1. Introduction

Differential equations appear in numerous problems of physics, engineering, and other sciences. Hence, we need powerful mathematical tools to handle them. The integral transforms form one of the widely used techniques applied for solving differential equations. Generalizations of integral transforms turn them into more flexible tools to deal with various complicated problems. Some generalizations are based on the extension of transforms to multivariate cases. Other generalizations can be done by deforming a differential operator and, hence, an integral. Two leading kinds of deformations are fractional calculus and $q$-calculus. Some applications of fractional calculus can be found in [23], while the applications of $q$-calculus and fractional $q$-calculus are presented in [1].

In the present paper, we propose a new deformation of the natural integral transform. We define new extensions of some special functions and apply our deformed transform to these functions. Among other tools, a new extension of $q, \alpha$-Taylor series is proposed. We start here by recalling three integral transforms, namely, the Laplace, Sumudu, and natural transforms. Further, we develop a new deformation of the natural transform and show its applications to some deformed differential equations.

The Laplace transform is one of the most famous integral transforms. It is defined as follows:

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$ 

This transform is very useful in solving differential equations with given initial and boundary conditions. Moreover, it can be used in finding new identities for functions and integrals (see [18, 21]). One of its important features is the transformation from the time domain into the frequency domain.

In 1993, Watugala [20] proposed a new integral transform, named the Sumudu transform, which is defined as follows:

$$S\{f(t)\} = F_s(u) = \int_0^\infty \frac{1}{u}e^{-\frac{u}{s}}f(t)dt.$$ 

Among its properties, Watugala mentioned its easier visualization. This transform was later extensively studied by different researchers (see, e.g., [4, 9, 14] and the references therein). In his works, Belgacem (see [4] and
the references therein) indicated that the Sumudu transform is an ideal tool for solving various engineering problems. The Sumudu transform, unlike the Laplace transform, is not focused on the transformation into the frequency domain. One of its main features is preserving the units and scale under transformation [4].

In 2008, Khan and Khan [12] defined a new transform, first called the \( N \)-transform and then renamed into the natural transform, as follows:

\[
R(u, s) = N(f(t)) = \int_0^\infty f(ut)e^{-st}dt.
\]

It is easy to see that, for \( u = 1 \), we obtain the Laplace transform and, in the case \( s = 1 \), we get the Sumudu transform. Due to its dual nature and close relationship with both Laplace and Sumudu transforms, the natural transform is more flexible and makes it possible to easily choose preferable way for the solution of the problem in each specific case. In 2017, Kiliçman and Omran generalized this transform to the two-dimensional case [13].

For both Sumudu and Laplace transforms, their \( q \)-anlogs were obtained and studied. The \( q \)-anlogs of the Sumudu transform based on the Jackson \( q \)-derivative and \( q \)-integral were studied by D. Albayrak and other researchers (see [3] and the references therein).

The \( q \)-anlogs of the Laplace transform based some on the Jackson and some on the Tsallis \( q \)-derivatives and \( q \)-integrals were studied in [6, 10, 15–17, 19]. Recently, a \( q \)-analog was also proposed for the natural transform [2].

In the present paper, we define and study a deformation of the natural transform based on the conformable fractional \( q \)-derivative defined by Chung [7]. This deformation is actually a generalization of the \( q \)-deformation based on the Jackson \( q \)-derivative. In case where some parameters of the transform are equal to 1, it proposes another definition for the \( q \)-Sumudu transform different from [3]. Moreover, our transform generalizes some results for the \( q \) analog of the natural transform defined in [13]. Finally, we demonstrate some applications of the \( q \)-deformed conformable fractional natural transform.

2. Definitions and Some Properties of the Conformable \( q \)-Derivative

We start from the definition of conformable fractional \( q \)-derivative \( D^{\alpha}_x \) given by Chung in [7], namely,

\[
D^{\alpha}_x f(x) = \frac{[\alpha](f(x) - f(qx))}{x^\alpha (1 - q^\alpha)} = x^{1-\alpha}D^q_x f(x),
\]

where \( [\alpha] = \frac{1 - q^\alpha}{1 - q} \) is the \( q \)-number of \( \alpha \) and \( D^q_x \) is the Jackson \( q \)-derivative with respect to the variable \( x \).

It is easy to see that the operator \( D^{\alpha}_x \) defined by (2.1) is a linear operator. One can show that, for \( \alpha = 1 \), this differential operator coincides with the Jackson \( q \)-derivative. The following notation is widely used in \( q \)-calculus:

\[
(a + b)_q^n = \prod_{j=0}^{n-1} (a + q^j b).
\]

According to this definition, we get

\[
(a + b)_q^n = \begin{cases} 
\prod_{j=0}^{n-1} (a + q^{\alpha j} b) & \text{for integer } n > 0, \\
1 & \text{for } n = 0.
\end{cases}
\]
Another notation closely related to $q$-calculus is a $q$-Pochhammer symbol

$$ (a; q)_n = \begin{cases} 
\prod_{j=0}^{n-1} (1 - aq^j) & \text{for integer } n > 0, \\
1 & \text{for } n = 0 
\end{cases} 
$$

and

$$ (a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j). $$

The conformable fractional $q$-integral is the inverse operation for the conformable fractional $q$-derivative

$$ I_x^{q, \alpha} f(x) = \frac{1}{\alpha} \left(1 - q^\alpha \right)x^\alpha \sum_{j \geq 0} q^{\alpha j} f(q^j x) = I_x^q (x^{\alpha - 1} f(x)), $$

where $I_x^q$ is the Jackson $q$-integral. Then, for an $\alpha$-monomial $x^{\alpha n}$, we have

$$ D_x^{q, \alpha} x^{\alpha n} = [n\alpha] x^{\alpha (n-1)}, \quad I_x^{q, \alpha} x^{\alpha n} = \frac{x^{\alpha (n+1)}}{([n+1] \alpha)}, $$

(2.3)

It can be shown that the Leibnitz rule for the conformable fractional $q$-derivative has the following form:

$$ D_x^{q, \alpha} (f(x)g(x)) = f(qx)D_x^{q, \alpha} g(x) + (D_x^{q, \alpha} f(x)) g(x). $$

(2.4)

Therefore, by integrating both sides of (2.4), we obtain a rule for integrating by parts:

$$ \int (D_x^{q, \alpha} f(x)) g(x) d_{q, \alpha} x = f(x)g(x) - \int f(qx)D_x^{q, \alpha} g(x) d_{q, \alpha} x. $$

(2.5)

Chung also defined a conformable fractional $q$-exponential function

$$ e_{q, \alpha}(x) = \sum_{j \geq 0} \frac{x^{\alpha j}}{[j\alpha]!} = \left( (1 - q)x^{\alpha}; q^{\alpha} \right)_\infty, $$

(2.6)

where

$$ [n\alpha]! = [\alpha][2\alpha] \ldots [n\alpha], $$

with the property

$$ D_x^{q, \alpha} e_{q, \alpha}(ax) = a^{\alpha} e_{q, \alpha}(ax). $$

(2.7)

Note that, as usual, $[0]! = 1$.

