Greenberger-Horne-Zeilinger states and few-body Hamiltonians

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The generation of Greenberger-Horne-Zeilinger (GHZ) states is a crucial problem in quantum information. We derive general conditions for obtaining GHZ states as eigenstates of a Hamiltonian. In general, degeneracy cannot be avoided if the Hamiltonian contains \( m \)-body interaction terms with \( m \leq 2 \) and a number of qubits strictly larger than 4. As an application, we explicitly construct a two-body 4-qubit Hamiltonian and a three-body 5-qubit Hamiltonian that exhibit a GHZ as a nondegenerate eigenstate.

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The use of quantum mechanics for improving tasks such as communication, computation and cryptography is based on the availability of highly entangled states. It is therefore of primary importance to obtain reliable strategies for their generation. Among others, GHZ states represent a paradigmatic example of multipartite entangled states. In particular, in the case of three qubits, these states contain purely tripartite entanglement and do not retain any bipartite entanglement when one of the qubits is traced out, thus maximizing the residual tangle.

The experimental realization of GHZ states, most recently with 14 qubits, has paved the way towards realistic implementation of quantum protocols. In these experiments a bottom-up approach is employed, whereby individual quantum systems (trapped particles, photons, cavities) are combined and manipulated. As the number of controllable qubits increases, the generation of GHZ states require the use of quantum operations, whose feasibility strongly depends on the physical system used (optical, semiconductor or superconductor based). In the case of the recent trapped-ion implementation, the problem is additionally complicated by the presence of correlated Gaussian phase noise, which provokes “superdecoherence”, by which decay scales quadratically with the number of qubits. It becomes therefore necessary to manipulate and control state fidelity and dynamics over sufficiently long timescales.

In principle, an alternative scheme for the implementation of GHZ states would consist in its encoding into one of the eigenstates (possibly the fundamental one) of a suitable Hamiltonian. For instance, in it was shown that for the quantum Ising model in a transverse field the ground state is approximately a GHZ state if the strength of the field goes to infinity. Moreover, a proper choice of local fields for an Heisenberg-like spin model can yield a ground state which is, again, approximately GHZ.

On the other hand, it would be interesting to understand what are the requirements to obtain an exact GHZ state as an eigenstate of a quantum Hamiltonian. In this Letter we will address this problem and find rigorous conditions for the encoding of GHZ states into one of the eigenstates of a Hamiltonian that contains few-body coupling terms.

Let

\[
|G_m^n\rangle = \frac{1}{\sqrt{2}} (|0\rangle^\otimes n \pm |1\rangle^\otimes n)
\]

be GHZ states, where \( \sigma^\alpha|i\rangle = (-1)^{|i|} |i\rangle \) defines the computational basis, with \( i = 0, 1 \) and \( \sigma^\alpha \) the third Pauli matrix. As a preliminary remark, we notice that it is trivial to find Hamiltonians involving \( n \)-body interaction terms, whose nondegenerate ground state is \( |G_m^n\rangle \): the simplest example is \( E_0|G_m^n\rangle\langle G_m^n| \), with \( E_0 < 0 \). On the other hand, we can ask whether it is possible for \( |G_m^n\rangle \) to be the nondegenerate ground state, even if the Hamiltonian involves at most \( m \)-body interaction terms (with \( m < n \)). One can easily see that this is not possible. The reason lies in the fact that \( |G_m^n\rangle \) and \( |G_m^n\rangle \) share the same \( m \)-body reduced density matrices, and thus the same expectation values on \( m \)-body interaction terms. If \( |G_m^n\rangle \) is a ground state, also \( |G_m^n\rangle \) must be a ground state. This is a special case of a result proved in [19].

