SIMPLE LABELED GRAPH $C^*$-ALGEBRAS ARE ASSOCIATED TO DISAGREEABLE LABELED SPACES

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Abstract. By a labeled graph $C^*$-algebra we mean a $C^*$-algebra associated to a labeled space $(E, L, \mathcal{E})$ consisting of a labeled graph $(E, L)$ and the smallest normal accommodating set $\mathcal{E}$ of vertex subsets. Every graph $C^*$-algebra $C^*(E)$ is a labeled graph $C^*$-algebra and it is well known that $C^*(E)$ is simple if and only if the graph $E$ is cofinal and satisfies Condition (L). Bates and Pask extend these conditions of graphs $E$ to labeled spaces, and show that if a set-finite and receiver set-finite labeled space $(E, L, \mathcal{E})$ is cofinal and disagreeable, then its $C^*$-algebra $C^*(E, L, \mathcal{E})$ is simple. In this paper, we show that the converse is also true.

1. Introduction

A class of $C^*$-algebras associated to directed graphs including the Cuntz-Krieger algebras [9] was introduced in [18, 19], and since then its generalizations have attracted much attention of many authors. The $C^*$-algebras associated to ultragraphs, infinite matrices, higher-rank graphs, subshifts, Boolean dynamical systems, and labeled spaces are examples of the generalizations (see [1, 3, 4, 7, 10, 11, 20, 22] among many others).

One of the main topics dealt with in the study of these generalized Cuntz-Krieger algebras is to describe the ideal structure of a $C^*$-algebra in question in terms of structural properties of the object to which the $C^*$-algebra is associated. The ideal structure of a graph $C^*$-algebra is now well understood, and if we recall it for a row-finite graph $E$ with no singular vertices, it says that there exists a one to one correspondence between the gauge-invariant ideals of the graph $C^*$-algebra $C^*(E)$ and the hereditary saturated vertex subsets of the graph $E$ ([2, 4, 10]), and moreover $C^*(E)$ is simple if and only if $E$ is cofinal and satisfies Condition (L) ([10, 18]). Here the gauge action is the action of the unit circle on a graph $C^*$-algebra which always exists because of the universal property of a graph $C^*$-algebra. Many authors put a great deal of effort to extend this result to the classes of generalized Cuntz-Krieger algebras, and in this paper we will look at the labeled graph $C^*$-algebras and focus on the question of when these algebras are simple.

If $(E, L)$ is a labeled graph, that is, $L : E^1 \to A$ is a labeling map of the edges $E^1$ onto an alphabet $A$, then as we will review in the next section, one can consider a collection $B$ consisting of certain vertex subsets so that a universal family of projections $\{p_A : A \in B\}$ and partial isometries $\{s_a : a \in A\}$ satisfying the relations imposed

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by the triple \((E, \mathcal{L}, \mathcal{B})\) exists and thus one can form the \(C^*-\)algebra \(C^*(E, \mathcal{L}, \mathcal{B})\) generated by this universal family of operators \(\{p_A, s_a\}\). We call \(C^*(E, \mathcal{L}, \mathcal{B})\) the \(C^*-\)algebra of a labeled space \((E, \mathcal{L}, \mathcal{B})\). Particularly, if \(\mathcal{E}\) is the smallest normal accommodating set, we will simply call \(C^*(E, \mathcal{L}, \mathcal{E})\) the labeled graph \(C^*-\)algebra of \((E, \mathcal{L})\) for convenience. In this paper, we will be mostly interested in these labeled graph \(C^*-\)algebras \(C^*(E, \mathcal{L}, \mathcal{E})\).

Every graph \(C^*-\)algebra is a labeled graph \(C^*-\)algebra (\cite[Example 5.1]{1}) and the class of Morita equivalence classes of \(C^*-\)algebras of labeled spaces strictly contains the class of Morita equivalence classes of graph \(C^*-\)algebras (see \cite[Remark 5.2]{1} and \cite[Theorem 3.7]{15}). By the universal property of a labeled graph \(C^*-\)algebra \(C^*(E, \mathcal{L}, \mathcal{B})\), there exists a gauge action of the unit circle on \(C^*(E, \mathcal{L}, \mathcal{B})\), and it is known \cite[14]{14} that if \(E\) has no sinks and \((E, \mathcal{L}, \mathcal{B})\) is a set-finite and receiver set-finite normal labeled space, there is a one to one correspondence between the gauge-invariant ideals of \(C^*(E, \mathcal{L}, \mathcal{B})\) and the hereditary saturated subsets of \(\mathcal{B}\). (A gauge invariant uniqueness theorem \cite[Theorem 5.3]{3} used in \cite[14]{14} was turned out to be incorrect, but was corrected in \cite[Theorem 2.7]{6} for normal labeled spaces and in \cite[11]{11} for general labeled spaces.)

Bates and Pask \cite{4} considered the question of when a \(C^*-\)algebra \(C^*(E, \mathcal{L}, \mathcal{E})\) of a set-finite and receiver set-finite labeled space \((E, \mathcal{L}, \mathcal{E})\) is simple, and proved that \(C^*(E, \mathcal{L}, \mathcal{E})\) is simple if \((E, \mathcal{L}, \mathcal{E})\) is cofinal and disagreeable. The notion of a disagreeable labeled space \((E, \mathcal{L}, \mathcal{E})\) introduced in \cite{4} is an analogue of Condition (L) of usual directed graphs. The cofinal condition for \((E, \mathcal{L}, \mathcal{E})\) used in \cite{4} needs to be modified to obtain the simplicity result for \(C^*(E, \mathcal{L}, \mathcal{E})\) as noted in \cite[Remark 3.15]{12} where a condition called strongly cofinal was used instead. The definition of a strongly cofinal labeled space given in \cite{12} is weaker than the one in this paper (see Definition \cite[2.10]{2} or \cite[Section 2.5]{16}), and throughout the present paper we mean Definition \cite[2.10]{2} if we mention strong cofinality. It then follows from \cite[Theorem 3.16]{12} that \(C^*(E, \mathcal{L}, \mathcal{E})\) is simple whenever \((E, \mathcal{L}, \mathcal{E})\) is disagreeable and strongly cofinal.

