REALIZATIONS OF SEIFERT MATRICES BY HYPERBOLIC KNOTS

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Abstract. Recently Kearton showed that any Seifert matrix of a knot is $S$–equivalent to the Seifert matrix of a prime knot. We show in this note that such a matrix is in fact $S$–equivalent to the Seifert matrix of a hyperbolic knot. This result follows from reinterpreting this problem in terms of Blanchfield pairings and by applying results of Kawauchi.

1. Introduction

We say that a square integral matrix $A$ is of Seifert type if $\det(A - A^t) = 1$. Let $A$ be a square integral matrix, then for any column vector $v$ the matrices

\[
\begin{pmatrix}
A & 0 & 0 \\
v^t & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\text{ and }
\begin{pmatrix}
A & v & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

are called elementary enlargements of $A$. We also say that $A$ is an elementary reduction of any of its elementary enlargements. Two matrices are $S$–equivalent if they can be connected by a chain of elementary enlargements, elementary reductions and unimodular congruences.

Let $K \subset S^3$ be a knot and $F$ a Seifert surface. Given a basis for $H_1(F)$ we can then define the Seifert matrix $A$ of $K$. It is well–known that $A$ is of Seifert type. It is shown in [Mu65, Theorem 3.1] (cf. also [Le70, Theorem 1]) that the $S$–equivalence class of the Seifert matrix is a knot invariant.

It is well–known that any matrix of Seifert type is the Seifert matrix of a knot. In [Ke04] Kearton showed that any matrix of Seifert type is $S$–equivalent to the Seifert matrix of a prime knot.

In this note we prove the following:

**Theorem 1.1.** Let $A$ be a matrix of Seifert type, then there exist infinitely many hyperbolic knots $K_i, i \in \mathbb{N}$ such that $A$ is $S$–equivalent to a Seifert matrix of $K_i$.

The proof relies on a reformulation of the $S$–equivalence class in terms of Blanchfield pairings and on realization results of Kawauchi.
2. Proof of the theorem

2.1. S–equivalence and Blanchfield forms. Given a knot $K \subset S^3$ we write $X(K) = S^3 \setminus \nu K$, the knot exterior. In the following we let $\Lambda = \mathbb{Z}[t, t^{-1}]$ and $Q(\Lambda) = \mathbb{Q}(t)$ the quotient field of $\Lambda$. We view $\Lambda = \mathbb{Z}[t, t^{-1}]$ with the involution $p \mapsto \bar{p}$ induced by $t \mapsto t^{-1}$.

Consider the following sequence of $\Lambda$–homomorphisms

$$H_1(X(K); \Lambda) \xrightarrow{\cong} H_1(X(K), \partial X(K); \Lambda) \xrightarrow{\cong} H^2(X(K); \Lambda) \xrightarrow{\cong} \text{Ext}^1_{\Lambda}(H_1(X(K); \Lambda), \Lambda) \xleftarrow{\cong} \text{Hom}(H_1(X(K); \Lambda), Q(\Lambda)/\Lambda).$$

Here the first map comes from the long exact sequence of the pair $(X(K), \partial X(K))$, and is easily seen to be an isomorphism. The second homomorphism is Poincaré duality, the third homomorphism comes from the universal coefficient spectral sequence (and is an isomorphism by [Le77, Proposition 3.2]) and finally the last homomorphism comes from the long exact Ext–sequence corresponding to the short exact sequence of coefficients

$$0 \to \Lambda \to Q(\Lambda) \to Q(\Lambda)/\Lambda \to 0.$$ 

This sequence of homomorphisms defines the Blanchfield pairing

$$\lambda(K) : H_1(X(K); \Lambda) \times H_1(X(K); \Lambda) \to Q(\Lambda)/\Lambda.$$ 

This pairing is non–singular and $\Lambda$–hermitian. Furthermore if $A$ is a Seifert matrix for $K$ of size $k \times k$, then the Blanchfield pairing is isometric to the pairing

$$\Lambda^k/(At - A^t)\Lambda^k \times \Lambda^k/(At - A^t)\Lambda^k \to Q(\Lambda)/\Lambda$$

$$(v, w) \mapsto \bar{v}(t-1)(At - A^t)^{-1}w.$$ 

In particular the $(S$–equivalence class of a) Seifert matrix determines the Blanchfield pairing of a knot. By [Tr73] (and also by comparing [Ke75] with [Le70]) the converse holds as well, more precisely, the following theorem holds true.

