The Kerr-Newman-Gödel Black Hole

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Abstract

By applying a set of Hassan-Sen transformations and string dualities to the Kerr-Gödel solution of minimal $D = 5$ supergravity we derive a four parameter family of five dimensional solutions in type II string theory. They describe rotating, charged black holes in a rotating background. For zero background rotation, the solution is $D = 5$ Kerr-Newman; for zero charge it is Kerr-Gödel. In a particular extremal limit the solution describes an asymptotically Gödel BMPV black hole.

1 Introduction

Two particularly interesting solutions of minimal supergravity in $D = 5$ are the maximally supersymmetric Gödel type universe [1] and the half supersymmetric BMPV black hole [2]. The former is a homogeneous space with Closed Timelike Curves through every point in spacetime. Such Closed Timelike curves can be made to vanish by uplifting the solution to higher dimensions [3], the reason being that these Gödel type universes are U-dual to pp-waves [4]. This fact makes the Gödel type universe even more interesting, due to the possibility of quantizing strings in this background and its relation to a certain limit of Super Yang Mills [5]. On the gravitational side the pp-waves dual to the Gödel universe arise as Penrose limits of (D-brane configurations) near horizon geometries.

The BMPV black hole is the only asymptotically flat, rotating, supersymmetric black hole with a regular, finite size horizon and thus finite entropy, known in string theory. See [22] for a uniqueness theorem concerning this solution. Its existence is made possible due to the particular Chern-Simons coupling of $D = 5$ minimal supergravity [6] and the fact that the spacetime dimension allows imposing a self-duality condition on the exterior derivative of the rotation one-form [7]. Embedded in type IIB string theory it corresponds to a deformation of the $D1$-$D5$ system for which one can compute the degeneracy of states at high level and find agreement with the classical Bekenstein-Hawking formula. On the string theory side angular momentum corresponds to R-charge carried

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by the quantum states $^2$. The BMPV spacetime can only be interpreted as a black hole for sufficiently small values of angular momentum; otherwise there is no horizon and the spacetime becomes a non-singular repulson $^8$ with Closed Timelike Curves passing through any point. On the string theory side unitarity is violated when this happens $^7$.

In this letter we will show that these two remarkable spacetimes can be superimposed as a solution to type II string theory; there is a BMPV-Gödel black hole. That black holes can exist in the Gödel universe, was shown in $^3$, where an extreme Reissner-Nordstrom black hole was discussed. Subsequently Gimon and Hashimoto $^{14}$ constructed a Kerr black hole which is asymptotically Gödel. We will use their result as the starting point for a well known procedure of generating charged black holes in string theory. Thus we will generate a Kerr-Newman-Gödel black hole. In the extremal limit this yields the advertised BMPV-Gödel solution.

The Gödel ‘deformation’ of the BMPV black hole becomes irrelevant close to the horizon. Thus, the near horizon geometry and the entropy are unchanged from the asymptotically flat case. In AdS/CFT this means that the background rotation is a sub-leading effect in $\alpha'$, in the decoupling limit $^3$.

This spacetime is not a solution of $D=5$ minimal supergravity, which explains why it was not found in $^1$, but it reduces to a solution of this theory both for vanishing ‘mass’ and for vanishing background rotation, when it yields the Gödel and BMPV backgrounds, respectively. In general, our solution can be characterized as a type IIA solution on $K3 \times S^1$, although it can certainly be dualized to yield a heterotic or type IIB solution.

From the viewpoint of the BPS spectrum of string theory this is qualitatively a new state, which from the five dimensional gravitational viewpoint includes both asymptotically decaying and asymptotically dominating ‘angular momentum’ or, more accurately, inertial frame dragging effects.

