Quantum theory of feedback of bosonic gases

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A quantum theory of feedback of bosonic many-atom systems is formulated. The feedback-induced many-atom correlations are treated by use of a parameterized correlation function, for which closed equations of motion are derived. Therefrom the dynamics of any additive property of the system, i.e., properties derived from the reduced single-atom density operator, can be obtained. An example is given that indicates the correlation effects of feedback.

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I. INTRODUCTION

Feedback is widely used in engineering to control certain properties of a system. Motivated by optical homodyne measurements, a theory of feedback has been formulated for cavity fields [1]. Many applications of feedback have been proposed for single quantum systems, such as, besides single-mode cavity fields, also for single atoms or generic harmonic oscillators. Moreover, extensions to time-delayed non-Markovian feedback have been derived [2] and coherent feedback [3] and feedback using weak measurements [4] have been studied. Furthermore, alternative approaches to feedback have been presented in the context of quantum-state estimation and quantum control [5].

Besides laser cooling [6], many advances have been accomplished in the last years in cooling atoms to the ultimate limits. Among them are Bose–Einstein condensation of dilute gases of alkali species [7,8,9,10] and atomic hydrogen [11], the experimental realization of atom lasers [12,13,14] and the implementation of micro- and nanostructured atom-optical devices [15,16,17]. This experimental progress draws attention to the investigation of many-atom systems and their condensate features from a quantum-optical viewpoint. In view of the experimental capabilities up to date, it is quite promising to engage in developing methods for controlling complex many-atom systems, such as produced nowadays in the laboratory.

The use of feedback methods acting on accessible macroscopic observables of the system appears particularly desirable. An experimental implementation of it would not require the active intervention of the experimenter, but would act as a self-automated basic technological tool, such as it is now in engineering. Unfortunately, for the case of feedback acting on a multi-mode many-atom system, the existing descriptions of feedback are in general not applicable: When trying to access arbitrary properties of the system, these methods become intractable, due to the required vast number of degrees of freedom.

Typically employed mean-field approximations, could be used to factorize or truncate higher-order many-atom correlations, in order to simplify the problem. In the considered case, however, the feedback operation contains the measurement of a macrostate observable of the many-atom system, to which all atoms contribute. The collective access to the atoms in such a measurement induces strong correlations between them. These correlations represent a main ingredient of the feedback and should be taken care of in a non-approximate way. The formation of correlations becomes clear when considering that an observed measurement outcome of a macrostate observable can be produced by a vast number of different microstates, that all enter the feedback operation in a coherent superposition. Such superposition states may contain strong correlations between the atoms and one may therefore expect that in general many-atom quantum correlations play a dominant role in the dynamics of the feedback system.

Correlations of this type are usually generated by interactions between atoms, such as for example two-body collisions. In the case considered here, such explicit interactions will not be taken into account. The many-atom correlations will solely be generated by the measurement process. An interaction between the atoms can in fact be identified in the measurement. The measurement process can be thought of as a probe field interacting with all the atoms with a subsequent projection of the probe by the final readout of the measurement result. Thus we may imagine an effective interaction between the atoms being mediated via the probe field.

In the following an exact method is presented for describing the reduced single-atom dynamics under the influence of feedback operations acting on macroscopic observables of a bosonic multi-mode many-atom system. The feedback loop consists of a measurement with predefined measurement uncertainty $\sigma$ and a subsequent unitary operation that again acts on all atoms of the system. The feedback loop is assumed to act instantaneously without time delay between measurement and operation. As will be shown, the many-atom correlations, that are generated by the measurement, can be formally incorporated into the single-atom dynamics by introducing a dynamically evolving additional degree of freedom. Thus the free evolution intermitted by feedback operations can be formulated in terms of a quasi single-atom dynamics, revealing modifications to the feedback dynamics of a true single-atom system.

The paper is structured as follows: In Sec. II the macroscopic observables needed for the feedback are introduced. Furthermore, the effect of the feedback on the many-atom
quantum state is described and a first indication of the importance of atom-atom correlations is given. Section II proceeds by introducing a specially designed correlation function that is shown to allow for a closed description of the feedback. Besides the dynamical evolution during the free evolution between feedback operations, also an interpretation of this correlation function is presented. An application of the developed method to the case of a condensed bosonic gas is given in Sec. IV and finally a summary and conclusions are found in Sec. V.