As for the converse of Bates and Pask’s simplicity result, the strong cofinality of \((E, \mathcal{L}, \mathcal{E})\) can be derived as in \cite[Theorem 3.8]{12} by slightly modifying the proof there (see Theorem \cite[2.11]{2}). On the other hand, it is not clear whether the labeled space \((E, \mathcal{L}, \mathcal{E})\) has to be disagreeable when its \(C^*-\)algebra \(C^*(E, \mathcal{L}, \mathcal{E})\) is simple, and this is the question we will consider in this paper. From a recent result \cite[Theorem 9,16]{8} on simplicity of a \(C^*-\)algebra associated to a Boolean dynamical system, we know that for a labeled space \((E, \mathcal{L}, \mathcal{E})\) whose Boolean dynamical system satisfies a sort of domain condition, the \(C^*-\)algebra \(C^*(E, \mathcal{L}, \mathcal{E})\) is simple if and only if \((E, \mathcal{L}, \mathcal{E})\) has no cycles without an exit and there are no nonempty hereditary saturated subsets of \(\mathcal{E}\). The first condition of having no cycles without an exit is always satisfied whenever the labeled space is disagreeable while the converse does not hold in general (Proposition \cite[3.2]{3}), and the second condition of having no nonempty saturated hereditary subsets is equivalent to the absence of gauge-invariant proper ideals in \(C^*(E, \mathcal{L}, \mathcal{E})\). Thus the question of whether the converse of Bates and Pask’s simplicity result holds true is not answered directly from \cite{8} while it is known \cite[Theorem 3.14]{12} that the converse holds if \(\mathcal{E}\) contains \(\{v\}\) for every vertex \(v\) in \(E\).
The purpose of the present paper is, as mentioned above, to figure out whether the converse of Bates and Pask’s simplicity result holds and it is proved in Theorem 3.7 that the labeled space \((E, L, \mathcal{E})\) is always disagreeable if \(C^*(E, L, \mathcal{E})\) is simple. This establishes the following: a labeled graph \(C^*\)-algebra \(C^*(E, L, \mathcal{E})\) is simple if and only if \((E, L, \mathcal{E})\) is strongly cofinal and disagreeable.

2. Preliminaries

In this section we set up notation and review definitions and basic results we need in the next section. For more details, we refer the reader to [1] or [16].

A directed graph \(E = (E^0, E^1, r, s)\) consists of the vertex set \(E^0\) and the edge set \(E^1\) together with the range, source maps \(r, s : E^1 \to E^0\). We call a vertex \(v \in E^0\) a sink (a source, respectively) if \(s^{-1}(v) = \emptyset\) \((r^{-1}(v) = \emptyset\), respectively). If every vertex in \(E\) emits only finitely many edges, \(E\) is called row-finite.

For each \(n \geq 1\), \(E^n\) denotes the set of all paths of length \(n\), and the vertices in \(E^0\) are regarded as finite paths of length zero. The maps \(r, s\) naturally extend to the set \(E^* = \bigcup_{n \geq 0} E^n\) of all finite paths, especially with \(r(v) = s(v) = v\) for \(v \in E^0\). By \(E^\infty\) we denote the set of all infinite paths \(x = \lambda_1 \lambda_2 \cdots\), where we define \(s(x) := s(\lambda_1)\). For \(A, B \subset E^0\) and \(n \geq 0\), we use the following notation

\[
AE^n := \{ \lambda \in E^n : s(\lambda) \in A \}, \quad E^n B := \{ \lambda \in E^n : r(\lambda) \in B \},
\]

and \(AE^n B := AE^n \cap E^n B\) with \(E^n v := E^n \{v\}\), \(v E^n := \{v\} E^n\). Also the sets of paths like \(E^{\geq k}\), \(AE^{\geq k}\), and \(AE^\infty\) which have their obvious meaning will be used.

A loop is a finite path \(\lambda \in E^{\geq 1}\) such that \(r(\lambda) = s(\lambda)\), and an exit of a loop \(\lambda\) is a path \(\delta \in E^{\geq 1}\) such that \(|\delta| \leq |\lambda|\), \(s(\delta) = s(\lambda)\), and \(\delta \neq \lambda_1 \cdots \lambda_{|\delta|}\). A graph \(E\) is said to satisfy Condition (L) if every loop has an exit.

Let \(A\) be a countable alphabet and let \(A^*\) (\(A^\infty\), respectively) denote the set of all finite words (infinite words, respectively) in symbols of \(A\). A labeled graph \((E, L)\) over \(A\) consists of a directed graph \(E\) and a labeling map \(L : E^1 \to A\) which is always assumed to be onto. Given a graph \(E\), one can define a so-called trivial labeling map \(L_{id} := id : E^1 \to E^1\) which is the identity map on \(E^1\) with the alphabet \(E^1\).

The labeling map naturally extends to any finite and infinite labeled paths, namely if \(\lambda = \lambda_1 \cdots \lambda_n \in E^n\), then \(L(\lambda) := L(\lambda_1) \cdots L(\lambda_n) \in L(E^n) \subset A^*\), and similarly to infinite paths. We often call these labeled paths just paths for convenience if there is no risk of confusion, and use notation \(L^*(E) := L(E^{\geq 1})\). For a vertex \(v \in E^0\) and a vertex subset \(A \subset E^0\), we set \(L(v) := v\) and \(L(A) := A\), respectively. A subpath \(\alpha_i \cdots \alpha_j\) of \(\alpha = \alpha_1 \alpha_2 \cdots \alpha_{|\alpha|} \in L^*(E)\) is denoted by \(\alpha_{[i,j]}\) for \(1 \leq i \leq j \leq |\alpha|\), and each \(\alpha_{[1,j]}\), \(1 \leq j \leq |\alpha|\), is called an initial path of \(\alpha\). The range and source of a path \(\alpha \in L^*(E)\) are defined to be the following sets of vertices

\[
r(\alpha) = \{ r(\lambda) : \lambda \in E^{\geq 1}, L(\lambda) = \alpha \},
\]

\[
s(\alpha) = \{ s(\lambda) : \lambda \in E^{\geq 1}, L(\lambda) = \alpha \},
\]

and the relative range of \(\alpha \in L^*(E)\) with respect to \(A \subset E^0\) is defined by

\[
r(A, \alpha) = \{ r(\lambda) : \lambda \in AE^{\geq 1}, L(\lambda) = \alpha \}.
\]

A collection \(\mathcal{B}\) of subsets of \(E^0\) is said to be closed under relative ranges for \((E, L)\) if \(r(A, \alpha) \in \mathcal{B}\) whenever \(A \in \mathcal{B}\) and \(\alpha \in L^*(E)\). We call \(\mathcal{B}\) an accommodating
set for \((E, \mathcal{L})\) if it is closed under relative ranges, finite intersections and unions and contains the ranges \(r(\alpha)\) of all paths \(\alpha \in \mathcal{L}^*(E)\). A set \(A \in \mathcal{B}\) is called minimal (in \(\mathcal{B}\)) if \(A \cap B\) is either \(A\) or \(\emptyset\) for all \(B \in \mathcal{B}\).

