Dunkl Operators at Infinity and Calogero–Moser Systems

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We define the Dunkl and Dunkl–Heckman operators in infinite number of variables and use them to construct the quantum integrals of the Calogero–Moser–Sutherland (CMS) problems at infinity. As a corollary, we have a simple proof of integrability of the deformed quantum CMS systems related to classical Lie superalgebras. We show how this naturally leads to a quantum version of the Moser matrix, which in the deformed case was not known before.

1 Introduction

The usual Calogero–Moser, or Calogero–Moser–Sutherland (CMS), system describes the interaction of \(N\) particles with equal masses on the line with the inverse square potential or, in the trigonometric version, with the inverse \(\sin^2\) potential [6]. The corresponding quantum Hamiltonian has the form

\[ H_N = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \frac{2k(k+1)}{(x_i - x_j)^2} \]  

(1)

Received February 6, 2014; Revised December 29, 2014; Accepted January 5, 2015

Communicated by Igor Krichever

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in the rational case and

$$H_N = - \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \frac{2k(k+1)}{\sin^2(x_i - x_j)}$$  \hfill (2)$$

in the trigonometric case. There is also a very important elliptic case, but we will not consider it in this paper.

The CMS systems admit natural generalizations related to root systems and simple Lie algebras [20], and, at the quantum level only, nonsymmetric integrable versions called deformed CMS systems [7], which were shown to be related to basic classical Lie superalgebras in [21]. In particular, in the case of Lie superalgebra $\mathfrak{sl}(m, n)$ we have two groups of particles with two different masses described by the following Hamiltonian:

$$H_{n,m} = - \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) - k \left( \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_m^2} \right) + \sum_{i<j} \frac{2k(k+1)}{\sin^2(x_i - x_j)}$$

$$+ \sum_{i<j} \frac{2(k^{-1} + 1)}{\sin^2(y_i - y_j)}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{2(k+1)}{\sin^2(x_i - y_j)}.$$  \hfill (3)$$

The importance of the deformed CMS systems became clear after the discovery of their deep relations with the theory of generalized discriminants and coincident root loci [22, 23] and with the representation theory of Lie superalgebras [26, 27] as well as of an intriguing link with the theory of logarithmic Frobenius structures [12].

The integrability of the deformed CMS systems turned out to be a quite nontrivial question. The standard methods like the Dunkl operator technique are not working in the general deformed case (for special values of parameters see recent Feigin’s paper [11]). For the classical series $A(n, m)$ and $BC(n, m)$, the integrability was proved in [21] by explicit construction of the quantum integrals. The recurrent procedure was a guesswork based on formulas from Matsuo [16] and the result was proved by straightforward lengthy calculations.

The goal of this paper is to give probably the simplest explanation of these integrals. Our main tool is the Dunkl operator at infinity, which seems to be not considered before. We show that although it does not allow to construct the Dunkl operators in the deformed case, it naturally leads to the quantum Moser matrix for the deformed CMS system.

This gives an interpretation of the integrals of the deformed CMS systems from [21] in terms of the quantum Lax pair, which, for the usual CMS system was first considered by Ujino et al. [31] and Wadati et al. [32]. Note that in contrast to the usual case
for the deformed CMS systems, there is no classical Lax pair to quantize, since their classical counterparts are believed to be nonintegrable.

For the deformed CMS system (3), the quantum Moser matrix is the following $(n + m) \times (n + m)$ matrix with the noncommuting entries

\[ L_{ii} = k^{p(i)} \frac{\partial}{\partial x_i}, \quad L_{ij} = k^{1-p(j)} \cot(x_i - x_j), \quad i \neq j, \]

(4) where $x_{n+i} := y_j, \quad j = 1, \ldots, m$, and $p(i) = 0$ for $i = 1, \ldots, n$, $p(i) = 1$ for $i = n+1, \ldots, n+m$. The quantum integrals of (3) can be constructed as the “deformed total trace” of the powers of $L$

\[ I_r = \sum_{i,j=1}^{n+m} k^{-p(i)} (L^r)_{ij}, \quad r = 1, 2, \ldots \]

(5) (see Section 5). We present similar formulae in the $BC$ case as well.

Another result of the paper is the new formulae for the quantum CMS integrals at infinity in both rational and trigonometric cases for types $A$ and $BC$. In the trigonometric case of type $A$, some formulae for the quantum integrals at infinity were recently found in [18] by Nazarov and Sklyanin. We comment on the relation with our results in the last section.

2 Dunkl Operator at Infinity: Rational Case

The usual Dunkl operators in dimension $N$ have the form

\[ D_{i,N} = \frac{\partial}{\partial x_i} - k \sum_{j \neq i}^{N} \frac{1}{x_i - x_j} (1 - \sigma_{ij}), \quad i = 1, 2, \ldots, N, \]

(6) where $\sigma_{ij}$ acts on the functions $f(x)$ by permuting variables $x_i$ and $x_j$. Their main property is the commutativity [9]

\[ [D_{i,N}, D_{j,N}] = 0. \]

Heckman [13] made an important observation that the differential operators

\[ L_N^{(r)} = \text{Res}(D_{1,N}^r + \cdots + D_{N,N}^r), \]

(7) where Res means the operation of restriction on the space of symmetric polynomials, commute, and give the integrals for the quantum CMS system. More precisely, $L_N^{(2)} = \mathcal{H}_N$, where

\[ \mathcal{H}_N = \text{Res}(D_{1,N}^2 + \cdots + D_{N,N}^2). \]
where the operator $H_N$ is the gauged (and opposite sign) version of the CMS operator (1) given by

$$
H_N = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i<j} \frac{2k}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right). \tag{8}
$$

Note that the operator (8) preserves the algebra of symmetric polynomials $\Lambda_N = \mathbb{C}[x_1, \ldots, x_N]^{S_N}$, generated (not freely) by $p_j(x) = x_1^j + \cdots + x_N^j$, $j \in \mathbb{Z}_{>0}$.

Let $\Lambda$ be the algebra of symmetric functions defined as the inverse limit of $\Lambda_N$ in the category of graded algebras (see [15]). We consider also the larger algebra $\tilde{\Lambda} = \Lambda[p_0]$, which is the commutative algebra with free generators $p_i$, $i \in \mathbb{Z}_{\geq 0}$. The dimension $p_0 = 1 + 1 + \cdots + 1 = N$ does not make sense in infinite-dimensional case, so we add $p_0$ as an additional variable (cf. [24, 25]). $\tilde{\Lambda}$ has a natural grading, where the degree of $p_i$ is $i$.

Define now the infinite-dimensional Dunkl operator $D_\infty : \tilde{\Lambda}[x] \to \tilde{\Lambda}[x]$ by

$$
D_\infty = \partial - k\Delta, \tag{9}
$$

where the derivation $\partial$ in $\tilde{\Lambda}[x]$ is defined by the formulae

$$
\partial(x) = 1, \quad \partial(p_l) = lx^{l-1}, \quad l \in \mathbb{Z}_{\geq 0},
$$

and the operator $\Delta : \tilde{\Lambda}[x] \to \tilde{\Lambda}[x]$ is defined by

$$
\Delta(x^l f) = \Delta(x^l) f, \quad \Delta(1) = 0, \quad f \in \tilde{\Lambda}, \quad l \in \mathbb{Z}_{\geq 0}
$$

and

$$
\Delta(x^l) = x^{l-1} p_0 + x^{l-2} p_1 + \cdots + x p_{l-2} + p_{l-1} - lx^{l-1}. \quad l > 0.
$$

The motivation is given by the following proposition. Let

$$
\varphi_{i,N} : \tilde{\Lambda}[x] \longrightarrow \Lambda_N[x_i]
$$

be the homomorphism defined by

$$
\varphi_{i,N}(x) = x_i, \quad \varphi_{i,N}(p_l) = x_1^l + \cdots + x_N^l, \quad l \in \mathbb{Z}_{\geq 0}.
$$
Proposition 2.1. The following diagram
\[ \begin{array}{ccc}
\tilde{\Lambda}[x] & \xrightarrow{D_{\infty}} & \tilde{\Lambda}[x] \\
\downarrow \varphi_{i,N} & & \downarrow \varphi_{i,N} \\
\Lambda_N[x_l] & \xrightarrow{D_{i,N}} & \Lambda_N[x_l]
\end{array} \tag{10} \]
where \( D_{i,N} \) are the Dunkl operators (6), is commutative.

\[ \square \]

Proof. We have
\[ \varphi_{i,N} \circ \Delta = \left( \sum_{j \neq i} \frac{1}{x_l - x_j} (1 - \sigma_{ij}) \right) \circ \varphi_{i,N} \]
since
\[ \sum_{j \neq i} \frac{1}{x_l - x_j} (1 - \sigma_{ij}) x_l^j = \sum_{j \neq i} \frac{x_l^j - x_l^j}{x_l - x_j} = x_l^{l-1} N + x_l^{l-2} p_l + \cdots + x_l p_{l-2} + p_{l-1} - l x_l^{l-1}. \]
The relation \( \varphi_{i,N} \circ \partial = \partial_i \circ \varphi_{i,N} \) is obvious. \[ \square \]

Introduce also a linear operator \( E : \tilde{\Lambda}[x] \longrightarrow \tilde{\Lambda} \) by the formula
\[ E(x_l^f) = p_l f, \quad f \in \tilde{\Lambda}, \quad l \in \mathbb{Z}_{\geq 0} \tag{11} \]
and define the operators \( \mathcal{L}^{(r)} : \tilde{\Lambda} \longrightarrow \tilde{\Lambda}, \quad r \in \mathbb{Z}_+ \) by
\[ \mathcal{L}^{(r)} = \text{Res} E \circ D_{\infty}^r, \tag{12} \]
where Res means the restriction to \( \tilde{\Lambda} \).

