Secondary Faraday waves in microgravity

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Abstract. Recent microgravity experiments have demonstrated that Faraday waves can arise in a secondary instability over the primary columnar patterns that develop after the frozen wave instability. While some numerical studies have investigated this phenomenon, theoretical analyses are only found in the works of Shevtsova et al. (2016) [1] and Lyubimova et al. (2019) [2]. Here, we extend these efforts by analysing the stability of a three-layer system, and derive the critical onset of Faraday waves, which appear via Hopf bifurcation. Numerical simulations — based on a model that reproduces the frozen wave mode with lowest wavenumber — are carried out to test this result and to analyse the character of the bifurcation. The predicted Hopf bifurcation is confirmed, which constitutes the first observation of modulated secondary Faraday waves. The abrupt growth of these modulated waves above onset indicates that the primary bifurcation is subcritical and is accompanied by a saddle-node bifurcation of periodic orbits that stabilises the (branch of) unstable solutions created in the subcritical Hopf bifurcation. Further above onset, these modulated waves are destroyed via a saddle-node heteroclinic bifurcation.

Results for an N-layer configuration, which represents a more general frozen wave pattern, are also presented and compared with the three-layer case.

1. Introduction

Gravity is a dominant force in most of the common fluid systems on Earth and plays a crucial role both in defining their familiar (vertically stratified) equilibrium configurations and as a driving or restoring force for instabilities [3, 4]. In weightless environments, however, the normal preference for flat density contours and interfaces is lost. In its absence, surface tension can lead to minimum surface configurations, as with spherical drops, while contact forces may cause a liquid to fully cover (wet) a solid boundary that it makes contact with.

When fluids are subjected to vibration in microgravity, this inertial forcing competes with existing surface and contact forces. Indeed, since gravity is not there to resist large-scale motion and changes in the fluid shape, relatively small vibrations can have an amplified effect on the shape and position of the liquid. In addition to this modified equilibrium, described by the theory of vibroequilibria [6, 7, 8, 9, 10], the absence of a gravitational restoring force allows for a more rapid and dramatic growth of many fluid instabilities and increases the possibility of interaction between them.

In particular, the behaviour of the frozen wave instability in microgravity differs markedly from that observed under normal gravity. Instead of small-amplitude waves, the Kelvin–Helmholtz type instability triggers the immediate growth of large columnar structures [1, 5, 11,
Figure 1: Snapshots showing the evolution of Faraday waves on opposite sides of a frozen wave columnar structure in microgravity. The experimental liquids are FC-40 (light) and 20 cSt silicone oil (dark), subjected to vibrations of frequency $\omega/\pi = 22$ Hz and amplitude $A = 1.76$ mm. For further details about the experiment, the reader is referred to Ref. [5].

Experimentally, this type of periodic stripe (columnar) pattern perpendicular to the axis of vibration was first found by Beysens and co-workers in 2009 [17] in supercritical CO$_2$.

The effects of finite domain size on the frozen wave instability were studied by Salgado Sánchez et al. [14]. It was found that near onset (i.e., at low vibrational forcing) an unstable flat interface develops into a single “pair” of columns, with the lighter fluid located in the centre and the heavier one divided equally and adjacent to the lateral walls. This is referred to as the $\langle K \rangle = 1$ mode, where $\langle K \rangle = L/\lambda$ represents the number of column pairs along the container length (identifying the lateral walls). As the vibrational velocity is increased, there is a cycle of column growth that characterises the transitions to higher $\langle K \rangle$ modes. Each of these modes is found over a certain interval of vibrational velocities, which widens as the wave number grows.

Experiments carried out by Shevtsova and collaborators in 2016 [1] demonstrated that Faraday waves can occur as a secondary instability of the columnar patterns that result from the primary frozen wave instability driven by horizontal vibration in microgravity. A theoretical estimate [5] of the critical amplitude $A_c$ can be obtained by assuming weak damping and forcing and neglecting interaction between columnar interfaces:

$$A(2\omega) = \frac{1}{\omega k} \left( \frac{\rho_1 + \rho_2}{\rho_1 - \rho_2} \right) \sqrt{\left( \frac{\omega^2 - \frac{\sigma}{\rho_1 + \rho_2} k^3}{\rho_1 - \rho_2} \right)^2 + 16k^4 \omega^2 \left( \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2} \right)^2},$$

