The self-interacting Dirac fields in FLRW spacetime

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Abstract. We prove the existence of global in time solution with the small initial data for the semilinear equation of the spin-$\frac{1}{2}$ particles in the Friedmann–Lemaître–Robertson–Walker spacetime. Moreover, we also prove that if the initial function satisfies the Lochak–Majorana condition, then the global solution exists for arbitrary large initial value. The solution scatters to free solution for large time. The mass term is assumed to be complex-valued. The conditions on the imaginary part of mass are discussed by proving nonexistence of the global solutions if certain relation between scale function and the mass are fulfilled.

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Introduction

The behavior of particles obeying the covariant Dirac equation in the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime has given rise to many investigations ([2,8,9,13,21,22,32]). On the other hand, the solution of the semilinear Dirac equation describing self-interacting Dirac fields in general FLRW spacetime according to our best knowledge, haven’t been discussed in literature. In this article we study solutions of the semilinear Dirac equation in the curved spacetime of the FLRW models of cosmology. In particular, we focus on the relationship between the mass term, scale factor, nonlinear term, and initial function, which provides a global in time existence or an estimate on the lifespan of the solution of the Dirac equation in the expanding universe.

The review of some results ([4,5,7,16]) on small and large amplitude global solutions of the semilinear Dirac equation in Minkowski space is given in [27].
The metric tensor of the spatially flat FLRW spacetime in Cartesian coordinates is

\[
(g_{\mu\nu}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -a^2(t) & 0 & 0 \\
0 & 0 & -a^2(t) & 0 \\
0 & 0 & 0 & -a^2(t)
\end{pmatrix}, \quad \mu, \nu = 0, 1, 2, 3, \quad (0.1)
\]

where the scale factor \(a(t) = a_0 t^\ell\), \(a_0, \ell \in \mathbb{R}\), \(t > 0\), and \(x \in \mathbb{R}^3\), \(x_0 = t\). If \(\ell < 0\) the spacetime is contracting. In the case of \(\ell > 1\) the expansion is accelerating (with horizon), while for \(0 < \ell < 1\) the expansion is decelerating. In particular, the Einstein–de Sitter space with the scale factor \(a(t) = t^{2/3}\) is modeling the expanding matter dominated universe, if \(a(t) = t^{1/2}\) is radiation dominated universe (see, e.g., [17,18]). The scale factor \(a(t) = t\) describes the Milne model [6,11,22]. The scalar curvature of the space with metric (0.1) is

\[
R = \frac{12}{a^2(t)}.
\]

The Dirac equation in the space with metric (0.1) is

\[
\left( i \gamma^0 \partial_0 + i \frac{1}{a(t)} \sum_{k=1,2,3} \gamma^k \partial_k + i \frac{3 \dot{a}(t)}{2a(t)} \gamma^0 - m \mathbb{I}_4 \right) \Psi = f, \quad (0.2)
\]

(see, e.g., [2]), where \(\partial_0 = \partial/\partial t\), \(\partial_k = \partial/\partial x_k\), and \(\dot{a}(t)\) is the Hubble parameter (see, e.g., [6, Ch.8],[11, Sec.11.4]), while

\[
\gamma^0 = \begin{pmatrix} 0 & 0 \\ 0 & -0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} \mathbb{O}_2 & \sigma^k \\ -\sigma^k & -\mathbb{O}_2 \end{pmatrix}, \quad k = 1, 2, 3,
\]

\[
\gamma^5 := -i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} \mathbb{O}_2 & -\mathbb{I}_2 \\ -\mathbb{I}_2 & \mathbb{O}_2 \end{pmatrix}.
\]

Here \(\sigma^k\) are Pauli matrices

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and \(\mathbb{I}_n, \mathbb{O}_n\) denote the \(n \times n\) identity and zero matrices, respectively.

Consider the Dirac equation

\[
\left( i \gamma^0 \partial_0 + i \frac{1}{a(t)} \sum_{k=1,2,3} \gamma^k \partial_k + i \frac{3 \dot{a}(t)}{2a(t)} \gamma^0 - m \mathbb{I}_4 + \gamma^0 V(x,t) \right) \Psi = 0,
\]

where \(V\) is the matrix-valued potential \(V(x,t) : \mathbb{R}^3 \times (0, \infty) \rightarrow M_4(\mathbb{C})\). The last equation can be written in the equivalent form as following,

\[
\left( \partial_0 + \frac{1}{a(t)} \sum_{j=1,2,3} \alpha^j \partial_j + \frac{3 \dot{a}(t)}{2a(t)} \mathbb{I}_4 + i m \gamma^0 - i V(x,t) \right) \Psi = 0,
\]

where \(\alpha^j = \gamma^0 \gamma^j\) are self-adjoint matrices, \((\alpha^j)^* = \alpha^j\). The semilinear Dirac equation, which describes the self-interacting field, can be obtained by introducing into the equation the nonlinear term. We characterize that term by the next condition.
Theorem 0.1. Let $F = F(\psi) \in C^3(\mathbb{C}^4; \mathbb{C}^4)$ be the Lipschitz continuous with exponent $\alpha > 0$ in the space $H_{(k)}(\mathbb{R}^3)$, $k \geq 3$, function and the potential $V \in \mathcal{B}^{(0,k)}$ is self-adjoint, $V(x,t) = V^*(x,t)$. The problem

$$
\begin{aligned}
&\left\{ \begin{array}{ll}
\partial_t + t^{-\ell} \sum_{j=1,2,3} \alpha^j \partial_j + \frac{3\ell}{2} t^{-1} I_4 + i mt^{-1} \gamma^0 - iV(x,t) \\
\psi(x,1) = \psi_0(x),
\end{array} \right.
\end{aligned}
$$

with

$$
3\ell > 2|\Im(m)| \quad \text{and} \quad \alpha > 2(3\ell - 2|\Im(m)|)^{-1}
$$

for sufficiently small $\varepsilon$ and $\psi_0 \in H_{(k)}$, $k \geq 3$, $\|\psi_0\|_{(k)} \leq \varepsilon$, has a global solution $\psi \in X(2\varepsilon, k, \ell, m)$. 

Condition (L) The function $F = F(\psi) \in C^3(\mathbb{C}^4; \mathbb{C}^4)$ is Lipschitz continuous with exponent $\alpha$ in the space $H_{(s)}(\mathbb{R}^3)$, that is, there is a constant $C > 0$ such that

$$
\| F(\psi_1) - F(\psi_2) \|_{H_{(s)}(\mathbb{R}^3)} \leq C \| \psi_1 - \psi_2 \|_{H_{(s)}(\mathbb{R}^3)} \left( \| \psi_1 \|_{H_{(s)}(\mathbb{R}^3)}^\alpha + \| \psi_2 \|_{H_{(s)}(\mathbb{R}^3)}^\alpha \right)
$$

for all $\psi_1, \psi_2 \in H_{(s)}(\mathbb{R}^3)$.
For the real-valued mass, $m \in \mathbb{R}$, the condition is $\alpha > 2/(3\ell)$. In particular, $\alpha > 1$ if $\ell = 2/3$, while $\alpha > \frac{4}{3}$ if $\ell = 1/2$. It is interesting, that for the Klein–Gordon equation with $m = 0$ for $\ell = 2/3$ if the semilinear term is positive and the Lipschitz continuous with exponent $\alpha < 2$, then, the blowup occurs (see, [10]). The same is true, if $\ell = 1/2$ and $m = 0$ if $\alpha \leq 4/3$. For the general $\ell$ the blowup occurs if $\alpha < 2/(3-3\ell)$ or $\alpha < (4-\ell)/(4-3\ell)$. Later in [19,20,26] various proofs of this result were proposed.

The Eq. (0.4) is a symmetric hyperbolic system, and the local existence of the solution is known (see, e.g., [25]). Thus, the local Cauchy problem for (0.4) is well posed in $C^0([1,T]; (H_s(\mathbb{R}^3))^4)$, $s \geq 3$, for some $T > 1$ (see, e.g., [15]). The next theorem states that some local solutions of the large data can be continued to the global ones.

**Theorem 0.2.** Let $m \in \mathbb{R}$, the potential $V \in \mathcal{B}^{(\infty,\infty)}$ is self-adjoint, $V^*(x,t) = V(x,t)$, and $V^T(x,t)\gamma^2 + \gamma^2V(x,t) = 0$. Suppose that the scale factor $a(t) = a_0 t^\ell$ has

$$\ell > 1/3.$$

Assume also that the function $F = F(\xi,\eta)$, $F \in C^\infty(\mathbb{R}^2; \mathbb{C}^4)$, has the form

$$F(\psi^*\gamma^0\psi, \psi^*\gamma^0\gamma^5\psi) = \alpha(\psi^*\gamma^0\psi, \psi^*\gamma^0\gamma^5\psi) I + i\beta(\psi^*\gamma^0\psi, \psi^*\gamma^0\gamma^5\psi) \gamma^5,$$

where $\alpha$ and $\beta$ are real-valued functions, with

$$\alpha(\xi,\eta) = O(|\xi| + |\eta|), \quad \beta(\xi,\eta) = O(|\xi| + |\eta|), \quad |\xi| + |\eta| \to 0.$$

Assume also that the function $\Psi_0 = \Psi_0(x) \in C^\infty_0(\mathbb{R}^3; \mathbb{C}^4)$ satisfies the Lochak–Majorana condition

$$\rho^2(\Psi_0(x)) := |\Psi_0^*(x)\gamma^0\Psi_0(x)|^2 + |\Psi_0^*(x)\gamma^0\gamma^5\Psi_0(x)|^2 = 0 \quad \text{for all} \quad x \in \mathbb{R}^3.$$

Then for $\chi_0 \in C^\infty_0(\mathbb{R}^3; \mathbb{C}^4)$ there is $\varepsilon_0 > 0$ such that the Cauchy problem

$$\begin{cases}
(\mathcal{D}_{FLRW}(t,\partial_t,\partial_x) + \gamma^0V(x,t)) \psi = F(\psi^*\gamma^0\psi, \psi^*\gamma^0\gamma^5\psi) \psi, \quad t > 1, \\
\psi(x,1) = \Psi_0(x) + \varepsilon\chi_0(x),
\end{cases}$$

with $0 < \varepsilon < \varepsilon_0$ has a unique solution $\psi = \psi(x,t)$ such that $\psi \in C^1_{t,x}([1,\infty) \times \mathbb{R}^3)$. The solution scatters to free solution as $t \to +\infty$.

The last assertion of the previous theorem follows from Theorem 4.1 that states that if $F(\psi^*\gamma^0\psi, \psi^*\gamma^0\gamma^5\psi)\psi$ is the Lipschitz continuous function with exponent $\alpha > 0$ in the space $H_{(3)}(\mathbb{R}^3)$, where $3\alpha > 2$, then for every solution $\bar{\psi} = \psi(x,t)$ of the problem (0.10) given by Theorem 0.2 there exists a solution $\tilde{\psi}(x,t)$ of the free Dirac equation

$$(\mathcal{D}_{FLRW}(t,\partial_t,\partial_x) + \gamma^0V(x,t)) \tilde{\psi} = 0,$$
such that
\[
\lim_{t \to +\infty} \left( \| \psi(x, t) - \tilde{\psi}(x, t) \|_{(H^3(\mathbb{R}^3))^4} + \| \partial_t \psi(x, t) - \partial_t \tilde{\psi}(x, t) \|_{(H^3(\mathbb{R}^3))^4} \right) = 0.
\]
Moreover, if \( V(x, t) = 0 \), then
\[
\tilde{\psi}(x, t) = D^\text{co}(t, \partial_t, \partial_x) \gamma^0 \left( \frac{\mathcal{K}_1(x, t, D_x; m; 1) I_2}{\mathcal{K}_1(x, t, D_x; -m; 1) I_2} \right) \tilde{\psi}(x, 1), \quad t > 1,
\]
with the operators \( D^\text{co}(t, \partial_t, \partial_x) \) and \( \mathcal{K}_1(x, t, D_x; m; 1) \) written in the explicit way (see [31] or Sect. 1.3). Henceforth, depending on the context, we use notation \( \psi \in H^3(\mathbb{R}^3) \) or \( \psi \in (H^3(\mathbb{R}^3))^4 \) for the spinor-valued functions belonging to the Sobolev spaces.

The rest of this paper is organized as follows. In Sect. 1, we prove Theorem 0.1 by the energy estimate (Sect. 1.1) and fixed point argument (Sect. 1.2). The asymptotics on the positive half-line of time (Theorem 1.2) and the representation of the solution of the free Dirac equation in the FLRW spacetime are given in Sect. 1.3. In Sect. 1.4, to make the text self-contained, we prove that the classical solutions of the semilinear Dirac equation obey the finite propagation speed property. In Sect. 2, we analyze the Lochak–Majorana condition in the FLRW spacetime and obtain its time-evolution. In Sect. 3, we prove Theorem 0.2 except the asymptotics part that follows from Theorem 4.1 of Sect. 4. In Sect. 4, we also give the asymptotics at infinity for the solutions of Theorem 0.2. In Sect. 5, we demonstrate the blow-up result (Theorem 5.2) for the solution of the Dirac equation in FLRW spacetime.

1. Small data global existence

1.1. Energy estimate

The metric tensor (0.1) allows us to develop the energy estimate that is the main tool to prove global existence of small amplitude solutions.

