On LAO Testing of Multiple Hypotheses for Many Independent Objects

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Abstract

The problem of many hypotheses logarithmically asymptotically optimal (LAO) testing for a model consisting of three or more independent objects is solved. It is supposed that M probability distributions are known and each object independently of others follows to one of them. The matrix of asymptotic interdependencies (reliability–reliability functions) of all possible pairs of the error probability exponents (reliabilities) in optimal testing for this model is studied.

This problem was introduced (and solved for the case of two objects and two given probability distributions) by Ahlswede and Haroutunian. The model with two independent objects with M hypotheses was explored by Haroutunian and Hakobyan.

Index Terms - Hypothesis testing, multiple hypotheses, logarithmically asymptotically optimal (LAO) tests, two independent objects, error probability, reliability function.

I. Introduction

In [1] (see also [2], [3]) Ahlswede and Haroutunian formulated an ensemble of new problems on multiple hypotheses testing for many objects and on identification of hypotheses. Noted problems are extentions of those investigated in the books [4] and [5]. Problems of identification of distribution and of distributions ranking for one object were solved in [2] completely. Also the problem of hypotheses testing for the model consisting of two independent or two strictly dependent objects (when they cannot admit the same distribution) with two possible hypothetical distributions was investigated in [2]. In this paper we study the model consisting of K(≥ 3) objects which independently follow to one of given M (≥ 2) probability distributions. The problem is a generalization of those investigated in papers [6] – [10], and for testing of many hypotheses concerning one object in [11]. The case of two independent objects with three hypotheses was examined in [12]. Recently Tuncel [13] published an interesting consideration of the problem of multiple hypothesis optimal testing, which differs from the approach of [11], [14].

Let $\mathcal{P}(\mathcal{X})$ be the space of all probability distributions (PDs) on finite set $\mathcal{X}$. There are given $M$ PDs $G_m \in \mathcal{P}(\mathcal{X}), m = \overline{1, M}$.

Let us recall main definitions from [11] for the case of one object. The random variables (RV) $X$ taking values on $\mathcal{X}$ follows to one of the M PDs $G_m, m = \overline{1, M}$. The statistician
must accept one of $M$ hypotheses $H_m : G = G_m$, $m = \overline{1,M}$, on the base of a sequence of results of $N$ observations of the object $x = (x_1, ..., x_n, ..., x_N)$, $x_n \in \mathcal{X}$, $n = \overline{1,N}$. The procedure of decision making is a non-randomized test $\varphi_N$, which can be defined by division of the sample space $\mathcal{X}^N$ on $M$ disjoint subsets $\mathcal{A}_{l}^N = \{x : \varphi^N(x) = l\}$, $l = \overline{1,M}$. The set $\mathcal{A}_{l}^N$ contains all vectors $x$ for which the hypothesis $H_l$ is adopted. The probability $\alpha_m|l(\varphi_N)$ of the erroneous acceptance of hypothesis $H_l$ provided that $H_m$ is true, is equal to $G_m^N(\mathcal{A}_{l}^N)$, $l \neq m$. The probability to reject $H_m$, when it is true, is

$$\alpha_m|m(\varphi_N) \triangleq \sum_{l \neq m} \alpha_m|l(\varphi_N).$$

(1)

The error probability exponents of the sequence of tests $\varphi$, which it is convenient to call "reliabilities", are defined as

$$E_m|l(\varphi) \triangleq \lim_{N \to \infty} \frac{1}{N} \log \alpha_m|l(\varphi_N), \; m, l = \overline{1,M}.$$ 

(2)

It follows from (1) that

$$E_m|m(\varphi) = \min_{l \neq m} E_m|l(\varphi), \; m = \overline{1,M}.$$ 

(3)

The matrix $E(\varphi) = \{E_m|l(\varphi)\}$ is the reliability matrix of the sequence $\varphi$ of tests. It was studied in [11].

**Definition:** We call the sequence of tests $\varphi^*$ logarithmically asymptotically optimal (LAO) if for given positive values of $M - 1$ diagonal elements of the matrix $E(\varphi^*)$ maximal values to all other elements of it are provided.

