Lower Bounds for the Generalized $h$-Vectors of Centrally Symmetric Polytopes

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Abstract

In a previous article, we proved tight lower bounds for the coefficients of the generalized $h$-vector of a centrally symmetric rational polytope using intersection cohomology of the associated projective toric variety. Here we present a new proof based on the theory of combinatorial intersection cohomology developed by Barthel, Brasselet, Fieseler and Kaup. This theory is also valid for nonrational polytopes when there are no corresponding toric varieties. So we can establish our bounds for centrally symmetric polytopes even without requiring them to be rational.

Introduction

In [St1], R. Stanley proved tight lower bounds for the $h$-vector of a simplicial centrally symmetric polytope. The entries of the $h$-vector are linear combinations of face numbers of the polytope, and they determine the face numbers completely.

Stanley also introduced the generalized $h$-vector of an arbitrary convex polytope (see [St2]). It is a combinatorial invariant of the polytope defined by recursion over its faces, and in the simplicial case it coincides with the usual $h$-vector.

If the polytope $P$ is rational then there is an associated projective toric variety $X_P$, and the coefficients of the generalized $h$-vector of $P$ have a topological interpretation as Betti numbers of the intersection cohomology of $X_P$. Let $n$ denote the dimension of the polytope. It follows from Poincaré-duality that the generalized $h$-vector $(h_0,\ldots,h_n)$ is palindromic (i.e. $h_j = h_{n-j}$ for all $j$). The Hard Lefschetz Theorem for intersection cohomology implies that the $h$-vector is unimodal (i.e. the coefficients increase up to the middle coefficient(s) and then decrease again).

In the article [AC], we considered among others centrally symmetric rational polytopes and asked for combinatorial conditions imposed on the generalized $h$-vector by the existence of the central symmetry. Using the topological interpretation via intersection...
cohomology, we proved tight lower bounds for the coefficients of the generalized $h$-vector of a centrally symmetric rational polytope. In the simplicial case these are precisely the bounds of Stanley (see [St1]).

The aim of this article is to show that the same bounds remain valid even if we do not assume the centrally symmetric polytope to be rational. Our proof is based on the theory of combinatorial intersection cohomology of fans developed by G. Barthel, J.-P. Brasselet, K. Fieseler and L. Kaup (see [BBFK2]) and independently by P. Bressler and Lunts (see [BL]).

They discovered that one can completely characterize the intersection cohomology of a toric variety by combinatorial and algebraic data associated to the corresponding fan, namely in terms of a minimal extension sheaf on the fan considered as a topological space where the subfans are the open subsets (see [BBFK1]). Associating an analogous object to a non–rational fan, they define a combinatorial intersection cohomology satisfying similar formal properties as the usual intersection cohomology.

Barthel, Brasselet, Fieseler and Kaup conjectured that for the combinatorial intersection cohomology of a fan arising from a polytope, a combinatorial version of the Hard Lefschetz theorem holds. Moreover they proved that if such a Hard Lefschetz theorem is true then the odd Betti numbers of the combinatorial intersection cohomology of a polytopal fan vanish and the even Betti-numbers are precisely the coefficients of the generalized $h$-vector of the corresponding polytope (see [BBFK2]).

The Hard Lefschetz theorem in this context was recently proved by Kalle Karu (see [Ka]). The fact that a combinatorial Hard Lefschetz theorem holds has striking consequences. For example, the coefficients of generalized $h$-vector of an arbitrary polytope are non-negative which is not at all clear from the definition. Moreover, the generalized $h$-vector of an arbitrary polytope is unimodal.

