Weakening Total Coloring Conjecture: Weak TCC and Hadwiger’s Conjecture on Total Graphs

Manu Basavaraju\textsuperscript{1}, L. Sunil Chandran\textsuperscript{2}, Mathew C. Francis\textsuperscript{3}, Ankur Naskar\textsuperscript{2}

\textsuperscript{1}Department of Computer Science and Engineering, National Institute of Technology Karnataka, Surathkal - 575025, India
manub@nitk.ac.in
\textsuperscript{2}Department of Computer Science and Automation, Indian Institute of Science, Bangalore - 560012, India
\{sunil,ankurnaskar\}@iisc.ac.in
\textsuperscript{3}Computer Science Unit, Indian Statistical Institute, Chennai Centre, Chennai - 600029, India
mathew@isichennai.res.in

Abstract

The total graph of a graph $G$, denoted by $T(G)$, is defined on the vertex set $V(G) \cup E(G)$ with $c_1, c_2 \in V(G) \cup E(G)$ adjacent whenever $c_1$ and $c_2$ are adjacent to (or incident on) each other in $G$. The total chromatic number $\chi''(G)$ of a graph $G$ is defined to be the chromatic number of its total graph. The well-known Total Coloring Conjecture or TCC states that for every simple finite graph $G$ having maximum degree $\Delta(G)$, $\chi''(G) \leq \Delta(G) + 2$. In this paper, we consider two ways to weaken TCC:

1. **Weak TCC**: This conjecture states that for a simple finite graph $G$, $\chi''(G) = \chi(T(G)) \leq \Delta(G) + 3$. While weak TCC is known to be true for 4-colorable graphs, it has remained open for 5-colorable graphs. In this paper, we settle this long pending case.

2. **Hadwiger’s Conjecture for total graphs**: We can restate TCC as a conjecture that proposes the existence of a strong $\chi''$-bounding function for the class of total graphs in the following way: If $H$ is the total graph of a simple finite graph, then $\chi''(H) \leq \omega(H) + 1$, where $\omega(H)$ is the clique number of $H$. A natural way to relax this question is to replace $\omega(H)$ by the Hadwiger number $\eta(H)$, the number of vertices in the largest clique minor of $H$. This leads to the Hadwiger’s Conjecture (HC) for total graphs: if $H$ is a total graph then $\chi(H) \leq \eta(H)$. We prove that this is true if $H$ is the total graph of a graph with sufficiently large connectivity.

A second motivation for studying Hadwiger’s conjecture for total graphs is the following: Consider the class of split graphs whose vertex set is partitioned into an independent set $A$ and a clique $B$, with the following additional constraints: (1) Each vertex in $B$ has exactly 2 neighbours in $A$; (2) No two vertices in $B$ have the same neighbourhood in $A$. It is known that if Hadwiger’s conjecture is proved for the squares of this special class of split graphs, then it holds also for the general case. Of course, proving the conjecture for this special case is indeed difficult, and therefore it is natural to consider the difficulty level of Hadwiger’s conjecture for the squares of graph classes defined by slightly modifying the above class of graphs. A natural structural modification is to assume that both $A$ and $B$ are independent sets, keeping everything else same. It turns out that the squares of this modified class of graphs is exactly the class of total graphs. From this perspective, it is not really surprising that HC on Total Graphs is also challenging. On the other hand, we show that weak TCC implies HC on total graphs. This perhaps suggests that the latter is an easier problem than the former.

Keywords: graph minors, Hadwiger’s conjecture, total coloring, total chromatic number, total coloring conjecture, total graphs, squares of graphs
1 Introduction

1.1 Total Coloring Conjecture

Let $G$ be a simple finite graph. For a vertex $v \in V(G)$ in $G$, we define $N_G(v) := \{u \in V(G) : u$ is adjacent to $v \text{ in } G\}$ and $E_G(v) := \{e \in E(G) : e$ is incident on $v \text{ in } G\}$. The degree of a vertex $v \in V(G)$ is defined as $d(v) = |N_G(v)| = |E_G(v)|$. The maximum degree of $G$, $\Delta(G) = \max_{v \in V(G)} d(v)$.

A vertex coloring is called a proper vertex-coloring if no two adjacent vertices are assigned the same color. The minimum number of colors required to achieve a proper vertex-coloring of $G$ is called the chromatic number of $G$, and is denoted by $\chi(G)$. Like the vertex-coloring problem, the problem of coloring the edges of a graph is also one that has received much attention in the literature. A coloring of the edges of a graph is called a proper edge-coloring if no two adjacent edges are assigned the same color. The minimum number of colors required in any proper edge-coloring of a graph $G$ is called its edge-chromatic number or chromatic index, and is denoted by $\chi'(G)$. The following theorem was proved by Vizing:

**Theorem 1.1** (Vizing’s Theorem). For a simple finite graph $G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Vizing’s theorem suggests a classification of simple finite graphs into two classes. A graph $G$ is said to be in class I if $\chi'(G) = \Delta(G)$ and in class II if $\chi'(G) = \Delta(G) + 1$.

The notion of line graphs allows us to view the edge coloring problem as a vertex coloring problem. Given a graph $G$, the line graph of $G$ is the graph $L(G) = (V', E')$ where $V' = E(G)$ and $e_1, e_2 \in E'$ form an edge $e_1e_2 \in E'$ whenever $e_1$ and $e_2$ are adjacent edges in $G$. Clearly $\chi'(G) = \chi(L(G))$. The clique number of $G$, denoted by $\omega(G)$, is the number of vertices in a maximum clique of $G$. Clearly $\omega(G) \leq \chi(G)$ for all graphs. It is easy to see that $\omega(L(G)) \geq \Delta(G)$, and therefore Vizing’s theorem guarantees that the chromatic number of any line graph is at most 1 more than its clique number. Given a class of graphs $\mathcal{F}$, if there exists a function $f$ such that $\forall G \in \mathcal{F}, \chi(G) \leq f(\omega(G))$, then $\mathcal{F}$ is a $\chi$-bounded class and $f$ is a $\chi$-bounding function for $\mathcal{F}$. $\chi$-boundedness of graph classes is an extensively studied topic, see the survey by Scott and Seymour [26] for further references. Thus Vizing’s Theorem guarantees a very strong $\chi$-bounding function for the class of line graphs.

After vertex and edge colorings it was natural for graph theorists to consider coloring vertices and edges simultaneously. A total coloring of a graph $G$ is a coloring of all its elements, that is, vertices and edges, such that no two adjacent elements are assigned the same color. (When we refer to two adjacent elements of a graph, they are either adjacent to each other or incident on each other as the case may be.) That is, it is an assignment $c$ of colors to $V(G) \cup E(G)$ such that $c_{|V(G)}$ is a proper vertex-coloring, $c_{|E(G)}$ is a proper edge-coloring, and $c(uv) \notin \{c(u), c(v)\}$ for any edge $uv \in E(G)$. The minimum possible number of colors in any total coloring of $G$ is called the total chromatic number of $G$, and is denoted by $\chi''(G)$.

The total coloring conjecture (or TCC) was proposed by Behzad [3] and Vizing [27] independently between 1964 and 1968.

**Conjecture 1.2** (Total coloring conjecture). For any simple, finite graph $G$,

$$\chi''(G) \leq \Delta(G) + 2$$

**Remark.** If we use multigraphs instead of simple graphs, the above statement may not hold. This is true of Theorem 1.1 also. In this paper we consider only simple finite graphs.

Even though many researchers have examined TCC over the years, it remains unsolved till date and is considered one of the hardest open problems in graph coloring. Bollobás and Harris [6] proved that $\chi''(G) \leq \frac{5}{4}\Delta(G)$, when $\Delta(G)$ is sufficiently large. Later Kostochka [17] proved that $\chi''(G) \leq \frac{7}{6}\Delta(G)$, when $\Delta(G) \geq 6$. The best known result to date for the general case was obtained by Molloy and Reed [19]:

**Theorem 1.3.** $\chi''(G) \leq \Delta(G) + C$, where $C$ is equal to $10^{26}$.

**Remark.** Molloy and Reed have mentioned in [19] that though they only show a value $10^{26}$ for the constant $C$, with much more effort it can be brought down to 500.
While TCC remains unsolved for the general case, it is known to hold for some special classes of graphs. For example, it is easy to see that TCC is true for all complete graphs and bipartite graphs [4]. Another case is the class of graphs with maximum degree at most 5.

**Theorem 1.4** ([22], [28], [16], [15]). TCC holds for all graphs $G$ with maximum degree at most 5.

For planar graphs, the total coloring conjecture is known to hold except when the maximum degree is 6.