### 2 The $D=5$ Kerr-Newman and Kerr-Gödel

The minimal supergravity theory in five spacetime dimensions was constructed in $^9$ $^{10}$. We take the action to be

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R - F^2 - \frac{2}{3\sqrt{3}} \tilde{\epsilon}^{\alpha\beta\gamma\delta\mu} F_{\alpha\beta} F_{\gamma\delta} A_\mu \right),$$

(2.1)

where $F = dA$, $\tilde{\epsilon}$ is the Levi-Civita tensor, related to the Levi-Civita tensor density by $\tilde{\epsilon}^{\alpha\beta\gamma\delta\mu} = \epsilon^{\alpha\beta\gamma\delta\mu} / \sqrt{-g}$ and we use a ‘mostly plus’ signature. This theory admits as an uncharged solution the five dimensional Kerr; it is parameterized by three parameters (mass and two angular momenta): $m, a, b$. The metric reads $^{11}$

$$ds^2 = -dt^2 + \frac{r^2 \Delta dr^2}{(r^2 + a^2)(r^2 + b^2)} + \Delta d\theta^2 + \sin^2 \theta (r^2 + a^2) d\phi^2 + \cos^2 \theta (r^2 + b^2) d\psi^2$$

$$+ \frac{2m}{\Delta} \left[ dt - (a \sin^2 \theta d\phi + b \cos^2 \theta d\psi) \right]^2,$$

(2.2)

and the gauge potential is zero. The function $\Delta$ is defined as

$$\Delta \equiv r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta.$$  

(2.3)
For $m = 0$ we have Minkowski space. If $a = b = 0$ the spatial part is $\mathbb{R}^4$ written in spherical bipolar coordinates. That is, we take two orthogonal 2-planes in $\mathbb{R}^4$ each parameterized by an azimuthal coordinate ($\phi$ and $\psi$ each with range $2\pi$); $\theta$ is the polar coordinate between the 2-planes with range $\pi/2$. If $a, b \neq 0$, the spatial part is written in oblate bipolar coordinates, which are adequate for describing squashed 3-spheres rather than round ones. In this Kerr metric, all squashing is contained in the one-form

$$a \sin^2 \theta d\phi + b \cos^2 \theta d\psi = \frac{a + b}{4} \sigma_L^3 + \frac{b - a}{4} \sigma_R^3.$$  \hspace{1cm} (2.4)

We have introduced a particular left ($\sigma_L^3$) and right ($\sigma_R^3$) one-form of SU(2) for which we follow the conventions of [3].\footnote{The relation between the bipolar coordinates above ($\theta, \phi, \psi$) and the Euler angles ($\tilde{\theta}, \tilde{\phi}, \tilde{\psi}$) used in [3] is $\tilde{\theta} = 2\theta$, $\tilde{\phi} = \phi + \psi$, $\tilde{\psi} = \psi - \phi$.} Explicitly, in terms of the bipolar coordinates used in (2.2),

$$\sigma_L^3 = 2(\cos^2 \theta d\psi + \sin^2 \theta d\phi), \quad \sigma_R^3 = 2(\cos^2 \theta d\psi - \sin^2 \theta d\phi).$$ \hspace{1cm} (2.5)

One thinks of $a(b)$ as the parameter associated with angular momentum in the $\phi(\psi)$ 2-plane, $J_\phi(J_\psi)$. One the other hand, since spherical symmetry in $4+1$ dimensions is an $SO(4) \simeq SU(2)_L \times SU(2)_R$ symmetry, it is natural to define left, $J_L$, and right, $J_R$, angular momentum as the Casimir operators of $SU(2)_L$ and $SU(2)_R$. The presence of $\sigma_L^3$ ($\sigma_R^3$) in (2.4) turns on $J_L$ ($J_R$). Thus we can roughly say that

$$J_L = J_\psi + J_\phi, \quad J_R = J_\psi - J_\phi.$$ \hspace{1cm} (2.6)

The Kerr solution with either $J_L = 0$ or $J_R = 0$ simplifies dramatically. Defining a new radial coordinate $R^2 \equiv \Delta = r^2 + a^2$, the ‘Left Kerr’ ($a = b$) and ‘Right Kerr’ ($a = -b$) are written as

$$ds^2 = -dt^2 + \frac{R^4dR^2}{R^4 - 2m(R^2 - a^2)} + \frac{R^2}{4}((\sigma_L^1)^2 + (\sigma_L^2)^2 + (\sigma_L^3)^2) + \frac{2m}{R^2}(dt + \frac{a}{2}\sigma_L^3)^2.$$ \hspace{1cm} (2.7)

We use the convention that upper (lower) signs refer to the right (left) solution. Of course Right Kerr and Left Kerr are the same solution; explicitly they are interchanged by

$$a \rightarrow -a, \quad \phi \rightarrow -\phi.$$ \hspace{1cm} (2.8)

