The microlensing rate and mass function vs. dynamics of the Galactic bar

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ABSTRACT

With the steady increase of the sample size of observed microlenses towards the central regions of the Galaxy, the main source of the uncertainty in the lens mass will shift from the simple Poisson noise to the intrinsic non-uniqueness of our dynamical models of the inner Galaxy, particularly, the Galactic bar. We use a set of simple self-consistent bar models to investigate how the microlensing event rate varies as a function of axis ratio, bar angle and velocity distribution. The non-uniqueness of the velocity distribution of the bar model adds a significant uncertainty (by about a factor of 1.5) to any prediction of the lens mass. Kinematic data and self-consistent models are critical to lift the non-uniqueness. We discuss the implications of these results for the interpretation of microlensing observations of the Galactic bulge. In particular we show that Freeman bar models scaled to the mass of the Galactic bulge/bar imply a typical lens mass of around $0.8M_\odot$, a factor of 3-5 times larger than the value from other models.

Key words: dark matter - gravitational lensing - galactic centre

1 INTRODUCTION

Five years of searches for gravitational microlenses, which significantly increase the brightness of a background source star when they come to the same line of sight by chance, have produced roughly 200 possible microlensing events towards the Galactic bulge/bar from several surveys (e.g., Udalski et al. 1994; Alard et al. 1995; Alcock et al. 1997; The expectation is that with more and more events coming in at a steady rate, we will obtain a better understanding of the structure of the Galaxy, and of the mass spectrum of lenses (see Paczyński 1996 for a review). Microlensing is perhaps the only direct method to probe the mass function in the Galactic bulge/bar, in particular the fraction of low luminosity objects (brown dwarfs/M-dwarfs) just below or just above the hydrogen burning limit. These objects are so faint intrinsically ($M_V > 10$mag) and so far away (distance modulus of about 14.5 mag) that they are missing in even the deepest star count studies of the bulge/bar with the Hubble Space Telescope (Gould, Bahcall & Flynn 1996).

Several attempts have been made to estimate the typical mass of the observed microlenses (Kiraga & Paczyński 1994; Zhao, Spergel & Rich 1994; Han & Lee 1996). On the basis that events last longer for more massive lenses, with $m \propto t^2$, where $m$ is the mass of a single point lens and $t$ is the time for a source on the Einstein ring (where the amplification factor is 1.34) to move to exactly behind the lens. The time scale $t$ can be derived by fitting the light curve of the amplified source, but to convert it to the lens mass $m$ requires knowing other system parameters, including the distances and transverse velocities of the lens and the source. These, unfortunately, are often poorly known from observations. Nevertheless the problem can be partially circumvented if one has a well-determined dynamical model of the inner Galaxy. The missing information can then be simulated by drawing random samples of the lens and the source in a Monte-Carlo fashion from the model phase-space distribution of the inner Galaxy.

With the increase of the event sample size, it becomes meaningful to ask the question whether the whole mass spectrum of the lenses can be determined in the above way (Han & Lee 1997). This requires an understanding of the uncertainty of the underlying dynamical model. Models of the inner Galaxy are, unfortunately, still far from being well-determined and they are subject to constant modifications and improvements, driven by new observations (see e.g., reviews by de Zeeuw 1993; Gerhard 1996). How much the Galactic bar differs from a simple oblate rotator is an unsettled issue. Generally speaking the density distribution of...
the Galactic bar is constrained only up to a one-parameter sequence by the integrated COBE map, with the bar angle being a free parameter (Binney, Gerhard & Spergel 1997; Zhao 1997). Various velocity structures of the bar could also be consistent with the same bar potential (Pfenniger 1984). Schwarzschild-type models for the inner Galaxy (Zhao 1996; Binney et al. 1997) which attempt to fit simultaneously the photometric and kinematic data measurements are still rare.

The observed velocity field of a bar is the result of its tumbling motion, the internal streaming motion on top of the pattern rotation, and the velocity anisotropy. Bars of different pattern speed and/or orbital compositions are indistinguishable in their projected density and optical depth maps, but they differ at the level of event rate and event duration distribution. This is easily seen from the basic equation of microlensing,

\[
\frac{1}{t} = \frac{V}{R_E} = V \left( \frac{4GmD}{c^2} \right)^{-1/2}, \quad D = \frac{D_l}{D_s} (D_s - D_l),
\]

where \( G \) is the gravitational constant, \( c \) is the velocity of light, \( R_E \) is the Einstein ring radius, \( t \) is the event duration (one Einstein radius crossing time), \( D_l \) and \( D_s \) are the distances to the lens and the source, \( V \) is the relative transverse speed between the two, and \( m \) is the mass of the lens. For most microlensing events one observes only their duration, \( t \), from which one can obtain a lensing probability (optical depth) \( \tau \) and an event rate \( \Gamma \) observationally. Other system parameters, such as the speed \( V \) and distances \( D_l \) and \( D_s \) can at best be inferred statistically from a phase-space density model of the Galaxy. As a result the rate is related to the dynamical model as follows (cf. eq. [7] of Paczyński 1986)

\[
\frac{\Gamma}{\tau} = \frac{2}{\pi} \left( \frac{1}{t} \right) = \frac{c}{\pi \sqrt{GmD}} \langle V \rangle, \tag{2}
\]

where the brackets indicate averages. In order to emphasize the effect of the velocity distribution we have fixed the mass of the lenses \( m \) and the positions of the lens and source.

Eq. (3) can also be interpreted as a constraint on lens mass. If the event rate is fixed at the observed value and the lens volume density model is fixed to fit the observed optical depth, then eq. (2) implies that

\[
m \propto \langle V \rangle^2, \tag{3}
\]

so that the inferred lens mass is sensitive to the velocity distribution of the dynamical model. An uncertainty of 1.2 – 2 in velocity translates to a factor of 1.4 – 4 in the lens mass, in which case one could mistake, e.g., an M-dwarf dominated model for a brown-dwarf dominated model.