**Theorem 1.5** ([7], [32], [24]). Let $G$ be a planar graph. If $\Delta(G) \neq 6$, then TCC holds for $G$, otherwise $\chi''(G) \leq \Delta(G) + 3$.

Considering the difficulty level of TCC, it makes sense to study relaxations of TCC. As the title of the paper indicates, we intend to consider two ways to weaken TCC. First one is the most obvious way: Increase the upper bound appearing in the statement of TCC. This leads to weak-TCC and $(k)$-TCC; see Section 2 for more details. Now we will develop the background for the second one.

Just like we defined line graphs and used it to perceive the edge coloring problem on $G$ as a vertex coloring problem on $L(G)$, we can define a similar structure in the context of total coloring also. Given a graph $G = (V, E)$ the total graph of $G$ is the graph $T(G) = (V'', E'')$ where $V'' = V \cup E$, the set of all elements of $G$, and $e_{1}e_{2} \in E''$ whenever elements $e_{1}, e_{2}$ are adjacent in $G$. Given a class $F$ of graphs, we define $T(F) := \{T(G) : G \in F\}$. We use $T$ to denote the class of total graphs i.e. $T = T(G)$, where $G$ is the class of all simple finite graphs.

**Note.** We shall call vertices in $T(G)$ that correspond to vertices in the original graph $G$ as $v$-vertices and vertices that correspond to edges in $G$ as $e$-vertices. For ease of notation, we denote an element of $G$, that is, an edge or a vertex in $G$, and its corresponding vertex in $T(G)$, by the same letter. Thus, the $v$-vertex in $T(G)$ corresponding to $x \in V(G)$ is also denoted by $x$.

**Observation 1.** Any total coloring of a graph $G$ corresponds to a proper vertex coloring of $T(G)$ and vice-versa. Thus, the total chromatic number of $G$ is equal to the chromatic number of $T(G)$ that is, $\chi''(G) = \chi(T(G))$.

**Observation 2.** For every vertex $x \in V(G)$, the $v$-vertex corresponding to $x$ and the $e$-vertices corresponding to the edges in $E_{G}(x)$ form a clique of order $d_{G}(x) + 1$ in $T(G)$. It follows that there exists a clique of order $\Delta(G) + 1$ in $T(G)$. Therefore, $\Delta(G) + 1 \leq \omega(T(G)) \leq \chi(T(G)) = \chi''(G)$. In fact, it is not difficult to see that the clique number of the total graph of $G$, $\omega(T(G)) = \Delta(G) + 1$, when $\Delta(G) \geq 2$.

It is easy to see that if $\Delta(G) \leq 1$, then $\chi''(G) = \omega(T(G))$. Thus, in view of Observation 2, Conjecture 1.2 can be thought to be suggesting the existence of a very strong $\chi$-bounding function for the class of total graphs, and can be restated as follows:

**Conjecture 1.6** (Restatement of total coloring conjecture). $\forall H \in T, \chi(H) \leq \omega(H) + 1$

The total coloring conjecture was intensely studied by several brilliant graph theorists for more than 50 years, but still remains unsolved and has earned the reputation of being one of the toughest problems in graph coloring. Therefore it makes sense to study relaxed versions of this question. From the structural perspective, one obvious way to relax Conjecture 1.6 is to replace clique number $\omega(H)$ by Hadwiger number $\eta(H)$, the number of vertices in the largest clique minor of $H$. This directly leads us to the study of Hadwiger’s conjecture on Total Graphs, hereafter abbreviated as “HC on $T$”. (See next section for the formal statement of Hadwiger’s Conjecture.)

**Problem 1.7** (Hadwiger’s Conjecture on Total Graphs or “HC on $T$”).

$$\chi(H) \leq \eta(H), \forall H \in T$$

**Remark.** Careful readers may object that if we replace $\omega(H)$ by $\eta(H)$ in Conjecture 1.6, the new inequality should be $\chi(H) \leq \eta(H) + 1, \forall H \in T$. But we will show in Theorem 2.3 that Conjecture 1.6 indeed implies HC on $T$ as stated in Problem 1.7.
Needless to say, Hadwiger’s conjecture is an even more celebrated and long standing conjecture than TCC in graph theory, owing its origin to the Four Color Theorem itself. While the general case of Hadwiger’s conjecture remains unsolved, it was proved for several special classes of graphs. For example, Reed and Seymour [20] proved Hadwiger’s conjecture for line graphs more than 15 years ago. Chudnovsky and Fradkin [10] even generalized the result to quasi-line graphs. But to the best of our knowledge, for the class of total graphs, whose definition is similar in spirit to that of line of graphs, Hadwiger’s conjecture has not yet been proved or studied.

1.2 Hadwiger’s Conjecture: The connection between the general case and the case of total graphs

Given an edge \( e = uv \) in \( G \), the contraction of the edge \( e \) involves the following: deleting vertices \( u \) and \( v \), introducing a new vertex \( w_e \), and making \( w_e \) adjacent to all vertices in the set \( (N_G(u) \cup N_G(v)) \setminus \{u, v\} \). The new graph thus obtained is denoted by \( G/e \).

A graph \( H \) is said to be a minor of \( G \) if a graph isomorphic to \( H \) can be obtained from \( G \), by performing a sequence of operations involving only vertex deletions, edge deletions, and edge contractions. If \( G \) contains \( H \) as a minor, then we write \( H \preceq G \).

The celebrated Hadwiger’s conjecture is a far-reaching generalization of the Four Color Theorem. It was proposed by Hugo Hadwiger [13] in 1943.

**Conjecture 1.8 (Hadwiger’s Conjecture).** Given any graph \( G \) and \( t > 0 \),

\[
\chi(G) \geq t \implies K_t \preceq G
\]

In other words, every graph has either a clique minor on \( t \) vertices (\( K_t \)-minor) or a proper vertex-coloring using \((t-1)\) colors.

For \( t \leq 3 \), Hadwiger’s conjecture is easy to prove. The \( t = 4 \) case was proved by Hadwiger himself in [13]. The Four Color Theorem was proved by Appel and Haken [12] in 1977. Using a result of Wagner [29], it can be shown that the Four Color Theorem is equivalent to Hadwiger’s conjecture for \( t = 5 \). Robertson, Seymour, and Thomas [21] proved in 1993 that Hadwiger’s conjecture holds true for \( t = 6 \). The conjecture remains unsolved for \( t \geq 7 \). So far, Hadwiger’s conjecture has been proved for several classes of graphs; see [5, 10, 15, 20, 30, 31].

Now we describe a curious observation from [8]. Given a graph \( G \), its square \( G^2 \) is defined on the vertex set \( V(G) \) with \( u \) and \( v \) being adjacent in \( G^2 \) whenever the distance between \( u \) and \( v \) in \( G \) is at most 2. For a class of graphs \( F \), let \( F^2 \) denote the set of graphs \( \{G^2 : G \in F\} \). Recall that a split graph is a graph whose vertex set can be partitioned into an independent set and a clique. Let \( S \) denote the special class of split graphs, whose vertex set is partitioned into an independent set \( A \) and a clique \( B \) with the following extra constraints: (1) Each vertex in \( B \) has exactly 2 neighbours in \( A \). (2) There are no two vertices in \( B \) having the same neighbourhood in \( A \).

The following fact is from [8]. (In [8], the following fact is not stated explicitly in this detail, but can be easily read out from the proof of Theorem 1.2, therein.)

**Fact.** Proving the general case of Hadwiger’s conjecture is equivalent to proving HC for the class \( S^2 \).

Obviously proving HC for \( S^2 \) is extremely difficult, despite its tantalizingly specialized appearance. From this perspective, it is natural to consider classes of graphs that can be obtained by simple structural modifications of \( S \) and see how difficult it is to prove HC for the squares of such classes. One such modification is to assume that both \( A \) and \( B \) are independent sets, keeping everything else same, i.e. the class of bipartite graphs with parts \( A \) and \( B \) with the extra constraints: (1) Each vertex in \( B \) has exactly 2 neighbours in \( A \). (2) No two vertices in \( B \) have the same neighbourhood in \( A \). In other words get a bipartite graph from each split graph in \( S \) by converting the clique \( B \) to an independent set. Let this new class be denoted by \( \hat{S} \). How difficult is it to prove HC for \( \hat{S}^2 \)? Careful inspection reveals that \( \hat{S} \) is a familiar class of graphs, the sub-divided graphs.
A graph $G$ is a subdivided graph if it can be obtained from another graph $H$ by subdividing each edge $uv$ in $H$; that is, replacing $uv$ by a path $uwv$, where $w$ is a new vertex. Now if we consider the set of newly introduced vertices as $A$ and the original vertices of $H$ as $B$, it is easy to see that $G \in \hat{S}$. The converse is also true: it is easy to verify that if a graph $G$ belongs to $\hat{S}$, then there exists another graph $H$ such that $G$ is obtained by subdividing all the edges of $H$. Interestingly, the class of squares of subdivided graphs is the same as the class of total graphs.

