Identification of the stress-energy tensor through conformal restriction in SLE and related processes

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Abstract

We derive the Ward identities of Conformal Field Theory (CFT) within the framework of Schramm-Loewner Evolution (SLE) and some related processes. This result, inspired by the observation that particular events of SLE have the correct physical spin and scaling dimension, and proved through the conformal restriction property, leads to the identification of some probabilities with correlation functions involving the bulk stress-energy tensor. Being based on conformal restriction, the derivation holds for SLE only at the value $\kappa = 8/3$, which corresponds to the central charge $c = 0$ and the case when loops are suppressed in the corresponding $O(n)$ model.

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1 Introduction

The description of two-dimensional statistical models at their critical points in terms of Conformal Field Theories (CFT) is one of the most fruitful achievements of theoretical physics [1] (for a pedagogical account, see [2]). In the past twenty years, remarkable exact results on universal quantities like critical exponents have been obtained within this framework. However, some issues have been only partially understood. Besides the lack of mathematical rigor in relating statistical models to CFT, the language of CFT is not best suited to the description of the geometrical aspects of conformal symmetry, being formulated upon the concept of local operators. Moreover, a clear and rigorous geometrical definition of local conformal invariance (and of its breaking by an anomaly, quantified by the ‘central charge’ $c$) is missing in CFT. An important progress in filling these gaps has recently been achieved in the context of probability theory and stochastic analysis, with a new approach to critical phenomena centered on Schramm-Loewner Evolution (SLE) [3, 4] (for a review aimed at theoretical physicists, see [5]). In a nutshell, SLE is a way of constructing measures on random curves which satisfy the expected properties of domain walls of critical statistical systems in the continuum limit. It turns out that such measures form a family described by one real parameter $\kappa$. Different values of $\kappa$ are expected to correspond to different statistical systems. The chordal version of SLE, which is the only one considered in detail in this paper, defines a measure on curves conditioned to start and end at distinct points on the boundary of a simply connected domain in $\mathbb{C}$, which can be conventionally chosen to be the upper half plane $\mathbb{H}$ by virtue of conformal invariance.

A natural question which arises is the precise relation between SLE and CFT. A first step in this direction was made by noticing [7] that the Fokker-Planck-type equations obtained from SLE are closely related to second order differential equations satisfied by certain CFT correlation functions involving the so-called ‘boundary condition changing operators’ $\phi_{2,1}$ [6]. This implies a precise relation between probabilities in SLE and correlation functions in CFT with the boundary operator $\phi_{2,1}$ inserted at the points where the SLE curve starts and ends. An important consequence of the above identification is the relation between the parameter $\kappa$ and the central charge $c$ of CFT:

$$c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}. \quad (1.1)$$

However, a deeper insight requires the identification of correlation functions involving other kinds of operators, inserted not only at the boundary but especially in the bulk of the domain. Particularly significant in CFT are holomorphic operators, which transform non-trivially under only one of the two copies of the underlying Virasoro algebra. Among holomorphic operators, the most important is the stress-energy tensor, the generator of conformal transformations, whose Ward identities are equivalent to the statement that scaling operators should be classified according to highest-weight representations of the Virasoro algebra.

This paper deals with the identification of some probabilities in the SLE context with CFT correlation functions involving the bulk stress-energy tensor $T$. More precisely, we consider the joint probability that the SLE intersects a number of short segments in the bulk, of lengths
\{\epsilon_j\}$ and centered about points $\{w_j\}$, at inclinations to some fixed axis characterized by angular variables $\{\theta_j\}$. One can then investigate the features of the Fourier components of this probability with respect to $\theta$, which are labelled by a variable $n$ which has the properties of conformal “spin”. We study the leading behavior of each component as $\epsilon \to 0$, which we assume is a power law. For instance, the leading power of the spin zero Fourier component is $2 - d_f$, where $d_f$ is the fractal dimension of SLE, rigorously computed in [8]. It is natural to guess a relation between the spin-2 Fourier component and the holomorphic stress-energy tensor, which is an operator carrying spin 2. The central result of the paper is the justification of this correspondence by proving that the second Fourier components of the above described probabilities satisfy the so-called conformal Ward identities, which are the mathematical formalization of the fact that the stress energy tensor generates conformal transformations. The instrumental tool in our proof is the so-called conformal restriction property [9], which refers not only to SLE, but also to more general random processes on the plane. Actually, conformal restriction has already been used in [10] to derive the Ward identities on the boundary, but we shall implement it differently using a method which is not restricted to work on the boundary only. As a by-product of our analysis, we obtain slightly more general results for the boundary case itself, with a more accurate interpretation in terms of CFT correlators.

The paper is organized as follows. In Section 2 we present our assumptions and the main result of the paper. In Section 3 we describe the SLE problem under consideration, analyzing the probability that a curve passes between the ending points of a segment and the cases in which it suggests the identification with a holomorphic operator in CFT. We also discuss the analogies and differences between the events of passing between the ending points of the segment or intersecting the segment. Section 4 contains the proof of our main result, valid for conformal restriction measures. In Section 5 we specialize the result to $\text{SLE}_{8/3}$ and we interpret it in the CFT language, showing that it corresponds to the Ward identities at $c = 0$. Section 6 discusses the boundary case and possible generalizations of our result to CFT with $c < 0$. Finally, in Section 7 we present our conclusions. The paper also includes four Appendices which present technical results useful for the general discussion.

2 Assumptions and main results

We first state our assumptions. Consider a measure on a random connected set $K \subset \mathbb{H}$ with $\{0, \infty\} \subset \bar{K}$ satisfying conformal restriction. Conformal restriction measures were defined and studied in [9], and their properties will be summarized in Section 4. In particular, they are characterized by a real number $h$, called restriction exponent, which will be defined in [12] and which will explicitly appear in our main result. Many properties of conformal restriction measures are known, but we need to assume some “smoothness” properties of probabilities. Although, to our knowledge, these properties were not fully assessed yet in the literature, it is our expectation that their proof, in the case of SLE, is a matter of a technical analysis, and that for other conformal restriction measures, they are essential for the definition of a local
stress-energy tensor.

More precisely, consider indicator events associated to the set $K$ depending on points $z_j \in \mathbb{H}$: events that the set $K$ is to the right of $z_j$, to the left of $z_j$, or that $z_j$ is inside $K$ (i.e. $z_j$ is included in the filling of $K$), and consider the generic probability

$$P(K \subset \mathbb{H} \setminus D_k, z_1, \ldots, z_l)$$

where $D_k \subset \mathbb{H}$, $\{0, \infty\} \subset \mathbb{H} \setminus D_k$, $\mathbb{H} \setminus D_k$ is topologically the upper half plane with $k$ holes and the boundary $\partial D_k$ is piecewise smooth. Commas represent intersection of events and $z_1, \ldots, z_l$ represent indicator events (the probability is not expected to be a holomorphic function of $z_1, \ldots, z_l$, but for notational convenience we will not write explicitly its dependence on $\bar{z}_1, \ldots, \bar{z}_l$).

By conformal restriction, this only depends on the coordinates in the moduli space of $\mathbb{H} \setminus (D_k \cup \{z_1, \ldots, z_l\})$ with 0 and $\infty$ fixed. These coordinates can be taken as the positions of $l$ points and the central positions and lengths of $k$ horizontal slits in $\mathbb{H}$ (up to an overall scale transformation).

Then, our assumptions amount to a statement of smoothness in moduli space, precisely:

**Assumptions 2.1** With $P(K \subset \mathbb{H} \setminus D_k, z_1, \ldots, z_l)$ and its coordinates in the moduli space as described above, we have

1. The singularities of the first derivative of $P(K \subset \mathbb{H} \setminus D_k, z_1, \ldots, z_l)$ in the moduli space may occur on the hyper-planes corresponding to the situations where 1) any slit or any point touches the real axis, 2) any two or more of the slits or points enter in contact with each other, or 3) the length of any slit is sent to zero. In any direction at points in the moduli space away from the singular planes, and on the singular planes parallel to them, the probability is differentiable at least once.

2. The limit towards a singular plane commutes with the derivative in any direction parallel to the singular plane at that point.

3. The probability behave, when the length $\epsilon$ of one of the slit is sent to 0, as the same probability with this slit missing (denoting the corresponding domain by $D_{k-1}$) plus a correction which is a power of the length of the slit:

$$P(K \subset \mathbb{H} \setminus D_k, z_1, \ldots, z_l) - P(K \subset \mathbb{H} \setminus D_{k-1}, z_1, \ldots, z_l) = O(\epsilon^{2-d})$$

for some $d < 2$. (In the case of $\text{SLE}_{8/3}$, which is a conformal restriction measure, $d$ is known to be $4/3$, the fractal dimension of the curve.)

These assumptions can easily be verified for the particular case of the probability $P(z)$ in $\text{SLE}_{8/3}$ using the arguments of Schramm [4] (here, $P(z)$ can be the probability that the SLE curve be to the right of the point $z$, or the probability that it be to its left). An idea of the general proof (at least for Point 1) could be as follows. Consider first the case of $\text{SLE}_{8/3}$. This is the unique conformal restriction measure that is supported on simple curves. It can be constructed by dynamically growing a curve using a Loewner map with a time-dependent driving term.
proportional to a one-dimensional standard Brownian motion started at 0. This measure has the property that when it is restricted to the curve having a given shape \( \Gamma \) from 0 to any point inside \( \mathbb{H} \), then it is equal to the measure obtained by a conformal transformation, through a Loewner map, from \( \mathbb{H} \setminus \Gamma \) to \( \mathbb{H} \). Ito’s calculus then tells us that the derivative, in the moduli space, of the probabilities considered above in the direction specified by a small Loewner map \( z \mapsto z + dt/z \) exists: this is at the basis of the derivation of the “SLE equation” for such probabilities. In fact, Ito’s calculus tells us more: for every curve \( \Gamma \), the Loewner maps of all sub-curves starting at 0 define a path in the moduli space. Then, the derivatives in the moduli space along all these paths exist. The proof would need to show that taking all curves \( \Gamma \) which can restrict the measure of SLE\(_{8/3}\), one can describe paths such that at any non-singular point in the moduli space, all directions occur. We expect that Assumptions 2.1 also hold for other conformal restriction measures. In those cases, one needs other explicit constructions along the lines discussed above. It is worth noting that Point 3 is probably the most delicate: the set \( K \) cannot be space-filling. We will briefly come back to this in the context of a certain conformal restriction measure constructed (by adding brownian bubbles to SLE) in [9].

In order to state our results, we need to introduce some objects and some notations. We first define a family \( \mathcal{E} \) of simply connected domains in \( \mathbb{H} \) whose members \( E_{w,\epsilon,\theta}[D] \) are parametrized by \( w \in \mathbb{H} \) (that is, with \( \text{Im}(w) > 0 \), \( \epsilon > 0 \) and \( \theta \in [0,2\pi] \), as well as a simply connected domain \( D \) of a certain type, with \( \partial D \) piecewise smooth. More precisely, the members of \( \mathcal{E} \) are defined by

\[
g_{w,\epsilon,\theta}(\mathbb{H} \setminus S_{w,\epsilon}(D)) = \mathbb{H} \setminus E_{w,\epsilon,\theta}[D] \tag{2.1}
\]

where \( S_{w,\epsilon} \) is a conformal map that scales by \( \epsilon \) with center at \( w \): \( S_{w,\epsilon}(z) = w + \epsilon(z - w) \) with \( S_{w,\epsilon}(D) \in \mathbb{H} \). The conformal transformations \( g_{w,\epsilon,\theta} \) are defined by

\[
g_{w,\epsilon,\theta}(z) = z + \frac{\epsilon^2}{16} \frac{e^{2i\theta}}{w - z} + \frac{\epsilon^2}{16} \frac{e^{-2i\theta}}{\overline{w} - z} - \frac{\epsilon^2}{16} \frac{e^{2i\theta}}{w} + \frac{\epsilon^2}{16} \frac{e^{-2i\theta}}{\overline{w}} \tag{2.2}
\]

and we can take any simply connected domain \( D \in S_{w,\epsilon}^{-1}(\mathbb{H}) \) such that the right-hand side of Eq. 2.1 indeed is a subset of \( \mathbb{H} \). In particular, the domain \( S_{w,\epsilon}(D) \) must include the branch points of \( g_{w,\epsilon,\theta} \) that are situated in \( \mathbb{H} \), which means that \( D \) must include the points at the positions \( w \pm \frac{i}{4} e^{i\theta} + O(\epsilon) \) for small \( \epsilon \). Note also that for any \( D \) which strictly contains the disk of radius 1/4 centered at \( w \), there exists an \( \epsilon[D] \) such that for all \( 0 < \epsilon < \epsilon[D] \), \( E_{w,\epsilon,\theta}[D] \) exists. If \( D \) is a disk centered at \( w \) of radius \( b/4 \) for some \( b > 1 \), then the boundary of \( E_{w,\epsilon,\theta}[D] \) describes an ellipse centered at \( w \) of major axis \( (b + \frac{1}{b}) \frac{b}{2} \) and minor axis \( (b - \frac{1}{b}) \frac{b}{2} \), plus a deformation of order \( \epsilon^2 \) of this ellipse:

\[
g_{w,\epsilon,\theta} \left( w + \frac{be}{4} e^{i\alpha+i\theta} \right) = w + \frac{\epsilon}{4} e^{i\theta + i\pi/2} \left( b + \frac{1}{b} \right) \sin \alpha + \frac{\epsilon}{4} e^{i\theta} \left( b - \frac{1}{b} \right) \cos \alpha \tag{2.3}
\]

\[
- \frac{\epsilon^2}{16} \frac{e^{2i\theta}}{w} - \frac{\epsilon^2}{16} \frac{e^{-2i\theta}}{\overline{w}} + \frac{\epsilon^2}{16} \frac{e^{2i\theta}}{\overline{w} - w} + \frac{\epsilon^3}{64} \frac{e^{i\alpha+i\theta}}{(w - \overline{w})^2} + O(\epsilon^4),
\]

