Growth of maps, distortion in groups and symplectic geometry

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Abstract

In the present paper we study two sequences of real numbers associated to a symplectic diffeomorphism:

• The uniform norm of the differential of its $n$-th iteration;

• The word length of its $n$-th iteration, where we assume that our diffeomorphism lies in a finitely generated group of symplectic diffeomorphisms.

We find lower bounds for the growth rates of these sequences in a number of situations. These bounds depend on the symplectic geometry of the manifold rather than on the specific choice of a diffeomorphism. They are obtained by using recent results of Schwarz on Floer homology. As an application, we prove non-existence of certain non-linear symplectic representations for finitely generated groups.

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1 Introduction and main results

1.1 Growth and distortion

Given a diffeomorphism $f$ of a smooth compact connected manifold $M$, define its growth sequence

$$\Gamma_n(f) = \max \left( \max_{x \in M} |d_x f^n|, \max_{x \in M} |d_x f^{-n}| \right), \ n \in \mathbb{N}.$$ 

Here $|d_x f|$ stands for the operator norm of the differential $d_x f$ calculated with respect to a Riemannian metric on $M$. Though the explicit value of $\Gamma_n(f)$ depends on the choice of metric, the appropriately defined growth type is an invariant of $f$ under conjugations in Diff$(M)$. Here is the definition. Given two positive sequences $a_n$ and $b_n$, we write $a_n \preceq b_n$ if there exists $c > 0$ so that $a_n \geq cb_n$ for all $n \in \mathbb{N}$, and $a_n \sim b_n$ if $a_n \geq b_n$ and $b_n \geq a_n$. With this language the growth type of a diffeomorphism $f$ is simply the equivalence class of the sequence $\Gamma_n(f)$.

The interest to the growth type is caused by a number of reasons. Using this notion, one can imitate the fundamental trichotomy hyperbolic-parabolic-elliptic in the context of diffeomorphisms (cf. [HaK]). We say that $f$ is hyperbolic if $\Gamma_n(f)$ is growing exponentially fast, $f$ is elliptic if $\Gamma_n(f)$ is bounded and $f$ is parabolic otherwise. This definition of course agrees with the classical one for Möbius transformations acting on the circle. Sometimes the type of a diffeomorphism reflects its important dynamical features. Here are several examples: Existence of an invariant measure with a positive Lyapunov exponent yields hyperbolicity. If $f$ can be included into an action of a compact group then it is elliptic. Integrable systems of classical mechanics often give rise to parabolic diffeomorphisms with $\Gamma_n(f) \sim n$ (see an example in 1.4.C below).

An interesting problem, which goes back to D’Ambra and Gromov [DAG] is to study restrictions on the growth type for various classes of diffeomorphisms. In the present paper we give some answers in the symplectic category. In particular, we prove a lower bound for the growth type of symplectic
diffeomorphisms of closed symplectic manifolds with vanishing $\pi_2$ (see 1.4 below). If $M$ is a closed oriented surface endowed with an area form $\omega$, the lower bounds for the growth type look as follows. Denote by $\text{Symp}_0(M, \omega)$ the group of all area-preserving diffeomorphisms of $M$ isotopic to the identity map $\mathbb{I}$.

**Theorem 1.1.A** Let $M$ be a closed oriented surface of genus $\geq 2$. Then $\Gamma_n(f) \geq n$ for every $f \in \text{Symp}_0(M, \omega) \setminus \{\mathbb{I}\}$.

**Theorem 1.1.B** [PS]. Let $M = \mathbb{T}^2$ be the 2-torus, and let $f \in \text{Symp}_0(\mathbb{T}^2, \omega) \setminus \{\mathbb{I}\}$ be a symplectic diffeomorphism with a fixed point. Then $\Gamma_n(f) \geq n$.

We refer to 1.4 and 1.5 below for further discussion and generalizations to higher dimensions.

From a different, more geometric, viewpoint the function

$$\log \Gamma_1 : \text{Diff}(M) \to [0; +\infty)$$

is a pseudo-norm on $\text{Diff}(M)$. Then the growth type of $f$ reflects the distortion of the cyclic subgroup $\{f^n\} \subset \text{Diff}(M)$ with respect to this pseudo-norm. This observation serves as a motivation for the study of distortion in finitely generated groups of symplectic diffeomorphisms (see 1.6 and §4 below). Here is a sample result for surfaces. We write $||f||$ for the word length of an element of a finitely generated group.

**Theorem 1.1.C** Let $M$ be a closed oriented surface of genus $\geq 2$, and let $\mathcal{G} \subset \text{Symp}_0(M, \omega)$ be a finitely generated subgroup. Then $||f^n|| \geq \sqrt{n}$ for every $f \in \mathcal{G} \setminus \{\mathbb{I}\}$.

Theorem [1.1.C] and related results (see 1.6 and §4 below) have a number of applications to the Zimmer program of studying non-linear representations of
discrete groups $\mathbb{Z}$. We refer to papers [111,112,113,114,115] for recent
exciting developments in this direction. Our first applications deal with
the group $G$ of all (not necessarily isotopic to identity) smooth symplectic
diffeomorphisms of a closed oriented surface $M$ of genus $\geq 2$. The next
corollary was explained to us by Marc Burger.

**Corollary 1.1.D** Let $G$ be an irreducible non-uniform lattice in a semisim-
ple real Lie group of real rank at least two. Assume that the Lie group is
connected, without compact factors and with finite center. Then every homo-
morphism $G \to G$ has finite image.

The proof is given in 1.6 below. A prototype example of such a lattice is
$SL(n,\mathbb{Z}) \subset SL(n,\mathbb{R})$ for $n \geq 3$.

Next, consider the Baumslag-Solitar group

$$BS(q,p) = \langle a,b \mid a^q = ba^p b^{-1} \rangle,$$

where $q, p \in \mathbb{Z}$, $q \neq 0$, $p \neq 0$, $|p| < |q|$.

**Theorem 1.1.E** For every homomorphism $\phi : BS(q,p) \to G$ the element
$\phi(a)$ is of finite order.

The proof is given in 4.7 below. In 1.6 the reader will find generalizations
to actions in higher dimensions. We refer to [116] for the study of actions of
$BS(q,p)$ on 1-dimensional manifolds.

This circle of problems is quite sensitive to the class of smoothness of
diffeomorphisms in question, see for instance [117] for results on real-analytic
actions of lattices on surfaces. Throughout the paper we work with $C^\infty$-
diffeomorphisms, however our results and proofs should be valid in the $C^1$-
case (see 4.6 below for an outline of such an extension).

Our approach to growth and distortion is based on some properties of
the action spectrum of Hamiltonian diffeomorphisms which were obtained
by Schwarz [118] with the use of Floer homology. Interestingly enough, the
bounds we get in higher dimensions substantially depend on the fundamental
group of $M$. In order to state our results we need the notion of the symplectic filling function (cf. [G2], [Si]) which will be introduced right now.

### 1.2 Symplectic filling function

Let $(M, \omega)$ be a closed symplectic manifold with $\pi_2(M) = 0$. Denote by $\tilde{\omega}$ the lift of the symplectic structure $\omega$ to the universal cover $\tilde{M}$ of $M$. The condition $\pi_2(M) = 0$ guarantees that $\tilde{\omega}$ is exact on $\tilde{M}$. Let $L$ be the space of all 1-forms on $\tilde{M}$ whose differential equals $\tilde{\omega}$. Fix any Riemannian metric, say $\rho$, on $M$ and write $\tilde{\rho}$ for its lift to $\tilde{M}$. Pick a point $x \in \tilde{M}$ and denote by $B(s)$ the Riemannian ball of radius $s > 0$ centered at $x$. Put

$$u(s) = \inf_{\alpha \in L} \sup_{z \in B(s)} |\alpha_z|_{\tilde{\rho}}.$$

Clearly the function $s \mapsto su(s)$ is strictly increasing. Let $v : (0; +\infty) \to (0; +\infty)$ be its inverse. We call $v$ the symplectic filling function of $(M, \omega)$. It is easy to check (see 3.1 below) that if $v'$ is the symplectic filling function associated to another Riemannian metric $\rho'$ on $M$ and a base point $x' \in \tilde{M}$ then $c^{-1}v \leq v' \leq cv$ for some $c > 0$ (which is denoted by $v \sim v'$).

**Example 1.2.A** Consider the standard symplectic torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the symplectic form $\omega = \sum_{j=1}^{n} dp_j \wedge dq_j$. We claim that $u(s) \sim s$, and therefore $v(s) \sim \sqrt{s}$. Indeed, take $x = 0 \in \mathbb{R}^{2n}$, and let $\rho$ be the Euclidean metric. Then $\tilde{\omega} = d\left(\sum_{j=1}^{n} p_j dq_j\right)$, so

$$u(s) \leq \sup_{|p|^2 + |q|^2 \leq s^2} \left|\sum_{j=1}^{n} p_j dq_j\right| = \sup_{|p|^2 + |q|^2 \leq s^2} |p| = s.$$

On the other hand, consider the 2-disc $D(s)$ of radius $s$ in the $(p_1, q_1)$-plane. Note that for every primitive $\alpha \in L$

$$\text{length}\partial D(s) \cdot \sup_{z \in B(s)} |\alpha_z| \geq \int_{\partial D(s)} \alpha = \int_{D(s)} \tilde{\omega} = \pi s^2,$$
and hence 

\[ \sup_{z \in B(s)} |\alpha_z| \geq \frac{\pi s^2}{2\pi s} = \frac{s}{2}. \]

Thus \( u(s) \geq s/2 \) and the claim follows.

**Example 1.2.B** Let \((M, \omega)\) be a closed oriented surface of genus \( \geq 2 \). We claim that \( u(s) \) is bounded, and therefore \( v(s) \sim s \). To prove the claim, represent \( M \) as \( \mathbb{H}/K \), where \( \mathbb{H} = \{ p + iq \in \mathbb{C} \mid q > 0 \} \) is the hyperbolic upper half-plane, and \( K \) is a discrete group of isometries. The hyperbolic metric \( \tilde{\rho} \) on \( \mathbb{H} \) is given by \((dp^2 + dq^2)/q^2\). Assume without loss of generality that the lift \( \tilde{\omega} \) of the symplectic form coincides with the hyperbolic area form: \( \tilde{\omega} = (dp \wedge dq)/q^2 \). Note that \( \tilde{\omega} = d\alpha \) for \( \alpha = dp/q \). Take \( z = p + iq \in \mathbb{H} \) and calculate

\[ |\alpha_z|_{\tilde{\rho}} = \sup_{(\xi, \eta) : \xi^2 + \eta^2 = q^2} \frac{|\eta|}{q} = 1. \]

Hence \( u(s) \leq 1 \) and the claim follows. This example motivates the next definition.

**Definition 1.2.C** A closed symplectic manifold \((M, \omega)\) with \( \pi_2(M) = 0 \) is called *symplectically hyperbolic* if the function \( u(s) \) is bounded (and therefore \( v(s) \sim s \)). For instance surfaces of genus \( \geq 2 \), their products, and, more generally, Kähler hyperbolic manifolds [G1], are symplectically hyperbolic.

### 1.3 Fixed points of symplectic diffeomorphisms

Let \((M, \omega)\) be a closed symplectic manifold. Denote by \( \text{Symp}_0(M, \omega) \) the identity component of the group of all symplectic diffeomorphisms of \( M \).

