Radion stabilization in the Randall-Sundrum model
with quadratic and quartic potentials

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Abstract
In this report we investigate the Goldberger-Wise (GW) mechanism of radion stabilization with quartic potential on the hidden brane and quadratic potential on the visible brane. The advantage of our simplified scenario over the original GW mechanism is that the modulus potential can be evaluated for finite $\lambda_v$ and $\lambda_h$. This enables us to probe how the modulus potential behaves over the entire range of $\lambda_h$. By staying away from the GW limit of $\frac{k}{\lambda_v v_v^2} \to 0$ and $\frac{k}{\lambda_h v_h^2} \to 0$ we show that it is possible to choose the parameters of the model so that the potential exhibits a minimum at $kr_c \approx 12$ and this adjustment does not involve any extreme fine tuning of parameters.
Several radical proposals based on higher dimensional theories have been recently put forward to explain the large hierarchy between the weak scale and the Planck scale. Among them the Randall-Sundrum (RS) [1] scenario is particularly attractive since it explains the hierarchy in terms of a small extra dimensions. Unlike theories with large extra dimensions, in the RS model there is no large hierarchy between the compactification scale $\frac{1}{r_c}$ and the fundamental Planck mass $M$. The reason behind this difference is that the hierarchy is explained in terms of an exponential warp factor that appears in the non-factorizable metric of the five dimensional RS world.

$$ds^2 = e^{-2kr_c|\phi|} \eta_{\mu\nu} dx^\mu dx^\nu - r_c^2 d\phi^2.$$  \hspace{1cm} (1)

Here $k$ is a parameter of the order of $M$. $-\pi \leq \phi \leq \pi$ is the coordinate of the single $S^1/Z_2$ orbifold extra dimension. The points $(x, \phi)$ and $(x, -\phi)$ are therefore identified. The hidden 3 brane is located at $\phi = 0$ and the visible brane is located at $\phi = \pi$.

In the RS scenario the compactification radius $r_c$ was associated with the vacuum expectation value (vev) of a four dimensional massless scalar field $T(x)$. However $T(x)$ had zero potential and $r_c$ was not stabilized by some dynamics. Goldberger and Wise (GW) [2] showed that it is possible to generate a potential $V(r_c)$ for the modulus field by introducing a bulk scalar field $\chi(x, \phi)$ with interaction potentials localized on the two branes. They also showed that the minimum of the modulus potential can be adjusted to occur at $kr_c \approx 12$ without fine tuning the parameters of the model. In their original proposal GW assumed the interaction potentials on the two branes to be quartic functions of $\chi$. As a result they could determine the modulus potential $V(r_c)$ only in the limit of infinite $\lambda_v$ and $\lambda_h$. For large but finite $\lambda_v$ and $\lambda_h$ the shifts in the vevs $\delta \chi(\pi)$ and $\delta \chi(0)$ from their values $v_v$ and $v_h$ at infinite $\lambda_v$ and $\lambda_h$ are given by [2]

$$\delta \chi(\pi) = -\frac{k}{\lambda_v v_v^2} (v_v - v_h e^{-ekr_c \pi}).$$  \hspace{1cm} (2a)
\[ \delta \chi(0) = -\frac{k}{\lambda_h v_h^2} e^{-(4+\epsilon)kr_c \pi} (v_v - v_h e^{-\epsilon kr_c \pi}). \] (2b)

Therefore the GW limit assumes that \( \frac{k}{\lambda_v v_v} \) and \( \frac{k}{\lambda_h v_h^2} \) have been tuned to very small values which may not be stable under fluctuations in the background metric or quantum corrections. It is therefore worthwhile to avoid this fine tuning and determine the modulus potential for finite \( \lambda_v \) and \( \lambda_h \). However for quartic brane potentials with finite \( \lambda_v \) and \( \lambda_h \) it is difficult to determine the background field configuration \( \chi(\phi) \) that satisfies the appropriate boundary conditions on the branes analytically. Therefore in this report we shall consider a simplified scenario with quartic interaction potential for \( \chi \) on the hidden brane and quadratic potential on the visible brane. This set up will enable us to determine the modulus potential analytically for finite \( \lambda_v \) and \( \lambda_h \). We find that the modulus potential \( V(r_c) \) for this scenario can be adjusted to yield a minimum both for finite and infinite \( \lambda_h \) at \( kr_c \approx 12 \) without fine tuning the parameters of the model.

