Widths of weighted Sobolev classes on a John domain*

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1 Introduction

Denote by \( \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{R}, \mathbb{R}_+ \) the sets of natural, integer, nonnegative integer, real and nonnegative real numbers, respectively.

Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain (a domain is an open connected set), and let \( g, v : \Omega \to \mathbb{R}_+ \) be measurable functions. For each measurable vector-valued function \( \varphi : \Omega \to \mathbb{R}^m, \varphi = (\varphi_k)_{1 \leq k \leq m} \), and for each \( p \in [1, \infty] \) put

\[
\|\varphi\|_{L^p(\Omega)} = \left\| \max_{1 \leq k \leq m} |\varphi_k| \right\|^p.
\]

Let \( \overline{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_d := (\mathbb{N} \cup \{0\})^d, |\overline{\beta}| = \beta_1 + \ldots + \beta_d \). For any distribution \( f \) defined on \( \Omega \) we write \( \nabla^r f = \left( \partial^r f/\partial x^{\overline{\beta}} \right)_{|\overline{\beta}|=r} \) (here partial derivatives are taken in the sense of distributions) and denote by \( m_r \) the number of components of the vector-valued distribution \( \nabla^r f \). We also write

\[
W^r_{p,g}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid \exists \varphi : \Omega \to \mathbb{R}^{m_r} : \|\varphi\|_{L^p(\Omega)} \leq 1, \nabla^r f = g \cdot \varphi \right\}
\]

(we denote the corresponding function \( \varphi \) by \( \nabla^r f/g \),

\[
\|f\|_{L^q,v(\Omega)} = \|f\|_{q,v} = \|fv\|_{L^q(\Omega)} , \quad L_{q,v}(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \|f\|_{q,v} < \infty \} .
\]

For \( x \in \mathbb{R}^d \) and \( \rho > 0 \) we shall denote by \( B_\rho(x) \) a closed euclidean ball of radius \( \rho \) in \( \mathbb{R}^d \) centered at the point \( x \).

**Definition 1.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain, and let \( a > 0 \). We say that \( \Omega \in \text{FC}(a) \) if there exists a point \( x_* \in \Omega \) such that for any \( x \in \Omega \) there exists a curve \( \gamma_x : [0, T(x)] \to \Omega \) with the following properties:

1. \( \gamma_x \in AC[0, T(x)], |\dot{\gamma}_x| = 1 \ a.e. \),
2. \( \gamma_x(0) = x, \gamma_x(T(x)) = x_* \),

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3. $B_{nt} (\gamma_x(t)) \subset \Omega$ holds for any $t \in [0, T(x)]$.

**Definition 2.** We say that $\Omega$ satisfies the John condition (and call $\Omega$ a John domain) if $\Omega \in FC(a)$ for some $a > 0$.

For a bounded domain the John condition is equivalent to the flexible cone condition (see the definition in [1]).

Reshetnyak in the papers [2][3] constructed the integral representation for functions defined on a John domain $\Omega$ (see the definition in [1]).

Definition 2. We say that $\Omega \in FC(a)$ if $\Omega \subset \mathbb{R}^d$ is an embedding operator of a space $X$ and by the linear $n$-width of a set $M \subset X$ in the space $X$ we mean the quantity

$$d_n(M, X) = \inf_{L \in \mathcal{L}_n(X)} \sup_{x \in M} \sup_{y \in L} \|x - y\|_X,$$

and by the linear $n$-width the quantity

$$\lambda_n(M, X) = \inf_{A \in L(X, X), \text{rk} A \leq n} \sup_{x \in M} \|x - Ax\|_X.$$

The approximation numbers of an operator $A \in L(X, Y)$ are defined by

$$A_n(A) = \inf \{\|A - A_n\|_{X \to Y} : \text{rk} A_n \leq n\}.$$

If $A$ is an embedding operator of a space $X$ in a space $Y$ and if $M \subset X$ is a unit ball, then we write $A_n(A) = A_n(M, Y)$. If the operator $A$ is compact, then from Heinrich’s result [23] follows that

$$A_n(M, Y) = \lambda_n(A(M), Y). \quad (1)$$

Let $X, Y$ be sets, $f_1, f_2 : X \times Y \to \mathbb{R}_+$. We write $f_1(x, y) \lesssim f_2(x, y)$ (or $f_2(x, y) \gtrsim f_1(x, y)$) if for any $y \in Y$ there exists $c(y) > 0$ such that $f_1(x, y) \leq c(y)f_2(x, y)$ for each $x \in X$; $f_1(x, y) \preceq f_2(x, y)$ if $f_1(x, y) \lesssim f_2(x, y)$ and $f_2(x, y) \lesssim f_1(x, y)$. 

For properties of weighted Sobolev spaces and their generalizations, see the books [6][11] and the survey paper [12]. Sufficient conditions of boundedness and compactness, whose dimension does not exceed $d$, were obtained by Kudryavtsev [13], Kufner [6], Triebel [7], Lizorkin and Otelbaev [14], Gurka and Opic [15], Besov [16][19], Antoci [20], Gol’dstein and Ukhlov [21], and other authors.

Let $(X, \| \cdot \|_X)$ be a linear normed space, let $n \in \mathbb{Z}_+$, $\mathcal{L}_n(X)$ be the family of subspaces of $X$ whose dimension does not exceed $n$. Denote by $L(X, Y)$ the space of continuous linear operators from $X$ into a normed space $Y$, by $\text{rk} A$ the dimension of the image of the operator $A : X \to Y$, and by $\|A\|_{X \to Y}$ its norm. By the Kolmogorov $n$-width of a set $M \subset X$ in the space $X$ we mean the quantity

$$d_n(M, X) = \inf_{L \in \mathcal{L}_n(X)} \sup_{x \in M} \|x - y\|_X,$$

and by the linear $n$-width the quantity

$$\lambda_n(M, X) = \inf_{A \in L(X, X), \text{rk} A \leq n} \sup_{x \in M} \|x - Ax\|_X.$$
In the 1960–1970s authors investigated problems concerning the values of the widths of function classes in $L_q$ (see [24–30, 32, 38] and also [40, 41, 42]) and of finite-dimensional balls $B^n_p$ in $l^n_q$. Here $l^n_q$ ($1 \leq q \leq \infty$) is the space $\mathbb{R}^n$ with the norm
\[ \| (x_1, \ldots, x_n) \|_q \equiv \| (x_1, \ldots, x_n) \|_{l^n_q} = \begin{cases} (|x_1|^q + \cdots + |x_n|^q)^{1/q}, & \text{if } q < \infty, \\ \max\{|x_1|, \ldots, |x_n|\}, & \text{if } q = \infty, \end{cases} \]
$B^n_p$ is the unit ball in $l^n_q$. For $p \geq q$, Pietsch [33] and Stesin [41] found the precise values of $d_n(B^n_p, l^n_q)$ and $\lambda_n(B^n_p, l^n_q)$. In the case of $p < q$, Kashin [28], Gluskin [45] and Garnaev, Gluskin [46] determined order values of the widths of finite-dimensional balls up to quantities depending on $p$ and $q$ only.

Order estimates for widths of non-weighted Sobolev classes on a segment were obtained by Tikhomirov, Ismagilov, Makovoz and Kashin [24, 25, 27, 28, 30]. In the case of multidimensional cube the upper estimate of widths (which is not always precise) was first obtained by Birman and Solomyak [31]. After publication of Kashin’s result in [28] estimates for widths of Sobolev classes on a multidimensional torus and their generalizations were found by Temlyakov and Galeev [32–36]. In papers of Kashin [37], (for $d = 1$) and Kulanin [38] (for $d > 1$) estimates of widths were found in the case of “small-order smoothness”. Here the upper estimate was not precise in the case $d > 1$ (with a logarithmic factor). The correct estimate follows from embedding theorems between Sobolev and Besov spaces and from the estimate of widths for embeddings of Besov classes (see, e.g., [39]). Let us formulate the final result.

Let $r \in \mathbb{N}$, $1 \leq p, q \leq \infty$. Denote $\eta_{pq} = \frac{1}{2} \cdot \frac{\frac{1}{p} \cdot \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}} \neq \left( \frac{r}{d} + \frac{1}{q} - \frac{1}{p} \right)^{-1}.$

**Theorem A.** Denote
\[ \theta_{p,q,r,d} = \begin{cases} \frac{r}{d}, & \text{if } p \geq q \text{ or } (2 \leq p < q \leq \infty, \ \frac{r}{d} \geq \eta_{pq}), \\ \frac{r}{d} + \frac{1}{q} - \frac{1}{p}, & \text{if } 1 \leq p < q \leq 2, \\ \frac{r}{d} + \frac{1}{q} - \frac{1}{p}, & \text{if } 1 < p < 2 < q \leq \infty \text{ and } \frac{r}{d} \geq \frac{1}{p}, \\ \frac{d}{p} \left( \frac{r}{d} + \frac{1}{q} - \frac{1}{p} \right), & \text{if } (p < 2 < q, \ \frac{r}{d} < \frac{1}{p} \text{ or } (2 \leq p < q, \ \frac{r}{d} < \eta_{pq}), \end{cases} \]
\[ \tilde{\theta}_{p,q,r,d} = \theta_{p,q,r,d} \text{ for } \frac{r}{d} + \frac{1}{q} \geq 1, \ \tilde{\theta}_{p,q,r,d} = \theta_{q',q',r,d} \text{ for } \frac{r}{d} + \frac{1}{q} < 1. \]

Suppose that $\frac{r}{d} \neq \frac{1}{p}$ holds in the case $1 \leq p < 2 < q \leq +\infty$, and $\frac{r}{d} \neq \eta_{pq}$ holds in the case $2 \leq p < q \leq +\infty$. Then
\[ d_n(W_p^r[0, 1]^d, L_q[0, 1]^d) \asymp n^{-\theta_{p,q,r,d}}. \]

Suppose that $\frac{r}{d} \neq \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$ holds in the case $1 \leq p < 2 < q \leq +\infty$. Then
\[ \lambda_n(W_p^r[0, 1]^d, L_q[0, 1]^d) \asymp n^{-\tilde{\theta}_{p,q,r,d}}. \]

Let us formulate the main result of this paper. Denote by $L_*^{\pm}(\Omega)$ the class of functions $w : \Omega \to \mathbb{R}^+$ such that there exists a sequence of functions $w_n : \Omega \to \mathbb{R}^+$, $n \in \mathbb{N}$, with the following properties:

3
• \( 0 \leq w_n(x) \leq w(x) \) for any \( x \in \Omega \);

• there exists a finite family of non-overlapping cubes \( K_{n,i} \subset \Omega \), \( 1 \leq i \leq N_n \), such that \( w_n|_{K_{n,i}} = \text{const} \), \( w_n(x) = 0 \) for \( x \in \Omega \setminus \bigcup_{i=1}^{N_n} K_{n,i} \);

• \( w_n(x) \to w(x) \) a.e. on \( \Omega \).

For sets \( A, B \subset \mathbb{R}^d \) and for a point \( x \in \mathbb{R}^d \) we write \( |x| = \|x\|_{l_2} \), dist \((x, A) = \inf_{y \in A} |x - y| \), dist \((A, B) = \inf_{y \in A, z \in B} |y - z| \).

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain such that \( \Omega \in \mathcal{FC(a)} \), let \( r \in \mathbb{N} \), \( 1 < p \leq \infty, 1 \leq q < \infty \), and let \( \frac{1}{q} + \frac{1}{r} - \frac{1}{p} > 0 \). Let \( \Gamma', \Gamma'' \subset \partial \Omega \) be closed sets, let \( g(x) = g_0(x)\tilde{g}(x), v(x) = v_0(x)\tilde{v}(x), x \in \Omega \), \( g_0 \in L_\alpha(\Omega, \mathbb{R}^+), v_0 \in L_\beta(\Omega, \mathbb{R}^+) \), \( 1 < \alpha, \beta \leq \infty, \beta > q \), \( \frac{1}{p} + \frac{1}{a} < 1, \frac{1}{r} := \frac{1}{d} + \frac{1}{q} - \frac{1}{p} - \frac{1}{a} \geq 0 \); suppose that if \( \frac{1}{r} = 0 \), then \( \tilde{g} = \tilde{v} = 1 \), and if \( \frac{1}{r} > 0 \), then \( \tilde{g}(x) \in \mathcal{L}_\omega(\Omega) \),

\[
\tilde{g}(x) = \varphi_{\tilde{g}}(\text{dist}(x, \Gamma')), \quad \tilde{v}(x) = \varphi_{\tilde{v}}(\text{dist}(x, \Gamma''));
\]

here the function \( \varphi_{\tilde{g}} : (0, +\infty) \to \mathbb{R}_+ \) decreases, the function \( \varphi_{\tilde{v}} : (0, +\infty) \to \mathbb{R}_+ \) increases, and there exists a number \( c_0 \geq 1 \) such that for any \( m \in \mathbb{Z}, t, s \in [2^{m-1}, 2^{m+1}] \)

\[
c_0^{-1} \leq \frac{\varphi_{\tilde{g}}(t)}{\varphi_{\tilde{g}}(s)} \leq c_0, \quad c_0^{-1} \leq \frac{\varphi_{\tilde{v}}(t)}{\varphi_{\tilde{v}}(s)} \leq c_0. \tag{3}
\]

1. Suppose that \( \frac{1}{q} \neq \frac{1}{p} \) holds for \( 1 < p < 2 < q < +\infty \), and \( \frac{1}{q} \neq \eta_{pq} \) holds for \( 2 \leq p < q < +\infty \). Then

\[
\lim_{n \to \infty} n^\theta_{p,q,r,a} d_n(W_{p,q}(\Omega), L_{q,v}(\Omega)) \lesssim \|gv\|_\mathcal{X}.
\]

2. Suppose that \( \frac{1}{q} \neq \max\left\{ \frac{1}{p}, \frac{1}{q}\right\} \) holds for \( 1 < p < 2 < q < +\infty \). Then

\[
\lim_{n \to \infty} n^\theta_{p,q,r,a} d_n(W_{p,q}(\Omega), L_{q,v}(\Omega)) \lesssim \|gv\|_\mathcal{X}.
\]

If \( g, v \in \mathcal{L}_+(\Omega) \), then

\[
\lim_{n \to \infty} n^\theta_{p,q,r,a} d_n(W_{p,q}(\Omega), L_{q,v}(\Omega)) \gtrsim \|gv\|_\mathcal{X},
\]

\[
\lim_{n \to \infty} n^\theta_{p,q,r,a} d_n(W_{p,q}(\Omega), L_{q,v}(\Omega)) \lesssim \|gv\|_\mathcal{X}.
\]
For common domains and \( r = 1, p = q \) Evans, Edmunds and Harris [22,47] obtained a sufficient condition under which the approximation numbers have the same orders as for a cube. In addition, note the results of Evans, Harris, Lang and Solomyak [48,49] on approximation numbers of weighted Sobolev classes on a metric graph for \( r = 1, p = q \). Also the author knows the recent Besov’s result on coincidence of orders of widths
\[
d_n(W_p^r(K_\sigma), L_q(K_\sigma)) \asymp d_n(W_p^r([0, 1]^d), L_q([0, 1]^d)),
\]
where
\[
K_\sigma = \{(x_1, \ldots, x_{d-1}, x_d) : |(x_1, \ldots, x_{d-1})|^{1/\sigma} < x_d < 1\},
\]
\( \sigma > 1, r - [\sigma(d - 1) + 1] \left( \frac{1}{p} - \frac{1}{q} \right)_+ > 0 \).

2 Notations

We denote by \( \overline{A} \), or \( \text{int} A \), or \( \text{mes} A \), or \( \text{card} A \) the closure of the set \( A \), or its interior, or its Lebesgue measure or its cardinality, respectively. If the set \( A \) is contained in some subspace \( L \subset \mathbb{R}^d \) of dimension \( (d - 1) \), then we denote by \( \text{int}_{d-1} A \) the interior of the set \( A \) with respect to the induced topology of the space \( L \). We say that the sets \( A, B \subset \mathbb{R}^d \) do not overlap if \( A \cap B \) has the Lebesgue measure zero. For a convex set \( A \) we denote by \( \dim A \) the dimension of the affine span of the set \( A \).

