A FRAÎSSÉ THEORETIC APPROACH TO THE JIANG–SU ALGEBRA

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Abstract. In this paper, we give a Fraïssé theoretic proof of the result of X. Jiang and H. Su that the Jiang–Su algebra is the unique monotracial simple C*-algebra among all inductive limits of prime dimension drop algebras. The proof presented here is self-contained and quite elementary, and does not depend on any K-theoretic technology. We also partially recover the fact that every unital endomorphism of the Jiang–Su algebra is approximately inner.

1. Introduction

The Jiang–Su algebra was originally introduced by Jiang and Su in [JS99] as a C*-algebraic analog of the hyperfinite type II₁-factor. This algebra is characterized as the unique simple monotracial C*-algebra among all inductive limits of prime dimension drop algebras (i.e. certain algebras of matrix-valued continuous functions on the closed interval [0, 1]). In addition to being simple and having unique tracial state, it is separable, nuclear and infinite-dimensional, and have the same K-theory as the complex numbers \( \mathbb{C} \). Furthermore, every unital endomorphism of this algebra is approximately inner, and it tensorially absorbs itself, i.e. \( \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z} \).

Because of these properties, it plays a central role in the Elliott’s classification program of separable nuclear C*-algebras via K-theoretic invariants [ET08].

The degree of difficulty in proving the properties of the Jiang–Su algebra varies from one to another. It is immediate from the construction that the Jiang–Su algebra is a unital simple separable nuclear infinite-dimensional C*-algebra with a unique tracial state and has the same K-theory as \( \mathbb{C} \). Compared with that, the proofs of the other properties that were presented in [JS99] are relatively difficult. For example, the uniqueness result of the Jiang–Su algebra among the inductive limits of prime dimension drop algebras is implied as a corollary of the complete classification of the unital infinite-dimensional simple inductive limits of direct sums of dimension drop algebras via K-groups and trace spaces [JS99, Theorem 6.2]. The construction of isomorphisms in the classification result is carried over by the standard approximate intertwining argument, and for this, one has to find an appropriate sequence of morphisms between direct sums of dimension drop algebras which is convergent to the isomorphism between the limits. This sequence of morphisms are obtained by studying the KK-theory of dimension drop algebras and observing how to lift a morphism from a KK-element. Such an observation is also used to prove that every unital endomorphism of the Jiang–Su algebra is approximately inner.

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It is true that, since KK-theory is a powerful tool, calculating KK-groups of dimension drop algebras is significant, and that the classification theorem is fairly interesting on its own right. On the other hand, it would be natural to ask whether there are more elementary proofs of these properties of the Jiang–Su algebra, taking the fundamental importance of the algebra into account.

In this paper, we give an alternative proof that a simple monotracial inductive limit of prime dimension drop algebras is unique up to isomorphism and every unital endomorphism of this inductive limit is approximately inner. The proof does not rely on KK-theory, but instead it uses Fraïssé theory.

Fraïssé theory is a topic in model theory where a bijective correspondence between certain classes consisting of finitely generated structures and countable structures with a certain homogeneity property is established. By definition, a countable structure is said to be ultra-homogeneous if every isomorphism between its two finitely generated substructures extends to an automorphism. The class corresponding to an ultra-homogeneous structure $M$ in the context of Fraïssé theory is $\text{Age} M$, the class of all finitely generated structures which are embeddable into $M$. Fraïssé’s theorem characterizes the classes obtained in this way, and produces a method to recover the original ultra-homogeneous structures. Namely, if $\{\iota_{n,m}: A_n \to A_m\}$ and $\{\eta_{n,m}: B_m \to B_n\}$ are generic inductive systems of members of such a class, then, passing to subsystems if necessary, one can find sequences $\{\varphi_n: A_n \to B_n\}$ and $\{\psi_n: B_n \to A_{n+1}\}$ of embeddings such that the triangles in the following diagram commute.

Consequently, the sequences $\{\varphi_n\}$ and $\{\psi_n\}$ provide embeddings between the inductive limits which are inverses of each other and so are isomorphisms. In fact, the resulting inductive limits are isomorphic to the original ultra-homogeneous structure. The class and the ultra-homogeneous structure in this context are called a Fraïssé class and its Fraïssé limit respectively.

Like other topics in model theory, this theory has been a target of generalization to the setting of metric structures ([Sch07], [Ben15]). In [Eag16], a variant of the theory presented in [Ben15] was used to recognize a part of AF algebras including the UHF algebras, the hyperfinite type II$_1$ factor, and the Jiang–Su algebra as generic limits of suitable Fraïssé classes. The author also gave an alternative proof of the same result on the Jiang–Su algebra as [Eag16], and realized the UHF algebras as Fraïssé limit in a different way [Mas16a].

The idea in [Eag16] and [Mas16a] of realizing the Jiang–Su algebra as a Fraïssé limit is essentially coming from the observation that the approximate intertwining argument, which is used to construct an isomorphism in the classification result for inductive limits of direct sums of dimension drop algebras, is a variant of the back-and-forth argument. That is, in order to construct suitable sequences of embeddings

\[
\begin{array}{cccccc}
A_1 & \xrightarrow{\iota_{1,2}} & A_2 & \xrightarrow{\iota_{2,3}} & A_3 & \xrightarrow{\iota_{3,4}} & \cdots \\
\varphi_1 & \quad & \varphi_2 & \quad & \varphi_3 & \quad & \cdots \\
B_1 & \xleftarrow{\eta_{1,2}} & B_2 & \xleftarrow{\eta_{2,3}} & B_3 & \xleftarrow{\eta_{3,4}} & \cdots \\
\psi_1 & \quad & \psi_2 & \quad & \psi_3 & \quad & \cdots \\
\end{array}
\]
between direct sums of dimension drop algebras, it would be suffice to verify that a suitable class of direct sums of dimension drop algebras forms a Fraïssé class, which does not necessarily mean that one needs KK-theory.

Thus, in short, the result on the Jiang–Su algebra in [Eag16] and [Mas16a] seems to be a Fraïssé theoretic expression of the fact that the algebra is the unique simple monotracial inductive limits of prime dimension drop algebras. Therefore, this fact should be recovered from the Fraïssé limit construction, and that is our strategy in this paper.

This paper consists of three sections. In the next section we briefly give a description of Fraïssé theory for metric structures. The Fraïssé limit construction of the Jiang–Su algebra is analyzed in the last section.

2. Approximate isomorphisms and Fraïssé limits

In this section, we briefly sketch the theory of Fraïssé limits used in [Eag16] and [Mas16a]. It is a slight modified version of the theory presented in [Ben15], and detailed proofs can be found in [Mas16b].

By definition, a language is a set \( L \) such that each element is exactly one of the following symbols:

- constant symbols;
- \( n \)-ary function symbols (\( n = 1, 2, 3, \ldots \));
- \( n \)-ary predicate symbols (\( n = 1, 2, 3, \ldots \)).

A metric \( L \)-structure \( \mathfrak{A} \) consists of a complete metric space \( A \), which is called the domain of \( \mathfrak{A} \) and denoted by \( |\mathfrak{A}| \), together with an interpretation \( S \mapsto S^{\mathfrak{A}} \) of the symbols in \( L \).

- If \( c \) is a constant symbol, then \( c^{\mathfrak{A}} \) is an element in \( A \).
- If \( f \) is an \( n \)-ary function symbol, then \( f^{\mathfrak{A}} \) is a continuous function from \( |\mathfrak{A}|^n \) into \( |\mathfrak{A}| \).
- If \( R \) is an \( n \)-ary predicate symbol, then \( R^{\mathfrak{A}} \) is a continuous \( \mathbb{R} \)-valued function on \( |\mathfrak{A}|^n \).

An isometry \( \iota \) from a metric \( L \)-structure \( \mathfrak{A} \) into another \( L \)-structure \( \mathfrak{B} \) is called an (\( L \))-embedding if

\[
\iota \left( f^{\mathfrak{A}}(a_1, \ldots, a_n) \right) = \iota \left( f^{\mathfrak{B}}(a_1, \ldots, a_n) \right) \quad (a_1, \ldots, a_n \in |\mathfrak{A}|)
\]

for any \( n \)-ary function symbol \( f \) in \( L \), and

\[
\iota \left( P^{\mathfrak{A}}(a_1, \ldots, a_n) \right) = \iota \left( P^{\mathfrak{B}}(a_1, \ldots, a_n) \right) \quad (a_1, \ldots, a_n \in |\mathfrak{A}|)
\]

for any \( n \)-ary predicate symbol \( P \) in \( L \).

In this paper, we are interested in the language \( L_{\text{TC}} \) of unital tracial C*-algebras, which consists of the following:

- two constant symbols 0 and 1;
- an unary function symbol \( \lambda \) for each \( \lambda \in \mathbb{C} \), which are to be interpreted as multiplication by \( \lambda \);
- an unary function symbol \( * \) for involution;
- a binary function symbol + and \( \cdot \).
• an unary predicate symbol $tr$.

Then every unital C*-algebra with a distinguished trace can be considered as a metric $L_1\text{C}^*$-structure. Note that the distance we adopt is the norm distance, and that a map between unital C*-algebras with fixed traces are $L$-embeddings if and only if it is a trace-preserving injective $*$-homomorphism.

**Definition 2.1.** Let $\mathcal{K}$ be a category of finitely generated separable metric $L$-structures and $L$-embeddings.

