TWO WEIGHT NORM INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATORS AND COMMUTATORS

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Contents

1. Introduction 2
2. Preliminaries 3
3. Dyadic operators 5
   Dyadic grids 7
   Dyadic operators 10
   Sparse operators 13
4. Digression: one weight inequalities 15
   The fractional maximal operator 16
   The fractional integral operator 20
   Commutators 22
5. Testing conditions 22
   Two weight inequalities 22
   Testing conditions for fractional maximal operators 24
   Testing conditions for fractional integral operators 28
   Local and global testing conditions 32
   Testing conditions for commutators 33
6. Bump conditions 34
   The $A_{p,q}^\alpha$ condition 34
   Young functions and Orlicz norms 40
   The $A_{p,q}^\alpha$ bump conditions 42
   Bump conditions for fractional maximal operators 44
   Bump conditions for fractional integral operators 46
   Bump conditions for commutators 47
7. Separated bump conditions 50
   The Muckenhoupt-Wheeden conjectures 50
   Separated bump conditions for fractional integral operators 51
   Two conjectures for commutators 53
References 54
1. Introduction

In these lecture notes we describe some recent work on two weight norm inequalities for fractional integral operators, also known as Riesz potentials, and for commutators of fractional integrals. Our point of view is strongly influenced by the groundbreaking work on dyadic operators that led to the proof of the $A_2$ conjecture by Hytönen [43] and the simplification of that proof by Lerner [58, 59]. (See also [42] for a more detailed history and bibliography of this problem.) Fractional integrals are of interest in their own right and have important applications in the study of Sobolev spaces and PDEs. They are positive operators and in many instances proofs are much easier for fractional integrals than they are for Calderón-Zygmund singular integrals. But as we will see, in many cases they are more difficult to work with, and we will give several examples of results which are known to hold for singular integrals but remain conjectures for fractional integrals.

After giving some preliminary results in Section 2, in Section 3 we lay out the abstract theory of dyadic grids and show how inequalities for fractional integrals and commutators can be reduced to the study of dyadic operators. All of these ideas were implicit in the classical Calderón-Zygmund decomposition but in recent years the essentials have been extracted, yielding a substantially new perspective.

In Section 4 we show how the dyadic approach can be used to simplify the proof of one weight norm inequalities for fractional integrals and commutators. The purpose of this digression is two-fold. First, it provides a nice illustration of the power of these dyadic methods, as the proofs are markedly simpler than the classical proofs. Second, we will use these proofs to illustrate the technical obstacles we will encounter in trying to prove two weight inequalities.

There are two approaches to two weight inequalities for fractional integrals: the testing conditions, first introduced by Sawyer [88, 91], and the “$A_p$ bump” conditions introduced by Neugebauer [76] and Pérez [79]. Both approaches have their advantages. In Section 5 we consider testing conditions. The fundamental result we discuss is due to Lacey, Sawyer and Uriarte-Tuero [55], but we will present a beautiful simplification of their proof due to Hytönen [42]. We conclude this section with a conjecture concerning testing conditions for commutators of fractional integrals.

In Sections 6 and 7 we will discuss bump conditions. Besides the work of Pérez cited above, the contents of these sections are based on recent work by the author and Moen [23, 24, 25]. We conclude the last section with several open problems.

Throughout these lecture notes we assume that the reader is familiar with real analysis (e.g., as presented by Royden [86]) and with classical harmonic analysis including the basics of the theory of Muckenhoupt $A_p$ weights and one weight norm inequalities (e.g., the first seven chapters of Duoandikoetxea [35]). Additional references include the classic books by Stein [94] and García-Cuerva and Rubio de Francia [38] and the
more recent books by Grafakos \([40, 41]\). Many of the results we give for weighted norm inequalities for fractional integrals are scattered through the literature—there is unfortunately no single reference for this material. We will provide copious references throughout, including historical ones. Some of the material in these notes is new and has not appeared in the literature before.

These notes are based on three lectures delivered at the 6th International Course of Mathematical Analysis in Andalucía, held in Antequera, Spain, September 8–12, 2014. They are, however, greatly expanded to include both new results and many details that I did not present in my lectures due to time constraints. In addition, I have taken this opportunity to correct some (relatively minor) mistakes in the proofs I sketched in the lectures. I am grateful to the organizers for the invitation to present this work. I would also like to thank Kabe Moen, my principal collaborator on fractional integrals (or Riesz potentials, as he prefers), and Carlos Pérez, who introduced me to bump conditions and has shared his insights with me for many years. It has been a privilege to work with both of them.

2. Preliminaries

In this section we gather some essential definitions and a few background results. Hereafter, we will be working in \(\mathbb{R}^n\), and \(n\) will always denote the dimension. We will denote constants by \(C, c, \text{ etc.} \) and the value may change at each appearance. If necessary, we will denote the dependence of the constants parenthetically: e.g., \(C = C(n, p)\). The letters \(P\) and \(Q\) will be used to denote cubes in \(\mathbb{R}^n\). By a weight we will always mean a non-negative, measurable function that is positive on a set of positive measure.

Averages of functions will play a very important role in these notes, so we introduce some useful notation. Given any set \(E, 0 < |E| < \infty\), we define

\[
\frac{1}{|E|} \int_E f(x) \, dx.
\]

More generally given a non-negative measure \(\mu\), we define

\[
\frac{1}{\mu(E)} \int_E f(x) \, d\mu.
\]

In other words, an average is always with respect to the measure. If we have a measure of the form \(\sigma \, dx\), where \(\sigma\) is a weight, we will write \(d\sigma\), as in \(\int_E f \, d\sigma\), to emphasize this fact. We will also use the following more compact notation, particularly when the set is a cube \(Q\):

\[
\langle f \rangle_Q, \quad \langle f \rangle_{Q, \sigma}.
\]
We now define the two operators we will be focusing on. Given $0 < \alpha < n$ and a measurable function $f$, we define the fractional integral operator $I_\alpha$ by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy.$$ 

Given a function $b \in \text{BMO}$, the space of functions of bounded mean oscillation, we define the commutator

$$[b, I_\alpha] f(x) = b(x)I_\alpha f(x) - I_\alpha (bf)(x) = \int_{\mathbb{R}^n} (b(x) - b(y)) \frac{f(y)}{|x-y|^{n-\alpha}} \, dy.$$

The fractional integral operator is classical: it was introduced by M. Riesz [85]. Commutators are more recent and were first considered by Chanillo [8]. The following are some of the basic properties of these operators; unless otherwise noted, see Stein [94, Chapter V] for details.

1. $I_\alpha$ is a positive operator: if $f(x) \geq 0$ a.e., then $I_\alpha f(x) \geq 0$. Note, however, that $[b, I_\alpha]$ is not positive.

2. For $1 < p < \frac{n}{\alpha}$, if we define $q$ by $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, then

$$I_\alpha : L^p \rightarrow L^q,$$

and for all $b \in \text{BMO}$,

$$[b, I_\alpha] : L^p \rightarrow L^q.$$

See Chanillo [8].

3. When $p = 1$, $q = \frac{n}{n-\alpha}$, then $I_\alpha$ satisfies the weak type inequality

$$I_\alpha : L^p \rightarrow L^{q,\infty},$$

but commutators are more singular and do not satisfy a weak $(1, \frac{n}{n-\alpha})$ inequality. For a counter-example and a substitute inequality, see [15].

4. We can define fractional powers of the Laplacian via the Fourier transform using the fractional integral operator: for all Schwartz functions $f$ and $0 < \alpha < n$,

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = c I_\alpha f(x).$$

We also have that for all $f \in C_c^\infty$,

$$|f(x)| \leq I_1(|\nabla f|)(x).$$

Fractional integrals have found wide application in the study of PDEs. Here we mention a few results. Recall the Sobolev embedding theorem (see [1, Chapter V]): if $f$ is contained in the Sobolev space $W^{1,p}$, then for $1 \leq p < n$ and $p^* = \frac{np}{n-p}$,

$$\|f\|_{L^{p^*}} \leq C \|\nabla f\|_{L^p}.$$
When $p > 1$ this is an immediate consequence of the inequality relating $I_1$ and the gradient, and the strong type norm inequality for $I_1$. When $p = 1$ it can be proved using the weak type inequality for $I_1$ and a decomposition argument due to Maz’ya [65, p. 110] (see also Long and Nie [64] and [21, Lemma 4.31]).

Two weight norm inequalities for $I_\alpha$ also yield weighted Sobolev embeddings. In particular, they can be used to prove inequalities of the form

$$
\|f\|_{L^p(u)} \leq C \|\nabla f\|_{L^p}.
$$

These were introduced by Fefferman and Phong [36] in the study of the Schrödinger operator. Such inequalities can also be used to prove that weak solutions of the elliptic equations with non-smooth coefficients are strong solutions: see, for example, Chiarenza and Franciosi [9] and [27]. For additional applications we refer to the paper by Sawyer and Wheeden [93] and the many references it contains. (We remark in passing that this paper has been extremely influential in the study of two weight norm inequalities for fractional integrals.)

Closely related to the fractional integral operator is the fractional maximal operator: given $0 < \alpha < n$ and $f \in L^1_{loc}$, define

$$
M_\alpha f(x) = \sup_Q |Q|^2 \int_Q |f(y)| \, dy \cdot \chi_Q(x),
$$

where the supremum is taken over all cubes with sides parallel to the coordinate axes. The fractional maximal operator was introduced by Muckenhoupt and Wheeden [67] in order to proved one weight norm inequalities for $I_\alpha$ via a good-\lambda inequality. This result is the analog of the one linking the Hardy-Littlewood maximal operator and Calderón-Zygmund singular integrals proved by Coifman and Fefferman [12].

For $1 < p < \frac{n}{\alpha}$, $M_\alpha$ satisfies the same strong $(p,q)$ inequality as $I_\alpha$. In addition, it satisfies the upper endpoint estimate $M_\alpha : L^\infty \rightarrow L^{\frac{n}{\alpha}}$. In contrast, if $f \in L^\infty$, then $I_\alpha f$ need not be bounded, but does satisfy an exponential integrability condition. See, for instance, Ziemer [104, Theorem 2.9.1].

Our approach to norm inequalities for the fractional integral operator will avoid $M_\alpha$; however, we will use it as a model operator since it has many features in common with $I_\alpha$ but is usually easier to work with. We note in passing that there is an Orlicz fractional maximal operator that plays a similar role for commutators of fractional integrals: see [15]. (This operator also plays a role in the study of two weight, weak $(1,1)$ inequalities for $I_\alpha$: see Section 7.)

3. Dyadic operators

In this section we explain the machinery of dyadic grids and dyadic operators. These ideas date back to the 1950’s and the seminal work of Calderón and Zygmund [4], and have played a prominent role in harmonic analysis since then. In
the past fifteen years they have been reformulated and taken on a new prominence because of their connection with the $A_2$ conjecture. A important early presentation of this new point of view was the lecture notes on dyadic harmonic analysis by C. Pereyra [78]. As she described them:

> These notes contain what I consider are the main actors and universal tools used in this area of mathematics. They also contain an overview of the classical problems that lead mathematicians to study these objects and to develop the tools that are now considered the $abc$ of harmonic analysis. The modern twist is the connection to a parallel dyadic world where objects, statements and sometimes proofs are simpler, but yet illuminated enough to guarantee that one can translate them into the non-dyadic world.

The major advance since this was written was the realization that not only could dyadic operators illuminate what was going on with their non-dyadic counterparts, but in fact the solution of non-dyadic problems could be reduced to proving the corresponding results for dyadic operators. Our understanding of this approach continues to evolve: see for instance, the very recent lecture notes on dyadic approximation by Lerner and Nazarov [60].

This philosophy of dyadic operators can be summarized by paraphrasing the title of the hit song from Irving Berlin’s 1946 musical, *Annie Get Your Gun*:

> **Anything you can do, I can do better (dyadically)!**

**Figure 1.** Ethel Merman as Annie Oakley, 1946
Dyadic grids. We begin by recalling the classical dyadic grid. This is the countable collection of cubes that are dyadic translates and dilations of the unit cube, \([0, 1]^n\):

\[ \Delta = \{ Q = 2^k([0,1]^n + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^n \}. \]

These cubes have a number of important properties: any cube in \(\Delta\) has side-length a power of two; any two cubes in \(\Delta\) are disjoint or one is contained in the other; given any \(k \in \mathbb{Z}\), the subcollection \(\Delta_k\) of cubes with side-length \(2^k\) forms a partition of \(\mathbb{R}^n\).

The importance of dyadic cubes lies in the Calderón-Zygmund cubes, which give a very powerful decomposition of a function. For proof of this result, see García-Cuerva and Rubio de Francia [38, Chapter II] and [21, Appendix A].

**Proposition 3.1.** Let \(f \in L^1_{\text{loc}}\) be such that \((f)_Q \to 0\) as \(|Q| \to \infty\) (e.g., \(f \in L^p, 1 \leq p < \infty\)). Then for each \(\lambda > 0\) there exists a collection of disjoint cubes \(\{Q_j\} \subset \Delta\) such that

\[ \lambda < \int_{Q_j} |f(x)| \, dx \leq 2^n \lambda. \]

Moreover, given \(a \geq 2^{n+1}\), for each \(k \in \mathbb{Z}\) let \(\{Q_j^k\}\) be the collection of cubes gotten by taking \(\lambda = a^k\) above. Define

\[ \Omega_k = \bigcup_j Q_j^k, \quad E_j^k = Q_j^k \setminus \Omega_{k+1}. \]

Then for all \(j\) and \(k\), the sets \(E_j^k\) are pairwise disjoint and \(|E_j^k| \geq \frac{1}{2} |Q_j^k|\).

These cubes are closely related to the dyadic maximal operator: given \(f \in L^1_{\text{loc}}\), define the operator \(M^d (\text{I})\) by

\[ M^d f(x) = \sup_{Q \in \Delta} \int_Q |f(y)| \, dy \cdot \chi_Q(x). \]

Then for each \(\lambda > 0\), if we form the cubes \(Q_j\) from the first part of Proposition 3.1,

\[ \{ x \in \mathbb{R}^n : M^d f(x) > \lambda \} = \bigcup_j Q_j. \]

The Calderón-Zygmund cubes were introduced by Calderón and Zygmund in [4]. The essential idea underlying the sets \(E_j^k\) from the second half of Proposition 3.1 is due to Calderón [3] (working with balls in a space of homogeneous type). This idea was applied to Calderón-Zygmund cubes by García-Cuerva and Rubio de Francia [38, Chapter IV] in their proof of the reverse Hölder inequality. It appears to have first been explicitly stated and proved as a property of Calderón-Zygmund cubes by Pérez [81].

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1In the notation we will introduce below, we would call this operator \(M^\Delta\). Here we prefer to use the classical notation. As Emerson said, “Foolish consistency is the hobgoblin of little minds.”
Given the specific example of the Calderón-Zygmund cubes, we make the following two definitions that extract their fundamental properties.

**Definition 3.2.** A collection of cubes $D$ in $\mathbb{R}^n$ is a dyadic grid if:

1. If $Q \in D$, then $\ell(Q) = 2^k$ for some $k \in \mathbb{Z}$.
2. If $P, Q \in D$, then $P \cap Q \in \{P, Q, \emptyset\}$.
3. For every $k \in \mathbb{Z}$, the cubes $D_k = \{Q \in D : \ell(Q) = 2^k\}$ form a partition of $\mathbb{R}^n$.

**Definition 3.3.** Given a dyadic grid $D$, a set $S \subset D$ is sparse if for every $Q \in S$,

$$\left| \bigcup_{P \in S \atop P \subset Q} P \right| \leq \frac{1}{2} |Q|.$$ 

Equivalently, if we define $E(Q) = Q \setminus \bigcup_{P \in S \atop P \subset Q} P$,

then the sets $E(Q)$ are pairwise disjoint and $|E(Q)| \geq \frac{1}{2} |Q|$.

It is immediate that the classical dyadic cubes $\Delta$ are a dyadic grid. By Proposition 3.1, given a function $f \in L^1_{\text{loc}}$, if we form the cubes $\{Q^k_j\}$, then they are a sparse subset of $\Delta$ with $E(Q^k_j) = E^k_j$. Because of this fact, given a fixed dyadic grid $D$, we will often refer to cubes in it as dyadic cubes.

Clearly, we can get dyadic grids by taking translations of the cubes in $\Delta$. The importance of this is that every cube in $\mathbb{R}^n$ is contained in a cube from a fixed, finite collection of such dyadic grids.

**Theorem 3.4.** There exist dyadic grids $D^k$, $1 \leq k \leq 3^n$, such that given any cube $Q$, there exists $k$ and $P \in D^k$ such that $Q \subset P$ and $\ell(P) \leq 3\ell(Q)$.

The origin of Theorem 3.4 is obscure but we believe that credit should be given to Okikiolu [77] and, for a somewhat weaker version, to Chang, Wilson and Wolff [7].(2)

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(2) Theorem 3.4 and variations of it have recently been attributed to Christ in [70] and also to Garnett and Jones in [54, Section 2.2]. In particular, some people suggested that it was in the paper by Garnett and Jones on dyadic $BMO$ [39]. It is not. Moreover, these authors have told me and others that this result did not originate with them, though they knew and shared it. The earliest appearance of a version of Theorem 3.4 in print seems to be in Okikiolu [77, Lemma 1b]. Earlier, Chang, Wilson and Wolff [7, Lemma 3.2] had a weaker but substantially similar version. They showed that given the set $\Delta_A = \{Q \in \Delta, \ell(Q) \leq 2^A\}$, then there exists a finite collection of translates of $\Delta$ such that given any $Q \in \Delta_A$, $3Q$ is contained in a cube of comparable size from one of these translated grids. A refined version of this lemma later appeared in Wilson [102, Lemma 2.1].

The basic idea underlying the proof of Theorem 3.4 is sometimes referred to as the “one-third trick” (e.g. in [56, 62]). This idea has been variously attributed [56, 66] to Garnett or Garnett.
The total number of dyadic grids needed can be reduced, though at the price of increasing the constant \( C \) relating the size of the cubes. Hytönen and Pérez [44, Theorem 1.10] showed that \( 2^n \) dyadic grids suffice, with \( C = 6 \). (For details of the proof, see [59, Proposition 2.1].) Conde [13] proved that only \( n+1 \) grids are necessary, and this bound is sharp, but with a constant \( C \approx n \).

**Proof.** We will use the following \( 3^n \) translates of the standard dyadic grid \( \Delta \):

\[
D^t = \{ 2^j([0,1]^n + m + t) : j \in \mathbb{Z}, m \in \mathbb{Z}^n \}, \quad t \in \{ 0, \pm 1/3 \}^n.
\]

Now fix a cube \( Q \); then there exists a unique \( j \in \mathbb{Z} \) such that

\[
\frac{2^j}{3} \leq \ell(Q) < \frac{2^{j+1}}{3}.
\]

At most \( 2^n \) cubes in \( \Delta \) of sidelength \( 2^j \) intersect \( Q \); let \( P \) be one such that \( |P \cap Q| \) is maximal.

