Floquet Theory for Quaternion-valued Differential Equations

Dong Cheng∗a, Kit Ian Kou†a, and Yong Hui Xia ‡b

aDepartment of Mathematics, Faculty of Science and Technology, University of Macau, Macao, China
bDepartment of Mathematics, Zhejiang Normal University, Jinhua, China

Abstract

This paper describes the Floquet theory for quaternion-valued differential equations (QDEs). The Floquet normal form of fundamental matrix for linear QDEs with periodic coefficients is presented and the stability of quaternionic periodic systems is accordingly studied. As an important application of Floquet theory, we give a discussion on the stability of quaternion-valued Hill’s equation. Examples are presented to illustrate the proposed results.

Keywords: Floquet theory, periodic systems, quaternion, non-commutativity, Hill’s equation.

Mathematics Subject Classification (2010): 34D08, 34B30, 20G20.

1 Introduction

The theory of quaternion-valued differential equations (QDEs) has gained a prominent attention in recent years due to its applications in many fields, including spatial kinematic modelling and attitude dynamics [1, 2], fluid mechanics [3, 4], quantum mechanics [5, 6], etc. A feature of quaternion skew field is that the multiplication of quaternion numbers is noncommutative, this property brings challenges to the study of QDEs. Therefore, although QDEs appear in many fields, the mathematical researches in QDEs are not so many. Leo and Ducati [7] solved some simple second order quaternionic differential equations by using the real matrix representation of left/right acting quaternionic operators. Applying the topological degree methods, Campos and Mawhin [8] initiated a study of the T-periodic solutions of quaternion-valued first order differential equations. Later, Wilczynski [9, 10] presented some sufficient conditions for the existence of at least two periodic solutions of the quaternionic Riccati equation and the existence of at least one periodic solutions of the quaternionic polynomial equations. The existence of periodic orbits, homoclinic loops, invariant tori for 1D autonomous homogeneous QDE ˙q = aq^n, (n = 2, 3) was proposed by Gasull et al. [11]. The study of Zhang [12] is devoted to the global struture of 1D quaternion Bernoulli equations. Recently, the basic theory and fundamental results of linear QDEs was established by Kou and Xia [13, 14, 15]. They proved that the algebraic structure of the solutions to QDEs is different from the classical case. Moreover, for lack of basic theory such as fundamental

∗chengdong720@163.com
†kikou@umac.mo
‡xiadoc@163.com
theorem of algebra, Vieta’s formulas of quaternions, it is difficult to solve QDEs. In [13, 14, 15, 16], the authors proposed several new methods to construct the fundamental matrices of linear QDEs.

As a generalization, QDEs have many properties similar to ODEs. At the same time, for the relatively complicated algebraic structure of quaternion, one may encounter various new difficulties when studying QDEs.

1. Factorization theorem and Vieta’s formulas (relations between the roots and the coefficients) for quaternionic polynomials are not valid (see e. g. [17, 18, 19]).

2. A quaternion matrix usually has infinite number of eigenvalues. Besides, the set of all eigenvectors corresponding to a non-real eigenvalue is not a module (see e. g. [20, 21]).

3. The study of quaternion matrix equations is of intricacy (see e. g. [22, 23]).

4. Even the quaternionic polynomials are not “regular” (an analogue concept of holomorphic). This fact leads to noticeable difficulties for studying analytical properties of quaternion-valued functions (see e. g. [24, 9]).

Up to present, the theory of QDEs remains far from systemic. To the best of authors’ knowledge, there was virtually nonexistent study about the stability theory of QDEs. Based on this fact, we are motivated to investigate the stability of the linear QDEs

\[ \dot{x} = A(t)x \]  

where \( A \) is a smooth \( n \times n \) quaternion-matrix-valued function. In particular, we will focus on the important special cases where \( A \) is a quaternionic constant or periodic quaternion-valued function. In the real-valued systems, the well-known Floquet theory indicates that the case where \( A \) is a periodic matrix-valued function is reducible to the constant case (see e. g. [25, 26]). Floquet theory is an effective tool for analyzing the periodic solutions and the stability of dynamic systems. Owing to its importance, Floquet theory has been extended in different directions. Johnson [27] generalized the Floquet theory to the almost-periodic systems. In [28, 29, 30], the authors extended the Floquet theory to the partial differential equations. Recently, the Floquet theory has been extensively explored for dynamic systems on time scales (see e. g. [31, 32, 33, 34]).

As a continuation of [13, 14, 15], we generalize the Floquet theory to QDEs in this paper. Specifically, the contributions of this paper are summarized as follows.

1. We show that the stability of constant coefficient homogeneous linear QDEs is determined by the standard eigenvalues of its coefficient matrix.

2. Floquet normal form of the fundamental matrix for linear QDEs with periodic coefficients is presented.

3. The monodromy matrix, characteristic multiplier and characteristic exponent for QDEs are defined. Moreover, the stability of quaternionic periodic systems is discussed.

4. We propose some sufficient conditions for the existence of periodic solution of quaternionic periodic systems.

5. Without question, there are some results of ODEs are inevitably invalid for QDEs. We will discuss some of these results. Specifically, we will discuss the stability of quaternion-valued Hill’s equation.
The rest of the paper is organized as follow. In Section 2, some basic concepts of quaternion algebra are reviewed. Besides, several lemmas of quaternion matrices are derived. Section 3 is devoted to the stability of constant coefficient linear homogeneous QDEs. In Section 4, we establish the Floquet theory for QDEs. Specifically, Floquet normal form of the fundamental matrix for quaternionic periodic systems is presented. Some important concepts such as monodromy matrix, characteristic multiplier and characteristic exponent for QDEs are defined and the stability of quaternionic periodic systems is accordingly studied. The stability of quaternion-valued Hill’s equation is discussed in Section 5. Finally, conclusions are drawn at the end of the paper.

2 Preliminaries

2.1 Quaternion algebra

The quaternions were first described by Hamilton in 1843 [24]. The algebra of quaternions is denoted by
\[ \mathbb{H} := \{ q = q_0 + q_1 i + q_2 j + q_3 k \} \]
where \( q_0, q_1, q_2, q_3 \) are real numbers and the elements \( i, j \) and \( k \) obey Hamilton’s multiplication rules:
\[ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1. \]

For every quaternion \( q = q_0 + iq_1 + jq_2 + kq_3 \), the scalar and vector parts of \( q \), are defined as \( R(q) = q_0 \) and \( V(q) = q_1 i + q_2 j + q_3 k \), respectively. If \( q = V(q) \), then \( q \) is called pure imaginary quaternion. The quaternion conjugate is defined by
\[ q^* = q_0 - iq_1 - jq_2 - kq_3, \]
and the norm \( |q| \) of \( q \) defined as
\[ |q|^2 = q q^* = q_0^2 + q_1^2 + q_2^2 + q_3^2. \]

Using the conjugate and norm of \( q \), one can define the inverse of \( q \) by \( q^{-1} = q^* |q|^2 \). For each fixed unit pure imaginary quaternion \( \varsigma \), the quaternion has subset
\[ C_{\varsigma} := \{ a + b\varsigma : a, b \in \mathbb{R} \}. \]
The complex number field \( \mathbb{C} \) can be viewed as a subset of \( \mathbb{H} \) since it is isomorphic to \( \mathbb{C} i \). Therefore we will denote \( \mathbb{C} i \) by \( \mathbb{C} \) for simplicity.

2.2 Matrices of quaternions

The quaternion exponential function \( \exp(A) \) for \( A \in \mathbb{H}^{m \times n} \) is defined by means of an infinite series as
\[ \exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!}. \]

When \( n = 1 \) and \( A = q \in \mathbb{H} \), analogous to the complex case one may derive a closed-form representation:
\[ e^q = \exp(q) = e^{q_0} \left( \cos |V(q)| + \frac{V(q)}{|V(q)|} \sin |V(q)| \right). \]

Every quaternion matrix \( A \in \mathbb{H}^{m \times n} \) can be expressed uniquely in the form of
\[ A = A_1 + A_2 j, \quad \text{where} \quad A_1, A_2 \in \mathbb{C}^{m \times n}. \]

Then the complex adjoint matrix [35, 20] of the quaternion matrix \( A \) is defined as
\[ \chi_A = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}. \quad (2.1) \]
By using the complex adjoint matrix, the $q$-determinant of $A$ is defined by

$$|A|_q := |\chi_A|,$$  \hspace{1cm} (2.2)

where $|\cdot|$ is the conventional determinant for complex matrices. By direct computations, it is easy to see that $|A|_q = |A|^2$ when $A$ is a complex matrix.

