PRODUCTS OF $H$-SEPARABLE SPACES IN THE LAVER MODEL

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Abstract. We prove that in the Laver model for the consistency of the Borel’s conjecture, the product of any two $H$-separable spaces is $M$-separable.

1. Introduction

This paper is devoted to products of $H$-separable spaces. A topological space $X$ is said [3] to be $H$-separable, if for every sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of $X$, one can pick finite subsets $F_n \subset D_n$ so that every nonempty open set $O \subset X$ meets all but finitely many $F_n$’s. If we only demand that $\bigcup_{n \in \omega} F_n$ is dense we get the definition of $M$-separable spaces introduced in [14]. It is obvious that second-countable spaces (even spaces with a countable $\pi$-base) are $H$-separable, and each $H$-separable space is $M$-separable. The main result of our paper is the following

Theorem 1.1. In the Laver model for the consistency of the Borel’s conjecture, the product of any two countable $H$-separable spaces is $M$-separable.

Consequently, the product of any two $H$-separable spaces is $M$-separable provided that it is hereditarily separable.

It worth mentioning here that by [12, Theorem 1.2] the equality $b = \aleph$ which holds in the Laver model implies that the $M$-separability is not preserved by finite products of countable spaces in the strong sense.

Let us recall that a topological space $X$ is said to have the Menger property (or, alternatively, is a Menger space) if for every sequence $\langle U_n : n \in \omega \rangle$ of open covers of $X$ there exists a sequence $\langle V_n : n \in \omega \rangle$ such that each $V_n$ is a finite subfamily of $U_n$ and the collection $\{ \bigcup V_n : n \in \omega \}$ is a cover of $X$. This property was introduced by Hurewicz, and the current name (the Menger property) is used because Hurewicz proved in [7] that for metrizable spaces his property is equivalent to a certain property of a base considered by Menger in [10]. If in the definition above we additionally require that $\{ n \in \omega : x \notin \bigcup V_n \}$ is finite for each $x \in X$, then we obtain the definition of the Hurewicz property introduced in [8]. The original idea behind the

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Menger’s property, as it is explicitly stated in the first paragraph of [10], was an application in dimension theory, one of the areas of interest of Mardešić. However, this paper concentrates on set-theoretic and combinatorial aspects of the property of Menger and its variations.

Theorem 1.1 is closely related to the main result of [13] asserting that in the Laver model the product of any two Hurewicz metrizable spaces has the Menger property. Let us note that our proof in [13] is conceptually different, even though both proofs are based on the same main technical lemma of [9]. Regarding the relation between Theorem 1.1 and the main result of [13], each of them implies a weak form of the other one via the following duality results: For a metrizable space \( X \), \( C_p(X) \) is \( M \)-separable (resp. \( H \)-separable) if and only if all finite powers of \( X \) are Menger (resp. Hurewicz), see [14, Theorem 35] and [3, Theorem 40], respectively. Thus Theorem 1.1 (combined with the well-known fact that \( C_p(X) \) is hereditarily separable for metrizable separable spaces \( X \)) implies that in the Laver model, if all finite powers of metrizable separable spaces \( X_0, X_1 \) are Hurewicz, then \( X_0 \times X_1 \) is Menger. And vice versa: The main result of [13] implies that in the Laver model, the product of two \( H \)-separable spaces of the form \( C_p(X) \) for a metrizable separable \( X \), is \( M \)-separable.

The proof of Theorem 1.1, which is based on the analysis of names for reals in the style of [9], unfortunately seems to be rather tailored for the \( H \)-separability and we were not able to prove any analogous results even for small variations thereof. Recall from [6] that a space \( X \) is said to be \( wH \)-separable if for any decreasing sequence \( \langle D_n : n \in \omega \rangle \) of dense subsets of \( X \), one can pick finite subsets \( F_n \subset D_n \) such that for any non-empty open \( U \subset X \) the set \( \{ n \in \omega : U \cap F_n \neq \emptyset \} \) is co-finite. It is clear that every \( H \)-separable space is \( wH \)-separable, and it seems to be unknown whether the converse is (at least consistently) true. Combining [6, Lemma 2.7(2) and Corollary 4.2] we obtain that every countable Fréchet-Urysohn space is \( wH \)-separable, and to our best knowledge it is open whether countable Fréchet-Urysohn spaces must be \( H \)-separable. The statement “finite products of countable Fréchet-Urysohn spaces are \( M \)-separable” is known to be independent from \( ZFC \): It follows from the PFA by [2, Theorem 3.3], holds in the Cohen model by [2, Corollary 3.2], and fails under \( CH \) by [1, Theorem 2.24]. Moreover\(^1\), \( CH \) implies the existence of two countable Fréchet-Urysohn \( H \)-separable topological groups whose product is not \( M \)-separable, see [11, Corollary 6.2]. These results motivate the following

**Question 1.2.**

(1) Is it consistent that the product of two countable \( wH \)-separable spaces is \( M \)-separable? Does this statement hold in the Laver model?

