Formal Expansions in Stochastic Model for Wave Turbulence 1: Kinetic Limit

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Abstract: We consider the damped/driven (modified) cubic NLS equation on a large torus with a properly scaled forcing and dissipation, and decompose its solutions to formal series in the amplitude. We study the second order truncation of this series and prove that when the amplitude goes to zero and the torus’ size goes to infinity the energy spectrum of the truncated solutions becomes close to a solution of the damped/driven wave kinetic equation. Next we discuss higher order truncations of the series.

Contents

1. Introduction ........................................ 951
2. Formal Decomposition of Solutions in Series in $\rho$ ................ 963
3. Limiting Behaviour of Second Moments .................... 966
4. Wave Kinetic Integrals and Equations ...................... 972
5. Quasisolutions ..................................... 975
6. Proof of Proposition 5.6 ................................ 981
7. Energy Spectra of Quasisolutions and Wave Kinetic Equation .... 987
8. Energy Spectra of Solutions (2.2) ........................ 990
9. Proof of Theorem 3.3 .................................. 992
10. Oscillating Sums Under the Limit (1.17) .................... 1000
11. Wave Kinetic Integrals and Equations: Proofs .................. 1006
12. Addenda ........................................... 1010

1. Introduction

1.1. The setting. The wave turbulence (WT) was developed in 1960’s as a heuristic tool to study small-amplitude oscillations in nonlinear Hamiltonian PDEs and Hamiltonian systems on lattices. We start with recalling basic concepts of the theory in application to the cubic non-linear Schrödinger equation (NLS).
Classical setting. Consider the cubic NLS equation

$$\frac{\partial}{\partial t} u + i \Delta u - i \lambda |u|^2 u = 0, \quad x \in \mathbb{T}_L^d = \mathbb{R}^d / (L\mathbb{Z}^d),$$

(1.1)

where $\Delta = (2\pi)^{-2} \sum_{j=1}^d (\partial^2 / \partial x_j^2)$, $d \geq 2$, $L \geq 1$ and $0 < \lambda \leq 1$. Denote by $H$ the space $L_2(\mathbb{T}_L^d; \mathbb{C})$, given the $L_2$-norm with respect to the normalised Lebesgue measure:

$$\|u\|^2 = \|u\|_{L_2(\mathbb{T}_L^d)}^2 = \langle u, u \rangle, \quad \langle u, v \rangle = \mathcal{F}^{-1} \mathcal{F} u \mathcal{F} v dx.$$  

The NLS equation is a hamiltonian system in $H$ with two integrals of motion—the Hamiltonian and $\|u\|^2$. The equation with the Hamiltonian $\lambda \|u\|^4$ is $\frac{\partial}{\partial t} u - 2i \lambda \|u\|^2 u = 0$ and its flow commutes with that of NLS. We modify the NLS equation by subtracting $\lambda \|u\|^4$ from its Hamiltonian, thus arriving at the equation

$$\frac{\partial}{\partial t} u + i \Delta u - i \lambda (\|u\|^2 - 2\|u\|^2)u = 0, \quad x \in \mathbb{T}_L^d.$$  

(1.2)

This modification is used by mathematicians, working with hamiltonian PDEs, since it keeps the main features of the original equation, reducing some non-crucial technicalities. It is also used by physicists, studying WT; e.g. see [24], pp. 89–90. Below we work with eq. (1.2) and write its solutions as functions $u(t, x) \in \mathbb{C}$ or as curves $u(t) \in H$.

The objective of WT is to study solutions of (1.1) and (1.2) when

$$\lambda \to 0 \quad \text{and} \quad L \to \infty$$

(1.3)

on large time intervals.

We will write the Fourier series for an $u(x)$ as

$$u(x) = L^{-d} \sum_{s \in \mathbb{Z}_L^d} v_s e^{2\pi i s \cdot x}, \quad \mathbb{Z}_L^d = L^{-1} \mathbb{Z}^d,$$

(1.4)

where the vector of Fourier coefficients $v = \{v_s, s \in \mathbb{Z}_L^d\}$ is the Fourier transform of $u(x)$:

$$v = \hat{u} = \mathcal{F}(u), \quad v_s = \hat{u}(s) = L^{-d/2} \int_{\mathbb{T}_L^d} u(x) e^{-2\pi i s \cdot x} dx.$$  

(1.5)

Given a vector $v = \{v_s, s \in \mathbb{Z}_L^d\}$ we will regard the sum in (1.4) as its inverse Fourier transform $\mathcal{F}^{-1}(v)(x)$, which we will also write as $(\mathcal{F}^{-1} v_s)(x).$ I.e.,

$$u(x) = (\mathcal{F}^{-1} v_s)(x) = (\mathcal{F}^{-1} v)(x).$$

Then $\|u\|^2 = \sum_s |v_s|^2$, where for a complex sequence $(w_s, s \in \mathbb{Z}_L^d)$ we denote

$$\sum_{s \in \mathbb{Z}_L^d} w_s = L^{-d} \sum_{s \in \mathbb{Z}_L^d} w_s$$

---

1. Note in addition that if $u(t, x)$ satisfies eq. (1.2), then $u' = e^{2i\lambda \|u\|^2} u$ is a solution of (1.1).

2. The symmetric form of the Fourier transform which we use—with the same scaling factor $L^{-d/2}$ for the direct and the inverse transformations—is convenient for the heavy calculation below in the paper.
(this equals to the integral over $\mathbb{R}^d$ of $w_x$, extended to a function, constant on the cells of the mesh in $\mathbb{R}^d$ of size $L^{-1}$). By $h$ we will denote the Hilbert space $h = L_2(\mathbb{Z}_L^d; \mathbb{C})$, given the norm
$$\|w\|^2 = \sum_{}|w_s|^2.$$ Abusing a bit notation we denote by the same symbol the norms in the spaces $H$ and $h$. This is justified by the fact that the Fourier transform (1.5) defines an isometry $\mathcal{F} : H \rightarrow h$.

Equations (1.1), (1.2) and other NLS equations on the torus $\mathbb{T}_L^d$ with fixed $L$ are intensively studied by mathematicians, e.g. see the book [2] and references in it. The limit $\lambda \rightarrow 0$ with $L$ fixed was rigorously treated in a number of publications, e.g. see [13]. But there are just a few mathematical works, addressing the limit (1.3). In the paper [11] $d = 2$ and the limit (1.3) is taken in a specific regime, when $L \rightarrow \infty$ much slower than $\lambda^{-1}$. The elegant description of the limit, obtained there, is far from the prediction of WT, and rather should be regarded as a kind of averaging. In recent paper [4] the authors study eq. (1.1) with random initial data $u(0, x) = u_0(x)$ such that the phases $\{v_{0x}, s \in \mathbb{Z}_L^d\}$ of components of the vector $v_0 = \mathcal{F}(u_0)$ are independent uniformly distributed random variables. In the notation of our work they prove that under the limit (1.3), if $L$ goes to infinity slower than $\lambda^{-1}$ but not too slow, then for the values of time of order $\lambda^{-1}L^{-\delta}, \delta > 0$, the energy spectrum $n_s(\tau)$ approximately satisfies the WKE, linearised on $u_0(x)$ and scaled by the factor $\lambda$. The authors of [23] start with eq. (1.1), replace there the space-domain $\mathbb{T}_L^d$ by the discrete torus $\mathbb{Z}_L^d/(L\mathbb{Z}_L^d)$, modify the discrete Laplacian on $\mathbb{Z}_L^d/(L\mathbb{Z}_L^d)$ to a suitable operator, diagonal in the Fourier basis, and study the obtained equation, assuming that the distribution of the initial data $u(0, x)$ is given by the Gibbs measure of the equation. See [10] for a related result.

From other hand, there are plenty of physical works on equations (1.1) and (1.2) under the limit (1.3); many references may be found in [24,25,27]. These papers contain some different (but consistent) approaches to the limit. Non of them was ever rigorously justified, despite the strong interest in physical and mathematical communities to the questions, addressed by these works.

Zakharov–L’vov setting. When studying eq. (1.2), members of the WT community talk about “pumping the energy to low modes and dissipating it in high modes”. To make this rigorous, Zakharov–L’vov [26] (also see [3], Section 1.2) suggested to consider the NLS equation, dumped by a (hyper) viscosity and driven by a random force:
$$\frac{\partial}{\partial t} u + i\Delta u - i\lambda(|u|^2 - 2\|u\|^2)u = -\nu \mathfrak{A}(u) + \sqrt{\nu} \frac{\partial}{\partial t}\eta^{\omega}(t, x),$$
$$\eta^{\omega}(t, x) = \mathcal{F}^{-1}\left(b(s)\beta^{\omega}_{s}(t)\right)(x) = L^{-d/2}\sum_{s \in \mathbb{Z}_L^d} b(s)\beta^{\omega}_{s}(t)e^{2\pi is \cdot x}.$$ (1.6)

Here $0 < \nu \leq 1/2, \{\beta_{s}(t), s \in \mathbb{Z}_L^d\}$ are standard independent complex Wiener processes, $b$ is a positive Schwartz function on $\mathbb{R}^d \supset \mathbb{Z}_L^d$ and $\mathfrak{A}$ is the dissipative linear operator
$$\mathfrak{A}(u(x)) = L^{-d/2}\sum_{s \in \mathbb{Z}_L^d} \gamma_s v_s e^{2\pi i s \cdot x} ,$$
$$ v = \mathcal{F}(u), \quad \gamma_s = \gamma^0(|s|^2),$$

---

3 i.e. $\beta_{s} = \beta^{1}_{s} + i\beta^{2}_{s}$, where $\{\beta^{j}_{s}, s \in \mathbb{Z}_L^d, j = 1, 2\}$ are standard independent real Wiener processes.
where $\gamma^0(y)$ is a smooth increasing function of $y \in \mathbb{R}$ that has at most a polynomial growth at infinity together with its derivatives of all orders, such that $\gamma^0 \geq 1$ and

$$c(1+y)^{r_0} \leq \gamma^0(y), \quad |\partial^k \gamma^0(y)| \leq C(1+y)^{r_0-k} \quad \text{for} \ 0 \leq k \leq 3, \ \forall \ y \geq 0. \quad (1.7)$$

The exponent $r_0 \geq 0$ and $c, C$ are positive constants.\(^4\)

It is convenient to pass to the slow time $\tau = \nu t$ and re-write eq. (1.6) as

$$\dot{u} + i\nu^{-1} \Delta u + \mathcal{A}(u) = i\rho \left(|u|^2 - 2\|u\|^2\right) u + \dot{\eta}^\nu(\tau, x), \quad \eta^\nu(\tau, x) = L^{-d/2} \sum_{s \in \mathbb{Z}_L} b(s) \beta_s(\tau)e^{2\pi ix \cdot s}, \quad (1.8)$$

where $\rho = \lambda \nu^{-1}$. Here and below the upper-dot stands for $\partial/\partial \tau$ and $\{\beta_s(\tau)\}$ is another set of standard independent complex Wiener processes.

Solutions $u(t)$ and $u(\tau)$ are random processes in the space $H$. If $r_0 \gg 1$, then equations (1.6) and (1.8) are well posed. In the context of equation (1.8), the objective of WT is to study its solutions when

$$\nu \to 0, \quad L \to \infty. \quad (1.9)$$

Below we show that the main characteristics of the solutions have non-trivial behaviour under the limit above if $\rho \sim \nu^{-1/2}$. We will choose

$$\rho = \nu^{-1/2} \varepsilon^{1/2}, \quad (1.10)$$

and will examine the limiting characteristics of the solutions when $\varepsilon$ is a fixed small positive constant $\varepsilon$. We will show that $\sqrt{\varepsilon}$ measures the properly scaled amplitude of the solutions, so indeed it should be small for the methodology of WT to apply. The parameter $\lambda = \nu \rho = \nu^{1/2} \varepsilon^{1/2}$ will be used only in order to discuss the obtained results in the original scaling (1.6). Note that as $\lambda \sim \sqrt{\nu}$, then we study eq. (1.6)=(1.8) in the regime when it is a perturbation of the original NLS equation (1.2) by dissipative and stochastic terms much smaller than the terms of (1.2). We will examine solutions of (1.8) on the time-scale $\tau \sim 1$. For the original fast time $t$ in eq. (1.6) it corresponds to the scale $t \sim \nu^{-1} \sim \lambda^{-2} = \text{(size of the nonlinearity)}^{-2}$. This is exactly the time-scale usually considered in WT, see in [23,24,27].

Applying Ito’s formula to a solution $u$ of (1.8) and denoting $B = \sum_s b(s)^2$ we arrive at the balance of energy relation:

$$\mathbb{E}\|u(\tau)\|^2 + 2\mathbb{E} \int_0^\tau \|\mathcal{A}(u(s))\|^2 ds = \mathbb{E}\|u(0)\|^2 + 2B\tau. \quad (1.11)$$

The quantity $\mathbb{E}\|u(\tau)\|^2$—the “averaged energy per volume” of a solution $u$—should be of order one, see [24,25,27]. This agrees well with (1.11) since there $B \sim \int b^2 \, dx \sim 1$ as $L \to \infty$, and we immediately get from (1.11) that

$$\mathbb{E}\|u(\tau)\|^2 \leq B + (\mathbb{E}\|u(0)\|^2 - B)e^{-2\tau},$$

uniformly in $\nu$, $\rho$ and $L$.

Using (1.4) and the relations

$$\tilde{u}_1\tilde{u}_2(s) = L^{-d/2} \sum_{s_1} \tilde{u}_1(s_1)\tilde{u}_2(s-s_1), \quad \tilde{u}(s) = \tilde{u}_{-s},$$

\(^4\) For example, if $\gamma_\varepsilon = (1 + |s|^2)^{r_0}$, then $\mathfrak{A} = (1 - \Delta)^{r_0}$. \(\triangleq\)
we write (1.8) as the system of equations
\[
\dot{v}_s - i v^{-1} |s|^2 v_s + \gamma s v_s = i \rho L^{-d} \left( \sum_{s_1, s_2} \delta_{s_3}^{s_1} v_{s_1} v_{s_2} \bar{v}_{s_3} - |v_s|^2 v_s \right) + b(s) \dot{\beta}_s, \tag{1.12}
\]

\( s \in \mathbb{Z}^d_L \), where
\[
\delta_{s_3}^{s_1} = \begin{cases} 
1, & \text{if } s_1 + s_2 = s_3 + s \text{ and } \{s_1, s_2\} \neq \{s_3, s\}, \\
0, & \text{otherwise.}
\end{cases}
\]

Note that if \( \{s_1, s_2\} \cap \{s_3, s\} \neq \emptyset \), then \( \delta_{s_3}^{s_1} = 0 \).

In view of the factor \( \delta_{s_3}^{s_1} \), in the double sum in (1.12) the index \( s_3 \) is a function of \( s_1, s_2, s \).

Now using the interaction representation
\[
v_s = \exp(i v^{-1} \tau |s|^2) a_s, \quad s \in \mathbb{Z}^d_L, \tag{1.13}
\]
we re-write (1.12) as
\[
\dot{a}_s + \gamma s a_s = i \rho \left( \mathcal{Y}_s(a, v^{-1} \tau) - L^{-d} |a_s|^2 a_s \right) + b(s) \dot{\beta}_s, \quad s \in \mathbb{Z}^d_L,
\]
\[
\mathcal{Y}_s(a; t) = L^{-d} \sum_{s_1} \sum_{s_2} \delta_{s_3}^{s_1} a_{s_1} a_{s_2} \bar{a}_{s_3} e^{i \omega L_{s_3}}. \tag{1.14}
\]

Here \( \{\beta_s\} \) is another set of standard independent complex Wiener processes, and
\[
\omega_{s_3}^{12} = |s_1|^2 + |s_2|^2 - |s_3|^2 - |s|^2 = 2(s_1 - s) \cdot (s - s_2) \tag{1.15}
\]
(the second equality holds since \( s_3 = s_1 + s_2 - s \) due to the factor \( \delta_{s_3}^{s_1} \).

Despite stochastic models of WT (including (1.6)) are popular in physics, it seems that the only their mathematical studies were performed by the authors of this work and their collaborators, and in paper [10]. In the same time the models, given by Hamiltonian chains of equations with stochastic perturbations, have received significant attention from the mathematical community, e.g. see [6,9,18] and references in these works. The equations in the corresponding works are related to the Zakharov–L’vov model, written as the perturbed chain of hamiltonian equations (1.12), but differ from it significantly since there, in difference with (1.12), the interaction between the equations is local. This leads to rather different results, obtained by tools, rather different from those in our paper.

1.2. The results. The energy spectrum of a solution \( u(\tau) \) of eq. (1.8) is the function
\[
\mathbb{Z}^d_L \ni s \mapsto \mathcal{N}_s(\tau) = \mathcal{N}_s(\tau; v, L) = \mathbb{E}|v_s(\tau)|^2 = \mathbb{E}|a_s(\tau)|^2. \tag{1.16}
\]
Traditionally in the center of attention of the WT community is the limiting behaviour of the energy spectrum \( \mathcal{N}_s \), as well as of correlations of solutions \( a_s(\tau) \) and \( v_s(\tau) \) under the limit (1.9). Exact meaning of the latter is unclear since no relation between the small parameters \( v \) and \( L^{-1} \) is postulated by the theory. In [13,22] it was proved that for \( \rho \) and \( L \) fixed, eq. (1.8) has a limit as \( v \to 0 \), called the limit of discrete turbulence, see [16,24] and Appendix 12.1. Next it was demonstrated in [21] on the physical level of rigour that if we scale \( \rho \) as \( \varepsilon \sqrt{L}, \varepsilon \ll 1 \), then the iterated limit \( L \to \infty \) leads to a kinetic equation
for the energy spectrum. Attempts to justify this rigorously so far failed. Instead in this work we specify the limit (1.3) as follows:

\[ \nu \to 0 \text{ and } L \geq \nu^{-2-\epsilon} \text{ for some } \epsilon > 0, \]

or first \( L \to \infty \) and next \( \nu \to 0 \) \hspace{1cm} (1.17)

(the second option formally corresponds to the first one with \( \epsilon = \infty \)). To present the results it is convenient for the moment to regard \( \rho \) as an independent parameter, however later we will choose it to be of the form (1.10) with a fixed small positive constant \( \epsilon \).

Accordingly to (1.17), everywhere in the introduction we assume that

\[ L \geq \nu^{-2-\epsilon} \geq 1, \quad \epsilon > 0. \] \hspace{1cm} (1.18)

Let us supplement equation (1.8) = (1.14) with the initial condition

\[ u(-T) = 0, \] \hspace{1cm} (1.19)

for some \( 0 < T \leq +\infty \), and in the spirit of WT decompose a solution of (1.14), (1.19) to formal series in \( \rho \):

\[ a = a^{(0)} + \rho a^{(1)} + \ldots. \] \hspace{1cm} (1.20)

Substituting the series in the equation we get that \( a^{(0)} \) satisfies the linear equation

\[ \dot{a}^{(0)}_s + \gamma_s a^{(0)}_s = b(s) \hat{\beta}_s, \quad s \in \mathbb{Z}_L, \] \hspace{1cm} (1.21)

so this is the Gaussian process

\[ a^{(0)}_s(\tau) = b(s) \int_{-T}^{\tau} e^{-\gamma_s(\tau-l)} d\beta_s(l), \] \hspace{1cm} (1.21)

while \( a^{(1)} \) satisfies

\[ \dot{a}^{(1)}_s + \gamma_s a^{(1)}_s = i \mathcal{Y}_s(a^{(0)}_s(\tau), v^{-1} \tau) - i L^{-d} |a^{(0)}_s(\tau)|^2 a^{(0)}_s(\tau), \quad \tau > -T, \]

so

\[ a^{(1)}_s(\tau) = i L^{-d} \int_{-T}^{\tau} e^{-\gamma_s(\tau-l)} \left( \sum_{s_1,s_2} \delta^{(12)}_{s_3} (a^{(0)}_s a^{(0)}_{s_1} a^{(0)}_{s_2}) (l) e^{i v^{-1} I_{n_3}^{(12)}} \right. \]

\[ \left. - |a^{(0)}_s(l)|^2 a^{(0)}_s(l) \right) dl \] \hspace{1cm} (1.22)

is a Wiener chaos of third order (see [15]). Similarly, for \( n \geq 1 \)

\[ a^{(n)}_s(\tau) = i L^{-d} \int_{-T}^{\tau} e^{-\gamma_s(\tau-l)} \sum_{n_1+n_2+n_3=n-1} \left( \sum_{s_1,s_2} \delta^{(12)}_{s_3} (a^{(n_1)}_{s_1} a^{(n_2)}_{s_2} a^{(n_3)}_{s_3}) (l) e^{i v^{-1} I_{n_3}^{(12)}} - (a^{(n_1)}_s a^{(n_2)}_s a^{(n_3)}_s) (l) \right) dl \] \hspace{1cm} (1.23)

is a Wiener chaos of order \( 2n + 1 \).

**Quasisolutions.** It is traditional in WT to retain the quadratic in \( \rho \) part of the decomposition (1.20) and analyse it, postulating that it well approximates small amplitude
solutions. Thus motivated we start our analysis with the quadratic truncations of the series (1.20), which we call the quasisolutions. Let

\[ \mathcal{A}(\tau) = (\mathcal{A}_s(\tau), s \in \mathbb{Z}_L^d), \quad \mathcal{A}_s(\tau) = a_s^{(0)}(\tau) + \rho a_s^{(1)}(\tau) + \rho^2 a_s^{(2)}(\tau), \] (1.24)

where \( a_s^{(0)}, a_s^{(1)} \) and \( a_s^{(2)} \) were defined above. The energy spectrum of a quasisolution \( \mathcal{A}(\tau) \) is

\[ n_s(\tau) = |\mathcal{A}_s(\tau)|^2, \quad s \in \mathbb{Z}_L^d, \] (1.25)

where

\[ n_s = n_s^{(0)} + \rho n_s^{(1)} + \rho^2 n_s^{(2)} + \rho^3 n_s^{(3)} + \rho^4 n_s^{(4)}, \quad s \in \mathbb{Z}_L^d, \] (1.26)

with

\[ n_s^{(k)} = \sum_{k_1 + k_2 = k, k_1, k_2 \leq 2} \mathbb{E} a_s^{(k_1)} a_s^{(k_2)} \] (1.27)

In particular, \( n_s^{(0)} = |\mathcal{A}_s^{(0)}|^2 \) and we easily derive from (1.21) that

\[ n_s^{(0)}(\tau) = |\mathcal{A}_s^{(0)}(\tau)|^2 = B(s)(1 - e^{-2\gamma_s(T+\tau)}), \quad B(s) = \frac{b(s)^2}{\gamma_s}. \] (1.28)

So \( n_s^{(0)} \) extends to a Schwartz function of \( s \in \mathbb{R}^d \), uniformly in \( \tau \geq -T \). Also it is not hard to see that \( n_s^{(1)} = 0 \), see in Section 2. The coefficients \( n_s^{(k)} \) with \( k \geq 2 \) are more complicated.

The processes \( a_s^{(r)}(\tau) \) and the functions \( n_s^{(r)}(\tau), r \geq 0 \), depend on \( v \) and \( L \). When it will be needed to indicate this dependence, we will write them as \( a_s^{(r)}(\tau) = a_s^{(r)}(\tau; v, L) \), etc. The dependence of the objects on \( T \) will not be indicated.

It was explained on the heuristic and half-heuristic level in many physical works concerning various models of WT that the term \( \rho^2 n_s^{(2)}(\tau) \) is the crucial non-trivial component of the energy spectrum, while the terms \( \rho^3 n_s^{(3)}(\tau) \) and \( \rho^4 n_s^{(4)}(\tau) \) make its perturbations. Our results justify this insight on the energy spectra of quasisolutions. Firstly we consider \( n_s^{(2)} = \mathbb{E}|a_s^{(1)}|^2 + 2\mathbb{E} a_s^{(2)} a_s^{(0)} \). The two terms on the right are similar. Consider the first one, \( \mathbb{E}|a_s^{(1)}|^2 \). The theorem below describes its asymptotic behaviour under the limit (1.17), where for simplicity we assume that \( T = \infty \).

We set \( B(s_1, s_2, s_3) := B(s_1) B(s_2) B(s_3) \) and denote by \( C^#(s) \) various positive continuous functions of \( s \) which decay as \( |s| \to \infty \) faster than any negative degree of \( |s| \).

The constants \( C, C_1 \) etc and the functions \( C^#(s) \) never depend on \( v, L, \rho, \varepsilon \) and on the times \( T, \tau \), (1.29)

unless the dependence is indicated.

**Theorem A.** Let in (1.19) \( T = \infty \). Then for any \( \tau \) and any \( s \in \mathbb{Z}_L^d \),

\[
\left| \mathbb{E}|a_s^{(1)}(\tau)|^2 - \frac{\pi v}{\gamma_s} \int_{\Sigma_s} \frac{B(s_1, s_2, s_1 + s_2 - s)}{\sqrt{|s_1 - s|^2 + |s_2 - s|^2}} \, ds_1 ds_2 |\Sigma_s| \right| \leq \left( v^2 + L^{-2}v^{-2} \right) C^#(s). \] (1.30)

Here \( \Sigma_s \) is the quadric \( \{(s_1, s_2) : (s_1 - s) \cdot (s_2 - s) = 0\} \) and \( ds_1 ds_2 |\Sigma_s \) is the volume element on it, corresponding to the Euclidean structure on \( \mathbb{R}^{2d} \). If \( d = 2 \), then the term \( C^#(s)v^2 \) in the r.h.s. of (1.30) should be replaced by \( C^#(s; \mathcal{N}) v^{2-\mathcal{N}} \), where \( \mathcal{N} \) is arbitrary positive number.
See Theorems 3.1, 3.3 and Corollary 3.4. Due to (1.15), \( \Sigma_s = \{(s_1, s_2, s_3) : s_1 + s_2 = s_3 + s \text{ and } \omega_{3s}^{12} = 0\} \). This quadric is the set of resonances for eq. (2.1).

Denote \( F_s(s_1, s_2) := (s_1 - s) \cdot (s_2 - s) = -\frac{1}{2} \omega_{3s}^{12} \). Then \( |\partial F_s| \) equals the divisor of the integrand in (1.30). So the integral in (1.30) is exactly what physicists call the integral of the size \( n \) of the integrand in (1.30). So the integral in (1.30) is exactly what physicists call the

\[ \frac{2\pi}{\gamma_s} \int B(s_1, s_2, s_3) \delta(\omega_{12}^{3s}) \delta_{3s}^{12} ds_1 ds_2 ds_3 \]  

(1.31)

(since \( \int \ldots \delta_{3s}^{12} ds_1 ds_2 ds_3 = \int \cdots |s_3 = s_1 + s_2 - s| ds_1 ds_2 \).)

Theorem A and its variations play important role in our work since the terms, quadratic in \( a^{(1)} \) and in its increments, as well as quadratic terms, linear in \( a^{(0)}, a^{(2)} \) and in their increments, play a leading role in the analysis of the energy spectrum \( n_s(\tau) \). It turns out that their asymptotic behaviour under the limit (1.17) is described by integrals, similar to (1.30). These results imply that

\[ n_s^{(2)} \sim \nu, \]  

(recall (1.18)). The terms \( n_s^{(3)} \) and \( n_s^{(4)} \) and their variations which appear in our analysis are smaller:

\[ |n_s^{(3)}|, |n_s^{(4)}| \leq C^#(s) \nu^2. \]  

(1.33)

If \( d = 2 \) the estimate for \( n_s^{(3)} \) is slightly weaker: \( |n_s^{(3)}| \leq C^#(s) \nu^2 \ln \nu^{-1} \). Upper bounds (1.33) may be obtained by annoying but rather straightforward analysis of certain integrals with oscillating exponents, based on results from Sections 5 and 10, but instead we get them from a much deeper result, presented below in Theorem D.

Estimates (1.32) and (1.33) justify the scaling (1.10) since for such a choice of \( \rho \)

\[ n_s = n_s^{(0)} + \epsilon \tilde{n}_s^{(2)} + O(\epsilon^2) \]  

(1.34)

(if \( \nu \ll 1 \)), where \( \tilde{n}_s^{(2)} = \nu^{-1} n_s^{(2)} \sim 1 \). Thus, the principal non-trivial component of \( n_s \) is given by \( \rho^2 n_s^{(2)} = \epsilon \tilde{n}_s^{(2)} \sim \epsilon \), while the other terms in (1.26)=(1.34) are smaller, of the size \( O(\epsilon^2) \). This also shows that the small parameter \( \sqrt{\epsilon} \) measures the properly scaled amplitude of the oscillations.

Since equation (1.14) for the processes \( a_s(\tau) \) fast oscillates in time, then the task to describe the behaviour of \( n_s(\tau) \) has obvious similarities with the problem, treated by the Krylov–Bogolyubov averaging (see in [1]). Accordingly it is natural to try to study the required limiting behaviour following the suit of the Krylov–Bogolyubov theory. That is, by considering the increments \( n_s(\tau + \theta) - n_s(\tau) \) with \( \nu \ll \theta \ll 1 \) and passing firstly to the limit as \( \nu \rightarrow 0 \) and next—to the limit \( \theta \rightarrow 0 \). That insight was exploited heuristically in many works on WT (e.g. see [24], Section 7), while in [13,22] it was rigorously applied to pass to the limit of discrete turbulence \( \nu \rightarrow 0 \) with \( L \) and \( \rho \) fixed. In this work we also argue as it is customary in the classical averaging and analyse the increments \( n_s(\tau + \theta) - n_s(\tau) \), using the asymptotical results like Theorem A and estimates like (1.33). This analysis shows that due to the important role, played by the integrals like (1.30), the leading nonlinear contribution to the increments is described
by the cubic wave kinetic integral operator $K$, sending a function $y(s), s \in \mathbb{R}^d$, to the function

$$K_s(y(\cdot)) = 2\pi \int_{\Sigma_s} ds_1 ds_2 |\Sigma_s| y_1 y_2 y_3 y_s \left( \frac{1}{y_1} + \frac{1}{y_3} - \frac{1}{y_1} - \frac{1}{y_2} \right),$$

where $y_s = y(s)$ and for $j = 1, 2, 3$ we denote $y_j = y(s_j)$ with $s_3 = s_1 + s_2 - s$. Using the notation (1.31), $K_s(y(\cdot))$ may be written as

$$4\pi \int y_1 y_2 y_3 y_s \left( \frac{1}{y_1} + \frac{1}{y_3} - \frac{1}{y_1} - \frac{1}{y_2} \right) \delta(\omega_s) d\omega_s ds_1 ds_2 ds_3.$$

This is exactly the wave kinetic integral which appears in physical works on WT to describe the 4-waves interaction, see [27], p. 71 and [24], p. 91.

The operator $K$ is defined in terms of the measure $\mu_s = 4\pi \int y_1 y_2 y_3 y_s \left( \frac{1}{y_1} + \frac{1}{y_3} - \frac{1}{y_1} - \frac{1}{y_2} \right) \delta(\omega_s) d\omega_s ds_1 ds_2 ds_3$.

Theorem B. For any $r > d$ the operator $K$ defines a continuous 3-homogeneous mapping $K : C_r(\mathbb{R}^d) \to C_{r+1}(\mathbb{R}^d)$.