To illustrate this point, it is best to take one step backwards to consider the simplest axisymmetrical models. If one observes the two discs shown in Figure 1 edge-on, then the distribution of transverse relative source-lens speed \( V = |v_s - v_l| \) in the simplest case where the lens and the source are at the front or on the back side of an annulus.

Figure 1. Two simple discs. Models to the right (R-type model) are rotating models where all stars (lenses and sources) move in the clockwise direction, and models to the left are non-rotating with half of the stars (lenses and sources) orbiting in the clockwise sense, half anti-clockwise. The transverse velocity of the lens and the source at the line of sight to the center (dotted lines) is \( \pm V_c \). The lower panel draws schematic distributions of the relative source-lens speed \( V = |v_s - v_l| \) in the simplest case where the lens and the source are at the front or on the back side of an annulus.
models. A side result is to give an error bar for predictions of the lens mass due to the non-uniqueness of the bar.

The rigorous way to survey microlensing properties of Galactic bar models is to build many sequences of three-dimensional bar models such as in Zhao (1996) which cover the multi-parameter space. But such heavy numerical modelling would be very inefficient. For a first study it is more interesting to gauge the underlying physical effects with simple theoretical models. In this paper we show which insights can be gained from studying the microlensing properties of the analytical two-dimensional Freeman (1966) bars, which surprisingly capture most of the microlensing effects of a realistic bar fairly well.

The outline of the paper is as follows. In §2, we summarize the properties of the Freeman bars, and we derive the optical depth for self-lensing. In §3, we derive basic scaling relations for the event rate distribution. In §4 we show the run of event rate as functions of bar parameters. In §5 we show how to distinguish Freeman bars with the same projected density and optical depth map, and in §6 we derive the rate for extremely short events. Finally, we discuss implications for the Galactic bar in §7. Some mathematical details are given in the Appendix.

2 SELF-LENSING OF THE FREEMAN BARS

Freeman (1966, hereafter F66) discovered self-consistent tumbling bar models with a known analytical distribution function. These two-dimensional bars have been widely used to gain insight into the structure of general self-consistent bars (Hunter 1974; Tremaine 1976; Weinberg & Tremaine 1983). Despite the two-dimensional nature and the special distribution functions, this class of easy-to-build bars contains the main factors by which bars can affect the microlensing rate. These include its elongated density contours and similarly elongated velocity ellipsoid, plus the pattern rotation and streaming motion. Our aim is to vary these factors and “observe” the bar both at a variable line-of-sight angle from its long axis and at a variable projected radius.

2.1 Freeman bars

The Freeman bar is a two-dimensional inhomogeneous elliptical bar of finite extent, with surface density $\Sigma$ given by

$$\Sigma(X, Y) = \frac{3M}{2\pi ab} (1 - \phi)^2, \quad \phi = \frac{X^2}{a^2} + \frac{Y^2}{b^2},$$

and a potential

$$\Phi(X, Y) = \frac{GM}{2(ab)^{3/2}} [A_2(q)X^2 + B_2(q)Y^2], \quad q \equiv \frac{b}{a},$$

where $M$ is the total mass, $a$ and $b$ are the semi-major ($X$) and semi-minor ($Y$) axes of the bar, and $A_2(q)$, $B_2(q)$ are dimensionless function of the axis ratio $q$ given in equation (2) of F66. The symmetry axes ($X$ and $Y$) rotate with an angular speed $\Omega$ with respect to the rest frame. For a given density, there is a sequence of self-consistent models with different amounts of pattern rotation. The sequence is a function of $\Omega/\Omega_J$, where

$$\Omega_J^2(M, a, b) \equiv \frac{GM}{(ab)^2} \frac{q^{-1}A_2 - qB_2}{q^{-1} - q},$$

is so defined that when $\Omega = \Omega_J$ the bar is of Jacobi type, that is, with neither internal streaming motion nor velocity anisotropy.

The potential (5) is that of a two-dimensional anisotropic harmonic oscillator with generally incommensurable frequencies in the $X$ and $Y$ directions. The motion is separable in canonical coordinates $(P_1, P_2, Q_1, Q_2)$ defined by F66. Accordingly, all orbits have two independent isolating integrals of motion, and they are in fact rectangular Lissajous figures in the $(Q_1, Q_2)$-plane. In the corotating $(X, Y)$-frame, the orbits show a more interesting variety of shapes, similar to what is seen in more general bar potentials. We illustrate these in Figure 2. The boundary curves of each orbit can be worked out analytically. We give the equation in Appendix A. For the fast tumbling bar, the orbits range from direct to retrograde. As the tumbling frequency approaches zero, all orbits become rectangular boxes. By properly weighting orbits with different size and axis ratio, F66 was able to produce a self-consistent elliptical bar. Its distribution function, which is positive definite, has the simple form

$$f(J) = f_0(1 - J)^{-2}, \quad \text{if } 0 \leq J \leq 1,$$

$$= 0 \quad \text{otherwise.}$$

\[ \begin{array}{c}
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\[ \rho(X, Y, Z) = \frac{\Sigma(X, Y)}{\Delta}, \quad \Delta \ll \min(a, b). \] (8)