We apply these results to a centrally symmetric polytope $P$ of dimension $n$. Denoting its generalized $h$-vector by $(h_0, \ldots, h_n)$, we prove the following for the polynomial $h_P := \sum_{j=0}^{n} h_j x^j$ (see Theorem 4.2):

**Theorem.** If a polytope $P$ of dimension $n$ admits a central symmetry then the polynomial $h_P(x) - (1 + x)^n$ has nonnegative, even coefficients, it is palindromic and unimodal. That means that we have the following bounds for the coefficients $h_j$ of $h_P$:

\[ h_j - h_{j-1} \geq \binom{n}{j} - \binom{n}{j-1} \quad \text{for} \quad j = 1, \ldots, n/2. \]

Note that $(1+x)^n$ occurs as the $h$-polynomial of the $n$-dimensional cross-polytope. We can reformulate the lower bounds given by the theorem in terms of the partial ordering on
real polynomials of degree $n$ defined by coefficientwise comparison, i.e. $a = \sum_{j=0}^{n} a_j x^j \leq b = \sum_{j=0}^{n} b_j x^j$ if and only if $a_j \leq b_j$ for all $j$. The $h$-polynomial of the $n$-dimensional cross-polytope is minimal in this sense and in fact this is the only polytope realizing the minimum (see Corollary 4.3).

**Corollary.** Let $P$ be an $n$-dimensional centrally symmetric polytope. Then

$$h_P \geq (1 + x)^n.$$  

Moreover, equality holds if and only if the polytope $P$ is affinely equivalent to the $n$-dimensional cross-polytope.

## 1 Preliminaries

Let $P$ denote a convex polytope of dimension $n$ in $V := \mathbb{R}^n$. Assume that zero lies in the interior of $P$. Then the polytope $P$ defines a complete fan in $V$ consisting of the cones through its proper faces:

$$\Delta_P := \{ \text{cone}(F); F \text{ proper face of } P \} \cup \{0\},$$

Moreover, this fan is equipped with a *strictly concave support function*, that means a concave function whose restriction to any cone in $\Delta_P$ is linear and such that for any two different maximal cones the linear functions obtained by restriction are different. To define this function, consider the dual polytope $P^* := \{ u \in V^*; \langle u, v \rangle \leq -1 \text{ for all } v \in P \}$ of $P$. There is an order–reversing one–to–one–correspondence between the proper faces of $P$ and the proper faces of $P^*$ defined by

$$F \mapsto s_F := \{ u \in P^*; \langle u, v \rangle = -1 \text{ for all } v \in F \}.$$  

So the vertices of $P^*$ are of the form $s_F$, where $F$ is a one–codimensional face of $P$, and $s_F$ defines a linear function on cone($F$). These linear functions glue together to a well–defined concave function $s_P: V \rightarrow \mathbb{R}$.

The *generalized $h$-vector* of the polytope $P$ is a combinatorial invariant defined by recursion over the faces of $P$ (see [St2]). In fact, this invariant only depends on the fan $\Delta_P$, and it makes sense to define a generalized $h$-vector for arbitrary complete fans using the same recursion formulae. So let us state the definition here in terms of fans. A *fan* in a real vector space $V$ is a nonempty set $\Delta$ of strictly convex polyhedral cones intersecting pairwise in common faces and such that if a cone belongs to the set $\Delta$ then all its faces also belong to $\Delta$. The fan $\Delta$ is called *complete* if its support $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$ equals $V$, and $\Delta$ is *rational* with respect to a lattice $N$ in $\mathbb{R}$, if all the cones are generated by vectors in $N$. For a given cone $\sigma$, let $V_\sigma$ denote the linear span of $\sigma$ in $V$. Let $\Lambda_\sigma$ denote the fan that we obtain by projecting the boundary of $\sigma$ to $V_\sigma/L$, where $L$ is a one–dimensional subspace generated by a vector in the relative interior of $\sigma$.

We introduce two polynomials, namely $h_\Delta$ for each complete fan $\Delta$ and $g_\sigma$ for each strictly convex polyhedral cone $\sigma$, satisfying the following recursion:
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\begin{enumerate}
  \item $g_0 \equiv 1$
  \item $h_\Delta(x) = \sum_{\sigma \in \Delta} (x - 1)^{\dim \Delta - \dim \sigma} g_\sigma(x)$
  \item $g_\sigma(x) = \tau_{\leq \left\lfloor \left( \frac{\dim \sigma}{2} \right) \right\rfloor}((1 - x) h_{\Lambda_\sigma}(x))$
\end{enumerate}

where $\tau_{\leq r}$ denotes the truncation operator $\tau_{\leq r}(\sum_{i=0}^n a_i x^i) := \sum_{i=0}^r a_i x^i$. The vector formed by the coefficients of the polynomial $h_\Delta$ is called the generalized $h$-vector of the fan $\Delta$. For a polytope $P$, we set $h_P := h_{\Delta_P}$.