![Figure 2. The construction of \( P' \) containing \( Q \).](image)

To get the desired cube we translate \( P \), acting on each coordinate in succession. If a face of \( P \) (i.e. a \( n-1 \) dimensional hyper-plane on the boundary) perpendicular to the \( j \)-th coordinate axis intersects the interior of \( Q \), translate \( P \) parallel to the \( j \)-th coordinate axis in the direction of the closest face of \( Q \) a distance \( \frac{2^j}{3} \). Because of the maximality of \( P \), this direction is away from the interior of \( P \). Hence, this moves the face out of \( Q \), and the opposite face remains outside as well, so more of \( Q \) is contained in the interior of \( P \). Thus, after at most \( n \) steps we will have a cube \( P' \) that is contained in one of the grids \( D^t \), \( \ell(P') = \ell(P) \leq 3\ell(Q) \), and such that \( Q \subset P' \). 

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and Jones, Davis, and Wolff. The earliest unambiguous appearance appears to be in Wolff [103, Lemma 1.4]; Wolff attributes this lemma to S. Janson.
Though we do not consider it here, we want to note that there is another important approach to dyadic grids. Nazarov, Treil and Volberg [71, 72, 74] have shown that random dyadic grids (i.e., translates of $\Delta$ where the translation is taken according to some probability distribution) are very well behaved “on average.” This approach was central to Hytönen’s original proof of the $A_2$ conjecture [43].

**Dyadic operators.** We can now introduce the dyadic operators that we will use in place of the fractional maximal and integral operators and commutators. We begin with the fractional maximal operator. Given $0 < \alpha < n$, a dyadic grid $D$ and $f \in L^1_{\text{loc}}$, define

$$M^D_\alpha f(x) = \sup_{Q \in D} |Q|^\frac{\alpha}{n} \int_Q |f(y)| \, dy \cdot \chi_Q(x).$$

**Proposition 3.5.** There exists a constant $C(n, \alpha)$ such that for every function $f \in L^1_{\text{loc}}$ and $1 \leq t \leq 3^n$,

$$M^{D_t}_\alpha f(x) \leq M_\alpha f(x) \leq C(n, \alpha) \sup_t M^{D_t}_\alpha f(x),$$

where the grids $D^t$ are defined by (3.1).

Proposition 3.5 is stated in [24] without proof; when $\alpha = 0$ this was proved in [44, Proof of Theorem 1.10] and the proof we give for $\alpha > 0$ is essentially the same.

**Proof.** The first inequality is immediate. To prove the second, fix $x$ and a cube $Q$ containing $x$. Then by Theorem 3.4 there exists $t$ and $P \in D^t$ such that $Q \subset P$ and $|P| \leq 3^n |Q|$. Therefore,

$$|Q|^\frac{\alpha}{n} \int_Q |f(y)| \, dy \leq 3^{n-\alpha} |Q|^\frac{\alpha}{n} \int_P |f(y)| \, dy \leq C(n, \alpha)M^{D_t}_\alpha f(x) \leq C(n, \alpha) \sup_t M^{D_t}_\alpha f(x).$$

If we take the supremum over all cubes $Q$ containing $x$, we get the desired inequality. $\square$

Because we are working with a finite number of dyadic grids, we have that

$$\sup_t M^{D_t}_\alpha f(x) \approx \sum_{t=1}^{3^n} M^{D_t}_\alpha f(x),$$

and the constants depend only on $n$. In other words, we can dominate any sub-linear expression for $M_\alpha$ by a sum of expressions involving $M^{D_t}_\alpha$. The same will be true for $I_\alpha$. Hereafter, we will use this equivalence without comment.

The dyadic analog of the fractional integral operator is defined as an infinite sum: given $0 < \alpha < n$ and a dyadic grid $D$, for all $f \in L^1_{\text{loc}}$, let

$$I^D_\alpha f(x) = \sum_{Q \in D} |Q|^\frac{\alpha}{n} \langle f \rangle_Q \cdot \chi_Q(x).$$
The dyadic fractional integral operator (with $D = \Delta$) was introduced by Sawyer and Wheeden [93] who showed that averages over an infinite family of dyadic grids dominated $I_\alpha$. Here we show that only a finite number of grids is necessary; this was proved in [25, Proposition 2.2].

**Proposition 3.6.** There exist constants $c(n, \alpha)$, $C(n, \alpha)$ such that for every non-negative function $f \in L^1_{\text{loc}}$ and $1 \leq t \leq 3^n$,

$$c(n, \alpha) I_\alpha^D f(x) \leq I_\alpha f(x) \leq C(n, \alpha) \sup_t I_\alpha^D f(x),$$

where the grids $D^t$ are defined by (3.1).

**Proof.** To prove the first inequality, fix a dyadic grid $D = D^t$, a non-negative function $f$, and $x \in \mathbb{R}^n$. Without loss of generality we may assume that $f$ is bounded: since $I_\alpha$ and $I_\alpha^D$ are positive operators, the inequality for unbounded $f$ follows by the monotone convergence theorem.

Let $\{Q_k\}_{k \in \mathbb{Z}} \subset D$ be the unique sequence of dyadic cubes such that $\ell(Q_k) = 2^k$ and $x \in Q_k$. Then for every integer $N > 0$,

$$\sum_{Q \in D, \ell(Q) \leq 2^N} |Q|^\frac{n}{n-\alpha} \langle f \rangle_Q \cdot \chi_Q(x)$$

$$= \sum_{k = -\infty}^N |Q_k|^\frac{n-1}{n-\alpha} \int_{Q_k \setminus Q_{k-1}} f(y) \, dy + \sum_{k = -\infty}^N |Q_k|^\frac{n-1}{n-\alpha} \int_{Q_{k-1}} f(y) \, dy$$

$$\leq c(n, \alpha) \sum_{k = -\infty}^N \int_{Q_k \setminus Q_{k-1}} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy + 2^{n-n} \sum_{Q \in D, \ell(Q) \leq 2^N} |Q|^\frac{n}{n} \langle f \rangle_Q \cdot \chi_Q(x)$$

$$= c(n, \alpha) \int_{Q_N} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy + 2^{n-n} \sum_{Q \in D, \ell(Q) \leq 2^N} |Q|^\frac{n}{n} \langle f \rangle_Q \cdot \chi_Q(x).$$

Because $f$ is bounded, the last sum is finite. Therefore, since $2^{n-n} < 1$, we can rearrange terms and take the limit as $N \to \infty$ to get

$$c(n, \alpha) I_\alpha^D f(x) \leq I_\alpha f(x).$$

To prove the second inequality, let $Q(x, r)$ be the cube of side-length $2r$ centered at $x$. Then

$$I_\alpha f(x) = \sum_{k \in \mathbb{Z}} \int_{Q(x, 2^k) \setminus Q(x, 2^{k-1})} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy \leq 2^{n-\alpha} \sum_{k \in \mathbb{Z}} 2^{-k(n-\alpha)} \int_{Q(x, 2^k)} f(y) \, dy.$$
By Theorem 3.4, for each \( k \in \mathbb{Z} \) there exists a grid \( \mathcal{D}^t \), \( 1 \leq t \leq 3^n \), and \( Q_t \in \mathcal{D}^t \) such that \( Q(x, 2^k) \subset Q_t \) and 
\[
2^{k+1} = \ell(Q(x, 2^k)) \leq \ell(Q_t) \leq 6\ell(Q(x, 2^k)) = 12 \cdot 2^k.
\]

Since \( \ell(Q_t) = 2^j \) for some \( j \), we must have that \( 2^{k+1} \leq \ell(Q_t) \leq 2^{k+3} \). Hence,
\[
2^n - \alpha \sum_{k \in \mathbb{Z}} (2^{-k})^{n-\alpha} \int_{Q(x, 2^k)} f(y) \, dy
\]
\[
\leq C(n, \alpha) \sum_{k \in \mathbb{Z}} \sum_{t=1}^{3^n} \sum_{Q \in \mathcal{D}^t, 2^{k+1} \leq \ell(Q) \leq 2^{k+3}} |Q|^{\frac{n}{n-\alpha}} (f)_{Q} \cdot \chi_{Q}(x)
\]
\[
\leq C(n, \alpha) \sum_{t=1}^{3^n} \sum_{Q \in \mathcal{D}^t} |Q|^{\frac{n}{n-\alpha}} (f)_{Q} \cdot \chi_{Q}(x)
\]
\[
\leq C(n, \alpha) \sum_{t=1}^{3^n} I_{D}^\alpha f(x)
\]
\[
\leq C(n, \alpha) \sup_t I_{D}^\alpha f(x).
\]

If we combine these two estimates we get the second inequality. \( \square \)

Intuitively, the dyadic version of the commutator \([b, I_{\alpha}]\) is the operator \([b, I_{\alpha}^D]\). However, recall that this operator is not positive: we cannot prove the pointwise bound
\[
| [b, I_{\alpha}] f(x) | \leq C \sup_t | [b, I_{\alpha}^D] f(x) |,
\]
even for \( f \) non-negative. (We are not certain whether this inequality is in fact true.) But if we pull the absolute values inside the integral we do get a useful dyadic approximation of the commutator. The following result was implicit in [23]; the proof is essentially the same as the proof of the second inequality in Proposition 3.6.

**Proposition 3.7.** There exists a constant \( C(n, \alpha) \) such that for every non-negative function \( f \in L_{loc}^1 \) and \( b \in BMO \),
\[
| [b, I_{\alpha}] f(x) | \leq C(n, \alpha) \sup_t C_{b}^D f(x),
\]
where the grids \( \mathcal{D}^t \) are defined by (3.1) and
\[
C_{b}^D f(x) = \sum_{Q \in \mathcal{D}^t} |Q|^{\frac{n}{n-\alpha}} \int_Q |b(x) - b(y)| f(y) \, dy \cdot \chi_{Q}(x).
\]
Sparse operators. We now come to another important reduction: we can replace the dyadic operators $M^D_\alpha$ and $I^D_\alpha$ with operators defined on sparse families. For the fractional maximal operator we replace it with a linear operator that resembles the fractional integral operator. Given a dyadic grid $\mathcal{D}$, a sparse set $\mathcal{S} \subset \mathcal{D}$ and $f \in L^1_{\text{loc}}$, define the operator $L^S_\alpha$ by

$$L^S_\alpha f(x) = \sum_{Q \in \mathcal{S}} |Q|^{\frac{n}{\alpha}} \langle f \rangle_Q \cdot \chi_{E(Q)}(x).$$

The idea for this linearization was implicit in Sawyer [87]; for the maximal operator see also de la Torre [34]. The following result was given without proof in [24].

**Proposition 3.8.** Given a dyadic grid $\mathcal{D}$ and a non-negative function $f$ such that $\langle f \rangle_Q \rightarrow 0$ as $|Q| \rightarrow \infty$, there exists a sparse set $\mathcal{S} = \mathcal{S}(f) \subset \mathcal{D}$ and a constant $C(n, \alpha)$ independent of $f$ such that for every $x \in \mathbb{R}^n$,

$$L^S_\alpha f(x) \leq M^D_\alpha f(x) \leq C(n, \alpha) L^S_\alpha f(x).$$

**Proof.** The sets $E(Q)$ are pairwise disjoint and for every $x \in E(Q)$, $|Q|^{\frac{n}{\alpha}} \langle f \rangle_Q \leq M^D_\alpha f(x)$, so the first inequality follows at once. To prove the second inequality, fix $a = 2^{n+1-\alpha}$ and for each $k \in \mathbb{Z}$, let $\Omega_k = \{x \in \mathbb{R}^n : M^D_\alpha f(x) > a^k\}$. For every $x \in \Omega_k$ there exists $Q \in \mathcal{D}$ such that $|Q|^{\frac{n}{\alpha}} \langle f \rangle_Q > a^k$. Let $\mathcal{S}_k$ be the collection of maximal, disjoint cubes with this property. Such maximal cubes exist by our assumption on $f$. Further, by maximality we must also have that for each $P \in \mathcal{S}_k$, $a^k < |P|^{\frac{n}{\alpha}} \langle f \rangle_P \leq 2^{-\alpha} a^k$, and

$$\Omega_k = \bigcup_{P \in \mathcal{S}_k} P.$$

Let $\mathcal{S} = \bigcup_k \mathcal{S}_k$; we claim that $\mathcal{S}$ is sparse. Clearly these cubes are nested: if $P' \in \mathcal{S}_{k+1}$, then there exists $P \in \mathcal{S}_k$ such that $P' \subset P$. Therefore, if we fix $k \in \mathbb{Z}$ and $P \in \mathcal{S}_k$, and consider the union of cubes $P' \in \mathcal{S}$ with $P' \subset P$, we may restrict the union to $P' \in \mathcal{S}_{k+1}$. Clearly these cubes satisfy $|P'| \leq 2^{-n}|P|$. Hence,

$$\left(3.2\right) \left| \bigcup_{P' \in \mathcal{S}} P' \right| = \sum_{P' \in \mathcal{S}_{k+1}} |P'|^{\frac{n}{\alpha}} \leq \frac{1}{a^{k+1}} \sum_{P' \in \mathcal{S}_{k+1}} |P'|^{\frac{n}{\alpha}} \int_{P'} f(y) \, dy$$

$$\leq \frac{2^{-\alpha}}{a^{k+1}} |P|^{\frac{n}{\alpha}} \int_P f(y) \, dy \leq \frac{2^{n-2\alpha}}{a} |P| = 2^{-\alpha-1}|P|.$$
To get the desired estimate, note first that by the definition of the cubes in $S$, for each $k \in \mathbb{Z}$,

$$
\Omega_k \setminus \Omega_{k+1} = \bigcup_{P \in S_k} E(P).
$$

Therefore, we have that for each $x \in \mathbb{R}^n$, there exists $k$ such that $x \in \Omega_k \setminus \Omega_{k+1}$, and so there exists $P \in S_k$ such that

$$
M_\alpha^D f(x) \leq a^{k+1} \leq a|P|^N (f)_P \cdot \chi_{E(P)} = C(n, \alpha) \sum_{P \in S} |P|^N (f)_P \cdot \chi_{E(P)}.
$$

□

The sparse operator associated with $I_D^\alpha$ is nearly the same as $L_S^\alpha$ except that the characteristic function is for the entire cube $Q$. Given a dyadic grid $D$ and a sparse set $S \subset D$, we define

$$
I_S^\alpha f(x) = \sum_{Q \in S} |Q|^N (f)_Q \cdot \chi_Q(x).
$$

When $\alpha = 0$, this operator becomes the sparse Calderón-Zygmund operator that plays a central role in Lerner’s proof of the $A_2$ conjecture [58, 59]. The operators $I_S^\alpha$ were implicit in Sawyer and Wheeden [93], Pérez [79] and Lacey, et al. [51], and first appeared explicitly in [25], where the following result was proved.

**Proposition 3.9.** Given a dyadic grid $D$ and a non-negative function $f$ such that $(f)_Q \to 0$ as $|Q| \to \infty$, there exists a sparse set $S = S(f) \subset D$ and a constant $C(n, \alpha)$ independent of $f$ such that for every $x \in \mathbb{R}^n$,

$$
I_S^\alpha f(x) \leq I_D^\alpha f(x) \leq C(n, \alpha) I_S^\alpha f(x).
$$

**Proof.** The first inequality is immediate for any subset $S$ of $D$. To prove the second inequality, we first construct the sparse set $S$. The argument is very similar to the construction in Proposition 3.5. Let $a = 2^{n+1}$. For each $k \in \mathbb{Z}$ define

$$
Q_k = \{ Q \in D : a^k < (f)_Q \leq a^{k+1} \}.
$$

Then for every $Q \in D$ such that $(f)_Q \neq 0$, there exists a unique $k$ such that $Q \in Q_k$.

Now define $S_k$ to be the maximal disjoint cubes contained in

$$
\{ P \in D : (f)_P > a^k \}.
$$

Such maximal cubes exist by our hypothesis on $f$. It follows that given any $Q \in Q_k$, there exists $P \in S_k$ such that $Q \subset P$. Furthermore, these cubes are nested: if $P' \in S_{k+1}$, then it is contained in some $P \in S_k$. If we let $S = \bigcup_k S_k$, then arguing as in inequality (3.2) we have that $S$ is sparse.
We now prove the desired inequality. Fix \( x \in \mathbb{R}^n \); then
\[
I^\alpha f(x) = \sum_{k} \sum_{Q \in Q_k} |Q|^\frac{\alpha}{n} \langle f \rangle_Q \cdot \chi_Q(x) \leq \sum_{k} a^{k+1} \sum_{P \in S_k} \sum_{Q \subseteq P} |Q|^\frac{\alpha}{n} \cdot \chi_Q(x).
\]
The inner sum can be evaluated:
\[
\sum_{Q \subseteq P} |Q|^\frac{\alpha}{n} \cdot \chi_Q(x) = \sum_{r=0}^{\infty} \sum_{Q \in Q_k : Q \subseteq P, \ell(Q) = 2^{-r} \ell(P)} |Q|^\frac{\alpha}{n} \cdot \chi_Q(x) = \frac{1}{1 - 2^{-\alpha}} |P|^\frac{\alpha}{n} \cdot \chi_P(x).
\]
Thus we have that
\[
\sum_{k} a^{k+1} \sum_{P \in S_k} \sum_{Q \subseteq P} |Q|^\frac{\alpha}{n} \cdot \chi_Q(x) \leq C(\alpha) \sum_{k} a^{k+1} \sum_{P \in S_k} |P|^\frac{\alpha}{n} \cdot \chi_P(x)
\]
\[
\leq C(n, \alpha) \sum_{k} \sum_{P \in S_k} |P|^\frac{\alpha}{n} \langle f \rangle_P \cdot \chi_P(x) = C(n, \alpha) I^S f(x).
\]
If we combine these estimates we get the desired inequality. \( \square \)

We conclude this section with a key observation:

**In light of Propositions 3.5, 3.6, 3.8 and 3.9, when proving necessary and/or sufficient conditions for weighted norm inequalities for fractional maximal or integral operators, it suffices to prove the analogous inequalities for either the associated dyadic or sparse operators.**

In the subsequent sections we will use this fact repeatedly. The ability to pass to a dyadic operator will considerably simplify the proofs. The choice to use the dyadic or sparse operator will be determined by the details of the proof.

Matters are more complicated for commutators. It is possible to reduce estimates for the dyadic commutator, or more precisely, the dyadic operator \( C^D_b \) defined in Proposition 3.7, to estimates for a sum defined over a sparse set. However, this reduction does not yield a pointwise inequality and is dependent on the particular result to be proved. For an example of this argument, we refer the reader to [23, Theorem 1.6]. This difficulty plays a role in some of the open problems which we will discuss below.

4. **Digression: one weight inequalities**

In this section we briefly turn away from the main topic of these notes, two weight norm inequalities, to present some basic results on one weight norm inequalities. We do so for two reasons. First, in this setting it is easier to see the advantages of the
reduction to dyadic operators; second, a closer examination of the proofs in the one weight case will highlight where the major obstacles will be in the two weight case.

The fractional maximal operator. We first consider the fractional maximal operator. The governing weight class is a generalization of the Muckenhoupt $A_p$ weights, and was introduced by Muckenhoupt and Wheeden [67].

**Definition 4.1.** Given $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, and $q$ such that $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, we say that a weight $w$ such that $0 < w(x) < \infty$ a.e. is in $A_{p,q}$ if

$$[w]_{A_{p,q}} = \sup_Q \left( \int_Q w^q \, dx \right)^{\frac{1}{q}} \left( \int_Q w^{-p'} \, dx \right)^{\frac{1}{p'}} < \infty,$$

where the supremum is taken over all cubes $Q$. When $p = 1$ we say $w \in A_{1,q}$ if

$$[w]_{A_{1,q}} = \sup_Q \sup_{x \in Q} \left( \int_Q w^q \, dx \right)^{\frac{1}{q}} w(x)^{-1} < \infty.$$

The $A_{1,q}$ condition is equivalent to assuming that $M_q w(x) = M(w^q)(x)^{1/q} \leq [w]_{A_{1,q}} w(x)$, that is, $w^q \in A_1$. (For a proof of this when $q = 1$, see [38, Section 5.1].) More generally, if $p > 1$, we have that $w \in A_{p,q}$ if and only if $w^q \in A_{1+\frac{p}{q}}$; this follows at once from the definition. By symmetry we have that $w \in A_{p,q}$ if and only if $w^{-1} \in A_{q',p'}$, and this is equivalent to $w^{-p'} \in A_{1+\frac{q'}{p'}}$.