See Theorem 4.1. Now consider the damped/driven wave kinetic equation (WKE):

$$\dot{m}(\tau, s) = -2y_s m(\tau, s) + \varepsilon K(m(\tau, \cdot))(s) + 2b(s)^2, \quad m(-T) = 0. \quad (1.35)$$

In Theorem 4.4 we easily derive from Theorem B that for small $\varepsilon$ this equation has a unique solution $m$. The latter can be written as $m = m^0(\tau, s) + \varepsilon m^1(\tau, s)$, where $m^0, m^1 \sim 1, m^0$ is a solution of the linear equation $\dot{m} - \varepsilon K(m(\cdot))(s) + 2b(s)^2 = 0$. Analysing the increments $n_\varepsilon(\tau + \theta) - n_\varepsilon(\tau)$ using the results, discussed above, in Section 5 we show that they have approximately the same form as the increments of solutions for eq. (1.35). Next in Section 7, arguing by analogy with the classical averaging theory, we get the stated below main result of this work (recall agreement (1.29)):

Theorem C (Main theorem). The energy spectrum $n_\varepsilon(\tau) = n_\varepsilon(\tau; v, L)$ of the quasisolution $A_\varepsilon(\tau)$ of (1.14), (1.19) satisfies the estimate $n_\varepsilon(\tau) \leq C\varepsilon(\tau)$ and is close to the solution $m(\tau, s)$ of WKE (1.35). Namely, under the scaling (1.10) for any $r$ there exists $\varepsilon_r > 0$ such that for $0 < \varepsilon \leq \varepsilon_r$ we have

$$|n_\varepsilon(\tau) - m(\tau, \cdot)|_r \leq C_r \varepsilon^2 \quad \forall \tau \geq -T, \quad (1.36)$$

if $0 < v \leq v_r(r)$ for a suitable $v_r(r) > 0$, and if $L$ satisfies (1.18). Moreover, the limit $n_\varepsilon(\tau; v, \infty)$ of $n_\varepsilon(\tau; v, L)$ as $L \to \infty$ exists, is a Schwartz function of $s \in \mathbb{R}^d$ and also satisfies the estimate above for any $r$. 
Since the energy spectrum $n_s$ is defined for $s \in \mathbb{Z}_L^d$ with finite $L$, then the norm in (1.36) is understood as $|f|_r = \sup_{s \in \mathbb{Z}_L^d} |f(s)| |\langle s \rangle|$. 

For $\varepsilon = 0$ eq. (1.35) has the unique steady state $m^0_s = m^0_s = b^2_s / \gamma_s$, which is asymptotically stable. By the inverse function theorem, for $\varepsilon < \varepsilon_r$ ($\varepsilon_r > 0$), eq. (1.35) has a unique steady state $m^\varepsilon \in C_r(\mathbb{R}^d)$, close to $m^0$. It is also asymptotically stable. Jointly with Theorem C this result describes the asymptotic in time behaviour of the energy spectrum $n_s$:

$$|n(\tau) - m^\varepsilon_s|_r \leq |m^0_s|_r e^{-\tau - T} + C_r\varepsilon^2, \quad \forall \tau \geq -T.$$  

(1.37)

See in Section 7.

Due to Theorem A and some modifications of this result, the iterated limit

$$\lim_{\nu \to 0} \lim_{L \to \infty} \nu^{-1} n_s^{(2)}(\tau; \nu, L)$$

exists and is non-zero. It is hard to doubt (however, we have not proved this yet) that a similar iterated limit also exists for $\nu^{-2} n_s^{(4)}$ (cf. estimate (1.33)). Then, in view of (1.33) for $n_s^{(3)}$, under the scaling (1.10) exists a limit $n_s(\tau)$, then for some diagrams $F$ we have

$$\lim_{\nu \to 0} \lim_{L \to \infty} n_s(\tau) = n_s(\tau; \nu, L).$$

If so, then $n_s(\tau; 0, \infty)$ also satisfies the assertion of Theorem C and obeys the time-asymptotic (1.37).

Theorems A–C are proved in Sections 2–7 and Sections 9–11, where Sections 9–11 contain a demonstration of Theorem A as well as of some lemmas, needed to prove Theorems B, C in Sections 2–7.

Section 8 presents the results of our second paper [8] on formal expansions (1.20) in series in $\rho$. There we decompose the spectrum $\mathcal{N}_s(\tau)$, defined in (1.16), in formal series,

$$\mathcal{N}_s(\tau) = n^0_s(\tau) + \rho n^1_s(\tau) + \rho^2 n^2_s(\tau) + \ldots$$

(1.38)

where $n^0_s = n^0_s(0)$ and for $k \geq 1$ $n^k_s$ is given by the formulas (1.27) with the restriction $k_1, k_2 \leq 2$ being dropped, i.e. $n^k_s = \sum_{k_1+k_2=k} \mathbb{E}\tilde{a}_s^{(k_1)}\tilde{a}_s^{(k_2)}$, so $n^j_s$ equals $n^{(j)}_s$ for $j \leq 2$, but not for $j = 3, 4$. Still the estimates (1.33) remain true for $n^2_s$ and $n^4_s$. The bounds on $n^0_s, \ldots, n^4_s$ suggest that $|n^k_s| \lesssim \nu^{k/2}$. The results of Section 8 put some light on this assumption. Namely, it is proved there that $n^k_s(\tau)$ may be approximated by a finite sum

$$\sum_{\mathcal{F}_k} I^k_s(\mathcal{F}_k)$$

of integrals $I^k_s(\mathcal{F}_k)$, naturally parametrised by a certain class of Feynman diagrams $\mathcal{F}_k$. These integrals satisfy the following assertion, where $[x]$ stands for the smallest integer $\geq x$.

**Theorem D.** For each $k$,

(a) every integral $I^k_s(\mathcal{F}_k)$ satisfies

$$|I^k_s(\mathcal{F}_k)| \leq C^\#(s; k) \max(\nu^{[k/2]}, \nu^d),$$

(1.39)

if $L$ is so big that $L^{-2}\nu^{-2} \leq \max(\nu^{[k/2]}, \nu^d)$. If $d = 2$ and $k = 3$ then the maximum in the r.h.s. above should be multiplied by $\ln \nu^{-1}$.

(b) estimate (1.39) is sharp in the sense that if $k > 2d$ (so that $\max(\nu^{[k/2]}, \nu^d) = \nu^d$), then for some diagrams $\mathcal{F}_k$ we have

$$|I^k_s(\mathcal{F}_k)| \sim C^\#(s; k) \nu^d \gg C^\#(s; k) \nu^{[k/2]}.$$  

(1.40)

Theorem D is a new result on the integrals with fast oscillating quadratic exponents (see the integral in (8.9)). We hope that the theorem and its variations (cf. [7] and the last section of [8]) will find applications outside the framework of WT.
As \( n_s^k \) is approximated by a finite sum of integrals \( I_s^k(\mathcal{F}_k) \), it also obeys (1.39). Since \( d \geq 2 \), then (1.39) implies estimate (1.33), needed to prove Theorem C. Also, by (1.39)

\[
|n_s^k| \lesssim \nu^{k/2} \quad \text{if} \quad k \leq 2d.
\]

Validity of inequality \(|n_s^k| \lesssim \nu^{k/2}\) for \( k > 2d \) is a delicate issue. Assertion (1.40) implies that the inequality does not hold for all summands, forming \( n_s^k \), but still it may hold for \( n_s^k = \sum_{\mathcal{F}_k} I_s^k(\mathcal{F}_k) \) due to cancellations. And indeed, we prove that some cancellations do happen in the sub-sum over a certain subclass \( \mathfrak{F}_k^B \) of diagrams \( \mathcal{F}_k \) which give rise to the biggest integrals \( I_s^k(\mathcal{F}_k) \). Namely, for any \( \mathcal{F}_k \in \mathfrak{F}_k^B \) we have (1.40) but

\[
|\sum_{\mathcal{F}_k \in \mathfrak{F}_k^B} I_s^k(\mathcal{F}_k)| \leq C^\#(s; k)\nu^{k-1} \leq C^\#(s; k)\nu^{k/2}.
\]

This does not imply the validity of (1.41) for all \( k \), which remains an open problem:

**Problem 1.1.** Prove that for any \( k \in \mathbb{N} \)

\[
|n_s^k(\tau)| \leq C^\#(s; k)\nu^{k/2} \quad \forall s, \quad \forall \tau \geq -T,
\]

if \( L \) is sufficiently big in terms of \( \nu^{-1} \).

If the conjecture (1.42) is correct, then under the scaling (1.10), for any \( M \geq 2 \) the order \( M \) truncation of the series (1.38), namely \( N_s,M(\tau) = \sum_{0 \leq k \leq M} \rho^k n_s^k(\tau) \), also meets the assertion of Theorem C, i.e. satisfies the WKE with the accuracy \( \epsilon^2 \).

It is unclear for us if \( N_s,M \) satisfies the equation with better accuracy, i.e. if it better approximates a solution of (1.35) than \( n_s(\tau) \). On the contrary, if (1.42) fails in the sense that for some \( k \) we have \( \|n_s^k\| \gtrsim C\nu^k \) with \( k' < k/2 \), then under the scaling (1.10) the sum (1.38) is not a formal series in \( \sqrt{\nu} \), uniformly in \( \nu \) and \( L \).

1.3. Conclusions.

- If in eq. (1.8) \( \rho \) is chosen to be \( \rho = \nu^{-1/2}\epsilon^{1/2} \) with \( 0 < \epsilon \leq 1 \), then the energy spectra \( n_s(\tau) \) of quasisolutions for the equation (i.e. of quadratic in \( \rho \) truncations of solutions \( u \), decomposed in formal series in \( \rho \)) under the limit (1.17) satisfy the damped/driven wave kinetic equation (1.35) with accuracy \( \epsilon^2 \). If we write the equation which we study using the original fast time \( t \) in the form (1.6), then the kinetic limit exists if there \( \lambda \sim \sqrt{\nu} \). The time, needed to arrive at the kinetic regime is \( t \sim \lambda^{-2} \).
- If (1.42) is true, then the energy spectra of higher order truncations for decompositions of solutions for (1.8) in series in \( \rho \) also satisfy (1.35) at least with the same accuracy \( \epsilon^2 \).
- Similar, if the energy spectrum \( N_s(\tau) = \mathbb{E}|v_s|^2 \) admits the second order truncated Taylor decomposition in \( \rho \) of the form

\[
N_s(\tau) = n_s^{(0)} + \rho^2 n_s^{(2)} + O(\rho^3 \nu^{3/2}),
\]

then under the scaling (1.10) and the limit (1.17) the spectrum \( N_s(\tau) \) satisfies the WKE (1.35) with accuracy \( \epsilon^{3/2} \). At this point we recall that different NLS equations and their damped/driven versions appear in physics as models for small oscillations in various media, obtained by neglecting in the exact equations terms of higher order of smallness. So it is not impossible that the kinetic limit holds for the energy spectra of the second order jets in \( \rho \) of solutions \( v(\tau) \) (which we call quasisolutions), but not for the solutions themselves since the former are closer to the physical reality.
To prove our results we have developed in this work and in its second part [8] new analytic and combinatoric techniques. They apply to quasisolutions of equations (1.6) under the WT limit (1.9) and for this moment give no non-trivial information about the exact solutions. Still we believe that these techniques make a basis for further study of the damped/driven NLS equations under the WT limit, as well as of other stochastic models of WT. In particular, we hope that being applied to some other models they will give there stronger and “more final” results. To verify this belief is our next goal.

1.4. Notation. By $\mathbb{R}_+^n$ and $\mathbb{R}_-^n$ we denote, respectively, the sets $[0, \infty)^n$ and $(-\infty, 0]^n$. For a vector $v$ we denote by $|v|$ its Euclidean norm and by $v \cdot u$ its Euclidean scalar product with a vector $u$. We write $(v) = (1 + |v|^2)^{1/2}$. For a real number $x$, $[x]$ stands for the smallest integer $\geq x$. We denote by $\chi_d(v)$ the constant

$$
\chi_d(v) = \begin{cases} 
1, & d \geq 3, \\
\ln v^{-1}, & d = 2.
\end{cases}
$$

(1.43)

The exponent $\nu_d$ is zero if $d \geq 3$ and is any positive number if $d = 2$. For an integral $I = \int_{\mathbb{R}_+^n} f(z) \, dz$ and a domain $M \subset \mathbb{R}_+^n$, open or closed, we write

$$
\langle I, M \rangle = \int_M f(z) \, dz.
$$

(1.44)

Similar we write $\langle |I|, M \rangle = \int_M |f(z)| \, dz$.

We denote $\sum_{s_1, \ldots, s_k \in \mathbb{Z}_L^d} := L^{-kd} \sum_{s_1, \ldots, s_k \in \mathbb{Z}_L^d}$. Following the tradition of WT, we often abbreviate $v_j$, $a_j$, $\gamma_j$, $\alpha_j$, $\gamma_j$, and abbreviate the sums $\sum_{s_1, \ldots, s_k \in \mathbb{Z}_L^d}$ and $\sum_{s_1, \ldots, s_k \in \mathbb{Z}_L^d}$ to $\sum_{1, \ldots, k}$ and $\sum_{1, \ldots, k}$. By $\delta_{3s}^{12}$ we denote the Kronecker delta of the relation $s_1 + s_2 = s_3 + s$.

Finally, $C^1(\cdot)$, $C^2(\cdot)$, ... stand for various non-negative continuous functions, fast decaying at infinity:

$$
0 \leq C^H(x) \leq C_N(x)^{-N} \quad \forall x,
$$

(1.45)

for every $N$, with suitable constants $C_N$. By $C^H(x; a)$ we denote a function $C^H$, depending on a parameter $a$. Below we discuss some properties of the functions $C^H$.

Functions $C^H$. For any Schwartz function $f$, $|f(x)|$ may be written as $C^H(x)$. If $\mathcal{L} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is a linear isomorphism, then the function $g(y) = C^H(\mathcal{L}y)$ may be written as $C^H(y)$. Next, for any $C^H(x, y)$, where $(x, y) \in \mathbb{R}^{d_1+d_2}$, $d_1, d_2 \geq 1$, there exist $C^H_1(x)$ and $C^H_2(y)$ such that

$$
C^H(x, y) \leq C^H_1(x)C^H_2(y).
$$

Indeed, consider $C^H_0(t) = \sup_{(x, y) \geq t} C^H(x, y)$, $t \geq 0$. This is a non-increasing continuous function, satisfying (1.45). Then

$$
C^H(x, y) \leq C^H_0\left(\frac{1}{\sqrt{2}}(|x| + |y|)\right) \sqrt{C^H_0\left(\frac{1}{\sqrt{2}}(|x| + |y|)\right)} \leq C^H_1(x)C^H_2(y),
$$

where $C^H_j(z) = \left(\frac{C^H_0\left(\frac{1}{\sqrt{2}}|z|\right)}{1/2}\right)^{1/2}$, $j = 1, 2$. Finally, for any $C^H_1(x)$ and $C^H_2(y)$ the function $C^H_1(x)C^H_2(y)$ may be written as $C^H(x, y)$. 

2. Formal Decomposition of Solutions in Series in $\rho$

2.1. Approximate $a$-equation. Despite that the term $L^{-d} |a_s|^2 a_s$ is only a small perturbation in equation (1.14), it is rather inconvenient for our analysis. Dropping it we consider the following more convenient equation:

$$\dot{a}_s + \gamma_s a_s = i \rho \gamma_s (a_s, v^{-1} \tau) + b(s) \dot{\beta}_s, \quad a_s(-T) = 0, \quad s \in \mathbb{Z}_L^d. \tag{2.1}$$

Similar to the process $a_s$, we decompose $a_s$ to the formal series in $\rho$:

$$a = a^{(0)} + \rho a^{(1)} + \ldots. \tag{2.2}$$

Here $a_s^{(0)}(\tau) = a_s^{(0)}(\tau)$ is the Gaussian process given by (1.21), while the processes $a_s^{(n)}(\tau)$ with $n \geq 1$, where

$$a_s^{(1)}(\tau) = i L^{-d} \int_{-T}^{\tau} e^{-\gamma_s(\tau - l)} \sum_{s_1, s_2} \delta_{3s}^{12} (a^{(0)}_{s_1} a^{(0)}_{s_2} a^{(0)}_{s_3})(l) e^{i v^{-1} \omega_{3s}^2}dl \tag{2.3}$$

and for $n \geq 1$

$$a_s^{(n)}(\tau) = i L^{-d} \int_{-T}^{\tau} e^{-\gamma_s(\tau - l)} \sum_{n_1 + n_2 + n_3 = n-1} \sum_{s_1, s_2} \delta_{3s}^{12} (a^{(n_1)}_{s_1} a^{(n_2)}_{s_2} a^{(n_3)}_{s_3})(l) e^{i v^{-1} \omega_{3s}^2}dl, \tag{2.4}$$

are Wiener chaoses of order $2n + 1$. In Section 12.4 we prove that the processes $a^{(n)}(\tau)$ and $a^{(n)}(\tau)$ are $L^{-d}$-close:

**Proposition 2.1.** For any $n \geq 0$,

$$\mathbb{E}|a_s^{(n)}(\tau) - a_s^{(n)}(\tau)|^2 \leq L^{-2d} C^#(s, n), \quad s \in \mathbb{Z}_L^d, \tag{2.5}$$

uniformly in $\tau \geq -T$ and $v$.

Since $\mathbb{E}|a^{(n)}(\tau)|^2$, $\mathbb{E}|a_s^{(n)}(\tau)|^2 \leq C^#(s, n)$ for any $n$ (this follows from Theorem 5.1 jointly with (2.5)), Proposition 2.1 and the Cauchy inequality imply

**Corollary 2.2.** For any $m, n \geq 0$,

$$\left| \mathbb{E} a_s^{(n)}(\tau_1) a_s^{(m)}(\tau_2) - \mathbb{E} a_s^{(n)}(\tau_1) a_s^{(m)}(\tau_2) \right| \leq L^{-d} C^#(s, m, n), \quad s \in \mathbb{Z}_L^d, \tag{2.6}$$

uniformly in $\tau_1, \tau_2$ and $v$.

We define a quasisolution $A(\tau)$ of equation (2.1) as in definition (1.24), where the processes $a^{(n)}$ are replaced by $a^{(n)}$, and define its energy spectrum as $n_s(\tau) = \mathbb{E}|A_s(\tau)|^2$, cf. (1.25). Due to Corollary 2.2 it suffices to prove the results, formulated in the introduction, replacing in their statements processes $a^{(k)}(\tau)$ and $n(\tau)$ with $a^{(k)}(\tau)$ and $n(\tau)$. Indeed, for Theorem A this assertion holds since $d \geq 2$. For Theorem C it follows from (1.17) as $|n_s - n_s| \leq \rho^4 L^{-d} C^#(s) \leq v^{-2} L^{-d} C^#(s)$, where we recall that $\rho$ is given by (1.10). For other results the argument is similar.

Below we work with the processes $a^{(j)}(\tau)$ and $n(\tau)$, and never with $a^{(j)}(\tau)$ and $n(\tau)$. 
2.2. Nonlinearity $\mathcal{Y}_s$. The cubic nonlinearity $\mathcal{Y}$ in (2.1) defines the 3-linear over real numbers mapping $(u, v, w) \mapsto \mathcal{Y}(u, v, w; t)$, where
\[
\mathcal{Y}_s(u, v, w; t) = L^{-d} \sum_{s_1, s_2} \delta^{1/2}_{3s} u_{s_1} v_{s_2} w_{s_3} e^{it\alpha_{3s}},
\]
so $\mathcal{Y}_s(a; t) = \mathcal{Y}_s(a, a, a; t)$. Often it will be better to use the symmetrisation
\[
\mathcal{Y}^{\text{sym}}_s(u, v, w; t) = \frac{L^{-d}}{3} \sum_{s_1, s_2} \delta^{1/2}_{3s} (u_{s_1} v_{s_2} w_{s_3} + v_{s_1} w_{s_2} u_{s_3} + w_{s_1} u_{s_2} v_{s_3}) e^{it\alpha_{3s}}. \tag{2.7}
\]
Clearly $\mathcal{Y}^{\text{sym}}_s(u, v, w) = \mathcal{Y}(v)$. Besides,
\[
\mathcal{Y}_s(v^0 + \delta v^1 + \delta^2 v^2) = \mathcal{Y}_s(v^0) + 3\delta \mathcal{Y}^{\text{sym}}_s(v^0, v^0, v^1) + 3\delta^2 \left( \mathcal{Y}^{\text{sym}}_s(v^0, v^1, v^1) + \mathcal{Y}^{\text{sym}}_s(v^0, v^0, v^2) \right) + O(\delta^3).
\]

2.3. Correlations of first terms in the formal decomposition. Let us go back to the formal decomposition (2.2). Correlations of the processes $a^{(n)}_s(\tau)$ are important for what follows. Since $a^{(0)}_s = a^{(0)}$, then by (1.21) for any $\tau_1, \tau_2$,
\[
\mathbb{E}a^{(0)}_s(\tau_1) a^{(0)}_s(\tau_2) = \mathbb{E}a^{(0)}_s(\tau_1) \tilde{a}^{(0)}_s(\tau_2) = 0,
\]
\[
\mathbb{E}a^{(0)}_s(\tau_1) \tilde{a}^{(0)}_s(\tau_2) = \delta_s^{b(s)} \frac{b(s)^2}{\gamma_s} \left( e^{-\gamma_s |\tau_1 - \tau_2|} - e^{-\gamma_s (2T + \tau_1 + \tau_2)} \right). \tag{2.8}
\]
Indeed, the first relations are obvious. To prove the last we may assume that $-T \leq \tau_1 \leq \tau_2$. Then the l.h.s. vanishes if $s \neq s'$, while for $s = s'$ it equals
\[
b(s)^2 \mathbb{E} \left( \int_{-T}^{\tau_1} e^{-\gamma_s (\tau_1 - l)} d\beta_s(l_1) \right) \left( \int_{-T}^{\tau_1} e^{-\gamma_s (\tau_2 - l)} d\beta_s(l_2) + \int_{-T}^{\tau_2} e^{-\gamma_s (\tau_2 - l)} d\beta_s(l_2) \right).
\]
The expectation of the second term vanishes, and that of the first equals $2 b(s)^2 \int_{-T}^{\tau_1} e^{-\gamma_s (\tau_1 - l)} d\beta_s(l_1)$, which is the r.h.s. of (2.8) (recall that $d\beta \cdot d\beta = 2dt$).

For the process $a^{(1)}$ we have

**Lemma 2.3.** For any $\tau_1, \tau_2$ and $s', s''$,

(i) $\mathbb{E}a^{(1)}_s(\tau_1) a^{(1)}_s(\tau_2) = 0$;

(ii) $\mathbb{E}a^{(1)}_s(\tau_1) \tilde{a}^{(1)}_s(\tau_2) = 0$ if $s' \neq s''$;

(iii) $\mathbb{E}a^{(1)}_s(\tau_1) a^{(0)}_s(\tau_2) = \mathbb{E}a^{(1)}_s(\tau_1) \tilde{a}^{(0)}_s(\tau_2) = 0$.

**Proof.** Let us verify ii). Due to (2.3), the expectation we examine is a sum over $s_1, s_2$ and $s'_1, s'_2$ of integrals of functions
\[
\mathbb{E} \left( \delta^{l/2}_{3s} (a^{(0)}_1 a^{(0)}_2 \tilde{a}^{(0)}_3)(l) \delta^{l/2}_{3s'} (\tilde{a}^{(0)}_1 \tilde{a}^{(0)}_2 \tilde{a}^{(0)}_3)(l') \right),
\]
multiplied by some density-functions. Since each $a^{(0)}_s$ is a Gaussian process, then Wick’s theorem applies. By (2.8) and due to the the factor $\delta^{l/2}_{3s} - a^{(0)}_1$ must be coupled with $\tilde{a}^{(0)}_1$ or with $a^{(0)}_2$, $a^{(0)}_3$—with $\tilde{a}^{(0)}_2$ or $\tilde{a}^{(0)}_3$, and $\tilde{a}^{(0)}_3$—with $a^{(0)}_3$. So $s' = s_1 + s_2 - s_3 = s'_1 + s'_2 - s'_3 = s''$ and ii) follows. The proof of i) and iii) is similar. □
2.4. Second moments $\mathbb{E}a_s^{(1)}(\tau)\bar{a}_s^{(1)}(\tau)$. In the center of attention of WT are the limiting, as $L \to \infty, \nu \to 0$, correlations of solutions $a_s(\tau)$ and $v_s(\tau)$). Accordingly we should analyse limiting correlations of the processes $a_s^{(n)}(\tau)$. To give an idea what we should expect there, let us consider the second moments of the process $a_s^{(1)}(\tau)$. The tools, needed for this analysis, will be systematically used later.

By the Wick theorem

$$
\sum_{s_1, s_2} \mathbb{E}\left( \delta_{3s}^{12} (a_{12}^{(0)} \bar{a}_{3}^{(0)}) (l_1) (a_{3}^{(0)} \bar{a}_{2}^{(0)} a_{3}^{(0)}) (l_2) e^{i\nu (l_1 a_{3}^{(0)} - l_2 a_{3}^{(0)})} \right) = 2 \prod_{j=1}^{3} \mathbb{E}a_{j}^{(0)}(l)\bar{a}_{j}^{(0)}(l'),
$$

where we used that $\mathbb{E}a_{j}^{(0)}(l)\bar{a}_{j}^{(0)}(l') = \mathbb{E}a_{j}^{(0)}(l)\bar{a}_{j}^{(0)}(l').$ Then, recalling the notation $\Sigma$ introduced in Section 1.4, by (2.8) we get

$$
\mathbb{E}|a_s^{(1)}(\tau)|^2 = 2 \sum_{1,2} \delta_{3s}^{12} \int_{-T}^{T} dl_1 \int_{-T}^{T} dl_2 B_{123}
\times \prod_{j=1}^{3} \left( e^{-\gamma_j l_1 l_2} - e^{-\gamma_j (2T l_1 + l_2)} e^{\gamma_s l_1 l_2 - 2\tau + i\nu^{-1} a_{3}^{(0)} l_1 l_2} \right),
$$

where we denoted

$$
B_{123} = B_1 B_2 B_3, \quad B_r = b(s_r)^2 / \gamma_s, \quad \text{for } r = 1, 2, 3. \quad (2.9)
$$

To simplify the computations, we first assume that $T = \infty$. In this case $\mathbb{E}|a_s^{(1)}(\tau)|^2$ does not depend on $\tau$ and equals $\Sigma_s$, where

$$
\Sigma_s = 2 \sum_{1,2} \delta_{3s}^{12} \int_{-\infty}^{0} dl_1 \int_{-\infty}^{0} dl_2 B_{123} e^{-l_1 l_2} (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_s (l_1 l_2) + i \nu^{-1} a_{3}^{(0)} l_1 l_2). \quad (2.10)
$$

Since for $a, b > 0, c \in \mathbb{R}$ we have

$$
\int_{-\infty}^{0} dl_1 \int_{-\infty}^{0} dl_2 e^{-a l_1 l_2 + b (l_1 + l_2) + i c (l_1 - l_2)} = \frac{a + b}{b((a + b)^2 + c^2)},
$$

then

$$
\Sigma_s = \frac{2 \nu^2}{\gamma_s} \sum_{1,2} \delta_{3s}^{12} B_{123} \frac{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_s}{(a_{3}^{(0)} l_1 l_2)^2 + \nu^2 (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_s)^2}. \quad (2.11)
$$

For the reason of equality (2.11), below we call expressions like those in the r.h.s. of (2.10) “sums”, meaning that they become sums after the explicit integrating over $dl_j$. 

By a similar argument it can be shown that $\mathbb{E}a_s^{(m)}(\tau)\bar{a}_s^{(n)} = 0$ for any $m, n$ and $s', s''$, while $\mathbb{E}a_s^{(m)}\bar{a}_s^{(n)} = 0$ for any $m, n$ and $s' \neq s''$. Moreover, $\mathbb{E}a_s^{(1)}\bar{a}_s^{(0)} = 0$ (while Corollary 2.2 implies only that this expectation is of the size $L^{-d}$), so, as it is claimed in the introduction, $n_s^{(1)} = 0$. We do not prove and do not use these observations.
3. Limiting Behaviour of Second Moments

In this section we study the asymptotic behaviour of the sum $\Sigma_s$ (see (2.10)=(2.11)) as $L \to \infty$ and $\nu \to 0$, assuming that $L \gg \nu^{-1} \gg 1$. The latter inequality will be specified later.

3.1. Approximation of the sums $\Sigma_s$ by integrals. Let us naturally extend $\gamma_s = \gamma^0(|s|^2)$ to a function on $\mathbb{R}^d$ and denote

$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_s =: \Gamma(s_1, s_2, s_3, s), \quad s_j, s \in \mathbb{R}^d.$$  

We also extend $B_s, s \in \mathbb{Z}^d_L$, to the function $B(s) = b(s)^2/\gamma_s, s \in \mathbb{R}^d$, and extend $B_{123}$ to $B(s_1, s_2, s_3) := B(s_1)B(s_2)B(s_3), s_j \in \mathbb{R}^d$. Recalling (1.15) we see that

$$\Sigma_s$$ naturally extends to a function on $\mathbb{R}^d \ni s,$  

both in the form (2.10) and (2.11). Doing that we understand the relation $s_1 + s_2 = s_3 + s$, following from the factor $\delta_{3s}^{12}$ as the rule “substitute $s_3 = s_1 + s_2 - s$”. In this case in (2.10) and (2.11) the indices $s_1, s_2$ belong to $\mathbb{Z}^d_L$, while $s$ and $s_3$ are vectors in $\mathbb{R}^d$. Here and in similar situations below we will keep denoting the extended functions by the same letters.

Considering (2.11) with $s \in \mathbb{R}^d$ we may replace there the sum $\sum_{1,2}$ by the integral over $\mathbb{R}^d \times \mathbb{R}^d$, thus getting the function $I_s$, defined as

$$I_s = \frac{\nu^2}{2\gamma_s} \int_{\mathbb{R}^d \times \mathbb{R}^d} ds_1 ds_2 \frac{\delta_{3s}^{12} B(s_1, s_2, s_3) \Gamma(s_1, s_2, s_3, s)}{((s_1 - s) \cdot (s_2 - s))^2 + \left(\frac{1}{2} \nu \Gamma'(s_1, s_2, s_3, s)\right)^2}.  \tag{3.1}$$

Here $\delta_{3s}^{12}$ is the Kronecker delta of the relation $s_1 + s_2 = s_3 + s$, so the factor $\delta_{3s}^{12}$ in the integrands means that there $s_3 := s_1 + s_2 - s$. By $F_s$ we denoted the positive Schwartz function on $\mathbb{R}^{3d}$

$$F_s(s_1, s_2) = \Gamma(s_1, s_2, s_3, s)B(s_1, s_2, s_3)/2\gamma_s, \quad s_3 := s_1 + s_2 - s. \tag{3.2}$$

Reverting the transformation, used to get (2.11) from (2.10) we find that

$$I_s = 2 \int_{\mathbb{R}^d \times \mathbb{R}^d} ds_1 ds_2 \int_{-T}^{0} dl_1 \int_{-T}^{0} dl_2 \delta_{3s}^{12} B(s_1, s_2, s_3) \times e^{-|l_1-l_2|(|\gamma_1+\gamma_2+\gamma_3|+|\gamma_s|l_1+l_2)+|l_1-l_2|} \tag{3.3}$$

Our goal in this section is to estimate the difference between the sum $\Sigma_s$ and the integral $I_s$, while in the next section we will study asymptotical behaviour of the latter as $\nu \to 0$. Moreover, we consider a bit more general sums, needed in the work [8].

