Scaling and conformal symmetries
for plane gravitational waves

P.-M. Zhang\textsuperscript{1,2*}, M. Cariglia\textsuperscript{3†}, M. Elbistan\textsuperscript{2,4‡}, P. A. Horvathy\textsuperscript{2,4§}

\textsuperscript{1} School of Physics and Astronomy, Sun Yat-sen University, Zhuhai, China
\textsuperscript{2} Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou, China
\textsuperscript{3} DEFIS, Universidade Federal de Ouro Preto, MG-Brasil,
\textsuperscript{3} Institut Denis Poisson, Tour University – Orléans University, UMR 7013 (France).

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Abstract

Isometries of an exact plane gravitational wave spacetime are symmetries for both massive and massless particles. Their conformal extensions, among which Chrono-Projective transformations play a distinguished role, are symmetries for massless particles. Homotheties are universal symmetries for any profile. The generically 5 parameter isometry group is extended Chrono-Projectively to a 6 or 7 parameter symmetry group under special circumstances.

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\* e-mail:zhpm@impcas.ac.cn
\† e-mail: marco.cariglia@ufop.edu.br
\‡ mailto:mahmut.ELbistan@lmpt.univ-tours.fr
\§ mailto:horvathy@lmpt.univ-tours.fr
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I. INTRODUCTION

Recent insight into the “Memory Effect” for gravitational waves [1–9] was brought about by a better understanding of the underlying symmetries. For exact plane waves the implicitly known isometry group [10–14], which maps geodesics to geodesics and yields conserved quantities, could be identified as Lévy-Leblond’s “Carroll” group with broken rotations [15–19]. However homotheties, \( h : \mathcal{M} \to \mathcal{M} \),

\[
U \to U, \quad X \to \chi X, \quad V \to \chi^2 V, \quad \chi = \text{const.} \tag{I.1}
\]

(where \( X \in \mathbb{R}^2 \) and \( U, V \) are the transversal resp. light-cone coordinates on the gravitational wave space-time) play also an important role, namely for the integrability of the geodesic equations [13, 20–22]. In fact, (I.1) carries lightlike geodesics to lightlike geodesics, as illustrated on fig.1. The homothety is not an isometry though; it is a conformal transformation.

It was noticed some time ago that special gravitational waves namely such that carry a covariantly constant null “vertical vector” \( \xi \) [23] are convenient to describe non-relativistic physics in one lower dimensions; conversely, non-relativistic physics can be “Eisenhart-Duval (E-D) lifted” to such a “Bargmann space” [24–27].

Null geodesics “above” are of particular interest, because they are the E-D lifts of non-relativistic motions “below”. Along null geodesics a conserved quantity \( Q \) is associated with any conformal vectorfield

\[
L_Y g_{\mu \nu} = 2\omega g_{\mu \nu} \tag{I.2}
\]

[24, 25]. If \( Y \) preserves, in addition \( \xi \), \( L_Y \xi = 0 \), then \( Q \) does not depend on the vertical coordinate and therefore projects to a conserved quantity for the underlying non-relativistic dynamics “downstairs” [24, 25]. However for null geodesics “upstairs” a conserved quantity \( Q \) is associated with any conformal vector field, even if it does not preserve \( \xi \); such a \( Q \) simply does not project to a well-defined quantity downstairs.

The homothety does not preserve the vertical vector; it is a conformal transformation of the gravitational wave with an additional special property

\[
L_Y \xi = \psi \xi \tag{I.3}
\]

for some real function \( \psi \), see (II.23b) below. Such transformations were introduced by Duval
et al. [25, 28–30], who called it a “Chrono-Projective transformation”\(^1\). The conserved charge generated by a Chrono-Projective transformation does not project to a conserved quantity downstairs, but it does “almost” \(^2\) [33], as it will be recalled below.

The aim of this paper is to clarify the rôle and the status of various conformal extensions and of Chrono-Projective transformations in particular. Our strategy is to turn around the E-D correspondence and take advantage of non-relativistic physics in 2+1 dimensions to describe null geodesic motion in a 4D gravitational wave space-time.

Our paper is organized as follows: after a reminder on exact plane waves and their homotheties, in sec. IIA we briefly outline the Bargmann [alias Eisenhart-Duval] approach. Then Chrono-Projective transformations are introduced. To make our paper self-contained, in sec. III the various conformal transformations are spelled out in the flat case.

A “pre-Noetherian” point of view initiated by Jacobi in his 1843 lectures [34] is presented in sec. IV. The main result which plays an important role in our subsequent investigations says that a Chrono-Projective transformation is a symmetry for null (but not for timelike) geodesics \(^2\). Homotheties are worked out in sec. IVD and are shown to generate a new type of conserved quantity typical for Chrono-Projective transformations.

The Chrono-Projective transformations of exact plane gravitational waves are identified in sec. V, using alternatively Brinkmann [23] and Baldwin-Jeffery-Rosen (BJR) [36] coordinates.

Our general theory is illustrated on various examples, sec. VI. In sec. VII we comment on the relation to Newton-Cartan theory [37–39].

\(^1\) See also eqn. \# (4.4) of [25] or \# (5.17)-(5.21) of [30]. In [31] it was rebaptized as the conformal Newton-Cartan group; in [32] it was called the “enlarged Schrödinger group”. In this paper we return to the original terminology proposed in [28]. The paradigm of a Chrono-Projective transformation is provided by Kepler’s third law [28, 33].

\(^2\) Symmetries of timelike geodesics were considered recently using non-local conservation laws [35].
II. EXACT PLANE GRAVITATIONAL WAVES

The metric of an exact gravitational plane wave in 4 dimensions is given, in Brinkmann coordinates \((X^\mu) = (U, X, V)\) [23], by

\[
g_{\mu\nu}dX^\mu dX^\nu = \delta_{ij}dX^i dX^j + 2d UdV + K_{ij}(U)X^i X^j dU^2, \tag{II.1a}
\]

\[
K_{ij}(U)X^i X^j = \frac{1}{2}A_+(U)\left((X^1)^2 - (X^2)^2\right) + A_\times(U)X^1 X^2, \tag{II.1b}
\]

where \(A_+\) and \(A_\times\) are the + and \(\times\) polarization-state amplitudes [11, 12, 14, 23].

For generic profile \(K_{ij}\), the isometries of (II.1) i.e. diffeomorphisms of spacetime, \(f : \mathcal{M} \rightarrow \mathcal{M}\) s.t.,

\[
f^*g_{\mu\nu} = g_{\mu\nu} \quad \text{infinitesimally} \quad L_Y g_{\mu\nu} = 0 \tag{II.2}
\]

span a 5-parameter group [10–14], which is in fact the subgroup of the Carroll group in 2 + 1 dimensions with broken rotations [10, 15–18]. However the homothety (I.1) [13, 20] i.e.,

\[
\text{hom} : \mathcal{M} \rightarrow \mathcal{M}, \quad U \rightarrow U, \quad X \rightarrow \chi X, \quad V \rightarrow \chi^2 V, \quad \chi = \text{const.} \tag{II.3}
\]

generated by the vector field

\[
Y_{\text{hom}} = X^i \partial_i + 2V \partial_V, \tag{II.4}
\]

is not an isometry but a conformal transformation of the pp-wave metric (II.1) \(^3\),

\[
f^*g_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \text{infinitesimally} \quad L_Y g_{\mu\nu} = 2\omega g_{\mu\nu}. \tag{II.5}
\]

(For (II.3) \(\Omega^2 = \chi^2 = \text{const.}\). Its role may be understood by looking at the geodesic motion for (II.1),

\[
\frac{d^2X}{dU^2} - \frac{1}{2} \begin{pmatrix} A_+ & A_\times \\ A_\times & -A_+ \end{pmatrix} X = 0, \tag{II.6a}
\]

\[
\frac{d^2V}{dU^2} + \frac{1}{4} \frac{dA_+}{dU} \left((X^1)^2 - (X^2)^2\right) + A_+ \left(X^1 \frac{dX^1}{dU} - X^2 \frac{dX^2}{dU}\right)
+ \frac{1}{2} \frac{dA_\times}{dU} X^1 X^2 + A_\times \left(X^2 \frac{dX^1}{dU} + X^1 \frac{dX^2}{dU}\right) = 0. \tag{II.6b}
\]

\(^3\) I.e., a diffeomorphism under which the metric pulls back to an (in general position dependent) positive multiple of itself. This should be distinguished from a “Weyl rescaling” of the metric by which the metric is replaced by a positive, in general position dependent, multiple of itself at the same point, i.e. in the same coordinate system.
The homothety (II.3) multiplies the $X$ - equation by $\chi$ and the $V$-equation by $\chi^2$ ; trajectories are therefore taken to trajectories. Alternatively, the geodesic Lagrangian

$$L_{geo} = \frac{1}{2} \delta_{ij} \dot{X}^i \dot{X}^j + \dot{U} \dot{V} + \frac{1}{2} K_{ij}(U) \dot{X}^i \dot{X}^j \dot{U}^2,$$

(II.7)

where the dot means derivation w.r.t. an arbitrary parameter $^4$scales under (II.3) as,

$$L_{geo} \rightarrow \chi^2 L_{geo},$$

(II.8)

again implying that the trajectories go into trajectories : The geodesic motion in such a background is thus scale invariant. We note that all of these $4D$ trajectories project to the same curve $X(U)$ in the transverse plane. Let us record for further use that for null geodesics

$$L_{geo} = 0$$

(II.9)

which are thus homothety-invariant by (II.8).

We record for later use an interesting property of the geodesic eqns (II.6). The transverse equations (II.6a) are decoupled from the “vertical” one, (II.6b), and can be solved separately. Once $X(U)$ has been determined, the result should be inserted into (II.6b) which then can be integrated. Analytic solutions are difficult to find, and therefore the best is to use numerical integration [8]. As it will be further discussed in sec.IV.D and in the Conclusion, VIII, the “new” conserved charge $Q_{hom}$ associated with the homothety provides an alternative way to derive the vertical motion.

A. The “Bargmann” point of view

Further insight can be gained using the “Bargmann” framework [24, 25]. We first recall that the space-time of a 4-dimensional gravitational wave with metric (II.1) we denote by $(\mathcal{M}, g_{\mu\nu})$ can be viewed as the “Bargmann space” for a non-relativistic system in $2 + 1$ dimensional non-relativistic spacetime, obtained by factoring out the integral curves of the covariantly constant “vertical” vector $\xi = \partial_V$. The factor space has coordinates $(U, X)$ with $U$ playing the rôle of non-relativistic time. The clue is that classical motions “downstairs" are the projections of the null geodesics “upstairs" ; see [24, 25] for precise definitions and details.