The action for our scenario is given by

\[ S_b = \frac{1}{2} \int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{G} (G^{AB} \partial_A \chi \partial_B \chi - m^2 \chi^2). \] (3a)

\[ S_h = -\int d^4x \int d\phi \sqrt{-g_h} \lambda_h (\chi^2 - v_h^2)^2 \frac{\delta(\phi)}{r_c}. \] (3b)

\[ S_v = -\int d^4x \int d\phi \sqrt{-g_v} \lambda_v v_v^2 (\chi^2 - v_v^2)^2 \frac{\delta(\phi - \pi)}{r_c}. \] (3c)

In this work we shall ignore the variation of \( \chi \) parallel to the 3 branes. This is justified since we are interested only in the stability of the size of the extra dimension or the interbrane separation. It can be shown that away from the boundaries \( \phi = 0, \pi \) the classical configuration of the bulk field \( \chi \) is given by

\[ \chi(\phi) = A e^{(2+\nu)\sigma} + B e^{(2-\nu)\sigma}. \] (4)
where \( \sigma = kr_c|\phi| \) and \( \nu = \sqrt{4 + \frac{m^2}{k^2}} \). In particular we shall consider the case \( m \ll k \) so that \( \nu \approx 2 + \epsilon \) where \( \epsilon \approx \frac{m^2}{4kr_c^2} \). The constants \( A \) and \( B \) are determined by the boundary conditions imposed by the interaction potentials on the two branes. Using the relations \( \chi'(0 + \epsilon) = -\chi'(0 - \epsilon) \) and \( \chi'(\pi + \epsilon) = -\chi'(\pi - \epsilon) \) the boundary conditions can be written as

\[
\left[(2 + \nu)e^{(2+\nu)kr_c\pi}A + (2 - \nu)e^{(2-\nu)kr_c\pi}B\right] + \lambda_v v^2\chi(\pi) = 0. \tag{5}
\]

and

\[
\left[(2 + \nu)A + (2 - \nu)B\right] - 2\lambda_h\chi(0)[\chi^2(0) - v_h^2] = 0. \tag{6}
\]

Here \( \chi(0) = A + B \) and \( \chi(\pi) = Ae^{(2+\nu)kr_c\pi} + Be^{(2-\nu)kr_c\pi} \). In order to solve (5) and (6) for \( A \) and \( B \) we shall assume that \( kr_c > 1 \) so that we can neglect higher powers of \( e^{-2\nu kr_c \pi} \). We then get

\[
A = \pm \sqrt{a}e^{-2\nu kr_c \pi}[1 + \{c(1 + \frac{2 + \nu}{\epsilon}) - 1\}be^{-2\nu kr_c \pi}]. \tag{7}
\]

and

\[
B = \pm \sqrt{a}[1 + \{c(1 + \frac{2 + \nu}{\epsilon}) - 1\}be^{-2\nu kr_c \pi}]. \tag{8}
\]

where \( a = v_h^2 - \frac{k\epsilon}{2\lambda_h}, b = \frac{k\epsilon - \lambda_v v_h^2}{k(2+\nu) + \lambda_v v_h^2} \) and \( c = \frac{k\epsilon}{4\lambda_h v_h^2 - \frac{1}{2\lambda_h}} \). Putting the solution (4) back into the action and integrating over \( \phi \) yields the following four dimensional potential for the modulus

