Bayesian nonparametric estimation of the spectral density of a long memory Gaussian time series

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Abstract: Let \( X = \{X_t, t = 1, 2, \ldots \} \) be a stationary Gaussian random process, with mean \( \mu \) and covariance function \( \gamma(\tau) = E(X_t - \mu)(X_{t+\tau} - \mu) \). Let \( f(\lambda) \) be the corresponding spectral density; a stationary Gaussian process is said to be long-range dependent, if the spectral density \( f(\lambda) \) can be written as the product of a slowly varying function \( \tilde{f}(\lambda) \) and the quantity \( \lambda^{-2d} \). In this paper we propose a novel Bayesian nonparametric approach to the estimation of the spectral density of \( X \). We prove that, under some specific assumptions on the prior distribution, our approach assures posterior consistency both when \( f(\cdot) \) and \( d \) are the objects of interest.

The rate of convergence of the posterior sequence depends in a significant way on the structure of the prior; we provide some general results and also consider the fractionally exponential (FEXP) family of priors (see below). Since it has not a well founded justification in the long memory set-up, we avoid using the Whittle approximation to the likelihood function and prefer to use the true Gaussian likelihood.

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1. Introduction

Let $X = \{X_t, t = 1, 2, \ldots \}$ be a stationary Gaussian random process, with mean $EX_t = \mu$ and covariance function $\gamma(\tau) = E(X_t - \mu)(X_{t+\tau} - \mu)$. Let $f(\lambda)$ be the corresponding spectral density, which satisfies the relation

$$\gamma(\tau) = \int_{-\pi}^{\pi} f(\lambda)e^{it\lambda} d\lambda \quad (\tau = 0, \pm 1, \pm 2, \ldots).$$

A stationary Gaussian process is said to be long-range dependent, if there exist a positive number $C$ and a value $d$ $(0 < d < 1/2)$ such that

$$\lim_{\lambda \to 0} \frac{f(\lambda)}{C\lambda^{-2d}} = 1.$$ 

Alternatively, one can define a long memory process as one such that its spectral density $f(\lambda)$ can be written as the product of a slowly varying function $\tilde{f}(\lambda)$ and the quantity $\lambda^{-2d}$ which causes the presence of a pole of $f(\lambda)$ at the origin.

Interest in long-range dependent time series has increased enormously over the last fifteen years; (4) provides a comprehensive introduction and the book edited by (10) explores in depth both theoretical aspects and various applications of long-range dependence analysis in several different disciplines, from telecommunications engineering to economics and finance, from astrophysics and geophysics to medical time series and hydrology.

Pioneering work on long memory process is due to (21), Mandelbrot and Wallis (1969) and others. Fully parametric maximum likelihood estimates of $d$ were introduced in the Gaussian case by (11) and (8) and they have recently been developed in much greater generality by (16); a regression approach to the estimation of the spectral density of long memory time series is provided in (12); generalised linear regression estimates were suggested by (3). However,
parametric inference can be highly biased under mis-specification of the true model: this fact has suggested semiparametric approaches; see for instance (25). Due to factorization of the spectral density \( f(\lambda) = \lambda^{-2d} \tilde{f}(\lambda) \), a semiparametric approach to inference seems particularly appealing in this context. One needs to estimate \( d \) as a measure of long-range dependence while no particular modeling assumptions on the structure of the covariance function at short ranges are necessary: (19) consider a Bayesian approach for this problem, while (2) provides an exhaustive review on the classical approaches.

Practically all the existing procedures either exploit the regression structure of the log-spectral density in a reasonably small neighborhood of the origin (25) or use an approximate likelihood function based on the so called Whittle’s approximation (27), where the original data vector \( \mathbf{X}_n = (X_1, X_2, \ldots, X_n) \) gets transformed into the periodogram \( I(\lambda) \) computed at the Fourier frequencies \( \lambda_j = 2\pi j/n, \quad j = 1, 2, \ldots, n \), and the “new” observations \( I(\lambda_1), \ldots, I(\lambda_n) \) are, under a short range dependence, approximately independent, each \( I(\lambda_j)/f(\lambda_j) \) having an exponential distribution. This is for example the approach taken in (7), which develop a Bayesian nonparametric analysis for the spectral density of a short memory time series. Unfortunately, the Whittle’s approximation fails to hold in the presence of long range dependence, at least for the smallest Fourier frequencies.

In this paper we propose a Bayesian nonparametric approach to the estimation of the spectral density of the stationary Gaussian process: we avoid the use of the Whittle approximation and we deal with the true Gaussian likelihood function.

The literature on Bayesian nonparametric inference has increased tremendously in the last decades, both from a theoretical and a practical point of view. Much of this literature has dealt with the independent case, mostly when the observations are identically distributed. The theoretical perspective was mainly dedicated to either construction of processes used to define the prior distribution with finite distance properties of the posterior, in particular when such a prior is conjugate, see for instance (15) for a review on this, or to consistency and rates of convergence properties of the posterior, see for instance (13) or (26).

The dependent case has hardly been considered from a theoretical perspective apart from (7), who deal with Gaussian weakly dependent data and, in a more general setting, (14). In this paper we study the asymptotic properties...
of the posterior distributions for Gaussian long-memory processes, where the unknown parameters are the spectral density and the long-memory parameter $d$. General consistency results are given and a special type of prior, namely the FEXP prior as it is based on the FEXP model, is studied. From this, consistency of Bayesian estimators of both the spectral density and the long memory parameter are obtained. To understand better the link between the Bayesian and the frequentist approaches we also study the rates of convergence of the posterior distributions, first in a general setup and then in the special case of FEXP priors. The approach considered here is similar to what is often used in the independent and identically distributed case, see for instance (13). In particular we need to control prior probability on some neighborhood of the true spectral density and to control a sort of entropy of the prior (see Section 3); however the techniques are quite different due to the dependence structure of the process.

The gist of the paper is to provide a fully nonparametric Bayesian analysis of long range dependence models. In this context there already exist many elegant and maybe more general (in the sense of being valid even without the Gaussian assumption) classical solutions. However we believe that a Bayesian solution would be still important because of the following reasons.

i) By definition, our scheme allows to include in the analysis some prior information which may be available in some applications.

ii) While classical solutions are, in a way or another, based on some asymptotic arguments, our Bayesian approach relies only on the (finite sample size) observed likelihood function (and prior information).

iii) We are able to provide a valid approximation to the “true” posterior distribution of the main parameters of interest in the model, namely the long memory parameter $d$ or the global spectral density.

Also, on a more theoretical perspective, we believe that this paper can be useful to clarify the intertwines between Bayesian and frequentist approaches to the problem.

The paper is organized as follows: in the next section we first introduce the necessary notation and mathematical objects; then we provide a general theorem which states some sufficient condition to ensure consistency of the posterior distribution. We also discuss in detail a specific class of priors, the FEXP prior,
which takes its name after the fractional exponential model which has been introduced by (23) (see also (24) to model the spectral density of a covariance stationary long-range dependent process. The FEXP model can be seen as a generalization of the exponential model proposed by (5) and it allows for semiparametric modeling of long range dependence; see also (4) or (17). In Section 3 we study the rate of convergence of the posterior distribution first in the general case and then in the case of FEXP priors. The final section is devoted to discussion of related problems.

2. Consistency results

We observe a set of $n$ consecutive realizations $\mathbf{X}_n = (X_1, \ldots, X_n)$ from a Gaussian stationary process with spectral density $f_0$, where $f_0(\lambda) = |\lambda|^{-2d_0} \tilde{f}_0(\lambda)$. Because of the Gaussian assumption, the density of $\mathbf{X}_n$ can be written as

$$\varphi_{f_0}(\mathbf{X}_n) = e^{-\frac{1}{2} \mathbf{X}_n^T T_n^{-1}(f_0) \mathbf{X}_n} / \left(2\pi\right)^{n/2},$$

where $T_n(f_0) = [\gamma(j-k)]_{1 \leq j, k \leq n}$ is the covariance matrix with a Toeplitz structure. The aim is to estimate both $\tilde{f}_0$ and $d_0$ using Bayesian nonparametric methods.