**Lemma 1.1.** Assume that \( a(t) = a_0 t^\ell \) and \( f \in C([0, \infty); (H^3(\mathbb{R}^3))^4) \), the potential \( V \in \mathcal{B}^{(0,k)} \) is self-adjoint, \( V(x, t) = V^*(x, t) \). Then, for the solution of
\[
\left( \partial_0 + \frac{1}{a(t)} \sum_{j=1,2,3} a_j^\gamma \partial_j + \frac{3\dot{a}(t)}{2a(t)} I_4 + imt^{-1} \gamma^0 - iV(x, t) \right) \psi = f(x, t),
\]
in the Sobolev space $H_{(k)}(\mathbb{R}^3)$ one has
\[
\|\psi(t)\|_k \leq ct^{-3\ell/2+|\Im(m)|} s^{3\ell/2-|\Im(m)|} \|\psi(s)\|_k \\
+ ct^{-3\ell/2+|\Im(m)|} \int_s^t \tau^{3\ell/2-|\Im(m)|} \|f(\tau)\|_k d\tau, \quad 1 \leq s \leq t, \tag{1.1}
\]
\[
\|\psi(t)\|_k \leq ct^{-3\ell/2-|\Im(m)|} s^{3\ell/2+|\Im(m)|} \|\psi(s)\|_k \\
+ ct^{-3\ell/2-|\Im(m)|} \int_t^s \tau^{3\ell/2+|\Im(m)|} \|f(\tau)\|_k d\tau, \quad 1 \leq t \leq s. \tag{1.2}
\]

**Proof.** Consider the energy integral
\[
E(t) = \int_{\mathbb{R}^3} |\psi(x, t)|^2 dx.
\]
Then,
\[
\frac{d}{dt} E(t) = \int_{\mathbb{R}^3} \left( -3\ell t^{-1} |\psi(x, t)|^2 + 2t^{-1} (\Im(m)) \psi^*(x, t) \gamma^0 \psi(x, t) \right) dx \\
+ \int_{\mathbb{R}^3} 2\Re(f^*(x, t) \psi(x, t)) dx.
\]
Furthermore,
\[
-2t^{-1} |\Im(m)| |\psi(x, t)|^2 \leq 2t^{-1} (\Im(m)) \psi^*(x, t) \gamma^0 \psi(x, t) \leq 2t^{-1} |\Im(m)| |\psi(x, t)|^2.
\]
Denote
\[
\delta_+(t) := (-3\ell + 2|\Im(m)|) t^{-1}, \quad \delta_-(t) := (-3\ell - 2|\Im(m)|) t^{-1}.
\]
Then
\[
\delta_- (t) |\psi(x, t)|^2 \leq -3\ell t^{-1} |\psi(x, t)|^2 + 2t^{-1} (\Im(m)) \psi^*(x, t) \gamma^0 \psi(x, t) \leq \delta_+(t) |\psi(x, t)|^2
\]
and
\[
\delta_- (t) E(t) + \int_{\mathbb{R}^3} 2\Re(f^*(x, t) \psi(x, t)) dx \leq \frac{d}{dt} E(t) \leq \delta_+(t) E(t) \\
+ \int_{\mathbb{R}^3} 2\Re(f^*(x, t) \psi(x, t)) dx. \tag{1.3}
\]
In particular,
\[
\frac{d}{dt} \left( E(t) \exp \left( - \int_1^t \delta_+ (\tau) d\tau \right) \right) \leq 2 \exp \left( - \int_1^t \delta_+ (\tau) d\tau \right) \|f(x, t)\| \|\psi(x, t)\|.
\]
We integrate the last inequality from $s$ to $t$, $s < t$, and obtain
\[
E(t) \exp \left( - \int_1^t \delta_+ (\tau) d\tau \right) \leq E(s) \exp \left( - \int_1^s \delta_+ (\tau) d\tau \right) \\
+ 2 \int_s^t \exp \left( - \int_1^r \delta_+ (\tau_1) d\tau_1 \right) \|f(x, \tau)\| \|\psi(x, \tau)\| d\tau.
\]
\[
\tag{1.4}
\]
We fix \( s \) and for \( k = 0 \) denote
\[
y(t) := \max_{\tau \in [s, t]} \exp \left( -\frac{1}{2} \int_1^\tau \delta_+ (\tau_1) \, d\tau_1 \right) \|\psi(\tau)\|_k,
\]
then
\[
y^2(t) = \max_{\tau \in [s, t]} \exp \left( -\int_1^\tau \delta_+ (\tau_1) \, d\tau_1 \right) \|\psi(\tau)\|^2_k.
\]
Hence, (1.4) implies
\[
y^2(t) \leq y^2(s) + 2y(t) \int_s^t \exp \left( -\frac{1}{2} \int_1^\tau \delta_+ (\tau_1) \, d\tau_1 \right) \|f(\tau)\| \, d\tau.
\]
Furthermore,
\[
y(t) \leq cy(s) + c \int_s^t \exp \left( -\frac{1}{2} \int_1^\tau \delta_+ (\tau_1) \, d\tau_1 \right) \|f(\tau)\| \, d\tau.
\]
Thus, we have
\[
\|\psi(t)\|_k \leq c \exp \left( \frac{1}{2} \int_s^t \delta_+ (\tau_1) \, d\tau_1 \right) \|\psi(s)\|_k + c \int_s^t \exp \left( \frac{1}{2} \int_1^\tau \delta_+ (\tau_1) \, d\tau_1 \right) \|f(\tau)\| \, d\tau,
\]
for all \( 1 \leq s \leq t \). Next we choose the left hand side of (1.3), rewrite it as
\[
\exp \left( -\int_1^t \delta_- (\tau) \, d\tau \right) \int_{\mathbb{R}^3} 2\Re(f^*(x, t)\psi(x, t)) \, dx \leq \frac{d}{dt} \left( E(t) \exp \left( -\int_1^t \delta_- (\tau) \, d\tau \right) \right),
\]
integrate from \( t \) to \( s \), \( s \geq t \geq 1 \), and obtain
\[
-2 \int_t^s \exp \left( -\int_1^\tau \delta_- (\tau_1) \, d\tau_1 \right) \|f(\tau)\| \|\psi(\tau)\| \, d\tau
\leq E(s) \exp \left( -\int_1^s \delta_- (\tau_1) \, d\tau_1 \right) - E(t) \exp \left( -\int_1^t \delta_- (\tau_1) \, d\tau_1 \right)
\]
that implies
\[
E(t) \exp \left( -\int_1^t \delta_- (\tau_1) \, d\tau_1 \right)
\leq E(s) \exp \left( -\int_1^s \delta_- (\tau_1) \, d\tau_1 \right)
+ 2 \int_t^s \exp \left( -\int_1^\tau \delta_- (\tau_1) \, d\tau_1 \right) \|f(\tau)\| \|\psi(\tau)\| \, d\tau, \quad 1 \leq t \leq s.
\]
If we fix \( s \) and for \( k = 0 \) denote
\[
z(t) := \max_{\tau \in [t, s]} \exp \left( -\frac{1}{2} \int_1^\tau \delta_- (\tau_1) \, d\tau_1 \right) \|\psi(\tau)\|_k,
\]
then
\[
z^2(t) = \max_{\tau \in [t, s]} \exp \left( -\int_1^\tau \delta_- (\tau_1) \, d\tau_1 \right) \|\psi(\tau)\|^2_k.
\]
The inequality
\[ z^2(t) \leq z^2(s) + 2z(t) \int_t^s \exp \left( -\frac{1}{2} \int_1^{\tau} \delta_-(\tau_1) \, d\tau_1 \right) \|f(\tau)\|_k \, d\tau, \quad 1 \leq t \leq s, \]
implies
\[ z(t) \leq cz(s) + c \int_t^s \exp \left( -\frac{1}{2} \int_1^{\tau} \delta_-(\tau_1) \, d\tau_1 \right) \|f(\tau)\|_k \, d\tau, \quad 1 \leq t \leq s, \]
and, consequently,
\[ \exp \left( -\frac{1}{2} \int_1^{t} \delta_-(\tau_1) \, d\tau_1 \right) \|\psi(t)\|_k \leq c \exp \left( -\frac{1}{2} \int_1^{s} \delta_-(\tau_1) \, d\tau_1 \right) \|\psi(s)\|_k + c \int_t^s \exp \left( -\frac{1}{2} \int_1^{\tau} \delta_-(\tau_1) \, d\tau_1 \right) \|f(\tau)\|_k \, d\tau, \]
for all \( 1 \leq t \leq s \). Thus, for all \( 1 \leq t \leq s \) we have
\[ \|\psi(t)\|_k \leq c \exp \left( \frac{1}{2} \int_1^{t} \delta_-(\tau_1) \, d\tau_1 \right) \|\psi(s)\|_k + c \int_t^s \exp \left( \frac{1}{2} \int_1^{\tau} \delta_-(\tau_1) \, d\tau_1 \right) \|f(\tau)\|_k \, d\tau. \]
For every \( k > 0 \) the inequalities of the lemma can be obtained from the case of \( k = 0 \) by differentiation. The lemma is proved. \( \square \)

1.2. Global existence: small data: Proof of Theorem 0.1
Set \( s = 1 \) in (1.1), then due to the condition \((\mathcal{L})\) for \( k > 3/2 \) and all \( t > 1 \) one has
\[ \|\psi(t)\|_k \leq ct^{-3\ell/2+|\Im(m)|} \|\psi(1)\|_k + ct^{-3\ell/2+|\Im(m)|} \int_1^t \tau^{3\ell/2-|\Im(m)|} \|F(\psi)(\tau)\|_k \, d\tau. \]
(1.5)

Hence,
\[ \sup_{t \in [1, \infty)} t^{3\ell/2-|\Im(m)|} \|\psi(t)\|_k \leq c \|\psi(1)\|_s + c \left( \sup_{t \in [1, \infty)} t^{3\ell/2-|\Im(m)|} \|\psi(t)\|_k \right)^{1+\alpha} \int_1^t \left( \tau^{3\ell/2-|\Im(m)|} \right)^{-\alpha} \, d\tau. \]

The last integral is
\[ \int_1^t \tau^{-\alpha(3\ell/2-|\Im(m)|)} \, d\tau = \frac{1}{1 - \alpha (\frac{3\ell}{2} - |\Im(m)|)} \left( t^{1-\alpha (\frac{3\ell}{2} - |\Im(m)|)} - 1 \right). \]

Hence, the integral is bounded for all \( t \in [1, \infty) \) because of the condition \((0.5)\).

Denote by \( S(t, s) \) the propagator (fundamental solution for the Cauchy problem), that is, an operator-valued solution of the problem
\[
\begin{cases}
\partial_0 + t^{-\ell} \sum_{j=1,2,3} \alpha_j \partial_j + \frac{3\ell}{2} t^{-1} \mathbb{I}_4 + i\text{mt}^{-1} g^0 - iV(x,t) & S(t,s) = 0, \quad t, s \in \mathbb{R}_+, \\
S(s,s) = I \text{(identity operator)}. & (1.6)
\end{cases}
\]
Then the solution of the problem
\[
\begin{cases}
\partial_0 + t^{-\ell} \sum_{j=1,2,3} \alpha_j \partial_j + \frac{3\ell}{2} t^{-1} \mathbb{1}_4 + i m t^{-1} \gamma^0 - i V(x, t) \psi(x, t) = f(x, t), \quad t \geq s > 0, \\
\psi(x, s) = \psi_0(x),
\end{cases}
\]
is given by Duhamel’s principle
\[
\psi(x, t) = S(t, s) \psi_0(x) + \int_s^t S(t, \tau) f(x, \tau) \, d\tau.
\]

It is known (see, e.g., [15, 24]) that for \( \psi_0 \in H_k(\mathbb{R}^3) \) and \( f \in C([0, \infty); H_k(\mathbb{R}^3)) \), \( k > 5/2 \), the unique solution \( \psi \in C([1, \infty); H_k(\mathbb{R}^3)) \cap C^1([1, \infty); H_{k-1}(\mathbb{R}^3)) \) exists. Next, we define the operator \( S \) by
\[
S\psi(x, t) := S(t, 1) \psi_0(x) + \int_1^t S(t, \tau) F(\psi(x, \tau)) \, d\tau.
\]

We are going to prove that \( S \) is a contraction
\[
S : X(R, k, \ell, m) \longrightarrow X(R, k, \ell, m)
\]
for sufficiently small \( R \). The inequality (1.5) proves that the operator \( S \) maps \( X(R, k, \ell, m) \) into itself, if \( \psi_0 \in H_k(\mathbb{R}^3), \| \psi_0 \|_{H_k} < \varepsilon \), and \( \varepsilon \) and \( R \) are sufficiently small, namely, if
\[
\varepsilon + \left( \frac{1}{\alpha(\frac{3}{2} \ell - |\Im(m)|)} - 1 \right) R^{1+\alpha} < R.
\]

In order to verify a contraction property, we write
\[
S\psi_1(x, t) - S\psi_2(x, t) = \int_1^t S(t, \tau) (F(\psi_1(x, \tau)) - F(\psi_2(x, \tau))) \, d\tau
\]
and use the condition (\( \mathcal{L} \)) and Lemma 1.1 to estimate the norm
\[
\| S\psi_1(x, t) - S\psi_2(x, t) \|_k \\
\leq c t^{-3\ell/2 + |\Im(m)|} \int_1^t \tau^{3\ell/2 - |\Im(m)|} \| F(\psi_1(\tau)) - F(\psi_2(\tau)) \|_k \, d\tau \\
\leq c t^{-3\ell/2 + |\Im(m)|} \int_1^t \tau^{3\ell/2 - |\Im(m)|} \| \psi_1(\tau) - \psi_2(\tau) \|_k \left( \| \psi_1(\tau) \|_k^2 + \| \psi_2(\tau) \|_k^2 \right) \, d\tau.
\]
It follows
\[
\sup_{t \in [1, \infty)} t^{3\ell/2 - |\lambda(m)|} \|S\psi_1(x, t) - S\psi_2(x, t)\|_k
\leq c \left( \sup_{\tau \in [1, \infty)} \tau^{3\ell/2 - |\lambda(m)|} \|\psi_1(\tau) - \psi_2(\tau)\|_k \right) \int_{1}^{t} \left( \|\psi_1(\tau)\|_k^\alpha + \|\psi_2(\tau)\|_k^\alpha \right) d\tau
\leq c \left( \sup_{\tau \in [1, \infty)} \tau^{3\ell/2 - |\lambda(m)|} \|\psi_1(\tau) - \psi_2(\tau)\|_k \right)
\times \left( \sup_{\tau \in [1, \infty)} \tau^{3\ell/2 - |\lambda(m)|} \left( \|\psi_1(\tau)\|_k + \|\psi_2(\tau)\|_k \right) \right) \int_{1}^{t} \tau^{-\alpha(3\ell/2 - |\lambda(m)|)} d\tau
\leq C \|\psi_1 - \psi_2\|_{X(R, k, \ell, m) \mathbb{R}^\alpha}.
\]

Here we have used the condition (0.5). Then we choose R such that \(CR^\alpha < 1\). The Banach fixed-point theorem completes the proof of Theorem 0.1. \(\square\)

1.3. Large time asymptotics

For the equation with \(\ell \neq 1\) we define the forward and backward light cones as the boundaries of \(D_+(x_0, t_0)\) and \(D_-(x_0, t_0)\), respectively, where
\[
D_\pm (x_0, t_0) := \{(x, t) \in \mathbb{R}^{3+1}; |x - x_0| \leq \pm (\phi(t) - \phi(t_0))\}, \tag{1.7}
\]
and \(\phi(t) := \frac{1}{\ell} t^{1-\ell}\). Any intersection of \(D_-(x_0, t_0)\) with the hyperplane \(t = \text{const}, 0 < \text{const} < t_0\), determines the so-called dependency domain for the point \((x_0, t_0)\), while the intersection of \(D_+(x_0, t_0)\) with the hyperplane \(t = \text{const}, \text{const} > t_0 > 0\) is the so-called domain of influence of the point \((x_0, t_0)\). The Dirac equation (0.2) is non-invariant with respect to time inversion and its solutions have different properties in different directions of time.