1. For $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, we set
   \[ JE_{\mathcal{K}}(\mathcal{A}, \mathcal{B}) := \{(\iota, \eta) \mid \exists C \in \text{Obj}(\mathcal{K}), \iota \in \text{Mor}_{\mathcal{K}}(\mathcal{A}, C) & \eta \in \text{Mor}_{\mathcal{K}}(\mathcal{B}, C)\} \]
   and call each member of $JE_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ a joint $\mathcal{K}$-embedding of $\mathcal{A}$ and $\mathcal{B}$. The category $\mathcal{K}$ is said to satisfy the joint embedding property (JEP) if $JE_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ is nonempty for any objects $\mathcal{A}, \mathcal{B}$ of $\mathcal{K}$.

2. The category $\mathcal{K}$ is said to satisfy the near amalgamation property if for any objects $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{K}$, any morphisms $\iota_i : \mathcal{A}_i \to \mathcal{B}_i$, any finite subset $G \subseteq \mathcal{A}$ and any $\varepsilon > 0$, there exists a joint $\mathcal{K}$-embedding $(\eta_1, \eta_2) \in JE_{\mathcal{K}}(\mathcal{B}_1, \mathcal{B}_2)$ such that the inequality
   \[ d(\eta_1 \circ \iota_1(a), \eta_2 \circ \iota_2(a)) \leq \varepsilon \]
   holds for all $a \in G$.

In the sequel, we fix a category $\mathcal{K}$ of finitely generated separable metric $L$-structures and $L$-embeddings with JEP and NAP.

**Definition 2.2.** (1) Let $\mathcal{A}, \mathcal{B}$ be objects of $\mathcal{K}$ and $\varphi : [\mathcal{A}] \times [\mathcal{B}] \to [0, \infty]$ be a bi-Katětov map, that is, a map satisfying
   \[
   \varphi(a, b) \leq d(a, a') + \varphi(a', b), \quad d(a, a') \leq \varphi(a, b) + \varphi(a', b),
   \]
   \[
   \varphi(a, b) \leq d(b, b') + \varphi(a, b'), \quad d(b, b') \leq \varphi(a, b) + (a, b')
   \]
   for all $a, a' \in [\mathcal{A}]$ and $b, b' \in [\mathcal{B}]$. Then $\varphi$ is called an approximate $\mathcal{K}$-isomorphism from $\mathcal{A}$ to $\mathcal{B}$ if for any finite subsets $A_0 \subseteq [\mathcal{A}]$ and $B_0 \subseteq [\mathcal{B}]$ and any $\varepsilon > 0$, there exists $(\iota, \eta) \in JE_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ such that the inequality
   \[ d(\iota(a), \eta(b)) \leq \varphi(a, b) + \varepsilon \]
   holds for all $a \in A_0$ and $b \in B_0$. We denote by $\text{Apx}_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ the set of all approximate $\mathcal{K}$-isomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

2. A $\mathcal{K}$-structure is a metric $L$-structure $\mathcal{M}$ together with a distinguished inductive system
   \[ \mathcal{A}_1 \xrightarrow{\iota_1} \mathcal{A}_2 \xrightarrow{\iota_2} \mathcal{A}_3 \xrightarrow{\iota_3} \cdots \]
   in $\mathcal{K}$ such that the inductive limit of the system as a metric $L$-structure is $\mathcal{M}$.

3. By definition, an approximate $\mathcal{K}$-isomorphism from a $\mathcal{K}$-structure $\mathcal{M} = \bigsqcup_n \mathcal{A}_n$ to another $\mathcal{K}$-structure $\mathcal{N} = \bigsqcup_m \mathcal{B}_m$ is a bi-Katětov map $\varphi : [\mathcal{M}] \times [\mathcal{N}] \to [0, \infty]$ such that the restriction $\varphi|_{\mathcal{A}_n \times \mathcal{B}_m}$ is in $\text{Apx}_{\mathcal{K}}(\mathcal{A}_n, \mathcal{B}_m)$ for all $n$ and $m$. We denote by $\text{Apx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N})$ the set of all approximate $\mathcal{K}$-isomorphisms from $\mathcal{M}$ to $\mathcal{N}$.
The following two examples of approximate $\mathcal{K}$-isomorphisms are of most importance.

**Example 2.3.** (1) For $L$-embeddings $\iota: \mathcal{A} \rightarrow C$ and $\eta: \mathcal{B} \rightarrow C$, we set
\[
\varphi_{\iota,\eta}(a, b) := d(\iota(a), \eta(b)) \quad (a \in \mathcal{A}, \ b \in \mathcal{B}).
\]
If both $\iota$ and $\eta$ are morphisms of $\mathcal{K}$, then $\varphi_{\iota,\eta}$ is an approximate $\mathcal{K}$-isomorphism between objects of $\mathcal{K}$. It is simply written as $\varphi_\iota$ when $C$ is equal to $\mathcal{B}$ and $\eta$ is the identity map.

Even if $\iota$ and $\eta$ are not morphisms of $\mathcal{K}$, the bi-Katětov map $\varphi_{\iota,\eta}$ can be an approximate $\mathcal{K}$-isomorphism. An $L$-embedding $\iota$ between $\mathcal{K}$-structures is said to be $\mathcal{K}$-admissible if $\varphi_\iota$ is an approximate $\mathcal{K}$-isomorphism.

(2) Let $M_1 = \bigcup_i A_i$, $M_2 = \bigcup_m B_m$ and $M_3 = \bigcup_n C_n$ be $\mathcal{K}$-structures. If $\varphi$ is an approximate $\mathcal{K}$-isomorphism from $M_1$ to $M_2$ and $\psi$ is an approximate $\mathcal{K}$-isomorphism from $M_2$ to $M_3$, then
\[
\psi \varphi(a, c) := \inf_{b \in |M_3|} [\varphi(a, b) + \psi(b, c)] \quad (a \in |M_1|, \ c \in |M_2|)
\]
is an approximate $\mathcal{K}$-isomorphism from $M_1$ to $M_3$. For a proof, see [Mas16b, Proposition 3.4].

Now, for each $n \in \mathbb{N}$, set
\[
\mathcal{K}_n := \{ \langle \mathcal{A}, \bar{a} \rangle \mid \mathcal{A} \in \text{Obj}(\mathcal{K}) \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \&
(2) \( M \) is approximately \( \mathcal{K} \)-homogeneous: For any member \( A \) of \( \mathcal{K} \), any \( \mathcal{K} \)-admissible \( L \)-embeddings \( \iota, \eta: A \to M \), any finite subset \( A_0 \subseteq |A| \) and any \( \epsilon > 0 \), there exists a \( \mathcal{K} \)-admissible \( L \)-automorphism \( \alpha \) of \( M \) such that \( d(\alpha \circ \iota(a), \eta(a)) < \epsilon \) holds for all \( a \in A_0 \).

**Theorem 2.6.** Every Fraïssé category admits a limit. Moreover, its limit is unique up to \( \mathcal{K} \)-admissible isomorphisms.

We conclude this section by characterizing the limit of a Fraïssé category in terms of approximate isomorphisms. For this, we need to introduce the concepts of strictness and totality.

Let \( A \subseteq A' \) and \( B \subseteq B' \) be subspaces of metric spaces and \( \varphi: A \times B \to [0, \infty] \) be a bi-Katětov map. Then the *trivial extension* \( \varphi|_{A' \times B'} \) of \( \varphi \) is the bi-Katětov map on \( A' \times B' \) defined by

\[
\varphi|_{A' \times B'}(a', b') := \inf_{a \in A, b \in B} [d(a', a) + \varphi(a, b) + d(b, b')] \quad (a' \in A', b' \in B').
\]

Note that if \( \psi \) is another bi-Katětov map on \( A' \times B' \), then \( \psi \leq \varphi|_{A' \times B'} \) is equivalent to \( \psi|_{A \times B} \leq \varphi \).

**Definition 2.7.** Let \( \varphi \) be an approximate \( \mathcal{K} \)-isomorphism from \( A \) to \( B \).

(1) The approximate \( \mathcal{K} \)-isomorphism \( \varphi \) is said to be *strict* if there exist another approximate isomorphism \( \psi \) from \( A \) to \( B \), a positive real number \( \epsilon > 0 \) and finite subsets \( A_0 \subseteq |A| \) and \( B_0 \subseteq |B| \) such that the inequality

\[
(\psi|_{A_0 \times B_0})|_{A 	imes B}(a, b) + \epsilon \leq \varphi(a, b)
\]

holds for all \( a \in |A| \) and \( b \in |B| \).

(2) The approximate \( \mathcal{K} \)-isomorphism \( \varphi \) is *\( \epsilon \)-total* on a subset \( A_0 \subseteq |A| \) if the inequality

\[
\varphi \circ \varphi(a, a') \leq d(a, a') + 2\epsilon,
\]

holds for all \( a, a' \in A_0 \), or equivalently,

\[
\inf_{b \in B} \varphi(a, b) \leq \epsilon
\]

for all \( a \in A_0 \).

The following theorem is a weaker version of [Mas16b, Lemma 4.6].