Note that in bipolar coordinates

$$(\sigma_L^1)^2 + (\sigma_L^2)^2 + (\sigma_L^3)^2 = (\sigma_L^1)^2 + (\sigma_L^2)^2 + (\sigma_L^3)^2 = 4(d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2).$$ \hspace{1cm} (2.9)

The five-dimensional Left and Right Kerr-Newman have an extra charge parameter $\delta$. The metrics are [12]

$$ds^2 = -\frac{R^2(R^2 - 2m)}{\Sigma^2}dt^2 + \frac{R^2 \Sigma dR^2}{R^4 - 2m(R^2 - a^2)} + \frac{\Sigma}{4}((\sigma_L^1)^2 + (\sigma_L^2)^2 + (\sigma_L^3)^2)$$

$$-\frac{2ma}{\Sigma} [R^2(1 + \cosh^3 \delta - \sinh^3 \delta) + 2m \sinh^3 \delta] \sigma_L^3 dt$$

$$+ \frac{ma^2}{\Sigma} \left[ \frac{R^2}{2} - m \sinh^3 \delta \left\{ 2 \sinh^3 \delta \pm 2 \cosh^3 \delta + 3 \sinh \delta \right\} \right] (\sigma_L^3)^2,$$ \hspace{1cm} (2.10)
where we defined
\[ \Sigma \equiv R^2 + 2m \sinh^2 \delta. \tag{2.11} \]
The gauge potential is
\[ A = \pm \frac{\sqrt{3}m \cosh \delta \sinh \delta}{\Sigma} \left[ dt + \frac{a}{2} (\sinh \delta \pm \cosh \delta) \sigma^3_{L,R} \right]. \tag{2.12} \]
The left and right solutions are now related by
\[ a \to -a, \quad \phi \to -\phi, \quad \delta \to -\delta. \tag{2.13} \]
The physical mass, charge and angular momentum for the Left Kerr-Newman are
\[ M = 3m \cosh (2\delta), \quad Q = m \sinh (2\delta), \quad J_L = 8ma (\cosh^3 \delta - \sinh \delta), \quad J_R = 0. \tag{2.14} \]
This geometry has a curvature singularity at \( \Sigma = 0 \); for sufficiently small \( a \) there is an outer event horizon at
\[ R^2 = m + \sqrt{m^2 - 2ma^2}. \tag{2.15} \]
A special case of this solution is the extremal limit:
\[ a, b, m \to 0, \quad \delta \to -\infty, \tag{2.16} \]
keeping fixed
\[ me^{-2\delta}/2 \equiv \mu, \quad ae^{-\delta}/2 \equiv \omega. \tag{2.17} \]
In this limit
\[ \Sigma \to R^2 + \mu, \tag{2.18} \]
and the Left Kerr-Newman becomes
\[ ds^2 = - \left( 1 + \frac{\mu}{R^2} \right)^{-2} \left[ dt + \frac{\mu \omega}{R^2} \sigma^3_L \right]^2 + \left( 1 + \frac{\mu}{R^2} \right) \left[ dR^2 + \frac{R^2}{4} \left( (\sigma^1_L)^2 + (\sigma^2_L)^2 + (\sigma^3_L)^2 \right) \right], \tag{2.19} \]
\[ A = \frac{\sqrt{3}}{2} \left( 1 + \frac{\mu}{R^2} \right)^{-1} \left[ dt + \frac{\mu \omega}{R^2} \sigma^3_L \right]. \]
This is the BMPV solution (first derived in [2]), in isotropic coordinates. The right Kerr-Newman yields, in this extremal limit, (2.19) with \( \omega = 0 \). Thus, solely the Left angular momentum survives in this extremal limit. If instead we would have taken the \( \delta \to \infty \) in the extremal limit (2.16) with the associated modifications for the fixed quantities, we would have found a BMPV solution from the Right Kerr-Newman; only Right angular momentum survives in such extremal limit.