We claim that these operators give the quantum CMS integrals at infinity. More precisely, we have the following result.

Theorem 2.2. The operator \( \mathcal{L}^{(r)} \) is a differential operator in \( \tilde{\Lambda} \) of order \( r \), which commutes with others:
\[ [\mathcal{L}^{(r)}, \mathcal{L}^{(s)}] = 0. \]

The operator \( \mathcal{L}^{(2)} \) has the following explicit form:
\[ \mathcal{L}^{(2)} = \sum_{a,b \geq 1} P_a P_{a+b-2} \partial_a \partial_b - k \sum_{a,b \geq 0} P_a P_b \partial_{a+b+2} + (1 + k) \sum_{a \geq 2} (a - 1) P_{a-2} \partial_a \tag{13} \]
with \( \partial_a = a \partial / \partial p_a \), and defines an infinite-dimensional version of the rational CMS operator (8) at infinity. \[ \square \]
Proof. To show that \( \mathcal{L}^{(r)} \) is a differential operator of order \( r \), we can essentially repeat Heckman’s arguments from [13]. Recall that \( \mathcal{L} \) is a differential operator in \( \tilde{A} \) of order \( r \) if \( \text{ad}(f)^{r+1} \mathcal{L} = 0 \) for any \( f \in \tilde{A} \), but in general \( \text{ad}(f)^r \mathcal{L} \neq 0 \). Since \( E \) and \( \Delta \) commute with multiplication by \( f \), we have

\[
\text{ad}(f)^{r+1}(\mathcal{L}^{(r)}) = \text{Res} E \circ \text{ad}(f)^{r+1}(D_\infty^r).
\]

Since

\[
\text{ad}(f)(D_\infty) = \text{ad}(f)(\partial) = -\partial f,
\]

which implies \( \text{ad}(f)^2(D_\infty) = 0 \) and hence \( \text{ad}(f)^{r+1}(D_\infty^r) = 0 \). On the other hand, \( \text{ad}(f)^r(D_\infty^r) = r!(\partial f)^r \), which implies that \( \text{ad}(p_1)\mathcal{L}^{(r)}(p_1)^r \neq 0 \), so \( \mathcal{L}^{(r)} \) is a differential operator of order \( r \).

The explicit form (13) easily follows from a direct calculation. As far as we know such a formula first appeared in [24, 25], although in the trigonometric case a similar formula (see (38)) was essentially known to Stanley [30]. An important advantage of the trigonometric case is stability, which means that in that case we do not need an additional variable \( p_0 \).

In order to prove the commutativity, we consider the finite-dimensional reductions. For every natural \( N \), there is a homomorphism \( \varphi_N : \tilde{A} \to \Lambda_N \) defined by

\[
\varphi_N(p_j) = x_1^j + \cdots + x_N^j, \quad j \in \mathbb{Z}_{\geq 0}.
\]

We have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{A} & \xrightarrow{\mathcal{L}^{(r)}} & \tilde{A} \\
\downarrow \varphi_N & & \downarrow \varphi_N \\
\Lambda_N & \xrightarrow{\mathcal{L}_N^{(r)}} & \Lambda_N
\end{array}
\]

where \( \mathcal{L}_N^{(r)} \) are the CMS integrals given by Heckman’s construction (7). Indeed, for any \( f \in \tilde{A} \) we have \( D_{\infty}^r(f) = \sum_l x_l^j g_l, g_l \in \tilde{A} \), where the sum is finite. We have, by Proposition 2.1,

\[
D_{i,N}^r \circ \varphi_N(f) = \varphi_{i,N} \circ D_{\infty}^r(f) = \sum_l x_l^j \varphi_N(g_l),
\]

\[
\sum_{i=1}^{N} D_{i,N}^r \circ \varphi_N(f) = \sum_{i=1}^{N} \sum_l x_l^j \varphi_N(g_l) = \sum_l \varphi_N(p_i) \varphi_N(g_l) = \varphi_N(E(D_{\infty}^r(f))).
\]
which proves the commutativity of the diagram. This implies that
\[
\varphi_N([L^{(r)}, L^{(s)}](f)) = [L^{(r)}_N, L^{(s)}_N](\varphi_N(f)) = 0
\]
since the integrals (7) commute [14]. To conclude the proof, we need the following lemma.

**Lemma 2.3.** Let \( f \) be an element of \( \hat{A} \). If \( \varphi_N(f) = 0 \) for all \( N \); then \( f = 0 \).

**Proof.** By definition, \( f \) is a polynomial in a finite number of generators \( p_r \), \( 1 \leq r \leq M \) for some \( M \) with coefficients polynomially depending on \( p_0 \). Take \( N \) bigger than this number \( M \). Since the corresponding \( \varphi_N(p_r) \) with \( 1 \leq r \leq M \) are algebraically independent in \( \Lambda_N \) and \( \varphi_N(f) = 0 \), all the coefficients of \( f \) are zero at \( p_0 = N \). Since this is true for all \( N > M \), the coefficients are identically zero, and therefore \( f = 0 \).

Applying lemma, we have the commutativity \([L^{(r)}, L^{(s)}] = 0\).

### 3 Deformed CMS Operators: Rational Case

The deformed CMS operators in the rational case have the form
\[
H_{n,m} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + k \sum_{i=1}^{m} \frac{\partial^2}{\partial y_i^2} - \sum_{i<j}^{n} \frac{2k(k+1)}{(x_i - x_j)^2} - \sum_{i<j}^{m} \frac{2(k^{-1}+1)}{(y_i - y_j)^2} - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{2(k+1)}{(x_i - y_j)^2}.
\](16)

They describe the interaction of two groups of particles on the line with masses 1 and \( 1/k \), respectively. When \( k = 1 \), we have the usual CMS system with \( n + m \) particles, hence the terminology.

The operators (16) for \( m = 1 \) were introduced and studied by Chalykh et al. in [7]; for general \( m \) they were considered by Berest and Yakimov [2]. Their integrability was first proved in [21], where the quantum integrals were constructed by a recursive procedure. We are going to show now that this procedure has a simple explanation in terms of Dunkl operators.

Let \( \mathcal{H}_{n,m} = \psi_0 L_{n,m} \psi_0^{-1} \) be the gauged form of (16) with
\[
\psi_0 = \prod_{i<j}^{n} (x_i - x_j)^k \prod_{i<j}^{m} (y_i - y_j)^{\frac{1}{k}} \prod_{i}^{n} (x_i - y_j).
\]

\[
\mathcal{H}_{n,m} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + k \sum_{i=1}^{m} \frac{\partial^2}{\partial y_i^2} - \sum_{i<j}^{n} \frac{2k}{(x_i - x_j)^2} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)
- \sum_{i<j}^{m} \frac{2}{(y_i - y_j)^2} \left( \frac{\partial}{\partial y_i} - k \frac{\partial}{\partial y_j} \right) - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{2}{(x_i - y_j)^2} \left( \frac{\partial}{\partial x_i} - k \frac{\partial}{\partial y_j} \right).
\](17)
It is convenient to define \( y_j = x_{n+j}, \ j = 1, \ldots, m, \) and introduce the parity function \( p(i) = 0, \ i = 1, \ldots, n \) and \( p(i) = 1, \ i = n+1, \ldots, n+m. \)

Following [21], consider the operators \( \partial_i^{(r)} \) defined recursively by

\[
\partial_i^{(r)} = \partial_i^{(1)} \partial_i^{(r-1)} - \sum_{j \neq i} \frac{k^1 - p(j)}{x_i - x_j} (\partial_i^{(r-1)} - \partial_j^{(r-1)})
\]

with \( \partial_i^{(1)} = k^{p(i)} \frac{\partial}{\partial x_i} \) (cf. [21, Formula (17)]). One can easily check that the operator

\[
L_{n,m}^{(2)} = \sum_{i=1}^{n+m} k^{p(i)} \partial_i^{(2)}
\]

coincides with the deformed CMS operator (17). In [21], it was proved by a lengthy but direct calculation that the operators

\[
L_{n,m}^{(r)} = \sum_{i=1}^{n+m} k^{p(i)} \partial_i^{(r)}
\]

commute with each other, and in particular are the quantum integrals of the deformed CMS system. The recursion formulae (18) were guesswork based on the recursive Matsuo’s formulae [16].

