Randomly coupled minimal models

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ABSTRACT

Using 1-loop renormalisation group equations, we analyze the effect of randomness on multi-critical unitary minimal conformal models. We study the case of two randomly coupled $M_p$ models and found that they flow in two decoupled $M_{p-1}$ models, in the infra-red limit. This result is then extend to the case with $M$ randomly coupled $M_p$ models, which will flow toward $M$ decoupled $M_{p-1}$.

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1 Introduction

Recently, many theoretical models have been proposed in studying the effect of randomness in two-dimensional systems (see for example [1], [2]). In a conformal invariant pure model, quenched impurities introduce a perturbation term which generally breaks conformal invariance. The effect of a weak disorder can be anticipated by the Harris criterion [3]. The relevance of disorder in the critical regime of a system can be evaluated by power counting. The naive dimensional analysis of the impurities induced term perturbing the (conformal invariant) Hamiltonian of the pure system give us the importance of disorder near the transition point. One of the first results in the case of uncorrelated quenched disorder was obtained in [4, 5, 6, 7] in the context of the Ising model with random bonds and in [8] for the Baxter model. These cases correspond to a marginal perturbation of the original Hamiltonian. The problem becomes more complicated when quenched impurities induce a relevant perturbation term. Such an example is found in the $q$-state Potts model with random bonds [7], [9], [10]. Using the replica method, it was argued that the system reach a non-trivial fixed point giving new critical $q$-dependent exponents. More recently, in [2, 11, 12, 13] it was shown that in some cases impurities can drift a first order phase transition to a continuous one, which is in many cases of Ising-like type. It was pointed out that, for the $q$-state Potts model, the $q$-dependence of exponents for different kinds of disorder is not obvious. The absence of non-perturbative results for uncorrelated disorder forbid us to give a definitive answer.

On the other hand, there exist powerful method in studying conformal field theories perturbed by some particular operators [14]. Some exact results have been obtained by perturbing unitary minimal models. These models corresponds to multi-critical two-dimensional statistical models [15], [16]. It is interesting to study these models when the perturbation corresponds to an addition of disorder. This is what we will do in this paper. The model that we study consists in taking two copies of such models and add a random energy density depending coupling between them. Our result is that under particular conditions, our systems will have a large scale behavior corresponding to that found in [14], namely they flow, in the infrared limit, in a new minimal model. In section
(2) we show how disorder is implemented in our initial pure systems. In the context of the replica method we show which perturbing term we have to consider in the effective Hamiltonian of our replicated system. Then, in section (3), we specialize in the case of two randomly coupled minimal models. We study the 1-loop Renormalisation Group (RG) behavior of the system for an arbitrary number of replica and the quenched case is developed in section (4). In section (5) we present a generalization to the case with \(M\) randomly coupled \(M_p\) models with \(M \geq 2\) arbitrary. Section (6) is devoted to discussions and conclusions.

2 Minimal models perturbed by randomness

We are interested in conformal invariant multi-critical models perturbed by randomness. Our pure system will then consist in unitary minimal models \(M_p\) of central charge \(c = 1 - \frac{6}{p(p+1)}\) \([15], [16]\). The conformal dimension of any operator in the conformal grid of the \(M_p\) model is \(\Delta_{n',n} = \frac{(pm'-(p+1)n)^2-1}{4p(p+1)}\) (physical dimension is \(2\Delta_{n',n}\)). For the unitary minimal models that we consider here, the energy operator correspond to \(\varepsilon = \phi_{2,1}\) and has conformal dimension \(\Delta_{2,1} = \frac{1}{4} - \frac{\epsilon}{4}\) with \(\epsilon = \frac{3}{p+1}\).

