ASYMPTOTIC STABILITY OF RAREFACTION WAVES FOR A HYPERBOLIC SYSTEM OF BALANCE LAWS

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Abstract. This paper is concerned with the rarefaction waves for a model system of hyperbolic balance laws in the whole space and in the half space. We prove the asymptotic stability of rarefaction waves under smallness assumptions on the initial perturbation and on the amplitude of the waves. The proof is based on the $L^2$ energy method.

1. Introduction. We consider the following model system of hyperbolic balance laws:

\[
\begin{align*}
    u_t + f(u)_x + q_x &= 0, \\
    q_t + q + u_x &= 0.
\end{align*}
\]

(1)

Here $u$ and $q$ are unknown real valued functions of $x \in \mathbb{R}$ and $t > 0$, and the flux $f$ is a given smooth function of $u$. We assume that $f$ is strictly convex with respect to $u$, that is, $f''(u) > 0$ for any $u$ under consideration. We consider the system (1) in the whole space $\mathbb{R}$ and in the half space $\mathbb{R}_+ = (0, \infty)$. In the whole space we prescribe the initial condition

\[
    (u,q)(x,0) = (u_0,q_0)(x), \quad x \in \mathbb{R}.
\]

(2)

On the other hand, in the half space we prescribe the boundary and initial conditions as follows:

\[
    u(0,t) = u_b, \quad t > 0,
\]

\[
    (u,q)(x,0) = (u_0,q_0)(x), \quad x \geq 0,
\]

(3)

where the boundary data $u_b$ is the constant state determined uniquely by $f''(u_b) = 0$.

Our system (1) is very simple but is interesting in the sense that (1) can be regarded as a model system in kinetic theory. In fact, $u$ and $q$ are considered as the...
variables describing macroscopic and microscopic states, respectively. In this case, by applying the Chapman-Enskog expansion to (1), we have \( q = 0 \) and
\[
 u_t + f(u)_x = 0 \tag{4}
\]
as the first order approximation. Note that (4) is regarded as a model of the compressible Euler equation. Also, as the second order approximation, we have
\[
 u_t + f(u)_x = u_{xx}, \tag{5}
\]
which is considered as a model of the compressible Navier-Stokes equation.

In this paper we are interested in solutions of (1), (2) in a neighborhood of a given rarefaction wave of (1). Our rarefaction wave of (1) is the function of the form \((u_r, 0)\), where \( u_r \) is the centered rarefaction wave of (4) which connects the constant states \( u_+ \) with \( u_- < u_+ \). Namely, \( u_r \) is the continuous weak solution of the Riemann problem for (4) with the Riemann data
\[
 u(x, 0) = u_0(x) := \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{6}
\]
Notice that \( u_r \) is given explicitly as
\[
 u_r(x,t) = \begin{cases} u_-, & x/t \leq f'(u_-), \\ (f')^{-1}(x/t), & f'(u_-) \leq x/t \leq f'(u_+), \\ u_+, & f'(u_+) \leq x/t. \end{cases} \tag{7}
\]

For the half space problem (1), (3) we have to modify (6) and (7) slightly. We choose \( u_+ \) such that \( u_b < u_+ \), and determine \( u_- \) uniquely by the relation \( f'(u_-) = -f'(u_+) \). Note that we have
\[
 u_- < u_b < u_+, \quad f'(u_-) = -f'(u_+) < 0 = f'(u_b) < f'(u_+). \tag{8}
\]
For this choice of \( u_\pm \), the centered rarefaction wave \( u^r \) in (7), which is the continuous weak solution of the Riemann problem (4), (6), satisfies
\[
 u^r(x,t) = \begin{cases} (f')^{-1}(x/t), & 0 \leq x/t \leq f'(u_+), \\ u_+, & f'(u_+) \leq x/t. \end{cases} \tag{9}
\]
in the half space \( x \geq 0 \). Consequently, \( u^r \) in (9) satisfies the boundary condition in (3), namely, \( u^r(0,t) = u_b \) for \( t > 0 \).

The main purpose in this paper is to show the asymptotic stability of the rarefaction wave \((u^r, 0)\) defined above. More precisely, we assume that the initial data \((u_0, q_0)\) is close to \((u^r_0, 0)\) in a suitable sense and the amplitude \( \delta = |u_+ - u_-| \) (resp. \( \delta = |u_+ - u_b| \)) of the rarefaction wave \((u^r, 0)\) is small. Then it will be shown that a unique solution \((u, q)\) to the problem (1), (2) (resp. (1), (3)) exists globally in time and approaches the rarefaction wave \((u^r, 0)\) uniformly in \( x \) as \( t \to \infty \). Namely, we have
\[
 \|(u - u^r, q)(t)\|_{L^\infty} \to 0 \quad \text{as} \quad t \to \infty.
\]
See Theorems 2.1 and 2.2 for the details.

There are a lot of works concerning the asymptotic stability of rarefaction waves for physically interesting systems. The pioneering work was done by Il’in and Oleinik [3] in 1960 for the scalar viscous conservation law (5). The rate of convergence toward the rarefaction waves for (5) was investigate in [1, 2]. For the
half space problem for (5), Liu, Matsumura and Nishihara [10] first proved the asymptotic stability of rarefaction waves and the corresponding convergence rate was obtained by Nakamura [15].

The asymptotic stability of rarefaction waves for the compressible Navier-Stokes equation (barotropic model) was first proved by Matsumura and Nishihara [12] in 1986. This stability result was improved in [13] for large data and in [9] for the half space problem. A similar asymptotic stability result is known also for the full system of the compressible Navier-Stokes equation. See [5].

The asymptotic stability of rarefaction waves was shown also for related systems. We refer the reader to [11, 6] for the Broadwell model in the discrete kinetic theory and to [7] for a model system of radiating gas.

There are also many works on the asymptotic stability of viscous shock waves for physically interesting systems. Here we only mention the following works which treat hyperbolic model systems: [11] for the Broadwell model and [16] for our system (1).

The contents of this paper are as follows. In Section 2 we give the precise statements of our main results concerning the asymptotic stability of rarefaction waves in the whole space (Theorem 2.1) and in the half space (Theorem 2.2). In Section 3, following to [12, 14], we construct smooth approximations of our rarefaction waves. Then in Section 4, we reformulate our problems as the perturbations from the smooth approximations of the rarefaction waves. The precise statements of the asymptotic stability results for the reformulated problems (Theorems 4.1 and 4.3) and the corresponding a priori estimates of solutions (Propositions 4.2 and 4.4) are also given in Section 4. Finally, in Sections 5 and 6, we prove our a priori estimates of solutions in the whole space (Section 5) and in the half space (Section 6). Each proof is based on the $L^2$ energy method which makes use of the energy form of the problem.

**Notation.** Let $\Omega = \mathbb{R}$ or $\Omega = \mathbb{R}_+ := (0, \infty)$. Let $1 \leq p \leq \infty$. Then $L^p = L^p(\Omega)$ denotes the usual Lebesgue space over $\Omega$ with the norm $\| \cdot \|_{L^p}$. For a nonnegative integer $s$, $W^{s,p} = W^{s,p}(\Omega)$ denotes the $s$-th order Sobolev space over $\Omega$ in the $L^p$ sense, equipped with the norm $\| \cdot \|_{W^{s,p}}$. When $p = 2$, we use the abbreviation $H^s = W^{s,2}$. We note that $L^p = W^{0,p}$ and $L^2 = H^0$. Let $I$ be an interval in $[0, \infty)$ and $X$ be a Banach space over $\Omega$. Then, for a nonnegative integer $k$, $C^k(I; X)$ denotes the space of $k$-times continuously differential functions on $I$ with values in $X$. Also, $L^2(I; X)$ denotes the space of $L^2$ functions on $I$ with values in $X$.

Finally, in this paper, we use $C$ or $c$ to denote various positive constants without confusion.

2. Main results. We state our main results concerning the asymptotic stability of rarefaction waves for (1). First we consider the initial value problem for (1) in the whole space $\mathbb{R}$. Let $(u^r, 0)$ be the rarefaction wave for (1). That is, $u^r$ is the centered rarefaction wave of (4) and connects the constant state $u_-$ with $u_- < u_+$.

Note that $u^r$ is the continuous weak solution of the Riemann problem (4), (6) and is given explicitly by (7). Then our stability result in the whole space is stated as follows.

**Theorem 2.1** (Stability in the whole space). Let $u_- < u_+$ and put $\delta = |u_+ - u_-|$. Assume that the initial data $(u_0, q_0)$ satisfies $u_0 - u^r \in L^2$, $(u_0)_x \in H^1$ and $q_0 \in H^2$, and put

$$I_0 = \| u_0 - u^r \|_{L^2} + \| (u_0)_x \|_{H^1} + \| q_0 \|_{H^2},$$

Then

$$u_0 \to u^r \text{ as } t \to +\infty.$$
where $u_0'$ is the Riemann data in (6). If $I_0 + \delta$ is suitably small, then the initial value problem (1), (2) has a unique global solution $(u, q)$ in an appropriate sense. Moreover, this solution approaches the rarefaction wave $(u^r, 0)$ specified above uniformly in $x \in \mathbb{R}$ as $t \to \infty$:

$$
\|(u - u^r, q)(t)\|_{L^\infty} \to 0 \quad \text{as} \quad t \to \infty. \tag{10}
$$

Next we consider the half space problem (1), (3). To state the result precisely, we need to consider the corresponding compatibility condition. To this end, we suppose that $(u, q)$ is a $C^1$-solution of (1), (3) over $(x, t) \in [0, \infty) \times [0, \infty)$. Then we see that the initial data $(u_0, q_0)$ should satisfy

$$
u_0(0) = u_b, \quad u_1(0) = 0, \tag{11}
$$

where $u_1$ is defined by $u_1(x) := u_t(x, 0)$ with $u_t = -f(u)x - q_x$. Namely, $u_1 = -f(u_0)x - (q_0)x$. The conditions in (11) are called the compatibility conditions up to order 1 for the initial-boundary value problem (1), (3).