Two new deformations of trigonometric functions were proposed in [7]:

$$ e_{q, \alpha}(i^{\frac{1}{\alpha}} x) = c_{q, \alpha}(x) + is_{q, \alpha}(x), $$

(2.8)
where
\[
c_{q,\alpha}(x) = \sum_{n \geq 0} \frac{(-1)^n}{[2n\alpha]!} x^{2n\alpha}, \quad s_{q,\alpha}(x) = \sum_{n \geq 0} \frac{(-1)^n}{(2n + 1)\alpha} x^{\alpha(2n+1)}.
\]

From (2.8), we obtain
\[
c_{q,\alpha}(x) = \frac{1}{2} \left( e_{q,\alpha}(\frac{i}{\alpha} x) + e_{q,\alpha}(\frac{-1}{\alpha} x) \right), \quad \text{(2.9)}
\]
\[
s_{q,\alpha}(x) = \frac{1}{2i} \left( e_{q,\alpha}(\frac{i}{\alpha} x) - e_{q,\alpha}(\frac{-1}{\alpha} x) \right). \quad \text{(2.10)}
\]

By applying (2.7), it is easy to show that
\[
D_{x}^{q,\alpha} c_{q,\alpha}(x) = -s_{q,\alpha}(x), \quad D_{x}^{q,\alpha} s_{q,\alpha}(x) = c_{q,\alpha}(x).
\]

By using the definition of deformed conformable fractional derivative (2.1), we evaluate the conformable derivative of a function \( \frac{1}{e_{q,\alpha}(ax)} \):
\[
D_{x}^{q,\alpha} \frac{1}{e_{q,\alpha}(ax)} = x^{1-\alpha} D_{x}^{q,\alpha} \frac{1}{e_{q,\alpha}(ax)}
\]
\[
= x^{1-\alpha} \frac{1}{e_{q,\alpha}(ax)} - \frac{1}{e_{q,\alpha}(qax)}
\]
\[
= x^{1-\alpha} \frac{e_{q,\alpha}(qax) - e_{q,\alpha}(ax)}{e_{q,\alpha}(qax)e_{q,\alpha}(ax)(x - qx)}
\]
\[
= - \frac{D_{x}^{q,\alpha} e_{q,\alpha}(ax)}{e_{q,\alpha}(qax)e_{q,\alpha}(ax)}. \quad \text{(2.11)}
\]

Thus, by applying (2.7), we obtain
\[
D_{x}^{q,\alpha} \frac{1}{e_{q,\alpha}(ax)} = - \frac{a^{\alpha} e_{q,\alpha}(ax)}{e_{q,\alpha}(qax)e_{q,\alpha}(ax)} = - \frac{a^{\alpha}}{e_{q,\alpha}(qax)}. \quad \text{(2.11)}
\]

It is easy to see that \( D_{x}^{q,\alpha} C = 0 \), where \( C \) is a constant (\( C \) does not depend on \( x \)). Indeed, it follows from (2.1) that
\[
D_{x}^{q,\alpha} C = x^{1-\alpha} D_{x}^{q,\alpha} C = 0.
\]

We can see that \( D_{x}^{q,\alpha} x^{\alpha} = \lfloor \alpha \rfloor \). Further, we construct a sequence of polynomials \( P_{0}(x), P_{1}(x), \ldots, P_{n}(x) \) of degrees \( 0, \alpha, \ldots, n\alpha \), respectively, such that
\[
D_{x}^{q,\alpha} P_{n}(x) = P_{n-1}(x),
\]
with the initial condition \( P_0(x) = 1 \). Therefore, the polynomial \( P_1(x) \) has the form

\[
P_1(x) = \frac{x^\alpha - a}{[\alpha]}.
\]

Obviously,

\[
P_1(a^{\frac{1}{\alpha}}) = \left( (a^{\frac{1}{\alpha}})^\alpha - a \right) / [\alpha] = 0 \quad \text{and} \quad D_{x}^{q,\alpha}P_1(x) = 1 = P_0(x).
\]

**Proposition 2.1.** For all \( n \in \mathbb{N} \),

\[
D_{x}^{q,\alpha}(x^\alpha - a)^n_q = [n\alpha](x^\alpha - a)^{n-1}_q.
\]

**Proof.** We perform the proof by induction on \( n \). It is easy to see that, for \( n = 1 \), we get

\[
D_{x}^{q,\alpha}(x^\alpha - a)^1_q = D_{x}^{q,\alpha}(x^\alpha - a) = [\alpha],
\]

and the statement holds. We assume that the required statement holds for some integer \( k \). It is necessary to prove it for \( k + 1 \). By virtue of (2.2), we get

\[
(x^\alpha - a)^{k+1}_q = (x^\alpha - a)^k_q (x^\alpha - q^\alpha k a).
\]

Further, by applying (2.4), we obtain

\[
D_{x}^{q,\alpha}(x^\alpha - a)^{k+1}_q = D_{x}^{q,\alpha} \left( (x^\alpha - a)^k_q (x^\alpha - q^\alpha k a) \right)
\]

\[
= (q^\alpha x^\alpha - q^\alpha k a) [k\alpha] (x^\alpha - a)^k_q + [\alpha] (x^\alpha - a)^k_q
\]

\[
= q^\alpha (x^\alpha - q^\alpha (k-1) a) [k\alpha] (x^\alpha - a)^k_q + [\alpha] (x^\alpha - a)^k_q
\]

\[
= q^\alpha [k\alpha] (x^\alpha - a)^k_q + [\alpha] (x^\alpha - a)^k_q
\]

\[
= (x^\alpha - a)^k_q \left( q^\alpha \frac{1 - q^{\alpha k}}{1 - q} + \frac{1 - q^\alpha}{1 - q} \right)
\]

\[
= (x^\alpha - a)^k_q \frac{q^\alpha - q^{\alpha(k+1)} + 1 - q^\alpha}{1 - q}
\]

\[
= (x^\alpha - a)^k_q \frac{1 - q^{\alpha(k+1)}}{1 - q}
\]

\[
= [(k + 1)\alpha] (x^\alpha - a)^k_q,
\]

which completes the induction.
Proposition 2.2. For all \( n \in \mathbb{N} \),
\[
D_x^{q,\alpha}(a - x^\alpha)^n_{q^\alpha} = -[n\alpha](a - q^{\alpha}x^\alpha)^{n-1}_{q^\alpha}.
\]

Proof. By virtue of (2.2), we get
\[
(a - x^\alpha)_n^{q^\alpha} = (a - x^\alpha)(a - q^{\alpha}x^\alpha)(a - q^{2\alpha}x^\alpha) \ldots (a - q^{(n-1)\alpha}x^\alpha)
\]
\[
= (a - x^\alpha)q^\alpha(q^{-\alpha}a - x^\alpha)q^{2\alpha}(q^{-2\alpha}a - x^\alpha) \ldots q^{(n-1)\alpha}(q^{-(n-1)\alpha}a - x^\alpha)
\]
\[
= (-1)^n q^{\alpha(n-1)/2} (x^\alpha - q^{-(n-1)\alpha}a) \ldots (x^\alpha - q^{-2\alpha}a)(x^\alpha - q^{-\alpha}a)(x^\alpha - a)
\]
\[
= (-1)^n q^{\alpha(n-1)/2} (x^\alpha - q^{-(n-1)\alpha}a)^n_{q^\alpha}.
\]