In this letter, we will consider a model consisting in 2 minimal models \(M_p\) randomly coupled in the following way: if \(H_{0,1}\) and \(H_{0,2}\) are respectively the Hamiltonians of the two pure models, our total Hamiltonian is:

\[
H = H_{0,1} + H_{0,2} + \int d^2x \, q(x) \, \varepsilon_1(x) \varepsilon_2(x)
\] (2.1)

where \(q(x)\) is a random coupling term between the two models. The case of a non-random \(q(x)\) was already shown to be integrable and leads to a massive theory \([17]\). In the context of the replica method, we take the average of the \(n^{th}\) power of the partition function:

\[
\overline{Z^n} = \int \prod_x dP[q(x)]Z^n
\] (2.2)

where \(dP[q(x)]\) is a symmetric normalized probability distribution for \(q(x)\). (2.2) gives naively the effective Hamiltonian:

\[
H = \sum_{i=1}^{n} (H_{0,1}^i + H_{0,2}^i) - \sigma \sum_{i,j=1}^{n} \int d^2x \, (\varepsilon_1^i \varepsilon_2^j \varepsilon_1^i \varepsilon_2^j) (x) + \cdots
\] (2.3)
Here we just have to write the second cumulant of the distribution $P[q(x)]$. If $p$ is large enough, the dimension of the energy operator will be close to $\frac{1}{2}$ and naively, higher cumulant terms will be irrelevant operators. The quenched case will be obtained in the limit $n \to 0$. As a first stage we will study the model (2.3) for generic $n$ specializing then in the quenched case.

We still have to make an important remark, before going to the detailed calculations. We can see that in both cases, the operator algebra (OA) of the energy operator contains the operator $\phi_{3,1}$ and his descendants [10]:

$$\varepsilon \varepsilon \to [I] + [\phi_{3,1}]$$

The conformal dimension of $\phi_{3,1}$ is given by $\Delta_{3,1} = 1 - \frac{2\epsilon}{3}$ and so it is a relevant operator. In fact, in the interaction terms displayed above, terms with same replica indices or coming from higher cumulants of the probability distribution will produce, apart from trivial or irrelevant contributions, a term of the form:

$$\int d^2 x \sum_{i=1}^{n} \left( \Phi_1^{i}(x) + \Phi_2^{i}(x) \right)$$

Here we have denoted by $\Phi$ the $\phi_{3,1}$ operator. The problem of one $M_p$ model perturbed by the $\phi_{3,1}$ operator has been studied extensively in [14]. There is a non trivial infra-red (IR) fixed point, and it has been shown that the system flows to the $M_{p-1}$ model. This fixed point will also be present in the (RG) behavior of our particular model and will be of particular interest in the quenched case.

### 3 Generic case

We first consider the case where we have $n$ replicas and thus our model consists in $2n$ minimal models coupled together. The idea is to study the (RG) behavior of (2.3) for $n > 2$ and to identify the different fixed points that can appear. As explained in the previous section, the model that we consider in the following is described by the more general Hamiltonian:

$$H = \sum_{i=1}^{n} H_{i,0,1}^{i} + H_{i,0,2}^{i} + \lambda \int d^2 x \sum_{i=1}^{n} \left( \Phi_1^{i}(x) + \Phi_2^{i}(x) \right) + g \int d^2 x \sum_{i \neq j}^{n} \left( \varepsilon_1^{i} \varepsilon_2^{i} \varepsilon_1^{j} \varepsilon_2^{j} \right) (x) \quad (3.1)$$
where $H^i_{0,1}$ and $H^i_{0,2}$ are the Hamiltonians of the unperturbed systems, each of them corresponding to a minimal model $M_p$. Note that we have replaced $\sigma$ by $-g$, thus the physical case for a random model corresponds to $g < 0$. The 1-loop RG equations can be easily obtained from the operator algebra of the perturbing field $s$:

$$
\sum_{i \neq j}^{n} (\epsilon_i^1 \epsilon_2^j \epsilon_1^j \epsilon_2^i) (x) \sum_{k \neq l}^{n} (\epsilon_k^1 \epsilon_2^k \epsilon_1^k \epsilon_2^l) (y) \rightarrow 4(n - 2)|x - y|^{-2+2\epsilon} \sum_{i \neq j}^{n} (\epsilon_i^1 \epsilon_2^j \epsilon_1^j \epsilon_2^i) (y)
$$

$$
+ 4(n - 1)C_{\phi,\epsilon} \epsilon |x - y|^{-2+\frac{8\epsilon}{3}} \sum_{i = 1}^{n} (\Phi_i (y) + \Phi_j (y)) + \cdots
$$

