Finite-Length and Asymptotic Analysis of Correlogram for Undersampled Data

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Abstract

This paper studies a spectrum estimation method for the case that the samples are obtained at a rate lower than the Nyquist rate. The method is referred to as the correlogram for undersampled data. The algorithm partitions the spectrum into a number of segments and estimates the average power within each spectral segment. This method is able to estimate the power spectrum density of a signal from undersampled data without essentially requiring the signal to be sparse. We derive the bias and the variance of the spectrum estimator, and show that there is a tradeoff between the accuracy of the estimation, the frequency resolution, and the complexity of the estimator. A closed-form approximation of the estimation variance is also derived, which clearly shows how the variance is related to different parameters. The asymptotic behavior of the estimator is also investigated, and it is proved that this spectrum estimator is consistent. Moreover, the estimation made for different spectral segments becomes uncorrelated as the signal length tends to infinity. Finally, numerical examples and simulation results are provided, which approve the theoretical conclusions.

Index Terms

Spectral analysis, correlogram, undersampling, consistency.

I. INTRODUCTION

Spectrum estimation from a finite set of noisy measurements is a classical problem with wide applications in communications, astronomy, seismology, radar, sonar signal processing, etc. [1],

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Classical methods such as the periodogram, the correlogram, the multiple signal classification (MUSIC) method \[3\], and the estimation of signal parameters via rotational invariance techniques (ESPRIT) \[4\] estimate the spectrum based on the Nyquist samples (samples obtained at the Nyquist rate). In practice, the rate at which the measurements are collected can be restricted. Examples include the case when the speed of the sampling hardware is limited or the case when samples of a data record are missing. Therefore, it is desirable to make spectrum estimation from measurements obtained at a rate lower than the Nyquist rate.

In \[5\] and \[6\], authors have studied signal reconstruction from sub-Nyquist samples which are obtained by nonuniform sampling. The methods in these works consider band-limited and multi-band signals with the prior knowledge of the spectral support of the signal, i.e., the position of the frequency bands. In \[7\], algorithms for signal recovery from undersampled data without the prior knowledge of the spectral support except for the number and the widths of the frequency bands have been proposed. The methods in \[5\]–\[7\] aim at reconstructing the signal, whereas depending on the application, e.g., cognitive radio systems \[8\], one might be only interested in recovering the spectral information of the signal. In \[9\], authors have shown that for signals with sparse Fourier representations, i.e., signals which have only a few nonzero coefficients in the Fourier basis, the Fourier coefficients can be estimated using a subset of the Nyquist samples. In \[10\], power spectral density (PSD) estimation based on compressive sensing (CS) techniques \[11\], \[12\] with applications in wideband cognitive radios has been introduced. In \[13\] and \[14\], the possibility of recovering signals sparse in the discrete-time Fourier transform (DTFT) domain from compressive samples obtained at a rate lower than the Nyquist rate has been demonstrated. In \[15\] and \[16\], the super-resolution problem has been addressed where the position of a few sparse sources is resolved with infinite precision from only samples of the low-frequency end of the spectrum. In the super-resolution methods, the information of the high-frequency portion of the spectrum is extrapolated based on the samples of the low-frequency part. However, this is only possible for sparse sources with the additional constraint that the distance between any two sources be larger than a minimum value, i.e., the sources be well-separated.

For all of the above mentioned methods, the sparsity of the signal is a requirement for successful recovery of the spectrum. In \[17\], PSD estimation from a subset of the Nyquist samples has been considered. The introduced method is able to estimate the PSD from undersampled data without essentially requiring the signal to be sparse. We will show in this paper that this is
achieved with a trade-off between the spectral resolution and the estimation accuracy. We refer to this method as the correlogram for undersampled data. In this method, samples are collected using multiple channels, each operating at a rate $L$ times lower than the Nyquist rate. This method of sampling is known as the multi-coset sampling [18]. The correlogram for undersampled data partitions the spectrum into $L$ segments (subbands), and it estimates the average power within each spectral segment. The frequency resolution of the estimator is given by the width of each spectral segment. In this paper, we equivalently use the number of spectral segments $L$ as the frequency resolution of the estimator (with larger $L$ meaning higher resolution or narrower segments). In [19], PSD estimation based on sub-Nyquist samples is also considered. The main difference to [17], however, is that in [19], the introduced method estimates samples of the PSD, whereas in the correlogram for undersampled data, the average power within subbands is estimated. As a result, the correlogram for undersampled data is less computationally complex [17].

The advantage of the correlogram for undersampled data as mentioned above is its ability in estimating the PSD from sub-Nyquist samples without necessarily imposing sparsity conditions on the signal. This is not, however, achieved without paying a price, and it is, therefore, of significant importance to know the associated tradeoffs. The focus of this paper is to analyze the performance of the correlogram for undersampled data and to formulate the associated tradeoffs.

We first study the correlogram for undersampled data by computing the bias of the estimator. Next, the covariance matrix of the estimator is derived, and using our derivations, we show that for finite-length signals, there exists a tradeoff between the estimation accuracy, the frequency resolution, and the complexity of the estimator[1]. For the case of a white Gaussian process, we derive a closed-form expression for the estimation variance, which clearly shows how the variance is related to different parameters. Moreover, we prove that the estimation bias and variance tend to zero asymptotically. Therefore, the correlogram for undersampled data is a consistent estimator. This is in contrast with the conventional correlogram which does not enjoy the consistency property [21]. Besides, we show that similar to the conventional correlogram, the correlogram for undersampled data makes uncorrelated estimations for different spectral

1Note that the complexity is a critical issue in a number of applications, for example, for fighting the curse of dimensionality for data acquisition in exploration seismology [20].
segments as the signal length goes to infinity.

The rest of the paper is organized as follows. The correlogram for undersampled data is revised in Section III. Specifically, we introduce a practical implementation of the filters used in the estimator. In Section III, the bias and the covariance matrix of the correlogram for undersampled data are derived, and a closed-form expression for the estimation variance is given. Section IV presents some numerical examples on the estimation bias and variance of the correlogram method for finite-length signals. Finally, Section V concludes the paper. The proofs and derivations are given in Appendices. This paper is reproducible research and the software needed to generate the numerical results will be provided to the IEEE Xplore together with the paper.

II. CORRELOGRAM FOR UNDERSAMPLED DATA

Consider a wide-sense stationary (WSS) stochastic process $x(t)$ bandlimited to $W/2$ Hz with power spectral density (PSD) $P_x(f)$. Let $x(t)$ be sampled using the multi-coset (MC) sampler as described in [17]. Samples are collected by a multi-channel system. The $i$-th channel ($1 \leq i \leq q$) samples $x(t)$ at the time instants $t = (nL+c_i)T$ for $n = 0, 1, 2, \ldots$, where $T$ is the Nyquist period ($T = 1/W$), $L$ is a suitable positive integer, and $q < L$ is the number of sampling channels. The time offsets $c_i$ ($1 \leq i \leq q$) are distinct non-negative integer numbers less than $L$, and the set $\{c_i\}$ is referred to as the sampling pattern. Let the output of the $i$-th channel be denoted by $y_i(n) = x((nL+c_i)T)$. The $i$-th channel can be implemented by a system that shifts $x(t)$ by $c_iT$ seconds and then samples uniformly at a rate of $1/(LT)$ Hz. The samples obtained in this manner form a subset of the Nyquist samples. The average sampling rate is $q/(LT)$ Hz, and it is less than the Nyquist rate since $q < L$.

