Standard and hypergeometric
representations for loop diagrams
and the photon-photon scattering

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By using the example of photon-photon scattering box diagrams, different representations for loop integrals are discussed. A connection between hypergeometric representation and dilogarithms is derived. An explicit analytic continuation formula to large values of Mandelstam variables is constructed.

1. When studying many important physical problems (calculation of radiative corrections to elementary particles interactions, examination of operator expansions, etc.), we are often confronted with various types of Feynman loop diagrams. The greatest success has been achieved in evaluating massless propagator-type diagrams (depending on one external momentum only). For such diagrams, this dependence is power-like, and the only problem is how to calculate numerical factors at these powers. Some powerful methods (Gegenbauer polynomial technique \cite{1}, integration by parts \cite{2}, the “uniqueness” method \cite{3}, and some others) have been developed to calculate these numbers, and for some classes of diagrams the results have been obtained for any numbers of loops (see, e.g., \cite{4}).

The problem gets essentially more complicated, however, when we consider diagrams with masses and (or) with larger number of external lines. In this cases, dimensionless combinations can be constructed from squared external momenta and masses. As a result, we can get complicated functions of these variables (instead of numbers), and the calculation becomes non-trivial even on one-loop level. Thus, the problem is how to describe the occurring functions in all possible regions of momenta and masses. There are different approaches to this problem.

Commonly-used ”standard” representations for loop diagrams usually involve special functions like polylogarithms (Li_N), generalized Clausen’s functions, etc. Explicit results of such type are known for one-loop diagrams up to four external lines (see, e.g., in \cite{5, 6}) and for some particular cases of two-loop diagrams (see, e.g., \cite{7, 8}). For some more complicated cases, however, only parametric integral representations are known (see, e.g., \cite{9}).

There exists also another way to represent results for loop diagrams – in terms of hypergeometric functions. In the convergence regions, these functions can be represented as simple

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power series \([10]\). Some results of such type have been presented in \([11]\). Moreover, in such a way it was possible to obtain explicit formulae for the general case of one-loop diagrams with arbitrary number of external lines \([12]\). Note that a special approach \([13]\) makes it also possible to represent integrals with tensor structure in numerators in the same form. If one need to know results in other regions of variables, it is possible to use analytic continuation formulae for occurring hypergeometric functions (see, e.g., in \([14]\)). On the other hand, for complicated cases there is a problem how to construct analytic continuation to all regions of interest.

To our opinion, both representations can be useful in realistic calculations. Because of this, it is also important to know how they are connected (how to go from one representation to another). Usually it is possible to use parametric integral representations for hypergeometric functions, and then to try to evaluate them in terms of known special functions (for example, polylogarithms). Such a procedure has been already considered by taking some examples of triangle diagrams \([15, 16]\). In the present paper we would like to illustrate it via an example of box diagrams contributing to photon-photon scattering.

2. Photon-photon scattering \((\gamma \gamma \rightarrow \gamma \gamma)\) is the well-known example of non-linear effects in QED (see, e.g., \([17]\) - \([19]\) and references therein). Note that the processes \(gg \rightarrow \gamma \gamma\) and \(gg \rightarrow gg\) are also described by the same diagrams. The lowest order of this process contains the box diagram presented in Fig. 1 as well as those obtained by permutations \((k_3 \leftrightarrow k_4)\) and \((k_2 \leftrightarrow k_3)\). Solid internal lines correspond to massive fermions (electrons or quarks) with the mass \(m\), while dashed external lines denote massless particles (photons or gluons):

\[k_2^2 = k_3^2 = k_4^2 = 0.\] (1)

To simplify the analysis, in what follows we shall restrict ourselves by scalar integrals corresponding to these box diagrams, because it is known that integrals with tensor numerators can
be reduced to scalar ones (see, e.g., [20, 13]). Formally, such scalar integrals correspond also to massless particles scattering in the Wick-Cutkosky model.

