Total subspaces in dual Banach spaces which are not norming over any infinite dimensional subspace
by
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1. Introduction.

Abstract. The main result: the dual of separable Banach space $X$ contains a total subspace which is not norming over any infinite dimensional subspace of $X$ if and only if $X$ has a nonquasireflexive quotient space with the strictly singular quotient mapping.

Let $X$ be a Banach space and $X^*$ be its dual space.
Let us recall some basic definitions.
A subspace $M$ of $X^*$ is said to be total if for every $0 \neq x \in X$ there is an $f \in M$ such that $f(x) \neq 0$.
A subspace $M$ of $X^*$ is said to be norming over a subspace $L \subset X$ if for some $c > 0$ we have

$$(\forall x \in L)(\sup_{f \in S(M)} |f(x)| \geq c \|x\|),$$

where $S(M)$ is the unit sphere of $M$. If $L = X$ then $M$ is called norming.

The following natural questions arise:
1) How far could total subspaces be from norming ones? (Of course, there are many different concretizations of this question.)
2) What is the structure of Banach spaces, whose duals contain total “very” non-norming subspaces?
3) What is the structure of total subspaces?
These questions was studied by many authors: [Al], [B, p. 208–216], [BDH], [DJ], [DL], [D], [F], [G], [Ma], [Mc], [M1], [M2], [O1], [O2], [P], [PP], [S1], [S2]. The obtained results find applications in the theory of Frechet spaces [BDH], [DM], [MM1], [MM2], [M2]; in the theory of improperly posed problems [O3], [PP, pp. 185–196] and in the theory of universal bases [Pl, p. 31].

The present paper is devoted to the following natural class of subspaces which are far from norming ones. A subspace $M$ of $X^*$ is said to be nowhere norming if it is not norming over every infinite dimensional subspace of $X$. If $X$ is such that $X^*$ contains a total nowhere norming subspace then we shall write $X \in TNNS$. This class was introduced by W.J.Davis and W.B.Johnson in [DJ], where the first example of a total nowhere norming subspace was constructed. In the same paper it was noted that J.C.Daneman proved that every infinite dimensional subspace of $l_1$ is norming over some infinite dimensional subspace of $c_0$. In [O2] a class of the spaces with $TNNS$ property was discovered. A.A.Albanese [Al] proved that the spaces of type $C(K)$ are not in $TNNS$. The problem of description of Banach spaces with $TNNS$ property arises in a natural way.

Our main result (Theorem 2.1) states that for a separable Banach space $X$ we have $X \in TNNS$ if and only if for some nonquasireflexive Banach space $Y$ there exists a surjective strictly singular operator $T : X \to Y$.\[1] Mathematics Subject Classification: Primary 46B20
Section 3 is devoted to the proof of the auxiliary Theorem 2.4. Using the same method we are able to prove the following result (Theorem 3.1):

A Banach space $M$ is isomorphic to a total nonnorming subspace of the dual of some Banach space if and only if $M^*$ contains a closed norming subspace of infinite codimension.

Thus the class of total nonnorming subspaces coincide with the class of Banach spaces that gives a negative solution to the J.J.Schaffer’s problem [Sc, p. 358] (see [DJ, p. 366]).

Section 4 is devoted to several remarks concerning general (not necessarily separable) spaces, in particular, we show that Banach spaces with the Pelczynski property are not $T_{NNS}$.

Section 5 is devoted to an example of a nonquasireflexive separable Banach space without the Pelczynski property and without $T_{NNS}$ too.

We hope that our notation is standard and self-explanatory. For a subset $A$ of a Banach space $X$, $\text{lin}A$, $A^\perp$ and $\text{cl}A$ are, respectively, the linear span of $A$, the set $\{x^* \in X^* : (\forall x \in A)(x^*(x) = 0)\}$ and the closure of $A$ in the strong topology. For a subset $A$ of a dual Banach space $X^*$, $w^* - \text{cl}A$ and $A^\top$ are, respectively, the closure of $A$ in the weak$^*$ topology and the set $\{x \in X : (\forall x^* \in A)(x^*(x) = 0)\}$. For an operator $T : X \to Y$ the notation $T|_Z$ denotes the restriction of $T$ to the subspace $Z$ of $X$.

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2. Main result.

Our sources for Banach space basic concepts and results are [DS, LT, W]. The unit ball and the unit sphere of Banach space $X$ are denoted by $B(X)$ and $S(X)$ respectively. The term “operator” means a bounded linear operator and the term “subspace” means a closed linear subspace.

Let us recall some definitions.

A Banach space $X$ is called quasireflexive if its canonical image has finite codimension in $X^{**}$. The number $\text{dim}(X^{**}/X)$ is called the order of quasireflexivity of $X$ and is denoted by $\text{Ord }X$.

An operator $T : X \to Y$ is called strictly singular if the restriction of $T$ to any infinite dimensional subspace of $X$ is not an isomorphism.

The main result of this paper is the following.

