Symmetry Type Graphs on 4-Orbit Maps

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It is well known that there exist twenty two symmetry type graphs associated to 4-orbit maps. For this ones we give the feasible values taken by the degree of the vertices and the number appropriate of edges in the boundary of each face of the map, by introducing the concepts of vertex type graph, face type graph and characteristic system.

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1 Introduction

The concept of map on a surface $S$, comes from the ancient idea of a map of the Earth. The surface $S$ is decomposed into a countries “faces” where every border “edge” belongs to exactly two countries. The points where tree or more countries are incident correspond to the vertices of the map. In each face we can choose an interior point and define a triangulation of the surface $S$, these triangles are called the flags of the map.

Every map has associated a symmetry group “automorphism” and a pregraph “symmetry type graph”, built as the quotient of its flag graph under the action of the automorphism group, this object has been investigated by Cunningham at all [CDRFHT15], Ković [Kov11], and Del Rio [Fra17]. As the automorphism group of the map acts on the set conformed by all flags of the map, if the action defines $k$ classes, then the map is said to be a $k$-orbit map. In [CM80, Chapter 8] H. S. M. Coxeter and W. O. J. Moser called to the 1-orbit maps, regular maps and a class of 2-orbit maps, irreflexible maps. Regular and irreflexible maps have been studied widely by several authors as e.g., Wilson [Wil02], D’Azevedo, Jones and Schulte [DJS11], and the first and third author jointly with Valdez [AMV17] among others. Mostly, the $k$-orbit maps on surfaces are interesting for their large number of implications and because in this subject converge topics as algebraic geometric, combinatorics and topology, reason for which it has attracted the attention of numerous researchers, see e.g., the work by Del Rio [DRF14], Helfand [Hel13], Cunningham and Pellicer [CP18]. In this paper we focus on 4-orbit
maps. Specifically, from the symmetry type graph associated to a 4-orbit map, we give the feasible values taken by the degree of the vertices and the number appropriate of edges in the boundary of each face of the map.

This article is organized as follows. In Section 2 we introduce some elements of the theory of maps as their flags, k-orbit map and automorphism and monodromy groups. In Section 3 we explore the concept of a symmetry type graph associated to a map, and we show the twenty two pregraphs, which could be symmetry type graph of any 4-orbit map. Finally, in Section 4 we introduce the concept of characteristic system of a vertex and a face. Also, we associate suitably to each vertex and each face of any 4-orbit map a pregraph. From these elements we summarize through a table, the feasible values taken by the degree of the vertices and the number appropriate of edges in the boundary of each face of 4-orbit maps.

2 Some review on maps

Along this paper, the term surface means a connected 2-dimensional topological real manifold with empty boundary, and it will be denoted as $S$. In particular, the transition functions of the corresponding atlas are only required to be continuous. It is important to remark that we do not require $S$ to be a compact topological space. A map $\mathcal{M}$ on a surface $S$ is a finite 2-cell embedding $i : G \hookrightarrow S$ of a locally finite simple graph $G$ into $S$. In other words, only a finite number of edges are incident in each vertex of $G$, the endpoints of each edge are in different vertices and the function $i$ is a topological embedding, such that each connected component of $S \setminus i(G)$ is homeomorphic to an open disk, whose boundary is the image under $i$ of a closed finite path in $G$.

Each connected component of $S \setminus i(G)$ is called a face of the map $\mathcal{M}$. A vertex of the map $i(v)$ is the image under $i$ of a vertex $v$ in $G$. Likewise, an edge of the map $i(e)$ is the image under $i$ of an edge $e$ in $G$. The degree of $i(v)$ with $v \in V(G)$ is the degree of $v$. The size of a face $f$ of the map $\mathcal{M}$ is the number of edges conforming its boundary.

On the other hand, a flag $\Phi$ of the map $\mathcal{M}$ is a triangle on the surface $S$ whose vertices are a vertex $i(v)$, the “midpoint” of an edge $i(e)$ incident to $i(v)$, and an interior point of a face $f \in S \setminus i(G)$ whose boundary contains $i(e)$. All flags contained in the closure of the face $f$ share the same vertex. Hence, each map $\mathcal{M}$ induces a triangulation of the surface $S$. From a combinatoric point of view, one can identify each flag $\Phi$ of the map $\mathcal{M}$ with an ordered incident triplet conformed by a vertex, edge and face of the map.

\[\text{For us } G \text{ will be the geometric realization of an abstract graph.}\]
\[ \Phi = (i(v), i(e), f) \]. To each flag \( \Phi \) of the map \( M \), there exists a unique adjacent flag \( \Phi^0 \) of the map \( M \) that differs from \( \Phi \) barely on the vertex, and in the same manner, there exist unique adjacent flags \( \Phi^1 \) and \( \Phi^2 \) that differ from \( \Phi \) on the edge and on the face, respectively. The flag \( \Phi^j \) will be called the \( j \)-adjacent flag of \( \Phi \), with \( j \in \{0, 1, 2\} \). We shall denote by \( F(M) \) the set conformed by all flags of the map \( M \). In Figure 1, we show an example of a map on the torus with some flags marked with their name.

![Figure 1: A map on the torus divided into flags.](image)

2.1 Automorphism and monodromy groups

An automorphism \( h \) of a map \( M \) is an automorphism of the graph \( G \), such that it can be extended to a homeomorphism \( \tilde{h} \) of the surface \( S \) to itself, this is \( i \circ h = \tilde{h} \circ i \). The automorphism set of a map \( M \), which will be denoted by \( Aut(M) \) has a group structure with the composition operation. We remark that the automorphism group of the map \( M \) is a subgroup of the group of automorphism of the graph \( G \), \( Aut(M) \leq Aut(G) \). Moreover, the automorphism group \( Aut(M) \) acts on the set of flags \( F(M) \). This action is free, each element of \( Aut(M) \) is completely determined by the image of a given flag (see [GW97, Lemma 3.1]). Hence, \( O_\Phi \) will denote the orbit of each flag \( \Phi \in F(M) \) under the action of the automorphism group \( Aut(M) \), and we denote by

\[
(1) \quad Orb(M) := \{ O_\Phi \mid \Phi \in F(M) \}
\]

the set conformed by the orbits defined by the action of \( Aut(M) \) on \( F(M) \).

A map \( M \) is called \( k \)-orbit map if the action of its automorphism group \( Aut(M) \) induces \( k \) orbits on the set of flags \( F(M) \), for some \( k \in \mathbb{N} \) (see [OPW10, Section 3]).
In the literature, a map $M$ is called **regular**, respectively **chiral**, if the the action of $\text{Aut}(M)$ on $F(M)$ induces one orbit on the set of flags, respectively if the action of $\text{Aut}(M)$ on $F(M)$ induces two orbits on the set of flags, with the property that adjacent flags belong to different orbits (see e.g., [MS02]).