From [13], we know that $\mathbb{H}^n$ over the division ring $\mathbb{H}$ is a right $\mathbb{H}$-module (a similar concept to linear space) and $\eta_1, \eta_2, \cdots, \eta_k \in \mathbb{H}^n$ are right linearly independent if

$$\eta_1 \alpha_1 + \eta_2 \alpha_2 + \cdots + \eta_k \alpha_k = 0, \alpha_i \in \mathbb{H} \text{ implies that } \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0.$$

Let $A \in \mathbb{H}^{n \times n}$, a nonzero $\eta \in \mathbb{H}^{n \times 1}$ is said to be a right eigenvector of $A$ corresponding to the right eigenvalue $\lambda \in \mathbb{H}$ provided that

$$A\eta = \eta \lambda$$

holds. A matrix $A_1$ is said to be similar to a matrix $A_2$ if $A_2 = S^{-1} AS$ for some nonsingular matrix $S$. In particular, we say that two quaternions $p, q$ are similar if $p = \alpha^{-1}q\alpha$ for some nonzero quaternion $\alpha$. By Theorem 2.2 in [20], we know that the similarity of quaternions defines an equivalence relation. The set

$$[q] := \{ p = \alpha^{-1}q\alpha : \alpha = \mathbb{H} \setminus \{0\} \}$$

is called an equivalence class of $q$. It is easy to see that $[q]$ can also be recognized by

$$[q] := \{ p \in \mathbb{H} : \mathcal{R}(p) = \mathcal{R}(q), |\mathcal{V}(p)| = |\mathcal{V}(q)| \}.$$ 

It follows that any equivalence class $[q]$ has one and only one complex-valued element with nonnegative imaginary part.

We recall some basic results about quaternion matrices which can be found, for instance, in [20, 36, 21].

**Theorem 2.1** Let $A \in \mathbb{H}^{n \times n}$, then the following statements hold.

1. $A$ has exactly $n$ right eigenvalues (including multiplicity) which are complex numbers with nonnegative imaginary parts. These eigenvalues are called standard eigenvalues of $A$.

2. If $A$ is a complex matrix and its eigenvalues are $\lambda_1 = \alpha_1 + i|\beta_1|, \lambda_2 = \alpha_2 + i|\beta_2|, \ldots, \lambda_n = \alpha_n + i|\beta_n|$ (repeated according to their multiplicity). Then the standard eigenvalues of $A$ are $\tilde{\lambda}_1 = \alpha_1 + i|\beta_1|, \tilde{\lambda}_2 = \alpha_2 + i|\beta_2|, \cdots, \tilde{\lambda}_n = \alpha_n + i|\beta_n|$. In particular, $|\tilde{\lambda}_j| = |\lambda_j|$ for $j = 1, 2, \cdots, n$.

3. $A$ is invertible if and only if $\chi_A$ is invertible.

4. If $A$ is (upper or lower) triangular, then the only eigenvalues are the diagonal elements (and the quaternions similar to them).

Let $\Omega$ be the totality of all $2n \times 2n$ partitioned complex matrices which have form of (2.1). It has been shown in [37, 21] that $\Omega$ is closed under addition, multiplication and inversion. Furthermore, each $A \in \mathbb{H}^{n \times n}$ has a Jordan form in $\mathbb{C}^{n \times n}$.

**Lemma 2.2** [37] Let $A, B \in \mathbb{H}^{n \times n}$. Then $\chi_A + \chi_B = \chi_{A+B} \in \Omega$ and $\chi_A \chi_B = \chi_{AB} \in \Omega$. Moreover, if $A$ is invertible, then $\chi_A^{-1} = \chi_{A^{-1}} \in \Omega$. 

Lemma 2.3 [37, 21] Let $A \in \mathbb{H}^{n \times n}$. Then there exists a $P \in \mathbb{H}^{n \times n}$ such that

$$
\chi_P^{-1} \chi_A \chi_P = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}
$$

is a Jordan canonical form of $\chi_A$, where and $J \in \mathbb{C}^{n \times n}$ has all its diagonal entries with nonnegative imaginary parts. Consequently, $P^{-1}AP = J$ is a Jordan canonical form of $A$ in $\mathbb{C}^{n \times n}$.

Remark 2.4 The diagonal entries of $J$ are actually the standard eigenvalues of $A$.

If $\lambda$ is a standard eigenvalue of $A \in \mathbb{H}^{n \times n}$, its algebraic multiplicity is defined by the number of its occurrences in the Jordan canonical form $J$. Since the totality of solutions for $A \eta = \eta \lambda$ is not a $\mathbb{H}$-module. Thus we could not use dimensionality of ‘eigenspace’ to define the geometric multiplicity for $\lambda$. Note that $\lambda$ is a eigenvalue of $\chi_A$ and motivated by Lemma 2.3, we may define the geometric multiplicity for the standard eigenvalues of quaternion matrices as follows.

Definition 2.5 Let $\lambda$ be a standard eigenvalue of $A \in \mathbb{H}^{n \times n}$, the geometric multiplicity for $\lambda$ is defined as the dimensionality of the (complex) linear space $\{x \in \mathbb{C}^n : (J - \lambda I)x = 0\}$, where $J$ is the Jordan canonical form of $A$ in $\mathbb{C}^{n \times n}$.

Employing above lemmas, it is not difficult to verify that $\Omega$ is also closed under exponential.

Lemma 2.6 Let $A, C \in \mathbb{H}^{n \times n}$, where $C$ is invertible. Then $e^{\chi_A} = \chi e^A \in \Omega$ and there is a $B \in \mathbb{H}^{n \times n}$ such that $e^B = C$.

Proof. By Lemma 2.3, there is a $P \in \mathbb{H}^{n \times n}$ such that $P^{-1}AP = J \in \mathbb{C}^{n \times n}$. Observe that $\exp(J) = \exp(\chi)$ and therefore

$$
\chi_P^{-1}e^{\chi_A} \chi_P = e^{\chi_P^{-1} \chi_A \chi_P}
$$

$$
= e^{\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}} = e^{\begin{pmatrix} e^J & 0 \\ 0 & e^J \end{pmatrix}} = \chi e^J.
$$

Hence $e^{\chi_A} = \chi P e^J \chi_P^{-1} = \chi P e^J P^{-1} = \chi e^{PJP^{-1}} = \chi e^A$.

For quaternion matrix $C$, there is a $S \in \mathbb{H}^{n \times n}$ such that $S^{-1}CS = K \in \mathbb{C}^{n \times n}$. Since $C$ is invertible, then $K$ is nonsingular. Moreover, there exists a complex matrix $D$ such that $K = e^D$ by Theorem 2.82 in [25]. Therefore

$$
\chi C = \chi S \chi K \chi_S^{-1}
$$

$$
= \chi S \chi e^D \chi_S^{-1} = \chi S e^{\chi D} \chi_S^{-1} = e^{\chi S \chi D} \chi^{-1} = e^{\chi S \chi D} \chi^{-1} = \chi e^{S \chi D} \chi^{-1}.
$$

Thus $C = e^{S \chi D \chi^{-1}}$. Set $B = S \chi D \chi^{-1}$, we complete the proof.

By Lemma 2.6 and Theorem 2.1, we obtain the following spectral mapping theorem.

Theorem 2.7 If $A \in \mathbb{H}^{n \times n}$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the standard eigenvalues of $A$ repeated according to their multiplicity, then $e^{\lambda_1}, e^{\lambda_2}, \cdots, e^{\lambda_n}$ are the standard eigenvalues of $e^A$, where $\tilde{\lambda}_j$ $(j = 1, 2, \cdots, n)$ is defined by

$$
\tilde{\lambda}_j := \begin{cases} 
\lambda_j, & \text{if } e^{\lambda_j} \text{ has nonnegative imaginary part;} \\
\overline{\lambda_j}, & \text{otherwise.}
\end{cases}
$$
The norm respectively defined by \( \| \cdot \| \) whenever submultiplicativity holds, that is

\[
\| \phi(t, t_0, x_0) \| < \delta \quad \text{implies} \quad \| \phi(t, t_0, x_0) - \phi(t, t_0, x_0) \| < \epsilon \quad \text{for all} \quad t \geq t_0.
\]

The solution \( \phi(t, t_0, x_0) \) is called asymptotically stable if there is a \( \delta > 0 \) such that \( \sup_{t \geq t_0} \| \phi(t, t_0, x_0) \| < \delta \). Thus it is permissible to simply say that system \((A, B) = (\xi_1, \xi_2, \ldots, \xi_n)^T \in \mathbb{R}^n \), the norm of \( A \) and \( \eta \) are respectively defined by

\[
\| A \| = \sum_{i,j=1}^{n} |a_{ij}|, \quad \| \eta \| = \sum_{k=1}^{n} |\eta_k|.
\]

The norm \( \| \cdot \| \) defined for \( A \) is a matrix norm. It is easy to verify that for any \( A, B \in \mathbb{R}^{n \times n} \), the submultiplicativity holds, that is

\[
\| AB \| \leq \| A \| \| B \|.
\]

By similar arguments to Theorem 1.1 in [38], we see that the stability of zero solution of (1.1) implies the stability of any other solutions. Thus it is permissible to simply say that system (1.1) is stable (or unstable).