where $\rho_1$ and $\rho_2$ ($< \rho_1$) are the fluid densities, $\mu_1$ and $\mu_2$ are the viscosities, $\sigma$ is the interfacial tension, $2\omega$ is the applied vibration frequency, and $k$ is the wavenumber of the frozen wave mode. In the case of miscible liquids ($\sigma = 0$), minimizing this expression over $k$ yields a
simplified expression [1] for the critical amplitude $A_c$:

$$A_c = \frac{2}{\rho_2 - \rho_1} \sqrt{\frac{(\rho_1 + \rho_2)(\mu_1 + \mu_2)}{2\omega}}. \quad (2)$$

Subsequent experiments performed by Salgado Sánchez et al. [5] observed a $\pi/2$ phase shift in the Faraday waves appearing on neighbouring interfaces. This phase difference is explained by the change in sign of the forcing at successive interfaces due to the alternating density differences. For subharmonic Faraday waves, this antiphase forcing implies a $\pi/2$ shift in the interface oscillations.

Lyubimova et al. [2] analysed the development of Faraday waves on periodic columnar patterns under zero gravity in near critical fluids. The Faraday instability that appeared in the thin band patterns was studied after taking the interaction between the interfaces into account. An analytical solution was obtained under the assumptions of low viscosity and low surface tension. In addition to that, the waves on neighbouring interfaces were assumed to be equal except for the $\pi/2$ phase shift (a half period of the driving frequency), in accord with numerical simulations. The following expression for the instability threshold was then obtained:

$$A_c = \frac{2\sqrt{2e^{-2H} + \sqrt{6e^{-2H} + 1}}}{(\rho_2 - \rho_1)(1 - e^{-2H})} \sqrt{\frac{(\rho_1 + \rho_2)(\mu_1 + \mu_2)}{2\omega}}, \quad (3)$$

where $H = kh$ is the dimensionless spatial period of the band pattern. A comparison of analytical results with numerical simulations and experimental data showed good agreement.

The main objective of the present study is to obtain a theoretical model to predict the threshold of the Faraday waves that may appear on the columnar patterns under vibration without assuming that the wave amplitudes on each interface are equal apart from a known phase shift and, thus, to allow for modulations induced by their interaction. In order to do that, two different stability problems, analogous to the classical Faraday problem, are solved. A three-layer configuration is chosen to represent the $\langle K \rangle = 1$ mode of the frozen wave instability. Once the Mathieu equations have been obtained, a separation of timescales method is used to derive the corresponding amplitude equations and a bifurcation analysis is conducted.

In order to test the theoretical model developed and further investigate the behaviour of Faraday waves, numerical simulations are carried out using COMSOL Multiphysics. The stability threshold that is determined numerically is compared with the theoretical prediction, while the simulations are also used to reveal whether the primary bifurcation is supercritical or subcritical and to provide evidence for a subsequent saddle-node heteroclinic bifurcation, which extinguishes the modulations at higher forcing.

An N-layer configuration is then studied in an analogous way, first obtaining the corresponding Mathieu equations and relying on a separation of timescales to derive amplitude equations and determine the stability threshold. Analytical results are compared with the ones obtained in the three-layer problem.

2. Three-layer problem
2.1. Analytical solution

Here, we consider three layers of immiscible and incompressible fluids of heights $h_1$, $h_2$ and $h_3$ in the presence of a gravitational field (that will later be set to zero). The lightest fluid layer, with density $\rho_3$ and viscosity $\mu_3$, is located at the top and the heaviest one, with density $\rho_1$ and viscosity $\mu_1$, is at the bottom, so that the system is gravitationally stable. The fluids are confined between two horizontal plates and subjected to vertical oscillations with vibrational amplitude $A$ and frequency $2\omega$ (the factor of two is included here for convenience so that $\omega$ is the subharmonic frequency appearing in the final equations).
Fluid 1: \( \rho_1, \mu_1 \)
Fluid 2: \( \rho_2, \mu_2 \)
Fluid 3: \( \rho_3, \mu_3 \)

\( h_1 \)
\( h_2 \)
\( h_3 \)
\( \xi_A \)
\( \xi_B \)
\( g \)
\( x \)
\( z \)

Figure 2: Sketch of the three-layer system considered. The initial configuration, which is gravitationally stable (\( \rho_1 > \rho_2 > \rho_3 \)), is subjected to vertical vibrations of amplitude \( A \) and frequency \( 2\omega \) perpendicular to the unperturbed interfaces.