The maximally supersymmetric Gödel type universe [1] has \( J_R = 0 \). This latter solution reads
\[ ds^2 = -\left[ dt + jR^2 \sigma^3_L \right]^2 + ds^2(\mathbb{R}^4), \quad A = \frac{\sqrt{3}}{2} jR^2 \sigma^3_L. \tag{2.20} \]
Gimon and Hashimoto showed that we can embed a Kerr black hole in this homogeneous space\footnote{14}. Naturally, the simplest case is to embed the ‘Left Kerr’. Then the Kerr-Gödel solution reads
\[
\begin{align*}
\dot{s}^2 &= -\left[dt + jR^2\sigma^3_L\right]^2 + \frac{R^4 dR^2}{R^4 - 2m(R^2 - a^2) + 8jmR^2(a + 2jm)} + \frac{R^2}{4} \left((\sigma^1_L)^2 + (\sigma^2_L)^2 + (\sigma^3_L)^2\right) \\
&\quad + \frac{2m}{R^2} \left[dt - \frac{a}{2}\sigma^3_L\right]^2 - 2mj^2R^2(\sigma^3_L)^2 ,
\end{align*}
\]
\[A = \frac{\sqrt{3}}{2}jR^2\sigma^3_L .\quad (2.21)\]

This solution has a curvature singularity at \(R = 0\), which, at least for sufficiently small \(a\) and \(j\) is hidden behind an outer event horizon at
\[R^2 = m - 4jm(a + 2jm) + \sqrt{m - 4jm(a + 2jm)^2 - 2ma^2} .\quad (2.22)\]

3 The Kerr-Newman-Gödel solution

Consider the following six dimensional truncation of the heterotic string low energy effective field theory (in the string frame)
\[S_{Het} = \int d^6x \sqrt{-G} e^{-\phi} \left(R + \partial_{\mu}\phi \partial^{\mu}\phi - \frac{1}{12} \tilde{H}_{\mu\nu\lambda} \tilde{H}^{\mu\nu\lambda} - \frac{1}{8} F_{\mu\nu} F^{\mu\nu}\right) .\quad (3.1)\]
The fundamental fields are the string frame metric \(G\), the NS field \(B\), one abelian gauge field \(A\) and the dilaton \(\phi\). The fields strengths present in the action are related with the potentials by
\[F = dA , \quad H = dB , \quad \tilde{H} = H - \frac{1}{4} A \wedge F .\quad (3.2)\]
Here, \(\tilde{H}\) and \(F\) are the gauge invariant objects; the gauge transformations are the standard one for the \(B\) field, \(B^{(2)} \rightarrow B^{(2)} + d\Lambda^{(1)}\), but for the \(A\) field we need
\[A^{(1)} \rightarrow A^{(1)} + d\Lambda^{(0)} , \quad \text{and} \quad B^{(2)} \rightarrow B^{(2)} + \Lambda^{(0)} F^{(2)} .\quad (3.3)\]
We have written the rank of the form as a superscript. The Bianchi identities are
\[dF = 0 , \quad d\tilde{H} = -\frac{1}{4} F \wedge F .\quad (3.4)\]

The following theorem provides a way of generating new solutions (charged) of this theory from known ones (uncharged):

**Theorem (Hassan-Sen)**\footnote{13}: Let \((G, B, A, \phi)\) be a solution of (3.1), which is independent of time, \(t\), and a set of spatial coordinates, \(x^i\). Denote \(a = 0, i\). Denote the \((i+1)\) dimensional square matrices \(G_{ab}\) and \(B_{ab}\) by \(\hat{G}\) and \(\hat{B}\). Denote the \((i+1)\) dimensional column vector \(A_a\) by \(\hat{A}\). Assume the metric and NS field obey \(G_{\mu a} = B_{\mu a} = 0\), for \(\mu \neq 0, i\). Define the Hassan-Sen matrix as
\[
\mathcal{M} = \begin{pmatrix}
(K^T - \eta)\hat{G}^{-1}(K - \eta) & (K^T - \eta)\hat{G}^{-1}(K + \eta) & - (K^T - \eta)\hat{G}^{-1}\hat{A} \\
(K^T + \eta)\hat{G}^{-1}(K - \eta) & (K^T + \eta)\hat{G}^{-1}(K + \eta) & - (K^T + \eta)\hat{G}^{-1}\hat{A} \\
- \hat{A}^T\hat{G}^{-1}(K - \eta) & - \hat{A}^T\hat{G}^{-1}(K + \eta) & \hat{A}^T\hat{G}^{-1}\hat{A}
\end{pmatrix} ,\quad (3.5)
\]
where
\[ K = -\hat{B} - \hat{G} - \frac{1}{4} \hat{A} \hat{A}^T, \quad \eta = \text{diag}(-1, 1, \ldots, 1). \] (3.6)

