Culf maps and edgewise subdivision

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Abstract

We show that, for any simplicial space $X$, the $\infty$-category of culf maps over $X$ is equivalent to the $\infty$-category of right fibrations over $Sd(X)$, the edgewise subdivision of $X$ (when $X$ is a Rezk complete Segal space or 2-Segal space, this is the twisted arrow category of $X$). We give two proofs of independent interest; one exploiting comprehensive factorization and the natural transformation from the edgewise subdivision to the nerve of the category of elements, and another exploiting a new factorization system of ambifinal and culf maps, together with the right adjoint to edgewise subdivision. Using this main theorem, we show that the $\infty$-category of decomposition spaces and culf maps is locally an $\infty$-topos.

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1 Introduction

Background

1.1. Decomposition spaces (2-Segal spaces). Decomposition spaces \cite{25, 26, 27} (the same thing as 2-Segal spaces \cite{18}; see \cite{21}) are simplicial \(\infty\)-groupoids (simplicial spaces) subject to an exactness condition weaker than the Segal condition. Technically the condition says that certain simplicial identities are pullback squares; equivalently, a simplicial space is a decomposition space when every slice and every coslice is a Segal space.

Where the Segal condition expresses composition, the weaker condition expresses decomposition. The motivation of Gálvez–Kock–Tonks \cite{25, 26, 27} for introducing and studying decomposition spaces was that they have incidence coalgebras and Möbius inversion. The motivation of Dyckerhoff and Kapranov \cite{18} came rather from homological algebra and representation theory. In both lines of development, an important example of a decomposition space is Waldhausen’s S-construction \cite{55}, of an abelian category \(\mathcal{A}\), say. Recall that \(S(\mathcal{A})\) is a simplicial groupoid which is contractible in degree 0, has the objects of \(\mathcal{A}\) in degree 1, and short exact sequences in degree 2, etc. Wide-ranging generalizations of Waldhausen’s construction resulted from the decomposition-space viewpoint; see \cite{6, 8, 9}, culminating with the discovery that every decomposition space arises from a certain generalized Waldhausen construction, which takes as input certain double Segal spaces.

1.2. Edgewise subdivision. The edgewise subdivision of a simplicial space \(X\), first introduced by Segal \cite{48}, is a new simplicial space \(Sd(X)\) (of the same homotopy type) with \(Sd(X)_n = X_{2^n+1}^n\). Formally (cf. \cite{5.2} below), \(Sd := Q^*\), for \(Q: \Delta \rightarrow \Delta\) given by \([n] \mapsto [n]^\text{op} \star [n] = [2n+1]\). When \(X\) is the nerve of a category, \(Sd(X)\) is the nerve of the twisted arrow category. A significant example of edgewise subdivision is the fact (due to Waldhausen \cite{55}) that the edgewise subdivision of the Waldhausen S-construction is the Quillen Q-construction \cite{42}, in this way relating the two main approaches to K-theory of categories.

Decomposition spaces can be characterized in terms of edgewise subdivision, by a theorem of Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer \cite{7}: \(X\) is decomposition if and only if \(Sd(X)\) is Segal. In this paper we explore similar viewpoints, not just on simplicial spaces but also on simplicial maps.

1.3. Culf maps. The most important class of simplicial maps for decomposition spaces — those that induce coalgebra homomorphisms — are the culf maps (standing for “conservative” and “unique-lifting-of-factorization”). The culf condition is weaker than being a right (or left) fibration. For \(\infty\)-categories, the culf maps are the same thing as the conservative exponentiable fibrations studied by Ayala and Francis \cite{4}. For 1-categories, culf functors are also called discrete Conduché fibrations \cite{31}.

A technically convenient formulation of the culf condition states that certain squares are pullbacks (cf. \cite{3.2} below). While that condition will feature in all our proofs, it is useful to know (cf. \cite{5.3}) that a simplicial map \(p\) is culf if and
only if \( \text{Sd}(p) \) is a right fibration. (For 1-categories, where edgewise subdivision is just the twisted arrow category, this result goes back to Lamarche and Bunge–Niefeld \[14\].)

Further interpretations can be given in analogy with right (or left) fibrations. Recall that a functor \( p: \mathcal{E} \to \mathcal{B} \) is a right fibration when for every object \( x \in \mathcal{E} \), the induced functor on slices \( p_x: \mathcal{E}_x \to \mathcal{B}_{p_x} \) is an equivalence. Similarly, \( p: \mathcal{E} \to \mathcal{B} \) is a left fibration if every induced map on coslices is an equivalence. The culf condition is weaker: \( p \) is culf when for every \( x \in \mathcal{E} \) the induced map on coslices is a right fibration, or equivalently, the induced map on slices is a left fibration.

1.4. Interval preservation, and culf maps in combinatorics. The data over which to slice and then coslice, or coslice and then slice, is just a 1-simplex \( f: x \to y \). The slice of the coslice (or the coslice of the slice) is then precisely Lawvere’s notion of interval of \( f \), denoted \( I(f) \). Intuitively, the interval of an arrow \( f \) is the category of its factorizations. Yet another characterization of culf maps is that they are the maps that induce equivalences on all intervals (cf. \[3.9\]). This is the original viewpoint on culf maps of Lawvere \[36\].

The notion of interval of a 1-simplex is central to the combinatorial theory of decomposition spaces \[27\], \[24\], \[23\], since it generalizes the notion of intervals in a poset, which form the basis for the incidence coalgebra of the poset. Just as the comultiplication map in classical incidence coalgebras splits poset intervals, the general notion of incidence coalgebra of decomposition spaces is about splitting decomposition-space intervals, or equivalently, summing over factorizations. The interpretation of the culf condition from the viewpoint of combinatorics is thus to preserve interval structure, or to preserve decomposition structure, loosely speaking.

1.5. Culf maps in dynamical systems and process algebra. Lawvere’s original motivation, both for the notion of interval and the notion of culf map, came from dynamical systems and the general theory of processes \[36\] (part of his long-time effort to understand continuum mechanics categorically). In this theory, the general role of culf maps is to express abstract notions of duration and synchronization, but depending on the situation they are also given interpretation in terms of “response” and “control.” The interval of an arrow, thought of as a process, is then the space of trajectories, or executions, of the process. It is important that the culf condition is weaker than left fibrations (discrete opfibrations) or right fibrations (discrete fibrations): where left or right fibrations express determinism, namely unique evolution forward or backward from a given state (object) (see \[57\] for a development of this viewpoint in computer science), the culf condition only expresses synchronization of a given process, or control of it, by a scheduling.

Brown and Yetter \[12\] interpreted the culf condition as preservation of more abstract notions of dynamics in the theory of \( C^* \)-algebras. Melliès \[40\] and Eberhart–Hirschowitz–Laouar \[19\] exploit similar viewpoints in game semantics.

1.6. Lamarche conjecture. Working on abstract notions of processes in computer science, at a time when presheaf semantics was gaining importance
to model concurrency (see for example Cattani–Winskel [16]), Lamarche (1996) made the conjecture that for any category $C$, the category $\text{Cat}^{\text{culf}}/C$ of culf maps over $C$ is a topos. It was soon discovered, though, that the conjecture is false in general, by famous counterexamples by Johnstone [31], Bunge–Niefield [14], and Bunge–Fiore [13]. (For the interesting history of this conjecture, see [35].)

The categories $C$ for which $\text{Cat}^{\text{culf}}/C$ is a topos are very special, expressing a certain local linear time evolution [13] (see Fiore [22] for further analysis). This includes the nonnegative reals, the monoid $\mathbb{N}$, and more generally free categories on a graph — these were the examples of importance to Lawvere [36] for dynamical systems. From the viewpoint of computer science the condition expresses a strict interleaving property (covering models such as labeled transition systems and synchronization trees [57]), but comes short in capturing more general notions of concurrency.

### 1.7. Kock–Spivak theorem

Decomposition spaces were first considered in connection with process algebra when Kock and Spivak [35] discovered that Lamarche’s conjecture is actually true in general, if just categories are replaced by decomposition spaces: they showed that for any discrete decomposition space $D$ (i.e. a simplicial set rather than a simplicial space), there is a natural equivalence of categories

$$\text{Decomp}_D / D \simeq \text{PrSh}(\text{SD} D).$$

This result shows that not only are culf maps natural to consider in connection with decomposition spaces, but that also decomposition spaces are a natural setting for culf maps: even if the base $D$ is actually a category, the nicely behaved class of culf maps into it is from decomposition spaces rather than from categories. From the viewpoint of processes, the lack of composability is something that occurs naturally in applications: Schultz and Spivak [46] observe that even if time intervals compose, processes over them do not necessarily compose, since constraints (called "contracts") may not extend over time.

### Contributions of this paper

One version of our main theorem is the following $\infty$-version of the Kock–Spivak result:

**Theorem D** (Theorem [9.3]). The $\infty$-category of decomposition spaces and culf maps is locally an $\infty$-topos. More precisely, for $X$ a decomposition space, we have an equivalence

$$\text{Decomp}_X / X \simeq \text{RFib}(\text{SD} X) \simeq \text{RFib}(\hat{\text{SD}} X) \simeq \text{PrSh}(\hat{\text{SD}} X).$$

Here $\hat{\text{(}}$ denotes the Rezk completion of a Segal space. We also will explain in Proposition [9.12] that if $X$ itself is Rezk complete as a decomposition space, then $\text{SD}(X)$ is Rezk complete as a Segal space, and we can write $\text{Decomp}_X / X \simeq \text{PrSh}(\text{SD} X)$ directly.

The substantial part of the result is the first equivalence in the display, which we establish as a special case of the following general theorem:
**Theorem C** (Theorem 7.1 & Theorem 8.12). *For any simplicial space \( X \), there is a natural equivalence*

\[
\text{Culf}(X) \cong \text{RFib}(\text{Sd} X).
\]

Theorem D follows from this since anything culf over a decomposition space is again a decomposition space, so for \( X \) a decomposition space, we have \( \text{Culf}(X) \cong \text{Decomp}/X \).

We give two proofs of Theorem C. The first uses the ideas of the proof of the Kock–Spivak theorem in the discrete case, but develops these ideas into more formal and conceptual arguments (as often required when upgrading a 1-categorical argument to \( \infty \)-categories). In particular we (prove and) exploit the comprehensive factorization system (final, right-fibration) in the \( \infty \)-category of simplicial spaces, extending the one for \( \infty \)-categories.

We show that Waldhausen’s last-vertex map \( \text{Nel}(X) \to X \) from the nerve of the \( \infty \)-category of elements back to a simplicial space \( X \) is final (Lemma 4.5). This was shown by Lurie and Cisinski for simplicial sets by combinatorial constructions. Here we give a conceptual high-level proof.

We then exploit the natural transformation \( \lambda : \text{Nel} \Rightarrow \text{Sd} \) first studied by Thomason [52], and show that it is cartesian on culf maps (Lemma 5.11).

With these preparations, we can exhibit an inverse to the displayed equivalence: it is given essentially by pullback along \( \lambda \) (modulo some identifications involving Nel).

The second proof is completely new, and involves the right adjoint to edgewise subdivision. It also involves a new factorization system of ambifinal maps and culf maps. This factorization system restricts to the stretched-culf factorization system on the \( \infty \)-category of intervals of [27], which in turn restricts to the active-inert factorization system on \( \Delta \). Indeed, the class of ambifinal maps is the saturation of the class of active maps between representables.

The second proof of Theorem C follows from several small lemmas of independent interest:

First we study the \( Q_! \dashv Q^* \) adjunction, and show that its unit is final on representables (Corollary 8.2) while its counit is ambifinal on representables (Proposition 8.3).

Moving on to the \( Q^* \dashv Q_* \) adjunction, we show that just as \( Q^* \) takes culf maps to right fibrations (Lemma 5.3), its right adjoint \( Q_* \) takes right fibrations to culf maps (Proposition 8.4). The key properties are now that the unit for the \( Q^* \dashv Q_* \) adjunction is cartesian on culf maps (Lemma 8.7) and that the counit is cartesian on right fibrations (Lemma 8.8).

After these preparations, the inverse to the equivalence displayed in Theorem C is shown to be given by first applying \( Q_* \) to get a culf map, and then pullback along the unit \( \eta' \) of the \( Q^* \dashv Q_* \) adjunction.

Lemma 5.3 together with the theorem of Bergner et al. [7] shows that edgewise subdivision is a key aspect of decomposition spaces and culf maps. The lemmas just quoted show that conversely, the classical notion of edgewise subdivision inevitably leads to culf maps and ambifinal maps, which are much more recent notions.
It should be noted that there is another convention for edgewise subdivision and twisted arrow category, which relates to the functor \(Q': \Delta \to \Delta\) given by \([n] \mapsto [n] \star [n]^{\text{op}}\) (instead of \([n] \mapsto [n]^{\text{op}} \star [n]\)). That convention is also widely used in the literature; see for example \([39]\). By taking opposites, we arrive at the following alternative version of Theorem D: For any simplicial space \(X\), there is a natural equivalence \(\text{Culf}(X) \cong \text{LFib}(\text{Sd}' X)\).

**Motivation and related work**

1.8. **∞-aspects in process algebra?** Viewing our Theorem D as an \(\infty\)-version of the Kock–Spivak theorem, it is natural to ask if it has any implications in process algebra. At the moment we don’t know of any, but rather than writing it off, we prefer to think that the theorem is a little bit ahead of its time, as category theory applied to computer science is still in the process of upgrading to \(\infty\)-categories. In the light of homotopy type theory \([53]\), where set-based semantics is routinely being replaced by semantics in \(\infty\)-groupoids, this upgrade seems inevitable.

Assuming this, Theorem D does have potential for applications. In process algebra there is usually a base to slice over, playing the role of time, or template for evolution, and the importance of being a topos — or even an \(\infty\)-topos — is the use of internal logic for them, as demonstrated by Schultz–Spivak \([46]\) and Schultz–Spivak–Vasilakopoulou \([47]\). Since every \(\infty\)-topos interprets homotopy type theory with the univalence axiom (by a recent breakthrough result of Shulman \([49]\)), this logic now becomes available as an internal language to reason in any slice. The notion of temporal type theory, introduced by Schultz and Spivak \([46]\) for the purpose of dynamical systems, is still formulated in ordinary sheaf semantics as in 1-toposes, but as the theory develops and constructive concerns impose themselves, it is to be expected that identity types and higher structures will creep in, thus necessitating \(\infty\)-sheaf semantics in the setting of \(\infty\)-toposes.

Even without reference to homotopy type theory, simplicial methods can be useful in process algebra and concurrency to overcome non-strict situations (as already occurs in combinatorics). Recently it was shown \([34]\) that processes of a Petri net rather easily assemble into a simplicial groupoid which is Segal, whereas it is very subtle to actually assemble them into a ordinary category.

While we do hope our theorem can find use in these contexts, our own motivations for it were very different:

1.9. **Free decomposition spaces.** Our motivations for Theorem D originate in combinatorics. In fact, the proof of Theorem D grew out of work on a more specific problem, whose solution is presented in the companion paper \([30]\), and which is now an application of the theorem.

For \(j: \Delta_{\text{inert}} \to \Delta\) the inclusion of the subcategory of inert maps in \(\Delta\), we show in \([30]\) that the simplicial space given by left Kan extension along \(j\) is always a decomposition space, and that the left Kan extension of any map is always culf. More precisely we establish
Theorem F (31). Left Kan extension along $j$ induces a canonical equivalence of $\infty$-categories

$$PrSh(\Delta_{\text{inert}}) \simeq \text{Decomp}_{/BN}.$$ 

Here $BN$ is the classifying space of the natural numbers, appearing here because $BN \simeq j_!(1)$. This theorem can be derived as a corollary of Theorem D of the present paper, via the neat identification

$$\Delta_{\text{inert}} \simeq Sd(BN).$$

Some more work is involved (in particular to identify the general equivalence with left Kan extension), and there is some machinery to set up. The proof of Theorem D and Theorem C grew out of an attempt at optimizing the original proof of Theorem F.

