Second order hydrodynamics for a special class of gravity duals

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Abstract

The sound mode hydrodynamic dispersion relation is computed up to order $q^3$ for a class of gravitational duals which includes both Schwarzschild $AdS$ and Dp-Brane metrics. The implications for second order transport coefficients are examined within the context of Israel-Stewart theory. These sound mode results are compared with previously known results for the shear mode. This comparison allows one to determine the third order hydrodynamic contributions to the shear mode for the class of metrics considered here.
1 Introduction

Hydrodynamics describes the behavior of a fluid on length and time scales which are much longer than any microscopic scale. The hydrodynamic stress-energy tensor is constructed as a derivative expansion in the fluid velocity, and is required to respect equilibrium thermodynamics and the symmetries in the problem. Such a derivative expansion will contain unknown coefficients (transport coefficients). In first order hydrodynamics (an expansion of the energy-momentum tensor which contains at most one derivative) without a conserved charge, the transport coefficients which enter are the shear viscosity $\eta$, and the bulk viscosity $\zeta$. See [1, 2] for a more complete introduction to hydrodynamics.

The plasma created at the Relativistic Heavy Ion Collider (RHIC) appears to be both strongly coupled and well described by hydrodynamics [3, 4, 5]. Transport coefficients are necessary input for hydrodynamic simulations of the RHIC plasma; it is desirable to calculate them, but the strong coupling renders conventional perturbative calculations unreliable.

The AdS/CFT correspondence [6, 7, 8] (or more generally, ‘gauge/gravity duality’ or ‘holography’) has become a useful tool in describing strongly coupled systems, and has enjoyed arguably its greatest success calculating hydrodynamic transport coefficients. There are many ways to calculate the transport coefficients of a strongly coupled gauge theory using an extra-dimensional gravity dual. One can compute correlation functions of the stress-energy tensor and use Kubo formulas or examine the poles of such correlators [9, 10, 11, 12, 13, 14]. Alternatively, one can examine the behavior of the gravitational background under perturbations and determine the dispersion relation for such perturbations by applying appropriate boundary conditions. Comparison with the expected dispersion relation from perturbations of the energy-momentum tensor yields formulas for the transport coefficients [15, 16, 17]. In addition, the black hole membrane paradigm has been employed to calculate the hydrodynamic properties of the stretched horizon of a black hole. In many cases, the transport coefficients calculated on the stretched horizon coincide with the transport coefficients in the dual gauge theory [18, 19, 20, 21, 22, 23]. Recently, the work of [24] provides yet another way to compute hydrodynamic transport coefficients by deriving the equations of fluid dynamics directly from gravity. This work has proved quite influential, and has led to much subsequent research [25, 26, 27, 28, 29, 30]. We use the gravitational perturbation approach similar to [15, 16], as this work is an extension of a calculation in [17].

The quintessential example of the success of gauge/gravity duality is the case of the shear viscosity. In [18] a formula for $\eta/s$ ($s$ is the entropy density) applicable to a wide variety of gravitational duals was derived. Application of the formula continually resulted in

$$\frac{\eta}{s} = \frac{1}{4\pi}. \quad (1)$$

It was later shown that this relation holds for all theories of Einstein gravity [23, 31] (assuming the dual gauge theory is infinitely strongly coupled). It is quite remarkable that this result holds for both conformal and non-conformal theories, and is independent of the number of dimensions. In [18], the relation led to the conjecture that $\eta/s \geq 1/4\pi$ for all
substances, which has observable consequences for the plasma created at RHIC, even though the gravitational dual to quantum chromodynamics (QCD) is not currently known. It is thus desirable to find other such universal behavior from gauge/gravity duality in hopes that it may have implications for heavy ion collisions at RHIC and at the Large Hadron Collider (LHC).

In the past few years, much work has been done to extend previous analyses to second order hydrodynamics [21, 22, 25, 26, 27, 28, 29, 30, 32, 33], which attempts to repair technical problems in first order hydrodynamics regarding causality. Most of the work on second order hydrodynamics so far has focused on conformal theories. It is notable that a universal relation between second order hydrodynamic transport coefficients of a conformal theory was presented in [34], though it is not known whether this relation still holds for non-conformal theories.

In this work, we examine second order hydrodynamics for a special class of gravitational backgrounds. This class is not necessarily conformal, and includes both Schwarzschild AdS (conformal), and Dp-Brane (non-conformal) backgrounds. Specifically, we extend the analysis of [17] to the next hydrodynamic order by computing the sound mode dispersion relation for such a class of backgrounds. We also discuss implications for second order transport coefficients within the context of the Israel-Stewart formulation of second order hydrodynamics [35].

The paper is organized as follows. In section 2, we detail the specific type of gravity dual on which we focus in this work. In section 3, we add hydrodynamic perturbations on top of this background, and review the calculation of [17], since this work is an extension thereof. In section 4, we extend the calculation to the next hydrodynamic order. The hydrodynamic dispersion relation $w(q)$ in the sound mode can be written

$$w(q) = w_1q + w_2q^2 + w_3q^3 + ...$$ (2)

The main result of this section (and indeed the paper as a whole) is the calculation of $w_3$. In section 5, we discuss the implications of our formula for $w_3$ for second order transport coefficients within the context of Israel-Stewart theory. In section 6, we compare our results to those of a different hydrodynamic mode, the shear mode. It was shown in [33] that the shear mode dispersion relation contains contributions from (currently unformulated) third order hydrodynamics. Still, by comparing the shear mode and the sound mode we can determine the value of these extra contributions for the class of metrics we consider here. Finally, in the last section we summarize our results and present prospects for future investigation.

