Curious congruences for cyclotomic polynomials

Shigeki Akiyama and Hajime Kaneko

Abstract

Let $\Phi^{(k)}_n(x)$ be the $k$th derivative of the $n$th cyclotomic polynomial. We are interested in the values $\Phi^{(k)}_n(1)$ for fixed positive integers $n$. D. H. Lehmer proved that $\Phi^{(k)}_n(1)/\Phi_n(1)$ is a polynomial of the Euler totient function $\phi(n)$ and the Jordan totient functions and gave it explicit formula. In this paper, we give a quick proof that $\Phi^{(k)}_n(1)/\Phi_n(1)$ is a polynomial of them without giving the explicit form. In the final section, we deduce some curious congruences: $2\Phi^{(3)}_n(1)$ is divisible by $\phi(n) - 2$. Moreover, if $k$ is greater than 1, then $\Phi^{(2k+1)}_n(1)$ is divisible by $\phi(n) - 2k$. The proof depends on a new combinatorial identity for general self-reciprocal polynomials over $\mathbb{Z}$, which gives rise to a formula that expresses the value $\Phi^{(k)}_n(1)$ as a $\mathbb{Z}$-linear combination of the coefficients in the minimal polynomial of $2\cos(2\pi/n) - 2$. As a supplement, we show the monotonic increasing property of $\Phi_n(x)$ on $[1, \infty)$ in two ways.

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1 Introduction

The $n$th cyclotomic polynomial

$$\Phi_n(x) = \prod_{0 < d | n \atop (d, n) = 1} \left( x - \exp \left( \frac{2\pi di}{n} \right) \right)$$

is the minimal polynomial of the $n$th primitive roots of unity over $\mathbb{Q}$. It is an irreducible polynomial in $\mathbb{Z}[x]$ of degree $\phi(n)$ where $\phi$ is the Euler totient function. From the relation $x^n - 1 = \prod_{d | n} \Phi_d(x)$, the well known formula

$$\Phi_n(x) = \prod_{d | n} \left( x^d - 1 \right)^{\mu(n/d)}$$

is derived by Möbius inversion. Here $\mu$ is the Möbius function. Motose [8,9] surmised that $\Phi_n(x)$ is an increasing function\(^1\) for $x > 1$. We start with a simple proof of this fact. It is probably known but we did not find it in the literature.

\(^1\)He did not give a proof of this fact, see the sentence before Theorem 3 in [8].
Theorem 1 For $j = 1, \ldots, \phi(n)$ we have

$$\Phi^j_N(1) > 0.$$ 

Consequently $\Phi^k_N(x)$ is strictly increasing for $x \geq 1$ and $k = 0, 1, \ldots, \phi(n) - 1$.

Proof Since $\Phi^1_N(1) = 1$ and $\Phi^2_N(1) = 1$, we may assume that $n \geq 3$. Then we have

$$\Phi_N(x) = \prod_{0 < d < n/2 \atop (d, n) = 1} \left( x - \exp \left( \frac{2\pi di}{n} \right) \right) \left( x - \exp \left( \frac{2\pi (n - d)i}{n} \right) \right) = \prod_{0 < d < n/2 \atop (d, n) = 1} \left( x^2 - 2 \cos \left( \frac{2\pi d}{n} \right) x + 1 \right).$$

Since all coefficients of

$$(x + 1)^2 + b(x + 1) + 1 = x^2 + (b + 2)x + b + 2$$

with $b \in (-2, 2)$ are positive, the expansion $\Phi_N(x + 1) = \sum_{j=0}^{d} \Phi^j_N(1)/j! \cdot x^j$ at $x = 0$ have positive coefficients $\Phi^j_N(1)/j!$ for $j \leq d = \phi(n)$. This proves the theorem.

Remark 1 The inequality $x \geq 1$ in Theorem 1 is sharp. If $p$ is an odd prime, then

$$\Phi_p(1) = \frac{1 - (-x)^p}{1 + x}.$$ 

It is easy to confirm

$$\Phi'_p \left( 1 - \frac{1}{\sqrt{p}} \right) = \frac{\left( 2p - \sqrt{p} + \frac{1}{\sqrt{p}} \right) \left( 1 - \frac{1}{\sqrt{p}} \right)^{p-1} - 1}{\left( 2 - \frac{1}{\sqrt{p}} \right)^2} < 0. \quad (3)$$

Thus there exists no $\varepsilon > 0$ that $\Phi_p(x)$ is increasing on $[1 - \varepsilon, \infty)$ for any $n \geq 1$.