We will study sums with the summation index $(s_1, \ldots, s_k) =: z \in \mathbb{Z}^{kd}_L, k \geq 1$, and integrals with the integrating variable $z = (s_1, \ldots, s_k) \in \mathbb{R}^{kd}$. For $s \in \mathbb{R}^d$ let us
consider the union $D_s = \bigcup_{j=1}^p D_s^j$ of $p \geq 0$ affine subspaces $D_s^j$ in $\mathbb{R}^{kd}$ (if $p = 0$ then $D_s$ is empty),

$$D_s^j = \{ z = (s_1, \ldots, s_k) \in \mathbb{R}^{kd} : c_0^j s + \sum_{i=1}^k c_i^j s_i = 0 \in \mathbb{R}^d \},$$

where $c_i^j$ are some real numbers. We assume that the $k$-vectors $(c_1^j, \ldots, c_k^j) \neq 0 \in \mathbb{R}^k$ for all $j$, so the subspaces $D_s^j$ have dimension $d(k-1)$ for every $s$. Consider the sum/integral

$$S_s = \int_{\mathbb{R}^m} \sum_{z \in \mathbb{Z}_L^{kd} \setminus D_s} G_s(z, \theta; \nu) \, d\theta, \quad m \in \mathbb{N} \cup \{0\}, \quad s \in \mathbb{R}^d,$$

(if $m = 0$, then there is no integration over $\mathbb{R}^m$), where $G_s$ is a measurable function of $(z, \theta, \nu) \in \mathbb{R}^{kd} \times \mathbb{R}^m \times (0, 1/2]$, $C^2$-smooth in $z$ and satisfying

$$|\partial_z^\alpha G_s(z, \theta; \nu)| \leq v^{-|\alpha|} C^#(s) C^#(z) C^#(\theta) \quad \text{if} \quad 0 \leq |\alpha| \leq 2, \quad (3.5)$$

for all values of the arguments. Our goal is to compare $S_s$ with the integral

$$J_s = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{kd}} G_s(z, \theta; \nu) \, dz \, d\theta, \quad s \in \mathbb{R}^d.$$

**Theorem 3.1.** Under the assumption $(3.5)$,

$$|S_s - J_s| \leq C^#(s) v^{-2} L^{-2}, \quad s \in \mathbb{R}^d. \quad (3.6)$$

Theorem 3.1 applies to the sum $\Sigma_s$ where, due to the factor $\delta'$, we take the summation over $s_1, s_2 \neq s$. Indeed, in this case $D_s = D_s^1 \cup D_s^2$ with $D_s^1 = \{(s_1, s_2) : s - s_j = 0\}$.

**Proof.** Denote by $\hat{S}_s$ the sum $S_s$, where $D_s$ is replaced by the empty set. Firstly we claim that

$$|S_s - \hat{S}_s| \leq C^#(s) L^{-d} \leq C^#(s) L^{-2}, \quad (3.7)$$

where in the last inequality we used that $d \geq 2$. Indeed, due to $(3.5)$ with $\alpha = 0$, $|S_s - \hat{S}_s| \leq C^#(s) L^{-kd} \sum_{z \in D_s \cap \mathbb{Z}_L^{kd}} C^#(z)$, and the claim follows from the fact that the affine subspaces $D_s^j$ have dimension $d(k-1)$.

Now we consider a mesh in $\mathbb{R}^d$, formed by cubes of size $L^{-1}$, centred at the points of the lattice $\mathbb{Z}_L^d$. For any $l \in \mathbb{Z}_L^d$ denote by $m(l)$ the cell of the mesh with the centre in $l$, and consider the measurable mapping

$$\Pi : \mathbb{R}^d \mapsto \mathbb{Z}_L^d, \quad \Pi(x) = \begin{cases} l, & \text{if } x \text{ is interior of } m(l), \ l \in \mathbb{Z}_L^d, \\ 0, & \text{if } x \in \partial m(l) \text{ for some } l' \in \mathbb{Z}_L^d. \end{cases}$$

Let $\Pi^k = \Pi \times \ldots \times \Pi$, where the product is taken $k$ times. Then

$$\hat{S}_s = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{kd}} G_s(z, \theta; \nu) \circ (\Pi^k \times \text{id}) \, dz \, d\theta.$$

Setting $G_s^\Delta = G_s - G_s \circ (\Pi^k \times \text{id})$, we see that

$$J_s - \hat{S}_s = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{kd}} G_s^\Delta(z, \theta; \nu) \, dz \, d\theta.$$
For any \( l = (l_1, \ldots, l_k) \in \mathbb{Z}^d \) let us denote \( m(l) = m(l_1) \times \ldots \times m(l_k) \), restrict \( G_s^\Delta \) to the cell \( m(l) \) and write it as

\[
G_s^\Delta(z, \theta) = \partial_z G_{s}(l, \theta) \cdot (z - l) + T_s(z, \theta), \quad z \in m(l).
\]

Here \( |T_s(z, \theta)| \leq CL^{-2} sup_{\xi \in m(l)} |\partial_\xi^2 G_s(\xi, \theta)| \) for \( z \in m(l) \). Then

\[
\left| \int_{m(l)} G_s^\Delta(z, \theta) dz \right| \leq L^{-kd} L^{-2} \nu^{-2} C^\#(s) C^\#(l) C^\#(\theta),
\]

and accordingly

\[
|J_s - \hat{S}_s| \leq \nu^{-2} L^{-2} C^\#(s) \int_{\mathbb{R}^m} \int_{\mathbb{R}^{kd}} C^\#(z) C^\#(\theta) dz d\theta \leq C^\#(s) \nu^{-2} L^{-2}.
\] (3.8)

The theorem is proved. \( \square \)

**Remark 3.2.** Theorem 3.1 remains true for \( d = 1 \) if we replace (3.6) by the weaker estimate

\[
|S_s - J_s| \leq C^\#(s) (\nu^{-2} L^{-2} + L^{-1}), \quad s \in \mathbb{R}^d.
\]

Indeed, in the proof of the theorem relation \( d \geq 2 \) was used only once, to get the second inequality in (3.7).

3.2. Limiting behaviour of the integrals \( I_s \). Here we study the integral \( I_s \), written in the form (3.2), when \( \nu \to 0 \). With the notation \( \Gamma_s(s_1, s_2) = \frac{1}{2} \Gamma(s_1, s_2, s_1 + s_2 - s, s) \) the integral takes the form

\[
I_s = \nu^2 \int_{\mathbb{R}^d} ds_1 ds_2 \frac{F_s(s_1, s_2)}{((s_1 - s) \cdot (s_2 - s))^2 + (\nu \Gamma_s(s_1, s_2))^2}.
\]

We study its asymptotical behaviour when \( \nu \to 0 \) in an abstract setting and do not use the explicit forms of the functions \( F_s \) and \( \Gamma_s \). Instead we assume that they are \( C^2 \)-smooth and \( C^3 \)-smooth real functions, correspondingly, satisfying certain restrictions on their behaviour at infinity. Namely, it suffices to assume that

\[
|\partial_{x_1, x_2, s}^\alpha F_s(s_1, s_2)| \leq C^\#(s, s_1, s_2) \quad \forall s_1, s_2, s \in \mathbb{R}^d, \quad \forall |\alpha| \leq 2,
\] (3.9)

and, for some real numbers \( r_1 \geq 0 \) and \( K > 0 \),

\[
|\Gamma_s(s_1, s_2)| \geq K^{-1}, \quad |\partial_{x_1, x_2, s}^\alpha \Gamma_s(s_1, s_2)| \leq K((s_1, s_2, s))^{r_1 - |\alpha|}
\] (3.10)

for all \( s_1, s_2, s \) and all \( |\alpha| \leq 3 \).

Note that function \( F_s \) from (3.3) satisfies (3.9), while function \( \Gamma_s = \frac{1}{2} \Gamma(s_1, s_2, s_1 + s_2 - s, s) \) satisfies (3.10) with \( r_1 = \max(2r_a, 3) \), as well as the functions \( \Gamma_s(s_1, s_2) = \gamma^0(|s_1|^2), i = 1, 2, 3, 4 \), where \( s_3 = s_1 + s_2 - s \) and \( s_4 = s \).

In the theorem below we denote by \( ^s \Sigma \) the quadric

\[
^s \Sigma = \{(s_1, s_2) : \omega_{s_3}^{1/2} = 0\}, \quad s_3 = s_1 + s_2 - s.
\] (3.11)

The latter has a singularity at the point \( (s, s) \), and we denote by \( ^s \Sigma_0 \) its smooth part \( ^s \Sigma_0 = ^s \Sigma \setminus \{(s, s)\} \).
Theorem 3.3. As $\nu \to 0$, the integral $I_s, s \in \mathbb{R}^d$, may be written as

$$I_s = \nu I_s^0 + \nu^2 I_s^\Delta,$$

(3.12)

$$I_s^0 = \pi \int_{\Sigma_s} \frac{F_s(s_1, s_2)}{\sqrt{|s - s_1|^2 + |s - s_2|^2} \Gamma_s(s_1, s_2)} \, ds_1 \, ds_2 \, |\Sigma_s|. \quad (3.13)$$

Here $ds_1 ds_2 |\Sigma_s$ is the volume element on $s\Sigma_s$, induced from $\mathbb{R}^{2d}$, and the integral for $I_s^0$ converges absolutely. The functions $I_s^0$ and $I_s^\Delta$ satisfy the estimates

$$|I_s^0| \leq C\#(s), \quad |I_s^\Delta| \leq C\#(s; \aleph_d) \nu^{-\aleph_d},$$

where $\aleph_d = 0$ if $d \geq 3$, while for $d = 2$, $\aleph_d$ is any positive number.

The theorem is proved in Section 9. If $F_1$ and $\Gamma_s$ do not depend on $s$, then the theorem holds under related (but milder) restrictions on $F$ and $\Gamma$, and in that case $|I_s^\Delta| \leq C\chi_d(\nu)$, where $\chi_d$ is defined in (1.43), see [19].

Theorem 3.3 implies that

$$|I_s| \leq \nu C\#(s). \quad (3.14)$$

In Appendix 12.3 we show that this inequality may be obtained easier and under weaker restrictions on the functions $F_1$ and $\Gamma_s$. This observation is important since later in the text we use various generalisations of inequality (3.14) in situations, where analogies of the asymptotic expansion (3.12) are not known for us.

Applying Theorems 3.1 and 3.3 to the sum $E|a_s^{(1)}(\tau)|^2 = \Sigma_s$ in (2.11) and recalling that $\Sigma_s$ was extended to a function on $\mathbb{R}^d$, we get

Corollary 3.4. Assume $T = \infty$. Then for $s \in \mathbb{R}^d$,

$$\left| E|a_s^{(1)}(\tau)|^2 - \nu \frac{\pi}{\gamma_s} \int_{\Sigma_s} \frac{B(s_1, s_2, s_1 + s_2 - s)}{\sqrt{|s_1 - s|^2 + |s_2 - s|^2}} \, ds_1 \, ds_2 \, |\Sigma_s| \right|$$

$$\leq C\#(s; \aleph_d)(L^{-2}v^{-2} + \nu^{2-\aleph_d}).$$

It is convenient to pass in (3.13) from the variables $(s_1, s_2, s)$ to

$$(x, y, s) = (s_1 - s, s_2 - s, s), \quad (x, y) =: z. \quad (3.15)$$

Then the quadric $s\Sigma$ becomes

$$\Sigma = \{z : x \cdot y = 0\} \subset \mathbb{R}_x^d \times \mathbb{R}_y^d = \mathbb{R}_z^{2d}.$$

The locus of $\Sigma$ is the point $(0, 0)$, and the regular part is $\Sigma_* = \{(x, y) \neq 0 : x \cdot y = 0\}$. Now we write the integral $I_s^0$ as

$$\int_{\Sigma_*} f(z)|z|^{-1} \, dz \, |\Sigma_*|, \quad (3.16)$$

where $f = \pi F_s / \Gamma_s$.

Integrals of the form (3.16) are important for what follows. In next section we discuss some their properties.
3.3. Integrals (3.16). Let us extend the measure \(|z|^{-1} dz|\Sigma_x|\) to a measure \(\mu^x\) on \(\Sigma,\) where \(\mu^x(\{0\}) = 0,\) and next extend \(\mu^x\) to a Borel measure on \(\mathbb{R}^d,\) supported by \(\Sigma,\) keeping for the latter the same name. Then the integrals (3.16) may be written as \(\int_{\Sigma_x} f(z) \mu^x(dz),\) or as \(\int_{\Sigma} f(z) \mu^x(dz),\) or as \(\int_{\mathbb{R}^d} f(z) \mu^x(dz).\)

For any real number \(r\) let \(C_x(\mathbb{R}^d)\) be the space of continuous complex functions on \(\mathbb{R}^d\) with the finite norm
\[
|f|_r = \sup_{z} |f(z)|r^r.
\]

(3.17)

**Proposition 3.5. Integral (3.16) as a function of \(f\) defines a continuous linear functional on the space \(C_x(\mathbb{R}^d)\) if \(r > 2d - 1.\)**

The proposition is proved in Section 9.9. To study further the measure \(\mu^x\) we consider the projection
\[
\Pi : \Sigma_x \to \mathbb{R}^d, \quad z = (x, y) \mapsto x.
\]
(3.18)

It defines a fibering of \(\Sigma_x,\) where the fiber \(\Pi^{-1}0 = \{0\} \times (\mathbb{R}^d \setminus \{0\})\) is singular, while for any non-zero \(x\) the fiber \(\Pi^{-1}x\) equals \(\{x\} \times x^\perp,\) where \(x^\perp\) is the orthogonal complement to \(x\) in \(\mathbb{R}^d.\) So the restriction of \(\Pi\) to the domain \(\Sigma^x = \Sigma \setminus \{(0) \times \mathbb{R}^d\}\) is a smooth euclidean vector bundle over \(\mathbb{R}^d \setminus \{0\}.\)

Let us us abbreviate to \(\mu\) the volume element \(dz|\Sigma_x.\) Since the measure \(\mu^x|\Sigma_x\) is absolutely continuous with respect to \(\mu,\) then \(\mu^x(\Sigma_x \setminus \Sigma^x) = \mu(\Sigma_x \setminus \Sigma^x) = 0;\) so to calculate the integrals (3.16) it suffices to know the restriction of \(\mu^x\) to \(\Sigma^x.\) The result below shows how to disintegrate the measures \(\mu|\Sigma_x\) and \(\mu^x|\Sigma_x\) with respect to \(\Pi,\) and allows to integrate explicitly over them.

**Theorem 3.6. The measures \(\mu|\Sigma_x\) and \(\mu^x|\Sigma_x\) disintegrate as follows:**

\[
\mu(dz) = |x|^{-1} dx|z| d_{x^\perp} y,
\]

(3.19)

\[
\mu^x(dz) = |x|^{-1} dx d_{x^\perp} y,
\]

(3.20)

where \(d_{x^\perp} y\) is the volume element on the space \(x^\perp\) (the orthogonal complement to \(x\) in \(\mathbb{R}^d_y).\)

We recall that equality (3.19) means that for any continuous function \(f\) on \(\Sigma^x\) with compact support
\[
\int_{\Sigma^x} f(z) \mu(dz) = \int_{\mathbb{R}^d \setminus \{0\}} |x|^{-1} dx \int_{x^\perp} f(z) |z| d_{x^\perp} y.
\]

(3.21)

**Proof.** It suffices to verify (3.21) for all continuous functions \(f,\) supported by a compact set \(K,\) for every \(K \subseteq (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d.\) For \(x' \in \mathbb{R}^d \setminus \{0\}\) and \(m \in \mathbb{N}\) we denote \(r' = |x'| > 0\) and set \(U_{x'} = \{x : |x - x'| < \frac{1}{2} r'\}\) and \(U^m = \{y : |y| < m\}.\) Since any \(K\) as above can be covered by a finite system of domains \(U_{x'} \times U^m,\) it suffices to prove (3.21) for any set \(U_{x'} \times U^m \subseteq U\) and any \(f \in C_0(U),\) where \(C_0(U)\) is the space of continuous compactly supported functions on \(U.\)

Now we construct explicitly a trivialisation of the linear bundle \(\Pi\) over \(U_{x'}.\) To do this we fix in \(\mathbb{R}^d\) a coordinate system such that

\[
x_1 \geq \kappa > 0 \quad \text{for any} \quad x = (x_1, x_2, \ldots, x_d) =: (x_1, \bar{x}) \in U_{x'}.
\]

(3.22)
We denote $Q = \Pi^{-1}U_{x'} \subset \Sigma^x$ and construct a linear in the second argument $\hat{\eta}$ coordinate mapping $\Phi : U_{x'} \times \mathbb{R}^{d-1} \to Q$ of the form

$$\Phi(x, \hat{\eta}) = (x, \Phi_x(\hat{\eta})), \quad \Phi_x(\hat{\eta}) = (\varphi(x, \hat{\eta}), \hat{\eta}).$$

The function $\varphi$ should be such that $\Phi_x(\hat{\eta}) \in x^\perp$. That is, it should satisfy $x \cdot \Phi_x(\hat{\eta}) = x_1 \varphi + \hat{x} \cdot \hat{\eta} = 0$. From here we find that $\varphi = -\hat{x}/x_1$. Thus obtained mapping $\Phi_x$ is linear in $\hat{\eta}$, and the image of $\Phi$ is the set $Q$. In the coordinates $(x, \hat{\eta}) \in U_{x'} \times \mathbb{R}^{d-1}$ the hypersurface $\Sigma^x$ is embedded in $\mathbb{R}^{2d}$ as a graph of the function $\varphi(x, \hat{\eta})$. Accordingly in these coordinates the volume element $\mu$ on $\Sigma^x$ reads $\mu = \tilde{p}(x, \hat{\eta})dx d\hat{\eta}$, where

$$\tilde{p}(x, \hat{\eta}) = (1 + |\partial \varphi(x, \hat{\eta})|^2)^{1/2} = (1 + x_1^{-2}((x_1^{-1}\hat{x} \cdot \hat{\eta})^2 + |\hat{\eta}|^2 + |\hat{x}|^2))^{1/2}.$$

So

$$\int_{U \subset Q} f(z)\mu(dz) = \int_{U_{x'}} \left( \int_{\mathbb{R}^{d-1}} f(x, \Phi_x(\hat{\eta})) \tilde{p}(x, \hat{\eta}) d\hat{\eta} \right) dx.$$

Passing from the variable $\hat{\eta}$ to $y = \Phi_x(\hat{\eta}) \in x^\perp$ we write the measure $\tilde{p}(x, \hat{\eta})d\hat{\eta}$ as $p(z)dx dy$ with

$$p(z) = p(x, y) = \tilde{p}(x, \Phi_x^{-1} y)|\det \Phi_x|^{-1}.$$

Then

$$\int_{U} f(z)\mu(dz) = \int_{U_{x'}} \left( \int_{U \cap \Sigma^x} f(z)p(z)dx dy \right) dx. \tag{23.32}$$

The smooth function $p$ in the integral above is defined on $U \cap \Sigma^x$ in a unique way and does not depend on the trivialisation of $\Pi$ over $U_{x'}$, used to obtain it. Indeed, if $p_1(z)$ is another smooth function on $U \cap \Sigma^x$ such that (23.32) holds with $p := p_1$, then

$$\int dx \int_{x^\perp} f(z)(p(z) - p_1(z))dx dy = 0 \quad \forall f \in C_0(U),$$

which obviously implies that $p = p_1$. To establish (3.21) it remains to verify that in (23.32)

$$p(x_*, y_*) = |x_*|^{-1}|z_*| \quad \forall z_* = (x_*, y_*) \in U \cap \Sigma^x. \tag{23.24}$$

To prove this equality let us choose in $\mathbb{R}^d$ euclidean coordinates with the first basis vector $e_1 = x_*/R_x$, $R_x = |x_*| \geq 1/2$. In these coordinates condition (3.22) holds, $x_* = (R_x, 0)$ and $y_* = (0, \hat{\eta}_*)$, $\hat{\eta}_* \in \mathbb{R}^{d-1}$. Then

$$\tilde{p}(x_*, \Phi_x^{-1} y_*) = \tilde{p}(x_*, \hat{\eta}_*) = R_x^{-1}(R_x^2 + |\hat{\eta}_*|^2)^{1/2} = |x_*|^{-1}|z_*|,$$

and (3.24) follows since $\det \Phi_x = 1$. This proves (3.19). Relation (3.20) follows from (3.19) and the definition of the measure $\mu_{\Sigma_x}$. □

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5 Indeed, denoting $\xi = (x, \hat{\eta})$ we see that the first fundamental form $I^\xi$ of $\Sigma^x$ is given by $I^\xi_{ij} = (\delta_{i,j} + \theta_i \theta_j)$, where $\theta = \partial_{\xi} \varphi \in \mathbb{R}^{2d-1}$. So for $X \in \mathbb{R}^{2d-1}$,

$$I^\xi(X, X) = \sum_{i,j} X_i^2 + \sum_{i,j} X_i \theta_i X_j \theta_j = \sum_i X_i^2 + (X \cdot \theta)^2.$$

Choosing in $\mathbb{R}^{2d-1}$ a coordinate system with the first basis vector $\theta/|\theta|$ we find that $I^\xi(X, X) = X_1^2(1 + |\theta|^2) + \sum_{j \geq 2} X_j^2$. So $\det I^\xi = 1 + |\partial_{\xi} \varphi|^2$, which implies the formula for the density $\tilde{p}$. 
4.1. Wave kinetic integrals. Let us calculate the wave kinetic integral 
$$\int_\Sigma \frac{1}{|x|^\delta} \, dz$$
where now the variable $z = (x, y)$, $x$ is restricted to the domain $\Sigma^y \setminus \{0\}$, and $y \in \mathbb{R}^d \setminus \{0\}$.

Example 3.7. Let us calculate the wave kinetic integrals of the balls $B_R^d = \{|z| \leq R\}$. Denoting by $A_n$ (by $V_n$) the area of the unit sphere (the volume of the unit ball) in $\mathbb{R}^n$, we have
$$\mu^\Sigma(B_R^d) = \int_{|x| \leq R} \frac{1}{|x|} \int_{|y| \leq \sqrt{R^2 - |x|^2}} \frac{1}{r} \, A_d \, V_d \, V_{d-1} \, r^{d-2} \, (R^2 - r^2)^{(d-1)/2} \, dr.$$
If $d = 3$ this equals $A_3 \, V_2 \, \frac{1}{4} \, R^4 = \pi^2 \, R^4$. If $d = 2$, this equals $A_2 \, V_1 \, \int_0^R \sqrt{R^2 - r^2} \, dr = \pi^2 \, R^2$.

4. Wave Kinetic Integrals and Equations

4.1. Wave kinetic integrals. For a complex function $v(s)$, $s \in \mathbb{R}^d$, and $s_1, s_2, s_3, s_4 \in \mathbb{R}^d$ with $s_4 = s$ we denote $v_j = v(s_j)$, $j = 1, 2, 3, 4$. In this section we study the wave kinetic integral $(K v)(s)$, defined as follows:

$$\begin{align*}
(K v)(s) &= 2\pi \int_{\Sigma_s} \frac{ds_1 \, ds_2 \, ds_3}{\sqrt{|s_1 - s|^2 + |s_2 - s|^2}} \, \delta_{\Sigma_2}^{12} \left( v_1 v_2 v_3 v_4 \right) \\
&= \left( \frac{1}{v_4} + \frac{1}{v_3} - \frac{1}{v_2} - \frac{1}{v_1} \right),
\end{align*}$$

where $\Sigma_s$ is the quadric (3.11) without the singular point $(s, s)$, and $s_4 = s$. Passing to the variable $z = (x, y) = (s_1 - s, s_2 - s)$ we write $K$ as an integral over $\Sigma_s$ with respect to the measure $\mu^\Sigma = |z|^{-1} \, dz$. (see (3.16)):

$$\begin{align*}
(K v)(s) &= 2\pi \int_{\Sigma_s} \frac{d\mu^\Sigma(z)}{\Sigma_s} \left( v_1 v_2 v_3 v_4 \right) (z) \\
&= : K_4(s) + K_3(s) + K_2(s) + K_1(s),
\end{align*}$$

where now $v_1, v_2, v_3$ should be written as functions of $z$ and $s_4 = s$ (note the minus-signs for $K_2$ and $K_1$).

Now evoking Theorem 3.6 we will show that the wave kinetic integral $K$ defines 1-smoothing continuous operators in the complex spaces $C_r(\mathbb{R}^d)$ (see (3.17)) with $r$ not too small.

Theorem 4.1. If $v \in C_r(\mathbb{R}^d)$ with $r > d$, then $K(v) \in C_{r+1}(\mathbb{R}^d)$ and

$$|K(v)|_{r+1} \leq C_r |v|^3,$$

where $C_r$ is an absolute constant.

That is, the kinetic integral defines a continuous complex-homogeneous mapping of third degree $K : C_r(\mathbb{R}^d) \to C_{r+1}(\mathbb{R}^d)$ if $r > d$. We will derive the theorem’s assertion from an auxiliary lemma, stated below and proved later in Section 11.1.

For $l = 1, \ldots, 4$ let $J_l(u^1, u^2, u^3, u^4)$ be the complex poly-linear operator of the third order, which does not depend on $u^l$ and sends the quadruple of complex functions
Let $u_j$, $j = 1, \ldots, 4$, be the functions.

\begin{equation}
\begin{split}
    v_1 &:= u_1(x + s), \quad v_2 := u_2(y + s), \quad v_3 := u_3(x + y + s), \quad v_4 := u_4(s),
\end{split}
\end{equation}

in accordance to the relation between the coordinates $(s_1, s_2, s_3, s_4)$ and $(x, y)$. That is,

\begin{equation}
    J_l(u^1, u^2, u^3, u^4)(s) = \int_{\Sigma_z} d\mu^\Sigma(z) \prod_{1 \leq i \leq 4, i \neq l} u^i, \quad l = 1, \ldots, 4,
\end{equation}

where $u^1, \ldots, u^4$ depend on the argument $z = (x, y)$ as in (4.4). Then for $l = 1, \ldots, 4$

\begin{equation}
    (K_l v)(s) = 2\pi \sigma_l J_l(v, v, v, v)(s),
\end{equation}

where $\sigma_1, \sigma_2 := -1$ and $\sigma_3, \sigma_4 := 1$. Theorem 4.1 is an easy consequence of the following assertion:

**Lemma 4.2.** Let $u_1, \ldots, u_4 \in C_r(\mathbb{R}^d)$ where $r > d$. Then for $s \in \mathbb{R}^d$ and $1 \leq l \leq 4$ the integral, defining $J_l(u^1, u^2, u^3, u^4)(s) =: J_l(s)$ converges absolutely and satisfies

\begin{equation}
    |J_l|_{r+1} \leq C_r \prod_{j \neq l} |u^j|_r.
\end{equation}

Theorem 4.1 is an easy consequence of the following assertion:

**Corollary 4.3.** If $v_1, v_2 \in C_r(\mathbb{R}^d)$ are such that $|v^1|_r, |v^2|_r \leq R$, where $r > d$, then

\begin{equation}
    |K_l(v^1) - K_l(v^2)|_{r+1} \leq C_r R^2 |v^1 - v^2|_r, \quad l = 1, \ldots, 4.
\end{equation}

4.2. Wave kinetic equations. Now we pass to the main topic of this section—the wave kinetic equation:
where $0 < \varepsilon \leq 1$, $K(u)(\tau, s) = K(u(\tau, \cdot))(s)$ is the wave kinetic integral (4.1) and $\mathcal{L}$ is the linear operator
\[ (\mathcal{L}u)(s) = 2\gamma_s u(s), \quad s \in \mathbb{R}^d. \] (4.11)

This operator defines in the spaces $C_r$ semigroups of contractions:
\[ \| \exp(-t\mathcal{L}) \|_{C_r(\mathbb{R}^d)}, C_r(\mathbb{R}^d) \leq \exp(-2t), \quad \forall t \geq 0, \forall r. \] (4.12)

We denote by $X_r$ the space of continuous curves $u : [0, \infty) \to C_r(\mathbb{R}^d)$, given the uniform norm $\| u \|_r = \sup_{t \geq 0} |u(t)|_r$.

**Theorem 4.4.** If $r > d$, then

1. for any $u_0 \in C_r(\mathbb{R}^d)$, $f \in X_r$ and any $\varepsilon$ the problem (4.9), (4.10) has at most one solution in $X_r$.

2. If
\[ |u_0|_r \leq C_*, \quad \| f \|_r \leq C_\ast \] (4.13)
for some constant $C_\ast$, then there exist positive constants $\varepsilon_\ast = \varepsilon_\ast(C_\ast, r)$ and $R = R(C_\ast, r)$ such that if $0 < \varepsilon \leq \varepsilon_\ast$, then the problem (4.9), (4.10) has a unique solution $u \in X_r$, and $\| u \|_r \leq R$. Moreover, if $(u_{01}, f_1)$ and $(u_{02}, f_2)$ are two sets of initial data, satisfying (4.13), and $u^1, u^2$ are the corresponding solutions, then
\[ \| u^1 - u^2 \|_r \leq C(r)(\| u_{01} - u_{02} \|_r + \| f_1 - f_2 \|_r). \] (4.14)

The first assertion is obvious in view of the contraction property (4.12), since the mapping $K$ is locally Lipschitz by Corollary 4.3, cf. the proof of Proposition 4.6 below. The second result follows elementary from Theorem 4.1. Details are given in Section 11.2.

Let $u^0(\tau, s)$ solves (4.9), (4.10) with $\varepsilon := 0$. Writing the solution $u$, constructed in Theorem 4.4, as $u^0 + \varepsilon v$ we get for $v$ the equation $\dot{v} = -\mathcal{L}v + K(u^0 + \varepsilon v)$, $v(0) = 0$. So $\| v \|_r \leq C(C_\ast, r)$ and Corollary 4.3 implies that $K(u^0 + \varepsilon v) = K(u^0) + O(\varepsilon)$. Accordingly, $v = u^1 + O(\varepsilon)$, where
\[ \dot{u}^1 = -\mathcal{L}u^1 + K(u^0), \quad u^1(0) = 0. \]

We have seen that the solution $u(\tau, s)$, built in Theorem 4.4, may be written as
\[ u(\tau, s) = u^0(\tau, s) + \varepsilon u^1(\tau, s) + O(\varepsilon^2). \] (4.15)

where $u^0$ and $u^1$ are defined above.

Let us fix any $r_0 > d$ and denote $\varepsilon(f) = \varepsilon_\ast(\| f \|_{r_0}, r_0)$.

**Corollary 4.5.** Let $u_0 = 0$, $f \in X_r \forall r$ and $0 \leq \varepsilon \leq \varepsilon(f)$. Then the problem (4.9), (4.10) has a unique solution $u$ such that $u \in X_r \forall r$. Its norms $\| u \|_r$ are bounded by constants, depending only on $r$ and $\| f \|_r$.