Then, a new, inequivalent solution of (3.1), \((G', B', A', \phi')\) is obtained as
\[ \mathcal{M}' = \Omega \mathcal{M} \Omega^T, \]
\[ \phi' = \phi + \ln \sqrt{\det \hat{G}' / \det \hat{G}}, \] (3.7)

where \(\Omega\) is defined as
\[ \Omega = \begin{pmatrix} S \\ R \end{pmatrix}, \] (3.8)

with \(S, R\) being \(O(1, i)\) and \(O(1, i + 1)\) matrices respectively. The \(S\) matrix describes Lorentz transformations in the \(t, x^i\)-plane. The \(R\) matrix describes Lorentz transformations in the \(t, x^i, z\) plane, where \(z\) is the internal direction associated to the gauge field \(A\).

We rewrite the Hassan-Sen matrix in the notation
\[ \mathcal{M} = \begin{pmatrix} \text{---} & \text{--} & \text{+} & \text{-} \\ \text{--} & \text{++} & \text{+} & \text{-} \\ \text{+} & \text{-} & \text{+} & \text{+} \\ \text{-} & \text{+} & \text{+} & \text{-} \end{pmatrix} \quad (X^T \quad X^T \quad X^T \quad \chi). \] (3.9)

Equating (3.9) with (3.5) defines the matrices \(M^{\pm \pm}\), the vectors \(X^\pm\) and the scalar \(\chi\).

We can compute the fields \(\hat{G}, \hat{A}, \hat{B}\) in terms of the entries of the matrix \(\mathcal{M}\). The metric is computed as
\[ \hat{G}^{-1} = \frac{1}{4} \eta (M^{++} + M^{--} - M^{+-} - M^{-+}) \eta, \] (3.10)

the gauge field follows as
\[ \hat{A} = \frac{1}{2} \hat{G} \eta (X^- - X^+), \] (3.11)

and the Neveu-Schwarz field can be computed from
\[ \hat{B} = \frac{1}{4} \hat{G} \eta (M^{--} - M^{++} + M^{-+} - M^{+-}) - \hat{G} - \frac{1}{4} \hat{A} \hat{A}^T. \] (3.12)

### 3.1 Generating the solution

The Kerr-Newman-Gödel solution is generated as follows:

i) Uplift the Kerr-Gödel solution (2.21) to six dimensions as a solution of heterotic string theory (3.1). This is achieved by the following ansatz of the six dimensional fields \((g, B, \phi, A)\) in terms of the five dimensional ones \((\tilde{g}, \tilde{A})\) [3],

\[ ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu + \left(dy + \frac{2}{\sqrt{3}} \tilde{A}\right)^2, \quad \phi = 0, \quad A = 0, \]
\[ H = dB = \frac{2}{\sqrt{3}} d\tilde{A} \wedge \left(dy + \frac{2}{\sqrt{3}} \tilde{A}\right) - \frac{2}{\sqrt{3}^2} d\tilde{A}. \] (3.13)
Here, \( y \) is the sixth dimension and \( \tilde{\star} \) is Hodge duality with respect to \( \tilde{g} \). We take the following \( B \) within its gauge equivalence class

\[
B = j(r^2 - 2m)dt \wedge \sigma_L^3 + j r^2 \sigma_L^3 \wedge dy + j ml \cos \theta d\phi \wedge d\psi.
\]

(3.14)

Note that since the dilaton is zero the string metric \( G \) coincides with \( g \).

ii) The previous solution is independent of \((t, \phi, \psi, y)\). Take \( \hat{G} \) and \( \hat{B} \) to be the restriction of the six dimensional fields to this subspace. Then, take the matrices \( S \) and \( R \) to be boosts in the \( y \) direction with parameters \(-\alpha\) and \(\alpha\) respectively

\[
S = \begin{pmatrix}
\cosh \alpha & - \sinh \alpha \\
- \sinh \alpha & \cosh \alpha
\end{pmatrix}
,\quad
R = \begin{pmatrix}
\cosh \alpha & \sinh \alpha \\
\sinh \alpha & \cosh \alpha
\end{pmatrix}
.
\]

(3.15)

