Recent Modification of Homotopy Perturbation Method for Solving System of Third Order PDEs

Luma Naji Mohammed Tawfiq 1, Huda Altaie 2

1,2 Department of Mathematics, College of Education for Pure Science, Ibn Al-Haitham, University of Baghdad

1 luma.n.m@ihcoedu.uobaghdad.edu.iq , 2 huda.al_taie@unice.fr

Abstract. This paper presents new modification of HPM to solve system of 3rd order PDEs with initial condition, for finding suitable accurate solutions in a wider domain. The modification based on new coupled that is LA-transform combined with the HPM for solving system of PDEs. The reliability of suggested method illustrated by solving some examples.

Keywords: System of PDEs, HPM, LA-transform, coupled method.

1. Introduction
The research is focus on systems of PDEs. Its useful model for describing natural phenomena of science especially physics and engineering fields, such as the geochemistry, heat flow, plasma physics, fluid mechanics, solid mechanics and the wave propagation phenomena are well described by PDEs [1–4].

For importance of the problem we suggest new efficient technique based on coupled two efficient methods such HPM and LA-transform defined by Luma and Alaa in [5].

The systems of nonlinear PDEs have been also noticed to arise in many applications [6]. Most of engineering problems are nonlinear, and it is difficult to solve them analytically. The importance of the exact solution of system of PDEs is still a significant problem that needs efficient methods to getting exact or approximate solutions. Various powerful high accuracy methods have been proposed for obtaining exact, approximate and analytic solutions. Some of the classic analytic methods are perturbation techniques [8].

In recent years, many research study the solutions of non-linear system of PDEs by using various methods such that homotopy analysis method (HAM) [9], ADM [10], the VIM [11-12], the Fourier transform method [13], the HPM [14-16], collocation method depending on Cubic trigonometric Bspline (CuTBS) [17], and Laplace decomposition method [18,19].

Here, we will use the new coupled method based on LA-transform with HPM which we will say the LATHPM to solve 3rd order nonlinear system of PDEs with initial condition.
2. LA-Transform

LA-transform was introduced by Luma and Alaa [5] to solve many differential equations and engineering problems with some advantages of LA-transform (LAT) over other integral transforms such as accuracy and simplicity.

The LA-transform of a function \( f(t) \), defined by

\[
\mathcal{T}[f(t)] = \int_{0}^{\infty} e^{-st} f(t) dt.
\]

Here some basic properties of the LAT are introduced:

1. Linearity Property: \( \mathcal{T}[af(t) + bg(t)] = a\mathcal{T}[f(t)] + b\mathcal{T}[g(t)] \)
2. Convolution Property: \( \mathcal{T}[f(t)g(t)] = \int_{0}^{\infty} f(\tau)g(t-\tau) d\tau \)
3. Differentiation Property: \( \mathcal{T}[f'(t)] = v(t)f(0) \)

For more details see [5]

3. Coupled Method for Solving System of Nonlinear PDE

This section consist the procedure of the coupled method based on combine LAT algorithm with the HPM and denoted by LATHPM to solve a system of nonlinear equation:

\[
\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} = 0
\]

To explain the main idea of LATHPM firstly writes the nonlinear PDE as follow:

\[
L[u(x,t)] + R[u(x,t)] + N[u(x,t)] = 0
\]

with initial condition (IC)

\[
u(x,0) = f(x)
\]

where \( x \in \mathbb{R}, L \) is a linear differential operator \( L = \frac{\partial}{\partial t} \), \( R \) is a remained of the linear operator, \( N \) is a nonlinear differential operator.

First step is construct a Homotopy as: \( u(x, t; p) : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R} \), using the HP technique which satisfies

\[
H(u(x,t), p) = (1 - p)\left[L(u(x,t)) - L(u(x,0))\right] + p[N(u(x,t))] = 0
\]

where \( p \in [0, 1] \) is an embedding parameter and the operator A defined as:

\[
A = L + R + N
\]

Obviously, if \( p = 0 \), equation (6) becomes

\[
L(u(x,t)) = L(u(x,0)) \quad \text{and} \quad L(v(x,t)) = L(v(x,0))
\]

It is clear that, if \( p = 1 \) then the homotopy equations (6) convert to the main differential equations (4). In topology, this deformation is called homotopic. Substitute equations (5) in equations (6) and rewrite it as:

\[
L(u(x,t)) - L(f(x)) + p[L(f(x))] = 0
\]