where \( \alpha \in [0,2\pi] \). The major axis of the ellipse makes an angle \( \theta \) with respect to the positive imaginary direction. The ellipse becomes, as \( b \to 1 \), a segment of length \( \epsilon \) centered at \( w \) and of angle \( \theta \) with respect to the imaginary direction.
We will consider the event that the set $K$ intersects a member $E_{w,\epsilon,\theta}[D]$ of the family $\mathcal{E}$ described above. In fact, since our results will be independent of the exact form of $D$, we will drop the explicit dependence on $D$. Let us then denote, for $n_1, \ldots, n_k \in \mathbb{Z}$:

$$Q_{n_1, \ldots, n_k}^{(k, l)}(w_1, \ldots, w_k, z_1, \ldots, z_l) = \left(\frac{8}{\pi}\right)^k \lim_{\epsilon_1, \ldots, \epsilon_k \to 0} e^{-|n_1|} \cdots e^{-|n_k|} \int_0^{2\pi} d\theta_1 e^{-i n_1 \theta_1} \cdots \int_0^{2\pi} d\theta_k e^{-i n_k \theta_k} \cdot P(K \cap E_{w_1,\epsilon_1,\theta_1} \neq \emptyset, \ldots, K \cap E_{w_k,\epsilon_k,\theta_k} \neq \emptyset, z_1, \ldots, z_l), \quad (2.4)$$

whenever this limit exists. Although we expect that it does exist for all $n_1, \ldots, n_k \in \mathbb{Z}\setminus\{0\}$, we will only need a subset of these (and our main theorem applies only to a particular case); we introduce this general notation in order to make contact with the motivations which led to our result. In particular, we expect that the numbers $n_i$ correspond to the “spin” and that their absolute values $|n_i|$ correspond to the “scaling dimension” of holomorphic (or antiholomorphic) operators in CFT\(^1\). Hence, we will use the terminology “spin” when referring to the discrete variables labelling Fourier components. We will always denote by $w$ (possibly with an index) the positions of domains of the type described above, and by $z$ (again possibly with an index) the positions of indicator events. We do not assume a priori that these objects are holomorphic functions of $w_1, \ldots, w_k$, but, as for the variables $z_1, \ldots, z_l$, we will omit the dependence on $w_1, \ldots, w_k$ for notational convenience. We also define

$$Q^{(0, l)}(z_1, \ldots, z_l) = P(z_1, \ldots, z_l), \quad Q^{(0, 0)} = 1. \quad (2.5)$$

We then have the following theorem, which is our main result:

**Theorem 2.1** Let $P(K \cap E_{w_1,\epsilon_1,\theta_1} \neq \emptyset, \ldots, K \cap E_{w_k,\epsilon_k,\theta_k} \neq \emptyset, z_1, \ldots, z_l)$ denote a probability of intersection of events in a conformal restriction measure with exponent $h$ on connected subsets $K \in \mathbb{H}$, $\{0, \infty\} \in \mathbb{K}$, with $E_{w,\epsilon,\theta}$ subsets of $\mathbb{H}$ as defined in (2.1), (2.2), and $z_1, \ldots, z_l$ representing $l$ indicator events. With the assumptions 2.1, we have that the limit (2.4) for $n_1 = n_2 = \ldots = n_k = 2$ exists for all $k \geq 0$ and $l \geq 0$, and that it satisfies the following recursion relations:

$$Q_{2, \ldots, 2}^{(k+1, l)}(w_1, \ldots, w_{k+1}, z_1, \ldots, z_l) =$$

$$\sum_{i=1}^{k} \left(\frac{1}{w_{k+1} - w_i} \frac{1}{w_i} \right) \frac{\partial}{\partial w_i} + \sum_{i=1}^{k} \left(\frac{2}{(w_{k+1} - w_i)^2} \right) +$$

$$\sum_{i=1}^{l} \left(\frac{1}{w_{k+1} - z_i} \frac{1}{z_i} \right) \frac{\partial}{\partial z_i} + \sum_{i=1}^{l} \left(\frac{1}{w_{k+1} - \bar{z}_i} \frac{1}{\bar{z}_i} \right) \frac{\partial}{\partial \bar{z}_i} + \frac{h}{w_{k+1}} \cdot Q_{2, \ldots, 2}^{(k, l)}(w_1, \ldots, w_k, z_1, \ldots, z_l) \quad (2.6)$$

for all $k \geq 0$ and $l \geq 0$. In particular, $Q_{2, \ldots, 2}^{(k, l)}(w_1, \ldots, w_k, z_1, \ldots, z_l)$ are meromorphic functions of $w_1, \ldots, w_k$.

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\(^1\)More precisely, in the CFT language, one associates to primary operators the conformal dimensions (real numbers) $h$ and $\bar{h}$, in terms of which the spin is $h - \bar{h}$ and the scaling dimension is $h + \bar{h}$. Holomorphic field are those for which $\bar{h} = 0$ and anti-holomorphic fields are those for which $h = 0$. 

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Remark 2.1 In fact, Theorem 2.1 could be stated in a still more general fashion, in two ways.

First, the map \((2.2)\) above, being part of the definition of the family \(E\) of regions \(E_{w,\epsilon,\theta}\) considered in the theorem, can be modified by adding to it any (finite) number of terms of the type
\[
A \epsilon^p \left( \frac{f(\theta)}{(w-z)^q} + \frac{(f(\theta))^*}{(\bar{w}-z)^q} \right) - \frac{f(\theta)}{w^q} - \frac{(f(\theta))^*}{\bar{w}^q}
\]
for any \(A \in \mathbb{C}, \ 2 < p \in \mathbb{R}, \ q \in \mathbb{N}\) with \(p \geq q + 1\) and any function \(f\) finite on \([0, 2\pi]\). If \(D\) is chosen to be a disk centered at \(w\) of radius strictly larger than \(1/4\), the boundary of \(E_{w,\epsilon,\theta}\) still describes the same ellipse \((2.3)\) as \(\epsilon \to 0\), plus, this time, an additional deformation \(O(\epsilon^{p-q})\). If \(p > q + 1\) for all terms added, the additional deformations are sub-leading, but if \(p = q + 1\) for some terms, this may not be true anymore. Given the freedom in the choice of the initial domain \(D\), it is not clear for us to which extent more freedom is provided by such terms, but it will be clear that our proof below is not affected by the presence of these terms.

Second, we could have replaced the scaling map \(S_{w,\epsilon}\) in \((2.1)\) by the scaling map \(S_{w,\epsilon^r}\) for any \(r > 0\), without affecting the proof of Theorem 2.1. Then, with such a scaling map, we could have added terms to the conformal map \(g\) \((2.2)\) as above, but with the condition \(p \geq q + 1\) replaced by \(p \geq r(q + 1)\) (this is a weaker condition for \(r < 1\)). This provides much more freedom, and generically the circumference of the boundary of the associated domains \(E_{w,\epsilon,\theta}\) will then be proportional to \(\epsilon^r\) as \(\epsilon \to 0\). We will not go into further analysis of this possibility.

3 Motivations from SLE

The aim of this section is to illustrate the ideas which lead to the identification of the stress-energy tensor within the SLE language. The arguments presented here are not rigorous, but have the advantage of applying to other kinds of operators in CFT as well. More complete and rigorous arguments for the identification of the stress-energy tensor will be given in the next sections through the tool of conformal restriction.

Differently to the rest of the paper, where we consider probabilities of \textit{intersecting} some domains included in \(\mathbb{H}\), here we will examine the event of \textit{passing between} the ending points of a segment. The reason is that the SLE equation for the corresponding probability can be easily obtained for any value of \(\kappa\), even in cases where conformal restriction does not hold. We will discuss at the end of the section which are the analogies and differences between the two cases.

Let us consider a chordal SLE\(_\kappa\) process (for \(0 < \kappa < 8\)) on the upper half plane \(\mathbb{H}\) described by complex coordinates \(w, \bar{w}\). Let us also consider the probability\(^2\) \(\mathcal{P}(w_1, w_2, \bar{w}_1, \bar{w}_2)\) of any event that can be fully characterized by two points \(w_1, w_2\) on the upper half plane, in the sense that it is characterized, after any conformal transformation \(G\), by the two points \(G(w_1), G(w_2)\).

\(^2\)Notice that we use the calligraphic style (\(\mathcal{P}\) here, and \(\mathcal{Q}\) below) when referring to events fully characterized by two points, in order to make clear their distinction from events of intersecting some domain.
From Ito’s formula, \( P(w_1, w_2, \bar{w}_1, \bar{w}_2) \) satisfies the equation
\[
\left\{ \frac{\kappa}{2} (\partial_w + \partial_{\bar{w}})^2 + \frac{2}{w} \partial_w + \frac{2}{w} \partial_{\bar{w}} - \left( \frac{1}{w^2} + \frac{1}{w^2} \right) \right\} \partial_x + \left( \frac{1}{w^2} - \frac{1}{w^2} \right) i \partial_{\theta} + O(\epsilon^2) \right\} P(w, \bar{w}, \epsilon, \theta) = 0 . \tag{3.1}
\]

It will be more convenient to parameterize the event by the middle point \( w \) of a straight segment, by its length \( \epsilon \) and by the angle \( \theta \) that it makes with the positive imaginary direction:
\[
w_1 = w - \frac{\epsilon}{2} e^{i \theta} , \quad w_2 = w + \frac{\epsilon}{2} e^{i \theta} .
\]

We will now analyze the leading contributions to the expectation \( P(w, \bar{w}, \epsilon, \theta) \) of such an event as \( \epsilon \to 0 \). Assuming that each of the Fourier modes of the probability, parameterized by the “spin” \( n \) and defined as
\[
\tilde{Q}_n(w, \bar{w}, \epsilon) \equiv \int_0^{2\pi} d\theta e^{-in\theta} P(w, \bar{w}, \epsilon, \theta) ,
\]
vanishes with a power law \( \epsilon^{\alpha_n} \) as \( \epsilon \to 0 \), we can use \( \partial_x = O(\epsilon^{-1}) \) to extract the leading order of eq. 3.1:
\[
\left\{ \frac{\kappa}{2} (\partial_w + \partial_{\bar{w}})^2 + \frac{2}{w} \partial_w + \frac{2}{w} \partial_{\bar{w}} - \left( \frac{1}{w^2} + \frac{1}{w^2} \right) \right\} \tilde{Q}_n(w, \bar{w}, \epsilon) = 0 . \tag{3.3}
\]

Performing a Fourier transform diagonalizes the operator \( \partial_\theta \), so that the Fourier modes satisfy, to leading order,
\[
\left\{ \frac{\kappa}{2} (\partial_w + \partial_{\bar{w}})^2 + \frac{2}{w} \partial_w + \frac{2}{w} \partial_{\bar{w}} - \left( \frac{1}{w^2} + \frac{1}{w^2} \right) \right\} \tilde{Q}_n(w, \bar{w}, \epsilon) = 0 , \tag{3.4}
\]
where the corrections will be described below. It is easy to check that
\[
\tilde{Q}_n(w, \bar{w}, \epsilon) = c_n \epsilon^{\alpha_n} w^{\beta_n} (w - \bar{w})^{\gamma_n} , \tag{3.5}
\]
with
\[
\alpha_n = \frac{\kappa - 8 - n}{2\kappa} , \quad \beta_n = \frac{\kappa - 8 + n}{2\kappa} , \quad \gamma_n = \frac{(8 - \kappa)^2 - \kappa^2 n^2}{8\kappa} , \quad x_n = 1 - \frac{\kappa}{8} + \frac{\kappa}{8} n^2 , \tag{3.6}
\]
satisfies eq. 3.4. In Appendix A, we justify this choice of solution for the events that the SLE curve passes between the two points (that is, to the left of \( w_1 \) and to the right of \( w_2 \), or viceversa). Note that this does not determine the actual probability corresponding to each of these events until one can fix the constants \( c_n \). As expected, the lowest scaling exponent is \( x_0 = 2 - d_f \), where \( d_f = 1 + \frac{\kappa}{8} \) is the fractal dimension of SLE.

The function gives the correct solution for the \( n \)-th Fourier component up to \( O(\epsilon^{\alpha_n}) \) only if the terms neglected in 3.3 contribute to higher order in \( \epsilon \). This is not automatically

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\( ^3 \)We use the notation *”* to indicate that we keep the full \( \epsilon \)-dependence of the Fourier components, contrary to taking the limit for \( \epsilon \to 0 \) as in 3.1.
guaranteed, since the discarded terms induce a mixing of Fourier components. By inspecting the structure of equation (3.1), it is easy to see that the corrections to (3.3) only contain terms of the form $\left(\varepsilon^2 e^{\pm 2i\theta}\right)^m$, with $m \geq 1$. As a consequence, (3.4) gets additional contributions of the form $\varepsilon^m D\tilde{Q}_n \equiv 2m$, where $D$ is some differential operator of order $O(1)$. Therefore, (3.5, 3.6) is the actual solution only for the values of $n$ such that

$$x_n < 2m + x_{n-2m} \quad \forall \ m \geq 1 \quad \text{such that} \quad c_{n-2m} \neq 0 .$$

When the relation

$$\kappa = \frac{8}{n+1}$$

holds, (3.5) simplifies to the purely holomorphic function

$$\tilde{Q}_n(w, \varepsilon) = \text{const} \times \left(\frac{\varepsilon}{w}\right)^n ,$$

where spin and scaling dimension are equal. This suggests a CFT interpretation of the leading order in $\varepsilon$ of the event in terms of purely holomorphic fields, whose physical meaning may be inferred from relation (3.8). For instance, the holomorphic probability with $n = 1$ appears at $\kappa = 4$, which is known to represent the level lines of a free boson, where the current is a holomorphic field with precisely spin 1. Another interesting example is given by $n = \frac{1}{2}$ and $\kappa = \frac{16}{3}$, suggestive of a fermionic field in the Fortuin-Kasteleyn representation of the Ising model. From the SLE point of view, the latter value of the spin can naturally occur by imposing conditions on the winding of the SLE curve around the two points; this has the effect of increasing the range of $\theta$ beyond which the probability is periodic. Depending on these conditions, the Fourier modes of the probability $P(w, \bar{w}, \varepsilon, \theta)$ may be nonzero only for even spins, or only for integer spins, or only for half-integer spins, etc.