Now, recalling that $f'(u_b) = 0$, we choose the constant states $u_{\pm}$ satisfying (8). For this choice of $u_{\pm}$, we consider the centered rarefaction wave $u^r$ which is the continuous weak solution of the Riemann problem (4), (6) and is given explicitly by (9). Also we know that $u^r(0, t) = u_b$ for $t > 0$. For this $u^r$, we call $(u^r, 0)$ the rarefaction wave of (1) in the half space with the boundary condition $u^r(0, t) = u_b$. Our stability result for the half space problem is then stated as follows.

**Theorem 2.2** (Stability in the half space). Let $f'(u_b) = 0$ and $u_b < u_+$, and put $\delta = |u_+ - u_b|$. Assume that the initial data $(u_0, q_0)$ satisfies $(u_0 - u_+, q_0) \in H^2$ and put $I_0 = \|(u_0 - u_+, q_0)\|_{H^2}$. Also we assume the compatibility conditions in (11). If $I_0 + \delta$ is suitably small, then the initial-boundary value problem (1), (3) has a unique global solution $(u, q)$ satisfying $(u - u_+, q) \in C^0([0, \infty); H^2) \cap C^1([0, \infty); H^1)$. Moreover, the solution approaches the rarefaction wave $(u^r, 0)$ specified above uniformly in $x \in [0, \infty)$ as $t \to \infty$:

$$
\|(u - u^r, q)(t)\|_{L^\infty} \to 0 \quad \text{as} \quad t \to \infty. \tag{12}
$$

3. **Smooth approximation of rarefaction waves.** Our rarefaction wave $u^r$ is the continuous weak solution of (4) and is not smooth. Following to [12, 14], we construct a smooth approximation of the rarefaction wave $u^r$.

To this end, we recall that if $u$ is a solution of (4), then $w := f'(u)$ satisfies the inviscid Burgers equation

$$
w_t + \left( \frac{1}{2}w^2 \right)_x = 0. \tag{13}
$$

Consequently, we see that for our rarefaction wave $u^r$, the function $w^r := f'(u^r)$ becomes a weak solution of the Riemann problem for (13) with the Riemann data

$$
w(x, 0) = w_0'(x) := \begin{cases} w_-, & x < 0, \\ w_+, & x > 0, \end{cases} \tag{14}
$$

where $w_{\pm} := f'(u_{\pm})$ with $u_- < u_+$, i.e., $w_- < w_+$. We see that this $w^r$ is given explicitly as

$$
w^r(x, t) = \begin{cases} w_-, & x/t \leq w_- , \\ x/t, & w_- \leq x/t \leq w_+, \\ w_+, & w_+ \leq w/t, \end{cases}
$$
This is the centered rarefaction wave of (13) which connects the constant states $w_\pm$ with $w_- < w_+$. For our purpose we first construct a smooth approximation of the rarefaction wave $w^r$. To this end, following to [12, 14], we consider (13) with the following smooth initial data:

$$w(x,0) = w_0(x) := \frac{1}{2}(w_+ + w_-) + \frac{1}{2}(w_+ - w_-) \tanh(\varepsilon x),$$

(15)

where $\varepsilon \in (0, 1]$ is a parameter. (In this paper we only use the case $\varepsilon = 1$.) It is known ([12, 14]) that the problem (13), (15) has a unique smooth solution $w$. We state this result in the following lemma.

**Lemma 3.1** ([12, 14]). Let $w_- < w_+$ and put $\delta = |w_+ - w_-|$. Let $\delta_0 > 0$ be any fixed constant and assume that $\delta \leq \delta_0$. Then the problem (13), (15) has a unique smooth solution $w$ with the following properties:

(i) $w_- < w(x,t) < w_+$ and $w_x(x,t) > 0$ for $x \in \mathbb{R}$ and $t \geq 0$.

(ii) $\|w_x(t)\|_{L^p} \leq \min \left\{ C\varepsilon^{1/p} \delta, C\delta^{1/p} (1 + t)^{-\left(1 - 1/p\right)} \right\}$ for $t \geq 0$, where $1 \leq p \leq \infty$.

(iii) $\|\partial_t^k w(t)\|_{L^p} \leq \min \left\{ C\varepsilon^{k-1/p} \delta, C\varepsilon^{(k-1)-1/p} (1 + t)^{-1} \right\}$ for $t \geq 0$, where $1 \leq p \leq \infty$ and $k = 2, 3, 4$.

(iv) $\|(w - w^r)(t)\|_{L^\infty} \to 0$ as $t \to \infty$, where $w^r$ is the rarefaction wave of (13).

We give the outline of the proof of Lemma 3.1 (iii) in Appendix A.

Now, following [12, 13], we define a smooth approximation $U^R$ of our rarefaction wave $w^r$ for (4) by the formula

$$U^R(x,t) := (f')^{-1}((w(x,t))), \quad \text{i.e.,} \quad f'(U^R(x,t)) = w(x,t),$$

(16)

where $w$ is the smooth solution of (13), (15) with $w_\pm = f'(u_\pm)$ and $\varepsilon = 1$. Note that this $U^R$ satisfies

$$\begin{cases}
U^R_t + f(U^R)_x = 0, \\
U^R(x,0) = U^R_0(x) := (f')^{-1}(w_0(x)),
\end{cases}$$

(17)

where $w_0$ is given by (15) with $w_\pm = f'(u_\pm)$ and $\varepsilon = 1$. As an easy consequence of Lemma 3.1 with $\varepsilon = 1$, we have the following result for our smooth approximation $U^R$.

**Lemma 3.2** (Smooth approximation in the whole space [12, 14]). Let $u_- < u_+$ and put $\delta = |u_+ - u_-|$. Let $\delta_0 > 0$ be any fixed constant and assume that $\delta \leq \delta_0$. Then the smooth approximation $U^R$ defined by (16) with $\varepsilon = 1$ satisfies the following properties:

(i) $u_- < U^R(x,t) < u_+$ and $U^R(x,t) > 0$ for $x \in \mathbb{R}$ and $t \geq 0$.

(ii) $\|U^R_x(t)\|_{L^p} \leq \min \left\{ C\delta, C\delta^{1/p} (1 + t)^{-\left(1 - 1/p\right)} \right\}$ for $t \geq 0$, where $1 \leq p \leq \infty$.

(iii) Let $\theta \in [0, 1]$. Then $\|\partial_t^k U^R(t)\|_{L^p} \leq C\delta^{\theta} (1 + t)^{-\left(1 - \theta\right)}$ for $t \geq 0$, where $1 \leq p \leq \infty$ and $k = 2, 3, 4$. Here the constant $C$ is independent of $\theta$.

(iv) $\|(U^R - w^r)(t)\|_{L^\infty} \to 0$ as $t \to \infty$, where $w^r$ is the rarefaction wave of (4).

Next we construct a smooth approximation of the rarefaction wave for the half space problem. Our rarefaction wave $w^r$ in the half space is given by (9), which is defined by the rarefaction wave $u^r$ in the whole space (see (7)) with the constant states $u_\pm$ satisfying (8). To construct a smooth approximation of the above rarefaction wave $u^r$ in the half space, we consider the smooth solution $w$ in Lemma
with the constant states \( w_\pm \) determined by \( w_\pm = f'(u_\pm) \). Since \( u_\pm \) satisfy (8), we have
\[
    w_+ > 0, \quad w_- = -w_+.
\]
Notice that the corresponding initial data \( w_0 \) in (15) is reduced to
\[
    w_0(x) = w_+ \tanh(x).
\]
In this case, as a modification of Lemma 3.1, we have:

**Lemma 3.3** (cf. [12, 14]). Let \( w_+ > 0 \) and \( w_- = -w_+ \), and put \( \delta = |w_+| \). Let \( \delta_0 > 0 \) be any fixed constant and assume that \( \delta \leq \delta_0 \). Then the problem (13), (15) has a unique smooth solution \( w \) with the following properties:

(i) \( -w_+ < w(x, t) < w_+ \) and \( w_+(x, t) > 0 \) for \( x \in \mathbb{R} \) and \( t \geq 0 \). Moreover, we have \( w(0, t) = 0 \) for \( t \geq 0 \).

(ii) \( \|w_x(t)\|_{L^p} + \|w_t(t)\|_{L^p} \leq \min \{C\varepsilon^{1-1/p} \delta, C\delta^{1/p}(1 + t)^{-(1-1/p)}\} \) for \( t \geq 0 \), where \( 1 \leq p \leq \infty \).

(iii) \( \|\partial_t \partial_x^j w(t)\|_{L^p} \leq \min \{C\varepsilon^{j+k-1/p} \delta, C\varepsilon^{j+k-1}(1 + t)^{-(1-1/p)}\} \) for \( t \geq 0 \), where \( 1 \leq p \leq \infty \), and \( j \) and \( k \) are nonnegative integers satisfying \( j + k = 2, 3, 4 \).

(iv) \( \|(w - w^r)(t)\|_{L^\infty} \to 0 \) as \( t \to \infty \), where \( w^r \) is the rarefaction wave of (13) which connects \( w_\pm \) with \( w_- = -w_+ \).

Now, as in the case of the whole space, we define \( U^R \) by the formula (16), where \( w \) is the smooth solution of (13), (15) with \( w_\pm = f'(u_\pm) \) and \( \varepsilon = 1 \). Here \( u_\pm \) are assumed to satisfy (8). We note that this \( U^R \) is a smooth approximation of the rarefaction wave \( w^r \) for (4) with \( w'(0, t) = u_0 \) in the half space (see (9)) and satisfies
\[
\begin{cases}
    U^R_t + f(U^R)_x = 0, \\
    U^R(0, t) = u_0, \\
    U^R(x, 0) = U^R_0(x) := (f')^{-1}(w_0(x)),
\end{cases}
\]
where \( w_0 \) is given by (15) with \( w_\pm = f'(u_\pm) \) and \( \varepsilon = 1 \). As a simple corollary of Lemma 3.3 we have:

**Lemma 3.4** (cf. [12, 14]). Let \( f'(u_0) = 0 \) and \( u_0 < u_+ \). Let \( \delta_0 > 0 \) be any fixed constant and assume that \( \delta \leq \delta_0 \). Then the smooth approximation \( U^R \) defined by (16) with \( \varepsilon = 1 \) satisfies the following properties:

(i) \( u_- < U^R(x, t) < u_+ \) and \( U^R_x(x, t) > 0 \) for \( x \in \mathbb{R} \) and \( t \geq 0 \). Moreover, we have \( U^R(0, t) = u_0 \) for \( t \geq 0 \).

