AERODYNAMICS OF FLYING SAUCERS

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Abstract. We identify various structures on the configuration space $C$ of a flying saucer, moving in a three-dimensional smooth manifold $M$. Always $C$ is a five-dimensional contact manifold. If $M$ has a projective structure, then $C$ is its twistor space and is equipped with an almost contact Legendrean structure. Instead, if $M$ has a conformal structure, then the saucer moves according to a CR structure on $C$. With yet another structure on $M$, the contact distribution in $C$ is equipped with a cone over a twisted cubic. This defines a certain type of Cartan geometry on $C$ (more specifically, a type of ‘parabolic geometry’) and we provide examples when this geometry is ‘flat,’ meaning that its symmetries comprise the split form of the exceptional Lie algebra $\mathfrak{g}_2$.

0. Introduction

Throughout this article $M$ will be a 3-dimensional smooth oriented manifold. For $x \in M$, a non-zero element $\omega \in T^*_x M$ defines an oriented 2-plane $\{ X \in T_x M \mid X \downarrow \omega = 0 \}$ at $x$.

Thus, we may realise the bundle of oriented two-planes in $TM$ as

$$\frac{\{ \omega \in T^*M \setminus \{ \text{the zero section} \} \}}{\omega \sim \lambda \omega \text{ for } \lambda > 0} = \text{Gr}^+_2(TM) \xrightarrow{\pi} M.$$

We shall write $C \xrightarrow{\pi} M$ for this configuration space of oriented saucers in $M$ (flying saucers are oriented as they traditionally have a cockpit).

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The basic intrinsic structure on $C$ is a contact distribution $H \subset TC$. We shall see that a saucer moves along a path in $C$ that is everywhere tangent to $H$ if and only if its motion in $M$ is in directions taken from its own disc. With this specification, arbitrary ‘rolls’ are allowed but, with more structure on $M$, these rolls are constrained and this gives rise to differential geometries on $C$ expressed in terms of various algebraic structures on $H$.

This article is organised as follows. Section 1 discusses the contact geometry on $C$. In Section 2 it is supposed that $M$ has a projective structure and consequently we shall find an almost contact Legendrean structure on $C$ (originally due to Takeuchi [17] using methods due to Tanaka). Then $M$ is projectively flat if and only if $C$ has maximal symmetry $\mathfrak{sl}(4, \mathbb{R})$. Usual CR structures emerge in Section 3 (following LeBrun [9]). In each case there are links with twistor theory. Section 4 explains how to endow $C$ with a geometric structure modelled on the contact homogeneous space for the split form of the exceptional Lie group $G_2$ and we present some examples for which this structure turns out to be ‘flat,’ meaning that it is locally isomorphic to the flat model. Section 5 further investigates the geometry on $M$ that is needed to construct this ‘$G_2$ contact structure’ on $C$.

This article is concerned only with the geometry of $C$ and especially its construction from, and relationship to, various geometrical features on $M$. In a companion article [5], we simply start with Euclidean space $M = \mathbb{R}^3$ and explain how the various aerodynamic options considered here are reflected in the aerobatic manoeuvres available to a pilot flying according to these options.

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1. The contact structure on $C$

In fact, the intrinsic contact structure on $C$ is, in addition, filtered. Specifically, we shall find canonically defined subbundles

$$TC \supset H \supset V,$$

where $H$ is the contact distribution and $V$ is the vertical subbundle of $\pi : C \to M$. To define $H$, we note the canonical identification

$$TC/V = \pi^*TM$$

and observe that a point in $C$ is precisely a point $x \in M$ together with an oriented 2-plane in $T_x M$. In other words, we have a tautologically defined rank 2 subbundle $P \subset \pi^*TM$ recording this subspace and we
may define $H$ as the inverse image of $P$ under $TC \to \pi^*TM$. In summary, we have a canonical filtration

$$TC = L + P + V = H$$

where $L$ is, by definition, the line bundle $TC/H$ and we are recording here the composition factors, with the rightmost bundle $V$ being the natural subbundle.

It remains to see that $H$ is, indeed, a contact distribution. This is a calculation in local co"ordinates. Specifically, we recall that the cotangent bundle $T^*M$ of any smooth manifold is equipped with the well-known tautological 1-form $\theta$. In 'canonical co"ordinates' $(x^a, p_a)$ on $T^*M$, we have $\theta = p_a \, dx^a$ (for details, see [3]). On $C$, we may use an affine chart $(x, y, z, a, b) \mapsto (x, y, z, a, b, 1)$ to embed $C$ in $T^*M$ and pull-back $\theta$ to the 1-form $a \, dx + b \, dy + dz$ whose kernel is $H$. Then

$$d(a \, dx + b \, dy + dz) = da \wedge dx + db \wedge dy$$

is the Levi form on $H$, which is manifestly non-degenerate.

Alternatively, the Levi form on $H$ may be seen as arising from the canonical symplectic form on $T^*M$ as follows. In canonical co"ordinates $(x^a, p_a)$ on $T^*M$, the symplectic form is $d\theta = dp_a \wedge dx^a$. It means that if we use canonical co"ordinates on the total space of $T^*M \rightarrow M$ to split its tangent bundle as

$$T(T^*M) = \nu^*TM \oplus \nu^*T^*M \ni \begin{bmatrix} X^a \\ \omega_a \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ p_a \end{bmatrix} = \theta$$

then the symplectic form is

$$\begin{bmatrix} X^a \\ \omega_a \end{bmatrix} \otimes \begin{bmatrix} \tilde{X}^b \\ \tilde{\omega}_b \end{bmatrix} \mapsto X^b \tilde{\omega}_b - \tilde{X}^b \omega_b,$$

(and is independent of choice of co"ordinates (as we shall see in the next section by a different argument)). Viewing $C$ as in (1), we see its tangent bundle as

$$\left\{ \begin{bmatrix} X^a \\ \omega_a \end{bmatrix} \right\} / \omega_a \sim \omega_a + tp_a \quad \text{for } t \in \mathbb{R}$$

and $H$ as the subbundle for which $X^a p_a = 0$. Evidently, the form (4) descends to $H$ (and is easily verified to be the Levi form).

Flying tangent to $H$ in $C$ is saying exactly that the velocity of the saucer in space is constrained to lie in its own disc. Otherwise, the pilot is free to make arbitrary 'rolls' and the Chow–Rashevskii Theorem [13].
in this context implies that a pilot flying with these manœuvres may park her craft in an arbitrary location and orientation.

Finally in this section, we consider the abstract structure on $C$ arising from its being a configuration space. Recall from (2) that $C$ is equipped with a filtration on its tangent bundle

\[ TC = L + P + V \]  

in which $H = P + V$ is contact and the two-dimensional subbundle $V$ is integrable. In fact, there are no local invariants of this arrangement.

**Theorem 1.** Suppose $C$ is a five-dimensional contact manifold with contact distribution $H$. Suppose $V$ is a rank two integrable subbundle of $H$. Then we may find local coördinates $(x, y, z, a, b)$ on $C$ so that

- $H$ is defined by the 1-form $\lambda \equiv dz + a\,dx + b\,dy$,
- $V$ is defined by $\lambda$ and the two 1-forms $dx$ and $dy$.

Thus, it is as if $C$ were defined by $M$ with local coördinates $(x, y, z)$.