The value $n = 2$ corresponds to the case of interest in the present paper. In the next Sections we shall analyse the case $n = 2$ and $\kappa = \frac{8}{3}$, and we shall justify the identification of $\tilde{Q}_2$ with a CFT correlation function involving the stress-energy tensor.

As already anticipated, however, in the following we will be interested in the probability $P_{\text{segm}}(w, \bar{w}, \varepsilon, \theta)$ of intersecting the small segment (instead of passing in between its two ending points), and its Fourier components $\tilde{Q}_{\text{segm}}^n(w, \bar{w}, \varepsilon)$, defined as in (3.2). At leading order in $\varepsilon \to 0$, $P_{\text{segm}}(w, \bar{w}, \varepsilon, \theta)$ satisfies eq. (3.3) as well. This can be seen by acting on $P_{\text{segm}}(w, \bar{w}, \varepsilon, \theta)$ with the Loewner map and using Ito’s formula, together with the transformation property

$$P_{\text{segm}}(w, \bar{w}, \varepsilon, \theta) \to P_{\text{segm}}\left(G(w), G(\bar{w}), |\partial G(w)|\varepsilon, \theta + \text{arg}(\partial G(w))\right)$$

which holds at leading order in $\varepsilon$ for any conformal map $G$, since locally, a conformal transformation is a combination of a translation, a rotation and a scale transformation. Obviously, deformations of the segment induced by the conformal mapping will alter the higher order structure of eq. (3.3), but they do not affect the leading order behaviour of $\tilde{Q}_{\text{segm}}^2$ for $\kappa = 8/3$. By Theorem 2.1 this is true if the segment is replaced by a region $E_{w,\varepsilon,\theta}$, which can be chosen to be a very elongated ellipse, close to a segment of length $\varepsilon$, plus deformations of order $\varepsilon^2$, as described in the previous section. In Appendix B we argue that these deformations do not affect the leading order of $\tilde{Q}_{\text{segm}}^2$. 
4 The general result from conformal restriction

The case $n = 2$ of the result discussed in Section 3 is particularly interesting, since the value 2 is the spin of the stress-energy tensor in conformal field theory. The corresponding value $\kappa = \frac{8}{3}$ is also peculiar, being the one at which SLE enjoys the property of conformal restriction. We will show that this property alone implies Eqs. (2.6), which are of the nature of the conformal Ward identities found in conformal field theory. In Section 5 we will use this and other results in order to relate these objects to certain type of correlation functions in conformal field theory involving the stress-energy tensor. For now, we first recall a more general family of measures satisfying conformal restriction, of which one member is SLE$_{8/3}$.

4.1 Conformal restriction measures

Consider a measure $\mu$ on connected subsets $K \subset \mathbb{H}$ with $\{0, \infty\} \subset \bar{K}$. The measure satisfies conformal restriction if

$$S \cdot \mu = \mu$$

$$\mu|_{K \subset \mathbb{H} \setminus D} = \Phi_D^{-1} \cdot \mu$$

(4.1)

where $S$ is a scale transformation with center at 0, $D \subset \mathbb{H}$ is such that $\mathbb{H} \setminus D$ is simply connected and contains 0 and $\infty$, and $\Phi_D : \mathbb{H} \setminus D \to \mathbb{H}$ is a conformal map which removes $D$ and preserves 0 and $\infty$. By normalizing the map $\Phi_D$ such that $\Phi_D(z) \sim z$ as $z \to \infty$, (4.1) implies

$$P(K \subset \mathbb{H} \setminus D) = [\Phi_D(0)]^h$$

(4.2)

where $h$ is called the restriction exponent of $K$. In particular, SLE$_{8/3}$ has been proven to satisfy conformal restriction with $h = \frac{5}{8}$.

It is important to realize that conformal restriction can be seen as a combination of conformal invariance and a restriction property. Indeed, if we use the symbol $\mu_{\mathbb{H}}$ to represent measures on connected sets $K \subset \mathbb{H}$ connecting 0 to $\infty$, then it is natural to take conformal invariance to state that $\mu_{\mathbb{H} \setminus D} = \Phi_D^{-1} \cdot \mu_{\mathbb{H}}$, and restriction to state that $\mu_{\mathbb{H} \setminus D} = \mu_{\mathbb{H} \setminus D}'$. In other words, conformal invariance and the restriction property can be seen as two different ways of relating probabilities defined on the domains $\mathbb{H}$ and $\mathbb{H} \setminus D$, and the fact that these two ways should lead to the same result gives a strong constraint on the measure, which is conformal restriction.

In the following, however, we will need to consider the case when $\mathbb{H} \setminus D$ is not simply connected. The conformal restriction property has recently been considered in multiply-connected domains of the type $\mathbb{H} \setminus D$ [11][12]. It was verified that

$$\mu|_{K \subset \mathbb{H} \setminus D} = G^{-1} \cdot \mu|_{K \subset \mathbb{H} \setminus D'}$$

(4.3)

if $D \subset \mathbb{H}$ and $D' \subset \mathbb{H}$ are related by $G(\mathbb{H} \setminus D) = \mathbb{H} \setminus D'$ for some conformal transformation $G$ preserving 0 and $\infty$ (both points also included in $\mathbb{H} \setminus D$). This can again be viewed as a combination of conformal invariance and restriction: $\mu_{\mathbb{H} \setminus D} = G^{-1} \cdot \mu_{\mathbb{H} \setminus D'}$ and $\mu_{\mathbb{H} \setminus D} = \mu_{\mathbb{H} \setminus D}$.
From the viewpoint of statistical models, this is very natural since lattice models certainly admit a description on multiply connected domains. For instance, the continuum limit of the critical $O(n)$ model at $n = 0$, if it exists, should still satisfy conformal invariance for conformal transformations relating domains of this type, and should exhibit the restriction property relating probabilities on $\mathbb{H} \setminus D$ to conditioned probabilities on $\mathbb{H}$.

In this case, conformal invariance and the restriction property do not form two different ways of relating the same pair of domains, since the image $\mathbb{H} \setminus D'$ of $\mathbb{H} \setminus D$ under a conformal transformation cannot be anymore the whole $\mathbb{H}$. However, their combination still provides non-trivial constraints, essentially because there are more conformal transformations $\mathbb{H} \setminus D \to \mathbb{H} \setminus D'$ relating domains of this type than there are conformal transformations preserving $\mathbb{H}$. From a pragmatic point of view, one can define by restriction probabilities on $\mathbb{H} \setminus D$ where $D \subset \mathbb{H}$ and one can verify that the defined probabilities are related to each other by conformal invariance. Note that such a definition of probabilities on multiply-connected domains would also be possible for any measure, not necessarily having the conformal restriction property, like SLE$_\kappa$ for generic $\kappa$. But for $\kappa \neq 8/3$, we would not expect conformal invariance to hold on the resulting probabilities (for conformal transformations that do not map $\mathbb{H}$ to itself).

In much the same way that (4.1) implies (4.2), it was shown [12] that (4.3) implies

$$P(K \subset \mathbb{H} \setminus D) = [G'(0)]^h P(K \subset \mathbb{H} \setminus D')$$

(4.4)

where $G : \mathbb{H} \setminus D \to \mathbb{H} \setminus D'$ is such that $G(0) = 0$ and $G(z) \sim z$ as $z \to \infty$.

### 4.2 Single slit

We now show Theorem 2.1 in the case $k = 0$, under the assumptions 2.1. The proof requires the use of the conformal transformation (2.2), which is singular at the location $w$ of the center of the ellipse. This is natural from the intuition that the insertion of a stress-energy tensor inside a correlation function, in conformal field theory, can be seen as resulting from a (non-globally defined) conformal transformation that is the identity at infinity and that has a pole at the point of insertion.

**Proof of Theorem 2.1 in the case $k = 0$.** We will begin by using (4.4) to calculate $Q_{2}^{(1,0)}$ and thus prove (2.6) for both $k = 0$ and $l = 0$. With $G = g_{w,\epsilon,\theta}$ and $D$ replaced by $D_{w,\kappa} = S_{w,\kappa}(D)$ as in Sect. 2, Eq. (4.4) reads

$$P(K \subset \mathbb{H} \setminus D_{w,\kappa}) = [g'_{w,\epsilon,\theta}(0)]^h \{1 - P(w, \epsilon, \theta)\}$$

where we have introduced the more compact notation $P(w, \epsilon, \theta) = P(K \cap E_{w,\epsilon,\theta} \neq \emptyset)$ (this probability is not a holomorphic function of $w$, but for notational convenience, here an below we do not write explicitly the dependence on $\bar{w}$). By applying $\int d\theta e^{-2i\theta}$ to both sides of this equation, using the fact that the left hand side is independent of $\theta$ and expanding in $\epsilon$ (with point 3 of our assumptions 2.1), we obtain

$$Q_{2}^{(1,0)}(w) = \frac{h}{w^2}.$$  

(4.5)
Note that this leading behavior as \( \epsilon \to 0 \) has the same dependence on \( w \) as that of \( \tilde{Q}_2(w, \epsilon) \) given by (3.9) at \( \kappa = \frac{8}{3} \) (up to a normalization). In a similar fashion we obtain, for a generic \( n \in \mathbb{Z} \setminus \{0\} \), that

\[
\int_0^{2\pi} d\theta e^{-2in\theta} P(w, \epsilon, \theta) = O(\epsilon^{2|n|}), \quad \int_0^{2\pi} d\theta e^{-i(2n+1)\theta} P(w, \epsilon, \theta) = 0, \tag{4.6}
\]

which implies that \( Q^{(1,0)}_{2n}(w) \) exists.

Consider again the conformal transformation (2.2). From invariance of the restricted probabilities under conformal mappings, we have (we denote \( \{z\} = z_1, \ldots, z_l \) and \( P(\cdots | K \subset \mathbb{H} \setminus D) = P(\cdots) \))

\[
P(\{z\})_{\mathbb{H} \setminus D_{w,\epsilon}} = P(g_{w,\epsilon,\theta}(\{z\}))_{\mathbb{H} \setminus E_{w,\epsilon,\theta}} = P(\{z\})_{\mathbb{H} \setminus E_{w,\epsilon,\theta}} + \frac{\epsilon^2}{16} \sum_i \left( \frac{1}{w - z_i} \partial_{z_i} + \frac{1}{w - \bar{z}_i} \partial_{\bar{z}_i} - \frac{1}{w} (\partial_{z_i} + \partial_{\bar{z}_i}) \right) P(\{z\})
\]

\[+ \frac{\epsilon^2}{16} e^{-2i\theta} \sum_i \left( \frac{1}{w - z_i} \partial_{z_i} + \frac{1}{w - \bar{z}_i} \partial_{\bar{z}_i} - \frac{1}{w} (\partial_{z_i} + \partial_{\bar{z}_i}) \right) P(\{z\}) + o(\epsilon^2).
\]

In order to understand the second step, consider the expression

\[
\frac{P(g_{w,\epsilon,\theta}(\{z\}))_{\mathbb{H} \setminus E_{w,\epsilon,\theta}} - P(\{z\})_{\mathbb{H} \setminus E_{w,\epsilon,\theta}}}{\epsilon^2}.
\]

The limit as \( \epsilon \to 0 \) exists by point 1 of our assumptions (2.1). Also, the limit as \( \epsilon' \to 0 \) and the limit as \( \epsilon \to 0 \) are independent by point 2. Hence, we can send first \( \epsilon' \to 0 \) in order to evaluate the expression using point 3; this gives the terms with derivatives with respect to \( \{z\} \). But we obtain the same value setting first \( \epsilon' = \epsilon \) then sending \( \epsilon \to 0 \). This explains the second step.

From the definition of restricted probabilities, we can write

\[
P(\{z\})_{\mathbb{H} \setminus E_{w,\epsilon,\theta}} = \frac{P(\{z\}) - P(\{z\}, w, \epsilon, \theta)}{1 - P(w, \epsilon, \theta)},
\]

where we have introduced the more compact notation \( P(\{z\}, w, \epsilon, \theta) = P(\{z\}, K \cap E_{w,\epsilon,\theta} \neq \emptyset) \).