Decomposition spaces arising from left Kan extension along $j$ are called free. We show in [30] that virtually all comultiplications of deconcatenation type in combinatorial coalgebras arise as incidence coalgebras of free decomposition spaces. In particular the Hopf algebra of quasisymmetric functions arises in this way, and the universal map it receives (as terminal object in the category of combinatorial coalgebras equipped with a zeta function [11]) is given an interpretation in terms of free decomposition spaces.

For the theory of free decomposition spaces, Theorem D may be regarded as somewhat of an overkill, but it has a second motivation coming from combinatorial Hopf algebras:

1.10. Implications in conjunction with the Gálvez–Kock–Tonks conjecture. Theorem D acquires further interest in connection with the so-called Gálvez–Kock–Tonks conjecture, from [27]. Lawvere’s interval construction and the universal Hopf algebra of intervals [37] was shown to be the incidence bialgebra of a decomposition space $U$ of all intervals [27]. It was conjectured that $U$ enjoys the following universal property: for any decomposition space $X$, we have $\text{Map}(X, U) \simeq 1$. The mapping space is the space of all culf maps. This would explain in which sense Lawvere’s Hopf algebra is universal. This is almost like saying that $U$ is a terminal object in $\text{Decomp}$, but size issues prevent this interpretation. However, the whole construction and the conjecture can be restricted to the case of Möbius decomposition spaces [26], certain decomposition spaces satisfying a finiteness condition ensuring that the general Möbius inversion principle admits a homotopy cardinality. Most decomposition spaces from combinatorics are Möbius.

The decomposition space of Möbius intervals $U^{\text{Möbius}}$ is small, so as to constitute a genuine terminal object in $\text{Decomp}^{\text{Möbius}}$, according to the conjecture. This is where Theorem D comes in: if a decomposition space $X$ is itself Möbius, then everything culf over $X$ is Möbius again, so that

$$\text{Decomp}^{\text{Möbius}}_{/X} \simeq \text{Decomp}_{/X}.$$ 

By Theorem D, the latter slice is an $\infty$-topos, and if $X$ is taken to be $U^{\text{Möbius}}$, and we assume the conjecture is true, then

$$\text{Decomp}^{\text{Möbius}} \simeq \text{Decomp}_{/U^{\text{Möbius}}} \simeq PrSh(Sd(U^{\text{Möbius}})),$$
so that $\text{Decomp}^{\text{M"obius}}$ itself will be an $\infty$-topos!

The current status of the conjecture is the following (see Forero [23] for a detailed exposition of the conjecture’s history and motivation). The work of Lawvere (suitably upgraded to the present context) shows that $\text{Map}(X,U)$ is inhabited: it contains the interval construction $f \mapsto I(f)$ from [36]. Gálvez–Kock-Tonks [27] proved that it is also connected: every culf map $X \to U$ is homotopy equivalent to $I$. The finer property of being contractible is the full homotopy uniqueness statement, that not only is every map equivalent to $I$: it is so uniquely (in a coherent homotopy sense). Forero [23] has proved the conjecture in the discrete case (where $X$ is a simplicial set). In this case there is a shift in categorical dimension: the universal $U$ for discrete decomposition spaces is not itself discrete but rather a simplicial groupoid. This shift in categorical dimension is unavoidable in the truncated situation, but goes away in the untruncated situation.

The prospective of a universal decomposition space (which cannot exist in truncated settings) was one of the motivations for Gálvez, Kock, and Tonks to develop the theory of decomposition spaces in the $\infty$-setting (see the introduction of [25]), although most examples in combinatorics are 0- or 1-truncated [24].

2 Comprehensive Factorization

2.1. Conventions and setting. In this paper we work with $\infty$-categories in a model-independent fashion. We assume a (large) $\infty$-category $\text{Cat}_{\infty}$ of all small $\infty$-categories, with a full sub-$\infty$-category $S$ of $\infty$-groupoids, which we call spaces. We are in particular concerned with simplicial spaces within the given model of $\infty$-categories: by definition $sS$ is the functor $\infty$-category $\text{Fun}(\Delta^{\text{op}}, S)$. There is a fully faithful nerve functor

$$N: \text{Cat}_{\infty} \longrightarrow sS$$

$$\mathcal{C} \longmapsto \text{Map}(\mathcal{C},-)$$

whose essential image is the subcategory of Rezk complete Segal spaces [33], which can therefore be considered an internal model of $\infty$-categories within the given model.

As a specific choice, one can take $\infty$-category to mean quasi-category in the sense of Joyal [32] (simply called $\infty$-categories by Lurie [38]). For our emphasis on synthetic reasoning, we recommend Riehl–Verity [45] as a background reference for $\infty$-categories, and also Ayala–Francis [4] for more specific results on fibrations, and Anel–Biedermann–Finster–Joyal [2] for factorization systems.

All concepts in this paper are the relevant equivalence-invariant ones, which are the only versions which make sense in this context. For instance, diagrams only commute up to equivalence, pullbacks are homotopy pullbacks, “unique” lifts means that the space of lifts is contractible, and so on.

2.2. Right fibrations of simplicial spaces. A simplicial map $f: Y \to X$ is called a right fibration when it is right orthogonal to all terminal-object-preserving
maps $\ell: \Delta^m \to \Delta^n$; or equivalently, considered as a natural transformation, $f$ is cartesian on all last-point-preserving maps $\ell: [m] \to [n]$. The diagram on the left expresses the right orthogonality; the diagram on the right expresses the equivalent cartesian condition:

$$
\begin{array}{ccc}
\Delta^m & \xrightarrow{\ell} & Y \\
\downarrow \cong & & \downarrow f \\
\Delta^n & \xrightarrow{\ell} & X
\end{array}
\quad
\begin{array}{ccc}
Y_n & \xrightarrow{\ell^*} & Y_m \\
\downarrow f_n & & \downarrow f_m \\
X_n & \xrightarrow{\ell^*} & X_m
\end{array}
$$

The following three lemmas are exercises using pullbacks.

**Lemma 2.3.** A simplicial map is a right fibration if and only if it is cartesian on each last-point inclusion $[0] \to [n]$.

**Lemma 2.4.** For a right fibration $Y \to X$, if $X$ is a Segal space (resp. a Rezk complete Segal space) then also $Y$ is a Segal space (resp. a Rezk complete Segal space).

**Lemma 2.5.** A simplicial map between Segal spaces $Y \to X$ is a right fibration if and only if it is cartesian on the coface map $d^0: [0] \to [1]$; that is, the square

$$
\begin{array}{ccc}
Y_0 & \xleftarrow{d_0} & Y_1 \\
\downarrow & & \downarrow \\
X_0 & \xleftarrow{d_0} & X_1
\end{array}
$$

is a pullback.

Thus our definition of right fibration recovers the usual one for Segal spaces from [10]. There is an evident dual notion of left fibration of simplicial spaces using initial-object-preserving maps between representables; restricted to Segal spaces one recovers the notion of left fibration from [54 2.1.1].

**2.6. Décalage.** Recall that the upper déccalage $\text{Dec}_\top X$ of a simplicial space $X$ is obtained by deleting $X_0$ as well as the top face and degeneracy maps, and shifting all spaces one degree down: $(\text{Dec}_\top X)_k = X_{k+1}$. More formally, as we shall exploit, let $\Delta^\top$ denote the category of ordinals with a top element, and top-preserving monotone maps, with forgetful functor $u: \Delta^\top \to \Delta$ and left adjoint $i: \Delta \to \Delta^\top$. (One can think of a $(\Delta^\top)^{op}$-diagram as a simplicial object with missing top face maps.) The upper déccalage comonad on $s\mathcal{S}$ can now be described as $\text{Dec}_\top = i^* \circ u^*$. Similarly, there is a lower déccalage $\text{Dec}_\bot X$ which deletes the bottom face and degeneracy maps.

**2.7. Slices.** The notion of slice makes sense for general simplicial spaces $X$ (not just for Segal spaces): for $x \in X_0$, the slice $X_{/x}$ is defined as the pullback

$$
\begin{array}{ccc}
X_{/x} & \xrightarrow{d_0} & \text{Dec}_\top X \\
\downarrow & & \downarrow \\
1 & \xrightarrow{r_x} & X_0
\end{array}
$$
Here the simplicial spaces in the bottom row are constant, and \( d_0 : \text{Dec}_\top X \to X_0 \) denotes the canonical augmentation sending an \((n+1)\)-simplex in \( X \) to its last vertex. Note also that \( \text{Dec}_\top(X) \) (and hence \( X_{/x} \)) comes with a canonical splitting, given by the original top degeneracy maps, making it into a \((\Delta^I)^{\text{op}}\)-diagram. More precisely, the pullback square above is \( i^* \) applied to the following pullback square of \( \Delta^I \)-presheaves

\[
\begin{array}{ccc}
(u^*X)_{/x} & \longrightarrow & u^*X \\
\downarrow & & \downarrow \\
1 & \longrightarrow & X_0
\end{array}
\]

where the bottom row consists of constant \( \Delta^I \)-presheaves and the right map is the augmentation. Here, \( u^*X \) simply deletes the top face maps of \( X \). (We can more generally take slices of an arbitrary \( \Delta^I \)-presheaf, and each such presheaf is the sum over all of its slices.)

**2.8. Warning.** The canonical projection \( X_{/x} \to X \) is not in general a right fibration. (It is a right fibration when \( X \) is Segal, of course, and it is culf (cf. §3 when \( X \) is a decomposition space [25, Proposition 4.9].)

**Lemma 2.9.** A simplicial map \( p : Y \to X \) is a right fibration if and only if for every \( y \in Y_0 \), the induced simplicial map \( Y_{/y} \to X_{/py} \) is a (levelwise) equivalence.

**Proof.** To say that \( p \) is a right fibration means that for all \( n \geq 0 \) the square

\[
\begin{array}{ccc}
Y_0 & \longrightarrow & Y_{n+1} \\
\downarrow & & \downarrow \\
X_0 & \longrightarrow & X_{n+1}
\end{array}
\]

is a pullback (Lemma 2.3). This in turn is equivalent to saying that the induced map on fibers is an equivalence for every \( y \in Y_0 \). But this map is precisely \((Y_{/y})_n \to (X_{/py})_n\). \( \Box \)

**2.10. Remark.** As a variation of the lemma, we have also that \( p : Y \to X \) is a right fibration if and only if for each \( y \in Y_0 \) the induced map \((u^*Y)_{/y} \to (u^*X)_{/py}\) is a levelwise equivalence of \( \Delta^I \)-presheaves. We shall use this in the proof of Lemma 2.12.

**2.11. Final maps.** A simplicial map is called final if it is left orthogonal to every right fibration. Note that every terminal-object-preserving map between representables \( \ell : \Delta^m \to \Delta^n \) is final.

**Lemma 2.12.** If a simplicial map \( f : B \to A \) between simplicial spaces with a terminal vertex preserves those terminal vertices, then it is final.
**Proof.** Since $A$ and $B$ have terminal vertices and $f$ preserves them, we have the following commutative square, where $b \in B_0$ and $a \in A_0$ are terminal objects.

\[
\begin{array}{ccc}
B/b & \xrightarrow{\sim} & B \\
\downarrow & & \downarrow^f \\
A/a & \xrightarrow{\sim} & A
\end{array}
\]

The left-hand map is in the image of the left adjoint $i^*$ of the décalage adjunction (see [2.6])

\[
i^* : \text{Fun}((\Delta^t)^{op}, S) \xhookrightarrow{\sim} \text{Fun}(\Delta^{op}, S) : u^*,
\]

following [2.7]. It thus suffices to check there is a contractible space of lifts for each square of simplicial spaces on the left below, where $p : Y \to X$ is a right fibration.

\[
\begin{array}{ccc}
i^*((u^*B)/b) & \to & Y \\
\downarrow & & \downarrow^p \\
i^*((u^*A)/a) & \to & X
\end{array} \quad \begin{array}{ccc}
(u^*B)/b & \to & u^*(Y) \\
\downarrow & & \downarrow^{u^*(p)} \\
(u^*A)/a & \to & u^*(X)
\end{array}
\]

By adjunction, this is equivalent to there being a unique lift for the diagram of $\Delta^t$-presheaves on the right. In the next paragraph we explain why this holds.

Since $\Delta^t$ has an initial object, by a standard lemma (see, for instance, [27, 1.15]) the $\infty$-category $\text{PrSh}((\Delta^t)^t) = \text{Fun}((\Delta^t)^{op}, S)$ has a factorization system where the left class consists of those natural transformations which are equivalences at the initial object of $\Delta^t$, and where the right class consists of the cartesian natural transformations. Now $(u^*B)/b \to (u^*A)/a$ is in the left class, as both of these presheaves have a contractible space as augmentation object (that is, the value at the initial object of $\Delta^t$), and $u^*(p)$ is in the right class: it is cartesian because $p$ is a right fibration. Thus there is a contractible space of lifts in the square above-right, as required.

**Theorem 2.13.** The $\infty$-category of simplicial spaces admits the comprehensive factorization system, whose left class consists of the final maps, and whose right class consists of the right fibrations.

**Proof.** Let $\Sigma$ be the set of last-vertex-preserving maps between representables (alternatively: the set of bottom coface maps, or the set of last-vertex inclusions $\Delta^0 \to \Delta^n$) and let $\Sigma^+$ be the saturated class generated by $\Sigma$. Since the $\infty$-category of simplicial spaces is presentable, it follows from [2, Proposition 3.1.18] that $(\Sigma, \Sigma^+)$ is a factorization system. By definition, $\Sigma^+$ is the class of right fibrations. Since this is a factorization system, $^+(\Sigma^+) = \Sigma$ by [38, Proposition 5.2.8.11] or [2, Lemma 3.1.9], hence $\Sigma$ is the class of final maps.

**2.14. Remarks.** The comprehensive factorization system for 1-categories is classical, due to Street and Walters [51]. For $\infty$-categories, the result appears
in Joyal [32, 169–173] and a proof can be found in Ayala–Francis [4, §6.3].
The straightforward generalization of the notion of right fibration to general
simplicial spaces may have been considered first in [25, 4.12]. The comprehensive
factorization system for simplicial spaces is also implicit in Rasekh [43, 5.29–
5.33], in a model-categorical setting, where the final maps are defined as certain
contravariant equivalences.

2.15. Comprehensive factorization for ∞-categories. The next proposition
will justify the usage of the name “final” for these simplicial maps. Recall
that a final functor $f : C \to B$ between ∞-categories is a functor so that for any
other functor $B \to Z$, the map colim($C \to B \to Z$) → colim($B \to Z$) exists and
is an equivalence if either colimit exists [1, Definition 6.1.1]. This is the left class
in a comprehensive factorization system on $\text{Cat}_\infty$, whose right class consists of the right fibrations, those functors $\pi : E \to C$ whose space of lifts for any square
as below is contractible.

\[
\begin{array}{ccc}
\Delta^0 & \longrightarrow & E \\
\downarrow & & \downarrow \pi \\
\Delta^1 & \longrightarrow & C
\end{array}
\]

Proposition 2.16. The comprehensive factorization system on simplicial spaces
restricts to the comprehensive factorization system on ∞-categories.

Proof. Temporarily write $(\mathcal{L}, \mathcal{R})$ for the comprehensive factorization system on
simplicial spaces, where $\mathcal{L}$ is the class of final maps and $\mathcal{R}$ is the class of right
fibrations. Let $N : \text{Cat}_\infty \to sS$ be the inclusion of ∞-categories into simplicial
spaces, where we may think of the former as the Rezk complete Segal spaces. By
Lemma 2.5 we have that $\mathcal{R} \cap \text{Cat}_\infty$ is the usual class of right fibrations between
∞-categories, as in [1]. Denoting the usual final-rightfibration factorization
system on $\text{Cat}_\infty$ by $(\mathcal{L}', \mathcal{R}')$, we have

\[
\mathcal{L} \cap \text{Cat}_\infty \subseteq \mathcal{L}'(\mathcal{R} \cap \text{Cat}_\infty) = \mathcal{R}' = \mathcal{L}'
\]

since $N$ is fully faithful. We wish to show that this inclusion is an equivalence.