## 2 Background fields

Because this work builds upon the paper [17], we use this section to review the setup and notation presented therein.
2.1 Black brane background

In [17], the sound mode was analyzed for black brane gravitational backgrounds in \( p + 2 \) dimensions. The matter supporting the metric was assumed to be one or more scalar fields. The form of the action is assumed to be

\[
S = \frac{1}{16\pi G_{p+2}} \int d^{p+2}x \sqrt{-g} \left( R - \frac{1}{2} \sum_{k=1}^{n} \partial_{\mu} \phi_{k} \partial^{\mu} \phi_{k} - U(\phi_{1}, \phi_{2}...\phi_{n}) \right). \tag{3}
\]

The energy-momentum tensor derived from the action is

\[
8\pi G_{p+2} T_{\mu\nu} = \frac{1}{2} \sum_{k=1}^{n} \left( \partial_{\mu} \phi_{k} \partial_{\nu} \phi_{k} - g_{\mu\nu} \mathcal{L}_{\phi k} \right) \tag{4}
\]

\[
\mathcal{L}_{\phi k} = \frac{1}{2} \sum_{k=1}^{n} \partial_{\lambda} \phi_{k} \partial^{\lambda} \phi_{k} + U(\phi_{1}, \phi_{2}...\phi_{n}). \tag{5}
\]

The metric takes the form

\[
ds^2 = g_{00}(r) dt^2 + g_{xx}(r) dx_j dx^j + g_{rr}(r) dr^2, \tag{6}
\]

where \( j = 1, 2...p \), and \( t \) is the time coordinate. If this metric is presumed to be dual to a strongly coupled gauge theory in \( p + 1 \) dimensions, such a theory would live on the boundary at \( r \rightarrow \infty \). The position of a horizon is assumed at \( r = r_0 \), and the metric components are assumed to behave in the standard way near a black brane horizon, namely that

\[
g_{00}(r) \approx -\gamma_0(r - r_0) + \mathcal{O}(r - r_0)^2 \tag{7}
\]

\[
g_{rr}(r) \approx \frac{\gamma_r}{r - r_0} + \mathcal{O}(1) \tag{8}
\]

\[
g_{xx}(r) \approx g_{xx}(r_0) + \mathcal{O}(r - r_0). \tag{9}
\]

The quantities \( \gamma_0, \gamma_r \) and \( g_{xx}(r_0) \) are independent of \( r \), though they may depend on \( r_0 \). The Hawking temperature of this metric is

\[
T = \frac{1}{4\pi} \sqrt{\frac{\gamma_0}{\gamma_r}}. \tag{10}
\]

For future convenience, we define the function

\[
F(r) \equiv -g_{00}(r) g^{xx}(r), \tag{11}
\]

and we use the following notation for the logarithmic derivative

\[
\mathcal{D}_L[X(r)] \equiv \frac{X'(r)}{X(r)}. \tag{12}
\]

The prime denotes derivatives with respect to \( r \) unless otherwise noted. Throughout this work, our general relativistic conventions are those of [36].
2.2 Restrictions on the metric

The sound mode gauge invariant equations for the type of backgrounds mentioned above were derived in \[17\] in much more generality than is necessary for this work. Instead, we will work with a special class of backgrounds; let us assume that the metric is generated by a single scalar field, and that the metric components satisfy the following constraints

\[
g_{rr}(r) = c_1 g_{xx}(r)^{p+1} F(r) D_L [F(r)]^2, \quad (13)
\]

\[
F(r) = 1 - \left( \frac{g_{xx}(r_0)}{g_{xx}(r)} \right)^{c_2}. \quad (14)
\]

Here \(c_1, c_2\) are constants.\(^1\) The first of these constraints is a consequence of the fact that the particular combination of Ricci tensor components

\[
R^0_0 - R^2_2 = 0 \quad (15)
\]

for any metric that is generated by \(r\) dependent scalar fields, as shown in \[17\]. The second of these constraints \((14)\) is imposed to simplify the calculations. It should be noted that the Schwarzschild AdS metric and the Dp-Brane metric satisfy both constraints, and the above parametrization conveniently allows us to compute the hydrodynamic dispersion relation for both of these important special cases.

The Hawking temperature for this metric is given by the relation

\[
\frac{1}{(4\pi T)^2} = c_1 g_{xx}(r_0)^p \quad (16)
\]

2.3 Scalar field and potential

Before proceeding with the calculation, it is worthwhile to ask what implications the above choice of metric has for the scalar field and the scalar potential. The combination of background Einstein equations

\[
g^{00} G_{00} - g^{rr} G_{rr} = -8\pi G_{p+2} \left( g^{00} T_{00} - g^{rr} T_{rr} \right) \quad (17)
\]

can be simplified to

\[
\phi'(r)^2 = 2g_{rr} \left( R^0_0(r) - R^r_r(r) \right). \quad (18)
\]

Explicitly computing the right side for the special metric chosen yields a relationship between \(\phi\) and \(g_{xx}\).

\[
\phi(r) = \kappa \log[g_{xx}(r)] + \phi_0 \quad (19)
\]

where \(\phi_0\) is an integration constant, and

\[
\kappa \equiv \pm \sqrt{\frac{p}{2} (p + 1 - 2c_2)}. \quad (20)
\]

\(^1\)This choice of metric is the same as the example worked out in \[17\], with the correspondence \(c_2 = a_0\), and the additional constraint \(a_2 = p - a_0\) has been imposed.
One can determine the form of the potential by considering the following combination of background Einstein equations

\[ g^{00}G_{00} + g^{rr}G_{rr} = -8\pi G_{p+2} \left( g^{00}T_{00} + g^{rr}T_{rr} \right) = U(\phi). \]  

(21)

Explicitly computing the left hand side for our special metric gives

\[ U(\phi(r)) = -\frac{p}{2c_1 c_2} g_{xx}(r_0)^{-2c_2} g_{xx}(r)^{2c_2-p-1} = -\frac{p}{2c_1 c_2} g_{xx}(r_0)^{-2c_2} g_{xx}(r)^{-\frac{2}{p} \kappa^2}. \]  

(22)

Using (19), we find

\[ U(\phi) = -\frac{p}{2c_1 c_2} g_{xx}(r_0)^{-2c_2} \exp \left( -\frac{2}{p} \kappa(\phi - \phi_0) \right). \]  

(23)

Thus, the sort of metrics we are considering are those generated by a potential which contains a single exponential, similar to the Chamblin-Reall backgrounds [37] examined in [38, 39]. Note that since the potential must be independent of temperature, it is required that the constant \( c_2 \), and the combination \( c_1 g_{xx}(r_0)^{2c_2} \) must themselves be independent of \( r_0 \).

