HYPOELLIPTICITY OF THE $\bar{\partial}$-NEUMANN PROBLEM AT EXPONENTIALLY DEGENERATE POINTS

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Abstract. We prove local hypoellipticity of the complex Laplacian $\square$ in a domain which has compactness estimates, is of finite type outside a curve transversal to the CR directions and for which the holomorphic tangential derivatives of a defining function are subelliptic multipliers in the sense of Kohn.

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1. Introduction

For the pseudoconvex domain $\Omega \subset \mathbb{C}^n$ whose boundary is defined in coordinates $z = x + iy$ of $\mathbb{C}^n$, by

$$2x_n = \exp \left( -\frac{1}{\left( \sum_{j=1}^{n-1} |z_j|^2 \right)^{\frac{1}{2}}} \right), \quad s > 0,$$

the tangential Kohn Laplacian $\square_b = \bar{\partial}_b \partial_b^* + \bar{\partial}_b^* \partial_b$ as well as the full Laplacian $\square = \partial \partial^* + \partial^* \partial$ show very interesting features especially in comparison with the “tube domain” whose boundary is defined by

$$2x_n = \exp \left( -\frac{1}{\left( \sum_{j=1}^{n-1} |x_j|^2 \right)^{\frac{1}{2}}} \right), \quad s > 0.$$

(Here $z_j$ have been replaced by $x_j$ at exponent.) Energy estimates are the same for the two domains. For the problem on the boundary $\partial \Omega$, they come as

$$\|(\log \Lambda)^{1/2} u\|_{\partial \Omega} \lesssim \|\bar{\partial}_b u\|_{\partial \Omega}^2 + \|\bar{\partial}_b^* u\|_{\partial \Omega}^2 + \|u\|_{\partial \Omega}^2$$

for any smooth compact support form $u \in C^\infty_c(\partial \Omega)$ of degree $k \in [1, n-2]$. Here $\log \Lambda$ is the tangential pseudodifferential operator with symbol $\log(1 + |\xi'|^2)^{\frac{1}{2}}$, $\xi' \in \mathbb{R}^{2n-1}$, the dual real tangent space. As for the problem on the domain $\Omega$, one has simply to replace $\bar{\partial}_b, \bar{\partial}_b^*$ by $\partial, \partial^*$ and take norms over $\Omega$ for forms $u$ in $D_{\partial^*}$, the domain of $\partial^*$, of degree $1 \leq k \leq n-1$; this can be seen, for instance, in [9]. In particular, these are superlogarithmic (resp. compactness) estimates if $s < 1$ (resp. for
any $s > 0$). A related problem is that of the local hypoellipticity of the Kohn Laplacian $\square_b$ or, with equivalent terminology, the local regularity of the inverse (modulo harmonics) operator $N_b = \square_b^{-1}$. Similar is the notion of hypoellipticity of the Laplacian $\square$ or the regularity of the inverse Neumann operator $N = \square^{-1}$. It has been proved by Kohn in [12] that superlogarithmic estimates suffice for local hypoellipticity of the problem both in the boundary and in the domain. (Note that hypoellipticity for the domain, [12] Theorem 8.3, is deduced from microlocal hypoellipticity for the boundary, [12] Theorem 7.1, but a direct proof is also available, [7] Theorem 5.4.) In particular, for (1.1) and (1.2), there is local hypoellipticity when $s < 1$.

As for the more delicate hypoellipticity, in the uncertain range of indices $s \geq 1$, only the tangential problem has been studied and the striking conclusion is that the behavior of (1.1) and (1.2) split. The first stays always hypoelliptic for any $s$ (Kohn [11]) whereas the second is not for $s \geq 1$ (Christ [4]). When one tries to relate $(\bar{\partial}_b, \bar{\partial}_b^*)$ on $b\Omega$ to $(\bar{\partial}, \bar{\partial}^*)$ on $\Omega$, estimates go well through (Kohn [12] Section 8 and Khanh [7] Chapter 4) but not regularity. In particular, the two conclusions about tangential hypoellipticity of $\square_b$ for (1.1) and non-hypoellipticity for (1.2) when $s \geq 1$, cannot be automatically transferred from $b\Omega$ to $\Omega$. Now, for the non-hypoellipticity in $\Omega$ in case of the tube (1.2) we have obtained with Baracco in [1] a result of propagation which is not equivalent but intimately related. The real lines $x_j$ are propagators of holomorphic extendibility from $\Omega$ across $b\Omega$. What we prove in the present paper is hypoellipticity in $\Omega$ for (1.1) when $s \geq 1$.