ZM use a Galacto-centric Cartesian coordinate system \((x, y, z)\) with the \(x\)-axis pointing away from the Sun, making an angle \(\alpha\) with the bar's long \((X)\) axis. Similar but more convenient is the following coordinate system \((\tilde{x}, y, z)\) with
\[ \tilde{x} \equiv x - Cy, \quad C = \frac{(q - q^{-1})\sin\alpha\cos\alpha}{q\cos^2\alpha + q^{-1}\sin^2\alpha}, \] (9)
so that it is obtained by applying a shear to the original \((x, y, z)\) coordinate in the line-of-sight direction, as illustrated in Figure 3. In these coordinates the face-on density of the Freeman bar can be rewritten as
\[ \Sigma(\tilde{x}, y) = \frac{3M}{2\pi ab} \left( 1 - \frac{\tilde{x}^2}{d_0^2} - \frac{y^2}{y_0^2} \right)^{\frac{1}{2}}, \] (10)
where \(2y_0\) and \(2d_0\) are the projected width and the central depth of the bar. Both quantities are functions of \(a, b\) and \(\alpha\), given by
\[ y_0 = (ab\bar{\gamma})^{\frac{1}{2}}, \quad d_0 = \left( \frac{ab}{\bar{\gamma}} \right)^{\frac{1}{2}}, \] (11)
with
\[ \gamma(a, q) \equiv q^{-1}\sin^2\alpha + q\cos^2\alpha. \] (12)

A line of sight with an impact parameter \(y\) intersects with the bar’s near end at \(\tilde{x} = -d(y)\) and far end at \(\tilde{x} = +d(y)\), where
\[ d(y) = d_0 \left( 1 - \frac{y^2}{y_0^2} \right)^{\frac{1}{2}}, \] (13)
so that \(2d(y)\) is the depth of the bar at \(y\). For an axisymmetric disc, \(a = b = y_0 = d_0 = \sqrt{d(y)^2 + y^2}\), and \(\gamma = q = 1\).

The transverse velocity distribution depends on the velocity field of the bar, the orientation and the impact parameter of the line of sight. It is given by a box car distribution:
\[ P(v_y) = \int f(J)dv_x \] (14)
\[ = \frac{1}{2w} \quad \text{if } |v_y - \bar{v}| \leq w, \]
\[ = 0 \quad \text{otherwise}, \]
where the local transverse streaming velocity \(\bar{v}\) is a linear function of \((x, y)\), and the half width \(w\) is \(\sqrt{3}\) times the local transverse dispersion, which is proportional to \((1 - \phi)^{\frac{3}{2}}\). \(\bar{v}\) can be split into a component \(\tilde{v}\) proportional to \(\tilde{x}\) and a (less useful) component proportional to \(y\). More rigorously
\[ \frac{\tilde{v}}{\sigma(y)} = \lambda \frac{\tilde{x}}{d(y)}, \quad \frac{w}{\sigma(y)} = \sqrt{3} \left( 1 - \frac{x^2}{d^2(y)} \right)^{\frac{1}{2}}, \] (15)

\[ \dagger \] The non-axisymmetric perturbation in the COBE/DIRBE map of the Galaxy has a thickness no less than the thin disc \((\Delta > 2 \times 200\text{pc})\) and semi-minor axis no greater than the corotation radius \((b < a < 4\text{kpc})\) so the requirement that \(\Delta/b \ll 1\) is barely satisfied.

\[ \dagger \]
where we have rescaled the $\tilde{x}$ coordinate by $d(y)$, the $y$ coordinate by $y_0$ and the velocities by $\sigma(y)$, which is defined as
\[
\sigma(y) \equiv \sigma_0 \left( 1 - \frac{y^2}{y_0^2} \right)^{1/2}.
\]
and $\sigma_0 = \sigma(y = 0)$ is the central transverse dispersion. The parameter $\lambda$ is defined by
\[
\lambda \left( \frac{b}{a} \right) \equiv \frac{\tilde{v}_{\text{max}}}{\sigma_0},
\]
where $\tilde{v}_{\text{max}}$ is the transverse rotation speed at $\tilde{x} = \pm d_0$ along the $y = 0$ line. So $\lambda$ is a dimensionless global (independent of $y$) indicator of the amount of rotational support in the system (similar to the conventional $V_{\text{rot}}/\sigma$ parameter, e.g., Binney & Tremaine 1987). Both $\sigma_0$ and $\lambda$ are lengthy functions of the bar axis ratio and pattern speed, which are given in Freeman (1966) in the case that the bar angle $\alpha = 0^\circ$ or $90^\circ$. The dependence on $\alpha$ is then easily worked out with
\[
\left[ \begin{array}{c} \sigma_2^2(\alpha) \\ (\lambda \sigma_0)(\alpha) \end{array} \right] = \left[ \begin{array}{c} \sigma_2^2(0) \\ (\lambda \sigma_0)(0) \end{array} \right] \left( \frac{\cos^2 \alpha}{\sin^2 \alpha} \right).
\]
For our purpose it is only important to know that for a nonrotating bar $\lambda = 0$ and the velocity ellipsoid is aligned with the bar; for a Jacobi-type rotator ($\Omega = \Omega_f$), stars have an isotropic dispersion on top of a solid body pattern rotation with respect to the rest frame.

More relevant to the microlensing is the lens-source relative transverse speed $V$ and its distribution $F(V)$, which is a convolution of the distribution $P(v_y)$ of eq. (14) for the transverse velocity distribution of the lens or the source.
\[
F(V) = \int P(v_y) \left[ P(v_y - V) + P(v_y + V) \right] \, dv_y,
\]
where we have integrated over the source velocity $v_y$. The integration can be carried out, but the result is somewhat lengthy, and is given in eq. (21). We remark that $F(V)$ satisfies the normalization $\int_0^\infty F(V) \, dV = 1$.

