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Rectangularization of the survival curve reconsidered:
The maximum inner rectangle approach

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Rectangularization of the survival curve—a key analytical framework in mortality research—relies on assumptions that have become partially obsolete in high-income countries due to mortality reductions among the oldest old. We propose refining the concept to adjust for recent and potential future mortality changes. Our framework, the ‘maximum inner rectangle approach’ (MIRA) considers two types of rectangularization. Outer rectangularization captures progress in mean lifespan relative to progress in maximum lifespan. Inner rectangularization captures progress in lifespan equality relative to progress in mean lifespan. Empirical applications show that both processes have generally increased since 1850. However, inner rectangularization has displayed country-specific patterns since the onset of sustained old-age mortality declines. Results from separating premature and old-age mortality, using the MIRA, suggest there has been a switch from reducing premature deaths to extending the premature age range; a shift potentially signalling a looming limit to the share of premature deaths.

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Introduction

Rectangularization of the survival curve is one of the key analytical frameworks in mortality research. We argue that the canonical understanding of rectangularization is outdated. Instead of abolishing the concept completely, we suggest extending the framework to accommodate recent developments in mortality.

While the process of rectangularization had been recognized earlier (e.g., Pearl and Miner 1935; Comfort 1956), Fries’ (1980) interpretation of the concept has arguably become the commonly accepted view. Mortality developments in high-income countries over recent decades have, however, rendered several of Fries’ assumptions obsolete. Most importantly, potential reductions in mortality among the oldest old were not anticipated in the classical rectangularization framework. Mortality for people aged 80 and above has been declining in a number of countries since the 1960s (see, e.g., Kannisto 1994; Kannisto et al. 1994; Vaupel 1997; Rau et al. 2008). This trend in turn has invalidated several (partly implicit) assumptions of Fries’ theory: the idea that life expectancy has ‘looming limits’ has been rejected (Oeppen and Vaupel 2002; Vallin and Meslé 2009), the contributions of improvements in premature mortality to increases in life expectancy have become negligible (Christensen et al. 2009), and Fries’ (1980, p. 130) assessment that ‘[...] there has been no detectable change in the number of people living longer than 100 years [...]’ has been disproven (e.g., Vaupel and Jeune 1995; Vaupel 2010). Although we have not witnessed any increase in maximum observed lifespan since the death of Jeanne Calment at age 122 in 1997 (Robine and Allard 1998), Fries’ (1980, p. 133) prediction that ‘human life span may not be fixed but may be slowly increasing, perhaps a month or so each century’ had already been exceeded by more than an order of magnitude with the increases that occurred between 1980 and 1997 (Wilmoth et al. 2000).

Furthermore, in Fries’ framework, life expectancy gains can only be generated by a decrease in lifespan variability (see, e.g., Nagnur 1986; Nusselder and
Mackenbach 1996). This compression would be completed when, ‘under ideal conditions’ (Fries 1980, p. 132), lifespans were scattered in a normal distribution around a mean of 85.6 years, with a standard deviation of about four years. Later, Fries (1989) assumed wider intervals. But since the second half of the twentieth century, both stagnating variability and increasing life expectancy have been observed simultaneously (Kannisto 1996; Bongaarts 2005; Canudas-Romo 2008; Bergeron-Boucher et al. 2015). This phenomenon is commonly labelled the ‘shifting of mortality’. Some authors go one step further and discuss the possibility of life expectancy gains in the presence of increasing variability in lifespans; this is called the ‘expansion of mortality’ (Myers and Manton 1984; Rothenberg et al. 1991; Cheung et al. 2005; Engelman et al. 2010, 2014). It is, however, clear that the almost constant difference between the mean and the modal age at death (Canudas-Romo 2010) are not compatible with normally distributed deaths across age, even with historically low levels of premature mortality.

It may come as a surprise that we do not suggest entirely discarding the concept of rectangularization. Indeed, we find the simplicity and intuitiveness of the concept appealing, and recognize that rectangularization is one of the few theoretical frameworks that incorporates the relationship between (average) length of life and lifespan variability. Thus, rather than rejecting the concept, we wish to extend it to incorporate recent mortality changes, as well as future developments.

The first step in extending rectangularization is to detach the framework from its static perspective. It needs to capture mortality changes dynamically at all ages, and not just in the premature age range. It would also be beneficial if a measurement approach could differentiate between and quantify changes in premature and old-age mortality. Furthermore, it would be desirable if an extension of Fries’ framework still allowed us to assess the potentially impending limits to lifespan.