**Proof.** The following argument pertains locally. Choose 1-forms $\lambda, \mu, \nu$ so that

\[ H = \lambda^\perp \quad \text{and} \quad V = (\lambda, \mu, \nu)^\perp. \]

Integrability of $V$ ensures, by Frobenius, that we can find coördinates $(x, y, z, u, v)$ so that

\[ \lambda, \mu, \nu \in \text{span}\{dx, dy, dz\} \]

and, since $\lambda \neq 0$, we may rescale it and subtract appropriate multiples thereof from $\mu$ and $\nu$ to suppose, without loss of generality, that

\[ \lambda = dz + a\,dx + b\,dy \quad \mu = p\,dx + q\,dy \quad \nu = r\,dx + s\,dy, \]

for suitable smooth functions $(a, b, p, q, r, s)$. Now, since $H$ is contact,

\[ 0 \neq \lambda \wedge d\lambda \wedge d\lambda = 2\,dx \wedge dy \wedge dz \wedge da \wedge db \]

so $(x, y, z, a, b)$ may be used as local coördinates instead. Finally,

\[ 0 \neq \lambda \wedge \mu \wedge \nu = (ps - qr)\,dx \wedge dy \wedge dz \]

so we may replace $\{\mu, \nu\}$ by $\{dx, dy\}$ without changing their span. \(\square\)

2. The almost contact Legendrean geometry on $C$

Firstly, we revisit the splitting (3), now using torsion-free connections instead of choosing coördinates. As is well-known \[11\], a connection on $T^*M$ may be viewed as a splitting (3) of $T(T^*M)$, into horizontal and vertical subbundles. If we change connections, say

\[ \tilde{\nabla}_a X^c = \nabla_a X^c + \Gamma^{c}_{ab} X^b, \]
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(using Penrose’s abstract index notation [15]) then the splitting changes according to

\[
\begin{bmatrix}
X^b \\
\omega_b
\end{bmatrix}
= \begin{bmatrix}
X^b \\
\omega_b + X^a \Gamma_{ab}^c p_c
\end{bmatrix}.
\]

Now, if we insist on using torsion-free connections, as we may, then the skew form (1) is manifestly invariant because \(\Gamma_{ab}^c\) is symmetric.

In order to navigate in \(M\), we now suppose that this manifold is endowed with a projective differential geometric structure. A detailed discussion, specifically in 3 dimensions, may be found in [4]. We shall therefore be brief in recalling the salient features. Although a projective structure may be viewed as a type of path geometry (eminently suitable for flying in \(M\)) an operational viewpoint on projective structures is as an equivalence class of torsion-free connections, where the notion of equivalence is that

\[
\nabla_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a
\]

for an arbitrary 1-form \(\Upsilon_a\). In (7) it means that

\[
\Gamma_{ab}^c = \Upsilon_a \delta_b^c + \Upsilon_b \delta_a^c
\]

where \(\delta_b^c\) is canonical pairing between vectors and covectors. Hence, with (9) in place, the formula (8) for the change in splitting becomes

\[
\begin{bmatrix}
X^b \\
\omega_b
\end{bmatrix}
= \begin{bmatrix}
X^b \\
\omega_b + X^a \Upsilon_a p_b + X^c p_c \Upsilon_b
\end{bmatrix}.
\]

But with a chosen connection and hence a chosen splitting in place,

\[P\] is the subspace of \(\pi^*TM\) given by \(\{X^b \mid X^b p_b = 0\}\)
and, in any case \(V\) is the quotient of \(\pi^*T^*M\) given by

\[\{\omega_b\}/\sim \text{ where } \omega_a \sim \omega_b + tp_a \forall t \in \mathbb{R}.
\]

From (10) it follows at once that the splitting \(H = P \oplus V\) is projectively invariant. In summary, we have proved the following.

**Theorem 2.** A projective structure on \(M\) gives rise to extra structure on its configuration space \(C\). Specifically, the contact distribution \(H\) canonically splits as

\[
H = P \oplus V.
\]

Both \(P\) and \(V\) are null with respect to the Levi form \(\mathcal{L}: \wedge^2 H \to L\), which otherwise restricts to a non-degenerate pairing \(P \otimes V \to L\).
In general, if \( C \) is a manifold with contact distribution \( H \subset TC \), then a splitting \( H = P \oplus V \) into null subspaces for the Levi form is a type of parabolic geometry \([2]\) called *almost contact Legendrean*. Projective differential geometry is another type of parabolic geometry and the construction of this section may be viewed in Dynkin diagram notation as \( \rightsquigarrow \rightarrow \rightarrow \rightarrow \rightarrow \). Furthermore, \( TC = L + \frac{P}{V} = \oplus \frac{1}{0} \). 

and the harmonic curvature splits into 3 pieces

\[
H^2(\mathfrak{g}_{-1}, \mathfrak{sl}(4, \mathbb{R})) = \oplus\left\{ \begin{array}{c}
\begin{array}{c}
\frac{3}{1} \oplus \frac{3}{2} \\
\frac{4}{1} \oplus \frac{1}{2}
\end{array}
\end{array}\right\}
\]

Meanwhile, as detailed in \([4]\), the harmonic curvature of 3-dimensional projective geometry (usually known as the projective *Weyl curvature*) lies in \( \frac{4}{1} \). One can easily check that, in case \( C \) is constructed from such a 3-dimensional projective \( M \), as above, then the harmonic curvature of \( C \) lies only in \( \frac{4}{1} \oplus \frac{1}{2} \) and that it is the pull-back of the Weyl curvature. In particular, the contact Legendrean structure on \( C \) is flat if and only if the projective structure on \( M \) is flat.

Aerobatics may now be restricted by requiring, not only that the trajectory in \( C \) be everywhere tangent to \( H \), but also that the tangent vector be null with respect to the neutral signature conformal metric on \( H \) given by the non-degenerate pairing \( P \otimes V \to L \). Some special manoeuvres are permitted. Firstly, there is the option of remaining stationary in \( M \) whilst changing the saucer orientation arbitrarily. In other words, since the fibres of \( \pi : C \to M \) are null, it is permitted to move along them as one wishes. The second option is to move along a projective geodesic in \( M \), with any initial orientation, lifting this curve to \( C \) in accordance with the projectively invariant splitting \([11]\). It is a common experience in usual aerobatics, that one carries along one’s own frame of reference! Indeed, any curve starting at \( x \in M \) with an initial choice of orientation in \( \pi^{-1}(x) \) can be uniquely lifted into \( C \) in accordance with \([11]\). This may be viewed as the difference between ‘gliding’ and ‘powered flight.’ In any case, null manoeuvring now has the geometric interpretation that, when moving in \( M \), ‘rolls’ are restricted to be about one’s axis of flight (the ‘slow roll’ in usual
aerobatics). Using only the two special manoeuvres of stationary rolling and gliding, as above, it is already clear that a pilot may park her craft in an arbitrary location and orientation.

3. A CR structure on $C$

In the previous section we saw that a projective structure on $M$ is exactly what is needed to define what might be called ‘attack mode,’ in which a saucer is permitted only to make rolls about its axis of flight. In coming in to land, however, this type of manoeuvre is unsuitable, even dangerous! More suitable for landing is the motion often observed in falling leaves, whereby rolls are constrained to be about axes orthogonal to the direction of flight. To make sense to this ‘landing mode,’ one clearly needs a notion of orthogonality in the disc of the saucer. It is natural to suppose that this notion is induced from $M$ itself. In other words, we shall suppose that $M$ is endowed with a conformal metric.

If two Riemannian metrics $g_{ab}$ and $\hat{g}_{ab}$ are conformally related, it is convenient to write $\hat{g}_{ab} = \Omega^2 g_{ab}$ for a smooth positive function $\Omega$. We shall suppose that $M$ is oriented and write $\epsilon_{abc}$ for the volume form associated to the metric $g_{ab}$. A conformal change of metric $\hat{g}_{ab} = \Omega^2 g_{ab}$ induces a change of volume form $\hat{\epsilon}_{abc} = \Omega^3 \epsilon_{abc}$ (we say that $\epsilon_{abc}$ has conformal weight 3) and the corresponding Levi-Civita connections are related according to

$$\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Gamma_a \phi_b - \Gamma_b \phi_a + \Gamma^c \phi_c g_{ab},$$

where $\Gamma_a = \nabla_a \log \Omega$. We may choose a metric in the conformal class and use its Levi-Civita connection to write

$$T(T^*M) = \bigoplus_{\nu^* T^* M} \left[ \begin{array}{c} X^a \\ \omega_a \end{array} \right],$$

According to (8) and (13), if $\hat{g}_{ab} = \Omega^2 g_{ab}$, then

$$\left[ \begin{array}{c} X^b \\ \omega_b \end{array} \right] = \left[ \begin{array}{c} X^b \\ \omega_b + X^a \Gamma_a \phi_b + X^c \phi_c \Gamma \phi_c - X_b \Gamma \phi_c \end{array} \right].$$