### 3 Hydrodynamic fluctuations

One can access the hydrodynamic regime of the dual gauge theory by examining perturbations of the gravitational background, and following the prescription of [15]. Here we briefly review this method

#### 3.1 Method for determining \( w(q) \)

In the case at hand, one should allow for fluctuations \( g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \) and \( \phi \to \phi + \delta \phi \). The space-time dependence of the fields is presumed to be

\[ \delta g_{\mu\nu}(t, z, r) = e^{i(qz-\omega t)} h_{\mu\nu}(r), \]

(24)

\[ \delta \phi(t, z, r) = e^{i(qz-\omega t)} \delta \phi(r). \]

(25)

Here we use the coordinate \( z \) to denote one of the spatial coordinates: \( z \equiv x_p \), and \( \omega \) and \( q \) are the energy and momentum of the perturbation. For the sound mode, in the gauge where \( h_{\mu\nu} = 0 \), the only non-zero fluctuations are [11, 16]: \( h_{00}(r), \frac{1}{p-1} \sum_{i=1}^{p-1} h_{ii}(r), h_{zz}(r), h_{0z}(r), \) and \( \delta \phi(r) \).

Turning on these perturbations, and expanding the background equations of motion to first order in the perturbation leads to a set of linearized equations for the perturbations mentioned above. To proceed, one should take appropriate combinations of the resulting linearized equations and construct equations involving only gauge invariant variables, those which do not transform under the diffeomorphism

\[ h_{\mu\nu} \to h_{\mu\nu} - \nabla_\mu^{(0)} \xi_\nu - \nabla_\nu^{(0)} \xi_\mu, \]

(26)

\[ (\delta \phi) \to (\delta \phi) - \xi_\mu (\partial^\mu \phi) \]

(27)
for any vector $\xi_\mu = \xi_\mu(r) e^{i(qz-ut)}$. (Here, $\nabla^{(0)}_\mu$ is the covariant derivative with respect to the background metric).

Solving the resulting gauge invariant equations perturbatively in the hydrodynamic regime $w, q \ll T$, and imposing an incoming wave boundary condition at the horizon and a Dirichlet boundary condition at the boundary leads to the hydrodynamic dispersion relation $w(q)$. One can then compare this dispersion relation to the expected hydrodynamic form to relate the transport coefficients to the field components in the gravity dual.

### 3.2 First order hydrodynamics for this metric

The gauge invariant equations which need to be solved for this type of metric were derived in [17]. They involve two gauge invariant variables $Z_0$ and $Z_\phi$. The equations are

\[
\frac{g_{rr}}{\sqrt{-g}} \alpha^2 F^2 \partial_r \left[ \sqrt{-g} g^{rr} Z_0 \right] + Z_0 \left( \mathcal{D}_L[F] \mathcal{D}_L[F \alpha] - g_{rr} \left( w^2 g^{00} + q^2 g^{xx} \right) \right) + 2Z_\phi F' \left( \alpha \partial_r \left[ \frac{1}{\alpha} \left( \frac{w^2}{F} - q^2 \right) \right] + \frac{q^2 \mathcal{D}_L[F]}{p \mathcal{D}_L[g_{xx}]} \mathcal{D}_L \left[ \sqrt{-g} g^{rr} \phi' \right] \right) = 0, \tag{28}
\]

\[
\frac{g_{rr}}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} Z_\phi' \right] - Z_\phi g_{rr} \left( q^2 g^{xx} + w^2 g^{00} \right) = 0, \tag{29}
\]

where

\[
\alpha(r) \equiv q^2 \left( (p-1) + \frac{\mathcal{D}_L[g_{00}(r)]}{\mathcal{D}_L[g_{xx}(r)]} \right) - \frac{pw^2}{F(r)}. \tag{30}
\]

Solving these equations perturbatively with appropriate boundary conditions leads to the dispersion relation $w(q)$. Classically, waves can enter the black hole’s horizon, but cannot be emitted from there. The standard way to apply this ‘incoming wave’ boundary condition is to make the ansatz [15, 16]

\[
Z_0(r) = F(r) \frac{iw(q)}{4\pi r} \left( Y_0(r) + qY_1(r) + q^2 Y_2(r) + \ldots \right) \tag{31}
\]

\[
Z_\phi(r) = F(r) \frac{iw(q)}{4\pi r} \left( Y_{\phi 0}(r) + qY_{\phi 1}(r) + q^2 Y_{\phi 2} + \ldots \right) \tag{32}
\]

\[
w(q) = w_1 q + w_2 q^2 + w_3 q^3 + \ldots \tag{33}
\]

The functions $Y$ must be regular at the horizon in order for the incoming wave boundary condition to be satisfied. One then inserts this ansatz into the gauge invariant equations, expands the resulting equation in powers of $q$, and solves for the functions $Y$. Finally, applying Dirichlet boundary conditions at the boundary ($r \to \infty$) gives the dispersion relation.