If \(\mathcal{B}\) is accommodating for \((E, \mathcal{L})\), the triple \((E, \mathcal{L}, \mathcal{B})\) is called a labeled space. We say that a labeled space \((E, \mathcal{L}, \mathcal{B})\) is set-finite (receiver set-finite, respectively) if for every \(A \in \mathcal{B}\) and \(k \geq 1\) the set \(\mathcal{L}(AE^k)\) (\(\mathcal{L}(E^kA)\), respectively) is finite. A labeled space \((E, \mathcal{L}, \mathcal{B})\) is said to be weakly left-resolving if

\[
r(\alpha, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)
\]

holds for all \(A, B \in \mathcal{B}\) and \(\alpha \in \mathcal{L}^*(E)\). If \(\mathcal{B}\) is closed under relative complements, we call \((E, \mathcal{L}, \mathcal{B})\) a normal labeled space.

**Notation 2.1.** For \(A \in \mathcal{E}\), we will use the following notation

\[
A \cap \mathcal{B} := \{B \in \mathcal{E} : B \subset A\}.
\]

**Assumptions.** Throughout this paper, we assume that graphs \(E\) have no sinks and sources, and labeled spaces \((E, \mathcal{L}, \mathcal{B})\) are weakly left-resolving, set-finite, receiver set-finite, and normal.

**Definition 2.2.** A representation of a labeled space \((E, \mathcal{L}, \mathcal{B})\) is a family of projections \(\{p_A : A \in \mathcal{B}\}\) and partial isometries \(\{s_a : a \in \mathcal{A}\}\) such that for \(A, B \in \mathcal{B}\) and \(a, b \in \mathcal{A}\),

(i) \(p_\emptyset = 0, p_{A \cap B} = p_A p_B,\) and \(p_{A \cup B} = p_A + p_B - p_{A \cap B}\),

(ii) \(p_A s_a = s_a p_{r(A, a)}\),

(iii) \(s_a^* s_a = p_{r(a)}\) and \(s_a^* s_b = 0\) unless \(a = b\),

(iv) \(p_A = \sum_{a \in \mathcal{L}(AE^1) s_a p_{r(A, a)}} s_a^*\).

It is known [11, 34] that given a labeled space \((E, \mathcal{L}, \mathcal{B})\), there exists a \(C^*\)-algebra \(C^*(E, \mathcal{L}, \mathcal{B})\) generated by a universal representation \(\{s_a, p_A\}\) of \((E, \mathcal{L}, \mathcal{B})\), so that if \(\{t_a, q_A\}\) is a representation of \((E, \mathcal{L}, \mathcal{B})\) in a \(C^*\)-algebra \(B\), there exists a \(\ast\)-homomorphism

\[
\phi : C^*(E, \mathcal{L}, \mathcal{B}) \to B
\]

such that \(\phi(s_a) = t_a\) and \(\phi(p_A) = q_A\) for all \(a \in \mathcal{A}\) and \(A \in \mathcal{B}\).

**Definition 2.3.** We call the \(C^*\)-algebra \(C^*(E, \mathcal{L}, \mathcal{B})\) generated by a universal representation of \((E, \mathcal{L}, \mathcal{B})\) the \(C^*\)-algebra of a labeled space \((E, \mathcal{L}, \mathcal{B})\).

The \(C^*\)-algebra \(C^*(E, \mathcal{L}, \mathcal{B})\) is unique up to isomorphism, and we simply write

\[
C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A)
\]

to indicate the generators \(s_a, p_A\) that are nonzero for all \(a \in \mathcal{A}\) and \(A \in \mathcal{B}\), \(A \neq \emptyset\).

**Remark 2.4.** Let \((E, \mathcal{L}, \mathcal{B})\) be a labeled space with \(C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A)\). By \(\epsilon\), we denote a symbol (not in \(\mathcal{L}^*(E)\)) such that \(ae = ea, r(\epsilon) = E^0\), and \(r(A, \epsilon) = A\) for all \(a \in \mathcal{A}\) and \(A \subset E^0\). We write \(\mathcal{L}^\#(E)\) for the union \(\mathcal{L}^*(E) \cup \{\epsilon\}\). Let \(s_\epsilon\)
denote the unit of the multiplier algebra of $C^*(E, \mathcal{L}, \mathcal{B})$. Then one can easily check the following

\[
(s_\alpha p_A s_\beta)(s_\gamma p_B s_\delta) = \begin{cases} 
  s_{\alpha\gamma'} p_{(A, \gamma') \cap B} s_{\beta}^*, & \text{if } \gamma = \beta' \\
  s_{\alpha A \cap (B, \beta')} s_{\beta}^*, & \text{if } \beta = \gamma' \\
  s_{\alpha A \cap B} s_{\delta}^*, & \text{if } \beta = \gamma \\
  0, & \text{otherwise},
\end{cases}
\]

for $\alpha, \beta, \gamma, \delta \in \mathcal{L}(E)$ and $A, B \in \mathcal{B}$ (see [3, Lemma 4.4]). Since $s_\alpha p_A s_\beta \neq 0$ if and only if $A \cap r(\alpha) \cap r(\beta) \neq \emptyset$, we have

\[
C^*(E, \mathcal{L}, \mathcal{B}) = \overline{\text{span}} \{ s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}(E) \text{ and } A \subseteq r(\alpha) \cap r(\beta) \}.
\]

For a labeled graph $(E, \mathcal{L})$, there are many accommodating sets to be considered to form a labeled space, and the $C^*$-algebras $C^*(E, \mathcal{L}, \mathcal{B})$ are not necessarily isomorphic to each other, in general. By $\mathcal{E}$ we denote the smallest accommodating set for which $(E, \mathcal{L}, \mathcal{E})$ is a normal labeled space. We are mostly interested in the $C^*$-algebras of these labeled spaces $(E, \mathcal{L}, \mathcal{E})$ throughout this paper.

For each $l \geq 1$, the relation $\sim_l$ on $E^0$ given by $v \sim_l w$ if and only if $\mathcal{L}(E^{\leq l}v) = L(E^{\leq l}w)$ is an equivalence relation, and the equivalence class $[v]_l$ of $v \in E^0$ is called a generalized vertex (or a vertex simply). If $k > l$, then $[v]_l \subseteq [v]_k$ is obvious and $[v]_l = \bigcup_{i=1}^n [v]_{l+1}$ for some vertices $v_1, \ldots, v_m \in [v]_l$ ([4, Proposition 2.4]). Moreover, we have

\[
\mathcal{E} = \left\{ \bigcup_{i=1}^n [v_i] : v_i \in E^0, \ l \geq 1, n \geq 0 \right\},
\]

with the convention $\sum_{i=1}^0 [v_i] := \emptyset$ by [13, Proposition 2.3].

Recall that a Cuntz-Krieger $E$-family for a graph $E$ is a representation of the labeled space $(E, \mathcal{L}_{id}, \mathcal{E})$ with the trivial labeling, and the Cuntz-Krieger uniqueness theorem for graph $C^*$-algebras says that if $E$ satisfies Condition (L), then every Cuntz-Krieger $E$-family of nonzero operators generates the same $C^*$-algebra $C^*(E)$ up to isomorphism (for example, see [5, Theorem 3.1], [10, Corollary 2.12], and [18, Theorem 3.7]). A condition of a labeled space corresponding to Condition (L) of a directed graph was suggested in [4] as below, and it is shown there in [4, Lemma 5.3] that for a graph $E$, the labeled space $(E, \mathcal{L}_{id}, \mathcal{B})$ with the trivial labeling is disagreeable if and only if $E$ satisfies Condition (L).