**Definition.** Let \( x \) be a fixed point of a symplectic diffeomorphism \( f \in \text{Symp}_0(M, \omega) \). We say that \( x \) is of *contractible type* if there exists a path \( \{f_t\}_{t \in [0,1]} \) of symplectic diffeomorphisms with \( f_0 = \mathbb{1}, f_1 = f \) such that the loop \( \{f_t x\}_{t \in [0,1]} \) is contractible in \( M \).
Example 1.3.A Every Hamiltonian diffeomorphism of a closed symplectic manifold with $\pi_2 = 0$ has a fixed point of contractible type. This is an immediate consequence of Floer’s famous proof of the Arnold conjecture (see [Fl]; we refer to [P1] for background on Hamiltonian diffeomorphisms and to 2.2 below for the definition.)

Example 1.3.B Consider the standard symplectic torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ endowed with the symplectic form $dp \wedge dq$. Let $f \in \text{Symp}_0(\mathbb{T}^{2n})$ be a symplectic diffeomorphism. We claim that every fixed point $x$ of $f$ is of contractible type. Indeed, take a path $\{f_t\}_{t \in [0;1]}$ of symplectic diffeomorphisms of $\mathbb{T}^{2n}$ such that $f_0 = \mathbb{1}$ and $f_1 = f$. Let $\tilde{f}_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be its lift to the universal cover. Pick a lift $\tilde{x}$ of $x$. Then $\tilde{f}_1 \tilde{x} = \tilde{x} + e$ for some $e \in \mathbb{Z}^{2n}$. Consider a symplectic flow $g_t : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ defined by

$$g_t z = z - te \pmod{1}, \quad z \in \mathbb{T}^{2n}.$$  

Put $h_t = g_t f_t$ for $t \in [0;1]$. Since $g_0 = g_1 = \mathbb{1}$ we have $h_0 = \mathbb{1}$ and $h_1 = f$. Further, the lift $\{\tilde{h}_t\}$ of $\{h_t\}$ to $\mathbb{R}^{2n}$ satisfies $\tilde{h}_t \tilde{x} = \tilde{f}_t \tilde{x} - te$, and in particular $\tilde{h}_1 \tilde{x} = \tilde{x}$. Hence the loop $\{h_t x\}_{t \in [0;1]}$ is contractible on $\mathbb{T}^{2n}$. This completes the proof of the claim.

Example 1.3.C Let $M$ be a closed surface of genus $\geq 2$. We claim that every $f \in \text{Symp}_0(M)$ has a fixed point of contractible type. Indeed, write $M = \mathbb{H}/K$ were $\mathbb{H}$ is the hyperbolic upper half-plane and $K$ is a discrete group of Möbius transformations. Take a symplectic isotopy $\{f_t\}_{t \in [0;1]}$ with $f_0 = \mathbb{1}$, $f_1 = f$ and lift it to $\mathbb{H}$. We get an isotopy $\{\tilde{f}_t\}$. Put $\tilde{f} = \tilde{f}_1$. Clearly it suffices to show that $\tilde{f}$ has a fixed point on $\mathbb{H}$. Assume on the contrary that $\tilde{f}x \neq x$ for all $x \in \mathbb{H}$. Define a smooth vector field $\eta$ on $\mathbb{H}$ as follows: $\eta(x)$ is the unit tangent vector to the geodesic ray $[x; \tilde{f}(x))$. Since $\tilde{f}$ commutes with elements of $K$, the field $\eta$ is $K$-invariant and hence descends to a unit vector field on $M$. Since the Euler characteristic of $M$ does not vanish we get a contradiction. The claim follows.
1.4 A lower bound for the growth type

We are ready now to state our main result. Let $(M, \omega)$ be a closed connected symplectic manifold with $\pi_2(M) = 0$. Let $v$ be its symplectic filling function.

**Theorem 1.4.A** Let $f \in \text{Symp}_0(M,\omega) \setminus \{1\}$ be a symplectic diffeomorphism with a fixed point of contractible type. Then $\Gamma_n(f) \succeq v(n)$.

The proof is given in §3 below. Several remarks are in order.

1.4.B. As a consequence we get that $\Gamma_n(f) \succeq \sqrt{n}$ if $(M, \omega)$ is the standard symplectic torus (see 1.2.A). Theorem 1.1.B refines this estimate for $n = 2$. Further, $\Gamma_n(f) \succeq n$ if $(M, \omega)$ is symplectically hyperbolic (see 1.2.B, 1.2.C). In particular, Theorem 1.1.A is an immediate consequence of 1.4.A, 1.3.C and 1.1.B. Patrice LeCalvez informed us that he can prove Theorems 1.1.A and 1.1.B by a different method. In higher dimensions, however, I am not aware of any alternative to the symplecto-topological approach.

1.4.C. On every compact symplectic manifold $(M, \omega)$ one can find a Hamiltonian diffeomorphism $f \neq 1$ such that $\Gamma_n(f) \sim n$. Indeed, suppose that $\dim M = 2m$. Put $N_\epsilon = T^m \times D^m(\epsilon)$, $\epsilon > 0$ where $T^m = \mathbb{R}^m(q_1, \ldots, q_m)/\mathbb{Z}^m$ and $D^m(\epsilon) = \{ p \in \mathbb{R}^m : |p| \leq \epsilon \}$. Endow $N_\epsilon$ with the standard symplectic form $\Omega = \sum_{i=1}^m dp_i \wedge dq_i$. It is well known that for $\epsilon > 0$ small enough there exists a symplectic embedding $j : (N_\epsilon, \Omega) \rightarrow (M, \omega)$. Fix such $\epsilon$ and $j$, and take any function $H : D^m(\epsilon) \rightarrow \mathbb{R}$ which vanishes near $\partial D^m(\epsilon)$, and is not identically zero. Consider the Hamiltonian flow of $H = H(p)$ on $N_\epsilon$:

$$h_t(p, q) = (p, q + t \frac{\partial H}{\partial p}(p))$$

(this flow represents the simplest integrable system of Classical Mechanics). Obviously, $\Gamma_n(h_1) = \Gamma_1(h_n) \sim n$. Now define a Hamiltonian diffeomorphism $f : M \rightarrow M$ as follows:

$$f \equiv 1$$ on $M \setminus j(N_\epsilon)$, and $f = jh_1j^{-1}$ on $j(N_\epsilon)$. 

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Clearly, $\Gamma_n(f) \sim n$ as required.

We conclude that the inequality $\Gamma_n(f) \geq n$ on $\mathbb{T}^2$ and on any symplectically hyperbolic manifold is sharp and cannot be improved.

**Open problem.** Find a closed symplectic manifold $M$ with $\pi_2(M) = 0$ and a Hamiltonian diffeomorphism $f \neq \mathbb{I}$ of $M$ which violates the inequality $\Gamma_n(f) \geq n$.

**1.4.D.** The condition $\pi_2(M) = 0$ cannot be removed. Indeed consider the standard $S^1$-action $\{f_t\}$ on $S^2$ (rotation around the vertical axis). For each $t$ the diffeomorphism $f_t$ is Hamiltonian, has fixed points of contractible type at the poles, and obviously the sequence $\Gamma_n(f_t)$ is bounded. (Still, it sounds likely that $\Gamma_n(f) \geq n$ for every non-identical area-preserving map $f : S^2 \to S^2$ with at least 3 fixed points.)

**1.4.E.** The fixed point condition of Theorem 1.4.A cannot be removed. Indeed, $\{\Gamma_n(f)\}$ is bounded if $f$ is a translation of $\mathbb{T}^2$. More sophisticated counterexamples are given by the next theorem.

**Theorem 1.4.F** For every $\beta \in (0; 1)$ there exists a $C^\infty$-function $\psi : S^1 \to \mathbb{R}$ and an irrational number $\alpha$ so that the map

$$f : \mathbb{T}^2 \to \mathbb{T}^2, (x, y) \mapsto (x + \alpha, y + \psi(x))$$

satisfies the following:

(i) $\Gamma_n(f) \leq n^\beta \log n$;

(ii) $\Gamma_{n_i}(f) \geq n_i^\beta$ for a subsequence $n_i \to +\infty$.

The proof is quite technical and will be given elsewhere. Let us emphasize that the growth bound 1.4.A is in general not true if $f$ has fixed points but none of them is of contractible type (see Appendix).

**1.4.G.** Let $G$ be a group of diffeomorphisms acting on a compact manifold $M$. We say that an increasing function $w : (0; +\infty) \to (0; +\infty)$, $w(s) \to +\infty$ as $s \to +\infty$ is a **growth bound** for $G$ if $\Gamma_n(f) \geq w(n)$ for all $f \in G \setminus \{\mathbb{I}\}$. 
Examples:

1.4.G(i). Suppose that $G$ is the group of Hamiltonian diffeomorphisms of a closed symplectic manifold $M$. Then the symplectic filling function $v(s)$ gives a growth bound for $G$ in view of Theorem [1.4.A]. One can take $w(s) = s$ for a symplectically hyperbolic manifold.

1.4.G(ii). Fix a point $x_0$ on a closed symplectic manifold $M$. Let $G$ be the identity component of the group $\{ f \in \text{Symp}_0(M) \mid f(x_0) = x_0 \}$. The conclusions of the previous example are still valid for $G$ in view of [1.4.A].

1.4.G(iii). Fix a point $x_0$ on any compact manifold $M$. Let $G$ be the identity component of the group $\{ f \in \text{Diff}_0(M) \mid f(x_0) = x_0 \}$. It is proved in [PSc] that $G$ admits no growth bound. In fact, for every increasing function $w : (0; +\infty) \rightarrow (0; +\infty)$ with $w(s) \rightarrow +\infty$ as $s \rightarrow +\infty$ there exists a diffeomorphism $f \in G\{1\}$ and a subsequence $n_i \rightarrow +\infty$ so that $\Gamma_{n_i}(f) \preceq w(n_i)$.

1.4.G(iv). Let $M$ be a closed manifold endowed with a volume form, and let $x_0$ be a point on $M$. Define $G$ as the identity component of the group of all volume-preserving diffeomorphisms $f$ with $f(x_0) = x_0$.

**Open problem.** Does there exist a closed manifold $M$ of dimension $\geq 3$ such that $G$ admits a growth bound?

1.5 Growth and propagation

The growth type of a diffeomorphism $f$ of a compact manifold $M$ is related to the dynamics of its lift $\tilde{f}$ to the universal cover $\widetilde{M}$. Fix such a lift and consider a fundamental domain $D$ of $\widetilde{M}$. Take a Riemannian metric $\rho$ on $M$ and write $\tilde{\rho}$ for its lift to $\widetilde{M}$. For purposes of our discussion assume that $\tilde{f}x = x$ for some $x \in D$. Consider the quantity

$$d_n(f) = \sup_{z \in D} \text{distance}_{\tilde{\rho}}(x, \tilde{f}^nz),$$

where $d_n(f)$ represents the growth rate of the sequence $\{\tilde{f}^n\}$. This quantity provides a measure of how the volume of the fundamental domain changes under the action of $f$. In the study of growth, one often considers sequences of fundamental domains that are nested, meaning that $D_{n+1} \subset D_n$. The growth rate $d_n(f)$ can then be used to define a growth function $g_f(s)$, which is the supremum of $d_n(f)$ over all such sequences. The growth function captures the rate at which the volume of the domain increases under repeated applications of $f$.