**Theorem 3.2** Let \( M(t) \) be a fundamental matrix of (1.1). Then the system (1.1) is stable if and only if \( \| M(t) \| \) is bounded. The system (1.1) is asymptotically stable if and only if \( \lim_{t \to \infty} \| M(t) \| = 0 \).

**Proof.** Let \( L \) be an upper bound for \( \| M(t) \| \), \( L_1 = \| M^{-1}(t_0) \| \) and \( \phi(t, t_0, \xi) \) be the solution of (1.1) with \( \phi(t_0, t_0, \xi) = \xi = (\xi_1, \xi_2, \ldots, \xi_n)^T \). Then \( \phi(t_0, t_0, \xi) = M(t)M^{-1}(t_0)\xi \). For any \( \epsilon > 0 \), let \( \delta = \frac{\epsilon}{LL_1} \), then \( \| \phi(t, t_0, \xi) - 0 \| = \| M(t)M^{-1}(t_0)\xi \| \leq LL_1 \| \xi \| < \epsilon \) whenever \( \| \xi \| < \delta \). If for any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( \| M(t)M^{-1}(t_0)\xi \| < \epsilon \) for \( \| \xi \| < \delta \). Then

\[
\| M(t)M^{-1}(t_0) \| = n \| M(t)M^{-1}(t_0)\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T \|
\]

\[
\leq n \sup_{\| \eta \| \leq 1} \| M(t)M^{-1}(t_0)\eta \|
\]

\[
= n \sup_{\| \xi \| \leq \delta} \| M(t)M^{-1}(t_0)\delta^{-1}\xi \|
\]

\[
< n\epsilon\delta^{-1}
\]

Therefore \( \| M(t) \| < n\epsilon\delta^{-1}L_1^{-1} \) is bounded.
If \( \lim_{t \to \infty} \| M(t) \| = 0 \). Then \( \| \phi(t, t_0, \xi) - 0 \| = \| M(t)M^{-1}(t_0)\xi \| = \| M(t)\| L_1 \| \xi \| \) tends to 0 as \( t \to \infty \) whenever \( \| \xi \| < \delta \). Conversely, it is easy to see that if the zero solution is asymptotically stable, then \( \| M(t)\| \) has to be convergent to 0 as \( t \to \infty \).

By using the Jordan canonical form of \( A \in \mathbb{H}^{n \times n} \), we can obtain a matrix representation for \( e^{tA} \). Let \( P \) be a quaternion matrix such that \( P^{-1}AP = J \in C^{n \times n} \), then \( P^{-1}e^{tA}P = e^{tP^{-1}AP} = e^{tJ} \). Let \( \lambda_1, \lambda_2, \cdots, \lambda_k \) be the distinct standard eigenvalues of \( A \) that correspond to multiplicities \( n_1, n_2, \cdots, n_k \), respectively. Then \( J = \text{diag}(J_1, J_2, \cdots, J_k) \) where \( J_i = \lambda_i I + N_i \) with \( N_i^{n_i} = 0 \). Thus we have that

\[
e^{tJ} = e^{t(\lambda_1 I + N_1)} = e^{t\lambda_1} e^{tN_1} = e^{t\lambda_1} \left( I + tN_i + \frac{t^2}{2!}N_i^2 + \cdots + \frac{t^{n_i-1}}{(n_i-1)!}N_i^{n_i-1} \right).
\]

Note that \( e^{tJ} = \text{diag}(e^{tJ_1}, e^{tJ_2}, \cdots, e^{tJ_k}) \), then we obtain an explicit matrix representation for \( e^{tA} = Pe^{tJ}P^{-1} \). Moreover, this representation has a similar form with the cases where \( A \) is a real or complex matrix. Hence by similar arguments to Theorem 4.2 in [26], we have the following theorem.

**Theorem 3.3** The system \( \dot{x} = Ax \)

1. is stable if and only if the standard eigenvalues of \( A \) all have non-positive real parts and the algebraic multiplicity equals the geometric multiplicity of each standard eigenvalue with zero real part;

2. is asymptotically stable if and only if all the standard eigenvalues of \( A \) have negative real parts.

**Remark 3.4** Since any two similar quaternions possess the same scalar part, thus the phrase ”standard” in Theorem 3.3 can be removed.

**Example 3.5** Consider the system \( \dot{x} = Ax \), where

\[
A = \begin{pmatrix} i & j & j \\ k & 1 & k \\ 0 & 0 & 1 \end{pmatrix}
\]

The principal fundamental matrix at \( t = 0 \) (\( M(0) = I \)) is given by

\[
M(t) = \begin{pmatrix} 1-\frac{i}{2} + \frac{1+i}{2} \gamma_1 & \frac{k-j}{2} + \gamma_2 & \frac{j}{2} \gamma_3 + \gamma_4 - e^{t} \\ \frac{1+i}{2} - \frac{j}{2} \gamma_2 & i \gamma_3 - \frac{j}{2} \gamma_4 - (1-j-k) e^{t} & e^{t} \\ 0 & 0 & e^{t} \end{pmatrix},
\]

where \( \gamma_1 = e^{(1+i)t} \), \( \gamma_2 = \frac{i-k}{2} e^{(1-i)t} \), \( \gamma_3 = \frac{k-1-i-j}{2} \), \( \gamma_4 = e^{(1+i)t} \frac{1-i-j-k}{2} \). By straightforward computations, we have the result shown in Table 1.

**Table 1: Description of Example 3.5**

| Fundamental matrix | The standard eigenvalues of \( A \) | Stability |
|--------------------|-----------------------------------|-----------|
| \( \lim_{t \to \infty} \| M(t) \| = \infty \) | \( \lambda_1 = 0, \mathcal{R}(\lambda_1) = 0; \lambda_2 = 1, \mathcal{R}(\lambda_2) > 0; \lambda_3 = 1+i, \mathcal{R}(\lambda_3) > 0 \) | unstable |
Example 3.6 Consider the system \( \dot{x} = Ax \), where

\[
A = \begin{pmatrix}
i & 1 & 0 \\
0 & j & 0 \\
0 & 1 & k
\end{pmatrix}
\]

The principal fundamental matrix at \( t = 0 \) is given by

\[
M(t) = \begin{pmatrix}
e^{it} & \frac{t}{2} \left( e^{it} - ke^{-it} \right) + \frac{1+k}{2} \sin t & 0 \\
0 & e^{jt} & 0 \\
0 & \frac{t}{2} \left( e^{jt} + ike^{jt} \right) + \frac{1-i}{2} \sin t & e^{kt}
\end{pmatrix}.
\]

By straightforward computations, we have the result shown in Table 2. Notice that the standard fundamental matrix \( \|M(t)\| \) is unbounded and the eigenvalues of \( A \) are \( \lambda_1 = \lambda_2 = \lambda_3 = i \); \( \Re(\lambda_1) = 0 \); unstable.

### Table 2: Description of Example 3.6

| Fundamental matrix | The standard eigenvalues of \( A \) | Stability |
|--------------------|-------------------------------|-----------|
| \( \|M(t)\| \) is unbounded | \( \lambda_1 = \lambda_2 = \lambda_3 = i \); \( \Re(\lambda_1) = 0 \); unstable |

The eigenvalue \( \lambda = i \) has zero real part, we need to show its algebraic multiplicity is less than its algebraic multiplicity 3. By some basic calculations, we find a quaternion matrix

\[
P = \begin{pmatrix}
-1 + i & -2i & -k \\
0 & -2i - 2j & 0 \\
1 - i - j - k & -1 + i & 0
\end{pmatrix}
\]

such that

\[
P^{-1}AP = \begin{pmatrix}
i & 0 & 0 \\
0 & i & 1 \\
0 & 0 & i
\end{pmatrix}.
\]

This implies that the algebraic multiplicity of \( \lambda = i \) is 2.

Example 3.7 Consider the system \( \dot{x} = Ax \), where

\[
A = \begin{pmatrix}
-1 + 2j - k & -1 + 2i + j \\
-i + j + 2k & -2 - i + k
\end{pmatrix}
\]

The principal fundamental matrix \( M(t) \) at \( t = 0 \) is given by

\[
 \left( \begin{array}{ccc}
 3+i+j-k & 2-j-k & 1-i+2k \\
 1-i+2j & 3-i-j-k & 1+2i+j \\
 1+i+2j & 3-i-j-k & 1+i+2k
\end{array} \right) \gamma_1 + \left( \begin{array}{ccc}
 3+i+j-k & 2-j-k & -1+3i-j-k \\
 1-i+2j & 3-i-j-k & 2i+j-k \\
 1+i+2j & 3-i-j-k & 1+i+2k
\end{array} \right) \gamma_2 \]

where \( \gamma_1 = e^{-(3+3i)t} \) and \( \gamma_2 = e^{(3i-3)t} \). By straightforward computations, we have the result shown in Table 3.

