TOMAS-STEIN RESTRICTION ESTIMATES ON CONVEX COCOMPACT HYPERBOLIC MANIFOLDS. I

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ABSTRACT. In this paper, we investigate the Tomas-Stein restriction estimates on convex cocompact hyperbolic manifolds $\Gamma \backslash \mathbb{H}^{n+1}$. Via the spectral measure of the Laplacian, we prove that the Tomas-Stein restriction estimate holds when the limit set has Hausdorff dimension $\delta_{\Gamma} < n/2$. This provides an example for which restriction estimate holds in the presence of hyperbolic geodesic trapping.

1. Introduction

In $\mathbb{R}^d$, the Tomas-Stein restriction theorem [T, St] states that if $1 \leq p \leq p_c := 2(d+1)/(d+3)$, then

$$\|R_1 f\|_{L^2(S^{d-1}_1)} \leq A \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in C_0^\infty(\mathbb{R}^d),$$

where $A > 0$ depends only on $d$ and $p$. Here, the Fourier transfer restriction operator (associated with the unit sphere $S^{d-1}_1$) is defined as

$$R_1 f(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx \quad \text{for } \xi \in S^{d-1}_1.$$

Let $R_1^*$ be the adjoint of $R_1$. Since $R_1^* R_1 : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$ for $p' = p/(p-1)$, the Tomas-Stein restriction estimate (1.1) is equivalent to

$$\|R_1^* R_1\|_{L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)} \leq A^2. \quad (1.2)$$

Denote $\Delta_{\mathbb{R}^d}$ the (positive) Laplacian in $\mathbb{R}^d$. Then $\sqrt{\Delta_{\mathbb{R}^d}}$ has an absolutely continuous spectrum on $[0, \infty)$ and

$$\sqrt{\Delta_{\mathbb{R}^d}} = \int_0^\infty \lambda \, dE_{\sqrt{\Delta_{\mathbb{R}^d}}}(\lambda),$$

in which $dE_{\sqrt{\Delta_{\mathbb{R}^d}}}$ is the spectral measure of $\sqrt{\Delta_{\mathbb{R}^d}}$. Notice that $dE_{\sqrt{\Delta_{\mathbb{R}^d}}}(\lambda) = R_{\lambda}^* R_{\lambda}$, where $R_{\lambda}$ is the Fourier restriction operator associated with the sphere $S^{d-1}_\lambda$ with radius $\lambda$. A direct dilation argument yields

$$\|R_{\lambda}^* R_{\lambda}\|_{L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)} = \lambda^{d\left(\frac{1}{p} - \frac{1}{p'}\right)^{-1}} \|R_1^* R_1\|_{L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)}. \quad (1.3)$$

Then (1.1) and (1.2) are also equivalent to

$$\left\|dE_{\sqrt{\Delta_{\mathbb{R}^d}}}(\lambda)\right\|_{L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)} \leq A^2 \lambda^{d\left(\frac{1}{p} - \frac{1}{p'}\right)^{-1}} \quad \text{for } 1 \leq p \leq p_c. \quad (1.4)$$

The Tomas-Stein restriction problem can therefore be generalized to manifolds $\mathbb{M}$, via spectral measure of $\sqrt{\Delta_{\mathbb{M}}}$: We assume that the Laplacian $\Delta_{\mathbb{M}}$ is nonnegative and essentially self-adjoint on $C_0^\infty(\mathbb{M}) \subset L^2(\mathbb{M})$. (These conditions are automatically true on the convex cocompact manifold...
hyperbolic manifolds that we consider in this paper. See below for details of the geometric setting.)

**Problem 1** (Restriction estimates on manifolds via the spectral measure). Let $\mathcal{M}$ be a $d$-dim manifold. Is the following Tomas-Stein restriction estimate true for $\lambda > 0$?

$$\|dE_{\Delta_{\mathcal{M}}}(\lambda)\|_{L^p(\mathcal{M}) \to L^{p'}(\mathcal{M})} \leq C\lambda^{d\left(\frac{1}{p} - \frac{1}{p'}\right)^{-1}} \quad \text{for } 1 \leq p \leq p_c. \quad (1.5)$$

See also the discussion in Chen-Hassell [CH Section 1.2]. The parameter $\lambda$ (i.e. energy) here is important since the dilation structure in $\mathbb{R}^d$ may not be available on the manifold. We are concerned with whether the restriction estimate (1.5) holds for all $\lambda > 0$ on a manifold and how it is influenced by the underlying geometry.

If $\mathcal{M}$ is compact, then the Laplacian $\Delta_{\mathcal{M}}$ has a discrete spectrum of eigenvalues $0 \leq \lambda_0^2 \leq \lambda_1^2 \leq \cdots \to \infty$ with smooth eigenfunctions $\{u_j\}_{j=0}^\infty$. Formally, $\sqrt{\Delta_{\mathcal{M}}} = \sum_j \lambda_j \langle u_j, \cdot \rangle u_j$. So the spectral measure $dE_{\Delta_{\mathcal{M}}}(\lambda)$ is a sum of Dirac delta measures at $\lambda_j$’s. Therefore, the restriction estimate (1.5) can never hold at $\lambda_j$’s. Instead, the appropriate “discrete” version of restriction estimates in this case is for the spectral projection onto finite intervals in the spectrum, e.g. $[\lambda, \lambda + 1]$. These estimates in term imply the $L^p$ estimates of spectral clusters. See Sogge [So Chapter 5].