For example $h_n = 1$ and $h_{n-1} = |\{\sigma \in \Delta ; \dim \sigma = 1\}| - n$, where $n$ denotes the dimension of the fan.

If $\sigma$ is simplicial cone, then $g_{\sigma} = 1$. Hence if $\Delta = \Delta_P$ is the fan through the faces of a simplicial polytope $P$, then $h_P = h_\Delta = \sum_{\sigma \in \Delta} (x - 1)^{\dim \Delta - \dim \sigma}$, which is nothing but the usual $h$-vector. But for a general polytope, its generalized $h$-vector does not only depend on the face numbers but is more complicated.

Stanley showed that the generalized $h$-vector of an $n$-dimensional polytope is palindromic, $h_j = h_{n-j}$ for all $j$. To give an example, the $h$-vector of the 3-dimensional cross-polytope is $(1, 3, 3, 1)$ and the $h$-vector of its dual, the 3-dimensional cube is $(1, 5, 5, 1)$.

\section{Combinatorial Intersection Cohomology}

For later use, in this section we very briefly summarize the main results on the combinatorial intersection cohomology for fans that are presented in [BBFK2], and at the same time we fix the notation. Let $\Delta$ be a (not necessarily rational) fan in a real vectorspace $V = \mathbb{R}^n$.

If a subset of cones $\Lambda \subset \Delta$ is again a fan, then we speak of a subfan and write $\Lambda \prec \Delta$. In the rational case, where $\Delta$ defines a toric variety, the subfans of $\Delta$ correspond to the open invariant subsets of the toric variety, so they define a “$T$–stable topology”. That is the motivation for considering the set of all subfans of an arbitrary fan $\Delta$ together with the empty set as the open sets of a topology, namely the fan topology on $\Delta$. A basis of this topology is formed by the affine subfans, i.e. the subfans that are fans of faces of single cones. For a cone $\sigma \in \Delta$, we denote the fan of faces of $\sigma$ by $\langle \sigma \rangle$ and its boundary fan by $\partial \sigma$.

Let $A^\bullet := S^\bullet(V^*)$ denote the algebra of real–valued polynomial functions on $V$, together with the grading defined by associating to each linear function the degree 2. The algebra $A^\bullet$ defines a sheaf of graded algebras $A^\bullet$ on $\Delta$ (with the fan topology), where for $\sigma \in \Delta$ the algebra $A^\bullet(\langle \sigma \rangle) := A_\sigma$ consists of the elements of $S^\bullet(V_\sigma^*)$ viewed as polynomial functions on $\sigma$. The restriction homomorphisms of $A^\bullet$ are given by restriction of polynomial functions. For $\Lambda \prec \Delta$, the sections in $A^\bullet(\Lambda)$ correspond to those functions on $\Lambda$ that are conewise polynomial. Instead of $A^\bullet(\Lambda)$ we also write $A_\Lambda^\bullet$. 
Now consider a sheaf $E^\bullet$ of graded $A^\bullet$-modules on $\Delta$. To denote the sections $E^\bullet(\Lambda)$ of $E^\bullet$ on $\Lambda \prec \Delta$ we also write $E_\Lambda$, and we abbreviate $E_{(\sigma)}$ to $E_\sigma$. Let $m$ denote the unique homogeneous maximal ideal of $A^\bullet$. Then forming residue classes modulo $m$ we obtain a sheaf of graded real vector spaces $\overline{E}^\bullet$ on $\Delta$, where $\overline{E}^\bullet(\Lambda) := \overline{E}_\Lambda$.

The sheaf $E^\bullet$ is called a minimal extension sheaf if the following properties hold:

i) $E_0^\bullet \simeq \mathbb{R}^\bullet$, where $\mathbb{R}^\bullet$ denotes $\mathbb{R}$ viewed as a graded algebra with trivial zero grading.

ii) For every $\sigma \in \Delta$, the module $E_\sigma^\bullet$ is free over $A^\bullet_\sigma$.

iii) For each cone $\sigma \in \Delta \setminus \{0\}$, the restriction map $\rho_\sigma: E^\bullet_\sigma \to E^\bullet_{\partial\sigma}$ induces an isomorphism $\overline{\rho}_\sigma: \overline{E}^\bullet_\sigma \to \overline{E}^\bullet_{\partial\sigma}$ of graded real vector spaces.