**Theorem 1.1.** Let $\Omega$ be a pseudoconvex domain of $\mathbb{C}^n$ in a neighborhood of $z_0 = 0$ and assume that the $\bar{\partial}$-Neumann problem satisfies the following properties

1. there are local compactness estimates,
2. there are subelliptic estimates for $(z_1, ..., z_{n-1}) \neq 0$,
3. $\partial z_j r, j = 1, ..., n - 1$, are subelliptic multipliers (cf. [10]).

Then $\square$ is locally hypoelliptic at $z_0$.

The proof follows in Section 2. It consists in relating the system on $\Omega$ to the tangential system on $b\Omega$ along the guidelines of [12] Section 8, and then in using the argument of [11] simplified by the additional assumption (i).

**Remark 1.2.** The domain with boundary (1.1), but not (1.2), satisfies the hypotheses of Theorem 1.1 for any $s > 0$: (i) is obvious, and (ii) and (iii) are the content of [11] Section 4.
Notice that $\partial \Omega$ is given only locally in a neighborhood of $z_0$. We can continue $\partial \Omega$ leaving it unchanged in a neighborhood of $z_0$, making it strongly pseudoconvex elsewhere, in such a way that it bounds a relatively compact domain $\Omega \subset \subset \mathbb{C}^n$ (cf. [14]). In this situation $\square$ is hypoelliptic at every boundary point. Also, it is well defined a $H^0$ inverse Neumann operator $N = \square^{-1}$, and, by Theorem [1.1] the $\bar{\partial}$-Neumann solution operator $\bar{\partial}^* N$ preserves $C^\infty(\Omega)$-smoothness. It even preserves the exact Sobolev class $H^s$ according to Theorem 2.7 below. In other words, the canonical solution $u = \bar{\partial}^* N f$ of $\bar{\partial} u = f$ for $f \in \text{Ker } \bar{\partial}$ is $H^s$ exactly at the points of $b\Omega$ where $f$ is $H^s$. The Bergman projection $B$ also preserves $C^\infty(\bar{\Omega})$-smoothness on account of Kohn’s formula $B = \text{Id} - \bar{\partial}^* N \bar{\partial}$.

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2. HYPOELLIPTICITY OF $\square$ AND EXACT HYPOELLIPTICITY OF $\bar{\partial}^* N$

We state properly hypoellipticity and exact hypoellipticity of a general system $(P_j)$.

**Definition 2.1.** (i) The system $(P_j)$ is locally hypoelliptic at $z_0 \in b\Omega$ if

$$P_j u \in C^\infty(\bar{\Omega})^k_{z_0} \text{ for any } j \text{ implies } u \in C^\infty(\bar{\Omega})^k_{z_0},$$

where $C^\infty(\bar{\Omega})^k_{z_0}$ denotes the set of germs of $k$-forms smooth at $z_0$.

(ii) The system $(P_j)$ is exactly locally hypoelliptic at $z_0 \in b\Omega$ when there is a neighborhood $U$ of $z_0$ such that for any pair of cut-off functions $\zeta$ and $\zeta'$ in $C^\infty_c(U)$ with $\zeta'|_{\text{supp}(\zeta)} \equiv 1$ we have for any $s$ and for suitable $c_s$

$$||\zeta u||_s^2 \leq c_s \left( \sum_j ||\zeta' P_j u||_s^2 + ||u||_0^2 \right), \quad u \in C^\infty(\bar{\Omega})^k \cap D(P_j).$$

If $(P_j)$ happens to have an inverse, this is said to be locally regular and locally exactly regular in the situation of (i) and (ii) respectively.

**Remark 2.2.** By Kohn-Nirenberg [13] the assumption $u \in C^\infty$ can be removed from (2.1). Precisely, by the elliptic regularization, one can prove that if $\zeta' P_j u \in H^s$ and $\zeta' u \in H^0$, then $\zeta u \in H^s$ and satisfies (2.1). This motivates the word “exact”, that is, Sobolev exact. Not only the local $C^\infty$- but also the $H^s$-smoothness passes from $P_j u$ to $u$.