\( I_n \) denotes the \( n \)-dimensional identity. We have omitted the last row and column of \( R \) corresponding to the internal \( z \) direction; in this transformation nothing happens in such direction. A solution with non-trivial \( G, B \) and \( \phi \) is obtained. The gauge field \( A \) is still zero.

iii) Perform a strong-weak coupling string duality between toroidally compactified heterotic string theory and type IIA string theory on \( K3 \). The latter has string frame action

\[
S_{IIA} = \int d^6x \sqrt{-G'} \left\{ e^{-\phi'} \left( R' + \partial_{\mu}\phi' \partial^{\mu}\phi' - \frac{1}{12} H'^2 \right) - \frac{1}{8} F'^2 + \frac{1}{25} \epsilon^{\mu_1...\mu_6} B'_{\mu_1\mu_2} F'_{\mu_3\mu_4} F'_{\mu_5\mu_6} \right\}.
\]

(3.16)

More specifically, the IIA fields (primed) are related to the heterotic fields (unprimed) by [2]

\[
\phi' = -\phi \ , \quad G' = e^{-\phi} G \ , \quad H' = e^{-\phi} \star \tilde{H} \ , \quad A' = A \ .
\]

(3.17)

The Einstein frame metric \( g = \exp(-\phi/2)G \) is invariant in the process. The duality swaps equation of motion and Bianchi identity in going from \( \tilde{H} \) to \( H' \). In particular the Bianchi identity becomes the standard \( dH' = 0 \) which means \( H' = dB' \). The idea of the transformation is to swap the sign of the dilaton which we will be able to cancel with another boost. Note that the difference between these truncations of the type IIA and heterotic actions is at the level of the gauge field \( A(A') \). Since at this point this field is vanishing, the final solution is also a solution of six dimensional heterotic string theory.

iv) Perform a second Hassan-Sen boost, with parameter \( \beta \), this time along the internal direction \( z \). Thus take, \( S = I_4 \) and \( R \) to be

\[
R = \begin{pmatrix}
\cosh \beta & \sinh \beta \\
\sinh \beta & \cosh \beta
\end{pmatrix}
.
\]

(3.18)

By choosing \( \beta = 2\alpha \), the resulting solution has vanishing dilaton and \( g_{yy} = 1 \). But there is a non-trivial \( B \) and \( A \) fields.
v) Perform again string-string duality to obtain a type IIA solution, as in iii); Since now the dilaton is constant, the only non-trivial change is for the NS field. The type IIA field is

\[ H' = \ast \tilde{H}. \]

This defines the IIA NS field as \( H' = dB' \).

The resulting solution has the following six dimensional fields

\[ ds_6^2 = ds_5^2 + \{ dy + j [2m \sinh \delta + R^2 (\cosh \delta - \sinh \delta)] \sigma_L^2 \}^2, \]

\[ A' = \frac{4m \sinh \delta \cosh \delta}{\Sigma} \left\{ dt + \frac{1}{2} (a + 2jR^2) (\sinh \delta - \cosh \delta) \sigma_L^3 \right\}, \]

\[ B' = \frac{j R^2 (R^2 - 2m)(\cosh \delta - \sinh \delta)}{\Sigma} dt \wedge \sigma_L^3 + jm \cos \theta (a - 4jm \sinh^2 \delta) d\phi \wedge d\psi \]

\[ + \frac{m \sinh 2\delta dt + [4jm^2 \sinh^3 \delta + (\cosh \delta - \sinh \delta) (jR^4 + 4mjR^2 \sinh^2 \delta - ma \sinh 2\delta/2)] \sigma_L^3 \wedge dy. \]

The function \( \Sigma \) is the same as in (3.16). We have separated the five dimensional piece of the metric which will in fact be the final result for the metric. It can be written as

\[ ds_5^2 = \frac{-R^2 (R^2 - 2m)}{\Sigma^2} dt^2 + \frac{R^2 \Sigma dR^2}{R^4 - 2m(R^2 - a^2) + 8jmR^2 (a + 2jm)} + \frac{\Sigma}{4} (\sigma_L^1)^2 + (\sigma_L^2)^2 + (\sigma_L^3)^2 \]

\[ - \frac{2}{\Sigma^2} \left\{ jR^2 \left[ (\cosh \delta - \sinh \delta)(\Sigma^2 - m^2 \sinh^2 (2\delta)) + 2m \Sigma \sinh \delta \right] \right. \]