(2) Is the product of two countable Fréchet-Urysohn space \( M \)-separable in the Laver model?

\(^1\)We do not know whether the spaces constructed in the proof of [1, Theorem 2.24] are \( H \)-separable.
(3) Is the product of three (finitely many) countable $H$-separable spaces $M$-separable in the Laver model?

(4) Is the product of finitely many countable $H$-separable spaces $H$-separable in the Laver model?

2. Proof of Theorem 1.1

We need the following

Definition 2.1. A topological space $\langle X, \tau \rangle$ is called box-separable if for every function $R$ assigning to each countable family $\mathcal{U}$ of non-empty open subsets of $X$ a sequence $R(\mathcal{U}) = \langle F_n : n \in \omega \rangle$ of finite non-empty subsets of $X$ such that $\{ n : F_n \subset U \}$ is infinite for every $U \in \mathcal{U}$, there exists $U \subset \tau \{ \emptyset \}$ of size $|U| = \omega_1$ such that for all $U \in \tau \{ \emptyset \}$ there exists $U \in U$ such that $\{ n : R(\mathcal{U})(n) \subset U \}$ is infinite.

Any countable space is obviously box-separable under CH, which makes the latter notion uninteresting when considered in arbitrary ZFC models. However, as we shall see in Lemma 2.3, the box-separability becomes useful under $b > \omega_1$. Here $b$ denotes the minimal cardinality of a subspace $X$ of $\omega^\omega$ which is not eventually dominated by a single function, see [4] for more information on $b$ and other cardinal characteristics of the reals.

The following lemma is the key part of the proof of Theorem 1.1. We will use the notation from [9] with the only difference being that smaller conditions in a forcing poset are supposed to carry more information about the generic filter, and the ground model is denoted by $V$.

A subset $C$ of $\omega_2$ is called an $\omega_1$-club if it is unbounded and for every $\alpha \in \omega_2$ of cofinality $\omega_1$, if $C \cap \alpha$ is cofinal in $\alpha$ then $\alpha \in C$.

Lemma 2.2. In the Laver model every countable $H$-separable space is box-separable.

Proof. We work in $V[G_{\omega_2}]$, where $G_{\omega_2}$ is $\mathbb{P}_{\omega_2}$-generic and $\mathbb{P}_{\omega_2}$ is the iteration of length $\omega_2$ with countable supports of the Laver forcing, see [9] for details. Let us fix an $H$-separable space of the form $\langle \omega, \tau \rangle$ and a function $R$ such as in the definition of box-separability. By a standard argument (see, e.g., the proof of [5, Lemma 5.10]) there exists an $\omega_1$-club $C \subset \omega_2$ such that for every $\alpha \in C$ the following conditions hold:

(i) $\tau \cap V[G_\alpha] \in V[G_\alpha]$ and for every sequence $\langle D_n : n \in \omega \rangle \in V[G_\alpha]$ of dense subsets of $\langle \omega, \tau \rangle$ there exists a sequence $\langle K_n : n \in \omega \rangle \in V[G_\alpha]$ such that $K_n \in [D_n]^\omega$ and for every $U \in \tau \setminus \emptyset$ the intersection $U \cap K_n$ is non-empty for all but finitely many $n \in \omega$;

(ii) $R(\mathcal{U}) \in V[G_\alpha]$ for any $\mathcal{U} \in [\tau \setminus \{ \emptyset \}]^\omega \cap V[G_\alpha]$; and

(iii) For every $A \in \mathcal{P}(\omega) \cap V[G_\alpha]$ the interior $\text{Int}(A)$ also belongs to $V[G_\alpha]$.

By [9, Lemma 11] there is no loss of generality in assuming that $0 \in C$. We claim that $U := [\tau \setminus \{ \emptyset \}]^\omega \cap V$ is a witness for $\langle \omega, \tau \rangle$ being box-separable.
Suppose, contrary to our claim, that there exists \( A \in \tau \setminus \{\emptyset\} \) such that \( R(\mathcal{U})(n) \not\subseteq A \) for all but finitely many \( n \in \omega \) and \( \mathcal{U} \in \mathcal{U} \). Let \( \hat{A} \) be a \( \mathbb{P}_{\omega_2} \)-name for \( A \) and \( p \in \mathbb{P}_{\omega_2} \) a condition forcing the above statement. Applying [9, Lemma 14] to the sequence \( \langle \hat{a}_i : i \in \omega \rangle \) such that \( \hat{a}_i = \hat{A} \) for all \( i \in \omega \), we get a condition \( p' \leq p \) such that \( p'(0) \not\leq 0 \ p(0) \), and a finite set \( \mathcal{U}_s \subset \mathcal{P}(\omega) \) for every \( s \in p'(0) \) with \( p'(0)(0) \leq s \), such that for each \( n \in \omega \), \( s \in p'(0) \) with \( p'(0)(0) \leq s \), and for all but finitely many immediate successors \( t \) of \( s \) in \( p'(0) \) we have

\[
p'(0)_t \upharpoonright [1, \omega_2] \models \exists U \in \mathcal{U}_s \ (\hat{A} \cap n = U \cap n).
\]