### 2.3 Optical depth

Following ZM, we assume that the sources are distributed with a number density proportional to the mass density $\rho_s$, which is in turn proportional to $\Sigma_s$. Then the lensing optical depth averaged over all sources along the line of sight with impact parameter $y$ is
\[
\tau(y) = \int_{-d(y)}^{+d(y)} \int_{-d(y)}^{+d(y)} d\tilde{x}_s \Sigma_s \, d\tau_s / \int_{-d(y)}^{+d(y)} d\tilde{x}_s \Sigma_s
\]
where
\[
d\tau_s = \frac{4\pi G}{c^2} (\tilde{x}_s - \tilde{x}_l) \rho_l \, d\tilde{x}_l,
\]
is the optical depth for a source located at $(\tilde{x}_s, y, 0)$, and $\rho_l$ is the volume density at the lens position. Here the subscripts $s$ and $l$ denote quantities of the source and the lens, respectively. For the Freeman bar:
\[
\tau(y) = \frac{128G M}{15\pi c^2} \frac{1 - y^2}{y_0^2} \left( 1 - \frac{y^2}{y_0^2} \right)^{3/2}.
\]

### 3 EVENT RATE DISTRIBUTION

#### 3.1 Definitions

The duration of a microlensing event is related to the lens mass $m$ and to the relative distance $\tilde{x}_s - \tilde{x}_l$ and velocity $V$ between the lens and the source by (cf. eq. 19)
\[
t \approx \frac{1}{V} \sqrt{\frac{4Gm}{c^2} (\tilde{x}_s - \tilde{x}_l)},
\]
where we have made the approximation $D \approx \tilde{x}_s - \tilde{x}_l$ for self-lensing of a far-away bar. If one fixes the lens mass $m$ and the source position $\tilde{x}_s$, then the probability of observing an event with any duration is the optical depth $\tau_s$ (cf. eq. 39). But only the fraction $F(V)dV$ of the lenses with relative velocity $V$ to $V + dV$ contributes to the event with duration $t$ to $t + dt$, where $F(V)$ is the relative velocity distribution given in eq. (13). So the contribution $d\Gamma$ to the event rate is
\[
d\Gamma = \frac{2}{\pi t} d\tau_s F(V) dV,
\]
where $2/(\pi t)$ is the average frequency of an event with time scale $t$, and $2/\pi$ is the ratio of the Einstein diameter to the area in dimensionless units (Paczynski 1991). It follows that the differential duration distribution is given by
\[
\frac{d\Gamma}{d\log t} = \frac{2\ln 10}{\pi t} d\tau_s F(V) .
\]
Just as for the case of the optical depth (20), the observable rate should be averaged over the source distribution along the line of sight. We define the microlensing duration distribution profile $f(\log t)$ normalized by the optical depth as
\[
f(\log t) \equiv \frac{d\Gamma}{\tau d\log t}
\]
so that
\[
\int f(\log t) d\log t = \frac{\Gamma}{\tau},
\]
where $\Gamma$ is the total event rate. Then the source density averaged duration profile is given by (cf. eqs 20 and 21)
\[
f(\log t) = \frac{2 \ln 10}{\pi t} \frac{1}{\tau} \int d\tilde{x}_s \int d\tilde{x}_l (\tilde{x}_s - \tilde{x}_l) \Sigma_s \Sigma_l \frac{V F(V)}{4Gm V^2}.
\]
Note that $f(\log t)$ is independent of the bar thickness $\Delta$.

#### 3.2 Scaling relations

The normalized event duration distribution $f(\log t)$ should have the dimension of a to-be-defined typical frequency $\nu$, multiplied by a function of the dimensionless time $\nu t$. We write this concisely as
\[
f(\log t) = \nu g_\nu(\nu t),
\]
where
\[
\nu \equiv \frac{\sigma(y)}{\sqrt{\sigma^2 - d(y)}} \equiv \frac{\sigma_0}{\sqrt{\sigma_0^2 - d_0}} \left( 1 - \frac{y^2}{y_0^2} \right)^{1/2}.
\]
and 1/ν is a typical time scale for the events, λ is the ratio of rotation vs. random motion in the model, and g_λ(μ) is a dimensionless function of μ whose functional form depends only on λ, which is a dimensionless function of b/a, Ω/Ω_f, and α. The exact derivation of how f(log t) depends on λ is given in Appendix B.

Eq. (29) can be understood intuitively as follows. Consider a line of sight with impact parameter y to a stationary Freeman bar model. The lens-source distance (x_s − x_l) scales with the bar’s depth d(y), and the transverse velocity V scales with the dispersion σ(y) (cf. eq. [3]). So the Einstein radius R_E ≈ [md(y)]^{1/2} and the event rate scale with \( \dot{\tau} = \frac{V}{R_E} \approx \frac{\sigma(y)}{[md(y)]^{1/2}} \propto \nu \). As a result, the event duration distribution depends on the bar density parameters, angle and impact parameter only through ν. For rotating bars, the profiles also depend on the dimensionless quantity λ.

A surprising but direct result of the scaling relation eq. (29) is that all non-rotating Freeman bars have the same “shape”, as illustrated in Fig. 4. This means that although the log t vs. log f diagram of models with the same λ have different median event duration and total rate, they will coincide after shifting the zero points in both axes by a constant log ν. In particular, there is a one-to-one correspondence between a non-rotating Freeman bar with a non-rotating Kalnajs (1976) disc, which is an axisymmetric version of the Freeman bar.

Interestingly the event duration distributions of all Freeman bars with the same λ also should have the same “shape”. The shape is a function of λ only. For non-rotating models λ = Ω = 0. One can always find a Kalnajs disc with the same rotation-to-dispersion ratio (λ) and the same shape of the microlensing event duration distribution. Just as for the dimensionless \( \frac{\tilde{a}}{\tilde{a}, \alpha, \tilde{y}} \), the width of the event duration distribution depends on the bar angle and axis ratio directly, but depends on the size and mass of the bar only through the normalized pattern speed \( \tilde{v}/v_f(M, a, b) \).

The shape of the event duration profiles is also invariant when the same bar model is viewed at different impact parameters because the global parameter λ is independent of the impact parameter y (cf. Fig. 4).

