Monogamy and trade-off relations for correlated quantum coherence

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One of the fundamental differences between classical and quantum mechanics is in the ways correlations can be distributed among the many parties that compose a system. While classical correlations can be shared among many subsystems, in general quantum correlations cannot be freely shared. This unique property is known as monogamy of quantum correlations. In this work, we study the monogamy properties of the correlated coherence for the $l_1$-norm and relative entropy measures of coherence. For the $l_1$-norm the correlated coherence is monogamous for a particular class of quantum states. For the relative entropy of coherence, and using maximally mixed state as the reference incoherent state, we show that the correlated coherence is monogamous for tripartite pure quantum systems.

Keywords: Trade-off relations; Monogamy relations; Measures of quantum coherence; Correlated coherence

I. INTRODUCTION

In quantum mechanics, the state of a physical system can be described as a vector in a Hilbert space $\mathcal{H}$. One of the features of such vector space is that any linear combination of vectors also belongs to $\mathcal{H}$, thus allowing superposition of states [1]. This superposition of states is crucial to explain interference patterns in multiple-slit experiments, that otherwise can’t be explained by classical physics. One special kind of quantum superposition is quantum coherence, which became an important physical resource in quantum information and quantum computation [2]. More recently, it was shown that quantum coherence is a natural generalization of visibility for quantifying the wave aspect of a quanton in multiple-slit experiments [3–8]. It also has an important role in several research fields, such as quantum biology [9, 10] and quantum metrology [11]. An important step towards the quantification of quantum coherence was given by Baumgratz et al. [12]. They established reasonable conditions that a measure of coherence must satisfy to be considered a bona fide measure: Nonnegativity, monotonicity under incoherent completely positive and trace preserving maps (ICPTP), monotonicity under selective incoherent operations on average, and convexity under mixing of states. In the same work, they showed that the $l_1$-norm and the relative entropy of coherence are bona fide measures of coherence, meanwhile the Hilbert-Schmidt (or $l_2$-norm) coherence is not a coherence monotone, i.e., it is not monotone under ICPTP. Later, an equivalent and rigorous framework for quantifying coherence was given in [13].

Another fundamental difference between classical and quantum mechanics is in the ways of sharing the correlations between many parties. Classical correlations can be shared among many parties, while quantum ones cannot be freely shared, e.g., if a pair of q-bits $A$ and $B$ are maximally entangled, then the system $A$ (or $B$) cannot be entangled to a third system $C$ [14, 15]. Thus, the more a q-bit is entangled with another q-bit, the less it can be entangled with a third one. This indicates that there is a limitation in the distribution of entanglement [16]. This unique property, known as entanglement monogamy, has received a lot of attention by researchers [17–21]. Mathematically, for a tripartite quantum system described by the density matrix $\rho_{A,B,C}$, the monogamy relation for an arbitrary quantum correlation measure $Q$ is expressed by

$$Q(\rho_{A|BC}) \geq Q(\rho_{AB}) + Q(\rho_{AC}),$$

where $\rho_{AB}$, $\rho_{AC}$ are the reduced states of $\rho_{A,B,C}$, and $Q(\rho_{A|BC})$ denotes the quantum correlation $Q$ between A and BC as a whole [22]. Relation (1) was first obtained in [14], using the squared concurrence as the correlation (entanglement) measure and became known as the Coffman-Kundu-Wooters relation.