Now we are going to give a much simpler proof of this together with more conceptual explanation of these formulae.

Define \( \varphi_{n,m}^{(i)} : \Lambda[x] \rightarrow \mathbb{C}[x_1, \ldots, x_{n+m}] \) by \( \varphi_{n,m}^{(i)}(x) = x_i \) and

\[
\varphi_{n,m}^{(i)}(p_l(x, k)) := \sum_{i=1}^{n+m} k^{-p(i)} x_i^j = \sum_{i=1}^{n} x_i^j + \frac{1}{k} \sum_{i=n+1}^{n+m} x_i^j,
\]

for all \( l \in \mathbb{Z}_{\geq 0}. \)

Denote by \( \Lambda_{n,m} \) the subalgebra in \( \mathbb{C}[x_1, \ldots, x_{n+m}] \) generated by the deformed power sums \( p_l(x, k), \ l \in \mathbb{Z}_{\geq 0}. \) We will show that the operators \( \partial_i^{(r)} \) map the algebra \( \Lambda_{n,m} \) into \( \Lambda_{n,m}[x_i] \) (see diagram (23)).

In the deformed case, we do not have the commutative diagram similar to (10), but we have the following important relation.

**Proposition 3.1.** The following relation holds on \( \Lambda[x]: \)

\[
\varphi_{n,m}^{(i)} \circ D_\infty = k^{p(i)} \frac{\partial}{\partial x_i} \circ \varphi_{n,m}^{(i)} - \sum_{j \neq i} \frac{k^1 - p(j)}{x_i - x_j} (\varphi_{n,m}^{(i)} - \varphi_{n,m}^{(j)}).
\]

\[\square\]
Proof. For any \( f \in \tilde{\Lambda} \), we have
\[
\varphi_{n,m}^{(i)} \circ (\partial - k \Delta)(x^f) = \varphi_{n,m}^{(i)}(lx^{f-1}f + x^f \partial f) = k(lx^{f-1}p_0 + x^{f-2}p_1 + \cdots + xp_{t-2} + p_{t-1} - lx^{f-1})f) = lx^{f-1}(1 + k)\varphi_{n,m}(f) + x^f \varphi_{n,m}(\partial f)
\]

On the other hand,
\[
k^{p(i)} \frac{\partial}{\partial x_i} \circ \varphi_{n,m}^{(i)}(x^f) - \sum_{j \neq i} k^{1-p(j)} \varphi_{n,m}^{(j)}(x^f) = k^{p(i)}lx^{f-1}(1 + k)\varphi_{n,m}(f) + x^f \varphi_{n,m}(\partial f) = k^{p(i)}lx^{f-1}(1 + k)\varphi_{n,m}(f) + x^f \varphi_{n,m}(\partial f)
\]

Since \( k^{p(i)} + k^{1-p(i)} = 1 + k \) for all \( i = 1, \ldots, n + m \), we only need to show that
\[
\varphi_{n,m}^{(i)}(\partial f) = k^{p(i)} \partial_i \varphi_{n,m}^{(i)}(f).
\]

Since both \( \partial \) and \( \partial_i \) are the derivations, it is enough to check this for \( f = p_i \), which is obvious.

Proposition 3.2. The following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{\Lambda} & \xrightarrow{D^r_{\infty}} & \tilde{\Lambda}[x] \\
\downarrow \varphi_{n,m}^{(i)} & & \downarrow \varphi_{n,m}^{(i)} \\
\Lambda_{n,m} & \xrightarrow{\partial_{\infty}^{(i)}} & \Lambda_{n,m}[x]
\end{array}
\]

Proof. We use the induction in \( r \). When \( r = 1 \), this follows from (22). For \( r > 1 \), we have
\[
\varphi_{n,m}^{(i)}(D^r_{\infty}(f)) = \varphi_{n,m}^{(i)}(D_\infty(g)),
\]

where \( g = D^r_{\infty}f \in \tilde{\Lambda}[x] \). By previous proposition,
\[
\varphi_{n,m}^{(i)}(D_\infty(g)) = \partial_i^{(1)} \circ \varphi_{n,m}^{(i)}(g) - \sum_{j \neq i} k^{1-p(j)} \varphi_{n,m}^{(j)}(g) - \varphi_{n,m}^{(j)}(g)).
\]
By inductive assumption \( \varphi_{n,m}^{(i)}(g) = \partial_i^{(r-1)} \varphi_{n,m}^{(i)}(f) \) and thus
\[
\varphi_{n,m}^{(i)}(D_\infty(g)) = \left( \partial_i^{(1)} \partial_i^{(r-1)} - \sum_{j \neq i} \frac{k^{1-p(j)}}{x_i - x_j} (\partial_j^{(r-1)} - \partial_j^{(r-1)}) \right) \varphi_{n,m}^{(i)}(f),
\]
where we have used that, for \( f \in \tilde{\Lambda} \), \( \varphi_{n,m}^{(i)}(f) = \varphi_{n,m}^{(j)}(f) \). Thus,
\[
\varphi_{n,m}^{(i)}(D_\infty^r(f)) = \partial_i^{(r)} \varphi_{n,m}^{(i)}(f),
\]
which concludes the proof.

Define the homomorphism \( \varphi_{n,m} : \tilde{\Lambda} \to \Lambda_{n,m} \) by
\[
\varphi_{n,m}(l) = \sum_{i=1}^{n+m} k^{-\rho(i)} x_i^l, \quad l \in \mathbb{Z}_{\geq 0}.
\]

**Theorem 3.3.** The following diagram is commutative:
\[
\begin{array}{ccc}
\tilde{\Lambda} & \xrightarrow{L^{(r)}} & \tilde{\Lambda} \\
\downarrow \varphi_{n,m} & & \downarrow \varphi_{n,m} \\
\Lambda_{n,m} & \xrightarrow{L^{(r)}_{n,m}} & \Lambda_{n,m}
\end{array}
\]
(24)
where \( L^{(r)} \) are the CMS integrals (12) at infinity and \( L^{(r)}_{n,m} \) are the operators (19). In particular, the latter operators commute:
\[
[L^{(r)}_{n,m}, L^{(s)}_{n,m}] = 0
\]
for all \( r, s \in \mathbb{Z}_{>0} \), and thus give the quantum integrals of the deformed CMS system.

**Proof.** For any \( f \in \Lambda \), we have
\[
D_\infty^r(f) = \sum_l x_l^l g_l, \quad g_l \in \Lambda,
\]
where the sum in the right-hand side is finite. By proposition 3.2,
\[
\partial_i^{(r)}(\varphi_{n,m}(f)) = \varphi_{n,m}^{(i)}(D_\infty^r(f)) = \sum_l x_l^l \varphi_{n,m}(g_l).
\]
Hence, we have
\[
L^{(r)}_{n,m} \varphi_{n,m}(f) = \sum_{i=1}^{n+m} k p^{(i)} \varphi_{n,m}(f) = \sum_{i=1}^{n+m} k p^{(i)} \sum_{l} x_l^i \varphi_{n,m}(g_l)
\]
\[
= \sum_{l} \varphi_{n,m}(p(x, k)) \varphi_{n,m}(g_l) = \varphi_{n,m}(L^{(r)}(f)).
\]
This proves the commutativity of the diagram. The commutativity of the operators (19) now follows from the commutativity of the CMS integrals (12) at infinity. ■

4 Quantum Moser Matrix for the Deformed CMS System

In contrast to the usual CMS case, the classical version of the deformed CMS system is believed to be not integrable; see [7]. This means that there is no proper replacement for following Moser matrix in the classical case

\[
L = \begin{pmatrix}
p_1 & \frac{k}{q_1 - q_2} & \frac{k}{q_1 - q_3} & \cdots & \cdots & \frac{k}{q_1 - q_n} \\
\frac{k}{q_1 - q_2} & p_2 & \frac{k}{q_2 - q_3} & \cdots & \cdots & \frac{k}{q_2 - q_n} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{k}{q_1 - q_n} & \frac{k}{q_2 - q_n} & \frac{k}{q_3 - q_n} & \cdots & \frac{k}{q_{n-1} - q_n} & p_n
\end{pmatrix}
\]