The hydrodynamic equations of this system, which assume ideal fluids and zero initial vorticity [18], can be written as

\[
\left( \partial_{zz} - k^2 \right) w_j = 0, \tag{4}
\]

for each horizontal mode with wavenumber \( k \). Here, \( w_j \) is the vertical velocity in the \( j \)th fluid layer and the operator \( \partial_{zz} \) denotes the second order partial derivative with respect to the vertical coordinate \( z \).

The problem is solved with no-flow boundary conditions at the plates:

\[
w_1 = 0 \quad \text{at} \quad z = 0, \tag{5a}
\]
\[
w_3 = 0 \quad \text{at} \quad z = h_1 + h_2 + h_3, \tag{5b}
\]

while at the free interface \( \xi_A \) we have

\[
w_1 = w_2 = \dot{\xi}_A, \quad \rho_2 \partial_z \dot{w}_2 - \rho_1 \partial_z \dot{w}_1 = -\left[ (\rho_2 - \rho_1)(g - 4\omega^2 A \cos 2\omega t) - \sigma_A k^2 \right] k^2 \xi_A, \tag{6}
\]

and, similarly, at \( \xi_B \)

\[
w_2 = w_3 = \dot{\xi}_B, \quad \rho_3 \partial_z \dot{w}_3 - \rho_2 \partial_z \dot{w}_2 = -\left[ (\rho_3 - \rho_2)(g - 4\omega^2 A \cos 2\omega t) - \sigma_B k^2 \right] k^2 \xi_B, \tag{7}
\]

where the overdot denotes a time derivative. These are applied at \( z = h_1 \) and \( z = h_1 + h_2 \), respectively, since the deformation of the interfaces is assumed to be small [18] with \( \xi_A, \xi_B \ll k^{-1} \).

By searching for solutions of the type \( w_j = A_j \cosh k z + B_j \sinh k z \) with the boundary conditions (5)–(7), one arrives at the system of Mathieu equations

\[
\ddot{\xi}_A - \rho_A \ddot{\xi}_B + 2\gamma_A \dot{\xi}_A + \Omega_A^2 (1 + f_A \cos 2\omega t) \dot{\xi}_A = 0, \tag{8a}
\]
\[
\ddot{\xi}_B - \rho_B \ddot{\xi}_A + 2\gamma_B \dot{\xi}_B + \Omega_B^2 (1 + f_B \cos 2\omega t) \dot{\xi}_B = 0, \tag{8b}
\]

where damping terms (those with coefficients \( \gamma_A \) and \( \gamma_B \)) have been included in the manner of Ref. [18].
The dynamics of Eqs. (8) depend on the natural subharmonic frequencies
\[
\Omega_A^2 = \frac{(\rho_1 - \rho_2) g_k^3}{\rho_1 \coth k h_1 + \rho_2 \coth k h_2}, \quad \Omega_B^2 = \frac{(\rho_2 - \rho_3) g_k^3}{\rho_2 \coth k h_2 + \rho_3 \coth k h_3},
\]
on the damping coefficients
\[
\gamma_A = \frac{2 k^2 (\mu_1 \coth k h_1 + \mu_2 \coth k h_2)}{\rho_1 \coth k h_1 + \rho_2 \coth k h_2}, \quad \gamma_B = \frac{2 k^2 (\mu_2 \coth k h_2 + \mu_3 \coth k h_3)}{\rho_2 \coth k h_2 + \rho_3 \coth k h_3},
\]
on the scaled forcing parameters
\[
f_A = \frac{4 (\rho_1 - \rho_2) A \omega^2}{\sigma_1 k^2}, \quad f_B = \frac{4 (\rho_2 - \rho_3) A \omega^2}{\sigma_2 k^2},
\]
and on the coupling terms
\[
\rho_A = \frac{\rho_2 / \sinh k h_2}{\rho_1 \coth k h_1 + \rho_2 \coth k h_2}, \quad \rho_B = \frac{\rho_2 / \sinh k h_2}{\rho_2 \coth k h_2 + \rho_3 \coth k h_3}.
\]