Let $\mathcal{F} = \{f, f \text{ symmetric on } [-\pi, \pi], \int |f| < \infty \}$ and $\mathcal{F}_+ = \{f \in \mathcal{F}, f \geq 0 \}$; then $\mathcal{F}_+$ denotes the set of spectral densities. We first define three types of pseudo-distances on $\mathcal{F}_+$. The Kullback-Leibler divergence for finite $n$ is defined as

$$KL_n(f_0; f) = \int_{\mathbb{R}^n} \varphi_{f_0}(\mathbf{X}_n) \left[ \log \varphi_{f_0}(\mathbf{X}_n) - \log \varphi_f(\mathbf{X}_n) \right] d\mathbf{X}_n = \frac{1}{2n} \left\{ \operatorname{tr} \left( T_n(f_0) T_n^{-1}(f) - \mathbf{id} \right) - \log \det(T_n(f_0)) \right\}$$

where $\mathbf{id}$ represents the identity matrix of the appropriate order. Letting $n \to \infty$, we can define, when it exists, the quantity

$$KL_\infty(f_0; f) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ \frac{f_0(\lambda)}{f(\lambda)} - 1 - \log \frac{f_0(\lambda)}{f(\lambda)} \right] d\lambda.$$

We also define two symmetrized version of $KL_n$, namely

$$h_n(f_0, f) = KL_n(f_0; f) + KL_n(f; f_0); d_n(f_0, f) = \min\{KL_n(f_0; f), KL_n(f; f_0)\}.$$
and their corresponding limits as \( n \to \infty \):

\[
h(f_0, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \frac{f_0(\lambda)}{f(\lambda)} + \frac{f(\lambda)}{f_0(\lambda)} - 2 \right] d\lambda;
\]

\[
d(f_0, f) = \min\{KL_\infty(f_0; f), KL_\infty(f; f_0)\}.
\]

We also consider the \( L_2 \) distance between the logarithms of the spectral densities, namely

\[
\ell(f, f') = \int_{-\pi}^{\pi} (\log f(\lambda) - \log f'(\lambda))^2 d\lambda.
\] (2.2)

This distance has been considered in particular by (22). This is quite a natural distance in the sense that it always exists, whereas the \( L_2 \) distance between \( f \) and \( f' \) need not, at least in the types of models considered in this paper. Let \( \pi \) be a prior probability distribution on the set

\[
\tilde{F} = \{ f \in \mathcal{F}, f(\lambda) = |\lambda|^{-2d}\tilde{f}(\lambda), \tilde{f} \in C^0, -\frac{1}{2} < d < \frac{1}{2} \}, \quad \tilde{F}_+ = \{ f \in \tilde{F}, f \geq 0 \},
\]

where \( C^0 \) is the set of continuous functions on \([-\pi, \pi]\).

Let \( A_\varepsilon = \{ f \in \tilde{F}_+; d(f, f_0) \leq \varepsilon \} \). Our first goal is to prove the consistency of the posterior distribution of \( f_0 \), that is, we show that, as \( n \to \infty \),

\[
P^\pi[A_\varepsilon^c|X_n] \to 0, \quad f_0 \quad \text{a.s.},
\]

where \( P^\pi[|X_n] \) denotes the posterior distribution associated with the prior \( \pi \).

From this, we will be able to deduce the consistency of some Bayes estimators of the spectral density \( f \) and of the long memory parameter \( d \). We first state and prove the strong consistency of the posterior distribution under very general conditions both on the prior and on the true spectral density. Then, building on these results, we will obtain the consistency of a class of Bayes estimates of the spectral density, together with the consistency of the Bayes estimates of the long memory parameter \( d \). The already introduced FEXP class of prior will be then proposed, and its use will be explored in detail.

### 2.1. The main result

In this section we derive the main result about consistency of the posterior distribution. We also discuss the asymptotic behavior of the posterior point estimates
of some parameter of major interest, such as the long memory parameter $d$ and the global spectral density.

Consider the following two subsets of $\mathcal{F}$

$$
\mathcal{G}(d, M, m, L, \rho) = \left\{ f \in \tilde{\mathcal{F}}_{+}: f(\lambda) = |\lambda|^{-2d} \tilde{f}(\lambda), m \leq \tilde{f}(\lambda) \leq M, |\tilde{f}(x) - \tilde{f}(y)| \leq L|x - y|^\rho \right\},
$$

where $-1/2 < d < 1/2$, $m, M, \rho > 0$;

$$
\mathcal{F}(d, M, L, \rho) = \left\{ f \in \tilde{\mathcal{F}}: f(\lambda) = |\lambda|^{-2d} \tilde{f}(\lambda), |\tilde{f}(\lambda)| \leq M, |\tilde{f}(x) - \tilde{f}(y)| \leq L|x - y|^\rho \right\}.
$$

The boundedness constraint on $\tilde{f}$ in the definition of $\mathcal{G}(d, M, m, L, \rho)$ is here to guarantee the identifiability of $d$, while the Lipschitz-type condition on $\tilde{f}$, in both definitions, are actually needed to ensure that normalized traces of products of Toeplitz matrices, that typically appear in the distances considered previously, will converge. We also consider the following set of spectral densities, which is of interest in the study of rates of convergence: let

$$
\mathcal{L}^*(M, m, L) = \{ h(\cdot) \geq 0, 0 < m \leq h(\cdot) \leq M, |h(x) - h(y)| \leq L|x - y||(x|\wedge|y||)^{-1} \}
$$

and

$$
\mathcal{L}(d, M, m, L) = \{ f = |\lambda|^{-2d} \tilde{f}(\lambda), \tilde{f} \in \mathcal{L}^*(M, m, L) \}.
$$

Note that $\mathcal{G}$ and $\mathcal{L}$ are similar, with only a slight modification on the Lipschitz condition. The set $\mathcal{L}$ has been considered in particular in (22).

We now consider the main result on the consistency of the posterior distribution. Let

$$
\mathcal{G}(t, M, m, L, \rho) = \cup_{-1/2 + t \leq d \leq 1/2 - t} \mathcal{G}(d, M, m, L, \rho)
$$

and

$$
\mathcal{L}(t, M, m, L) = \cup_{-1/2 + t \leq d \leq 1/2 - t} \mathcal{L}(d, M, m, L).
$$

In the following theorem and in its proof we consider spectral densities in sets either in the form $\mathcal{G}$ or in the form $\mathcal{L}$. To simplify the presentation we give results for densities in $\mathcal{G}$ only, however the results remain valid for densities in $\mathcal{L}$, the only difference being that in the conditions in the form $4|d - d_0| \leq \gamma$ where $\gamma = \rho \wedge \rho_0 \wedge 1/2$ the quantities $\rho, \rho_0$ can be chosen equal to 1 if the corresponding spectral densities belong to $\mathcal{L}$.
Theorem 2.1. Assume that there exist \((t_0, M_0, m_0, L_0)\) such that we have

- either \(f_0 \in \mathcal{G}(t_0, M_0, m_0, L_0, \rho_0)\) with \(0 < \rho_0 \leq 1\)
- or \(f_0 \in \mathcal{L}(t_0, M_0, m_0, L_0) = \bigcup_{-1/2 < t_0 \leq d \leq 1/2 - t_0} \mathcal{L}(d, M_0, m_0, L_0)\).

Let \(t, m, M, L\) be positive reals with \(t < (\rho \land \rho_0)/4\). Let \(\pi\) be a prior distribution on either \(\mathcal{G}(t, M, m, L, \rho)\), \(\rho > 0\) or \(\mathcal{L}(t, m, M, L)\). If the prior satisfies:

i) For all \(\varepsilon > 0\), small enough, there exists \(\mathcal{F}_n \subset \mathcal{H}_n\), such that \(\pi(\mathcal{F}_n) \leq e^{-nr}\) and a smallest possible net \(\mathcal{H}_n \subset \mathcal{G}(t, M, m, L, \rho)\); \(d(f, d_0) > \varepsilon/2\) (resp. \(\mathcal{L}(t, M, m, L)\)) such that when \(n\) is large enough, \(\forall f \in \mathcal{F}_n \cap A^c\), \(\exists f_i \in \mathcal{H}_n\), \(f_i = |x|^{-2d}, f_i(x) \geq f\) such that \(4|d_i - d| \leq \rho \land 1/2\).

- If \(4|d - d_0| \leq \gamma - t\),
  \[
  \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(f_i - f)(x)}{f_0(x)} dx \leq h(f_0, f_i)/4
  \]

- If \(4|d - d_0| > \gamma - t\)
  \[
  \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(f_i - f)^2(x)}{f^2(x)} dx \leq b(f_0, f_i) |\log \varepsilon|^{-1}
  \]

- If \(4|d_0 - d| > \gamma - t\)
  \[
  \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(f_i - f)^2(x)}{f_0^2(x)} dx \leq b(f_i, f_0), \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(f_i - f)(x)}{f(x)} dx \leq b(f_i, f_0) |\log \varepsilon|^{-1}
  \]

Denote by \(N_n\) the logarithm of the cardinality of the smallest possible net \(\mathcal{H}_n\). Then, if

\[
N_n \leq nc_1, \quad \text{with} \quad c_1 \geq \varepsilon |\log \varepsilon|^{-2}
\]

then

\[
P^n [A_x | X_n] \to 1, \quad f_0 \text{ a.s.} \quad (2.5)
\]

Proof. See Appendix B. \(\Box\)

The above theorem is important to clarify which conditions on the prior distribution \(\pi\) are really crucial in a long memory setting, where the techniques usually adopted in the i.i.d. case, cannot be used and even the adoption of a
Whittle approximation is not legitimate in this setting (at least at the lowest frequencies). From a practical perspective, however, the hardest part of the program is actually to verify whether a specific type of priors actually meets the conditions listed in Theorem 2.1. We will discuss in detail these issues in the context of the FEXP prior in §2.3.