Given the MC samples, the first step of the correlogram for undersampled data method is to undo the time shift that each channel imposes on the signal. Let $z_i(n)$ be defined as $y_i(n)$ delayed by a fractional delay equal to $c_i/L$. Let also $a$ and $b$ denote two channel indices. It is shown in [17] that the cross-correlation function $r_{z_az_b}(k) = E\{z_a(n+k)z_b^*(n)\}$ at $k = 0$ is given by

$$r_{z_az_b}(0) = \sum_{l=1}^{L} e^{-j2\pi L(c_a-c_b)m_l} P_x(m_l)$$

(1)

where $E\{\cdot\}$ stands for the expectation operator, $L$ is an odd number, $m_l = -\frac{1}{2}(L+1) + l$, and
$P_x(m_l)$ is defined as

$$P_x(m_l) \triangleq \int_{W/2L}^{W} P_x\left(f - \frac{W}{L}m_l\right) df. \tag{2}$$

Consider partitioning the bandwidth of $x(t)$ into $L$ equal segments. Then, for a given $m_l$, $\frac{W}{L}P_x(m_l)$ is equal to the average power of the process $x(t)$ within the spectral segment $[\frac{W}{2} - \frac{W}{L}, \frac{W}{2} - \frac{W}{L}(l-1)]$.

Let us arrange the elements of the cross-correlation function $r_{zaz_b}(0) \ (1 \leq a, b \leq q)$ in a matrix $R_z \in \mathbb{C}^{q \times q}$ such that $[R_z]_{a,b} = r_{zaz_b}(0)$. Note that $R_z$ is a Hermitian matrix with equal diagonal elements. Then, it is sufficient to let the indices $a$ and $b$ just refer to the elements of the upper triangle and the first diagonal element of $R_z$. Therefore, there are $Q = q(q-1)/2 + 1$ equations of type (1). In matrix-vector form, (1) can be rewritten as

$$u = \Psi v \tag{3}$$

where $v = [v_1, v_2, \ldots, v_L]^T \in \mathbb{R}^{L \times 1}$ consists of the elements $v_l = P_x(m_l)$, $(\cdot)^T$ stands for the transposition operator, $u = [u_1, u_2, \ldots, u_Q]^T \in \mathbb{C}^{Q \times 1}$ is composed of $u_1 = [R_z]_{1,1}$ and $u_2, \ldots, u_Q$ corresponding to the elements of the upper triangle of $R_z$, and $\Psi \in \mathbb{C}^{Q \times L}$ consists of the elements given by

$$[\Psi]_{k,l} = e^{-j\omega_km_l} \tag{4}$$

where $\omega_k = \frac{2\pi}{L}(c_a - c_b)$, $(1 \leq l \leq L$ and $1 \leq k \leq Q)$. Note that $a$ and $b$ are obtained from $k$ based on the arrangement of the elements of $R_z$ in $u$.

Since the elements of $v$ are real-valued, the number of equations in (3) can be doubled by solving $\tilde{\mathbf{u}} = \bar{\Psi} \mathbf{v}$, where $\tilde{\mathbf{u}} \triangleq [\text{Re}(\mathbf{u}), \text{Im}(\mathbf{u})]^T \in \mathbb{R}^{2Q \times 1}$ and $\bar{\Psi} \triangleq [\text{Re}(\Psi), \text{Im}(\Psi)]^T \in \mathbb{R}^{2Q \times L}$.

Suppose $\bar{\Psi}$ is full rank and $2Q \geq L$. Then, $\tilde{\mathbf{u}} = \bar{\Psi} \mathbf{v}$ is an overdetermined system and $\mathbf{v}$ can be obtained using the pseudoinverse of $\bar{\Psi}$ as

$$\mathbf{v} = (\bar{\Psi}^T \bar{\Psi})^{-1} \bar{\Psi}^T \tilde{\mathbf{u}}. \tag{5}$$

The cross-correlation function $r_{zaz_b}(k)$ can be estimated from a finite number of samples as

$$\hat{r}_{zaz_b}(k) = \frac{1}{N} \sum_{n=0}^{N-|k|-1} \hat{z}_a(n+k)\hat{z}_b(n) \tag{6}$$

$^2$Doubling the number of equations is beneficial in turning an underdetermined system of equations into an overdetermined system.

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where \( N \) is the number of samples obtained from each channel, and \( \hat{z}_a(n + k) \) and \( \hat{z}_b(n) \) are obtained by delaying \( y_a(n + k) \) and \( y_b(n) \) for \( c_a/L \) and \( c_b/L \) fractions, respectively. Next, the elements of the matrix \( R_z \) are estimated as

\[
[\hat{R}_z]_{a,b} = \hat{r}_{za \ast z}(0) = \frac{1}{N} \sum_{n=0}^{N-1} \hat{z}_a(n) \hat{z}_b^\ast(n). \tag{7}
\]

The fractional delays \( c_a/L \) and \( c_b/L \) can be implemented by fractional delay (FD) filters. In [17], authors consider using ideal FD filters which have infinite impulse responses. Then, for the purpose of implementation, these filters are truncated using a rectangular window whose width is twice the signal length \( N \). Consequently, the length of the filters can be quite large as \( N \) increases. Here, we consider using causal finite impulse response (FIR) filters which have two practical advantages [22]: first, the length of the filters are fixed, and second, they enjoy causality. As for the analysis, we will use a general formulation for the FIR FD filters, and for numerical examples, we will use the Lagrange interpolator [23].

FIR FD filters perform the best when the total delay is approximately equal to half of the order of the filter [24]. The fractional delays \( c_a/L \) and \( c_b/L \) are positive numbers less than one, and the performance of the FIR FD filters is very poor with such delays. To remedy this problem, a suitable integer delay can be added to the fractional part. Note that \( \hat{r}_{za \ast z}(k) \) is the inverse discrete-time Fourier transform (DTFT) of \( (1/N) \hat{Z}_a(e^{j2\pi fL/W}) \hat{Z}_b^\ast(e^{j2\pi fL/W}) \), where \( \hat{Z}_a(e^{j2\pi fL/W}) \) and \( \hat{Z}_b(e^{j2\pi fL/W}) \) are the DTFT of \( \hat{z}_a(n) \) and \( \hat{z}_b(n) \), respectively [25]. Then, considering that

\[
\hat{Z}_a(e^{j2\pi fL/W}) \hat{Z}_b^\ast(e^{j2\pi fL/W}) = \left[ \hat{Z}_a(e^{j2\pi fL/W}) e^{-jD2\pi fL/W} \right] \left[ \hat{Z}_b(e^{j2\pi fL/W}) e^{-jD2\pi fL/W} \right]^\ast \tag{8}
\]

we can rewrite (7) as

\[
[\hat{R}_z]_{a,b} = \frac{1}{N} \sum_{n=0}^{N-1} \hat{z}_a(n - D) \hat{z}_b^\ast(n - D) \tag{9}
\]

where \( D \) is a suitable integer number close to half of the order of the FD filter.