On non-compact and complete manifolds, the restriction estimate (1.5)1 has been proved in various settings. We mention Guillarmou-Hassell-Sikora [GHS] for asymptotically conic manifolds and Chen-Hassell [CH] for asymptotically hyperbolic manifolds, which are the motivation and also main resources for our investigation in the current paper. In both of these two cases, a geodesic non-trapping condition is assumed, that is, there is no geodesic which is contained in some compact region of $\mathcal{M}$; it in particular requires that there are no closed geodesics in $\mathcal{M}$.

Furthermore, Guillarmou-Hassell-Sikora [GHS Section 8C] remarked that if there is an elliptic closed geodesic $l \subset \mathcal{M}$, then the restriction estimate (1.5) fails. In this case, one can construct well approximated eigenfunctions (i.e. quasimodes) associated with $l$. See Babich-Lazutkin [BL] and Ralston [R]. Precisely, there are $\lambda_j \to \infty$ and $u_j \in L^2(\mathcal{M})$ such that

$$\|\lambda_j (\Delta_{\mathcal{M}} - \lambda_j^2) u_j\|_{L^2(\mathcal{M})} \leq C_N \lambda_j^{-N} \|u_j\|_{L^2(\mathcal{M})} \quad \text{for all } N \in \mathbb{N} \text{ as } j \to \infty.$$

In fact, the construction of such quasimodes associated with $l$ is local around the geodesic, i.e. $u_j \in L^2(K)$ for some compact $K \ni l$. The existence of these quasimodes ensures that following statement is *invalid* for all $1 \leq p < 2$ and $M > 0$ [GHS Proposition 8.7].

$$\exists C > 0, \exists \lambda_0, \forall \lambda \geq \lambda_0, \|dE_{\sqrt{\Delta_{\mathcal{M}}}}(\lambda)\|_{L^p(\mathcal{M}) \to L^{p'}(\mathcal{M})} \leq C\lambda^M.$$ 

So the question arises naturally, c.f. [GHS Remark 1.5].

Can the restriction estimate (1.5) hold in the presence of non-elliptic closed geodesics?

We focus on hyperbolic closed geodesics in this paper and remark that the (non-)existence of well approximated eigenfunctions as above but associated with a hyperbolic closed geodesic is not completely understood. It is a major problem in the study of Quantum Chaos; see Christianson [Chr] and Zelditch [Z Section 5]. Nevertheless, in this paper, we are able to treat the restriction estimate in Problem 1 on certain hyperbolic manifolds, where all closed geodesics are hyperbolic. To the author’s knowledge, these manifolds are the first examples with geodesic trapping for which the restriction estimate (1.5) holds.

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1See also the recent work of Huang-Sogge [HS], which includes spectral projection estimates on hyperbolic spaces $\mathbb{H}^{n+1}$. The restriction estimates in (1.5) can be derived from [HS Equation 1.16].
Geometric setting. Denote $\mathbb{H}^{n+1}$ the $(n+1)$-dim hyperbolic space. Let $\mathcal{M} = \Gamma \setminus \mathbb{H}^{n+1}$ be a convex cocompact hyperbolic manifold, i.e. $\Gamma$ is a discrete group of orientation preserving isometries of $\mathbb{H}^{n+1}$ that consists of hyperbolic elements and $\mathcal{M}$ is geometrically finite and has infinite volume. The set of closed geodesics in $\mathcal{M}$ corresponds to the conjugacy classes within the group $\Gamma$.

The size of the geodesic trapped set is characterized by the limit set $\Lambda_\Gamma$ of $\Gamma$. The limit set $\Lambda_\Gamma \subset \partial \mathbb{H}^{n+1}$ is the set of accumulation points on the orbits $\Gamma z$, $z \in \mathbb{H}^{n+1}$. The Hausdorff dimension of $\Lambda_\Gamma$, $\delta_\Gamma := \dim_{\text{H}} \Lambda_\Gamma \in [0,n)$. Then the trapped set of the geodesic flow in the unit tangent bundle $SM$ has Hausdorff dimension $2\delta_\Gamma + 1$. See Patterson \cite{P} and Sullivan \cite{Su}.

Example. The simplest example of convex cocompact hyperbolic manifolds is the hyperbolic cylinder $\Gamma \setminus \mathbb{H}^{n+1}$, in which $\Gamma = \mathbb{Z}$ acts on $\mathbb{H}^{n+1}$ by powers of a fixed dilation. In this case, the limit set $\Lambda_\Gamma = \{0,\infty\}$. There is only one closed geodesic. On non-elementary convex cocompact hyperbolic manifolds, however, there can be infinitely many closed geodesics.

It is now well-known by Lax-Phillips \cite{LP1,LP2} that the spectrum of Laplacian $\Delta_M$ consists of at most finitely many eigenvalues in the interval $(0,n^2/4)$ and absolutely continuous spectrum $[n^2/4,\infty)$ with no embedded eigenvalues. It is hence convenient in notation to consider the restriction estimates for the operator

$$P_M = \left(\Delta_M - \frac{n^2}{4}\right)^\frac{1}{2},$$

where $(\cdot)_+ = \max\{\cdot,0\}$. The operator $P_M$ has an absolutely continuous spectrum $[0,\infty)$.

