Quantum rings for beginners II: Bosons versus fermions

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The purpose of this overview article, which can be viewed as a supplement to our previous review on quantum rings, [S. Viefers et al, Physica E 21 (2004), 1-35], is to highlight the differences of boson and fermion systems in one-dimensional (1D) and quasi-one-dimensional (Q1D) quantum rings. In particular this involves comparing their many-body spectra and other properties, in various regimes and models, including spinless and spinful particles, finite versus infinite interaction, and continuum versus lattice models. Our aim is to present the topic in a comprehensive way, focusing on small systems where the many-body problem can be solved exactly. Mapping out the similarities and differences between the bosonic and fermionic cases is of renewed interest due to the experimental developments in recent years, allowing for more controlled fabrication of both fermionic and bosonic quantum rings.

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I. INTRODUCTION

Strictly 1D systems are unusual in that they provide us with one of the rare cases of exactly solvable models for a quantum mechanical many-particle system. Thus, a tremendous amount of work has been dedicated to such systems. We mention a few selected highlights illustrating the history of the field: Tonks\cite{1} studied the classical equation of state of elasical spheres in 1D, and the corresponding quantum mechanical derivation was given by Nagamiya\cite{2}. Girardeau\cite{3, 4} showed that there is an intimate relationship between impenetrable bosons and fermions in 1D. Nowadays, 1D gases with infinitely strong short range repulsion thus go by the name Tonks-Girardeau gases. Lieb and Liniger\cite{5} as well as Lieb\cite{6} and Young\cite{7} showed that a Bose gas with contact interactions has exact solutions. (This topic has later on been taken up by many authors, see for example, Refs. \cite{8–12}). The Hubbard model for the 1D fermi gas was solved exactly by Lieb and Wu\cite{13}. The quantum mechanics in 1D has been reviewed by Kolomeisky and Straley\cite{14} for fermions and by Girardeau and Wright\cite{15} for Tonks gases. More recently, Girardeau and Minguzzi\cite{16} have studied quite generally soluble models for 1D gas mixtures, and also discussed the fermionic Tonks-Girardeau gas\cite{17}. Infinitely long 1D fermion systems are usually described as Luttinger liquids\cite{18, 19}. In 1D the Fermi surface consists only of two points, which leads to a so-called Peierls instability\cite{20} and a breakdown of the Fermi liquid theory. In such a system the low energy excitations are collective, have a phonon-like linear dispersion and thus behave like bosons\cite{21, 22}. Arguably the most exotic property of a Luttinger liquid is spin-charge separation – spin and charge excitations of the interacting 1D system generally move at different velocities. In fact, in small finite quantum rings where the many-body spectrum can be solved exactly, the separation of phonon-like charge excitation and spin excitations, which can be described in the Heisenberg model, is seen in a very explicit manner\cite{24, 25}.

Particles confined in a quasi-one-dimensional ring can support quantum states exhibiting persistent currents in normal metals\cite{26, 27}, superconductors, quantum liquids and quantum gases. Persistent currents have mostly been studied in normal metals and superconductors (for reviews see\cite{25, 28, 29}). Superfluidity of helium liquids provides strongly interacting atomic systems where persistent currents were predicted\cite{30, 31} and subsequently observed for bosons in $^4\text{He}\cite{30, 32}$ and for fermions in $^3\text{He}\cite{33, 34}$. The possibility of experimentally realizing annular traps\cite{35–44} for cold atom condensates opened up new possibilities to study persistent currents in bosonic ring traps\cite{44–46}. Such systems have been analyzed in many theory works, most of them applying the Gross-Pitaevskii approximation, but also going beyond; see for example, Refs. \cite{47–55}.

In metallic or semiconducting quantum rings the current can be generated by a magnetic flux through a mesoscopic ring. For neutral atoms confined in a trap, with electromagnetic fields it is possible to create interactions which mimic the effect of a magnetic flux\cite{43, 56–59}. Also, microchip design for persistent current measurements has been suggested\cite{60}. In addition, there has been a lot of recent interest in atomic quantum gases with dipolar interactions (see\cite{61, 62} for reviews). In ring traps, there is an interesting interplay between the anisotropy of the dipolar two-body
interactions and the topology of the ring trap; we do not discuss such dipolar systems any further here, but refer to the recent works in Refs. [63–67].

The key difference between bosons and fermions is the symmetry of their many-body wave functions: Exchange of the (spatial and spin) coordinates switches the sign of the fermion wave function, but leaves the boson wave function unchanged. In general, the spin of bosons is an integer while that of fermions is a half-integer. In the case that the Hamiltonian is independent of spin, the spin only adds an additional degree of freedom which has to be taken into account in requiring the proper symmetry of the many-body wave function. It is often illustrative to talk about pseudo- or isospin for systems with different components of particles. We can have one-component fermions, for example, by polarizing an electron gas so that each electron is in the spin-up state. In this case one usually speaks of “spinless” fermions, in analogy to bosons with spin zero. In the case of cold atom clouds one can prepare a bosonic condensate which has atoms in two different hyperfine states. This is a two-component boson system, and if we choose to denote the different hyperfine states as different (pseudo) spin states, we effectively have bosons with pseudospin 1/2.

The purpose of this review is to highlight the differences of boson and fermion systems in one-dimensional (1D) and quasi-one-dimensional (Q1D) quantum rings. (For harmonic traps set rotating, a comparison between fermion and boson quantum droplets is given in Refs. [68, 69], also discussing the formation of vortices).

Mapping out the similarities and differences between bosonic and fermionic rings is of renewed interest due to the experimental progress during recent years, in particular concerning trapping techniques for cold atoms. While the original experiments on quantum rings were performed on electronic systems (such as in semiconductor heterostructures), bosonic as well as fermionic counterparts can nowadays be fabricated with appropriately trapped (charge-neutral) cold atoms. These systems allow for a much larger degree of control of physical parameters (such as interaction strength) than is the case for electrons in semiconductor structures. Thus, while a lot of interesting work has been done on Coulomb-interacting electronic rings [70–73], we here mainly have in mind atomic systems where interactions are typically short-range.

The main emphasis in this paper is put on small systems where the many-body problem can be solved exactly. Our intention is to keep the theory as simple as possible, and avoid going into too much detail of advanced many-particle descriptions. The present paper provides a natural supplement to our previous review on quantum rings, Ref. [25], which we will frequently refer back to.

The outline of the paper is as follows. We first consider one-component boson and fermion systems. We start with particles interacting with infinitely strong delta function (contact) interactions. In section [II] we study the exact solution of the two-particle case, and in sections [III] and [IV] we generalize the solution to many particles and show how the boson and fermion spectra are similar for odd numbers of particles, but differ for even numbers of
particles. In Section V we introduce the concept of persistent currents mentioned previously, and demonstrate the relation between the current and angular momentum in small rings. In Section VI we discuss lattice models for rings with only one component of particles. Section VII considers two-component bosons and fermions. In section VIII we abandon the ideal case of infinitely strong interactions, and study the effect of finite interparticle interactions. Quasi-one-dimensional systems are considered in Section IX.