Recall that Moser has shown that the equations of motion of the classical CMS system with
\[
H = \sum_{i=1}^{n} p_i^2 - \sum_{i<j}^{n} \frac{2k^2}{(q_i - q_j)^2}
\]
can be rewritten in the Lax form as
\[
\dot{L} = [L, M],
\]
where
\[
M = -2k \begin{pmatrix}
a_{11} & \frac{1}{(q_1 - q_2)^2} & \frac{1}{(q_1 - q_3)^2} & \cdots & \cdots & \frac{1}{(q_1 - q_n)^2} \\
\frac{1}{(q_1 - q_2)^2} & a_{22} & \frac{1}{(q_2 - q_3)^2} & \cdots & \cdots & \frac{1}{(q_2 - q_n)^2} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{1}{(q_1 - q_n)^2} & \frac{1}{(q_2 - q_n)^2} & \frac{1}{(q_3 - q_n)^2} & \cdots & \frac{1}{(q_{n-1} - q_n)^2} & a_{nn}
\end{pmatrix}
\]
with $a_{ii} = -\sum_{i \neq j}^{n} \frac{1}{(q_i - q_j)^2}$ (see [17]). Note that the last condition means that $Me = e^T M = 0$, where $e = (1, \ldots, 1)^T$.

The Dunkl operator approach from the previous section naturally leads to the following quantum Moser matrix $L$ for the deformed CMS system. The quantum analog of the Lax pair for the usual CMS systems was proposed in 1992 by Ujino et al. [31] and Wadati et al. [32] (see also Shastry and Sutherland [29]), but the quantum version of Moser matrix $L$ was used already in 1975 by Calogero et al. in [5] to produce the quantum integrals of the CMS system.

To derive the quantum Moser matrix in the deformed case, rewrite the relation (21) in the matrix form as

$$
\Phi_{n,m} \circ D_\infty = L \circ \Phi_{n,m},
$$

(25)

where $\Phi_{n,m}$ is the vector with components $\phi_{n,m}^{(i)}, \ i = 1, \ldots, n + m$ and $L$ is an $(n + m) \times (n + m)$ matrix with the (noncommuting) entries

$$
L_{ii} = k^{p(i)} \frac{\partial}{\partial x_i} - \sum_{j \neq i}^{n+m} \frac{k^{1-p(j)}}{x_i - x_j}, \quad L_{ij} = \frac{k^{1-p(j)}}{x_i - x_j}, \ i \neq j.
$$

(26)

The quantum Moser matrix $L$ is the following gauged version of $L$:

$$
L_{ii} = k^{p(i)} \frac{\partial}{\partial x_i}, \quad L_{ij} = \frac{k^{1-p(j)}}{x_i - x_j}, \ i \neq j.
$$

(27)

Note that the operators $L_{ii}$ and $L_{ii}$ are conjugated by the multiplication operator by

$$
\Psi_0 = \prod_{i<j}^{n+m} (x_i - x_j)^{k^{1-p(i)-p(j)}}.
$$

Introduce the corresponding matrix $M$ as the following simple modification of Moser’s matrix:

$$
M_{ij} = \frac{2k^{1-p(j)}}{(x_i - x_j)^2}, \ i \neq j, \quad M_{ii} = -\sum_{j \neq i}^{n+m} \frac{2k^{1-p(j)}}{(x_i - x_j)^2}.
$$

(28)

Note that this matrix has the properties $Me = 0$ (like in the usual case), and $e^* M = 0$, where $e^* = (1, \ldots, 1, \frac{1}{k}, \ldots, \frac{1}{k})$ (or, $e^*_i = k^{-p(i)}, \ i = 1, \ldots, n + m$) is the deformed dual to $e$.

Introduce also the “matrix Hamiltonian” $H$ which is a diagonal $(n + m) \times (n + m)$ matrix with the deformed CMS operator (16) on the diagonal:

$$
H_{ii} = H_{n,m}, \quad H_{ij} = 0, \ i \neq j.
$$
The commutator $[L, H]$ has the entries $[L, H]_{ij} = [L_{ij}, H_{n,m}]$, which can be considered as a quantum version of $\hat{L}$ (cf. [31, 32]).

**Theorem 4.1** (Quantum Lax pair for the deformed CMS system). We have the following identity:

$$[L, H] = [L, M]. \tag{29}$$

The proof is by direct calculation similarly to the usual case [31, 32].

**Corollary 4.2.** The operators $L_{n,m}^{(r)} = e^rLre$ are the quantum integrals of the deformed CMS system (16). These integrals coincide with the integrals from the previous section modulo conjugation by $\Psi_0$.

Indeed, from (29) we have $[L, H - M] = 0$, and hence $[L^r, H - M] = 0$, or

$$[L^r, H] = [L^r, M].$$

This implies

$$[L^{(r)}_{n,m}, H_{n,m}] = 0,$$

since $Me = e^rM = 0$ (cf. the case of the usual CMS system in [32]). This proves that $L_{n,m}^{(r)}$ are the integrals of the deformed CMS system. One can check that $L_{n,m}^{(r)} = H_{n,m}$. The integrals $L_{n,m}^{(r)}$ can be interpreted as the “deformed total trace” (cf. [32]) of the powers of quantum Moser matrix:

$$L_{n,m}^{(r)} = \sum_{i,j=1}^{m+n} k^{-p_i(L^r)}_{ij}. \tag{30}$$

Note that the fact that $L_{n,m}^{(r)}$ commute with each other does not follow from the Lax approach. In our case, this follows from the results of the previous section since $L_{n,m}^{(r)}$ are the gauged versions of $\mathcal{L}^{(r)}_{n,m}$. Alternatively, one can show first the commutativity of the diagram (24) with $\mathcal{L}^{(r)}_{n,m}$ defined as $e^r\mathcal{L}re$. Then the commutativity of $\mathcal{L}^{(r)}_{n,m}$ (and hence of $L_{n,m}^{(r)}$) follows from the commutativity of the operators $\mathcal{L}^{(r)}$.

5 **Trigonometric Case: Dunkl–Heckman Operator at Infinity**

In this section, we follow mainly our paper [28], where a more general Laurent case is considered. Since it is largely parallel to the rational case, we will omit most of the proofs. We will be also using the same letters to denote the similar quantities as in the rational case; hopefully, this will not lead to much of confusion.
In the trigonometric (hyperbolic) case, we have the following CMS operator:

\[ H_N = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - \sum_{i<j}^{N} \frac{2k(k+1)}{\sinh^2(z_i - z_j)}. \]

It has an eigenfunction \( \Psi_0 = \prod_{i<j}^{N} \sinh^{-k}(z_i - z_j) \) with the eigenvalue \( \lambda_0 = -k^2 N(N - 1)/4 \). Its gauged version \( \frac{1}{4} \frac{\partial}{\partial z_i} (L_N - \lambda_0) \Psi_0 \) in the exponential coordinates \( x_i = e^{2z_i} \) has the form

\[ \mathcal{H}_N = \sum_{i=1}^{N} \left( x_i \frac{\partial}{\partial x_i} \right)^2 - k \sum_{i<j}^{N} \frac{x_i + x_j}{x_i - x_j} \left( \frac{x_i}{\partial x_i} - \frac{x_j}{\partial x_j} \right). \quad (30) \]

The corresponding version of the Dunkl operator in this case was first introduced by Heckman \([14]\) and has the form

\[ D_{i,N} = \partial_i - \frac{k}{2} \sum_{j \neq i}^{N} \frac{x_i + x_j}{x_i - x_j} (1 - \sigma_{ij}), \quad \partial_i = x_i \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, N, \quad (31) \]

where as before \( \sigma_{ij} \) is a transposition, acting on the functions by permuting the coordinates \( x_i \) and \( x_j \). The main problem with these operators is that they do not commute. (We should note that there are commuting versions of Dunkl operators in the trigonometric case due to Cherednik \([8]\), but however, they lack the symmetry properties of Heckman’s versions, which are essential for us.) However, Heckman \([14]\) managed to show that the differential operators

\[ \mathcal{L}^{(r)}_N = \text{Res}(D_{1,N}^r + \cdots + D_{N,N}^r), \quad (32) \]

where \( \text{Res} \) means the operation of restriction on the space of symmetric polynomials, do commute with each other

\[ [\mathcal{L}^{(r)}_N, \mathcal{L}^{(s)}_N] = 0. \quad (33) \]

Since \( \mathcal{L}^{(2)}_N = \mathcal{H}_N \), they are the integrals of the quantum CMS system (30).