2.2. Consistency of estimates for some quantities of interest

We now discuss the problem of consistency for the Bayes estimates of the spectral density. The usual quadratic loss function for the class of functions $F$ is not the natural one for this problem, since there exist some spectral densities in $F$ that are not square integrable (i.e. if $d > 1/4$). A more reasonable loss function is the quadratic loss on the logarithm of $f$, as defined by (2.2), which is always integrable, at least in the framework considered in the paper. The Bayes estimator of $f$ associated with the loss $\ell$ and the prior $\pi$ is given by

$$\hat{f}(\lambda) = \exp\{E_{\pi}[\log f(\lambda)|X_n]\} = |\lambda|^{-2d}\exp\{E_{\pi}[\log \tilde{f}(\lambda)|X_n]\},$$

where $\hat{d} = E_{\pi}[d|X_n]$. Note also that the Bayes estimator of $f$ associated with the loss $h(\cdot,\cdot)$ and the prior $\pi$ is given by

$$\hat{f}_2^2(\lambda) = \sqrt{\frac{E_{\pi}[f(\lambda)|X_n]}{E_{\pi}[f^{-1}(\lambda)|X_n]}}.$$  \hspace{1cm} (2.6)

Also, in many applications of long memory processes, the real parameter of interest is just $d$, the long memory exponent. It is possible to deduce, from Theorem 2.1, that the posterior mean of $d$, that is the Bayes estimator associated with the quadratic loss on $d$, is actually consistent. Let

$${\mathcal{F}}_+(t,M,m) = \bigcup_{-1/2+t \leq d \leq 1/2-t}\{f \in {\mathcal{F}}_+, f(\lambda) = |\lambda|^{-2d}\tilde{f}(\lambda), 0 < m \leq \tilde{f} \leq M\}.$$

**Corollary 1.** Under the assumptions of Theorem 2.1, for all $\epsilon > 0$, as $n \to \infty$,

$$\pi \left[ \{|f|^{-2d}\tilde{f}; |d - d_0| > \epsilon\} |X_n\right] \to 0 \quad f_0 \text{ a.s}$$

and $\hat{d} \to d_0, \quad f_0 \text{ a.s.}$

**Proof.** The result comes from the fact that, when $|d - d_0| > \epsilon$, there exists a positive constant $\epsilon'$ such that for all $f, f_0 \in {\mathcal{F}}_+(t,M,m)$, $f = |\lambda|^{-2d}\tilde{f}$ and
\[ f_0 = |\lambda|^{-2d_0} \tilde{f}_0, \ h(f, f_0) > \epsilon' \]. In fact, assume without loss of generality, that \( d > d_0 \), then for all \( A > 4M/m \)

\[
0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f/f_0 + f_0/f - 2)(\lambda d\lambda \\
\geq \frac{m}{8\pi M} \int_{|\lambda|^{-2(d-d_0)} > A} |\lambda|^{-2(1-2d_0)}d\lambda.
\]

The above quantity is infinite if \( 2(d-d_0) \geq 1 \), otherwise

\[
h(f, f_0) \geq \frac{m}{4\pi M(1 - 2(d-d_0))} A^{1-1/(2(d-d_0))} = \epsilon'.
\]

This implies that

\[
\pi[A_{\epsilon'}|X] \geq \pi \left\{ f = |\lambda|^{-2d} \tilde{f}, |d-d_0| > \epsilon \right\}|X_n| \to 0, \ f_0 \text{ a.s.}
\]

Since \( d \) is bounded, a simple application of the Jensen’s inequality gives

\[
(d_0 \tilde{d})^2 \leq E^\pi [(d-d_0)^2 |X_n] \to 0, \ f_0 \text{ a.s.}
\]

It is also possible to derive consistency results for the point estimate of the whole spectral density:

**Corollary 2.** Under the assumptions of Theorem 2.1, if \( \hat{f}_n^\pi \) is as defined in (2.6), as \( n \to \infty \),

\[
h(f_0, \hat{f}_n^\pi) \to 0, \ f_0 \text{ a.s.}
\]

**Proof.** To simplify the notations we set \( C \) to be a generic positive constant. Let \( H(x) = x + x^{-1} - 2 \), then for any \( a > 0 \)

\[
h(f_0, \hat{f}_n^\pi) = C \int \left( \frac{\sqrt{E^\pi [f/f_0(\lambda)]|X_n|}}{\sqrt{E^\pi [f_0/f(\lambda)]|X_n|}} + \frac{\sqrt{E^\pi [f_0/f(\lambda)]|X_n|}}{\sqrt{E^\pi [f/f_0(\lambda)]|X_n|}} - 2 \right) d\lambda
\]

\[
\leq C \int_{\lambda > a} E^\pi [H(f/f_0(\lambda))|X_n|] + C \int_{\lambda < a} H \left[ \frac{\sqrt{E^\pi [f/f_0(\lambda)]|X_n|}}{\sqrt{E^\pi [f_0/f(\lambda)]|X_n|}} \right] d\lambda
\]

\[
= I_1 + I_2
\]

We have:

\[
I_1 \leq C \epsilon + CE^\pi \left[ |\lambda| h(f, f_0) > \epsilon \right] \int_{\lambda > a} H(f/f_0(\lambda))d\lambda|X_n|
\]

\[
I_2 \leq C \epsilon^2 + CE^\pi \left[ \frac{1}{\lambda} h(f, f_0) > \epsilon \right] \int_{\lambda > a} H(f/f_0(\lambda))d\lambda|X_n|
\]

\[
\Rightarrow h(f_0, \hat{f}_n^\pi) \to 0, \ f_0 \text{ a.s.}
\]
Now, consider the test $\phi_n$ defined in the proof of Theorem 2.1 and the same type of inequality as those used in the proof of the same theorem: for all $f \in A'_c$

$$E_f[1 - \phi_n] \leq e^{-nc|\log \epsilon|^{-1}}.$$ 

Then we choose a small $\delta > 0$ such that

$$P^n_0[I_1 > 2C\epsilon] \leq E^n_0[\phi_n] + \frac{C}{n^3} + \frac{C\epsilon^{\delta}}{\epsilon} \int_{\lambda > a} H(f/f_0(\lambda))E^n_f[1 - \phi_n]d\pi(f)d\lambda \leq Cn^{-3} + Ce^{-nc|\log \epsilon|^{-1}/2} \left[ 1 + \int_{\lambda > a} H(f/f_0(\lambda))d\pi(f)d\lambda \right] \leq Cn^{-3} + Ce^{-nc|\log \epsilon|^{-1}/2}a^{-2}.$$

Let $a = \exp(-nc|\log \epsilon|^{-1}/8)$ then

$$P^n_0[I_1 > 2C\epsilon] \leq Ce^{-nc|\log \epsilon|^{-1}/4}.$$ 

We also have

$$I_2 = \int_0^a H \left[ \frac{E^n[f/f_0(\lambda)X_n]}{E^n[f_0/f(\lambda)X_n]} \right] d\lambda \leq C \int_{0<\lambda<a} \frac{E^n[\lambda^{-2(d-\tilde{d})}X_n]}{E^n[\lambda^{-2(d_0-\tilde{d})}X_n]} + \frac{E^n[\lambda^{-2(d_0-\tilde{d})}X_n]}{E^n[\lambda^{-2(d-\tilde{d})}X_n]} d\lambda \leq C \int_{0<\lambda<\bar{a}} \lambda^{2\tilde{d}\lambda - \tilde{d}} E^n[\lambda^{-2\tilde{d}}X_n] d\lambda + C \int_{0<\lambda<a} \lambda^{-2d_0\lambda + \tilde{d}} E^n[\lambda^{2\tilde{d}}X_n] d\lambda$$

Let $A$ be the set where $\tilde{d} = E^n[d|X_n]$ converges to $d_0$; then $P^n_0[A] = 1$ and $\forall \tilde{d} > 0$ and $n$ large enough,

$$I_2 \leq C \int_{0<\lambda<\bar{a}} \lambda^{d_0 - \frac{1}{2} + \tilde{d}} d\lambda + C \int_{0<\lambda<a} \lambda^{-d_0 - \frac{1}{2} - \tilde{d}} d\lambda \leq C(a^{1/2 + d_0 - \tilde{d}} + a^{1/2 - d_0 - \tilde{d}}) \leq e^{-nc|\log \epsilon|^{-1}},$$

for some $c > 0$. \hfill \Box

**Corollary 3.** Under the assumptions of Theorem 2.1, as $n \to \infty$,

$$\ell(f_0, \hat{f}^n) \to 0, \quad f_0 \text{ a.s.}$$

**Proof.** Note that for all $x \in \mathbb{R}$, $e^x + e^{-x} - 2 \geq x^2$. Then $h(f, f_0) \geq l(f, f_0)$ and

$$P^n[f; l(f, f_0) > \epsilon |X_n|] \leq P^n[A'_c |X_n].$$
This implies, together with the fact that \( l(f, f_0) \) is bounded when \( f \in G(t, m, M, \rho) \), that \( \forall \epsilon > 0, \)

\[
l(\hat{f}, f_0) \leq E^\pi[l(f, f_0)|X_n] \leq \epsilon + CP^\pi[A^\epsilon_n|X_n].
\]

\[
Since the conditions stated in Theorem 2.1 are somewhat non standard, they need to be carefully checked for the specific class of priors one is dealing with. Here we consider the class of Fractionally Exponential priors (FEXP), and we show that these priors actually fulfill the above conditions.