In [BBFK2], the authors prove that for any given fan $\Delta$, a minimal extension sheaf exists and is unique up to isomorphism. If the fan is rational, then we have an associated toric variety $X_\Delta$. The equivariant intersection cohomology of open subsets of $X_\Delta$ defines a minimal extension sheaf on $\Delta$ by the assignment $\Lambda \mapsto IH^*_T(X_\Lambda; \mathbb{R})$ for $\Lambda \prec \Delta$, so in particular $E^\bullet_\Delta \simeq IH^*_T(X_\Delta; \mathbb{R})$. Moreover, $\overline{E}^\bullet_\Delta$ is isomorphic to the intersection cohomology $IH^*(X_\Delta; \mathbb{R})$ of $X_\Delta$ (see [BBFK1]).

If the fan $\Delta$ is not rational, then there is no way of associating a toric variety to it. But the construction of the sheafs $E^\bullet$ and $\overline{E}$ still makes sense, and the authors call it the combinatorial intersection cohomology of $\Delta$.

Now let us assume that $\Delta = \Delta_P$ arises from a polytope $P$, and let $s_P$ denote the corresponding strictly concave support function on $\Delta$. Then the following holds (see [BBFK2]):

2.1 Theorem. (Barthel, Brasselet, Fieseler, Kaup) $E^\bullet_\Delta$ is a free $A^\bullet$-module, and therefore $E^\bullet_\Delta = A^\bullet \otimes_{\mathbb{R}} \overline{E}^\bullet_\Delta$.

A weak version of the Hard Lefschetz Theorem asserts the following (see [Ka]):

2.2 Theorem. (Karu) The map $\overline{\rho}^{2q}: \overline{E}^{2q}_\Delta \to \overline{E}^{2q+2}_\Delta$ induced by the multiplication with $s_P \in A^2(\Delta)$ is injective for $2q \leq n - 1$ and surjective for $2q \geq n - 1$.

In [BBFK2] the authors showed that if 2.2 is true then the Betti-numbers of the combinatorial intersection cohomology are precisely given by the coefficients of the $h$-polynomial of the fan:

2.3 Theorem. (Barthel, Brasselet, Fieseler, Kaup)

$$h_\Delta(t^2) = \sum_{q=0}^{2n} (\dim \overline{E}^q_\Delta) t^q.$$
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The fact that the $h$-polynomial is palindromic is reflected by a combinatorial version of Poincaré-duality, also proved by the four authors.

For $E^\bullet_\Delta$ and $E^{\bullet*}_\Delta$ we can define a Poincaré–series $v_\Delta$ and a Poincaré–polynomial $u_\Delta$ respectively as follows:

\[ v_\Delta(t) := \sum_{q \geq 0} (\dim E^q_\Delta) t^q \quad \text{and} \quad u_\Delta(t) := \sum_{q \geq 0} (\dim E^{\bullet*}_\Delta) t^q. \]

Note that here we do not follow the convention used in [BBFK2] in order to be consistent with the notation used in [AC]. One obtains the Poincaré–series used in [BBFK2] from ours by viewing it as a function in $t^2$.

As a consequence of the first part of the above theorem we obtain:

\[ v_\Delta(t) = \frac{1}{(1-t^2)^n} u_\Delta(t). \]

3 Refined Poincaré–Series

From now on let $\Delta$ denote a complete fan, and assume that for every $\sigma \in \Delta$ also $-\sigma \in \Delta$, in other words assume that $\Delta$ is centrally symmetric. Let $\varphi = -\text{id}_V$ denote the central symmetry. Being an invertible linear transformation, $\varphi$ induces an $\mathbb{R}$–linear automorphism of the graded algebra $A^\bullet = S^\bullet(V^*)$. Note that for every cone $\sigma \in \Delta$, we have $V_\sigma = V_{-\sigma}$.