Remark 2 There is an alternative proof that only works for $k = 0$, giving a starting point for this paper. Since $\Phi_1(x) = x - 1$, we may assume that $n \geq 2$. From

$$\sum_{d \mid n} \mu(d) = 0 \quad (4)$$

for $n \geq 2$, we may replace $x^d - 1$ by $(x^d - 1)/(x - 1)$ in (2). Taking $x \to 1$, we have

$$\Phi_N(1) = \prod_{d \mid n} d^{\mu(n/d)}$$

which is a positive integer. We see this is rewritten as $\Phi_N(1) = \exp(\Lambda(n))$ with the von Mangoldt function

$$\Lambda(n) := \begin{cases} \log p & n = p^e \ (p \text{ prime}), \\ 0 & \text{otherwise}, \end{cases}$$
which plays a crucial role in analytic number theory. The fact above was proved by Lebesgue [5]. Using (4), we also have

\[
\log \Phi_n(x) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \log \left(\frac{x^d - 1}{x - 1}\right)
\]

\[
\Phi_n'(x) / \Phi_n(x) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \left(\frac{(d - 1)x^{d-2} + (d - 2)x^{d-3} + \cdots + 1}{x^{d-1} + x^{d-2} + \cdots + 1}\right).
\] (5)

Letting \(x \to 1\) and using (4) again, we obtain

\[
\Phi_n'(1) / \Phi_n(1) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \frac{d - 1}{2} = \frac{1}{2} \sum_{d \mid n} \mu\left(\frac{n}{d}\right) d = \frac{\phi(n)}{2}.
\]

Thus we see that

\[
\Phi_n'(1) = \frac{1}{2} \phi(n) \Phi_n(1) \geq 1 > 0,
\] (6)

which was proved by Hölder [4] (c.f. [1, Lemma 10]). Now we consider \(\Phi_n(z)\) as a polynomial of complex variable \(z \in \mathbb{C}\). Recalling Gauss–Lucas theorem, any root of \(\Phi_n(z)\) lies in the convex hull of the roots of \(\Phi_n(z)\) in the complex plane. Therefore from (1) and \(n \geq 2\), the real function \(\Phi_n'(x)\) has no root in \(x \geq 1\). This implies \(\Phi_n'(x) > 0\) for \(x \geq 1\) since \(\Phi_n'\) is continuous.

**Remark 3** Let \(p\) be an odd prime. Then (3) and (6) imply that there exists a real root of \(\Phi_{2p}'(x)\) in the interval \((1 - 1/\sqrt{p}, 1)\).

Jordan totient function is defined by \(J_k(n) = \sum_{d \mid n} \mu(n/d)d^k\). This is multiplicative and we have

\[
J_k(n) = n^k \prod_{p \mid n} \left(1 - \frac{1}{p^k}\right)
\]

where \(p\) runs over prime divisors of \(n\). Clearly \(J_k(n)\) is a generalization of the Euler totient function \(\phi(n) = J_1(n)\). The name came from C. Jordan who studied linear groups over \(\mathbb{Z}/n\mathbb{Z}\) and deduced, e.g.,

\[
\text{Card}(GL_k(\mathbb{Z}/n\mathbb{Z})) = n^{k(k-1)/2} \prod_{j=1}^{k} J_j(n).
\]

As we observed in Remark 2, the special values \(\Phi_n^{(k)}(1)\) give important arithmetic functions such as the von Mangoldt function and the Euler totient function. Lehmer [6] gave an explicit formula of \(\Phi_n^{(k)}(1)/\Phi_n(1)\) as a polynomial of \(\phi(n)\) and \(J_{2j}(n)\) over \(\mathbb{Q}\), using Stirling numbers and Bernoulli numbers, see [3,7,10,11] for related developments. Here we give a quick proof of this fact but without the explicit form of the polynomial.

**Theorem 2** (6) For \(n \geq 2\), \(\Phi_n^{(\ell)}(1)/\Phi_n(1)\) is expressed as a polynomial of \(\phi(n)\) and \(J_{2j}(n)\) \((1 \leq j \leq (\ell + 1)/2\) over \(\mathbb{Q}\), and its value is a positive integer\(^2\) for \(\phi(n) \geq \ell\).

