Bäcklund transformations of $Z_n$-Sine-Gordon systems

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Abstract

In this paper, from the algebraic reductions from the Lie algebra $gl(n, \mathbb{C})$ to its commutative subalgebra $Z_n$, we construct the general $Z_n$-Sine-Gordon and $Z_n$-Sinh-Gordon systems which contain many multi-component Sine-Gordon type and Sinh-Gordon type equations. Meanwhile, we give the Bäcklund transformations of the $Z_n$-Sine-Gordon and $Z_n$-Sinh-Gordon equations which can generate new solutions from seed solutions. To see the $Z_n$-systems clearly, we consider the $Z_2$-Sine-Gordon and $Z_3$-Sine-Gordon equations explicitly including their Bäcklund transformations, the nonlinear superposition formula and Lax pairs.

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1 Introduction

The Sine-Gordon equation and Sinh-Gordon equation are important integrable equations which describe many interesting phenomena including dynamics of coupled pendulums, Josephson junction arrays [1], models of nonlinear excitations in complex systems in physics and in living cellular structures, both intra-cellular and inter-cellular [2]. The Sine-Gordon equation also accounts for continuum limits of the crystalline lattices [3].

The scalar Sine-Gordon equation originates in differential geometry and has profound applications in physics and in life sciences which can be seen from a recent review [4]. Recently in [5], a dressing method was used for the vector Sine-Gordon equation and its soliton interactions.

In [6], a new hierarchy called as a $Z_m$-KP hierarchy which take values in a maximal commutative subalgebra of $gl(m, \mathbb{C})$ was constructed, meanwhile the relation between Frobenius manifolds and the dispersionless reduced $Z_m$-KP hierarchy was discussed. From the $Z_m$-KP hierarchy, one can derive some coupled equations like the coupled KdV equation which appears in [2, 7–9]. We consider the Hirota quadratic equation of the commutative version of extended multi-component Toda hierarchy in [10] which should be useful in Frobenius manifold theory. Because of logarithm terms, some extended Vertex operators are constructed in generalized Hirota bilinear equations which might be useful in topological field theory and Gromov-Witten theory. Later we defined a new multi-component BKP hierarchy which takes values in a commutative subalgebra of $gl(N, \mathbb{C})$.

After this, we give the gauge transformation of this commutative multi-component BKP (CMBKP) hierarchy [11]. Meanwhile we construct a new constrained CMBKP hierarchy which contains some new integrable systems including coupled KdV equations under a certain reduction. After this, the quantum torus symmetry and quantum torus constraint on the tau function of the commutative multi-component BKP hierarchy are constructed. Then a natural questions appears, i.e. what about the corresponding commutative multi-component version of other integrable systems like the well-known Sine-Gordon equation and Sinh-Gordon equation? In this paper, we will answer this question by several steps. These steps contain their Bäcklund transformations which were studied a lot for the systems of the scalar Sine-Gordon equation [12, 13].

This paper is arranged as follows. In Section 2, we recall some basic facts about the classical Sine-Gordon equation and then we give the construction of the general $Z_n$-Sine-Gordon equation, $Z_n$-Sinh-Gordon equations and their Bäcklund transformations in Section 3. What’s more, we study on the Bäcklund transformations and nonlinear superposition formula of the $Z_2$-Sine-Gordon equation in section 4 and 5. In Section 6, we give the Lax equations of the $Z_2$-Sine-Gordon equation. In Section 7, we give the corresponding results on the $Z_3$-Sine-Gordon equation.

2 The classical Sine-Gordon equation

To introduce the $Z_n$-Sine-Gordon equation, we firstly recall the original Sine-Gordon equation. The mathematical method of soliton equations was derived mainly through the zero curvature equation. The zero curvature equation of the original Sine-Gordon equation is as follows

$$M_t - N_x + [M, N] = 0, [M, N] = MN - NM. \quad (2.1)$$
One can export many soliton equations by choosing appropriate values of $M$ and $N$. M. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur considered the zero curvature equation of the original Sine-Gordon equation with

$$M = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix}, N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}.$$ (2.2)

Here the zero curvature equation (2.1) becomes:

$$\begin{align*}
A_x &= qc - rB, \\
q_t &= B_x + 2i\lambda B + 2qA, \\
r_t &= C_x - 2i\lambda C - 2rA.
\end{align*}$$ (2.3)

For example, by choosing appropriate values of $A$, $B$, $C$ as:

$$A = \frac{g}{\lambda}, B = \frac{b}{\lambda}, C = \frac{c}{\lambda},$$ (2.4)

where

$$g = \frac{i}{4} \cos u, b = c = \frac{i}{4} \sin u, q = r = -\frac{u_x}{2},$$ (2.5)

one can derive the well-known Sine-Gordon (SG) equation as following

$$u_{xt} = \sin u.$$ (2.6)

Supposing $u$ is a solution of eq.(2.6), and under the following transformation, $u'$ in the following will be another solution of eq.(2.6),

$$\frac{u' + u}{2}_x = a \sin \frac{u' - u}{2},$$ (2.7)

$$\frac{u' - u}{2}_t = \frac{1}{a} \sin \frac{u' + u}{2}.$$ (2.8)

This transformation is called the Bäcklund transformation of SG equation. If $u$ is a solution of SG eq.(2.6), we can get another new solution $u'$ of SG equation by solving the above first-order equations (eq.(2.7) and eq.(2.8)).

