Lie Group Action and Stability Analysis of Stationary Solutions for a Free Boundary Problem Modelling Tumor Growth

Shangbin Cui

Institute of Mathematics, Sun Yat-Sen University, Guangzhou, Guangdong 510275, People’s Republic of China. E-mail: cuib3@yahoo.com.cn

Abstract

In this paper we study asymptotic behavior of solutions for a multidimensional free boundary problem modelling the growth of nonnecrotic tumors. We first establish a general result for differential equations in Banach spaces possessing a local Lie group action which maps a solution into new solutions. We prove that a center manifold exists under certain assumptions on the spectrum of the linearized operator without assuming that the space in which the equation is defined is of either $D_A(\theta)$ or $D_A(\theta, \infty)$ type. By using this general result and making delicate analysis of the spectrum of the linearization of the stationary free boundary problem, we prove that if the surface tension coefficient $\gamma$ is larger than a threshold value $\gamma^*$ then the unique stationary solution is asymptotically stable modulo translations, provided the constant $c$ representing the ratio between the nutrient diffusion time and the tumor-cell doubling time is sufficiently small, whereas if $\gamma < \gamma^*$ then this stationary solution is unstable.

Keywords and phrases: Free boundary problem, tumor growth, asymptotic stability, center manifold, local Lie group.

AMS subject classification 34G20, 35B35, 35R35, 47H20, 76D27.

1 Introduction

This paper aims at studying asymptotic behavior of solutions of the following free boundary problem:

\begin{align*}
 c\partial_t \sigma &= \Delta \sigma - f(\sigma), \quad x \in \Omega(t), \quad t > 0, \\
 -\Delta p &= g(\sigma), \quad x \in \Omega(t), \quad t > 0, \\
 \sigma &= \bar{\sigma}, \quad x \in \partial \Omega(t), \quad t > 0, \\
 p &= \gamma \kappa, \quad x \in \partial \Omega(t), \quad t > 0, \\
 \mathbf{V} &= -\partial_n p, \quad x \in \partial \Omega(t), \quad t > 0, \\
 \sigma(x, 0) &= \sigma_0(x), \quad x \in \Omega_0, \\
 \Omega(0) &= \Omega_0.
\end{align*}

(1.1) - (1.7)
Here $\sigma = \sigma(x,t)$ and $p = p(x,t)$ are unknown functions defined on the space-time manifold $\cup_{t\geq0}(\Omega(t) \times \{t\})$, and $\Omega(t)$ is an a priori unknown bounded time-dependent domain in $\mathbb{R}^n$, whose boundary $\partial \Omega(t)$ has to be determined together with the unknown functions $\sigma$ and $p$. Besides, $f$ and $g$ are given functions, $c$, $\tilde{\sigma}$ are $\gamma$ are positive constants, $\kappa$, $V$ and $n$ are the mean curvature, the normal velocity and the unit outward normal vector of $\partial \Omega(t)$, respectively, and $\sigma_0$, $\Omega_0$ are given initial data of $\sigma = \sigma(\cdot,t)$ and $\Omega = \Omega(t)$, respectively. The sign of $\kappa$ is fixed on by the condition that $\kappa \geq 0$ at points where $\partial \Omega(t)$ is convex with respect to $\Omega(t)$.

The above problem arises from recently developed subject of tumor growth modelling. It models the growth of tumors cultivated in laboratory or so-called multilcellular spheroids ([11], [15], [7], [26], [27], [29], [32]). In this model $\Omega(t)$ represents the domain occupied by the tumor at time $t$, $\sigma$ and $p$ stand for the nutrient concentration and the tumor tissue pressure, respectively, and $f(\sigma)$, $g(\sigma)$ are the nutrient consumption rate and the tumor cell proliferation rate, respectively. It is assumed that all tumor cells are alive and dividable, and their density is constant, so that in $f$ and $g$ no cell density argument is involved. It is also assumed that the tumor is cultivated in a solution of nutrition materials whose concentration keeps constant in the process of cultivation, and $\tilde{\sigma}$ reflects this constant nutrient supply to the tumor. The term $\gamma \kappa$ on the right-hand side of (1.4) stands for surface tension of the tumor. The equation (1.5) reflects the fact that the normal velocity of the tumor surface is equal to the normal component of the movement velocity of tumor cells adjacent to the surface. For more details of the modelling we refer the reader to see the references [11], [6], [7], [9], [11], [14]–[16] and [26]. Here we point out that, by rescaling which we have pre-assumed and did not particularly mention, the constant $c$ represents the ratio between the nutrient diffusion time and the tumor-cell doubling time, so that $c \ll 1$, cf. [11], [6], and [7]. Finally, we make the following assumptions on the functions $f$ and $g$:

(A1) $f \in C^\infty[0,\infty)$, $f'(\sigma) > 0$ for $\sigma \geq 0$ and $f(0) = 0$.

(A2) $g \in C^\infty[0,\infty)$, $g'(\sigma) > 0$ for $\sigma \geq 0$ and there exists a number $\tilde{\sigma} > 0$ such that $g(\tilde{\sigma}) = 0$ ($\Rightarrow g(\sigma) < 0$ for $0 \leq \sigma < \tilde{\sigma}$ and $g(\sigma) > 0$ for $\sigma > \tilde{\sigma}$).

(A3) $\tilde{\sigma} < \sigma$.

These assumptions are based on biological considerations, see [11], [15] and [16].

Local well-posedness of the above problem has been recently established by the author in a more general framework in the reference [14] by using the analytic semigroup theory, which extends and modifies an earlier work of Escher [20] for the special case that $f(\sigma) = f(\sigma)$ but $g(\sigma) = \mu(\sigma - \tilde{\sigma})$. In this paper we consider the more difficult topic of asymptotic behavior of the solution. More precisely, from [11] and [15] we know that under the above assumptions (A1)–(A3), the system (1.1)–(1.5) has a radially symmetric stationary solution $(\sigma_s, p_s, \Omega_s)$, which is unique up to translations and rotations of the coordinate of $\mathbb{R}^n$ and globally asymptotically stable under radially symmetric perturbations. This paper aims at studying the following question: Is $(\sigma_s, p_s, \Omega_s)$ also asymptotically stable under non-symmetric perturbations?

We first make a short review to previous work on this topic. Rigorous analysis of free boundary problems of partial differential equations arising from tumor growth modelling has attracted a lot of attention during the past several years, and many interesting results have been systematically derived, cf. [4], [5], [6], [18], [20], [22]–[24], and the references cited therein. As far as the problem (1.1)–(1.7) and its certain more specific forms are concerned, we cite
the references [4], [5], [9], [11], [14]–[16], [20], [22]–[24]. In particular, in [23] Friedman and Reitich considered radially symmetric version of the problem (1.1)–(1.7) in the special case that 
\[ f(\sigma) = \lambda \sigma \quad \text{and} \quad g(\sigma) = \mu(\sigma - \bar{\sigma}). \]
Under the assumption (A3), they proved the following results:
1. The problem is globally well-posed.
2. There exists a unique stationary solution.
3. For \( c \) sufficiently small this stationary solution is globally asymptotically stable.
4. For \( c \) large the stationary solution is unstable.

The author of the present paper has recently extended the assertions (1), (2), (3) to the general case that \( f \) and \( g \) are general functions satisfying the conditions (A1)–(A3), see [14].

The general non-symmetric version of (1.1)–(1.7) in the special case that \( f(\sigma) = \lambda \sigma \) and \( g(\sigma) = \mu(\sigma - \bar{\sigma}) \) has also been systematically studied by Friedman and his collaborators. Bazaliy and Friedman investigated local well-posedness of the time-dependent problem in the reference [4]. In [5] they studied asymptotic behavior of the solution starting from a neighborhood of the unique radially symmetric stationary solution ensured by the above assertion (2), and proved that, for \( c = 1, \lambda = 1, \gamma = 1 \) and \( \mu \) sufficiently small, the radially symmetric stationary solution is (locally) asymptotically stable under non-radial perturbations.

This work was recently refined by Friedman and Hu [22]. They proved that, again for \( c = 1, \lambda = 1 \) and \( \gamma = 1 \), there exists a threshold value \( \mu^* > 0 \), such that for \( 0 < \mu < \mu^* \) the radially symmetric stationary solution is (locally) asymptotically stable under non-radial perturbations, while for \( \mu > \mu^* \) this stationary solution is unstable.

In a recent work of the present author jointly with Escher [16], the problem (1.1)–(1.7) with general functions \( f \) and \( g \) satisfying (A1)–(A3) but \( c = 0 \) was studied. We proved that there exists a threshold value \( \gamma_0 > 0 \), the supremum of all bifurcation points \( \gamma_k \) \( (k = 2, 3, \ldots) \), see [15]), such that if \( \gamma > \gamma_0 \) then the radially symmetric stationary solution \( (\sigma_s, p_s, \Omega_s) \) is (locally) asymptotically stable modulo translations, i.e., any solution starting from a small neighborhood of \( (\sigma_s, p_s, \Omega_s) \) is global and, as \( t \to \infty \), it converges to either \( (\sigma_s, p_s, \Omega_s) \) or an adjacent stationary solution \( (\sigma_s, p_s, \Omega_s') \) obtained by translating \( (\sigma_s, p_s, \Omega_s) \) (recall that any translation of \( (\sigma_s, p_s, \Omega_s) \) is still a stationary solution), whereas if \( \gamma < \gamma_0 \) then \( (\sigma_s, p_s, \Omega_s) \) is unstable.

In this paper we want to extend the above result of [16] for the degenerate case \( c = 0 \) to the more difficult non-degenerate case \( c \neq 0 \), assuming that \( c \) is sufficiently small. The main idea of analysis is the same with that of [16], namely, we shall first reduce the PDE problem into a differential equation in a Banach space and next use the abstract geometric theory for parabolic differential equations in Banach spaces to get the desired result. However, unlike in [16] where we used the well-developed center manifold theorem by Da Prato and Lunardi [19] to make the analysis, in this paper we shall have to first establish a new center manifold theorem, because the above-mentioned center manifold of Da Prato and Lunardi is not applicable to the case \( c \neq 0 \). The reason is as follows. Recall that the center manifold theorem of Da Prato and Lunardi requires the Banach space in which the differential equation is considered must be of the type either \( D_\Lambda(\theta) \), the continuous interpolation space, or \( D_\Lambda(\theta, \infty) \), the real interpolation space of the type \( (\theta, \infty) \) \( (0 < \theta < 1) \). Such spaces cannot be reflexive (cf. [3], [91]). In the degenerate case \( c = 0 \) the reduced equation contains only the unknown function \( \rho \) defining the free boundary \( \partial \Omega(t) \), which is a quasi-linear parabolic pseudo-differential equation on a compact manifold, so that no boundary conditions appear and we can thus work on the little Hölder space \( h^{m+\alpha} \) which is of the type \( D_\Lambda(\theta) \). In the present non-degenerate case \( c \neq 0 \), however, since the reduced equation contains not only \( \rho \) but also the unknown \( \sigma \), the Dirichlet boundary condition
for $\sigma$ renders it impossible for us to work on a space of the type either $D_A(\theta)$ or $D_A(\theta, \infty)$.

To remedy this deficiency, in this paper we shall first establish a new center manifold theorem which removes this very restrictive assumption on the space $X$, but instead we shall assume that the equation admits a local Lie group action by which a solution is mapped into new solutions. We shall show that the phase diagram of a differential equation possessing such a Lie group action has a very nice structure and its center manifold can be very easily obtained. In particular, this new center manifold theorem does not make any additional assumption on the structure of the space $X$. Since the differential equation reduced from the the problem (1.1)–(1.7) naturally possesses a Lie group action induced by translations of the coordinate of $\mathbb{R}^n$, by using this new center manifold result we are able to make analysis in the framework of Sobolev and Besov spaces. Our final result says that similar assertions as for the case $c = 0$ also hold for the case $c \neq 0$.

It should be noted that our center manifold theorem for differential equations in Banach spaces possessing Lie group action established in this paper not only works for the tumor model (1.1)–(1.7) as well as its special form of the case $c = 0$ but also applies to other problems such as the Hele-Shaw problem. Thus, the center manifold theorem established in this paper has its own theoretic importance. More applications of this result will be given in our future work.

To give a precise statement of our main result, let us first introduce some notation. Recall that the radially symmetric stationary solution $(\sigma_s, p_s, \Omega_s)$ of (1.1)–(1.5), where $\Omega_s = \{ r < R_s \}$ with $r = |x|$, is the unique solution of the following free boundary problem:

\begin{align}
\sigma''_s(r) + \frac{n-1}{r} \sigma'_s(r) &= f(\sigma_s(r)), \quad 0 < r < R_s, \\
p''_s(r) + \frac{n-1}{r} p'_s(r) &= -g(\sigma_s(r)), \quad 0 < r < R_s, \\
\sigma'_s(0) &= 0, \quad \sigma_s(R) = \bar{\sigma}, \\
p'_s(0) &= 0, \quad p_s(R_s) = \frac{\gamma}{R_s}, \\
p'_s(R_s) &= 0.
\end{align}

For $z \in \mathbb{R}^n$, we denote

$$
\sigma^z(x) = \sigma_s(|x - z|), \quad p^z_s(x) = p_s(|x - z|), \quad \Omega^z_s = \{ x \in \mathbb{R}^n : |x - z| < R_s \}.
$$

Clearly, for any $z \in \mathbb{R}^n$ the triple $(\sigma^z_s, p^z_s, \Omega_s^z)$ is a stationary solution of the system (1.1)–(1.5). If $|z|$ is sufficiently small then there exists a unique $p^z_s \in C^\infty(S^{n-1})$ which is sufficiently close to the constant function $R_s$, such that

$$
\Omega^z_s = \{ r < p^z_s(\omega), \ \omega \in S^{n-1} \}.
$$

Since we shall only consider solutions of (1.1)–(1.7) which are close to the stationary solution $(\sigma_s, p_s, \Omega_s)$, we can write $\Omega(t)$ as $\Omega(t) = \{ r < \rho(\omega, t), \ \omega \in S^{n-1} \}$ for some $\rho(\cdot, t) \in C(S^{n-1})$ for every $t > 0$, and, correspondingly, we write $\Omega_0$ as $\Omega_0 = \{ r < p_0(\omega), \ \omega \in S^{n-1} \}$, where
\( \rho_0 \in C(\mathbb{S}^{n-1}) \). Finally, from [15] we know that the linearization of the stationary version of (1.1)–(1.5) has infinite many eigenvalues \( \gamma_k, \ k = 2, 3, \ldots \), which are all positive and converge to zero as \( k \to \infty \). As in [16] we set

\[
\gamma_* = \max \{ \gamma_k, \ k = 2, 3, \ldots \}.
\]

The main result of this paper is as follows:

**Theorem 1.1** If \( \gamma > \gamma_* \) then there exists a corresponding \( c_0 > 0 \) such that for any \( 0 < c < c_0 \), the stationary solution \( (\sigma_s, p_s, \Omega_s) \) of (1.1)–(1.5) is asymptotically stable modulo translations in the following sense: There exists \( \varepsilon > 0 \) such that for any \( \rho_0 \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \) and \( \sigma_0 \in W^{m,q}(\Omega_0) \) (\( m \in \mathbb{N}, m \geq 5, n/(m - 4) < q < \infty \)) satisfying

\[
\|\rho_0 - R_s\|_{B_{qq}^{m-1/q}(\mathbb{S}^{n-1})} < \varepsilon, \quad \|\sigma_0 - \sigma_s\|_{W^{m,q}(\Omega_0)} < \varepsilon, \quad \sigma_0|_{\partial \Omega_0} = \bar{\sigma},
\]

the problem (1.1)–(1.7) has a unique solution \( (\sigma, p, \Omega) \) for all \( t \geq 0 \), and there exists \( z \in \mathbb{R}^n \) uniquely determined by \( \rho_0 \) and \( \sigma_0 \) such that

\[
\|\sigma(\cdot, t) - \sigma^*_s\|_{W^{m-1,q}(\Omega(t))} + \|p(\cdot, t) - p^*_s\|_{W^{m-3,q}(\Omega(t))} + \|\rho(\cdot, t) - \rho^*_s\|_{B_{qq}^{m-1/q}(\mathbb{S}^{n-1})} \leq Ce^{-\kappa t}
\]

for some \( C > 0, \kappa > 0 \) and all \( t \geq 0 \). If \( \gamma < \gamma_* \) then there also exists a corresponding \( c_0 > 0 \) such that for any \( 0 < c < c_0 \), \( (\sigma_s, p_s, \Omega_s) \) is unstable.

**Remark 1.1.** By the assertion (4) of Friedman and Reitich reviewed before, we see that the condition \( c < c_0 \) cannot be removed. Besides, as we mentioned earlier, though we only consider solutions in \( W^{m-1,q} \times W^{m-3,q} \times B_{qq}^{m-1/q} \), a similar result surely also holds for solutions in the space \( C^{m+\alpha} \times C^{m-2+\alpha} \times C^{m+\alpha} \). In addition, the conditions \( m \geq 5 \) and \( n/(m - 4) < q < \infty \) can be weakened up to \( m \geq 3 \) and \( n/(m - 2) < q < \infty \). To achieve this improvement we need a modified version of Theorem 2.1 of the next section; see Remark 2.1 in the end of Section 2.

The proof of the above theorem will be given in the last section of this paper, after step-by-step preparations in Sections 2–6. The layout of the rest part is as follows. In Section 2 we establish the general result for differential equations in Banach spaces mentioned earlier. In Section 2 we first use the so-called Hanzawa transformation to transform the problem (1.1)–(1.7) into an equivalent problem on the fixed domain \( \Omega_s \), which for simplicity of notation will be assumed to be the unit sphere \( \mathbb{B}^n \) later on, and next we further reduce the PDE problem into a differential equation in the Banach space \( W^{m-3,q}(\mathbb{B}^n) \times B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1}) \) for the unknowns \( (\sigma, \rho) \). In Section 4 we construct Lie group action for the reduced differential equation. In Section 5 we compute the linearization of the reduced equation. Section 6 aims at studying the spectrum of the linearized problem. In the last section we complete the proof of Theorem 1.1.

### 2 An abstract result

Let \( X \) and \( X_0 \) be two Banach spaces such that \( X_0 \hookrightarrow X \). \( X_0 \) need not be dense in \( X \). Let \( \mathcal{O} \) be an open subset of \( X_0 \). Let \( F \in C^{2-0}(\mathcal{O}, X) \), i.e. \( F \in C^1(\mathcal{O}, X) \) and \( F' = DF = \text{the Fréchet} \)
derivative of $F$) is Lipschitz continuous. In this section we consider the initial value problem

$$
\begin{align*}
    & u'(t) = F(u(t)), \quad t > 0, \\
    & u(0) = u_0,
\end{align*}
$$

where $u_0 \in \mathcal{O}$. By a \textit{solution} of (2.1) we mean a solution of the class $u \in C([0,T), X) \cap C((0, T), \mathcal{O}) \cap L^\infty((0, T), \mathcal{O}) \cap C^1((0, T), X)$ defined in a maximal existence interval $I = [0, T)$ ($0 < T \leq \infty$), which satisfies (2.1) in $[0, T)$ in usual sense and is not extendable. If $u$ satisfies the stronger condition $u \in C([0, T), \mathcal{O}) \cap C^1((0, T), X)$ then we call it a \textit{strict solution}. Later on we shall denote by $u(t, u_0)$ the solution of (2.1) when it exists and is unique. We always assume that for some $u_s \in \mathcal{O}$ there holds $F(u_s) = 0$, so that $u(t) = u_s$, $t \geq 0$, is a stationary solution of the equation $u' = F(u)$. We want to study asymptotic stability of $u_s$. Our first assumption is as follows:

$$(B_1) \quad A = F'(u_s) \text{ is a sectorial operator in } X \text{ with domain } X_0, \text{ and the graph norm of } A \text{ is equivalent to the norm of } X_0: \|u\|_{X_0} \sim \|u\|_X + \|Au\|_X.$$ 

Next, we consider some invariance property of $F$. Let $G$ be a \textit{local Lie group of dimension} $n$ in the sense of L. S. Pontrjagin [30]. Let $\mathcal{O}'$ be an open subset of $X$ such that $\mathcal{O} \subseteq \mathcal{O}'$. Let $\mathcal{O}_1$ be an open subset of $X_0$ contained in $\mathcal{O}$, and $\mathcal{O}'_1$ be an open subset of $X$ contained in $\mathcal{O}'$, such that $u_s \in \mathcal{O}_1 \subseteq \mathcal{O}'_1$. We assume that there is a continuous mapping $p: G \times \mathcal{O}'_1 \rightarrow \mathcal{O}'$, such that

(i) $p(G \times \mathcal{O}_1) \subseteq \mathcal{O};$

(ii) $p(e, u) = u$ for every $u \in \mathcal{O}'_1$, where $e$ denotes the unit of $G$, and $p(\sigma, p(\tau, u)) = p(\sigma \tau, u)$ for any $u \in \mathcal{O}'_1$ and $\sigma, \tau \in G$ such that $\sigma \tau$ is well-defined and $p(\sigma, u) \in \mathcal{O}'_1;$

(iii) If $\sigma, \tau \in G$ such that $p(\sigma, u) = p(\tau, u)$ for some $u \in \mathcal{O}'_1$ then $\sigma = \tau$.

(iv) For any $\sigma \in G$, the mapping $u \rightarrow p(\sigma, u)$ from $\mathcal{O}'_1$ to $\mathcal{O}'$ is Fréchet differentiable at every point in $\mathcal{O}_1$, and $|u \rightarrow D_\sigma p(\sigma, u)| \subseteq C(\mathcal{O}_1, L(X, X)).$

(v) For any $u \in \mathcal{O}_1$, the mapping $\sigma \rightarrow p(\sigma, u)$ from $G$ to $\mathcal{O}$ is continuously Fréchet differentiable when regarded as a mapping from $G$ to $X$ ($\quad \implies D_\sigma p(\sigma, u) \subseteq L(T_\sigma(G), X)$, and $|\sigma \rightarrow p(\sigma, u)| \subseteq C^1(G, X))$. Moreover, $\text{rank}D_\sigma p(\sigma, u) = n$ for every $\sigma \in G$ and $u \in \mathcal{O}_1$.