Thus, we relax our initial requirement and try to understand whether \( |G_m^n\rangle \) can be a nondegenerate excited eigenstate for some \( m \)-body Hamiltonian. More specifically, we search for a limiting value \( m_\ast \), depending on the number \( n \) of qubits in the system, such that, if the Hamiltonian involves \( m \)-body interaction terms (with \( m < m_\ast \)), \( |G_m^n\rangle \) cannot be a nondegenerate eigenstate, otherwise the task becomes possible. The most generic \( m \)-body Hamiltonian acting on the Hilbert space of \( n \) qubits can be written as

\[
H^{(m)} = \sum_{j_1=1}^{n} \ldots \sum_{j_m=1}^{n} \sum_{\alpha_1=1}^{m} \ldots \sum_{\alpha_m=1}^{m} J^{\alpha_1 \ldots \alpha_m} \sigma_{j_1}^{\alpha_1} \ldots \sigma_{j_m}^{\alpha_m}
\]

with \( \alpha_i = 0, x, y, z \), \( \sigma_i^0 \equiv 1 \), being the identity operator, \( \sigma_i^\alpha \) the Pauli matrices acting on the Hilbert space of qubit \( i \) and \( J \)'s real numbers. Terms involving only identities and an even number of 1's map \( |G_m^n\rangle \) on the subspace...
We now define a set of normalized state vectors, depending on a multi-index, whose elements range from 1 to \(n\) and satisfy
\[
|\Psi^{(m)}\rangle (G^n_+) = 0.
\]
Since the action of the Hamiltonian \(H\) consists in inverting spins and changing the relative sign of \(G^n_+\), the vector \(|\Psi^{(m)}\rangle\) can be expressed in a convenient way by introducing a new notation. Let
\[
\mathcal{N} = (1, 2, \ldots, n)
\]
be the ordered set of naturals from 1 to \(n\), and let
\[
\mathcal{I} = (i_1, i_2, \ldots, i_l)
\]
denote a multi-index, whose elements range from 1 to \(n\) and satisfy \(i_1 < i_2 < \ldots < i_l\). The cardinality \(|\mathcal{I}| = l\) verifies
\[
1 \leq |\mathcal{I}| \leq m < n.
\]
We now define a set of normalized state vectors, depending on the choice of the multi-index \(\mathcal{I}\) and on the sign \(\sigma = \pm\):
\[
|\tilde{G}^{m}_{\sigma, \mathcal{I}}\rangle = \frac{1}{\sqrt{2}} \left( \bigotimes_{i \in \mathcal{I}} |1\rangle_i \bigotimes_{j \in \mathcal{N}/\mathcal{I}} |0\rangle_j \right) + \sigma \left( \bigotimes_{i \in \mathcal{I}} |0\rangle_i \bigotimes_{j \in \mathcal{N}/\mathcal{I}} |1\rangle_j \right)
\]
The state \(\tilde{G}^{m}_{\sigma, \mathcal{I}}\) differs from \(G^n_+\) in that spins corresponding to the indices in \(\mathcal{I}\) are reversed in both computational basis vectors in the superposition \(G^n_+\). This means that \(\tilde{G}^{m}_{\sigma, \mathcal{I}} = G^n_+\) if \(\mathcal{I}\) is the empty set. Moreover, the relative phase of the two vectors can be positive or negative, according to the sign \(\sigma\). Thus, the vector \(|\Psi^{(m)}\rangle\) in Eq. (9) can be expressed as
\[
|\Psi^{(m)}\rangle = b_0 |G^n_+\rangle + \sum_{\mathcal{I}} \left( a_{\mathcal{I}} |\tilde{G}^{m}_{+\mathcal{I}}\rangle + b_{\mathcal{I}} |\tilde{G}^{m}_{-\mathcal{I}}\rangle \right).
\]
The coefficients \(a_{\mathcal{I}}, b_{\mathcal{I}}\) and \(b_0\) are functions of the parameters of the Hamiltonian (2). It is obvious that, if they can all be set to zero by a proper choice of \(H\), \(|\Psi^{(m)}\rangle\) will be an eigenstate of the Hamiltonian. A problem arises, however, if we take into account the antisymmetric state \(G^{(m)}_\sigma\). The action of \(H\) on this vector reads
\[
H^{(m)} |G^{(m)}_\sigma\rangle = \epsilon |G^{(m)}_\sigma\rangle + |\Phi^{(m)}\rangle,
\]
where \(|\Phi^{(m)}\rangle\) is orthogonal to \(G^n_\sigma\) and can be decomposed as
\[
|\Phi^{(m)}\rangle = b_0 |G^n_+\rangle + \sum_{\mathcal{I}} \left( a_{\mathcal{I}} |\tilde{G}^{m}_{-\mathcal{I}}\rangle + b_{\mathcal{I}} |\tilde{G}^{m}_{+\mathcal{I}}\rangle \right).
\]
If all the coefficients in Eq. (9) are set to zero, this will result in the cancellation of \(|\Phi^{(m)}\rangle\). As a consequence, \(G^n_+\) and \(G^n_-\) will be degenerate eigenstates (with eigenvalue \(\epsilon\)). Thus, if the sufficient conditions
\[
b_0 = 0,
\]
\[
a_{\mathcal{I}} = 0, \quad b_{\mathcal{I}} = 0
\]
are also necessary for \(G^n_+\) to be an eigenstate of \(H^{(m)}\), degeneracy is unavoidable. We notice that, since the following equality holds
\[
\langle G^n_- | G^n_{\sigma, \mathcal{I}}\rangle = 0 \quad \forall \mathcal{I} \quad \text{and} \quad \forall \sigma,
\]
Eq. (12) is always a necessary condition.