Since we are motivated here by recent microgravity experiments [5, 16] and simulations [14], we set \( g = 0 \) and restrict to the case where \( \rho_3 = \rho_1, \mu_3 = \mu_1, \sigma_1 = \sigma_2 = \sigma \) and \( h_3 = h_1 \), which corresponds to the \( \langle K \rangle = 1 \) columnar mode arising from the frozen wave instability [14]. The system of Mathieu equations then simplifies to
\[
\begin{align*}
\dddot{\xi}_A - \rho \ddot{\xi}_A + 2 \gamma \dot{\xi}_A + \Omega^2 (1 + f \cos 2 \omega t) \xi_A &= 0, \quad (9a) \\
\dddot{\xi}_B - \rho \ddot{\xi}_B + 2 \gamma \dot{\xi}_B + \Omega^2 (1 - f \cos 2 \omega t) \xi_B &= 0, \quad (9b)
\end{align*}
\]
where
\[
\rho = \frac{\rho_2 / \sinh k h_2}{\rho_1 \coth k h_1 + \rho_2 \coth k h_2}, \quad \gamma = \frac{2 k^2 (\mu_1 \coth k h_1 + \mu_2 \coth k h_2)}{\rho_1 \coth k h_1 + \rho_2 \coth k h_2},
\]
\[
\Omega^2 = \frac{\sigma k^3}{\rho_1 \coth k h_1 + \rho_2 \coth k h_2}, \quad f = \frac{4 (\rho_1 - \rho_2) A \omega^2}{\sigma k^2}.
\]

To determine the stability threshold, a separation of timescales method is used along with the ansatz
\[
(\xi_A, \xi_B) = (A, B) e^{i \omega t} + c.c.,
\]
where \( A, B \) are amplitudes that depend only on the slow time \( \tau = \varepsilon t \), where \( \varepsilon \ll 1 \) is a small parameter, \( \omega \) is the subharmonic frequency and \( c.c. \) denotes the complex conjugate.

We further assume weak forcing, damping and coupling and make the substitution \((f, \gamma, \rho) \rightarrow \varepsilon (f, \gamma, \rho)\). The excitation is also assumed to be near the subharmonic resonance so that \( \omega = \Omega + \varepsilon \nu \), with \( \nu \) a detuning parameter. The leading-order equations for the amplitudes \( A \) and \( B \) are then
\[
\begin{align*}
\dot{A} &= (-\gamma - i \nu) A + i F \dot{A} + i \mu B, \quad (11a) \\
\dot{B} &= (-\gamma - i \nu) B - i F \dot{B} + i \mu A, \quad (11b)
\end{align*}
\]
where \( F = f \Omega / 4, \mu = \rho \Omega / 2, \) and the overdot now denotes a derivative with respect to the slow time \( \tau \).
These amplitude equations are analogous to those studied in Ref. [19] in the case of antisymmetric forcing, where it was shown that the most unstable (complex conjugate) eigenvalues of the problem are

$$\lambda_{\pm} = -\gamma \pm i\mu + \sqrt{F^2 - \nu^2},$$

and thus, the primary instability is a Hopf bifurcation along the curve

$$F^2 = \gamma^2 + \nu^2$$

with a modulation (Hopf) frequency of

$$\Omega_H = \mu.$$ (14)

Using the definition of $\nu$ given above, one can express the critical vibrational amplitude $A_c$ to leading order as

$$A_c^2 = \frac{4k^5\sigma(\mu_1 \coth k h_1 + \mu_2 \coth k h_2)^2}{\omega^4(\rho_1 - \rho_2)^2(\rho_1 \coth k h_1 + \rho_2 \coth k h_2)} + \frac{(\omega - \Omega)^2k\sigma(\rho_1 \coth k h_1 + \rho_2 \coth k h_2)}{\omega^4(\rho_1 - \rho_2)^2}. \quad (15)$$

2.2. Numerical model

To investigate the secondary Faraday wave instability, numerical simulations are performed from an initial configuration of three columns, which one can think of as having developed following the frozen wave instability. Due to the vibroequilibria effect [4, 14, 20, 21], the lighter fluid is always located at the centre while the heavier one is divided equally along the two lateral walls. The basic problem considered is, thus, the application of horizontal (perpendicular) vibrations to a rectangular container holding three alternating vertical layers of immiscible liquids, as sketched in Fig. 3. The container has interior dimensions of $L \times 2H = 15 \text{ mm} \times 7.5 \text{ mm}$ and vibrates with amplitude $A$ and frequency $2\omega$. 