**Fact.** If $H$ is a graph and $G$ is obtained by subdividing every edge of $H$, then the graph $G^2$ is isomorphic to the total graph $T(H)$ of $H$. In other words, the class of total graphs $T = \hat{S}^2$.

Considering the structural closeness of $\hat{S}$ and $S$ and the fact from [8] that HC on $S^2$ is equivalent to the general case of HC, it is not very surprising that proving Hadwiger’s Conjecture for total graphs turns out to be challenging.

**Remark.** Let $G$ be a graph and let $T(G)$ be its total graph. Denote the set of $v$-vertices and the set of $e$-vertices in $T(G)$ by $V$ and $E$, respectively. Note that the subgraph of $T(G)$ induced by $E$ is isomorphic to the original graph $G$, the subgraph of $T(G)$ induced by $E$ is isomorphic to the line graph $L(G)$ of $G$, and the bipartite graph induced by the edges between the sets $V$ and $E$ is precisely the subdivided graph $S(G)$ of $G$.

### 1.3 Our contributions to Hadwiger’s conjecture on total graphs

In Section 2 and Section 3, we explore the difficulty level of $HC$ on $T$ in comparison with $TCC$. Our first result is Theorem 2.1 which states that if $TCC$ is true for a class $F$ of graphs that is closed under taking subgraphs, then Hadwiger’s conjecture holds true for the class $T(F)$. Taking $F = G$, the class of simple finite graphs, we infer that $TCC$ implies $HC$ on $T$, thus justifying our claim that Problem 1.7 is a relaxation of Conjecture 1.2 and its restatement, Conjecture 1.6. But is $TCC$ indeed a stronger statement than $HC$ on $T$? Our next result confirms this.

We will refer to the statement “$\forall G \in G, \chi''(G) \leq \Delta(G) + 3$” as $weak TCC$. In Theorem 3.1, we show that if $weak TCC$ is true for a class $F$ of graphs that is closed under taking subgraphs, then Hadwiger’s conjecture is true for the class $T(F)$. Thus the weaker hypothesis $weak TCC$ implies $HC$ on $T$; therefore $HC$ on $T$ is strictly easier than $TCC$. A graph $G$ is $t$-total critical if $\forall e \in E(G), \chi''(G - e) < \chi''(G) = t$. We show that for every $(\Delta(G) + 3)$-total critical graph $G$, $T(G)$ has a clique minor of order $\Delta(G) + 3$.

For each fixed positive integer $k \geq 2$, let $(k)$-$TCC$ be the statement, “$\forall G \in G, \chi''(G) \leq \Delta(G) + k$.” Thus $2$-$TCC$ is same as $TCC$ and $3$-$TCC$ is same as $weak-TCC$. Can we prove that the weaker hypothesis $k$-$TCC$ (for some fixed positive integer $k \geq 4$) implies $HC$ on $T$? We do not know the answer, but our approach of showing the existence of a $(\Delta(G) + k)$ clique minor in $T(G)$, assuming that $G$ is $(\Delta + k)$-total critical, does not seem to work when $k \geq 4$. More sophisticated approaches may be needed to answer this question.

However, if we have the additional guarantee that the graph $G$ has a high vertex connectivity, then Hadwiger’s conjecture will hold for $T(G)$ if $(k)$-$TCC$ is true for $G$. We show in Theorem 3.3 that if $(k)$-$TCC$ is true, then Hadwiger’s conjecture is true for $T(F)$, where $F := \{ G : \kappa(G) \geq 2k - 1 \}$ (here, $\kappa(G)$ denotes the vertex-connectivity of $G$). By combining this with the upper bound of Molloy and Reed [19], we get that there exists a constant $C'$ such that Hadwiger’s conjecture is true for $T(F)$, where $F := \{ G : \kappa(G) \geq C' \}$. Here $C' = 2C - 1$, where $C$ is the constant from Theorem 1.3 in [19] Molloy and Reed proved $C = 10^{26}$, but mention that with more detailed analysis, $C$ can be brought down to 500.

### 1.4 Our contributions to the total coloring literature

In this section we concentrate on the first (and the more straightforward) relaxation of $TCC$ mentioned earlier: $weak-TCC$.

The first reason that motivated us to go through the known literature on total coloring to find out the important special classes $F$ of graphs for which $weak TCC$ is known to hold (while $TCC$ may still remain unsolved), is the promise of Theorem 3.1 that $HC$ is true for $T(F)$ for such $F$. We realized that there are
some important graph classes in this category, for example planar graphs. Unfortunately, for general graphs, it seems the chance of either TCC or weak TCC getting proved in the near future is very less.

Is it possible that weak TCC is a more reasonable conjecture than TCC? It is remarkable that another very well-known and widely believed conjecture, the list coloring conjecture, implies the weak TCC, whereas TCC seems beyond its reach. TCC is indeed a bold conjecture, in the sense that at times it makes us think that it is not unreasonable to look for counterexamples. For example, despite intense research for decades, the conjecture could not be proved for planar graphs with maximum degree 6, though it has been found to hold good for all other cases of planar graphs. On the other hand, it is very easy to prove weak TCC for all planar graphs: If \( G \) is a planar graph, then \( \chi''(G) \leq \Delta(G) + 3 \). Moreover, weak TCC is in fact true for a much wider class, namely 4-colorable graphs, which properly includes planar graphs, whereas proving TCC on 4-colorable graphs seems to be extremely difficult. This led us to survey the status of weak TCC for a much wider class, namely 4-colorable graphs, which properly includes planar graphs, whereas proving TCC is not known to hold for the entire class of 5-colorable graphs till date.

The most important result in this direction was proved by Hind [14].

\[ \chi''(G) \leq \chi'(G) + 2 \left\lceil \sqrt{\chi'(G)} \right\rceil \leq \Delta(G) + 1 + 2 \left\lceil \sqrt{\chi(G)} \right\rceil. \]

We note that for small values of \( \chi(G) \) (such as \( \leq 9 \)), Theorem 1.9 implies \( \chi''(G) \leq \Delta(G) + 7 \). Naturally many researchers have attempted to improve this result using adaptations of the technique used in Hind’s proof. One such result was presented by Sánchez-Arroyo [23] and a slightly better bound was obtained by Chew [9].

**Theorem 1.10 (Chew).** For any connected multigraph \( G \),

\[
\chi''(G) \leq \begin{cases} 
\chi'(G) + \left\lceil \frac{\chi(G)}{3} \right\rceil + 1 & \text{if } \chi(G) \equiv 2 \pmod{3} \\
\chi'(G) + \left\lceil \frac{\chi(G)}{3} \right\rceil + 1 & \text{otherwise}
\end{cases}
\]

This implies that \( \chi''(G) \leq \chi'(G) + 3 \), for \( \chi(G) \leq 5 \). Although weak TCC for class-I 5-colorable graphs follows from this result, weak TCC is not known to hold for the entire class of 5-colorable graphs till date. In Theorem 5.1, we prove the following long-pending result:

\[ \text{Weak TCC is true for 5-colorable graphs.} \]
Remark. The method used in our proof is completely different from the proofs of Theorem 1.8 and Theorem 1.10.

Note. Proving TCC for 5-colorable graphs is likely to be a considerably harder problem. In fact, TCC remains to be proved even for 4-colorable graphs. As mentioned before the $\Delta = 6$ case for planar graphs is still open even after decades of research.

2 Hadwiger’s conjecture and total coloring

Definition 2.1 (total-critical graph). A graph $G$ is said to be $t$-total-critical if $\chi''(G) = t$ and $\chi''(G - e) \leq t - 1$, for any edge $e$ of $G$.

Note that any graph that contains at least one edge has total chromatic number at least 3.

Lemma 2.2. If a connected graph $H$ is $t$-total-critical, where $t \geq \Delta(H) + 2$, then $H$ has no cut-vertices.

Proof. Let $H$ be a connected $t$-total-critical graph, where $t \geq \Delta(H) + 2$. Let us choose a vertex $v$ in $H$ with $d_H(v) = d$ (say). We claim that $v$ is not a cut-vertex in $H$. Otherwise if $v$ is a cut-vertex then $2 \leq d \leq \Delta(H)$.