### 3 The $Z_n$-Sine-Gordon equation and $Z_n$-Sinh-Gordon equation

To construct new multicomponent Sine-Gordon and Sinh-Gordon systems which might have potential applications in biology such as DNA structural dynamics, we will consider the case when $u$ take values in a commutative algebra $Z_n = \mathbb{C}[\Gamma]/(\Gamma^n)$ and $\Gamma = (\delta_{i,j+1})_{ij} \in gl(n, \mathbb{C})$. This will lead to the case of $Z_n$-Sine-Gordon equation which is also equivalent to another $Z_n$-Sinh-Gordon equation. Let us firstly introduce the following lemma.

**Lemma 1.** The following identity holds

$$\sin \begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix} = \begin{pmatrix} b_0 & 0 & 0 & \cdots & 0 \\ b_1 & b_0 & 0 & \cdots & 0 \\ b_2 & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_0 \end{pmatrix},$$ (3.9)
where,

\[ b_k = \sum_{i_1 k_1 + i_2 k_2 + \cdots + i_j k_j = k} \frac{a_{i_1}^{k_1} \cdot a_{i_2}^{k_2} \cdots \cdot a_{i_j}^{k_j}}{k_1! \cdot k_2! \cdots \cdot k_j!} \cdot \sin \left( a_0 + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right). \] (3.10)

Proof. In order to prove the above conclusion, a direct calculation can lead to

\[ \sin (a_0 + a_1 \Gamma_1 + \cdots + a_n \Gamma^n) = (a_0 E + a_1 \Gamma_1 + \cdots + a_n \Gamma^n)^3 + \cdots. \] (3.11)

Here we have used the results as

\[ \Gamma^k = (\delta_{i,j+k})_{ij}, \quad k < n; \Gamma^n = 0_{n \times n}. \] (3.12)

Through a direct calculation, we can finish the proof by choosing the specific k-th diagonal position of both matrices of two sides of the identity (3.9). \( \square \)

By a tedious calculation, we can get the following \( Z_n \)-Sine-Gordon equation:

\[ (a_k)_{xt} = \sum_{i_1 k_1 + i_2 k_2 + \cdots + i_j k_j = k} \frac{a_{i_1}^{k_1} \cdot a_{i_2}^{k_2} \cdots \cdot a_{i_j}^{k_j}}{k_1! \cdot k_2! \cdots \cdot k_j!} \cdot \sin \left( a_0 + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right), \quad 0 \leq k \leq n. \] (3.13)

Also by a tedious calculation, we can derive the Bäcklund transformation of the \( Z_n \)-Sine-Gordon equation in the following theorem.

**Theorem 1.** The \( Z_n \)-Sine-Gordon equation (3.13) has the following Bäcklund transformation

\[
\begin{align*}
\left( \frac{u'_k + u_k}{2} \right)_x &= a \sum_{i_1 k_1 + i_2 k_2 + \cdots + i_j k_j = k} (u'_{i_1} - u_{i_1})^{k_1} \cdot (u'_{i_2} - u_{i_2})^{k_2} \cdots \cdot (u'_{i_j} - u_{i_j})^{k_j} \frac{2^{k_1 + k_2 + \cdots + k_j} \cdot k_1! \cdot k_2! \cdots \cdot k_j!}{2^{k_1 + k_2 + \cdots + k_j} \cdot k_1! \cdot k_2! \cdots \cdot k_j!} \cdot \sin \left( \frac{u'_0 - u_0}{2} + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right), \\
\left( \frac{u'_k - u_k}{2} \right)_t &= \frac{1}{a} \sum_{i_1 k_1 + i_2 k_2 + \cdots + i_j k_j = k} (u'_{i_1} - u_{i_1})^{k_1} \cdot (u'_{i_2} - u_{i_2})^{k_2} \cdots \cdot (u'_{i_j} - u_{i_j})^{k_j} \frac{2^{k_1 + k_2 + \cdots + k_j} \cdot k_1! \cdot k_2! \cdots \cdot k_j!}{2^{k_1 + k_2 + \cdots + k_j} \cdot k_1! \cdot k_2! \cdots \cdot k_j!} \cdot \sin \left( \frac{u'_0 - u_0}{2} + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right).
\end{align*}
\] (3.14)