Before we state the main theorem, we remark that the range $1 \leq p \leq p_c$ in the restriction estimate \cite{L} can be extended to $1 \leq p < 2$ if $\mathcal{M} = \mathbb{H}^{n+1}$ (more generally, $\mathcal{M}$ is a non-trapping asymptotically hyperbolic manifold, see Chen-Hassell \cite{CH} Theorem 1.6 and Remark 1.7). This range is larger than the one on $\mathbb{R}^d$ in \cite{L} and is related to the Kunze-Stein theory \cite{KS} of harmonic analysis on semisimple Lie groups. The extended range of $p$ for restriction estimate persists on the hyperbolic manifolds considered here.

Our main theorem states

**Theorem 2** (Restriction estimates on convex cocompact hyperbolic manifolds). Let $\mathcal{M} = \Gamma \setminus \mathbb{H}^{n+1}$ be a convex cocompact hyperbolic manifold for which $\delta_\Gamma < n/2$. Then there exists $C' > 0$ depending on $\mathcal{M}$ and $p$ such that at high energy $\lambda \geq 1$,

$$\|dE_M(\lambda)\|_{L^p(\mathcal{M})} \leq \begin{cases} C\lambda^{(n+1)(\frac{1}{p'} - \frac{1}{p})^{-1}} & \text{for } 1 \leq p \leq p_c = \frac{2(n+2)}{n+4}, \\ C\lambda^{n(\frac{1}{p'} - \frac{1}{2})} & \text{for } p_c \leq p < 2. \end{cases}$$

Some remarks on the proof of the theorem and further investigations are in order.

**Remark 3** (Restriction estimates at low energy). Under the condition in Theorem 2 the resolvent (acting on appropriate spaces, see e.g. Bourgain-Dyatlov \cite{BD})

$$R_\lambda := (\Delta_M - n^2/4 - \lambda^2)^{-1}$$

is holomorphic in the half complex plane $\{\lambda \in \mathbb{C} : \text{Im}\lambda > -(n/2 - \delta_\Gamma)\}$ by the Patterson-Sullivan theory \cite{P,Su}. So in this half plane, there are no resonances, which are the poles of $R_\lambda$ in $\mathbb{C}$. (That is, there is a spectral gap of size at least $n/2 - \delta_\Gamma > 0$.) In particular, there is no resonance at the bottom of the continuous spectrum $[0,\infty)$ of $\Delta_M - n^2/4$. This condition guarantees that the restriction estimates at low energy $\lambda \leq 1$ in Chen-Hassell \cite{CH}.
1.5 and 1.6] remain valid. That is, at low energy \( \lambda \leq 1 \),

\[
\| dE_{P_M}(\lambda) \|_{L^p(M) \to L^{p'}(M)} \leq C\lambda^2 \quad \text{for } 1 \leq p < 2.
\]

**Remark 4** (Critical \( \delta_\Gamma \) for the restriction estimate). Our method in this paper cannot treat the restriction estimate in Problem 1 on \( M = \Gamma \setminus \mathbb{H}^{n+1} \) for which \( \delta_\Gamma \geq n/2 \). It is not yet clear whether the restriction estimate (1.5) holds on such manifolds with large limit sets (and thus with large hyperbolic trapped sets). Notice that in the extreme case when \( M \) is compact, \( \Lambda_\Gamma = \partial \mathbb{H}^{n+1} \) (so \( \delta_\Gamma = n \)) and (1.5) fails. It is interesting to find the “critical” dimension \( n/2 \leq \delta_c \leq n \) of the limit sets for which (1.5) fails for the corresponding hyperbolic manifolds. We plan to investigate this problem in a future work. Some relevant spectral information on hyperbolic surfaces (i.e. \( \dim M = 2 \)) when \( \delta_\Gamma \geq 1/2 \) has recently been proved, in particular, Bourgain-Dyatlov [BD] established an essential spectral gap for the resolvent \( R_\lambda \) in \( \mathbb{C} \).

**Remark 5** (More general geometries for which the hyperbolic trapped sets are small). The proof of Theorem 2 is inspired by Burq-Guillarmou-Hassell [BGH, Theorem 1.1], in which they studied the Strichartz estimates for Schrödinger equation on the convex cocompact hyperbolic manifolds for which \( \delta_\Gamma < n/2 \). In the same paper, the authors also treated more general classes of manifolds, including manifolds that contain small sets of hyperbolic trapped sets but not necessarily with constant negative curvature. Instead of using the Hausdorff dimension of the limit set to characterize the size of trapped set, they used the topological pressure condition. It is interesting to see if Theorem 2 can be generalized to such setting.

### 2. Proof of Theorems 2

The main tool to prove the Tomas-Stein restriction estimates in Theorem 2 is the abstract spectral theory by Guillarmou-Hassell-Sikora [GHS, Theorem 3.1]. See also Chen [Che].

**Theorem 6.** Let \( (X, d, \mu) \) be a metric measure space and \( L \) be an abstract nonnegative self-adjoint operator on \( L^2(X, \mu) \). Assume that the spectral measure \( dE_{\sqrt{\tau}}(\lambda) \) has a Schwartz kernel \( dE_{\sqrt{\tau}}(\lambda)(x, y) \) for \( x, y \in X \). Suppose that there is a subset \( I \subset [0, \infty) \) such that for \( \lambda \in I \),

\[
\left| \frac{d^j}{d\lambda^j} dE_{\sqrt{\tau}}(\lambda)(x, y) \right| \leq C\lambda^{m-1-j} (1 + \lambda d(x, y))^{-(m-1)/2+j},
\]

in which

(i). \( j = 0, j = m/2 - 1, \) and \( j = m/2 \) if \( m \) is even,

(ii). \( j = m/2 - 3/2 \) and \( j = m/2 + 1/2 \) if \( m \) is odd.

