Application of water based drilling clay-nanoparticles in heat transfer of fractional Maxwell fluid over an infinite flat surface

Muhammad Imran Asjad, Rizwan Ali, Azhar Iqbal, Taseer Muhammad & Yu-Ming Chu

In the present paper, unsteady free convection flow of Maxwell fluid containing clay-nanoparticles is investigated. These particles are hanging in water, engine oil and kerosene. The values for nanofluids based on the Maxwell-Garnett and Brinkman models for effective thermal conductivity and viscosity are calculated numerically. The integer order governing equations are being extended to the novel non-integer order fractional derivative. Analytical solutions of temperature and velocity for Maxwell fluid are build using Laplace transform technique and expressed in such a way that they clearly satisfied the boundary conditions. To see the impact of different flow parameters on the velocity, we have drawn some graphs. As a result, we have seen that the fractional model is superior in narrate the decay property of field variables. Some limiting solutions are obtained and compared with the latest existing literature. Moreover, significant results can be observed for clay nanoparticles with different base fluids.

Since the last decades, in various disciplines, fractional calculus theory has brought greater attention for the researchers. In fact, it was found that use of fractional derivatives is very helpful in modifying many process related to thermal transport processes, engineering sciences, circuit analysis, Biotechnology and signal processing. There are so many other applications related to heat and mass transfer and fluid dynamics can be perceived in the references. Different books related to fractional derivatives were written by Jagdev et al., Kolade and Atangana and Baleanu et al. and they also discuss the application of fractional derivatives and also about their operators.

In literature different concepts related to fractional derivatives can be find out. In physics fractional derivative plays an important role in designing different phenomena. Yet, it was mentioned in Caputo and Fabrizio, that different circumstances belonging to material heterogeneities cannot be well-modeled using fractional derivatives introduced by Riemann-Liouville or Caputo. Due to this fact, new fractional derivative related to non-singular kernel were introduced by Caputo and Fabrizio. It was important that, rather than power law and exponential decay function the kernel Mittag-Leffler function is more general. Therefore, both Riemann-Liouville and Caputo-Fabrizio are special cases of Atangana-Baleanu fractional operators. In applied sciences, some new kinds of derivatives exist which are known as fractal derivative. Therefore there was a need to redefine the concept of expressing velocity in fractal media for example, scaling time in fractal \((x, t^\alpha)\). Baleanu et al. in 2020 introduced a new fractional operator using power law and are called hybrid fractional derivatives. This derivative is linear combination of constant proportional and Caputo type fractional derivative.

Ali investigated the Atangana-Baleanu derivative with a novel approach. Circuits with fractional derivatives were developed by Hammouch and Mekkaoui and they also discussed its behavioral dynamics. It is known that for description of price of opinion can be given by using Time Fractional Black Scholes Equation (TFBSE) with a time derivative of real order. The investigation of heat dissipation in transmission line of electrical circuit is given in Abro et al. An analysis of generalized Jeffery nanofluid in a rotating frame with non-singular fractional derivative is given in Ali et al. The behavior realted to heat transfer in different model with singular and non-singular is given in articles.
To control the entropy in the flow of heat is one of the main concerns of the industrial sectors. Since the problem had a large amplitude so a lot of researchers pay attention to solve it. They use different methods for different fluids to enhance the thermal conductivity. Bejan et al. came to the conclusion that viscous dissipation, heat transfer, mass transfer and chemical reaction are the main reasons are entropy enhancement in thermal systems. To overcome the problem the first meaningful contribution was made by Choi, when he gave the idea of nanofluids. This was the revolution in the flow of heat and has given the answers to the many unsolved problems. He just added nanosized particles of different type to solve the problem of entropy. Entropy in magnetohydrodynamics (MHD) by having exact analysis is investigated by Khan et al. A definition regarding the Bejan number which is useful to predict the power of magnetic field and entropy of fluid friction due to heat transfer is given by Awad. Saouli and Aïboud-Saouli has used an inclined plate and liquid film to analyzed entropy generation. Mahmud et al. reported the experiment with added magnetic field influence. Selimefendigil et al. has explored entropy for natural convection flow of a nanofluid. made a useful contribution in this regard.