Further, by using Proposition 2.1, we obtain
\[
D_x^{q,\alpha}(a - x^\alpha)^n_{q^\alpha} = D_x^{q,\alpha} \left((-1)^n q^{\alpha(n-1)/2} (x^\alpha - q^{-(n-1)\alpha}a)^n_{q^\alpha}\right)
\]
\[
= [n\alpha](-1)^n q^{\alpha(n-1)/2} (x^\alpha - q^{-(n-1)\alpha}a)^{n-1}_{q^\alpha}
\]
\[
= (-1)^n [n\alpha] q^{\alpha}q^{2\alpha} \ldots q^{(n-1)\alpha}(x^\alpha - q^{-(n-1)\alpha}a) \ldots (x^\alpha - q^{-2\alpha}a)(x^\alpha - q^{-\alpha}a)
\]
\[
= -[n\alpha] q^{\alpha}q^{2\alpha} \ldots q^{(n-1)\alpha}(q^{-(n-1)\alpha}a - x^\alpha) \ldots (q^{-2\alpha}a - x^\alpha)(q^{-\alpha}a - x^\alpha)
\]
\[
= -[n\alpha] (a - q^{-(n-1)\alpha}x^\alpha) \ldots (a - q^{2\alpha}x^\alpha)(a - q^{\alpha}x^\alpha)
\]
\[
= -[n\alpha] (a - q^{\alpha}x^\alpha)^{n-1}_{q^\alpha},
\]
and the proof is completed.

We can now prove that
\[
P_n(x) = \frac{(x^\alpha - a)^n_{q^\alpha}}{[n\alpha]!}.
\]

Indeed, \( P_n(a^{1/\alpha}) = 0 \) and
\[
D_x^{q,\alpha}P_n(x) = D_x^{q,\alpha} \frac{(x^\alpha - a)^n_{q^\alpha}}{[n\alpha]!}
\]
\[
= \frac{[n\alpha](x^\alpha - a)^{n-1}_{q^\alpha}}{[n\alpha]!} = \frac{(x^\alpha - a)^{n-1}_{q^\alpha}}{[(n-1)\alpha]!} = P_{n-1}(x).
\]

Therefore, by using the [11] (Theorems 2.1 and 8.1), we can formulate the following result:
**Theorem 2.1.** Any polynomial or formal power-series function \( f(x) \) can be expressed via the generalized conformable fractional \( q \)-Taylor expansion about \( x = a \) as follows:

\[
f(x) = \sum_{n \geq 0} \left(D_{q,a}^x\right)^n f(x) \left(\frac{1}{n!} \right)^n \frac{(x - a)^n}{n!}.
\]

We now define one more \( q \)-deformed conformable fractional exponential function

\[
E_{q,a}(x) = \sum_{j \geq 0} q^{\alpha(j-1)/2} \frac{x^{\alpha j}}{[j\alpha]!} = (1 + (1 - q)x^\alpha)^\infty.
\]

It is easy to see that \( E_{q,a}(0) = 1 \). By using (2.3), we obtain

\[
D_{q,a} E_{q,a}(ax) = \sum_{j \geq 1} q^{\alpha(j-1)/2} a^{\alpha j} x^{\alpha(j-1)} \frac{1}{[(j-1)\alpha]!} = \sum_{j \geq 0} q^{\alpha(j+1)/2} a^{\alpha(j+1)} x^{\alpha j} \frac{1}{[j\alpha]!} = a^\alpha E_{q,a}(qx).
\]

For some \( n \geq 1 \), we define a \( q \)-deformed conformable fractional gamma-function:

\[
\Gamma_{q,a}(n + 1) = \int_0^\infty x^{\alpha n} \frac{1}{e_{q,a}(qx)} \, dx.
\]

**Proposition 2.3.** For all \( n > 0 \), the function \( \Gamma_{q,a}(n + 1) \) defined by (2.13) satisfies the recurrence relation

\[
\Gamma_{q,a}(n + 1) = [n\alpha] \Gamma_{q,a}(n)
\]

with the initial condition \( \Gamma_{q,a}(1) = 1 \).

**Proof.** We perform the proof by induction on \( n \). For \( n = 0 \), we have

\[
\Gamma_{q,a}(1) = \int_0^\infty \frac{1}{e_{q,a}(qx)} \, dx = - \int_0^\infty D_{q,a} x \frac{1}{e_{q,a}(x)} \, dx = - \frac{1}{e_{q,a}(x)} \bigg|_0^\infty = 1.
\]

We now assume that the claim holds for \( k - 1 \) and prove it for \( k \). Thus, we consider the function \( \Gamma_{q,a}(k + 1) \) for some \( k \). By virtue of (2.13), we get

\[
\Gamma_{q,a}(k + 1) = \int_0^\infty x^\alpha \frac{1}{e_{q,a}(qx)} \, dx.
\]
Hence, as a result of rearrangement, by using \( (2.4) \), we get

\[
\Gamma_{q,\alpha}(k + 1) = - \int_{0}^{\infty} x^{\alpha k} \left( D_{x}^{q,\alpha} \frac{1}{e_{q,\alpha}(x)} \right) d_{q,\alpha}x
\]

\[
= -x^{\alpha k} \frac{1}{e_{q,\alpha}(x)} \bigg|_{0}^{\infty} + \int_{0}^{\infty} \left( D_{x}^{q,\alpha} x^{\alpha k} \right) \frac{1}{e_{q,\alpha}(qx)} d_{q,\alpha}x
\]

\[
= [k\alpha] \int_{0}^{\infty} x^{\alpha(k-1)} \frac{1}{e_{q,\alpha}(qx)} d_{q,\alpha}x = [k\alpha] \Gamma_{q,\alpha}(k),
\]

which completes the proof.