Suppose $f : C \to B$ is a final functor between ∞-categories, that is, a mor-
phism in $\mathcal{L}'$. Form the factorization of simplicial maps below left

\[
\begin{array}{ccc}
X & \to & C \\
\ell \uparrow & & \downarrow f \\
Nf & \to & NB
\end{array}
\]

with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$. By Lemma 2.4, $X$ is a Rezk complete Segal space,
$X \simeq NE$ for some $E \in \text{Cat}_\infty$, and we may regard the triangle above left as the
nerve of the triangle above right. Since $\ell$ is a final functor by [2], it follows that
$\tilde{r}$ is in $\mathcal{L}' \cap \mathcal{R}'$, hence is an equivalence. Thus $r$ is also an equivalence, and we
conclude that $Nf : NE \to NB$ is in $\mathcal{L}$. Thus $\mathcal{L}' = \mathcal{L} \cap \text{Cat}_\infty$.

\[
\square
\]
2.17. Categories of right fibrations. For a simplicial space $X$, denote by $RFib(X)$ the full subcategory of $sS/X$ spanned by the right fibrations. By the left cancellation property satisfied by right classes, the morphisms in $RFib(X)$ are again right fibrations, so $RFib(X)$ can also be described as the slice over $X$ of the $\infty$-category of simplicial spaces and right fibrations. In light of Lemma 2.4 and Proposition 2.16, if $C$ is an $\infty$-category and $RFib(C) \subset Cat_{\infty/C}$ is the usual $\infty$-category of right fibrations over $C$, then the nerve functor induces an equivalence $RFib(C) \simeq RFib(\mathbf{N}C)$.

Essentially equivalent to the existence of the factorization system is the fact that the inclusion functor $RFib(X) \rightarrow sS/X$ has a left adjoint (reflection): it sends an arbitrary simplicial map $Y \rightarrow X$ to the right-fibration part of its comprehensive factorization. The left part is the unit of the adjunction. (This reflection is the main ingredient in the construction of the factorization system (cf. proof of [38, 5.5.5.7]; see in particular [38, 5.5.4.15]).)

2.18. Base change. For a simplicial map $F: X' \rightarrow X$, there is a canonical base-change functor $F^*: RFib(X) \rightarrow RFib(X')$ given by pullback along $F$, sending $p: Y \rightarrow X$ to $p'$ as in the diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{j} & Y \\
\downarrow{p'} & & \downarrow{p} \\
X' & \xrightarrow{F} & X.
\end{array}
\]

The following is a special case of [2, Proposition 3.1.22].

2.19. Cobase change. The base-change functor $F^*$ has a left adjoint cobase-change functor $F_!: RFib(X') \rightarrow RFib(X)$ given by pullback along $F$, then taking factorization into final map followed by right fibration, and finally returning the right fibration, as $q' \mapsto q$ in the diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{\text{final}} & Y \\
\downarrow{q'} & & \downarrow{\text{f.r.fib.}} \\
X' & \xrightarrow{F} & X.
\end{array}
\]

(For $\infty$-categories, see also [3, Remark 6.1.10] and [17, 6.1.14].)

The unit (which we shall not need) is given by the universal property of the pullback. The counit $F^*F(p) \rightarrow p$ is given by the universal property of the final-rightfibration factorization system, as exemplified in Lemma 4.8.

3 Culf maps and ambifinal maps

3.1. The active-inert factorization system. The category $\Delta$ has an active-inert factorization system: the active maps, written $g: [k] \rightarrow [n]$, are those that preserve end-points, $g(0) = 0$ and $g(k) = n$; the inert maps, written $f: [m] \rightarrow [n]$,
are those that are distance preserving, \( f(i+1) = f(i) + 1 \) for \( 0 \leq i \leq m - 1 \). The active maps are generated by the codegeneracy maps and the inner coface maps; the inert maps are generated by the outer coface maps \( d^k \) and \( d^T \). This orthogonal factorization system is an instance of the important general notion of generic-free factorization system of Weber [56] who referred to the two classes as generic and free. The active-inert terminology is due to Lurie [39].

### 3.2. Culf maps.
Recall (from [25, §4]) that a simplicial map \( p : Y \to X \) is culf when it is right orthogonal to every active map \( \Delta^k \to \Delta^n \), or equivalently, when it is cartesian on active maps. The picture on the left expresses the right orthogonality; the picture on the right expresses the equivalent cartesian condition:

\[
\begin{array}{ccc}
\Delta^k & \to & Y \\
\downarrow \mathbf{3}^k & & \downarrow p \\
\Delta^n & \to & X
\end{array}
\quad \quad \quad
\begin{array}{ccc}
Y_n & \xrightarrow{g^*} & Y_k \\
\downarrow p_n & & \downarrow p_m \\
X_n & \xrightarrow{g^*} & X_k
\end{array}
\]

Note that every left or right fibration is culf.

#### 3.3. Remark.
Culf stands for “conservative” and “unique lifting of factorizations” (cf. Lawvere [38]), as the notion recovers these conditions in the case of strict nerves of ordinary categories. In this case the notion of culf functor is also the same as discrete Conduché fibration [31]. In the case of \( \infty \)-categories, the culf maps are the same thing as the conservative exponentiable fibrations studied by Ayala and Francis [4].

**Lemma 3.4.** [25, Lemma 4.1] A simplicial map is culf if and only if it is cartesian on each active map of the form \([1] \to [n]\).

A simplicial map is culf if and only if it is cartesian on degeneracy maps and inner face maps. The next lemma gives that only the inner face maps are necessary.

**Lemma 3.5.** To check that a general simplicial map \( F : Y \to X \) is culf, it is enough to check that it is cartesian on active maps of the form \([1] \to [n]\) for \( n \geq 1 \); it is then automatically cartesian also on \([1] \to [0]\). (In other words, ulf implies culf.)

**Proof.** By Lemma 3.4 it only remains to check that \( F \) is cartesian on the codegeneracy map \([0] \leftarrow [1]\). This is the leftmost (=rightmost) square in the commutative \((\Delta^1 \times \Delta^1 \times \Delta^2)\)-diagram

\[
\begin{array}{cccc}
Y_0 & \xrightarrow{s_0} & Y_1 & \xrightarrow{d_0} & Y_0 \\
\downarrow s_0 & & \downarrow s_1 & & \downarrow s_0 \\
Y_1 & \xrightarrow{s_0} & Y_2 & \xrightarrow{d_0} & Y_1 \\
\downarrow s_0 & & \downarrow s_1 & & \downarrow s_0 \\
X_0 & \xrightarrow{s_0} & X_1 & \xrightarrow{d_0} & X_0 \\
\downarrow s_0 & & \downarrow s_1 & & \downarrow s_0 \\
X_1 & \xrightarrow{s_0} & X_2 & \xrightarrow{d_0} & X_1
\end{array}
\]
which exhibits the square as a retract of the middle square. The middle square in turn is already known to be a pullback, since \( s_1 \) is a section to the active face map \( d_1 \). Since pullbacks are stable under retracts, it follows that also the leftmost square is a pullback.

\[ \]

\[\]

3.6. Remark. Lawvere and Menni [37, Lemma 4.4] proved this for the case of 1-categories in which all identities are indecomposable. Their proof works more generally for all decomposition spaces that are split [26 §5]. Gálvez, Kock, and Tonks [25 Proposition 4.2] proved the result for general decomposition spaces, but their prism-lemma proof does not work for general simplicial spaces. The above retract argument giving the general case is directly inspired by the proof by Feller et al. [21] that all 2-Segal spaces are unital. The result in full generality was found independently by Barkan and Steinebrunner (see [5]).

3.7. Intervals. An interval is a simplicial space with an initial and a terminal object. To every 1-simplex \( f: a \to b \) in a simplicial space \( X \) there is associated an interval \( I(f) := (X/b)_f \) (with initial object \( s_0(f) \) and terminal object \( s_1(f) \)). This simplicial space can also be described as \( \text{Dec}_\perp \text{Dec}_\top (X) \times_{X_1} \{f\} \). Usually (see [27]), this notion is considered only when \( X \) is a decomposition space, in which case \( I(f) \) is always a Segal space. Here we consider the more general case only to be able to state the following result (which goes back to Lawvere [36] in the case where \( X \) is the strict nerve of a 1-category).

3.8. Warning. For general simplicial spaces \( X \), the canonical projection \( I(f) \to X \) (given on objects by taking a 2-simplex with long edge \( f \) to its middle vertex) is not always culf. (It is culf when \( X \) is a decomposition space [27, §3].)

Lemma 3.9. A simplicial map \( p: Y \to X \) is culf if and only if for every \( f \in Y_1 \), the corresponding map of intervals \( I(f) \to I(pf) \) is a (levelwise) equivalence.

Proof. By Lemma 3.5, to say that \( p \) is culf means that for all \( n \geq 0 \) the square

\[
\begin{array}{c}
Y_1 \\
\downarrow \\
X_1
\end{array}
\quad
\begin{array}{c}
Y_{n+2} \\
\downarrow \\
X_{n+2}
\end{array}
\quad
\begin{array}{c}
\text{long} \\
\text{long}
\end{array}
\]

is a pullback. (The horizontal maps return the long edge of a simplex.) This in turn means that for each \( f: a \to b \) in \( Y_1 \) the induced map on fibers \( (Y_{n+2})_f \to (X_{n+2})_{pf} \) is an equivalence. But this map is precisely \( ((Y/b)_f)_n \to ((X/pb)_{pf})_n \), which is the \( n \)-component of the map on intervals \( I(f) \to I(pf) \).

3.10. Ambifinal maps. A simplicial map is called ambifinal if it is left orthogonal to every culf map.

In close analogy with Lemma 2.12 we have the following result, which we treat only briefly as it is not necessary in what follows.
Lemma 3.11. If a simplicial map between simplicial spaces with both an initial and a terminal vertex preserves those initial and terminal vertices, then it is ambifinal.

Proof sketch. The proof is analogous to that of Lemma 2.12 but using instead the adjunction

\[ i^* : \text{Fun}(\Delta^{t,b}_{\text{op}}, S) \cong \text{Fun}(\Delta^{op}, S) : u^* \]

from [27 §§2-3], whose induced comonad \( i^*u^* \) is double décalage (both upper and lower). Here, \( \Delta^{t,b} \) is the category of ordinals with distinct top and bottom elements, and monotone maps which preserve these.

Theorem 3.12. The classes of ambifinal maps and culf maps form a factorization system on \( sS \).

Proof. The proof is completely analogous to the proof of Theorem 2.13, but with generating set \( \Sigma \) of the left class now being the set of active maps \( \Delta^1 \to \Delta^n \) between representables; then the saturated class is the class of ambifinal maps.

3.13. The generating set \( \Sigma \). In the proof of the preceding theorem, one can choose many possible alternate generating sets \( \Sigma \) for the left class. For instance, from Lemma 3.4 we know that we could take only the active maps \( \Delta^1 \to \Delta^n \). Instead, one could take active maps \( \Delta^1 \to \Delta^{2n+1} \) landing in odd-dimensional simplices. This is because the active map \( \Delta^1 \to \Delta^{2n+1} \) into an even-dimensional simplex, for \( n > 0 \), is a retract of \( \Delta^1 \to \Delta^{2n+1} \). Lemma 3.5 takes care of the remaining map \( \Delta^n \to \Delta^0 \). In fact, one could take the active maps \( \Delta^1 \to \Delta^{n_k} \) for any infinite collection of non-negative integers \( n_k \).

4 The last-vertex map

4.1. Categories of elements. Let \( X : \mathcal{C}^{\text{op}} \to S \) be a presheaf. The category of elements of \( X \) is by definition \( \text{el}(X) := \mathcal{C} \downarrow X \), the domain of the right fibration corresponding to \( X \) under the basic straightening-unstraightening equivalence of \( \infty \)-categories \( \text{Rfib}(\mathcal{C}) \simeq \text{PrSh}(\mathcal{C}) \) (due to Lurie [35]; see [4, Thm 3.4.6] for a model-independent statement).

For a simplicial space \( X : \Delta^{\text{op}} \to S \), we shall be concerned with the nerve of its category of elements, written \( \text{Nel}(X) \). It is thus a simplicial space again. The \( k \)-simplicies of \( \text{Nel}(X) \) are configurations

\[ \Delta^{n_0} \to \Delta^{n_1} \to \cdots \to \Delta^{n_k} \to X. \]

4.2. The last-vertex map \( \xi : \text{Nel}(X) \to X \). For any simplicial space \( X \), the last-vertex map

\[ \xi_X : \text{Nel}(X) \to X \]
is given on objects by sending an $n$-simplex $\sigma: \Delta^n \to X$ to its last vertex

$$\Delta^0 \xrightarrow{\text{last}} \Delta^n \xrightarrow{\sigma} X.$$ The action of $\xi$ on higher simplices is given, with reference to the general combinatorial lower-segments construction below, by sending

$$f: \Delta^{n_0} \to \cdots \Delta^{n_k} \to X \in (\text{Nel} X)_k$$

to

$$\Delta^k \xrightarrow{\beta(f_k)} \Delta^{n_k} \to X \in X_k.$$ In a moment we will explain why this is natural in simplicial operators, that is, why $\xi_X$ is a map of simplicial spaces. The construction is natural in simplicial spaces $X$, defining altogether a natural transformation

$$\xi: \text{Nel} \Rightarrow \text{id}.$$ 4.3. Lower-segments construction. (see [35, Lemma 3.2]). For any $k$, let $f \in (\text{N}(\Delta))_k$ denote a sequence of maps $[n_0] \xrightarrow{f_1} [n_1] \xrightarrow{f_2} \cdots \xrightarrow{f_k} [n_k]$ in $\Delta$. Then there is a unique commutative diagram $B(f)$ of the form

$$
\begin{array}{cccccc}
[0] & \xrightarrow{d^T} & [1] & \cdots & \xrightarrow{d^T} & [k] \\
& \downarrow & & & \downarrow & \\
n_0 & \xrightarrow{f_i} & n_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_k} n_k
\end{array}
$$

for which all the vertical maps are last-point-preserving, and all the maps in the top row are $d^T$. Indeed, building the diagram from the left to the right, in each step it remains to define the next $\beta$ map on the last vertex, and here the value is determined by the requirement that it be last-point-preserving. 

The resulting map $\beta(f)$ can be described explicitly as

$$
\begin{array}{c}
[k] \mapsto [n_k] \\
i \mapsto f_k \cdots f_{i+1}(n_i),
\end{array}
$$

where $n_i$ denotes the last element of $[n_i]$. 

The lower-segments construction and the last-vertex map are natural in maps in $\Delta$: as an illustration we do the case of a coface map $d^i: [k-1] \to [k]$. The effect on the chain $f$ is to omit $[n_i]$ and compose the adjacent maps to $f_{i+1} \circ f_i$. The effect on the resulting right-most map $\beta(f)$ is therefore to omit the $i$th vertex, that is to precompose with $d^i: [k-1] \to [k]$, in turn precisely the effect of the face operator $d_i$ for the simplicial space $X$. Note that in this calculation we allow $i = k$, in which case $f_{i+1}$ must be interpreted as the $n_k$-simplex $\sigma: \Delta^{n_k} \to X$. 

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Lemma 4.4. The natural transformation $\xi : \text{Nel} \Rightarrow \text{id}$ is cartesian on right fibrations. That is, for $p : Y \to X$ a right fibration between simplicial spaces, the naturality square

\[
\begin{array}{ccc}
\text{Nel}(Y) & \xrightarrow{\xi_Y} & Y \\
p' \downarrow & & \downarrow p \\
\text{Nel}(X) & \xrightarrow{\xi_X} & X
\end{array}
\]

is a pullback.