II. FEEDBACK LOOP

The feedback loop shall be considered as a measurement of a macroscopic observable \( \hat{A} \) of the system and the use of the so obtained information, i.e., the measured value \( \hat{A} \), for applying a unitary operation, \( \hat{U} (A) \), on the system. Information is therefore gained by measurements and used to manipulate the system, cf. Fig. 1. For the macroscopic observable we may think of any extensive quantity of the system, i.e., a quantity that has additive contributions from each atom in the system. Thus measurements are not performed on single atoms but on large numbers of atoms.

First let us consider the single-atom observable \( \hat{A} \) and its canonical conjugate \( \hat{B} \) with commutator \( (\hbar = 1) \)

\[
[\hat{A}, \hat{B}] = i.
\]  

(1)

Note that from now on for denoting single-atom Schrödinger operators calligraphic symbols are used. We now turn to a second-quantized description of the many-atom system by introducing the bosonic atom-field operator \( \hat{\phi} (x) \) with commutator

\[
[\hat{\phi} (x), \hat{\phi}^{\dagger} (x')] = \delta (x - x'),
\]  

(2)

Using an expansion into a set of complete and orthonormal modes \( u_\lambda (x) \),

\[
\hat{\phi} (x) = \sum_\lambda \hat{\phi}_\lambda u_\lambda (x),
\]  

(3)

the resulting commutator relation for the annihilation and creation operators of atoms in mode \( \lambda \), \( \hat{\phi}_\lambda \) and \( \hat{\phi}^{\dagger}_\lambda \), respectively, reads

\[
[\hat{\phi}_\lambda, \hat{\phi}^{\dagger}_\mu] = \delta_{\lambda\mu}.
\]  

(4)

The macroscopic observable \( \hat{A} \) that is used for the measurement in the feedback loop is a sum of contributions of the individual atoms. In the formalism of second quantization \( \hat{A} \) is therefore bilinear in the atom-field operators and reads

\[
\hat{A} = \sum_{\mu\lambda} \hat{\phi}^{\dagger}_\mu \hat{\phi}_\lambda \langle \mu | \hat{A} | \lambda \rangle.
\]  

(5)

Here \( \langle \mu | \hat{A} | \lambda \rangle \) are the matrix elements of the corresponding single-atom operator \( \hat{A} \) with the single-atom Schrödinger modes \( | \lambda \rangle \) being defined by: \( \langle x | \lambda \rangle = u_\lambda (x) \).

For implementing a feedback operation \( \hat{U} (A) \) that is capable of changing the measured property \( A \), the canonically conjugate variable of \( \hat{A} \) is needed. Whereas for a single atom this is trivially given by the operator \( \hat{B} \), cf. Eq. (1), for a general many-atom system a conceptual problem arises. Clearly, the proper canonical conjugate is given by

\[
\hat{B} = \frac{1}{\hat{N}} \sum_{\mu\lambda} \hat{\phi}^{\dagger}_\mu \hat{\phi}_\lambda \langle \mu | \hat{B} | \lambda \rangle,
\]  

(6)

where again \( \langle \mu | \hat{B} | \lambda \rangle \) are the matrix elements of the corresponding single-atom operator \( \hat{B} \). Moreover, \( \hat{N} \) is the operator of the total number of atoms in the system,

\[
\hat{N} = \sum_\lambda \hat{\phi}_\lambda^{\dagger} \hat{\phi}_\lambda.
\]  

(7)

With this definition the sought commutator relation is established:

\[
[\hat{A}, \hat{B}] = i,
\]  

(8)

as in the case of a single atom, cf. Eq. (1).