$^4$ We mostly choose $\{\cdot\} = d/dU.$
For example, the null geodesics of 4D flat Minkowski spacetime written in light-cone coordinates project to free non relativistic motions in (2+1) dimensions. More generally, let us consider
\begin{equation}
\text{ds}^2 = dX^2 + 2dUdV - 2\Phi(U,X)dU^2 \tag{II.11}
\end{equation}
The geodesics are described by the action
\begin{equation}
S = \int \mathcal{L}_{\text{geo}}d\sigma, \quad \mathcal{L}_{\text{geo}} = \frac{1}{2}(\ddot{X})^2 + \dot{U}\dot{V} - \Phi(U,X)(\dot{U})^2, \tag{II.12}
\end{equation}
where the “dot” denotes derivation w.r.t. an affine parameter $\sigma$, $\{\cdot\} \equiv \frac{d}{d\sigma}$. The equations of motion are
\begin{align}
\ddot{X} &= -(\ddot{U})^2 \frac{\partial \Phi}{\partial X}, \tag{II.13a} \\
\ddot{U} &= 0, \tag{II.13b} \\
\frac{d}{d\sigma}(\dot{V} - 2\Phi \dot{U}) &= -\frac{\partial \Phi}{\partial U} \dot{U}^2, \tag{II.13c}
\end{align}
which identifies $\Phi(X,U)$ as a [possibly “time”-dependent] scalar potential in one lower dimension. Adding an arbitrary constant to $V$ leaves the equations unchanged and the projected motion is governed by the single equation (II.13a). We have immediately that $\dot{U}$ is a constant of the motion. The related geodesic Hamiltonian in 4D is
\begin{equation}
\mathcal{H} = \frac{1}{2}P^2 + P_UP_V + \Phi(X,U)P_V^2. \tag{II.14}
\end{equation}
The Hamiltonian (II.14) and the Lagrangian (II.12) are in fact identical. As they do not depend explicitly upon $\sigma$, we also have the constraint
\begin{equation}
\dot{X}^2 + 2\dot{U}\dot{V} - 2\Phi(X)(\dot{U})^2 = -\epsilon, \tag{II.15}
\end{equation}
where $\epsilon = 1$ for timelike geodesics and $\epsilon = 0$ for null geodesics.

Focusing our attention at null-geodesics $\epsilon = 0$ by requiring $\mathcal{H} \equiv 0$ and $P_V = M$ yields the non-relativistic Hamiltonian “downstairs”,
\begin{equation}
H_{\text{NR}} = \frac{P^2}{2M} + M\Phi(X,U) = -P_U. \tag{II.16}
\end{equation}

\footnote{In 4D, the metric of any solution of the vacuum Einstein equations $R_{\mu\nu} = 0$ which is conformal to some vacuum Einstein solution can be brought to the form \cite{23,25},
\begin{equation}
\text{ds}^2 = G_{ij}(U,X)dX^i dX^j + 2dUdV - 2\Phi(U,X)dU^2 \tag{II.10}
\end{equation}
where $G_{ij}(U,X)$ is a possibly $U$ [but not $V$] dependent metric on transverse space. This is however not true in $D \geq 5$ dimensions, allowing for more freedom \cite{23,25}.



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\( \mathcal{H} = 0 \) implies, in terms of velocities,
\[
\frac{1}{2} \dot{X}^2 + \dot{U} \dot{V} - \Phi(X, U) \dot{U}^2 = 0. \tag{II.17}
\]
Expressing \( \dot{V} \) from here, on the r.h.s. we recognize (minus) the non-relativistic Lagrangian,
\[
\dot{V} = -\dot{U} \left( \frac{1}{2} \frac{\dot{X}^2}{\dot{U}^2} - \Phi(X, U) \right) = - \left( \frac{1}{2} M \dot{X}^2 - M \Phi(X, U) \right) = - L_{NR}. \tag{II.18}
\]
Therefore the vertical coordinate is essentially (minus) the classical action along the path \( X(\sigma) \),
\[
V = V_0 - S, \quad S = \int L_{NR} d\sigma, \tag{II.19}
\]
as noticed already by Eisenhart [27].

When the potential \( \Phi \) happens not to depend on \( U \) explicitly, \( \partial \Phi / \partial U = 0 \), eqn. (II.13c) implies that \( (\dot{V} - 2 \Phi \dot{U}) \) is also conserved; eliminating \( \dot{V} \) using (II.18) yields the constant of the motion \( E = \frac{1}{2} \left( \dot{X} / \dot{U} \right)^2 + \Phi \), identified as the conserved energy for unit mass of the projected motion. One may think of \( U = \sigma \) as Newtonian time. The special choice (II.1),
\[
\Phi(U, X) = -\frac{1}{2} K_{ij}(U) X^i X^j \tag{II.20}
\]
where \( K_{ij} \) is a traceless symmetric matrix, represents, in Bargmann terms, a time-dependent anisotropic (attractive or repulsive) harmonic oscillator in the transverse plane [3, 8, 16].

For a general Bargmann space, those isometries (resp. conformal transformations) which preserve in addition the vertical vector \( \xi = \partial_V \), i.e., which satisfy (II.2) resp. (II.5), with the additional condition
\[
f_* \xi = \xi \quad \text{infinitesimally} \quad L_Y \xi = 0 \tag{II.21}
\]
span the [generalized] Bargmann (alias extended Galilei) resp. the [generalized] extended Schrödinger group/algebra. One can prove that the conformal factors \( \Omega \) resp. \( \omega \) depend only on \( U \) [24, 25].

The homothety (II.3) does not preserve the the vertical vector \( \xi = \partial_V \) however it preserves its direction,
\[
h_* \xi = \chi^{-2} \xi. \tag{II.22}
\]
Therefore it does not belong to the extended Schrödinger group; it belongs in fact to the [extended] Chrono-Projective group [25, 29, 30]. The latter is a further 1-parameter (non-central) extension of the (centrally extended) Schrödinger group with the constraint (II.21)
weakened,

\[ f^*g_{\mu\nu} = \Omega^2(U)g_{\mu\nu} \quad \text{infinitesimally} \quad L_\gamma g_{\mu\nu} = 2\omega(U)g_{\mu\nu} \quad (\text{II.23a}) \]

\[ f_*\xi = \Psi\xi \quad \text{infinitesimally} \quad L_Y\xi = \psi\xi \quad (\text{II.23b}) \]

where \( \Psi \) is some positive function.

Most of the sections which follow are devoted to study Chrono-Projective transformations.

III. THE MINKOWSKI CASE

We first recall how things work in flat Minkowski spacetime written in light-cone coordinates, \( dX^2 + 2dUdV \), (II.1) with \( K_{ij} \equiv 0 \). Plane gravitational waves with non-trivial profile \( K_{ij} \) will be studied in sec. V.

A. Poincaré and Bargmann

The isometries (resp. conformal transformations) of Minkowski spacetime span the 10-parameter Poincaré group \( P \equiv P_4 \) (resp. the 15 parameter [relativistic] conformal group \( \text{O}(4,2) \)). Those isometries which leave the “vertical” vector \( \xi = \partial_V \) invariant, (II.21), span “the” Bargmann (alias centrally extended Galilei) group, which also includes \( V \)-translations; it has 7 parameters \([24, 25]\) and can be represented by the matrices \(^6\)

\[
\begin{pmatrix}
1 & 0 & 0 & e \\
b & R & 0 & f \\
-\frac{1}{2}b^2 - b^T \cdot R & 1 & h \\
0 & 0 & 0 & 1
\end{pmatrix}
\quad (\text{III.1})
\]

where \( R \in \text{O}(2) \), \( b, f \in \mathbb{R}^2 \) and \( e, h \in \mathbb{R} \).

The conformal transformations of the Bargmann space(time) which preserve \( \xi = \partial_V \) span the centrally extended Schrödinger group, which has \( 7 + 2 = 9 \) parameters. The additional non-isometric transformations are non-relativistic dilations and expansions \([40]\) lifted to

---

\(^6\) Here we consider \( b, f \), etc as column vectors; \( b^T \) is thus a row-vector; the “cdot,” \( \cdot \), means scalar product. \( b^2 = b^T \cdot b \); \( R \) is a \( 2 \times 2 \) rotation matrix, which can act on a two-component column vector by matrix multiplication.
Bargmann space [24, 25].

\[ U \rightarrow \ell^2 U, \quad X \rightarrow \ell X, \quad V \rightarrow V, \quad \Omega = \ell \quad \text{Sch dilation} \quad (\text{III.2a}) \]

\[ U \rightarrow \Omega U, \quad X \rightarrow \Omega X, \quad V \rightarrow V + \frac{k}{2} \Omega X^2, \quad \Omega = \frac{1}{1 - kU} \quad \text{Sch expansion} \quad (\text{III.2b}) \]

\( \ell, k = \text{const.} \). We record for further use that the conformal transformations (III.2) are generated by the vector fields

\[ Y_D = (2U) \partial_U + X^i \frac{\partial}{\partial X^i}, \quad (\text{III.3a}) \]

\[ Y_K = U^2 \partial_U + UX^i \frac{\partial}{\partial X^i} - \frac{1}{2} X^2 \partial_V. \quad (\text{III.3b}) \]

The associated conserved quantities will be reviewed in sec. IV.