Later on we denote $S_\sigma(u) = p(\sigma, u)$ for $\sigma \in G$ and $u \in \mathcal{O}_1$. Our second assumption is as follows:

$$(B_2) \quad \text{There is a local Lie group } G \text{ satisfying the properties (i)–(v), such that for any } u \in \mathcal{O}_1 \text{ and } \sigma \in G \text{ there holds}$$

$$F(S_\sigma(u)) = DS_\sigma(u)F(u). \quad (2.2)$$

This assumption has some obvious inferences. First, it implies that for any $u_0 \in \mathcal{O}_1$ and $\sigma \in G$ there holds $u(t, S_\sigma(u_0)) = S_\sigma(u(t, u_0))$, namely, if $t \rightarrow u(t)$ is a solution of the equation $u' = F(u)$ with initial value $u_0$, then $t \rightarrow S_\sigma(u(t))$ is also a solution of this equation, with
initial value $S_\sigma(u_0)$. In particular, for any $\sigma \in G$, $S_\sigma(u_s)$ is a stationary solution of $u' = F(u)$.

Next, if $u_s$ is more regular than $(v)$ in the sense that $[\sigma \to p(\sigma, u_s)] \in C^1(G, X_0)$ (so that $D_\sigma p(\sigma, u_s) \in L(T_\sigma(G), X_0)$ for any $\sigma \in G$), then by differentiating the relation $F(S_\sigma(u_s)) = 0$ in $\sigma = e$ we see that $DF(u_s)D_\sigma p(e, u_s)\xi = 0$ for any $\xi \in T_e(G)$, so that $A = DF(u_s)$ is degenerate, and $\dim \ker A \geq n$. We now assume that

\[ (B_3) \quad [\sigma \to p(\sigma, u_s)] \in C^1(G, X_0), \quad \dim \ker A = n, \text{ and the induced operator } \bar{A} : X_0/\ker A \to X/\ker A \text{ of } A \text{ is an isomorphism.} \]

Here and throughout this paper, by isomorphism from a Banach space $X_1$ to another Banach space $X_2$ we mean a linear mapping $T : X_1 \to X_2$ such that it is an 1-1 correspondence, and both $T$ and $T^{-1}$ are continuous (i.e., $T$ is not merely a linear isomorphism, but a topological homeomorphism as well). Finally, we assume that

\[ (B_4) \quad \omega_- \equiv -\sup\{\text{Re}\lambda : \lambda \in \sigma(A) \setminus \{0\}\} = -\sup\{\text{Re}\lambda : \lambda \in \sigma(\bar{A})\} > 0. \]

We point out that the condition $(B_3)$ is equivalent to the following condition:

\[ (B_3') \quad \dim \ker A = n, \text{ Range } A \text{ is closed in } X, \text{ and } X = \ker A \oplus \text{Range } A. \]

The main result of this section is as follows:

**Theorem 2.1** Let the assumptions $(B_1)$--$(B_4)$ be satisfied. Then there exists a neighborhood $O_2$ of $u_s$, $O_2 \subseteq O_1$, such that the following assertions hold:

1. For any $u_0 \in O_2$ the problem $(2.1)$ has a unique solution $u(t, u_0)$ which exists for all $t \geq 0$, and if furthermore $F(u_0) \in X_0$, then $u(t, u_0)$ is a strict solution.

2. The center manifold of the equation $u' = F(u)$ in $O_2$ is given by $M^c = \{S_\sigma(u_s) : \sigma \in G\} \cap O_2$, which is a $C^{2-0}$ manifold of dimension $n$ and consists of all stationary solutions of this equation in $O_2$.

3. There exists a $C^{2-0}$ submanifold $M^s \subseteq O_2$ of codimension $n$ in $X_0$ passing $u_s$, such that for any $u_0 \in M^s$ there holds $\lim_{t \to \infty} u(t, u_0) = u_s$ and vice versa, i.e. $M^s$ is the stable manifold of $u_s$ in $O_2$.

4. For every $u_0 \in O_2$ there exist a unique $\sigma \in G$ and a unique $v_0 \in M^s$ such that $u_0 = S_\sigma(v_0)$, and we have

\[ \lim_{t \to \infty} u(t, u_0) = S_\sigma(u_s). \]

Moreover, for any $0 < \omega < \omega_-$ there exists corresponding $C = C(\omega) > 0$ such that

\[ \|u(t, u_0) - S_\sigma(u_s)\|_{X_0} \leq C e^{-\omega t}\|u_0 - S_\sigma(u_s)\|_{X_0} \text{ for all } t \geq 0. \]

To prove this theorem, we need a preliminary lemma. Let $X$ be a Banach space. Let $\alpha \in (0, 1)$ and $T > 0$. Recall that $C^\alpha_0((0, T], X)$ is the Banach space of bounded mappings $u : (0, T] \to X$ such that $t^\alpha u(t)$ is uniformly $\alpha$-Hölder continuous for $0 < t \leq T$, with norm

\[ \|u\|_{C^\alpha_0((0, T], X)} = \sup_{0 < t \leq T} \|u(t)\|_X + \sup_{0 < s < t \leq T} \frac{\|t^\alpha u(t) - s^\alpha u(s)\|_X}{(t - s)^\alpha}. \]
For $\omega > 0$, $C^\alpha([T, \infty), X, -\omega)$ is the Banach space of bounded mappings $u : [T, \infty) \to X$ such that $e^{w t} u(t)$ is uniformly $\alpha$-Hölder continuous for $t \geq T$, with norm

$$
\|u\|_{C^\alpha([T, \infty), X, -\omega)} = \sup_{t \geq T} \|e^{w t} u(t)\|_X + \sup_{t, s \geq T, t > s} \frac{\|e^{w t} u(t) - e^{w s} u(s)\|_X}{(t - s)^\alpha}.
$$

**Lemma 2.2** Let $X$ and $X_0$ be two Banach spaces such that $X_0 \hookrightarrow X$. Let $A$ be a sectorial operator in $X$ with domain $X_0$. Assume that $\omega_0 = -\sup\{\Re \lambda : \lambda \in \sigma(A)\} > 0$ and $f \in C^{\alpha}_0((0, 1], X) \cap C^\alpha([1, \infty), X, -\omega)$, where $\alpha \in (0, 1)$ and $\omega \in (0, \omega_0)$. Let $u(t) = e^{t A} u_0 + \int_0^t e^{(t-s) A} f(s) ds$, where $u_0 \in X_0$. Then $u \in C^\alpha_0((0, 1], X_0) \cap C^\alpha([1, \infty), X_0, -\omega)$, and there exists constant $C = C(\alpha, \omega) > 0$ independent of $f$ and $u_0$ such that

$$
\|u\|_{C^\alpha_0((0,1],X_0)} + \|u\|_{C^\alpha([1,\infty),X_0,-\omega)} \leq C(\|u_0\|_{X_0} + \|f\|_{C^\alpha_0((0,1],X_0)} + \|f\|_{C^\alpha([1,\infty),X_0,-\omega)}).
$$

**Proof:** By Theorem 4.3.5 and Corollary 4.3.6 (ii) of [28] we have $\|u\|_{C^\alpha_0((0,1],X_0)} \leq C(\|u_0\|_{X_0} + \|f\|_{C^\alpha_0((0,1],X)}$, and by Proposition 4.4.10 (i) of [28] we have $\|u\|_{C^\alpha([1,\infty),X_0,-\omega)} \leq C(\|u_0\|_{X} + \|f\|_{L^1([0,1],X)} + \|f\|_{C^\alpha([1,\infty),X_0,-\omega)}$. Hence (2.5) holds. \qed

**Proof of Theorem 2.1:** Without loss of generality we assume that $u_s = 0$. Since we are studying solutions of (2.1) in a neighborhood of 0, by the assumption $(B_1)$ and a standard perturbation result, we may assume that $F'(u)$ is a sectorial operator for every $u \in \mathcal{O}$ (with domain $X_0$), and the graph norm of $F'(u)$ is equivalent to the norm of $X_0$. It follows by a standard result (cf. Theorem 8.1.1 of [28]) that for any $u_0 \in X_0$, the problem (2.1) has a unique local solution $u \in C([0, T], X) \cap C((0, T], X_0) \cap L^\infty((0, T), X) \cap C^1([0, T], X) \cap C^\alpha_0((0, T], X_0)$, and if further $F(u_0) \in X_0$ then $u \in C([0, T], X) \cap C^1([0, T], X) \cap C^\alpha_0((0, T], X_0)$, where $T > 0$ depends on $u_0$ and $\alpha$ is an arbitrary number in $(0, 1)$. Moreover, denoting by $T^*(u_0)$ the supreme of all such $T$, we know that there exists a constant $\varepsilon > 0$ independent of $u_0$ such that if $\|u(t, u_0)\|_{X_0} < \varepsilon$ for all $t \in [0, T^*(u_0))$, then $T^*(u_0) = \infty$ (cf. Proposition 9.1.1 of [28]).

Next we denote $\sigma_-(A) = \sigma(A) \setminus \{0\}$. Let $\Gamma$ be a closed smooth curve in the complex plane which encloses 0 and separates it from $\sigma_-(A)$, and let $P$ be the projection operator in $X$ defined by

$$
P = \frac{1}{2\pi i} \int_\Gamma R(\lambda, A) d\lambda.
$$

Since $X = \ker A \oplus \operatorname{Range} A$, we have $PX = PX_0 = \ker A$, $(I - P)X = \operatorname{Range} A$ (cf. Proposition A.2.2 of [28]), and $AP = 0$. Let $A_* = (I - P) A|_{(I - P)X_0} : (I - P)X_0 \to (I - P)X$. Then $\sigma(A_*) = \sigma(A) \setminus \{0\}$, so that $\sup\{\Re \lambda : \lambda \in \sigma(A_*)\} = -\omega_- < 0$. Besides, by the assumption $(B_3)$ we see that $A_* : (I - P)X_0 \to (I - P)X$ is an isomorphism.

Let $\mathcal{M}^c = \{S_\sigma(0) : \sigma \in G\}$. By $(v)$ in the assumption $(B_2)$ we see that $\mathcal{M}^c$ is a $C^1$ submanifold of $X$, and $\dim \mathcal{M}^c = n$. The equation $u = S_\sigma(0)$ ($\sigma \in G$) gives a parametrization of $\mathcal{M}^c$ by $G$. We can also give a parametrization of $\mathcal{M}^c$ by $PX$ as follows. For $u \in \mathcal{O}$ let $x = Pu$ and $y = (I - P)u$. Take two sufficiently small numbers $\delta > 0$ and $\delta' > 0$ such that $x \in B_1(0, \delta)$
and \( y \in B_2(0, \delta') \) imply that \( u = x + y \in \mathcal{O}_1 \), where

\[
B_1(0, \delta) = \{ x \in PX : \|x\|_{x_0} < \delta \} \quad \text{and} \quad B_2(0, \delta') = \{ y \in (I - P)X_0 : \|y\|_{x_0} < \delta' \}.
\]

For \( (x, y) \in B_1(0, \delta) \times B_2(0, \delta') \) we denote \( \mathcal{F}_1(x, y) = PF(x + y) \) and \( \mathcal{F}_2(x, y) = (I - P)F(x + y) \). We have \( \mathcal{F}_2 \in C^{2,0}(B_1(0, \delta) \times B_2(0, \delta'), (I - P)X) \), \( \mathcal{F}_2(0, 0) = 0 \), and \( D_y \mathcal{F}_2(0, 0) = A_- \). Since \( A_- : (I - P)X_0 \to (I - P)X \) is an isomorphism, by the implicit function theorem we infer that if \( \delta \) is sufficiently small then there exists \( \varphi \in C^{2,0}(B_1(0, \delta), B_2(0, \delta')) \) such that \( \varphi(0) = 0 \), \( \mathcal{F}_2(x, \varphi(x)) = 0 \) for every \( x \in B_1(0, \delta) \), and for \( (x, y) \in B_1(0, \delta) \times B_2(0, \delta') \), \( \mathcal{F}_2(x, y) = 0 \) if and only if \( y = \varphi(x) \). It follows that the equation \( \mathcal{F}_2(x, y) = 0 \) defines a \( C^{2,0} \) submanifold \( \mathcal{M}_0 \) of dimension \( n \). Since \( F(S_\sigma(0)) = 0 \) for every \( \sigma \in G \), which particularly implies that \( (I - P)F(S_\sigma(0)) = 0 \) for every \( \sigma \in G \), we conclude that \( \mathcal{M}^c \cap B_1(0, \delta) \times B_2(0, \delta') = \mathcal{M}_0 \). Hence, the equation \( y = \varphi(x) \) gives a parametrization of \( \mathcal{M}^c \) by \( PX \). Furthermore, from this argument we also see that \( \mathcal{F}_1(x, \varphi(x)) = P F(S_\sigma(0)) = 0 \) for every \( x \in B_1(0, \delta) \). Note that since \( D_x \mathcal{F}_2(0, 0) = (I - P)AP = 0 \), we have \( \varphi'(0) = -[D_y \mathcal{F}_2(0, 0)][1]D_x \mathcal{F}_2(0, 0) = 0 \).

Let \( N(u) = F(u) - Au \) (for \( u \in \mathcal{M}_1 \)), \( N_1(x, y) = PN(x + y) \) and \( N_2(x, y) = (I - P)N(x + y) \) (for \( (x, y) \in B_1(0, \delta) \times B_2(0, \delta') \)). Let \( x_0 = Pu_0 \) and \( y_0 = (I - P)u_0 \). Then (2.1) is equivalent to the following problem:

\[
\begin{cases}
  x' = N_1(x, y), & x(0) = x_0, \\
y' = A_- y + N_2(x, y), & y(0) = y_0.
\end{cases}
\]

(2.6)

Let \( (x, y) = (x(t), y(t)) \) be the solution of (2.6) defined in a maximal interval \([0, T*) \) such that it exists for all \( t \in [0, T*) \) and lies in \( B_1(0, \delta) \times B_2(0, \delta') \). Since \( (x, y) = (0, 0) \) is a solution defined for all \( t \geq 0 \), by continuous dependence of solutions on initial data, we see that there exists a neighborhood \( \mathcal{O}_2 \) of \( 0 \) contained in \( B_1(0, \delta) \times B_2(0, \delta') \), such that for any \( u_0 \in \mathcal{O}_2 \) there holds \( T_* > 1 \). In the sequel we assume that \( u_0 \in \mathcal{O}_2 \) so that \( T_* > 1 \). Let \( v(t) = y(t) - \varphi(x(t)) \). Since \( A_- \varphi(x) + N_2(x, \varphi(x)) = \mathcal{F}_2(x, \varphi(x)) = 0 \) and \( N_1(x, \varphi(x)) = F_1(x, \varphi(x)) = 0 \) for all \( x \in B_1(0, \delta) \), we have

\[
v'(t) = A_- v(t) + [N_2(x(t), y(t)) - N_2(x(t), \varphi(x(t)))] - \varphi'(x(t))[N_1(x(t), y(t)) - N_1(x(t), \varphi(x(t)))] \equiv A_- v(t) + \mathcal{G}(t),
\]

so that

\[
v(t) = e^{t A_-} v(0) + \int_0^t e^{(t - s) A_-} \mathcal{G}(s) ds.
\]

It follows by Lemma 2.2 that for any \( 0 < \alpha < 1 \) and \( \omega \in (0, \omega_-) \) we have

\[
\|v\|_{C^\alpha_\omega([0,1],X_0)} + \|v\|_{C^\alpha([1,T_*],X_0,-\omega)} \leq C (\|v(0)\|_{X_0} + \|\mathcal{G}\|_{C^\alpha_\omega([0,1],X)}) + \|\mathcal{G}\|_{C^\alpha([1,T_*],X,-\omega)},
\]

(2.7)

where \( C^\alpha([1, T_*], X, -\omega) \) is defined similarly as \( C^\alpha([1, \infty), X, -\omega) \), with \( \infty \) replaced with \( T_* \). Note that all assertions in Lemma 2.2 clearly hold when \( \infty \) is replaced by any \( T_* \in (1, \infty) \). By a similar argument as in the proof of Theorem 9.1.2 (more precisely, as in Line 24, Page 342 through Line 10, Page 343) of [28], we have

\[
\|\mathcal{G}\|_{C^\alpha_\omega([0,1],X)} \leq C (\sup_{0 < t \leq 1} \|u(t)\|_{X_0} + \sup_{0 < t \leq 1} \|\mathcal{G}(t)\|_{X_0}) \|v\|_{C^\alpha_\omega([0,1],X)}
\]

(2.8)
It is clear that $N$ is sufficiently small then for any $u_0 \in \mathcal{O}_2$ we have $T_\epsilon = T^*(u_0) = \infty$. This proves the assertion (1).

Similarly as in the proof of (2.11), for any $s > t \geq 0$ we have

$$
\|x(t) - x(s)\|_{X_0} \leq C \int_t^s \|\mathcal{G}_1(\tau)\|_{X_0} d\tau \leq C \int_t^s \|v(\tau)\|_{X_0} d\tau \leq C (e^{-\omega t} - e^{-\omega s}) \|v(0)\|_{X_0}. 
$$

(12)

Hence $\lim_{t \to \infty} x(t)$ exists. Let $\ddot{x} = \lim_{t \to \infty} x(t)$ and $\dddot{u} = \dddot{x} + \varphi(\ddot{x})$. Then $\dddot{u} \in \mathcal{M}_c$, so that it is a stationary point of the equation $u' = F(u)$. Moreover, by the facts that $\lim_{t \to \infty} x(t) = \ddot{x}$ and $\lim_{t \to \infty} v(t) = 0$ in $(I - P)X_0$ we see that $\lim_{t \to \infty} u(t) = \dddot{u}$ in $X_0$. Letting $s \to \infty$ in (12) we see that

$$
\|x(t) - \ddot{x}\|_{X_0} \leq C e^{-\omega t} \|v(0)\|_{X_0}. 
$$

(13)

From (10) and (13) we obtain

$$
\|u(t) - \dddot{u}\|_{X_0} \leq C e^{-\omega t} \|v(0)\|_{X_0}. 
$$

(14)

Hence $\mathcal{M}_c$ is the unique center manifold of the equation $u' = F(u)$ in a neighborhood of the origin. This proves the assertion (2).

Next we note that the equation $u' = F(u)$ can be rewritten as $u' = Au + N(u)$. Besides, it is clear that $N \in C^{2-0}(\mathcal{O}_1, X)$, $N(0) = 0$, and $N'(0) = 0$, so that $\|N(u)\|_X \leq C\|u\|_X^2$ and
We introduce a mapping $u \in \mathcal{M} = \{ (x,y) \in \mathbb{R}^2 : \|u\| \leq R \}$ for any $x \in \mathbb{R}$. Given $y \in B_2(0, \delta')$, we consider the initial value problem
\begin{equation}
\label{eq:15}
u'(t) = Au(t) + N(u(t)) \quad \text{for} \quad t > 0, \quad \text{and} \quad (I - P)u(0) = y.
\end{equation}
We assert that this problem has a unique solution defined for all $t \geq 0$ and converging to 0 as $t \to \infty$, provided $\delta'$ is sufficiently small. To prove existence let $\alpha$ and $\omega$ be as before, and for a positive number $R$ to be specified later we introduce a metric space $(\mathcal{M}, d)$ by letting
\begin{align*}
\mathcal{M} = \{ u \in C([0, \infty), X_0) \cap C([0, 1], X_0) \cap C([1, \infty), X_0, -\omega) : \|u\| \leq R \},
\end{align*}
where
\begin{align*}
\|u\| = \|u\|_{C([0,1], X_0)} + \|u\|_{C([1,\infty), X_0, -\omega)},
\end{align*}
and $d(u, v) = \|u - v\|$. We define a mapping $\Psi_y : \mathcal{M} \to C([0, \infty), X_0)$ by letting $\Psi_y(u) = v$ for every $u \in \mathcal{M}$, where
\begin{align*}
v(t) = e^{tA} - y + \int_0^t e^{(t-s)A} (I - P)N(u(s))ds - \int_t^\infty PN(u(s))ds.
\end{align*}
Using Lemma 2.2, we can easily prove that for sufficiently small $R$, $\delta'$ and for any $y \in B_2(0, \delta')$, $\Psi_y$ is well-defined, maps $\mathcal{M}$ into itself and is a contraction mapping. Hence, $\Psi_y$ has a unique fixed point in $\mathcal{M}$ which we denote by $u_y$. Since $AP = 0$ so that $e^{(t-s)AP} = P$, it is clear that $u_y$ is a solution of (2.15), and $\lim_{t \to \infty} \|u_y(t)\|_{X_0} = 0$. This proves existence. To prove uniqueness, for any $(x, y) \in B_1(0, \delta) \times B_2(0, \delta')$ we denote by $u(t, x, y)$ the unique solution of the equation $u' = F(u)$ satisfying the initial conditions $Pu(0) = x$ and $(I - P)u(0) = y$. By Assertion (1) we know that $u(t, x, y)$ exists for all $t \geq 0$. Using the fact $AP = 0$ we can easily deduce that $\lim_{t \to \infty} u(t, x, y) = 0$ if and only if
\begin{equation}
\label{eq:16}x + \int_0^\infty PN(u(s, x, y))ds = 0.
\end{equation}
We introduce a mapping $\mathcal{F} : B(0, \delta) \times B(0, \delta') \to PX$ by letting
\begin{align*}
\mathcal{F}(x, y) = x + \int_0^\infty PN(u(s, x, y))ds
\end{align*}
for $(x, y) \in B_1(0, \delta) \times B_2(0, \delta')$. $\mathcal{F}$ is well-defined. Indeed, we know that for any $(x, y) \in B_1(0, \delta) \times B_2(0, \delta')$, $\bar{u} = \lim_{t \to \infty} u(t, x, y)$ exists and it belongs to $\mathcal{M}$. Let $\bar{x} = P\bar{u}$ and $\bar{y} = (I - P)\bar{u}$. Then $\bar{y} = \varphi(\bar{x})$, so that $PN(\bar{u}) = \mathcal{F}(\bar{x}, \bar{y}) = \mathcal{F}(\bar{x}, \varphi(\bar{x})) = 0$. Thus we have
\begin{align*}
\|PN(u(s, x, y))\|_X = \|PN(u(s, x, y)) - PN(\bar{u})\|_X \leq C\|u(s, x, y) - \bar{u}\|_{X_0} \leq C(x, y) e^{-\omega s}.
\end{align*}
Hence, the integral in the definition of $\mathcal{F}$ is convergent. By a similar argument we can show that $\mathcal{F} \in C^{2,0}(B(0, \delta) \times B(0, \delta'), PX)$. Since $u(t, 0, 0) = 0$ and $N(0) = N'(0) = 0$, we have $\mathcal{F}(0, 0) = 0$ and $D_2\mathcal{F}(0, 0) = id$. Thus, by the implicit function theorem we conclude that the solution of (2.16) is unique for fixed $y \in B_2(0, \delta')$, provided $\delta'$ is sufficiently small. This proves uniqueness.