Then the following Tomas-Stein restriction estimate holds for all \( \lambda \in I \) and \( 1 \leq p \leq p_c \).

\[
\| dE_{\sqrt{\tau}}(\lambda) \|_{L^p(M) \to L^{p'}(M)} \leq C\lambda^{m\left(\frac{1}{p'} - \frac{1}{p}\right)}^{-1}.
\]

In application, we substitute \( L = (\Delta_M - n^2/4)_{+} \) into the above theorem to prove Theorem 2. We begin from the estimates on the hyperbolic space \( \mathbb{H}^{n+1} \). For notational simplicity, from now on we denote

\[
\mathbb{H} = \mathbb{H}^{n+1}.
\]

Let

\[
P_{\mathbb{H}} = \left( \Delta_{\mathbb{H}} - \frac{n^2}{4} \right)_{+}^{\frac{1}{2}}.
\]

The topological pressure condition reduces to the condition about Hausdorff dimension of the limit set if the manifold has constant negative curvature. See [BGH, Lemma 3.5].
The following pointwise estimates are from Chen-Hassell [CH, Equations (1.9) and (1.10)]. They actually proved these estimates on asymptotically hyperbolic manifolds with geodesic non-trapping condition.

**Proposition 7** (Pointwise estimates of the spectral measure on $\mathbb{H}$). The Schwartz kernel of $dE_{P_\Gamma}(\lambda)(x,y)$ for $\lambda \geq 1$ and $x,y \in \mathbb{H}(= \mathbb{H}^{n+1})$ satisfies

$$
\left| \frac{d^j}{d\lambda^j} dE_{P_\Gamma}(\lambda)(x,y) \right| \leq \begin{cases} 
C\lambda^{n-j}(1 + \lambda d_{\mathbb{H}}(x,y))^{-n/2+j} & \text{for } d_{\mathbb{H}}(x,y) \leq 1; \\
C\lambda^{n/2} d_{\mathbb{H}}(x,y)^j e^{-nd_{\mathbb{H}}(x,y)/2} & \text{for } d_{\mathbb{H}}(x,y) \geq 1.
\end{cases}
$$

Here, $d_{\mathbb{H}}$ is the hyperbolic distance in $\mathbb{H}$.

**Remark.** The pointwise upper bounds in the two distance ranges above reflect two different behaviors of the spectral measure on hyperbolic spaces.

1. When $d_{\mathbb{H}}(x,y) \leq 1$, the estimate is similar to the one in $\mathbb{R}^d$:

$$
\left| \frac{d^j}{d\lambda^j} dE_{\sqrt{\Delta_{\mathbb{H}}}}(\lambda)(x,y) \right| = \frac{d^j}{d\lambda^j} \int_{S_{\lambda}^{-1}} e^{i(x-y) \cdot d}\xi \lambda^{d-1-j}(1 + \lambda d_{\mathbb{R}^d}(x,y))^{-d+1+j},
$$

following the standard non-stationary phase asymptotics. See e.g. Stein [St].

2. When $d_{\mathbb{H}}(x,y) \geq 1$, the exponential estimate is different with the one in $\mathbb{R}^d$ and is related to the exponential volume growth in radii of geodesic balls in the hyperbolic space.

**Remark 8.** In Proposition 7, the distance range cutoff at $d_{\mathbb{H}}(x,y) = 1$ is rather arbitrary. Give a convex cocompact group $\Gamma$. For each $\gamma \in \Gamma$, there is a unique hyperbolic line in $\mathbb{H}$, called the axis of $\gamma$, which is invariant under $\gamma^k$, $k \in \mathbb{N}$. Then $l_{\gamma} := d(z,\gamma z)$ for all $z$ on the axis and is called the displacement length of $\gamma$. Moreover, $l_{\gamma} = \min_{z \in \mathbb{H}} d_{\mathbb{H}}(z,\gamma z)$. Denote

$$
l_0 = \min_{\gamma \in \Gamma \setminus \{Id\}} \{l_{\gamma}\}. \quad (2.3)
$$

We know that $l_0 > 0$ since $\Gamma$ is a discrete group. In the following, we instead use the spectral measure pointwise estimates on $\mathbb{H}$ at the distance range cutoff $d_{\mathbb{H}}(x,y) = l_0/2$:

$$
\left| \frac{d^j}{d\lambda^j} dE_{P_\Gamma}(\lambda)(x,y) \right| \leq \begin{cases} 
C\lambda^{n-j}(1 + \lambda d_{\mathbb{H}}(x,y))^{-n/2+j} & \text{for } d_{\mathbb{H}}(x,y) < l_0/2; \\
C\lambda^{n/2} d_{\mathbb{H}}(x,y)^j e^{-nd_{\mathbb{H}}(x,y)/2} & \text{for } d_{\mathbb{H}}(x,y) \geq l_0/2.
\end{cases}
$$

But of course now the constant $C$ depends on $l_0$ (therefore on $\Gamma$).