Let \( \gamma \) be a rectifiable curve in \( \mathbb{R}^d \). We shall denote by \( |\gamma| \) its length.

Let \( K \) be a family of closed cubes in \( \mathbb{R}^d \) with axes parallel to coordinate axes. For a cube \( K \in \mathcal{K} \) and for \( s \in \mathbb{Z}_+ \) we denote by \( \Xi_s(K) \) the set of \( 2^s \) closed non-overlapping cubes of the same size that form a partition of \( K \), and write \( \Xi(K) := \bigcup_{s \in \mathbb{Z}_+} \Xi_s(K) \). We note the following property of \( \Xi(K), \ K \in \mathcal{K} \): if \( \Delta_1, \Delta_2 \in \Xi(K) \), then either \( \Delta_1 \) and \( \Delta_2 \) do not overlap or we have either \( \Delta_1 \in \Xi(\Delta_2) \) or \( \Delta_2 \in \Xi(\Delta_1) \).

Denote by \( \chi_E \) an indicator function of a set \( E \).

We recall some definitions from graph theory. Throughout, we assume that the graphs have neither multiple edges nor loops.

Let \( \Gamma \) be a graph which contains no more than a countable number of vertices. We shall denote by \( \mathbf{V}(\Gamma) \) and by \( \mathbf{E}(\Gamma) \) the set of vertices and the set of edges of \( \Gamma \), respectively. Two vertices are called adjacent if there is an edge between them. We shall identify pairs of adjacent vertices with edges connecting them. If a vertex is an endpoint of an edge, we say that these vertex and edge are incident. If \( v_i \in \mathbf{V}(\Gamma), 1 \leq i \leq n \), the vertices \( v_i \) and \( v_{i+1} \) are adjacent for any \( i = 1, \ldots, n - 1 \), then the sequence \( (v_1, \ldots, v_n) \) is called a path of length \( n - 1 \). If all vertices \( v_i \) are distinct, then such a path is called simple. If \( n \geq 4, (v_1, \ldots, v_{n-1}) \) is a simple path and \( v_1 = v_n \), then such a path is called a cycle. We say that a path \( (v_1, \ldots, v_{n-1}, v_n) \) is almost simple, if the path \( (v_1, \ldots, v_{n-1}) \) is simple (in particular, simple paths and cycles are almost simple). Let \( \Gamma \) be a directed graph, let \( v_i \) be a head of the arc \( (v_i, v_{i+1}) \), and let \( v_{i+1} \) be its tail for any \( i = 1, \ldots, n - 1 \). Then we say that the path \( (v_1, \ldots, v_n) \) is directed; here \( v_1 \) is the origin and \( v_n \) is the destination of this path. We say that a graph is
connected if there is a finite path from any vertex to any other vertex in the graph. If a connected graph has no cycles, then it is called a tree.

Let \((T, v_0)\) be a tree with a distinguished vertex (or a root) \(v_0\). Then a partial order on \(V(T)\) is introduced as follows: we say that \(v' > v\) if there exists a path \((v_0, v_1, \ldots, v_n, v')\) such that \(v = v_k\) for some \(k \in 0, n\). In this case we put \(\rho(v, v') = \rho(v', v) = n + 1 - k\) and call this value the distance between \(v\) and \(v'\). In addition, put \(\rho(v, v) = 0\). If \(v' > v\) or \(v' = v\), then we write \(v' \geq v\) and put \([v, v'] := \{v'' \in V(T) : v \leq v'' \leq v'\}\). Denote by \(V_1(v)\) the set of vertices \(v' > v\) such that \(\rho(v, v') = 1\). Let \(v \in V(T)\). Denote by \(T_v = (T_v, v)\) a subtree of \(T\) with a set of vertices

\[
\{v' \in V(T) : v' \geq v\}. \tag{4}
\]

The introduced partial order on \(T\) induces a partial order on its subtree.

We notice the following property of a tree \((T, v_0)\): if its vertices \(v'\) and \(v''\) are incomparable, then \(T_{v'} \cap T_{v''} = \emptyset\).

Let \(T\) be a tree. Denote by \(ST(T)\) the set of subtrees in \(T\). If \(T_1, T_2 \in ST(T)\) and \(V(T_1) \subset V(T_2)\), then we say that \(T_1 \subset T_2\).

Let \(W \subset V(T)\). We say that \(W \in \text{VST}(T)\) if \(W = V(T')\) for some \(T' \in \text{ST}(T)\). Notice that \(V(T_1) \cap V(T_2) \subset \text{VST}(T)\) holds for any trees \(T_1, T_2 \in \text{ST}(T)\). Denote by \(T_1 \cap T_2\) a subtree with a set of vertices \(V(T_1) \cap V(T_2)\). If \(V(T_1) \cup V(T_2) \in \text{VST}(T)\) (or \(V(T_1) \setminus V(T_2) \in \text{VST}(T)\), then we denote by \(T_1 \cup T_2\) (\(T_1 \setminus T_2\), respectively) a subtree with a set of vertices \(V(T_1) \cup V(T_2)\) (or \(V(T_1) \setminus V(T_2)\), respectively). If \(V(T_1) \cap V(T_2) = \emptyset\), then we write \(T_1 \cup T_2 = T_1 \setminus T_2\).

Let \(T, T_1, \ldots, T_k\) be trees that have no common vertices, and let \(v_1, \ldots, v_k \in V(T), w_j \in V(T_j), j = 1, \ldots, k\) \((k \in \mathbb{N} \cup \{\infty\})\). Denote by

\[
J(T, T_1, \ldots, T_k; v_1, w_1, \ldots, v_k, w_k)
\]
a tree obtained from \(T, T_1, \ldots, T_k\) by an edge connecting the vertex \(v_j\) with the vertex \(w_j\) for \(j = 1, \ldots, k\).

### 3 Auxiliary assertions

Let \(\Theta \subset \Xi([0, 1]^d)\) be a set of non-overlapping cubes.

**Definition 3.** Let \(G\) be a graph, and let \(F : V(G) \to \Theta\) be a one-to-one mapping. We say that \(F\) is consistent with the structure of the graph \(G\) if the following condition holds: for any adjacent vertices \(v', v'' \in V(G)\) the set \(\Gamma_{v', v''} := F(v') \cap F(v'')\) has dimension \(d - 1\).

**Remark.** If the mapping \(F\) is consistent with the structure of a graph \(G\), the vertices \(v'\) and \(v''\) are adjacent and \(\text{mes} F(v') \geq \text{mes} F(v'')\), then \(F(v') \cap F(v'')\) is a \((d - 1)\)-dimensional face of the cube \(F(v'')\).

Let \((T, v_u)\) be a tree, and let \(F : V(T) \to \Theta\) be a one-to-one mapping consistent with the structure of the tree \(T\). For any adjacent vertices \(v'\) and \(v''\) we set \(\Gamma_{v', v''} = \ldots\)
\[ \Omega_{T',F} = \left( \bigcup_{v \in V(T')} \text{int} F(v) \right) \cup \left( \bigcup_{(v',v'') \in E(T')} \Gamma_{v',v''} \right). \]  

If \( v \in V(T) \) and \( \Delta = F(v) \), then we denote \( \Omega_{v,\Delta} = \Omega_{[v,v],F}. \)

Let \( v', v'' \) be adjacent vertices of \( T \), let \( \Gamma_{v',v''} \) coincide with a \((d-1)\)-dimensional face of \( F(v') \) (then \( \text{mes} F(v') \leq \text{mes} F(v'') \)), and let \( x', x'' \) be centers of the cubes \( F(v') \) and \( F(v'') \), respectively. Denote by \( y \) the orthogonal projection of the point \( x' \) onto \( \Gamma_{v',v''} \), and set

\[
\gamma_{v'v''}(t) = \left\{ \begin{array}{ll}
\frac{|x'-y|-t}{|x'-y|} x' + \frac{t}{|x'-y|} y, & 0 \leq t \leq |x' - y|, \\
\frac{|x''-y|-t}{|x''-y|} x'' + \frac{t}{|x''-y|} y, & |x' - y| \leq t \leq |x'' - y| + |x' - y|,
\end{array} \right.
\]

\[
\gamma_{v''v'}(t) = \left\{ \begin{array}{ll}
\frac{|x''-y|-t}{|x''-y|} x'' + \frac{t}{|x''-y|} y, & 0 \leq t \leq |x'' - y|, \\
\frac{|x'-y|-t}{|x'-y|} x' + \frac{t}{|x'-y|} y, & |x'' - y| \leq t \leq |x'' - y| + |x' - y|.
\end{array} \right.
\]

Let \( \Delta \in \Xi([0,1]^d) \). Denote by \( m(\Delta) \) such \( m \in \mathbb{N} \) that \( \Delta \in \Xi_m([0,1]^d) \). For any vertex \( v \in V(T) \) put \( m_v = m(F(v)) \).

**Lemma 1.** Let \((T, v_*)\) be a tree, and let \( F : V(T) \to \Theta \) be consistent with the structure of \( T \). Suppose that there exist \( l_*, k_* \in \mathbb{N} \) such that for any vertices \( v', v'' \in V(T) \), \( v' > v'' \)

\[ l_*(m_{v'} - m_{v''}) \geq \rho(v', v'') - k_. \]  

Then there exists \( \hat{a} = \hat{a}(k_*, l_*, d) \) such that for any subtree \( T' \) of \( T \) the set \( \Omega_{T',F} \) is a domain belonging to the class \( \text{FC} (\hat{a}) \). Here the curve \( \gamma_x \) from Definition [7] can be chosen so that

\[ B_{\hat{a}}(\gamma_x(t)) \subset \Omega_{v,F(w)}, \text{ if } x \in F(w). \]

In addition,

\[ \text{mes} \Omega_{T',F} \geq \hat{a},d \text{ mes } F(v), \]

where \( v \) is the minimal vertex of \( T' \).

**Proof.** Let \( T' \) be a subtree in \( T \), let \( v \) be the minimal vertex of \( T' \), let \( x_v \) be the center of the cube \( F(v) \), and let \( z \in \Omega_{T',F} \). Then \( z \in F(v') \), where \( v' \) is a vertex of \( T' \). Define the curve \( \gamma_z \) with the starting point \( z \) and the endpoint \( x_v \) as follows. Let \( v_1 > v_2 > \ldots > v_k \) be a sequence of vertices in \( T' \) such that \( v_1 = v' \), \( v_k = v \), \( \rho(v_j, v_{j+1}) = 1 \), \( j = 1, \ldots, k - 1 \). Denote by \( x_j \) the center of \( F(v_j) \), \( s_j = |\gamma_{v_{j-1}v_j}| \), \( \tau_1 = |z - x_1| \), \( \tau_j = \tau_{j-1} + s_j, 2 \leq j \leq k \),

\[
\gamma(t) = \left\{ \begin{array}{ll}
\frac{t}{\tau_1} z + \frac{1}{\tau_1} x_1, & 0 \leq t \leq \tau_1, \\
\gamma_{v_{j-1}v_j}(t - \tau_{j-1}), & \tau_{j-1} \leq t \leq \tau_j, 2 \leq j \leq k,
\end{array} \right.
\]
\[ E_t = \begin{cases} F(v_1), & 0 \leq t \leq \tau_1, \\ F(v_{j-1}) \cup F(v_j), & \tau_{j-1} \leq t \leq \tau_j, \quad 2 \leq j \leq k. \end{cases} \]

For any \( t \in [0, \tau_k] \) denote by \( a_t \) the maximal radius of an open ball centered at \( \gamma(t) \) that is contained in \( E_t \). Show that

\[ a_t \geq t. \tag{9} \]

This will imply the first assertion of Lemma and (7).

Denote by \( \sigma_j \) the length of the side of \( F(v_j) \). Then \( \tau_1 \leq d \sigma_1, s_j \lesssim d \max \{ \sigma_j, \sigma_{j-1} \} \).

Notice that for \( t \in [0, \tau_1] \) the inequality \( a_t \geq t \) holds. Let \( j \geq 2, \tau_{j-1} \leq t \leq \tau_j \), and let \( \tilde{t}_j \in [\tau_{j-1}, \tau_j] \) be such that \( \gamma(\tilde{t}_j) \in \Gamma_{v_{j-1}, v_j} \). We have

\[
\tau_{j-1} = \tau_1 + \sum_{i=2}^{j-1} s_i \lesssim \tau_1 + \sum_{i=2}^{j-1} \max \{ \sigma_i, \sigma_{i-1} \} \lesssim \sum_{i=1}^{j-1} 2\sigma_i = \sum_{i=1}^{j-1} 2^{-m_{v_i}+1} = \]

\[
= 2^{-m_{\nu_{j-1}+1}} \sum_{i=1}^{j-1} 2^{-m_{v_i}+m_{v_{j-1}}} \lesssim_{d,k,\nu,d} \sigma_{j-1} \sum_{i=1}^{j-1} \frac{2^{-m_i}}{1} \lesssim \sigma_{j-1}. \tag{10}
\]

Show that

\[ a_t \geq \max \{ t - \tilde{t}_j, \sigma_{j-1} \}. \tag{11} \]

Indeed, if \( t \in [\tau_{j-1}, \tilde{t}_j) \), then \( \max \{ t - \tilde{t}_j, \sigma_{j-1} \} = \sigma_{j-1} \); if \( t \in [\tilde{t}_j, \tau_j) \), then \( a_t \geq \frac{t-\tilde{t}_j}{2(\tau_j-\tau_{j-1})} \sigma_j \geq \frac{t-\tilde{t}_j}{2(\gamma(\tilde{t}_j)-\gamma(\tilde{t}_j))} \sigma_j \geq \frac{t-\tilde{t}_j}{2} \tilde{t}_j \). From (10) follows that \( \min \{ \sigma_j, \sigma_{j-1} \} \geq d,k,\nu,d \sigma_{j-1} \).

Finally, \( a_t \geq \min \{ \sigma_j, \sigma_{j-1} \} \), which yields (11).

Let us prove (9). If \( t - \tilde{t}_j \leq \sigma_{j-1} \), then

\[ t \leq \tilde{t}_j + \sigma_{j-1} = \tau_{j-1} + (\tilde{t}_j - \tau_{j-1}) + \sigma_{j-1} \lesssim_{d,k,\nu} \sigma_{j-1} \lesssim_{d,k,\nu} a_t. \]

Let \( t - \tilde{t}_j > \sigma_{j-1} \). If \( t - \tilde{t}_j \leq \frac{t}{2} \), then

\[ t \leq 2\tilde{t}_j = 2\tau_{j-1} + 2(\tilde{t}_j - \tau_{j-1}) \lesssim_{d,k,\nu} \sigma_{j-1} \lesssim_{d,k,\nu} a_t. \]

If \( t - \tilde{t}_j > \frac{t}{2} \), then \( a_t \geq_{d,k,\nu,d} t - \tilde{t}_j > \frac{t}{2} \).

The relation (8) follows from the inclusion \( F(v) \subset \Omega_{\Gamma, F} \) and from the estimate \( \text{diam} \Omega_{\Gamma, F} \approx \text{diam} F(v) \) (which is the consequence of the definition of \( \text{FC}(\hat{a}) \)).

\[ \Box \]

**Definition 4.** Let \( \Gamma \) be a finite direct graph, let \( v_* \in V(\Gamma) \) and \( k \in \mathbb{N} \). We say that \((\Gamma, v_*) \in \Phi_k \) if for any \( v \in V(\Gamma) \) there exists a simple directed path with the origin \( v_* \) and the destination \( v \) such that the length of this path does not exceed \( k \).
**Lemma 2.** Let $\Gamma$ be a finite directed graph, let $v_* \in V(\Gamma)$, $k \in \mathbb{N}$, and let $(\Gamma, v_*) \in \mathcal{G}_k$. In addition, suppose that $v_*$ is the head of all edges incident to $v_*$. Then there exists a tree $T$ rooted at $v_*$, which is a subgraph of $\Gamma$ such that $V(T) = V(\Gamma)$ and for any $v \in V(T)$ the inequality $\rho(v, v_*) \leq k$ holds.