\[ + ma \left[ (\cosh^3 \delta - \sinh^3 \delta) R^2 + 2m \sinh^3 \delta \right] \} dt \sigma_L^3 \]

\[ - \frac{m(a + 2jR^2)}{2 \Sigma^2} \left\{ 2jR^4 (1 - \sinh 2\delta + 4 \cosh \delta \sinh^2 \delta \cosh \delta - \sinh \delta) + 16jm^2 \sinh^5 \delta \cosh \delta - \sinh \delta \right. \]

\[ + 4jmR^2 \sinh^2 \delta \left( 2 + 5 \sinh^2 \delta - \sinh 2\delta - 6 \sinh^3 \delta \cosh \delta - \sinh \delta \right) \]

\[ - a \left[ R^2 - 2m \sinh^3 \delta (2 \sinh^3 \delta - 2 \cosh^3 \delta + 3 \sinh \delta) \right] \} (\sigma_L^3)^2 \]

\[ - j^2 \left[ 2m \sinh \delta + R^2 (\cosh \delta - \sinh \delta) \right]^2 (\sigma_L^3)^2, \]

The fields (3.20)-(3.22), together with \( \phi = 0 \) can be verified to solve the equations of motion that follow from (3.16). For completeness we write down these equations (in the Einstein frame)

\[ R'_{\mu\nu} = \frac{1}{4} \left\{ \partial_{\mu} \phi' \partial_{\nu} \phi' + e^{-\phi'} \left[ H'_{\mu\alpha\beta} H'_{\nu}^{\alpha\beta} - \frac{1}{6} g'_{\mu\nu} H'^2 \right] + e^{\phi'/2} \left[ F'_{\mu\nu} F'_{\nu}^{\alpha} - \frac{1}{8} g'_{\mu\nu} F'^2 \right] \right\}, \]

\[ D_{\mu} (e^{-\phi'} H'^{\mu\alpha\beta}) = -\frac{1}{16} \varepsilon^{\alpha\beta\mu_1 \ldots \mu_4} F'_{\mu_1 \mu_2} F'_{\mu_3 \mu_4}, \quad D_{\mu} (e^{\phi'/2} F'^{\mu\nu}) = -\frac{1}{12} \varepsilon^{\nu \mu_1 \ldots \mu_5} F'_{\mu_1 \mu_2} H'^{\mu_3 \mu_4 \mu_5}, \]

\[ \square \phi' = -\frac{1}{6} e^{-\phi'} H'^2 + \frac{1}{8} e^{\phi'/2} F'^2. \]
vi) Perform Kaluza-Klein reduction along the y direction to five dimensions. Besides the metric, 
\( ds_5^2 \) we obtain a five dimensional NS field, \( B_5 \), and three gauge fields, \( A' \) already present in 
6 dimensions, \( A_g \) from the metric and \( A_B \) from the NS field. The KK ansatz for the metric 
and NS field are the standard ones

\[
\begin{align*}
 ds_6'^2 &= ds_5^2 + [dy + A_g]^2, \\
 B' &= B_5 + A_B \wedge dy .
\end{align*}
\]

(3.27)

The final five dimensional fields are therefore \( ds_5^2, A', B_5, A_g \) and \( A_B \), which are a solution to IIA 
string theory on \( K3 \times S^1 \). Note that \( A' \) is the Ramond-Ramond one-form.

For \( j = 0 \), the metric (3.23) reduces to (2.10), the gauge fields \( A' \) and \( A_B \) reduce to (2.12) (up 
to a redefinition by a constant) and \( A_g \) and \( B_5 \) vanish. Thus we recover the Kerr-Newman black 
hole. For \( \delta = 0 \), (3.23) reduces to (2.21), \( A' \) vanishes, \( A_B \) and \( A_g \) reduce to (2.21) (up to constant) 
and \( B_5 \) contains the same information as \( A_g \) via

\[
\begin{align*}
 dB_5 &= A_g \wedge A_g + \star dA_g .
\end{align*}
\]

(3.28)

Thus we recover the Kerr-Gödel black hole. In this sense we dub our five dimensional solution as 
the Kerr-Newman-Gödel black hole. We will postpone for somewhere else a more detailed analysis 
of the full solution.