Let $\mathcal{F} \subset \mathbb{H}$ be a fundamental domain of $\mathbb{M} = \Gamma \setminus \mathbb{H}$. Then for $x,y \in \mathcal{F}$,

$$
dE_{P_\Gamma}(\lambda)(x,y) = \sum_{\gamma \in \Gamma} dE_{P_\Gamma}(\lambda)(x,\gamma y). \quad (2.5)
$$

**Remark** (Spectral measure on Euclidean cylinders). We remark that the convex cocompact group structure of $\Gamma$ on $\mathbb{H}$ is crucial for the restriction estimates on $\Gamma \setminus \mathbb{H}$. For example, take $\mathbb{M} = \Gamma \setminus \mathbb{R}^2$ as a Euclidean cylinder. Here, $\Gamma = \mathbb{Z}$ acts on $\mathbb{R}^2$ by powers of a fixed translation $x \to x + l$, $l \in \mathbb{R}^2 \setminus \{0\}$. Then by (2.2),

$$
\frac{d}{d\lambda} dE_{\sqrt{\Delta_{\mathbb{H}}}}(\lambda)(x,y) = \sum_{k \in \mathbb{Z}} dE_{\sqrt{\Delta_{\mathbb{H}}}}(\lambda)(x,y + kl) \\
\sim \sum_{k \in \mathbb{Z}} (1 + \lambda d_{\mathbb{R}^d}(x,y + kl))^{\frac{1}{2}} \\
\geq \lambda^\frac{1}{2} |l|^{\frac{1}{2}} \sum_{k \in \mathbb{Z}} |k|^{\frac{1}{2}}
$$
clearly fails the estimate in Theorem 8 when \( m = 2 \) and \( j = 1 \). On the other hand, there are elliptic closed geodesics \( \{(x + tl) : t \in [0, 1]\} \) and by Guillarmou-Hassell-Sikora [GHS] Section 8C the restriction estimate \((1.5)\) fails.

We control the summation in the right-hand-side of \((2.5)\) by the Patterson-Sullivan theory [PL Su]. In particular, the Patterson-Sullivan theory concludes that the Poincaré series

\[
G_s(x, y) := \sum_{\gamma \in \Gamma} e^{-sd_H(x, \gamma y)}
\]

is convergent if and only if \( s > \delta_\Gamma \). In fact, by the triangle inequalities

\[
d_H(y, \gamma y) - d_H(x, y) \leq d(x, \gamma y) \leq d_H(x, y) + d_H(y, \gamma y),
\]

we have that

\[
e^{-sd_H(x, y)} e^{-sd_H(y, \gamma y)} \leq e^{-sd_H(x, \gamma y)} \leq e^{sd_H(x, y)} e^{-sd_H(y, \gamma y)}.
\]

Summing over \( \gamma \in \Gamma \),

\[
e^{-sd_H(x, y)} G_s(y, y) \leq G_s(x, y) \leq e^{sd_H(x, y)} G_s(y, y).
\]

So the convergence of the Poincaré series \((2.6)\) is independent of \( x \) and \( y \). When the series \( G_s(x, y) \) converges, that is, \( s < \delta_\Gamma \), we need a quantitative estimate of it that is sufficient for our purpose.

Following Borthwick [B] Section 2.5.2], if \( s < \delta_\Gamma \), then

\[
\sum_{\gamma \in \Gamma} e^{-sl_\gamma} < C_s,
\]

in which \( C_s \) depends on \( s \) and \( \Gamma \). It immediately follows that for all \( R > 0 \),

\[
N(R) := \#\{\gamma \in \Gamma : l_\gamma \leq R\} \leq C_R,
\]

in which \( C_R \) depends on \( R \) and \( \Gamma \).

**Lemma 9.** Let \( \mathcal{F} \) be a fundamental domain of \( \mathbb{M} = \Gamma \setminus \mathbb{H} \). There are constants \( R, C > 1 \) such that for all \( \gamma \in \Gamma \) with \( l_\gamma > R \) and any \( k \in \mathbb{N} \), we have that

\[
e^{-d_\mathcal{F}(x, y)} \leq C e^{-l_\gamma} \min\{1, d_\mathcal{F}(x, y)^{-k}\} \quad \text{for all } x, y \in \mathcal{F}.
\]

Here, \( d_\mathcal{F}(x, y) \) is the distance between \( x \) and \( y \) in \( \mathcal{F} \).

**Proof.** We use the Poincaré ball model \( \mathbb{B} \) of the hyperbolic space \( \mathbb{H} \) and denote \( |z| \) the Euclidean norm of \( z \in \mathbb{B} \). From Guillarmou-Moroianu-Park [GMP] Lemma 5.2, there are positive constants \( R \) and \( C \) such that for all \( \gamma \in \Gamma \) with \( l_\gamma > R \) and all \( x, y \in \mathcal{F} \),

\[
e^{-d_\mathcal{F}(x, y)} \leq C e^{-l_\gamma} (1 - |x|^2)(1 - |y|^2) \leq C e^{-l_\gamma}.
\]

Notice that \( d_\mathcal{F}(x, y) = d_\mathcal{B}(\gamma_0 x, \gamma_0 y) \) for any hyperbolic isometry \( \gamma_0 \) of \( \mathbb{B} \). Choose \( \gamma_0 \) such that \( \gamma_0 z = e \), where \( e \) is the origin in \( \mathbb{B} \). Therefore without loss of generality, we can assume that \( \mathcal{F} \ni e \) and \( x = e \). Note that

\[
d_\mathcal{F}(e, y) = \log \left(\frac{1 + |y|}{1 - |y|}\right) \leq C \log \left(\frac{1}{1 - |y|}\right).
\]