2.1. Theorem. Let $X$ be a separable Banach space. Then $X \in T_{NNS}$ if and only if for some nonquasireflexive Banach space $Y$ there exists a surjective strictly singular operator $T : X \to Y$.

Proof. Let us suppose that such an operator $T$ exists. We may assume without loss of generality that $T$ is a quotient map. Then $T^*$ is an isometric embedding of $Y^*$ into $X^*$. The subspace $M_1 := T^*(Y^*)$ is nowhere norming because $T$ is strictly singular. This space is also not total. Our aim is to find an isomorphism $Q : X^* \to X^*$ which is a small perturbation of the identity operator and is such that under its action $M_1$ becomes a total subspace but remains a nowhere norming one.

The space $Y$ is nonquasireflexive, hence, by [DJ, p. 360] there exist a weak$^*$ null basic sequence $\{y_n\} \subset Y^*$, a bounded sequence $\{g_n\} \subset Y^{**}$ and a partition $\{I_n\}_{n=1}^\infty$ of
the integers into pairwise disjoint infinite subsets such that
\[
g_k(y_n) = \begin{cases} 
1, & \text{if } n \in I_k; \\
0, & \text{if } n \not\in I_k.
\end{cases}
\]

Let us denote the vector \( T^* y_n \) by \( u^*_n \). The operator \( T^* \) is weak* continuous and isometric hence \( \{ u^*_n \}_{n=1}^{\infty} \) is a weak* null basic sequence in \( X^* \) and in \( X^{**} \) there exists a bounded sequence \( \{ v^*_n \}_{n=1}^{\infty} \) such that
\[
v^*_k(u^*_n) = \begin{cases} 
1, & \text{if } n \in I_k; \\
0, & \text{if } n \not\in I_k.
\end{cases}
\]

Let \( \{ s^*_k \}_{k=1}^{\infty} \) be a normalized sequence spanning a total subspace in \( X^* \). Let the operator \( Q : X^* \to X^* \) be given by
\[
Q(x^*) = x^* + \sum_{k=1}^{\infty} 4^{-k} v^*_k(x^*) s^*_k / \| v^*_k \|.
\]

It is clear that \( Q \) is an isomorphism. Let \( M = Q(M_1) \). We shall show that \( M \) is a total nowhere norming subspace. Let \( 0 \neq x \in X \) and let \( k \in \mathbb{N} \) be such that \( s^*_k(x) \neq 0 \). Since \( \{ u^*_n \}_{n=1}^{\infty} \) is weak* null then we can choose \( n \in I_k \) such that \( |u^*_n(x)| < 4^{-k} s^*_k(x) / \| v^*_k \| \). We have
\[
(Q(u^*_n))(x) = u^*_n(x) + 4^{-k} s^*_k(x) / \| v^*_k \| \neq 0.
\]
Hence \( M \) is total.

Recall that if \( U, V \) are subspaces of a Banach space, then the number
\[
\delta(U, V) = \inf \{ \| u - v \| : u \in S(U), v \in V \}
\]
is called the inclination of \( U \) to \( V \).

We shall prove that \( M \subset X^* \) is nowhere norming. Let us suppose that it is not the case and let an infinite dimensional subspace \( L \subset X \) be such that \( M \) is norming over \( L \). By strict singularity of \( T \) there is no any infinite dimensional subspace of \( X \) with nonzero inclination to \( \ker T \). Using standard reasoning with basic sequences (see [Gu]) we can find in \( L \) a normalized basic sequence \( \{ z_i \} \) such that for some sequence \( \{ t_i \} \subset \ker T \) we shall have \( \| z_i - t_i \| \leq 2^{-i} \), and moreover we may require that
\[
(\forall n \in \mathbb{N}) (\lim_{i \to \infty} s^*_n(t_i) = 0). \tag{1}
\]
Let \( c > 0 \) be such that
\[
(\forall x \in L) (\exists f \in S(M)) (|f(x)| \geq c \| x \|).
\]
In particular
\[
(\forall i \in \mathbb{N}) (\exists f_i \in S(M)) (|f_i(z_i)| \geq c).
\]
By the definition of \( M \) we can find such \( y^*_i \in Y^* \) that
\[
f_i = T^* y^*_i + \sum_{k=1}^{\infty} 4^{-k} v^*_k(T^* y^*_i) s^*_k / \| v^*_k \|.
\]
¿From this equality we obtain
\[ ||f_i|| \geq (2/3)||T^*y^*_i||. \]

Using this inequality we obtain for every positive integer \( i \) that
\[
c \leq |f_i(z_i)| \leq |f_i(z_i - t_i)| + |f_i(t_i)| \leq \\
2^{-i} + \sum_{k=1}^{\infty} 4^{-k} v^*_k(T^*y^*_i) s_k^*(t_i)/||v^*_k|| \leq \\
2^{-i} + (3/2) \sum_{k=1}^{\infty} 4^{-k} |s_k^*(t_i)|.
\]

Using (1) and the boundedness of sequences \( \{s_n^*\} \) and \( \{t_i\} \) we arrived at a contradiction. Hence \( M \) is nowhere norming.