The two-dimensional magnetohydrodynamics (MHD) flow of Casson fluid and heat transfer find out by Hamid et al., mainly we focus the study is to survey the linear thermal radiation influence on dual solutions for both the steady and unsteady flow of Casson fluid under the effect of uniform magnetic field. The blood flow connecting nanoparticles through porous blood vessels in the occurrence of magnetic field with the help of collocation and least squares techniques inspected by Usman et al. Blood is a non-Newtonian fluid having nanoparticles which are used for different models to find the viscosity of the nanofluids. Hamid et al. discussed the unsteady MHD flow of Williamson nanofluid between the permeable channel with heat source/sink. The effect of molybdenum disulphide (MoS$_2$) nanoparticles forms on circiling flow of nanofluid along an elastic stretched sheet. This nanofluid flow is measured in the existance of magnetic fields, thermal radiation and variable thermal conductivity. Hamid et al. discussed the different types of nanoparticles like Platelet, cylindrical and brick forms. Usman et al. evaluate the flow of ethylene glycol and water based copper (Cu) nanoparticles between two squeezed parallel disks.

Newly, the industrialists are interested theoretically and experimentally studying the advancement of nanofluids. The industrialists are elaborated to find thermophysical properties (heat capacitance, thermal conductivity, density, electrical conductivity, thermal expansion, and viscosity) of unlike nanoparticles and base fluids using several procedures because the next generations fluid is a nanofluid for heat transport which can deal additional thermal presentation in various industrial sectors like power generation, transportation, hyperthermia and air conditioning.

Inspiriting from the above-discussed literature, this study aims to focus on the applications of nanofluid in the drilling process. Different type of nanofluid is used in drilling activities such as oil-based drilling mud (OBM), water-based drilling mud. For this purpose we have used the Polymers nanoparticles and clay nanoparticles.

It has been observed that for stability in temperature, least value of torque, prevention of fluid loss, to control the rheological possessions for scrubbing the hole and to filter the quality of cake, clay nanoparticles helps a lot. Khan et al. has used three different base fluids to clean water. Nisar et al. has studied the entropy of clay nanoparticles.

The main purpose of this paper is to extend the idea of Imran et al. in which analytical solutions are obtained for viscous fluid. They used the Laplace transform method to obtain the solutions for temperature and velocity fields respectively. For the moment there is no such results regarding Maxwell fluid containing clay nanoparticles therefore, we have applied the most recent hybrid fractional operator for a Maxwell fluid of caly-water base nanofluids over an infinite vertical surface moving with constant velocity and obtained solutions with Laplace transform method. Some limiting solutions are also obtained and justified through graphical comparison and presented in the graphical section. Thermophysical properties of nanomaterials are defined in Table 1.

### Mathematical formulation and solution

Let water, kerosene and engine oils are base fluids to carve-up in the flow of clay nanoparticles. The flow of the fluid is in the region $y_1 > 0$, close to a heated flat vertical plate. The plate is normal to $y$-axis and is fixed. In the beginning it is assumed that the fluid is at rest on the plate having surrounding temperature $T_\infty$. This ambient temperature changes from $T_\infty$ to $T_w$ in no time causing the motion in the plate with velocity $V_0$ forcing the fluid to move in $x$-direction as shown in Fig. 1. The governing equations are given as follows:

| Material       | Symbol | $\rho \left( \frac{m}{m^2} \times kg \right)$ | $C_p \left( \frac{cal}{m^2} \times J \right)$ | $K_\eta \left( \frac{cm}{s} \times W \right)$ | $\frac{d}{d^2}$ | Pr   |
|----------------|--------|---------------------------------------------|----------------|---------------------------------|----------------|------|
| Clay Nanoparticles | N/P | 6320                                      | 531.8          | 76.5                            | 1.80                   | –    |
| Water          | $H_2O$ | 997                                       | 4797           | 0.613                           | 21                        | 6.2   |
| Kerosene oil   | KO     | 783                                       | 2090           | 0.145                           | 99                        | 21    |
| Engine oil     | EO     | 884                                       | 1910           | 0.114                           | 70                        | 500   |

**Table 1.** Thermophysical properties of nanofluids.
Continuity Eq.66

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \]  

where \( \rho \) is the density of fluid, \( \nabla \) is the divergence operator and \( \mathbf{v} \) is the velocity of fluid. For incompressible fluid \( \nabla \cdot \mathbf{v} = 0 \).

Cauchy stress tensor for Maxwell model of the form67

\[ T = -pI + S, \quad S + \lambda_1 \frac{\delta S}{\delta t} = \mu A_1, \]  

where \( -p \) is the pressure, \( I \) is the identity matrix, \( \mu \) is the viscosity, and \( A_1 \) is the first Rivlin-Eriksen tensor defined as

\[ A_1 = \nabla \mathbf{v} + \nabla \mathbf{v}^T, \]  

\[ \frac{\delta S}{\delta t} = \frac{DS}{Dt} - LS - SL^T. \]

where \( \frac{\partial}{\partial t} \) is the material time derivative, and \( L \) is gradient of the velocity.