Let us start considering the case in which the Hamiltonian (2) contains interaction terms up to \(m\)-body such that
\[
m < m_n^* \equiv [(n + 1)/2]
\]
with \([\cdot]\) denoting the integer part. Following Eq. (17), the sum in the decomposition of \(|\Psi^{(m)}\rangle\) and \(|\Phi^{(m)}\rangle\) runs over all the multi-indices whose length satisfies
\[
1 \leq |\mathcal{I}| \leq m < m_n^*.
\]
If this inequality holds, the following orthogonality relations are verified:
\[
\langle \tilde{G}^{m}_{\sigma_1, \mathcal{I}_1} | \tilde{G}^{m}_{\sigma_2, \mathcal{I}_2}\rangle = 0 \quad \text{if} \quad \mathcal{I}_1 \neq \mathcal{I}_2 \quad \text{or} \quad \sigma_1 \neq \sigma_2.
\]
Thus, Eq. (13) is a necessary condition to cancel \(|\Psi^{(m)}\rangle\) and make \(G^n_+\) an eigenstate of \(H^{(m)}\). In this case, however, \(G^n_+\) and \(G^n_-\) are eigenstates corresponding to the same eigenvalue. We can conclude that, if the Hamiltonian of a qubit system involves terms coupling less than \(m_n^*\) spins, the GHZ state \(G^n_+\), and any equivalent state by local unitaries, cannot be a nonequivalent eigenstate. If \(G^n_+\) is an eigenstate for some Hamiltonian \(H^{(m)}\), it must be at least two-fold degenerate.

On the other hand, if \(m = m_n^*\) degeneracy can be avoided. Actually, in this case some conditions in Eq. (13) are no longer necessary and, therefore, the orthogonality relations in Eq. (17) hold if inequality (16) is satisfied. However, a new relation emerges connecting \(|\tilde{G}^{m}_{\sigma, \mathcal{I}}\rangle\) states corresponding to multi-indices of length \(m_n^*\) and \((n - m_n^*)\) (which is equal to \(m_n^*\) for even \(n\) and to \(m_n^* - 1\) for odd \(n\)). Indeed, reversing \(m_n^*\) spins in \(G^n_+\) is completely equivalent to reversing the other \(n - m_n^*\) ones.
Instead, if the same operations are applied on the anti-symmetric state \( |G^n_{-} \rangle \), they will differ only by an overall sign. Thus, we have the following relations

\[
|G^n_{\pm,\pm} \rangle = \pm \tilde{G}^n_{\pm,\mp} \quad \text{if } |\mathcal{I}| = m^*_n, n - m^*_n, (18)
\]