Since $A^\bullet_\sigma$ is the algebra of polynomial functions on $V_\sigma$ restricted to $\sigma$, the algebras $A^\bullet_\sigma$ and $A^{\bullet*\sigma}$ are not identical, but canonically isomorphic. The action of $\varphi$ on $V_\sigma$ induces an isomorphism of graded algebras from $A^\bullet_\sigma$ to $A^{\bullet*\sigma}$ that is compatible with this canonical isomorphism. Moreover, the induced isomorphisms are compatible with the restriction homomorphisms $\rho^\tau_\sigma: A^\bullet_\sigma \to A^\bullet_\tau$ for every $\tau < \sigma$. That means that in fact $\varphi$ defines a natural automorphism of $\Delta$ as a ringed space equipped with the sheaf of graded algebras $A^\bullet$. We can also define an action of $\varphi$ on the minimal extension sheaf $E^\bullet_\Delta$ on $\Delta$.

3.1 Lemma. There are isomorphisms of graded vector spaces

\[ \varphi: E^\bullet_\sigma \to E^{\bullet*\sigma}_\Delta \]

for every $\sigma \in \Delta$ that are equivariant with respect to the module structure over $A^\bullet_\sigma$ and $A^{\bullet*\sigma}$ respectively and compatible with the restriction homomorphisms of the sheaf $E^\bullet_\Delta$.

Proof. To define the required isomorphisms, we proceed by recursion over the $k$–skeleton $\Delta^{\leq k}$ of $\Delta$ following the recursive construction of $E^\bullet$ as in Section 1 of [BBFK2]. On $E^\bullet_0 = \mathbb{R}^\bullet$, where 0 denotes the zero cone, $\varphi$ acts as the identity. Now assume that the isomorphisms have been defined for $\Delta^{\leq k}$, and let $\sigma \in \Delta^k$. We can assume that $E^\bullet_\sigma = A^\bullet_\sigma \otimes \mathbb{R} E^{\bullet*}_{\partial \sigma}$. By induction, we already have an isomorphism $\varphi: E^{\bullet*}_{\partial \sigma} \to E^{\bullet*}_{-\partial \sigma}$, and since the maximal ideal $m$ of $A^\bullet$ is $\varphi$–stable, $\varphi$ induces an isomorphism $\varphi: E^{\bullet*}_{\partial \sigma} \to E^{\bullet*}_{-\partial \sigma}$. Together...
with the map from $A^*_\sigma$ to $A^-_{-\sigma}$ determined by $\varphi$, that provides us with an isomorphism of graded vector spaces $\varphi: E^*_\sigma \to E^-_{-\sigma}$. By construction, this map is equivariant as a map from an $A^*_\sigma$–module to an $A^-_{-\sigma}$–module.

In the construction of $E^\bullet$, the restriction homomorphism from $E^*_\sigma = A^*_\sigma \otimes_\mathbb{R} E^-_{\partial\sigma}$ to $E^\bullet_{\partial\sigma}$ is defined using the restriction homomorphism $\rho^\sigma_{\partial\sigma}: A^*_\sigma \to A^\bullet_{\partial\sigma}$ and an $R$–linear section $s^\sigma: E^-_{\partial\sigma} \to E^\bullet_{\partial\sigma}$ of the residue class map $E^-_{\partial\sigma} \to E_{\partial\sigma}$. The section $s^\sigma$ can be chosen freely. So we can assume that for any pair $\sigma, -\sigma$ of antipodal cones in $\Delta$ the corresponding sections have been chosen such that the following diagram is commutative:

\[
\begin{array}{ccc}
E^-_{\partial\sigma} & \xrightarrow{s^\sigma} & E^\bullet_{\partial\sigma} \\
\downarrow & & \downarrow \varphi \\
E^-_{-\partial\sigma} & \xrightarrow{s_{-\sigma}} & E^\bullet_{-\partial\sigma}
\end{array}
\]

That implies compatibility of $\varphi$ with the restriction homomorphisms of $E^\bullet$. □