However, it is known that entanglement is not the only quantum correlation existing in multipartite quantum systems [23–27]. For example, quantum discord is a type of quantum correlation that describes the incapacity of a local observer to obtain information about a subsystem without perturbing it [28]. In [18], the authors showed that quantum discord and entanglement of formation obey the same monogamy relationship for tripartite pure cases. But, it’s known that, in general, quantum discord is not monogamous, except when the quantum states

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and the state of the subsystem \( A_m \), which is obtained by tracing over the other subsystems, is given by
\[
\rho_{A_m} = \sum_{i_m,j_m} \rho_{i_m,j_m}^{A_m} \langle i_m | A_m \langle j_m | = \sum_{i_m,j_m} \sum_{i_n \neq m} \rho_{i_1,...,i_n,i_1,...,j_n} | i_m \rangle_{A_m} \langle j_m | ,
\]
where \( \sum_{i_\alpha, \forall \alpha \neq m} \) means summation for all \( i_\alpha \) such that \( \alpha \neq m \) for a given \( m \). Also, we already omitted the upper limits of the summations for convenience. The following theorem holds for any multipartite quantum system:

**Theorem 1.** Let \( \rho_{A_1,...,A_n} \) be a multipartite quantum state and let \( \{|i_m\rangle_{A_m}\}_{i_m=0}^{d_{A_m}-1} \) be an orthonormal local basis for the Hilbert space \( \mathcal{H}_{A_m} \), with \( m = 1,...,n \). Then \( C_{C_1}(\rho_{A_1,...,A_n}) \geq 0 \), where \( C_{C_1} \) is the correlated coherence using the \( l_1 \)-norm as coherence measure.

**Proof.** By definition
\[
C_{C_1}(\rho_{A_1,...,A_n}) := C_{C_1}(\rho_{A_1,...,A_n}) - \sum_{m=1}^{n} C_{C_1}(\rho_{A_m})
\]
and using maximally mixed state as the reference incoherent state, we show that the correlated coherence is monogamous for tripartite pure quantum systems. Another interesting finding is that the relative entropy of correlated coherence is equal to the mutual quantum information. We also present some trade-off relations between tripartite and bipartite quantum systems using correlated coherence.
\[
\sum_{(i_1, \ldots, i_n) \neq (j_1, \ldots, j_n)} \equiv \sum_{i_1 \neq j_1} + \sum_{i_2 \neq j_2} + \ldots + \sum_{i_n \neq j_n} + \sum_{i_1 \neq j_1} + \sum_{i_2 \neq j_2} + \ldots + \sum_{i_n \neq j_n}.
\]

Using the fact that \(\sum_i z_i \leq \sum_i |z_i| \quad \forall \quad z_i \in \mathbb{C}\), we have

\[
C_{l_1}^c(\rho_{A_1, \ldots, A_n}) \geq \sum_{(i_1, \ldots, i_n) \neq (j_1, \ldots, j_n)} |\rho_{i_1 \ldots i_n} - \rho_{i_1 \ldots i_n}| - \sum_{m=1}^{n} \sum_{i_m \neq j_m} \sum_{\alpha \neq m} |\rho_{i_1 \ldots i_n} - \rho_{i_1 \ldots i_n}| \geq 0,
\]

once we have the sum of non-negative real numbers.

It’s worth pointing out that the theorem stated above was first proved in [29] for the bipartite case, and implicitly proved in [30] for the three-qubit case, since they proved the following result: \(C_{l_1}(\rho_{A,B,C}) \geq C_{l_1}(\rho_A) + C_{l_1}(\rho_B) + C_{l_1}(\rho_C)\).

Now, let’s restrict ourselves to a bipartite quantum system. Then, for a separable uncorrelated quantum state \(\rho_{A,B} = \rho_A \otimes \rho_B = \sum_{i,k,l} \rho_{i,k}^{(A)} \rho_{j,l}^{(B)} |i,j\rangle_A \langle k,l|\), the correlated coherence is not necessarily zero:

\[
C_{l_1}^c(\rho_{A} \otimes \rho_B) = \sum_{(i,j) \neq (k,l)} |\rho_{A}^{(i)}| |\rho_{B}^{(j)}| - \sum_{i \neq k} |\rho_{A}^{(i)}| - \sum_{j \neq l} |\rho_{B}^{(j)}| = \sum_{i \neq k} |\rho_{A}^{(i)}| |\rho_{B}^{(j)}| = C_{l_1}(\rho_A)C_{l_1}(\rho_B) \geq 0,
\]