(ii) \( \|U^R_x(t)\|_{L^p} \leq \min \{C\delta, C\delta^{1/p}(1 + t)^{-(1-1/p)}\} \) for \( t \geq 0 \), where \( 1 \leq p \leq \infty \).

(iii) Let \( \theta \in [0, 1] \). Then \( \|\partial_t \partial_x^j U^R(t)\|_{L^p} \leq C\delta^\theta(1 + t)^{-(1-\theta)} \) for \( t \geq 0 \), where \( 1 \leq p \leq \infty \), and \( j \) and \( k \) are nonnegative integers satisfying \( j + k = 2, 3, 4 \). Here the constant \( C \) is independent of \( \theta \).

(iv) \( \|U^R - w^r(t)\|_{L^\infty} \to 0 \) as \( t \to \infty \), where \( w^r \) is the rarefaction wave of (4) with \( w'(0, t) = u_0 \) in the half space (see (9)).

4. Reformulation of the problems.

4.1. **Reformulation in the whole space.** We consider the initial value problem (1), (2). Let \( w^r \) be the centered rarefaction wave for (4) which is given in (7), and let \( U^R \) be the smooth approximation of \( w^r \). This \( U^R \) is constructed in Lemma 3.2 and
satisfies (17). We regard \((U^R, -U^R)\) as a smooth approximation of the rarefaction wave \((u^*, 0)\) for (1), and look for solutions \((u, q)\) of the problem (1), (2) in the form
\[
u = U^R + \phi, \quad q = -U_x^R + r.
\] (19)
Our problem (1), (2) is then rewritten in the following form for the perturbation \((\phi, r)\):
\[
\phi_t + (f(U^R + \phi) - f(U^R))_x + r_x = U^R_{xx},
\]
\[
r_t + r + \phi_x = -f(U^R)_{xx},
\]
\[
(\phi, r)(x, 0) = (\phi_0, r_0)(x), \quad x \in \mathbb{R},
\]
where
\[
\phi_0 = u_0 - U^R_0, \quad r_0 = q_0 + (U^R_0)_x.
\]
For this reformulated problem (20), (21), we obtain the following result on the global existence and asymptotic stability.

**Theorem 4.1** (Global existence and stability in the whole space). Let \(u_- < u_+\) and put \(\delta = |u_+ - u_-|\). Assume that \((\phi_0, r_0) \in H^2\) and put \(E_0 = \|(\phi_0, r_0)\|_{H^2}\). Then there is a positive constant \(\delta_1\) such that if \(E_0 + \delta \leq \delta_1\), then the problem (20), (21) has a unique global solution \((\phi, r)\) satisfying
\[
(\phi, r) \in C^0([0, \infty); H^2) \cap C^1([0, \infty); H^1),
\]
\[
\phi_x \in L^2(0, \infty; H^1), \quad r \in L^2(0, \infty; H^2).
\]
Moreover, the solution \((\phi, r)\) decays to \((0, 0)\) uniformly in \(x \in \mathbb{R}\) as \(t \to \infty\):
\[
\|(\phi, r)(t)\|_{W^{1,\infty}} \to 0 \quad \text{as} \quad t \to \infty. \tag{22}
\]

The key to the proof of Theorem 4.1 is to show the desired a priori estimate of solutions to the problem (20), (21). To state our a priori estimate, we introduce the energy norm \(E(t)\) and the corresponding dissipation norm \(D(t)\) as follows:
\[
E(t) := \sup_{0 \leq \tau \leq t} \|(\phi, r)(\tau)\|_{H^2},
\]
\[
D(t)^2 := \int_0^t \sqrt{U^R_x \phi(\tau)}^2_{L^2} + \|\phi_x(\tau)\|_{H^1}^2 + \|r(\tau)\|_{H^2}^2 \, d\tau. \tag{23}
\]

Then the result on our a priori estimate is stated as follows.

**Proposition 4.2** (A priori estimate in the whole space). Let \(T > 0\) and let \((\phi, r)\) be a solution to the problem (20), (21) such that
\[
(\phi, r) \in C^0([0, T]; H^2) \cap C^1([0, T]; H^1).
\]
Then there is a positive constant \(\delta_2\) not depending on \(T\) such that if \(E(T) + \delta \leq \delta_2\), then the solution \((\phi, r)\) verifies the a priori estimate
\[
E(t)^2 + D(t)^2 \leq C(E_0^2 + \delta^{2\theta}) \tag{24}
\]
for \(t \in [0, T]\), where \(\theta \in (0, 1/4)\) is a fixed number and \(C\) is a positive constant independent of \(T\).

We will give the proof of Proposition 4.2 in Section 5.
Proof of Theorem 4.1. The global existence result in Theorem 4.1 can be proved by the standard method which is based on the local existence result combined with the a priori estimate stated in Proposition 4.2. Here we omit the details on the proof of the global existence result and only give the proof of the convergence (22).

To this end, we first note that our global solution $(\phi, r)$ satisfies the energy estimate (24) for any $t \geq 0$. This together with (20) yields the estimate for the time derivatives:

$$
\int_0^t \|\phi_{x\tau}(\tau)\|_{L^2}^2 + \|r_t\|_{L^2}^2 \, d\tau \leq C(E_0^2 + \delta^9)
$$

(25) for any $t \geq 0$. To prove (22) for $\phi$, we put $\Phi(t) := \|\phi_x(t)\|_{L^2}^2$. We see that $\Phi \in L^1(0, \infty)$ by (24). Also we observe that $|\Phi'| \leq 2\|\phi_x\|_{L^2} \|\phi_{xx}\|_{L^2}$. Therefore we find that $\Phi' \in L^1(0, \infty)$ by (24) and (25). Thus we have $\Phi \in W^{1,1}(0, \infty)$, which shows the convergence $\Phi(t) = \|\phi_x(t)\|_{L^2}^2 \to 0$ as $t \to \infty$. This together with Sobolev’s inequality yields

$$
\|\phi\|_{L^\infty} \leq C\|\phi\|_{L^2}^{1/2} \|\phi_x\|_{L^2}^{1/2} \to 0,
$$

$$
\|\phi_x\|_{L^\infty} \leq C\|\phi_x\|_{L^2}^{1/2} \|\phi_{xx}\|_{L^2}^{1/2} \to 0
$$

as $t \to \infty$, where we also used (24). Thus we have proved $\|\phi(t)\|_{W^{1,\infty}} \to 0$ as $t \to 0$.

We can prove (22) for $r$ in a similar way. We put $R(t) := \|r(t)\|_{H^1}^2$. Then, using (24) and (25), we know that $R \in W^{1,1}(0, \infty)$, which shows $R(t) = \|r(t)\|_{H^1} \to 0$ as $t \to \infty$. This together with Sobolev’s inequality and (24) gives the convergence $\|r(t)\|_{W^{1,\infty}} \to 0$ as $t \to 0$. Thus the proof of Theorem 4.1 is complete. □

Finally in this subsection, we give the proof of Theorem 2.1 by using Theorem 4.1.

Proof of Theorem 2.1. We assume the smallness condition in Theorem 2.1. Namely, we assume that $I_0 + \delta$ is suitably small, where $I_0 = \|u_0 - u_0^R\|_{L^2} + \|(u_0)_{x}\|_{H^1} + \|q_0\|_{H^2}$. For the initial data $(\phi_0, r_0)$ in Theorem 4.1, we see that

$$
\|\phi(0, r_0)\|_{L^2} \leq \|(u_0 - u_0^R, q_0)\|_{L^2} + \|(u_0^R - U_0^R, (U_0^R)_{x})\|_{L^2} \leq I_0 + C\delta,
$$

$$
\|\partial_x(\phi_0, r_0)\|_{H^1} \leq \|\partial_x(u_0, q_0)\|_{H^1} + \|\partial_x(U_0^R, (U_0^R)_{x})\|_{H^1} \leq I_0 + C\delta.
$$

Therefore we have $E_0 \leq I_0 + C\delta$. Since $I_0 + \delta$ is assumed to be small, we see that $E_0 + \delta$ is also small. Consequently, by applying Theorem 4.1, we get a unique global solution $(\phi, r)$ to the problem (20), (21). Then the function $(u, q)$ defined by (19) becomes the desired global solution to the original problem (1), (2).

Finally, we show the convergence (10) by using (22). We see that

$$
\|(u - u^r, q)(t)\|_{L^\infty} \leq \|(U^R - u^r, -U^R_{x})(t)\|_{L^\infty} + \|\phi, r(t)\|_{L^\infty} \to 0
$$

as $t \to \infty$, where we also used Lemma 3.2. This completes the proof of Theorem 2.1. □

4.2. Reformulation in the half space. We consider the initial-boundary value problem (1), (3) in the half space. Let $u^r$ be the centered rarefaction wave for (4) with $u^r(0, t) = u_0$ which is given in (9), and let $U^R$ be the smooth approximation of $u^r$. This $U^R$ is constructed in Lemma 3.4 and satisfies (18). As in the case of the whole space, we regard $(U^R, -U^R_{x})$ as a smooth approximation of the rarefaction wave $(u^r, 0)$ for (1) in the half space, and look for solutions $(u, q)$ of the problem (1),
(3) in the form of (19). Then our problem (1), (3) is reformulated in the following form for the perturbation \((\phi, r)\):

\[
\phi_t + (f(U^R + \phi) - f(U^R))_x + r_x = U^R_{xx}, \tag{26a}
\]

\[
r_t + r + \phi_x = -f(U^R)_{xx}, \tag{26b}
\]

\[
\phi(0, t) = 0, \quad t > 0, \tag{27}
\]

where \(\phi_0 = u_0 - U^R_0\) and \(r_0 = q_0 + (U^R_0)_x\). Note that (26) is just the same as (20).

