Diophantine Approximation on varieties I: Algebraic distance and metric Bézout Theorem

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Abstract

For two properly intersecting effective cycles in projective space $X, Y$, and their intersection product $Z$, the metric Bézout Theorem relates the degrees, heights of $X, Y$, and $Z$, as well as their distances and algebraic distances to a given point $\theta$. Applications of this Theorem are in the area of Diophantine Approximation, giving estimates for approximation properties of $Z$ with respect to $\theta$ against the ones of $X$, and $Y$.

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1 Introduction

Let \( k \) be a number field with ring of integers \( \mathcal{O}_k \) and \( \mathcal{X} \) a flat, integral, quasi projective scheme of finite type over \( \mathcal{O}_k \). The dimension of \( \mathcal{X} \) will be denoted by \( t + 1 \), and the base extension of \( \mathcal{X} \) to \( k \) by \( X \). Let further \( \mathcal{L} \) be an ample metrized line bundle on some projective closure \( \bar{\mathcal{X}} \) of \( \mathcal{X} \), and \( \sigma : k \hookrightarrow \mathbb{C} \) a fixed embedding which gives rise to a base extension \( \psi : X_{\sigma} = \mathcal{X} \times_{\mathcal{O}_k} \text{Spec} \mathcal{O}_k \to \mathcal{X} \). On \( X_{\sigma}(\mathbb{C}) \) choose any metric \( | \cdot, \cdot | \) that induces the usual topology on \( X_{\sigma}(\mathbb{C}) \).

For a point \( \alpha \in \mathcal{X}(\bar{k}) \) denote \( \kappa(\alpha) = \mathcal{O}_{X,\alpha}/m_\alpha \) the residue field of \( \alpha \) which is a finite extension of \( k \). The stalk \( \mathcal{L}_\alpha \) of \( \mathcal{L} \) at \( \alpha \) is a one dimensional \( \mathcal{O}_{X,\alpha} \)-module, thus \( \mathcal{L}_\alpha/m_\alpha \mathcal{L}_\alpha \) is a one dimensional \( \kappa(\alpha) \)-vector space. Hence, with \( f_0 \in \Gamma(\bar{\mathcal{X}}, \mathcal{L}) \) such that \( f_0|_{\alpha} \notin m_\alpha \mathcal{L}_\alpha \), one gets a \( k \)-linear map

\[
r_D : \Gamma(\mathcal{X}, \mathcal{L} \otimes D) \to \kappa(\alpha), \quad f \mapsto a \quad \text{such that} \quad f_0 a (f_0)^\otimes_D \in m_\alpha,
\]

whose image does not depend on the choice of \( f_0 \).

Define \( X_D(\bar{k}) \) as the set of \( \alpha \in \mathcal{X}(\bar{k}) \) such that \( r_D \) is a surjection onto \( \kappa(\alpha) \), and \( X_{D,H}(\bar{k}) \) as the set of \( \alpha \in \mathcal{X}(\bar{k}) \) such that the image of the global sections \( f \in \Gamma(\mathcal{X}, \mathcal{L} \otimes D) \) with \( |f| = \log |f|_{L^2(X)} \leq H \) under \( r_D \) spans \( \kappa(\alpha) \) as a \( k \)-vector space.

To a point \( \theta \in X_{\sigma}(\mathbb{C}) \) we attach several numbers: With \( x = \psi(\theta) \), set \( \kappa(x) = \mathcal{O}_{X,x}/m_x \) which is a finitely generated extension of \( k \), and set \( t(\theta) \in \mathbb{N} \) equal to the transcendence degree of \( \kappa(x) \) which equals the dimension of the closure of \( x \).

The other numbers attached to \( \theta \) are defined via the approximability of \( \theta \) by algebraic points and hypersurfaces. For \( D \in \mathbb{N} \) a \( D \)-frame \( F \) is a pair \((\alpha, f)\) consisting of a point \( \alpha \in X_D(\bar{k}) \) together with a nonzero \( f \in \Gamma(\bar{\mathcal{X}}, \mathcal{L} \otimes D) \) such that \( f_\alpha \neq 0 \). For \( D \in \mathbb{N} \), \( H \in \mathbb{R}_{\geq 0} \), a \((D, H)\)-frame \( F \) consists of an \( \alpha \in X_{D,H}(\bar{D}) \) together with a nonzero \( f \in \Gamma(\mathcal{X}, \mathcal{L} \otimes D) \) of logarithmic length at most \( H \) such that \( f_\alpha \neq 0 \). Denote by \( \mathcal{F}_D, \mathcal{F}_{D,H} \) the set of \( D \)-frames and \((D, H)\)-frames respectively.

For \( F = (\alpha, f) \) an element of \( \mathcal{F}_D \) or \( \mathcal{F}_{D,H} \) and \( \theta \in X(\mathbb{C}_\sigma) \) a nonalgebraic point, define

\[
D(F, \theta) := \max(\log |\alpha, \theta|, \log |f_\theta|), \quad h(\alpha) \quad \text{the height of} \ \alpha,
\]

and further the lower and upper approximational degrees

\[
\bar{l}_1(\theta) := \sup \left\{ s \in \mathbb{R} \mid \limsup_{D \to \infty} \left( - \inf_{F = (\alpha, f) \in \mathcal{F}_D} \left\{ \frac{D(F, \theta)}{(h(\alpha) + \deg \alpha)(\deg \alpha)^s} \right\} \right) = \infty \right\} \in \mathbb{R}^\geq, \\
\bar{l}_2(\theta) := \sup \left\{ s \in \mathbb{R} \mid \limsup_{D \to \infty} \left( - \inf_{F = (\alpha, f) \in \mathcal{F}_D} \left\{ \frac{D(F, \theta)}{(D + \log |f|)D^s} \right\} \right) = \infty \right\} \in \mathbb{R}^\geq, \\
\bar{t}_2(\theta) := \sup \left\{ s \in \mathbb{R} \mid \liminf_{D \to \infty} \left( - \inf_{F = (\alpha, f) \in \mathcal{F}_D} \left\{ \frac{D(F, \theta)}{(D + \log |f|)D^s} \right\} \right) = \infty \right\} \in \mathbb{R}^\geq, \\
\bar{t}_4(\theta) := \sup \left\{ s \in \mathbb{R} \mid \limsup_{D \to \infty} \left( - \inf_{F = (\alpha, f) \in \mathcal{F}_{D,a}} \left\{ \frac{D(F, \theta)}{a(\deg \alpha)^{s+1}} \right\} \right) = \infty \right\} \in \mathbb{R}^\geq,
\]

and for \( a \in \mathbb{R}^+ \) the upper and lower approximational \( a \)-degree
\[ t_1^a(\theta) := \sup \left\{ s \in \mathbb{R} \mid \limsup_{D \to \infty} \left( \inf_{F=(\alpha,f) \in F_{D,aD}} \left\{ \frac{D(F,\theta)}{aD^{s+1}} \right\} \right) = \infty \right\} \in \mathbb{R}^\geq, \]
\[ \bar{t}_2^a(\theta) := \sup \left\{ s \in \mathbb{R} \mid \liminf_{D \to \infty} \left( \inf_{F=(\alpha,f) \in F_{D,aD}} \left\{ \frac{D(F,\theta)}{D^{s+1}} \right\} \right) = \infty \right\} \in \mathbb{R}^\geq, \]

The main objective of this series of papers is to prove the

1.1 Theorem

Let \( X \) be any flat, integral, quasi projective scheme over \( \mathcal{O}_k \), choose an embedding \( \sigma : k \to \mathbb{C} \), let \( \mathcal{L} \) be a metrized line bundle on \( X \), and choose a metric on \( X_{\sigma}(\mathbb{C}) \) that induces the usual topology. With \( a \gg 0 \), for every \( \theta \in X(\mathbb{C}_\sigma) \),

1. \( t_1(\theta) t_2(\theta) \geq t_1^a(\theta) t_2^a(\theta) \geq 1 \geq t_1^a \bar{t}_2(\theta) \geq t_1(\theta) \bar{t}_2(\theta) \).

2. If \( a \) is sufficiently big, for every \( \theta \in X_{\sigma} \)

\[ \limsup_{D \to \infty} \left( \inf_{F=(\alpha,f) \in F_{D}} \left\{ \frac{D(F,\theta)}{a(\text{deg} \alpha)^{1+\frac{1}{t}}} \right\} \right) \geq b > 0, \]

where \( b \) only depends on \( t \) and the degree and height of \( X \); hence, the inequalities

\[ t_1(\theta) \geq t_1^a(\theta) \geq \frac{1}{t(\theta)} \]

hold, and consequently

\[ t(\theta) \geq \bar{t}_2^a(\theta) \geq \bar{t}_2(\theta). \]

3. If \( Y \) is any subscheme of \( X \), then for all \( \theta \in Y_{\sigma}(\mathbb{C}) \) with the exception of a subset of measure zero, the equalities

\[ t(\theta) = \frac{1}{t_1(\theta)} = \frac{1}{t_2^a(\theta)} = t_2(\theta) = \bar{t}_2(\theta) = t_2^a(\theta) = \bar{t}_2^a(\theta) \]

hold.

The lower bounds for \( t_1 \) and \( t_1^a \) in part two are statements about the approximability of \( \theta \) by algebraic points. It entails

1.2 Theorem In the situation of the previous Theorem, there exists a positive real number \( b \) only depending on \( t(\theta) \) and the degree and height of the closure of \( \psi(\theta) \), such that for any sufficiently big real number \( a \), there is an infinite subset \( M \subset \mathbb{N} \) such that for all \( D \in M \) there exists an algebraic point \( \alpha_D \in X(\mathbb{Q}) \) fulfilling

\[ \text{deg}(\alpha_D) \leq D^t, \quad h(\alpha_D) \leq aD^t, \quad \text{and} \quad \log |\alpha_D, \theta| \leq -abD^{t+1}, \]

where \( h(\alpha_D) \) denotes the height of \( \alpha_D \), and \( |\cdot, \theta| \) the distance to the point \( \theta \) with respect to any metric on \( X(\mathbb{C}_\sigma) \).
The lower bound for $t(\theta)$ from Theorem 1.1.2 may serve to prove lower bounds for transcendence degrees and will be exploited in the fifth paper of this series. Theorem 1.1 will be proved in part 3 of this series ([Ma2]). The first inequality in its second part is best possible in the following sense: There is a constant $B > 0$ such that the set of points $\theta \in X(\mathbb{C}_b)$ fulfilling the theorem with $b$ replaced by $B$ has measure 0. For $t = 1$, the theorem in slightly different formulation was already proved in [RW].

The quantities $|X, \theta|$ and $|f_\theta|$ defining $D(F, \theta)$ are important for applications, but rather ill-suited for proofs as they don’t fit into the framework of algebraic geometry: $|f_\theta|$ describes “distances” of points $\theta \in X_\sigma$ only to cycles of codimension one, namely to div $f$ for $f \in \Gamma(X, \mathcal{L})$, and $|X, \theta|$ has no functorial properties and does not behave well with respect to intersections.

To overcome these deficiencies, various authors (Philippon [Ph1], Nesterenko [Nes]) developed a new kind of distance between an effective cycles and a point in projective space that semi officially has been called algebraic distance. Although this lead to proofs of hitherto unknown approximation Theorems ([Ph2]), strong theorems for approximation of transcendental points, that have good geometrical interpretation and thus tie in well with the calculus of distances $|x, \theta|$, $|X, \theta|$ of points $\theta$ to points or cycles could only be made in the one dimensional case ([RW]).

This paper taking the concepts and results of [RW] as a starting point presents a geometrical approach to the concept of algebraic distances in the framework of Arakelov geometry. It allows to prove amply stronger versions of arithmetic Bézout Theorems in the higher dimensional case, that accordingly have many new applications. In the present context, the arithmetic Bézout Theorem will be used to prove upper estimates for the algebraic distance of the intersection of two cycles to a point $\theta$, given that both cycles have small algebraic distances to $\theta$ compared with their degrees and heights, as well as upper estimates for the distances of points $\alpha$ to $\theta$ when $\alpha$ belongs to a certain frame $(\alpha, f)$. Also, it will give upper bounds for the approximational degrees as stated in the last two inequalities of Theorem 1.1.1.

Proving the existence of the points $\alpha$ claimed in the Theorems above due to the metric Bézout Theorem reduces to finding sufficiently many hyperplanes with small algebraic distance to a point $\theta$ that intersect properly, hence produce cycles of higher codimension, in particular points, with small algebraic distance to $\theta$. These hyperplanes can be found once one has good explicit upper bounds for arithmetic Hilbert functions which will be proved in the second paper of this series.

## 2 The main results

Let $E = \mathbb{Z}^{t+1}$, and equip $E_{\mathbb{C}} := E \otimes_{\mathbb{Z}} \mathbb{C}$ with the standard hermitian product. It induces a hermitian metric on the line bundle $O(1)$ on the projective space

$$\mathbb{P}^t = \text{Proj}(\text{Sym}(\hat{E})).$$
which in turn defines an $L^2$-norm on the space of global sections $\Gamma(\mathbb{P}^t(\mathbb{C}), O(D))$, and a height $h(X)$ for any effective cycle $X$ on $\mathbb{P}^t$. A subscheme of $\mathbb{P}^t$ of dimension greater zero that has nonempty generic fibre will be called a subvariety.

Let further $|\cdot, \cdot|$ denote the Fubini-Study metric on $\mathbb{P}^t(\mathbb{C})$, and $\mu$ its Kähler form which also is the Chern form of $O(1)$. Finally for any projective subspace $\mathbb{P}(W) \subset \mathbb{P}^t(\mathbb{C})$, denote by $\rho_{\mathbb{P}(W)}$ the function

$$\rho_{\mathbb{P}(W)} : \mathbb{P}^t(\mathbb{C}) \setminus \mathbb{P}(W) \to \mathbb{R}, \ x \mapsto \log |x, \mathbb{P}(W)|.$$  

For $X \in Z^p(\mathbb{P}^t)$ an effective cycle of pure codimension $p$, and $\theta$ a point in $\mathbb{P}^t(\mathbb{C}) \setminus \text{supp}(X)$ the logarithm of the distance $\log |\theta, X|$ is defined to be the minimum of the restriction of $\rho_{\theta}$ to $X$. There are various different definitions of the algebraic distance of $\theta$ to $X = X'(\mathbb{C})$ all identical modulo a constant times $\deg X$; the simplest being

2.1 Definition With the above notations, and $\Lambda_{\mathbb{P}(W)}$ the Levine form of a projective subspace $\mathbb{P}(W)$, define the algebraic distance of $\theta \notin \text{supp}(X)$ to $X$ as

$$D(\theta, X) := \sup_{\mathbb{P}(W)} \int_{X(\mathbb{C})} \Lambda_{\mathbb{P}(W)} - \deg X \sum_{n=1}^{q} \sum_{m=0}^{t-q} \frac{1}{m+n},$$

where the supremum is taken over all spaces $\mathbb{P}(W) \subset \mathbb{P}^t$ of codimension $t + 1 - p$ that contain $\theta$. In case $\text{supp}(X) = \emptyset$, the algebraic distance $D(\theta, X)$ is defined to be zero.

2.2 Proposition Let $X \in Z_{\text{eff}}(\mathbb{P}^t)_{\mathbb{C}}$ be an effective cycle and $\theta \in \mathbb{P}^t(\mathbb{C})$ a point that is not contained in the support of $X$. There are effectively computable constants $c, c'$ only depending on $t$ such that

$$\deg(X) \log |\theta, X(\mathbb{C})| \leq D(\theta, X) + c \deg X \leq \log |\theta, X(\mathbb{C})| + c' \deg X.$$  

2.3 Proposition Let $f \in \Gamma(\mathbb{P}^t_{\mathbb{C}}, O(D))$, and $X = \text{div} f$. Then,

$$h(X) \leq \log |f_D|_{L^2} + D\sigma_t, \quad \text{and}$$

$$D(\theta, X) + h(X) = \log |f_\theta| + D\sigma_t,$$

where the $\sigma_t'$s are certain constants.
2.4 Theorem (Metric Bézout Theorem) Let $p + q \leq t + 1$, and $X, Y \in \mathbb{P}^t_{\mathbb{C}}$ be effective cycles of pure codimensions $p$ and $q$ respectively intersecting properly. There is an effectively computable positive constant $e$, only depending on $t, p, q$, and for each $\theta$ a point in $\mathbb{P}^t(\mathbb{C}) \setminus (\text{supp}(X_\square \cup Y_\square))$ there exists a map

$$f_{X,Y}: \deg X + \deg Y \to \deg X \times \deg Y$$

from the set of natural numbers less or equal $\deg X + \deg Y$ to the set of natural numbers less or equal $\deg X \times \deg Y$ such the maps $pr_1 \circ f_{X,Y}, pr_2 \circ f_{X,Y}$ are monotonously increasing and surjective, $f_{X,Y}$ is a right inverse to the sum and for every $T \in \deg X + \deg Y$, with $(\nu, \kappa) = f_{X,Y}(T)$, the inequality

$$\nu \kappa \log |\theta, X + Y| + D(\theta, X.Y) + h(\mathcal{X}, \mathcal{Y}) \leq \
\kappa D(\theta, X) + \nu D(\theta, Y) + \deg Y h(\mathcal{X}) + \deg X h(\mathcal{Y}) + e \deg X \deg Y$$

holds.

For $t = 1$, this Theorem has been proved in [RW].

2.5 Corollary In the situation of the Theorem,

1. if either $D(X, \theta) \leq \log |Y, \theta|$ or $|X, \theta| \leq |Y, \theta|$, then with $e'$ an effectively computable constant only depending on $t, p, q$,

$$D(\theta, X.Y) + h(\mathcal{X}, \mathcal{Y}) \leq D(\theta, Y) + \deg Y h(\mathcal{X}) + \deg X h(\mathcal{Y}) + e' \deg X \deg Y.$$

2. in any case

$$D(\theta, X.Y) + h(\mathcal{X}, \mathcal{Y}) \leq \max(D(\theta, X), D(\theta, Y)) + \deg Y h(\mathcal{X}) + \deg X h(\mathcal{Y}) + e' \deg X \deg Y.$$

A variant of this corollary has been proved in [Ph1] in case $\mathcal{X}$ is a hypersurface.

The crucial idea for proving the Theorem is to express the algebraic distance of an effective cycle $\mathcal{X}$ of pure codimension $p$ to a point $\theta$ as the sum of the distances of $\theta$ to certain points lying on $\mathcal{X}$, namely the points forming the intersection of $\mathcal{X}$ with a suitable projective subspace of $\mathbb{P}^t$ of dimension $p = \text{codim } \mathcal{X}$. More precisely,

2.6 Theorem For $p \leq t$ there are effectively computable constants $c, c'$ only depending on $p$ and $t$, such that for all effective cycles $X$ of pure codimension $p$ in $\mathbb{P}^t_{\mathbb{C}}$, and points $\theta \in \mathbb{P}^t(\mathbb{C})$ not contained in $\text{supp}(X)$, there is a subspace $\mathbb{P}(F) \subset \mathbb{P}^t_{\mathbb{C}}$ of codimension $t - p$ containing $\theta$, and properly intersecting $X$, such that

$$D(\theta, X) \leq \sum_{x \in \text{supp}(X, \mathbb{P}(F))} n_x \log |x, \theta| + c \deg X \leq D(\theta, X) + c' \deg X,$$

where the $n_x$ are the intersection multiplicities of $X$ and $\mathbb{P}(F)$ at $x$.

This Theorem is proved in section 4.
3 Arakelov varieties

This section mainly collects various well known facts about Arakelov varieties, most of which can be found in \[SABK\], \[GS1\], or \[BGS\].

Let \(X\) a regular, flat, projective scheme of relative dimension \(d\) over \(\text{Spec } \mathbb{Z}\). Such a scheme is called a projective arithmetic variety over \(\text{Spec } \mathbb{Z}\) The base extensions of \(X\) to \(\mathbb{Q}\) and \(\mathbb{C}\), as well as their \(\mathbb{C}\) valued point \(X(\mathbb{C})\) will all be denoted by \(X\) if no confusion arises. On the \(\mathbb{C}\)- valued points \(X(\mathbb{C})\) we have the space of smooth forms of type \((p,p)\) invariant under complex conjugation \(F_\infty\) denoted by \(A_{p,p}\), the space \(\tilde{A}_{p,p} := A_{p,p}/(\text{Im } \partial + \text{Im } \bar{\partial})\), and the space of currents \(D_{p,p}\) which is the space of Schwartz continuous linear functionals on \(A^{d-p,d-p}\), and \(\tilde{D}_{p,p} = D_{p,p}/(\text{Im } \partial + \text{Im } \bar{\partial})\).

On \(D_{p,p}\) the maps \(\partial, \bar{\partial}, d = \partial + \bar{\partial}, d^c = \partial - \bar{\partial}\) are defined by duality.

A cycle \(Y \subset Z^p(X(\mathbb{C}))\) of pure codimension \(p\) defines a current \(\delta_Y \in D^{p-p}\) by

\[
\omega \in A^{d-p,d-p} \mapsto \sum_i n_i \int_{Y_i} \omega,
\]

where \(Y = \sum_i n_i Y_i\) is the decompositions into irreducible components leading an embedding \(\iota : A^{p,p} \hookrightarrow D^{p,p}, \omega \mapsto [\omega]\). The integrals are defined by resolution of singularities, see \[GS1\] or \[SABK\]. A Green current \(g_Y\) for \(Y\) is a current of type \((p-1,p-1)\) such that

\[
d d^c g_Y + \delta_Y \in \iota(A^{p,p}).
\]

A densely defined form \(g_Y\) on \(X\) is called a Green form for \(Y\) if \([g_Y] := \iota(g_Y)\) exists and is a Green current for \(Y\); it is called of logarithmic type along \(Y\) if it has only logarithmic singularities at \(Y\) (see \[SABK\] Def. II.2.3).

