One contribution to a special feature ‘Quantum gravity, branes and M-theory’ dedicated to Michael J. Duff on the occasion of his 70th birthday organized by Leron Borsten, Alessio Marrani, Christopher Pope and Kellogg Stelle.
by [4,5], where the STU black holes and their entropies were related to quantum information theory. In these papers, the relation between quantum entanglement in a three-qubit system, Alice, Bob and Charlie, and three-moduli STU black holes was discussed.

The recent stage of this story is linked to a renewed interest in mathematical aspects of black holes in string theory/supergravity as studied in [6–9]. The relation between STU black holes and the Bhargava cube was observed and discussed previously in [10–12]. We will add to the recent advances in all these papers the analysis of the triality symmetry, which exists for these black holes in addition to the well-known and well-studied U-duality \([\text{SL}(2, \mathbb{Z})]^3\) symmetry. Basically triality symmetry is a statement that Alice, Bob and Charlie are on an equal footing. The aspects of the Bhargava cube related to the properties of the Cayley hyperdeterminant will be discussed here. We will clarify the concept of equivalence of black holes with the same entropy with U-duality symmetry \([\text{SL}(2, \mathbb{Z})]^3 \rtimes S_3\).

It was noted in [5] that the black holes in \(N = 8\) 4d supergravity can be brought to a canonical basis. Their entropy formula defined in general by 56 charges in the quartic Cartan–Cremmer–Julia \(E_{7(7)}\) invariant, in the canonical basis, depends only on eight charges and coincides with the Cayley hyperdeterminant defining the STU black hole area of the horizon/entropy. In the Bhargava cube terminology, this \([\text{SL}(2, \mathbb{Z})]^3\) invariant is a discriminant of the associated binary quadratic forms.

The \([\text{SL}(2, \mathbb{Z})]^3 \rtimes S_3\) symmetry of the Cayley hyperdeterminant/Bhargava cube is also a symmetry following from the Kähler potential, which is given by

\[
K_{3\text{mod}} = -\sum_{i=1}^{3} \log \left( T_i^2 + \bar{T}_i^2 \right).
\]

STU black holes can be associated with M-theory first truncated to seven moduli, \(T_i, i = 1, \ldots, 7\) with \([\text{SL}(2, \mathbb{Z})]^7\) and \(S_7\) symmetry and the Kähler potential given by

\[
K_{7\text{mod}} = -\sum_{i=1}^{7} \log \left( T_i^2 + \bar{T}_i^2 \right).
\]

When four of the seven moduli are truncated we have the remaining \([\text{SL}(2, \mathbb{Z})]^3\) duality as well as the triality permutation symmetry \(S_3\), and we recover the kinetic term of the \(N = 2\) supergravity STU model. A detailed derivation of the STU model from string theory/10d supergravity was performed in [2].

The second story of this paper is about the new ideas in cosmology based on the seven-moduli model of M-theory compactified on a manifold with \(G_2\) holonomy and with \([\text{SL}(2, \mathbb{Z})]^7\) symmetry and the Kähler potential in equation (1.2). Mike Duff was the first to point out in [13] that the maximal supersymmetry of M-theory is spontaneously broken down to \(N = 1\) supersymmetry in 4d when compactified on a manifold with \(G_2\) holonomy. More recently 11d M-theory/supergravity compactified on a twisted 7-tori with holonomy group \(G_2\) was investigated in [14].

During the last few years I have studied the issues in cosmology initiated by discussions with Sergio Ferrara, which resulted in our paper [15]. This work, in turn, originated from Ferrara’s work with Mike Duff and his collaborators [11,16–19]. One of the central ideas in all of these studies is based on the fact that \(E_{7(7)}(\mathbb{R})\) symmetry of \(N = 8\) 4d supergravity has a subgroup \([\text{SL}(2, \mathbb{R})]^7\). For the discrete subgroups this becomes the following relation:

\[
E_{7(7)}(\mathbb{Z}) \supset [\text{SL}(2, \mathbb{Z})]^7.
\]

When the relevant cosmological models were constructed in [15,20–22], seven targets for Early Universe future searches of gravitational waves from inflation were proposed. These are shown in figure 1 by seven purple lines for Poincaré discs.

The theoretical underpinning of the cosmological models in [15,20–22] was very recently proposed in my paper [24] with Gunaydin, Linde and Yamada. The entangled seven-qubit system
corresponds to seven parties—Alice, Bob, Charlie, Daisy, Emma, Fred and George—and it is related to seven imaginary units of octonions.

Mike Duff had a long and deep appreciation of the fact that there are four normed division algebras: the real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and octonions ($\mathbb{O}$). He and his collaborators have developed many new aspects of the relations between octonions and physics; see for example [11]. I will show here how octonions, Fano planes and error-correcting Hamming (7,4) codes help to build cosmological models which will be tested by future cosmological observations.

I will also show that the mass eigenvalues of heavy scalars in cosmological models in [24] described by a pair of cubic equations

\[ x^3 - 7x - 7 = 0, \]
\[ y^3 - 7x - 7 = 0 \]

have a particular relation to the exceptional Jordan1 eigenvalue problem [26–30]. There is an interesting connection between the product of the mass eigenvalues of fermions in cosmological models and the entropy of the STU black holes. Both correspond to a determinant of a certain relevant in each case of the Jordan matrix.