We now introduce a mapping $\psi : (I - P)X_0 \to PX$ by define
\begin{align*}
\psi(y) = Pu_y(0) = -\int_0^\infty PN(u_y(s))ds \quad \text{for} \quad y \in B_2(0, \delta').
\end{align*}
Clearly, \( x = \psi(y) \) is the implicit function solving the equation \( \mathcal{F}(x, y) = 0 \), so that \( \psi \in C^{2,0}(B_2(0, \delta'), P_X) \). Letting \( \mathcal{M}^s = \text{graph}\psi \), we see that all requirements of the assertion (3) are satisfied. This proves the assertion (3).

Finally, given \( u_0 \in \mathcal{O}_3 \) let \( \bar{u} \) be as in (2.14). Since \( \bar{u} \in \mathcal{M}^s \), there exists a unique \( \sigma \in G \) such that \( S_\sigma(0) = \bar{u} \). Let \( v_0 = S_{\sigma^{-1}}(u_0) = S_{\sigma^{-1}}(u_0) \). Then we have

\[
\lim_{t \to -\infty} u(t, v_0) = \lim_{t \to -\infty} S_{\sigma^{-1}}^{-1}(u(t, u_0)) = S_{\sigma^{-1}}^{-1}(S_{\sigma}(0)) = 0,
\]

so that \( v_0 \in \mathcal{M}^s \). Noticing that \( u_0 = S_{\sigma}(v_0) \) and (2.4) is an immediate consequence of (2.14), we get the assertion (4). This completes the proof. \( \Box \)

**Remark 2.1.** Checking the proof of Theorem 2.1, we see that the condition on the Lie group action \( p \) can be weakened, that is, \( p \) need not to act on the space \( X \); an action on \( X_0 \) is sufficient.

### 3 Reduction of the problem

In this section we shall reduce the problem (1.1)–(1.7) into an initial value problem of an abstract differential equation in some Banach space. The reduction will be fulfilled in two steps: First we use the Hanzawa transformation to convert the free boundary problem (1.1)–(1.7) into an initial-boundary value problem on the fixed domain \( \Omega_s \). Next we solve the equations for \( p \) in terms of \( \sigma \) and \( \rho \), the function defining the free boundary \( \partial \Omega(t) \), to reduce this initial-boundary value problem into a purely evolutionary type and regard it as a differential equation in a suitable Banach space, which will be the desired abstract equation. For simplicity of notation, later on we always assume that \( R_s = 1 \). Note that this assumption is reasonable because the general case can be reduced into this special case by making suitable rescaling. It follows that

\[
\Omega_s = \mathbb{B}^n = \{ x \in \mathbb{R}^n : |x| < 1 \} \quad \text{and} \quad \partial \Omega_s = \partial \mathbb{B}^n = S^{n-1}.
\]

Besides, throughout this paper we assume that the initial domain \( \Omega_0 \) is a small perturbation of \( \Omega_s = \mathbb{B}^n \), so that \( \partial \Omega_0 \) is contained in a small neighborhood of \( \partial \Omega_s = S^{n-1} \).

To perform the first step of reduction let us first consider the Hanzawa transformation.

Fix a positive number \( \delta \) such that \( 0 < \delta < 1 \), and denote

\[
\mathcal{O}_\delta(S^{n-1}) = \{ \rho \in C^1(S^{n-1}) : \|\rho\|_{C^1(S^{n-1})} < \delta \}.
\]

Given \( \rho \in \mathcal{O}_\delta(S^{n-1}) \), we define a mapping \( \theta_\rho : S^{n-1} \to \mathbb{R}^n \) by letting \( \theta_\rho(\xi) = (1 + \rho(\xi))\xi \) for \( \xi \in S^{n-1} \), and denote

\[
\Gamma_\rho = \text{Im}(\theta_\rho) = \{ x \in \mathbb{R}^n : x = (1 + \rho(\xi))\xi, \ \xi \in S^{n-1} \}.
\]

Clearly, \( \Gamma_\rho \) is a closed \( C^1 \)-hypersurface diffeomorphic to \( S^{n-1} \), and \( \theta_\rho \) is a \( C^1 \)-diffeomorphism from \( S^{n-1} \) onto \( \Gamma_\rho \). We denote by \( \Omega_\rho \) the domain enclosed by \( \Gamma_\rho \). In the following we always assume that \( \partial \Omega_0 \) is of \( C^1 \) class and is contained in the \( \delta \)-neighborhood of \( S^{n-1} \). More precisely, we assume that there exists \( \rho_0 \in \mathcal{O}_\delta(S^{n-1}) \) such that \( \partial \Omega_0 = \Gamma_{\rho_0} \), and, accordingly, \( \Omega_0 = \Omega_{\rho_0} \).
Let $m$ be an integer, $m \geq 2$, and let $n/(m-1) < q < \infty$. Then we have $B_{qq}^{m-1/q}(S^{n-1}) \subseteq C^1(S^{n-1})$. The well-known trace theorem ensures that the trace operator $\text{tr}(u) = u|_{S^{n-1}}$ from $C^\infty(B^n)$ to $C^\infty(S^{n-1})$ can be extended to $W^{m,q}(B^n)$ such that it maps $W^{m,q}(B^n)$ into $B_{qq}^{m-1/q}(S^{n-1})$ and is bounded and surjective. We introduce a right inverse $\Pi$ of this operator as follows: Given $\rho \in B_{qq}^{m-1/q}(S^{n-1})$, let $u \in W^{m,q}(B^n)$ be the unique solution of the boundary value problem

$$\Delta u = 0 \quad \text{in} \quad B^n, \quad \text{and} \quad u = \rho \quad \text{on} \quad S^{n-1},$$

and define $\Pi(\rho) = u$. Then clearly $\text{tr}(\Pi(\rho)) = \rho$ for $\rho \in B_{qq}^{m-1/q}(S^{n-1})$, and the standard $L^p$ estimate and the maximum principle yield the following relations:

$$\|\Pi(\rho)\|_{W^{m,q}(B^n)} \leq C\|\rho\|_{B_{qq}^{m-1/q}(S^{n-1})} \quad \text{and} \quad \sup_{x \in B^n} |\Pi(\rho)(x)| = \max_{x \in S^{n-1}} |\rho(x)|.$$

Note that since $W^{m,q}(B^n) \hookrightarrow C^1(B^n)$, the first relation implies that

$$\|\Pi(\rho)\|_{C^1(B^n)} \leq C_0\|\rho\|_{B_{qq}^{m-1/q}(S^{n-1})}. \quad (3.1)$$

Here we use the special notation $C_0$ to denote the constant in (3.1) because later on this constant will play a special role. We now introduce

$$\mathcal{O}_\delta^{m,q}(S^{n-1}) = \{ \rho \in B_{qq}^{m-1/q}(S^{n-1}) : \|\rho\|_{B_{qq}^{m-1/q}(S^{n-1})} < \delta, \|\rho\|_{C^1(S^{n-1})} < \delta \}.$$

In the sequel we further assume that $\delta < \min\{1/5, (3C_0)^{-1}\}$. Take a function $\phi \in C^\infty(\mathbb{R}, [0, 1])$ such that

$$\phi(\tau) = 1 \quad \text{for} \quad |\tau| \leq \delta, \quad \phi(\tau) = 0 \quad \text{for} \quad |\tau| \geq 3\delta, \quad \text{and} \quad \sup |\phi'| < 2/3\delta^{-1}.$$

Given $\rho \in \mathcal{O}_\delta^{m,q}(S^{n-1})$, we define the Hanzawa transformation $\Theta_\rho : \overline{B^n} \to \overline{\Omega}_\rho$ by

$$\Theta_\rho(x) = x + \phi(|x| - 1)\Pi(\rho)(x)\omega(x) \quad \text{for} \quad x \in \overline{B^n},$$

where $\omega(x) = x/|x|$ for $x \in \mathbb{R}^n \setminus \{0\}$, and $\omega(0) = 0$. The choice of $\delta$ and the inequality (3.1) ensures that for fixed $\omega \in S^{n-1}$, the function $r \mapsto r + \phi(r - 1)\Pi(\rho)(r\omega)$ is strictly monotone increasing for $0 \leq r \leq 1$, so that $\Theta_\rho$ is a bijection from $\overline{B^n}$ onto $\overline{\Omega}_\rho$. In fact, since the derivative of this function is strictly positive, it can be easily shown that $\Theta_\rho \in W^{m,q}(\mathbb{B}^n, \Omega_\rho)$ and $\Theta_\rho^{-1} \in W^{m,q}(\Omega_\rho, \mathbb{B}^n)$. Besides, it is clear that $\Theta_\rho|_{S^{n-1}} = \theta_\rho$. Since $W^{m,q}(\mathbb{B}^n) \subseteq C^1(\mathbb{B}^n)$ and $W^{m,q}(\Omega_\rho) \subseteq C^1(\overline{\Omega}_\rho)$, we see that $\Theta_\rho$ is particularly a $C^1$-diffeomorphism from $\overline{B^n}$ onto $\overline{\Omega}_\rho$.

As usual we denote by $\Theta_\rho'$ and $\Theta_\rho^*$ respectively the push-forward and pull-back operators induced by $\Theta_\rho$, i.e., $\Theta_\rho'u = u \circ \Theta_\rho^{-1}$ for $u \in C(\overline{\Omega}_\rho)$, and $\Theta_\rho^*u = u \circ \Theta_\rho$ for $u \in C(\overline{\Omega}_\rho)$. Similarly, $\theta_\rho^*$ denotes the pull-back operator induced by $\theta_\rho$, i.e., $\theta_\rho^*u(\xi) = u(\theta_\rho(\xi))$ for $u \in C(\Gamma_\rho)$ and $\xi \in S^{n-1}$. Later, we shall need the following result:

**Lemma 3.1** Let $m$ be an integer and $1 < q < \infty$. Let $\Omega_1$ and $\Omega_2$ be two open subsets of $\mathbb{R}^n$. Let $\Phi$ be a diffeomorphism from $\Omega_1$ to $\Omega_2$ such that $\Phi \in W^{m,q}(\Omega_1, B^n)$ and $\Phi^{-1} \in W^{m,q}(\Omega_2, B^n)$. Assume that $m \geq 2$ and $q > n/(m-1)$. Then for any $0 \leq k \leq m$ we have

$$\Phi_* \in L(W^{k,q}(\Omega_1), W^{k,q}(\Omega_2)) \quad \text{and} \quad \Phi^* \in L(W^{k,q}(\Omega_2), W^{k,q}(\Omega_1)).$$
In particular, for any \( \rho \in B_{qq}^{m-1/q}(S^{n-1}) \) and \( 0 \leq k \leq m \) we have
\[
\Theta^k_{\rho} \in L(W^{k,q}(\mathbb{B}^n), W^{k,q}(\Omega_{\rho})) \quad \text{and} \quad \Theta^k_{\rho} \in L(W^{k,q}(\Omega_{\rho}), W^{k,q}(\mathbb{B}^n)).
\]

**Proof:** See the proof of Lemma 2.1 of [14] for the case \( k = m \). Proofs for the rest cases \( 0 \leq k \leq m - 1 \) are similar and simpler. \( \square \)

Next we introduce some notations.

In the sequel we assume that \( m \geq 2 \) and \( q > n/(m-1) \). As in [14], for \( \rho \in \mathcal{O}_m^{m,q}(S^{n-1}) \) we introduce a second-order partial differential operator \( A(\rho) : W^{m,q}(\mathbb{B}^n) \to W^{m-2,q}(\mathbb{B}^n) \) by
\[
A(\rho) = \Theta^1_{\rho} \Delta (\Theta^0_{\rho} u) \quad \text{for} \quad u \in W^{m,q}(\mathbb{B}^n).
\]

By Lemma 3.1 we see that \( A(\rho) \in L(W^{m,q}(\mathbb{B}^n), W^{m-2,q}(\mathbb{B}^n)) \). We also introduce nonlinear operators \( F \) and \( G : W^{m,q}(\mathbb{B}^n) \to W^{m,q}(\mathbb{B}^n) \) respectively by
\[
F(u) = f \circ u, \quad G(u) = g \circ u \quad \text{for} \quad u \in W^{m,q}(\mathbb{B}^n).
\]

Since the condition \( q > n/(m-1) > n/m \) implies that \( W^{m,q}(\mathbb{B}^n) \) is an algebra, we see that these definitions make sense and we have \( F, G \in C^\infty(W^{m,q}(\mathbb{B}^n), W^{m,q}(\mathbb{B}^n)) \). Given \( \rho \in \mathcal{O}_m^{m,q}(S^{n-1}) \) we denote
\[
\psi_\rho(x) = |x| - 1 - \rho(\omega(x)) \quad \text{for} \quad x \in \mathcal{R} \equiv \{ x \in \mathbb{R}^n : 1 - 4\delta < |x| < 1 + 4\delta \}.
\]

Clearly, \( \psi_\rho \in B_{qq}^{m-1/q}(\mathcal{R}) \). Since \( \Gamma_\rho = \{ x \in \mathcal{R} : \psi_\rho(x) = 0 \} \), we see that the unit outward normal field \( n \) on \( \Gamma_\rho \) is given by \( n(x) = \nabla \psi_\rho(x)/|\nabla \psi_\rho(x)| \) for \( x \in \Gamma_\rho \). We introduce a first-order trace operator \( D(\rho) : W^{m,q}(\mathbb{B}^n) \to B_{qq}^{m-1-1/q}(S^{n-1}) \) by
\[
D(\rho) = \Theta^1_{\rho} (\text{tr}_{\Gamma_\rho} (\nabla (\Theta^0_{\rho} u) \cdot \nabla \psi_\rho)) \quad \text{for} \quad u \in W^{m,q}(\mathbb{B}^n),
\]

where \( \text{tr}_{\Gamma_\rho} \) denotes the usual trace operator from \( \overline{\mathcal{R}} \cap \mathcal{R} \) to \( \Gamma_\rho \), i.e., \( \text{tr}_{\Gamma_\rho}(u) = u|_{\Gamma_\rho} \) for \( u \in C(\overline{\mathcal{R}} \cap \mathcal{R}) \). It can be easily seen that \( D(\rho) \) maps \( W^{m,q}(\mathbb{B}^n) \) into \( B_{qq}^{m-1-1/q}(S^{n-1}) \), and \( D(\rho) \in L(W^{m,q}(\mathbb{B}^n), B_{qq}^{m-1-1/q}(S^{n-1})) \) for any \( \rho \in \mathcal{O}_m^{m,q}(S^{n-1}) \). Similarly, given \( (\rho, u) \in \mathcal{O}_m^{m,q}(S^{n-1}) \times W^{m,q}(\mathbb{B}^n) \), we introduce a first-order pseudo-differential operator \( P(\rho, u) : W^{m,q}(\mathbb{B}^n) \to W^{m-1,q}(\mathbb{B}^n) \) as follows:
\[
P(\rho, u)v = \mathcal{M}(\rho, u) \cdot \Pi(D(\rho)v) \quad \text{for} \quad v \in W^{m,q}(\mathbb{B}^n).
\]

Here we used the same notation \( \Pi \) as before to denote the bounded right inverse of the trace operator \( \text{tr} : W^{m-1,q}(\mathbb{B}^n) \to B_{qq}^{m-1-1/q}(S^{n-1}) \) such that its restriction on \( B_{qq}^{m-1-1/q}(S^{n-1}) \) is equal to the previous \( \Pi \), and
\[
\mathcal{M}(\rho, u)(x) = \phi(|x| - 1)(\Theta^0_{\rho} \nabla \Theta^1_{\rho} u)(x, \omega(x)) \quad \text{for} \quad x \in \mathbb{B}^n,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^n \). We note that \( \mathcal{M}(\rho, u) \in W^{m-1,q}(\mathbb{B}^n) \) and the mapping \( u \to \mathcal{M}(\rho, u) \) is a first-order partial differential operator. Since \( [v \to \Pi(D(\rho)v)] \in L(W^{m,q}(\mathbb{B}^n), W^{m-1,q}(\mathbb{B}^n)) \) and the condition \( q > n/(m-1) \) implies that \( W^{m-1,q}(\mathbb{B}^n) \) is an algebra, we see that \( P(\rho, u) \in L(W^{m,q}(\mathbb{B}^n), W^{m-1,q}(\mathbb{B}^n)) \). Finally, we define the transformed mean curvature operator \( K : C^2(S^{n-1}) \cap \mathcal{O}_m(S^{n-1}) \to C(S^{n-1}) \) by
\[
K(\rho) = \Theta^0_{\rho}(K_{\rho}^1),
\]
where $\kappa_{\Gamma_\rho}$ denotes the mean curvature of the hypersurface $\Gamma_\rho$ (recall that $\kappa_{\Gamma_\rho} \in C(\Gamma_\rho, \mathbb{R})$ for $C^2$ class hypersurface $\Gamma_\rho$). Later we shall restrict $\mathcal{K}$ in $\mathcal{O}_{\delta}^{m,q}(S^{n-1})$ and shall see that $\mathcal{K} \in C(\mathcal{O}_{\delta}^{m,q}(S^{n-1}), B_{\delta}^{m-2-1/q}(S^{n-1}))$.

Let $T$ be a given positive number and consider a function $\rho : [0,T] \to \mathcal{O}_{\delta}^{m,q}(S^{n-1})$. We assume that $\rho \in C([0,T], \mathcal{O}_{\delta}^{m,q}(S^{n-1}))$. Given a such $\rho$, we denote

$$
\Gamma_\rho(t) = \Gamma_{\rho(t)}, \quad \Omega_\rho(t) = \Omega_{\rho(t)} \quad (0 \leq t \leq T).
$$

Later on in case no confusion can be produced we shall occasionally abbreviate $\Gamma_\rho(t)$ and $\Omega_\rho(t)$ respectively as $\Gamma_\rho$ and $\Omega_\rho$. We shall briefly write the families of operators $t \to \mathcal{A}(\rho(t))$ and $t \to \mathcal{D}(\rho(t))$ $(0 \leq t \leq T)$ as $\mathcal{A}(\rho)$ and $\mathcal{D}(\rho)$, respectively, and for $u, v : [0,T] \to W^{m,q}(\mathbb{B}^n)$, we briefly write the families of functions $\mathcal{F}(\rho(u),v(t))$, $\mathcal{G}(\rho(u),v(t))$ and $\mathcal{M}(\rho(u),v(t)v(t))$ $(0 \leq t \leq T)$ respectively as $\mathcal{F}(\rho,u)$, $\mathcal{G}(\rho,u)$ and $\mathcal{M}(\rho,u)v$. Besides, we shall identify a function $\rho : [0,T] \to C(S^{n-1})$ (resp. $u : [0,T] \to C(S^{n-1})$) with the corresponding function on $\mathbb{S}^{n-1} \times [0,T]$ (resp. $\mathbb{S}^{n-1} \times [0,T]$) defined by $\rho(\xi,t) = \rho(t)(\xi)$ (resp. $u(x,t) = u(t)(x)$), where $t \in [0,T]$ and $\xi \in S^{n-1}$ (resp. $x \in \mathbb{S}^{n-1}$), and vice versa.

With the above notations, it is not hard to verify that if we denote

$$
u(x,t) = \sigma(\Theta_{\rho(t)}(x),t), \quad v(x,t) = p(\Theta_{\rho(t)}(x),t),
$$

then the Hanzawa transformation transforms (1.1)–(1.7) into the following system of equations:

\begin{align}
\frac{c}{\kappa_{\Gamma_\rho}}u - \mathcal{A}(\rho)u + c\mathcal{P}(\rho,u)v &= -\mathcal{F}(u) \quad \text{in } \mathbb{B}^n \times (0,T], \quad (3.2) \\
-\mathcal{A}(\rho)v &= \mathcal{G}(u), \quad \text{in } \mathbb{B}^n \times (0,T], \quad (3.3) \\
u &= \sigma \quad \text{on } \mathbb{S}^{n-1} \times (0,T], \quad (3.4) \\
v &= \gamma \kappa(\rho) \quad \text{on } \mathbb{S}^{n-1} \times (0,T], \quad (3.5) \\
\partial_t \rho + \mathcal{D}(\rho)v &= 0 \quad \text{on } \mathbb{S}^{n-1} \times (0,T], \quad (3.6) \\
u(0) &= u_0 \quad \text{on } \mathbb{B}^n, \quad (3.7) \\
\rho(0) &= \rho_0 \quad \text{on } \mathbb{S}^{n-1}, \quad (3.8)
\end{align}

where $u_0 = \Theta_{\rho_0}^* \sigma_0$. Indeed, it is immediate to see that (3.3), (3.4), (3.5), (3.7) and (3.8) are transformations of (1.2), (1.3), (1.4), (1.6) and (1.7), respectively. For the proof that the transformation of (1.5) is (3.6), we refer the reader to see the deduction of (2.19) in [14] and (2.8) in [20]. Finally, (3.2) is obtained from transforming (1.1) and using (3.6).

To establish properties of the operator $\mathcal{K}$, we need the following lemma:

**Lemma 3.2**  (i) Let $k$, $m$ be nonnegative integers, and $p,q \in [1,\infty]$. Let $\Omega$ be an open subset of $\mathbb{R}^n$ with a smooth boundary. Assume that $k \geq m$ and either $1 \leq p \leq n/m$, $k > n/q$ or $p > n/m$, $k - n/q \geq m - n/p$. Then we have

$$
\|uv\|_{W^{m,p}(\Omega)} \leq C\|u\|_{W^{k,q}(\Omega)}\|v\|_{W^{m,p}(\Omega)}. \quad (3.9)
$$
(ii) Let $s, t > 0$ and $p, q, r_1, r_2 \in [1, \infty]$. Let $\Omega$ be as before. Assume that $t \geq s$ and either $1 \leq p \leq n/s, t > n/q$ or $p > n/s, t - n/q \geq s - n/p$. Then we have
\[
\|uv\|_{B_{pr_1}^s(\Omega)} \leq C\|u\|_{B_{qr_2}^t(\Omega)}\|v\|_{B_{pr_1}^s(\Omega)}.
\] (3.10)

Here $r_1, r_2$ are arbitrary numbers in $[1, \infty]$ in case $t > s$, and $1 \leq r_2 \leq r_1 \leq \infty$ if $t = s$.