**Proof.** Notice that the graph $\Gamma$ is connected. Denote by $\mathcal{S}$ the set of directed almost simple paths with the origin $v_*$ (this set is finite). Let $v \neq v_*$. Denote by $\mathcal{S}_v$ the set of paths belonging to $\mathcal{S}$ with the destination $v$. We say that the path belongs to $\mathcal{S} = \mathcal{S}(\Gamma)$ if it belongs to $\mathcal{S}_v$ for some $v \neq v_*$ and $\text{card} \mathcal{S}_v \geq 2$.

Show that

1. if $\mathcal{S} = \emptyset$, then $\Gamma$ is a tree;

2. if $\mathcal{S} \neq \emptyset$, then there exists a graph $\tilde{\Gamma}$, which is obtained from $\Gamma$ by removing an edge (while preserving the vertices), such that $(\tilde{\Gamma}, v_*) \in \mathcal{G}_k$.

Employing the preceding statements, by induction we attain the assertion of the Lemma. Indeed, set $\Gamma_0 = \Gamma$. Let for some $n \in \mathbb{Z}_+$ the graph $\Gamma_n$ be constructed by removing $n$ edges in $\Gamma$ such that $(\Gamma_n, v_*) \in \mathcal{G}_k$. If $\mathcal{S}(\Gamma_n) = \emptyset$, then $\Gamma_n$ is the desired tree, otherwise we apply assertion 2 to $(\Gamma_n, v_*)$ and define the graph $\Gamma_{n+1}$. Since the graph $\Gamma$ is finite, then there exists $m \in \mathbb{N}$ such that $\mathcal{S}(\Gamma_m) = \emptyset$.

Let us prove assertion 1. Assume that $\Gamma$ is not a tree. Then $\Gamma$ has a cycle. Hence, there exists a vertex $v$, which can be connected with $v_*$ via two distinct simple paths (not necessarily directed). Since $(\Gamma, v_*) \in \mathcal{G}_k$, then there exists a simple directed path $s = (v_*, v_1, \ldots, v_n)$ with the origin $v_*$ and the destination $v = v_n$. Let

$$\bar{s} = (v_*, v'_1, \ldots, v'_l, v_{m+1}, \ldots, v_n)$$

be another simple path connecting $v_*$ and $v$, and let $v'_l \neq v_m$. Put $v'_{l+1} = v_{m+1}$, $v'_{l+2} = v_m$, $v'_l = v_*$. Let $W \subset \{v'_0, \ldots, v'_l, v'_{l+1}\}$ be the set of vertices $v'_j$ that are heads of $(v'_j, v'_{j+1})$. By the condition of Lemma, $v'_0 \in W$. In addition, $v'_{l+1} \notin W$ (since the path $s$ is directed and its origin is $v_*$). Let

$$\hat{l} = \max\{j \in \overline{0,l}: v'_j \in W\}.$$

Show that $\text{card} \mathcal{S}_{v_1}^{v_{l+1}} \geq 2$. Indeed, by the condition of Lemma, there exist directed simple paths $\sigma' \in \mathcal{S}_{v'_l}$ and $\sigma'' \in \mathcal{S}_{v'_{l+1}}$. Appending the vertex $v'_{l+1}$, we obtain two paths which belong to $\mathcal{S}_{v'_l}^{v_{l+1}}$ (notice that $v'_{l+1} \notin \{v'_l, v'_{l+2}\}$). In order to prove that these paths are different, it is sufficient to check that $v'_{l} \neq v'_{l+2}$. Indeed, if $\hat{l} \leq l - 1$, then it is true, since the path $\bar{s}$ is simple; if $\hat{l} = l$, then it follows from the condition $v'_l \neq v_m$.

Prove assertion 2. Actually, there exists the path $(v_*, v_1, \ldots, v_k) \in \mathcal{S}$ such that

$$k_0 = \max\{|s|: s \in \mathcal{S}\},$$

(12)

where $|s|$ is the length of the path $s$. Since $\text{card} \mathcal{S}_{v_{k_0}} \geq 2$, then by (12) and by the condition $\Gamma \in \mathcal{G}_k$, there exists a direct path $s' = (v_*, v'_1, \ldots, v'_k, v_k, \ldots, v_{k_0}) \in \mathcal{S}_{v_{k_0}}$ such that

$$|s'| = \min\{|s|: s \in \mathcal{S}_{v_{k_0}}\}$$

(13)
and \( v' \neq v_{k_1} \). Then the path \( s' \) is simple. Remove the edge \((v_{k_1}, v_{k_1})\) and obtain the graph \( \Gamma' \). Show that \((\Gamma', v_s) \in \Theta_k\). Let \( v \in V(\Gamma) \). Then there exists a direct simple path \( s_v \in \mathcal{S}_v \) in \( \Gamma \) such that \(|s_v| \leq k\). If \( s_v \) does not contain the edge \((v_{k_1}, v_{k_1})\), then it is a path in \( \Gamma' \). Let \( s_v \) contain the edge \((v_{k_1}, v_{k_1})\), that is, \( s_v \) is composed of two paths
\[
s_1 = (v_s, w_1, \ldots, w_{l_v}, v_{k_1}, v_{k_1}) \quad \text{and} \quad s_2 = (v_{k_1}, u_1, \ldots, u_{l_v}, v).
\]
Consider the path
\[
s'' = (v_s, v'_1, \ldots, v'_r, v_{k_1}, u_1, \ldots, u_{l_v}, v).
\]
Then \( s'' \) does not contain the edge \((v_{k_1}, v_{k_1})\). Removing if necessary cyclic sections in \( s'' \), we obtain a simple path \( s'' \in \mathcal{S}_v \) without the edge \((v_{k_1}, v_{k_1})\). It remains to prove that \(|s''| \leq k\). Indeed,
\[
|s''| \leq |s''| \leq l + 1 + |s_2| \leq |s_1| + |s_2| = |s_v| \leq k.
\]
This completes the proof. \( \square \)

Formulate the Whitney covering theorem (see, e.g., [50], page 562).

**Theorem B.** Let \( \Omega \subset (0, 1)^d \) be an open set. Then there exists a family of closed pairwise non-overlapping cubes \( \Theta(\Omega) = \{\Delta_j\}_{j \in \mathbb{N}} \subset \Xi([0, 1]^d) \) with the following properties:

1. \( \Omega = \bigcup_{j \in \mathbb{N}} \Delta_j \);
2. \( \text{dist} (\Delta_j, \partial \Omega) \asymp 2^{-m(\Delta_j)} \);
3. for any \( j \in \mathbb{N} \)
\[
\text{card} \{i \in \mathbb{N} : \text{dim}(\Delta_i \cap \Delta_j) = d - 1\} \leq 12^d. \quad (14)
\]

**Lemma 3.** Let \( a > 0 \), \( \Omega \in \mathbf{FC}(a) \), \( \Omega \subset (0, 1)^d \). Then there exists a tree \((\mathcal{T}, v_0)\), a mapping \( F : V(\mathcal{T}) \to \Theta(\Omega) \) consistent with the structure of \( \mathcal{T} \), and numbers \( k_\ast = k_\ast(a, d) \in \mathbb{N} \), \( l_\ast = l_\ast(a, d) \in \mathbb{N} \) such that for any vertices \( v' > v'' \) the inequality \( (6) \) holds.

**Proof.** Index the cubes \( \{\Delta_{i,j}\}_{j \in \mathbb{N}^+, 1 \leq i \leq r_j} \) from the set \( \Theta(\Omega) \) so that \( r_0 = 1 \), \( m(\Delta_{i,j}) = \mu_j \in \mathbb{Z}^+ \), \( 1 \leq i \leq r_j \), and
\[
\mu_0 \leq \mu_1, \quad \mu_j < \mu_{j+1}, \quad j \in \mathbb{N}. \quad (15)
\]
Let \( x_\ast \) be the center of \( \Delta_{1,0} \). Since \( \Omega \in \mathbf{FC}(a) \), then there exists \( b = b(a, d) \) such that for any \( x \in \Omega \) there exists a polygonal arc \( \gamma_x : [0, T_0(x)] \to \Omega \) with the following properties:

a) \( \gamma_x \in AC[0, T_0(x)], |\dot{\gamma}_x| = 1 \text{ a.e.} \);
b) \( \gamma_x(0) = x, \gamma_x(T_0(x)) = x_\ast \);
c) for any \( t \in [0, T_0(x)] \) the inclusion \( B_{b}(\gamma_x(t)) \subset \Omega \) holds;
d) arcs of $\gamma_x$ are not parallel to any $d - 1$-dimensional coordinate planes and for each $\Delta \in \Theta(\Omega)$ the set $\gamma_x([0, T_0(x)])$ does not intersect $\Delta$ at any point of $k$-dimensional faces, $k \leq d - 2$.

Construct by induction sequences of trees $\{T_n\}_{n \in \mathbb{Z}_+}$ rooted at $v_0$, families of cubes $\Theta_n \subset \Theta(\Omega)$ and one-to-one mappings $F_n : V(T_n) \rightarrow \Theta_n$ consistent with the structure of $T_n$ and satisfying the following conditions:

1. $T_n$ is a subtree of $T_{n+1}$, $F_{n+1}|_{V(T_n)} = F_n$, $\Theta_n \supset \{\Delta_{i,n}\}_{1 \leq i \leq n}$, $n \in \mathbb{N}$, $F_0(v_0) = \Delta_{1,0}$;
2. there exists $c_s(d, a) \in \mathbb{N}$ such that for any vertex $v$ of the tree $T_n$ the inequality $m_v \leq \mu_n + c_s(d, a)$ holds;
3. there exists $c_{ss}(d, a) \in \mathbb{N}$ such that if $v, v' \in T_n \setminus T_{n-1}$, $v' > v$, then $\rho(v', v) \leq c_{ss}(d, a)$;
4. $T_n$ satisfies the condition of Lemma 1 with

\[
l_s = 1 + (c_{ss}(d, a) + 1)(c_s(d, a) + 2),
\]

\[
k_s = l_sc_s(d, a) + (c_{ss}(d, a) + 1)(c_s(d, a) + 2).
\]

As $T_0$ we take a tree consisting of the unique vertex $v_0$; put $\Theta_0 = \{\Delta_{1,0}\}$, $F_0(v_0) = \Delta_{1,0}$. The property 1 holds by construction, the property 2 follows from the equality $m_{v_0} = m(\Delta_{1,0}) = \mu_0$, properties 3 and 4 for $n = 0$ are trivial.

Let $n \in \mathbb{N}$, and let the tree $T_{n-1}$, the family of cubes $\Theta_{n-1}$ and the mapping $F_{n-1}$ satisfying the conditions 1-4 be defined. Construct $T_n$, $\Theta_n$ and $F_n$.

If $\{\Delta_{i,n}\}_{i=1}^n \subset \Theta_{n-1}$, then put $T_n = T_{n-1}$, $\Theta_n = \Theta_{n-1}$, $F_n = F_{n-1}$. Let

\[
I_n := \{i = 1, \ldots, r_n : \Delta_{i,n} \notin \Theta_{n-1}\} \neq \emptyset.
\]

Consider $i \in I_n$. Denote by $x_{i,n}$ the center of the cube $\Delta_{i,n}$ and set $\tau_{i,n} = T_0(x_{i,n})$, $\gamma_{i,n} = \gamma_{x_{i,n}}$;

\[
t_{i,n} = \min\{t \in [0, \tau_{i,n}] : \gamma_{i,n}(t) \in \Omega_{T_{n-1}, F_{n-1}}\}.
\]

Then

\[
t_{i,n} < \tau_{i,n}.
\]

Let $k = k(i)$, $\{Q_1^i, \ldots, Q_k^i\} \subset \Theta(\Omega) \setminus \Theta_{n-1}$ be the set of all cubes such that $\gamma_{i,n}([0, t_{i,n}]) \cap Q_j^i \neq \emptyset$. Then

\[
m(Q_j^i) \geq \mu_n
\]

by the property 1 of the tree $T_{n-1}$ and by (15). On the other hand, there exists $c_1(d, a) \in \mathbb{N}$ such that

\[
m(Q_j^i) \leq \mu_n + c_1(d, a).
\]
Indeed, if $Q_j^i = \Delta_{i,n}$, then it follows from the definition of $\mu_n$. If $Q_j^i \neq \Delta_{i,n}$, then for any $t$ such that $\gamma_{i,n}(t) \in Q_j^i$, the inequality $|x_{i,n} - \gamma_{i,n}(t)| \geq 2^{-\mu_n - 1}$ holds. Therefore, $t \geq 2^{-\mu_n}$. It remains to apply the property c) of the polygonal arc $\gamma_{i,n}$ and Theorem [13].

Construct the sequence of different cubes $Q_{j_1}, \ldots, Q_{j_{\nu}}$, $\nu = \nu(t)$, $1 \leq j \leq k$, such that $Q_{j_1}^i = \Delta_{i,n}$, $\gamma_{i,n}(t_{j_{\nu}}) \in Q_{j_{\nu}}$ and $\dim(Q_j^i \cap Q_{j_{\nu+1}}) = d - 1$, as well as the sequence $\tilde{t}_1 < \cdots < \tilde{t}_{\nu}$ such that $\gamma_{i,n}(\tilde{t}_s) \in Q_{j_s}^i$, $1 \leq s \leq \nu$. Set $\tilde{t}_0 = 0, \tilde{t}_1 = \max\{t \in [0, t_{i,n}] : \gamma_{i,n}(t) \in Q_{j_1}^i\}$.

Let the cubes $Q_{j_1}^i = \Delta_{i,n}$, $Q_{j_2}^i$, $\ldots$, $Q_{j_{\nu}}^i$ and numbers $\tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_s$ be constructed, with

$$\tilde{t}_s = t_{i,n}, \text{ then the construction is completed. Suppose that } \tilde{t}_s < t_{i,n}. \text{ Denote by } J_s \text{ the set of indices } j' \in \{1, \ldots, k\}\{j_1, \ldots, j_{\nu}\} \text{ such that}$$

$$\dim(Q_{j_s}^i \cap Q_{j_{\nu}}^i) = d - 1.$$ 

Prove that $J_s \neq \emptyset$. Indeed, the property d) of the polygonal arc $\gamma_{i,n}$ and (21) imply that $\gamma_{i,n}(t_s) \in \text{int}_{d-1}(Q_{j_s}^i \cap Q_{j_{\nu}}^i)$ for some $Q \in \Theta(\Omega), Q \neq Q_{j_{\nu}}$. Moreover, $Q \notin U_{i,n}^{d-1}Q_{j_s}^i$ by (21). Finally, from the definition of $t_{i,n}$ and from the condition $t_s < t_{i,n}$ follows that $Q \notin \Theta_{i,n}$. Hence, $Q \in \{Q_{j_1}^i, \ldots, Q_{j_k}^i\}\{Q_{j_1}^i, \ldots, Q_{j_{\nu}}^i\}$.

Set $\tilde{t}_{s+1} = \max\{t \in [\tilde{t}_s, t_{i,n}] : \gamma_{i,n}(t) \in \bigcup_{j' \in J_s} Q_{j'}^i\}$. Define $j_{s+1} \in J_s$ by the inclusion $\gamma_{i,n}(\tilde{t}_{s+1}) \in Q_{j_{s+1}}^i$. Show that $j_{s+1} \in J_s$ is well-defined. Actually, let $\gamma_{i,n}(\tilde{t}_{s+1}) \in Q_{j'}^i \cap Q_{j''}^i$. Indeed, the property d) of $\gamma_{i,n}$ implies that $\gamma_{i,n}(\tilde{t}_{s+1}) \in \text{int}_{d-1}(Q_{j'}^i \cap Q_{j''}^i)$. By (18), we have $\tilde{t}_{s+1} < \tau_{i,n}$. Therefore, there exists $\delta \in (0, \tau_{i,n} - \tilde{t}_{s+1})$ such that $\gamma_{i,n}(\tilde{t}_{s+1} + \delta) \in \text{int} Q_{j'}^i \cup \text{int} Q_{j''}^i$ (it follows again from the property d) of $\gamma_{i,n}$). Hence, $\tilde{t}_{s+1} < t_{i,n}$, and we get the contradiction with the definition of $\tilde{t}_{s+1}$. Notice that $\tilde{t}_{s+1} = \max\{t \in [\tilde{t}_s, t_{i,n}] : \gamma_{i,n}(t) \in Q_{j_{s+1}}^i\}$.