A retarded fundamental solution (a retarded inverse) for the Dirac operator (0.3) is a four dimensional matrix with the distribution-valued entries \(\mathcal{E}_\text{ret} = \mathcal{E}_\text{ret}(x, t; x_0, t_0; m)\) that solves the equation
\[
\varphi_{FLRW}(t, \partial_t, \partial_x) \mathcal{E}(x, t; x_0, t_0; m) = \delta(x-x_0)\delta(t-t_0)\mathbb{1}_4, \quad (x, t), (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}_+,
\tag{1.8}
\]
and with the support in the chronological future (causal future) \(D_+(x_0, t_0)\) of the point \((x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}_+\). The advanced fundamental solution (propagator) \(\mathcal{E}_\text{adv} = \mathcal{E}_\text{adv}(x, t; x_0, t_0; m)\) solves the equation (1.8) and has the support in the chronological past (causal past) \(D_-(x_0, t_0)\).

Following [31] we introduce two more \(\gamma\)-matrices (projection operators), the upper left corner and lower right corner matrices,
\[
\gamma^U = \begin{pmatrix} \mathbb{1}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} = \frac{1}{2}(\mathbb{1}_4 + \gamma^0), \quad \gamma^L = \begin{pmatrix} \mathbb{0}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{1}_2 \end{pmatrix} = \frac{1}{2}(\mathbb{1}_4 - \gamma^0).
\]

Next we define the right co-factor
\[
\varphi_{FLRW}^\alpha(t, \partial_t, \partial_x) := it^{-\frac{\ell}{2}}\gamma^0(t^{im}\gamma^U + t^{-im}\gamma^L)\frac{\partial}{\partial t} + it^{-\frac{2\ell}{2}} \sum_{k=1}^{3} \gamma^k(t^{im}\gamma^U + t^{-im}\gamma^L) \frac{\partial}{\partial x_k}
\]
of the Dirac operator $\mathcal{D}_{FLRW}(t, \partial_t, \partial_x)$ of (0.3). The composition $\mathcal{D}_{FLRW}(t, \partial_t, \partial_x) \mathcal{D}^0_{FLRW}(t, \partial_t, \partial_x)$ is a diagonal matrix of operators (see [31]).

Denote $\mathcal{E}^w_t(x, t)$ the fundamental solution to the Cauchy problem for the wave equation in the Minkowski space, that is, the solution of the problem

$$\mathcal{E}^w_t - \Delta \mathcal{E}^w = 0, \quad \mathcal{E}^w(x, 0) = \delta(x), \quad \mathcal{E}_t^w(x, 0) = 0.$$ 

Here $\Delta$ is the Laplace operator in $\mathbb{R}^3$. Henceforth, $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function (see, e.g., [3]). For $n = 3$ (see, e.g., [23])

$$\mathcal{E}^w(x, t) := \frac{1}{4\pi} \frac{\partial}{\partial t} \frac{1}{t} \delta(|x| - t).$$

The distribution $\delta(|x| - t)$ is defined by

$$\langle \delta(| \cdot | - t), \psi(\cdot) \rangle = \int_{|x|=t} \psi(x) \, dx \quad \text{for} \quad \psi \in C_0^\infty(\mathbb{R}^3).$$

According to Theorem 1.1 [31], for every $x_0 \in \mathbb{R}^3, t, t_0 \in \mathbb{R}_+$, the retarded fundamental solution of the Dirac operator $\mathcal{E}^w_{ret}(x, t; x_0, t_0; m)$ is given as follows

$$\mathcal{E}^w_{ret}(x, t; x_0, t_0; m) = -2t_0^{\frac{\ell - im}{2}} \mathcal{D}^0_{FLRW}(t, \partial_t, \partial_x) \times \int_0^{\phi(t) - \phi(t_0)} \left( \begin{array}{cc} E(r, t; t_0; m) I_2 & 0 \\ 0 & E(r, t; t_0; -m) I_2 \end{array} \right) \mathcal{E}^w(x - x_0, r) \, dr.$$ 

Here the kernel is defined as follows (see [29,31])

$$E(r, t; t_0; m) := 2^{\frac{\ell + im}{2}} (1 - \ell)^{\frac{\ell}{4}} \phi(t_0) \frac{\ell + im}{1 - \ell} \left( (\phi(t) + \phi(t_0))^2 - r^2 \right)^{-\frac{im}{1 - \ell}} \times F\left( i \frac{m}{1 - \ell}, i \frac{m}{1 - \ell}; 1; \frac{\phi(t) - \phi(t_0)}{\phi(t) + \phi(t_0)} \right).$$

The fundamental solution $\mathcal{E}_+(x, t; x_0; m; \varepsilon)$ to the Cauchy problem for the Dirac equation is given by Theorem 1.2 [31]. It states that $\mathcal{E}_+(x, t; x_0; m; \varepsilon)$, that is, a distribution satisfying

$$\begin{cases} \mathcal{D}_{FLRW}(t, \partial_t, \partial_x) \mathcal{E}_+(x, t; x_0; m; \varepsilon) = \mathbb{O}_4, \\ \mathcal{E}_+(x, \varepsilon; x_0; m; \varepsilon) = \delta(x - x_0) \mathbb{I}_4, \end{cases}$$

is given by

$$\mathcal{E}_+(x, t; x_0; m; \varepsilon) = -i \gamma^0 \mathcal{D}_{FLRW}(x, t, \partial_t, \partial_x) \phi(\varepsilon) \times \int_0^{\phi(t) - \phi(\varepsilon)} \left( \begin{array}{cc} K_1(r, t; m; \varepsilon) I_2 & 0 \\ 0 & K_1(r, t; -m; \varepsilon) I_2 \end{array} \right) \mathcal{E}^w(x - x_0, r) \, dr,$$

where

$$K_1(r, t; m; \varepsilon) := 2^{\frac{im}{1 - \ell}} (1 - \ell)^{-\frac{m}{2}} \phi(\varepsilon)^{2i \frac{m}{1 - \ell} - 1} \left( (\phi(t) + \phi(\varepsilon))^2 - r^2 \right)^{-\frac{im}{1 - \ell}}$$

$$\times F\left( i \frac{m}{1 - \ell}, i \frac{m}{1 - \ell}; 1; \frac{\phi(t) - \phi(\varepsilon)}{\phi(t) + \phi(\varepsilon)} \right).$$
In order to write the solution to the Cauchy problem we introduce the operator

\[ G(x, t; m) : = -2 \int_{\varepsilon}^{t} b^{\frac{\ell}{2} - im} db \int_{0}^{\frac{\Phi(t) - \Phi(b)}{E(r, t; b; m)}} E(r, t; b; m) \times \int_{\mathbb{R}^3} E^w(x - y, r) f(y, b) dy dr, \quad f \in C_0^\infty(\mathbb{R}^{n+1}), \]

and the operator \( K_1(x, t; D_x; m; \varepsilon) \) as follows:

\[ K_1(x, t; D_x; m; \varepsilon)[\varphi](x, t) : = -i\varepsilon^{1 + \frac{\ell}{2} - im} (1 - \ell)^{-1} \int_{0}^{\frac{\Phi(t) - \Phi(\varepsilon)}{E^w(x - y, r)}} \varphi(y) dy dr, \quad \varphi \in C_0^\infty(\mathbb{R}^n). \]

According to Theorem 1.3 [31] the solution to the Cauchy problem

\[
\begin{aligned}
\bigl\{ (D_{\text{FLRW}}(t, \partial_t, \partial_x))^2 \Psi(x, t) = F(x, t), \quad t > \varepsilon > 0, \\
\Psi(x, \varepsilon) = \Psi_\varepsilon(x),
\end{aligned}
\]

with \( m \in \mathbb{C} \), is given as follows:

\[
\Psi(x, t) = D_{\text{FLRW}}^{\psi}(t, \partial_t, \partial_x) \left\{ \left( G(x, t; D_x; m) \right)^2_2 \quad \left( G(x, t; D_x; -m) \right)^2_2 \right\} [F](x, t) + \gamma^0 \left( K_1(x, t; D_x; m; \varepsilon) \right)^2_2 \quad \left( K_1(x, t; D_x; -m; \varepsilon) \right)^2_2 [\Psi_\varepsilon](x, t), \quad t > \varepsilon > 0.
\]

Let \( S(t, s) \) be the propagator (fundamental solution for the Cauchy problem) of the Dirac equation, that is, an operator-valued solution of the problem (1.6). Hence,

\[
\psi(x, t) = S(t, s) \psi(x, s), \quad t, s \in (0, \infty), \quad x \in \mathbb{R}^3.
\]

For the case of \( V(x, t) = 0 \) the operator \( S(t, s) \) is written in the explicit form in [31].

**Theorem 1.2.** Assume that

\[
2|\Im(m)| + \alpha \left( |\Im(m)| - \frac{3}{2} \ell \right) < -1.
\]

Let the function \( \psi = \psi(x, t) \in X(2\varepsilon, k, \ell, m), \quad k \geq 3 \), be the solution of the problem (0.4) given by Theorem 0.1. Then the limit

\[
\lim_{t \to \infty} \int_{1}^{t} S(1, \tau) F(\psi(x, \tau)) d\tau
\]

exists in the space \( (H_{(k)}(\mathbb{R}^3))^4 \). Furthermore, the solution \( \tilde{\psi}(x, t) \) of the Cauchy problem for the free Dirac equation

\[
\begin{aligned}
\bigl\{ (D_{\text{FLRW}}(t, \partial_t, \partial_x) + \gamma^0 V(x, t))^2 \tilde{\psi}(x, t) = 0, \\
\tilde{\psi}(x, 1) = \psi_0^+(x),
\end{aligned}
\]

where

\[
\psi_0^+(x) = \psi_0(x) + \lim_{t \to \infty} \int_{1}^{t} S(1, \tau) F(\psi(x, \tau)) d\tau
\]

exists in the space \( (H_{(k)}(\mathbb{R}^3))^4 \). Furthermore, the solution \( \tilde{\psi}(x, t) \) of the Cauchy problem for the free Dirac equation
satisfies
\[ \lim_{t \to \infty} \left\| \psi(x, t) - \tilde{\psi}(x, t) \right\|_{(H^1_k)(\mathbb{R}^3)^4} = 0, \]
and \( S : \psi \mapsto \tilde{\psi} \) is a continuous operator in \( X(2\varepsilon, k, \ell, m) \), while \( S_0 : \psi_0 \mapsto \psi^+ \) is a continuous operator in \( H^1_k(\mathbb{R}^3) \).

Moreover, if \( V(x, t) = 0 \), then
\[ \tilde{\psi}(x, t) = \mathcal{S}^\diamond(t, \partial_t, \partial_x)^\gamma_0 \left( K_1(x, t, D_x; m; 1)I_2, K_1(x, t, D_x; -m; 1)I_2 \right) \psi_0(x, t), \quad t \geq 1. \]

**Proof.** According to Lemma 1.1
\[ \| S(1, \tau) \psi(\tau) \|_k \leq c\tau^{3\ell/2+|\mathbb{S}(m)|}\| \psi(\tau) \|_k, \quad 1 \leq \tau. \]
Then the existence of the limit follows from
\[ \| S(1, \tau) F(\psi(x, \tau)) \|_k \leq c\tau^{3\ell/2+|\mathbb{S}(m)|}\| F(\psi(x, \tau)) \|_k \leq c\tau^{3\ell/2+|\mathbb{S}(m)|}\| \psi(x, \tau) \|_k^{1+\alpha} \leq c\tau^{3\ell/2+|\mathbb{S}(m)|} \left[ \tau^{-3\ell/2+|\mathbb{S}(m)|} \right]^{1+\alpha} \leq c\tau^{3\ell}(2+\alpha)-3\ell\alpha/2, \]
where
\[ |\mathbb{S}(m)|(2 + \alpha) - \frac{3}{2}\ell\alpha = 2|\mathbb{S}(m)| + \alpha \left( |\mathbb{S}(m)| - \frac{3}{2}\ell \right) < -1 \]
according to the condition of the theorem.