**Theorem 2.8.** Let \( M = \bigcup_n A_n \) be a \( \mathcal{K} \)-structure. Then \( M \) is the Fraïssé limit of \( \mathcal{K} \) if and only if for any object \( B \) of \( \mathcal{K} \), any strict approximate \( \mathcal{K} \)-isomorphism \( \varphi \) from \( B \) to \( A_n \), any finite subset \( B_0 \subseteq |B| \) and any \( \epsilon > 0 \), there exist \( N > n \) and an approximate \( \mathcal{K} \)-isomorphism \( \psi \) from \( B \) into \( A_N \) which is \( \epsilon \)-total on \( B_0 \) and dominated by \( \varphi|_{B \times A_N} \).
3. The Jiang–Su algebra

In this paper, we shall denote by $\mathbb{M}_n$ the $C^*$-algebra of all $n$-by-$n$ complex matrices. For natural numbers $p$, and $q$, the dimension drop algebra $\mathbb{Z}_{p,q}$ is defined as the $C^*$-algebra of all $\mathbb{M}_p \otimes \mathbb{M}_q$-valued continuous functions $f$ on the closed interval $[0, 1]$ such that $f(0)$ and $f(1)$ are contained in $\mathbb{M}_p \otimes 1_q$ and $1_p \otimes \mathbb{M}_q$ respectively. It is said to be prime if $p$ and $q$ are coprime. Note that if $(e_{ij})_{i,j}$ and $(f_{kl})_{i,j}$ are systems of matrix units of $\mathbb{M}_p$ and $\mathbb{M}_q$ respectively, then $(e_{ij} \otimes f_{kl})_{i,j,j,l}$ is a system of matrix units, so $\mathbb{M}_p \otimes \mathbb{M}_q$ is isomorphic to $\mathbb{M}_{pq}$. We denote by $c^p_q$ the map from $\mathbb{M}_p \otimes 1_q$ to $\mathbb{M}_p$ defined by $a \otimes 1 \mapsto a$. The map $c^p_q : 1_p \otimes \mathbb{M}_q \to \mathbb{M}_q$ is defined similarly. When no confusion arises, these maps are simply denoted by $c$. Also, for $t \in [0, 1]$, we denote by $e_t$ the evaluation map at $t$. The following proposition is a trivial modification of [Mas16a, Proposition 3.5].

**Proposition 3.1.** Let $\iota : \mathbb{Z}_{p,q} \rightarrow \mathbb{Z}_{p',q'}$ be a unital $\ast$-homomorphism. Then the following statements hold.

1. There exist integers $a, b$ with $0 \leq a < q$ and $0 \leq b < p$, continuous maps $t_1, \ldots, t_k$ from $[0, 1]$ into $[0, 1]$ and a family $\{v_s\}_{s \in [0,1]}$ of unitary matrices of size $p'q'$ such that $\iota$ is of the form

$$\iota(f)(s) = \text{Ad}(v_s)(\text{diag}[c(f(0)), \ldots, c(f(0)), f(t_1(s)), \ldots, f(t_k(s)), c(f(1)), \ldots, c(f(1))])$$

for $f \in \mathbb{Z}_{p,q}$ and $s \in [0, 1]$, where $\text{Ad}(v)$ denotes the inner automorphism associated to $v$, and $\text{diag}[a_1, \ldots, a_n]$ is the block diagonal matrix with $a_i$ as its $i$-th block.

2. Suppose that $t_1, \ldots, t_k$ are as in (1). Then for any finite $G \subseteq \mathbb{Z}_{p,q}$ and any $\varepsilon > 0$, there exists a continuous path $u : [0, 1] \rightarrow \mathbb{M}_{p',q'}$ of unitaries such that the $\ast$-homomorphism $\iota' : \mathbb{Z}_{p,q} \rightarrow \mathbb{Z}_{p',q'}$ defined by

$$\iota'(f)(s) = \text{Ad}(u(s))(\text{diag}[c(f(0)), \ldots, c(f(0)), f(t_1(s)), \ldots, f(t_k(s)), c(f(1)), \ldots, c(f(1))])$$

satisfies $||u(g) - \iota'(g)|| < \varepsilon$ for all $g \in G$.

The family $t_1, \ldots, t_k$ of continuous maps and the integers $a$ and $b$ that appeared in Proposition 3.1 are called an eigenvalue pattern and the remainder indices of the $\ast$-homomorphism $\iota$. An eigenvalue pattern $t_1, \ldots, t_k$ is said to be normalized if it satisfies the inequality $t_1 \leq \cdots \leq t_k$. Note that the normalized eigenvalue pattern is unique for each $\ast$-homomorphism. Also, if $\mathbb{Z}_{p,q}$ is prime, then the remainder indices depend only on the integers $p$, $q$, $p'$ and $q'$. Indeed, if $\eta$ is another $\ast$-homomorphism from $\mathbb{Z}_{p,q}$ into $\mathbb{Z}_{p',q'}$, and if $a_\eta$ and $b_\eta$ are the remainder indices
of \( \eta \), then the congruence equation
\[
p a + q b \equiv p' q' \equiv p a_\eta + q b_\eta \pmod{p q}
\]
holds, so that
\[
a \equiv a_\eta \pmod{q}, \quad b \equiv b_\eta \pmod{p},
\]
as \( p \) and \( q \) are coprime.

Let \( \iota \) be a \( * \)-homomorphism between dimension drop algebras with an eigenvalue pattern \( t_1, \ldots, t_k \). We shall denote by \( V(t_1, \ldots, t_k) \) the maximum of the diameters of the images of \( t_1, \ldots, t_k \), and call it the variation of the eigenvalue pattern. The infimum of the variations of the eigenvalue patterns is clearly equal to the variation of the normalized eigenvalue pattern, which is called the variation of \( \iota \) and denoted by \( V(\iota) \). The next proposition is a variant of [Mas16a, Proposition 4.4].

**Proposition 3.2.** Let \( p, q \in \N \) be coprime and \( \varepsilon \) be a positive real number. Then there exists \( M \in \N \) such that if \( p', q' \) are larger than \( M \), then there exists a unital embedding of \( \Z_{p,q} \) into \( \Z_{p',q'} \) with its variation less than \( \varepsilon \).

**Proof.** Since \( p \) and \( q \) are coprime, there exists \( M \in \N \) with \( M \geq pq(1/\varepsilon + 2) \) such that if \( p', q' > M \), then
\[
p a + pq k + q b = p' q'
\]
for some \( a, b, k \in \N \). Without loss of generality, we may assume \( 0 \leq a < q \) and \( 0 \leq b < p \). Also, one can find \( l^0, m^0 \in [0, q) \) and \( l^1, m^1 \in [0, p) \) such that
\[
\begin{align*}
p l^0 & \equiv p' \pmod{q}, \quad p m^0 \equiv q' \pmod{q}, \\
q l^1 & \equiv p' \pmod{p}, \quad q m^1 \equiv q' \pmod{p}.
\end{align*}
\]
Then,
\[
\begin{align*}
p q' l^0 & \equiv p p' m^0 \equiv p' q' \equiv p a \pmod{q}, \\
q q' l^1 & \equiv q p' m^1 \equiv p' q' \equiv q b \pmod{p},
\end{align*}
\]
so
\[
q' l^0 \equiv p' m^0 \equiv a \pmod{q}, \quad q' l^1 \equiv p' m^1 \equiv b \pmod{p}.
\]
We set
\[
\begin{align*}
n_0^0 & \equiv \frac{q' l^0 - a}{q}, \quad n_0^1 \equiv \frac{q' l^1 - b}{p}, \\
n_0^0 & \equiv \frac{p' m^0 - a}{q}, \quad n_0^1 \equiv \frac{p' m^1 - b}{p},
\end{align*}
\]
so that
\[
\begin{align*}
a + q n_0^0 & \equiv b + p n_0^1 \equiv 0 \pmod{q'}, \\
a + q n_1^0 & \equiv b + p n_1^1 \equiv 0 \pmod{p'}.
\end{align*}
\]
We claim that
\begin{itemize}
  \item \( n_0^0 + n_1^1 \) and \( n_1^0 + n_1^1 \) are smaller than \( k \);
  \item \( k - n_0^0 - n_1^1 \) and \( k - n_1^0 - n_1^1 \) are multiples of \( q' \) and \( p' \) respectively; and
  \item \( (k - n_0^0 - n_0^1)/q' \) and \( (k - n_1^0 - n_1^1)/p' \) are larger than \( 1/\varepsilon \).
\end{itemize}
Indeed, we have
\[ n_0^0 + n_0^1 = \frac{q'(p_0^0 - a)}{q} + \frac{q'l_1^1 - b}{p} \]
\[ = \frac{q'(p_0^0 + q'l_1^1) - pa - qb}{pq} \]
\[ < \frac{2q'pq - p'q' + pqk}{pq} < k. \]

Also, note that
\[ pq(k - n_0^0 - n_0^1) = p'q' - p'q'l_0^0 - qq'l_1^1 = q'(p' - p_0^0 - ql_1^1). \]

Since \( p \) and \( q \) divide \( p' - ql_1^1 \) and \( p' - p_0^0 \) respectively, and since \( p \) and \( q \) are coprime, it follows that \( pq \) divides \( p' - p_0^0 - ql_1^1 \), so \( q' \) divides \( k - n_0^0 - n_0^1 \); and
\[ \frac{k - n_0^0 - n_0^1}{q'} = \frac{p' - p_0^0 - ql_1^1}{pq} \]
\[ > \frac{p' - 2pq}{pq} > \frac{1}{\epsilon}. \]

Similarly, it follows that \( n_1^0 + n_1^1 \) is smaller than \( k \), that \( p' \) divides \( k - n_1^0 + n_1^1 \), and that \( (k - n_1^0 - n_1^1)/p' \) is larger than \( 1/\epsilon \).

From the claim in the previous paragraph, one can easily construct a family \( t_1, \ldots, t_k \) of continuous maps from \([0, 1]\) into \([0, 1]\) such that
- the union of the images of \( t_1, \ldots, t_k \) is equal to \([0, 1]\);
- the diameter of the image of \( t_i \) is smaller than \( \epsilon \) for all \( i \);
- \( \#\{i \mid t_i(x) = y\} = n_x^y \) for \( x, y = 0, 1 \); and
- for each \( y \) with \( 0 < y < 1 \), the integers \( q' \) and \( p' \) divide \( \#\{i \mid t_i(0) = y\} \) and \( \#\{i \mid t_i(1) = y\} \) respectively.