This implies

\[
P(\{z\})_{\mathbb{H} \setminus D_{w,\epsilon}} = P(\{z\}) - P(\{z\}, w, \epsilon, \theta) + P(w, \epsilon, \theta) P(\{z\}) + \sum_{n=1}^{\infty} P(\{z\}) P(w, \epsilon, \theta)^n - \sum_{n=1}^{\infty} P(\{z\}, w, \epsilon, \theta) P(w, \epsilon, \theta)^n
\]

\[+ \frac{\epsilon^2}{16} e^{2i\theta} \sum_i \left( \frac{1}{w - z_i} \partial_{z_i} + \frac{1}{w - \bar{z}_i} \partial_{\bar{z}_i} - \frac{1}{w} (\partial_{z_i} + \partial_{\bar{z}_i}) \right) P(\{z\})
\]

\[+ \frac{\epsilon^2}{16} e^{-2i\theta} \sum_i \left( \frac{1}{w - z_i} \partial_{z_i} + \frac{1}{w - \bar{z}_i} \partial_{\bar{z}_i} - \frac{1}{w} (\partial_{z_i} + \partial_{\bar{z}_i}) \right) P(\{z\}) + o(\epsilon^2).
\]
Applying to eq. (4.7) the integral \(2\pi \int_0^\theta d\theta e^{-2i\theta}\) and using the fact that the left-hand side is independent of \(\theta\), we obtain
\[
0 = -\int_0^{2\pi} d\theta e^{-2i\theta} P(\{z\}, w, \epsilon, \theta) + \frac{\pi}{8} \epsilon^2 Q_2^{(1,0)}(w) P(\{z\}) + \frac{\pi}{8} \epsilon^2 \sum_i \left( \frac{1}{w - z_i} \partial z_i + \frac{1}{w - \bar{z}_i} \partial \bar{z}_i - \frac{1}{w} (\partial z_i + \partial \bar{z}_i) \right) P(\{z\}) + o(\epsilon^2)
\]
where we used (4.6) (for \(n = 1\)) in the first line. We used the fact that the second line of (4.7) contributes only to \(o(\epsilon^2)\). In order to see this, consider the first sum and expand \(P(w, \epsilon, \theta)\) in its Fourier modes (in the variable \(\theta\)). Under \(2\pi \int_0^{2\pi} d\theta e^{-2i\theta}\), the terms left are those whose total spin (the sum of the spins of their factors) is 2. By (4.6), the leading of these terms as \(\epsilon \to 0\) are those for which all factors have zero spin except one factor; this gives a contribution \(o(\epsilon^2)\) since there is at least two factors (and using point 3 of the assumptions (2.1)). Consider now the second sum on the second line of (4.7). Again using Fourier modes, now the leading terms will be those for which the total spin of the Fourier components of \(P(w, \epsilon, \theta)\) is 2, 0 or -2. In the case 2 and -2, using (4.6) and point 3 of assumptions (2.1), the contributions are \(o(\epsilon^2)\). In the case 0, the contributions are \(o(\epsilon) \cdot \int_0^{2\pi} d\theta e^{-2i\theta} P(\{z\}, w, \epsilon, \theta)\) which is of higher order than the first term in the first line of (4.8) and hence gives contributions to \(o(\epsilon^2)\). Using further the result (4.5), we finally obtain
\[
Q_2^{(1,0)}(w, \{z\}) = \sum_i \left( \frac{1}{w - z_i} \partial z_i + \frac{1}{w - \bar{z}_i} \partial \bar{z}_i - \frac{1}{w} (\partial z_i + \partial \bar{z}_i) \right) P(\{z\}) + o(\epsilon^2),
\]
which is the special case \(k = 0\) of (2.6).

### 4.3 Multiple slits

In order to prove Theorem 2.1 for \(k \geq 1\), we derive the way by which the quantity
\[
Q_{2,...,2}^{(k,l)}(w_1, \ldots, w_k, \{z\} | K \subset \mathbb{H} \setminus D),
\]
for some simply connected \(D \subset \mathbb{H}\) bounded away from \(w_1, \ldots, w_k\) with \(\partial D\) piecewise smooth\(^4\) (with a straightforward extension of the notation introduced in (2.1)), transforms under a conformal transformation that maps \(\mathbb{H} \setminus D\) to a subset of \(\mathbb{H}\). More precisely, we show below the following proposition.

**Proposition 4.1** The following transformation property holds
\[
Q_{2,...,2}^{(k,l)}(w_1, \ldots, w_k, \{z\} | K \subset \mathbb{H} \setminus D) = \left( \prod_{i=1}^{k} [G'(w_i)]^2 \right) Q_{2,...,2}^{(k,l)}(G(w_1), \ldots, G(w_k), \{G(z)\} | K \subset \mathbb{H} \setminus D'),
\]
for \(G : \mathbb{H} \setminus D \to \mathbb{H} \setminus D'\).

\(^{4}\)as assumed for the domains considered in Section 2.
Let us first prove Theorem 2.1 in the general case using this proposition.

**Proof of Theorem 2.1 in the general case.** Proposition 4.1 is enough to prove 2.6 in the general case. Indeed, we just have to repeat the derivation of equation (2.6) done in the previous sub-section in the case \( k = 0 \), but using \( Q_{2}^{(k,l)}(w_{1}, \ldots , w_{k}, \{ z \} | K \subset \mathbb{H} \setminus D_{w_{k+1}}, \varepsilon) \) instead of \( P(\{ z \} | K \subset \mathbb{H} \setminus D_{w}, \varepsilon) \) as a starting object, and using (4.10) with \( G = g_{w,\varepsilon,\theta} \) instead of invariance under the transformation \( g_{w,\varepsilon,\theta} \) as a starting step. The rest of the derivation goes along similar lines, using our assumptions 2.1 in order to obtain derivatives with respect to \( w_{1}, \ldots , w_{k} \) as well as with respect to \( z_{1}, \ldots , z_{l} \), and we immediately find

\[
Q_{2,\ldots ,2}^{(k+1,l)}(w_{1}, \ldots , w_{k}, w_{k+1}, z_{1}, \ldots , z_{l}) = \sum_{i=1}^{k} \left( \frac{1}{w_{k+1} - w_{i}} - \frac{1}{w_{k+1}} \right) \frac{\partial}{\partial w_{i}} + \sum_{i=1}^{k} \left( \frac{1}{w_{k+1} - \bar{w}_{i}} - \frac{1}{w_{k+1}} \right) \frac{\partial}{\partial \bar{w}_{i}} + \sum_{i=1}^{k} \frac{2}{(w_{k+1} - w_{i})^2} \cdot \sum_{i=1}^{l} \left( \frac{1}{w_{k+1} - z_{i}} - \frac{1}{w_{k+1}} \right) \frac{\partial}{\partial z_{i}} + \sum_{i=1}^{l} \left( \frac{1}{w_{k+1} - \bar{z}_{i}} - \frac{1}{w_{k+1}} \right) \frac{\partial}{\partial \bar{z}_{i}} + \frac{h}{w_{k+1}^2} \right). 
\]

Recursively using the fact that \( Q_{2,\ldots ,2}^{(k,l)}(w_{1}, \ldots , w_{k}, z_{1}, \ldots , z_{l}) \) is analytic in \( w_{1}, \ldots , w_{k} \), we obtain 2.6 and Theorem 2.1.

**Remark 4.2** It is worth mentioning that an alternative proof of the multiple Ward identity that mimics the proof of the single Ward identity in sub-Section 4.2 could be obtained along the following lines. First, find a conformal map with simple poles at the positions \( w_{1}, \ldots , w_{k} \) and parameterized by the variables \( \epsilon_{1}, \ldots , \epsilon_{k} \) and \( \theta_{1}, \ldots , \theta_{k} \) in such a way that the domain \( \mathbb{H} \setminus (D_{1} \cup \cdots \cup D_{k}) \), for some \( D_{1}, \ldots , D_{k} \) disjoint simply connected regions of \( \mathbb{H} \), is mapped into \( \mathbb{H} \setminus (E_{w_{1},\epsilon_{1},\theta_{1}} \cup \cdots \cup E_{w_{k},\epsilon_{k},\theta_{k}}) \). Then, apply the techniques of sub-Section 4.2 by taking the spin-2 Fourier components for all variables \( \theta_{1}, \ldots , \theta_{k} \) and by looking at the leading order when \( \epsilon_{1} \to 0, \ldots , \epsilon_{k} \to 0 \) independently (this should be allowed by the conformal map). Finally, observe the multiple Ward identity (2.6) by comparing what is obtained with \( k \to k+1 \) and what is obtained with \( k \). In Appendix C we present a part of the proof along these lines by giving the conformal map that gives the multiple Ward identity for \( k = 2 \). Unfortunately, we were as of yet unable to show that this conformal map is able to produce the region \( \mathbb{H} \setminus (E_{w_{1},\epsilon_{1},\theta_{1}} \cup E_{w_{2},\epsilon_{2},\theta_{2}}) \); we believe that for this, one needs to use the freedom of the choice of conformal maps along the lines of Remark 2.1.

**Proof of Proposition 4.1** We must first derive some general properties of maps \( f \) from boundaries \( \partial D \) of disjoint unions of simply connected domains \( D = \bigcup D_{i} \in \mathbb{H} \) (such that \( \partial D_{i} \) are piecewise smooth) to the complex numbers, defined by

\[
f(\partial D) = P(\{ z \}, K \subset \mathbb{H} \setminus D | K \subset \mathbb{H} \setminus \mathcal{D})
\]
for simply connected \( D \in \mathbb{H} \) bounded away from 0, from \( \infty \), and from \( D \). The first property is as follows. From our assumptions the following limit exists:

\[
\lim_{\eta \to 0} \frac{f((\text{id} + \eta H)(\partial D)) - f(\partial D)}{\eta} = \int_0^1 ds \, H(\partial D(s)) (\Delta_s f)(\partial D) + c.c. \tag{4.12}
\]

where \( H \) is any real-analytic conformal map that maps \( \mathbb{H} \setminus D \) to to another domain of \( \mathbb{H} \) of the same topology (with one hole). In fact, this limit can be written as appropriate derivatives with respect to the coordinates \( x_i \) in the moduli space of \( \mathbb{H} \setminus (\{z\} \cup D \cup D) \) with 0 and \( \infty \) fixed. There is a finite number of derivatives, and, choosing appropriate coordinates, every derivative \( \partial / \partial x_i \) can be obtained by an appropriate small and smooth deformations \( \eta H_i \) of \( \partial D \). The coefficients of these derivatives are linear in \( H \) (since, first, they are not singular when \( H \) is zero anywhere on \( \partial D \), and second, one can replace \( \eta \mapsto q\eta \) to see that the result scales linearly with \( H \)) and they depend on \( H \) only through the image of \( \partial D \) under \( H \). Hence, they are linear functionals of \( H \) supported on \( \partial D \) and can be written as integrals on \( \partial D \) of \( H \) times appropriate functions making the projection onto \( H_i(\partial D) \). Putting these integrals together, we can write

\[
\lim_{\eta \to 0} \frac{f((\text{id} + \eta H)(\partial D)) - f(\partial D)}{\eta} = \int_0^1 ds \, H(\partial D(s)) (\Delta_s f)(\partial D) + c.c. \tag{4.13}
\]

where \( s \) is the normalized length along \( \partial D \) starting from any point on \( \partial D \) and going counterclockwise on each component in a fixed order, normalized to a total length of 1, and \( \partial D(s) \) is the associated value of \( \partial D \). This equation essentially defines the new maps \( \Delta_s f, \tilde{\Delta}_s f \) (for all \( s \)).

For the second property that we will need, consider, for \( G : \mathbb{H} \setminus \tilde{D} \to \mathbb{H} \setminus \tilde{D}' \) for some \( \tilde{D} \subset D \),

\[
f(G((\text{id} + \eta H)(\partial D))) = f(F(G(\partial D))) \tag{4.14}
\]

where

\[
F = G \circ (\text{id} + \eta H) \circ G^{-1} = \text{id} + \eta(G'(G^{-1})(H \circ G^{-1}) + O(\eta^2)) \tag{4.15}
\]

We can write

\[
\lim_{\eta \to 0} \frac{f(F(G(\partial D))) - f(G(\partial D))}{\eta} = \int_0^1 ds \, [G'(\partial D(s))]^2 \, H(\partial D(s)) (\Delta_s f)(G(\partial D)) + c.c. \tag{4.16}
\]

where c.c. means “complex conjugate” and we used

\[
[G(\partial D)(\tilde{s})] = G(\partial D(s)) \Rightarrow d\tilde{s} = G'(\partial D(s))ds \tag{4.17}
\]

Hence, the map \( f \circ G \) has the same property as \( f \), that is,

\[
\lim_{\eta \to 0} \frac{(f \circ G)((\text{id} + \eta H)(\partial D)) - (f \circ G)(\partial D)}{\eta} = \int_0^1 ds \, H(\partial D(s)) (\Delta_s (f \circ G))(\partial D) + c.c. \tag{4.18}
\]

with

\[
\Delta_s (f \circ G) = [G'(\partial D(s))]^2 (\Delta_s f) \circ G \tag{4.19}
\]
Now, using (4.13) with $\mathcal{D} = D_{w, \epsilon}$ and with $\text{id} + \eta H = g_{w, \epsilon, \theta}$, $\eta = \epsilon^2$, we can easily derive an expression similar to (4.20) for the quantity $Q_2^{(1, l)}(w, \{z\}, \gamma \subset \mathbb{H} \setminus D)$. In fact, it is convenient to keep the starting point $a \in \mathbb{R}$ of the curve arbitrary for now, so that we have

$$Q_2^{(1, l)}(w, \{z\}, \gamma \subset \mathbb{H} \setminus D; a)$$

$$= \left[ \sum_i \left( \frac{1}{w - z_i} \partial G_i + \frac{1}{w - z_i} \partial G_i \right) + \frac{1}{w - a} (\partial_a + \partial_a) \right] P(\{z\}, \gamma \subset \mathbb{H} \setminus D; a) +$$

$$\int_0^1 ds \frac{1}{w - \partial D(s)} (\Delta_s P)(\{z\}, \gamma \subset \mathbb{H} \setminus D; a) + \int_0^1 ds \frac{1}{w - \partial D(s)} (\Delta_s P)(\{z\}, \gamma \subset \mathbb{H} \setminus D; a).$$

We can obtain a similar expression for $Q_2(w, \{G(z)\}, \gamma \subset G(\mathbb{H} \setminus D); G(a))$ where $G$ is a real analytic conformal transformation that maps $\mathbb{H} \setminus D$ to $\mathbb{H} \setminus D'$ for some $D' \subset \mathbb{H}$ simply connected, with $G(z) \sim z$ at $z \to \infty$ (but generically, $G(a) \neq a$):

$$Q_2^{(1, l)}(w, \{G(z)\}, \gamma \subset G(\mathbb{H} \setminus D); G(a))$$

$$= \left[ \sum_i \left( \frac{1}{w - G(z_i)} \partial G(z_i) + \frac{1}{w - G(z_i)} \partial G(z_i) \right) + \frac{1}{w - G(a)} (\partial G(a) + \partial G(a)) + \frac{h}{(w - G(a))^2} \right] \cdot$$

$$P(\{G(z)\}, \gamma \subset G(\mathbb{H} \setminus D); G(a)) +$$

$$\int_0^1 ds \frac{G'(\partial D(s))}{w - G(\partial D(s))} (\Delta_s P)(\{G(z)\}, \gamma \subset G(\mathbb{H} \setminus D); G(a)) +$$

$$\int_0^1 ds \frac{G'(\partial D(s))}{w - G(\partial D(s))} (\Delta_s P)(\{G(z)\}, \gamma \subset G(\mathbb{H} \setminus D); G(a))$$