However, due to the appearance of the atom-number operator \( \hat{N} \) in Eq. (6), the operator \( \hat{B} \) is not bidiagonal in the atom-field operators. It is an intensive quantity and not an extensive one and thus is not a simple sum of contributions of the individual atoms. Thus it cannot be experimentally realized by application of an external field (e.g. laser field) that interacts with each individual atom. In other words, the problem is, that the total number of atoms is in principle unknown without its particular measurement. For the feedback action we need to know how many atoms have contributed to the previously measured quantity \( A \). The lack of this knowledge is a potential source of imperfection of the overall feedback action.

This problem can be overcome in two ways: Either in each feedback loop not only \( \hat{A} \) but also the atom number \( \hat{N} \) is measured, or the atom number is estimated. We will here concentrate on the latter approach, where the estimated number of atoms \( \hat{N} \) may be derived from a priori knowledge about the system, without additional measurements.
It is worth noting, that neglecting this issue by replacing $\hat{N}$ by its average $(\hat{N})$ in Eq. (3), but still assuming the commutator relation (5), results in the neglect of atom-number fluctuations. Such an approach is only valid when the atom number is exact. Considering the case where only part of the system is subject to the feedback, say only atoms being located in a predefined spatial region, an exact atom number in this region is rather doubtful even when the total number is precisely known.

Such an approximation can be avoided by consistently defining the operator $B$ by use of the estimated atom number $N_e$,

$$
\hat{B} = \frac{1}{N_e} \sum_{\mu \lambda} \hat{\phi}_\mu^\dagger \hat{\phi}_\lambda \langle \mu | \hat{B} | \lambda \rangle,
$$

and using the resulting commutator relations

$$
[\hat{A}, \hat{B}] = i\hat{N}/N_e, \quad [\hat{A}, \hat{N}] = [\hat{B}, \hat{N}] = 0.
$$

The measurement outcome is then a pure addition of all single-atom contributions. And the subsequent unitary operation will feed back the appropriately estimated part to each atom.

Let us now consider the action of the feedback loop on the quantum state of the many-atom system. Let $t^+$ be times infinitesimally before and after the time $t$ when the feedback is applied. The action of the feedback on the many-atom density operator $\hat{\rho}$ then reads

$$
\hat{\rho}(t^+) = \int d\hat{A} \hat{U}(A) \hat{M}(A - \hat{A}) \hat{\rho}(t^-) \hat{M}^\dagger(A - \hat{A}) \hat{U}^\dagger(A).
$$

It expresses the relation between the many-atom density operators $\hat{\rho}(t^-)$ and $\hat{\rho}(t^+)$ before and after the feedback, respectively. The function $\hat{M}(A)$ is the resolution-amplitude [21] that describes in its operator form the measurement process with a measurement uncertainty $\sigma$ determined by the width of $|\hat{M}(A)|^2$. For simplicity we may take it to be a Gaussian. The integration signifies the classical averaging over all possible measurement outcomes. Note that Eq. (11) can hardly be solved due to the large number of degrees of freedom of the many-atom density operator.

An important class of feedback mechanisms is defined by the following action of $\hat{U}(A)$ on the observable $\hat{A}$,

$$
\hat{U}^\dagger(A) \hat{A} \hat{U}(A) = \hat{A} + f(A) \hat{N}/N.
$$

It is realized by choosing for the unitary feedback operation

$$
\hat{U}(A) = \exp \left[ -i f(A) \hat{B} \right].
$$

Here the function $f(A)$ determines the response of the feedback loop to a measured value $A$. It encodes the desired action of the feedback and is a numerical function chosen by the experimenter.

An example may illustrate the feedback operation given in Eq. (13): For a system containing exactly $N$ atoms ($N_e = N$) and choosing for example $f(A) = s (A + A_0)$, the transformation (12) corresponds to positive ($s > 0$) or negative ($s < 0$) feedback with an offset $A_0$. In the case of negative feedback and vanishing offset ($s = -1$, $A_0 = 0$) we may derive from Eq. (11) relations of the single-atom quantities $A$ and $B$ before and after the feedback. That is, the system is measured and fed back as a whole, and we regard the effects on the average properties of an individual atom in the system.