\(^2\)Clearly \(\Phi_n^{(\ell)}(x) = 0\) for \(\phi(n) < \ell\).
Proof  Applying Leibniz formula to (5),
\[ \frac{\Phi_n^{(k+1)}(x)}{\Phi_n(x)} = \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{\Phi_n^{(\ell)}(x)}{\Phi_n(x)} \sum_{d|n} \mu \left( \frac{n}{d} \right) \frac{\partial^{k-\ell}}{\partial x^{k-\ell}} \left( \frac{1 - dx^{d-1} + dx^d - x^d}{(x-1)(x^d - 1)} \right). \]

Substituting \( x \) by \( 1 + t \), we get the Taylor expansion at \( t = 0 \):
\[ \frac{(d-1)t - 1(t + 1)^{d-1} + 1}{t ((t + 1)^d - 1)} = \frac{d - 1}{2} + \left( \frac{d^2 - 6d + 5}{12} \right) t + \left( \frac{-d^2 + 4d - 3}{8} \right) t^2 + O(t^3). \] (7)

Regarding \( d \) as a real variable, we see that the numerator of (7) has the expansion at \( t = 0 \) of the form
\[ \sum_{j \geq 0} df_j t^{j+2} \quad f_j \in \mathbb{Q}[d], \quad \text{deg}(f_j) = j + 1, \quad f_0 = (d - 1)/2. \]

Similarly, the denominator has the form
\[ \sum_{j \geq 0} dg_j t^{j+2} \quad g_j \in \mathbb{Q}[d], \quad \text{deg}(g_j) = j, \quad g_0 = 1. \]

Thus, the \( \ell \)th Taylor coefficient of their quotient is a polynomial of \( d \) whose degree does not exceed \( \ell + 1 \). Using these Taylor coefficients, we recursively obtain the explicit formula for \( \Phi_n^{(\ell)}(1)/\Phi_n(1) \). Thus \( \Phi_n^{(\ell)}(1)/\Phi_n(1) \) a polynomial on \( J_1(n) \), \( J_2(n) \), \ldots, \( J_{\ell+1}(n) \) over \( \mathbb{Q} \). Moreover since
\[ \frac{(d-1)t - 1(t + 1)^{d-1} + 1}{t ((t + 1)^d - 1)} - \frac{d}{2(t + 1)} = \frac{d}{2(t + 1)(t + 1)^d - 1} - \frac{1}{t} \]
is an even function on \( d \), the terms \( d^{2k+1} \) with \( k = 1, 2, \ldots \) do not show, i.e., \( J_{2k+1}(n) \) \( (k = 1, 2, \ldots) \) never appear. By Theorem 1, \( \Phi_n^{(\ell)}(1)/\Phi_n(1) > 0 \) for \( \phi(n) \geq \ell \). Since
\[ \Phi_n(1) = \exp(\Lambda(n)) = \begin{cases} p & n = p^e (p : \text{prime}), \\ 1 & \text{otherwise}, \end{cases} \]
it suffices to show \( \Phi_n^{(\ell)}(1) \equiv 0 \pmod p \). By
\[ \Phi_p(x) = \frac{x^p - 1}{x^{p-1} - 1} = \Phi_p(x^{p-1}), \]
the case \( e > 1 \) is plain and the case \( e = 1 \) remains to be settled. Indeed we have,
\[ \Phi_p^{(\ell)}(1) = \sum_{j=0}^{p-1} j(j - 1) \cdots (j - \ell + 1) = \frac{p(p - 1) \cdots (p - \ell)}{\ell + 1} \equiv 0 \pmod p. \]
\[ \square \]
Corollary 1 We have