Of course, any \( p'' \leq p' \) also has the property above with the same \( \mathcal{U}_s \)'s. However, the stronger \( p'' \) is, the more elements of \( \mathcal{U}_s \) might play no role any more. Therefore throughout the rest of the proof we shall call \( U \in \mathcal{U}_s \) void for \( p'' \leq p' \) and \( s \in p''(0) \), where \( p''(0)(0) \leq s \), if there exists \( n \in \omega \) such that for all but finitely many immediate successors \( t \) of \( s \) in \( p''(0) \) there is no \( q \leq p''(0)_t \cap p'' \upharpoonright [1, \omega_2) \) with the property \( q \models \hat{A} \cap n = U \cap n \). Note that for any \( p'' \leq p' \) and \( s \in p''(0) \), \( p''(0)(0) \leq s \), there exists \( U \in \mathcal{U}_s \) which is non-void for \( p'' \). Two cases are possible.

a) For every \( p'' \leq p' \) there exists \( s \in p''(0) \), \( p''(0)(0) \leq s \), and a non-void \( U \in \mathcal{U}_s \) for \( p'' \), \( s \) such that \( \text{Int}(U) \neq \emptyset \). In this case let \( \mathcal{U} \in \mathcal{U} \) be any countable family containing \( \{\text{Int}(U) : U \in \bigcup_{s \in p''(0), p''(0)(0) \leq s} \mathcal{U}_s\} \setminus \{\emptyset\} \). It follows from the above that \( p \) forces \( R(\mathcal{U})(k) \not\subseteq \hat{A} \) for all but finitely many \( k \in \omega \). Let \( p'' \leq p' \) and \( m \in \omega \) be such that \( p'' \) forces \( R(\mathcal{U})(k) \not\subseteq \hat{A} \) for all \( k \geq m \). Fix a non-void \( U \) for \( p'' \), \( s \), where \( s \in p''(0) \) and \( p''(0)(0) \leq s \), such that \( \text{Int}(U) \neq \emptyset \) (and hence \( \text{Int}(U) \in \mathcal{U} \)). It follows from the above that there exists \( k \geq m \) such that \( R(\mathcal{U})(k) \subset \text{Int}(U) \subset U \). Let \( n \in \omega \) be such that \( R(\mathcal{U})(k) \subset n \). By the definition of being non-void there are infinitely many immediate successors \( t \) of \( s \) in \( p''(0) \) for which there exists \( q_t \leq p''(0)_t \cap p'' \upharpoonright [1, \omega_2) \) with the property \( q_t \models \hat{A} \cap n = U \cap n \). Then for any \( q_t \) as above we have that \( q_t \) forces \( R(\mathcal{U})(k) \subset \hat{A} \) because \( R(\mathcal{U})(k) \subset U \cap n \), which contradicts the fact that \( q_t \leq p'' \) and \( p'' \models \hat{R}(\mathcal{U})(k) \not\subseteq \hat{A} \).

b) There exists \( p'' \leq p' \) such that for all \( s \in p''(0) \), \( p''(0)(0) \leq s \), every \( U \in \mathcal{U} \) with \( \text{Int}(U) \neq \emptyset \) is void for \( p'' \), \( s \). Note that this implies that every \( U \in \mathcal{U}_s \) with \( \text{Int}(U) \neq \emptyset \) is void for \( q, s \) for all \( q \leq p'' \) and \( s \in q(0) \) such that \( q(0)(0) \leq s \).

Let \( \langle D_k : k \in \omega \rangle \in V \) be a sequence of dense subsets of \( (\omega, \tau) \) such that for every \( U \in \bigcup_{s \in p''(0), p''(0)(0) \leq s} \mathcal{U}_s \), if \( \text{Int}(U) = \emptyset \), then \( \omega \setminus U = D_k \) for infinitely many \( k \in \omega \). Let \( \langle K_k : k \in \omega \rangle \in V \) be such as in item (i) above. Then \( p'' \) forces that \( K_k \cap \hat{A} \neq \emptyset \) for all but finitely many \( k \in \omega \). Passing to a stronger condition, we may additionally assume if necessary, that there exists \( m \in \omega \) such that \( p'' \models \forall k \geq m \ (K_k \cap \hat{A} \neq \emptyset) \).