While conditions (13) still hold for \( |\mathcal{I}| < \min(m^*_n, n - m^*_n) \), for larger values of \( |\mathcal{I}| \) one should use

\[
\begin{cases}
a_T = -a_{N/\mathcal{I}} \\
b_T = b_{N/\mathcal{I}}
\end{cases}
\quad \text{if } |\mathcal{I}| = m^*_n, n - m^*_n. (19)
\]

Thus, in order to cancel \( |\Psi^{(m)}\rangle \), it is no longer necessary to set all the coefficient \( a_T = 0 \) and \( b_T = 0 \) in Eq. (9) because this would give a degeneracy (remember that, by the same conditions, one would have \( |\Phi^{(m)}\rangle = 0 \)). Instead, by using Eq. (10), we have the vector \( |\Phi^{(m)}\rangle \) in Eq. (11) becomes

\[
|\Phi^{(m)}\rangle = \sum_{|\mathcal{I}|=m^*_n, n-m^*_n} \left( a_T \tilde{G}^n_{\mathcal{I},\mp} + b_T \tilde{G}^n_{\mathcal{I},\pm} \right), (20)
\]

which is generally different from the null vector. If, for some values of the parameters in the Hamiltonian (2), the conditions (13) are satisfied without cancelling \( |\Phi^{(m)}\rangle \), the GHZ state \( |G^4_{+}\rangle \) can, at least in principle, be a non-degenerate eigenstate of an Hamiltonian with interaction terms coupling no more than \( m^*_n = (n + 1)/2 \) qubits.

As a further remark, we notice that this result does not ensure that \( |G^4_{+}\rangle \) is non-degenerate. The absence of degeneracy can be excluded only by the explicit solution of the Hamiltonian.

The case \( m^*_n < m < n \) is analogous to the previous one, since conditions of the type (13) hold for multi-indices \( \mathcal{I} \) that satisfy \( n - m \leq |\mathcal{I}| \leq m \). Following the same procedure as in the case \( m = m^*_n \), we find that the degeneracy of the eigenspace spanned by \( |G^4_{+}\rangle \) and \( |G^4_{-}\rangle \) can still be avoided.

We will now consider an interesting application of the previous results. In the following example, we will focus our attention on the symmetric GHZ state \( |G^4_{+}\rangle \) and the case

\[
n = 4, \quad m = m^*_4 = 2 \quad (21)
\]

and will show that state \( |G^4_{+}\rangle = (|0000\rangle + |1111\rangle) / \sqrt{2} \) can be a non-degenerate eigenstate of a two-body Hamiltonian. We restrict our attention to Hamiltonians involving only two body coupling along the \( x \) and \( z \) axes, and we add the condition of nearest-neighbour couplings on a ring:

\[
H^{(2)} = \sum_{i=1}^{4} (J^x_i \sigma^x_i \sigma^x_{i+1} + J^z_i \sigma^z_i \sigma^z_{i+1}). (22)
\]

In Eq. (22) we have used periodic boundary conditions \( \sigma_5 = \sigma_1 \). We notice that interaction terms of the form \( \sigma^z_i \sigma^z_{i+1} \) leave \( |G^4_{+}\rangle \) invariant, while the terms \( \sigma^x_i \sigma^x_{i+1} \) reverse two nearest-neighbour spins. Since, as in Eq. (18), we have

\[
|G^4_{\pm,(1,2)}\rangle = \pm \tilde{G}^4_{\pm,(3,4)}, \quad |G^4_{\pm,(2,3)}\rangle = \pm \tilde{G}^4_{\pm,(1,4)}, (23)
\]

the action of \( H^{(2)} \) on the four-qubits GHZ state \( |G^4_{+}\rangle \) reads

\[
H^{(2)} |G^4_{+}\rangle = \left( \sum_{i=1}^{4} J^x_i \right) |G^4_{+}\rangle + \frac{2}{2} \sum_{i=1}^{4} (J^z_i + J^z_{i+2}) |G^4_{\pm,(i,i+1)}\rangle. (24)
\]