In particular, we obtain an induced automorphism $\varphi$ on the module $E^\bullet_\Delta$ of global sections of $E^\bullet$, and though the automorphism is not canonical, the dimensions of the eigenspaces in each graded piece are uniquely determined and therefore the so–called refined Poincaré–series for the action of $\varphi$ on $E^\bullet$ depends only on $\Delta$. This series is defined as a polynomial over the group ring $\mathbb{Z}[G]$ of the character group $G := \{\pm 1\}$ of the group generated by $\varphi$ in $\text{GL}(V)$, namely:

\[
v^\varphi_\Delta(t) := \sum_{q \geq 0} (\dim(E^+_\Delta)^+ + \dim(E^-_\Delta)^-) \chi^q t^q,
\]

where $\chi$ denotes the element corresponding to $-1$ in $\mathbb{Z}[G]$, and the superscripts $+, -$ indicate the eigenspaces for the eigenvalues $+1$ and $-1$ respectively. The refined Poincaré–polynomial $u^\varphi_\Delta$ for the action of $\varphi$ on $E^\bullet$ is defined analogously.

By Theorem 2.1, $E^\bullet_\Delta$ is a free $A^\bullet$–module. The action of $\varphi$ on $E^\bullet_\Delta$ is induced by taking residue classes. So if we choose a homogeneous basis for $E^\bullet_\Delta$ and preimages under the residue class map in $E^\bullet_\Delta$ to define an isomorphism

\[
E^\bullet_\Delta \to E^\bullet_\Delta \otimes_\mathbb{R} A^\bullet,
\]

then this isomorphism is automatically compatible with the action of $\varphi$. That implies

\[
v^\varphi_\Delta(t) = \frac{1}{(1 - \chi t^2)^n} \cdot u^\varphi_\Delta(t).
\]

To obtain a relation between the Poincaré–series $v_\Delta$ and its refined version $v^\varphi_\Delta$, we can use the fact that the minimal extension sheaf $E^\bullet$ as a sheaf of real vector spaces can be written as a direct sum of simpler subsheafs. Note that $E^\bullet$ is a flabby sheaf on $\Delta$. Here that means that the restriction homomorphism $\rho^\sigma_{\partial\sigma}: E^\bullet_\sigma \to E^\bullet_{\partial\sigma}$ is surjective for all $\sigma \in \Delta$. 
For $\sigma \in \Delta$, let $J_\sigma$ denote the *characteristic sheaf* of $\sigma$ defined on $\Lambda \prec \Delta$ by

$$J_\sigma(\Lambda) := \begin{cases} \mathbb{R} & \text{if } \sigma \in \Lambda \\ 0 & \text{otherwise} \end{cases}.$$ 

Then there is an isomorphism of sheaves of graded real vector spaces

$$E^\bullet \simeq \bigoplus_{\sigma \in \Delta} J_\sigma \otimes \mathbb{R} K_\sigma,$$

where $K_\sigma$ denotes the kernel of the restriction homomorphism $\rho^\sigma_{\partial \sigma} : E^\bullet_{\partial \sigma} \to E^\bullet_{\partial \sigma}$ (see Section 3, [BBFK2]).

For every $\sigma \in \Delta$, $\varphi$ induces a map from $J_\sigma$ to $J_{-\sigma}$ that is compatible with the action of $\varphi$ on $E^\bullet$, and using these maps together with the action of $\varphi$ on $E^\bullet$, we obtain an induced $\varphi$–action on the direct sum on the right-hand side of (3), where $\varphi$ maps $J_\sigma$ to $J_{-\sigma}$ and $K_\sigma$ to $K_{-\sigma}$. Modifying the proof of the decomposition theorem from [BBFK2], we can show the following:

**3.2 Lemma.** The isomorphism of sheaves of graded real vector spaces on $\Delta$ in (3) can be chosen as $\varphi$–equivariant.

**Proof.** We prove our claim by induction on the number of antipodal pairs of cones in $\Delta$. Suppose that there is a $\varphi$–equivariant decomposition of $E^\bullet$ into $\varphi$–stable flabby sheaves

$$E^\bullet \simeq F \oplus \left( \bigoplus_{\sigma \in \Lambda} J_\sigma \otimes \mathbb{R} K_\sigma \right),$$

where the sum is taken over a $\varphi$–stable subset $\Lambda$ of $\Delta$ (that is not necessarily a subfan), such that $F(\sigma) = 0$ for all $\sigma \in \Lambda$.