On $H$, since $\omega_b$ is only defined modulo $p_b$, and since $X^c \phi_c = 0$, we can drop two of these terms to obtain

$$\left[ \begin{array}{c} X^b \\ \omega_b \end{array} \right] = \left[ \begin{array}{c} X^b \\ \omega_b - X_b \Gamma \phi_c \phi_c \end{array} \right] \text{ on } H = \bigoplus \text{ in the presence of } g_{ab}.$$

Instead of $\omega_b$ up to multiples of $p_b$, we may use the conformal metric to suppose that $\omega^a p_a = 0$ (i.e., normalise by $\omega_b \mapsto \omega_b - (\omega^c \phi_c / p^a \phi_a) p_b$).
The change in splitting respects this normalisation (since $X^b p_b = 0$).
So now we have, for a chosen metric in the conformal class,
$$H = \left\{ \begin{bmatrix} X^b \\ \omega^b \end{bmatrix} \text{ s.t. } X^b p_b = 0 \text{ and } \omega^b p_b = 0 \right\},$$
where $\omega^b$ has conformal weight $-2$ and, if $\tilde{g}_{ab} = \Omega^2 g_{ab}$, then (14) applies.
We define $J : H \to H$ by
$$\left[ \begin{bmatrix} X^b \\ \omega^b \end{bmatrix} \right] \mapsto -\frac{1}{\sqrt{p^a p_d}} \left[ e^{abc} X_a p_c e^{ade} \omega_a p_e \right].$$
It respects the change (14) and is hence well-defined. Since
$$\epsilon^{abc} \epsilon_{ade} = \delta_d^b \delta_e^c - \delta_e^b \delta_d^c$$
it follows that $J^2 = -\text{Id}$ and we have defined an almost CR structure.
In fact, we may check that this almost CR structure is integrable as follows. Since $P$ and $V$ are both two-dimensional, we need only check that, for the Nijenhuis tensor $N(\omega, \omega),$
$$N \left( \begin{bmatrix} X^a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \omega_b \end{bmatrix} \right) = 0.$$
This requirement expands to the vanishing of
$$-\omega_b \partial^b X^a - J(\omega_b \partial^b (JX^a)) - J(\partial^b \omega_b \partial^b \omega^a) + (J\omega_b) \partial^b (JX^a)$$
$$X^a \nabla_a \omega_b + J(X^a \nabla_a (J\omega_b)) + J((JX^a) \nabla_a \omega_b) - (JX^a) \nabla_a (J\omega_b)$$
where $\partial^a = \partial/\partial p_a$ and may be verified as follows. Firstly,
$$\partial^b (JX^a) = \partial^b \left( \frac{1}{\sqrt{p^a p_c}} e^{cad} X_c p_d \right),$$
which may be expanded by the Leibniz rule as
$$\frac{1}{\sqrt{p^a p_c}} e^{cad} (\partial^b X_c) p_d - \frac{p^b}{(p^a p_c)^{3/2}} e^{cad} X_c p_d + \frac{1}{\sqrt{p^a p_c}} e^{abc} \omega_b X_c$$
Therefore
$$\omega_b \partial^b (JX^a) = J(\omega_b \partial^b X^a) + \frac{1}{\sqrt{p^a p_c}} e^{abc} \omega_b X_c$$
so
$$J(\omega_b \partial^b (JX^a)) = -\omega_b \partial^b X^a + \frac{1}{p^b p_c} (g^{ab} g^{cd} - g^{ac} g^{bd}) \omega_b X_c p_d,$$
which, bearing in mind that $X^d p_d = 0$ and $\omega^d p_d = 0,$ reduces to
$$J(\omega_b \partial^b (JX^a)) = -\omega_b \partial^b \omega^a.$$
and all terms in the first line of \((16)\) cancel. For the second line, it is evident that
\[
X^a \nabla_a (J \omega_b) = J(X^a \nabla_a \omega_b) \quad \text{and} \quad (JX^a) \nabla_a (J \omega_b) = J((JX^a) \nabla_a \omega_b)
\]
and, again, all terms cancel.

The ‘landing mode,’ informally described at the beginning of this section is now formally defined by the restriction \(X^b J \omega_b = 0\), noting from \((14)\) and \((15)\) that this constraint is invariantly defined. Such a manoeuvre is very much at odds with the ‘attack mode’ of the previous section. A conformal structure on \(M\) does not allow slow rolls to be defined: one sees from \((14)\) that the restriction \(X^b \omega_b = 0\) is always ill-defined unless one restricts to constant rescalings of the metric. In both modes, however, flying is restricted by requiring that the allowed curves in \(C\) are not only tangent to \(H\), but also that they be null for an appropriately defined neutral signature metric on \(H\) (with values in the line bundle \(L\)). These metrics are

\[
(17) \quad \| \begin{bmatrix} X \\ \omega \end{bmatrix} \|^2 = X \perp \omega \quad \text{(attacking)} \quad \| \begin{bmatrix} X \\ \omega \end{bmatrix} \|^2 = X \perp J \omega \quad \text{(landing)}.
\]

As tensors on \(H\), we have the usual compatibility \(J \alpha_\beta = \Omega_\alpha^\gamma g^{\beta \gamma}\) in which any two of the Levi form \(\Omega_\alpha^\beta\), the inverse metric \(g^{\alpha \beta}\), and the endomorphism \(J \alpha_\beta\) determine the third. In landing mode, we have \(J^2 = -\text{Id}\). In attacking mode, the endomorphism \(J : H \to H\) instead satisfies \(J^2 = \text{Id}\), being the identity on \(V\) and minus the identity on \(P\) (for any given metric in the conformal class on \(M\)). For either of these geometries, stationary rolling is allowed since \(V\) is null in either case. A metric on \(M\) induces a splitting

\[
0 \to V \longrightarrow H \xrightarrow{\pi} P \to 0
\]

and, as in Section \([2]\) the ‘gliding’ manoeuvre is now available. In other words, we may use a horizontal lift to arrive at \(\omega = 0\) in either of the neutral signature metrics \((17)\). Therefore, parking in an arbitrary location and orientation is easily achievable in the CR case. The only difference is that the splitting, and hence the particular manoeuvring to be used, depends on choosing a metric in the conformal class.

The Dynkin diagram notation for this construction is

\[
\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
\Gamma & \longrightarrow & \Gamma
\end{array}
\]

The harmonic curvature in three-dimensional conformal geometry is the Cotton tensor in \(\mathbb{T}^{\mathbb{5} \times 4}\), which pulls back to \(\mathbb{T}^{\mathbb{3} \times 4}\). For more details concerning this construction, and especially a characterisation of the 5-dimensional CR manifolds that arise in this way, see \([9]\) (and
also [10] for some very interesting consequences in the real-analytic setting). The flat model of this construction is when \( M = S^3 \hookrightarrow S^4 \) is the standard inclusion of round spheres and
\[
C \hookrightarrow \mathbb{CP}_3 \\
\tau \downarrow \\
S^3 \hookrightarrow S^4
\]
where \( \tau : \mathbb{CP}_3 \rightarrow S^4 \) is the twistor fibration and \( C \hookrightarrow \mathbb{CP}_3 \) is the Levi indefinite hyperquadric.

Finally, we remark that if we would like to have at our disposal both the ‘attack mode’ of Section 2 and the ‘landing mode’ of the current section, then we require compatible projective and conformal structures on \( M \). If \( \nabla_a \) represents a projective structure and \( g_{ab} \) is a metric, then one can check that the 1-form
\[
\omega_a \equiv 4g^{bc}\nabla_b g_{ac} - \epsilon^{bcd}\nabla_a \epsilon_{bcd},
\]
where \( \epsilon_{abc} \) is the volume form of \( g_{ab} \), is projectively invariant. To find a metric in the conformal class of \( g_{ab} \) whose Levi-Civita connection is in the given projective class, it is firstly necessary that \( \omega_a \) be exact. In this case, a further necessary and sufficient condition is that
\[
40\nabla_a g_{bc} + 2\omega_a g_{bc} - 3\omega_b g_{ac} - 3\omega_c g_{ab} = 0,
\]
where \( \nabla_a \) has been chosen from the projective class so that \( \nabla_a \epsilon_{bcd} = 0 \) (cf. [12]). In this case, we are obliged to have a Riemannian metric on \( M \) defined up to homothety, i.e. only constant rescalings are allowed. More severely, as discussed in [14], the generic projective structure does not arise from a metric at all.