Let \( h_a(n) \) be the impulse response of a causal filter that delays a signal for \( c_a/L + D \). Furthermore, let us assume that the length of \( h_a(n) \) is large enough, so that its deviation from an ideal FD filter can be ignored. Therefore, \( z_a(n - D) \) can be written as

\[
z_a(n - D) = \sum_{r=0}^{N_b - 1} h_a(r)y_a(n - r) \tag{10}
\]
where \( N_h \) is the length of the filter’s impulse response. For a limited number of samples, we have

\[
\hat{z}_a(n-D) = \sum_{r=0}^{N_h-1} h_a(r)y_a(n-r)W_D(n-r)
\]

\[
= \sum_{r=n-N_h+1}^{n} h_a(n-r)y_a(r)W_D(r)
\]

(11)

where \( W_D(n) \) is a window of length \( N \) which equals 1 for \( 0 \leq n \leq N - 1 \) and is equal to zero elsewhere. Using the elements of \( \hat{\mathbf{R}}_z \), the vector \( \hat{\mathbf{u}} \) is formed as an estimation for \( \mathbf{u} \). Next, \( \hat{\mathbf{v}} \) (the estimation for \( \mathbf{v} \)) is formed by replacing \( \mathbf{u} \) with \( \hat{\mathbf{u}} \) in (5) as

\[
\hat{\mathbf{v}} = (\hat{\mathbf{\Psi}}^T \hat{\mathbf{\Psi}})^{-1}\hat{\mathbf{\Psi}}^T \hat{\mathbf{z}}.
\]

(12)

Finally, let us define \( \hat{\mathbf{p}} \in \mathbb{R}^{L \times 1} \) as

\[
\hat{\mathbf{p}} \triangleq \frac{L}{W} \hat{\mathbf{v}}.
\]

(13)

The elements of \( \hat{\mathbf{p}} \) give an estimation for the average power within each spectral segment.

III. BIAS AND VARIANCE OF CORRELATION FOR UNDERSAMPLED DATA

Consider a Gaussian WSS signal \( x(t) \) bandlimited to \( W/2 \) Hz, and let \( x(m) \) be the samples of the signal obtained at the Nyquist rate \( (m \in \mathbb{Z}) \). Let also \( r_x(k) = E\{x(m+k)x^*(m)\} \) and \( P_x(e^{j2\pi f/W}) = \text{DTFT}\{r_x(k)\} \) be the autocorrelation function and the PSD of \( x(m) \), respectively. Furthermore, consider a zero-mean Gaussian random process \( e(t) \) bandlimited to \( W/2 \) Hz with a flat PSD \( P_e(f) = \sigma^2/W \). The autocorrelation function of \( e(t) \) is \( r_e(\tau) = \sigma^2 \text{sinc}(W \tau) \). Let \( e(m) \) be the samples of \( e(t) \) obtained at the Nyquist rate. Then, the autocorrelation function of \( e(m) \) is given by

\[
r_e(k) = \sigma^2 \text{sinc}(Wk/W) = \sigma^2 \delta(k)
\]

(14)

where \( \delta(k) \) is the Kronecker delta. Therefore, the PSD of \( e(m) \) is given by \( P_e(e^{j2\pi f/W}) = \sigma^2 \). Now, consider a filter \( h_x(m) \) such that \( \sigma^2|H_x(e^{j2\pi f/W})|^2 \) is equal to \( P_x(e^{j2\pi f/W}) \), where \( H_x(e^{j2\pi f/W}) \) is the DTFT of \( h_x(m) \). Therefore, we have

\[
P_x(e^{j2\pi f/W}) = |H_x(e^{j2\pi f/W})|^2 P_e(e^{j2\pi f/W}).
\]

(15)
As a result, \( x(m) \) can be considered as \( e(m) \) filtered by \( h_x(m) \) since the output of the filter has the same PSD as \( P_x(e^{j2\pi f/W}) \). Then, the output of the \( i \)-th sampling channel can be written as

\[
y_i(n) = x(nL + c_i) = \sum_{m \in \mathbb{Z}} h_x(m)e(nL + c_i - m).
\]

(16)

Let \( a \) and \( b \) denote two channel indices. The cross-correlation function \( r_{y_ay_b}(k) = E\{y_a(n + k)y_b^*(n)\} \) is given by

\[
r_{y_ay_b}(k) = \sum_{m \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} h_x(m)h_x^*(l)E\{e((n + k)L + c_a - m)e^*(nL + c_b - l)\}
\]

\[
= \sum_{m \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} h_x(m)h_x^*(l)r_e(kL + l - m + c_a - c_b)
\]

\[
= \sigma^2 \sum_{m \in \mathbb{Z}} h_x(m)h_x^*(-kL + m + c_b - c_a).
\]

(17)

Furthermore, using (10), \( [R_z]_{a,b} \) can be written as

\[
[R_z]_{a,b} = E\{z_a(n - D)z_b^*(n - D)\}
\]

\[
= \sum_{r=0}^{N_h-1} \sum_{p=0}^{N_h-1} h_a(r)h_b(p)r_{y_ay_b}(p - r).
\]

(18)

A. Bias Analysis

The bias of the correlogram for undersampled data estimator is given by

\[
E\{\hat{p}\} - p = \frac{L}{W} (E\{\hat{v}\} - v)
\]

(19)

where \( p = (L/W)v \). The expected value of \( \hat{\theta} \) is obtained using (12) as

\[
E\{\hat{\theta}\} = (\hat{\Psi}^T \hat{\Psi})^{-1} \hat{\Psi}^T E\{\hat{u}\}.
\]

(20)

Computing \( E\{\hat{u}\} \) requires finding the expected value of the real and imaginary parts of \( \hat{R}_z \). The expectation operation can be performed before taking the real or imaginary parts of \( \hat{R}_z \), as these operators are linear. Moreover, \( \hat{\theta} \) is used to form \( \hat{R}_z \). Taking expectation from both
sides of (9) along with using (11) results in

\[
E\{\hat{\mathbf{R}}_{\alpha,\beta}\} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{r=0}^{N_{h}-1} \sum_{p=0}^{N_{h}-1} h_{a}(r)h_{b}(p)W_{D}(n-r)W_{D}(n-p)E\{y_{a}(n-r)y_{b}^{*}(n-p)\}
\]

\[
= \sum_{r=0}^{N_{h}-1} \sum_{p=0}^{N_{h}-1} h_{a}(r)h_{b}(p)r_{y_{a}y_{b}}(p-r) \times \frac{1}{N} \sum_{n=0}^{N-1} W_{D}(n-r)W_{D}(n-p) \tag{21}
\]

With the assumption that the number of samples \( N \) is larger than the length of the fractional delay filters \( N_{h} \), the last summation of (21) can be simplified to

\[
\sum_{n=0}^{N-1} W_{D}(n-r)W_{D}(n-p) = N - \max(r,p). \tag{22}
\]

Therefore, (21) can be rewritten as

\[
E\{\hat{\mathbf{R}}_{\alpha,\beta}\} = [\mathbf{R}_{\alpha,\beta}] - \frac{1}{N} \sum_{r=0}^{N_{h}-1} \sum_{p=0}^{N_{h}-1} h_{a}(r)h_{b}(p)r_{y_{a}y_{b}}(p-r)\max(r,p) \tag{23}
\]

where \([\mathbf{R}_{\alpha,\beta}]\) is given by (18).

It can be seen from (23) that as \( N \) tends to infinity, \( E\{[\hat{\mathbf{R}}_{\alpha,\beta}]\} \) tends to \([\mathbf{R}_{\alpha,\beta}]\). Therefore, \( \hat{\mathbf{R}}_{\alpha,\beta} \) is an asymptotically unbiased estimator of \( \mathbf{R}_{\alpha,\beta} \). Since \( \hat{\mathbf{u}} \) consists of the elements of \( \hat{\mathbf{R}}_{\alpha,\beta} \) and the operation of taking the real and imaginary parts are linear, it follows that \( \hat{\mathbf{u}} \) is also an asymptotically unbiased estimator of \( \mathbf{u} \). Furthermore, letting the number of samples tend to infinity in (20) and using (5), we find that

\[
\lim_{N \rightarrow \infty} E\{\hat{\mathbf{x}}\} = (\hat{\mathbf{\Psi}}^{T}\hat{\mathbf{\Psi}})^{-1}\hat{\mathbf{\Psi}}^{T}\lim_{N \rightarrow \infty} E\{\hat{\mathbf{u}}\}
\]

\[
= (\hat{\mathbf{\Psi}}^{T}\hat{\mathbf{\Psi}})^{-1}\hat{\mathbf{\Psi}}^{T}\hat{\mathbf{u}} = \mathbf{v}. \tag{24}
\]

In other words, \( \hat{\mathbf{x}} \) is also an asymptotically unbiased estimator of \( \mathbf{v} \). Finally, it can be concluded from (19) that the correlogram for undersampled data estimator \( \hat{\mathbf{p}} \) is asymptotically unbiased.