Let $\vartheta$ be the formal adjoint of $\bar{\partial}$ and $\Delta = \bar{\partial} \vartheta + \vartheta \bar{\partial}$ the Laplacian; it acts on forms by the action of the usual Laplacian on its coefficients.
If \( u \in D_{\Box} \), then \( \Box u = \Delta u \). We first prove exact hypoellipticity of the system \( (\bar{\partial}, \bar{\partial}^*, \Delta) \); hypoellipticity of \( \Box \) itself will follow by the method of Boas-Straube.

**Theorem 2.3.** In the situation of Theorem 1.1, we have, for a neighborhood \( U \) of \( z_0 \) and for any couple of cut-off \( \zeta \) and \( \zeta' \) with \( \zeta' \mid_{\text{supp} \zeta} \equiv 1 \)

\[
|\zeta u|^2_s \lesssim |\zeta' \bar{\partial} u|^2_s + |\zeta' \bar{\partial}^* u|^2_s + |\zeta' \Delta u|^2_{s-2} + ||u||^2_0, \quad u \in D_{\bar{\partial}}.
\]

In particular, the system \((\bar{\partial}, \bar{\partial}^*, \Delta)\) is exactly locally hypoelliptic at \( z_0 = 0 \).

**Remark 2.4.** The hypoellipticity of \( \Box_b \) under (ii) and (iii) of Theorem 1.1 is proved by Kohn in [11]. It does not require (i) but it is not exact hypoellipticity (the neighborhood \( U \) of (2.1) depends on \( s \)). However, inspection of his proof shows that, if (i) is added, then in fact (2.1) holds for \((P_j) = \Box_b\). Our proof consists in a reduction to the tangential system.

**Proof.** We proceed in several steps which are highlighted in two intermediate propositions. We use the standard notation \( Q(u, u) \) for

\[
||\bar{\partial} u||^2_0 + ||\bar{\partial}^* u||^2_0 \text{ and some variants as, for an operator } Op, Q_{Op}(u, u) := ||Op \bar{\partial} u||^2_0 + ||Op \bar{\partial}^* u||^2_0; \text{ most often, in our paper, } Op \text{ is chosen as } \Lambda^s \zeta'.
\]

We decompose a form \( u \) as

\[
\begin{cases}
  u = u^\tau + u^\nu, \\
  u^\tau = u^{\tau^+} + u^{\tau^-} + u^{\tau^0},
\end{cases}
\]

where the first is the decomposition in tangential and normal component and the second is the microlocal decomposition \( u^{\tau^\pm} = \Psi^{\tau^\pm} u^{\tau} \) in which \( \Psi^{\tau^\pm} \) are the tangential pseudodifferential operators whose symbols \( \psi^{\tau^\pm} \) are a conic decomposition of the unity in the space dual to \( \mathbb{R}^{2n-1} \) the real orthogonal to \( \partial r \) (cf. Kohn [12]). We begin our proof by remarking that any of the forms \( u^\# = u^\nu, u^{\tau^-}, u^{\tau^0} \) enjoys elliptic estimates

\[
|\zeta u^\#|^2_s \lesssim |\zeta' \bar{\partial} u^\#|^2_s + |\zeta' \bar{\partial}^* u^\#|^2_s + |\zeta' \Delta u^\#|^2_{s-2} + ||u^\#||^2_0, \quad s \geq 2.
\]

We refer to [6] formula (1) of Main theorem as a general reference but also give an outline of the proof. For this, we have to call into play the tangential \( s \)-Sobolev norm which is defined by \( |||u|||_s = |||\Lambda^s u|||_0 \). We start from

\[
|\zeta u^\#|^2_s \lesssim Q(\zeta u^\#, \zeta u^\#) + ||u^\#||^2_0,
\]

\[
|\zeta u^\#|^2_s \lesssim |||u^\#|||^2_s \lesssim Q(\zeta u^\#, \zeta u^\#) + ||u^\#||^2_0,
\]
this is the basic estimate for \( u^\nu \) (which vanishes at \( b\Omega \)) whereas it is Lemma 8.6 for \( u^{\tau -} \) and \( u^{\tau 0} \). Applying (2.4) to \( \zeta'\Lambda^{s-1}\zeta u^\# \) one gets the estimate of tangential norms for any \( s \). Finally, by non-characteristicity of \((\bar{\partial}, \bar{\partial}^*)\) one passes from tangential to full norms along the guidelines of Theorem 1.9.7. The version of this argument for \( s \) is the first which is the most central