Figure 3: Sketch illustrating the setup of the two-dimensional numerical model in microgravity.
Table 1: Density ($\rho$) and kinematic viscosity ($\nu$) of Fluids 1 and 2, which are similar to FC-40 and 20 cSt silicone oil, respectively, but with increased density (silicone oil) and viscosity (FC-40).

| Fluid | $\rho$ (kg/m$^3$) | $\nu$ (m$^2$/s) |
|-------|------------------|-----------------|
| 1     | 1855             | $5 \times 10^{-6}$  |
| 2     | 1050             | $20 \times 10^{-6}$ |

The fluid parameters used are shown in Table 1 and are similar to those of FC-40 (Fluid 1) and 20 cSt silicone oil (Fluid 2) but with increased density (silicone oil) and viscosity (FC-40). The interfacial tension is $\sigma = 6.021$ mN/m [22]. A contact angle of $\beta = \pi/2$ is used so as to be consistent with an initial interface configuration that is completely perpendicular to the applied vibrations.

The interfacial dynamics are simulated using a level-set-based formulation of the Navier-Stokes equations [23, 24], which are solved using the finite element method with the software COMSOL Multiphysics. We note that this model has been validated elsewhere against experiments [5, 16] and has already been used to analyse different features of the frozen wave instability [4, 14, 15]; the reader is directed to these references for further details.

2.3. Results

The Faraday wave stability threshold is determined from a series of simulations beginning from a configuration with a central column that is twice the width of the two lateral columns ($h_2 = 2h_1$), which corresponds to the first columnar mode ($\langle K \rangle = 1$) generated by the frozen wave instability [14, 15]. The procedure used to obtain this threshold is an iterative process where the frequency is fixed at a given value while the vibrational amplitude is varied. Simulations are performed at each frequency value, noting the presence or absence of Faraday waves, until the threshold is determined to within 0.0625 mm in amplitude.

Figure 4 compares the numerically obtained threshold to the analytical prediction of Eq. (15) for the secondary Faraday instability. The numerical results (dashed black curve) are closer to the analytical prediction (solid black curve), which is calculated by assuming zero detuning (i.e., using $\Omega(k) = \omega$ to determine $k$), in the range of higher frequencies.

For an additional comparison, the threshold for Faraday waves obtained numerically in Salgado Sánchez et al. [5] is included (dashed blue curve with markers). Although some of the fluid parameters differ in that reference from the ones used in this study, both threshold curves have the same shape. The mild deviation at higher frequencies may be due to the transition to a higher columnar frozen waves modes ($\langle K \rangle > 1$), something which is not considered here. The threshold obtained by minimizing Eq. (1) over $k$ is also shown (solid blue curve) and clearly deviates from Eq. (15) in the low frequency regime where the coupling between the waves on the two interfaces is relevant.

The character of the bifurcation is investigated in the case of $\omega/\pi = 25$ Hz by increasing the forcing amplitude $A$ beyond the stability threshold and measuring the maximum Faraday wave amplitude $A$. These results are shown with black markers in Fig. 5. Another series of simulations (grey markers) are carried out by decreasing the vibrational amplitude from the value $A = 1.625$ mm and, after waiting a certain amount of time for the system to stabilize, measuring the Faraday wave amplitude $A$. 


Figure 4: Theoretical (solid black curve) and numerical (dashed black curve) thresholds for the secondary Faraday wave instability. The blue curves show the numerical results (dashed curve with open markers) obtained in Ref. [5] and the minimization of Eq. (1) (solid curve).

Figure 5: Faraday wave amplitude $A$ (in mm) as a function of the vibrational amplitude $A$ (in mm) for increasing (black) and decreasing (grey) forcing. The simulations use a fixed forcing frequency of $\omega/\pi = 25$ Hz.
As the forcing is increased beyond threshold, Faraday waves quickly grow in amplitude, settling at a relatively large value. In the case of decreasing forcing, the finite-amplitude solutions persist below threshold, which demonstrates hysteresis and indicates that the Faraday wave instability is subcritical and (presumably) accompanied by a secondary saddle-node bifurcation. Although supercritical bifurcations are expected to occur for some values of the excitation frequency, they were not clearly observed in the numerical simulations performed here.