Such growth functions are fundamental in the study of dynamical systems and have applications in various fields, including geometric group theory and the study of geometric structures on manifolds.
which measures the rate of propagation of the trajectories of $\tilde{f}$ on the universal cover (cf. [P2],[BPS]). Obviously,

\[(1.5.A) \quad \Gamma_n(f) \succeq d_n(\tilde{f}).\]

In some situations this inequality combined with an information about fixed points of $f$ gives rise to a lower bound for the growth type of $f$. Assume for instance that there exists a point $x' \in D$ such that $\tilde{f} x' = T x'$, where $T$ is the deck transformation corresponding to an element $\alpha \in \pi_1(M)$. Then obviously $d_n(\tilde{f}) \succeq \|\alpha^n\|$ where $\|\|$ stands for the word length in $\pi_1(M)$. Thus $\Gamma_n(f) \succeq \|\alpha^n\|$.

Let $(M,\omega)$ be a closed symplectic manifold with $\pi_2 = 0$. Consider a Hamiltonian diffeomorphism $f$ of $M$. In his famous work [Fl] on Arnold’s conjecture Floer proved that every Hamiltonian path $\{f_t\}$ with $f_0 = 1$ and $f_1 = f$ has a contractible 1-periodic orbit. Consider the lift $\tilde{f}$ of $f$ to the universal cover $\tilde{M}$ associated to the path $\{f_t\}$. It follows that the lift $\tilde{f}$ does not depend on the specific choice of a Hamiltonian path joining the identity with $f$. We will call $\tilde{f}$ the canonical lift of $f$. Note also that a contractible 1-periodic orbit of $\{f_t\}$ corresponds to a fixed point of $\tilde{f}$. Thus the sequence $\{d_n(\tilde{f})\}$ is well defined and its growth type is an invariant of the Hamiltonian diffeomorphism $f$.

We say that $f$ does not propagate if $d_n(\tilde{f})$ is bounded. One may have the impression that this property is not too useful since the estimate (1.5.A) becomes trivial. Paradoxically, in the Hamiltonian category the lack of propagation guarantees at least linear growth of the differential.

**Theorem 1.5.B** Let $f \neq 1$ be a Hamiltonian diffeomorphism of a closed symplectic manifold with $\pi_2(M) = 0$. Assume that $f$ does not propagate. Then $\Gamma_n(f) \succeq n$.

The proof is given in §3 below.
1.6 Symplectic filling function and distortion in finitely generated groups

Let \((M, \omega)\) be a closed connected symplectic manifold with \(\pi_2(M) = 0\), and let \(\mathcal{G}\) be a finitely generated subgroup of \(\text{Symp}_0(M, \omega)\). Fix a system of generators in \(\mathcal{G}\) and write \(\|f\|\) for the word length of an element \(f \in \mathcal{G}\). We are interested in the distortion of the cyclic subgroup \(\{f^n\} \subset \mathcal{G}\), that is in the growth type of the sequence \(\|f^n\|\) (see Ch. 3 in [G2] for discussion on the distortion). Our main results in this direction are given in the next two theorems.

**Theorem 1.6.A** Let \(\mathcal{G} \subset \text{Ham}(M, \omega)\) be a finitely generated subgroup. Then
\[
\|f^n\| \geq v(n) \text{ for all } f \in \mathcal{G}\setminus\{1\}.
\]

**Theorem 1.6.B** Assume that the fundamental group \(\pi_1(M)\) has trivial center. Let \(\mathcal{G} \subset \text{Symp}_0(M, \omega)\) be a finitely generated subgroup. Then
\[
\|f^n\| \geq \min(v(n), \sqrt{n})
\]
for all \(f \in \mathcal{G}\setminus\{1\}\).

The assumption on \(\pi_1\) in Theorem 1.6.B cannot be removed. For instance, the group \(\text{Symp}_0(T^{2m}, dp \wedge dq)\) contains a translation \(f\) of a finite order, thus \(\{\|f^n\|\}\) is a bounded sequence. Theorem [1.1.C] is an immediate consequence of 1.6.B. Theorems [1.6.A] and [1.6.B] are proved in §4. Some remarks are in order.

**1.6.C.** Let \((M, \omega)\) be a symplectically hyperbolic manifold. Then \(v(n) \sim n\) and we conclude that every element \(f \neq 1\) of a finitely generated subgroup \(\mathcal{G} \subset \text{Ham}(M, \omega)\) is undistorted: \(\|f^n\| \sim n\). This follows from [1.6.A] and the obvious upper bound \(\|f^n\| \leq n\).

**1.6.D.** Theorem 1.6.A can be rephrased as follows. Let \(\mathcal{G}\) be an abstract finitely generated discrete subgroup. Assume that \(g \in \mathcal{G}\) is an element such
that \( \|g^{n_i}\|/v(n_i) \to 0 \) as \( n_i \to +\infty \). Then \( \phi(g) = 1 \) for every homomorphism \( \phi : \mathcal{G} \to \text{Ham}(M, \omega) \). Theorem 1.6.B, of course, admits a similar reformulation.

Let us illustrate this statement. Following [LMR] we call an element \( x \in \mathcal{G} \) a U-element if it is of infinite order and
\[
\liminf \frac{\log \|x^n\|}{\log n} = 0.
\]

U-elements appear in a number of interesting groups. We present two examples.

**Example 1.6.E.** Consider the Baumslag-Solitar group
\[
BS(q, p) = \langle a, b \mid a^q = ba^p b^{-1} \rangle,
\]
where \( q, p \in \mathbb{Z} \), \( q \neq 0 \), \( p \neq 0 \), \( |p| < |q| \).

It is known that the element \( a \) has logarithmic distortion: \( \|a^n\| \lesssim \log(n+1) \).

Here is a simple argument which we learned from Zlil Sela. Assume for simplicity that \( q > p > 0 \). Define a function \( \varphi : \mathbb{N} \to \mathbb{Z} \) as follows: \( \varphi(k) \) is the integer lying in \( \left[ \frac{p}{q} k - 1; \frac{p}{q} k \right] \). Note that \( a^{qk} = ba^{pk} b^{-1} = ba^{q\varphi(k)+i_k} b^{-1} \), where \( i_k \in [0; q-1] \). Put \( u_k = \|a^{qk}\| \). Then \( u_k \leq q + 1 + u_{\varphi(k)} \) for all \( k \in \mathbb{N} \) which readily yields \( u_k \leq \text{const} \cdot \log(k+1) \) for all \( k \in \mathbb{N} \). But \( \|a^{qk+j}\| \leq q - 1 + u_k \) for all \( k \in \mathbb{N}, j \in \{1; \cdots; q-1\} \). This yields \( \|a^n\| \lesssim \log(n+1) \) as required.

**Example 1.6.F.** Let \( \mathcal{G} \) be an irreducible lattice in a semisimple real Lie group of real rank at least two. We assume that the Lie group is connected, without compact factors and with finite center. If \( \mathcal{G} \) is non-uniform, a classical result due to Kazhdan and Margulis implies that it contains a unipotent element of infinite order. Lubotzky-Mozes-Raghunathan [LMR] proved that it must be a U-element.

**Corollary 1.6.G** Let \( G \) be one of the following groups:

(i) \( \text{Ham}(M, \omega) \) where \( (M, \omega) \) is either the standard symplectic \( 2m \)-dimensional torus or a symplectically hyperbolic manifold;

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(ii) \( \text{Symp}_0(M, \omega) \) where \( M \) is a product of surfaces of genus \( \geq 2 \) endowed with the split symplectic structure.

Let \( \mathcal{G} \) be a finitely generated group containing a \( U \)-element \( x \). Then \( \phi(x) = 1 \) for every homomorphism \( \phi : \mathcal{G} \to G \).

For \( \mathcal{G} = B(q, p) \) with \( p | q \) and \( G = \text{Ham}(M, \omega) \) this result can be extended to all closed manifolds with \( \pi_2 = 0 \).

**Proposition 1.6.H** Assume that \( p \) divides \( q \). Let \( (M, \omega) \) be an arbitrary closed symplectic manifold with \( \pi_2(M) = 0 \). Then \( \phi(a) = 1 \) for every homomorphism \( \phi : BS(q, p) \to \text{Ham}(M, \omega) \).

This result is easier than the previous ones. It is proved in 2.6 below.

**1.6.I. Proof of Corollary 1.1.D:**

Let \( G \) be the group of all symplectic diffeomorphisms of a closed oriented surface \( M \), and let \( \mathcal{G} \) be a non-uniform lattice as in 1.6.F. Consider any homomorphism \( \phi : \mathcal{G} \to G \). It is proved in [FM] (cf. [KM]) that there exists a normal subgroup \( K \subset \mathcal{G} \) of finite index such that \( \phi(K) \subset \text{Symp}_0(M) \).

Denote by \( x \) a \( U \)-element of \( \mathcal{G} \) (see [LMR] and 1.6.F). Choose \( p \in \mathbb{N} \) such that \( x^p \in K \). It follows from Proposition 2.2 of [LMR] that \( x^p \) is a \( U \)-element in \( K \). Applying Corollary 1.6.G we conclude that \( \phi(x^p) = 1 \). Therefore the kernel \( Q \) of \( \phi \) contains an element of infinite order, and thus \( Q \subset \mathcal{G} \) is an infinite normal subgroup. By Margulis finiteness theorem we get that \( Q \) is of finite index in \( \mathcal{G} \), and hence \( \phi \) has finite image. \( \Box \)

**1.6.J.** Let \( \mathcal{G} \) be a non-uniform irreducible lattice as in 1.6.F. Let \( (M, \omega) \) be a closed symplectic manifold with \( \pi_2 = 0 \). Suppose that the symplectic filling function \( v(s) \) of \( M \) satisfies \( v(s) \geq cs^\epsilon \) for some \( c, \epsilon > 0 \). (Think, for instance, about the standard symplectic torus \( T^{2m} \)). Write \( G = \text{Symp}_0(M, \omega) \) and \( G_0 = \text{Ham}(M, \omega) \). We claim that every homomorphism \( \phi : \mathcal{G} \to G \) has finite image. Here is the proof. It is known (see e.g. [Ba], [MS], [LMP]) that
there exists a countable subgroup $E \subset H^1(M, \mathbb{R})$ and a homomorphism\footnote{The subgroup $E$ is called the flux subgroup of $(M, \omega)$. It is known \cite{LMP} to be discrete provided $\pi_2(M) = 0$, though we do not use it. With the notation of Subsection 2.2 below $E = \text{Image}(\Delta)$. The homomorphism $\text{Flux}$ is defined in 2.2 in the simplest case when $E = \{0\}$. In the general case one modifies it in the obvious way.} \[ \text{Flux} : G \to H^1(M, \mathbb{R})/E \]

whose kernel equals $G_0$. It follows from Margulis finiteness theorem that the kernel $K$ of $\text{Flux} \circ \phi$ is a normal subgroup of finite index in $\mathcal{G}$. Note that $\phi(K)$ is contained in $G_0$.