We decompose

\[
\tau^+ = u^{\tau+}(h) + u^{\tau+}(0),
\]

where \( u^{\tau+}(h) \) is the “harmonic extension” in the sense of Kohn and \( u^{\tau+}(0) \) is just the complementary part. We denote by \( \bar{\partial}^r \) the extension of \( \bar{\partial}_b \) from \( b\Omega \) to \( \Omega \) which stays tangential to the level surfaces \( r \equiv \text{const} \). It acts on tangential forms \( u^\tau \) and it is defined by \( \bar{\partial}^ru^\tau = (\bar{\partial}u^\tau)^r \). We denote by \( \bar{\partial}^ru^\tau \) its adjoint; thus \( \bar{\partial}^ru^\tau = \bar{\partial}r(u^\tau) \). We use the notations \( \Box^r \) and \( Q^r \) for the corresponding Laplacian and energy. We notice that over a tangential form \( u^\tau \) we have a decomposition

\[
Q = Q^r + ||\bar{L}_nu^\tau||_0^2.
\]

The proof of (2.2) for \( u^{\tau+} \) requires two crucial technical results. Here is the first which is the most central

**Proposition 2.5.** For the harmonic extension \( u^{\tau+}(h) \) we have

\[
|||\zeta u^{\tau+}(h)|||_s^2 \lesssim Q_{\Lambda^s}\zeta(u^{\tau+}(h), u^{\tau+}(h)) + ||u^{\tau+}(h)||_0^2.
\]

**Proof.** We apply compactness estimates (cf. e.g. Section 6) for \( \zeta'\Lambda^s\zeta u^{\tau+}(h) \),

\[
||\zeta'\Lambda^\ast\zeta u^{\tau+}(h)||^2_2 \leq \epsilon Q(\zeta'\Lambda^s\zeta u^{\tau+}(h), \zeta'\Lambda^s\zeta u^{\tau+}(h)) + c_\epsilon ||\zeta'\Lambda^s\zeta u^{\tau+}(h)||^2_{-1}.
\]

We decompose \( Q \) according to (2.5). We calculate \( Q^r \) over \( \zeta'\Lambda^s\zeta u^{\tau+}(h) \) and compute errors coming from commutators \([Q^r, \zeta'\Lambda^s\zeta]\). In this calculation we assume that the cut off functions are of product type \( \zeta(z')\zeta(t) \) where \( z' \) (resp. \( t \)) are complex (resp. totally real) tangential coordinates in \( T_{z_0}b\Omega \). We have

\[
Q^r(\zeta'\Lambda^s\zeta u^{\tau+}(h), \zeta'\Lambda^s\zeta u^{\tau+}(h)) \lesssim Q^r(\zeta'\Lambda^s\zeta u^{\tau+}(h), u^{\tau+}(h)) + ||\zeta'\tau^+(h)||_{-1}^2 + ||\zeta'\tau^+(h)||_{-1}^2
\begin{align*}
+ \left( ||(|\dot{\zeta}(z')| + |\dot{\zeta}(z')|)\Lambda^s\tau^+(h)||_0^2 + || \sum_{j=1}^{n-1} |r_{z^j}[(|\dot{\zeta}(t)| + |\dot{\zeta}(t)|)\Lambda^s\tau^+(h)||_0^2 \right).
\end{align*}

We explain (2.8). First, the commutators $[\bar{\partial}^r, \zeta' \Lambda^s \zeta]$ (and similarly as for $[\bar{\partial}^r, \zeta' \Lambda^s \zeta]$) are decomposed by Jacobi identity as

$$[\bar{\partial}^r, \zeta' \Lambda^s \zeta] = [\bar{\partial}^r, \zeta'] \Lambda^s \zeta + \zeta'[\bar{\partial}^r, \Lambda^s] \zeta + \zeta' \Lambda^s [\bar{\partial}^r, \zeta].$$

The central commutator $[\bar{\partial}^r, \Lambda^s]$ produces the error term $|||\zeta u^{r+\langle h \rangle}|||^2_n$.