Another interesting feature of our cosmological models [24] is the symmetry of the fermion mass matrix at Minkowski vacua. It is invariant under the $O(7)$ symmetry and its subgroups. The discrete subgroup of it is the Weyl group $W(E_7)$. We show the Coxeter plane of the root system of $W(E_7)$ in §5. When one imposes the invariance of the octonion algebra on the transformations one obtains a finite subgroup of $G_2$, the adjoint Chevalley group $G_2(2)$ of order 12 096 as discussed in [31–34]. This is interesting since it is expected that neutrino physics will require an extension of the standard model. Some of these extensions might include discrete subgroups of $G_2$; see for example [35,36].

Thus, both of these stories, STU black holes and M-theory cosmology seven-moduli models, have an interesting connection to $E_7$ symmetry. I would like to note here that the current status of 4d $\mathcal{N} = 8$ supergravity and its perturbative ultraviolet (UV) behaviour remain puzzling. Some heroic efforts were made by Bern et al. [37] in amplitude loop computations. They have shown that maximal supergravity behaves in UV much better than expected. It was suggested in [38–40] that $E_7(7)$ symmetry together with the maximal supersymmetry of perturbative maximal supergravity in 4d might explain the cancellation of UV infinities observed in ‘theoretical experiments’ as described in [37]. It would be very interesting to learn more about these exceptional symmetries and their role in physics.

1Studies of octonions and Jordan algebras are based on the work of [25].
Figure 2. The $2 \times 2 \times 2$ hypermatrix corresponding to supergravity black holes given in fig. 2 of [5]. It represents the STU black hole solution in [3] with eight charges $p^A = p^0, p^1, p^2, p^3$ and $q_A = q_0, q_1, q_2, q_3$, where three moduli $z^1, z^2, z^3$ at the black hole horizon are functions of these charges.

2. STU black holes, triality and the Bhargava cube

A significant effort has been dedicated over the years to understanding the properties of black holes in M-theory/string theory/supergravity. The STU black holes are sufficiently simple: there are exact analytic solutions in classical $\mathcal{N} = 2$ supergravity with the prepotential

$$F = \frac{d_{ijk}X^i X^j X^k}{X^0} = \frac{X^1 X^2 X^3}{X^0}$$

(2.1)

in the so-called double extreme approximation, when the values of the three moduli, $z^i = X^i / X^0$, near the horizon are the same as those far away from the black hole, $z^i|_{\text{inf}} = z^i|_{\text{hor}}$. The solution depends on eight charges: four electric and four magnetic. The area of the horizon/the entropy of these black holes was computed in [3] in terms of the eight black hole charges $(p^A, q_A)$, $A = 0, 1, 2, 3$, shown as corners of the $2 \times 2 \times 2$ hypermatrix in figure 2. The entropy is the function of charges

$$S = \left( W(p^A, q_A) \right)^{1/2},$$

(2.2)

where

$$W(p^A, q_A) = -(p \cdot q)^2 + 4((p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^2 q_2)(p^3 q_3))$$

$$-4p^0 q_1 q_2 q_3 + 4q_0 p^1 p^2 p^3$$

(2.3)

and

$$p \cdot q = (p^0 q_0) + (p^1 q_1) + (p^2 q_2) + (p^3 q_3).$$

(2.4)

The function $W(p^A, q_A)$ is manifestly symmetric under transformations

$$p^1 \leftrightarrow p^2 \leftrightarrow p^3, \quad q_1 \leftrightarrow q_2 \leftrightarrow q_3.$$  

(2.5)

Under $[\text{SL}(2, \mathbb{Z})]^3$ transformations the charges and the moduli transform but the entropy is invariant.

The values of the three complex moduli near the horizon, for each $i = 1, 2, 3$, were computed in [3]

$$z^i = \frac{B^i}{2A_i} = \frac{i \sqrt{B^i - 4A_i C_i}}{2A_i},$$

(2.6)

where for each $i = 1, 2, 3$

$$A_i = p^0 q_i - 3d_{ijk} p^j p^k, \quad B^i = p \cdot q - 2p^i q_i, \quad C^i = -(p^i q_0 + 3d_{ijk} q_j q_k)$$

(2.7)
Figure 3. The Bhargava cube.

and

\[ W = -D = B_1^3 - 4A_1C_1 = B_2^3 - 4A_2C_2 = B_3^3 - 4A_3C_3. \] (2.8)

It was pointed out in [4] that the classical expression for the entropy of the STU black holes
\[ W(p^A, q_A) \] (2.3) can be represented in a very beautiful form,

\[ S_{BPS} = \pi \sqrt{W} = \frac{\pi}{2} \sqrt{\text{Det } \psi}, \quad \text{Det } \psi < 0, \] (2.9)

where \( \text{Det } \psi \) is the Cayley hyperdeterminant of the vector with components \( \psi_{ijk} \), constructed in 1845. The dictionary between the eight charges \( p^A \) and \( q_A \) and the components of \( \psi_{ijk} \) is the following:

\[ \begin{array}{cccccccc}
p^0 & p^1 & p^2 & p^3 & q_0 & q_1 & q_2 & q_3 \\ \psi_{000} & -\psi_{001} & -\psi_{010} & -\psi_{100} & \psi_{111} & \psi_{110} & \psi_{101} & \psi_{011} \end{array} \] (2.10)

The Cayley hyperdeterminant of the \( 2 \times 2 \times 2 \) hypermatrix \( \psi_{ijk} \) is defined as follows:

\[ \text{Det } \psi = -\frac{1}{2} e^{i\ell'} e^{j\ell} e^{k\ell'} e^{m\ell'} e^{n\ell} \psi_{ijkl'} \psi_{ijkl'}. \] (2.11)