If we define a \( * \)-homomorphism \( \eta \) from \( Z_{p,q} \) into \( C([0, 1], \mathbb{M}_{p',q'}) \) by
\[ \eta(f)(s) = \left( \text{diag}[c(f(0)), \ldots, c(f(0))], \right. \]
\[ \left. f(t_1(s)), \ldots, f(t_k(s)), c(f(1)), \ldots, c(f(1)) \right), \]

then one can easily verify from the construction of \( t_1, \ldots, t_k \) that the images of \( e_0 \circ \eta \) and \( e_1 \circ \eta \) are included in isomorphic copies of \( \mathbb{M}_{p'} \otimes 1_{q'} \) and \( 1_p \otimes \mathbb{M}_{q'} \) respectively, so there is a unitary \( u \in C([0, 1], \mathbb{M}_{p',q'}) \) with \( \text{Im Ad}(u) \circ \eta \subseteq Z_{p',q'} \). \( \square \)

Note that the integers \( a \) and \( b \) in the proof of the previous proposition is the reminder indices of the embedding that is constructed. In particular, both of the indices are equal to 0 if \( pq \) divides \( p'q' \).

The next proposition is also a slight modification of [Mas16a, Lemma 4.9].

Recall that a modulus of uniform continuity of a function \( f \) on \([0, 1]\) is a map \( \Delta_f : (0, 1] \rightarrow (0, 1] \) such that \( |s - s'| < \Delta_f(\epsilon) \) implies \( ||f(s) - f(s')|| \leq \epsilon \).

**Proposition 3.3.** Let \( p, q \) be coprime positive integers, \( t_1, t_2 : Z_{p,q} \rightarrow Z_{p',q'} \) be unital \( * \)-homomorphisms with eigenvalue patterns \( t_1^1, \ldots, t_k^1 \) and \( t_1^2, \ldots, t_k^2 \) respectively, \( G \) be a finite subset of \( Z_{p,q} \), and \( \epsilon \) be a positive real number. If the inequality
\[ \max_i ||t_i^1 - t_i^2||_\infty < \min_{g \in G} \Delta_g(\epsilon) \]
holds, where $\Delta_0$ is a modulus of uniform continuity of $g$, then there exists a unitary $w \in \mathbb{Z}_{p,q}$ with
\[ \| \text{Ad}(w) \circ \iota_1(g) - \iota_2(g) \| < 5\varepsilon \]
for all $g \in G$.

**Proof.** By Proposition 3.1, we may assume without loss of generality that $\iota_j$ is of the form
\[ \iota_j(f)(s) = \text{Ad}(u^j(s)) \left( \frac{\text{diag}[c(f(0)), \ldots, c(f(0)), f(t^j_1(s)), \ldots, f(t^j_n(s)), c(f(1)), \ldots, c(f(1))]}{b} \right) \]
for $f \in \mathbb{Z}_{p,q}$, where $u^j \in C([0,1], M_{p,q})$ is a unitary and $t^j_1 \leq \cdots \leq t^j_n$. Also, we may assume that $||g|| \leq 1$ for all $g \in G$.

Let $n^0_0$ and $n^1_0$ be the least non-negative integers such that
\[ a + qn^0_0 \equiv b + pn^1_0 \equiv 0 \pmod{q^*}. \]
Then, from the condition $\iota_j(f) \in M_{p,q} \otimes 1_{q^*}$, it easily follows that
\[ 0 = t^j_{n^0_0}(0) \leq t^j_{n^0_0+1}(0) = \cdots = t^j_{n^0_0+q}(0) \]
\[ \leq t^j_{n^0_0+q+1}(0) = \cdots = t^j_{n^0_0+2q}(0) \]
\[ \leq \cdots \leq t^j_{k-n^1_0+1}(0) = 1, \]
and there exists a unitary $v^j_0 \in M_{p,q}$ such that
\[ c(\iota_j(f)(0)) = \text{Ad}(v^j_0) \left( \frac{\text{diag}[c(f(0)), \ldots, c(f(0)), f(t^j_{n^0_0+q}(0)), f(t^j_{n^0_0+2q}(0)), \ldots, f(t^j_{k-n^1_0}(0)), c(f(1)), \ldots, c(f(1))]}{b} \right) \]
for some non-negative integers $a'$ and $b'$. Similarly, for suitable non-negative integers $n^1_0, n^1_1, a''$ and $b''$ and a unitary $v^j_1 \in M_{p,q}$, we have
\[ c(\iota_j(f)(1)) = \text{Ad}(v^j_1) \left( \frac{\text{diag}[c(f(0)), \ldots, c(f(0)), f(t^j_{n^1_0+p}(1)), f(t^j_{n^1_0+2p}(1)), \ldots, f(t^j_{k-n^1_1}(p)), c(f(1)), \ldots, c(f(1))]}{b''} \right). \]
Thus, if $v$ is a path of unitaries connecting $[v^2_0(v^1_0)_{\ast}] \otimes 1_{q^*}$ to $1_{q^*} \otimes [v^2_1(v^1_1)_{\ast}]$, then $v$ is in $\mathbb{Z}_{p,q}$ and $(\text{Ad}(v) \circ \iota_1)(f)(s) = (\text{Ad}(u^2(u^1)_{\ast}) \circ \iota_1)(f)(s)$ for $s = 0, 1$. Therefore, considering $\text{Ad}(v) \circ \iota_1$ instead of $\iota_1$ if necessary, we may assume from the outset that $u^2(0)u^1(0)^{\ast}$ and $u^2(1)u^1(1)^{\ast}$ commutes with every matrix in the image of $e_0 \circ \iota_1$ and $e_1 \circ \iota_1$. 
Now, take $\delta > 0$ so that $|s - s'| < \delta$ implies $|t^j_1(s) - t^j_1(s')| < \min_{g \in G} \Delta_g(\varepsilon)$ and $\|u^j(s) - u^j(s')\| < \varepsilon$. Let $w: [0, 1] \to \mathbb{M}_{pq'}$ be a path of unitaries such that

- $w_{[0,\delta/2]}$ connects $1_{pq'}$ to $u^{2}(0)u^{1}(0)^*$ within the commutant of the image of $e_0 \circ t_1$;
- $w(s) = u^2(2s - \delta')u^1(2s - \delta)^*$ for $s \in [\delta/2, \delta]$;
- $w(s) = u^2(s)u^1(s)^*$ for $s \in [\delta, 1 - \delta]$;
- $w(s) = u^2(2s - 1 + \delta)u^1(2s - 1 + \delta)^*$ for $s \in [1 - \delta, 1 - \delta/2]$; and
- $w_{[1-\delta/2,1]}$ connects $u^2(1)u^1(1)^*$ to $1_{pq'}$ within the commutant of the image of $e_0 \circ t_1$.

Then it is not difficult to see that this $w$ has the desired property. \hfill \Box

Now, let $\mu$ be a probability Radon measure on $[0, 1]$. Then

$$f \mapsto \int_0^1 \text{tr}(f(t)) \, d\mu(t) \quad (f \in \mathcal{Z}_{pq})$$

is a tracial state on $\mathbb{Z}_{pq}$, where $\text{tr}$ denotes the normalized trace on $\mathbb{M}_{pq}$. One can easily show that every tracial state on a dimension drop algebra is of this form. Henceforth, we identify probability Radon measures on $[0, 1]$ with tracial states on dimension drop algebras and use the same adjectives for measures and traces in common. Thus, for example, a trace is said to be atomless if its corresponding measure is atomless.

The following lemma is useful for dealing with faithful measures on $[0, 1]$. A proof can be found in [Mas16a, Lemma 3.1 and Proposition 3.2], for example.

**Lemma 3.4.** If $\lambda$ is an atomless faithful probability measure on $[0, 1]$, then for any faithful probability measure $\tau$ on $[0, 1]$, there exists a non-decreasing continuous surjection $\beta$ from $[0, 1]$ onto $[0, 1]$ with $\beta_*(\lambda) = \tau$, so that $\beta^*$ is a trace-preserving embedding of $(\mathbb{Z}_{pq}, \lambda)$ into $(\mathbb{Z}_{pq}, \tau)$.

We shall define the category $\mathcal{K}_Z$ as following.

- $\text{Obj}(\mathcal{K}_Z)$ is the class of all the pairs $(\mathbb{Z}_{pq}, \tau)$, where $p, q$ are coprime and $\tau$ is a faithful tracial state on $\mathbb{Z}_{pq}$.
- Every $L_{TF}$-embedding between objects of $\mathcal{K}_Z$ is a morphism of $\mathcal{K}_Z$.

The following theorem was first proved in [Eag16]. The proof presented here is essentially the same as that of [Mas16a].

**Theorem 3.5.** The category $\mathcal{K}_Z$ is a Fraïssé category.

**Proof.** In view of Lemma 3.4, one can easily modify the proof of Proposition 3.2 to show the following claim: For any coprime integers $p$ and $q$ and any faithful tracial state $\tau$ on $\mathbb{Z}_{pq}$, there exists a natural number $M$ such that if $p'$ and $q'$ are larger than $M$ and $pq$ divides $p'q'$, then we can construct a trace-preserving $*$-homomorphism from $(\mathbb{Z}_{pq}, \tau)$ into $(\mathbb{Z}_{p'q'}, \lambda)$, where $\lambda$ is any atomless tracial state on $\mathbb{Z}_{p'q'}$. (For a precise proof of this claim, see [Mas16a, Proposition 4.4].) So $\mathcal{K}_Z$ satisfies JEP. Also, a combination of Propositions 3.1, 3.2 and 3.3 immediately yields a proof of NAP. Next, fix an atomless measure $\lambda$ on $[0, 1]$. Then any object $(\mathbb{Z}_{pq}, \tau)$ of $\mathcal{K}_Z$ can be embedded into $(\mathbb{Z}_{pq}, \lambda)$ by Lemma 3.4, so WPP follows. Finally, CCP is
Lemma 3.6. Suppose that $p$ and $q$ are coprime and $\mathbb{Z}_{p,q}$ is embeddable into $\mathbb{Z}_{p',q'}$. Then there exists a tracial state $\lambda_{p',q'}$ on $\mathbb{Z}_{p,q}$ with the following properties.