We can assume that for some $s < d$, $s$ neighbours of $v$ say, $v_1, v_2, \ldots, v_s$ lie in one connected component $C_1$, while the remaining $d - s$ neighbours, $v_{s+1}, \ldots, v_d$ lie in the remaining connected components $C_2, \ldots, C_k$ ($k \geq 2$) of $H - v$. Let the edge $vv_i$ be denoted by $e_i$, for all $i \leq d$. Define $T_1 := H[V(C_1) \cup \{v\}]$ and $T_2 := H - V(C_1)$. Both $T_1$ and $T_2$ can be total colored with $(t - 1)$ colors since $H$ is $t$-total-critical.

Consider a total coloring $\varphi : V(T_1) \cup E(T_1) \rightarrow [t - 1]$ of $T_1$. Note that in $\varphi$, the colors on the edges $e_1, e_2, \ldots, e_s$ and the vertex $v$ are all different. We can assume without loss of generality (by renaming the colors if necessary) that $\varphi(v) = 1$ and $\varphi(e_i) = i + 1$ for each $i \in [s]$. Now consider a total coloring $\psi : V(T_2) \cup E(T_2) \rightarrow [t - 1]$ of $T_2$. As in the previous case, in $\psi$, the colors on the edges $e_{s+1}, e_{s+2}, \ldots, e_d$ and the vertex $v$ are all different. Again by renaming colors if necessary, we can assume that $\psi(v) = 1$ and $\psi(e_i) = i + 1$ for each $i \in \{s + 1, s + 2, \ldots, d\}$. Notice that the renaming of colors in both $\varphi$ and $\psi$ is possible because $d + 1 \leq \Delta(H) + 1 \leq t - 1$. We now construct a total coloring $\phi : V(H) \cup E(H) \rightarrow [t]$ of $H$ by combining the colorings $\varphi$ of $T_1$ and $\psi$ of $T_2$ (i.e. $\phi(v) = 1$ and for every other element (vertex or edge) $x$ of $H$, $\phi(x) = \varphi(x)$ if $x$ belongs to $T_1$ and $\phi(x) = \psi(x)$ if $x$ belongs to $T_2$). It is easy to verify that $\phi$ is a total coloring of $H$ using just $t - 1$ colors. This contradicts the fact that $H$ is a $t$-total-critical graph.

Lemma 2.3. Let $G$ be a connected $(\Delta(G) + k)$-total-critical graph on at least 3 vertices, for some $2 < k \leq \Delta(G)$. Then the minimum degree of $G$ is at least $k$.

Proof. Assume that there exists a vertex $w \in G$ with $d_G(w) \leq k - 1$, and let $\Delta(G) = \Delta$. Since $G$ has at least three vertices and there are no cut-vertices in $G$ (Lemma 2.2), $w$ cannot have degree 1.

Let the neighbours of $w$ be $v_1, v_2, \ldots, v_j$, where $2 \leq j \leq k - 1$. Call the edge $v_iw$ as $e_i$, for each $i \in [j - 1]$ and $v_jw$ as $f$. Then $G - f$ has a total coloring using $\Delta + k - 1$ colors, which we shall assume to be the set $[\Delta + k - 1]$.

Assume that $v_i$ is assigned color $\alpha_i$ for each $i \in [j]$ and the edge $e_i$ is assigned color $\beta_i$ for each $i \in [j - 1]$. Call the color assigned to $w$ as $c$.

If $\alpha_j = c$, pick a color $c' \in \{1, 2, \ldots, \Delta + k - 1\} \setminus \{\alpha_j : i \in [j - 1]\} \cup \{\beta_i : i \in [j - 1]\} \cup \{c\}$ (such a $c'$ exists because $k \leq \Delta$ implies $\Delta + k - 1 > 2(k - 2) + 1$), and assign it to the vertex $w$. Now, the vertices $w$ and $v_j$ have different colors on them. Else if $\alpha_j \neq c$, let $c' := c$.

Since $d_G(v_j) \leq \Delta$, and therefore $d_{G-j}(v_j) \leq \Delta - 1$, there exists a list $L$ of at least $(k - 1)$ colors that are not assigned to $v_j$ or its incident edges. If at least one of the colors in $L$ is not in $\{c'\} \cup \{\beta_i : i \in [j - 1]\}$, assign it to the edge $f$ giving $G$ a $(\Delta + k - 1)$-total coloring. Otherwise since $|L| \geq k - 1$ and $w$ has at most $k - 2$ incident edges in $G - f$, we have the set of colors that appear on $w$ and its incident edges is exactly the list $L$; that is, $\{c'\} \cup \{\beta_i : i \in [j - 1]\}$ = $L$. Choose a color $c'' \in \{1, 2, \ldots, \Delta + k - 1\} \setminus \{\alpha_j : i \in [j - 1]\} \cup \{\beta_i : i \in [j - 1]\}$ (such a $c''$ exists as $j \leq k - 1 < \Delta$ implies $\Delta + k - 1 > 2(k - 2) + 2$), and assign it to the vertex $w$ and then assign color $c'$ to the edge $f$, giving a $(\Delta + k - 1)$-total coloring of $G$.

This is a contradiction as $\chi''(G) = \Delta + k$ by hypothesis. Hence, $d_G(w) \geq k$ for each vertex $w \in G$.  

Proof. Assume that TCC holds for all graphs in \( F \). Let \( G \in F \) and \( \Delta := \Delta(G) \). Therefore by Observation \([2]\) \[ \Delta + 1 \leq \chi''(G) \leq \Delta + 2. \] Note that the last inequality follows because TCC holds for \( F \).

Case 1: \( \chi''(G) = \Delta + 1 \). \( T(G) \) contains a clique of size \( \Delta + 1 \) (Observation \([2]\)). Therefore, this case is trivial.

Case 2: \( \chi''(G) = \Delta + 2 \). Let \( H \) be a \((\Delta + 2)\)-total critical subgraph of \( G \). We then have \( \chi''(H) = \Delta + 2 \). It is clear that \( \Delta(H) = \Delta \), otherwise since \( H \) also belongs to \( F \), TCC would imply that \( \chi''(H) \leq \Delta + 1 \). Choose a vertex \( v_\Delta \in H \) such that \( d_H(v_\Delta) = \Delta \). Let \( e_1, e_2, \ldots, e_\Delta \) be the edges incident on \( v_\Delta \) in \( H \). From Lemma \([22]\) we know \( H - v_\Delta \) is connected. This implies that the subgraph \( X \) of \( T(G) \) induced by the elements of \( H - v_\Delta \) is connected. In \( T(G) \), contract the subgraph \( X \) into a single vertex \( w \). Since the subgraph \( X \) contains the \( v \)-vertex corresponding to every neighbour of \( v_\Delta \), the vertex \( w \) is now adjacent to the \( e \)-vertices \( e_1, \ldots, e_\Delta \) and the \( v \)-vertex \( v_\Delta \), forming a \((\Delta + 2)\)-clique minor in \( T(G) \). \( \square \)

3 Weak total coloring conjecture

We may state weaker versions of Conjecture \([1, 2]\) by relaxing the upper bound on \( \chi''(G) \) as follows:

Conjecture 3.1 ((\( k \))-TCC). Let \( k \geq 2 \) be a fixed positive integer. For any graph \( G \),

\[ \chi''(G) \leq \Delta(G) + k \]

Note. \((2)\)-TCC is the same as Conjecture \([1, 2]\) : that is, the total coloring conjecture. We shall refer to \((3)\)-TCC as the \textit{weak total coloring conjecture} or \textit{weak-TCC} for short.

We first make a general observation about graphs that do not contain cut-vertices.

Claim 3.2. Suppose that \( G \) is a 2-connected graph and \( v \) is a vertex in \( G \). Then there exists a vertex \( w \in N(v) \) such that \( \{w, v\} \) is not a separator of \( G \).

Proof. Suppose for contradiction that each set in \( \{(v, w)\}_{w \in N(v)} \) is a separator of \( G \). Since \( G \) has no cut-vertices, we can conclude that each set in \( \{(v, w)\}_{w \in N(v)} \) is a minimum separator of \( G \). For \( w \in N(v) \), let \( F(v, w) \) be the number of vertices in the smallest component in \( G - \{v, w\} \). Choose \( w \in N(v) \) such that \( F(v, w) \) is as small as possible. Let \( S \) be the smallest component in \( G - \{v, w\} \). Since \( \{v, w\} \) is a minimum separator, \( v \) has a neighbour in each component of \( G - \{v, w\} \), and so also in \( S \). Let \( x \in S \cap N(v) \). Now consider the graph \( G - \{w, x\} \). Since \( w \) had a neighbour in each connected component of \( G - \{v, w\} \) (as \( \{v, w\} \) is a minimum separator), all the components of \( G - \{v, w\} \), together with \( w \), now form one connected component of \( G - \{w, x\} \). Thus the other connected components of \( G - \{w, x\} \) must all be subgraphs of \( S \). Since there is at least one other connected component and it cannot contain \( x \in S \), we can conclude that there is a connected component of \( G - \{w, x\} \) that is smaller than \( S \). This contradicts the choice of \( w \) and \( S \). \( \square \)

Lemma 3.3. Let \( H \) be a \((\Delta + 3)\)-total-critical graph having maximum degree \( \Delta \). Then \( T(H) \) contains a clique minor of order \( \Delta + 3 \).