Now we begin to prove the converse statement. We need the following result, which easily follows from the arguments of [DJ, p. 358].

\textbf{2.2. Lemma.} Let \( X \) be a separable Banach space and let \( N \) be a subspace of \( X^* \) such that the strong closure of the canonical image of \( X \) in \( N^* \) is of infinite codimension. Then \( N \) contains a weak* null basic sequence \( \{u_n^*\} \) such that for some bounded sequence \( \{v^*_k\} \) in \( X^{**} \) and some partition \( \{I_k\} \) of the positive integers into pairwise disjoint infinite subsets we have
\[
v^*_k(u_n^*) = \begin{cases} 1, & \text{if } n \in I_k; \\ 0, & \text{if } n \not\in I_k. \end{cases}
\]

It turns out that a total nowhere norming subspace need not satisfy the condition of Lemma 2.2.

\textbf{2.3. Proposition.} There exists a total nowhere norming subspace \( L \) of \((l_1)^*\) such that the canonical image of \( l_1 \) is dense in \( L^* \).

Proof. Let a Banach space \( X \) be such that \( X^* \) is separable, contains closed norming subspaces of infinite codimension and does not contain subspaces isomorphic to \( l_1 \). We may take e.g. \( X = (\sum \oplus J)_2 \), where \( J \) is James’ space [LT, p. 25].

We need the following definition. Let \( a \geq 0, b \geq 0 \). We shall say that subset \( A \subset X^* \) is \((a,b)\)-norming if the following conditions are satisfied:
\[
(\forall x \in X)(\sup\{|x^*(x)| : x^* \in A\} \geq a||x||);
\]
\[
\sup\{|x^*| : x^* \in A\} \leq b.
\]

Let \( K \subset X^* \) be a closed norming subspace of infinite codimension. Let \( \alpha : l_1 \to K \) be some quotient mapping. Hence the set \( \alpha(B(l_1)) \) is \((c,1)\)-norming for some \( c > 0 \). Let a sequence \( \{z_i\}_{i=1}^{\infty} \subset X^* \) be such that its image under the quotient map \( X^* \to X^*/K \) is minimal. Let us introduce the operator \( \beta : l_1 \to X^* \) in the following way:
\[
\beta(\{a_i\}_{i=1}^{\infty}) = (c/2) \sum_{i=1}^{\infty} a_i z_i/||z_i|| + \alpha(\{a_i\}_{i=1}^{\infty}).
\]
It is clear that $\beta$ is injective and that the set $\beta(B(l_1))$ is $(c/2, 1 + c/2)$-norming. Let (finite or infinite) sequence $\{y_i\}_{i=1}^k \subset X^*$ be such that its image under the quotient mapping $X^* \to X^*/\gamma \beta(l_1)$ is injective. Then $X^* = \text{cl}(\text{lin}(\{y_i\}_{i=1}^k \cup \beta(l_1)))$. Let us represent $l_1$ as $l_1 \oplus l_1^2$ (or $l_1 \oplus l_1$ if $k$ equals infinity) and define the operator $\gamma : l_1 \oplus l_1^2 \to X^*$ in the following way:

$$\gamma(\{a_i\}_{i=1}^\infty, \{b_i\}_{i=1}^k) = \beta(\{a_i\}_{i=1}^\infty) + \sum_{i=1}^k b_i y_i/\|y_i\|.$$ 

It is clear that $\gamma$ is injective, its image is dense in $X^*$ and that $\gamma(B(l_1))$ is $(c/2, 1 + c/2)$-norming. Besides this $\gamma$ is a strictly singular operator since $X^*$ does not contain subspaces isomorphic to $l_1$.

Let $L = \gamma^*(X) \subset (l_1)^*$. This subspace is total since $\gamma$ is injective. Since $\gamma(B(l_1))$ is $(c/2, 1 + c/2)$-norming then $\gamma^*|_X$ is an isomorphic embedding. Therefore the strict singularity of $\gamma$ implies that $L$ is nowhere norming. On the other hand it is easy to check that the canonical image of $l_1$ in $L^*$ may be identified with $\gamma(l_1)$ and therefore is dense. The proposition is proved.

In order to make Lemma 2.2 applicable for our purposes we need the following result.

2.4. Theorem. Let $X$ be a Banach space and $M$ be a total nowhere norming subspace of $X^*$. Then there exists an isomorphic embedding $E : M \to X^*$ such that $E(M)$ is a also nowhere norming subspace and the closure of $E^*(X)$ in the strong topology has infinite codimension in $M^*$.

We postpone the proof until section 3.

Let $X \in TNNS$ and let $M \subset X^*$ be a total nowhere norming subspace. Applying Theorem 2.4 we find an embedding $E : M \to X^*$ such that $N = E(M)$ is a nowhere norming subspace satisfying the condition of Lemma 2.2. Let $\{u_n^*\}_{n=1}^\infty \subset E(M)$, $\{u_k^*\}_{k=1}^\infty \subset X^*$ and $\{I_k\}_{k=1}^\infty$ be sequences obtained by application of Lemma 2.2 to $N = E(M)$.