Fix \( U \in \mathcal{U}_p''(0)(0) \) non-void for \( p'', p''(0)(0) \). Then \( \text{Int}(U) = \emptyset \) by the choice of \( p'' \) and hence there exists \( k \geq m \) such that \( \omega \setminus U = D_k \). It follows that \( K_k \cap U = \emptyset \) because \( K_k \subset D_k \). On the other hand, since \( U \) is non-void
for \( p'' , p''(0)/{0} \), for \( n = \max K_k + 1 \) we can find infinitely many immediate successors \( t \) of \( p''(0)/{0} \) in \( p''(0) \) for which there exists \( q_t \leq p''(0)\mathcal{U}p'' \mid \{1, \omega_2\} \) forcing \( \dot{A} \cap n = U \cap n \). Then any such \( q_t \) forces \( K_k \cap \dot{A} = \emptyset \) (because \( K_k \subset n \) and \( K_k \cap U = \emptyset \)), contradicting the fact that \( p'' \geq q_t \) and \( p'' \vDash K_k \cap \dot{A} \neq \emptyset \).

Contradictions obtained in cases a) and b) above imply that \( U := [\tau \setminus \{\emptyset\}]^\omega \cap V \) is a witness for \( \langle \omega, \tau \rangle \) being box-separable, which completes our proof. \( \, \square \)

Theorem 1.1 is a direct consequence of Lemma 2.2 combined with the following

**Lemma 2.3.** Suppose that \( b > \omega_1 \), \( X \) is box-separable, and \( Y \) is H-separable. Then \( X \times Y \) is \( M \)-separable provided that it is separable.

**Proof.** Let \( \langle D_n : n \in \omega \rangle \) be a sequence of countable dense subsets of \( X \times Y \). Let us fix a countable family \( \mathcal{U} \) of open non-empty subsets of \( X \) and a partition \( \omega = \bigcup_{U \in \mathcal{U}} \mathcal{O}_U \) into infinite pieces. For every \( n \in \Omega_U \) set \( D^U_n = \{ y \in Y : \exists x \in U \langle (x, y) \in D_n \rangle \} \) and note that \( D^U_n \) is dense in \( Y \) for all \( n \in \omega \). Therefore there exists a sequence \( \langle L^U_n : n \in \omega \rangle \) such that \( L^U_n \in \big[ D^U_n \big]^\omega \) and for every open non-empty \( V \subset Y \) we have \( L^U_n \cap V \neq \emptyset \) for all but finitely many \( n \). For every \( n \in \Omega_U \) find \( K^U_n \in [U]^\omega \) such that for every \( y \in L^U_n \) there exists \( x \in K^U_n \) such that \( \langle x, y \rangle \in D_n \), and set \( R(U) = \langle K^U_n : n \in \omega \rangle \). Note that \( R \) is as such in the definition of box-separability because \( K^U_n \subset U \) for all \( n \in \Omega_U \) and the latter set is infinite. Since \( X \) is box-separable there exists a family \( \mathcal{U} \) of countable collections of open non-empty subsets of \( X \) of size \( |\mathcal{U}| = \omega_1 \), and such that for every open non-empty \( U \subset X \) there exists \( \mathcal{U} \in \mathcal{U} \) with the property \( R(\mathcal{U})(n) \subset U \) for infinitely many \( n \). Since each \( D_n \) is countable and \( |\mathcal{U}| < b \), there exists a sequence \( \langle F_n : n \in \omega \rangle \) such that \( F_n \in [D_n]^\omega \) and for every \( \mathcal{U} \in \mathcal{U} \) we have \( F_n \supset (K^U_n \times L^U_n) \cap D_n \) for all but finitely many \( n \in \omega \).

We claim that \( \bigcup_{n \in \omega} F_n \) is dense in \( X \times Y \). Indeed, let us fix open non-empty subset \( X \times Y \) of the form \( X \times V \) and find \( \mathcal{U} \in \mathcal{U} \) with the property \( R(\mathcal{U})(n) = K^U_n \cap U \) for infinitely many \( n \), say for all \( n \in I \in B^\omega \). Passing to a co-finite subset of \( I \), we may assume if necessary, that \( F_n \supset (K^U_n \times L^U_n) \cap D_n \) for all \( n \in I \). Finally, fix \( n \in I \) such that \( L^U_n \cap V \neq \emptyset \) and pick \( y \in L^U_n \cap V \). By the definition of \( D^U_n \) and \( L^U_n \subset D^U_n \) we can find \( x \in K^U_n \) such that \( \langle x, y \rangle \in D_n \). Then \( \langle x, y \rangle \in U \times V \) and \( \langle x, y \rangle \in F_n \) because \( \langle x, y \rangle \in K^U_n \times L^U_n \) and \( \langle x, y \rangle \in D_n \). This completes our proof. \( \, \square \)

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