As for the two others, we have

$$[\bar{\partial}^r, \zeta(z') \zeta(t)] = [\bar{\partial}^r, \zeta(z')] \zeta(t) + \zeta(z')[\bar{\partial}^r, \zeta(t)],$$

and similarly for $\zeta$ replaced by $\zeta'$ and $\partial^r$ by $\bar{\partial}^r$. Now,

$$(2.9) \quad [\bar{\partial}^r, \zeta(z')] \sim \zeta(z').$$

On the other hand, we first notice that it is not restrictive to assume that $\partial_{z_1}, ..., \partial_{z_{n-1}}$ are a basis of $T_0^{1.0} \Omega$ for otherwise, owing to (iii), we have subelliptic estimates from which local regularity readily follows. Thus, each $\tilde{L}_j, \ j = 1, ..., n-1$, is of type $\tilde{L}_j = r_{z_j} \partial_{z_n} - r_{z_n} \partial_{z_j}$, and then

$$[\bar{\partial}^r, \zeta(t)] \sim \sum_{j=1}^{n-1} [\tilde{L}_j, \zeta(t)] \sim \sum_{j=1}^{n-1} r_{z_j} \dot{\zeta}(t). \quad (2.10)$$

By combining (2.9) with (2.10) (and using the analogous for $\zeta'$ and $\bar{\partial}^r$), we get the last line of (2.8). This establishes (2.8). Next, since $(\bar{\partial}^r, \bar{\partial}^r)$ has subelliptic estimates, say $\eta$-subelliptic, for $z' \neq 0$ and hence in particular over $\text{supp} \ \dot{\zeta}(z')$ and $\text{supp} \ \dot{\zeta}'(z')$ and since the $r_{z_j}$ are, say, $\eta$-subelliptic multipliers even at $z' = 0$, then the last line of (2.8) is estimated by $|||\zeta'' \Lambda^{s-\eta} \zeta u^{r+\langle h \rangle}|||^2$ where $\zeta'' \equiv 1$ over $\text{supp} \ \zeta'$. This shows, using iteration over increasing $k$ such that $k\eta > s$ and over decreasing $j$ from $s - 1$ to 0, that (2.7) and (2.8) imply (2.6) provided that we add on the right side the extra term $|||\tilde{L}_n \zeta' \Lambda^s \zeta u^{r+\langle h \rangle}|||^2$. Note that, as a result of the inductive process, we have to replace $Q_{C' \Lambda^s \zeta}$ in (2.8) by $Q_{\Lambda^s \zeta'}$ in (2.6).

Up to this point the argument is the same as in [11] and does not make any use of the specific properties of the harmonic extension $u^{r+\langle h \rangle}$. We start the new part which is dedicated to prove that $|||\tilde{L}_n \zeta' \Lambda^s \zeta u^{r+\langle h \rangle}|||^2$ can be removed from the right of (2.6). For this we have to use the main property of this extension expressed by [12] Lemma 8.5, that is,

$$(2.11) \quad |||\tilde{L}_n \zeta u^{r+\langle h \rangle}|||^2_0 \lesssim \sum_{j=1}^{n-1} |||\tilde{L}_j \zeta u^{r+\langle h \rangle}|||^2_{b_{n-1}^{-\frac{1}{2}}} + |||u^{r+\langle h \rangle}|||^2_0.$$
Note that (2.11) differs from [12] Lemma 8.5 by $[\bar{L}_n, \Psi^{\dagger}]$; but this is an error term which can be taken care of by $u^\tau 0$ to which elliptic estimates apply. Applying (2.11) to $\zeta' \Lambda^s \zeta u^\tau + (h)$ (for the first inequality below), and using the classical inequality $|| \cdot ||_{b, -\frac{1}{2}}^2 \leq c_i || | \zeta^0 || + \epsilon || | \partial_{\tau} \cdot ||_{-1}^2$ (cf. e.g. [8] (1.10)) together with the splitting $\partial_{\tau} = \bar{L}_n + Tan$ (for the second), we get

$$|| \bar{L}_n \zeta' \Lambda^s \zeta u^\tau + (h) ||_0^2 \leq \sum_{j=1}^{n-1} || \bar{L}_j \zeta' \Lambda^s \zeta u^\tau + (h) ||_{b, -\frac{1}{2}}^2 + || \zeta' \Lambda^s \zeta u^\tau + (h) ||_0^2$$

$$\leq c_i \sum_{j=1}^{n-1} || \bar{L}_j \zeta' \Lambda^s \zeta u^\tau + (h) ||_0^2 + \epsilon \sum_{j=1}^{n-1} || \bar{L}_n \bar{L}_j \zeta' \Lambda^s \zeta u^\tau + (h) ||_{-1}^2$$

$$+ \epsilon \sum_{j=1}^{n-1} || || Tan \bar{L}_j \zeta' \Lambda^s \zeta u^\tau + (h) ||_{-1}^2 + || \zeta' \Lambda^s \zeta u^\tau + (h) ||_0^2.$$