It thus follows that for all \( k \in \mathbb{N} \),

\[
(1 - |y|^2)(1 - |y|^2) = 1 - |y|^2 \leq C \left[\log \left(\frac{1}{1 - |y|}\right)\right]^{-k} \leq C d_\mathcal{F}(e, y)^{-k}.
\]
Hence,

\[ e^{-d_H(e, \gamma y)} \leq C e^{-l_0} d_F(e, y)^{-k}. \]

\[ \square \]

Remark 10. Before proving Theorem 2, we remark that Chen-Ouhabaz-Sikora-Yan [COSY] developed an abstract system that includes some characterization of the restriction estimates by certain dispersive estimates. In particular, by [COSY, Section II.2], one can deduce the restriction estimates (1.5) in certain range of \( p \) from the dispersive estimate

\[ \| e^{it \Delta} \|_{L^1(M) \to L^\infty(M)} \leq C |t|^{-k} \]

for some \( k > 0 \).

However, as seen in Burq-Guillarmou-Hassell [BGH, Theorem 1.1], such dispersive estimate on hyperbolic manifolds in general is not sufficient to imply the restriction estimates in the range of \( 1 \leq p \leq p_c \). On a manifold, the relations between spectral measure estimates in (1.5), dispersive estimates, and also Strichartz estimates for Schrödinger equation are not yet clear. See Burq-Guillarmou-Hassell [BGH, Remark 1.3].

We now proceed to prove Theorem 2 by Theorem 6. Fix \( x, y \in M = \Gamma \setminus \mathbb{H} \), we choose the Dirichlet domain of the point \( y \) for the representation of \( M \):

\[ D = D_y := \{ z \in \mathbb{H} : d_\mathbb{H}(z, y) < d_\mathbb{H}(z, \gamma y) \text{ for all } \gamma \in \Gamma \setminus \{ \text{Id} \} \}. \]

Also, the distance of \( x, y \) in \( D \) equals \( d_\mathbb{H}(x, y) \).

To estimate the summation in (2.5), we first take \( \gamma = \text{Id} \).

Case I. \( d_\mathbb{H}(x, y) < l_0 / 2 \). Then the spectral measure pointwise estimate on \( \mathbb{H} \) in the first distance range of (2.4) applies. But it coincides with (2.1) in Theorem 6.

Case II. \( d_\mathbb{H}(x, y) \geq l_0 / 2 \). Then the spectral measure pointwise estimate on \( \mathbb{H} \) in the second distance range of (2.4) applies. It is straightforward to see that

\[ \left| \frac{d}{d\lambda} dE_{\Phi_\lambda}(\lambda)(x, y) \right| \leq C \lambda^{n/2} d_\mathbb{H}(x, y)^j e^{-n d_\mathbb{H}(x, y)/2} \]

\[ \leq C \lambda^{n-j} (1 + \lambda d_\mathbb{H}(x, y))^{-n/2+j}. \]

In both of these cases for \( \gamma = \text{Id} \), the corresponding term \( dE_{\Phi_\lambda}(\lambda)(x, \gamma y) \) in the summation (2.5) satisfies the condition (2.1) in Theorem 6. We then discuss \( \gamma \in \Gamma \setminus \{ \text{Id} \} \), in the same two cases as above.

Case I. \( d_\mathbb{H}(x, y) < l_0 / 2 \). Then \( d_\mathbb{H}(x, \gamma y) \geq l_0 / 2 \) for all \( \gamma \in \Gamma \setminus \{ \text{Id} \} \). If not, i.e. \( d_\mathbb{H}(x, \gamma y) < l_0 / 2 \), then triangle inequality implies that

\[ l_\gamma = \min_{z \in \mathbb{H}} d_\mathbb{H}(z, \gamma z) \leq d_\mathbb{H}(y, \gamma y) \leq d_\mathbb{H}(x, y) + d_\mathbb{H}(x, \gamma y) < l_0, \]

contradicting the fact that \( l_0 = \min_{\gamma \in \Gamma \setminus \{ \text{Id} \}} \{ l_\gamma \} \) defined in (2.3).

Case II. \( d_\mathbb{H}(x, y) \geq l_0 / 2 \). Then by the definition of the Dirichlet domain,

\[ d_\mathbb{H}(x, \gamma y) > d_\mathbb{H}(x, y) \geq l_0 / 2 \] for all \( \gamma \in \Gamma \setminus \{ \text{Id} \} \).

Up to this point, to estimate the summation in (2.5), we only need to estimate the terms for \( \gamma \in \Gamma \setminus \{ \text{Id} \} \). Moreover, in (2.4), the spectral measure pointwise estimate in second distance range applies only.

1Majority of [COSY] requires that the geometry satisfies volume doubling condition, which the hyperbolic manifolds clearly do not. However, the results in [COSY, Section II.2] are valid on all metric spaces.
We write the proof for the restriction estimates \((1.5)\) when \(\dim M = n + 1\) is even (so \(n\) is odd). To this end, we verify \((2.1)\) with \(\sqrt{L} = P_{M}\) for \(j = 0, j = (n - 1)/2,\) and \(j = (n + 1)/2.\) The case when \(\dim M = n + 1\) is odd proceeds with little modification.