The new aspect of the STU black holes associated with the Bhargava cube developed in [6–9] is the following. It is possible to attach a triple of quadratic forms

\[ A_i x^2 + B_i xy + C_i y^2 \] (2.12)

of the same discriminant \( D = B_1^2 - 4A_1C_1 \) to a cube, with the corners given by an octuple \( a, b, c, d, e, f, g, h \). We show this cube in figure 3. Even when only two of the forms are available, one can construct the third one as well as the Bhargava cube. The dictionary between the eight black hole charges in figure 2 and the Bhargava octuple in figure 3 is

\[ \begin{array}{cccccccc}
p^2 & p^0 & p^3 & q_1 & q_3 & p^1 & q_0 & -q_2 \\ a & b & c & d & e & f & g & h \end{array} \] (2.13)

The cube has a three-way slicing—up–down, left–right, front–back—and many interesting properties. The discriminant of the cube is given by the following expression:

\[ D_{Bha} = a^2 h^2 + b^2 g^2 + c^2 f^2 + d^2 e^2 - 2(abgh + cdef + acfh + bdeg + aedh + bfeg) + 4(adfg + bceh). \] (2.14)
relevant charges are values of the moduli in the prepotential given in equation (2.1), the entropy is shown in equation (2.3) and the some of these eight charges can be related to each other by a U-duality symmetry $[\text{SL}(2, \mathbb{Z})]^3$. Examples with triality symmetry were given in [9]. Here, we discuss a general case of triality symmetry in the context of the Bhargava cube. Therefore, the three symmetries of the discriminant of the Bhargava cube which reflect the symmetry between the three moduli $z^i$ at the horizon and the relevant charges are

\begin{align}
& z^1 \leftrightarrow z^2 : \quad p^1 \leftrightarrow p^2 \quad q_1 \leftrightarrow q_2 \\
& z^2 \leftrightarrow z^3 : \quad p^2 \leftrightarrow p^3 \quad q_2 \leftrightarrow q_3 \\
& z^3 \leftrightarrow z^1 : \quad p^3 \leftrightarrow p^1 \quad q_3 \leftrightarrow q_1.
\end{align}

Therefore, the three symmetries of the discriminant of the Bhargava cube which reflect the corresponding black hole symmetries are

\begin{align}
& f \leftrightarrow a \quad c \leftrightarrow h \\
& a \leftrightarrow d \quad h \leftrightarrow e \\
& d \leftrightarrow f \quad e \leftrightarrow c.
\end{align}

We show these by the red diagonal lines in figure 4.

(a) The issue of black hole equivalence

Supersymmetric STU black holes are defined by their entropy as well as by the values of the three moduli near the horizon. In the basis, where all three moduli $z^i$ are on an equal footing in the prepotential given in equation (2.1), the entropy is shown in equation (2.3) and the values of the moduli $z^i$ are given in equation (2.6). The $\mathcal{N} = 2$ supergravity in this basis and the black hole solution both have this symmetry. The symmetry of solutions is presented in equation (2.16).

The permutation symmetry for black holes, a triality symmetry, is important when the physical question is asked: what kind of STU black holes are equivalent? It is known that the entropy might be the same for a different set of eight charges, for example $(p^i, q^i)$ and $(p'^i, q'^i)$. However, some of these eight charges can be related to each other by a U-duality symmetry $[\text{SL}(2, \mathbb{Z})]^3 \ltimes S_3$ transformation. In such a case, these two sets of eight charges belong to the same U-duality orbit.

\^Examples with triality symmetry were given in [9]. Here, we discuss a general case of triality symmetry in the context of the Bhargava cube.
If, however, they are not related to each other by an \([\text{SL}(2, \mathbb{Z})]^3 \ltimes S_3\) transformation, they belong to different orbits.

There has been significant progress in understanding the discrete properties of the Bhargava cube that may be useful in the context of string theory counting of states associated with supersymmetric black holes with integer charges. To use these properties, it would be nice to take into account systematically also the triality symmetry \(S_3\) of the discriminant of the Bhargava cube, in addition to modular \([\text{SL}(2, \mathbb{Z})]^3\) symmetry, which has already been studied extensively.

In the basis \(a, b, c, d, e, f, g, h\), which is standard in the Bhargava cube literature, this \(S_3\) symmetry (2.17) of the discriminant in equation (2.14) is not obvious since it does not appear to be related to a supergravity \(S_3\) invariant prepotential (2.1). However, it is present there. The metric, and therefore the entropy, of the STU black hole solution is U-duality invariant. The \(S_3\) symmetry is therefore manifest in equation (2.3) since it follows from the triality invariant prepotential.

3. M-theory cosmology, octonions and error-correcting codes

A short summary of the recent paper [24] suitable for this set-up is the following. We have proposed an expression for the effective \(N = 1\) 4d supergravity following from M-theory/11d supergravity compactified on a manifold with \(G_2\) structure. Starting with general type \(G_2\)-structure manifolds one finds Minkowski vacua only in the cases that the twisted 7-tori are \(G_2\)-holonomy manifolds. Here, again, it was a crucial early insight of Mike Duff that the maximal supersymmetry of M-theory is spontaneously broken by compactification to minimal \(N = 1\) supersymmetry in 4d [13] when the compactification manifold has a \(G_2\) holonomy.

Our choice of the superpotential is based on a split of the seven-qubit system—Alice, Bob, Charlie, Daisy, Emma, Fred, George—into three-qubits and four-qubits. The three qubits codify the multiplication table of octonions; there are seven associated triads there. The seven complementary four-qubits define our superpotential. The automorphism group of octonions is \(G_2\), therefore it is natural to define the superpotentials using octonions.