2.3. The FEXP prior

Consider the set of the spectral densities with the form

\[
f(\lambda) = |1 - e^{i\lambda}|^{-2d} \hat{f}(\lambda),
\]

where \( \log \hat{f}(\lambda) = \sum_{j=0}^{K} \theta_j \cos(j\lambda) \), for some finite \( k \in \mathbb{N} \), and assume that the true log spectral density satisfies \( \log \hat{f}_0(\lambda) = \sum_{j=0}^{\infty} \theta_{0j} \cos(j\lambda) \) (in other words, it is equal to its Fourier series expansion), with

\[
|\hat{f}_0(\lambda) - \hat{f}_0(\lambda')| \leq L \frac{|\lambda - \lambda'|}{|\lambda| \wedge |\lambda'|}, \quad \sum_j |\theta_{0j}| < \infty,
\]

for all \( \lambda \) and \( \lambda' \) in \([-\pi, \pi]\). In this section our base model is presented in a slightly different way: however it comes to the same thing since \( |1 - e^{i\lambda}|/|\lambda| \) is continuous and strictly positive on \([-\pi, \pi]\).

This class of densities has been considered, from a frequentist perspective, in (17). Note that there exists an alternative and equivalent way of writing a FEXP spectral density in which the first coefficient of the series expansion \( \theta_0 \) is explicitly expressed in terms of the variance of the process, that is \( \sigma^2 = 2\pi e^{\theta_0} \). We will use both the parameterizations according to notational convenience.

A prior distribution on \( f \) can then be expressed as a prior on the parameters \( (d, K, \theta_0, ..., \theta_K) \) in the form \( p(K)\pi(d|K)\pi(\theta|d, K) \), where \( \theta = (\theta_0, ..., \theta_K) \), and \( K \) represents the (random) order of the FEXP model. Usually, \( d \) is set independent of \( \theta \) for given \( K \) and it is also independent of \( K \) itself. Let \( \pi(d) > 0 \) on \([-1/2 + t, 1/2 - t]\), for some \( t > 0 \), arbitrarily small. Let \( K \) be a priori Poisson distributed and, conditionally on \( K \), in order to obtain a Lipschitz condition on \( \sum_{j=1}^{K} \theta_j \cos(j\lambda) \) we consider \( \theta \)'s such that \( \sum_{j=1}^{K} j|\theta_j| \leq B \), where
$B$ is large but finite. This implies in particular, that the terms $\sum_{j=1}^{K} |\theta_j|$ are uniformly bounded over the supports of $\pi_K$. A possible way to formalize it, is to assume that, for given $K$, the quantity $S_K = \sum_{j} j |\theta_j|$ has a finite support distribution; then, setting $V_j = j |\theta_j|/S_K$, $j = 1, \ldots, K$, one may consider a distribution on the set $\{ z \in \mathbb{R}^K; z = (z_1, \ldots, z_K), \sum z_i = 1, z_i \geq 0 \}$ for example:

$$
(V_1, \ldots, V_K) \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_K),
$$

Since the variance of the $|\theta_j|$'s should be decreasing as $j$ increases, we may assume, for example, that, for all $j$'s, $\alpha_j = O((1 + j)^{-2})$. Note that if we further assume that $S_K$ has a Gamma distribution with mean $\sum \alpha_j$ and variance $\sum \alpha_j^2$, then we are approximately assuming (modulo the truncation at $A$) that $(|\theta_1|, \ldots, |\theta_K|)$ are independent Gamma$(1, \alpha_j)$ random variables. Alternative parameterization are also available here; for example one can assume that $(V_1, \ldots, V_K)$ follows a logistic normal distribution (1), which allows for a more flexible elicitation. Under the above conditions on the prior, the posterior distribution is strongly consistent, in terms of the distance $d(\cdot, \cdot)$, the estimator $\hat{f}$ as described in the previous section is almost surely consistent and so is the estimator $\hat{\theta}$. To prove this, we need to show that the FEXP prior satisfies assumptions (i) and (ii). First, we check assumption (i): let $K_\varepsilon$ be such that $\sum_{j=K_\varepsilon+1}^{\infty} |\theta_{0j}| \leq \sqrt{\varepsilon}/4$, then $h(f_0, f_{0\varepsilon}) \leq \varepsilon/8$, where

$$
f_{0\varepsilon} = |1 - e^{i\lambda}|^{-2d\varepsilon} \exp \left\{ \sum_{j=0}^{K_\varepsilon} \theta_{0j} \cos j \lambda \right\},
$$

Let $\theta = (\theta_0, \ldots, \theta_{K_\varepsilon})$ be such that $\sum_{j=0}^{K_\varepsilon} |\theta_{0j} - \theta_j| \leq \sqrt{\varepsilon}/4$ $j = 1, \ldots, K_\varepsilon$ If $|d - d_0| < \varepsilon c$, with $c \leq \left( \int_\pi^\pi |1 - e^{i\lambda}|^{-1/4} d\lambda \right)^{-1} / (8\pi)$, then

$$
h(f, f_0) \leq \varepsilon \int_\pi^\pi |1 - e^{i\lambda}|^{-\varepsilon/4} d\lambda + e^{\sqrt{\varepsilon}/2} \frac{\varepsilon}{4} \leq \varepsilon
$$

for $\varepsilon > 0$ small enough. Also $\pi_K, \{ \{ \theta : |\theta_j - \theta_{0j}| < \sqrt{\varepsilon}/(8K_\varepsilon), \forall j \leq K_\varepsilon \}$ $> 0$, as soon as $A > \sum \theta_{0j}$. Thus assumption (i) of Theorem 2.1 is satisfied.

Now we verify assumption (ii). Let $\varepsilon > 0$ and set

$$
f_{k, d, \theta}(\lambda) = |1 - e^{-i\lambda}|^{-2d} \exp \left\{ \sum_{j=0}^{k} \theta_j \cos (j \lambda) \right\},
$$

where the $\theta_j$'s satisfy the above constraint. Consider

$$
F_n = \{ f_{k, d, \theta}, d \in [-1/2 + t, 1/2 - t], k \leq k_n \},
$$
where $k_n = k_0 n/\log n$. Since $\pi(K \geq k_n) < e^{-nr}$, for some $r$ depending on $k_0$, we have that $\pi(F_n^c \geq k_n) < e^{-nr}$. Now consider spectral densities in the form,

$$f_i(\lambda) = (1 - \cos \lambda)^{-d_i} \exp\{-d_i \log(2) + \sum_{j=0}^{k} \theta_{ij} \cos j\lambda\}.$$  

Consider

$$f(\lambda) = (1 - \cos \lambda)^{-d} \exp\{-d \log(2) + \sum_{j=0}^{k} \theta_j \cos j\lambda\},$$

where $d_i - c_1 \epsilon \leq d \leq d_i$, $\theta_0 - c_2 \epsilon \leq \theta_0 + (d_i - d) \log(2) \leq \theta_0 - c_3 \epsilon$, and $\sum_{j=1}^{k} |\theta_j - \theta_{ij}| \leq c_0 \epsilon$. Then

$$\frac{f(\lambda)}{f_i(\lambda)} = (1 - \cos \lambda)^{d_i - d} \exp\{(d_i - d) \log(2) + \sum_{j=0}^{k} (\theta_j - \theta_{ij}) \cos j\lambda\} \leq 1$$

and

$$\frac{f(\lambda)}{f_i(\lambda)} \geq (1 - \cos \lambda)^{-c_1 \epsilon} e^{-(c_2 + c_3) \epsilon}$$

Hence by choosing $c_0, c_1, c_2$ small enough, $f_i - f$ verifies the three inequalities considered in assumption (ii) of Theorem 2.1. The covering number of $F_n$ with balls defined by the above inequalities can be bounded by

$$\exp(N_n) \leq k_n (Ck_n/\epsilon)^{k_n+2} \leq e^{2k_0 n(-\log \epsilon - \log \log n)}$$

so that if $n$ is large enough

$$N_n \leq n\epsilon|\log \epsilon|^{-2}$$

and assumption (ii) is satisfied.

### 3. Rates of convergence

In this section we first provide a general theorem relating rates for the posterior distribution to conditions on the prior. These conditions are, in essence, similar to the conditions obtained in the i.i.d. case; in other words there is a condition on the the prior mass of Kullback-Leibler neighbourhoods of the true spectral density and an entropy condition on the support of the prior. We then present the results in the case of the FEXP prior.
3.1. Main result

We now present the general Theorem on convergence rates for the posterior distribution.