On the other hand, we denote as $s_j : F(M) \to F(M)$, for every $j \in \{0, 1, 2\}$, the permutation on the set of flags $F(M)$ of the map $M$, which sends each flag $\Phi$ to its $j$-adjacent flag $\Phi^j$.

\begin{equation}
\Phi \mapsto \Phi \cdot s^j := \Phi^j.
\end{equation}

We remark that $s_j$ is an involution, it means $$(\Phi \cdot s_j) \cdot s_j = \Phi^j \cdot s_j = (\Phi^j)^j = \Phi,$$ for each flag in $M$.

Moreover, $s_j$ is not an automorphism of the map $M$ because it does not induce a homeomorphism of the surface $S$, it is merely a bijection in the set of flags (see e.g. [CPR+15, Section 2]).

The **monodromy group**\(^2\) $\text{Mon}(M)$ of the map $M$ is the subgroup of the permutations group of the set of flags $F(M)$, which is generated by the elements $s_0, s_1$ and $s_2$, i.e.,

\begin{equation}
\text{Mon}(M) := \langle s_0, s_1, s_2 \rangle.
\end{equation}

Let $\Phi$ be a flag of the map $M$ and let $j_0$ and $j_1$ be index in the set $\{0, 1, 2\}$, then by equation (2) we introduce the following notation

\begin{equation}
(\Phi \cdot s_{j_0} \cdot s_{j_1}) = (\Phi^{j_0}) \cdot s_{j_1} := \Phi^{j_0j_1}.
\end{equation}

Hence, one can naturally define the right action of $\text{Mon}(M)$ on $F(M)$ as follows

\begin{equation}
\alpha(w, \Phi) := \Phi \cdot w,
\end{equation}

for each $\Phi \in F(M)$ and each $w \in \text{Mon}(M)$. We remark that for each $w \in \text{Mon}(M)$ there are integers $j_0, j_1, \ldots, j_k \in \{0, 1, 2\}$, for any $k \in \mathbb{N}$, such that $w = s_{j_k} \circ s_{j_1} \circ \ldots \circ s_{j_0}$, then the equation (5) can be written as

$$
\Phi \cdot w = \Phi \cdot (s_{j_k} \circ s_{j_1} \circ \ldots \circ s_{j_0}) = \Phi^{j_0j_1\ldots j_k}.
$$

Moreover, this group satisfies the following properties (see [HOIW09]).

(1) The group $\text{Mon}(M)$ is transitive on $F(M)$.

(2) The elements $s_0, s_1$ and $s_2$ are fixed-point free involutions.

(3) $s_0 \circ s_2 = s_2 \circ s_0$, and $s_0 \circ s_2$ is fixed-point free involution.

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\(^2\)Stephen E. Wilson in [Wil94, pag. 540] calls this group as the *connection group*. 
3 Pregraph and Symmetry type graph

Given an edge colouring \( C \) of the graph \( G \), and a partition\(^3\) \( B \) of the set of vertices \( V(G) \) of \( G \), then the **pregraph of \( G \) with respect to \( B \)**, denoted as \( G_B \), is the graph such that its set of vertices is \( B \), and two classes \([u], [v] \in B \) define an edge with colour \( k \), if and only if, there exist \( \hat{u} \in [u] \) and \( \hat{v} \in [v] \) such that there is an edge with colour \( k \) from \( u \) to \( v \). It could be that an edge with colour \( k \) of the pregraph \( G_B \) has the same endpoints \( i.e., [u] = [v] \), in this case the edge “loop” will be called **semi-edge** with colour \( k \) and it will be thought as is shown in Figure 2. For more details, we refer the reader to [CDRFHT15, Section 3].

![Figure 2: An edge of the pregraph \( G_B \) with the same endpoints.](image)

We will denote an edge of \( G_B \) having colour \( k \), and endpoints the class \([v] \) and \([u] \) as \(([u],[v])_k \). Similarly, we will denote an semi-edge of \( G_B \) having colour \( k \), and endpoints the class \([v] \) as \(([v])_k \).

Given a map \( M \), then the **flag graph** \( G_M \) associated to \( M \) is the graph whose set of vertices is conformed by the flags of the map \( M \), and two flags \( \Phi, \Psi \in \mathcal{F}(M) \) define an edge if they are adjacent. The flag graph \( G_M \) is 3-regular \( i.e., \) each vertex of \( G_M \) has degree three, because each flag \( \Phi \) of \( M \) is only adjacent to three flags: \( \Phi^0 \), \( \Phi^1 \) and \( \Phi^2 \), respectively. Thus, we consider the three different colours \( k_1, k_2 \) and \( k_3 \), and define the edge colouring of \( G_M \)

\[
C : E(G_M) \rightarrow \{k_0, k_1, k_2\},
\]

which sends the edge with endpoints the flag \( \Phi, \Psi \) to the colour either \( k_0 \), \( k_1 \) or \( k_2 \), if them differ by a vertex, an edge or a face, respectively.

We remark that the function \( C \) sends the edges of \( G_M \) with endpoints \( \Phi \) and \( \Phi^j \) to the colour \( k_j \), with \( j \in \{0, 1, 2\} \), for each \( \Phi \in \mathcal{F}(M) \). Moreover, the colour preserving

\(^3\)In the sense of set theory.
automorphism group of $G_M$ is isomorphic to the automorphism group of $M$ (see [BVCP13, Subsection 1.4]).

Given a map $M$, let $C$ be the edge colouring of the flag graph $G_M$ defined in equation (6), and $Orb(M)$ the set of orbits defined in equation (1). Then the **symmetry type graph** $T(M)$ **associated to** $M$ is the pregraph of $G_M$ with respect to $Orb(M)$. It means, $T(M)$ is the graph such that its set of vertices is $Orb(M)$, and two orbits $O_\Phi, O_\Psi \in Orb(M)$ define an edge with colour $k_j$, with $j \in \{0, 1, 2\}$, if and only if, there exist $\hat{\Phi} \in O_\Phi$ and $\hat{\Psi} \in O_\Psi$ such that there is an edge with colour $k_j$ from $\Phi$ to $\Psi$. We note that if $M$ is $k$-orbit map, then $T(M)$ has exactly $k$ vertices.

For example, the symmetry type graph $T(M)$ of a regular map $M$ has a single vertex $O_\Phi$ and three semi-edges $\{O_\Phi\}_j$ having colours $k_j$ respectively, for each $j \in \{0, 1, 2\}$, because the automorphism $Aut(M)$ defined only one orbit $O_\Phi$, and for any flag $\Phi$ of the map $M$ and its respective $j$-adjacent flag $\Phi^j$ belong to the orbit $O_\Phi$ (see Figure 3-a). On the other hand, the symmetry type graph $T(M)$ of a chiral map $M$ has exactly two vertices $O_\Phi$ and $O_\Psi$ joined by three edges $\{O_\Phi, O_\Psi\}_j$ having colours $k_j$ respectively, for each $j \in \{0, 1, 2\}$, because the automorphism $Aut(M)$ defined two orbits $O_\Phi$ and $O_\Psi$ such that two adjacent flags belong to different orbits (see Figure 3-b).