For each $s \in \mathbb{N}$, $\nu - 1$ denote by $\tilde{J}_s$ the set of indices $j' \in \{1, \ldots, k\}\{j_1, \ldots, j_{s+1}\}$ such that $Q_{j_{s+1}}^i \cap Q_{j'}^i \neq \emptyset$. By (19) and (20) (or by Theorem [13]), there exists $c_2(d, a) \in \mathbb{N}$ such that card $\tilde{J}_s \leq c_2(d, a)$. Let us prove that for any $s \in \mathbb{N}$ such that $\tilde{t}_{s+c_2(d,a)+2} < t_{i,n}$ the following inequality holds:

$$\tilde{t}_{s+c_2(d,a)+3} - \tilde{t}_{s+1} \geq 2^{-\mu_n - c_1(d,a)}.$$ 

Indeed, let $j_{\sigma} \in \tilde{J}_s$ for any $\sigma \in \{s + 2, \ldots, s + c_2(d, a) + 3\}$. Since all indices $j_{\sigma}$ are different, we have card $\tilde{J}_s \geq c_2(d, a) + 1$, which leads to a contradiction. Assume that there exists $\sigma \in \{s + 2, \ldots, s + c_2(d, a) + 3\}$ such that $j_{\sigma} \notin \tilde{J}_s$. Then $Q_{j_{s+1}}^i \cap Q_{j_{\sigma}}^i = \emptyset$.

Since $\gamma_{i,n}(\tilde{t}_{s+1}) \in Q_{j_{s+1}}^i$ and $\gamma_{i,n}(\tilde{t}_{\sigma}) \in Q_{j_{\sigma}}^i$, then $\tilde{t}_{\sigma} - \tilde{t}_{s+1} \geq |\gamma_{i,n}(\tilde{t}_{\sigma}) - \gamma_{i,n}(\tilde{t}_{s+1})| \geq 2^{-\mu_n - c_1(d,a)}$, which implies (22).

On the other hand, since $\gamma_{i,n}(t_{i,n}) \in Q_{j_{\nu}}^i$, then by (19) and Theorem [13] there exists $c_3(d) \in \mathbb{N}$ such that $\text{dist}(\gamma_{i,n}(t_{i,n}), \partial \Omega) \leq 2^{-\mu_n + c_3(d)}$. Therefore, $t_{i,n} \leq 2^{-\mu_n + c_3(d)}$ for
some \(c_4(d, a) \in \mathbb{N}\) (see the property c) of \(\gamma_{i,n}\). This together with \((22)\), yield that \(\nu \leq c_5(d, a)\) for some \(c_5(d, a) > 0\).

Set \(\Theta^I_n = \{Q^i_{j,i}, i \in I_n, 1 \leq s \leq \nu(i)\}\). Construct a directed graph \(G\) (without multiple edges and loops) and a one-to-one mapping \(\Phi : V(G) \rightarrow \Theta^I_n\): we regard the vertices \(v', v''\) as adjacent, regard \(v''\) as the head of \((v'', v')\) and regard \(v'\) as the tail of \((v'', v')\) if and only if there exists \(i \in I_n\) and \(s \in 1, \ldots, \nu(i) - 1\) such that \(\Phi(v') = Q^i_{j,i}\), \(\Phi(v'') = Q^i_{j,i+1}\). Since \(\dim(Q^i_{j,i} \cap Q^i_{j,i+1}) = d - 1\), then \(\Phi\) is consistent with the structure of \(G\).

Let \(\{\Phi^{-1}(Q^i_{j,i+i})\}_{i \in I_n} = \{w_1, \ldots, w_{r_i}\}\). Add to the graph \(G\) a vertex \(\hat{v}\) and connect it with vertices \(w_{v'}, 1 \leq i' \leq r_{v'}\), considering that \(\hat{v}\) is the head of these edges. Thereby, we obtain a directed graph \(\hat{G}\) such that \((\hat{G}, \hat{v}) \in \Theta_{c_5(d, a)}\) (see Definition 4). By Lemma 2 there exists a tree \(T'\) rooted at \(\hat{v}\), which is a subgraph of \(\hat{G}\) such that \(V(T') = V(\hat{G})\) and for any \(v \in V(T')\) the following inequality holds:

\[\rho(\hat{v}, v) \leq c_5(d, a).\] (23)

Since \(\gamma_{i,n}(t_{i,n}) \in Q^i_{j,i+i} = \Phi(w_{v'})\) for some \(i' \in \{1, \ldots, r_{v'}\}\), then the definition of \(t_{i,n}\) and the property d) of \(\gamma_{i,n}\) yield that there exists a vertex \(u_{v'} \in V(T_{n-1})\) and a cube \(\hat{Q}_{v'} = F_{n-1}(w_{v'})\) such that \(\dim(\hat{Q}_{v'} \cap \Phi(w_{v'})) = d - 1\). Put

\[T_n = \{T_{n-1}, (T')_{w_1}, \ldots, (T')_{w_{r_n}}; u_1, w_1, \ldots, u_{r_n}, w_{r_n}\},\]

\[F_n(v) = \begin{cases} F_{n-1}(v), & \text{if } v \in T_{n-1}, \\
\Phi(v), & \text{if } v \in V((T')_{w_i}), 1 \leq i \leq r_v.\end{cases}\]

\(\Theta_n = F_n(V(T_n))\). Then the mapping \(F_n\) is bijective and consistent with the structure of \(T_n\) (it follows from the construction, from the definition of the graph \(G\) and from the induction assumption).

The properties 1–2 of \(T_n\) follow from the construction, \((15)\) and \((20)\), the property 3 follows from \((22)\). Check the property 4. Let \(v', v'' \in V(T_n), v' > v''\). If \(v', v'' \in V(T_{n-1})\), then it holds by the induction assumption. Let \(v' \in V(T_n) \setminus V(T_{n-1})\). If for any \(v \in [v'', v']\) the inequality \(m_v \geq \mu_n\) holds, then \(v'' \in V(T_n) \setminus V(T_{n-c_5(d,a)-2})\). Indeed, if \(v \in V(T_{n-j})\) and \(m_v \geq \mu_n\), then

\[\mu_{n-j} + j - 1 \leq \mu_{n-j+1} + j - 1 \leq \mu_{n-j+2} + j - 2 \leq \cdots \leq \mu_n \leq m_v \leq \mu_{n-j} + c_5(d, a)\]

(the last inequality follows from the property 2 of the tree \(T_{n-j}\)), which implies \(j \leq 1 + c_5(d, a)\). Hence, by the property 3 of the trees \(T_k, 1 \leq k \leq n\), we have

\[\rho(v', v'') \leq (c_5(d, a) + 1)(c_5(d, a) + 2);\] (24)

by the property 2 of \(T_n\), the inequality \(m_{v'} - m_{v''} \geq \mu_n - (\mu_n + c_5(d, a)) = -c_5(d, a)\) holds, so

\[l_s(m_{v'} - m_{v''}) \geq -l_s c_5(d, a) \Rightarrow (c_5(d, a) + 1)(c_5(d, a) + 2) - k_s \geq \rho(v', v'') - k_s.\]
Assume now that \( v' \in V(T_n) \setminus V(T_{n-1}) \) and \( W := \{ v \in [v'', v'] : m_v \leq \mu_n - 1 \} \neq \emptyset \). Set \( v_* = \max W \). Then it follows from (19) that
\[
v_* \in V(T_{n-1}), \quad v_* < v'.
\] (25)

Denote by \( v_* \) the vertex in \([v_*, v']\) that is the successor of \( v_* \). By the inequality (24) applied to \( v'' := v_* \), by (25) and by the induction assumption we have
\[
l_*(m_{v'} - m_{v''}) = l_*(m_{v'} - m_{v_*}) + l_*(m_{v_*} - m_{v''}) \geq l_* + \rho(v_*, v'') - k_* =
\]
\[
= (l_* - 1 - \rho(v', v_*)) + \rho(v', v'') - k_* \geq
\]
\[
\geq (l_* - 1 - (c_*(d, a) + 1)(c_*(d, a) + 2)) + \rho(v', v'') - k_* \geq \rho(v', v'') - k_*.
\]

It remains to take as \( T \) the tree which is obtained by countable repeating of induction steps, and to define \( F \) by \( F|_{T_n} = F_n \).

**Corollary 1.** Let \( \Omega \subset (0, 1)^d, \Omega \in FC(a) \), let \( T \) and \( F : V(T) \to \Theta(\Omega) \) be the tree and the mapping constructed in Lemma 3. Then for any subtree \( T' \) in \( T \) we have
\[
\Omega_{T', F} \in FC(b_*) \quad \text{where} \quad b_* = b_*(a, d) > 0.
\] (26)

Here
\[
\text{card} V_1(v) \leq 1, \quad v \in V(T)
\] (27)

(see the notation on the page 6),
\[
\text{mes} \Omega_{T', F} \asymp \text{mes} F(v), \quad \text{where} \quad v \text{ is the minimal vertex in } T'.
\] (28)

**Proof.** Indeed, Lemma 1 implies that
\[
\Omega_{T', F} \in FC(\hat{a}(\hat{k}_*(a, d), l_*(a, d), d));
\]
(27) and (28) follow from (14) and (8), respectively.

Let \( T_1, \ldots, T_l \in ST(T), \cup_{j=1}^l T_j = T \) and \( T_i \cap T_j = \emptyset \) for any \( i \neq j \). Then we call \( \{T_1, \ldots, T_l\} \) a partition of the tree \( T \). If \( \mathcal{G} \) is a partition of \( T \) and \( \mathcal{A} \in ST(T) \), then put
\[
\mathcal{G}|_A = \{ A \cap T' : T' \in \mathcal{G}, \mathcal{A} \cap T' \neq \emptyset \}.
\]

Let \( T \) be a tree, and let \( \Psi : 2^V(T) \to \mathbb{R}_+ \). Throughout, we denote \( \Psi(T') := \Psi(V(T')) \) for \( T' \in ST(T) \).

**Lemma 4.** Let \( (T, v_*) \) be a tree, and let \( \Psi : 2^V(T) \to \mathbb{R}_+ \) satisfy the following conditions:
\[
\Psi(V_1 \cup V_2) \geq \Psi(V_1) + \Psi(V_2), \quad V_1, V_2 \subset V(T), \quad V_1 \cap V_2 = \emptyset;
\] (29)
Define the partition Σ(T) = \{v \in V(T) : \Psi(T_v) > \Psi(T) − γ\}. Let γ > 0, Ψ(T) > 2γ. Then there exists a unique vertex \(\hat{v}\) such that
\[\Psi(T_{\hat{v}}) > \Psi(T) - γ\] and for any \(v' \in V_1(\hat{v})\)
\[\Psi(T_{v'}) \leq \Psi(T) - γ.\]

**Proof.** Denote
\[E = \{v \in V(T) : \Psi(T_v) > \Psi(T) - γ\}.\]
Since \(v_1 \in E\), then \(E \neq \emptyset\). Show that \(E\) is a finite chain. At first check that any two vertices in \(E\) are comparable. Indeed, let \(v', v'' \in E\) be incomparable. Then \(T_{v'} \cap T_{v''} = \emptyset\) and
\[\Psi(T) \geq 2\Psi(T_{v'}) + \Psi(T_{v''}) > 2\Psi(T) - 2γ,\]
i.e. \(\Psi(T) < 2γ\). This contradicts the hypothesis of Lemma. Therefore, \(E\) is a chain. Prove that \(E\) is finite. Indeed, otherwise there exists a sequence \(\{v_n\}_{n \in \mathbb{N}} \subset E\) such that \(v_1 < \cdots < v_n < \cdots\) and \(\Psi(T_{v_n}) > \Psi(T) - γ \geq γ\). This contradicts to (30).

As \(\hat{v}\) we take the maximal vertex in \(E\). Since \(E\) is a chain, the vertex satisfying (31) and (32) is unique.

Let the conditions of Lemma 4 hold, let \(\hat{v}\) be a vertex satisfying (31) and (32). Define the partition \(\Sigma(T, γ)\) of the tree \(T\) by
\[\Sigma(T, γ) = \{T \setminus T_{\hat{v}}\} \cup \{\hat{v}\} \cup \{T_{v'}\}_{v' \in V_1(\hat{v})}.\]

Let \(x \in \mathbb{R}\). Denote by \([x]\) the nearest integer to \(x\) from above.

In the following Lemma we construct a special partition of a tree. Notice that a similar partition of a metric tree was constructed in [49].

**Lemma 5.** Let \(k \in \mathbb{N}\). Then for any tree \(T\) rooted at \(v_1\) such that
\[\text{card} V_1(v) \leq k, \ v \in V(T),\]
for any mapping \(\Psi : 2^V(T) \to \mathbb{R}_+\) satisfying (24) and (31) and for any \(γ > 0\) there exists a partition \(\mathcal{G}(T, γ)\) of \(T\) with the following properties:

1. let \(u \in V(T)\), \(\mathcal{G}(T_u, γ) \subset \mathcal{G}(T, γ)\); if \(\Psi(T_u) > (k + 1)γ\),
\[
\Sigma(T_u, γ) = \{T_u \setminus T_{\hat{v}_u}\} \cup \{\hat{v}_u\} \cup \{T_{v'}\}_{v' \in V_1(\hat{v}_u)},
\]
then
\[
\mathcal{G}(T_u, γ) = \{T_u \setminus T_{\hat{v}_u}\} \cup \{\hat{v}_u\} \bigcup_{v' \in V_1(\hat{v}_u)} \mathcal{G}(T_{v'}, γ)
\]
(35)

with \(\Psi(T_{\hat{v}_u}) > \Psi(T_u) - γ, \Psi(T_u \setminus T_{\hat{v}_u}) < γ\); if \(\Psi(T_u) \leq (k + 1)γ\), then \(\mathcal{G}(T_u, γ) = \{T_u\}\).
2. if $T' \in \mathcal{S}(T, \gamma)$ and $\Psi(T') > (k + 2)\gamma$, then $\text{card } V(T') = 1$;

3. if $\left\lceil \frac{\Psi(T)}{\gamma} \right\rceil \geq k + 2$, then $\text{card } \mathcal{S}(T, \gamma) \leq (k + 2) \left\lceil \frac{\Psi(T)}{\gamma} \right\rceil - (k + 1)(k + 2)$, otherwise $\text{card } \mathcal{S}(T, \gamma) = 1$;

4. let $v > v_*$; then $\text{card } \mathcal{S}(T, \gamma)|_{T_v} \leq (k + 2) \left( \left\lceil \frac{\Psi(T_v)}{\gamma} \right\rceil + 1 \right)$;

5. if $A \in \mathcal{S}(T, \gamma)$ and $\text{card } V(A) \geq 2$, then either $A = T_v$ for some $v \in V(T)$ or $A = T_v \setminus T_w$ for some $v, w \in V(T)$, $w > v$; here in the second case $\Psi(A) < \gamma$ and $\Psi(T_w) > \Psi(T_v) - \gamma$.

**Proof.** Let $m \in \mathbb{Z}_+$, and let $T$ be a tree. We write $(T, \Psi) \in \mathcal{R}_{m, \gamma}$ if $(m - 1)\gamma < \Psi(T) \leq m\gamma$.

If $v \in V(T)$, then set $\mu(v) = \mu(v, T) = \left\lceil \frac{\Psi(T_v)}{\gamma} \right\rceil$. Therefore, $(T_v, \Psi) \in \mathcal{R}_{\mu(v), \gamma}$.