The concept of LAO test was introduced by L. Birge [10] and also elaborated in [11], [12] and [13]. Now let us consider the model with three objects. Let $X_1$, $X_2$ and $X_3$ be independent RV taking values in the same finite set $\mathcal{X}$ with one of $M$ PDs, they are characteristics of corresponding independent objects. The random vector $(X_1, X_2, X_3)$ assumes values $(x^1, x^2, x^3) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}$.

Let $(x_1, x_2, x_3) = ((x_{11}, x_{12}, x_{13}), ..., (x_{n1}, x_{n2}, x_{n3}), ..., (x_{N1}, x_{N2}, x_{N3}))$, $x_n \in \mathcal{X}$, $k = \overline{1,3}$, $n = \overline{1,N}$, be a sequence of results of $N$ independent observations of the vector $(X_1, X_2, X_3)$. It is necessary to define unknown PDs of the objects on the base of observed data. The decision for each object must be made from the same set of hypotheses: $H_m : G = G_m$, $m = \overline{1,M}$.

We call this procedure the test for three objects and denote it by $\Phi_N$. It can be considered as three sequences of tests $\varphi^1$, $\varphi^2$, $\varphi^3$ by one for each object. We will denote the compound test sequence by $\Phi$. When we have $K$ independent objects the test $\Phi$ is composed of $K$ sequences of tests $\varphi^1$, $\varphi^2$, ..., $\varphi^K$.

Let $\alpha_{m_1,m_2,m_3|m_1,l_2,l_3}(\Phi_N)$ be the probability of the erroneous acceptance by the test $\Phi_N$ of the hypotheses triple $(H_{l_1}, H_{l_2}, H_{l_3})$ provided that the triple $(H_{m_1}, H_{m_2}, H_{m_3})$ is true, where $(m_1, m_2, m_3) \neq (l_1, l_2, l_3)$, $m_i, l_i = \overline{1,M}$, $i = \overline{1,3}$. The probability to reject a true triple of hypotheses $(H_{m_1}, H_{m_2}, H_{m_3})$ by analogy with (1) is the following:

$$\alpha_{m_1,m_2,m_3|m_1,m_2,m_3}(\Phi_N) = \sum_{(l_1,l_2,l_3) \neq (m_1,m_2,m_3)} \alpha_{m_1,m_2,m_3|m_1,l_2,l_3}(\Phi_N).$$

(4)
We study corresponding limits \( E_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi) \) of error probability exponents of the sequence of tests \( \Phi \), called reliabilities

\[
E_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi) \triangleq \lim_{N \to \infty} - \frac{1}{N} \log \alpha_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi_N), \quad m_i, l_i = 1, M, \quad i = 1, 3. \tag{5}
\]

It follows from (5) that (compare with (3))

\[
E_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi) = \min_{(l_1,l_2,l_3) \neq (m_1,m_2,m_3)} E_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi). \tag{6}
\]

For the case of \( K \) objects the error probability and reliability are considered also in papers [11 – 3]. The test sequence \( \Phi^* \) is called LAO for the model with \( K \) objects if for given positive values of certain \( K(M-1) \) elements of the reliability matrix \( E(\Phi^*) \) the procedure provides maximal values for all other elements of it.

Our aim in this paper is to analyze the reliability matrix \( E(\Phi^*) = \{E_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi^*)\} \) of LAO tests for three objects. The first idea was to study matrix \( E(\Phi) \) by renumbering the triples of PD-s from 1 to \( M^3 \). We can give \( M^3 - 1 \) diagonal elements of matrix \( E(\Phi) \) and apply Theorem 1. In this case the number of preliminary given elements of the matrix \( E(\Phi) \) will be grater and the procedure of calculations will be longer than in the algorithm presented in Section 3.

The generalization of the problem for \( K \) independent objects will be discussed during the text and in Section 4.

II. LAO TESTING OF HYPOTHESES FOR ONE OBJECT

We define the divergence (Kullback-Leibler distance) \( D(Q||G) \) for PDs \( Q, G \in \mathcal{P}(\mathcal{X}) \), as usual (see [13]):

\[
D(Q||G) = \sum_x Q(x) \log \frac{Q(x)}{G(x)}.
\]

We need to remind the Theorem and its Corollaries from [11] for the convenience to have notations.