The operator

\[ \Delta_{i,N} := \sum_{j \neq i}^{N} \frac{x_i + x_j}{x_i - x_j} (1 - \sigma_{ij}) \quad (34) \]

acts trivially on the algebra of symmetric polynomials \( \Lambda_N \) and has the property

\[ \Delta_{i,N}(x_i^j) = \sum_{j \neq i}^{N} \frac{x_i + x_j}{x_i - x_j} (1 - \sigma_{ij})(x_i^j) = \sum_{j \neq i}^{N} \frac{x_i + x_j}{x_i - x_j} (x_i^j - x_i^j) = x_i^j N + 2x_i^{j-1} p_1 + \cdots + 2x_i p_{-1} + p_1 - 2l x_i. \quad (35) \]
Define the \textit{infinite-dimensional Dunkl–Heckman operator} \( D_\infty : \tilde{\Lambda}[x] \to \tilde{\Lambda}[x] \) by

\[
D_\infty = \partial - \frac{1}{2} k \Delta,
\]

where the derivation \( \partial \) in \( \tilde{\Lambda}[x] \) is defined by the formulae

\[
\partial(x) = x, \quad \partial(p_l) = lx^l, \quad l \in \mathbb{Z}_{\geq 0},
\]

and the operator \( \Delta : \tilde{\Lambda}[x] \to \tilde{\Lambda}[x] \) is defined by

\[
\Delta(x^l f) = \Delta(x^l) f, \quad \Delta(1) = 0, \quad f \in \tilde{\Lambda}, \quad l \in \mathbb{Z}_{\geq 0}
\]

and

\[
\Delta(x^l) = x^l p_0 + 2x^{l-1} p_1 + \cdots + 2xp_{l-1} + p_l - 2lx^l, \quad l > 0.
\]

One can check that the following diagram

\[
\begin{array}{ccc}
\tilde{\Lambda}[x] & \xrightarrow{D_\infty} & \tilde{\Lambda}[x] \\
\downarrow \varphi_{i,N} & & \downarrow \varphi_{i,N} \\
\Lambda_N[x_i] & \xrightarrow{D_{i,N}} & \Lambda_N[x_i]
\end{array}
\]

is commutative, where \( D_{i,N} \) are the Dunkl–Heckman operators (31), and \( \varphi_{i,N}(x) = x_i \), \( \varphi_{i,N}(p_l) = x_i^l + \cdots + x_i^N \), \( l \geq 0 \), as before.

Let \( E : \tilde{\Lambda}[x] \longrightarrow \tilde{\Lambda} \) be the same as above: \( E(x^l f) = p_l f \), \( f \in \tilde{\Lambda}, \quad l \in \mathbb{Z}_{\geq 0} \). Define the operators \( \mathcal{L}^{(r)} : \tilde{\Lambda} \longrightarrow \tilde{\Lambda}, \quad r \in \mathbb{Z}_+ \) by

\[
\mathcal{L}^{(r)} = \text{Res } E \circ D_{\infty}^r,
\]

where \( \text{Res} \) means that the action of the right-hand side is restricted to \( \tilde{\Lambda} \).

The operator \( \mathcal{L}^{(2)} \) has the following explicit form:

\[
\mathcal{L}^{(2)} = \sum_{a,b > 0} p_{a+b} \partial_a \partial_b - k \sum_{a,b > 0} p_a p_b \partial_{a+b} + (1 + k) \sum_{a > 0} a p_a \partial_a - k p_0 \sum_{a > 0} p_a \partial_a,
\]

where \( \partial_a = a \frac{\partial}{p_a} \), and is known to be the (trigonometric) CMS operator at infinity (see [1, 24, 30]).

Note that the dependence on \( p_0 \) in the trigonometric case can be easily eliminated since \( \sum_{a > 0} p_a \partial_a \) is the total momentum, which corresponds to the stability property of the CMS operator in this case (see the discussion in [24]).
The claim is that the operators (37) commute:

\[ [L^{(r)}, L^{(s)}] = 0, \]  

(39)

and thus are the quantum CMS integrals at infinity. This follows from the commutativity of Heckman’s integrals (32), Lemma 2.3, and the commutativity of the diagram

\[
\Lambda_N \xrightarrow{L_N^{(r)}} \Lambda_N \xrightarrow{\varphi_N} \Lambda_N
\]

where \(L_N^{(r)}\) are the CMS integrals given by (32) and the homomorphism \(\varphi_N : \tilde{\Lambda} \to \Lambda_N\) is defined by \(\varphi_N(p_l) = x_1^l + \cdots + x_N^l, \ l \geq 0\).

Consider now the deformed CMS operator [21], which in the exponential coordinates has the form

\[
H_{n,m} = \sum_{i=1}^{n} \left( x_i \frac{\partial}{\partial x_i} \right)^2 + k \sum_{j=1}^{m} \left( y_j \frac{\partial}{\partial y_j} \right)^2 - \sum_{i<j}^{n} \frac{2k(k+1)x_i x_j}{(x_i - x_j)^2} - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{2(k+1)x_i y_j}{(x_i - y_j)^2}.
\]  

(40)

or, in the gauged form,

\[
\mathcal{H}_{n,m} = \sum_{i=1}^{n} \left( x_i \frac{\partial}{\partial x_i} \right)^2 + k \sum_{j=1}^{m} \left( y_j \frac{\partial}{\partial y_j} \right)^2 - k \sum_{1 \leq i < j \leq n} \frac{2(k+1)x_i x_j}{x_i - x_j} \left( x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right) - \sum_{1 \leq i < j \leq m} \frac{y_i + y_j}{y_i - y_j} \left( y_i \frac{\partial}{\partial y_i} - y_j \frac{\partial}{\partial y_j} \right) - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{x_i + y_j}{x_i - y_j} \left( y_i \frac{\partial}{\partial y_i} - k y_j \frac{\partial}{\partial y_j} \right).\]  

(41)

We use the notation from Section 3: let \(y_j = x_{n+j}, \ j = 1, \ldots, m,\) and \(p(i)\) be the parity function, \(\varphi_{n,m}^{(i)} : \Lambda[x] \to \mathbb{C}[x_1, \ldots, x_{n+m}]\) be defined by (20) and \(\Lambda_{n,m}\) be the subalgebra in \(\mathbb{C}[x_1, \ldots, x_{n+m}]\) generated by the deformed power sums

\[
p(x, k) = \sum_{i=1}^{n} x_i + \frac{1}{k} \sum_{i=n+1}^{n+m} x_i.
\]

One can check that the following equality is valid in \(\Lambda[x]s:\)

\[
\varphi_{n,m}^{(i)} \circ D_\infty = k^{p(i)} \frac{\partial}{\partial x_i} \varphi_{n,m}^{(i)} - \frac{1}{2} \sum_{j \neq i} k^{1-p(j)} \frac{x_i + x_j}{x_i - x_j} (\varphi_{n,m}^{(i)} - \varphi_{n,m}^{(j)}).\]  

(42)
After the gauge transformation with
\[ \Psi_0 = \prod_{i<j} \left( \frac{x_i x_j}{(x_i - x_j)^2} \right) \]
this leads us to the following quantum version of Moser's matrix in the deformed trigonometric case
\[ L_{ii} = k^{p(i)} \partial_i, \quad L_{ij} = \frac{1}{2} k^{1-p(j)} \frac{x_i + x_j}{x_i - x_j}, \quad i \neq j. \tag{43} \]

Define also \((n + m) \times (n + m)\) matrix \(M\) by
\[ M_{ij} = \frac{2k^{1-p(j)} x_i x_j}{(x_i - x_j)^2}, \quad i \neq j, \quad M_{ii} = -\sum_{j \neq i} \frac{2k^{1-p(j)} x_i x_j}{(x_i - x_j)^2}. \tag{44} \]

Let \( e \) and \( e^* \) be the same as in Corollary 4.2, and \( H \) is defined by \( H_{ii} = H_{n,m}, \quad H_{ij} = 0, \quad i \neq j, \) with \( L_{n,m} \) defined by (40). One can check that, as in the rational case, we have the quantum Lax relation
\[ [L, H] = [L, M], \]
leading to the following set of integrals.