1. There exists a trace-preserving embedding from $(\mathbb{Z}_{p,q}, \lambda_{p',q'})$ into $(\mathbb{Z}_{p',q'}, \lambda)$, where $\lambda$ corresponds to the Lebesgue measure on $[0, 1]$.

2. If $\tau$ is a tracial state on $\mathbb{Z}_{p,q}$ of the form $i^*(\tau')$ for some embedding $i$ of $\mathbb{Z}_{p,q}$ into $\mathbb{Z}_{p',q'}$ and some tracial state $\tau'$ on $\mathbb{Z}_{p',q'}$, then there exists a non-decreasing continuous map $\beta$ from $[0, 1]$ onto $[0, 1]$ with $\beta_{*}(\lambda_{p,q}) = \tau$.

Proof. Since $\mathbb{Z}_{p,q}$ is embeddable into $\mathbb{Z}_{p',q'}$, there is an embedding $\rho$ of $\mathbb{Z}_{p,q}$ into $\mathbb{Z}_{p',q'}$ of the form

$$
\rho(f)(s) = Ad(u(s))\left(\text{diag}\{c(f(0)), \ldots, c(f(1))\}\right),
$$

where $t_{1}, \ldots, t_{k}$ are piecewise strictly monotone functions such that the union of the images is equal to $[0, 1]$. We shall set $\lambda_{p',q'} := \rho^{*}(\lambda)$. Note that if $\lambda_{p',q'} = \lambda_{p',q'}^{d} + \lambda_{p',q'}^{c}$ is the Lebesgue decomposition, then the discrete measure $\lambda_{p',q'}^{d}$ is equal to $(ap\delta_{0} + bq\delta_{1})/p'q'$, where $\delta_{0}$ and $\delta_{1}$ are the Dirac measures supported on 0 and 1 respectively, and the support of the atomless measure $\lambda_{p',q'}^{c}$ is $[0, 1]$.

If $\tau$ is of the form $i^*(\tau')$ for some embedding $i$ of $\mathbb{Z}_{p,q}$ into $\mathbb{Z}_{p',q'}$ and some tracial state $\tau'$ on $\mathbb{Z}_{p',q'}$, then necessarily $\tau = \lambda_{p',q'}^{d} + \mu$ for a suitable measure $\mu$ on $[0, 1]$, and $||\lambda_{p',q'}^{c}|| = ||\mu||$. Since $\lambda_{p',q'}^{c}$ is continuous, there exists a non-decreasing continuous map $\beta$ from $[0, 1]$ onto $[0, 1]$ with $\beta_{*}(\lambda_{p',q'}) = \mu$, so $\beta_{*}(\lambda_{p',q'}) = \tau$, as desired.

Lemma 3.7. Let $i: \mathbb{Z}_{p,q} \to \mathbb{Z}_{p',q'}$ be a unital $*$-homomorphism, $\beta: [0, 1] \to [0, 1]$ be a non-decreasing continuous surjection, $G$ be a finite subset of $\mathbb{Z}_{p,q}$, and $e$ be a positive real number. Suppose that the inequality $V(i) < \min_{g \in G} \Delta_{g}(e)$ holds, where $\Delta_{g}$ denotes a modulus of uniform continuity of $g$. Then there exists a unitary $w \in \mathbb{Z}_{p,q}$ with $||Ad(w) \circ \beta^{*} \circ i(g) - i(g)|| < 5e$ for all $g \in G$.

Proof. Note that if $t_{1} \leq \cdots \leq t_{k}$ is the normalized eigenvalue pattern of $i$, then $||t_{i} - i_{*}\beta||_{\infty} < \min_{g \in G} \Delta_{g}(e)$. Thus, the claim is immediate from Proposition 3.3.

At first sight, the proof of the following proposition might seem to be complicated. However, the underlying idea is very simple; see Remark 3.9.
Proposition 3.8. An inductive system \( \{ (\mathbb{Z}_{p,q}, \tau_2), \mu_{n,m} \} \) of prime dimension drop algebras with distinguished traces is regular if \( \lim_{n \to \infty} V(\mu_{n,m}) = 0 \) for all \( m \).

Proof. We shall apply Theorem 2.8. Let \( (\mathbb{Z}_{p,q}, \tau) \) be a prime dimension drop algebra with a fixed faithful trace, \( F \) be a finite subset of \( \mathbb{Z}_{p,q} \), and \( \varphi \) be a strict approximate \( K \)-isomorphism from \( (\mathbb{Z}_{p,q}, \tau) \) into \( (\mathbb{Z}_{p,q}, \tau_n) \). Our goal is to find an approximate \( K \)-isomorphism \( \psi \) from \( (\mathbb{Z}_{p,q}, \tau) \) into \( (\mathbb{Z}_{p,q}, \tau_n) \) for some \( N > n \) such that

1. \( \psi(f, g) \leq \varphi(f, g) \) for \( f \in \mathbb{Z}_{p,q} \) and \( g \in \mathbb{Z}_{p,q} \), and
2. \( \psi \) is \( \epsilon \)-total on \( F \) for a given \( \epsilon > 0 \).

By the definition of strict approximate isomorphisms, there exist finite subsets \( G_1 \subseteq \mathbb{Z}_{p,q} \) and \( G_2 \subseteq \mathbb{Z}_{p,q} \), morphisms \( \theta_1, \theta_2 \) from \( (\mathbb{Z}_{p,q}, \tau) \) and \( (\mathbb{Z}_{p,q}, \tau_n) \) into some \( (\mathbb{Z}_{r,s}, \sigma) \), and a positive real number \( \delta \) such that

\[
\varphi \geq (\varphi_{\theta_1, \theta_2}[G_1 \times G_2])^{\mathbb{Z}_{p,q} \times \mathbb{Z}_{p,q}} + \delta.
\]

Here, we may assume the following. Fix an arbitrary positive real number \( \gamma \).

1. The subset \( G_1 \) includes \( \mathbb{Z}_{p,q} \). This is because we may replace \( G_1 \) with a larger subset.
2. There exist \( m < n \) and a finite subset \( G'_2 \subseteq \mathbb{Z}_{p,q} \) such that \( \mu_{n,m}[G'_2] = G_2 \) and \( V(\mu_{n,m}) < \Delta_n(\gamma) \) for all \( g \in G'_2 \), where \( \Delta_n \) is a modulus of uniform continuity for \( \varphi \). This is because, taking our goal into account, we may replace \( \varphi \) with \( \varphi_{\theta_1, \theta_2}[G_1 \times G_2] \) for \( l > n \), and we have

\[
\varphi[\mathbb{Z}_{p,q} \times \mathbb{Z}_{p,q}] \geq (\varphi_{\theta_1, \theta_2}[G_1 \times G_2])^{\mathbb{Z}_{p,q} \times \mathbb{Z}_{p,q}} + \delta.
\]

3. The embedding \( \theta_1 \) satisfies \( V(\theta_1) < \Delta_l(\gamma) \) for all \( f \in G_1 \), by Proposition 3.2.
4. The tracial state \( \sigma \) is atomless, by Lemma 3.4.

Now, take sufficiently large \( N \) so that there exists an embedding \( \zeta \) of \( \mathbb{Z}_{r,s} \) into \( \mathbb{Z}_{p,q} \) with \( V(\zeta) < \Delta_n(\gamma) \) for all \( g \in \theta_1[G_1] \). Let \( \lambda \) be the tracial state on \( \mathbb{Z}_{p,q} \), corresponding to the Lebesgue measure, \( \alpha \) be the nondecreasing surjective continuous map from \( [0, 1] \) to \( [0, 1] \) with \( \alpha_\lambda(\lambda) = \tau_N \), and \( \Sigma_\alpha \) be the closed subset of \( [0, 1] \) such that \( f \in \mathbb{Z}_{p,q} \) is in the image of \( \alpha^\# \) if and only if \( f \) is constant on \( \Sigma_\alpha \). Also, let \( \lambda_{\alpha, \tau_N} \) be the tracial state on \( \mathbb{Z}_{p,q} \) as in Lemma 3.6, and set

\[
\alpha' := \zeta'(\alpha), \quad \tau' := \theta_1^*(\alpha'), \quad \tau'_n := \theta'_2(\alpha').
\]