In our proofs we will keep track of the dependence on the constant $[w]_{A_{p,q}}$; however, our proofs will not yield sharp results. For the exact dependence, see [23, 51].

**Theorem 4.2.** Given $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, $q$ such that $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, and a weight $w$, the following are equivalent:

1. $w \in A_{p,q}$;
2. for any $f \in L^p(w^p)$,

$$\sup_{t>0} t w^q(\{x \in \mathbb{R}^n : M_\alpha f(x) > t\})^{\frac{1}{q}} \leq C(n, \alpha) [w]_{A_{p,q}} \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)^p \, dx \right)^{\frac{1}{p}}.$$

The sufficiency of the $A_{p,q}$ condition was first proved in [67]. Our proof is basically the same as theirs, but using the sparse operator $L^S_\alpha$ obviates the need for a covering lemma argument—this is “hidden” in the construction of the sparse operator. The necessity of the $A_{p,q}$ condition was not directly considered but was implicit in their results for the fractional integral. Our argument below is adapted from the case $\alpha = 0$ in [38, Section 5.1].

**Proof.** To show the sufficiency of the $A_{p,q}$ condition, without loss of generality we may assume $f$ is non-negative. It is straightforward to show that if the sequence
\{f_k\} increases pointwise a.e. to \(f\), then \(M_\alpha f_k\) increases to \(M_\alpha f\), so we may assume that \(f\) is bounded and has compact support. (For the details of this argument when \(\alpha = 0\), see [16, Lemma 3.30].) Further, it will suffice to fix a dyadic grid \(\mathcal{D}\) and prove the weak type inequality for \(M^D_\alpha\).

We first consider the case when \(p > 1\). Fix \(t > 0\). If \(x \in \mathbb{R}^n\) is such that \(M^D_\alpha f(x) > t\), then there exists a cube \(Q \in \mathcal{D}\) such that \(|Q|^{\alpha/n} \langle f \rangle_Q > t\). Let \(\mathcal{Q}\) be the set of maximal disjoint cubes in \(\mathcal{D}\) with this property. (Such cubes exist by our assumptions on \(f\).) Then by H"{o}lder’s inequality,

\[
\begin{align*}
t^q w^q(\{x \in \mathbb{R}^n : M^D_\alpha f(x) > t\}) &= \sum_{Q \in \mathcal{Q}} w^q(Q) \\
&\leq \sum_{Q \in \mathcal{Q}} w^q(Q) (|Q|^{\alpha/n} \langle f \rangle_Q)^q \\
&\leq \sum_{Q \in \mathcal{Q}} |Q|^{\alpha/q - q} w^q(Q) \left( \int_Q f(y)w(y)^{-1} \, dy \right)^q \\
&\leq \sum_{Q \in \mathcal{Q}} |Q|^{\alpha/q - q} w^q(Q) \left( \int_Q w(y)^{-p'} \, dy \right)^{\frac{q}{p'}} \left( \int_Q f(y)^p w(y)^p \, dy \right)^{\frac{q}{p}};
\end{align*}
\]

by our choice of \(q\), \(q - q^\alpha_n = 1 + \frac{q}{p}\), so by the \(A_{p,q}\) condition,

\[
\begin{align*}
&\leq [w]_{A_{p,q}}^q \sum_{Q \in \mathcal{Q}} \left( \int_Q f(y)^p w(y)^p \, dy \right)^{\frac{q}{p'}} \\
&\leq [w]_{A_{p,q}}^q \left( \sum_{Q \in \mathcal{Q}} \int_Q f(y)^p w(y)^p \, dy \right)^{\frac{q}{p'}} \\
&\leq [w]_{A_{p,q}}^q \left( \int_{\mathbb{R}^n} f(y)^p w(y)^p \, dy \right)^{\frac{q}{p'}}.
\end{align*}
\]

The second to last inequality holds because \(\frac{q}{p} \geq 1\) and the final inequality since the cubes in \(\mathcal{Q}\) are pairwise disjoint by maximality. This completes the proof of the weak type inequality when \(p > 1\).

When \(p = 1\) the same proof works, omitting H"{o}lder’s inequality and using the pointwise inequality in the \(A_{1,q}\) condition.

To prove the necessity of the \(A_{p,q}\) condition, we again first consider the case \(p > 1\). Fix a cube \(Q\) and let \(f = w^{-p'} \chi_Q\). Then for \(x \in Q\), \(M_\alpha f(x) \geq |Q|^{\alpha/n} (w^{-p'})_Q\). Then
for all $t < |Q|^\frac{n}{q} \langle w^{-p'} \rangle_Q$, the weak type inequality implies that

\[
t^q w^q(Q) \leq C \left( \int_Q f(x)^p w(x)^p \, dx \right) \frac{q}{p} = C |Q|^\frac{n}{p} \left( \int_Q w(x)^{-p'} \, dx \right)^\frac{q}{p}.
\]

Taking the supremum over all such $t$ yields

\[
|Q|^\frac{n}{q} \int_Q w(x)^q \left( \int_Q w(x)^{-p'} \, dx \right)^q \leq C |Q|^\frac{n}{p} \left( \int_Q w(x)^{-p'} \, dx \right)^\frac{n}{p},
\]

and rearranging terms we get the $A_{p,q}$ condition on $Q$ with a uniform constant.

When $p = 1$ we repeat the above argument but now with $f = \chi_P$, where $P \subset Q$ is any cube. Then we get

\[
\int_Q w(x)^q \, dx \leq C \left( \int_P w(x)^p \, dx \right)^\frac{n}{p}.
\]

Let $x_0$ be a Lebesgue point of $w^p$ in $Q$, and take the limit as $P \to \{x_0\}$; by the Lebesgue differentiation theorem we get

\[
\int_Q w(x)^q \, dx \leq C w(x_0)^q.
\]

The $A_{1,q}$ condition follows at once. \hfill \square

The weak type inequality and its proof have two consequences. First, the proof when $p = 1$, holds for all $p$ and we can replace the cube $P$ by any measurable set $E \subset Q$. Doing this yields an $A_\infty$ type inequality:

\[
(4.1) \quad \frac{|E|}{|Q|} \leq [w]_{A_{p,q}} \left( \frac{w^q(E)}{w^q(Q)} \right)^\frac{1}{q}.
\]

Second, though we assumed a priori in the definition of the $A_{p,q}$ condition that $0 < w(x) < \infty$ a.e., we can use this inequality to show that this in fact is a consequence of the weak type inequality. For the details of the proof when $\alpha = 0$, see [38, Section 5.1]. We note in passing that the usual $A_\infty$ condition, which exchanges the roles of $w^q$ and Lebesgue measure in (4.1), is more difficult to prove since it also requires the reverse Hölder inequality.

To prove the strong type inequality we could use the fact that $w^q \in A_{1+\frac{\varepsilon}{q},\frac{n}{q}}$ implies $w^q \in A_{1+\frac{\varepsilon}{q},\frac{n}{q}-\epsilon}$ for some $\epsilon > 0$ to apply Marcinkiewicz interpolation. This is the approach used in [67] and it requires the reverse Hölder inequality.

Instead, here we are going to give a direct proof that avoids the reverse Hölder inequality. It is based on an argument for the Hardy-Littlewood maximal operator due to Christ and Fefferman [10] that only uses (4.1). We also introduce an auxiliary operator, a weighted dyadic fractional maximal operator. Such weighted operators
when $\alpha = 0$ have played an important role in the proof of sharp constant inequalities: see [20, 57, 59]. Given a non-negative Borel measure $\sigma$ and a dyadic grid $D$, define

$$M_{\sigma,\alpha}^D f(x) = \sup_{Q \in D} |Q|^\frac{\alpha}{n} \int_Q |f(y)| \, d\sigma \cdot \chi_Q(x).$$

If $\alpha = 0$ we simply write $M_{\sigma}^D$.

**Lemma 4.3.** Given $0 \leq \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, $q$ such that $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, a dyadic grid $D$, and a non-negative Borel measure $\sigma$, $\sigma(Q)^\frac{\alpha}{n} \int_Q |f(x)|^p \, d\sigma \leq \frac{1}{t} \int_{R^n} |f(x)|^p w(x)^p \, dx \leq C(n, \alpha, p, [w]_{A_{p,q}}) \int_{R^n} |f(x)|^p w(x)^p \, dx \leq \|f\|_{L^p_{\alpha,\sigma}(\sigma)}$.

Furthermore, if $p > 1$, $\int_{R^n} M_{\sigma,\alpha}^D f(x)^q w(x)^q \, dx \leq C(p, q) \int_{R^n} |f(x)|^p \, d\sigma$.

**Proof.** The proof of the weak $(1, q)$ inequality for $M_{\sigma,\alpha}^D$ is essentially the same as the proof of Theorem 4.2. By Hölder’s inequality we have for any cube $Q \in D$,

$$\sigma(Q)^{\frac{\alpha}{n}} \int_Q |f(x)| \, d\sigma \leq \sigma(Q)^{\frac{\alpha}{n}} \left( \int_Q |f(x)|^\frac{n}{\alpha} \, d\sigma \right)^{\frac{n}{\alpha}} \leq \|f\|_{L^\frac{n}{\alpha}_{\sigma,\alpha}(\sigma)}.$$

which immediately implies that $M_{\sigma,\alpha}^D : L^\frac{n}{\alpha}_{\sigma,\alpha}(\sigma) \rightarrow L^\infty$. The strong $(p, q)$ inequality then follows from off-diagonal Marcinkiewicz interpolation [96, Chapter V, Theorem 2.4].

**Theorem 4.4.** Given $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, $q$ such that $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, and a weight $w$, the following are equivalent:

1. $w \in A_{p,q}$;
2. for any $f \in L^p(w^p)$,

$$\left( \int_{R^n} M_{\alpha} f(x)^q w(x)^q \, dx \right)^{\frac{1}{q}} \leq C(n, \alpha, p, [w]_{A_{p,q}}) \left( \int_{R^n} |f(x)|^p w(x)^p \, dx \right)^{\frac{1}{p}}.$$

**Proof.** Since the strong type inequality implies the weak type inequality, necessity follows from Theorem 4.2. To prove sufficiency we can again assume $f$ is non-negative, bounded and has compact support, and so it is enough to prove the strong type inequality for $L^\frac{n}{\alpha}_{\sigma} f$, where $S$ is any sparse subset of a dyadic grid $D$.

Let $\sigma = w^{-p'}$. Since the sets $E(Q)$, $Q \in S$ are disjoint, we have that

$$\|L^\frac{n}{\alpha}_{\sigma} f\|_q^q = \sum_{Q \in S} |Q|^\frac{\alpha}{n} \langle f \rangle^q_Q w^q(E(Q))$$
\[
\leq \sum_{Q \in S} (\sigma(Q)^{\frac{n}{q}} (f \sigma^{-1})_Q)_Q^q |Q|^q w^q(Q) \sigma(Q)^{q-q^*}
\]
\[
= \sum_{Q \in S} (\sigma(Q)^{\frac{n}{q}} (f \sigma^{-1})_Q)_Q^q |Q|^{-\frac{n}{p}} w^q(Q) \sigma(Q)^{\frac{q}{p}} \sigma(Q);
\]
by inequality (4.1), the properties of sparse cubes, the definition of \( A_{p,q} \) and Lemma 4.3,
\[
\leq C([w]_{A_{p,q}}) \sum_{Q \in S} (\sigma(Q)^{\frac{n}{q}} (f \sigma^{-1})_Q)_Q^q \sigma(E(Q))
\]
\[
\leq C([w]_{A_{p,q}}) \int_{E(Q)} M_{\sigma,\alpha}^D (f \sigma^{-1})(x)^q d\sigma
\]
\[
\leq C([w]_{A_{p,q}}) \int_{\mathbb{R}^n} M_{\sigma,\alpha}^D (f \sigma^{-1})(x)^q d\sigma
\]
\[
\leq C(p, q, [w]_{A_{p,q}}) \left( \int_{\mathbb{R}^n} f(x)^p \sigma(x)^{-p} \sigma(x) \, dx \right)^{\frac{q}{p}}
\]
\[
= C(p, q, [w]_{A_{p,q}}) \left( \int_{\mathbb{R}^n} f(x)^p w(x)^p \, dx \right)^{\frac{q}{p}}.
\]

The above proof has several features that we want to highlight. First, since the sets \( E(Q), Q \in S \) are pairwise disjoint, we are able to pull the power \( q \) inside the summation. For dyadic fractional integrals (even sparse ones) this is no longer the case. As we will see below, the standard technique for avoiding this problem is to use duality. Second, a central obstacle is that we have a sum over cubes \( Q \) that are not themselves disjoint, so we need some way of reducing the sum to the sum of integrals over disjoint sets. Here we use that the cubes in \( S \) are sparse, and then use the \( A_{\infty} \) property given by inequality (4.1). In the two weight setting we will no longer have this property. To overcome this we will pass to a carefully chosen subfamily of cubes that are sparse with respect to some measure induced by the weights (e.g., \( d\sigma \) in the proof above).

**The fractional integral operator.** We now turn to one weight norm inequalities for the fractional integral operator. We will give a direct proof of the strong type inequality that appears to be new, though it draws upon ideas already in the literature: in particular, the two weight bump conditions for the fractional integral due to Pérez [79] (see Theorem 6.9 below). The original proof of this result by Muckenhoupt and Wheeden [67] used a good-\( \lambda \) inequality; another proof using sharp maximal function estimates and extrapolation was given in [17] (see also [21, Chapter 9]). One
important feature of these approaches is that they also yield weak type inequalities for the fractional integral. It would be very interesting to give a proof of the weak type inequalities using the techniques of this section as it would shed light on several open problems: see Section 7.

**Theorem 4.5.** Given \(0 < \alpha < n\), \(1 < p < \frac{n}{\alpha}\), \(q\) such that \(\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}\), and a weight \(w\), the following are equivalent:

1. \(w \in A_{p,q}\);
2. for any \(f \in L^p(w^p)\),
   \[
   \left( \int_{\mathbb{R}^n} I_\alpha f(x)^q w(x)^q \, dx \right)^{\frac{1}{q}} \leq C(n, \alpha, p, [w]_{A_{p,q}}) \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)^p \, dx \right)^{\frac{1}{p}}.
   \]

**Proof.** By the pointwise inequality \(M_\alpha^D f(x) \leq I_\alpha^D f(x)\), the necessity of the \(A_{p,q}\) condition follows from Theorem 4.4.

To prove sufficiency, we may assume \(f\) is non-negative. Furthermore, by the monotone convergence theorem, if \(\{f_k\}\) is any sequence of functions that increases pointwise a.e. to \(f\), then for each \(x \in \mathbb{R}^n\), \(I_\alpha f_k(x)\) increases to \(I_\alpha f(x)\). Therefore, we may also assume that \(f\) is bounded and has compact support. Thus, it will suffice to prove this result for the sparse operator \(I_\alpha^S\), where \(S\) is any sparse subset of a dyadic grid \(D\).

Let \(v = w^q\) and \(\sigma = w^{-p'}\) and estimate as follows: there exists \(g \in L^{q'}(w^{-q'})\), \(\|gw^{-1}\|_{q'} = 1\), such that

\[
\|(I_\alpha^S f)w\|_q = \int_{\mathbb{R}^n} I_\alpha f(x) g(x) \, dx \\
= \sum_{Q \in S} |Q|^{\frac{\alpha}{n}} <f>_{Q} \int_{Q} g(x) \, dx \\
= \sum_{Q \in S} |Q|^{\frac{\alpha}{n} - 1} \sigma(Q)v(Q)^{1 - \frac{\alpha}{n}} <f\sigma^{-1}>_{Q,v}v(Q)^\frac{\alpha}{n} (gv^{-1})_{Q,v}.
\]

Since \(1 - \frac{\alpha}{n} = \frac{1}{p'} + \frac{1}{q}\), by the definition of the \(A_{p,q}\) condition and inequality (4.1) (applied to both \(v\) and \(\sigma\)), we have that

\[
|Q|^{\frac{\alpha}{n} - 1} \sigma(Q)v(Q)^{1 - \frac{\alpha}{n}} \leq [w]_{A_{p,q}} \sigma(Q)^{\frac{1}{p'}} v(Q)^{\frac{1}{p'}} \leq C([w]_{A_{p,q}}) \sigma(E(Q))^{\frac{1}{p'}} v(E(Q))^{\frac{1}{p'}}.
\]

If we combine these two estimates, by Hölder’s inequality and Lemma 4.3 we get that

\[
\|(I_\alpha^S f)w\|_q \\
\leq C([w]_{A_{p,q}}) \sum_{Q \in S} <f\sigma^{-1}>_{Q,v} \sigma(E(Q))^{\frac{1}{p'}} v(Q)^{\frac{\alpha}{n}} (gv^{-1})_{Q,v} v(E(Q))^{\frac{1}{p'}}
\]
\[
\begin{align*}
&\leq C([w]_{A,p,q}) \left( \sum_{Q \in S} (f^{\sigma-1})_{Q,\sigma}(E(Q)) \right)^{\frac{1}{p'}} \left( \sum_{Q \in S} [v(Q)]_{\alpha}^a (gv^{-1})_{Q,v} \right)^{\frac{1}{p}} v(E(Q)) \right) \frac{1}{p}
\end{align*}
\]

Commutators. We conclude this section with the statement of the one weight norm inequality for the commutator \([b, I_\alpha]\). This was proved in [23] using a Cauchy integral formula technique due to Chung, Pereyra and P´erez [11]. We refer the reader there for the details of the proof.

**Theorem 4.6.** Given \(0 < \alpha < n\), \(1 < p < \frac{n}{\alpha}\), \(q\) such that \(\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}\), \(b \in \text{BMO} \) and a weight \(w\), then for any \(f \in L^p(w^p)\),

\[
\left( \int_{\mathbb{R}^n} [b, I_\alpha] f(x)^q w(x)^q \, dx \right)^{\frac{1}{q}} \leq C(n, \alpha, p, [w]_{A,p,q}, \|b\|_{\text{BMO}}) \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)^p \, dx \right)^{\frac{1}{p}}.
\]

5. Testing conditions

In this section we turn to our main topic: two weight norm inequalities for fractional maximal and integral operators and for commutators. We will consider one of the two dominant approaches to this problem: the Sawyer testing conditions.

**Two weight inequalities.** Before discussing characterizations of two weight inequalities, we first reformulate them in a way that works well with arbitrary weights. We are interested in weak and strong type inequalities of the form

\[
\sup_{t > 0} t u(\{x \in \mathbb{R}^n : |T f(x)| > t\})^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p u(x) \, dx \right)^{\frac{1}{p}}
\]

\[
\left( \int_{\mathbb{R}^n} |T f(x)|^q v(x) \, dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) \, dx \right)^{\frac{1}{p}}
\]
where $1 < p \leq q < \infty$ and $T$ is one of $M_\alpha$, $I_\alpha$, or $[b, I_\alpha]$. For the weak type inequality we can also consider the (more difficult) endpoint inequality when $p = 1$. For two weight inequalities we no longer assume that there is a relationship among $p$, $q$ and $\alpha$. This allows us to consider “diagonal” inequalities: e.g., $I_\alpha : L^p(v) \to L^p(u)$. For this reason it is more convenient to write the weights as measures (e.g., "$u \, dx$") rather than as “multipliers” as we did in the previous section for one weight norm inequalities.