Proof. We check it in each simplicial degree separately. In simplicial degree $k$ we have

\[
\text{Nel}(Y)_k = \sum_{[n_0] \to \cdots \to [n_k]} Y_{n_k}.
\]

If the chain of maps $[n_0] \to \cdots \to [n_k]$ is called $f$, then the lower-segments construction \[\Box\] gives us a last-point-preserving map $\beta_f : [k] \to [n_k]$, and $\xi_Y$ is given in the $f$-summand by

\[
Y_{n_k} \xrightarrow{\beta_f^*} Y_k.
\]

Altogether, the square we want to show is a pullback (in degree $k$) is identified with

\[
\begin{array}{ccc}
\sum_f Y_{n_k} & \xrightarrow{\beta_f^*} & Y_k \\
\downarrow & & \downarrow p_k \\
\sum_f X_{n_k} & \xrightarrow{\beta_f^*} & X_k
\end{array}
\]

(with $\beta_f$ varying with $f$, so generally different in each summand). The left vertical map respects $f$ (since it is a morphism of right fibrations over $\Delta$), so the pullback property can be established separately for each summand. For a fixed chain $f$, the square is thus

\[
\begin{array}{ccc}
Y_{n_k} & \xrightarrow{\beta_f^*} & Y_k \\
\downarrow & & \downarrow p_k \\
X_{n_k} & \xrightarrow{\beta_f^*} & X_k
\end{array}
\]

and this square is a pullback since $\beta_f$ is last-point-preserving and $p$ is a right fibration.

\[
\square
\]

Lemma 4.5. For any simplicial space $A$, the last-vertex map $\xi : \text{Nel}(A) \to A$ is final.
Proof. To show that $\xi_A : \text{Nel}(A) \to A$ is final, we need to show that there is a contractible space of lifts for the square

$$
\begin{array}{ccc}
\text{Nel}(A) & \xrightarrow{f} & Y \\
\downarrow{\xi_A} & & \downarrow{p} \\
A & \xrightarrow{g} & X
\end{array}
$$

for any right fibration $p : Y \to X$ and arbitrary simplicial maps $f$ and $g$.

By Lemma 4.4 we have the pullback square

$$
\begin{array}{ccc}
\text{Nel}(Y) & \xrightarrow{\xi_Y} & Y \\
\downarrow{p'} & & \downarrow{p} \\
\text{Nel}(X) & \xrightarrow{\xi_X} & X.
\end{array}
$$

This pullback square is the outer square of the following diagram, where the two distorted squares are the outer square of the following diagram, where the two distorted squares are (3) and naturality of $\xi$ with respect to $g$.

$$
\begin{array}{ccc}
\text{Nel}(Y) & \xrightarrow{\xi_Y} & Y \\
\downarrow{p'} & & \downarrow{p} \\
\text{Nel}(A) & \xrightarrow{\xi_A} & A \\
\downarrow{g'} & & \downarrow{g} \\
\text{Nel}(X) & \xrightarrow{\xi_X} & X
\end{array}
$$

The universal property of the pullback now gives the dashed arrow $h'$.

All three small triangles in the following diagram commute, hence the outer triangle commutes as well.

$$
\begin{array}{ccc}
\text{Nel}(A) & \xrightarrow{h'} & \text{Nel}(Y) \\
\downarrow{g'} & & \downarrow{p'} \\
\text{Nel}(X) & \xrightarrow{\xi_X} & \text{Nel}(X)
\end{array}
$$

Since the nerve functor is fully faithful, this implies that $h'$ is the nerve of a right fibration $\text{el}(A) \to \text{el}(Y)$ over $\Delta$, hence $h' = \text{Nel}(h)$ for a unique simplicial map $h : A \to Y$, as in the solid triangle

$$
\begin{array}{ccc}
\text{Nel}(A) & \xrightarrow{f} & Y \\
\downarrow{\xi_A} & & \downarrow{p} \\
A & \xrightarrow{g} & X
\end{array}
$$
Meanwhile, the upper triangle is the outer triangle in the commutative diagram

\[
\begin{array}{ccc}
\text{Nel}(A) & \xrightarrow{h'} \text{Nel}(Y) & \xrightarrow{\xi_Y} Y \\
\downarrow{\xi_A} & & \downarrow{h} \\
A & & \\
\end{array}
\]

so \(h\) is a lift in the square.

4.6. Alternate interpretation of proof. We want to prove that \(\text{Map}(A, Y) \to \text{Map}(\xi_A, p)\) is an equivalence. But the statement of Lemma 4.4 is that

\[
\text{Map}(\text{Nel}(A), \text{Nel}(Y)) \to \text{Map}(\text{Nel}(A), Y) \times_{\text{Map}(\text{Nel}(A), X)} \text{Map}(\text{Nel}(A), \text{Nel}(X))
\]

is an equivalence. Now we have

\[
\text{Map}(\xi_A, p) = \text{Map}(\text{Nel}(A), Y) \times_{\text{Map}(\text{Nel}(A), X)} \text{Map}(A, X),
\]

and \(\text{Map}(A, X) \subset \text{Map}(\text{Nel}(A), \text{Nel}(X))\) consists of those maps living over \(N\Delta\). So this proof is about identifying the two subspaces on the top row:

\[
\begin{array}{ccc}
\text{Map}(A, Y) & \xrightarrow{i} & \text{Map}(\text{Nel}(A), Y) \times_{\text{Map}(\text{Nel}(A), X)} \text{Map}(A, X) \\
\downarrow{j} & & \downarrow{j} \\
\text{Map}(\text{Nel}(A), \text{Nel}(Y)) & \xrightarrow{\sim} & \text{Map}(\text{Nel}(A), Y) \times_{\text{Map}(\text{Nel}(A), X)} \text{Map}(\text{Nel}(A), \text{Nel}(X))
\end{array}
\]

But we’re exhibiting a map in the opposite direction on the top by composing \(j\) with a homotopy inverse on the bottom, and then showing that it lands in \(\text{Map}(A, Y)\) (or factors through \(i\)). In that case the exhibited map is automatically a homotopy inverse since the downward arrows are monomorphisms of \(\infty\)-groupoids.

4.7. Remark. The last-vertex map \(\xi : \text{Nel}(A) \to A\) already has some history. It was used by Waldhausen [55], and more recently exploited by Lurie [38, 4.2.3.14] who also established that it is final, in the special case of simplicial sets, by a rather involved combinatorial construction (see also [17, 7.3.9]).

Lemma 4.8. For \(\xi : \text{Nel}(X) \to X\) the last-vertex map of 4.2, the counit \(\xi_! \xi^* \Rightarrow \text{Id}\) is an equivalence. In particular \(\xi^*\) is fully faithful.

Proof. For any right fibration \(p : Y \to X\), the pullback diagram

\[
\begin{array}{ccc}
\text{Nel}(Y) & \xrightarrow{\xi_Y} & Y \\
\downarrow{p'} \downarrow{j} & & \downarrow{p} \\
\text{Nel}(X) & \xrightarrow{\xi_X} & X
\end{array}
\]

of Lemma 4.4, together with the fact that \(\xi_Y\) is final (Lemma 4.5), shows that \(p\) is already the right-fibration part of the final-rightfibration factorization of \(\xi_X \circ p'\), so \(\varepsilon_p : \xi_!(\xi^*)(p) \to p\) is the identity.

\[\square\]
Corollary 4.9. For a simplicial space $X$, we have a natural equivalence $\xi_X^* \simeq \Nel_X$ of functors from $RFib(X)$ to $RFib(\Nel(X))$.

5 Edgewise subdivision and the natural transformation $\lambda$

Consider the functor

$$Q: \Delta \to \Delta$$

$$[n] \mapsto [n]^{op} \star [n] = [2n+1].$$

With the following special notation (following Waldhausen [55]) for the elements of the ordinal $[n]^{op} \star [n] = [2n+1]$,

$$\begin{array}{c}
0 \to 1 \to \cdots \to n \\
\uparrow \\
0' \leftarrow 1' \leftarrow \cdots \leftarrow n',
\end{array}$$

the functor $Q$ is described on arrows by sending a coface map $d^i: [n-1] \to [n]$ to the monotone map that omits the elements $i$ and $i'$, and by sending a codegeneracy map $s^i: [n] \to [n-1]$ to the monotone map that repeats both $i$ and $i'$.

The next lemma follows from the definition.

Lemma 5.1. The functor $Q: \Delta \to \Delta$ sends last-point-preserving maps to active maps, giving this commutative square:

$$\begin{array}{ccc}
\Delta & \xrightarrow{Q} & \Delta \\
\downarrow & \uparrow & \downarrow \\
\Delta^t & \xrightarrow{Q} & \Delta_{act}
\end{array}$$

5.2. Edgewise subdivision functor. For $X: \Delta^{op} \to S$ a simplicial space, the edgewise subdivision is given by precomposing with $Q: \Delta \to \Delta$:

$$\text{Sd}(X) := Q^* X = X \circ Q.$$

At the level of right fibrations over $\Delta$, this is simply the pullback

$$\begin{array}{ccc}
Q^*(\text{el} X) & \xrightarrow{\omega} & \text{el}(X) \\
\downarrow & \downarrow & \downarrow \\
\Delta & \xrightarrow{Q} & \Delta.
\end{array}$$

This means that we have the identification

$$Q^*(\text{el} X) = \text{el}(\text{Sd} X).$$
The top horizontal map ω: el(Sd X) → el(X) has been named because it features in Lemma 5.13 below. On objects it is given by sending an n-simplex Δ^n → Sd(X) to the corresponding map under adjunction Q_i Δ^n → X, which is a (2n + 1)-simplex of X.

**Lemma 5.3.** A simplicial map f: Y → X is culf if and only if Sd(f): Sd(Y) → Sd(X) is a right fibration.

**Proof.** Suppose f: Y → X is culf. By Lemma 2.3 to check that Sd(Y) → Sd(X) is a right fibration is to show that for every last-vertex inclusion ℓ: [0] → [n] there is a unique lift for the diagram on the left,

\[
\begin{array}{ccc}
\Delta^0 & \to & \text{Sd}(Y) \\
\ell & \downarrow & f \\
\Delta^n & \to & \text{Sd}(X)
\end{array}
\]

or equivalently, by adjunction, a unique lift for the diagram on the right. But if f is culf then there is such a unique lifting, because Q_i(ℓ) is active (by Lemma 5.1).

Conversely, suppose Sd(Y) → Sd(X) is a right fibration. To check that Y → X is culf, we should find a lift against any active map Δ^1 → Δ^k, and by Lemma 3.5 it is enough to treat k > 0. For odd values of k, this is the same adjunction argument in reverse. For even k > 0, it is enough to observe that every active map of the form Δ^1 → Δ^2n (n > 0) is a retract of Δ^1 → Δ^{2n+1}, like those appearing in the adjunction argument, so if f is right orthogonal to Δ^1 → Δ^{2n+1} it is also right orthogonal to Δ^1 → Δ^{2n}.

**5.4. Remark.** In the special case of 1-categories, this result is due to Bunge and Niefield [14, Proposition 4.4].

We will need also the following corollary which generalizes Lemma 5.1.

**Corollary 5.5.** If f: B → A is final, then Q_i(f): Q_i(B) → Q_i(A) is ambifinal.

**Proof.** We need to check that there is a unique lift in the diagram on the left (for every culf map p), but by the adjunction argument this is equivalent to having a unique lift in the diagram on the right:

\[
\begin{array}{ccc}
Q_i(B) & \to & \text{Y} \\
Q_i(f) & \downarrow & f \\
Q_i(A) & \to & \text{X}
\end{array}
\]

\[
\begin{array}{ccc}
B & \to & \text{Q^*(Y)} \\
P & \downarrow & Q^*(p) \\
A & \to & \text{Q^*(X)}
\end{array}
\]

and this lift exists uniquely since Q^*(p) is a right fibration by Lemma 5.3.

**Proposition 5.6 (Kock–Spivak).** There is a natural transformation λ: Nel → Sd

\[
\lambda: \text{Nel} \Rightarrow \text{Sd}
\]

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whose component on a simplicial space $X$ is given in simplicial degree 0 by sending $\Delta^n \to X$ (a 0-simplex of Nel($X$)) to the long edge $\Delta^1 \to \Delta^n \to X$ (considered as a 0-simplex of Sd($X$)).

The action of $\lambda$ on higher simplices is given, with reference to the general combinatorial middle-segments construction below, by sending

$$f : \Delta^{n_0} \to \cdots \to \Delta^{n_k} \to X \quad \in \text{(Nel } X)_k$$

to

$$Q! \Delta^k \xrightarrow{\alpha(f)} \Delta^{n_k} \to X \quad \in \text{(Sd } X)_k.$$ 

5.7. Middle-segments construction [Cf. [35, Lemma 3.3]]. This is a two-sided variant of the lower-segments construction: For any $k$, let $f \in (N \Delta)_k$ denote a sequence of maps $[n_0] \xrightarrow{f_1} [n_1] \xrightarrow{f_2} \cdots \xrightarrow{f_k} [n_k]$ in $\Delta$. Then there is a unique commutative diagram $A(f)$ of the form

$$
\begin{array}{cccc}
Q[0] & Q[1] & \cdots & Q[k] \\
\downarrow & \downarrow & \cdots & \downarrow \\
[n_0] & [n_1] & \cdots & [n_k] \\
& f_1 & \cdots & f_k \\
& & & \\
& f_k & \cdots & f_{i+1}(n_i) \\
& & & \\
& f_k & \cdots & f_{i+1}(0).
\end{array}$$

i.e. for which all the vertical maps are active and all the maps in the top row are of the form $Q(d^?)$.

Indeed, building the diagram from the left to the right, in each step it remains to define the next vertical map on the first and the last vertex, and here the value is determined by the requirement that it be active.

The resulting map $\alpha(f)$ can be described explicitly as

$$Q[k] \quad \xrightarrow{i} \quad [n_k]$$

$$i \quad \xrightarrow{f_k \cdots f_{i+1}(n_i)}$$

$$i' \quad \xrightarrow{f_k \cdots f_{i+1}(0)}.$$

This defines the components $(\text{Nel } X)_k \to (\text{Sd } X)_k$.

We should check that it is natural in maps in $\Delta$: as an illustration we do the case of a coface map $d^i : [k-1] \to [k]$. The effect on the chain $f$ is to omit $[n_i]$ and compose the adjacent maps to $f_{i+1} \circ f_i$. The effect on the resulting right-most map $\alpha(f)$ is therefore to omit the two vertices $i'$ and $i$ (since these were picked out by the first- and last-elements of $[n_i]$). But this is precisely to precompose with $Q(d^i) : Q[k-1] \to Q[k]$, in turn precisely the effect of the face operator $d_i$ for the simplicial space Sd($X$). Note that in this calculation we allow $i = k$, in which case $f_{i+1}$ must be interpreted as the $n_k$-simplex $\sigma : \Delta^{n_k} \to X$. 

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5.8. Remark. In the $k = 1$ case,

$$Q[0] \xrightarrow{Q(d^T)} Q[1]$$

$$\downarrow \quad \downarrow \alpha(f)$$

$$[m] \xrightarrow{f} [n],$$

the map $\alpha(f): [3] \rightarrow [n]$ is described explicitly for any $f: [m] \rightarrow [n]$ as follows:

$$0 \mapsto 0, \quad 1 \mapsto f(0), \quad 2 \mapsto f(m), \quad 3 \mapsto n.$$

Note in particular that if $f$ itself is active, so that $f(0) = 0$ and $f(m) = n$, then the 3-simplex $\alpha(f)$ is doubly degenerate, in the sense that it factors through $Q(s^0)$ as in the following diagram:

$$\begin{array}{ccc}
[1] & \xrightarrow{Q(s^0)} & [3] \\
\downarrow \alpha(f) & & \downarrow \alpha(f) \\
[m] & \xrightarrow{f} & [n].
\end{array}$$

Corollary 5.9. The natural transformation $\lambda: \text{Nel} \Rightarrow \text{Sd}$ sends 1-simplices in $\text{Nel}(X)$ lying over active maps in $\Delta$ to equivalences in $\text{Sd}(X)$:

$$\Delta^{n_0} \rightarrow \Delta^{n_1} \rightarrow X \xrightarrow{\cdot}$$

5.10. Remark. In the category case, the natural transformation $\lambda$ goes back to Thomason’s notebooks [52, p.152]; it was exploited by Gálvez–Neumann–Tonks [28] to exhibit Baues–Wirsching cohomology as a special case of Gabriel–Zisman cohomology. The following cartesian property was established for discrete decomposition spaces in [35].