which implies that \(C_{l_1}(\rho_{A} \otimes \rho_B) \neq C_{l_1}(\rho_A) + C_{l_1}(\rho_B)\), as mentioned before (for a related discussion, see [31]). Analogously, for a separable state \(\rho_{A,B} = \sum_{\alpha} \rho_{\alpha} \otimes \rho_{\alpha}^{(B)} = \sum_{\alpha} \sum_{i,j,k,l} \rho_{\alpha,ik} \rho_{\alpha,jl} |i,j\rangle_A \langle k,l|\), we have

\[
C_{l_1}^c(\rho_{A,B}) = \sum_{j \neq l} \sum_{\alpha} \rho_{\alpha,ik} |\rho_{\alpha,jl}| \geq 0.
\]

However, using the relative entropy of coherence as measure of correlated coherence [32], since \(S(\bigotimes_m \rho_{A_m}) = \sum_m S(\rho_{A_m})\), we have \(C_{rel}(\bigotimes_m \rho_{A_m}) = \sum_m C_{rel}(\rho_{A_m})\), and therefore the relative entropy of correlated coherence satisfies \(C_{rel}(\rho_{A_1, \ldots, A_n}) = 0\) for \(\rho_{A_1, \ldots, A_n} = \bigotimes_m \rho_{A_m}\). More generally, it is worth mentioning that Rényi’s entropy is also additive [33, 34], hence correlated coherence measures based on Rényi’s entropy must satisfy such relation.

III. MONOGAMY AND TRADE-OFF RELATIONS OF CORRELATED COHERENCE

A. \(l_1\)-norm correlated coherence

In [30], the authors proved that the conjecture \(C(\rho_{A,B,C}) \geq C(\rho_{A,B}) + C(\rho_{A,C})\), made by Yao et al. [27], for the \(l_1\)-norm is invalid. They considered a counter example using the following quantum state: \(|\Psi\rangle = a_{000} |0,0,0\rangle_{A,B,C} + a_{100} |1,0,0\rangle_{A,B,C}\). However, it is interesting to note that for the correlated coherence this type of trade-off relation holds for the quantum state mentioned above. Since \(C^c_{l_1}(\rho_{A,B,C}) \geq C^c_{l_1}(\rho_{A,B}) + C^c_{l_1}(\rho_{A,C})\) implies \(C_{l_1}(\rho_{A,B,C}) + C_{l_1}(\rho_{A,C}) \geq C_{l_1}(\rho_{A,B}) + C_{l_1}(\rho_{A,C})\), and \(C_{l_1}(\rho_{A,B,C}) = C_{l_1}(\rho_A) = C_{l_1}(\rho_{A,B}) = C_{l_1}(\rho_{A,C})\), we have the following theorem:

**Theorem 2.** Let \(\rho_{A,B,C}\) be a tripartite quantum state and let \(\{|i\rangle_A\}_{i=0}^{d_A-1}, \{|j\rangle_B\}_{j=0}^{d_B-1}, \{|k\rangle_C\}_{k=0}^{d_C-1}\) be an orthonormal local basis of \(\mathcal{H}_A, \mathcal{H}_B, \) and \(\mathcal{H}_C\), respectively. If the reduced quantum system \(\rho_A = Tr_{B,C} \rho_{A,B,C}\) satisfies

\[
C_{l_1}(\rho_A) = \sum_{i \neq l} \sum_{j,k} |\rho_{i,j,k}| = \sum_{i \neq l} \sum_{j,k} |\rho_{i,j,k}|,
\]

then...
then
\[ C_{l_1}^c(\rho_{A,B,C}) \geq C_{l_1}^c(\rho_{A,B}) + C_{l_1}^c(\rho_{A,C}). \] (14)