Proof. To prove (3.9), we first note that since $k > n/q$, we have $W^{k,q}(\Omega) \hookrightarrow L^\infty(\Omega)$, so that
\[
\|uv\|_{L^p(\Omega)} \leq \|u\|_{L^\infty(\Omega)}\|v\|_{L^p(\Omega)} \leq C\|u\|_{W^{k,q}(\Omega)}\|v\|_{W^{m,p}(\Omega)}.
\] (3.11)

Next let $\alpha \in \mathbb{Z}_+^m$ be an arbitrary $n$-index of length $m$, i.e., $|\alpha| = m$. We write the Leibnitz formula:
\[
\partial^\alpha (uv) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta! (\alpha - \beta)!} \partial^\beta u \partial^{\alpha - \beta} v.
\]

For every $n$-index $\beta \leq \alpha$ we take $r_1, r_2 \in [1, \infty]$ as follows:
\[
\begin{cases}
\frac{1}{r_1} = \frac{1}{q} - \frac{k-|\beta|}{n}, & \frac{1}{r_2} = \frac{1}{p} - \frac{k-|\beta|}{n} \quad \text{if } |\beta| > k - \frac{n}{q}, \\
\frac{1}{r_1} = \frac{1}{\varepsilon}, & \frac{1}{r_2} = \frac{1}{p} - \varepsilon \quad \text{if } |\beta| = k - \frac{n}{q}, \\
\frac{1}{r_1} = \infty, & \frac{1}{r_2} = p \quad \text{if } |\beta| < k - \frac{n}{q},
\end{cases}
\]

where $\varepsilon$ is a small positive number. Note that since $|\beta| \leq m \leq k$, we have $\frac{1}{p} - \frac{1}{q} + \frac{k-|\beta|}{n} \geq 0$. Clearly
\[
\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{p}, \quad \|\partial^\beta u\|_{L^{r_1}(\Omega)} \leq C\|u\|_{W^{k,q}(\Omega)}\|v\|_{W^{m,p}(\Omega)}.
\]

Hence
\[
\|\partial^\alpha (uv)\|_{L^p(\Omega)} \leq C \sum_{\beta \leq \alpha} \|\partial^\beta u\|_{L^{r_1}(\Omega)}\|\partial^{\alpha - \beta} v\|_{L^{r_2}(\Omega)} \leq C\|u\|_{W^{k,q}(\Omega)}\|v\|_{W^{m,p}(\Omega)}.
\] (3.12)

Combining (3.11) and (3.12), we get (3.9).

Having proved (3.9), (3.10) easily follows by interpolation.  

Corollary 3.3 Assume that $m \geq 2$ and either $q > n/(m-1), 0 < s \leq m - 1 - 1/q$ or $q > \max\{2n/(m+n-2), n/(m-1)\}$ and $-1/q \leq s \leq 0$. Then we have
\[
\|uv\|_{B_{qq}^s(\mathbb{S}^{n-1})} \leq C\|u\|_{B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1})}\|v\|_{B_{qq}^s(\mathbb{S}^{n-1})}.
\] (3.13)

Proof. If $s > 0$ then the desired assertion follows immediately from Lemma 3.2 (ii), because we can easily verify that all conditions of Lemma 3.2 (ii) are satisfied when we replace $t$ with $m - 1 - 1/q, p$ with $q$ and $n$ with $n - 1$. Next we consider the case $-1/q \leq s \leq 0$. We can also easily verify that in this case all conditions of Lemma 3.2 (ii) are satisfied when we replace $t$ with $m - 1 - 1/q, s$ with $1/q, p$ with $q'$, and $n$ with $n - 1$, so that
\[
\|uv\|_{B_{q'q'}^{1/q}(\mathbb{S}^{n-1})} \leq C\|u\|_{B_{q'q'}^{m-1-1/q}(\mathbb{S}^{n-1})}\|v\|_{B_{q'q'}^{1/q}(\mathbb{S}^{n-1})}.
\]

By dual, this implies that
\[
\|uv\|_{B_{qq}^{-1/q}(\mathbb{S}^{n-1})} \leq C\|u\|_{B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1})}\|v\|_{B_{qq}^{-1/q}(\mathbb{S}^{n-1})}.
\]
Interpolating this inequality with (3.22) for $s > 0$, we see that (3.22) also holds for $-1/q \leq s \leq 0$ under the prescribed conditions. □

**Lemma 3.4** Let $m \geq 2$ and $q > n/(m - 1)$. Then for any $2 \leq k \leq m$ we have the following assertions:

\[ A \in C^\infty(O_\delta^{m,q}(S^{n-1}), L(W^{k,q}(\mathbb{B}^n), W^{k-2,q}(\mathbb{B}^n))), \]

\[ D \in C^\infty(O_\delta^{m,q}(S^{n-1}), L(W^{k,q}(\mathbb{B}^n), B_{qq}^{k-1-1/q}(S^{n-1}))), \]

\[ \mathcal{P} \in C^\infty(O_\delta^{m,q}(S^{n-1}) \times W^{m,q}(\mathbb{B}^n), L(W^{k,q}(\mathbb{B}^n), W^{k-1,q}(\mathbb{B}^n))), \]

and for any $k > n/q$ we have

\[ \mathcal{F}, \mathcal{G} \in C^\infty(W^{k,q}(\mathbb{B}^n), W^{k,q}(\mathbb{B}^n)). \]

**Proof:** (3.14) is an immediate consequence of Lemma 3.1 and the fact that $\Delta \in L(W^{k,q}(\Omega_\rho), W^{k-2,q}(\Omega_\rho))$ for any $k$. (3.15) is an immediate consequence of Lemmas 3.1, 3.2 and the fact that $\nabla \in L(W^{k,q}(\Omega_\rho), W^{k-1,q}(\Omega_\rho, \mathbb{R}^n))$ for any $k$. (3.16) follows from similar reasons as for (3.15). Finally, (3.17) follows from the fact that $W^{k,q}(\Omega_\rho)$ is an algebra under the condition $k > n/q$, as we mentioned earlier. □

**Lemma 3.5** (i) The mean curvature operator $\mathcal{K}(\rho)$ has the following splitting:

\[ \mathcal{K}(\rho) = \mathcal{L}(\rho)\rho + \mathcal{K}_1(\rho), \]

where $\mathcal{L}(\rho)$ is a second-order elliptic linear partial differential operator on $S^{n-1}$, with coefficients being functions of $\rho$ and its first-order derivatives, and $\mathcal{K}_1(\rho)$ is a first-order partial nonlinear differential operator on $S^{n-1}$.

(ii) Assume that $m \geq 3$ and $q > \max\{2n/(m+n-2), n/(m-1)\}$. Then we have

\[ \mathcal{L} \in C^\infty(O_\delta^{m,q}(S^{n-1}), L(B_{qq}^{k-1/q}(S^{n-1}), B_{qq}^{k-2-1/q}(S^{n-1}))), \]

\[ 2 \leq k \leq m, \]

\[ \mathcal{K}_1 \in C^\infty(O_\delta^{m,q}(S^{n-1}), B_{qq}^{m-1-1/q}(S^{n-1})), \]

so that

\[ \mathcal{K} \in C^\infty(O_\delta^{m,q}(S^{n-1}), B_{qq}^{m-2-1/q}(S^{n-1})). \]

**Proof:** The Assertion (i) is an immediate consequence of the mean curvature formula, see [20] and [21]. Next, since the condition $q > n/(m-1)$ implies that $B_{qq}^{m-1-1/q}(S^{n-1})$ is an algebra, (3.20) easily follows from the fact that $\mathcal{K}_1$ is a first-order nonlinear partial differential operator. Similarly, (3.19) follows from Corollary 3.3 and the facts that $B_{qq}^{m-1-1/q}(S^{n-1})$ is an algebra and $\mathcal{L}(\rho)$ is a second-order partial differential operator with coefficients being smooth functions of $\rho$ and its first-order partial derivatives. Finally, (3.21) follows readily from (3.18)–(3.20). □

In order to perform the second step of reduction, we need the following lemma:
**Lemma 3.6** Let $m \geq 2$, $q > n/(m - 1)$ and $2 \leq k \leq m$. Given $\rho \in \mathcal{O}_{\delta}^{m,q}(\mathbb{S}^{n-1})$ and $(w, \eta) \in W^{k-2,q}(\mathbb{B}^n) \times B_{qq}^{k-1/q}(\mathbb{S}^{n-1})$, the problem

$$
\begin{align*}
-\mathcal{A}(\rho)u &= w \quad \text{in } \mathbb{B}^n, \\
\quad u &= \eta \quad \text{on } \mathbb{S}^{n-1}
\end{align*}
$$

has a unique solution $u \in W^{k,q}(\mathbb{B}^n)$, and it has the following expression:

$$
u = \mathcal{S}(\rho)w + \mathcal{T}(\rho)\eta,$$

where

\begin{align*}
\mathcal{S} &\in C^\infty(\mathcal{O}_{\delta}^{m,q}(\mathbb{S}^{n-1}), L(W^{k-2,q}(\mathbb{B}^n), W^{k,q}(\mathbb{B}^n))), \\
\mathcal{T} &\in C^\infty(\mathcal{O}_{\delta}^{m,q}(\mathbb{S}^{n-1}), L(B_{qq}^{k-1/q}(\mathbb{S}^{n-1}), W^{k,q}(\mathbb{B}^n))).
\end{align*}

**Proof.** All assertions easily follow from the standard theory of elliptic partial differential equations, cf. the proof of Lemma 3.1 of [16]. $\square$

In the sequel we perform the second step of reduction.

By Lemmas 3.5 and 3.6 we see that given $u \in W^{m-1,q}(\mathbb{B}^n)$ and $\rho \in B_{qq}^{n-1/q}(\mathbb{S}^{n-1})$, the solution of Eq. (3.3) subject to the boundary value condition (3.5) is given by

$$v = \gamma \mathcal{T}(\rho)\mathcal{L}(\rho)\rho + \gamma \mathcal{T}(\rho)\mathcal{K}_1(\rho) + \mathcal{S}(\rho)\mathcal{G}(u).$$

Substitute this expression into (3.2) and (3.6), we see that the problem (3.2)–(3.8) is reduced into the following problem:

\begin{align*}
\partial_t u - c^{-1}\mathcal{A}(\rho)u - \mathcal{Q}(\rho, u)\rho &= \mathcal{F}_1(\rho, u) \quad \text{in } \mathbb{B}^n \times (0, \infty), \\
\partial_t \rho - \mathcal{B}(\rho)\rho &= \mathcal{G}_1(\rho, u) \quad \text{on } \mathbb{S}^{n-1} \times (0, \infty), \\
\quad u &= \bar{\sigma} \quad \text{on } \mathbb{S}^{n-1} \times (0, \infty), \\
\quad u(0) &= u_0 \quad \text{on } \mathbb{B}^n, \\
\quad \rho(0) &= \rho_0 \quad \text{on } \mathbb{S}^{n-1},
\end{align*}

where $\mathcal{A}(\rho)$ is as before, and

\begin{align*}
\mathcal{B}(\rho)\zeta &= -\gamma \mathcal{D}(\rho)\mathcal{T}(\rho)\mathcal{L}(\rho)\zeta, \\
\mathcal{Q}(\rho, u)\zeta &= \mathcal{M}(\rho, u) \cdot \Pi(\mathcal{B}(\rho)\zeta), \\
\mathcal{F}_1(\rho, u) &= -c^{-1}\mathcal{F}(u) - \gamma \mathcal{P}(\rho, u)\mathcal{T}(\rho)\mathcal{K}_1(\rho) - \mathcal{P}(\rho, u)\mathcal{S}(\rho)\mathcal{G}(u) \\
&= -c^{-1}\mathcal{F}(u) - \mathcal{M}(\rho, u) \cdot \Pi(\mathcal{G}_1(\rho, u)), \\
\mathcal{G}_1(\rho, u) &= -\gamma \mathcal{D}(\rho)\mathcal{T}(\rho)\mathcal{K}_1(\rho) - \mathcal{D}(\rho)\mathcal{S}(\rho)\mathcal{G}(u).
\end{align*}
To homogenize the boundary condition (3.26) we define

\[ C(\rho, u) = Q(\rho, u + \bar{\sigma}), \quad \mathcal{F}_2(\rho, u) = \mathcal{F}_1(\rho, u + \bar{\sigma}), \quad \mathcal{G}_2(\rho, u) = \mathcal{G}_1(\rho, u + \bar{\sigma}). \]

Replacing \( Q, \mathcal{F}_1 \) and \( \mathcal{G}_1 \) in (3.24) and (3.25) with \( C, \mathcal{F}_2 \) and \( \mathcal{G}_2 \), respectively, we see that the inhomogeneous boundary value condition (3.26) is replaced by the homogeneous boundary value condition

\[ u = 0 \quad \text{on} \quad S^{n-1} \times (0, \infty). \quad (3.29) \]

We now denote

\[ U = \begin{pmatrix} u \\ \rho \end{pmatrix}, \quad \mathcal{A}(U) = \begin{pmatrix} c^{-1} A(\rho) & C(\rho, u) \\ 0 & B(\rho) \end{pmatrix}, \quad \mathcal{F}_0(U) = \begin{pmatrix} \mathcal{F}_2(\rho, u) \\ \mathcal{G}_2(\rho, u) \end{pmatrix}, \quad U_0 = \begin{pmatrix} \sigma_0 - \bar{\sigma} \\ \rho_0 \end{pmatrix}, \]

and

\[ \mathcal{F}(U) = \mathcal{A}(U)U + \mathcal{F}_0(U). \]

We also denote

\[ X = W^{m-3,q}(\mathbb{B}^n) \times B_{qq}^{m-3-1/q}(S^{n-1}), \quad X_0 = (W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)) \times B_{qq}^{m-1/q}(S^{n-1}), \]

\[ Y = W^{m-2,q}(\mathbb{B}^n) \times B_{qq}^{m-2-1/q}(S^{n-1}), \]

and

\[ \mathcal{O} = (W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)) \times \mathcal{O}_d^{m,q}(S^{n-1}). \]

Then the equations (3.24), (3.25) (with \( Q, \mathcal{F}_1, \mathcal{G}_1 \) respectively replaced with \( C, \mathcal{F}_2, \mathcal{G}_2 \)) and (3.29) are reduced into the following abstract differential equation in the Banach space \( X \):

\[ \frac{dU}{dt} = \mathcal{F}(U), \quad (3.30) \]

and the problem (3.24)–(3.28) is reduced into the following initial value problem:

\[
\begin{aligned}
U'(t) &= \mathcal{F}(U(t)) \quad \text{for} \quad t > 0, \\
U(0) &= U_0
\end{aligned}
\quad (3.31)
\]

Clearly, \( X, X_0 \) and \( Y \) are Banach spaces, \( X_0 \hookrightarrow X \), \( Y \) is an intermediate space between \( X \) and \( X_0 \), and \( \mathcal{O} \) is an open subset of \( X_0 \). From (3.14)–(3.21) and (3.22)–(3.20) we see that

\[ \mathcal{A} \in C^\infty(\mathcal{O}, L(X_0, X)), \quad \mathcal{F}_0 \in C^\infty(\mathcal{O}, Y) \subseteq C^\infty(\mathcal{O}, X), \]

so that \( \mathcal{F} \in C^\infty(\mathcal{O}, X) \). We note that for \( m = 3 \), \( X_0 \) is dense in \( X \), while for \( m \geq 4 \) the closure of \( X_0 \) in \( X \) is given by

\[ \bar{X}_0 = (W^{m-3,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)) \times B_{qq}^{m-3-1/q}(S^{n-1}). \]
4 The Lie group action

For \( \varepsilon > 0 \) we denote by \( \mathbb{B}_ \varepsilon^n \) the ball in \( \mathbb{R}^n \) centered at the origin with radius \( \varepsilon \). Regarding \( \mathbb{B}_ \varepsilon^n \) as a neighborhood of the unit element 0 of the commutative Lie group \( \mathbb{R}^n \), we see that \( G = \mathbb{B}_ \varepsilon^n \) is a local Lie group of dimension \( n \). In this section we introduce an action \( S^* \) of this (local) Lie group \( G \) to some open subset \( \mathcal{O}' \) of \( X \), \( \mathcal{O}' \cap X_0 = \mathcal{O} \), such that the relation

\[
F(S_z^*(u)) = DS_z^*(u)\mathcal{F}(u), \quad z \in G, \quad u \in \mathcal{O}
\]

is satisfied.

Given \( z \in \mathbb{R}^n \), we denote by \( S_z \) the translation in \( \mathbb{R}^n \) induced by \( z \), i.e.,

\[
S_z(x) = x + z \quad \text{for} \quad x \in \mathbb{R}^n.
\]

Let \( \rho \in \mathcal{C}^1(\mathbb{S}^{n-1}) \) such that \( \|\rho\|_{\mathcal{C}^1(\mathbb{S}^{n-1})} \) is sufficiently small, say, \( \|\rho\|_{\mathcal{C}^1(\mathbb{S}^{n-1})} < \delta \) for some small \( \delta > 0 \). For any \( z \in \mathbb{B}_\varepsilon^n \), where \( \varepsilon \) is sufficiently small, consider the image of the hypersurface \( r = 1 + \rho(\omega) \) under the translation \( S_z \), which is still a hypersurface. This hypersurface has the equation \( r = 1 + \tilde{\rho}(\omega) \) with \( \tilde{\rho} \in \mathcal{C}^1(\mathbb{S}^{n-1}) \), and \( \tilde{\rho} \) is uniquely determined by \( \rho \) and \( z \). We denote

\[
\tilde{\rho} = S_z^*(\rho).
\]

Let \( r_0 = |z| \) and \( \omega_0 = z/|z| \). Then the explicit expression of \( \tilde{\rho} \) is as follows:

\[
\tilde{\rho}(\omega') = \sqrt{[1 + \rho(\omega)]^2 + r_0^2 + 2r_0[1 + \rho(\omega)]\omega \cdot \omega_0} - 1,
\]

where \( \omega' \in \mathbb{S}^{n-1} \) and \( \omega \in \mathbb{S}^{n-1} \) are connected by the following relation:

\[
\omega' = \frac{[1 + \rho(\omega)]\omega + r_0\omega_0}{\sqrt{[1 + \rho(\omega)]^2 + r_0^2 + 2r_0[1 + \rho(\omega)]\omega \cdot \omega_0}}.
\]

In the sequel, the notations \( \mathcal{O}_\delta(\mathbb{S}^{n-1}) \) and \( \mathcal{O}'_{\delta,\eta}(\mathbb{S}^{n-1}) \) have same meaning as in the previous section.

**Lemma 4.1** If \( \varepsilon \) and \( \delta \) are sufficiently small then for any \( z \in \mathbb{B}_\varepsilon^n \) and \( \rho \in \mathcal{O}_\delta(\mathbb{S}^{n-1}) \), \( S_z^*(\rho) \) is well-defined, and

\[
S_z^* \in C(\mathcal{O}_\delta(\mathbb{S}^{n-1}), C^1(\mathbb{S}^{n-1})) \cap C^1(\mathcal{O}_\delta(\mathbb{S}^{n-1}), C(\mathbb{S}^{n-1})).
\]

**Proof:** Let \( f_z(\rho, \omega) \) be the expression in the right-hand side of (4.3). We first prove that if \( \varepsilon \) is sufficiently small then for any \( z \in \mathbb{B}_\varepsilon^n \) the mapping \( \omega \to \omega' = f_z(\rho, \omega) \) from \( \mathbb{S}^{n-1} \) to itself is an injection. Assume that \( f_z(\rho, \omega_1) = f_z(\rho, \omega_2) \) for some \( \omega_1, \omega_2 \in \mathbb{S}^{n-1} \). Then there exists \( \lambda > 0 \) such that

\[
[1 + \rho(\omega_2)]\omega_2 + r_0\omega_0 = \lambda[(1 + \rho(\omega_1)]\omega_1 + r_0\omega_0).
\]

Let \( \lambda = 1 + \mu, \omega_2 = \omega_1 + \xi \) and \( \rho(\omega_2) = \rho(\omega_1) + \eta \), where \( \mu \in \mathbb{R}, \xi \in \mathbb{R}^n \) and \( \eta \in \mathbb{R} \). Substituting these expressions into (4.4) we get

\[
[1 + \rho(\omega_1)]\xi + \omega_2\eta = \mu[(1 + \rho(\omega_1)]\omega_1 + r_0\omega_0),
\]
which yields $\xi = \mu \omega_1 + \zeta$, where $\zeta = (\mu r_0 \omega_0 - \omega_2 \eta)/(1 + \rho(\omega_1))$. Since $|\rho(\omega_1)| < \delta$ and $|r_0| < \varepsilon$, from the expression of $\zeta$ we see that $|\zeta| \leq 2(\varepsilon|\mu| + |\eta|)$ if $\delta \leq 1/2$. Since $\max_{\omega \in S^{n-1}} |\nabla_\omega \rho(\omega)| < \delta$, by the mean value theorem we easily deduce that $|\eta| \leq \delta|\xi|$, so that

$$|\zeta| \leq 2(\varepsilon|\mu| + \delta|\xi|). \quad (4.5)$$

From the relation $\xi = \mu \omega_1 + \zeta$ we have

$$|\xi| \leq |\mu| + |\zeta|. \quad (4.6)$$

Substituting the relation $\xi = \mu \omega_1 + \zeta$ into $\omega_2 = \omega_1 + \xi$ we get $\omega_2 = (1 + \mu)\omega_1 + \zeta$, or $(1 + \mu)\omega_1 = \omega_2 - \zeta$. From this relation and the fact that $|\mu| < 1$ (for $\varepsilon$ and $\delta$ sufficiently small) we obtain

$$|\mu| \leq |\zeta|. \quad (4.7)$$

From (4.5)–(4.7) we can easily deduce that $|\zeta| = |\xi| = |\mu| = 0$ for sufficiently small $\varepsilon$ and $\delta$, which proves the desired assertion.