**B. The Chrono-Projective group**

The Chrono-Projective group is obtained as the subgroup of the [relativistic] conformal group, (II.5), in the 4D Minkowski context it is a 10-parameter subgroup of \( O(4, 2) \) [25, 29, 30], implemented as

\[
\begin{pmatrix} U \\ X \\ V \end{pmatrix} \rightarrow \begin{pmatrix} \frac{dU + e}{kU + g} \\ \frac{R X + bU + f}{kU + g} \\ V + v + \frac{k}{2} \left( \frac{R X + bU + f}{kU + g} \right)^2 - b \cdot R X - \frac{1}{2} U b^2 \end{pmatrix}
\]

(III.4)

where \( R \in SO(2); b, f \in \mathbb{R}^2; \ d > 0, e, v, k, g \in \mathbb{R} \) and

\[
D = \begin{pmatrix} d & e \\ k & g \end{pmatrix} \in \text{GL}(2, \mathbb{R}).
\]

(III.5)

The corresponding conformal factors are therefore

\[ \Omega = \frac{1}{kU + g} \quad \& \quad \Psi = \det(D) = dg - ke \neq 0. \quad (\text{III.6}) \]

The novelty w.r.t. (III.2) is that time and space dilatation are now unrelated,

\[ U \rightarrow fU, \quad X \rightarrow \ell X, \quad V \rightarrow (\ell^2/f) V. \quad (\text{III.7}) \]

Putting \( f = d/g, \ell = 1/g \), we have \( \Omega^2 = \ell^2 \) and \( \Psi = dg = f/\ell^2 \).
The Chrono-Projective transformation (III.4) belongs to the (extended) Schrödinger group when $\Psi = \det(D) = dg - ek = 1$, i.e., when the matrix in (III.5) belongs to $SL(2, \mathbb{R})$. This happens when $f = \ell^2$ i.e., when time- and space dilations are as $2 : 1$. Expansions with parameter $k$ are obtained in turn for $R = I$, $b = f = 0$, $e = v = 0$, $g = d = 1$ i.e., $\ell = f = \frac{1}{1 + kU}$, consistent with (III.2b).

The extended group also contains the homothety (II.3), which is recovered for $d = g = 1/\chi \Rightarrow \mu = 1, \lambda = \chi, \Psi = \chi^{-2}$ cf. (II.22) ; therefore it adds one more parameter to the [extended] Schrödinger group.

$U - V$ boosts. A special Chrono-Projective transformation arises when a non-relativistic dilation (III.2a) with parameter $\ell = \mu^{1/2}$ is followed by a homothety with parameter $\chi = \mu^{-1/2}$, which yields

$$U \rightarrow \mu U, \quad X \rightarrow X, \quad V \rightarrow \mu^{-1}V.$$  

For Minkowski spacetime this is an element of the Poincaré group i.e. an isometry. Moreover, writing $2d UdV = dz^2 - dt^2$ (III.8) becomes $(z, t) \rightarrow (z - \beta t, t - \beta z)$ with $\mu = \gamma(1 - \beta)$ and is thus an $U-V$ boost. It belongs nevertheless to the Chrono-Projective (but not to the Schrödinger) subgroup, because the vertical coordinate, $V$, is not left invariant : in other words, (III.8) is a Chrono-Projective isometry,

$$L_{Y_{UV}} g = 0, \quad L_{Y_{UV}} \xi = \xi.$$  

We note for later use that for the Schrödinger $\rightarrow$ Chrono-Projective extension, homotheties (II.3) can be traded for these $U - V$ boosts and vice versa.

C. Carroll, Conformal Carroll & Chrono-Carroll

A $(d + 1)$ dimensional manifold $C$ is said to carry a weak Carroll structure if it is endowed with a twice-symmetric covariant, positive, tensor field $G$, whose kernel is generated by the nowhere vanishing, complete vector field $\xi$. The strong definition requires that it carries in addition a symmetric affine connection $\nabla$ that parallel-transport both $G$ and $\xi$ [17, 43, 44].

7 (III.8) is the lift to 4D Bargmann space of “time dilation alone” in 2+1 dimensional non-relativistic spacetime, $U \rightarrow \mu U, X \rightarrow X$ encountered before, e.g., in hydrodynamics [41, 42].
The Bargmann framework, whose primary aim is to provide a "relativistic" description of non-relativistic physics, has also additional bonuses. One of them is to consider, instead of projecting from 4 to 3 dimensions, the pull-back of a given Bargmann metric to the 3-dimensional \( U = 0 \). Then \( C \) has coordinates \((X, V)\) and carries indeed a Carroll structure.

\( \xi = \partial_V \); the coordinate \( V \) is interpreted as "Carrollian time" [17, 25].

When \((M, g_{\mu\nu})\) is Minkowski spacetime, then the subgroup of Poincaré which stabilizes both the vertical vector \( \xi = \partial_V \) and the submanifold \( C \),

\[
f^*g = g, \quad f_*\xi = \xi, \quad f(C) \subset C,
\]

is the Carroll group [15, 17, 18, 25]. It is spanned by Bargmann matrices (III.1) with no \( U \)-translations,

\[
U = 0 \quad \Rightarrow \quad e = 0.
\]

In coordinate terms, the Carroll manifold \( C \) (resp. the Carroll group) are embedded into Bargmann spacetime (resp. Bargmann group) as

\( \iota : \begin{pmatrix} X \\ V \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ X \\ V \\ 1 \end{pmatrix}, \) (III.12a)

\( \begin{pmatrix} R & 0 & f \\ -b^T \cdot R & 1 & h \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & e = 0 \\ b & R & 0 & f \\ -\frac{1}{2}b^2 - b^T \cdot R & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix} \) (III.12b)

where \( R \in O(2), b, f \in \mathbb{R}^2, h \in \mathbb{R} \). The Carroll and Bargmann matrices act affinely by matrix action ; \( h \) translates the Carrollian time, \( V \). We record for later use that under a Carroll boost the position is fixed and only "Carrollian time", \( V \), is boosted,

\( X \to X' = X, \) (III.13a)

\( V \to V' = V - b \cdot X. \) (III.13b)

\footnote{The embedding \( U = \text{const.} \) would yield an equivalent construction.}
Turning to the conformal extension of the Carroll group in $2 + 1$ dimensions, we first relax, by analogy with the Bargmann → Schrödinger extension, the $\xi$-constraint in (III.10),

$$f^* g_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad f_* \xi = \xi, \quad f(\mathcal{C}) \subset \mathcal{C}. \quad \text{(III.14)}$$

This yields the subgroup generated by the 8-parameter vectorfield on $\mathcal{C}$,

$$(\omega_i^j X^j + \gamma^i + \lambda X^i) \frac{\partial}{\partial X^i} + T(X) \partial_V, \quad T(X) = \nu - \beta \cdot X + \frac{1}{2} \kappa X^2, \quad \text{(III.15)}$$

where $(\omega_{ij}) \in \mathfrak{so}(2), \beta, \gamma \in \mathbb{R}^2$ and $\nu, \lambda, \kappa \in \mathbb{R}$ [43]. $T(X)$ here is called, borrowing the BMS-inspired terminology [43, 44], supertranslations. They are $U$-independent and span a 4 parameter subgroup, which, besides $V$-translations and boost, also involves Schrödinger expansions with infinitesimal parameter $\kappa$, – which however act on Carrollian time only but not on space; on the contrary, only $X$ is dilated. The subalgebra (III.15) called the Schrödinger-Carroll algebra [43] has no non-trivial central extension.

**Chrono-Carroll**

A look at the homothety (II.3) shows that it is a sort of “Carrollian counterpart” of non-relativistic dilations, (III.2a), with the rôles of the light-cone coordinates interchanged, $U \leftarrow V$. Eqn. (III.15) shows also that it does not belong to the Schrödinger-Carroll group: (II.3) dilates the “Carrollian time” coordinate $V$, and therefore does not leave the vertical vector $\xi = \partial_V$ invariant. Therefore including (II.3) requires further extension of the Schrödinger-Carroll group, achieved by generalizing the $\xi$-constraint $f_* \xi = \xi$ in (III.14) as in (II.23b), i.e.,

$$f_* \xi = \Psi \xi, \quad \text{infinitesimally} \quad L_Y \xi = \psi \xi. \quad \text{(III.16)}$$

This amounts to considering the restriction of the $e = 0$ subgroup of the Chrono-Projective action (III.4) to the embedded Carroll manifold $\mathcal{C}$, yielding a 9-parameter group action,

$$\begin{pmatrix} 0 \\ X \\ V \end{pmatrix} \to \begin{pmatrix} 0 \\ \frac{R X + f}{g} \\ \frac{1}{gd} \left[ V + h + \frac{k}{2} \frac{(RX + f)^2}{g} - b \cdot RX \right] \end{pmatrix} \quad \text{(III.17)}$$
where \( R \in \text{SO}(2) \); \( b, f \in \mathbb{R}^2 \); \( h, k, \in \mathbb{R}, d, g > 0 \) we may call Chrono-Projective-Carroll or in short Chrono-Carroll. It differs from the Schrödinger-type extension (III.15) by that the determinant of \( \det(D) = gd \) may now take any positive value, \( \Psi = \det(D)^{-1} \), adding 3 conformal parameters to those 6 of the Carroll group:

1. Choosing \( g = 1, k = h = 0 \) yields a one-parameter subgroup parametrized with \( d > 0 \),

\[
\begin{pmatrix}
0 \\
X \\
V
\end{pmatrix} \rightarrow \begin{pmatrix}
0 \\
X \\
\frac{1}{d}V
\end{pmatrix}
\]  \hspace{1cm} (III.18)

interpreted, in this context, as “dilation of Carrollian time alone”. It is consistent with the restriction to \( U = 0 \) of (III.8), we called \( U - V \) boost, or “dilation of Newtonian time alone” [41, 42].

2. The homothety lifted to Bargmann space (II.3), corresponds to the choice \( g = d = \chi^{-1} \).

3. The Schrödinger-type expansion is in turn recovered for \( g = d = 1 \) and parameter \( k \in \mathbb{R} \),

\[
U = 0 \rightarrow U = 0, \quad X \rightarrow X, \quad V \rightarrow V + \frac{k}{2}X^2.
\]  \hspace{1cm} (III.19)

Alternatively, one may trade the homotheties (II.3) by \( U - V \) boosts (III.8) or for non-relativistic dilations, (III.2a) (all of them restricted to \( U = 0 \)).

Infinitesimally, the conf-Carroll supertranslations (III.15) are generalized to

\[
Y^V = \psi V + \left( \nu - \beta \cdot X + \frac{1}{2} \kappa X^2 \right)_{T(X,V)},
\]  \hspace{1cm} (III.20)

which may now depend also on \( V \).