Denote by $x$ a $U$-element of $\mathcal{G}$ (see \cite{LMR} and 1.6.F). Choose $p \in \mathbb{N}$ such that $x^p \in K$. It follows from Proposition 2.2 of \cite{LMR} that $x^p$ is a $U$-element in $K$. Applying 1.6.D we conclude that $\phi(x^p) = \mathbb{I}$. Therefore the kernel $Q$ of $\phi$ contains an element of infinite order, and thus $Q \subset \mathcal{G}$ is an infinite normal subgroup. By Margulis finiteness theorem we get that $Q$ is of finite index in $\mathcal{G}$, and hence $\phi$ has finite image. The claim follows.$\square$

1.6.K. The previous claim is in general not true when one replaces the group $\text{Symp}_0(M, \omega)$ with $\text{Diff}_0(M)$. One can give a counterexample already when $M$ is the 2-torus $\mathbb{T}^2$. Consider the group $\mathcal{G} = \text{PSL}(2, \mathbb{Z}[\sqrt{2}])$ of projectivized matrices with determinant 1 and with the entries of the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Z}$. Call a number $a - b\sqrt{2}$ to be conjugate to $a + b\sqrt{2}$ in $\mathbb{Z}[\sqrt{2}]$. Given a matrix $A \in \mathcal{G}$, denote by $\bar{A}$ the matrix with conjugate entries. Consider a monomorphism

$$\psi : \mathcal{G} \to \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}), \ A \mapsto (A, \bar{A}).$$

One can show (see 2.12 in \cite{LMR} and \cite{vdG}) that $\psi(\mathcal{G})$ is an irreducible non-uniform lattice in the Lie group $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$. The real rank of this Lie group equals 2. Therefore it follows from 1.6.J that every homomorphism $\mathcal{G} \to \text{Symp}_0(\mathbb{T}^2)$ has finite image. On the other hand, $\mathcal{G}$ embeds to $\text{Diff}_0(\mathbb{T}^2)$ in the obvious way:

$$\mathcal{G} \to \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}) \to \text{Diff}_0(S^1) \times \text{Diff}_0(S^1) \to \text{Diff}_0(S^1 \times S^1).$$
1.6.L. Let us emphasize that the group of Hamiltonian diffeomorphisms of a closed symplectic manifold with $\pi_2(M) = 0$ has no torsion. This follows immediately from Theorem 1.6.A (see also the paragraph following Proposition 2.6.A below for a direct proof). If in addition $\pi_1(M)$ has trivial center then the same is valid for the group $\text{Symp}_0(M, \omega)$ (use Theorem 1.6.B). Looking at groups $\text{Ham}(S^2)$ and $\text{Symp}_0(T^2)$ which contain torsion elements we conclude that the topological assumptions cannot be removed.

2 A review of the symplectic action

In this section we sum up some known facts on the symplectic action which will be used for the proof of results stated in the introduction. Unless otherwise stated, all symplectic manifolds below are assumed to be connected.

2.1 Action difference

Let $(P, \Omega)$ be a symplectic manifold with $\pi_1(P) = \pi_2(P) = 0$ (and hence $P$ is necessarily non-closed). Let $\varphi : P \to P$ be a symplectic diffeomorphism. Given two fixed points $x$ and $y$ of $\varphi$, define their action difference $\delta(\varphi; x, y)$ as follows. Take any curve $\gamma : [0; 1] \to P$ with $\gamma(0) = x$, $\gamma(1) = y$ and take a disc $\Sigma \subset P$ with $\partial \Sigma = \varphi \gamma - \gamma$ (here $\gamma$ is considered as a 1-chain in $P$). Put

$$\delta(\varphi; x, y) = \int_{\Sigma} \Omega. \quad (2.1.A)$$

Let us verify that this definition is correct, that is $\delta(\varphi; x, y)$ does not depend on the choice of $\gamma$ and $\Sigma$. Indeed, let $\gamma', \Sigma'$ be another choice. Since $P$ is simply connected, there exists a disc $\Delta$ with $\partial \Delta = \gamma' - \gamma$. Note that the 2-chain $\Pi = \Sigma - \Sigma' + \varphi \Delta - \Delta$ represents a 2-sphere in $P$, and hence $\int_{\Pi} \Omega = 0$ since $\pi_2(P) = 0$. But this yields

$$\int_{\Sigma} \Omega - \int_{\Sigma'} \Omega = \int_{\Delta} \Omega - \int_{\varphi \Delta} \Omega = 0$$

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since \( \varphi \) preserves \( \Omega \), and the claim follows. The action difference behaves nicely under iterations of \( \varphi \).

**Proposition 2.1.B** \( \delta(\varphi^n; x, y) = n\delta(\varphi; x, y) \) for all \( n \in \mathbb{Z} \).

**Proof.** Assume for simplicity that \( n > 0 \). Take a curve \( \gamma \) joining \( x \) with \( y \), and let \( \Sigma \) be a disc with \( \partial \Sigma = \varphi \gamma - \gamma \). Put \( \Delta = \Sigma + \varphi \Sigma + \cdots + \varphi^{n-1} \Sigma \). Clearly \( \Delta \) is a disc with \( \partial \Delta = \varphi^n \gamma - \gamma \). Then \( \delta(\varphi^n; x, y) = \int_{\Delta} \Omega = n \int_{\Sigma} \Omega = n\delta(\varphi; x, y) \).

\( \square \)

The main result of the present section is as follows.

**Theorem 2.1.C** Let \((M, \omega)\) be a closed symplectic manifold with \( \pi_2(M) = 0 \), and let \( f \in \text{Symp}_0(M, \omega) \) be a symplectic diffeomorphism with a fixed point of contractible type. Then \( f \) admits a lift \( \widetilde{f} \) to the universal cover \((\widetilde{M}, \widetilde{\omega})\) of \((M, \omega)\) such that \( \delta(\widetilde{f}; x, y) \neq 0 \) for some fixed points \( x \) and \( y \) of \( \widetilde{f} \).

The proof is given in 2.5 below.

**Remark 2.1.D** If \( f \) as above is Hamiltonian, the lift \( \widetilde{f} \) coincides with the canonical lift of \( f \) defined in 1.5.

### 2.2 Symplectic and Hamiltonian

Let \((M, \omega)\) be a symplectic manifold (not necessarily closed) with \( \pi_2(M) = 0 \). Consider a path \( \{f_t\}_{t \in [0; 1]} \) of symplectic diffeomorphisms with \( f_0 = \text{id}, f_1 = f \).

Let \( \xi_t \) be the corresponding time-dependent vector field on \( M \):

\[
\frac{d}{dt} f_t x = \xi_t(f_t x) \quad \text{for all } x \in M, \ t \in [0; 1].
\]

Since the Lie derivative \( L_{\xi_t} \omega \) vanishes we get that the 1-forms \( \lambda_t = -i_{\xi_t} \omega \) are closed. Write \([\lambda_t]\) for the cohomology class of \( \lambda_t \). The quantity

\[
\text{Flux}\{f_t\} = \int_0^1 [\lambda_t]dt \in H^1(M, \mathbb{R})
\]

(2.2.A)
is called the flux of the path \( \{f_t\} \). A path \( \{f_t\} \) is called Hamiltonian if the 1-forms \( \lambda_t \) are exact for all \( t \). In this case there exists a smooth function \( F : M \times [0;1] \to \mathbb{R} \) so that \( \lambda_t = dF_t \), where \( F_t(x) \) stands for \( F(x,t) \). The function \( F \) is called the Hamiltonian function generating the flow \( \{f_t\} \). Note that \( F_t \) is defined uniquely up to an additive time-dependent constant.

A symplectic diffeomorphism \( f : M \to M \) is called Hamiltonian if there exists a Hamiltonian path \( \{f_t\}_{t \in [0;1]} \) with \( f_0 = 1 \) and \( f_1 = f \). Hamiltonian diffeomorphisms form a group denoted by \( \text{Ham}(M,\omega) \). The next statement is proved in [Ba],[MS].

**Proposition 2.2.B** Let \( (M,\omega) \) be a closed symplectic manifold. Let \( \{f_t\}_{t \in [0;1]} \) be a path of symplectic diffeomorphisms with \( f_0 = 1 \) and \( \text{Flux}\{f_t\} = 0 \). Then the diffeomorphism \( f_1 \) is Hamiltonian.

It is well known (see [Ba],[MS]) that \( \text{Flux}\{f_t\} \) does not change under a homotopy of the path \( \{f_t\} \) with fixed end points. Thus one can define a homomorphism

\[
\Delta : \pi_1(\text{Symp}_0(M,\omega)) \to H^1(M,\mathbb{R})
\]

by \( \Delta(a) = \text{Flux}\{f_t\} \), where \( \{f_t\} \) is a loop \( (f_0 = f_1 = 1) \) of symplectic diffeomorphisms representing an element \( a \in \pi_1(\text{Symp}_0(M,\omega)) \). Sometimes \( \Delta \) vanishes identically. In this case, consider a map

\[
\text{Flux} : \text{Symp}_0(M,\omega) \to H^1(M,\mathbb{R}) ,
\]

which sends a diffeomorphism \( f \in \text{Symp}_0(M,\omega) \) to \( \text{Flux}\{f_t\} \), where \( \{f_t\} \) is any symplectic path with \( f_0 = 1 \), \( f_1 = f \). The condition \( \Delta \equiv 0 \) guarantees that \( \text{Flux} \) is well defined. Moreover, it is a group homomorphism. We refer to [MS] for a detailed discussion of the flux. For the proof of Theorem 1.6.B we shall need the next result.

**Proposition 2.2.D** Assume that \( (M,\omega) \) is a closed symplectic manifold with \( \pi_2(M) = 0 \). Suppose in addition that the fundamental group \( \pi_1(M) \)
has trivial center. Then \( \Delta \) vanishes and hence the homomorphism

\[
\text{Flux} : \text{Symp}_0(M, \omega) \rightarrow H^1(M, \mathbb{R})
\]

is well defined.

**Proof:** This fact can be easily extracted e.g. from [LMP]. For the reader’s convenience we present the argument. Take any loop \( \{f_t\} \) of symplectic diffeomorphisms representing an element \( a \in \pi_1(\text{Symp}_0(M, \omega)) \). Fix a point, say \( y_\ast \), on \( M \) and denote by \( b \) the orbit \( \{f_t y_\ast\} \). Let \( \beta \in \pi_1(M, y_\ast) \) be the element represented by \( b \).

Let \( c : [0;1] \rightarrow M \) be any closed curve with \( c(0) = c(1) = y_\ast \). Write \( \gamma \) for the element represented by \( c \) in \( \pi_1(M, y_\ast) \) and \( \overline{\gamma} \) for the homology class of \( c \) in \( H_1(M, \mathbb{Z}) \).

Consider the map \( [0;1] \times [0;1] \rightarrow M, \)

\[
(s,t) \mapsto f_t(c(s)).
\]

It defines a 2-torus, say \( \Sigma \), in \( M \). Since \( b \) and \( c \) lie on the 2-torus the elements \( \beta \) and \( \gamma \) commute in \( \pi_1(M, y_\ast) \). This remains true for any choice of \( c \). Therefore \( \beta \) belongs to the center of the fundamental group. Since the center is trivial by our assumption, we conclude that \( \beta = 1 \). In other words, the orbits of \( \{f_t\} \) are contractible in \( M \).

It is well known (see [LMP], [MS]) that

\[
\langle \Delta(a), \overline{\gamma} \rangle = -\int_\Sigma \omega.
\]

Filling the cycle \( b \subset \Sigma \) by a 2-disc we get that the torus \( \Sigma \) is homologous to a 2-sphere in \( M \). Thus the integral above vanishes since \( \pi_2(M) = 0 \). This proves that \( \Delta \) vanishes. \( \square \)

**Remark 2.2.E** It follows from [2.2.3] that \( \text{Ker(Flux)} = \text{Ham}(M, \omega) \).
Remark 2.2.F The triviality of the center of $\pi_1$ has another consequence which will be used below. Observe that exactly the same argument as in the proof of 2.2.D shows that all orbits of any loop of diffeomorphisms of $M$ are contractible in $M$. Take any diffeomorphism $g \in \text{Diff}_0(M)$ and any path $\{g_t\}$ with $g_0 = 1$, $g_1 = g$. Consider the lift $\{\tilde{g}_t\}$ of this path to the universal cover $\tilde{M}$ so that $\tilde{g}_0 = 1$. The observation above implies that the lift $\tilde{g}_1$ does not depend on the choice of the path $\{g_t\}$. In particular, every $g \in \text{Diff}_0(M)$ has a canonical lift $\tilde{g}$ to $\tilde{M}$. Of course, if $g$ is a Hamiltonian diffeomorphism then $\tilde{g}$ coincides with its canonical lift defined in 1.5 above.