II. TWO SPINLESS PARTICLES WITH INFINITELY STRONG CONTACT INTERACTIONS

We start with the simplest case, namely two spinless particles confined in an infinitely narrow ring and interacting with an infinitely strong repulsive contact interaction that has the form of a \( \delta \)-function, \( \delta(x_1 - x_2) \). When speaking of "spinless fermions", what one typically has in mind is fully polarized electrons (all in the same spin state, say spin-up). The Hamiltonian for such a system is (in atomic units \( \hbar = m = 1 \))

\[
H_2 = -\frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + g_\infty \delta(x_1 - x_2),
\]

where \( x \) is the coordinate along the ring. For convenience we choose \( R = 1 \) and thus \( x \) can be considered as an angle between 0 and \( 2\pi \). We assume the limit \( g_\infty \to \infty \), so that the wave function is zero whenever \( x_1 = x_2 \) (which for fermions is automatically satisfied due to the Pauli principle). With the change of variables \( u = x_1 - x_2 \) and \( v = (x_1 + x_2)/2 \) we get

\[
H_2 = -\left( \frac{\partial^2}{\partial u^2} + \frac{1}{4} \frac{\partial^2}{\partial v^2} \right) + g_\infty \delta(u).
\]

The solution can be written as

\[
\psi(u, v) = B(u) \sin(\frac{nu}{2}) e^{imv},
\]

where \( B \) is a piecewise constant function needed for bosons and explained below. The periodic boundary condition \( \psi(u, v) = \psi(u, v + 2\pi) \) requires that \( m \) is an integer, and the condition \( \psi(0, v) = \psi(2\pi, v) = 0 \) requires that \( n \) is a positive integer (a negative integer would only change the sign of the wave function). Using the original coordinates we can write the wave function as

\[
\psi(x_1, x_2) = B(x_1 - x_2) \sin \left( \frac{n}{2} (x_1 - x_2) \right) e^{im(x_1 + x_2)/2}.
\]

We first consider fermions. In this case the function \( B \) is not needed (or \( B(u) \equiv 1 \)) since the rest of the wave function is already antisymmetric with respect to coordinate change. We have an additional requirement of periodicity

\[
\psi(x_1, x_2) = \psi(x_1 + 2\pi, x_2) = \psi(x_1, x_2 + 2\pi),
\]

which is fulfilled only if \( n + m \) is even. It is easy to see that in this case the solution is a single Slater determinant

\[
\psi(x_1, x_2) \propto \exp \left[ i \frac{n + m}{2} x_1 \right] \exp \left[ i \frac{m - n}{2} x_2 \right] - \exp \left[ i \frac{n - m}{2} x_1 \right] \exp \left[ i \frac{m + n}{2} x_2 \right],
\]

(5)
consisting of single-particle states with integer angular momenta \((m + n)/2\) and \((m - n)/2\).

The case of bosons is more cumbersome since we have to require that the wave function is symmetric with respect to particle exchange. This can be done with a piecewise constant function \(B\) which has constant absolute value but changes sign at \(x_1 - x_2 = 2\pi k\) where \(k\) is an integer. Note that the discontinuity of \(B\) is not a problem since at these points the wave function is zero due to the sine function. The required function is then a square wave \(B(u) = \text{sgn}(\sin(u/2))\), where \(\text{sgn}\) is the sign function. The requirement that \(\psi(x_1, x_2) = \psi(x_1 + 2\pi, x_2)\) is then fulfilled for bosons only if \(n + m\) is odd, since adding \(2\pi\) either to \(x_1\) or to \(x_2\) changes the sign of \(B(u)\).

Note that the bosonic wave function is not analytic at \(x_1 = x_2\) (for odd \(n\)), but has a cusp. However, this is allowed since we assumed that the contact interaction is infinite and, consequently, the wave function has a node at that point.

In general, we can now write the eigenvalues for the two-particle system with infinitely strong contact interaction as

\[
\epsilon_{n,m} = \frac{1}{4}(n^2 + m^2),
\]

where \(n + m\) has to be even for fermions and odd for bosons.

The total angular momentum of the state is \(m\), since

\[
\hat{L}\psi = -i \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \psi = -i \frac{\partial}{\partial u} \psi = m\psi.
\]

The wave function thus consists of an internal part, \(\sin(\nu u/2)\), and a center of mass part, \(\exp(i m v)\). The latter corresponds to a rigid rotation of the two-particle system, while the internal part corresponds to an internal vibrational mode. Note that due to symmetry restrictions, we get the same selection rules as for two-atomic molecules: Purely rotational states \((n = 1)\) are allowed for fermions only with odd angular momenta, and for bosons only for even angular momenta. Figure 1 shows the energy spectra displaying the alternating allowed fermion and boson states for each of the vibrational states.

The ground state of two bosons \((m = 0, n = 1)\) has zero angular momentum, energy \(E = 1/4\) and, interestingly, it naively looks like a function of single-particle states with \(half-integer\) angular momenta (or half of the mass),

\[
\Psi_{2\text{bosons}}^{\text{GS}} = \text{sgn} \left[ \sin \left( \frac{x_1 - x_2}{2} \right) \sin \left( \frac{x_1 - x_2}{2} \right) \right] = \text{sgn} \left[ \sin \left( \frac{x_1 - x_2}{2} \right) \right] \frac{1}{2i} \left[ e^{ix_1/2} e^{-ix_2/2} - e^{-ix_1/2} e^{ix_2/2} \right].
\]

These, however, are not single-particle eigenstates of the ring. Fourier expansion in terms of the true single-particle eigenstates gives an infinite Fourier series (denoting, as earlier, \(x_1 - x_2 = u\))

\[
\Psi_{2\text{bosons}}^{\text{GS}} = \frac{2}{\pi} + \sum_{m=1}^{\infty} \frac{4 \cos(mu)}{\pi(1 - 4m^2)}
\]

\[
= \frac{2}{\pi} + \sum_{m=0}^{\infty} \frac{2}{\pi(1 - 4m^2)} \left[ \cos(mx_1) e^{-imx_2} + \cos(mx_1) e^{imx_2} \right].
\]
In the case of spinless fermions the antisymmetry dictates that a single Slater determinant is always a solution for contact interactions, as pointed out above. The ground state, found by setting \( n = m = 1 \) in Eq. (5), equals \( \exp(ix_1) - \exp(ix_2) \). It has angular momentum 1 and energy \( E = 1/2 \), while the state with zero angular momentum \( (m = 0, n = 2) \) is

\[
\Psi_{L=0}^{\text{fermions}} = e^{ix_1}e^{-ix_2} - e^{-ix_1}e^{ix_2}
\]

and has higher energy \( E = 1 \). Both energies are higher than the ground state energy for bosons \( (E = 1/4) \), as illustrated in Fig. 1.

In contrast to spinless fermions, for bosons the interaction has a true effect. This allows the bosonic wave function to 'take advantage' of all single-particle states, like in Eq. (10), for lowering the energy.

III. SPINLESS FERMIONS WITH CONTACT INTERACTIONS (NONINTERACTING FERMIONS)

In this section we discuss the so-called yrast states, i.e. lowest energy states for given total angular momentum, of non-interacting spinless fermions in a strictly 1D ring. Most of the detailed analysis is presented for an odd number of particles, with the corresponding results for even \( N \) stated at the end.

The ground state of \( N \) noninteracting fermions is a Slater determinant of the lowest \( N \) single-particle states. In a strictly one-dimensional (1D) ring with radius \( R = 1 \) the single-particle energies and wave functions are

\[
\epsilon_\ell = \frac{\ell^2}{2}, \quad \phi(x) = e^{i\ell x},
\]

where the \( \ell \) is the single-particle angular momentum and \( x \) the coordinate along the ring (equal to the angle in polar coordinates). In the ground state the \( N \) lowest states are occupied. For an odd \( N \) the single-particle states from \( \ell = 0 \)
to \( \ell = \pm (N - 1)/2 \) are occupied and the energy becomes

\[
E^{\text{ODD}}_{\text{gs}}(N) = 2 \sum_{\ell = 1}^{(N - 1)/2} \frac{\ell^2}{2} = \frac{1}{24} N(N^2 - 1). \tag{12}
\]

The angular momentum of the ground state is zero. The corresponding ground state wave function for an odd number \( N \) of fermions is the Slater determinant

\[
\Psi_{\text{GS}}(x_1, x_2, \ldots, x_N) = \begin{vmatrix}
   e^{-inx_1} & e^{-inx_2} & \cdots & e^{-inx_N} \\
   e^{-i(n-1)x_1} & e^{-i(n-1)x_2} & \cdots & e^{-i(n-1)x_N} \\
   \vdots & \vdots & \ddots & \vdots \\
   e^{inx_1} & e^{inx_2} & \cdots & e^{inx_N}
\end{vmatrix},
\]

where \( n = (N - 1)/2 \). Using the mathematical properties of a determinant we can recast this as

\[
\Psi_{\text{GS}} = e^{-inx_1}e^{-inx_2}\cdots e^{-inx_N} \prod_{i<j}^N (e^{ix_i} - e^{ix_j}),
\]

where in the last step we have identified the determinant as a so-called Vandermonde determinant \( \text{75} \).