Construct by induction on $m \in \mathbb{Z}_+$ partitions $\mathcal{S}(T, \gamma)$ for all $T$ such that $(T, \Psi) \in \mathcal{R}_{m, \gamma}$. If $m \leq k + 1$, then set $\mathcal{S}(T, \gamma) = \{T\}$. Let

$$m \geq k + 1,$$

and let partitions $\mathcal{S}(T, \gamma)$ satisfying properties 1–5 be constructed for all $(T, \Psi) \in \mathcal{R}_{m', \gamma}$, $m' \leq m$. Construct partitions for all $(T, \Psi) \in \mathcal{R}_{m + 1, \gamma}$. Here

$$m\gamma < \Psi(T) \leq (m + 1)\gamma.$$  

Consider the partition $\Sigma(T, \gamma)$ defined by (33). Then

$$\Psi(T \setminus T_\delta) \leq \Psi(T) - \Psi(T_\delta) < \gamma.$$  

Since for any $v' \in V_1(\hat{v})$

$$\Psi(T_{v'}) \leq \Psi(T) - \gamma \leq m\gamma, \quad \mu(v') = \mu(v', T) \leq m,$$

then by induction assumption the partition $\mathcal{S}(T_{v'}, \gamma)$ is defined. Set

$$\mathcal{S}(T, \gamma) = \{T \setminus T_\delta\} \cup \{\hat{v}\} \bigcup_{v' \in V_1(\hat{v})} \mathcal{S}(T_{v'}, \gamma).$$  

On this induction step we get that in the case $\Psi(T_\delta) \in (m\gamma, (m + 1)\gamma]$

if $\Sigma(T_\delta, \gamma) = \{\hat{v}\} \cup \{T_{v'}\}_{v' \in V_1(\hat{v})}$, then $\mathcal{S}(T_\delta, \gamma) = \{\hat{v}\} \bigcup_{v' \in V_1(\hat{v})} \mathcal{S}(T_{v'}, \gamma),$

if $T_\delta \setminus T_w \in \Sigma(T_\delta, \gamma)$ for some $w > \hat{v}$, then $T_\delta \setminus T_w \in \mathcal{S}(T_\delta, \gamma).$
By the induction assumption, for the case $Ψ(\mathcal{T}_v) \leq mγ$ (41) and (12) hold as well.

Prove the property 1. If $u = v_*$, then the assertion follows from the construction and (38). If $u > v_*$, then by (40) we get that $\mathcal{T}_u \subset \mathcal{T}_v$ for some $v' \in \mathcal{V}_1(\hat{v})$ or $u = \hat{v}$. In the first case the property 1 holds by the induction assumption. Consider the second case. If $Ψ(\mathcal{T}_v) \leq (k+1)γ$, then $\mathcal{S}(\mathcal{T}_v, γ) = \{\hat{v}\}$ by construction (see the base of induction).

Hence, $\mathcal{S}(\mathcal{T}_v, γ) \not\subseteq \mathcal{S}(T, γ)$. Let $Ψ(\mathcal{T}_v) > (k+1)γ$. Then (12), (40) and the inclusion $\mathcal{S}(\mathcal{T}_v, γ) = \mathcal{S}(\mathcal{T}_u, γ) \subset \mathcal{S}(T, γ)$ imply that $Σ(\mathcal{T}_v, γ) = \{\hat{v}\} \cup \{\mathcal{T}_v\}_{v' \in \mathcal{V}_1(\hat{v})}$. Therefore, (35) follows from (41).

The property 2 follows from (38), (40) and the induction assumption.

Prove the property 3. By definition of $μ(v')$,

$$\sum_{v' \in \mathcal{V}_1(\hat{v})} μ(v')γ \leq \sum_{v' \in \mathcal{V}_1(\hat{v})} (Ψ(\mathcal{T}_{v'}) + γ) \leq Ψ(T) + kγ \leq (m+1)γ + kγ,$$

that is

$$\sum_{v' \in \mathcal{V}_1(\hat{v})} μ(v') \leq m + 1 + k. \tag{43}$$

Set $\mathcal{V}' = \{v' \in \mathcal{V}_1(\hat{v}) : μ(v') \geq k + 2\}$, $\mathcal{V}'' = \{v' \in \mathcal{V}_1(\hat{v}) : μ(v') < k + 2\}$. If $\text{card } \mathcal{V}' \geq 2$, then by the induction assumption

$$\text{card } \mathcal{S}(\mathcal{T}, γ) \leq 2 + \sum_{v' \in \mathcal{V}'} ((k+2) · μ(v') - (k+1)(k+2)) + \text{card } \mathcal{V}'' \leq 2 - 2(k+1)(k+2) + (k+2)(m+1 + k) + k = (k+2)(m+1) - (k+1)(k+2).$$

If $\text{card } \mathcal{V}' = 1$, then by the induction assumption

$$\text{card } \mathcal{S}(\mathcal{T}, γ) \leq 2 + (k+2)m - (k+1)(k+2) + k = (k+2)(m+1) - (k+1)(k+2).$$

If $\text{card } \mathcal{V}' = 0$, then

$$\text{card } \mathcal{S}(\mathcal{T}, γ) \leq 2 + k = (k+2)(k+2) - (k+1)(k+2) \leq (k+2)(m+1) - (k+1)(k+2). \tag{36}$$

Prove the property 4. If $v > \hat{v}$, then $\mathcal{T}_v \subset \mathcal{T}_{v'}$ for some $v' \in \mathcal{V}_1(\hat{v})$, and the assertion follows from the induction assumption. If $v, \hat{v}$ are incomparable, then $\mathcal{T}_v \subset \mathcal{T} \setminus \mathcal{T}_v$ and $\text{card } \mathcal{S}(\mathcal{T}, γ)|_{\mathcal{T}_v} = 1$. If $v \leq \hat{v}$, then card $\mathcal{S}(\mathcal{T}, γ)|_{\mathcal{T}_v} \leq \text{card } \mathcal{S}(\mathcal{T}, γ)$, and the assertion follows from the property 3 (which is already proved) and from inequalities $Ψ(\mathcal{T}_v) \geq Ψ(\mathcal{T}_v) > Ψ(\mathcal{T}) - γ$.

Prove the property 5. Since $\text{card } \mathcal{V}(\mathcal{A}) > 1$, then by (40) there are two alternatives: a) $\mathcal{A} \in \mathcal{S}(\mathcal{T}_v, γ)$ for some $v' \in \mathcal{V}_1(\hat{v})$ (then the property 5 holds by the induction assumption); b) $\mathcal{A} = \mathcal{T} \setminus \mathcal{T}_v$ (then the property 5 follows from this equality and (38)).
Lemma 6. Let conditions of Lemma 5 hold. Then there exists $c(k) > 0$ such that for any $\gamma > 0$ and
\[ \gamma' \geq \frac{\gamma}{2} \] (44)
each element of the partition $\mathcal{G}(T, \gamma)$ intersects with no more than $c(k)$ elements of $\mathcal{G}(T, \gamma')$.

Proof. Let $\tilde{T}$ be an element of $\mathcal{G}(T, \gamma)$, let $l$ be the number of elements of $\mathcal{G}(T, \gamma')$ that intersect with $\tilde{T}$. Suppose that $l > 1$. Then $\tilde{T}$ contains at least two vertices. By Lemma 5 (see property 5), there are two cases.

Case 1. Let $\tilde{T} = T^v$ for some $v \in T$. Apply Lemma 5. By property 2, $\Psi(\tilde{T}) \leq (k + 2)\gamma$. Hence, by properties 3 and 4,
\[ l = \text{card } \mathcal{G}(T, \gamma)|_{T^v} \leq (k + 2) \left( \left\lceil \frac{(k + 2)\gamma}{\gamma'} \right\rceil + 1 \right) \leq (k + 2)(2k + 5). \]

Case 2. Let $\tilde{T} = T^v \setminus T^w$, $v, w \in V(T)$, $w > v$, and
\[ \Psi(T^v \setminus T^w) < \gamma. \] (45)

Put
\[ \tilde{E} = \{ u \in V(T) : T^v \subset T^u, \mathcal{G}(T^u, \gamma') \subset \mathcal{G}(T, \gamma') \}. \]
This set is nonempty, since it contains $v_*$. Further, $\tilde{E}$ is a chain. Indeed, if vertices $u_1$ and $u_2$ are incomparable, then $T^u_1$ and $T^u_2$ do not intersect. Finally, $\rho(v_*, v) \geq \rho(v_*, u)$ for any $u \in \tilde{E}$. Therefore, $\tilde{E}$ contains the maximal element $u_0$.

Since $l > 1$, then by Lemma 5 (see the property 1), there exists a vertex $\hat{u} \geq u_0$ such that
\[ \mathcal{G}(T^u_0, \gamma') = \{ T^u_0 \setminus T^u \} \cup \{ \hat{u} \} \bigcup \left( \bigcup_{v \in V_1(\hat{u})} \mathcal{G}(T^v, \gamma') \right). \] (46)

Show that
\[ v \leq \hat{u}. \] (47)

Indeed, if $v$ and $\hat{u}$ are incomparable, then $T^v \subset T^u_0 \setminus T^u$, and
\[ l = \text{card } \mathcal{G}(T, \gamma')|_{\tilde{T}} \leq \text{card } \mathcal{G}(T, \gamma')|_{T^v} = \text{card } \mathcal{G}(T^u_0, \gamma')|_{T^v} = 1. \]

If $v > \hat{u}$, then $v \in T^{v'}$ for some $v' \in V_1(\hat{u})$, that is $v' \in \tilde{E}$ by (46) and $v' > u_0$. This contradicts the fact that $u_0 = \max \tilde{E}$.

Let $u \in V(T)$, $\mathcal{G}(T^u, \gamma') \subset \mathcal{G}(T, \gamma')$, $u \geq v$. Denote by $l_u$ the number of elements of the partition $\mathcal{G}(T^u, \gamma')$ that intersect with $T^v \setminus T^w$. 

18
Set
\[ U = \{ u \geq v : \mathcal{S}(T_u, \gamma') \subset \mathcal{S}(T, \gamma'), l_u > 1 \}. \quad (48) \]

**Assertion (\( \ast \)).** There exists a mapping \( \varphi : U \to V(T) \) such that \( \varphi(u) > u \) for any \( u \in U \),
\[ \mathcal{S}(T_{\varphi(u)}, \gamma') \subset \mathcal{S}(T_u, \gamma'), \quad \Psi(T_{\varphi(u)}) \leq \Psi(T_u) - \gamma' \text{ and } l_u \leq k + 1 + l_{\varphi(u)}, \quad (49) \]
and if \( l_{\varphi(u)} > 1 \), then
\[ \varphi(u) < w. \quad (50) \]

**Proof of Assertion (\( \ast \)).** By Lemma 5 (see the property 1), for any \( u \in U \) there exists a vertex \( \hat{u} \geq u \) such that
\[ \mathcal{S}(T_u, \gamma') = \{ T_u \setminus T_{\hat{u}} \} \cup \{ \hat{u} \} \bigcup \bigcup_{v' \in V_1(\hat{u})} \mathcal{S}(T_{v'}, \gamma') \]
with \( \Psi(T_{v'}) \leq \Psi(T_u) - \gamma' \) for any \( v' \in V_1(\hat{u}) \). Hence, \( l_u \leq 2 + \sum_{v' \in V_1(\hat{u})} l_{v'} \). In order to satisfy first two expressions in (49), it is sufficient to choose \( \varphi(u) \in V_1(\hat{u}) \).

Prove that if \( l_{v'} > 1 \), then \( v' < w \). If \( v' \geq w \), then \( l_{v'} = 0 \). If \( v' \) and \( w \) are incomparable, then \( T_{v'} \subset T_u \setminus T_w \) (since \( v' > u \geq v \) and \( T_w \cap T_{v'} = \emptyset \)). Then
\[ \Psi(T_{v'}) \leq \Psi(T_u \setminus T_w) < \gamma \leq 2\gamma'. \]
Therefore, \( l_{v'} \leq \text{card} \mathcal{S}(T_{v'}, \gamma') = 1 \) (see the property 3 in Lemma 5).

Set \( A_u = \{ v' \in V_1(\hat{u}) : l_{v'} > 1 \} \). From card \( \{ v' \in V_1(\hat{u}) : v' < w \} \leq 1 \) follows that card \( A_u \leq 1 \). If card \( A_u = 1 \), then we take as \( \varphi(u) \) an element of \( A_u \) (then (50) holds). If \( A_u = \emptyset \), then we take as \( \varphi(u) \) an arbitrary element of the set \( V_1(\hat{u}) \). In both cases the last inequality in (49) holds. This completes the proof of the Assertion (\( \ast \)).

From (46) follows that \( \mathcal{S}(T_{u'}, \gamma') \subset \mathcal{S}(T, \gamma') \) for any \( u' \in V_1(\hat{u}) \) and \( u' > \hat{u} \geq v \). Hence, if \( l_{u'} > 1 \), then \( u' \in U \).

By (46) and the condition \( u_0 \in \hat{E} \), we have \( l \leq 2 + \sum_{u' \in V_1(\hat{u})} l_{u'} \). Estimate \( l_{u'} \) for each \( u' \in V_1(\hat{u}) \) such that \( l_{u'} > 1 \). For this we use (49). If \( l_{\varphi(u')} \leq 1 \), then \( l_{u'} \leq k + 2 \). Let \( l_{\varphi(u')} > 1 \). Then \( \varphi(u') \in U \). Applying (49) once again, we get \( l_{u'} \leq k + 1 + k + 1 + l_{\varphi(u')} \) with
\[ \Psi(T_{\varphi(u')}) \leq \Psi(T_{\varphi(u')}) - \gamma' \leq \Psi(T_{u'}) - 2\gamma' \leq \Psi(T_{u'}) - \gamma. \]
If \( l_{\varphi(u')} > 1 \), then \( \varphi(u') < w \), and \( \Psi(T_{u'}) \leq \Psi(T_{\varphi(u')}) \). Then we get the contradictory chain of inequalities
\[ \Psi(T_{u'}) - \gamma < \Psi(T_{u'}) \leq \Psi(T_{\varphi(u')}) \leq \Psi(T_{u'}) - \gamma \leq \Psi(T_{u'}) - \gamma. \]
Hence, \( l_{\varphi(u')} \leq 1, l_{w} \leq 2k + 3 \text{ and } l \leq 2 + k(2k + 3) \). \qed
Corollary 2. Let conditions of Lemma 5 hold, and let $\Psi(T) > 0$. Then there exists a number $C(k) > 0$ such that for any $n \in \mathbb{N}$ there exists a partition $\mathcal{G}_n = \mathcal{G}(T, \Psi(T)/n)$ of the tree $T$ into at most $C(k)n$ subtrees $T_j$ such that $\Psi(T_j) \leq \frac{(k+2)\Psi(T)}{n^2}$ for any $j$ that satisfies the condition $\text{card} \, V(T_j) \geq 2$. In addition, there exists $C_1(k) > 0$ such that if $m \leq 2n$, then each element $\mathcal{G}_n$ intersects with at most $C_1(k)$ elements of $\mathcal{G}_m$.

The partition of a cube. Let $\Phi$ be a nonnegative function defined on Lebesgue measurable subsets of $\mathbb{R}^d$ and satisfying the conditions

$$\Phi(A_1) + \Phi(A_2) \leq \Phi(A_1 \cup A_2), \quad \text{where } A_1, A_2 \subset \mathbb{R}^d \text{ do not overlap};$$

if $\text{mes} A_n \to 0$, then $\Phi(A_n) \to 0 \quad (n \to \infty).$ (51)

Denote by $\mathcal{R}$ the family of sets $K \setminus K'$, where $K \in \mathcal{K}$, $K' \in \Xi(K)$, $K' \neq K$.

Lemma 7. Let $K \in \mathcal{K}$, and let the function $\Phi$ satisfy (51) and (52). Then for any $n \in \mathbb{N}$ there exists a partition $T_n = T_n(K)$ of the cube $K$ with the following properties:

1. the number of elements of $T_n$ does not exceed $2^d n$;
2. for any $\Delta \in T_n$ the inequality $\Phi(\Delta) \leq 3n^{-1}\Phi(K)$ holds;
3. each element of $T_n$ belongs to $\Xi(K)$ or to $\mathcal{R}$;
4. there exists $C(d) \in \mathbb{N}$ such that for $l \leq 2m$ ($l, m \in \mathbb{N}$) each element of $T_m$ overlaps with at most $C(d)$ elements of $T_l$.

A similar partition is constructed in [52]. As proved, this partition satisfies properties similar to 1–3 (with other constants estimating the number of elements and the value $\Phi(\Delta)$).