Lemma 5.11. The natural transformation $\lambda: \text{Nel} \Rightarrow \text{Sd}$ is cartesian on culf maps. In other words, for every culf map $F: Y \rightarrow X$ of simplicial spaces, the naturality square

$$\begin{array}{ccc}
\text{Nel}(Y) & \xrightarrow{\lambda_Y} & \text{Sd}(Y) \\
\downarrow \text{Nel}(F) & & \downarrow \text{Sd}(F) \\
\text{Nel}(X) & \xrightarrow{\lambda_X} & \text{Sd}(X)
\end{array}$$

is a pullback.
Proof. This proof follows the same idea as that of Lemma 4.4, but using the middle-segments construction 5.7 instead of the lower-segments construction 4.3. We check it in each simplicial degree separately. In simplicial degree $k$ we have

$$\text{Nel}(Y)_k = \sum_{[n_0] \to \cdots \to [n_k]} Y_{n_k}.$$ 

If the chain of maps $[n_0] \to \cdots \to [n_k]$ is called $f$, then the middle-segments construction gives us an active map $\alpha_f : [2k+1] \to [n_k]$, and $\lambda_Y$ is given in the $f$-summand by

$$Y_{n_k} \xrightarrow{\alpha_f^*} Y_{2k+1}.$$ 

Altogether, the square we want to show is a pullback (in degree $k$) is identified with

$$\sum_f Y_{n_k} \xrightarrow{\alpha_f^*} Y_{2k+1}$$
$$\sum_f X_{n_k} \xrightarrow{\alpha_f^*} X_{2k+1}$$

(with $\alpha_f$ varying with $f$, so generally different in each summand). The left vertical map respects $f$ (since it is a morphism of right fibrations over $\Delta$), so the pullback property can be established separately for each summand. For a fixed chain $f$, the square is thus

$$Y_{n_k} \xrightarrow{\alpha_f^*} Y_{2k+1}$$
$$X_{n_k} \xrightarrow{\alpha_f^*} X_{2k+1}$$

and this square is a pullback since $\alpha_f$ is active and $F$ is culf.

**Lemma 5.12.** The lower-segments construction $B$ 4.3 and the middle-segments construction $A$ 5.7 are related by

$$Q(B(f)) = A(Q(f)).$$

**Proof.** This is a direct computation. For details see [35].

The following is the analogue of [35, Lemma 3.8] for simplicial spaces.

**Lemma 5.13.** There is a natural commutative diagram of simplicial spaces

$$\begin{array}{ccc}
\text{Nel}(X) & \xrightarrow{N(\omega_X)} & Q^*(\text{Nel} X) \\
\lambda_X & \downarrow \xi_{\text{Sd}(X)} & \uparrow \xi_{\text{Sd}(X)} \\
\text{Sd}(X) & \xrightarrow{\xi_{\text{Sd}(X)}} & \text{Sd}(X)
\end{array}$$

**Proof.** This is a direct calculation, involving the middle-segments construction for $\lambda$ and the lower-segments construction for $\xi$. The equality of the two ways around is a consequence of Lemma 5.12. (For details, see [35].)
6 Culfy and righteous maps

For the proof of Theorem 7.1 below, it is helpful to recast the notions of culf maps and right fibrations from the category of simplicial spaces $\text{PrSh}(\Delta)$ to the equivalent category $\text{Rfib}(\Delta)$ of right fibrations over $\Delta$.

6.1 Categories of elements. Recall that for a presheaf $X: \mathcal{C}^{\text{op}} \to S$, the category of elements $\text{el}(X) := \mathcal{C} \downarrow X$ has as objects and arrows, respectively, diagrams of the form

\[
\begin{array}{ccc}
m & \xrightarrow{\alpha} & m' \\
\downarrow{\tau} & & \downarrow{\tau} \\
X & \xrightarrow{\gamma} & X
\end{array}
\]

with $m$ and $m'$ objects in $\mathcal{C}$ (that is, representables). (We shall be interested in the case $\mathcal{C} = \Delta$, so we use letters such as $n$ to denote the objects in $\mathcal{C}$.)

**Lemma 6.2.** A natural transformation

\[
\begin{array}{ccc}
\mathcal{C}^{\text{op}} & \xrightarrow{\psi} & S \\
\downarrow{f} & & \downarrow{\alpha} \\
X & \xrightarrow{\tau} & Y
\end{array}
\]

is cartesian if and only if the corresponding right fibration

\[
el(Y) \xrightarrow{p} \text{el}(X)
\]

is also a left fibration.

**Proof.** The proof amounts to spelling out what the two conditions mean. To say that $p: \text{el}(Y) \to \text{el}(X)$ is a left fibration means that given an object $\tau \in \text{el}(Y)$ and an arrow $\theta: p\tau \to \sigma$, there is a “unique” lift $\overline{\theta}: \tau \to \gamma$ (that is, $p(\overline{\theta}) = \theta$). See for example Riehl–Verity [45, Proposition 5.5.6]. Uniqueness means that there is a contractible space of such $\overline{\theta}$, as we now detail. The data given, the arrow $\theta: p\tau \to \sigma$, amounts to the following diagram:

\[
\begin{array}{ccc}
m & \xrightarrow{\alpha} & n \\
\downarrow{\tau} & & \downarrow{\sigma} \\
Y & \xrightarrow{f} & X.
\end{array}
\]

To find $\overline{\theta}$ such that $p(\overline{\theta}) = \theta$ means to find $\gamma: n \to Y$ making these two triangles commute:

\[
\begin{array}{ccc}
m & \xrightarrow{\alpha} & n \\
\downarrow{\tau} & & \downarrow{\sigma} \\
Y & \xrightarrow{f} & X.
\end{array}
\]
To say that $p$ is a left fibration is to say that for any given square $\theta$, the space of lifts $\gamma$ is contractible. More formally this means that the following diagram of spaces is a pullback:

$$
\begin{array}{ccc}
\text{Map}(n,Y) & \xleftarrow{\text{pre } \alpha} & \text{Map}(m,Y) \\
\downarrow{\text{post } f} & & \downarrow{\text{post } f} \\
\text{Map}(n,X) & \xleftarrow{\text{pre } \alpha} & \text{Map}(m,X).
\end{array}
$$

But this square is nothing more than the condition

$$
\begin{array}{ccc}
Y_n & \xrightarrow{\alpha^*} & Y_m \\
\downarrow{f_n} & & \downarrow{f_m} \\
X_n & \xrightarrow{\alpha^*} & X_m,
\end{array}
$$

which states precisely that $f: Y \Rightarrow X$ is cartesian. \hfill \Box

**Corollary 6.3.** A simplicial map

$$
\Delta^{op} \xrightarrow{\psi f} S
$$

is culf if and only if the corresponding right fibration $p$

$$
\text{el}(Y) \xrightarrow{p} \text{el}(X) \rightarrow \Delta
$$

becomes a left fibration after restriction to $\Delta_{\text{act}} \subset \Delta$, the subcategory of active maps.

We shall call such functors $p$ **culfy maps** of right fibrations over $\Delta$. We also use the term culfy map for the corresponding notion for right fibrations of simplicial spaces over $N\Delta$. That is, a culfy map between right fibrations over $N\Delta$ is one whose pullback along $N\Delta_{\text{act}} \rightarrow N\Delta$ is a left fibration as well. This is illustrated in the following diagram, where the middle row of the diagram concerns right fibrations between $\infty$-categories, rather than between simplicial spaces.

$$
\begin{array}{ccc}
\text{Culf}(X) & \xleftarrow{\text{culfy } \Delta} & \text{RFib}_{\Delta}(\text{el } X) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\text{Culf}_{N\Delta}(\text{Nel } X) & \xleftarrow{\text{culfy } N\Delta} & \text{RFib}_{N\Delta}(\text{Nel } X)
\end{array}
$$
Corollary 6.4. A simplicial map

\[ \Delta^\text{op} \xrightarrow{\psi f} S \]

is a right fibration if and only if the corresponding right fibration \( p \)

\[ \text{el}(Y) \xrightarrow{p} \text{el}(X) \longrightarrow \Delta \]

becomes a left fibration after restriction to \( \Delta^t \subset \Delta \), the subcategory of last-point-preserving monotone maps (that is, presheaves on \( \Delta^t \) are “simplicial objects with missing top face maps” — see [2.6]).

We call such functors \( p \) righteous maps of right fibrations over \( \Delta \). We also use the term righteous map for the corresponding notion for right fibrations of simplicial spaces over \( N\Delta \). The following diagram summarizes the notions. The middle row concerns right fibrations of \( \infty \)-categories rather than of simplicial spaces.

\[ \begin{array}{ccc}
RFib(X) & \xleftarrow{\approx} & sS_{/X} \\
& \approx & \\
\text{righteous}_\Delta(\text{el} X) & \xleftarrow{\approx} & RFib_\Delta(\text{el} X) \\
& \approx & \\
\text{Righteous}_{N\Delta}(\text{Nel} X) & \xleftarrow{\approx} & RFib_{N\Delta}(\text{Nel} X) \\
& \approx & \\
N & \approx & \\
\end{array} \]

\[ Y \rightarrow X \]

\[ \text{el}(Y) \rightarrow \text{el}(X) \rightarrow \Delta \]

\[ \text{Nel}(Y) \rightarrow \text{Nel}(X) \rightarrow N\Delta \]

7 Main theorem via pullback along \( \lambda \)

We know by Lemma 5.3 that the edgewise subdivision of a culf map is a right fibration. The following main theorem gives an inverse construction.

Theorem 7.1 (Theorem C). For \( X \) a simplicial space, the functor

\[ \text{Sd}_X : \text{Culf}(X) \rightarrow RFib(\text{Sd}X) \]

\[ Y \rightarrow X \rightleftharpoons \text{Sd}(Y) \rightarrow \text{Sd}(X) \]

is an equivalence. The inverse equivalence is given essentially by pullback along \( \lambda_X : \text{Nel}(X) \rightarrow \text{Sd}(X) \), as detailed in the proof.

Henceforth, subscripts on functors, such as \( \text{Sd}_X \), indicate the functors induced on slices (or subcategories of slices).
Proof of Theorem 7.1. We have the following commutative triangles:

\[
\begin{array}{ccc}
Culf(X) & \xrightarrow{Sd_X} & RFib(Sd X) \\
\downarrow \text{Nel}_X & & \downarrow \text{Nel}_{Sd X} = (\xi_{Sd X})^* \\
RFib(\text{Nel}(X)) & \xrightarrow{Q^*} & RFib(Q^* \text{Nel} X) \simeq RFib(\text{Nel} Sd X)
\end{array}
\]

The upper left triangle commutes because of Lemma 5.11, which says that applying Sd and then pulling back along \(\lambda\) is the same as applying Nel.

The lower right triangle involves some canonical identifications, first of all \(Q^*(\text{Nel} X) \simeq \text{Nel}(Sd X)\). Note further that since \(\xi\) is cartesian on right fibrations (Lemma 4.4), the pullback square of Lemma 4.4 gives the identification \(\text{Nel} Sd X = (\xi_{Sd X})^*\) indicated. The triangle now commutes by Lemma 5.13.

The point is that all the downgoing maps — including \(\lambda X^*\) — actually land in culfy and righteous maps, as indicated in the diagram

\[
\begin{array}{ccc}
Culf(X) & \xrightarrow{Sd_X} & RFib(Sd X) \\
\downarrow \text{Nel}_X & & \downarrow \text{Nel}_{Sd X} = (\xi_{Sd X})^* \\
\text{Culfy}_{N\Delta}(\text{Nel}(X)) & \xrightarrow{Q^*} & \text{Righteous}_{N\Delta}(\text{Nel}(Sd X))
\end{array}
\]

Here, as in Section 6, \(\text{Culfy}_{N\Delta}(\text{Nel}(X)) \subset RFib_{N\Delta}(\text{Nel}(X))\) is the \(\infty\)-category of maps of right fibrations \(Y \rightarrow \text{Nel}(X) \rightarrow N\Delta\) that become also left fibrations after base change along \(N\Delta_{act} \subset N\Delta\). Similarly,

\(\text{Righteous}_{N\Delta}(\text{Nel}(Sd X)) \subset RFib_{N\Delta}(\text{Nel}(Sd X))\)

is the \(\infty\)-category of maps of right fibrations \(Y \rightarrow \text{Nel}(Sd X) \rightarrow N\Delta\) that become also left fibrations after base change along \(N\Delta_{act} \subset N\Delta\).

Note that \(Q^*\) sends culfy maps to righteous maps, by the same argument as Lemma 5.3. Indeed, the statement is that pullback along \(Q\) followed by restriction to \(\Delta^i\) is a left fibration, but this is true because this is the same as first restricting to \(\Delta_{act}\) and then pulling back along \(Q\) (as in Lemma 5.1); now restriction to \(\Delta_{act}\) gives a left fibration since we started with a culf map, and finally pulling back preserves left fibrations.

We also already know that Nel sends culf maps to culfy maps and right fibrations to righteous maps (by definition of these classes of maps). We have to check that also \(\lambda^*\) lands in culfy maps. Let \(Y \rightarrow Sd(X)\) be a right fibration. We need to check that after pullback along \(\lambda\) and restriction to \(\Delta_{act}\) the result
is a left fibration. The relevant diagram is

\[
\begin{array}{cccc}
\Delta^0 \ar[d]^{d^1} & \ar[r]^{r y^\lambda} \ar[d] & \cdot \ar[r] \ar[d] & Y \ar[d] \\
\Delta^1 \ar[r]_{r e^\lambda} & \Nel(X)_{\text{act}} \ar[r]_{\lambda X} & \Nel(X) \ar[r]_{\lambda X} & \Sd(X).
\end{array}
\]

By Lemma 2.4, all objects in the second column are Segal spaces; by the dual of Lemma 2.5, to prove the map of interest is a left fibration it is enough to prove it is orthogonal to \(d^1: \Delta^0 \to \Delta^1\). So we need to show, for any maps \(r y^\lambda\) and \(r e^\lambda\) making the leftmost square commute, that the space of lifts indicated is contractible. As the two squares in the upper right are given by pullback, this space of lifts is equivalent to the space of lifts of the long composite rectangle. But since the arrow \(e\) in \(\Nel(X)\) is active, \(\lambda X\) takes it to an equivalence in \(\Sd(X)\) (by Corollary 5.9), so it lifts uniquely, since \(Y \to \Sd(X)\) is a right fibration.

Finally, having established that the downward arrows land in culfy and righteous maps as indicated, since the vertical maps are equivalences onto these images, we see that \(\lambda X^*\) has up-to-homotopy inverses on both sides, so these three maps are actually equivalences.

\[\square\]

8 Main theorem via right Kan extension

In this section we give a very different description of the inverse featured in the main theorem, in terms of the right adjoint to edgewise subdivision. In a sense this description is much more direct than the pullback-along-\(\lambda\) description given in Theorem 7.1, but in practice the right adjoint is not easy to compute.

The edgewise subdivision of an ordinary category \(\mathcal{C}\) is just the twisted arrow category, so its objects are the arrows of \(\mathcal{C}\). In the special case \(\mathcal{C} = \Delta^n\), which is a poset, an arrow is completely specified by its endpoints, so we denote by \((i, j)\) the object in \(\Sd(\Delta^n)\) corresponding to the arrow \(i \to j\) in \(\Delta^n\).

We will arrive shortly at the \(Q^* \dashv Q_*\) adjunction, but first we need a few preliminary results on the \(Q_! \dashv Q^*\) adjunction.