By noticing that each \( x_i \) appears in \( 2n \) factors of the product \( (2n = N - 1) \), we can distribute the exponentials in front into these factors and get \( \text{76} \)

\[
\Psi_{\text{GS}} = e^{-inx_1}e^{-inx_2}\cdots e^{-inx_N} \prod_{i<j}^N (e^{ix_i} - e^{ix_j}) = e^{-inx_1}e^{-inx_2}\cdots e^{-inx_N} \prod_{i<j}^N \sin \left( \frac{x_i - x_j}{2} \right). \tag{13}
\]

Since we are working with unnormalized wave functions, the overall constant factor will be ignored in the following.

It is easy to verify that multiplying the wave function of this (or any other) many-particle eigenstate with the symmetric function \( \exp(i \sum x_i) \) will again generate a solution of the many-particle problem, with an energy increase \( E \to E + M + N/2 \) where \( M \) is the total angular momentum of the original state. This operation corresponds to an increase of the angular momentum of each single-particle state by one. Exciting the ground state this way (recalling that its total angular momentum is zero) will thus increase the angular momentum by \( \Delta M = N \), and the total energy by \( N/2 = \Delta M^2/2N = \Delta M^2/2I \). This term in fact has the same form as the classical rotational energy of a rigid body, with moment of inertia \( N \) (we have chosen the radius and mass to be equal to one). For this reason we will refer to it, somewhat sloppily, as "classical rotation energy", although of course it results from a purely quantum mechanical derivation. More generally, we find that the lowest energy states for total angular momenta \( M = mN \) are
obtained through rigid rotation of the ground state, with wave function
\[
\Psi_{[M=mN]} = \prod_{i}^{N} e^{imx_i} \prod_{i<j}^{N} \sin \left( \frac{x_i - x_j}{2} \right),
\]
and corresponding energy
\[
E_{ODD}^{L=mN}(N) = \frac{1}{24} N(N^2 - 1) + \frac{M^2}{2N},
\]
where the last term is again recognized as corresponding to rigid rotation of the entire ring.

For angular momenta different from \(mN\) we can also easily determine the lowest energy from the single Slater determinant of each state. For example, the lowest state for \(M = 1\) is obtained by exciting the topmost single-particle from \(\ell = (N - 1)/2\) to \(\ell = (N + 1)/2\). In terms of wave functions this means multiplying the ground state with a symmetric polynomial of exponentials exp\((ix_j)\),
\[
\Psi_{[M=1]} = \left( \sum_{i}^{N} e^{ix_i} \right) \prod_{i<j}^{N} \sin \left( \frac{x_i - x_j}{2} \right),
\]
with energy \(E_{M=1} = E_{GS} + N/2\). We note that, curiously, the energy is the same for \(M = 1\) and \(M = N\). This energy increase, \(N/2\), is much larger than what the classical rotational energy, \(1/2N\), would be. The energy difference is a vibrational energy of the quantum state.

It is straightforward to calculate the lowest vibrational energy for any angular momentum \(\nu\) (\(1 \leq \nu \leq M\)), i.e. the energy difference between the true quantum mechanical energy and the 'classical' rotational energy:
\[
\Delta E_{\nu}(\nu) = \frac{\nu(N + 1 - \nu)}{2} - \frac{\nu^2}{2N},
\]
where the first term is the lowest excitation energy from the \(M = 0\) to the \(M = \nu\) state, obtained by exciting the particle at \(\ell = (N - 1)/2 - (\nu - 1)\) to \(\ell = (N - 1)/2 + 1\) (see Fig.1 in Ref.\[25\]), and the second term is the subtraction of the classical rotation energy. Since any state can be multiplied by the rigid rotation \(\exp(im\sum x_i)\), we obtain that the rotational spectrum has 'a period of \(N\)' in the sense that the internal structure of the state does not change when the angular momentum is increased by \(N\) \[25\]. This means that the vibrational energy must be the same for \(M = \nu\) and \(M = \nu + mN\) (\(m\) integer). The above 'phonon' spectrum, taking the form of an inverted parabola, is analogous to a classical row of particles interacting with \(1/d^2\) interaction.\[25, 77–79\]. In quantum mechanics the effective \(1/d^2\) interaction results from the kinetic energy: A particle confined between the neighboring particles is essentially a particle in an infinite box with kinetic energy proportional to \(1/d^2\). Classical particles interacting only with a contact interaction would not have vibrational states.

We can now construct the lowest energy for each angular momentum, i.e. the yrast states, by adding the ground state energy for \(M = 0\), i.e. Eq. \[12\], the 'classical' rotational energy \(M^2/2N\), and the vibrational energy of Eq. \[16\], taking into account that it is the same for \(\nu\) and \(\nu + mN\). The result can be written as
\[
E_{ODD}^{yrast}(N, M) = \frac{1}{2}(N + 1)(M \mod N) - \frac{1}{2}(1 + \frac{1}{N})(M \mod N)^2 + \frac{M^2}{2N} + \frac{1}{24} N(N^2 - 1),
\]
as may be independently verified by a numerical computation of the quantum mechanical spectrum. Higher 'vibrational' excited states can be derived from the single-particle energies in a similar manner.

Finally, let us state the corresponding main results for an even number of spinless fermions. In this case the ground state can again be expressed as a Slater determinant, but now the angular momentum is not zero but rather $M = \pm N/2$. Manipulations similar as those leading to Eq. (13) give the wave functions for the two degenerate ground states

$$\Psi_{\text{fermions}}^{\text{GS}} = \prod_i e^{\pm ix_i/2} \prod_{i<j} \sin \left( \frac{x_i - x_j}{2} \right) ,$$

where total angular momentum can be immediately read off from the first product. An analysis similar to the one leading to Eq. (17) can be carried out to find the yrast energies,

$$E_{\text{yrast}}^{\text{EVEN}}(N, M) = \frac{1}{2}(N+1)((M + N/2) \mod N) - \frac{1}{2}(1 + \frac{1}{N})((M + N/2) \mod N)^2 + \frac{M^2}{2N} + \frac{1}{24} N(N^2 - 1).$$

**IV. BOSONS WITH INFINITELY STRONG CONTACT INTERACTIONS**

For odd numbers of bosons the solutions of energy levels are equal to those of spinless fermions. This can be most easily shown in a lattice model which we will return to in section VI. The ground state wave function can be obtained from the fermion solution by multiplying it with an appropriate product of sign functions, to make the state symmetric with respect to particle coordinate change. Like in the two-particle case, the boson wave function is not analytic, since at the points $x_i = x_j$ the partial derivatives are not continuous. However, this is not required since we assume an infinitely strong repulsive interaction (in fact even a finite contact interaction causes a discontinuity in the derivative of the wave function[5]). For example, the ground state for an even number of bosons is a direct generalization of the two-particle case [N],

$$\Psi_{\text{bosons}}^{\text{GS}} = \prod_{i<j} \text{sgn} \left[ \sin \left( \frac{x_i - x_j}{2} \right) \right] \sin \left( \frac{x_i - x_j}{2} \right) ,$$

and has angular momentum $M = 0$.

In the boson case the energy of the yrast line was recently studied in Ref.[5]. It is qualitatively similar for odd and even numbers of particles and is expressed in both cases by Eq. (17). This means that for odd numbers of particles the yrast lines for bosons and fermions are identical to each other, while for even numbers they differ qualitatively. Figure 2 shows the energy spectra for 8 spinless fermions and bosons with infinitely strong contact interactions. Like in the case of two particles, the boson and fermion states have different 'selection rules' which allow combination of the angular momenta of the 'vibrational states'.

Again we find that the energy spectrum is periodic in $N$ in the following sense: For any energy eigenvalue $E_{[M]}$ we have $E_{[M'=M+kN]} = E_{[M]} + k^2 N/2$, and at the yrast line $E_{[M=1]} = E_{[M=N]} = E_{\text{GS}} + N/2$. In the case of bosons this
FIG. 2. Energy spectra for 8 spinless fermions (blue) and for 8 bosons interacting with infinitely strong contact interactions (pink) in a 1D quantum ring of radius $R = 1$. The small dots show forbidden states. The blue and pink lines show the yrast energies for fermions and bosons.

holds both for odd and even numbers of particles.