Fractional stress tensor for Maxwell fluid with constant proportional Caputo time fractional derivative14

\[ \tau + \lambda_1 \frac{\partial^{\alpha} \tau}{\partial t^{\alpha}} = \mu n_f \frac{\partial \mathbf{v}}{\partial y_1}, \quad 0 < y < h, \quad t > 0 \]  

where \( \tau = S_{xy} \) is the non zero component of extra stress tensor and \( \frac{\partial^{\alpha} \tau}{\partial t^{\alpha}} \) is the constant proportional Caputo derivative of non integer order \( \alpha \). The innovative fractional derivative is defined and given in14.

\[ \text{CPC} D^{\alpha}_t h(t) = \left[ \Gamma(1 - \alpha) \right]^{-1} \int_0^t \frac{\left( L_1(\alpha) h(\tau) + L_0(\alpha) h(\tau) \right)}{(t - \tau)^\alpha} d\tau, \quad 0 < \alpha < 1. \]  

The Laplace transform of constant proportional Caputo is given as14

\[ L \{ \text{CPC} D^{\alpha}_t h(t) \} = \left\{ \frac{L_1(\alpha)}{s} + L_0(\alpha) \right\} s^{\alpha} - L_0(\alpha) s^{\alpha - 1} h(0). \]  

Navier–Stokes Eq.66

\[ \rho n_f \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \frac{\partial \tau}{\partial y_1}. \]  

Multiplying Eq. (9) by \( 1 + \lambda_1 \frac{\partial^{\alpha} \tau}{\partial t^{\alpha}} \)
\[\rho_{nf}\left(\frac{\partial v}{\partial t} + v \cdot (\nabla v)\right)\left(1 + \lambda_1^{\alpha} \frac{\partial^\alpha}{\partial t^\alpha}\right) = (-\nabla p)\left(1 + \lambda_1^{\alpha} \frac{\partial^\alpha}{\partial t^\alpha}\right) + k_{nf} \frac{\partial \tau}{\partial y_1}\left(1 + \lambda_1^{\alpha} \frac{\partial^\alpha}{\partial t^\alpha}\right) + g(\rho \beta_T)(T(y_1, t) - T_\infty)\left(1 + \lambda_1^{\alpha} \frac{\partial^\alpha}{\partial t^\alpha}\right),\]  

(10)

In the absence of pressure gradient and convective term, and by using Eq. (6) into Eq. (10), we have 

\[\rho_{nf}\frac{\partial v(y_1, t)}{\partial t} + \rho_{nf} \lambda_1^{\alpha} \frac{\partial^\alpha v(y_1, t)}{\partial t^\alpha + 1} = \mu_{nf} \frac{\partial^2 v(y_1, t)}{\partial y_1^2} + \left(1 + \lambda_1^{\alpha} \frac{\partial^\alpha}{\partial t^\alpha}\right)g(\rho \beta_T)(T(y_1, t) - T_\infty).\]  

(11)

The constitutive relation for thermal flux is given as 

\[(\rho C_p)_{nf} \frac{\partial T(y_1, t)}{\partial t} = -\frac{\partial q(y_1, t)}{\partial y_1},\]  

(12)

In order to find the fractional energy equation, applying the operator \(1 + \lambda_1^{\alpha} \frac{\partial^\alpha}{\partial t^\alpha}\) on both sides of Eq. (10)\(^{65,64}\), 

\[(\rho C_p)_{nf} \left(1 + \lambda_1^{\alpha} \frac{\partial^\alpha}{\partial t^\alpha}\right) \frac{\partial T(y_1, t)}{\partial t} = -\frac{\partial}{\partial y_1} \left(1 + \lambda_1^{\alpha} \frac{\partial^\alpha}{\partial t^\alpha}\right)q(y_1, t),\]  

(13)

Generalization of fractional Cattaneo’s law\(^{68}\) 

\[\left(1 + \lambda_1^{\alpha} \frac{\partial^\alpha}{\partial t^\alpha}\right)q(y_1, t) = -K_{nf} \frac{\partial^2 T(y_1, t)}{\partial y_1^2},\]  

(14)

Using Eq. (14) into Eq. (13), We have 

\[(\rho C_p)_{nf} \frac{\partial T(y_1, t)}{\partial t} + (\rho C_p)_{nf} \lambda_1^{\alpha} \frac{\partial^\alpha T(y_1, t)}{\partial t^\alpha + 1} = K_{nf} \frac{\partial^2 T(y_1, t)}{\partial y_1^2},\]  

(15)

where \(v = v(y_1, t), T = T(y_1, t), \lambda, q, \rho_{nf}, \mu_{nf}, \beta_T, g, (\rho C_p)_{nf}, K_{nf}\) are respectively the fluid velocity, temperature, Maxwell parameter, heat flux, density, the dynamic viscosity, volumetric thermal expansion coefficient, gravitational acceleration, heat capacitance, thermal conductivity of nanofluids.