Next we prove that if $\varepsilon$ and $\delta$ are sufficiently small then $D_\omega f_\omega(\rho, \omega) : T_\omega(S^{n-1}) \to T_{\omega}(S^{n-1})$ is non-degenerate for any $\omega \in S^{n-1}$ and $\rho \in O_\delta(S^{n-1})$. Note that since $\rho \in C^1(S^{n-1})$, we have $f_\omega(\rho, \cdot) \in C^1(S^{n-1}, S^{n-1})$. Let $a = [1 + \rho(\omega)]\omega + \tau_0 \omega_0$ and $b = [1 + \rho(\omega)]\xi + |\nabla \rho(\omega) : \xi|\omega$, where $\xi \in T_\omega(S^{n-1})$. Then a simple calculation shows that for any $\xi \in T_\omega(S^{n-1})$ we have

$$D_\omega f_\omega(\rho, \omega) \xi = \frac{|a|^2 b - (a \cdot b)a}{|a|^3}.$$  

Since $a = \omega + O(\delta + \varepsilon)$, $b = \xi + O(\delta)|\xi|$ and $\omega \cdot \xi = 0$, from the above expression we see that $D_\omega f_\omega(\rho, \omega) \xi = \xi + O(\delta + \varepsilon)|\xi|$, so that the desired assertion holds.

It follows that for any $\rho \in O_\delta(S^{n-1})$ and $z \in B_0^\rho$, the mapping $f_\omega(\rho, \cdot) : S^{n-1} \to S^{n-1}$ is open. As a result, $\text{Im} f_\omega(\rho, \cdot)$ is an open subset of $S^{n-1}$. Since $f_\omega(\rho, \cdot)$ is continuous, $\text{Im} f_\omega(\rho, \cdot)$ is also closed in $S^{n-1}$. Thus, $f_\omega(\rho, \cdot) : S^{n-1} \to S^{n-1}$ must be a surjection.

Now let $g_\omega(\rho, \cdot)$ be the inverse of $f_\omega(\rho, \cdot)$. By the inverse function theorem we know that $g_\omega(\rho, \cdot) \in C^1(S^{n-1}, S^{n-1})$. Let $F_\omega(\rho, \omega)$ denote the right-hand side of (4.2). Substituting $\omega = g_\omega(\rho, \omega')$ into (4.2) we see that

$$\hat{\rho}(\omega') = F_\omega(\rho, g_\omega(\rho, \omega')) \quad \text{for} \; \omega' \in S^{n-1}. \quad (4.8)$$

Hence, the mapping $S^*_{\omega}$ is well-defined, and $S^*_{\omega}(\rho) = F_\omega(\rho, g_\omega(\rho, \cdot))$.

Finally, it is clear that $F_\omega \in C^1(O_\delta(S^{n-1}) \times S^{n-1}, \mathbb{R})$ and $f_\omega \in C^1(O_\delta(S^{n-1}) \times S^{n-1}, S^{n-1})$. By the implicit function theorem, we also have $g_\omega \in C^1(O_\delta(S^{n-1}) \times S^{n-1}, S^{n-1})$. Thus the mapping $(\rho, \omega) \to S^*_{\omega}(\rho)(\omega)$ from $O_\delta(S^{n-1}) \times S^{n-1}$ to $\mathbb{R}$ is of $C^1$ class. Hence, we have $S^*_{z} \in C(O_\delta(S^{n-1}), C^1(S^{n-1})) \cap C(O_\delta(S^{n-1}), C(S^{n-1})).$ This completes the proof. \[\square\]

From the proof of Lemma 4.1 we can see that if $\rho \in O^m_\delta(S^{n-1}) = C^m(S^{n-1}) \cap O_\delta(S^{n-1})$ for some $m \geq 2$, then $F_\omega$ and $f_\omega$ are of $C^m$ class, which implies that $g_\omega$ and the mapping $(\rho, \omega) \to S^*_{\omega}(\rho)(\omega) = F_\omega(\rho, g_\omega(\rho, \omega))$ are of $C^m$ class, so that $S^*_{z} \in C^{k}(O^m_\delta(S^{n-1}), C^{m-k}(S^{n-1}))$ for any $0 \leq k \leq m$. This particularly implies that

$$S^*_{z} \in C^{\infty}(C^{\infty}(S^{n-1}) \cap O_\delta(S^{n-1}), C^{\infty}(S^{n-1})). \quad (4.9)$$
Similarly, if $\rho \in \mathcal{O}^{m+\mu}(\mathbb{S}^{n-1}) = C^{m+\mu}(\mathbb{S}^{n-1}) \cap \mathcal{O}_{\delta}(\mathbb{S}^{n-1})$ for some $m \geq 1$ and $0 < \mu \leq 1$, then $S^*_x \in C^k(\mathcal{O}^{m+\mu}(\mathbb{S}^{n-1}), C^{m-k+\mu}(\mathbb{S}^{n-1}))$ for any $0 \leq k \leq m$. To establish a similar result for the space $B^{m-1/q}_{qq}(\mathbb{S}^{n-1})$, we need the following lemma:

**Lemma 4.2** Let $\Omega_1$, $\Omega_2$ be two bounded smooth open subset of $\mathbb{R}^n$. Let $m \geq 2$ and $q > n/(m - 1)$. Let $\Phi$ be a diffeomorphism from $\overline{\Omega}_1$ to $\overline{\Omega}_2$. Assume that $\Phi \in W^{m,q}(\Omega_1, \mathbb{R}^n)$. Then $\Phi^{-1} \in W^{m,q}(\Omega_2, \mathbb{R}^n)$. Moreover, given $\varepsilon > 0$, the mapping $\Phi \mapsto \Phi^{-1}$ from the set

$$\{ \Phi \in W^{m,q}(\Omega_1, \mathbb{R}^n) : |\det D\Phi(x)| \geq \varepsilon \text{ for any } x \in \Omega_1 \}$$

to $W^{m,q}(\Omega_2, \mathbb{R}^n)$ is $C^\infty$, and there exists a continuous function $C_\varepsilon : [0, \infty) \to [0, \infty)$ such that

$$\|\Phi^{-1}\|_{W^{m,q}(\Omega_2, \mathbb{R}^n)} \leq C_\varepsilon(\|\Phi\|_{W^{m,q}(\Omega_1, \mathbb{R}^n)}).$$

**Proof** We first note that the assumptions on $m$ and $q$ imply that $W^{m,q}(\Omega_1, \mathbb{R}^n) \hookrightarrow C^1(\overline{\Omega}_1, \mathbb{R}^n)$, and there exists constant $C > 0$ such that $\|\Phi\|_{C^1(\overline{\Omega}_1, \mathbb{R}^n)} \leq C\|\Phi\|_{W^{m,q}(\Omega_1, \mathbb{R}^n)}$. In the sequel we denote by $x$ the variable in $\Omega_1$, and by $y$ the variable in $\Omega_2$. We also denote $\Psi = \Phi^{-1}$. Then we have

$$D\Psi(y) = [D\Phi(x)]^{-1} = [\det D\Phi(x)]^{-1} D^*\Phi(x),$$

where $D^*\Phi(x)$ denotes the co-matrix of the matrix $D\Phi(x)$. By this formula, the Leibnitz rule and the Gagliardo-Nirenberg inequality we can easily deduce that for any $\alpha \in \mathbb{Z}_+^n$ such that $0 < |\alpha| \leq m$ and any $\varepsilon > 0$ such that $|\det D\Phi(x)| \geq \varepsilon$ for all $x \in \Omega_1$, we have

$$\|\partial^\alpha \Psi\|_{L^q(\Omega_2, \mathbb{R}^n)} \leq C\varepsilon^{-|\alpha|}\|D\Phi\|_{L^\infty(\Omega_1, \mathbb{R}^n)}^{n|\alpha|-2} \sum_{|\beta| = |\alpha|} \|\partial^\beta \Phi\|_{L^q(\Omega_1, \mathbb{R}^n)} \leq C\varepsilon^{-|\alpha|}\|\Phi\|_{W^{m,q}(\Omega_1, \mathbb{R}^n)}^{n|\alpha|-1},$$

Hence (4.10) holds. The assertion that the mapping $\Phi \mapsto \Phi^{-1}$ is smooth is an immediate consequence of the above argument. \qed

**Lemma 4.3** Let $m$ and $q$ be as in lemma 4.2. Then we have the following assertions:

(i) For $\delta > 0$ sufficiently small and for $z \in \mathbb{B}_\varepsilon^n$ with $\varepsilon$ sufficiently small, we have

$$S^*_z \in C^k(\mathcal{O}^{m,q}(\mathbb{S}^{n-1}), B^{m-k-1/q}_{qq}(\mathbb{S}^{n-1})), \quad 0 \leq k \leq m - 1.$$

In particular, $S^*_z \in C(\mathcal{O}^{m,q}(\mathbb{S}^{n-1}), B^{m-k-1/q}_{qq}(\mathbb{S}^{n-1})) \cap C^1(\mathcal{O}^{m,q}(\mathbb{S}^{n-1}), B^{m-k-1/q}_{qq}(\mathbb{S}^{n-1}))$. Moreover, for any $\rho \in \mathcal{O}^{m,q}(\mathbb{S}^{n-1})$ and $1 \leq k \leq m - 1$, the operator $D^*S^*_z(\rho)$ from $B^{m-k-1/q}_{qq}(\mathbb{S}^{n-1})$ to $B^{m-k-1/q}_{qq}(\mathbb{S}^{n-1})$ can be uniquely extended to the space $B^{m-k-1/q}_{qq}(\mathbb{S}^{n-1})$, such that

$$D^*S^*_z(\rho) \in L(B^{m-k-1/q}_{qq}(\mathbb{S}^{n-1}), B^{m-k-1/q}_{qq}(\mathbb{S}^{n-1})).$$

(ii) For any $z, w \in \mathbb{B}_\varepsilon^n$ with $\varepsilon$ sufficiently small, we have

$$S^*_z \circ S^*_w = S^*_{z+w}, \quad S^*_0 = \text{id}, \quad \text{and} \quad (S^*_z)^{-1} = S^*_{-z}.$$
(iii) The mapping $S^*: z \to S^*_z$ from $\mathbb{B}^n_\varepsilon$ to $C(\mathcal{O}^{n,q}_\delta(S^{n-1}), B^{m-1/q}_{qq}(S^{n-1}))$ is an injection, and

$$S^* \in C^k(\mathbb{B}^n, C^l(\mathcal{O}^{m,q}_\delta(S^{n-1}), B^{m-k-l-1/q}_{qq}(S^{n-1}))), \quad k \geq 0, \ l \geq 0, \ k+l \leq m-1.$$ 

(iv) Finally assume that $2 \leq k < m, q > n/(k-1)$ and define $p: \mathbb{B}^n_\varepsilon \times \mathcal{O}^{n,q}_\delta(S^{n-1}) \to B^{m-1/q}_{qq}(S^{n-1})$ by $p(z, \rho) = S^*_z(\rho)$. Then for any $\rho \in \mathcal{O}^{n,q}_\delta(S^{n-1})$ the mapping $z \to p(z, \rho)$ from $\mathbb{B}^n_\varepsilon$ to $B^{m-1/q}_{qq}(S^{n-1})$ is Fréchet differentiable when regarded as a mapping from $\mathbb{B}^n_\varepsilon$ to $B^{k-1/q}_{qq}(S^{n-1})$, and we have rank $D_zp(z, \rho) = n$ for any $z \in \mathbb{B}^n_\varepsilon$ and $\rho \in \mathcal{O}^{m,q}_\delta(S^{n-1})$. If furthermore $\rho \in C^\infty(S^{n-1})$ then $[z \to p(z, \rho)] \in C^\infty(\mathbb{B}^n_\varepsilon, C^\infty(S^{n-1})) \subset C^\infty(\mathbb{B}^n_\varepsilon, B^{m-1/q}_{qq}(S^{n-1})).$

Proof. We first note that the assumptions on $m$ and $q$ imply that $B^{m-1/q}_{qq}(S^{n-1}) \hookrightarrow C^1(S^{n-1})$, so that by Lemma 4.1, $S^*_z(\rho)$ makes sense for $\rho \in \mathcal{O}^{m,q}_\delta(S^{n-1})$. Next, by (4.8) we see that $S^*_z(\rho) = F_z(\rho, \varepsilon, \omega)$. Considering (4.2) and (4.3), for given $z = r_0\omega_0 \in \mathbb{B}^n_\varepsilon$ and any $u \in W^m,q(\mathbb{B}^n)$ such that $\|u\|_{C^1(\mathbb{B}^n)} < \delta$ we define

$$\bar{u}(x') = \sqrt{[1 + u(x)]^2 + r_0^2 + 2r_0[1 + u(x)]x \cdot \omega_0 - 1}, \quad x \in \mathbb{B}^n,$$ 

where $x'$ and $x$ are related by

$$x' = \frac{[1 + u(x)]x + r_0\omega_0}{[1 + u(x)]x + r_0\omega_0} |x| \phi(|x| - 1) + [1 - \phi(|x| - 1)]x, \quad x \in \mathbb{B}^n,$$ 

where $\phi$ is as in Section 3. As before we use the notation $F_z(u, x)$ to denote the expression on the right-hand side of (4.11). Since the assumptions on $m$ and $q$ imply that $W^m,q(\mathbb{B}^n)$ is an algebra, it is clear that $F_z(u, \cdot) \in W^m,q(\mathbb{B}^n)$, and the mapping $u \to F_z(u, \cdot)$ is $C^\infty$. We also use the same notation $f_z(u, x)$ as before to denote the expression on the right-hand side of (4.12), because if we particularly take $u = \Pi(\rho)$ and $x = \omega \in S^{n-1}$ then we get $f_z(\rho, \omega)$ defined before. It can be easily shown that if $\varepsilon$ and $\delta$ are sufficiently small then the mapping $\Phi_u: x \to x' = f_z(u, x)$ is a diffeomorphism of $\mathbb{B}^n$ to itself and det $D\Phi_u(x) = 1 + O(\varepsilon + \delta)$. Moreover, since $W^m,q(\mathbb{B}^n)$ is an algebra, we have $\Phi_u \in W^m,q(\mathbb{B}^n, \mathbb{R}^n)$ and it is clear that the mapping $u \to \Phi_u$ is $C^\infty$. By Lemma 4.2 we infer that $\Phi_u^{-1} \in W^m,q(\mathbb{B}^n, \mathbb{R}^n)$, and the mapping $\Phi_u \to \Phi_u^{-1}$ is $C^\infty$. Substituting $x = \Phi_u^{-1}(x')$ into the right-hand side of (4.11) and using Lemma 3.1, we see that $\bar{u} = F_z(\Phi_u^{-1}(\cdot)) \in W^m,q(\mathbb{B}^n)$. Now, clearly if $u = \Pi(\rho)$ for some $\rho \in \mathcal{O}^{m,q}_\delta(S^{n-1})$ then we have $\bar{u}|_{S^{n-1}} = S^*_z(\rho)$, so that we have proved that $S^*_z(\rho) \in B^{m-1/q}_{qq}(S^{n-1})$ for any $\rho \in \mathcal{O}^{m,q}_\delta(S^{n-1})$. We note that though both the mappings $u \to F_z(u, \cdot)$ and $u \to \Phi_u^{-1}$ are $C^\infty$, the mapping $u \to \bar{u} = F_z(u, \Phi_u^{-1}(\cdot))$ is, however, not necessarily $C^\infty$, because $F_z(u, x)$ is generally not $C^\infty$ in $x$. Despite this inconvenience, we still can ensure that the mapping $u \to \bar{u} = F_z(u, \Phi_u^{-1}(\cdot))$ from $W^m,q(\mathbb{B}^n) \cap \{u \in C^1(\mathbb{B}^n) : \|u\|_{C^1(\mathbb{B}^n)} < \delta\}$ to $W^m,q(\mathbb{B}^n)$ is continuous, because both $(u, x) \to F_z(u, x)$ and $u \to \Phi_u^{-1}$ are continuous. Thus $S^*_z \in C(\mathcal{O}^{m,q}_\delta(S^{n-1}), B^{m-1/q}_{qq}(S^{n-1}))$. Next, since $S^*_z(\rho) = \Gamma F_z(\Pi(\rho), \Phi_u^{-1}(\cdot))$, where $\Gamma$ denotes the trace operator, we have

$$DS^*_z(\rho)\eta = \Gamma D_1 F_z(\Pi(\rho), \Phi_u^{-1}(\cdot))\Pi(\eta) + \Gamma D_2 F_z(\Pi(\rho), \Phi_u^{-1}(\cdot))D_u\Phi_u^{-1}(\cdot)\Pi(\eta)$$ 

$$\equiv I(\rho)\eta + II(\rho)\eta,$$ 

(4.13)
where $D_1$ and $D_2$ represent the Fréchet derivatives in the first and the second arguments, respectively, and $D_u \Phi_{\Pi(\rho)}^{-1} = D_u \Phi_{\Pi(\rho)}^{-1}|_{u=\Pi(\rho)}$. By Lemma 3.2 (i) it is obvious that

$$I \in C(O_\delta^{m,q}(S^{n-1}), L(B_{qq}^{m-1/q}(S^{n-1}), B_{qq}^{m-1/q}(S^{n-1})))$$

$$\cap C(O_\delta^{m,q}(S^{n-1}), L(B_{qq}^{m-k-1/q}(S^{n-1}), B_{qq}^{m-k-1/q}(S^{n-1}))), \ 1 \leq k \leq m - 1.$$ 

To treat II we denote $G_z(t, y) = \sqrt{(x + t)^2 + \rho_0^2 + 2\rho_1(1 + t)y \cdot \omega_0} - 1$ for $t \in \mathbb{R}$ and $y \in \mathbb{B}^n$. Then $F_z(u, x) = G_z(u(x), x)$ for $u \in W^{m,q}(\mathbb{B}^n)$ and $x \in \mathbb{B}^n$, so that

$$D_2 F_z(u, x) = D_1 G_z(u(x), x) Du(x) + D_2 G_z(u(x), x).$$

Given $u \in W^{m,q}(\mathbb{B}^n)$, from the above expression of $D_2 F_z(u, x)$ we see that $D_2 F_z(u, \cdot) = [x \rightarrow D_2 F_z(u, x)] \in W^{m-1,q}(\mathbb{B}^n, L(\mathbb{R}^n, \mathbb{R}^n))$. Besides, since

$$D_u \Phi_u^{-1} \in L(W^{m,q}(\mathbb{B}^n), W^{m,q}(\mathbb{B}^n, \mathbb{R}^n)) \cap L(W^{m-k,q}(\mathbb{B}^n), W^{m-k,q}(\mathbb{B}^n, \mathbb{R}^n))$$

$(1 \leq k \leq m - 1)$, for any $\eta \in B_{qq}^{m-k-1/q}(S^{n-1})$ we have $D_u \Phi_u^{-1} \pi(\eta) = [x \rightarrow D_u \Phi_u^{-1}(x) \pi(\eta)] \in W^{m-k,q}(\mathbb{B}^n, \mathbb{R}^n)$, and if $\eta \in B_{qq}^{m-k-1/q}(S^{n-1})$ for some $1 \leq k \leq m - 1$ then we have $D_u \Phi_u^{-1} \pi(\eta) \in W^{m-k,q}(\mathbb{B}^n, \mathbb{R}^n)$. Hence, given $\rho \in B_{qq}^{m-k-1/q}(S^{n-1})$, for any $\eta \in B_{qq}^{m-k-1/q}(S^{n-1})$ we have

$$D_2 F_z(\pi(\rho), \pi(\rho)) D_u \Phi_u^{-1} \pi(\rho) \pi(\eta) = [x \rightarrow D_2 F_z(\pi(\rho), \pi(\rho)) D_u \Phi_u^{-1} \pi(\rho) \pi(\eta)] \in W^{m-1,q}(\mathbb{B}^n),$$

and if $\eta \in B_{qq}^{m-k-1/q}(S^{n-1})$ for some $1 \leq k \leq m - 1$ then we have

$$D_2 F_z(\pi(\rho), \pi(\rho)) D_u \Phi_u^{-1} \pi(\rho) \pi(\eta) \in W^{m-k,q}(\mathbb{B}^n).$$

This implies that for $\rho \in B_{qq}^{m-k-1/q}(S^{n-1})$, if $\eta \in B_{qq}^{m-k-1/q}(S^{n-1})$ then $II(\rho) \eta \in B_{qq}^{m-k-1/q}(S^{n-1})$, whereas if $\eta \in B_{qq}^{m-k-1/q}(S^{n-1})$ for some $1 \leq k \leq m - 1$ then $II(\rho) \eta \in B_{qq}^{m-k-1/q}(S^{n-1})$. A similar analysis shows that

$$II \in C(O_\delta^{m,q}(S^{n-1}), L(B_{qq}^{m-1/q}(S^{n-1}), B_{qq}^{m-1/q}(S^{n-1})))$$

$$\cap C(O_\delta^{m,q}(S^{n-1}), L(B_{qq}^{m-k-1/q}(S^{n-1}), B_{qq}^{m-k-1/q}(S^{n-1}))), \ 1 \leq k \leq m - 1.$$ 

Hence, $S_z^* \in C^1(O_\delta^{m,q}(S^{n-1}), B_{qq}^{m-1/q}(S^{n-1}))$, and

$$DS_z^* \in C(O_\delta^{m,q}(S^{n-1}), L(B_{qq}^{m-1/q}(S^{n-1}), B_{qq}^{m-1/q}(S^{n-1})))$$

$$\cap C(O_\delta^{m,q}(S^{n-1}), L(B_{qq}^{m-k-1/q}(S^{n-1}), B_{qq}^{m-k-1/q}(S^{n-1}))), \ 1 \leq k \leq m - 1.$$ 

Furthermore, by an induction argument we see that $S_z^* \in C^k(O_\delta^{m,q}(S^{n-1}), B_{qq}^{m-k-1/q}(S^{n-1}))$ for any $0 \leq k \leq m-1$. This proves Assertion (i). Assertion (ii) is obvious. The first part of Assertion (iii) is evident, and the second part follows by checking more carefully the argument in the proof of Assertion (i), which we omit here. From the proof of Assertion (i) we see that for any integers $2 \leq k < m$ and $q > n/(k - 1)$, the mapping $p : \mathbb{B}_e^m \times O_\delta^{m,q}(S^{n-1}) \rightarrow B_{qq}^{k-1/q}(S^{n-1})$ defined by $p(z, \rho) = S_z^*(\rho)$ is continuously differentiable at any point $(z, \rho) \in \mathbb{B}_e^m \times O_\delta^{m,q}(S^{n-1})$. Moreover,
a simple calculation shows that \( D_1 p(0,0) z = z \cdot \omega \). Here \( z \cdot \omega \) represents the function \( \omega \rightarrow z \cdot \omega \) on \( S^{n-1} \). This shows that \( \text{rank} D_1 p(0,0) = n \). By continuity, we infer that \( \text{rank} D_1 p(z, \rho) = n \) for any \((z, \rho) \in \mathbb{B}_\varepsilon^n \times \mathcal{O}_0^\varepsilon(S^{n-1})\), provided \( \varepsilon \) and \( \delta \) are sufficiently small. Finally, if \( \rho \in C^\infty(S^{n-1}) \) then from the construction of \( S^*_z \) it is clear that \([z \rightarrow p(z, \rho)] \in C^\infty(\mathbb{B}_{\varepsilon}^n, C^\infty(S^{n-1}))\). Hence, Assertion \( (iv) \) follows. This completes the proof. \( \square \)

By Lemma 4.3 we see that the mapping \( S^*_z \) provides an action of the local group \( G = \mathbb{B}_\varepsilon^n \) to some open subset of \( B_{q q}^{m-1/q}(S^{n-1}) \). We note that if \( c = 0 \) then by some similar arguments as in Section 3 we can reduce the problem (1.1)–(1.5) into a differential equation \( \rho' = A_\varepsilon(\rho(t)) \) in the Banach space \( B_{q q}^{m-3-1/q}(S^{n-1}) \) in the unknown function \( \rho = \rho(t) \) only, where \( A_\varepsilon \) is defined in some open subset of \( B_{q q}^{m-1/q}(S^{n-1}) \). It can be shown that in this case the reduced equation satisfies a similar relation as that in (4.1) under the above action of \( G \) (cf. the proof of Lemma 4.6 below). For the equation (3.30), however, \( G \) has to act on some open set in \( X \). This is fulfilled in the following paragraph. In the sequel, the notations \( X, X_0 \) and \( \mathcal{O} \) have the same meaning as introduced in the end of Section 3.