**Theorem 5.1.** The operators
\[ L_{n,m}^{(r)} = e^* L^r e = \sum_{i,j=1}^{m+n} k^{-p(i)} (L^r)_{ij} \tag{45} \]
are the commuting quantum integrals of the deformed CMS system (40). \( \square \)

To prove the commutativity, we should consider the gauged version \( \mathcal{L} \) of the matrix \( L \)
\[ \mathcal{L}_{ii} = k^{p(i)} \partial_i - \frac{1}{2} \sum_{j \neq i} k^{1-p(j)} \frac{x_i + x_j}{x_i - x_j}, \quad \mathcal{L}_{ij} = \frac{1}{2} k^{1-p(j)} \frac{x_i + x_j}{x_i - x_j}, \quad i \neq j \tag{46} \]
and define the operators
\[ \mathcal{L}_{n,m}^{(r)} = e^* \mathcal{L}^r e \tag{47} \]
with \( \mathcal{L}_{n,m}^{(2)} \) being the quantum Hamiltonian of the deformed CMS system (41). Similarly to the rational case, one can show that the diagram (24) is commutative. Since the operators \( \mathcal{L}_{n,m}^{(r)} \) commute with each other, the same is true for \( \mathcal{L}_{n,m}^{(r)} \), and hence for \( L_{n,m}^{(r)} \).
6 Rational B-Type Case

The proofs are pretty similar to the type A case, so we only provide the modified formulae.

The rational CMS operator of type $B_N$ has the form

$$H_N = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \sum_{i<j}^{N} \frac{2k(k+1)}{(x_i-x_j)^2} - \sum_{i<j}^{N} \frac{2k(k+1)}{(x_i+x_j)^2} - \sum_{i=1}^{N} \frac{q(q+1)}{x_i^2}$$

and depends on two parameters $k$ and $q$. Its gauged version $\mathcal{H}_N = \delta H_N \delta^{-1}$ with

$$\delta = \prod_{i<j}^N (x_i - x_j)^k (x_i + x_j)^k \prod_i^N x_i^q$$

is

$$\mathcal{H}_N = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - \sum_{i<j}^{N} \frac{2k}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) - \sum_{i<j}^{N} \frac{2k}{x_i + x_j} \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) - \sum_{i=1}^{N} \frac{2q}{x_i} \frac{\partial}{\partial x_i}. \quad (48)$$

The CMS operator (48) preserves the algebra of symmetric polynomials $\Lambda_N = \mathbb{C}[x_1, \ldots, x_N]^{W_N}$ with respect to the group $W_N = S_N \rtimes \mathbb{Z}^N_2$ generated (not freely) by $p_j(x) = x_i^{2j} + \cdots + x_N^{2j}$, $j \in \mathbb{Z}_{\geq 0}$. The group $W_N$ is generated by the reflections

$$\sigma_{ij}^+ : (x_i, x_j) \to (x_j, x_i), \quad \sigma_{ij}^- : (x_i, x_j) \to (-x_j, -x_i), \quad 1 \leq i < j \leq N$$

and

$$\tau_i : x_i \to -x_i, \quad i = 1, \ldots, N$$

(leaving the other coordinates untouched).

The Dunkl operators in this case have the form

$$D_{i,N} = \frac{\partial}{\partial x_i} - k \sum_{j \neq i} \left( \frac{1}{x_i - x_j} (1 - \sigma_{ij}^+) + \frac{1}{x_i + x_j} (1 - \sigma_{ij}^-) \right) - \frac{P}{x_i} (1 - \tau_i). \quad (49)$$

where $i = 1, 2, \ldots, N$. These operators commute [9] and can generate the quantum integrals of the corresponding CMS system as

$$\mathcal{L}_N^{(2r)} = \text{Res} \left( D_{1,N}^{2r} + \cdots + D_{N,N}^{2r} \right), \quad (50)$$

where Res means the restriction to $\Lambda_N$ with $\mathcal{L}_N^{(2)} = \mathcal{H}_N$ given by (48) (see [14]).
Let $\Lambda$ be the same as before and define the infinite-dimensional Dunkl operator of $B$-type $D_\infty : \Lambda[x] \rightarrow \Lambda[x]$ by

$$D_\infty = \partial - 2k\Delta - \frac{q}{x}(1 - \tau).$$

Here the derivation $\partial$ in $\Lambda[x]$ is defined by the formulae

$$\partial(x) = 1, \quad \partial(p_l) = 2lx^{2l-1}, \quad l \in \mathbb{Z}_{\geq 0},$$

the operator $\Delta : \Lambda[x] \rightarrow \Lambda[x]$ is defined by

$$\Delta(x^l f) = \Delta(x^l) f, \quad \Delta(1) = 0, \quad f \in \Lambda, \quad l \in \mathbb{Z}_{\geq 0}$$

with

$$\Delta(x^l) = x^{2l-1} p_0 + x^{2l-3} p_1 + \cdots + x^3 p_{l-2} + xp_{l-1} - lx^{2l-1},$$

$$\Delta(x^{2l-1}) = x^{2l-2} p_0 + x^{2l-4} p_1 + \cdots + x^2 p_{l-2} + p_{l-1} - lx^{2l-2}, \quad l > 0,$$

and the involution $\tau$ is defined by

$$\tau(x^l f) = (-x)^l f, \quad f \in \Lambda.$$

Let $\varphi_{i,N} : \Lambda[x] \rightarrow \Lambda_N[x_i]$ be the homomorphism such that

$$\varphi_{i,N}(x) = x_i, \quad \varphi_{i,N}(p_l) = x_1^{2l} + \cdots + x_N^{2l}, \quad l \in \mathbb{Z}_{\geq 0}.$$

One can show that the following diagram

$$\begin{array}{ccc}
\Lambda[x] & \xrightarrow{D_\infty} & \Lambda[x] \\
\downarrow \varphi_{i,N} & & \downarrow \varphi_{i,N} \\
\Lambda_N[x_i] & \xrightarrow{D_{i,N}} & \Lambda_N[x_i]
\end{array}$$

where $D_{i,N}$ are the Dunkl operators (49), is commutative.

Define a linear operator $E : \Lambda[x] \rightarrow \Lambda$ by the formulae

$$E(x^{2l} f) = p_l f, \quad E(x^{2l+1} f) = 0, \quad f \in \Lambda, \quad l \in \mathbb{Z}_{\geq 0},$$

and the operators $L^{(r)} : \Lambda \rightarrow \Lambda, \quad r \in \mathbb{Z}_+$ by

$$L^{(r)} = \text{Res } E \circ D_{\infty}^{2r},$$

where as before Res means the restriction to $\Lambda$. 
The claim is that these operators give the quantum CMS integrals at infinity in the rational $B$-case.

**Theorem 6.1.** The differential operators $L^{(r)}$ commute with each other:

$$[L^{(r)}, L^{(s)}] = 0.$$  

The operator $L^{(2)}$ has the following explicit form:

$$L^{(2)} = 8 \sum_{a,b \geq 1} p_{a+b} \partial_a \partial_b - 4k \sum_{a,b \geq 0} p_a p_b \partial_{a+b+1} + 4k \sum_{a \geq 0} (a+1) p_a \partial_{a+1}$$

$$+ 2 \sum_{a \geq 0} (2a+1) p_a \partial_{a+1} - 4q \sum_{a \geq 0} p_a \partial_{a+1}$$  

(54)

with $\partial_a = a\partial_0 p_a$, and coincides with the rational CMS operator of $B$-type at infinity. $\square$

The explicit form (54) is in agreement with [23, Formula (32)] and follows from the relations

$$(E \circ \Delta \circ \partial)(p_a) = 2a(p_{a-1} P_0 + \cdots + P_0 p_{a-1} - ap_{a-1}),$$

$$\left( E \circ \frac{1}{x} (1 - \tau) \circ \partial \right)(p_a) = 4ap_{a-1}, \quad E \circ \partial^2(p_a) = 2a(2a - 1)p_{a-1},$$

$$(E \circ \partial^2)(p_a p_b) = 2a(2a - 1)p_{a-1} p_b + 2b(2b - 1)p_{b-1} p_a + 8abp_{a+b-1}.$$

Now let us apply this to the deformed case. The deformed rational CMS operator of type $B_{n,m}$ has the form [21]

$$H_{n,m} = - \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) - k \left( \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_m^2} \right)$$

$$+ \sum_{i<j} \frac{2k(k+1)}{(x_i - x_j)^2} + \frac{2k(k+1)}{(x_i + x_j)^2} + \sum_{i<j} \frac{2(k^{-1} + 1)}{(y_i - y_j)^2} + \frac{2(k^{-1} + 1)}{(y_i + y_j)^2}$$

$$+ \sum_{i=1}^n \sum_{j=1}^m \frac{2(k+1)}{(x_i - y_j)^2} + \frac{2(k+1)}{(x_i + y_j)^2} + \sum_{i=1}^n \frac{q(q+1)}{x_i^2} + \sum_{j=1}^m \frac{ks(s+1)}{y_j^2},$$  

(55)

where the parameters $k, q, s$ satisfy the relation

$$2q + 1 = k(2s + 1).$$  

(56)

Let $x_{n+i} = y_i, i = 1, \ldots, m$ as before and introduce the multiplicity function $m(i) = q$ for $i = 1, \ldots, n$ and $m(i) = s$ for $i = n+1, \ldots, n+m$. 