Proof. Recall that if \( H \) is not connected, then \( H \) contains at most one connected component that is not an isolated vertex. Since the maximum degree of this component is also \( \Delta \) and it is also \((\Delta + 3)\)-total-critical, we will be done if we prove the statement of the lemma for this component. So from here on, we assume that \( H \) is connected.
Choose a vertex $v_\Delta$ in $H$, with $d_H(v_\Delta) = \Delta$. By Lemma 2.2, $H$ does not contain any cut-vertices. Then by Claim 3.2 there exists a $w \in N_H(v_\Delta)$, such that $\{v_\Delta, w\}$ is not a separator in $H$. Recall that $TCC$ holds for all graphs with maximum degree at most 5 (Theorem 1.1). Therefore, $\Delta \geq 6$ since $\chi''(H) = \Delta + 3$ by assumption. It follows from Lemma 2.3 that degree of $w$ in $H$ is at least 3, that is, there are at least two edges between $w$ and $H \setminus \{v_\Delta, w\}$. Let $e'$ and $e''$ be the $e$-vertices in $T(H)$ corresponding to these two edges.

Our goal is to construct a clique minor of order $\Delta + 3$ ($K_{\Delta+3}$-minor) in $T(H)$. For this we use the following branch sets:

1. The set $S^0$ consisting of the $v$-vertices that correspond to $\{v_\Delta, w\}$. Note that $S^0$ is connected in $T(H)$ and can be contracted to a vertex $z$.

2. The singleton $\{e\}$ for each $e$-vertex $e \in T(H)$ that corresponds to an edge in $E_H(v_\Delta)$. Let $S^1$ denote the set of these $\Delta$ $e$-vertices.

3. $S^2 = S' \cup \{e'\}$, where $S'$ is the set of $e$-vertices corresponding to the edges in $E(H \setminus \{v_\Delta, w\})$. Note that $S^2$ is connected and can be contracted to a single vertex $y$.

4. $S^3 = S'' \cup \{e''\}$, where $S''$ is the set of $v$-vertices corresponding to the vertices in $V(H \setminus \{v_\Delta, w\})$. $S^3$ is also connected and can be contracted to a single vertex $x$.

Now we show that $S^1 \cup \{x\} \cup \{y\} \cup \{z\}$ forms the desired clique of order $\Delta + 3$ in the resulting graph.

Let the $e$-vertex corresponding to the edge $v_\Delta w$ be $f$. First, $(S^1 \setminus \{f\}) \cup \{x\} \cup \{y\}$ forms a clique of order $\Delta + 1$ since $H \setminus \{v_\Delta, w\}$ is connected and contains end-points of all edges incident on $v_\Delta$ other than $v_\Delta w$.

Now obviously $f$ is adjacent to each vertex in $S^1 \setminus \{f\}$ and also to $e'$ and $e''$ in $T(H)$. It follows that $f$ is adjacent to $x, y$ and each member of $S^1 \setminus \{f\}$.

Now $z$ is obviously adjacent to all vertices in $S^1$. Moreover $z$ is adjacent to $x$, since $e''$ is adjacent to $w$ in $T(H)$, and adjacent to $y$, since $e'$ is adjacent to $w$ in $T(H)$.

**Theorem 3.4.** Let $\mathcal{F}$ be a class of graphs that is closed under the operation of taking subgraphs. Then, Hadwiger’s conjecture holds for the class $T(\mathcal{F})$, if weak $TCC$ is true for graphs in $\mathcal{F}$.

**Proof.** Let $G$ be a graph in $\mathcal{F}$. Let $\Delta = \Delta(G)$. Since weak $TCC$ holds for $G$, we know that $\chi''(G) \leq \Delta + 3$. Together with Observation 2 this implies that $\chi''(G) \in \{\Delta + 1, \Delta + 2, \Delta + 3\}$.

**Case 1:** $\chi''(G) = \Delta + 3$

Take a $(\Delta + 3)$-total-critical subgraph $H$ of $G$. Clearly, $\Delta(H) = \Delta$; otherwise if $\Delta(H) < \Delta$, since $H$ also belongs to $\mathcal{F}$, $H$ would have a $(\Delta + 2)$-total coloring by weak $TCC$, which is a contradiction. Now by Lemma 3.3 $T(H)$ contains a clique minor of order $\Delta + 3$. As $T(H)$ is a subgraph of $T(G)$, we are done.

**Case 2:** $\chi''(G) = \Delta + 2$

Let $H$ be a $(\Delta + 2)$-total critical subgraph of $G$. Since $H$, being a subgraph of $G$, belongs to $\mathcal{F}$, and therefore satisfies weak $TCC$, we have that $\Delta(H) \geq \Delta - 1$. As observed earlier, we assume that $H$ is connected, since if not, we can just take the only component of $H$ that is not an isolated vertex to be $H$. If $\Delta(H) = \Delta$, then choose a vertex $v_\Delta$ of $H$ such that $d_H(v_\Delta) = \Delta$. By Lemma 2.2 we know that $H - v_\Delta$ is connected. Then in $T(H)$, we can contract the $v$-vertices corresponding to vertices in $V(H) - v_\Delta$ to a single vertex $z$ (as in the proof of Theorem 2.1) so that the $e$-vertices corresponding to the edges incident on $v_\Delta$, the $v$-vertex corresponding to $v_\Delta$, and the vertex $z$ form a clique of order $\Delta + 2$ in the resulting graph. On the other hand, if $\Delta(H) = \Delta - 1$, then since $H$ is $(\Delta(H) + 3)$-total-critical, we have by Lemma 3.3 that $T(H)$ contains a clique minor of order $(\Delta - 1) + 3 = \Delta + 2$.

**Case 3:** $\chi''(G) = \Delta + 1$

As we can see from Observation 2 this case is trivial.

**Corollary 3.5.** Hadwiger’s conjecture holds true for total graphs of all planar graphs.
Proof. Follows from Theorems 1.4 and 3.4.

Given a graph \( G \) with a list \( L_e \) of colors assigned to each edge \( e \), a list edge coloring is a proper coloring of \( E(G) \) such that each edge \( e \) is assigned a color from the list \( L_e \). A graph \( G \) is said to be \( k \)-edge-choosable if a list edge coloring of \( G \) exists for any assignment of lists of size \( k \) to the edges of \( G \). The smallest positive integer \( k \) such that \( G \) is \( k \)-edge-choosable is called the edge choosability of \( G \), which is denoted as \( ch'(G) \). The following is a well-known fact.

Lemma 3.6. For any graph \( G \), \( \chi''(G) \leq \chi'(G) + 2 \).

The list coloring conjecture is as follows:

**Conjecture 3.7** (List coloring conjecture). For any graph \( G \), \( ch'(G) = \chi'(G) \).

**Theorem 3.8.** Let \( \mathcal{F} \) be a class of graphs closed under the operation of taking subgraphs. Hadwiger’s conjecture holds for the class \( T(\mathcal{F}) \), if \( \mathcal{F} \) satisfies the list coloring conjecture.

Proof. Since \( \mathcal{F} \) satisfies the list coloring conjecture, for each \( G \in \mathcal{F} \), \( \chi''(G) \leq \chi'(G) + 2 = \chi'(G) + 2 \leq \Delta(G) + 3 \) (by Lemma 3.6 and Theorem 1.1), that is, \( \mathcal{F} \) satisfies weak TCC. The theorem then follows from Theorem 3.4.

Thus, the validity of the list coloring conjecture would imply that Hadwiger’s conjecture is true for all total graphs.

## 4 Connectivity and Hadwiger’s conjecture for total graphs

Given a partition of the vertex set of a graph, an edge of the graph is said to be a cross-edge if its endpoints belong to different sets of the partition.

Our next result makes use of the following well-known theorem:

**Theorem 4.1** (Tutte; Nash-Williams; See [11]). A multigraph contains \( k \) edge-disjoint spanning trees if and only if every partition \( P \) of its vertex set contains at least \( k(|P| - 1) \) cross-edges.