Thus \( |G^4_{+}\rangle \) is an eigenstate of \( H^{(2)} \) if and only if

\[
\begin{cases}
J^x_1 = -J^x_3 \\
J^z_1 = -J^z_3.
\end{cases} (25)
\]

Under these conditions, the Hamiltonian acts on the antisymmetric combination \( |G^4_{-}\rangle = (|0000\rangle - |1111\rangle) / \sqrt{2} \) as

\[
H^{(2)} |G^4_{-}\rangle = \left( \sum_{i=1}^{4} J^x_i \right) |G^4_{-}\rangle + \frac{2}{2} \sum_{i=1}^{4} J^z_i |G^4_{\pm,(i,i+1)}\rangle. (26)
\]

If \( J^x_1 \neq 0 \) or \( J^z_2 \neq 0 \), \( |G^4_{+}\rangle \) is not an eigenstate. We explicitly solve a simple model, with \( J^x_i \equiv J^x / 4 \) for all \( i \) and \( J^z_1 = J^z_2 = J^z / 4 \). The coupling constants are not zero and \( J^x \neq J^z \). \( |G^4_{+}\rangle \) is a nondegenerate eigenstate, corresponding to the eigenvalue \( J^z \). It is remarkable, however, that this model has three other eigenstates which are equivalent to \( |G^4_{+}\rangle \) by local unitaries, corresponding to the eigenvalues \( -J^z \) and \( J^z \). As expected, none of them can be the non-degenerate ground state, since the ground-state energy is \( \epsilon_0 = (J^x)^2 + (J^z)^2 \). \( |G^4_{+}\rangle \) is the (nondegenerate) first excited state if \( J^z < 0 \) and \( -1 < J^z/J^x < 1 \). Incidentally, for different ranges of the parameters the first excited state of this Hamiltonian is one of the three eigenstates which are locally equivalent to \( |G^4_{+}\rangle \).

The five-qubit GHZ state \( |G^5_{+}\rangle = (|00000\rangle + |11111\rangle) / \sqrt{2} \) needs at least three-body interactions to be the nondegenerate eigenstate of any Hamiltonian \( (m^*_n = 3) \). It is straightforward to check that \( |G^5_{+}\rangle \) is an eigenstate of

\[
H^{(3)} = J^z \frac{5}{5} \sum_{i=1}^{5} \sigma^z_i \sigma^z_{i+1} + J^z \frac{5}{5} \sum_{i=1}^{5} (\sigma^x_i \sigma^x_{i+1} \sigma^x_{i+2} - \sigma^z_i \sigma^z_{i+1}), (27)
\]

with eigenvalue \( J^z \) (periodic boundary conditions are assumed). The conditions for \( |G^5_{+}\rangle \) to be a nondegenerate eigenstate are easily worked out by diagonalizing \( H^{(3)} \). It is the nondegenerate first excited eigenstate if \( J^z < 0 \), \( J^x \neq 0 \) and

\[
-2 + \frac{2}{\sqrt{3}} < \frac{J^z}{J^x} < \frac{1}{\sqrt{3}} \left[ \sqrt{2(75 + 7\sqrt{5})} - (7 + \sqrt{5}) \right]. (28)
\]
\(|G^e_n\rangle\) can be the ground state only if \(J^x = 0\), but in this case it is twice degenerate.

In conclusion, we investigated general conditions such that GHZ states \(|\text{GHZ}\rangle\) are nondegenerate in the spectrum of a Hamiltonian. We showed that if the Hamiltonian acting on the Hilbert space of \(n\) qubits involves terms that couple at most \(m\) qubits, it is impossible to have a nondegenerate GHZ eigenstate if \(m < m^*_n = \lceil (n + 1)/2 \rceil\). If \(m \geq m^*_n\), degeneracy can in principle be absent.

The difficulty in obtaining GHZ states as ground states (or even eigenstates) of Hamiltonians that involve only few-body interactions is in accord with previous results \cite{20} and seems to be a characteristic trait of multipartite entanglement. It would be interesting, also in view of applications, to investigate the existence of general conditions for obtaining approximate GHZ states for an arbitrary number of qubits by making use of few-body Hamiltonians.

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