Next, we consider the case that the input signal \( x(t) \) is equal to the white Gaussian random process \( e(t) \). It is shown in Appendix A that

\[
E\{\hat{\mathbf{p}}\} = H_{1}\mathbf{p} = H_{1}\frac{\sigma^{2}}{W}\mathbf{1}_{L} \tag{25}
\]
where $1_L$ is the column vector of length $L$ with all its elements equal to 1, and $H_1$ is given by

$$H_1 = \frac{1}{N} \sum_{r=0}^{N_h-1} (N - r) h^2_1(r).$$

(26)

Therefore, the bias of the correlogram for undersampled data estimator in this case is given by

$$E\{\hat{p}\} - p = (1 - H_1) \frac{\sigma^2}{W} 1_L.$$ (27)

B. Variance Analysis

The covariance matrix of the correlogram for undersampled data is given by

$$C_{\hat{p}} = E\{(\hat{p} - E\{\hat{p}\})(\hat{p} - E\{\hat{p}\})^T\}$$

$$= E\{\hat{p}\hat{p}^T\} - E\{\hat{p}\}E\{\hat{p}\}^T.$$ (28)

The diagonal elements of $C_{\hat{p}}$ are the estimation variance of each spectral segment. The off-diagonal elements of $C_{\hat{p}}$ represent the correlation between pairs of the estimations made for different spectral segments.

It follows from (12) and (13) that

$$E\{\hat{p}\hat{p}^T\} = \left(\frac{L}{W}\right)^2 (\hat{\psi}^T \hat{\psi})^{-1} \hat{\psi}^T U \hat{\psi} \hat{\psi}^T (\hat{\psi}^T \hat{\psi})^{-1}$$

(29)

where $U = E\{\hat{u}\hat{u}^T\} \in \mathbb{R}^{2Q \times 2Q}$. Computation of the elements of $U$ involves taking expectation of the multiplication of the real or imaginary parts of the elements of $\hat{R}_z$. We will use the following lemma [26] for interchanging the expectation and the operation of taking real or imaginary parts.

**Lemma 1.** Let $x$ and $y$ be two arbitrary complex numbers. The following equations hold

$$Re(x)Re(y) = \frac{1}{2} (Re(xy) + Re(xy^*))$$

(30)

$$Im(x)Im(y) = -\frac{1}{2} (Re(xy) - Re(xy^*))$$

(31)

$$Re(x)Im(y) = \frac{1}{2} (Im(xy) - Im(xy^*))$$.

(32)

The elements of $U$ can be easily obtained using $E\{[\hat{R}_z]_{a,b}[\hat{R}_z]_{c,d}\}$, $E\{[\hat{R}_z]_{a,b}^* [\hat{R}_z]_{c,d}\}$, and Lemma 1, where $[\hat{R}_z]_{a,b}$ and $[\hat{R}_z]_{c,d}$ are the elements of $\hat{R}_z$ used for forming $\hat{u}$. Let the outputs
of the sampling channels be given by (16). Using (9) and (11), we obtain

$$E\{ [\hat{R}_z]_{a,b} [\hat{R}_z]_{c,d} \} = $$

$$\frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{r=(n-N_h+1)}^{(n-N_h+1)} \sum_{p=(u-N_h+1)}^{(u-N_h+1)} \sum_{u=0}^{N-1} \sum_{m=(u-N_h+1)}^{(u-N_h+1)} h_a(n-r)h_b(n-p)h_c(u-s)h_d(u-m) \times$$

$$W_D(r)W_D(p)W_D(s)W_D(m) \times$$

$$E\{ y_a(r)y_b^*(p)y_c(s)y_d^*(m) \} = $$

$$\frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{r=\max}^{(0,n-N_h+1)} \sum_{p=\max}^{(0,n-N_h+1)} \sum_{u=0}^{N-1} \sum_{m=\max}^{(0,n-N_h+1)} h_a(n-r)h_b(n-p)h_c(u-s)h_d(u-m) \times$$

$$(r_{ya,yb}(r-p)r_{yc,yd}(s-m) + r_{ya,yd}(r-m)r_{yc,yb}(s-p)) .$$

(33)

The last line in (33) is obtained using the forth-order moment of Gaussian random processes.

In a similar way, $E\{ [\hat{R}_z]_{a,b}^* [\hat{R}_z]_{c,d} \}$ can be obtained as

$$E\{ [\hat{R}_z]_{a,b}^* [\hat{R}_z]_{c,d} \} = $$

$$\frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{r=\max}^{(0,n-N_h+1)} \sum_{p=\max}^{(0,n-N_h+1)} \sum_{u=0}^{N-1} \sum_{m=\max}^{(0,n-N_h+1)} h_a(n-r)h_b(n-p)h_c(u-s)h_d(u-m) \times$$

$$(r_{ya,yb}(r-p)r_{yc,yd}(m-s) + r_{ya,yd}(r-s)r_{yc,yb}(m-p)) .$$

(34)

The details of simplifying $U$ for the case that the input signal $x(t)$ is equal to the white Gaussian random process $e(t)$ are given in Appendix B. It is shown that in this case, $U$ is a diagonal matrix with

$$[U]_{1,1} = \frac{\sigma^4}{N^2} \left( N^2 H_1^2 + (N - 2N_h + 2)G_1 + \Sigma_1 \right)$$

$$[U]_{Q+1,Q+1} = 0$$

$$[U]_{k,k} = \frac{\sigma^4}{2N^2} \left( (N - 2N_h + 2)G_k + \Sigma_k \right)$$

(35)
where $G_1, \Sigma_1, G_k, $ and $\Sigma_k$ ($2 \leq k \leq 2Q$ and $k \neq Q + 1$) are independent of the signal length and depend on the FD filters.

The equations for computing the covariance matrix $C_p$ as given by (28) to (35) are in the matrix form. Next, we simplify these formulas to show the dependence of the estimation variance on different parameters more clearly. It is shown in Appendix C that for the white Gaussian process, the diagonal elements of $C_p$ can be approximated by

$$([C_p]_{l,l} \approx \frac{\sigma^4}{2W^2N_x^2} \left( \frac{L^3}{Q} + L \right) \times ((N_x - 2N_hL + 2L)G_1 + L\Sigma_1))$$

(36)

where $N_x$ is the number of Nyquist samples. Considering a large enough $N_x$, it can be seen from (36) that the estimation variance is a cubic function of the number of spectral segments $L$ as $(L^3/Q + L)$. Moreover, the variance is inversely proportional to $Q$, which means that the variance decreases quadratically with the number of sampling channels $q$. Furthermore, at a fixed average sampling rate $(q/L)W$ and a given signal length $N_x$, the variance increases almost linearly with the number of spectral segments. Finally, it can be seen that the estimation variance decreases as the signal length increases at an approximate rate of $1/N_x$.