The framework we propose, which we call the ‘maximum inner rectangle approach’ (MIRA), is designed to address these issues. Our approach uses two dimensions of rectangularization. We call the classical perspective ‘outer rectangularization’ because it relates the survival curve and, accordingly, life expectancy, to the ‘maximum living potential’. Hence, it compares the current experience with the current theoretical maximum, if everyone survived to and then died at the actual maximum lifespan. There is, however, another perspective that has so far largely been neglected, which we call ‘inner rectangularization’. Defined as the rectangle under the survival curve with the largest possible area, this perspective relates the current inequality in lifespan to current life expectancy. The basic idea is illustrated in Figure 1. The bold solid line denotes a hypothetical survival curve starting from a radix of one and reaching zero at the highest attainable age, \( \omega \). The dotted line denotes the classic reference used to estimate the advancement of rectangularization—the outer rectangle—which also expresses the maximum living potential. The dashed line depicts our newly proposed concept: the maximum inner rectangle.

**Maximum inner rectangle approach (MIRA)**

The MIRA is based on different areas and specific ages that will be introduced in the following section. Table 1 provides an overview of all MIRA quantities.

In the MIRA, we distinguish between inner and outer rectangularization. Outer rectangularization is the standard perspective of rectangularization, and captures progress in mean lifespan relative to progress in maximum lifespan. Hence, the outer frame of the survival curve serves as a reference point. We denote the area of the outer rectangle as \( \omega \), because it is determined by the maximum attainable age (\( \omega \)) and the radix of the survival function (\( l_0 \)), which we set to one. The maximum age should be able to move forwards or backwards depending on the underlying mortality development. In the empirical application we link \( \omega \) to a specific survival proportion, \( k \)—such as the age at which 1 per cent of the population is still alive—so that \( l_0 = k \).

In a population, \( \omega \) can be interpreted as the maximum living potential. It counts the hypothetical number of person-years that could be lived in a population if everyone survived to the maximum age and then died. In comparison, the actual number of person-years lived in a population corresponds to the area under the survival curve, and determines mean lifespan. The ratio of mean to maximum lifespan serves to capture the degree of outer rectangularization of the survival curve. Thus, we define the outer rectangle ratio (ORR) as

\[
\text{ORR} = \frac{\int_0^\omega l_0 \, da}{\omega} \leq \frac{e_0}{\omega},
\]  

with \( e_0 \) denoting life expectancy (or mean lifespan) and \( \omega \) maximum age. By definition, \( 0 \leq \text{ORR} \leq 1 \). The ratio relates the observed number of person-years lived in a population to the maximum person-
years possible. For example, if ORR = 0.8, then current living conditions are allowing the population to exploit 80 per cent of its current maximum living potential.

**Inner rectangularization** adds a new perspective. In contrast to the outer rectangle, we seek the largest rectangle under the survival curve. Any inner rectangle (IR) under the survival curve is defined horizontally by age, \( x \), and defined vertically by survival to that age, \( l_x \). Consequently, the corresponding area is \( \text{IR}_x = x \times l_x \). The first age derivative of \( \text{IR}_x \) then identifies the age, \( x^* \), that corresponds to the maximum inner rectangle (MIR) with an area of

\[
\text{MIR} = x^* \times l_{x^*}
\]

as the solution to

\[
\frac{d \text{IR}_x}{dx} = 0
\]

which simplifies to

\[
x^* = \frac{1}{\mu_{x^*}}.
\]

Although there is no closed form solution, \( x^* \) can be found numerically, given that

\[
\mu_x > 0 \quad \forall \ x \in [0, \omega],
\]

where \( \mu_x \) denotes the force of mortality. A proof for a unique maximum in the case of increasing mortality with age is included in the supplementary material (section A).

MIR counts the ‘maximum uniformly shared person-years’. It is determined by the maximum shared lifespan (\( x^* \)) and the survival proportion up to this lifespan (\( l_{x^*} \)). At ages below \( x^* \), the share of the population living for \( x \) years (\( l_x \)) would be larger than \( l_{x^*} \), but the number of years lived per individual would be smaller than \( x^* \). Likewise, at ages above \( x^* \), the number of years lived per individual would be larger than \( x^* \), but the share of the population living for \( x \) years (\( l_x \)) would be smaller than \( l_{x^*} \). In either case, the total number of uniformly shared person-years, as indicated by MIR, would be reduced.