The effective \(N = 1\) 4d supergravity following from M-theory/11d supergravity is defined as follows. The Kähler potential is given in equation (1.2). The superpotential in general is given by a sum over the seven four-qubits \(\{ijkl\}\) of the form

\[
W_O = \sum_{\{ijkl\}} (T^i - T^j) (T^k - T^l) = \frac{1}{2} M_{ij} T^i T^j. \tag{3.1}
\]

It appears to have 28 terms of the form \(T^i T^j\); however, half of them cancel and we are left with 14 terms. For example, for Cartan–Shouten–Coxeter octonion conventions [41,42]

\[
W_O = \sum_{r=0}^{6} (T^{r+2} - T^{r+4}) (T^{r+5} - T^{r+6}). \tag{3.2}
\]

We can see these \(7 \times 4\) terms on the right-hand side of equation (3.5). But actually, the formula simplifies to 14 terms

\[
W_O = - \sum_{r=1}^{7} T^r (T^{r+1} - T^{r+2}). \tag{3.3}
\]

Explicitly the 14 terms are

\[
W_O = - \left( T^1 (T^2 - T^3) + T^2 (T^3 - T^4) + T^3 (T^4 - T^5) + T^4 (T^5 - T^6) + T^5 (T^6 - T^7) + T^6 (T^7 - T^1) + T^7 (T^1 - T^2) \right). \tag{3.4}
\]

The set of seven terms in the superpotential in the form (3.2) is easy to compare with seven octonion associate triads, with seven quadruples and with seven codewords of the cyclic (7,4) Hamming error-correcting code. We show this relation in equation (3.5).
Figure 5. An oriented Fano plane (fig. 1 in [43]). On each of the seven lines (including the circle), there are three points, e.g. 1, 2 and 4 on a circle. The octonian multiplication rule is built into the Fano table. For example, one can see from the oriented circle that $e_1 \cdot e_2 = e_4$.

| triads | codewords | quadruples | $\mathcal{WO}$ |
|--------|-----------|------------|----------------|
| (137)  | 1 0 1 0 0 0 1 | (2456) | $(T^2 - T^4)(T^5 - T^6)$ |
| (241)  | 1 1 0 1 0 0 0 | (3567) | $(T^3 - T^5)(T^6 - T^7)$ |
| (352)  | 0 1 1 0 1 0 0 | (4671) | $(T^4 - T^6)(T^7 - T^1)$ |
| (463)  | 0 0 1 1 0 1 0 | (5712) | $(T^5 - T^7)(T^1 - T^2)$ |
| (574)  | 0 0 0 1 1 0 1 | (6123) | $(T^6 - T^1)(T^2 - T^3)$ |
| (615)  | 1 0 0 0 1 1 0 | (7234) | $(T^7 - T^2)(T^3 - T^4)$ |
| (726)  | 0 1 0 0 1 1 0 | (1345) | $(T^1 - T^3)(T^4 - T^5)$ |

(3.5)

Let us show how the octonion triads are represented in the oriented Fano plane. Each of the seven lines has three points; the arrows show the order, with possible cyclic permutations. For example, the first one in equation (3.5) is 137; we see it as the internal line going up and to the right—it shows as 371. The next one is 241; it is a set of points on a circle. One more: 352 is the one at the bottom; going to the left, it shows as 523 etc. (figure 5).

We studied Minkowski vacua in seven-moduli models with octonionic superpotentials (3.1). We found that these models have a Minkowski minimum at

$$T^1 = T^2 = T^3 = T^4 = T^5 = T^6 = T^7 \equiv T$$

(3.6)

with one flat direction.

There are 480 different octonion conventions. We have presented a general formula of the superpotential for any octonion convention in [24]. In all these cases, the matrix $M_{ij}$ in equation (3.1) can be computed either using the general formula or by performing a change of variables. It is, therefore, not surprising that the eigenvalues of these matrices for all possible octonions are always the same. We will discuss these eigenvalues and their relation to $3 \times 3$ octonionic Hermitian matrices and to black hole entropy in the next section.

We can cut from the superpotential (3.2) some terms according to the rules specified via error-correcting codes (figure 6). The related kinetic terms for the inflaton fields corresponding to all these models with one and two flat directions are

$$K = -m \log \left( T_{(1)} + \bar{T}_{(1)} \right) - n \log \left( T_{(2)} + \bar{T}_{(2)} \right)$$

(3.7)

with $m + n = 7$ and cases like $m = 0, n = 7$; $m = 1, n = 6$; $m = 2, n = 5$; $m = 3, n = 4$. The superpotentials $\mathcal{WO}_{m,n}$ at the vacuum have the following properties:

$$\mathcal{WO}_{m,n} = 0, \quad \partial_i \mathcal{WO}_{m,n} = 0$$

(3.8)

at

$$T^1 = \cdots = T^m \equiv T_{(1)}, \quad T^{m+1} = \cdots = T^n \equiv T_{(2)}.$$  

(3.9)
Based on these M-theory Minkowski vacua, we have built $\mathcal{N}=1$ supergravity phenomenological models with the potential

$$V = F(T, \bar{T}) \left( 1 + \frac{|\mathcal{W}_{\text{oct}}|^2}{W_0^2} \right) + \sum_{i=1}^{7} (T^i + \bar{T}^i)^2 |\partial_i \mathcal{W}_{\text{oct}}|^2, \quad (3.10)$$

where

$$\mathcal{W}_{\text{oct}} \equiv \frac{1}{\sqrt{\prod_{i=1}^{7} (2 T^i)}} \mathcal{W}_O. \quad (3.11)$$

Along the supersymmetric Minkowski flat directions we have $\mathcal{W}_{\text{oct}} = \partial_j \mathcal{W}_{\text{oct}} = 0$. Therefore, the full expression for the inflaton potential, for example in the simplest T-models, is given by an inflationary potential for the $\alpha$-attractor models and a cosmological constant

$$V = \Lambda + m^2 \tanh^2 \sqrt{\frac{1}{6\alpha}} \phi. \quad (3.12)$$

Inflation along various flat directions with these kinetic terms leads to $3\alpha = 7, 6, 5, 4, 3, 2, 1$ and therefore to seven possible values of the tensor to scalar ratio $r = 12\alpha/N^2$ in the range $10^{-2} \gtrsim r \gtrsim 10^{-3}$, which should be accessible to future cosmological observations. They are shown by the seven purple lines in figure 1, which is taken from the LiteBIRD satellite mission forecast.