**Theorem 2.1.** Let $K_1, K_2 \subset S^3$ be knot. Then $K_1$ and $K_2$ have $S$–equivalent Seifert matrices if and only if the Blanchfield pairings $\lambda(K_1)$ and $\lambda(K_2)$ are isometric.

2.2. Kawauchi’s realization results. Before we continue we recall that the derived series $G(n), n \in \mathbb{N}$ of a group $G$ is defined inductively by $G(0) = G$ and $G(n+1) = [G(n), G(n)]$, the commutator of $G(n)$. We recall the following hyperbolic realization result by Kawauchi.

**Theorem 2.2.** Let $L \subset S^3$ be any link, then for any $V \in \mathbb{R}$ there exists a hyperbolic link $\tilde{L} \subset S^3$ together with a map $f : (S^3, \tilde{L}) \to (S^3, L)$ such that the following hold:

1. $\text{Vol}(S^3 \setminus \tilde{L}) > V$, 

2. $\tilde{L}$ is a link in the complement of a knot.

3. $f$ induces an isomorphism of the fundamental groups.

4. The induced map on the knot complements is an isomorphism.

5. The induced map on the knot complements is a homeomorphism.

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30. The induced map on the knot complements is a homeomorphism.
(2) the induced map \( \pi_1(S^3 \setminus \tilde{L})/\pi_1(S^3 \setminus \tilde{L})^{(n)} \to \pi_1(S^3 \setminus L)/\pi_1(S^3 \setminus L)^{(n)} \) is an isomorphism for any \( n \).

The theorem follows from the theory of almost identical imitations of Kawauchi. More precisely, the theorem follows from combining [Ka89b, Theorem 1.1] with [Ka89a, Properties I and V, p. 450] (cf. also [Ka89c]).

2.3. Conclusion of the proof of the theorem. Let \( K \subset S^3 \) be a knot and \( V \in \mathbb{R} \). Let \( \tilde{K} \) be as in Theorem 2.2. Since we can choose \( V \) arbitrarily large it follows from Theorem 2.1 that it is enough to show that the Blanchfield pairings \( \lambda(K) \) and \( \lambda(\tilde{K}) \) are isometric.

First note that by Theorem 2.2 (2), applied to \( n = 1 \), we have a commutative diagram

\[
\begin{array}{ccc}
\pi_1(X(\tilde{K})) & \xrightarrow{f_*} & \pi_1(X(K)) \\
\downarrow & & \downarrow \\
\mathbb{Z} & & \mathbb{Z}
\end{array}
\]

In particular we get induced maps \( H_i(X(\tilde{K}); \Lambda) \to H_i(X(K); \Lambda) \). Write \( X = X(K) \) and \( \tilde{X} = X(\tilde{K}) \). We then get the following commutative diagram

\[
\begin{array}{ccc}
H_1(\tilde{X}; \Lambda) & \to & H_1(X, \partial X; \Lambda) \to H^2(\tilde{X}; \Lambda) \to \text{Ext}_\Lambda^1(H_1(\tilde{X}; \Lambda), \Lambda) \xrightarrow{\sim} \text{Hom}(H_1(\tilde{X}; \Lambda), Q(\Lambda)/\Lambda) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_1(X; \Lambda) & \to & H_1(X, \partial X; \Lambda) \to H^2(X; \Lambda) \to \text{Ext}_\Lambda^1(H_1(X; \Lambda), \Lambda) \xrightarrow{\sim} \text{Hom}(H_1(X; \Lambda), Q(\Lambda)/\Lambda).
\end{array}
\]

This means that we get a commutative diagram

\[
\begin{array}{ccc}
H_1(X(\tilde{K}); \Lambda) & \times & H_1(X(\tilde{K}); \Lambda) \\
\downarrow & & \downarrow \\
H_1(X(K); \Lambda) & \times & H_1(X(K); \Lambda)
\end{array}
\]

But it follows from Theorem 2.2 (2), applied to \( n = 2 \), that the induced map \( H_1(X(\tilde{K}); \Lambda) \to H_1(X(K); \Lambda) \) is an isomorphism of \( \Lambda \)-modules. In particular \( \lambda(\tilde{K}) \) is isometric to \( \lambda(K) \).

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