Using relation (4.19), the last two lines can be written

$$\int_0^1 ds \frac{1}{G'(\partial D(s))} \frac{1}{G'(\partial D(s))} (\Delta_s (P \circ G))(\{z\}, \gamma \subset \mathbb{H} \setminus D; a) +$$

$$\int_0^1 ds \frac{1}{G'(\partial D(s))} \frac{1}{G'(\partial D(s))} (\Delta_s (P \circ G))(\{z\}, \gamma \subset \mathbb{H} \setminus D; a)$$

where $P \circ G$ means the map from $\{z\}, \partial D, a$ to $[0, 1]$ given by $P(\{G(z)\}, \gamma \subset G(\mathbb{H} \setminus D); G(a))$, and as before, for the purpose of the symbol $\Delta_s$, it is regarded as a function of $\partial D$. Now consider $G$ such that $G(a) = a$ so that we can use (4.19): $P(\{G(z)\}, \gamma \subset G(\mathbb{H} \setminus D); G(a)) = (G'(a))^{-h} P(\{z\}, \gamma \subset \mathbb{H} \setminus D; a)$. Hence, we have

$$(G'(a))^h Q_2^{(1, l)}(w, \{G(z)\}, \gamma \subset G(\mathbb{H} \setminus D); a)$$

$$= \left[ \sum_i \left( \frac{1}{w - G(z_i)} G'(z_i) \partial z_i + \frac{1}{w - G(z_i)} G'(z_i) \partial z_i \right) +$$

$$+ \frac{1}{w - a} G'(a) (\partial_a + \partial_a) - \frac{h}{w - a} G'(a)^2 + \frac{h}{(w - a)^2} \right] P(\{z\}, \gamma \subset \mathbb{H} \setminus D; a) +$$

$$\int_0^1 ds \frac{1}{w - G(\partial D(s))} \frac{1}{G'(\partial D(s))} (\Delta_s P)(\{z\}, \gamma \subset \mathbb{H} \setminus D; a) +$$

$$\int_0^1 ds \frac{1}{w - G(\partial D(s))} \frac{1}{G'(\partial D(s))} (\Delta_s P)(\{z\}, \gamma \subset \mathbb{H} \setminus D; a)$$

(4.20)
Consider the analytical properties in $w$ of the last expression. It gives a real-analytic function of $w$ in $G(\mathbb{C} \setminus (D \cup \bar{D})) = \mathbb{C} \setminus (E \cup \bar{E})$ with simple poles at $G(z_i)$’s and $G(\bar{z}_i)$’s, and a double pole at $a$ (one can check that there is no pole at $\infty$), the residues being directly read off. For $w \in E$ or $w \in \bar{E}$, the expression gives an analytic function. The difference between the expression near $\partial E$ (at $w = G(\partial D(s))$, say) outside of $E$ and the expression near $\partial E$ inside of $E$ is

$$2\pi i [(G^{-1})'(w)]^2 [\partial D'(s)]^{-1} (\Delta_s P)(\{z\}, \gamma \subset \mathbb{H} \setminus D; a)$$

A similar result holds near $\partial \bar{E}$. These properties completely determine the analytic functions of $w$ on both sides of the cuts at $\partial E$ and at $\partial \bar{E}$.

Finally, consider the expression

$$[(G^{-1})'(w)]^2 \cdot \left\{ \sum_i \left( \frac{1}{G^{-1}(w) - z_i} \frac{1}{\partial z_i} + \frac{1}{G^{-1}(w) - \bar{z}_i} \frac{1}{\partial \bar{z}_i} \right) + \frac{1}{G^{-1}(w) - a} (\partial_a + \partial_{\bar{a}}) + \frac{h}{(G^{-1}(w) - a)^2} \right\} \cdot P(\{z\}, \gamma \subset \mathbb{H} \setminus D; a) + \int_0^1 ds \frac{1}{G^{-1}(w) - \partial D(s)} (\Delta_s P)(\{z\}, \gamma \subset \mathbb{H} \setminus D; a) + \int_0^1 ds \frac{1}{G^{-1}(w) - \partial \bar{D}(s)} (\bar{\Delta}_s P)(\{z\}, \gamma \subset \mathbb{H} \setminus D; a) \right\}.$$ 

It is a simple matter to check that it has the same singularity and cut structure as (4.20), hence it is the same function of $w$. This immediately leads to

$$Q_2^{(1,1)}(w, \{z\}, K \subset \mathbb{H} \setminus D) = [G'(w)]^2 [G'(0)]^h Q_2(G(w), \{G(z)\}, K \subset \mathbb{H} \setminus D') .$$

(4.21)

Specializing $D$ to be simply connected, this gives (4.10) in the case $k = 1$. Note that (4.21) can also be written

$$\lim_{\epsilon \to 0} \frac{8}{\pi \epsilon^2} \int_0^{2\pi} d\bar{\theta} e^{-2i\bar{\theta}} P(\{G(z)\}, K \cap G(E_{w,\epsilon,\bar{\theta}}) \neq \emptyset, K \subset \mathbb{H} \setminus D')$$

$$= [G'(w)]^2 Q_2^{(1,1)}(G(w), \{G(z)\}, K \subset \mathbb{H} \setminus D') .$$

If we take one component of $D$ to be itself some $E_{w,\epsilon,\bar{\theta}}$ and if we integrate over $\bar{\theta}$ with the factor $8e^{-2i\pi \bar{\theta}}/\pi$, we can use this same equation to derive

$$Q_{2,2}^{(2,1)}(w_1, w_2, \{z\}, K \subset \mathbb{H} \setminus D) = [G'(w_1)]^2 [G'(w_2)]^2 [G'(0)]^h Q_{2,2}^{(2,1)}(G(w_1), G(w_2), \{G(z)\}, K \subset \mathbb{H} \setminus D') .$$

(4.22)

Repeating the process, dividing the left-hand side by $P(K \subset \mathbb{H} \setminus D)$ and the right-hand side by $P(K \subset \mathbb{H} \setminus D')$ and using (4.14), we obtain (4.10) for arbitrary $k$.

5 CFT interpretation

In this section, we shall interpret Theorem 2.1 from the point of view of CFT, showing that it represents the Ward identities, hence $Q_2^{(k,1)}$ can be identified with correlation functions involving
the stress-energy tensor. Recall that being based on conformal restriction, Theorem 2.1 holds for SLE only at the particular value $\kappa = 8/3$, which corresponds to a CFT with central charge $c = 0$, as discussed below. It is natural that the stress-energy tensor is identified with a local event in SLE only for $\kappa = 8/3$, since this corresponds to the limit $n \to 0$ of the $O(n)$ model, which is the only limit where the loops disappear and where the domain wall is sufficient to describe the full CFT. Possible generalizations to CFT with $c \neq 0$ will be mentioned in Section 6.

Let us consider in detail the application of Theorem 2.1 to SLE$_{8/3}$. The corresponding restriction exponent $h = \frac{5}{8}$ coincides in CFT with the conformal weight of the boundary operator $\phi_{2,1}$ at $c = 0$, which is the value associated to $\kappa = \frac{8}{3}$ in the identification (1.1). This particular operator has already been understood to play an important role in the correspondence between probabilities in SLE and correlation functions in CFT, being the one inserted at the points where the SLE curve starts and ends.

Therefore, eq. (4.9) (i.e. Theorem 2.1 for $k = 0$) takes the form of the conformal Ward identity which links the CFT correlation functions

$$P(\{z\}) = \frac{\langle \phi_{2,1}(0) \phi_{2,1}(\infty) \prod_i O_i(z_i) \rangle}{\langle \phi_{2,1}(0) \phi_{2,1}(\infty) \rangle},$$

$$Q_{2}^{(1, l)}(w, \{z\}) = \frac{\langle \phi_{2,1}(0) \phi_{2,1}(\infty) \prod_i O_i(z_i) T(w) \rangle}{\langle \phi_{2,1}(0) \phi_{2,1}(\infty) \rangle},$$

where $O_i$ are operators with zero scaling dimension and $T$ is the bulk stress-energy tensor. Recalling the results of Section 3 (and Appendix B), this means that the spin-2 Fourier component of the SLE probability of intersecting a segment of length $\epsilon$ is associated to the operator $\frac{5}{8} \epsilon^2 T$ as $\epsilon \to 0$. Similarly, Theorem 2.1 for $k > 0$ has the form of a multiple Ward identity at $c = 0$, where $Q_{2}^{(k, l)}(w_1, \ldots, w_k, \{z\})$ is a correlation function involving $k$ insertions of $T$:

$$Q_{2}^{(k, l)}(w_1, \ldots, w_k, \{z\}) = \frac{\langle \phi_{2,1}(0) \phi_{2,1}(\infty) \prod_i O_i(z_i) T(w_1) \cdots T(w_k) \rangle}{\langle \phi_{2,1}(0) \phi_{2,1}(\infty) \rangle}$$

(5.3)

Notice that the transformation property (3.10) itself identifies $Q_{2}^{(k, l)}(w_1, \ldots, w_k, \{z\})$ with a correlation function involving $k$ primary operators with spin 2 and scaling dimension 2, plus $l$ dimensionless primary operators. In general, the stress energy tensor is not a primary operator, since an extra term appears in its transformation property (the so-called Schwarzian derivative). However, this term is proportional to the central charge $c$, and therefore it disappears in the present case $c = 0$.

A further argument in favor of the above correspondence can be obtained by generalizing eq. (3.4) to multiple segments and extending it to the class of shapes $E_{w, \epsilon, \theta}$. The resulting equation

$$\left\{ \frac{4}{3} \left( \sum_{i=1}^{k} \partial_{w_i} \right)^2 - 2 \sum_{i=1}^{k} \left( \frac{2}{w_i} - \frac{1}{w_i^2} \partial_{w_i} \right) \right\} Q_{2}^{(k, 0)}(w_1, \ldots, w_k) = 0$$

(5.4)
precisely corresponds to the null-vector equation obtained in CFT by acting with the appropriate combination of Virasoro differential operators $L_n$ on the correlation function of interest [1]:

$$\left(4 - \frac{2}{3}L_{-1} - 2L_{-2}\right) \frac{\langle \phi_{2,1}(0) \phi_{2,1}(\infty) T(w_1) \cdots T(w_k) \rangle}{\langle \phi_{2,1}(0) \phi_{2,1}(\infty) \rangle} = 0.$$ 

6 Boundary stress-energy tensor and Ward identities

In this Section, we will briefly review the same problem discussed in the rest of the paper, but in the simpler situation in which the slits are connected to the boundary of the domain. In this case, there is no concept of spin as before, and the segments can be considered to be vertical. It is now natural to look for some correspondence between probabilities with scaling behavior $\epsilon^2$ and the boundary stress-energy tensor, which is an operator of scaling dimension 2.

This problem has been already analyzed in [10], where the boundary Ward identities have been proven through conformal restriction. However, it is worth to study it along the lines of our previous discussion, in order to get a more general result and a clearer CFT interpretation.

In [10], the Ward identities were obtained by directly exploiting the conformal map

$$\Phi(z) = \sqrt{(z-x)^2 + \epsilon^2} - \sqrt{x^2 + \epsilon^2},$$

which removes the vertical segment $[x, x+i\epsilon]$ from the upper half plane $\mathbb{H}$. Inserted in (4.2), this map produces the result

$$P(x, \epsilon) = \frac{h}{2} \frac{\epsilon^2}{x^2} + o(\epsilon^2)$$

for the probability that a restriction set (with restriction exponent $h$) intersects a single segment connected to the boundary.

We will now derive the same result of [10] in a slightly different way, which is actually the only one generalizable to the bulk case. We will exploit another kind of conformal map, similar to (2.2), which has a pole at the location $x$ of the segment, i.e. where the stress-energy tensor is inserted in the correlation functions. Let us therefore introduce the singular conformal transformation

$$g_{x,\epsilon}(z) = z + \frac{\epsilon^2}{4} \frac{1}{x-z},$$

which preserves the boundary and maps the semidisk $D_{x,\epsilon}$ of radius $\frac{\epsilon}{2}$ around $x \in \mathbb{R}$ to the vertical segment $[x, x+i\epsilon]$. By implementing (6.2) and using restriction we obtain

$$P(\{z\})_{\mathbb{H} \setminus D_{x,\epsilon}} = P(\{z\}) - P(\{z\}, x, \epsilon) + P(\{z\}) P(x, \epsilon) +
\frac{\epsilon^2}{4} \sum_{i} \left( \frac{1}{x - z_i} \partial_{z_i} + \frac{1}{x - \bar{z}_i} \partial_{\bar{z}_i} - \frac{1}{x} (\partial_{z_i} + \partial_{\bar{z}_i}) \right) P(\{z\}) + o(\epsilon^2)$$

where the notation has the same meaning as in the bulk case. Since now $\mathbb{H} \setminus D_{x,\epsilon}$ is simply connected, we can map it to the upper half plane through the function

$$\Psi(z) = z - \frac{\epsilon^2}{4} \frac{1}{x-z}.$$
Therefore, conformal restriction also implies
\[ P(\{z\})_{\mathbb{H} \setminus D_x, \varepsilon} = P(\Psi(\{z\})) = P(\{z\}) - \frac{\varepsilon^2}{4} \sum_i \left( \frac{1}{x - z_i} \partial_{z_i} + \frac{1}{x - \bar{z}_i} \partial_{\bar{z}_i} - \frac{1}{x} (\partial_{z_i} + \partial_{\bar{z}_i}) \right) P(\{z\}) + o(\varepsilon^2), \]
and this leads to the final relation
\[ P(\{z\}, x, \varepsilon) = \sum_i \left( \frac{1}{x - z_i} \partial_{z_i} + \frac{1}{x - \bar{z}_i} \partial_{\bar{z}_i} - \frac{1}{x} (\partial_{z_i} + \partial_{\bar{z}_i}) \right) + \frac{h}{x^2} \right] P(\{z\}) + o(\varepsilon^2). \quad (6.3) \]

A result analogous to (2.6) for the probabilities of intersecting multiple slits can be obtained along the same lines discussed for the bulk case:
\[ P(\{z\}, x_1, \varepsilon_1, \ldots, x_k, \varepsilon_k) = \sum_i \left( \frac{1}{x_k - z_i} \partial_{z_i} + \frac{1}{x_k - \bar{z}_i} \partial_{\bar{z}_i} - \frac{1}{x_k} (\partial_{z_i} + \partial_{\bar{z}_i}) \right) + \frac{h}{x_k^2} \right] P(\{z\}) + o(\varepsilon^2). \quad (6.4) \]

(restriction also implies that \( P(\{z\}, x_1, \varepsilon_1, \ldots, x_k, \varepsilon_k) = O(\varepsilon_1^2 \cdots \varepsilon_k^2) \)).