Theorem 3.1. Let \((\rho_n)_n\) be a sequence of positive numbers decreasing to zero, and \(B_n\) a ball belonging to \(G(t, M, m, L, \rho) \cup \tilde{L}(t, M, m, L)\), defined as

\[
B_n(\delta) = \{ f(x) = |x|^{-2(d-d_0)} \tilde{f}(x); KL_n(f_0; f) \leq \rho_n/4, b_n(f_0, f) \leq \rho_n, |d-d_0| \leq \delta \},
\]

for some \(\rho \in (0, 1]\). Let \(\pi\) be a prior which satisfies conditions (i) and (ii) of Theorem 2.1. Assume that:

(i). There exists \(\delta > 0\) such that \(\pi(B_n(\delta)) \geq \exp(-n\rho_n/2)\).

(ii). For all \(\epsilon > 0\) small enough, there exists a positive sequence \((\epsilon_n)_n\) decreasing to zero and \(F_n \subset \tilde{F}_* \cap \{ f, d(f, f_0) \leq \epsilon \}, \) such that \(\pi(F_n \cap \{ f, d(f, f_0) \leq \epsilon \}) \leq e^{-2n\rho_n}\).

(iii). Let

\[
S_{n,j} = \{ f \in \tilde{F}_n; \varepsilon_n^2 j \leq h_n(f_0, f) \leq \varepsilon_n^2 (j + 1) \},
\]

with \(J_n \geq j \geq J_0\), with fixed \(J_0 > 0\) and \(J_n = [\varepsilon_n^2 / \varepsilon_n^2] \cdot \forall J_0 \leq j \leq J_n\), there exists a smallest possible net \(\tilde{N}_{n,j} \subset S_{n,j}\) such that \(\forall f \in S_{n,j} \exists f_i \geq f \in \tilde{N}_{n,j}\) satisfying \(\text{tr} (T_n(f)^{-1} T_n(f_i) - \text{id}) / n \leq h_n(f_0, f_i)/8\), and \(\text{tr} (T_n(f_i - f) T_n^{-1}(f_0)) / n \leq h_n(f_0, f_i)/8\). Denote by \(\tilde{N}_{n,j}\) the logarithm of the cardinality of the smallest possible net \(\tilde{N}_n\).

Then, there exist \(M, C, C' > 0\) such that if \(\rho_n \leq \varepsilon_n^2\) and \(n\) is large enough

\[
E_{\theta}^n \left[ \pi \left( f; h_n(f_0, f) \geq M\varepsilon_n^2 | X \right) \right] \leq \max \left( e^{-n\varepsilon_n^2 C', C' \rho_n^2} \right).
\]  \hspace{1cm} (3.1)

Proof. Throughout the proof \(C\) denotes a generic constant. We have

\[
\pi \left( f; h_n(f_0, f) \geq M\varepsilon_n^2 | X \right) = \frac{\int_{f; h_n(f_0, f) \geq M\varepsilon_n^2} \varphi_f(X_n)/\varphi_{f_0}(X_n) d\pi(f)}{\int \varphi_f(X_n)/\varphi_{f_0}(X_n) d\pi(f)} = \frac{\int_{f; h_n(f_0, f) \geq M\varepsilon_n^2} \varphi_f(X_n)/\varphi_{f_0}(X_n) d\pi(f)}{\int \varphi_f(X_n)/\varphi_{f_0}(X_n) d\pi(f)} + \frac{\int \varphi_f(X_n)/\varphi_{f_0}(X_n) d\pi(f)}{\int \varphi_f(X_n)/\varphi_{f_0}(X_n) d\pi(f)} = \frac{N_n}{D_n} + R_{n,2}.
\]
for some $\varepsilon > 0$. Theorem 2.1 implies that $P_0 \left[ R_{n,2} > e^{-n\delta} \right] \leq \frac{C}{n^\delta}$, for some constants $C, \delta > 0$. Then we consider the first term of the right hand side of the above equation. Using an argument similar to the one used in the previous proof, let

$$N_{n,j} = \int_{f: \varepsilon_{2,j} \leq h_n(f_0,f) \leq \varepsilon_{2,j+1}} \frac{\varphi f(X_n)}{\varphi f_0(X_n)} d\pi(f)$$

and

$$E^n_0 \left[ \frac{N_{n,j}}{D_n} \right] \leq \sum_{j \geq M} E^n_0 [\varphi_{n,j}] + E^n_0 \left[ (1 - \varphi_{n,j}) \frac{N_{n,j}}{D_n} \right],$$

where $\varphi_{n,j} = \max_{f_i \in \mathcal{R}_{n,j}} \varphi_{i,j}$, and $\varphi_i$ is a test function defined as in the previous Section, that is $\varphi_i = 1_{D_i}$, where

$$D_i = \left\{ \bar{X}'_n (T_n^{-1}(f_i) - T_n^{-1}(f_0)) X_n \geq \text{tr} (\text{id} - T_n(f_0)T_n^{-1}(f_i)) + h_n(f_0, f_i)/4 \right\}.$$

Then, (B.3) implies that

$$E^n_0 [\varphi_{n,j}] \leq \sum_{i: f_i \in \mathcal{R}_{n,j}} e^{-Cn\varepsilon_{2,j}^2} \leq \bar{N}_{n,j} e^{-Cn\varepsilon_{2,j}^2} \leq e^{-Cn\varepsilon_{2,j}^2}.$$

We also have that

$$E^n_0 \left[ (1 - \varphi_{n,j}) \frac{N_{n,j}}{D_n} \right] \leq P^n_0 \left[ D_n \leq e^{-n\rho_n}/2 \right] + 2e^{n\rho_n} \pi(\mathcal{F}^n \cap \{ f : d(f, f_0) \leq \varepsilon \}) + 2e^{n\rho_n} \int_{\mathcal{S}_{n,j}} E^n_0 [1 - \varphi_{n,j}] d\pi(f)$$

$$\leq 2e^{-n\rho_n} + 2e^{n\rho_n} e^{-nC\varepsilon_{2,j}^2} + P^n_0 \left[ D_n \leq e^{-n\rho_n}/2 \right].$$

Moreover, using the same calculations as in the proof of theorem 2.1

$$P^n_0 \left[ D_n \leq e^{-n\rho_n}/2 \right] \leq \frac{P^n_0 \left[ D_n \leq e^{-n\rho_n/2}\pi(B_n) \right]}{\int_{B_n} P^n_0 \left[ \Omega_{n,1}^c (f) \right] d\pi(f)} \pi(B_n),$$

where $\Omega_{n,1} = \{ (X_n,f); X'_n(T_n^{-1}(f) - T_n^{-1}(f_0))X_n - \log \det[A(f_0, f)] \leq n\rho_n/2 \}$. Using the exponential bound similar to (B.3), we obtain, if $f \in B_n$, that

$$P^n_0 [\Omega_n^c] \leq \exp \left( -n\rho_n \left( \frac{\rho_n}{16b_n(f_0, f)} \wedge \frac{1}{8} \right) \right) \leq e^{-n\rho_n/16},$$

on $B_n$ and Theorem 3.1 is proved.
The conditions given in Theorem 3.1 are similar in spirit to those considered for rates of convergence of the posterior distribution in the i.i.d. case. The first one is a condition on the prior mass of Kullback-Leibler neighborhoods of the true spectral density, the second one is necessary to allow for sets with infinite entropy (some kind of non compactness) and the third one is an entropy condition. The inequality (3.1) obtained in Theorem 3.1 is non asymptotic, in the sense that it is valid for all \( n \). However, the distances considered in Theorem 3.1 heavily depend on \( n \) and, although they express the impact of the differences between \( f \) and \( f_0 \) on the observations, they are not of great practical use. For these reasons, the entropy condition is awkward and it cannot be directly transformed into some more common entropy conditions. To state a result involving distances between spectral densities that might be more useful, we need to consider some specific class of priors, namely the FEXP priors, as defined in Section 2.3. For this class we obtain rates of convergence in terms of the \( L_2 \) distance between the logarithm of the spectral densities, \( l(f, f') \). The rates obtained are the optimal rates up to a \( \log n \) term, at least on certain classes of spectral densities. It is to be noted that the calculations used when working on these classes of priors are actually more involved than those used to prove Theorem 3.1. This is quite usual when dealing with rates of convergence of posterior distributions, however this is emphasized here by the fact that distances involved in Theorem 4 are strongly dependent on \( n \). The method used in the case of the FEXP prior can be extended to other types of priors.