For given positive elements \( E_{1|1}, E_{2|2}, \ldots, E_{M-1|M-1} \) we denote

\[
\mathcal{R}_l \triangleq \{Q : D(Q||G_l) \leq E_{l|l}\}, \quad l = 1, M-1, \tag{7.a}
\]

\[
\mathcal{R}_M \triangleq \{Q : D(Q||G_l) > E_{l|l}, \quad l = 1, M-1\} = \mathcal{P}(\mathcal{X}) - \bigcup_{l=1}^{M-1} \mathcal{R}_l, \tag{7.b}
\]

and consider the following values:

\[
E^{*}_{l|l} = E^{*}_{l|l}(E_{l|l}) \triangleq E_{l|l}, \quad l = M-1, \tag{8.a}
\]

\[
E^*_{m|l} = E^*_{m|l}(E_{l|l}) \triangleq \inf_{Q \in \mathcal{R}_l} D(Q||G_m), \quad m = 1, M, \quad m \neq l, \quad l = 1, M-1, \tag{8.b}
\]

\[
E^*_{m|M} = E^*_{m|M}(E_{1|1}, \ldots, E_{M-1|M-1}) \triangleq \inf_{Q \in \mathcal{R}_M} D(Q||G_m), \quad m = 1, M-1, \tag{8.c}
\]
\[ E_{M|M}^* = E_{M|M}^*(E_{1|1}, \ldots, E_{M-1|M-1}) \triangleq \min_{l=1, M-1} E_{M|l}^*. \tag{8.d} \]

**Theorem 1** [11]: If the distributions \( G_m, m = 1, M \), are different, that is all elements of the matrix \( \{D(G_l||G_m)\} \), are strictly positive, then two statements hold:

a) when the given numbers \( E_{1|1}, E_{2|2}, \ldots, E_{M-1|M-1} \) satisfy conditions

\[ 0 < E_{1|1} < \min_{l=2, M} D(G_l||G_1), \tag{9.a} \]

\[ 0 < E_{m|m} < \min_{l=1, M-1} \min_{l=1, M} E_{m|l}^*(E_{l|l}), \min_{l=m+1, M} D(G_l||G_m), \quad m = 2, M-1, \tag{9.b} \]

then there exists a LAO sequence of tests \( \varphi^* \), the reliability matrix of which \( \mathbf{E}(\varphi^*) = \{E_{m|l}^*\} \) is defined in (8) and all elements of it are strictly positive;

b) even if one of conditions (9) is violated, then the reliability matrix of any such test includes at least one element equal to zero (that is the corresponding error probability does not tend to zero exponentially).

**Corollary 1** [12]: It can be proved that

\[ E_{m|m}^* = E_{m|M}^*, \quad m = 1, M-1, \text{and} \quad E_{m|m}^* \neq E_{m|l}^*, \quad l \neq m, M. \tag{10} \]

**Proof:** Applying theorem of Kuhn-Tucker in (8.b) we can derive that the elements \( E_{l|l}^* \), \( l = 1, M-1 \) may be determined by elements \( E_{m|l}^* \), \( m \neq l, m = 1, M \), by the following inverse function

\[ E_{l|l}^*(E_{m|l}^*) = \inf_{Q: D(Q||G_m) \leq E_{m|l}^*} D(Q||G_l). \]

From conditions (9) we see that \( E_{m|m}^* \) can be equal only to one among \( E_{m|l}^* \), \( l = m+1, M \). Assume that (10) is not true, that is \( E_{m|m}^* = E_{m|l}^* \), for \( l = m+1, M-1 \). From (8.b) it follows that

\[ E_{l|l}^*(E_{m|l}^*) = \inf_{Q: D(Q||G_m) \leq E_{m|l}^*} D(Q||G_l) = \inf_{Q: D(Q||G_m) \leq E_{m|m}^*} D(Q||G_l) \]

\[ = E_{l|l}^*, \quad m = 1, M-1, l = 1, M-1, m < l, \]

but from conditions (9) it follows that \( E_{l|l}^* < E_{l|m}^* \) for \( m = 1, l-1 \). Our assumption is not correct, hence (10) is valid.