We need the following definition [JR].

A sequence $\{x_n^*\}_{n=1}^\infty \subset X^*$ is called weak* basic provided that there is a sequence $\{x_n\}_{n=1}^\infty \subset X$ so that $\{x_n, x_n^*\}$ is biorthogonal and for each $x^* \in w^* - \text{cl}(\text{lin}(x_n^*)_{n=1}^\infty)$,

$$x^* = w^* - \lim_{n \to \infty} \sum_{i=1}^n x^*(x_i)x_i^*.$$ 

By [JR, p. 82] (see also [LT, p. 11]) every bounded away from 0 weak* null sequence in the dual of a separable Banach space has a weak* basic subsequence. Therefore, we may select a weak* basic subsequence $\{u_{n(j)}^*\}_{j=1}^\infty \subset \{u_n^*\}_{n=1}^\infty$. Moreover, by arguments of [JR] we may suppose that the intersection $I_k \cap \{n(j)\}_{j=1}^\infty$ is infinite for every $k \in \mathbb{N}$.

Let $Y = X/(\{u_{n(j)}^*\}_{j=1}^\infty)$ and let $T : X \to Y$ be the quotient map. The dual $Y^*$ may be naturally identified with $w^* - \text{cl}(\text{lin}(u_{n(j)}^*)_{j=1}^\infty)$. The space $Y$ is nonquasireflexive because intersections $I_k \cap \{n(j)\}_{j=1}^\infty$ ($k \in \mathbb{N}$) are infinite. By the well-known properties of weak* basic sequences [LT, p. 11] it follows that for some $\lambda < \infty$ we have

$$B(Y^*) \subset \lambda w^* - \text{cl}(B(\text{lin}\{u_{n(j)}^*\}_{j=1}^\infty)) \subset \lambda w^* - \text{cl}(B(E(M))).$$
Therefore $Y^*$ is nowhere norming subspace of $X^*$. Hence $T$ is strictly singular. The proof of Theorem 2.1 is complete.

2.5. Corollary. If $X$ is a separable Banach space which contains a complemented subspace $Y$ with $Y \in TNNS$ then $X \in TNNS$.

Proof. The composition of a strictly singular surjection $T : Y \rightarrow Z$ and any projection $P : X \rightarrow Y$ is a required surjection.

2.6. Corollary. For every separable Banach space $X$ we have $X \oplus l_1 \in TNNS$.

To prove this we need only to recall that there is a surjective strictly singular operator $T : l_1 \rightarrow c_0$ [LT, p. 75, 108].

2.7. Remark. There exists a space $X$ for which $X \in TNNS$ but $X$ does not contain any subspaces isomorphic to $l_1$. In fact, by the James-Lindenstrauss theorem [LT, p. 26] there exists a separable space $Z$ for which $Z^*/Z$ is isomorphic to $c_0$ and $Z^{**}$ is isomorphic to $Z^* \oplus l_1$. Let $X = Z^*$. Then there exists a quotient map $T : X \rightarrow c_0$. This map must be strictly singular because if not then $X$ must contain a subspace isomorphic to $c_0$ [LT, p. 53] but this contradicts the fact that $X$ is a separable dual space [LT, p. 103]. At the same time $X$ does not contain subspaces isomorphic to $l_1$ because $X^*$ is separable.

3. Total nonnorming subspaces in dual Banach spaces.

In this section we shall prove Theorem 2.4 and the following characterization of total nonnorming subspaces.

3.1. Theorem. A Banach space $M$ is isomorphic to a total nonnorming subspace of the dual of some Banach space if and only if $M^*$ contains a closed norming subspace of infinite codimension.

We need the following lemmas.

3.2. Lemma [B, p. 39]. If $U$ and $V$ are Banach spaces and $P : U \rightarrow V$ is an operator with nonclosed image then the closure of $P(B(U))$ in the strong topology does not contain interior points.

3.3. Lemma [LT, p. 79]. If $P : U \rightarrow V$ is an operator with nonclosed image and $F : U \rightarrow V$ is a finite rank operator then the operator $(P + F)$ has nonclosed image.

3.4. Lemma. Let $P : U \rightarrow V$ be an operator with nonclosed image and let $\varepsilon > 0$. Then there exist a functional $f \in V^*$ and an operator $P_1 : U \rightarrow V$ such that $f$ does not vanish on $\text{im } P$, the image of $(P - P_1)$ is one-dimensional, $\|P - P_1\| \leq \varepsilon$ and $\text{im } P_1 \subset \text{ker } f \cap \text{cl}(\text{im } P)$.