Using this definition of MIR allows us to add an inner perspective to the process of rectangularization. In an analogy to the maximum living potential (\( \omega \)), we can interpret life expectancy (\( e_0 \)) as a population’s current theoretical maximum number of life years that could be shared uniformly. Accordingly, with perfectly uniform lifespans, 100 per cent of individuals in a population would share a lifespan of length \( e_0 \). With actual lifespan inequality as measured by MIR, however, a maximum survival fraction of \( l_{x^*} < 100 \) per cent shares a uniform lifetime of at most \( x^* \) years. Thus, by relating MIR to \( e_0 \), we define inner rectangularization as the process of a population approaching its current lifespan equality potential. It is measured by the inner rectangle ratio (IRR), which is given by

\[
\text{IRR} = \frac{\text{MIR}}{e_0}.
\]

The IRR captures a trend that differs from that of the ORR, because changes in the MIR do not require a change in the mean or the maximum lifespan. Indeed, the trend could be characterized by a constant mean and a falling maximum lifespan, or by an increasing mean but a constant maximum lifespan, or even by a falling mean and a falling maximum lifespan. Though closely related, the IRR differs from...
the ORR because it is essentially an index of lifespan equality, while the ORR is an index of exploiting maximum living potential. Accordingly, if IRR = 0.8, then current living conditions are allowing the population to exploit 80 per cent of its current lifetime equality potential.

The two indices can be combined into a single index to measure total rectangularization. We define the total rectangle ratio (TRR) as

\[ \text{TRR} = \frac{\text{TRR}}{\omega} = \frac{\text{MIR}}{\omega} \quad (7) \]

The TRR measures achieved lifespan equality in relation to maximum possible equality. Accordingly, if TRR = 0.8, then current living conditions are allowing the population to achieve 80 per cent of its maximum possible lifespan equality at present.

**Data and estimation procedure**

We computed MIRA quantities using period life tables, which we estimated from death counts and corresponding exposures from the Human Mortality Database (2015). In this paper, we choose to highlight the trajectories of Swedish, Danish, and Italian females because these countries provide three exemplary mortality developments. Furthermore, all three countries have sufficient data coverage over time. In estimating \( x^* \) and \( l_x^* \), a key challenge we faced was that the data are only available in discrete integer units, but \( x \) and \( l_x \) need to be continuous. Therefore, we estimated \( x^* \) in two steps. First, we smoothed the product of \( x \) and \( l_x \) with cubic splines using R’s `splinefun()` function (R Core Team 2015), which allowed us to evaluate the function value with arbitrary precision. Second, we used R’s general-purpose univariate optimization function `optimize()` to find the maximum. A similar two-step approach with splines has been used previously in mortality research to estimate the modal age at death (Ouellette and Bourbeau 2011). We calculated other age estimates, such as \( \omega \) and the threshold ages discussed in the next section, using the same procedure.

In several empirical studies on rectangularization, the maximum age (\( \omega \)) is not set at the actual age at which there are no survivors left in the life table population. Wilmoth and Horiuchi (1999), for instance, set the cut-off age at the point at which 0.1 per cent of the population were still alive. Rossi et al. (2013) used the 10 per cent threshold and, most recently, Schalkwijk et al. (2016) used the 0.1, 1, and 10 per cent thresholds. In our study, we opted for a threshold of 1 per cent. Sensitivity analysis revealed that the actual choice of value for \( l_\omega \) had only minor effects on the results. As the maximum age changes with varying survival fractions, the estimates of TRR and ORR change quantitatively. However, the patterns of the ratios remain stable over time.

**Illustration of the inner, the outer, and the total rectangle ratio**

Figure 2 shows the three ratios for females in Italy (upper left panel), Denmark (upper right), and Sweden (lower). It depicts the IRR (black line), the ORR (light grey), and the TRR (dark grey). Figure 2 illustrates the following key points:

1. The TRR and the ORR have been developing almost in parallel for about 160 years. This suggests that Fries’ concept of rectangularization needs to be revised. If Fries’ ideas were correct, we would have expected to witness a ‘catching-up period’ of the TRR to the ORR until his ‘ideal conditions’ with