Proof.
\[ C_{l_1}^c(\rho_{A,B,C}) - C_{l_1}^c(\rho_{A,B}) - C_{l_1}^c(\rho_{A,C}) = C_{l_1}(\rho_{A,B,C}) + C_{l_1}(\rho_{A}) - C_{l_1}(\rho_{A,B}) - C_{l_1}(\rho_{A,C}) \] (15)
\[
\geq \left( \sum_{i \neq l} \sum_{j \neq m} \sum_{k \neq n} |\rho_{ijk,lmn}| + \sum_{i \neq l} \sum_{j,k} |\rho_{ijk,ljk}| - \sum_{i \neq l} \sum_{(i,j) \neq (l,m)} |\rho_{ijk,lmn}| - \sum_{(i,k) \neq (l,n)} |\rho_{ijk,ljn}| \right) \geq 0, \] (16)

Above we used the fact that \( \sum_i |z_i| \leq \sum_i |z_i|, \forall z_i \in \mathbb{C}. \) This completes the proof. \( \square \)

Now, following the same reasoning:

**Theorem 3.** If the reduced quantum systems \( \rho_A, \rho_B, \rho_C \) satisfy
\[ C_{l_1}(\rho_A) = \sum_{i \neq l} \left| \sum_{j,k} \rho_{ijk,ljk} \right| = \sum_{i \neq l} \left| \rho_{ijk,ljk} \right|, \] (20)
\[ C_{l_1}(\rho_B) = \sum_{j \neq m} \left| \sum_{j,k} \rho_{ijk,imk} \right| = \sum_{j \neq m} \left| \rho_{ijk,imk} \right|, \] (21)
\[ C_{l_1}(\rho_C) = \sum_{k \neq n} \left| \sum_{i,j} \rho_{ijk,ijn} \right| = \sum_{k \neq n} \left| \rho_{ijk,ijn} \right|, \] (22)
then
\[ C_{l_1}^c(\rho_{A,B,C}) \geq C_{l_1}^c(\rho_{A,B}) + C_{l_1}^c(\rho_{A,C}) + C_{l_1}^c(\rho_{B,C}). \] (24)

Proof.
\[ C_{l_1}^c(\rho_{A,B,C}) - C_{l_1}^c(\rho_{A,B}) - C_{l_1}^c(\rho_{A,C}) - C_{l_1}^c(\rho_{B,C}) = C_{l_1}(\rho_{A,B,C}) + \sum_{\alpha=A,B,C} C_{l_1}(\rho_{\alpha}) - \sum_{\alpha<\beta=A,B,C} C_{l_1}(\rho_{\alpha,\beta}) \] (25)
\[
\geq \left( \sum_{i \neq l} \sum_{j \neq m} \sum_{k \neq n} |\rho_{ijk,lmn}| + \sum_{i \neq l} \sum_{j,k} |\rho_{ijk,ljk}| + \sum_{j \neq m} \sum_{i,k} |\rho_{ijk,imk}| + \sum_{k \neq n} \sum_{i,j} |\rho_{ijk,ijn}| \right) \geq 0, \] (27)

This kind of trade-off relation is equivalent to the monogamy relation expressed by the Eq. (1) for the correlated coherence.
Figure 1: The function $M := C_{t_i}^c(\rho_{A|BC}) - C_{t_i}^c(\rho_{A,B}) - C_{t_i}^c(\rho_{A,C})$ as function of $p$ and $\epsilon$ for the state $|\Phi(p,\epsilon)\rangle$.

**Theorem 4.** A tripartite quantum system that satisfies the trade-off relation
\begin{equation}
C^c(\rho_{A,B,C}) \geq C^c(\rho_{A,B}) + C^c(\rho_{A,C}) + C^c(\rho_{B,C})
\end{equation}
also satisfies the monogamy relation
\begin{equation}
C^c(\rho_{A|BC}) \geq C^c(\rho_{A,B}) + C^c(\rho_{A,C}),
\end{equation}
for any coherence measure, where $C_{t_i}^c(\rho_{A|BC}) := C_{t_i}^c(\rho_{A,B,C}) - C_{t_i}(\rho_A) - C_{t_i}(\rho_{B,C})$ denotes the correlated coherence between A and BC.