**Proof.** By the theorem the problem has a unique solution $u \in X_r$ with $r = r_0$. By Corollary 4.3, $K(u) \in X_{r+1}$ and we get from the equation (4.9) that also $u \in X_{r+1}$. Iterating this argument we see that $u \in \cap X_r$. The second assertion follows from the theorem. □
Now let us assume that in (4.9) the function \( f \) does not depend on \( \tau \), so \( f(\tau, s) = f_s \), where

\[
f \in C_r, \quad r > d. \tag{4.16}
\]

Then for \( \epsilon = 0 \) the only steady state of (4.9), i.e. a solution of the equation \(-L u + f = 0\), is \( u^0 = L^{-1} f \); it is asymptotically stable. By the implicit function theorem, there exists \( \epsilon' \) such that for \( 0 < \epsilon \leq \epsilon' \), equation (4.9) has a unique steady state \( u^\epsilon \), close to \( u^0 \), \(-L u^\epsilon + \epsilon K(u^\epsilon) + f = 0\), and \( |u^\epsilon - u^0|_r \leq C_r \epsilon \).

**Proposition 4.6.** Let \( f \) and \( r \) be as in (4.16) and (4.13) holds for some \( C_* \). Let \( \epsilon \leq \epsilon_*(C_*, r) \) and \( u(\tau) \) be a solution of (4.9), (4.10). Then there exists \( \epsilon' = \epsilon'(C_*, r) \) such that if \( 0 < \epsilon \leq \min(\epsilon_*, \epsilon') \), then

\[
|u(\tau) - u^\epsilon|_r \leq |u_0 - u^\epsilon|_r e^{-\tau} \text{ for } \tau \geq 0.
\]

**Proof.** Denoting \( w(\tau) = u(\tau) - u^\epsilon \), we find

\[
w(\tau) = e^{-Lt}w(0) + \epsilon \int_0^\tau e^{-(\tau-t)L} (K(u(t)) - K(u^\epsilon)) dt.
\]

Since \( |u^0|_r \leq |f|_r \leq C_* \), then we may assume that \( |u^\epsilon|_r \leq 2C_* \) since \( \epsilon \) is sufficiently small. Moreover, by Theorem 4.4, \( |u(\tau)|_r \leq R(C_*, r) \) for all \( \tau \geq 0 \). In particular, this implies that \( |w(0)|_r \leq C(C_*, r) \). Then, in view of Corollary 4.3 and (4.12), we find

\[
|w(\tau)|_r \leq e^{-2\tau} |w(0)|_r + \epsilon C_r R^2 \int_0^\tau e^{-2(\tau-t)} |w(t)|_r dt.
\]

This relation and Gronwall’s lemma, applied to the the function \( e^{2\tau}|w(t)|_r \), imply that

\[
|w(\tau)|_r \leq |w(0)|_r e^{-(2-\epsilon C_r R^2) \tau}.
\]

Choosing \( \epsilon \) so small that \( \epsilon C_r R(C_*, r)^2 \leq 1 \) we obtain the desired estimate. \( \Box \)

5. **Quasisolutions**

In this section we start to study quasisolutions \( A(\tau) = A(\tau; \nu, L) \) of eq. (2.1) (where \( r_* > 0 \)), which are second order truncations of series (2.2):

\[
A(\tau) = (A_s(\tau), s \in \mathbb{Z}_L^d), \quad A_s(\tau) = a_s^{(0)}(\tau) + \rho a_s^{(1)}(\tau) + \rho^2 a_s^{(2)}(\tau). \tag{5.1}
\]

Our main goal is to examine the energy spectrum of \( A_s \),

\[
n_s(\tau) = n_s(\tau; \nu, L) = \mathbb{E}|A_s(\tau)|^2, \quad s \in \mathbb{Z}_L^d,
\]

when \( L \) is large and \( \nu \) is small and to show that \( n_s(\tau) \) approximately satisfies the wave kinetic equation (WKE) (1.35). The energy spectrum \( n_s(\tau) \) is a polynomial in \( \rho \) of degree four,

\[
n_s = n_s^{(0)} + \rho n_s^{(1)} + \rho^2 n_s^{(2)} + \rho^3 n_s^{(3)} + \rho^4 n_s^{(4)}, \quad s \in \mathbb{Z}_L^d, \tag{5.2}
\]

where \( n_s^{(k)}(\tau) = n_s^{(k)}(\tau; \nu, L) \),

\[
n_s^{(k)} = \sum_{k_1 + k_2 = k, k_1, k_2 \leq 2} \mathbb{E}a_s^{(k_1)} a_s^{(k_2)'}, \quad 0 \leq k \leq 4.
\]
The term \( n_s^{(0)} \) is given by (1.28) while by Lemma 2.3,
\[
n_s^{(1)} = 2n!\mathbb{E}\tilde{a}_s^{(0)} a_s^{(1)} = 0. \tag{5.3}
\]
Writing explicitly \( n_s^{(i)} \) with \( 2 \leq i \leq 4 \), we find that
\[
n_s^{(2)} = \mathbb{E} \left( |a_s^{(1)}|^2 + 2n!\tilde{a}_s^{(0)} a_s^{(2)} \right), \quad n_s^{(3)} = 2n!\mathbb{E}\tilde{a}_s^{(1)} a_s^{(2)}, \quad n_s^{(4)} = \mathbb{E}|a_s^{(2)}|^2. \tag{5.4}
\]
We decompose
\[
n_s = n_s^{\leq 2} + n_s^{\geq 3},
\]
where
\[
n_s^{\leq 2} = n_s^{(0)} + \rho^2 n_s^{(2)} \quad \text{and} \quad n_s^{\geq 3} = \rho^3 n_s^{(3)} + \rho^4 n_s^{(4)}.
\]
Let us extend \( n_s^{(0)} \) to the Schwartz function on \( s \in \mathbb{R}^d \) given by (1.28). Iterating formula (2.4) we write \( a_s^{(n)}(\tau), n \in \mathbb{N} \), as an iterative integral of polynomials of \( a_s^{(0)}(\tau') \), \( s' \in \mathbb{Z}_L^d, \tau' \leq \tau \). Since each \( a_s^{(0)}(\tau') \) is a Gaussian random variable (1.21), the Wick formula applies to every term \( \mathbb{E}a_s^{(k_1)} \tilde{a}_s^{(k_2)} \) and we see that
\[
\text{for any } 0 \leq k_1, k_2 \leq 2 \text{ and any } v, L \text{ the second moment } \mathbb{E}a_s^{(k_1)} \tilde{a}_s^{(k_2)}
\]
naturally extends to a Schwartz function of \( s \in \mathbb{R}^d \), (5.5) cf. (3.1).

The function \( n_s^{(0)} \) is of order one, and \( n_s^{(1)} \equiv 0 \) by (5.3). Consider the function \( \mathbb{R}^d \ni s \mapsto n_s^{(2)}(\tau) \). It is made by two terms (see (5.4)). By Corollary 3.4 the first may be written as
\[
\mathbb{E}|a_s^{(1)}(\tau; v, L)|^2 = v\Phi_1(s, \tau) + O(v^{-2-\mathbb{R}_d})\mathbb{C}^\#(s; \mathbb{R}_d) \quad \text{if} \quad L \geq v^{-2},
\]
where \( \Phi_1 \) is a Schwartz function of \( s \), independent from \( L \) and \( v \). This relation was proved for \( T = \infty \), but it remains true for any finite \( T \) due to a similar argument. Similarly if \( L \geq v^{-2} \), then \( \mathbb{E}\mathbb{R}(\tilde{a}_s^{(0)} a_s^{(2)}(\tau; v, L)) = v\Phi_2(s, \tau) + O(v^{-2-\mathbb{R}_d})\mathbb{C}^\#(s; \mathbb{R}_d) \), so
\[
n_s^{(2)}(\tau; v, L) = v\Phi(s, \tau) + O(v^{-2-\mathbb{R}_d})\mathbb{C}^\#(s; \mathbb{R}_d) \quad \text{if} \quad L \geq v^{-2}, \tag{5.6}
\]
where \( s \mapsto \Phi(s, \tau) \) is a Schwartz function (we are not giving a complete proof of (5.6) since this relation is used only for motivation and discussion). To understand the limiting behaviour of \( n_s^{(3)} \) and \( n_s^{(4)} \) we will use another result, proved in [8], where \( a_s^{(i)}(\tau) \) denote the terms of the series (2.2). Recall that the function \( \chi_d \) is defined in (1.43) and \([-\cdot]\) in Notation.

**Theorem 5.1.** For any \( k_1, k_2 \geq 0 \) and \( k := k_1 + k_2 \), we have
\[
\left| \mathbb{E}_{d}(k_1) \tilde{a}_s^{(k_2)}(\tau_2) \right| \leq C^\#(s; k)(v^{-2}L^{-2} + \max(v^{[k/2]}, v^d) \chi_d^k(v)) \tag{5.7}
\]
for any \( s \in \mathbb{Z}_L^d \), uniformly in \( \tau_1, \tau_2 \geq -T \), where \( \chi_d^k(v) = \chi_d(v) \) if \( k = 3 \) and \( \chi_d^k(v) \equiv 1 \) otherwise. The second moment \( \mathbb{E}_{d}(k_1) \tilde{a}_s^{(k_2)}(\tau_2) \) extends to a Schwartz function of \( s \in \mathbb{R}^d \supset \mathbb{Z}_L^d \) which satisfies the same estimate (5.7).
Note that for $k \leq 2$ the theorem’s assertion follows from the preceding discussion since $d \geq 2$ (and recall that for $k = 1$ the l.h.s. of (5.7) vanishes by Lemma 2.3). A short direct proof of (5.7) with $k_1 = k_2 = 1$ and $T = \infty$ is given in Addendum 12.3. In Section 8 we discuss a strategy of the theorem’s proof for any $k$, given in [8]. By Theorem 5.1, for any $s \in \mathbb{R}^d$

\[ |n_s^{(3)}| \leq C^\# (s) v^2 \chi_d (v), \quad |n_s^{(4)}| \leq C^\# (s) v^2 \quad \text{if} \quad L \geq v^{-2}. \]  

Choosing the parameter $\rho$ in the form (1.10), we will examine the energy spectrum $n_s$ under the limit (1.17). Due to the discussion above, under this limit

\[ n_s^{(0)} = B(s) \left( 1 - e^{-2\gamma_s (T + \tau)} \right), \quad \rho n_s^{(1)} \equiv 0, \quad \rho^2 n_s^{(2)} \to \varepsilon \Phi (s, \tau), \quad \rho^3 n_s^{(3)} \to 0. \]

Concerning the term $\rho^n n_s^{(4)}$, our results do not allow to find its asymptotic under the limit, but only imply that $|\rho^n n_s^{(4)}| \leq \varepsilon^2 C^\# (s)$. Accordingly our goal is to examine the energy spectrum $n_s$ under the limit (1.17) and the scaling (1.10) with precision $\varepsilon^2 C^\# (s)$, regarding the constant $\varepsilon \leq 1$ (which measures the size of solutions for (1.8) under the proper scaling) as a fixed small parameter.

5.1. Increments of the energy spectra $n_s^{2, 3}$ and the reminder $n_s^{\geq 3}$. We will show that the process $n_s^{2, 3}$ approximately satisfies the WKE (1.35), while the reminder $n_s^{\geq 3}$ is small. This will imply that $n_s (\tau)$ is an approximate solution of the WKE. We always assume (1.10) and that $L \geq 1, 0 < \nu \leq 1/2$.

Now for $u \in \mathcal{C}_r (\mathbb{R}^d)$, $r > d$, and for $\tau \in (0, 1]$, we consider the kinetic integral $K^\tau (u) = ((K^\tau u) (s), s \in \mathbb{R}^d)$:

\[ K^\tau (u) = \int_0^\tau e^{-t \mathcal{L}} K (u) \, dt, \]

where the operator $K = K_1 + \cdots + K_4$ is defined in Section 4 and the linear operator $\mathcal{L}$ is introduced in (4.11). That is,

\[ (K^\tau u) (s) = \frac{1 - e^{-2\gamma_s \tau}}{2\gamma_s} (K u) (s) = \frac{1 - e^{-2\gamma_s \tau}}{2\gamma_s} \sum_{j=1}^4 (K_j u) (s). \]  

The result below is the main step in establishing the wave kinetic limit. There, using (5.5), we regard $n_s^{2, 3} (\tau)$ as a Schwartz function of $s \in \mathbb{R}^d$.

**Theorem 5.2.** For any $0 < \tau \leq 1$ we have

\[ n_s^{2, 3} (\tau) = e^{-\tau \mathcal{L}} n_s^{2, 3} (0) + 2 \int_0^\tau e^{-t \mathcal{L}} b^2 \, dt + \varepsilon K^\tau (n_s^{2, 3} (0)) + \mathcal{R}, \]

where $b^2 = \{ b^2 (s), s \in \mathbb{R}^d \}$ and the reminder $\mathcal{R} (\tau, s)$ satisfies

\[ |\mathcal{R} (\tau)|_r \leq C_{r, \mathcal{R}_d} \varepsilon \left( \nu^{1-\mathcal{R}_d} + \nu^{-3} L^{-2} + \tau^2 + \varepsilon \tau \right), \quad \forall r, \]

and $\mathcal{R}_d$ is defined as in Theorem 3.3.
Proof of Theorem 5.2 is given in Section 5.2. Since for any $\tau' \geq -T$ the process $\tau \to (A_s(\tau'+\tau), s \in \mathbb{Z}_L^d)$ is a quasisolution of the problem (2.1), (1.19) with $T := T + \tau'$ and $\beta_s(\tau) := \beta_s(\tau + \tau')$, then the theorem applies to study the increments of $n^{\leq 2}$ from $\tau'$ to $\tau'+\tau$, for any $\tau' \geq -T$. That is, (5.11) remains true if we replace $n^{\leq 2}(0)$ by $n^{\leq 2}(\tau')$ and $n^{\leq 2}(\tau)$ by $n^{\leq 2}(\tau' + \tau)$.

We also need an estimate on the reminder $n_\tau^3$. It is a part of the assertion below, which is an immediate consequence of (5.7)-(5.8) since $d \geq 2$:

**Proposition 5.3.** If $L \geq \nu^{-2}$, then for $k = 0, 1, 2, 4$ we have
\[
|n_s^{(k)}(\tau)| \leq C^\#(s) \nu^{k/2},
\]
and $|n_s^{(3)}(\tau)| \leq C^\#(s) \nu^2 \chi_d(\nu)$, uniformly in $\tau \geq -T$. So
\[
|n_s^{\leq 2}(\tau)| \leq C^\#(s)
\]
and if $\nu(\chi_d(\nu))^{1/2} \leq \varepsilon$, then
\[
|n_s^{\geq 3}(\tau)| \leq C^\#(s) \varepsilon^2.
\]

In accordance with (1.17) we will study the energy spectrum $n_s(\tau)$ under the two limiting regimes: $L \gg \nu^{-1}$ when $\nu \to 0$; or first $L \to \infty$ and then $\nu \to 0$. To treat the latter we will need the following result.

**Proposition 5.4.** For any $\nu \in (0, 1/2]$, $\tau_1, \tau_2 \geq -T$, $k_1, k_2 \geq 0$ and $s \in \mathbb{R}^d$ the moment $E a_s^{(k_1)}(\tau_1 \nu, L) a_s^{(k_2)}(\tau_2 \nu, L)$ admits a finite limit as $L \to \infty$. The limit is a Schwartz function of $s$.

In particular, this result implies that any $n_s^{(k)}(\tau; \nu, L)$ converges, as $L \to \infty$, to a Schwartz function $n_s^{(k)}(\tau; \nu, \infty)$ of $s \in \mathbb{R}^d$. The proposition can be obtained directly by iterating the Duhamel formula (2.4) and using Theorem 3.1 to replace the corresponding sum by an $L$-independent integral (cf. Section 3.1, where the moments with $k_1 = k_2 = 1$ are approximated by integrals $I_s$). We do not give here a proof since it follows from a stronger result in [8], discussed in Section 8 (see there (8.7) and (8.9)).

5.2. Proof of Theorem 5.2. It is convenient to decompose the processes $a_s^{(i)}(\tau)$, $\tau \geq 0$, as
\[
a_s^{(i)}(\tau) = c_s^{(i)}(\tau) + \Delta a_s^{(i)}(\tau), \quad i = 0, 1, 2, \quad s \in \mathbb{Z}_L^d,
\]
where
\[
c_s^{(i)}(\tau) = e^{-\gamma_s \tau} a_s^{(i)}(0)
\]
and $\Delta a_s^{(i)}$ is defined via (5.16). That is, $c_s(\tau) := c_s^{(0)}(\tau) + \rho c_s^{(1)}(\tau) + \rho^2 c_s^{(2)}(\tau)$ with $\tau \geq 0$ is a solution of the linear equation (2.1)$_{p=0, b(s)=0}$, equal $A(0)$ at $\tau = 0$, and $\Delta a_s(\tau)$ equals $A(\tau) - c_s(\tau)$. By (5.5), for $0 \leq i, j \leq 2$ the functions
\[
E c_s^{(i)} c_s^{(j)}, \quad E c_s^{(i)} \Delta a_s^{(j)}, \quad E \Delta a_s^{(i)} \Delta a_s^{(j)}
\]
naturally extend to Schwartz functions of $s \in \mathbb{R}^d$. (5.17)
Due to (5.3) and (5.4),
\[ e^{-2\gamma_s \tau} n_s^{\leq 2}(0) = \mathbb{E}[c_s^{(0)}(\tau)]^2 + \rho^2 \mathbb{E}[|c_s^{(1)}(\tau)|^2 + 2\mathfrak{H} \tilde{c}_s^{(0)}(\tau) c_s^{(2)}(\tau)], \quad \forall s \in \mathbb{R}^d. \]

Also,
\[ n_s^{\leq 2}(\tau) - e^{-2\gamma_s \tau} n_s^{\leq 2}(0) = \mathbb{E}\left(|a_s^{(0)}(\tau)|^2 - |c_s^{(0)}(\tau)|^2 + \rho^2 \left(|a_s^{(1)}(\tau)|^2 - |c_s^{(1)}(\tau)|^2 + 2\mathfrak{H}(a_s^{(2)} \tilde{a}_s^{(0)}(\tau) - c_s^{(2)} \tilde{c}_s^{(0)}(\tau))\right)\right). \tag{5.18} \]

Writing explicitly processes \( \Delta a_s^{(i)}(\tau), s \in \mathbb{Z}_L^d \), from eq. (2.1), we find
\[
\begin{align*}
\Delta a_s^{(0)}(\tau) &= b(s) \int_0^\tau e^{-\gamma_s(\tau-l)} d\beta_s(l), \\
\Delta a_s^{(1)}(\tau) &= i \int_0^\tau e^{-\gamma_s(\tau-l)} \mathcal{Y}_s^{(i)}(a^{(0)}, v^{-1}l) dl, \\
\Delta a_s^{(2)}(\tau) &= i \int_0^\tau e^{-\gamma_s(\tau-l)} 3 \gamma_s^{sym}(a^{(0)}, a^{(0)}, a^{(1)}; v^{-1}l) dl,
\end{align*}
\tag{5.19}
\]
where \( a^{(0)} = a^{(0)}(l) \) and we recall that \( \gamma_s^{sym} \) is defined at the beginning of Section 2. Let us note that to get explicit formulas for \( c_s^{(i)}(\tau), i = 0, 1, 2 \), it suffices to replace in the r.h.s.’s of the relations in (5.19) the range of integrating from \([0, \tau]\) to \([-T, 0]\). For example,
\[ c_s^{(0)}(\tau) = e^{-\gamma_s \tau} a_s^{(0)}(0) = b(s) \int_0^\tau e^{-\gamma_s(\tau-l)} d\beta_s(l). \]

Using that \( \mathbb{E}c_s^{(i)} \Delta a_s^{(0)} = \mathbb{E}c_s^{(i)} \mathbb{E} \Delta a_s^{(0)} = 0 \) for any \( i \) and \( s \), we obtain
\[ \mathbb{E}\left(a_s^{(2)} \tilde{a}_s^{(0)}(\tau) - c_s^{(2)} \tilde{c}_s^{(0)}(\tau)\right) = \mathbb{E}\Delta a_s^{(2)}(\tau), \tag{5.20} \]
and from (5.16) we get that
\[ |a_s^{(1)}(\tau)|^2 - |c_s^{(1)}(\tau)|^2 = |\Delta a_s^{(1)}(\tau)|^2 + 2\mathfrak{H} \Delta a_s^{(1)} \tilde{c}_s^{(1)}(\tau), \]
\[ \mathbb{E}|a_s^{(0)}(\tau)|^2 - \mathbb{E}|c_s^{(0)}(\tau)|^2 = \mathbb{E}|\Delta a_s^{(0)}(\tau)|^2. \tag{5.21} \]

Then, inserting (5.20) and (5.21) into (5.18), we find
\[ n_s^{\leq 2}(\tau) - e^{-2\gamma_s \tau} n_s^{\leq 2}(0) = \mathbb{E}|\Delta a_s^{(0)}(\tau)|^2 + \rho^2 Q_s(\tau), \quad s \in \mathbb{R}^d, \]
where
\[ Q_s(\tau) := \mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 + 2\mathfrak{H}\mathbb{E}\left(\Delta a_s^{(1)}(\tau) \tilde{c}_s^{(1)}(\tau) + \Delta a_s^{(2)}(\tau) \tilde{a}_s^{(0)}(\tau)\right), \tag{5.22} \]
and we recall (5.17). Since
\[ \mathbb{E}|\Delta a_s^{(0)}(\tau)|^2 = \frac{b(s)^2}{\gamma_s} (1 - e^{-2\gamma_s \tau}) = 2 \int_0^\tau e^{-\mathcal{L} b^2(s) dt,} \]
then
\[ n_s^{\leq 2}(\tau) - e^{-\mathcal{L} b^2} n_s^{\leq 2}(0) = 2 \int_0^\tau e^{-\mathcal{L} b^2} dt + \rho^2 Q(\tau), \]
for \( n_s^{\leq 2} = (n_s^{\leq 2}, s \in \mathbb{R}^d) \). So the desired formula (5.11) is an immediate consequence of the assertion below:
Proposition 5.5. We have
\[ \rho^2 Q_s(\tau) = \varepsilon K^\tau (n^{\leq 2}(0))(s) + R(\tau, s), \quad s \in \mathbb{R}^d, \]
where the reminder \( R \) satisfies (5.12).

Proof. Below we abbreviate \( n_s^{(0)}(0) \) to \( n_s^{(0)} \).

Since \( \rho^2 \nu = \varepsilon \), then we should show that for any \( r \),
\[
\left| Q(\tau) - \nu K^\tau (n^{\leq 2}(0))\right|_r \leq C_{r, N_{\mathbf{d}}}(\nu^{2-N_{\mathbf{d}}} + \nu^{-2}L^{-2} + \nu \tau^2 + \varepsilon \nu \tau). \tag{5.23}
\]

To this end, iterating formula (2.4), we will express the processes \( \Delta a_s^{(2)}, \Delta a_s^{(1)} \) and \( c_k^{(1)} \), entering the definition (5.22) of \( Q_s \), through the processes \( a_k^{(0)} \). Then, applying the Wick formula, we will see that \( Q_s \) depends on the quasisolution \( A_s \) only through the correlations of the form \( \mathbb{E} a_k^{(0)}(l)\bar{a}_k^{(0)}(l') \). We will show that the main input to \( Q_s \) comes from those terms which depend only on the correlations with the times \( l, l' \) satisfying \( 0 \leq l, l' \leq \tau \). Then, approximating these correlations by their values at \( l = l' = 0 \), we will see that \( Q_s(\tau) \) is close to a sum \( Z_s \) from Proposition 5.6 below, which depends only on \( \tau \) and the energy spectrum \( n_{k}^{(0)}(0) = \mathbb{E}|a_k^{(0)}(0)|^2 \). Next we will approximate the sum \( Z_s \) by its asymptotic as \( \nu \to 0 \) and \( L \to \infty \), which is given by the kinetic integral \( \nu K^\tau (n^{(0)}(0)) \). Finally, replacing \( \nu K^\tau (n^{(0)}(0)) \) with \( \nu K^\tau (n^{\leq 2}(0)) \) and estimating the difference of the two kinetic integrals we will get (5.23).

We will derive (5.23) from the following result:

Proposition 5.6. We have
\[
\left| Q_s(\tau) - Z_s \right| \leq C^\#(s)(\nu^2 \chi_d(\nu) + \nu^{-2}L^{-2} + \nu \tau^2), \quad s \in \mathbb{R}^d, \tag{5.24}
\]
where
\[
Z_s := 2\sum_{1, 2} \delta_{3s}^{12}(Z^4 n_1^{(0)} n_2^{(0)} n_3^{(0)} + Z^3 n_1^{(0)} n_2^{(0)} n_3^{(0)} - 2Z^1 n_2^{(0)} n_3^{(0)} n_s^{(0)}) = \pm 2S_1^1 + 2S_2^2 - 4S_3^3, \tag{5.25}
\]
and the terms \( Z^i = Z^j(s_1, s_2, s_3, s, \tau) \) have the following form:
\[
Z^4 = \frac{e^{i\nu^{-1}a_{3s}^{12}\tau} - e^{-\gamma_s \tau}}{\gamma_s^2 + (\nu^{-1}a_{3s}^{12})^2}, \quad Z^j = 2\frac{1 - e^{-\gamma_s \tau}}{\gamma_s} \frac{\gamma_j}{\gamma_j^2 + (\nu^{-1}a_{3s}^{12})^2} \quad \text{for } j = 1, 2, 3. \tag{5.26}
\]

The proposition is proved in the next section.

To deduce the desired estimate (5.23) from (5.24) we will approximate the sums \( S_s^j \) \( (j = 1, 2, 3) \) in (5.25) by their asymptotic as \( \nu \to 0 \) and \( L \to \infty \):

The sum \( S_s^1 \). It has the form (10.2) with \( F_s(s_1, s_2) = n_s^{(0)} n_s^{(0)} n_s^{(0)} \). So by (10.4) and Theorem 10.2,
\[
\left| S_s^1 - \nu \frac{1 - e^{-2\gamma_s \tau}}{4\gamma_s} (K_4 n^{(0)}(s)) \right| \leq C^\#(s; N_{\mathbf{d}})(\nu^{2-N_{\mathbf{d}}} + \nu^{-2}L^{-2}) \tag{5.27}
\]
for all \( s \in \mathbb{R}^d \), where we recall that the integral \( K_4 \) is defined in (4.2).
The sum $S_2^2$. Let us set $F_s(s_1, s_2) := \gamma_3 n_s^{(0)} n_{s_2}^{(0)} n_{s_3}^{(0)}$ and $\Gamma_s(s_1, s_2) = \gamma_3 / 2$. Then the sum takes the form

$$S_2^2 = \frac{1 - e^{-\gamma s \tau}}{2\gamma_s} \nu^2 \sum_{s_1, s_2} \delta_{s_3}^{12} \frac{F_s(s_1, s_2)}{(v_{\Gamma_s})^2 + (\omega_{s_3}^2 / 2)^2}.$$ 

Applying Theorems 3.1 and 3.3 we get

$$\left| S_2^2 - \nu \frac{1 - e^{-\gamma s \tau}}{2\gamma_s} (K_3 n^{(0)}) (s) \right| \leq C^\# (s; \mathbb{R}_d) (v^{2-\mathbb{N}_d} + v^{-2} L^{-2}). \quad (5.28)$$

The sum $S_3^3$. Let us note that the function $Z_1$ equals to $Z_3^3$, if we there replace $\gamma_3$ by $\gamma_1$. Then, repeating the argument used to analyse the sum $S_2^2$, we get

$$\left| - S_3^3 - \nu \frac{1 - e^{-\gamma s \tau}}{2\gamma_s} (K_1 n^{(0)}) (s) \right| \leq C^\# (s; \mathbb{R}_d) (v^{2-\mathbb{N}_d} + v^{-2} L^{-2}). \quad (5.29)$$

Using the symmetry of the integral $K_1$ with respect to the transformation $(s_1, s_2) \mapsto (s_2, s_1)$ in its integrand, we see that $K_1(n^{(0)}) = K_2(n^{(0)})$.

In (5.27) the sum $S_1^3$ is approximated by the integral $\frac{1}{2} K_4(n^{(0)})$, multiplied by the factor $\frac{1 - e^{-2\gamma s \tau}}{2\gamma_s}$ which also arises in (5.10), while for the sums $S_2^2$ and $S_3^3$ the corresponding factors are slightly different, see (5.28) and (5.29). To handle this difficulty we consider $K_j(n^{(0)}) =: \eta$, where $j$ is 1 or 3. By (1.28) and Lemma 4.2, $|\eta|_r \leq C_r$ for all $r$. Denote

$$\xi_s := \left( \frac{1 - e^{-\gamma s \tau}}{\gamma_s} - \frac{1 - e^{-2\gamma s \tau}}{2\gamma_s} \right) \eta_s, \quad s \in \mathbb{R}^d.$$ 

Since $\frac{1 - e^{-\gamma s \tau}}{\gamma_s} - \frac{1 - e^{-2\gamma s \tau}}{2\gamma_s} = (1 - e^{-\gamma s \tau})^2 \leq \tau^2 \gamma_s \, \eta_s$, we find

$$|\xi|_r \leq \tau^2 |\eta|_{r+2r_s} \leq \tau^2 C_r.$$ 

This estimate allows to replace in (5.28) and (5.29) $\nu \frac{1 - e^{-\gamma s \tau}}{2\gamma_s} K_j(n^{(0)})$ by $\nu \frac{1 - e^{-2\gamma s \tau}}{4\gamma_s} K_j(n^{(0)})$ with accuracy $v \tau^2$. So recalling the definition of $K^\tau$ in (5.10), combining (5.27), (5.28), (5.29) and using that $K_1(n^{(0)}) = K_2(n^{(0)})$, for any $r$ we get

$$|Z_3 - \nu (K^\tau n^{(0)}) (s)|_r \leq C_{r, \mathbb{N}_d} (v^{2-\mathbb{N}_d} + v^{-2} L^{-2}) + C_r v \tau^2. \quad (5.30)$$

Finally, since by Proposition 5.3 together with (5.3) $|n_s^{2/2} (s) - n_s^{(0)} (s) | = \rho^2 |n_s^{(0)} (s)| \leq C^\# (s) \varepsilon$, then $|K^\tau (n^{(0)}) - K^\tau (n^{(0)})|_{r+1} \leq C_r \varepsilon \tau$ in view of Corollary 4.3 and (5.9). This inequality jointly with estimates (5.30) and (5.24) imply the desired relation (5.23).