D. The conformal algebra \( o(4,2) \) in light-cone coordinates

An oversight of the interrelations can be gained from the “map” of the conformal group \( O(4,2) \) written in light-cone coordinates.
\[ Y_U = \partial_U, \quad Y_T^i = -\partial^i, \quad Y_V = -\partial_V, \quad \text{translations,} \quad (\text{III.21a}) \]
\[ Y_{12} = X^1\partial^2 - X^2\partial^1, \quad X^1 - X^2 \text{ rotation,} \quad (\text{III.21b}) \]
\[ Y_B^i = U\partial^i - X^i\partial_V, \quad \text{Galilean boosts,} \quad (\text{III.21c}) \]
\[ Y_{AB}^i = X^i\partial_U - V\partial^i, \quad \text{“antiboosts”,} \quad (\text{III.21d}) \]
\[ Y_{UV} = U\partial_U - V\partial_V, \quad \text{U-V boost,} \quad (\text{III.21e}) \]
\[ Y_D = 2U\partial_U + X^i\partial^i, \quad \text{Sch dilatation,} \quad (\text{III.21f}) \]
\[ Y_K = U^2\partial_U + UX^i\partial^i - \frac{X^2}{2}\partial_V, \quad \text{Sch expansion,} \quad (\text{III.21g}) \]
\[ Y_{C1} = \frac{X^2}{2}\partial_U - VX^i\partial^i - V^2\partial_V, \quad C_1, \quad (\text{III.21h}) \]
\[ Y_{C2}^i = X^iU\partial_U + X^iV\partial_V - \left( \frac{X^2}{2} + UV \right)\partial^i + X^i(X^j\partial^j) \quad C_2^i. \quad (\text{III.21i}) \]

cf. [41], which allow us to identify the various subalgebras/subgroups. The following statements are verified by inspection:

- The 4D **Poincaré group** \( P_4 \) is the 10-parameter group of isometries generated by
  \[ \left\{ Y_U, Y_T^i, Y_V, Y_{12}, Y_B^i, Y_{AB}, Y_{UV} \right\}, \quad (\text{III.22}) \]
  some of which do not preserve \( \partial_V \), \([Y_{AB}^i, \partial_V] = \partial^i, \quad [Y_{UV}, \partial_V] = \partial_V. \]

- The isometries which *do* preserve the vertical vector provide us with the 7-parameter **Bargmann group** (III.1), whose Lie algebra defined by \((\mathcal{L}_Y g)_{\mu\nu} = 0, \quad [Y, \partial_V] = 0\) is spanned by
  \[ \left\{ Y_U, Y_T^i, Y_V, Y_{12}, Y_B^i \right\}. \quad (\text{III.23}) \]

- The **extended Schrödinger group** includes such conformal transformations which preserve the vertical vector \( \partial_V \), \((\mathcal{L}_Y g)_{\mu\nu} = \Omega^2(U) \ g_{\mu\nu}, \quad [Y, \partial_V] = 0\). The two additional generators \( Y_D \) and \( Y_K \), which act as in (III.3), satisfy,
  \[ (\mathcal{L}_{Y_D} g)_{\mu\nu} = 2\Lambda \ g_{\mu\nu}, \quad [Y_D, \partial_V] = 0, \quad (\text{III.24a}) \]
  \[ (\mathcal{L}_{Y_K} g)_{\mu\nu} = 2\Lambda U \ g_{\mu\nu}, \quad [Y_K, \partial_V] = 0. \quad (\text{III.24b}) \]

The extended Schrödinger group has thus 9 parameters, namely,
\[ \left\{ Y_U, Y_T^i, Y_V, Y_{12}, Y_B^i, Y_D, Y_K \right\}. \quad (\text{III.25}) \]
• The **Chrono-Projective group** \([25, 28–31]\) has 10 parameters; it is defined by the “Chrono-projective” condition \([Y, \partial_V] = \psi \partial_V\), cf. (II.23b). It is spanned by

\[
\{ Y_U, Y^i_T, Y_V, Y_{12}, Y^i_B, Y_D, Y_K, Y_{UV} \}. \tag{III.26}
\]

The generator \(Y_{UV}\) satisfies (II.23) with \(\omega = 0\) and \(\psi = 1\), respectively. The homothety \(Y_{\text{hom}} = Y_D - 2Y_{UV}\) belongs therefore to the Chrono-projective algebra.

• The restriction of the Bargmann group \(P_4\) to the 3D submanifold \(C\) defined by the constraint \(U = 0\) is a 6 parameter subgroup embedded into \(P_4\) by the constraint \(e = 0\) cf. (III.11)-(III.12), and is identified with the \((2+1)\)D **Carroll group**. \(U\)-translations \(Y_U\) are not more allowed. Boosts act “vertically”, as in (III.13). Its generators are,

\[
\{ Y^i_T, Y_V, Y^i_B, Y_{12} \}, \quad U = 0. \tag{III.27}
\]

• The **Schrödinger-Carroll group** is the conformal extension of the Carroll group within the conformal group \(O(4,2)\), obtained by relaxing the isometry condition in (III.10) but still requiring that \(\partial_V\) be preserved,

\[
(L_Y g)_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad [Y, \partial_V] = 0, \quad U = 0, \tag{III.28}
\]

which is the infinitesimal version of (III.14). Therefore we have \(6 + 2 = 8\) generators, namely those of the Carroll isometries augmented by non-relativistic dilations and expansions,

\[
\{ Y^i_T, Y_V, Y^i_B, Y_{12}, Y_D, Y_K \}, \quad U = 0. \tag{III.29}
\]

• The **Chrono-Carroll** group is a 1-parameter extension of the Schrödinger-Carroll group by with the weakened condition (II.23b), \([Y, \partial_V] = \psi \partial_V\). This adds \(Y_{UV}\) to the Schrödinger-Carroll algebra, yielding 9 generators

\[
\{ Y^i_T, Y_V, Y^i_B, Y_{12}, Y_D, Y_K, Y_{UV} \}, \quad U = 0. \tag{III.30}
\]

The homothety, as a combination of \(Y_D\) and \(Y_{UV}\) belongs to the Chrono-Carroll sub-algebra.

**IV. GEODESICS AND THEIR SYMMETRIES**

In this section we revisit, for the reader’s possible convenience, some aspects of geodesics and the conserved quantities associated with Killing and reps. conformal Killing vectors.
A. Affinely parametrised geodesics

A fully covariant action for a particle (and the only one for a massless particle) is

\[ S = \int g_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma} d\sigma. \]  

(IV.1)

Variation w.r.t. \( X^\mu \) gives the geodesic equations in the form

\[ \frac{d^2 X^\mu}{d\sigma^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dX^\alpha}{d\sigma} \frac{dX^\beta}{d\sigma} = 0. \]  

(IV.2)

Here the \( \Gamma^{\mu}_{\alpha\beta} \) are the Christoffel symbols of the metric \( g_{\mu\nu} \). Because there is no explicit dependence on \( \sigma \), we have the constraint

\[ g_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma} = \epsilon = \text{const}. \]  

(IV.3)

Choosing the constant to be \(-m^2 \leq 0\),

- When \( m \neq 0 \), one sees that
  \[ |m| d\sigma = d\tau, \]  
  where \( m \) is the relativistic mass of the geodesic and \( \tau \) is proper time along the curve \( X^\mu = X^\mu(\sigma) \).
- However when our geodesic is massless, \( m^2 = 0 \), then \( \sigma \) is called an affine parameter, and is defined only up to an affine transformation.

The constraint (IV.3) may be written as \( g^{\mu\nu} P_\mu P_\nu = -m^2 \), where \( P_\mu = g_{\mu\nu} \frac{dx^\nu}{d\sigma} \) is the 4-momentum. If \( m^2 \neq 0 \) one has \( P_\mu = |m| g_{\mu\nu} \frac{dx^\nu}{d\sigma} \). In flat space one may set \( P_0 = -E \), and one obtains the well known formulae \( E^2 - P^2 = m^2 \), \( E = \sqrt{m^2 + P^2} \).

For the metric of a general plane gravitational wave the constraint is

\[ -K_{ij}(U) X^i X^j P_V^2 + 2 P_U P_V + m^2 + P_i P_i = 0. \]  

(IV.5)

In general the only conserved quantity is \( P_V \), which is given by

\[ K_{ij}(U) X^i X^j P_V = P_U \pm \sqrt{P_U^2 + K_{ij}(U) X^i X^j (m^2 + P_i P_i)}. \]  

(IV.6)

If in addition \( K_{ij} \) is independent of \( U \), we have an additional conserved quantity,

\[ P_U = \frac{1}{2P_V} \left( K_{ij} X^i X^j P_V^2 - m^2 - P_I P_I \right). \]  

(IV.7)
An affine parameter can be obtained by starting with a general parameter $\lambda$, by two successive integrals. It is unique up to an affine transformation $\sigma \to a\sigma + b$ where $a, b$ are integration constants.

We mention for completeness that null geodesics lying in the null hypersurfaces $U = \text{const.}$, referred to as the \textit{null geodesic generators of the null hypersurfaces} $U = \text{const.}$; they may be related to lifts of isotropic geodesics in Newton-Cartan spacetimes [28, 43] for which $V$ is an affine parameter.

\section{B. Killing resp. conformal Killing vectors}

We first recall what happens for \textit{Killing vectors}. If we define the tangent vector of a curve with general parameter $\lambda$ by $T^\mu = \frac{dX^\mu}{d\lambda}$, then a geodesic satisfies

$$T^\alpha T^\mu_{\; ;\alpha} = h(\lambda) T^\mu$$

for some function $h(\lambda)$, where the “semicolon $;\alpha$” denotes covariant derivative.

\textbf{Killing vectors.}

Suppose first that $Y^\mu$ is a \textit{Killing vector field}; then it satisfies Killing’s equations

$$Y_{\mu;\alpha} + Y_{\alpha;\mu} = 0.$$  \hfill (IV.9)

It follows that

$$\mathcal{E} = Y_\mu T^\mu = g_{\mu\nu} T^\mu Y^\nu$$

satisfies

$$\mathcal{E}_{;\alpha} T^\alpha = \frac{d\mathcal{E}}{d\lambda} = h(\lambda) \mathcal{E}.$$  \hfill (IV.10)

Taking $\lambda = U$ enables us to calculate $f(\lambda)$ and hence the affine parameter $\sigma$. Then we get a conserved quantity for the geodesic motion,

$$Q_Y = g_{\mu\nu} \frac{dX^\mu}{d\sigma} Y^\nu = g_{\mu\nu} \frac{dX^\mu}{d\lambda} \frac{d\lambda}{d\sigma} Y^\nu, \quad \frac{dQ_Y}{d\sigma} = 0.$$  \hfill (IV.11)

Translations along the “vertical” vector $\xi = \partial_V$ are isometries for any metric of the form (II.1); the associated conserved quantity is the mass downstairs, $M = P_V$. 