### Table 1 Quantities of the maximum inner rectangle approach (MIRA)

| Name                  | Acronym | Expression             | Interpretation                                                                 |
|-----------------------|---------|------------------------|-------------------------------------------------------------------------------|
| Inner rectangle       | IR \(_x\) | \( x \times l_x \)   | Age-specific uniformly shared person-years (PY)                              |
| Maximum shared lifespan | \( x^* \) | \( \max[x \times l_x] \) | Maximum number of uniformly shared life years by largest number of survivors |
| Maximum proportion    | \( l_x^* \) | \( l_x^* \)           | Largest proportion alive at the maximum shared lifespan                       |
| Maximum inner rectangle | MIR     | \( x^* \times l_x^* \) | Population’s current maximum number of uniformly shared PY                    |
| Life expectancy       | \( e_0 \) | \( \int_0^\omega l_x \, dx \) | Population’s current number of PY, i.e., mean lifespan                        |
| Outer rectangle ratio | ORR     | \( \omega \times l_\omega \) | Maximum possible PY                                                           |
| Inner rectangle ratio | IRR     | \( \text{MIR}/e_0 \)  | Proportion of uniformly shared PY from all PY lived                           |
| Total rectangle ratio | TRR     | \( \text{MIR}/\omega \) | Proportion of uniformly shared PY lived of maximum possible PY                |
life expectancy of about 85 years were reached. None of the selected countries has reached this level of life expectancy yet. Consequently, we should see a continued narrowing of the gap, but this is not the case.

(2) Inner rectangularization describes a different dimension of mortality progress. Its trajectory is decoupled from those of the ORR and the TRR, mainly because its frame of reference is not maximum lifespan, but life expectancy. In each of the examples we can see an increase over time, with a trend change occurring sometime during the 1950s when the slope becomes shallower. This break can likely be attributed to the shift in survival improvements from younger to older ages (e.g., Christensen et al. 2009).

(3) We can see that the IRR in each of these three countries has followed a different trajectory over the last half century, with a steady increase in Italy, a slow increase in Sweden, and a slight dip in Denmark during the 1970s and 1980s. Even though the IRR has generally increased over time, the country-specific patterns suggest that the

\[ \text{Figure 2} \quad \text{Inner rectangle ratio, outer rectangle ratio, and total rectangle ratio for females, Italy (1872–2012), Denmark (1850–2011), and Sweden (1850–2014)} \]

Note: All calculations are based on period life tables in the respective year.
Source: Human Mortality Database (2015).
forces behind this development vary. For instance, the steady increase in Italy suggests that a rise in life expectancy accompanies a faster growth of the MIR. The almost stagnating IRR in Sweden between 1960 and 1990 suggests that life expectancy has increased similarly to the MIR. Denmark’s unusual dynamics suggest a declining MIR while life expectancy stagnated.

(4) The IRR highlights how evenly the magnitude of and trends in age-specific mortality changes are spread over age. We claim that this provides a new perspective on lifespan variability. Our suggestion is strengthened by the correlation between the IRR and other summary measures of lifespan variability. Table 1 in the supplementary material (section B) shows that the IRR is less correlated with common measures of lifespan variability (shown in Table 2 in the supplementary material) than those measures are correlated with each other. This pattern is especially pronounced for the time period when gains in premature survival were instrumental for the increase in life expectancy (the period 1850–1950 in our analysis).

**Applying the MIRA to separate premature from old-age mortality**

Premature and old-age mortality are terms that are frequently used in mortality research, but they are loosely defined, which may be sufficient for many applications. However, in analysing rectangularization, these definitions are a crucial issue. In Fries’ description, premature mortality plays a central role. He argues that declines in premature mortality drive the process of rectangularization, and implicitly assumes that only these improvements are generating life expectancy increases. Although, Fries remains unclear in his definition of premature ages, his descriptions most evidently suggest that he is referring to life expectancy as a threshold. We argue that $x^*$ can be interpreted as an age that allows us to separate premature from old-age mortality. With $x^*$, we provide an approach that is embedded within our rectangularization framework, and which quantifies the threshold. Accordingly, the threshold in MIRA is based on the longest lifespan that is shared by the largest fraction of the population.

Even though existing approaches, such as those by Zhang and Vaupel (2009) and Gillespie et al. (2014), rest on different lifespan variability measures, their respective threshold ages result from a proportional perturbation of age-specific mortality and, hence, they rely on the same perturbation/definition. In both approaches, the threshold refers to a specific age, such that proportional mortality reductions before this age would result in a decrease in lifespan variability, whereas reductions at higher ages would lead to an increase in lifespan variability (see section F of the supplementary material for more details). Hence, $x^*$ could serve as an alternative definition of a threshold age separating premature and old-age mortality, based on the maximum shared lifespan.