However, there are some problems with this formulation. For instance, since $I_\alpha$ is self-adjoint, a strong type inequality also implies a dual inequality. For instance, at least formally, the dual inequality to $I_\alpha : L^p(v) \to L^p(u)$ is

$$I_\alpha : L^q(u^{1-q'}) \to L^{q'}(v^{1-p'}).$$

To make sense of this we need to assume either that $0 < v(x) < \infty$ a.e. (which precludes weights that have compact support) or deal with weights that are measurable functions but equal infinity on sets of positive measure. This is possible, but it requires some care to consistently evaluate expressions of the form $0 \cdot \infty$. For a careful discussion of the details in one particular setting, see [21, Section 7.2].

To avoid these problems we adopt a point of view first introduced by Sawyer [88, 89]. We introduce a new weight $\sigma = u^{1-p'}$ and replace $f$ by $f\sigma$; then we can restate the weak and strong type inequalities as

$$\sup_{t>0} tu \left( \{ x \in \mathbb{R}^n : |T(f\sigma)(x)| > t \} \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \sigma(x) \, dx \right)^{1/p},$$

$$\left( \int_{\mathbb{R}^n} |T(f\sigma)(x)|^q u(x) \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \sigma(x) \, dx \right)^{1/p}.$$

With this formulation, the dual inequality becomes much more natural: for example, for $I_\alpha$, the dual of

$$(5.1) \quad I_\alpha(\cdot \sigma) : L^p(\sigma) \to L^q(u)$$

is given by

$$(5.2) \quad I_\alpha(\cdot u) : L^q(u) \to L^{p'}(\sigma).$$

Hereafter, in a slight abuse of terminology, we will refer to inequalities like (5.2) as the dual of (5.1) even if the operator involved (e.g., $M_\alpha$) is not self-adjoint or even linear.

Another advantage of this formulation (though not one we will consider here) is that in this form one can take $u$ and $\sigma$ to be non-negative measures. See for instance, Sawyer [88], or more recently, Lacey [49].
Finally, we note in passing that two weight inequalities when $q < p$ are much more difficult and we will not discuss them. For more information on such inequalities for $I_\alpha$, we refer the reader to Verbitsky [100] and the recent paper by Tanaka [97]. We are not aware of any analogous results for $M_\alpha$ or $[b, I_\alpha]$.

**Testing conditions for fractional maximal operators.** Our first approach to characterizing the pairs of weights $(u, \sigma)$ for which a two weight inequality hold is via testing conditions. The basic idea of a testing condition is to show that an operator $T$ satisfies the strong $(p, q)$ inequality $T(\sigma) : L^p(\sigma) \to L^q(u)$ if and only if $T$ satisfies it when restricted to a family of test functions: for instance, the characteristic functions of cubes, $\chi_Q$. This approach to the problem is due to Sawyer, who first proved testing conditions for maximal operators [87], the Hardy operator [89], and fractional integrals [88, 91]. For this reason, these are often referred to as Sawyer testing conditions.

Testing conditions received renewed interest in the work of Nazarov, Treil and Volberg [73, 75, 101]; they first made explicit the conjecture that testing conditions were necessary and sufficient for singular integral operators, beginning with the Hilbert transform. (Even this case is an extremely difficult problem which was only recently solved by Lacey, Sawyer, Shen and Uriarte-Tuero [50, 53].) They also pointed out (see [101]) the close connection between testing conditions and the David-Journé $T1$ theorem that characterizes the boundedness of singular integrals on $L^2$. This was not immediately obvious in the original formulation of the $T1$ theorem, but became clear in the version given by Stein [95].

We first consider the testing condition that characterizes the strong $(p, q)$ inequality for the fractional maximal operator. As we noted, this was first proved by Sawyer [87]. Here we give a new proof based on ideas of Hytönen [42] and Lacey, et al. [53]. For a related proof that avoids duality and is closer in spirit to the proof of Theorem 4.4, see Kairema [45].

**Theorem 5.1.** Given $0 \leq \alpha < n$, $1 < p \leq q < \infty$, and a pair of weights $(u, \sigma)$, the following are equivalent:

1. $(u, \sigma)$ satisfy the testing condition

   $$M_\alpha = \sup_Q \sigma(Q)^{-1/p} \left( \int_Q M_\alpha(\chi_Q \sigma)(x)^q u(x) \, dx \right)^{1/q} < \infty;$$

2. for every $f \in L^p(\sigma)$,

   $$\left( \int \left( \int_{\mathbb{R}^n} M_\alpha(f \sigma)(x)^q u(x) \, dx \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \leq C(n, p, \alpha) M_\alpha \left( \int |f(x)|^p \sigma(x) \, dx \right)^{\frac{1}{p}}.$$
To overcome the fact that the weights \( u \) and \( \sigma \) need not satisfy the \( A_\infty \) condition (which was central to the proof in the one weight case) we introduce a stopping time argument referred to as the corona decomposition. This technique was one of the tools introduced into the study of the \( A_2 \) conjecture by Lacey, Petermichl and Reguera [52]. The terminology goes back to David and Semmes [32, 33], but the construction itself seems to have first been used by Muckenhoupt and Wheeden [69] in one dimension, where they constructed “principal intervals.” (See also [18, 90].)

Before proving Theorem 5.1 we first describe the corona construction in more general terms. Given a fixed dyadic cube \( Q_0 \) in a dyadic grid \( \mathcal{D} \), a family of dyadic cubes \( \mathcal{T} \subset \mathcal{D} \) all contained in \( Q_0 \), a non-negative, locally integrable function \( f \), and a weight \( \sigma \), we define a subfamily \( \mathcal{F} \subset \mathcal{T} \) inductively. Let \( \mathcal{F}_0 = \{Q_0\} \). For \( k \geq 0 \), given the collection of cubes \( \mathcal{F}_k \), and \( F \in \mathcal{F}_k \) let \( \eta_{\mathcal{F}}(F) \) be the collection of maximal disjoint subcubes \( Q \) of \( F \) such that \( \langle f \rangle_{Q, \sigma} \geq 2 \langle f \rangle_{F, \sigma} \). (This collection could be empty; if it is the construction stops.) Then set

\[
\mathcal{F}_{k+1} = \bigcup_{F \in \mathcal{F}_k} \eta_{\mathcal{F}}(F)
\]

and define

\[
\mathcal{F} = \bigcup_k \mathcal{F}_k.
\]

We will refer to \( \mathcal{F} \) as the corona cubes of \( f \) with respect to \( \sigma \).

Given any cube \( Q \in \mathcal{T} \), then by construction it is contained in some cube in \( \mathcal{F} \). Let \( \pi_{\mathcal{F}}(Q) \) be the smallest cube in \( \mathcal{F} \) such that \( Q \subset \pi_{\mathcal{F}}(Q) \). We will refer to the cubes \( \eta_{\mathcal{F}}(F) \) as the children of \( F \) in \( \mathcal{F} \), and \( \pi_{\mathcal{F}}(Q) \) as the parent of \( Q \) in \( \mathcal{F} \).

The cubes in \( \mathcal{F} \) have the critical property that they are sparse with respect to the measure \( d\sigma \). Given any \( F \in \mathcal{F} \), if we compute the measure of the children of \( F \) we see that

\[
\sum_{F' \in \eta_{\mathcal{F}}(F)} \sigma(F') \leq \frac{1}{2} \sum_{F' \in \eta_{\mathcal{F}}(F)} \frac{(f\sigma)(F')}{\langle f \rangle_{\sigma,F}} \leq \frac{1}{2} \frac{(f\sigma)(F)}{\langle f \rangle_{\sigma,F}} \leq \frac{1}{2} \sigma(F).
\]

Therefore, if we define the set

\[
E_{\mathcal{F}}(F) = F \setminus \bigcup_{F' \in \eta_{\mathcal{F}}(F)} F',
\]

then

\[
\sigma(E_{\mathcal{F}}(F)) \geq \frac{1}{2} \sigma(F).
\]

We will refer to this as the \( A_\infty \) property of the cubes in \( \mathcal{F} \).

\(^3\)In the literature, the notation \( \text{ch}_{\mathcal{F}}(F) \) is often used for the children of \( F \). We wanted to use Greek letters to denote both sets. The letter \( \eta \) seemed appropriate since it is the Greek “h”, and in Spanish the cubes in these collections are called hijos and padres.
Below we will perform this construction not just on a single cube $Q_0$ but on each cube in a fixed set of disjoint cubes. We will again refer to the collection of all the cubes that result from this construction applied to each cube in this set as $\mathcal{F}$.

**Proof of Theorem 5.1.** The necessity of the testing condition is immediate if we take $f = \chi_Q$.

To prove the sufficiency of the testing condition, first note that arguing as we did in the proof of Theorem 4.2 we may assume that $f$ is non-negative, bounded and has compact support. Therefore, it will suffice to show that given any dyadic grid $\mathcal{D}$ and sparse set $\mathcal{S} \subset \mathcal{D}$, the strong type inequality holds for $L_\alpha^S$ assuming the testing condition holds for $L_\alpha^S$. Here we use the fact that given $f$ there exists a sparse subset $\mathcal{S}$ such that $M_\mathcal{D}^f(x) \lesssim L_\alpha^Sf(x)$, and that for every such sparse set, $L_\alpha^S(\chi_Q\sigma)(x) \leq M_\mathcal{D}^S(\chi_Q\sigma)(x)$.

Fix $\mathcal{D}$, $\mathcal{S}$ and $f$. Then there exists a function $g \in L^{q'}(u)$, $\|g\|_{L^{q'}(u)} = 1$, such that

$$\|L_\alpha^S(f\sigma)\|_{L^{q'}(u)} = \int_{\mathbb{R}^n} L_\alpha^S(f\sigma)(x)g(x)u(x) \, dx = \sum_{Q \in \mathcal{S}} |Q|^{\frac{n}{p'}} \langle f\sigma \rangle_Q \int_{E(Q)} g(x)u(x) \, dx.$$

To estimate the right-hand side, fix $N \geq 0$ and let $\mathcal{S}_N$ be the maximal disjoint cubes $Q$ in $\mathcal{S}$ such that $\ell(Q) \leq 2^N$. Then by the monotone convergence theorem it will suffice to prove that

$$\sum_{Q \in \mathcal{S}_N} |Q|^{\frac{n}{p'}} \langle f\sigma \rangle_Q \int_{E(Q)} g(x)u(x) \, dx \leq C(n,p,\alpha)M_\alpha\|f\|_{L^p(\sigma)}.$$

For each cube $Q \in \mathcal{S}_N$, form the corona decomposition of $f$ with respect to $\sigma$. Then we can rewrite the sum above as

$$\sum_{Q \in \mathcal{S}_N} |Q|^{\frac{n}{p'}} \langle f\sigma \rangle_Q \int_{E(Q)} g(x)u(x) \, dx = \sum_{F \in \mathcal{F}} \sum_{Q \in \mathcal{S}_N \atop \pi_F(Q) = F} |Q|^{\frac{n}{p'}} \langle f\sigma \rangle_Q \int_{E(Q)} g(x)u(x) \, dx.$$

Fix a cube $F$ and $Q$ such that $\pi_F(Q) = F$. Then given any $F' \in \eta_F(F)$, we must have that $F' \cap Q = \emptyset$ or $F' \subset Q$. If the latter, then, since $\mathcal{S}$ is sparse, we must have that $F' \cap E(Q) = \emptyset$. Therefore,

$$\int_{E(Q)} g(x)u(x) \, dx = \int_{E(Q) \cap E_F(F)} g(x)u(x) \, dx + \sum_{F' \in \eta_F(F)} \int_{E(Q) \cap F'} g(x)u(x) \, dx = \int_{E(Q) \cap E_F(F)} g(x)u(x) \, dx.$$
Let \( g_F(x) = g(x) \chi_{E(F)} \) and argue as follows: by the definition of the corona cubes, the testing condition, and Hölder’s inequality,

\[
\sum_{F \in \mathcal{F}} \sum_{Q \in S_N \pi_F(Q) = F} |Q|^{\frac{n}{n'}} \langle f \sigma \rangle_Q \int_{E(Q)} g(x) u(x) \, dx
\]

\[
= \sum_{F \in \mathcal{F}} \sum_{Q \in S_N \pi_F(Q) = F} |Q|^{\frac{n}{n'}} \langle f \sigma \rangle_Q \int_{E(Q)} g_F(x) u(x) \, dx
\]

\[
\leq 2 \sum_{F \in \mathcal{F}} \langle f \rangle_{F, \sigma} \sum_{Q \in S_N \pi_F(Q) = F} |Q|^{\frac{n}{n'}} \langle \sigma \rangle_Q \int_{E(Q)} g_F(x) u(x) \, dx
\]

\[
\leq 2 \sum_{F \in \mathcal{F}} \langle f \rangle_{F, \sigma} \int_{F} L_{\alpha}(\sigma \chi_F)(x) g_F(x) u(x) \, dx
\]

\[
\leq 2 \sum_{F \in \mathcal{F}} \langle f \rangle_{F, \sigma} \| L_{\alpha}(\sigma \chi_F) \|_{L^q(u)} \| g_F \chi_F \|_{L^{q'}(u)}
\]

\[
\leq 2 \mathcal{M}_\alpha \sum_{F \in \mathcal{F}} \langle f \rangle_{F, \sigma} \sigma(F)^{1/p} \| g_F \chi_F \|_{L^{q'}(u)}
\]

\[
\leq 2 \mathcal{M}_\alpha \left( \sum_{F \in \mathcal{F}} \langle f \rangle_{F, \sigma}^p \sigma(F) \right)^{1/p} \left( \sum_{F \in \mathcal{F}} \| g_F \chi_F \|_{L^{q'}(u)}^p \right)^{1/p}.
\]

We estimate each of these sums separately. For the first we use the \( A_\infty \) property of cubes in \( \mathcal{F} \) and Lemma 4.3:

\[
\left( \sum_{F \in \mathcal{F}} \langle f \rangle_{F, \sigma}^p \sigma(F) \right)^{1/p} \leq 2^{\frac{1}{p}} \left( \sum_{F \in \mathcal{F}} \langle f \rangle_{F, \sigma}^p \sigma(E_F(F)) \right)^{1/p}
\]

\[
\leq 2^{\frac{1}{p}} \left( \sum_{F \in \mathcal{F}} \int_{E_F(F)} M_{\sigma}^D f(x)^p \, d\sigma \right)^{1/p} \leq 2^{\frac{1}{p}} \left( \int_{\mathbb{R}^n} M_{\sigma}^D f(x)^p \, d\sigma \right)^{1/p} \leq C(n, p) \| f \|_{L^p(\sigma)}.
\]

To estimate the second sum we use the fact that \( q' \leq p' \):

\[
\left( \sum_{F \in \mathcal{F}} \| g_F \chi_F \|_{L^{q'}(u)}^{p'} \right)^{1/q'} \leq \left( \sum_{F \in \mathcal{F}} \| g_F \chi_F \|_{L^{q'}(u)}^{q'} \right)^{1/q'}
\]

\[
= \left( \int_{E_F(F)} g(x)^{q'} u(x) \, dx \right)^{1/q'} \leq \left( \int_{\mathbb{R}^n} g(x)^{q'} u(x) \, dx \right)^{1/q'} = 1.
\]

If we combine these two estimates we get the desired inequality. \( \square \)
One consequence of this proof is that a weaker condition on the operator is actually sufficient. At the point we apply the testing condition, we could replace $L_S^\alpha(\sigma \chi_F)$ with the smaller, localized operator

$$L_{\alpha,F}^\text{S,In} \sigma(x) = \sum_{Q \in S, Q \subset F} |Q|^{\alpha \langle \sigma \rangle_Q \chi_E(Q)(x)}.$$  

The discarded portion of the sum contains no additional information: for all $x \in F$,

$$\sum_{Q \in S, F \subset Q} |Q|^{\alpha \langle \sigma \chi_F \rangle_Q \chi_E(Q)(x)} \leq \sigma(F) \sum_{k=1}^\infty |F|^{\alpha \langle \sigma \rangle} 2^{\alpha-n} \chi_F(x) \leq C(n, \alpha)|F|^\frac{\alpha}{\alpha-1} \chi_F(x).$$

The final characteristic function is over $F$ instead of $E(F)$, but this yields a finite overlap and so does not substantially affect the rest of the estimate. We will consider such local testing conditions again for the fractional integral operator below.

**Testing conditions for fractional integral operators.** We now prove a testing condition theorem for fractional integrals. If we try to modify the proof of Theorem 5.1 we quickly discover the main obstacle: since the sum defining $I_\alpha^S$ is over the characteristic functions $\chi_Q$ and not $\chi_E(Q)$, the definition of the function $g_F$ must change. There are additional terms in the sum and the estimate for the norm of $g_F$ no longer works. Another condition is required to evaluate this sum.

The need for such a condition is natural: while a testing condition for $I_\alpha$ is clearly necessary, Sawyer [88] constructed a counter-example showing that by itself it is not sufficient. Motivated by work of Muckenhoupt and Wheeden [69] that suggested duality played a role, Sawyer [91] showed that the testing condition plus the testing condition derived from the dual inequality for $I_\alpha$ is necessary and sufficient. Necessity follows immediately: if $I_\alpha(\cdot, \sigma) : L^p(\sigma) \to L^q(u)$, then, since $I_\alpha$ is a self-adjoint linear operator, we have that $I_\alpha(\cdot, u) : L^q(\sigma) \to L^p(u)$. Moreover, it turns out that this “dual” testing condition is the right one for the weak type inequality.

**Theorem 5.2.** Given $0 \leq \alpha < n$, $1 < p \leq q < \infty$, and a pair of weights $(u, \sigma)$, then the following are equivalent:

1. The testing condition

$$I_\alpha = \sup_Q \sigma(Q)^{-\frac{1}{q'}} \left( \int_Q I_\alpha(\chi_Q \sigma)(x)^q u(x) \, dx \right)^\frac{1}{q} < \infty,$$

and the dual testing condition

$$I_\alpha^* = \sup_Q u(Q)^{-\frac{1}{p'}} \left( \int_Q I_\alpha(\chi_Q u)(x)^p \sigma(x) \, dx \right)^\frac{1}{p} < \infty,$$
Norm inequalities for fractional integrals hold;

(2) For all \( f \in L^p(\sigma) \),

\[
\left( \int_{\mathbb{R}^n} |I_\alpha(f\sigma)(x)|^q u(x) \, dx \right)^{\frac{1}{q}} \leq C(n, p, q)(\mathcal{I}_\alpha + \mathcal{I}_\alpha^*) \left( \int_{\mathbb{R}^n} |f(x)|^p \sigma(x) \, dx \right)^{\frac{1}{p}}.
\]

The dual testing condition is equivalent to the weak type inequality

\[
\sup_{t>0} t u(\{x \in \mathbb{R}^n : |I_\alpha(f\sigma)(x)| > t\})^{\frac{1}{q}} \leq C(n, p, q) \mathcal{I}_\alpha^* \left( \int_{\mathbb{R}^n} |f(x)|^p \sigma(x) \, dx \right)^{\frac{1}{p}}.
\]

The equivalence between the dual testing condition and the weak type inequality has the following very deep corollary relating the weak and strong type inequalities.