**Proof.** The proof follows directly from the definition
\begin{equation}
C^c(\rho_{A|BC}) - C^c(\rho_{A,B}) - C^c(\rho_{A,C}) = C(\rho_{A,B,C}) + \sum_{\alpha=A,B,C} C(\rho_{\alpha}) - \sum_{\alpha<\beta,A,B,C} C(\rho_{\alpha,\beta})
\end{equation}
\begin{equation}
= C(\rho_{A,B,C}) - C^c(\rho_{A,B}) - C^c(\rho_{A,C}) - C^c(\rho_{B,C}) \geq 0.
\end{equation}
\hfill \Box

For instance, the pure quantum state $|G,H,Z,W\rangle_{A,B,C} = \lambda_1 |0,0,0\rangle_{A,B,C} + \lambda_2 |0,0,1\rangle_{A,B,C} + \lambda_3 |0,1,0\rangle_{A,B,C} + \lambda_4 |1,0,0\rangle_{A,B,C} + \lambda_5 |1,1,1\rangle_{A,B,C}$, with $|\lambda_1|^2 + |\lambda_2|^2 + |\lambda_3|^2 + |\lambda_4|^2 + |\lambda_5|^2 = 1$, satisfies the monogamy relation for the correlated coherence. Also, for the state $|\Phi\rangle_{A,B,C} = a_{000} |0,0,0\rangle_{A,B,C} + a_{101} |1,0,1\rangle_{A,B,C} + a_{111} |1,1,1\rangle_{A,B,C}$ such that $|a_{000}|^2 + |a_{101}|^2 + |a_{111}|^2 = 1$, we have $C_{t_i}^c(\rho_{A|BC}) - C_{t_i}^c(\rho_{A,B}) - C_{t_i}^c(\rho_{A,C}) = 2|a_{000}a_{111}| \geq 0$. This state was considered by Giorgi [18] in the form $|\Phi(p,\epsilon)\rangle = \sqrt{p} |0,0,0\rangle_{A,B,C} + \sqrt{(1-p)/2} |1,1,1\rangle_{A,B,C} + \sqrt{(1-p)/2} |1,0,1\rangle_{A,B,C}$, with $p,\epsilon \in [0,1]$. Hence, for the correlated coherence, $|\Phi(p,\epsilon)\rangle$ is monogamous for any value of $p, \epsilon \in [0,1]$. In Fig. 1, we plotted $M := C_{t_i}^c(\rho_{A|BC}) - C_{t_i}^c(\rho_{A,B}) - C_{t_i}^c(\rho_{A,C})$ as function of $p$ and $\epsilon$.

Also, let’s consider the following state $|\Psi\rangle_{A,B,C} = \lambda_1 |0,0,0\rangle_{A,B,C} + \lambda_2 |0,1,1\rangle_{A,B,C} + \lambda_3 |1,0,0\rangle_{A,B,C} + \lambda_4 |1,1,1\rangle_{A,B,C}$, with $|\lambda_1|^2 + |\lambda_2|^2 + |\lambda_3|^2 + |\lambda_4|^2 = 1$ [35], such that $C_{t_i}(\rho_A) = 2|\lambda_1\lambda_3^* + \lambda_2\lambda_4^*| \neq 2(|\lambda_1\lambda_3^*| + |\lambda_2\lambda_4^*|)$. Then
\begin{equation}
C_{t_i}^c(\rho_{A|BC}) - C_{t_i}^c(\rho_{A,B}) - C_{t_i}^c(\rho_{A,C}) = 2(|\lambda_1\lambda_3^*| + |\lambda_2\lambda_4^*|) + 2(|\lambda_1\lambda_3^* + \lambda_2\lambda_4^*| - |\lambda_1\lambda_3^*| - |\lambda_2\lambda_4^*|)
\end{equation}
\begin{equation}
+ 2(|\lambda_1\lambda_3^* + \lambda_3\lambda_4^*| - |\lambda_1\lambda_3^* + \lambda_3\lambda_4^*|)
\end{equation}
\begin{equation}
\geq 2(|\lambda_1\lambda_3^*| + |\lambda_2\lambda_4^*|) + 2(|\lambda_1\lambda_3^* + \lambda_2\lambda_4^*| - |\lambda_1\lambda_3^*| - |\lambda_2\lambda_4^*|)
\end{equation}
\begin{equation}
\geq 0.
\end{equation}