4. Properties of the mass matrix in octonion cosmological models

The octonion superpotentials for models in [24] with $G_2$ holonomy and seven moduli have 14 terms in the form

$$\mathcal{W}_O = \frac{1}{2} M_{ij} T^i T^j. \quad (4.1)$$

The matrix $M_{ij}$ for the simplest case $\mathcal{W}_O$ for Cartan–Shouten–Coxeter octonion notations is

$$M_{\mathcal{W}_O} = \begin{pmatrix}
0 & -1 & 1 & 0 & 0 & 1 & -1 \\
-1 & 0 & -1 & 1 & 0 & 0 & 1 \\
1 & -1 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 1 & -1 \\
1 & 0 & 0 & 1 & -1 & 0 & -1 \\
-1 & 1 & 0 & 0 & 1 & -1 & 0
\end{pmatrix}. \quad (4.2)$$

One can see that it has the property $M_{ij} = \sum_j M_{ij} = 0, \forall i$. In a Minkowski vacuum with $\mathcal{W}_O = \mathcal{W}_{\text{oct}}, i = 0$, the fermion mass matrix is

$$\frac{1}{2} e^{\frac{x}{2}} X^i M_{ij} X^j. \quad (4.3)$$
The non-vanishing six eigenvalues of the $\mathbb{M}$ matrix, defining the fermion mass eigenstates in Minkowski vacua, solve a double set of cubic equations

$$x^3 - 7x - 7 = 0, \quad y^3 - 7y - 7 = 0.$$ (4.4)

The eigenvalues of the fermion mass matrix are $M_{\text{EV}}^{\text{W}0} = \begin{pmatrix}
x_1 & 0 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 0 & 0 & 0 \\
x_3 & 0 & 0 & 0 & 0 & 0 \\
0 & y_1 & 0 & 0 & 0 & 0 \\
0 & 0 & y_2 & 0 & 0 & 0 \\
0 & 0 & 0 & y_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$, (4.5)

where $x_a = y_a$ with $a = 1, 2, 3$ are solutions of the cubic equation (4.4). Numerically this gives, for a set of $x_1, y_1; x_2, y_2; x_3, y_3$ and a massless one, the following values

$$3.0489, 3.0489; -1.6920, -1.6920; -1.3569, -1.3569; 0$$ (4.6)

as shown in [24]. It looks like the numerical sum of all three eigenvalues vanishes,

$$3.0489 - 1.6920 - 1.3569 \approx 0.$$ (4.7)

Meanwhile, we can also solve equation (4.4) analytically. With $z_a = x_a = y_a$ and $\theta = \text{Arc tan } 3^{-3/2}$ the solutions are

$$z_1 = 2\sqrt[3]{\frac{7}{3}} \text{Re } e^{i(\theta/3)},$$
$$z_2 = 2\sqrt[3]{\frac{7}{3}} \text{Re } e^{i(\theta+2\pi)/3},$$
$$z_3 = 2\sqrt[3]{\frac{7}{3}} \text{Re } e^{i(\theta+4\pi)/3}.$$ (4.8)

Also one finds that

$$z_1 + z_2 + z_3 = 0,$$ (4.9)

which means that indeed the sum of the three eigenvalues of the fermion mass matrix vanishes exactly. A number of other relations can be seen in the exact solution

$$z_1z_2 + z_2z_3 + z_1z_3 = -7,$$
$$z_1z_2z_3 = 7,$$
$$z_1^2 + z_2^2 + z_3^2 = 2 \cdot 7,$$
$$z_1^3 + z_2^3 + z_3^3 = 3 \cdot 7,$$
$$z_1^4 + z_2^4 + z_3^4 = 2 \cdot 7^2$$
$$z_1^5 + z_2^5 + z_3^5 = 5 \cdot 7^2.$$ (4.10)

We can compare the eigenvalues of the $3 \times 3$ part of the fermion mass matrix with the eigenvalues of the $3 \times 3$ octonionic Hermitian matrix studied in [28,29], which defines the supersymmetric black hole entropy in 5d. This entropy was shown in [44] to be equal to a square root of the cubic invariant $I_3$ of $E_{6(6)}$. In [28,29], it was shown how this cubic invariant is related to the Jordan algebra $J^O_3$ of the $3 \times 3$ Hermitian matrices over the composition algebra of octonions $O$. 
A generic element \( J \) of \( J^O_3 \) has the form

\[
J = \begin{pmatrix}
    \alpha_1 & \alpha_3 & \alpha_2 \\
    \alpha_3^* & \alpha_2 & \alpha_1 \\
    \alpha_2 & \alpha_1^* & \alpha_3
\end{pmatrix},
\]

where \( \alpha_a \) are real numbers and \( \alpha_a \) with \( a = 1, 2, 3 \) are elements of \( O \). The automorphism of the split exceptional Jordan algebra is the non-compact \( F_4(4) \) group. In the case of the non-split octonions, the automorphism group is \( F_4 \). An element of \( J^O_3 \) can be brought to a diagonal form by an \( F_4 \) rotation \([26–28,30]\). In the case of the black holes, the generic element of \( J \) has eigenvalues \( \lambda_a, a = 1, 2, 3 \), and the cubic norm of \( J^O_3 \) is as shown in \( [28] \).