Conformal Killing vectors.

Now we suppose that we have instead a conformal Killing vector \( Y^\mu \), i.e., one for which
\[
Y_{\mu\nu} + Y_{\nu\mu} = 2\omega g_{\mu\nu}
\]
(IV.12)
for some function \( \omega \). If \( \omega = \) constant, \( Y^\mu \) is called a homothetic Killing vector since it generates a homothety. For a timelike geodesic with tangent vector \( T^\alpha = \frac{dx^\alpha}{d\tau} \) where \( \tau \) is proper time along the geodesic so that \( T^\alpha T_\alpha = -1 \), we have instead
\[
(Y_\alpha T^\alpha)_{;\mu} T^\mu = -\omega.
\]
(IV.13)
Thus in general the quantity (IV.10) i.e. \( Y^\alpha T^\alpha = \frac{1}{m} Y^\mu P_\mu \) (where \( P_\mu = m T_\mu \) is the momentum of a particle of mass \( m \)) is not constant along the world line. From the point of view of the covariant Hamiltonian treatment, \( Y^\mu P_\mu \) is the moment map generating the lift to the co-tangent bundle of the conformal transformation of the base manifold.

In the special case of a homothety when \( \omega = \omega_0 = \) const., we find that \( \frac{d(Y_\alpha T^\alpha)}{d\tau} = -\omega_0 \Rightarrow Y_\alpha T^\alpha = -\omega_0 \tau - \omega_1 \). Alternatively, deriving again, we have \( d^2(Y_\alpha T^\alpha)/d\tau^2 = 0 \), which is a covariant version of the Lagrange-Jacobi identity [34].

Conformal Killing vectors do not generate symmetries for timelike geodesics. However, as observed by Jacobi [34], while \( Y_\alpha T^\alpha \) is not in general conserved, the two constants of integration above yield, in modern language, the conserved quantities (IV.17) associated with non-relativistic dilations and expansions (III.2) and (III.3), respectively [37].

C. Conserved quantities for null geodesics

By contrast, if one considers an affinely parametrised null geodesic with tangent vector \( l^\alpha = dx^\alpha/d\sigma \) that satisfies
\[
g_{\mu\nu} l^\mu l^\nu = 0, \quad l^\alpha_{;\mu} l^\nu = 0,
\]
(IV.14)
we do obtain a constant of the motion,
\[
Q_Y = Y_\mu l^\mu, \quad \frac{dQ_Y}{d\sigma} = 0.
\]
(IV.15)
• As a first illustration, we rederive the conserved quantities associated with Schrödinger
dilations and expansions in (III.3) from this viewpoint. Adopting the Hamiltonian point of
view, the geodesic Hamiltonian associated with the action (II.12) is
\[ H = \frac{1}{2} P_i P_i + P_U P_V + \Phi(U, X) P_V^2 \] with \( P_V = \dot{U}, \ P_U = \dot{V} - 2\Phi \dot{U}. \) (IV.16)
For null geodesics \( m = 0, \ P_V = M \) is associated with the “vertical” Killing vector and
is therefore conserved and identified with the non-relativistic mass “downstairs”, and we
recover the Bargmann interpretation given in sec II A.

For a free particle, or for the inverse-square potential \( \Phi(X) \propto |X|^{-2} \), the generators
(III.3) are conformal Killing vectors for the lifted metric, and (IV.15) yields the conserved
Schrödinger quantities \([40]\),
\[ D = P_i X^i - 2EU \quad \text{dilation} \quad \text{(IV.17a)} \]
\[ K = -EU^2 + UP_i X^i - \frac{M}{2} X_i X_i \quad \text{expansion} \quad \text{(IV.17b)} \]
These quantities are conserved for null geodesics “upstairs”, in Bargmann space; they do
not depend on \( V \) and project therefore to well-defined conserved quantities for the projected
non-relativistic motion. In fact \( D, K \) close, with the projected Hamiltonian \( H_{NR} \) to an \( o(2, 1) \)
algebra \([40]\).

We note also for later use that \( P_V = M = dU/d\sigma \), so that \( U \) is the proper time along the
geodesic.

For motion along null geodesics eqn. (IV.15) associates a conserved quantity; if the latter
preserves in addition also the “vertical” vector \( \xi = \partial_V, \ L_V \xi = 0 \) cf. (II.21), this quantity (we
call of the Schrödinger type) does not depend on the vertical coordinate \( V \), and therefore
projects to a conserved quantity for the underlying non-relativistic dynamics “downstairs”
— this is in fact the original idea of the Bargmann framework \([24, 25]\).

Let us underline, however, that (IV.15) yields a conserved quantity for null geodesic
motion for any conformal vector field, with or without extra condition on \( \xi \). However for
a Chrono-Projective transformation \( L_V \xi = \psi \xi \) cf. (II.23) with \( \psi \neq 0 \) a new, “Chrono-
Projective” type of conserved quantity (first noticed in \([33]\) ) is associated. See secs. (IV D)
and \( V \) for details.

We would also like to stress that for non-trivial profile \( K_{ij}(U) \) most generators of \( o(4, 2) \)
and even of the Poincaré group are broken.
D. Scale transformations

Postponing our general investigations to sec.V, first we spell out what happens for scale transformations; we start with the homothety (II.3) – (II.4). Being a conformal vector for the gravitational wave spacetime \(^9\), (IV.15) provides us with

\[
Q_{\text{hom}} = X^i P_i + 2V P_V
\]  

(IV.18)

where \(P_V\), associated with the “vertical” Killing vector \(\partial_V\), \(Q_{\text{hom}}\) is conserved for null, not for timelike geodesics, as confirmed also by using the equations of motion (II.6b)

The conservation of \(Q_{\text{hom}}\) allows us to determine the evolution of the “vertical coordinate” assuming that the transverse motion \(X^i(\sigma)\) had already been determined,

\[
V(\sigma) = \frac{Q_{\text{hom}}}{2P_V} - \frac{X^i(\sigma)P_i(\sigma)}{2P_V} = \frac{Q_{\text{hom}}}{2P_V} - \frac{1}{4P_V} \frac{d}{d\sigma} \left(X^i(\sigma)X_i(\sigma)\right) .
\]  

(IV.19)

As explained in sec. IIA, the null dynamics in 4D projects to an underlying non-relativistic system in \(2 + 1\) D whereas \(P_V\) becomes the mass, \(M\), \(U\) the non-relativistic time. \(\dot{U} = M\); the non-relativistic Hamiltonian and Lagrangian are recovered as in (II.16) and (II.18), allowing us to express \(Q_{\text{hom}} = Q_{\text{NR}} + 2MV_0\)

\[
Q_{\text{NR}} = X^i P_i - 2 \int^U L_{\text{NR}}(u)du ,
\]  

(IV.20)

whose conservation can be confirmed directly using the eqns of motion.

More generally, the anisotropic rescaling

\[
U \to \mu^b U, \quad X^i \to \mu^a X^i, \quad V \to \mu^c V , \quad \mu = \text{const}.
\]  

(IV.21)

induces, for the Brinkmann metric (II.1),

\[
g_{\mu\nu}dX^\mu dX^\nu \to \mu^{2a} \left(\delta_{ij}dX^i dX^j + \mu^{-2a+b+c} 2dUdV + K_{ij}(\mu^b U)\mu^{2b} X^i X^j dU^2\right)
\]

which is conformal provided

\[
c = 2a - b, \quad K_{ij}(\mu^b U) = \mu^{-2b} K_{ij}(U) .
\]  

(IV.22)

If these conditions hold, then, for any \(b\),

\[
g_{\mu\nu}dX^\mu dX^\nu \to \Omega^2 g_{\mu\nu}dX^\mu dX^\nu , \quad \Omega = \mu^a .
\]  

(IV.23)

\(^9\) The free (Minkowski) case is that of \(K_{ij} = 0\).
The associated conserved quantity

\[ Q_{a,b} = aX^i P_i + b U P_U + (2a - b) V P_V \]  
  \[ = aX^i P_i - b U E - (2a - b) \left( \int U L_{NR} + V_0 P_V \right) \]  

(IV.24)

1. For \( a = b = c = 1 \) we would get the relativistic (isotropic) dilation \( U \to \mu U, X^i \to \mu X^i, V \to \mu V \); when \( b = 2a \) we get Schrödinger dilations.

2. When \( b = 0 \) we recover, for any profile \( K_{ij}(U) \), the homothety (II.3).

3. If \( K_{ij} \) is \( U \)-independent (as for Brdička metric (VI.6) below), then \( b = 0 \);

4. \( b \neq 0 \) could be obtained for the [singular] non-trivial profile [47]

\[ K_{ij}(U) = \frac{K_{ij}^0}{U^2}, \quad K_{ij}^0 = \text{const.} \]  

(IV.25)

Choosing \( a = 0 \) then \( c = -b \) and we obtain a Chrono-Projective isometry – namely our \( U-V \) boost, \( U \to \mu U, X^i \to X^i, V \to \mu^{-1} V \) which is in (III.8). Its conserved charge is “Chrono-Projective”

\[ Q_{UV} = U P_U - V P_V = -U E + \int U L_{NR} + V_0 P_V. \]  

(IV.26)

Choosing instead \( b = 0, c = 2a \) we recover (as said above) the homothety (II.3)

10. Thus this example has again a a maximal i.e. a 7-parameter Chrono-Projective algebra.

\( z = b/a \) is also called the dynamical exponent [41, 42, 45, 46]. The typical relativistic value is \( z = 1 \); for Schrödinger-type expressions \( z = 2 \); the “Chrono-Projective” contribution with the action integral arises when \( b \neq 2a \) i.e., when \( z \neq 2 \).

So far we analysed what happens for a possibly time-dependent quadratic-in-\( \mathbf{X} \) potential profile \( K(U) \). We note that a similar analysis can be carried out for different homogeneous potentials [33].

\[ \text{The profile (IV.25) is symmetric also w.r.t. Schrödinger dilations. This is not a surprise, though, because a Schrödinger dilation is a combination of an } U \text{ boost and of homothety, as seen before.} \]
V. CHRONO-PROJECTIVE TRANSFORMATIONS: GENERAL STUDY

Now we turn to the systematic study of Chrono-Projective transformations of the gravitational wave metric (II.1) with a non-trivial profile $K_{ij}(U)$.