To prove the last passage, let’s suppose, by contradiction, that $|\lambda_1\lambda_3^*| + |\lambda_2\lambda_4^*| \geq |\lambda_1\lambda_3^*| + |\lambda_2\lambda_4^*| + |\lambda_1\lambda_3^* + \lambda_2\lambda_4^*|$. Squaring this expression, we obtain
\begin{equation}
0 \geq |\lambda_1\lambda_3^*|^2 + |\lambda_2\lambda_4^*|^2 + 2(|\lambda_1\lambda_3^*| + |\lambda_2\lambda_4^*|)|\lambda_1\lambda_3^* + \lambda_2\lambda_4^*| + 2\Re(\lambda_1\lambda_3^*\lambda_2\lambda_4^*)
\end{equation}
\begin{equation}
= |\lambda_1\lambda_3^* + \lambda_2\lambda_4^*|^2 + 2(|\lambda_1\lambda_3^*| + |\lambda_2\lambda_4^*|)|\lambda_1\lambda_3^* + \lambda_2\lambda_4^*|,
\end{equation}
\begin{equation}
\geq 0.
\end{equation}
which is an absurd because the right-hand side is the sum of positive real numbers. On the other hand, if the coefficients \( \{\lambda_i\}_{i=1}^4 \) are real and positive, the monogamy is obviously satisfied. We can check this considering the state in the form \( |\Psi(p, \epsilon)\rangle = \sqrt{p} |0, 0\rangle_{A,B,C} + \sqrt{1-p} \left( |1, 1\rangle_{A,B,C} + \sqrt{1-p}/2 |0, 1\rangle_{A,B,C} + |0, 1\rangle_{A,B,C} \right) \), with \( p, \epsilon \in [0,1] \), where, in Fig. 2, we plotted \( M := C_{i_1}^c(\rho_{A|BC}) - C_{i_1}^c(\rho_{A,B}) - C_{i_1}^c(\rho_{A,C}) \) as function of \( p \) and \( \epsilon \). Therefore, the conditions expressed by the equations (20), (21), and (22) are sufficient but not necessary, and seems reasonably to conjecture that the monogamy relation \( C_{i_1}^c(\rho_{A|BC}) \geq C_{i_1}^c(\rho_{A,B}) + C_{i_1}^c(\rho_{A,C}) \) holds for any tripartite pure quantum state. Finally, it is possible to establish a weaker trade-off relation for an arbitrary tripartite quantum state, i.e.,

\[
C_{i_1}^c(\rho_{A,B,C}) \geq \frac{1}{2} \left( C_{i_1}^c(\rho_{A,B}) + C_{i_1}^c(\rho_{A,C}) + C_{i_1}^c(\rho_{B,C}) \right),
\]

once

\[
C_{i_1}^c(\rho_{A,B,C}) - \frac{1}{2} \left( C_{i_1}^c(\rho_{A,B}) + C_{i_1}^c(\rho_{A,C}) + C_{i_1}^c(\rho_{B,C}) \right) = C_{i_1}(\rho_{A,B,C}) - \frac{1}{2} \left( C_{i_1}(\rho_{A,B}) + C_{i_1}(\rho_{A,C}) + C_{i_1}(\rho_{B,C}) \right) 
\geq \left( \sum_{i \neq l} + \frac{1}{2} \sum_{j \neq m} + \sum_{k \neq n} \right) |\rho_{ijklmn}| 
\geq 0.
\]