We next consider the asymptotic behavior of the correlogram for undersampled data for the case of a white Gaussian process. The following theorem studies the covariance matrix of the estimator as the length of the signal tends to infinity. The proof of the theorem is given in Appendix D.

**Theorem 1:** In the case of a white Gaussian process, the correlogram estimation based on undersampled data is a consistent estimator of the average power in each spectral segment. Furthermore, the estimations made for different spectral segments are asymptotically uncorrelated.

### IV. Numerical Examples

In this section, we investigate the behavior of the correlogram for undersampled data for finite-length signals based on the analytical results obtained in Section III and Monte Carlo simulations.

The estimation bias and variance of the correlogram method depends on the number of sampling channels $q$, the number of spectral segments $L$, and the number of samples per channel $N$. Here, the Nyquist sampling rate is considered to be $W = 1000$ Hz. The time offsets $c_i$
are distinct positive integer numbers less than \( L \) which are generated with equal probability for each \((L, q)\)-pair. After generating the time offsets \( c_i \), the matrix \( \tilde{\Psi} \) is formed and its rank is checked. In the case that \( \tilde{\Psi} \) is rank deficient, a new set of time offsets is generated until a full rank matrix \( \tilde{\Psi} \) is obtained or a maximum number of tries is performed. In the latter case, the given \((L, q)\)-pair is considered as unfeasible. Once a full rank matrix \( \tilde{\Psi} \) is obtained, it is kept unchanged for different signal lengths.

We present six examples to illustrate the bias and variance of the correlogram for undersampled data. For the first four examples, we consider a white Gaussian process with its PSD equal to \( \sigma^2/W = 1 \). For the last two examples, a filtered Gaussian process is used.

The estimation bias is investigated first. We consider the case when the average sampling rate \((q/L)W\) is kept unchanged. Therefore, for a given number of Nyquist samples, the overall number of samples available for estimation is the same for different \((L, q)\)-pairs. Fig. 1 depicts the bias of the estimator versus the number of Nyquist samples \( N_x \). The curve marked with squares is obtained by Monte Carlo simulations for comparison with the theoretical results. The rest of the curves are obtained from (27). Referring to (26) and (27), it can be seen that the bias is proportional to the inverse of the signal length \( N_x \) (consider multiplying (26) by \( L/L \), and note that \( N_x = NL \)). Moreover, at a given signal length, the bias increases linearly with the number of spectral segments. It can also be seen that the estimation bias tends to zero as the length of the signal tends to infinity.

Fig. 2 depicts the variance of the estimator \( [\hat{C}_p]_{1,1} \) versus the number of sampling channels \( q \) for different values of spectral segments \( L \). The signal length is fixed at \( N_x = 10^5 \). The curves drawn with solid lines represent the exact variance obtained from (28) to (35), and the curves plotted with dashed lines are the approximate values obtained from (36). Increasing \( q \) at a fixed \( L \) is equivalent to increasing the average sampling rate \((q/L)W\). According to the approximate variance as given in (36), the variance decreases quadratically with the number of sampling channels \( q \). Therefore, the performance of the estimator improves by increasing \( q \), but this comes at the price of adding to the complexity of the system by using more sampling channels.

Fig. 3 shows the variance of the estimator \( [\hat{C}_p]_{1,1} \) versus the number of spectral segments \( L \) for different numbers of sampling channels \( q \). The signal length is fixed at \( N_x = 10^5 \). Again, the curves drawn with solid lines are obtained from (28) to (35), and the curves plotted with
dashed lines are obtained from (36). According to the approximate variance as given in (36), the variance increases cubically with the number of spectral segments $L$. Therefore, at a fixed signal length and fixed number of sampling channels, the performance of the estimator is degraded by increasing the number of spectral segments $L$, i.e., by increasing the frequency resolution.

The variance of the estimator $[C_p]_{1,1}$ versus the signal length $N_x$ is illustrated in Fig. 4. Here, the average sampling rate $(q/L)W$ is kept unchanged. Therefore, for a given number of Nyquist samples, the overall number of samples available for estimation is the same for different $(L, q)$-pairs. The curve marked with squares is obtained by Monte Carlo simulations for comparison with the theoretical results. Again, the curves drawn with solid lines are obtained from (28) to (35), and the curves plotted with dashed lines are obtained from (36). Referring to the approximate variance as given in (36), the variance is almost proportional to the inverse of the signal length $N_x$. From the curves corresponding to the $(51,12)$, $(101,25)$, and $(201,50)$-pairs in Fig. 4, it can be seen that the performance of the estimator degrades when increasing the number of spectral segments, i.e., when increasing the frequency resolution. The average sampling rate is kept almost the same in this scenario. It can also be seen that the estimation variance tends to zero as the length of the signal tends to infinity.

For the next two examples, we consider a more general case with a filtered Gaussian process. The signal is obtained by passing a white Gaussian signal through a bandlimited filter with cutoff frequencies set at $W/10$ and $W/5$ Hz. Through our experiments, we found that the estimation variance at each spectral segment depends not only on the power of signal at that frequency band, but also it is dependant on the power of the signal at other spectral segments. As noticed from the analytical derivations for the white Gaussian process (see (29), (35), and (36)), the estimation variance is proportional to the square of the signal power ($\sigma^4/W^2$). Therefore, we set the gain of the filter so that the square of the power averaged over all spectral segments for both the white Gaussian process at the input of the filter and the filtered signal is the same.

In Fig. 5 the variance of the estimator $[C_p]_{1,1}$ versus the number of spectral segments $L$ is depicted. The number of sampling channels is set to $q = 45$, and the signal length is fixed at $N_x = 10^5$. The curve for the white Gaussian signal is based on (28) to (35), and the curve for the filtered Gaussian signal is obtained by Monte Carlo simulations. The latter curve is the average estimation variance of the spectral segments that pass through the filter. It can be seen in Fig. 5 that the variance of the estimator for the white and the filtered signals are close to each other.
Finally, the variance of the estimator $[\mathcal{C}_p]_{1,1}$ versus the signal length $N_x$ for the white and the filtered signals is investigated. The number of spectral segments is set to $L = 101$, and the number of sampling channels is set to $q = 25$. Again, the curve for the white Gaussian signal is based on (28) to (35), and the curve for the filtered Gaussian signal is obtained by Monte Carlo simulations. Similar to the previous example, it can be seen in Fig. 6 that the estimation variance for the white and the filtered signals are close to each other. It can also be seen that the estimation variance tends to zero as the length of the signal tends to infinity.

V. CONCLUSION

We considered the correlogram for undersampled data which estimates the spectrum from a subset of the Nyquist samples. This method has been analyzed in this paper by computing the bias and the variance of the estimator. It has been shown that the bias and the variance of the method tend to zero asymptotically. Therefore, this method is a consistent estimator. Furthermore, it has been shown that the estimation made for different spectral segments becomes uncorrelated as the signal length goes to infinity.

The behavior of the estimator for finite-length signals has also been investigated. It has been shown that at a given signal length, the estimation accuracy increases as the average sampling rate is increased (either by decreasing the frequency resolution $L$ or by increasing the complexity of the system $q$). It has also been shown that at a fixed average sampling rate, the performance of the estimator degrades for the estimation with higher frequency resolution. To sum up, it has been illustrated that there is a tradeoff between the accuracy of the estimator (the estimation variance), the frequency resolution (the number of spectral segments), and the complexity of the estimator (the number of sampling channels).