**Corollary 2** [12]: If one preliminary given element \( E_{m|m}^*, m = 1, M-1 \), of the reliability matrix of an object is equal to zero, then the corresponding elements of the matrix determined as functions of \( E_{m|m}^* \), will be define as in the case of Stain’s lemma [15]:

\[ E_{l|m}^*(E_{m|m}^*) = D(G_m||G_l), \quad l = 1, M, \quad l \neq m. \tag{11} \]

and the remaining elements of the matrix are defined by \( E_{l|l}^* > 0, l \neq m, l = 1, M-1 \), as follows from Theorem 1.

**Remark 1:** The number of elements \( E_{m|m}^* \) equal to zero may be any between 1 and \( M-1 \). Generalization of Corollary 2 is straightforward.
III. LAO Testing of Hypotheses for Three Independent Objects

Now let us consider the case of three independent objects and $M$ hypotheses. The compound test $\Phi$ may be composed from three separate tests $\varphi^1$, $\varphi^2$, $\varphi^3$.

Let us denote by $E(\varphi^i)$ the reliability matrices of the sequences of tests $\varphi^i$, $i = 1, 3$, for each of the objects. The following Lemma is a generalization of Lemma from [11] and [12].

**Lemma 1:** If elements $E_{m,l}(\varphi^i)$, $m, l = 1, M$, $i = 1, 3$, are strictly positive, then the following equalities hold for $\Phi = (\varphi^1, \varphi^2, \varphi^3)$:

\[
E_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi) = \sum_{i=1}^{3} E_{m_i|l_i}(\varphi^i), \quad \text{if } m_i \neq l_i, \quad i = 1, 3, \quad (12.a)
\]

\[
E_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi) = \sum_{i} E_{m_i|l_i}(\varphi^i), \quad m_k = l_k, \quad m_i \neq l_i, \quad i \neq k, \quad i, k = 1, 3, \quad (12.b)
\]

\[
E_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi) = E_{m_i|l_i}(\varphi^i), \quad m_k = l_k, \quad m_i \neq l_i, \quad i \neq k, \quad k, i = 1, 3. \quad (12.c)
\]

Equalities (12.a) are valid also if $E_{m,l}(\varphi^i) = 0$ for several pairs $(m, l)$ and several $i$.

**Proof:** It follows from the independence of the objects that

\[
\alpha_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi_N) = \prod_{i=1}^{3} \alpha_{m_i|l_i}(\varphi_N^i), \quad \text{if } m_i \neq l_i, \quad (13.a)
\]

\[
\alpha_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi_N) = (1 - \alpha_{m_k|l_k}(\varphi_N^k)) \prod_{i \neq k} \alpha_{m_i|l_i}(\varphi_N^i), \quad m_k = l_k, \quad m_i \neq l_i, \quad i, k = 1, 3, \quad (13.b)
\]

\[
\alpha_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi_N) = \alpha_{m_i|l_i}(\varphi_N^i) \prod_{i \neq k} (1 - \alpha_{m_k|l_k}(\varphi_N^k)), \quad m_k = l_k, \quad m_i \neq l_i, \quad k, i = 1, 3. \quad (13.c)
\]

Remark that here we consider also the probabilities of right (not erroneous) decisions. According to the definitions (4) and (5) from equalities (13) we obtain relations (12).

Now we shall show how we can find LAO test from the set of compound tests $\{\Phi = (\varphi^1, \varphi^2, \varphi^3)\}$ when strictly positive elements $E_{m,m,m|M,m,m}, E_{m,m,m|M,m,m}$ and $E_{m,m,m|M,m,m}$, $m = 1, M - 1$, of the reliability matrix are given.