**Definition 2.5.** ([4, Definition 5.2]) A path $\alpha \in \mathcal{L}^*(E)$ with $s(\alpha) \cap [v]_l \neq \emptyset$ is called agreeable for $[v]_l$ if $\alpha = \beta \alpha' = \alpha' \gamma$ for some $\alpha', \beta, \gamma \in \mathcal{L}^*(E)$ with $|\beta| = |\gamma| \leq l$. Otherwise $\alpha$ is called disagreeable. We say that $[v]_l$ is disagreeable if there is an $N \geq 1$ such that for all $n > N$ there is an $\alpha \in \mathcal{L}(E^{\leq n})$ which is disagreeable for $[v]_l$.

A labeled space $(E, \mathcal{L}, \mathcal{B})$ is said to be disagreeable if for every $v \in E^0$, there is an $L_v \geq 1$ such that every $[v]_{l}$ is disagreeable for all $l \geq L_v$.

It is then natural to ask whether every loop in a disagreeable labeled space must have an exit, which first leads us to try to seek a right definition for a loop in a labeled space and then to work on whether the important results known for graph $C^*$-algebras $C^*(E)$ which involve the loop structure of $E$ can be generalized to
labeled graph $C^*$-algebras. We take the following definition and will see in the next section that every loop in a disagreeable labeled space has an exit.

**Definition 2.6.** ([13, Definition 3.2]) Let $(E, L, E)$ be a labeled space. For a path $\alpha \in L^*(E)$ and a nonempty set $A \in \mathcal{E}$, we call $(\alpha, A)$ a loop if

$$A \subset r(A, \alpha).$$

We say that a loop $(\alpha, A)$ has an exit if one of the following holds:

(I) there exists a path $\beta \in L(AE \geq 1)$ such that $|\beta| = |\alpha|$, $\beta \neq \alpha$,

(II) $A \subset r(A, \alpha)$.

In [8], the notion of cycle was introduced to define Condition $(L_B)$ for a labeled space $(E, L, B)$ (more generally for Boolean dynamical systems) which can be regarded as another condition analogous to Condition (L) for usual directed graphs.

**Definition 2.7.** ([8, Definition 9.5]) For $\alpha \in L^*(E)$ and a nonempty $A \in \mathcal{E}$, the pair $(\alpha, A)$ is called a cycle if $B = r(B, \alpha)$ for all $B \in A \cap \mathcal{E}$.

Clearly every cycle is a loop, and if $(\alpha, A)$ is a cycle with an exit, then the exit must be of type (I).

If $(E, L, \mathcal{E})$ is a labeled space satisfying our standing assumptions and if, in addition, for each path $\alpha \in L^*(E)$,

$$r(D_\alpha, \alpha) = r(\alpha)$$

(1)

(that is, every path has a domain in $\mathcal{E}$), then the labeled graph $C^*$-algebra $C^*(E, L, \mathcal{E})$ can be regarded as a $C^*$-algebra associated to a Boolean dynamical system as discussed in [8] Example 11.1]. A Boolean dynamical system on a Boolean algebra $B$ is said to satisfy Condition $(L_B)$ if it has no cycle without an exit: if this is the case for the Boolean dynamical system induced from a labeled space $(E, L, \mathcal{E})$, we will simply say that $(E, L, \mathcal{E})$ satisfies Condition $(L_\mathcal{E})$.

Theorem 2.8 below is the Cuntz-Krieger uniqueness theorem for labeled graph $C^*$-algebras: if $(E, L, \mathcal{E})$ is disagreeable, it satisfies Condition $(L_\mathcal{E})$ (see Proposition 3.2 or [16 Proposition 3.7]). We need to understand Condition $(L_\mathcal{E})$ and disagreeability of labeled spaces not only to answer the simplicity question of labeled graph $C^*$-algebras, but also to be able to apply this useful uniqueness theorem for labeled graph $C^*$-algebras.

**Theorem 2.8.** ([3, Theorem 5.5], [8, Theorem 9.9]) Let $\{t_a, q_A\}$ be a representation of a labeled space $(E, L, \mathcal{E})$ such that $q_A \neq 0$ for all nonempty $A \in \mathcal{E}$. If $(E, L, \mathcal{E})$ satisfies condition $(L_\mathcal{E})$, in particular if $(E, L, \mathcal{E})$ is disagreeable, then the canonical homomorphism $\phi : C^*(E, L, \mathcal{E}) = C^*(s_a, p_A) \to C^*(t_a, q_A)$ such that $\phi(s_a) = t_a$ and $\phi(p_A) = q_A$ is an isomorphism.
Definition 2.9. ([14] Definition 3.4) A subset $H$ of $\mathcal{E}$ is hereditary if it is closed under subsets, finite unions, and relative ranges in $\mathcal{E}$. A hereditary set $H$ is saturated if $A \in H$ whenever $A \in \mathcal{E}$ and $r(A, \alpha) \in H$ for all $\alpha \in \mathcal{L}^*(E)$.

Let $\overline{\mathcal{L}(E^\infty)}$ be the set of all infinite sequences $x \in \mathcal{A}^\infty$ such that every finite words of $x$ occurs as a labeled path in $(E, \mathcal{L})$, namely

$$\overline{\mathcal{L}(E^\infty)} := \{x \in \mathcal{A}^\infty \mid x_{[1,n]} \in \mathcal{L}(E^n) \text{ for all } n \geq 1\}.$$  

Clearly $\mathcal{L}(E^\infty) \subset \overline{\mathcal{L}(E^\infty)}$, and the notation $\overline{\mathcal{L}(E^\infty)}$ comes from the fact that $\overline{\mathcal{L}(E^\infty)}$ is the closure of $\mathcal{L}(E^\infty)$ in the totally disconnected perfect space $\mathcal{A}^\infty$ which is equipped with the topology that has a countable basis of open-closed cylinder sets $Z(\alpha) := \{x \in \mathcal{A}^\infty : x_{[1,n]} = \alpha\}, \alpha \in \mathcal{A}^n$, $n \geq 1$ (see Section 7.2 of [17]).