The CFT interpretation of this results is similar to the one presented in Section 5: in the case of SLE8/3, (6.3) and (6.4) correspond to the Ward identities if we associate the segment of length \( \varepsilon \) to the insertion of \( \frac{1}{2} \varepsilon^2 T \).

### 6.1 SLE\( \kappa \) with \( \kappa < \frac{8}{3} \)

Actually, the result obtained in [10] holds for any restriction measure, and it was also applied to an explicit random set \( K \), which is constructed by adding ‘Brownian bubbles’ to SLE [9]. Although SLE\( \kappa \) does not satisfy restriction for \( \kappa \neq \frac{8}{3} \), the resulting set \( K \) enjoys this property if the Brownian bubbles are attached to the SLE curve with an intensity \( \lambda \) chosen as
\[ \lambda = -\frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}, \quad (6.5) \]
which is the negative of the central charge \( c \) in [11]. Since \( \lambda \) must be positive and the SLE curve must be a simple curve, this construction only works for \( \kappa < \frac{8}{3} \), which correspond to \( c < 0 \).

As a matter of fact, our result as stated in Theorem 2.1 cannot be extended to this construction, because Point 3 in Assumptions 2.1 does not hold for the set \( K \) described above. However, if we restrict the analysis to the boundary case considered in this section, then the procedure is unaffected by relaxing Point 3, since we know from (6.1) that the probability \( P(x, \varepsilon) \) that \( K \) intersects a segment connected to the boundary vanishes as \( \varepsilon^2 \) when \( \varepsilon \to 0 \) (the generalization of this result to the probability \( P(\{z\}, x, \varepsilon) \) is straightforward).
The result (6.4) can be interpreted from the CFT point of view in the following way. The restriction exponent associated to the set $K$ is

$$ h = \frac{6 - \kappa}{2\kappa}, \quad (6.6) $$

and it coincides with the conformal weight of the boundary operator $\phi_{2,1}$ at generic $\kappa$. Therefore, the interpretation of eq. (6.4) goes as for $\kappa = 8/3$. The correspondence is not immediately clear, however, for the case of multiple slits, because eq. (6.4) does not display the terms proportional to the central charge which are now expected since $c \neq 0$. In particular, from CFT one would expect (6.3) to be modified as

$$ P(\{z\}, x_1, \epsilon_1, \ldots, x_{k+1}, \epsilon_{k+1}) = \frac{\epsilon_{k+1}^2}{2} \left[ \sum_{i=1}^{k} \left( \frac{1}{x_{k+1} - x_i} - \frac{1}{x_{k+1} + x_i} \right) \partial x_i + \sum_{i=1}^{k} \left( \frac{2}{(x_{k+1} - x_i)^2} + \frac{1}{(x_{k+1} + x_i)^2} \right) \partial x_i + \frac{h}{x_{k+1}^2} \right] \cdot P(\{z\}, x_1, \epsilon_1, \ldots, x_k, \epsilon_k) + \cdots \quad (6.7) $$

where $\hat{x}_j$ indicates that the coordinate $x_j$ is missing.

The apparent contradiction is solved by identifying probabilities with connected correlation functions in CFT. Intuitively, this can be understood by noticing that the set $K$ is connected itself, therefore probabilities of intersecting regions at large distance from its starting or ending point vanish instead of factorizing, a property which is realized by connected correlation functions in QFT. The same idea is valid also at $\kappa = \frac{8}{3}$, when $K$ reduces to the SLE curve; in that case, however, connected correlation functions are equal to unconnected ones, due to the vanishing of the central charge.

Let us define the connected correlation functions as

$$ \langle T_1 \ldots T_k O \rangle_c = \langle T_1 \ldots T_k O \rangle - \sum_{j=2}^{k} \sum_{(\alpha) \subset \{1, \ldots, k\}} \sum_{(\beta) \subset \{1, \ldots, k\} \setminus (\alpha)} \langle T_{\alpha_1} \ldots T_{\alpha_j} \rangle \langle T_{\beta_1} \ldots T_{\beta_{k-j}} O \rangle_c, \quad (6.8) $$

with $T_i \equiv T(x_i)$ and $O = \frac{\phi_{2,1}(0) \phi_{2,1}(\infty)}{\phi_{2,1}(0) \phi_{2,1}(\infty)}$. It is easy to prove (see Appendix D) that, if $\langle T_1 \ldots T_k O \rangle$ satisfies the conformal Ward identities at $c \neq 0$, then $\langle T_1 \ldots T_k O \rangle_c$ satisfies

$$ \langle T_1 \ldots T_k T_{k+1} O \rangle_c = \left\{ \sum_{i=1}^{k} \left[ \left( \frac{1}{x_{k+1} - x_i} - \frac{1}{x_{k+1} + x_i} \right) \partial x_i + \frac{2}{(x_{k+1} - x_i)^2} \right] + \frac{h}{x_{k+1}^2} \right\} \langle T_1 \ldots T_k O \rangle_c, \quad (6.9) $$

which are precisely the Ward identities without anomaly as obtained from restriction in (6.3). The same equation can be obtained for $O = \prod_{i=1}^{k} \frac{\phi_{2,1}(x_i) \phi_{2,1}(0) \phi_{2,1}(\infty)}{\phi_{2,1}(0) \phi_{2,1}(\infty)}$. Therefore, it is natural to
suggest the identification

\[
P(\{z\}, x_1, \epsilon_1, \ldots, x_k, \epsilon_k) = \frac{\epsilon_1 \cdot \cdots \cdot \epsilon_k}{2^k} \left( \phi_{2,1}(0) \phi_{2,1}(\infty) \prod_i \mathcal{O}_i(z_i) T(x_1) \cdots T(x_k) \right)_c.
\]

(6.10)

However, in order to fully justify the identification (6.10) we should rule out two other possibilities: one is that we are actually looking again at a CFT with central charge \( c = 0 \), and the other is that \( P(\{z\}, x_1, \epsilon_1, \ldots, x_k, \epsilon_k) \) corresponds to a non-connected correlation function at \( c \neq 0 \) involving primary spin-2 operators instead of the stress-energy tensor. This can be done by looking at the analog of the SLE equation (5.4) for multiple slits, when we consider the random process defined by SLE + loops [10]:

\[
\left\{ \frac{\kappa}{2} \left( \sum_{i=1}^k \partial x_i \right)^2 - 2 \sum_{i=1}^k \left( \frac{2}{x_i^2} - \frac{1}{x_i} \partial x_i \right) \right\} P(x_1, \epsilon_1, \ldots, x_k, \epsilon_k) + \lambda \sum_{j=1}^k \sum_{\{\alpha\} \subset \{1, \ldots, k\}, \{\beta\} \subset \{1, \ldots, k\} \setminus \{\alpha\}} \frac{\epsilon_{\alpha_1} \cdot \cdots \cdot \epsilon_{\alpha_j}}{2^j} T_j(x_{\alpha_1}, \ldots, x_{\alpha_j}) P(x_{\beta_1}, \epsilon_{\beta_1}, \ldots, x_{\beta_{k-j}}, \epsilon_{\beta_{k-j}}) = 0
\]

(6.11)

where

\[
T_j(x_1, \ldots, x_j) = \sum_{s \in \sigma_j} \frac{1}{x_{s(1)}^2 (x_{s(2)} - x_{s(1)})^2 \cdots (x_{s(j)} - x_{s(j-1)})^2 x_{s(j)}^2},
\]

(6.12)

with \( \sigma_j \) indicating the permutations of \( j \) numbers. The meaning of eq. (6.11) can be understood by noticing that (6.12) is the probability that a Brownian bubble intersects \( j \) of the \( k \) slits.

It can be easily checked that (6.11) coincides with the CFT null-vector equation

\[
\left( \frac{\kappa}{2} \mathcal{L}^{-1}_{-2} - 2 \mathcal{L}_{-2} \right) \langle T_1 \cdots T_k \mathcal{O} \rangle_c = 0.
\]

in a CFT with central charge \( c = -\lambda \) (the proof is presented in Appendix D). The need for connected correlation functions can be understood as follows: the null-vector equation for \( \langle T_1 \cdots T_k \mathcal{O} \rangle \) only reproduces the terms in (6.11) with \( j = 1 \), i.e. it only takes into account the cases when the Brownian bubble intersects a single slit. The additional terms in (6.11) precisely generate the events of the Brownian bubble intersecting more slits.

This corroborates (6.10) and the identification of the density \( \lambda \) as the negative of the central charge.

It is now worth to comment possible extension of this result to the bulk case discussed in the rest of the paper. As we already mentioned, Theorem 2.1 cannot be directly applied to the SLE + bubbles construction, since the corresponding measure does not satisfy point 3 in Assumptions 2.1. A natural interpretation of this fact is that the measure on Brownian bubbles, although it satisfies conformal restriction, does not exhibit anymore the "Markov property" as SLE; that is, we cannot partially restrict the random set and say that the rest is obtained by conformal transformation from the initial domain. Therefore, a description in terms of a local field theory as for SLEs/3 does not seem possible anymore. At this point, it may seem puzzling that things work for the boundary case, as shown in [10] and further elaborated in this section.
However, one should notice that if a connected set intersects a segment of height $\epsilon$ connected to the boundary, then the outer boundary of the connected set necessarily intersects the segment as well. We think that there exists a correct description giving bulk connected correlation functions at $c < 0$ starting from the outer boundary of the SLE+Brownian bubbles.

7 Conclusions

In this paper, we have shown that suitable probabilities in SLE and related processes can be associated to certain correlation functions containing the holomorphic stress-energy tensor $T(w)$ of CFT with central charge $c = 0$. Our result can be conceptually stated as 1) the identification between a particular random variable and the stress-energy tensor:

$$\frac{8}{\pi} \lim_{\epsilon \to 0} \epsilon^{-2} \int_0^{2\pi} d\theta e^{-2i\theta} v(w, \epsilon, \theta) \leftrightarrow T(w)$$

where $v(w, \epsilon, \theta)$ is 1 when the random set intersects a segment centered at $w$ of length $\epsilon$ and of angle $\theta$ with respect to the imaginary direction, and 0 otherwise; and 2) the identification between the stochastic average of such random variables (in the random processes considered) and correlation functions in CFT with $c = 0$. This result adds to previous ones in the understanding of the connection between SLE and CFT: the boundary stress-energy tensor was already identified in [10], and the end-points of the SLE curve where first identified with $\phi_{2,1}$ boundary operators of CFT in [7]. It can be generalized in three main directions.

One is the application of our methods to other conformally invariant processes on the plane or on other Riemann surfaces, like self-avoiding loops and the Conformal Loop Ensemble (CLE), whose formalization based on conformal restriction is at present an active research topic in the mathematical community. In particular, this should give access to CFT with $c > 0$, therefore to a rigorous derivation of (a wide range of models of) CFT in terms of stochastic processes. Moreover, an appropriate generalization of the SLE + Brownian bubbles construction is the natural candidate for the description of CFT with $c < 0$, as we have seen for the boundary case.

The second natural extension of the present work is the identification of other kinds of holomorphic operators, which, as we have seen, naturally emerge at some values of $\kappa$. To justify their correspondence with local CFT operators one should prove appropriate functional relations analogous to the Ward identities derived here.

Finally, another possible direction is the identification of other primary scaling operators. These can be specified, for example, by requiring that the SLE curve pass between two given points, separated by a distance $\epsilon$, in a prescribed manner. Correlation functions with insertions of these operators will correspond to the coefficients of given powers of $\epsilon$ in the expansion of the associated probability as the points approach each other. One would like to show that the local operators generated in this way then form a closed operator algebra, and compute the OPE coefficients directly. This would lead to a construction of at least one sector of the full CFT from the viewpoint of conformally invariant measures on planar sets.
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A SLE probabilities in the disk geometry

The ansatz
\[
\tilde{Q}_n(w, \bar{w}, \epsilon) = c_n \epsilon^{x_n} w^{\alpha_n} \bar{w}^{\beta_n} (w - \bar{w})^{\gamma_n}
\]
(A.1)
solves eq. (3.4) for two different choices of the parameters:
\[
\alpha_n = -\frac{2n}{\kappa - 4}, \quad \beta_n = \frac{2n}{\kappa - 4}, \quad \gamma_n = -\frac{2\kappa n^2}{(\kappa - 4)^2}, \quad x_n = \frac{2\kappa n^2}{(\kappa - 4)^2}
\]
(A.2)
and
\[
\alpha_n = \frac{\kappa - 8}{2\kappa} - \frac{n}{2}, \quad \beta_n = \frac{\kappa - 8}{2\kappa} + \frac{n}{2}, \quad \gamma_n = \frac{(8 - \kappa)^2 - \kappa^2 n^2}{8\kappa}, \quad x_n = 1 - \frac{\kappa}{8} + \frac{\kappa}{8} n^2.
\]
(A.3)