B. Relative entropy of correlated coherence

First, we'll obtain a upper bound for the correlated coherence of a bipartite quantum system \( \rho_{A,B} \):

\[
C_{re}^c(\rho_{A,B}) = C_{re}(\rho_{A,B}) - C_{re}(\rho_A) - C_{re}(\rho_B) = S(\rho_{A,B_{diag}}) - S(\rho_{A,diag}) + S(\rho_{A}) - S(\rho_{B_{diag}}) + S(\rho_{B}) 
\leq S(\rho_A) + S(\rho_B) - S(\rho_{A,B}),
\]

since \( S(\rho_{A,B_{diag}}) - S(\rho_{A,diag}) - S(\rho_{B_{diag}}) \leq 0 \) (by the subadditivity of von Neumann’s entropy [2]). Thus

\[
0 \leq C_{re}^c(\rho_{A,B}) \leq S(\rho_A) + S(\rho_B) - S(\rho_{A,B}) := \iota_{A,B},
\]

where \( \iota_{A,B} \) is the quantum mutual information, which is the total amount of correlations in any bipartite quantum state \([36, 37]\). Also, it’s known that the quantification of quantum coherence depends on a particular reference basis, i.e., these quantum coherence measures are basis-dependent \([38]\). Following the authors in \([39, 40]\), we’ll define the basis-independent relative entropy of coherence, or intrinsic relative entropy of quantum coherence (IREQC). Since the maximally mixed state, \( I/d \), is the only basis-independent incoherent state, the IREQC is defined by taken the maximally mixed state as the reference incoherent state, i.e.,

\[
C_{re}^{I}(\rho) := S(\rho||I/d) = \log d - S(\rho),
\]
where \( S(\rho|I/d) = \text{Tr}(\rho \ln \rho - \rho \ln I/d) \) is the relative entropy. Now, defining the intrinsic relative entropy of correlated coherence (IRECC) for a bipartite quantum system as

\[
C_{rc}^{I}(\rho_{A,B}) := C_{re}^{I}(\rho_{A,B}) - C_{re}^{I}(\rho_{A}) - C_{re}^{I}(\rho_{B}),
\]

we see it is equal to the total amount of correlations in any bipartite quantum system:

**Theorem 5.** The intrinsic relative entropy of correlated coherence of a bipartite quantum system, \( C_{rc}^{I}(\rho_{A,B}) \), is equal to the quantum mutual information \( I_{A,B} = S(\rho_{A}) + S(\rho_{B}) - S(\rho_{A,B}) \).

**Proof.** The result follows directly from the definition:

\[
C_{rc}^{I}(\rho_{A,B}) = C_{re}^{I}(\rho_{A,B}) - C_{re}^{I}(\rho_{A}) - C_{re}^{I}(\rho_{B}) = S(\rho_{A}) + S(\rho_{B}) - S(\rho_{A,B})
\]

(49)

\[
C_{rc}^{I}(\rho_{A,B}) = S_{A,B}.
\]

(50)

Now, by considering a tripartite quantum system, the IRECC is given by

\[
C_{rc}^{I}(\rho_{A,B,C}) = C_{re}^{I}(\rho_{A,B,C}) - \sum_{\alpha=A,B,C} C_{re}^{I}(\rho_{\alpha}).
\]

Hence, we have the following trade-off relation

\[
C_{rc}^{I}(\rho_{A,B,C}) - C_{rc}^{I}(\rho_{A,B}) - C_{rc}^{I}(\rho_{A,C}) = C_{re}^{I}(\rho_{A,B,C}) + C_{re}^{I}(\rho_{A}) - C_{re}^{I}(\rho_{A,B}) - C_{re}^{I}(\rho_{A,C})
\]

(51)

\[
= \log(d_{A}d_{B}d_{C}) - S(\rho_{A,B,C}) + \log(d_{A}) - S(\rho_{A}) - \log(d_{A}d_{B}) = S(\rho_{A,B}) - S(\rho_{A,C}) - S(\rho_{A})
\]

(52)

\[
\geq 0,
\]

(53)

since \( S(\rho_{A,B}) + S(\rho_{A,C}) - S(\rho_{A}) \geq 0 \) (by the strong subadditivity of von Neumann’s entropy [2]).