![Figure 3: Classic examples of symmetry type graphs.](image)

There are twenty two symmetry type graphs associated to 4-orbit maps (see [OPW10]). It means, the symmetry type graph of any 4-orbit map is isomorphic to one of those pregraph shown in Figure 4.
Symmetry Type Graphs on 4-Orbit Maps

Figure 4: Symmetry type graphs associated to the 4-orbit maps, with edges and semi-edges with colours $k_j$ for $j \in \{0, 1, 2\}$.

If $\mathcal{M}$ is a map and $C$ is the edge colouring of the flag graph $G_M$, then the set of edges of $G_M$ with colour $j$ forms a perfect matching, for $j \in \{0, 1, 2\}$ (see [HdRFOP13, Section 2]). Thence, the graph $G_{ij}^j$ conformed by the edges of $G_M$ with colours $j$ and $i$, such that $j \neq i \in \{0, 1, 2\}$, is a subgraph of $G_M$ whose connected components are even cycles. The graph $G_{ij}^j$ is called a 2-factor of $G_M$.

Given that the permutation $s_0 \circ s_2$ of the monodromy group $\text{Mon}(\mathcal{M})$ is fixed-point free involution, then the cycles of the subgraph $G_{ij}^{0,2} \subset G_M$ are colourable alternately with colours $k_0$ and $k_2$, and all them have length four. Hence, if $\Phi$ is a flag of $\mathcal{M}$, then the elements in the sequence of flags $\Phi, \Phi^0, \Phi^{0,2}, \Phi^{0,2,0}$ are the vertices of a cycle of $G_{ij}^{0,2}$, where the second coordinate of the flags $\Phi, \Phi^0, \Phi^{0,2}, \Phi^{0,2,0}$ are the same. Therefore, there is a biunique correspondence between the set of edges of $\mathcal{M}$ and the cycles of $G_{ij}^{0,2}$. This correspondence is given by the orbits of $\mathcal{F}(\mathcal{M})$ under the action of the subgroup of $\text{Mon}(\mathcal{M})$ generated by $s_0$ and $s_1$, i.e.,

(7) \[ i(e) \rightarrow \{\Phi, \Phi^0, \Phi^{0,2}, \Phi^{0,2,0}\} := O_{\Phi}^{(s_0,s_2)} = \{\Phi \cdot w : w \in \langle s_0, s_2 \rangle\}, \]

being $\Phi$ an flag of $\mathcal{M}$, such that its second coordinate is $i(e)$. We will say that the orbit $O_{\Phi}^{(s_0,s_2)}$ is around the edge $i(e)$ (see Figure 5). Therefore, the cycle of $G_{ij}^{0,2}$ such that its vertices are the flags belong to $O_{\Phi}^{(s_0,s_2)}$ will be denoted as $C_{i(e)}$.

Analogously, the permutation $s_1 \circ s_2$ of the monodromy group $\text{Mon}(\mathcal{M})$ is fixed-point free, and it has finite order, then the cycles of the subgraph $G_{ij}^{1,2} \subset G_M$ are colourable
alternately with colours $k_1$ and $k_2$, and all them have length even. Hence, if $\Phi$ is a flag of $\mathcal{M}$, then the elements in the finite sequence of flags $\Phi, \Phi^1, \Phi^{1,2}, \ldots, \Phi^{1,2,\ldots,1}$ are the vertices of a cycle of $G_{\mathcal{M}}^{1,2}$, where the first coordinate of the flags $\Phi, \Phi^1, \Phi^{1,2}, \ldots, \Phi^{1,2,\ldots,1}$ are the same. Therefore, there is a biunique correspondence between the set of vertices of $\mathcal{M}$ and the cycles of $G_{\mathcal{M}}^{1,2}$. This correspondence $\varphi_{1,2}$ is given by the orbits of $\mathcal{F}(\mathcal{M})$ under the action of the subgroup of $\text{Mon}(\mathcal{M})$ generated by $s_1$ and $s_2$, i.e.,

$$(8) \quad i(\nu) \to \{\Phi, \Phi^1, \Phi^{1,2}, \ldots, \Phi^{1,2,\ldots,1}\} := O_{\Phi}^{(s_1,s_2)} = \{\Phi \cdot w : w \in \langle s_1, s_2 \rangle\},$$

being $\Phi$ an flag of $\mathcal{M}$, such that its first coordinate is $i(\nu)$. We will say that the orbit $O_{\Phi}^{(s_1,s_2)}$ is around the vertex $i(\nu)$ (see Figure 6). Therefore, the cycle of $G_{\mathcal{M}}^{1,2}$ such that its vertices are the flags belong to $O_{\Phi}^{(s_1,s_2)}$ will be denoted as $C_{i(\nu)}$.

Likewise, the permutation $s_0 \circ s_1$ of the monodromy group $\text{Mon}(\mathcal{M})$ is fixed-point free, and it has finite order, then the cycles of the subgraph $G_{\mathcal{M}}^{0,1} \subset G_{\mathcal{M}}$ are colourable alter-
nantely with colours $k_0$ and $k_1$, and all them have length even. Hence, if $\Phi$ is a flag of $\mathcal{M}$, then the elements in the finite sequence of flags $\Phi, \Phi^0, \Phi^{0,1}, \ldots, \Phi^{0,1,\ldots,0}$ are the vertices of a cycle of $G_{\mathcal{M}}^{0,1}$ where the third coordinate of the flags $\Phi, \Phi^0, \Phi^{0,1}, \ldots, \Phi^{0,1,\ldots,0}$ are the same. Therefore, there is a biunique correspondence between the set of faces of $\mathcal{M}$ and the cycles of $G_{\mathcal{M}}^{0,1}$. This correspondence is given by the orbits of $\mathcal{F}(\mathcal{M})$ under the action of the subgroup of $\text{Mon}(\mathcal{M})$ generated by $s_0$ and $s_1$, i.e.,

\begin{equation}
(9) \quad f \rightarrow \{\Phi, \Phi^0, \Phi^{0,1}, \ldots, \Phi^{0,1,\ldots,0}\} := O^{(s_0,s_1)}_{\Phi} = \{\Phi \cdot w : w \in \langle s_0, s_1 \rangle\},
\end{equation}

being $\Phi$ an flag of $\mathcal{M}$, such that its third coordinate is $f$. We will say that the orbit $O^{(s_0,s_1)}_{\Phi}$ is around the face $f$, (see Figure 7). Therefore, the cycle of $G_{\mathcal{M}}^{0,1}$ such that its vertices are the flags belong to $O^{(s_0,s_1)}_{\Phi}$ will be denoted as $C_f$.