The last proposition immediately yields the following result:

**Corollary 2.1.** For all \( n \in \mathbb{N} \), the following relation is true:

\[
\Gamma_{q,\alpha}(n + 1) = [n\alpha]!.
\]

The function \( \Gamma_{q,\alpha}(n) \) defined by \( (2.13) \) is a \( q \)-deformed conformable fractional extension of the \( \Gamma \)-function. It is well known that the \( \Gamma \)-function is closely related to the \( B \)-function, i.e., for the \( B \)-function defined as

\[
B(m, n) = \int_{0}^{1} x^{m-1}(1-x)^{n-1} dx,
\]

we have

\[
B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{2.14}
\]

We now define a function

\[
B_{q,\alpha}(m, n) = \int_{0}^{1} x^{\alpha(m-1)}(1-q^{\alpha}x^{\alpha})^{n-1} d_{q,\alpha}x.
\]

**Proposition 2.4.** For all natural \( m, n, \)

\[
B_{q,\alpha}(m, n) = \frac{\Gamma_{q,\alpha}(m)\Gamma_{q,\alpha}(n)}{\Gamma_{q,\alpha}(m+n)}.
\]

**Proof.** By using the notation

\[
f(qx) = (1-q^{\alpha}x^{\alpha})^{n-1} \quad \text{and} \quad D_{x}^{q,\alpha}g(x) = x^{\alpha(m-1)} d_{q,\alpha}x,
\]
we obtain
\[ f(x) = (1 - x^\alpha)^{n-1} \quad \text{and} \quad g(x) = \frac{x^{\alpha m}}{\alpha m}. \]

Therefore, by virtue of Proposition 2.2, we get
\[ D_x^{q,\alpha} f(x) = -[(n-1)\alpha](1 - q^\alpha x^\alpha)^{n-2}. \]

Applying (2.5) with our notation, we find
\[ B_{q,\alpha}(m, n) = (1 - x^\alpha)^{n-1} \frac{x^{\alpha m}}{\alpha m} \bigg|_0^1 \]
\[ + \int_0^1 [(n-1)\alpha](1 - q^\alpha x^\alpha)^{n-2} \frac{x^{\alpha m}}{\alpha m} d_{q,\alpha} x \]
\[ = \frac{[(n-1)\alpha]}{\alpha m} \int_0^1 x^{\alpha m}(1 - q^\alpha x^\alpha)^{n-2} d_{q,\alpha} x. \]

Thus, by assuming that \( m \) and \( n \) are natural numbers, we obtain
\[ B_{q,\alpha}(m, n) = \int_0^1 x^{\alpha (m-1)}(1 - q^\alpha x^\alpha)^{n-1} d_{q,\alpha} x \]
\[ = \frac{[(n-1)\alpha]}{\alpha m} \int_0^1 x^{\alpha m}(1 - q^\alpha x^\alpha)^{n-2} d_{q,\alpha} x \]
\[ = \frac{[(n-1)\alpha]}{\alpha m} \frac{[(n-2)\alpha]}{\alpha (m+1)} \int_0^1 x^{\alpha (m+1)}(1 - q^\alpha x^\alpha)^{n-3} d_{q,\alpha} x \]
\[ \ldots = \frac{[(n-1)\alpha]}{\alpha m} \frac{[(n-2)\alpha]}{\alpha (m+1)} \ldots \frac{[2\alpha]}{\alpha (m+n-3)} \]
\[ \times \int_0^1 x^{\alpha (m+n-3)}(1 - q^\alpha x^\alpha)^{n-1} d_{q,\alpha} x \]
\[ = \frac{[(n-1)\alpha]}{\alpha m} \frac{[(n-2)\alpha]}{\alpha (m+1)} \ldots \frac{[2\alpha]}{\alpha (m+n-3)} \]
\[ \times \left( \frac{x^{\alpha (m+n-2)}}{((m+n-2)\alpha)}(1 - x^\alpha)^{n-1} + \int_0^1 \frac{x^{\alpha (m+n-2)}}{((m+n-2)\alpha)} d_{q,\alpha} x \right) \]
\[ + \int_0^1 x^{\alpha (m+n-2)} d_{q,\alpha} x \]
\[ = \left( \frac{x^{\alpha (m+n-2)}}{((m+n-2)\alpha)} \right)(1 - x^\alpha)^{n-1} + \int_0^1 \frac{x^{\alpha (m+n-2)}}{((m+n-2)\alpha)} d_{q,\alpha} x \]
\[
\begin{align*}
\frac{\Gamma_q(n+1)\Gamma_q(m+1)}{\Gamma_q(n+1+m+1)} &= \left[\frac{(n-1)!}{(m+1)!}\right] \left[\frac{(m-1)!}{(n+1)!}\right] = \frac{(n-1)!}{(m+n-1)!} \\
&= \Gamma_q(n)\Gamma_q(m) \Gamma_q(m+n) \Gamma_q(n+m),
\end{align*}
\]

(2.15)

Proposition 2.4 is proved.

**Remark 2.1.** It is easy to see that, for \(\alpha = q = 1\), relation (2.15) turns into (2.14). Thus, the functions \(\Gamma_q,\alpha\) and \(B_q,\alpha\) are \(q\)-deformed conformable fractional extensions of the well-known \(\Gamma\) and \(B\)-functions, respectively. This proposition can be extended for all positive \(m\) and \(n\).

3. \(q\)-Deformed Conformable Fractional Natural Transform

We now define a \(q\)-deformed conformable fractional natural transform as

\[
N_q,\alpha(f(t)) = \int_0^\infty f(ut) \frac{1}{e_{q,\alpha}(qst)} d_{q,\alpha}t, \quad s > 0.
\]

(3.1)

Thus, we get

\[
N_q,\alpha(1) = \int_0^\infty \frac{1}{e_{q,\alpha}(qst)} d_{q,\alpha}t
\]

\[
= -\frac{1}{s^\alpha} \int_0^\infty D_t^q,\alpha \frac{1}{e_{q,\alpha}(st)} d_{q,\alpha}t = -\frac{1}{s^\alpha} \frac{1}{e_{q,\alpha}(st)} \bigg|_0^\infty = \frac{1}{s^\alpha}.
\]

Further, we obtain the transform of an \(\alpha\)-monomial.

**Proposition 3.1.** For all integer \(N \geq 0\),

\[
N_q,\alpha(t^{\alpha N}) = \frac{u^{\alpha N}}{s^{\alpha(N+1)}} \Gamma_q,\alpha(N+1).
\]

**Proof.** By definition (3.1), we have

\[
N_q,\alpha(t^{\alpha N}) = \int_0^\infty u^{\alpha N} t^{\alpha N} \frac{1}{e_{q,\alpha}(qst)} d_{q,\alpha}t
\]

\[
= -\frac{1}{s^\alpha} u^{\alpha N} \int_0^\infty \left( D_t^q,\alpha \frac{1}{e_{q,\alpha}(st)} \right) t^{\alpha N} d_{q,\alpha}t.
\]

Integrating the last equation by parts, we find

\[
N_q,\alpha(t^{\alpha N}) = -\frac{u^{\alpha N}}{s^\alpha} \left\{ \frac{1}{e_{q,\alpha}(st)} t^{\alpha N} \bigg|_0^\infty - \int_0^\infty \frac{1}{e_{q,\alpha}(qst)} (D_t^q,\alpha t^{\alpha N}) d_{q,\alpha}t \right\}
\]
Thus, by (3.2), we obtain

\[
N_q,\alpha(t^{\alpha N}) = \frac{u^\alpha}{s^\alpha} [N\alpha] \frac{u^\alpha}{s^\alpha} [(N-1)\alpha] \ldots \frac{u^\alpha}{s^\alpha} [\alpha] \frac{1}{s^\alpha} = \frac{u^{\alpha N}}{s^\alpha (N+1)} [N\alpha]!.
\]  

(3.3)

Applying Corollary 2.1, we complete the proof.

We now consider the transform of two deformed exponential functions.