**Corollary 4.2** (See [11]). Every \( 2k \)-edge-connected multigraph \( G \) has \( k \) edge-disjoint spanning trees.

**Theorem 4.3.** Let \( G \) be a \((2k - 1)\)-connected graph. Then, the total graph \( T(G) \) contains a clique minor of size \( \Delta + k \), where \( \Delta = \Delta(G) \).

Proof. Let \( v_\Delta \) be a vertex of maximum degree in \( G \). Let the set of \( e \)-vertices in \( T(G) \) corresponding to the edges in \( E_G(v_\Delta) \) be denoted by \( Z \). Note that \( Z \) induces a \( \Delta \)-clique (call it \( K_Z \)) in \( T(G) \). Let \( H \) be the graph obtained from \( G \) by removing \( v_\Delta \). We will show that there exist \( k \) disjoint but pair-wise adjacent connected induced subgraphs of \( T(H) \), so that by contracting these induced subgraphs we get \( k \) more vertices that can be added to the existing \( \Delta \)-clique \( K_Z \) thus achieving the desired \((\Delta + k)\)-clique minor in \( T(G) \).

Since \( G \) is \((2k - 1)\)-connected, the vertex-connectivity of \( H \) is at least \( 2k - 2 \). It follows that, edge-connectivity of \( H \) is also at least \( 2k - 2 \). Hence, by Corollary 4.2, \( H \) has \((k - 1)\) pair-wise edge-disjoint spanning trees.

Let \( T_1, \ldots, T_{k-1} \) be the pair-wise edge-disjoint spanning trees in \( H \). Let \( E'_i \) be the set of \( e \)-vertices in \( T(G) \) corresponding to the edges in \( E(T_i) \), for \( i \in [k - 1] \). Clearly the induced subgraph of \( T(G) \) on \( E'_i \) is isomorphic to the line graph of \( T_i \) and is connected and therefore, can be contracted to a single vertex, say \( y_i \). Let \( Y := \{ y_i : i \in [k - 1] \} \). The set of \( v \)-vertices of \( T(G) \) is also connected since they induce a subgraph isomorphic to \( H \) itself, and can be contracted to a single vertex \( v^* \). Let \( F \) be the graph so obtained. Since \( T_1, \ldots, T_{k-1} \) are all spanning trees of \( H \), it follows that \( y_i y_j \in E(F) \) for distinct \( i, j \in [k - 1] \) and also that \( v^* y_i \in E(F) \) for all \( i \in [k - 1] \). Further, it can be seen that in \( F \), each \( y_i \in Y \) and \( v^* \) are adjacent to all the \( e \)-vertices in \( Z \). Therefore, \( Y \cup Z \cup \{ v^* \} \) forms a clique of order \( \Delta + k \) in \( F \), implying that \( T(G) \) contains a \((\Delta + k)\)-clique minor.
Corollary 4.4. If a \((2k - 1)\)-connected graph \(G\) satisfies \((k)\)-TCC, then Hadwiger’s conjecture is true for \(T(G)\).

Proof. Since \(G\) satisfies \((k)\)-TCC, we have \(\chi(T(G)) = \chi''(G) \leq \Delta(G) + k\). As \(G\) is \((2k - 1)\)-connected, we have from Theorem 1.3 that \(T(G)\) contains a clique minor of size \(\Delta(G) + k\). Thus Hadwiger’s Conjecture is true for \(T(G)\). 

By Theorem 1.3, we have that \((C)\)-TCC is true, where \(C\) is the constant in Theorem 1.3. We thus have the following corollary.

Corollary 4.5. If \(G\) is a \((2C - 1)\)-connected graph, then \(T(G)\) satisfies Hadwiger’s conjecture, where \(C\) is the constant from Theorem 1.3.

5 Chromatic number and total coloring

Notational notes for this section. Given a coloring of the vertices and edges of a graph, we call a vertex \(v\) or an edge \(e\) which is assigned color \(c\), a \(c\)-vertex, or a \(c\)-edge. Given a set of colors \(C\), we call a vertex \(v\) or an edge \(e\) having a color from \(C\) a \(C\)-vertex or a \(C\)-edge, respectively. We say that the vertex \(v\) “sees” the color \(c\), if \(v\) is a \(c\)-vertex, or \(v\) has a \(c\)-edge incident on it. Note that the situation that a \(c\)-vertex \(u\) is adjacent to \(v\) is not referred to by the phrase, “\(v\) sees the color \(c\)”.

For a vertex \(v\), we define \(S_v\) to be the set of colors that are “seen” by the vertex \(v\) with respect to a given coloring \(\varphi\). For positive integers \(i, j\) such that \(i \leq j\), we denote by \([i, j]\) the set \(\{i, i + 1, \ldots, j\}\). A trail in a graph is a sequence of vertices \(v_1v_2\ldots v_k\) such that for \(i \in \{1, 2, \ldots, k - 1\}\), \(v_iv_{i+1}\) is an edge of the graph, and the edges \(v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\) are all pairwise distinct. Note that a vertex can appear more than once on a trail. Given two subsets of edges \(A\) and \(B\), we say that a trail \(v_1v_2\ldots v_k\) in the graph is an \((A, B)\)-trail if \(v_iv_{i+1} \in A\) when \(i\) is odd and \(v_iv_{i+1} \in B\) when \(i\) is even.

Theorem 5.1. Let \(G\) be a simple finite connected graph such that \(\chi(G) \leq 5\). Then \(\chi''(G) \leq \Delta(G) + 3\).

Proof. We begin by coloring \(V(G)\) with colors \([5] = \{1, 2, 3, 4, 5\}\), and \(E(G)\) with colors \([3, \Delta + 3] = \{3, 4, 5, 6, \ldots, \Delta + 3\}\), where \(\Delta := \Delta(G)\). Recall that such an edge coloring is possible by Theorem 1.1. Note that this coloring of \(V(G) \cup E(G)\), which we shall denote by \(\alpha\), need not be a total coloring.

Clearly, in this coloring, no two adjacent vertices have the same color, and no two adjacent edges have the same color. But there could exist edges of the form \(e = uv\) such that \(\alpha(e) = \alpha(v)\) or \(\alpha(e) = \alpha(u)\). Note that if that happens, then \(\alpha(e) \in \{3, 4, 5\}\). We shall call such an edge a “conflicting edge”.

We now create another coloring \(\pi : V(G) \cup E(G) \rightarrow [\Delta + 3]\), by slightly modifying the coloring \(\alpha\) if necessary, so that the following property holds for \(\pi\).

Property A. There is no 5-vertex \(v\) having a 3-edge, 4-edge and 5-edge incident on it.

The coloring \(\alpha\) can be transformed into such a coloring \(\pi\) by repeatedly doing the following: Consider each 5-vertex \(v\) having a 3-edge, 4-edge and 5-edge incident on it. Since \(v\) can now see at most \(\Delta - 3\) colors from \([6, \Delta + 3]\), there exists a color \(c \in [6, \Delta + 3]\) that is not seen by \(v\). We assign the color \(c\) to \(v\). Note that no neighbors of \(v\) have the color \(c\) since the 5-vertices of \(\alpha\) formed an independent set and the color \(c\) was not assigned to any vertex in \(\alpha\).

We call this coloring \(\pi\) the “original coloring” of \(G\). We emphasize that:

(i) \(\pi|_{V(G)}\) and \(\pi|_{E(G)}\) are proper colorings of \(V(G)\) and \(E(G)\), respectively.

(ii) \(\pi\) satisfies Property A.

(iii) Any conflicting edge in \(\pi\) is also a conflicting edge in \(\alpha\). Thus, if \(e\) is a conflicting edge in \(\pi\), then \(\pi(e) \in \{3, 4, 5\}\).
Our goal now is to start with the coloring \( \pi \) of \( G \) and then recolor each conflicting edge so that it is no longer a conflicting edge in the resulting coloring. We will do so in three phases and in each phase, we select a color \( i \in \{3, 4, 5\} \) and recolor the conflicting \( i \)-edges one by one. Our recoloring strategy will rely on the following lemma.

**Lemma 5.2.** Let \( \varphi \) be a coloring of \( V(G) \cup E(G) \) at a certain stage of the recoloring process. Let \( i \in \{3, 4, 5\} \) and let \( A_i = \{1, 2, \ldots, i - 1\} \). Suppose \( \varphi \) satisfies the following properties:

1. \( \varphi|_{V(G)} = \pi|_{V(G)} \).
2. \( \varphi|_{E(G)} \) is a proper edge coloring of \( G \).
3. There are no conflicting \( A_i \)-edges in \( G \).
4. Every \( A_i \)-edge in \( G \) has original color in \( \{3, 4, \ldots, i\} \).
5. Every \( A_i \)-edge in \( G \) that has original color \( i \) has an endpoint colored \( i \), that is, it was originally a conflicting \( i \)-edge, and
6. If \( e' \) is a conflicting \( i \)-edge in \( G \), then it has original color \( i \), that is, \( \pi(e') = \varphi(e') = i \).