6. Proof of Proposition 5.6

To conclude the proof of Theorem 5.2, it remains to establish Proposition 5.6. The proof of the proposition is somewhat cumbersome since we have to consider a number of different terms and different cases. During the proof we will often skip the upper index (0), so by writing $a$ and $a_s$ we will mean $a^{(0)}$ and $a_s^{(0)}$. 
We recall that $Q_s$ is given by formula (5.22) and first consider the term $\mathbb{E}\Delta a_s^{(2)}(\tau)\tilde{a}_s(\tau)$. Inserting the identity $a^{(1)}(l) = c^{(1)}(l) + \Delta a^{(1)}(l)$ into formula (5.19) for $\Delta a_s^{(2)}$, we obtain

$$
\mathbb{E}\Delta a_s^{(2)}(\tau)\tilde{a}_s(\tau) = N_s + \tilde{N}_s,
$$

where

$$
N_s := i \mathbb{E}\left( \tilde{a}_s(\tau) \int_0^\tau e^{-\gamma_s(\tau-l)}3Y_s^{sym}(a, a, \Delta a^{(1)}; v^{-1}l) dl \right)
$$

(6.1)

and

$$
\tilde{N}_s := i \mathbb{E}\left( \tilde{a}_s(\tau) \int_0^\tau e^{-\gamma_s(\tau-l)}3Y_s^{sym}(a, c^{(1)}; v^{-1}l) dl \right).
$$

Thus,

$$
Q_s = \mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 + 2\Re N_s + 2\Re \mathbb{E}\Delta a_s^{(1)}(\tau)\tilde{c}_s^{(1)}(\tau) + 2\Re \tilde{N}_s, \quad s \in \mathbb{R}^d.
$$

So we have to analyze the four terms in the r.h.s. above.

6.1. The first term of $Q_s$. First we will show that the term $\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2$ can be approximated by the first sum $2S_1$ from (5.25). Indeed, due to (5.19), we have

$$
\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 = \mathbb{E} \int_0^\tau dl \int_0^\tau dl' e^{-\gamma_s(2\tau-l-l')}Y_s(a, v^{-1}l)Y_s(a, v^{-1}l').
$$

(6.2)

Writing the functions $Y$ explicitly and applying the Wick theorem, in view of (2.8) we find

$$
\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 = 2\sum_{1,2,3s}^{12} \delta_{3s}^{12} \mathbb{E}a_1(l)\tilde{a}_1(l') \mathbb{E}a_2(l)\tilde{a}_2(l') \mathbb{E}a_3(l)\tilde{a}_3(l').
$$

(6.3)

Denoting $g_{s123}(l, l', \tau) = -\gamma_s(2\tau-l-l')+iv^{-1}\omega_{3s}^{12}(l-l')$ and computing the time integrals of the exponent above, we obtain

$$
\int_0^\tau dl \int_0^\tau dl' e^{g_{s123}(l, l', \tau)} = Z^4,
$$

where $Z^4$ is defined in (5.26). Together with the sum in (6.3) we consider a sum obtained from the latter, without factor 2, by replacing the processes $a_k(l), \tilde{a}_k(l')$ by their value at zero $a_k(0)$:

$$
\sum_{1,2,3s}^{12} \delta_{3s}^{12} Z^4 \mathbb{E}|a_1(0)|^2 \mathbb{E}|a_2(0)|^2 \mathbb{E}|a_3(0)|^2 = \sum_{1,2,3s}^{12} \delta_{3s}^{12} Z^4 n_1^{(0)} n_2^{(0)} n_3^{(0)} = S^4_3.
$$

(6.4)

Our goal in this section is to show that

$$
|\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 - 2S^4_3| \leq C^\#(s)(v^{-2}L^{-2} + \tau^2 v).
$$

(6.5)

Due to (6.3) and (2.8),

$$
\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 = 2\sum_{1,2,3s}^{12} \delta_{3s}^{12} \int_0^\tau dl \int_0^\tau dl' e^{g_{s123}(l, l', \tau)} B_{123h_{123}}(l, l'),
$$

(6.6)
where $B_{123}$ is the function defined in (2.9), extended from $\mathbb{Z}_+^3$ to $\mathbb{R}^3$, and $h_{123}(l, l') = \prod_{j=1}^3 (e^{-\gamma_j |l-l'|} - e^{-\gamma_j (2T+4s+l')})$ (also viewed as a function on $\mathbb{R}^3$). Let $f_{123}$ denote the increment of the function $h_{123}$, that is

$$f_{123}(l, l') = h_{123}(l, l') - h_{123}(0, 0).$$

It is straightforward to see that

$$|\partial_{s_1, s_2, s_3}^\alpha f_{123}(l, l')| \leq C_{|\alpha|} \langle (s_1, s_2, s_3) \rangle^{m_{|\alpha|}} \tau \quad \forall |\alpha| \geq 0, \tag{6.7}$$

for appropriate constants $C_k, m_k > 0$, uniformly in $0 \leq l, l' \leq \tau$. Since $n_1^{(0)} n_2^{(0)} n_3^{(0)} = B_{123} h_{123}(0, 0)$, from (6.6) and (6.4) we see that $\mathbb{E} |\Delta s_1^{(1)}|^2 - 2S_1 = 2S_\Delta^\ast$ with

$$S_\Delta = \sum_{l, 2} S_{1, 2} \int_0^\tau dl \int_0^\tau dl' f_{123}(l, l') B_{123}(l, l'),$$

Now in the expression above we replace $\sum_{l, 2}$ by $\int ds_1 ds_2$ and denote the obtained integral by $I_\Delta$,

$$I_\Delta = \int_0^\tau dl \int_0^\tau dl' \int_{\mathbb{R}^d} ds_1 ds_2 e^{g_{123}(l, l', \tau)} B_{123}(l, l'), \quad s_3 = s_1 + s_2 - s. \tag{6.8}$$

Since $B_{123}$ with $s_3 = s - s_1 - s_2$ is a Schwartz function of $s, s_1, s_2$ and the function $f_{123}$ satisfies (6.7), then Theorem 3.1 applies and we find

$$|S_\Delta - I_\Delta| \leq C^\#(s) v^{-2} L^{-2}. \tag{6.9}$$

To establish (6.5) it remains to prove that $|I_\Delta| \leq C^\#(s) \tau^2 v$. To do that we divide the external integral (over $dl dl'$) in (6.8) to two integrals:

**Integral over** $|l-l'| \leq v$. In view of (6.7) with $\alpha = 0$, in this case the internal integral (over $ds_1 ds_2$) is bounded by $C^\#(s) \tau$, so $I_\Delta$ is bounded by $C^\#(s) \tau^2 v$.

**Integral over** $|l-l'| \geq v$. If $\tau < v$ than this integral vanishes, so we assume that $\tau \geq v$. Since $\omega^{12}_{3s} = 2(s_1 - s) \cdot (s - s_2) \in \mathbb{R}^2 d$, for any $l, l'$ from the considered domain the integral over $ds_1 ds_2 = dz$ in (6.8) has the form (12.2) with $v := v|l-l'|^{-1} \leq 1$ and $n = 2d$. In view of (6.7), estimate (12.4) together with (12.5) implies that this integral is bounded by $C^\#(s) \tau v^d |l-l'|^{-d}$. So

$$|I_\Delta| \leq C^\#(s) \tau v^d \int_0^\tau dl \int_0^\tau dl' |l-l'|^{-d} \chi_{|l-l'| \geq v}$$

$$\leq C^\#(s) \tau^2 v^d \int_v^\tau x^{-d} dx \leq C^\#(s) \tau^2 v.$$

We saw that $|I_\Delta| \leq C^\#(s) \tau^2 v$. This relation and (6.9) imply (6.5).
6.2. The second term of $Q_s$. To study the term $2\Re N_s$ we use the same strategy as above. Namely, expressing in (6.1) the function $3\mathcal{Y}_{s}^{sym}$ via $\mathcal{Y}_s$, we write $N_s$ as $N_s = N_s^1 + 2N_s^2$, $s \in \mathbb{R}^d$, where

$$
N_s^1 = i \mathbb{E}\left(\tilde{a}_s(\tau) \int_0^\tau e^{-\gamma_s(\tau-l)} \mathcal{Y}_s(a, a, \Delta a^{(1)}; v^{-1}l) \, dl\right),
$$

$$
N_s^2 = i \mathbb{E}\left(\tilde{a}_s(\tau) \int_0^\tau e^{-\gamma_s(\tau-l)} \mathcal{Y}_s(\Delta a^{(1)}, a, a; v^{-1}l) \, dl\right).
$$

We will show that the terms $2\Re N_s^1$ and $4\Re N_s^2$ can be approximated by the second and the third sums from (5.25).

**Term $N_s^1$.** Let us start with the term $N_s^1$: writing explicitly the function $\mathcal{Y}_s$ and then $\Delta \tilde{a}^{(1)}$ we get

$$
N_s^1 = i L^{-d} \sum_{1,2} \delta_{12}^3 \int_0^\tau \mathbb{E}\left((a_1(l)a_2(l)\Delta a_3^{(1)}(l)\tilde{a}_s(\tau))\right)
$$

$$
= L^{-2d} \sum_{1,2} \delta_{12}^3 \delta_{12}' \int_0^\tau \int_0^l \mathbb{E}\left((a_1(l)a_2(l)\tilde{a}_1(l')\tilde{a}_2(l')a_3(l')\tilde{a}_s(\tau))\right).
$$

By the Wick theorem, we need to take the summation only over $s_1'$, $s_2'$, $s_3'$ satisfying $s_1' = s_1$, $s_2' = s_2$, $s_3' = s$ or $s_1' = s_2$, $s_2' = s_1$, $s_3' = s$. Since in the both cases we get $\delta_{12}' = \delta_{12}^3$ and $\omega_{12}' = \omega_{12}^3$, we find

$$
N_s^1 = 2 \sum_{1,2} \delta_{12}^3 \int_0^\tau \int_0^l \mathbb{E}\left((a_1(l)a_2(l)\tilde{a}_1(l')\tilde{a}_2(l')a_3(l')\tilde{a}_s(\tau))\right).
$$

Replacing in (6.11) the processes $a_k^{(0)}(l)$, $a_k^{(0)}(l')$ and $\tilde{a}_k^{(0)}(\tau)$ by their value at zero, we get instead of $N_s^1$ the sum

$$
\hat{N}_s^1 := 2 \sum_{1,2} \delta_{12}^3 T_{12} n_1^{(0)} n_2^{(0)} n_3^{(0)},
$$

where $T_{12}$ denotes the integral of the exponent above

$$
T_{12} := \int_0^\tau \int_0^l \mathbb{E}(e^{-\gamma_1(l')-\gamma_3(l')-i\omega_{12}^3(l'-l')}).
$$

Arguing as in Section 6.1 we find

$$
|N_s^1 - \hat{N}_s^1| \leq C^\#(s)(v^{-2}L^{-2} + \tau^2v).
$$

The term $2\Re \hat{N}_s^1$ is not equal to the second sum from (5.25) yet. To extract the latter from the former we write the integral over $dl'$ in (6.12) as $\int_0^l = \int_{-\infty}^l - \int_{-\infty}^0$. We get

$$
T_{12} = T_{12}' + T_{12}''.
$$
where

\[ T_{12} = e^{-\gamma_s \tau} \int_0^\tau dl e^{l(y_3-\gamma_3+i\nu-1\omega_{3s}^2)} \int_{-\infty}^l dl' e^{l'(y_3-\gamma_3+i\nu-1\omega_{3s}^2)} = \frac{1 - e^{-\gamma_s \tau}}{(\gamma_3 - i\nu-1\omega_{3s}^2)\gamma_s} \]

and

\[ T'_{12} = -\int_0^\tau dl \int_0^0 dl' e^{-\gamma_s(\tau-l)-y_3(l-l')i\nu-1\omega_{3s}^2(l-l')} \tag{6.15} \]

Computing the real part of \( T_{12} \), we find that

\[ 2\Re T_{12} = 2 \frac{1 - e^{-\gamma_s \tau}}{\gamma_s} \frac{\gamma_3}{\gamma_3^2 + (\nu-1\omega_{3s}^2)^2} = Z^3, \tag{6.16} \]

where \( Z^3 \) is defined in (5.26). On the other hand,

\[ \left| \hat{N}_s^1 - 2 \sum_{1,2} \delta_{3s}^1 T_{12} \bar{n}_1(0) n_2(0) n_s(0) \right| = 2 \left| \sum_{1,2} \delta_{3s}^1 T'_{12} \bar{n}_1(0) n_2(0) n_s(0) \right| \tag{6.17} \]

by (6.15) equals to

\[ 2 \left| \sum_{1,2} \delta_{3s}^1 \int_0^\tau dl \int_0^0 dl' e^{l' + i\nu-1\omega_{3s}^2(l-l')} F_s(s_1, s_2, l, l', \tau) \right| \tag{6.18} \]

with \( F_s = e^{-\gamma_s(\tau-l)-y_3(l-l')i\nu-1\omega_{3s}^2} n_1(0) n_2(0) n_s(0) \). Since by (1.28) \( n_r(0) = n_r(0) \) is a Schwartz function of \( r \in \mathbb{R}^d \) and \( y_3 - 1 \geq 0 \), then Theorem 10.3 applies and implies that (6.18) is bounded by \( C^*(s)(\nu^2\chi_d(\nu) + \nu^2L^{-2}) \), so the l.h.s. of (6.17) also is.

Thus, due to (6.13) and (6.16) we arrive at the relation

\[ 2\Re N_s^1 - 2 \sum_{1,2} \delta_{3s}^1 Z^3 n_1(0) n_2(0) n_s(0) \leq C^*(s)(\nu^2\chi_d(\nu) + \nu^2L^{-2} + \tau^2\nu). \tag{6.19} \]

**Term \( N_s^2 \).** Finally, we study the term \( N_s^2 \) by literally repeating the argument we have applied to \( N_s^1 \). We find that

\[ N_s^2 = i L^{-d} \sum_{1,2} \delta_{3s}^1 \int_0^\tau dl e^{-\gamma_s(\tau-l)+i\nu-1\omega_{3s}^2l} \mathbb{E} \Delta(a_1(l) \bar{a}_2(l) \bar{a}_s(l) \bar{a}_s(\tau)) \]

\[ = -L^{-2d} \sum_{1,2} \sum_{1',2'} \delta_{3s}^1 \delta_{3s}^2 \delta_{3s}^2 \int_0^\tau dl \int_0^l dl' e^{-\gamma_s(\tau-l)+i\nu-1\omega_{3s}^2l} e^{-\gamma_1(l-l')+i\nu-1\omega_{3s}^2l'} \]

\[ \times \mathbb{E}(a_1(l') \bar{a}_2(l') \bar{a}_s(l') \bar{a}_3(l) \bar{a}_s(l) \bar{a}_s(\tau)). \tag{6.20} \]

By the Wick theorem we should take summation either under the condition \( s_1 = s_3 \), \( s_2 = s \), \( s_3 = s_2 \) or \( s_1 = s, s_2 = s_3, s_3 = s_2 \). Since in both cases \( \delta_{3s}^1 = \delta_{3s}^2 \) and \( \omega_{3s}^2 = -\omega_{3s}^1 \), then

\[ N_s^2 = -2 \sum_{1,2} \delta_{3s}^1 \int_0^\tau dl \int_0^l dl' e^{-\gamma_1(l-l')-y_3(l-l')i\nu-1\omega_{3s}^2(l-l')} \]

\[ \times \mathbb{E}(a_2(l) \bar{a}_2(l') \bar{a}_3(l) \bar{a}_s(l) \bar{a}_s(l')). \tag{6.21} \]
We set

\[ M_{12} := \int_0^\tau dl \int_0^l dl' e^{-\gamma_1(\tau-l)-\gamma_1(l-l') + iv^{-1} \omega_3^2(l-l')} \]

and note that \( M_{12} \) equals to \( T_{12} \), defined in (6.12), if replace \( \gamma_3 \) by \( \gamma_1 \). Then, as in (6.14)–(6.16) we get \( M_{12} = M_{12} + M'_{12} \), where \( 2n_1 M_{12} = z_1 = 1 \). Arguing again as in Section 6.1, we replace in (6.21) the processes \( a_k^{(0)}(l), a_k^{(0)}(l') \) and \( a_k^{(0)}(\tau) \) by their value at zero \( a_k^{(0)}(0) \). Next, using Theorem 10.3 we show that the input to the resulting sum of the term corresponding to \( M'_{12} \) is small. Finally, similarly to (6.19), we get

\[ \left| 4n_1 n_2^2 + 4 \sum_{1,2,3} n_2^{(0)} n_3^{(0)} n_s^{(0)} \right| \leq C^\#(s)(v^2 \chi_d(v) + v^{-2} L^2 + \tau^2 v), \]

where the sign “+” in the l.h.s. is due to the sign “−” in (6.21) (cf. (6.11)).

6.3. The last two terms of \( Q_s \). In Sections 6.1 and 6.2 we have seen that the first two terms of \( Q_s \) approximate the sum \( Z_s \). So to get assertion of the proposition it suffices to show that

\[ |\mathbb{E} \Delta a_s^{(1)}(\tau) \bar{c}_s^{(1)}(\tau)|, |\bar{N}_s| \leq C^\#(s)(v^2 \chi_d(v) + v^{-2} L^2). \tag{6.22} \]

We have

\[ \mathbb{E} \Delta a_s^{(1)} \bar{c}_s^{(1)}(\tau) = \mathbb{E} \int_0^\tau e^{-\gamma_s(\tau-l)} \mathcal{Y}_s(a, v^{-1}l) dl \int_0^0 e^{-\gamma_s(\tau-l')} \mathcal{Y}_s(a, v^{-1}l') dl'. \]

This expression coincides with (6.2) in which the integral \( \int_0^\tau dl' \) is replaced by \( \int_{-\tau}^0 dl' \). Then, \( \mathbb{E} \Delta a_s^{(1)} \bar{c}_s^{(1)}(\tau) \) has the form (6.3) where the same replacement is done. Since the correlations \( \mathbb{E} \Delta a_j(l) \bar{a}_j(l') \) are given by (2.8), Theorem 10.3 applies (see a discussion after its formulation) and we get the first inequality from (6.22).

Expressing the function \( \mathcal{Y}_s^{sym} \) through \( \mathcal{Y}_s \), for the term \( \bar{N}_s \) we find \( \bar{N}_s = \bar{N}_s^1 + 2 \bar{N}_s^2 \), where

\[ \bar{N}_s^1 = i \mathbb{E} \left( \bar{a}_s(\tau) \int_0^\tau e^{-\gamma_s(\tau-l)} \mathcal{Y}_s(a, a, c^{(1)}; v^{-1}l) dl \right) \]

and

\[ \bar{N}_s^2 = i \mathbb{E} \left( \bar{a}_s(\tau) \int_0^\tau e^{-\gamma_s(\tau-l)} \mathcal{Y}_s(c^{(1)}, a, a; v^{-1}l) dl \right). \]

Expressing \( c^{(1)} \) through \( a^{(0)} \), we see that the terms \( \bar{N}_s^1 \) and \( \bar{N}_s^2 \) have the forms (6.10) and (6.20) correspondingly, where the integral \( \int_0^l dl' \) is replaced by \( \int_{-\tau}^0 dl' \). Then the second inequality from (6.22) again follows from Theorem 10.3.  \( \square \)
7. Energy Spectra of Quasisolutions and Wave Kinetic Equation

Everywhere in this section in addition to (1.10) we assume that

\[ L \geq v^{-2-\epsilon}, \quad \epsilon > 0. \]  

(7.1)

Let us consider equation (4.9) with \( f(s) = 2b(s)^2 \) for \( \tau \geq -T \):

\[ \dot{z}(\tau, s) = -L\dot{z} + \epsilon K(z) + 2b(s)^2, \quad s \in \mathbb{R}^d, \]  

(7.2)

with the initial condition

\[ \dot{z}(-T) = 0. \]  

(7.3)

Fix any \( r_0 > d \). Denoting \( \epsilon(b) = \epsilon_\ast(2\|b^2(s)\|_{r_0}, r_0) > 0 \), where \( \epsilon_\ast \) is the constant from Theorem 4.4, we get from Corollary 4.5 that for \( 0 \leq \epsilon \leq \epsilon(b) \) a solution \( \dot{z} \) of (7.2), (7.3) exists, is unique and \( \dot{z} \in X_r \) for each \( r \), so that

\[ |\dot{z}(\tau)|_r \leq C_r, \quad \forall r, \]  

(7.4)

uniformly in \( \tau \). Our goal is to compare \( n^{\leq 2}_r(\tau) \) (extended to a function on \( \mathbb{R}^d = \{s\} \)) with \( \dot{z} \) in the spaces \( \dot{C}_r(\mathbb{R}^d) \), and we recall that, due to (5.14),

\[ |n^{\leq 2}(\tau)|_r \leq C_r \quad \forall r, \]  

(7.5)

uniformly in \( \nu, L, \tau \). The constants \( C_r \) below vary from formula to formula and we often skip the dependence on \( r \) writing simply \( C \).

Since both curves \( n^{\leq 2} \) and \( \dot{z} \) vanish at \(-T\), then their difference \( w = n^{\leq 2} - \dot{z} \) also does. Let us estimate the increments of \( w \).

**Proposition 7.1.** If \( \rho, \nu, L \) satisfy (1.10), (7.1), \( r > d \) and \( \epsilon \leq C_{1r}^{-1} \leq \epsilon(b) \), then

\[ |w(\tau + \tau')|_r \leq (1 - \tau/2)|w(\tau')|_r + C_2r\tau W \quad \forall \tau' \geq -T, \quad 0 \leq \tau \leq 1/2, \]  

(7.6)

where \( W = \epsilon(\tau + \epsilon + \tau^{-1}\nu^{1-\aleph_0}d + \tau^{-1}\nu^{-3}L^{-2}) \) and \( \aleph_0 \) is defined as in Theorem 3.3. The constants \( C_{1r}, C_{2r} \) do not depend on \( \tau, \tau', T \) and \( \nu, L, \epsilon \), but \( C_{2r} \) depends on \( \aleph_0 \).

**Proof.** The calculation below does not depend on \( \tau' \geq -T \) and to simplify presentation we take \( \tau' = 0 \). Then

\[ \dot{z}(\tau) = e^{-\tau L}z(0) + 2\int_0^\tau e^{-tL}b^2 dt + \epsilon \int_0^\tau e^{-(\tau-t)L}K(\dot{z}(t)) dt. \]

From here and (5.11) we have

\[ w(\tau) = e^{-\tau L}w(0) + \epsilon \Delta + R, \]  

(7.7)

where \( \Delta = [K^\tau(n^{\leq 2}(0)) - \int_0^\tau e^{-(\tau-t)L}K(\dot{z}(t)) dt] \). Now we will estimate \( \Delta \). By (5.9), it is made by the sum of four terms

\[ \int_0^\tau e^{-tL}K_j(n^{\leq 2}(0)) dt - \int_0^\tau e^{-(\tau-t)L}K_j(\dot{z}(t)) dt, \quad 1 \leq j \leq 4. \]

Their estimating is very similar. Consider, for example, the term with \( j = 1 \) and write it as

\[ \int_0^\tau e^{-(\tau-t)L}(K_1(n^{\leq 2}(0)) - K_1(\dot{z}(0))) dt \]
In view of Corollary 4.3 and (4.12), $|\Delta^1|_r \leq C \tau |w(0)|_r$, where $C$ depends on the norms $|n^{\pm 2}(0)|_r$ and $|\tilde{z}(0)|_r$. By (7.5) and (7.4) the two norms are bounded by constants, so the constant $C = C_r$ is absolute. Consider $\Delta^2$. From (7.4) and the equation on $\tilde{z}$ we have that $|\tilde{z}|_r \leq C_r$, so $|\tilde{z}(0) - \tilde{z}(\tau)|_r \leq C_r \tau$. Hence, by Corollary 4.3 and (4.12), $|\Delta^2|_r \leq C_r \tau^2$.

We have seen that

$$|\Delta|_r \leq C'_{1r} \tau (\tau + |w(0)|_r).$$

(7.8)

Since $0 \leq \tau \leq 1/2$, then by (4.12) we have $|e^{-\tau L}w(0)|_r \leq (1 - \tau)|w(0)|_r$. So using the bound (5.12) for $|\mathcal{R}|_r$, by (7.7) we get

$$|w(\tau)|_r \leq (1 - \tau)|w(0)|_r + C'_{1r} \epsilon \tau |w(0)|_r + C_{2r} \tau W$$

with $W$ as in the proposition. This relation implies the assertion with $C_{1r}^{-1} = \min((2C'_{1r})^{-1}, \epsilon(b))$. □

Denote $\tilde{w}(\tau) = e^{-\tau L}w(0)$. Then $\frac{d}{d\tau} \tilde{w}(\tau) = -L\tilde{w}(\tau)$, so $|\tilde{w}(\tau) - w(0)|_r \leq \int_0^\tau |L\tilde{w}(\tau)|_r \, dt \leq C \tau$, where in the latter inequality we used (7.4) and (7.5). Note also that, due to (5.12), $|\mathcal{R}|_r \leq C \tau$ provided that (7.1) holds and $v^{1-\delta_d} \leq C_1 \tau$. These relations, equality (7.7) and estimate (7.8) on the term $\Delta$ imply

**Corollary 7.2.** Assume that $v^{1-\delta_d} \leq C \tau$ for some constant $C > 0$. Then, under the assumptions of Proposition 7.1,

$$|w(\tau') + v - w(\tau)|_r \leq C_1 \tau \delta_d \tau, \quad \forall \tau' \geq -T.$$

(7.9)

Now we state and prove the main result of our work. Let us again fix any $r_0 > d$ and set

$$\epsilon_r = C_1^{-1} \max(r, r_0),$$

where $C_{1r}$ is the constant from Proposition 7.1. We recall that the energy spectrum $n_s(\tau) = n_s(\tau; v, L)$ of a quasisolution $A(\tau)$ naturally extends to a Schwartz function of $s$, see (5.5).

**Theorem 7.3.** Let $v$ and $L$ satisfy (7.1), let $A(\tau)$ be the quasisolution (5.1), corresponding to $\rho^2 = ev^{-1}$, let $n_s(\tau) = E[A_s(\tau)]^2$ be its energy spectrum and $\tilde{z}(\tau, s)$ be the (unique) solution of (7.2), (7.3). Then for any $r$ and for $0 < \epsilon \leq \epsilon_r$ there exists $v_\epsilon(r) > 0$ such that for $v \leq v_\epsilon(r)$ we have

$$|n(\tau) - \tilde{z}(\tau)|_r \leq C_r \epsilon^2 \quad \forall \tau \geq -T, \quad \forall r.$$

(7.10)

The constant $C_r$ does not depend on $\tau$ and $T$.

Note that $n(\tau)$ has the form (5.2), where $n^{(1)} = 0$ and the first nontrivial term $\rho^2 n^{(2)}$ is of order $\epsilon$, which is much bigger than the r.h.s. of (7.10) if $\epsilon \ll 1$. Similarly, $\tilde{z}(\tau)$ has the form (4.15), where the first nontrivial term $\epsilon u^1$ is also of the size $\epsilon$. 


Proof. Since $| \cdot |_{r'} \leq | \cdot |_r$ for $r' \leq r$, then estimate (7.10) for $r < r_0$ follows from (7.10) with $r = r_0$. Assume now that $r \geq r_0$. Since $w(t) = n^{-2}(t) - \zeta(t)$, then in view of (5.15) it suffices to establish that

$$|w(t')|_r \leq C \epsilon^2 \quad \forall t' \geq -T$$  

(7.11)

(we assume $\nu_\epsilon \ll 1$). Let us fix some $0 < \nu_\epsilon \ll 1$ and any time-step $\tau$, satisfying

$$C_0 \nu_\epsilon^{-1} \nu_\epsilon^{1-\nu_\epsilon} \leq \tau \leq C_0^{-1} \epsilon^2,$$  

(7.12)

for a sufficiently large constant $C_0 > 0$. We claim that

$$w_n := |w(-T + n\tau)|_r \leq CC_2r \epsilon^2 \quad \forall n \in \mathbb{N} \cup \{0\},$$  

(7.13)

where $C_2r = C_{2r, \nu_\epsilon}$ is the constant from Proposition 7.1. Indeed, let us fix any $N \in \mathbb{N}$ and let $w_n, n = 0, \ldots, N$, attains its maximum at a point $n$, which we write as $n = k_0 + 1$. If $k_0 + 1 = 0$, then $w_k \equiv 0$. Otherwise, in view of (7.6),

$$w_{k_0} \leq w_{k_0 + 1} \leq (1 - \tau/2)w_{k_0} + C_{2r} \tau W;$$

so $w_{k_0} \leq 2C_{2r} W$. From here $\max_{0 \leq k \leq N} |w(-T + n\tau)|_r = w_{k_0 + 1} \leq 3C_{2r} W$ for any $N$, and (7.13) follows in view of (7.12) and (7.1).

Since for any $t' \geq -T$ we can find an $n$ such that $t' \in [-T + n\tau, -T + (n + 1)\tau]$, then (7.11) follows from (7.13) and (7.9). □

By Proposition 5.4, when $L \to \infty$ and $v$ stays fixed, the energy spectrum $n_s(\tau; v, L)$ admits a limit $n_s(\tau; v, \infty)$, which is a Schwartz function of $s \in \mathbb{R}^d$. Since estimate (7.10) is uniform in $v, L$, we immediately get

**Corollary 7.4.** For $0 < \epsilon \leq \epsilon_r, \rho = \epsilon^{1/2} \nu^{-1/2}$ and $\nu \leq \nu_\epsilon(r)(\nu_\epsilon(r)$ as in Theorem 7.3), the limiting energy spectrum $n_s(\tau; v, \infty)$ satisfies estimate (7.10).

Jointly with Proposition 4.6, Theorem 7.3 implies exponential convergence of $n(\tau)$ to an equilibrium, modulo $\epsilon^2$:

**Corollary 7.5.** For $r > d$ there exists $\epsilon' = \epsilon'(r, |b^2|_r)$ such that if $0 < \epsilon \leq \min(\epsilon', C_1^{-1})$, then eq. (7.2) has a unique steady state $\zeta^{\epsilon}$ close to $\zeta^0 = 2L^{-1} b^2$, and

$$|n(\tau) - \zeta^{\epsilon}|_r \leq e^{-\tau - T} |\zeta^{\epsilon}|_r + C_r \epsilon^2, \quad \forall \tau \geq -T.$$  

In view of Corollary 2.2, Theorem C from the introduction follows from Theorem 7.3 and Corollary 7.4. Similarly, the asymptotic (1.37) follows from Corollary 7.5.

By (4.15), the solution $\zeta$ of (7.2), (7.3) may be written as $\zeta = \zeta^0 + \epsilon \zeta^1 + O(\epsilon^2)$, where

$$\zeta^0 = -\mathcal{L} \zeta^0 + 2b(s)^2, \quad \zeta^0(0-T) = 0,$$

$$\zeta^1 = -\mathcal{L} \zeta^1 + K(\zeta^0), \quad \zeta^1(0-T) = 0.$$  

From the first equation we see that $\zeta^0(s) = n_s^{(0)}(s)$ (see (1.28)), and from the second—that

$$\zeta^1(\tau) = \int_{-T}^{\tau} e^{-(T-l)\mathcal{L}} K(n_s^{(0)}(l)) \, dl.$$  

Since, by (5.15), $n(\tau) = n_s^{(0)}(\tau) + \epsilon (\nu^{-1} n_s^{(2)}(\tau)) + O(\epsilon^2)$, then (7.10) implies that

$$n_s^{(2)}(\tau) = \nu \int_{-T}^{\tau} e^{-(T-l)\mathcal{L}} K(n_s^{(0)}(l)) \, dl + O(\nu \epsilon),$$  

(7.14)

where $|O(\nu \epsilon)|_r \leq \nu \epsilon C_r$ for every $r$. Other way round, now, after the exact form of the operator $K$ in (7.2) is established, the validity of the presentation (7.14) (obtained by some direct calculation), jointly with estimate (5.15) would imply Theorem 7.3.
8. Energy Spectra of Solutions (2.2)

Results of Sections 5 and 7 concerning the quasisolutions suggest a natural question if they extend to higher order truncations of the complete decomposition (2.2) in series in \( \rho \). In this section we discuss the corresponding positive and negative results, obtained in [8]. For a complex number \( z \) we denote by \( z^* \) either \( z \) or \( \bar{z} \).