**Proposition 3.2.** The \(q\)-deformed conformable natural transforms of \(q\)-deformed conformable exponential functions are given by the formulas

\[
N_q,\alpha(e_{q,\alpha}(at)) = \frac{1}{s^\alpha - a^\alpha u^\alpha},
\]

\[
N_q,\alpha(E_{q,\alpha}(at)) = \sum_{n=0}^{\infty} q^{\alpha(n-1)} \frac{(ua)^{\alpha n}}{s^\alpha(n+1)}.
\]

**Proof.** Applying transform (3.1) to the deformed exponential function (2.6), we obtain

\[
N_q,\alpha(e_{q,\alpha}(at)) = \int_0^\infty e_{q,\alpha}(stu) \frac{1}{e_{q,\alpha}(qst)} d_{q,\alpha}t
\]

\[
= \int_0^\infty \sum_{n=0}^{\infty} \frac{a^{\alpha n} u^{\alpha n} t^{\alpha n}}{[n\alpha]!} \frac{1}{e_{q,\alpha}(qst)} d_{q,\alpha}t
\]

\[
= \sum_{n=0}^{\infty} \frac{a^{\alpha n}}{[n\alpha]!} \int_0^\infty u^{\alpha n} t^{\alpha n} \frac{1}{e_{q,\alpha}(qst)} d_{q,\alpha}t
\]

\[
= \sum_{n=0}^{\infty} \frac{a^{\alpha n}}{[n\alpha]!} N_q,\alpha(t^{\alpha n}) = \sum_{n=0}^{\infty} \frac{a^{\alpha n} u^{\alpha n}}{[n\alpha]! s^\alpha(n+1) [n\alpha]!}
\]

\[
= \frac{1}{s^\alpha} \sum_{n=0}^{\infty} \frac{(au)^{\alpha n}}{s^\alpha} = \frac{1}{s^\alpha - a^\alpha u^\alpha}.
\]
Further, applying transform (3.1) to the deformed exponential function (2.12), we get

\[
N_{q,\alpha}(E_{q,\alpha}(at)) = \int_{0}^{\infty} E_{q,\alpha}(uat) \frac{1}{e_{q,\alpha}(qst)} d_{q,\alpha}t
\]

\[
= \sum_{n=0}^{\infty} q^{\alpha(n-1)} \frac{a^{\alpha n} u^{\alpha n} t^{\alpha n}}{[n\alpha]!} \frac{1}{e_{q,\alpha}(qst)} d_{q,\alpha}t
\]

\[
= \sum_{n=0}^{\infty} q^{\alpha(n-1)} \frac{u^{\alpha n}}{[n\alpha]!} \int_{0}^{\infty} t^{\alpha n} \frac{1}{e_{q,\alpha}(qst)} d_{q,\alpha}t
\]

\[
= \sum_{n=0}^{\infty} q^{\alpha(n-1)} \frac{u^{\alpha n}}{[n\alpha]!} s^{\alpha(n+1)} [n\alpha]!
\]

\[
= \sum_{n=0}^{\infty} q^{\alpha(n-1)} \frac{(ua)^{\alpha n}}{s^{\alpha(n+1)}}.
\]

The proof is completed.

We now consider the transforms of the deformed trigonometric functions (2.9) and (2.10).

**Proposition 3.3.** The deformed conformable fractional natural transform of the deformed trigonometric functions defined by (2.9) and (2.10) is given by

\[
N_{q,\alpha}(c_{q,\alpha}(t)) = \frac{s^{\alpha}}{s^{2\alpha} + u^{2\alpha}},
\]

\[
N_{q,\alpha}(s_{q,\alpha}(t)) = \frac{u^{\alpha}}{s^{2\alpha} + u^{2\alpha}}.
\]

**Proof.** It follows from definition (2.9) and the linearity of the transform \(N_{q,\alpha}\) that

\[
N_{q,\alpha}(c_{q,\alpha}(at)) = \frac{1}{2} \left( N_{q,\alpha}\left(e_{q,\alpha}(\frac{1}{2} at) - \frac{1}{2} at\right) + N_{q,\alpha}\left((-1)^{\frac{1}{2}} at\right)\right)
\]

\[
= \frac{1}{2} \left( \frac{1}{s^{\alpha} - i(\frac{1}{2})^{\alpha} u^{\alpha}} + \frac{1}{s^{\alpha} - i(-1)^{\frac{1}{2}} a^{\alpha} u^{\alpha}} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{s^{\alpha} - i(au)^{\alpha}} + \frac{1}{s^{\alpha} + i(au)^{\alpha}} \right)
\]

\[
= \frac{1}{2} \left( s^{\alpha} + i(au)^{\alpha} + s^{\alpha} - i(au)^{\alpha} \right)
\]

\[
= \frac{s^{\alpha}}{2(s^{\alpha} - i(au)^{\alpha})(s^{\alpha} + i(au)^{\alpha})}
\]

\[
= \frac{s^{\alpha}}{s^{2\alpha} + (au)^{2\alpha}}.
\]
In the same way, from definition (2.10) and the linearity of the transform $N_{q,\alpha}$, we get

$$N_{q,\alpha}(s_{q,\alpha}(at)) = \frac{1}{2i} \left( N_{q,\alpha}\left( e_{q,\alpha}(\frac{i}{\alpha} at) - N_{q,\alpha}(e_{q,\alpha}((-i)^{\frac{1}{\alpha}} at)) \right) \right)$$

$$= \frac{1}{2i} \left( \frac{1}{s^\alpha - (i^{\frac{1}{\alpha}} a)^\alpha u^\alpha} - \frac{1}{(s^\alpha - ((-i)^{\frac{1}{\alpha}} a)^\alpha u^\alpha} \right)$$

$$= \frac{1}{2i} \left( \frac{1}{s^\alpha - i(au)^\alpha} - \frac{1}{s^\alpha + i(au)^\alpha} \right)$$

$$= \frac{1}{2i} \left( s^\alpha + i(au)^\alpha - s^\alpha + i(au)^\alpha \right)$$

$$= \frac{(au)^\alpha}{s^{2\alpha} + (au)^{2\alpha}}.$$ 

This completes the proof.

Suppose that a function $f(t)$ has a polynomial or formal power series expansion in $\alpha$-monomials $t^{\alpha n}$. We denote this function $f(t)$ by $f_\alpha(t)$ and consider the transform of its derivative $D_t^{q,\alpha} f_\alpha(t)$:

$$N_{q,\alpha} \left( D_t^{q,\alpha} f_\alpha(t) \right) = \int_0^\infty (D_t^{q,\alpha} f_\alpha) (ut) \frac{1}{e_{q,\alpha}(qst)} d_{q,\alpha}t$$

$$= \frac{1}{u^\alpha} \int_0^\infty (D_t^{q,\alpha} f_\alpha) (y) \frac{1}{e_{q,\alpha}(q \frac{y}{u})} d_{q,\alpha}y$$

$$= \frac{1}{u^\alpha} f_\alpha(y) \left[ \frac{1}{e_{q,\alpha}(q \frac{y}{u})} \right]_0^\infty + \frac{1}{u^\alpha w^\alpha} \int_0^\infty f_\alpha(y) \frac{1}{e_{q,\alpha}(q \frac{y}{u})} d_{q,\alpha}y$$

$$= -\frac{1}{u^\alpha} f_\alpha(0) + \frac{s^\alpha}{u^\alpha} \int_0^\infty f_\alpha(ut) \frac{1}{e_{q,\alpha}(qst)} d_{q,\alpha}t$$

$$= -\frac{1}{u^\alpha} f_\alpha(0) + \frac{s^\alpha}{u^\alpha} N_{q,\alpha}(f_\alpha(t)).$$