Appropriate initial and boundary conditions are 

\[v(y_1, 0) = 0, \quad T(y_1, 0) = T_\infty, \quad \text{for all} \quad y_1 \geq 0,\]  

(16)

\[v(0, t) = V_0 H(t), \quad T(0, t) = T_w, \quad t > 0,\]  

(17)

\[v(\infty, t) \to 0, \quad T(\infty, t) \to T_\infty, \quad t > 0,\]  

(18)

where \(H(t)\) is a Heaviside unit step function.

Thermo-physical properties are defined in\(^{39}\) as follows:

\[\frac{\rho_{nf} - \phi \rho_s}{(1 - \phi)\rho_s} = 1, \quad \frac{\mu_{nf} - (1 - \phi)\mu_f}{\mu_f} = 1, \quad \frac{(\rho C_p)_{nf} - \phi (\rho C_p)s}{(1 - \phi)(\rho C_p)s} = 1, \quad K_{nf} = K_s + 2K_f - 2\phi(K_f - K_s),\]  

\[K_f = K_s + 2K_f - 2\phi(K_f - K_s), \quad \frac{\rho(\beta T)_{nf} - \phi (\rho \beta_T)s}{(1 - \phi)(\rho \beta_T)s} = 1.\]

where \(\phi\) is the nanoparticle volume fraction, \(\rho_f, \rho_s\) are the density of the base fluid and nanoparticle, \(\beta_f, \beta_s\) are the volumetric coefficients of thermal expansion of nanoparticle and base fluid, \((C_p)_s, (C_p)_f\) are the specific heat capacities of nanoparticle and base fluid at constant pressure and . Here \(K_f, K_s\) are thermal conductivities of base fluid and nanoparticle. \(\mu_f, \mu_s\) are the dynamic viscosity of the base fluid. 

Presenting non-dimensional variables and functions 

\[\gamma_* = \frac{V_0}{V_f} y_1, \quad t_* = \frac{V_0^2}{V_f^2} t, \quad v_* = \frac{v}{V_0}, \quad \psi_* = \frac{T - T_\infty}{T_w - T_\infty},\]  

(19)

into Eqs. (11), (15) and Eqs. (16)–(18) and reducing \(\psi_*, v_*\), we have 

\[b_0 \left(\frac{\partial v(y_1, t)}{\partial t} + \lambda_1^{\alpha} \frac{\partial^\alpha v(y_1, t)}{\partial t^\alpha + 1}\right) = b_1 \frac{\partial^2 v(y_1, t)}{\partial y_1^2} + b_2 \frac{\partial^2 v(y_1, t)}{\partial y_1^2} \left(1 + \lambda_1^{\alpha} \frac{\partial^\alpha}{\partial t^\alpha}\right) \phi(y_1, t),\]  

(20)
taking Laplace inverse of Eq. (31), we have
\[
\psi(t, s) = \left\{ \begin{array}{ll}
\frac{b_0}{s} & \text{for } s > 0,
\frac{b_0}{s} & \text{for } s < 0,
\end{array} \right.
\]
\[
\psi(0, s) = 0, \quad \psi(0, s) = 0, \quad \text{as } s \to \infty.
\]

The expression appears in Eq. (29) in exponential form is complicated and difficult to obtain analytically, so we express this form in its equivalent form:
\[
\psi(t, s) = \sum_{n=0}^{\infty} \left( \frac{b_n}{s} \right) e^{-n \frac{\phi}{\alpha}}
\]
\[
\psi(0, s) = 0, \quad \psi(0, s) = 0, \quad \text{as } s \to \infty.
\]

where \( b_0, b_1 \) and \( k_0, k_1 \) are constants

Solution of Eq. (26) subject to Eqs. (27), we have
\[
\psi(t, s) = \sum_{n=0}^{\infty} \left( \frac{b_n}{s} \right) e^{-n \frac{\phi}{\alpha}}
\]
\[
\psi(0, s) = 0, \quad \psi(0, s) = 0, \quad \text{as } s \to \infty.
\]

where \( \alpha \) and \( \beta \) are fractional parameters defined in "Pr" is the Prandtl number and "Gr" is the thermal Grashof number. Taking Laplace transform of Eqs.

\[
\psi(t, s) = \sum_{n=0}^{\infty} \left( \frac{b_n}{s} \right) e^{-n \frac{\phi}{\alpha}}
\]
\[
\psi(0, s) = 0, \quad \psi(0, s) = 0, \quad \text{as } s \to \infty.
\]
The Eq. (33) can be expressed in series form so that we can easily find its inverse Laplace analytically.