**Corollary 5.3.** Given \( 0 < \alpha < n \) and \( 1 < p \leq q < \infty \),

\[
\|I_\alpha(\cdot \sigma)\|_{L^p(\sigma) \to L^q(u)} \approx \|I_\alpha(\cdot \sigma)\|_{L^p(\sigma) \to L^q,\infty(u)} + \|I_\alpha(\cdot \sigma)\|_{L^{q'}(u) \to L^{p'},\infty(\sigma)}.
\]

It is conjectured that a similar equivalence holds for singular integrals. However, this is a much more difficult problem and was only recently proved for the Hilbert transform on weighted \( L^2 \) by Lacey, et al. [53].

Theorem 5.2 was first proved by Sawyer [88, 91] (see also [93]). The proof of the weak type inequality is relatively straightforward and readily adapts to the case of dyadic operators (see [55]). We will omit this proof and refer the reader to these papers. The proof of the strong type inequality is more difficult and even for the dyadic fractional integral operator was initially quite complex: see Lacey, Sawyer and Uriarte-Tuero [55]. Recently, however, Hytönen has given a much simpler proof that relies on the corona decomposition and which is very similar to the proof given above for the fractional maximal operator. Besides its elegance, this proof has the advantage that it makes clear why two testing conditions are needed: it provides a means of evaluating a summation over non-disjoint cubes \( Q \) instead of over disjoint sets \( E(Q) \) as we did for the fractional maximal operator. We give this proof below. Another proof that takes a somewhat different approach is due to Treil [98].

**Proof of Theorem 5.2.** As we already discussed, the necessity of the two testing conditions is immediate. To prove sufficiency, we will follow the outline of the proof of Theorem 5.1, highlighting the changes.

First, by arguing as we did in the proof of Theorem 4.6 we can assume that \( f \) is non-negative, bounded and has compact support. Further, it will suffice to prove the strong type inequality for the dyadic operator \( I_D^\alpha \), where \( D \) is any dyadic grid, assuming that the testing condition holds for this operator. (We could in fact pass to the sparse operator \( I_S^\alpha \), but unlike for the fractional maximal operator, sparseness with respect to Lebesgue measure does not simplify the proof.)
Fix a dyadic grid $\mathcal{D}$ and for each $N > 0$ let $\mathcal{D}_N$ be the collection of dyadic cubes $Q$ in $\mathcal{D}$ such that $\ell(Q) \leq 2^N$. Then by duality and the monotone convergence theorem, it will suffice to prove that for any $g \in L^{q'}(u), \|g\|_{L^{q'}(u)} = 1$,

$$\sum_{Q \in \mathcal{D}_N} \frac{Q}{|Q|} \langle f \sigma \rangle_Q \int_Q g(x)u(x) \, dx \leq C(n, p, q) (I_\alpha + I_\beta) \|f\|_{L^p(\sigma)}.$$  

We now form two “parallel” corona decompositions. For each cube in $\mathcal{D}_N$ of side-length $2^N$ form the corona decomposition of $f$ with respect to $\sigma$; denote the union of all of these cubes by $\mathcal{F}$. (Since $f$ has compact support we in fact only form a finite number of such decompositions.) Simultaneously, on the same cubes form the corona decomposition of $g$ with respect to $u$; denote the union of these sets of cubes by $\mathcal{G}$.

We now decompose the sum above as follows:

$$\sum_{Q \in \mathcal{D}_N} \frac{Q}{|Q|} \langle f \sigma \rangle_Q \int_Q g(x)u(x) \, dx = \sum_{F \in \mathcal{F}} \sum_{\substack{G \in \mathcal{G} \\ \pi_F(G) = F}} \sum_{\pi_G(Q) = G} + \sum_{G \in \mathcal{G}} \sum_{F \in \mathcal{F}} \sum_{\substack{Q \in \mathcal{D}_N \\ \pi_F(Q) = F}} \sum_{\pi_G(Q) = G}$$

$$= \Sigma_1 + \Sigma_2.$$

We first estimate $\Sigma_1$. Fix $F, G \subset F$ and $Q$ such that $\pi_F(Q) = F$ and $\pi_G(Q) = G$. (If no such $G$ or $Q$ exists, then this term in the sum is vacuous and can be disregarded.) Let $F' \in \eta_F(F)$ be such that $Q \cap F' \neq \emptyset$. We cannot have $Q \subseteq F'$, since this would imply that $\pi_F(Q) \subseteq F' \subseteq F$, a contradiction. Hence, $F' \subsetneq Q \subset G$. We now define the function $g_F$ by

$$\int_Q g(x)u(x) \, dx = \int_{Q \cap E_F(F)} g(x)u(x) \, dx + \sum_{F' \in \eta_F(F)} \int_{Q \cap F'} g(x)u(x) \, dx$$

$$= \int_Q \left( g(x) \chi_{E_F(F)} + \sum_{F' \in \eta_F(F)} \langle g \rangle_{F', u} \chi_{F'}(x) \right) u(x) \, dx = \int_Q g_F(x)u(x) \, dx.$$  

Moreover, in the definition of $g_F$, the sum is over $F' \subsetneq Q \subset G$, so we can actually restrict the sum to be over $F'$ in the set

$$\eta^*_F(F) = \{ F' \in \eta_F(F) : \pi_F(F') \subseteq F \}.$$
We can now argue as follows: by the definition of corona cubes, the testing condition and Hölder’s inequality,

\[
\Sigma_1 \leq 2 \sum_{F \in F} \langle f^{-1} \rangle_{F, \sigma} \sum_{Q \in \mathcal{P}_F} |Q|^{\frac{n}{n}} \langle \sigma \rangle_Q \int_Q g_F(x) u(x) \, dx
\]

To estimate the second sum, we use the properties of the corona cubes in $F$ and $G$, the definition of $\eta_F(F')$, and Lemma 4.3:

\[
\sum_{F \in F} \sum_{F' \in \eta_F(F)} \langle g \rangle_{F', u}^q (F') = \sum_{F \in F} \sum_{G \in \mathcal{G}_F} \sum_{F' \in \eta_F(F)} \langle g \rangle_{F', u}^q (F')
\]

The first sum in the last term we estimate exactly as we did in the proof of Theorem 5.1, getting that it is bounded by $C(n, p) \|f\|_{L^p(\sigma)}$. To estimate the second sum we use the fact that $q' \leq p'$ and divide it into two parts to get

\[
\left( \sum_{F \in F} \|g\|_{L^{q'}(u)}^{p'} \right)^{\frac{1}{p'}} \leq \left( \sum_{F \in F} \|g\|_{L^{q'}(u)}^{q'} \right)^{\frac{1}{q'}}
\]

We again estimate the first sum as we did in the proof of Theorem 5.1, getting that it is bounded by 1. To bound the second sum, we use the properties of the corona cubes in $F$ and $G$, the definition of $\eta_F(F')$, and Lemma 4.3:
\[
\leq 2^{q'} \sum_{F \in F} \sum_{G \subseteq F} \langle g \rangle^{q'}_{G,u}(G)
\]
\[
\leq 2^{q'+1} \sum_{F \in F} \sum_{G \subseteq F} \sum_{G \in G} \langle g \rangle^{q'}_{G,u}(E_G(G))
\]
\[
\leq 2^{q'+1} \sum_{F \in F} \sum_{G \subseteq F} \sum_{G \in G} \int_{E_G(G)} M^D_u g(x)^q u(x) \, dx
\]
\[
\leq 2^{q'+1} \int_{\mathbb{R}^n} M^D_u g(x)^q u(x) \, dx
\]
\[
\leq C(q) \int_{\mathbb{R}^n} g(x)^q u(x) \, dx
\]
\[
= C(q).
\]

This completes the estimate of \(\Sigma_1\).

The estimate for \(\Sigma_2\) is exactly the same, exchanging the roles of \((f, \sigma)\) and \((g, u)\) and using the dual testing condition which yields the constant \(I^*_\alpha\). This completes the proof. \(\square\)

**Local and global testing conditions.** An examination of the proof of Theorem 5.2 shows that we did not actually need the full testing conditions on the operator \(I^D_\alpha\); rather, we used the following localized testing conditions:

\[
I^L_{D, in} = \sup_Q \sigma(Q)^{-\frac{1}{p}} \left( \int_Q I^D_{\alpha,Q}(\chi_Q \sigma)(x)^p u(x) \, dx \right)^{\frac{1}{p}} < \infty,
\]
\[
I^*_{D, in} = \sup_Q u(Q)^{-\frac{1}{q'}} \left( \int_Q I^D_{\alpha,Q}(\chi_Q u)(x)^{q' \sigma}(x) \, dx \right)^{\frac{1}{p'}} < \infty,
\]

where for \(x \in Q\),

\[
I^D_{\alpha,Q}(\chi_Q \sigma)(x) = \sum_{P \subseteq Q} \left| P \right|^{\frac{\alpha}{n}} \langle \sigma \rangle_P \chi_P(x).
\]

Similarly, the weak type inequality is equivalent to the dual local testing condition (i.e., the condition that \(I^L_{D, in} < \infty\)). This fact is not particular to the dyadic fractional integrals: it is a general property of positive dyadic operators and reflects the fact that they are, in some sense, local operators. See Lacey et al. [55].
Somewhat surprisingly, when \( p < q \) the local testing conditions can be replaced with global testing conditions:

\[
I_{D, \text{out}} = \sup_Q |Q|^{-\frac{1}{p}} \left( \int_{\mathbb{R}^n} |I_{D, \text{out}}^{\alpha}(\chi_Q \sigma)(x)|^p u(x) \, dx \right)^{\frac{1}{p}} < \infty,
\]

\[
I_{D, \text{out}}^* = \sup_Q |Q|^{-\frac{1}{q'}} \left( \int_{\mathbb{R}^n} |I_{D, \text{out}}^{\alpha}(\chi_Q u)(x)|^{q'} \sigma(x) \, dx \right)^{\frac{1}{q'}} < \infty,
\]

where for \( x \in Q \),

\[
I_{D, \text{out}}^{\alpha}(\chi_Q u)(x) = \sum_{\substack{P \in D \smallsetminus Q \subset P \colon \left| P \right| = \alpha \left| Q \right|}} \langle \sigma \chi_Q \rangle_P \chi_P(x).
\]

We record this fact as theorem; we will discuss one of its consequences in Section 7. For a proof, see [55].

**Theorem 5.4.** Given \( 0 \leq \alpha < n \), \( 1 < p < q < \infty \), a dyadic grid \( D \), and a pair of weights \((u, \sigma)\), then:

1. \( \| I_D^{\alpha}(\sigma) \|_{L^p(\sigma) \to L^q(u)} \approx I_{D, \text{out}} + I_{D, \text{out}}^* \);
2. \( \| I_D^{\alpha}(\sigma) \|_{L^p(\sigma) \to L^q, \infty(u)} \approx I_{D, \text{out}}^* \).

**Testing conditions for commutators.** We conclude this section by considering testing conditions and commutators. This problem is completely open but we give some conjectures and also sketch some possible approaches and the problems which will be encountered.

In light of the testing conditions in Theorems 5.1 and 5.2, it seems reasonable to conjecture that for \( 1 < p \leq q < \infty \), \( 0 < \alpha < n \) and \( b \in \text{BMO} \), the following two testing conditions,

\[
C_\alpha = \sup_Q \sigma(Q)^{-\frac{1}{p}} \left( \int_Q [b, I_\alpha](\chi_Q \sigma)(x)^p u(x) \, dx \right)^{\frac{1}{q}} < \infty,
\]

\[
C_\alpha^* = \sup_Q u(Q)^{-\frac{1}{q'}} \left( \int_Q [b, I_\alpha](\chi_Q u)(x)^{q'} \sigma(x) \, dx \right)^{\frac{1}{p'}} < \infty,
\]

are necessary and sufficient for the strong type inequality \([b, I_\alpha](\cdot, \sigma) : L^p(\sigma) \to L^q(u)\), and that the dual testing condition (i.e., \( C_\alpha^* < \infty \)) is necessary and sufficient for the weak type inequality. The necessity of both testing conditions for the strong type inequality is immediate. The necessity of the dual testing condition follows by duality: see, for instance, Sawyer [88] for the proof of necessity for \( I_\alpha \) which adapts immediately to this case.

A significant obstacle for proving sufficiency is that we cannot pass directly to dyadic operators, such as the operator \( C_b^D \) defined in Proposition 3.7. The first problem is that since \([b, I_\alpha] \) is not a positive operator, we do not have an obvious
pointwise equivalence between $[B, I_\alpha]$ and $C^D_b$. Therefore, we cannot pass from a testing condition for the commutator to a dyadic testing condition as we did in the proof of Theorem 5.2. This means that we will be required to work directly with the non-dyadic testing conditions. This is very much the same situation as is encountered for the Hilbert transform, and we suspect that the same (sophisticated) techniques used there may be applicable to this problem. In addition, the recent work of Sawyer, et al. [92] on fractional singular integrals in higher dimensions should also be relevant.

An intermediate result would be to prove that testing conditions for the operator $C^D_b$ are necessary and sufficient for that operator to be bounded, which would yield a sufficient condition for $[b, I_\alpha]$. In this case the parallel corona decomposition used in the proof of Theorem 5.2 should be applicable, but there remain some significant technical obstacles. In particular, it is not clear how to use the fact that $b$ is in $BMO$ in a way which interacts well with the corona decomposition.

6. Bump conditions

In this section we discuss the second approach to two weight norm inequalities, the $A_p$-bump conditions. These were first introduced by Neugebauer [76], but they were systematically developed by Pérez [79, 81]. They are a generalization of the Muckenhoupt $A_p$ and Muckenhoupt-Wheeden $A_{p,q}$ conditions. Compared to testing conditions they have several relative strengths and weaknesses. They only provide sufficient conditions—they are not necessary, though examples show that they are in some sense sharp (see [28]). On the other hand, they are “universal” sufficient conditions: they give conditions that hold for families of operators and are not conditioned to individual operators. (This property is much more important in the study of singular integrals than it is for the study of fractional integrals.) The bump conditions are geometric conditions on the weights and do not involve the operator, so in practice it is easier to check whether a pair of weights satisfies a bump condition. In addition, there exists a very flexible technique for constructing pairs that satisfy a given condition: the method of factored weights which we will discuss below. Finally, since the bump conditions are defined with respect to cubes, they work well with the Calderón-Zygmund decomposition and with dyadic grids in general.

The $A^\alpha_{p,q}$ condition. We begin by defining the natural generalization of the one weight $A_{p,q}$ condition given in Definition 4.1. To state it we introduce the following notation for normalized, localized $L^p$ norms: given $1 \leq p < \infty$ and a cube $Q$,

$$\|f\|_{p,Q} = \left(\int_Q |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$
Definition 6.1. Given $1 < p \leq q < \infty$ and $0 \leq \alpha < n$, we say that a pair of weights $(u, \sigma)$ is in the class $A_{p,q}^\alpha$ if

$$[u, \sigma]_{A_{p,q}^\alpha} = \sup_Q |Q|^{\frac{n+\frac{\alpha}{2}}{q} - \frac{\alpha}{p}} \|u\|^\frac{1}{q}_Q \|\sigma\|^\frac{1}{p}_Q < \infty.$$  

We can extend this definition to the case $p = 1$ by using the $L^\infty$ norm. However, in this case it makes more sense to express the endpoint weak type inequality in terms of pairs $(u, v)$ as originally discussed in Section 5. We will consider these endpoint inequalities in Section 7.

The two weight $A_{p,q}^\alpha$ characterizes weak type inequalities for $M_\alpha$. This result is well-known but a proof has never appeared in the literature since it is very similar to the proof of Theorem 4.2; we also omit the details. For a generalization to non-homogeneous spaces whose proof adapts well to dyadic grids, see García-Cuerva and Martell [37].

Theorem 6.2. Given $1 < p \leq q < \infty$, $0 < \alpha < n$, and a pair of weights $(u, \sigma)$, the following are equivalent:

1. $(u, \sigma) \in A_{p,q}^\alpha$;
2. for any $f \in L^p(\sigma)$,

$$\sup_{t>0} t u(\{x \in \mathbb{R}^n M_\alpha(f\sigma)(x) > t\}) \leq C(n, \alpha)[u, \sigma]_{A_{p,q}^\alpha} \left(\int_{\mathbb{R}^n} |f(x)|^p \sigma(x) \, dx\right)^\frac{1}{p}.$$

While the $A_{p,q}^\alpha$ condition characterizes the weak type inequality, it is not sufficient for the strong type inequality. This fact has been part of the folklore of the field, but a counter-example was not published until recently [24]. When $\alpha = 0$, a counter-example for the Hardy-Littlewood maximal operator was constructed by Muckenhoupt and Wheeden [68]. However, this example does not extend to the case $\alpha > 0$ and our construction is substantially different from theirs.

Example 6.3. Given $1 < p \leq q < \infty$ and $0 < \alpha < n$, there exists a pair of weights $(u, \sigma) \in A_{p,q}^\alpha$ and a function $f \in L^p(\sigma)$ such that $M_\alpha(f\sigma) \notin L^q(u)$.

To construct Example 6.3 we will make use of the technique of factored weights. Factored weights are generalization of the easier half of the Jones $A_p$ factorization theorem: given $w_1, w_2 \in A_1$, then for $1 < p < \infty$, $w_1 w_2^{1-p} \in A_p$. (See [35, 38]; in [21] this was dubbed reverse factorization.) Precursors of this idea have been well-known since the 1970s (cf. the counter-example in [68]) but it was first systematically developed (in the case $p = q$) in [21, Chapter 6]. The following lemma was proved in [24].
Lemma 6.4. Given $0 < \alpha < n$, suppose $1 < p \leq q < \infty$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{n}$. Let $w_1, w_2$ be locally integrable functions, and define

$$u = w_1(M_\gamma w_2)^{-\frac{q'}{p'}}, \quad \sigma = w_2(M_\gamma w_1)^{-\frac{q'}{q}}$$

where

$$\gamma = \frac{\alpha + 1}{n} - \frac{1}{p}.$$  

Then $(u, \sigma) \in A_{p,q}^\alpha$ and $[u, \sigma]_{A_{p,q}^\alpha} \leq 1$.

Proof. By our assumptions on $p$, $q$ and $\alpha$, $0 \leq \gamma \leq \alpha$. Fix a cube $Q$. Then

$$|Q|^{\frac{\alpha + 1}{n} - \frac{1}{p}} \left( \int_Q w_1(x)(M_\gamma w_2(x)^{-\frac{q'}{p'}} dx \right)^{\frac{1}{q'}} \left( \int_Q w_2(x)(M_\gamma w_1(x)^{-\frac{q'}{q}} dx \right)^{\frac{1}{q'}} 

\leq |Q|^{\frac{\alpha + 1}{n} - \frac{1}{p}} \left( \int_Q w_1(x) dx \right)^{\frac{1}{q'}} \left( |Q|^{\frac{1}{q}} \left( \int_Q w_2(x) dx \right) \right)^{-\frac{1}{q'}} 

\times \left( \int_Q w_2(x) dx \right)^{\frac{1}{q'}} \left( |Q|^{\frac{1}{q}} \left( \int_Q w_1(x) dx \right) \right)^{-\frac{1}{q'}} 

= |Q|^{\frac{\alpha + 1}{n} - \frac{1}{p} - \frac{1}{q} (1 + \frac{1}{q} - \frac{1}{p})} 

= 1.$$  

Construction of Example 6.3. To construct the desired example, we need to consider two cases. In both cases we will work on the real line, so $n = 1$.