2.3 Symplectic action

Let $(M, \omega)$ be a symplectic manifold with $\pi_2(M) = 0$. Let $\{f_t\}$ be a Hamiltonian path with $f_1 = f$ generated by a Hamiltonian function $F : M \times [0; 1] \to \mathbb{R}$. Let $x$ be a fixed point of $f$ such that its orbit $\alpha = \{f_t x\}_{t \in [0;1]}$ is contractible in $M$. Take any 2-disc $\Sigma \subset M$ with $\partial \Sigma = \alpha$, and define the symplectic action

\[(2.3.A) \quad A(F, x) = \int_{\Sigma} \omega - \int_{0}^{1} F_t(f_t x) dt .\]

Since $\pi_2(M) = 0$ the integral $\int \omega$ does not depend on the choice of the disc $\Sigma$. The following deep fact is proved in [Sch] by using Floer homology.

**Proposition 2.3.B** Let $(M, \omega)$ be a closed symplectic manifold with $\pi_2(M) = 0$. Let $\{f_t\}$, $f_0 = 1$, $f_1 = f$ be a Hamiltonian path on $M$ generated by a Hamiltonian function $F$. Assume that $f \neq 1$. Then $f$ has a pair of fixed points $x$ and $y$ so that their orbits $\{f_t x\}$ and $\{f_t y\}$ are contractible and $A(F, y) - A(F, x) \neq 0$.

This proposition is the key ingredient from “hard” symplectic topology we use in this paper.
2.4 Action difference revisited

Let us return to the situation described in 2.1 above. Let \((P, \Omega)\) be a symplectic manifold with \(\pi_1(P) = \pi_2(P) = 0\). Note that any path \(\{\varphi_t\}\) of symplectic diffeomorphisms of \(P\) is automatically Hamiltonian since \(H^1(P, \mathbb{R}) = 0\). Take such a path and write \(\Phi\) for the Hamiltonian function. Write \(\varphi = \varphi_1\).

**Proposition 2.4.A** \(\delta(\varphi; x, y) = \mathcal{A}(\Phi, y) - \mathcal{A}(\Phi, x)\).

This justifies the wording "action difference".

**Proof.** Consider the orbits \(\alpha_x = \{\varphi_t x\}_{t \in [0; 1]}\) and \(\alpha_y = \{\varphi_t y\}_{t \in [0; 1]}\) of \(x\) and \(y\), and choose discs \(\Sigma_x, \Sigma_y\) in \(P\) so that \(\partial \Sigma_x = \alpha_x\) and \(\partial \Sigma_y = \alpha_y\). Choose a curve \(\gamma : [0; 1] \rightarrow P\) with \(\gamma(0) = x\) and \(\gamma(1) = y\). Define a 2-chain \(\Delta : [0; 1] \times [0; 1] \rightarrow P\) by \(\Delta(t, s) = \varphi_t \gamma(s)\). Note that \(\partial \Delta = -\gamma + \varphi \gamma - \alpha_y + \alpha_x\), where we assume that the boundary of the square \([0; 1] \times [0; 1]\) is oriented counter-clockwise. Thus the boundary of the topological disc \(\Pi = \Delta + \Sigma_y - \Sigma_x\) equals \(\varphi \gamma - \gamma\). Therefore

\[
\delta(\varphi; x, y) = \int_{\Pi} \Omega.
\]

Denote by \(\xi_t\) the vector field of the flow \(\varphi_t\) (see 2.2 above). Then

\[
\Delta^* \Omega = \Omega \left( \xi_t(\varphi_t \gamma(s)), \frac{\partial}{\partial s} \varphi_t \gamma(s) \right) dt \wedge ds = -d\Phi_t \left( \frac{\partial}{\partial s} \varphi_t \gamma(s) \right) dt \wedge ds.
\]

Hence

\[
\int_{\Delta} \Omega = \int_{[0; 1] \times [0; 1]} \Delta^* \Omega = - \int_{0}^{1} dt \int_{0}^{1} ds \int_{0}^{1} d\Phi_t \left( \frac{\partial}{\partial s} \varphi_t \gamma(s) \right) = \int_{0}^{1} \Phi_t(\varphi_t x) dt - \int_{0}^{1} \Phi_t(\varphi_t y) dt.
\]
Therefore,
\[
\delta(\varphi; x, y) = \int_\Pi \Omega = \int_\Delta \Omega + \int_{\Sigma_y} \Omega - \int_{\Sigma_x} \Omega = \mathcal{A}(\Phi, y) - \mathcal{A}(\Phi, x).
\]

The proof is complete. \qed

### 2.5 Proof of Theorem 2.1.C

The proof splits into two cases.

**Case I: \( f \) is Hamiltonian.** Let \( \{f_t\} \) be a Hamiltonian path on \( M \) with \( f_0 = \mathbb{1}, f_1 = f \). Denote by \( F \) the Hamiltonian function. Let \( \tilde{f}_t, \tilde{f}, \tilde{F} \) be the lifts of \( f_t, f, F \) to the universal cover \((\tilde{M}, \tilde{\omega})\) respectively.

Proposition 2.3.B guarantees that \( f \) has two fixed points \( x, y \) such that their orbits are contractible and \( \mathcal{A}(F, y) - \mathcal{A}(F, x) \neq 0 \). Let \( \tilde{x}, \tilde{y} \) be any lifts of \( x \) and \( y \) to \( \tilde{M} \). The contractibility of the orbits yields \( \tilde{f}\tilde{x} = \tilde{x} \) and \( \tilde{f}\tilde{y} = \tilde{y} \). Further,
\[
\mathcal{A}(\tilde{F}, \tilde{x}) = \mathcal{A}(F, x) \quad \text{and} \quad \mathcal{A}(\tilde{F}, \tilde{y}) = \mathcal{A}(F, y).
\]

Combining this with 2.4.A we get
\[
\delta(\tilde{f}, x, y) = \mathcal{A}(\tilde{F}, y) - \mathcal{A}(\tilde{F}, x) = \mathcal{A}(F, y) - \mathcal{A}(F, x) \neq 0.
\]

This proves the statement of the theorem in Case I.

**Case II: \( f \) is not Hamiltonian.** Let \( x \in M \) be a fixed point of contractible type of \( f \). Thus there exists a path of symplectic diffeomorphisms \( \{f_t\} \) such that \( f_0 = \mathbb{1}, f_1 = f \) and the orbit \( \{f_t x\} \) is contractible. Note that \( \text{Flux}\{f_t\} \neq 0 \) in view of Proposition 2.2.B. Thus there exists an element \( \alpha \in \pi_1(M, x) \) such that \( \langle \text{Flux}\{f_t\}, \overline{\alpha} \rangle \neq 0 \), where \( \overline{\alpha} \) stands for the image of \( \alpha \) in \( H_1(M, \mathbb{Z}) \) under the Hurewitz homomorphism.

Let \( \{\tilde{f}_t\} \) be the lift of \( \{f_t\} \) to the universal cover \( \tilde{M} \). Put \( \tilde{f} = \tilde{f}_1 \). Denote by \( \tilde{F} \) the Hamiltonian function of \( \{\tilde{f}_t\} \) (recall from 2.4 that every path of symplectic diffeomorphisms on \( \tilde{M} \) is Hamiltonian). Note that \( d\tilde{F}_t = \tau^*\lambda_t \)
where \( \tau : \widetilde{M} \to M \) stands for the natural projection, and \( \{ \lambda_t \} \) is the family of 1-forms associated to \( \{ f_t \} \) as in 2.2 above. Let \( T : \widetilde{M} \to \widetilde{M} \) be the deck transformation corresponding to \( \alpha \in \pi_1(M, x) \). We claim that

\[
\tilde{F}_t(Tz) - \tilde{F}_t(z) = \langle [\lambda_t], \alpha \rangle \quad \text{for all} \quad t \in [0; 1], z \in \widetilde{M}.
\]

(2.5.A)

Indeed, choose any path \( \gamma : [0; 1] \to \widetilde{M} \) with \( \gamma(0) = z \) and \( \gamma(1) = Tz \). Then

\[
\tilde{F}_t(Tz) - \tilde{F}_t(z) = \int_\gamma \tau^* \lambda_t = \int_\tau \gamma([\lambda_t], \alpha),
\]

and (2.5.A) follows.

Let \( \widetilde{x} \) be a lift of \( x \). Then \( \widetilde{f} \widetilde{x} = \widetilde{x} \) since \( \{ f_t x \} \) is contractible. Moreover, \( \widetilde{f}_t \) commutes with \( T \) for every \( t \) so that \( \widetilde{f}T \widetilde{x} = T \widetilde{x} \). We claim that

\[
\delta(\widetilde{f}; \widetilde{x}, T \widetilde{x}) = -\langle \text{Flux}\{ f_t \}, \alpha \rangle.
\]

(2.5.B)

Indeed,

\[
\delta(\widetilde{f}, \widetilde{x}, T \widetilde{x}) = \mathcal{A}(\widetilde{F}, T \widetilde{x}) - \mathcal{A}(\widetilde{F}, \widetilde{x}).
\]

Choose a disc \( \Sigma \subset \widetilde{M} \) spanning \( \{ \widetilde{f}_t \widetilde{x} \} \). Then \( T \Sigma \) spans \( \{ \widetilde{f}_t T \widetilde{x} \} \). Applying (2.3.A) we get that

\[
\mathcal{A}(\widetilde{F}, T \widetilde{x}) - \mathcal{A}(\widetilde{F}, \widetilde{x}) = \int_{T \Sigma} \tilde{\omega} - \int_{\Sigma} \tilde{\omega}
\]

\[
- \int_0^1 \left[ \tilde{F}_t(T \widetilde{f}_t x) - \tilde{F}_t(\widetilde{f}_t x) \right] dt
\]

\[
\overset{2.5.A}{=} - \int_0^1 \langle [\lambda_t], \alpha \rangle dt \overset{2.2.A}{=} -\langle \text{Flux}\{ f_t \}, \alpha \rangle.
\]

This proves (2.5.B). Recall now that \( \langle \text{Flux}\{ f_t \}, \alpha \rangle \neq 0 \). Hence \( \delta(\widetilde{f}; \widetilde{x}, T \widetilde{x}) \neq 0 \) as required. This completes the proof of Theorem 2.1.C. \( \square \)
2.6 Action spectrum of Hamiltonian diffeomorphisms

Let \((M, \omega)\) be a closed symplectic manifold with \(\pi_2(M) = 0\). For a Hamiltonian diffeomorphism \(f \in \text{Ham}(M, \omega)\) take a Hamiltonian path \(\{f_t\}\) of symplectic diffeomorphisms with \(f_0 = \text{Id}, f_1 = f\). Let \(F(x, t)\) be the corresponding Hamiltonian function normalized so that

\[
\int_M F(x, t) \, d(\text{volume}) = 0
\]

for all \(t \in [0, 1]\). Let \(\text{Fix}_0\) be the set of all fixed points \(x\) of \(f\) such that the orbit \(\{f_t x\}\) is contractible. Define the action spectrum of \(f\) as

\[
\text{spectrum}(f) = \{A(F, x) \mid x \in \text{Fix}_0\} \subset \mathbb{R}.
\]

It is known (see [Sch]) that the action spectrum of \(f\) does not depend on the choice of a Hamiltonian path generating \(f\), and is invariant under conjugations in the group of all symplectomorphisms of \(M\). Moreover, this set is compact [HZ],[Sch]. Define the following invariant:

\[
\text{width}(f) = \max_{\alpha, \beta \in \text{spectrum}(f)} |\alpha - \beta|.
\]

Lifting the path \(\{f_t\}\) to the universal cover \(\widetilde{M}\) and applying Proposition 2.4.A we get

\[
\text{width}(f) = \max_{x, y} \delta(\tilde{f}, x, y),
\]

where \(\tilde{f}\) is the canonical lift of \(f\) to \(\widetilde{M}\) and \(x, y\) run over the set of fixed points of \(\tilde{f}\). Combining this with 2.1.B, 2.1.C and 2.1.D we get the following inequality.