Proof. If \((u_k, 0 \leq k \leq n)\) are solutions of the \( Z_n \)-Sine-Gordon equation, we will prove that under the transformations (3.12), \((u'_k, 0 \leq k \leq n)\) are also solutions of the eq. (3.13). Here we give the proof of the above argument by direct calculation using the transformation
Then we can say that the transformation (3.14) is the Bäcklund transformation of the
therefore we can derive the
(3.14),
Take the seed solution as
new solutions of the
\[ a \sum_{i_1 k_1 + i_2 k_2 + \ldots + i_j k_j = k} \left( u'_{i_1} - u_{i_1} \right)^{k_1} \cdots \left( u'_{i_j} - u_{i_j} \right)^{k_j} \sin \left( \frac{u'_0 - u_0}{2} + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right) t \]
\[ = \sum_{i_1 k_1 + i_2 k_2 + \ldots + i_j k_j = k} \frac{\left( u'_{i_1} - u_{i_1} \right)^{k_1} \cdots \left( u'_{i_j} - u_{i_j} \right)^{k_j} \left( u'_0 - u_0 \right) t}{2^{k_1+k_2+\ldots+k_j} \cdot k_1! \cdot k_2! \cdots \cdot k_j!} \cos \left( \frac{u'_0 - u_0}{2} + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right) \]
\[ = \frac{1}{2} \sum_{k=1}^{n} \sum_{i_1 k_1 + i_2 k_2 + \ldots + i_j k_j = k} \frac{u'_{i_1} u_{i_2} \cdots u'_{i_j}}{k_1! \cdot k_2! \cdots \cdot k_j!} \sin \left( u_0 + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right) \Gamma^k \]
\[ + \frac{1}{2} \sum_{k=1}^{n} \sum_{i_1 k_1 + i_2 k_2 + \ldots + i_j k_j = k} \frac{u_{i_1} u_{i_2} \cdots u_{i_j}}{k_1! \cdot k_2! \cdots \cdot k_j!} \sin \left( u'_0 + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right) \Gamma^k. \]

Then because \( u_k \) are solutions of the \( Z_n \)-Sine-Gordon equation, i.e.
\[ u_{kxt} = \sum_{k=1}^{n} \sum_{i_1 k_1 + i_2 k_2 + \ldots + i_j k_j = k} \frac{u_{i_1}^k \cdot u_{i_2}^k \cdots u_{i_j}^k}{k_1! \cdot k_2! \cdots \cdot k_j!} \sin \left( u_0 + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right), \]
therefore we can derive the \( u'_k \) are also solutions of the \( Z_n \)-Sine-Gordon equation
\[ u'_{kxt} = \sum_{k=1}^{n} \sum_{i_1 k_1 + i_2 k_2 + \ldots + i_j k_j = k} \frac{u_{i_1}^k \cdot u_{i_2}^k \cdots u_{i_j}^k}{k_1! \cdot k_2! \cdots \cdot k_j!} \sin \left( u'_0 + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right). \]
Then we can say that the transformation (3.14) is the Bäcklund transformation of the
\( Z_n \)-Sine-Gordon equation.

By the Bäcklund transformation of the \( Z_n \)-Sine-Gordon equation, we can get other
new solutions of the \( Z_n \)-Sine-Gordon equation. One can see it in the following example.

**Example 1.** Take the seed solution as \( a_k = 0, \quad 0 \leq k \leq n \), then new solutions \( a'_k \) satisfy
\[ a'_{kx} = 2a \sum_{i_1 k_1 + i_2 k_2 + \ldots + i_j k_j = k} \frac{a_{i_1}^k \cdot a_{i_2}^k \cdots a_{i_j}^k}{k_1! \cdot k_2! \cdots \cdot k_j!} \sin \left( a_0 + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right), \quad (3.15) \]
\[ a'_{kt} = \frac{2}{a} \sum_{i_1 k_1 + i_2 k_2 + \ldots + i_j k_j = k} \frac{a_{i_1}^k \cdot a_{i_2}^k \cdots a_{i_j}^k}{k_1! \cdot k_2! \cdots \cdot k_j!} \sin \left( a_0 + \frac{\pi}{2} \sum_{i=1}^{j} k_i \right). \quad (3.16) \]
In some special cases, the results are as follows when \( n = 0 \), \( a_0' = 4 \arctan e^{ax+a^{-1}t} \), when \( n = 1 \), \( a_1' = 2c e^{(ax+2a^{-1}t)+1} \), when \( n = 2 \), \( a_2' = -\left( \frac{1}{e^{(ax+2a^{-1}t)+1}} + c \right) e^{(ax+a^{-1}t)} \).

It is well known that the classical Sine-Gordon equation is equivalent to Sinh-Gordon equation. Therefore now let us also consider the case of the \( Z_n \)-Sinh-Gordon equation in similar ways as in the following lemma.

**Lemma 2.** The following identity holds

\[
\begin{pmatrix}
    a_0 & 0 & 0 & \cdots & 0 \\
    a_1 & a_0 & 0 & \cdots & 0 \\
    a_2 & a_1 & a_0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \\
\end{pmatrix} = \begin{pmatrix}
    c_0 & 0 & 0 & \cdots & 0 \\
    c_1 & c_0 & 0 & \cdots & 0 \\
    c_2 & c_1 & c_0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    c_n & c_{n-1} & c_{n-2} & \cdots & c_0 \\
\end{pmatrix},
\]

(3.17)

where

\[
c_k = \sum_{i_1k_1+i_2k_2+\cdots+i_jk_j=k} \frac{a_{i_1}^{k_1} \cdot a_{i_2}^{k_2} \cdots \cdot a_{i_j}^{k_j} \cdot e^{a_0} - (-1)^{\sum_{l=1}^{j} k_l} \cdot e^{-a_0}}{k_1! \cdot k_2! \cdots \cdot k_j!},
\]

(3.18)

Also by a tedious calculation, we can derive the \( Z_n \)-Sinh-Gordon equation

\[
(a_k)_{xt} = \sum_{i_1k_1+i_2k_2+\cdots+i_jk_j=k} \frac{a_{i_1}^{k_1} \cdot a_{i_2}^{k_2} \cdots \cdot a_{i_j}^{k_j} \cdot e^{a_0} - (-1)^{\sum_{l=1}^{j} k_l} \cdot e^{-a_0}}{k_1! \cdot k_2! \cdots \cdot k_j!}, \quad 0 \leq k \leq n, \quad (3.19)
\]

and its Bäcklund transformation in the following theorem.