As usual, we define the Mandelstam variables as
\[ s \equiv (k_1 + k_2)^2, \quad t \equiv (k_2 - k_3)^2, \quad u \equiv (k_1 - k_3)^2; \quad s + t + u = 0. \] (2)

Note that only two of them (for example, \( s \) and \( t \)) are independent. So, if we denote the integral corresponding to Fig. 1 as \( J(s, t; m) \) then the whole amplitude (including diagrams with permutations) is proportional to the sum
\[ J(s, t; m) + J(s, u; m) + J(u, t; m). \] (3)

Hypergeometric representation for \( J \) in four dimensions can be easily obtained from general formulae of the papers [12] (see also [18, 10]) by putting \( k_i^2 = 0 \),
\[ J(s, t; m) = \frac{i\pi^2}{6m^4} F_3 \left( 1, 1, 1; \frac{5}{2}, \frac{s}{4m^2}, \frac{t}{4m^2} \right), \] (4)
where
\[ F_3 (\alpha, \alpha', \beta, \beta'; \gamma| x, y) \equiv \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{x^j y^l}{j! l!} \frac{(\alpha)_j (\alpha')_l (\beta)_j (\beta')_l}{(\gamma)_{j+l}} \] (5)
is Appell’s hypergeometric function of two variables (see, e.g., [14]), and
\[ (\alpha)_j \equiv \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)} \] (6)
denotes the Pochhammer symbol. Note that the convergence region of the double series (3) is \( x < 1, \ y < 1 \). So, formulae (4)-(5) yield asymptotic expansion of \( J(s, t; m) \) for small values of \( x \equiv s/(4m^2) \) and \( y \equiv t/(4m^2) \).

To obtain "standard" representation for \( J(s, t; m) \), it is convenient to use the known parametric integral representation for \( F_3 \) (see, e.g., [14]):
\[ F_3 (\alpha, \alpha', \beta, \beta'; \gamma| x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\beta') \Gamma(\gamma - \beta - \beta')} \times \int \int \frac{d\xi \ d\eta}{1 - \xi - \eta} \xi^{\beta - 1} \eta^{\beta' - 1} (1 - \xi - \eta)^{\gamma - \beta - \beta' - 1} (1 - \xi x)^{-\alpha} (1 - \eta y)^{-\alpha'} \] (7)
\[ \xi \geq 0, \eta \geq 0, \xi + \eta \leq 1 \]
(it is valid for \( \text{Re} \beta > 0, \text{Re} \beta' > 0, \text{Re}(\gamma - \beta - \beta') > 0 \)). For the case of (4) we find, from (7), that
\[ F_3 (1, 1, 1; 5/2| x, y) = \frac{3}{4} \int \int \frac{d\xi \ d\eta}{(1 - \xi x)(1 - \eta y)\sqrt{1 - \xi - \eta}}. \] (8)
Evaluating the integrals on the r.h.s. of (8), we arrive at the following result:

\[
F_3(1, 1, 1, 1; 5/2 \mid x, y) = \frac{3}{4xy \beta_{xy}} \left\{ 2 \ln^2 \left( \frac{\beta_{xy} + \beta_x}{\beta_{xy} + \beta_y} \right) + \ln \left( \frac{\beta_{xy} - \beta_x}{\beta_{xy} + \beta_x} \right) \ln \left( \frac{\beta_{xy} - \beta_y}{\beta_{xy} + \beta_y} \right) - \frac{\pi^2}{2} \right. \\
+ \sum_{i=x, y} \left[ 2 \text{Li}_2 \left( \frac{\beta_i - 1}{\beta_{xy} + \beta_i} \right) - 2 \text{Li}_2 \left( -\frac{\beta_{xy} - \beta_i}{\beta_i + 1} \right) - \ln^2 \left( \frac{\beta_i + 1}{\beta_{xy} + \beta_i} \right) \right] \right\},
\]

where

\[
\beta_x = \sqrt{1 - \frac{1}{x}}, \quad \beta_y = \sqrt{1 - \frac{1}{y}}, \quad \beta_{xy} = \sqrt{1 - \frac{1}{x} - \frac{1}{y}}.
\]

By simple transformations of dilogarithms, it is easy to show that the result (4), (9) coincides with the well-known expression (see, e.g., [18]). Our expression, however, is more compact and contains four dilogarithms only (instead of eight ones in [18]).

3. To obtain the formula of analytic continuation to the variables \(1/x\) and \(1/y\), it is convenient to use double Mellin–Barnes representation for \(F_3\),

\[
F_3(\alpha, \alpha', \beta, \beta'; \gamma \mid x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\alpha') \Gamma(\beta) \Gamma(\beta')} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} da \, db \, (-x)^a (-y)^b \\
\times \Gamma(-a) \, \Gamma(-b) \, \frac{\Gamma(-a + b) \, \Gamma(\alpha + a + b) \, \Gamma(\beta + a) \, \Gamma(\beta' + b)}{\Gamma(\gamma + a + b)},
\]