In the case of non-split octonions, we also start with the element \( (4.11) \) and diagonalizes it using \( F_4 \) transformation \([26]\). The three eigenvalues in this case were shown to satisfy a certain characteristic cubic equation \([30]\)

\[
-\det(J - \lambda I) = \lambda^3 - (\text{Tr}J) \lambda^2 + \text{Tr}(J \times J) \lambda - (\det J) I = 0,
\]

where in the notation of \([27]\)

\[
J \times J = J^{-1} \det J.
\]

In our cosmological model, the analogous cubic equation \( x^3 - 7x - 7 = 0 \) corresponds to the choice

\[
\text{Tr} J = 0, \quad \text{Tr} J^{-1} = -I, \quad \det J = 7.
\]

The choice \( \text{Tr} J = 0 \) according to \([27]\) means that our matrix \( (4.11) \) depends on only 26 parameters and therefore it is a 26-dimensional representation of \( F_4 \). It is also explained in \([27]\) that the invariants of \( F_4 \) are

\[
\text{Tr} J, \quad \text{Tr}(J \times J), \quad \det J.
\]

Thus we find that our fermion mass matrix eigenvalues are defined by a cubic equation \( x^3 - 7x - 7 = 0 \) of the kind which defines the exceptional Jordan matrix eigenvalues \([30]\) with special \( F_4 \) invariant properties,

\[
\text{Tr} J = 0, \quad (J \times J) = -I, \quad \det J = 7.
\]

Meanwhile, the relation between the det of the fermion mass matrix \( \det M_W \) and black hole entropy in the diagonal basis \( \sqrt{I_3} \) is

\[
\det M_W = x_1 x_2 x_3 \quad I_3 = \lambda_1 \lambda_2 \lambda_3 = p^1 p^2 p^3.
\]

In 5d black holes, the values of magnetic charges, \( p^1, p^2, p^3 \), are less restricted; they do not satisfy a cubic equation of the kind \( x^3 - 7x - 7 = 0 \). In fact, the entropy of the 4d STU black holes we started with in equation (2.3) is the same as the one in 5d under the condition that \( p^0 = q_1 = q_2 = q_3 = 0 \).

Thus, in addition to numerous relations between various BPS and non-BPS black holes, we have observed here an interesting relation to octonion-based cosmological models and the fermion mass matrix.

5. Discrete symmetry of fermions in cosmological models

The fermion mass matrix in equation (4.3) at Minkowski vacua in cosmological models \([24]\) can be brought to a diagonal form as shown in equations (4.5) and (4.6). Since it is a \( 7 \times 7 \) matrix, its eigenvalues are invariant under the \( O(7) \) symmetry and its subgroups. The discrete subgroup of it is the Weyl group \( W(E_7) \). It is isomorphic to a finite subgroup of \( O(7) \), which is the direct product of \( Z_2 \times SO_7(2) \). The group \( SO_7(2) \) is the adjoint Chevalley group of order \( 1451 \, 520 \). The Weyl group \( W(E_7) \) has \( 2903 \, 040 \) symmetries. The root system of the Weyl group \( W(E_7) \) cannot be visualized since it is an object in seven dimensions, but the two-dimensional projections of them, the Coxeter planes, are well known. We present them in figures 7 and 8.

However, the Weyl group \( W(E_7) \) does not preserve the octonion algebra. When one imposes the invariance of the octonion algebra on the transformations of the \( E_7 \) roots one obtains a finite
subgroup of $G_2$, as expected—the adjoint Chevalley group $G_2(2)$ of order 12 096. We now review the analysis of $E_7$ roots and its $G_2(2)$ symmetry following [31–34] and show that it applies to the fermions in the cosmological models of [24]. First, we note that $E_8$ roots can be defined by the integral octonions of the following form. We take Cartan–Shoten–Coxeter octonion conventions, which we used in equation (3.5). The triples are 124, 235, 346, 457, 561, 672, 713 and the quadruples are 3567, 4671, 5712, 6123, 7234, 1345, 2456. The set of 240 integer octonions is

\[ \pm 1, \pm e_i, \]
\[ \frac{1}{2}(\pm 1 \pm e_i \pm e_j \pm e_k), \] (5.1)
\[ \frac{1}{2}(\pm e_i \pm e_j \pm e_k \pm e_l). \] (5.2)

Here in (5.2) $ijk$ belong to triples, so we have $7 \times 16 = 112$, and in (5.3) $ijkl$ belong to complementary quadruples, so we have again $7 \times 16 = 112$. To this, we add 16 from equation (5.1). This gives the total of 240 integral octonions, which make the $E_8$ roots, also called Cayley integers or octavians. From the set of integral octonions above, we keep only the ones in

\[ \pm e_i, \frac{1}{2}(\pm e_i \pm e_j \pm e_k \pm e_l). \] (5.4)

There are $14 + 7 \times 16 = 126$ integral octonions. It was shown in [33] that the set of transformations which preserve the octonion algebra of the root system of $E_7$ in (5.4) is the adjoint Chevalley group $G_2(2)$. It is possible to decompose these 126 imaginary octonions into 18 sets of seven imaginary octonionic units that can be transformed to each other by the finite subgroup of matrices. These lead to 18 sets of 7, which we see in figures 7 and 8.
Figure 8. The Coxeter projections of all exceptional root systems are given by Tamás Görbe (https://tamasgorbe.wordpress.com/2015/10/28/coxeter-projection-of-exceptional-root-systems/), including the $E_7$ case shown here. As in figure 7, we can see seven circles with 18 points each, a total of 126 points representing a root system of $E_7$. (Online version in colour.)