By Lemmas 3.6 and 3.7 and assumption (2) in the first paragraph, there exists a morphism \( \eta \) from \( (\mathbb{Z}_{p,q}, \tau_n) \) to \( (\mathbb{Z}_{p,q}, \lambda_{\alpha, \tau_N}) \) with \( \|\eta(g) - g\| < 5\gamma \) for all \( g \in G_2 \). Similarly, there exists a morphism \( \eta' \) from \( (\mathbb{Z}_{p,q}, \tau_n) \) to \( (\mathbb{Z}_{p,q}, \lambda_{\alpha, \tau_N}) \) with \( \|\eta'(g) - g\| < 5\gamma \) for all \( g \in G_2 \). Also, by Lemmas 3.4 and 3.7 and assumption (3) in the first paragraph, there exists a morphism \( \rho \) from \( (\mathbb{Z}_{r,s}, \sigma') \) to \( (\mathbb{Z}_{r,s}, \sigma) \) with \( \|\rho(f) - f\| < 5\gamma \) for all \( f \in \theta_1[G_1] \). Finally, by Lemma 3.6, one can find a morphism \( \iota \) from \( (\mathbb{Z}_{p,q}, \lambda_{\alpha, \tau_N}) \) to \( (\mathbb{Z}_{p,q}, \lambda) \). Here, by Proposition 3.3 and assumption (2) in the first paragraph, we can modify \( \iota \) and \( \zeta \) by inner automorphisms so that the
inequalities \(\|\alpha^* \circ \iota_{N,\alpha}(g) - \iota \circ \eta(g)\| < 5\gamma\) and \(\|\zeta \circ \theta_2(g) - \iota \circ \eta'(g)\| < 5\gamma\) for all \(g \in G_2\). Finally, by Proposition 3.1, we may assume that \(\zeta\) is of the form

\[
\zeta(f)(s) = \text{Ad}(u(s))\bigl(\text{diag}[c(f(0)), \ldots, c(f(0))], \bigr)
\]

where \(f_1, \ldots, f_k\) is the normalized eigenvalue pattern of \(\zeta\). Since \(V(\zeta) < \Delta_\gamma(\gamma)\) for all \(f \in \theta_1[G_1]\), and since

\[
||\zeta \circ \theta_2(g) - \alpha^* \circ \iota_{N,\alpha}(g)|| < ||\zeta \circ \theta_2(g) - \iota \circ \eta'(g)|| + ||\iota \circ \eta'(g) - \alpha^* \circ \iota_{N,\alpha}(g)|| < 20\gamma,
\]

for all \(g \in G_2\), one can easily modify the unitary \(u\) as in the last paragraph of the proof of Proposition 3.3 so that \(u\) is constant on \(\Sigma\) while \(\zeta\) still satisfies the inequality \(\|\zeta \circ \theta_2(g) - \alpha^* \circ \iota_{N,\alpha}(g)|| < 100\gamma\) for all \(g \in G_2\). Then, since \(u\) is constant on \(\Sigma\) and \(V(\zeta) < \Delta_\gamma(\gamma)\) for all \(f \in \theta_1[G_1]\), the inequality

\[
\inf_{g \in \mathbb{Z}_{pN,qN}} ||\zeta \circ \theta_1(f) - \alpha^*(g)|| < \gamma
\]

holds for all \(f \in G_1\).

Set \(\psi := \varphi_{\zeta,\alpha^*} \varphi_{\theta_1,\rho}\). Then, for \(f \in G_1\) and \(g \in G_2\), we have

\[
\psi(f, \iota_{N,\alpha}(g)) \leq \varphi_{\theta_1,\rho}(f, \theta_1(f)) + \varphi_{\zeta,\alpha^*}(\theta_1(f), \iota_{N,\alpha}(g))
\]

\[
= ||\theta_1(f) - \rho \circ \theta_1(f)|| + ||\zeta \circ \theta_1(f) - \alpha^* \circ \iota_{N,\alpha}(g)||
\]

\[
\leq ||\zeta \circ \theta_1(f) - \zeta \circ \theta_2(g)|| + 105\gamma
\]

\[
= \varphi_{\theta_1,\theta_2}(f, g) + 105\gamma.
\]

Also, since \(||\theta_1(f) - \rho \circ \theta_1(f)|| < 5\gamma\) and \(\inf_{\gamma} ||\zeta \circ \theta_1(f) - \alpha^*(g)|| < \gamma\), one can easily see that \(\psi\) is \(6\gamma\)-total on \(G_1\). Since \(\gamma\) was arbitrary, we may assume that \(\gamma < \min\{\epsilon/6, \delta/105\}\) so that \(\psi\) has the desired property.

\(\square\)

**Remark 3.9.** Here, for the reader’s better understanding, we shall present a simpler version of the proof above in a certain special case. Let \(\langle \mathbb{Z}_{p,q}, \tau \rangle\) be an object of
\( X \), \( F \) be a finite subset of \( \mathbb{Z}_{p,q} \), and \( \varphi \) be a strict approximate \( X \)-isomorphism from \( \langle \mathbb{Z}_{p,q}, \tau \rangle \) to \( \langle \mathbb{Z}_{p,q}, \tau_n \rangle \). Then there exist finite subsets \( G_1 \subseteq \mathbb{Z}_{p,q} \) and \( G_2 \subseteq \mathbb{Z}_{p,q} \), a joint \( X \)-embedding \((\theta_1, \theta_2)\) of \( \langle \mathbb{Z}_{p,q}, \tau \rangle \) and \( \langle \mathbb{Z}_{p,q}, \tau_n \rangle \) into some \( \langle \mathbb{Z}_{r,s}, \sigma \rangle \) and \( \delta > 0 \) such that

\[
\varphi \geq (\varphi_{\theta_1, \theta_2}|_{G_1 \times G_2})\mathbb{Z}_{p,q} \times \mathbb{Z}_{p,q} + \delta.
\]

Without loss of generality, we may assume that \( G_1 \) includes \( F \).

Now, assume that there happens to be a trace-preserving \( * \)-homomorphism \( \zeta' \) from \( \langle \mathbb{Z}_{r,s}, \sigma \rangle \) to \( \langle \mathbb{Z}_{p,q}, \tau_N \rangle \) for sufficiently large \( N \). Since \( V(\iota_{N,m}) \to 0 \) as \( N \to \infty \) for each \( m \), we may assume that both \( V(\iota_{N,m}) \) and \( V(\zeta' \circ \theta_2) \) are smaller than \( \delta/5 \), whence there is a unitary \( u \) in \( \mathbb{Z}_{p,q}^{\times \mathbb{Z}_{p,q}} \) with

\[
\| (\text{Ad}(u) \circ \zeta' \circ \theta_2)(g) - \iota_{N,m}(g) \| < \delta
\]

for all \( g \in G_2 \). Now, set \( \psi := \varphi_{\zeta' \circ \theta_2} \). Then we have

\[
\psi(f, \iota_{N,m}(g)) = \| \zeta' \circ \theta_1(f) - \iota_{N,m}(g) \|
\]

\[
\leq \| \zeta' \circ \theta_1(f) - \zeta' \circ \theta_2(g) \| + \| \zeta' \circ \theta_2(g) - \iota_{N,m}(g) \|
\]

\[
< \varphi_{\theta_1, \theta_2}(f, g) + \delta,
\]

so \( \psi \leq \varphi|_{\mathbb{Z}_{p,q}^{\times \mathbb{Z}_{p,q}}} \). Of course, \( \psi \) is \( \varepsilon \)-total for any \( \varepsilon > 0 \), since clearly

\[
\inf_{g \in \mathbb{Z}_{p,q}} \psi(f, g) = 0
\]

for all \( f \). This was what we would like to show, in view of Theorem 2.8.

In general, \( \langle \mathbb{Z}_{r,s}, \sigma \rangle \) is not necessarily embeddable into some \( \langle \mathbb{Z}_{p,q}, \tau_N \rangle \), however. This is why we need to approximate the measures \( \tau_N \) and \( \sigma \) by \( \lambda \) and \( \sigma' \) in the original proof above, which causes all the other additional steps.

In the sequel, we fix an inductive system \( \iota_{m,n} : \mathbb{Z}_m \to \mathbb{Z}_n \) of prime dimension drop algebras and write its limit by \( \mathbb{Z}_0 \). Note that every \( * \)-homomorphism between prime dimension drop algebras is automatically unital and injective, and \( \mathbb{Z}_0 \) admits a tracial state. We also let \( \iota_{m,n}^1 \leq \cdots \leq \iota_{k(m,n)}^{m,n} \) be the normalized eigenvalue pattern of \( \iota_{m,n} \).

**Lemma 3.10.** The following two conditions are equivalent.

1. The limit \( \mathbb{Z}_0 \) is simple.
2. For any \( \varepsilon > 0 \), any \( y \in [0, 1] \) and any \( m \in \mathbb{N} \), there exists \( n > m \) such that if \( x \in [0, 1] \) satisfies \( \iota_{m,n}^i(x) = y \) for some \( i \), then the Hausdorff distance between \( \{ \iota_{1,n}^i(x), \ldots, \iota_{k(m,n)}^{m,n}(x) \} \) and \([0, 1]\) is less than \( \varepsilon \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose that (2) does not hold. Then there exist \( \varepsilon > 0 \), \( y_0 \in [0, 1] \) and \( m_0 \in \mathbb{N} \) such that for any \( n > m_0 \) there is \( x_n \in [0, 1] \) with \( \iota_{m_0,n}^i(x_n) = y_0 \) for some \( i \) and

\[
d(\{ \iota_{m_0,n}^1(x_n), \ldots, \iota_{k(m_0,n)}^{m_0,n}(x_n) \}, [0, 1]) \geq \varepsilon.
\]

Take \( N \in \mathbb{N} \) so that \( 1/N < \varepsilon/2 \). For each \( n > m_0 \) there is \( a(n) \in \{0, \ldots, N\} \) with

\[
U(a(n)/N, 1/N) \cap \{ \iota_{m_0,n}^1(x_n), \ldots, \iota_{k(m_0,n)}^{m_0,n} \} = \emptyset,
\]
where $U(z, \delta)$ denotes the open ball of center $z$ and radius $\delta$. Passing to a subsystem if necessary, we may assume that $a(n)$ is constant, say $a$. Put $U := U(a/N, 1/N)$. 