If \(y\) is a point in \(X^{(p-1)}\), that is \(Y = \{y\}\) is a closed integral sub scheme of codimension \(p\) in \(X\), a rational function \(f \in k(y)^*\) gives rise to the Green form of log type \(- \log |f|^2\) for \(\text{div}(f)\). (\[SABK\], III.1)

The group of arithmetic cycles \(\tilde{Z}^p(X)\) consisting of the pairs \((Y, g_Y)\) where \(Y\) is a cycle of pure codimension \(p\) and \(g_Y\) a Green current for \(Y(\mathbb{C})\), thus contains the subgroup \(\tilde{R}^p(Z)\) generated by the pairs \((\text{div}(f), -|\log |f|^2|), with f \in k(y)^*, y \in X^{(p-1)}\) and \((0, \partial(u) + \bar{\partial}(v))\), where \(u\) and \(v\) are currents of type \((p-2, p-1)\) and \((p-1, p-2)\) respectively. If we set \(\tilde{Z}^p(X) := \tilde{Z}^p(X)/(\text{Im } \partial + \text{Im } \bar{\partial}), and \tilde{R}^p(X) := \tilde{R}^p(X)/(\text{Im } \partial + \text{Im } \bar{\partial})\), we get

**3.1 Definition** The arithmetic Chow group \(\tilde{CH}^p(X)\) of codimension \(p\) of \(X\) is defined as the quotient \(\tilde{Z}^p(X)/\tilde{R}^p(X)\).

**3.2 Examples**
1. For $S = \text{Spec} \mathbb{Z}$, it is easily calculated (see [BGS], 2. 1. 3)

\[
\hat{CH}^0(S) \cong \mathbb{Z}, \quad \hat{CH}^1(S) \cong \mathbb{R}, \quad \hat{CH}^p(S) = 0, \quad \text{if } p > 1.
\]

The isomorphism of $\hat{CH}^1(S)$ to $\mathbb{R}$ is usually denoted $\hat{deg}$.

2. If $\bar{L}$ is an ample metric line bundle on $X$, and $f \in \Gamma(X, L)$ is a global section

then $[-\log |f|^2]$, by the Poincaré-Lelong formula, is a Green current for $\text{div} f$; we have $dd^c[-\log |f|^2] + \delta_{\text{div} f} = c_1(L)$ the Chern form of $L$. The class

\[
\hat{c}_1(\bar{L}) := (\text{div} f, [-\log |f|^2]) \in \hat{CH}^1(X)
\]

is called the first Chern class of $\bar{L}$.

For the purposes of this paper a subgroup of the arithmetic Chow group, the Arakelov Chow group will play a much more important role. Suppose $X(\mathbb{C})$ is equipped with a Kähler metric with Kähler form $\mu$. The pair $(X, \mu)$ is denoted $\bar{X}$, and called an Arakelov variety. If $H^{p,p}(X)$ denotes the harmonic forms with respect to the chosen metric, and

\[
H : D^{p,p}(X) \to H^{p,p}(X)
\]

the harmonic projection, define

\[
Z^p(\bar{X}) := \{(Y, g_Y)|dd^c g_Y + \delta_Y \in H^{p,p}(X)\} \subset \hat{Z}^p(X).
\]

As $\hat{R}^p(\bar{X}) \subset Z^p(\bar{X})$, one can define the Arakelov Chow group

\[
CH^p(\bar{X}) := Z^p(\bar{X})/\hat{R}^p(\bar{X}).
\]

For $[(Y, g_Y)] \in CH^p(\bar{X})$ we have

\[
dd^c g_Y + \delta_Y = \omega_Y = H(\omega_Y) = H(dd^c g_Y) + H(\delta_Y) = H(\delta_Y).
\] (1)

A Green $g_Y$ current with $dd^c g_Y + \delta_Y = H(\delta_Y)$ i. e. $(X, g_X) \in Z(\bar{X})$ is called admissible.

In [GS1] Gillet and Soulé define the star product

\[
[g_Y] * g_Z := [g_Y] \wedge \delta_Z + g_Z \wedge \omega_Y
\]

of a Green current $[g_Y]$ coming from a Green form $g_Y$ of log type along $Y$ with a Green current $g_Z$ for properly intersecting $Y$, and $Z$. 

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3.3 Proposition If $dd^c[g_Y] + \delta_Y = [\omega_Y]$, and $dd^c g_Z + \delta_Z = [\omega_Z]$, then

$$dd^c([g_Y] * g_Z) + \delta_{Y,Z} = [\omega_Y \wedge \omega_Z].$$

Further, the star product is commutative and associative modulo $\text{Im} \partial + \text{Im} \bar{\partial}$. As by [GST] every cycle has a Green form of log type, there is an intersection product

$$\widehat{CH}^p(\mathcal{X}) \times \widehat{CH}^q(\mathcal{X}) \to \widehat{CH}^{p+q}(\mathcal{X}), \quad ([\mathcal{Y}, [g_Y]], [\mathcal{Z}, g_Z]) \mapsto ([\mathcal{Y}, \mathcal{Z}], [g_Y] * g_Z),$$

where $\mathcal{Y}, \mathcal{Z}$ are chosen to intersect properly, which is commutative and associative. If on $\mathcal{X}$ the product of two harmonic forms is always harmonic, the above product makes $CH(\mathcal{X})^*$ into a subring of $\widehat{CH}^*(\mathcal{X})$.

Proof [GST], II, Theorem 4.

3.4 Proposition Let $\mathcal{X}, \mathcal{Y}$ be arithmetic varieties over $\text{Spec } \mathbb{Z}$, and $f : \mathcal{X} \to \mathcal{Y}$ a morphism.

For $(Z, g_Z) \in \mathcal{Z}^p(Y)$ we have $dd^c f^* g_Z + \delta_{f^*(Z)} = f^* \omega_Z$, and the densely defined map

$$f^* : \mathcal{Z}^p(Y) \to \mathcal{Z}^p(X), \quad (Z, g_Z) \mapsto (f^*(Z), f^* g_Z),$$

induces a multiplicative pull-back homomorphism $f^* : \widehat{CH}^p(\mathcal{Y}) \to \widehat{CH}^p(\mathcal{X})$.

If $f$ is proper, $f_Q : X_Q \to Y_Q$ is smooth, and $X, Y$ are equidimensional, then $dd^c f_* g_Z + \delta_{f_*(Z)} = f_* \omega_Z$ for any $(Z, g_Z) \in \mathcal{Z}^p(\mathcal{X})$. This induces a push-forward homomorphism

$$f_* : \widehat{CH}^p(\mathcal{X}) \to \widehat{CH}^{p-\delta}(\mathcal{Y}), \quad (\delta := \dim Y - \dim Z).$$

If $f_*(f^* \text{ respectively})$ map harmonic forms to harmonic forms, they induce homomorphisms of the Arakelov Chow groups.

Proof [SABK], Theorem III. 3.

The following Proposition enables calculations in $\widehat{CH}^*(\mathcal{X})$ and $CH^*(\mathcal{X})$.

3.5 Proposition Let $a : A^{p-1,p-1}(X) \to \widehat{CH}^p(\mathcal{X})$ be the map $\eta \mapsto [(0, \eta)]$, and $\zeta : \widehat{CH}^p(\mathcal{X}) \to \text{CH}(\mathcal{X})$ the map $[(\mathcal{Y}, g_Y)] \mapsto [\mathcal{Y}]$. With $\mathcal{Z}^p(\mathcal{X}) = \mathcal{Z}^p(\mathcal{X})/(\text{Im} \partial + \text{Im} \bar{\partial})$, the diagram
is commutative, and the rows are exact.

**Proof** [GS1]

If $\mathcal{Y} \in Z^p(\mathcal{X})$ and $g_\mathcal{Y}, g'_\mathcal{Y}$ are two admissible Green currents for $Y$, the exactness of the first row implies $g_\mathcal{Y} - g'_\mathcal{Y} = \eta \in H^{p-1,p-1}(X)$. Hence, the projection of $g_\mathcal{Y}$ to the orthogonal complement of $H^{p-1,p-1}$ in $\tilde{D}^{p-1,p-1}$ is independent of $g_\mathcal{Y}$, and one can define the map $s : Z^p(\mathcal{X}) \rightarrow Z^p(\tilde{\mathcal{X}})$ by $Y \mapsto (Y, g_\mathcal{Y})$ where $g_\mathcal{Y}$ is the unique Green form of log type for $Y$ which is orthogonal to $H^{p-1,p-1}(X)$. Then, $s$ defines a splitting of the first exact sequence, and induces a pairing

\[
\tilde{CH}^p(\mathcal{X}) \times Z^q(\mathcal{X}) \rightarrow \tilde{CH}^{p+q}(\mathcal{X}), \quad (y, Z) \mapsto (y|Z) = y.s(Z). \tag{2}
\]

A Green current $g_\mathcal{Y}$ with $(\mathcal{Y}, g_\mathcal{Y}) = s(\mathcal{Y})$ i.e. $(\mathcal{Y}, g_\mathcal{Y}) \in Z^p(\tilde{\mathcal{X}})$, and $g_\mathcal{Y}$ is orthogonal to $H^{d-p,d-p}$ is called $(\mu)$-normalized. It is unique modulo $\text{Im} \partial + \text{Im} \bar{\partial}$.

**3.6 Definition** For $\pi : \mathcal{X} \rightarrow \text{Spec} \mathbb{Z}$ a projective arithmetic variety, $\tilde{\mathcal{L}}$ a metrized ample line bundle on $\mathcal{X}$ with Chern form $c_1(\mathcal{L})$ equal to $\mu$, the height of an effective cycle $Z \in Z^p(\mathcal{X})$ is defined as

\[
h(Z) := \deg(\pi_*(\tilde{c}_1(\mathcal{L})^{d-p+1}|Z)) \in \mathbb{R}.
\]

If $c_1^{d+1-p}(\mathcal{L}) = [(\mathcal{Y}, g_\mathcal{Y})]$, and $\text{supp}(\mathcal{Y}) \cap \text{supp}(Z) = 0$, this is equal to

\[
\frac{1}{2} \int_{Z(\mathcal{C})} g_\mathcal{Y}.
\]
3.7 Proposition Let $\mathcal{X}, \mathcal{Y}$ be regular, projective, flat schemes over $\text{Spec} \, \mathbb{Z}$, and $f : \mathcal{X} \rightarrow \mathcal{Y}$ a morphism. Further, let $p, q$ be natural numbers with $p + q = d + 1 = \dim \mathcal{X}$, and $(\mathcal{Z}, g_\mathcal{Z}) \in \widetilde{CH}^p(\mathcal{X})$, $\mathcal{W} \in \mathbb{Z}^q(\mathcal{Y})$. If $\dim f(\mathcal{W}) = \dim(\mathcal{W})$, we have

$$(f^*(\mathcal{Z}, g_\mathcal{Z})|\mathcal{W}) = (f_*(\mathcal{Z}, g_\mathcal{Z}))|\mathcal{W}).$$

If $f$ is flat, and surjective, has smooth restriction to every component of $X_{\mathbb{Q}}$, and $\mathcal{X}, \mathcal{Y}$ have constant dimension, then with $\delta = \dim \mathcal{X} - \dim \mathcal{Y}$, $(\mathcal{Z}, g_\mathcal{Z}) \in \widetilde{CH}^p(\mathcal{X})$, $\mathcal{W} \in \mathbb{Z}^{d+1-p-\delta}(\mathcal{Y})$, we have

$$(\mathcal{Z}, g_\mathcal{Z})|f^*(\mathcal{W})) = (f_*(\mathcal{Z}, g_\mathcal{Z})|\mathcal{W}).$$

Proof [BGS], Proposition 2.3.1, (iv),(v).

3.8 Proposition

1. Let $\mathcal{Y}$ be an effective cycle of pure codimension $p$ on $\mathcal{X}$, $\mathcal{L}$ an ample line bundle on $\mathcal{X}$, and $f$ a global section of $\mathcal{L}^\otimes D$ on $\mathcal{X}$, whose restriction to $Y$ is nonzero. Then,

$$h(\mathcal{Y}.\text{div} f) = Dh(\mathcal{Y}) + \int_{\mathcal{X}(\mathbb{C})} \log ||f||^d |\mu^{d-p}\delta_Y|.$$  

2. Assume that on the variety $\bar{X}$ the product of two harmonic forms is always harmonic, and $\mathcal{Y}, Z$ are effective cycles of pure codimensions $p$ and $q$ respectively, intersecting properly. With $s(\mathcal{Y}) = (\mathcal{Y}, g_\mathcal{Y})$,

$$s(\mathcal{Y}.\mathcal{Z}) = s(\mathcal{Y}).s(\mathcal{Z}) - \frac{1}{2}a(H(g_\mathcal{Y}\delta_Z c_1(\mu^{d+1-p,q}))),$$

and consequently,

$$h(\mathcal{Y}.\mathcal{Z}) = \pi_*(\hat{c}_1(\mathcal{L})^{d+1-p-q}.s(\mathcal{Y}).s(\mathcal{Z}) + a(H(g_\mathcal{Y}\delta_Z))).$$

Proof 1. [BGS], Proposition 3.2.1 (iv).

2. Let $s(\mathcal{Y}) = (\mathcal{Y}, g_\mathcal{Y})$, and $s(\mathcal{Z}) = (\mathcal{Z}, g_\mathcal{Z})$. Then,

$$s(\mathcal{Y}).s(\mathcal{Z}) = (\mathcal{Y}.\mathcal{Z}, g_\mathcal{Y}\delta_Z + H(\delta_\mathcal{Y})g_\mathcal{Z}).$$

As the form

$$dd^c(g_\mathcal{Y}\delta_Z + H(\delta_\mathcal{Y})g_\mathcal{Z}) + \delta_\mathcal{Y}.\mathcal{Z} = H(\delta_\mathcal{Y})H(\delta_Z)$$

is harmonic by assumption, $(\mathcal{Y}.\mathcal{Z}, g_\mathcal{Y}\delta_Z + H(\delta_\mathcal{Y})g_\mathcal{Z}) \in \mathbb{Z}^{p+q}(\bar{X})$, and

$$s(\mathcal{Y}.\mathcal{Z}) = (\mathcal{Y}.\mathcal{Z}, g_\mathcal{Y}\delta_Z + H(\delta_\mathcal{Y})g_\mathcal{Z} - H(g_\mathcal{Y}\delta_Z + H(\delta_\mathcal{Y})g_\mathcal{Z})).$$
Since multiplication with a harmonic forms leaves the space of harmonic forms invariant, it also leaves the space of forms orthogonal to the harmonic forms invariant; hence $H(\delta_Y)g_Z$ is orthogonal to the space of harmonic forms, and $H(H(\delta_Y)g_Z) = 0$, implying
\[ s(Y).s(Z) = s(Y,Z) + a(H(g_Y\delta_Z)). \]

The claim about the heights follows by multiplying the last equality with $\hat{c}_1(\mathcal{L})^{d-p-q+1}$ and applying $\pi_\ast$.

An important tool for making estimates is the concept of positive Green forms: A smooth form $\eta$ of type $(p,p)$ on a complex manifold is called positive if for any complex sub manifold $\iota: V \to X$ of dimension $p$, the volume form $\iota^*g_Y$ on $V$ is nonnegative, i.e. for each point $v \in V$, the local form $(\varphi^*g_Y)_v$ is either zero or induces the canonical local orientation at $v$.

3.9 Lemma Let $X, Y$ be complex manifolds, and $\eta$ a positive form of type $(p,p)$ on $X$.

1. For any holomorphic map $f: Y \to X$, the form $f^*\eta$ is positive.

2. If $g: X \to Y$ is a smooth holomorphic map whose restriction to the support of $\eta$ is proper, the form $g_\ast\eta$ is positive.

3. For any positive form $\omega$ of type $(1,1)$ the form $\omega \wedge \eta$ is positive.

Proof [BGS], Proposition 1.1.4.

3.1 Projective Space

Let $M$ be a free $\mathbb{Z}$ module of rank $t + 1$, and $\mathbb{P}^t = Proj(Sym(\tilde{M}))$ the projective space with structural morphism $\pi: \mathbb{P}^t \to Spec \mathbb{Z}$. If $M_\mathbb{C} = \mathbb{C} \otimes \mathbb{Z}^{t+1}$ is equipped with a hermitian product, this induces a metric on the line bundle $O(1)$ on $\mathbb{P}^t_\mathbb{C}$ and the Fubini-Study metric on $\mathbb{P}^t(\mathbb{C})$. The Chern form $\mu = c_1(O(1))$ equals the Kähler form corresponding to this metric.

For any torsion free submodule $N \subset M$ define $\deg(N) = \deg(N \otimes \mathbb{Q})$ as minus the logarithm of the covolume of $N$ in $N_\mathbb{C} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We will always assume that the hermitian product chosen in such a way that $\deg(M) = 0$.

Then, the height of a projective subspace $\mathbb{P}(F) \subset \mathbb{P}^t$ of dimension $p$ equals
\[ h(\mathbb{P}(E)) = \deg(\tilde{F}) + \sigma_p, \quad (3) \]

where the number
\[ \sigma_p := \frac{1}{2} \sum_{k=1}^{p} \sum_{m=1}^{k} \frac{1}{m} \]
is called the $p$th Stoll number ([BGS], Lemma 3.3.1). The height of any effective cycle $Z \in Z^*(\mathbb{P}^t)$ is nonnegative ([BGS], Proposition 3.2.4). The space of harmonic forms $H^{p,p}(\mathbb{P}^t)$ with respect to the chosen metric is one dimensional with generator $\mu^p$. By Proposition 3.5, together with the definition of the map $s$, an element in $CH^{p}(\mathbb{P}^t)$ may be written as $\alpha \hat{\mu}^p + \beta a(\mu^p)$ with $\hat{\mu} := \hat{c}_1(O(1)), \alpha \in \mathbb{Z}, \beta \in \mathbb{R}$. As $\zeta \circ s = id$, the degree of any $\alpha \hat{\mu}^p + \beta a(\mu^p)$ equals $\alpha$. One easily calculates $a(\mu^p)\alpha(\mu^p) = 0$, and $\hat{\mu}^p a(\mu^2) = a(\mu^{p+q})$. This, together with Proposition 3.8.2 implies the Proposition ([BGS], 5. 4.3).

3.10 Proposition Let $\mathcal{X}, \mathcal{Y}$ be effective cycles of pure codimension $p$ and $q$ respectively in $\mathbb{P}^t$ intersecting properly. With $(\mathcal{X}, g_X) = s(\mathcal{X}),$

1. 

$$h(\mathcal{X}, \mathcal{Y}) = \deg Xh(\mathcal{Y}) + \deg Yh(\mathcal{X}) - \frac{1}{2} \int_{\mathbb{P}^t} g_X \delta Z \mu^{t+1-p-q} - \sigma_t \deg X \deg Y.$$ 

2. With $c_1(p, q, t) := \sigma_{p+q-t} + \sigma_t - \sigma_p - \sigma_q$ and $c_1(p, q, t) = c_1(p, q, t) + \frac{p+q-t-1}{2}$, the inequality

$$-\frac{1}{2} \int_{\mathbb{P}^t} g_X \delta Z \mu^{t+1-p-q} \leq c_1 \deg X \deg Y$$ 

holds.

PROOF 1. Assume

$$[s(X)] = \alpha \hat{\mu}^p + \beta a(\mu^p), \text{ and } [s(Y)] = \alpha' \hat{\mu}^q + \beta' a(\mu^q).$$

Then $\alpha = \deg X, \alpha' = \deg Y$, and

$$h(X) = \pi_\ast(\mu^{t+1-p}(\alpha \hat{\mu}^p + \beta \mu^p)) = \alpha \pi_\ast(\hat{\mu}^{t+1}) + \beta \pi_\ast(a(\mu^{t+1})) = \alpha \sigma_t + \beta.$$ 

Similarly $h(Y) = \alpha' \sigma_t + \beta'$. Next, by Proposition 3.8.2,

$$h(X, Y) = \pi_\ast(\hat{\mu}^{t+1-p-q}(\alpha \hat{\mu}^p + \beta \mu^p)(\alpha' \hat{\mu}^q + \beta' \mu^q)) - \frac{1}{2} \int_{\mathbb{P}^t} g_Y \delta Z \mu^{t+1-p-q}.$$ 

The proposition thus follows from the calculation

$$\hat{\mu}^{t+1-p-q}(\alpha \hat{\mu}^p + \beta \mu^p)(\alpha' \hat{\mu}^q + \beta' \mu^q) = \hat{\mu}^{t+1-p-q}(\alpha \alpha' \hat{\mu}^{p+q} + \alpha' \hat{\mu}^p a(\mu^q) + \alpha' \hat{\mu}^q a(\mu^p)) = \alpha \alpha' \hat{\mu}^{t+1} + (\alpha \beta' + \alpha' \beta) a(\mu^{t+1}),$$

and thus

$$h(\mathcal{X}, \mathcal{Y}) = \alpha \alpha' \sigma_t + \alpha \beta' + \alpha' \beta - \frac{1}{2} \int_{\mathbb{P}^t} g_Y \delta Z \mu^{t+1-p-q}$$

$$= \deg Xh(Y) + \deg Yh(X) - \alpha \alpha' \sigma_t - \frac{1}{2} \int_{\mathbb{P}^t} g_Y \delta Z \mu^{t+1-p-q}. $$
2. [BGS], Proposition 5.1.1 together with the proof of Theorem 5.4.4.(ii).