First we solve for the functions $Y_{\phi i}$. Inserting the above ansatz into (29) and expanding the result in powers of $q$ leads to

\[
\partial_r \left[ \sqrt{-g} g^{rr} Y'_{\phi 0} \right] + q \partial_r \left[ \sqrt{-g} g^{rr} \left( Y'_{\phi 1} - \frac{iw_1}{2\pi T} \mathcal{D}_L[F] Y_{\phi 0} \right) \right] + \mathcal{O}(q^2). \tag{34}
\]
It should be noted that we have omitted some terms which are proportional to \((R_0^2 - R_x^2)\) since they vanish by the background equations of motion.

Solving the equation for \(Y_{\phi 0}\) (and using the fact that \(\sqrt{-gg^{rr}} \propto D_L[F]\) from (13)) yields a solution
\[
Y_{\phi 0}(r) = k_0 + k_1 \log(F(r))
\] (35)
but only the constant term can contribute due to the assumption of regularity at the horizon, and presumed near horizon behavior of the metric (7). Finally the constant must be set to zero by the Dirichlet boundary condition at infinity. Proceeding now to higher orders in \(q\), one finds that the equations always reduce to the same as that for \(Y_0\), and as a result \(Z_{\phi} = 0\) to all orders in \(q\).

Proceeding with the calculation one must now insert the incoming wave ansatz into the remaining gauge invariant equation (28),
\[
\frac{g_{rr}}{\sqrt{-g}} \alpha^2 F^2 \partial_r \left[ \frac{\sqrt{-gg^{rr}}}{\alpha^2 F^2} Z_0' \right] + Z_0 \left[ D_L[F] D_L[F \alpha] - g_{rr} \left( w^2 g^{00} + q^2 g^{xx} \right) \right] = 0,
\] (36)
solve for the \(Y_i\) functions, and impose regularity at the horizon and a Dirichlet boundary condition at \(r \to \infty\). These steps were completed up to \(O(q)\) in [17]. The results for the type of metric we consider here are summarized below.

\[
Y_0(r) = y_0 \left( F(r) - 1 \right), \quad (37)
\]
\[
Y_1(r) = y_1 \left( F(r) - 1 \right), \quad (38)
\]
\[
w_1 = \pm \sqrt{\frac{2c_2}{p} - 1}, \quad (39)
\]
\[
w_2 = \frac{-i(p - c_2)}{2\pi T p}. \quad (40)
\]

Here \(y_0\) and \(y_1\) are constants.

Comparing these results to the dispersion relation expected from first order hydrodynamics
\[
w_1 = \pm v_s, \quad (41)
\]
\[
w_2 = -i \eta \left( \frac{p - 1}{p} + \frac{\zeta}{2\eta} \right), \quad (42)
\]
one gains knowledge of \(v_s\) (speed of sound), \(\eta\) (shear viscosity), and \(\zeta\) (bulk viscosity).

\[
\eta/s = 1/4\pi, \quad (43)
\]
\[
v_s = \sqrt{\frac{2c_2}{p} - 1}, \quad (44)
\]
\[
\zeta/\eta = 2 \left( \frac{1}{p} - v_s^2 \right). \quad (45)
\]
Here \( s = (\epsilon + P)/T \) is the entropy density, \( \epsilon \) is the equilibrium energy density, and \( P \) is the equilibrium pressure. Note that the conjectured bulk viscosity bound of Buchel \[40\] is saturated for metrics of the type we consider. It was also shown in \[17\] that the above results agree with previous calculations for the Schwarzschild AdS metric (with the choice \( c_2 = (p+1)/2 \)). These results also agree with the calculation of \[16\] for the Dp-Brane metric with the choice \( c_2 = p(7-p)/(9-p) \).

4 Solution for \( w_3 \)

It is the purpose of this work to extend the above calculation to the next hydrodynamic order, and thus to determine the next coefficient in the dispersion relation \( w_3 \).

4.1 Equation for \( Y_2 \)

We have already shown that \( Z_\phi \) vanishes to all orders in \( q \), so then it remains to return to \( (36) \), insert the incoming wave ansatz, expand in powers of \( q \), and insert the solutions \( (37 - 40) \). Completing these steps, one finds the following differential equation which must be solved for \( Y_2 \):

\[
\partial_r \left[ \frac{Y''_2}{D_L[F](1+F)^2} \right] + \frac{F'}{(1+F)^2} \left\{ Y_2 + \frac{y_0}{(4\pi T)^2} \left[ x_0 + \frac{1-F^2}{F} \left( w_1^2 + \frac{g_{xx}(r)}{g_{xx}(r_0)} \left( F - w_1^2 \right) \right) \right] \right\} = 0. \quad (46)
\]

where

\[
x_0 \equiv 2 \left( 5w_1^2 - 1 - \frac{2w_1w_3(4\pi T)^2}{1-w_1^2} \right). \quad (47)
\]

In writing the above expression, we have replaced all occurrences of the constant \( c_2 \) which appears in the metric with \( w_1 \) due to the relation \( (39) \), and have removed the constant \( c_1 \) in favor of \( T \) using \( (16) \).

4.2 Solution for \( Y_2 \)

4.2.1 Solution to the associated homogeneous equation

The associated homogeneous equation for \( (46) \) is the same as the homogeneous part of the equation for \( Y_1 \) which was considered in \[17\]. The solution can easily be found by first making the ansatz \( Y_2(r) = y_{2a}(1 - F(r)) \), and then using the technique of reduction of order once this solution is found. The general solution to the homogeneous part of the above equation is

\[
Y_{2h}(r) = y_{2a}(1 - F(r)) + y_{2b} \left[ (1 - F(r)) \log[F(r)] + 4 \right]. \quad (48)
\]

Here the subscript \( h \) stands for homogeneous, and \( y_{2a}, y_{2b} \) are arbitrary constants.
4.2.2 Particular solution

In order to find the general solution for $Y_2$, we must now find a particular solution to the inhomogeneous equation. Since the homogeneous solution is known, one can construct a particular solution using the method of variation of parameters. For completeness, this method is outlined in Appendix A. Applying the method to the case at hand gives the following solution (details are given in Appendix B).