For each $m, n \in \mathbb{N}$ with $m_0 \leq m \leq n$, set
\[ C_{m_0, n} := \{ x \in [0, 1] \mid r_i^{m_0,n}(x) \notin U \text{ for any } i \}, \]
\[ C_{m, n} := \{ r_i^{m,n}(x) \mid x \in C_{m, n} \}, \]
\[ C_m := \bigcap_{n \geq m} C_{m, n}. \]

Note that $C_{m_0, n}$ is nonempty, since $x_n$ is in $C_{m_0, n}$. Also, if $y$ is in $C_{m, n+1}$, then there exists $x$ in $C_{m_0, n+1}$ with $r_i^{m_0,n+1}(x) = y$ for some $i$. Now, since $t_{n+1,m} = t_{n+1,0} \circ t_{n,m}$, there are some $i, j$ with $r_i^{m_0,n+1}(x) = r_j^{m,n}(r_j^{m,n+1}(x))$. On the other hand, $r_i^{m_0,n}(r_j^{m,n+1}(x))$ is not in $U$ for any $i$, because $x$ is in $C_{m_0, n+1}$. Therefore, $r_j^{m,n+1}(x)$ is in $C_{m, n}$, whence $y = r_j^{m,n}(r_j^{m,n+1}(x))$ is in $C_{m, n}$. Consequently, $C_m$ is a nonempty closed subset of $[0, 1]$.

We shall show
\[
(*) \quad C_m = \bigcup_{i=1}^{k(m,n)} \{ \alpha_i \}.
\]

Clearly, the right-hand side is included in the left-hand side. To see the opposite inclusion, let $y$ be in $C_m$. Then, for each $l \geq n$, there is $z_l$ in $C_{m,l}$ with $r_i^{l,n}(z_l) = y$ for some $i$. By the pigeonhole principle, there is $i_0$ with $r_i^{l,n}(z_l) = y$ for infinitely many $l$. Let $z$ be a limit point of such $z_l$’s. Then clearly $z$ is in $C_n$ and $r_i^{m_0,n}(z) = y$.

For each $m \geq m_0$, set
\[
I_{m_0} := \{ f \in \mathbb{Z}_{p^m,q_m} \mid f|_{C_m} \equiv 0 \} \subseteq \mathbb{Z}_{p^m,q_m}.
\]

Then, by $(*)$, we have $I_{m+1} = I_{m_0} \cap I_{m+1} = I_{m+1,m}[I_m]$, so the sequence $\{ I_n \}$ defines a closed ideal $I$ of $\mathbb{Z}_0$. Since $I_{m_0}$ includes $\{ f \mid \text{supp } f \subseteq U \}$, the ideal $I$ is nontrivial, so $\mathbb{Z}_0$ is not simple.

$(2) \Rightarrow (1)$. Let $I$ be a proper ideal of $\mathbb{Z}_0$, and set
\[
I_m := I \cap \mathbb{Z}_{p^m,q_m},
\]
\[
C_m := \{ x \mid f(x) = 0 \text{ for all } f \in I_m \}
\]

It suffices to show that $C_m$ coincides with $[0, 1]$. For this, we may assume $I_m \subseteq \mathbb{Z}_{p^m,q_m}$, so $C_m$ is nonempty. Let $y$ be in $C_m$. By assumption, for any $\varepsilon > 0$ there is $n_0 > m$ such that if $r_i^{m,n_0}(x) = y$, then
\[
d((l_1^{m,n_0}(x)), \ldots, l_n^{m,n_0}(x)), [0, 1]) < \varepsilon.
\]

However, since $C_m = \bigcup_{i=1}^{k(m,n_0)} l_i^{m,n_0}[C_{n_0}]$ by construction, we can find $x \in C_{n_0}$ with $r_i^{m,n_0}(x) = y$ for some $i$, and
\[
\{ l_i^{m,n_0}(x), \ldots, l_{k(m,n_0)}^{m,n_0}(x) \} \subseteq C_m.
\]

Consequently, it follows that the Hausdorff distance between $C_m$ and $[0, 1]$ is less than arbitrary $\varepsilon$, so $C_m = [0, 1]$. \qed
For $y \in [0, 1]$ and $\varepsilon > 0$, we set 
\[
a_{m,n}(y, \varepsilon) := \max\{i \mid \max t_{i}^{m,n} \leq y + \varepsilon\},
\]
\[
b_{m,n}(y, \varepsilon) := \max\{i \mid \min t_{i}^{m,n} < y - \varepsilon\}
\]
\[
c_{m,n}(y, \varepsilon) := \max\{b_{m,n}(y, \varepsilon) - a_{m,n}(y, \varepsilon), 0\}
\]
\[
= \#\{i \mid \min t_{i}^{m,n} < y - \varepsilon \land \max t_{i}^{m,n} > y + \varepsilon\}.
\]

Lemma 3.11. The following are equivalent.

1. The limit $Z_{0}$ is montracial.
2. For any $y$, any $\varepsilon$ and any $m$,
\[
\lim_{n \to \infty} c_{m,n}(y, \varepsilon)/k(m,n) = 0.
\]

Proof. (1) $\Rightarrow$ (2). Suppose (2) does not hold. Then, passing to a subsystem if necessary, we may assume that there exist $y \in [0, 1]$, $\varepsilon > 0$ and $\delta > 0$ with $c_{m,n}(y, \varepsilon)/k(m,n) \geq \delta$ for all $n > m$. Let $x_{1,n}, x_{2,n} \in [0, 1]$ be such that
\[
t_{1,n}^{m,n}(x_{1,n}) > y + \varepsilon, \quad t_{2,n}^{m,n}(x_{2,n}) < y - \varepsilon,
\]
and $\tau_{1}, \tau_{2}$ be limit points of the tracial states $t_{1,n}^{m,n}(\delta_{x_{1,n}}), t_{2,n}^{m,n}(\delta_{x_{2,n}})$ respectively. We note that these are restrictions of some tracial states on $Z_{0}$. Now, if $f \in C[0, 1]$ is taken so that
\[
f\mid_{[0,y-\varepsilon]} \equiv 0, \quad f\mid_{[y+\varepsilon,1]} \equiv 1, \quad 0 \leq f \leq 1,
\]
then
\[
\tau_{1} \geq \lim\sup\{1 - a_{m,n}(y, \varepsilon)/k(m,n)\},
\]
\[
\tau_{2} \leq \lim\inf\{1 - b_{m,n}(y, \varepsilon)/k(m,n)\},
\]
whence
\[
\tau_{1}(f) - \tau_{2}(f) \geq \lim c_{m,n}(y, \varepsilon)/k(m,n) \geq \delta.
\]

Consequently, $Z_{0}$ is montracial.

(2) $\Rightarrow$ (1). Suppose (2) holds. We shall first show that, given $m \in \mathbb{N}$, $\delta > 0$ and $\varepsilon > 0$, one can find $n > m$ with
\[
\#\{i \mid \text{Im} t_{i}^{m,n} > \delta/k(m,n) < \varepsilon.
\]
Indeed, take $N \in \mathbb{N}$ with $1/N < \delta/3$, and let $n(j)$ be sufficiently large so that
\[
\frac{c_{m,n}(j/N, 1/N)}{k(m,n(j))} < \frac{\varepsilon}{N} \quad (j = 1, \ldots, N - 1).
\]
Set $n := \max j n(j)$. If $\text{Im} t_{i}^{m,n} > \delta$, then
\[
\min t_{i}^{m,n} < (j - 1)/N, \quad \max t_{i}^{m,n} > (j + 1)/N
\]
for some $j$, so the desired inequality follows.

We shall next show that, for $f \in C[0, 1], m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $n > m$ with
\[
\sup_{x,x' \in [0,1]} \left| t_{m,n}(x') - t_{m,n}(x)\right| \leq \varepsilon.
\]
For this, we may assume $\|f\| \leq 1$. Take $\delta > 0$ so that $|y - y'| < \delta$ implies $\|f(y) - f(y')\| \leq \varepsilon/3$, and put $J := \{i \mid \text{Im} t_{i}^{m,n} > \delta\}$. By what we proved in the
Proof. As mentioned in the preceding paragraph, there exists \( n > m \) with \( \#J / k(m, n) < \varepsilon / 3 \). Then, for \( x, x' \in [0, 1] \), we have

\[
\left| \left[ \tau_{n,m}^*(\delta_x) - \tau_{n,m}^*(\delta_{x'}) \right](f) \right| \\
= \frac{1}{k(m, n)} \left| \sum_i f(t_i^{m,n}(x)) - f(t_i^{m,n}(x')) \right| \\
\leq \frac{1}{k(m, n)} \left( \sum_{i \in J} + \sum_{i \notin J} \right) \left| f(t_i^{m,n}(x)) - f(t_i^{m,n}(x')) \right| \\
\leq \varepsilon,
\]

as desired.

Finally, we shall show that \( \mathcal{Z}_0 \) is monocracial. Let \( \tau, \tau' \) be tracial states on \( \mathcal{Z}_0 \) and \( \mu, \mu' \) be the corresponding measures on \([0, 1]\). Fix an element \( f \) in the center of \( \mathcal{Z}_{\rho_n, \delta_n} \), which is canonically identified with an element of \( C[0, 1] \), and take sufficiently large \( m > n \) so that

\[
\sup_{x, x' \in [0, 1]} \left| \left[ \tau_{n,m}^*(\delta_x) - \tau_{n,m}^*(\delta_{x'}) \right](f) \right| \leq \varepsilon / 3.
\]

Since the convex combinations of the Dirac measures are weakly* dense in the set of probability measures, we can find \( x_1, \ldots, x_l, x'_1, \ldots, x'_l \) in \([0, 1]\) with

\[
\left| \left( \mu_x - \sum_j \delta_{x_j} / l \right)(f \circ t_i^{m,n}) \right| < \varepsilon / 3,
\]

\[
\left| \left( \mu_{x'} - \sum_j \delta_{x'_j} / l \right)(f \circ t_i^{m,n}) \right| < \varepsilon / 3
\]

for all \( i \). Consequently,

\[
|\tau(f) - \tau(f')| \\
= \frac{1}{k(m, n)} \left| \sum_i \mu_x(f \circ t_i^{m,n}) - \mu_x(f \circ t_i^{m,n}) \right| \\
\leq \frac{2}{3} \varepsilon + \frac{1}{k(m, n)} \left| \sum_{i,j} \delta_{x_j} (f \circ t_i^{m,n}) - \delta_{x'_j} (f \circ t_i^{m,n}) \right| \\
= \frac{2}{3} \varepsilon + \frac{1}{l} \left| \sum_j \left[ \tau_{n,m}^*(\delta_x) - \tau_{n,m}^*(\delta_{x'}) \right](f) \right| \\
\leq \varepsilon.
\]

Since \( \varepsilon \) was arbitrary, \( \tau(f) = \tau'(f) \), and so \( \tau = \tau' \).

\[\Box\]

**Proposition 3.12.** The limit C*-algebra \( \mathcal{Z}_0 \) is simple and monocracial if and only if \( \lim_n V(t_{n,m}) = 0 \) for each \( m \).