**Theorem 2.** The \( Z_n \)-Sinh-Gordon equation (3.19) has the following Bäcklund transformation

\[
\begin{cases}
    \left( \frac{u'_i+u_k}{2} \right)_x = a \sum_{i_1k_1+i_2k_2+\cdots+i_jk_j=k} \frac{(u_{i_1}'-u_{i_1})^{k_1} \cdot (u_{i_2}'-u_{i_2})^{k_2} \cdots \cdot (u_{i_j}'-u_{i_j})^{k_j} \cdot e^{\frac{u_{i_1}'-u_0}{2} - (-1)^{\sum_{l=1}^{j} k_l} \cdot e^{-\frac{u_0'-u_0}{2}}}}{2^{k_1+k_2+\cdots+k_j} \cdot k_1! \cdot k_2! \cdots \cdot k_j!}, \\
    \left( \frac{u'_i-u_k}{2} \right)_t = \frac{1}{a} \sum_{i_1k_1+i_2k_2+\cdots+i_jk_j=k} \frac{(u_{i_1}'-u_{i_1})^{k_1} \cdot (u_{i_2}'-u_{i_2})^{k_2} \cdots \cdot (u_{i_j}'-u_{i_j})^{k_j} \cdot e^{\frac{u_{i_1}'-u_0}{2} - (-1)^{\sum_{l=1}^{j} k_l} \cdot e^{-\frac{u_0'-u_0}{2}}}}{2^{k_1+k_2+\cdots+k_j} \cdot k_1! \cdot k_2! \cdots \cdot k_j!}.
\end{cases}
\]

(3.20)

The proof of this theorem is similar as the proof of the Theorem [1]. We will skip it here.

To see the \( Z_n \)-Sine-Gordon systems clearly, we will take take \( n = 2 \) and \( n = 3 \) as examples in the following several sections.

### 4 The \( Z_2 \)-Sine-Gordon equation and its Bäcklund transformation

In this section, we will construct the \( Z_2 \)-Sine-Gordon equation in the commutative algebra \( Z_2 = \mathbb{C}[\Gamma]/(\Gamma^2) \) and \( \Gamma = (\delta_{i,j+1})_{ij} \in gl(2, \mathbb{C}) \). By a direct computation using Taylor expansion, we can get the following lemma.
**Lemma 3.** The matrix form of $\sin u$ and $\cos u$ are as following through the summation of matrices

$$
\begin{align*}
\sin \begin{pmatrix} u_0 & 0 \\ u_1 & u_0 \end{pmatrix} &= \begin{pmatrix} \sin u_0 & 0 \\ u_1 \cos u_0 & \sin u_0 \end{pmatrix}, \\
\cos \begin{pmatrix} u_0 & 0 \\ u_1 & u_0 \end{pmatrix} &= \begin{pmatrix} \cos u_0 & 0 \\ -u_1 \sin u_0 & \cos u_0 \end{pmatrix}.
\end{align*}
$$

(4.21) (4.22)

In this matrix case, after a denotation as $u_0 := u, u_1 := v$, we can derive the following new $Z_2$-Sine-Gordon equation

$$
\begin{cases}
u_{xt} = \sin u, \\
v_{xt} = v \cos u.
\end{cases}
$$

(4.23)

**Theorem 3.** The $Z_2$-Sine-Gordon equation

$$
\begin{cases}
u_{0xt} = \sin u_0, \\
u_{1xt} = u_1 \cos u_0,
\end{cases}
$$

(4.24)

has the following Bäcklund transformation

$$
\begin{cases}
\frac{u'_0 + u_0}{2} x = a \sin \frac{u'_0 - u_0}{2}, \\
\frac{u'_1 + u_1}{2} x = \frac{u'_1}{2} \cos \frac{u'_0 - u_0}{2}, \\
\frac{u_0 - u_0}{2} t = \frac{1}{a} \sin \frac{u'_0 + u_0}{2}, \\
\frac{u'_1 - u_1}{2} t = \frac{1}{a} \frac{u_1 + u_1}{2} \cos \frac{u'_0 + u_0}{2}.
\end{cases}
$$

(4.25)