In summary, for a Riemannian metric on \( M \) there are two possible flying modes, namely the ‘attacking mode’ of §2, which sees only the induced projective structure on \( M \), and the ‘landing mode’ of §3, which sees only the induced conformal structure on \( M \). In this case, on the configuration space \( C \rightarrow M \), there are two different neutral signature conformal metrics on the contact distribution \( H \subset TC \). The vertical bundle \( V \hookrightarrow H \) is null for either of these conformal metrics on \( H \), as is the horizontal bundle \( P \hookrightarrow H \) (defined only by the projective structure on \( M \)). So, when \( M \) is Riemannian, a flying saucer may be fitted with a switch that enables its pilot to toggle between attacking and landing.

4. A \( G_2 \) contact structure on \( C \)

Recall that the contact distribution \( H \subset TC \) on a five-dimensional contact manifold \( C \) is endowed with its Levi form \( \Lambda^2 H \rightarrow L \equiv TC/H \), a non-degenerate symplectic form defined up to scale, thereby reducing
the structure group of frames for $H$ from $\text{GL}(4, \mathbb{R})$ to the conformal symplectic group $\text{CSp}(4, \mathbb{R})$. A $G_2$ contact structure on $C$ is a further reduction of structure group to $\text{GL}(2, \mathbb{R}) \subset \text{CSp}(4, \mathbb{R})$, realised by the representation of $\text{GL}(2, \mathbb{R})$ on the third symmetric power $\bigodot^3 \mathbb{R}^2$ of the standard representation of $\text{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$. As $\bigodot^3 \mathbb{R}^2$ is 4-dimensional, we have $\text{GL}(2, \mathbb{R}) \hookrightarrow \text{GL}(4, \mathbb{R})$ and, since

$$\wedge^2 \bigodot^3 \mathbb{R}^2 = (\bigodot^4 \mathbb{R}^2 \otimes \wedge^2 \mathbb{R}^2) \oplus (\wedge^2 \mathbb{R}^2 \otimes \bigodot^3 \mathbb{R}^2 \otimes \wedge^2 \mathbb{R}^2),$$

the second summand of which is 1-dimensional, this homomorphism maps to $\text{CSp}(4, \mathbb{R}) \subset \text{GL}(4, \mathbb{R})$. (See [1] for a discussion of similar reductions for the frame bundle of a four-dimensional manifold.)

Equivalently, a $G_2$ structure on a five-dimensional contact manifold $C$ is a rank two vector bundle $S \rightarrow C$ together with an identification of vector bundles, compatible with the Levi form,

$$\bigodot^3 S = H,$$

where $H$ is the contact distribution and $\bigodot^3 S$ is the third symmetric power of $S$. It is named for a ‘flat model’ in parabolic geometry, namely the homogeneous space $\mathbb{K}^\times$ for the split real form of the Lie group $G_2$. The tangent bundle of this homogeneous space is an extension

$$T(\mathbb{K}^\times) = 0 \mathbb{K}^\times + \mathbb{K}^\times^{-1},$$

and $H$ has the required form for $S = \mathbb{K}^\times^{-1/3}$. For this flat model, the local infinitesimal symmetries are isomorphic to the split real form of the exceptional Lie algebra $G_2$.

Yet a third interpretation of a $G_2$ contact structure is as a field of twisted cubic cones inside $H$ (akin to the interpretation of a Lorentzian conformal structure as a field of quadratic cones, i.e. the null vectors). Specifically, writing $H$ in the form (19) defines a cone

$$s \odot s \odot s \in \bigodot^3 S = H$$

of simple vectors in each fibre of $H$. It is easy to check that this cone determines the structure. Geometrically, it can be regarded as a smoothly varying family of twisted cubics in the bundle $\mathbb{P}(H)$ of three-dimensional projective spaces (again, see [1] for the corresponding geometry in four dimensions).

In any case, our aim in this section is to equip the configuration space of a flying saucer with a $G_2$ contact structure.

In Sections 2 and 3, the structures on the configuration space $C$ were nicely determined by suitable differential geometric structures on $M$, specifically projective in Section 2 and conformal in Section 3. Now we shall firstly suppose that $M$ has a projective structure and also, for the
moment, a fixed volume form $\epsilon_{bcd}$. In this case, we can specify a unique connection from the projective class by insisting that $\nabla_a \epsilon_{bcd} = 0$. In any case, as in Section 2, we have a well-defined splitting $H = P \oplus V$.

To complete the geometric structure on $M$, we suppose that we are given two linearly independent 1-forms, say $\phi$ and $\psi$. These forms are sufficient to define a $G_2$ contact structure on (an open subset of) $C$ as follows.

For $P \hookrightarrow C$, define a frame $e^1, e^2 \in \Gamma(P)$ by requiring that
\[
e^1 \int \pi^* \phi = 1 \quad e^1 \int \pi^* \psi = 0 \\
e^2 \int \pi^* \phi = 0 \quad e^2 \int \pi^* \psi = 1.
\]
This is legitimate wherever $\pi^* \phi$ and $\pi^* \psi$ are linearly independent when restricted to $H \subset TC$ (and defines an open subset of $C$). Next, recall the canonical short exact sequence
\[
0 \to P \to \pi^* TM \to L \to 0
\]
on $C$ and hence a canonical identification
\[
L^* = \pi^* \wedge^3_M \otimes \wedge^2 P.
\]
Thus, we may use the nowhere vanishing sections $\pi^* \epsilon_{bcd} \in \Gamma(\pi^* \wedge^3_M)$ and $\pi^* \phi \wedge \pi^* \psi \in \Gamma(\wedge^2 P)$ to trivialise $L^*$. With this trivialisation, the Levi form $P \otimes V \to L$ identifies $V = P^*$ and we take $e_1, e_2 \in \Gamma(V)$ to be the dual frame to $e^1, e^2 \in \Gamma(P)$. Finally, we define a twisted cubic
\[
\mathbb{RP}_1 \hookrightarrow \mathbb{P}(P \oplus V) = \mathbb{P}(H)
\]
by
\[
(21) \quad [s, t] \mapsto [s^3 e^1 + s^2 t e^2 + t^3 e_1 - 3st^2 e_2] \in \mathbb{P}(H).
\]
The seemingly peculiar choice of constants here is to ensure that this cubic induce the existing Levi form on $H$, specifically that
\[
(s \hat{t} - t \hat{s})^3 = (s^3 e^1 + s^2 t e^2) \mathcal{J}(\hat{t}^3 e_1 - 3\hat{s}\hat{t}^2 e_2) - (\hat{s}^3 e^1 + \hat{s}^2 \hat{t} e^2) \mathcal{J}(t^3 e_1 - 3st^2 e_2)
\]
in accordance with (4).

There is some ‘gauge freedom’ associated with this arrangement, i.e. changes in the data $(\epsilon, \phi, \psi)$ on $M$ that do not affect the associated $G_2$ projective structure on $C$. Specifically, if we replace $(\epsilon_{bcd}, \phi_b, \psi_b)$ by
\[
(22) \quad \hat{\epsilon}_{bcd} = \Omega^4 \epsilon_{bcd} \quad \hat{\phi}_b = h^3 \phi_b \quad \hat{\psi}_b = \Omega \psi_b,
\]
for arbitrary smooth nowhere vanishing functions $\Omega$ and $h$ on $M$, then
\[
\hat{e}^1 = h^{-3} e^1 \quad \hat{e}^2 = \Omega^{-1} e^2 \quad \hat{e}_1 = h^6 \Omega^{-3} e_1 \quad \hat{e}_2 = h^3 \Omega^{-2} e_2,
\]
giving rise to the twisted cubic
\[
[s, t] \mapsto [h^{-3} s^3 e^1 + \Omega^{-1} s^2 \hat{t} e^2 + h^6 \Omega^{-3} e_1 - 3h^3 \Omega^{-2} \hat{s}\hat{t} e_2],
\]
which is simply a reparameterisation, namely \([s, \hat{t}] = [hs, h^{-2}\Omega t]\), of the original cubic (21). Using the language of projective weights, e.g. as in [4], we may regard \(\epsilon_{bcd}\) as the tautologically defined section of \(\wedge^3(4)\) and the true data as a 1-form \(\varphi \in \Gamma(\wedge^1)\), defined only up to scale, and \(\psi \in \Gamma(\wedge^1(1))\), a 1-form of projective weight 1.