The relations for the mean values before and after the feedback read

$$
\langle \hat{A} \rangle_{t^+} = 0, \quad \langle \hat{B} \rangle_{t^+} = \langle \hat{B} \rangle_{t^-}.
$$

Thus on average the quantity $\hat{A}$ is perfectly compensated to zero without affecting $\hat{B}$, as expected for this type of feedback. Conceptual differences from the behavior of a true single-atom system are revealed when regarding the variances of these properties. The variance of $\hat{B}$ reflects the expected increase of noise due to the back-action of the measurement:

$$
\langle (\Delta \hat{B})^2 \rangle_{t^+} = \langle (\Delta \hat{B})^2 \rangle_{t^-} + \left( \frac{1}{2 \sigma^2} \right)^2.
$$

However, the variance of $\hat{A}$, i.e., of the single-atom version of the measured observable, does not correspond to that expected for the feedback acting on a single-atom system:

$$
\langle (\Delta \hat{A})^2 \rangle_{t^+} = \langle (\Delta \hat{A})^2 \rangle_{t^-} - \frac{1}{N^2} \langle (\Delta \hat{A})^2 \rangle_{t^-} + \left( \frac{\sigma}{N} \right)^2.
$$

For $N = 1$ atom in the system we have $\hat{A} \rightarrow \hat{A}$ and therefore the first two terms on the r.h.s. cancel each other, leaving as variance only the measurement uncertainty $\sigma^2$. This is the result that we expect for the single-atom case. In the general $N$ atom case, however, the first two terms do not cancel each other. Equation (16) can be interpreted as follows: The uncertainty of the single-atom observable $\hat{A}$ is reduced by the gain of knowledge on the macroscopic observable $\hat{A}$. However, the uncertainty of the single-atom quantity $\hat{A}$ is in general not identical to the uncertainty of the macroscopic observable $\hat{A}$ divided by the number of atoms. And therefore the first two terms do not compensate and in this way increase the noise above the measurement uncertainty term.

The second term in Eq. (16) contains atom-atom correlations of the type $\langle \hat{\phi}_\mu^\dagger \hat{\phi}_\lambda \hat{\phi}_\mu \hat{\phi}_\lambda \rangle$, as can be seen from the definition (5). Therefore, the amount of compensation of noise in the single-atom properties due to the feedback depends on many-atom quantum correlations. The determination of these correlations requires a hierarchy of equations for higher-order atom correlations. To incorporate all these correlations in a consistent way is therefore required even for obtaining single-atom properties.

### III. Solution in Terms of Correlation Functions

Our goal is to describe expectation values of arbitrary extensive properties of the system, such as for example the total
energy. Any such quantity can be expressed in terms of the so-called single-atom density matrix $\rho_{\mu\lambda}$, that is defined by

$$\rho_{\mu\lambda} = \langle \hat{\phi}_{\lambda}^\dagger \hat{\phi}_\mu \rangle. \quad (17)$$

This density matrix describes the average properties of single atoms in the system and is normalized to the total average number of atoms

$$\sum_{\lambda} \rho_{\lambda\lambda} = \langle \hat{N} \rangle. \quad (18)$$

However, for the purpose of including the necessary higher-order atom correlations and for obtaining a closed set of equations we study a slightly different quantity. Namely, we consider the change due to a feedback operation of the correlation function,

$$D_{\mu\lambda}(z) = \langle \hat{\phi}_{\lambda}^\dagger \hat{T}(z) \hat{\phi}_\mu \rangle. \quad (19)$$

The operator $\hat{T}(z)$ is yet unknown, but we choose it to satisfy the requirement

$$\lim_{z \to 0} \hat{T}(z) = 1. \quad (20)$$

In this way the single-atom density matrix results as the value for $z = 0$,

$$D_{\mu\lambda}(0) = \langle \hat{\phi}_{\lambda}^\dagger \hat{\phi}_\mu \rangle. \quad (21)$$

As already mentioned, with the introduction of $\hat{T}(z)$ higher-order many-atom correlations are contained in $D_{\mu\lambda}(z)$, which is the clue to the solution.