### 3.3 Asymptotic behaviour

Now we consider the asymptotic behavior of the event time scale distribution. The asymptotic power-law profiles are evident in Figure 4. This is because very short events come from a pair of lens and source which are very close in distance, and very long events happen if they are very close in proper motion. Mao & Paczyński (1996) show that this results in two generic power-law profiles for \( f(\log t) \) at very small or large values of t for three-dimensional models. Similar to these authors, we found the following asymptotic relations for the two-dimensional Freeman bar.

\[
\frac{1}{\nu} \cdot f(\log t) = g_\lambda(\nu t) \\
\rightarrow \frac{108 \ln 10}{35\pi} (\nu t)^3 \quad t \to 0, \tag{31}
\]

\[
\rightarrow \frac{108 \ln 10}{35\pi} \xi(\lambda)^5 (\nu t)^{-2} \quad t \to \infty, \tag{32}
\]

where ξ is a dimensionless function of λ with ξ(0) = 0.94 (cf. eq. 36).

The full profile of \( f(\log t) \) can be constructed, approximately, by interpolating between the asymptotic relations

\[
\frac{1}{\nu} \cdot f(\log t) = g_\lambda(\nu t) \\
\approx \frac{108 \ln 10}{35\pi} (\mu)^3 \left[ 1 + \left( \frac{\mu}{\xi} \right)^{\frac{1}{5}} \right]^{-5p}, \quad \mu \equiv \nu t, \tag{34}
\]

where p is a measure of the width of the distribution. Both p and ξ are weakly decreasing functions of λ. We find that

\[
p(\lambda) \approx 0.36 \left( 1 - \frac{\lambda^2}{2} \right), \quad \xi(\lambda) \approx 0.94 \left( 1 + \frac{\lambda^2}{5} \right)^{-\frac{1}{5}}, \tag{35}
\]

\( 5 \) for very long duration, \( f(\log t) \) generally is proportional to \( (\nu t)^{-dim} \), where dim is the dimension of the system. So for three-dimensional bars, \( \frac{1}{\nu} f(\log t) \propto (\nu t)^{-3} \) when \( t \to \infty \).
together with eq. (35), give a reasonably good approximation to $f(\log t)$, good within 10% for non-rotating Freeman bars ($\Omega = \lambda = 0$) (cf. Fig. 4). Using this interpolation, we find that $\log t$ has a mean at approximately $-0.14 - \log \nu$ dex with an rms width approximately 0.32 dex; both the mean and the width are also weak functions of $\lambda$.

The function $F(\log t)$ is independent of $\lambda$ or rotation for very short events because the lens and source of these events are very close in distance, so their relative rotation speed is always zero. The short duration events are particularly useful for constraining the mass of the lenses in the bar because lenses in the foreground disc are generally sufficiently far away from the sources in the bar that they do not contribute significantly to the short events.

Compared to non-rotating models, the $F(\log t)$ profiles are slightly narrower for models with increasing rotation. This can be understood as a smaller spread in the velocities of lens and source in rotating models leads to a smaller spread of the event duration. However, the variation of the width is small (see Fig 4), and to good approximation Freeman bars all have very similar $F(\log t)$ profiles.

### 4 THE TOTAL EVENT RATE AS FUNCTION OF THE BAR PARAMETERS

#### 4.1 The size and mass of the bar

The total event rate per optical depth is given by eqs. (27) and (29):

$$\frac{\Gamma}{\tau} = \int \nu g_{\lambda}(\nu t) d(\log t) = \xi_1(\lambda) \nu,$$

(37)

where $\xi_1$ is a dimensionless increasing function of $\lambda$. We find

$$\xi_1(\lambda) \approx 0.4 \left(1 + \frac{\lambda^2}{5}\right).$$

(38)

Eq. (17) shows that $F(\log t)$ is a function of $\nu$ and $\lambda$ only, and it is proportional to $\nu$. We now show that several useful relations follow from eqs. (47) and (29).

The event rate scales with the lens mass $m$, the bar mass $M$, size $L \equiv \sqrt{ab}$, and thickness $\Delta$. We define the following characteristic lens-source velocity $V^*$, Einstein radius $R_E^*$, and frequency $\nu^*$,

$$V^* = \sqrt{\frac{G M}{L}},$$

$$R_E^* = \sqrt{\frac{G m}{c^2 L}},$$

$$\nu^* = \frac{V^*}{R_E^*} = \frac{c}{L} \sqrt{\frac{M}{m}}.$$  

Then

$$\frac{\Gamma}{\tau} \propto \nu^* = \frac{c}{\sqrt{ab}} \sqrt{\frac{M}{m}}.$$  

(40)

Upon substitution in eq. (29), we find the following scaling relation for the optical depth and the event rate,

$$\tau \propto \tau^* = \frac{G M}{c^2 \Delta}, \quad \Gamma \propto \Gamma^* = \frac{G M}{c \sqrt{ab} \Delta} \sqrt{\frac{M}{m}}.$$  

(41)

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\[\text{Figure 5.}\ \text{The optical depth, event rate, and event rate per optical depth (scaled with values at } y = 0\) as functions of the impact parameter } y.\ \text{The behavior is the same for all Freeman bars.}\]

For a compact bar with high $M$ or low $ab$, the velocity dispersion is high, so the event duration is shortened, and the event rate is increased as expected.

#### 4.2 The impact parameter of the line of sight of the observer

Moving away from the center within a model, the event rate drops faster than the optical depth, mostly as a result of a lower escape velocity at large radii, so that the mean event duration $\propto \nu^{-1}$ shifts towards larger values (cf. eq. 29 and Fig. 4). Specifically:

$$\frac{\tau}{\tau^*} \propto \left(1 - \frac{y^2}{y_0^2}\right) \frac{1}{4},$$

(42)

$$\frac{\Gamma}{\Gamma^*} \propto \left(1 - \frac{y^2}{y_0^2}\right) \frac{1}{4},$$

(43)

$$\frac{\Gamma}{\tau^* \nu^*} \propto \left(1 - \frac{y^2}{y_0^2}\right) \frac{1}{4}.$$  

(44)

However, the shape of the duration distribution profile is always independent of the impact parameter (cf. eq. 29 and Fig. 4).