We rewrite this expression as follows:

$$N_{q,\alpha}(D_t^{q,\alpha} f_\alpha(t)) = \frac{s^\alpha}{u^\alpha} N_{q,\alpha}(f_\alpha(t)) - \frac{1}{u^\alpha} f_\alpha(0). \quad (3.4)$$

Therefore,

$$N_{q,\alpha}\left( (D_t^{q,\alpha})^2 f_\alpha(t) \right) = \frac{s^\alpha}{u^\alpha} N_{q,\alpha}(D_t^{q,\alpha} f_\alpha(t)) - \frac{1}{u^\alpha} (D_t^{q,\alpha} f_\alpha)(0)$$
with an initial condition

\[ f(0) = 1, \quad D_t^{q,\alpha} f(t) = 0, \quad (D_t^{q,\alpha})^2 f(0) = 5. \]

Thus, we can formulate the following result:

**Theorem 3.1.** Suppose that a function \( f_\alpha(t) \) has polynomials or formal power series expansion in \( \alpha \)-monomials \( t^\alpha \). Then, for all integer \( n > 0 \), the following relation is true:

\[
N_{q,\alpha} \left( (D_t^{q,\alpha})^n f_\alpha(t) \right) = \left( \frac{s^\alpha}{u^\alpha} \right)^n N_{q,\alpha}(f_\alpha(t)) - \frac{1}{u^\alpha} \sum_{j=0}^{n-1} \left( \frac{s^\alpha}{u^\alpha} \right)^{n-1-j} (D_t^{q,\alpha})^j f_\alpha(0).
\]

**Proof.** We perform the proof by induction on \( n \). The statement of the theorem is true for \( n = 1 \) as shown in (3.4). We assume that the required formula holds for \( k \), and prove it for \( k + 1 \). By using (3.4), we obtain

\[
N_{q,\alpha} \left( (D_t^{q,\alpha})^{k+1} f_\alpha(t) \right) = \frac{s^\alpha}{u^\alpha} N_{q,\alpha} \left( (D_t^{q,\alpha})^k f_\alpha(t) \right) - \frac{1}{u^\alpha} ((D_t^{q,\alpha})^k f_\alpha)(0),
\]

and, by the induction assumption, we get

\[
N_{q,\alpha} \left( (D_t^{q,\alpha})^{k+1} f_\alpha(t) \right) = -\frac{1}{u^\alpha} ((D_t^{q,\alpha})^k f_\alpha)(0) + \frac{s^\alpha}{u^\alpha} \left( \frac{s^\alpha}{u^\alpha} \right)^k N_{q,\alpha}(f_\alpha(t)) - \frac{1}{u^\alpha} \sum_{j=0}^{k-1} \left( \frac{s^\alpha}{u^\alpha} \right)^{k-1-j} (D_t^{q,\alpha})^j f_\alpha(t) - \frac{1}{u^\alpha} ((D_t^{q,\alpha})^k f_\alpha)(0)
\]

which completes the proof.

We now consider two examples of application of conformable fractional \( q \)-deformed natural transforms for the solution of differential equations.

**Example 3.1.** This example is an extension of Example 4.2.4 in [8]. We consider a differential equation

\[
\left( (D_t^{q,\alpha})^3 + (D_t^{q,\alpha})^2 - 6D_t^{q,\alpha} \right) f(t) = 0,
\]

with an initial condition

\[
f(0) = 1, \quad D_t^{q,\alpha} f(0) = 0, \quad (D_t^{q,\alpha})^2 f(0) = 5.
\]
We apply our conformable fractional natural \( q \)-transform to this differential equation and, by using Theorem 3.1, obtain

\[
\left( \frac{s^\alpha}{u^\alpha} \right)^3 \tilde{f} - \frac{1}{u^\alpha} \sum_{j=0}^{2} \left( \frac{s^\alpha}{u^\alpha} \right)^{2-j} (D_t^{q,\alpha})^j f(0)
+ \left( \frac{s^\alpha}{u^\alpha} \right)^2 \tilde{f} - \frac{1}{u^\alpha} \sum_{j=0}^{1} \left( \frac{s^\alpha}{u^\alpha} \right)^{1-j} (D_t^{q,\alpha})^j f(0) - 6 \cdot \frac{s^\alpha}{u^\alpha} \tilde{f} + \frac{6}{u^\alpha} f(0) = 0,
\]

where

\[ \tilde{f} = N_{q,\alpha}(f(t)). \]

Let \( w = \frac{s^\alpha}{u^\alpha} \). Thus, we get

\[
w^3 \tilde{f} - \frac{w^2}{u^\alpha} f(0) - \frac{w}{u^\alpha} D_t^{q,\alpha} f(0) - \frac{1}{u^\alpha} (D_t^{q,\alpha})^2 f(0)
+ w^2 \tilde{f} - \frac{w}{u^\alpha} f(0) - \frac{1}{u^\alpha} D_t^{q,\alpha} f(0) - 6w \tilde{f} + \frac{6}{u^\alpha} f(0) = 0.
\]

Further, by applying the initial conditions, we obtain

\[
\tilde{f} = \frac{1}{u^\alpha} \frac{w^2 + w - 1}{w(w^2 + w - 6)}
= \frac{1}{u^\alpha} \left( \frac{1}{6w} + \frac{1}{3(w+3)} + \frac{1}{2(w-2)} \right)
= \frac{1}{6} \frac{1}{s^\alpha} + \frac{1}{3} \frac{1}{s^\alpha + 3u^\alpha} + \frac{1}{2} \frac{1}{s^\alpha - 2u^\alpha}
= \frac{1}{6} \frac{1}{s^\alpha} + \frac{1}{3} \frac{1}{s^\alpha - \left(-3 \frac{\pi}{\alpha}\right)^\alpha u^\alpha} + \frac{1}{2} \frac{1}{s^\alpha - \left(2 \frac{\pi}{\alpha}\right)^\alpha u^\alpha}.
\]

Thus, by using the results of Proposition 3.2, we can find the original function \( f(t) \) as follows:

\[
f(t) = \frac{1}{6} + \frac{1}{3} e_{q,\alpha} \left((-3 \frac{\pi}{\alpha}) t\right) + \frac{1}{2} e_{q,\alpha} \left(2 \frac{\pi}{\alpha} t\right).
\]