$$
\sum_{i = 1}^{n} (\Phi_2^i (x) + \Phi_2^j (x)) \sum_{i = 1}^{n} (\Phi_1^i (y) + \Phi_2^j (y)) \rightarrow C_{\phi,\phi} \phi |x - y|^{-2+\frac{4\epsilon}{3}} \sum_{i = 1}^{n} (\Phi_i (y) + \Phi_j (y)) + \cdots
$$

$$
\sum_{i = 1}^{n} (\Phi_2^i (x) + \Phi_2^j (x)) \sum_{i \neq j}^{n} (\epsilon_i^1 \epsilon_2^j \epsilon_1^j \epsilon_2^i) (y) \rightarrow 4C_{\phi,\epsilon} \epsilon |x - y|^{-2+\frac{8\epsilon}{3}} \sum_{i \neq j}^{n} (\epsilon_i^1 \epsilon_2^j \epsilon_1^j \epsilon_2^i) (y) + \cdots
$$

(3.2)

where we have omitted the descendent terms. $C_{\phi,\phi}$ and $C_{\phi,\epsilon}$ are the structure constants of the model $M_p$. They are symmetric under permutation of the three indices and their values can be obtained from [16]:

$$
C_{\phi,\epsilon} = \frac{\sqrt{3}}{2} + O(\epsilon) \quad C_{\phi,\phi} = \frac{4}{\sqrt{3}} + O(\epsilon)
$$

Using the formula of the 1-loop RG equation [1], [14]:

$$
\dot{g} = (2 - dim(g_i))g_i - \pi K_{ijk} g_j g_k
$$

(3.4)
\[ \lambda = 4\epsilon \quad ; \quad g = 0 \]  \hspace{1cm} (3.5)

\[ g = -\frac{2\epsilon \sqrt{2}}{\sqrt{2(n-2)^2 + 9(n-1)}} \quad ; \quad \lambda = 2\epsilon\left(1 + \frac{\sqrt{2}(n-2)}{\sqrt{2(n-2)^2 + 9(n-1)}}\right) \]

\[ g = \frac{2\epsilon \sqrt{2}}{\sqrt{2(n-2)^2 + 9(n-1)}} \quad ; \quad \lambda = 2\epsilon\left(1 - \frac{\sqrt{2}(n-2)}{\sqrt{2(n-2)^2 + 9(n-1)}}\right) \]  \hspace{1cm} (3.6)

Note that \((3.4)\) and \((3.5)\) are the two fixed points present in the work of Zamolodchikov [14], who found that a pure \(M_p\) model, perturbed by a \(\phi_{3,1}\) operator flows to the point \((3.5)\) which correspond to the \(M_{p-1}\) model. The next step is to study the stability of each of these fixed points. This is done by linearizing \((3.3)\) around the solutions given above \(g = g^* + \delta g\); \(\lambda = \lambda^* + \delta \lambda\) and getting a linear system:

\[ \begin{pmatrix} \delta \dot{g} \\ \delta \dot{\lambda} \end{pmatrix} = A \begin{pmatrix} \delta g \\ \delta \lambda \end{pmatrix} \]

The eigenvalues of the matrix \(A\) for each of the cases \((3.4)\) to \((3.6)\) give us information about the stability of these points. It is easy to see that for \((3.4)\) both eigenvalues are positives indicating that this fixed point is unstable in all directions, while for \((3.5)\) we have a stable fixed point in both directions. For \((3.6)\) and \((3.6)\) we obtain one real negative and one real positive eigenvalue, that is, these points are stable in one direction and unstable in the other. Thus, they can be reached only if we fine tune the values of \(g\) and \(\lambda\) to keep our system in the stable line of these points. Studying in detail the flow diagram of \((3.3)\) we can see that the initial conditions \(\lambda = 0\); \(g \neq 0\) will flow far from our fixed points toward either a massive theory or another fixed point which can not be seen at this order in perturbation theory.