#### 4.3 The bar axis ratio and orientation angle

For the same bar axis ratio, and the same amount of rotational support (say $\lambda = 0$ as for stationary bars), the optical depth and the event rate increase if the bar points closer to the line of sight, but the events are longer (cf. Fig. 4). This is because when the bar points towards us, it is longest in the line of sight, and smallest in transverse velocity (motions are primarily along the long axis). As a result, both the Einstein radius and the event duration are at their maximum. Figure 8 also shows that the event rate is significantly higher for a rotating model than for a stationary bar with the same axis ratio and bar angle.

At a fixed angle less than $45^\circ$, there is an optimal axis ratio (near $b/a = \tan \alpha$) for the bar to yield the highest rate.
Figure 6. The optical depth (upper panel), event rate $\Gamma$ (middle panel) and event rate per optical depth $\Gamma/\tau$ (lower panel) as functions of the axis ratio of the bar at several values of the bar angle for a line of sight through the center of the bars. The shaded areas show the spread due to different pattern speed. $\Gamma$ and $\tau$ have been scaled with $\Gamma^*$ and $\tau^*$ (cf. eq. 41).

and optical depth (see also Fig. 2a of ZM, and Fig 6 here). At angles larger than $45^\circ$, the rate and the optical depth are highest for the axisymmetric model.

Finally, the event rate $\Gamma$ largely follows the trend of $\tau$ as expected. This is because

$$\Gamma \propto \nu \propto M^{\frac{1}{2}} \left( 1 - \frac{y^2}{y^2_{0}} \right)^{\frac{1}{2}},$$

where $\gamma(\alpha, \frac{y}{y_0})$ is given in eq. (2), and is generally a “weaker” function of the bar parameters $(M, a, b)$ and the line-of-sight parameters $(\alpha, y)$ than is the optical depth

$$\tau \propto M \gamma^{-1} \left( 1 - \frac{y^2}{y^2_{0}} \right)^{\frac{3}{2}}.$$  

5 TOTAL EVENT RATE FOR MODELS WITH THE SAME OPTICAL DEPTH AND PROJECTED DENSITY

Now we return to the question raised in the Introduction: how does the event rate vary for models with the same optical depth and projected density? This is interesting because the projected density from COBE maps and the observed optical depth of a bar do not uniquely specify the volume density of the bar. Also the velocity structure of the bar is not uniquely constrained by self-consistency. If one tries to derive the lens mass by fitting the observed microlensing event rate with any specific model, the result will be model-dependent. Here we try to study this systematic effect.

5.1 Pattern speed

First, one can generate a sequence of Freeman bars with the same density but different pattern speeds. Fixing the bar’s volume density and the angle to the bar, the event rate is generally (at least for models which are bigger in depth than width) larger for rotating models than for non-rotating models (cf. Fig. 7 and Fig. 6). This is because rotation increases the relative lens-source speed. The effect can be as strong as changing the event rate by 25%, as illustrated in Figure 7.

Since the event distribution is narrower and the rate is higher for rotating models (Fig. 4), when fitting these models to the observed event distribution, one would derive a broader mass distribution with more massive stars as compared to a slightly narrower spectrum with more low mass objects which would be inferred if, instead, one adopts non-rotating models (cf. eq. 3). The change of the mass spectrum occurs because the observed spread in log $t$ is partly due to the spread in lens mass and partly to the width in $f$ (log $t$) of a single lens mass model.

5.2 Shear

One can build a sequence of bar models by shearing along the line of sight (Fig. 8). Each model of the sequence has the same surface density and optical depth for an observer located sufficiently far away from the bar, but all have different event rates. At least in the limit of a bar sheared to a needle-shape, it would have infinitely shallow potential well, hence infinitely small lens or source velocity, and would predict very few events, but these would last extremely long.

The sequence of sheared bar models is characterized by a ratio between the central depth of the bar and the projected size of the bar, $d_0/y_0$. The sequence that contains the
Figure 7. The event rate as a function of the axis ratio for a sequence of three bars related by a shear along the line of sight with a fixed depth to width ratio $d_0/y_0 = 1/\gamma$. The shaded areas show the spread by bars with different pattern speed. The rates are all for a line of sight through the center of the bar. The optical depth and the surface density is invariant in the sequence (cf. eq. 20).

The axisymmetric model satisfies $d_0/y_0 = \gamma$. A sequence with $b/a = \tan^2 \alpha$ and the optical depth is the same as for the axisymmetric model, while bars with maximum optical depth have $b/a = \tan \alpha$, and the optical depth is enhanced by $1/\sin 2\alpha$ (ZM).

Figure 8. A bar model made by shearing an axisymmetric model along the line of sight direction. For any line of sight, the shear moves the centroid of the mass distribution without changing the amount of lens and source, and their relative distance in the line of sight, so the projected density and the optical depth do not change from model to model. The lower panel draws schematic distributions of the transverse speed at the center (the plus symbol) for a non-rotating bar or disc. The elongated bar has a shallower potential and lower escape velocity $\propto \sqrt{2\Phi_{\text{bar}}(0)}$ than the disc ($\propto \sqrt{2\Phi_{\text{disc}}(0)}$), where $\Phi$ is the gravitational potential.

d_0/y_0 > 1$ produces more optical depth than an axisymmetric model, while a sequence with $d_0/y_0 = 1/\gamma < 1$ produces less. This is because $\tau/\tau^* \propto 1/\gamma = d_0/y_0$ (cf. eqs [2] and [20]). The observed high optical depth towards the bulge suggests that $d_0/y_0 > 1$ for the Galactic bar, that is, it is longer in depth than across.