APPENDIX A

BIAS SIMPLIFICATION

In the case that $x(t)$ is equal to $e(t)$, we have $h_x(m) = \delta(m)$. Then, using (17), the cross-correlation function $r_{y_a y_b}(k)$ is given by

$$r_{y_a y_b}(k) = \sigma^2 \delta(k) \delta(a - b).$$

(37)
Applying (37) to (23), we find that
\[
E\{[\hat{R}_z]_{a,b}\} = [R_z]_{a,b} - \frac{1}{N} \sum_{r=0}^{N_b-1} h_a^2(r) r \sigma^2 \delta(a - b).
\] (38)

Next, \([R_z]_{a,b}\) is obtained using (18) and (37) as
\[
[R_z]_{a,b} = \sum_{r=0}^{N_b-1} h_a^2(r) \sigma^2 \delta(a - b).
\] (39)

Replacing (39) into (38) results in
\[
E\{[\hat{R}_z]_{a,b}\} = 0
\] (40)

for \(a \neq b\), and
\[
E\{[\hat{R}_z]_{a,b}\} = \sigma^2 \frac{1}{N} \sum_{r=0}^{N_b-1} (N - r) h_a^2(r) = H_a \sigma^2
\] (41)

for \(a = b\), where
\[
H_a \triangleq \frac{1}{N} \sum_{r=0}^{N_b-1} (N - r) h_a^2(r).
\] (42)

Recalling that the first diagonal element of \(\hat{R}_z\) is used in \(\hat{u}\) and taking the real and imaginary parts of (40) and (41), \(E\{\hat{u}\}\) can be obtained as
\[
E\{\hat{u}\} = H_1 \sigma^2 e_1
\] (43)

where \(e_1\) is a column vector of length \(q(q - 1) + 2\) with all its elements equal to zero except for the first element which is 1. The expected value of \(\hat{v}\) can be found using (20) and (43) as
\[
E\{\hat{v}\} = H_1 \sigma^2 (\hat{\Psi}^T \hat{\Psi})^{-1} \hat{\Psi}^T e_1.
\] (44)

Next, consider the fact that \(x(t)\) has equal power in all spectral segments (the elements of \(v\) are all the same). Since \(\hat{v}\) is asymptotically unbiased, it follows that the elements of \(\lim_{N \to \infty} E\{\hat{v}\}\) are also equal.

Replacing the true values in (1) with the estimated values for \(a = b = 1\), taking expectation from both sides, and letting the number of samples tend to infinity, we obtain that
\[
\lim_{N \to \infty} E\{[\hat{R}_z]_{1,1}\} = \sum_{l=1}^{L} \lim_{N \to \infty} E\{\hat{v}_l\}
\]
\[
= 1^T \lim_{N \to \infty} E\{\hat{v}\}
\] (45)
where \( \hat{v}_l \) \((1 \leq l \leq L)\) are the elements of \( \hat{v} \). Considering normalized FD filters \( \sum_{r=0}^{N_b-1} h_a^2(r) = 1 \) and referring to (42), we also find that

\[
\lim_{N \to \infty} H_a = 1.
\]  

(46)

Therefore, using (41), we can find that

\[
\lim_{N \to \infty} E\{[\hat{R}_z]_{1,1}\} = \sigma^2.
\]  

(47)

Combining (45) with (47) results in

\[
\lim_{N \to \infty} E\{\hat{v}\} = \frac{\sigma^2}{L} 1_L.
\]  

(48)

Letting the number of samples tend to infinity in (44) and using (48), we obtain

\[
\lim_{N \to \infty} E\{\hat{v}\} = \sigma^2 (\hat{\Psi}^T \hat{\Psi})^{-1} \hat{\Psi}^T e_1 = \frac{\sigma^2}{L} 1_L.
\]  

(49)

It follows from (49) that all the elements of the first column of \( (\hat{\Psi}^T \hat{\Psi})^{-1} \hat{\Psi}^T \) are equal to \( 1/L \). Therefore, (44) can be simplified as

\[
E\{\hat{v}\} = H_1 \frac{\sigma^2}{L} 1_L.
\]  

(50)

Finally, using (13), we have

\[
E\{\hat{p}\} = H_1 \frac{\sigma^2}{W} 1_L.
\]  

(51)

**APPENDIX B**

**VARIANCE SIMPLIFICATION**

In the case that \( x(t) \) is equal to \( e(t) \), we have \( h_x(m) = \delta(m) \). Then, the cross-correlation functions in (33) are simplified as

\[
E_1 \triangleq r_{y_a y_b}(r - p) r_{y_c y_d}(s - m) + r_{y_a y_d}(r - m) r_{y_c y_b}(s - p)
= \sigma^4 \left( \delta(r - p) \delta(a - b) \delta(s - m) \delta(c - d) + \delta(r - m) \delta(a - d) \delta(s - p) \delta(c - b) \right).
\]  

(52)

Similarly, the cross-correlation functions in (34) are simplified as

\[
E_2 \triangleq r_{y_a y_b}(r - p) r_{y_d y_c}(m - s) + r_{y_a y_c}(r - s) r_{y_d y_b}(m - p)
= \sigma^4 \left( \delta(r - p) \delta(a - b) \delta(m - s) \delta(d - c) + \delta(r - s) \delta(a - c) \delta(m - p) \delta(d - b) \right).
\]  

(53)
Recalling that only the first diagonal element of $\hat{\mathbf{R}}_z$ is present in $\hat{\mathbf{u}}$, $E_1$ can be found to be equal to

$$E_1 = \sigma^4 (\delta(r - p)\delta(s - m) + \delta(r - m)\delta(s - p))$$

(54)

for $a = b = c = d = 1$, and it equals to zero otherwise. Similarly, $E_2$ can be found to be equal to

$$E_2 = \sigma^4 \delta(r - s)\delta(m - p)$$

(55)

for $a = c$ and $b = d$, and it equals zero otherwise (excluding the case when $a = b = c = d = 1$ since $[\hat{\mathbf{R}}_z]_{1,1}$ is real-valued, and therefore, we do not need to compute (34)). Noting that $E\{[\hat{\mathbf{R}}_z]_{a,b}[\hat{\mathbf{R}}_z]_{c,d}\}$ and $E\{[\hat{\mathbf{R}}_z]_{a,b}[\hat{\mathbf{R}}_z]_{c,d}^*\}$ are real-valued and using (32), (54), and (55), we can find that all the off-diagonal elements of $\mathbf{U}$ are equal to zero.

Let us start computing the diagonal elements of $\mathbf{U}$ by setting $a = b = c = d = 1$. It follows from (33) and (54) that

$$E\{[\hat{\mathbf{R}}_z]_{1,1}[\hat{\mathbf{R}}_z]_{1,1}\} = \frac{\sigma^4}{N^2} \left( \sum_{n=0}^{N-1} \sum_{r=\max(0, n-Nh+1)}^{n} \sum_{u=0}^{u=\max(0, u-Nh+1)} h^2_1(n - r)h^2_1(u - s) + \sum_{n=0}^{N-1} S_1(n) \right)$$

(56)

where $S_1(n)$ is defined as

$$S_1(n) \triangleq \sum_{r=\max(0, n-Nh+1)}^{n} \sum_{p=\max(0, n-Nh+1)}^{n} \sum_{u=0}^{\max(0, u-Nh+1)} \sum_{m=\max(0, u-Nh+1)}^{u} \delta(r - m)\delta(s - p)h_1(n - r)h_1(n - p)h_1(u - s)h_1(u - m).$$