![Figure 7: Orbit $O^{(s_0,s_1)}_{\Phi}$ around the face $f$.](image)

## 4 Characteristic system of vertices and faces

In this section we discusses about the local combinatorial nature of 4-orbit maps, from the point of view of their symmetry type graph, characteristic system of a vertex and characteristic system of a face.

### 4.1 Characteristic system of a vertex

To start let us consider a 4-orbit map $\mathcal{M}$, let $G_{\mathcal{M}}$ be the flag graph associated to $\mathcal{M}$ and let $C$ be the edge colouring defined in equation (6). If $i(v)$ is a vertex of the map $\mathcal{M}$, then there is a cycle $C_{i(v)}$ of $G_{\mathcal{M}}$ around $i(v)$ (see equation (8)), having length even and being two colourable alternately with colours $k_1$ and $k_2$. From this properties is motivated the following definition.
Definition 4.1  Let $M$ be a 4-orbit map, let $T(M)$ be the symmetry type graph of $M$ and let $i(v)$ be a vertex of $M$. The ordered triplet

$$(2m_v, k_1, k_2)$$

associated to $i(v)$ is called the characteristic system of the vertex $i(v)$, where $2m_v$ is the length of the two colourable alternately cycle $C_{i(v)}$, with colours $k_1$ and $k_2$, for some $m_v \in \mathbb{N}$.

We remark that the positive integer $m_v$ corresponds to the degree of the vertex $i(v)$.

Given that the characteristic system of the vertex $i(v)$ is determined by three parameters, if we consider a symmetry type graph $T(M)$ described in Figure 4, and we remove the edges and semi-edges having colour $k_0$, then we hold a new pregraph $T_0(M)$ isomorphic to one of those twenty two pregraphs shown in Figure 8.

![Figure 8: Pregraphs $T_0(M)$ associated to the 4-orbit maps without the edges and semi-edges with colour $k_0$.](image)

Remark 4.1  The pregraph $T_0(M)$ is conformed by at most three connected components.

We remark that each connected component of $T_0(M)$ is isomorphic to one of those eight pregraphs shown in Figure 9, which we denote as $v_x$, for some $x$ in the set of index $\{1a, 2a, 2b, 2c, 3a, 4a, 4b, 4c\}$. 
Now, if we fix a vertex \( i(v) \) of the map \( M \), we hold the cycle \( C_{i(v)} \) of \( G_M \) around \( i(v) \), and remember that \( C_{i(v)} \) is a two colourable alternately cycle, then we can introduce the set of orbits \( \text{Orb}(C_{i(v)}) \) defined by the action of \( \text{Aut}(M) \) on \( \mathcal{F}(M) \) restricted to the flags that conformed the vertices of \( C_{i(v)} \). Using the pregraph definition in Section 3 we hold that the \( v_x \) pregraph of the cycle \( C_{i(v)} \) with respect to \( \text{Orb}(C_{i(v)}) \) induces the following definition.

**Definition 4.2** Consider the pregraph \( \overline{C}_{i(v)} \) which is contained into a connected component of \( T_0(M) \) (by construction), \( \overline{C}_{i(v)} \subset v_x \), for some \( x \) in the set of index \( \{1a, 2a, 2b, 2c, 3a, 4a, 4b, 4c\} \). This connected component is called the **vertex type graph** \( T(i(v)) \) of the vertex \( i(v) \).

**Lemma 4.1** Let us consider a 4-orbit map and let \( i(v) \) be a vertex of the map. If \( v_{2a} \) is the vertex type graph of \( i(v) \), then the degree of the vertex is even. Moreover, the characteristic system of the vertex is \((4n, k_1, k_2)\) for some \( n \geq 2 \).

**Proof** Let us consider an edge \( i(e) \) incidents to \( i(v) \) and \( f \) a face of the map \( M \) such that \( i(e) \) belongs to its boundary. We denote as \( \Phi \) the flag of the map \( M \) conformed by the triplet

\[
\Phi := (i(v), i(e), f).
\]

Suppose that there is a flag \( \Psi \) on the map, such that the classes \( O_\Phi \) and \( O_\Psi \) are vertices of the pregraph \( v_{2a} \) (see Figure 10-a). We will count the number of elements in the set \( O^{(s_1, s_2)}_\Phi \) applying the vertex type graph \( v_{2a} \).

Considering the action of \( \langle s_1, s_2 \rangle \) on the flags set, by equation (8), the class

\[
O^{(s_1, s_2)}_\Phi = \{ \Phi \cdot w : w \in \langle s_1, s_2 \rangle \}
\]

contains all the flags around the vertex \( i(v) \), it means that

\[
O^{(s_1, s_2)}_\Phi = \{ \Phi, \Phi^1, \Phi^2, \Phi^{1^2}, \Phi^{2^1}, \Phi^{1^2}, \Phi^{2^1}, \Phi^{2^1}, \Phi^{2^1}, \ldots \}.
\]
Given that the set \( \{O_\Phi\}_1 \) is an edge of the pregraph \( v_{2a} \), then the flag \( \Phi^1 \) is belonged to the orbit \( O_\Phi \). Then without lost of generality we can assume that \( \Phi = \Phi_1 \) and \( \Phi^1 = \Phi_2 \). Analogously, as the sets \( \{O_\Phi,O_\Psi\}_2 \), \( \{O_\Psi\}_1 \) and \( \{O_\Phi,O_\Psi\}_2 \) are edges of the pregraph \( v_{2a} \), then the flags \( \Phi^{1,2}, \Phi^{1,2,1} \) and \( \Phi^{1,2,1,2} \) are belonged to the orbit \( O_\Psi, O_\Psi \) and \( O_\Phi \), respectively. Hence, we can assume that \( \Phi^{1,2} = \Psi_1, \Phi^{1,2,1} = \Psi_2 \) and \( \Phi^{1,2,1,2} = \Phi_3 \). Following with this construction we obtain the finite sequence \( \Phi_1, \Phi_2, \Psi_1, \Psi_2, \Phi_3, ..., \Phi_{l-1}, \Phi_l, \Psi_{l-1}, \Psi_l \) (see Figure 10-b), where \( \Phi_l \in O_\Phi \) and \( \Psi_l \in O_\Psi \) for \( l \in \{1, 2, 3, ..., n\} \).

As the number of flags in \( O_{\Phi}^{(s_1,s_2)} \) is finite, then by construction the number of incident edges on the vertex \( i(v) \) of \( M \) must be even and greater than two, it means \( m_v = 2n \) for some natural number \( n \geq 2 \). From this, it holds that the characteristic system of the vertex is \( (4n, k_1, k_2) \).