Proof. By Lemma 3.2 the closed convex set $\text{cl}(P(B(U)))$ does not have interior points in the subspace $V_0 = \text{cl}(\text{im } P) \subset V$. Therefore there exists a functional $f_0 \in S(V_0^*)$ such that

$$(\forall v \in P(B(U)))(|f_0(v)| \leq \varepsilon/2).$$

It is clear that $f_0$ does not vanish on $\text{im } P$. Let $v_0 \in V_0$ be such that $f_0(v_0) = 1$ and $|v_0| \leq 2$. Let us define an operator $P_1 : U \rightarrow V$ by $P_1(u) = P(u) - f_0(P(u))v_0$. Let $f$ be any continuous extension of $f_0$ onto the whole $V$. It can be directly verified that $P_1$ and $f$ defined in such a way satisfy all the requirements of Lemma 3.4.

Let $X$ be a Banach space and let $M$ be a subspace of its dual. Every element of $X$ may be considered as a functional on $M$. So there is a natural map of $X$ into $M^*$. We shall denote this map by $H$. 

6
3.5. Proposition. Let $X$ be a Banach space and let $M$ be a total nonnorming subspace in $X^*$. Then there exists an isomorphic embedding $E : M \to X^*$ such that the closure of the $E^*(X)$ in the strong topology is of infinite codimension in $M^*$ and the difference $(E^*|_X - H)$ is a nuclear operator.

Proof. In our case the map $H$ is injective because $M$ is total and is not an isomorphic embedding because $M$ is nonnorming. By the open mapping theorem the image of $H$ is nonclosed.

Let us apply Lemma 3.4 to $P = H$ and $\varepsilon = 1/4$, and denote the obtained functional by $f_1$ and the obtained operator by $H_1$. By Lemma 3.3 the operator $H_1$ also has nonclosed image. Applying Lemma 3.4 to $P = H_1$ and $\varepsilon = 1/8$ we find functional $f_2$ and operator $H_2$. We continue in an obvious way.

We have $||H_{i-1} - H_i|| < 2^{-i-1}$. Therefore the sequence $\{H_i\}_{i=1}^\infty$ is uniformly convergent. Let us denote by $R$ its limit.

The operator $(R - H)$ is nuclear and satisfies the inequality

$$||R - H|| < 2^{-1}$$

It is clear that the set $H(B(X))$ is $(1,1)$-norming. By this and by inequality (2) we obtain that the set $R(B(X))$ is $(1/2,3/2)$-norming. Moreover we have $\text{cl}(\text{im} R) \subset \cap_{i=1}^\infty \ker f_i$. The sequence $\{f_i\}_{i=1}^\infty$ is linearly independent because it is constructed in such a way that $f_{i+1}$ does not vanish on $\cap_{k=1}^i \ker f_k$. Therefore $\text{cl}(\text{im} R)$ is of infinite codimension in $M^*$. Let us introduce an operator $E : M \to X^*$ by $(E(m))(x) = (R(x))(m)$. This operator is an isomorphic embedding because the set $R(B(X))$ is $(1/2,3/2)$-norming.

It is easy to see that the restriction of $E^*$ onto $X$ coincides with $R$. Therefore $(E^*|_X - H)$ is nuclear and $\text{cl}(E^*(X))$ is of infinite codimension in $M^*$. The proof is complete.

Proof of Theorem 2.4. A nowhere norming subspace is of course nonnorming. So we can apply Proposition 3.5. It should be noted that for nowhere norming $M$ the operator $H$ is strictly singular.

Let $E : M \to X^*$ be the operator constructed in Proposition 3.5. We need only to check that $E(M)$ is nowhere norming. But this follows immediately from the fact that $E^*(X) = R = H + (R - H)$ is strictly singular as a sum of two strictly singular operators [LT, p. 76].

Proof of Theorem 3.1. The necessity follows immediately from Proposition 3.5 and the fact that $\text{cl}(E^*(X))$ is norming subspace in $M^*$.

Let us suppose that $M$ is a Banach space for which there exists a closed norming subspace $V \subset M^*$ of infinite codimension. Let $\{z_i\}_{i=1}^\infty$ be a normalized basic sequence in $M^*/V$ and let $m_i^* \in M^*(i \in \mathbb{N})$ be such that $||m_i^*|| \leq 2$ and $Q(m_i^*) = z_i$, where $Q : M^* \to M^*/V$ is a quotient map.

Let $X = V \oplus l_1$. Let us define an operator $H : X \to M^*$ by

$$H(v, \{a_i\}_{i=1}^\infty) = v + \sum_{i=1}^\infty (a_i/i)m_i^*.$$ 

It is clear that this operator is injective but is not an isomorphic embedding.
The restriction of $H^*$ onto $M$ is an isomorphic embedding because $V$ is a norming subspace. The subspace $H^*(M) \subset X^*$ is total because $H$ is injective and is not norming because $H$ is not an isomorphic embedding. This completes the proof of Theorem 3.1.

3.6. Corollary. If $M$ is a total subspace of $X^*$ and $M$ is quasireflexive then $X$ is quasireflexive and $\text{Ord}(X) = \text{Ord}(M)$.