Write
\[
\sum_{\gamma \in \Gamma \setminus \{Id\}} dE_{P_{M}}(\lambda)(x, \gamma y) = \sum_{\gamma \in \Gamma \setminus \{Id\}, l_{\gamma} \leq R} dE_{P_{M}}(\lambda)(x, \gamma y) + \sum_{\gamma \in \Gamma \setminus \{Id\}, l_{\gamma} > R} dE_{P_{M}}(\lambda)(x, \gamma y),
\]
in which \(R\) is from Lemma \([9]\).

2.1. The estimate for \(j = 0.\) Using the fact that \(e^{-st} \leq C t^{-k}\) for any \(k \in \mathbb{R}\) uniformly on \(t \in (0, \infty),\)
\[
\sum_{\gamma \in \Gamma \setminus \{Id\}, l_{\gamma} \leq R} |dE_{P_{M}}(\lambda)(x, \gamma y)| \leq C \sum_{\gamma \in \Gamma \setminus \{Id\}, l_{\gamma} \leq R} \lambda^{\frac{n}{2}} e^{-\frac{d_{\mathcal{H}}}{2}(x, \gamma y)} \leq CN(R) \lambda^{\frac{n}{2}} e^{-\frac{d_{\mathcal{H}}}{2}(x, y)} \leq C \lambda^{n}(1 + \lambda d_{\mathcal{H}}(x, y))^{-\frac{n}{2}}.
\]
Here, \(C\) depends on \(R.\)

Set \(s\) such that \(0 < \delta_{\Gamma} < s < n/2.\) Since \(l_{\gamma} > R,\) we apply Lemma \([9]\) to compute that
\[
\sum_{\gamma \in \Gamma \setminus \{Id\}, l_{\gamma} > R} |dE_{P_{M}}(\lambda)(x, \gamma y)| \leq C \sum_{\gamma \in \Gamma \setminus \{Id\}, l_{\gamma} > R} \lambda^{\frac{n}{2}} e^{-\frac{d_{\mathcal{H}}}{2}(x, \gamma y)} \leq C \lambda^{\frac{n}{2}} \sum_{\gamma \in \Gamma \setminus \{Id\}, l_{\gamma} > R} e^{-sd_{\mathcal{H}}(x, \gamma y)} \leq C \lambda^{\frac{n}{2}} d_{\mathcal{H}}(x, y)^{-k} \sum_{\gamma \in \Gamma} e^{-s l_{\gamma}} \leq C \lambda^{n}(1 + \lambda d_{\mathcal{H}}(x, y))^{-\frac{n}{2}},
\]
by choosing \(k\) large enough. Here, we used \((2.8)\) so the constant \(C\) here depends on \(s\) and \(R.\)

The above two estimates together imply that
\[
\sum_{\gamma \in \Gamma \setminus \{Id\}} |dE_{P_{M}}(\lambda)(x, \gamma y)| \leq C \lambda^{n}(1 + \lambda d_{\mathcal{H}}(x, y))^{-\frac{n}{2}}.
\]

2.2. The estimate for \(j = (n - 1)/2.\) Set \(s\) such that \(0 < \delta_{\Gamma} < s < n/2.\) First we have that
\[
\sum_{\gamma \in \Gamma \setminus \{Id\}, l_{\gamma} \leq R} \left| \frac{d^{(n-1)/2}}{d(\lambda^{(n-1)/2})} dE_{P_{M}}(\lambda)(x, \gamma y) \right| \leq C \sum_{\gamma \in \Gamma \setminus \{Id\}, l_{\gamma} \leq R} \lambda^{\frac{n}{2}} d_{\mathcal{H}}(x, \gamma y)^{\frac{n-1}{2}} e^{-\frac{d_{\mathcal{H}}}{2}(x, \gamma y)} \leq C \lambda^{\frac{n}{2}} \sum_{\gamma \in \Gamma \setminus \{Id\}, l_{\gamma} \leq R} e^{-sd_{\mathcal{H}}(x, \gamma y)} \leq CN(R) \lambda^{\frac{n}{2}} e^{-sd_{\mathcal{H}}(x, y)} \leq C \lambda^{\frac{n}{2} + 1}(1 + \lambda d_{\mathcal{H}}(x, y))^{-\frac{1}{2}}.
\]
Then for \( l_\gamma > R \), we apply Lemma 9 to compute that
\[
\sum_{\gamma \in \Gamma \setminus \{\text{Id}\}, l_\gamma > R} \left| \frac{d^{(n-1)/2}}{d\lambda^{(n-1)/2}} dE_{P_\lambda}(\lambda)(x, \gamma y) \right| \leq C \sum_{\gamma \in \Gamma : l_\gamma > R} \lambda^\frac{n}{2} d_H(x, \gamma y) \frac{\gamma y}{\lambda} e^{-\frac{\gamma}{2} d_H(x, \gamma y)} \\
\leq C \lambda^\frac{n}{2} \sum_{\gamma \in \Gamma : l_\gamma > R} e^{-s d_H(x, \gamma y)} \\
\leq C \lambda^\frac{n}{2} d_H(x, y)^{-k} \sum_{\gamma \in \Gamma} e^{-s l_\gamma} \\
\leq C \lambda^\frac{n}{2} (1 + \lambda d_H(x, y))^{-\frac{1}{2}},
\]

by choosing \( k \) large enough.