**Proof.** To see the Bäcklund transformation of the $Z_n$-Sine-Gordon equation clearly, here we give the proof of the above theorem by a direct calculation in terms of matrices

$$
\begin{align*}
&\left(\begin{array}{ccc}
\frac{u'_0 + u_0}{2} & 0 \\
\frac{u'_1 + u_1}{2} & \frac{u'_0 + u_0}{2}
\end{array}\right)_{xt} \\
&= a \left(\begin{array}{ccc}
\frac{\sin \frac{u'_0 - u_0}{2} \cos \frac{u'_0 - u_0}{2}}{\sin \frac{u'_0 - u_0}{2}} & 0 \\
\frac{\frac{u'_1 - u_1}{2} \cos \frac{u'_0 - u_0}{2}}{\sin \frac{u'_0 - u_0}{2}} & 0
\end{array}\right)_{t} \\
&= a \left(\begin{array}{ccc}
\frac{1}{a} (u'_1 + u_1) \cos \frac{u'_0 - u_0}{2} \cos \frac{u'_0 - u_0}{2} - \frac{u'_1 - u_1}{2} \cos u'_0 \sin \frac{u'_0 - u_0}{2} \sin \frac{u'_0 - u_0}{2} \cos \frac{u'_0 - u_0}{2} & 0 \\
\frac{1}{a} \frac{u'_1 + u_1}{2} \cos \frac{u'_0 - u_0}{2} \sin \frac{u'_0 - u_0}{2} \cos \frac{u'_0 - u_0}{2} & 0
\end{array}\right)_{t} \\
&= \left(\begin{array}{ccc}
\frac{1}{2} (\sin u'_0 + \sin u_0) & 0 \\
\left(\frac{u_1 \sin u_0 + u_1' \sin u_0}{2}\right) \frac{1}{2} (\sin u'_0 + \sin u_0) & 0
\end{array}\right)_{t} \\
&= \left(\begin{array}{ccc}
\frac{1}{2} \left(\sin u'_0 + \sin u_0\right) & 0 \\
\left(\frac{u_1 \cos u_0 + u_1' \cos u'_0}{2}\right) \frac{1}{2} (\sin u'_0 + \sin u_0) & 0
\end{array}\right)
\end{align*}
$$

Then because $u_0, u_1$ are solutions of the $Z_2$-Sine-Gordon equation, i.e.

$$
u_{0xt} = \sin u_0, \quad u_{1xt} = u_1 \cos u_0,
$$

7
therefore we can derive the $u'_0, u'_1$ are also solutions of the $Z_2$-Sine-Gordon equation

$$u'_{0xt} = \sin u'_0, \quad u'_{1xt} = u'_1 \cos u'_0.$$  

Then we can say that the transformation (4.25) is the Bäcklund transformation of the $Z_2$-Sine-Gordon equation.

Next we give a simple example with seed solutions $u_0, u_1$; and we can get another new solution of the $Z_2$-Sine-Gordon equation by solving the first-order equation of the above Bäcklund transformation (4.25).

**Example 2.** Let $u_0 = u_1 = 0$, then the new solutions from the Bäcklund transformation (4.25) becomes

\[
\begin{cases}
(u'_0/2)_x = a \sin \frac{u'_0}{2}, \\
(u'_0/2)_t = \frac{1}{a} \sin \frac{u'_0}{2},
\end{cases}
\quad (4.26)
\]

\[
\begin{cases}
(u'_1/2)_x = a \frac{u'_1}{2} \cos \frac{u'_0}{2}, \\
(u'_1/2)_t = \frac{1}{a} \frac{u'_1}{2} \cos \frac{u'_0}{2}.
\end{cases}
\quad (4.27)
\]

Here the solutions of two sets of equations will be derived by integral calculations and we get the following conclusions:

$$u'_0 = 4 \arctan e^{ax+a^{-1}t}.$$  

According to the relation of $\cos 2a$ and $\tan a$ as

$$\cos 2a = \frac{1 - \tan^2 a}{1 + \tan^2 a},$$

we derive

$$\cos \frac{u'_0}{2} = \cos [2 \arctan e^{ax+a^{-1}t}] = \frac{1 - e^{2ax+2a^{-1}t}}{1 + e^{2ax+2a^{-1}t}}.$$  

So the second equation of equations (4.27) becomes:

\[
\begin{cases}
(\ln u'_1/2)_x = \frac{1 - e^{2ax+2a^{-1}t}}{1 + e^{2ax+2a^{-1}t}}, \\
(\ln u'_1/2)_t = \frac{1 - e^{2ax+2a^{-1}t}}{a (1 + e^{2ax+2a^{-1}t})},
\end{cases}
\quad (4.28)
\]

which further leads to

$$u'_1 = 2e^{(ax+a^{-1}t)} e^{(2ax+2a^{-1}t)} + 1.$$
4.1 The $Z_2$-Sinh-Gordon equation and its Bäcklund transformation

In this subsection, we will use the similar method in the last section to consider the Bäcklund transformation of the $Z_2$-Sinh-Gordon equation. Basing on the well-known Sinh-Gordon equation as following

$$u_{xt} = \sinh u,$$  \hspace{1cm} (4.29)

we will consider the following $Z_2$-Sinh-Gordon equation

$$\begin{cases} u_{xt} = \sinh u, \\
v_{xt} = v \cosh u. \end{cases}$$  \hspace{1cm} (4.30)