**Lemma 8.1.** The unit for the \(Q_! \dashv Q^*\) adjunction in given on representables by

\[
\eta_{\Delta^n}: \Delta^n \to \Delta^n = \Sd(\Delta^{2n+1})
\]

\[i \mapsto (n-i, n+i+1).\]

**Proof.** For a general simplicial space \(X\), the adjunction equivalence \(\Map(Q_! \Delta^n, X) \cong \Map(\Delta^n, Q^* X)\) acts by sending a \((2n+1)\)-simplex in \(X\) to the same simplex, but reinterpreted as an \(n\)-simplex in \(\Sd(X)\). Now instantiate at \(X = Q_! \Delta^n\), and

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consider the identity map $\text{id} : Q_1 \Delta^n \to Q_1 \Delta^n$. The unit will be the corresponding map under the adjunction equivalence. This is now the $n$-simplex

$$
\begin{array}{c}
n+1 \\ n+2 \\ \vdots \\ 2n+1 \\
n \\ n-1 \\ \vdots \\ 0
\end{array}
$$

where clearly the $i$th vertex is the arrow $(n-i, n+i+1)$.

**Corollary 8.2.** The unit $\eta_{\Delta^n} : \Delta^n \to Q^*Q_1\Delta^n$ is final.

**Proof.** It sends the terminal object $n \in \Delta^n$ to the terminal object $(0, 2n+1) \in \text{Sd}(\Delta^{2n+1})$. By Lemma 2.12, $\eta_{\Delta^n}$ is final.

**Proposition 8.3.** The counit $\varepsilon_{\Delta^n} : Q_1^*Q_1^{*}\Delta^n \to \Delta^n$ is ambifinal.

**Proof.** By Corollary 5.5, we know that $Q_1$ sends final maps to ambifinal maps; combining this with Corollary 8.2, we see that $Q_1(\eta_{\Delta^n})$ is ambifinal. A triangle identity gives that $\varepsilon_{Q_1^*\Delta^n} \circ Q_1(\eta_{\Delta^n}) \simeq \text{id}_{Q_1^*\Delta^n}$, so by right cancellation, $\varepsilon_{Q_1^*\Delta^n} \simeq \varepsilon_{\Delta^{2n+1}}$ is ambifinal. We thus have the result for odd-dimensional simplices. But $\Delta^{2n}$ is a retract of $\Delta^{2n+1}$, and the left class in any factorization system is closed under retracts, so $\varepsilon_{\Delta^{2n}}$ is ambifinal as well.

We now come to the right adjoint $Q_*$ to edgewise subdivision $Q^*$.

**Proposition 8.4.** The right Kan extension functor $Q_* : \text{sS} \to \text{sS}$ takes right fibrations to culf maps.

**Proof.** The edgewise subdivision functor $Q^*$ takes active maps $\Delta^n \to \Delta^m$ to last-point-preserving maps, which are then final by Lemma 2.12. By a standard argument concerning adjunctions and factorization systems, the right adjoint $Q_*$ takes right fibrations to culf maps.

The following general result should be well known to experts, but we could not find a suitable reference for it. Note that the relationship between $\varepsilon$ and $\eta'$ is not dual (via taking opposite categories) to the relationship between $\varepsilon'$ and $\eta$.

**Lemma 8.5.** Suppose

$$
\begin{array}{ccc}
D & \overset{L}{\leftarrow} & E \\
\text{id}_{D} & \overset{R}{\Rightarrow} & \text{id}_{E}
\end{array}
$$

is a string of adjoint functors, then the units and counits

$$
\begin{array}{c}
\eta : \text{id}_E \Rightarrow FL \\
\varepsilon : LF \Rightarrow \text{id}_D \\
\eta' : \text{id}_D \Rightarrow RF \\
\varepsilon' : FR \Rightarrow \text{id}_E
\end{array}
$$

are related by the following:

$$
\begin{array}{c}
\text{Map}(b, FRc) \overset{\sim}{\longrightarrow} \text{Map}(Lb, Rc) \\
\text{Map}(d, e) \overset{\text{Map}(\varepsilon, d)}{\longrightarrow} \text{Map}(LFd, e)
\end{array}
$$

$$
\begin{array}{c}
\text{Map}(b, c) \overset{\text{Map}(\eta, c)}{\longrightarrow} \text{Map}(FLb, c) \\
\text{Map}(d, e) \overset{\text{Map}(\varepsilon, d)}{\longrightarrow} \text{Map}(Fd, Fe)
\end{array}
$$

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where \(b, c \in C\) and \(d, e \in D\).

**Proof.** For clarity, we temporarily write \([-,-]\) instead of \(\text{Map}(-,-)\). The first square in the statement commutes because it arises from the following pasting.

\[
\begin{array}{cccc}
[b, FRe] & \xrightarrow{L} & [Lb, LFRe] & \xrightarrow{\varepsilon_{Re, \circ}} [Lb, Rc] \\
\downarrow & & \downarrow F & \\
[b, FRe] & \xrightarrow{F \varepsilon_{Re, \circ}} [b, FFLRe] & \xrightarrow{\varepsilon'_{\circ}} [FLb, Fr] & \xrightarrow{L \eta} [b, FLc] \\
\downarrow & & \downarrow & \\
[b, c] & \xleftarrow{\varepsilon'_{\circ}} [FLb, c] & \xleftarrow{L \eta} [FLb, Fr] & \xleftarrow{\varepsilon'_{\circ}} [FLb, c]
\end{array}
\]

One of the triangles in the top left is a triangle identity, and the other commutes since \(\eta\) is a natural transformation \(id_C \Rightarrow FL\). The square in the upper left uses that \(F\) is a functor, while the other two commute by associativity of composition.

The second square in the statement commutes because it arises from the following pasting.

\[
\begin{array}{cccc}
[d, e] & \xrightarrow{- \circ \varepsilon_d} & [LFd, e] & \\
\downarrow \eta'_{\circ} & \downarrow & \downarrow \varepsilon'_{\circ} & \\
[Ld, LFRe] & \xrightarrow{id} & [LFd, LFe] & \\
\downarrow \varepsilon'_{\circ} & \downarrow L & \downarrow & \\
[d, RF] & \xrightarrow{F} [Fd, FRFe] & \xleftarrow{L \varepsilon_{Re, \circ}} [Fd, Fr] & \xleftarrow{\varepsilon'_{\circ}} [Fd, F]
\end{array}
\]

The top region commutes because \(\varepsilon\) is a natural transformation \(LR \Rightarrow id_C\). The bottom left (resp. bottom right) region commutes since \(LF\) (resp. \(L\)) is a functor.

The triangle is \(L\) applied to the triangle identity of \(L \dashv F\). \(\Box\)

**Corollary 8.6.** The unit \(\eta^!'\) of the \(Q^* \dashv Q_*\) adjunction can be computed in simplicial degree \(n\) as

\[
\text{Map}(\varepsilon_{\Delta^n}, X) : \text{Map}(\Delta^n, X) \to \text{Map}(Q!Q^*\Delta^n, X) = \text{Map}(\Delta^n, Q_+Q^*X).
\]

**Lemma 8.7.** The unit \(\eta^!'\) of the \(Q^* \dashv Q_*\) adjunction is cartesian on culf maps.

**Proof.** Let \(p : Y \to X\) be culf. By Lemma 5.3 and Proposition 8.4 \(Q_+Q^*(p)\) is
culf as well. We will show that the diagram

\[
\begin{array}{ccc}
  Y & \xrightarrow{f_Y} & Q_+ Q^* Y \\
  \downarrow & & \downarrow_{Q_+ Q^* p} \\
  X & \xrightarrow{f_X} & Q_+ Q^* X
\end{array}
\]

is a pullback in each simplicial degree \( n \). Using Corollary 8.6, this becomes

\[
\begin{array}{ccc}
  \text{Map}(\Delta^n, Y) & \xrightarrow{\text{Map}(\epsilon_{\Delta^n}, Y)} & \text{Map}(Q_+ Q^* \Delta^n, Y) \\
  \downarrow & & \downarrow \\
  \text{Map}(\Delta^n, X) & \xrightarrow{\text{Map}(\epsilon_{\Delta^n}, X)} & \text{Map}(Q_+ Q^* \Delta^n, X)
\end{array}
\]

where \( \epsilon \) is the counit for the \( Q_+ \dashv Q^* \) adjunction. By Proposition 8.3, \( \epsilon_{\Delta^n} \) is ambifinal, so this square is a pullback. \( \square \)

**Lemma 8.8.** The counit \( \epsilon' \) of the \( Q^* \dashv Q_+ \) adjunction is cartesian on right fibrations.

**Proof.** Let \( f: Y \to X \) be a right fibration. We show that the diagram

\[
\begin{array}{ccc}
  Q^* Q_+ Y & \xrightarrow{\epsilon'_Y} & Y \\
  Q^* Q_+ f \downarrow & & \downarrow f \\
  Q^* Q_+ X & \xrightarrow{\epsilon'_X} & X
\end{array}
\]

is a pullback in each simplicial degree \( n \). Using Lemma 8.5, this becomes

\[
\begin{array}{ccc}
  \text{Map}(\Delta^n, Q^* Q_+ Y) & \xrightarrow{\text{Map}(\eta_{\Delta^n}, Y)} & \text{Map}(Q^* Q_+ \Delta^n, Y) \\
  \text{Map}(\Delta^n, Q^* Q_+ X) & \xrightarrow{\text{Map}(\eta_{\Delta^n}, X)} & \text{Map}(Q^* Q_+ \Delta^n, X)
\end{array}
\]

where \( \eta \) is the unit for the \( Q_+ \dashv Q^* \) adjunction. Since \( \eta_{\Delta^n} \) is final by Corollary 8.2 and \( f \) is a right fibration, this is a pullback. \( \square \)

**8.9. Slicing adjunctions.** Recall that given an adjunction

\[
\begin{array}{ccc}
  \mathcal{D} & \xleftarrow{F} & \mathcal{E} \\
  \xrightarrow{G} & & \\
  \mathcal{E} & \xleftarrow{F_d} & \mathcal{D}_{/d}
\end{array}
\]

where \( \mathcal{D} \) has pullbacks, we can obtain an adjunction

\[
\begin{array}{ccc}
  \mathcal{D}_{/d} & \xleftarrow{F_d} & \mathcal{E}_{/F_d} \\
  \xrightarrow{G} & & \\
  \mathcal{E}_{/F_d} & \xleftarrow{F_d} & \mathcal{D}_{/d}
\end{array}
\]

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whose right adjoint is given by applying $G_{Fd}: C_{/Fd} \rightarrow D_{/GFd}$ and then pulling back along the unit $\eta_d: d \rightarrow GFd$ (see [38 Proposition 5.2.5.1]).

Instantiating to the $Q^* \dashv Q_*$ adjunction, we get the sliced adjunction

$$
\begin{array}{c}
sS_{/X} \ \\
\downarrow \ sS_{/\text{Sd}(X)} \hspace{1cm} \ \ \ \ \\
\downarrow \ (\eta'_{dX})^* \circ Q_*
\end{array}
$$

By Lemma 5.3 and Proposition 8.4, this adjunction restricts to an adjunction

$$
\begin{array}{c}
\text{Culf}(X) \ \\
\downarrow \ \ \\
\text{RFib}(\text{Sd}X)
\end{array}
$$

In detail, the right adjoint acts on a given right fibration $W \rightarrow Q^*X$ by first applying $Q_*$ to get a culf map $Q_*W \rightarrow Q_*Q^*X$ (by Proposition 8.4), and then pulling back along the unit $\eta'_{dX}$ of the $Q^* \dashv Q_*$ adjunction to get a culf map $Y \rightarrow X$ as in

$$
\begin{array}{c}
Y \ \\
\downarrow \ j \hspace{1cm} \ \\
X \ \\
\downarrow \ \eta'_{dX} \hspace{1cm} \ \\
Q_*Q^*X
\end{array}
$$

The next two lemmas together show that this adjunction is an adjoint equivalence, so that in particular the functor $(\eta'_{dX})^* \circ Q_*$ is an alternative description of the inverse to the equivalence displayed in Theorem C.

**Lemma 8.10.** The natural transformation

$$
\begin{array}{c}
\text{RFib}(\text{Sd}X) \ \\
\downarrow \ id \ \\
\text{Culf}(\eta'_{dX})^* \ \\
\downarrow \ \\
\text{Culf}(X) \ \\
\downarrow \ Q^* \ \\
\text{RFib}(\text{Sd}X)
\end{array}
$$

is an equivalence.

**Proof.** Let $p: W \rightarrow Q^*X$ be a right fibration. Applying $Q_*$ and pulling back along $\eta'_{dX}$ gives the culf map $q$ on the left in

$$
\begin{array}{c}
Y \ \\
\downarrow \ q \hspace{1cm} \ \\
X \ \\
\downarrow \ \eta'_{dX} \hspace{1cm} \ \\
Q_*Q^*X
\end{array}
$$

This is sent by $Q^*$ to the left square in the following diagram, which is again a
pullback since \(Q^*\) is also a right adjoint.

\[
\begin{array}{ccc}
Q^*Y & \xrightarrow{j} & Q^*Q_*W \\
\downarrow Q^*q & & \downarrow Q^*Q_*p \\
Q^*X & \xrightarrow{Q^*(\eta'_X)} & Q^*Q_*Q^*X \\
\downarrow \text{id} & & \downarrow Q^*Q_*\varepsilon_X \\
\end{array}
\]

The right square is a pullback by Lemma 8.8. Since the large rectangle is thus a pullback, with bottom edge an identity, it follows that the top edge \(Q^*Y \to W\) is an equivalence. Hence \(Q^*q \simeq p\), as desired.

\[\square\]

**Lemma 8.11.** The natural transformation

\[
\begin{array}{ccc}
\text{Culf}(X) & \xrightarrow{Q^*} & \text{RFib}(\text{Sd}X) \\
\downarrow \text{id} & & \downarrow \text{id} \\
\text{Culf}(Q_*Q^*X) & \xrightarrow{(\eta'_X)^*} & \text{Culf}(X)
\end{array}
\]

is an equivalence.

**Proof.** If \(p: Y \to X\) is culf, then

\[
\begin{array}{ccc}
Y & \xrightarrow{\eta_Y} & Q_*Q^*Y \\
\downarrow p & & \downarrow Q_*Q^*p \\
X & \xrightarrow{\eta_X} & Q_*Q^*X
\end{array}
\]

is a pullback by Lemma 8.7.

\[\square\]

Combining the previous two lemmas, we have established our second proof of Theorem C:

**Theorem 8.12** (Theorem C). The adjunction \(\text{Sd}X: S_S/X \rightleftarrows S_S/\text{Sd}X\) restricts to an equivalence \(\text{Culf}(X) \simeq \text{RFib}(\text{Sd}X)\).

### 9 Decomposition spaces and Rezk completeness

**9.1. Decomposition spaces/2-Segal spaces.** A decomposition space \([25]\) (or 2-Segal space \([18]\)) is a simplicial \(\infty\)-groupoid \(X: \Delta^{op} \to S\) that takes active-inert pushouts in \(\Delta\) to pullbacks in \(S\).

**Lemma 9.2** (Bergner et al. \([7]\)). A simplicial space \(X\) is a decomposition space if and only if \(\text{Sd}(X)\) is a Segal space.

The following is Theorem D from the introduction.
Theorem 9.3 (Theorem [1], Theorem D). The $\infty$-category of decomposition spaces and culf maps is locally an $\infty$-topos. More precisely, for $X$ a decomposition space, we have an equivalence

$$\text{Decomp}_X \simeq \text{RFib}(\text{Sd} X) \simeq \text{RFib}(\text{Sd} \hat{X}) \simeq \text{PrSh}(\text{Sd} \hat{X}).$$

Here $\hat{(-)}$ denotes the Rezk completion of a Segal space.

Proof. The first step in the equivalence is Theorem C. The second step is Proposition 9.4 below, which says that Rezk completion of Segal spaces does not affect right fibrations — note that Sd$(X)$ is a Segal space since $X$ is a decomposition space. The last step is straightening/unstraightening for complete Segal spaces.