**Theorem 6.** If a tripartite quantum system \( \rho_{A,B,C} \) is pure, then \( \rho_{A,B,C} \) satisfies the trade-off relation

\[
C_{rc}^{I}(\rho_{A,B,C}) = C_{re}^{I}(\rho_{A,B}) + C_{re}^{I}(\rho_{A,C}) + C_{re}^{I}(\rho_{B,C}),
\]

(55)

and consequently the monogamy relation

\[
C_{rc}^{I}(\rho_{A|BC}) = C_{re}^{I}(\rho_{A,B}) + C_{re}^{I}(\rho_{A,C}).
\]

(56)

**Proof.**

\[
C_{rc}^{I}(\rho_{A,B,C}) - \sum_{\alpha<\beta=A,B,C} C_{re}^{I}(\rho_{\alpha,\beta}) = C_{re}^{I}(\rho_{A,B,C}) + \sum_{\alpha=A,B,C} C_{re}^{I}(\rho_{\alpha}) - \sum_{\alpha<\beta=A,B,C} C_{re}^{I}(\rho_{\alpha,\beta})
\]

(57)

\[
= S(\rho_{A,B}) + S(\rho_{A,C}) + S(\rho_{B,C}) - S(\rho_{A,B}) - S(\rho_{A}) \]

(58)

\[
= T_{A,B,C},
\]

(59)

since \( T_{A,B,C} \) is the interaction information [41], and \( T_{A,B,C} = 0 \) for tripartite pure states, since \( S(\rho_{A,B,C}) = 0 \), and \( S(\rho_{A,B}) = S(\rho_{C}) \), \( S(\rho_{A,C}) = S(\rho_{B}) \), \( S(\rho_{B,C}) = S(\rho_{A}) \) [42]. This completes the proof.

Consequently, \( C_{rc}^{I}(\rho_{A|BC}) = C_{re}^{I}(\rho_{A,B}) + C_{re}^{I}(\rho_{A,C}) \) is equivalent to \( I_{A|BC} = I_{A,B} + I_{A,B} \) [43], once that \( C_{rc}^{I}(\rho_{A|BC}) = I_{A|BC} \), \( C_{re}^{I}(\rho_{X,Y}) = I_{X,Y}, \forall X,Y = A,B,C \) such that \( X \neq Y \).

**IV. CONCLUSIONS**

Monogamy relations are an important feature of quantum correlations, as they tell us that a specific quantum resource cannot be shared freely. Recently, the correlated coherence was used as a resource for remote state preparation and quantum teleportation [44]. Hence, it is important to know if the correlated coherence satisfies monogamy relations. In this paper, we have studied the monogamy properties of the correlated coherence for the \( l_1 \)-norm and relative entropy measures. For the \( l_1 \)-norm, the correlated coherence is monogamous for a given class of quantum states, and we conjectured that it is monogamous, at least, for tripartite pure quantum states. For the relative entropy of coherence, and using a maximally mixed state as the reference incoherent state, we showed that the correlated coherence is monogamous for tripartite pure quantum system. Another interesting finding is that the intrinsic relative entropy of correlated coherence (IRECC) is equal to quantum mutual information. Finally, we also established some trade-off relations between tripartite and bipartite quantum systems, and proved that the trade-off relation \( C^{I}(\rho_{A,B,C}) \geq C^{I}(\rho_{A,B}) + C^{I}(\rho_{A,C}) + C^{I}(\rho_{B,C}) \) is equivalent to the monogamy relation \( C^{I}_{l_1}(\rho_{A|BC}) \geq C^{I}_{l_1}(\rho_{A,B}) + C^{I}_{l_1}(\rho_{A,C}) \).
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