The above two estimates together imply that
\[
\sum_{\gamma \in \Gamma \setminus \{\text{Id}\}} \left| \frac{d^{(n-1)/2}}{d\lambda^{(n-1)/2}} dE_{P_\lambda}(\lambda)(x, \gamma y) \right| \leq C \lambda^\frac{n+1}{2} (1 + \lambda d_H(x, y))^{-\frac{1}{2}}.
\]

2.3. The estimate for \( j = (n+1)/2 \). Set \( s \) such that \( 0 < \delta_\Gamma < s < n/2 \). First similarly as in the above subsection we have that
\[
\sum_{\gamma \in \Gamma \setminus \{\text{Id}\}, l_\gamma \leq R} \left| \frac{d^{(n-1)/2}}{d\lambda^{(n-1)/2}} dE_{P_\lambda}(\lambda)(x, \gamma y) \right| \leq C \lambda^\frac{n}{2} (1 + \lambda d_H(x, y))^{\frac{1}{2}}.
\]

Then for \( l_\gamma > R \), we apply Lemma 9 to compute that
\[
\sum_{\gamma \in \Gamma \setminus \{\text{Id}\}, l_\gamma > R} \left| \frac{d^{(n+1)/2}}{d\lambda^{(n+1)/2}} dE_{P_\lambda}(\lambda)(x, \gamma y) \right| \leq C \sum_{\gamma \in \Gamma : l_\gamma > R} \lambda^\frac{n}{2} d_H(x, \gamma y) \frac{\gamma y}{\lambda} e^{-\frac{\gamma}{2} d_H(x, \gamma y)} \\
\leq C \lambda^\frac{n}{2} \sum_{\gamma \in \Gamma : l_\gamma > R} e^{-s d_H(x, \gamma y)} \\
\leq C \lambda^\frac{n}{2} d_H(x, y)^{-k} \sum_{\gamma \in \Gamma} e^{-s l_\gamma} \\
\leq C \lambda^\frac{n}{2} (1 + \lambda d_H(x, y))^{\frac{1}{2}}.
\]

The above two estimates together implies that
\[
\sum_{\gamma \in \Gamma \setminus \{\text{Id}\}} \left| \frac{d^{(n+1)/2}}{d\lambda^{(n+1)/2}} dE_{P_\lambda}(\lambda)(x, \gamma y) \right| \leq C \lambda^\frac{n+1}{2} (1 + \lambda d_H(x, y))^{\frac{1}{2}}.
\]

By the abstract theory of restriction estimates in Theorem 6, Theorem 2 for the range \( 1 \leq p \leq p_c \) follows in even dimensions. The case for odd dimension is similar and we omit it here.

2.4. Proof of Theorem 2 for \( p_c \leq p < 2 \). We argue the restriction estimates in the range \( p_c \leq p < 2 \) similarly as in Chen-Hassell [CH, Section 2.2], i.e. Theorem 2 for \( p_c \leq p < 2 \) follows
\[
\left| \frac{d^j}{d\lambda^j} dE_{P_\lambda}(\lambda)(x, y) \right| \leq C \lambda^\frac{n}{2} \quad \text{for all } j \geq 1.
\]
Again we only need to estimate the summation in (2.5) for $\gamma \neq \text{Id}$ and apply the spectral measure pointwise estimate (2.4) in second distance range. Set $s$ such that $0 < \delta_\Gamma < s < n/2$. First we have that

$$\sum_{\gamma \in \Gamma \setminus \{\text{Id}\} : l_\gamma \leq R} \left| \frac{d}{d\lambda_j} dE_{P_k}(\lambda)(x, \gamma y) \right| \leq C \sum_{\gamma \in \Gamma \setminus \{\text{Id}\} : l_\gamma \leq R} \lambda^\frac{n}{2} d_\text{H}(x, \gamma y)^j e^{-\frac{n}{2} d_\text{H}(x, \gamma y)}$$

$$\leq C \lambda^\frac{n}{2} \sum_{\gamma : l_\gamma \leq R} e^{-sd_\text{H}(x, \gamma y)}$$

$$\leq CN(R) \lambda^\frac{n}{2}$$

$$\leq C \lambda^\frac{n}{2}.$$

Then for $l_\gamma > R$, we apply $e^{-d_\text{H}(x, \gamma y)} \leq Ce^{-l_\gamma}$ from Lemma 9 to compute that

$$\sum_{\gamma \in \Gamma \setminus \{\text{Id}\} : l_\gamma > R} \left| \frac{d}{d\lambda_j} dE_{P_k}(\lambda)(x, \gamma y) \right| \leq C \sum_{\gamma \in \Gamma \setminus \{\text{Id}\} : l_\gamma > R} \lambda^\frac{n}{2} d_\text{H}(x, \gamma y)^j e^{-\frac{n}{2} d_\text{H}(x, \gamma y)}$$

$$\leq C \lambda^\frac{n}{2} \sum_{\gamma \in \Gamma : l_\gamma > R} e^{-sl_\gamma}$$

$$\leq C \lambda^\frac{n}{2} \sum_{\gamma \in \Gamma} e^{-sl_\gamma}$$

$$\leq C \lambda^\frac{n}{2}.$$

The above two estimates together imply that

$$\sum_{\gamma \in \Gamma \setminus \{\text{Id}\}} \left| \frac{d}{d\lambda_j} dE_{P_k}(\lambda)(x, \gamma y) \right| \leq C \lambda^\frac{n}{2}.$$

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