From Eq. (11) the correlation $D_{\mu\lambda}(z, t^+)$ right after a feedback operation can be expressed as

$$D_{\mu\lambda}(z, t^+) = \int dA \left\langle \hat{M}_\lambda^\dagger (A - A) \hat{U}_\lambda^\dagger (A) \hat{T}(z) \hat{\phi}_\mu \right. \times \left. \hat{U}(A) \hat{M}(A - \hat{A}) \right\rangle_{t^+}. \quad (22)$$

Now the operator function $\hat{T}(z)$ is chosen in such a way, that the r.h.s. of this equation can again be expressed in terms of the correlation functions. More precisely, in terms of $D_{\mu\lambda}(z, t^-)$ at the time $t^-$ before the feedback operation. It can be shown that the specific Ansatz

$$\hat{T}(z) = \exp(i z \cdot \hat{Z}), \quad (23)$$

is appropriate for that purpose. Here the vector parameter is $z^T = (\alpha, \beta, \gamma)$ [22] and $\hat{Z}$ is defined as the vector operator

$$\hat{Z} = \left( \begin{array}{c} \hat{A} \\ \hat{B} \\ \hat{N}/N_c \end{array} \right). \quad (24)$$

Using this Ansatz, from Eq. (22) we obtain a closed set of equations that connects the correlations $D_{\mu\lambda}(z)$ immediately after the feedback operation with those before the feedback:

$$D_{\mu\lambda}(z, t^+) = \int d\alpha d\beta d\gamma F_{\mu\lambda}^{\alpha', \beta', \gamma'}(z, z') D_{\mu'\lambda'}(z', t^(-)). \quad (25)$$

The function $F_{\mu\lambda}^{\alpha', \beta', \gamma'}(z, z')$ represents the action of the feedback mechanism and will be denoted in the following as feedback kernel. It is defined as

$$F_{\mu\lambda}^{\alpha', \beta', \gamma'}(z, z') = \delta(\beta - \beta') \frac{1}{4\pi z} \int dA \int dA' \int dA'' \int dk \delta[\gamma - \gamma' + \alpha f(A) + \beta k] \times \exp[i(\alpha - \alpha')(A - A') + ikA''] \langle \mu | \hat{U}(A) \hat{M}(A' + A'' - \hat{A}) | \mu' \rangle \langle \lambda' | \hat{M}_\lambda^\dagger (A' - A'' - \hat{A}) \hat{U}_\lambda^\dagger (A) | \lambda \rangle. \quad (26)$$

This kernel can easily be calculated since it only contains matrix elements of the single-atom measurement operator, that is obtained from the resolution amplitude $M(A)$ by substituting for the argument the single-atom operator $A - \hat{A}$, i.e.,

$$\hat{M}(A - \hat{A}) = M(A - B) \big|_{B = \hat{A}}. \quad (27)$$

Moreover, it contains the single-atom unitary operation that reads

$$\hat{U}(A) = \exp[-if(A) \hat{B}/N_c]. \quad (28)$$

The kernel [25] can be interpreted in terms of single-atom properties as follows: First the single atom’s observable $\hat{A}$ is measured with outcome $A' \pm A''$ and then this observable is shifted by $f(A)/N_c$, as would be expected when equally distributing the collective shift $f(A)$ among the estimated num-
ber of atoms. The connection between \( A \) and \( A' + A'' \), however, is determined by the mapping of the additionally introduced degrees of freedom, \( z \) and \( z' \). This latter mechanism is the way by which all orders of many-atom correlations are incorporated in the mapping and signifies the deviation from the true single-atom case.