**Proposition 2.6.A** \(\text{width}(f^n) \geq n\) for every \(f \in \text{Ham}(M, \omega)\)\{\text{Id}\}.

This fact turns out to be very useful for studying group-theoretic properties of \(\text{Ham}(M, \omega)\). For instance, let us show that for a closed symplectic manifold \((M, \omega)\) with \(\pi_2 = 0\) the group \(\text{Ham}(M, \omega)\) has no torsion. Indeed,
suppose that $f$ is a Hamiltonian diffeomorphism of finite order. Then the sequence $\text{width}(f^n)$ is bounded, and so Proposition 2.6.A yields $f = \mathbb{1}$.

As another immediate application, we prove Proposition 1.6.H.

**Proof of 1.6.H:** Assume first that that $f^m$ is conjugate to $f$ in $\text{Ham}(M, \omega)$ for some $m \in \mathbb{Z}$ with $|m| > 1$. Then $f^{mk}$ is conjugate to $f$ for all $k \in \mathbb{N}$. Hence

$$\text{width}(f^{mk}) = \text{width}(f)$$

for all $k \in \mathbb{N}$ which contradicts Proposition 2.6.A. Thus we proved the proposition for $B(m, 1)$.

Consider now the general case of $B(q, p)$ where $p$ divides $q$. Assume that $q = pm$ with $|m| > 1$ and $f^{pm}$ is conjugate to $f^p$. Applying the argument above to $f^p$ we see that $f^p = \mathbb{1}$. The absence of torsion yields $f = \mathbb{1}$ as required.

\[\square\]

3 Proofs of lower bounds for growth

In 3.2–3.4 below we prove Theorems 1.1.B, 1.4.A and 1.5.B.

3.1 Remarks on the symplectic filling function

We use notations of subsection 1.2.

**Lemma 3.1.A** Symplectic filling functions $v_1$ and $v_2$ associated to Riemannian metrics $\rho_1$ and $\rho_2$ on $M$ and to the same base point $x \in \tilde{M}$ are equivalent: $v_1 \sim v_2$.

**Proof.** There exists $c > 1$ such that

$$B_{\tilde{\rho}_2}(c^{-1}s) \subset B_{\tilde{\rho}_1}(s) \subset B_{\tilde{\rho}_2}(cs)$$

and

$$c^{-1}|\xi|_{\tilde{\rho}_2} \leq |\xi|_{\tilde{\rho}_1} \leq c|\xi|_{\tilde{\rho}_2}$$

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for all \( s \geq 0, \xi \in T^*\widetilde{M} \). Put \( u_i(s) = \inf_{\alpha \in \mathcal{L}} \sup_{z \in B_{\rho_i}(s)} |\alpha_z|_{\rho_i} \) and \( w_i(s) = su_i(s) \) for \( i = 1, 2 \). Then \( c^{-1}u_1(c^{-1}s) \leq u_2(s) \leq cu_1(cs) \) for all \( s \), and so
\[
w_1(c^{-1}s) \leq w_2(s) \leq w_1(cs) .
\]
Since \( v_1, v_2 \) are inverse to \( w_1(s), w_2(s) \) respectively we conclude that \( c^{-1}v_1(s) \leq v_2(s) \leq cv_1(s) \) as required. \( \square \)

**Lemma 3.1.B** Symplectic filling functions \( v_1 \) and \( v_2 \) associated to Riemannian metrics \( \rho_1 \) and \( \rho_2 \) and to base points \( x_1, x_2 \in \widetilde{M} \) are equivalent.

**Proof.** The group \( \mathrm{Symp}_0(M, \omega) \) acts transitively on \( M \). Hence there exists a symplectomorphism \( \varphi : \widetilde{M} \to \widetilde{M} \) which commutes with the action of \( \pi_1(M) \) on \( \widetilde{M} \) so that \( \varphi(x_1) = x_2 \). Consider the symplectic filling function \( v_3(s) \) associated to the metric \( \rho_3 = \varphi^*\rho_1 \) and to the base point \( x_2 \). Note that \( \varphi \) establishes a 1-to-1 map \( \mathcal{L} \to \mathcal{L} \) which sends \( \alpha \) to \( \varphi^*\alpha \). Hence \( v_3(s) \equiv v_1(s) \). But \( v_3 \sim v_2 \) in view of 3.1.A. This completes the proof. \( \square \)

**Lemma 3.1.C** \( v(cs) \sim v(s) \) for all \( c > 0 \).

**Proof.** It suffices to show that
\[
v(s) \leq v(cs) \leq cv(s)
\]
for all \( s > 0, c > 1 \). The inequality \( v(cs) \geq v(s) \) holds since \( v \) is increasing. By definition, \( v(s)u(v(s)) = s \). Hence
\[
cv(s) = \frac{cs}{u(v(s))}
\]
and
\[
v(cs) = \frac{cs}{u(v(cs))} .
\]
But \( v(cs) \geq v(s) \), so \( u(v(cs)) \geq u(v(s)) \) since \( u \) is non-decreasing. It follows that
\[
v(cs) \leq \frac{cs}{u(v(s))} = cv(s)
\]
as claimed. \( \square \)
3.2 Starting the proofs

Let $(M,\omega)$ be a closed symplectic manifold with $\pi_2(M) = 0$. Let $f \in \text{Symp}_0(M,\omega) \setminus \{1\}$ be a symplectic diffeomorphism with a fixed point of contractible type. Choose its lift $\tilde{f}$ as in Theorem 2.1.C. Let $x, y \in \tilde{M}$ be fixed points of $\tilde{f}$ with non-vanishing action difference: $|\delta(\tilde{f}; x, y)| = c > 0$. Then 2.1.B yields

$$|\delta(\tilde{f}^n; x, y)| = nc$$

for all $n \in \mathbb{N}$.

Take any curve $\gamma : [0; 1] \to \tilde{M}$ joining $x$ and $y$ on $\tilde{M}$. Denote by $\ell_n$ the loop formed by $\gamma$ and $\tilde{f}^n \gamma$. Then

$$\text{(3.2.B)} \quad nc = \left| \int_{\ell_n} \alpha \right|$$

for every primitive $\alpha$ of $\tilde{\omega}$. Denote by $b$ the length of $\gamma$. Clearly,

$$\text{(3.2.C)} \quad \text{length}(\ell_n) \leq b(1 + \Gamma_n(f)) \leq 2b\Gamma_n(f)$$

(by definition, $\Gamma_n(f) \geq 1$). Without loss of generality assume that the base point appearing in the definition of the symplectic filling function $v$ is $x$.

3.3 Proof of 1.4.A and 1.5.B

Denote by $B(s)$ the ball of radius $s$ centered at $x$. Pick any positive $s > b$.

Case I. Assume that $\tilde{f}^n(\gamma) \subset B(s)$. Then 3.2.B yields

$$nc \leq \text{length}(\ell_n) \cdot \sup_{z \in B(s)} |\alpha_z| .$$

Taking into account 3.2.C together with the definition of $u$ we get $nc \leq 2b\Gamma_n(f)u(s)$. Thus

$$\text{(3.3.A)} \quad \Gamma_n(f) \geq \frac{nc}{2bu(s)} .$$
Case II. Assume that there exists a point $z \in \gamma$ so that $\tilde{f}^n(z)$ lies outside $B(s)$. Then obviously

\begin{equation}
\Gamma_n(f) \geq \frac{s}{b}.
\end{equation}
(3.3.B)

Given $n$, the choice of $s$ is in our hands. Choose it (assuming that $n$ is large enough) so that

\[ \frac{s}{b} = \frac{nc}{2bu(s)}, \]

that is $s = v\left(\frac{1}{2}nc\right)$. Then (3.3.A) and (3.3.B) yield

\[ \Gamma_n(f) \geq \frac{s}{b} = \frac{1}{b} v\left(\frac{1}{2}nc\right). \]

Applying Lemma 3.1.C we get $\Gamma_n(f) \geq v(n)$. This completes the proof of Theorem 1.4.A.

Assume now that $f$ is a Hamiltonian diffeomorphism which does not propagate. Choose $s_0 > 0$ large enough and argue as above. The only possible case for all $n \in \mathbb{N}$ is Case I. Thus (3.3.A) reads

\[ \Gamma_n(f) \geq \frac{nc}{2bu(s_0)} \geq n, \]

which proves Theorem 1.5.B. \hfill \Box

3.4 Proof of 1.1.B (following [PS1])

Here we assume that $M = \mathbb{T}^2$. An additional ingredient is an isoperimetric inequality which was proved (in a much more general context) by Bonk and Eremenko [BE]. Let $\mathbb{R}^2$ be the Euclidean plane and $L \subset \mathbb{R}^2$ a lattice. There exists a constant $\kappa = \kappa(L) > 0$ such that for every piece-wise smooth curve $\beta : S^1 \to \mathbb{R}^2 \setminus L$ which is contractible in $\mathbb{R}^2 \setminus L$

\begin{equation}
\text{area}(\beta) \leq \kappa \cdot \text{length}(\beta).
\end{equation}
(3.4.A)
Here \( \text{area}(\beta) = \inf_{\varphi} \int_{D^2} |\varphi^* \tilde{\omega}| \), where \( \tilde{\omega} \) is the Euclidean area form and \( \varphi \) runs over all piece-wise smooth maps \( D^2 \to \mathbb{R}^2 \setminus L \) with \( \varphi|_{S^1} = \beta \). We refer to [PSi] for a different proof of (3.4.A). In order to prove 1.1.B start arguing as in 3.2. Further, choose a vector \( e \in \mathbb{Z}^2 \) and a natural number \( N \) so that the curves \( \gamma \) and \( \tilde{f}\gamma \) are homotopic with fixed end points in \( \mathbb{R}^2 \setminus L \). Since \( L \) consists of fixed points of \( \tilde{f} \) we see that \( \tilde{f}^n\gamma \) is homotopic to \( \tilde{f}^{n-1}\gamma \) with fixed end points in \( \mathbb{R}^2 \setminus L \). Therefore the loop \( \ell_n \) is contractible in \( \mathbb{R}^2 \setminus L \) which yields

\[
\text{(3.4.B)} \quad \text{area}(\ell_n) \leq \kappa \cdot \text{length}(\ell_n).
\]