The first term on the right of the last inequality is controlled by

$$\sum_{j=1}^{n-1} || \zeta' \Lambda^s \zeta \bar{L}_j u^\tau + (h) ||^2 + || \zeta u^\tau + (h) ||_s^2 + || \zeta' u^\tau + (h) ||_{s-1}^2$$

by the first part of the proposition; moreover, we have the immediate estimate

$$\sum_{j=1}^{n-1} || \zeta' \Lambda^s \zeta \bar{L}_j u^\tau + (h) ||^2 \sim Q_{\Lambda^s \zeta'}(u^\tau + (h), u^\tau + (h)).$$

The term which carries $\epsilon Tan$, after $Tan$ has been annihilated by the Sobolev norm of index $-1$, has the same estimate as the first term. It remains to control the second term in the right which involves $\epsilon \bar{L}_n$. We rewrite $\bar{L}_n \bar{L}_j = \bar{L}_j \bar{L}_n + [\bar{L}_n, \bar{L}_j]$; when $\bar{L}_j$ moves in first position, it is annihilated by $-1$ and what remains is absorbed in the left. As for the commutator, we have

$$|| || \bar{L}_n, \bar{L}_j || \zeta' \Lambda^s \zeta u^\tau + (h) ||_{s-1}^2 \leq || \zeta u^\tau + (h) ||_s^2 + || \partial_{\tau} \zeta' \Lambda^s \zeta u^\tau + (h) ||_{s-1}^2$$

$$\leq \epsilon || \zeta u^\tau + (h) ||_s^2 + || \bar{L}_n \zeta' \Lambda^s \zeta u^\tau + (h) ||_{s-1}^2,$$

where we have used the splitting $\partial_{\tau} = Tan + \bar{L}_n$ in the second inequality. Again, the term with $\bar{L}_n$, which now comes in $-1$ norm, is absorbed in the left of (2.12). Summarizing up, we have got

$$|| \bar{L}_n \zeta' \Lambda^s \zeta u^\tau + (h) ||_0^2 \sim c_i Q_{\Lambda^s \zeta'}(u^\tau + (h), u^\tau + (h))$$

$$+ || \zeta u^\tau + (h) ||_s^2 + || \zeta' u^\tau + (h) ||_{s-1}^2.$$

But $|| \bar{L}_n \cdot ||_0^2$ comes with a factor $\epsilon$ of compactness and hence the term in $s$-norm in the last line can be absorbed in the left of the initial
inequalities \eqref{2.7} or \eqref{2.6}. Finally, we use an inductive argument to go down from \( s - 1 \) to 0. This concludes the proof of the proposition.

We remark now that
\[
|||\zeta u^{\tau + (h)}|||^0_0 \lesssim |||\zeta u^{\tau +}_b|||^2_{b, -\frac{1}{2}} \\
\lesssim |||\zeta u^{\tau +}|||^2_0 + |||\partial_r \zeta u^{\tau +}|||^2_{-1} \\
\lesssim |||\zeta u^{\tau +}|||^2_0 + |||\bar{L}_n \zeta u^{\tau +}|||^2_{-1} + |||\Tan \zeta u^{\tau +}|||^2_{-1} \\
\lesssim Q_{\Lambda - 1\zeta}(u^{\tau +}, u^{\tau +}) + |||\zeta u^{\tau +}|||^2_{0}.
\] (2.14)

The same inequality also holds for \( u^{\tau + (h)} \) replaced by \( u^{\tau + (0)} \) on account of the identity \( u^{\tau + (0)} = u^{\tau +} + u^{\tau + (h)} \). We need another preparation result