$$\hat{Q}(r) = \int (1 + F(\tilde{r}))^{-2} F'(\tilde{r}) \left[ w_1^2 + g_{xx}(\tilde{r})^{p} (F(\tilde{r}) - w_1^2) \right] d\tilde{r},$$

(50)

Thus, the general solution for $Y_2$ is given by

$$Y_2(r) = Y_{2h}(r) + Y_{2p}(r).$$

(51)

To proceed, it is convenient to change coordinates. We define the coordinate $u$ by

$$u^2 \equiv \left( \frac{g_{xx}(r_0)}{g_{xx}(r)} \right)^{c_2} = \left( \frac{g_{xx}(r_0)}{g_{xx}(r)} \right)^{\frac{p}{2}(w_1^2+1)},$$

(52)

so that

$$F(u) = 1 - u^2,$$

(53)

and the horizon is now located at $u = 1$, and the boundary is located at $u = 0$. In terms of the new coordinates, $Y_2$ is given by

$$Y_{2h}(u) = y_{2a} u^2 + y_{2b} \left[ u^2 \log[1 - u^2] + 4 \right],$$

(54)

$$Y_{2p}(u) = -\frac{y_0}{(4\pi T)^2} \left\{ x_0 + 4 u^2 \int (2 - u^2)^2 \int u v^5 \left[ w_1^2 + v^{-4/(1+w_1^2)} (1 - v^2 - w_1^2) \right] \frac{d\tilde{v} dw}{(2 - v^2)^2(1 - v^2)} \right\}$$

(55)

The inner integral can be written in terms of Hypergeometric functions, but the result is not particularly enlightening, so we do not reproduce it here. The full analytical form of this function is not needed to determine the dispersion relation.

4.3 Boundary conditions

4.3.1 Regularity at horizon

The first boundary condition which must be applied on $Y_2$ is the condition of regularity at the horizon. In order to do so, one must extract the coefficient of the logarithmic divergence

\footnote{Here the assumption is that $c_2 > 0$, and that $g_{xx}(r \to \infty) \sim r^n$ with $n > 0$. These assumptions hold for the Schwarzschild AdS metric for any positive $p$, and for the Dp-Brane metric provided $p < 7$.}
in the particular solution \(Y_{2p}\). To do so, we first expand the integrand in powers of \((u - 1)\), and look for the coefficient of the \((u - 1)^{-1}\) term. After integration, this term will lead to the logarithmic divergence.

With the aid of Mathematica, we find the nested integral can be expanded near the horizon as

\[
\int_0^u v^5 \left[ w_1^2 + v^{-4/(1+w_1^2)} (1 - v^2 - w_1^2) \right] dv 
\approx \frac{1}{2} \left\{ \frac{1 - 3w_1^2}{1 - w_1^2} - w_1^2 \left[ 2 + i\pi - H_n \left( \frac{-2}{1 + w_1^2} \right) \right] \right\} + \mathcal{O}(u - 1)
\]  

(56)

where \(H_n(\alpha)\) is the ‘Harmonic Number’ defined as

\[
H_n(\alpha) \equiv \int_0^1 \frac{1 - x^\alpha}{1 - x} dx.
\]

(57)

Using this expansion in \(Y_{2p}\), we find that near the horizon,

\[
Y_{2p}(u \to 1) \approx -\frac{y_0}{(4\pi T)^2} \left\{ x_0 + 4 \int \frac{du}{4(1-u)} \left[ \frac{1 - 3w_1^2}{1 - w_1^2} - w_1^2 \left[ 2 + i\pi - H_n \left( \frac{-2}{1 + w_1^2} \right) \right] \right] \right\} + \mathcal{O}(1).
\]

(58)

Going back to the general solution for \(Y_2\), one thus finds

\[
Y_2(u \to 1) \approx \log[1 - u] \left\{ y_{2b} + \frac{y_0}{(4\pi T)^2} \left[ \frac{1 - 3w_1^2}{1 - w_1^2} - w_1^2 \left[ 2 + i\pi - H_n \left( \frac{-2}{1 + w_1^2} \right) \right] \right] \right\} + \mathcal{O}(1).
\]

(59)

The requirement of regularity at \(u = 1\) thus gives

\[
y_{2b} = \frac{y_0}{(4\pi T)^2} \left[ w_1^2 \left( 2 + i\pi - H_n \left( \frac{-2}{1 + w_1^2} \right) \right) - \frac{1 - 3w_1^2}{1 - w_1^2} \right].
\]

(60)

For future convenience, we make use of the identity

\[
H_n(\alpha) = H_n(\alpha + 1) - \frac{1}{1 + \alpha}
\]

(61)

to write

\[
H_n \left( \frac{-2}{1 + w_1^2} \right) = H_n \left( \frac{2w_1^2}{1 + w_1^2} \right) - \frac{1 - 2w_1^2 - 3w_1^4}{2w_1^2(1 - w_1^2)}.
\]

(62)

Substituting this into the equation for \(y_{2b}\) we find

\[
y_{2b} = \frac{y_0}{(4\pi T)^2} \left[ w_1^2 \left( 2 + i\pi - H_n \left( \frac{2w_1^2}{1 + w_1^2} \right) \right) - \frac{1}{2} (1 - 3w_1^2) \right].
\]

(63)
4.3.2 Dirichlet boundary condition at $u = 0$

Finally, we must apply the Dirichlet boundary condition at $u = 0$. We proceed as above, by expanding the integrand of $Y_2$ near $u = 0$. Mathematica gives

$$\int v^5 \left[ w_1^2 + v^{-4/(1+w_1^2)} (1-v^2 - w_1^2) \right] dv \approx -\frac{w_1^2}{2} (2 + i\pi) + O(u),$$

so that

$$Y_2(u \to 0) \approx \left( 4y_{2b} + O(u) \right) - \frac{y_0}{(4\pi T)^2} \left[ x_0 - 8u^2w_1^2(2 + i\pi) \int \left[ \frac{1}{w_1^3} + O(u^{-2}) \right] du \right].$$