Considering the case \( z = 0 \) (\( \alpha = \beta = \gamma = 0 \)), from Eqs (25) and (26) it becomes immediately clear that for the single-atom density matrix \( \langle \hat{\phi}_A^\dagger \hat{\phi}_\mu \rangle \) no closed mapping is obtained:

\[
\langle \hat{\phi}_A^\dagger \hat{\phi}_\mu \rangle_{t+} = \frac{1}{2\pi} \sum_{\mu' \lambda'} \int d\alpha' \int dA \int dA' e^{-i\alpha'(A-A')} \langle \mu | \hat{U}(A) \hat{M}(A' - \hat{A}) | \mu' \rangle \\
\times \langle \lambda' | \hat{M}^\dagger(A' - \hat{A}) \hat{U}(A) | \lambda \rangle D_{\mu' \lambda'}(\alpha', t^-),
\]

(29)

where

\[
D_{\mu' \lambda'}(\alpha', t^-) = D_{\mu' \lambda'}(z', t^-)|_{z'=-(\alpha',0,0)}.
\]

On the r.h.s. of Eq. (29) higher-order many-atom correlations always remain in \( D_{\mu' \lambda'}(\alpha', t^-) \) due to the non-vanishing integration variable \( \alpha' \). That is, even when considering the single-atom dynamics, the mapping of the full correlations \( D_{\mu\lambda}(z) \) has to be employed. Afterwards \( z \) can be set to zero to obtain the sought single-atom density matrix.

It is worth mentioning, that an approximate closed mapping for the single-atom density matrix in fact exists in the limit \( \sigma \to \infty \). Then the measurement resolution amplitude \( M \) only slowly depends on its argument and the \( A' \) integration in (29) leads to non-vanishing contributions only for \( \alpha' \approx 0 \). That shows that for very bad measurement resolutions, almost no correlations are generated between the atoms.

In the special case when exactly one atom is present in the system, the correlation function will naturally reduce to the single-atom density operator \( D_{\mu\lambda}(z) \to \langle \hat{\phi}_A^\dagger \hat{\phi}_\mu \rangle \). Therefore, it does no longer depend on \( z \). In this true single-atom case, the \( \alpha' \) integration in the mapping (29) acts only on the kernel and results in the usual relation of the form given in Eq. (11). Now, however, for the single-atom density operator of a true single-atom system. Thus, the need of the parameter \( z \) signifies the presence of many-atom correlations in the feedback mapping.

Between two feedback operations the many-atom system will evolve according to the Hamiltonian \( H \). Assuming non-interacting atoms it is of the form

\[
\hat{H} = \sum_{\mu \lambda} \langle \mu | \hat{H} | \lambda \rangle \hat{\phi}_\mu^\dagger \hat{\phi}_\lambda.
\]

(31)

Without loss of generality the mode functions \( u_\lambda(x) = \langle x | \lambda \rangle \) are chosen as energy eigenstates of the single-atom Hamiltonian \( \hat{H} \) with energies \( E_\lambda \). Given the unitary time-evolution operator

\[
\hat{U}(t) = \exp(-i\hat{H}t),
\]

(32)

we may employ the time-dependent version of the operator \( \hat{T}(z) \),

\[
\hat{T}(z, t) = \hat{U}^\dagger(t) \hat{T}(z) \hat{U}(t) = \exp \left[ i \zeta^T \cdot \hat{Z}(t) \right],
\]

(33)

where \( \hat{Z}(t) \) reads

\[
\hat{Z}(t) = \hat{U}^\dagger(t) \hat{Z} \hat{U}(t).
\]

(34)

Using these definitions we arrive at the expression for the correlation \( D_{\mu\lambda}(z) \) propagated according to the Hamiltonian (31) from time \( t' \) to \( t \),

\[
D_{\mu\lambda}(z, t) = \langle \hat{\phi}_A^\dagger \hat{T}(z, t-t') \hat{\phi}_\mu \rangle e^{-i\epsilon(E_\mu - E_\lambda)(t-t')}.
\]

(35)

Now an assumption has to be made with respect to the free time evolution of the operators contained in \( \hat{Z} \). We assume that \( \hat{Z}(t) \) can be given as

\[
\hat{Z}(t) = \hat{V}_z(t-t') \cdot \hat{Z}(t'),
\]

(36)

where the \( 3 \times 3 \) real-valued matrix has the property \( \hat{V}_z(-t) = \hat{V}_z^{-1}(t) \). Note that the free time evolution of many physical systems can be cast into this form. Representative examples are free or harmonically trapped atoms when \( \hat{A} \) and \( \hat{B} \) are center-of-mass position and total momentum or vice versa. In such cases the correlations evolve as

\[
D_{\mu\lambda}(z, t) = \int \hat{d}z' W_{\mu\lambda}(z, z', t-t') \cdot D_{\mu\lambda}(z', t'),
\]

(37)

where in energy representation the kernel is given as

\[
W_{\mu\lambda}(z, z', t) = \delta^{(3)} [z' - \hat{V}_z^T(t) \cdot z] e^{-i\epsilon(E_\mu - E_\lambda)t},
\]

(38)

which again can be evaluated rather easily.