Let $F_C \subset M_C$ be a sub vector space of codimension $p$ with orthogonal complement $F_C^\perp$, and $pr_{F^\perp}$ the projection to the orthogonal complement. Then, on $M_C \setminus \{0\}$ (resp. $M_C \setminus F_C$) the functions $\rho(x) = \log |x|^2$ (resp. $\tau(x) = \log |pr_{F^\perp}(x)|^2$) are defined, and give rise to the $(1,1)$-forms $\mu_M := dd^c \rho$ on $\mathbb{P}(E_C)$, and $\lambda_{M,F} := dd^c \tau$ on $\mathbb{P}(M_C) \setminus \mathbb{P}(F_C)$ and a function $\rho - \tau$ on $\mathbb{P}(M_C) \setminus \mathbb{P}(F_C)$. Here, $\mu_M$ is just the Chern form of the metrized line bundle $O(1)$, and $(\rho - \tau)(x)$ is $-\frac{1}{2}$ times the logarithm of the Fubini-Study distance of $x$ to $\mathbb{P}(F_C)$. With these notations, the so-called Levine form

$$\Lambda_{\mathbb{P}(F)} := (\rho - \tau) \sum_{i+j=p-1}^{p-1} \mu_M^i \lambda_{M,F}^j$$

is a positive admissible Green form for $\mathbb{P}(F_C)$, (see [BGS], example 1. 2. (v)), that is (BGS. Prop. 1.4.1)

$$dd^c[\Lambda_{\mathbb{P}(F)}] + \delta_{\mathbb{P}(F)} = \mu_M^p,$$

and the harmonic projection of $\Lambda_{\mathbb{P}(E)}$ with respect to $\mu$ equals

$$H(\Lambda_{\mathbb{P}(E)}) = \sum_{n=1}^{p} \sum_{m=0}^{t-p} \frac{1}{m+n} \mu^{p-1} = 2(\sigma_t - \sigma_{p-1} - \sigma_{t-p}),$$

that is

$$\int_{\mathbb{P}(E_C)} \Lambda_{\mathbb{P}(E)} \mu^{t-p+1} = \sum_{n=1}^{p} \sum_{m=0}^{t-p} \frac{1}{m+n}.$$  

([BGS],(1.4.1), (1.4.2))

3.11 Lemma Denote by $|f|_\infty$ the sup norm of an element $f \in E_D$. Then

$$\log |f|_\infty - \frac{D}{2} \sum_{m=1}^{t} \frac{1}{m} \leq \int_{\mathbb{P}(E_C)} \log |f| \mu^t \leq \log |f|_{L^2} \leq \log |f|_\infty.$$  

The first inequality is an equality iff $f$ is a power of a linear form.

PROOF [BGS], Prop. 1. 4. 2, and formula (1.4.10).

3.12 Theorem Let $\mathcal{X}, \mathcal{Y}$ be effective cycles in $\mathbb{P}^t$ of codimension $p$, and $q$ respectively each being at most $t$, and assume that $p+q \leq t+1$, and that $\mathcal{X}$ and $\mathcal{Y}$ intersect properly. Then,

$$h(\mathcal{X}, \mathcal{Y}) \leq \deg(\mathcal{Y}) h(\mathcal{X}) + \deg(\mathcal{X}) h(\mathcal{Y}) + (c_1 - \sigma_t) \deg(\mathcal{X}) \deg(\mathcal{Y}).$$

PROOF Follows immediately from Proposition 3.10.
3.2 Flag varieties

For \( r \leq t \) and \( 0 < q_1 < q_2 < \cdots < q_r < q_{r+1} = t+1, q_i \in \mathbb{N} \), let \( \mathcal{F} = \mathcal{F}_{(q_1, \ldots, q_r)} \) be the flag variety over \( \text{Spec} \mathbb{Z} \) which assigns to each field \( k \) the set of flags

\[
F: \quad 0 \subset V_1 \subset \cdots \subset V_r \subset k^{t+1},
\]

where each \( V_i \) is a \( q_i \)-dimensional subspace of \( k^{t+1} \); denote by \( d+1 = 1 + \sum_{i=1}^{r} (q_i+1 - q_i)q_i \) the dimension of \( \mathcal{F} \).

We abbreviate \( (q_1, \ldots, q_r) \) by \( \tilde{q} \). On \( \mathcal{F}_{\tilde{q}} \), there are the canonical quotient bundles \( Q_i, i = 1, \ldots, r \), whose highest exterior powers \( L_i, i = 1, \ldots, r \) are ample generators of \( \text{Pic}(\mathcal{F}_{\tilde{q}}) \). Set \( \mathcal{L} = L_\mathcal{F} := \bigotimes_{i=1}^{r} L_i \). A hermitian product on \( \mathcal{C}^{t+1} \) canonically induces a hermitian metric on the base extensions \( L_i \) and \( L \). We will always assume that the canonical hermitian product on \( \mathcal{C}^{t+1} \) was chosen, and \( U(t+1) \) the unitary group with respect to this product. Set \( \mu = \mu_\mathcal{F} := c_1(L) \).

Since \( U(t+1) \) operates transitively on \( F_\mathcal{C} \), there is a unique \( U(t+1) \)-invariant volume form \( \omega_\mathcal{F} \) on \( F_\mathcal{C} \) that is positive with respect to the canonical orientation on \( F_\mathcal{C} \) and gives \( F_\mathcal{C} \) the volume one. For an arbitrary point \( P_0 \in F(\mathcal{C}) \), by [BGS], 6.2, example (ii), there is a positive green form of log type \( g_0 \) for \( P_0 \) such that \( dd^c g_0 + \delta P_0 = \omega_\mathcal{F} \).

For each point \( P \in F(\mathcal{C}) \) choose an \( h_p \in U(t+1) \) such that \( P = h_p P_0 \), and define \( g_P := (h_p^{-1})^* g_0 \). Since the metric on \( L \), and thereby \( \mu_\mathcal{F} \) is \( U(t+1) \)-invariant, \( g_P \) is positive for every \( P \), and

\[
c_3(\tilde{q}) := \frac{1}{2} \int_{F(\tilde{q})} g_P \mu_\mathcal{F}(\tilde{q})
\]

does not depend on \( P \).

On \( E_D = \Gamma(F_{\tilde{q}}, L^{\otimes D}) \) the space of global sections of \( L^{\otimes D} \), there are the norms

\[
|f|_\infty = \sup_{P \in F} |f_P|, \quad |f|_m := \left( \int_{F} |f|^m \omega_\mathcal{F} \right)^{\frac{1}{m}}, \quad |f|_0 := \exp \left( \int_{F} \log |f| \omega_\mathcal{F} \right).
\]

3.13 Proposition The above norms fulfill the relations

\[
||f||_0 \leq ||f||_m \leq ||f||_\infty \leq \exp(c_3(\tilde{q})D)||f||_0.
\]

Proof The first two inequalities are valid for every probability space. For the third inequality, let \( f \in E_D \), and \( P \) be a point in \( G \). Then, \( -\log |f|^2 \) is a Green form for \( \text{div}(f) \). Let \( g_P \) be the positive green form for \( P \) on \( F \) from above. By the commutativity of the star product,

\[
[-\log |f|^2] \delta_P + [g_P] D \mu_\mathcal{F} = [-\log |f|^2] \omega_\mathcal{F} + g_P \delta_{\text{div}(f)} \mod \text{Im} \partial + \text{Im} \bar{\partial}.
\]
Integrating over $F$ gives
\[-\log |f_P|^2 + 2c_3(q)D = -\log |f|^2 + \int_{\text{div}(f)} g_P.\]

As $g_P$ is positive, $\int_{\text{div}(f)} g_P \geq 0$, hence
\[\log |f_P|^2 - 2c_3 D \leq \log |f|^2\]
for every $P \in F$ which implies
\[\log |f|_{\infty} \leq c_3 D + \log |f|_0.\]

### 3.3 Grassmannians

With the notations of the previous section, assume that $\bar{q} = (q)$ is a single number. Then $\mathcal{F}_q = \mathcal{G}_{t+1,q}$ or $\mathcal{G}_q$ or $\mathcal{G}$ for short is the Grassmannian that assigns to each field $k$ the set of flags $0 \subset V_q \subset k^{t+1}$ consisting of only one space $V_q$ of dimension $q$; in particular $\mathbb{P}^t = \mathcal{G}_{t+1,q}$. The Picard group of $\mathcal{G}$ is generated by the determinant $\mathcal{L} = \mathcal{L}_G$ of the canonical quotient bundle $Q$, hence $c_1(\mathcal{L})$ is a generator for $CH^1(\mathcal{G}) \cong \mathbb{Z}$. The intersection product of every effective cycle $X \in Z_\mathbb{P}(G_{q})$ with $c_1(L)^p$ is nonzero and defined as the degree of $X$. Further, $\mu_G = c_1(L)$ is the Kähler form for the Kähler metric induced by the canonical metric on $\mathbb{C}^{t+1}$ and the harmonic forms are exactly the forms that are invariant under $U(t+1)$. Hence, the product of two harmonic forms is again harmonic, and by Proposition 3.3, $CH^*(\mathcal{G})$ is a subring of $\widehat{CH}^*(\mathcal{G})$. In particular $H^{0,0}(G(\mathbb{C}))$ are the constant functions, and $H^{d,d}(G(\mathbb{C}))$ are the multiples of the volume form $\omega_G = \omega_{F_{(q)}}$.

With a fixed complete flag
\[
\{0\} \subset V_1 \subset \cdots \subset V_t \subset V_{t+1} = k^{t+1}
\]
and numbers $1 \leq i_1 < \cdots < i_q \leq t + 1$ define the Schubert cell $S_{(i_1,\ldots,i_q)}^q(k)$ in $G_q$ as the set of subspaces $W$ of dimension $q$ such that
\[
\dim V_{i_l} \cap W = l \quad \forall \ l = 1,\ldots,q.
\]
Then the closure $S_{(i_1,\ldots,i_q)}$ of the Schubert cell is the set of subspaces $W$ of dimension $q$ such that
\[
\dim V_{i_l} \cap W \geq l \quad \forall \ l = 1,\ldots,q.
\]
The Schubert cells are algebraic subvarieties with $\dim S_{(i_1,\ldots,i_q)} = \sum_{l=1}^q (i_l - l)$ and
\[
G_q = \bigcup_{(i_1,\ldots,i_q)} S_{(i_1,\ldots,i_q)}.
\]
the union being disjoint. Further,
\[ Ch(G) = \bigoplus_{1 \leq i_1 < \cdots < i_q \leq t+1} \mathbb{Z}[S(i_1, \ldots, i_q)]. \]

In particular, for \( p \leq q \) the Schubert cell \( G_{V_p} = S_{1, \ldots, p, t+2-(q-p), \ldots, t+1} \) is the sub Grassmannian consisting of the sub spaces \( W \) that contain \( V_p \), and for \( p \geq q \) the Schubert cell \( G_{V_p}^q = S_{p+1-q, \ldots, p} \) is the sub Grassmannian of the sub spaces \( W \) that are contained in \( V_p \). Further, \( S_{1, \ldots, q} \) represents the unique 0-dimensional cycle class and \( S_{(t+1-q, t+3-q, \ldots, t+1)} \) represents the unique \( q(t+1-q) \)-1-dimensional cycle class.

Denote \( \sigma(i_1, \ldots, i_q) = [S(i_1, \ldots, i_q)] \in CH(G_q) \), and the harmonic projection of \( \delta S_{i_1, \ldots, i_q} \) with \( \eta_{i_1, \ldots, i_q} \).

### 3.14 Proposition

1. If \( \sigma(i_1, \ldots, i_q) \) and \( \sigma(j_1, \ldots, j_q) \) have complementary dimension, that is \( dim \sigma(i_1, \ldots, i_q) + dim \sigma(j_1, \ldots, j_q) = \sum_{l=1}^q i_l - l + \sum_{l=1}^q j_l - l = q(t + 1 - q) \), then

\[
\sigma(i_1, \ldots, i_q), \sigma(j_1, \ldots, j_q) = \begin{cases} 
\sigma(1, \ldots, q) & \text{if } i_l + j_{q+1-l} = t + 2 \quad \forall l = 1, \ldots, q \\
0 & \text{otherwise.}
\end{cases}
\]

2. The harmonic projections \( \omega_p : CH^p(G_q)_\mathbb{C} \to H^{p,p}(G_q), [X] \mapsto H(\delta X) \) combine to a ring isomorphism \( \omega : CH(G_q)_\mathbb{C} \cong H(G_q) \). In particular \( H^{1,1}(G_q) \) and \( H^{(t+1-q), (t+1-q)}(G_q) \) are one dimensional.

3. The Arakelov Chow group \( CH^p(G_q) \) is isomorphic to the direct sum \( CH^p(G_q)_\mathbb{C} \oplus H^{p-1,p-1}(G_q) \) via the isomorphism

\[
CH^p(G_q)_\mathbb{C} \oplus H^{p-1,p-1}(G_q) \to CH^p(G_q), \quad ([X], \eta) \mapsto s(X) + a(\eta).
\]

4. With \( d := \sum_{l=1}^q j_l - l = q(t + 1 - q) - (\sum_{l=1}^q i_l - l) = dim S_{i_1, \ldots, i_q} \),

\[
\deg \sigma(i_1, \ldots, i_q) = \frac{d!}{(i_1 - 1)! \cdots (i_q - 1)!} \prod_{l<k} (i_k - i_l).
\]

5. The restriction of \( \eta_{i_1, \ldots, i_q} \) to \( S_{j_1, \ldots, j_k} \) equals zero unless \( i_l + j_{q+1-l} = t+1 \) for every \( l = 1, \ldots, q \), in which case it equals the unique volume form \( \omega \) on \( S_{i_1, \ldots, i_q} \) such that \( S \) has volume one with respect to \( \omega \). Consequently,

\[
\int_G \eta_{i_1, \ldots, i_q} \eta_{j_1, \ldots, j_q} = \int_{S_{i_1, \ldots, i_q}} \eta_{j_1, \ldots, j_q} = \delta_{i_1, \ldots, i_q, j_1, \ldots, j_q}.
\]

and

\[
\int_{S_{i_1, \ldots, i_q}} \mu^d_G = \deg S_{i_1, \ldots, i_q} = \frac{d!}{(i_1 - 1)! \cdots (i_q - 1)!} \prod_{l<k} (i_k - i_l).
\]

In particular, \( \mu^d_{G(t+1-q)} = \deg G_q \omega_G = \frac{1!2! \cdots (q-1)! (q(t+1-q))!}{(t+2-q)! \cdots (t+1)!} \omega_G \).
1. [Fu], p. 271.
2. [Mai], Théorème 2.2.1.
3. Follows from part 2 and the Definition of the section $s$.
4. [Fu], Example 14.7.11.
5. The restriction of $\eta = \eta_{j_1, \ldots, j_q}$ to $S$ is a $U(i_1) \times U(i_2 - i_1) \times \cdots \times U(i_q - i_{q-1})$-invariant volume form on $S$. Hence $\eta|_S = a\omega$ with $a \in \mathbb{R}$. Since there is a Green current $g$ for $S_{j_1, \ldots, j_q}$ with $$dd^c g + \delta_{S_{j_1, \ldots, j_q}} = \eta,$$ one gets $$\int_S \eta = (dd^c g)(\eta) + (\delta_S, \delta_{S_{j_1, \ldots, j_q}})(1) = g(dd^c \eta) + (\delta_S, \delta_{S_{j_1, \ldots, j_q}})(1),$$ which because of $dd^c \eta = 0$ and part 1 equals 1.

For the second equality, shorten $(i_1, \ldots, i_q) = I$, and $i_1 + \cdots + i_q = |I|$. Because of part 2,
$$\mu^d = c_1(\bar{L})^d = H(c_1(L)^d) = H(\sigma_{t+1-q,t+3-q, \ldots, t+1})^d = H(\sigma'_{t+1-q,t+3-q, \ldots, t+1}).$$
Hence, with $\sigma_{t+1-q,t+3-1, \ldots, t+1} = \sum_{|I|=(t+1)(t+1-q)-d-(q+1)q/2} a_I \sigma_I$,
$$\int_S \mu^d = \sum_{|I|=(t+1)(t+1-q)-d-(q+1)q/2} \int_S a_I \eta_I,$$
which by the above equals
$$a_{j_1, \ldots, j_q} = [S]. \sigma^d_{t+1-q,t+3-1, \ldots, t+1} = [S].c_1(L)^d = \deg S,$$
and the claim follows from part 4.

For $\bar{q} = (q_1, \ldots, q_r)$ the canonical maps $\varphi_{q,q_i} : \mathcal{F}_q \rightarrow \mathcal{G}_{q_i} = \mathcal{G}_{t+1,q_i}$, $i = 1, \ldots, r$ that forget every subspace except the $i$th, are projective bundle maps, hence flat, proper, smooth, surjective, and projective. For $\mathcal{F} = \mathcal{G}_{(1, \ldots, t)}$ the full flag variety, and $g_p : \mathcal{F} \rightarrow \mathcal{G}_p$, $p = 1, \ldots, t$ the form $(g_p)_* \omega_F$ is $U(t+1)$ invariant, and $\int_{\mathcal{G}_p} (g_p)_* \omega = \int_F \omega = 1$, hence $(g_p)_* \omega_F = \omega_p := \omega_{\mathcal{G}_p}$.

For $\bar{q} = (q, p)$, there is the correspondence
$$\mathcal{F}_q \xrightarrow{\varphi_{q,p}} \mathcal{G}_q \xleftarrow{\varphi_p} \mathcal{G}_p$$
between $\mathcal{G}_q = \mathcal{G}_{t+1,q}$, and $\mathcal{G}_p = \mathcal{G}_{t+1,p}$. For later applications we will need the fact that the maps $\varphi_q$ and $\varphi_p$ preserve the dimension of certain Schubert cells.
3.15 Lemma  With the above notations

1. let $q = 1$. Then,

$$\dim(\varphi_p(\varphi_1^{-1}S_{t+1-p})) = \dim(\varphi_p(\varphi_1^{-1}P^{t-p})) = \dim(\varphi_1^{-1}P^{t-p}),$$

and with $W$ a space of dimension $p$,

$$\dim(\varphi_1(\varphi_p^{-1}S_{t+3-p,...,t+1})) = \dim(\varphi_1(\varphi_p^{-1}W)) = \dim(\varphi_1^{-1}W).$$

2. The maps $(\varphi_p)_* \circ \varphi_q^*, (\varphi_q)_* \circ \varphi_p^*$ map harmonic forms to harmonic forms.

Proof. 1. For $[v] \in \mathbb{P}^t$ a point, by Proposition 3.14,

$$\dim \varphi_1^{-1}([v]) = \dim S_{(1,t+3-p,...,t+1)} = (t + 1 - p)(p - 1),$$

hence

$$\dim \varphi_1^{-1}P^{t-p} = \dim \mathbb{P}^{t-p} + (p - 1)(t + 1 - p) = t - p + (p - 1)(t + 1 - p),$$

$$\dim(\varphi_p(\varphi_1^{-1}P^{t-p})) =$$

$$\dim S_{t+1-p,t+3-p,...,t+1} = t + 1 - p - 1 + \sum_{l=2}^{p} t + 1 + l - p - l = t - p + (p - 1)(t + 1 - p),$$

and also,

$$\dim(\varphi_p^{-1}W) = \dim S(p) = 1 \cdot (p - 1),$$

and

$$\dim(\varphi_1(\varphi_p^{-1}W)) = \dim \mathbb{P}(W) = p - 1.$$

2. Follows from the facts, that the harmonic forms on $G_p, G_q$ are exactly the forms that are invariant under $U(t + 1)$ and that $\varphi_p, \varphi_q$ are $U(t + 1)$-invariant.

3.16 Corollary Let $X$ be an effective cycle of pure codimension $t + 2 - p$ in $\mathbb{P}^t$. Then, $(\varphi_p)_* \varphi_1^*X$ is an effective cycle that has pure codimension 1 and the same dimension as $\varphi_1^*X$. We will use the abbreviation $V_X = (\varphi_p)_* \varphi_1^*X$.

Proof. Follows from the Lemma together with $[X] = \deg X[S_{t+1-p}]$.