The $Z_2$-Sinh-Gordon equation has the following Bäcklund transformation

$$\begin{cases} (u' + u) x = a \sinh \frac{u' - u}{2}, \\
(\frac{v' + v}{2}) x = \frac{v' - v}{2} \cosh \frac{u' - u}{2}, \\
(\frac{u' - u}{2}) t = \frac{1}{a} \sinh \frac{u' + u}{2}, \\
(\frac{v' - v}{2}) t = \frac{1}{a} \frac{v' + v}{2} \cosh \frac{u' + u}{2}. \end{cases}$$  \hspace{1cm} (4.31)

5 Nonlinear superposition formula of $Z_2$-Sine-Gordon equation

The above Bäcklund transformation in the last section gives us an important method to derive new solutions from known solutions. But sometimes it is not easy to solve the first order equation. Besides Bäcklund transformations, the non-linear superposition formula is also useful to derive new solutions by algebraic calculations. We suppose $(c_0, c_1)$ are solutions of $Z_2$-Sine-Gordon equation with respect to a parameter $h_1$ from the seed solution $(b_0, b_1)$, $(d_0, d_1)$ are solutions of $Z_2$-Sine-Gordon equation with respect to a parameter $h_2$ from the seed solution $(b_0, b_1)$. Also we suppose $(e_0, e_1)$ are solutions of $Z_2$-Sine-Gordon equation with respect to a parameter $h_2$ from the seed solution $(c_0, c_1)$ and the $(e_0, e_1)$ should also be solutions of $Z_2$-Sine-Gordon equation with respect to a parameter $h_1$ from the seed solution $(d_0, d_1)$.

Then the Bäcklund transformation formula can be expressed as

$$\begin{cases} \left( \frac{c_0 + b_0}{2} \right)_x = h_1 \sin \frac{c_0 - b_0}{2}, \\
\left( \frac{c_1 + b_1}{2} \right)_x = h_1 \frac{c_1 - b_1}{2} \cos \frac{c_0 - b_0}{2}, \end{cases}$$  \hspace{1cm} (5.32)

$$\begin{cases} \left( \frac{c_0 + c_0}{2} \right)_x = h_2 \sin \frac{c_0 - c_0}{2}, \\
\left( \frac{c_1 + c_1}{2} \right)_x = h_2 \frac{c_1 - c_1}{2} \cos \frac{c_0 - c_0}{2}, \end{cases}$$  \hspace{1cm} (5.33)

$$\begin{cases} \left( \frac{d_0 + b_0}{2} \right)_x = h_2 \sin \frac{d_0 - b_0}{2}, \\
\left( \frac{d_1 + b_1}{2} \right)_x = h_2 \frac{d_1 - b_1}{2} \cos \frac{d_0 - b_0}{2}, \end{cases}$$  \hspace{1cm} (5.34)

$$\begin{cases} \left( \frac{e_0 + d_0}{2} \right)_x = h_1 \sin \frac{e_0 - d_0}{2}, \\
\left( \frac{e_1 + d_1}{2} \right)_x = h_1 \frac{e_1 - d_1}{2} \cos \frac{e_0 - d_0}{2}. \end{cases}$$  \hspace{1cm} (5.35)
By adding up eq. (5.32) and eq. (5.35), we can get:

\[
\begin{pmatrix}
  c_0 + b_0 + d_0 \\
  c_1 + b_1 + d_1 \\
\end{pmatrix}
\times
\begin{pmatrix}
  0 \\
  c_0 + b_0 + c_1 + d_1 \\
\end{pmatrix}
= h_1 \left( \frac{c_1}{2} \cos \frac{c_0 - b_0}{2} + \frac{c_1 - b_1}{2} \cos \frac{c_0 - d_0}{2} \sin \frac{c_0 - b_0}{2} + \sin \frac{c_0 - d_0}{2} \right).
\]

By adding up eq. (5.33) and eq. (5.34), we can get:

\[
\begin{pmatrix}
  c_0 + c_0 + d_0 + b_0 \\
  c_1 + c_1 + d_1 + b_1 \\
\end{pmatrix}
\times
\begin{pmatrix}
  0 \\
  c_0 + c_0 + d_0 + b_0 \\
\end{pmatrix}
= h_2 \left( \frac{c_1}{2} \cos \frac{c_0 - c_0}{2} + \frac{c_1 - c_1}{2} \cos \frac{d_0 - b_0}{2} \sin \frac{c_0 - c_0}{2} + \sin \frac{d_0 - b_0}{2} \right).
\]

According to the above two equations, we can get a system of algebraic equations

\[
\begin{align*}
  h_1(\sin \frac{c_0 - b_0}{2} + \sin \frac{e_0 - d_0}{2}) &= h_2(\sin \frac{c_0 - c_0}{2} + \sin \frac{d_0 - b_0}{2}), \\
  h_1(\frac{c_1 - b_1}{2} \cos \frac{c_0 - b_0}{2} + \frac{c_1 - d_1}{2} \cos \frac{c_0 - d_0}{2}) &= h_2(\frac{c_1 - c_1}{2} \cos \frac{c_0 - c_0}{2} + \frac{d_1 - d_1}{2} \cos \frac{d_0 - b_0}{2}).
\end{align*}
\]