For this reformulated problem (26), (27), the associated compatibility conditions are given by

\[
\phi_0(0) = 0, \quad \phi_1(0) = 0, \tag{28}
\]

where \(\phi_1\) is defined by \(\phi_1(x) := \phi_t(x, 0)\). It follows from (26a) that

\[
\phi_1 = -(f(U^R + \phi_0) - f(U^R)_x) - (r_0)_x + (U^R_{0})_{xx}.
\]

Note that (28) is corresponding to (11).

For the above reformulated problem (26), (27), we obtain the following result on the global existence and asymptotic stability.

**Theorem 4.3** (Global existence and stability in the half space). Let \(f'(u_b) = 0\) and \(u_b < u_+\), and put \(\delta = |u_+ - u_b|\). Assume that \((\phi_0, r_0) \in H^2\) and put \(E_0 = \|(\phi_0, r_0)\|_{H^2}\). Also we assume the compatibility conditions in (28). Then there is a positive constant \(\delta_1\) such that if \(E_0 + \delta \leq \delta_1\), then the problem (26), (27) has a unique global solution \((\phi, r)\) satisfying

\[
(\phi, r) \in \bigcap_{j=0}^2 C^j([0, \infty); H^{2-j}),
\]

\[
\phi_x \in L^2(0, \infty; H^1), \quad r \in L^2(0, \infty; H^2),
\]

\[
\partial_t^j(\phi, r) \in L^2(0, \infty; H^{2-j}), \quad j = 1, 2.
\]

Moreover, the solution \((\phi, r)\) decays to \((0, 0)\) uniformly in \(x \in \mathbb{R}\) as \(t \to \infty\):

\[
\|(\phi, r)(t)\|_{W^{1, \infty}} \to 0 \quad \text{as} \quad t \to \infty. \tag{29}
\]

The key to the proof of Theorem 4.3 is to show the desired a priori estimate of solutions to the problem (26), (27). To state our a priori estimate, we introduce the energy norm \(E(t)\) and the corresponding dissipation norm \(D(t)\) as follows:

\[
E(t) := \sum_{j=0}^2 \sup_{0 \leq \tau \leq t} \|\partial_t^j(\phi, r)(\tau)\|_{H^{2-j}},
\]

\[
D(t)^2 := \int_0^t \|U_x^R\phi(\tau)|_{L^2}^2 + \|\phi_x(\tau)|_{L^2}^2 + \|r(\tau)|_{L^2}^2
\]

\[
+ \sum_{j=1}^2 \|\partial_t^j(\phi, r)(\tau)\|_{H^{2-j}}^2, d\tau. \tag{30}
\]

Then the result on our a priori estimate is stated as follows.
Proposition 4.4 (A priori estimate in the half space). Let $T > 0$ and let $(\phi, r)$ be a solution to the problem (26), (27) such that

$$(\phi, r) \in \bigcap_{j=0}^{2} C^{j}([0, T]; H^{2-j}).$$

Then there is a positive constant $\delta_2$ not depending on $T$ such that if $E(T) + \delta \leq \delta_2$, then the solution $(\phi, r)$ verifies the a priori estimate

$$E(t)^2 + D(t)^2 \leq C(E_0^2 + \delta^{2\theta})$$

for $t \in [0, T]$, where $\theta \in (0, 1/4)$ is a fixed number and $C$ is a positive constant independent of $T$.

We will prove Proposition 4.4 in Section 6.

Proof of Theorem 4.3. Our problem (26), (27) is an initial-boundary value problem for a symmetric hyperbolic system with a non-singular boundary matrix. Therefore we can show the local existence of a unique solution by applying the standard theory. We omit the details and refer to [18, 8]. Combining this local existence result and the a priori estimate in Proposition 4.4, we can obtain the desired global solution $(\phi, r)$ to the problem (26), (27). This global solution satisfies the energy estimate (31) for any $t \geq 0$. Once we get this uniform estimate (31), we can show the convergence (29) just in the same way as in the proof of Theorem 4.1. Thus the proof of Theorem 4.3 is complete.

Finally in this subsection, we give the proof of Theorem 2.2 by using Theorem 4.3.

Proof of Theorem 2.2. We assume the smallness condition in Theorem 2.2. Namely, we assume that $I_0 + \delta$ is suitably small, where $I_0 = \| (u_0 - u_+, q_0) \|_{H^2}$. For the initial data $(\phi_0, r_0)$ in Theorem 4.3, we see that

$$\|(\phi_0, r_0)\|_{H^2} \leq \| (u_0 - u_+, q_0)\|_{H^2} + \| (u_+ - U_0^R, (U_0^R)_x)\|_{H^2} \leq I_0 + C\delta.$$ 

Therefore we have $E_0 \leq I_0 + C\delta$. Since $I_0 + \delta$ is assumed to be small, we know that $E_0 + \delta$ is also small. Consequently, by applying Theorem 4.3, we get a unique global solution $(\phi, r)$ to the problem (26), (27). Then the function $(u, q)$ defined by (19) becomes the desired global solution to the original problem (1), (3). The convergence (12) follows from (29). In fact, we have

$$\|(u-u^r, q)(t)\|_{L^\infty} \leq \|(U^R - u^r, -U_x^R)(t)\|_{L^\infty} + \|(\phi, r)(t)\|_{L^\infty} \to 0$$

as $t \to \infty$, where we also used Lemma 3.4. This completes the proof of Theorem 2.2.

5. A priori estimate in the whole space. The aim of this section is to prove Proposition 4.2 on the a priori estimate of solutions to the problem (20), (21) in the whole space. In this section we assume that the solution $(\phi, r)$ satisfies the additional regularity

$$(\phi, r) \in C^0([0, T]; H^3) \cap C^1([0, T]; H^2).$$

This can be realized by using the Friedrichs mollifier with respect to $x \in \mathbb{R}$. Also we assume that

$$E(T) + \delta \leq \delta_0,$$  

(32)
where $\delta_0 > 0$ is a fixed constant. In this section $\theta$ denotes a fixed number satisfying $\theta \in (0, 1/4)$.

First we show the basic energy estimate.

**Lemma 5.1.** We have

$$
\|(\phi, r)(t)\|_{L^2}^2 + \int_0^t \left( \|\sqrt{U^R_x} \phi(\tau)\|_{L^2}^2 + \|r(\tau)\|_{L^2}^2 \right) d\tau \leq CE_0^2 + C\delta^\theta E(t)^{1/2} D(t)^{1/2}. \tag{33}
$$

**Proof.** We multiply (20a) and (20b) by $\phi$ and $r$, respectively, and add these two equalities. After a technical computation we obtain

$$
\left\{ \frac{1}{2} (\phi^2 + r^2) \right\}_t + \left\{ (f(U^R + \phi) - f(U^R))\phi - \int_0^\phi (f(U^R + \eta) - f(U^R))d\eta + \phi r \right\}_x + \left\{ f(U^R + \phi) - f(U^R) - f'(U^R)\phi \right\} U^R_x + r^2 = \phi U^R_x - r f(U^R)_{xx}. \tag{34}
$$

We integrate (34) over $\mathbb{R} \times (0, t)$. Then, using $\{ f(U^R + \phi) - f(U^R) - f'(U^R)\phi \} U^R_x \geq cU^R_x \phi^2$, we obtain

$$
\|(\phi, r)(t)\|_{L^2}^2 + \int_0^t \left( \|\sqrt{U^R_x} \phi(\tau)\|_{L^2}^2 + \|r(\tau)\|_{L^2}^2 \right) d\tau \leq CE_0^2 + C \int_0^t R^{(0)}(\tau) d\tau, \tag{35}
$$

where

$$
R^{(0)} = \int_{\mathbb{R}} |\phi| |U^R_{xx}| + |r||f(U^R)_{xx}| \, dx \tag{36}
$$

Here we can show that

$$
\int_0^t R^{(0)}(\tau) d\tau \leq C\delta^\theta E(t)^{1/2} D(t)^{1/2}. \tag{37}
$$

Once this is verified, the desired estimate (33) follows from (35) and (37).

We verify the estimate (37). Applying Sobolev’s inequality, we have

$$
\int_{\mathbb{R}} |\phi||U^R_{xx}| \, dx \leq \|\phi\|_{L^\infty} \|U^R_{xx}\|_{L^1} \leq C \|\phi\|_{L^\infty}^{1/2} \|\phi_x\|_{L^2}^{1/2} \|U^R_{xx}\|_{L^1}^{1/2} \leq C\delta^\theta E(t)^{1/2} \|\phi_x\|_{L^2}^{1/2} (1 + \tau)^{-\left(1-\theta\right)},
$$

where we used Lemma 3.2. Therefore we obtain

$$
\int_0^t \int_{\mathbb{R}} |\phi||U^R_{xx}| \, dx \, d\tau \leq C\delta^\theta E(t)^{1/2} \int_0^t \|\phi_x(\tau)\|_{L^2}^{1/2} (1 + \tau)^{-\left(1-\theta\right)} \, d\tau
$$

$$
\leq C\delta^\theta E(t)^{1/2} \left( \int_0^t \|\phi_x(\tau)\|_{L^2}^2 \, d\tau \right)^{1/4} \left( \int_0^t (1 + \tau)^{-\frac{4}{3}(1-\theta)} \, d\tau \right)^{3/4}
$$

$$
\leq C\delta^\theta E(t)^{1/2} D(t)^{1/2},
$$

where we used the Hölder inequality and the fact that $\frac{4}{3}(1-\theta) > 1$ for $\theta \in (0, 1/4)$. Another term in $R^{(0)}$ (see (36)) is estimated similarly and we obtain (37). This completes the proof of Lemma 5.1. \qed

Next we show the energy estimate for the derivatives.