Definition 2.10. ([16] Section 2.5) We say that a labeled space $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal if for each $x \in \overline{\mathcal{L}(E^\infty)}$ and $[v]_l \in \mathcal{E}$, there exist an $N \geq 1$ and a finite number of paths $\lambda_1, \ldots, \lambda_m \in \mathcal{L}^*(E)$ such that

$$r(x_{[1,N]}) \subseteq \bigcup_{i=1}^m r([v]_l, \lambda_i).$$

The above definition of a strongly cofinal labeled space is stronger than the one given in [12]: for example, in the following labeled space

\[
\begin{array}{ccccccccc}
\ldots & \bullet & a & \bullet & a & \bullet & a & \bullet & \ldots \\
\downarrow v_{-4} & v_{-3} & v_{-2} & v_{-1} & v_0 & v_1 & v_2 & v_3 & v_4 \\
\end{array}
\]

if $x := a^\infty \in \overline{\mathcal{L}(E^\infty)} \setminus \mathcal{L}(E^\infty)$ and $N, n, l \geq 1$, then

$$r(x_{[1,N]}) = r(a^N) = r(a) = \{v_{-k} : k \geq 0\} \not\subseteq \bigcup_{i=1}^m r([v]_l, \lambda_i)$$

for any paths $\lambda_1, \ldots, \lambda_m \in \mathcal{L}^*(E)$, namely the labeled space is not strongly cofinal although it is in the sense of [12]. The result [12] Theorem 3.8 can be improved as below with a slightly modified proof which we provide here for the sake of readers’ convenience.

Theorem 2.11. If $C^*(E, \mathcal{L}, \mathcal{E})$ is simple, then $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal.

Proof. Suppose to the contrary that there exist $[v]_l$ and $x \in \overline{\mathcal{L}(E^\infty)}$ such that

$$r(x_{[1,N]}) \not\subseteq \bigcup_{i=1}^m r([v]_l, \lambda_i)$$

for all $N \geq 1$ and any finite number of labeled paths $\lambda_1, \ldots, \lambda_m$. Let $I$ be the ideal generated by the projection $p_{[v]_l}$ and let $p_{x_1} := p_{r(x_1)}$. Since $C^*(E, \mathcal{L}, \mathcal{E})$ is simple, we must have $p_{x_1} \in I$ and thus there is an element $\sum_{j=1}^m c_j (s_{\alpha_j} p_{A_j} s_{\beta_j}^*) p_{[v]_l} (s_{\gamma_j} p_{B_j} s_{\delta_j}^*)$ in $I$ with $c_j \in \mathbb{C}$ such that

$$\| \sum_{j=1}^m c_j (s_{\alpha_j} p_{A_j} s_{\beta_j}^*) p_{[v]_l} (s_{\gamma_j} p_{B_j} s_{\delta_j}^*) - p_{x_1} \| < 1$$

(3)
and the paths $\delta_j$'s have the same length $|\delta_j| = N_0 \geq 1$. Then
\[
1 > \left\| \sum_j c_j(s_{\alpha_j} p_{A_j} s_{\beta_j}^* p_{[v]} (s_{\gamma_j} p_{B_j} s_{\delta_j}) - p_{x_1}) \right\|
\]
\[
\geq \left\| \sum_j c_j(s_{\alpha_j} p_{A_j} s_{\beta_j}^* p_{[v]} (s_{\gamma_j} p_{B_j} s_{\delta_j}) p_{x_1} - p_{x_1}) \right\|
\]
\[
= \left\| \sum_j c_j(s_{\alpha_j} p_{A_j} s_{\beta_j}^* p_{[v]} (s_{\gamma_j} p_{r([v], \gamma_j)} (x_1 \delta_j) s_{\delta_j}) - p_{x_1}) \right\|.
\]

We first show that for each $j = 1, \ldots, m$,
\[
r(x_1 \delta_j) \subset \bigcup_{i=1}^m r([v], \gamma_i).
\]
(4)
If $r(x_1 \delta_j) \not\subset \bigcup_{i=1}^m r([v], \gamma_i)$ for some $j$, then $r(x_1 \delta_j) \setminus \bigcup_{i=1}^m r([v], \gamma_i) \neq \emptyset$ hence $p_j := p_{r(x_1 \delta_j) \setminus \bigcup_{i=1}^m r([v], \gamma_i)} \neq 0$. Then with $J := \{i \mid \delta_i = \delta_j\}$,
\[
1 > \left\| \left( \sum_i c_i(s_{\alpha_i} p_{A_i} s_{\beta_i}^* p_{[v]} (s_{\gamma_i} p_{r([v], \gamma_i)} (x_1 \delta_i) s_{\delta_i}) - p_{x_1}) s_{\delta_i} \right) \right\|
\]
\[
= \left\| \sum_{i \in J} c_i(s_{\alpha_i} p_{A_i} s_{\beta_i}^* p_{[v]} s_{\gamma_i} p_{r([v], \gamma_i)} (x_1 \delta_i) - p_{x_1} s_{\delta_i}) \right\|
\]
\[
= \left\| \sum_{i \in J} c_i(s_{\alpha_i} p_{A_i} s_{\beta_i}^* p_{[v]} s_{\gamma_i} p_{r([v], \gamma_i)} (x_1 \delta_i) - s_{\delta_i} p_{r(x_1 \delta_j)} ) \right\|
\]
\[
\leq \left\| \sum_{i \in J} c_i(s_{\alpha_i} p_{A_i} s_{\beta_i}^* p_{[v]} s_{\gamma_i} p_{r([v], \gamma_i)} (x_1 \delta_i) p_j - s_{\delta_i} p_{r(x_1 \delta_j)} p_j \right\|.
\]
\[
= \left\| s_{\delta_j} p_j \right\| = 1,
\]
which is a contradiction and (4) follows. Also $\delta_i \neq x_{[2, N_0+1]}$ for each $1 \leq i \leq m$. In fact, if $\delta_i = x_{[2, N_0+1]}$ for some $i$, then by (4),
\[
r(x_{[1, N_0+1]}) = r(x_1 \delta_i) \subset \bigcup_{j=1}^m r([v], \gamma_j),
\]
which is not possible because of (2). Thus $s_{\delta_i}^* s_{x_{[2, N_0+1]}} = 0$ for $i = 1, \ldots, m$. Then the partial isometry $y := p_{x_1} s_{x_{[2, N_0+1]}} = s_{x_{[2, N_0+1]}} p_{r(x_{[1, N_0+1]})}$ is nonzero since $s_{x_{[2, N_0+1]}}^* y = p_{r(x_{[1, N_0+1]})} \neq 0$, and $s_{\delta_i}^* y = s_{\delta_i}^* p_{x_1} s_{x_{[2, N_0+1]}} = p_{r(x_1 \delta_i)} s_{\delta_i}^* s_{x_{[2, N_0+1]}} = 0$ for all $i$. From (3), we have
\[
1 > \left\| \sum_{i=1}^m c_i(s_{\alpha_i} p_{A_i} s_{\beta_i}^* p_{[v] i} (s_{\gamma_i} p_{B_i} s_{\delta_i}).yy^* - p_{x_1} yy^* \right\| = \left\| yy^* \right\| = 1,
\]
a contradiction, and we conclude that $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal. \hfill \Box

3. **Disagreeable labeled spaces of simple labeled graph $C^*$-algebras**

In this section we prove that simplicity of a labeled graph $C^*$-algebra implies that the labeled space is disagreeable. For this, we will use condition (c) of the following lemma which is equivalent to disagreeability of a labeled space since the original definition of disagreeability seems a little complicated as recalled in Definition 2.5.
Lemma 3.1. ([16] Proposition 3.2) For a labeled space \((E, \mathcal{L}, \mathcal{E})\), the following are equivalent:

(a) \((E, \mathcal{L}, \mathcal{E})\) is disagreeable.
(b) \([v]_l \) is disagreeable for all \(v \in E^0\) and \(l \geq 1\).
(c) For any nonempty set \(A \in \mathcal{E}\) and a path \(\beta \in L^*(E)\), there is an \(n \geq 1\) such that \(L(AE^{[\beta | n]} \neq \{\beta^n\}\).