In order to select the correct set of parameters, it is convenient to map our problem onto the unit disk \( \mathbb{D} \), through the transformation \( z' = \frac{z-w}{z-\bar{w}} \) for \( z \in \mathbb{H} \) and \( z' \in \mathbb{D} \). This transformation maps the point \( w \) to the center of the disk, the length \( \epsilon \) to \( \epsilon/|w - \bar{w}| \), and it shifts the angle \( \theta \) by an angle of \( \pi/2 \). Also, the point 0 is mapped to \( w/\bar{w} \) on the boundary of the disk, and the point \( \infty \) to 1. We are then describing an SLE curve on the unit disk started at \( w/\bar{w} \) and required to end at 1. Fixing the power of \( \epsilon/|w - \bar{w}| \) to be some number \( x_n \) (the “scaling dimension”), we are left, after integration over \( \theta \) as in (3.2), with a second order ordinary differential equation in the angle \( \alpha = \arg(w/\bar{w}) \in [0, 2\pi] \). This equation is the eigenvalue equation for an eigenfunction of the two-particle Calogero-Sutherland Hamiltonian with eigenvalue (energy) \( 2x_n/\kappa \) and with total momentum \( n \) \[13\]. For generic \( \kappa \), the Calogero-Sutherland Hamiltonian admits only two types of series expansions \( C\alpha^\omega([\alpha^2]) \) (with \( C \neq 0 \)) as \( \alpha \to 0^+ \) for its eigenfunctions: one with a leading power \( \omega = 8/\kappa - 1 \), the other with a leading power \( \omega = 0 \). It admits the same two types of series expansions \( C'(2\pi - \alpha)^\omega' \((2\pi - \alpha)^2\)) \( with \( C' \neq 0 \) as \( \alpha \to 2\pi^- \). Allowing only one type of series expansion at 0 and only one at \( 2\pi \) (the possibilities give the Calogero-Sutherland system in the fermionic sector \( \omega = \omega' = 8/\kappa - 1 \), bosonic sector \( \omega = \omega' = 0 \) or mixed sector, \( \omega \neq \omega' \)), the Calogero-Sutherland Hamiltonian has a discrete set of eigenfunctions, with eigenvalues bounded from below (since it is a self-adjoint operator on the space of functions with these asymptotic conditions). The lowest eigenvalue is obtained for the eigenfunction (the ground state) with the least number of nodes (zeros of the eigenfunction). If the leading powers \( \omega \) and \( \omega' \) are chosen...
equal to each other, then the ground state (in the sector with total momentum $n$) is described by the solutions (A.1) with (A.2) (for $\omega = 0$) or (A.3) (for $\omega = 8/\kappa - 1$), which, in the coordinates of the disk, take the form

\[ \tilde{Q}_n(|w - \bar{w}|, \alpha, \epsilon) = \tilde{c}_n \left( \frac{\epsilon}{|w - \bar{w}|} \right)^{x_n} e^{i \frac{\alpha_n - \beta_n}{2} \alpha} \left( \sin \frac{\alpha}{2} \right)^{\gamma_n + x_n}. \] (A.4)

The probabilities that we are considering require the curve to pass by the center of the disk. Hence, they vanish when the starting point of the SLE curve is brought toward its ending point on the disk, from any direction; this fixes the power to be $8/\kappa - 1$ (for $\kappa < 8$) at both values $\alpha = 0, 2\pi$, and therefore selects the solution in the fermionic sector (A.3). Note that since the probability could be given by an excited state in the fermionic sector (which corresponds to a higher value in place of the exponent $x_n$), we do not have the condition that $\tilde{c}_n$ is nonzero.

B Deformation of the segment

In this Appendix, we will show that Theorem 2.1 can be used to conclude that the second Fourier component of the probability $P_{\text{segm}}(w, \bar{w}, \epsilon, \theta)$ that the SLE$_{8/3}$ curve intersects a segment is given by

\[ \tilde{Q}_{\text{segm}}(w, \bar{w}, \epsilon) = \frac{\pi}{8} \epsilon^2 \frac{h}{w^2} + o(\epsilon^2). \] (B.1)

This means in particular that (B.1) is equal, at leading order in $\epsilon$, to the second Fourier component of the probability $P(w, \bar{w}, \epsilon, \theta)$ of passing between the ending points of the segment as in (3.9) with $n = 2$, up to an overall constant.

First, let us recall that the result (2.6), and in particular (4.5), applies to the case when the considered shapes are deformed segments, which correspond to $b \to 1$ in (2.3):

\[ g_{w,\epsilon,\theta} \left( w + \frac{\epsilon}{4} e^{i\alpha + i\theta} \right) = w + \frac{\epsilon^2}{2} e^{i\theta + i\pi/2} \sin \alpha + \frac{\epsilon^2}{16} \frac{e^{2i\theta}}{w} - \frac{\epsilon^2}{16} \frac{e^{-2i\theta}}{\bar{w}} + \frac{\epsilon^2}{16} \frac{e^{-2i\theta}}{w - \bar{w}} + \frac{\epsilon^3}{64} \frac{e^{i\alpha + i\theta}}{(w - \bar{w})^2} + O(\epsilon^4), \] (B.2)

where $\alpha \in [0, 2\pi]$. As we have discussed in the main text, the probability of intersecting a straight segment, corresponding to the first line of (B.2), satisfies at leading order eq. (3.3), and its Fourier components satisfy eq. (3.4), which coincide with the equations for the probability of passing in between the two ending points of the segment. We now have to show that the deformations described in the second line of (B.2) do not affect the leading order behaviour in (3.3).

Let us first analyze the effect of the $\epsilon^2$ terms in (B.2). Since they do not depend on $\alpha$, they merely correspond to a change in the central position of the segment:

\[ w \to w - \frac{\epsilon^2}{16} \frac{e^{2i\theta}}{w} - \frac{\epsilon^2}{16} \frac{e^{-2i\theta}}{\bar{w}} + \frac{\epsilon^2}{16} \frac{e^{-2i\theta}}{w - \bar{w}}. \]

Therefore, their effect on the differential equation (3.3) for $P_{\text{segm}}(w, \bar{w}, \epsilon, \theta)$ translates into the introduction of terms of the type $\epsilon^2 \partial_w \tilde{Q}_{\text{segm}}(w, \bar{w}, \epsilon)$ and $\epsilon^2 \partial_{\bar{w}} \tilde{Q}_{\text{segm}}(w, \bar{w}, \epsilon)$ in the equation.
for $\tilde{Q}^\text{segm}_2(w, \tilde{w}, \epsilon)$. Since each Fourier component is assumed to vanish with a power law as $\epsilon \to 0$, these corrections turn out to be of order $o(\epsilon^2)$.

The remaining terms in (B.2), of order $\epsilon^3$ and higher, depend on $\alpha$, therefore they induce a change not only in the position of the segment, but also in its length and inclination. However, these can only introduce in contributions of the form $(\epsilon^3 + o(\epsilon^3))\partial_\epsilon^3 \tilde{Q}^\text{segm}_m(w, \tilde{w}, \epsilon)$, which give corrections of order $o(\epsilon^2)$ to $\tilde{Q}^\text{segm}_2(w, \tilde{w}, \epsilon)$. Furthermore, the segment gets distorted by these terms in (B.2), so that it develops higher moments besides the dipole one. However, assuming smoothness of the probabilities, these contributions are also of order $o(\epsilon^2)$.

C  The double Ward identities

In this appendix, we will sketch a possible proof of the multiple Ward identities (2.6) alternative to the one presented in Section 4.3 as mentioned in Remark 4.2. The discussion is not rigorous, but it displays interesting features that are worth to comment. For simplicity, we will just consider the case of two slits, but the following arguments can be easily extended to $k$ slits.

The basic idea is to consider the generalization $g \equiv g_{w_1, \epsilon_1, \theta_1, w_2, \epsilon_2, \theta_2}$ of the conformal map (2.2) which is singular at the two points $w_1$ and $w_2$ and satisfies

$$g [\mathbb{H} \setminus (D_{w_1, \epsilon_1} \cup D_{w_2, \epsilon_2})] = \mathbb{H} \setminus (E_{w_1, \epsilon_1, \theta_1} \cup E_{w_2, \epsilon_2, \theta_2}) \ , \quad \text{(C.1)}$$

where the notation is the same as in Section 4.4 $D_{w_i, \epsilon_i} = S_{w_i, \epsilon_i}(D_i)$ and $E_{w_i, \epsilon_i, \theta_i} \in \mathcal{E}$. We can now slightly extend (C.1) to write

$$P(\{z\}, K \subset \mathbb{H} \setminus (E_{w_1, \epsilon_1, \theta_1} \cup E_{w_2, \epsilon_2, \theta_2})) = \left[ (g^{-1})'(0) \right]^h P(\{g^{-1}(z)\}, K \subset \mathbb{H} \setminus (D_{w_1, \epsilon_1} \cup D_{w_2, \epsilon_2})) \ . \quad \text{(C.2)}$$

Since

$$P(\{z\}, K \subset \mathbb{H} \setminus (E_{w_1, \epsilon_1, \theta_1} \cup E_{w_2, \epsilon_2, \theta_2})) = 1 - P(\{z\}, K \cap E_{w_1, \epsilon_1, \theta_1} \neq \emptyset) - P(\{z\}, K \cap E_{w_2, \epsilon_2, \theta_2} \neq \emptyset) + P(\{z\}, K \cap E_{w_1, \epsilon_1, \theta_1} \neq \emptyset, K \cap E_{w_2, \epsilon_2, \theta_2} \neq \emptyset) \ , \quad \text{(C.3)}$$

eq \text{eq. (C.2)} \text{ implies}

$$Q^{(2, l)}_{2, 2}(w_1, w_2, \{z\}) = \left( \frac{8}{\pi} \right)^2 \lim_{\epsilon_1, \epsilon_2 \to 0} \epsilon_1^2 \epsilon_2^2 \int_0^{2\pi} d\theta_1 e^{-2i\theta_1} \int_0^{2\pi} d\theta_2 e^{-2i\theta_2} \cdot [(g^{-1})'(0)]^h P(\{g^{-1}(z)\}, K \subset \mathbb{H} \setminus (D_{w_1, \epsilon_1} \cup D_{w_2, \epsilon_2})) \ . \quad \text{(C.4)}$$

Therefore $Q^{(2, l)}_{2, 2}$ will be expressed as a differential operator acting on $P(\{z\})$, and the operator is obtained by expanding the map $g^{-1}$ in $\epsilon_1$ and $\epsilon_2$.

The lack of rigor in our considerations is due to the fact that, although we know that the map $g$ exists, we do not know its explicit form. However, we can approximate it with another
conformal map \( \hat{g} \), associated to a family of shapes \( \mathcal{E} \) and defined through its inverse as

\[
\hat{g}^{-1}(z) = z - \frac{\epsilon_1^2 e^{2i\theta_1}}{16} \left( \frac{1}{w_1 - z} - \frac{1}{w_1} \right) - \frac{\epsilon_2^2 e^{-2i\theta_1}}{16} \left( \frac{1}{w_1 - z} - \frac{1}{w_1} \right) + (C.5)
\]

\[
- \frac{\epsilon_2^2 e^{2i\theta_2}}{16} \left( \frac{1}{w_2 - z} - \frac{1}{w_2} \right) - \frac{\epsilon_2^2 e^{-2i\theta_2}}{16} \left( \frac{1}{w_2 - z} - \frac{1}{w_2} \right) +
\]

\[
+ \frac{\epsilon_1^2 \epsilon_2^2 e^{2i\theta_1} e^{2i\theta_2}}{16(16)^2} \left( \frac{1}{w_1 - z} + \frac{1}{w_1} \right) \left( \frac{1}{w_2 - z} - \frac{1}{w_2} \right) +
\]

\[
+ \frac{\epsilon_1^2 \epsilon_2^2 e^{-2i\theta_1} e^{-2i\theta_2}}{16(16)^2} \left( \frac{1}{w_1 - z} + \frac{1}{w_1} \right) \left( \frac{1}{w_2 - z} - \frac{1}{w_2} \right).
\]

Let us define the domains \( \hat{E}_1 \) and \( \hat{E}_2 \) as two disjoint simply connected domains such that

\[
\hat{g} \left( \mathbb{H} \setminus \left( D_{w_1, \epsilon_1} \cup D_{w_2, \epsilon_2} \right) \right) = \mathbb{H} \setminus \left( \hat{E}_1 \cup \hat{E}_2 \right) \quad (C.6)
\]