With $\mathbb{P}(W) \subset \mathbb{P}^t(\mathbb{C})$ a subspace of dimension $q$, define the map

$$\pi_{W^\perp}: \mathbb{P}^t \setminus W \to W^\perp, \quad [v + w] \mapsto [w], \quad v \in \mathbb{P}(W), w \in \mathbb{P}(W^\perp).$$

If $X$ is an effective cycle in $\mathbb{P}^t$ whose support does not meet $\mathbb{P}(W)$, define

$$X_W := \overline{\pi^*(\varphi_1^*X)}.$$ (9)
3.17 Proposition

1. For \( p > q \), consider the 3 correspondences on complex manifolds

\[
\varphi_1 \xrightarrow{F_{1,p}} \varphi_p \quad \psi_1 \xrightarrow{F_{1,p-q}} \psi_{p-q} \quad \varphi_{p-q} \xrightarrow{F_{p-q,p}} \varphi_p,
\]

let \( \mathbb{P}(W) \subset \mathbb{P}^t \) be a \( q \)-dimensional subspace, \( X \in Z^p_{\text{eff}}(\mathbb{P}^t) \) an effective cycle whose support does not intersect \( \mathbb{P}(W) \), and \( G_W \subset G_p \) the Schubert cell consisting of the subspaces that contain \( W \). Then

\[
(\varphi_{p-q})^*(\varphi_p)^*(G_W.(\varphi_p)^*(\varphi_1^*X)) = (\psi_{p-q})^*(\psi_1^*(X_W)), \tag{10}
\]

\[
\dim(\varphi_{p-q})^*(\varphi_p)^*(G_W.(\varphi_p)^*(\varphi_1^*X)) = \varphi_p^*(G_W.(\varphi_p)^*(\varphi_1^*X)).
\]

2. For \( p \leq q \), with the correspondences on complex manifolds

\[
\varphi_1 \xrightarrow{F_{1,p}} \varphi_p \quad \psi_1 \xrightarrow{F_{1,p+q}} \psi_{p+q} \quad \varphi_{p+q} \xrightarrow{F_{p+q,p}} \varphi_p,
\]

let \( \mathbb{P}(F) \subset \mathbb{P}^t \) be a \( q \)-codimensional subspace, \( X \in Z^p_{\text{eff}}(\mathbb{P}^t) \) an effective cycle intersecting \( \mathbb{P}(F) \) properly, and \( G^F \subset G_p \) the sub Grassmannian of subspaces that are contained in \( F \). Then

\[
(\varphi_{p+q})^*(\varphi_p)^*(G^F.(\varphi_p)^*(\varphi_1^*X)) = (\psi_{p+q})^*(\psi_1^*(X.F.(\mathbb{P}(F))), \tag{11}
\]

where \( G^F \subset G_p \) is the subset of spaces being contained in \( F \). Further,

\[
\dim(\varphi_{p+q})^*(\varphi_p)^*(G^F.(\varphi_p)^*(\varphi_1^*X)) = \dim \varphi_p^*(G^F.(\varphi_p)^*(\varphi_1^*X)).
\]

**Proof** For \( X \) a variety, the equalities of proof for the equalities (10) and (11) are easy exercises in linear algebra, thus hold for arbitrary cycles by linearity. For the equalities of dimensions,

1. Since \( G_W \) is isomorphic to the Grassmannian of \( p - q \)-dimensional subspaces in \( \mathbb{C}^{t+1-q} \) and \( \text{codim}V_X = \text{codim}(\varphi_p)^*(\varphi_1^*X)) = 1 \), we have \( \dim G_W.(\varphi_p)^*(\varphi_1^*X)) = (p - q)(t + 1 - p) - 1 \). Further, the fibre of \( F_{p-q,p} \) above each point \( E \) in \( G_p \) consists of the \( (p - q) \)-dimensional subspaces of \( E \), and has thus dimension \( q(p - q) \). Hence

\[
\dim \varphi_p^*(\dim G_W.(\varphi_p)^*(\varphi_1^*X)) = (p - q)(t + 1 - p) - 1 + \dim G_{p-q} = (p - q)(t + 1 - p) - 1 + q(p - q) = (p - q)(t + 1 + q - p) - 1.
\]
On the other hand,

\[ \dim V_{X,W} = \dim (\psi_{p,q})_*(\psi'_1 X_W) = \dim G_{p,q} - 1 = (t + 1 + q - p)(p - q) - 1, \]

hence, the two dimensions are equal.

2. Again, \( \dim (G^F.(\varphi_p)_*(\varphi'_p X)) = \dim G^F - 1 = p(t + 1 - p - q) - 1, \) and a fibre of \( \tilde{\varphi}_p \) has dimension \( q(t + 1 - p - q). \) Hence, \( \dim \tilde{\varphi}_p^*(G^F.\varphi_p)_*(\varphi'_p X)) = (p + q)(t + 1 - p - q) - 1. \) Since also

\[ \dim (\psi_{p+q})_*(\psi'_q(X.\mathbb{P}(F))) = \dim V_{X,\mathbb{P}(F)} = G_{p+q} - 1 = (p + q)(t + 1 - p - q) - 1, \]

the dimensions are equal.

### 3.4 Chow divisor

Let \( \hat{\mathbb{P}}^t \) be the dual projective space. The \( p \) projections \( pr_i : (\hat{\mathbb{P}}^t)^p \to \hat{\mathbb{P}}^t, \) define line bundles \( O_i(1) = pr_i^*(O(1)) \), and \( O(D_1, \ldots, D_p) = O(1)^{\otimes D_1} \otimes \cdots \otimes O(1)^{\otimes D_p}. \)

A dual inner product on \( \hat{\mathbb{P}}^t \) defines metrics on \( O(D) \), and \( O(D_1, \ldots, D_p) \), and a Kähler metric on \( \hat{\mathbb{P}}^t \), and \( (\hat{\mathbb{P}}^t)^p. \) The corresponding Kähler forms are \( \bar{\mu} = c_1(O(1)) \), and \( \bar{\mu} = \bar{\mu}_1 + \cdots + \bar{\mu}_p = pr_1^* \bar{\mu} + \cdots + pr_p^* \bar{\mu} = c_1(O(1, \ldots, 1)) \). The space of harmonic forms on \( (\mathbb{P}^t)^p \) is the linear span of the products \( \prod_{i=1}^p \bar{\mu}_i^{k_i} \), \( 1 \leq k_i \leq t \ \forall i = 1, \ldots, p \).

Let \( \delta : \hat{\mathbb{P}}^t \to (\hat{\mathbb{P}}^t)^p \) be the diagonal, and define the correspondence

\[
\begin{align*}
\begin{array}{ccc}
\mathbb{P}^t & \xrightarrow{f} & \mathbb{P}^t \\
\hat{\mathbb{P}}^t & \xrightarrow{g} & (\hat{\mathbb{P}}^t)^p \end{array}
\end{align*}
\]

where \( \mathcal{C} \) is the sub scheme of \( (\mathbb{P}^t)^p \times (\hat{\mathbb{P}}^t)^p \) assigning to each \( t + 1 \) dimensional vector space \( V \) over a field \( k \) the set

\[
\{(v_1, \ldots, v_p, \tilde{v}_1, \ldots, \tilde{v}_p) | v_i \in V, \ \tilde{v}_i \in \tilde{V}, \ \tilde{v}_i(v_i) = 0, \ \forall i = 1, \ldots, p \} \]

The maps \( f : \mathcal{C} \to (\mathbb{P}^t)^p, g : \mathcal{C} \to (\hat{\mathbb{P}}^t)^p \) are just the restrictions of the projections. They are flat, projective, surjective, and smooth.

Let \( \mathcal{X} \in Z_{eff}^{t-1-p}(\mathbb{P}^t) \), and define the Chow divisor \( \text{Ch}(\mathcal{X}) \subset (\hat{\mathbb{P}}^t)^p \) as \( \text{Ch}(\mathcal{X}) := g_* \circ f^* \circ \delta_*(\mathcal{X}) \).

### 3.18 Proposition

1. The Chow divisor has codimension one; it is the divisor corresponding to a global section \( f_X \in \Gamma((\hat{\mathbb{P}}^t)^p, O(\deg X, \ldots, \deg X)) \) such that

\[
d^c \log |f_X|^2 + \delta_{\text{Ch}(\mathcal{X})} = \deg X \bar{\mu}.
\]
Consequently, \(-\log |f_X|^2\) is an admissible Green form of log type for \(Ch(X)\), and for all \(i = 1, \ldots, p\) the multi degrees of \(Ch(X)\), that is the numbers 
\[c_1(O(1))^t \cdot \ldots \cdot c_1(O_{i-1}(1))^t \cdot c_1(O_i(1))^{t-1} \cdot \ldots \cdot c_1(O_{p+1}(1))^t \cdot [Ch(X)]\]
all equal \(\deg X\), i.e. \([Ch(X)] = (\deg X, \ldots, \deg X) \in \mathbb{Z}^p = CH^1((\mathbb{P}^t)^p)\).

Further \(\dim X = \dim \delta(X)\), and \(\dim g^\ast \delta_s X = \dim f(g^\ast \delta_s X)\).

2. If \(\mathbb{P}(W)\) does not meet \(X\), then \(\mathbb{P}(W)\), and \(Ch(X)\) intersect properly. Further if \(\hat{w}_1, \ldots, \hat{w}_p \in \hat{W}\) are \(p\) linearly independent vectors, and \(\mathbb{P}(W)\) the intersection of their kernels, then \(\dim g^{-1}(\hat{w}_1, \ldots, \hat{w}_p) = \dim f(g^{-1}(\hat{w}_1, \ldots, \hat{w}_p))\), and \(\delta^\ast f_g^\ast g^\ast(\hat{w}_1, \ldots, \hat{w}_p) = \mathbb{P}(W)\).

3. The maps \(g_\ast \circ f^\ast, f_\ast \circ g^\ast\) map harmonic forms to harmonic forms.

4. For \(k \leq t + 1 - p\) let \(\hat{v}_1, \ldots, \hat{v}_k\) be orthonormal vectors, \(E_i = \ker(\hat{v}_i), i = 1, \ldots, k\), and \(E = E_1 \cap \cdots \cap E_k\). Let further \(g\) be a normalized green form for \(E_1 \times \cdots \times E_k \times (\mathbb{P}^t)^{t+1-p-k}\) in \((\mathbb{P}^t)^p\), and \(i^{-1}(g)\) its restriction to \(\mathbb{P}^t\) via the diagonal embedding \(\delta\). The number 
\[c_5 := \int_{\mathbb{P}^t} i^{-1}(g) \mu^{t+1-k}\]
only depends on \(t, p, k\), and \(i^{-1}(g) - c_5 \mu^t\) is a normalized green form for \(E\).

PROOF 1. One only has to check that a generator \([\mathbb{P}(V)] \in CH^{t+1-p}(\mathbb{P}^t)\) by \(g_\ast f^\ast \delta_s\) is mapped to \((1, \ldots, 1) \in CH^1((\mathbb{P}^t)^p)\), which is obvious.

2. is obvious.

3. Follows from the fact that the harmonic forms on \((\mathbb{P}^t)^p\) and \((\mathbb{P})^p\) are exactly the forms invariant under \(U(t + 1)^p\), and the \(U(t + 1)\) equivariance of \(f\) and \(g\).

4. Follows immediately from the fact that \(U(t + 1)\) acts transitively on the set orthonormal \(k\)-tupels.

The canonical quotient bundle \(Q\) on \((\mathbb{P}^t)^p\) carries a canonical metric as well. Let \(\hat{c}(\hat{Q})\) be its total arithmetic Chern class. (See [SABK]). Define the height of a divisor \(D\) in \((\mathbb{P}^t)^p\) as 
\[h(D) = \pi_\ast(\hat{c}(\prod_{i=1}^p \hat{Q}_i)|D)\].

3.19 Proposition For all effective cycles \(X \in Z_{eff}^{t+1-p}(\mathbb{P}^t)\), 
\[h(X) = h(Ch(X))\].
With \((d_1, \ldots, d_p) \in \mathbb{N}^p\) the space \(E_{d_1, \ldots, d_p} = \Gamma(\mathbb{P}^t, O(d_1, \ldots, d_p))\) carries norms \(\| \cdot \|_0, \| \cdot \|_r, \| \cdot \|_\infty, r \in \mathbb{R}_{>0}\) just like \(\Gamma(\mathbb{P}^t, O(D))\), and the analogous estimates hold.

**3.20 Lemma** There is a positive constant \(c_7\), only depending on \(t\) such that for any \(f \in E_{d_1, \ldots, d_p}, r \in \mathbb{R}_{>0}\),

\[
\log \| f \|_\infty - c_7 \sum_{i=1}^p d_i \leq \log \| f \|_0 \leq \log \| f \|_r \leq \log \| f \|_\infty.
\]

**Proof** \([BGS],\) Corollary 1. 4. 3.

### 4 The algebraic distance

In this section all varieties, and cycles are assumed to be defined over \(\mathbb{C}\). All integrals over sub varieties of smooth projective varieties are defined via resolutions of singularities (cp. \([SABK],\) II.1.2.)

**4.1 Definitions and fundamental properties**

Let \(X\) be a smooth projective variety over \(\mathbb{C}\), and fix a Kähler metric with Kähler form \(\mu\) on \(X\). For \(p + q \leq t + 1\), define the pairing

\[
Z^p(X_\mathbb{C}) \times Z^q(X_\mathbb{C}) \to \mathcal{D}^{t,t}(X_\mathbb{C}), \quad (Y, Z) \mapsto (Y|Z) := [g_Y](\delta Z - H(\delta Z))\mu^{t+1-p-q}
\]

on the cycle group, where \(g_Y\) is an admissible Green form of log type for \(Y\).

**4.1 Lemma and Definition** The above pairing is well defined and symmetric modulo \(\text{Im} \partial + \text{Im} \overline{\partial}\). If \(X\) and \(Y\) intersect properly, the algebraic distance

\[
D(X, Y) := -\frac{1}{2} \int_X (Y|Z)
\]

is finite.

**Proof** Let \(g'_Y\) be another admissible Green form for \(Y\). By Proposition 3.3, \(g_Y - g'_Y\) modulo \(\text{Im} \partial + \text{Im} \overline{\partial}\) equals a harmonic form \(\eta\). Thus,

\[
[g_Y](\delta Z - H(\delta Z))\mu^{t+1-p-q} - [g'_Y](\delta Z - H(\delta Z))\mu^{t+1-p-q} =
\]

\[
(\delta Z - H(\delta Z))\eta \mu^{t+1-p-q} = -(dd^c[g_Z])\eta \mu^{t+1-p-q} =
\]

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\[ d^r[g_z]d(\eta \mu^{t+1-p-q}) \mod \text{Im} d \subset \text{Im} \partial + \text{Im} \bar{\partial}. \]

The last expression equals zero, since harmonic forms are contained in the kernel of \( d \). It follows that the pairing is well defined.

For the symmetry, we have to prove

\[ [g_Y](\delta_Z - H(\delta_Z)) \mu^{t+1-p-q} = [g_Z](\delta_Y - H(\delta_Y)) \mu^{t+1-p-q} \mod \text{Im} \partial + \text{Im} \bar{\partial} \]

for admissible Green forms \( g_Y, g_Z \). This is equivalent to

\[ [g_Y] \delta_Z + [g_Z] H(\delta_Y) = [g_Z] \delta_Y + [g_Y] H(\delta_Z) \mod \text{Im} \partial + \text{Im} \bar{\partial}, \]

which just is the commutativity of the star product of Green currents.

The last claim of the Lemma follows from the fact that \( g_Y \) is of log type along \( Y \).

One immediately observes

**4.2 Fact** If one of the cycles \( X, Y \) is the zero cycle, then \( D(X,Y) = 0 \). For \( p+q \leq t+1 \) the map \( D : Z^p_{eff}(X) \times Z^q_{eff}(X) \to \mathbb{R} \) is bilinear on the subset on which it is defined, i.e. for properly intersecting cycles.

**4.3 Remark** If \( g_Y \) is normalized, then

\[ D(Y, Z) = -\frac{1}{2} \int_{X(\mathbb{C})} g_Y \delta_Z. \]

If \( X \) is the base extension of an arithmetic variety \( X \) and on \( X(\mathbb{C}) \) the product of harmonic forms is again harmonic, then \( H(g_Y(\delta_Z - H(\delta_Z))) = H(g_Y \delta_Z) \), and by Proposition 3.8.2,

\[ h(Y, Z) = \pi_*(\hat{c}_1(L)^{t-p-q+1}s(Y)s(Z)) + D(Y, Z). \]

In case \( X = \mathbb{P}^t \), Proposition 3.10.1 reformulates as

\[ h(Y, Z) = \deg Y h(Z) + \deg Z h(Y) + D(Y, Z) - \sigma_i \deg Y \deg Z, \]

and Theorem 3.12 reformulates as

\[ D(X, Y) \leq \tilde{c}_1 \deg X \deg Y. \]

**4.4 Lemma** Let \( Y \in Z^{t+1}_{eff}(G) \), with \( G = G_{t,p} \) the Grassmannian, and \( V \in G(\mathbb{C}) \) a point not contained in the support of \( Y \). Then, with \( V \subset \mathbb{C}^{t+1} \) a subspace of dimension \( p \), i.e. a point in \( G \),

\[ D(Y, V) \leq c_3(t, p) \deg_{L_G} Y. \]
Proof Let $f \in \Gamma(G, L_G)$ such that $Y = \text{div} f$. Since with $d = \dim G$,
\[
D(Y, V) = \log |f_V| - \int_G \log |f| \mu_G^d = \log |f_V| - \log ||f||_0,
\]
the Lemma immediately follows from Proposition 3.13.

The following two propositions supply the essential techniques for calculating algebraic distances. Let $Z, W$ be projective Kähler varieties of dimensions $r, s$ with Kähler structures $\mu_Z, \mu_W$, and $C$ a projective regular variety over $\mathbb{C}$ of pure dimension $d$, and consider a correspondence

\[
\begin{array}{ccc}
Z & \overset{f}{\longrightarrow} & C \\
\downarrow & & \downarrow \\
W & \overset{g}{\longrightarrow} & \end{array}
\]  

\hspace{1cm} (13)

with $f, g$ flat, surjective and projective. For $X \in Z_{e\text{ff}}(Z), Y \in Z_{e\text{ff}}(W)$ define $C_*(X) := g_*(f^*(X)), C^*(Y) := f_*(g^*(Y))$.

4.5 Proposition (Functoriality) In the above situation, let $q \geq p$ and $X \in Z_{e\text{ff}}^p(Z), Y \in Z_{e\text{ff}}^q(W)$ be such that $C_*(X)$ and $C^*(Y)$ are both nonzero, and the intersections of $X$ with $C^*(Y)$ and of $C_*(X)$ with $Y$ are both proper.

1. If $C^* = f_* \circ g^*$ maps harmonic forms to harmonic forms, then $C_* = g_* \circ f^*$ maps normalized Green forms to normalized Green forms.

2. If $p + q = d + 1$ and $C^* = f_* \circ g^*$ maps harmonic forms to harmonic forms, then
\[
D(X, C^*Y) = D(C_*X, Y).
\]

Proof 1. Let $g_X$ be a normalized Green form for $X \in Z_{e\text{ff}}^p(Z)$, and $\eta \in H^{d+1-p,d+1-p}(W)$. Since by [SABK], Lemma II.2 (ii),
\[
g_*[f^*gz] = [g_*(f^*gz)],
\]
\[
\int_W C_*(g_X \eta) = \int_W g_*(f^*g_X) \eta = \int_C f^*g_X g^* \eta = \int_Z g_X f_*(g^* \eta) = 0,
\]
as $f_* \circ g^* \eta$ is harmonic, that is $C_*g_X$ is orthogonal to $H^{d+1-p,d+1-p}(W)$.

2. By Remark 4.3 with $g_X$ a normalized Green form for $X$,
\[
D(X, C^*Y) = -\frac{1}{2} \int_{C^*Y} g_X = [\mathbb{C}(g^{-1}Y) : \mathbb{C}(f(g^{+1}(Y)))] \int_{f(g^{-1}(Y))} g_X = \int_{g^{-1}(Y)} f^*g_X,
\]

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which by Fubini’s Theorem equals
\[
\int_Y (g_\ast \circ f^\ast) g_X.
\]
As, by the first part \((g_\ast \circ f^\ast) g_z\) is a normalized Green form, this in turn equals
\(D(C_\ast X, Y)\).

**4.6 Proposition** Let \(W\) be a projective algebraic Kähler variety of dimension \(t\) with Kähler form \(\mu\), and for \(p + q + r \leq t + 1\), let \(X, Y, Z\) be effective cycles of pure codimension \(p, q, r\) on \(W\) such that \(X.Y, Y.Z, X.Y, Y.Z.W\) are of pure codimension \(p, q, r, p + r, p + q + r\) respectively. Then,

1. If \(g_Y, g_Z\) are admissible Green forms for \(Y\) and \(Z\)
\[
[g_Y] \wedge \delta_X.Z + H(\delta_Y) \wedge [g_Z] \wedge \delta_X = \delta_X.Y \wedge [g_Z] + H(\delta_Z) \wedge [g_Y] \wedge \delta_X \mod \text{Im}\partial + \text{Im}\bar{\partial}.
\]
In particular, for \(W = \mathbb{P}^t\) projective space,
\[
[g_Y] \wedge \delta_X.Z + \deg Y \mu^q \wedge [g_Z] \wedge \delta_X = \delta_X.Y \wedge [g_Z] + \deg Z \mu^r \wedge [g_Y] \wedge \delta_X \mod \text{Im}\partial + \text{Im}\bar{\partial}.
\]
2.
\[
D(Y, X.Z) - \frac{1}{2} \int_X [g_Z] H(\delta_Y) \mu^{t+1-p-q-r} =
\]
\[
D(X.Y, Z) - \frac{1}{2} \int_X [g_Y] H(\delta_Z) \mu^{t+1-p-q-r}.
\]
In particular, in projective space,
\[
D(Y, X.Z) + \deg Y D(X, Z) = D(X.Y, Z) + \deg Z D(X, Y).
\]

**Proof** 1. \cite{GS1}, Theorem 2.2.2.
2. In 1, let \(g_Y, g_Z\) be the \(\mu\)-normalized Green forms of \(Y\), and \(Z\). Multiplying the equality of 1 with \(\mu^{t+1-p-q-r}\), integrating over \(X\), and dividing by \(-2\) leads the first equality. The second equality follows from \(H(\delta_Y) = \deg Y \mu^{t+1-q}, H(\delta_Z) = \deg Z \mu^{t+1-r}\) in case of projective space, and \(g_Y, g_Z\) normalized and hence admissible.