To prove continuity of the operator \( S : \psi \mapsto \tilde{\psi} \) consider two solutions \( \psi_1 = \psi_1(x, t) \) and \( \psi_2 = \psi_2(x, t) \) given by Theorem 0.1 and the corresponding functions \( \tilde{\psi}_1(x, t), \psi_1^+(x) \), and \( \tilde{\psi}_2(x, t), \psi_2^+(x) \). Then by Lemma 1.1 we obtain
\begin{align*}
t^{3\ell/2-|\mathbb{S}(m)|}\| \tilde{\psi}_1(t) - \tilde{\psi}_2(t) \|_k &\leq c\| \tilde{\psi}_1(1) - \tilde{\psi}_2(1) \|_k \\
&\leq c\| \psi_1(1) - \psi_2(1) \|_k + \lim_{t \to \infty} \int_1^t \| S(1, \tau) [F(\psi_1(x, \tau)) - F(\psi_2(x, \tau))] \|_k \, d\tau \\
&\leq c\| \psi_1(1) - \psi_2(1) \|_k + \lim_{t \to \infty} \int_1^t c\tau^{3\ell/2+|\mathbb{S}(m)|} \| F(\psi_1(x, \tau)) - F(\psi_2(x, \tau)) \|_k \, d\tau \\
&\leq c\| \psi_1(1) - \psi_2(1) \|_k \\
&\quad + \lim_{t \to \infty} \int_1^t c\tau^{3\ell/2+|\mathbb{S}(m)|} \| \psi_1(x, \tau) \|_k - \| \psi_2(x, \tau) \|_k \| (\| \psi_1(x, \tau) \|_k^\alpha + \| \psi_2(x, \tau) \|_k^\alpha) \, d\tau \\
&\leq c\| \psi_1(1) - \psi_2(1) \|_k \\
&\quad + c \left( \sup_{\tau \in [1, \infty)} \tau^{3\ell/2-|\mathbb{S}(m)|} \| \psi_1(\tau) - \psi_2(\tau) \|_k \right) \int_1^t (\| \psi_1(\tau) \|_k^\alpha + \| \psi_2(\tau) \|_k^\alpha) \, d\tau \\
&\leq c\| \psi_1(1) - \psi_2(1) \|_k \\
&\quad + c \left( \sup_{\tau \in [1, \infty)} \tau^{3\ell/2-|\mathbb{S}(m)|} \| \psi_1(\tau) - \psi_2(\tau) \|_k \right) \\
&\leq c\| \psi_1(1) - \psi_2(1) \|_k. 
\end{align*}
This completes the proof of the last statement of theorem.

\[ \times \left( \sup_{\tau \in [1, \infty)} \tau^{3\ell/2 - |\Im(m)|} \left( \|\psi_1(\tau)\| + \|\psi_2(\tau)\| \right) \right)^\alpha. \]

\[ 2|\Im(m)| + \alpha (|\Im(m)| - 1) < -1 \iff |\Im(m)| < 1 \quad \text{and} \quad \alpha > 1 + 3 \frac{|\Im(m)|}{1 - |\Im(m)|}. \]

Corollary 1.3. (i) For the matter dominated universe \( \ell = 2/3 \), the condition of the theorem implies

\[ 2|\Im(m)| + \alpha (|\Im(m)| - 1) < -1 \iff |\Im(m)| < 1 \quad \text{and} \quad \alpha > 1 + 3 \frac{|\Im(m)|}{1 - |\Im(m)|}. \]

(ii) For the radiation dominated universe \( \ell = 1/2 \), the condition of the theorem implies

\[ 2|\Im(m)| + \alpha (|\Im(m)| - \frac{3}{4}) < -1 \iff |\Im(m)| < \frac{3}{4} \quad \text{and} \quad \alpha > 1 + \frac{1}{4} + 3 \frac{|\Im(m)|}{\frac{3}{4} - |\Im(m)|}. \]

1.4. Finite speed propagation property

In this section we prove that the dependence domain for the classical solution \( u(t, x) \) at the point \((T, x_0)\) of the semilinear equation coincide with the one of the corresponding linear equation. We present the proof here in order to make the paper self-contained. For \((T, x_0) \in (1, \infty) \times \mathbb{R}^3\) let

\[ \Sigma_-(T, x_0) := \left\{ (t, x) \in [1, T] \times \mathbb{R}^3 \mid |x - x_0| = \phi(T) - \phi(t) \right\} \]

be a part of the backward “curved light cone” (nullcone), where \( \phi(t) := \frac{1}{1 - \ell} t^{1-\ell} \) if \( \ell \neq 1 \) and \( \phi(t) := \ln(t) \) if \( \ell = 1 \). Let also

\[ D_-(T, x_0) = \left\{ (t, x) \in [1, T] \times \mathbb{R}^3 \mid |x - x_0| \leq \phi(T) - \phi(t) \right\} \]

be the region defined in (1.7), whose boundary contains \( \Sigma_-(T, x_0) \). The outline of the proof of the next theorem is similar to one of [14].

Theorem 1.4. Let \( \psi \) be a \( C^1 \) solution of the equation

\[
\left( \partial_0 + \frac{1}{a(t)} \sum_{j=1,2,3} \alpha^j \partial_j + \frac{3\dot{a}(t)}{2a(t)} \mathbb{I}_4 + imt^{-1} \gamma^0 - iV(x,t) \right) \psi = F(\psi), \quad (1.9)
\]

in the backward curved light cone \( D_-(T, x_0) \) through \((T, x_0) \in (1, \infty) \times \mathbb{R}^3\). Assume that the potential \( V \in C([0, \infty) \times \mathbb{R}^3) \) and the nonlinear term \( F(\psi) \in C^1 \) is such that

\[ F(0) = 0. \quad (1.10) \]

If

\[ \psi(x, 1) = 0 \quad \text{for all} \quad x \in D_-(T, x_0) \cap \{t = 1\}, \quad (1.11) \]

then \( \psi \) vanishes in \( D_-(T, x_0) \).

Proof. We consider the Eq. (1.9) in the proper time (see, e.g., [11,17,18])

\[ \tau := \phi(t) - \phi(1) \geq 0, \quad \tau_0 := \phi(T) - \phi(1) > 0, \]
then \( a(t) \partial_0 = \partial_\tau \) and the equation reads
\[
\left( \partial_\tau + \sum_{j=1,2,3} \alpha^j \partial_j + \ell b(\tau) \frac{3}{2} \Gamma_4 + i mb(\tau) \gamma^0 - ic(\tau) V(x,t(\tau)) \right) \psi = c(\tau) F(\psi),
\]
where
\[
\begin{cases}
  b(\tau) := [(1 - \ell) (\tau + \phi(1))]^{-1}, & c(\tau) := [(1 - \ell) (\tau + \phi(1))]^{\frac{\ell}{1-\ell}} \text{ if } \ell \neq 1, \\
  b(\tau) := 1, & c(\tau) := e^{\tau} \text{ if } \ell = 1.
\end{cases}
\]

For \( 0 \leq s < \tau_0 \) let
\[
\rho(s, x) = \tau_0 - \left( (\tau_0 - s)^2 + \tau_0^{-2} (2\tau_0 s - s^2) |x - x_0|^2 \right)^{1/2}.
\]
Then
\[
\rho(0, x) = 0, \quad \text{and} \quad \lim_{s \to \tau_0} \rho(s, x) = \tau_0 - |x - x_0|
\]

Define
\[
R_s = \{ (\tau, x) | 0 \leq \tau \leq \rho(s, x), |x - x_0| < \tau_0 - \tau \},
\]
then in the proper time the interior of \( D_-(T, x_0) \) is
\[
\Lambda_-(\tau_0, x_0) := \{ (\tau, x) \in [0, \tau_0) \times \mathbb{R}^n \mid |x - x_0| < \tau_0 - \tau \} = \bigcup_{0 \leq s < \tau_0} R_s.
\]

We fix \( s_0 \) such that \( 0 < s_0 < \tau_0 \), then with some number \( \theta(s_0) \) we have
\[
|\nabla_x \rho(s, x)| = \frac{\tau_0^{-2} (2\tau_0 s - s^2) |x - x_0|}{\left( (\tau_0 - s)^2 + \tau_0^{-2} (2\tau_0 s - s^2) |x - x_0|^2 \right)^{1/2}} \leq \theta(s_0) < 1
\]
if \( 0 \leq s \leq s_0 \). Next, we consider the surface defined by
\[
\Lambda_s := \{ (\tau, x) \mid \tau = \rho(s, x), |x - x_0| < \tau_0 \}.
\]

Note that the outward unit normal at \( (\rho(s, x), x) \in \Lambda_s \) is
\[
\frac{(1, -\nabla_x \rho)}{\sqrt{1 + |\nabla_x \rho|^2}}.
\]

To apply the energy method to the Eq. (1.9), we use the identity
\[
\partial_\tau |\psi|^2 + \sum_{j=1,2,3} \partial_j (\psi^* \alpha^j \psi) + 3\ell b(\tau) |\psi|^2 - 2\Im (m b(\tau) \psi^* \gamma^0 \psi + 2\psi^* c(\tau) \Im (V(x,t(\tau))) \psi = c(\tau) (\psi^* F(\psi) + F^*(\psi) \psi).
\]
The divergence theorem and the vanishing initial data (1.11) imply
\[
\int_{R_s} \left( \partial_\tau |\psi|^2 + \sum_{j=1,2,3} \partial_j (\psi^* \alpha^j \psi) \right) d\tau dx
\]
\[
= \int_{\Lambda_s} \left( |\psi|^2 - \sum_{j=1,2,3} (\psi^* \alpha^j \psi) \partial_j \rho(s, x) \right) \frac{1}{\sqrt{1 + |\nabla_x \rho(s, x)|^2}} d\sigma.
\]
Therefore,

\[
\int_{\Lambda_s} \left( |\psi|^2 - \sum_{j=1,2,3} (\psi^* \alpha^j \psi) \partial_j \rho(s, x) \right) \frac{1}{\sqrt{1 + |\nabla_x \rho(s, x)|^2}} d\sigma \\
+ \int_{R_s} (3b(\tau)|\psi|^2 - 2\Im(m)b(\tau)\psi^* \gamma^0 \psi + 2c(\tau)\psi^* \Im(V(x,t(\tau)))\psi) \, d\tau \, dx \\
= \int_{R_s} c(\tau) (\psi^* F(\psi) + F^*(\psi)\psi) \, d\tau \, dx.
\]

(1.12)

The hermitian matrix

\[
A = I_4 - \sum_{j=1,2,3} \alpha^j a_j, \quad a_j := \partial_j \rho(s, x), \quad j = 1, 2, 3,
\]

for all \( s \in [0, s_0] \) has two double positive eigenvalues \( 1 - \sqrt{a_1^2 + a_2^2 + a_3^2} > 0 \) and \( 1 + \sqrt{a_1^2 + a_2^2 + a_3^2} \), since \( |\nabla_x \rho(s, x)| \leq \theta(s_0) < 1 \). Hence, there is \( \delta(s_0) > 0 \) such that

\[
|\psi|^2 - \sum_{j=1,2,3} (\psi^* \alpha^j \psi) \partial_j \rho(s, x) \geq \delta(s_0)|\psi|^2 \quad \text{for all} \quad s \in [0, s_0].
\]

The Eq. (1.12) and condition (1.10) yield

\[
\int_{\Lambda_s} |\psi|^2 \frac{1}{\sqrt{1 + |\nabla_x \rho(s, x)|^2}} d\sigma \\
\leq C(s_0) \int_0^s \int_{\Lambda_\lambda} |\psi|^2 \frac{\partial_\lambda \rho(\lambda, x)}{\sqrt{1 + |\nabla_x \rho(\lambda, x)|^2}} d\sigma d\lambda \quad \text{for all} \quad s \in [0, s_0].
\]

Now we set

\[
I(s) = \int_{\Lambda_s} |\psi|^2 \frac{1}{\sqrt{1 + |\nabla_x \rho|^2}} d\sigma,
\]

and derive

\[
I(s) \leq C(s_0) \left( \max_{0 \leq s \leq s_0} |\partial_s \rho(s, x)| \right) \int_0^s I(\lambda) \, d\lambda \quad \text{for all} \quad 0 \leq s \leq s_0.
\]

It remains to apply Gronwall’s inequality.

\[ \square \]

2. Lochak–Majorana condition in FLRW spacetime

Consider the Dirac equation with the matrix-valued potential function

\[
A(x, t) = \alpha(x, t)I_4 + i\beta(x, t)\gamma^5,
\]

(2.1)

where \( \alpha(x, t), \beta(x, t) \) are any real-valued continuous functions.
Lemma 2.1. For the solution $\psi \in C^1([1, \infty); L^2(\mathbb{R}^3)) \cap C^0([1, \infty); H^1(\mathbb{R}^3))$ of the Dirac equation

$$
i \gamma^0 \partial_0 + it^{-\ell} \sum_{j=1,2,3} \gamma^j \partial_j + i \frac{3\ell}{2} t^{1-\gamma^0} - mt^{-1} \mathbb{I}_4 + \gamma^0 V(x,t) \psi = -A\psi,$$

the following energy identity holds

$$||\psi(\cdot, t)||^2_{L^2(\mathbb{R}^3)} = t^{-3\ell} ||\psi(\cdot, 1)||^2_{L^2(\mathbb{R}^3)} + 2\Im(m)t^{-3\ell} \int_1^t s^{3\ell-1} \int_{\mathbb{R}^3} \psi^*(x,s)\gamma^0 \psi(x,s) \, dx \, ds$$

$$-2t^{-3\ell} \int_1^t s^{3\ell} \int_{\mathbb{R}^3} \psi^*(x,s)\Im(V(x,s)\psi(x,s)) \, dx \, ds \quad \text{for all} \quad t \geq 1.$$  

Proof. We write the Eq. (2.2) in the equivalent form

$$\left(\partial_0 + t^{-\ell} \sum_{j=1,2,3} \alpha^j \partial_j + i \frac{3\ell}{2} t^{-1} \mathbb{I}_4 + imt^{-1} \gamma^0 - iV(x,t)\right) \psi = i\gamma^0 A\psi.$$

Hence,

$$\partial_0 ||\psi||^2 + \sum_{j=1,2,3} \partial_j \left(t^{-\ell} \psi^* \alpha^j \psi\right) + 3\ell t^{-1} ||\psi||^2 - 2\Im(m)t^{-1} \psi^* \gamma^0 \psi + 2\psi^* \Im(V(x,t)) \psi$$

$$= \psi^* i\gamma^0 A\psi + (i\gamma^0 A\psi)^* \psi.$$

The terms with the potential $A$ can be easily evaluated:

$$\psi^* i\gamma^0 A\psi + (i\gamma^0 A\psi)^* \psi = 0.$$ 

Therefore,

$$\partial_0 ||\psi||^2 = -\sum_{j=1,2,3} \partial_j \left(t^{-\ell} \psi^* \alpha^j \psi\right) - 3\ell t^{-1} ||\psi||^2 + 2\Im(m)t^{-1} \psi^* \gamma^0 \psi - 2\psi^* \Im(V(x,t)) \psi.$$

The finite propagation speed property implies

$$\frac{d}{dt} ||\psi||^2_{L^2(\mathbb{R}^3)} = -3\ell t^{-1} ||\psi||^2_{L^2(\mathbb{R}^3)} + 2\Im(m)t^{-1} \int_{\mathbb{R}^3} \psi^* \gamma^0 \psi \, dx - 2 \int_{\mathbb{R}^3} \psi^* \Im(V(x,t)) \psi \, dx$$

and, consequently,

$$\frac{d}{dt} \left(t^{3\ell} ||\psi||^2_{L^2(\mathbb{R}^3)}\right) = 2\Im(m)t^{-1} t^{3\ell} \int_{\mathbb{R}^3} \psi^* \gamma^0 \psi \, dx - 2t^{3\ell} \int_{\mathbb{R}^3} \psi^* \Im(V(x,t)) \psi \, dx.$$

It follows (2.3). Lemma is proved. 