\[ \tilde{v}(y_1, s) = \frac{1}{s} e^{-y_1 \sqrt{c_1 [1 + \lambda \{ K_1(\alpha) \}^k - K_0(\alpha)] s^k}} \]

\[ + \frac{b_2 Gr}{b_0} \left( 1 + \lambda \{ K_1(\alpha) \} s^k \right) \left[ b_0 \left( 1 + \lambda \{ K_1(\alpha) \} s^k \right) \left( s - c_2 \left( 1 + \beta \{ K_1(\alpha) \} s^k \right) \right) \right] \]

\[ \left\{ e^{-y_1 \sqrt{c_1 [1 + \lambda \{ K_1(\alpha) \}^k - K_0(\alpha)] s^k}} - e^{-y_1 \sqrt{c_1 [1 + \lambda \{ K_1(\alpha) \}^k - K_0(\alpha)] s^k}} \right\} \]

The Eq. (33) can be expressed in series form so that we can easily find its inverse Laplace analytically.

\[ \tilde{v}(y_1, s) = \frac{1}{s} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-y_1 \sqrt{c_1 [1 + \lambda \{ K_1(\alpha) \}^k - K_0(\alpha)] s^k})^i}{i! j! k! (K_0(\alpha))^k - s^k - (1 - \lambda) s^k - \frac{\alpha}{1 - \lambda} s^k} \cdot \Gamma \left( \frac{j}{2} + 1 \right) \Gamma \left( \frac{k}{2} + 1 \right) \]

\[ + \frac{b_2 Gr}{b_0} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \sum_{m_4=0}^{\infty} \frac{(-y_1 \sqrt{c_1 [1 + \lambda \{ K_1(\alpha) \}^k - K_0(\alpha)] s^k})^{m_1 m_2}}{m_1! m_2! m_3! m_4! (K_0(\alpha))^{m_1 m_2 - 1}} \cdot \Gamma \left( m_1 + m_2 + m_3 + 1 \right) \]

\[ = \frac{b_2 Gr}{b_0} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \frac{(-y_1 \sqrt{c_1 [1 + \lambda \{ K_1(\alpha) \}^k - K_0(\alpha)] s^k})^{n_1 n_2}}{n_1! n_2! n_3! n_4! (b_0)^{n_1 n_2} (K_0(\alpha))^{n_1 n_2 - n_3}} \cdot \Gamma \left( n_2 + n_3 + 1 \right) \]

\[ = \frac{b_2 Gr}{b_0} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \frac{(-y_1 \sqrt{c_1 [1 + \lambda \{ K_1(\alpha) \}^k - K_0(\alpha)] s^k})^{n_1 n_2}}{n_1! n_2! n_3! n_4! (b_0)^{n_1 n_2} (K_0(\alpha))^{n_1 n_2 - n_3}} \cdot \Gamma \left( n_2 + n_3 + 1 \right) \]

(34)

Taking Laplace inverse of Eq. (34), we have
This paper deals with the investigation of Clay nanoparticles in free convection of a Maxwell fluid. The analytical solutions satisfy the initial and boundary conditions. The solutions are obtained with the application of novel fractional derivative and Laplace transformation. The influence of nanoparticles as well fractional parameter are discussed through some graphs.

**Numerical outcomes and analysis**

This paper deals with the investigation of Clay nanoparticles in free convection of a Maxwell fluid. The analytical solutions satisfy the initial and boundary conditions. The solutions are obtained with the application of novel fractional derivative and Laplace transformation. The influence of nanoparticles as well fractional parameter are discussed through some graphs.

**Figure 3.** Velocity profile against $y_1$ due to $\alpha$ for large time, when: $t = 0.1$, $\phi = 0.01$, $Pr = 6.2$, $Gr = 0.1$ and $\lambda = 0.5$.

**Figure 4.** Velocity profile against $y_1$ due to $\phi$, when: $t = 0.5$, $Gr = 0.1$, $Pr = 6.2$, $\lambda = 0.01$ and $\alpha = 0.5$.