Figure 3(a) illustrates the relationship between $x^*$ (horizontal axis) and $l_{x^*}$ (vertical axis); that is, the coordinates to measure the number of maximum uniformly shared person-years; again, this is shown for females in Italy, Denmark, and Sweden. The grey contour lines depict the number of life years lived in the MIR. The two time periods 1850–1950 and 1951–2014 are illustrated by dashed and solid lines, respectively. Generally, two trends can be distinguished in Figure 3(a): a ‘vertical’ development (until 1950) and a ‘horizontal’ development (after 1950).

The share of the life table population dying at older ages is denoted by $l_{x^*}$. Consequently, $1 - l_{x^*}$ equals the proportion dying prematurely. Premature mortality improvements drove progress until 1950, as illustrated by the increasing share of survivors ($l_{x^*}$). With improving old-age mortality, $x^*$ shows an accelerated movement towards higher ages; whereas the corresponding survival fraction at $x^*$ ($l_{x^*}$) shows only small gains. This pattern is similar to that of the modal age at death, which is also robust to mortality changes at lower ages, but sensitive to changes at higher ages (Canudas-Romo 2010). To compare the trajectories resulting from our approach with alternative approaches, we also plot the relationship between the threshold ages proposed by Gillespie et al. (2014; Figure 3(b)) and Zhang and Vaupel (2009; Figure 3(c)), which are based on the variance and the number of life years lost, respectively, and the corresponding survival proportions. These estimates show similar shifts in the trend. The slopes of the curves in each of the three parts of Figure 3 have become shallower in recent decades. This development could point to the existence of a limit to premature mortality that cannot be lowered any further. Fries (1980) argued that premature mortality would be almost eliminated,
declining to a maximum share of 2–5 per cent of all deaths. The three measures presented here suggest that there is a limit of at least 10–15 per cent (the minimum is from the MIRA measure).

It should, however, be noted that we use different vertical scales within Figure 3. If we instead used the scale from our measure in Figure 3(a) for the other two measures, we would obtain almost horizontal lines for those measures. Within the 160 years of life expectancy development contained in the figure, the proportion dying at old age has changed relatively little under the threshold ages suggested by Zhang and Vaupel (2009) and Gillespie et al. (2014). In both cases, the change amounts to less than 15 percentage points; a shift we consider to be rather small. In contrast, our measure shows a shift of about 65 percentage points, from 20 per cent dying at old age in 1850 to about 85 per cent dying.

**Figure 3** Scatterplot of the threshold age and survival proportion at the threshold age according to: (a) MIRA; (b) Gillespie et al. (2014); and (c) Zhang and Vaupel (2009), women, Italy (1872–2012), Denmark (1850–2011), and Sweden (1850–2014).

Notes: The trend lines are based on a locally weighted smoothing to highlight the patterns only. Additionally, contour lines in panel (a) visualize the corresponding number of person-years lived in equality (MIR), since this is determined by the product of both. The approaches of Zhang and Vaupel (2009) and Gillespie et al. (2014) and their calculation are explained in more detail in the supplementary material.

Source: As for Figure 2.
at old age in the most recent years. These numbers seem to be more in line with the findings of, for example, Christensen et al. (2009), who estimated that almost 80 per cent of recent gains in life expectancy for Japanese women were caused by survival improvements among older people.

**Discussion and conclusion**

Rectangularization is one of the established analytical frameworks in mortality research. We propose refining the classical concept to adjust for recent changes in survival improvements, and to allow for the incorporation of anticipated mortality trajectories in the near future. This new framework, which we call the maximum inner rectangle approach (MIRA), rests on two theoretically distinct types of rectangularization: inner rectangularization and outer rectangularization.

**Outer rectangularization** relates the number of life years that are currently lived (i.e., life expectancy) to a theoretical maximum where everyone dies at the same (maximum) age. We extend this standard definition of rectangularization by introducing the concept of **inner rectangularization**. This novel concept corresponds to the largest rectangle *under* the survival curve. This rectangle captures the largest number of life years lived by the largest proportion of the population. Thus, it measures the proportion of lifespan equality at the current level of life expectancy. By contrast, outer rectangularization measures the degree of living potential exploited, using maximum lifespan as a reference point.