The construction above is almost captured geometrically as follows. Firstly, the projective structure on \(M\) splits the contact distribution as \(H = P \oplus V\), which, in the bundle \(\mathbb{P}(H) \to M\) of 3-dimensional projective spaces, may be viewed as a family of skew lines

\[
\mathbb{P}(V_x) \quad \mathbb{P}(H_x)
\]

and the family of twisted cubics (21) looks like this:

intersecting \(\mathbb{P}(P)\) and \(\mathbb{P}(V)\) tangentially at \([e^1]\) and \([e_1]\), respectively, as can be seen from (21). Requiring that the twisted cubic be compatible with the Levi form, as we do, is insufficient to fix it. The gauge freedom explained above says that there is just one more scalar-valued piece of information required at each point \(x \in M\) and this may be interpreted as our requiring \(\psi \in \Gamma(\wedge^1(1))\).

As is detailed in [5], there are two examples of this construction for which \(C\) turns out to be (locally) the flat model \(\mathbb{R}^3\) for the split real form of the Lie group \(G_2\). For both, we start with the standard flat projective structure on \(\mathbb{R}^3\) with usual coordinates \((x, y, z)\) and take

\[
\epsilon_{abc} = dx \wedge dy \wedge dz,
\]

the standard volume form.
**First example.** We take \( \phi = dx \) and \( \psi = dy \).

With slightly different coördinates, this example is due to Engel \([6]\). In \([5]\) we write down all its 14-dimensional symmetries.

**Second example.** We take \( \phi = x \, dy - y \, dx \) and \( \psi = y^{-1} \, dy \).

The geometry on \( M \) appears to be different from the first example but we shall see in Theorem \([3]\) below that the induced \( G_2 \) contact structure on \( C \) is \( G_2 \)-flat and therefore isomorphic.

5. The geometry on \( M \)

In this section we speculate on the geometry on \( M \) that is needed to generate the \( G_2 \) contact structure on \( C \rightarrow M \), as in \([4]\). We have already seen in \([22]\) that \( \psi \) should have projective weight 1 whilst \( \phi \) may be arbitrarily rescaled. In fact, we shall suppose that \( \phi \) has projective weight 2. There are several reasons for this, the most naive of which is as follows. For a 1-form of projective weight \( w \), the formula \([9]\) for projective change becomes

\[
(23) \quad \tilde{\nabla}_a \phi_b = \nabla_a \phi_b + (w - 1) \Upsilon_a \phi_b - \Upsilon_b \phi_a
\]

and when \( w = 2 \), we see that \( \nabla_{(a} \phi_{b)} \) is invariant, where the round brackets mean to take the symmetric part. We may, therefore, assume that \( \nabla_{(a} \phi_{b)} = 0 \) as a sort of compatibility between \( \phi_b \) and the projective structure defined by \( \nabla_a \). We shall come back to this shortly but an immediate and congenial consequence of imposing \( \nabla_{(a} \phi_{b)} = 0 \) is that \( \phi_b \) is then determined up to an overall constant:

\[
\nabla_{(a} (f \phi_{b)}) = f \nabla_{(a} \phi_{b)} + (\nabla_{(a} f) \phi_{b)} = 0 \Rightarrow \nabla_a f = 0.
\]

In summary, the data we are supposing on \( M \) is as follows.

- A projective structure, determined by \( \nabla_a \),
- \( \phi_b \in \Gamma(M, \Lambda^1(2)) \), such that \( \nabla_{(a} \phi_{b)} = 0 \),
- \( \psi_b \in \Gamma(M, \Lambda^1(1)) \),

and we note that the two examples from \([4]\) above satisfy \( \nabla_{(a} \phi_{b)} = 0 \), as requested. There are various projective invariants that we may generate from this data. The **concircularity** operator

\[
\theta^b \mapsto (\nabla_a \theta^b)_\circ = \nabla_a \theta^b - \frac{1}{3} \delta^b_a \nabla_c \theta^c
\]

(where \( \circ \) means to take the trace-free part) is projectively invariant if \( \theta^a \) has projective weight \(-1\). Meanwhile, the tautological form \( \epsilon^{bcd} \) has...
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projective weight $-4$ so $\epsilon^{bcd}\phi_c\psi_d$ has projective weight $-1$ and hence $(\nabla_a(\epsilon^{bcd}\phi_c\psi_d))_a$ is invariant. It may be rewritten as

$$\nabla_a(\phi_b\psi_c) - \nabla_{[a}(\phi_{b]}\psi_c).$$

It is easily verified that this expression vanishes for the two examples given at the end of the previous section. When $w = 1$ the projective change (23) reads

$$\tilde{\nabla}_b\psi_c = \nabla_b\psi_c - \Upsilon_{c\psi_b} \quad \text{whence} \quad \psi_{[a}\nabla_{b]}\psi_c \text{ is invariant}$$

and also vanishes for our two examples.

**Theorem 3.** For the data (24) to define a flat $G_2$ contact structure on $C$, it is necessary and sufficient that

(25) $\nabla_a$ be projectively flat, $\nabla_a(\phi_b\psi_c) = \nabla_{[a}(\phi_{b]}\psi_c), \psi_{[a}\nabla_{b]}\psi_c = 0.$

The proof will be given shortly but, firstly, some discussion. This theorem leads us to the following.

**Third example.** We take the standard flat projective structure on $\mathbb{R}^3$ with usual coordinates $(x, y, z)$ and

$$\phi = xd\psi - y dx \quad \text{and} \quad \psi = zy^{-1} dy - dz.$$  

It is easily verified that all conditions (25) hold and the corresponding $G_2$ contact structure on the configuration space $C$ is, therefore, flat.

**Remarks on these three examples.** Although these examples seem naïvely to be distinct (and from the Riemannian viewpoint, this is true), they are, in fact, projectively equivalent. Specifically, if we set

$$\hat{x} = -x/y, \quad \hat{y} = -1/y, \quad \hat{z} = -z/y$$

then

$$d\hat{x} = y^{-2}(xdy - y dx) \quad \text{and} \quad d\hat{y} = y^{-2}dy$$

and, taking into account that $\phi$ should have projective weight 2 and $\psi$ should have projective weight 1, it follows that the first example, ($\phi = d\hat{x}, \psi = d\hat{y}$) is converted into ($\phi = xd\psi - y dx, \psi = y^{-1} dy$), which is the second example. Similarly, the projective change

$$\hat{x} = -x/y, \quad \hat{y} = -z/y, \quad \hat{z} = 1/y$$

converts ($\phi = d\hat{x}, \psi = d\hat{y}$) into ($\phi = xd\psi - y dx, \psi = zy^{-1} dy - dz$), which is the third example.