In order to more easily observe the frequency of the Hopf bifurcation in numerical simulations, the width of the central column (h₂) is decreased, which increases the coupling between the interfaces; note that this corresponds to fluid layers of unequal depth prior to the frozen wave instability. Figure 6 shows the decrease of the Hopf frequency with column width and (generally) with forcing frequency (equivalently, k); in both cases, a large value of kh₂ suppresses coupling. We select the value h₂ = L/8 because it allows for the observation of the Hopf frequency over a relatively short period of time. We note, however, that the coupling in this case is substantial and may violate the assumption of weak coupling made in Sec. 2.1.

The results of the simulations performed for L/8 and ω/π = 10 Hz are shown in Fig. 7. The modulation of the Faraday wave amplitude at each interface is clearly visible with a frequency near that of the Hopf frequency f_H and a (modulation) phase shift of π between each interface. A supplementary video is available (3layer_Lover8_A_3p45mm_f_10Hz.avi) to illustrate the modulated solutions observed at A = 3.45 mm [panel (a)] over half of the modulation period.

For the case of h₂ = L/8 and ω/π = 10 Hz, when the forcing amplitude is increased beyond the Hopf bifurcation, the modulation period also increases, as seen in Fig. 8. This increase in modulation period, obtained from simulations, is in good agreement with the inverse square root scaling expected of a saddle-node heteroclinic bifurcation [19]. Above this global bifurcation, which is estimated from the fitting to occur at A = 3.749 mm, the modulation ceases and only steady solutions are observed, as in Fig. 7(d).
Figure 7: Time series showing modulated Faraday waves for \( h_2 = L/8 \) and \( \omega/\pi = 10 \) Hz. Each line corresponds to the location of the interface at \( x = 0 \) mm. The transition from approximately sinusoidal modulation to more irregular modulation with increasing period and, finally, a fixed-amplitude solution, can be seen with the increasing forcing amplitudes: (a) \( A = 3.45 \) mm, (b) \( 3.65 \) mm, (c) \( 3.7 \) mm and (d) \( 3.75 \) mm.
3. Periodic alternating layers

3.1. Analytical solution

We now consider a liquid system formed by alternating layers of two immiscible and incompressible fluids of densities $\rho_1$ and $\rho_2$ and viscosities $\mu_1$ and $\mu_2$ in microgravity, as sketched in Fig. 9.

This configuration is relevant to the higher modes, with multiple columns, that form after the frozen wave instability when larger forcing is applied [14]. The heights of the alternating fluid layers are assumed to be $h_1$ and $h_2$, respectively, and to be the same for each successive pair. To simplify the otherwise large system of $N - 1$ equations, we assume that each initially identical interface behaves identically throughout the time period considered and, thus, that the dynamics is periodic in $z$. In this case, in the limit of large $N$, it is sufficient to consider a two-layer system with periodic boundary conditions at the upper and lower interfaces. The interfaces between the fluids are initially perpendicular to the direction of vibration, which is again characterised by the amplitude $A$ and the frequency $2\omega$.

The governing hydrodynamic equations are as in Sec. 2, including the boundary conditions at the interface $\xi_A (z = 0)$:

$$w_1 = w_2 = \dot{\xi}_A, \quad \rho_2 \partial_z \dot{w}_2 - \rho_1 \partial_z \dot{w}_1 = - [(\rho_1 - \rho_2)4\omega^2 A \cos 2\omega t - \sigma_A k^2] k^2 \dot{\xi}_A.$$ (16)

Analogous equations hold at $\xi_B (z = h_2)$, while the full system is closed using periodic boundary conditions at $z = -h_1, h_2$, expressed as

$$w_1(-h_1) = w_2(h_2).$$ (17)
As before, we assume solutions of the form \( w_j = A_j \cosh k z + B_j \sinh k z \) and arrive at a system of Mathieu equations

\[
\ddot{\xi}_A - \rho \ddot{\xi}_B + 2 \gamma \dot{\xi}_A + \Omega^2 (1 + f \cos 2\omega t) \xi_A = 0, \tag{18a}
\]

\[
\ddot{\xi}_B - \rho \ddot{\xi}_A + 2 \gamma \dot{\xi}_B + \Omega^2 (1 - f \cos 2\omega t) \xi_B = 0, \tag{18b}
\]

where

\[
\rho = \frac{\rho_1 / \sinh k h_1 + \rho_2 / \sinh k h_2}{\rho_1 \coth k h_1 + \rho_2 \coth k h_2}.
\]