Definition of functions $\Phi$ and $\Psi$. Let $\alpha_1, \ldots, \alpha_{l_*} > 0$, $\sum_{j=1}^{l_*} \alpha_j = 1$, let $\mu_1, \ldots, \mu_{l_*}$ be finite absolutely continuous measures on $\Omega$. From the Radon – Nikodym theorem and from the absolute continuity of the Lebesgue integral follows that

$$\mu_j(A) \to 0 \quad \text{as } \text{mes} \, A \to 0.$$ (53)

For a Lebesgue measurable set $A \subset \mathbb{R}^d$ we put

$$\Phi(A) = \prod_{j=1}^{l_*} (\mu_j(A \cap \Omega))^{\alpha_j}.$$ (54)

The conditions (53) and $\alpha_j > 0$ imply (52). Further, by the Hölder inequality we get

$$\prod_{j=1}^{l_*} (b_j + c_j)^{\alpha_j} \geq \prod_{j=1}^{l_*} b_j^{\alpha_j} + \prod_{j=1}^{l_*} c_j^{\alpha_j}, \quad b_j \geq 0, \ c_j \geq 0,$$ (55)
which yields (51).

Let $T$ and $F$ be the tree and the mapping constructed in Lemma 3. Define the mapping $\Psi : 2^V(T) \to \mathbb{R}_+$ by formula

$$\Psi(W) = \Phi(\cup_{v \in W} F(v)), \quad W \subset V(T).$$

(56)

Then (51) implies (29). Prove (30). Let $\{v_j\}_{j \in \mathbb{N}} \subset V(T), v_1 < \cdots < v_n < \cdots$. From (51) follows that $m_{v_n} \to \infty$. Hence, $\text{mes} F(v_n) \to 0$, and by (28) we get $\text{mes} \Omega_{T_{v_n}, F} \to 0$. Therefore, (30) follows from (52).

**Lemma 8.** For any $n \in \mathbb{N}$ there exists a family of partitions $\{B_{n,m}\}_{m \in \mathbb{Z}_+}$ of the domain $\Omega$ with the following properties:

1. $\text{card } B_{n,m} \lesssim d^{m n}$;
2. if $E \in B_{n,m}$, then
   
   (a) either $E = \Omega_{T', F}$ for some subtree $T' \subset T$ or $E \subset F(w)$ for some vertex $w \in V(T)$ and $E \in \Xi(F(w)) \cup \mathcal{R}$;
   
   (b) $\Phi(E) \lesssim d^{\frac{\Phi(\Omega)}{2^{m n}}}$;
3. there exists $C_*(d)$ such that each element of $B_{n,m}$ overlaps with at most $C_*(d)$ elements of $B_{n,m \pm 1}$.

**Proof.** We suppose that $\Phi(\Omega) > 0$ (otherwise set $B_{n,m} = \{\Omega_{T,F}\}$). From (27) follows that for any vertex $v \in V(T)$ we have $\text{card } V(T) \lesssim 1$. By Corollary 2 for any $n \in \mathbb{N}, m \in \mathbb{Z}_+$ there exists a partition $\mathcal{G}_{2^n m} = \{T_{j}^{m,n}\}_{j=1}^{j_*(m,n)}$ of $T$ with the following properties:

1. $T_{j}^{m,n}$ is a tree;
2. $j_*(m, n) \lesssim d^{m n}$;
3. if $\text{card } V(T_{j}^{m,n}) \geq 2$, then $\Psi(T_{j}^{m,n}) \lesssim \frac{\Psi(T)}{2^{m n}}$;
4. there exists $\hat{C}(d) > 0$ such that each element of $\mathcal{G}_{2^n m}$ intersects with at most $\hat{C}(d)$ elements of $\mathcal{G}_{2^n m \pm 1}$.

Denote

$$J_{m,n} = \left\{ j \in 1, j_*(m, n) : \text{card } V(T_{j}^{m,n}) = 1, \quad \Psi(T_{j}^{m,n}) \geq \frac{\Psi(T)}{2^{m n}} \right\}. \quad (57)$$

Let $j \in J_{m,n}$, $V(T_{j}^{m,n}) = \{v_j^{m,n}\}$. Put $\Delta_j^{m,n} := F(\{v_j^{m,n}\})$, $l_{j,m,n} = \left\lfloor \frac{2^{m n} \Phi(\Delta_j^{m,n})}{\Phi(\Omega)} \right\rfloor$. \quad (58)
Then
\[
\sum_{j \in J_{m,n}} l_{j,m,n} \leq j_*(m, n) + \sum_{j \in J_{m,n}} \frac{2^{m,n} \Phi(\Delta_j^{m,n})}{\Phi(\Omega)} \leq j_*(m, n) + 2^{m,n} \lesssim 2^m n. \tag{59}
\]

Let \( T_{j,m,n}(\Delta_j^{m,n}) \) be the partition of the cube \( \Delta_j^{m,n} \) defined in Lemma 7. Set
\[
\mathcal{B}_{n,m} = \{ \Omega T_{j,m,n}^{m,n} \} \cup \bigcup_{j \in J_{m,n}} T_{j,m,n}(\Delta_j^{m,n}).
\]

Check the property 1:
\[
\text{card } \mathcal{B}_{n,m} \leq j_*(m, n) + \sum_{j \in J_{m,n}} 2^d l_{j,m,n} \lesssim 2^m n. \tag{59}
\]

Check the property 2. Item a) follows from the construction and item 3 of Lemma 7. Check item b). Let \( E = \Omega T_{j,m,n}^{m,n} \) with \( j \notin J_{m,n} \). From the property 3 of \( \mathcal{S}_{2^m,n} \) and from the definition of \( J_{m,n} \) follows that
\[
\Phi(E) \leq \Phi \left( \bigcup_{v \in \mathcal{V}(T_j^{m,n})} F(v) \right) \leq \Psi(T_j^{m,n}) \leq \frac{\Phi(\Omega)}{2^{m,n}}. \tag{59}
\]

Let \( E \in T_{j,m,n}(\Delta_j^{m,n}) \) for some \( j \in J_{m,n} \). From item 2 of Lemma 7 follows that
\[
\Phi(E) \leq 3I_{j,m,n}^{-1} \Phi(\Delta_j^{m,n}) \leq 3\Phi(\Omega) \quad \lesssim \frac{3\Phi(\Omega)}{2^{m,n}}. \tag{59}
\]

Check the property 3. Let \( E = \Omega T_{j,m,n}^{m,n} \) with \( j \notin J_{m,n} \). Denote
\[
I_j^+ = \{ i \in J_{m \pm 1,n} : v_i^{m \pm 1,n} \in \mathcal{V}(T_j^{m,n}) \}. \tag{61}
\]

Then
\[
\text{card } I_j^+ \cdot \frac{\Psi(T_j^{m,n})}{2^{m,n}} \leq 2 \sum_{i \in I_j^+} \Psi(\{ v_i^{m \pm 1,n} \}) \leq 2\Psi(T_j^{m,n}) \lesssim \frac{\Phi(\Omega)}{2^{m,n}}. \tag{59}
\]

which implies
\[
\text{card } I_j^+ \lesssim \frac{1}{d}. \tag{62}
\]

Therefore, by item 1 of Lemma 7 and by the property 4 of \( \mathcal{S}_{2^m,n} \), we have
\[
\text{card } \{ E' \in \mathcal{B}_{n,m \pm 1} : (\text{int } E') \cap (\text{int } E) \neq \emptyset \} \leq \hat{C}(d) \sum_{i \in I_j^+} 2^{d} l_{i,m \pm 1,n} \lesssim \frac{\Phi(\Omega)}{2^{m,n}}. \tag{59}
\]
\begin{align*}
\leq C(d) + 2^d & \sum_{i \in I_j^\pm} \left( \frac{2^{m \pm 1} n \Phi(\Delta_i^{m \pm 1, n})}{\Phi(\Omega)} + 1 \right) \overset{53, 62}{\lesssim} d \\
\lesssim 1 & + \sum_{i \in I_j^\pm} \frac{2^{m \pm 1} n \Psi(\{v_i^{m \pm 1, n}\})}{\Psi(T)} \overset{29, 61}{\lesssim} \frac{2^{m \pm 1} n \Psi(T_{j,m,n})}{\Psi(T)} \overset{60}{\lesssim} 1.
\end{align*}

Let \( E \in T_{l_{j,m,n}(\Delta_j^{m,n})} \) for some \( j \in J_{m,n} \) and let \( E \) overlap with at least two elements of \( B_{n,m \pm 1} \). Then \( \Delta_j^{m,n} = \Delta_i^{m \pm 1, n} \) for some \( i \in J_{m \pm 1, n} \) and

\[
\text{card} \{ E' \in B_{n,m \pm 1} : (\text{int} E') \cap (\text{int} E) \neq \emptyset \} = \text{card} \{ E' \in T_{l_{i,m \pm 1, n}(\Delta_i^{m \pm 1, n})} : (\text{int} E') \cap (\text{int} E) \neq \emptyset \} = \text{card} \{ E' \in T_{l_{i,m \pm 1, n}(\Delta_j^{m,n})} : (\text{int} E') \cap (\text{int} E) \neq \emptyset \} \leq C(d)
\]

by item 4 of Lemma 7, by the definition of \( l_{j,m,n} \) and \( l_{i,m \pm 1, n} \) and by the inequality \( \lceil 2^a \rceil \leq 2 \lceil a \rceil \).

\[\Box\]

### 4 The spline approximation and the estimate of widths

In the papers \cite{2,3} the integral representation for smooth functions defined on a John domain in terms of their \( r \)th derivatives was obtained. Here we shall formulate this result for functions that vanish on some ball and then we shall repeat more accurately some steps of the proof from \cite{2,3}.

In \cite{2} the following equivalent definition of a John domain was given.

**Definition 5.** A domain \( \Omega \) satisfies the John condition if there exist \( 0 < \rho < R \) and \( x_\ast \in \Omega \) such that for any \( x \in \Omega \) there exists a curve \( \gamma_x \) with properties 1 and 2 from Definition 7 such that \( T(x) \leq R \) and for any \( t \in [0, T(x)] \) the following inequality holds:

\[
\text{dist} (\gamma_x(t), \partial \Omega) \geq \frac{\rho}{T(x)} t.
\]

Let us check the equivalence of Definitions 1 and 5. Indeed, if the domain \( \Omega \) satisfies the John condition in the sense of Definition 5, then \( \Omega \in \mathbf{FC}(a) \) for \( a < \frac{\rho}{R} \). Conversely, let \( \Omega \in \mathbf{FC}(a) \). Without loss of generality we may assume that \( a < 1 \). Let \( R_0 = \text{dist} (x_\ast, \partial \Omega) \). Then for any \( x \in \Omega \) we have \( a \cdot T(x) \leq R_0 \). Set

\[
R = \frac{R_0}{a}, \quad \rho = \frac{aR_0}{2}.
\]

Check (63). If \( T(x) \geq \frac{R_0}{2} \), then

\[
\text{dist} (\gamma_x(t), \partial \Omega) \geq at = \frac{2\rho}{R_0} t \geq \frac{\rho}{T(x)} t.
\]

23
If \( T(x) < \frac{R_0}{2} \), then \(|\gamma_x(t) - x_*| < \frac{R_0}{2}\) for any \( t \in [0, T(x)]\). Hence,

\[
\text{dist} (\gamma_x(t), \partial \Omega) \geq \frac{R_0}{2} \geq \frac{aR_0}{2} \cdot \frac{t}{T(x)} = \frac{\rho}{T(x)} t.
\]

Notice that

\[
\frac{\rho}{R} = \frac{a^2}{2}.
\] 

(65)

**Theorem C.** Let \( \Omega \in FC(a) \), let the point \( x_* \) and the curves \( \gamma_x \) be such as in Definition 1, \( R_0 = \text{dist} (x_*, \partial \Omega) \), \( r \in \mathbb{N} \). Then there exist measurable functions \( H_j : \Omega \times \Omega \to \mathbb{R} \), \( \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}^d_+ \), \( |\beta| = r \), such that the inclusion \( \text{supp} H_j(x, \cdot) \subset \cup_{t \in [0, T(x)]} B_{\hat{\rho}}(\gamma_x(t)) \) and the inequality \( |H_j(x, y)| \lesssim |x - y|^{-d} \) hold for any \( x \in \Omega \), and for any function \( f \in C^\infty(\Omega) \), \( f|_{B_{\hat{\rho}/2}(x_*)} = 0 \) the following representation holds:

\[
f(x) = \sum_{|\beta| = r} \int_{\Omega} H_j(x, y) \nabla^\beta f(y) dy.
\]

(66)

**Proof.** Without loss of generality we may assume that \( a < 1 \). From formulas (5.10), (5.14), (5.20) and (5.22) of the paper [2] follows that

\[
f(x) = \int_{B_{\hat{\rho}(x_*)}} f(y) \theta(y - x_*) dy + \sum_{j=1}^d \int_{\Omega} H_j(x, y) \frac{\partial f}{\partial y_j}(y) dy,
\]

where \( \theta(\cdot) \) is some smooth function whose support is contained in the ball \( B_{\hat{\rho}(0)} \),

\[
\hat{\rho} = \frac{\rho}{2(1+\sqrt{d})} = \frac{aR_0}{2(1+\sqrt{d})} \leq \frac{R_0}{2},
\]

\( H_j(x, y) = \sum_{\nu \in \mathbb{N}} \omega_\nu(x) H_{j,\nu}(x, y) \), \( \omega_\nu \) is a smooth partition of unity, and measurable functions \( H_{j,\nu} \) are represented as

\[
H_{j,\nu}(x, y) = \int_0^{T(x)} \psi_{j,\nu}(x, y, t) \theta \left( \frac{Ry - \gamma_x(t)}{t} \right) dt
\]

(the functions \( \psi_{j,\nu} \) are defined in the formula (5.14)); here

\[
H_j(x, y) \lesssim \left( \frac{R}{\rho} \right)^d |x - y|^{1-d} \lesssim \left( \frac{2}{a^2} \right)^d |x - y|^{1-d}.
\]

Since \( \hat{\rho} \leq \frac{R_0}{2} \), then for any smooth function \( f \) such that \( f|_{B_{\hat{\rho}/2}(x_*)} = 0 \) we get

\[
f(x) = \sum_{j=1}^d \int_{\Omega} H_j(x, y) \frac{\partial f}{\partial y_j}(y) dy.
\]
If $|y - \gamma_x(t)| \geq at$ for any $t \in [0, T(x)]$, then the assumption $a < 1$ implies that

$$|y - \gamma_x(t)| \geq \frac{a^2}{4(1 + \sqrt{d})} t \hat{\rho} R t;$$

since $\text{supp} \theta \subset B_\delta(0)$, then $H_{j,\nu}(x, y) = 0$. Hence, $\text{supp} H_j(x, \cdot) \subset \cup_{t \in [0, T(x)]} B_{at}(\gamma_x(t))$.

Let us prove the theorem for arbitrary $r \in \mathbb{N}$ following the arguments from [3]. Let

$$\varphi(x, \xi) = \sum_{|\beta| \leq r - 1} \frac{(x - \xi)^\pi}{\beta!} \nabla^\beta f(\xi).$$

Then for $f|_{B_{R/2}(x^\ast)} = 0$ we have $\varphi(x, \cdot)|_{B_{R/2}(x^\ast)} = 0$, and (66) implies

$$\varphi(x, \xi) = \sum_{j=1}^d \int_\Omega H_j(\xi, y) \frac{\partial \varphi}{\partial y_j}(x, y) dy.$$

Taking $\xi = x$, we get

$$f(x) = \sum_{j=1}^d \int_\Omega H_j(x, y) \frac{\partial \varphi}{\partial y_j}(x, y) dy.$$

It remains to apply the formula (5) from the paper [3]:

$$\frac{\partial \varphi}{\partial y_j}(x, y) = \sum_{|\beta| = r - 1} \frac{(x - y)^\pi}{\beta!} \nabla^\beta \delta_j f(y),$$

where $\delta_j = (\delta_{j1}, \ldots, \delta_{jd})$ and $\delta_{j,\nu}$ is the Kronecker symbol.