Thus it appears that the cosmological models in [24] derived from compactification of 11d supergravity on a manifold with $G_2$ holonomy have some hidden $E_7$ symmetry. It would be nice to understand the relation of all this to the maximal 4d supergravity with $E_{7(7)}$ symmetry.

6. Discussion

The entanglement of seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George) was important in Mike Duff’s studies of black holes in M-theory. In the context of M-theory cosmology, it is not surprising that the concept of seven qubits and the related tools like octonions, $G_2$ symmetry, Fano planes and (7,4) Hamming correcting codes also play an important role. It would be interesting to develop more understanding of both black holes and cosmological models in M-theory and of the role of octonions in physics.

Neutrino physics may also require some new ideas to satisfy the current and future data. It was advocated in [35,36] that some discrete subgroups of $G_2$, like $PSL_2(13)$, might be useful for this purpose.

A nice feature of our cosmological models in [24] is that they describe a case of maximal supersymmetry spontaneously broken down to a minimal supersymmetry. These models will be tested by future cosmological observations, as we show in figure 1. The most recent forecast of the CMB-S4 in [45] suggests that the ground-based stage 4 experiments will achieve the science goals of detecting primordial gravitational waves for $r > 0.003$ at greater than $5\sigma$, or, in the absence of a detection, of reaching an upper limit of $r < 0.001$ at a 95% confidence limit. Therefore, the benchmark targets of cosmological models in [24] will be tested during the next decade or two.

Data accessibility. This article has no additional data.
Competing interests. I declare I have no competing interests.
Funding. I am supported by SITP and the US National Science Foundation (grant no. PHY-1720397) and by the Simons Foundation Origins of the Universe program (Modern Inflationary Cosmology collaboration).
Acknowledgements. I am grateful to G. Dall’Agata, M. Duff, S. Ferrara, S. Kachru, A. Linde, A. Van Proeyen and Y. Yamada for stimulating discussions. I am especially grateful to M. Gunaydin for clarification of the relevant issues with octonions and exceptional groups.