3.2. The FEXP prior - rates of convergence

Here we apply Theorem 3.1 to the FEXP priors, which we define through a slightly different parameterization. In particular, \( f(\lambda) = |1 - e^{i\lambda}|^{-2d}\tilde{f}(\lambda) \), and \( \log \tilde{f}(\lambda) = \sum_{j=0}^{K} \theta_j \cos j\lambda \). Then the prior can be written in terms of a prior on \((d, K, \theta_0, ..., \theta_K)\). Define now the classes of spectral densities

\[ S(\beta, L_0) = \{ h \geq 0; \log h \in L^2[-\pi, \pi], \log h(x) = \sum_{j=0}^{\infty} \theta_j \cos jx, \sum_j \theta_j^2 (1+j)^{2\beta} \leq L_0 \}, \]

with \( \beta > 0 \). Also, assume that there exists a real value \( \beta > 0 \) such that \( \tilde{f}_0 \in L^*(M, m, L) \cap S(\beta, L_0) \). We can then write \( f_0 \) as

\[ f_0(\lambda) = |1 - e^{i\lambda}|^{-2d_0} \exp \left\{ \sum_{j=0}^{\infty} \theta_{j,0} \cos j\lambda \right\}. \]
Note that $\beta$ is a smoothness parameter. Classes similar to $S(\beta, L_0)$ are considered by \cite{22}. We now describe the construction of the FEXP prior, so that it can be adapted to $S(\beta, L_0)$. Let $S_K$ be a r.v. with density $g_A(\cdot)$, which is positive in the interval $[0, A]$, let $\eta_j = \theta_j j^\beta$ and suppose that the prior on $(\eta_1/S_K, \ldots, \eta_K/S_K)$ has positive density on the set

\[ S_{K+1} = \{ x = (x_1, \ldots, x_{K+1}); \sum_{j=1}^{K+1} j^2 = 1 \} \]

We denote this class as the class of FEXP($\beta$) priors. Note that if $\beta > 1/2$ then there exists a constant $M$ and $\rho < 2\beta - 1$ such that for all $f \in S(\beta, L_0)$ associated with the parameters $(k, \theta_0, \ldots, \theta_k)$ then

\[ \sum_{i=0}^k |\theta_i| \leq M, \quad |\log \hat{f}(x) - \log \hat{f}(y)| \leq M|x - y|^{\rho}. \]

first,

\[ \sum_{i=0}^k |\theta_i| \leq \sum_{i=0}^k i^{2\beta} \theta_i^2 + \sum_{i=0}^k |\theta_i| \sum_{j \geq (1+i)^{2\beta}} \theta_i^j \]

\[ \leq L_0 + \sum_{i=0}^\infty (1 + i)^{-2\beta}, \]

and, second, since $\sum_j j^\rho |\theta_j|$ is uniformly bounded,

\[ |\log \hat{f}(x) - \log \hat{f}(y)| \leq |x - y|^{\rho} \left( 2 \sum_{j} |\theta_j| \sum_{j \geq 1} |\theta_j| j^\rho \right) \]

\[ \leq |x - y|^{\rho} \left( 2 \sum_{j} |\theta_j| \sum_{j \geq 1} |\theta_j| j^\rho \right) \]

\[ \leq M|x - y|^{\rho}. \]

Therefore the prior lies in $\mathcal{G}(t, m, M, L, \rho)$ for some positive constant $m, M, L, \rho$.

We now give the rates of convergence associated with the FEXP($\beta$) priors, when the true spectral density belongs to $S(\beta, L_0)$.

**Theorem 3.2.** Assume that there exists $\beta > \frac{1}{2}$ s.t. $\tilde{f}_0 \in \mathcal{L}^*(e^{L_0}, e^{-L_0}, L) \cap S(\beta, L_0)$. Let $\pi$ be a FEXP($\beta$) prior and assume that

i) $K \sim \text{Poi}(\mu)$;
ii) the prior on $d$ is strictly positive on $[-1/2 + t, 1/2 - t]$, with $t > 0$;
iii) $S_K$ has a positive density on $(0, A)$ with $A$ such that $A^2 \geq L_0$.

Then there exist $C, C' > 0$ such that, for $n$ large enough

$$P^n \left[ \{ f \in F^+ : l(f, f_0) > Cn^{-2\beta/(2\beta+1)} \log n^{(2\beta+3)/(2\beta+1)} \} | X_n \right] \leq \frac{C'}{n^2} \tag{3.2}$$

and

$$E^n_0 \left[ l(\hat{f}, f_0) \right] \leq 2Cn^{-2\beta/(2\beta+1)} \log n^{(2\beta+3)/(2\beta+1)}, \tag{3.3}$$

where $\log \hat{f}(\lambda) = E^n [ \log f(\lambda) | X_n ]$.

Proof. Throughout the proof, $C$ denotes a generic constant. The proof of the theorem is divided in two parts; in the first part, we prove that

$$E^n_0 \left[ P^n \left( \{ f : h_n(f, f_0) \geq n^{-2\beta/(2\beta+1)} \log n^{(2\beta+3)/(2\beta+1)} \} | X_n \right) \right] \leq \frac{C}{n^2}, \tag{3.4}$$

while in the second part we prove that

$$h_n(f, f_0) \leq Cn^{-2\beta/(2\beta+1)} \log n^{1/\beta} \Rightarrow l(f, f_0) \leq C'n^{-2\beta/(2\beta+1)} \log n^{2\beta+3}, \tag{3.5}$$

for some constant $C' > 0$, when $n$ is large enough. The latter inequality implies that

$$E^n_0 \left[ l(f, f_0) | X_n \right] \leq C'n^{-2\beta/(2\beta+1)} \log n^{2\beta+3} + \int_{A(n, \beta)} l(f, f_0) d\pi(f | X_n)$$

$$\leq 2C'n^{-2\beta/(2\beta+1)} \log n^{2\beta+3},$$

for large $n$, where $A(n, \beta) = \{ h_n(f, f_0) > Cn^{-2\beta/(2\beta+1)} \log n^{2\beta+3} \}$. This would imply Theorem 3.2. To prove (3.4), we need to show that conditions (i)-(iii) of Theorem 3.1 are fulfilled. Condition (ii) is obvious because the prior has the same form as in Section 2.3 and, because $S(\beta, L) \subset \tilde{G}(t, m, M', \rho)$, with $t, m, L', \rho$ positive constant depending on $\beta, L$. Thus we can choose

$$\bar{F}_n = \left\{ f(\lambda) = |1 - e^{i\lambda}|^{-2d} \exp \left( \sum_{j=0}^{K} \theta_j \cos (j\lambda) \right) \right\},$$

with $K \leq K_n, |d - d_0| \leq \delta, \sum j^{2\beta} \theta_j^2 \leq L_0$, leading to

$$\pi(\bar{F}_n \cap \{ f, h(f, f_0) < \epsilon \}) \leq \pi(K \geq K_n) \leq e^{-K_n \log K_n}.$$
for $K_n$ large enough. By choosing $K_n = k_n n^{1/(2\beta+1)} \log n^{2/(2\beta+1)}$, we obtain

$$\pi \left( \tilde{F}_n \cap \{ f, h(f, f_0) < \epsilon \} \right) \leq e^{-k_n n^{1/(2\beta+1)} \log n^{1-2\beta)/(2\beta+1)}}.$$

Hence, letting $\rho_n = \epsilon_n^2 / n^{2\beta/(2\beta+1)} \log n^{(2\beta+3)/(2\beta+1)}$, condition (ii) is satisfied. We now show that assumption (i) of Theorem 3.1 is satisfied. Let $d \leq d_0 \leq d + \epsilon_n / \log n^{3/2}$ and, for all $l = 0, \ldots, K_n$,

$$|\theta_l - \theta_{0l}| \leq (l + 1)^{-\beta/(\beta+1)}(\log (l + 1))^{-1} \epsilon_n / \log n^{3/2}.$$

Since $f_0 \in S(\beta, L_0)$, $\exists t_0 > 0$ such that

$$\sum_{l \geq K_n} \theta_{0l}^2 \leq L_0 K_n^{-2\beta} \leq C \epsilon_n^2 / (\log n)^{-3}, \quad \sum_{l \geq K_n} |\theta_{0l}| \leq K_n^{-t_0} \quad (3.6)$$

Since

$$KL_n(f_0; f) \leq h_n(f_0, f) = \frac{1}{2n} \text{tr} \left( T_n(f_0 - f)T_n^{-1}(f)T_n(f_0 - f)T_n^{-1}(f_0) \right),$$

it is enough to prove the assumption under the above conditions for $h_n(f, f_0) \leq C \epsilon_n^2$. The difficulty here comes from the strong dependence on $n$ of the distance $h_n$. Let

$$f_{0n}(\lambda) = |1 - e^{i\lambda}|^{-2d_0} \exp \left( \sum_{l=0}^{K_n} \theta_{0l} \cos l \lambda \right), \quad b_n(\lambda) = 1 - \exp \left( - \sum_{l \geq K_n+1} \theta_{0l} \cos l \lambda \right),$$

and $g_n = f_{0n}^{-1}(f_0 - f)$; then $f_0 - f = f_0 b_n + f_0 g_n$ and

$$nh_n(f_0, f) \leq \text{tr} \left( T_n(f_0 b_n)T_n^{-1}(f)T_n(f_0 b_n)T_n^{-1}(f_0) \right) + \text{tr} \left( T_n(f_0 g_n)T_n^{-1}(f)T_n(f_0 g_n)T_n^{-1}(f_0) \right). \quad (3.7)$$