(57)
For \( N_h - 1 \leq n \leq N - N_h \), \( S_1(n) \) is given by

\[
S_1(n) = \sum_{u=\min(n,n-N_h)}^{n+N_h-1} \left[ \sum_{r=\min(n,n-N_h)}^{n} \sum_{m=\max(0,u-N_h+1)}^{u} \delta(r-m) \delta(s-p) \times \right.
\]

\[
h_1(n-r)h_1(u-m) \times h_1(n-p)h_1(u-s) \right].
\]

(58)

Note that the summations in the brackets are equivalent to each other, which leads to the following simplification

\[
S_1(n) = \sum_{u=\min(n,n-N_h)}^{n+N_h-1} \left[ \sum_{r=\min(n,n-N_h)}^{n} \sum_{m=\max(0,u-N_h+1)}^{u} \delta(r-m) \times \right.
\]

\[
h_1(n-r)h_1(u-m) \right]^2
\]

(59)

Next, a change of variable \((g = u - n + N_h - 1)\) is used, which results in

\[
S_1(n) = \sum_{g=0}^{2N_h-2} \left[ \sum_{r=\max(0,g-N_h+1)+n-N_h+1}^{\min(g,N_h-1)+n} \sum_{p=\max(0,g-N_h+1)}^{\min(g,N_h-1)} \right.
\]

\[
h_1(n-r)h_1(n-r+g-N_h+1) \right]^2.
\]

(60)

With another change of variable \((p = n - r + g - N_h + 1)\), we obtain the following

\[
S_1(n) = \sum_{g=0}^{2N_h-2} \left[ \sum_{p=\max(0,g-N_h+1)}^{\min(g,N_h-1)} \sum_{r=\max(0,g-N_h+1)+n-N_h+1}^{\min(0,g-N_h+1)+n} \right.
\]

\[
h_1(p-g+N_h-1)h_1(p) \right]^2
\]

(61)
which is equal to
\[
G_1 \triangleq S_1(n) = \sum_{g=0}^{2N_h-2} [h_1(i) * h_1(N_h - 1 - i)]_g^2
\]  
(62)

where * denotes the convolution operation. Note that \( G_1 \) is not a function of \( n \). In a similar way, \( S_1(n) \) for \( 0 \leq n < N_h - 1 \) is given by
\[
S_1(n) = \sum_{g=0}^{n+N_h-1} [(h_1(i)W_n(i)) * h_1(N_h - 1 - i)]_g^2
\]  
(63)

where \( W_n(i) \) is equal to 1 for \( 0 \leq i \leq n \) and zero elsewhere. For \( N - N_h < n \leq N - 1 \), \( S_1(n) \) is given by
\[
S_1(n) = \sum_{g=0}^{N-n+N_h-2} [h_1(i) * h_1(N_h - 1 - i)]_g^2.
\]  
(64)

Next, (56) can be rewritten as
\[
E\{[\hat{R}_z]_{1,1}[\hat{R}_z]_{1,1}\} = \frac{\sigma_1^4}{N^2} \times \\
\left( \sum_{n=0}^{N-1} \sum_{r=\text{max}(0,n-N_h+1)}^{n} h_1^2(n-r) \sum_{u=0}^{N-1} \sum_{s=\text{max}(0,u-N_h+1)}^{u} h_1^2(u-s) + \\
(N - 2N_h + 2)G_1 + \sum_{n=0}^{N_h-2} S_1(n) + \sum_{n=N-N_h+1}^{N-1} S_1(n) \right). 
\]  
(65)

Using (42), we have
\[
\sum_{n=0}^{N-1} \sum_{r=\text{max}(0,n-N_h+1)}^{n} h_1^2(n-r) = \sum_{r=0}^{N_h-1} (N - r)h_1^2(r) = NH_1.
\]  
(66)

Therefore, (65) can be simplified as
\[
E\{[\hat{R}_z]_{1,1}[\hat{R}_z]_{1,1}\} = \frac{\sigma_1^4}{N^2} \left( N^2H_1^2 + (N - 2N_h + 2)G_1 + \Sigma_1 \right) 
\]  
(67)

where \( \Sigma_1 \triangleq \sum_{n=0}^{N_h-2} S_1(n) + \sum_{n=N-N_h+1}^{N-1} S_1(n) \). Note that \( [\hat{R}_z]_{1,1} \) is real-valued. Therefore, \( [U]_{1,1} \) is equal to \( E\{[\hat{R}_z]_{1,1}[\hat{R}_z]_{1,1}\} \) as given in (67) and \( [U]_{Q+1,Q+1} \) equals zero since the imaginary part of \( [\hat{R}_z]_{1,1} \) is zero.

For the rest of the diagonal elements of \( U \), \( E\{[\hat{R}_z]_{a,b}[\hat{R}_z]_{a,b}^*\} \) equals zero, as \( E_1 \) is zero. Therefore, \( [U]_{k,k} \) (\( 2 \leq k \leq 2Q \) and \( k \neq Q + 1 \)) can be obtained using (30) and (31) as
\[
[U]_{k,k} = \frac{1}{2} Re \left( E\{[\hat{R}_z]_{a,b}[\hat{R}_z]_{a,b}^*\} \right).
\]  
(68)
From (34) and (55) we have

\[
[U]_{k,k} = \frac{\sigma^4}{2N^2} \sum_n S_k(n)
\]  

(69)

where \( S_k(n) \) is defined as

\[
S_k(n) \triangleq \sum_{r=\max(0,n-Nh+1)}^{n} \sum_{p=\max(0,n-Nh+1)}^{n} \sum_{u=0}^{N-1} \sum_{m=\max(0,u-Nh+1)}^{u} \delta(r-s)\delta(m-p)h_a(n-r)h_a(n-p)h_a(u-s)h_b(u-m).
\]

(70)

It can be shown that for \( Nh - 1 \leq n \leq N - Nh \), \( S_k(n) \) is given by

\[
G_k \triangleq S_k(n) = \sum_{g=0}^{2Nh-2} (h_a(i) * h_a(Nh - 1 - i)) |_g \times
\]

\[
(h_b(i) * h_b(Nh - 1 - i)) |_g.
\]

(71)

For \( 0 \leq n < Nh - 1 \), \( S_k(n) \) is given by

\[
S_k(n) = \sum_{g=0}^{N+n-Nh-1} ((h_a(i)W_n(i)) * h_a(Nh - 1 - i)) |_g \times
\]

\[
((h_b(i)W_n(i)) * h_b(Nh - 1 - i)) |_g.
\]

(72)

For \( N - Nh < n \leq N - 1 \), \( S_k(n) \) is given by

\[
S_k(n) = \sum_{g=0}^{N-n+Nh-2} (h_a(i) * h_a(Nh - 1 - i)) |_g \times
\]

\[
(h_b(i) * h_b(Nh - 1 - i)) |_g.
\]

(73)

Thus, (69) can be rewritten as

\[
[U]_{k,k} = \frac{\sigma^4}{2N^2} ((N - 2Nh + 2)G_k + \Sigma_k)
\]

(74)

where \( \Sigma_k \triangleq \sum_{n=0}^{Nh-2} S_k(n) + \sum_{n=N-Nh+1}^{N-1} S_k(n) \).