Lemma 2.2. Assume that

$$V^T(x,t)\gamma^2 + \gamma^2 V(x,t) = 0.$$  

For the solution $\psi \in C^1([0, \infty); L^2(\mathbb{R}^3)) \cap C^0([0, \infty); H^1(\mathbb{R}^3))$ of the Dirac equation (2.2) we have

$$\int_{\mathbb{R}^3} \psi^T(x,t)\gamma^2 \psi(x,t) \, dx = t^{-3\ell} \int_{\mathbb{R}^3} \psi^T(x,1)\gamma^2 \psi(x,1) \, dx.$$
In particular, for \( z \in \mathbb{C}, \ |z| = 1, \)
\[
\int_{\mathbb{R}^3} 2\Re(z\psi^T(x,t)\gamma^2\psi(x,t))
\ dx = t^{-3\ell} \int_{\mathbb{R}^3} 2\Re(z\psi^T(x,1)\gamma^2\psi(x,1))
\ dx .
\]

**Proof.** We multiply the Eq. (2.2) by \(-i\psi^T\gamma^2\gamma^0\) and obtain
\[
\psi^T\gamma^2\partial_0\psi = -t^{-\ell} \sum_{j=1,2,3} \psi^T\gamma^2\gamma^0\gamma^j \partial_{x_j} \psi
- \frac{3\ell}{2} t^{-1} \psi^T\gamma^2\psi
- imt^{-1}\psi^T\gamma^2\gamma^0\psi + \psi^T\gamma^2iV(x,t)\psi + i\psi^T\gamma^2\gamma^0A\psi .
\]

Here
\[
\gamma^2\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & -\sigma_2 \\ -\sigma_2 & \mathbb{1}_2 \end{pmatrix}, \quad \gamma^2\gamma^0\gamma^1 = \begin{pmatrix} -i\sigma_3 & \mathbb{1}_2 \\ \mathbb{1}_2 & i\sigma_3 \end{pmatrix},
\]
\[
\gamma^2\gamma^0\gamma^2 = \gamma^0, \quad \gamma^2\gamma^0\gamma^3 = \begin{pmatrix} i\sigma_1 & \mathbb{1}_2 \\ \mathbb{1}_2 & -i\sigma_1 \end{pmatrix},
\]
\[
\gamma^0\gamma^5 = \begin{pmatrix} \mathbb{1}_2 & -\mathbbm{i} \mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix} = -\gamma^5\gamma^0, \quad \gamma^5\gamma^2 + \gamma^2\gamma^5 = 0 .
\]

Now
\[
\partial_0\psi^T = -t^{-\ell} \sum_{j=1,2,3} \partial_j\psi^T(\gamma^j)^T\gamma^0 - \frac{3\ell}{2} t^{-1} \psi^T - imt^{-1}\psi^T\gamma^0 + i\psi^T V^T(x,t) + i\psi^T A\gamma^0,
\]
and, consequently,
\[
\partial_0(\psi^T\gamma^2\psi) = \left(-t^{-\ell} \sum_{j=1,2,3} \partial_j\psi^T(\gamma^j)^T\gamma^0 - \frac{3\ell}{2} t^{-1} \psi^T - imt^{-1}\psi^T\gamma^0 + i\psi^T A\gamma^0\right)\gamma^2\psi
- t^{-\ell} \sum_{j=1,2,3} \psi^T\gamma^2\gamma^0\gamma^j \partial_j \psi
- \frac{3\ell}{2} t^{-1} \psi^T\gamma^2\gamma^0\psi
- imt^{-1}\psi^T\gamma^2\gamma^0\psi
+ i\psi^T(V^T(x,t)\gamma^2 + \gamma^2V(x,t))\psi + i\psi^T\gamma^2\gamma^0A\psi .
\]

Consider now the terms with the potential \(A\):
\[
(i\psi^TA\gamma^0)\gamma^2\psi + \psi^T\gamma^2(i\gamma^0A\psi) = i\psi^T\left(\alpha I + i\beta\gamma_5\right)\gamma^0\gamma^2\psi + i\psi^T\gamma^2\gamma^0\left(\alpha I + i\beta\gamma_5\right)\psi
= i\psi^T\left(\alpha\gamma^0\gamma^2 + i\beta\gamma_5\gamma^0\gamma^2 + \alpha\gamma^2\gamma^0 + i\beta\gamma^2\gamma^2\gamma^0\right)\psi
= -\psi^T\beta\left(\gamma^5\gamma^0\gamma^2 + \gamma^2\gamma^5\gamma^0\right)\psi = 0 .
\]

Thus, from (2.4) and (2.5) we obtain
\[
\partial_0(\psi^T\gamma^2\psi) = -3\ell t^{-1}\psi^T\gamma^2\psi - t^{-\ell} \sum_{j=1,2,3} \partial_j\psi^T(\gamma^j)^T\gamma^0\gamma^2\psi
- t^{-\ell} \sum_{j=1,2,3} \psi^T\gamma^2\gamma^0\gamma^j \partial_j \psi
= -3\ell t^{-1}\psi^T\gamma^2\psi - t^{-\ell} \sum_{j=1,2,3} \left\{ (\partial_j\psi^T)(\gamma^j)^T\gamma^0\gamma^2\psi + \psi^T\gamma^2\gamma^0\gamma^j \partial_j \psi \right\} .
\]

For the sum in the last equation we have
\[
(\gamma^1)^T\gamma^0\gamma^2 = \gamma^2\gamma^0\gamma^1, \quad (\gamma^2)^T\gamma^0\gamma^2 = \gamma^2\gamma^0\gamma^2, \quad (\gamma^3)^T\gamma^0\gamma^2 = \gamma^2\gamma^0\gamma^3 .
\]
It follows
\[ \partial_t (\psi^T \gamma^2 \psi) = -3 \ell t^{-1} \psi^T \gamma^2 \psi - t^{-\ell} \sum_{j=1,2,3} \partial_j \left( \psi^T \gamma^2 \gamma_j \psi \right) \]
and, consequently,
\[ \frac{d}{dt} \int_{\mathbb{R}^3} \psi^T(x,t) \gamma^2 \psi(x,t) \, dx = -3 \ell t^{-1} \int_{\mathbb{R}^3} \psi^T(x,t) \gamma^2 \psi(x,t) \, dx . \]
Thus, the first statement of the lemma is proved. To prove the second statement, we use \( m = -\gamma_2 \) and recall the formula from [1]:
\[ |\psi - z \gamma^2 \overline{\psi}|^2 = 2 |\psi|^2 + 2 \Re (\overline{\psi}^T \gamma^2 \psi) , \tag{2.6} \]
where \( z \in \mathbb{C}, |z| = 1 \). The lemma is proved. \( \square \)

**Lemma 2.3.** Assume that \( V^*(x,t) = V(x,t) \) and (2.4) if fulfilled. For the solution \( \psi \in C^1([1, \infty); L^2(\mathbb{R}^3)) \cap C^0([1, \infty); H_1(\mathbb{R}^3)) \) of the Dirac equation (2.2) one has
\[ \int_{\mathbb{R}^3} |\psi(x,t) - z \gamma^2 \overline{\psi(x,t)}|^2 \, dx = t^{-3\ell} \left( \int_{\mathbb{R}^3} |\psi(x,1) - z \gamma^2 \overline{\psi(x,1)}|^2 \, dx \right) \]
\[ + 4 \Im(m) t^{-3\ell} \int_{1}^{t} \int_{\mathbb{R}^3} \psi^*(x,s) \gamma^0 \psi(x,s) \, dx \, ds . \]

**Proof.** From Lemmas 2.1, 2.2, and (2.6)
\[ \int_{\mathbb{R}^3} |\psi(x,t) - z \gamma^2 \overline{\psi(x,t)}|^2 \, dx = \int_{\mathbb{R}^3} \left( 2 |\psi|^2 + 2 \Re (\overline{\psi}^T \gamma^2 \psi) \right) \, dx \]
\[ = 2 \left( t^{-3\ell} \| \psi(.,1) \|^2_{L^2(\mathbb{R}^3)} \right) \]
\[ + 2 \Im(m) t^{-3\ell} \int_{1}^{t} \int_{\mathbb{R}^3} \psi^*(x,s) \gamma^0 \psi(x,s) \, dx \, ds \]
\[ + t^{-3\ell} \int_{\mathbb{R}^3} 2 \Re (\overline{\psi}^T(x,1) \gamma^2 \psi(x,1)) \, dx . \]
That is,
\[ \int_{\mathbb{R}^3} |\psi(x,t) - z \gamma^2 \overline{\psi(x,t)}|^2 \, dx = 2 t^{-3\ell} \left( \| \psi(.,1) \|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \Re (\overline{\psi}^T(x,1) \gamma^2 \psi(x,1)) \, dx \right) \]
\[ + 4 \Im(m) t^{-3\ell} \int_{1}^{t} \int_{\mathbb{R}^3} \psi^*(x,s) \gamma^0 \psi(x,s) \, dx \, ds . \]
The lemma is proved. \( \square \)

The last statement of the next corollary contains (0.9) written as follows:
\[ \rho^2(\psi) = (|\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2)^2 + (2 \Im(\psi_1 \overline{\psi}_3) + 2 \Im(\psi_2 \overline{\psi}_4))^2 . \]

**Corollary 2.4.** Assume that (2.4) if fulfilled. (i) If \( \Im(m) = 0 \), then
\[ \int_{\mathbb{R}^3} |\psi(x,t) - z \gamma^2 \overline{\psi(x,t)}|^2 \, dx = t^{-3\ell} \left( \int_{\mathbb{R}^3} |\psi(x,1) - z \gamma^2 \overline{\psi(x,1)}|^2 \, dx \right) . \]
(ii) If \( \psi(x,1) - z \gamma^2 \overline{\psi(x,1)} = 0 \) and \( \Im(m) \neq 0 \), then
\[ \int_{\mathbb{R}^3} |\psi(x,t) - z \gamma^2 \overline{\psi(x,t)}|^2 \, dx = 4 \Im(m) t^{-3\ell} \int_{1}^{t} \int_{\mathbb{R}^3} \psi^*(x,s) \gamma^0 \psi(x,s) \, dx \, ds . \]
(iii) If $\psi(x, 1) - z\gamma^2 \overline{\psi}(x, 1) = 0$, then
\[
\int_{\mathbb{R}^3} |\psi(x, t) - z\gamma^2 \overline{\psi}(x, t)|^2 \, dx \leq 4|\Im(m)|t^{-3\ell} \int_1^t s^{3\ell-1} \int_{\mathbb{R}^3} \rho(x, s) \, dx \, ds.
\]

Proof. It is evident. \qed

Proposition I.1 [1] implies that $\psi - z\gamma^2 \overline{\psi} = 0$ is equivalent to $\rho^2(\psi) = 0$ if $z \in \mathbb{C}$, $|z| = 1$.

3. Large amplitude global solution: Proof of Theorem 0.2

We are going to prove the existence of the global solution by the successive approximations. We follow approach developed by Bachelot [1] that appeals to the estimates obtained by the replacing the generators of the Poincaré group with the Fermi operators. Energy estimates obtained in Sect. 1.1 allow us to expand the result from [1] to the Dirac equation in the FLRW spacetime.

Let $\Psi$ be the solution of
\[
\begin{cases}
(i\gamma^0 \partial_0 + it^{-\ell} \sum_{j=1,2,3} \gamma^j \partial_j + i\frac{3\ell}{2} t^{-\ell-1} \gamma^0 - mt^{-\ell-1} \mathbb{1}_4 + i\gamma^0 V(x, t)) \Psi(x, t) = 0, \\
\Psi(x, 1) = \Psi_0(x).
\end{cases}
\]

(3.1)

Here $V^*(x, t) = V(x, t)$. The finite propagation speed property implies that for every $t > 1$ the support of $\Psi = \Psi(x, t) \in C^\infty(\mathbb{R}^3 \times (1, \infty); \mathbb{C}^4)$ is in some compact subset of $\mathbb{R}^3$. Then according to Lemma 1.1
\[
\|\Psi(t)\|_k \leq ct^{-3\ell/2 + |\Im(m)|} \|\Psi_0\|_k \quad \text{for all} \quad t \geq 1. \tag{3.2}
\]

We look for the solution of (0.10) in the form
\[
\psi = \Psi + \chi, \tag{3.3}
\]

where $\Psi$ solves (3.1).