**Lemma 2:** These elements can be strictly positive only in the following three subsets of tests $\{\Phi = (\varphi^1, \varphi^2, \varphi^3)\}$:

$A \triangleq \{\Phi : E_{m|m}(\varphi^i) > 0, m = 1, M - 1, \quad i = 1, 3\}$,

$B \triangleq \{\Phi : \text{one or several } m' \text{ from } [1, M - 1] \text{ exist such that } E_{m|m'}(\varphi^i) = 0 \text{ for two } i,$

but $E_{m|m'}(\varphi^i) > 0, \quad i \neq j, \quad \text{and for for other } m < M, \quad E_{m|m}(\varphi^i) > 0, \quad i, j = 1, 3\}$,

$C \triangleq \{\Phi : \text{one or several } m' \text{ from } [1, M - 1] \text{ exist such that } E_{m|m'}(\varphi^i) = 0, \quad \text{and for other } m < M, \quad E_{m|m}(\varphi^i) > 0, \quad i = 1, 3\}$.

**Proof:** When $E_{m|m}(\varphi^i) > 0$, then

\[\lim_{N \to \infty} \frac{-1}{N} \log(1 - \alpha_{m|m}(\varphi^i)) = 0, \quad m = 1, M, \quad i = 1, 3.\]
and when \( E_{m|m}(\varphi^i) = 0 \), then
\[
\lim_{N \to \infty} \frac{-1}{N} \log(1 - \alpha_{m|m}(\varphi^i)) > 0, \quad m = \frac{1}{M}, \quad i = 1,3.
\]

From these equalities and inequalities, keeping in mind (5), (10) and (13) we obtain that the given elements are positive for \( \Phi \in A \cup B \cup C \). Proposition 1 is proved.

Let us define the following family of decision sets for given positive elements \( E_{m,m,m|{m,M,m}}, \ E_{m,m,m|{m,M,m}} \) and \( E_{m,m,m|{m,M,m}}, \ m = \frac{1}{M - 1} \):
\[
\mathcal{R}^{(i)}_m \triangleq \{ Q : D(Q||G_m) \leq E_{m,m,m_1,m_2,m_3}, \ m_i = M, \ m_j = m, \ i \neq j \}, \ m = \frac{1}{M - 1}, \ i = 1,3,
\]
\[
\mathcal{R}^{(i)}_M \triangleq \{ Q : D(Q||G_m) > E_{m,m,m_1,m_2,m_3}, \ m_i = M, \ m_j = m, \ i \neq j, \ m = \frac{1}{M - 1} \}, \ i = 1,3.
\]

Let also
\[
E^*_{m,m,m|{m,M,m}} \triangleq E_{m,m,m|{m,M,m}},
\]
\[
E^*_{m,m,m|{m,M,m}} \triangleq E_{m,m,m|{m,M,m}},
\]
\[
E^*_{m,m,m|{m,M,m}} \triangleq E_{m,m,m|{m,M,m}},
\]

\[
E^*_{m,m,m|{m,M,m}} \triangleq \inf_{Q, Q^i \in \mathcal{R}^{(i)}_i} D(Q||G_m), \ m_k = l_k, \ m_i \neq l_i, i \neq k, \ i, k = 1,3, \tag{14.a}
\]

\[
E^*_{m,m,m|{m,M,m}} \triangleq \sum_{i \neq k} \inf_{Q, Q^i \in \mathcal{R}^{(i)}_i} D(Q||G_m), \ m_k = l_k, \ m_i \neq l_i, \ i, k = 1,3, \tag{14.b}
\]

\[
E^*_{m,m,m|{m,M,m}} \triangleq E^*_{m,m,m|{m,M,m}} + E^*_{m,m,m|{m,M,m}} + E^*_{m,m,m|{m,M,m}}, \ m_i \neq l_i, \tag{14.c}
\]

The result of the present paper is formulated in Theorem 2: If all distributions \( G_m, \ m = \frac{1}{M} \), are different, and consequently \( D(G_l||G_m) > 0, \ l \neq m, \ l, m = \frac{1}{M} \), then the following three statements are valid:

a) when given strictly positive elements \( E_{m,m,m|{m,M,M}}, E_{m,m,m|{m,M,M}} \) and \( E_{m,m,m|{m,M,m}, m = \frac{1}{M - 1} \}, \) meet the following conditions
\[
\max(E_{1,1,1|M,1,1}, E_{1,1,1|M,1,1}, E_{1,1,1|M,1,1}, E_{1,1,1|M,1,1}) < \min_{l=2,M} D(G_l||G_1), \tag{15.a}
\]
\[
E_{m,m,m|{m,M,m}} < \min_{l=1,m-1} \min_{l=m+1,M} E^*_{m,m,m|{m,M,m}}, \min_{l=m+1,M} D(G_l||G_m), \ m = \frac{2}{M - 1}, \tag{15.b}
\]
\[
E_{m,m,m|{m,M,m}} < \min_{l=1,m-1} \min_{l=m+1,M} E^*_{m,m,m|{m,M,m}}, \min_{l=m+1,M} D(G_l||G_m), \ m = \frac{2}{M - 1}, \tag{15.c}
\]
\[
E_{m,m,m|{m,M,m}} < \min_{l=1,m-1} \min_{l=m+1,M} E^*_{m,m,m|{m,M,m}}, \min_{l=m+1,M} D(G_l||G_m), \ m = \frac{2}{M - 1}, \tag{15.d}
\]
then there exists a LAO test sequence \( \Phi^* \in A \), the reliability matrix of which \( \mathbf{E}(\Phi^*) = \{ E_{m,m,m|{m,M,m}}(\Phi^*) \} \) is defined in (14) and all elements of it are positive,

b) when even one of the inequalities (15) is violated, then there exists at least one element of the matrix \( \mathbf{E}(\Phi^*) \) equal to 0,
c) for given strictly positive numbers $E_{m,m,m|M,m,m}$, $E_{m,m,m|M,M,m}$, and $E_{m,m,m|m,M,M}$; $m = 1, M - 1$ the reliability matrix $E(\Phi)$ of the tests $\Phi$ from the defined in Lemma 2 families $\mathcal{B}$ and $\mathcal{C}$ necessarily contains elements equal to zero.

**Proof:** a) Conditions (15) imply that inequalities analogous to (9) hold simultaneously for the case of three objects. Really, using equalities (10) we can rewrite inequalities (9) for three objects as follows:

$$\max(E_{1|M}(\varphi^1), E_{1|M}(\varphi^2), E_{1|M}(\varphi^3)) < \min_{l=2,M} D(G_l||G_1), \quad (16.a)$$

$$E_{m|M}(\varphi^i) < \min_{l=1,m-1} E_{m|l}(\varphi^i), \quad \min_{l=m+1,M} D(G_l||G_m)], \quad i = 1,3, \quad m = 2, M - 1, \quad (16.b)$$

We shall prove, for example, the inequalities (16.b), for $i = 2$ which are the consequence of the inequalities (15.c). Let us consider the tests $\Phi \in \mathcal{A}$ such that $E_{m,m,m|m,M,m}(\Phi) = E_{m,m,m,m|M,M,m}$ and $E_{m,m,m|m,M,m}(\Phi) = E^{*}_{m,m,m|m,M,m}$, $l = 1, m - 1$, $m = 1, M - 1$. The corresponding error probabilities $\alpha_{m,m,m,M,M}(\Phi_N)$ and $\alpha_{m,m,m,m,l}(\Phi_N)$ are given as products defined by (13.c). Because $\Phi \in \mathcal{A}$, then

$$\lim_{N \to \infty} -\frac{1}{N} \log(1 - \alpha_{m|m}(\varphi^i_N)) = 0, \quad i = 1,3. \quad (17)$$

According to (5), (13.c) and (17) we obtain that

$$E^{*}_{m,m,m|m,M,m}(\Phi) = E^{*}_{m|M}(\varphi^2), \quad m = 2, M - 1, \quad (18.a)$$

$$E^{*}_{m,m,m|m,M,m}(\Phi) = E^{*}_{m|l}(\varphi^2), \quad m = 2, M - 1. \quad (18.b)$$

So (16.c) is consequence of (15.c).

As we noted in the beginning of the proof it follows from (10) and (16) that conditions (9) of Theorem 1 take place for each of three objects. According to Theorem 1 there exists LAO sequences of tests $\varphi^{*,1}$, $\varphi^{*,2}$ and $\varphi^{*,3}$ three objects such that the elements of the matrices $E(\varphi^{*,i})$, $i = 1,3$, are determined according to (8). We consider the sequence of tests $\Phi^*$, which is composed of three sequences of tests $\varphi^{*,1}$, $\varphi^{*,2}$, $\varphi^{*,3}$ and we will show that $\Phi^*$ is LAO and other elements of the matrix $E(\Phi^*)$ are determined according to (14).