Proposition 9.4. Suppose $X$ is a Segal space. Then pulling back along the completion map $X \to \hat{X}$ induces an equivalence $\text{RFib}(\hat{X}) \to \text{RFib}(X)$.

This is proved in the appendix.

We finish the paper with the observation (Proposition 9.12) that in many cases it is not necessary to Rezk complete, namely when the decomposition space itself is Rezk complete, as is usually the case for decomposition spaces of combinatorial origin. For this we first need a few results about Rezk completeness for decomposition spaces.

9.5. Equivalences. Let $X$ be a decomposition space. An arrow $f \in X_1$ is called an equivalence if there exist $\sigma \in X_2$ such that $d_2(\sigma) = f$ and $d_1(\sigma) = s_0d_1(f)$ and there exists $\tau \in X_2$ such that $d_0(\tau) = f$ and $d_1(\tau) = s_0d_0(f)$. We denote by $X_1^{eq} \subset X_1$ the full sub $\infty$-groupoid spanned by the equivalences. Note that degenerate arrows are always equivalences.

9.6. Remark. Feller [20] proposes a stronger notion of equivalence for 1-simplices in a decomposition space, which are instead witnessed by maps from the nerve of the free-living isomorphism. For Segal spaces, this makes no difference, but generally this change will result in a potentially larger class of Rezk complete decomposition spaces than the following definition from [26, 5.13].

9.7. Rezk completeness. A decomposition space $X$ is called Rezk complete when the canonical map $s_0 : X_0 \to X_1^{eq}$ is a homotopy equivalence.

9.8. Remark. The Rezk completeness condition as formulated here only refers to 1-simplices. However, since decomposition spaces have the property that degeneracy can be detected on principal edges [26 §2], the 1-dimensional condition implies other conditions corresponding to degeneracies. We will not attempt at distilling this observation into a general statement, but only prove the following illustrative case, which we will actually need in the proof of Proposition 9.12. Intuitively it says that the space of 3-simplices whose first and third principal edges are equivalences is homotopy equivalent to $X_1$. 

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Lemma 9.9. The square

\[
\begin{array}{ccc}
X_1 & \xrightarrow{s_\perp s_\top} & X_3 \\
\downarrow (s_0 d_1, \text{id}, s_0 d_0) & & \downarrow (d_\top d_\top, d_\top d_\bot, d_\bot d_\bot) \\
X_1^\text{eq} \times X_1 \times X_1^\text{eq} & \longrightarrow & X_1 \times X_1 \times X_1
\end{array}
\]

is a pullback.

Proof. By Rezk completeness in the sense of 9.7, the square is homotopy equivalent to

\[
\begin{array}{ccc}
X_1 & \xrightarrow{s_\perp s_\top} & X_3 \\
\downarrow (d_1, \text{id}, d_0) & & \downarrow (d_\top d_\top, d_\top d_\bot, d_\bot d_\bot) \\
X_0 \times X_1 \times X_0 & \longrightarrow & X_1 \times X_1 \times X_1
\end{array}
\]

which is a pullback by the characterization of decomposition spaces given in Proposition 6.9 (3) of [25].

Proposition 9.10. If \(X\) is a Rezk complete decomposition space, and \(Y \to X\) is culf, then also \(Y\) is a Rezk complete decomposition space.

Proof. By [25, Lemma 4.6], we know that \(Y\) is a decomposition space. Further, since pullbacks of monomorphisms are monomorphisms, \(s_0 : Y_0 \to Y_1\) is a monomorphism of \(\infty\)-groupoids.

Suppose \(X\) is Rezk complete. Equivalences are preserved by arbitrary maps of simplicial spaces, so \(Y_1^\text{eq} \to Y_1 \to X_1\) lands in \(X_1^\text{eq}\). The culf property guarantees that the square in the following diagram is a pullback

\[
\begin{array}{ccc}
Y_1^\text{eq} \ar{dr}[r] & & Y_0 \ar{d}{s_0} \ar{r}{s_0} & Y_1 \ar{d} \ar{d} \\
X_1^\text{eq} \ar{d}[r][r] & \ar{r} & Y_0 \ar{d}{s_0} \ar{r}{s_0} & X_1 \ar{d} \ar{d}
\end{array}
\]

hence we have a map \(r : Y_1^\text{eq} \to Y_0\) which we hope is an equivalence. Here, \(s_0 r\) is equivalent to the inclusion \(Y_1^\text{eq} \to Y_1\). We thus have the commutative diagram of \(\infty\)-groupoids

\[
\begin{array}{ccc}
Y_0 \ar{dr}[r][r] & & Y_1^\text{eq} \ar{d}[r][r] \ar{r}{s_0} & Y_0 \ar{d} \ar{d} \\
& & \ar{d}[r][r] & \ar{d} \ar{d} \\
& & Y_1 \ar{d}[r][r] \ar{r}{s_0} & Y_1 \ar{d} \ar{d}
\end{array}
\]

Since the non-horizontal maps are monomorphisms of \(\infty\)-groupoids, it follows from the following general Lemma 9.11 that the inclusion \(Y_0 \to Y_1^\text{eq}\) is an equivalence.

\[\square\]
Lemma 9.11. If

\[
\begin{array}{c}
A \xrightarrow{k} B \xrightarrow{r} A \\
\downarrow i \quad \downarrow j \quad \downarrow i \\
C
\end{array}
\]

is a commutative diagram of spaces with \(i\) and \(j\) monomorphisms of \(\infty\)-groupoids, then \(k\) and \(r\) are equivalences.

**Proof.** First take path components of each of the spaces in question. Since \(\pi_0(i)\) and \(\pi_0(j)\) are injections of sets, we have the same is true of \(\pi_0(k)\) and \(\pi_0(r)\). Hence both \(\pi_0(k)\) and \(\pi_0(r)\) are bijections of sets.

Now suppose \(W \in \pi_0(A)\) is some path component of \(A\), the element \(W' \in \pi_0(B)\) is its image under \(\pi_0(k)\), and \(Z \in \pi_0(C)\) is the image of \(W\) under \(\pi_0(i)\). By the assumption that \(i\) and \(j\) are monomorphisms of \(\infty\)-groupoids, we have the two diagonal legs in the diagram

\[
\begin{array}{c}
W \xrightarrow{k|W} W' \\
\downarrow i|W \quad \downarrow j|W \\
Z
\end{array}
\]

are equivalences, hence \(k|_W\) is an equivalence as well. It follows that \(k: A \to B\) is an equivalence, hence \(r: B \to A\) is an equivalence. \(\Box\)

**Proposition 9.12.** If \(X\) is a Rezk complete decomposition space, then \(Sd(X)\) is a Rezk complete Segal space.

**Proof.** We already know from Lemma 9.2 that \(Sd(X)\) is a Segal space, so it remains to check that \(Sd(X)\) is Rezk complete. So assume that \(\tau \in (Sd X)_1\) is an equivalence, meaning that there exist \(\alpha \in (Sd X)_2\) such that \(d_\tau^0(\alpha) = \tau\) and \(d_\tau^1(\alpha) = s_\tau^0 d_\tau^0(\tau)\) and there exists \(\beta \in (Sd X)_2\) such that \(d_\tau^0(\beta) = \tau\) and \(d_\tau^1(\beta) = s_\tau^0 d_\tau^0(\tau)\). Spelling out everything in terms of the original face and degeneracy maps of \(X\), we have \(\alpha \in X_5\) such that \(d_\tau^0(\alpha) = \tau\) and \(d_{\perp+1}d_{\perp+1}(\alpha) = s\perp s_\tau d_{\perp+1}d_{\perp+1}(\tau)\) as well as \(\beta \in X_5\) such that \(d_{\perp+2}d_{\perp+2}(\beta) = \tau\) and \(d_{\perp+1}d_{\perp-1}(\beta) = s\perp s_\tau d_{\perp+1}d_{\perp-1}(\tau)\). We make the following picture of \(\tau\):

\[
\begin{array}{ccc}
a & \tau & b \\
\downarrow & \uparrow & \downarrow \\
\perp & \tau & \perp \\
\downarrow & \uparrow & \downarrow \\
u & v & u
\end{array}
\]

just to have the notation \(u := d_\tau d_{\perp+1}(\tau)\) and \(v := d_{\perp+1}d_{\perp}(\tau)\). In order to show that \(\tau\) is equivalent to an element in the image of \(s\perp s_\tau: X_1 \to X_3\), we should show
that \( u \) and \( v \) are themselves equivalences; this is because Lemma 9.9 gives that the diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{s_\bot s_\top} & X_3 \\
\downarrow & & \downarrow d_\top d_\top d_\bot d_\bot d_\bot \\
X_1^{\mathrm{eq}} \times X_1 \times X_1^{\mathrm{eq}} & \longrightarrow & X_1 \times X_1 \times X_1
\end{array}
\]

is a pullback.

The relevant 2-simplices to show that \( u \) and \( v \) are equivalences are extracted from \( \alpha \) and \( \beta \): applying \( d_\top d_\top d_\top \) to the 5-simplices \( \alpha \) and \( \beta \) we get the required 2-simplices for \( u \), and applying \( d_\bot d_\bot d_\bot \) to the 5-simplices \( \alpha \) and \( \beta \) we get the required 2-simplices for \( v \). There are thus four cases. Just as an illustration of how the argument goes, let us consider \( d_\top d_\top d_\top (\alpha) \). Here is a picture of \( \alpha \):

![Diagram](attachment:image.png)

The curved dotted lines illustrate the edges obtained by applying \( d_\bot d_\bot d_\bot d_\bot d_\bot \). By assumption, the resulting 3-simplex is doubly degenerate. Precisely,

\[
d_\bot d_\bot d_\bot d_\bot d_\bot (\alpha) = s_\bot s_\top d_\bot d_\top (\tau) = s_\bot s_\top (a).
\]

The fact that this whole 3-simplex is doubly degenerate in this way implies that also the curved edges are degenerate individually. The lower triangle in the picture is \( d_\top d_\top d_\top (\alpha) \) and it is thus degenerate in precisely the way required to exhibit \( u \) as an equivalence (from one side). With similar arguments applied to \( \beta \) we see that \( u \) is also an equivalence from the other side, and finally by Rezk completeness we can therefore conclude that \( u \) is degenerate. The analogous conclusion for \( v \) is reached using \( d_\bot d_\bot d_\bot \) instead of \( d_\top d_\top d_\top \). \( \square \)

9.13. Remark. In the special case where \( X \) is a Rezk complete Segal space (and not just a Rezk complete decomposition space), this result was proven by Mukherjee [41] in the complete Segal space model structure for bisimplicial sets.
A Relative complete maps and Rezk completion

by Philip Hackney, Joachim Kock, and Jan Steinebrunner

We prove the following:

Proposition [A.22] For $X$ a Segal space, we have $RFib(X) \simeq RFib(LX)$,

where $LX$ is the Rezk completion of $X$ (previously denoted $\hat{X}$). This result should be attributed to Boavida, who proved it in the setting of model categories [10].

Our proof is synthetic and a bit more conceptual, deriving the result from the following:

Proposition [A.14] Rezk completion is a semi-left-exact localization.

This is of independent interest. For example, it readily implies that there is a factorization system consisting of the Dwyer–Kan equivalences and the relative Rezk complete maps (Proposition [A.15]). We will also use this to show that relative complete Segal spaces over a Segal space $X$ correspond to complete Segal spaces over $LX$:

Proposition [A.16] For any Segal space $X$: $\text{Seg}_{/X}^{rc} \simeq \text{CSS}_{/LX}$.

Before coming to these results, we need to set up some terminology, notation, and a few preliminary results.

Let $E(1)$ denote the strict nerve of the contractible groupoid with two objects. By [44, Theorem 6.2] the space of equivalences $X_1^{eq} \subset X_1$ of a Segal space $X$ is equivalent to

$$X_1^{eq} \simeq \text{Map}(E(1), X).$$

Recall (from [44, §6]) that a Segal space $X$ is called (Rezk) complete when either (and hence both) of the maps

$$X_0 \overset{d_1}{\leftarrow} X_1^{eq} \overset{d_0}{\rightarrow}$$

is an equivalence (this is equivalent to the definition from [9, 7]).

The full inclusion $\text{CSS} \hookrightarrow \text{Seg}$ of complete Segal spaces into all Segal spaces has a left adjoint (reflection)

$$L : \text{Seg} \rightarrow \text{CSS}.$$ 

An explicit formula was given by Rezk [44, §14] (see also [3, Proposition 2.6] for a model independent account). We shall not need the explicit formula. What we do need is Theorem [A.2] (due to Rezk) characterizing the class of maps inverted by $L$ as the Dwyer–Kan equivalences, as we now recall.
Recall that any Segal space $X$ has mapping spaces:

$$\text{Map}_X(x, x') \to X_1 \xrightarrow{\partial} \ast \to X_0 \times X_0$$

There is an associated homotopy category $\text{ho}(X)$, with set of objects $\pi_0(X_0)$ and $\text{hom}(x, x') = \pi_0(\text{Map}_X(x, x'))$. See [44, §5].

**Definition A.1** ([44, 7.4]). A map $f: Y \to X$ between Segal spaces is a Dwyer–Kan equivalence if

1. $f$ is essentially surjective, that is, the induced map $\text{ho}(f): \text{ho}(Y) \to \text{ho}(X)$ is essentially surjective, and
2. $f$ is fully faithful, that is, for each $y, y' \in Y$, the induced map on mapping spaces $\text{Map}_Y(y, y') \to \text{Map}_X(fy, fy')$ is an equivalence.

Note that being fully faithful is equivalent to the assertion that the square

$$Y_0 \times X_0 \to X_0 \times X_0$$

is a pullback for $n = 1$ or equivalently for all $n$.

**Theorem A.2** ([44 Theorem 7.7]). The Dwyer–Kan equivalences between Segal spaces are precisely the maps that are inverted by the completion functor $L: \text{Seg} \to \text{CSS}$.

We now introduce the notion of a relative complete map between Segal spaces, which is a variation on what was called a fiberwise complete Segal space in [11, 2.2] and [10, §1.4]. (A good notion for maps between general simplicial spaces would utilize arbitrary maps $E(n) \to E(m)$.)

**Definition A.3.** Suppose $Y$ and $X$ are Segal spaces. A map $Y \to X$ is relative complete if it is right orthogonal to both morphisms $E(0) \to E(1)$.

Since there is an automorphism of $E(1)$ that permutes the two morphisms $E(0) \to E(1)$ it suffices to check the orthogonality against only one of them. Spelling this out we see that $Y \to X$ is relative complete if and only if the square

$$Y_1^{\text{eq}} \to X_1^{\text{eq}} \xrightarrow{\partial} \ast \to X_0$$
is a pullback for $i = 0$ or $i = 1$.

We begin by recording some properties of relative complete maps that are also fully faithful or essentially surjective.

**Lemma A.4.** If $Y \to X$ is a map between Segal spaces which is relative complete and fully faithful, then it is levelwise a monomorphism (of ∞-groupoids).

**Proof.** We first show that $Y_0 \to X_0$ is a monomorphism. Relative completeness implies that the square

$$
\begin{array}{ccc}
Y_0 & \to & Y^{E(1)} \\
\downarrow & & \downarrow \\
X_0 & \to & X^{E(1)}
\end{array}
$$

is a pullback of spaces. We thus have the composite pullback

$$
\begin{array}{ccc}
Y_0 & \to & Y_1^{eq} & \to & Y_1 & \to & Y_0^{x^2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_0 & \to & X_1^{eq} & \to & X_1 & \to & X_0^{x^2}
\end{array}
$$

where the right pullback square says that $Y \to X$ is fully faithful and the middle square uses conservativity of $\text{ho}(Y) \to \text{ho}(X)$. The fiber of the diagonal $Y_0 \to Y_0^{x^2}$ at $(y, y')$ is the space of paths from $y$ to $y'$, so this being a pullback shows that $\text{Path}_{Y_0}(y, y') \to \text{Path}_{X_0}(f_0 y, f_0 y')$ is an equivalence for all $y, y'$. This implies $f_0$ is a monomorphism ($\pi_0$-injective and an equivalence of spaces on each path component).