To some extent, this projective equivalence of our three examples justifies our request that the 1-form $\phi$ should have projective weight 2 but there is another good reason for this, as follows. Recall, in (22), that $\phi$ may be arbitrarily rescaled without effecting the resulting $G_2$...
contact geometry on \( C \). Its kernel \( D \subset TM \) is therefore canonically defined and we may write

\[
0 \rightarrow D \rightarrow TM \xrightarrow{\phi} \xi \rightarrow 0,
\]
a short exact sequence tautologically defining a line bundle \( \xi \) on \( M \). Let us temporarily suppose that \( D \) is a contact distribution (even though this is false in our three examples). In three dimensions, the Levi form then provides a canonical isomorphism \( \wedge^2 D = \xi \) and, feeding this back into (26), we may identify \( \wedge^3 TM = \xi^2 \). That \( M \) is oriented allows us to identify \( \xi \) as the bundle of densities of projective weight 2 (whether or not \( M \) has a projective structure). In summary, when \( D \) is a contact distribution we are forced to regard \( \phi \in \Gamma(M, \wedge^0(2)) \) and, even when \( D \) is integrable, we may choose to do this. As already observed, in the presence of a projective structure \( [\nabla_a] \), we may also insist that \( \nabla_a(\phi b) = 0 \). In case \( D \) is a contact distribution, this is exactly the compatibility required between \( D \) and a projective structure in order that the pair \( ([\nabla_a], D) \) define a contact projective structure in the sense of Harrison [8] and/or Fox [7]. It is a type of parabolic geometry, the flat model of which is \( \mathbb{RP}_3 \) under the action of \( \text{Sp}(4, \mathbb{R}) \).

It would be nice to construct a flat \( G_2 \) contact structure starting with this flat contact projective structure. Unfortunately, this seems to be impossible. Specifically, in standard coördinates \((x, y, z)\) on \( \mathbb{R}^3 \), we may arrange that

\[
\phi = x\, dy - y\, dx + dz.
\]

On the other hand, the general solution of the concircularity equation \( (\nabla_a \theta b)_c = 0 \) is

\[
\theta = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} + e \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right),
\]

in which case

\[
\theta b \phi b = c + bx - ay + ez.
\]

Therefore, we cannot find \( \psi_d \neq 0 \) such that \( (\nabla_de^{bcd}\phi_c\psi_d)_a = 0 \), which is the second condition from (25). As we shall see in the proof of Theorem 4 below, the quantity \( e^{bcd}\phi_c\psi_d \) naturally arises in constructing the \( G_2 \) contact geometry on \( C \) and, even if Theorem 3 is too restrictive, one would expect the projective invariant \( (\nabla_a(e^{bcd}\phi_c\psi_d))_c \) to be part of the harmonic curvature of the \( G_2 \) contact geometry on \( C \) (which, in general, is a binary septic: in fact, a section of \( \langle 7 - 4 \rangle \)).

Taking \( \phi \in \Gamma(M, \wedge^1(2)) \) and \( \psi \in \Gamma(M, \wedge^1(1)) \), allows us to write the twisted cubic cone (21) inside the contact distribution \( H \subset TC \) more invariantly than was done in \( \S 4 \). We obtain the following.
Theorem 4. Suppose $M$ is a three-dimensional smooth manifold and write $C \xrightarrow{\pi} M$ for the configuration space of flying saucers in $M$. Given data in the form (24) on $M$, we may canonically construct a $G_2$ contact structure on a suitable open subset of $C$ so that the contact distribution $H \subset TM$ is written as

$$H = \bigodot^3 S,$$

where $S = \pi^* \wedge^0 (2/3) \oplus \pi^* \wedge^0 (-1/3)$.

Proof. Recall from (12) that

$$P = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Also, from (12) we have

$$0 \to \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \to \pi^* TM \to \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \to 0,$$

a canonical short exact sequence on $C$ and, in particular, canonical surjections

$$\pi^* \wedge^1_M \to \left( \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \right)^* = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \pi^* \wedge^1_M (w) \to \begin{pmatrix} w-2 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

for any projective weight $w$. Writing $\pi^!$ for the pullback $\pi^*$ followed by this surjection, firstly gives

$$\Theta \equiv \pi^! \phi \land \pi^! \psi \in \Gamma(C, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}) = \Gamma(C, \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix})$$

and then, on the open set where $\Theta$ is non-vanishing,

$$E_1 \equiv \Theta \pi^! \phi \in \Gamma(C, \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}) \quad E_2 \equiv \Theta \pi^! \psi \in \Gamma(C, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix})$$

and

$$E^1 \equiv -\Theta^{-1} \pi^! \psi \in \Gamma(C, \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}) \quad E^2 \equiv \Theta^{-1} \pi^! \phi \in \Gamma(C, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}).$$

Bearing in mind that

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} = V \otimes \wedge^0_C (1) \quad \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = V$$

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = P \otimes \wedge^0_C (-2) \quad \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix} = P \otimes \wedge^0_C (-1),$$

where $\wedge^0_C (w) = \begin{pmatrix} w & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} = \pi^* \wedge^0_M (w)$, we conclude that

$$S \equiv \pi^* \wedge^0 (2/3) \oplus \pi^* \wedge^0 (-1/3) \ni (\sigma, \tau)$$

$$\downarrow$$

$$\sigma^3 E^1 + \sigma^2 \tau E^2 + \tau^3 E_1 - 3 \sigma \tau^2 E_2 \in P \oplus V = H$$

is well defined. It is an invariant formulation of (21) whose range defines a twisted cubic cone in $\mathbb{P}(H)$ compatible with the Levi form and hence a $G_2$ contact structure on $C$. \hfill \Box
We remark that, although the projective weights $2/3$ and $-1/3$ in the identification of $S$ may look contrived, they yield
\[ \wedge^2 S \otimes \wedge^2 S \otimes \wedge^2 S = \pi^* \wedge^0(1) \]
and therefore, in accordance with the vector bundle version of (18), that $\pi^* \wedge^0(w) = w$, as one might expect.

**Proof of Theorem 3.** Since $\nabla_a$ is projectively flat we may suppose, without loss of generality, that our manifold is $\mathbb{R}^3 \hookrightarrow \mathbb{RP}_3$ with $\nabla_a$ the standard flat connection. The operator $\phi_b \mapsto \nabla_a(\phi_b)$ is the first BGG operator
\[ \begin{array}{ccc}
0 & 1 & 0 \\
\nabla & -2 & 2 \\
- & 0 & 0 \\
\end{array} \]
don $\mathbb{RP}_3$ with kernel the irreducible representation $\mathbb{RP}_3 = \wedge^2 \mathbb{R}^4$ of $\text{SL}(4, \mathbb{R})$ (acting by projective transformations on $\mathbb{RP}_3$). There are two non-zero orbits for the action of $\text{SL}(4, \mathbb{R})$ on $\wedge^4 \mathbb{R}^4$ depending on rank and the non-degenerate case is represented by $\phi = x \, dy - y \, dx + dz$, which we have already seen to be incompatible with the second condition of (25). It follows that, without loss of generality, we may suppose $\phi = dx$.

The operator $\theta^b \mapsto (\nabla_a \theta^b)_a$ is also a first BGG operator
\[ \begin{array}{ccc}
0 & 1 & 0 \\
\nabla & -2 & 1 \\
- & 1 & 1 \\
\end{array} \]
whose solution space is $\mathbb{R}^4 = \mathbb{RP}_3$ as an $\text{SL}(4, \mathbb{R})$-module. The degenerate 2-form corresponding to $\phi$ specifies a 2-plane in $\mathbb{R}^4$ and so there are just two cases for $\theta^b$ depending on whether the corresponding vector in $\mathbb{R}^4$ lies in this plane or not. This is exactly whether $\theta^a \phi_a$ vanishes or not and, with $\theta^a$ being of the form $\epsilon^{abc} \phi_b \psi_c$, it must vanish. Therefore, without loss of generality $\theta^a = \partial/\partial z$. We have reached the following normal forms for $\phi$ and $\psi$:
\[ \phi = dx \quad \text{and} \quad \psi = \xi(x, y, z) \, dx + dy \]
where $\xi(x, y, z)$ is an arbitrary smooth function. It remains to consider the remaining condition from (25), namely that $\psi^a \nabla_b \psi_c = 0$. It means that $\xi = \xi(x, y)$, a function of $(x, y)$ alone, and that $\xi_x = \xi_y = \xi_z$. In the computation that follows, we shall find $\xi \xi_y - \xi_x$ as the only non-trivial component of the harmonic curvature for the associated $G_2$ contact structure and our proof will be complete.