### 4 Quenched system

We now turn to the case \(n = 0\) which correspond to the quenched system of two randomly coupled minimal models. By just putting \(n = 0\) in \((3.3)\) we get our new RG equations:

\[ \dot{g} = 2\epsilon g + 2g^2 - \lambda g + \cdots \]

\[ \dot{\lambda} = \frac{4\epsilon}{3} \lambda - \frac{\lambda^2}{3} + \frac{3}{2} g^2 + \cdots \]  \hspace{1cm} (4.1)
Solutions (3.4) and (3.5) are still valid with the same kind of stability but now there is no more fixed point solutions with $g \neq 0$ (points (3.6) and (3.6) became complex.) So, (3.4) and (3.5) are the only fixed points at this order in perturbation theory. Assuming that higher loop corrections to (4.1) will not change the qualitative behavior of the flow near our two fixed points, we can see that a system with initial conditions $\lambda_0 = 0; \ g_0 < 0$ will flow toward the point (3.5). This is supported by the numerical calculation of the RG flow (4.1) in figure 1 for different values of $g_0$.

Initial conditions $\lambda_0 = 0; \ g_0 < 0$ are precisely what we expect for the case of a quenched system, since the term in the bare Hamiltonian proportional to the $\Phi$ operator will have a factor $n$, coming from the contraction of the other pairs of energy operators in (2.3). Then, in the limit $n = 0$, $\lambda_0$ cancels and $g_0 = -\sigma < 0$. So, adding a small random coupling through the energy densities of two $M_p$ minimal model will drive our system at large scales to two decoupled $M_{p-1}$ models.
5 Generalization to $M$ coupled models

In this section we will consider the generalization to $M$ coupled models. The hamiltonian (3.1) is thus replaced by

$$H = \sum_{a=1}^{M} \sum_{i=1}^{n} H_{0,a} + \lambda \int d^2x \sum_{a=1}^{M} \sum_{i=1}^{n} (\Phi_{a}^{i}(x)) + \rho \int d^2x \sum_{a\neq b,c\neq d}^{M} \sum_{i\neq j}^{n} (\varepsilon_{a}^{i} \varepsilon_{b}^{j} \varepsilon_{c}^{j} \varepsilon_{d}^{i})(x) \quad (5.1)$$

In the following computations, it will be more convenient to express the last term like

$$\rho \int d^2x \sum_{a\neq b,c\neq d}^{M} \sum_{i\neq j}^{n} (\varepsilon_{a}^{i} \varepsilon_{b}^{j} \varepsilon_{c}^{j} \varepsilon_{d}^{i})(x) = \rho \int d^2x \sum_{a\neq b,c\neq d}^{M} \sum_{i\neq j}^{n} (\varepsilon_{a}^{i} \varepsilon_{b}^{j} \varepsilon_{c}^{j} \varepsilon_{d}^{i})(x) + \sigma \int d^2x \sum_{a,b,c,d}^{M} \varepsilon_{a}^{i} \varepsilon_{b}^{j} \varepsilon_{c}^{j} \varepsilon_{d}^{i} \quad (5.2)$$

Here $<a,b,c,d>$ means all the summation over $a,b,c,d$ which take different values two by two. Under the renormalisation group transformations, each of these terms is going to behave differently and thus there will be two different equations for $g$ and $\sigma$. In fact this is only true for $M \geq 4$. For $M < 4$, the $\sigma$ term is absent. From the set of equations with $g, \sigma$ and $\lambda$, we will still be able to recover the case $M = 2$ and $M = 3$ by simply suppressing $\sigma$. The case $M = 2$ was already considered in the previous section and the case $M = 3$ will be mentioned at the end of this section. The renormalisation group equations for the parameters $g, \sigma$ and $\lambda$ are trivial to compute by generalizing computations of previous sections. We obtain

$$\dot{g} = 2\epsilon g - g^2(M(M-1)(n-2) + 4(M-2)^2) - g\sigma(M-2)(M-3) - \lambda g + \cdots$$

$$\dot{\sigma} = 2\epsilon \sigma - \frac{3}{2}g^2(M-4)(M-5) - \frac{3}{2}g^2(n-1)M(M-1) - \lambda \sigma + \cdots$$

$$\dot{\lambda} = \frac{4\epsilon}{3} \lambda - \frac{1}{3} \lambda^2 - \frac{3}{4}g^2(n-1)M(M-1)^2 - \frac{1}{4}g^2(M-1)(M-2)(M-3) + \cdots$$