**Proof.** It is clear from Lemmas 3.10 and 3.11 that if \( \lim_n V(t_{n,m}) = 0 \) for all \( m \), then \( \mathcal{Z}_0 \) is simple and monocracial. For the opposite implication, first note that if \( \mathcal{Z}_0 \) is simple, then

\[\lim_n \text{diam} \text{ Im } t_{1}^{m,n} = 0\]
for each $m$. Indeed, for any $\varepsilon > 0$, there exists sufficiently large $n$ such that if $t_i^{m,n}(x) = \varepsilon$ for some $i$ and $x \in [0, 1]$, then

$$d([t_i^{m,n}(x), \ldots, t_{(k,m)}^{m,n}(x)], [0, 1]) < \varepsilon,$$

by Lemma 3.10. This implies that $\varepsilon \notin \text{Im} t_i^{m,n}$, and since $0 \in \text{Im} t_i^{m,n}$, it follows that $t_i^{m,n} \subseteq [0, \varepsilon)$.

Next, for each $m, n \in \mathbb{N}$, let $\Delta_{m,n}$ be a map from $[0, 1]$ to $[0, 1]$ such that $|x - x'| \leq \Delta_{m,n}(\varepsilon)$ implies $|t_i^{m,n}(x) - t_i^{m,n}(x')| \leq \varepsilon$ for all $i$. Passing to a subsystem if necessary, we may assume that $\text{Im} t_i^{m,n+1} = \text{Im} t_i^{m,n+1}$. For a fixed $m \in \mathbb{N}$, set

$$F_n := \text{Im} t_i^{m,n+1} \circ \ldots \circ t_i^{m,n+1}$$

and take $y_0 \in \bigcap_n F_n$. By Lemma 3.10, there is $n > m$ such that if $x \in [0, 1]$ satisfies $t_i^{m,n}(x) = y_0$ for some $i$, then the distance between $[t_i^{m,n}(x), \ldots, t_{(k,m)}^{m,n}(x)]$ and $[0, 1]$ is less than $\varepsilon/2$. On the other hand, by definition of $F_n$, we can find $x \in \text{Im} t_i^{m,n+1} \subseteq [0, \Delta_{m,n+1}(\varepsilon/2)]$ with $t_i^{m,n}(x) = y_0$ for some $i$. Consequently, it follows that for any $y \in [0, 1]$ there exists $i$ with $\text{Im} t_i^{m,n} \circ t_i^{m,n+1} \subseteq [y - \varepsilon, y + \varepsilon]$. However, there can be only finitely many such pairs $(n', n'')$, because otherwise $f$ cannot be uniformly continuous. Thus,

$$\delta := \min\{|a_{n'} - b_{n''}| \mid \forall n, a_{n'} \leq a_n \leq b_{n''} \text{ or } a_{n'} \leq b_n \leq b_{n''}\}$$

has the desired property.

Finally, suppose that there is $m \in \mathbb{N}$ with $\lim_n V(t_{n,m}) > 0$. Without loss of generality, we may assume $m = 1$. Also, by passing to a subsystem if necessary, we may assume that there is $y \in [0, 1]$ and $\varepsilon > 0$ with the following property: For any $n$, there exists $i$ such that the image of $t_i^{1,n}$ includes $[y - \varepsilon, y + \varepsilon]$. By what we proved in the second paragraph, it is easy to find $n_0, i_1, i_2 \in \mathbb{N}$ with

$$\text{Im} t_i^{1,2}_{i_1} \circ t_i^{2,3}_{i_2} \circ t_i^{3,4}_{n_0} \circ t_i^{4,n+1} \subseteq [y - \varepsilon/2, y + \varepsilon/2].$$

We may assume $n_0 = 3$. Set

$$F := \{t_i^{1,2} \circ t_i^{2,3} \circ t_i^{3,4} \mid \text{Im} t_i^{1,2} \circ t_i^{2,3} \circ t_i^{3,4} \subseteq [y - \varepsilon, y + \varepsilon]\}$$

and take $\delta > 0$ so that if $f$ is in $F$ and if the image of $f \circ g$ includes $[y - \varepsilon, y + \varepsilon]$, then $\text{diam } g \geq \delta$. Since $Z_0$ is monotracial, we may assume

$$\#(t_i^{A,5} \mid \text{diam } t_i^{A,5} \geq \delta)/k(4, 5) < 1/|F|.$$
For any natural number $n$, let $\psi \in A_n$ be a regular sequence with the following property, the existence of which follows from Theorem 3.13:

1. $\psi$ divides $p_nq_n$ and $\tau_n$ is atomless for all $n$.
2. For any natural number $a$, there exists sufficiently large $n$ such that $a$ divides $p_nq_n$.

We shall first show that if $\rho$ is a $\mathcal{K}$-admissible endomorphism of $(\mathcal{Z}, \tr)$, then for any finite subset $F \subseteq Z_{p_nq_n}$ and any $\varepsilon > 0$, there exists a morphism $\iota$ from $(\mathcal{Z}_{p_nq_n}, \tau_n)$ to $(\mathcal{Z}_{p_nq_n}, \tau_n)$ with $\|\rho(f) - \iota(f)\| < \varepsilon$ for all $f \in F$. Take sufficiently large $m$ and so that for any $f \in F$, there exists $f' \in Z_{p_mq_m}$ with $\|\rho(f) - f'\| < \varepsilon/4$. We shall fix such $f'$ for each $f \in F$ and set $F' := \{f' \mid f \in F\}$. Put

$\psi := (\varphi_{\rho})|_{F \times F'}|^{Z_{p_mq_m} \times Z_{p_mq_m}} + \varepsilon/4$

and note that this is a strict approximate $\mathcal{K}$-isomorphism, as $\rho$ is $\mathcal{K}$-admissible. Since $\psi$ is strict, there exists a joint $\mathcal{K}$-embedding $(\theta_1, \theta_2)$ of $(\mathcal{Z}_{p_nq_n}, \tau_n)$ and $(\mathcal{Z}_{p_mq_m}, \tau_m)$ into some object $(\mathcal{Z}_{r,s}, \sigma)$ with $\varphi_{\theta_1, \theta_2} \leq \psi$, whence $\|\theta_1(f) - \theta_2(f')\| \leq \varepsilon/2$. Now by Proposition 3.2, one can embed $Z_{r,s}$ into $Z_{p_{m'}q_{m'}}$ for some $m' > m$. By assumption (2), we may assume that $rs$ divides $p_{m'}q_{m'}$, so the remainder indices vanish. Consequently, since $\tau_{m'}$ is atomless by assumption (1), one can easily find
a morphism $\eta$ from $(\mathbb{Z}_{e,s,n})$ to $(\mathbb{Z}_{p_n,q_n,m'})$. Since $V(t_{m'},N) \to 0$ as $N \to \infty$ by Theorem 3.13, one can find $N > m'$ and a unitary $u$ in $\mathbb{Z}_{p_N,q_N}$ with

$$
\|\left(\operatorname{Ad}(u) \circ t_{m'} \circ \zeta \circ \theta_2\right)(f') - t_{m'}(f')\| < \varepsilon/4
$$

for all $f' \in F'$, by Proposition 3.3. We set $v := \operatorname{Ad}(u) \circ t_{m'} \circ \zeta \circ \theta_1$. Then, for $f \in F$, we have

$$
\|\rho(f) - \iota(f)\| \leq \|\rho(f) - f'\| + \|t_{m'}(f') - (\operatorname{Ad}(u) \circ t_{m'} \circ \zeta \circ \theta_2)(f')\| + \|\left(\operatorname{Ad}(u) \circ t_{m'} \circ \zeta \circ \theta_2\right)(f') - (\operatorname{Ad}(u) \circ t_{m'} \circ \zeta \circ \theta_1)(f)\|
$$

$$
< \varepsilon,
$$

as desired.

Now, since $V(t_{m,N}) \to 0$ as $M \to \infty$, there exists sufficiently large $M$ and a unitary $v$ in $\mathbb{Z}_{p_M,q_M}$ with $\|\left(\operatorname{Ad}(v) \circ \iota(f) - f\right\| < \varepsilon$ for all $f \in F$, by Proposition 3.3. This implies $\|\rho(f) - \operatorname{Ad}(v^*)(f)\| < 2\varepsilon$, so $\rho$ is approximately inner, which completes the proof. \qed

Remark 3.15. It was shown by the author that every UHF algebra can be recognized as a Fraïssé category of $C^*$-algebras of all matrix-valued functions on cubes with distinguished faithful traces and diagonalizable morphisms, and that every endomorphism of UHF algebra is automatically admissible [Mas16b, Theorems 5.4 and 5.10]. In view of this fact, one should be able to show that every endomorphism of $\mathbb{Z}$ is $\mathcal{K}$-admissible, although the author could not do that.

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