As already mentioned in the introduction, we here stress again that boson systems with finite contact interaction have been solved exactly already long time ago by Lieb and Liniger [5, 6] and [7] and later on many others (see for example, Refs. [8–12, 50, 54], this list being far from complete).

V. PERSISTENT CURRENT AND ANGULAR MOMENTUM

In the case of electrons, or other charged particles, a persistent current can be introduced by a magnetic flux through the ring. Neutral particles obviously do not respond to magnetic fields in the same way. However, as mentioned in the Introduction, there are ways of mimicking the effect of a magnetic flux on, e.g., charge neutral bosonic atoms [43, 56–59]. Thus, the following discussion is relevant also to such systems, although we will stick to the standard terminology of ’real’ magnetic fields.

The physics of persistent currents becomes particularly transparent if one chooses the magnetic field at the ring radius to zero, while the flux through the ring is nonzero. This can be modeled by an appropriate choice of vector potential [25]. In such a system the single-particle wave functions of the ideal 1D ring do not change due to the flux, while the single-particle energies are modified (for a derivation, see e.g. [25]),

$$
\epsilon_m(\Phi) = \frac{1}{2} \left( m - \frac{\Phi}{\Phi_0} \right)^2,
$$

where $\Phi$ is the flux through the ring and $\Phi_0$ the flux quantum $\Phi_0 = h/e$ ($\Phi_0 = 2\pi$ in our units where $\hbar = e = 1$).

The many-particle state for noninteracting electrons (or spinless electrons with contact interactions) is still the same
single Slater determinant as without the flux, but the total many-particle energy becomes

\[ E_M(\Phi) = E_M(0) - M \left( \frac{\Phi}{\Phi_0} \right) + \frac{N}{2} \left( \frac{\Phi}{\Phi_0} \right)^2. \]  \hspace{1cm} (21)

In the case of bosons interacting with an infinitely strong contact interaction, the many-particle state consists of many permanents made of the single-particle wave functions. Since the flux does not change the single-particle states and each permanent must have the same angular momentum, the flux does not change the many-body wave functions. The many-particle energy for bosons will also be described with Eq. (21) above.

![Energy spectra for 8 spinless fermions (blue) and for 8 bosons interacting with infinitely strong contact interactions (pink) in a 1D quantum ring of radius \( R = 1 \). A magnetic flux \( \Phi = \Phi_0 \) passes through the ring. The small dots show forbidden states. The blue and pink lines show the yrast energies for fermions and bosons.](image)

The current (density) for a single-particle state \( \phi_\ell \) in a 1D system, with a flux \( \Phi \) passing through the ring (of radius \( R = 1 \) and in units \( \hbar = e = 1 \)), is[25]

\[ j_\ell = \frac{-i}{4\pi} \left( \phi_\ell \left( \frac{\partial}{\partial x} - i \frac{\Phi}{\Phi_0} \right) \phi_\ell(x) - \phi_\ell \left( \frac{\partial}{\partial x} + i \frac{\Phi}{\Phi_0} \right) \phi_\ell^*(x) \right) = \ell \frac{\Phi}{\Phi_0} - \frac{\Phi}{\Phi_0^2}. \]  \hspace{1cm} (22)

It follows that the current of the many-particle state, consisting of Slater determinants or permanents, will be

\[ J = M/\Phi_0 - N\Phi/\Phi_0^2. \]  \hspace{1cm} (23)

This is because each determinant (permanent) has the same total angular momentum \( M \) and consists of \( N \) single-particle states.

Generally, it can be shown that the current caused by the flux can be derived from the flux dependence of the total energy as[25, 81]

\[ J = -\frac{\partial E}{\partial \Phi}. \]  \hspace{1cm} (24)

Clearly, when applied to Eq. (21) the above equation gives Eq. (23). At finite temperatures the current can be derived by replacing the total energy \( E \) with the Helmholtz free energy \( F \) in Eq. (24).
FIG. 4. Energy spectra for 8 spinless fermions (blue) and for 8 bosons (pink) as a function of the flux passing through the ring (left), and the corresponding persistent current (right). Energies and currents corresponding to different angular momenta are shown as dashed lines and those corresponding to the ground state as solid lines. The particles interact with an infinitely strong contact interaction.

Comparing figures 2 and 3 we notice that the effect of the flux is to tilt the spectrum so that energy states with higher angular momenta become the lowest energy states. When we determine the persistent current for the ground state we notice that we jump from one angular momentum state to another when the flux increases. For 8 bosons these states are 0, 8, 16 etc., and for fermions 4, 12, 20 etc. (see left panel in Fig. 4). The current as a function of the flux for 8 particles interacting with infinitely strong contact interaction is shown in the right panel of Fig. 4.

VI. HUBBARD MODEL WITH INFINITE INTERACTION ($t$-MODEL)

A. Tight-binding basis and spinless particles

The Hubbard model [81] has been extensively used for studying persistent currents in quantum rings (see e.g. [25, 82–84]). The basis of the Hubbard model is the tight binding (TB) model: The electrons are predominantly localized on atoms at lattice sites, but are allowed to hop between neighbouring sites. In a 1D lattice the simple TB model is, somewhat surprisingly, also closely related to the free electron model. A numerical solution of the free electron Hamiltonian in a lattice leads to the simple TB model with one state per lattice site [85]. This means that the low energy spectrum of the TB model is identical with that of the free-electron case. The simple TB Hamiltonian for the ring is

$$H = -t \sum_{i=1}^{L} a_{i}^{\dagger} a_{i+1} + \text{HC},$$  \hspace{1cm} (25)$$

where $a_{i}^{\dagger}$ and $a_{i}$ are creation and annihilation operators for site $i$, HC denotes Hermitian conjugate, and the periodic boundary condition requires that $a_{L+1}^{\dagger} = a_{1}^{\dagger}$. The corresponding Hamiltonian matrix has nonzero matrix elements
$H_{ij} = -t$ only if $i$ and $j$ are nearest neighbours. The eigenvalues for the Hamiltonian matrix for a ring of $L$ sites and radius $R = 1$ are (assuming $L > 2$)

$$\epsilon(n) = -2t \cos \left( \frac{2\pi}{L} n \right),$$

and the corresponding eigenvectors (wave functions at site $j$)

$$\phi_n(j) = e^{i2\pi nj/L}. \quad (27)$$

From the periodic boundary condition it follows that $\epsilon(n) = \epsilon(L - n) = \epsilon(-n)$ and $\phi_n(j) = \phi_{L-n}(-j) = \phi_{-n}(-j)$.

We notice that in a large ring, with $L \gg n$ the low energy eigenvalues are those of free electrons, $\epsilon(n) \approx -2t + tk^2$, with the wave vector $k = 2\pi n/L$.

We will first consider the many-body problem of spinless particles interacting with an infinitely strong contact interaction in quantum rings made of $L$ lattice sites. The infinitely strong repulsive short range interaction prevents two or more particles from occupying the same lattice site. The many-body Hamiltonian describing such a system is the Hubbard model

$$H = -t \sum_{ij} a_i^\dagger a_j + \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1), \quad (28)$$

where "nn" means that the sum goes only over nearest neighbors, $\hat{n}_i = a_i^\dagger a_i$ and $U$ the onsite interaction. Note that for spinless fermions the interaction term vanishes since $\hat{n}_i^2 = \hat{n}_i$. Also in the case of bosons with $U \to \infty$ we can neglect the interaction term of the Hamiltonian if we restrict the many-particle basis to states with only zero or single occupancy in each site. The Hamiltonian then reduces to the tight binding Hamiltonian, Eq. (25). This model is sometimes called the $t$-model.

We describe the many-body states in the localized basis and write the basis vectors as $|n_1 n_2 n_3 \cdots n_L\rangle$, where occupation numbers $n_i$ are now restricted to be 0 or 1 (also for bosons) and the total number of particles is $\sum_i n_i = N$.