where the integration contours are chosen so as to separate ”right” and ”left” series of poles of gamma functions in the integrand. If \(|x| < 1, \, |y| < 1\), we should close both contours to the right, and we obtain the formula (3). On the other hand, for large \(|x|\) and \(|y|\) we should close the contours to the left. As a result, we obtain the following known analytic continuation formula (see, e.g., [14]):

\[
F_3(\alpha, \alpha', \beta, \beta'; \gamma \mid x, y) = \sum_{\{\lambda, \mu, \rho, \sigma\}} \frac{\Gamma(\gamma) \Gamma(\rho - \lambda) \Gamma(\sigma - \mu) \Gamma(\rho) \Gamma(\sigma) \Gamma(\gamma - \lambda - \mu)}{\Gamma(\gamma - \lambda - \mu)} \left( -\frac{1}{x} \right)^\lambda \left( -\frac{1}{y} \right)^\mu \\
\times F_2 \left( \lambda + \mu - \gamma + 1, \lambda, \mu; \lambda - \rho + 1, \mu - \sigma + 1 \; \bigg| \frac{1}{x}, \frac{1}{y} \right),
\]

where \(F_2\) is another Appell’s hypergeometric function of two variables,

\[
F_2(\alpha, \beta, \beta'; \gamma, \gamma' \mid u, v) \equiv \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{u^j v^l}{j! \, l!} \frac{\Gamma(\alpha)_{j+l} \, \Gamma(\beta)_{j+l} \, \Gamma(\beta')_{l}}{\Gamma(\gamma)_{j+l} \, \Gamma(\gamma')_{l}}.
\]
and the sum (in (12)) extends over the following four sets of \{\lambda, \mu, \rho, \sigma\}: \{\alpha, \alpha', \beta, \beta'\}, \{\beta, \alpha', \alpha, \beta'\} and \{\beta, \beta', \alpha, \alpha'\}. Note that the radius of convergence of the function (13) is \(u + v < 1\).

The case \(\alpha = \alpha' = \beta = \beta' = 1, \gamma = 5/2\) corresponds to so-called ’logarithmic’ case of analytic continuation, because in this case gamma functions in separate terms on the r.h.s. of (12) have singularities (which cancel in the whole sum). Considering this limit of the expression (12), we obtain the following analytic continuation formula:

\[
F_3(1, 1, 1; 5/2 \mid x, y) = \frac{3}{4xy} \left\{ \left[ \ln(-4x) \ln(-4y) - \frac{\pi^2}{2} \right] F_2 \left( 1/2, 1, 1; 1, 1 \mid \frac{1}{x}, \frac{1}{y} \right) \right. \\
- \left. \left[ \ln(-4x) + \ln(-4y) \right] \partial_\alpha F_2 - \ln(-4y) \partial_\gamma F_2 - \ln(-4x) \partial_\gamma F_2 \right) + \partial_\alpha^2 F_2 + \partial_\alpha \partial_\gamma F_2 + \partial_\alpha \partial_\gamma' F_2 + \partial_\gamma \partial_\gamma' F_2 \},
\]

(14)

where we have introduced notation for parametric derivatives of the function (13), for example:

\[
\partial_\alpha F_2 \equiv \frac{\partial}{\partial \alpha} F_2 \left( \alpha, \beta, \beta'; \gamma, \gamma' \mid \frac{1}{x}, \frac{1}{y} \right) \bigg|_{\alpha=1/2, \beta=\beta'=\gamma=\gamma'=1},
\]

(15)

and so on. Note that the series representations of these parametric derivatives (obtained by differentiating the formula (13)) will contain \(\psi\)-functions,

\[
\psi(z) \equiv \frac{d}{dz} \ln \Gamma(z),
\]

(16)

and their derivatives, because

\[
\frac{\partial}{\partial \alpha} (\alpha)_j = (\alpha)_j \left( \psi(\alpha + j) - \psi(\alpha) \right),
\]

(17)

etc. The representation (14) gives us asymptotic expansion of \(J(s, t; m)\) (4) for large \(|x|\) and \(|y|\) (with due regard for \(\ln x\) and \(\ln y\) terms).