The following theorem is proved in [4, 5]; see also [50] (page 566) and [51] (page 51).

**Theorem D.** Let $1 < \tilde{p} < \tilde{q} < \infty$, $d \in \mathbb{N}$, $r > 0$, $\frac{r}{d} + \frac{1}{\tilde{q}} - \frac{1}{\tilde{p}} = 0$.

$$T f(x) = \int_{\mathbb{R}^d} f(y) |x - y|^{r-d} dy.$$ 

Then the operator $T : L_{\tilde{q}}(\mathbb{R}^d) \to L_{\tilde{q}}(\mathbb{R}^d)$ is bounded.

**Lemma 9.** Let $r, d \in \mathbb{N}$, let $\Omega \in \text{FC}(a)$ be a bounded domain in $\mathbb{R}^d$, let $1 < \tilde{p} \leq \infty$, $1 \leq \tilde{q} < \infty$, $\frac{1}{\tilde{p}} = \frac{1}{\tilde{q}} + \frac{1}{q} - \frac{1}{p} \geq 0$, let the functions $\tilde{g}, \tilde{v} : \Omega \to \mathbb{R}_+$ satisfy the conditions of Theorem [7]. Let $\mathcal{T}$, $F : \mathcal{V}(\mathcal{T}) \to \Theta(\Omega)$ be the tree and the mapping defined in Lemma 3, let $\tilde{T}$ be a subtree in $\mathcal{T}$, $\tilde{\Omega} = \Omega_{\tilde{T}, F}$. Then for any function $f \in W_{\tilde{p}, \tilde{q}}(\tilde{\Omega})$ there exists a polynomial $P_f$ of degree not exceeding $r - 1$ such that

$$\|f - P_f\|_{L_{\tilde{q}, \delta}(\Omega)} \lesssim_{\tilde{p}, \tilde{q}, r, d, a, c_0} \|\tilde{g}\tilde{v}\|_{L_{\tilde{p}, \delta}(\tilde{\Omega})} \left\|\nabla^r f \right\|_{L_{\tilde{q}, \delta}(\tilde{\Omega})}.$$ 

(67)

Here the mapping $f \mapsto P_f$ is linear.
Proof. By Corollary 11 we have \( \tilde{\Omega} \in \mathbf{FC}(b_*) \) with \( b_* = b_*(a, d) > 0 \). Let \( \gamma_x : [0, T(x)] \to \tilde{\Omega} \) be the curve from Definition 11 \( \gamma_x(T(x)) = x_* \). By Lemma 11 \( \gamma_x \) can be chosen so that

\[
\cup_{t \in [0, T(x)]} B_{b_*(t)}(\gamma_x(t)) \subset \tilde{\Omega} \subset F(w), \quad \text{if } x \in F(w). \tag{68}
\]

The set \( C^\infty(\Omega) \cap W^{r, q}_{p, \tilde{g}}(\Omega) \) is dense in \( W^{r, q}_{p, \tilde{g}}(\Omega) \) (it can be proved similarly as in the non-weighted case, see, e.g., [34], page 16). Hence, it is sufficiently to check (67) for smooth functions.

**Step 1.** Let \( R_0 = \text{dist} (x_*, \partial \tilde{\Omega}) \). Prove that (67) follows from the estimate

\[
\|f\|_{L^q(\tilde{\Omega})} \leq \|\tilde{g} \tilde{v}\|_{L^p(\tilde{\Omega})} \left\| \frac{\nabla^r f}{\tilde{g}} \right\|_{L^p(\tilde{\Omega})}, \quad f \in C^\infty(\Omega), \quad f|_{B_{R_0/2}(x_*)} = 0. \tag{69}
\]

Indeed, it was proved in [34] (see also [34]) that for any function \( f \in C^\infty(\Omega) \) there exists a polynomial \( P_f \) of degree not exceeding \( r - 1 \) such that

\[
\|\nabla^k (f - P_f)\|_{L^q(B_{3R_0/4}(x_*))} \leq R_0^{r-k+\frac{d}{q} - \frac{d}{p}} \|\nabla^r f\|_{L^p(B_{3R_0/4}(x_*))}, \quad 0 \leq k \leq r - 1, \tag{70}
\]

and the mapping \( f \mapsto P_f \) is linear. Let \( \psi : \mathbb{R}^d \to [0, 1], \psi \in C^\infty(\mathbb{R}^d), \text{supp} \psi \subset B_{3/4}(0), \psi|_{B_{1/2}(0)} = 1, \psi(x) = \psi\left(\frac{\|x - x_*\|}{R_0}\right) \). Then

\[
\|\psi(f - P_f)\|_{L^q(\tilde{\Omega})} \leq \|f - P_f\|_{L^q(B_{3R_0/4}(x_*))} \leq R_0^{r+\frac{d}{q} - \frac{d}{p}} \|\nabla^r f\|_{L^p(B_{3R_0/4}(x_*))}. \tag{71}
\]

From (2) and (3) follows that

\[
\frac{\tilde{g}(x)}{\tilde{g}(y)} \sim_{c_0, d} 1, \quad \frac{\tilde{v}(x)}{\tilde{v}(y)} \sim_{c_0, d} 1, \quad x, y \in B_{3R_0/4}(x_*). \tag{72}
\]

Therefore,

\[
\|f - P_f\|_{L^q(\tilde{\Omega})} \leq \|\psi(f - P_f)\|_{L^q(\tilde{\Omega})} + \|(1 - \psi)(f - P_f)\|_{L^q(\tilde{\Omega})} \leq R_0^{r+\frac{d}{q} - \frac{d}{p}} \|\nabla^r f\|_{L^p(\tilde{\Omega})}. \tag{69, 71, 72}
\]

\[
\leq \|\tilde{g} \tilde{v}\|_{L^p(\tilde{\Omega})} \left(2 \left\| \frac{\nabla^r f}{\tilde{g}} \right\|_{L^p(\tilde{\Omega})} + \left\| \frac{\nabla^r (1 - \psi)(f - P_f)}{\tilde{g}} \right\|_{L^p(\tilde{\Omega})} \right) \leq R_0^{r+\frac{d}{q} - \frac{d}{p}} \|\nabla^r f\|_{L^p(\tilde{\Omega})}. \tag{70, 72}
\]

\[
\leq \|\tilde{g} \tilde{v}\|_{L^p(\tilde{\Omega})} \left(2 \left\| \frac{\nabla^r f}{\tilde{g}} \right\|_{L^p(\tilde{\Omega})} + \sum_{k=0}^r R_0^{k-r} \left\| \frac{\nabla^k (f - P_f)}{\tilde{g}} \right\|_{L^p(B_{3R_0/4}(x_*))} \right) \leq R_0^{r+\frac{d}{q} - \frac{d}{p}} \|\nabla^r f\|_{L^p(\tilde{\Omega})}.
\]

\(^1\)Here \( C^\infty(\Omega) \) is the space of functions that are smooth on the open set \( \Omega \) yet not necessarily extendable to smooth functions on the whole space \( \mathbb{R}^d \).
Step 2. Let \( f \in C^\infty(\Omega), f|_{B_{R_0/2}(x_*)} = 0 \), \( \varphi(x) = \frac{\vert \nabla f(x) \vert}{g(x)} \). From Theorem \( \Box \) follows that for any \( x \in \hat{\Omega} \) there exists a set \( G_x \subset \bigcup_{t \in [0,T(x)]} B_{b_0}(\gamma_x(t)) \) such that \( \{(x, y) \in \hat{\Omega} \times \hat{\Omega} : y \in G_x \} \) is measurable and
\[
\vert f(x) \vert \lesssim_{r,d,a} \int_{G_x} \vert x - y \vert^{r-d} \hat{g}(y) \varphi(y) \, dy.
\]
Hence, in order to prove (69), it is sufficient to obtain the estimate
\[
\left( \int_{\hat{\Omega}} \hat{g}(x) \left( \int_{G_x} \vert x - y \vert^{r-d} \hat{g}(y) \varphi(y) \, dy \right)^{\frac{q}{\hat{q}}} \, dx \right)^{1/\hat{q}} \lesssim_{\hat{p},\hat{q},r,d,c,a} \Vert \hat{g} \varphi \Vert_{L_{\hat{p}}(\hat{\Omega})} \Vert \varphi \Vert_{L_{\hat{p}}(\hat{\Omega})}.
\]

Step 3. Let \( \frac{1}{r} > 0 \). Denote by \( w_* \) the minimal vertex of the tree \( \hat{T} \). Let \( T', T'' \) be subtrees in \( \hat{T} \) rooted at \( w_* \), and let \( E' = \hat{\Omega} \setminus \hat{\Omega}_{T',F}, E'' = \hat{\Omega}_{T'',F} \). Show that (73) holds for \( \hat{g} = \chi_{E'}, \hat{v} = \chi_{E''} \). Actually, \( E' \cap E'' = \cup_{i} E_i \), where \( E_i = \hat{\Omega}_{A_i,F}, A_i \subset \hat{T} \) are subtrees with pairwise incomparable minimal vertices. From (68) follows that for any \( x \in \Omega_{T',F} \) the inclusion \( G_x \subset \Omega_{T',F} \) holds and \( G_x \cap E' = \emptyset \), and for any \( x \in E_i \) the inclusion \( G_x \cap E' \subset E_i \) holds. Hence, the left-hand side of (73) does not exceed
\[
\left( \sum_{i} \int_{E_i} \left( \int_{E_i} \vert x - y \vert^{r-d} \varphi(y) \, dy \right)^{\frac{\hat{q}}{q}} \, dx \right)^{1/\hat{q}} =: M.
\]
By Corollary \( \Box \) \( E_i \in FC(b_*) \), which implies \( \text{diam } E_i \)^d \( \lesssim_{a,d} \text{mes } E_i \). Applying Theorem \( \Box \) and the Hölder inequality, we get
\[
M \lesssim_{\hat{p},\hat{q},r,d,a} \left( \sum_{i} \left( \text{mes } E_i \right)^{\frac{\hat{q}}{q}} \Vert \varphi \Vert_{L_{\hat{p}}(E_i)} \right)^{\frac{1}{\hat{q}}} \lesssim \left( \text{mes } (E' \cap E'') \right)^{1/\hat{q}} \Vert \varphi \Vert_{L_{\hat{p}}(\hat{\Omega})} = \Vert \hat{g} \varphi \Vert_{L_{\hat{p}}(\hat{\Omega})} \Vert \varphi \Vert_{L_{\hat{p}}(\hat{\Omega})}
\]
(for \( \hat{p} \leq \hat{q} \) the second inequality follows from \( \left( \sum_{i} a_i^{\hat{q}} \right)^{\frac{1}{\hat{q}}} \leq \left( \sum_{i} a_i^{\hat{p}} \right)^{\frac{1}{\hat{p}}} \), and for \( \hat{p} > \hat{q} \) it follows from the Hölder inequality and from \( \left( \sum_{i} a_i^{\hat{p}} \right)^{\frac{1}{\hat{p}}} \leq \left( \sum_{i} a_i^{\hat{q}} \right)^{\frac{1}{\hat{q}}} \)).

Step 4. Let \( \frac{1}{r} > 0 \). Show that there exist two sequences of subtrees \( \{T'_j\}_{j \in \mathbb{Z}^+} \) and \( \{T''_j\}_{j \in \mathbb{Z}^+} \) in the tree \( T \) and there are functions \( \hat{g}, \hat{v} : \hat{\Omega} \to \mathbb{R}_+ \) with the following properties:

1. \( T'_0 = T''_0 = \emptyset, T'_j \subset T'_{j+1}, T''_j \subset T''_{j+1}, j \in \mathbb{Z}^+, \cup_{j \in \mathbb{Z}^+} T'_j = \cup_{j \in \mathbb{Z}^+} T''_j = T; \)
2. \( \hat{g}|_{\Omega_{T'_j,F} \setminus \Omega_{T'_{j-1},F}} = C'_j, \hat{v}|_{\Omega_{T''_j,F} \setminus \Omega_{T''_{j-1},F}} = C''_j, j \in \mathbb{N}; \)
3. the sequence \( \{C'_j\}_{j \in \mathbb{N}} \) increases and the sequence \( \{C''_j\}_{j \in \mathbb{N}} \) decreases;
4. \( \{C''_j\}_{j \in \mathbb{N}} \) decreases;
4. \( \hat{g}(x) \asymp \hat{g}(x), \ \hat{v}(x) \asymp \hat{v}(x). \)

Let us construct the function \( \hat{g} \) (the function \( \hat{v} \) can be constructed similarly). Let \( \Gamma' \subset \partial \Omega \) be the set from the conditions of Theorem \( \text{II} \) let \( T' \) be a subtree in \( T \), let \( w' \) be the minimal vertex of \( T' \), \( m \in \mathbb{Z} \), \( \text{dist} \ (F(w'), \ \Gamma') \in [2^{-m}, 2^{-m+1}) \). Denote by \( S_{T'} \) the maximal tree in the sense of inclusions from the set of trees \( S' \subset T' \) rooted at \( w' \) and satisfying

\[
\text{dist} \ (F(w), \ \Gamma') \geq 2^{-m}, \ w \in V(S').
\]

Show that for any \( w \in V(S_{T'}), x \in F(w) \) we have

\[
\text{dist} \ (x, \ \Gamma') \lesssim 2^{-m}.
\]

Indeed, choose \( x' \in F(w') \) such that \( \text{dist} \ (x', \ \Gamma') < 2^{-m+1} \). Since \( \Omega_{S_{T'}, F} \in \mathcal{F}(b_s(a, d)) \) (see Corollary \( \text{II} \), then Definition \( \text{II} \) yields that \( |x - x'| \lesssim 2^{-m(F(w'))}. \) By Theorem \( \text{II} \)

\[
2^{-m(F(w'))} \approx (x', \partial \Omega).
\]

Therefore,

\[
\text{dist} \ (x, \ \Gamma') \approx |x - x'| + \text{dist} \ (x', \ \Gamma') + 2^{-m+1} \leq \text{dist} \ (x', \ \Gamma') + 2^{-m+1} \leq 2^{-m+2}.
\]

The trees \( T'_j \) are constructed by induction on \( j \in \mathbb{Z}_+ \). Set \( T'_0 = \emptyset \). Let the trees \( T'_j \) be constructed for \( i \in \{0, \ldots, j\} \), and let the numbers \( C'_i = \varphi_{\beta}(2^{-m_i}) \) be defined (see \( \text{(2)} \)), where \( m_i \in \mathbb{Z}, i \in \{1, \ldots, j\}, m_1 \leq \ldots \leq m_j \). In addition, suppose that \( T = T'_j \cup \left( \bigcup_{s=1}^{m_j} T'_{j,s} \right) \), where \( T'_{j,s} \) are trees rooted at \( w_{j,s}, s_0(j) \in \mathbb{N} \cup \{\infty\}, w_{j,s} \) are adjacent to some vertices of \( T'_j \) and

\[
\text{dist} \ (F(w_{j,s}), \ \Gamma') \in [2^{-m_{j,s}}, 2^{-m_{j,s}+1}), \ m_{j,s} \in \mathbb{Z}, \ m_{j,s} \geq m_j + 1.
\]

Set \( m_{j+1} = \min_{1 \leq s \leq s_0(j)} m_{j,s}, C'_j = \varphi_{\beta}(2^{-m_{j+1}}), I_j = \left\{ s \in \mathbb{N}_0(j) : m_{j,s} = m_{j+1} \right\}, T'_j = T'_j \cup \left( \bigcup_{s \in I_j} S_{T'_{j,s}} \right), \hat{g}|_{\Omega_{T'_j, F} \setminus \Omega_{T'_j, F} = C'_j + 1}. \) Then the properties 1 and 2 hold by the construction, the property 3 holds since the function \( \varphi_{\beta} \) decreases and the sequence \( \{m_j\}_{j \in \mathbb{N}} \) increases. The property 4 follows from \( \text{(2)}, \text{(3)}, \text{(73)} \) and \( \text{(75)} \).