Suppose first that $\frac{1}{p} - \frac{1}{q} > \alpha$. Let $f = \sigma = \chi_{[-2,1]}$ and let $u = x^t \chi_{[0,\infty)}$, where $t = q(1 - \alpha) - 1$. Given any $Q = (a, b)$, $Q \cap \text{supp}(u) \cap \text{supp}(\sigma) = \emptyset$ unless $a < -1$ and $b > 0$. In this case we have that

$$|Q|^{\alpha + \frac{1}{q} - \frac{1}{p}} \|u^\frac{1}{q} \|_{q,Q} \|\sigma^\frac{1}{p'} \|_{p',Q} 

\leq b^{\alpha + \frac{1}{q} - \frac{1}{p}} \left( \frac{1}{b} \int_0^b x^t dx \right)^{\frac{1}{q'}} \left( \frac{1}{b} \int_{-b}^{-1} dx \right)^{\frac{1}{q'}} \lesssim b^{\alpha + \frac{t+1}{q} - 1} = 1.$$  

Hence, $(u, \sigma) \in A_{p,q}^\alpha$. On the other hand, for all $x > 1$, 

$$M_\alpha(f\sigma)(x) \approx x^{\alpha - 1},$$

and so

$$\int_{\mathbb{R}} M_\alpha(f\sigma)(x)^q u(x) dx \gtrsim \int_1^\infty x^{\alpha(\alpha - 1)} x^{q(1 - \alpha) - 1} dx = \int_1^\infty \frac{dx}{x} = \infty.$$
Now suppose \( \frac{1}{p} - \frac{1}{q} \leq \alpha \). Fix \( \gamma \) as in Lemma 6.4. We first construct a set \( E \subset [0, \infty) \) such that \( M_\gamma(\chi_E)(x) \approx 1 \) for \( x > 0 \). Let

\[
E = \bigcup_{j \geq 0} [j, j + (j + 1)^{-\gamma}).
\]

Suppose \( x \in [k, k + 1) \); if \( k = 0 \), then it is immediate that if we take \( Q = [0, 2] \), then \( M_\gamma(\chi_E) \geq 3 \cdot 2^{-\gamma} \approx 1 \). If \( k \geq 1 \), let \( Q = [0, x] \); then

\[
M_\gamma(\chi_E)(x) \geq x^{\gamma - 1} \sum_{0 \leq j \leq [x]} (j + 1)^{-\gamma} \approx (k + 1)^{\gamma - 1} (k + 1)^{1 - \gamma} = 1.
\]

To prove the reverse inequality we will show that \( |Q|^{\gamma - 1} |Q \cap E| \lesssim 1 \) for every cube \( Q \). If \( |Q| \leq 1 \), then

\[
|Q|^{\gamma - 1} |Q \cap E| \leq |Q|^{\gamma} \leq 1,
\]

so we only have to consider \( Q \) such that \( |Q| \geq 1 \). In this case, given \( Q \) let \( Q' \) be the smallest interval whose endpoints are integers that contains \( Q \). Then \( |Q'| \leq |Q| + 2 \leq 3|Q| \), and so \( |Q|^{\gamma - 1} |E \cap Q| \approx |Q'|^{\gamma - 1} |E \cap Q'| \). Therefore, without loss of generality, we may assume that \( Q = [a, a + h + 1] \), where \( a, h \) are non-negative integers. Then

\[
|Q|^{\gamma - 1} |Q \cap E| = (1 + h)^{\gamma - 1} \sum_{a \leq j \leq a + h} (j + 1)^{-\gamma} \approx (1 + h)^{\gamma - 1} \int_a^{a + h} (t + 1)^{-\gamma} dt \approx (1 + h)^{\gamma - 1} ((a + h + 1)^{1 - \gamma} - (a + 1)^{1 - \gamma}).
\]

To estimate the last term suppose first that \( h \leq a \). Then by the mean value theorem the last term is dominated by

\[
(1 + h)^{\gamma - 1} (1 + h)(a + 1)^{-\gamma} \leq 1.
\]

On the other hand, if \( h > a \), then the last term is dominated by

\[
(1 + h)^{\gamma - 1} (a + h + 1)^{1 - \gamma} \leq 2^{1 - \gamma} \approx 1.
\]

This completes the proof that \( M_\gamma(\chi_E)(x) \approx 1 \).

We can now give our desired counter example. Let \( w_1 = \chi_E \) and \( w_2 = \chi_{[0,1]} \). Then for all \( x \geq 2 \),

\[
M_\gamma w_1(x) \approx 1, \quad M_\gamma w_2(x) = \sup_Q |Q|^{\gamma - 1} \int_Q w_2(y) dy \approx x^{\gamma - 1}.
\]

Define

\[
u = w_1(M_\gamma w_2)^{-\frac{q}{p}}, \quad \sigma = w_2(M_\gamma w_1)^{-\frac{p}{q}};
\]
then by Lemma 6.4, \((u, \sigma) \in A_{p,q}^\alpha\). Moreover, for \(x \geq 2\), we have that
\[
u(x) \approx x^{(1-\gamma)p} \chi_E(x), \quad \sigma(x) \approx \chi_{[0,1]}(x).
\]

Fix \(f \in L^p(\sigma)\): without loss of generality, we may assume \(\text{supp}(f) \subset [0,1]\). Then \(f\sigma\) is locally integrable, and for \(x \geq 2\) we have that
\[
M_\alpha(f\sigma)(x) \geq x^{\alpha-1} \|f\sigma\|_1 \approx x^{\alpha-1}.
\]
Therefore, for \(x \geq 2\),
\[
M_\alpha(f\sigma)(x)^q u(x) \gtrsim x^{(\alpha-1)q \gamma} x^{(1-\gamma)p} \chi_E(x).
\]
By the definition of \(\gamma\),
\[
\gamma \left(\frac{1}{q} + \frac{1}{p'}\right) = \gamma \left(1 + \frac{1}{q} - \frac{1}{p}\right) = \alpha + \frac{1}{q} - \frac{1}{p} = \alpha - 1 + \frac{1}{q} + \frac{1}{p'};
\]
equivalently,
\[
(\gamma - 1) \left(\frac{q}{p'} + 1\right) = q(\alpha - 1),
\]
and so
\[
(\alpha - 1)q + (1 - \gamma) \frac{q}{p'} = \gamma - 1.
\]
Therefore, to show that \(M_\alpha(f\sigma) \notin L^q(u)\), it will be enough to prove that
\[
\int_2^\infty x^{\gamma-1} \chi_E(x) \, dx = \infty,
\]
but this is straightforward:
\[
\int_2^\infty x^{\gamma-1} \chi_E(x) \, dx = \sum_{j=2}^\infty \int_j^{j+(j+1)-\gamma} x^{\gamma-1} \, dx \geq \sum_{j=2}^\infty (j + (j+1)^{-\gamma})^{\gamma-1} (j+1)^{-\gamma}
\]
\[
\geq \sum_{j=2}^\infty (j+1)^{-1} (j+1)^{-\gamma} \geq \sum_{j=2}^\infty (j+1)^{-1} = \infty.
\]

If we combine Example 6.3 with the pointwise inequalities in Section 3, we see that the \(A_{p,q}^\alpha\) condition is also not sufficient for the fractional integral operator to satisfy the strong type inequality. This condition is also not sufficient for the weak \((p,q)\) inequality. A counter-example when \(p = q = n = 2\) and \(\alpha = \frac{1}{2}\) using measures was constructed by Kerman and Sawyer \[46\]. Here we construct a general counter-example that holds for all \(p, q\) and \(\alpha\). For simplicity we construct the example for \(n = 1\), but it can be modified to work in all dimensions. We want to thank E. Sawyer for useful comments on an earlier version of this construction.
Example 6.5. Let \( n = 1 \). Given \( 1 < p \leq q < \infty \) and \( 0 < \alpha < 1 \), there exists a pair of weights \( (u, \sigma) \in A^\alpha_{p,q} \) and a non-negative function \( f \in L^p(\sigma) \) such that
\[
\sup_{t>0} t u(\{ x \in \mathbb{R} : I_\alpha(f\sigma)(x) > t \})^{\frac{1}{q}} = \infty.
\]
Proof. Fix \( p, q \) and \( \alpha \) and let \( u = \chi_{[-1,1]} \). We will first construct a non-negative weight \( \sigma \) such that \([u, \sigma]_{A^\alpha_{p,q}} < \infty\). We will then find a non-negative function \( f \in L^p(\sigma) \) such that \( I_\alpha(f\sigma)(x) = \infty \) for all \( x \in (0,1) \). Then we have that
\[
\sup_{t>0} t u(\{ x \in \mathbb{R}^n : I_\alpha(f\sigma)(x) > t \})^{\frac{1}{q}} \geq \sup_{t>0} t u([0,1])^{\frac{1}{q}} = \infty.
\]
Let \( \sigma = |x|^{-r} \chi_{\{|x|>1\}} \), where \( r \) is defined by
\[
\alpha - \frac{1}{p} = \frac{r}{p'}.\]

Given that \( u \) and \( \sigma \) are symmetric around the origin and have disjoint supports, it is immediate that to check the \( A^\alpha_{p,q} \) condition it suffices to check it on intervals \( Q = [0,t], \ t > 1 \). But in this case,
\[
|Q|^{\alpha + \frac{1}{q} - \frac{1}{p'}} \left( \int_Q u(x) \, dx \right)^{\frac{1}{q}} \left( \int_Q \sigma(x) \, dx \right)^{\frac{1}{p'}} = t^{\alpha + \frac{1}{q} - \frac{1}{p'}} \left( \frac{1}{t} \int_1^t x^{-r} \, dx \right)^{\frac{1}{p'}}.
\]

If \( r < 1 \) then \( x^{-r} \) is locally integrable at the origin, and so by our choice of \( r \), the right hand term is bounded by
\[
t^{\alpha + \frac{1}{q} - \frac{1}{p'} - \frac{1}{q} - \frac{r}{p'}} = 1.
\]

On the other hand, if \( r > 1 \), then \( x^{-r} \in L^1(\mathbb{R}) \), and so the right hand side is bounded by
\[
t^{\alpha + \frac{1}{q} - \frac{1}{p'} - \frac{1}{q} - \frac{r}{p'}} = t^{\alpha - 1} \leq 1.
\]

Hence, \([u, \sigma]_{A^\alpha_{p,q}} < \infty\).

We now construct \( f \) with the desired properties. Let
\[
f(x) = \frac{x^{r-\alpha}}{\log(e^x \chi_{(1,\infty)}(x))};
\]
then
\[
f(x)^p \sigma(x) = \frac{x^{(r-\alpha)p-r}}{\log(e^x)^p \chi_{(1,\infty)}(x)}.
\]

By our definition of \( r \),
\[
\alpha - \frac{1}{p} = r \left( 1 - \frac{1}{p'} \right),
\]
or equivalently,
\[
\frac{r}{p} - \frac{1}{p} = r - \alpha,
\]
which in turn implies that \(p(r - \alpha) = r - 1\). Hence, since \(p > 1\),
\[
f(x)^p \sigma(x) = \frac{1}{x \log(e^x)^p} \chi_{(1, \infty)}(x) \in L^1(\mathbb{R}).
\]

On the other hand, for \(x \in (0, 1)\),
\[
I_\alpha(f\sigma)(x) = \int_1^\infty \frac{f(y)\sigma(y)}{|x - y|^{1 - \alpha}} dy
\]
\[
= \int_1^\infty \frac{dy}{y^\alpha(y - x)^{1 - \alpha} \log(e^y)} \geq \int_1^\infty \frac{dy}{y \log(e^y)} = +\infty.
\]
This completes the proof. \(\square\)

Though it does not matter for our proof, we note in passing that in this example we actually have that \(I_\alpha(f\sigma)(x) = \infty\) for all \(x\).

**Young functions and Orlicz norms.** Given the failure of the \(A^\alpha_{p,q}\) condition to be sufficient for strong type norm inequalities for fractional maximal and integral operators, our goal is to generalize this condition to get one that is sufficient, resembles the \(A^\alpha_{p,q}\) condition and shares its key properties. In particular, the condition should be “geometric” in the sense that, unlike the testing conditions in Section 5, it does not involve the operator itself, and it should interact well with dyadic grids. Our approach will be to replace the \(L^q\) and \(L^{p'}\) norms in the definition with larger norms.

For \(A_p\) weights this was first done by Neugebauer, who replaced the \(L^p\) and \(L^{p'}\) norms with \(L^{rp}\) and \(L^{rp'}\) norms, \(r > 1\). Pérez [79, 81] greatly extended this idea by showing that Orlicz norms that lie between \(L^p\) and \(L^{p'}\) for any \(r > 1\) will also work.

To formulate his approach we first need to introduce some basic ideas about Young functions and Orlicz norms. For complete information see [47, 82]. A function \(B : [0, \infty) \to [0, \infty)\) is a Young function if it is continuous, convex and strictly increasing, if \(B(0) = 0\), and if \(B(t)/t \to \infty\) as \(t \to \infty\). \(B(t) = t\) is not properly a Young function, but in many instances what we say applies to this function as well. It is convenient, particularly when computing constants, to assume \(B(1) = 1\), but this normalization is not necessary. A Young function \(B\) is said to be doubling if there exists a positive constant \(C\) such that \(B(2t) \leq CB(t)\) for all \(t > 0\).

Given a Young function \(B\) and a cube \(Q\), we define the normalized Luxemburg norm of \(f\) on \(Q\) by
\[
\|f\|_{B,Q} = \inf \left\{ \lambda > 0 : \int_Q B\left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.
\]
When \( B(t) = t^p \), \( 1 \leq p < \infty \), the Luxemburg norm coincides with the normalized \( L^p \) norm:
\[
\|f\|_{B,Q} = \left( \int_Q |f(x)|^p \, dx \right)^{1/p} = \|f\|_{p,Q}.
\]

If \( A(t) \leq B(ct) \) for all \( t \geq t_0 > 0 \), then there exists a constant \( C \), depending only on \( A \) and \( B \), such that for all cubes \( Q \) and functions \( f \), \( \|f\|_{A,Q} \leq C \|f\|_{B,Q} \).

Given a Young function \( B \), the associate Young function \( \bar{B} \) is defined by
\[
\bar{B}(t) = \sup_{s>0} \{ st - B(s) \}, \quad t > 0;
\]
\( B \) and \( \bar{B} \) satisfy
\[
t \leq B^{-1}(t) \bar{B}^{-1}(t) \leq 2t.
\]
Note that the associate of \( \bar{B} \) is again \( B \). Using the associate Young function, Hölder’s inequality can be generalized to the scale of Orlicz spaces: given any Young function \( B \), then for all functions \( f \) and \( g \) and all cubes \( Q \),
\[
\int_Q |f(x)g(x)| \, dx \leq 2 \|f\|_{B,Q} \|g\|_{\bar{B},Q}.
\]
More generally, if \( A, B \) and \( C \) are Young functions such that for all \( t \geq t_0 > 0 \),
\[
B^{-1}(t)C^{-1}(t) \leq cA^{-1}(t),
\]
then
\[
\|fg\|_{A,Q} \leq K \|f\|_{B,Q} \|g\|_{C,Q}.
\]

Below we will need to impose a growth condition on Young functions that compares them to powers of \( t \). This condition was first introduced by Pérez [81]. Given \( 1 < p < \infty \), we say that a Young function \( B \) satisfies the \( B_p \) condition if
\[
\int_1^\infty \frac{B(t)}{t^p} \frac{dt}{t} < \infty.
\]
Frequently, we will want to make an assumption of the form \( \bar{B} \in B_p \). If both \( B \) and \( \bar{B} \) are doubling, then this is equivalent to
\[
(6.2) \quad \int_1^\infty \left( \frac{t^p}{B(t)} \right)^{p-1} \frac{dt}{t} < \infty.
\]
(See [21, Proposition 5.10].) There are two important examples of functions that satisfy the \( B_p \) condition. If \( B(t) = t^{rp} \), \( r > 1 \), or if \( B(t) = t^p \log(e + t)^{p-1+\delta} \), \( \delta > 0 \), then \( \bar{B} \in B_p \). For reasons that will be clear below, we will refer to these as power bumps and log bumps. One essential property of this condition is that if \( \bar{B} \in B_p \), then \( \bar{B} \lesssim t^p \) and \( B \gtrsim t^p \). Note in particular that if \( B(t) = t^p \), then \( \bar{B}(t) = t^p \) is not in \( B_p \).
The $B_p$ condition was introduced by Pérez to characterize the boundedness of the Orlicz maximal operator. Given a Young function $B$ and a measurable function $f$, define

$$M_B f(x) = \sup_Q \|f\|_{B,Q} \chi_Q(x).$$

**Proposition 6.6.** Given a Young function $B$ and $1 < p < \infty$, the following are equivalent:

1. $B \in B_p$;
2. for all $f \in L^p$,

$$\left(\int_{\mathbb{R}^n} M_B f(x)^p \, dx\right)^{\frac{1}{p}} \leq C(n,p) \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

As given in [81], Proposition 6.6 included the assumption that $B$ was doubling. However, this assumption was only included to use the $B_p$ condition in the form of (6.2) to prove sufficiency. This was correctly noted in [21], but we made the incorrect assertion that it was not needed for the proof of necessity in [81]. However, Liu and Luque [63] recently gave a proof that it is necessary without assuming doubling.

We can also define a fractional Orlicz maximal operator $M_{B,\alpha}$: see Section 7 below. There is also a corresponding $B_p^\alpha$ condition which is useful in determining sharp constants estimates for the fractional integral operator: see [24] for details.

**The $A_{p,q}^\alpha$ bump conditions.** Using the machinery introduced above, we can now state our generalizations of the $A_{p,q}^\alpha$ condition. Given $0 < \alpha < n$, $1 < p \leq q < \infty$, Young functions $A$ and $\bar{A}$, $\bar{A} \in B_{q'}$ and $\bar{B} \in B_p$, and a pair of weights $(u, \sigma)$, we define

$$[u, \sigma]_{A_{p,q}^\alpha, B} = \sup_Q |Q|^\frac{n}{\alpha} + \frac{1}{q} - \frac{1}{p} \left\|u\right\|_{L^{q,Q}} \left\|\sigma\right\|_{B,Q} < \infty,$$

$$[u, \sigma]_{A_{p,q}^\alpha, A}^{\ast} = \sup_Q |Q|^\frac{n}{\alpha} + \frac{1}{q} - \frac{1}{p} \left\|u\right\|_{L^{q,Q}} \left\|\sigma\right\|_{B,Q} < \infty.$$

By our hypotheses on $A$ and $B$ both of these quantities are larger than $[u, \sigma]_{A_{p,q}^{\infty}}$: we have “bumped up” one of the norms in the scale of Orlicz spaces. For this reason we refer to these as $A_{p,q}^\alpha$ bump conditions.