(\( \hat{E}_1 \) and \( \hat{E}_2 \) are disjoint for \( \epsilon_1 \) and \( \epsilon_2 \) small enough). Both domains \( \hat{E}_1 \) and \( \hat{E}_2 \) depend on the variables \( w_1, \epsilon_1, \theta_1, w_2, \epsilon_2, \theta_2 \) (as well, of course, as on the initial domains \( D_1 \) and \( D_2 \)). It can be easily checked, however, that \( \hat{E}_1 \) is given at leading order by \( E_{w_1, \epsilon_1, \theta_1} \), plus higher order corrections which also depend on \( w_2, \epsilon_2 \) and \( \theta_2 \) (and that the converse is true for \( \hat{E}_2 \)). If we assume that the \( \theta_2 \)-dependence of \( P(\{z\}, K \cap \hat{E}_1 \neq \emptyset) \) and that the \( \theta_1 \)-dependence of \( P(\{z\}, K \cap \hat{E}_2 \neq \emptyset) \) contribute to the double integration in \( \theta_1 \) and \( \theta_2 \) at higher order in \( \epsilon_1 \epsilon_2 \) than \( P(\{z\}, K \cap \hat{E}_1 \neq \emptyset, K \cap \hat{E}_2 \neq \emptyset) \), we can still use \( (C.4) \) to obtain

\[
\hat{Q}^{(2,1)}_{2,2}(w_1, w_2, \{z\}) = (\mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4) P(\{z\}) ,
\]

where

\[
\hat{Q}^{(2,1)}_{2,2}(w_1, w_2, \{z\}) = \left( \frac{8}{\pi} \right)^2 \lim_{\epsilon_1, \epsilon_2 \to 0} \left. \epsilon_1^{-2} \epsilon_2^{-2} \int_0^{2\pi} d\theta_1 e^{-2i\theta_1} \int_0^{2\pi} d\theta_2 e^{-2i\theta_2} P(\{z\}, K \cap \hat{E}_1 \neq \emptyset, K \cap \hat{E}_2 \neq \emptyset) ,
\]

\[
\mathcal{D}_1 = \sum_{ij} \left( \frac{1}{w_1 - z_i} - \frac{1}{w_1} \right) \left( \frac{1}{w_2 - z_j} - \frac{1}{w_2} \right) \partial_i \partial_j + \sum_{ij} \left( \frac{1}{w_1 - z_i} - \frac{1}{w_1} \right) \left( \frac{1}{w_2 - z_j} - \frac{1}{w_2} \right) \bar{\partial}_i \bar{\partial}_j +
\]

\[
+ \sum_{ij} \left( \frac{1}{w_1 - z_i} - \frac{1}{w_1} \right) \left( \frac{1}{w_2 - z_j} - \frac{1}{w_2} \right) \partial_i \bar{\partial}_j + \sum_{ij} \left( \frac{1}{w_1 - z_i} - \frac{1}{w_1} \right) \left( \frac{1}{w_2 - z_j} - \frac{1}{w_2} \right) \bar{\partial}_i \partial_j ,
\]

\[
\mathcal{D}_2 = \frac{1}{(w_1 - w_2)^2} \sum_i \left[ \left( \frac{1}{w_1 - z_i} + \frac{1}{w_2 - z_i} - \frac{1}{w_1} - \frac{1}{w_2} \right) \partial_i + \left( \frac{1}{w_1 - z_i} + \frac{1}{w_2 - z_i} - \frac{1}{w_1} - \frac{1}{w_2} \right) \bar{\partial}_i \right] ,
\]

\[
\mathcal{D}_3 = \frac{h}{w_1^2} \sum_i \left[ \left( \frac{1}{w_2 - z_i} - \frac{1}{w_2} \right) \partial_i + \left( \frac{1}{w_2 - z_i} - \frac{1}{w_2} \right) \bar{\partial}_i \right] +
\]

\[
+ \frac{h}{w_2^2} \sum_i \left[ \left( \frac{1}{w_1 - z_i} - \frac{1}{w_1} \right) \partial_i + \left( \frac{1}{w_1 - z_i} - \frac{1}{w_1} \right) \bar{\partial}_i \right] ,
\]

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\[ D_4 = \frac{h^2}{w_1 w_2} + \frac{2h}{w_1 w_2(w_1 - w_2)^2}. \]

The result (6.17) is the symmetrized form of the Ward identities (2.6) for \( k = 1 \) with the right-hand side expanded using the Ward identity (2.6) for \( k = 0 \). To make this argument a proof of (2.6) one would need to rigorously justify the assumption before (6.17) and to show that \( \hat{Q}^{(2,l)}_2 \) coincides with \( Q^{(2,l)}_2 \). In order to do this, it could be useful to exploit the freedom in the choice of conformal maps as commented in Remark 2.1.

### D Properties of connected correlation functions in CFT

In this Appendix we will explicitly prove that appropriate connected correlation functions in CFT, defined in (6.8), satisfy equations (6.9) and (6.11) presented in the main text.

Let us first notice that solving the recursion in definition (6.8) we obtain

\[
\langle T_1 \cdots T_{k+1} \mathcal{O} \rangle_c = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \sum_{\cup_{i=0}^{n} J_i = \{1, \ldots, k\}} \langle T_{J_0} \mathcal{O} \rangle \langle T_{J_1} \rangle \cdots \langle T_{J_n} \rangle =
\]

where \( T_{J_i} \equiv T_{\alpha_1} \cdots T_{\alpha_{|J_i|}} \) with ordered \( \alpha_l \in J_i \) and, in the last equation, \( T_0 = \mathcal{O} \) by definition.

Note that the last equation is completely symmetric: nothing makes the operator \( \mathcal{O} \) particular with respect to the \( T \)'s, so that we could as well have correlation functions connected to any of these \( T \)'s. In the SLE context, \( \mathcal{O} \) stands for

\[
\mathcal{O} = \frac{\phi_{2,1}(0) \phi_{2,1}(\infty)}{\langle \phi_{2,1}(0) \phi_{2,1}(\infty) \rangle}
\]

and we are in the boundary CFT on the half-plane.

We will now show by induction that these connected correlation functions of energy-momentum tensors in CFT satisfy (6.9), which can be compactly written as

\[
\langle T_1 \cdots T_{k+1} \mathcal{O} \rangle_c = \mathcal{L}_{-2}(x_{k+1}) \langle T_1 \cdots T_k \mathcal{O} \rangle_c,
\]

where we have defined the operator

\[
\mathcal{L}_{-2}(x) = \sum_{i=1}^{k} \left[ \left( \frac{1}{x-x_i} - \frac{1}{x} \right) \frac{\partial}{\partial x} + \frac{2}{(x-x_i)^2} \right] + \frac{h}{x^2}.
\]

Assume that the insertion of the operator \( T_I \) in \( \langle T_1 \cdots T_I \mathcal{O} \rangle_c \) is implemented by applying the operator \( \mathcal{L}_{-2}(x_I) \) as in (D.2) on the correlation function \( \langle T_1 \cdots T_{I-1} \mathcal{O} \rangle_c \) for all \( I \leq k \). From CFT, we know that

\[
\langle T_1 \cdots T_{k+1} \mathcal{O} \rangle = \mathcal{L}_{-2}(x_{k+1}) \langle T_1 \cdots T_k \mathcal{O} \rangle + \sum_{j=1}^{k} \langle T_j T_{k+1} \rangle \langle T_1 \cdots \hat{T}_j \cdots T_k \mathcal{O} \rangle,
\]

where

\[
\hat{T}_j \equiv T_{\alpha_1} \cdots T_{\alpha_j} | J_j |	ext{ with ordered } \alpha_l \in J_j \text{ and, in the last equation, } T_0 = \mathcal{O} \text{ by definition.}
\]

Note that the last equation is completely symmetric: nothing makes the operator \( \mathcal{O} \) particular with respect to the \( T \)'s, so that we could as well have correlation functions connected to any of these \( T \)'s. In the SLE context, \( \mathcal{O} \) stands for

\[
\mathcal{O} = \frac{\phi_{2,1}(0) \phi_{2,1}(\infty)}{\langle \phi_{2,1}(0) \phi_{2,1}(\infty) \rangle}
\]

and we are in the boundary CFT on the half-plane.

We will now show by induction that these connected correlation functions of energy-momentum tensors in CFT satisfy (6.9), which can be compactly written as

\[
\langle T_1 \cdots T_{k+1} \mathcal{O} \rangle_c = \mathcal{L}_{-2}(x_{k+1}) \langle T_1 \cdots T_k \mathcal{O} \rangle_c,
\]

where we have defined the operator

\[
\mathcal{L}_{-2}(x) = \sum_{i=1}^{k} \left[ \left( \frac{1}{x-x_i} - \frac{1}{x} \right) \frac{\partial}{\partial x} + \frac{2}{(x-x_i)^2} \right] + \frac{h}{x^2}.
\]

Assume that the insertion of the operator \( T_I \) in \( \langle T_1 \cdots T_I \mathcal{O} \rangle_c \) is implemented by applying the operator \( \mathcal{L}_{-2}(x_I) \) as in (D.2) on the correlation function \( \langle T_1 \cdots T_{I-1} \mathcal{O} \rangle_c \) for all \( I \leq k \). From CFT, we know that

\[
\langle T_1 \cdots T_{k+1} \mathcal{O} \rangle = \mathcal{L}_{-2}(x_{k+1}) \langle T_1 \cdots T_k \mathcal{O} \rangle + \sum_{j=1}^{k} \langle T_j T_{k+1} \rangle \langle T_1 \cdots \hat{T}_j \cdots T_k \mathcal{O} \rangle,
\]

where

\[
\hat{T}_j \equiv T_{\alpha_1} \cdots T_{\alpha_j} | J_j |	ext{ with ordered } \alpha_l \in J_j \text{ and, in the last equation, } T_0 = \mathcal{O} \text{ by definition.}
\]
where the symbol $\hat{T}_j$ means that the operator $T_j$ has been removed from the correlation function. Applying $\mathcal{L}_{-2}(x_{k+1})$ on $\langle T_1 \cdots T_k \mathcal{O}\rangle_c$, using (D.3) and noticing that the inductive hypothesis implies

$$
\mathcal{L}_{-2}(x_{k+1})\langle T_{a_1} \cdots T_{a_j} \rangle_{T_{b_1} \cdots T_{b_{k-j}} \mathcal{O}} = \langle T_{a_1} \cdots T_{a_j} \rangle_{T_{b_1} \cdots T_{b_{k-j}} T_{k+1} \mathcal{O}} + \langle T_{a_1} \cdots T_{a_j} T_{k+1} \rangle_{T_{b_1} \cdots T_{b_{k-j}} \mathcal{O}} - \sum_{l=1}^j \langle T_{a_k} T_{l+1} \rangle_{T_{a_1} \cdots T_{a_l} T_{a_j}} \langle T_{b_1} \cdots T_{b_{k-j}} \mathcal{O} \rangle_c
$$

we indeed find (D.1). It is easy to check explicitly that this formula is valid for $k = 1$, hence the induction is complete.

In order to prove that $\langle T_1 \cdots T_j \mathcal{O}\rangle_c$ also satisfies eq. (6.11), we have to preliminary identify the CFT correlation function corresponding to $T_j(x_1, \ldots, x_j)$. By adapting the inductive argument presented above to the case $\mathcal{O} = T(0)$, it is straightforward to check that

$$
\langle T_1 \cdots T_j T_{j+1} T(0)\rangle_c = \left\{ \sum_{i=1}^j \left[ \frac{1}{x_{j+1} - x_i} - \frac{1}{x_i} \right] \partial_i + \frac{2}{\lambda (x_{j+1} - x_i)^2} \right\} \langle T_1 \cdots T_j T(0)\rangle_c .
$$

Since $\langle T(x)T(0)\rangle = \frac{c/2}{x^4}$, the only solution to the recursion is

$$
\langle T_1 \cdots T_j T(0)\rangle_c = \frac{c}{2} T_j(x_1, \ldots, x_j)
$$

with $T_j$ defined in (6.12).

Therefore, eq. (6.11) can be written as

$$
\left\{ \mathcal{D} - \sum_{i=1}^k \frac{4}{x_i^2} \right\} \langle T_1 \cdots T_k \mathcal{O}\rangle_c + 2 \lambda \frac{c}{2} \sum_{j=1}^k \sum_{\{\alpha\} \subset \{1, \ldots, k\}} \langle T_{\alpha_1} \cdots T_{\alpha_j} T(0)\rangle_c \langle T_{\beta_1} \cdots T_{\beta_{k-j}} \mathcal{O}\rangle_c = 0 ,
$$

where we have defined the differential operator

$$
\mathcal{D} = \frac{k}{2} \left( \sum_{i=1}^k \partial_i \right)^2 + \sum_{i=1}^k \frac{2}{w_i} \partial_i .
$$

We know from CFT that

$$
\left\{ \mathcal{D} - \sum_{i=1}^k \frac{4}{x_i^2} \right\} \langle T_1 \cdots T_k \mathcal{O}\rangle - c \sum_{i=1}^k \frac{1}{x_i} \langle T_1 \cdots \hat{T}_i \cdots T_k \mathcal{O}\rangle = 0 \quad (D.4)
$$

By using the induction hypothesis for $k - j < k$ we have

$$
\mathcal{D} \langle T_{\beta_1} \cdots T_{\beta_{k-j}} \mathcal{O}\rangle_c = -2 \lambda \sum_{l=1}^{k-j} \left( \sum_{\gamma \subset \{\beta_1, \ldots, \beta_{k-j}\}} \langle T_{\gamma} \cdots T_{\mathcal{H}} T(0)\rangle_c \langle T_{\delta_1} \cdots T_{\delta_{k-j-l}} \mathcal{O}\rangle_c + \sum_{l=1}^{k-j} \frac{4}{x_{\beta_l}} \langle T_{\beta_1} \cdots T_{\beta_{k-j}} \mathcal{O}\rangle_c ,
$$

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while CFT tells us that
\[
\mathcal{D} \langle T_{\alpha_1}...T_{\alpha_j} \rangle = \\
= -2 \left[ \langle T(0)T_{\alpha_1}...T_{\alpha_j} \rangle - \sum_{\ell=1}^{j} \langle T(0)T_{\ell} \rangle \langle T_{\alpha_1}...\hat{T}_{\ell}...T_{\alpha_j} \rangle \right] + \left( \sum_{\ell=1}^{j} \frac{4}{x_{\alpha_\ell}^2} \right) \langle T_{\alpha_1}...T_{\alpha_j} \rangle .
\]

Therefore we have
\[
\left\{ \mathcal{D} - \sum_{i=1}^{k} \frac{4}{x_i^2} \right\} \sum_{\alpha,\beta} \langle T_{\alpha_1}...T_{\alpha_j} \rangle \langle T_{\beta_1}...T_{\beta_{k-j}} \mathcal{O} \rangle_c = \\
- 2 \sum_{j=2}^{k} \sum_{\alpha,\beta} \langle T(0)T_{\alpha_1}...T_{\alpha_j} \rangle \langle T_{\beta_1}...T_{\beta_{k-j}} \mathcal{O} \rangle_c + \\
+ 2 \sum_{j=2}^{k} \sum_{\alpha,\beta} \sum_{\ell=1}^{j} \langle T(0)T_{\ell} \rangle \langle T_{\alpha_1}...\hat{T}_{\ell}...T_{\alpha_j} \rangle \langle T_{\beta_1}...T_{\beta_{k-j}} \mathcal{O} \rangle_c - \\
- \frac{2\lambda}{c} \sum_{j=2}^{k} \sum_{\alpha,\beta} \sum_{l=1}^{k-j} \langle T_{\alpha_1}...T_{\alpha_j} \rangle \langle T_{\gamma_1}...T_{\gamma_l} T(0) \rangle_c \langle T_{\delta_1}...T_{\delta_{k-j-l}} \mathcal{O} \rangle_c
\]
and the thesis follows if \( \lambda = -c \). The induction is then completed by checking explicitly (6.11) in the case \( k = 2 \).

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