Then choose a pair of antipodal cones $\sigma, -\sigma \in \Delta \setminus \Lambda$ of minimal dimension $k$ with $F(\sigma) \neq 0 \neq F(-\sigma)$. We have to show that we can write $F$ as a direct sum of $\varphi$–stable flabby subsheaves $F = G \oplus H$ such that $H(\sigma) = 0$ and if $\sigma = 0$ then $G \simeq J_0 \otimes \mathbb{R} K_0$ and if $\sigma \neq 0$ then $G \simeq (J_\sigma \otimes \mathbb{R} K_\sigma) \oplus (J_{-\sigma} \otimes \mathbb{R} K_{-\sigma})$. We define $G$ and $H$ on the $k$–skeleton $\Delta^{\leq k}$ by

$$G(\tau) := \begin{cases} K_\tau & \text{if } \tau = \pm \sigma \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H(\tau) := \begin{cases} 0 & \text{if } \tau = \pm \sigma \\ F(\tau) & \text{otherwise} \end{cases}.$$ 

Now suppose, that $F = G \oplus H$ is already defined on $\Delta^{\leq m}$, and consider a pair of antipodal cones $\pm \tau$ of dimension $m + 1$. If $\sigma$ is neither a face of $\tau$ nor of $-\tau$, then set $G(\pm \tau) = 0$ and $H(\pm \tau) := F(\pm \tau)$. Otherwise say $\sigma \prec \tau$ and $-\sigma \prec -\tau$. Note that $\tau$ cannot contain both $\sigma$ and $-\sigma$ as a face.

By assumption, we have a decomposition $F(\partial \tau) = G(\partial \tau) \oplus H(\partial \tau)$, and $G(-\partial \tau) = \varphi(G(\partial \tau))$ and $H(-\partial \tau) = \varphi(H(\partial \tau))$. Since $F$ is flabby, we can choose a decomposition $F(\tau) = U \oplus W$ such that the restriction homomorphism $\rho^\tau_{\partial \tau} : F(\tau) \to F(\partial \tau)$ induces an isomorphism $U \to G(\partial \tau)$ and a surjective homomorphism from $W$ to $H(\partial \tau)$.
Since the action of $\varphi$ is compatible with the restriction homomorphisms, for the decomposition $\mathcal{F}(-\tau) = \varphi(U) \oplus \varphi(W)$ the following holds: The restriction homomorphism $\rho_{-\partial \tau}: \mathcal{F}(-\tau) \to \mathcal{F}(-\partial \tau)$ induces an isomorphism $\varphi(U) \to \mathcal{G}(-\partial \tau)$ and a surjective homomorphism from $\varphi(W)$ to $\mathcal{H}(-\partial \tau)$. Now set $\mathcal{G}(\tau) = U$ and $\mathcal{G}(-\tau) = \varphi(U)$, $\mathcal{H}(\tau) = W$ and $\mathcal{H}(-\tau) = \varphi(W)$. Then $\mathcal{G}$ and $\mathcal{H}$ have the required properties. □

Now consider the action of $\varphi$ on the direct sum on the righthandside of (3). Apparently, for every cone $\sigma \neq 0$, the action of $\varphi$ interchanges the summands $J_\sigma \otimes_R K_\sigma$ and $J_{-\sigma} \otimes_R K_{-\sigma}$. So for every $q$, in $\bigoplus_{\sigma \neq 0} J_\sigma \otimes_R K_\sigma$ the eigenvalue $+1$ and $-1$ occur with the same multiplicity. We obtain the relation

\begin{equation}
(4) \quad v^*_\varphi(t) - 1 = \frac{1}{2}((v_\Delta(t) - 1) + (v_\Delta(t) - 1) \cdot \chi) = \frac{1}{2}(1 + \chi)(v_\Delta(t) - 1).
\end{equation}

4 Lower Bounds for the Generalized $h$–Vector

Summarizing the considerations in the previous section, we obtain the following description of the refined Poincaré–polynomial:

4.1 Proposition. Let $\Delta$ be a centrally symmetric complete fan of dimension $n$. Then

$$u^*_\varphi(t) = \frac{1}{2}(u_\Delta(t) + (1 + t^2)^n) + \frac{1}{2} \chi(u_\Delta(t) - (1 + t^2)^n).$$