The harmonic curvature is computed in Theorem 5 below. To use it we must specify the $G_2$ contact structure on $C$ in terms of an adapted co-frame (30). Starting with
\[ \omega^1 = \phi = dx \quad \text{and} \quad \omega^2 = \psi = \xi(x, y) \, dx + dy \]
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on \( M = \mathbb{R}^3 \) we may take

\[
\omega^0 = dz - a \, dx - b \, dy \quad \omega^3 = -\frac{1}{3} \, db \quad \omega^4 = da - \xi(x, y) \, db
\]

in local coordinates \((x, y, z, a, b)\) on \( C \), as in \( \text{[11]} \). These satisfy \( \text{[30]} \) with \( \chi \equiv 1 \). We compute

\[
\begin{align*}
d\omega^1 &= 0 \\
d\omega^2 &= d\xi \wedge dx = -\xi_y \omega^1 \wedge \omega^2 \\
d\omega^3 &= 0 \\
d\omega^4 &= -d\xi \wedge db = 3(\xi_x - \xi_y) \omega^1 \wedge \omega^3 + 3\xi_y \omega^2 \wedge \omega^3
\end{align*}
\]

and we see that the only non-zero coefficients in \( \text{[31]} \) are

\[
\begin{align*}
a^2_{12} &= -\xi_y \\
a^4_{13} &= 3(\xi_x - \xi_y) \\
a^4_{23} &= 3\xi_y.
\end{align*}
\]

Substituting into \( \text{[32]} \) gives \( \psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = \psi_5 = \psi_7 = 0 \) and \( \psi_6 = 6(\xi\xi_y - \xi_x) \), as claimed. \( \square \)

**Remarks.** The partial differential equation \( \xi\xi_y = \xi_x \) has plenty of local solutions. Indeed, if \( F(t) \) is an arbitrary smooth function and we define \( \xi(x, y) \) implicitly by the equation

\[
F(\xi) = x\xi + y,
\]

then \( \xi\xi_y = \xi_x \). In particular, the foliation of \( \mathbb{R}^3 \) defined by \( \psi = \xi \, dx + dy \) need not be the planar foliation exhibited in the three examples above. Therefore, we have found many projectively inequivalent examples of structures given by data of the form \( \text{[21]} \) that all give rise to the same (flat) \( G_2 \) contact structure on the associated configuration space.

**Appendix:** harmonic curvature of a \( G_2 \) contact structure

As already mentioned (see \( \text{[2]} \) for the general theory), the harmonic curvature of a \( G_2 \) contact structure is a section of the bundle \( \mathbb{R}^7 \to \mathbb{R}^4 \). This binary septic may be obtained by the Cartan equivalence method.

Specifically, a \( G_2 \) contact structure on a five-dimensional manifold \( C \) may be specified in terms of an adapted co-frame as follows. Firstly, we choose a 1-form \( \omega^0 \) whose kernel is the contact distribution \( H \subset TC \). Non-degeneracy of the contact distribution says that \( \omega^0 \wedge d\omega^0 \wedge d\omega^0 \neq 0 \). The \( G_2 \) structure is then determined by completing \( \omega^0 \) to a co-frame \( \omega^0, \omega^1, \omega^2, \omega^3, \omega^4 \) so that

\[
d\omega^0 = \chi(\omega^1 \wedge \omega^4 - 3\omega^2 \wedge \omega^3) \mod \omega^0
\]

for some smooth function \( \chi \). Specifically, if \( X_0, X_1, X_2, X_3, X_4 \) is the dual frame, then \( H = \operatorname{span}\{X_1, X_2, X_3, X_4\} \) and the twisted cubic \( \text{[21]} \) may be given as

\[
(s, t) \mapsto s^3X_1 + s^2tX_2 + st^2X_3 + t^3X_4,
\]
compatibility with the Levi form being a consequence of (27). Imposing the structure equations (27) leaves precisely the following freedom in choice of co-frame:

\[
\begin{bmatrix}
\omega^0 \\
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4
\end{bmatrix}
= \begin{bmatrix}
t_9 & 0 & 0 & 0 & 0 \\
t_{10} & t_5^3 & 3t_5^2t_6 & 3t_5t_6^2 & t_6^3 \\
t_{11} & t_5^2t_7 & t_5(t_5^2t_8 + 2t_6t_7) & t_6(2t_5t_8 + t_6t_7) & t_6^2t_8 \\
t_{12} & t_5t_7^2 & t_7(2t_5t_8 + t_6t_7) & t_8(t_5t_8 + 2t_6t_7) & t_6t_8^2 \\
t_{13} & t_7^3 & 3t_7^2t_8 & 3t_7t_8^2 & t_8^3
\end{bmatrix}
\begin{bmatrix}
\omega^0 \\
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4
\end{bmatrix}
\]

(29)

for arbitrary functions \(t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\) on \(C\) subject only to \(t_9(t_5t_8 - t_6t_7) \neq 0\). (The functions \(t_5, t_6, t_7, t_8\) correspond to \((s, t) \mapsto (s, t) \begin{pmatrix} t_5 & t_7 \\ t_6 & t_8 \end{pmatrix}\) as a change of parameterisation in (28) whilst \(t_9, t_{10}, t_{11}, t_{12}, t_{13}\) modify the co-frame with multiples of \(\omega^0\).) To proceed with Cartan’s method of equivalence, we now pass to the bundle \(\tilde{C} \to C\) whose sections are frames adapted according to (27). It is a \(G\)-principal bundle where \(G\) is the 9-dimensional Lie subgroup of \(\text{GL}(5, \mathbb{R})\) given in (29) and comes tautologically equipped with 1-forms \(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4\) whose pull-backs along a section are \(\omega^0, \omega^1, \omega^2, \omega^3, \omega^4\), the co-frame on \(C\) corresponding to that section. Cartan’s aim is to extend this to an invariant co-frame on \(\tilde{C}\) by making various normalisations. For our purposes we need not take these normalisations too far. For calculation, we choose a co-frame on \(C\) adapted according to (27) so that \(\tilde{C}\) is identified as \(G \times C\) and the forms \(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4\) are given as

\[
\begin{bmatrix}
\theta^0 \\
\theta^1 \\
\theta^2 \\
\theta^3 \\
\theta^4
\end{bmatrix}
= \begin{bmatrix}
t_9 & 0 & 0 & 0 & 0 \\
t_{10} & t_5^3 & 3t_5^2t_6 & 3t_5t_6^2 & t_6^3 \\
t_{11} & t_5^2t_7 & t_5(t_5^2t_8 + 2t_6t_7) & t_6(2t_5t_8 + t_6t_7) & t_6^2t_8 \\
t_{12} & t_5t_7^2 & t_7(2t_5t_8 + t_6t_7) & t_8(t_5t_8 + 2t_6t_7) & t_6t_8^2 \\
t_{13} & t_7^3 & 3t_7^2t_8 & 3t_7t_8^2 & t_8^3
\end{bmatrix}
\begin{bmatrix}
\omega^0 \\
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4
\end{bmatrix}
\]

(30) Normalise the co-frame on \(C\) so that

\[d\omega^0 = \chi(\omega^1 \wedge \omega^4 - 3\omega^2 \wedge \omega^3).\]

This is easily achieved by the freedom

\[\tilde{\omega}^1 = \omega^1 + t_{10}\omega^0, \quad \tilde{\omega}^2 = \omega^1 + t_{11}\omega^0, \quad \tilde{\omega}^3 = \omega^1 + t_{12}\omega^0, \quad \tilde{\omega}^4 = \omega^1 + t_{13}\omega^0\]

and determines the functions \(t_{10}, t_{11}, t_{12}, t_{13}\).

**Step 1** Find \(\theta^5\) such that

\[d\theta^0 = -6\theta^0 \wedge \theta^5 + \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3.\]
This may be achieved by setting \(-t_9 = \Delta^3/\chi\), where \(\Delta = t_6t_7 - t_5t_8\) and then
\[
\theta^5 = -\frac{1}{6} \frac{d\chi}{\chi} + \frac{1}{2} \frac{d\Delta}{\Delta} + \chi \left( \frac{t_{13}\theta^1}{6} - \frac{t_{12}\theta^2}{2} + \frac{t_{11}\theta^3}{2} - \frac{t_{10}\theta^4}{6} \right) + s\theta^0
\]
for some function \(s\).