Going directly to the quenched case, we found

$$\dot{g} = 2\epsilon g - g^2(2(M-2)^2 - M(M-1)) - g\sigma(M-2)(M-3) - \lambda g + \cdots$$

$$\dot{\sigma} = 2\epsilon \sigma - \frac{3}{2}g^2(M-4)(M-5) + \frac{3}{2}g^2M(M-1) - \lambda \sigma + \cdots$$

$$\dot{\lambda} = \frac{4\epsilon}{3} \lambda - \frac{1}{3} \lambda^2 + \frac{3}{4}g^2M(M-1)^2 - \frac{1}{4}g^2(M-1)(M-2)(M-3) + \cdots$$
The next step consists in computing the fixed points associated with these equations. Using an $\epsilon$ expansion, we got the following points: first, because $\dot{g} = g(...)$, one set of solutions is given by $g = 0$ and, after some additional computations

\begin{align*}
\sigma &= 0 \quad ; \quad \lambda = 0 \quad (5.5) \\
\sigma &= 0 \quad ; \quad \lambda = 4\epsilon \quad (5.6) \\
\sigma &= \epsilon x \quad ; \quad \lambda = 2\epsilon(1 - y) \quad (5.7) \\
\sigma &= -\epsilon x \quad ; \quad \lambda = 2\epsilon(1 + y) \quad (5.8)
\end{align*}

where we have defined the following quantities

\begin{align*}
x &= \frac{4}{\sqrt{3}} [(M - 1)(M - 2)(M - 3) + 3(M - 4)^2(M - 5)^2]^{\frac{1}{2}} \\
y &= \frac{3}{4} x (M - 4)(M - 5)
\end{align*}

In addition, there is a second set of solutions with $g \neq 0$ (with some extra conditions on $M$, see below.) After some more tedious computations, we get

\begin{align*}
g &= -\epsilon X_M^+ \quad ; \quad \sigma = -\epsilon Z_M^+ X_M^+ \quad (5.9) \\
g &= -\epsilon X_M^- \quad ; \quad \sigma = -\epsilon Z_M^- X_M^- \quad (5.10) \\
g &= \epsilon X_M^+ \quad ; \quad \sigma = \epsilon Z_M^+ X_M^+ \quad (5.11) \\
g &= \epsilon X_M^- \quad ; \quad \sigma = \epsilon Z_M^- X_M^- \quad (5.12)
\end{align*}

and $\lambda = 2\epsilon - g(2(M - 2)^2 - M(M - 1)) - \sigma(M - 2)(M - 3)$. We also used the following definitions

\begin{align*}
Z_M^\pm &= \frac{-(2(M - 2)^2 - M(M - 1)) \pm 2\sqrt{f_M}}{2((M - 2)(M - 3) - \frac{3}{2}(M - 4)(M - 5))} \\
X_M^\pm &= \frac{\sqrt{3}}{2} \left( \frac{1}{3} (Z_M^\pm (M - 2)(M - 3) + (2(M - 2)^2 - M(M - 1)))^2 \\
&+ \frac{1}{4} (Z_M^\pm)^2 (M - 1)(M - 2)(M - 3) - \frac{3}{4} M(M - 1)^2 \right)^{\frac{1}{2}} \\
f_M &= M^4 - 17M^3 + 65M^2 - 64M + 16
\end{align*}
These last four solutions do not exist for every \( M \). We have some additional constraints: \( f_M \) is positive, and the fixed points real, only for \( M \geq 13 \); \( X_M^+ \) is real only for \( M \leq 66 \). Thus the four solutions (5.9-5.12) are real for \( 13 \leq M \leq 66 \) while for \( M \geq 67 \) only solutions (5.10,5.12) do exist. The next step is to study the stability of these solutions.