Doing the integral, one finally has

$$Y_2(u \to 0) \approx 4y_{2b} - \frac{y_0}{(4\pi T)^2} \left[ x_0 + 4w_1^2 (2 + i\pi) \right] + O(u),$$

and applying $Y_2(u \to 0) = 0$ gives

$$4y_{2b} = \frac{y_0}{(4\pi T)^2} \left[ x_0 + 4w_1^2 (2 + i\pi) \right],$$

Using (63),

$$x_0 = -4 \left[ w_1^2 H_n \left( \frac{2w_1^2}{1 + w_1^2} \right) + \frac{1}{2} (1 - 3w_1^2) \right].$$

Finally, (67) allows us to solve for $w_3$. The result is

$$w_3 = \frac{w_1(1 - w_1^2)}{(4\pi T)^2} \left[ 1 + H_n \left( \frac{2w_1^2}{1 + w_1^2} \right) \right].$$

This is our main result. To summarize, we have computed the coefficient of the $q^3$ term in the sound mode hydrodynamic dispersion relation for a specific class of metrics (see (14)). This class of metrics contains two constants which we denote $c_1$ and $c_2$. Our expression for $w_3$ should necessarily depend on these constants, but we have eliminated $c_1$ in favor of $T$ using (16) and $c_2$ in favor of $w_1^2 = v_s^2$ due to the relation (44).

5 Transport coefficients in Israel-Stewart theory

Now that we have computed the dispersion relation to $O(q^3)$ in the sound mode, we are in a position to examine the implications for second order hydrodynamic transport coefficients. Second order hydrodynamics attempts to repair some technical problems regarding causality in the first order theory. This subject was first broached by Müller [41], and later by Israel and Stewart [35]. Recently, the formulation of second order hydrodynamics presented in [33] has gained popularity, though at present it is only applicable to conformal theories.
should be noted that recently, some progress has been made in generalizing the work of [33] to non-conformal theories [42]). The metrics we consider are not necessarily conformal, and thus we will use the Israel-Stewart formulation.

Israel introduced five new transport coefficients that appear in the hydrodynamic expansion of the energy momentum tensor. In what follows, we use the same notations and conventions as [32]. Three of these five transport coefficients are relaxation times associated with the diffusive, shear, and sound mode, and are denoted by $(\tau_J, \tau_\pi, \tau_\Pi)$ respectively. There are two other transport coefficients which are related to coupling between the different modes, $\alpha_0, \alpha_1$.

In [32], the sound mode dispersion relation was computed in terms of these transport coefficients.

\begin{align*}
  w_1 &= \pm v_s \\
  w_2 &= \pm \frac{i}{Ts} \left( \frac{p-1}{p} \eta + \frac{\zeta}{2} \right) \\
  w_3 &= \pm \frac{\eta}{2v_s Ts} \left[ \frac{p-1}{p} \left( 2v_s^2 \tau_\pi - \left( 1 - \frac{1}{p} \right) \frac{\eta}{Ts} \right) + \frac{\zeta}{\eta} \left( v_s^2 \tau_\Pi - \left( 1 - \frac{1}{p} \right) \frac{\eta}{Ts} - \frac{\zeta}{4Ts} \right) \right]
\end{align*}

These relations were derived within the context of a ‘decoupled ansatz’ which presumes the background contains no R-charge. The cases which we will consider below (Schwarzschild AdS and Dp-Brane) fit this criteria. Our backgrounds are generated by scalar fields only, and thus any gauge field necessary to provide R-charge is absent from the cases we consider here. A more complete list of assumptions regarding this dispersion relation can be found in [32].

Comparing (72) to our main result (69), and eliminating $\eta$ and $\zeta$ from the relations (43, 45) gives the relation

\begin{equation}
\tau_\pi + \eta \frac{(1-pv^2_s)}{(p-1) \tau_\Pi} - \left( 1 - \frac{1}{p} \right) \eta \left( v_s^2 \tau_\Pi - \left( 1 - \frac{1}{p} \right) \eta - \frac{\zeta}{4Ts} \right) = 0
\end{equation}

As expected, the coefficients $\tau_\pi$ and $\tau_\Pi$ cannot in general be determined separately using this method. Still, if one of these coefficients is known, the above relation allows us to determine the other.

Let us now explicitly check that our results agree with other calculations in the case of a conformal background. The Schwarzschild AdS black hole metric in the near horizon limit takes the form

\begin{align*}
  ds^2 &= \frac{r^2}{L^2} \left[ -F(r) dt^2 + dx_j dx^j \right] + \frac{L^2 dr^2}{r^2 F(r)} \\
  F(r) &= 1 - \left( \frac{r_0}{r} \right)^{p+1}
\end{align*}
where $L$ is the radius of curvature of the $AdS$ space. For this metric, the parameter $c_2 = (p + 1)/2$ and thus $v_s^2 = 1/p$. In this case, (73) gives

$$\tau_{\pi}^{AdS} = \frac{1}{4\pi T} \left[ \frac{p+1}{2} + H_n \left( \frac{2}{p+1} \right) \right],$$

(76)

which is in agreement with [26], [27]. This is a non-trivial check on our calculation; the cited results were arrived at by completely different methods than those we employ here. Furthermore, it should be noted that (76) confirms a conjecture made by Natsuume in [22].