If we consider any vertex \( i(v) \) of the 4-orbit map \( M \) and we suppose that its vertex type graph \( T \left( i(v) \right) \) associated is \( v_x \), for any \( x \) in the set of index \( \{1a, 2a, 2b, 2c, 3a, 4a, 4b, 4c\} \), then following the same ideas in the proof of the lemma 4.1 it is easy to find the degree of \( i(v) \) and the characteristic system of \( i(v) \). These results are collected in Table 1. In Figure 11 are represented the eight different vertex type graphs.
Table 1: Degree and characteristic system of a vertex \( i(v) \) from its vertex type graph.

| Vertex type graph of the vertex \( i(v) \) | Degree of the vertex \( i(v) \) | Characteristic System of the vertex \( i(v) \) |
|------------------------------------------|----------------|---------------------------------|
| \( v_{1a} \)                           | \( n \)       | \( n \geq 3 \) \( (2n, k_1, k_2) \) |
| \( v_{2a} \)                           | \( 2n \)       | \( n \geq 2 \) \( (4n, k_1, k_2) \) |
| \( v_{2b} \)                           | \( 2n \)       | \( n \geq 2 \) \( (4n, k_1, k_2) \) |
| \( v_{2c} \)                           | \( n \)       | \( n \geq 3 \) \( (2n, k_1, k_2) \) |
| \( v_{3a} \)                           | \( 3n \)       | \( n \geq 1 \) \( (6n, k_1, k_2) \) |
| \( v_{4a} \)                           | \( 2n \)       | \( n \geq 2 \) \( (4n, k_1, k_2) \) |
| \( v_{4b} \)                           | \( 4n \)       | \( n \geq 1 \) \( (8n, k_1, k_2) \) |
| \( v_{4c} \)                           | \( 4n \)       | \( n \geq 1 \) \( (8n, k_1, k_2) \) |

Figure 11: Sequence follow by the flags around a vertex \( i(v) \) from its vertex type graph.

**Remark 4.2** Let \( i(v_1) \), \( i(v_2) \) be vertices of 4-orbit map \( M \) and let \( C_{i(v_1)}, C_{i(v_2)} \) be the cycles of the flag grap \( G_M \) associated to the vertices \( i(v_1) \) and \( i(v_2) \), respectively. Suppose that the pregraphs \( \bar{C}_{i(v_1)} \) and \( \bar{C}_{i(v_2)} \) are contained to some connected component of \( T_0(M) \). Given that the automorphism group \( \text{Aut}(M) \) acts on the flags set \( F(M) \), then the vertices \( i(v_1) \) and \( i(v_2) \) have the same degree, and their characteristic systems
are the same

\[(2m_{v_1}, k_1, k_2) = (2m_{v_2}, k_1, k_2).\]

However, if the pregraphs \( \overline{C}_{i(v_1)} \) and \( \overline{C}_{i(v_2)} \) belong to different connected component of \( T_0(M) \) but they are isomorphic, then the degree of the vertices \( i(v_1) \) and \( i(v_2) \) are multiples of \( s \), for any \( s \in \mathbb{N} \).

If \( l \) is number of connected component of \( T_0(M) \), for any \( l \in \{1, 2, 3\} \), then there are \( n_l \) positive integers, with \( i \in \{1, \ldots, l\} \) such that the characteristic system of any vertex of the map is either \((2n_1, k_1, k_2), \ldots, (2n_l, k_1, k_2)\).

### 4.2 Characteristic system of a face

Let \( f \) be a face of the 4-orbit map, let \( G_M \) be the flag graph associated to \( M \) and let \( C \) be the edge colouring defined in equation (6). If \( f \) is a face of the map \( M \), then there is a cycle \( C_f \) of \( G_M \) around \( f \) (see equation (9)), having length even and being two colourable alternately with colours \( k_0 \) and \( k_1 \). From this properties is motivated the following definition.

**Definition 4.3** Let \( M \) be a 4-orbit map, let \( T(M) \) be the symmetry type graph of \( M \) and let \( f \) be a face of \( M \). The ordered triplet

\[(2m_f, k_0, k_1)\]

associated to \( f \) is called the **characteristic system of the face** \( f \), where \( 2m_f \) is the length of the two colourable alternately cycle \( C_f \), with colours \( k_0 \) and \( k_1 \), for some \( m_f \in \mathbb{N} \).

We remark that the positive integer \( m_f \) corresponds to the number of edges of the map \( M \) in the boundary of the face \( f \). This number will be called the **size of the face** \( f \).

Given that the characteristic system of the face \( f \) is determined by three parameters, if we consider a symmetry type graph \( T(M) \) described in Figure 4, and we remove the edges and semi-edges having colour \( k_2 \), then we hold a new pregraph \( T_2(M) \) isomorphic to one of those twenty two pregraphs shown in Figure 12.

**Remark 4.3** The pregraph \( T_2(M) \) is conformed by at most three connected components.
Figure 12: Pregraphs $T_2(M)$ associated to the 4-orbit maps without the edges and semi-edges with colour $k_2$.

Figure 13: Pregraphs $f_x$.

We remark that each connected component of $T_2(M)$ is isomorphic to one of these eight pregraphs shown in Figure 15, which we denote as $f_x$, for some $x$ in the set of index \{1a, 2a, 2b, 2c, 3a, 4a, 4b, 4c\}.

Now, if we fix a face $f$ of the map $M$, we hold the cycle $C_f$ of $G_M$ such that its vertices are all flags having a vertex in the interior of $f$, and remember that $C_f$ is a two colourable alternately cycle, then we can introduce the set of orbits $\text{Orb}(C_f)$ defined by the action of $\text{Aut}(M)$ on $\mathcal{F}(M)$ restricted to the flags that conformed the vertices of $C_f$. Using the pregraph definition in Section 3 we hold that the $\overline{C_f}$ pregraph of the cycle $C_f$ with respect to $\text{Orb}(C_f)$ induces the following definition.

**Definition 4.4** Consider the pregraph $\overline{C_f}$ which is contained into a connected component of $T_2(M)$ (by construction), $\overline{C_f} \subset f_x$, for some $x$ in the set of index \{1a, 2a, 2b, 2c, 3a, 4a, 4b, 4c\}. This connected component is called the **face type graph** $T(f)$ of
the face $f$.

**Lemma 4.2**  Let us considerer a 4-orbit map and let $f$ be a face of the map. If $f_{3a}$ is the face type graph of $f$ then the boundary of $f$ is conformed by $3n$ edges, for any $n \geq 1$. In other words, $f$ has size $3n$. Moreover, the characteristic system of the face $f$ is $(6n, k_0, k_1)$.