It follows from (16), (10) and (9) that the requirements of Lemma 1 are fulfilled. Applying Lemma 1 we can deduce that the reliability matrix $E(\Phi^*)$ can be obtained from matrices $E(\varphi^{*,i})$ as in (12).

When conditions (15) take place, we obtain (14) according to (12), (8), (10) and (18). The equality in (14.e) is a particular case of (6). From (14) it follows that all elements of $E(\Phi^*)$ are positive.

Now it is easy to verify that the compound test $\Phi^*$ for three objects is LAO.

b) When one of the inequalities (15) is violated, then from (14.b) we see, that some elements in the matrix $E(\Phi^*)$ must be equal to zero.

c) When $\Phi \in \mathcal{B}$, then from (10) and (12.a) we can see that the elements $E_{m',m',m'|M,M,M} = \sum_{i=1}^{3} E_{m'|M}(\varphi) = 0.$
Let $\Phi = (\varphi^1, \varphi^2, \varphi^3) \in C$. For example $E_{m'|m'}(\varphi^1) > 0$, $E_{m'|m'}(\varphi^2) = E_{m'|m'}(\varphi^3) = 0$, then

$$E_{m',m',m'|M,M} = \lim_{N \to \infty} \frac{1}{N} \log(1 - \alpha_{m'|m'}(\varphi^1)) + E_{m'|m'}(\varphi^2) + E_{m'|m'}(\varphi^3) = 0.$$  

Theorem 2 is proved.

Remark 2: For every test $\Phi$ from noted in Lemma 2 subset $C$ (for the case of subset $B$ resonemets are similar) for given $3(M - 1)$ elements of matrix $E(\Phi)$, using independence of three objects, the definition (4) and equalities (10) we can determine all other elements of $E(\Phi)$ in the following way:

$$E_{m',m',m|M,m',m'}(\Phi) = \lim_{N \to \infty} \frac{1}{N} \log(1 - \alpha_{m'|m'}(\varphi^1)) =$$

$$= \frac{1}{2} [E_{m',m',m'|M,m',m'}(\Phi) + E_{m',m',m'|M,m',m'}(\Phi) - E_{m',m',m'|M,m',m'}(\Phi)], \quad (19.a)$$

$$E_{m',m',m'|M,m',m'}(\Phi) = \lim_{N \to \infty} \frac{1}{N} \log(1 - \alpha_{m'|m'}(\varphi^2)) =$$

$$= \frac{1}{2} [E_{m',m',m'|M,m',m'}(\Phi) + E_{m',m',m'|M,m',m'}(\Phi) - E_{m',m',m'|M,m',m'}(\Phi)], \quad (19.b)$$

$$E_{m',m',m'|M,m',m'}(\Phi) = \lim_{N \to \infty} \frac{1}{N} \log(1 - \alpha_{m'|m'}(\varphi^3)) =$$

$$= \frac{1}{2} [E_{m',m',m'|M,m',m'}(\Phi) + E_{m',m',m'|M,m',m'}(\Phi) - E_{m',m',m'|M,m',m'}(\Phi)], \quad (19.c)$$

From these equalities it follows that

$$E_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi) = E_{m_1|l_1} + E_{m_2|l_2} + E_{m_3|l_3}, \quad m_1 \neq l_i, \ i = 1, 2, 3, \quad (20.a)$$

$$E_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi) = \sum_{i: i \neq k} E_{m_i|l_i}(\varphi^3), \quad m_i \neq l_i, \ m_k = l_k, k \neq i. \quad (20.b)$$