To be sure that the analytic continuation (14) corresponds to the same function (4), we may compare the results for \(x \leq 0, y \leq 0\). To obtain ”standard” representations for the functions on the r.h.s. of (14), it is convenient to use the well-known reduction formula,

\[
F_2(\alpha, \beta, \beta'; \beta, \gamma' \mid u, v) = (1 - u)^{-\alpha} 2F_1 \left( \frac{\alpha, \beta'}{\gamma'} \mid \frac{v}{1 - u} \right),
\]

(18)

where \(2F_1\) is a usual Gauss hypergeometric function. Using the well-known parametric integral representation for \(2F_1\),

\[
2F_1 \left( \frac{\alpha, \beta}{\gamma} \mid z \right) = \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma - \beta)} \int_0^1 d\xi \frac{\xi^{\beta-1}(1 - \xi)^{\gamma-\beta-1}}{(1 - \xi z)^\alpha},
\]

(19)
and its parametric derivatives, it is easy to obtain the following results:

\[ F_2(1/2, 1; 1, 1 | u, v) = \frac{1}{\sqrt{1-u-v}} = \frac{1}{\beta_{xy}}, \quad (20) \]

\[ \partial_\alpha F_2 = -\frac{2}{\beta_{xy}} \ln \beta_{xy}, \quad (21) \]

\[ \partial_\gamma F_2 = -\frac{2}{\beta_{xy}} \ln \left( \frac{\beta_{xy} + \beta_y}{2\beta_{xy}} \right), \quad (22) \]

\[ \partial_\alpha \partial_\gamma F_2 = \frac{2}{\beta_{xy}} \left\{ 2\ln \beta_{xy} \ln \left( \frac{\beta_{xy} + \beta_y}{2\beta_{xy}} \right) + \text{Li}_2 \left( \frac{\beta_{xy} - \beta_y}{\beta_{xy} + \beta_y} \right) - \text{Li}_2 \left( -\frac{\beta_{xy} - \beta_y}{\beta_{xy} + \beta_y} \right) \right\}. \quad (23) \]

Note that analogous results for \( \partial_\gamma F_2 \) and \( \partial_\alpha \partial_\gamma F_2 \) can be obtained from (22) and (23) by substitution \( x \leftrightarrow y \).

To calculate \( \partial_\gamma \partial_\gamma' F_2 \), it is impossible to use (18), and we need to employ a double integral representation for \( F_2 \) [14]. Differentiating it with respect to \( \gamma \) and \( \gamma' \) and putting \( \alpha = 1/2, \beta = \beta' = \gamma = \gamma' = 1 \), we get

\[ \partial_\gamma \partial_\gamma' F_2 = \frac{3}{4} uv \int_0^1 \int_0^1 d\xi d\eta \ln(1-\xi) \ln(1-\eta) (1-u\xi - v\eta)^{-5/2}. \quad (24) \]

Evaluating the integrals yields

\[ \partial_\gamma \partial_\gamma' F_2 = \frac{2}{\beta_{xy}} \sum_{i=x,y} \left\{ \ln^2 \left( \frac{\beta_{xy} + \beta_i}{2\beta_{xy}} \right) - \frac{1}{2} \ln^2 \left( \frac{\beta_i + 1}{\beta_{xy} + \beta_i} \right) \right\} \]

\[ + \text{Li}_2 \left( \frac{\beta_i - 1}{\beta_{xy} + \beta_i} \right) - \text{Li}_2 \left( -\frac{\beta_{xy} - \beta_i}{\beta_i + 1} \right) - \text{Li}_2 \left( \frac{\beta_{xy} - \beta_i}{\beta_{xy} + \beta_i} \right) + \text{Li}_2 \left( -\frac{\beta_{xy} - \beta_i}{\beta_{xy} + \beta_i} \right) \}. \quad (25) \]

Inserting the expressions (20)-(23) and (25) into (14), we obtain the same result as in (3). This fact confirms the correctness of analytic continuation from the variables \( x \) and \( y \) to \( 1/x \) and \( 1/y \).

4. In the present paper we have discussed different representations for Feynman loop diagrams by taking, as an example, the box diagram contributing to the scattering of photons by photons. Hypergeometric representation (4) is convenient for studying asymptotic expansions and constructing analytic continuation to other variables (14). Standard representation (3) (involving dilogarithms of rather complicated arguments) is convenient in numerical calculations. Anyway, the formula of connection (3) enables one to pass from one representation to another (when it is necessary). The formulae (3) and (14) explicitly give all terms of asymptotic expansions for small and large values of \( |s| \) and \( |t| \), respectively. Note that these formulae do
not cover all possible regions of Mandelstam variables (in other regions we need to use other analytic continuation formulae). For example, in the papers [10, 11] analytic continuation to physical thresholds has been studied. We consider the presented analytic continuation as an interesting example of using the multiple hypergeometric functions theory in loop diagrams calculation.

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