**Proof**  We consider the vertex $i(v)$ and the edge $i(e)$ of the 4-orbit map $M$ such that $i(e)$ incidents to $i(v)$ and $i(e)$ belongs to the boundary of the face $f$. Then we denote as $\Phi$ the flag of the map $M$ conformed by the triplet

$$\Phi := (i(v), i(e), f).$$

Suppose that there are flags $\Upsilon, \Omega$ on the map such that the classes $O_\Phi, O_\Upsilon$ and $O_{\Omega}$ are the vertices of the pregraph $f_{3a}$ (see Figure 14-a). We will count the number of elements in the set $O_\Phi^{(s_0, s_1)}$ applying the face type graph $f_{3a}$.

Considering the action of $\langle s_1, s_2 \rangle$ on the flags set, by equation (9), the class

$$O_\Phi^{(s_0, s_1)} = \{ \Phi \cdot w : w \in \langle s_0, s_1 \rangle \}$$

contains all the flags having a vertex in the interior of the face $f$, it means

$$O_\Phi^{(s_0, s_1)} = \{ \Phi, \Phi^0, \Phi^1, \Phi^{0,1}, \Phi^{1,0}, \Phi^{0,1,0}, \Phi^{1,0,1}, \Phi^{0,1,0,1}, \ldots \}$$

Given that the set $\{O_\Phi\}_0$ is an edge of the pregraph $f_{3a}$, then the flag $\Phi^0$ is belonged to the orbit $O_\Phi$. Then without lost of generality we can assume that $\Phi = \Phi_1 \ y \ \Phi^0 = \Phi_2$. Analogously, as the sets $\{O_\Phi, O_\Upsilon\}_1, \{O_\Upsilon, O_{\Omega}\}_0, \{O_{\Omega}, O_{\Upsilon}\}_0$ and $\{O_{\Upsilon}, O_\Phi\}_1$ are edges of the pregraph $f_{3a}$, then the flags $\Phi^{0,1}, \Phi^{0,1,0}, \Phi^{0,1,0,1}, \Phi^{0,1,0,1,0}$ and $\Phi^{0,1,0,1,0,1}$ are belonged to in the orbit $O_\Upsilon, O_{\Omega}, O_{\Upsilon}$ $y$ $O_{\Phi}$, respectively. Then we can assume that $\Phi^{0,1} = \Upsilon_1, \Phi^{0,1,0} = \Omega_1, \Phi^{0,1,0,1} = \Omega_2, \Phi^{0,1,0,1,0} = \Upsilon_2 \ y \ \Phi^{0,1,0,1,0,1} = \Phi_3$. Following with this construction we obtain the finite sequence $\Phi_1, \Phi_2, \Upsilon_1, \Omega_1, \Omega_2, \Upsilon_2, \Phi_3, \ldots, \Phi_{l-1}, \Phi_l, \Upsilon_{l-1}, \Omega_{l-1}, \Omega_l, \Upsilon_l$ (see Figure 14-b), where $\Phi_l \in O_\Phi, \Upsilon_l \in O_\Upsilon$ and $\Omega_l \in O_{\Omega}$ for $l \in \{1, 2, 3, \ldots, n\}$. From this it holds

$$O_\Phi^{(s_0, s_1)} = \{ \Phi, \Phi^0, \Phi^1, \Phi^{0,1}, \Phi^{0,1,0}, \Phi^{0,1,0,1}, \Phi^{0,1,0,1,0}, \Phi^{0,1,0,1,0,1}, \ldots \} = \{ \Phi_1, \Phi_2, \Upsilon_1, \Omega_1, \Omega_2, \Upsilon_2, \Phi_3, \ldots, \Phi_{l-1}, \Phi_l, \Upsilon_{l-1}, \Omega_{l-1}, \Omega_l, \Upsilon_l \}.$$
As the number of flags in $O_{\Phi}^{(s_0,s_1)}$ is finite, then the number of edges in the boundary of each face $f$ of $M$ must be even and greater than two, and the number of flags in the class $O_{\Phi}^{(s_0,s_1)}$ is $6n$ for some $n \geq 1$. This implies that the number of edges conforming the boundary of $f$ is $3n$, for $n \geq 1$. From this, it holds that the characteristic system of the face $f$ is $(6n, k_0, k_1)$.

If we consider any face $f$ of the 4-orbit map $M$ and we suppose that its face type graph $T(f)$ associated to $f$, for any $x$ in the set of index $\{1a, 2a, 2b, 2c, 3a, 4a, 4b, 4c\}$, then following the same ideas in the proof of lemma 4.2, it is easy to find the number of edges conforming the boundary of $f$. These results are collected in Table 2. In Figure 15 are represented the face type graphs.

| Face type graph of the face $f$ | Size of $f$ | Characteristic System of the face $f$ |
|---------------------------------|------------|--------------------------------------|
| $f_{1a}$                        | $n$        | $n \geq 3$ (2$n$, $k_0$, $k_1$)      |
| $f_{2a}$                        | $2n$       | $n \geq 2$ (4$n$, $k_0$, $k_1$)      |
| $f_{2b}$                        | $2n$       | $n \geq 2$ (4$n$, $k_0$, $k_1$)      |
| $f_{2c}$                        | $n$        | $n \geq 3$ (2$n$, $k_0$, $k_1$)      |
| $f_{3a}$                        | $3n$       | $n \geq 1$ (6$n$, $k_0$, $k_1$)      |
| $f_{4a}$                        | $2n$       | $n \geq 2$ (4$n$, $k_0$, $k_1$)      |
| $f_{4b}$                        | $4n$       | $n \geq 1$ (8$n$, $k_0$, $k_1$)      |
| $f_{4c}$                        | $4n$       | $n \geq 1$ (8$n$, $k_0$, $k_1$)      |

Table 2: Size and characteristic system of a face $f$ from its face type graph.
Remark 4.4 Let $f_1, f_2$ be faces of 4-orbit map $M$ and let $C_{f_1}, C_{f_2}$ be the cycles of the flag graph $G_M$ associated to the faces $f_1$ and $f_2$, respectively. Suppose that the pregraphs $C_{f_1}$ and $C_{f_2}$ are contained to some connected component of $T_2(M)$. Given that the automorphism group $\text{Aut}(M)$ acts on the flags set $F(M)$, then the faces $f_1$ and $f_2$ have the same number of edges in its boundary, and their characteristic systems are same

$$(2m_{f_1}, k_0, k_1) = (2m_{f_2}, k_0, k_1).$$

However, if the pregraphs $C_{f_1}$ and $C_{f_2}$ belong to different connected component of $T_2(M)$ but they are isomorphic, then the size of the faces $f_1$ and $f_2$ are multiples of $s$, for any $s \in \mathbb{N}$.