One can easily see that, for \( q = 1, \, \alpha = 1 \), this solution becomes

\[
f(t) = \frac{1}{6} + \frac{1}{3} e^{-3t} + \frac{1}{2} e^{2t},
\]

which coincides with the solution in [8].
Example 3.2. Consider an extension of the differential equation appearing in Example 4 of [22]:

\[ D_f^{q,\alpha} f(t) + 3 f(t) = 13 s_{q,\alpha} \left( 2^{\frac{1}{\alpha}} t \right), \]

with an initial condition \( f(0) = 6 \). Again, let \( \tilde{f} = N_{q,\alpha}(f(t)) \). By applying the integral transform to this differential equation, we arrive at the equation

\[ \frac{s^\alpha}{u^\alpha} \tilde{f} - \frac{1}{u^\alpha} f(0) + 3 \tilde{f} = 13 \cdot \frac{2 u^\alpha}{s^{2\alpha} + 4 u^{2\alpha}}. \]

By applying the initial condition, this equation can be rewritten as

\[ \frac{s^\alpha + 3 u^\alpha}{u^\alpha} \tilde{f} = \frac{26 u^\alpha}{s^{2\alpha} + 4 u^{2\alpha}} + \frac{6}{u^\alpha}. \]

We can now express the transformation \( \tilde{f} \) as follows:

\[ \tilde{f} = \frac{1}{s^\alpha + 3 u^\alpha} \left( \frac{5 - u^{2\alpha} + 6 s^{2\alpha}}{s^{2\alpha} + 4 u^{2\alpha}} \right) = \frac{A s^\alpha + C u^\alpha}{s^{\alpha} + 3 u^\alpha} + \frac{B s^\alpha + C u^\alpha}{s^{2\alpha} + 4 u^{2\alpha}}. \] (3.5)

The unknown constants \( A, B, \) and \( C \) can be found by comparing two expressions for \( \tilde{f} \). One can easily show that \( A = 8, \) \( B = -2, \) and \( C = 6. \)

Therefore, (3.5) can be rewritten as

\[ \tilde{f} = \frac{8}{s^\alpha + 3 u^\alpha} - 2 \frac{s^\alpha}{s^{2\alpha} + 4 u^{2\alpha}} + 3 \frac{2 u^\alpha}{s^{2\alpha} + 4 u^{2\alpha}}. \]

Thus, by Propositions 3.2 and 3.3, we can obtain the original function \( f(t) \) in the following form:

\[ f(t) = 8 e_{q,\alpha} \left( (-3)^{\frac{1}{\alpha}} t \right) - 2 c_{q,\alpha} \left( 2^{\frac{1}{\alpha}} t \right) + 3 s_{q,\alpha} \left( 2^{\frac{1}{\alpha}} t \right). \]

Note that, for \( q = 1 \) and \( \alpha = 1 \), we obtain the solution of [22].

We have already considered the transforms of functions and their derivatives. We now consider the derivatives of transforms. We would like to emphasize that a function \( f(t) \) is actually a polynomial or a formal power series in the \( t^\alpha \)-monomials. By \( R_{q,\alpha}(u, s) \) we denote the conformable fractional \( q \)-deformed natural transform (3.1). By using the notation

\[ e_{q,\alpha}^{-1}(t) = \frac{1}{e_{q,\alpha}(t)}, \]

we can formulate the following lemma:

**Lemma 3.1.** For all integer \( n > 0, \)

\[ \left( D_s^{q,\alpha} \right)^n e_{q,\alpha}^{-1} \left( q^{-\alpha (n-1)} st \right) = (-1)^n t^{\alpha n} q^{-\frac{\alpha n}{2}} e_{q,\alpha}^{-1}(q st). \]
By using Lemma 3.1, we get

\[ (D_s^{q,\alpha})^n e_{q,\alpha}^{-1}(q^{-(n-1)}st) = (-q^{-(n-1)}t) \alpha \cdots (-q^{-1}t) \alpha D_s^{q,\alpha} e_{q,\alpha}^{-1}(st). \]

Applying (2.11) once again, we get

\[ (D_s^{q,\alpha})^n e_{q,\alpha}^{-1}(q^{-(n-1)}st) = (-q^{-(n-1)}t) \alpha \cdots (-q^{-1}t) \alpha (-t) \alpha e_{q,\alpha}^{-1}(qst) \]

\[ = (-1)^n t^\alpha q^{-\binom{n}{2}} \alpha e_{q,\alpha}^{-1}(qst). \]

This yields the statement of the lemma.

**Proposition 3.4.** Suppose that a function \( f_\alpha(t) \) has polynomials or formal power series expansion in the \( \alpha \)-monomials \( t^\alpha \). Then, for all integer \( n > 0 \),

\[ N_{q,\alpha}(t^\alpha f_\alpha(t)) = (-1)^n q^{\binom{n}{2}} \alpha u^\alpha (D_s^{q,\alpha})^n R_{q,\alpha}(u, q^{-n}s), \]

where \( R_{q,\alpha}(u, s) = N_{q,\alpha}(f_\alpha(t)) \).

**Proof.** We have

\[ (D_s^{q,\alpha})^n R_{q,\alpha}(u, q^{-n}s) = (D_s^{q,\alpha})^n \int_0^\infty f_\alpha(ut) e_{q,\alpha}^{-1}(q^{-(n-1)}st)d_q \alpha t \]

\[ = \int_0^\infty f_\alpha(ut) (D_s^{q,\alpha})^n e_{q,\alpha}^{-1}(q^{-(n-1)}st)d_q \alpha t. \]

By using Lemma 3.1, we get

\[ (D_s^{q,\alpha})^n R_{q,\alpha}(u, q^{-n}s) = \int_0^\infty f_\alpha(ut)(-1)^n t^\alpha q^{-\binom{n}{2}} \alpha e_{q,\alpha}^{-1}(qst)d_q \alpha t \]

\[ = (-1)^n q^{-\binom{n}{2}} \alpha u^\alpha \int_0^\infty (ut)^\alpha f_\alpha(ut)e_{q,\alpha}^{-1}(q^{-(n-1)}st)d_q \alpha t \]

\[ = (-1)^n q^{-\binom{n}{2}} \alpha u^\alpha N_{q,\alpha}(t^\alpha f_\alpha(t)). \]

The rearrangement of the last equation completes the proof.