**Proposition 2.6.** We have
\[
Q^r_{\Lambda - 1\zeta'}(u^{\tau + (h)}, u^{\tau + (h)}) \lesssim Q^r_{\Lambda - 1\zeta'}(u^{\tau +}, u^{\tau +}) + Q^r_{\partial_r \Lambda^{s - 1}\zeta'}(u^{\tau +}, u^{\tau +})
\] (2.15)
and
\[
|||\zeta' u^{\tau + (0)}|||^2_s \lesssim Q^r_{\Lambda^{s - 1}\zeta'}(u^{\tau +}, u^{\tau +}) + Q^r_{\partial_r \Lambda^{s - 2}\zeta'}(u^{\tau +}, u^{\tau +}) \\
+ |||\zeta' \Delta u^{\tau +}|||^2_{s - 2} + ||u^{\tau +}||^2_0.
\] (2.16)

**Proof.** The proof of \eqref{2.15} is an immediate combination of the formulas \( |||\zeta' u^{\tau + (h)}|||^0 \lesssim |||\zeta' u^{\tau +}_b|||^2_{b, -\frac{1}{2}} \), and \( |||\zeta' u^{\tau +}|||^2_{b, -\frac{1}{2}} \lesssim ||\zeta' u^{\tau +}||^0 + |||\partial_r \zeta' u^{\tau +}|||^2_{-1} \).

We prove now \eqref{2.16}. By elliptic estimate for \( u^{\tau + (0)} \) (which vanishes at \( \partial \Omega \)) with respect to the order 2 elliptic operator \( \Delta \), we have
\[
|||\zeta' u^{\tau + (0)}|||^2_s \lesssim |||\zeta' \Delta u^{\tau + (0)}|||^2_{s - 2} + ||u^{\tau + (0)}||^2_0.
\] (2.17)

This result of Sobolev regularity at the boundary is very classical: it is formulated, for functions in \( H^1_0 \) such as the coefficients of \( u^{\tau + (0)} \), e.g., in Evans [5] Theorem 5 p. 323. Owing to the identity \( \Delta u^{\tau + (0)} = \Delta u^{\tau +} + P^1 u^{\tau + (h)} \) for a 1-order operator \( P^1 \) (cf. [12] p. 241), we can replace \( \Delta u^{\tau + (0)} \) by \( \Delta u^{\tau +} \) on the right side of \eqref{2.17} putting the contribution of \( P^1 \) into an error term of type \( |||\zeta' u^{\tau + (h)}|||^0_{s - 1} + |||\zeta' \partial_r u^{\tau + (h)}|||^0_{s - 2} \), which can be estimated, on account of the splitting \( \partial_r = \bar{L}_n + \Tan \), by \( |||\zeta' u^{\tau + (h)}|||^0_{s - 1} + |||\zeta'' u^{\tau + (h)}|||^0_{s - 2} + Q^r_{\Lambda^{s - 2}\zeta'}(u^{\tau + (h)}, u^{\tau + (h)}) \). We write the terms of order \( s - 1 \) and \( s - 2 \) as a common \( |||\zeta'' u^{\tau + (h)}|||^0_{s - 1} \) that we can estimate, using \eqref{2.6} and \eqref{2.15}, by
\[
|||\zeta'' u^{\tau + (h)}|||^2_{s - 1} \lesssim Q^r_{\Lambda^{s - 1}\zeta''}(u^{\tau +}, u^{\tau +}) + Q^r_{\Lambda^{s - 2}\partial_r \zeta''}(u^{\tau +}, u^{\tau +}).
\]
This brings down from $s - 1$ to 0 the Sobolev index in the error term. This 0-order term $||u^{\tau + (h)}||^2_0$, together with its companion $||u^{\tau + (0)}||^2_0$ in the right of (2.17), is estimated, because of (2.14), by $||u^{\tau +}||^2_0$ up to a term $Q_{\Lambda - 1, \zeta}$ which is controlled by the right side of (2.16). This concludes the proof of (2.16).