Given \( z \in \mathbb{B}_\varepsilon^n \) and \( \rho \in \mathcal{O}_0(S^{n-1}) \), let \( P_{z, \rho} : C(\mathbb{B}_\varepsilon^n) \rightarrow C(\mathbb{B}_\varepsilon^n) \) be the mapping

\[
P_{z, \rho}(u)(x) = u(\Theta_{\rho}^{-1}(\Theta_{S^*_z(\rho)}(x) - z)) \quad \text{for} \quad u \in C(\mathbb{B}_\varepsilon^n).
\]

Clearly, \( P_{z, \rho} \in L(C(\mathbb{B}_\varepsilon^n), C(\mathbb{B}_\varepsilon^n)) \). Moreover, if \( \rho \in B_{q q}^{m-1/q}(S^{n-1}) \) then \( S^*_z(\rho) \in B_{q q}^{m-1/q}(S^{n-1}) \), so that by Lemma 3.1 we have \( P_{z, \rho} \in L(W^{m, q}(\mathbb{B}_\varepsilon^n), W^{m, q}(\mathbb{B}_\varepsilon^n)) \). For \( u \in C(\mathbb{B}_\varepsilon^n) \), \( \rho \in \mathcal{O}_0(S^{n-1}) \) and \( z \in \mathbb{B}_\varepsilon^n \) we denote

\[
S^*_z(u, \rho) = \begin{pmatrix} P_{z, \rho}(u) \\ S^*_z(\rho) \end{pmatrix}.
\]

Note that \( S^*_z(\cdot, \cdot) \) is id.

**Lemma 4.4** Let \( m \geq 5 \) and \( q > n/(m - 4) \). Let

\[
\mathcal{O}' = W^{m-3}(\mathbb{B}_\varepsilon^n) \times (B_{q q}^{m-3-1/q}(S^{n-1}) \cap \mathcal{O}_0(S^{n-1})) \quad (\implies \mathcal{O} = X_0 \cap \mathcal{O}')
\]

For sufficiently small \( \varepsilon > 0 \) and \( \delta > 0 \) we have the following assertions:

(i) For any \( \varepsilon \in \mathbb{B}_\varepsilon^n \) we have \( S^*_z \in C(\mathcal{O}', X) \cap C(\mathcal{O}, X_0) \). Moreover, regarded as a mapping from \( \mathcal{O}' \) to \( X \), \( S^*_z \) is Fréchet differentiable at every point in \( \mathcal{O} \), and \( DS^*_z \in C(\mathcal{O}, L(X, X)) \).

(ii) For any \( z, w \in \mathbb{B}_\varepsilon^n \) we have

\[
S^*_z \circ S^*_w = S^*_{z+w}, \quad S^*_0 = id, \quad \text{and} \quad (S^*_z)^{-1} = S^*_{-z}.
\]

(iii) The mapping \( S^*: z \rightarrow S^*_z \) from \( \mathbb{B}_\varepsilon^n \) to \( C(\mathcal{O}', X) \) is an injection, and

\[
S^* \in C^k(\mathbb{B}_\varepsilon^n, C(\mathcal{O}, W^{m-k-l-1/q}(\mathbb{B}_\varepsilon^n) \times B_{q q}^{m-k-l-1/q}(S^{n-1}))) \quad , \quad k \geq 0, \quad l \geq 0, \quad k + l \leq m - 1.
\]

(iv) Define \( p: \mathbb{B}_\varepsilon^n \times \mathcal{O}' \rightarrow X \) by \( p(z, U) = S^*_z(U) \). Then for any \( U \in \mathcal{O} \) we have \( p(\cdot, U) \in C^1(\mathbb{B}_\varepsilon^n, X) \), and \( \text{rank} D_z p(z, \rho) = n \) for every \( z \in \mathbb{B}_\varepsilon^n \) and \( U \in \mathcal{O} \). If furthermore \( U \in X^\infty = C^\infty(\mathbb{B}_\varepsilon^n) \times C^\infty(S^{n-1}) \) then \( p(\cdot, U) \in C^\infty(\mathbb{B}_\varepsilon^n, X^\infty) \).
Proof. All assertions of this lemma follow readily from the corresponding assertions in Lemma 4.3. □

In the sequel, for \( \rho = \rho(t) \), \( u = u(x,t) \) and \( U = (u(x,t)) \), we denote by \( P_{z,\rho}(u) \) the function \( \tilde{u}(x,t) = u(\Theta^{-1}_{\rho(t)}(\Theta S^*_z(\rho(t)))(x) - z), t) \), by \( S^*_z(\rho) \) the function \( \tilde{\rho}(t) = S^*_z(\rho(t)) \), and by \( S^*_z(U) \) the vector function \( (P_{z,\rho}(u)) = (\tilde{u}(x,t)) \).

**Lemma 4.5** If \( U = \binom{u}{\rho} \) is a solution of the equation (3.30) such that \( \|\rho\|_{C^1([a-1])} \) is sufficiently small, then for any \( z \in \mathbb{R}^n \) such that \( |z| \) is sufficiently small, \( S^*_z(U) = \binom{P_{z,\rho}(u)}{S^*_z(\rho)} \) is also a solution of (3.30).

Proof. It is easy to see that if a triple \((\sigma,p,\Omega)\) is a solution of the problem (1.1)–(1.5), then for any \( z \in \mathbb{R}^n \) the triple \((\tilde{\sigma},\tilde{p},\tilde{\Omega})\) defined by

\[
\tilde{\sigma}(x,t) = \sigma(x-z,t), \quad \tilde{p}(x,t) = p(x-z,t), \quad \tilde{\Omega}(t) = \Omega(t) + z,
\]

is also a solution of that problem. From this fact one can easily verify that if \( U = \binom{u}{\rho} \) is a solution of the equation (3.30) then \( \tilde{U} = \binom{\tilde{u}}{\tilde{\rho}} \), where

\[
\tilde{u}(x,t) = u(\Theta^{-1}_{\rho(t)}(\Theta S^*_z(\rho(t)))(x) - z), t), \quad \tilde{\rho}(t) = S^*_z(\rho(t)),
\]

is also a solution of that equation, which is the desired assertion. □

**Lemma 4.6** The following relation holds for any \( z \in \mathbb{R}^n \) and any \( U = \binom{u}{\rho} \in \mathcal{O} \), provided \( \varepsilon \) and \( \delta \) are sufficiently small:

\[
\mathbb{F}(S^*_z(U)) = DS^*_z(U)\mathbb{F}(U). \tag{4.14}
\]

Proof. By Theorem 1.1 of [14], given any \( U = \binom{u}{\rho} \in X_0 \) there exists \( \delta > 0 \) such that the equation (3.30) has a unique solution \( V = V(t) \) for \( 0 \leq t \leq \delta \), which belongs to \( C([0,\delta],X) \cap C((0,\delta],\mathcal{O}) \cap L^\infty((0,\delta],X_0) \cap C^1((0,\delta],X) \) and satisfies the initial condition \( V(0) = U \) (This result also follows from Corollary 5.3 in the next section and a standard existence theorem that we used in the proof of Theorem 2.1). Let \( \tilde{V}(t) = S^*_z(V(t)) \) for \( 0 \leq t \leq \delta \). By Lemma 4.5, \( \tilde{V} \) is also a solution of (3.30), satisfying the initial condition \( \tilde{V}(0) = S^*_z(U) \). The fact that \( \tilde{V} \) is the solution of (3.30) implies that

\[
\frac{d\tilde{V}(t)}{dt} = \mathbb{F}(\tilde{V}(t)) \quad \text{for} \quad 0 < t \leq \delta.
\]

On the other hand, since \( \tilde{V}(t) = S^*_z(V(t)) \), we have

\[
\frac{d\tilde{V}(t)}{dt} = DS^*_z(V(t))\frac{dV(t)}{dt} = DS^*_z(V(t))\mathbb{F}(V(t)) \quad \text{for} \quad 0 < t \leq \delta.
\]

Thus \( \mathbb{F}(\tilde{V}(t)) = DS^*_z(V(t))\mathbb{F}(V(t)) \) for \( 0 < t \leq \delta \). If \( V(t) \) is a strict solution then clearly \( \tilde{V}(t) \) is also a strict solution, so that by directly letting \( t \to \delta^+ \) we get (4.14). If \( V(t) \) is not
a strict solution then we appeal to the quasi-linear structure of \( \mathcal{F}(U) \) to prove (4.14): Since \( V \in L^{\infty}((0, \delta), X_0) \cap C([0, \delta], X) \) and \( V(0) = U \), we infer that \( V(t) \) weakly converges to \( U \) in \( X_0 \) as \( t \to 0^+ \). Similarly \( \tilde{V}(t) \) weakly converges to \( S^*_2(U) \) in \( X_0 \). Since \( \mathcal{F}(U) = \mathcal{A}(U)U + \mathcal{F}_0(U) \), we have

\[
\mathcal{F}(V(t)) - \mathcal{F}(U) = [\mathcal{A}(V(t)) - \mathcal{A}(U)]V(t) + [\mathcal{A}(U)[V(t) - U] + [\mathcal{F}_0(V(t)) - \mathcal{F}(U)]
\]

\[= I(t) + II(t) + III(t). \]

We have \( \|I(t)\|_X \leq C\|\mathcal{A}(V(t)) - \mathcal{A}(U)\|_{L(X_0, X)} \), so that \( \lim_{t \to 0^+} \|I(t)\|_X = 0 \), because \( \mathcal{A} \) maps \( X_0 \) compactly into \( L(X_0, X) \). We also have \( \lim_{t \to 0^+} \|III(t)\|_X = 0 \) by a similar reason. In addition, it is evident that \( II(t) \) weakly converges to 0 in \( X \) as \( t \to 0^+ \). Therefore, \( \mathcal{F}(V(t)) \) weakly converges to \( \mathcal{F}(U) \) in \( X \). Similarly, \( \mathcal{F}(\tilde{V}(t)) \) weakly converges to \( \mathcal{F}(S^*_2(U)) \) in \( X \). Finally, from the expression of \( DS^*_2 \) (cf. (4.13)) we can easily find that \( DS^*_2 \) maps \( X_0 \) compactly into \( L(X, X) \). Thus by a similar argument as above we infer that \( DS^*_2(V(t))\mathcal{F}(V(t)) \) weakly converges to \( DS^*_2(U)\mathcal{F}(U) \) in \( X \) as \( t \to 0^+ \). Hence (4.14) holds. \( \square \)

Lemma 4.6 has some obvious corollaries. First, let \( \mathcal{F}_2 \) be the second component of \( \mathcal{F} \). Taking the second components of both sides of (4.14) we get

\[
\mathcal{F}_2(S^*_2(U)) = DS^*_2(\rho)\mathcal{F}_2(U), \tag{4.15}
\]

where \( \rho \) is the second component of \( U \). Next, let \( u_s = \sigma_s - \bar{\sigma} \) and \( U_s = \binom{u_s}{0}. \) \( U_s \) is a stationary point of the equation (3.30), i.e., \( \mathcal{F}(U_s) = 0 \). Taking \( U = U_s \) in (4.14) we get \( \mathcal{F}(S^*_2(U_s)) = 0 \) for any \( z \in B^n \). Since clearly \( U_s \in X^\infty \), we have \( [z \to S^*_2(U_s)] \in C^\infty(B^n, X^\infty). \) Thus, differentiating the equation \( \mathcal{F}(S^*_2(U_s)) = 0 \) in \( z \) at \( z = 0 \), we obtain

\[
\mathcal{F}'(U_s)W_j = 0, \quad W_j = \left[ [\phi(r - 1)r - 1]\sigma'_s(r)\omega_j \right], \quad j = 1, 2, \ldots, n, \tag{4.16}
\]

i.e., 0 is an eigenvalue of \( \mathcal{F}'(U_s) \) of (geometric) multiplicity \( n \), and the corresponding linearly independent eigenvectors are \( W_1, W_2, \ldots, W_n \).

5 Calculation of \( \mathcal{F}'(U_s) \)

In this section we calculate the Fréchet derivative of \( \mathcal{F} \) at the stationary point \( U_s \). Since \( \mathcal{F}(U) = \mathcal{A}(U)U + \mathcal{F}_0(U) \), we have

\[
\mathcal{F}'(U_s)V = \mathcal{A}(U_s)V + [\mathcal{A}'(U_s)V]U_s + \mathcal{F}'_0(U_s)V \quad \text{for} \quad V \in X_0. \tag{5.1}
\]

Recall that \( \mathcal{A} \in C^\infty(O, L(X_0, X)) \), so that \( \mathcal{A}(U_s) \in L(X_0, L(X_0, X)) \), and \( \mathcal{A}'(U_s)V \in L(X_0, X) \) for \( V \in X_0 \). Since \( U_s = \binom{u_s}{0} \), simple calculations show that for any \( V = \binom{v}{\eta} \in X_0 \) we have

\[
\mathcal{A}(U_s)V = \begin{pmatrix} c^{-1}A(0)v + C(0, u_s)\eta \\ B(0)\eta \end{pmatrix}, \quad [\mathcal{A}'(U_s)V]U_s = \begin{pmatrix} c^{-1}[A'(0)\eta]u_s \\ 0 \end{pmatrix}, \tag{5.2}
\]
Hence we have

\[ F_0(U_s)V = \begin{pmatrix} D_u \mathcal{F}_2(0, u_s)v + D_\rho \mathcal{F}_2(0, u_s)\eta \\ D_u \mathcal{G}_2(0, u_s)v + D_\rho \mathcal{G}_2(0, u_s)\eta \end{pmatrix}, \tag{5.3} \]

where \( D_u \mathcal{F}_2 \) and \( D_\rho \mathcal{F}_2 \) represent Fréchet derivatives of \( \mathcal{F}_2(\rho, u) \) in \( u \) and \( \rho \), respectively, and similarly for \( D_u \mathcal{G}_2 \) and \( D_\rho \mathcal{G}_2 \). Clearly,

\[ A(0)v = \Delta v, \tag{5.4} \]

and a simple computation shows that

\[ C(0, u_s)\eta = \phi(r - 1)\sigma_s'(r)\Pi(B(0)\eta). \tag{5.5} \]

To compute \( B(0)\eta = -\gamma D(0)T(0)\mathcal{L}(0)\eta \) we first note that, clearly,

\[ D(0)v = \left. \frac{\partial v}{\partial r} \right|_{r=1} \quad \text{and} \quad T(0)\eta = \Pi(\eta). \]

Next, recall that

\[ K(\rho) = \mathcal{L}(\rho)\rho + \mathcal{K}_1(\rho), \quad \text{so that} \quad K'(0)\eta = \mathcal{L}(0)\eta + \mathcal{K}_1'(0)\eta. \tag{5.6} \]

On the other hand, from [25] we know that

\[ \mathcal{K}(\varepsilon \eta) = 1 - \varepsilon[\eta(\omega) + \frac{1}{n-1}\Delta_\omega \eta(\omega)] + o(\varepsilon), \]

which implies that \( \mathcal{K}(0) = 1 \) and \( \mathcal{K}'(0)\eta = -[\eta + \frac{1}{n-1}\Delta_\omega \eta] \). Comparing these expressions with those in (5.6), we obtain

\[ \mathcal{L}(0)\eta = -\frac{1}{n-1}\Delta_\omega \eta, \quad \mathcal{K}_1(0) = 1, \quad \text{and} \quad \mathcal{K}_1'(0)\eta = -\eta. \]

Hence we have

\[ B(0)\eta = -\gamma D(0)T(0)\mathcal{L}(0)\eta = \left. \frac{\gamma}{n-1} \frac{\partial}{\partial r} \Pi(\Delta_\omega \eta) \right|_{r=1}. \tag{5.7} \]

We denote \( u^s_{\varepsilon, \eta} = \Theta^s_{\varepsilon, \eta} \sigma_s - \bar{\sigma} \). Then we have

\[ A(\varepsilon \eta)u^s_{\varepsilon, \eta} = \mathcal{F}(u^s_{\varepsilon, \eta} + \bar{\sigma}), \]

so that

\[ [A(\varepsilon \eta) - A(0)]u^s_{\varepsilon, \eta} + A(0)(u^s_{\varepsilon, \eta} - u_s) = \mathcal{F}(u^s_{\varepsilon, \eta} + \bar{\sigma}) - \mathcal{F}(u_s + \bar{\sigma}). \]

Dividing both sides with \( \varepsilon \) and letting \( \varepsilon \to 0 \), we get

\[ [A'(0)\eta]u_s + A(0)[\mathcal{M}(0, \sigma_s)\Pi(\eta)] = \mathcal{F}'(\sigma_s)[\mathcal{M}(0, \sigma_s)\Pi(\eta)]. \]

Here we used the fact that \( \lim_{\varepsilon \to 0} \varepsilon^{-1}(u^s_{\varepsilon, \eta} - u_s) = \mathcal{M}(0, \sigma_s)\Pi(\eta) \) \( (= \phi(r - 1)\sigma_s'(r)\Pi(\eta)) \). Hence,

\[ [A'(0)\eta]u_s = -A(0)[\mathcal{M}(0, \sigma_s)\Pi(\eta)] + \mathcal{F}'(\sigma_s)[\mathcal{M}(0, \sigma_s)\Pi(\eta)] 
= -\Delta [\phi(r - 1)\sigma_s'(r)\Pi(\eta)] + f'(\sigma_s(r))\phi(r - 1)\sigma_s'(r)\Pi(\eta) 
= -[\Delta - f'(\sigma_s(r))][\phi(r - 1)\sigma_s'(r)\Pi(\eta)]. \tag{5.8} \]
To compute $F_0'(U_s)$, we first note that since $P(\rho, u)v = M(\rho, u)\Pi(D(\rho)v)$, we have
\[
F_2(\rho, u) = -c^{-1}F(u + \sigma) - \gamma P(\rho, u + \sigma)T(\rho)K_1(\rho) - P(\rho, u + \sigma)S(\rho)G(u + \sigma)
= -c^{-1}F(u + \sigma) - \gamma M(\rho, u + \sigma)\Pi[D(\rho)T(\rho)K_1(\rho)]
- M(\rho, u + \sigma)\Pi[D(\rho)S(\rho)G(u + \sigma)].
\]
Differentiating this expression in $u$ at $(\rho, u) = (0, u_s)$ yields
\[
\begin{align*}
D_uF_2(0, u_s)v &= -c^{-1}f'(\sigma_s(r))v - \gamma M(0, v)\Pi[D(0)T(0)K_1(0)]
- M(0, v)\Pi[D(0)S(0)g(\sigma_s(r))] - M(0, u_s)\Pi[D(0)S(0)g'(\sigma_s(r))v].
\end{align*}
\]
We have $D(0)T(0)K_1(0) = D(0)T(0)1 = D(0)1 = 0$, and, by denoting $v_s(r) = p_s(r) - p_s(1)$,
$D(0)S(0)g(\sigma_s(r)) = D(0)v_s = p_s'(1)1 = 0$. Hence,
\[
D_uF_2(0, u_s)v = -c^{-1}f'(\sigma_s(r))v - M(0, u_s)\Pi[D(0)S(0)g'(\sigma_s(r))v]
= -c^{-1}f'(\sigma_s(r))v - \phi(r - 1)\sigma'_s(r)\Pi[D(0)S(0)g'(\sigma_s(r))v]. \quad (5.9)
\]
In order to compute $D_\rho F_2(0, u_s)$ we write
\[
\begin{align*}
D(0)T(0)K_1'(0)\eta &= -D(0)T(0)\eta = -D(0)\Pi(\eta) = -\frac{\partial \Pi(\eta)}{\partial r} \bigg|_{r=1}, \\
D(0)[T'(0)\eta]K_1(0) &= D(0)[T'(0)\eta]1 = 0 \quad \text{(because $T(\varepsilon\eta)1 = T(0)1 = 1$)}, \\
[D'(0)\eta]T(0)K_1(0) &= [D'(0)\eta]T(0)1 = [D'(0)\eta]1 = 0 \quad \text{(because $D(\varepsilon\eta)1 = D(0)1 = 0$)}, \\
[D'(0)\eta]S(0)g(\sigma_s(r)) &= [D'(0)\eta]v_s = p_s'(1)1\eta = 0, \\
D(0)[S'(0)\eta]g(\sigma_s(r)) &= D(0)[S(0)[A'(0)\eta]S(0)g(\sigma_s(r))] = D(0)[S(0)[A'(0)\eta]v_s]. \quad (5.10)
\end{align*}
\]
In getting (5.10) we used the identity $S'(0)\eta = S(0)[A'(0)\eta]S(0)$, which follows from the fact
that $A(\rho)S(\rho) = -i\sigma$ for any $\rho \in C^2(S^{n-1})$. By a similar argument as in the proof of (5.8) we see that
\[
[A'(0)\eta]v_s = -A(0)[M(0, p_s)\Pi(\eta)] - g'(\sigma_s)[M(0, \sigma_s)\Pi(\eta)]
= -\Delta[\phi(r - 1)p_s'(r)\Pi(\eta)] - g'(\sigma_s(r))\phi(r - 1)\sigma'_s(r)\Pi(\eta). \quad (5.11)
\]
Substituting (5.11) into (5.10) we get
\[
D(0)[S'(0)\eta]g(\sigma_s(r)) = D(0)[\phi(r - 1)p_s'(r)\Pi(\eta)] - D(0)[S(0)[g'(\sigma_s(r))\phi(r - 1)\sigma'_s(r)\Pi(\eta)]
= -g(\sigma)\eta - D(0)S(0)[g'(\sigma_s(r))\phi(r - 1)\sigma'_s(r)\Pi(\eta)].
\]
Using these results and the relations $D(0)T(0)K_1(0) = 0$ and $D(0)S(0)g(\sigma_s(r)) = 0$, we see that
\[
\begin{align*}
D_\rho F_2(0, u_s)\eta &= -\gamma D_\rho M(0, \sigma_s)\eta \cdot \Pi(D(0)T(0)K_1(0)) - \gamma M(0, \sigma_s)\Pi(D(0)T(0)K_1'(0)\eta)
- \gamma M(0, \sigma_s)\Pi(D(0)[T'(0)\eta]K_1(0)) - \gamma M(0, \sigma_s)\Pi(D'(0)\eta)T(0)K_1(0))
- D_\rho M(0, \sigma_s)\eta \cdot \Pi(D(0)S(0)g(\sigma_s(r))) - M(0, \sigma_s)\Pi(D(0)[S'(0)\eta]g(\sigma_s(r)))
- M(0, \sigma_s)\Pi([D'(0)\eta]S(0)g(\sigma_s(r)))
= \gamma \phi(r - 1)\sigma'_s(r)\Pi \left(\frac{\partial \Pi(\eta)}{\partial r} \bigg|_{r=1}\right) + \phi(r - 1)\sigma'_s(r)\Pi(D(0)S(0)[g'(\sigma_s(r))]
\times \phi(r - 1)\sigma'_s(r)\Pi(\eta)) + g(\sigma)\phi(r - 1)\sigma'_s(r)\Pi(\eta). \quad (5.12)
\end{align*}
\]
Finally, differentiating \( G_2(\rho, u) = -\gamma D(\rho) T(\rho) K_1(\rho) - \gamma D(\rho) S(\rho) \mathcal{G}(u + \sigma) \) in \( u \) at \( (\rho, u) = (0, u_s) \) yields
\[
D_u G_2(0, u_s) v = -\gamma D(0) S(0) [g'(\sigma_s(r)) v],
\] (5.13)
and differentiating in \( \rho \) gives
\[
D_\rho G_2(0, u_s) \eta = -\gamma D(0) T(0) K'_1(0) \eta - \gamma D(0) [T'(0) \eta] K_1(0) - \gamma [D'(0) \eta] T(0) K_1(0) - \gamma [D'(0) \eta] S(0) [g'(\sigma_s(r)) - \gamma D(0) S(0) \eta g'(\sigma_s(r))]
\]
\[
= \gamma \frac{\partial \Pi(\eta)}{\partial r} \bigg|_{r=1} + D(0) S(0) [g'(\sigma_s(r)) \phi(r - 1) \sigma'_s(r) \Pi(\eta)] + g(\sigma) \eta. \] (5.14)