Note that the second condition is the “dual” of the first, in the sense that

$$[u, \sigma]_{A_{p,q}^\alpha, A}^{\ast} = [\sigma, u]_{A_{q',p}^{\infty}, A}.$$

As we will see below, this condition will play a role analogous to that of the dual testing conditions discussed in Section 5. Informally, it is common to refer to the $[u, \sigma]_{A_{p,q}^\alpha, B}$ condition as having a bump on the right, and the $[u, \sigma]_{A_{p,q}^\alpha, A}$ as having a bump on the left, and collectively we refer to these as separated bump conditions.
We can combine these conditions by putting a bump on both norms simultaneously:

\[ [u, \sigma]_{A^p_{q,A,B}} = \sup_Q |Q|^{\alpha + \frac{1}{q} - \frac{1}{p}} \|u^\frac{1}{q}\|_{A,Q} \|\sigma^\frac{1}{p}\|_{B,Q} < \infty. \]

We refer this as a conjoined bump condition. Clearly, it is larger than either of the separated bump conditions. In fact, assuming the conjoined bump condition is stronger than assuming both separated bump conditions. The following example (with \( \alpha = 0, p = 2 \)) was constructed in [2].

**Example 6.7.** Given \( 0 \leq \alpha < n \) and \( 1 < p \leq q < \infty \), there exists a pair of Young functions \( A \) and \( B \), \( \tilde{A} \in B_{q'} \) and \( \tilde{B} \in B_p \), and a pair of weights \( (u, \sigma) \), such that \( [u, \sigma]_{A^p_{q,A,B}} \), \( [u, \sigma]_{A^p_{q,A,B}}^* \) \( < \infty \), but \( [u, \sigma]_{A^p_{q,A,B}} = \infty \).

**Proof.** We construct our example on the real line, so \( 0 < \alpha < 1 \). Define the Young functions

\[ A(t) = t^q \log(e + t)^q, \quad B(t) = t^{q'} \log(e + t)^{q'}. \]

Then \( \tilde{A} \in B_{q'} \) and \( \tilde{B} \in B_p \). By rescaling, if we let \( \Psi(t) = t \log(e + t)^q, \Phi(t) = t \log(e + t)^{q'} \), then for any pair \( (u, \sigma) \),

\[ \|u^\frac{1}{q}\|_{A,Q} \approx \|u\|_{\Psi,Q}, \quad \|\sigma^\frac{1}{q'}\|_{B,Q} \approx \|\sigma\|_{\Phi,Q}. \]

Therefore, it will suffice to estimate the norms of \( u \) and \( \sigma \) with respect to \( \Psi \) and \( \Phi \). Similarly, we can replace the localized \( L^q \) and \( L^{q'} \) norms of \( u^\frac{1}{q} \) and \( \sigma^\frac{1}{q} \) with the \( L^1 \) norms of \( u \) and \( \sigma \).

Before we define \( u \) and \( \sigma \) we first construct a pair \( (u_0, \sigma_0) \) which will be the basic building block for our example. Fix an integer \( k \geq 2 \) and define \( Q = (0, k) \), \( \sigma_0 = \chi_{(0,1)} \) and \( u_0 = K^q_k \chi_{(k-1,k)} \), where \( K_k = k^{1-\alpha} \log(e + k)^{-\frac{3}{2}} \). Since \( \Psi^{-1}(t) \approx t \log(e + t)^{-q} \), \( \Phi^{-1}(t) \approx t \log(e + t)^{-q'} \), by the definition of the Luxemburg norm,

\[ \|u_0\|_{1,Q} = \frac{K_k}{k^\frac{1}{q}}, \quad \|u_0\|_{\Psi,Q} \approx \frac{K_k \log(e + k)}{k^\frac{1}{q}}, \quad \|\sigma_0\|_{1,Q} = \frac{1}{k^\frac{1}{p'}}, \quad \|\sigma_0\|_{\Phi,Q} \approx \frac{\log(e + k)}{k^\frac{1}{p'}}. \]

Therefore, we have that

\[ |Q|^{\alpha + \frac{1}{q} - \frac{1}{p}} \|u_0\|_{1,Q} \|\sigma_0\|_{\Psi,Q} \|\sigma_0\|_{\Phi,Q} \approx \frac{1}{\log(e + k)^{\frac{1}{2}}}, \]

but

\[ |Q|^{\alpha + \frac{1}{q} - \frac{1}{p}} \|u_0\|_{\Psi,Q} \|\sigma_0\|_{\Phi,Q} \approx \log(e + k)^{\frac{1}{2}}. \]

We now define \( u \) and \( \sigma \) as follows:

\[ u(x) = \sum_{k \geq 2} K^q_k \chi_{I_k}(x), \quad \sigma(x) = \sum_{k \geq 2} \chi_{J_k}(x). \]
where $I_k = (e^k + k - 1, e^k + k)$ and $J_k = (e^k, e^k + k)$. Since the above computations are translation invariant, we immediately get that if $Q_k = (e^k, e^k + k)$, then
\[
|Q_k|^{\alpha + \frac{1}{q} - \frac{1}{p}} \|u\|_{L^q(Q_k)} \frac{\|\sigma\|_{L^p(Q_k)}}{|Q_k|} \approx \log(e + k)^{\frac{1}{2}},
\]
and so $[u, \sigma]_{A_p^\alpha, A, B} = \infty$.

We will now prove that $[u, \sigma]_{A_p^\alpha, B}$ and $[\sigma, u]_{A_p^\alpha, A}$ are both finite. We will show $[u, \sigma]_{A_p^\alpha, A} < \infty$; the argument for the first condition is essentially the same. Fix an interval $Q$; we will show that $|Q|^{\alpha + \frac{1}{q} - \frac{1}{p}} \|u\|_{L^q(Q)} \frac{\|\sigma\|_{L^p(Q)}}{|Q|} \leq 1$ is uniformly bounded. Let $N$ be an integer such that $N - 1 \leq |Q| \leq N$. We need to consider those values of $k$ such that $Q$ intersects either $I_k$ or $J_k$.

Suppose that for some $k \geq N + 2$, $Q$ intersects $I_k$. But in this case it cannot intersect $J_j$ for any $j$ and so $\|\sigma\|_{L^1(Q)} = 0$. Similarly, if $Q$ intersects $J_k$, then $\|u\|_{L^1(Q)} = 0$.

Now suppose that for some $k < N + 2$, $Q$ intersects one of $I_k$ or $J_k$. If $\log(N) \lesssim k$ (more precisely, if $N < e^k - e^{k-1} - 1$), then for any $j \neq k$, $Q$ cannot intersect $I_j$ or $J_j$.

In this case $|Q|^{\alpha + \frac{1}{q} - \frac{1}{p}} \|u\|_{L^q(Q)} \frac{\|\sigma\|_{L^p(Q)}}{|Q|} \neq 0$ only if $Q$ intersects both $I_k$ and $J_k$, and will reach its maximum when $N \approx k$. But in this case we can replace $Q$ by $(e^k, e^k + k)$ and the above computation shows that $|Q|^{\alpha + \frac{1}{q} - \frac{1}{p}} \|u\|_{L^q(Q)} \frac{\|\sigma\|_{L^p(Q)}}{|Q|} \leq 1$.

Finally, suppose $Q$ intersects one or more pairs $I_k$ and $J_k$ with $k \lesssim \log(N)$. Then $|\text{supp}(u) \cap Q| \lesssim \log(N)$ and $\|u\|_{L^\infty(Q)} \approx K_{\log(N)}^q \lesssim \log(N)^{\theta(1 - \alpha)}$. Therefore, for any $r > 1$,
\[
\|u\|_{L^q(Q)}^{\frac{1}{r}} \lesssim \|u\|_{L^q(Q)} \leq \|u\|_{L^\infty(Q)} \left(\frac{|\text{supp}(u) \cap Q|}{|Q|}\right)^{\frac{1}{\theta}} \lesssim \frac{\log(N)^{1 - \alpha + \frac{1}{r}}}{N^{\frac{1}{\theta}}},
\]
A similar calculation shows that
\[
\|\sigma\|_{L^p(Q)}^{\frac{1}{r}} \lesssim \left(\frac{\log(N)}{N}\right)^{\frac{1}{\theta}}.
\]
Hence, we have that
\[
|Q|^{\alpha + \frac{1}{q} - \frac{1}{p}} \|u\|_{L^q(Q)} \frac{\|\sigma\|_{L^p(Q)}}{|Q|} \lesssim N^{\alpha + \frac{1}{q} - \frac{1}{p} - \frac{1}{\theta} + \frac{1}{\theta}} \log(N)^{1 - \alpha + \frac{1}{r} + \frac{1}{\theta}}.
\]
Since $\alpha < 1$, if we fix $r > 1$ sufficiently close to 1 we have that the exponent on $N$ is negative, and so this quantity will be uniformly bounded for all $N$. We thus have that $[u, \sigma]_{A_p^\alpha, A} < \infty$ and our proof is complete.  

**Bump conditions for fractional maximal operators.** There is a parallel between bump conditions and the testing conditions described in Section 5. For maximal operators, only a single testing condition is needed for the strong type inequality;
similarly, only a single bump (on the right) is required to get a sufficient condition. The following result is due to Pérez [79, 81] and our proof is based on his.

**Theorem 6.8.** Given $0 \leq \alpha < n$, $1 < p \leq q < \infty$, and a Young function $B$ such that $B \in B_p$, suppose the pair of weights $(u, \sigma)$ is such that $[u, \sigma]_{A_{p,q,B}^\alpha} < \infty$. Then for every $f \in L^p(\sigma)$,

$$
\left( \int_{\mathbb{R}^n} M_\alpha(f\sigma)(x)^q u(x) \, dx \right)^{\frac{1}{q}} \leq C(n, p, q)[u, \sigma]_{A_{p,q,B}^\alpha} \left( \int_{\mathbb{R}^n} |f(x)|^p \sigma(x) \, dx \right)^{\frac{1}{p}}.
$$

Note that while our proof shows directly that the constant depends linearly on $[u, \sigma]_{A_{p,q,B}^\alpha}$, in fact this is always true in two weight inequalities. This is an observation due to Sawyer: see [23, Remark 1.4].

**Proof.** Arguing as we did in the proof of Theorem 4.4, we may assume that $f$ is non-negative, bounded and has compact support, and it will suffice to prove the desired inequality for $L^S_\alpha$, where $S$ is a sparse subset of a dyadic grid $D$. Indeed, we begin as we did there, using the fact that the sets $E(Q)$ are disjoint. But instead of the $A_\infty$ property we will use the generalized Hölder’s inequality to introduce the Orlicz maximal operator. This allows us to sum over the cubes in $S$ and apply Proposition 6.6 to get the desired estimate:

$$
\|L^S_\alpha(f\sigma)^q\|_{L^q(u)} = \sum_{Q \in S} |Q|^\frac{\alpha}{n} \langle f\sigma \rangle_Q^q u(E(Q))
$$

$$
= \sum_{Q \in S} |Q|^\frac{\alpha}{n} + 1 - \frac{q}{p} \langle f\sigma \rangle_Q^q \langle u \rangle_Q |Q|^\frac{q}{p}
$$

$$
\leq 2^{\frac{q}{p}+1} \sum_{Q \in S} |Q|^\frac{\alpha}{n} + 1 - \frac{q}{p} \langle u \rangle_Q \|\sigma\|_{B,Q}^{\frac{1}{p}} \|f\sigma\|_{B,Q}^{\frac{1}{q}} |E(Q)|^\frac{q}{p}
$$

$$
\leq 2^{\frac{q}{p}+1} [u, \sigma]_{A_{p,q,B}^\alpha} \sum_{Q \in S} \|f\sigma\|_{B,Q}^{\frac{1}{q}} |E(Q)|^\frac{q}{p}
$$

$$
\leq 2^{\frac{q}{p}+1} [u, \sigma]_{A_{p,q,B}^\alpha} \left( \sum_{Q \in S} \|f\sigma\|_{B,Q}^{\frac{1}{p}} |E(Q)| \right)^\frac{q}{p}
$$

$$
\leq 2^{\frac{q}{p}+1} [u, \sigma]_{A_{p,q,B}^\alpha} \left( \sum_{Q \in S} \int_{E(Q)} M_B(f\sigma^{\frac{1}{p}})(x)^p \, dx \right)^\frac{q}{p}
$$

$$
\leq 2^{\frac{q}{p}+1} [u, \sigma]_{A_{p,q,B}^\alpha} \left( \int_{\mathbb{R}^n} M_B(f\sigma^{\frac{1}{p}})(x)^p \, dx \right)^\frac{q}{p}.
$$
Bump conditions for fractional integral operators. We now consider bump conditions for the fractional integral operator. For the strong type condition, we need two bumps, analogous to the fact that you need two testing conditions. Our first result is for conjoined bumps; we will discuss separated bump conditions in Section 7 below. Theorem 6.9 was originally proved by Pérez [79] and our proof is modeled on his.

Theorem 6.9. Given $0 < \alpha < n$, $1 < p \leq q < \infty$, and Young functions $A, B$ such that $\bar{A} \in B^q_{p'}$ and $\bar{B} \in B^p_q$, suppose the pair of weights $(u, \sigma)$ is such that $[u, \sigma]_{A^\alpha_{p,q,A,B}} < \infty$. Then for every $f \in L^p(\sigma)$,

$$\left( \int_{\mathbb{R}^n} |I_\alpha(f\sigma)(x)|^q u(x) \, dx \right)^{\frac{1}{q}} \leq C(n, p)[u, \sigma]_{A^\alpha_{p,q,A,B}} \left( \int_{\mathbb{R}^n} |f(x)|^p \sigma(x) \, dx \right)^{\frac{1}{p}}.$$

We note that Theorem 6.9 was very influential in the study of two weight norm inequalities, and it led to the conjecture that an analogous result held for singular integral operators. This problem was solved recently by Lerner [58]; for prior results see [19, 20, 30].

Proof. Arguing as we did in the proof of Theorem 4.6, we may assume that $f$ is non-negative, bounded and has compact support. Further, it will suffice to prove the desired inequality for $I^S_\alpha$, where $S$ is a sparse subset of a dyadic grid $D$.

We begin as in the one weight case by applying duality. But here we use the generalized Hölder’s inequality to introduce two Orlicz maximal operators and use these to sum over cubes in $S$. We can then apply Proposition 6.6 twice. More precisely, by duality there exists $g \in L^{q'}(u)$, $\|g\|_{L^{q'}(u)} = 1$, such that

$$\|I^S_\alpha(f\sigma)\|_{L^{q'}(u)} \leq \left( \int_{\mathbb{R}^n} |I^S_\alpha(f\sigma)(x)|^q u(x) \, dx \right)^{\frac{1}{q}} \leq C(n, p)[u, \sigma]_{A^\alpha_{p,q,A,B}} \left( \int_{\mathbb{R}^n} |f(x)|^p \sigma(x) \, dx \right)^{\frac{1}{p}}.$$
\[ \leq 2^{\frac{1}{p} + \frac{1}{q} + 2} [u, \sigma]_{A_{p,q,A,B}} \sum_{Q \in S} \| f \sigma^{\frac{1}{p}}\|_{B,Q} |E(Q)|^{\frac{1}{p}} \| g u^{\frac{1}{q}}\|_{A,Q} |E(Q)|^{\frac{1}{q}}. \]

By Hölder’s inequality and the fact that \( q' \leq p' \), we have that
\[
\leq 16[u, \sigma]_{A_{p,q,A,B}} \left( \sum_{Q \in S} \| f \sigma^{\frac{1}{p}}\|_{B,Q} |E(Q)| \right)^{\frac{1}{p}} \left( \sum_{Q \in S} \| g u^{\frac{1}{q}}\|_{A,Q} |E(Q)| \right)^{\frac{1}{q}}
\]
\[
\leq 16[u, \sigma]_{A_{p,q,A,B}} \left( \sum_{Q \in S} \int_{E(Q)} M_B(f \sigma^{\frac{1}{p}})(x)^p \, dx \right)^{\frac{1}{p}} \left( \sum_{Q \in S} \int_{E(Q)} M_A(g u^{\frac{1}{q}})(x)^{q'} \, dx \right)^{\frac{1}{q'}}
\]
\[
\leq 16[u, \sigma]_{A_{p,q,A,B}} \left( \int_{\mathbb{R}^n} M_B(f \sigma^{\frac{1}{p}})(x)^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^n} M_A(g u^{\frac{1}{q}})(x)^{q'} \, dx \right)^{\frac{1}{q'}}
\]
\[
\leq C(n, p, q)[u, \sigma]_{A_{p,q,A,B}} \left( \int_{\mathbb{R}^n} f(x)^p \sigma(x) \, dx \right)^{\frac{1}{p}}.
\]

\[ \square \]

**Bump conditions for commutators.** Finally, we prove a conjoined bump condition for commutators. Because commutators are more singular, we need stronger bump conditions. To use the fact that \( b \) is a \( BMO \) function, it is most natural to state these in terms of log bumps. This result was originally proved in [23]; our proof is a simplification of the argument given there.

**Theorem 6.10.** Given \( 0 < \alpha < n \), \( 1 < p \leq q < \infty \), and \( b \in BMO \), suppose the pair of weights \((u, \sigma)\) is such that \([u, \sigma]_{A_{p,q,A,B}} < \infty\), where
\[
A(t) = t^q \log(e + t)^{2q - 1 + \delta}, \quad B(t) = t^{p'} \log(e + t)^{2p' - 1 + \delta}, \quad \delta > 0.
\]

Then for every \( f \in L^p(\sigma) \),
\[
\left( \int_{\mathbb{R}^n} |[b, I_\alpha](f \sigma)(x)|^p u(x) \, dx \right)^{\frac{1}{p}} \leq C(n, p) \| b \|_{BMO}[u, \sigma]_{A_{p,q,A,B}} \left( \int_{\mathbb{R}^n} |f(x)|^p \sigma(x) \, dx \right)^{\frac{1}{p}}.
\]

The proof requires one lemma, which generalizes a result due to Sawyer and Wheeden [93, p. 829] and is proved in much the same way.
Lemma 6.11. Fix $0 < \alpha < n$, a dyadic grid $D$, and a Young function $\Phi$. Then for any $P \in D$ and any function $f$,

$$\sum_{Q \in D} |Q|^{\frac{\alpha}{n}} |Q| \|f\|_{\Phi,Q} \leq C(\alpha) |P|^{\frac{\alpha}{n}} |P| \|f\|_{\Phi,P}.$$

Proof. To prove this we need to replace the Luxemburg norm with the equivalent Amemiya norm [82, Section 3.3]:

$$\|f\|_{\Phi,P} \leq \inf_{\lambda > 0} \{ \lambda \int_P 1 + \Phi \left( \left| \frac{f(x)}{\lambda} \right| \right) dx \} \leq 2 \|f\|_{\Phi,P}.$$

By the second inequality, we can fix $\lambda_0 > 0$ such that the middle quantity is less than $3 \|f\|_{\Phi,P}$. Then by the first inequality,

$$\sum_{Q \in D} |Q|^{\frac{\alpha}{n}} |Q| \|f\|_{\Phi,Q} = \sum_{k=0}^{\infty} \sum_{Q \subset P, \ell(Q) = 2^{-k} \ell(P)} |Q|^{\frac{\alpha}{n}} |Q| \|f\|_{\Phi,Q}
\leq |P|^{\frac{\alpha}{n}} \sum_{k=0}^{\infty} 2^{-k\alpha} \sum_{Q \subset P, \ell(Q) = 2^{-k} \ell(P)} \lambda_0 \int_Q 1 + \Phi \left( \left| \frac{f(x)}{\lambda_0} \right| \right) dx
= C(\alpha) |P|^{\frac{\alpha}{n}} \lambda_0 \int_P 1 + \Phi \left( \left| \frac{f(x)}{\lambda_0} \right| \right) dx
\leq C(\alpha) |P|^{\frac{\alpha}{n}} |P| \|f\|_{\Phi,P}.$$

Proof. Fix $b \in BMO$. We first make some reductions. Since $[b, I_\alpha]$ is linear, by splitting $f$ into its positive and negative parts we may assume $f$ is non-negative. By Fatou’s lemma we may assume that $f$ is bounded and has compact support. Finally, by Proposition 3.7 it will suffice to prove this result for the dyadic operator $C^D_b$, where $D$ is any dyadic grid.