Define $\varphi^{(i)}_{n,m} : \tilde{\Lambda}[x] \to \mathbb{C}[x_1, \ldots, x_{n+m}]$ by $\varphi^{(i)}_{n,m}(x) = x_i$ and

$$\varphi^{(i)}_{n,m}(p_i) = \sum_{l=1}^{n+m} k^{-p^{(i)}_l} x_i^{2l} = \sum_{l=1}^{n} x_i^{2l} + \frac{1}{k} \sum_{l=n+1}^{n+m} x_i^{2l}$$  \hspace{1cm} (57)

for all $l \in \mathbb{Z}_{\geq 0}$. As before, let also $\tau_i$ be the automorphism of $\mathbb{C}[x_1, \ldots, x_{n+m}]$ changing the sign of $x_i$.

**Proposition 6.2.** We have the following relation on $\tilde{\Lambda}[x]$:  

$$\varphi^{(i)}_{n,m} \circ D_\infty = k^{p^{(i)}} \frac{\partial}{\partial x_i} \circ \varphi^{(i)}_{n,m} - \frac{k^{p^{(i)}} m^{(i)}}{x_i} (1 - \tau_i) \varphi^{(i)}_{n,m}$$

$$- \sum_{j \neq i} \frac{k^{1-p^{(j)}}}{x_i - x_j} (\varphi^{(i)}_{n,m} - \varphi^{(j)}_{n,m}) - \sum_{j \neq i} \frac{k^{1-p^{(j)}}}{x_i + x_j} (\varphi^{(i)}_{n,m} - \tau_j \varphi^{(j)}_{n,m})$$  \hspace{1cm} (58)  

\[\square\]

Rewrite (58) in the matrix form as  

$$\Phi_{n,m} \circ D_\infty = L \circ \Phi_{n,m},$$  

where $\Phi_{n,m} = (\varphi^{(1)}_{n,m}, \ldots, \varphi^{(n+m)}_{n,m})^T$, $\tau_1 \varphi^{(1)}_{n,m}, \ldots, \tau_{n+m} \varphi^{(n+m)}_{n,m})^T$ and $L$ has the block form

$$L = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}$$

with the following $(n+m) \times (n+m)$ matrices $A$ and $B$:

$$A_{ii} = k^{p^{(i)}} \frac{\partial}{\partial x_i} - \frac{k^{p^{(i)}} m^{(i)}}{x_i} - \sum_{j \neq i} \frac{k^{1-p^{(j)}}}{x_i - x_j} - \sum_{j \neq i} \frac{k^{1-p^{(j)}}}{x_i + x_j}, \quad A_{ij} = \frac{k^{1-p^{(j)}}}{x_i - x_j}, \quad i \neq j.$$  \hspace{1cm} (59)

$$B_{ii} = \frac{k^{p^{(i)}} m^{(i)}}{x_i}, \quad B_{ij} = \frac{k^{1-p^{(j)}}}{x_i + x_j}, \quad i \neq j.$$  \hspace{1cm} (60)

After a suitable conjugation, we have the following quantum Moser matrix $L$ for the deformed rational CMS system of $B$-type:

$$L = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}$$

with

$$A_{ii} = k^{p^{(i)}} \frac{\partial}{\partial x_i}, \quad A_{ij} = \frac{k^{1-p^{(j)}}}{x_i - x_j}, \quad i \neq j,$$  \hspace{1cm} (61)

$$B_{ii} = \frac{k^{p^{(i)}} m^{(i)}}{x_i}, \quad B_{ij} = \frac{k^{1-p^{(j)}}}{x_i + x_j}, \quad i \neq j.$$  \hspace{1cm} (62)
In the nondeformed case \( m = 0 \), it reduces to the quantum Moser matrix proposed by Yamamoto in [33] as a version of Lax matrices found in the classical case by Olshanetski and Perelomov [19].

Let \( e = (1, \ldots, 1)^T \) and \( e_i^* = e_{n+m+i}^* = k^{-p(i)} \) for \( i = 1, \ldots, (n + m) \). Similarly to the \( A \)-type case, one can prove the following result, which gives an explanation of the recursive formulas found in [21].

**Theorem 6.3.** The operators

\[
L^{(i)}_{n,m} = e^* L^2 e
\]

are the commuting quantum integrals of the deformed CMS system in the rational \( B_{n,m} \) case (55). □

### 7 Trigonometric BC Case

The trigonometric \( BC_N \) CMS operator depends on three parameters \( k, p, q \) and in the exponential coordinates has the form

\[
H_N = \sum_{i=1}^{N} (x_i \frac{\partial}{\partial x_i})^2 - \sum_{1 \leq i < j \leq N} \frac{2k(k+1)x_ix_j}{(x_i - x_j)^2} + \frac{2k(k+1)x_ix_j}{(x_i x_j - 1)^2} - \sum_{i=1}^{N} \left( \frac{p(p+2q+1)x_i}{(x_i - 1)^2} + \frac{4q(q+1)}{(x_i^2 - 1)^2} \right),
\]

or, after a gauge transformation and using \( \partial_i = x_i \frac{\partial}{\partial x_i} \),

\[
\mathcal{H}_N = \sum_{i=1}^{N} \partial_i^2 - k \sum_{1 \leq i < j \leq N} \frac{x_i + x_j}{x_i - x_j} (\partial_i - \partial_j) - k \sum_{1 \leq i < j \leq N} \frac{x_i x_j + 1}{x_i x_j - 1} (\partial_i + \partial_j)
- \sum_{i=1}^{N} \left( \frac{p x_i + 1}{x_i - 1} + 2q \frac{x_i^2 + 1}{x_i^2 - 1} \right) \partial_i.
\]

The operator \( \mathcal{H}_N \) preserves the algebra \( A^W_N \) of \( W_N \)-invariant Laurent polynomials, where the action of Weyl group \( W_N = S_N \ltimes \mathbb{Z}_2^N \) on \( \mathbb{C}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}] \) is generated by \( s_{ij}^\pm \) and \( t_i, \ i = 1, \ldots, N \), acting according to

\[
s_{ij}^\pm(x_i, x_j) = (x_j^{\pm 1}, x_i^{\pm 1}), \quad t_i(x_i) = x_i^{-1}, \quad i = 1, \ldots, N
\]

(other coordinates are unchanged). The algebra \( A^W_N \) is generated by the invariants

\[
p_l = x_1^l + x_1^{-l} + \cdots + x_N^l + x_N^{-l}, \quad l \in \mathbb{Z}_{>0}.
\]
The corresponding Dunkl–Heckman operators $D_{i,N}$ acting on Laurent polynomials $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ have the form

$$D_{i,N} = \partial_i - \frac{1}{2} k \sum_{j \neq i}^N \left( \frac{x_i + x_j}{x_i - x_j} (1 - s_i^j) + \frac{x_i x_j + 1}{x_i x_j - 1} (1 - s_i^j) \right)$$

$$- \frac{1}{2} p \frac{x_i + 1}{x_i - 1} (1 - t_i) - q \frac{x_i^2 + 1}{x_i^2 - 1} (1 - t_i).$$

(66)

Note that $D_{i,N}$ preserves the subalgebra $\Lambda^W_N[x_i, x_i^{-1}] \subset \mathbb{C}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$. The quantum CMS integrals can be given by Heckman’s formula [14]

$$\mathcal{L}_N^{(2r)} = \text{Res}(D_{1,N}^{2r} + \cdots + D_{N,N}^{2r}),$$

where Res means the restriction to $\Lambda^W_N$. One can check that $\mathcal{L}_N^{(2)} = \mathcal{H}_N$ is the CMS operator (65).