\[ \frac{\Phi_n^{(2)}(1)}{\Phi_n(1)} = \frac{j_2(n)}{12} + \frac{\phi(n)^2}{4} - \frac{\phi(n)}{2}, \]
\[ \frac{\Phi_n^{(3)}(1)}{\Phi_n(1)} = \frac{(\phi(n) - 2)(j_2(n) + \phi(n)(\phi(n) - 4))}{8}, \]
\[ \frac{\Phi_n^{(4)}(1)}{\Phi_n(1)} = \frac{1}{240} \left( 30j_2(n)\phi(n)^2 - 180j_2(n)\phi(n) + 5j_2(n)^2 + 220j_2(n) - 2j_4(n) \right. \]
\[ \left. + 15\phi(n)^4 - 180\phi(n)^3 + 660\phi(n)^2 - 720\phi(n) \right), \]
\[ \frac{\Phi_n^{(5)}(1)}{\Phi_n(1)(\phi(n) - 4)} = \frac{1}{96} \left( 3\phi(n)^4 - 48\phi(n)^3 + 10j_2(n)\phi(n)^2 + 228\phi(n)^2 \right. \]
\[ \left. - 80j_2(n)\phi(n) - 288\phi(n) + 5j_2(n)^2 + 100j_2(n) - 2j_4(n) \right). \]

Let

\[ \Phi_n(x + 1) := \sum_{h=0}^{\phi(n)} b_n(h)x^h \quad \text{with} \quad b_n(h) = \frac{1}{h!} \Phi_n^{(h)}(1) \in \mathbb{Z}. \]

Lehmer [6] further stated an interesting observation on the coefficients \(b_n(h)\). For a real \(R\), set \(R[\ell] := R(R - 1) \cdots (R - \ell + 1)\). For a positive integer \(r\), let \(t_r := j_r(n)/(2r)\). We define Bernoulli numbers \(B_m\) \((m \geq 0)\) by

\[ \frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \]

Under the setting above, he claimed that

\[ \frac{b_n(h)}{\Phi_n(1)} = t_1^{[h]} + 2 \sum_{\ell=1}^{\infty} B_{2\ell} \left( \frac{h}{2\ell} \right) (t_1 - \ell)^{[h-2\ell]} \Omega_\ell, \]

but the general form of \(\Omega_\ell\) is not given. He only wrote the first few terms:

\[ \Omega_1 = t_2, \]
\[ \Omega_2 = t_4 - 5t_2^{[2]}, \]
\[ \Omega_3 = t_6 - 7t_4(t_2 - 1) + \frac{35}{3} t_2^{[3]} + \frac{14}{3} t_2, \]
\[ \Omega_4 = t_8 - \frac{20}{3} t_6(t_2 - 1) - \frac{7}{3} t_4^{[2]} + \frac{70}{3} t_4(t_2 - 1)^{[2]} \]
\[ - \frac{175}{3} t_2^{[4]} + \frac{10}{3} t_6 - \frac{280}{9} t_2^{[2]} + \frac{290}{9} t_2. \]

Both Corollary 1 and this observation suggest the following:

Conjecture 1 For any non-negative integer \(k\), \(\Phi_n^{(2k+1)}(1)/\Phi_n(1)\) is divisible by \(\phi(n) - 2k\) in the polynomial ring \(\mathbb{Q}(\phi(n), j_2(n), j_4(n), \ldots, j_{2(\ell+1)/2}(n))\).

We checked its validity for \(k \leq 15\). The goal of this paper is to prove intimately related divisibility:

\[ \Phi_n^{(2k+1)}(1) \text{ is divisible by } \phi(n) - 2k \text{ in } \mathbb{Z}, \]
for \( k \geq 1 \), see Theorem 3. (The dividend should be doubled for the case \( k = 1 \).) We did not find yet a special meaning for this divisibility. For a fixed \( n \), such divisibility is proved using Theorem 2 and Proposition 1 below. However, such an individual proof does not seem to extend to the general case. Note that \( \Phi_n^{(2k+1)}(1)/\Phi_n(1) \) is likely to be divisible by \( \phi(n) - 2k \) for \( k > 1 \) but it can not be shown by our method.

Let \( \lambda(m) \) be the Carmichael lambda function, i.e., the exponent of \((\mathbb{Z}/m\mathbb{Z})^*\), the unit group of the ring \( \mathbb{Z}/m\mathbb{Z} \) (c.f. [2,10]). For an odd prime \( p \), \( \lambda(p^e) = \phi(p^e) \) holds since \((\mathbb{Z}/p^e\mathbb{Z})^* \) is cyclic. From
\[
(\mathbb{Z}/2^e\mathbb{Z})^* \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^{e-2}\mathbb{Z}
\]
for \( e \geq 2 \), we have
\[
\lambda(2^e) = \begin{cases} 
1 & e = 1, \\
2 & e = 2, \\
2^{e-2} & e \geq 3.
\end{cases}
\]
For a prime \( p \) and a positive integer \( e \), we write \( \lambda(p^e) \parallel k \) if both \( \lambda(p^e) \mid k \) and \( \lambda(p^{e+1}) \not\mid k \) hold. Proposition 1 may be known, but we give a proof for self-containedness.