In case the sum of the codimensions of two cycles is bigger than the dimension of the space plus one, the definition of their algebraic distance is not as straightforward. On projective space however there are even several possibilities to define the algebraic definition in a useful way.

**4.7 Definition** For \(p + q \geq t + 1\) let \(\mathbb{P}(W) \subset \mathbb{P}^t\) a subspace of codimension \(q\), and \(X\) an effective cycle of pure codimension \(p\) not meeting \(\mathbb{P}(W)\) in \(\mathbb{P}^t\).
1. Let $G_W$ be the sub Grassmannian of the Grassmannian $G_{t+1,t+1-p}$ consisting of the subspaces of codimension $t + 1 - p$ that contain $W$, and $V_X$ as in Corollary 3.16. Define

$$D(\mathbb{P}(W), X) = D_G(\mathbb{P}(W), X) := \frac{1}{\deg G_W} D(G_W, V_X).$$

2. In the same situation as in 1, define

$$D_\infty(\mathbb{P}(W), X) := \text{sup}_{V \in G_W} D(V, V_X) = \text{sup}_{V \in G_W} D(\mathbb{P}(V), X).$$

The equality holds by the Lemma 3.15 and 4.5.

3. Let $Ch(X) \subset (\mathbb{P}^t)^{t+1-p}$ be the Chow divisor of $X$, and

$$\binom{(t + 1 - p)(q - 1)}{q - 1, \ldots, q - 1} = \frac{(t + 1 - p)(q - 1)!}{(q - 1)!^{t+1-p}}$$

the multinomial coefficients. With $\mathbb{P}(\tilde{W}) \subset \tilde{\mathbb{P}^t}$ the subspace dual to $\mathbb{P}(W)$, define

$$D_{Ch}(\mathbb{P}(W), X) := \frac{1}{\binom{(t+1-p)(q-1)}{q-1,\ldots,q-1}} D(Ch(X), \mathbb{P}(\tilde{W})^{t+1-p}).$$

4.8 Fact For $p + q \geq t + 1$, and $\mathbb{P}(W)$ a fixed subspace of codimension $q$, the maps

$$D_0(\mathbb{P}(W), \cdot), D_{Ch}(\mathbb{P}(W), \cdot) : Z_{eff}^{p}(\mathbb{P}^t) \to \mathbb{R}$$

are additive when defined.

4.9 Theorem

1. For $p + q \geq t + 1$, any $X \in Z_{eff}^{p}(\mathbb{P}^t)$, and $\mathbb{P}(W)$ a subspace of codimension $q$ not meeting $X$,

$$D_G(\mathbb{P}(W), X) \leq D_\infty(\mathbb{P}(W), X) \leq c_3(t + 1 - q, t + 1) \deg X + D_G(\mathbb{P}(W), X),$$

2. With $c_5, c_7$ the constants from Proposition 3.18 and Lemma 3.20, for all $X$, $\mathbb{P}(W)$ as above,

$$D_\infty(X, \mathbb{P}(W)) - (c_7 + c_5) \deg X \leq D_{Ch}(X, \mathbb{P}(W)) \leq$$

$$D_\infty(X, \mathbb{P}(W)) - c_5 \deg X.$$
PROOF 1. Let \( f_X \in \Gamma(G, \mathcal{L}^{\otimes \text{deg} X}) \). be a of norm \( \log \| f_X \|_0 = 0 \) such that \( V_X = \text{div}(f_X) \). Then,

\[
D_G(\mathbb{P}(W), X) = \frac{1}{\deg G_W} \int_{G_W} \log |f_X|\mu_G^{(p+q-t-1)(2t+2-p-q)} = \\
\int_{G_W} \log |f_X|\omega_{G_W} = \log \| f_X \|_{G_W}. \]

By Proposition 3.13 this is less or equal

\[
\log \| f_X \|_{G_W} = D_\infty(\mathbb{P}(W), X),
\]

and this in turn by the same Proposition is less or equal

\[
\log \| f_X \|_{G_W} + c_0(t+1-q, t+1) \deg X = D_G(\mathbb{P}(W), X) + c_3(t+1-q, t+1) \deg X.
\]

2. Let \( f \in \Gamma((\mathbb{P}^t)^{t+1-p}, O(\deg X, \ldots, \deg X)) \) such that \( \text{Ch}(X) = \text{div} f \). Then,

\[
|f_{[\check{v}_1, \ldots, \check{v}_{t+1-p}]}| = \frac{f(\check{v}_{t}^{\deg X}, \ldots, \check{v}_{t+1-p}^{\deg X})}{|\check{v}_1|^{\deg X} \ldots |\check{v}_{t+1-p}|^{\deg X}},
\]

and \( f(\check{v}_1, \ldots, \check{v}_{t+1-p}) = 0 \) if \( \check{v}_i = \check{v}_j \) for some \( i, j \). Hence \( |f_{[\check{v}_1, \ldots, \check{v}_{t+1-p}]| \) takes it’s supremum at some \( [\check{v}_1, \ldots, \check{v}_{t+1-p}] \) with \( \check{v}_i \perp \check{v}_j \) for every \( i \neq j \).

Further, by Proposition 3.18 and Lemma 4.5,

\[
D(\text{Ch}(X), [\check{v}_1, \ldots, \check{v}_{t+1-p}]) = D(X^{t+1-p}, \ker \check{v}_1 \times \cdots \times \ker \check{v}_{t+1-p}),
\]

which for \( \check{v}_1, \ldots, \check{v}_{t+1-p} \) pairwise orthogonal by Proposition 3.18 equals

\[
D(X, \ker \check{v}_1 \cap \cdots \cap \ker (\check{v}_{t+1-p})) - c_3(t, p, t+1-p) \deg X.
\]

Hence,

\[
\log |f|_{\mathbb{P}(W)^{t+1-p}} - \log |f_X|_0 = D_\infty(\mathbb{P}(W), X) - c_5 \deg X.
\] 

Since

\[
D_{\text{Ch}}(\mathbb{P}(W), X) = \frac{1}{(t+1-p)(q-1)} D(\mathbb{P}(W)^{t+1-p}, \text{Ch}(X)) = \\
\frac{1}{(t+1-p)(q-1)} \int_{(\mathbb{P}^t)^{t+1-p}} \log |f_X|\tilde{\mu}_{(t-q)(t+1-p)} - \\
\frac{1}{(t+1-p)(q-1)} \int_{(\mathbb{P}^t)^{t+1-p}} \log |f_X|\tilde{\mu}_1^q \cdots \tilde{\mu}_{t+1-p}^q \tilde{\mu}_{(t-q)(t+1-p)},
\]

which by Lemma 3.20 is less or equal

\[
\sup_{[\check{v}_1, \ldots, \check{v}_{t+1-p}] \in W^{t+1-p}} \log |(f_X)_{[\check{v}_1, \ldots, \check{v}_{t+1-p}]|} - \log |f_X|_0 = \log |f_X|_{\mathbb{P}(W)^{t+1-p}} - \log |f_X|_0,
\]

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which in turn by the same proposition is less or equal
\[
\frac{1}{(t+1-p)(q-1)} \int_{\mathbb{P}(W)^{t+1-p}} \log |f_X| \mu^{(t-q)(t+1-p)} -
\]
\[
\frac{1}{(t+1-p)(q-1)} \int_{(\mathbb{P})^t} \log |f_X| \mu_1^q \cdots \mu_{t+1-p}^q \mu^{(t-q)(t+1-p)} + c_7 \deg X.
\]
Together with equation (14), this implies the claim.

### 4.2 Linear varieties

For linear cycles, the algebraic distance is easy to calculate. Let $\mathbb{P}(V), \mathbb{P}(W) \subset \mathbb{C}^{t+1}$ be properly intersecting subspaces of dimension $p, q$, define $r := \dim V \cap W - 1 = \max(-1, p+q-t)$, and let $v_0, \ldots, v_{p+q+1-r} \in \mathbb{C}^{t+1}$ such that $(v_0, \ldots, v_r), (v_0, \ldots, v_p)$, $(v_0, \ldots, v_r, v_{p+1}, \ldots, v_{p+q+1-r})$ are bases for $V \cap W, V, W$ respectively. Define

\[
|V, W| := \frac{\text{vol}(v_0, \ldots, v_{p+q+1-r}) \text{vol}(v_0, \ldots, v_r)}{\text{vol}(v_0, \ldots, v_p) \text{vol}(v_0, \ldots, v_r, v_{p+1}, \ldots, v_{p+q+1-r})}.
\]

Clearly, $\log |V, W| \leq 0$, and for $V, W$ both one dimensional $|V, W|$ equals the Fubini-Study distance of the points $V, W$ in $\mathbb{P}(\mathbb{C})$.

### 4.10 Proposition

Let $\mathbb{P}(V), \mathbb{P}(W) \subset \mathbb{P}^t$ be projective subspaces of dimensions $p, q$ respectively that intersect properly, and with $r \leq t + 1$ let $G_r$ be the Grassmannian of $r$-dimensional subspaces of $\mathbb{C}^{t+1}$. With the constants constants $c_1, c_2$ and some positive constants $c_4, c_5$ only depending on $p, q, r, t$,

1. For $p + q \geq t - 1$,

\[
D(\mathbb{P}(V), \mathbb{P}(W)) = \log |V, W| + c_1(t, p, q) \leq c_1(t, p, q).
\]

The inequality is an equality, iff the orthogonal complements of $V \cap W$ in $V$ and $W$ are orthogonal to each other.

2. There are positive constant $c_4, c_6$ only depending on $p, q, t$ such that for $p + q < t$,

\[
D_G(\mathbb{P}(V), \mathbb{P}(W)) - c_4(p, q, t) = D_\infty(\mathbb{P}(V), \mathbb{P}(W)) = D_{Ch}(\mathbb{P}(V), \mathbb{P}(W)) - c_6(p, q, t) = \log |V, W| + c_2(t, q) \leq c_2(t, q).
\]

The inequality is an equality iff $V, W$ are orthogonal to each other. The algebraic distance of two projective spaces is thus symmetric modulo $c_2(t, q) - c_2(t, p) + c_4(c_6)$.  

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3. If \( p + q + 2 \leq r < t + 1 \), then \( G_V, G_W \) in \( G_r \) fulfill

\[
D(G_V, G_W) = \frac{\deg G_r}{\deg G_{V+W}} \log |V, W| + c_8(t, p, q) \leq c_8(t, p, q).
\]

The inequality is an equality, iff \( V \) and \( W \) are orthogonal to each other.

4. For \( p + q + 1 - t \geq r \), the varieties \( G^F, G^E \) in \( G_r \) fulfill

\[
D(G^F, G^E) = \frac{\deg G_r}{\deg G_{F \cap E}} D(\mathbb{P}(F), \mathbb{P}(E)) + c_8(t, p, q) \leq c_8(t, p, q) + c_1(t, p, q).
\]

The inequality is an equality iff the orthogonal complements of \( E \cap F \) in \( E \) and \( F \) are orthogonal.

**Proof**

Let \( v_0, \ldots, v_{p+q+1-r} \) as in the Definition of the distance of \( V \) and \( W \). To shorten formulas, always assume that the bases of \( V \cap W, V \) and \( W \) are orthonormal. Then,

\[
|V, W| = \text{vol}(v_0, \ldots, v_{p+q+1-r}).
\]

Further, since distances and algebraic distances are invariant under the unitary group, one may assume \( v_i = e_i \) the standard unit vector for \( i = 0, \ldots, p \), hence \( \mathbb{P}(V) \) is the space where, \( x_{p+1}, \ldots, x_t \) vanish.

1. Use complete induction over the codimension of \( \mathbb{P}(V) \). For \( t - p = \text{codim} \mathbb{P}(V) = 1 \), we have \( \log |V, W| = \log |v_t^\perp| = \log |(x_t)|_{v_t^\perp}| \), where is \( v_t^\perp \) projection of \( v_t \) to the orthogonal complement of \( V = \langle e_0, \ldots, e_{t-1} \rangle \). By Lemma 3.11

\[
D(\mathbb{P}(V), \mathbb{P}(W)) = \int_{\mathbb{P}(W)} \log |x_t| - \int_{\mathbb{P}^t} \log |x_t|
\]

\[
= \sup_{P \in \mathbb{P}(W)} \log |(x_t)|_P - \sum_{j=1}^q \frac{1}{j} - \int_{\mathbb{P}^t} \log |x_t|
\]

\[
= \log |(x_t)|_{v_t^\perp}| - \frac{1}{2} \sum_{j=1}^q \frac{1}{j} + \frac{1}{2} \sum_{j=1}^t \frac{1}{j},
\]

which by the above equals

\[
\log |V, W| + \sum_{j=q+1}^t \frac{1}{j} = \log |V, W| + \sigma_t - \sigma_{t-1} - \sigma_q + \sigma_{q-1} = \log |V, W| + c_1(t, t-1, q).
\]

Assume now the claim has been proved for \( \text{codim} \mathbb{P}(V) = t - p \), and let \( \text{codim} \mathbb{P}(V) = t + 1 - p \), that is \( \dim V = p \). Again, because of the invariance of distance and algebraic distance under the unitary group, one may assume that
$v_{p+1}, \ldots, v_{t-1}$ are contained in $\langle e_0, \ldots, e_{t-1} \rangle$. Let $\mathbb{P}(F)$ be the vanishing set of $x_t$, and $\Lambda_W$ as in (4). Then, by (7),

$$D(\mathbb{P}(V), \mathbb{P}(W)) = -\frac{1}{2} \int_{\mathbb{P}(V)} \Lambda_{\mathbb{P}(W)} \mu^{t+1-p-q} - \sigma_{t-q} + \sigma_{q-1} - \sigma_t =$$

$$-\frac{1}{2} \int_{\mathbb{P}(W)} \Lambda_{\mathbb{P}(W)} \mu^{t+1-p-q} + \frac{1}{2} \int_{\mathbb{P}(F)} \Lambda_{\mathbb{P}(W)} \mu^{t-p-q} \left( -\frac{1}{2} \int_{\mathbb{P}(F)} \Lambda_{\mathbb{P}(W)} \mu^{t-p-q} + \sigma_{t-q} + \sigma_{q-1} - \sigma_t =
\right.$$

$$\left. -\frac{1}{2} \int_{\mathbb{P}(W)} \Lambda_{\mathbb{P}(W)} \mu^{t+1-p-q} + \frac{1}{2} \int_{\mathbb{P}(F)} \Lambda_{\mathbb{P}(W)} \mu^{t-p-q} + D(\mathbb{P}(F), \mathbb{P}(W)). \right)$$

The first two summands are the algebraic distance of $\mathbb{P}(V)$ and $\mathbb{P}(W)$ as subvarieties of $\mathbb{P}(F)$. As $\mathbb{P}(V)$ has codimension $t-p$ in $\mathbb{P}(F)$, the induction hypothesis implies that this sum equals

$$\log |\mathbb{P}(V), \mathbb{P}(W) \cap \mathbb{P}(F)| + \sigma_{p+q-t-1} + \sigma_{t-1} - \sigma_{p-1} - \sigma_{q-1} =$$

$$\log \text{vol} \langle v_0, \ldots, v_{t-1} \rangle + \sigma_{p+q-t-1} + \sigma_{t-1} - \sigma_{p-1} - \sigma_{q-1}.$$ Further, as seen above

$$D(\mathbb{P}(F), \mathbb{P}(W)) = \log |\mathbb{P}(F), \mathbb{P}(W)| + \sigma_t - \sigma_{t-1} - \sigma_{p} + \sigma_{q} =$$

$$\log |v_t^+| + \sigma_t - \sigma_{t-1} - \sigma_{q} + \sigma_{q-1},$$

where $v_t^+$ denotes the projection of $v_t$ to the orthogonal complement of $\mathbb{P}(F)$. Since $\text{vol} \langle v_0, \ldots, v_{p+q+1-r} \rangle = \text{vol} \langle v_0, \ldots, v_{t} \rangle = \text{vol} \langle v_0, \ldots, v_{t-1} \rangle \cdot |v_t^+|$, the claim follows with $c_1(t, p, q) = \sigma_{p+q-t} + \sigma_t - \sigma_p - \sigma_q$.

2. By Definition and part one,

$$D_{\infty}(\mathbb{P}(V), \mathbb{P}(W)) = \sup_{\mathbb{P}(F)} D(\mathbb{P}(F), \mathbb{P}(W)) = \sup_F \log |F, W| + c_1(t, t + 1 - q, q),$$

where $\mathbb{P}(F)$ runs over the superspaces of $\mathbb{P}(V)$ of codimension $q+1$. As $\sup_F \log |F, W| = \log |V, W|$, and the supremum is attained at $F$ a direct sum of $W$ and an orthogonal complement of $V + W$, the claim about $D_{\infty}$ follows with $c_2(t, q) = c_1(t, t + 1 - q, q) = \sigma_1 + \sigma_t - \sigma_{t+1-q} - \sigma_q$.

Next, let $f \in \Gamma(G_{t+1-t-q-1}, L_G)$ be such that $V_{\mathbb{P}(W)} = \text{div} f$. This implies $f|_{G_V} \in \Gamma(G_V, L_{G_V})$. Since the fix group of $G_V$ inside the unitary group operates transitively on the unit sphere in $\Gamma(G_V, L_{G_V})$, the number

$$c_4 := \int_{G_V} \log |f| \omega_{G_V} - \sup_{P \in G_V} \log |f_P|$$

do not depend on $f, V, W$ but only on $t, p, q$, and we have

$$D_G(\mathbb{P}(V), \mathbb{P}(W)) = \frac{1}{\deg G_V} D(G_V, V_{\mathbb{P}(W)}) = \int_{G_V} \log |f| \omega_{G_V} - \int_G \log |f| \omega_G =$$

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\[ c_4 + \sup_{p \in G_V} \log |fp| - \int_G \log |f| \omega_G = c_4 + D_\infty(\mathbb{P}(V), \mathbb{P}(W)), \]

finishing the proof for \( D_G(\bullet, \bullet) \). The claim about \( D_{CH}(\bullet, \bullet) \) is proved similarly.

3. Assume first that \( p + q + 2 = r \), and let \( E \) be a subspace of dimension \( t + 1 - r \) that is orthogonal to \( V + W \). Since \( U(t + 1) \) acts transitively on spaces \( F, E \) of given dimensions with \( F \perp E \),

\[ D(G_V, V_{\mathbb{P}(E)}) = c, \quad D(G_V.G_W, V_{\mathbb{P}(E)}) = D(G_{V+W}, V_{\mathbb{P}(E)}) = c' \]

with constants \( c, c' \) only depending on \( p, q, r, t \). In Proposition 3.17 replace \( p, q, p-q \) by \( r, q+1, p+1 \). Then, by Lemma 4.5

\[
D(G_V.V_{\mathbb{P}(E)}, G_W) = D(G_V(\varphi_r)_{*}^\frac{1}{\mathbb{P}(E)}, (\varphi_r)_{*}^\frac{1}{\mathbb{P}(E)}') W = D(\psi_{q+1}^+, \psi_{q+1}^-)\mathbb{P}(E)V, W = D(\mathbb{P}(E)V, \psi_{q+1}^-)\mathbb{P}(E)V, W = \log |V+E, W| + c_1(t, t + p - r + 1, q) = \log |V, W| + c_1(t, t + p - r + 1, q).
\]

The last inequality holding since \( E \) is orthogonal to \( V + W \). Hence, by Proposition 4.6 and Lemma 4.5

\[
D(G_V, G_W) = -D(G_V.G_W, V_{\mathbb{P}(E)}) + D(G_V.V_{\mathbb{P}(E)}, G_W) + \frac{1}{\deg G_W}D(G_V, V_{\mathbb{P}(E)}) = -c' + \log |V, W| + c_1(t, t + p - r + 1, q) + \frac{1}{\deg G_W}D(G_V, V_{\mathbb{P}(E)})
\]

and the claim holds with \( c_8(t, p, q) = c_8'(t, p, q) := -c' + c_1(t, t + p - r + 1, q) + \frac{c}{\deg G_W} \).

Let now \( p + q + 2 < r \) and \( E \) a space of dimension \( r - p - q - 1 \) that is orthogonal to \( V + W \). Then, using again a transitive action of \( U(t + 1) \), one gets \( D(G_V, G_E) = c'' \) with \( c'' \) only depending on \( p, t \). Further, by the first half of the proof

\[
D(G_{V+W}, G_E) = \log |V+W, E| + c_8'(t, p+q+1, r-p-q-2) = c_8(t, p+q+1, r-p-q-2),
\]

and

\[
D(G_{V+E}, G_W) = \log |V+E, W| + c_8'(t, t - q, q) = \log |V, W| + c_8'(t, t - q, q).
\]

Hence, by Proposition 4.6

\[
\frac{\deg G_{V+W}}{\deg G_r}D(G_V, G_W) = \frac{1}{\deg G_E}D(G_V, G_W) = -D(G_V.G_W, G_E) + \frac{1}{\deg G_W}D(G_V, G_E) + D(G_V.G_E, G_W) = -D(G_{V+W}, G_E) + D(G_V, G_E) + D(G_{V+E}, G_W) = c_8(t, p+q+1, t-p-q-2) + \frac{c''}{\deg G_W} + c_8'(t, t - q, q) + \log |V, W|,
\]

finishing the proof.