Proof. Inserting (1) in (4), we obtain

$$v^*_\varphi(t) = \frac{1}{2}(1 + \chi)v_\Delta(t) + \frac{1 - \chi}{2}.$$

Using (2), that implies

$$u^*_\varphi(t) = \frac{1}{2} \frac{(1 + \chi)(1 - \chi t^2)^n}{(1 - t^2)^n} u_\Delta(t) + \frac{1 - \chi}{2} (1 - \chi t^2)^n.$$

Note that since $\chi^2 = 1$, we have $(1 - \chi t^2)(1 + \chi) = (1 - t^2)(1 + \chi)$ and $(1 - \chi t^2)(1 - \chi) = (1 + t^2)(1 - \chi)$. This implies

$$u^*_\varphi(t) = \frac{1 - \chi}{2}(1 + t^2)^n + \frac{1 + \chi}{2} u_\Delta(t).$$

We now apply this proposition to polytopal centrally symmetric fans.
4.2 Theorem. Let $P$ be a centrally–symmetric polytope of dimension $n$. Then the polynomial
\[ h_P(x) - (1 + x)^n, \]
has nonnegative, even coefficients, it is palindromic and unimodal. In particular, we have the following bounds for the coefficients $h_j$ of $h_P$:
\[ h_j - h_{j-1} \geq {n \choose j} - {n \choose j-1} \quad \text{for } j = 1, \ldots, n/2. \]

Proof. As before let $\Delta := \Delta_P$ denote the fan through the faces of $P$ and let $s_P$ denote the $\Delta$–strictly convex support function defined by $P$. By Theorem 2.3, $u_\Delta(t) = h_P(t^2)$. The symmetry follows immediately from the combinatorial Poincaré–duality. Moreover, by the above proposition, we have
\[ \frac{1}{2}(u_\Delta(t) - (1 + t^2)^n) = \sum_{q \geq 0} (\dim(\overline{E}_\Delta^q)) t^q. \]
This implies in particular, that all the coefficients of $p(t) := u_\Delta(t) - (1 + t^2)^n$ are nonnegative and even.

Since the support function $s_P \in \mathcal{A}^2(\Delta)$ is invariant under $\varphi$, we have
\[ s_P \cdot (\overline{E}_\Delta^q)^- \subseteq (\overline{E}_\Delta^{q+2})^- \cdot \]
Now it follows from the combinatorial Hard Lefschetz theorem (see Theorem 2.2) that
\[ \dim((\overline{E}_\Delta^{2q})^-) \leq \dim((\overline{E}_\Delta^{2q+2})^-) \quad \text{for } 2q \leq n - 1, \]
and that means that the polynomial $p$ is unimodal.

Note that $(1+x)^n$ occurs as the $h$-polynomial of the $n$-dimensional cross-polytope. We can reformulate the lower bounds given by the theorem in terms of the partial ordering on real polynomials of degree $n$ defined by coefficientwise comparison, i.e. $a = \sum_{j=0}^n a_j x^j \leq b = \sum_{j=0}^n b_j x^j$ if and only if $a_j \leq b_j$ for all $j$. The $h$-polynomial of the $n$-dimensional cross-polytope is minimal in this sense and in fact this is the only polytope realizing the minimum.

4.3 Corollary. Let $P$ be an $n$-dimensional centrally symmetric polytope. Then
\[ h_P \geq (1 + x)^n. \]
Moreover, equality holds if and only if the polytope $P$ is affinely equivalent to the $n$-dimensional cross-polytope.

Proof. Suppose that $h_P = (1 + x)^n$ for a centrally symmetric polytope $P$. Then in particular, $h_{n-1} = n$. On the other hand, as mentioned in Section 1, we always have, $h_{n-1} = |\{\text{vertices of } P\}| - n$. So $P$ has $2n$ vertices. Let us choose a facet $F$ of $P$. Since $F$ contains at least $n$ vertices, and it is disjoint from its opposite facet $-F'$, we obtain that $P$ is the convex hull of $F$ and $-F$. That means that $P$ is affinely equivalent to the $n$-dimensional cross-polytope.
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