We begin by applying duality: there exists $g \in L^{q'}(u), \|g\|_{L^{q'}(u)} = 1$, such that

$$\|C^D_b (f \sigma)\|_{L^{q'}(u)} = \int_{\mathbb{R}^n} C^D_b (f \sigma) g(x) u(x) \, dx
= \sum_{Q \in D} |Q|^{\frac{n}{q'}} \int_Q \int_Q |b(x) - b(y)| f(y) \sigma(y) \, dy \, g(x) u(x) \, dx
\leq \sum_{Q \in D} |Q|^{\frac{n}{q'}} \int_Q \int_Q |b(x) - b(y)| g(x) u(x) \, dx \, \langle f \sigma \rangle_Q |Q|.$$
We will estimate the first term; the estimate for the second is exactly the same, exchanging the roles of $f$, $\sigma$ and $g$, $u$. Arguing as we did in the proof of Theorem 5.2, if we let $\mathcal{D}_N$ be the set of all dyadic cubes $Q$ in $\mathcal{D}$ with $\ell(Q) = 2^N$, then it will suffice to bound this sum with $\mathcal{D}$ replaced by $\mathcal{D}_N$ and with a constant independent of $N$. Form the corona decomposition of $f \sigma$ with respect to Lebesgue measure for each cube in $\mathcal{D}_N$. Let $\mathcal{F}$ denote the union of all these cubes.

Let $\Phi(t) = t \log(e + t)$; then $\Phi(t) \approx e^t - 1$, and so by the generalized Hölder’s inequality and the John-Nirenberg inequality,

$$\int_Q |b(x) - \langle b \rangle_Q|g(x)u(x)\,dx \leq 2 \|b - \langle b \rangle_Q\|_{BMO} \|gu\|_{\Phi,Q} \leq C(n)\|b\|_{BMO} \|gu\|_{\Phi,Q}.$$  

Furthermore, if we define

$$C(t) = \frac{t^{q'}}{\log(e + t)^{1+(q'-1)\beta}},$$

then $C \in B_{q'}$ and $A^{-1}(t)C^{-1}(t) \leq \Phi^{-1}(t)$. (See [23, Lemma 2.12] for the details of this calculation.) We also have that $\tilde{B} \in B_p$.

If we combine all of these facts and use Lemma 6.11 and the generalized Hölder’s inequality twice, we get that

$$\sum_{Q \in \mathcal{D}_N} |Q|^{\frac{\alpha}{n}} \int_Q |b(x) - \langle b \rangle_Q|g(x)u(x)\,dx \langle f \sigma \rangle_Q |Q|$$

$$\leq C(n)\|b\|_{BMO} \sum_{F \in \mathcal{F}} \langle f \sigma \rangle_F \sum_{\pi_F(Q) = F} |Q|^{\frac{\alpha}{n}} |Q| \|gu\|_{\Phi,F}$$

$$\leq C(n, \alpha)\|b\|_{BMO} \sum_{F \in \mathcal{F}} F^{\frac{\alpha}{n}} \|gu\|_{\Phi,F}$$

$$\leq C(n, \alpha)\|b\|_{BMO} \sum_{F \in \mathcal{F}} |F|^{\frac{\alpha}{n}} \|u^{\frac{1}{2}}\|_{A,F} \|\sigma^{\frac{1}{2}}\|_{B,F} \|f \sigma^{\frac{1}{2}}\|_{B,F} \|gu^{\frac{1}{2}}\|_{C,F} |E_F(F)|$$

$$\leq C(n, \alpha)\|b\|_{BMO} \sum_{F \in \mathcal{F}} \|f \sigma^{\frac{1}{2}}\|_{B,F} \|gu^{\frac{1}{2}}\|_{C,F} |E_F(F)|^{\frac{1}{p} + \frac{1}{q'}}.$$

We can now apply Hölder’s inequality and use the fact that $\tilde{B} \in B_p$ and $C \in B_{q'}$ to finish the argument exactly as we did in the proof of Theorem 6.9. \qed
7. Separated bump conditions

We conclude with a discussion of some very recent work and some additional open problems for fractional integral operators and their commutators. To put these into context, we will first review the Muckenhoupt-Wheeden conjectures for singular integral operators and their relation to bump conditions. For a more detailed overview of these conjectures, see [20, 21].

The Muckenhoupt-Wheeden conjectures. In the late 1970’s while studying two weight norm inequalities for the Hilbert transform, Muckenhoupt and Wheeden made a series of conjectures relating this problem to two weight norm inequalities for the maximal operator. These conjectures were quickly extended to general singular integral operators. Restated in terms of weights \((u, \sigma)\) instead of weights \((u, v)\) as they were originally framed, they conjectured that for \(1 < p < \infty\), a sufficient condition for a singular integral operator to satisfy \(T(\cdot, \sigma) : L^p(\sigma) \to L^p(u)\) is that the maximal operator satisfy

\[
M(\cdot, \sigma) : L^p(\sigma) \to L^p(u)
\]

and the dual inequality

\[
M(\cdot, u) : L^p(u) \to L^p(\sigma).
\]

They further conjectured that the weak type inequality \(T : L^p(\sigma) \to L^{p, \infty}(u)\) holds if the maximal operator only satisfies the dual inequality. (Note the parallels between these conjectures and the testing conditions described in Section 5.) Finally, they conjectured that the following weak \((1, 1)\) inequality holds:

\[
\sup_{t>0} t \cdot u\{x \in \mathbb{R}^n : |Tf(x)| > t\} \leq C \int_{\mathbb{R}^n} |f(x)| M u(x) \, dx.
\]

In the one weight case (i.e., with Muckenhoupt \(A_p\) weights) all of these conjectures are true, and with additional assumptions on the weights (e.g., \(u, v \in A_\infty\)) they are true in the two weight case. However, all three conjectures were recently shown to be false. The weak \((1, 1)\) conjecture was disproved by Reguera and Thiele [84]; the strong \((p, p)\) conjecture by Reguera and Scurry [83]; and building on this the weak \((p, p)\) conjecture was disproved in [31].

On the other hand, an “off-diagonal” version of this conjecture is true [22]: if \(1 < p < q < \infty\), and the maximal operator satisfies

\[
M(\cdot, \sigma) : L^p(\sigma) \to L^q(u)
\]

and the dual inequality

\[
M(\cdot, u) : L^q(u) \to L^q(\sigma),
\]

I first learned these conjectures from Pérez, and later learned some of their history directly from Muckenhoupt. However, they do not appear to have ever been published until they appeared in [21]. The weak \((1, 1)\) conjecture appeared shortly before this in [61].
then \( T : L^p(\sigma) \to L^q(u) \). If the dual inequality holds, then the weak \((p,q)\) inequality \( T : L^p(\sigma) \to L^{q,\infty}(u) \) holds as well. Examples of such weights can be easily constructed using the Sawyer testing condition (Theorem 5.1 with \( \alpha = 0 \)). For instance, \( u = \chi_{[0,1]} \) and \( \sigma = \chi_{[2,3]} \) work for all \( p > 1 \).

It follows from Theorem 6.8 (with \( \alpha = 0 \)) that these two off-diagonal inequalities for the maximal operator are implied by a pair of separated bump conditions, \([u,\sigma]_{A^0_{p,q,B}}^{p,q,A} < \infty\). When \( p = q \) this leads to the separated bump conjectures for singular integrals: if \([u,\sigma]_{A^0_{p,p,B}}^{p,p,A} < \infty\), then a singular integral satisfies the strong \((p,p)\) inequality, and if the dual condition holds, it satisfies the weak \((p,p)\) inequality. This conjecture is due to Pérez: his study of bump conditions was partly motivated by the Muckenhoupt-Wheeden conjectures. It was first published, however, in [31], where it was proved for log bumps: \( A(t) = t^p \log(e + t)^{p-1+\delta} \), \( B(t) = t^{p'} \log(e + t)^{p'-1+\delta} \), \( \delta > 0 \) and some closely related bump conditions (the so-called “loglog” bumps). The proof was quite technical, relying on a “freezing” argument and a version of the corona decomposition. For another, simpler proof that also holds in spaces of homogeneous type, see [2]. It is not clear if the separated bump conjecture is true for singular integrals only assuming bumps that satisfy the \( B_p \) condition. For very recent work that suggests it may be false, see Lacey [48] and Treil and Volberg [99].

**Separated bump conditions for fractional integral operators.** Though never addressed by Muckenhoupt and Wheeden, their conjectures for singular integrals extend naturally to fractional integrals as well. Such a generalization was first considered by Carro, et al. [5], who showed that the analog of the Muckenhoupt weak \((1,1)\) conjecture,

\[
\sup_{t>0} t u(\{ x \in \mathbb{R}^n : \left| I_\alpha f(x) \right| > t \}) \leq C \int_{\mathbb{R}^n} |f(x)| M_\alpha u(x) \, dx,
\]

is false.

In [24] we made the following conjectures: given \( 0 < \alpha < n \) and \( 1 < p \leq q < \infty \), suppose the fractional maximal operator satisfies

\[
M_\alpha(\cdot) : L^p(\sigma) \to L^q(u)
\]

and the dual inequality

\[
M_\alpha(\cdot u) : L^q(u) \to L^{p'}(\sigma).
\]

Then the strong \((p,q)\) inequality holds, and if the dual inequality holds, the weak \((p,q)\) inequality holds. Analogous to the case of singular integrals, both of these conjectures are true in when \( p < q \); this was proved in [24]. Earlier, in [25] we proved a weaker version of this conjecture for separated bump conditions when \( \frac{1}{p} - \frac{1}{q} \approx \frac{\alpha}{n} \).
Theorem 7.1. Given $0 < \alpha < n$ and $1 < p < q < \infty$, suppose the pair of weights $(u, \sigma)$ are such that (7.2) and (7.3) hold. Then $I_\alpha : L^p(\sigma) \to L^q(u)$. If (7.3) holds, then $I_\alpha : L^p(\sigma) \to L^{q, \infty}(u)$.

Proof. It will suffice to prove this for the dyadic fractional integral operator $I^D_\alpha$, where $D$ is any dyadic grid. We will show that the desired inequalities follow immediately from Theorem 5.4. To see this we will first consider the testing condition

$$
I_{D,\text{out}} = \sup_Q \sigma(Q)^{-\frac{1}{p}} \left( \int_{\mathbb{R}^n} I^D_{\alpha,Q}(\sigma \chi_Q)(x) q u(x) \, dx \right)^{\frac{1}{q}} < \infty.
$$

Fix a cube $Q$ and $x \in \mathbb{R}^n$ such that there exists a dyadic cube $P \in D$ with $x \in P$ and $Q \subset P$. (If no such cube exists then $I^D_{\alpha,Q}(\sigma \chi_Q)(x) = 0$.) Let $Q_0$ be the smallest such cube, and for $k \geq 1$ let $Q_k$ be the unique dyadic cube such that $Q_0 \subset Q_k$ and $\ell(Q_k) = 2^k \ell(Q_0)$. Then

$$
I^D_{\alpha,Q}(\sigma \chi_Q)(x) = \sum_{k=0}^{\infty} |Q_k|^\frac{n}{n} \langle \sigma \chi_Q \rangle_{Q_k} \chi_{Q_k}(x)
$$

$$
= |Q_0|^\frac{n}{n} \langle \sigma \chi_Q \rangle_{Q_0} \sum_{k=0}^{\infty} 2^{k(\alpha-n)} \leq C(n, \alpha) |Q_0|^\frac{n}{n} \langle \sigma \chi_Q \rangle_{Q_0} \leq C(n, \alpha) M_\alpha(\sigma \chi_Q)(x).
$$

Therefore, we can replace $I^D_{\alpha,Q}$ by $M_\alpha$ in the testing condition, and if (7.2) holds, then we immediately get that $I_{D,\text{out}} < \infty$. Similarly, if we assume (7.3), then we get that the dual testing condition satisfies $I^*_{D,\text{out}} < \infty$. The strong and weak type inequalities then follow from Theorem 5.4. \qed

We do not know whether Theorem 7.1 is true when $p = q$, though the failure of the Muckenhoupt-Wheeden conjectures for singular integrals suggests that it is false. However, it is not clear where to look for a counter-example. One possibility is to modify the example of Reguera and Scurry [83]. However, this example depends strongly on the cancellation in the Hilbert transform, which is not present in the fractional integral, and it is not certain how this would affect the example. An alternative would be to consider the counter-example to (7.1) in [5].

When $p = q$ there is a weaker conjecture that we believe is true. As we noted above, by Theorem 6.8 we have that (7.2) holds if $[u, \sigma]_{A^p_{p,p,B}} < \infty$, $B \in B_p$, and (7.3) holds if $[u, \sigma]_{A^p_{p,p,A}} < \infty$, $A \in B_{p'}$. We therefore conjecture that that if $[u, \sigma]_{A^p_{p,p,B}} < \infty$, then $I_\alpha(\cdot, \sigma) : L^p(\sigma) \to L^p(u)$, and if $[u, \sigma]_{A^p_{p,p,A}} < \infty$, then $I_\alpha(\cdot, \sigma) : L^p(\sigma) \to L^{p, \infty}(u)$.

This conjecture is the analog of the separated bump conjecture for singular integrals. For fractional integrals, this conjecture is only known for “double” log bumps:
i.e., $A(t) = t^p \log(e + t)^{2p-1-\delta}$, $B(t) = t^{p'} \log(e + t)^{2p'-1+\delta}$. In [21, Theorem 9.42] it was shown that for this choice of $A$ the weak $(p,p)$ inequality is true if $[u, \sigma]_{A_{p,p,A}}^p < \infty$.

We therefore also have that the weak $(p',p')$ inequality is true if $[u, \sigma]_{A_{p',p,B}}^p < \infty$. Then by Corollary 5.3 we have that the two bump conditions together imply the strong $(p,p)$ inequality.

The proof that the bump condition implies the weak type inequality has two steps. First, using a sharp function estimate and two weight extrapolation, we prove a weak $(1,1)$ inequality similar to (7.1):

$$
\sup_{t>0} t u(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \leq C \int_{\mathbb{R}^n} |f(x)| M_{\Phi,\alpha} u(x) \, dx,
$$

where $\Phi(t) = t \log(e + t)^{1+\epsilon}$, $\epsilon > 0$, and $M_{\Phi,\alpha}$ is the Orlicz fractional maximal operator

$$
M_{\Phi,\alpha} u(x) = \sup_Q |Q|^{\frac{1}{n'}} \|u\|_{\Phi,\alpha} \chi_Q(x).
$$

The weak $(p,p)$ inequality then follows by again applying two weight extrapolation and a two weight norm inequality for $M_{\Phi,\alpha}$.

We conjecture that the weak $(1,1)$ inequality is true if we replace $\Phi$ with $\Psi(t) = t \log(e + t)^{\epsilon}$. If this were the case, then the same extrapolation argument would yield the weak $(p,p)$ inequality for log bumps, and the strong type inequality would follow as before. The analogous weak $(1,1)$ inequality is true for singular integrals: this was proved by Pérez [80]. (See also [2].) Unfortunately, every attempt to adapt these proofs to fractional integrals has failed.

An alternate approach would be to prove the weak $(p,p)$ inequality directly using the testing conditions in Theorem 5.2. One way to do this would be to adapt the corona decomposition argument used in [31] to fractional integrals. We tried to do this, but our proof in [25] only worked if $\frac{1}{p} - \frac{1}{q} \approx \frac{\alpha}{n}$. More recently, we have shown [26] that it can be modified to work provided $p < q$; but again the argument fails when $p = q$. We strongly believe that the separated bump conjecture is true for log bumps, and suspect that it is true in general. However, it is clear that either new ideas or a non-trivial adaptation of existing ones will be needed to prove it.

**Two conjectures for commutators.** We conclude with two conjectures for commutators of fractional integrals. The first is a separated bump conjecture. A close examination of the proof of Theorem 6.10 shows that we actually proved something stronger: we showed that for $0 < \alpha < n$ and $1 < p \leq q < \infty$, if a pair of weights $(u, \sigma)$ satisfies

$$
(7.4) \quad \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u\|_{A,Q}^\frac{1}{p} \|\sigma\|_{B,Q}^\frac{1}{q} < \infty,
$$
with \(A(t) = t^q \log(e + t)^{2q - 1 + \delta}\), and \(B(t) = t^{p'} \log(e + t)^{p' - 1 + \delta}\), and
\[
(7.5) \quad \sup_Q |Q|^{\frac{n}{2} + \frac{1}{q} - \frac{1}{p}} \|u\|_{C,Q} \|\sigma\|_{D,Q}^{\frac{1}{2}} < \infty,
\]
with \(C(t) = t^q \log(e + t)^{q - 1 + \delta}\), and \(D(t) = t^{p'} \log(e + t)^{2p' - 1 + \delta}\), then the strong \((p,q)\) inequality \([b, I_\alpha] \sigma : L^p(\sigma) \to L^q(u)\) holds.

There is no comparable result known for the weak \((p,q)\) inequality. However, in [29] two weight weak type inequalities were proved for singular integral operators and we believe that the proofs could be adapted to prove that \([b, I_\alpha] \sigma : L^p(\sigma) \to L^q(u)\) provided that \((7.5)\) holds with \(C(t) = t^{rq}, r > 1\), and \(D(t) = t^{p'} \log(e + t)^{p'}\). Further, using ideas from [14], we could in fact take \(C\) to be from a family of Young functions called exponential log bumps.

We conjecture that the following separated bump conditions are sufficient: the strong \((p,q)\) inequality holds if \((u, \sigma)\) satisfy \((7.4)\) and \((7.5)\) but with \(B(t) = t^{p'}\) and \(C(t) = t^q\). Similarly, the weak \((p,q)\) inequality holds if \((7.4)\) holds with \(B(t) = t^{p'}\). To prove these conjectures, it would suffice to prove them for the dyadic operator \(C^D_b\) in Proposition 3.7. It will probably be easier to prove these conjectures in the off-diagonal case when \(p < q\). One approach in this case would be to prove a “global” version of the testing condition conjectures for commutators given at the end of Section 5. This might be done by adapting the arguments in [55]. Further, though it would probably not yield the full conjecture, it would be interesting to see if the proof in [21, Theorem 9.42] for fractional integrals could be modified to prove a non-optimal weak type inequality for commutators. It seems possible that this approach would yield the weak type inequality with \(A(t) = t^q \log(e + t)^{3q - 1 + \delta}\).

The second conjecture concerns the necessity of \(BMO\) for commutators to be bounded. In [8] Chanillo showed that if \([b, I_\alpha] : L^p \to L^q, \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}\), and \(n - \alpha\) is an even integer, then \(b \in BMO\). (Very recently, this restriction was removed by Chaffee [6].) At the end of the meeting in Antequera, J. L. Torrea asked if anything could be said about \(b\) if there exists a pair of weights \((u, \sigma)\) (or perhaps a family of such pairs) such that \([b, I_\alpha] \sigma : L^p(\sigma) \to L^q(u)\). Nothing is known about this question, but it merits further investigation.

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