**APPENDIX C**

**Variance Approximation**

Referring to (29), computation of the \( l \)-th diagonal element of the covariance matrix requires the knowledge of the elements of the \( l \)-th row of \( A \triangleq (\tilde{\Psi}^T \tilde{\Psi})^{-1} \tilde{\Psi}^T \). The diagonal elements
of $U$ for $2 \leq k \leq 2Q$ and $k \neq Q + 1$ as given by (35) differ from each other in $G_k$ and $\Sigma_k$. However, the values of $G_k$ and $\Sigma_k$ for different values of $k$ almost remain the same as they are related to the energy of the FD filters which are normalized to one. Let us approximate $G_k$ and $\Sigma_k$ by $G_1$ and $\Sigma_1$. Then, $[U]_{k,k}$ can be approximated by

$$
\gamma \triangleq \frac{\sigma^4}{2N^2} \left( (N - 2N_h + 2)G_1 + \Sigma_1 \right). \tag{75}
$$

The approximation in (75) relaxes the problem of computing the $l$-th diagonal element of the covariance matrix to just finding the Euclidean norm of the $l$-th row of $A$. The squared norm of the $l$-th row of $A$ can be obtained as

$$
\phi_l \triangleq \left[ AA^T \right]_{l,l} = \left[ (\hat{\Psi}^T \hat{\Psi})^{-1} \right]_{l,l} = \left[ (\text{Re}(\Psi^H \Psi))^{-1} \right]_{l,l}. \tag{76}
$$

Referring to (4), the diagonal elements of $\text{Re}(\Psi^H \Psi)$ are all equal to $Q$, and the off-diagonal elements are given as

$$
[\text{Re}(\Psi^H \Psi)]_{i,j} = 1 + \sum_{k=2}^{Q} \cos((i - j)\omega_k) \tag{77}
$$

where $1 \leq j \leq L$ and $i \neq j$. Noting that the frequencies $\omega_k$ are randomly obtained based on the sampling pattern, the value of the off-diagonal elements of $\text{Re}(\Psi^H \Psi)$ are negligible compared to the value of the diagonal elements. Therefore, $\text{Re}(\Psi^H \Psi)$ can be approximated by a diagonal matrix with elements equal to $Q$, which results in

$$
\phi_l \approx \frac{1}{Q}. \tag{78}
$$

It is shown in Appendix A that all the elements of the first column of $A$ are equal to $1/L$. Furthermore, all the elements of the $(Q + 1)$-th column of $A$ are equal to zero, as all the elements of the $(Q + 1)$-th row of $\hat{\Psi}$ are zero. Then, using (25), (28), (29), and (75), $[C_p]_{l,l}$ can be approximated as

$$
[C_p]_{l,l} \approx \left( \frac{L}{W} \right)^2 \left[ \gamma \phi_l + \frac{1}{L^2} ([U]_{1,1} - \gamma) \right] - \left( H_1 \frac{\sigma^2}{W} \right)^2. \tag{79}
$$

Next, using (35), (75), and (78), we can simplify (79) to

$$
[C_p]_{l,l} \approx \frac{\sigma^4}{2W^2N_x^2} \left( \frac{L^3}{Q} + L \right) \times ((N_x - 2N_hL + 2L)G_1 + L\Sigma_1) \tag{80}
$$

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where \( N_x \triangleq NL \) is the number of Nyquist samples.

**APPENDIX D**

**PROOF OF THEOREM 1**

Letting the number of samples tend to infinity in (28) yields

\[
\lim_{N \to \infty} C_{\hat{p}} = \lim_{N \to \infty} E\{\hat{p}\hat{p}^T\} - \lim_{N \to \infty} E\{\hat{p}\} E\{\hat{p}\}^T. \tag{81}
\]

Since the correlogram for undersampled data estimator is asymptotically unbiased, we have

\[
\lim_{N \to \infty} E\{\hat{p}\} = p = \frac{\sigma^2}{W} 1_L. \tag{82}
\]

From (29), we obtain

\[
\lim_{N \to \infty} E\{\hat{p}\hat{p}^T\} = \left(\frac{L}{W}\right)^2 \left(\hat{\Psi}^T \hat{\Psi}\right)^{-1} \hat{\Psi}^T \left(\lim_{N \to \infty} U\right) \hat{\Psi} \left(\hat{\Psi}^T \hat{\Psi}\right)^{-1}. \tag{83}
\]

Recall that all the off-diagonal elements of \( U \) are zeros, and the first diagonal element of \( U \) is given by (35). Letting the number of samples tend to infinity in (35), we obtain

\[
\lim_{N \to \infty} E\{U\}_{1,1} = \sigma^4. \tag{84}
\]

The \((Q + 1)\)-th element of \( U \) is zero, and if the number of samples tend to infinity in (74), \( \lim_{N \to \infty} [U]_{k,k} = 0 \). Therefore, all the elements of \( \lim_{N \to \infty} U \) are equal to zero except for its first diagonal element which is equal to \( \sigma^4 \).

In order to further simplify (83), only the elements of the first column of \( \left(\hat{\Psi}^T \hat{\Psi}\right)^{-1} \hat{\Psi}^T \) are required. We have shown in Appendix A that these elements are all equal to \( 1/L \). Therefore, (83) can be simplified to

\[
\lim_{N \to \infty} E\{\hat{p}\hat{p}^T\} = \left(\frac{L}{W}\right)^2 \left(\frac{\sigma^4}{L^2}\right) 1_{LL} = \left(\frac{\sigma^4}{W^2}\right) 1_{LL}. \tag{85}
\]

where \( 1_{LL} \) is an \( L \times L \) matrix with all its elements equal to 1. It follows from (81), (82), and (85) that

\[
\lim_{N \to \infty} C_{\hat{p}} = 0. \tag{86}
\]

In other words, the variance of the correlogram for undersampled data tends to zero as the number of samples goes to infinity, which proves the consistency of the estimator. Moreover, all the elements of \( C_{\hat{p}} \) tend to zero, which implies that the estimations made for different spectral segments are asymptotically uncorrelated.
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Fig. 1. Bias versus Nyquist signal length $N_x$. The average sampling rate $(q/L)W$ for the $(L, q) = (51, 12)$, $(101, 25)$, and $(201, 50)$ pairs are 235Hz, 247Hz, and 248Hz, respectively. The curve marked with squares is obtained by Monte Carlo simulations. The rest of the curves are based on (27).

Fig. 2. Variance $[C_{pp}]_{1,1}$ versus number of sampling channels $q$ at a fixed number of spectral segments $L$. The number of Nyquist samples is set to $N_x = 10^5$. Solid lines are based on (28) to (35) and dashed lines are based on (36).
Fig. 3. Variance $[\mathcal{C}^p_{1,1}]$ versus number of spectral segments $L$ at a fixed number of sampling channels $q$. The number of Nyquist samples is set to $N_x = 10^5$. Solid lines are based on (28) to (35) and dashed lines are based on (36).

Fig. 4. Variance $[\mathcal{C}^p_{1,1}]$ versus Nyquist signal length $N_x$. The average sampling rate $(q/L)W$ for the $(L,q) = (51,12)$, $(101,25)$, and $(201,50)$ pairs are 235Hz, 247Hz, and 248Hz, respectively. The curve marked with squares is obtained by Monte Carlo simulations. Solid lines are based on (28) to (35), and dashed lines are based on (36).
Fig. 5. Variance $[C_p]_{1,1}$ versus number of spectral segments $L$ at a fixed number of sampling channels $q = 45$. The number of Nyquist samples is set to $N_x = 10^5$. The curve for the white Gaussian signal is based on (28) to (35), and the curve for the filtered Gaussian signal is obtained by Monte Carlo simulations.

Fig. 6. Variance $[C_p]_{1,1}$ versus Nyquist signal length $N_x$ for $(L, q) = (101, 25)$ pair. The curve for the white Gaussian signal is based on (28) to (35), and the curve for the filtered Gaussian signal is obtained by Monte Carlo simulations.