**Lemma 5.2.** We have

$$
\|\partial_x(\phi, r)(t)\|_{H^1} + \int_0^t \|r_x(\tau)\|_{H^1}^2 \, d\tau \leq CE_0^2 + C (\delta + E(t)) D(t)^2 + C\delta^\theta D(t). \tag{38}
$$
We will show that
\[ \int Integrating this equality over \( R \) for \( k = 1, 2 \) to (39) and obtain
\[ \partial_x^k \phi_t + f'(U^R + \phi) \partial_x^k \phi_x + \partial_x^k r_x = g^k + \partial_x^k U^R, \] (41a)
\[ \partial_x^k r_t + \partial_x^k r + \partial_x^k \phi_x = -\partial_x^k f(U^R)_{xx}, \] (41b)
where
\[ g^k = -[\partial_x^k, f'(U^R + \phi)] \phi_x + \partial_x^k g. \] (42)

Here \([\cdot, \cdot]\) denotes the commutator. We multiply (41a) and (41b) by \( \partial_x^k \phi \) and \( \partial_x^k r \), respectively, and add the resulting equalities. This yields
\[ \frac{1}{2} \left( (\partial_x^k \phi)^2 + (\partial_x^k r)^2 \right) + \frac{1}{2} f'(U^R + \phi) (\partial_x^k \phi)^2 + \partial_x^k \phi \cdot \partial_x^k r + (\partial_x^k r)^2 \]
\[ = \frac{1}{2} f'(U^R + \phi)_x (\partial_x^k \phi)^2 + \partial_x^k \phi (g^k + \partial_x^k U^R_{xx}) - \partial_x^k r \cdot \partial_x^k f(U^R)_{xx}. \]

Integrating this equality over \( \mathbb{R} \times (0, t) \), we obtain
\[ \| \partial_x^k (\phi, r)(t) \|_{L^2}^2 + \int_0^t \| \partial_x^k r(\tau) \|_{L^2}^2 d\tau \leq CE_0^2 + C \int_0^t R^{(k)}(\tau) d\tau \] (43)
for \( k = 1, 2 \), where
\[ R^{(k)} = \int_\mathbb{R} |U^R| + |\phi_x| |\partial_x^k \phi|^2 + |g^k| |\partial_x^k \phi| + |\partial_x^k \phi| |\partial_x^k U^R| + |\partial_x^k r| |\partial_x^k f(U^R)_{xx}| dx. \] (44)

We will show that
\[ \int_0^t R^{(k)}(\tau) d\tau \leq C(\delta + E(t)) D(t)^2 + C\delta^k D(t) \] (45)
for \( k = 1, 2 \). Once this is done, we substitute (45) into (43) and add for \( k = 1, 2 \). This yields the desired estimate (38).

It remains to prove (45). Concerning the first term in (44), we observe that
\[ \int_0^t \int_\mathbb{R} \|U^R\| + |\phi_x| |\partial_x^k \phi|^2 \, dx \, d\tau \leq C(\delta + E(t)) \int_0^t \| \partial_x^k \phi(\tau) \|_{L^2}^2 d\tau \leq C(\delta + E(t)) D(t)^2 \] for \( k = 1, 2 \). Next we estimate the second term in (44). Recalling (42), we see that
\[ ||[\partial_x^k, f'(U^R + \phi)] \phi_x ||_{L^2} \leq C(\delta + E(t)) || \phi_x ||_{H^1} \] for \( k = 1, 2 \). Therefore we obtain
\[ \int_0^t \int_\mathbb{R} ||[\partial_x^k, f'(U^R + \phi)] \phi_x ||^2 \, dx \, d\tau \leq C(\delta + E(t)) \int_0^t || \phi_x(\tau) ||^2_{H^1} d\tau \leq C(\delta + E(t)) D(t)^2 \] for \( k = 1, 2 \). Also we apply a direct computation to \( g \) in (40) and find that
\[ || \partial_x^k g ||_{L^2} \leq C\delta^k E(t)(1 + \tau)^{-(1-\theta)} + C\delta || \phi_x ||_{H^1}, \]
for \( k = 1, 2 \), where we used Lemma 3.2. Therefore we obtain
\[
\int_0^t \int_{\mathbb{R}} |\partial_x^k \phi| |\partial_x^k \phi| \, dx \, d\tau \leq C\delta E(t) \int_0^t \|\partial_x^k \phi(\tau)\|_{L^2}(1 + \tau)^{-1} \, d\tau + C\delta \int_0^t \|\phi_x(\tau)\|^2_{H^1} \, d\tau \leq C\delta E(t)D(t) + C\delta D(t)^2
\]
for \( k = 1, 2 \). Moreover, we see easily that
\[
\int_0^t \int_{\mathbb{R}} |\partial_x^k \phi| |\partial_x^k U_{xx}^R| + |\partial_x^k r| |\partial_x^k f(U_R)_{xx}| \, dx \, d\tau \leq C\delta \int_0^t \|\partial_x^k (\phi, r)(\tau)\|_{L^2}(1 + \tau)^{-1} \, d\tau \leq C\delta D(t)
\]
for \( k = 1, 2 \). All these estimates give the desired estimate (45). Thus the proof of Lemma 5.2 is complete.

Finally, we show the dissipative estimate for \( \phi_x \).

**Lemma 5.3.** We have
\[
\int_0^t \|\phi_x(\tau)\|^2_{H^1} \, d\tau \leq CE_0^2 + C\|\phi_x(\tau)\|^2_{H^2} + C \int_0^t \|r(\tau)\|^2_{H^2} \, d\tau + C(\delta + E(t))D(t)^2 + C\delta D(t).
\]

**Proof.** We use (41) for \( k = 0, 1 \). (Note that (41) with \( k = 0 \) coincides with (39).)

To create the dissipative estimate for \( \partial_x^k \phi_x \), we multiply (41b) and (41a) by \( \partial_x^k \phi_x \) and \( -\partial_x^k r_x \), respectively, and add these two equalities. This gives
\[
(\partial_x^k \phi_x, \partial_x^k r) + (\partial_x^k \phi_x, \partial_x^k r)_x + (\partial_x^k \phi_x, \partial_x^k \phi_x)_x + (\partial_x^k r, \partial_x^k r) - (f'(U^R + \phi)\partial_x^k \phi_x + \partial_x^k r_x) \partial_x^k r_x
\]
\[
= -\partial_x^k r_x (g^k + \partial_x^k U_{xx}^R) - \partial_x^k \phi_x \partial_x^k f(U_R)_{xx}.
\]

Integrating over \( \mathbb{R} \times (0, t) \), we obtain
\[
\int_0^t \|\partial_x^k \phi_x(\tau)\|^2_{L^2} \, d\tau \leq CE_0^2 + C\|\partial_x^k (\phi, r)(\tau)\|^2_{H^1} + C \int_0^t \|\partial_x^k r(\tau)\|^2_{H^1} \, d\tau + C \int_0^t S^{(k)}(\tau) \, d\tau
\]
for \( k = 0, 1 \), where
\[
S^{(k)} = \int |g^k||\partial_x^k r_x| + |\partial_x^k r_x||\partial_x^k U_{xx}^R| + |\partial_x^k \phi_x||\partial_x^k f(U_R)_{xx}| \, dx.
\]

We will show that
\[
\int_0^t S^{(k)}(\tau) \, d\tau \leq C(\delta + E(t))D(t)^2 + C\delta D(t)
\]
for \( k = 0, 1 \). Once this is verified, we substitute (49) into (47) and add for \( k = 0, 1 \). Then we obtain the desired estimate (46).

To complete the proof, we need to show (49). Recalling (42), we observe that
\[
\|\partial_x^k, f'(U^R + \phi)\|_{L^2} \leq C(\delta + E(t))\|\phi_x\|_{L^2}
\]
for \( k = 0, 1 \). (The commutator vanishes
we find that $\|_{\text{norm of the initial data:}}$ of several energy estimates. To state these energy estimates, we introduce a new A priori estimate in the half space.

6. Proof of Proposition 4.2. We add (33) and (38) to get

$$\|(\phi, r)(t)\|_{H^2} + \int_0^t \|U_x^R \phi(\tau)\|_{L^2}^2 + \|r(\tau)\|_{H^2}^2 \, d\tau \leq CE_0^2 + C(\delta + E(t))D(t)^2 + C\delta^0(E(t) + D(t)).$$

(50)

We substitute (50) into (46). This gives

$$\int_0^t \|\phi_x(\tau)\|_{H^1} \, d\tau \leq CE_0^2 + C(\delta + E(t))D(t)^2 + C\delta^0(E(t) + D(t)).$$

(51)

Adding (50) and (51), we arrive at the inequality

$$E(t)^2 + D(t)^2 \leq CE_0^2 + C(\delta + E(t))D(t)^2 + C\delta^0(E(t) + D(t)).$$

This inequality is reduced to $E(t)^2 + D(t)^2 \leq (CE_0^2 + \delta^2) + C(\delta + E(t))D(t)^2$, which yields the desired estimate (24), provided that $E(T) + \delta$ is suitably small. Thus the proof of Proposition 4.2 is complete.

6. A priori estimate in the half space. In this section we prove Proposition 4.4 on the a priori estimate of solutions to the problem (26), (27). Our proof consists of several energy estimates. To state these energy estimates, we introduce a new norm of the initial data:

$$\tilde{E}_0 = \sum_{j=0}^2 \|\phi_j, r_j\|_{H^{2-j}},$$

where $(\phi_j, r_j)(x) := \partial^j_t (\phi, r)(x, 0)$ for $j = 0, 1, 2$. Here $(\phi_j, r_j)$ are given inductively as follows:

$$\begin{cases}
\phi_1 = -(f(U_0^R + \phi_0) - f(U_0^R))_{xx} - (r_0)_{xx} + (U_0^R)_{xxx}, \\
r_1 = -r_0 - (\phi_0)_{x} - f(U_0^R)_{xx},
\end{cases}$$

for $k = 0$.) Therefore we obtain

$$\int_0^t \int_\mathbb{R} \|\partial_x^k f'(U_x^R + \phi)\|_{L^2} \|\partial_x^k r_x\| \, dx \, d\tau \leq C(\delta + E(t)) \int_0^t \|\phi_x(\tau)\|_{L^2} \|\partial_x^k r_x(\tau)\|_{L^2} \, d\tau \leq C(\delta + E(t))D(t)^2$$

for $k = 0, 1$. Concerning the term $g$ in (40), we see that $|g| \leq C U_x^R |\phi|$. Therefore we find that $\|g\|_{L^2} \leq C \delta^{1/2} \|U_x^R \phi\|_{L^2}$. Thus we have

$$\int_0^t \int_\mathbb{R} \|g\|_{L^2} \, dx \, d\tau \leq C \delta^{1/2} \int_0^t \|U_x^R \phi(\tau)\|_{L^2} \|r_x(\tau)\|_{L^2} \, d\tau \leq C \delta^{1/2} D(t)^2.$$
and

\[
\begin{aligned}
\phi_2 &= - \{f'(U_0^R + \phi_0)\phi_1 + (f'(U_0^R + \phi_0) - f'(U_0^R))U_1^R\}_x - (r_1)_x + (U_1^R)_{xx}, \\
r_2 &= -r_1 - (\phi_1)_x - (f'(U_0^R)U_1^R)_{xx},
\end{aligned}
\]

where \(U_1^R(x) := \partial_t U^R(x, 0)\). Note that \(\tilde{E}_0 \leq C(E_0 + \delta)\).