**Step 2** Find \(\theta^7, \theta^8, \theta^9\) such that
\[
E^1 \equiv d\theta^1 - (6\theta^0 \wedge \theta^0 - 3\theta^1 \wedge \theta^5 - 3\theta^1 \wedge \theta^8 + 3\theta^2 \wedge \theta^7)
\]
is of the form \(c^1_{\mu\nu} \theta^\mu \wedge \theta^\nu\) for \(\mu, \nu = 0, 1, 2, 3, 4\). It follows that
\[
E^1 \wedge \theta^0 \wedge \theta^1 \wedge \theta^2 = c^1_{34} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4 \wedge \theta^5.
\]

It turns out that
\[
c^1_{34} = \frac{t_{57}}{\Delta^5} \left( \psi_0 + \psi_1 s + \psi_2 s^2 + \psi_3 s^3 + \psi_4 s^4 + \psi_5 s^5 + \psi_6 s^6 + \psi_7 s^7 \right),
\]
where \(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7\) are functions on \(C\). These are the coefficients of the invariantly defined harmonic curvature. In fact, if
\[
d\omega^0 = \chi(\omega^1 \wedge \omega^4 - 3\omega^2 \wedge \omega^3)
\]
on \(C\) and we write
\[
d\omega^i = \sum_{1 \leq j < k \leq 4} a^i_{jk} \omega^j \wedge \omega^k \mod \omega^0, \text{ for } i = 1, 2, 3, 4
\]
then
\[
\begin{align*}
\psi_0 &= a^1_{34} \\
\psi_1 &= -2a^1_{24} + 3a^2_{34} \\
\psi_2 &= 3a^1_{14} + a^1_{23} - 6a^2_{24} + 3a^3_{34} \\
\psi_3 &= -2a^1_{13} + 9a^2_{14} + 3a^2_{23} - 6a^3_{24} + a^4_{34} \\
\psi_4 &= a^1_{12} - 6a^2_{13} + 9a^3_{14} + 3a^3_{23} - 2a^4_{24} \\
\psi_5 &= 3a^2_{12} - 6a^3_{13} + 3a^4_{14} + a^4_{23} \\
\psi_6 &= 3a^3_{12} - 2a^4_{13} \\
\psi_7 &= a^4_{12}.
\end{align*}
\]

From the general theory of parabolic geometry \[2, \text{Theorem 3.1.12}\] we find the following characterisation of flat \(G_2\) contact structures.

**Theorem 5.** The local symmetry algebra of the \(G_2\) contact structure specified by a co-frame adapted according to (31) is the split exceptional Lie algebra \(G_2\) if and only if \(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7\) given by (32) all vanish. Moreover, in this case, the manifold \(C\) is locally isomorphic to the homogeneous model.
Remarks. The formulæ (31) and (32) may alternatively be derived as follows. According to (20), the exterior derivative $d : \mathcal{D} \rightarrow \mathcal{D}$ gives rise, via the diagram

\[
\begin{array}{cccc}
0 & \to & 0 \ominus 1 & \to & \mathcal{D} \\
\downarrow d & & \downarrow \phi & & \downarrow 0 \\
0 & \to & 3 \ominus 1 & \to & 3 \ominus 2 & \to & 0
\end{array}
\]

\[
0 \to 3 \ominus 2 \to \mathcal{D} \to 0
\]

to an invariantly defined first order differential operator

\[
\nabla : 3 \ominus 2 \to 4 \ominus 3.
\]

In fact, this is nothing more than the second operator in the Rumin complex (10), which depends only on the contact structure on $\mathcal{D}$. The exterior derivative, on the other hand, may be written as $\omega_b \mapsto \nabla_{[\omega_b]}$ for any torsion-free connection $\nabla_a$ on $\mathcal{D}$. Thus, the Rumin operator (33) may be written with spinor indices (15), adapted to our cause, as

\[
\omega_{ABC} \mapsto \nabla (\pi_C \pi_D \pi_H).
\]

One may readily check that the formulæ (31) and (32) amount to the stipulation that

\[
\psi_{ABCDEF} = \pi^A \pi^B \pi^C \pi^D \pi^E \pi^F \pi^G \nabla \nabla (\pi_C \pi_D \pi_H)
\]

for all sections $\pi_A$ of $S^* = 1 \ominus 2/3$. Note, by the Leibniz rule

\[
\pi^A \nabla \pi^B \pi^C \pi^D \nabla (f \pi_C \pi_D \pi_H) = f \pi^A \pi^B \pi^C \pi^D \nabla (\pi_C \pi_D \pi_H) + \underbrace{\pi^A \pi^B \pi^C \pi^D \pi_C \pi_D \pi_H \nabla f}_{\equiv 0},
\]

that the right hand side of (35) is homogeneous of degree 7 over the functions and, therefore, automatically of the form given on the left. It follows that $\psi_{ABCDEF}$ is the obstruction to writing (33) as

\[
\omega_{ABC} \mapsto \mathcal{D}_{(AB} H \omega_{CD)H},
\]

where $\mathcal{D}_{ABC}$ is induced by a connection on $S^* = 1 \ominus 2/3$, because the Leibniz rule for such a connection would imply that

\[
\mathcal{D}_{AB} H (\pi_C \pi_D \pi_H) = \pi_C \pi_D \mathcal{D}_{AB} H \pi_H + \pi_C \pi_H \mathcal{D}_{AB} H \pi_D + \pi_D \pi_H \mathcal{D}_{AB} H \pi_C
\]

and the right hand side of (35) would therefore vanish. In other words, the formulæ (31) depends on $\nabla_a$ being torsion-free and $\psi_{ABCDEF}$ may be seen as some invariant part of the partial torsion of a freely chosen spinor connection $\mathcal{D}_a : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$.

More specifically, suppose $\mathcal{D}_a : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ is any connection and define its partial torsion $T_{(ABCDEF)} = T_{(ABCD)(EF)}$ according to

\[
T_{(ABE)D} (T_{EFG}) f = T_{(ABCD)(EF)} (T_{(EFG)} D_{EFG} f), \ \forall \text{ smooth functions } f.
\]
Changing the connection $\mathcal{D}(a)$, leads to a change of partial connection $\mathcal{D}_{ABC} : S^* \rightarrow \mathbb{R}^2 \otimes S^*$ according to

$$\hat{\mathcal{D}}_{ABC} \pi_D = \mathcal{D}_{ABC} \pi_D - \Gamma_{ABCD}^E \pi_E,$$

where $\Gamma_{ABCD}^E = \Gamma_{(ABC)D}^E$ and, therefore, an induced change on $\mathbb{R}^2 = \bigodot^3 S^*$, namely

$$\hat{\mathcal{D}}_{ABC} \omega_{DEF} = \mathcal{D}_{ABC} \omega_{DEF} - 3 \Gamma_{ABC(D}^E \omega_{EF)G}.$$

It follows that

$$\hat{\mathcal{D}}_{AB}^E \hat{\mathcal{D}}_{CDE} f = \mathcal{D}_{AB}^E \mathcal{D}_{CDE} f - 3 \Gamma_{AB}^E (C^G \mathcal{D}_{DE} f) G f,$$

and, therefore, that

$$\hat{\mathcal{D}}_{(AB}^E \hat{\mathcal{D}}_{CD)E} f = \mathcal{D}_{(AB}^E \mathcal{D}_{CD)E} f - 2 \Gamma_{(AB}^E (C^F \mathcal{D}_{D)EF} f + \Gamma_{H(AB}^H (E^G \mathcal{D}_{D)G} f.$$n

whence the partial torsion of $\mathcal{D}(a)$ changes according to

$$\hat{T}_{ABCD}^{EFG} = T_{ABCD}^{EFG} - 2 \Gamma_{(AB}^E (C^F \delta_D^G) + \Gamma_{H(AB}^H (E^G \delta_C^F \delta_D^G).$$

In particular, the trace-free part of $T_{ABCD}^{EFG}$, equivalently

$$\psi_{ABCDEF} \equiv T_{(ABCDEFG)},$$

is an invariant of the $G_2$ contact structure.
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