Combining this with 3.2.B and 3.2.C we get

\[
n \cdot c = \left| \int_{\ell_n} \alpha \right| \leq \text{area}(\ell_n) \leq \kappa \cdot \text{length}(\ell_n) \leq 2\kappa b \Gamma_n(f).
\]

We conclude that \( \Gamma_n(f) \geq n \). This completes the proof. \( \square \)

4 Proofs of lower bounds for distortion

4.1 Measurements on the group of symplectomorphisms

Let \((M, \omega)\) be a closed symplectic manifold. Throughout §4 we fix a compatible Riemannian metric on \((M, \omega)\), that is a metric of the form \( \omega(\xi, J\eta) \) where \( J \) is an almost complex structure on \( M \). An important feature of such a metric is the equality \( |\nabla F| = |\xi| \), where \( F \) is any smooth function on \( M \) with the Hamiltonian field \( \xi \). Let \( \{f_t\}, \ t \in [0; 1] \) be a smooth path of symplectomorphisms. Write \( \xi_t \) for the time-dependent vector field on \( M \) which generates this path. Put

\[
L\{f_t\} = \int_0^1 \max_{x \in M} |\xi_t(x)| dt.
\]

If \( \{f_t\} \) is a Hamiltonian path, write \( F(x, t) = F_t(x) \) for its Hamiltonian function. We always assume that the Hamiltonian function is normalized
so that the mean value of $F_t$ with respect to the canonical measure on $M$ vanishes for all $t \in [0; 1]$. Put

$$\Lambda\{f_t\} = \int_0^1 \max_{x \in M} |F_t(x)| \, dt.$$ 

Both $L$ and $\Lambda$ are right-invariant length structures on groups $\text{Symp}_0(M, \omega)$ and $\text{Ham}(M, \omega)$ respectively associated to norms $\max |\xi|$ and $\max |F|$ on their Lie algebras. The right-invariance, of course, means that $L\{f_{th}\} = L\{f_t\}$ and $\Lambda\{f_{th}\} = \Lambda\{f_t\}$. In fact $\Lambda$ is left-invariant as well and is a version of Hofer’s length structure on the group of Hamiltonian diffeomorphisms (see e.g. [P1]). The length structure $L$ has an $L_2$-cousin which is used in hydrodynamics. Let us emphasize that, as it should be for length structures, both $L$ and $\Lambda$ do not change under reparameterization of paths, and are additive with respect to juxtaposition.

Given $f \in \text{Symp}_0(M, \omega)$ set

$$\alpha(f) = \inf L\{f_t\},$$

where the infimum is taken over all symplectic paths $\{f_t\}$ with $f_0 = \mathbb{I}$ and $f_1 = f$.

Given $f \in \text{Ham}(M, \omega)$ set

$$\beta(f) = \inf (L\{f_t\} + \Lambda\{f_t\}),$$

where the infimum is taken over all Hamiltonian paths $\{f_t\}$ with $f_0 = \mathbb{I}$ and $f_1 = f$.

One readily checks that both

$$\alpha : \text{Symp}_0(M, \omega) \to [0; +\infty)$$

and

$$\beta : \text{Ham}(M, \omega) \to [0; +\infty)$$

are norms. In other words, they have the following properties:
• triangle inequality: 
  \[ \alpha(fg) \leq \alpha(f) + \alpha(g) \text{ and } \beta(fg) \leq \beta(f) + \beta(g); \]

• symmetry: \( \alpha(f^{-1}) = \alpha(f) \) and \( \beta(f^{-1}) = \beta(f) \);

• non-degeneracy: \( \alpha(f) > 0 \) and \( \beta(f) > 0 \) for all \( f \neq \mathbb{1} \).

The non-degeneracy follows from the obvious inequalities 
\[ \alpha(f) \geq \max_{x \in M} \text{dist}(x, fx), \]
and 
\[ \beta(f) \geq \max_{x \in M} \text{dist}(x, fx). \]

Let us mention that the triangle inequality is the only property of \( \alpha \) and \( \beta \) we need for the proof of Theorems 1.6.A,B.

### 4.2 Geometric inequalities

Assume now that \( \pi_2(M, \omega) = 0 \). Let \( u \) be the function defined in subsection 1.2 and let \( d \) be the diameter of \( M \).

**Lemma 4.2.A** Let \( f \in \text{Ham}(M, \omega) \) be a Hamiltonian diffeomorphism with \( \beta(f) = b \). Then 
\[ \text{width}(f) \leq 2(b + bu(d + b)). \]

**Lemma 4.2.B** Assume in addition that \( \pi_1(M) \) has trivial center. Let \( f \in \text{Ham}(M, \omega) \) be a Hamiltonian diffeomorphism with \( \alpha(f) = a \). Then 
\[ \text{width}(f) \leq 2(a^2 + da + au(d + a)). \]

The proofs are postponed till 4.5.
4.3 Proof of Theorem 1.6.A

Let $G \subset \text{Ham}(M,\omega)$ be a finitely generated group with generators $h_1, \ldots, h_N$. Put $\kappa = \max_j \beta(h_j)$. Then $\beta(f) \leq \kappa ||f||$ for all $f \in G$. Put $c_n = ||f^n||$ and $w_n = \text{width}(f^n)$. Applying Lemma 4.2.A to $f^n$ we get that

$$w_n \leq 2\kappa c_n + 2\kappa c_n u(d + \kappa c_n).$$

Thus the exists a constant $\mu > 0$ independent on $n$ so that

$$w_n \leq \mu c_n u(\mu c_n).$$

This yields

$$\mu c_n \geq v(w_n),$$

where $v$ is the symplectic filling function. Recall from 2.6.A that $w_n \geq n$. Applying Lemma 3.1.C we conclude that $c_n \geq v(n)$. This completes the proof. $\square$

4.4 Proof of Theorem 1.6.B

Let $G \subset \text{Symp}_0(M,\omega)$ be a finitely generated group with generators $h_1, \ldots, h_N$. Take $f \in G$.

Case 1: $f$ is Hamiltonian. Put $\kappa = \max_j \alpha(h_j)$. Then $\alpha(f) \leq \kappa ||f||$ for all $f \in G$. Put $c_n = ||f^n||$ and $w_n = \text{width}(f^n)$. Applying Lemma 4.2.B to $f^n$ we get that

$$w_n \leq 2\kappa^2 c_n^2 + 2d\kappa c_n + 2\kappa c_n u(d + \kappa c_n).$$

Thus the exists a constant $\mu > 0$ independent on $n$ so that

$$2w_n \leq \mu c_n^2 + \mu c_n u(\mu c_n).$$

Therefore for every $n$ either $\mu c_n^2 \geq w_n$ or $\mu c_n u(\mu c_n) \geq w_n$. The last inequality is equivalent to $\mu c_n \geq v(w_n)$, where $v$ is the symplectic filling function. Recall from 2.6.A that $w_n \geq n$. Applying Lemma 3.1.C we conclude that

$$c_n \geq \min \left( \sqrt{n}, v(n) \right).$$
This completes the proof for a Hamiltonian $f$.

**Case 2: f is not Hamiltonian.** We shall use the flux homomorphism

$$\text{Flux} : \text{Symp}_0(M, \omega) \to H^1(M, \mathbb{R})$$

defined in 2.2.C, 2.2.D. Assume that $f^n = h_{i_n} \cdot \ldots \cdot h_{i_1}$. Then

$$n \cdot \text{Flux}(f) = \sum_{j=1}^m \text{Flux}(h_{i_j}).$$

Choose any norm on $H^1(M, \mathbb{R})$ and put

$$\kappa = \max_{j \in \{1, \ldots, N\} } |\text{Flux}(h_{j})|. $$

We get

$$n|\text{Flux}(f)| \leq m \kappa,$$

so

$$\|f^n\| \geq \frac{|\text{Flux}(f)|}{\kappa} \cdot n.$$ 

Since $\text{Flux}(f) \neq 0$ we conclude that

$$\|f^n\| \succeq n.$$ 

This completes the proof. \(\square\)

4.5 Proof of geometric inequalities

We start with the following situation which is common for 4.2.A and 4.2.B. Let $x_* \in \tilde{M}$ be the base point of $\tilde{M}$. Let $D \subset \tilde{M}$ be the ball of radius $d$ centered at $x_*$. Note that $D$ projects onto the whole $M$. Let $\{f_t\}, t \in [0; 1]$ be any path of symplectic diffeomorphisms of $M$ with $f_0 = \mathbb{1}$ and $f_1 = f$. Lift it to the path $\{h_t\}$ on $\tilde{M}$, and write $\xi_t$ for the vector field generating $h_t$. Denote $h = h_1$.

Suppose that $h$ is the canonical Hamiltonian lift of $f$ defined in 1.5. Recall that the path $\{h_t\}$ on $\tilde{M}$ is always Hamiltonian since $\tilde{M}$ is simply
connected. Let $H(x, t)$ be any Hamiltonian function on $\tilde{M}$ generating \{$h_t$\} (the choice of a time-dependent additive constant is in our hands). It follows from Proposition 2.4.A that

$$\text{width}(f) = \max \delta(h; y_1, y_2) = \max (A(H, y_1) - A(H, y_2)),$$

where $y_1, y_2$ run over fixed points of $h$ and

$$A(H, y) = \int_{\{h\cdot y\}} \lambda - \int_0^1 H(h_t y, t) dt,$$

where $\lambda$ is a primitive of $\tilde{\omega}$. Thus our purpose is to estimate $A(H, y)$, where $y$ is a fixed point of $h$.

Note also that

$$A(H, y) - A(H, Ty) = \delta(h; y, Ty) = 0$$

for every deck transformation $T$ of the covering $\tilde{M} \to M$. To see this, consider any Hamiltonian path on $M$ joining 1 with $f$. Applying Proposition 2.4.A to the lift of this path we obtain $\delta(h; y, Ty) = 0$. Summing up this discussion, we can assume without loss of generality that $y \in D$.

The “symplectic area” term of $A(H, y)$ can be estimated as follows. Denote by $\gamma$ the orbit \{$h_t y$\}, $t \in [0; 1]$. Abbreviate $c = L\{f_t\}$. Obviously

$$\text{length}(\gamma) = \int_0^1 |\xi_t(h_t y)| dt \leq c. \quad (4.5.A)$$

Therefore $\gamma$ is contained in the ball of radius $d + c$ with the center at $x_\ast$. Hence

$$\int_{\gamma} \lambda \leq cu(d + c) \quad (4.5.B)$$

for every primitive $\lambda$ of $\tilde{\omega}$.

It remains to estimate the “Hamiltonian” term of $A(H, y)$. At this point the proofs of 4.2.A and 4.2.B split up.
Proof of Lemma 4.2.A: Suppose that \( \{f_t\} \) is a Hamiltonian path which joins \( \mathbb{I} \) with \( f \). Therefore \( h \) is the canonical lift of \( f \). Write \( F(x,t) \) for the normalized Hamiltonian of \( \{f_t\} \), and choose \( H \) to be the lift of \( F \) to \( \tilde{M} \). Note that
\[
\left| \int_0^1 H(h_t x, t) dt \right| \leq \Lambda \{f_t\}.
\]
Applying (4.5.B) we obtain
\[
A(h, y) \leq \Lambda \{f_t\} + cu(d + c)
\]
with \( c = L\{f_t\} \). Since this is true for every Hamiltonian path \( \{f_t\} \) which joins \( \mathbb{I} \) with \( f \) we get
\[
A(h, y) \leq b + bu(d + b).
\]
The same action bound holds for any other fixed point of \( h \). Therefore
\[
\text{width}(f) \leq 2(b + bu(d + b)),
\]
which proves 4.2.A.