To derive our energy estimates, we may assume that the solution \((\phi, r)\) has the additional regularity

\[(\phi, r) \in \bigcap_{j=0}^2 C^j([0,T]; H^{3-j}).\]

This can be realized by the standard technique for initial-boundary value problems, which makes use of the Friedrichs mollifier with respect to the time variable \(t\). With this additional regularity we have

\[
\partial_t^j \phi(0, t) = 0, \quad j = 0, 1, 2,
\]

for \(t > 0\). Moreover, as in Section 5, we assume that

\[E(T) + \delta \leq \delta_0,\]

where \(\delta_0 > 0\) is a fixed constant, and \(E(T)\) is defined in (30). In what follows \(\theta\) denotes a fixed number satisfying \(\theta \in (0, 1/4)\).

As in the case of the whole space, we first show the basic energy estimate.

**Lemma 6.1.** We have

\[
\| (\phi, r)(t) \|^2_{L^2} + \int_0^t \| U_x^R \phi(\tau) \|^2_{L^2} + \| r(\tau) \|^2_{L^2} d\tau \leq C E_0^2 + C \delta \theta E(t)^{1/2} D(t)^{1/2}. \tag{54}
\]

**Proof.** Since our system (26) is just the same as (20), we have the energy equality (34). We integrate (34) over \(\mathbb{R}_+ \times (0, t)\) and use the boundary condition \(\phi(0, t) = 0\) in (27) (or (52) for \(j = 0\)). Then we obtain the energy inequality (35), where the error term \(R(0)\) is given by (36) in which the integration over \(\mathbb{R}\) is replaced by \(\mathbb{R}_+\). This error term can be estimated as in (37) and therefore we arrive at the desired estimate (54). This completes the proof of Lemma 6.1.

Next we derive the energy estimates for the time derivatives \(\partial_t^j (\phi, r)\) for \(j = 1, 2\) by using (52) for \(j = 1, 2\).

**Lemma 6.2.** For \(j = 1, 2\) we have

\[
\| \partial_t^j (\phi, r)(t) \|^2_{L^2} + \int_0^t \| \partial_t^j r(\tau) \|^2_{L^2} d\tau \leq C E_0^2 + C(\delta + E(t))D(t)^2 + C \delta \theta D(t). \tag{55}
\]

**Proof.** We rewrite (26) in the form of (39) and apply \(\partial_t^j (j \leq 2)\) to (39). This yields

\[
\begin{aligned}
\partial_t^j \phi_t + f'(U^R + \phi) \partial_t^j \phi_x + \partial_t^j r_x &= \tilde{g}^j + \partial_t^j U_x^R, \\
\partial_t^j r_t + \partial_t^j r + \partial_t^j \phi_x &= -\partial_t^j f(U^R)_{xx},
\end{aligned}
\]

where

\[
\tilde{g}^j = -[\partial_t^j, f'(U^R + \phi)]\phi_x + \partial_t^j g, \tag{57}
\]
and $g$ is given in (40). Here $[\cdot, \cdot]$ denotes the commutator. We multiply (56a) and (56b) by $\partial_t^j \phi$ and $\partial_t^j r$, respectively, and add the resulting equalities. This gives
\[
\left\{ \frac{1}{2} \left( (\partial_t^j \phi)^2 + (\partial_t^j r)^2 \right) \right\}_t + \left\{ \frac{1}{2} f'(U^R + \phi)(\partial_t^j \phi)^2 + \partial_t^j \phi \cdot \partial_t^j r \right\} + (\partial_t^j r)^2 
= \frac{1}{2} f'(U^R + \phi)x(\partial_t^j \phi)^2 + \partial_t^j \phi (\partial_t^j U^R_{xx}) - \partial_t^j r \cdot \partial_t^j f(U^R)_{xx}.
\]
Integrating this equality over $\mathbb{R}_+ \times (0, t)$ and using (52), we obtain
\[
\|\partial_t^j (\phi, r)(t)\|_{L^2}^2 + \int_0^t \|\partial_t^j r(\tau)\|_{L^2}^2 d\tau \leq C\tilde{E}_0^2 + C \int_0^t \bar{R}^{(j)}(\tau) d\tau
\] (58)
for $j = 1, 2$, where
\[
\bar{R}^{(j)} = \int_{\mathbb{R}_+} \left( |U_x^R| + |\phi_x| \right) |\partial_t^j \phi|^2 + |\tilde{g}^j| |\partial_t^j \phi| + |\partial_t^j \phi| |\partial_t^j U_{xx}^R| + |\partial_t^j r| |\partial_t^j f(U^R)_{xx}| dx.
\]
We estimate the error term $\bar{R}^{(j)}$ just in the same way as $R^{(k)}$ in (44) and obtain
\[
\int_0^t \bar{R}^{(j)}(\tau) d\tau \leq C(\delta + E(t))D(t)^2 + C\delta^g D(t)
\] (59)
for $j = 1, 2$. We substitute (59) into (58) and obtain the desired estimate (55) for $j = 1, 2$. This completes the proof of Lemma 6.2.

We need to show the energy estimates for the spatial derivatives.

Lemma 6.3. We have
\[
\sum_{j=0}^1 \|\partial_t^j \partial_x^2 (\phi, r)(t)\|_{L^2}^2 + \|\partial_t^j (\phi, r)(t)\|_{L^2}^2 \leq C\tilde{E}_0^2 + C(\delta + E(t))D(t)^2 + C\delta^g E(t) + D(t)
\] (60)
Proof. To show the estimate for $\partial_t^j \partial_x^2 (\phi, r)$, we use (56). First it follows from (56b) that $\partial_t^j \phi_x = -\partial_t^j r_t - \partial_t^j r - \partial_t^j f(U^R)_{xx}$. We multiply this equality by $\partial_t^j \phi_x$ and integrate over $\mathbb{R}_+$. This yields
\[
\|\partial_t^j \phi_x(t)\|_{L^2}^2 \leq C(||\partial_t^j r_t(t)||_{L^2}^2 + ||\partial_t^j r(t)||_{L^2}^2) + CP^{(j)}_1(t)
\] (61)
for $j = 0, 1$, where
\[
P^{(j)}_1 = \int_{\mathbb{R}_+} |\partial_t^j \phi_x| |\partial_t^j f(U^R)_{xx}| dx.
\]
This error term is estimated as $P^{(j)}_1(t) \leq C\delta E(t)$. Therefore, using (54) and (55) in (61), we obtain
\[
\|\partial_t^j \phi_x(t)\|_{L^2}^2 \leq C\tilde{E}_0^2 + C(\delta + E(t))D(t)^2 + C\delta^g E(t) + D(t)
\] (62)
for $j = 0, 1$. Next we have from (56a) that $\partial_t^j r_x = -\partial_t^j \phi_t - f'(U^R + \phi)\partial_t^j \phi_x + \tilde{g}^j + \partial_t^j U^R_{xx}$. We multiply this equality by $\partial_t^j r_x$ and integrate over $\mathbb{R}_+$. This yields
\[
\|\partial_t^j r_x(t)\|_{L^2}^2 \leq C(||\partial_t^j \phi_t(t)||_{L^2}^2 + ||\partial_t^j \phi_x(t)||_{L^2}^2) + CQ^{(j)}_1(t)
\] (63)
for $j = 0, 1$, where
\[
Q^{(j)}_1 = \int_{\mathbb{R}_+} |\tilde{g}^j| |\partial_t^j r_x| + |\partial_t^j r_x| |\partial_t^j U^R_{xx}| dx.
\]
This error term can be estimated as \( Q^{(j)}_1(t) \leq C\delta E(t)^2 + C\delta E(t) \). Therefore, using (55) and (62) in (63), we obtain
\[
\|\partial_t^j r_\phi(t)\|_{L^2}^2 \leq C\bar{E}_0^2 + C(\delta + E(t))D(t)^2 + C\delta^0(E(t) + D(t)) \tag{64}
\]
for \( j = 0, 1 \).