**Proposition 1** (Trivial congruence) For \( k \geq 3 \) and \( n \geq k + 2 \), we have
\[
J_k(n) \equiv 0 \pmod{\prod_{\lambda(p^e) \parallel k} p^e}.
\]

For \( M > \prod_{\lambda(p^e) \parallel k} p^e \) and any \( n_0 \in \mathbb{N} \), there exists \( n \geq n_0 \) such that \( J_k(n) \not\equiv 0 \pmod{M} \).

**Proof** There are only finitely many prime \( p \) such that \( \lambda(p^e) \parallel k \). For a prime factor \( q \) of \( n \), \( q^k - 1 \) is a factor of \( J_k(n) \). The condition \( \lambda(p^e) \mid k \) implies \( q^k - 1 \equiv 0 \pmod{p^e} \) for each \( q \) which is coprime with \( p \). Assume that
\[
n > \max\{p : \lambda(p^e) \parallel k\}.
\]
If \( n \) has two distinct prime factors \( p_1 \) and \( p_2 \), then \( J_k(n) \) is divisible by \( (p_1^k - 1)(p_2^k - 1) \). We see \( p_1^k - 1 \) is divisible by \( p^e \) with \( p \neq p_1 \) if \( \lambda(p^e) \mid k \). This implies that \( (p_1^k - 1)(p_2^k - 1) \) is divisible by \( \prod_{\lambda(p^e) \parallel k} p^e \). Thus we may assume that \( n = \) a power of a prime \( q \) and \( \lambda(q^e) \parallel k \), i.e., \( n = q^e \) and \( \ell \geq 2 \). In this case, \( J_k(n) \) is divisible by \( q^\ell k - q^{(\ell-1)k} = q^{(\ell-1)k}(q^k - 1) \). We see
\[
\prod_{\lambda(p^e) \parallel k, p \neq q} p^e \mid q^k - 1.
\]
When \( q \neq 2 \), since \( e \leq \lambda(q^e) \leq k \), we see \( q^e \) divides \( q^{(\ell-1)k} \) and the required congruence holds.

For \( q = 2 \) we only have \( e - 1 \leq \lambda(2^e) \leq k \) and hence \( e \leq 2k \). So additionally if \( \ell > 2 \), \( q^e \mid q^{(\ell-1)k} \) holds. Therefore our discussion fails only when \( n = 2^e, \lambda(2^e) \parallel k \) and \( e > k \). This happens when \( 2k - 1 \leq k \), that is, \( k \leq 2 \). Summing up if \( k \geq 3 \), (8) implies our congruence. Moreover (8) holds if \( n > k + 1 \), because the worst case happens when \( k + 1 \) is an odd prime.

Take \( M > \prod_{\lambda(p^e) \parallel k} p^e \) and any \( n_0 \in \mathbb{N} \). There exists a prime power factor \( p^{e+1} \) of \( M \) such that \( \lambda(p^{e+1}) \) does not divide \( k \). From the definition of the exponent, there exists \( t \in \mathbb{N} \) which is coprime to \( p \) that \( t^k \equiv 1 \mod p^{e+1} \). By Dirichlet’s theorem, there exists a prime \( q \geq n_0 \) such that \( q \equiv t \pmod{p^{e+1}} \). Then \( J_k(q) = q^k - 1 \not\equiv 0 \mod M \). \( \square \)

Here is a table of the first numbers appearing in Proposition 1, see also [12].
2 Congruences for self-reciprocal polynomials

Let $q$ be a positive integer and $t_j$ be a complex number for $1 \leq j \leq q$. Let

$$\sum_{h=0}^{2q} b(t; h)x^h := \prod_{j=1}^{q}(x^2 + t_jx + t_j)$$

and

$$\sum_{\ell=0}^{q} a(t; \ell)y^\ell := \prod_{j=1}^{q}(y + t_j),$$

where $t = (t_j)_{1 \leq j \leq q}$. Our key result is a special combinatorial equality between $b(t; h)$ and $a(t; \ell)$.