Proof. It is known [CY] that $\text{Ord}(X^*) = \text{Ord}(X)$ and that the order of quasireflexivity of a subspace is not greater than the order of quasireflexivity of the whole space.

It is well-known and is easy to see that the duals of quasireflexive spaces do not contain norming subspaces of infinite codimension. Therefore by Theorem 3.1 the subspace $M \subset X^*$ is norming. Hence $X$ is isomorphic to a subspace of $M^*$. Using the abovementioned result we obtain $\text{Ord}(X) \leq \text{Ord}(M^*) = \text{Ord}(M)$.

Using the abovementioned result once more we obtain $\text{Ord}(M) \leq \text{Ord}(X^*) = \text{Ord}(X)$. The proof is complete.

3.7. Remark. By [DJ, p. 355] nonquasireflexivity of $X$ does not yield the existence in $X^*$ of an infinite codimensional norming subspace. Therefore there exist nonquasireflexive spaces which are not isomorphic to total nonnorming subspaces.

4. Remarks on the nonseparable case and on spaces with the Pelczynski property.

Theorem 2.1 and Corollaries 2.5 and 2.6 are not valid in nonseparable case. In order to prove this let us show that the space $X = l_1 \oplus l_2(\Gamma)$ does not have $TNNS$ property if $\text{card}(\Gamma) > 2^c$.

Let $M$ be a total subspace in $X^*$. Then $2^{\text{card}(M)} \geq \text{card}(X) \geq \text{card}(\Gamma)$. Consequently $\text{card}(M) > c$. Therefore $M$ contains a set of functionals of cardinality greater than $c$, whose restrictions to $l_1$ coincide. Therefore the intersection of $M$ with the subspace of $X^*$ which vanishes on $l_1$ is an infinite dimensional subspace in $\{0\} \oplus l_2(\Gamma)$. If we “transfer” this subspace into $X$ then we shall obtain a subspace over which $M$ is norming.

Problem. Characterize $TNNS$ in the nonseparable setting.

At the moment it is known [Al] that $C(K) \notin TNNS$ for every compact $K$.

4.1. Proposition. Let Banach space $X$ be such that every strictly singular operator $T : X \to Y$ is weakly compact. Then $X \notin TNNS$.

Proof. Evidently it is sufficient to consider the case when $X$ is nonreflexive. Let $M$ be a total nowhere norming subspace in $X^*$. Let $X_M$ be the completion of $X$ under the norm $||x||_M = \sup\{|f(x)| : f \in S(M)\}$.

Let $T : X \to X_M$ be the natural embedding. The operator $T$ is strictly singular because $M$ is nowhere norming. Hence $T$ is weakly compact.

The subspace $M \subset X^*$ may be considered also as a subspace of $(X_M)^*$. Moreover the restriction of $T^* : (X_M)^* \to X^*$ onto $M$ is an isometry.

The operator $T^*$ is weakly compact by V.Gantmacher’s theorem [DS, VI.4.8]. Therefore $M$ is reflexive, hence the subspace $M$ is weak* closed in $X^*$ by the M. Krein–V. Smulian theorem [DS, V.5.7]. Since $M$ is a total subspace of $X^*$ we obtain $M = X^*$. This contradiction completes the proof.
4.2. Remark. For separable spaces this proposition follows immediately from Theorem 2.1.

The conditions of Proposition 4.1 are satisfied by spaces with the Pelczynski property. Let us recall the definition.

A Banach space $X$ has the Pelczynski property if for every subset $K \subset X^*$ that is not relatively weakly compact there exists a weakly unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ in $X$ such that
\[
\inf_n \sup_{x^* \in K} x^*(x_n) > 0.
\]

This property was introduced by A. Pelczynski in [Pe] (under the name “property (V)”). In the same paper it was proved that for any compact Hausdorff space $S$ the space $C(S)$ has property (V). For other spaces with the Pelczynski property see [W, pp. 166-172].

The fact that spaces with the Pelczynski property satisfy the conditions of Proposition 4.1 follows from the next proposition.

4.3. Proposition [W, p. 172]. Suppose $X$ has the Pelczynski property. Then for every operator $T : X \to Y$ that is not weakly compact there exists a subspace $X_1 \subset X$ such that $X_1$ is isomorphic to $c_0$ and the restriction of $T$ onto $X_1$ is an isomorphic embedding.

The spaces which have no TNNS property need not have the Pelczynski property and need not satisfy the conditions of Proposition 4.1.

A corresponding example is given by James’ space $J$. Let us recall its definition [LT, p. 25]. The space $J$ consists of all sequences of scalars $x = (a_1, a_2, \ldots, a_n, \ldots)$ for which
\[
||x|| = \sup 2^{-1/2}((a_{p_1} - a_{p_2})^2 + (a_{p_2} - a_{p_3})^2 + \ldots + (a_{p_{m-1}} - a_{p_m})^2)^{1/2} < \infty
\]
where the supremum is taken over all choices of $m$ and $p_1 < p_2 < \ldots < p_m$; and
\[
\lim_{n \to \infty} a_n = 0.
\]

It is easy to see that the operator $T : J \to c_0$ which maps every sequence from $J$ onto the same sequence in $c_0$ is a non-weakly compact strictly singular operator. On the other hand any total subspace in $J^*$ is norming over $J$. This follows from well-known properties of quasireflexive spaces.