4 Extremal limit

The extremal limit of the Kerr-Newman was taken in (2.16) and (2.17). Now we have an extra 
parameter, \( j \). Consider first (3.23) with \( m = 0 \). We find (2.20) with \( j \) replaced by \( j(\cosh \delta - 
\sinh \delta) \). Thus, to get an asymptotic Gödel solution we take the following extremal limit of our five 
dimensional solution:

\[
\begin{align*}
 \delta \to -\infty , & \quad j, m, a \to 0 , \\
 J & \equiv je^{-\delta} , \quad \mu \equiv me^{-2\delta}/2 , \quad \omega \equiv ae^{-\delta}/2 .
\end{align*}
\]

(4.1)

(4.2)

The metric we obtain is

\[
\begin{align*}
 ds_5^2 &= -\left(1 + \frac{\mu}{R^2}\right)^{-2} \left[ dt + \left(JR^2 + 2\mu J + \frac{\mu \omega}{R^2}\right) \sigma_L^3 \right]^2 \\
 & \quad + \left(1 + \frac{\mu}{R^2}\right) \left[ dR^2 + \frac{R^2}{4} \left((\sigma_L^1)^2 + (\sigma_L^2)^2 + (\sigma_L^3)^2\right) \right],
\end{align*}
\]

(4.3)

and the form fields are

\[
\begin{align*}
 A' &= 2\mu \frac{-dt + (\omega + JR^2)\sigma_L^3}{R^2 + \mu} , \quad A_g = JR^2\sigma_L^3 , \quad A_B = -\mu dt + (JR^4 + 2\mu R^2 J + \mu \omega)\sigma_L^3 , \quad (4.4)
\end{align*}
\]

and

\[
B_5 = \frac{JR^4}{R^2 + \mu} dt \wedge \sigma_L^3 .
\]

(4.5)

For \( J = 0 \), (4.3) reduces to (2.19), \( B_5 \) and \( A_g \) vanish and \( A' \) and \( A_B \) give (2.19) up to a gauge 
transformation and an overall factor. For \( \mu = 0 \), (4.3) reduces to (2.20), \( A' \) vanishes, \( A_g = A_B \) reduce
to (2.20) up to constant and $B_5$ is related to $A_g$ by (3.28). The general solution has a coordinate singularity at $R^2 = 0$. For $4\omega^2/\mu < 1$, this surface is a null surface and can be interpreted as the black hole event horizon, exactly as for the usual (asymptotically flat) BMPV black hole. This is easily seen by changing to a Schwarzschild like radial coordinate defined by $r^2 = R^2 + \mu$; the coefficient of $\sigma^3_L$ in the metric becomes

$$r^2 - \frac{(Jr^4 - J\mu^2 + \mu\omega)^2}{r^4},$$

and thus the J dependence vanishes at $r^2 = \mu$. More precisely one can introduce a set of regular coordinates on the horizon following [3] and show that for $4\omega^2/\mu < 1$, $r^2 = \mu$ is a null surface. The black hole entropy is then

$$S = \frac{\pi^2 \mu}{2G_5} \sqrt{\mu - 4\omega^2}. \quad (4.7)$$

There is a true physical singularity (timelike and point-like) at $r^2 = 0$, since the Ricci scalar diverges there, as $\mu^3/r^8$.

5 Conclusions

In this letter we have derived a new solution to type II string theory in five dimensions, interpreted as a Kerr-Newman black hole in a Gödel type universe. This was achieved by a set of Hassan-Sen transformations and string dualities. Recently [18], a Hassan-Sen transformation was used to give charge to the rotating black ring of Emparan and Reall [15]; this ring is another surprise of five dimensional gravity, with no four dimensional equivalent, as the BMPV black hole and the Gödel type universe discussed herein. Perhaps the five dimensional Petrov classification [16] [17] will be a useful tool in understanding and classifying these various solutions.

The most interesting special case of our new solution is an extremal limit which is BPS. The BMPV black hole has a ten dimensional interpretation as a D1-D5-Brinkmann wave system [7]; the Gödel spacetime has a ten dimensional interpretation as a pp-wave [4]. It will therefore be interesting to understand in more detail the ten dimensional interpretation of the BMPV-Gödel solution, in particular in relation to the issue of existence of horizons in pp-waves [19] [20] [21].

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