The only difference between the bosons and fermions is thus whether commutation or anticommutation relations are used for the operators $a$ and $a^\dagger$. Consider one term of the Hamiltonian, $a_i^\dagger a_{i+1}$, where $i < L$, operating on a state with $n_{i+1} = 1$ and $n_i = 0$. Both for fermions and bosons we get

$$a_i^\dagger a_{i+1} |\cdots 01 \cdots \rangle = + |\cdots 10 \cdots \rangle,$$

independent of the occupation of the other states. This is because the creation and annihilation operators operate on neighbouring positions and, consequently, the phase factors for fermion operators (powers of $(-1)$) always add to an even number. A difference can appear when $i = L$, since $a_L^\dagger a_{L+1} = a_L^\dagger a_1$. In this case the operation with $a_{L+1} = a_1$ will give no phase factor but the subsequent operation with $a_L^\dagger$ gives a phase $(-1)^{N-1}$ for fermions. Clearly this is
+1 if \( N \) is odd and \(-1 \) if \( N \) is even. The periodic boundary conditions thus give

\[
a_L^\dagger a_1|1\cdots0\rangle = \begin{cases} 
-|0\cdots1\rangle & \text{for even number of fermions} \\
|0\cdots1\rangle & \text{otherwise.} 
\end{cases}
\] (30)

The important result is that if the number of particles is odd, the Hamiltonian matrices for bosons and fermions are identical, leading to the same eigenvalues and eigenvectors. However, two things have to be noticed: (i) the eigenvectors are the same only if the localized single-particle basis is used for constructing the many-body states and (ii) the meaning of the many-body basis vector \( |n_1n_2n_3\cdots n_L\rangle \) is different for bosons and fermions. In the case of fermions it is an antisymmetric Slater determinant while in the case of bosons it is a symmetric function called permanent.

Remembering that at low energies the TB model is equal to the free particle model, it is easy now to reconsider the particles in a continuous ring studied in sections III and IV. In the case of odd numbers of particles the energy spectra of bosons and fermions are identical, while for even numbers of particles, as shown in Figs. 1 and 2, they are still related, but qualitatively different.

![FIG. 5. Energy spectrum of three spinless particles in a Hubbard ring with 10 sites and infinite \( U \). The left panel shows the lowest energy levels (dots) as a function of the angular momentum \( M \). The yrast levels are connected with a line to guide the eye. The spectrum is identical for bosons and fermions. The right panels show the occupation of the TB single-particle states of the ring for the ground state (A) and for an excited state (B), blue for fermions and pink for bosons. The fermion states are single determinants with only three of the states occupied, while the boson states consist of numerous permanents.](image)

We could also use the single-particle states of the TB model, Eq. (27), when constructing the many-particle basis for bosons and fermions. In this case all fermion wave functions would be single Slater determinants, as we noted above (the contact interaction has no meaning for spinless fermions). However, a typical boson state would be a complicated linear combination of several permanents. This difference is illustrated in Figure 5 which shows the energy spectrum for three particles in a ring of ten sites (\( N = 3, \ L = 10 \)). The spectrum is identical for bosons and fermions. The state vectors are also identical in the occupation number representation when the localized basis is
used. However, in the TB basis, where the fermion wave functions reduce to single Slater determinants, the boson wave functions remain complicated as shown in Fig. 5 where the occupations of the TB single-particle states are shown for the ground state and for an excited state. In the case of fermions only three TB states are occupied. In the case of bosons all single-particle states have some occupation. Note that the infinitely strong contact interaction leads to a significant depletion of the single particle ground state, which in the non-interacting case would be occupied by all \( N \) bosons.

The rotational spectrum of a Hubbard ring is characterized by a periodicity of \( L \). In the case when \( N \ll L \) the yrast line also shows a periodicity of \( N \) in a similar way as in the case of continuous rings. This is illustrated in Fig. 5 showing the yrast line for \( N = 3 \) which has minima at angular momenta -3, 0, and 3. Lattice quantum rings thus have all the same features as continuous rings when particles with strong contact interaction are considered.

### B. Magnetic flux and current in a lattice

The effect of a magnetic flux piercing a ring of \( L \) lattice sites is to add a phase in the hopping term of the TB Hamiltonian

\[
H = -t \sum_i \left( e^{-i2\pi\Phi/(\Phi_0 L)} a_i^{\dagger} a_{i+1} + e^{i2\pi\Phi/(\Phi_0 L)} a_{i+1}^{\dagger} a_i \right) .
\]

In the lattice we can define the current operator between two points \( i \) and \( j \) generally as

\[
J_{ij} = -i \frac{2\pi}{\Phi_0 L} \left( t_{ij} e^{i\Phi_{ij}} a_j^{\dagger} a_i - t_{ji} e^{i\Phi_{ji}} a_i^{\dagger} a_j \right) ,
\]

where \( t_{ij} \) is the hopping parameter and \( \Phi_{ij} \) the phase change between the two points. In a simple ring \( \Phi_{i+1,i} = 2\pi\Phi/(\Phi_0 L) = -\Phi_{i+1,i} \), and the current is the same between any neighbouring sites. Alternatively, we can again determine the current as the derivative of the total energy, using Eq. (31). Note that when the ring is quasi-one-dimensional, Eq. (32) is still valid, but for calculating the total current one has to sum the currents through different paths.

### C. Particle-hole symmetry

In a Hubbard ring the persistent current can be determined as the derivative of the total energy as a function of the flux (Eq. (24)) or by computing the expectation value of the current operator between two adjacent points, Eq. (32).

In the case of bosons with infinitely strong contact interaction \( U \to \infty \) the Hamiltonian is the same for particles and holes, i.e. \( H = \sum a_i^{\dagger} a_j = \sum a_j a_i^{\dagger} \) since the operators commute when \( i \neq j \). This means that the ground state energy and the persistent current are symmetric with respect to particles and holes, irrespective of the symmetry of
the single-particle spectrum. The situation is different for fermions due to the anticommutation rule, which changes the sign of the Hamiltonian for holes. Consequently, in the case of fermions the many-body energy and the persistent current of the ring are symmetric with respect to particles and holes only if the single-particle spectrum is symmetric.

VII. TWO-COMPONENT SYSTEMS (SPIN-1/2)

A. Hubbard model with infinite \( U \)

Let us now consider particles with spin (e.g. normal spin for fermions and two different hyperfine states for bosons), but still apply the same simple Hubbard model with \( U \rightarrow \infty \). The spin increases the size of the Fock space since all spin configurations should be included. Since we still assume the infinite \( U \)-, or \( t \)-model, only one particle is allowed in a lattice site, independent of its spin. The energy spectrum for bosons and fermions is the same if both of the spin components have an odd number of particles, or if the total number of particles is odd. In the former case we can arrange the single-particle states as \( |n_{1\uparrow}, n_{2\uparrow}, \ldots, n_{L\uparrow}; n_{1\downarrow}, \ldots n_{L\downarrow} \rangle \). Acting with the fermion operator \( a_{i,\sigma}^\dagger a_{i+1,\sigma} \) does not change the sign. In the latter case we arrange the basis states as \( |n_{1\uparrow}, n_{1\downarrow}, n_{2\uparrow}, n_{2\downarrow}, \ldots, n_{L\uparrow}, n_{L\downarrow} \rangle \), and again \( a_{i,\sigma}^\dagger a_{i+1,\sigma} \) can not change the sign (note that only one of \( n_{i+1,\uparrow} \) and \( n_{i+1,\downarrow} \) can be nonzero since only one particle is allowed in any site.

The case when both \( N_{\uparrow} \) and \( N_{\downarrow} \) are even, is more complicated. However, also in this case all the eigenvalues are still the same for bosons and fermions, but the degeneracies of the eigenvalues can differ.