Then

\[
L(u(x,t)) - L(f(x)) + p[L(g(x))] = 0
\]

Since \( f(x) \) and \( g(x) \) are independent of the variable \( t \) and the linear operator \( L \) dependent on \( t \), therefore

\[
L(u(x,t)) = 0 \quad \text{and} \quad L(g(x)) = 0
\]

Thus, equations (7) becomes:

\[
L(u(x,t)) = 0 \quad \text{and} \quad L(g(x)) = 0
\]
According to the classical perturbation technique, the solution of eq. (8) can be written as a power series of embedding parameter $p$, in the form:

$$ u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) $$

$$ v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) \quad (9) $$

It is clear that, the series form in eq. (9) is convergent and the convergent rate depends on the nonlinear parts $N(u(x,t))$ and $N(v(x,t))$.

Taking the LAT (with respect to the variable $t$) for the equations (8) to get:

$$ T[u(x)] + p T[R(u) + N(u)] = 0 $$

$$ T[v(x)] + p T[R(v) + N(v)] = 0 \quad (10) $$

Now by using the differentiation property of LAT and IC in equations (5), equations (10) becomes:

$$ sT[u] - uf(x) + p T[R(u) + N(u)] = 0 $$

$$ sT[v] - vg(x) + p T[R(v) + N(v)] = 0 \quad (11) $$

Hence:

$$ T[u] = g(x) + p \left[ \frac{1}{s} T[\{ -R(u) - N(u) \}] \right] $$

## $$ T[v] = g(x) + p \left[ \frac{1}{s} T[\{ -R(v) - N(v) \}] \right] \quad (12) $$

By taking the inverse of LA- transform on both sides of equations (12), to get:

$$ u(x,t) = f(x) + p T^{-1} \left[ \frac{1}{s} T[\{ -R(u(x,t)) - N(u(x,t)) \}] \right] $$

$$ v(x,t) = g(x) + p T^{-1} \left[ \frac{1}{s} T[\{ -R(v(x,t)) - N(v(x,t)) \}] \right] \quad (13) $$

Then, substitute equations (9) in equations (13), to obtain:

$$ \sum_{n=0}^{\infty} p^n u_n = f(x) + p T^{-1} \left[ \frac{1}{s} T\left\{ -R(\sum_{n=0}^{\infty} p^n u_n) - N(\sum_{n=0}^{\infty} p^n u_n) \right\} \right] $$

$$ \sum_{n=0}^{\infty} p^n v_n = g(x) + p T^{-1} \left[ \frac{1}{s} T\left\{ -R(\sum_{n=0}^{\infty} p^n v_n) - N(\sum_{n=0}^{\infty} p^n v_n) \right\} \right] \quad (14) $$

The nonlinear part can be decomposed, as will be explained later, by substituting equation (9), in it as:

$$ N(u) = N(\sum_{n=0}^{\infty} p^n u_n(x,t)) = \sum_{n=0}^{\infty} p^n A_n $$

Then equations (14) become:

$$ \sum_{n=0}^{\infty} p^n u_n = f(x) + p T^{-1} \left[ \frac{1}{s} T\left\{ -R(\sum_{n=0}^{\infty} p^n u_n) - \sum_{n=0}^{\infty} p^n A_n \right\} \right] $$

$$ \sum_{n=0}^{\infty} p^n v_n = g(x) + p T^{-1} \left[ \frac{1}{s} T\left\{ -R(\sum_{n=0}^{\infty} p^n v_n) - \sum_{n=0}^{\infty} p^n B_n \right\} \right] \quad (16) $$

By comparing the coefficient with the same power of $p$, in both sides of the equations (16) we get:
and so on. According to the series solution in equations (16), then at \( p=1 \) we can get

\[
\begin{align*}
  u(x, t) &= u_0(x, t) + u_1(x, t) + \cdots + \sum_{n=2}^{\infty} u_n(x, t) \\
  v(x, t) &= v_0(x, t) + v_1(x, t) + \cdots + \sum_{n=2}^{\infty} v_n(x, t)
\end{align*}
\]

(10)

4. Applications

In this section we investigate the procedure of the coupled method LATHPM in mathematical physics, especially in systems of 3rd order nonlinear equations of interest.