We consider the nonlinear term. The Lochak–Majorana condition (0.9) is equivalent to $\rho(\Psi_0) = 0$ (see [1]). First we note, that according to Corollary 2.4, (0.8), and (0.9), if $\Im(m) = 0$, then
\[
F(\Psi^*(x, t) \gamma^0 \Psi(x, t), \Psi^*(x, t) \gamma^0 \gamma^5 \Psi(x, t)) = 0 \quad \text{for all} \quad t \geq 1, \quad x \in \mathbb{R}^3.
\]

Further, (0.7), where $\alpha$ and $\beta$ are real-valued functions, can be written as follows
\[
F(\xi, \eta) = \alpha(\xi, \eta) I + i\beta(\xi, \eta) \gamma^5, \quad \xi = \overline{\psi}^T \gamma^0 \psi \in \mathbb{R}, \quad \eta = \overline{\psi}^T \gamma^0 \gamma^5 \psi \in \mathbb{R}.
\]

It is evident that with the solution $\Psi(x, t)$ we can write
\[
\begin{align*}
| (\Psi(x, t) + \chi(x, t))^T \gamma^0 (\Psi(x, t) + \chi(x, t)) - \overline{\Psi(x, t)}^T \gamma^0 \Psi(x, t) |
\leq C|\chi(x, t)| |(\chi(x, t) + |\Psi(x, t)|), \\
| (\Psi(x, t) + \chi(x, t))^T \gamma^5 (\Psi(x, t) + \chi(x, t)) - (\Psi)^T \gamma^0 \gamma^5 (\Psi(x, t)) |
\leq C|\chi(x, t)| |(\chi(x, t) + |\Psi(x, t)|)
\end{align*}
\]
and, consequently,
\[
\begin{align*}
\alpha (\psi^{*} \gamma^{0} \psi , \psi^{*} \gamma^{0} \gamma^{5} \psi ) \psi &= \alpha ((\Psi + \chi)^{*} \gamma^{0} (\Psi + \chi), (\Psi + \chi)^{*} \gamma^{0} \gamma^{5} (\Psi + \chi)) \\
&= \alpha (\Psi^{*} \gamma^{0} \Psi , \Psi^{*} \gamma^{0} \gamma^{5} \Psi ) \psi + \alpha_{1} (\chi, \Psi) \\
&= \alpha_{1} (\chi, \Psi), \\
\beta (\psi^{*} \gamma^{0} \psi , \psi^{*} \gamma^{0} \gamma^{5} \psi ) \psi &= \beta ((\Psi + \chi)^{*} \gamma^{0} (\Psi + \chi), (\Psi + \chi)^{*} \gamma^{0} \gamma^{5} (\Psi + \chi)) \\
&= \beta (\Psi^{*} \gamma^{0} \Psi , \Psi^{*} \gamma^{0} \gamma^{5} \Psi ) \psi + \beta_{1} (\chi, \Psi) \\
&= \beta_{1} (\chi, \Psi),
\end{align*}
\]

where \( \alpha_{1}, \beta_{1} \in C^{\infty} (\mathbb{C}^{8}; \mathbb{C}^{4}) \), and, as the functions of \( \chi \),
\[
\begin{align*}
|\alpha_{1} (\chi, \Psi(x, t))| &= O\left(||\chi|| (||\chi|| + ||\Psi(x, t)||)\right) \quad \text{as } ||\chi|| \rightarrow 0 , \quad (3.4) \\
|\beta_{1} (\chi, \Psi(x, t))| &= O\left(||\chi|| (||\chi|| + ||\Psi(x, t)||)\right) \quad \text{as } ||\chi|| \rightarrow 0 . \quad (3.5)
\end{align*}
\]

Further, the Cauchy problem becomes
\[
\begin{cases}
(\gamma^{0} \partial_{0} + it^{\ell} \sum_{j=1,2,3} \gamma^{j} \partial_{j} + \frac{i3\ell}{2} t^{\ell-1} \gamma^{0} - m t^{-1} I_{4} + i \gamma^{0} V(x, t)) \chi = f_{1} (\chi, \Psi) \quad (3.6) \\
\chi(x,1) = \varepsilon \chi_{0}(x),
\end{cases}
\]

where, in view of (3.1), we have denoted
\[
\begin{align*}
f_{1} (\chi, \Psi) := F ((\Psi + \chi)^{*} \gamma^{0} (\Psi + \chi), (\Psi + \chi)^{*} \gamma^{0} \gamma^{5} (\Psi + \chi)) \Psi \\
+ \frac{F ((\Psi + \chi)^{*} \gamma^{0} (\Psi + \chi), (\Psi + \chi)^{*} \gamma^{0} \gamma^{5} (\Psi + \chi)) \chi}
\end{align*}
\]

that can be rewritten as \( f_{1} (\chi, \Psi) = - A \chi \) similar to (2.1). Hence, \( f_{1} \in C^{\infty} (\mathbb{C}^{8}; \mathbb{C}^{4}) \), while (3.4) and (3.5) imply
\[
|f_{1} (\chi, \Psi(x, t))| = O\left(||\chi|| (||\chi|| + ||\Psi(x, t)||)^{2}\right) \quad \text{as } ||\chi|| \rightarrow 0 .
\]

We look for the function \( \chi \) as a limit of the sequence \( \{\chi^{(k)}\}_{1}^{\infty} \) that is defined by
\[
\begin{cases}
(\gamma^{0} \partial_{0} + it^{\ell} \sum_{j=1,2,3} \gamma^{j} \partial_{j} + \frac{i3\ell}{2} t^{\ell-1} \gamma^{0} - m t^{-1} I_{4} + i \gamma^{0} V(t)) \chi^{(k)} = f_{1} (\chi^{(k-1)} , \Psi), \\
\chi^{(k)}(x,1) = \varepsilon \chi_{0}(x), \quad k = 1,2,\ldots,
\end{cases}
\]

and \( \chi^{(0)}(x, t) \equiv 0 \). The finite propagation speed property implies that for every \( t > 0 \) the supports of the functions \( \chi^{(k)} = \chi^{(k)}(x, t) \) are in the same compact subset of \( \mathbb{R}^{3} \) for all \( k \geq 0 \). Lemmas 1.1, 2.1 and
\[
||\chi^{(1)}(t)||_{s} \leq c t^{-3\ell / 2} ||\chi^{(1)}(1)||_{s} + c t^{-3\ell / 2} \int_{1}^{t} \tau^{3\ell / 2} ||f_{1} (\chi^{(0)}(\tau), \Psi(\tau))||_{s} d\tau , \quad t \geq 1,
\]

imply
\[
||\chi^{(1)}(t)||_{s} \leq c t^{-3\ell / 2} \varepsilon ||\chi_{0}||_{s} , \quad t \geq 1 .
\]
Corollary 6.4.5 [12] and Lemma 1.1 imply for $k = 2, 3, \ldots$ the following estimate

$$
\|\chi^{(k)}(t)\|_s \leq ct^{-3\ell/2} \|\chi_0(t)\|_s + ct^{-3\ell/2} \int_1^t \tau^{3\ell/2} \|f_1(\chi^{(k-1)}, \Psi)(\tau)\|_s d\tau
$$

$$
\leq ct^{-3\ell/2} \|\chi_0(t)\|_s + ct^{-3\ell/2} \int_1^t \tau^{3\ell/2} \|\chi^{(k-1)}(\tau)\|_s \left(\|\chi^{(k-1)}(\tau)\|_s + \|\Psi(\tau)\|_s\right) d\tau
$$

$$
+ |\Psi(\tau)|_{[\frac{s}{2}] Map}^2 d\tau, \quad t \geq 1,
$$

(3.7)

where $[\frac{s}{2}] Map$ is the integer part of $\frac{s}{2}$, while

$$
|\Psi(t)|_{[\frac{s}{2}]} := \sup_{|\alpha| \leq s} \|\partial^\alpha_x \Psi(x, t)\|_{L^\infty(\mathbb{R}^3)}.
$$

Now we apply Sobolev embedding theorem and Lemma 1.1 to the function $\Psi$:

$$
|\Psi(t)|_{[\frac{s}{2}]} \leq \|\Psi(t)\|_s \leq ct^{-3\ell/2} \|\Psi_0\|_s \quad \text{for all} \quad t \geq 1.
$$

For the given $s$ and $n$ we define

$$
a_n(t) := \sup_{1 \leq \tau \leq t, \ 0 \leq k \leq n} \tau^{3\ell/2} \|\chi^{(k)}(\tau)\|_s, \quad A_n := \sup_{t \in [1, \infty)} a_n(t).
$$

Then for $s \geq 3$ by (3.7) we derive

$$
\|\chi^{(k)}(t)\|_s \leq ct^{-3\ell/2} \|\chi_0(0)\|_s + ct^{-3\ell/2} \int_1^t \tau^{3\ell/2} \|\chi^{(k-1)}(\tau)\|_s \left(\|\chi^{(k-1)}(\tau)\|_s + \|\Psi(\tau)\|_s\right) d\tau
$$

$$
\leq ct^{-3\ell/2} \left\{\|\epsilon\|_s \|\chi_0(0)\|_s
$$

$$
+ \int_1^t \tau^{-3\ell/2} \|\chi^{(k-1)}(\tau)\|_s \left(\tau^{3\ell/2} \|\chi^{(k-1)}(\tau)\|_s + \tau^{3\ell/2} \|\Psi(\tau)\|_s\right) d\tau\right\}, \quad t \geq 1,
$$

and

$$
a_n(t) \leq c\epsilon \|\chi_0(0)\|_s + c \int_1^t \tau^{-3\ell/2} \|\chi^{(n-1)}(\tau)\|_s \left(\tau^{3\ell/2} \|\chi^{(n-1)}(\tau)\|_s + \tau^{3\ell/2} \|\Psi(\tau)\|_s\right) d\tau
$$

$$
\leq c\epsilon \|\chi_0(0)\|_s + c \int_1^t \tau^{-3\ell} \left(\tau^{3\ell/2} \|\chi^{(n-1)}(\tau)\|_s \right) \left(\tau^{3\ell/2} \|\chi^{(n-1)}(\tau)\|_s + \|\Psi_0\|_s\right) d\tau.
$$

Hence,

$$
a_n(t) \leq c\epsilon \|\chi_0(0)\|_s + c \int_1^t \tau^{-3\ell} a_{n-1}(\tau) (1 + A_{n-1})^2 d\tau
$$

$$
\leq c\epsilon \|\chi_0(0)\|_s + c (1 + A_{n-1})^2 \int_1^t \tau^{-3\ell} a_{n-1}(\tau) d\tau
$$

$$
\leq c\epsilon \|\chi_0(0)\|_s + c (1 + A_{n-1})^2 \int_1^t \tau^{-3\ell} a_n(\tau) d\tau.
$$

(3.8)

Denote

$$
y(t) := \int_1^t \tau^{-3\ell} a_n(\tau) d\tau \quad \text{then} \quad y'(t) = t^{-3\ell} a_n(t)
$$

and according to (3.8) we obtain

$$
y'(t) \leq t^{-3\ell} c\epsilon \|\chi_0(0)\|_s + 2c (1 + A_{n-1})^2 t^{-3\ell} y(t),
$$

(3.9)
which implies
\[
\frac{d}{dt} \left( \exp\left( -2c(1 + A_{n-1})^2 \int_1^t \tau^{-3\ell} \, d\tau \right) y(t) \right)
\leq \exp\left( -2c(1 + A_{n-1})^2 \int_1^t \tau^{-3\ell} \, d\tau \right) t^{-3\ell} c\varepsilon \|\chi_0\|_s.
\]

Since \( y(1) = 0 \), after evaluating the integral we obtain
\[
\exp\left( -2c(1 + A_{n-1})^2 \int_1^t \tau^{-3\ell} \, d\tau \right) y(t)
\leq c\varepsilon \|\chi_0\|_s \int_1^t \exp\left( -2c(1 + A_{n-1})^2 \int_1^{\tau_1} \tau^{-3\ell} \, d\tau \right) \tau_1^{-3\ell} \, d\tau_1
\]
and
\[
y(t) \leq c\varepsilon \|\chi_0\|_s \exp\left( 2c(1 + A_{n-1})^2 \int_1^t \tau^{-3\ell} \, d\tau \right) \times \int_1^t \exp\left( -2c(1 + A_{n-1})^2 \int_1^{\tau_1} \tau^{-3\ell} \, d\tau \right) \tau_1^{-3\ell} \, d\tau_1.
\]

Hence,
\[
y(t) \leq c\varepsilon \|\chi_0\|_s \int_1^t \exp\left( 2c(1 + A_{n-1})^2 \int_1^{\tau_1} \tau^{-3\ell} \, d\tau \right) \tau_1^{-3\ell} \, d\tau_1,
\]
and according to (3.9)
\[
t^{-3\ell} a_n(t) \leq t^{-3\ell} c\varepsilon \|\chi_0\|_s + 2c(1 + A_{n-1})^2 t^{-3\ell} y(t)
\leq t^{-3\ell} c\varepsilon \|\chi_0\|_s + 2c(1 + A_{n-1})^2 t^{-3\ell} c\varepsilon \|\chi_0\|_s
\times \int_1^t \exp\left( 2c(1 + A_{n-1})^2 \int_1^{\tau_1} \tau^{-3\ell} \, d\tau \right) \tau_1^{-3\ell} \, d\tau_1,
\]
that is,
\[
a_n(t) \leq c\varepsilon \|\chi_0\|_s \left\{ 1 + 2c(1 + A_{n-1})^2 \int_1^t \exp\left( 2c(1 + A_{n-1})^2 \int_1^{\tau_1} \tau^{-3\ell} \, d\tau \right) \tau_1^{-3\ell} \, d\tau_1 \right\},
\]
and
\[
A_n \leq c\varepsilon \|\chi_0\|_s \left\{ 1 + 2c(1 + A_{n-1})^2 \int_1^t \exp\left( 2c(1 + A_{n-1})^2 \int_1^{\tau_1} \tau^{-3\ell} \, d\tau \right) \tau_1^{-3\ell} \, d\tau_1 \right\}.
\]