To derive the estimate for \( \partial_x^2(\phi, r) \), we use (41a) and (41b) for \( k = 1 \). It follows from (41b) for \( k = 1 \) that \( \partial_x \phi_x = -\partial_x r_x - \partial_x f(U^R)_{xx} \). We multiply this equality by \( \partial_x \phi_x \) and integrate over \( \mathbb{R}_+ \). This yields
\[
\|\partial_x \phi_x(t)\|_{L^2}^2 \leq C(\|\partial_x r_x(t)\|_{L^2}^2 + \|\partial_x r(t)\|_{L^2}^2) + CP^{(1)}_2(t), \tag{65}
\]
where
\[
P^{(1)}_2 = \int_{\mathbb{R}_+} |\partial_x \phi_x||\partial_x f(U^R)_{xx}| \, dx.
\]
This error term is estimated as \( P^{(1)}_2(t) \leq C\delta E(t) \). Therefore, using (64) in (65), we obtain
\[
\|\partial_x \phi_x(t)\|_{L^2}^2 \leq C\bar{E}_0^2 + C(\delta + E(t))D(t)^2 + C\delta^0(E(t) + D(t)). \tag{66}
\]
Finally, we have from (41a) for \( k = 1 \) that \( \partial_x r_x = -\partial_x \phi_t - f'(U^R + \phi)\partial_x \phi_x + g^1 + \partial_x R_x \). We multiply this equality by \( \partial_x r_x \) and integrate over \( \mathbb{R}_+ \). This gives
\[
\|\partial_x r_x(t)\|_{L^2}^2 \leq C(\|\partial_x \phi(t)\|_{L^2}^2 + \|\partial_x \phi_x(t)\|_{L^2}^2) + CQ^{(1)}_2(t), \tag{67}
\]
where
\[
Q^{(1)}_2 = \int_{\mathbb{R}_+} |g^1||\partial_x r_x| + |\partial_x r_x||\partial_x U^R_{xx}| \, dx.
\]
This error term can be estimated as \( Q^{(1)}_2(t) \leq C\delta E(t)^2 + C\delta E(t) \). Therefore, using (62) and (65) in (67), we obtain
\[
\|\partial_x r_x(t)\|_{L^2}^2 \leq C\bar{E}_0^2 + C(\delta + E(t))D(t)^2 + C\delta^0(E(t) + D(t)). \tag{68}
\]
The desired estimate (60) now follows from (62), (64), (66) and (68). Thus the proof of Lemma 6.3 is complete. \( \square \)

Finally, we show the dissipative estimates for \( \partial_t^j \partial_x(\phi, r) \), \( \partial_t^j \phi_t \), \( j = 0, 1 \), and \( \partial_x^2(\phi, r) \).

**Lemma 6.4.** We have
\[
\sum_{j=0}^{1} \int_0^t \|\partial_t^j \partial_x(\phi, r)(\tau)\|_{L^2}^2 + \|\partial_t^j \phi_t(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \|\partial_x^2(\phi, r)(\tau)\|_{L^2}^2 \, d\tau \leq C\bar{E}_0^2 + C(\delta + E(t))D(t)^2 + C\delta^0(E(t) + D(t)). \tag{69}
\]

**Proof.** We integrate (61) over \((0, t)\) to get
\[
\int_0^t \|\partial_t^j \phi_x(\tau)\|_{L^2}^2 \, d\tau \leq C \int_0^t \|\partial_t^j r_x(\tau)\|_{L^2}^2 + \|\partial_t^j r(\tau)\|_{L^2}^2 \, d\tau + C \int_0^t P^{(j)}_1(\tau) \, d\tau \leq C\bar{E}_0^2 + C(\delta + E(t))D(t)^2 + C\delta^0(E(t) + D(t)) \tag{70}
\]
for \( j = 0, 1 \), where we used (54), (55) and \( \int_0^t P^{(j)}_1(\tau) \, d\tau \leq C\delta^0 D(t) \).
Lemma 5.3. We multiply (56a) and (56b) by $\frac{1}{t}$ and add the resulting two equalities. This yields

$$-(\partial_t^2 \phi_x \partial_t r_x) + (\partial_t^2 \phi_x \partial_t^2 r_x) + (\partial_t^2 r_x)^2 + f'(U^R + \phi) \partial_t^2 \phi_x \partial_t^2 r_x - \partial_t^2 \phi_x \partial_t r
$$

$$-(\partial_t^2 \phi_x)^2 = \partial_t^2 r_x \left( \tilde{g}^j + \partial_t^2 U_{xx}^R \right) + \partial_t^2 \phi_x \partial_t^2 f(U^R)_{xx}.$$  

Integrating over $\mathbb{R}_+ \times (0, t)$ and using (52), we obtain

$$\int_0^t \|\partial_t^2 r_x(t)\|_{L^2}^2 \, dt \leq C E_0^2 + C \|\partial_t^2 (\phi, r)(t)\|_{H^1}^2,$$

$$+ C \int_0^t \|\partial_t^2 r(t)\|_{L^2}^2 + \|\partial_t^2 \phi_x(t)\|_{L^2}^2 \, dt + C \int_0^t \tilde{S}^{(2)}(\tau) \, d\tau$$

for $j = 0, 1$, where

$$\tilde{S}^{(2)} = \int_{\mathbb{R}_+} |\tilde{g}| \|\partial_t^2 r_x| + |\partial_t^2 r_x| |\partial_t^2 U_{xx}^R| + |\partial_t^2 \phi_x| |\partial_t^2 f(U^R)_{xx}| \, dx.$$  

Here the error term can be estimated as

$$\int_0^t \tilde{S}^{(2)}(\tau) \, d\tau \leq C(\delta + E(t)) D(t) + C \delta \theta D(t)$$

for $j = 0, 1$. Therefore, substituting (54), (55), (60) and (70) into (71), we arrive at the estimate

$$\int_0^t \|\partial_t^2 r_x(t)\|_{L^2}^2 \, dt \leq C E_0^2 + C(\delta + E(t)) D(t) + C \delta \theta (E(t) + D(t))$$

(72)

for $j = 0, 1$.

Next we use (56a) to estimate $\partial_t^2 \phi_t$. We have $\partial_t^2 \phi_t = -f'(U^R + \phi) \partial_t^2 \phi_x - \partial_t^2 r_x + \tilde{g}^j + \partial_t^2 U_{xx}^R$. We multiply this equality by $\partial_t^2 \phi_t$ and integrate over $\mathbb{R}_+ \times (0, t)$. Then we have

$$\int_0^t \|\partial_t^2 \phi_t(t)\|_{L^2}^2 \, dt \leq C \int_0^t \|\partial_t^2 \phi_x(t)\|_{L^2}^2 \, dt + C \int_0^t \tilde{Q}^{(2)}_1(\tau) \, d\tau$$

(73)

for $j = 0, 1$, where

$$\tilde{Q}^{(2)}_1 = \int_{\mathbb{R}_+} |\tilde{g}| \|\partial_t^2 \phi_t| + |\partial_t^2 \phi_t| \|\partial_t^2 U_{xx}^R| \, dx.$$  

We see that

$$\int_0^t \tilde{Q}^{(2)}_1(\tau) \, d\tau \leq C(\delta + E(t)) D(t) + C \delta \theta D(t)$$

for $j = 0, 1$. Also we have (70) and (72). Substituting all these estimates into (73), we obtain

$$\int_0^t \|\partial_t^2 \phi_t(t)\|_{L^2}^2 \, dt \leq C E_0^2 + C(\delta + E(t)) D(t) + C \delta \theta (E(t) + D(t))$$

(74)

for $j = 0, 1$.

Finally, we estimate $\partial_t^2 (\phi, r)$. We integrate (65) over $(0, t)$ to get

$$\int_0^t \|\partial_t^2 (\phi, r)(t)\|_{L^2}^2 \, dt \leq C \int_0^t \|\partial_t^2 r_x(t)\|_{L^2}^2 + \|\partial_t^2 r(t)\|_{L^2}^2 \, dt + C \int_0^t P_2^{(1)}(\tau) \, d\tau$$

$$\leq C E_0^2 + C(\delta + E(t)) D(t) + C \delta \theta (E(t) + D(t)),$$
where we used (72) and \( \int_0^t P_2^{(1)}(\tau) \, d\tau \leq C\delta^6 D(t) \). Also, integrating (67) over \((0, t)\), we have
\[
\int_0^t \|\partial_x r_x(\tau)\|_{L^2}^2 \, d\tau \leq C \int_0^t \|\partial_x \phi_x(\tau)\|_{L^2}^2 + \|\partial_x \phi_x(\tau)\|_{L^2}^2 \, d\tau + C \int_0^t Q_2^{(1)}(\tau) \, d\tau
\]
\[
\leq C\bar{E}_0^2 + C(\delta + E(t)) D(t)^2 + C\delta^6 (E(t) + D(t)),
\]
where we used (70) with \( j = 1 \), (75) and
\[
\int_0^t Q_2^{(1)}(\tau) \, d\tau \leq C(\delta + E(t)) D(t)^2 + C\delta^6 D(t).
\]
The desired estimate (69) follows from (70), (72), (74), (75) and (76). Thus the proof of Lemma 6.4 is complete.

**Proof of Proposition 4.4.** We add all the estimates (54), (55), (60) and (69), and obtain
\[
E(t)^2 + D(t)^2 \leq C\bar{E}_0^2 + C(\delta + E(t)) D(t)^2 + C\delta^6 (E(t) + D(t)).
\]
Since \( \bar{E}_0 \leq C(E_0 + \delta) \), the above inequality is reduced to \( E(t)^2 + D(t)^2 \leq C(E_0^2 + \delta^2) + C(\delta + E(t)) D(t)^2 \), which yields the desired estimate (31), provided that \( E(T) + \delta \) is suitably small. Thus the proof of Proposition 4.4 is complete.

**Appendix A. Proof of Lemma 3.1.** In this Appendix we prove Lemma 3.1 (iii) concerning the estimates for higher order derivatives of the solution \( w \) to the problem (13), (15).