\[
\tilde{v}(y_1, t) = 1 + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^i \sqrt{\Gamma(i)} \Gamma(K_1(\alpha))^k t^k - \frac{1}{2} - aj}{i!j!k!} \frac{\Gamma(\frac{1}{2} + 1)\Gamma(j + 1)}{\Gamma(\frac{1}{2} - j + 1)\Gamma(1 + 1 - \frac{1}{2} - aj + k)} \frac{\Gamma(\frac{1}{2} + 1)\Gamma(j + 1)}{\Gamma(1 - \frac{1}{2} - \alpha j + k)} \\
+ b_2Gr \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \sum_{m_4=0}^{\infty} \frac{(-1)^{m_1} (c_1)^m_2 (-\lambda)^m_3 (K_1(\alpha))^m_4 t^{m_2} - \frac{m_4}{2} + m_2 - m_3}{m_1!m_3!m_4!(b_0)m_2(K_0(\alpha))^m_4 - \frac{m_2}{2} - m_2 - m_3} \\
\frac{\Gamma(m_2 + m_3 + 1)\Gamma(m_2 + m_3 + 1)}{\Gamma(m_3)\Gamma(m_2 + m_3 + 1 - m_4)\Gamma(2 - \frac{m_2}{2} + 1 - \alpha m_2 - \alpha m_3 + m_4)} \\
- b_2Gr \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \frac{(-1)^n_1 \sqrt{\Gamma(i)} \Gamma(n_1(\alpha))^n_2 t^{n_2} - \frac{n_2}{2} - \alpha n_2 + n_4}{n_1!n_3!n_4!(b_0)n_2(K_0(\alpha))^n_2 - \frac{n_2}{2} - \alpha n_2 + n_4} \\
\frac{\Gamma(n_2 + 1)\Gamma(n_2 + 1)}{\Gamma(n_2 + n_3 + 1)\Gamma(n_2 + n_3 + 1 - n_4)\Gamma(2 - \frac{n_2}{2} + 1 - \alpha n_2 - \alpha n_3 + n_4)}. \\
(35)
\]
Figure 5. Velocity profile against $y_1$ due to $\text{Gr}$, when: $t = 0.25$, $\phi = 0.04$, $\text{Pr} = 6.2$, $\alpha = 0.2$ and $\lambda = 0.01$.

Figure 6. Comparison of velocity profile with different base fluids, when: $t = 0.005$, $\phi = 0.42$, $\text{Gr} = 5$, $\alpha = 0.1$ and $\lambda = 0.51$.

Figure 7. Comparison of velocity profile when $\lambda = 0$ with Imran et al.\cite{61}, when: $t = 0.1$, $\phi = 0.04$, $\text{Pr} = 6.2$, $\text{Gr} = 0.1$ and $\alpha = 0.2$. 
Figure 2 depicted to see the variation of fractional parameter $\alpha$. The maximum decay in velocity can be obtained for small time. It is clear from the Fig. 2 by increasing values of $\alpha$ velocity exhibits the maximum decay for small time. This behavior can be reversed for large values of time is shown in Fig. 3. Further, fractional parameter can be used to control the momentum boundary layer thickness. The relationship of velocity of clay nanoparticles and volume fraction $\phi$ is discussed in Fig. 4. For large values of $\phi$ fluid velocity will decrease as

| $y_1$ | $\alpha = 0.99$ | $\alpha = 0.99$ |
|-------|---------------|----------------|
|       | Present paper when $\lambda = 0$ | Imran et al. 61 |
| 0.0   | 1             | 1              |
| 0.1   | 0.942         | 0.941          |
| 0.2   | 0.885         | 0.882          |
| 0.3   | 0.828         | 0.824          |
| 0.4   | 0.772         | 0.767          |
| 0.5   | 0.717         | 0.712          |
| 0.6   | 0.664         | 0.658          |
| 0.7   | 0.612         | 0.606          |
| 0.8   | 0.562         | 0.556          |
| 0.9   | 0.514         | 0.509          |
| 1.0   | 0.468         | 0.463          |

Table 2. Effect of fractional parameter on dimensionless velocity.

Figure 8. Comparison of velocity profile of fractional Maxwell fluid with viscous fluid 59, when: $t = 1.5$, $\phi = 0.4$, $Pr = 6.2$, $\lambda = 0.1$, $Gr = 0.1$ and $\alpha = 0.8$.

| $y_1$ | $\alpha = 0.4$ | $\alpha = 0.7$ | $\alpha = 0.9$ |
|-------|----------------|----------------|----------------|
|       | Fractional Maxwell | Viscous 59 | Fractional Maxwell | Viscous 59 | Fractional Maxwell | Viscous 59 |
| 0.0   | 1              | 1              | 1              | 1              | 1              | 1              |
| 0.1   | 0.939          | 0.958          | 0.946          | 0.958          | 0.950          | 0.958          |
| 0.2   | 0.888          | 0.916          | 0.893          | 0.916          | 0.901          | 0.916          |
| 0.3   | 0.822          | 0.874          | 0.840          | 0.874          | 0.851          | 0.874          |
| 0.4   | 0.766          | 0.833          | 0.787          | 0.833          | 0.802          | 0.833          |
| 0.5   | 0.711          | 0.791          | 0.736          | 0.791          | 0.753          | 0.791          |
| 0.6   | 0.659          | 0.751          | 0.686          | 0.751          | 0.705          | 0.751          |
| 0.7   | 0.608          | 0.710          | 0.637          | 0.710          | 0.658          | 0.710          |
| 0.8   | 0.560          | 0.671          | 0.590          | 0.671          | 0.611          | 0.671          |
| 0.9   | 0.514          | 0.632          | 0.543          | 0.632          | 0.566          | 0.632          |
| 1.0   | 0.470          | 0.594          | 0.499          | 0.594          | 0.521          | 0.594          |

Table 3. Effect of fractional parameter on dimensionless velocity.