From (5.1)–(5.5), (5.7)–(5.9) and (5.12)–(5.14) we obtain
\[
\mathcal{F}'(U_s) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\] (5.15)
where, by denoting \( m(r) = \phi(r - 1) \sigma'_s(r), \)
\[
A_{11} v = e^{-1}[\Delta - f'(\sigma_s(r))] v - m(r) \Pi[D(0) S(0) [g'(\sigma_s(r)) v]],
\]
\[
A_{12} \eta = m(r) \Pi \left[ \gamma \frac{\partial \Pi(\eta)}{\partial r} \left( \eta + \frac{1}{n-1} \Delta \omega \eta \right) \bigg|_{r=1} + g(\sigma) \eta \right] - e^{-1}[\Delta - f'(\sigma_s(r))] [m(r) \Pi(\eta)]
\]
\[
+ m(r) \Pi[D(0) S(0) [g'(\sigma_s(r)) m(r) \Pi(\eta)]],
\]
\[
A_{21} v = -D(0) S(0) [g'(\sigma_s(r)) v],
\]
\[
A_{22} \eta = \gamma \frac{\partial \Pi(\eta)}{\partial r} \left( \eta + \frac{1}{n-1} \Delta \omega \eta \right) \bigg|_{r=1} + g(\sigma) \eta + D(0) S(0) [g'(\sigma_s(r)) m(r) \Pi(\eta)].
\]

We summarize the above result in the following lemma:

**Lemma 5.1** The Fréchet derivative \( \mathcal{F}'(U_s) \) is given by (5.15).

In Section 4 we proved, by using the relation (4.14), that \( W_j \) \( (j = 1, 2, \cdots, n) \) given in (4.16) are eigenvectors of \( \mathcal{F}'(U_s) \) corresponding to the eigenvalue 0. We can easily reprove this result by using the expression (5.15) of \( \mathcal{F}'(U_s) \).

**Lemma 5.2** The operator \( \mathcal{F}'(U_s) \), regarded as an unbounded linear operator in \( X \) with domain \( X_0 \), is a sectorial operator.

**Proof.** Recalling (5.1), we see that \( \mathcal{F}'(U_s) = A + B \), where \( A = \mathcal{A}(U_s) \) and \( BV = [\mathcal{A}'(U_s) V] U_s + \mathcal{F}'(U_s) V \) for \( V \in X_0 \). By Lemma 4.1 of [14] we know that \( A \) is a sectorial operator in \( X \) with domain \( X_0 \). Next we consider \( B \). Since \( \mathcal{F}'(U_s) \in C^\infty(\mathcal{O}, Y) \), we have \( \mathcal{F}'(U_s) \in L(X_0, Y) \). Besides, from the second relation in (5.2) and the result obtained in (5.8) we easily see that the mapping \( V \to [\mathcal{A}'(U_s) V] U_s \) also belong to \( L(X_0, Y) \). Hence, in conclusion we have \( B \in L(X_0, Y) \). Since \( Y \) is clearly an intermediate space between \( X_0 \) and \( X \), by a standard result we get the desired assertion. \( \square \)

By a standard perturbation result, we have

**Corollary 5.3** If the neighborhood \( \mathcal{O} \) of \( U_s \) (in \( X_0 \)) is sufficiently small, then for any \( U \in \mathcal{O} \), \( \mathcal{F}'(U) \) is a sectorial operator.
Later on we shall assume that the number \( \delta \) is so small that the open set \( O \) defined in the end of Section 3 satisfies the condition of the above corollary.

## 6 The spectrum of \( \mathbb{F}'(U_s) \)

Given a closed linear operator \( B \) in a Banach space \( X \), we denote by \( \rho(B) \) and \( \sigma(B) \) the resolvent set and the spectrum of \( B \), respectively. In the sequel we study \( \sigma(\mathbb{F}'(U_s)) \).

We introduce the operator \( A_0 : W^{m-1,q}(\mathbb{B}^n) \to W^{m-3,q}(\mathbb{B}^n) \) by

\[
A_0v = [\Delta - f'(\sigma_s(r))]v \quad \text{for } v \in W^{m-1,q}(\mathbb{B}^n),
\]

the operator \( Q : B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \to W^{m,q}(\mathbb{B}^n) \) by

\[
Q\eta = m(r)\Pi(\eta) = \phi(r-1)\sigma_s'(r)\Pi(\eta) \quad \text{for } \eta \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1}),
\]

and the operator \( J : W^{m-1,q}(\mathbb{B}^n) \to B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \) by

\[
Jv = -\mathcal{D}(0)\mathcal{S}(0)[g'(\sigma_s(r))v] = \frac{\partial}{\partial r}\left\{ \Delta^{-1}[g'(\sigma_s(r))v]\right\}_{r=1} \quad \text{for } v \in W^{m-1,q}(\mathbb{B}^n)
\]

Here \( \Delta^{-1} \) denotes the inverse of the Laplacian under the homogeneous Dirichlet boundary condition. Let \( \Pi_0 : B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \to W^{m,q}(\mathbb{B}^n) \) be the operator \( \Pi_0(\eta) = v \), where for given \( \eta \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \), \( v \in W^{m,q}(\mathbb{B}^n) \) is the solution of the boundary value problem

\[
\Delta v - f'(\sigma_s(r))v = 0 \quad \text{in } \mathbb{B}^n \quad \text{and } v = \eta \quad \text{on } \mathbb{S}^{n-1}.
\]

Note that this definition implies that \( A_0\Pi_0 = 0 \). We define \( \mathcal{B}_\gamma : B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \to B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1}) \) by

\[
\mathcal{B}_\gamma \eta = \gamma \frac{\partial}{\partial r}\left\{ \Pi\left( \eta + \frac{1}{n-1}\Delta_\theta \eta \right) \right\}_{r=1} + g(\sigma)\eta - \sigma_s'(1)J\Pi_0(\eta)
\]

\[
= \frac{\partial}{\partial r}\left\{ \gamma \Pi\left( \eta + \frac{1}{n-1}\Delta_\theta \eta \right) - \sigma_s'(1)\Delta^{-1}\left( g'(\sigma_s(r))\Pi_0(\eta) \right) \right\}_{r=1} + g(\sigma)\eta,
\]

for \( \eta \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \).

Finally we define the operators \( \mathbb{M} : X_0 \to X \) and \( \mathbb{T} : X \to X \) respectively by

\[
\mathbb{M} = \begin{pmatrix} c^{-1}A_0 + \sigma_s'(1)\Pi_0 J & \sigma_s'(1)\Pi_0 \mathcal{B}_\gamma \end{pmatrix} \quad \text{and} \quad \mathbb{T} = \begin{pmatrix} I & \sigma_s'(1)\Pi_0 - Q \end{pmatrix}.
\]

Here the first \( I \) in \( \mathbb{T} \) represents the identity operator in \( W^{m-3,q}(\mathbb{B}^n) \), while the second \( I \) in \( \mathbb{T} \) represents the identity operator in \( B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1}) \). Note that \( (\sigma_s'(1)\Pi_0 - Q)\eta|_{\mathbb{S}^{n-1}} = 0 \) for any \( \eta \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \), so that \( \mathbb{T} \) maps \( X_0 \) to \( X_0 \).

**Lemma 6.1** For \( V \in X_0 \) and \( \lambda \in \mathbb{C} \), the relation \( \mathbb{F}'(U_s)V = \lambda V \) holds if and only if the relations \( \mathbb{M}W = \lambda W \) and \( W = \mathbb{T}V \) hold.
\textit{Proof:} Clearly,
\[
A_{11}v = e^{-1}A_0v + QJv, \quad A_{12}\eta = -e^{-1}A_0Q\eta + Q(B_\gamma + \sigma'(1)\mathcal{J}\Pi_0 - \mathcal{J}Q)\eta,
\]
\[
A_{21}v = Jv, \quad A_{22}\eta = (B_\gamma + \sigma'(1)\mathcal{J}\Pi_0 - \mathcal{J}Q)\eta.
\]

Using these relations and the fact that $A_0\Pi_0 = 0$ we can easily verify that
\[
\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I & Q - \sigma'_s(1)\Pi_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} e^{-1}A_0 + \sigma'_s(1)\Pi_0J & \sigma'_s(1)\Pi_0B_\gamma \\ J & B_\gamma \end{pmatrix} \begin{pmatrix} I & \sigma'_s(1)\Pi_0 - Q \\ 0 & I \end{pmatrix},
\]
or $\mathcal{F}'(U_s) = \mathbb{T}^{-1}\mathcal{M}$. From this relation the desired assertion follows immediately. \hfill $\Box$

Since $X_0$ is clearly compactly embedded in $X$, by Lemma 5.2 we see that $\sigma(\mathcal{F}'(U_s))$ consists entirely of eigenvalues. Hence, by Lemma 6.1 we have

\textbf{Corollary 6.2} \quad $\sigma(\mathcal{F}'(U_s)) = \sigma(\mathcal{M})$. \hfill $\Box$

We shall see that for sufficiently small $c$, $\sigma(\mathcal{B}_\gamma)$ plays a major role in determining $\sigma(\mathcal{M})$. Hence, in the sequel we first compute $\sigma(\mathcal{B}_\gamma)$. To this end we introduce some notation and recall some results of [15]. For every nonnegative integer $k$, let $Y_{kl}(\omega)$, $l = 1, 2, \cdots, d_k$, be the normalized orthogonal basis of the space of all spherical harmonics of degree $k$, where $d_k$ is the dimension of this space, i.e.

\[
d_0 = 1, \quad d_1 = n, \quad d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \quad (k \geq 2).
\]

It is well-known that
\[
\Delta_\omega Y_{kl}(\omega) = -\lambda_k Y_{kl}(\omega), \quad \lambda_k = k^2 + (n - 2)k \quad (k = 0, 1, 2, \cdots),
\]
and $\lambda_k$ ($k = 0, 1, 2, \cdots$) are the all eigenvalues of $\Delta_\omega$. We denote
\[
a_k = 2k + n - 1 \geq n - 1,
\]
and denote by $\bar{u}_k(r)$ the solution of the initial value problem
\[
\begin{aligned}
\bar{u}_k''(r) + \frac{ak}{r}u_k'(r) &= f'(\sigma_s(r))u_k(r), \\
\bar{u}_k(0) &= 1, \quad \bar{u}_k'(0) = 0,
\end{aligned}
\]
By using some ODE techniques we can show that this problem has a unique solution for all $r \in [0, R^*]$, where $[0, R^*]$ is the maximal existence interval of $\sigma_s(r)$. We also denote
\[
\gamma_k = \frac{n-1}{(\lambda_k - n + 1)k} \left[ g(\bar{\sigma}) - \frac{\sigma'_0(1)}{\bar{u}_k(1)} \int_0^1 g'(\sigma_0(\rho))\bar{u}_k(\rho)\rho^{a_k}d\rho \right] \quad (k \geq 2).
\]
From [15] we know that $\gamma_k$’s and $\gamma_1 = 0$ are the all eigenvalues of the linearization of the stationary version of the system (1.1)–(1.5) at the radially symmetric stationary solution $(\sigma_s, p_s, \Omega_s)$, $\gamma_k > 0$ for all $k \geq 2$, and $\lim_{k \to \infty} \gamma_k = 0$. Next we denote

$$\alpha_{k, \gamma} = -\frac{(\lambda_k - n + 1)k}{n - 1}(\gamma - \gamma_k), \quad k = 2, 3, \cdots.$$  

Note that $\alpha_{k, \gamma} \sim -\gamma k^3/(n-1)$ as $k \to \infty$. Finally, we denote $\alpha_{1, \gamma} = 0$ and

$$\alpha_{0, \gamma} = g(\bar{\sigma}) - \frac{\sigma_0'(R_s)}{\bar{u}_0(R_s)R_s^{n-1}} \int_0^{R_s} g'(\sigma_0(r)) \bar{u}_0(r) r^{n-1} dr.$$  

From [16] we know that $\alpha_{0, \gamma} < 0$ for all $\gamma > 0$.

**Lemma 6.3** $\mathcal{B}_\gamma$ is a Fourier multiplication operator of the following form: For any $\eta(\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} b_{kl} Y_{kl}(\omega) \in C^\infty(S^{n-1})$,

$$\mathcal{B}_\gamma \eta(\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} \alpha_{k, \gamma} b_{kl} Y_{kl}(\omega).$$  

(6.1)

As a result, we have $\sigma(\mathcal{B}_\gamma) = \{\alpha_{k, \gamma} : k \in \mathbb{N}, k \geq 2\} \cup \{0, \alpha_{0, \gamma}\}$.

**Proof**: It can be easily seen that $\mathcal{B}_\gamma$ has the same expression as that introduced in [16] with the same notation (but notice that $\mathcal{B}_\gamma$ in [16] is a mapping from $C^{m+\mu}(S^{n-1})$ to $C^{m-3+\mu}(S^{n-1})$ for some integer $m$ and $0 < \mu < 1$). Hence, by a similar calculation as in [16] we get (6.1). \qed

**Lemma 6.4** For any $\gamma > 0$ and $k \geq 2$, there exists corresponding $c_0 > 0$ such that for any $0 < c \leq c_0$, $\mathbb{M}$ has an eigenvalue $\lambda_{k, \gamma} = \alpha_{k, \gamma} + c\mu_{k, \gamma}(c)$, where $\mu_{k, \gamma}(c)$ is a bounded continuous function in $0 < c \leq c_0$. Moreover, the corresponding eigenvectors of $\mathbb{M}$ have the form $\left(\begin{array}{c} c_{a_{k, \gamma}}(r, c) \\ \cdot \cdot \cdot \\ 1 \end{array}\right) Y_{kl}(\omega)$ ($l = 1, 2, \cdots, d_k$), where $a_{k, \gamma}(r, c)$ is a smooth function in $r \in [0, 1]$ and is bounded and continuous in $0 < c \leq c_0$.

**Proof**: Let $U = \left(\begin{array}{c} c_{a_{k, \gamma}}(r, c) \\ \cdot \cdot \cdot \\ 1 \end{array}\right) Y_{kl}(\omega)$. Then the relation $\mathbb{M}U = \lambda_{k, \gamma} U$ holds if and only if the following relations hold:

$$A_0(a_{k, \gamma} Y_{kl}) + c\sigma_s'(1)\Pi_0 J(a_{k, \gamma} Y_{kl}) + c\phi_s'(1)\Pi_0 B_\gamma(Y_{kl}) = c\mu_{k, \gamma} a_{k, \gamma} Y_{kl} + c^2 \mu_{k, \gamma} a_{k, \gamma} Y_{kl},$$  

(6.2)

$$cJ(a_{k, \gamma} Y_{kl}) + B_\gamma(Y_{kl}) = c\lambda_{k, \gamma} Y_{kl} + \mu_{k, \gamma} Y_{kl}.$$  

(6.3)

Let $L_k$ be the second-order differential operator $L_k u(r) = u''(r) + \frac{n-1}{r} u'(r) - \frac{\lambda_k}{r^2} u(r)$, and $J_k$ be the operator $u \to v_k'(1)$, where for a given continuous function $u = u(r)$ ($0 \leq r \leq 1$), $v = v_k(r)$ is the solution of the boundary value problem:

$$\begin{cases}
  v''(r) + \frac{n-1}{r} v'(r) - \frac{\lambda_k}{r^2} v(r) = g'(\sigma_s(r)) u(r), & 0 < r < 1, \\
  v'(0) = 0, \quad v(1) = 0.
\end{cases}$$

Then we have $A_0(a_{k, \gamma} Y_{kl}) = L_k (a_{k, \gamma} Y_{kl})$ and $J(a_{k, \gamma} Y_{kl}) = J_k (a_{k, \gamma} Y_{kl})$. Besides, it can be easily seen that $\Pi_0(Y_{kl}) = w_k(r) Y_{kl}$, where $w_k(r)$ ($0 \leq r \leq 1$) is the solution of the boundary value
problem:
\[
\begin{cases}
w''_k(r) + \frac{n-1}{r} w'_k(r) - \left( \frac{\lambda}{r^2} + f'(\sigma(r)) \right) w_k(r) = 0, & 0 < r < 1, \\
w'_k(0) = 0, & w_k(1) = 0.
\end{cases}
\]

Using these facts and the relation \( B_\gamma(Y_{kl}) = \alpha_{k,\gamma} Y_{kl} \) (cf. (6.1)) we see that (6.2) and (6.3) reduce to the following system of equations:
\[
\begin{align*}
\mathcal{L}_k(a_{k,\gamma}) + c\sigma_s'(1) J_k(a_{k,\gamma}) w_k(r) + \sigma_s'(1) \alpha_{k,\gamma} w_k(r) &= c\alpha_{k,\gamma} a_{k,\gamma} + c^2 \mu_{k,\gamma} a_{k,\gamma}, \\
\mu_{k,\gamma} &= J_k(a_{k,\gamma}),
\end{align*}
\]
which can be further reduced to the following scalar equation in \( a_{k,\gamma} \):
\[
\mathcal{L}_k(a_{k,\gamma}) = -c\sigma_s'(1) J_k(a_{k,\gamma}) w_k(r) + c\alpha_{k,\gamma} a_{k,\gamma} + c^2 a_{k,\gamma} J_k(a_{k,\gamma}) - \sigma_s'(1) \alpha_{k,\gamma} w_k(r).
\]

By using a standard fixed point argument we can easily show that for \( c \) sufficiently small this equation complemented with the boundary value conditions \( \frac{\partial a_k}{\partial r} \big|_{r=0} = 0 \) and \( a_k \big|_{r=1} = 0 \) has a unique solution. By this assertion, the desired result follows immediately. \( \square \)

We denote
\[
\gamma_* = \max_{k \geq 2} \gamma_k \quad \text{and} \quad \alpha^*_\gamma = \max_{k \geq 2} \alpha_{k,\gamma}.
\]
Since \( \gamma_k > 0 \), \( \lim_{k \to \infty} \gamma_k = 0 \) and \( \lim_{k \to \infty} \alpha_{k,\gamma} = -\infty \), \( \gamma_* \) and \( \alpha^*_\gamma \) are both well-defined. Clearly, we have \( \alpha^*_\gamma < 0 \) for any \( \gamma > \gamma_* \), while \( \alpha^*_\gamma > 0 \) for any \( 0 < \gamma < \gamma_* \).

**Lemma 6.5** Given \( \gamma > \gamma_* \), there exists corresponding \( c_0 > 0 \) such that for any \( 0 < c \leq c_0 \) and any \( \lambda \in \mathbb{C}\{0\} \) satisfying \( \text{Re} \lambda \geq \frac{1}{2} \alpha^*_\gamma \), there holds \( \lambda \in \rho(\mathbb{M}) \), or equivalently, \( \sup \{ \text{Re} \lambda : \lambda \in \sigma(\mathbb{M}) \{0\} \} \leq \frac{1}{2} \alpha^*_\gamma < 0 \).