Both terms of the right hand side of (3.7) are treated similarly, using Lemma 3 we can bound them by

$$\text{tr} \left( T_n(f_0 b_n)T_n^{-1}(f)T_n(f_0 b_n)T_n^{-1}(f_0) \right) \leq C(\log n)^3 n |b_n|_2^2 + O(n^\delta).$$

$$\text{tr} \left( T_n(f_0 b_n)T_n^{-1}(f)T_n(f_0 b_n)T_n^{-1}(f_0) \right) \leq C(\log n)^3 n |g_n|_2^2 + O(n^\delta).$$

This implies that $h_n(f_0, f) \leq C \epsilon_n^2$, when $f$ satisfies the conditions described above and

$$\mathcal{B}_n \subset \left\{ f_{k,d,\theta}; k \geq K_n, d \leq d_0 \leq d + \frac{\epsilon_n}{(\log n)^{3/2}}, 0 \leq l \leq K_n, \right. \left. |\theta_l - \theta_{0l}| \leq \frac{(l + 1)^{-\beta/(\beta+1)} \epsilon_n}{(\log (l + 1)) \log n^{3/2}} \right\}.$$
The prior probability of the above set is bounded from below by

$$\pi(K_n)\mu_1\left(\{(\eta_1, \ldots, \eta_{K_n}) : |\eta_l - \eta_0| \leq C \frac{l^{-1/2}c_n}{(\log l) \log n^{3/2}}\right) \rho_n \log n^{-3/2},$$

where $\mu_1$ denotes the uniform measure on the set $\{(\eta_1, \ldots, \eta_{K_n}) : \sum l^2 \eta_l^2 \leq A\}$. We finally obtain that

$$\pi(B_n(\delta)) \geq e^{-CK_n \log n} \geq e^{-n^{\rho_n/2}}$$

by choosing $k_0$ small enough, and condition (i) of Theorem 3.2 is satisfied by the $\text{FEXP}(\beta)$ prior. We now verify condition (iii) of Theorem 3.2. Let $j_0 \leq j \leq J_n$, where $j_0$ is some positive constant, and consider $f \in S_{n,j}$, as defined in Theorem 3.1, where $f(\lambda) = f_{\theta,k} = |1 - e^{i\lambda}|^{-2d} \exp\{\sum_{l=1}^{k} \theta_l \cos(l \lambda)\}$. Consider

$$f_u(\lambda) = |1 - e^{i\lambda}|^{-2d} \exp\{\sum_{l=1}^{k} \theta_l^n \cos(l \lambda)\}; f(\lambda) = |1 - e^{i\lambda}|^{-2d} \exp\{\sum_{l=1}^{k} \theta_l \cos(l \lambda)\},$$

then, if $c \geq c_0 \sum_{l \geq 0} (l + 1)^{-\beta-1/2} \log (l + 1)^{-1}, c_0 > 0$ and $d_u \geq d \geq d_n - cc_n^2 j, |\theta_l - \theta_l^n| \leq c_0 (l + 1)^{-\beta-1/2} \log (l + 1)^{-1} c_n^2 j, l \geq 1, \theta_0^n - 4cc_n^2 j \leq \theta_0 - \theta_0^n - 3cc_n^2 j$, one obtains

$$1 \leq \frac{f_u(\lambda)}{f(\lambda)} \leq (1 - \cos \lambda)^{-2cc_n^2 j} e^{5cc_n^2 j}$$

and

$$\text{tr} \left( T_n^{-1}(f) T_n(f_u - f) \right) \leq 15cc_n^2 j \text{tr} \left( T_n^{-1}(f) T_n(f_u) \right) \leq Ccc_n^2 j \leq Cc h_n(f_0, f_u).$$

Choosing $c$ small enough one obtains that $\text{tr} \left( T_n^{-1}(f) T_n(f_u - f) \right) \leq nh_n(f_0, f_u)/8$. Similarly

$$\text{tr} \left( T_n^{-1}(f_0) T_n(f_u - f) \right) \leq 4cc_n^2 j \text{tr} \left( T_n^{-1}(f_0) T_n(f_u) \right) \leq Cc h_n(f_0, f_u)/8.$$
Let $K \leq K_n = K_0 n^{1/(2\beta+1)}(\log n)^{-1}$, the number of $f_u$ defined as above in the set $S_{n,j}$ is bounded by

$$N_{n,j} \leq K_n^{-1}c_n^{-2} \left(CK_n^{-1}c_n^{-2}\right)^{K_n}$$

and

$$\tilde{N}_{n,j} = \log N_{n,j} \leq c_j n c_n^2$$

where $c_j$ is decreasing in $j$. Hence by choosing $j_0$ large enough condition (iii) is verified by the FEXP($\beta$) prior. This achieves the proof of (3.4) and we obtain a rate of convergence, in terms of the distance $h_n(\cdot, \cdot)$. We now prove (3.5) to obtain a rate of convergence in terms of the distance $l(\cdot, \cdot)$. Consider $f$ such that

$$h_n(f_0, f) = \frac{1}{n} \text{tr} \left(T_n^{-1}(f_0)T_n(f - f_0)T_n^{-1}(f)T_n(f - f_0)\right) \leq c_n^2.$$

Equation (C.2) of Lemma 3 implies that

$$\frac{1}{n} \text{tr} \left(T_n(g_0)f_0^{-1}T_n(f - f_0)T_n(g_0)T_n(f - f_0)\right) \leq C c_n^2,$$

leading to

$$\frac{1}{n} \text{tr} \left(T_n(g_0)f_0^{-1}T_n(f - f_0)T_n(g_0)T_n(f - f_0)\right) \leq C c_n^2,$$

where $g_0 = (1 - \cos \lambda)d_0$, $g = (1 - \cos \lambda)d$.

We now prove that $\text{tr} \left(T_n(g_0)(f - f_0)T_n(g_0)(f - f_0)\right) \leq C c_n^2$: we use the same representation as in the treatment of $\gamma(b)$ in Appendix C. For the sake of simplicity we consider the case $d \geq d_0$

$$\Delta = \frac{1}{n} \text{tr} \left(T_n(g_0)(f - f_0)T_n(g_0)(f - f_0)\right)$$

$$- \frac{1}{n} \text{tr} \left(T_n(g_0)f_0^{-1}T_n(g_0)T_n(f - f_0)\right)$$

$$= \frac{1}{n} \int_{[-\pi, \pi]^4} \left( f - f_0 \right)(\lambda_2)g_0(\lambda_2)(f - f_0)(\lambda_4)g(\lambda_4) \left( \frac{g_0(\lambda_1)}{g_0(\lambda_2)} - 1 \right)$$

$$\times \Delta_n(\lambda_1 - \lambda_2)\Delta_n(\lambda_2 - \lambda_4)\Delta_n(\lambda_4 - \lambda_1)d\Delta$$

$$+ \frac{1}{n} \int_{[-\pi, \pi]^4} \left( f - f_0 \right)(\lambda_2)g_0(\lambda_1)(f - f_0)(\lambda_4)g(\lambda_4) \left( \frac{g(\lambda_3)}{g(\lambda_4)} - 1 \right)$$

$$\times \Delta_n(\lambda_1 - \lambda_2)\Delta_n(\lambda_2 - \lambda_3)\Delta_n(\lambda_3 - \lambda_4)\Delta_n(\lambda_4 - \lambda_1)d\Delta$$
\[ \leq \frac{C \log n}{n} \int_{[-\pi,\pi]^2} |\lambda_2\lambda_1^{-1+\delta}L_n(\lambda_1 - \lambda_2)d\lambda \]
\[ + \frac{C}{n} \int_{[-\pi,\pi]^4} |\lambda_1\lambda_2\lambda_3\lambda_4|^{-1} L_n(\lambda_1 - \lambda_2) L_n(\lambda_2 - \lambda_3) L_n(\lambda_3 - \lambda_4) L_n(\lambda_4 - \lambda_1)d\lambda \]
\[ \leq \frac{C \log n n^{2\delta}}{n} \]
\[ + \frac{C \log n n^\delta}{n} \int_{[-\pi,\pi]^3} |\lambda_1\lambda_2\lambda_3|^{-1} L_n(\lambda_1 - \lambda_2) L_n(\lambda_2 - \lambda_3) L_n(\lambda_3 - \lambda_1)d\lambda \]
\[ \leq \frac{C(\log n)^2}{n^{1-2\delta}}. \]

If \( \delta \geq 4(d - d_0) \), we have used inequality (C.6) together with inequality (C.5). This implies, together with (3.8) that

\[ \frac{1}{n} \text{tr} \left( T_n(g_0(f - f_0))T_n(g(f - f_0)) \right) \leq C \epsilon_n^2. \]

To finally obtain (3.5), we use equation (C.3) in Lemma 3 which implies that

\[ A_n = \text{tr} \left( T_n(g_0(f - f_0))T_n(g(f - f_0)) - \text{tr} \left( T_n(g_0g(f - f_0)^2) \right) \right) \]
\[ \leq Cn^{-1+\delta} + \log n \sum_{l=0}^{K_n} l|\theta_l| \left( \int_{[-\pi,\pi]} g_0g(f - f_0)^2(\lambda)d\lambda \right)^{1/2}. \]

Moreover

\[ \sum_{l=1}^{K_n} l|\theta_l| \leq \sum_{l=1}^{K_n} l^{2\beta + r} + \sum_{l=1}^{K_n} l^{-r/(2\beta - 1)} \]
\[ \leq CK_n^\beta + K_n^{-r/(2\beta - 1)}, \]

by choosing \( r = (2\beta - 1)/2\beta \), \( A_n/n \) is of order \( n^{-4\beta^2/2\beta} \) which is negligible compared to \( n^{-2\beta/(2\beta + 2)} \) so that if \( \beta \geq 1/2 \)

\[ \int_{[-\pi,\pi]} g_0g(f_0 - f)^2d\lambda \leq \epsilon_n^2, \]

which achieves the proof.