**Proposition 2** Let $t = (t_j)_{1 \leq j \leq q}$ be a sequence of complex numbers. Then, for any $h$ with $0 \leq h \leq 2q$, we have

$$b(t; h) = \sum_{\ell=\max\{0, h-q\}}^{\lfloor h/2 \rfloor} \binom{q-\ell}{h-2\ell} a(t; \ell).$$

**Proof** We have

$$\prod_{j=1}^{q}(x^2 + t_jx + t_j) = (x + 1)^q \prod_{j=1}^{q}\left(\frac{x^2}{x + 1} + t_j\right)$$

$$= (x + 1)^q \sum_{\ell=0}^{q} a(t; \ell) \left(\frac{x^2}{x + 1}\right)^\ell$$

$$= \sum_{\ell=0}^{q} a(t; \ell)x^{2\ell} (x + 1)^{q-\ell}$$

$$= \sum_{\ell=0}^{q} a(t; \ell)x^{2\ell} \sum_{h=0}^{q-\ell} \binom{q-\ell}{h} x^h$$

$$= \sum_{\ell=0}^{q} a(t; \ell) \sum_{h=\ell}^{q+\ell} \binom{q-\ell}{h-2\ell} x^h$$

$$= \sum_{h=0}^{2q} \left(\sum_{\ell=\max\{0, h-q\}}^{\lfloor h/2 \rfloor} \binom{q-\ell}{h-2\ell} a(t; \ell)\right) x^h.$$

Comparing the coefficients, we get the desired result. \qed

**Remark 4** Under the usual convention $\binom{m}{n} = 0$ for non negative integers $m, n$, with $m < n$, Proposition 2 is rephrased as

$$b(t; h) = \sum_{\ell=0}^{\lfloor h/2 \rfloor} \binom{q-\ell}{h-2\ell} a(t; \ell).$$
Example 1  Let $a$ be a complex number. Setting $t_j = a$ for $j = 1, \ldots, q$, we obtain
\[
(x^2 + ax + a)^q = \sum_{h=0}^{2q} \left( \sum_{\ell=0}^{\lfloor h/2 \rfloor} \binom{q - \ell}{h - 2\ell} \binom{q}{\ell} a^{q-\ell} \right) x^h.
\]
This is also shown directly by the binomial theorem.

Example 2  Setting $t_j = \pm j$ for $j = 1, \ldots, q$, we have
\[
\prod_{j=1}^{q}(x^2 - jx - j) = \sum_{h=0}^{2q} \left( \sum_{\ell=0}^{\lfloor h/2 \rfloor} \binom{q - \ell}{h - 2\ell} s(q + 1, \ell + 1) \right) x^h,
\]
\[
\prod_{j=1}^{q}(x^2 + jx + j) = \sum_{h=0}^{2q} \left( \sum_{\ell=0}^{\lfloor h/2 \rfloor} \binom{q - \ell}{h - 2\ell} |s(q + 1, \ell + 1)| \right) x^h,
\]
where $s(m, n)$ is the Stirling number of the 1st kind defined by
\[
x(x-1) \cdots (x-m+1) = \sum_{n=0}^{m} s(m, n)x^n.
\]

We apply Proposition 2 to general self-reciprocal polynomials of even degree. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with degree $2q$ ($q \geq 1$). Suppose that $f(x)$ is self-reciprocal, that is, $x^{2q} f(x^{-1}) = f(x)$. It is easily seen by induction on $q$ that there exists $g(y) \in \mathbb{Z}[y]$ such that $f(x) = x^q g(y)$ with $y = x + x^{-1}$. Let
\[
f(x + 1) := \sum_{h=0}^{2q} \beta(h)x^h, \quad g(y + 2) := \sum_{\ell=0}^{q} \alpha(\ell)y^\ell,
\]
where $\beta(2q) = \alpha(q)$. Then we have the following:

Proposition 3  For any $h$ with $0 \leq h \leq 2q$, we have
\[
\beta(h) = \sum_{\ell=0}^{\lfloor h/2 \rfloor} \binom{q - \ell}{h - 2\ell} \alpha(\ell).
\]