Now the right square in the above diagram is a pullback, so $f_1$ is a monomorphism. Since $Y$ and $X$ are Segal it follows that $f_n : Y_n \to X_n$ is a monomorphism for all $n$.

**Lemma A.5.** For each Segal space $X$ and $n \in \mathbb{N}$, the completion map

$$(\alpha_X)_n : X_n \to (LX)_n$$

is $\pi_0$-surjective.

**Proof.** We begin with the case $n = 0$. Since $X \to LX$ is a Dwyer–Kan equivalence it is essentially surjective. This means that every point in $LX_0$ is isomorphic in $LX$ to a point in the image of $X_0 \to LX_0$. But $LX$ is complete and hence isomorphic objects are isotopic, therefore $X_0 \to LX_0$ is $\pi_0$-surjective. For general $n$ we can use fully faithfulness to write $X_n \to LX_n$ as the base-change of $X_0^{x^{n+1}} \to LX_0^{x^{n+1}}$, which is $\pi_0$-surjective by the first part of the proof.

**Lemma A.6.** If $Y \to X$ is a map between Segal spaces which is relative complete and essentially surjective, then $\pi_0(Y_0) \to \pi_0(X_0)$ is surjective.
Proof. Given \( x \in X_0 \), since \( \text{ho}(Y) \to \text{ho}(X) \) is essentially surjective there exists \( e: E(1) \to X \) so that the following diagrams commute for some \( y \in Y_0 \).

\[
\begin{array}{ccc}
\Delta^0 & \xrightarrow{y} & Y \\
\downarrow & & \downarrow \\
E(1) & \xrightarrow{e} & X \\
\uparrow & & \uparrow \\
\Delta^0 & \xrightarrow{x} & X.
\end{array}
\]

Since \( Y \to X \) is relative complete, a unique lift exists in the top square. \( \square \)

**Proposition A.7.** A map \( Y \to X \) between Segal spaces is a levelwise equivalence if and only if it is a Dwyer–Kan equivalence and relative complete.

**Proof.** By Lemma A.4, \( Y_0 \to X_0 \) is a monomorphism of \( \infty \)-groupoids, and by Lemma A.6 it is also surjective on \( \pi_0 \), so altogether \( Y_0 \to X_0 \) is an equivalence. Since

\[
\begin{array}{ccc}
Y_1 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
Y_0 \times^2 & \longrightarrow & X_0 \times^2
\end{array}
\]

is a pullback, we see that also \( Y_1 \to X_1 \) is an equivalence, and since \( Y \) and \( X \) are Segal, all the higher \( Y_n \to X_n \) become equivalences too. \( \square \)

**Lemma A.8.** Dwyer–Kan equivalences that are \( \pi_0 \)-surjective in simplicial degree zero are stable under pullback.

By Lemma A.5 this lemma applies, in particular, to Dwyer–Kan equivalences whose codomain is a complete Segal space.

**Proof.** Suppose \( B \to A \) is such a Dwyer–Kan equivalence, and consider a pullback diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & A.
\end{array}
\]

This yields a cube of spaces, four of whose faces are given below:

\[
\begin{array}{cc}
\begin{array}{ccc}
Y_1 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
B_1 & \longrightarrow & A_1
\end{array} & \begin{array}{ccc}
Y_1 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
Y_0 \times^2 & \longrightarrow & X_0 \times^2
\end{array}
\end{array}
\]

\[
\begin{array}{cc}
\begin{array}{ccc}
B_0 \times^2 & \longrightarrow & A_0 \times^2 \\
\downarrow & & \downarrow \\
B_0 \times^2 & \longrightarrow & A_0 \times^2
\end{array}
\end{array}
\]

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The bottom left square is a pullback since $B \to A$ is fully faithful; it follows that
the top right square is a pullback as well. Hence $Y \to X$ is fully faithful.

Now if $\pi_0(B_0) \to \pi_0(A_0)$ is surjective, the same is true of $\pi_0(Y_0) \to \pi_0(X_0)$.
In particular, $\text{ho}(Y) \to \text{ho}(X)$ is surjective on objects, hence essentially surjective.

Let $L$ be the Rezk completion functor on Segal spaces, and $\alpha : \text{id} \Rightarrow L$ the
completion natural transformation.

**Lemma A.9.** The natural transformation $\alpha$ is cartesian on relative complete
maps.

**Proof.** Consider the diagram

```
Y \quad \alpha_Y
\downarrow f \quad \Rightarrow \quad \downarrow j
Q \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
```

Since $\alpha_X$ is a Dwyer–Kan equivalence, so too is $Q \to LY$ by Lemma A.8. Then
since $\alpha_Y$ is a Dwyer–Kan equivalence we have by 2-of-3 (see [44, Lemma 7.5])
that $Y \to Q$ is as well.

But now $Lf$ is relative complete since it’s a map between complete Segal
spaces, and so too is $Q \to X$, since relative complete maps are stable under pullback. The map $f$ was assumed relative complete, so $Y \to Q$ is also relative
complete by left cancellation. Proposition A.7 implies that $Y \to Q$ is an
equivalence.

The proof of the preceding lemma really shows that any square whose
vertical edges are relative complete and whose horizontal edges are Dwyer–Kan
equivalences must in fact be a pullback.

**Corollary A.10.** The following classes of maps between Segal spaces coincide:

1. relative complete maps,
2. those maps on which the natural transformation $\alpha$ is cartesian, and
3. pullbacks of maps between complete Segal spaces.

**Proof.** Every map between complete Segal spaces is relative complete, and
pullbacks of relative complete maps are relative complete, so class (3) is contained
in class (1). Lemma A.9 states that (1) is contained in (2), and the final
containment (2) $\subseteq$ (3) is immediate. 

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Definition A.11. Suppose \( C \rightarrow \mathcal{D} \) is a reflective subcategory of a finitely-complete \( \infty \)-category \( \mathcal{D} \), with left adjoint \( L: \mathcal{D} \rightarrow C \). We say that \( L \) is semi-left-exact if, for all pullback squares of \( \mathcal{D} \) below-left with \( S, T \in C \),

\[
\begin{array}{ccc}
Y & \rightarrow & T \\
\downarrow & & \downarrow \\
X & \rightarrow & S
\end{array}
\quad
\begin{array}{ccc}
LY & \rightarrow & T \\
\downarrow & & \downarrow \\
LY & \rightarrow & S
\end{array}
\]

the square above-right is also a pullback (in \( C \)).

This terminology agrees with that of Cassidy–Hébert–Kelly in the 1-categorical case [15, p.298]. Gepner and Kock [29, 1.2] also consider this condition in the case of locally cartesian closed \( \infty \)-categories, and call such left adjoints locally cartesian localizations.

Proposition A.12. Suppose \( L: \mathcal{D} \rightarrow C \) is a semi-left-exact reflector (where \( \mathcal{D} \) is finitely-complete), and \( \alpha : \text{id} \Rightarrow L \) is the unit of the reflection. Then there is a factorization system \( (L, R) \) on \( \mathcal{D} \), where \( L \) are those maps inverted by \( L \), and \( R \) is the class of maps on which the natural transformation \( \alpha \) is cartesian.

Proof. This is a straightforward generalization to \( \infty \)-categories of a classical theorem of Cassidy–Hébert–Kelly [15, Theorem 4.3]. In the setting of \( \infty \)-categories it is essentially Proposition 3.1.10 of [2] except that they have “left-exact” in place of “semi-left-exact.” Inspecting the second paragraph of their proof one sees that only the semi-left-exact condition is actually used. \( \square \)

Recall the following (for example from [50, Lemma 3.3]).

Lemma A.13. Consider a diagram of spaces

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
 & \rightarrow & Z
\end{array}
\]

where the left square is a pullback, the composite rectangle is a pullback, and the map \( f \) is \( \pi_0 \)-surjective. Then the right square is a pullback as well. \( \square \)

Proposition A.14. The completion functor \( L: \text{Seg} \rightarrow \text{CSS} \) is semi-left-exact.

Proof. Consider a pullback of Segal spaces

\[
\begin{array}{ccc}
Y & \rightarrow & T \\
\downarrow & & \downarrow \\
X & \rightarrow & S
\end{array}
\]
where $S$ and $T$ are complete Segal spaces. Since $T \to S$ is automatically relative complete, it follows that $Y \to X$ is relative complete. We then have the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\alpha_Y} & LY \\
\downarrow & & \downarrow \\
X & \xrightarrow{\alpha_X} & LX
\end{array}
\xrightarrow{\phi} 
\begin{array}{ccc}
T & \xrightarrow{\phi} & S
\end{array}
$$

where the left square is a pullback by Lemma A.9 and the outer square is a pullback by assumption. Our goal is to show that the right-hand square is a pullback as well. By Lemma A.5 the map $X_n \to LX_n$ is surjective on path components. Thus by Lemma A.13, for each $n$ the right square in

$$
\begin{array}{ccc}
Y_n & \xrightarrow{\alpha_{Y_n}} & LY_n \\
\downarrow & & \downarrow \\
X_n & \xrightarrow{\alpha_{X_n}} & LX_n
\end{array}
\xrightarrow{\phi} 
\begin{array}{ccc}
T_n & \xrightarrow{\phi} & S_n
\end{array}
$$

is a pullback. Thus $L(T \times_S X) \to T \times_SLX$ is an equivalence, as desired.

**Proposition A.15.** Dwyer–Kan equivalences and relative complete maps constitute a factorization system on $\text{Seg}$.

**Proof.** This is the factorization system guaranteed by Proposition A.12 using that the localization $L: \text{Seg} \to CSS$ is semi-left-exact (Proposition A.14). The Dwyer–Kan equivalences are precisely those maps between Segal spaces which are inverted by $L$. In Corollary A.10 we identified the relative complete maps as the right class in this factorization system.

**Proposition A.16.** Suppose $X$ is a Segal space. Then $\alpha^*: \text{Seg}_{/LX} \to \text{Seg}_{/X}$ restricts to an equivalence $CSS_{/LX} \to \text{Seg}^{rc}_{/X}$, into the full subcategory of the relative complete maps with codomain $X$. The inverse is given by the completion $\mathbb{L}$.

**Proof.** Recall (from 8.9 and §5.2.5.1) that the sliced localization functor $L_X: \text{Seg}_{/X} \to CSS_{/LX}$ has right adjoint

$$
CSS_{/LX} \xhookleftarrow{} \text{Seg}_{/LX} \xrightarrow{\alpha^*} \text{Seg}_{/X}
$$

with counit (at an object $A \to LX$) given by

$$
L(X \times_{LX} A) \xrightarrow{L(\text{pt}_2)} LA \xrightarrow{\varepsilon_A} A.
$$

The statement is that this counit is invertible. But since $L$ is semi-left-exact, the first map is just the projection $L(X \times_{LX} A) \cong LX \times_{LX} LA \to LA$ which is an equivalence, and the second map $\varepsilon_A$ is an equivalence since the inclusion $CSS \subset \text{Seg}$ is full. (This argument is borrowed from the proof of [29, Lemma 1.7], which however unnecessarily assumes the ambient $\infty$-category to be locally cartesian closed.) Corollary A.10 identifies the image of (7) as the relative complete maps.
Remark A.17. In their work on configuration categories, Boavida and Weiss\cite{BoavidaWeiss} Appendix B introduce a model category of relative complete Segal spaces over a Segal space $X$. Proposition A.16 shows that the $\infty$-category underlying this model category is equivalent to the slice category $\text{Cat}_{\infty/LX}$.

Corollary A.18. If $f: Y \to X$ is a Dwyer–Kan equivalence between Segal spaces, then $f^*: \text{Seg}_{/X}^{rc} \to \text{Seg}_{/Y}^{rc}$ is an equivalence.

Proof. Since $Lf: LY \to LX$ is an equivalence of simplicial spaces, we have the indicated equivalences in the following commutative square.

$$
\begin{array}{ccc}
\text{CSS}_{/LX} & \xrightarrow{(Lf)^*} & \text{CSS}_{/LY} \\
\alpha_X^* \downarrow & & \downarrow \alpha_Y^* \\
\text{Seg}_{/X}^{rc} & \xrightarrow{f^*} & \text{Seg}_{/Y}^{rc}
\end{array}
$$

It follows that $f^*$ is an equivalence as well. \qed

The proof of Proposition A.16 and its corollary did not use anything special about our situation, other than $L: \text{Seg} \to \text{CSS}$ being semi-left-exact. We conclude that the following proposition holds (and the case $Y \to LY$ recovers the statement of Proposition A.16).

Proposition A.19. Suppose $L: \mathcal{D} \to \mathcal{C}$ is a semi-left-exact reflector and $(\mathcal{L}, \mathcal{R})$ is the factorization system on $\mathcal{D}$ from Proposition A.12. If $f: Y \to X$ is in $\mathcal{L}$, then $f^*: \mathcal{R}_{/X} \to \mathcal{R}_{/Y}$ is an equivalence. \qed

We now examine the implications of these results to our maps of interest, right fibrations.

Lemma A.20. Right fibrations between Segal spaces are relative complete.

Proof. It is enough to observe that $1: \Delta^0 \to E(1)$ is a final map (by Lemma 2.12) since it preserves the last vertex. \qed

Proposition A.21. If $Y \to X$ is a right fibration between Segal spaces, then $LY \to LX$ is also a right fibration.

Proof. By Lemma A.9 and Lemma A.20, the following naturality square is a pullback.

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & LY \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & LX
\end{array}
$$

It follows that the right square below is a pullback, while the left square is a pullback since $Y \to X$ is a right fibration.

$$
\begin{array}{ccc}
Y_n & \xrightarrow{f_n} & Y_0 \to LY_0 \\
\downarrow & & \downarrow & & \downarrow \\
X_n & \xrightarrow{g_n} & X_0 \to LX_0
\end{array}
$$
We then have the left and outer squares of the following are pullbacks.

\[
\begin{array}{ccc}
Y_n & \longrightarrow & LY_n \\
\downarrow & & \downarrow \\
X_n & \longrightarrow & LX_n \\
\end{array}
\quad \text{and}
\begin{array}{ccc}
LY_n & \longrightarrow & LY_0 \\
\downarrow & & \downarrow \\
LX_n & \longrightarrow & LX_0 \\
\end{array}
\quad (8)
\]

We wish to deduce that the right hand square is a pullback. But by Lemma \[A.5\] \(X_n \to LX_n\) is surjective on path components, so by Lemma \[A.13\] the right square in \((8)\) is a pullback.

The following equivalence is a restriction of that of Proposition \[A.16\]. It more or less recovers \[10\] Corollary 5.6.

**Proposition A.22.** Suppose \(X\) is a Segal space. Then \(\alpha_X^* : \text{RFib}_{/LX} \to \text{RFib}_{/X}\) is an equivalence, with inverse given by the completion \(L\).

**Proof.** The fully-faithful functor \(\text{RFib}_{/X} \to \text{Seg}_{/X}\) factors through \(\text{Seg}^{\text{rc}}_{/X}\) by Lemma \[A.20\] Any right fibration whose codomain is a complete Segal space also has a complete Segal space as its domain. By Proposition \[A.21\] we have the following lift of \(L\).

\[
\begin{array}{ccc}
\text{RFib}_{/X} & \longrightarrow & \text{RFib}_{/LX} \\
\downarrow & \text{f.f.} & \downarrow \text{f.f.} \\
\text{Seg}^{\text{rc}}_{/X} & \sim & \text{CSS}_{/LX} \\
\end{array}
\]

Of course \(\alpha_X^*\) restricts to \(\text{RFib}_{/LX} \to \text{RFib}_{/X}\), and since the vertical maps are fully faithful, this implies the result.

The following more or less recovers \[10\] Proposition 5.5.

**Corollary A.23.** If \(f : Y \to X\) is a Dwyer–Kan equivalence between Segal spaces, then \(f^* : \text{RFib}_{/X} \to \text{RFib}_{/Y}\) is an equivalence.

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