Example 1

Consider a system of 3rd order homogeneous nonlinear PDEs

\[
\begin{align*}
  \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= 0 \\
  \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= 0
\end{align*}
\]

with initial condition (IC)

\[
\begin{align*}
  u(x, 0) &= 3 - 6 \tanh^2 \left( \frac{x}{2} \right) \\
  v(x, 0) &= -3\sqrt{2} \tanh \left( \frac{x}{2} \right)
\end{align*}
\]

\[
\begin{align*}
  p^0: u_0(x, t) &= 3 - 6 \tanh^2 \left( \frac{x}{2} \right), & p^0: v_0(x, t) &= -3\sqrt{2} \tanh \left( \frac{x}{2} \right) \\
  p^1: u_1 &= T^{-1} \left[ \frac{1}{6} T \left[ \frac{\partial^3 u_0}{\partial x^3} + \frac{\partial^3 v_0}{\partial y^3} + \frac{\partial^3 v_0}{\partial z^3} \right] \right] \\
  v_1 &= T^{-1} \left[ \frac{1}{6} T \left[ 12 \tanh \left( \frac{x}{2} \right) \sech^2 \left( \frac{x}{2} \right) + 12 \tanh \left( \frac{x}{2} \right) \sech^2 \left( \frac{x}{2} \right) + 18 \sqrt{2} \tanh \left( \frac{x}{2} \right) \sech^2 \left( \frac{x}{2} \right) \right] \right]
\end{align*}
\]
and so on, therefore
\[ u_{app} = 3 - 6 \tanh^2 \left( \frac{x}{2} \right) - \left( 6 \tanh \left( \frac{x}{2} \right) \sec \left( \frac{x}{2} \right) \right) t + \ldots \]
\[ v_{app} = -3 \sqrt{3} \tanh \left( \frac{x}{2} \right) - \left( 3 \sqrt{3} \tanh \left( \frac{x}{2} \right) \sec \left( \frac{x}{2} \right) \right) t + \ldots \]

The above series closed to exact solution is
\[ u = 3 - 6 \tanh^2 \left( \frac{x + t}{2} \right), \quad v = -3 \sqrt{2} \tanh^2 \left( \frac{x + t}{2} \right) \]

**Example 2**
Consider a system of 3rd order homogeneous nonlinear PDEs
\[
\begin{align*}
\frac{\partial^3 u}{\partial t^3} &- \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial x} = 0 \\
\frac{\partial^3 v}{\partial t^3} &- \frac{\partial v}{\partial x} + 2 \frac{\partial v}{\partial x} = 0
\end{align*}
\]
with IC:
\[
\begin{align*}
u(x, 0) &= -\tanh \left( \frac{x}{\sqrt{3}} \right) \\
v'(x, 0) &= -\frac{1}{6} \tanh \left( \frac{x}{\sqrt{3}} \right)
\end{align*}
\]

and so on, therefore
\[ u_{app} = -\tanh \left( \frac{x}{\sqrt{3}} \right) + \left( \frac{1}{6} \sec \left( \frac{x}{\sqrt{3}} \right) \right) t + \ldots \]
\[ v_{app} = -\frac{1}{2} \tanh^2 \left( \frac{x}{\sqrt{3}} \right) + \left( \frac{1}{\sqrt{3}} \tanh \left( \frac{x}{\sqrt{3}} \right) \right) \tanh \left( \frac{x}{\sqrt{3}} \right) t + \ldots \]

The above series closed to exact solution is
\[ u = -\tanh \left( \frac{x + t}{\sqrt{3}} \right), \quad v = -\frac{1}{6} \tanh^2 \left( \frac{x + t}{\sqrt{3}} \right) \]

**Example 3**
Consider the following 3rd order PDEs
\[
\begin{align*}
\frac{\partial^3 u}{\partial t^3} &- \frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial x} = 0 \\
\frac{\partial^3 v}{\partial t^3} &- \frac{\partial v}{\partial x} + 3 \frac{\partial v}{\partial x} = 0
\end{align*}
\]
with IC: ...
The above series closed to exact solution is

\[
\sum_{n=0}^\infty \frac{w^n}{n!} = \frac{w}{3} \tanh(w) + \frac{w^2}{6} \tanh^2(w) + \frac{w^3}{18} \tanh^3(w) + \ldots
\]

5. Conclusions
This research considers the suggested coupled method provides an effective and reliable way for solving wide types of nonlinear PDEs. The advantage of suggested technique is capability of combining two powerful tools for getting exact solutions for system of 3rd order homogenous nonlinear PDEs, where the HPM was disability to get the exact solution for the same problems and solved its approximately.

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