Denoting right sums from (19.a) by $A$, from (19.b) by $B$, from (19.c) by $C$ we obtain that

$$E_{m',m_2,m_3|l_2,l_3}(\Phi) = E_{m_2|l_2} + E_{m_3|l_3}, \quad m_2 \neq l_2, \ m_3 \neq l_3, \quad (20.c)$$

$$E_{m_1,m',m_3|l_1,l_3}(\Phi) = E_{m_1|l_1} + E_{m_3|l_3}, \quad m_1 \neq l_1, \ m_3 \neq l_3, \quad (20.c)$$

$$E_{m_1,m_2,m'|l_2,l_3}(\Phi) = E_{m_1|l_1} + E_{m_2|l_2}, \quad m_1 \neq l_1, \ m_2 \neq l_2, \quad (20.c)$$

$$E_{m',m_2,m_3|l_2,l_3}(\Phi) = E_{m_3|l_3} + E_{m',m'|m',m',M}, \quad (20.c)$$
\[ E_{m',m_2,m'|m',l_2,m'}(\Phi) = E_{m_3|m_3}(\varphi^3) + E_{m',m',m'|m',M,m'} \quad (20.d) \]
\[ E_{m_1,m',m'|l_1,l_2,m'}(\Phi) = E_{m_3|m_3}(\varphi^3) + E_{m',m',m'|M,m',m'} \]

Remember that in this case the elements \( E_{m',m',m'|M,M,M}(\Phi) = 0 \). From (20) we see, that LAO test \( \Phi'(\varphi^1, \varphi^2, \varphi^3) \), \( m' \in [1, M - 1] \) is composed of the tests \( \varphi^1 \), \( \varphi^2 \) and \( \varphi^3 \) discussed in Corollary 2.

**IV. Supplements for \( K(>3) \) Objects**

When we consider the model with \( K \) independent objects the generalization of Lemma 1 will take the following form:

*Lemma 3:* If elements \( E_{m_l|m}(\varphi^i) \), \( m, l = \overline{1, M} \), \( i = \overline{1, K} \), are strictly positive, then the following equalities hold for \( \Phi = (\varphi^1, \varphi^2, ..., \varphi^K) \):

\[ E_{m_1,m_2,...,m_K|l_1,l_2,...,l_K}(\Phi) = \sum_{i=1}^{K} E_{m_i|m_i}(\varphi^i), \text{ if } m_i \neq l_i, \ i = \overline{1, K} \]
\[ E_{m_1,m_2,...,m_K|l_1,l_2,...,l_K}(\Phi) = \sum_{i: i \neq j} E_{m_i|m_i}(\varphi^i), \text{ if } m_j = l_j, \ m_i \neq l_i, \ i_j, \ i, j = \overline{1, K} \]

For given \( K(M - 1) \) strictly positive elements \( E_{m,m,...,m|M,m,...,m}, E_{m,m,...,m|M,m,...,m}, ..., E_{m,m,...,m|m,M, ..., m}, m = \overline{1, M - 1} \) for the case of \( K \) independent objects. We can find the LAO test \( \Phi^* \) we will find as in the case of three independent objects. So the problem of many hypotheses testing of the model with \( K \) independent objects may be solved by constricting corresponding sets \( R_{m}^{(k)}, \ k = \overline{1, K}, \ m = \overline{1, M} \) as in (14) and formulating conditions analagical to (15).

It is interesting to analyse the generalization of Corollary 2 for the case of many objects.

**V. Example**

Let us consider an example on LAO testing hypotheses concerning one object and two objects. The set \( X = \{0, 1\} \) of two elements and the following probability distributions given on \( X \): \( G_1 = \{0, 10; 0, 90\} \), \( G_2 = \{0.85; 0, 14\} \), \( G_3 = \{0, 23; 0, 77\} \). In Fig. 1 and Fig. 2 the results of calculations of functions \( E_{2|1}(E_{1|1}) \) and \( E_{2,1|2,1}(E_{1,1|3,1}, E_{2,2|2,3}) \) are presented. For these distributions we have \( \min(D(G_2, G_1), D(G_3, G_1)) \approx 2, 2 \) and \( \min(E_{2,2|2,1}, D(G_3, G_2)) \approx 1, 4 \). We see that when the first inequality in (9.a) is violated then \( E_{2|1} = 0 \) and, when the inequality (15.b) and (15.c) are violated, then \( E_{2,1|1,2} = 0 \).
Fig. 1

Fig. 2
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