In general, for a fixed total number of particles, \( N = N_{\uparrow} + N_{\downarrow} \), all the energies of the spectrum are the same for any combination of nonzero \( N_{\uparrow} \) and \( N_{\downarrow} \), but the degeneracies can differ. To prove this, we first notice that a state with \( N_{\uparrow} = N_{\downarrow} \) (or \( N_{\uparrow} = N_{\downarrow} + 1 \) for odd number of particles) has \( S_z = 0 \) (or \( S_z = 1/2 \)), where \( S_z = N_{\uparrow} - N_{\downarrow} \) is the \( z \)-component of the total spin. Since the Hamiltonian is independent of the spin, the spectrum of \( S_z = 0 \) includes all energies for states with \( S_z = -N_{\downarrow}, -N_{\downarrow} + 1, \ldots, N_{\downarrow} \), i.e. for all possible values of \( S_z = N_{\uparrow} - N_{\downarrow} \). We then only have to show that \( N_{\uparrow} = N - 1 \) and \( N_{\downarrow} = 1 \) already gives all possible eigenvalues. This can be done with the help of the Bethe ansatz solution, as will be discussed in the case of an infinitely strong contact interaction in the section below.

It is interesting to observe, that adding to a one-component system just one particle of the other component will change the energy spectrum completely by allowing now all the states, which were forbidden by the wave function symmetry in the one-component case. For example, in the eight particle case of Fig. 2 all the energies shown with black dots would become allowed for both bosons and fermions if the 8 particles consisted of, say, 7 ”spin-up” particles and one ”spin-down” particle.
B. Bethe ansatz solution for particles with infinitely strong contact interactions

The Hubbard model for fermions in a 1D lattice was solved exactly by Lieb and Wu using the Bethe ansatz. The total energy of a given many-body state with \(N\) particles can be written as

\[
E = -2t \sum_{j=1}^{N} \cos k_j, \tag{33}
\]

where the numbers \(k_j\) are found by solving a set of algebraic (transcendental) Bethe equations (see e.g. [22]). In the limit of infinite \(U\) these numbers will be

\[
k_j = \frac{2\pi}{L} \left[ I_j - \frac{p}{N} \right], \tag{34}
\]

with

\[
p = -\sum_{\alpha=1}^{N_\uparrow} J_\alpha. \tag{35}
\]

The quantum numbers \(I_j\) and \(J_\alpha\) are related to the charge and spin degrees of freedom, respectively. Their possible values depend on whether \(N\) and \(N_\uparrow\) are even or odd. Here we will only consider even \(N\), in which case the one-component \((N_\uparrow = 0\) or \(N_\uparrow = N\)) boson and fermion systems have different spectra. In this case the \(I_j\):s are integers (half-odd integers), and the \(J_\alpha\):s are half-odd integers (integers) if \(N_\uparrow\) is even (odd). We note that \(p\) will always be an integer. Thus, in practice, all eigenvalues for \(0 < N_\uparrow < N\) can be found by letting \(p\) run over all integers and choosing all possible sets of \(I_j\):s with the restriction \(I_{\text{max}} - I_{\text{min}} < L\).

The total angular momentum depends on the above quantum numbers and can be determined as

\[
M = \sum_{j=1}^{N} I_j + \sum_{\alpha=1}^{N_\uparrow} J_\alpha. \tag{36}
\]

For \(N_\uparrow = 0\) the quantum numbers \(I_j\) are integers and \(p = 0\). In this case we see immediately that the many-body energy (33) is a sum of the single-particle energies (26). When \(L \to \infty\) we recover the result for particles in a continuous ring and, say for 8 particles, the spectrum shown with large blue dots in Fig. 2. Next, consider the case \(N_\uparrow = 1\). The number \(p\) already goes through all integers and, as predicted in the previous subsection, we thus get all possible energies, shown in small blue dots in Fig. 2. Increasing \(N_\uparrow\) from one thus can not give any new energy levels. A similar analysis for odd \(N\) also shows that with \(N_\uparrow = 1\) we already get all possible energy levels.

We have here considered fermions. In the case of bosons the Hubbard model has turned out to be more difficult to solve with the Bethe ansatz method. Fortunately, when we are only interested in the energy spectrum of a 1D ring with infinitely strong contact interaction, we do not need to solve the boson problem, since the fermion solution gives the same energies as long as \(0 < N_\uparrow < N\), as discussed in the previous section.
C. Noninteracting particles

The single-particle spectra of noninteracting particles were given by Eqs. (11) and (26) for a 1D continuous ring and a TB lattice ring, respectively. In the case of bosons the number of components (number of spin components) does not change the energy spectrum, since each single-particle state may be occupied by any number of bosons even in the single component case; only the degeneracies of the energy levels depend on the number of components. At zero angular momentum all the bosons occupy the $\ell = 0$ single-particle state. When the angular momentum is increased, the angular momentum state $\ell = 1$ starts to be occupied until all particles are in that single-particle state. Then we start to occupy the $\ell = 2$ state and so on. The yrast energy increases linearly with the total angular momentum $M$, but the slope has discontinuities at points $M = nN$, where the next single-particle angular momentum starts to be occupied. Figure 6 shows the beginning of the yrast spectrum for eight noninteracting bosons (as pink dots).

In the case of fermions the number of components has a marked effect on the spectrum of noninteracting particles. It is easy to construct the spectrum using the Pauli exclusion principle: Each single-particle state can accommodate only one (or zero) particle of each component. Figure 6 shows the yrast lines for 8 particles. The left panel shows the result for a two-component system (spin-1/2 fermions) with different numbers of spin-up and spin-down particles. The lowest energies are obtained when the number of spin-up and spin-down particles is the same. The left panel
shows the effect of the number of components on the yrast spectrum of fermions. When the number of components (spin of the particles) increases, the energy approaches that of bosons. When the number of components equals that of the number of particles, the fermion spectrum is identical to that of the bosons, since in this case \( N \) fermions can occupy the same single-particle state.

VIII. INTERACTION WITH FINITE STRENGTH

So far we have only considered the ideal limits of zero and infinite (contact) interaction. The more general case where the interaction strength is finite is naturally more complicated. Nevertheless, also in this case several exact solutions exist. The seminal paper for bosons was that of Lieb and Liniger\(^5\) who solved exactly the problem of interacting bosons in 1D, and for fermions that of Lieb and Wu\(^13\) who solved the 1D Hubbard model for spin-1/2 fermions. Instead of reviewing this important analytic work, we here choose to merely illustrate the qualitative differences between bosons and fermions, based on exact (numerical) solutions of small Hubbard systems.

A. Bosons with finite interaction

In previous sections we have encountered examples of energy spectra for quantum rings of bosons with both infinite (Fig.2) and zero interaction strength (Fig.6). The intermediate problem of bosons interacting with finite contact interaction was solved exactly by Lieb and Liniger\(^5\). Naturally, when the contact interaction increases from zero, the corresponding yrast lines smoothly interpolate between those of zero and infinitely strong interaction.

FIG. 7. Dependence of the yrast spectrum on the interaction strength \( U \) of the Hubbard model for one-component bosons. The ring has 8 sites and 4 bosons. The different curves represent different strengths of the interaction \( U \). From bottom to top \( U = 0, 0.3, 1, 3, 7, 20, \) and \( U \to \infty \). The results for \( U = 0 \) and \( U \to \infty \) are joined with solid lines, the others with dashed lines.

To illustrate this, consider a small Hubbard ring with eight sites (\( L = 8 \)) and four particles (\( N = 4 \)). Figure 7...
shows the yrast energy as a function of the angular momentum for different values of the interparticle interaction $U$ of the Hubbard model. The energy is a periodic function of the angular momentum $M$ due to the finite number of sites. Nevertheless, even in this small (lattice) ring we already see clearly the same features as for a smooth 1D ring: In the case of noninteracting bosons the energy increases linearly with the angular momentum, and in the case of $U \to \infty$ the energy has minima at $M = 0$ and at $M = N$. When $U$ increases from zero, the yrast energy smoothly approaches that for $U \to \infty$. The local minimum at $M = 4$ is reached with rather small interaction strength $U = 3$.

In the case of one-component (spinless) fermions the strength of the contact interaction is irrelevant, since the Pauli exclusion principle prevents the particles from "seeing" the interaction.