Figure 2 depicted to see the variation of fractional parameter $\alpha$. The maximum decay in velocity can be obtained for small time. It is clear from the Fig. 2 by increasing values of $\alpha$ velocity exhibits the maximum decay for small time. This behavior can be reversed for large values of time is shown in Fig. 3. Further, fractional parameter can be used to control the momentum boundary layer thickness. The relationship of velocity of clay nanoparticles and volume fraction $\phi$ is discussed in Fig. 4. For large values of $\phi$ fluid velocity will decrease as
Figure 9. Comparison of velocity profile with Danish et al.\(^\text{69}\), when: \(t = 0.02, \phi = 0.04, \text{Pr} = 6.2, \lambda = 0.001, \text{Gr} = 0.1, M = 0.1\) and \(\alpha = 0.2\).

| \(y_1\) | \(\alpha = 0.2\) | \(\alpha = 0.2\) | \(\alpha = 0.5\) | \(\alpha = 0.5\) |
|-------|----------------|----------------|----------------|----------------|
|       | Present paper | Danish et al.\(^\text{69}\) | Present paper | Danish et al.\(^\text{69}\) |
| 0.0   | 1             | 0.987          | 1              | 0.991          |
| 0.1   | 0.870         | 0.881          | 0.870          | 0.873          |
| 0.2   | 0.744         | 0.785          | 0.743          | 0.769          |
| 0.3   | 0.623         | 0.700          | 0.623          | 0.677          |
| 0.4   | 0.512         | 0.625          | 0.511          | 0.595          |
| 0.5   | 0.412         | 0.557          | 0.411          | 0.524          |
| 0.6   | 0.325         | 0.496          | 0.324          | 0.460          |
| 0.7   | 0.252         | 0.442          | 0.250          | 0.405          |
| 0.8   | 0.191         | 0.394          | 0.189          | 0.355          |
| 0.9   | 0.142         | 0.351          | 0.141          | 0.312          |
| 1.0   | 0.104         | 0.312          | 0.102          | 0.274          |

Table 4. Effect of fractional parameter on dimensionless velocity.

Figure 10. Comparison of velocity profile with Khan et al.\(^\text{59}\) and Imran et al.\(^\text{61}\), when: \(t = 0.095, \phi = 0.72, \text{Pr} = 6.2\) and \(\text{Gr} = 0.1\).
viscous forces became stronger with increasing $\phi$. Figure 5 high lights the influence of Grashof number $Gr$. Fluid velocity increases as we enhance the value of $Gr$. $Gr$ describes the influence of the thermal buoyancy force to the viscous force. If $Gr$ equal to zero, then there is no free convection current, if $Gr$ is greater than zero, plate is outwardly chilled and if $Gr$ is less than zero, plate is outside frenzied. It means larger plate is outwardly chilled increasing the velocity and effect is reverse of $Pr$. Figure 6 represents the contrast of velocity profile for three different types of base fluids (water, kerosene, and engine oil). It is observed that the velocity of water-based clay nanofluid fluid is larger than kerosene oil and engine oil-based clay nanofluids, respectively.

| $\gamma_1$ | Present result | Imran et al. | Khan et al. |
|------------|----------------|-------------|-------------|
|            | $\beta = 0, \alpha = 1$ | $\alpha = 1$ | $\alpha = 1$ |
| 0.0        | 0.998          | 1           |             |
| 0.1        | 0.944          | 0.939       | 0.944       |
| 0.2        | 0.888          | 0.882       | 0.888       |
| 0.3        | 0.833          | 0.825       | 0.833       |
| 0.4        | 0.778          | 0.771       | 0.778       |
| 0.5        | 0.725          | 0.717       | 0.725       |
| 0.6        | 0.673          | 0.666       | 0.673       |
| 0.7        | 0.622          | 0.616       | 0.622       |
| 0.8        | 0.573          | 0.568       | 0.573       |
| 0.9        | 0.526          | 0.523       | 0.526       |
| 1.0        | 0.481          | 0.479       | 0.481       |

Table 5. Comparisons of dimensionless velocity.