**Proof:** We denote
\[
\mathbb{M}_0 = \begin{pmatrix} c^{-1} A_0 & 0 \\ J & B_\gamma \end{pmatrix}, \quad \mathbb{N} = \begin{pmatrix} \sigma_s'(1) \Pi_0 J & \sigma_s'(1) \Pi_0 B_\gamma \\ \sigma_s'(1) \Pi_0 J & \Pi_0 B_\gamma \end{pmatrix}.
\]
Then \( \mathbb{M}_0 \in L(X_0, X) \), \( \mathbb{N} \in L(X_0, X) \), and \( \mathbb{M} = \mathbb{M}_0 + \mathbb{N} \). Since \( f'(\sigma(r)) \geq 0 \), from the standard theory of elliptic partial differential equations of the second-order we know that all eigenvalues of \( A_0 \) are negative and they make up a decreasing sequence tending to \( -\infty \). Let \( \nu_1 \) be the largest eigenvalue of \( A_0 \), and let \( c_0 = \nu_1 / \alpha^*_\gamma \). Then for any \( 0 < c \leq c_0 \) and any \( \lambda \in \mathbb{C}\{0\} \) such that \( \text{Re} \lambda \geq \frac{1}{2} \alpha^*_\gamma \) we have \( \text{Re}(c \lambda) \geq \frac{1}{2} \nu_1 \), so that both \( \lambda I - c^{-1} A_0 = c^{-1}(c \lambda I - A_0) \) and \( \lambda I - B_\gamma \) are invertible, which implies that \( \lambda I - \mathbb{M}_0 \) is invertible. In fact,
\[
(\lambda I - \mathbb{M}_0)^{-1} = \begin{pmatrix} (\lambda I - c^{-1} A_0)^{-1} & 0 \\ (\lambda I - B_\gamma)^{-1} J (\lambda I - c^{-1} A_0)^{-1} & (\lambda I - B_\gamma)^{-1} \end{pmatrix}.
\]
Hence
\[
\lambda I - \mathbb{M} = (\lambda I - \mathbb{M}_0) - \mathbb{N} = (\lambda I - \mathbb{M}_0)(I - c \mathbb{N}),
\]
where
\[ K = C^{-1} (\lambda I - \mathcal{M}_0)^{-1} \mathbb{N} \]

\[ = \begin{pmatrix} (c\lambda I - \mathcal{A}_0)^{-1} \sigma_s'(1) \Pi_0 J & (c\lambda I - \mathcal{A}_0)^{-1} \sigma_s'(1) \Pi_0 B_\gamma \\ (\lambda I - B_\gamma)^{-1} J (c\lambda I - \mathcal{A}_0)^{-1} \sigma_s'(1) \Pi_0 J & (\lambda I - B_\gamma)^{-1} J (c\lambda I - \mathcal{A}_0)^{-1} \sigma_s'(1) \Pi_0 B_\gamma \end{pmatrix}. \]

Since \( \mathcal{A}_0 \) is a self-adjoint sectorial operator and \( \nu_1 \) is the maximal eigenvalue of \( \mathcal{A}_0 \), we have
\[ \| (c\lambda I - \mathcal{A}_0)^{-1} \|_{L(W^{m-3,q}(\mathbb{B}^n),W^{m-3,q}(\mathbb{B}^n))} \leq \frac{C}{|c\lambda - \nu_1|} \leq 2C/\nu_1, \]
where \( C \) is a constant independent of \( c \) and \( \lambda \). Using this fact, the identity
\[ \mathcal{A}_0(c\lambda I - \mathcal{A}_0)^{-1} = c\lambda(c\lambda I - \mathcal{A}_0)^{-1} - I, \]
and the Agmon-Douglis-Nirenberg inequality, we obtain
\[ \| (c\lambda I - \mathcal{A}_0)^{-1} \|_{L(W^{m-3,q}(\mathbb{B}^n),W^{m-3,q}(\mathbb{B}^n))} \leq C + \frac{C|c\lambda|}{|c\lambda - \nu_1|} \leq C. \]

Similarly we have
\[ \| (\lambda I - B_\gamma)^{-1} \|_{L(B_{q,0}^{m-3-1/q}(\mathbb{B}^{n-1}),B_{q,0}^{m-1-1/q}(\mathbb{B}^{n-1}))} \leq C. \]

Using these estimates we can easily show that
\[ \| K \|_{L(X_0,X_0)} \leq C \]
for any \( 0 < c \leq c_0 \) and any \( \lambda \in \mathbb{C} \) such that \( \text{Re}\lambda \geq \frac{1}{2} \alpha_s^* \). It follows that if we take \( c_0 \) further small such that \( c_0 C < 1 \) then for \( c \) and \( \lambda \) in the set specified above, the operator \( \lambda I - \mathcal{M} \) is invertible and the inverse is continuous. Hence, the desired assertion follows. \( \square \)

7 The proof of Theorem 1.1

Proof of Theorem 1.1: We first assume that \( \gamma > \gamma_s \). By Lemma 5.2 we see that \( \mathbb{F}'(U_s) \) is a sectorial operator in \( X \) with domain \( X_0 \). In what follows we prove that the norm of \( X_0 \) coincides the graph norm of \( \mathbb{F}'(U_s) \). From Section 6 we see that \( \mathbb{F}'(U_s) = T^{-1} \mathcal{M} T \). Clearly,
\[ C\| U \|_X \leq \| T U \|_X \leq C^{-1} \| U \|_X \quad \text{and} \quad C\| U \|_{X_0} \leq \| T U \|_{X_0} \leq C^{-1} \| U \|_X \quad (7.1) \]
for some constants \( C > 0 \). Thus the graph norm of \( \mathbb{F}'(U_s) \) is equivalent to the graph norm of \( \mathcal{M} \). Next, let
\[ T_0 = \begin{pmatrix} I & \sigma_s'(1) \Pi_0 \\ 0 & I \end{pmatrix}. \]
Then we have $M = T_0M_0$. Clearly, all estimates in (7.1) still hold when $T$ is replaced by $T_0$. Hence the graph norm of $M$ is equivalent to the graph norm of $M_0$. Clearly, as an unbounded linear operator in $W^{m-3,q}(\mathbb{B}^n)$ with domain $W^{m-1,q}(\mathbb{B}^n)$, the graph norm of $A_0$ is equivalent to the norm of $W^{m-1,q}(\mathbb{B}^n)$. Also, we know that as an unbounded linear operator in $B^{m-3-1/q}_q(S^{n-1})$ with domain $B^{m-1/q}_q(S^{n-1})$, the graph norm of $\hat{B}_\gamma$ is equivalent to the norm of $B^{m-1/q}_q(S^{n-1})$ (cf. [16]). Besides, it is easy to see that $J$ maps $W^{m-3,q}(\mathbb{B}^n)$ continuously into $B^{m-2-1/q}_q(S^{n-1})$, so that it is a compact operator from $W^{m-3,q}(\mathbb{B}^n)$ to $B^{m-3-1/q}_q(S^{n-1})$. From these facts, we can easily show that the graph norm of $M_0$ is equivalent to the norm of $X_0$. Hence, the graph norm of $\mathbb{F}'(U_s)$ is equivalent to the norm of $X_0$. This verifies that $\mathbb{F}'(U_s)$ satisfies the condition $(B_1)$. By the results of Section 4 we see that $\mathbb{F}'(U_s)$ also satisfies the condition $(B_2)$. Next we consider the condition $(B_3)$. We first prove that $M$ satisfies this condition. To this end we denote by $H_1(S^{n-1})$ the linear space of all first-order spherical harmonics, and for every integer $k$ we introduce

$$\hat{B}^{k-1/q}_q(S^{n-1}) = \{ \rho \in B^{k-1/q}_q(S^{n-1}) : \rho \text{ is orthogonal to } H_1(S^{n-1}) \text{ in } L^2(S^{n-1}) \}. $$

We also denote $\hat{B}^{\infty}_q(S^{n-1}) = \bigcap_{k=1}^{\infty} \hat{B}^{k-1/q}_q(S^{n-1})$. It can be easily shown that $\hat{B}^{k-1/q}_q(S^{n-1})$ is a closed subspace of $B^{k-1/q}_q(S^{n-1})$, and

$$B^{k-1/q}_q(S^{n-1}) = \hat{B}^{k-1/q}_q(S^{n-1}) \oplus H_1(S^{n-1})$$

(for any integer $k$). By (6.1) we see that ker$(B_\gamma) = H_1(S^{n-1})$. We denote $\hat{B}_\gamma = B_\gamma|\hat{B}^{m-1/q}_q(S^{n-1})$, and split $J$ into $J_1 + J_2$ such that $J_1 \in B^{m-3-1/q}_q(S^{n-1})$ and $J_2 \in H_1(S^{n-1})$ for any $v \in W^{m-1,q}(\mathbb{B}^n) \cap W^{1,q}_0(\mathbb{B}^n)$. We correspondingly split $X_0$ and $X$ into $(W^{m-1,q}(\mathbb{B}^n) \cap W^{1,q}_0(\mathbb{B}^n)) \times \hat{B}^{m-1/q}_q(S^{n-1}) \times H_1(S^{n-1})$ and $W^{m-3,q}(\mathbb{B}^n) \times \hat{B}^{m-3-1/q}_q(S^{n-1}) \times H_1(S^{n-1})$, respectively. Then

$$\hat{M} = \begin{pmatrix} c^{-1}A_0 + \sigma'_s(1)\Pi_0(J_1 + J_2) & \sigma'_s(1)\Pi_0\hat{B}_\gamma & 0 \\
\hat{J}_1 & \hat{B}_\gamma & 0 \\
\hat{J}_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \hat{M} & 0 \\
\hat{J} & 0 \end{pmatrix},$$

where

$$\hat{M} = \begin{pmatrix} c^{-1}A_0 + \sigma'_s(1)\Pi_0(J_1 + J_2) & \sigma'_s(1)\Pi_0\hat{B}_\gamma \\
J_1 & \hat{B}_\gamma \end{pmatrix} = \begin{pmatrix} I & \sigma'_s(1)\Pi_0 \\
0 & I \end{pmatrix} \begin{pmatrix} c^{-1}A_0 + \sigma'_s(1)\Pi_0J_2 & 0 \\
J_1 & \hat{B}_\gamma \end{pmatrix} \equiv \hat{T}_0\hat{M}_1,$$

and $\hat{J} = (J_2 \ 0)$. We claim that $\hat{B}_\gamma$ is an isomorphism from $\hat{B}^{m-1/q}_q(S^{n-1})$ to $\hat{B}^{m-3-1/q}_q(S^{n-1})$. Indeed, from (6.1) and the fact that $B_\gamma$ maps $B^{m-1/q}_q(S^{n-1})$ to $B^{m-3-1/q}_q(S^{n-1})$ boundedly it is clear that $\hat{B}_\gamma$ maps $\hat{B}^{m-1/q}_q(S^{n-1})$ to $\hat{B}^{m-3-1/q}_q(S^{n-1})$ boundedly and is an injection. Next, from (6.1) we see immediately that for any $\zeta \in \hat{B}^{\infty}_q(S^{n-1})$ there exists a unique $\eta \in \hat{B}^{\infty}_q(S^{n-1})$ such that $B_\gamma\eta = \zeta$. Now assume that $\zeta \in \hat{B}^{m-3-1/q}_q(S^{n-1})$. Let $\zeta_j \in \hat{B}^{\infty}_q(S^{n-1}) \ (j = 1, 2, \cdots)$ be such that $\zeta_j \to \zeta$ in $\hat{B}^{m-3-1/q}_q(S^{n-1})$, and let $\eta_j \in \hat{B}^{\infty}_q(S^{n-1})$ be the solution of the equation
Take a real number \( s \) such that \( s < m - 3 - 1/q - (n-1)(\frac{1}{2} - \frac{1}{q}) \). Then \( B_{qq}^{m-3-1/q}(S^{n-1}) \hookrightarrow H^s(S^{n-1}) \), where \( H^s(S^{n-1}) \) stands for the usual Sobolev space. Thus \( \zeta_j \to \zeta \) in \( H^s(S^{n-1}) \). By (6.1) and the fact that \( \alpha_{k,\gamma} \sim Ck^3 \) we easily deduce that \( \{\eta_j\} \) is a Cauchy sequence in \( H^{s+3}(S^{n-1}) \). Let \( \eta \in H^{s+3}(S^{n-1}) \) be the limit of \( \{\eta_j\} \). By a standard argument we have

\[
\|\rho\|_{B_{qq}^{m-1/q}(S^{n-1})} \leq C\left(\|\rho\|_{H^{s+3}(S^{n-1})} + \|\mathcal{B}_\gamma \rho\|_{B_{qq}^{m-3-1/q}(S^{n-1})}\right).
\]

Applying this estimate to \( \rho = \eta_j - \eta \), we conclude that \( \eta_j \to \eta \) in \( B_{qq}^{m-1/q}(S^{n-1}) \). Since \( B_{qq}^{m-1/q}(S^{n-1}) \) is closed in \( B_{qq}^{m-1/q}(S^{n-1}) \), we get \( \eta \in B_{qq}^{m-1/q}(S^{n-1}) \). This shows that \( \mathcal{B}_\gamma \) is a surjection. Hence, by the Banach inverse mapping theorem we see that \( \mathcal{B}_\gamma \) is an isomorphism from \( B_{qq}^{m-1/q}(S^{n-1}) \) to \( B_{qq}^{m-3-1/q}(S^{n-1}) \), as desired. Next, since \( A_0 \) is an isomorphism from \( W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n) \) to \( W^{m-3,q}(\mathbb{B}^n) \) and clearly \( \sigma'_s(1)\Pi_0 J_2 \) is a bounded operator from \( W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n) \) to \( W^{m-3,q}(\mathbb{B}^n) \) (actually a compact operator), it follows that for \( c \) sufficiently small, \( c^{-1}A_0 + \sigma'_s(1)\Pi_0 J_2 \) is an isomorphism from \( W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n) \) to \( W^{m-3,q}(\mathbb{B}^n) \). By these results combined with the fact that \( J_1 \) is a bounded operator from \( W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n) \) to \( B_{qq}^{m-1/q}(S^{n-1}) \) (actually a compact operator), we immediately deduce that \( \mathcal{M}_1 \) is a self-isomorphism on \( W^{m-3,q}(\mathbb{B}^n) \times B_{qq}^{m-3-1/q}(S^{n-1}) \) to \( W^{m-3,q}(\mathbb{B}^n) \times B_{qq}^{m-3-1/q}(S^{n-1}) \). Since \( T_0 \) is clearly a self-isomorphism on \( W^{m-3,q}(\mathbb{B}^n) \times B_{qq}^{m-3-1/q}(S^{n-1}) \), we conclude that \( \mathcal{M} \) is an isomorphism from \( (W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)) \times B_{qq}^{m-1/q}(S^{n-1}) \) to \( W^{m-3,q}(\mathbb{B}^n) \times B_{qq}^{m-3-1/q}(S^{n-1}) \). This easily implies that \( \mathcal{M} \) satisfies the condition (B3). Now, since \( \mathbb{F}'(U_s) = T^{-1}\mathcal{M} T \), it follows immediately that \( \mathbb{F}'(U_s) \) also satisfies the condition (B3).

Finally, by Corollary 6.2 and Lemma 6.5 we see that

\[
\omega_- = -\sup \{\text{Re}\lambda : \lambda \in \sigma(\mathbb{F}'(U_s)) \setminus \{0\} \} > 0,
\]

so that the condition (B4) is also satisfied by \( \mathbb{F}'(U_s) \). Hence, by Theorem 2.1 we get the assertion (i) of Theorem 1.1.

Next we assume that \( 0 < \gamma < \gamma_* \). Then there exists \( k_0 \geq 2 \) such that \( \alpha_{k_0,\gamma} > 0 \). By Lemma 6.4 and Corollary 6.2, this implies that for sufficiently small \( c \), \( \mathbb{F}'(U_s) \) has a positive eigenvalue. Furthermore, if \( \alpha_{k_1,\gamma}, \alpha_{k_2,\gamma}, \ldots, \alpha_{k_N,\gamma} \) are the all positive eigenvalues of \( \mathcal{B}_\gamma \), then by Lemma 6.4 and a similar argument as in the proof of Lemma 6.5 we see that for \( c \) sufficiently small, \( \lambda_{k_j,\gamma} = \alpha_{k_j,\gamma} + c\mu_{k_j,\gamma}(c) \) (\( j = 1, 2, \ldots, N \)) are the all positive eigenvalues of \( \mathbb{F}'(U_s) \), and the following estimate holds:

\[
\sup \{\text{Re}\lambda : \lambda \in \sigma(\mathbb{M}) \setminus \{0, \lambda_{k_1,\gamma}, \lambda_{k_2,\gamma}, \ldots, \lambda_{k_N,\gamma}\} \} \leq \frac{1}{2} \max \{\alpha_k : k \geq 2, k \neq k_1, k_2, \ldots, k_N \} < 0.
\]

Thus by using Theorem 9.1.3 of [28], we obtain the assertion (ii) of Theorem 1.1. This completes the proof of Theorem 1.1. \( \square \)

**Acknowledgement.** This work is financially supported by the National Natural Science Foundation of China under the grant number 10471157.

**References**

[1] J. Adam and N. Bellomo, *A Survey of Models for Tumor-Immune System Dynamics*, Birkhäuser, Boston(1997).
[2] H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in *Function Spaces, Differential Operators and Nonlinear Analysis*, H. J. Schmeisser & H. Triebel, editors, Teubner (1993), 9–126.

[3] J. B. Baillon, Caractère borné de certains générateurs de semigroupes linéaires dans les espaces de Banach, *C. R. Acad. Sc. Paris Série A*, 290 (1980), 757–760.

[4] B. Bazaliy and A. Friedman, A free boundary problem for an elliptic-parabolic system: application to a model of tumor growth, *Comm. Part. Diff. Equas.*, 28 (2003), no. 3-4, 517–560.

[5] B. Bazaliy and A. Friedman, Global existence and asymptotic stability for an elliptic-parabolic free boundary problem: an application to a model of tumor growth. *Indiana Univ. Math. J.*, 52 (2003), no. 5, 1265–1304.

[6] H. M. Byrne, A weakly nonlinear analysis of a model of avascular solid tumor growth, *J. Math. Biol.*, 39 (1999), 59–89.

[7] H. M. Byrne and M. A. J. Chaplain, Growth of nonnecrotic tumors in the presence and absence of inhibitors, *Math. Biosci.*, 130 (1995), 151–181.

[8] X. Chen, S. Cui and A. Friedman, A hyperbolic free boundary problem modeling tumor growth: asymptotic behavior. *Trans. Amer. Math. Soc.*, 357 (2005), 4771-4804.

[9] S. Cui, Analysis of a mathematical model for the growth of tumors under the action of external inhibitors, *J. Math. Biol.*, 44 (2002), 395–426.

[10] S. Cui, Global existence of solutions for a free boundary problem modelling the growth of necrotic tumors, *Interfaces and Free Boundaries*, 7 (2005), 147–159.

[11] S. Cui, Analysis of a free boundary problem modelling tumor growth, *Acta Math. Sinica (English Series)*, 21 (2005), 1071-1083.

[12] S. Cui, Existence of a stationary solution for the modified Ward-King tumor growth model, *Advances in Appl. Math.*, 36 (2006), 421–445.

[13] S. Cui, Formation of necrotic cores in the growth of tumors: analytic results, *Acta Math. Sci. Engl. Series*, 26B (2006), 781–796.

[14] S. Cui, Well-posedness of a multidimensional free boundary problem modelling the growth of non-necrotic tumors, *J. Func. Anal.*, 245 (2007), 1-18.

[15] S. Cui and J. Escher, Bifurcation analysis of an elliptic free boundary problem modelling the growth of avascular tumors, *SIAM J. Math. Anal.*, 39 (2007), 210–235.

[16] S. Cui and J. Escher, Asymptotic behavior of solutions for a moving boundary problem modelling tumor growth, *Comm. Part. Diff. Equa.*, to appear.

[17] S. Cui and A. Friedman, A free boundary problem for a singular system of differential equations: an application to a model of tumor growth, *Trans. Amer. Math. Soc.*, 355 (2003), no.9, 3537-3590.

[18] S. Cui and X. Wei, Existence of solutions for a parabolic-hyperbolic free boundary problem, *Acta Math. Appl. Sinica Engl. Series*, 21 (2005), no.4, 597-614.

[19] G. Da Prato and A. Lunardi, Stability, instability and center manifold theorem for fully nonlinear autonomous parabolic equations in Banach spaces, *Arch. Rat. Mech. Anal.*, 101 (1988), 115–141.

[20] J. Escher, Classical solutions to a moving boundary problem for an elliptic-parabolic system, *Interfaces Free and Boundaries*, 6 (2004), 175–193.
[21] J. Escher and G. Simonett, Classical solutions for Hele-Shaw models with surface tension, *Adv. Diff. Equa.* **2** (1997), 619–643.

[22] A. Friedman and B. Hu, Asymptotic stability for a free boundary problem arising in a tumor model, *J. Diff. Equa.*, **227**(2006), 598–639.

[23] A. Friedman and F. Reitich, Analysis of a mathematical model for the growth of tumors, *J. Math. Biol.*, **38** (1999), 262–284.

[24] A. Friedman and F. Reitich, Symmetry-breaking bifurcation of analytic solutions to free boundary problems, *Trans. Amer. Math. Soc.*, **353**(2000), 1587–1634.

[25] A. Friedman and F. Reitich, On the existence of spatially patterned dormant malignancies in the model for the growth of non-necrotic vascular tumor, *Math. Models Appl. Sci.*, **11**(2000), 601–625.

[26] H. P. Greenspan, On the growth and stability of cell cultures and solid tumors, *J. Theor. Biol.*, **56**(1976), 229–242.

[27] J. B. Kim, R. Stein and M. J. O’Hare, Three-dimensional in vitro tissue culture models for breast cancer – a review, *Breast Cancer Research and Treatment*, **149**(2004), 1–11.

[28] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser Verlag, 1995.

[29] Mueller-Klieser, Three-dimensional cell cultures: from molecular mechanisms to clinical applications, *Amer. J. Cell Physiol.*, **273**(1997), 1109–1123.

[30] L. S. Pontryagin, *Continuous Groups* (in Russian), Gosudarstv. Izdat. Techn.- Teor. Lit., Moskow, 1954.

[31] G. Simonett, Center manifolds for quasilinear reaction diffusion systems, *Diff. Int. Equa.*, **8**(1995), 753–796.

[32] R. M. Sutherland, Cell and environmental interactions in tumor microregions: the multicell spheroid model, *Science*, **240**(1988), 177–184.