### B. Yrast-line of two-component bosons and fermions

Two-component systems have an effective spin $1/2$. In the case of fermions it can be the true spin of the particles, while in the case of bosons it can be the pseudospin, realized for example by two different hyperfine states of bosonic atoms. While a contact interaction has no effect for spinless fermions, it does play a role when both spin components are present, since the Pauli principle does not prevent particles with opposite spin from occupying the same site. We learned above that in the case of an infinite interaction the energy spectrum is the same for bosons and fermions as soon as we have at least one particle of the minority component. This not true for interactions of finite strength.

![FIG. 8. Dependence of the yrast spectrum on the interaction strength $U$ of the Hubbard model. The ring has 8 sites and 4 particles, 2 spin-up and 2 spin-down. The left panel shows the results for fermions and the right panel for bosons. The black points connected with solid lines show the results for the noninteracting case ($U = 0$), the blue points connected with solid lines correspond to $U \to \infty$. Note that this result is the same for bosons and fermions for reasons discussed in Section VII A. The upper curves with dashed lines show results for one-component systems. The boson result for the noninteracting case is plotted also in the left panel with pink points connected with dashed lines. The other plots in the fermion case correspond to $U = 1$ (red), $U = 3$ (blue), $U = 7$ (green), and in the boson case $U = 0.3$ (pink), $U = 1$ (red), $U = 3$ (blue).]

We will first consider the yrast spectrum for a Hubbard system with eight sites and four particles so that we
have two particles of each component. Figure 8 shows the results for several strengths of the interaction $U$. In the case of $U \to \infty$ the result is the same for fermions and bosons for reasons described in Section VII A. In the case of noninteracting systems the fermion and boson results differ, except at $M = 2$ which is the ground state for noninteracting fermions. When the interaction strength increases the yrast lines smoothly approach that of infinite $U$. In the case of fermions the local minimum at $M = 2$ remains to quite high values of $U$, it is still marked at $U = 7$. Only at very large $U$ the boson and fermion results will be qualitatively similar.

C. Spin-splitting of the yrast line

In the extreme case of infinite $U$, the effective Hubbard Hamiltonian is totally spin-independent, and thus the spectra are degenerate in the spin degree of freedom. As we shall see in this section, this is no longer the case for finite interaction strength $U$, where this degeneracy is split. First, however, let us return to the spin-independent limit for a two-component system, i.e. particles of (pseudo)spin 1/2. Different choices of numbers of particles in the up- and down components thus correspond to different $z$-components of the spin. For each total spin there will be a multiplet of states, with the same energy and orbital angular momentum, corresponding to different $z$-components of the total spin or, equivalently, different numbers of particles in each component. In the case of four particles, for example, the energy spectrum computed for two spin-up and two spin-down particles ($N_\uparrow = 2, N_\downarrow = 2$) also includes all energy levels for $N_\uparrow = 3, N_\downarrow = 1$ as well as for $N_\uparrow = 4$ and $N_\downarrow = 0$.

![Graph showing spin-dependence of the yrast line](image)

FIG. 9. Spin-dependence of the yrast spectra for four spin-1/2 fermions (solid lines) and (pseudo)spin-1/2 bosons (dashed lines). The results are given for the Hubbard model with 8 sites and $U = 20$. Different colors indicate the total (pseudo)spin: Red for $S = 0$, blue for $S = 1$ and green for $S = 2$. The dashed and solid red lines coincide. For infinite $U$ (dashed line with dots) the spin-splitting vanishes. For fermions the splitting at large but finite $U$ corresponds to that of the antiferromagnetic Heisenberg model.

The distribution $(N_\uparrow, N_\downarrow)$ also provides information on the total spin of a given state of the two-component system.
In the case of four particles we know that the total spin can be $S = 0, 1, \text{ or } 2$. The totally polarized case with $S_z = 2$, i.e. the one-component case, corresponds to $S = 2$. Note that the case $N_\uparrow = 3$, and $N_\downarrow = 1$ has the $z$-component of the spin $S_z = 1$, but the total spin can be $1$ or $2$. Similarly, in the case of $N_\uparrow = 2$ and $N_\downarrow = 2$, $S_z = 0$, while the total spin can be $0, 1$ or $2$.

In the case of an infinitely strong interaction, states with different total spin will be degenerate. Away from this limit, however, this is no longer true. In particular, in the case of fermions the Hubbard model approaches the antiferromagnetic Heisenberg model for large but finite $U$ \cite{88}, so the Hamiltonian will contain a spin-dependent term

$$H_J = \frac{J}{2} \sum_{i \neq j} S_i \cdot S_j.$$  \hspace{1cm} (37)

Here, the effective Heisenberg coupling is proportional to the inverse of $U$ and vanishes at $U \to \infty$ \cite{89}. Thus, while the energies are independent of the total spin of the quantum state in the infinite $U$ limit, this degeneracy will split for finite $U$. Note that this splitting is caused by the (spin-independent) inter-particle interaction $U$.

Figure 9 shows the spin splitting of the lowest energy states for four particles in a Hubbard ring with eight sites. The dashed line with dots shows the result for $U \to \infty$. This result is the same for bosons and fermions and naturally does not show any spin splitting. The colored lines show the energy levels for different spin states in the case when $U = 20$. In the case of fermions (solid lines) the spin splitting corresponds to that of the antiferromagnetic Heisenberg model. The lowest state for $M = 0$ has $S = 1$. This is a result of Hund’s first rule: The lowest single-particle state ($\ell = 0$) is filled with two particles with opposite spins, while the two remaining particles occupy the states $\ell = 1$ and $\ell = -1$ and prefer to have the same spin for removing the effect of the repulsive interparticle interaction. The energy of the $S = 2$ state for fermions equals that for infinitely strong interaction. This is because now the system is polarized and thus a one-component system where the interaction does not have any meaning.

The result for bosons is qualitatively different. Now the lowest state has $S = 2$, corresponding to a one-component system. The results for $S = 0$ are the same as for fermions. This is true for any value of $U$ as seen also in Fig. 8 where all the points for $M = 2$ are the same for bosons and fermions. In the case of four particles this result appears to be independent of the length of the ring. However, computations for six particles show that this is not a general property of bosons and fermions in Hubbard rings.

A major difference between the fermionic and bosonic cases lies in their respective magnetic states. It has been shown that the Bose-Hubbard model for spin-1/2 particles with large on-site repulsion can be approximated by a ferromagnetic Heisenberg model \cite{90}. Again for half-filling the coupling constant has the same value as in the case of fermions ($J = -4t^2/U$) but now the sign is opposite. In the case of four particles in a ring, the Heisenberg model (independent of the sign of $J$) is exactly solvable (see an exercise in\cite{60}). We notice that the splitting seen in Fig. 9 is already qualitatively the same as in the Heisenberg model, and that the boson and fermion cases have different magnetic order. Numerical diagonalization of the Hubbard and Heisenberg Hamiltonians for four and six fermions
and bosons shows that the coupling constant for large $U$ is the same for the antiferromagnetic fermion case and the ferromagnetic boson case for all filling fractions ($N \leq L$). For half-filling ($N/L = 1$), $|J| \approx 4t^2/U$, while for smaller fillings $|J|$ decreases with $N/L$[4].

D. Fixed points in fermion spectra

Considering particles with infinitely strong contact interactions we learned that in a two-component system it is sufficient to have only one particle of the minority component in order to obtain the entire rotational spectrum for any mixtures, while the degeneracies will depend on the numbers of particles in each component. In the case of one-component (spinless) fermions the contact interaction does not have any effect and is equivalent to the noninteracting case. At angular momenta $M = N/2 + kN$ for even number of fermions and at $M = kN$ for odd number of fermions ($k$ any integer), the lowest energy is the same for spinless fermions and for two-component systems with $N_\downarrow = 1$ interacting with contact interactions. Moreover, it is easy to discover that the same energy results also for noninteracting two-component fermions with $N_\downarrow = 1$, as seen in Fig. 6.