The conformal transformations of pp-waves have been determined some time ago [47]. Below we study them in our case of interest in a novel way. Occasionally we calculate in Brinkmann coordinates however mostly use Baldwin-Jeffery-Rosen (BJR) coordinates $(u, x, v)$ [36] which turned out useful for the isometries [3, 10, 16],

$$U = u, \quad X = P(u) x, \quad V = v - \frac{1}{4} x \cdot \dot{a}(u)x,$$

where $a(u) = P^\dagger(u)P(u)$, the $2 \times 2$ matrix $P = (P_{ij})$ being a solution of the Sturm-Liouville problem [3, 19]

$$\dddot{P} = KP, \quad P^\dagger \dddot{P} = \dddot{P}^\dagger P .$$

In BJR coordinates the metric takes the form,

$$g = a_{ij}(u)dx^i dx^j + 2dudv .$$

The infinitesimal “Chrono-Projective” conditions (II.23) require

$$\mathcal{L}_Y g = 2\omega g,$$

$$\mathcal{L}_Y \xi = \psi \xi$$

with $\omega = \omega(u)$ [24, 28]. We record for further use that the $\xi$-condition (V.4b) implies that the $v$-component $Y^v$ is at most linear in $v$,

$$Y = Y^u(x, u)\partial_u + Y^t(x, u)\partial_t + (b(x, u) - \psi v) \partial_v .$$

The conformal Killing equation (V.4a) requires, for $Y$ as in (V.5),

$$\partial_t Y^u = 0 ,$$

$$\partial_u Y^u = 2\omega + \psi ,$$

$$\partial_u Y^v = 0$$

$$\partial_t Y^v + (\partial_u Y^j)a_{ij} = 0 ,$$

$$Y^u(\partial_u a_{ij}) + a_{kj}(u)\partial_i(Y^k(x, u)) + a_{ki}(\partial_j Y^k(x, u)) = 2\omega(u)a_{ij} .$$
Eqn. (V.6a) implies that $Y^u = Y^u(u)$; then (V.6b) can be solved as

$$Y^u(u) = \epsilon + \int_0^u (2\omega(w) + \psi)dw.$$  \hspace{1cm} (V.7)

($\epsilon = \text{const.}$) Eqn. (V.6c) implies that $\partial_x b(x, u) = v \partial_u \psi$ which requires both sides to vanish. Thus $b$ and $\psi$ are $u$-independent, $b = b(x)$ and $\psi = \text{const.}$. Then $Y^i(x, u)$ can be written as

$$Y^i(x, u) = K^i(u) + F^i(x) + L^i(x, u).$$

Substituting into (V.6d) integrating from $u_0 = 0$ to $u$ and collecting and renaming terms, we get,

$$Y^k(x, u) = F^k(x) - H^{ki}(u) \partial_k b(x),$$ \hspace{1cm} (V.8)

where $H^{ki}(u)$ is Souriau’s $2 \times 2$ matrix [10, 16],

$$H^{ki}(u) = \int_0^u a^{ki}(w)dw,$$ \hspace{1cm} (V.9)

where $(a^{ij})$ is the inverse matrix, $a^{ij} a^{jk} = \delta^i_k$. Inserting this into (V.6e), the last condition can be written as

$$-2\omega(u) a_{ij}(u) + Y^u(u) (\partial_u a_{ij}(u)) +$$

$$a_{kj}(u) \left( \partial_i F^k(x) - H^{km}(u) \partial_i \partial_m b(x) \right) + a_{ki}(u) \left( \partial_j F^k(x) - H^{km}(u) \partial_j \partial_m b(x) \right) = 0.$$ \hspace{1cm} (V.10)

Collecting our results,

$$Y^u(u) = \epsilon + 2 \int_0^u \omega(w)dw + \psi u,$$ \hspace{1cm} (V.11a)

$$Y^i(x, u) = F^i(x) - H^{ij}(u) \partial_j b(x),$$ \hspace{1cm} (V.11b)

$$Y^v(x, v) = b(x) - \psi v.$$ \hspace{1cm} (V.11c)

Thus $\epsilon = \text{const.}$ is a time translation; the conformal resp. Chrono-Projective factors $\omega$ and $\psi$ contribute to time dilations.

The functions $F^i(x)$ and $b(x)$ should be determined from the constraint (V.10). Deriving (V.10) by the $x$-coordinates, rearranging, deriving by $u$ and again rearranging yields the relations

$$a_{kj}(\partial_n \partial_i F^k(x) - H^{km}(u) \partial_n \partial_j \partial_m h(x)) + a_{ki}(\partial_n \partial_j F^k(x) - H^{km}(u) \partial_n \partial_j \partial_m h(x)) = 0$$ \hspace{1cm} (V.12a)

$$2 \partial_n \partial_i \partial_s b(x) = a_{sij} a^{ij}(\partial_n \partial_j F^k(x) - H^{km}(u) \partial_n \partial_j \partial_m b(x)).$$ \hspace{1cm} (V.12b)
From the latter we infer that $\partial_n \partial_i \partial_s b(x) = 0$ and therefore $b(x)$ is at most quadratic in $x$,
\[ b(x) = b_{ij} x^i x^j - b_i x^i + h. \quad \text{(V.13)} \]

Thus eqn (V.12a) simplifies to
\[ \partial_n \left( \partial_i F^l(x) + (a_{kj}) \partial_j F^k(x) \right) = 0 \quad \text{for all } n. \quad \text{(V.14)} \]

This expression makes it likely that $F^i$ should be at most of the first order in $x^j$. So far we could check this statement for each example studied in the next section.

**Hamiltonian structure**

The geodesic Lagrangian resp. Hamiltonian are, in BJR coordinates,
\[ \mathcal{L} = \frac{1}{2} a_{ij}(u) \dot{x}^i \dot{x}^j + \dot{u} \dot{v}, \quad \mathcal{H} = \frac{1}{2} a^{ij} p_i p_j + p_u p_v, \quad \text{(V.15)} \]
where the canonical momenta $p_\mu = \partial \mathcal{L} / \partial \dot{x}^\mu$ are $p_u = \dot{v}, p_v = \dot{u}, p_i = a_{ij} \dot{x}^j \Rightarrow \dot{x}^i = a^{ij} p_j$.

By (IV.15) the conserved quantity associated with the conformal vectorfield $Y$ is,
\[ Q_Y = Y^\mu p_\mu = Y^u(x,u) p_u + Y^i(x,u) p_i + (b(x) - \psi v) p_v. \quad \text{(V.16)} \]

In terms of the Poisson bracket “upstairs”, \( \{ \mathcal{R}, \mathcal{T} \} = \frac{\partial \mathcal{R}}{\partial x^\mu} \frac{\partial \mathcal{T}}{\partial p_\mu} - \frac{\partial \mathcal{R}}{\partial p_\mu} \frac{\partial \mathcal{T}}{\partial x^\mu} \), the generating vector field is recovered as $Y^\mu \partial_\mu = \{ x^\mu, Q_Y \} \partial_\mu$. Rewriting the Hamiltonian as,
\[ \mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2} (a^{-1})^{ij} p_i p_j + p_u p_v = \frac{1}{2} (g^{-1})^{\mu\nu} p_\mu p_\nu, \quad \text{(V.17)} \]
we have
\[ \{ Q, \mathcal{H} \} = -\frac{1}{2} \mathcal{L} g^{\mu\nu} p_\mu p_\nu, \quad \text{and} \quad \{ Q, \xi \} = -\mathcal{L} \gamma(\xi). \quad \text{(V.18)} \]

By (V.4) a charge which is conserved for null geodesics should satisfy therefore
\[ \{ Q, \mathcal{H} \} = 2\omega \mathcal{H}, \quad \{ Q, \xi \} = -\psi \xi \quad \text{(V.19)} \]
with $\omega = \omega(u), \psi = \text{const.}$. These formulas come handy to check whether a given quantity is conserved or not.
Isometries

For a generic profile, setting $\omega = 0$ and $\psi = 0$ yields the 5 standard isometries,

$$T^i = \delta^i_j p_j, \quad i = 1, 2 \quad \text{x-translations} \quad (V.20a)$$
$$T_v = p_v, \quad i = 1, 2 \quad \text{v-translations} \quad (V.20b)$$
$$B^i = H^i_j p_j - x^i p_v, \quad \text{x-boosts} \quad (V.20c)$$

all commute with the geodesic Hamiltonian $H$; the only non-vanishing brackets are

$$\{ T_i, B^j \} = \delta^j_i p_v. \quad (V.21)$$

In the free case, we also have, in addition, time translations and x-rotations.

Homothety

First we check that the homothety (II.3) exported to BJR coordinates,

$$u \rightarrow u, \quad x \rightarrow \chi x, \quad v \rightarrow \chi^2 v \quad \text{generated by} \quad Y_{\text{hom}} = x^i \partial_i + 2v \partial_v \quad (V.22)$$

is Chrono-projective for any $a_{ij}(u) : g_{\mu\nu}dx^\mu dx^\nu \rightarrow \chi^2 g_{\mu\nu}dx^\mu dx^\nu, \mathcal{L}_Y g = 2g, \mathcal{L}_Y \xi = -2\xi$.

The induced conserved charge is identical to (IV.18),

$$Q_{\text{hom}} = x^i p_i + 2vp_v, \quad \{ Q_{\text{hom}}, H \} = 2H. \quad (V.23)$$

The Poisson brackets of the homothety $Q_{\text{hom}}$ with $T_i, T_v, B^i$ are in turn

$$\{ T_i, Q_{\text{hom}} \} = -T_i, \quad (V.24a)$$
$$\{ B^i, Q_{\text{hom}} \} = -B^i, \quad (V.24b)$$
$$\{ T_v, Q_{\text{hom}} \} = -2T_v. \quad (V.24c)$$

Let us record for further use that

$$\{ p_u, H \} = -\frac{1}{2} \delta^{ij} p_i p_j, \quad (V.25)$$

which confirms that $u$-translations are broken when the profile is $u$-dependent.