Example 3.8 Consider the system \( \dot{x} = Ax \), where

\[
A = \begin{pmatrix}
-1 + i - k & -i \\
1 + i - j + k & -2 - k
\end{pmatrix}
\]
Table 3: Description of Example 3.7

| Fundamental matrix | The standard eigenvalues of $A$ | Stability          |
|--------------------|---------------------------------|--------------------|
| $\|M(t)\|$ is bounded | $\lambda_1 = 0, \Re(\lambda_1) = 0$; $\lambda_2 = -3 + 3i, \Re(\lambda_1) < 0$ | stable but not asymptotically stable |

The principal fundamental matrix $M(t)$ at $t = 0$ is given by

$$
\begin{pmatrix}
\frac{1-i}{2}e^{it} + \frac{k-j}{2}e^{-it} & e^{-2t} + \frac{1+i+j-k}{2}e^{-t} \\
(1+i)(1-e^{(j-1)t})e^{-t} & \frac{1-i}{2}e^{(i-2)t} + \frac{1+i}{2}e^{-t}
\end{pmatrix}
$$

By straightforward computations, we have the result shown in Table 4.

Table 4: Description of Example 3.8

| Fundamental matrix | The standard eigenvalues of $A$ | Stability          |
|--------------------|---------------------------------|--------------------|
| $\lim_{t\to\infty} \|M(t)\| = 0$ | $\lambda_1 = -1, \Re(\lambda_1) < 0$; $\lambda_2 = -1 + \frac{i}{2}, \Re(\lambda_1) < 0$ | asymptotically stable |

4 Floquet theory for QDEs

We consider the quaternionic periodic systems

$$\dot{x} = A(t)x$$

(4.1)

where $A(t)$ is a $T$-periodic continuous quaternion-matrix-valued function. The following Floquet’s theorem gives a canonical form for fundamental matrices of (4.1).

**Theorem 4.1** If $M(t)$ is a fundamental matrix of (4.1). Then

$$M(t + T) = M(t)M^{-1}(0)M(T).$$

In addition, it has the form

$$M(t) = P(t)e^{tB}$$

(4.2)

where $P(t)$ is a $T$-periodic quaternion-matrix-valued function and $B$ satisfying

$$e^{TB} = M^{-1}(0)M(T).$$

**Proof.** Since $M(t)$ is a fundamental matrix of (4.1) and $A(t + T) = A(t)$, then

$$\dot{M}(t + T) = A(t + T)M(t + T) = A(t)M(t + T).$$

That means $M(t + T)$ is also a fundamental matrix. Therefore, there is a nonsingular quaternion matrix $C$ such that $M(t + T) = M(t)C$. By Lemma 2.6, there is a quaternion matrix $B$ such that $C = e^{TB}$. 

9
Let \( P(t) := M(t)e^{-tB} \), then
\[
P(t + T) = M(t + T)e^{-T \cdot tB} = M(t)Ce^{TB}e^{-tB} = M(t)e^{-tB} = P(t)
\]
and \( M(t) = P(t)e^{tB} \). By letting \( t = 0 \), we have \( e^{TB} = C = M^{-1}(0)M(T) \) which completes the proof. \( \square \)

**Remark 4.2** In the above proof, we used the fact that if \( A_1, A_2 \in \mathbb{H}^{n \times n} \) are commutable then \( e^{A_1}e^{A_2} = e^{A_1+A_2} \). We know that this assertion is true for complex matrices. We now verify that this result is also valid for quaternion matrices.

If \( A_1, A_2 \) are commutable, so are \( \chi_{A_1} \chi_{A_2} \). By applying Lemma 2.2 and 2.6, we have that
\[
\chi_{e^{A_1}e^{A_2}} = \chi_{e^{A_1}} \chi_{e^{A_2}} = e^{\chi_{A_1}} e^{\chi_{A_2}} = e^{\chi_{A_1}+\chi_{A_2}} = \chi_{e^{A_1}+A_2}.
\]
It follows that \( e^{A_1}e^{A_2} = e^{A_1+A_2} \).

**Corollary 4.3** Suppose that \( M_1(t), M_2(t) \) are fundamental matrices of (4.1) and \( e^{TB_1} = M_1^{-1}(0)M_1(T), e^{TB_2} = M_2^{-1}(0)M_2(T) \). Then \( e^{TB_1}, e^{TB_2} \) are similar and therefore they have the same standard eigenvalues.

**Proof.** Let \( M_0(t) \) be the fundamental matrix such that \( M_0(0) = I \), then \( M_1(t) = M_0(t)M_1(0) \) and \( M_2(t) = M_0(t)M_2(0) \) for every \( t \in \mathbb{R} \). Therefore \( M_1(T)M_1^{-1}(0) = M_2(T)M_2^{-1}(0) = M_0(t) \). Note that both \( M_1^{-1}(0)M_1(T) \) and \( M_2^{-1}(0)M_2(T) \) are similar with \( M_0(T) \). Thus \( e^{TB_1}, e^{TB_2} \) are similar and they possess the same standard eigenvalues. \( \square \)

The representation (4.2) is called a Floquet normal form for the fundamental matrix \( M(t) \). From this normal form, we accordingly define several concepts for quaternionic periodic system (4.1) as follows.

- For any fundamental matrix \( M(t) \), \( e^{TB} = M^{-1}(0)M(T) \) is called a monodromy matrix of (4.1). By Corollary 4.3, we see that any two monodromy matrices are similar.
- The standard eigenvalues of any monodromy matrix are called characteristic multipliers of (4.1). The totality of characteristic multipliers is denoted by \( CM \).
- A complex number \( \mu \) is called a characteristic exponent of (4.1), if \( \rho \) is a characteristic multiplier and \( e^{\mu T} = \rho \). The totality of characteristic exponents is denoted by \( CE \).

**Theorem 4.4** Consider system (4.1), suppose that \( M(t) = P(t)e^{tB} \) is a Floquet norm form for the fundamental matrix \( M(t) \). Let \( \mu_1, \mu_2, \ldots, \mu_n \) be the standard eigenvalues of \( B \). Then \( \tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_n \) are characteristic exponents, where \( \tilde{\mu}_j \) (\( j = 1, 2, \ldots, n \)) is defined by
\[
\tilde{\mu}_j := \begin{cases} 
\mu_j, & \text{if } e^{\mu_j T} \text{ has nonnegative imaginary part;} \\
\overline{\mu_j}, & \text{otherwise.}
\end{cases}
\]
If \( \mu \) is a characteristic exponent of (4.1), then there exists \( 1 \leq k \leq n \) such that \( \{e^{\mu_k T} \cap \{e^{\mu_k T}, e^{\overline{\mu_k T}} \} \neq \emptyset \) and \( \Re(\mu) = \Re(\mu_k) \).

**Proof.** If \( \mu_1, \mu_2, \ldots, \mu_n \) are the standard eigenvalues of \( B \), from Theorem 2.7, \( e^{\tilde{\mu}_j T} \) (\( j = 1, 2, \ldots, n \)) is a standard eigenvalue of \( e^{TB} \). That is, \( \{e^{\tilde{\mu}_j T} : j = 1, 2, \ldots, n \} = CM \). Therefore \( \tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_n \)
are characteristic exponents. If $\mu$ is a characteristic exponent, then $\rho = e^{\mu T}$ is a standard eigenvalue of $e^{TB}$. Hence there exists $1 \leq k \leq n$, such that $\rho = e^{\tilde{\mu}_k T}$. It follows that
$$\{e^{\mu T}\} \cap \{e^{\tilde{\mu}_k T}, e^{\tilde{\mu}_k T}\} \neq \emptyset,$$
and
$$e^{\mathcal{R}(\mu)T} = |e^{\mu T}| = |e^{\tilde{\mu}_k T}| = e^{\mathcal{R}(\tilde{\mu}_k)T} = e^{\mathcal{R}(\mu_k)T}.$$ 
Thus $\mathcal{R}(\mu) = \mathcal{R}(\mu_k)$. $\square$

As an immediate consequence of Theorem 4.4, we have the following result.

**Corollary 4.5** Consider system (4.1), Let $M(t) = P(t)e^{TB}$ be a Floquet norm form for the fundamental matrix $M(t)$. Then
$$\{\mathcal{R}(\mu) : \mu \in CE\} = \{\mathcal{R}(\mu) : \mu \in \sigma(B)\}$$
where $\sigma(B)$ is the totality of the standard eigenvalues of $B$.