Firstly let us return to Section 2. Iterating the Duhamel integral in the r.h.s. of (2.4) and expressing there iteratively \( a^{(n,j)}(l) \) with \( 1 \leq n_j < n \) via integrals (2.4) with \( n := n_j \), we will eventually represent each \( a^{(n)}_s(\tau) \) as a sum of iterated integrals of the form

\[
J_s(T) = J_s(\tau; n, T) = \int \ldots \int dl_1 \ldots dl_n L^{-nd} \sum_{s_1, \ldots, s_{3n}} (\ldots). \tag{8.1}
\]

The zone of integrating in \( l = (l_1, \ldots, l_n) \) is a convex polyhedron in the cube \([-T, \tau]^n\), and the summation is taken over the vectors \((s_1, \ldots, s_{3n}) \in (\mathbb{Z}_L^d)^{3n}\) subject to certain linear relations which follow from the factor \( \delta_{3s}^{12} \) in the definition (2.1) of \( \mathcal{Y}_s \). The summand in brackets in (8.1) is a monomial of exponents \( e^{-\gamma_\nu(l_k-l_j)}e^{\pm l\nu^{-1}\omega_{l_k}^0} \) and the processes \((a^{(0)}_s(l))^{*}(l_r)\), which has degree \( 2n + 1 \) with respect to the processes. Each integral \( J_s(\tau; n, T) \) corresponds to an oriented rooted tree \( T \) from a class \( \Gamma(n) \) of trees with the root at \( a^{(n)}_s(\tau) \), with random variables \((a^{(0)}_s(n))^{*}(l_r)\) at its leaves and with vertices labelled by \((a^{(n')}_s(n'))^{*}(l'_r)\) (\( 1 \leq n' < n \)). To any vertex \((a^{(n')}_s(n'))^{*}(l'_r)\) enters one edge of the tree and three edges outgo from it to the vertices or leaves, corresponding to some three specific terms \((a^{(n_1)}_s(n))^{*}, (a^{(n_2)}_s(n))^{*}, (a^{(n_3)}_s(n))^{*}\) in the decomposition (2.4) of \((a^{(n')}_s(n'))^{*}(l'_r)\). So

\[
a^{(n)}_s(\tau) = \sum_{T \in \Gamma(n)} J_s(\tau; n, T). \tag{8.2}
\]

Now let us consider the formal series

\[
N_s(\tau) = n^{0}_s(\tau) + \rho n^1_s(\tau) + \rho^2 n^2_s(\tau) + \ldots \tag{8.3}
\]

for the energy spectrum \( N_s = \mathbb{E}|a_s|^2 \) of a solution \( a_s(\tau) \), when the latter is written as the formal series (2.2). There \( n^0_s \sim 1, n^1_s = 0 \) and \( n^2_s \) are the same as \( n^{(0)}_s, n^{(1)}_s \) and \( n^{(2)}_s \) in the decomposition (5.2) of quasisolutions, but \( n^3_s \) and \( n^4_s \) are different. This small ambiguity should not cause a problem, and we will see below that the new \( n^3_s \) and \( n^4_s \) still meet the estimates (5.13). Let us consider any \( n^k_s(\tau) \). It equals

\[
n^k_s(\tau) = \mathbb{E} \sum_{k_1+k_2=k} a^{(k_1)}_s(\tau)\bar{a}^{(k_2)}_s(\tau). \tag{8.4}
\]

We analyse each expectation \( \mathbb{E} a^{(k_1)}_s(\tau)\bar{a}^{(k_2)}_s(\tau) \) separately. Due to (8.2),

\[
\mathbb{E} a^{(k_1)}_s(\tau)\bar{a}^{(k_2)}_s(\tau) = \sum_{T_1 \in \Gamma(k_1), T_2 \in \Gamma(k_2)} \mathbb{E} J_s(\tau; k_1, T_1) J_s(\tau; k_2, T_2), \tag{8.5}
\]

with

\[
\mathbb{E} J_s(\tau; k_1, T_1) J_s(\tau; k_2, T_2) = \int \ldots \int dl_1 \ldots dl_k L^{-kd} \sum_{s_1, \ldots, s_{3k}} \mathbb{E}(\ldots), \tag{8.6}
\]
where \( k = k_1 + k_2 \) and under the expectation sign stands a product of the terms in the brackets in the presentations \((8.1)\) for \( I_s(\tau; k_1, T_1) \) and \( I_s(\tau; k_2, T_2) \).

Since \( a^{(0)}_s(l) \) are Gaussian random variables whose correlations are given by \((2.8)\), then by the Wick theorem (see \([15]\)) each expectation \((8.6)\) is a finite sum over different Wick pairings between the conjugated and non-conjugated variables \( a^{(0)}_s(l'_j) \) and \( a^{(0)}_s(l''_j) \), labelling the leaves of \( T_1 \cup T_2 \). By \((2.8)\), for each Wick pairing, in the sum \( \sum_{s_1,\ldots,s_{3k}} \) from \((8.6)\) only those summands do not vanish for which indices \( s' \) and \( s'' \) of the Wick paired variables are equal, \( s' = s'' \). So we take the sum only over vectors \( s_1,\ldots,s_{3k} \) satisfying this restriction. The Wick pairings in \((8.6)\) can be parametrised by Feynman diagrams \( F \) from a class \( \mathcal{F}(k_1, k_2) \) of diagrams, obtained from the union \( T_1 \cup T_2 \) of various trees \( T_1 \in \Gamma(k_1) \) and \( T_2 \in \Gamma(k_2) \) by all pairing of conjugated leaves with non-conjugated leaves in \( T_1 \cup T_2 \). We denote the summands, forming the r.h.s. of \((8.6)\), by \( I_s(F) \), where \( F \in \mathcal{F}(k_1, k_2) \), and accordingly write \((8.5)\) as

\[
\mathbb{E}a^{(k_1)}_s(\tau)a^{(k_2)}_s(\tau) = \sum_{F \in \mathcal{F}(k_1, k_2)} I_s(F), \quad I_s(F) = I_s(\tau; k_1, k_2, F). \tag{8.7}
\]

Then, \((8.4)\) takes the form

\[
n_s(\tau) = \sum_{k_1+k_2=k} \sum_{F \in \mathcal{F}(k_1, k_2)} I_s(F). \tag{8.8}
\]

See \([8]\) for a detailed explanation of the formulas \((8.1)-(8.8)\).

Resolving all the restrictions on the indices \( s_1,\ldots,s_{3k} \) in \((8.6)\) which follow from the rules, used to construct the trees and the diagrams, we find that among those indices exactly \( k \) are independent. Suitably parametrizing them by vectors \( z_1,\ldots,z_k \in \mathbb{Z}_L^d \) we write the sum in \((8.6)\) as \( \sum_{z_1,\ldots,z_k} \). Approximating the latter by an integral over \( \mathbb{R}^{kd} \) using Theorem 3.1, we get for integrals \( I_s(F) \) with \( F \in \mathcal{F}(k_1, k_2) \) an explicit formula:

\[
I_s(F) = \int_{\mathbb{R}^k} dl \int_{\mathbb{R}^{kd}} dz \, F_{\mathcal{F}}(\tau, s, l, z) e^{i\sum_{i,j=1}^k \alpha^{F}_{ij}(l_i-l_j)z_i\cdot z_j} + O(L^{-2}v^{-2}), \tag{8.9}
\]

where \( k = k_1 + k_2, z = (z_1,\ldots,z_k) \), the function \( F_{\mathcal{F}} \) is smooth in \( s \) and fast decays in \( s \) and \( l \), while \( \alpha^F = (\alpha^F_{ij}) \) is a skew-symmetric matrix without zero lines and rows. Its rank is \( \geq 2 \) and for any \( k \) it may be equal to 2. In particular, \((8.7)\) together with \((8.9)\) implies Proposition 5.4 since the integrals in \((8.9)\) are independent from \( L \) and are Schwartz functions of \( s \).

Now let us go back to the series \((8.3)\). We know that \( n_s^0 \sim 1, n_s^1 = 0 \) and \( n_s^2 \sim C^\#(s)v \). By a direct (but long) calculation, similar to that in Appendix 12.3, it is possible to verify that \( |n_s^3| \leq C^\#(s)v^3/2 \chi_d(v) \leq C^\#(s)v^{3/2} \) and \( |n_s^4| \leq C^\#(s)v^2 \). This suggests to assume that

\[
|n_s^k| \leq C^\#(s; k) v^{k/2} \quad \text{for any } k. \tag{8.10}
\]

If this is the case, then under the “kinetic” scaling \( \rho = v^{-1/2} \varepsilon^{1/2} \) the series \((8.3)\) becomes a formal series in \( \sqrt{\varepsilon} \), uniformly in \( v \) and \( L \), and its truncation of any order \( m \geq 2 \) in \( \rho \) still is \( O(\varepsilon^2) \)-close to the solution \( \chi(\tau) \) of the wave kinetic equation \((7.2),(7.3)\). On the contrary, if this is not the case in the sense that \( |n_s^k| \geq C_k v^{k'} \) with \( k' < k/2 \) for some \( k \), then \((8.3)\) even is not a formal series, uniformly in \( v, L \) under the kinetic scaling above.

Analysing formula \((8.9)\), we obtain estimate \((5.7)\) of Theorem 5.1 for every integral \( I_s(F) \), which implies estimate \((5.7)\) itself (see Theorem 1.2 in \([8]\)). By \((8.4)\), estimate \((5.7)\) implies
Theorem 8.1. For each $k$,

$$|n_k^S| \leq C^#(s; k) \max(v^{[k/2]}, v^d) \chi_d^S(v), \quad (8.11)$$

provided that $L$ is so big that $C^#(s; k)L^{-2}v^{-2}$ is smaller than the r.h.s. of $(8.11)$.

Below in this section we always assume that $L$ is as big as it is required in the theorem. The theorem implies estimate $(8.10)$ only for $k \leq 2d$. In particular, for $k \leq 4$ since $d \geq 2$.

To get estimate $(8.11)$, we establish it for each integral $I_s(\mathcal{F})$ separately. Our next result shows that estimate $(8.10)$ can not be obtained by improving $(8.11)$ for every integral $I_s(\mathcal{F})$, since inequality $(8.11)$ is sharp for some of the integrals.

Let $\mathfrak{F}^B(k)$ be the set of Feynman diagrams in $\cup_{k_1+k_2=k}\mathfrak{F}(k_1, k_2)$ for which the matrix $\alpha^\mathcal{F}$ from (8.9) satisfies $\alpha^\mathcal{F}_{ij} = 0$ if $i \neq p$ and $j \neq p$ (so, only the $p$-th line and column of the matrix $\alpha^\mathcal{F}$ are non-zero). This set is not empty.

Proposition 8.2. If $k > 2d$, then for any $\mathcal{F} \in \mathfrak{F}^B(k)$ the corresponding integrals $I_s(\mathcal{F})$ satisfy $I_s(\mathcal{F}) \sim C^#(s; k)v^d$.

Proposition 8.2 shows that the estimate $(8.10)$ can be true for $k > 2d$ only if in the sum $(8.8)$ the large terms cancel each other. And indeed, we observe strong cancellations among the integrals from the set $\mathfrak{F}^B(k)$:

Proposition 8.3. For any $k$, $\left| \sum_{\mathcal{F} \in \mathfrak{F}^B(k)} I_s(\mathcal{F}) \right| \leq C^#(s; k)v^{k-1}$.

Since for $k \geq 2$ we have $v^{k-1} \leq v^{k/2}$, the estimate in the proposition agrees with $(8.10)$. In the proofs of Propositions 8.2 and 8.3 we use special structure of the set $\mathfrak{F}^B(k)$ and we do not have similar results for a larger set of diagrams, nevertheless we find it plausible that $(8.10)$ is true. Namely, that the decomposition of integrals $I_s(\mathcal{F})$ in asymptotic sum in $v$ is such that a few main order terms of the decomposition of the sum $(8.8)$ vanish due to a cancellation, so that $(8.10)$ holds (see Problem 1.1). We understand the mechanism of this cancellation, but do not know if it goes till the order $v^{k/2}$.

9. Proof of Theorem 3.3

9.1. Vicinity of the point $(s, s)$. Off the quadric $^s\Sigma$ the integral $I_s$ is small—of order $v^2$—and the main task is to examine it in the vicinity of $^s\Sigma$. First we will study $I_s$ near the locus $(s, s)$, and next—near the smooth part $^s\Sigma_\delta$.

Passing to the variables $(x, y, s) = (z, s)$ (see (3.15)) we write $F_s$ and $\Gamma_s$ as $F_s(z)$ and $\Gamma_s(z)$. The functions still satisfy (3.9) and (3.10) with $(s_1, s_2)$ replaced by $(x, y)$ since the map $(s_1, s_2, s) \mapsto (x, y, s)$ is a linear isomorphism.

Consider the domain

$$K_\delta = \{|x| \leq \delta, |y| \leq \delta\} \subset \mathbb{R}^d \times \mathbb{R}^d, \quad 0 < \delta \leq 1, \quad (9.1)$$

and the integral $\langle I_s, K_\delta \rangle$ (see Notation, formula (1.44)):

$$\langle I_s, K_\delta \rangle = v^2 \int_{K_\delta} \frac{F_s(z) \, dz}{(x \cdot y)^2 + (v \Gamma_s(z))^2}. \quad (9.2)$$

Obviously, everywhere in $K_\delta$, $|F_s(z)| \leq C^#(s)$ and $\Gamma_s(z) \geq K^{-1}$. Then, since the volume of $K_\delta$ is bounded by $C\delta^{2d}$,

$$\left| \langle I_s, K_\delta \rangle \right| \leq C^#(s)\delta^{2d}. \quad (9.3)$$
Next we pass to the global study of integral $I_s$, written similar to (9.2) as

$$I_s = v^2 \int_{\mathbb{R}^{2d}} \frac{F_s(z)}{(x \cdot y)^2 + (v \Gamma_s(z))^2} \, dz.$$  

(9.4)

9.2. The manifold $\Sigma_*$ and its vicinity. We denote by $\Sigma$ the quadric $\Sigma = \{z = (x, y) : x \cdot y = 0\}$ and $\Sigma_* = \Sigma \setminus (0, 0)$. The set $\Sigma_*$ is a smooth manifold of dimension $2d - 1$. Let $\xi \in \mathbb{R}^{2d-1}$ be a local coordinate on $\Sigma_*$ with the coordinate mapping $\xi \mapsto z_\xi = (x_\xi, y_\xi) \in \Sigma_*$. Abusing notation we write $|\xi| = |(x_\xi, y_\xi)|$. The vector $N(\xi) = (y_\xi, x_\xi)$ is a normal to $\Sigma_*$ at $\xi$ of length $|\xi|$, and

$$\langle N(\xi), (x_\xi, y_\xi) \rangle = 2x_\xi \cdot y_\xi = 0.$$  

For any $0 < R_1 < R_2$ we denote

$$S_{R_1} = \{z \in \mathbb{R}^{2d} : |z| = R_1\}, \quad \Sigma_{R_1} = \Sigma \cap S_{R_1},$$

$$S_{R_2} = \{z : R_1 < |z| < R_2\}, \quad \Sigma_{R_2} = \Sigma \cap S_{R_2},$$  

(9.5)

and for $t > 0$ denote by $D_t$ the dilation operator

$$D_t : \mathbb{R}^{2d} \to \mathbb{R}^{2d}, \quad z \mapsto tz.$$  

It preserves $\Sigma_*$, and we write $D_t \xi = t \xi$ for $\xi \in \Sigma_*$. The following lemma, specifying the geometry of $\Sigma_*$ and its vicinity, is proved in [19]:

**Lemma 9.1.** (1) There exists $\theta_0^* > 0$ such that for any $0 < \theta_0 \leq \theta_0^*$ a suitable neighbourhood $\Sigma^v = \Sigma^v(\theta_0)$ of $\Sigma_*$ in $\mathbb{R}^{2d} \setminus \{0\}$ may be uniquely parametrised as

$$\Sigma^v = \{\pi(\xi, \theta), \xi \in \Sigma_*, |\theta| < \theta_0\},$$  

(9.6)

where

$$\pi(\xi, \theta) = (x_\xi, y_\xi) + \theta N_\xi = (x_\xi, y_\xi) + \theta (y_\xi, x_\xi).$$

(2) For any vector $\pi := \pi(\xi, \theta) \in \Sigma^v$ its length equals

$$|\pi| = |\xi| \sqrt{1 + \theta^2}.$$  

The distance from $\pi$ to $\Sigma$ equals $|\xi||\theta|$, and the shortest path from $\pi$ to $\Sigma$ is the segment $S := [\pi(\xi, t\theta) : 0 \leq t \leq 1]$.  

(3) If $z = (x, y) \in S^R$ is such that $\text{dist}(z, \Sigma) \leq \frac{1}{2} R \theta_0$, then $z = \pi(\xi, \theta) \in \Sigma^v$, where

$$|\xi| \leq R \leq |\xi| \sqrt{1 + \theta_0^2}.$$  

(4) If $(x, y) = \pi(\xi, \theta) \in \Sigma^v$, then

$$x \cdot y = |\xi|^2 \theta.$$  

(9.7)

(5) If $(x, y) \in S^R \cap (\Sigma^v)^c$, then $|x \cdot y| \geq c|\theta_0|^2 R^2$ for some $c > 0$.  

The coordinates (9.6) are known as the normal coordinates, and their existence follows easily from the implicit function theorem. The assertion 1) is a bit more precise than the general result since it specifies the width of the neighbourhood $\Sigma^v$.

Let as fix any $0 < \theta_0 \leq \theta_0^*$, and consider the manifold $\Sigma^v(\theta_0)$. Below we provide it with additional structures and during the corresponding constructions decrease $\theta_0^*$, if needed. Consider the set $\Sigma^1$ (see (9.5)). It equals

$$\Sigma^1 = \{(x, y) : x \cdot y = 0, |x|^2 + |y|^2 = 1\},$$

and is a smooth compact submanifold of $\mathbb{R}^d$ of codimension 2. Let us cover it by some finite system of charts $\mathcal{N}_1, \ldots, \mathcal{N}_n, \mathcal{N}_j = \{\eta^j = (\eta_1^j, \ldots, \eta_{2d-2}^j)\}$. Denote by $m(d\eta)$ the volume element on $\Sigma^1$, induced from $\mathbb{R}^d$, and denote the coordinate maps as $\mathcal{N}_j \ni \eta^j \mapsto (x_{\eta^j}, y_{\eta^j}) \in \Sigma^1$. We will write points of $\Sigma^1$ both as $\eta$ and $(x_{\eta}, y_{\eta})$.

The mapping

$$\Sigma^1 \times \mathbb{R}^+ \to \Sigma_* \quad (\eta, t) \to D_t(x_{\eta}, y_{\eta}) \in \Sigma^t,$$

is 1-1, and is a local diffeomorphism; so this is a global diffeomorphism. Accordingly we can cover $\Sigma_*$ by the $\tilde{n}$ charts $\tilde{\mathcal{N}}_{\tilde{j}} \times \mathbb{R}^+$, with the coordinate maps $(\eta^j, t) \mapsto D_t(x_{\eta^j}, y_{\eta^j})$, $\eta^j \in \tilde{\mathcal{N}}_{\tilde{j}}, t > 0$. In these coordinates the volume element on $\Sigma^t$ is $t^{2d-2}m(d\eta)$. Since $\partial/\partial t$ and $\partial/\partial \theta$ form an orthonormal base of the orthogonal complement in $\mathbb{R}^d$ to $T_{(\eta, t, \theta=0)}\Sigma^t$, then in the domain $\Sigma^v$ the volume element $d\Sigma = dx \, dy$ may be written as

$$d\Sigma = t^{2d-2} \mu(\eta, t, \theta) m(d\eta) dt \, d\theta, \quad \mu(\eta, t, 0) = 1. \quad (9.9)$$

The transformation $D_r : (\eta, t, \theta) \mapsto (\eta, rt, \theta)$, $r > 0$, multiplies the form in the l.h.s. by $r^{2d}$, preserves $d\eta$ and $d\theta$, and multiplies $dt$ by $r$. Hence, $\mu$ does not depend on $t$, and we have got

**Lemma 9.2.** The coordinates $(\eta^j, t, \theta)$, where $\eta^j \in \mathcal{N}_j$, $t > 0$, $|\theta| < \theta_0$, and $1 \leq j < \tilde{n}$, define on $\Sigma^v$ coordinate systems, jointly covering $\Sigma^v$. In these coordinates the dilations $D_r$, $r > 0$, read as $D_r : (\eta, t, \theta) \mapsto (\eta, rt, \theta)$, and the volume element has the form (9.9), where $\mu$ does not depend on $t$.

Consider the mapping

$$\Pi : \Sigma^v \to \Sigma_*, \quad z = \pi(\xi, \theta) \mapsto \xi.$$

By Lemma 9.1.2, $|z| \leq |\Pi(z)| \leq 2|z|$.

For $0 < R_1 < R_2$ we will denote

$$\Sigma^{v}_{R_1} = \Sigma^v \cap (\Pi^{-1}(\mathbb{R}_{R_1}^{R_2})). \quad (9.10)$$

In coordinates $(\eta^j, t, \theta)$ this domain is $\{\eta^j, t, \theta) : R_1 < t < R_2, |\theta| < \theta_0\}$. 

A. Dymov, S. Kuksin
Let us consider functions $\Gamma$ and $F$ in the variables $(\eta, t, \theta)$. Consider first $\Gamma_s(z)$ with $z = \pi(\xi, \theta) \in \Sigma^v$. Since $\pi(\xi, \theta) = z \pm \theta N \xi$, then $\frac{\partial^k \Gamma_s(z)}{\partial \theta^k} = \partial_z^k \Gamma_s(z)(N(\xi), \ldots, N(\xi))$. As $|N(\xi)| = |\xi| \leq |z|$, then by (3.10),

$$|\Gamma_s| \geq K^{-1}, \quad \left| \frac{\partial^k}{\partial \theta^k} \Gamma_s \right| \leq C_1 K \langle (z, s) \rangle^{r_1-k} |N \xi|^k \leq C_2 K \langle (z, s) \rangle^{r_1}, \quad (9.11)$$

for $k \leq 3$. Similar, since $F$ satisfies (3.9), then for $z = \pi(\eta, t, \theta)$ we have

$$F_s(\eta, t, \theta) \in C^2 \quad \text{and} \quad \left| \frac{\partial^k}{\partial \theta^k} F_s \right| \leq C^#(t, s), \quad k \leq 2. \quad (9.12)$$

9.3. Integral over the complement to a neighbourhood of $\Sigma$. On $\mathbb{R}^n \times \mathbb{R}^d$ let us define the function

$$\Theta = \Theta(t, s) = \langle (t, s) \rangle^{-r_1} \leq 1$$

($r_1 \geq 0$ is the exponent in (3.10)), and consider a neighbourhood of $\Sigma$:

$$\Sigma^{nbh}(\theta_0) = \{\pi(\xi, \theta) : |\theta| \leq \theta_0 m \} \subset \Sigma^v(\theta_0), \quad \theta_0 m = \theta_0 \Theta(t, s).$$

Consider the integral over its complement, $\Upsilon^m_s(\theta_0) = \langle |I_s|, \mathbb{R}^{2d} \setminus \Sigma^{nbh}(\theta_0) \rangle$. Using the polar coordinates in $\mathbb{R}^{2d}$, we have

$$\Upsilon^m_s(\theta_0) \leq \langle |I_s|, \{|z| \leq v^b\} \rangle + v^2 C_d \int_{v^b}^\infty dr r^{2d-1} \int_{S^1 \setminus \Sigma^{nbh}(\theta_0)} \frac{|F_s(z)| d s_1}{(x \cdot y)^2 + (v \Gamma_s(z))^2},$$

where we choose $b = \frac{1}{2}$ and denote $d s_1$ is the normalised Lebesgue measure on $S^1$. By item 5) of Lemma 9.1 with $\theta_0$ replaced by $\theta_0 \Theta(t, s)$, the divisor of the integrand is $\geq C^{-2} r^4 \theta_0^4 \Theta^4$. So the internal integral is bounded by $C^#(r, s)r^{-4} \langle (t, s) \rangle^{r_1}$. Due to this, (9.12) and (9.3) the r.h.s. is bounded by

$$C_1^#(s) v^{2bd} + v^2 C^#(s) \int_{v^b}^\infty C^#(r) r^{2d-5} dr \leq C_1^#(s) v^2 \chi_d(v)$$

(recall the notation (1.43)). Accordingly,

$$\Upsilon^m_s(\theta_0) \leq C^#(s) v^2 \chi_d(v). \quad (9.13)$$

9.4. Integral over the vicinity of $\Sigma$. Let $\Sigma^{vm}_s$ be a neighbourhood of $\Sigma_s$ such that

$$\Sigma^{nbh}(\frac{1}{2} \theta_0) \subset \Sigma^{vm}_s \subset \Sigma^{nbh}(2 \theta_0). \quad (9.14)$$

Then in view of (9.13)

$$|\langle I_s, \Sigma^{nbh}(2 \theta_0) \rangle - \langle I_s, \Sigma^{vm}_s \rangle| \leq \langle |I_s|, \mathbb{R}^{2d} \setminus \Sigma^{nbh}(\frac{1}{2} \theta_0) \rangle$$

$$= \Upsilon^m_s(\theta_0) \leq C_1^#(s) v^2 \chi_d(v).$$
In view of this estimate and (9.13),

$$\left| \langle I_s, \mathbb{R}^{2d} \rangle - \langle I_s, \Sigma_s^{um} \rangle \right| \leq C_{1}^{#}(s) v^{2} \chi_{d}(v). \quad (9.15)$$

So to prove the theorem it suffices to calculate integrals $\langle I_s, \Sigma_s^{vm} \rangle$ over any domains $\Sigma_s^{vm}$ as in (9.14). Below we will do this for domains $\Sigma_s^{vm}$ of the form

$$\Sigma_s^{vm} = \{(\eta, t, \theta) : -\theta^{-}(t, s) < \theta < \theta^{+}(t, s)\},$$

with suitably defined functions $\theta^{\pm}_{0}(t, s) \in [\frac{1}{2}\theta_{0m}, 2\theta_{0m}]$.

We estimate the integrals $\langle I_s, \Sigma_s^{vm}(\theta_{0}) \rangle$ in 4 steps.

**9.5. Step 1: disintegration of $I_s$.** For any $0 < R_{1} < R_{2}$ we define domains $(\Sigma^{nbh}(\theta_{0}))_{R_{1}}^{R_{2}} = \Sigma^{nbh}(\theta_{0}) \cap (\Pi^{-1} \Sigma_{R_{1}}^{R_{2}})$ and using (9.9) write integral $\langle I_s, (\Sigma^{nbh}(\theta_{0}))_{R_{1}}^{R_{2}} \rangle$ as

$$v^{2} \int_{\Sigma^{1}} m(d\eta) \int_{R_{1}}^{R_{2}} dt \ t^{2d-1} \int_{-\theta_{0m}}^{\theta_{0m}} d\theta \ \frac{F_{s}(\eta, t, \theta) \mu(\eta, \theta)}{(x \cdot y)^{2} + (v\Gamma_{s}(\eta, t, \theta))^{2}}$$

$$= v^{2} \int_{\Sigma^{1}} m(d\eta) \int_{R_{1}}^{R_{2}} J_{s}(\eta, t) t^{2d-1} dt$$

(cf. (9.4)), where by (9.7)

$$J_{s}(\eta, t) = \int_{-\theta_{0m}}^{\theta_{0m}} d\theta \ \frac{F_{s}(\eta, t, \theta) \mu(\eta, \theta)}{t^{4}\theta^{2} + (v\Gamma_{s}(\eta, t, \theta))^{2}}.$$

To study $J_{s}$ let us write $\Gamma_{s}$ as

$$\Gamma_{s}(\eta, t, \theta) = h_{\eta, t, s}(\theta) \Gamma_{s}(\eta, t, 0).$$

The function $h_{\eta, t, s}(\theta) =: h(\theta)$ is $C^{3}$-smooth and in view of (9.11) satisfies

$$h(0) = 1, \quad \left| \frac{\partial^{k}}{\partial \theta^{k}} h(\theta) \right| \leq C_{\Theta}^{-1} \quad \forall \ 1 \leq k \leq 3, \quad (9.16)$$

for all $\eta, t, \theta, s$. Denoting $\varepsilon = vt^{-2} \Gamma_{s}(\eta, t, 0)$, we write $J_{s}$ as

$$J_{s} = t^{-4} \int_{-\theta_{0m}}^{\theta_{0m}} \frac{F_{s}(\eta, t, \theta) \mu(\eta, \theta) h^{-2}(\theta) d\theta}{\theta^{2} h^{-2}(\theta) + \varepsilon^{2}}. \quad (9.17)$$
9.6. Step 2: definition of domains $\Sigma_{s}^{vm}$. On the segment $\mathcal{I} = [-\theta_{0m}, \theta_{0m}] \subset [-\theta_{0}, \theta_{0}]$ consider the function $h$ and the function

$$f = f_{\eta,t,s} : \mathcal{I} \ni \theta \mapsto \zeta = \theta / h(\theta),$$

By (9.16), $\frac{2}{3} \leq h \leq \frac{3}{2}$ on $\mathcal{I}$, if $\theta_{0}$ is small enough. From here and (9.16), for $\theta \in \mathcal{I}$ we have $\frac{1}{2} \leq f'(\theta) \leq 2$ (if $\theta_{0}$ is small), and

$$\left| \frac{\partial^{k} f}{\partial \theta^{k}} \right| \leq C_1\Theta^{-(k-1)}(t,s) \quad \forall 1 \leq k \leq 3.$$  \hspace{1cm} (9.18)

So $f$ defines a $C^3$-diffeomorphism of $\mathcal{I}$ on $f(\mathcal{I})$, $\frac{1}{2}\mathcal{I} \subset f(\mathcal{I}) \subset 2\mathcal{I}$, such that $f'(0) = 1$ and $f^{-1}$ also satisfies estimates (9.18) (with a modified constant $C$).