In fact there are nonquasireflexive spaces of this type as is shown in the next section.

5. A nonquasireflexive separable Banach space without the Pelczynski property whose dual does not contain nowhere norming subspaces.

5.1. Theorem. There exists a nonquasireflexive Banach space $X \notin TNNS$ and such that there exists a strictly singular non-weakly compact operator $T : X \to c_0$.

Proof. Let $X = (\sum_{n=1}^{\infty} \oplus J)_2$. The unit vectors in $J$ we shall denote by $\{e_i\}_{i=1}^{\infty}$. It is known [LT, p. 25] that $\{e_i\}$ is shrinking basis of $J$, therefore its biorthogonal functionals $\{e_i^*\}_{i=1}^{\infty}$ form a basis of $J^*$.

It is clear that vectors
\[
e_{n,j} = (0, \ldots, 0, e_j, 0, \ldots)
\]
(where \( e_j \) is on the \( n \)-th place) after any numeration preserving order in sequences \( \{e_{n,j}\}_{j=1}^{\infty} \) form a basis of \( X \). We need the following two lemmas about \( X \) and its dual.

**5.2. Lemma.** Every weakly null sequence \( \{x_m\}_{m=1}^{\infty} \) in \( X \) for which \( \inf ||x_m|| > 0 \) contains a subsequence equivalent to the unit vector basis of \( l_2 \).

**5.3. Lemma.** Every infinite dimensional subspace of \( X^* \) contains a subspace isomorphic to \( l_2 \).

This lemmas easily follows by well-known arguments (see [An] and [HW]).

Let us consider an operator \( T : (\sum \oplus J)_2 \to c_0 = (\sum \oplus c_0)_0 \) defined by

\[
T(x_1, \ldots, x_n, \ldots) = (x'_1, \ldots, x'_n, \ldots),
\]

where \( (x_i) \) is a sequence of elements of \( J \) and \( (x'_i) \) is a sequence of elements of \( c_0 \) with the same coordinates. It is clear that \( T \) is a continuous operator. It is not weakly compact because for any \( n \in \mathbb{N} \) the sequence \( (T(\sum_{j=1}^{k} e_{n,j}))_{k=1}^{\infty} \) does not have limit points in weak topology. At the same time the operator \( T \) is strictly singular because by Lemma 5.2 the space \( X \) does not contain subspaces isomorphic to \( c_0 \).

Let us suppose that \( X \in \text{T NNS} \). Then by Theorem 2.1 there exists a surjective strictly singular operator \( T : X \to Z \) where \( Z \) is a certain Banach space. Consequently \( Z^* \) is isomorphic to a subspace of \( X^* \). By Lemma 5.3 the space \( Z^* \) contains a subspace isomorphic to \( l_2 \). Let us denote this subspace by \( U \). Let \( R \) be the quotient map \( R : Z \to Z/U^\perp \). The space \( (Z/U^\perp)^* \) may be in natural way identified with \( w^* - clU \). Since \( U \) is reflexive then by M.Krein-V.Smulian theorem [DS, V.5.7] we have \( w^* - cl(U) = U \). Therefore \( Z/U^\perp \) is isomorphic to \( l_2 \). Let \( \{u_i\}_{i=1}^{\infty} \) be sequence in \( Z/U^\perp \) equivalent to the unit vector basis of \( l_2 \). By Lemma 2 of [GR] we can find in \( X \) a weakly null sequence \( \{x_i\}_{i=1}^{\infty} \) for which \( \{RT x_i\}_{i=1}^{\infty} \) is a subsequence of \( \{u_i\} \). By Lemma 5.2 the sequence \( \{x_i\} \) contains a subsequence \( \{x_{n_i}\}_{i=1}^{\infty} \) which is equivalent to the unit vector basis of \( l_2 \). The restriction of \( RT \) onto the closed linear span of \( \{x_{n_i}\}_{i=1}^{\infty} \) is an isomorphism. Because \( T \) is strictly singular this gives us a contradiction.

**REFERENCES**

[Al] A.A.Albanese, On total subspaces in duals of spaces of type \( C(K) \) or \( L^1 \), preprint.

[An] A.Andrews, James’ quasi-reflexive space is not isomorphic to any subspace of its dual, Israel J. Math. 38(1981), 276–282.

[B] S.Banach, Theorie des operations lineaires, Monografje Matematyczne. No. 1 (Warszawa, 1932).

[BDH] E.Behrends, S.Dierolf and P.Harmand, On a problem of Bellenot and Dubinsky, Math. Ann. 275 (1986), 337–339.