If $l$ is number of connected component of $T_2(M)$, for any $l \in \{1, 2, 3\}$, then there are $l$ values to the size of the faces of $M$. Hence, there are $n_l$ positive integers, with $i \in \{1, \ldots, l\}$ such that the characteristic system of any face of the map is either $(2n_{i1}, k_0, k_1), \ldots, (2n_{il}, k_0, k_1)$. 

Figure 15: Sequence follow by the flags in the boundary of a face $f$, from its face type graph of type.
4.3 Main consequence

From previous discussion we shall study the 4-orbit maps having symmetry type graph $4_A$, and we shall summarize through a table the feasible values taken by the degree of the vertices and the number appropriate of edges in the boundary of each face of the 4-orbit map.

Proposition 4.1 If the symmetry type graph of a 4-orbit map $M$ is $4_A$, then

1. The pregraph $T_0(M)$ has only one component connected isomorphic to $v_{4b}$. If $i(v)$ is a vertex of $M$, then there is a positive integer $n$ such that the degree of $i(v)$ is $4n$, the characteristic system of $i(v)$ is $(8n, k_1, k_2)$, and its vertex type graph is $v_{4b}$.

2. The pregraph $T_2(M)$ is conformed by two component connected isomorphic to $f_{2b}$. If $f$ is a face of $M$, then there are positive integers $m, n$ such that the number of edges in the boundary of $f$ is either $2n$ or $2m$, the characteristic system of $f$ is either $(4n, k_0, k_1)$ or $(4m, k_0, k_1)$, and its face type pregraph is $f_{2b}$.

Proof If we remove the edges and semi-edges of $4_A$ having colour $k_0$, then the new pregraph $T_0(M)$ is conformed by a connected component isomorphic to $v_{4b}$ (see Figures 8 and 9). This implies that for each vertex $i(v)$ of the map $M$ it has vertex type graph $v_{4b}$ and characteristic system $(8n, k_1, k)$ (see Table 1). This properties are summarized in Table 3.

Analogously, if we remove the edges and semi-edges of $4_A$ having colour $k_2$, then the new pregraph $T_2(M)$ is conformed by two connected components isomorphic to $f_{2a}$ (see Figures 12 and 13). This implies that for each face $f$ of the map $M$ it has face type graph $f_{2a}$ and there are positive integers $m_1, m_2$ such that its characteristic system is either $(4m_1, k_0, k, 1)$ or $(4m_2, k_0, k, 1)$ (see Table 2). This properties are summarized in Table 3. □
| Pregraph  | Number of connected component of $T_0(M)$ | Vertex type graph of the vertex $i(v)$ | Degree of the vertex $i(v)$ | Characteristic System of the vertex $i(v)$ |
|-----------|------------------------------------------|----------------------------------------|-----------------------------|----------------------------------------------|
| $4_{(A_0)}$ | 1                                        | $v_{4b}$                               | $4n$                        | $(8n,k_1,k_2)$                              |
| Pregraph  | Number of connected component of $T_2(M)$ | Face type graph of the face $f$         | Size of the face $f$        | Characteristic System of the face $f$        |
| $4_{(A_2)}$ | 2                                        | $f_{2b}$                               | $2m_1$                      | $(4m_1,k_0,k_1)$                            |
|           |                                           | $f_{2b}$                               | $2m_2$                      | $(4m_2,k_0,k_1)$                            |

Table 3: Properties for 4-orbit maps with symmetry type graph $4_A$.

Following the same ideas that in the proof of the Proposition 4.1 for any other of the twenty one possible symmetry type graphs associated to the 4-orbit maps, the characterization in terms of Number of connected components of $T_0(M)$, vertex type graph, degree of a vertex and characteristic system of a vertex are given in the Table 4. Respectively, the characterization in terms of Number of connected components of $T_2(M)$, face type graph, size of a face and characteristic system of a face are given in the Table 5.
| Pregraph $\mathcal{T}_0(M)$ | Number of connected component of $\mathcal{T}_0(M)$ | Vertex type graph of the vertex $i(v)$ | Degree of the vertex $i(v)$ | Characteristic System of the vertex $i(v)$ |
|-----------------------------|---------------------------------|---------------------------------|------------------|---------------------------------|
| $4_{(A_3)}$                | 1                              | $v_{4b}$                        | $4n$             | $(8n, k_1, k_2)$               |
| $4_{(A_4d_3)}$             | 2                              | $v_{2b}$ $v_{2b}$              | $2n_1$ $n_1 \geq 2$ | $(4n_1, k_1, k_2)$              |
| $4_{(A_5d_3b)}$            | 1                              | $v_{4b}$                        | $4n$             | $(8n, k_1, k_2)$               |
| $4_{(B_3)}$                | 1                              | $v_{4c}$                        | $4n$             | $(8n, k_1, k_2)$               |
| $4_{(B_6d_3)}$             | 3                              | $v_{2b}$ $v_{1a}$ $v_{1a}$     | $2n_1$ $n_1 \geq 2$ | $(4n_1, k_1, k_2)$              |
| $4_{(B_6d_3b)}$            | 1                              | $v_{4c}$                        | $4n$             | $(8n, k_1, k_2)$               |
| $4_{(C_3)}$                | 1                              | $v_{4a}$                        | $2n$             | $(4n, k_1, k_2)$               |
| $4_{(C_6d_3)}$             | 2                              | $v_{2b}$ $v_{2b}$              | $2n_1$ $n_1 \geq 2$ | $(4n_1, k_1, k_2)$              |
| $4_{(C_6d_3b)}$            | 1                              | $v_{4a}$                        | $2n$             | $(4n, k_1, k_2)$               |
| $4_{(D_3)}$                | 2                              | $v_{3a}$ $v_{1a}$              | $3n_1$ $n_1 \geq 1$ | $(6n_1, k_1, k_2)$              |
| $4_{(D_6d_3)}$             | 1                              | $v_{4a}$                        | $2n$             | $(4n, k_1, k_2)$               |
| $4_{(D_6d_3b)}$            | 1                              | $v_{4a}$                        | $4n$             | $(8n, k_1, k_2)$               |
| $4_{(D_6d_3)}$             | 1                              | $v_{4a}$                        | $2n$             | $(4n, k_1, k_2)$               |
| $4_{(D_6d_3b)}$            | 1                              | $v_{4a}$                        | $4n$             | $(8n, k_1, k_2)$               |
| $4_{(E_3)}$                | 2                              | $v_{2a}$ $v_{2a}$              | $2n_1$ $n_1 \geq 2$ | $(4n_1, k_1, k_2)$              |
| $4_{(E_6d_3)}$             | 1                              | $v_{4a}$                        | $2n$             | $(4n, k_1, k_2)$               |
| $4_{(E_6d_3b)}$            | 1                              | $v_{4a}$                        | $4n$             | $(8n, k_1, k_2)$               |
| $4_{(F_3)}$                | 1                              | $v_{4a}$                        | $2n$             | $(4n, k_1, k_2)$               |
| $4_{(G_3)}$                | 2                              | $v_{2c}$ $v_{2c}$              | $n_1$ $n_1 \geq 3$ | $(2n_1, k_1, k_2)$              |
| $4_{(G_6d_3)}$             | 1                              | $v_{4a}$                        | $2n$             | $(4n, k_1, k_2)$               |
| $4_{(G_6d_3b)}$            | 1                              | $v_{4a}$                        | $4n$             | $(8n, k_1, k_2)$               |
| $4_{(H_3)}$                | 2                              | $v_{2c}$ $v_{2a}$              | $n_1$ $n_1 \geq 3$ | $(2n_1, k_1, k_2)$              |
| $4_{(H_6d_3)}$             | 1                              | $v_{4a}$                        | $2n$             | $(4n, k_1, k_2)$               |
| $4_{(H_6d_3b)}$            | 1                              | $v_{4a}$                        | $4n$             | $(8n, k_1, k_2)$               |
| $4_{(H_6d_3)}$             | 2                              | $v_{2c}$ $v_{2a}$              | $n_1$ $n_1 \geq 3$ | $(2n_1, k_1, k_2)$              |
| $4_{(H_6d_3b)}$            | 1                              | $v_{4a}$                        | $2n$             | $(4n, k_1, k_2)$               |