By finishing both sides of the first formula of the eq. (5.36), we can derive the following nonlinear superposition formula of Z2-Sine-Gordon equation

\[
\begin{align*}
  \tan \frac{c_0 - b_0}{4} &= \frac{h_2 + h_1}{h_2 - h_1} \tan \frac{c_0 - d_0}{4}, \\
  h_1(\frac{c_1 - b_1}{2} \cos \frac{c_0 - b_0}{2} + \frac{c_1 - d_1}{2} \cos \frac{c_0 - d_0}{2}) &= h_2(\frac{c_1 - c_1}{2} \cos \frac{c_0 - c_0}{2} + \frac{d_1 - d_1}{2} \cos \frac{d_0 - b_0}{2}).
\end{align*}
\]

From the known solutions \(b_0, b_1, c_0, c_1, d_0\) and \(d_1\), we can get the fourth solution \(e_0, e_1\) by eq. (5.37) instead of solving differential equations. From the following example, one can see it clearly.

**Example 3.** From the known solutions \(b_0 = b_1 = 0, c_0 = 4 \arctan e^{h_1x + h_1^{-1}t}, c_1 = 2c_0 e^{(h_1x + h_1^{-1}t)}, d_0 = 4 \arctan e^{h_2x + h_2^{-1}t}, d_1 = 2c_0 e^{2h_2x + 2h_2^{-1}t}\), by eq. (5.37), we can get the following new solution:

\[
\begin{align*}
  e_0 &= 4 \arctan \left( \frac{h_2 + h_1 \sinh \frac{1}{2}((h_1 + h_2)x + (h_1^{-1} + h_2^{-1})t)}{h_2 - h_1 \cosh \frac{1}{2}((h_1 + h_2)x + (h_1^{-1} + h_2^{-1})t)} \right), \\
  e_1 &= \frac{h_1 c_1 \cos \frac{c_0 - c_0}{2} + h_1 (c_1 - d_1) \cos \frac{c_0 - d_0}{2} - h_2 d_1 \cos \frac{d_0 - b_0}{2}}{h_2 \cos \frac{c_0 - c_0}{2}}.
\end{align*}
\]

**Proof.** Here we give the proof of the above argument by a simple calculation,

\[
\tan \frac{e_0 - b_0}{4} = \frac{h_2 + h_1}{h_2 - h_1} \tan \frac{e_0 - d_0}{4} = \frac{h_2 + h_1}{h_2 - h_1} \tan \left( \arctan e^{h_1x + h_1^{-1}t} - \arctan e^{h_2x + h_2^{-1}t} \right).
\]

Using the tangent formula:

\[
\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y},
\]

we can get

\[
e_0 = 4 \arctan \left( \frac{h_2 + h_1 \sinh \frac{1}{2}((h_1 + h_2)x + (h_1^{-1} + h_2^{-1})t)}{h_2 - h_1 \cosh \frac{1}{2}((h_1 + h_2)x + (h_1^{-1} + h_2^{-1})t)} \right).
\]
Now by considering
\[ h_1(\frac{c_1-b_1}{2} \cos \frac{c_0-b_0}{2} + \frac{e_1-d_1}{2} \cos \frac{e_0-d_0}{2}) = h_2(\frac{e_1-c_1}{2} \cos \frac{e_0-c_0}{2} + \frac{d_1-b_1}{2} \cos \frac{d_0-b_0}{2}), \]
we can derive
\[ e_1 = \frac{h_1 c_1 \cos \frac{c_0}{2} + h_1 (c_1 - d_1) \cos \frac{e_0}{2} - h_2 d_1 \cos \frac{d_0}{2}}{h_2 \cos \frac{e_0+c_0}{2}}. \]

In this way, by the algebraic iterated operation we can get many new solutions of the \( Z_2 \)-Sine-Gordon equation.

### 6 Lax equations of the \( Z_2 \)-Sine-Gordon equation

In the Lax equation of the original Sine-Gordon equation, in the (2.2) we suppose
\[
\begin{align*}
g &= \frac{i}{4} \begin{pmatrix} \cos u_0 & 0 \\ -u_1 \sin u_0 & \cos u_0 \end{pmatrix}, \\
b &= c = \frac{i}{4} \begin{pmatrix} \sin u_0 & 0 \\ u_1 \cos u_0 & \sin u_0 \end{pmatrix}, \\
q &= -r = -\frac{1}{2} \begin{pmatrix} u_{0x} & 0 \\ u_{1x} & u_{0x} \end{pmatrix}.
\end{align*}
\]

That is to say when:
\[
M = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix} = \begin{pmatrix} -i\lambda & 0 & -\frac{1}{2}u_{0x} & 0 \\ 0 & -i\lambda & -\frac{1}{2}u_{1x} & -\frac{1}{2}u_{0x} \\ \frac{1}{2}u_{0x} & 0 & i\lambda & 0 \\ \frac{1}{2}u_{1x} & \frac{1}{2}u_{0x} & 0 & i\lambda \end{pmatrix},
\]
\[
N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} = \begin{pmatrix} \frac{i}{4} \cos u_0 & 0 & \frac{i}{4} \sin u_0 & 0 \\ -\frac{i}{4} \sin u_0 & \frac{i}{4} \cos u_0 & \frac{i}{4} \sin u_0 & \frac{i}{4} \cos u_0 \\ \frac{i}{4} u_1 \cos u_0 & 0 & -\frac{i}{4} u_1 \sin u_0 & 0 \\ \frac{i}{4} u_1 \sin u_0 & \frac{i}{4} u_1 \cos u_0 & -\frac{i}{4} u_1 \sin u_0 & -\frac{i}{4} \cos u_0 \end{pmatrix},
\]
we can export the Lax equation of the \( Z_2 \)-Sine-Gordon equation.