Finally, we can also consider the case of the Dp-Branes. In the Einstein frame, the metric can be reduced to [18, 16]

$$ds^2 = \left( \frac{r}{L} \right)^{\frac{9-p}{p}} \left[ -F(r) dt^2 + dx_j dx^j \right] + \left( \frac{r}{L} \right)^{\frac{p^2-8p+9}{p}} \frac{dr^2}{F(r)},$$

(77)

$$F(r) = 1 - \left( \frac{r_0}{r} \right)^{7-p}.$$ 

(78)

for this metric, the parameter $c_2 = (7p - p^2)/(9 - p)$, and $v_s^2 = (5 - p)/(9 - p)$ [16]. Inserting this into (73) one finds

$$(9 - p)(1 - p)\tau_{\pi}^{Dp} = (3 - p)^2 \tau_{\pi}^{Dp} - \frac{p}{\pi T} \left( \frac{7 - p}{5 - p} \right) \left( 1 + \frac{5 - p}{5 - p} H_n \left( \frac{5 - p}{5 - p} \right) \right).$$

(79)

This formula agrees with previous computations for $p = 1$ and $p = 4$ in [32].

6 Comparison with the shear mode

Though we have exclusively dealt with the sound mode in this work, we now compare our results to similar ones from the shear mode. Let us define the shear mode dispersion relation in the same way as [32],

$$w(q)_{\text{shear}} = -iD_\eta q^2 - iD_\eta^2 \tau_{\text{shear}} q^4 + O(q^6)$$

(80)

where

$$D_\eta = \frac{\eta}{T_s} = \frac{1}{4\pi T}. $$

(81)

A formula for $\tau_{\text{shear}}$ was given in [21] which is applicable to a wide variety of metrics, including the special metrics we have considered in this note. It states that

$$\tau_{\text{shear}} = \frac{\sqrt{-g(r_0)}}{\sqrt{-g_{00}(r_0)g_{rr}(r_0)}} \int_{r_0}^\infty dr \frac{g_{rr}(r)}{\sqrt{-g(r)}} \left[ 1 - \left( \frac{D(r)}{D(r_0)} \right)^2 \right]$$

(82)

3In comparing with the results of [27], one needs to employ the identity (61) to see the agreement

4To be precise, the conjecture is confirmed for the case of $p \geq 2$; the case of $p = 1$ should probably be checked separately as the derivation of the gauge invariant equations in [10], [17] rely on at least 2 spatial dimensions. See [43], where first order hydrodynamics is examined for $p = 1$
where
\[ D(r) \equiv \frac{\sqrt{-g(r)}}{\sqrt{-g_0(r)g_{rr}(r)}} \int_r^\infty dr' \frac{-g_{00}(r')g_{rr}(r')}{\sqrt{-g(r')g_{xx}(r')}}. \] (83)

Using the special metric (13-14), and the relationship (16) in this formula yields
\[ \tau_{\text{shear}} = \frac{1}{4\pi T} H_n \left( 2 - \frac{p}{c_s^2} \right) = \frac{1}{4\pi T} H_n \left( \frac{2w_1^2}{1 + w_1^2} \right). \] (84)

This result agrees with the special cases of the Dp-Brane and the Schwarzschild AdS metrics as computed in [22].

Previously, it was thought that \( \tau_{\text{shear}} = \tau_\pi \), but recently the authors of [33] showed that this is not the case because the coefficient of the \( q^4 \) term contains not only \( \tau_\pi \), but also contributions from (currently unformulated) third order hydrodynamics. We can now determine these unknown contributions for metrics which obey (13,14).

Let us parametrize
\[ \tau_{\text{shear}} = \tau_\pi + \Delta \] (85)
where \( \Delta \) denotes the unknown contributions from third order hydrodynamics. Combining (84) and (73) allows one to solve for \( \Delta \). Evidently,
\[ \Delta = \left( \frac{pv_s^2 - 1}{p(1 - v_s^2)} \right)(\tau_\pi - \tau_\Pi) - \frac{1 + v_s^2}{8\pi T v_s^2}. \] (86)

The correction \( \Delta \) does not appear to be universal in the sense that the first term is not present in the case of a conformal theory. In the conformal case, \( \Delta \) is still not ‘universal’, because it depends on the number of dimensions of the theory. In the future, when the particular transport coefficients which comprise \( \Delta \) are known, it will be interesting to see whether there is any universal relationship between these unknown coefficients, \( \tau_\pi \), and \( \tau_\Pi \).

We can easily check that the formula (86) reproduces the results in the well known SAdS5 metric, which is dual to \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory at finite temperature. In this case, \( p = 3 \) and \( v_s^2 = 1/3 \). Immediately, we have
\[ \Delta^{SAdS5} = -\frac{1}{2\pi T}. \] (87)

In [32, 33], it was found that⁵
\[ \tau_{\text{shear}}^{SAdS5} = \frac{1 - \log(2)}{2\pi T}, \] (88)
\[ \tau_\pi^{SAdS5} = \frac{2 - \log(2)}{2\pi T}. \] (89)

It is clear that the relaxation time computed from the shear mode and the sound mode differ by the amount predicted by the formula (86); our results are in agreement with [32, 33].

⁵Even though the formalism of Baier et al. [32] is different than Israel-Stewart, one can check that the sound mode dispersion relations coincide in the limit of conformal theories (\( \zeta \to 0 \)). In an unfortunate clash of notations, the relaxation time introduced by Baier et al. is denoted by \( \tau_\Pi \), its Israel-Stewart counterpart is \( \tau_\pi \).
7 Conclusion and outlook

In this work, we have extended the calculation of \cite{17} to the next hydrodynamic order in $q$. The main result (69) is the coefficient of the $q^3$ term in the sound mode dispersion relation. This result is applicable to metrics which obey (13 - 14). These metrics are not necessarily conformal, and contain an arbitrary number of spatial dimensions $p$ with $p > 2$.