Let first note that the physical initial conditions are

\[
\lambda = 0 \quad ; \quad g = \sigma < 0 \quad (5.14)
\]

Because \( \dot{g} = g(\ldots) \), we have the condition \( g \leq 0 \). Then, the last two solutions (5.11) and (5.12) can be immediately discarded. Thus only 6 solutions can be relevant. The first one \( (g = \sigma = \lambda = 0) \) is obviously unstable and can also be discarded. The second one \( (g = \sigma = 0, \lambda = 4\epsilon) \) turns out to be stable \( (\delta\dot{g} = -2\epsilon\delta g, \delta\dot{\sigma} = -2\epsilon\delta\sigma, \delta\dot{\lambda} = -\frac{4}{3}\epsilon\delta\lambda) \) and this independently of \( M \). This point is just the stable fixed point (3.5) of the previous section. The next two solutions \( (g = 0, \sigma = \pm\epsilon x, \lambda = 2\epsilon(1 \mp y)) \) are also unstable. This can be seen by noticing that

\[
\begin{pmatrix}
\delta\dot{\sigma} \\
\delta\dot{\lambda}
\end{pmatrix}
= \epsilon A_{\pm}
\begin{pmatrix}
\delta\sigma \\
\delta\lambda
\end{pmatrix}
\quad (5.15)
\]

with

\[
A_{\pm} = \pm \frac{1}{2}x
\begin{pmatrix}
3(M-4)(M-5) & 2 \\
(M-1)(M-2)(M-3) & -2(M-4)(M-5)
\end{pmatrix}
\quad (5.16)
\]

and then

\[
\det A_{\pm} = -\frac{1}{4}x^2(6(M-4)^2(M-5)^2 + 2(M-1)(M-2)(M-3)) < 0
\quad (5.17)
\]

for all \( M \geq 4 \), which implies that there is two eigenvalues with opposite sign. Then we remain with the last two solutions, (5.9) and (5.10) for the previously mentioned values of \( M \). Again these two solutions will be unstable. Here, it is not possible to give an analytical proof for all \( M \). So we computed numerically the eigenvalues of the matrix \( A_{\pm} \) defined by

\[
\begin{pmatrix}
\delta\dot{g} \\
\delta\dot{\sigma} \\
\delta\dot{\lambda}
\end{pmatrix}
= \epsilon A_{\pm}
\begin{pmatrix}
\delta g \\
\delta\sigma \\
\delta\lambda
\end{pmatrix}
\quad (5.18)
\]

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We found that for each $13 \leq M \leq 1000$, at least one of the eigenvalue of $A_-$ is positive, the same being true for $A_+$ for each $13 \leq M \leq 66$.

Finally the last step is to check if under the renormalisation group, we are going to reach the only remaining fixed point, (5.6). Again this was done by constructing numerically the flow diagram. We construct it for every value of $M$ between 4 and 1000 and for each of these flows, we found that we reach the stable fixed point (5.6).

Before going to the discussion, let’s also mention the case $M = 3$. Renormalisation group equations for this case are obtained from (5.4) by suppressing the $\sigma$ terms. We found 2 solutions to these equations: the trivially unstable solution ($g = \lambda = 0$) and the stable solution ($g = 0, \lambda = 4 \epsilon$). Again, we remain with only one solution which corresponds to 3 decoupled $M_{p-1}$ models.

6 Discussion

In this paper, we have considered a very particular way of adding randomness for more general systems than the well studied Ising or Potts models with random bonds. We considered the case of two minimal $M_p$ models randomly coupled and our 1-loop calculation shows that (weak) randomness can be easily chosen such that criticality is maintained and our system will behave at large distances as two decoupled $M_{p-1}$ models. Then we generalized the study to $M$ randomly coupled $M_p$. Again we found that at the 1-loop order, these models will behave at large distance like $M$ decoupled $M_{p-1}$ models. In fact, the operator algebra of the perturbation induced by randomness produce in the effective action a supplementary term which drift our system to a unitary I.R. fixed point. These results seem to tell us that the critical behavior of some two dimensional systems in the presence of randomness depend crucially on the particular model and the kind of randomness we are considering, in contrast to what seems to happens in the cases studied in [2, 11, 12]. We expect that higher loop corrections in our renormalization group equations shouldn’t modify the qualitative behavior of the flow. However, an analytic solution for the coupling flow should make more concrete these conclusion.
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