### 7 The \( Z_3 \)-Sine-Gordon equation and its Bäcklund transformation

Now let us consider the case when \( u \) takes values in the commutative algebra \( Z_3 = \mathbb{C}[\Gamma]/(\Gamma^3) \) and \( \Gamma = (\delta_{ij+1})_{ij} \in gl(3, \mathbb{C}) \). The SG equation is generalized to the following \( Z_3 \)-Sine-Gordon equation
\[
\begin{align*}
z_{xt} &= \sin z, \\
v_{xt} &= v \cos z, \\
w_{xt} &= w \cos z - \frac{v^2 \sin z}{2},
\end{align*}
\]
and the following lemma holds.
Lemma 4. The following identity holds

\[
\sin \begin{pmatrix} z & 0 & 0 \\ v & z & 0 \\ w & v & z \end{pmatrix} = \begin{pmatrix} \sin z & 0 & 0 \\ v \cos z & \sin z & 0 \\ w \cos z - \frac{v^2 \sin z}{2} & v \cos z & \sin z \end{pmatrix}, \tag{7.45}
\]

\[
\cos \begin{pmatrix} z & 0 & 0 \\ v & z & 0 \\ w & v & z \end{pmatrix} = \begin{pmatrix} \cos z & 0 & 0 \\ 1 - v \sin z & \cos z & 0 \\ 1 - w \sin z - \frac{v^2 \cos z}{2} & 1 - v \sin z & \cos z \end{pmatrix}. \tag{7.46}
\]

Similarly, by the Bäcklund transformation of the SG equation (eq.(2.7) and eq.(2.8)), we can get another new solution of the \( Z_3 \)-SG equation.

Let \( z = v = w = 0 \), then the Bäcklund transformation will lead to the conditions of new solutions

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{z'}{2} &= a \sin \frac{z'}{2}, \\
\frac{v'}{2} &= \frac{1}{a} \sin \frac{v'}{2},
\end{array} \right. \tag{7.47}
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{v'}{2} &= a \cos \frac{v'}{2}, \\
\frac{w'}{2} &= \frac{1}{a} \cos \frac{w'}{2},
\end{array} \right. \tag{7.48}
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{w'}{2} &= a \left( \frac{w'}{2} \cos \frac{w'}{2} - \frac{v^2 \sin \frac{w'}{2}}{2} \right), \\
\frac{w'}{2} &= \frac{1}{a} \left( \frac{w'}{2} \cos \frac{w'}{2} - \frac{v^2 \sin \frac{w'}{2}}{2} \right),
\end{array} \right. \tag{7.49}
\end{align*}
\]

Then the following new solutions can be derived

\[
\begin{align*}
\left\{ \begin{array}{l}
z' &= 4 \arctan e^{ax+a^{-1}t}, \\
v' &= 2 \left( \frac{e^{(ax+a^{-1}t)} + 1}{e^{ax+a^{-1}t} + 1} \right), \\
w' &= \left( \frac{e^{(ax+a^{-1}t)} - 1}{e^{ax+a^{-1}t} + 1} \right) \left( \frac{e^{(ax+a^{-1}t)} - 1}{e^{ax+a^{-1}t} + 1} \right) + c \left( \frac{e^{(ax+a^{-1}t)} - 1}{e^{ax+a^{-1}t} + 1} \right).
\end{array} \right. \tag{7.50}
\end{align*}
\]

Similar to the second order matrix, by supposing

\[
\begin{align*}
g &= \frac{i}{4} \begin{pmatrix} \cos z & 0 & 0 \\ 1 - v \sin z & \cos z & 0 \\ 1 - w \sin z - \frac{v^2 \cos z}{2} & 1 - v \sin z & \cos z \end{pmatrix}, \tag{7.51}
\end{align*}
\]

\[
\begin{align*}
b &= \frac{i}{4} \begin{pmatrix} \sin z & 0 & 0 \\ v \cos z & \sin z & 0 \\ w \cos z - \frac{v^2 \sin z}{2} & v \cos z & \sin z \end{pmatrix}, \tag{7.52}
\end{align*}
\]

\[
\begin{align*}
q &= -r = -\frac{1}{2} \begin{pmatrix} z_x & 0 & 0 \\ v_x & z_x & 0 \\ w_x & v_x & z_x \end{pmatrix}, \tag{7.53}
\end{align*}
\]

one can derive the Lax equation of the \( Z_3 \)-Sin-Gordon equation (7.44).

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