Proof of Lemma 4.2.B: Here \( \{f_t\} \) is an arbitrary symplectic path joining \( \mathbb{I} \) with \( f \). The triviality of the center of \( \pi_1(M) \) guarantees that \( h \) is the canonical lift of \( f \) (see Remark 2.2.F above). Suppose that the Hamiltonian \( H(x,t) \) is normalized so that \( H_t(x_s) = 0 \) for all \( t \in [0;1] \). Warning: since the path \( \{f_t\} \) is not in general Hamiltonian, the function \( H \) does not come as a lift of any function on \( M \), and in particular \( H \) may be unbounded on \( \tilde{M} \). We will get around this difficulty by noticing that the differential of \( H_t \) is a bounded 1-form. We proceed as follows. Observe that
\[
|H_t(h_t y)| \leq |H_t(y)| + \text{length}(\gamma) \cdot \max_x |\nabla H_t(x)|,
\]
and
\[
|H_t(y)| \leq |H_t(x_s)| + d \cdot \max_x |\nabla H_t(x)|.
\]
Taking into account that $H_t(x_*) = 0$, $|\nabla F_t| = |\xi_t|$ and using (4.5.A) we get

$$|H_t(h_t x)| \leq (d + c) \max_x |\xi_t(x)|.$$  

Therefore

$$\left| \int_0^1 H_t(h_t x) dt \right| \leq (d + c) \cdot \int_0^1 \max_x |\xi_t(x)| dt = (d + c)c.$$

Thus

$$\mathcal{A}(H, y) \leq c^2 + cd + cu(d + c).$$

Since this is true for every symplectic path $\{f_t\}$ which joins $\mathbb{I}$ with $f$ we get

$$\mathcal{A}(H, y) \leq a^2 + ad + au(d + a).$$

The same action bound holds for any other fixed point of $h$. Therefore

$$\text{width}(f) \leq 2(a^2 + ad + au(d + a)),$$

which proves 4.2.B. \qed

### 4.6 A remark on smoothness

Here we outline an extension of our results above to $C^1$-smooth symplectic diffeomorphisms. Geometric inequalities 4.2.A,B remain valid for symplectomorphisms which are generated by $C^1$-smooth vector fields. Various topological facts about symplectic diffeomorphisms which appeared above should extend to the $C^1$-case without problems. One should simply use an appropriate approximation by $C^\infty$-diffeomorphisms.

A more delicate argument is needed to show that any Hamiltonian diffeomorphism $f \neq \mathbb{I}$ has two fixed points with strictly positive action difference (see crucial Proposition 2.3.B). We claim that this holds true for $C^1$-smooth Hamiltonian diffeomorphisms. The proof of the claim is based on a remarkable “energy-capacity” inequality in symplectic topology. We need a version
from [Sch]. Let $g$ be a $C^\infty$-smooth Hamiltonian diffeomorphism of $M$ which displaces an open subset $B \subset M$:

$$g(B) \cap B = \emptyset.$$ 

Then $g$ has two fixed points whose action difference is at least $c(B)$, where $c(B)$ is a strictly positive constant which depends only on $B$. Assume now that $f \neq \mathbb{1}$ is a $C^1$-smooth Hamiltonian diffeomorphism. Clearly, $f$ displaces a small ball, say $B$. Choose a sequence $g_i$ of $C^\infty$-smooth Hamiltonian diffeomorphisms which converges to $f$ in the $C^1$-sense. Then $g_iB \cap B = \emptyset$ for large $i$. Therefore each $g_i$ has a pair of fixed points, say $x_i$ and $y_i$, whose action difference is at least $c(B)$. By compactness, choose a subsequence $i_k \to \infty$ such that $x_{i_k}$ and $y_{i_k}$ converge to fixed points $x$ and $y$ of $f$ respectively. Since the action difference is continuous with respect to the $C^1$-convergence (use 2.1 above to see this), we conclude that the action difference of $x$ and $y$ is at least $c(B)$. This proves the claim.

### 4.7 Proof of Theorem 1.1.E

The proof is divided into 3 steps.

1) Let $G$ be the group of all symplectic diffeomorphisms of a closed oriented surface $M$ of genus $\geq 2$. Let $\phi : BS(q,p) \to G$ be a homomorphism. We assume for simplicity that $q > p > 0$. Denote by Mod the mapping class group of $M$, and let $\pi : G \to \text{Mod}$ be the natural projection. Farb-Lubotzky-Minsky theorem [FLM] implies that every non-torsion element in $\text{Mod}$ is undistorted (in the sense of 1.6.C above). Since the element $a \in BS(q,p)$ has logarithmic distortion (see 1.6.E) we conclude that $\pi(\phi(a))$ is of finite order in $\text{Mod}$. Therefore there exists $k \in \mathbb{N}$ such that $\phi(a^k)$ lies in $\text{Symp}_0(M)$. Denote $f = \phi(b), g = \phi(a^k)$. Then $g \in \text{Symp}_0$ and $g^q = fg^pf^{-1}$.

2) We claim that in fact $g$ lies in $\text{Ham}(M, \omega)$. Indeed, assume on the contrary that $\text{Flux}(g) \neq 0$. 

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Consider the isomorphism of $H^1(M, \mathbb{R})$ induced by $f$, and denote by $I$ its inverse. Then the equation $g^q = fg^p f^{-1}$ yields
\[ \overline{\text{Flux}}(g^q) = I \cdot \overline{\text{Flux}}(g^p). \]
Rewrite this as
\[ I \cdot \overline{\text{Flux}}(g) = \frac{q}{p} \cdot \overline{\text{Flux}}(g). \]
Therefore the number $q/p$ is a root of the characteristic polynomial $\chi(t)$ of $I$. Note now that since $I$ preserves the lattice $H^1(M, \mathbb{Z})$ all coefficients of $\chi$ are integers. Moreover the leading and the free coefficients of $\chi$ are equal to $1$. Hence the only rational roots of $\chi$ are $\pm 1$, so $q = \pm p$. This contradicts to the assumption $q > p > 0$. The claim follows.

3) Note that
\[ g^{q^m} = f^m g^{p^m} f^{-m} \]
for all $m \in \mathbb{N}$. Then
\[ \text{width}(g^{q^m}) = \text{width}(g^{p^m}). \]
Assume that $g \neq \mathbb{1}$. Then 2.6.A yields
\[ \text{width}(g^{q^m}) \geq q^m. \]
Applying 4.2.A we get
\[ \text{width}(g^{p^m}) \leq \beta(g^{p^m}) \leq p^m. \]
Combining these inequalities we conclude that $q^m \leq p^m$ which contradicts our choice of $p$ and $q$. Therefore $g = \mathbb{1}$, and so $\phi(a)^k = \mathbb{1}$. This completes the proof. \(\square\)

4.8 Epilogue

It is instructive to take a look at our approach to distortion from the viewpoint of global geometry of symplectomorphisms groups.
Let $(M, \omega)$ be a closed symplectic manifold with $\pi_2 = 0$ and such that $\pi_1$ has trivial center. Choose a compatible Riemannian metric on $M$ and choose any norm on $H^1(M, \mathbb{R})$. Define a norm

$$\gamma : \text{Symp}_0(M, \omega) \to [0; +\infty)$$

by

$$\gamma(f) = \alpha(f) + \left| \text{Flux}(f) \right|.$$

Inequality 4.2.B in conjunction with 2.6.A gives a bound for distortion of any non-trivial cyclic subgroup of $\text{Symp}_0(M, \omega)$ with respect to $\gamma$. Assume for instance that $(M, \omega)$ is symplectically hyperbolic. Then $\gamma(f^n) \geq n$ for every non-Hamiltonian diffeomorphism $f$ and $\gamma(f^n) \geq \sqrt{n}$ for every Hamiltonian $f$.

Similarly, inequality 4.2.A leads to distortion bounds for cyclic subgroups of $\text{Ham}(M, \omega)$ with respect to the norm $\beta$.

Of course, these distortion bounds give rise to obstructions for representations of finitely generated groups into symplectomorphisms groups. Let me mention also that the group of Hamiltonian diffeomorphisms carries in addition two remarkable norms which are invariant under conjugations – the Hofer norm and the commutator norm. It would be interesting to understand distortion bounds in these cases. We refer to [P1] for some results on distortion in the Hofer norm. As far as the commutator norm is concerned, we refer to [BG] and [E].

5 Appendix: an example

In this section we show that the growth bound 1.4.A is in general not true for symplectic diffeomorphisms which have fixed points but none of them is of contractible type. Namely, we will construct a diffeomorphism $f \in \text{Symp}_0(M, \omega)$ of certain symplectic manifold $(M, \omega)$ with $\pi_2(M) = 0$ which has the following properties:
* $f$ has fixed points, and none of them is of contractible type;

* $f^2 = \mathbb{1}$, so that $\{\Gamma_n(f)\}$ is a bounded sequence.

The construction goes as follows. Consider the standard symplectic torus $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ endowed with the symplectic form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. Let

$$
\gamma : T^4 \to T^4, \quad \gamma(p_1, q_1, p_2, q_2) = (p_1, q_1 + \frac{1}{2}, -p_2, -q_2) \mod 1
$$

be an involution of the torus. Clearly, $\gamma$ is symplectic and has no fixed points. Thus it generates a free symplectic action of $\mathbb{Z}_2$ on $T^4$. Let $(M, \omega)$ be the quotient space $T^4/\mathbb{Z}_2$, and let $\tau : T^4 \to M$ be the natural projection.

Consider the flow $f_t : T^4 \to T^4$, $(p_1, q_1, p_2, q_2) \mapsto (p_1, q_1 + \frac{t}{2}, p_2, q_2) \mod 1$.

It is a symplectic flow which commutes with $\gamma$. Hence it defines a symplectic flow $f_1$ on $M$. Put $f = f_1$ and note that $f^2 = \mathbb{1}$. Look now at fixed points of $f$. All of them have the form $\tau(z)$, where $z \in T^4$ satisfies $\overline{f}_1 z = \gamma z$, so that

$$
z = (p, q, m_1/2, m_2/2) \mod 1, \quad p, q \in \mathbb{R}, \quad m_1, m_2 \in \mathbb{Z}.
$$

We claim that all fixed points of $f$ are not of contractible type. To prove this, assume on the contrary that there exists another path, say $\{g_t\}_{t \in [0,1]}$ of symplectic diffeomorphisms of $M$ such that $g_0 = \mathbb{1}$, $g_1 = f$ and the loop $\{g_t \tau(z)\}$ is contractible. Consider a loop $\{h_t\}$ of diffeomorphisms of $M$ formed by $\{f_t\}$ and $\{g_{1-t}\}$: $h_t x = f_{2t} x$ if $t \in [0; 1/2]$ and $h_t x = g_{2-2t} x$ if $t \in [1/2; 1]$. Let $\overline{h}_t$ be its lift to $T^4$. Note that $\tau \circ \overline{h}_1 = \tau$, so either $\overline{h}_1 = \mathbb{1}$ or $\overline{h}_1 = \gamma$. Due to our construction one has $\overline{h}_1 z = \gamma z$, which rules out the first possibility. Therefore $\overline{h}_1 = \gamma$. But $\gamma$ acts non-trivially on $H_1(T^4)$ and thus cannot be isotopic to the identity. This contradiction proves the claim.
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