On the other hand,
\[
2c(1 + A_{n-1})^2 \int_1^t \exp\left( 2c(1 + A_{n-1})^2 \int_1^{\tau_1} \tau^{-3\ell} \, d\tau \right) \tau_1^{-3\ell} \, d\tau_1
= \exp\left( 2c(1 + A_{n-1})^2 \int_1^{\tau_1} \tau^{-3\ell} \, d\tau \right) - 1
\]
and the condition (0.6) lead to
\[
A_n \leq c\varepsilon \|\chi_0\|_s \exp\left\{ (1 + A_{n-1})^2 C \right\}, \quad C := \frac{2c}{1 - 3\ell},
\]
Then, with some other constant $\tilde{C}$ we obtain
\[ A_n \leq \tilde{C} \varepsilon \| \chi_0 \|_s \exp \{ \tilde{C} A_{n-1} (1 + A_{n-1}) \} . \] (3.10)

Let $\varepsilon_0$ be such that
\[ 2\tilde{C}^2 \varepsilon_0 \| \chi_0 \|_s (1 + 2\tilde{C} \varepsilon_0 \| \chi_0 \|_s) < \ln 2 . \]

If
\[ A_{n-1} \leq 2\tilde{C} \varepsilon \| \chi_0 \|_s \quad \text{and} \quad \varepsilon \leq \varepsilon_0 , \]
then due to (3.10) we obtain
\[ A_n \leq \tilde{C} \varepsilon \| \chi_0 \|_s \, \exp \{ \tilde{C} 2\tilde{C} \varepsilon \| \chi_0 \|_s (1 + 2\tilde{C} \varepsilon \| \chi_0 \|_s) \} \leq 2\tilde{C} \varepsilon \| \chi_0 \|_s . \]

Thus, for given $s$ and for all $n \geq 1$ we have proved the estimate
\[ \sup_{t \in [1, \infty)} \sup_{1 \leq \tau \leq t} \sup_{0 \leq k \leq n} \tau^{3t/2} \| (n)^{\nu}(\tau) \|_s \leq 2\tilde{C} \varepsilon \| \chi_0 \|_s < \infty \quad \text{for all} \quad n \geq 1 . \]

The last estimate, (3.2), and Sobolev inequality imply
\[ \sup_{n=0,1,2,...} \sup_{x \in \mathbb{R}^3} \sup_{t \in [1, \infty)} \left\{ t^{3t/2} |(n)^{\nu}(x,t)|, t^{3t/2} |\Psi(x,t)| \right\} = r < \infty . \] (3.11)

Hence, for all $(x,t) \in \mathbb{R}^3 \times [1, \infty)$ we have
\[ \left| f_1(\chi^{(k-1)}(x,t); \Psi(x,t)) - f_1(\chi^{(k-2)}(x,t); \Psi(x,t)) \right| \leq \left| \chi^{(k-1)}(x,t) - \chi^{(k-2)}(x,t) \right| \sup_{\xi, \eta \in \mathbb{C}} \sup_{|\xi|, |\eta| \leq r} |\nabla_{\xi, \eta} f_1(\xi, \eta)| . \] (3.12)

Consider
\[ \left( i\gamma^0 \partial_0 + it^{-\ell} \sum_{j=1,2,3} \gamma^j \partial_{x_j} + \frac{3\ell}{2} t^{-1} \gamma^0 - m \partial_4 + i\gamma^0 V(x,t) \right) \left( \chi^{(k)} - \chi^{(k-1)} \right) = f_1(\chi^{(k-1)}, \Psi) - f_1(\chi^{(k-2)}, \Psi) , \quad k = 1, 2, \ldots . \]

By Lemma 1.1 for the solution of the last equation taking into account (3.12) and the initial values, one has
\[ t^{3t/2} \| \chi^{(k)}(t) - \chi^{(k-1)}(t) \|_{L^2(\mathbb{R}^3)} \leq c \int_0^t s^{3t/2} \| f_1(\chi^{(k-1)}(x,t), \Psi(x,t)) - f_1(\chi^{(k-2)}(x,t), \Psi(x,t)) \|_{L^2(\mathbb{R}^3)} ds \leq c \left( \sup_{\xi, \eta \in \mathbb{C}, |\xi|, |\eta| \leq r} |\nabla_{\xi, \eta} f_1(\xi, \eta)| \right) \int_0^t s^{3t/2} \| \chi^{(k-1)}(x,t) - \chi^{(k-2)}(x,t) \|_{L^2(\mathbb{R}^3)} ds . \]
It follows
\[ t^{3t/2} \| \chi^{(k)}(t) - \chi^{(k-1)}(t) \|_{L^2(\mathbb{R}^3)} \]
\[ \leq c^2 \sup_{\xi, \eta \in \mathbb{C}, |\xi|, |\eta| \leq r} |\nabla \xi, \eta f_1(\xi, \eta)|^2 \]
\[ \times \int_0^t ds \int_0^s s^{3t/2} \| \chi^{(k-2)}(x, s_1) - \chi^{(k-3)}(x, s_1) \|_{L^2(\mathbb{R}^3)} ds_1 \]
and, consequently,
\[ t^{3t/2} \| \chi^{(k)}(t) - \chi^{(k-1)}(t) \|_{L^2(\mathbb{R}^3)} \leq C \| \chi_0 \|_{L^2(\mathbb{R}^3)} \frac{(Ct)_k}{k!}, \quad \text{for all } k = 1, 2, \ldots . \]

Hence, the sequence \( t^{3t/2} \chi^{(k)}(t) \) converges to some \( t^{3t/2} \chi \in C^0([1, \infty); (L^2(\mathbb{R}^3))^4) \), that is,
\[ \lim_{k \to \infty} t^{3t/2} \chi^{(k)}(t) = t^{3t/2} \chi(t) \]
uniformly on every compact subset of \( \mathbb{R}^3 \). By (3.11)
\[ \lim_{k \to \infty} f_1(\chi^{(k)}), \Psi) = f_1(\chi, \Psi) \quad \text{in } C^0([1, \infty); (L^2(\mathbb{R}^3))^4). \]

Thus, \( \chi \) solves (3.6) while \( \psi \) solves (0.10) and
\[ \sup_{t \in [1, \infty)} t^{3t/2} \| \psi(t) \|_s < \infty \quad \text{for } s \geq 3 \] (3.13)

implies
\[ \sup_{t \in [1, \infty)} t^{3t/2} \| \psi(x, t) \|_{s'} < \infty \quad \text{for } s' < s - 3/2. \]

It follows that for any integer \( s \geq 3 \) we have
\[ \left\| F \left( \bar{\psi}^T \gamma^0 \psi, \bar{\psi}^T \gamma^0 \gamma^5 \psi \right) \psi(t) \right\|_{(H^s(\mathbb{R}^3))^4} \leq C_s \| \psi(t) \|_{(H^s(\mathbb{R}^3))^4}. \] (3.14)

For every \( s \geq 3 \) the local Cauchy problem for (0.10) is well posed in \( C^0([0, T_s); (H^s(\mathbb{R}^3))^4) \) for some \( 1 < T_s \). According to (3.14) and
\[ \| \psi(x, t) \|_s \leq C_s t^{-3t/2} + C_s t^{-3t/2} \int_0^t \tau^{3t/2} \| \psi(x, \tau) \|_s d\tau, \quad \text{for } t \in [1, T_s), \]
from the last inequality we conclude that \( T_s = \infty \) and \( \psi(t) \in (C^0(\mathbb{R}^3))^4 \). The equation (0.10) implies \( \psi \in (C^{1, \infty}_{t,x}([1, \infty) \times \mathbb{R}^3))^4 \). Theorem is proved. \qed

4. Asymptotics at infinity

**Theorem 4.1.** Assume that all conditions of Theorem 0.2 are fulfilled. Let \( \psi = \psi(x, t) \) be a solution of the problem (0.10) given by Theorem 0.2. Suppose also that \( F (\psi^* \gamma^0 \psi, \psi^* \gamma^0 \gamma^5 \psi) \psi \) is the Lipschitz continuous function with exponent \( \alpha > 0 \) in the space \( H^{3\alpha}((1, \infty) \times \mathbb{R}^3), \) where \( 3\alpha > 2 \).
Then the limit
\[
\lim_{t \to \infty} \int_1^t S(1, \tau) F \left( \psi^*(x, \tau) \gamma^0 \psi(x, \tau), \, \psi^*(x, \tau) \gamma^0 \gamma^5 \psi(x, \tau) \right) \psi(x, \tau) \, d\tau
\]
exists in the space \((H^{(3)}(\mathbb{R}^3))^4\). Furthermore, the solution \(\tilde{\psi}(x, t)\) of the Cauchy problem for the free Dirac equation
\[
\begin{cases}
\left( D^{FLRW}(t, \partial_t, \partial_x) + \gamma^0 V(x, t) \right) \tilde{\psi}(x, t) = 0, \\
\tilde{\psi}(x, 1) = \tilde{\psi}(x),
\end{cases}
\]
where
\[
\tilde{\psi}(x) = \Psi_0(x) + \epsilon \chi_0(x) + \lim_{t \to \infty} \int_1^t S(1, \tau) F \left( \psi^*(x, \tau) \gamma^0 \psi(x, \tau), \, \psi^*(x, \tau) \gamma^0 \gamma^5 \psi(x, \tau) \right) \psi(x, \tau) \, d\tau,
\]
satisfies
\[
\lim_{t \to +\infty} \left\| \psi(x, t) - \tilde{\psi}(x, t) \right\|_{(H^{(3)}(\mathbb{R}^3))^4} = 0.
\]
Moreover, if \(V(x, t) = 0\), then
\[
\tilde{\psi}(x, t) = D^{co}(t, \partial_t, \partial_x) \gamma^0 \left( \begin{array}{c} \mathcal{K}_1(x, t, D_x; m; 1) I_2 \\ \mathcal{O}_2 \end{array} \right) \tilde{\psi}(x)(x, t), \quad t > 1.
\]
Proof. It is enough to prove the convergence in the space \((H^{(3)}(\mathbb{R}^3))^4\). The solution \(\psi\) can be written as follows (3.3)
\[
\psi = \Psi + \chi,
\]
where the function \(\Psi = \Psi(x, t)\) solves (3.1)
\[
\begin{cases}
\left( D^{FLRW}(t, \partial_t, \partial_x) + \gamma^0 V(x, t) \right) \Psi = 0, \\
\Psi(x, 1) = \Psi_0(x),
\end{cases}
\]
while the function \(\chi = \chi(x, t)\) solves (3.6)
\[
\begin{cases}
\left( D^{FLRW}(t, \partial_t, \partial_x) + \gamma^0 V(x, t) \right) \chi = f_1(\chi, \Psi), \\
\chi(x, 1) = \epsilon \chi_0(x),
\end{cases}
\]
where
\[
|f_1(\chi, \Psi)| = O\left(|\chi|(|\chi| + |\Psi|)^2\right).
\]
Since \(F \left( \psi^* \gamma^0 \psi, \, \psi^* \gamma^0 \gamma^5 \psi \right) \psi\) is a Lipschitz continuous function with exponent \(\alpha > 0\) and according to (3.13) and (1.2) of Lemma 1.1,
\[
\left\| F \left( \psi^* \gamma^0 \psi, \, \psi^* \gamma^0 \gamma^5 \psi \right) \psi(\tau) \right\|_{(H^{(3)}(\mathbb{R}^3))^4} \leq C_\epsilon \left\| \psi(\tau) \right\|_0 \leq C_\epsilon \tau^{-\frac{\alpha}{2}} \infty.
\]
At the same time
\[ \| S(1, \tau) F \left( \psi^* \gamma^0 \psi, \psi^* \gamma^0 \gamma^5 \psi \right) \psi(\tau) \|_{(H(3)(\mathbb{R}^3))^4} \leq C_s \tau^{\frac{3}{2} \ell} \| F \left( \psi^* \gamma^0 \psi, \psi^* \gamma^0 \gamma^5 \psi \right) \psi(\tau) \|_{(H(3)(\mathbb{R}^3))^4} \leq C_s \tau^{-\frac{3}{2} \ell \alpha}. \]

The last estimate due to the condition $3\ell \alpha > 2$ implies a convergence of (4.1) in $(H(3)(\mathbb{R}^3))^4$. Theorem is proved. \( \square \)

We skip the statement on the continuity of the scattering operator similar to one made in Theorem 1.2, since it can be easily extracted from the calculations written above, while a proof of such statement is really cumbersome and does not contain any new ideas.