The measurement of both constituent parts of the MIRA rests on simple ratios. To measure outer rectangularization, we use the well-known concept of the moving rectangle (Wilmoth and Horiuchi 1999; Rossi et al. 2013; Schalkwijk et al. 2016). As far as we know, there are no demographic predecessors to our concept of inner rectangularization. Thus, the age that maximizes the IR provides a novel point of reference, indicating maximum shared lifespan. This point represents the optimal trade-off between past lifetime and number of survivors in terms of lived person-years. Hence, the principle of inner rectangularization rests on identifying the optimal combination of two (inversely related) inputs—age and survival—which unify the biggest area under a curve representing their respective relationship. Such measures have previously been applied elsewhere. For instance, the Hirsch index (or h-index) measures the productivity and citation impact of scientists (Hirsch 2005). It depicts that *x* publications of a scientist have been cited at least *x* times. The geometric equivalent is a list of all publications by a scientist (*y*-axis) sorted by the number of citations (*x*-axis). This approach is similar to our application, where the survival curves could be interpreted as a sorted list of life lengths (*x*-axis) of the population (*y*-axis). Another example of such a maximum rectangle can be found in physics: the ‘maximum power point’ indicates the maximum power of a photovoltaic module, with a given current–voltage curve (Waszystezuk 1983). Our approach is also related to Cohen’s (2015) decomposition of life expectancy model, which derives Markov’s inequality and Chebyshev’s inequality for tail probabilities in a novel way. In this approach, Cohen decomposes life expectancy into three parts, one of these parts being a non-maximized version of the IR.

Our most important empirical findings are as follows. First, we found that outer rectangularization has shown continuous gains over time (see also Figures 1 and 2 in sections C and D of the supplementary material). This is a consequence of the straight linear increases in life expectancy (Oeppen and Vaupel 2002), which have been faster than the increase in the longest lifespans, as measured by $\alpha$. However, we also detected a considerably slower pace of outer rectangularization since the middle of the twentieth century. Second, we found that inner rectangularization also increased rather uniformly until around 1950; and that the patterns thereafter could not be summarized with a general trend because they are rather country-specific (see also Figures 1 and 2 in sections C and D of the supplementary material). These country-specific patterns appear to be attributable to differences in the onset of sustained mortality declines among the oldest old (Kannisto 1994), as well as by other factors, such as smoking among Danish women (e.g., Juel et al. 2000; Jacobsen et al. 2002; Lindahl-Jacobsen et al. 2016), and postponed reforms of the healthcare system in the Netherlands (Mackenbach et al. 2011; Peters et al. 2015).

If we interpret $x^*$, the age maximizing the IR, as a threshold age separating premature from old-age mortality, then the rises in $x^*$ (see also Figure 3 in section E of the supplementary material) and in corresponding survival, $l_{x^*}$, switched in around 1950, from a reduction in premature deaths to an extension of the premature age range. This dynamic also points to a potential minimum proportion of individuals dying prematurely. Depending on the underlying definition of threshold age, the share dying prematurely varies between 10–15 per cent (MIRA), 15–20 per cent (Gillespie et al. 2014), and 30–35 per
cent (Zhang and Vaupel 2009) under current mortality conditions. Hence, Fries’ prediction that premature mortality would be almost completely eradicated seems rather unlikely. We can, however, see that the definition and measurement of premature mortality are issues that have been unresolved at least since Lexis (1877).

Using $x^*$ as a reference point would enable us to extend the MIRA beyond the areas and ratios presented here. These areas above and below the survival curve would allow us to decompose life expectancy and maximum living potential because they capture all person-years apart from those included in the MIR. An application of the decomposition could provide a basis for a more detailed analysis of past and potential future developments of rectangularization. For instance, the non-uniform number of person-years lived of life expectancy $(e_x - \text{MIR})$ could be subdivided into the numbers for those dying prematurely and for those living longer than $x^*$. Such an analysis can show to what extent changes in life expectancy, and thus rectangularization, are determined by lifespan equality increases, premature mortality reductions, and longevity extensions. Generally, we would expect to see continuous gains in life expectancy if large shares of the population benefit from mortality improvements. This would result in rising lifespan equality. Indeed, the absolute number of uniformly shared person-years (MIR) has increased in almost all countries with continuously rising life expectancy (see Figure 3(a), for example). But relative to life expectancy, the country-specific patterns of the IRR after 1950 question this relationship (as shown in Figure 2). The rise in uniformity of person-years lived seems to be more detached from overall gains in life expectancy in some countries than in others. In the selected countries, for instance, Italian females were shown to be closest to the described scenario; whereas Danish females, with their convex IRR trend, were found to have a stronger degree of detachment. The extension opportunities offered by the MIRA should help us to gain deeper insights into these dynamics.

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Notes and acknowledgements

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