**Theorem 4.6** If $\rho_j = e^{\mu_j T}, j = 1, 2, \cdots, n$, are the characteristic multipliers of (4.1), then
$$\prod_{j=1}^{n} |\rho_j| = \exp \left( \int_0^T \mathcal{R}(\text{tr}A(\tau)) d\tau \right), \quad (4.3)$$
$$\mathcal{R} \left( \sum_{j=1}^{n} \mu_j \right) = \frac{1}{T} \left( \int_0^T \mathcal{R}(\text{tr}A(\tau)) d\tau \right). \quad (4.4)$$

**Proof.** Let $M(t)$ be a fundamental matrix of (4.1), by Liouville’s formula of QDEs (see [14]), we have
$$|M(t)|_q = \exp \left( 2 \int_{t_0}^{t} \mathcal{R}(\text{tr}A(\tau)) d\tau \right) |M(t_0)|_q. \quad (4.5)$$
Note that $\rho_j, j = 1, 2, \cdots, n$, are the standard eigenvalues of $M(T)M^{-1}(0)$, by the definition of $q$-determinant, we have
$$|M(T)|_q |M(0)|^{-1}_q = |M(T)M^{-1}(0)|_q = \prod_{j=1}^{n} |\rho_j|^2.$$ 
Taking $t = T, t_0 = 0$ in (4.5), we obtain
$$\prod_{j=1}^{n} |\rho_j|^2 = \exp \left( 2 \int_0^T \mathcal{R}(\text{tr}A(\tau)) d\tau \right),$$
and therefore (4.3) holds. Observe that $|\rho_j| = |e^{\mu_j T}| = e^{\mathcal{R}(\mu_j)T}$, then (4.3) implies that
$$\exp \left( \mathcal{R} \left( \sum_{j=1}^{n} \mu_j \right) T \right) = \exp \left( \int_0^T \mathcal{R}(\text{tr}A(\tau)) d\tau \right).$$
This proves the theorem. $\square$
If $\rho = e^{\mu T}$, where $\rho, \mu$ are complex numbers. Since $|\rho| = |e^{\mu T}| = e^{\Re(\mu) T}$, it is easy to see that the following assertions hold.

- $|\rho| = 1$ if and only if $\Re(\mu) = 0$.
- $|\rho| < 1$ if and only if $\Re(\mu) < 0$.
- $|\rho| > 1$ if and only if $\Re(\mu) > 0$.

The next result demonstrates that the stability of (4.1) is equivalent to the stability of the linear system with constant coefficients $\dot{y} = By$, where $B$ stems from the Floquet normal form (4.2).

**Theorem 4.7** Let $M(t) = P(t)e^{tB}$ is a Floquet norm form for the fundamental matrix $M(t)$ of (4.1). Then the following assertions hold.

1. The system (4.1) is stable if and only if the standard eigenvalues of $B$ all have non-positive real parts and the algebraic multiplicity equals the geometric multiplicity of each standard eigenvalue with zero real part; or equivalently, the characteristic multipliers of (4.1) all have modulus not larger than 1 ($\leq 1$) and the algebraic multiplicity equals the geometric multiplicity of each characteristic multiplier with modulus one.

2. The system (4.1) is asymptotically stable if and only if the standard eigenvalues of $B$ all have negative real parts; or equivalently, the characteristic multipliers of (4.1) all have modulus less than 1.

**Theorem 4.8** If $\mu$ is a characteristic exponent and $\rho = e^{\mu T}$ is a characteristic multiplier of (4.1), then there is a nontrivial solution of the form

$$x(t) = p(t)e^{\mu t}.$$ 

Moreover $p(t + T) = p(t)$ and $x(t + T) = x(t)\rho$.

**Proof.** Let $M(t) = P(t)e^{tB}$ is a Floquet norm form for the principal fundamental matrix $M(t)$ at $t = 0$. By Theorem 4.4, there is a standard eigenvalue $\mu_1$ of $B$ such that

$$\{e^{\mu T}\} \cap \{e^{\mu_1 T}, e^{\mu T}\} \neq \emptyset.$$ 

Without loss of generality, we assume that $\rho = e^{\mu T} = e^{\mu_1 T}$. Then there exists a $k \in \mathbb{Z}$ such that $\mu_1 = \mu + \frac{2k\pi i}{T}$. Let $\eta \neq 0$ be an eigenvector of $B$ corresponding to $\mu_1$. It follows that $B\eta = \eta \mu_1$ and therefore $e^{tB}\eta = \eta e^{\mu_1 t}$. Thus the solution $x(t) := M(t)\eta$ can also be represented in the form

$$x(t) = P(t)e^{tB}\eta = P(t)\eta e^{\frac{2k\pi i}{T} e^{\mu t}}.$$ 

Let $p(t) = P(t)\eta e^{\frac{2k\pi i}{T}}$. It is easy to see that $p(t)$ is a $T$-periodic function. Moreover

$$x(t + T) = p(t + T)e^{\mu(T+t)} = p(t)e^{\mu t}e^{\mu T} = x(t)\rho.$$ 

This completes the proof. \qed
**Theorem 4.9** If μ is a complex number, \( p(t + T) = p(t) \), and \( x(t) = p(t)e^{t\mu} \neq 0 \) is a nontrivial solution of (4.1), then one of \( \mu, \overline{\mu} \) is a characteristic exponent.

**Proof.** Let \( M(t) = P(t)e^{tB} \) be a Floquet norm form for the principal fundamental matrix \( M(t) \) at \( t = 0 \) and \( \eta = p(0) \), then \( \eta \neq 0 \). Otherwise, \( x(t) \equiv 0 \) is the trivial solution by uniqueness of solution. Note that both \( p(t)e^{t\mu} \) and \( P(t)e^{tB}\eta \) are solutions of (4.1) with the same initial value at \( t = 0 \), therefore

\[
p(t)e^{t\mu} = P(t)e^{tB}\eta
\]  

(4.6)

Taking \( t = T \) in (4.6) and note that \( p(T) = p(0) = \eta \), \( P(T) = P(0) = I \) by periodicity. It follows that

\[\eta e^{\mu T} = e^{TB}\eta.\]

Hence \( e^{\mu T} \) is a complex-valued eigenvalue of \( e^{TB} \). Thus, one of \( e^{\mu T}, e^{\overline{\mu} T} \) is a characteristic multiplier of (4.1). Therefore, one of \( \mu, \overline{\mu} \) is a characteristic exponent of (4.1). □

Next result is a direct consequence of Theorem 4.8 and 4.9.

**Corollary 4.10** There is a \( T \)-periodic solution of (4.1) if and only if there is a zero characteristic exponent; or equivalently, there is a characteristic multiplier \( \rho = 1 \). If there is a characteristic exponent of the form \( \mu = \frac{2k+1}{T}\pi i \) for some \( k \in \mathbb{Z} \), or equivalently, there is a characteristic multiplier \( \rho = -1 \), then there is a \( 2T \)-periodic solution of (4.1).

The following result shows that different characteristic multipliers will generate linearly independent solutions.

**Corollary 4.11** Assume that \( \mu_1, \mu_2 \) are characteristic exponents of (4.1) satisfying \( \rho_1 = e^{\mu_1 T}, \rho_2 = e^{\mu_2 T} \). If the characteristic multipliers \( \rho_1, \rho_2 \) are not equal, then there are \( T \)-periodic functions \( p_1(t), \ p_2(t) \) such that

\[x_1(t) = p_1(t)e^{\mu_1 t}\]

and

\[x_2(t) = p_2(t)e^{\mu_2 t}\]

are linearly independent solutions of (4.1).

**Proof.** Let \( M(t) = P(t)e^{tB} \) be a Floquet norm form for the principal fundamental matrix \( M(t) \) at \( t = 0 \) and \( \eta_1 = x_1(0), \ \eta_2 = x_2(0) \). By similar arguments of Theorem 4.9, we conclude that \( \eta_1, \ \eta_2 \) are eigenvectors of \( B \) corresponding to the standard eigenvalues \( \rho_1, \rho_2 \) respectively. Note that \( \rho_1 \neq \rho_2 \). It follows that \( x_1(0) \) and \( x_2(0) \) are linearly independent and therefore \( x_1(t) \) and \( x_2(t) \) are linearly independent solutions of (4.1). □

**Example 4.12** Consider the system (4.1), where \( A(t) \) is \( \pi \)-periodic function and given by

\[
A(t) = \begin{pmatrix} 1 & 1 \\ 0 & i + 2e^{2it}\end{pmatrix}
\]

Then the principal fundamental matrix is

\[
M(t) = \begin{pmatrix} e^t & -1+i-j-k \cdot e^{jt}\ \\
0 & -\frac{j}{20} \cdot e^{3i} e^{2jt} + \frac{-1-3i+2j+4i+2k}{10} e^{jt} \end{pmatrix}
\]
By straightforward computations, we have \( \lim_{t \to \infty} ||M(t)|| = \infty \). That is, \( ||M(t)|| \) is unbounded. Thus this system is unstable by Theorem 3.2. Observe that \( M(0) = I \) and

\[
M(\pi) = \begin{pmatrix} e^{\pi} & \frac{3-i+4j+2k}{10} (1 + e^{\pi}) \\ 0 & -1 \end{pmatrix}.
\]