Among bars related by a shear transformation, the rate becomes smaller when the bar’s axis ratio decreases ($b/a \to 0$), as a result of shallower potential and smaller velocities. The rate is largest when the axis ratio is maximum and the bar points towards or perpendicular to the observer (Fig 8). The velocity in a needle-shaped bar is close to zero, and so is the event rate. However, excluding such extremely elongated bars which are likely unstable, we find that for bars with $d_0/y_0 = 1/\gamma \geq 1$ the rate is only a weak function of axis ratio; the fractional change is less than 20%.

6 THE RATE OF EXTREMELY SHORT EVENTS

As mentioned in §3, the lenses in the bar dominate the events at the short duration end. The event distribution is a power-law for small $t$ (cf. eq. [2]). To quantify the rate of these
and the impact parameter $y \equiv K$ where for the Freeman bar the coefficient $K$ is given by

$$K = \frac{36}{35\pi} \nu^4 \tau.$$  

(48)

It follows that $\Gamma(<t)|_{t \to 0}$ is also a power-law of $t$ with slope 3 (cf. Mao & Paczyński 1996). The expression of $K$ shows two properties: (a) $\Gamma(<t)|_{t \to 0}$ is independent of the rotation parameter $\lambda$, (b) $\Gamma(<t)|_{t \to 0}$ is very sensitive to $\nu$.

It is straightforward to repeat the analysis of §4 for the total rate but now for $K$, since

$$\frac{K}{\tau} \propto \nu^4 \propto \frac{1}{m^2} \left( \frac{M}{ab} \right)^2 \gamma \left( 1 - \frac{y^2}{36} \right)^{-2}.$$  

(49)

For example,

$$\frac{K}{\tau} \propto \nu^4 \propto \frac{1}{m^2} \left( \frac{M}{ab} \right)^2 \gamma \left( 1 - \frac{y^2}{36} \right)^{-2}.$$  

(50)

where $\gamma(\alpha, \frac{y}{\rho})$ is given in equation (14). The general result for extremely short events is that both $\Gamma(<t)|_{t \to 0}$ and $K$ are very sensitive functions of the bar parameters $(M, a, b, \Omega, \alpha)$ and the impact parameter $y$.

The value of $K$ can easily vary by a factor of $(1.2 - 1.3)^4 \approx 2 - 3$ among models with the same projected density and optical depth (cf. Fig. 13). This implies that the mass moment $\langle m^2 \rangle$ of the lenses in the bar is poorly determined unless kinematic data are used to break the degeneracy of models with the same projected density and optical depth. To clarify this point, it is helpful to rewrite eq. (18) as

$$\left\langle \frac{1}{m^2} \right\rangle = \frac{35\pi}{36} \frac{G^2}{c^2} \frac{d^2}{\sigma_0^4} \left[ t^{-3} \Gamma(<t) \right]_{t \to 0, y \to 0} \propto \sigma_0^{-4}. $$  

(51)

7 CONCLUSIONS AND IMPLICATIONS FOR THE GALACTIC BAR

We have evaluated the event rate, as well as the event duration distribution and the optical depth, for a family of bar models known as the two-dimensional Freeman bars. We presented several analytical formulae which show the dependence of the optical depth and event rate on the bar mass, size, axis ratio, pattern speed, the bar angle and the impact parameter with a given line of sight. Models with the same optical depth and projected density make slightly different predictions on the event rate and the event duration distribution for a fixed lens mass. Here we consider the implications for the bar in the center of the Galaxy.
7.1 Event rate as functions of the bar parameters

Can we reliably generalize the results for the event rate of the Freeman bars to the Galactic bar? Realistic bars are three dimensional and their density structure cannot be modelled by the Freeman disc. Furthermore, the orbits in a realistic bar potential have more variety (e.g., Pfenniger 1984) than in the Freeman bars. For the Galactic bar, it is most interesting to consider models which can fit both the COBE map and the observed optical depth. The Freeman bars certainly are too simple to model these.

Nevertheless the Freeman bars should be able to offer some insights and serve as a reference to more complex systems. The trends found here for the event rate can be extrapolated with some modifications. In particular some of the results are due purely to geometry, e.g., the event rate and duration as functions of the bar angle. Some are generic, e.g., the asymptotic power-laws of $f(\log t)$; the slope at the short duration side is 3 for both two-dimensional and three-dimensional models. Others, e.g., the radial increase of the event duration, and radial fall-off of the event rate and optical depth are natural results of a density gradient and the radial fall-off of velocity dispersion; the gradient of $\frac{\sigma}{\Gamma}$ and $\tau$ will depend sensitively on the density gradient. Because the event rate is related to the bar dynamics by several integrations in the lens-source velocity and volume space, which smooth out details of the phase space, only geometrical or generic effects will show up in the event rate. So although true three-dimensional bars have phase-space distribution functions that may be quite different from those of the simple Freeman bars, we expect that the latter will match the basic microlensing physics in the more realistic models.

7.2 Uncertainty of predicting the lens mass

How severe is the non-uniqueness of the lens mass spectrum? The common way to derive the mass spectrum in the COBE bar using microlensing is to compute first an event distribution $f(\log t)$ for a lens with mass $m$ with a given dynamical model of the bar, and then to convolute with it with such a mass spectrum that the result fits the observed distribution (e.g., Mao & Paczyński 1996). Any non-uniqueness of the dynamical model propagates to $f(\log t)$, which in turn propagates to the mass spectrum.

The current best estimates of the lens masses, which use realistic models for the Galactic bulge or bar, indicate masses near $0.15 M_\odot$, with an uncertain but small < 30% fraction below the hydrogen burning limit (Zhao, Spergel & Rich 1995; Zhao, Rich & Spergel 1996; Han & Gould 1996; Han & Lee 1997). If these results were insensitive to the adopted density profile and flattening of the bar, then a comparable result would be expected from the Freeman bars. But if one simply scales up the Freeman bar models to the often quoted mass and size of the Galactic bulge/bar, and makes a straightforward prediction of the event distribution (cf. Fig. 2), one finds, quite surprisingly, that the typical lens mass is now boosted to around $0.8 M_\odot$, a factor of 3-5 times larger than the value in more realistic models. Fig. 2 also shows that the theoretical predictions can
not distinguish very well between a model with 0.075\(M_\odot\) lenses and a model with the lenses being twice as massive when different bulge/bar axis ratio, orientation and rotation speed are considered. We conclude that the fraction of brown dwarfs in the Bulge is also sensitive to uncertainties from the dynamical model.