**Proof of Lemma 3.1.** Following the similar argument as in \([12, 14]\), we prove (iii) of Lemma 3.1. Let \( w = w(x, t) \) be the solution to the initial value problem (13), (15) for the inviscid Burgers equation. Using the characteristic curve method, we have the solution formula:
\[
w(x, t) = w_0(x_0(x, t)), \tag{77a}
\]
\[
x = x_0(x, t) + w_0(x_0(x, t)) t, \tag{77b}
\]
where \( x_0 = x_0(x, t) \) is a point in \( \mathbb{R} \). We differentiate (77b) with respect to \( x \) and obtain
\[
\partial_x x_0 = \frac{1}{1 + w'_0(x_0)x_0 t}, \tag{78}
\]
where \( w'_0(x_0) = \frac{d}{dx_0} w_0(x_0) \). Notice that \( w'_0(x_0) > 0 \) by (15). (See (83) below.) Also, by differentiating (77a) with respect to \( x \) and using (78), we have
\[
w_x = w'_0(x_0) \cdot \partial_x x_0 = \frac{w'_0(x_0)}{1 + w'_0(x_0)x_0 t}. \tag{79}
\]
Moreover, differentiating (79) with respect to \( x \) inductively, we obtain
\[
w_{xx} = \frac{w''_0(x_0)}{(1 + w'_0(x_0)x_0 t)^3}; \tag{80}
\]
\[
\partial_x^2 w = \frac{w^{(3)}_0(x_0)}{(1 + w'_0(x_0)x_0 t)^4} - 3 \frac{w''_0(x_0)^2 x_0 t}{(1 + w'_0(x_0)x_0 t)^5}, \tag{81}
\]
\[
\partial_x^3 w = \frac{w^{(4)}_0(x_0)}{(1 + w'_0(x_0)x_0 t)^5} - 10 \frac{w^{(3)}_0(x_0) w''_0(x_0)x_0 t}{(1 + w'_0(x_0)x_0 t)^6} + 15 \frac{w''_0(x_0)^3 t^2}{(1 + w'_0(x_0)x_0 t)^6}. \tag{82}
\]
where \( \varepsilon \in (0,1] \) and \( \delta = |w_{+} - w_{-}| \) with \( w_{-} < w_{+} \). Also, differentiating (83) with respect to \( x_{0} \) and using (84), again, we see that
\[
 w''_{0}(x_{0}) = -2\varepsilon \tanh(\varepsilon x_{0}) w'_{0}(x_{0}).
\] 
(84)

Note that \( w''_{0}(x_{0}) < 0 \) for \( x_{0} > 0 \) and \( w''_{0}(x_{0}) > 0 \) for \( x_{0} < 0 \). Furthermore, differentiating (84) with respect to \( x_{0} \), we obtain
\[
 w^{(3)}_{0}(x_{0}) = -2\varepsilon^{2} \cosh^{-2}(\varepsilon x_{0}) w'_{0}(x_{0}) - 2\varepsilon \tanh(\varepsilon x_{0}) w''_{0}(x_{0})
\]
\[
= -2\varepsilon^{2} \cosh^{-2}(\varepsilon x_{0}) w'_{0}(x_{0}) + 4\varepsilon^{2} \tanh^{2}(\varepsilon x_{0}) w''_{0}(x_{0}),
\] 
(85)

\[
 w^{(4)}_{0}(x_{0}) = -8\varepsilon^{2} \cosh^{-2}(\varepsilon x_{0}) w''_{0}(x_{0}) + 4\varepsilon^{2} \tanh^{2}(\varepsilon x_{0}) w''_{0}(x_{0}).
\] 
(86)

We estimate \( \partial^{k}x w \) for \( k = 2,3,4 \). It follows from (80) that
\[
 |w_{xx}| \leq |w''_{0}(x_{0})|(1 + w'_{0}(x_{0})t)^{-3}.
\] 
(87)

Also we estimate (81) and (82) by using (84), (85) and (86). After tedious computations, we find that
\[
 |\partial^{2}_{x}w| \leq C\varepsilon |w''_{0}(x_{0})|(1 + w'_{0}(x_{0})t)^{-4} + C\varepsilon^{2} \cosh^{-2}(\varepsilon x_{0}) w'_{0}(x_{0})(1 + w'_{0}(x_{0})t)^{-4},
\] 
(88)

\[
 |\partial^{k}_{x}w| \leq C\varepsilon^{k-2} |w''_{0}(x_{0})|(1 + w'_{0}(x_{0})t)^{-5}.
\] 
(89)

Therefore, to get the desired \( L^{p} \) estimates, we need to consider the functions
\[
 I_{k} := \varepsilon^{k-2} |w''_{0}(x_{0})|(1 + w'_{0}(x_{0})t)^{-(k+1)},
\]
\[
 J_{k} := \varepsilon^{k-1} \cosh^{-2}(\varepsilon x_{0}) w'_{0}(x_{0})(1 + w'_{0}(x_{0})t)^{-(k+1)},
\] 
(90)

and prove the following estimate:
\[
 \|I_{k}\|_{L^{p}} + \|J_{k}\|_{L^{p}} \leq \min \left\{ C\varepsilon^{k-1/\theta} \delta, C\varepsilon^{(k-1)-1/\theta}(1 + t)^{-1} \right\},
\] 
(91)

where \( k \geq 2 \) and \( 1 \leq p < \infty \). In fact, we see from (87), (88) and (89) that \( |w_{xx}| \leq I_{2} \), \( |\partial^{2}_{x}w| \leq C(I_{3} + J_{3}) \) and \( |\partial^{4}_{x}w| \leq CI_{4} \). This fact together with (91) gives the desired estimates in (iii) of Lemma 3.1.

Therefore it suffices to show the estimate (91). We first show the estimate for \( p = \infty \). We have from (84) that \( |w''_{0}(x_{0})| \leq 2\varepsilon w'_{0}(x_{0}) \). We use this estimate in (90) and obtain
\[
 I_{k} + J_{k} \leq C\varepsilon^{k-1} w'_{0}(x_{0})(1 + w'_{0}(x_{0})t)^{-(k+1)} \leq C\varepsilon^{k-1} t^{-1}.
\]

Also we have \( w'_{0}(x_{0}) \leq \varepsilon \delta /2 \) by (83), so that we obtain \( I_{k} + J_{k} \leq C\varepsilon^{k} \delta \). Thus we have shown that
\[
 \|I_{k}\|_{L^{\infty}} + \|J_{k}\|_{L^{\infty}} \leq \min \left\{ C \varepsilon^{k} \delta, C \varepsilon^{k-1} t^{-1} \right\},
\]
which implies (91) for \( p = \infty \).

Next we show the estimate (91) for \( 1 \leq p < \infty \). Letting \( 1 \leq p < \infty \), we consider the \( L^{p} \) norm \( \|I_{k}\|_{L^{p}} \) which is given by an integral with respect to \( x \). We change the
integration variable by \( x_0 = x_0(x, t) \). Then we have \( dx = (1 + w'_0(x_0)t)dx_0 \) by (78). Consequently, we obtain
\[
\|I_k\|_{L^p}^p = \varepsilon^{p(k-2)} \int_{\mathbb{R}} |w''_0(x_0)|^p(1 + w'_0(x_0)t)^{1-p(k+1)} \, dx_0 \\
= 2\varepsilon^{p(k-2)} \int_{-\infty}^{0} |w''_0(x_0)|^p(1 + w'_0(x_0)t)^{1-p(k+1)} \, dx_0,
\]
where we used the fact that the integrand is an even function of \( x_0 \). Here we note that
\[
|w''_0(x_0)| \leq 2\varepsilon w'_0(x_0) = \varepsilon^2 \delta \cosh^{-2}(\varepsilon x_0),
\]
which follows from (84) and (83). Substituting this estimate into (92) and using \( 1 - p(k+1) \leq 0 \), we obtain
\[
\|I_k\|_{L^p}^p \leq 2\varepsilon^{p(k-1)} \delta^p \int_{-\infty}^{0} \cosh^{-2p}(\varepsilon x_0) \, dx_0 \leq C\varepsilon^{pk-1} \delta^p,
\]
which shows that \( \|I_k\|_{L^p} \leq C\varepsilon^{k-1/p} \delta \).

On the other hand, we change the integration variable in (92) by \( y = w'_0(x_0)t \). Then we have \( dy = w''_0(x_0)t \, dx_0 = |w''_0(x_0)|^t \, dx_0 \) for \( x_0 < 0 \). Also we have \( |w''_0(x_0)| \leq 2\varepsilon w'_0(x_0) = 2\varepsilon y^{-1} \), where we used (84). Consequently, we obtain
\[
\|I_k\|_{L^p}^p = 2\varepsilon^{p(k-2)} \int_{-\infty}^{0} |w''_0(x_0)|^{p-1}(1 + w'_0(x_0)t)^{1-p(k+1)}(|w'_0(x_0)|t) \, dx_0 \\
\leq C\varepsilon^{p(k-2)+p-1} t^{-1-p} \int_{0}^{w'_0(0)t} y^{p-1}(1 + y)^{1-p(k+1)} \, dy \\
\leq C\varepsilon^{p(k-1)-1} t^{-p},
\]
where we used the fact that \( p - 1 \geq 0 \) and \( 1 - p(k+1) + (p-1) = -pk \leq -2 \). This shows that \( \|I_k\|_{L^p} \leq C\varepsilon^{(k-1)-1/p} t^{-1} \). Thus we have shown (91) for \( \|I_k\|_{L^p} \) with \( 1 \leq p < \infty \).

Finally we estimate \( \|J_k\|_{L^p} \) for \( 1 \leq p < \infty \). Similarly to the derivation of (92), we have
\[
\|J_k\|_{L^p}^p = \varepsilon^{p(k-1)} \int_{\mathbb{R}} \cosh^{-2p}(\varepsilon x_0) w'_0(x_0)(1 + w'_0(x_0)t)^{1-p(k+1)} \, dx_0.
\]
(93)

Since \( w'_0(x_0) \leq \varepsilon \delta/2 \) by (83), we have
\[
\|J_k\|_{L^p}^p \leq C\varepsilon^{p(k-1)+p} \delta^p \int_{\mathbb{R}} \cosh^{-2p}(\varepsilon x_0) \, dx_0 \leq C\varepsilon^{pk-1} \delta^p,
\]
which shows that \( \|J_k\|_{L^p} \leq C\varepsilon^{k-1/p} \delta \). Also we can estimate (93) as follows:
\[
\|J_k\|_{L^p}^p \leq C\varepsilon^{p(k-1)} t^{-p} \int_{\mathbb{R}} \cosh^{-2p}(\varepsilon x_0) \, dx_0 \leq C\varepsilon^{p(k-1)-1} t^{-p},
\]
where we used \( 1 - p(k+1) + p = 1 - pk \leq 0 \). This implies \( \|J_k\|_{L^p} \leq C\varepsilon^{(k-1)-1/p} t^{-1} \). Thus we have shown (91) also for \( \|J_k\|_{L^p} \) with \( 1 \leq p < \infty \). This completes the proof of (iii) of Lemma 3.1.

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