If \((E, \mathcal{L}, \mathcal{E})\) is disagreeable, then it satisfies Condition \((L_\mathcal{E})\) as shown in [16] Proposition 3.7 and [8, Example 11.1]. But the the converse is not true in general as we see from the following proposition.

Proposition 3.2. Consider the following conditions of a labeled space \((E, \mathcal{L}, \mathcal{E})\).

(a) \((E, \mathcal{L}, \mathcal{E})\) is disagreeable.
(b) Every loop in \((E, \mathcal{L}, \mathcal{E})\) has an exit.
(c) \((E, \mathcal{L}, \mathcal{E})\) satisfies \((L_\mathcal{E})\), that is, every cycle has an exit.

Then \((a) \Rightarrow (b) \Rightarrow (c)\) hold. But the other implications are not true, in general.

Proof. \((b) \Rightarrow (c)\) is clear since every cycle is a loop.

\((a) \Rightarrow (b)\) Let \((A, \alpha)\) be a loop so that \(A \subseteq r(A, \alpha)\). If \(A \subseteq r(A, \alpha)\), then the loop has an exit of type \((\Pi)\). So we may assume that \(A = r(A, \alpha)\). By Lemma 3.1 there is an \(n \geq 1\) such that \(L(AE^{[\alpha | n]} \neq \{\alpha^n\}\). Choose \(\beta \in L(AE^{[\alpha | n]})\) with \(\beta \neq \alpha^n\). If \(\beta_{[1, |\alpha|]} \neq \alpha\), then \(L(AE^{[\alpha | n]}) \neq \{\alpha\}\) and \((A, \alpha)\) has an exit of type \((\Pi)\). If \(\beta := \alpha^k\delta\) for some \(1 \leq k \leq n - 1\) and \(\delta \in L^*(E)\) with \(\delta_{[1, |\alpha|]} \neq \alpha\), then the loop \((A, \alpha)\) has an exit of type \((\Pi)\) since from \(A = r(A, \alpha^k)\) we have

\[\alpha \neq \delta_{[1, |\alpha|]} \in L(r(A, \alpha^k)E^{[\alpha]}) = L(AE^{[\alpha]}).\]

\((b) \not\Rightarrow (a)\) The labeled space \((E, \mathcal{L}, \mathcal{E})\) of the following labeled graph is obviously not disagreeable while \((b)\) is trivially satisfied since it has no loops.

\[\cdots \bullet -4 \cdot \bullet -3 \cdot \bullet -2 \cdot \bullet -1 \bullet \alpha \bullet \alpha \bullet \alpha \bullet \alpha \bullet \cdots \]

\((c) \not\Rightarrow (b)\) and \((c) \not\Rightarrow (a)\) Note that the labeled space \((E, \mathcal{L}, \mathcal{E})\) of the following labeled graph, which is not disagreeable clearly, has loops \((A_i, \alpha)\), \(i = 0, 1\), where \(A_0 = \{v_0\}\) and \(A_1 := r(\alpha) = \{v_0, v_1, \ldots\}\). The loop \((A_1, \alpha)\) has no exits while \((A_0, \alpha)\) has an exit of type \((\Pi)\). Thus \((b)\) is not satisfied for \((E, \mathcal{L}, \mathcal{E})\). But the labeled space has no cycles and thus \((c)\) is trivially satisfied.

\[\cdots \bullet -4 \cdot \bullet -3 \cdot \bullet -2 \cdot \bullet -1 \bullet \alpha \bullet \alpha \bullet \alpha \bullet \alpha \bullet \cdots. \]

Lemma 3.3. ([16] Lemma 4.6) Let a labeled space \((E, \mathcal{L}, \mathcal{E})\) have a loop \((A, \beta)\) without an exit. If \(A\) is a minimal set in \(\mathcal{E}\), then the \(C^*\)-algebra \(C^*(E, \mathcal{L}, \mathcal{E})\) has a
hereditary subalgebra which is isomorphic to $M_n(C(T))$ for some $n \geq 1$, in particular $C^*(E, \mathcal{L}, \mathcal{E})$ is not simple.

**Lemma 3.4.** For a nonempty set $A \in \mathcal{E}$, let $H_A := \{ \bigcup_{i=1}^k C_i : k \geq 1, C_i \in r(A, \beta) \cap \mathcal{E}, \beta \in \mathcal{L}^\#(E) \}$. Then $H_A$ is a hereditary subset of $\mathcal{E}$ and $\bar{H}_A := \{ B \in \mathcal{E} : \exists n \geq 1 \text{ such that } r(B, \alpha) \in H_A \text{ for all } \alpha \in \mathcal{L}(E^{\geq n}) \}$ is a hereditary saturated subset of $\mathcal{E}$ with $H_A \subset \bar{H}_A$.

**Proof.** It is easy to check that $H_A$ is a hereditary set. For convenience, we write a number $n$ in the definition of $\bar{H}_A$ for $B \in \mathcal{E}$ as $n_B$ although it is not unique. Clearly $\bar{H}_A$ is closed under subsets. Let $B \in \bar{H}_A$, then $r(B, \alpha) \in H_A$ for all $\alpha \in \mathcal{L}(E^{\geq n_B})$. Hence $\bar{H}_A$ is closed under finite unions. Let $B \in \bar{H}_A$ and $|\sigma| \geq 1$. Then $r(r(B, \sigma), \alpha) = r(B, \sigma \alpha) \in H_A$ whenever $|\alpha| \geq n_B$ because then $|\sigma \alpha| \geq n_B$. Thus $\bar{H}_A$ is also closed under relative ranges, which shows that $\bar{H}_A$ is a hereditary subset of $\mathcal{E}$. To see that $\bar{H}_A$ is saturated, let $B \in \mathcal{E}$ satisfy $r(B, \alpha) \in H_A$ for all paths $\alpha$ with $|\alpha| \geq 1$. We have to show that $B \in \bar{H}_A$. Since our labeled space is assumed to be set-finite, there are only finitely many labeled edges, say $\delta_1, \ldots, \delta_k$, emitting out of $B$. Then $r(B, \delta_i) \in H_A$ for each $i$, and thus there is an $n_i \geq 1$ such that $r(r(B, \delta_i), \alpha) \in H_A$ for all $\alpha \in \mathcal{L}(E^{\geq n_i})$. For $n := \max_{1 \leq i \leq k} \{ n_i \}$, we then have $r(B, \delta_i \alpha) = r(r(B, \delta_i), \alpha) \in H_A$ whenever $|\alpha| \geq n$ and $1 \leq i \leq k$. This means that $r(B, \alpha) \in H_A$ for all $\alpha$ with $|\alpha| \geq n + 1$. Thus $B \in \bar{H}_A$ follows as desired. 