Information about second order transport coefficients was presented within the context of the Israel-Stewart theory (specifically within the formulation presented in \cite{32}). In general, a relationship (73) between two transport coefficients $\tau_\pi$ and $\tau_\Pi$ can be determined.

In the conformal case of the Schwarzschild AdS metric, the relation mentioned above allows the determination of the coefficient $\tau_\pi$. We have verified that our results agree with those calculated from different methods.

Finally, by comparing the sound mode dispersion relation to the shear mode discussed in \cite{21, 22}, we were able to determine the contribution of third order hydrodynamics to the shear mode (it was pointed out that such contributions would be present in \cite{33}).

As mentioned in the introduction, it is desirable to find other universal relations among transport coefficients, as such relations sometimes lead to observable consequences. The main results of this paper (69), (73) and the third order hydrodynamic contributions to the shear mode (86) are applicable to certain gravity duals which may or may not be conformal. However, despite the fact that these relations appear to be applicable to many theories, they do not seem to be universal in the same way as $\eta/s = 1/4\pi$, and the relation presented in \cite{34}. For example, the number of spatial dimensions $p$ enters explicitly into our formulas, whereas a universal relation should not depend on this quantity.

Furthermore, we stress that the class of theories examined herein is quite limited. It seems likely that any generalization will explicitly contain the bulk viscosity $\zeta$, which we were able to eliminate due to the relation (45).

Of course, it would be interesting to re-examine the results here for a broader class of gravity duals. Unfortunately, the gauge invariant equations derived in \cite{17} are difficult to solve analytically except in the special case presented here, though perhaps insight could be gained by approaching the problem numerically. It would also be useful to attempt to generalize the results of \cite{17} by including other matter fields in addition to the scalar fields considered there. It seems unlikely (though perhaps not impossible) that similar special cases would be analytically solvable after the addition of different kinds of matter. These are issues which should be investigated in the future.

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Appendices

A  The method of variation of parameters

One fundamental technique used in generating particular solutions to inhomogeneous differential equations is the method of ‘Variation of Parameters’. The method can be used to find a general solution to a second order linear differential equation if the solution to the associated homogeneous equation is known.

The theory behind the method can be found in any textbook on differential equations. Here, we simply present the essential formulas. Consider a differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x), \quad (90)$$

and assume that the functions $y_1(x)$ and $y_2(x)$ are linearly independent, and satisfy the associated homogeneous equation. That is,

$$y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) = 0, \quad (91)$$

and similarly for $y_2(x)$. It can be shown that the function $y_p(x)$ is a solution to the inhomogeneous equation, where

$$y_p(x) = y_2(x) \int \frac{y_1(x)g(x)}{W(x)} \, dx - y_1(x) \int \frac{y_2(x)g(x)}{W(x)} \, dx. \quad (92)$$

Here, $W(x)$ is the Wronskian

$$W(x) \equiv y_1(x)y_2'(x) - y_1'(x)y_2(x). \quad (93)$$

To arrive at the form of the function $y_p$ used in the text $[\text{50}]$, let us define

$$h(x) \equiv \frac{y_2(x)}{y_1(x)}. \quad (94)$$

Then,

$$W(x) = h'(x)y_1^2(x), \quad (95)$$

and

$$y_p(x) = y_1(x) \left[ h(x) \int \frac{g(x)}{y_1(x)h'(x)} \, dx - \int \frac{h(x)g(x)}{h'(x)y_1(x)} \right]. \quad (96)$$

One can see that this is equivalent to

$$y_p(x) = y_1(x) \int h'(x) \int x \frac{g(z)}{y_1(z)h'(z)} \, dz \, dx \quad (97)$$

by performing an integration by parts.
Particular solution for $Y_2(r)$

One can now apply the method detailed in Appendix A to the differential equation for $Y_2$. Using the notation from Appendix A (and changing independent variables from $x$ to $r$), we have

\begin{align*}
y_1(r) &= 1 - F(r), \\
h(r) &= \log[F(r)] + \frac{4}{1 - F(r)}, \\
g(r) &= -\frac{(F')^2}{F(1+F)} \frac{y_0}{(4\pi T)^2} \left[ x_0 + \frac{1 - F^2}{F} \left( w_1^2 + \frac{g_{xx}(r)^p}{g_{xx}(r_0)^p} (F - w_1^2) \right) \right].
\end{align*}  

(98, 99, 100)

After a bit of work, one can find the particular solution

\begin{align*}
Y_{2p}(r) &= -\frac{y_0}{(4\pi T)^2} (1 - F(r)) \left[ \int \frac{1 + F(r)}{1 - F(r)} \left( \frac{F'(r)}{F(r)} \right)^2 Q(r) \, dr \right] \\
Q(r) &= \int^r \frac{1 - F(z)}{(1 + F(z))^3} F'(z) \left[ x_0 + \frac{1 - F(z)^2}{F(z)} \left( w_1^2 + \frac{g_{xx}(z)^p}{g_{xx}(r_0)^p} (F(z) - w_1^2) \right) \right] \, dz.
\end{align*}  

(101, 102)

The term involving $x_0$ can be integrated directly, by changing variables from $z$ to $F$. This simplifies the solution to

\begin{align*}
Y_{2p}(r) &= -\frac{x_0 y_0}{(4\pi T)^2} + \frac{y_0}{(4\pi T)^2} (F(r) - 1) \left[ \int \frac{1 + F(r)}{1 - F(r)} \left( \frac{F'(r)}{F(r)} \right)^2 \hat{Q}(r) \, dr \right] \\
\hat{Q}(r) &= \int^r \left( \frac{1 - F(z)}{1 + F(z)} \right)^2 \frac{F'(z)}{F(z)} \left[ w_1^2 + \frac{g_{xx}(z)^p}{g_{xx}(r_0)^p} (F(z) - w_1^2) \right] \, dz,
\end{align*}  

(103, 104)

which is the form presented in the text.