4. Follows similarly to part 3, this time using the orthogonal complement \( E \) of \( V \cap W \) and Proposition 3.17.2.
4.3 Decompositions

With $W$ a projective Kähler variety of dimension $t$ and $Z \subset W$ a subvariety of codimension $r$, and $X,Y$ effective cycles of pure codimensions $p,q$, we have the algebraic distance $D(X,Y)$ if $p+q \leq t+1$ and $X,Y$ intersect properly. If for $p+q \leq t+r+1$ the supports of $X$ and $Y$ are contained in $Z$, and they intersect properly as cycles in $Z$, denote with $D^Z(X,Y)$ the algebraic distance of $X,Y$ as cycles in $Z$.

4.11 Proposition Let $t,p,r \leq q$ be natural numbers with $p+r \leq t+1$, further $X \in \mathcal{Z}_{eff}(\mathbb{P}^t)$ and $\mathbb{P}(W) \subset \mathbb{P}(F) \subset \mathbb{P}^t$ subspaces of codimensions $q,r$ such that the intersection of $\mathbb{P}(W)$ and $\mathbb{P}(F)$ with $X$ are proper.

1. If $p+q \leq t+1$, then
   \[ D(\mathbb{P}(W),X) = D^{\mathbb{P}(F)}(X,\mathbb{P}(F),\mathbb{P}(W)) + D(\mathbb{P}(F),X). \]

2. If $p+q > t+1$, and $X_W$ is the cycle defined in (9), then
   \[ D(\mathbb{P}(W),X) = D^{\mathbb{P}(F)}(\mathbb{P}(W),X,\mathbb{P}(F)) + D(X,\mathbb{P}(F)) - D(X_W,\mathbb{P}(F)) + c_{11} \deg X, \]
   where $c_{11}$ is a constant only depending on $p,q,$ and $t$.

3. If $p+q \leq t+1$, then in the Grassmannian $G = G_{p,t+1}$,
   \[ D(V_X,G^W) = \frac{\deg G^W}{\deg G^F} D(V_X,G^F) + D^{G^F}(V_X,G^F,G^W). \]

4. If $p+q \leq t+1$, then in the Grassmannian $G = G_{p,t+1}$,
   \[ D(V_X,G^F) = \deg G^F D(X,\mathbb{P}(F)) \leq d(t-q,p)c_2(t,p,q)\deg X, \]
   with $d$ a constant depending on $t-q$ and $p$.

Proof 1. Let $g_X$ be an admissible Green form for $X$. By [SABK], Lemma II.2, $g_X^{\mathbb{P}(F)} := g_X|_{\mathbb{P}(F)}$ is a Green form of log type for $X.\mathbb{P}(F)$ in $\mathbb{P}(F)$, which by Proposition 3.4 is admissible. Hence,
   \[ D(X,\mathbb{P}(W)) = -\frac{1}{2} \int_{\mathbb{P}(W)} g_X \mu^{t+1-p-q} + \frac{1}{2} \int_{\mathbb{P}(F)} g_X \mu^{t+1-p} = \]
   \[ -\frac{1}{2} \int_{\mathbb{P}(W)} g_X^{\mathbb{P}(F)} \mu^{t+1-p-q} + \frac{1}{2} \int_{\mathbb{P}(F)} g_X^{\mathbb{P}(F)} \mu^{t+1-p-r} - \frac{1}{2} \int_{\mathbb{P}(F)} g_X \mu^{t+1-p-r+} \]
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Further repeating the argument for (16) with $/C8$

Inserting into (15) gives

\[
\text{hence}
\]

and

\[
\text{X of}
\]

For dim $X > 0$, let $V \subset F$ be a subspace of dimension $p + q - 1 - 1$ such that $U := V + W$ is a direct sum and $\mathbb{P}(U)$ does not meet the support of $X$. One has $U = G_V \cap G_W$ in $G = G_{t+1,p}$. With $V_X$ as in Corollary 3.16 one has $H(\delta_{V_X}) = \deg X \mu_G$, and by Proposition 4.6 with $g_W$ a normalized Green form for $G_W$, and $\eta_W = H(\delta_{G_W})$,

\[
\deg X D(G_V, G_W) + D(V_X, U) = \frac{1}{2} \int_{G_V} g_W \mu_G + D(V_X, U) = D(G_W, V_X, G_V) - \frac{1}{2} \int_{V_X} g_W \eta_{G_V}. \quad (15)
\]

If $f \in \Gamma(G, L^{\deg X})$ with $V_X = \text{div} f$, the associativity of the star product implies

\[
-\frac{1}{2} \int_{V_X} g_W \eta_{G_V} = -\frac{1}{2} \deg X \int_{G_V} g_W \eta_{G_V} \mu_G + \int_{G_W} \log |f| \eta_{G_V} - \int_{G} \log |f| \eta_{G_W} \eta_{G_V}
\]

By Proposition 3.14 $\eta_{G_V} \eta_{G_W} = \omega_G$, and as $g_W$ is normalized, this equals

\[
\frac{1}{\deg G_W} D(G_W, V_X).
\]

Inserting into (15) gives

\[
\deg X D(G_V, G_W) + D(V_X, U) = D(G_W, V_X, G_V) + \frac{1}{\deg G_W} D(G_W, V_X).
\]

Thus, by (10), Proposition 3.17 Proposition 4.5 and the Definition of $D(\mathbb{P}(W), X), \deg X D(G_W, G_V) + D(X, \mathbb{P}(U)) = D(X_W, \mathbb{P}(V)) + D(\mathbb{P}(W), X). \quad (16)

By part one,

\[
D(X, \mathbb{P}(U)) = D_{\mathbb{P}(F)}(X, \mathbb{P}(F), \mathbb{P}(U)) + D(X, \mathbb{P}(F)),
\]

and

\[
D(X_W, \mathbb{P}(V)) = D_{\mathbb{P}(F)}(X_W, \mathbb{P}(F), \mathbb{P}(V)) + D(X_W, \mathbb{P}(F)),
\]

hence

\[
\deg X D(G_W, G_V) + D_{\mathbb{P}(F)}(X, \mathbb{P}(F), \mathbb{P}(U)) + D(X, \mathbb{P}(F)) = D_{\mathbb{P}(F)}(X_W, \mathbb{P}(F), \mathbb{P}(V)) + D(X_W, \mathbb{P}(F)) + D(\mathbb{P}(W), X). \quad (17)
\]

Further repeating the argument for (16) with $\mathbb{P}(F)$ instead of $\mathbb{P}^t$ and $X. \mathbb{P}(F)$ instead of $X$, one gets

\[
\deg X D^{GR}(G_V^F, G_W^F) + D_{\mathbb{P}(F)}(X, \mathbb{P}(F), \mathbb{P}(U)) =
\]
\[ D^{p(F)}((X, \mathbb{P}(F))_W, \mathbb{P}(V)) + D^{p(F)}(\mathbb{P}(W), X, \mathbb{P}(F)). \]  
(18)

Using \((X, \mathbb{P}(F))_W = X_W.\mathbb{P}(F)\) while subtracting (17) and (18) leads to

\[ \deg X \left( D(G_V, G_W) - D^{G^F}(G_V, G_W^F) \right) + D(X, \mathbb{P}(F)) = D(X, \mathbb{P}(F)) + D(\mathbb{P}(W), X) - D^{p(F)}(\mathbb{P}(W), X, \mathbb{P}(F)). \]

Finally, by Proposition \[4.10.3\],

\[ D(G_V, G_W) - c_s(t, p, q) = D^{G^F}(G_V^F, G_W^F) - c_s(p, p, q) = \log \| \mathbb{P}(V), \mathbb{P}(W) \|, \]

hence the claim follows with \(c_11(t, p, q) = c_s(p, p, q) - c_s(t, p, q)\).

3. Let \(f \in \Gamma(G, L^\otimes \deg X)\) such that \(V_X = \text{div} f\), and \(\eta_G^W = H(\delta_{G^W})\). Then,

\[ D(V_X, G^W) = \int_{G^W} \log |f| \mu_G^{p(t+1-q-p)} - \int_G \log |f| \eta_G^W \mu_G^{p(t+1-q-p)}, \]

which by Proposition \[3.14.5\] equals

\[ \int_{G^W} \log |f| \mu_G^{p(t+1-q-p)} - \deg G^W \int_G \log |f| \omega_G = \]

\[ \int_{G^W} \log |f| \mu_G^{p(t+1-r-p)} - \int_{G^F} \log |f| \eta_G^{G^F} \mu_G^{p(t+1-q-p)} + \]

\[ \int_{G^F} \log |f| \eta_G^{G^F} \mu_G^{p(t+1-q-p)} - \deg G^W \int_G \log |f| \omega_G, \]

where \(\eta_G^{G^F} = H^{G^F}(\delta_{G^W})\) with \(H^{G^F}\) the harmonic projection in the sub Grassmannian \(G^F \subset G\). Using Proposition \[3.14.5\] once more, this equals

\[ \int_{G^W} \log |f| \mu_G^{p(t+1-q-p)} - \int_{G^F} \log |f| \eta_G^{G^F} \mu_G^{p(t+1-q-p)} + \]

\[ \frac{\deg G^W}{\deg G^F} \int_{G^F} \log |f| \mu_G^{p(t+1-r-p)} - \frac{\deg G^W}{\deg G^F} \int_G \log |f| \eta_G^{G^F} \mu_G^{p(t+1-r-p)} = \]

\[ D^{G^F}(V_X, G^F, G^W) + \frac{\deg G^W}{\deg G^F} D(V_X, G^F). \]

4. Assume first, \(\mathbb{P}(F) = \mathbb{P}(W)\) has codimension \(t + 1 - p\). Then, \(G^W = G_W = W\) is a point, hence has degree one, and

\[ D(X, \mathbb{P}(W)) = D(V_X, W) = D(V_X, G_W) = D(V_X, G_W^F) \]  
(19)

follows from Lemma \[3.15\] and Lemma \[4.5\], and Remark \[4.3\].
In the general case, let \( \mathbb{P}(V) \subset \mathbb{P}(F) \) be a subspace of dimension \( p - 1 \) that does not meet the support of \( X \). By part one,

\[
D(X, \mathbb{P}(F)) = D(X, \mathbb{P}(V)) - D^{\mathbb{P}(F)}(X, \mathbb{P}(F), \mathbb{P}(V)).
\]

Thus, applying (19) once for \( \mathbb{P}(F) \) and \( G_{t+1} \), and once for \( \mathbb{P}(F) \) and \( G_{t+1} \),

\[
D(X, \mathbb{P}(F)) = D(V_X, V) - D^{G_{t+1}}(V_{X, \mathbb{P}(F)}, F) = D(V_X, V) - D^{G_{t+1}}(V_X, G_{t+1}, W),
\]

which by part 3 equals

\[
D(V_X, W) - \left( -\frac{1}{\deg G_{t+1}} D(G_{t+1}, V_X) + D(V_X, W) \right) = \frac{1}{\deg G_{t+1}} D(G_{t+1}, V_X).
\]

Since, by Remark 4.3, \( D(X, \mathbb{P}(F)) \leq c_2(t, p, q) \), the claim follows with \( d(t - q, p) = \deg G_{t+1} \).

### 4.4 Reduction to distances to points

**4.12 Lemma** For \( p, q \leq t + 1 \) let \( X \in Z_{eff}^p(\mathbb{P}^t) \), and \( \mathbb{P}(V) \supseteq \mathbb{P}^t \) a subspace of codimension \( q \) intersecting \( X \) properly, and \( s \) some natural number between 0 and \( t + 1 \). There is a positive constant \( C \), only depending on \( t, p, q, s \) such that,

1. if \( p + q = t + 1 \), then

\[
D(\mathbb{P}(E), X) \geq D(\mathbb{P}(V), X) - C(t, p, q, s) \deg X
\]

for every space \( \mathbb{P}(E) \) of dimension \( s \) such that \( \mathbb{P}(E) \subset \mathbb{P}(V) \) or \( \mathbb{P}(E) \supseteq \mathbb{P}(V) \).

2. if \( p + q < t + 1 \), then

\[
D(\mathbb{P}(E), X) \geq D(\mathbb{P}(V), X) - C(t, p, q, s) \deg X,
\]

for every subspace \( \mathbb{P}(E) \) of dimension \( s > t - q \) containing \( \mathbb{P}(V) \), and

\[
\sup_{E \in G_V^s} D(\mathbb{P}(E), X) \geq D(\mathbb{P}(V), X) - C(t, p, q, s) \deg X,
\]

for every \( s \leq t - q \).

3. if \( p + q > t + 1 \), then

\[
D(\mathbb{P}(E), X) \geq D(\mathbb{P}(V), X) - C(t, p, q, s) \deg X,
\]

for every subspace \( \mathbb{P}(E) \subset \mathbb{P}(V) \) of dimension \( s < t - q \), and

\[
\sup_{E \in (G_{t+1})^v} D(\mathbb{P}(E), X) \geq D(\mathbb{P}(V), X) - C(t, p, q, s) \deg X,
\]

for every \( s \geq t - q \).

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Proof

Let \( f \in \Gamma(G_p, L_G^{\deg X}) \) such that \( V_X = \text{div} f \).

1. Let first \( \mathbb{P}(E) \) be a subspace of dimension \( s \leq t + 1 - q \) such that \( \mathbb{P}(E) \subset \mathbb{P}(V) \). Then, with \( d(s, p, t) = \deg G_E \),

\[
D(\mathbb{P}(E), X) = \frac{1}{\deg G_E} D(G_E, V_X) =
\]

\[
\frac{1}{d(s, p, t)} \int_{G_E} \log |f| \mu_G^{(p-s)(t+1-p)} - \frac{1}{d(s, p, t)} \int_{G_p} \log |f| \eta_G \mu_G^{(p-s)(t+1-p)},
\]

which by Propositions 3.13 and 3.14 is greater or equal

\[
\log |f_V| - \int_{G_p} \log |f| \omega_G - c_3(s, t + 1) \deg X = D(V, V_X) - c_3(s, t + 1) \deg X,
\]

which by Proposition 4.5 equals

\[
D(\mathbb{P}(V), X) - c_3(s, t + 1) \deg X.
\]

If \( s > t + 1 - q \) and \( \mathbb{P}(E) \supset \mathbb{P}(V) \), then by Proposition 4.11,1,

\[
D(\mathbb{P}(E), X) = D(\mathbb{P}(V), X) - D^{\mathbb{P}(E)}(\mathbb{P}(V), X, \mathbb{P}(E)),
\]

which by Remark 4.3 is greater or equal

\[
D(\mathbb{P}(V), X) - c_1(t - s, t + 1 - p - s, p) \deg X.
\]

2. By Proposition 4.11.4, with \( d(p, q, t) = \deg G^V \),

\[
D(X, \mathbb{P}(V)) = \frac{1}{\deg G^V} D(V_X, G^V) =
\]

\[
\frac{1}{d(p, q, t)} \int_{G_p^V} \log |f| \mu_G^{p(t+1-q-p)} - \frac{1}{d(p, q, t)} \int_{G_p} \log |f| \mu_G^{p(t+1-p)},
\]

which by Propositions 3.13 and 3.14.5 is less or equal

\[
\sup_{W \in G_p^V} \log |f_W| - \frac{1}{d(t, q)} \int_{G_p} \log |f| \mu_G^{p(t+1-q-p)} = \sup_{W \in G_p^V} \log |f_W| - \int_{G_p} \log |f| \omega_G =
\]

\[
\sup_{W \in G_p^V} D(V_X, W) = \sup_{W \in G_p^V} D(\mathbb{P}(W), X).
\]

Let \( W_0 \in G_p^V \) be such that \( \sup_{W \in G_p^V} D(\mathbb{P}(W), X) = D(\mathbb{P}(W_0), X) \). Then, \( D(\mathbb{P}(W_0), X) \geq D(\mathbb{P}(V), X) \), any space containing \( \mathbb{P}(V) \) also contains \( \mathbb{P}(W_0) \), and for every \( s \leq t - q \), there is a space \( \mathbb{P}(E) \) of codimension \( s \) that either contains \( \mathbb{P}(W_0) \) or is contained in \( \mathbb{P}(W_0) \). Thus, the claim follows from part one.
3. By definition
\[
D(\mathbb{P}(V), X) = \frac{1}{\deg G_V} D(V_X, G_V) =
\]
\[
\frac{1}{\deg G_V} \int_{G_V} \log |f| \mu_G^{(p-q)(t+1-p)} - \frac{1}{\deg G_V} \int_{G_p} \log |f| \eta_{G_V} \mu_G^{(p-t+1-p)},
\]
which by Proposition 3.13 is less or equal
\[
\sup_{W \in (G_p)_V} \log |f| - \frac{1}{\deg G_V} \int_{G_p} \log |f| \eta_{G_V} \mu_G^{(p-q)(t+1-p)} =
\]
\[
\sup_{W \in (G_p)_V} D(V_X, W) = \sup_{W \in G_V} D(X, \mathbb{P}(W)).
\]

Choose \(W_0\) of dimension \(p\) such that \(D(X, \mathbb{P}(W_0)) = \sup_{W \in G_V} D(X, \mathbb{P}(W))\). Then, \(D(\mathbb{P}(W_0), X) \geq D(\mathbb{P}(V), X)\), any space contained in \(\mathbb{P}(V)\) is also contained in \(\mathbb{P}(W_0)\), and for every \(s \geq q\) there is a space \(\mathbb{P}(E)\) of codimension \(s\) that is either contained in \(\mathbb{P}(W_0)\) or contains \(\mathbb{P}(W_0)\). Thus the claim follows again from part 1.

4.13 Definition For \(p+q \geq t+1\), \(X \in Z_{eff}(\mathbb{P}^t_{\mathbb{Q}})\), and \(\mathbb{P}(W) \subset \mathbb{P}^t_{\mathbb{Q}}\) a subspace of codimension \(q\) not meeting \(X\), let \(X_W\) be the cycle defined in (9), and \(\mathbb{P}(F) \subset \mathbb{P}^t\) a subspace of dimension at least \(p\) containing \(\mathbb{P}(W)\) and intersecting \(X\) properly.

1. The space \(\mathbb{P}(F)\) is called \(c\)-admissible for \(X\) and \(\mathbb{P}(W)\) if
\[
D(\mathbb{P}(F), X) \geq -c \deg X \quad \text{and} \quad D(\mathbb{P}(F), X_W) \geq -c \deg X.
\]

2. If the dimension of \(\mathbb{P}(F)\) equals \(p\), define
\[
D_{\mathbb{P}(F)}(\mathbb{P}(W), X) := D_{\mathbb{P}^t}(\mathbb{P}(W), X, \mathbb{P}(F)) - (c_2 + c_4) \deg X = 
\sum_{x \in \text{supp}(\mathbb{P}(F), X)} n_x \log |\theta, x|,
\]
where the \(n_x\) are the intersection multiplicities of \(\mathbb{P}(F)\) and \(X\) at \(x\).

4.14 Theorem With the above notations,

1. let \(X \in Z^p(\mathbb{P}^t)\) and \(\mathbb{P}(W)\) a subspace of codimension \(q > t-p\) not meeting \(X\), and \(c_{10}(j, t+1) \sum_{j=1,j \neq i}^{t} c_{3}(j, t+1)\). For every \(s \geq t-p\), and every \(c \geq c_{10} + C(p,q,s,t)\), with \(C\) the constant from the previous Lemma, there is at least one space \(\mathbb{P}(F)\) of dimension \(r\) that is \(c\)-admissible for \(X\) and \(\theta\).
2. There are positive constants $e_1, e_2, e'_1, e'_2$ such that for all $p \leq t$ and every $X \in Z_{eff}^p(\mathbb{P}^t)$ and $\theta \in \mathbb{P}^t$ not contained in the support in $X$, if $\mathbb{P}(F)$ is a $c$-admissible subspace of dimension $q \geq p$ for $X$ and $\mathbb{P}(W)$, then

$$D(\theta, X) \leq D_{\mathbb{P}(F)}(\theta, X, \mathbb{P}(F)) + (c + e'_1) \deg X \leq D(\theta, X) + (2c + e'_1 + e'_2) \deg X.$$ 

In particular, if $q = p$,

$$D(\theta, X) \leq D_{\mathbb{P}(F)}(\theta, X) + c + e_1 \deg X \leq D(\theta, X) + (2c + e_1 + e_2) \deg X.$$ 

Hence, the algebraic distance of $\theta$ to $X$ essentially equals the weighted sum of the distances of $\theta$ to the points contained in the intersection of $X$ with some projective subspace $\mathbb{P}(F) \subset \mathbb{P}^t$ of codimension $t - p$ containing $\theta$.