Let us set $\zeta^+ = f(\theta_0\Theta)$ and $\zeta^- = -f(-\theta_0\Theta)$. Then $2^{-1}\theta_0\Theta \leq \zeta^\pm \leq 2\theta_0\Theta$, and passing in integral (9.17) from variable $\theta$ to $\zeta = f(\theta)$ we find that

$$J_s = t^{-4} \int_{-\zeta^-}^{\zeta^+} \frac{F_s(\eta, t, \theta)\mu(\eta, \theta)h^{-2}(\theta)(f^{-1}(\zeta))'}{\zeta^2 + \varepsilon^2} d\zeta.$$  \hspace{1cm} 

Denoting the nominator of the integrand as $\Phi_s(\eta, t, \zeta)$ and using (9.12) we see that this is a $C^2$-smooth function, satisfying

$$\left| \frac{\partial^{k} \Phi_s}{\partial \theta^{k}} \right| \leq C_2^\#(t,s) \quad \forall 0 \leq k \leq 2.$$  \hspace{1cm} (9.19)

Moreover, since $h(0) = 1$ and $(f^{-1}(0))' = f'(0) = 1$, then in view of (9.9) we have that

$$\Phi_s(\eta, t, 0) = F_s(\eta, t, 0).$$  \hspace{1cm} (9.19)

Now denote

$$\zeta_0 = \min(\zeta^+, \zeta^-) \in [\frac{1}{2}\theta_0\Theta, 2\theta_0\Theta],$$  \hspace{1cm} (9.20)

set

$$\theta_0^-(\eta, t, s) = -f_{\eta,t,s}^{-1}(\zeta_0), \quad \theta_0^+(\eta, t, s) = f_{\eta,t,s}^{-1}(\zeta_0),$$

and use these functions $\theta_0^\pm$ to define the domains $\Sigma_{s}^{vm}$. Then

$$\langle I_s, (\Sigma_{s}^{vm})_{R_1^{R_2}} \rangle = v^2 \int_{\Sigma_1} m(d\eta) \int_{R_1}^{|R_2|} J^m_s(\eta, t) t^{2d-1} dt,$$  \hspace{1cm} (9.21)

where

$$J^m_s(\eta, t) = \int_{-\theta_0^-}^{\theta_0^+} d\theta \frac{F_s(\eta, t, \theta)\mu(\eta, \theta)}{t^4\Theta^2 + (v\Gamma_s(\eta, t, \theta))^2}$$

$$= t^{-4} \int_{-\zeta_0}^{\zeta_0} \frac{F_s(\eta, t, \theta)\mu(\eta, \theta)h^{-2}(\theta)(f^{-1}(\zeta))'}{\zeta^2 + \varepsilon^2} d\zeta.$$
9.7. Step 3: calculating integrals \( \langle I_s, \Sigma_{s}^{\nu m} \rangle \). Clearly,

\[
|J^m_s| \leq C^\#(t, s) \frac{t^4}{t^4} \int_{-\zeta_0}^{\zeta_0} \frac{d\zeta}{\zeta^2 + \varepsilon^2} \leq C^\#(t, s)\theta_0 \Theta v^2 \Gamma_s^2 (\eta, t, 0) \leq v^{-2} C^\#(t, s)
\]

(here and below the constants may depend on \( \theta_0 \)). So if \( \alpha > 0 \), then

\[
(\langle I_s, (\Sigma_{s}^{\nu m})_{\nu-m}^{\infty} \rangle \leq v^{-2} v^2 C^\#(s) \int_{v^{-2}}^{\infty} t^{2d-1} C^\#(t) dt \leq C^\#(s) v^2. \tag{9.22}
\]

Similar, if \( |s| \geq v^{-\beta} \), \( \beta > 0 \), then

\[
(\langle I_s, (\Sigma_{s}^{\nu m})_{\nu-m}^{\infty} \rangle \leq C^\#(s) \int_{0}^{\infty} t^{2d-1} C^\#(t) dt \leq C^\#(s) v^2. \tag{9.23}
\]

To estimate \( J^m_s(\eta, t) \) for \( t \) and \( s \) not very large, let us consider the integral \( J^0_{s} \), obtained from \( J^m_s \) by froening \( \Phi_s \) at \( \zeta = 0 \):

\[
J^0_{s} = t^{-4} \int_{-\zeta_0}^{\zeta_0} \frac{\Phi_s(\eta, t, 0) d\zeta}{\zeta^2 + \varepsilon^2} = 2t^{-4} F_s(\eta, t, 0) \varepsilon^{-1} \tan^{-1} \frac{\zeta_0}{\varepsilon}
\]

(we use (9.19)). As \( 0 < \frac{\pi}{2} - \tan^{-1} \frac{1}{\varepsilon} < \varepsilon \) for \( 0 < \varepsilon \leq \frac{1}{2} \), then

\[
0 < \pi v^{-1} t^{-2} (\Gamma_s / \Gamma_s) |_{\theta=0} - J^0_{s} \leq \frac{2}{\zeta_0} t^{-4} F_s(\eta, t, 0), \tag{9.24}
\]

if \( vt^{-2} \Gamma_s(\eta, t, 0) \leq \frac{1}{2} \zeta_0 \), which holds if

\[
v \leq Ct^2 \Theta^2(t, s), \tag{9.25}
\]

in view of (9.11) and (9.20). Now we estimate the difference between \( J^m_s \) and \( J^0_{s} \). We have:

\[
(J^m_s - J^0_{s})(\eta, t) = t^{-4} \int_{-\zeta_0}^{\zeta_0} \frac{\Phi_s(\eta, t, \zeta) - \Phi_s(\eta, t, 0) d\zeta}{\zeta^2 + \varepsilon^2}.
\]

Since \( \Phi_s \) is a smooth function and its \( C^2 \)-norm is bounded by \( C^\#(t, s) \), then

\[
\Phi_s(\eta, t, \zeta) - \Phi_s(\eta, t, 0) = A_s(\eta, t) \zeta + B_s(\eta, t, \zeta) \zeta^2,
\]

where \( |A_s|, |B_s| \leq C^\#(t, s) \). From here

\[
|J^m_s - J^0_{s}| \leq C^\#(t, s) t^{-4} \int_{0}^{\zeta_0} \frac{\zeta^2 d\zeta}{\zeta^2 + \varepsilon^2} \leq C^\#(t, s) t^{-4}.
\]

Denote

\[
\mathcal{J}_s(\eta, t) = \pi t^{-2} (\Gamma_s / \Gamma_s)(\eta, t, 0). \tag{9.26}
\]

Then, jointly with (9.24), the last estimate tell us that

\[
|J^m_s - v^{-1} \mathcal{J}_s(\eta, t)| \leq C^\#(t, s) t^{-4} \quad \text{if (9.25) holds.} \tag{9.27}
\]
9.8. Step 4: end of the proof. Let us write $\langle I_s, \Sigma_s^{um} \rangle$ as

$$\langle I_s, (\Sigma_s^{um})_0^{\nu b} \rangle + \langle I_s, (\Sigma_s^{um})_0^{\nu-a} \rangle + \langle I_s, (\Sigma_s^{um})_0^{\infty} \rangle.$$

We will analyse the three terms, choosing properly positive constants $a, b$.

1. By (9.22),

$$\left| \langle I_s, (\Sigma_s^{um})_0^{\infty} \rangle \right| \leq C_a^\#(s)v^2.$$

Similar,

$$\left| v \int_{\Sigma^1} m(d\eta) \int_{\nu-a}^{\infty} dt \ t^{2d-1} J_s(\eta, t) \right| \leq C^\#(s)v \int_{\nu-a}^{\infty} t^{2d-3} C^\#(t) \ dt \leq C_a^\#(s)v^2.$$

2. Since $(\Sigma_s^{um})_0^\delta \subset K_{2\delta}$, estimate (9.3) with $\delta = v^b$ implies

$$\langle |I_s|, (\Sigma_s^{um})_0^{\nu b} \rangle \leq \langle |I_s|, K_{2v^b} \rangle \leq C^\#(s)v^{2bd}.$$

Besides,

$$v \int_{\Sigma^1} m(d\eta) \int_0^{\nu b} dt \ t^{2d-1} J_s(\eta, t) \leq vC^\#(s) \int_0^{\nu b} t^{2d-3} \ dt = C^\#(s)v^{1+2b(d-1)}.$$

3. Now consider

$$X_s := \left| \langle I_s, (\Sigma_s^{um})_0^{\nu-a} \rangle - v \int_{\Sigma^1} m(d\eta) \int_{\nu b}^{\nu-a} dt \ t^{2d-1} J_s(\eta, t) \right|. \quad (9.28)$$

We claim that

$$X_s \leq v^2 \chi_d(v)C^\#(s), \quad (9.29)$$

where $\chi_d(v)$ was defined in (1.43). Indeed, if $|s| \geq \nu^{-a}$, then by (9.23) the modulus of the first term in the r.h.s. of (9.28) is $\leq C^\#(s)v^2$. The second term also satisfies this estimate with some other $C^\#(s)$. So (9.29) is established if $|s| \geq \nu^{-a}$. By a similar (and even easier) argument the claimed estimate holds if $\nu_1 \leq v \leq 1$ for any fixed constant $\nu_1 > 0$.

Now let us consider the case $v \leq \nu_1, |s| \leq \nu^{-a}$. Then for $v^b \leq t \leq \nu^{-a}$ the r.h.s. of (9.25) is no smaller than $Y := C\nu^{2b} \nu^{-2a} - r_1$. If

$$2b + 2ar_1 < 1, \quad (9.30)$$

then $Y \geq v$ if $v \leq \nu_1$ and $\nu_1 > 0$ is small enough. Then assumption (9.25) holds, and (9.21) together with (9.27) implies (9.29):

$$X_s \leq v^2 \int_{\Sigma^1} m(d\eta) \int_{\nu b}^{\nu-a} dt \ t^{2d-1} C^\#(t, s)t^{-4} \leq v^2C^\#(s) \int_{\nu b}^{\nu-a} dt \ C^\#(t)t^{2d-5} \leq C_1^\#(s)v^2 \chi_d(v).$$

4. In the same time, in view of (9.26), for any $\nu_1 \leq A < B \leq \infty$ we have

$$\left| v \int_{\Sigma^1} m(d\eta) \int_{\nu b}^{\nu A} dt \ t^{2d-1} J_s(\eta, t) \right| \leq v \int_{\nu b}^{\nu A} dt \ t^{2d-1-2} C^\#(t, s) \leq C_1^\#(s)v, \quad (9.31)$$
since \( d \geq 2 \).

Now we get from 1-4 that
\[
\left| \langle I_s, \Sigma_s^{vm} \rangle - \nu \int_{\Sigma^1} m(d\eta) \int_0^\infty dt \ t^{2d-1} \mathcal{F}_s(\eta, t) \right|
\leq \left| \langle I_s, \Sigma_s^{vm} \rangle_{v^{-a}}^\infty \right| + \nu \int_{\Sigma^1} m(d\eta) \int_0^\infty dt \ t^{2d-1} |\mathcal{F}_s(\eta, t)|
\]
\[
+ \left| \langle I_s, (\Sigma_s^{vm})_{v^b} \rangle - \nu \int_{\Sigma^1} m(d\eta) \int_0^{v^b} dt \ t^{2d-1} \mathcal{F}_s(\eta, t) \right|
\]
\[
+ \left| \langle I_s, (\Sigma_s^{vm})_{v^b} \rangle - \nu \int_{\Sigma^1} m(d\eta) \int_0^{v^b} dt \ t^{2d-1} |\mathcal{F}_s(\eta, t)| \right|
\leq C^\#(s) \nu \int_{\Sigma^1} m(d\eta) \int_0^\infty dt \ t^{2d-1} |\mathcal{F}_s(\eta, t)| =: Z, \tag{9.32}
\]

if condition (9.30) holds for some \( a, b > 0 \). If \( d \geq 3 \), we choose \( b = \frac{1}{d} < 1 \). Then (9.30) holds for some \( a > 0 \), and \( Z \leq C^\#(s) \nu^2 \chi_d(v) \). If \( d = 2 \), then \( \nu^{2bd} = \nu^4b \) and \( \nu^{1+2b(d-1)} = \nu^{1+2b} \). We choose \( b = 1/2 - \frac{8}{4}, \frac{8}{4} > 0 \). Then again (9.30) holds for some \( a = 0 \), so \( Z \leq C_N^\#(s) \nu^{2-N} \).

By (9.31) the improper integral \( \int_{\Sigma^1} m(d\eta) \int_0^\infty dt \ t^{2d-1} \mathcal{F}_s \) converges absolutely and is bounded by \( C^\#(s) \). In view of (9.8) and (9.26) it may be written as
\[
\nu \int_{\Sigma^1} m(d\eta) \int_0^\infty dt \ t^{2d-1} \mathcal{F}_s(\eta, t) = \pi \nu \int_{\Sigma^1} \frac{F_s(z)}{\Gamma_s(z) \sqrt{|x|^2 + |y|^2}} dz |\Sigma^1|.
\]

This result jointly with the estimates (9.32) and (9.15) imply the assertion of Theorem 3.3. \( \square \)

9.9. Proof of Proposition 3.5. Denoting by \( B^R_r \) the spherical layer \( \{ r \leq |z| \leq R \} \), in view of (9.8) for \( R \geq 0 \) we have
\[
\int_{B^R_r} \mu^\Sigma(dz) \leq C \int_R^{R+1} |z|^{2d-2} dz = C_1(R + 1)^{2d-2}.
\]
So for any function \( f \in \mathcal{C}_m(\mathbb{R}^{2d}) \) with \( m > 2d - 1 \) the integral \( \int f(z) \mu^\Sigma(dz) \) is bounded by
\[
|f|_m \sum_{R=0}^\infty \int_{B^R_r} |z|^{-m} \mu^\Sigma(dz) \leq C_2 |f|_m \sum_{R=0}^\infty \frac{(R + 1)^{2d-2}}{R^m} = C_3 |f|_m.
\]
This proves the proposition.

10. Oscillating Sums Under the Limit (1.17)

10.1. Correlations between increments of \( a_s^{(1)} \). Correlations between the increments of \( a_s^{(1)} \) and some similar quantities, treated in Section 6 lead to sums of the following form:
\[
\Sigma_s^0(\tau) = \sum_{1,2} F_s(s_1, s_2) s_1^{12} \int_0^\tau \int_0^\tau d\theta_1 d\theta_2 e^{-i\pi(2\theta_1 - \theta_2)} + i\nu^{-1} \omega^{12}_s(\theta_1 - \theta_2), \tag{10.1}
\]
where $s \in \mathbb{R}^d$ and $\tau \in (0, 1]$ is a parameter. Our goal is to study their asymptotical behaviour as $\nu \to 0$, $L \to \infty$, uniformly in $\tau$. In (10.1) $F_s$ is a real $C^2$-function on $\mathbb{R}^{2d}$, satisfying (3.9) (for example, $F_s$ may be the function, defined in (3.3)). Integrating over $d\theta_1 \, d\theta_2$ we find that

$$\Sigma_s^0(\tau) = \sum_{1,2} F_s(s_1, s_2) \delta_{3S} \frac{|e^{i\nu^{-1}\omega_{3S}^2 \tau} - e^{-\gamma_s \tau}|^2}{\gamma_s^2 + (\nu^{-1}\omega_{3S}^2)^2}. \tag{10.2}$$

Note that the nominator in the fraction above equals

$$|e^{i\nu^{-1}\omega_{3S}^2 \tau} - e^{-\gamma_s \tau}|^2 = 1 + e^{-2\gamma_s \tau} - 2e^{-\gamma_s \tau} \cos(\nu^{-1}\tau \omega_{3S}^2). \tag{10.3}$$

We wish to study the asymptotical behaviour of the sum $\Sigma_s^0$ as $\nu \to 0$, $L \to \infty$ and, as before, will do this by comparing (10.1) with the integral

$$I_s^0 = \int d s_1 \int d s_2 \int_0^\tau \! d \theta_1 \int_0^\tau \! d \theta_2 \delta_{3S} M_s(s_1, s_2, s_3; \theta_1, \theta_2),$$

where

$$M_s = F_s(s_1, s_2) e^{-\gamma_s (2\tau - \theta_1 - \theta_2) + i\nu^{-1}\omega_{3S}(\theta_1 - \theta_2)}.$$

By Theorem 3.1,

$$|I_s^0 - \Sigma_s^0| \leq C^\#(s) \nu^{-2} L^{-2}. \tag{10.4}$$

So if $L \gg \nu^{-1}$, then to calculate the asymptotical behaviour of $\Sigma_s^0$ it suffices to calculate that of $I_s^0$. Integrating over $d\theta_1 \, d\theta_2$ in the expression for $I_s^0$ we obtain (10.2) with $\sum_j$ replaced by $\int ds_j$. In view of (10.3) we get that $I_s^0 = \nu^2 I_s^{0.1} + \nu^2 I_s^{0.2}$, where

$$I_s^{0.1} = (1 + e^{-2\gamma_s \tau}) \int d s_1 \int d s_2 \delta_{3S} F_s(s_1, s_2) \frac{\omega_{3S}^2}{\nu^2 \gamma_s^2 + (\omega_{3S}^2)^2},$$

$$I_s^{0.2} = -2e^{-\gamma_s \tau} \int d s_1 \int d s_2 \delta_{3S} F_s(s_1, s_2) \cos(\nu^{-1}\tau \omega_{3S}^2) \frac{\omega_{3S}^2}{\nu^2 \gamma_s^2 + (\omega_{3S}^2)^2}.$$

Consider first $I_s^{0.1}$. Applying Theorem 3.3 with $\Gamma_s = \gamma_s/2$, we find

$$I_s^{0.1} = \frac{1 + e^{-2\gamma_s \tau}}{2\gamma_s^2 \pi} \nu^{-1} \int_{\Sigma_s} F_s(z) \frac{d z | \Sigma_s + O(1)\nu^{-\delta} C^\#(s; \nu_0)}{|(s - s_1, s - s_2)|} d z | \Sigma_s \tag{10.5}$$

where $z = (s_1, s_2)$.

It remains to study the asymptotical behaviour of $I_s^{0.2}$. It is described by the following result, proved in Section 10.3 (also see [20]):

**Lemma 10.1.** For any $s \in \mathbb{R}^d$ and $\tau \in (0, 1]$,

$$\left| I_s^{0.2} + \gamma_s^{-1} \nu^{-1} e^{-2\gamma_s \tau} \pi \int_{\Sigma_s} \frac{F_s(z)}{|(s - s_1, s - s_2)|} d z | \Sigma_s \right| \leq C^\#(s) \chi_d(\nu). \tag{10.6}$$

Relations (10.5) and (10.6) imply the main result of this section:

**Theorem 10.2.** For any $s \in \mathbb{R}^d$ and $\tau \in (0, 1]$,

$$\left| I_s^0 - \frac{1 - e^{-2\gamma_s \tau}}{2\gamma_s} \nu\pi \int_{\Sigma_s} \frac{F_s(z)}{|(s - s_1, s - s_2)|} d z | \Sigma_s \right| \leq C^\#(s; \nu_0) \nu^{-2-\delta}.$$
10.2. Correlations between solutions and their increments. In this section we analyse
sums similar to (10.1) in which the integral \( \int_0^T d\theta_2 \) is replaced by the integral \( \int_0^0 d\theta_2 \). Sums of such form arise in Section 6, when studying the correlation

\[
\mathbb{E} \Delta a_s^{(1)}(\tau) c_s^{(1)}(\tau) = e^{-\gamma \tau} \mathbb{E} (a_s^{(1)}(\tau) - e^{-\gamma \tau} a_s^{(1)}(0)) \overline{a}_s^{(1)}(0)
\]

and similar quantities. We will show that the considered sums are negligible. The reason
for this is that for \( \omega_{23} \neq 0 \) the “fast” frequency \( \nu^{-1} \omega_{23}(\theta_1 - \theta_2) \) is of the size \( \lesssim 1 \) only
if \( |\theta_1|, |\theta_2| \lesssim \nu \), so the Lebesgue measure of such resonant vectors \((\theta_1, \theta_2)\) is only of
order \( \nu^2 \) (if \( \tau \sim 1 \)). That is why such sums are much smaller than those of the form
(10.1), where the measure of the resonant vectors \((\theta_1, \theta_2)\) is of order \( \nu \gg \nu^2 \).

We consider the sum

\[
S_\nu(\tau) = \sum_{1,2} \delta_{3s}^{(1)}(\tau) \int_0^\tau \int_0^0 d\theta_1 d\theta_2 e^{i \nu^{-1} (\theta_1 - \theta_2)} F(s, s_2, l, l', \tau), \quad s \in \mathbb{R}^d,
\]

where \( \nu > 0 \) and \( F_s \) is a real function, measurable in \( s_1, s_2, l, l' \), \( C^2 \)-smooth in \( s_1, s_2 \), and such that

\[
e^{i \nu l'} |a_{s_1, s_2}^\nu F_s| \leq C^\#(s) C^\#(s_1) C^\#(s_2) \quad \forall 0 \leq |\alpha| \leq d + 1,
\]

uniformly in \( l' \leq 0, 0 \leq \tau \leq 1, 0 \leq l \leq \tau \).

**Theorem 10.3.** Let \( 0 \leq \tau \leq 1 \). Then under the assumption (10.7) the sum \( S_\nu \) meets the estimate

\[
|S_\nu(\tau)| \leq C^\#(s) \left( \nu^2 \chi_d(\nu) + \nu^{-2} L^{-2} \right),
\]

where \( C^\#(s) \) depends only on \( d, \nu \) and the function \( C^\#(s) \) from (10.7).

Note that since we assume no smoothness in \( l' \) for function \( F_s \), then the theorem also applies to the sums

\[
S_\nu^T(\tau) = \sum_{1,2} \delta_{3s}^{(1)}(\tau) \int_0^\tau \int_0^0 d\theta_1 d\theta_2 e^{i \nu^{-1} (\theta_1 - \theta_2)} F_s(s, s_2, l, \tau),
\]

since we may extend \( F_s \) by zero for \( l' < -T \) and regard \( S_\nu^T \) as the sum \( S_\nu \).

**Proof.** It is convenient to change the variable \( \theta = -l' \), so that the sum \( S_\nu \) takes the form

\[
S_\nu(\tau) = \sum_{1,2} \delta_{3s}^{(1)}(\tau) \int_0^\tau d\theta \int_0^\tau d\theta_2 e^{-\gamma \theta} e^{i \nu^{-1} (\theta + l \theta_2)} F(s, s_2, \theta, l, \tau).
\]

As usual, to estimate \( S_\nu \) we first approximate it by the integral

\[
I_\nu(\tau) = \int_0^\tau d\theta \int_0^\tau d\theta_2 \int_0^\tau ds_1 ds_2 \delta_{3s}^{(1)}(\tau) e^{-\gamma \theta} e^{i \nu^{-1} (\theta + l \theta_2)} F(s, s_2, \theta, l, \tau).
\]

Applying Theorem 3.1 we get that

\[
|I_\nu(\tau) - S_\nu(\tau)| \leq C^\#(s) \nu^{-2} L^{-2}.
\]

To estimate \( I_\nu \) we write it as a sum of integrals over the domains \( \{ \nu^{-1} (l + \theta) \geq 1 \} = \{ l \geq v - \theta \} \) and \( \{ \nu^{-1} (l + \theta) \leq 1 \} = \{ 0 \leq l \leq v - \theta \} \).
Integral over \( \{ l \geq \nu - \theta \} \). Let us denote this integral \( I_s^1 \), and for any fixed \( \theta \geq 0 \) and \( l \geq \nu - \theta \) consider the internal integral over \( ds_1 ds_2 \):

\[
I_s^1(l, \theta) = e^{-\gamma \theta} \int ds_1 ds_2 \delta_{3s}^{12} e^{i(v^{-1}(l+\theta))\omega_{3s}^2 F_s}.
\]

Since \( \omega_{3s}^2 = 2(s_1 - s) \cdot (s - s_2) \) is a non-degenerate quadratic form and \( v^{-1}(l+\theta) \geq 1 \) on the zone of integrating, then the integral \( I_s^1(l, \theta) \) has the form (12.2) with \( \nu := \nu(l+\theta)^{-1}, \varphi = e^{-\gamma \theta} F_s \) and \( n = 2d \). So by (12.4), (12.5) and (10.7),

\[
|I_s^1(l, \theta)| \leq C^\#(s)v^d(l+\theta)^{-d} e^{-\gamma \theta},
\]

where \( C^\#(s) \) is as in the theorem. Accordingly,

\[
|I_s^1| \leq C \int_0^{\infty} d\theta e^{-\gamma \theta} \int_0^{\tau} d(l + \theta)^{-d} \chi_{[l \geq v - \theta]}, \quad C = C^\#(s)v^d.
\]

We split the integrating zone \( \{ \theta \geq 0, 0 \leq l \leq \tau, l \geq \nu - \theta \} \) to the part where \( \theta \geq \nu \) and its complement, and accordingly split the integral above as \( I_s^{1.1} + I_s^{1.2} \), where

\[
I_s^{1.1} = C \int_v^{\infty} d\theta e^{-\gamma \theta} \int_0^{\tau} d(l + \theta)^{-d},
\]

\[
I_s^{1.2} = C \int_{(v-\tau)\wedge 0}^{v} d\theta e^{-\gamma \theta} \int_0^{\tau} d(l + \theta)^{-d}.
\]

Consider first \( I_s^{1.1} \). Computing the internal integral and replacing \( \tau \) by \( \infty \) we find

\[
I_s^{1.1} \leq C C_1 \varepsilon \int_v^{\infty} \frac{e^{-\gamma \theta}}{\theta^{d-1}} d\theta \leq C_1 C^\#(s) v^2 \chi_d(v).
\]

Now consider \( I_s^{1.2} \). Replacing in the external integral \( (v - \tau) \wedge 0 \) by 0 and in the internal one \( \tau \) by \( \infty \) we find

\[
I_s^{1.2} \leq C C_1 \varepsilon \int_0^{v} d\theta v^{-d+1} = C_1^\#(s) v^2.
\]

Integral over \( \{ 0 \leq l \leq \nu - \theta \} \). Denoting this integral as \( I_s^2 \), we find

\[
|I_s^2| \leq \int_0^{v} d\theta \int_0^{(v-\theta)\wedge \tau} dl \int ds_1 ds_2 |F_s| \leq C^\#(s)v^2,
\]

since \( \int ds_1 ds_2 |F_s| \leq C^\#(s) \) and the area of integration over \( d\theta dl \) is bounded by \( v^2 \).

We have seen that \( |I_s| \leq C^\#(s)v^2 \chi_d(v) \). This and (10.8) jointly imply the assertion of the theorem. \( \square \)
10.3. Proof of Lemma 10.1. Let us denote \( \nu' = \frac{1}{2} v \gamma_s \). If \( \nu' > 1 \), then \( |s| \geq C v^{-1/2} \), so taking into account assumption (3.9) we see that the both summands in the l.h.s. of (10.6) are bounded by \( C^\#(s) \) and the result follows. So we may assume that \( \nu' \leq 1 \).

Let us write \( I_{s}^{0,2} \) as
\[
I_{s}^{0,2} = -\frac{1}{2} e^{-\gamma_s \tau} \int \int ds_1 ds_2 \delta_{3s} F_s(s_1, s_2) \cos(v^{-1} \omega_{12}^s)/\nu' + (\omega_{12}^s/2)^2.
\]

We will examine the integrals \( \langle I_{s}^{0,2}, K_{2r} \rangle, r \ll 1, \langle I_{s}^{0,2}, \Sigma^v \rangle \) and \( \langle I_{s}^{0,2}, R^{2d} \setminus \Sigma^v \rangle \) (see Notation, (9.1) and Lemma 9.1), and will derive the lemma from this analysis. The constants below do not depend on \( \tau, s \) and \( \nu' \).