Proof  We now reduce Proposition 3 to Proposition 2. Let
\[
g(y) := \beta(2q) \prod_{j=1}^{q} (y + \gamma_j),
\]
where $\gamma_1, \ldots, \gamma_q$ are complex numbers. Then we see
\[
f(x) = \beta(2q) \prod_{j=1}^{q} (x^2 + \gamma_j x + 1),
\]
and so
\[
f(x + 1) = \beta(2q) \prod_{j=1}^{q} (x^2 + (\gamma_j + 2)x + \gamma_j + 2).
\]
On the other hand, using
\[
f(x) = x^q \cdot \beta(2q) \prod_{j=1}^{q} \left( x + \frac{1}{x} + \gamma_j \right) = x^q g(y),
\]
we see
\[
g(y+2) = \beta(2q) \prod_{j=1}^{q} (y + \gamma_j + 2).
\]
Hence, (9) and (10) hold with \( t_j = \gamma_j + 2 \) for \( j = 1, 2, \ldots, q \). Therefore, Proposition 3 follows from Proposition 2 and Remark 4.

Proposition 3 leads to the following congruences.

**Proposition 4**

(i) \( 2f'''(1) \) is divisible by \( 2q - 2 \). Moreover, if \( q \) is even, then \( f'''(1) \) is divisible by \( 2q - 2 \).

(ii) Suppose that \( k \geq 2 \). Then \( f^{(2k+1)}(1) \) is divisible by \( 2q - 2k \).

**Proof**

For the proof of (i), we may assume that \( 2q \geq 3 \). Since \( 2f'''(1) = 12\beta(3) \), (i) follows from Proposition 3 and

\[
12 \left( \frac{q}{3} \right) = (2q - 2) \cdot q(q - 2).
\]

The latter part of (i) is similarly proved because \( q \) is even.

For the proof of (ii), we may assume that \( 2q \geq 2k + 1 \). Then (ii) follows from \( f^{(2k+1)}(1) = (2k + 1)!\beta(2k + 1) \) and Proposition 3.

Set

\[
c(\ell) := (2k + 1)! \left( \frac{q - \ell}{2k + 1 - 2\ell} \right).
\]

We shall show that \( c(\ell) \) is divisible by \( 2q - 2k \). We may assume that \( c(\ell) \neq 0 \). Note that \( (q - 2k + 1)(q - 2k) \) is even. If \( \ell = 0 \), then we see by \( k \geq 2 \) that

\[
c(0) = q(q - 1) \cdots (q - k) \cdots (q - 2k + 1)(q - 2k)
\]

is divisible by \( 2q - 2k \). Moreover, if \( 1 \leq \ell \leq k \), then

\[
c(\ell) = \frac{(2k + 1)!}{(2k + 1 - 2\ell)!} \cdot (q - \ell) \cdots (q - k) \cdots (q - 2k + \ell)
\]

is divisible by \( 2q - 2k \) because \( (2k + 1)!/(2k + 1 - 2\ell)! \) is even.

Recall that \( \Phi_n(x) \) is self-reciprocal. For \( f(x) = \Phi_n(x) \), the corresponding \( g(y) \) is the minimum polynomial of \( \zeta_n + \zeta_n^{-1} \), where \( \zeta_n \) is a primitive \( n \)-th root of unity. Proposition 3 gives a formula to express \( \Phi_n^{(k)}(1) \) as a \( \mathbb{Z} \)-linear combination of the coefficients of the minimal polynomial of \( \zeta_n + \zeta_n^{-1} - 2 \). Proposition 4 includes our curious congruences on cyclotomic polynomials.

**Theorem 3**

(i) \( 24\Phi_n'''(1) \) is divisible by \( \phi(n) - 2 \). In particular, if \( \phi(n) \) is divisible by \( 4 \), then \( \Phi_n'''(1) \) is divisible by \( \phi(n) - 2 \).

(ii) Suppose that \( k \geq 2 \). Then \( \Phi_n^{(2k+1)}(1) \) is divisible by \( \phi(n) - 2k \).

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**Note added in proof.** After our presentation at RIMS on 12 Oct 2022, T. Matsusaka informed us of a proof of Conjecture 1. G. Shibukawa told us that this proof of Theorem 1 is published later in Japanese textbooks for cyclotomic polynomials.
by K. Motose. We also got to know that G. Shibukawa introduced a similar method to Proposition 2 in Fibonacci Quart. 58 (2020), no. 5, 200-221. We are hoping to discuss these in a future work.

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