Further, let \( e \) be a conflicting \( i \)-edge in \( G \). Then \( G \) can be recolored to a coloring \( \psi \) so that \( e \) is no longer a conflicting edge in \( \psi \), and the conditions \[a] - \[f\] hold for this new coloring \( \psi \) too.

**Proof.** Let \( e = v_0v_1 \), where \( \varphi(e) = \varphi(v_1) = i \). By \[f\] we know that the original color of \( e \) is \( i \).

If there exists a color in \( A_i \setminus (S_\varphi(v_0) \cup S_\varphi(v_1)) \), we can recolor \( e \) with that color to get \( \psi \). It is easy to verify that all the properties \[a] - \[f\] hold for \( \psi \).

So we shall assume that \( A_i \subseteq S_\varphi(v_0) \cup S_\varphi(v_1) \). Let \( X \) be the set of edges incident on \( v_0 \) that have original color in \( [3, i] \). Since \([3, i] \) is a path and \( \pi|_{E(G)} \) is a proper edge coloring, we have \( |X| \leq i - 2 \). Since \( v_0v_1 \) has original color \( i \), we have \( |X| \leq i - 2 \), which implies that \( |X \setminus \{v_0v_1\}| \leq i - 3 \). Since \( \varphi(v_0v_1) = i \), it is not an \( A_i \)-edge. Thus there are at most \( i - 3 \) \( A_i \)-edges in \( X \). Since every \( A_i \)-edge has original color in \( [3, i] \) by \(d\), we know that every \( A_i \)-edge incident on \( v_0 \) is in \( X \). It follows that there are at most \( i - 3 \) \( A_i \)-edges incident on \( v_0 \). Considering the possibility that \( \varphi(v_0) \) also may be in \( A_i \), we have \( |S_\varphi(v_0) \cap A_i| \leq i - 2 \). This implies that there exists a color \( \gamma \in A_i \setminus S_\varphi(v_0) \). As \( A_i \subseteq S_\varphi(v_0) \cup S_\varphi(v_1) \), we have \( \gamma \in S_\varphi(v_1) \). Therefore, as \( \varphi(v_1) = i \), there exists an edge \( v_1v_2 \) such that \( \varphi(v_1v_2) = \gamma \). As \( \gamma \not\in S_\varphi(v_0) \), we have \( v_1v_2 \neq v_0v_1 \).

We call an edge \( f \) in \( G \) an \( i \)-edge if \( \varphi(f) \in A_i \setminus \{\gamma\} \) and the original color of \( f \) is \( i \). By \[e\] we know that one of the endpoints of every \( i \)-edge is colored \( i \). Let \( \Gamma \) and \( \Upsilon \) be the set of all \( \gamma \)- and \( i \)-edges in \( G \), respectively.

Now consider a maximal \((\Gamma, \Upsilon)\)-trail \( P = v_1v_2 \ldots v_k \), where \( k \geq 2 \), starting with the edge \( v_1v_2 \). It is easy to see that every edge on \( P \) is an \( A_i \)-edge. Then we have the following:

1. For each odd \( t \in \{1, 2, \ldots, k\} \), \( \varphi(v_t) = i \).
2. Note that \( \varphi(v_1) = i \). Suppose that for some odd \( t \in \{1, 2, \ldots, k - 2\} \), we have \( \varphi(v_t) = i \). Recall that \( \varphi|_{V(G)} = \pi|_{V(G)} \) and that \( \pi|_{V(G)} \) is a proper vertex coloring of \( G \). Then clearly \( \varphi(v_{t+1}) \neq i \), and since \( v_{t+1}v_{t+2} \) is an \( i \)-edge and an endpoint of every \( i \)-edge must be colored \( i \), we have \( \varphi(v_{t+2}) = i \).

(2) \( v_0P \) is a path.

Otherwise, some vertex must be repeated while traversing \( v_0P \). Let \( v \) be the first such vertex. If \( v \neq v_0 \), then it either has two edges having original color \( i \) or two \( \gamma \)-edges incident on it; as both \( \varphi|_{E(G)} \) and \( \pi|_{E(G)} \) are proper edge colorings of \( G \), this is a contradiction. Now, let \( v = v_0 \). Recall that by choice of \( \gamma \), the vertex \( v_0 \) has no incident \( \gamma \)-edge. Then, since \( v_0v_1 \) has original color \( i \), there must be two incident edges with original color \( i \) at \( v_0 \); another contradiction. It follows that \( v_0P \) is a path.

Note that the last vertex \( v_k \) on \( P \) satisfies the following properties:

(3a) If \( v_{k-1}v_k \) is an \( i \)-edge, \( \gamma \not\in S_\varphi(v_k) \).
(3b) If $v_{k-1}v_k$ is a $\gamma$-edge, then there exists $c' \in A_i \setminus S_\varphi(v_k)$. (Note that $c' \neq \gamma$.)

To see this, note that in the first case, we have that $k$ is odd and then by (1), $\varphi(v_k) = i \neq \gamma$. By the maximality of $P$, we have that $v_k$ has no incident $\gamma$-edges. In the second case, suppose $A_i \subseteq S_\varphi(v_k)$. This happens only if, for each $c \in A_i \setminus \{\varphi(v_k)\}$, $v_k$ has an $c$-edge incident on it. By property (d) and the fact that $\pi|_{E(G)}$ is a proper edge coloring, these $i-2$ edges have a distinct original color from the set $[3, i]$. Since $|[3, i]| = i - 2$, we can conclude that $v_k$ has an incident edge $v_kw$ with original color $i$ and $\varphi(v_kw) \in A_i$. Is it possible that $w = v_{k-1}$? Recall that in this case, $k-1$ is odd and $v_{k-2}v_{k-1}$ is an $i$-edge (when $k = 2$) or an $i'$-edge (when $k > 2$). As $\pi|_{E(G)}$ is a proper edge-coloring, $v_{k-1}v_k$ cannot have original color $i$ and so $w \neq v_{k-1}$. As $\varphi(v_{k-1}v_k) = \gamma$, we have $\varphi(v_kw) \neq \gamma$, and so $v_kw$ is an $\bar{\gamma}$-edge. This means $v_k$ has an incident $\bar{\gamma}$-edge outside of $P$. This violates the maximality of the trail $P$. So there exists a color $c' \in A_i \setminus S_\varphi(v_k)$.

Now we recolor the edges on $v_0P$ by the following color shift strategy:

- For each $1 \leq t \leq k-1$, assign $\varphi(v_tv_{t+1})$ to the preceding edge $v_{t-1}v_t$ by setting $\psi(v_{t-1}v_t) := \varphi(v_tv_{t+1})$.
- If the last edge $v_{k-1}v_k$ on $P$ is an $\bar{\gamma}$-edge, set $\psi(v_{k-1}v_k) := \gamma$.
- If $v_{k-1}v_k$ is a $\gamma$-edge, pick a color $c' \in A_i \setminus S_\varphi(v_k)$ and set $\psi(v_{k-1}v_k) := c'$. Such a $c'$ exists by (3b).

Finally, set $\psi(v) := \varphi(v)$, for each vertex $v \in V(G)$, and $\psi(e) := \varphi(e)$, for each edge $e \in E(G) \setminus E(v_0P)$.

It is easy to see that properties (a), (d), (e), and (f) hold for this new colorings $\psi$. We have not recolored any vertex, therefore (a) holds; Other than $v_0v_1$, all the edges that we have recolored are $A_i$-edges (with respect to $\varphi$) and they remain $A_i$-edges with respect to $\psi$ as well; The only new $A_i$-edge with respect to $\psi$ is $e = v_0v_1$. This is consistent with (d) and (e); the set of conflicting $i$-edges with respect to $\psi$ is the set of conflicting $i$-edges with respect to $\varphi$ minus $v_0v_1$, therefore (f) holds. We shall now proceed to show that the color shift strategy produces no new conflicts, ensuring that $\psi$ satisfies properties (b) and (c) also.