\[
V(r_c) = k\epsilon(v_h^2 - \frac{k\epsilon}{4\lambda_h}) + ka[(2 + \nu)b^2 - \epsilon(1 + 2b)]\bar{\phi}^4 + \lambda_v v^2[(1 + b)^2\bar{\phi}^4 + 2\epsilon - v^2\bar{\phi}^4]. \tag{9}
\]

where \( \bar{\phi} = e^{-kr_c \pi} \). In the above expression for \( V(r_c) \) we have retained only terms up to order \( e^{-2\nu kr_c \pi} \). Since \( e^{-2\nu kr_c \pi} \ll 1 \) even for \( kr_c \approx 1 \) we can trust the above expression...
for $V(r_c)$ as long as $kr_c \geq 1$. Note that as $\lambda_v \to \infty$, $b \approx -(1 - (2 + \nu)\frac{k}{\lambda_v v_v})$ and $\chi(\pi) \to 0$. The visible brane potential therefore approaches the value $-\lambda_v v_v^4$. Since the visible brane potential becomes unbounded from below for $\lambda_v \to \infty$ we shall restrict ourselves to finite values of $\lambda_v$ only.

The above expression for $V(r_c)$ will exhibit a non-trivial minimum provided the coefficient of the $\tilde{\phi}^{4+2\epsilon}$ term is positive. In this case as $\tilde{\phi}$ increases from zero the $\tilde{\phi}^4$ term ultimately begins to dominate causing $V(r_c)$ to increase and exhibit a non-trivial minimum. In the following we shall assume that $\lambda_v$ and $\lambda_h$ are finite. In particular we shall consider the case where $k \epsilon \text{over} \lambda_v v_v^2$ and $\frac{k \epsilon}{\lambda_h v_h^2}$ are of order unity. In other words we shall study the minima of $V(r_c)$ by staying away from the GW limit of $\frac{k \lambda_v v_v^2}{v_v^2} \to 0$ and $\frac{k \lambda_h v_h^2}{v_h^2} \to 0$.

Let us choose the parameters so that $v_h^2 = 2 k \epsilon \lambda_h + O(\epsilon^2)$ and $v_v^2 = 2 k \epsilon \lambda_v + O(\epsilon^2)$. It then follows that $a \approx \frac{3k \epsilon}{2 \lambda_h} + O(\epsilon^2)$ and $b \approx -\frac{\epsilon}{4} + O(\epsilon^2)$. Neglecting terms of $O(a \epsilon^2)$ and higher the modulus potential under this condition takes the following form

$$V(r_c) \approx \frac{3k^2 \epsilon^2}{4 \lambda_h} + k \epsilon[a \tilde{\phi}^{4+2\epsilon} - v_v^2 \tilde{\phi}^4].$$

(10)

The above expression for $V(r_c)$ exhibits a minimum at $\tilde{\phi}^{2\epsilon} \approx \frac{v_v^2}{a}$ or $2 k r_c \pi \approx \ln \frac{3v_h^2}{4v_v^2}$. For $kr_c \approx 12$ and $\epsilon \approx .01$ we need to adjust $v_v$ and $v_h$ so that $\frac{v_h^2}{v_v^2} \approx 2.8$, a condition that does not involve a large hierarchy between $v_v$ and $v_h$ and hence no fine tuning. On the other hand for $\epsilon \approx .1$ the minimum of the potential would occur naturally at $kr_c \approx 1$. We would like to note that the natural order of magnitude value of $\lambda_v$ and $\lambda_h$ is $k^{-2}$. Under this condition $\frac{v_v^2}{k} \approx O(\frac{v_h^2}{k}) \approx O(\epsilon) \ll 1$ and the back reaction of the bulk scalar field on the background metric can be neglected [2, 3]. The mass of the stabilized radion can also be derived from (10). It can be shown that

$$m_\phi^2 = \frac{1}{f^2} \frac{\partial^2 V}{\partial \phi^2} \approx 16 \epsilon^3 \frac{k^2}{f^2 \lambda_v} e^{-2kr_c \pi}.$$ 