Therefore the characteristic multipliers are \( \rho_1 = e^{\pi}, \rho_2 = -1 \). From Lemma 2.6, there is a quaternion-valued matrix

\[
B = \begin{pmatrix} 1 & \frac{1-2i+j+3k}{3} \\ 0 & \frac{5}{i} \end{pmatrix}
\]

such that \( M(\pi) = e^{\pi B} \). Applying the definition of exponential function,

\[
e^{tB} = \begin{pmatrix} e^{t} & \frac{3-i+4j+2k}{10} (e^{t} - e^{it}) \\ 0 & e^{it} \end{pmatrix}
\]

Then we obtain the Floquet norm form \( P(t)e^{tB} \) for \( M(t) \), where \( P(t) \) is given by

\[
P(t) = \begin{pmatrix} 1 & \frac{3-i+4j+2k}{10} + \frac{-1+i-j-k}{4} e^{it} e^{-it} + \frac{-1-3i-3j+k}{20} e^{3it} e^{-it} \\ 0 & \cos 2t + e^{2it} j \sin 2t \end{pmatrix}
\]

It is easy to see that \( P(t) \) is \( \pi \)-periodic as required. The standard eigenvalues of \( B \) are \( \mu_1 = 1, \mu_2 = i \) and the corresponding eigenvectors are \( \eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \eta_2 = \begin{pmatrix} -\frac{7+i+10j}{10} \\ 2+i \end{pmatrix} \).

Note that \( \mu_1, \mu_2 \) are characteristic exponents. By Theorem 4.8, there are two nontrivial solutions

\[
x_1(t) = M(t)\eta_1 = p_1(t)e^t \quad \text{and} \quad x_2(t) = M(t)\eta_2 = p_2(t)e^{it},
\]

where \( p_1(t), p_2(t) \) are \( \pi \)-periodic functions given by

\[
p_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad p_2(t) = \begin{pmatrix} -\frac{1+i-j-k}{4} e^{it} e^{-it} (2+i) + \frac{-1-3i-3j+k}{20} e^{3it} e^{-it} (2+i) \\ (2+i) \cos 2t + 2e^{2it} (j+k) \sin 2t \end{pmatrix}.
\]

By Corollary 4.10, \( x_2(t) \) is a \( 2\pi \)-periodic solution. To provide a direct description of the system, Table 5 is presented to visualize its properties.

| Fundamental matrix | Characteristic multipliers | The standard eigenvalues of \( B \) | Stability |
|---------------------|-------------------------|-----------------------------|----------|
| \( \|M(t)\| \) is unbounded | \( \rho_1 = e^{\pi}, |\rho_1| > 1; \rho_2 = -1, |\rho_2| = 1 \) | \( \mu_1 = 1, R(\mu_1) > 0; \mu_2 = i, R(\mu_2) = 0 \) | unstable |
Example 4.13 Consider the system (4.1), where \( A(t) \) is \( \pi \)-periodic function and is given by

\[
A(t) = \begin{pmatrix} k & 1 \\ 0 & i + 2e^{2it}j \end{pmatrix}
\]

Then the principal fundamental matrix \( M(t) \) is

\[
\begin{pmatrix}
e^{kt} & \frac{1-i-j+k}{4}e^{it}t + \frac{1+i-j-k}{4}e^{3jt}t + \frac{1-i-j+k}{16}(e^{3jt} - e^{-jt}) \\ 0 & \end{pmatrix}
\]

By straightforward computations, \( \|M(t)\| \) is unbounded. Thus this system is unstable by Theorem 3.2.

Observe that \( M(0) = I \) and \( M(\pi) = \begin{pmatrix} -1 & -1-i-j+k \\ 0 & -1 \end{pmatrix} \).

Therefore the characteristic multipliers are \( \rho_1 = \rho_2 = -1 \). There is a quaternion-valued matrix \( B = \begin{pmatrix} -i & 1+i+j+k \\ 0 & -k \end{pmatrix} \)

such that \( M(\pi) = e^{\pi B} \). The standard eigenvalues of \( B \) are \( \mu_1 = \mu_2 = i \). To provide a direct description of the system, Table 6 is presented to visualize its properties. By some basic calculations, we obtain the Jordan canonical form of \( B \):

\[
J = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}
\]

This implies that the geometric multiplicity of \( \mu = i \) is 1, which is less than its algebraic multiplicity.

| Fundamental matrix | Characteristic multipliers | The standard eigenvalues of \( B \) | Stability |
|--------------------|---------------------------|-----------------------------------|-----------|
| ||\( M(t) \)|| is unbounded | \( \rho_1 = \rho_2 = -1; \) \( |\rho_1| = |\rho_2| = 1 \) | \( \mu_1 = \mu_2 = i; \) \( R(\mu_1) = R(\mu_2) = 0 \) | unstable |

Example 4.14 Consider the system (4.1), where \( A(t) \) is \( \pi \)-periodic function and given by

\[
A(t) = \begin{pmatrix} k & e^{-2it} \\ 0 & i + 2j \cos 2t + 2k \sin 2t \end{pmatrix}
\]

Then the principal fundamental matrix \( M(t) \) is

\[
\begin{pmatrix}
e^{kt} & \frac{2+2i+5j-5k}{21}e^{-2it} + \frac{2+2i+3j+3k}{3}e^{2it} - \frac{1+2i+2j-k}{4}e^{3jt} + \frac{1+6i-6j-k}{35}e^{3jt} \\ 0 & \end{pmatrix}
\]

It is easy to see that \( \|M(t)\| \) is bounded but is not convergent to zero as \( t \) tends to infinity. Thus this system is stable (but not asymptotically) by Theorem 3.2. Observe that \( M(0) = I \) and \( M(\pi) = \begin{pmatrix} k & -\frac{3}{35} - \frac{12}{35}i + \frac{1}{9}j + \frac{2}{9}k \\ 0 & -1 \end{pmatrix} \).
Therefore the characteristic multipliers are \( \rho_1 = i \), \( \rho_2 = -1 \). There is a quaternion-valued matrix

\[
B = \begin{pmatrix}
\frac{k}{2} & 33 - 76i - 12j + 104k \\
0 & 105i
\end{pmatrix}
\]

such that \( M(\pi) = e^{\pi B} \). The standard eigenvalues of \( B \) are \( \mu_1 = \frac{i}{2} \), \( \mu_2 = i \). To provide a direct description of the system, Table 7 is presented to visualize its properties.

| Fundamental matrix | Characteristic multipliers | The standard eigenvalues of \( B \) | Stability |
|---------------------|---------------------------|----------------------------------|-----------|
| \( \|M(t)\| \) is unbounded but not convergent to 0 | \( \rho_1 = i \neq -1 = \rho_2; \) \( |\rho_1| = |\rho_2| = 1 \) | \( \mu_1 = \frac{i}{2} \neq i = \mu_2; \) \( R(\mu_1) = R(\mu_2) = 0 \) | stable but not asymptotically |

**Example 4.15** Consider the system (4.1), where \( A(t) \) is \( \pi \)-periodic function and given by

\[
A(t) = \begin{pmatrix}
\frac{i}{2} - 1 & e^{2jt}e^{-k\sin 2t} \\
2k \cos 2t - 1 & 0
\end{pmatrix}
\]

Then the principal fundamental matrix \( M(t) \) is

\[
\begin{pmatrix}
e^{\frac{j}{2}t}e^{-t} & \frac{1}{6}(e^{-(1+2i)t} - e^{(\frac{j}{2} - 1)t})(i - j) + \frac{1}{3}(e^{(\frac{j}{2} - 1)t} - e^{(2i-1)t})(i + j) \\
0 & e^{-t}e^{k\sin 2t}
\end{pmatrix}
\]

It is easy to see that \( \lim_{t \to \infty} \|M(t)\| = 0 \). Thus this system is asymptotically stable by Theorem 3.2. Observe that \( M(0) = I \) and

\[
M(\pi) = \begin{pmatrix}
ie^{-\pi} & -\frac{2 - 2i - 8j + 8k}{15}e^{-\pi} \\
0 & \frac{15}{e^{-\pi}}
\end{pmatrix}
\]

Therefore the characteristic multipliers are \( \rho_1 = ie^{-\pi}, \rho_2 = e^{-\pi} \). There is a quaternion-valued matrix

\[
B = \begin{pmatrix}
\frac{i}{2} - 1 & -\frac{1 + 4k}{15} \\
0 & -1
\end{pmatrix}
\]

such that \( M(\pi) = e^{\pi B} \). The standard eigenvalues of \( B \) are \( \mu_1 = \frac{i}{2} - 1, \mu_2 = -1 \). To provide a direct description of the system, Table 8 is presented to visualize its properties.