5. Nonexistence of global in time solution

Consider the semilinear Dirac equation
\[
\begin{pmatrix}
  i\gamma^0 \partial_0 + it^{-\ell} \sum_{j=1,2,3} \gamma^j \partial_j + i \frac{3\ell}{2} t^{-1} \gamma^0 - mt^{-1} \mathbb{I}_4 + \gamma^0 V(x,t) \\
\end{pmatrix}
\psi = iG(\psi) \gamma^0 \psi,
\]
where $m \in \mathbb{C}$, and the matrix-valued term $G(\psi)$ commutes with $\gamma^0$,
\[
\gamma^0 G(\psi) = G(\psi) \gamma^0.
\]

The equation can also be written in the equivalent form of the following symmetric hyperbolic system
\[
\begin{pmatrix}
  \partial_0 + t^{-\ell} \sum_{j=1,2,3} \alpha^j \partial_j + \frac{3\ell}{2} t^{-1} \mathbb{I}_4 + i mt^{-1} \gamma^0 - i V(x,t) \\
\end{pmatrix}
\psi = G(\psi) \psi.
\]

Consider the energy integral
\[
E(t) = \int_{\mathbb{R}^3} |\psi(x,t)|^2 \, dx.
\]

**Lemma 5.1.** Let $m \in \mathbb{C}$ and $V^*(x,t) = V(x,t)$. Then
\[
\frac{d}{dt} E(t) = \int_{\mathbb{R}^3} \left( 2 \Re(G_{jk}(\psi) \psi_k(x,t) \psi_j(x,t)) - 3\ell t^{-1} |\psi(x,t)|^2 + 2t^{-1} (3(m)) \psi^*(x,t) \gamma^0 \psi(x,t) \right) dx.
\]

**Proof.** The proof is similar to the proof of Lemma 1.1 and we skip it. \( \square \)

The next theorem gives the blowup result for the Dirac equation in the universe with the scale function $a(t) = t^\ell$, $\ell \in \mathbb{R}$.
Theorem 5.2. Consider the Cauchy problem
\[
\begin{aligned}
\left\{ \begin{array}{l}
(\gamma^0 \partial_0 + it^{-\ell} \sum_{j=1,2,3} \gamma^j \partial_j + i\frac{3\ell}{2} t^{-1} \gamma^0 - m t^{-1} \mathbb{I}_4 + \gamma^0 V(x,t)) \psi = G(\psi) \gamma^0 \psi, \\
\psi(x,1) = \psi_0(x)
\end{array} \right.
\end{aligned}
\]
with \(\psi_0(x)\) such that \(\text{supp} \, \psi_0(x) \subseteq \{x \in \mathbb{R}^3 \mid |x| \leq R\}\). Assume that
\[
G(\zeta) = O(|\zeta|), \quad \gamma^0 G(\zeta) = G(\zeta) \gamma^0, \quad R(\zeta^* G(\zeta)^* \zeta) \geq c_0 |\zeta|^{2+\alpha}, \quad c_0 > 0, \quad \alpha > 0.
\]
Then for arbitrary size initial data there is no global solution \(\psi\) of (5.1) that obeys the finite propagation speed property
\[
\text{supp} \, \psi(x,t) \subseteq \{x \in \mathbb{R}^3 \mid |x| \leq R + A(t)\}, \quad A(t) := \int_1^t \frac{1}{a(\tau)} \, d\tau,
\]
in the following cases:

- If \(\ell < 1\) and \(1 \geq \frac{3\alpha}{2} + \alpha |\Im(m)|\),
- If \(1 < \ell\) and \(1 \geq \frac{3\alpha}{2} \ell + \alpha |\Im(m)|\),
- If \(\ell = 1\) and \(1 \geq \frac{3\alpha}{2} + \alpha |\Im(m)|\).

Furthermore, in the following cases for the large initial data there is no global solution of (5.1) that obeys the finite propagation speed property:

- If \(\ell < 1\) and \(1 < \frac{3\alpha}{2} + \alpha |\Im(m)|\),
- If \(1 < \ell\) and \(1 < \frac{3\alpha}{2} \ell + \alpha |\Im(m)|\),
- If \(\ell = 1\) and \(1 < \frac{3\alpha}{2} + \alpha |\Im(m)|\).

The solution blows up no later than time \(T_{bu}\) depending on \(c_0, \alpha, m, \psi_0, \) and \(\ell\) such that
\[
(E(1))^{-\frac{\alpha}{2}} = \frac{\alpha}{2} c_0 \int_1^{T_{bu}} (R + A(t))^{-\frac{3\alpha}{2}} t^{-\ell \frac{3\alpha}{2} - \alpha |\Im(m)|} \, dt \quad (5.2)
\]
and
\[
\lim_{t \to T_{bu}} \int_{\mathbb{R}^3} |\psi(x,t)|^2 \, dx = \infty,
\]
provided that \(E(1)\) is sufficiently large.

Proof. According to Lemma 5.1 if \(m \in \mathbb{C}\), then
\[
\frac{d}{dt} E(t) \geq \int_{\mathbb{R}^3} \left( c_0 |\psi(x,t)|^{2+\alpha} - 3t^{-1} |\psi(x,t)|^2 + 2t^{-1} \Im(m) \psi^*(x,t) \gamma^0 \psi(x,t) \right) \, dx 
\]
\[
\geq c_0 \int_{\mathbb{R}^3} |\psi(x,t)|^{2+\alpha} \, dx - \int_{\mathbb{R}^3} \left( 3t^{-1} |\psi(x,t)|^2 + 2t^{-1} |\Im(m)| |\psi(x,t)|^2 \right) \, dx 
\]
\[
\geq c_0 \int_{\mathbb{R}^3} |\psi(x,t)|^{2+\alpha} \, dx - \int_{\mathbb{R}^3} \left( 3t^{-1} + 2t^{-1} |\Im(m)| \right) \int_{\mathbb{R}^3} |\psi(x,t)|^2 \, dx. \quad (5.3)
\]
Further, if the solution obeys the finite speed propagation property, we obtain
\[
\int_{\mathbb{R}^3} |\psi(x,t)|^2 \, dx \leq \left( \int_{\text{supp} \, \psi} |\psi(x,t)|^{2+\alpha} \, dx \right)^{2/(2+\alpha)} (R + A(t))^{3\alpha/(2+\alpha)}.
\]
It follows
\[(R + A(t))^{-3\alpha/2} \left( \int_{\mathbb{R}^3} |\psi(x, t)|^2 \, dx \right)^{(2+\alpha)/2} \leq \int_{\text{supp} \psi} |\psi(x, t)|^{2+\alpha} \, dx .\]

Then by (5.3)
\[
\frac{d}{dt} E(t) \geq c_0 \left( R + A(t) \right)^{-3\alpha/2} E(t)^{(2+\alpha)/2} - \left( 3\ell t^{-1} + 2t^{-1}|\Im(m)| \right) E(t) .
\]

Denote
\[
K(t) := c_0 \left( R + A(t) \right)^{-3\alpha/2} ,
\]
\[
A(t) := \int_1^t \left( 3\ell \tau^{-1} + 2\tau^{-1}|\Im(m)| \right) \, d\tau = \ln a(t)^3 - \ln a(1)^3 + 2|\Im(m)| \ln(t),
\]

then the inequality (5.4) reads
\[
\frac{d}{dt} E(t) \geq K(t) E(t)^{(2+\alpha)/2} - E(t) \frac{d}{dt} A(t) .
\]

Hence,
\[
\frac{d}{dt} \left( E(t)e^{A(t)} \right) \geq K(t)e^{-A(t)\alpha/2} \left( E(t)e^{A(t)} \right)^{(2+\alpha)/2} .
\]

For the function
\[
F(t) := E(t)e^{A(t)}
\]
the inequality leads to
\[
\frac{d}{dt} F(t) \geq K(t)e^{-A(t)\alpha/2} F(t)^{(2+\alpha)/2}
\]
or
\[
\frac{d}{dt} F(t)^{-\alpha/2} \leq -\frac{\alpha}{2} K(t)e^{-A(t)\frac{\alpha}{2}} .
\]

After integration we obtain
\[
F(t)^{-\frac{\alpha}{2}} \leq F(1)^{-\frac{\alpha}{2}} - \frac{\alpha}{2} c_0 \int_1^t \left( R + A(s) \right)^{-3\alpha/2} e^{-A(s)\frac{\alpha}{2}} \, ds .
\]

Furthermore,
\[
\int_1^t \left( R + A(s) \right)^{-3\alpha/2} e^{-A(s)\frac{\alpha}{2}} \, ds = \int_1^t \left( R + A(s) \right)^{-3\alpha/2} \left( \frac{a(1)}{a(s)} \right)^{\frac{3\alpha}{2}} s^{-\alpha|\Im(m)|} \, ds
\]
\[
= J(t, m, R, \alpha, \ell) ,
\]

where
\[
J(t, m, R, \alpha, \ell) := \int_1^t \left( R + A(s) \right)^{-3\alpha/2} s^{-\ell\frac{3\alpha}{2} - \alpha|\Im(m)|} \, ds .
\]

In the case of \( \ell < 1 \) and \( 1 = \frac{3\alpha}{2} + \alpha|\Im(m)| \) we obtain
\[
J(t, m, R, \alpha, \ell)
\]
\[
\approx C + \int_1^t \left( s^{-1} - \ell \right)^{-\frac{3\alpha}{2}} s^{-\ell\frac{3\alpha}{2} - \alpha|\Im(m)|} \, ds = C + \int_1^t s^{-1} \, ds = \ln(t) \to \infty \quad \text{for large } t .
\]
In the case of $\ell < 1$ and $1 > \frac{3\alpha}{2} + \alpha|\Im(m)|$ we obtain

$$J(t, m, R, \alpha, \ell) \approx C + \int_1^t (s^{-1-\ell})^{-\frac{3\alpha}{2}} s^{-\ell \frac{3\alpha}{2} - \alpha|\Im(m)|} ds \approx C + t^{1 - \frac{3\alpha}{2} - \alpha|\Im(m)|} \to \infty \quad \text{for large } t .$$

Thus, in these two cases the solution blows up for initial data with arbitrary size.

In the case of $\ell < 1$ and $1 > \frac{3\alpha}{2} + \alpha|\Im(m)|$ we obtain

$$J(t, m, R, \alpha, \ell) \approx C + \int_1^t (s^{-1-\ell})^{-\frac{3\alpha}{2}} s^{-\ell \frac{3\alpha}{2} - \alpha|\Im(m)|} ds \leq 2C \quad \text{for large } t .$$

Thus, the solution blows up for the large initial data. The solution blows up no later than time $T_{bu}$ such that (5.2) holds and $E(1)$ is sufficiently large.

In the case of $1 < \ell$ and $1 = \frac{3\alpha}{2} + \alpha|\Im(m)|$ we obtain

$$J(t, m, R, \alpha, \ell) \approx C + \int_1^t \left( R + \frac{1}{\ell - 1} s^{1-\ell} \right)^{-\frac{3\alpha}{2}} s^{-\ell \frac{3\alpha}{2} - \alpha|\Im(m)|} ds \approx C + \frac{2R^{-3\alpha/2}}{1 - \ell^{3\alpha/2} - \alpha|\Im(m)|} t^{1 - \ell \frac{3\alpha}{2} - \alpha|\Im(m)|} \to \infty \quad \text{for large } t .$$

In the case of $1 < \ell$ and $1 > \frac{3\alpha}{2} + \alpha|\Im(m)|$ we obtain

$$J(t, m, R, \alpha, \ell) \approx C + \int_1^t \frac{1}{1 - \ell^{3\alpha/2} - \alpha|\Im(m)|} t^{1 - \ell \frac{3\alpha}{2} - \alpha|\Im(m)|} \to \infty \quad \text{for large } t .$$

Thus, in the last two cases the solution blows up for initial data with arbitrary size.

In the case of $1 < \ell$ and $1 < \frac{3\alpha}{2} + \alpha|\Im(m)|$ we obtain

$$J(t, m, R, \alpha, \ell) \approx C + \frac{2R^{-3\alpha/2}}{1 - \ell^{3\alpha/2} - \alpha|\Im(m)|} t^{1 - \ell \frac{3\alpha}{2} - \alpha|\Im(m)|} \leq 2C \quad \text{for large } t .$$

Thus, the solution blows up for the large initial data. The solution blows up no later than time $T_{bu}$ such that (5.2) holds and $E(1)$ is sufficiently large.

In the case of $\ell = 1$ and $1 = \frac{3\alpha}{2} + \alpha|\Im(m)|$ we obtain (for $\alpha \neq 2/3$)

$$J(t, m, R, \alpha, \ell) \approx C + \int_2^t (\ln(s))^{-\frac{3\alpha}{2}} s^{-\ell \frac{3\alpha}{2} - \alpha|\Im(m)|} ds = C + \int_2^t (\ln(s))^{-\frac{3\alpha}{2}} s^{-1} ds = C + \frac{1}{1 - \frac{3\alpha}{2}} \left( (\ln(t))^{1 - \frac{3\alpha}{2}} - (\ln(2))^{1 - \frac{3\alpha}{2}} \right) \quad \text{for large } t .$$

Hence, there is no global solution for initial data with arbitrary size.

In the case of $\ell = 1$ and $1 < \frac{3\alpha}{2} + \alpha|\Im(m)|$ we obtain

$$J(t, m, R, \alpha, \ell) \approx C + \int_2^t (\ln(s))^{-\frac{3\alpha}{2}} s^{-\frac{3\alpha}{2} - \alpha|\Im(m)|} ds \leq 2C \quad \text{for large } t .$$

There is no global solution for large data. The solution blows up no later than time $T_{bu}$ such that (5.2) holds and $E(1)$ is sufficiently large.
In the case of $\ell = 1$ and $1 > \frac{3\alpha}{2} + \alpha|\Im(m)|$ we obtain

$$J(t, m, R, \alpha, \ell) \approx C + \int_{\frac{t}{2}}^{t} (\ln(s)) - \frac{3\alpha}{2} s^{-3\alpha} - \alpha|\Im(m)| \, ds \to \infty \quad \text{for large} \quad t.$$  

There is no global solution for initial data with arbitrary size. Theorem 5.2 is proved.

**Corollary 5.3.** For the matter dominated universe with $\ell = 2/3$, for the radiation dominated universe with $\ell = 1/2$, and for Milne model with $\ell = 1$, if $1 \geq \frac{3\alpha}{2} + \alpha|\Im(m)|$, then for arbitrary size initial data there is no global solution $\psi$ of (5.1) that obeys the finite propagation speed property. If $1 < \frac{3\alpha}{2} + \alpha|\Im(m)|$, then for the large initial data there is no global solution of (5.1) that obeys the finite propagation speed property.

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