![Graph](image)

**FIG. 10.** Dependence of the yrast spectrum on the interaction strength $U$ of the Hubbard model in the case with $N = 4$, $N_\uparrow = 3$, $N_\downarrow = 1$ and $L = 8$. The left panel shows the results for fermions, and the right panel for bosons. The curves from down to up correspond to $U = 0, 1, 3, 7,$ and $U \to \infty$, respectively. Note that in the case of fermions the energy at $M = 2$ and $M = 6$ is independent of $U$.

This gives us the interesting result that at these angular momenta ($M = N/2 + kN$ or $M = kN$), as long as $N_\downarrow = 1$, the lowest energy is completely independent of the strength of the contact interaction. Naturally, this result only holds for fermions as demonstrated in Fig. 10 showing the results for $N = 4$, $L = 8$ for bosons and fermions (now $N_\uparrow = 3$ and $N_\downarrow = 1$).
IX. QUASI-ONE-DIMENSIONAL RINGS

In experiments, as they were mentioned in the Introduction, the quantum rings will typically not be strictly one-dimensional, which means that the particles eventually may “pass” each other. Considering the ring as a quasi-one-dimensional (Q1D) wire, the perpendicular modes of the single-particle wave function can be separated from the longitudinal modes. If the excitation energy of the perpendicular modes is large, the many-particle state is composed mainly of the lowest perpendicular mode, and the system becomes rather similar to the strictly 1D case.

A. Continuous Q1D rings

A generic continuous Q1D ring can be modelled using a harmonic confinement,

\[ V(r,z) = \frac{1}{2}m \omega^2 (r - R)^2 + V_z \]

where \( R \) is the radius of the ring, \( r \) the radial coordinate on the plane of the ring, \( \omega \) the planar confinement frequency and \( m \) the mass of the particles. \( V_z \) indicates the confining potential perpendicular to the plane. Depending on the system considered, it might be harmonic or, e.g., correspond to the effective confinement experienced by the 2D electron gas in semiconductor heterojunctions. Frequently \( V_z \) is assumed to be so large that motion in the \( z \)-direction is completely frozen out, and thus the system is effectively two-dimensional. We will here only consider 2D systems and thus the second term in Eq. (38) is not needed.

Exact diagonalization techniques as well as density functional methods have been used to describe electrons in Q1D rings\[^24\,92\]. For reviews we refer to\[^25\,93\]. Exact diagonalization studies show that in narrow rings the spin and charge excitations separate: The former resemble those of an antiferromagnetic Heisenberg model, and the latter vibrational modes of localized electrons\[^24\,25\]. The results of density functional methods (making use of the local-spin-density approximation) show clearly the electron localization in narrow rings and give qualitatively correct persistent currents\[^94\,97\].

Bosons in annular traps have been studied using the Gross-Pitaevskii method (see for example,\[^47\,51\,53\,55\,63\,64\]) as well as with exact diagonalization (see for example,\[^49\,50\,54\,91\,98\,100\]).

Figure 11 shows the yrast spectrum for 17 bosons in a Q1D quantum ring, calculated by Bargi et al.\[^91\] using exact diagonalization techniques. In the case of a one-component system (red line) the result is qualitatively similar to that of particles interacting with an infinitely strong contact interaction (see Fig. 2). In the two-component case, with \( N_\uparrow = 15 \) and \( N_\downarrow = 2 \), a smoother curve is produced. In an infinitely narrow ring the black curve would be expected to be a parabola, for the reasons explained in Section VII A, independent of the number of particles in the minority component \( N_\downarrow \) (as long as it is nonzero). It is interesting to note that the effect of the finite width of the confinement
appears to be (qualitatively) stronger in the case of a two-component system than in the case of a one-component system.

B. Q1D rings in the Hubbard model

Quasi-one-dimensional quantum rings can also be made out of optical lattices which can be described with the Hubbard model. In the case of bosons the simple 1D Hubbard ring with finite $U$ already mimics the Q1D ring, since a finite $U$ means that the particles can pass each other like in a Q1D continuous ring. This is illustrated in Fig. 12: The 1D ring with a finite $U$ shown in the middle panel of the figure, gives a qualitatively similar low energy spectrum as a Q1D ring with infinite $U$, shown in the lowest panel.

In the case of fermions the situation is different. In a strictly 1D one-component system the Pauli exclusion principle prevents particles from passing each other even with zero $U$. However, in the case of two-component fermions the finite $U$ Hubbard model also mimics a Q1D ring [25]. This is illustrated in Fig. 13 which shows results for seven two-component fermions in a ring with 15 sites with the interaction strength $U = 5$. We know from the previous sections that for $U \to \infty$ the yrast line is the same for any $0 < N_\uparrow < N$. Figure 13 shows that this is not the case for finite $U$, but results for different $N_\uparrow$ differ qualitatively. It is interesting to note that the case $N_\uparrow = N - 2$ is qualitatively similar to the related case for bosons in a Q1D ring, shown in Fig. 11.
FIG. 12. Flux dependence of the lowest energy levels for four bosons in quantum rings. The uppermost panel shows the result for $U \to \infty$. The middle panel shows the result for $U = 3.4$. the lowest panel shows the result for $U \to \infty$ in the case of a Q1D Hubbard ring. The rings in each case are shown at the left.

The finite width of the quantum ring can change the energy spectrum markedly. In the case of four fermions in a 1D ring the ground state is degenerate and has a finite angular momentum. This is not necessarily the case if the
FIG. 13. Energy as a function of angular momentum for two-component fermions in a Hubbard ring with \( L = 15 \) sites, \( N_{\uparrow} + N_{\downarrow} = 7 \) particles and the interaction strength \( U = 5 \). \( N_{\uparrow} = 7 \) black; \( N_{\uparrow} = 6 \) green; \( N_{\uparrow} = 5 \) blue; \( N_{\uparrow} = 4 \) red. For comparison, the dashed line shows the \( U \to \infty \) result which the same for \( 0 < N_{\uparrow} < 7 \). Note the qualitative similarity of blue curve (\( N_{\uparrow} = N - 2 = 5 \)) with the black curve in Fig. 11.

ring is wide enough. The upper panel of Figure 14 shows that in a Q1D Hubbard ring the lowest energy state of the fermion spectrum is not degenerate and the energy increases with the flux, even if the system has four electrons. In the 1D case the ground state would be degenerate at zero flux, and increasing the flux would split the degeneracy, decreasing one of the energy levels.

Figure 14 also shows the result for a so-called Möbius ring, studied in detail by Ballon et al.\cite{101}, where the fermion ground state is degenerate. It is interesting to note that in the Möbius ring the boson spectrum is nearly identical with that of the normal Q1D ring, while the fermion spectrum becomes qualitatively different. Note that only the periodic boundary condition is changed in going from the normal ring (upper panel in the figure) to the Möbius ring (lower panel).

X. CONCLUSIONS

The theoretical description of quantum rings is a vast field, involving various types and strengths of interactions, continuum- versus lattice models, strictly 1D versus quasi-1D systems, presence or absence of magnetic fields (and thus persistent currents), spinful versus spinless particles – and of course the quantum statistics of the particles. Our hope is that the present paper has provided a comprehensive and fairly self-consistent overview of the similarities and differences between bosonic and fermionic quantum rings in all of the above contexts. As we have seen, in many cases the quantum statistics of the particles does make a big difference to the energy spectra, and thus the physical properties, of the system. Since both bosonic and fermionic quantum rings can be fabricated in the lab nowadays,
FIG. 14. Flux dependence of the lowest energy levels for one-component fermions (red) and bosons (blue) interacting with infinitely strong contact interactions for the different Hubbard rings shown left. In each case the system has four particles. The lower case is the M"obius ring.

these differences are important to be aware of. The list of references certainly does not exhaust all of the work done in the field, in particular since we have chosen to mainly focus on small systems.
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It is well-known that the Hubbard model for fermions at half-filling and in the limit of large $U$ approaches the antiferromagnetic Heisenberg model with effective coupling constant $4t^2/U$. In fact, the large $U$ limit can be described by the Heisenberg model for any filling which is smaller than half-filling, though with a different coupling constant.
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