Turning to further conformal extensions, we limit ourselves to the following observations.

\[\text{\textsuperscript{11}}\text{Note that this is not a central extension; the generators belong themselves to the algebra.}\]
Expressing the flat-case Schrödinger dilation (III.2a) and U-V boosts (III.8) in BJR co-
ordinates involves the (generally unknown) $P(u)$ matrix and will be considered in particular
cases, see the sect. VI.

In the free case the vector field (III.3b) generates special Schrödinger conformal trans-
formations [expansions]. It is actually more convenient to work in Brinkmann coordinates,
when the associated conserved charge is $K = U^2 P_U + U P_i X^i - \frac{1}{2} P V X_i X^i$. Then testing the
PB relation for the general Brinkmann Hamiltonian $H = \frac{1}{2} P_i P_i + P_U P_V - \frac{1}{2} K_{ij} X^i X^j P_V^2$ we find
\begin{equation}
\{K, H\} = 2U H + 2U \left( K_{ij} + \frac{1}{4} U \dot{K}_{ij} \right) X^i X^j P_V^2.
\end{equation}
Thus Schrödinger expansions are broken in general.

VI. EXAMPLES

Below we illustrate our general theory on selected examples.

A. Minkowski case

As first illustration, we recover the results of sec III for the flat metric $a_{ij} = \delta_{ij}$. The
Souriau matrix is $H^{ij}(u) = u \delta^{ij}$ and the constraint (V.10) requires
\begin{equation}
-2u \partial_i \partial_j b(x) + \left( \partial_i F^j(x) + \partial_j F^i(x) \right) = 2\omega(u) \delta_{ij}.
\end{equation}

1. Deriving the equation by $u$ shows that the conformal factor $\omega(u)$ can be at most linear
in $u$, $\omega(u) = \lambda + \kappa u$ $\lambda$, $\kappa = \text{const}$.

2. $d\omega/du$ can not be a function of $x$ and therefore $\partial_i \partial_j b = -\kappa \delta_{ij}$, implying
\begin{equation}
b(x) = -\frac{\kappa}{2} x^2 - b \cdot x + h, \quad b_i, \ h = \text{const}.. \tag{VI.2}
\end{equation}

3. It follows that $\partial_i F^j(x) + \partial_j F^i(x) = 2\lambda \delta_{ij}$, implying
\begin{equation}
F^i(x) = \omega^i_j x^j + \lambda x^i + f^i, \tag{VI.3}
\end{equation}
where $\omega^i_j$ is a constant antisymmetric matrix, $\lambda = \text{const.}$ and $f^i = \text{const}..
Inserting these expressions into (V.11) we end up with the BJR form of the free Chrono-projective Lie algebra studied sec. III,

\[ Y^u(u) = \epsilon + 2\lambda u + \kappa u^2 + \psi u \]  
\[ Y^i(x,u) = \omega^i_j x^j + f^i + ub_i + \lambda x^i + \kappa ux^i, \]  
\[ Y^v(x,v) = -\frac{\kappa}{2}x^2 - b \cdot x + h - \psi v, \]

which identifies \( \lambda \) and \( \kappa \) as the parameters of Schrödinger dilations and expansions, \( f^i \) as space translations, \( \omega^i_j \) as rotations, \( b_i \) as Galilei boosts. \( \psi Y_{uv} = \psi(u\partial_u - v\partial_v) \) generates u-v boosts (III.8). The homotheties (V.22) are hidden here as \( Y_{\text{hom}} = Y_D - 2Y_{UV} \) where \( Y_D = 2u\partial_u + x^i\partial_i \) generates Schrödinger dilations cf. sec. III.B. The u-v boosts generate again a “new type” of conserved charge,

\[ Q_{uv} = up_u - vp_v \Rightarrow Q_{\text{NR}}^{uv} = Q_{uv} - Mv_0 = -uH + \int_0^u L_{\text{free}}d\tau, \]  

which is however not really new because \( Q_{uv} = \frac{1}{2}(D - Q_{\text{hom}}) \).

Now we study some examples with non-trivial profile and with explicitly known, Brinkmann \( \rightarrow \) BJR transcription.

**B. The Brdička metric**

As a simple example with non-trivial profile, let us consider the \( U \)-independent linearly polarized gravitational wave metric in Brinkmann coordinates proposed by Brdička [48],

\[ dX_1^2 + dX_2^2 + 2dUdV - \Omega^2\left(X_1^2 - X_2^2\right) dU^2, \quad \Omega = \text{const.}. \]

Before proceeding to a systematic search for all Chrono-Projective transformations we mention that a quick inspection shows that

1. The Brdička profile \( K_{ij} \) in (VI.6) is \( U \)-independent, leaving us with the homothety (II.3) as the only rescaling symmetry, cf. sec.IVC.

2. A Schrödinger dilation (III.2a) i.e. \( U \rightarrow \Lambda^2 U, \ X \rightarrow \Lambda X \ V \rightarrow V \) scales the kinetic part by \( \Lambda^2 \) while he potential term scales by \( \Lambda^6 \); Schrödinger dilations are therefore broken.
3. U-V boosts with parameter $\mu$ (III.8) leave the kinetic part invariant but scale the potential term by $\mu^2$; they are therefore broken.

However that a Schrödinger dilation with $\Lambda$ followed by an U-V boost with $\mu = \Lambda^{-2}$ scales both the kinetic and the potential term by $\Lambda^2$ and combine therefore to the homothety (II.3) which is Chrono-Projective; cf. sec. III.B.

4. Expansions broken by (V.26).

A systematic study can be carried out by switching to BJR coordinates. The Sturm-Liouville equation (V.2) is solved by

$$P(u) = \text{diag}\left(\cos \Omega u, \cosh \Omega u\right).$$

(VI.7)

The induced Brinkmann $\rightarrow$ BJR transformation (V.1) i.e.

$$\left\{ \begin{array}{l}
U = u \\
X_1 = x^1 \cos(\Omega u) \\
X_2 = x^2 \cosh(\Omega u) \\
V = v + \frac{\Omega}{4}(x^1)^2 \sin(2\Omega u) - \frac{\Omega}{4}(x^2)^2 \sinh(2\Omega u)
\end{array} \right.$$  

(VI.8)

yields the BJR metric (V.3) resp. Souriau matrix

$$a(u) = \text{diag}\left(\cos^2(\Omega u), \cosh^2(\Omega u)\right).$$  

(VI.9a)

$$(H^{ij}(u)) = \Omega^{-1} \text{diag}\left(\tan(\Omega u), \tanh(\Omega u)\right).$$  

(VI.9b)

A "screw-type" isometry. When written in Brinkmann coordinates, the metric (VI.6) is $U$-independent, implying that the $U$-translations

$$U \rightarrow U + e$$  

(VI.10)

add a 6th manifest isometry to the 5 standard ones. It is redundant however instructive to see how this comes about in BJR coordinates, for which the symmetry (VI.10) is not

---

12 Indices are lifted by the transverse metric, $x^i = a^{ij} x_j$.
13 We borrowed the word from [8], where a broken $U$ translation combines with a broken rotation into a symmetry, which acts as a "screw" [9, 14, 16], as illustrated in fig.2 below.
manifest. However taking into account that \( P = P(u) \), yields

\[
\begin{align*}
u & \rightarrow u + e, & (VI.11a) \\
(x^1, x^2) & \rightarrow \left( x^1 \frac{\cos \Omega u}{\cos \Omega(u + e)}, x^2 \frac{\cosh \Omega u}{\cosh \Omega(u + e)} \right), & (VI.11b) \\
v & \rightarrow v - \frac{\Omega}{2} \left( (x^1)^2 \frac{\sin \Omega e \cos \Omega u}{\cos \Omega(u + e)} - (x^2)^2 \frac{\sinh \Omega e \cosh \Omega u}{\cosh \Omega(u + e)} \right), & (VI.11c)
\end{align*}
\]

which does preserve the BJR metric (V.3).

To find all Chrono-Projective vectorfields we follow the recipe outlined in sec.V. We start with the general equations (V.11); then a calculation similar to the free case shows that

\[
\begin{align*}
\ddot{\omega} + 2\Omega^2(2\omega + \psi) &= 0 \\
\ddot{\omega} - 2\Omega^2(2\omega + \psi) &= 0
\end{align*}
\] (VI.12)

whose consistency requires \( 2\omega + \psi = 0 \). Therefore \( \omega \) is a constant and \( Y^u \) is a mere \( u \)-translation, \( Y^u = \epsilon \). Then from (V.10) we deduce by some effort that

\[
b(x) = -\frac{\Omega^2}{2}((x^1)^2 - (x^2)^2) \epsilon - b_i x^i + h \quad \text{and} \quad F^i(x) = \omega x^i + f^i, \ f^i = \text{const.} \quad (VI.13)
\]

In conclusion, the most general homothetic Chrono-Projective vectorfield for the Brdička metric has 7 parameters; in BJR coordinates it is,

\[
Y_{\text{chr}} = \epsilon \left( \partial_u + \Omega \left( x^1 \tan(\Omega u) \partial_1 - x^2 \tanh(\Omega u) \partial_2 \right) - \frac{\Omega^2}{2} \left( (x^1)^2 - (x^2)^2 \right) \partial_v \right) + f^i \partial_i + h \partial_v + \left( \frac{1}{\Omega} \left( b_1 \tan(\Omega u) \partial_1 + b_2 \tanh(\Omega u) \partial_2 \right) - b_i x^i \partial_v \right) + \omega \left( x^i \partial_i + 2v \partial_v \right).
\] (VI.14)

Thus \( f^i \) and \( h \) generate space and vertical translations and the \( b_i \) generate the boosts. The Chrono-Projective factors are \( \omega = -\psi/2 = \text{const} \). The parameter \( \epsilon \in \mathbb{R} \) generates the additional isometry induced by \( u \)-translations which is in fact the infinitesimal version of (VI.11). Expressed in Brinkmann coordinates the “screw-charge” that this isometry generates,

\[
Q^{\text{scr}}_\epsilon = p_u + \Omega \left( \tan(\Omega u)x^1 p_1 - \tanh(\Omega u)x^2 p_2 \right) - \frac{\Omega^2}{2} \left( (x^1)^2 - (x^2)^2 \right)
\] (VI.15)

turns out to be (minus) the energy \( P_U \), as expected.