Table 4: Properties for 4-orbit maps from its pregraph $\mathcal{T}_0(M)$. 

Symmetry Type Graphs on 4-Orbit Maps
| Pregraph $\mathcal{T}_2(M)$ | Number of connected component of $\mathcal{T}_2(M)$ | Face type graph of the face $f$ | Size of the face $f$ | Characteristic System of the face $f$ |
|-----------------------------|---------------------------------|-------------------------------|---------------------|-------------------------------------|
| $4_{(A)}$                  | 2                               | $f_{2b}$                      | $2m_1$ $m_2$      | $(4m_1, k_0, k_1)$ $(4m_2, k_0, k_1)$ |
| $4_{(Cd)}$                 | 1                               | $f_{4b}$                      | $4m$               | $(8m, k_0, k_1)$                   |
| $4_{(Cd_2)}$               | 1                               | $f_{4b}$                      | $4m$               | $(8m, k_0, k_1)$                   |
| $4_{(B)}$                  | 1                               | $f_{4b}$                      | $4m$               | $(8m, k_0, k_1)$                   |
| $4_{(C)}$                  | 1                               | $f_{4b}$                      | $4m$               | $(8m, k_0, k_1)$                   |
| $4_{(D)}$                  | 1                               | $f_{4b}$                      | $4m$               | $(8m, k_0, k_1)$                   |
| $4_{(E)}$                  | 1                               | $f_{4b}$                      | $4m$               | $(8m, k_0, k_1)$                   |
| $4_{(F)}$                  | 1                               | $f_{4b}$                      | $4m$               | $(8m, k_0, k_1)$                   |
| $4_{(G)}$                  | 1                               | $f_{4b}$                      | $4m$               | $(8m, k_0, k_1)$                   |
| $4_{(H)}$                  | 1                               | $f_{4b}$                      | $4m$               | $(8m, k_0, k_1)$                   |

Table 5: Properties for 4-orbit maps from its pregraph $\mathcal{T}_2(M)$. 

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References

[AMV17] John A. Arredondo, Camilo Ramírez Maluendas, and Ferrán Valdez, On the topology of infinite regular and chiral maps, Discrete Math. 340 (2017), no. 6, 1180–1186.

[DJS11] Antonio Breda D’Azevedo, Gareth A. Jones, and Egon Schulte, Constructions of chiral polytopes of small rank, Canad. J. Math. 63 (2011), no. 6, 1254–1283.

[BVCP13] Gunnar Brinkmann, Nico Van Cleemput, and Tomáš Pisanski, Generation of various classes of trivalent graphs, Theoret. Comput. Sci. 502 (2013), 16–29.

[CPR + 15] Thierry Coulbois, Daniel Pellicer, Miguel Raggi, Camilo Ramírez, and Ferrán Valdez, The topology of the minimal regular covers of the Archimedean tessellations, Adv. Geom. 15 (2015), no. 1, 77–91.

[CM80] H. S. M. Coxeter and W. O. J. Moser, Generators and relations for discrete groups, 4th ed., Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 14, Springer-Verlag, Berlin-New York, 1980.

[CP18] Gabe Cunningham and Daniel Pellicer, Open problems on k-orbit polytopes, Discrete Math. 341 (2018), no. 6, 1645–1661.

[CDRFHT15] Gabe Cunningham, María Del Río-Francos, Isabel Hubard, and Micael Toledo, Symmetry type graphs of polytopes and maniplexes, Ann. Comb. 19 (2015), no. 2, 243–268.

[Fra17] María Del Río Francos, Truncation symmetry type graphs, Ars Combin. 134 (2017), 135–167.

[DRF14] María Del Río Francos, Chamfering operation on k-orbit maps, Ars Math. Contemp. 7 (2014), no. 2, 507–524.

[GW97] Jack E. Graver and Mark E. Watkins, Locally finite, planar, edge-transitive graphs, Mem. Amer. Math. Soc. 126 (1997), no. 601, vi+75.

[He13] Ilanit Helfand, Constructions of k-orbit Abstract Polytopes, ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–Northeastern University.

[HdRFOP13] Isabel Hubard, María del Río Francos, Alen Orbanic, and Tomáš Pisanski, Medial symmetry type graphs, Electron. J. Combin. 20 (2013), no. 3, Paper 29, 28.

[HOIW09] Isabel Hubard, Alen Orbanic, and Asia Ivić Weiss, Monodromy groups and self-invariance, Canad. J. Math. 61 (2009), no. 6, 1300–1324.
[Kov11] Jurij Kovič, *Symmetry-type graphs of Platonic and Archimedean solids*, Math. Commun. **16** (2011), no. 2, 491–507.

[MS02] Peter McMullen and Egon Schulte, *Abstract regular polytopes*, Encyclopedia of Mathematics and its Applications, vol. 92, Cambridge University Press, Cambridge, 2002.

[OPW10] Alen Orbanić, Daniel Pellicer, and Asia Ivić Weiss, *Map operations and k-orbit maps*, J. Combin. Theory Ser. A **117** (2010), no. 4, 411–429.

[Wil94] Stephen E. Wilson, *Parallel products in groups and maps*, J. Algebra **167** (1994), no. 3, 539–546.

[Wil02] Steve Wilson, *Families of regular graphs in regular maps*, J. Combin. Theory Ser. B **85** (2002), no. 2, 269–289.

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