| Fundamental matrix | Characteristic multipliers | The standard eigenvalues of \( B \) | Stability |
|---------------------|---------------------------|----------------------------------|-----------|
| \( \lim_{t \to \infty} \|M(t)\| = 0 \) | \( \rho_1 = ie^{-\pi}, |\rho_1| < 1; \rho_2 = e^{-\pi}, |\rho_2| < 1 \) | \( \mu_1 = \frac{i}{2} - 1, R(\mu_1) < 0; \mu_2 = -1, R(\mu_2) < 0 \) | asymptotically stable |

**Remark 4.16** Thanks to the assertion 2 of Theorem 2.1, the above results are coincide with the traditional results when \( A(t) \) is complex-valued.
5 Quaternion-valued Hill’s equations

For real-valued systems, the Floquet theory effectively depict the stability of Hill’s equation (see e.g. [25])

\[ \ddot{u} + a(t)u = 0, \ a(t) = a(t + T). \]

We will consider the quaternion case where \( a(t) \) is a quaternion-valued function. Let \( \mathbf{x} = (u, u')^T \), then quaternion-valued Hill’s equation is equivalent to the quaternionic periodic systems (4.1) with

\[ A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}. \]

Let \( M(t) \) be the principal fundamental matrix at \( t = 0 \). By Liouville’s formula of QDEs, we have

\[
|M(t)|_q = \exp \left( \frac{2}{T} \int_0^T R(\text{tr}A(\tau))d\tau \right) \left| M(0) \right|_q = 1.
\]

If \( a(t) \) is real-valued, then \( M(T) \) is a real-valued matrix. If \( \alpha = \alpha_1 + i\alpha_2 \) and \( \beta = \beta_1 + i\beta_2 \) are roots of the equation

\[
\lambda^2 - (\text{tr}M(T))\lambda + |M(T)|_q = \lambda^2 - (\text{tr}M(T))\lambda + 1 = 0.
\] (5.1)

Then \( \rho_1 = \alpha_1 + i|\alpha_2| \) and \( \rho_2 = \beta_1 + i|\beta_2| \) are characteristic multipliers of (4.1) and \( |\rho_1| = |\alpha|, \ |\rho_2| = |\beta| \). It is well-known that the stability of real-valued Hill’s equation depends on the value of \( \text{tr}M(T) \) (see e.g. [25]).

| Table 9: Description of Real-valued Hill’s equation |
|--------------------------------------------------|
| The value of \( \text{tr}M(T) \) | The roots of (5.1) | Stability of real-valued Hill’s equation |
|---------------------------------|-----------------|-------------------------------|
| \( \text{tr}M(T) < -2 \)       | \( \alpha < -1 < \beta < 0 \); \( |\alpha| = 1, \Im(\alpha) \neq 0 \); | unstable |
| \( -2 < \text{tr}M(T) < 2 \)   | \( \beta = \pi, |\alpha| = 1, \Im(\alpha) \neq 0 \); | stable but not asymptotically |
| \( \text{tr}M(T) = 2 \)        | \( \beta = \alpha = 1 \); | stable if and only if \( M(T) = I \) |
| \( \text{tr}M(T) > 2 \)        | \( 0 < \alpha < 1 < \beta \); | unstable |
| \( \text{tr}M(T) = -2 \)       | \( \beta = \alpha = -1 \); | stable if and only if \( M(T) = -I \) |

If \( a(t) \) is quaternion-valued, then \( M(T) \) is a quaternion matrix. Therefore we can not use (5.1) to find the characteristic multipliers (the standard eigenvalues of \( M(T) \)). In this case, \( \text{tr}M(T) \) is a quaternion. The structure of the set of zeros of quaternionic polynomials is more complicated than complex polynomials. It is natural to modify (5.1) to be

\[
\lambda^2 - R(\text{tr}M(T))\lambda + |M(T)|_q = 0.
\] (5.2)

This raises the question of whether the roots of (5.2) and characteristic multipliers possess the same absolute value. The answer is negative. This implies that even if we add \( R \) to the front of \( \text{tr}M(T) \), the stability of quaternion-valued Hill’s equation can not be determined by Table 9.

**Example 5.1** Consider the quaternion-valued Hill’s equation with

\[ a(t) = 2 + j\cos^2 2t + k\sin 2t. \]

Note that \( a(t) \) is a quaternion-valued \( \pi \)-periodic function. Based on the numerical methods, we obtain


\[ M(\pi) \approx \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}, \text{ where} \]

\[
\begin{align*}
m_1 &= -0.131186 + 0.037757i + 0.584454j - 0.418119k, \\
m_2 &= -0.607206 + 0.255374i - 0.025292j, \\
m_3 &= 1.900430 + 0.005637i + 0.173381j, \\
m_4 &= -0.131186 + 0.037757i + 0.584454j + 0.418119k.
\end{align*}
\]

Therefore, by direct computations, we have \( R(\text{tr}M(\pi)) \approx -0.262372 \in (-2, 2) \). The characteristic multipliers are \( \rho_1 \approx -0.197803 + 1.73905i \) and \( \rho_2 \approx -0.064569 + 0.567682i \). Note that \( |\rho_1| > 1 \), thus this equation is unstable. On the other hand, the roots of \( \lambda^2 - R(\text{tr}M(\pi)) + |M(\pi)|_q \approx \lambda^2 + 0.262372\lambda + 1 = 0 \) are \( \alpha \approx 0.131186 + 0.991358i \) and \( \beta = \pi \).

In fact, if \( \rho_1, \rho_2 \) are characteristic multipliers, we only have

\[
\begin{align*}
R(\rho_1) + R(\rho_2) &= R(\text{tr}M(T)), \\
|\rho_1||\rho_2| &= |M(T)|_q = 1.
\end{align*}
\]

If \( |R(\text{tr}M(T))| > 2 \), then one of \( |R(\rho_1)| \) and \( |R(\rho_2)| \) has to be larger than 1. In this case, the equation is unstable. By similar arguments, we could know the stability of quaternion-valued Hill’s equation when \( |R(\text{tr}M(T))| = 2 \). In summary, Table 10 is presented to visualize the stability of quaternion-valued Hill’s equation.

| The value of \( R(\text{tr}M(T)) \) | Stability of quaternion-valued Hill’s equation |
|-------------------------------|-----------------------------------------------|
| \( |R(\text{tr}M(T))| > 2 \) | unstable                                      |
| \( |R(\text{tr}M(T))| < 2 \) | undetermined                                  |
| \( R(\text{tr}M(T)) = 2 \)  | stable if and only if \( M(T) = I \)        |
| \( R(\text{tr}M(T)) = -2 \) | stable if and only if \( M(T) = -I \)        |

We use the following example to illustrate (5.3) and Table 10.

**Example 5.2** Consider the quaternion-valued Hill’s equation with \( a(t) = -1 + j \cos 2t + k \sin 2t \).

Based on the numerical methods, we obtain \( M(\pi) \approx \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \), where

\[
\begin{align*}
m_1 &= 13.6488 - 2.9075i - 1.1093j - 2.3529k, \\
m_2 &= 12.3192 - 2.2187i + 2.3529j, \\
m_3 &= 14.6721 - 2.2187i - 5.2605j, \\
m_4 &= 13.6488 - 2.9075i + 1.1093j + 2.3529k.
\end{align*}
\]

Therefore, by direct computation, we have the following result (Table 11).

For the case of \( |R(\text{tr}M(T))| < 2 \), the scalar part of \( \text{tr}M(T) \) is not enough to determine the stability of quaternion-valued Hill’s equation. To take the vector part of \( \text{tr}M(T) \) into account, however, we still can’t determine the stability of quaternion-valued Hill’s equation at this moment. This raises the
question: can we determine the stability of quaternion-valued Hill’s equation by $\text{tr}M(T)$ (including scalar and vector parts)? If yes, how to determine the stability of quaternion-valued Hill’s equation by $\text{tr}M(T)$?

Multiplying $M(T)$ by its conjugate transpose $M(T)^\dagger$ we construct a positive semidefinite matrix $K(T) := M(T)M(T)^\dagger$. It is easy to see that the eigenvalues of $K(T)$ are $\kappa_1 = |\rho_1|^2$, $\kappa_2 = |\rho_2|^2$. Note that $|\rho_1| |\rho_2| = 1$ and $\text{tr}K(T) = \|M(T)\|_F^2$ where $\|\cdot\|_F$ is the Frobenius norm. It follows that $\kappa_1, \kappa_2$ are solutions of $\lambda^2 - \|M(T)\|_F^2 \lambda + 1 = 0$. Then we have the following result.

**Theorem 5.3** If $\|M(T)\|_F^2 > 2$, then quaternion-valued Hill’s equation is unstable.

By direct computation, we have that $|\mathcal{R}(\text{tr}M(T))| < \|M(T)\|_F^2$. It turns out that $|\mathcal{R}(\text{tr}M(T))| > 2$ implies $\|M(T)\|_F^2 > 2$. In Example 5.1, $|\mathcal{R}(\text{tr}M(T))| = 0.262372 < 2$ and $\|M(T)\|_F^2 = 5.14637 > 2$. It means that the stability of Example 5.1 can be determined by Theorem 5.3. In fact, for some quaternion-valued Hill’s equations with $|\mathcal{R}(\text{tr}M(T))| < 2$, the corresponding $\|M(T)\|_F^2$ can be very large.