□

End of proof of Theorem 2.3. We prove (2.2) for $u^{\tau +}$; this implies the conclusion in full generality according to the first part of the proof. We have

\begin{equation}
|||\zeta u^{\tau + (h)}|||^2_s \lesssim Q^{\tau}_{\Lambda, \zeta'}(u^{\tau + (h)}, u^{\tau + (0)}) + ||u^{\tau + (h)}||^2_0
\end{equation}

by (2.19)

\begin{equation}
\lesssim Q^{\tau}_{\Lambda, \zeta'}(u^{\tau +}, u^{\tau +}) + Q^{\tau}_{\partial, \Lambda^{-1}, \zeta'}(u^{\tau +}, u^{\tau +}) + ||u^{\tau +}||^2_0.
\end{equation}

We combine (2.18) with (2.16); what we get is

\begin{equation}
|||\zeta u^{\tau +}|||^2_s \lesssim |||\zeta u^{\tau + (h)}|||^2_s + |||\zeta u^{\tau + (0)}|||^2_s
\end{equation}

\begin{equation}
\lesssim |||\zeta' \bar{\partial} u^{\tau +}|||^2_s + |||\zeta' \bar{\partial}^* u^{\tau +}|||^2_s + |||\zeta' \Delta u^{\tau +}|||^2_{s - 2} + ||u^{\tau +}||^2_0.
\end{equation}

By the non-characteristicity of $Q$, we can replace the tangential norm $||| \cdot |||_s$ by the full norm $|| \cdot ||_s$ in the left of (2.19). (The explanation of this point can be found, for example, in [12] second part of p. 245.) This proves (2.2) for $u^{\tau +}$ and thus also for a general $u$.

□

We modify $b\Omega$ outside a neighborhood of $z_0$ where it satisfies the hypotheses of Theorem 1.1 so that it is strongly pseudoconvex in the modified portion and bounds a relatively compact domain; in particular, there is well defined the $H^0$ inverse $N$ of $\Box$ in this domain. There is an immediate crucial consequence of Theorem 2.3

Theorem 2.7. We have that

\begin{equation}
\bar{\partial}^* N \text{ is exactly regular over } \text{Ker}\bar{\partial}
\end{equation}

and

\begin{equation}
\bar{\partial} N \text{ is exactly regular over } \text{Ker}\bar{\partial}^*.
\end{equation}
Proof. As for (2.20), we put \( u = \bar{\partial}^* Nf \) for \( f \in \text{Ker} \bar{\partial} \). We get
\[
\begin{align*}
\bar{\partial} u &= f, \\
\bar{\partial}^* u &= 0, \\
\Delta u &= (\partial \bar{\partial} + \bar{\partial} \partial) \bar{\partial}^* Nf \\
&= \partial (\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}) Nf + \bar{\partial} \partial \bar{\partial}^* Nf \\
&= \vartheta \Box Nf = \vartheta f.
\end{align*}
\]
Thus, by (2.2)
\[
||\zeta u||_s^2 \lesssim ||\zeta' f||_s^2 + ||\zeta' \vartheta f||_{s-2}^2 + ||u||_0^2
\]
(2.22)
\[
\lesssim ||\zeta' f||_s^2 + ||u||_0^2.
\]
To prove (2.21), we put \( u = \partial Nf \) for \( f \in \text{Ker} \bar{\partial} \). We have a similar calculation as above which leads to the same formula as (2.22) (with the only difference that \( \vartheta \) is replaced by \( \bar{\partial} \) in the intermediate inequality). Thus from (2.22) applied both for \( \bar{\partial}^* N \) and \( \bar{\partial} N \) on \( \text{Ker} \bar{\partial} \) and \( \text{Ker} \bar{\partial}^* \) respectively, we conclude that these operators are exactly regular.

We are ready for the proof of Theorem 1.1. This follows from Theorem 2.7 by the method of Boas-Straube.

Proof of Theorem 1.1. From the regularity of \( \bar{\partial}^* N \) it follows that the Bergman projection \( B \) is also regular. (Notice that exact regularity is perhaps lost by taking \( \partial \bar{\partial} \) in \( B \).) We exploit formula (5.36) in [15] in unweighted norms, that is, for \( t = 0 \):
\[
N_q = B_q(N_q \partial)(\text{Id} - B_{q-1})(\bar{\partial}^* N_q)B_q \\
+ (\text{Id} - B_q)(\bar{\partial}^* N_{q+1})B_{q+1}(N_{q+1} \partial)(\text{Id} - B_q).
\]
Now, in the right side, the \( \partial N \)'s and \( \bar{\partial}^* N \)'s are evaluated over \( \text{Ker} \partial \) and \( \text{Ker} \bar{\partial} \) respectively; thus they are exactly regular. The \( B \)'s are also regular and therefore such is \( N \). This concludes the proof of Theorem 1.1.

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