(11)
If \( \lambda_v \approx k^{-2} \) then \( m_\phi \approx \sqrt{\frac{2}{3} \epsilon^2 k \epsilon^{-kr_c \pi}} \approx 0.8 \text{ GeV.} \)

If we let \( \lambda_h \to \infty \) keeping all other parameters fixed then \( a \to v_h^2 \). The condition for getting a minimum at \( kr_c \approx 12 \) for \( \epsilon \approx 0.01 \) then becomes \( \frac{\nu^2}{\lambda_h^2} \approx 2.1 \). The requirement for minimum at \( kr_c \approx 12 \) for \( \epsilon \approx 0.01 \) therefore does not change significantly as \( \lambda_h \to \infty \) starting from some finite value.

We shall now determine the form of the modulus potential for \( kr_c \ll 1 \). Although a minimum at \( kr_c \ll 1 \) may not be useful from the point of view of generating the weak scale-Planck scale hierarchy it would be useful for comparing the condition for minimum at small \( kr_c \) with that for large \( kr_c \) particularly with regard to their compatibility. In this case one can expand the exponentials as a power series in \( kr_c \) and keep only the leading terms. We find that the constants \( A \) and \( B \) to leading order in \( kr_c \) are given by

\[
A \approx -\frac{x}{4} \sqrt{\frac{k}{\lambda_h}} (1 - \frac{3}{2}x) \quad \text{and} \quad B \approx \sqrt{\frac{k}{\lambda_h}} (1 + \frac{x}{2}).
\]

Here \( x = \nu kr_c \pi \ll 1 \). The modulus potential to leading order is given by

\[
V(r_c) \approx \frac{k^2 \epsilon^2}{\lambda_h} (2x^2 + 3 - 4x) - 4 \frac{k^2 \epsilon^2}{\lambda_v} (1 - 2x).  \tag{11}
\]

The above expression for modulus potential exhibits a local minimum for \( kr_c \ll 1 \) provided \( \lambda_v > 2\lambda_h \) and the minimum occurs at \( x = 1 - 2 \frac{\lambda_h}{\lambda_v} \). In order to prevent any minimum from occurring at small \( kr_c \) we could choose \( \lambda_v \) and \( \lambda_h \) so that \( \lambda_v < 2\lambda_h \). This choice will still allow a minimum at \( kr_c \approx 12 \) without any fine tuning.

In conclusion in this report we have studied the GW mechanism of radion stabilization for quadratic potential on the visible brane and quartic potential on the hidden brane. We have determined the form of the modulus potential both for small and large \( kr_c \). By choosing \( v_u^2 \approx \frac{2k}{\lambda_v} + O(\epsilon^2) \) and \( v_v^2 \approx \frac{2k}{\lambda_v} + O(\epsilon^2) \) we then showed that the minimum of the potential can be arranged to occur around \( kr_c \approx 12 \) without fine tuning the parameters of the model. We have also shown that by adjusting \( \lambda_v < 2\lambda_h \) it is possible to exclude the possibility of any minimum from occurring at small \( kr_c \). Our simplified model allows us to determine the modulus potential for finite values of \( \lambda_v \) and \( \lambda_h \) in contrast to their
infinite values assumed in the original GW model. Since our model is quite close to the original GW model our work provides some indication that it might be possible to obtain a minimum at $kr_c \approx 12$ for finite $\lambda_v$ and $\lambda_h$ even with quartic potentials on both branes. However as mentioned earlier an analytic solution then becomes quite difficult.

Note added: While this work was in progress Ref. [4] appeared which discusses GW mechanism with quadratic potential on both branes. This set up is even simpler than ours and a bit more distant from the GW model.

References

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