Now it becomes also clear, why the additional parameter \( \beta \) has been introduced. Clearly it is not necessary for describing the effect of a feedback operation on the single-atom density matrix, as can be seen from Eq. (29). For the free evolution, however, this parameter is needed to provide a complete set of dynamically evolving variables. Consider, for example, the case of harmonically trapped atoms where the feedback is performed on the total momentum. Then \( \alpha \) is associated with the total momentum, whereas \( \beta \) is associated with the center-of-mass coordinate. The two parameters are then coupled during the free evolution, which shows the need for the additional parameter \( \beta \). Note, that the parameter \( \gamma \) naturally emerged due to the commutator relation of \( \hat{A} \) and \( \hat{B} \).
It has been shown that by introduction of an additional (real-valued) vector parameter \( z \) a specifically chosen correlation function can be evolved in time by closed sets of equations of motion. Both for the feedback of a macroscopic observable and for the free evolution between the feedback operations [under the constraint \( \text{(55)} \)] the appropriate kernel functions can be given. The vector \( z \) could be formally interpreted as being an additional degree of freedom of the single atom, so that we may view the complete feedback dynamics as a quasi single-atom problem. This is justified also by the fact that all required kernel functions are obtained from matrix elements of single-atom operators.

The correlation function \( D_{\mu\lambda}(z) \) reveals another interesting aspect. Clearly, for \( z = 0 \) it converges to the single-atom density matrix \( \rho_{\mu\lambda} \) and describes the microscopic properties of the average single atom. It has however also a connection to macroscopic properties, as will be shown now. If we would neglect in the definition \( \text{(19)} \) the field operators \( \hat{\phi}_\lambda^\dagger \) and \( \hat{\phi}_\mu \), we could approximately write

\[
D_{\mu\lambda}(z) \sim D(z) \langle \hat{T}(z) \rangle. \quad (39)
\]

Given the definition of \( \hat{T}(z) \) in Eq. \( \text{(23)} \) it becomes clear that \( D(z) \) is the characteristic function (i.e. Fourier transform) of the probability of measuring simultaneously the observables \( \hat{A}, \hat{B}, \) and \( \hat{N}/N_e \). For a system with precise atom number, the latter observable can be omitted and we essentially arrive at the Fourier transform of the Wigner function \( W(A, B) \) of the pair of canonically conjugate macroscopic variables \( \hat{A} \) and \( \hat{B} \).

Turning back to the true definition \( \text{(19)} \) it can now be observed that the diagonal correlation \( D_{\lambda\lambda}(z) \) can also be interpreted as the Fourier transform of a Wigner-type function of the macroscopic phase space. However, it is that of a many-atom state, where one atom in mode \( \lambda \) has been removed. The off-diagonal elements \( D_{\mu\lambda}(z) \) \( (\lambda \neq \mu) \) represent the corresponding terms including single-atom coherences. Thus, the chosen correlation function covers both microscopic and macroscopic properties. Moreover, it accomplishes this in a specific combination, that allows for the solution of the feedback dynamics.

**IV. APPLICATION TO AN IDEAL BOSE GAS**

For illustrating the effects of atom correlations that can be treated with the presented method, let us consider feedback of a gas of \( N \) trapped bosonic atoms. We assume all the atoms to be initially in the ground state of the harmonic trap potential, e.g., a simple model for a non-interacting Bose–Einstein condensate. The feedback shall consist in measuring the total momentum \( P \) of all atoms and in subsequently shifting this observable to zero by use of the observed value and the center-of-mass operator \( \hat{Q} \) \( [\hat{A} \rightarrow \hat{P}, \; \hat{B} \rightarrow \hat{Q}, \; f(P) = -P, \) and \( N_e = N] \).