C. “Screw” for circularly polarized periodic (CPP) waves

We just mention that circularly polarized periodic waves, (II.1) with profile

\[
\begin{align*}
\mathcal{A}_+(U) &= \cos(\omega U), \\
\mathcal{A}_x(U) &= \sin(\omega U),
\end{align*}
\] (VI.16)
has, beyond the homothety and the usual 5 isometries also a 6th “screw” isometry obtained by combining broken rotations and broken $U$-translations [8, 9, 14, 16],

$$Y^{\text{scp}}_{CPP} = \partial_U + \frac{\omega}{2}(X^1 \partial_2 - X^2 \partial_1). \quad \text{(VI.17)}$$

Referring the reader to the literature for details, here we merely illustrate by plotting how trajectories are carried into trajectories under homotheties (fig. 1) and screw transformations (fig. 2).

![Diagram illustrating homothety and screw transformations](image)

**FIG. 1:** For the circularly polarized periodic profile (II.1) with $A_+(U) = \cos(U)$, $A_\times(U) = \sin(U)$ the **homothety** (II.3) takes the trajectory with initial condition $(U_0, X_0, V_0)$ [in magenta] into that with initial condition $(U_0, \chi X_0, \chi^2 V_0)$ [in green].

**D. “Screw” with expansion**

In [20, 21] Andrzejewski and Prencel investigate the memory effect for the linearly polarized gravitational wave with regular $U$-dependent profile [47]

$$K_{ij}(U) = \frac{\epsilon^2}{(U^2 + \epsilon^2)^2} \text{diag}(1, -1), \quad \text{(VI.18)}$$

whose explicit $U$-dependence breaks the $U$-translation symmetry. On the other hand, (IV.22) shows that rescalings are broken with the exception of the homothety. However combining
FIG. 2: Dropping the $V$-coordinate and unfolding the transverse CPP trajectory by adding $U$ yields spirals. The screw-transformation (VI.17) (in blue) carries the trajectory in magenta into another trajectory (in green).

the broken $U$-translation with a broken Schrödinger expansion (III.3b),

$$Y^{scr} = Y_K + \epsilon^2 \partial_U$$  \hspace{1cm} (VI.19)

we to call here a “screwed expansion” will be conserved for null geodesics. To check this we simply combine $\{K, H\}$ in (V.26) with

$$\{P_U, H\} = \frac{1}{2} \dot{K}_{ij} X^i X^j P^2_V$$  \hspace{1cm} (VI.20)

(which is valid for an arbitrary profile) and add ; a straightforward calculation shows that $2U\dot{K} + \frac{1}{2} U^2 \ddot{K} = -\frac{1}{2} \epsilon^2 \dot{K}$ for the profile (VI.18) canceling the symmetry-breaking terms in (V.26) \(^{14}\).

$$L_{Y^{scr}} g_{\mu \nu} = 2U g_{\mu \nu}, \quad L_{Y^{scr}} \xi = 0.$$  \hspace{1cm} (VI.21)

In conclusion, the Brinkmann metric with profile (VI.18) provides us with an example with 5 isometries and two conformal generators, one of them (namely the homothety) being Chrono-Projective and the other of the Schrödinger-type.

\(^{14}\) (VI.18) is in fact the only such profile.
E. “Screw” with U-V boost

Ilderton \cite{9, 49} mentions that for the [singular] profile

\[ K_{ij}(U) = \frac{K_{ij}^0}{(1 + U)^2}, \quad K_{ij}^0 = \text{const.} \]  

\text{(VI.22)}

the manifest breaking of U-translation invariance can be cured by “screw-combining” it with a (broken) boost. His statement is confirmed by calculating that

\[ Q_{UV}^{\text{scr}} = P_U + Q_{UV}, \quad \{ Q_{UV}^{\text{scr}}, H \} = \frac{1}{2}(2K_{ij} + (1 + U)\dot{K}_{ij})X^iX^jP^2_V; \]  

\text{(VI.23)}

But for (VI.22) the bracketed quantity vanishes. This conserved quantity is generated by \( Y_{UV}^{\text{scr}} = Y_U + Y_{UV} \). The chronoprojective factors are \( \omega = 0 \) and \( \psi = 1 \). Adding the homothety, we end up with a Chrono-isometry plus a Chrono-conformal transformation in addition to the standard 5 isometries.

Another way of understanding Ilderton’s statement is to observe that \( U \to U - 1 \) carries the profile (VI.22) to the form (IV.25), whose U-V boost symmetry was already established in sec.IV D.

Having reviewed these exceptional cases with high symmetry, we mention for completeness that for a generic wave no extra isometry arises. In ex. 2 [eqn $\# (4.5)$] of \cite{16}, for example, the manifest-$U$-dependent profile

\[ \frac{1}{2}K_{ij}(U)X^iX^j = \tan U [(X^2)^2 -(X^1)^2], \]  

\text{(VI.24)}

makes it plausible that no additional 6th isometry will arise. This is confirmed by solving the Sturm-Liouville eqn (V.2), \( P(u) = -\cos u \, \text{diag}(e^u, e^{-u}) \). Then a tedious calculation allows us to deduce that the Chrono-Projective algebra has just the standard 5-parameter isometry algebra augmented with the homothety (V.22); the Chrono-Projective factors are, \( \omega = -\psi/2 = \chi \).

VII. CONFORMAL CARROLL : THE NEWTON-CARTAN APPROACH

The various conformal extensions of the Galilei group can be conveniently studied in the Newton-Cartan framework; see e.g. \cite{28, 30, 31, 37–39, 45, 46} for details. Carroll manifolds and their conformal transformations can be introduced analogously \cite{17, 43, 44}. Skipping
technical details, we just mention that for a given positive integer $N$, the \textit{conformal Carroll transformations of level N} of a Carroll spacetime $C$ are generated by those vectorfields $Z$ on $C$ which satisfy \cite{43}, with some abuse of notation

$$L_Z G_{ij} = 2 \omega G_{ij} \quad \& \quad L_Z \xi = \psi \xi \quad \text{with} \quad 2 \omega + N \psi = 0. \quad (\text{VII.1})$$

Note that the definition is \textit{reminiscent of} the “Chrono-Projective” definition (II.23) except that it involves the space metric $G_{ij}$ only. The basic difference is that the space metric and the vertical vector (here the generator of Carrollian time) are only very loosely related.

Referring the reader to the literature for details, here we just recall that, in the flat case, the Conformal Carroll algebra of level $N$ is \cite{43}

$$Z = \left( \omega_j^i X^j + \gamma^i + \left( \chi - 2(\kappa \cdot X) \right) X^i + \kappa^i X^2 \right) \partial_i + \left( \frac{2}{N} \left( \chi - 2 \kappa \cdot X \right) V + F(X) \right) \partial_V \quad (\text{VII.2})$$

where $\omega_j^i \in \mathfrak{so}(2)$, $\gamma^i, \kappa^i \in \mathbb{R}^2$, $\chi \in \mathbb{R}$ and $F(X)$ is an \textit{arbitrary} function of the position alone, making our Lie algebra infinite-dimensional. It has conformal factors $\omega = \chi - 2 \kappa \cdot X$ and $\psi = -2 \omega/N$. Note that the conformal factor here \textit{can} depend on $X$. Comparison with (III.10) shows that the Carroll Lie algebra corresponds to choosing $\chi = 0$, $\kappa = 0$ and $T(X) = \nu - \beta \cdot X$.

The homothety (II.3) is extended in turn to the level-$N$ case as

$$Y_{hom}^{(N)} = \chi (X^i \partial_i + (2V/N) \partial_V). \quad (\text{VII.3})$$

\textbf{VIII. CONCLUSION}

Plane gravitational waves have long been known to have, generically, a 5-parameter isometry group \cite{10–14}, recently identified as Lévy-Leblond’s “Carroll” group with broken rotations \cite{15, 16}. It is also known that the isometry group can, in special instances, be enlarged to 6 parameters \cite{14, 47}. This is plainly the case when the profile is $U$-independent so that $U$-translations are isometries; a particular example is given by the Brdička metric (VI.6).

$U$-translations are manifestly broken when the profile depends on $U$, however these broken $U$-translations can, under special circumstances, be combined with another broken symmetry generator yielding an additional “screw-type” symmetry \cite{8, 9, 14, 47}. Examples are presented in (IV.25) and in sec.VI.
The memory problem requires finding the geodesics in the gravitational wave spacetime, and the integrability of the geodesic equations is related to \textit{conformal symmetry} [13, 20–22]. General theorems say that for non-trivial profile the maximum number of conformal vectors is 7 [14, 47] – a number in fact attained by the examples mentioned above. They all have a 6-parameter isometry group, augmented with just one “truly conformal generator” – namely the homothety, (II.3), which is a universal symmetry for all Brinkmann waves.

Identifying all conformal symmetries of a pp wave is a difficult task which requires a series of constraints to be satisfied whose solution depends on the chosen profile [47] and is found case-by-case only.

It is remarkable that all cases we studied in sec.VI have an additional property: for all of them the covariantly constant null vector $\xi$ characteristic for pp waves [23] is Lie-transported parallel to itself,

$$L_Y \xi = \psi \xi$$

(VIII.1)

for some function $\psi$, cf. (I.3) or (II.23b). Conformal transformations with such additional property were considered by Duval et al. some time ago [25, 28–31] under the name of \textit{Chrono-Projective transformations}.

Here we just mention that for an “antiboost” $Y \equiv Y^1 = X^1 \partial_U - V \partial^1$ (III.21d) [which is a generator of the Poincaré group which is \textit{not} Chrono-Projective], for example, the $UU$ component of the Killing equations does not vanish, $(L_Y g)_{UU} = X^1 \dot{K}_{ij} X^i X^j - 2V K_{1j} X^j \neq 0$ : an antiboost is not an isometry when the profile is non-trivial. An additional bonus of Chrono-Projective transformations is that they are much simpler to study as general conformal transformations.

The particular role of the homothety (II.3) is understood as follows. Remember first that the conserved quantities associated with the \textit{isometries} allow us to determine the transverse-space trajectory $X(\sigma)$ in (II.6a) [10, 16]. Knowing $X(\sigma)$ allows us to determine also the vertical coordinate $V(\sigma)$ by \textit{integrating} the $V$-equation (II.6b). Now the homothety is an additional symmetry for null geodesics which generates the conserved quantity $Q_{\text{hom}}$ in (IV.18) and $V(\sigma)$ is in fact determined by $Q_{\text{hom}}$ by eqn (IV.19).
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