**Step 5.** Let us prove \( \text{(75)} \). If \( \frac{1}{x z} = 0 \), then it follows from Theorem \( \text{II} \) (remind that by the condition of Theorem \( \text{II} \) in this case we have \( \hat{g} = 1 \) and \( \hat{v} = 1 \)). Let \( \frac{1}{z} > 0 \). We may assume that \( \hat{g} = \hat{g}, \ \hat{v} = \hat{v} \), where \( \hat{g} \) and \( \hat{v} \) are functions constructed at step 4. Applying the estimate which is obtained at step 3, we argue similarly as in the paper \( \text{[55]} \) (see Lemma 5.4 on the page 487). Notice that in the case \( \hat{g} = 1 \) the corresponding set \( G_y \) is defined as \( G_y = \{x \in \Omega : y \in G_x\} \). If \( y \in F(w), w \in V(T) \), then \( G_y \subset \Omega_{T_w, F} \).

Denote by \( \mathcal{P}_{r-1} (\mathbb{R}^d) \) the space of polynomials on \( \mathbb{R}^d \) of degree not exceeding \( r - 1 \). For a measurable set \( E \subset \mathbb{R}^d \) set \( \mathcal{P}_{r-1} (E) = \{f|_E : f \in \mathcal{P}_{r-1} (\mathbb{R}^d)\} \).

28
Let $G \subset \mathbb{R}^d$ be a domain and let $T = \{\Omega_i\}_{i=1}^{i_0}$ be its finite partition. Denote
\[ S_{r,T}(G) = \{S: G \to \mathbb{R}: S|_{\Omega_i} \in \mathcal{P}_{r-1}(\Omega_i), \ 1 \leq i \leq i_0\}; \quad (76) \]
for $f \in L_{q,v}(G)$ set
\[ \|f\|_{p,q,T,v} = \left( \sum_{i=1}^{i_n} \|f\|^q_{L_{q,v}(\Omega_i)} \right)^\frac{1}{q}, \quad (77) \]
where $\sigma_{p,q} = \min\{p, q\}$. Denote by $L_{p,q,T,v}(G)$ the space of functions $f \in L_{q,v}(G)$ with the norm $\|f\|_{p,q,T,v}$. Notice that $\|f\|_{p,q,T,v} \geq \|f\|_{L_{q,v}(G)}$.

**Proof of Theorem 1.** The lower estimate can be proved similarly as in [51]. In order to obtain the upper estimate, we shall prove that for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $n \geq N(\varepsilon)$, $m \in \mathbb{Z}_+$ there exists a partition $\tilde{T}_{m,n,\varepsilon} = \tilde{T}_{m,n,\varepsilon}(\Omega) = \{G_j^{m,n,\varepsilon}\}_{j=1}^{j_\varepsilon}$ of $\Omega$ with the following properties:

1. $\nu_{m,n} \leq 2^m n$;
2. for any function $f \in W^r_{\alpha,p}(\Omega)$ there exists a spline $\hat{S}_{m,n,\varepsilon}(f) \in S_{r,\tilde{T}_{m,n,\varepsilon}}(\Omega)$ such that
   \[ \|f - \hat{S}_{m,n,\varepsilon}(f)\|_{p,q,T,m,n,\varepsilon,v} \lesssim (\|gv\|_{x} + \varepsilon)(2^m n)^{-\frac{d}{2} + \beta} \frac{1}{\alpha}, \quad (78) \]
   and the mapping $f \mapsto \hat{S}_{m,n,\varepsilon}(f)$ is linear;
3. for any $G_j^{m,n,\varepsilon}$
   \[ \text{card } \{i \in \{1, \ldots, \nu_{m+1,n}\} : \text{mes } (G_j^{m,n,\varepsilon} \cap G_i^{m+1,n,\varepsilon}) > 0 \} \lesssim \frac{1}{d}. \]

Then by repeating arguments from the paper [53] (see pages 499–501), we get the desired upper estimate for widths.

**Step 1.** Let us consider the case $\alpha < \infty$ and $\beta < \infty$ only (if $\alpha = \infty$ or $\beta = \infty$, then arguments are similar with slight changes in the definition of the function $\Phi$). Let $\mu_1(E) = \int_E g_0^\beta(x) \, dx$, $\mu_2(E) = \int_E v_0^\beta(x) \, dx$. If $\frac{1}{p} := \frac{1}{d} + \frac{1}{q} - \frac{1}{p} > 0$, then we set $l_* = 3$, $\mu_3(E) = \int_E \tilde{g}^\beta(x) \tilde{v}^\beta(x) \, dx$,

\[ \alpha_1 = \frac{1}{d} + \frac{1}{q} - \frac{1}{p}, \quad \alpha_2 = \frac{1}{d} + \frac{1}{q} - \frac{1}{p}, \quad \alpha_3 = \frac{r}{d} + \frac{1}{q} - \frac{1}{p} - \frac{1}{\alpha} - \frac{1}{\beta}; \quad (79) \]

if $\frac{1}{d} + \frac{1}{q} - \frac{1}{p} - \frac{1}{\alpha} - \frac{1}{\beta} = 0$, then we set $l_* = 2$ and and define $\alpha_1$ and $\alpha_2$ by the formula (79). Define the function $\Phi$ by (54). In addition, set $\frac{1}{p} = \frac{1}{d} + \frac{1}{q} - \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{d} - \frac{1}{\beta}$, From conditions of the Theorem follows that $\tilde{p} > 1$ and $\tilde{q} < 1$. 29
Step 2. Let \((T, \omega_\ast)\) and \(F\) be the tree and the mapping defined in Lemma 3. For \(k \in \mathbb{N}\) we denote by \(T_{\leq k}\) a subtree in \(T\) such that \(V(T_{\leq k}) = \{w \in V(T) : \rho(\omega_\ast, w) \leq k\}\). Since \(\operatorname{card} V_1(w) < \infty\) for any \(w \in V(T)\), then the set \(V(T_{\leq k})\) is finite.

Fix \(\delta > 0\) and choose \(k \in \mathbb{N}\) such that
\[
\delta_k := \Phi(\Omega \setminus \Omega_{T_{\leq k}, F}) \leq \delta. \tag{80}
\]

Then \(T = T_{\leq k} \cup \left( \bigcup_{l=1}^{l_{0}(k)} T_{k,l} \right)\), where \(T_{k,l}\) are trees rooted at \(\hat{w}_{k,l}\), \(l_{0}(k) \in \mathbb{N}\). Set \(\delta_{k,l} = \Phi(\Omega_{T_{k,l}, F})\).

Step 3. If a domain \(U\) is a finite union of non-overlapping cubes,\(^2\) then for sufficiently large \(n\) the partition \(\hat{T}_{m,n,\varepsilon}(U)\) can be constructed in the same way as in (51). Define the partition \(\hat{T}_{m,n,\varepsilon/2,k} = \hat{T}_{m,n,\varepsilon/2}(\Omega_{T_{\leq k}, F})\) with properties similar to 1–3. In particular, for any function \(f \in W^r_{p,q}(\Omega)\) there exists a spline \(\hat{S}_{m,n,\varepsilon/2,k}(f) \in S_{r,T_{m,n,\varepsilon/2,k}}(\Omega_{T_{\leq k}, F})\) such that
\[
\|f - \hat{S}_{m,n,\varepsilon/2,k}(f)\|_{p,q,T_{m,n,\varepsilon/2,k},v} \lesssim \left(\|g v\|_\infty + \frac{\varepsilon}{\delta}\right) (2^m n)^{-\frac{1}{2} + \frac{\varepsilon}{2}} (\frac{1}{2})^+, \tag{81}
\]
and the mapping \(f \mapsto \hat{S}_{m,n,\varepsilon/2,k}(f)\) is linear.

Step 4. Let \(n \geq l_{0}(k)\). For each \(l \in \{1, \ldots, l_{0}(k)\}\) we set
\[
n_l = \begin{cases} \left\lfloor n \frac{\delta_k}{\delta_{k,l}} \right\rfloor, & \text{if } \delta_k > 0, \\ 1, & \text{if } \delta_k = 0. \end{cases} \tag{82}
\]

Hence, if \(\delta_k = 0\), then \(\sum_{l=1}^{l_{0}(k)} n_l = l_{0}(k) = n\), and if \(\delta_k > 0\), then
\[
\sum_{l=1}^{l_{0}(k)} n_l \leq n \sum_{l=1}^{l_{0}(k)} \frac{\delta_k}{\delta_{k,l}} + l_{0}(k) \leq n + l_{0}(k) \leq 2n. \tag{83}
\]

Step 5. Prove that for any \(l \in \{1, \ldots, l_{0}(k)\}, n \geq l_{0}(k), m \in \mathbb{Z}_+\) there exists a partition \(T_{m,n} = \{G_{j,l}^{m,n}\}_{j=1}^{\nu_{m,n,l}}\) of \(\Omega_{T_{k,l}, F}\) with the following properties: 1. \(\nu_{m,n,l} \lesssim 2^m n_l\), 2. for any function \(f \in W^r_{p,q}(\Omega)\) there exists a spline \(S_{m,n,l}(f) \in S_{r,T_{m,n}}(\Omega_{T_{k,l}, F})\) such that
\[
\|f - S_{m,n,l}(f)\|_{p,q,T_{m,n},v} \lesssim \left(\frac{\delta}{2^m n}\right)^{\frac{1}{2} - \frac{1}{p}} \left(\sum_{E \in T_{m,n}} \left\| \frac{\nabla f}{g} \right\|_{L_p(E)} \right)^{\frac{1}{p}} \tag{84}
\]
and the mapping \(f \mapsto S_{m,n,l}(f)\) is linear, 3. for any \(G_{j,l}^{m,n}\)
\[
\operatorname{card} \{i \in \{1, \ldots, \nu_{m \pm 1,n,l}\} : \operatorname{mes} (G_{j,l}^{m,n} \cap G_{i,l}^{m \pm 1,n}) > 0\} \lesssim 1. \tag{85}
\]

\(^2\)Here a cube is set that contains an open cube and is contained in closure of this cube.
Let $\mathcal{B}_{m,n} = \{E_{j,m,n}^{m,n}\}_{j=1}^{ν_{m,n}}$ be the partition of $Ω_{T_k,F}$ defined in Lemma 8. From item 1 follows that $ν_{m,n} ≤ 2^m n$. By item 2, a), either $E_{j,m,n}^{m,n} ⊂ F(w)$ (with the strict inclusion) and $E_{j,m,n}^{m,n} \in \mathcal{R} ∪ \Xi(F(w))$ for some $w = w_{j,m,n}^{m,n} \in \mathcal{V}(T_{k,l})$ (the set of such $j$ will be denoted by $J_{1,m,n}^{1}$), or $E_{j,m,n}^{m,n} = Ω_{j,m,n}^{m,n,l} F$ for some subtree $T_{j,m,n}^{m,n,l} ⊂ T_{k,l}$ (the set of such $j$ will be denoted by $J_{2,m,n}^{2}$). From item 2, b) follows that

$$\Phi(E_{j,m,n}^{m,n}) ≤ \frac{Φ(Ω_{T_k,F})}{2^m n} ≤ \frac{δ}{2^m n} ≤ \frac{δ}{2^m n}. \quad (86)$$

Let $j \in J_{1,m,n}^{1}$. Then Theorem 3 together with (2) and (3) imply that for any $x$, $y \in E_{j,m,n}^{m,n}$ we have $\frac{g(x)}{g(y)} ≥ 1$ and $\frac{v(x)}{v(y)} ≥ 1$. Hence, there exists a partition $Π_{j,l}^m$ of the set $E_{j,m,n}^{m,n}$ into at most $2d$ measurable subsets with the following property: for any function $f \in W_{p,g}(Ω)$ there exists a spline $S_{j,l}(f) \in S_rΠ_{j,l}(E_{j,m,n}^{m,n})$ such that

$$\|f - S_{j,l}(f)\|_{p,q,Π_{j,l}^{m,n}} ≤ \frac{Φ(E_{j,m,n}^{m,n})}{\hat{P}_{q,r,d,a,b,c}} \frac{∥\nabla f∥_{L_p(E_{j,m,n}^{m,n})}}{∥g∥_{L_p(E_{j,m,n}^{m,n})}} \leq \frac{δ}{2^m n} \frac{∥\nabla f∥_{L_p(E_{j,m,n}^{m,n})}}{∥g∥_{L_p(E_{j,m,n}^{m,n})}}. \quad (87)$$

(see the beginning of the proof of Theorem 3 in [51]).

Let $j \in J_{2,m,n}^{2}$. By Lemma 9 there exists a polynomial $P_{j,l}(f)$ of degree not exceeding $r - 1$ such that

$$\|f - P_{j,l}(f)\|_{L_{q,p}(E_{j,m,n}^{m,n})} ≤ \frac{Φ(E_{j,m,n}^{m,n})}{\hat{P}_{q,r,d,a,b,c}} \frac{∥v_0∥_{L_p(E_{j,m,n}^{m,n})}}{∥\nabla g∥_{L_p(E_{j,m,n}^{m,n})}} \frac{∥\nabla f∥_{L_{q,p}(E_{j,m,n}^{m,n})}}{∥g∥_{L_{q,p}(E_{j,m,n}^{m,n})}} \leq \frac{δ}{2^m n} \frac{∥\nabla f∥_{L_{q,p}(E_{j,m,n}^{m,n})}}{∥g∥_{L_{q,p}(E_{j,m,n}^{m,n})}}. \quad (79)$$

This together with the Hölder inequality yields

$$\|f - P_{j,l}(f)\|_{L_{q,p}(E_{j,m,n}^{m,n})} ≤ \frac{Φ(E_{j,m,n}^{m,n})}{\hat{P}_{q,r,d,a,b,c}} \frac{∥v_0∥_{L_p(E_{j,m,n}^{m,n})}}{∥\nabla g∥_{L_p(E_{j,m,n}^{m,n})}} \frac{∥\nabla f∥_{L_p(E_{j,m,n}^{m,n})}}{∥g∥_{L_p(E_{j,m,n}^{m,n})}} \frac{∥\nabla f∥_{L_{q,p}(E_{j,m,n}^{m,n})}}{∥g∥_{L_{q,p}(E_{j,m,n}^{m,n})}} \leq \frac{δ}{2^m n} \frac{∥\nabla f∥_{L_{q,p}(E_{j,m,n}^{m,n})}}{∥g∥_{L_{q,p}(E_{j,m,n}^{m,n})}}. \quad (88)$$

Set $T_{m,n}^l = (\bigcup_{j \in J_{1,m,n}^{1}} Π_{j,l}^m) ∪ \{E_{j,m,n}^{m,n}\}_{j \in J_{2,m,n}^{2}}$, $S_{m,n,l}(f)|_{E_{j,m,n}^{m,n}} = S_{j,l}(f)$, $j \in J_{1,m,n}^{1}$, $S_{m,n,l}(f)|_{E_{j,m,n}^{m,n,l}} = P_{j,l}(f)$, $j \in J_{2,m,n}^{2}$. Then the property 1 and (85) follow from items 1 and 3 of Lemma 8 and (88) imply (81).

**Step 6.** Put $\hat{T}_{m,n,ε} = \hat{T}_{m,n,ε/2;k} ∪ (\bigcup_{j=1}^{l_0(k)} T_{m,n}^l)$. $\hat{S}_{m,n,ε}(f)|_{\hat{T}_{m,n,ε}^l} = \hat{S}_{m,n,ε}(f)|_{\hat{T}_{m,n,ε}^l}$, $\hat{S}_{m,n,ε}(f)|_{\hat{T}_{m,n,ε}^l} = S_{m,n,l}(f)$. The property 1 follows from the estimate

$$\text{card} \hat{T}_{m,n,ε} = \text{card} \hat{T}_{m,n,ε/2;k} + \sum_{l=1}^{l_0(k)} ν_{m,n,l} ≤ d^{2m n} + \sum_{l=1}^{l_0(k)} 2^{2m n} κ \lesssim d^2 m n. \quad (89)$$
The inequality (78) follows from (81), (84) with sufficiently small $\delta > 0$, (89) and the Hölder inequality. The property 3 of the partition $\tilde{T}_{m,n,\varepsilon}$ follows from the property 3 of the partitions $\tilde{T}_{m,n,\varepsilon/2}$ and $T^d_{m,n}$. \hfill \Box

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