**Remark** If $X$ is a hypersurface, the stronger estimate

$$D(\theta, X) \leq \inf_{\mathbb{P}(F) \ni x, \dim \mathbb{P}(F) = 1} D_{\mathbb{P}(F)}(\theta, X) + e_3 \deg X,$$ 

holds. This is wrong in general as the following example shows: Let

$$t = 3, \quad \theta = [1, 0, 0, 0], \quad X = \text{div}(x^{D-1} - w - z^D).\text{div}(\varepsilon x - y), \quad \mathbb{P}(F) = \text{div}(w),$$

$$\mathbb{P}(V) = \text{div}(y - w).\text{div}(z - w), \quad \mathbb{P}(E) = \text{div}(z - w),$$

$$Y = \text{div}(z - w).\text{div}(x^{D-1} - w - z^D),$$

and $\zeta_{D-1}$ a primitive $(D - 1)$th root of unity. By Lemmas 3.11 and 4.12, since $[1, 0, 1, 1]$ lies in the support of $\text{div}z - w$,

$$D(\text{div}(z - w), \text{div}(x^{D-1} - y - z^D)) \geq D([1, 0, 1, 1], \text{div}(x^{D-1} - y - z^D)) - CD = - \log |1^{D-1} \cdot 0 - 1^D| - \log |x^{D-1} - y - z^D|_0 - CD \geq$$

$$- \log |x^{D-1} - y - z^D|_\infty - D \left( C + \frac{1}{2} \sum_{i=1}^t \frac{1}{i} \right) \geq - D \left( C + \frac{1}{2} \sum_{i=1}^t \frac{1}{i} \right).$$

Further, with $\mathbb{P}(E) = \text{div} y$ and $\pi$ the orthogonal projection $\pi : \mathbb{P}^3 \setminus [0, 1, 0, 0] \to \mathbb{P}(E)$, one has $Y = \pi^*(Y, \mathbb{P}(E))$, thus by [BGS], Proposition 5.1.1, $D(Y, \mathbb{P}(E)) = c_1 D > 0$, and consequently $D(Y, \text{div}(\varepsilon x - y)) \geq 0$ for $\varepsilon$ sufficiently small. Hence, by Proposition 4.6 and Remark 4.3

$$D(\mathbb{P}(E), X) = - D(\text{div}(x^{D-1} - w - z^D), \text{div}(\varepsilon x - y)) +$$

$$D(\text{div}(z - w), \text{div}(x^{D-1} - w - z^D)) + D(Y, \text{div}(\varepsilon x - y)) \geq - D \left( C + c_1 + \frac{1}{2} \sum_{i=1}^t \frac{1}{i} \right).$$
for \( \epsilon \) small, and since \( \theta \in \mathbb{P}(V) \), by Proposition 4.11.1, and 4.10.2

\[
D(\mathbb{P}(E), X) + D^{\mathbb{P}(E)}\left(\left[1, \epsilon, 0, 0\right] + \sum_{i=1}^{D-1} \left[1, \epsilon, \zeta_{D-1}^{i}, \zeta_{D-1}^{i}\right], \mathbb{P}(V)\right) \geq \\
D(\mathbb{P}(E), X) + \log \left|\left[1, \epsilon, 0, 0\right] + \sum_{i=1}^{D-1} \left[1, \epsilon, \zeta_{D-1}^{i}, \zeta_{D-1}^{i}\right], \mathbb{P}(V)\right| + (c_2 + c_4)D \geq \\
-D\left(C + \bar{c}_1 + c_2 + c_4 + \frac{1}{2} \sum_{i=1}^{t} \frac{1}{i}\right) + \log \frac{\epsilon}{\sqrt{1 + \epsilon^2}} + (D - 1) \log \frac{1}{2} \geq \log \epsilon - cD,
\]

for small \( \epsilon \), and some fixed constant \( c \).

On the other hand, \( X, \mathbb{P}(F) = D[1, \epsilon, 0, 0] \), hence

\[
D_{\mathbb{P}(F)}(\theta, X) = D \log \left|\left[1, 0, 0, 0\right], \left[1, \epsilon, 0, 0\right]\right| = D \log \frac{\epsilon}{\sqrt{1 + \epsilon^2}} \leq D \log \epsilon,
\]

and the inequality

\[
D(\mathbb{P}(F), \bar{X}_i) \leq \inf_{\mathbb{P}(F) \ni x, \dim \mathbb{P}(F) = 1} D_{\mathbb{P}(F)}(\theta, X) + c' \deg X
\]

for some fixed \( c' \) would imply

\[
\log \epsilon - cD \leq D \log \epsilon + c'D,
\]

which is clearly wrong for \( \epsilon \) sufficiently small and \( D \geq 2 \).

**4.15 Proposition** For \( i = 1, \ldots, t \) let \( X_i \in Z_{\text{eff}}(G_i) \) be an effective cycle of codimension one in the corresponding Grassmannian over \( \mathbb{C} \). There exists a flag of vector spaces

\[
\{0\} \subset V_1 \subset \cdots \subset V_t \subset \mathbb{C}^{t+1}
\]

with \( \dim V_i = i \) such that

\[
D(X_i, V_i) \geq -c_{10}(i) \deg X_i,
\]

where \( c_{10}(i) := \sum_{j \neq i} c_3(j) \).

**Proof** Let \( \bar{X}_i \) be the cycle \( \bar{X}_i = \left(\sum_{k \neq i} \deg X_k\right) \cdot X_i \). Then, \( D := \prod_{k=1}^{t} \deg X_k = \deg \bar{X}_i \). Next, let \( F \) be the complete flag variety, \( \varphi_i : F \to G_i, i = 1, \ldots, t \) the canonical projections, and \( Y_i := \varphi_i^* \bar{X}_i \). Then,

\[
\sum_{i=1}^{t} Y_i = c_1(L^{\otimes D}),
\]

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with \( L \) the line bundle on \( F \) introduced in section 3. Let \( f_i \in \Gamma(G_i, L_i^\otimes D) \) be such that \( X_i = \text{div } f_i \). Also assume that \( f_i \) is normalized, i.e. \( \int_{G_i} \log |f_i| \omega_i = 0 \). With \( f = \prod_{i=1}^t \varphi_i^* f_i \), we have \( f \in \Gamma(F, L^\otimes D) \) and \( \sum_{i=1}^t Y_i = \text{div } f \). Further, \( \sum_{i=1}^t \varphi_i^*(\log |f_i|) = \log |f| \), hence

\[
\int_F \log |f| \omega_F = \sum_{i=1}^t \int_F \varphi_i^*(\log |f_i|) \omega_F = \sum_{i=1}^t \int_{G_i} \log |f_i| (\varphi_i)_* \omega_F = \sum_{i=1}^{t+1} 0.
\]

The last equality holds because \((\varphi_i)_* \omega_F = \omega_i\) since \((\varphi_i)_* \omega_F\) is \( U(t+1) \)-invariant and \( \int_{G_i} \omega_i = \int_{\varphi_i^* G_i} \omega_F = \int_{G_i} (\varphi_i)_! \omega_i = \int_F \omega_F = 1 \). By Proposition 3.13, there is a point \( P \in F \) such that

\[
\log |f_P| \geq \int_F \log |f| \omega_F = 0,
\]

and

\[
\log |(f_i)_\varphi (P)| \leq \log |f_i|_\infty \leq c_3(i) D + \log |f_i|_0 = c_3(i) D, \quad i = 1, \ldots, t + 1,
\]

hence,

\[
- \int_F \varphi_i^*(\log |f_i|) \omega_F + \log |(\varphi_i^*(f_i))_P| = \sum_{j \neq i} \int_F \varphi_j^*(\log |f_j|) \omega_F - \int_F \log |f| \omega_F + \log |f_P| - \sum_{j \neq i} \log |(\varphi_j^*(f_j))_P| = \sum_{j \neq i} \left( \int_{G_j} \log |f_j| \omega_{G_j} - |(f_j)_\varphi_j(P)| \right) + \log |f_P| \geq -D \sum_{j \neq i} c_3(j) = -c_{10}(i) D.
\]

The point \( P \) corresponds to a complete flag \( \{0\} \subset V_1 \subset \cdots \subset V_t \subset \mathbb{C}^{t+1} \) with \( V_i = \varphi_i(P) \). Thus,

\[
D(X_i, V_i) = \log |(f_i)_V| - \int_{G_i} \log |f_i| \omega_i = \int_{G_i} \log |f_i| \delta_{\varphi_i, P} - \int_{G_i} \log |f_i| (\varphi_i)_* \omega_F = \int_F \varphi_i^*(\log |f_i|) \delta_P - \int_F \varphi_i^*(\log |f_i|) \omega_F \geq -c_{10}(i) D.
\]

Dividing the inequality by \( \prod_{k \neq i} \text{deg } X_i \) gives

\[
D(X_i, V_i) \geq -c_{10}(i) \text{deg } X_i
\]

as claimed.
4.16 Corollary Let $X \in Z_{eff}(\mathbb{P}^t)$ be an effective cycle in $\mathbb{P}^t$, and denote by $X_i$ its component of codimension $i$, $0 = 1, \ldots, t$. There is a flag of vector spaces

$$\{0\} \subset V_1 \subset \cdots \subset V_i \subset \mathbb{C}^{i+1}$$

with $\dim V_i = i$ such that

$$D(\mathbb{P}(V_j), X_i) \geq -c_{10} \deg X_i, \quad i = 0, \ldots, t, j = 1, \ldots, t.$$ 

Consequently, by Proposition 4.12

$$D(\mathbb{P}(V_j), X) \geq -(c_{10} + C) \deg X, \quad j = 1, \ldots, t.$$ 

Proof For $i$ arbitrary, by Proposition 4.5

$$D(X_i, \mathbb{P}(V_i)) = D(V_i, V_i),$$

and by the previous Proposition, there is a flag such that this is greater or equal $-c_{10}(i) \deg X_i$.

Proof of Theorem 4.14 By the Corollary, there are spaces $\mathbb{P}(V_{p-2}) \subset \mathbb{P}(V_{p-1})$ of dimensions $p - 2, p - 1$ respectively such that with $X_\theta$ as in (9),

$$D(X_\theta, \mathbb{P}(V_{p-2})) \geq -c_{10} \deg X_\theta = -c_{10} \deg X, \quad D(X, \mathbb{P}(V_{p-1})) \geq -c_{10} \deg X.$$ 

Choosing $\mathbb{P}(F)$ as any subspace containing $\mathbb{P}(V_{p-1})$ as well as $\theta$, the claim follows from Proposition 4.12.

2. Let $\mathbb{P}(F)$ be admissible for $X$ and $\theta$. By Proposition 4.11.2,

$$D(\theta, X) = D^{\mathbb{P}(F)}(\theta, X, \mathbb{P}(F)) + D(X, \mathbb{P}(F)) - D(X_\theta, \mathbb{P}(F)) + c_{11} \deg X.$$ 

The claim thus follows from Remark 4.3 and the fact that $\mathbb{P}(F)$ is admissible.

5 Joins

The proof of Theorem 2.4 will be using the construction of the join of cycles $X, Y \subset \mathbb{P}(E) = \mathbb{P}^t$ in projective space, which is defined as follows. Let $\pi_i : \mathbb{P}(E) \to \mathbb{P}(E) \times \mathbb{P}(E), i = 1, 2$ be the canonical embeddings, $\pi : \mathbb{P}(E \oplus E) \to \text{Spec } (\mathbb{Z})$ the structure maps, and $p_i : \mathbb{P}(E) \times \mathbb{P}(E) \to \mathbb{P}(E), i = 1, 2$ the canonical projections. Then $F := p_1^*O(-1) \oplus p_2^*O(-1)$ is a subbundle of $\pi^*(E \oplus E) = p_1^*\pi^*E \oplus p_2^*\pi^*E$

over $\mathbb{P}(E) \times \mathbb{P}(E)$. The inclusion defines a map from the projective bundle $\mathbb{P}_{\mathbb{P}(E) \times \mathbb{P}(E)}(F)$ to $\mathbb{P}_{\mathbb{P}(E) \times \mathbb{P}(E)}(\pi^*(E \oplus E)) = \mathbb{P}(E \oplus E) \times \mathbb{P}(E) \times \mathbb{P}(E)$, hence a map
$g$ to $\mathbb{P}^{2t+1} = \mathbb{P}(E \oplus E)$; the bundle map $f : \mathbb{P}(E) \times \mathbb{P}(E) \to \mathbb{P}(E) \times \mathbb{P}(E)$ is flat. Hence, for $\mathcal{X}, \mathcal{Y} \in \mathbb{P}(E)$, the expression

$$\mathcal{X} \# \mathcal{Y} := g_* f^*(\mathcal{X} \times \mathcal{Y})$$

is well defined and is called the join of $\mathcal{X}$ and $\mathcal{Y}$. We have

5.1 Proposition The degree and height of the join compute as

$$\deg(\mathcal{X} \# \mathcal{Y}) = \deg(\mathcal{X}) \deg(\mathcal{Y}), \quad \text{and}$$

$$h(\mathcal{X} \# \mathcal{Y}) = \deg X h(\mathcal{Y}) + \deg Y h(\mathcal{X}).$$

Proof [BGS], Proposition 4.2.2.

5.2 Proposition With $p + r \geq t + 1, q + s \geq t + 1$ let $X, Y, Z, W$ be effective cycles in $\mathbb{P}^t_C$ of codimensions $p, q, r, s$ respectively such that the $X$ and $Z$ as well as $Y$ and $W$ intersect properly. Then, $X \# Y$ and $Z \# W$ intersect properly, and the algebraic distance $D(X \# Y, Z \# W)$ computes as

$$D(X \# Y, Z \# W) = \deg X \deg Z D(Y, W) + \deg Y \deg W D(X, Z) +$$

$$(\sigma_{2t+1} - 2\sigma_t) \deg X \deg Y \deg Z \deg W.$$  

Let $\theta \in \mathbb{P}^t(C)$ be a point neither contained in the support of $X$ nor in that of $Y$, and $X_\theta, Y_\theta$ the varieties as in [4]. Further, $(\theta, \theta) \in \mathbb{P}^{2t+1}$ the intersection of $\theta \# \theta$ with the diagonal, and $(X \# Y)_{\theta, \theta}$ as in [4]. Then,

$$D((X \# Y)_{\theta, \theta}, Z \# W) = \deg X \deg Z D(Y_\theta, W) + \deg Y \deg W D(X_\theta, Z) +$$

$$(\sigma_{2t+1} - 2\sigma_t) \deg X \deg Y \deg Z \deg W.$$  

Proof It suffices to prove the Proposition for cycles that are irreducible varieties. Assume first that $X, Y, Z, W$ are obtained by base extension from subvarieties $\mathcal{X}, \mathcal{Y}, Z, W$ of $\mathbb{P}^t_C$. By Remark 4.3

$$D(X \# Y, Z \# W) = h((\mathcal{X} \# \mathcal{Y})(\mathcal{Z} \# \mathcal{W})) -$$

$$\deg(X \# Y) h(\mathcal{Z} \# \mathcal{W}) - \deg(Z \# W) h(\mathcal{X} \# \mathcal{Y}) + \sigma_{2t+1} \deg(X \# Y) \deg(Z \# W),$$

$$h((\mathcal{X}, \mathcal{Z}) \# (\mathcal{Y} \# \mathcal{W})) - \deg(X \# Y) h(\mathcal{Z} \# \mathcal{W}) - \deg(Z \# W) h(\mathcal{X} \# \mathcal{Y}) +$$

$$\sigma_{2t+1} \deg(X \# Y) \deg(Z \# W),$$

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which by the previous Proposition and Remark 4.3 equals
\[ \deg X \deg Z (\deg Yh(W) + \deg Wh(Y) + D(Y, W) - \sigma_t \deg Y \deg Z) + \]
\[ \deg Y \deg W (\deg Xh(Z) + \deg Zh(X) + D(X, Z) - \sigma_t \deg X \deg Z) \]
\[ - \deg X \deg Y (\deg Zh(W) + \deg Wh(Z)) - \deg Z \deg W (\deg Xh(Y) + \deg Yh(X)) + \]
\[ \sigma_{2t+1} \deg X \deg Y \deg Z \deg W = \]
\[ \deg X \deg Z D(Y, W) + \deg Y \deg WD(X, Z) + (\sigma_{2t+1} - 2\sigma_t) \deg X \deg Y \deg Z \deg W. \]

Since the subvarieties \( X \subset \mathbb{P}^t(\mathbb{Z}) \) form a dense subset of the subvarieties \( X \subset \mathbb{P}^t(\mathbb{C}) \), and the algebraic distance is a continuous function, the equation holds for arbitrary \( X, Y, Z, W \).

Since \( (X \# Y)_{(\theta, \theta)} = g_* f^*(X_{\theta} \times Y_{\theta}) \), by Lemma 4.5,
\[ D((X \# Y)_{(\theta, \theta)}, Z \# W) = D(X_{\theta} \# Y_{\theta}, Z \# W), \]
which by the above equals
\[ D((X \# Y)_{(\theta, \theta)}, Z \# W) = \deg X \deg Z D(Y_{\theta}, W) + \deg Y \deg WD(X_{\theta}, Z) + \]
\[ (\sigma_{2t+1} - 2\sigma_t) \deg X \deg Y \deg Z \deg W. \]

Let \( X, Y \) be cycles of pure codimension \( p, q \) with \( p + q \geq t + 1 \). They do intersect properly, do not intersect respectively, if if and only if \( X \# Y \) does intersect \( \mathbb{P}(\Delta) \) properly, does not intersect \( \mathbb{P}(\Delta) \). Thus, one may define

5.3 Definition For \( p + q \leq t + 1 \) define the algebraic distance
\[ \bar{D}(X, Y) := D(\mathbb{P}(\Delta), X \# Y)), \]
For \( p + 1 \geq t + 1 \) define
\[ \bar{D}_G(X, Y) := D_G(\mathbb{P}(\Delta), X \# Y), \quad \bar{D}_\infty(X, Y) := D_\infty(\mathbb{P}(\Delta), X \# Y), \]
\[ \bar{D}_{Ch}(X, Y) := D_{Ch}(\mathbb{P}(\Delta), X \# Y). \]
As \( (X + X') \# Y = (X \# Y) + (X' \# Y) \) it immediately follows form Lemma 4.8 that for \( p + q \geq t + 1 \) the maps
\[ \bar{D}_0, \bar{D}_{Ch} : Z_{eff}^p(\mathbb{P}^t) \times Z_{eff}^q(\mathbb{P}^t) \to \mathbb{R} \]
are bilinear.
5.4 Proposition There exist constants \( c, c_0, c_\infty, c_{Ch} \) only depending on \( p, q, \) and \( t \) such that in the situation of the Definition,

\[
D(X, Y) = D(X, Y) + c \deg X \deg Y,
\]

and for \( Y = \mathbb{P}(W) \) a projective subspace,

\[
\begin{align*}
\bar{D}_0(X, \mathbb{P}(W)) &= D_0(\mathbb{P}(W), X) + c_0 \deg X, \\
\bar{D}_\infty(X, \mathbb{P}(W)) &= D_\infty(\mathbb{P}(W), X) + c_\infty \deg X, \\
\bar{D}_{Ch}(X, \mathbb{P}(W)) &= D_{Ch}(\mathbb{P}(W), X) + c_{Ch} \deg X.
\end{align*}
\]

That is, the above Definition of algebraic distances coincide with the old definition modulo constants times \( \deg X \).

Proof The proof will be given in a different paper ([Ma6]).

Let \( \mathcal{X}, \mathcal{Y} \) be irreducible subschemes of \( \mathbb{P}^t \) whose generic fibre is not empty, and \([x], [y] \) closed points of \( X_\mathbb{C}, Y_\mathbb{C} \) represented by vectors \( x, y \in \mathbb{C}^{t+1} \) of length one. Then the closed points of the join \( x\# y \subset (\mathcal{X}\# \mathcal{Y})_\mathbb{C} \) are the points

\[
g(f^{-1}([x], [y])) = g(\lambda x, \mu y) = ([\lambda x, \mu y]),
\]

with \( \lambda, \mu \in \mathbb{C} \).

5.5 Lemma Let \( x, y, \theta \) be vectors of length one in \( \mathbb{C}^{t+1} \) with \([x] \neq [\theta] \neq [y] \). Then,

\[
\begin{align*}
\min(||[\theta], [x]||, ||[\theta], [y]||) &\leq ||x\# y\||, ([\theta], [\theta]) \leq \max(||[\theta], [x]||, ||[\theta], [y]||).
\end{align*}
\]

Proof Since \(|\langle([\theta], [\theta])\rangle, [x]\# [y]| = |\langle([\theta], [\theta])\rangle, [v]|\) where \([v]\) is the point in \( x\# y \) with minimal distance to \( ([\theta], [\theta]) \). Hence, it suffices to prove

\[
\begin{align*}
\min(||[\theta], [x]||, ||[\theta], [y]||) &\leq ||v||, (\theta, \theta) \leq \max(||[\theta], [x]||, ||[\theta], [y]||).
\end{align*}
\]

As any point \([w]\) in \( x\# y \in \mathbb{P}^t(\mathbb{C}) \) may be written as \([\lambda(x, 0) + \mu(0, y)] \) with \( \lambda, \mu \in \mathbb{C} \), and

\[
|\langle\lambda x, \mu y\rangle, (\theta, \theta)|^2 = 1 - \frac{|\langle\lambda x, \mu y\rangle(\theta, \theta)|^2}{2(|\lambda|^2 + |\mu|^2)};
\]

further, \(|x, \theta|^2 = 1 - |\langle x, \theta \rangle|^2, |y, \theta|^2 = 1 - |\langle y, \theta \rangle|^2\), we have to show that

\[
\begin{align*}
\min(|\langle x, \theta \rangle|^2, |\langle y, \theta \rangle|^2) &\leq \sup_{\lambda, \mu} \frac{|\langle\lambda x, \mu y\rangle(\theta, \theta)|^2}{2(|\lambda|^2 + |\mu|^2)} \leq \max(|\langle x, \theta \rangle|^2, |\langle y, \theta \rangle|^2).
\end{align*}
\]
Firstly,
\[
\frac{|\langle (\lambda x, \mu y) \rangle (\theta, \theta) |^2}{2(|\lambda|^2 + |\mu|^2)} \leq \frac{\lambda^2 |\langle x |\theta \rangle |^2 + 2 |\lambda \mu \langle x |\theta \rangle \langle y |\theta \rangle |^2 + |\mu^2 \langle y |\theta \rangle |^2}{2(|\lambda|^2 + |\mu|^2)} \leq \frac{|\lambda|^2 + 2 |\lambda \mu | + |\mu|^2}{2(|\lambda|^2 + |\mu|^2)} \max(|\langle x |\theta \rangle |^2, |\langle y |\theta \rangle |^2) \leq \max(|\langle x |\theta \rangle |^2, |\langle y |\theta \rangle |^2),
\]
as \[|\lambda|^2 + 2 |\lambda \mu | + |\mu|^2 \leq 2(|\lambda|^2 + |\mu|^2),\] whence the second inequality.