**Example 5.4** Consider the quaternion-valued Hill’s equation with $a(t) = -1 + je^{\cos 2t} + k \sin 2t$. Based on the numerical methods, we have the following result (Table 12).

| $|\mathcal{R}(\text{tr}M(\pi))| | \|M(T)\|_F^2 | \quad \kappa_1 | |\rho_1| > 1; \frac{\rho_2}{2} \approx 0.006425 + 0.024862i, |\rho_1| |\rho_2| \approx 1 | \quad$ unstable |
|-----------------|-----------------|-----------------|------------------|
| $|\mathcal{R}(\text{tr}M(\pi))| \approx 1.0394 < 2 | \kappa_1 | -1.03876 + 40.196i, |\rho_1| > 1; \frac{\rho_2}{2} \approx -0.006425 + 0.024862i, |\rho_1| |\rho_2| \approx 1 | 

### 6 Conclusions

The Floquet theory for QDEs is developed, which coincides with the classical Floquet theory when considering ODEs. The concepts of characteristic multipliers and characteristic exponents for QDEs are introduced. The newly obtained results are useful to determine the stability of quaternionic periodic systems. As an important example of applications of Floquet theory for QDEs, we discuss the stability of quaternion-valued Hill’s equation in detail. It is shown that some results of real-valued Hill’s equation are invalid for the quaternion-valued Hill’s equation. Throughout the paper, adequate examples are provided to support the results.

### References

[1] J. C. Chou, “Quaternion kinematic and dynamic differential equations,” *IEEE Trans. Robot. Autom.*, vol. 8, no. 1, pp. 53–64, 1992.
[2] S. Gupta, “Linear quaternion equations with application to spacecraft attitude propagation,” in *Aerospace Conference, 1998 IEEE*, vol. 1. IEEE, 1998, pp. 69–76.

[3] J. Gibbon, “A quaternionic structure in the three-dimensional Euler and ideal magnetohydrodynamics equations,” *Physica D: Nonlinear Phenomena*, vol. 166, no. 1, pp. 17–28, 2002.

[4] J. D. Gibbon, D. D. Holm, R. M. Kerr, and I. Roulstone, “Quaternions and particle dynamics in the euler fluid equations,” *Nonlinearity*, vol. 19, no. 8, p. 1969, 2006.

[5] S. L. Alder, “Quaternionic quantum field theory,” *Commun. Math. Phys.*, vol. 104, no. 4, pp. 611–656, 1986.

[6] S. L. Adler, *Quaternionic quantum mechanics and quantum fields*. Oxford Univ. Press, 1995.

[7] S. De Leo and G. C. Ducati, “Solving simple quaternionic differential equations,” *J. Math. Phys.*, vol. 44, no. 5, pp. 2224–2233, 2003.

[8] J. Campos and J. Mawhin, “Periodic solutions of quaternionic-valued ordinary differential equations,” *Ann. Mat. Pura Appl.*, vol. 185, pp. S109–S127, 2006.

[9] P. Wileczyński, “Quaternionic-valued ordinary differential equations. the Riccati equation,” *J. Differ. Equ.*, vol. 247, no. 7, pp. 2163–2187, 2009.

[10] ——, “Quaternionic-valued ordinary differential equations ii. coinciding sectors,” *J. Differ. Equ.*, vol. 252, no. 8, pp. 4503–4528, 2012.

[11] A. Gasull, J. Llibre, and X. Zhang, “One-dimensional quaternion homogeneous polynomial differential equations,” *J. Math. Phys.*, vol. 50, no. 8, p. 082705, 2009.

[12] X. Zhang, “Global structure of quaternion polynomial differential equations,” *Commun. Math. Phys.*, vol. 303, no. 2, pp. 301–316, 2011.

[13] K. I. Kou and Y.-H. Xia, “Linear quaternion differential equations: Basic theory and fundamental results,” *Stud. Appl. Math.*, vol. 141, no. 1, pp. 3–45, 2018.

[14] K. I. Kou, W. K. Liu, and Y. H. Xia, “Linear quaternion differential equations: Basic theory and fundamental results II,” *arXiv preprint arXiv:1602.01660*, 2016.

[15] Y. H. Xia, H. Huang, and K. I. Kou, “An algorithm for solving linear nonhomogeneous quaternion-valued differential equations,” *arXiv preprint arXiv:1602.08713*, 2016.

[16] D. Cheng, K. I. Kou, and Y. H. Xia, “A unified analysis of linear quaternion dynamic equations on time scales,” *J. Appl. Anal. Comput.*, vol. 8, no. 1, pp. 172–201, 2018.

[17] S. Eilenberg and I. Niven, “The fundamental theorem of algebra for quaternions,” *Bull. Amer. Math. Soc.*, vol. 50, no. 4, pp. 246–248, 1944.

[18] R. Serôdio and L.-S. Siu, “Zeros of quaternion polynomials,” *Appl. Math. Lett.*, vol. 14, no. 2, pp. 237–239, 2001.

[19] A. Pogorui and M. Shapiro, “On the structure of the set of zeros of quaternionic polynomials,” *Complex Variables, Theory and Application: An International Journal*, vol. 49, no. 6, pp. 379–389, 2004.
[20] F. Zhang, “Quaternions and matrices of quaternions,” *Linear Alg. Appl.*, vol. 251, pp. 21–57, 1997.

[21] L. Rodman, *Topics in quaternion linear algebra*. Princeton University Press, 2014.

[22] Q.-W. Wang, H.-X. Chang, and Q. Ning, “The common solution to six quaternion matrix equations with applications,” *Appl. Math. Comput.*, vol. 198, no. 1, pp. 209–226, 2008.

[23] Q.-W. Wang and C.-K. Li, “Ranks and the least-norm of the general solution to a system of quaternion matrix equations,” *Linear Alg. Appl.*, vol. 430, no. 5, pp. 1626–1640, 2009.

[24] A. Sudbery, “Quaternionic analysis,” in *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 85, no. 02. Cambridge Univ Press, 1979, pp. 199–225.

[25] C. Chicone, *Ordinary Differential Equations with Applications*. Springer Science & Business Media, 2006.

[26] J. K. Hale, *Ordinary Differential Equations*. Dover Publications, 2009.

[27] R. A. Johnson, “On a Floquet theory for almost-periodic, two-dimensional linear systems,” *J. Differ. Equ.*, vol. 37, no. 2, pp. 184–205, 1980.

[28] S.-N. Chow, K. Lu, and J. Malletparet, “Floquet theory for parabolic differential equations,” *J. Differ. Equ.*, vol. 109, no. 1, pp. 147–200, 1994.

[29] P. Kuchment, *Floquet Theory for Partial Differential Equations*. Springer Science & Business Media, 1993.

[30] ———, “On the behavior of Floquet exponents of a kind of periodic evolution problems,” *J. Differ. Equ.*, vol. 109, no. 2, pp. 309–324, 1994.

[31] C. D. Ahlbrandt and J. Ridenhour, “Floquet theory for time scales and Putzer representations of matrix logarithms,” *J. Differ. Equ. Appl.*, vol. 9, no. 1, pp. 77–92, 2003.

[32] J. J. DaCunha and J. M. Davis, “A unified Floquet theory for discrete, continuous, and hybrid periodic linear systems,” *J. Differ. Equ.*, vol. 251, no. 11, pp. 2987–3027, 2011.

[33] R. Agarwal, V. Lupulescu, D. O’Regan, and A. Younus, “Floquet theory for a Volterra integro-dynamic system,” *Appl. Anal.*, vol. 93, no. 9, pp. 2002–2013, 2014.

[34] M. Adivar and H. C. Koyuncuoglu, “Floquet theory based on new periodicity concept for hybrid systems involving q-difference equations,” *Appl. Math. Comput.*, vol. 273, pp. 1208–1233, 2016.

[35] H. Aslaksen, “Quaternionic determinants,” *Math. Intell.*, vol. 18, no. 3, pp. 57–65, 1996.

[36] A. Baker, “Right eigenvalues for quaternionic matrices: a topological approach,” *Linear Alg. Appl.*, vol. 286, no. 1, pp. 303–309, 1999.

[37] F. Zhang and Y. Wei, “Jordan canonical form of a partitioned complex matrix and its applications to real quaternion matrices,” *Commun. Algebr.*, vol. 29, no. 6, pp. 2363–2375, 2001.

[38] V. N. Afanasiev, V. Kolmanovskii, and V. R. Nosov, *Mathematical theory of control systems design*. Springer Science & Business Media, 2013.