We first show that for any $t \in \{1, 2, \ldots, k-1\}$, we have $\psi(v_{t-1}v_t) \neq \psi(v_tv_{t+1})$. For $t \leq k-2$, we know that $\psi(v_{t-1}v_t) = \varphi(v_{t-1}v_t)$, $\varphi(v_tv_{t+1}) = \varphi(v_{t+1}v_{t+2})$, and that $\varphi(v_tv_{t+1}) \neq \varphi(v_{t+1}v_{t+2})$ (as $\varphi$ satisfies (b)). It follows that $\psi(v_{t-1}v_t) \neq \psi(v_tv_{t+1})$. Next, if $t = k-1$, then $\psi(v_{k-2}v_{k-1}) = \varphi(v_{k-1}v_k)$ and by (3a) and (3b), $\psi(v_{k-1}v_k) \neq S_\varphi(v_k)$. Since $\varphi(v_{k-1}v_k) \in S_\varphi(v_k)$, this means that $\psi(v_{k-2}v_{k-1}) \neq \psi(v_{k-1}v_k)$. Thus in $\psi$, no two consecutive edges of $v_0P$ have the same color.

Since we are recoloring only the edges on $v_0P$, if (b) or (c) is violated in $\psi$, there must be an edge on $v_0P$ such that it conflicts either with one of its end-points or with an adjacent edge. Let $t$ be any index such that $v_tv_{t+1}$ is such an edge with respect to $\psi$. Let $c = \psi(v_tv_{t+1})$. We will say that the edge $v_tv_{t+1}$ has a conflict at the end-point $v_{t+1}$, if either $\psi(v_{t+1}) = c$ or $v_{t+1}$ has an incident edge $v_{t+1}u \neq v_tv_{t+1}$ with $\psi(v_{t+1}u) = c$. Otherwise, the conflict has to be at the end-point $v_t$.

We claim that $v_tv_{t+1}$ cannot have a conflict at the end-point $v_{t+1}$. If $t = k-1$, this is obvious from (3a) and (3b). For $t \leq k-2$, $\psi(v_tv_{t+1}) = \varphi(v_{t+1}v_{t+2}) = c$. Suppose that $\psi(v_{t+1}) = c$. Then since we are recoloring only edges, we have $\varphi(v_{t+1}) = c$, which contradicts the fact that $\varphi$ satisfies (c). Next, suppose that there exists an edge $v_{t+1}u \neq v_tv_{t+1}$ such that $\psi(v_{t+1}u) = c$. Since no two consecutive edges of $v_0P$ have the same color in $\psi$, we know that $u \neq v_{t+2}$. Then $v_{t+1}u$ is not an edge of $v_0P$ which means that $\varphi(v_{t+1}u) = \psi(v_{t+1}u) = c$, which implies that there are two edges with color $c$ incident at $v_{t+1}$ in $\varphi$. This contradicts the fact that $\varphi$ satisfied (b). Therefore, in $\psi$, the edge $v_tv_{t+1}$ cannot have a conflict at the end-point $v_{t+1}$.

Hence, $v_tv_{t+1}$ has a conflict at the end-point $v_t$. We consider the two possible cases separately.

Case 1: $\psi(v_tv_{t+1}) = \gamma$.

In this case the edge $v_tv_{t+1}$ is an $\bar{\gamma}$-edge (if $t > 0$) or $i'$-edge (if $t = 0$) before the recoloring. Note that this means that $t+1$ is odd. If $t = 0$, then by the choice of $\gamma$, we have $\gamma \notin S_\varphi(v_0)$ and thus the edge $v_0v_1$ has no conflict at the end-point $v_0$ in $\psi$. For $t \geq 2$, since $\varphi(v_{t-1}v_t) = \gamma$, and $\varphi$ satisfied (c) and (b), we know that $\psi(v_t) = \varphi(v_t) \neq \gamma$ and that no edge incident at $v_t$ other than $v_{t-1}v_t, v_tv_{t+1}$ has the color $\gamma$ in $\psi$. Since in $\psi$, no two consecutive edges on $v_0P$ have the same color, $\psi(v_{t-1}v_t) \neq \gamma$, and so we can conclude that $v_tv_{t+1}$ has no conflict at the end-point $v_t$ in $\psi$ in this case.
Case 2: $\psi (v_tv_{t+1}) \neq \gamma$.

In this case, $\varphi (v_{t+1}v_{t+2}) = \psi (v_tv_{t+1}) \neq \gamma$, which implies that $v_{t+1}v_{t+2}$ is an $t$-edge before the recoloring. Then $t$ is odd and $\varphi (v_tv_{t+1}) = \gamma$. By (1), $\psi (v_t) = \varphi (v_t) = i$. Since $\psi (v_tv_{t+1}) \in A_i$ and $v_tv_{t+1}$ has a conflict at the end-point $v_t$, this implies that there exists some edge $v_ix \neq v_tv_{t+1}$ such that $\psi (v_ix) = \psi (v_tv_{t+1})$. As no two consecutive edges on $v_0P$ have the same color in $\psi$, we have that $x \neq v_{t-1}$. Then $\varphi (v_ix) = \psi (v_ix) = \psi (v_tv_{t+1})$. Thus, the three edges $v_{t-1}v_t$, $v_tx$, and $v_tv_{t+1}$ are all $A_i$-edges with respect to $\varphi$, and by property (d) have original colors in $[3, i]$. Since $\pi|_{E(G)}$ is a proper edge-coloring, all these three edges should have distinct colors with respect to $\pi$. This can happen only if $||[3, i]|| \geq 3$, which happens only if $i = 5$. Then with respect to $\pi$, $v_t$ is a 5-vertex with a 3-edge, a 4-edge, and a 5-edge all incident on it. This contradicts the fact that $\pi$ satisfies Property A.

Thus, we conclude that in $\psi$, no edge on $v_0P$ has conflict at either of its end-points, which also implies that $e = v_0v_1$ is not a conflicting edge in $\psi$. Thus $\psi$ satisfies (b), and since $e$ is the only new $A_i$-edge in $\psi$, property (c) holds for $\psi$. This completes the proof.

With the help of the above lemma, we can start with the original coloring $\pi$ and proceed with the recoloring in three phases. In each phase, we recolor all the conflicting $i$-edges by invoking Lemma 5.2 first with $i = 3$ in phase I, then $i = 4$ in phase II, and finally $i = 5$ in phase III. Note that after the phase corresponding to $i$, there are no conflicting $i$-edges in $G$. Therefore, after the three phases there are no conflicts in $G$. Thus, we have a total coloring of $G$ using colors from $\{1, 2, 3, \ldots, \Delta + 3\}$ implying that, $\chi''(G) \leq \Delta + 3$.

6 Open problems

1. Corollary 4.5 states that Hadwiger’s conjecture is true for graphs with high vertex-connectivity, that is, at least $C'$ where $C'$ is a constant.

   It is interesting to see whether a similar statement can be made with respect to edge-connectivity. Note that Theorem 4.4 uses only edge-connectivity. Hence, it would be natural to expect Corollary 4.5 to be expressed in terms of edge-connectivity. Unfortunately, we could not find a way to do it.

   Of course, it would be most desirable to prove Hadwiger’s conjecture for all total graphs dropping any constraints.

2. In Theorem 5.4, we show that if weak $TCC$ holds for all graphs, then Hadwiger’s conjecture is true for all total graphs. However, improving this result to get the following remains challenging.

   “If $(k)$-TCC is true for all graphs, then Hadwiger’s conjecture is true for all total graphs.”

   for some fixed positive integer $k > 3$. The easiest case to try out is when $k = 4$ and we think that this case itself would be interesting for it may require a new idea compared to the one used for proving the $k = 3$ case.

3. Since we could not prove Hadwiger’s conjecture for all total graphs, it is interesting to try to prove Hadwiger’s conjecture for total graphs of specialized graph classes for which weak $TCC$ is not yet known to hold.

4. It remains hard to prove weak $TCC$ for the general case. In Theorem 5.1, we proved it for 5-colorable graphs. The current best upper bound on the total chromatic number of $k$-colorable graphs when $6 \leq k \leq 8$ is $\Delta + 4$. Proving weak $TCC$ for $k$-colorable graphs even for small values of $k$, say $k = 6, 7, \ldots$, remains open.
5. In the proof of Theorem 5.1, \(\chi'(G)\) colors are used to do the initial proper edge coloring and an additional two colors are used to get a proper vertex coloring. This suggests the question whether it is possible, when \(\chi'(G) = \Delta(G)\) (that is, when \(G\) is class I), to obtain the bound \(\chi''(G) \leq \Delta(G) + 2\), instead of the \(\Delta(G) + 3\) bound that we proved in Theorem 5.1. Unfortunately, our proof fails to achieve this since even when dealing with class I graphs, we need one extra color, the \((\Delta + 3)\)-th color, to establish the Property A for the original coloring. Property A is crucial in phase III of the recoloring process, that is, when conflicting 5-edges are recolored (see proof of Lemma 5.2).

It would be interesting to prove, \(\chi''(G) \leq \Delta(G) + 2\) when \(G\) is a class I 5-colorable graph, possibly by some clever tweaking of the proof of Theorem 5.1.

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