The initial correlation function of the degenerate bosonic gas reads

\[
D_{\mu\lambda}(z) = N \langle \mu|0\rangle\langle0|\lambda \rangle \exp[i\gamma(1 - 1/N)] \times \exp \left\{ -\frac{N-1}{2} \left( \Delta p_0 \right)^2 \sigma^2 + \frac{(\Delta q_0)^2}{N^2} \beta^2 \right\}. \quad (40)
\]

Here \( \Delta q_0 \) and \( \Delta p_0 \) are the position and momentum uncertainties, respectively, of the single-atom ground state \( |0\rangle \) in the trapping potential. This example shows quite well the above mentioned combination of microscopic and macroscopic descriptions. In this special case the correlation \( \text{(40)} \) is a product of the single atom density matrix \( \rho_{\mu\lambda} = N \langle \mu|0\rangle\langle0|\lambda \rangle \), a phase factor determined by the precise number of atoms, and a Gaussian distribution. The latter is the Fourier transform of the joint probability distribution for simultaneously observing the macroscopic values \( Q \) and \( P \) of \( N-1 \) atoms. That is, it is the Fourier transform of the Wigner function \( W(Q, P) \) of the center-of-mass degree of freedom of the macroscopic atomic gas with one atom being removed.

Though the correlation \( \text{(40)} \) could be propagated in time as described previously, we consider here only the result of a single feedback operation. In this way, the variances of single-atom quantities, such as position and momentum, can be studied in dependence on the atom number \( N \) and the measurement uncertainty \( \sigma \). Figure 2 shows the variance of the single-atom momentum in dependence on the atom number for three different measurement uncertainties \( \sigma \). One observes that for larger numbers of atoms, the momentum variance converges to the initial ground-state variance \( \Delta p_0 \), which establishes the regime of weak measurements where the single atom is only weakly affected by the measurement of the feedback loop. For only a few atoms the dependence on the chosen measurement resolution indicates an increasing amount of atom-atom correlations with decreasing measurement uncertainty. For larger measurement uncertainties a \( \sigma/N \) behavior will appear in the single-atom momentum variance, as would be expected classically when the measurement uncertainty is equally distributed among all atoms [cf. Eq. \( \text{(18)} \)].
FIG. 3: Uncertainty product of single-atom position and momentum, scaled in units of the ground-state uncertainty $\Delta q_0 \Delta p_0$, versus atom number $N$ (same parameters as in Fig. 2). The minimum uncertainty product corresponds to the value 1.

In Fig. 3 the uncertainty product of single-atom momentum and position, including now also the measurement-induced back-action noise in the position, is plotted versus the atom number for different measurement uncertainties. For small atom numbers atom-atom correlations cannot be built up, as in the case $\sigma / \Delta p_0 = 1$ (filled boxes), leading to minimum values at certain atom numbers. For larger atom numbers the measurement-induced noise dominates and the scaled value

$1 + (\Delta p_0 / \sigma)^2$ is reached. The presented features are those of a condensate state, for other highly correlated quantum states one may expect even more dramatic deviations from the classically expected behavior.

V. SUMMARY AND CONCLUSIONS

In conclusion a method has been presented for treating the dynamics of feedback of macroscopic observables of a bosonic multi-mode many-atom system. The introduced unique correlation function obeys closed equations of motion for the feedback operation and the free-evolution dynamics. Thus the dynamics intermitted by feedback operations at predefined times can be exactly described and calculated on a quasi single-atom level. The description contains the full many-atom correlations, that are generated by the feedback loop. In this way the dynamics of expectation values of any extensive property of the many-atom system, as for example the energy of a dilute atomic gas, can be obtained. The advantage of the method is its exactness combined with the treatment on a quasi single-atom level, vastly reducing the number of degrees of freedom, that would have to be employed for a solution of the full many-atom problem. Applications of the method can be found in the context of control and stochastic cooling of dilute bosonic gases [20] or feedback of a few atoms in an optical cavity [21].

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