For the first inequality, choose
\[
\lambda_0 = \sqrt{\frac{|\langle x |\theta \rangle |^2}{|\langle x |\theta \rangle |^2 + |\langle y |\theta \rangle |^2}}, \quad \mu_0 = \sqrt{\frac{|\langle y |\theta \rangle |^2}{|\langle x |\theta \rangle |^2 + |\langle y |\theta \rangle |^2}}.
\]

Then,
\[
\frac{|\langle (\lambda_0 x, \mu_0 y) \rangle (\theta, \theta) |^2}{2(|\lambda_0|^2 + |\mu_0|^2)} = \frac{1}{2} \left( |\langle \lambda_0 x, \mu_0 y \rangle, (\theta, \theta) |^2 \right) = \frac{(|\langle x |\theta \rangle |^2 + |\langle y |\theta \rangle |^2)^2}{2(|\langle x |\theta \rangle |^2 + |\langle y |\theta \rangle |^2)} = \frac{|\langle x |\theta \rangle |^2 + |\langle y |\theta \rangle |^2}{2} \geq \min(|\langle x |\theta \rangle |^2, |\langle y |\theta \rangle |^2),
\]
whence the first inequality.

6 Proofs for the main results

By Theorem [4.9], the results are true for one of the algebraic distances \(D_G, D_\infty, D_{Ch}\) if it holds for any of the others, only with different constants \(c, c', c, c'\).

Remember, that for a subvariety \(X \subset \mathbb{P}_k^t\), the height \(h(X)\) is defined for \(X\) over \(Z\), the degree is defined for the base extension \(X = X_Q\), and the algebraic distance to some point \(\theta \in \mathbb{P}^t(\mathbb{C})\) is defined for the \(\mathbb{C}\)-valued points of \(X\), denoted \(X_\infty\), or \(X\) if clear from the context.

**Proof of Proposition 2.2** Let \(p\) be the codimension of \(X\), and \(\mathbb{P}(F)\) a \(c_{10}\)-admissible subspace of dimension \(p\) for \(X\) and \(\theta\). Then,
\[
\deg X \log |\theta, X | \leq \sum_{x \in X \cap \mathbb{P}(F)} n_x \log |\theta, x | = D_{\mathbb{P}(F)}(\theta, X),
\]
which by Theorem [4.14] is less or equal \(D(\theta, X) + (c_{10} + e_1) \deg X\) proving the first inequality.

For the second inequality, let \(x_0 \in \text{supp}(X(\mathbb{C}))\) be such that \(|\mathbb{P}(W), X| = |\mathbb{P}(W), x_0|\).

By Corollary [4.16] there is a subspace \(\mathbb{P}(V) \subset \mathbb{P}^t\) of dimension \(p - 2\) such that \(D(\mathbb{P}(V), X_\theta) \geq -c_{10} \deg X\). With \(\mathbb{P}(F)\) a space of dimension \(p\) containing \(\mathbb{P}(V)\) as
well as θ and \(x_0\) and intersecting \(X\) as well as \(X_θ\) properly, Proposition 4.12 implies \(D(\mathbb{P}(F), X_θ) \geq -(c_{10} + C) \deg X\). Further by Proposition 4.12, 

\[
D(\theta, X) = D^{\mathbb{P}(F)}(\theta, X, \mathbb{P}(F)) + D(\mathbb{P}(F), X) - D(\mathbb{P}(F), X_θ) + c_{11} \deg X,
\]

which by the above and Remark 4.3 is less or equal

\[
D^{\mathbb{P}(F)}(\theta, X, \mathbb{P}(F)) + \left(\bar{c}_1 + \frac{c_{10}}{d(p - 2, t)} + C + c_{11}\right) \deg X,
\]

which in turn by Proposition 4.10 equals

\[
\sum_{x \in X \cap \mathbb{P}(F)} n_x \log |\theta, x| + \left(\bar{c}_1 + \frac{c_{10}}{d(p - 2, t)} + C + c_{11} + c_2 + c_4\right) \deg X =
\]

\[
n_{x_0} \log |\theta, x_0| + \sum_{x \in X \setminus \mathbb{P}(F)} n_x \log |\theta, x| + \left(\bar{c}_1 + \frac{c_{10}}{d(p - 2, t)} + C + c_{11} + c_2 + c_4\right) \deg X
\]

\[
\leq \log |\theta, X| + \left(\bar{c}_1 + \frac{c_{10}}{d(p - 2, t)} + C + c_{11} + c_2 + c_4\right) \deg X,
\]

finishing the proof.

**Proof of Proposition 2.3** Remember that in case of codimension one \(D_∞ = D_G = D_{\mathbb{C}b}\). To deduce the first inequality, firstly, by 3.8.1, and (3),

\[
h(\text{div } f) = Dσ_t + \int_{\mathbb{P}^t(\mathbb{C})} \log |f|\mu^t.
\]

Further, by Lemma 3.11

\[
\int_{\mathbb{P}^t(\mathbb{C})} \log |f|\mu^t \leq \frac{1}{2} \log \int_{\mathbb{P}^t(\mathbb{C})} |f|^2\mu^t = \log |f|_{L^2}.
\]

The two formulas together imply the first formula.

For the second formula,

\[
D(\theta, \text{div } f) = \log |f_θ| - \int_{\mathbb{P}^t_\mathbb{C}} \log |f|\mu^t,
\]

which by Proposition 3.8 and (3) equals

\[
\log |(f|_θ)| - h(\text{div } f) + Dσ_t.
\]

**Proof of Theorem 2.4** With \(c = c_{10} + C\), where \(c_{10}\) and \(C\) are the constants from Theorem 4.14 and Proposition 4.12, let \(\mathbb{P}(F), \mathbb{P}(F')\) be \(c\)-admissible subspaces of
dimensions \( p, q \) for \( X \) and \( \theta, Y \) and \( \theta \) respectively, and let \( x_1, \ldots, x_{\deg X}, y_1, \ldots, y_{\deg Y} \in \mathbb{P}^d(\mathbb{C}) \) such that

\[
|\theta, x_1| \leq |\theta, x_2| \leq \cdots \leq |\theta, x_{\deg X}|, \quad |\theta, y_1| \leq \cdots \leq |\theta, y_{\deg Y}|
\]

and

\[
X, \mathbb{P}(F) = \sum_{i=1}^{\deg X} x_i, \quad Y, \mathbb{P}(F') = \sum_{i=1}^{\deg Y} y_i.
\]

By Theorem 4.14

\[
\sum_{i=1}^{\deg X} \log |\theta, x_i| \leq D(\theta, X) + (c + e_2) \deg X, \quad (21)
\]

and

\[
\sum_{i=1}^{\deg Y} \log |\theta, y_i| \leq D(\theta, Y) + (c + e_2) \deg Y. \quad (22)
\]

Define the function \( f_{X,Y} \) subject to requirement that for every \( T \in \deg X + \deg Y \) and \((\nu, \kappa) = f_{X,Y}(t)\) the conditions

\[
\nu + \kappa = t, \quad \text{and} \quad |\theta, x_\nu| \leq |\theta, y_{\kappa+1}|, \quad |\theta, y_\nu| \leq |\theta, x_{\kappa+1}|
\]

hold.

The proof will be given in two steps.

1. \( \nu \kappa \log |\theta, X + Y| + D_G((\theta, \theta), X \# Y) + h(X \# Y) \leq \kappa D_G(\theta, X) + \nu D_G(\theta, Y) + \deg Y h(X) + \deg X h(Y) + (2\sigma_t - \sigma_{2t+1} + 3c + e_1 + e_2) \deg X \deg Y, \)

2. \( D_\infty(\theta, X, Y) + h(X, Y) \leq D_\infty((\theta, \theta), X \# Y) + h(X \# Y) + \left( \frac{3t + 2 - p - q}{2} \log 2 + \sigma_t - \sigma_{2t+1} \right) \deg X \deg Y. \)

As, by Theorem 4.14, \( D_\infty((\theta, \theta), X \# Y) \leq D_G((\theta, \theta), X \# Y) + c_3 \deg X \deg Y, \) and \( D_G(\theta, X, Y) \leq D_\infty(\theta, X, Y), \) the two inequalities together imply the claim with \( e = 2\sigma_t - 2\sigma_{2t+1} + 3c + e_1 + e_2 + \frac{3t + 2 - p - q}{2} \log 2. \)

1. Write \( D(\cdot, \cdot) \) for \( D_G(\cdot, \cdot). \) By Proposition 5.2

\[
D(X \# Y, \mathbb{P}(F) \# \mathbb{P}(F')) = \deg XD(Y, \mathbb{P}(F')) + \deg YD(X, \mathbb{P}(F)) + (\sigma_{2t+1} - 2\sigma_t) \deg X \deg Y \geq (\sigma_{2t+1} - 2\sigma_t - 2c) \deg X \deg Y;
\]
and
\[ D(X \# Y, \mathbb{P}(F) \# \mathbb{P}(F')) \leq c_1 \deg X \deg Y. \]

Further,
\[ D((X \# Y)_{(\theta, \theta)}, \mathbb{P}(F) \# \mathbb{P}(F')) = \]
\[ \deg X D(Y_{\theta}, \mathbb{P}(F')) + \deg Y D(X_{\theta}, \mathbb{P}(F)) + (\sigma_{2t+1} - 2\sigma_t) \deg X \deg Y \geq \]
\[ (\sigma_{2t+1} - 2\sigma_t - 2c) \deg X \deg Y, \]
and
\[ D((X \# Y)_{(\theta, \theta)}, \mathbb{P}(F) \# \mathbb{P}(F')) \leq c_1 \deg X \deg Y. \]

Hence, \( \mathbb{P}(F) \# \mathbb{P}(F') \) is a \((2\sigma_t - \sigma_{2t+1} + 2c)\)-admissible subspace for \( X \# Y \) and \((\theta, \theta)\). Since
\[ (\mathbb{P}(F) \# \mathbb{P}(F')).(X \# Y) = \sum_{i=1}^{\deg X} \sum_{j=1}^{\deg Y} x_i \# y_j, \]
Theorem 4.14 and Proposition 4.10.2 imply
\[ D((\theta, \theta), X \# Y) \leq D^{\mathbb{P}(F) \# \mathbb{P}(F')}((\theta, \theta), (X \# Y), (\mathbb{P}(F) \# \mathbb{P}(F'))) = \]
\[ + \sum_{x,y} n_x n_y \log |(\theta, \theta), x \# y| \]
\[ + (2\sigma_t - \sigma_{2t+1} + 2c + e_1) \deg X \# Y. \]

(23)

Next, since the logarithm of the Fubini-Study metric is nonpositive,
\[ \nu \kappa \log |\theta, X + Y| + \sum_{i=1}^{\deg X} \sum_{j=1}^{\deg Y} \log |x_i \# y_j, (\theta, \theta)| \leq \]
\[ \nu \kappa \log |\theta, X + Y| + \sum_{i=1}^{\nu} \sum_{j=1}^{\kappa} \log |x_i \# y_j, (\theta, \theta)| + \]
\[ \sum_{i=1}^{\nu} \sum_{j=\kappa+1}^{\deg Y} \log |x_i \# y_j, (\theta, \theta)| + \sum_{i=\nu+1}^{\deg X} \sum_{j=1}^{\kappa} \log |x_i \# y_j, (\theta, \theta)|. \]

(24)

Now, since \( \log |\theta, X + Y| \leq \min(\log |\theta, x|, \log |\theta, y|) \), for any \( x \in \text{supp } (X), y \in \text{supp } (Y) \), and by Lemma 5.5 also \( \log |(\theta, \theta), X \# Y| \leq \max(\log |\theta, x|, \log |\theta, y|) \), for all \( i \leq \nu, j \leq \kappa \), the inequality
\[ \log |\theta, X + Y| + \log |x_i \# y_j, (\theta, \theta)| \leq \log \min(|x_i, \theta|, |y_j, \theta|) + \]
\[ \log \max(|x_i, \theta|, |y_j, \theta|) = \log |x_i, \theta| + \log |y_j, \theta| \]
holds. Hence, the sum of the first two summands on the right hand side of (24) is less or equal than
\[ \kappa \sum_{i=1}^{\nu} \log |x_i, \theta| + \nu \sum_{j=1}^{\kappa} \log |y_j, \theta|. \] (25)

For \( i \leq \nu \) and \( j \geq \kappa + 1 \) we have \( |x_i, \theta| \leq |y_j, \theta| \), consequently, by Lemma 5.5
\( |x_i \# y_j, (\theta, \theta)| \leq |\theta, y_j| \). Using this, and the analogous inequality for \( i \geq \nu + 1, j \leq \kappa \),
the sum of the third and fourth summand of (24) is less or equal
\[ \sum_{i=1}^{\nu} \deg Y \log |y_j, \theta| + \sum_{i=1}^{\nu+1} \sum_{j=1}^{\kappa} \log |x_i, \theta| = \nu \sum_{j=1}^{\kappa+1} \log |y_j, \theta| + \kappa \sum_{i=\nu+1}^{\nu+1} \log |x_i, \theta|. \] (26)

Hence, (24) is less or equal than the sum of (25) and (26), which equals
\[ \kappa \sum_{i=1}^{\deg X} \log |x_i, \theta| + \nu \sum_{j=1}^{\deg Y} \log |y_j, \theta|, \]
which in turn, by (21), and (22) is less or equal
\[ \kappa D(\theta, X) + \nu D(\theta, Y) + (c_2 + c)(\nu \deg Y + \kappa \deg X). \]

Together with (23), this gives
\[ \nu \kappa \log |\theta, X + Y| + D((\theta, \theta), X \# Y) \leq \kappa D(\theta, X) + \nu D(\theta, Y) + ((2\sigma_t - \sigma_{2t+1} + 2c + e_1) \deg X \deg Y + (e_2 + c)(\kappa \deg Y + \nu \deg X) \leq
\nu D(\theta, Y) + \kappa D(\theta, X) + (2\sigma_t - \sigma_{2t+1} + 3c + e_1 + e_2) \deg X \deg Y, \]

Adding the equation \( h(X \# Y) = \deg(X) h(Y) + \deg(Y) h(X) \) of proposition 5.1.2 to
this inequality leads the desired inequality.

2. Let \( \Delta \subset \mathbb{C}^{2t+2} \) be the diagonal, and \( i : \mathbb{P}^t \to \mathbb{P}^{2t+1} \) the inclusion
\[ i : \mathbb{P}^t \to \mathbb{P}(\Delta) \subset \mathbb{P}^{2t+1}, \quad [v] \mapsto [(v, v)]. \]

Then, \( (X \# Y).\mathbb{P}^\Delta = i(X.Y), \) and since the restriction of \( O_{\mathbb{P}^{2t+1}}(1) \) to \( \mathbb{P}^t = \mathbb{P}(\Delta) \)
equals \( O_{\mathbb{P}^t}(1) \) with the norm multiplied with \( \sqrt{2} \), we have \( h((X \# Y).\mathbb{P}(\Delta)) = h(\mathcal{X}, \mathcal{Y}) + \frac{2t+2-p-q}{2} \log 2. \)

Let \( \mathbb{P}(V) \subset \mathbb{P}^t \) be a subspace of dimension \( p + q - 1 \) containing \( \theta \) such that
\( D(\mathbb{P}(V), X.Y) \) is maximal, i.e. \( D_\infty(\theta, X.Y) = D(\mathbb{P}(V), X.Y) \).

Since \( i : \mathbb{P}^t \to \mathbb{P}(\Delta) \) is an isometry, we get
\[ D_\infty(\theta, X.Y) = D(\mathbb{P}(V), X.Y) = D_{\mathbb{P}(\Delta)}(i(\mathbb{P}(V)), (X \# Y).\mathbb{P}(\Delta)), \]
which by Proposition 4.11.1 equals
\[ D(i(\mathbb{P}(V)), X \# Y) - D(\mathbb{P}(\Delta), X \# Y), \]
which in turn by Remark 4.3 equals
\[ D(i(\mathbb{P}(V)), X \# Y) + h(\mathcal{X} \# \mathcal{Y}) + \deg(X \# Y)h(\mathbb{P}(\Delta)) - h(\mathbb{P}(\Delta) \mathcal{X} \# \mathcal{Y}) - \sigma_{2t+1} \deg X \# Y. \]

Since \( h(\mathbb{P}(\Delta)) = \frac{t}{2} \log 2 + \sigma_t \), by (3), and \( h((\mathcal{X} \# \mathcal{Y}) \mathbb{P}(\Delta)) = h(\mathcal{X} \mathcal{Y}) + \frac{2t^2 - p - q - 1}{2} \log 2 \) from above,
\[ D_\infty(\theta, X.Y) + h(\mathcal{X} \mathcal{Y}) = \]
\[ D(i(\mathbb{P}(V)), X \# Y) + h(\mathcal{X} \# \mathcal{Y}) + \left(\frac{3t + 2 - p - q}{2} \log 2 + \sigma_t - \sigma_{2t+1}\right) \deg X \deg Y, \]
and the trivial estimate \( D(i(\mathbb{P}(V)), X \# Y) \leq D_\infty((\theta, \theta), X \# Y) \) implies
\[ D_\infty(\theta, X.Y) + h(\mathcal{X} \mathcal{Y}) \leq \]
\[ D_\infty((\theta, \theta), X \# Y) + h(\mathcal{X} \# \mathcal{Y}) + \left(\frac{3t + 2 - p - q}{2} \log 2 - \sigma_{2t+1}\right) \deg X \deg Y, \]
as was to be proved.

**Proof of Corollary 2.5.** 1. The way \( f_{X,Y} \) is defined in the proof of Theorem 2.4 above, it follows that \((\nu, \kappa)\) in Theorem 2.4 can be chosen such that \( \nu = 1 \). Then,
\[ \kappa \log |\theta, X + Y| + D(\theta, X.Y) + h(\mathcal{X} \mathcal{Y}) \leq \]
\[ \kappa D(\theta, X) + D(\theta, Y) + \deg Y h(X) + \deg X h(Y) + e \deg X \deg Y. \]
As by Theorem 2.4,
\[ \kappa D(\theta, X) \leq \kappa \log |\theta, X| + \kappa c(p, t) \deg X, \]
in either of the two cases \(|\theta, X| \leq |\theta, Y|\) or \( D(\theta, X) \leq \log |\theta, Y|\) one has
\[ \kappa D(\theta, X) \leq \kappa \log |\theta, X + Y| + \kappa c(p, t) \deg X. \]

Inserting this into the first inequality and subtracting \( \kappa \log |\theta, X + Y| \) from the resulting formula gives the claim with \( e' = c + e \).

2. Again, in Theorem 2.4 \((\nu, \kappa)\) can be either chosen equal to \((0,1)\) or equal to \((1,0)\). Without loss of generality, assume \((\nu, \kappa) = (0,1)\). Then,
\[ D(\theta, X.Y) + h(\mathcal{X} \mathcal{Y}) \leq D(\theta, X) + \deg Y h(X) + \deg X h(Y) + e \deg X \deg Y \leq \]
\[ \max(D(\theta, X), D(\theta, Y)) + \deg Y h(X) + \deg X h(Y) + e \deg X \deg Y. \]
6.1 Proposition For $X, Y$ arbitrary effective cycles in $\mathbb{P}(\mathbb{C})$, and $\theta \in \mathbb{P}(\mathbb{C})$ a point not contained in $\text{supp}X \cup \text{supp}Y$ there is a function $f_{X,Y}: \deg X + \deg Y \rightarrow \deg X \times \deg Y$ with the same properties as in Theorem 2.4 such that for every $T \in f_{X,Y}: \deg X + \deg Y$ and $(\nu, \kappa) = f_{X,Y}(T)$ the inequality

$$\nu \kappa \log |\theta, X + Y| + D(\theta, X.Y) + D(X, Y) \leq \kappa D(\theta, X) + \nu D(\theta, Y) + \bar{e} \deg X \deg Y.$$ 

Further, if $|\theta, X + Y| = |\theta, X|$, or $D(\theta, X) \leq \log |\theta, Y|$, then

$$D(\theta, X.Y) + D(X, Y) \leq D(\theta, Y) + \bar{e}' \deg X \deg Y,$$

and in general

$$D(\theta, X.Y) + D(X, Y) \leq \max(D(\theta, X), D(\theta, Y)) + \bar{e}' \deg X \deg Y.$$ 

If $X$ and $Y$ have pure complementary dimension, then $D(\theta, X.Y) = 0$, and the above implies the logarithmic triangle inequality

$$D(X, Y) \leq \max(D(\theta, X), D(\theta, Y)) + \bar{e}' \deg X \deg Y$$

holds.

Proof Just repeat the proofs of Theorem 2.4 and Corollary 2.5 without using the fact

$$D(\mathbb{P}(\Delta), X\#Y) =$$

$$h((X\#Y).\mathbb{P}(\Delta)) - h(X\#Y) - \deg(X\#Y)h(\mathbb{P}(\Delta)) + \sigma_{2t+1} \deg(X\#Y),$$

and instead use that $D(\mathbb{P}(\Delta), X\#Y)$, and $D(X, Y)$ only differ by a constant times $\deg X \deg Y$ by Proposition 5.3.

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