There are two sources of non-uniqueness of the dynamical model. The first is the shear transformation. Its effect on the distribution \(f(\log t)\) of event durations is generally weak, typically at the 10% level in terms of the total rate (cf. Fig. 3a). Also, as detailed analysis of the COBE map indicates, the shear transformation generally leaves detectable signatures in the left-to-right asymmetry map of the COBE map due to perspective effects (Binney et al. 1997, Zhao 1997 and references therein). The second source of non-uniqueness is the variety of velocity structures that are possible in the same bar. This non-uniqueness cannot be constrained from the COBE map. As shown also in Fig 3a, it can create typically about 20% difference in the event rate when one compares a stationary bar with a Jacobi-type rotating bar. This kind of non-uniqueness is potentially dangerous.

Most relevant to the lens mass function in the bar is the event duration distribution at the short event side. In particular, the mass moment \(\left< \frac{1}{\sin^2 \psi} \right>\) (cf. eq. (24)) is very sensitive to the velocity structure and the shear, and can easily vary by a factor of 2–3. This translates to an uncertainty of the lens mass (particularly at the lower end) by a factor about \(X\) in parametrized form using eqs (31–33) of F66. We find

\[
\begin{align*}
X(\phi) &= \mu \left[ 1 \pm \left( \frac{\mu^2}{A_\beta} + \frac{k_\mu A_\alpha}{k_\lambda A_\beta} \right)^{-\frac{1}{2}} \right], \\
Y(\phi) &= \nu \left[ 1 \pm \left( \frac{k_\mu^2 A_\beta}{k_\lambda^2 A_\beta} + \frac{\nu^2}{k_\lambda^2 A_\beta} \right)^{-\frac{1}{2}} \right],
\end{align*}
\]

where the \(\pm\) sign indicates the outer and inner boundaries, \(\mu = A_\alpha \cos \phi, \quad \nu = k_\mu A_\alpha \sin \phi\), and the parameter \(\phi\) is an angle ranging between 0 and 2\(\pi\). All quantities have the same meaning as in F66.

**APPENDIX B: DERIVATION OF THE EVENT DURATION DISTRIBUTION**

The distribution of the relative lens-source speed is integrated to be (cf. eq. [19])

\[
F(V) = \max[0, Q(V)] + \max[0, Q(-V)],
\]

where

\[
4w_s w_l Q(V) = \min\left[ w_s, w_l + \bar{e}_s - V \right] \quad + \min\left[ w_s, w_l - \bar{e}_s + V \right],
\]

and \(\bar{e}_s \equiv \bar{e}_l \equiv \bar{e}_s\). Apply the following change of variables to eq. (24)

\[
\bar{\psi}_s = d(y) \sin \psi_s, \quad \bar{\psi}_l = d(y) \sin \psi_l,
\]

define \(\lambda\) and \(\nu\) as in eq. (24), and write

\[
\delta \equiv \sin \psi_s - \sin \psi_l, \quad \mu \equiv \nu t,
\]

then

\[
w_{s,l} = \sqrt{\sigma(y)} \cos \psi_{s,l}, \quad \bar{e}_{s,l} = \lambda \sigma(y) \sin \psi_{s,l},
\]

and

\[
\Sigma_{s,l} = \frac{3M}{2 \pi a b} (1 - \frac{y^2}{y_0^2}) \cos \psi_{s,l}, \quad \frac{V}{\sigma(y)} = 2\mu^{-1} \sqrt{\delta}.
\]

It follows that (cf. eq. [24])

\[
\frac{1}{\nu} \int f(\log t) = \frac{2 \ln 10}{\pi (\nu t)^2} I_1(\lambda, \nu t).
\]

where

\[
I_1(\lambda, \mu) = \int_{-\pi/2}^{\pi/2} d\psi_s \int_{-\pi/2}^{\pi/2} d\psi_l \cos^2 \psi_s \cos^2 \psi_l
\]

\[
\times (\max[0, F_0(\lambda, \mu, \psi_s, \psi_l)]
\]

\[
+ \max[0, F_0(\lambda, -\mu, \psi_s, \psi_l)],
\]

and

\[
I_2 = \int_{-\pi/2}^{\pi/2} d\psi_s \int_{-\pi/2}^{\pi/2} d\psi_l \cos^2 \psi_s \cos^2 \psi_l = \frac{32}{45}.
\]

Here we have defined

\[
F_0 \equiv (\nu t) V Q(V)
\]

\[
= \sqrt{\delta} \sqrt{\frac{3}{6 \cos \psi_s \cos \psi_l}}
\]

\[
\times (\max[\cos \psi_s, \cos \psi_l - H_0]
\]

\[
+ \min[\cos \psi_s, \cos \psi_l + H_0]),
\]

where

\[
H_0 \equiv \frac{1}{\sqrt{3}} (\lambda \delta + \frac{2 \sqrt{\delta}}{\mu})
\]

Clearly \(I_1\) is generally a dimensionless function of \(\lambda\) and \(\nu t\) only, so that

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\[
\frac{1}{\nu} f(\log t) = g_2(\nu t) = \frac{45 \ln 10}{16\pi (\nu t)^2} I_1(\lambda, \nu t).
\] (B12)

The asymptotic relations (cf. eqs 32 and 33) can be derived by letting \( \delta \to 0 \) and \( \mu \to 0 \) for \( t \to 0 \), and \( \mu \to \infty \) for \( t \to \infty \).

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