Given that Eqs. (18) are identical in form to Eqs. (9), the instability in the infinite periodic case is also associated with a Hopf bifurcation along the curve

\[
F^2 = \gamma^2 + \nu^2. \tag{19}
\]

The coupling parameter \( \rho \), however, differs in the two cases, which leads to different modulation frequencies, as can be observed in Fig. 10. Although the dependence on vibrational frequency is similar, the Hopf frequency of the N-layer problem can be significantly higher than that of the three-layer problem in the range of low frequencies.

4. Conclusions

Mathieu equations describing the parametric instability producing Faraday waves in a three-layer fluid configuration were obtained. The problem was simplified by assuming that the layers correspond to an ideal \( \langle K \rangle = 1 \) mode triggered by the frozen wave instability in weightlessness. A separation of timescales method was applied and a primary Hopf bifurcation found in the case of weak forcing, damping, and coupling. Numerical simulations were carried out to test the theoretical predictions and a comparison of the calculated threshold with the numerically observed onset showed good agreement, although larger deviation was observed for low frequencies where the small coupling assumption is likely not valid.
The oscillatory character of the primary bifurcation was also investigated using numerical simulations, which confirm that a Hopf bifurcation occurs. The resulting modulated solutions were more easily observed by decreasing the central column width, which increases coupling and, thus, the Hopf frequency. We note that this is the first observation of modulation in secondary Faraday wave patterns on the columnar patterns arising from the frozen wave instability in microgravity. The fact that weak coupling, which is associated with thicker columns and larger $k$, leads to long modulation periods (see Fig. 6 for $h_2 = L/2$) and, at the same time, a smaller forcing interval where modulations can exist [19], helps to explain why they have not yet been observed in parabolic flight experiments where the observation period is too short.

It was also seen via simulations that if the forcing is increased beyond the Hopf bifurcation, the modulation period increases and appears to diverge in a manner indicative of a global (saddle-node heteroclinic) bifurcation. If forcing is decreased gradually, hysteresis is (often) observed, which indicates that the primary Hopf bifurcation is subcritical and suggests the existence of a saddle-node bifurcation that stabilizes the small-amplitude subcritical branch of oscillatory solutions at finite amplitude. Future work that includes a derivation of the nonlinear (cubic) coefficient would help clarify when this hysteresis can be expected.

An analogous type of modulated solutions was described by Salgado Sánchez et al. [19] in an open container of liquid subjected to horizontal excitation; see also [25]. Physically, such a system is quite different from that considered here and produces cross-waves, not Faraday waves. However, both systems involve weakly coupled parametrically forced waves excited by out-of-phase forcing — here, the coupling between successive interfaces occurs through the bulk liquid of each column, while cross-waves that are concentrated near the endwalls (wavemakers) interact across the more quiescent interior of the container. The appearance of subcritical primary bifurcations and the transition from the initially modulated solutions to constant-amplitude waves via a saddle-node heteroclinic bifurcation appear to be general features of both systems.

An N-layer configuration was also studied in the (infinite) limit of large N by assuming
periodicity in space. The corresponding Mathieu equations were obtained and the same procedure followed to derive the stability threshold and the Hopf frequency. The results of this case are quite similar to those of the simplified three-layer problem, with the same predicted onset. The expressions for the modulation frequency, however, are different and were compared in Fig. 10. In both cases, the Hopf frequency decreases with forcing frequency (larger $k$) except for thin layers in the range of low frequencies (where, as mentioned, the assumption of weak coupling may fail). It is over this range of low frequencies where the modulation of the N-layer system can be significantly faster than in the three-layer case.

Finally, we note that the theoretical analysis carried out here is more general than that of Lyubimova et al. [2] which, in addition to spatial periodicity, assumed that $A = iB$. This condition precludes the possibility of modulated solutions like those found here (see Fig. 7), where waves are sometimes located almost entirely on one interface and sometimes on both.

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