**Notation 3.5.** For a path $\beta := \beta_1 \cdots \beta_{|\beta|} \in \mathcal{L}^*(E)$, let $\bar{\beta} := \beta \beta \cdots$ denote the infinite repetition of $\beta$, namely $\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_3 \cdots = \beta_1 \cdots \beta_{|\beta|} \beta_1 \cdots \beta_{|\beta|} \cdots$. Then for each $j \geq 1$, we have $\bar{\beta}_j = \beta_k$ for some $1 \leq k \leq |\beta|$ with $k = j \pmod{|\beta|}$. The initial path $\bar{\beta}_1 \cdots \bar{\beta}_j$ of $\bar{\beta}$, $j \geq 1$, is denoted by $\bar{\beta}_{[1,j]}$ as before.

We will call a path $\beta \in \mathcal{L}^*(E)$ irreducible if it is not a repetition of its proper initial path. The following Lemma 3.6 will be used to derive a contradiction in the proof of Theorem 3.7 but then we see from Theorem 3.7 that there does not exist a labeled space $(E, \mathcal{L}, \mathcal{E})$ satisfying the assumptions of Lemma 3.6.

**Lemma 3.6.** Let $C^*(E, \mathcal{L}, \mathcal{E})$ be a simple $C^*$-algebra and $A_0 \in \mathcal{E}$ be a nonempty set. If there exists an irreducible path $\beta$ such that $\mathcal{L}(A_0 E^{\geq 1}) = \{ \bar{\beta}_{[1,n]} : n \geq 1 \} = \{ \beta^m \beta' : m \geq 0, \beta' \text{ is an initial path of } \beta \}$, (5) or equivalently $\mathcal{L}(A_0 E^{n|\beta|}) = \{ \beta^n \}$ for all $n \geq 1$, then the following hold.
(i) There is an \( N \geq 1 \) such that for all \( k \geq 1 \),

\[
\mathcal{L}(D^E) = \{ \beta_{[j_0+1,j_0+r]} \beta_{[j_0+r+1,|\beta|]} \beta_{[1,j_0]} \} = \{ \beta_{[j_0+1,|\beta|]} \beta_{[1,j_0]} \}.
\]

But then the subpaths

\[
\mu := \beta_{[j_0+1,|\beta|]} \quad \text{and} \quad \nu := \beta_{[j_0+r+1,|\beta|]} \beta_{[1,j_0]}
\]

of \( \beta \) satisfy \( \mu \nu = \nu \mu \), which contradicts to irreducibility of \( \beta \) (see [16] Lemma 3.1).

(i) Since \( \tilde{H}_{A_0} \) is a nonempty hereditary saturated set by Lemma 3.1 and \( C^*(E, \mathcal{L}, \mathcal{E}) \) is simple, it follows from [14] Theorem 5.2 that \( \mathcal{E} = \tilde{H}_{A_0} \). Suppose

\[
r(A_0, \beta_{[1,n]}) \cup \bigcup_{j=1}^{n-1} r(A_0, \beta_{[1,j]}) \neq \emptyset
\]

for infinitely many \( n \geq 1 \). Then by (7), \( r(A_0, \beta^n) \nsubseteq \bigcup_{j=1}^{n-1} r(A_0, \beta^j) \) for infinitely many \( n \), which implies that \( r(\beta^r) \nsubseteq \tilde{H}_{A_0} \) for all \( r \geq 1 \). In fact, if \( r(\beta^r) = \bigcup_{i=1}^{k} C_i \in \tilde{H}_{A_0} \) with some \( C_i \in r(A_0, \beta_{[1,m_i]}) \cap \mathcal{E} \), then each \( m_i \) must be a multiple of \( |\beta| \) by (7) and thus for \( m := \max \{ m_i \} / |\beta| \) we have \( r(\beta^r) \subset \bigcup_{i=1}^{m} r(A_0, \beta^i) \). But then for all sufficiently large number \( n > m |\beta| \),

\[
r(\beta^n) \subset r(\beta^r) \subset \bigcup_{i=1}^{m} r(A_0, \beta^i) \subset \bigcup_{j=1}^{n-1} r(A_0, \beta_{[1,j]}),
\]

which is not possible by (8). Hence \( r(\beta^r) \nsubseteq \tilde{H}_{A_0} \) for all \( r \geq 1 \), which then easily implies that \( r(\beta^r) \notin \tilde{H}_{A_0} \) for all \( r \geq 1 \). But this contradicts to \( \tilde{H}_{A_0} = \mathcal{E} \), and thus the left hand side of (8) must be empty for all but finitely many \( n \)'s. Therefore we see from (7) that there exists an \( N \geq 1 \) such that

\[
r(A_0, \beta_{[1,n]}) \subset \bigcup_{j=1}^{n-1} r(A_0, \beta_{[1,j]}) \quad \text{for all} \quad n \geq N.
\]

Then \( r(A_0, \beta_{[1,N+2]}) \subset \bigcup_{j=1}^{N+1} r(A_0, \beta_{[1,j]}) \subset \bigcup_{j=1}^{N} r(A_0, \beta_{[1,j]}) \) because \( r(A_0, \beta_{[1,N+1]}) \subset \bigcup_{j=1}^{N} r(A_0, \beta_{[1,j]}) \), and actually an induction gives

\[
r(A_0, \beta_{[1,N+k]}) \subset \bigcup_{j=1}^{N} r(A_0, \beta_{[1,j]})
\]

for all \( k \geq 1 \), which proves (i).
(ii) We can take \( N = |\beta|N_0 \), a multiple of \( |\beta| \) in (i). Then \( N_0 \) satisfies (6) by (i) and (7) since (7) implies that for each \( k \geq 1 \),
\[
  r(A_0, \tilde{\beta}_{[1,N+k]}) \subset \bigcup_{1 \leq j \leq N, j=k(\text{mod } |\beta|)} r(A_0, \tilde{\beta}_{[1,j]}).
\]
To show \( A = r(A, \beta) \) for \( A := \bigcup_{j=1}^{N_0} r(A_0, \beta^j) \), first note from (9) that
\[
  A \supset r(A, \beta) \supset r(A, \beta^2) \supset \cdots.
\]
Suppose \( B := A \setminus r(A, \beta) \neq \emptyset \). Then for \( l > k \geq 1 \),
\[
  r(B, \beta^k) \cap r(B, \beta^j) \subset r(B, \beta^k) \cap r(A, \beta^{k+1}) = r(B \cap r(A, \beta), \beta^k) = \emptyset.
\]
Thus \( r(B, \beta^n) \setminus \bigcup_{j=1}^{n-1} r(B, \beta^j) \neq \emptyset \) for infinitely many \( n \). But this contradicts to (6) with \( B \) in place of \( A_0 \) since \( \mathcal{L}(BE^{[\beta]}) = \{\beta^n\} \) for all \( n \geq 1 \), and we conclude that \( B = \emptyset \). \( \square \)

In the following Theorem 3.7, (a) \( \iff \) (c) is known in [8, Theorem 9.16] for the Boolean dynamical system induced from a labeled space with the domain property (11).