The natural transform is a function of two variables, namely, \( u \) and \( s \). The previous proposition establishes the relationship between the transform of the product of \( f_\alpha(t) \) with a positive power of \( \alpha \)-monomials \( t^\alpha \) and the \( q, \alpha \)-deformed derivative with respect to one of the variables, namely, \( s \), of the \( q, \alpha \)-transform. We now consider a derivative of the deformed transform with respect to another its variable \( u \).
Proposition 3.5. Suppose that a function \( f_\alpha(t) \) has the following expansion:

\[
f_\alpha(t) = \sum_{m=0}^{\infty} a_m t^{\alpha m}.
\]

Then, for all integer \( n > 0 \), the following relation is true:

\[
N_{q,\alpha}(t^{\alpha n} f_\alpha(t)) = \frac{u^{\alpha n}}{s^{\alpha n}} (D_u^q u^{\alpha})^n u^{\alpha n} R_{q,\alpha}(u, s).
\]

Proof. If

\[
f_\alpha(t) = \sum_{m=0}^{\infty} a_m t^{\alpha m},
\]

then

\[
N_{q,\alpha}(t^{\alpha n} f_\alpha(t)) = N_{q,\alpha} \left( \sum_{m=0}^{\infty} a_m t^{\alpha(m+n)} \right).
\]

By using the linearity of the transform and (3.3), we get

\[
N_{q,\alpha}(t^{\alpha n} f_\alpha(t)) = \sum_{m=0}^{\infty} \frac{u^{\alpha(n+m)}}{s^{\alpha(n+m+1)}} [(n + m)\alpha]! a_m
\]

\[
= \frac{u^{\alpha n}}{s^{\alpha n}} \sum_{m=0}^{\infty} \frac{u^{\alpha m}}{s^{\alpha(m+1)}} [(n + m)\alpha]! a_m
\]

\[
= \frac{u^{\alpha n}}{s^{\alpha n}} \sum_{m=0}^{\infty} (D_u^q u^{\alpha})^n \frac{[m\alpha]! a_m u^{\alpha(n+m)}}{s^{\alpha(m+1)}}
\]

\[
= \frac{u^{\alpha n}}{s^{\alpha n}} \sum_{m=0}^{\infty} (D_u^q u^{\alpha})^n \frac{u^{\alpha n} \cdot [m\alpha]! a_m u^{\alpha m}}{s^{\alpha(m+1)}}
\]

\[
= \frac{u^{\alpha n}}{s^{\alpha n}} (D_u^q u^{\alpha})^n u^{\alpha n} N_{q,\alpha}(f_\alpha(t)).
\]

The replacement \( N_{q,\alpha}(f_\alpha(t)) \) by \( R_{q,\alpha}(u, s) \) in the last equation completes the proof.

These results are in complete agreement with the results obtained for the nondeformed Sumudu and natural transform investigated by Belgacem, et al. (see [5] and the references therein). We now give another representation of the \( q,\alpha \)-deformed natural transform of the product of \( f_\alpha(t) \) with positive degree of the \( \alpha \)-monomial \( t^{\alpha} \). This proposition extends Theorem 4.2 in [5].

Proposition 3.6. Suppose that a function \( f_\alpha(t) \) has a polynomial or formal power series expansion in \( \alpha \)-monomials \( t^{\alpha} \). Then, for all integer \( n > 0 \),

\[
N_{q,\alpha}(t^{\alpha n} f_\alpha(t)) = \frac{u^{\alpha n}}{s^{\alpha n}} \sum_{k=0}^{n} b_{n,k} u^{\alpha k} (D_u^q u^{\alpha})^k R_{q,\alpha}(u, s),
\]
where the coefficients $b_{n,k}$ satisfy the recurrence relations

$$b_{n,k} = \begin{cases} [n\alpha]b_{n-1,0}, & k = 0, \\ [(n+k)\alpha]b_{n-1,k} + q^{\alpha(n-1+k)}b_{n-1,k-1}, & 0 < k < n, \\ q^{\alpha(2n-1)}b_{n-1,n-1}, & k = n, \end{cases}$$

with the initial condition $b_{0,0} = 1$.

**Proof.** We proceed by induction on $n$. For $n = 0$, we get

$$N_{q,\alpha}(f_\alpha(t)) = R_{q,\alpha}(u, s)$$

and, hence, $b_{0,0} = 1$. By the previous proposition, for $n = 1$, we find

$$N_{q,\alpha}(t^\alpha f_\alpha(t)) = \frac{u^\alpha}{s^\alpha} D_u q^\alpha u^\alpha R_{q,\alpha}(u, s).$$

By applying the deformed Leibnitz rule (2.4), this relation can be rewritten as

$$N_{q,\alpha}(t^\alpha f_\alpha(t)) = \frac{u^\alpha}{s^\alpha} ([\alpha] R_{q,\alpha}(u, s) + q^\alpha u^\alpha D_u q^\alpha R_{q,\alpha}(u, s)).$$

Thus,

$$b_{1,0} = [\alpha] = [\alpha]b_{0,0}, \quad b_{1,1} = q^\alpha = q^\alpha b_{0,0},$$

and the claim holds. We now assume that the claim holds for $m \leq n$ and prove that it is true for $m = n + 1$:

$$N_{q,\alpha}(t^{\alpha(n+1)} f_\alpha(t)) = N_{q,\alpha}(t^{\alpha(n+1)} f_\alpha(t))$$

$$= \frac{u^\alpha}{s^\alpha} D_u q^\alpha u^\alpha \sum_{k=0}^n b_{n,k} u^\alpha(D_u q^\alpha)^k R_{q,\alpha}(u, s)$$

$$= \frac{u^\alpha}{s^\alpha} D_u q^\alpha \sum_{k=0}^n b_{n,k} u^{\alpha(1+n+k)} \frac{s^\alpha}{s^\alpha} (D_u q^\alpha)^k R_{q,\alpha}(u, s)$$

$$= \frac{u^\alpha}{s^\alpha} \sum_{k=0}^n b_{n,k} [(n+k+1)\alpha]u^{\alpha(n+k)}(D_u q^\alpha)^k R_{q,\alpha}(u, s)$$

$$+ \frac{u^\alpha}{s^\alpha} \sum_{k=0}^n b_{n,k} q^{\alpha(1+n+k)}u^{\alpha(1+n+k)}(D_u q^\alpha)^{k+1} R_{q,\alpha}(u, s)$$

$$= \frac{u^\alpha}{s^\alpha(n+1)} \sum_{k=0}^n b_{n,k} [(n+k+1)\alpha]u^{\alpha k}(D_u q^\alpha)^k R_{q,\alpha}(u, s)$$
\[ u^{\alpha(n+1)} = \sum_{k=1}^{n-1} b_{n,k-1} q^{\alpha(n+k)} u^{\alpha k} (D_q^{\alpha})^k R_{q,\alpha}(t, s) \]

\[ = u^{\alpha(n+1)} \sum_{k=0}^{n+1} b_{n+1,k} u^{\alpha k} (D_q^{\alpha})^k R_{q,\alpha}(t, s), \]

where

\[ b_{n+1,0} = b_{n,0}(n + 1) \alpha, \]

\[ b_{n+1,k} = b_{n,k}(n + k + 1) \alpha + b_{n,k-1} q^{\alpha(n+k)}, \quad 1 \leq k \leq n, \]

\[ b_{n+1,n+1} = b_{n,n} q^{\alpha(2n+1)}, \]

which completes the proof.

At end of the paper, we make the following conclusion: Our new generalization of the natural transform also offers new generalizations for some other widely used integral transforms. By applying the techniques described in the present work, we can solve a \( k \)-order linear \( q \)-differential equation with constant coefficients. There is no need to find separately homogeneous solution and a particular solution. In order to solve a differential equation by applying the integral transform, it is necessary to know the integral transform of the right-hand side function of the differential equation

\[ \sum_{0 \leq j \leq k} a_j (D_x^{\alpha})^{k-j} f(x) = b(x) \]

and the initial conditions \( (D_x^{\alpha})^j y(0) = y_j \) for \( j = 0, \ldots, k - 1 \).

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