| $\beta$ | $Nu_{t=2}$ | $Nu_{t=3.5}$ | $Nu_{t=5}$ |
|---------|------------|-------------|------------|
| 0.1     | 1.908      | 1.767       | 1.708      |
| 0.2     | 1.855      | 1.677       | 1.592      |
| 0.3     | 1.797      | 1.587       | 1.480      |
| 0.4     | 1.735      | 1.498       | 1.374      |
| 0.5     | 1.668      | 1.411       | 1.272      |
| 0.6     | 1.598      | 1.325       | 1.177      |
| 0.7     | 1.525      | 1.242       | 1.085      |
| 0.8     | 1.450      | 1.161       | 1.004      |
| 0.9     | 1.374      | 1.084       | 0.926      |
| 1.0     | 1.297      | 1.011       | 0.856      |

Table 6. Statistically analysis of Nusselt number for the effect of fractional parameter.

| $\alpha$ | $C_f_{t=1}$ | $C_f_{t=2}$ | $C_f_{t=3.5}$ |
|----------|-------------|-------------|---------------|
| 0.1      | 1.066       | 1.437       | -61.378       |
| 0.2      | 1.049       | 1.413       | -63.535       |
| 0.3      | 1.030       | 1.388       | -65.760       |
| 0.4      | 1.007       | 1.363       | -68.057       |
| 0.5      | 0.981       | 1.338       | -70.427       |
| 0.6      | 0.954       | 1.313       | -72.873       |
| 0.7      | 0.923       | 1.287       | -75.399       |
| 0.8      | 0.890       | 1.262       | -78.007       |
| 0.9      | 0.855       | 1.236       | -80.699       |
| 1.0      | 0.819       | 1.211       | -83.479       |

Table 7. Statistically analysis of Skin friction for the effect of fractional parameter.
thermal conductivity of water is larger than that of kerosene oil and engine oil then the velocity of water-based clay nanofluid is larger than the others.

Figure 7 our obtained results by taking $\lambda = 0$ are compared with the results of Imran et al.65 while keeping other parameters values constant. Table 2 represents the velocity comparison of present paper when $\lambda = 0$ and Imran et al.65 for different values of parameter $\alpha$. We have seen that both Fig. 7 and Table 2 are in good agreement with each other. Figure 8 is plotted to see the velocity comparison of hybrid fractional derivative and Khan et al.99. Since the velocity obtained in99 is for viscous fluid and in the present study for Maxwell fluid. This figure shows the viscous fluid is stiffer than Maxwell fluid. The reason is that Maxwell fluid is non-Newtonian one and more thicker than viscous. Table 3 shows the velocity comparison of current paper and Khan et al.99 for different values of parameter $\alpha$. In both cases we have found that velocity is smaller for CPC fractional model. Physically, fractional operator is responsible for the history of the model and can have better control for momentum boundary layer. Figure 9 shows the velocity comparison between present result and Danish et al.98 with the solutions obtained with the same fractional operator for viscous fluid and we see that velocity is smaller for constant proportional Caputo model of Maxwell fluid over viscous fluid. Due to less viscosity of viscous fluid, the it flows faster than Maxwell fluid. The velocity contrast of present result and Danish et al.98 is shown in Table 4 and velocity is minimum for constant proportional Caputo model. The velocity comparison between present result when $\lambda = 0, \alpha = 1, \text{Imran et al.}65$ when $\alpha = 1$ and Khan et al.98 is shown in Fig. 10. We see that these results are in good agreement. Table 5 represents the velocity contrast of present result and published results and we see that results are same. The influence of fractional parameter $\beta$ on Nusselt number is studied numerically in Table 6. Nusselt number is decreasing function of fractional parameter $\beta$. The influence of fractional parameter $\alpha$ on Skin friction is evaluated numerically in Table 7. Skin friction is decreasing function of fractional parameter $\alpha$.

Conclusions

The convection heat transfer in clay nanofluid using Maxwell model is studied. Exact solutions for velocity and temperature are evaluated with help of the Laplace transform technique. We have drawn the comparisons with the published results and they are in good agreement. Key findings of current study are:

1. Velocity of drilling fluid, for small values of time shows decay behavior for increasing fractional parameter and concentration of nanoparticles.
2. Water based drilling nanofluids exhibit maximum velocity rather than oil based drilling fluids.
3. Different Comparisons of present result of velocity are drawn with Khan et al.59, Imran et al.61 and Danish et al.69.

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M.A.I and R. A. formulate the problem, M.A. I and Y. M. C and T. M. wrote the paper and review the paper, R. A. A. I. and M.A. I prepared figures and wrote discussion. A.I. and T.M. made the response to the reviewers comments and revised the main manuscript. All the authors reviewed and approved the final manuscript.

Competing interests
The authors declare no competing interests.

Additional information
Correspondence and requests for materials should be addressed to Y.-M.C.

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