Let us re-write \( I_{s}^{0,2} \), using the variables \( (x, y) = z \), see (3.15). Disregarding for a moment the pre-factor \( -\frac{1}{2} e^{-\gamma_s \tau} \) we examine the integral
\[
J_s = \int_{\mathbb{R}^{2d}} dz F_s(z) \cos \lambda x \cdot y \quad (x \cdot y)^2 + \nu'^2, \quad \lambda = \nu'^{-1} \gamma_s.
\]

Step 1. Since \( |\cos \lambda x \cdot y| \leq 1 \), an upper bound for \( \langle J_s, K_{2r} \rangle \), where \( r \ll 1 \), (see (9.1)) follows from (9.3):
\[
\langle |J_s|, K_{2r} \rangle \leq C^\#(s) \nu'^{-2} r^{2d}. \tag{10.9}
\]

Step 2. Integral over \( \Sigma^v \). We recall that \( (\Sigma^v)^R_1 \) is defined in (9.10). Passing to the variables \( (\eta, t, \theta) \) (see Lemma 9.2) and using (9.7) we disintegrate \( \langle J_s, (\Sigma^v)^R \rangle =: J_s^r \) as follows:
\[
J_s^r = \int_{\Sigma^1} m(d\eta) \int_{r}^{t} dt t^{2d-1} \int_{-\theta_0}^{t_0} d\theta \frac{F_s(\eta, t, \theta) \mu(\eta, \theta) \cos(\lambda x \cdot y)}{(t^2 \theta^2 + \nu'^2)}
\]
\[
= \int_{\Sigma^1} m(d\eta) \int_{r}^{t} dt t^{2d-1} \Upsilon_s(\eta, t), \tag{10.10}
\]
where
\[
\Upsilon_s(\eta, t) = t^{-4} \int_{-\theta_0}^{t_0} F_s(\eta, t, \theta) \mu(\eta, \theta) \cos(\lambda t^2 \theta) \frac{d\theta}{\theta^2 + \varepsilon^2}, \quad \varepsilon = \nu' t^{-2}.
\]

To estimate \( \Upsilon_s \), consider first the integral \( \Upsilon_s^0 \), obtained from \( \Upsilon_s \) by freezing \( F_s \mu \) at \( \theta = 0 \). Since \( \mu(\eta, 0) = 1 \), then \( \Upsilon_s^0 \) equals
\[
2t^{-4} F_s(\eta, t, 0) \int_0^{t_0} \cos(\lambda t^2 \theta) \frac{d\theta}{\theta^2 + \varepsilon^2} = 2 \nu'^{-1} t^{-2} F_s(\eta, t, 0) \int_0^{\theta_0/\varepsilon} \cos(\gamma_s \tau w) \frac{dw}{w^2 + 1}.
\]

Consider the integral
\[
2 \int_0^{\theta_0/\varepsilon} \cos(\gamma_s \tau w) \frac{dw}{w^2 + 1} = 2 \int_0^{\infty} \cos(\gamma_s \tau w) \frac{dw}{w^2 + 1} - 2 \int_{\theta_0/\varepsilon}^{\infty} \cos(\gamma_s \tau w) \frac{dw}{w^2 + 1} =: I_1 - I_2.
\]

Since
\[
2 \int_0^{\infty} \cos(\xi w) \frac{dw}{w^2 + 1} = \int_{-\infty}^{\infty} e^{i\xi w} \frac{dw}{w^2 + 1} = \pi e^{-|\xi|},
\]

then $I_1 = \pi e^{-\gamma \tau}$. For $I_2$ we have an obvious estimate $|I_2| \leq 2\varepsilon/\theta_0 = C_1 \nu' t^{-2}$. So

$$\gamma_s^0(\eta, t) = \nu'^{-1} \pi t^{-2} F_s(\eta, t, 0)(e^{-\nu' t} + \Delta t), \quad |\Delta t| \leq C \nu' t^{-2}. \quad (10.11)$$

Now we estimate the difference between $\gamma_s$ and $\gamma_s^0$. Writing $(F_s \mu)(\eta, t, \theta) - (F_s \mu)(\eta, t, 0)$ as $A_s(\eta, t, \theta) + B_s(\eta, t, \theta)\theta^2$, where $|A_s|, |B_s| \leq C^\#(s, t)$, we have

$$\gamma_s - \gamma_s^0 = t^{-4} \int_{-\theta_0}^{\theta_0} \frac{(A_s \theta + B_s \theta^2) \cos(\lambda \tau \theta) d\theta}{\theta^2 + \varepsilon^2}.$$

Since the first integrand is odd in $\theta$, then its integral vanishes, and

$$|\gamma_s - \gamma_s^0| \leq C^\#(s, t) t^{-4} \int_{-\theta_0}^{\theta_0} \frac{\theta^2 d\theta}{\theta^2 + \varepsilon^2} \leq 2C^\#(s, t) t^{-4} \theta_0.$$

So by (10.11)

$$|\gamma_s(\eta, t) - \nu'^{-1} \pi t^{-2} F_s(\eta, t, 0)e^{-\gamma_s \tau}| \leq C^\#(s, t) \left(t^{-4} + \nu'^{-1} t^{-2} \nu' t^{-2}\right) = C_1^\#(s, t) t^{-4}.$$

Then, by (10.10),

$$J^r_s = \int_{\Sigma^1} m(d\eta) \int_r^{\infty} dt \int t^{2d-1} \left[\nu'^{-1} \pi t^{-2} e^{-\gamma_s \tau} F_s(\eta, t, 0) + O(C^\#(s, t) t^{-4})\right],$$

since $\int_r^{\infty} t^{2d-1} C^\#(s, t) t^{-4} dt \leq C^\#(s)(r^d(r))$. Noting that

$$|\nu'^{-1} \pi e^{-\gamma_s \tau} \int_{\Sigma^1} m(d\eta) \int_0^r dt \int t^{2d-3} F_s(\eta, t, 0)| \leq C^\#(s) \nu'^{-1} r^{2d-2},$$

we arrive at the inequality

$$|J^r_s - \nu'^{-1} \pi e^{-\gamma_s \tau} \int_{\Sigma^1} m(d\eta) \int_0^\infty dt \int t^{2d-2} (t^{-1} F_s(\eta, t, 0))| \leq C^\#(s)(\nu'^{-1} r^{2d-2} + \chi_d(r)). \quad (10.12)$$

Here in view of (9.8)

$$\int_{\Sigma^1} m(d\eta) \int_0^{\infty} dt \int t^{2d-2} (t^{-1} F_s(\eta, t, 0)) = \int_{\Sigma^1} F_s(z)|z|^{-1}. \quad (10.13)$$

Since $(\Sigma^\nu)^r_0 \subset K_2r$ by Lemma 9.1.3, where $\theta_0$ is assumed to be sufficiently small, then by (10.9),

$$|J^r_s - \langle J_s, \Sigma^\nu \rangle| \leq C^\#(s) \nu'^{-2} r^{2d}. \quad (10.14)$$

**Step 3.** Final asymptotic. Consider the integral over the complement to $\Sigma^\nu$:

$$\left|\langle J_s, \mathbb{R}^{2d} \setminus \Sigma^\nu \rangle\right| \leq \left|\langle J_s, \{|z| \leq r\} \rangle\right| + C \int_r^{\infty} dt \int t^{2d-1} \int_{\mathbb{R}^{2d} \setminus \Sigma^\nu} \frac{|F_s(z)| d\Sigma^1}{\omega^2/4 + \nu'^2}.$$
By item 5) of Lemma 9.1, $|\omega(z)| \geq Ct^2$ in $S^l \setminus \Sigma^v$. Jointly with (10.9) it implies that
\[
\left| \langle J_s, \mathbb{R}^{2d} \setminus \Sigma^v \rangle \right| \leq C^#(s)\nu^{-2}r^{2d} + C^#(s) \int_r^\infty t^{2d-1}t^{-4}C^#(t)\,dt \\
\leq C^#(s)\left(\nu'^{-2}r^{2d} + \chi_d(r)\right). 
\] (10.15)
Finally by (10.14), (10.15), (10.12) and (10.13) with $r = \sqrt{\nu}$, we have
\[
\left| J_s - \nu'^{-1}e^{-\nu s^T} \pi \int_{\Sigma^v} F_s(z)z^{-1} \right| \leq C^#(s)\chi_d(\nu).
\]
That is, $\left| I_0^{0,2} + \nu'^{-1}e^{-2\nu s^T} \pi \int_{\Sigma^v} F_s(z)z^{-1} \right| \leq C^#(s)\chi_d(\nu)$, and the lemma is proved.

11. Wave Kinetic Integrals and Equations: Proofs

11.1. Proof of Lemma 4.2. To prove the lemma we may assume that $|u^1|_r = \cdots = |u^4|_r = 1$.

Consider first the integral $J_4$. By the above,
\[
|J_4(s)| \leq \int_{\Sigma^v} d\mu(z)(x + s)^{-r}(y + s)^{-r}(x + y + s)^{-r} =: J_4(s), \quad z = (x, y).
\]
We should verify that
\[
J_4(s) \leq C(s)^{-r-1} \quad \forall \, s \in \mathbb{R}^d. 
\] (11.1)
If $|s| \leq 2$, then it is not hard to check that
\[
(x + s)(y + s)(x + y + s) \geq C^{-1}(z)^2
\]
for a suitable constant $C$ independent from $s$. So the integrand for $J_4(s)$ is bounded by $C_1(z)^{-2r}$. Since $r > d$, then by Proposition 3.5 $J_4(s) \leq C$ if $|s| \leq 2$. This proves (11.1) if $|s| \leq 2$, and it remains to consider the case when
\[
R := |s| \geq 2.
\]
Assuming that a vector $s$ as above is fixed, let us consider the sets
\[
O_1 = \{\xi \in \mathbb{R}^d : |\xi| \leq \frac{9}{10}R\}, \quad O_2 = \{\xi : \frac{9}{10} \leq |\xi| \leq \frac{11}{10}R\}, \\
O_3 = \{\xi : |\xi| \geq \frac{11}{10}R\}, \quad O_{ij} = O_i \cup O_j, \quad i, j = 1, 2, 3,
\]
and define
\[
\Sigma^{i,j} = \Sigma^v \cap (O_i \times O_j), \quad \Sigma^{12,j} = \Sigma^v \cap (O_{12} \times O_j), \quad \text{etc.}
\]
Next we denote by $J_4^{i,j}(s)$ the part of the integral in (4.5) which comes from the integrating over $\Sigma^{i,j}$:
\[
J_4^{i,j}(s) = \int_{\Sigma^{i,j}} d\mu(z)\left(u^1(x + s)u^2(y + s)u^3(x + y + s)\right),
\]
and define similar integrals \( J_{4}^{i,j} (s) \), etc. Then \( J_{4} (s) = \sum_{i,j \in \{1,2,3\}} J_{4}^{i,j} (s) \) and \(|J_{4}^{i,j} (s)| \leq J_{4}^{i,j} (s)\), where

\[
J_{4}^{i,j} (s) = \int_{\Sigma_{i,j}} d\mu (z) (x + s)^{-r} (y + s)^{-r} (x + y + s)^{-r}.
\]

Clearly

\[
J_{4}^{i,j} (s) = J_{4}^{i,j} (s) \quad \forall \ (i, j),
\]

and \( J_{4}^{i,j} (s) = J_{4}^{1,j} (s) + J_{4}^{2,j} (s) \), etc. To verify (11.1) it remains to check that

\[
J_{4}^{i,j} (s) \leq CR^{-r-1} \quad \forall \ (i, j),
\]

when \( R = |s| \geq 2 \). Applying Theorem 3.6 we find that

\[
J_{4}^{i,j} (s) = \int_{O_{j}} dx \ |x|^{-1} (x + s)^{-r} \int_{x_{\perp} \cap O_{j}} d_{x_{\perp}} y \ (y + s)^{-r} (x + y + s)^{-r}.
\]  

(11.4)

By \( s_{x} \) (by \( s_{y} \)) we will denote the projection of \( s \) on the space \( x_{\perp} \) (on \( y_{\perp} \)), and by \( \hat{s}, \hat{x}, \hat{y} \)—the vectors \( s / R, x / R, y / R \); so \( |\hat{s}| = 1 \).

We will estimate the r.h.s. of (11.4) for each set \( \{ i, j \} \subset \{ 1, 2, 3 \} \), using the following elementary inequalities, related to the domains \( O_{j} \):

\[
\langle x + y + s \rangle \geq C^{-1} (|x| + |y|) \quad \forall \ z \in (O_{23} \times O_{23}) \cap \Sigma.
\]

(11.6)

Proof of (11.6) uses that \(|x + y| \geq (|x| + |y|) / \sqrt{2} \) for \( x, y \in \Sigma \). We will also use the integral inequalities below, where \(|s| = R \geq 2 \) and \( \hat{x}_{l} = \hat{x}_{l} (R) \) equals 1 if \( l \neq 0 \) and equals \( \ln R \) if \( l = 0 \):

\[
\int_{|\xi| \leq \frac{1}{10} R} |y + \xi|^{-p} dy \leq C \quad \forall \ \xi \in \mathbb{R}^{d} \quad \text{if} \quad p > d,
\]

(11.7)

\[
\int_{|x| \leq \frac{1}{10} R} |x|^{-1} (x + s)^{-\hat{r}} dx \leq C \hat{x}_{d-\hat{r}} R^{\max (1-d-1-\hat{r})}.
\]

(11.8)

\[
\int_{|x| \geq \frac{9}{10} R} |x|^{A} (x + s)^{-\hat{r}} dx \leq C \hat{x}_{d-\hat{r}} R^{\max (A+d-1-\hat{r})}.
\]

(11.9)

if \( A < \hat{r} - d \). We will use these relations with \( d := d \) or \( d := d - 1 \). The first inequality is obvious. To prove (11.8) note that there the integral equals

\[
R^{d-1} \int_{|y| \leq \frac{1}{10} R} |y|^{-1} (\hat{R} (y + \hat{s}))^{-\hat{r}} dy, \quad \hat{s} = s / R.
\]

It is straightforward to see that this integral (with the pre-factor), taken over the ball \(|y + \hat{s}| \leq 1 / 10 \) is \( \leq C \hat{x}_{d-\hat{r}} R^{\max (1-d-1-\hat{r})} \), while the integral over the ball’s complement is \( \leq C R^{d-1-\hat{r}} \). It implies (11.8). Proof of (11.9) is similar.

Estimating the integral \( J_{4}^{23,23} (s) \). By (11.4) and (11.6) the integral is bounded by

\[
C \int_{|x| \geq \frac{9}{10} R} dx \ |x|^{-1} (x + s)^{-\hat{r}} \int_{|y| \geq \frac{9}{10} R} d_{x_{\perp}} y \ (y + s)^{-r} (|x| + |y|)^{-r}.
\]
Since \(\langle y + s \rangle \geq \langle y + s_x \rangle\) and \(r > d\), the internal integral is less than
\[
CR^{-r} \int_{|y| \geq \frac{9}{10} R} d_x^+ y \langle y + s_x \rangle^{-r} \leq C_1 R^{-r},
\]
where we used (11.7) with \(d := d - 1\). So
\[
\mathcal{J}_{4}^{23,23}(s) \leq CR^{-r} \int_{|x| \geq \frac{9}{10} R} dx \, |x|^{-1} \langle x + s \rangle^{-r} \leq R^{-r-1}
\]
by (11.9) with \(A = -1\) and \(\hat{r} = r\). This implies (11.3) if \(i, j \in \{2, 3\}\).

Estimating the integral \(\mathcal{J}_{4}^{12,13}(s)\). By (11.4) and (11.5),
\[
\mathcal{J}_{4}^{12,13}(s) \leq CR^{-r} \int_{|x| \leq \frac{11}{10} R} dx \, |x|^{-1} \langle x + s \rangle^{-r} \int d_x^+ y \langle x + y + s \rangle^{-r}.
\]
Since \(\langle x + y + s \rangle \geq \langle y + s_x \rangle\), by (11.7) with \(d := d - 1\) the internal integral is bounded by a constant. So in view of (11.8) with \(\hat{r} = r\), we get \(\mathcal{J}_{4}^{12,13}(s) \leq CR^{-r-1}\). This implies (11.3) if \(i \in \{1, 2\}, j \in \{1, 3\}\).

Evoking the symmetry (11.2) we see that we have checked (11.3) for all \((i, j)\), and thus have proved (4.7) for \(l = 4\).

Other three integrals are easier. Let us first consider \(J_3(s)\),
\[
J_3(s) = u^4(s) \int_{\Sigma_s} d\mu_S(z) u^1(x + s) u^2(y + s),
\]
where \(|u^j|_r = 1\) for each \(j\). Then
\[
|J_3(s)| \leq \langle s \rangle^{-r} \int_{\Sigma_s} d\mu_S(z) \langle x + s \rangle^{-r} \langle y + s \rangle^{-r} : = J_3(s).
\]
If \(|s| \leq 2\), then \(J_3(s)\) is bounded by
\[
C \int_{\Sigma_s} d\mu_S(z) \langle x + s \rangle^{-r} \langle y + s \rangle^{-r} = C \int dx \, |x|^{-1} \langle x + s \rangle^{-r} \int_{x^+} d_x^+ y \langle y + s \rangle^{-r}.
\]
Applying (11.7) to the both internals, using that \(r > d\) and the integrability of \(|x|^{-1}\) over a neighbourhood of \(0 \in \mathbb{R}^d\), we see that \(|J_3(s)| \leq C\), which implies \(|J_3(s)| \leq C \langle s \rangle^{-r-1}\) for \(|s| \leq 2\).

Now we pass to the case \(R = |s| \geq 2\). Then
\[
J_3(s) \leq \langle s \rangle^{-r} \int dx \, |x|^{-1} \langle x + s \rangle^{-r} \int_{x^+} d_x^+ y \langle y + s \rangle^{-r}.
\]
Applying (11.7) we see that the internal integral is bounded by a constant, while by (11.8) and (11.9) with \(A = -1\) and \(\hat{r} = r\) the external is \(\leq CR^{-1}\). This proves (4.7) for \(l = 3\).

The integrals \(J_1\) and \(J_2\) are very similar and we only consider \(J_1\):
\[
J_1(s) = u^4(s) \int_{\Sigma_s} d\mu_S(z) u^2(y + s) u^3(x + y + s).
\]
Assume first $|s| \leq 2$. Using the disintegration of the measure $d \mu^\Sigma (z)$ in the form (3.25) we get

$$|J_1 (s)| \leq C \int dy \ |y|^{-1} (y + s)^{-r} \int _{y \perp} d_\perp \cdot (x + y + s)^{-r}.$$ 

Since $(x + y + s)^{-r} \leq (x + s)^{-r}$ and $r > d$, then using (11.7) we see that $|J_1 (s)| \leq C$ for $|s| \leq 2$.

If $R = |s| \geq 2$, then, in view of (3.25),

$$|J_1 (s)| \leq \langle y \rangle - r \int _{y \perp} d_\perp \cdot (x + y + s)^{-r}.$$ 

Again, using (11.7) we estimate the internal integral, using (11.8) and (11.9) estimate the external and get that $|J_1 (s)| \leq C R - r - 1$.

Thus the lemma’s assertion also holds for $l = 1, 2$, and Lemma 4.2 is proved.

11.2. Proof of Theorem 4.4. Let us denote by $\{S_t, t \geq 0\}$ the semigroup of the linear equation, i.e. $S_t = e^{-t L}$. Obviously, $|S_t|_{C_r} \leq e^{-2 t}$. The solution $u^0 (t, x)$ of the linear problem (4.9), (4.10) is given by the Duhamel integral

$$u^0 (t) = C (u_0, f)(t) := S_t u_0 + \int _0 ^t S_{t-s} f (s) \, ds,$$

so

$$||u^0||_r = ||C(u_0, f)||_r \leq |u_0|_r + \frac{1}{2} ||f||_r.$$  \hspace{1cm} (11.10)

A solution $u$ for the problem (4.9), (4.10) is a fixed point of the operator $u \mapsto C (u_0, f + \varepsilon K (u)) =: \mathcal{B} (u)$.

Assuming (4.13) and using (4.3) we see from (11.10) that

$$||\mathcal{B} (u)||_r \leq \frac{3}{2} C_r + \frac{\varepsilon}{2} C_r K ||u||^3,$$

where $C_r^K$ is the constant from (4.3). So the operator $\mathcal{B}$ preserves the ball

$$B_R = \{ u \in X_r, ||u||_r \leq R \}, \quad R = \frac{3}{2} C_r + 1,$$

if $\varepsilon \ll 1$. Using Corollary 4.3 we see that the operator $\mathcal{B}$ defines a contraction of the ball $B_R$ if $\varepsilon \ll 1$. Accordingly eq. (4.9) has a unique solution $u \in B_R$.

Finally, if $(u_{01}, f_1)$ and $(u_{02}, f_2)$ are two sets of initial data, satisfying (4.13), and $u^1, u^2$ are the corresponding solutions, then

$$||u^1 - u^2||_r \leq |u_{01} - u_{02}|_r + \frac{1}{2} ||f_1 - f_2||_r + \frac{3}{2} \varepsilon C_r^K R^2 ||u^1 - u^2||_r$$

for all $t \geq 0$, so

$$||u^1 - u^2||_r \leq (1 - \frac{3}{2} \varepsilon C_r^K R^2)^{-1} (|u_{01} - u_{02}|_r + \frac{1}{2} ||f_1 - f_2||_r).$$

This implies (4.14) if $\varepsilon \ll 1$ and the theorem is proved.
12. Addenda

12.1. Discrete turbulence. In order to study the double limit (1.9) it is natural to examine first the limit \( \nu \to 0 \) (with \( L \) and \( \rho \) fixed), known as the limit of discrete turbulence, see [24]. To do this consider the following effective equation:

\[
\dot{a}_s + \gamma_s a_s = i \rho L^{-d} \left( \sum_{s_1} \sum_{s_2} \delta_{3s}^{12} \delta(\omega) a_{s_1} a_{s_2} a_{s_3} - a_s |a_s|^2 \right) + b(s) \dot{\beta}_s, \quad s \in \mathbb{Z}_L^d, \tag{12.1}
\]

where \( \delta(\omega) \) is the delta-function of \( \omega = \omega_{3s}^{12} \) (equal 1 if \( \omega_{3s}^{12} = 0 \) and equal 0 otherwise). The following result is proven in [13, 22].

**Theorem 12.1.** If \( r_s \) is sufficiently big in terms of \( d \), then eq. (12.1) is well posed and mixing. When \( L \) and \( \rho \) are fixed and \( \nu \to 0 \), then

(i) solutions of (1.14) converge in distribution, on time intervals of order 1, to solutions of (12.1) with the same initial data at \( \tau = 0 \);

(ii) the unique stationary measure \( \mu_{\nu, L} \) of (1.14) weakly converges to the unique stationary measure of eq. (12.1).

12.2. Estimates for integrals with fast oscillating exponents, given by quadratic forms. Let \( \varphi \) be a complex \( L_1 \)-function on \( \mathbb{R}^n \) such that its Fourier transform \( \hat{\varphi} \in L_1(\mathbb{R}^n) \), where in this appendix we define \( \hat{\varphi}(\xi) \) as \( \int e^{-i x \cdot \xi} \varphi(x) \, dx \). Let \( Q \) be a symmetric non-degenerate real \( n \times n \)-matrix. Consider the integral

\[
I(\nu) = \int_{\mathbb{R}^n} e^{i \nu^{-1} x \cdot Qx/2} \hat{\varphi}(x) \, dx, \quad 0 < \nu \leq 1. \tag{12.2}
\]

The Fourier transform of the function \( e^{i \nu^{-1} x \cdot Qx/2} =: F_0 \) is

\[
\hat{F}_0 = (2\pi \nu)^{n/2} \xi |\det Q|^{-1/2} \hat{\varphi}(\xi) e^{-i \nu \xi \cdot Q^{-1} \xi/2},
\]

where \( \xi \) is some complex number of unit norm (see [5, 14]). Formally applying Parseval’s identity we get

\[
I(\nu) = (2\pi)^{-n} \int \hat{F}_0 \hat{\varphi} \, d\xi = \left( \frac{\nu}{2\pi} \right)^{n/2} \xi |\det Q|^{-1/2} \int \hat{\varphi}(\xi) e^{-i \nu \xi \cdot Q^{-1} \xi/2} d\xi. \tag{12.3}
\]

To justify the validity of (12.3) in our situation, we approximate \( F_0 \) by functions

\[
F_{\xi} = e^{i \nu^{-1} x \cdot \langle Q + i \varepsilon I \rangle x/2}, \quad \varepsilon > 0.
\]

These are Schwartz functions whose Fourier transforms

\[
\hat{F}_{\xi} = (2\pi \nu)^{n/2} |-\varepsilon I|^{-1/2} \hat{\varphi}(\xi) e^{-i \nu \xi \cdot \langle Q + i \varepsilon I \rangle^{-1} \xi/2}
\]

converge to \( \hat{F}_0(\xi) \) for each \( \xi \), as \( \varepsilon \to 0 \) (see [5, 14]). For every \( \varepsilon > 0 \) Parseval’s identity holds for \( F_0 \) replaced by \( F_{\xi} \). Passing there to the limit as \( \varepsilon \to 0 \) using Lebesgue’s theorem we recover (12.3). So

\[
|I(\nu)| \leq \left( \frac{\nu}{2\pi} \right)^{n/2} \det Q |\det Q|^{-1/2} |\hat{\varphi}|_{L^1}. \tag{12.4}
\]

We recall that

\[
|\hat{\varphi}|_{L^1} \leq C_m \|\varphi\|_{H^m(\mathbb{R}^n)} \quad \text{for any } m > n/2, \tag{12.5}
\]

where \( H^m(\mathbb{R}^n) \) denotes the Sobolev space on \( \mathbb{R}^n \).
12.3. Direct proof of estimate (3.14). In this appendix we estimate directly integral \( I_s \) in (3.4) with \( T = \infty \) (indirectly this integral in the form (3.2) was estimated in (3.14) via Theorem 3.3). Setting \( x = \sqrt{2}(s_1 - s) \) and \( y = \sqrt{2}(s_2 - s) \), in view of (1.15) we get \( \omega_{3s} = -x \cdot y \). Then, denoting \( z = (x, y) \), we obtain

\[
I_s = \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d_+} dl \; F_s(l, z)e^{i\nu^{-1}x \cdot y(l_2-l_1)}, \quad l = (l_1, l_2),
\]

(12.6)

where \( F_s(l, z) \) is the function

\[
2B(s_1, s_2, s_3)e^{-|l_1-l_2|(\gamma_1+\gamma_2+\gamma_3)+\gamma_4(l_1+l_2)}, \quad s_3 = s_1 + s_2 - s,
\]

written in the coordinates \( l, z \). Since \( B \) is a Schwartz function and \( \gamma_s \geq 1 \), for \( l \in \mathbb{R}^d_+ \) the function \( F_s \) satisfies the estimate

\[
|\partial^{\alpha}_{z} F_s(l, z)| \leq C^#(s)C^#(z)e^{l_1+l_2}, \quad \forall \alpha.
\]

(12.7)

Let \( N \) be the subset of \( \mathbb{R}^d_+ = \{ l = (l_1, l_2) \} \) where \( |l_1-l_2| \geq \nu \), and \( N^c \) be its complement. Then, bounding the complex exponent in (12.6) by one, we find

\[
|\langle I_s, N^c \rangle| \leq C^#(s)\nu,
\]

(12.8)

where we recall that the notation \( \langle I_s, N^c \rangle \) was introduced in (1.44).

To estimate the term \( \langle I_s, N \rangle \), we note that the integral over \( dz \) in (12.6) has the form (12.2) with \( \nu := \nu|l_1-l_2|^{-1} \) and \( n = 2d \). Then, due to (12.4) and (12.5),

\[
|\langle I_s, N \rangle| \leq \frac{1}{(2\pi)^d} \int_{N} |\nu^d| |\hat{F}_s(l, \cdot)|_{L^1} dl \leq C^#(s)\nu^d \int_{N} |\nu|_{l_1-l_2}^d dl,
\]

where in the last inequality we used (12.7) and the definition of the set \( N \). Since \( d \geq 2 \), this implies \( |\langle I_s, N \rangle| \leq C^#_1(s)\nu^d/\nu^{d-1} = C^#_1(s)\nu \). Combining the obtained inequality with (12.8), we get the desired estimate (3.14).

12.4. Proof of Proposition 2.1. We start by explaining the scheme of the proof. Let us express the process \( a^{(n)}_s \) through the processes \( a^{(0)}_k \) by iterating formula (2.4) \( n \) times and compute the expectation \( \mathbb{E}|a^{(n)}_s(\tau)|^2 \) applying the Wick theorem. It can be shown that when \( L \to \infty \) this expectation stays of size one. The reason for this is as follows: arguing by induction we see that an affine space of variables \( k \in L^d \) which serves as the set of indices over which we take summation in the expression for \( |a^{(n)}_s(\tau)|^2 \) through \( a^{(0)}_k \), \( \tilde{a}^{(0)}_k \) has dimension \( 4nd \) (indeed, for \( n = 0 \) this dimension obviously is zero, while for \( n = 1 \) due to (2.3) and the conjugated formula it is \( 4d \), etc.).

When computing the expectation \( \mathbb{E}|a^{(n)}_s(\tau)|^2 \) we make the Wick pairing of terms \( a^{(0)}_k \), \( \tilde{a}^{(0)}_k \) and it is possible to see that the dimension of the corresponding space of variables becomes \( 2nd \) (since in view of (2.8) for Wick-coupled terms \( a^{(0)}_k \) and \( \tilde{a}^{(0)}_k \) we should have \( k = k' \)). At the same time, after \( n \) iterations of (2.4) we get a factor

\[ \delta_{3s}^{12} \]

This is true for the most of Wick-pairings, while for some of them the affine space of variables may become empty because of the restrictions of the type \( \{s_1, s_2\} \neq \{s_3, s\} \) imposed by \( \delta_{3s}^{12} \).
Now we express the difference $\Delta_s^n(\tau) := a_s^{(n)}(\tau) - a_s^{(n)}(\tau)$ through the processes $a_k^{(0)}$
and write $\Delta_s^n(\tau)$ as a finite sum $\Delta_s^n = \sum \Delta_s^{n,j}$. Each term $\Delta_s^{n,j}$ is obtained by iterating
(2.4), where at least one iteration $\delta s_3^{12}$ is replaced by $-\delta s_3^{12} \delta s_3^{3}$ (this corresponds to the
"diagonal" term $-a_s^{(n_1)} a_s^{(n_2)} a_s^{(n_3)}$ in (1.23)).

This implies that dimension of the affine space of variables over which we take the summation in the formula for
(2.4), where at least one iteration $\delta s_3^{12}$ is replaced by $-\delta s_3^{12} \delta s_3^{3}$ (this corresponds to the
"diagonal" term $-a_s^{(n_1)} a_s^{(n_2)} a_s^{(n_3)}$ in (1.23)).

We give a complete proof only for the cases $n = 1, 2$ since in Theorem C which is
the main result of this paper we only deal with $a^{(n)}$ and $a^{(n)}$ such that $n \leq 2$. A general
case can be considered similarly, by analyzing the dimensions of the affine spaces of
variables over which we take the summation in the formula for $\Delta^n$. In particular, no
delicate cancellation argument is used. However, the proof is cumbersome due to the
notational difficulty, arising when expressing $\Delta^n$ through $a^{(0)}$.

By (1.22) and (2.3),

$$\Delta_s^1(\tau) = -i L^{-d} \int_{-T}^{\tau} e^{-\gamma_s(\tau-l)} |a_s^{(0)}(l)|^2 a_s^{(0)}(l) dl,$$

so in view of (2.8) the Wick theorem implies $\mathbb{E}|\Delta_s^1(\tau)|^2 \leq C_2(s) L^{-2d}$. Next, $\Delta_s^{2,1} =
2 \Delta_s^{2,1} + \Delta_s^{2,2}$, where the term

$$\Delta_s^{2,1}(\tau) = i L^{-d} \int_{-T}^{\tau} e^{-\gamma_s(\tau-l)} \left( \sum_{s_1, s_2} \delta s_3^{12} \delta s_3^{3} (\Delta_s^{1} a_s^{(0)} a_s^{(0)})(l) e^{i\omega_l l} \right)$$
$$- a_s^{(1)}(l) a_s^{(0)}(l) \right|^2 dl = \Delta_s^{2,1,1}(\tau) - \Delta_s^{2,1,2}(\tau)$$

(12.10)
corresponds to the choice $n_1 = 1$ and $n_2 = n_3 = 0$ in (1.23) with $n = 2$. The term $\Delta_s^{2,2}$
corresponds to the choice $n_1 = n_2 = 0, n_3 = 1$ and has a similar form. Below we only
discuss $\Delta_s^{2,1}$.

Because of the factor $L^{-d}$ in (12.10) it is straightforward to see that $\mathbb{E}|\Delta_s^{2,1,2}(\tau)|^2 \leq
C_2(s) L^{-2d}$. Let us study the term $\Delta_s^{2,1,1}$. By (12.10) joined with (12.9),

$$\mathbb{E}|\Delta_s^{2,1,1}(\tau)|^2 = L^{-4d} \int_{-T}^{\tau} dl \int_{-T}^{l} dl_1 \int_{-T}^{\tau} dl'_1 \int_{-T}^{l'} dl'_2 e^{-\gamma_s(2\tau-l-l')}$
$$\sum_{s_1, s_2, s'_1, s'_2} \delta s_3^{12} \delta s_3^{3} e^{-\gamma_{l'}(l'_1-l'_2)} e^{i\omega_l l} a_s^{(0)}(l) a_s^{(0)}(l'_1) a_s^{(0)}(l'_2) a_s^{(0)}(l'}) \right|^2.$$

(12.11)

Next we apply the Wick theorem. Due to (2.8), non-conjugated variables $a_k^{(0)}$ can be
coupled with conjugated variables $a_{k'}^{(0)}$ only, and $k$ should equals to $k'$. Consider e.g. the
case when $a_1^{(0)}$ is coupled with $\tilde{a}_1^{(0)}, \tilde{a}_1^{(0)}$ with $\tilde{a}_1^{(0)}, \tilde{a}_2^{(0)}$ with $\tilde{a}_2^{(0)}, a_3^{(0)}$ with $a_3^{(0)}$ and $a_4^{(0)}$.
with \( a_{l}^{(0)} \) (we write \(|a_k|^2\) as \(a_k \tilde{a}_k\)). Then, due to (2.8), the corresponding to this Wick pairing term in the expression for \( E|\Delta_{x,1}^{2;1,1}(\tau)|^2 \) does not exceed

\[
CL^{-4d} \sum_{s_1, s_2} \delta_{s_3}^{s_4} (B(s_1))^3 B(s_2) B(s_3) \leq L^{-2d} C^\#(s),
\]

where \( B(s) \) is defined in (1.28). The other Wick pairings can be considered similarly—the dimension of the space of variables over which we take the summation always does not exceed \(2d\), so the resulting estimate will be the same.

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