Consider the algebra $\tilde{\Lambda}$ freely generated by $p_i, i \in \mathbb{Z}_{\geq 0}$ as before. Define the Dunkl–Heckman operator of BC-type at infinity $D_\infty : \tilde{\Lambda}[x, x^{-1}] \rightarrow \tilde{\Lambda}[x, x^{-1}]$ as

$$D_\infty = \partial - \frac{1}{2} k \Delta - \frac{1}{2} \frac{x + 1}{x - 1} (1 - t) - \frac{q}{x} \frac{x^2 + 1}{x^2 - 1} (1 - t),$$

(67)

where the derivation $\partial$ is defined by $\partial(x) = x$, $\partial p_i = l(x^l - x^{-l})$, $l \in \mathbb{Z}_{\geq 0}$, and the homomorphisms of $\tilde{\Lambda}$-modules $\Delta$ and $t$ are defined by $\Delta(1) = 0$,

$$\Delta(x^l) = (p_0 - 2l - 1)x^l - 2 \sum_{j=1}^{l-1} x^{l-2j} - x^{-l} + 2 \sum_{j=1}^{l-1} p_j x^{-l+j} + p,$$

$$\Delta(x^{-l}) = -(p_0 - 2l - 1)x^{-l} + 2 \sum_{j=1}^{l-1} x^{l-2j} + x^l - 2 \sum_{j=1}^{l-1} p_j x^{l+j} - p, \quad l > 0,$$

$$t(x) = x^{-1}, \quad t(p_l) = p_l.$$

One can check that the diagram

$$\begin{array}{ccc}
\tilde{\Lambda}[x, x^{-1}] & \xrightarrow{D_\infty} & \tilde{\Lambda}[x, x^{-1}] \\
\downarrow \varphi_{i,N} & & \downarrow \varphi_{i,N} \\
\Lambda^W_N[x_i, x_i^{-1}] & \xrightarrow{D_{i,N}} & \Lambda^W_N[x_i, x_i^{-1}]
\end{array}$$

is commutative, where the homomorphisms $\varphi_{i,N}$ are defined by $\varphi_{i,N}(x) = x_i$ and

$$\varphi_{i,N}(p_l) = x_1^l + x_1^{-l} + \cdots + x_N^l + x_N^{-l}, \quad l \geq 0$$

(in particular, $p_0$ is specialized to $2N$).
Define the homomorphism of $\tilde{\Lambda}$-modules $E : \tilde{\Lambda}[x, x^{-1}] \to \tilde{\Lambda}$ by

$$E(x^j) = p_{|j|}, \quad j \in \mathbb{Z}.$$ 

The CMS integrals of $BC$-type at infinity can be defined now by the formula

$$\mathcal{L}^{(2r)} = \text{Res} E \circ D^{2r}_\infty,$$  

(68)

where Res means the restriction to $\tilde{\Lambda}$. For $r = 1$, we have the $BC$ operator at infinity

$$\mathcal{L}^{(2)} = 4 \sum_{a,b \geq 1} \left( p_{a+b} - p_{a-b} \right) \partial_a \partial_b + 2 \sum_{a \geq 1} (ak + a + k + h) p_a \partial_a$$

$$+ 2(k - q) \sum_{a \geq 2} \left( \sum_{j=1}^{a-1} p_{a-2j} \right) \partial_a - p \sum_{a \geq 2} \left( \sum_{j=1}^{2a-1} p_{a-j} \right) \partial_a - 2k \sum_{a \geq 2} \left( \sum_{j=1}^{a-1} p_j p_{a-j} \right) \partial_a,$$  

(69)

where as usual $\partial_a = a \frac{\partial}{\partial p_a}$ and we used the notation from [23]

$$h = -kp_0 - \frac{1}{2} p - q$$

and defined $p_j := p_{|j|}$ for all $j \in \mathbb{Z}$. Note that the comparison with the formulae in [23] is not easy since the variables there correspond to the different choice of invariants in $\Lambda^W_N$:

$$p_i = \sum u_i, \quad u_i = \frac{1}{2} (x_i + x_i^{-1} - 2).$$

Consider now briefly the deformed $BC$ case. The corresponding CMS operator [21] in the exponential coordinates has the form

$$H_{n,m} = \sum_{i=1}^{n} \left( x_i \frac{\partial}{\partial x_i} \right)^2 + k \sum_{j=1}^{m} \left( y_j \frac{\partial}{\partial y_j} \right)^2 - \sum_{i<j} \left( \frac{8k(k+1)x_i x_j}{(x_i - x_j)^2} + \frac{8k(k+1)x_i x_j}{(x_i x_j - 1)^2} \right)$$

$$- \sum_{i<j} \left( \frac{8(k^{-1} + 1)y_i y_j}{(y_i - y_j)^2} + \frac{8(k^{-1} + 1)y_i y_j}{(y_i y_j - 1)^2} \right) - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{8(k+1)x_i y_j}{(x_i - y_j)^2}$$

$$- \sum_{i=1}^{n} \left( \frac{4p(p + 2q + 1)x_i}{(x_i - 1)^2} + \frac{16q(q + 1)x_i^2}{(x_i^2 - 1)^2} \right)$$

$$- \sum_{j=1}^{m} \left( \frac{4kr(r + 2s + 1)y_j}{(y_j - 1)^2} + \frac{16ks(s + 1)y_j^2}{(y_j^2 - 1)^2} \right),$$  

(70)

where the parameters $k, p, q, r, s$ satisfy the relations

$$p = kr, \quad 2q + 1 = k(2s + 1).$$  

(71)
Define as before $x_{n+i} = y_i$, $i = 1, \ldots, m$, $\partial_j = x_j \frac{\partial}{\partial x_j}$ and introduce the multiplicity functions $\mu(i) = p$, $\nu(i) = q$ for $i = 1, \ldots, n$ and $\mu(i) = r$, $\nu(i) = s$ for $i = n+1, \ldots, n+m$.

The quantum Moser matrix in this case also has the block form

$$L = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}$$

with the following $(n+m) \times (n+m)$ matrices $A$ and $B$:

$$A_{ii} = k^{p(i)} \partial_i, \quad A_{ij} = \frac{k^{1-p(j)}(x_i + x_j)}{2(x_i - x_j)}, \quad i \neq j,$$

$$B_{ii} = \frac{k^{p(i)} \mu(i)(x_i + 1)}{2(x_i - 1)} + \frac{k^{p(i)} \nu(i)(x_i^2 + 1)}{x_i^2 - 1}, \quad B_{ij} = \frac{k^{1-p(j)}(x_i x_j + 1)}{2(x_i x_j - 1)}, \quad i \neq j.$$  \hspace{2cm} (72)

The commuting quantum integrals of the deformed CMS system (70) now can be constructed as

$$L_{n,m}^{(2l)} = e^* L^{2l} e,$$  \hspace{2cm} (74)

where as before $e = (1, \ldots, 1)^T$ and $e^* = e_{n+m+i}^* = k^{-p(i)}$ for $i = 1, \ldots, (n+m)$.

One can check that these integrals after a gauge transformation coincide with the integrals given by the recursive procedure in [21].

8 Concluding Remarks

We have shown how the Dunkl operator at infinity leads to the quantum Moser matrix and to the proof of integrability for the deformed CMS systems related to classical series of Lie superalgebras. A simple form of the corresponding quantum Moser matrix suggests that it might be possible to guess it for the deformed CMS systems related to the exceptional Lie superalgebras [21], for which the integrability is still to be studied. (As we have learnt from Oleg Chalykh, at least in the rational case, there is a relatively simple way to prove the integrability of all deformed CMS systems, including exceptional ones, using the theory of rational Cherednik algebras [3].)

Another open question is about elliptic version. The elliptic Dunkl operators were studied in [4] and were used to construct the integrals of the elliptic CMS systems in [10]. The construction is not as straightforward as in the trigonometric case and involves the integrals of the corresponding classical system. The question is whether the methods of our paper could be modified to this case.
Finally, it is interesting to understand the precise relation of our formulae for quantum CMS integrals at infinity in the trigonometric type A case with the results of the recent paper [18] by Nazarov and Sklyanin, whose main tool was the quantum Lax operator for the periodic quantum Benjamin–Ono equation, which they have introduced. (After the first version of the present paper had appeared in the ArXiv, Evgeni Sklyanin informed us that he also came with Maxim Nazarov to the idea of using Dunkl operator technique for construction of the CMS integrals at infinity (in the trigonometric type A case).) We believe that their integrals (which do not depend on $p_0$) are simply related to the stable integrals $H_k^{(r)}$ from our recent paper [28], which were constructed using the infinite-dimensional version of the Polychronakos operator (rather than the Dunkl–Heckman operator used in the present paper). The relation between $H_k^{(r)}$ and our quantum CMS integrals (37) is nontrivial (see the formulae in [28, Section 5]). Note also that the trigonometric A-type case is special in the sense that, only in this case, the dependence on the parameter $p_0$ at infinity can be eliminated (see [24]).

Funding

This work was partly supported by the EPSRC (grant EP/J00488X/1). Funding to pay the Open Access publication charges for this article was provided by Research Councils UK (RCUK) Block Grant at Loughborough University.

Acknowledgements

We are grateful to O. A. Chalykh and M. V. Feigin for useful discussions and to the referees for helpful comments. A.N.S. is grateful to Loughborough University for their hospitality during the autumn semesters 2012–14.

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