[CY] P.Civin and B.Yood, Quasi-reflexive spaces, Proc. Amer. Math. Soc. 8 (1957), 906–911.

[DJ] W.J.Davis and W.B.Johnson, Basic sequences and norming subspaces in non-quasi-reflexive Banach spaces, Israel J. Math. 14 (1973), 353–367.

[DL] W.J.Davis and J.Lindenstrauss, On total nonnorming subspaces, Proc. Amer. Math. Soc. 31 (1972), 109–111.

[DM] S.Dierolf and V.B.Moscatelli, A note on quojections, Funct. Approx. Comment. Math. 17 (1987), 131–138.
[D] J.Dixmier, Sur un theoreme de Banach, Duke Math. J. 15 (1948), 1057–1071.

[DS] N.Dunford and J.T.Schwartz, Linear operators. Part I: General theory (Interscience Publishers, New York - London, 1958).

[F] R.J.Fleming, Weak∗-sequential closures and the characteristic of subspaces of conjugate Banach spaces, Studia Math. 26 (1966), 307–313.

[G] B.V.Godun, On weak∗ derived sets of sets of linear functionals, Mat. Zametki 23 (1978), 607–616 (in Russian).

[GR] B.V.Godun and S.A.Rakov, Banach - Saks property and the three space problem, Mat. Zametki. 31 (1982), 61-74. (in Russian).

[Gu] V.I. Gurarii, On openings and inclinations of subspaces of a Banach space, Teor. Funktsii, Funktsional. Anal. i Prilozhen. 1 (1965), 194–204. (in Russian).

[HW] R.Herman and R.Whitley, An example concerning reflexivity, Studia Math. 28 (1967), 289–294.

[JR] W.B.Johnson and H.P.Rosenthal, On w∗-basic sequences and their applications to the study of Banach spaces, Studia Math. 43 (1972), 77–92.

[LT] J.Lindenstrauss and L.Tzafriri, Classical Banach spaces I. Sequence spaces (Springer, Berlin, 1977).

[Ma] S.Mazurkiewicz, Sur la derivee faible d’un ensemble de fonctionelles lineaires, Studia Math. 2 (1930), 68–71.

[Mc] O.C.McGehee, A proof of a statement of Banach about the weak∗ topology, Michigan Math. J. 15 (1968), 135–140.

[MM1] G.Metafune and V.B.Moscatelli, Generalized prequojections and bounded maps, Results in Math. 15 (1989), 172–178.

[MM2] G.Metafune and V.B.Moscatelli, Quojections and prequojections, in: Advances in the Theory of Frechet Spaces, T.Terzioglu (ed.), Kluwer, Dordrecht, 1989, 235–254.

[M1] V.B.Moscatelli, On strongly non-norming subspaces, Note Mat. 7 (1987), 311–314.

[M2] V.B.Moscatelli, Strongly nonnorming subspaces and prequojections, Studia Math. 95 (1990), 249–254.

[O1] M.I.Ostrovskii, w∗-derived sets of transfinite order of subspaces of dual Banach spaces, Dokl. Akad. Nauk Ukrain. SSR, Ser. A (1987), no. 10, 9–12 (in Russian).

[O2] M.I.Ostrovskii, On total nonnorming subspaces of a conjugate Banach space, Teor. Funktsii, Funktsional. Anal. i Prilozhen. 53 (1990), 119–123. (in Russian). Engl. transl.: J. Soviet Math. 58 (1992), no. 6, 577–579.

[O3] M.I.Ostrovskii, Regularizability of superpositions of inverse linear operators, Teor. Funktsii, Funktsional. Anal i Prilozhen. 55 (1991), 96–100 (in Russian). Engl. transl.: J. Soviet Math. 59 (1992), no. 1, 652–655.

[Pe] A.Pelczynski, Banach spaces on which every unconditionally converging operator is weakly compact, Bull. l’Acad. Polon. Sc. Ser. math. astr. phys. 10 (1962), 641–648.

[P] Yu.I.Petunin, Conjugate Banach spaces containing subspaces of zero characteristic, Dokl. Akad. Nauk SSSR, 154 (1964), 527–529, Engl. transl.: Soviet Math. Dokl. 5 (1964), 131–133.
[PP] Yu.I.Petunin and A.N.Plichko, The theory of characteristic of subspaces and its applications (Vyshcha Shkola, Kiev, 1980, in Russian).

[Pl] A.N.Plichko, On bounded biorthogonal systems in some function spaces, Studia Math. 84 (1986), 25–37.

[S1] D.Sarason, On the order of a simply connected domain, Michigan Math. J. 15 (1968), 129–133.

[S2] D.Sarason, A remark on the weak-star topology of $l^\infty$, Studia Math. 30 (1968), 355–359.

[Sc] J.J.Schaffer, Linear differential equatons and functional analysis. VI, Math. Ann. 145 (1962), 354–400.

[W] P.Wojtaszczyk, Banach spaces for analysts, Cambridge studies in advanced mathematics 25 (Cambridge University Press, 1991).

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