**Theorem 3.7.** Let \((E, \mathcal{L}, \mathcal{E})\) be a labeled space. Then the following are equivalent:

(a) \( C^*(E, \mathcal{L}, \mathcal{E}) \) is a simple \( C^* \)-algebra.

(b) \((E, \mathcal{L}, \mathcal{E})\) is strongly cofinal and disagreeable.

Also these conditions imply the following.

(c) \((E, \mathcal{L}, \mathcal{E})\) has no cycles without exits and there is no proper hereditary saturated subsets in \( \mathcal{E} \).

If \((E, \mathcal{L}, \mathcal{E})\) satisfies the domain condition (11), then (c) is equivalent to (a) and (b).

**Proof.** We only need to show that (a) implies that \((E, \mathcal{L}, \mathcal{E})\) is disagreeable.

Suppose \((E, \mathcal{L}, \mathcal{E})\) is not disagreeable. Then by Lemma 3.1 there exists a nonempty set \( A_0 \in \mathcal{E} \) and a path \( \beta \in \mathcal{L}^*(E) \) such that for all \( n \geq 1 \),
\[
  \mathcal{L}(A_0E^{[\beta]}n) = \{\beta^n\},
\]
where we assume \( \beta \) to be irreducible. Choose an integer \( N_0 \geq 1 \) such that
\[
  r(A_0, \beta^{N_0+k}) \subset \bigcup_{j=1}^{N_0} r(A_0, \beta^j)
\]
for all \( k \geq 1 \), which exists by Lemma 3.5(ii). Then for
\[
  A := \bigcup_{j=1}^{N_0} r(A_0, \beta^j),
\]
we have \( A = r(A, \beta) \) by the same lemma. A simple computation shows that the hereditary subalgebra \( p_A C^*(E, \mathcal{L}, \mathcal{E})p_A \) of \( C^*(E, \mathcal{L}, \mathcal{E}) \) generated by \( p_A \) is equal to
\[
  \text{Her}(p_A) := \operatorname{span}\{s_\mu p_BS_\nu^* : B \in r(A, \mu) \cap \mathcal{E}, \mu, \nu \in [\beta_{[1,j]}^*, 0 \leq j \leq |\beta|], \beta_{[1,j]}^* \}
\]
where we use notation
\[
  \beta^r_{[1,j]} := \{\beta^r \beta_{[1,j]} : r, j \geq 0 \} \quad \text{with } \beta^0 := \epsilon =: \beta_{[1,0]}.
\]
Let $A_1 \in A \cap \mathcal{E}$ be a nonempty subset. Then $\bigcup_{j=1}^{N} r(A_1, \beta^j) \subset A$ for all $N \geq 1$ since $A = r(A, \beta)$, but one can actually show that there exists an $N_1 \geq 1$ such that

$$A = \bigcup_{j=1}^{N_1} r(A_1, \beta^j).$$

(9)

In fact, an integer $N_1 \geq 1$ for which

$$\bigcup_{j=1}^{N_1} r(A_1, \beta^j) = \bigcup_{i=1}^{\infty} r(A_1, \beta^j)$$

holds ($N_1$ exists again by Lemma (3.6)(ii)) satisfies (9) because otherwise one can easily show that

$$\emptyset \neq A \setminus \bigcup_{j=1}^{N_1} r(A_1, \beta^j) \notin \tilde{H}_{A_1},$$

which is a contradiction to simplicity of $C^*(E, \mathcal{L}, \mathcal{E})$ (or to $\tilde{H}_{A_1} = \mathcal{E}$).

Now we claim that $Her(p_{A_1}) = Her(p_A)$ for any nonempty subset $A_1 \in A \cap \mathcal{E}$. The hereditary subalgebra generated by $p_{A_1}$ is also equal to

$$Her(p_{A_1}) = \overline{\text{span}}\{s_{\mu}p_{BS_\mu}^*: B \in r(A_1, \mu) \cap \mathcal{E}, \mu, \nu \in \beta_{[1,j]}, 0 \leq j \leq |\beta| \},$$

and for each positive element of the form $s_{\mu}p_{BS_\mu}^* \in Her(p_A)$ with $B \in r(A, \mu) \cap \mathcal{E}$, the following computation

$$s_{\mu}p_{BS_\mu}^* \leq s_{\mu}p_{r(A, \mu)}s^*_\mu = s_{\mu}P_{r(\bigcup_{i=1}^{N_1} r(A_1, \beta^i), \mu)}s^*_\mu \leq \sum_{i=1}^{N_1} s_{\mu}P_{r(A_1, \beta^i, \mu)}s^*_\mu$$

where we apply (9) for the second equality shows that $s_{\mu}p_{BS_\mu}^* \in Her(p_{A_1})$. Then for each $s_{\mu}p_{BS_\mu}^* \in Her(p_{A_1})$, the identity

$$s_{\mu}p_{BS_\mu}^* = (s_{\mu}p_{BS_\mu}^*)s_{\mu}p_{BS_\mu}^*(s_{\nu}p_{BS_\nu}^*)$$

proves that $s_{\mu}p_{BS_\mu}^* \in Her(p_{A_1})$ (for this, see [21, Theorem 3.2.2]). Thus $Her(p_{A_1}) = Her(p_A)$ follows for any nonempty subset $A_1 \in A \cap \mathcal{E}$. However, this is not possible if $A$ has a proper subset $A_1 \in A \cap \mathcal{E}$ since $p_A = p_{A_1} + p_{A \setminus A_1} \geq p_{A_1}$. Hence $A$ must be a minimal set. But then, by Lemma 3.3 the $C^*$-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ contains a nonsimple hereditary subalgebra (isomorphic to $C(\mathbb{T})$), and from this contradiction to simplicity of $C^*(E, \mathcal{L}, \mathcal{E})$, we finally conclude that $(E, \mathcal{L}, \mathcal{E})$ is disagreeable. \[ \square \]

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