References

1. Ferrara S, Kallosh R. 1996 Supersymmetry and attractors. Phys. Rev. D 54, 1514–1524. (doi: 10.1103/physrevd.54.1514)
2. Duff M, Liu JT, Rahmfeld J. 1996 Four-dimensional string-string-string triality. Nucl. Phys. B 459, 125. (doi: 10.1016/0550-3213(95)00555-2)
3. Behrndt K, Kallosh R, Rahmfeld J, Shmakova M, Wong WK. 1996 STU black holes and string triality. Phys. Rev. D 54, 6293–6301. (doi: 10.1103/PhysRevD.54.6293)
4. Duff M. 2007 String triality, black hole entropy and Cayley’s hyperdeterminant. Phys. Rev. D 76, 025017. (doi: 10.1103/PhysRevD.76.025017)
5. Kallosh R, Linde AD. 2006 Strings, black holes, and quantum information. Phys. Rev. D 73, 104033. (doi: 10.1103/PhysRevD.73.104033)
6. Benjamin N, Kachru S, Ono K, Rolen L. 2018 Black holes and class groups. (http://arxiv.org/abs/1807.00797)
7. Gunaydin M, Kachru S, Tripathy A. 2020 Black holes and Bhargava’s invariant theory. (http://arxiv.org/abs/1903.02323)
8. Banerjee N, Bhand A, Dutta S, Sen A, Singh RK. 2020 Bhargava’s cube and black hole charges. (http://arxiv.org/abs/2006.02494)
9. Borsten L, Duff M, Marrani A. 2020 Black holes and higher composition laws. (http://arxiv.org/abs/2006.03574)
10. Borsten L. 2008 E(7)(7) invariant measures of entanglement. Fortsch. Phys. 56, 842–848. (doi: 10.1002/prop.200810542)
11. Borsten L, Dahanayake D, Duff M, Ebrahim H, Rubens W. 2009 Black holes, qubits and octonions. Phys. Rep. 471, 113–219. (doi: 10.1016/j.physrep.2008.11.002)
12. Borsten L. 2010 Aspects of M-theory and quantum information. PhD Thesis, Imperial College London, London, UK.
13. Awada M, Duff M, Pope C. 1983 N = 8 supergravity breaks down to N = 1. Phys. Rev. Lett. 50, 294–297. (doi: 10.1103/PhysRevLett.50.294)
14. Dall’Agata G, Prezas N. 2005 Scherk-Schwarz reduction of M-theory on G2-manifolds with fluxes. J. High Energy Phys. 10, 103. (doi: 10.1088/1126-6708/2005/10/103)
15. Ferrara S, Kallosh R. 2016 Seven-disk manifold, α-attractors, and B modes. Phys. Rev. D 94, 126015. (doi: 10.1103/PhysRevD.94.126015)
16. Duff MJ, Ferrara S. 2007 E(7) and the tripartite entanglement of seven qubits. Phys. Rev. D 76, 025018. (doi: 10.1001/PhysRevD.76.025018)
17. Levay P. 2007 Strings, black holes, the tripartite entanglement of seven qubits and the Fano plane. Phys. Rev. D 75, 024024. (doi: 10.1103/PhysRevD.75.024024)
18. Levay P. 2010 Attractors, black holes and multiqubit entanglement. Springer Proc. Phys. 134, 850. (doi: 10.1007/978-3-642-70376-8_3)
19. Borsten L, Duff MJ, Levay P. 2012 The black-hole/qubit correspondence: an up-to-date review. Class. Quant. Grav. 29, 224008. (doi: 10.1088/0264-9381/29/22/224008)
20. Kallosh R, Linde A, Wrase T, Yamada Y. 2017 Maximal supersymmetry and B-mode targets. J. High Energy Phys. 04, 144. (doi: 10.1007/JHEP04(2017)144)
21. Kallosh R, Linde A, Roest D, Yamada Y. 2017 D3 induced geometric inflation. J. High Energy Phys. 07, 057. (doi: 10.1007/JHEP07(2017)057)
22. Kallosh R, Linde A. 2019 CMB targets after the latest Planck data release. Phys. Rev. D 100, 123523. (doi: 10.1001/PhysRevD.100.123523)
23. Lee A et al. 2019 LiteBIRD: an all-sky cosmic microwave background probe of inflation. Bull. Am. Astron. Soc. 51, 286. (doi: 10.1103/PhysRevD.100.123523)
24. Gunaydin M, Kallosh R, Linde A, Yamada Y. 2020 M-theory cosmology, octonions, error correcting codes. (http://arxiv.org/abs/2008.01494)
25. Freudenthal H. 1951 Oktaven, Ausnahmegruppen und Oktavengeometrie. Utrecht, The Netherlands: Mathematisch Instituut der Rijksuniversiteit te Utrecht.
26. Gunaydin M, Piron C, Ruegg H. 1978 Moufang plane and octonionic quantum mechanics. Commun. Math. Phys. 61, 69–85. (doi: 10.1007/BF01609468)
27. Dundarer R, Gursey F, Tze HC. 1984 Generalized vector products, duality and octonionic identities in $D = 8$ geometry. *J. Math. Phys.* 25, 1496–1506. (doi:10.1063/1.526321)
28. Ferrara S, Gunaydin M. 1998 Orbits of exceptional groups, duality and BPS states in string theory. *Int. J. Mod. Phys. A* 13, 2075–2088. (doi:10.1142/S0217751X98000913)
29. Ferrara S, Gunaydin M. 2006 Orbits and attractors for $N = 2$ Maxwell–Einstein supergravity theories in five dimensions. *Nucl. Phys. B* 759, 1. (doi:10.1016/j.nuclphysb.2006.09.016)
30. Dray T, Manogue CA. 1999 The exceptional Jordan eigenvalue problem. *Int. J. Theor. Phys.* 38, 2901–2916. (doi:10.1023/A:1026699850361)
31. Koca M, Ozdes N. 1989 Division algebras with integral elements. *J. Phys. A* 22, 1469. (doi:10.1088/0305-4470/22/10/006)
32. Karsch F, Koca M. 1990 G2(2) as the automorphism group of the octonionic root system of E7. *J. Phys. A* 23, 4739–4750. (doi:10.1088/0305-4470/23/21/016)
33. Koca M, Koc R, Koca NO. 2007 The Chevalley group G2(2) of order 12096 and the octonionic root system of E7. *Linear Algebr. Appl.* 422, 808–823. (doi:10.1016/j.laa.2006.12.011)
34. Anastasiou A, Hughes M. 2015 Octonionic $D = 11$ supergravity and `octavian integers’ as dilaton vectors. (http://arxiv.org/abs/1502.02578)
35. Ramond P. 2020 The Freund–Rubin coset, textures and group theory. *J. Phys. A* 53, 341001. (doi:10.1088/1751-8121/ab9d45)
36. Perez MJ, Rahat MH, Ramond P, Stuart AJ, Xu B. 2020 Tribimaximal mixing in the SU(5) × $T_{13}$ texture. *Phys. Rev. D* 101, 075018. (doi:10.1103/PhysRevD.101.075018)
37. Bern Z, Carrasco JJ, Chiodaroli M, Johansson H, Roiban R. 2019 The duality between color and kinematics and its applications. (http://arxiv.org/abs/1909.01358)
38. Beisert N, Elvang H, Freedman DZ, Kiermaier M, Morales A, Stieberger S. 2011 E7(7) constraints on counterterms in $N = 8$ supergravity. *Phys. Lett. B* 694, 265–271. (doi:10.1016/j.physletb.2010.09.069)
39. Kallosh R. 2019 The action with manifest E7 type symmetry. *J. High Energy Phys.* 05, 109. (doi:10.1007/JHEP05(2019)109)
40. Gunaydin M, Kallosh R. 2019 Supersymmetry constraints on U-duality invariant deformations of $N \geq 5$ supergravity. *J. High Energy Phys.* 09, 105. (doi:10.1007/JHEP09(2019)105)
41. Cartan E, Schouten JA. 1926 On Riemannian geometries admitting an absolute parallelism. *R. Acad. Amst. Proc. Sect. Sci.* 29, 933.
42. Coxeter HSM. 1946 Integral Cayley numbers. *Duke Math. J.* 13, 561–578. (doi:10.1215/S0012-7094-46-01347-6)
43. Baez JC. 2002 The octonions. *Bull. Am. Math. Soc.* 39, 145–205. (doi:10.1090/S0273-0979-01-00934-X)
44. Ferrara S, Kallosh R. 1996 Universality of supersymmetric attractors. *Phys. Rev. D* 54, 1525. (doi:10.1103/PhysRevD.54.1525)
45. COLLABCMB-S4 collaboration. 2020 CMB-S4: forecasting constraints on primordial gravitational waves. (http://arxiv.org/abs/2008.12619).