4. Discussion

In this paper we have considered the theoretical properties of our Bayesian procedure. A related and important problem, which deserves the same attention,
is its practical implementation. Due to the length of the present paper, we discuss these issues elsewhere; see for example (20); here we only sketch the main features of the proposed algorithm.

From a computational perspective, the practical implementation of a non-parametric Bayesian analysis based on a FEXP prior and a Gaussian likelihood, is plagued by two difficulties: i) the number of parameters to estimate varies with $K$ the number of terms in the FEXP expansion; ii) the likelihood function is quite expensive to evaluate, due to the Toeplitz structure of the covariance matrix.

After trying several approaches we finally recommend the use of the $D$-kernel Population MonteCarlo algorithm, presented and discussed in (9), and which can be easily adapted to the varying dimension set-up. For the evaluation of the inverse and of the determinant of the Toeplitz covariance matrix, we have used the algorithms proposed in (6).

**Appendix A: Lemmas 1 and 2**

We state two technical lemmas, which are extensions of (18) on uniform convergence of traces of Toeplitz matrices, and which are repeatedly used in the paper.

**Lemma 1.** Let $t > 0$, $M > 0$ and $\tilde{M}$ a positive function on $]0, \pi[$, let $p$ be a positive integer, and

$$\tilde{\mathcal{F}}(d, M, \tilde{M}) = \left\{ f \in \hat{F}, \forall u > 0, \sup_{|\lambda| > u} \frac{d\tilde{f}(\lambda)}{d\lambda} \leq \tilde{M}(u) \right\},$$

we have:

$$\sup_{p(d_1 + d_2) \leq 1/2 - t, f_i \in \tilde{\mathcal{F}}(d_1, M, \tilde{M}), g_i \in \tilde{\mathcal{F}}(d_2, M, \tilde{M})} \left| \frac{1}{n} \text{tr} \left( \prod_{i=1}^{p} T_n(f_i) T_n(g_i) \right) - (2\pi)^{2p-1} \int_{-\pi}^{\pi} \prod_{i=1}^{p} f_i(\lambda) g_i(\lambda) d\lambda \right| \to 0 \quad (A.1)$$

and let $L > 0$ and $\rho \in (0, 1]$.

$$\sup_{p(d_1 + d_2) \leq 1/2 - t, f_i \in F(d_1, M, L, \rho), g_i \in F(d_2, M, L, \rho)} \left| \frac{1}{n} \text{tr} \left( \prod_{i=1}^{p} T_n(f_i) T_n(g_i) \right) - (2\pi)^{2p-1} \int_{-\pi}^{\pi} \prod_{i=1}^{p} f_i(\lambda) g_i(\lambda) d\lambda \right| \to 0 \quad (A.2)$$
This lemma is an obvious adaptation from (18), and the only non-obvious part is the change from the condition of continuous differentiability in that paper to the Lipschitz condition of order $\rho$, considered equation A.2. This different assumption affects only equation (30) of (18), with $\eta_n$ replaced by $\eta_n^\rho$, which does not change the convergence results.

**Lemma 2.**

\[
\sup_{2p(d_1-d_2)\leq \rho_2 \wedge 1/2-t} \frac{1}{n} \left| \frac{1}{n} \text{tr} \left( \prod_{i=1}^{p} T_n(f_i)T_n(g_i)^{-1} \right) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{i=1}^{p} \frac{f_i(\lambda)}{g_i(\lambda)} d\lambda \right| \to 0,
\]

\[
\sup_{2p(d_1-d_2)\leq \rho_2 \wedge 1/2-t} \frac{1}{n} \left| \frac{1}{n} \text{tr} \left( \prod_{i=1}^{p} T_n(f_i)T_n(g_i)^{-1} \right) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{i=1}^{p} \frac{f_i(\lambda)}{g_i(\lambda)} d\lambda \right| \to 0.
\]

and

\[
\sup_{2p(d_1-d_2)\leq 1/2-t} \frac{1}{n} \left| \frac{1}{n} \text{tr} \left( \prod_{i=1}^{p} T_n(f_i)T_n(g_i)^{-1} \right) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{i=1}^{p} \frac{f_i(\lambda)}{g_i(\lambda)} d\lambda \right| \to 0.
\]

**Proof.** In this second lemma, the uniformity result is a consequence of the first lemma, as in (18); The only difference is in the proof of Lemma 5.2. of (8), i.e. in the study of terms in the form

\[
|\text{id} - T_n(g)^{1/2}T_n((4\pi^2 g)^{-1})T_n(g)^{1/2}|.
\]

Following Dahlhaus’s (8) proof, we obtain an upper bound of

\[
\left| \frac{g(\lambda_1)}{g(\lambda_2)} - 1 \right|
\]

which is different from (8). If $g \in G(d_2, m, M, \rho_2)$, the Lipschitz condition in $\rho$ implies that

\[
\left| \frac{g(x)}{g(y)} - 1 \right| \leq K \left( |x - y|^\rho + \frac{|x - y|^{1-\delta}}{|x|^{1-\delta}} \right).
\]

Calculations using $L_N$ as in (8) imply that

\[
|I - T_n(f)^{1/2}T_n((4\pi^2 f)^{-1})T_n(f)^{1/2}|^2 = O(n^{1-2\rho} \log n) + O(n^\delta), \quad \forall \delta > 0.
\]
If $g \in L^\star(M, m, L)$ as defined in Section 3.2, then
\[ \left| \frac{f(x)}{f(y)} - 1 \right| \leq K \left( \frac{|x - y|^{1 - 3\delta}}{(|x| \wedge |y|)^{1 - \delta}} \right) \leq K |x - y|^{1 - 3\delta} \left( \frac{1}{|x|^{1 - \delta}} + \frac{1}{|y|^{1 - \delta}} \right) \]
and (8) Lemma 5.2 is proved, leading to a constraint in the form $4p(d_1 - d_2) < 1$ (corresponding to $\rho = 1$).
Then, using again Dahlhaus’ (1989) calculations, we obtain that
\[ |A - B| = 0(n^{2(d_2 - d_1)}n^{1/2 - (\rho \wedge 1/2) + \delta}), \quad \forall \delta > 0 \]
and finally that
\[ \frac{1}{n} \text{tr} \left( \prod_{j=1}^{p} A_j - \prod_{j=1}^{p} B_j \right) = \sum_{k=1}^{p} O(n^{-1/2}n^{2(p-k)(d_2 - d_1)}n^{2(d_2 - d_1)n^{1/2} - \rho}) \]
\[ = \sum_{k=1}^{p} O(n^{2(p-k+1)(d_2 - d_1) - (\rho \wedge 1/2)}) \]
which goes to 0 when $2p(d_2 - d_1) < \rho \wedge 1/2$.

Appendix B: Proof of Theorem 2.1

Before giving the proof of Theorem 2.1, we give a few notations that are used throughout the paper: Let $f, f_1$ be spectral densities:

- $A(f_1, f) = T_n(f)^{-1}T_n(f_1)$
- $B(f_1, f) = T_n(f_1)^{1/2}[T_n(f)^{-1} - T_n(f_1)^{-1}]T_n(f_1)^{1/2}$
- $b_n(f_1, f) = \text{tr} \ (\text{id} - T_n(f_1)T_n(f)^{-1})^2 / n$
- $b(f_1, f) = (2\pi)^{-1} \int_{\pi}^{\pi} (f_1 / f - 1)^2(x)dx$.

Proof of Theorem 2.1

The proof follows the same ideas as in (13). The main difficulty here is to transform constraints on quantities such as $b_n(f, f_0)$ or $KL_n(f, f_0)$ in terms of distances between $f, f_0$ independent on $n$, uniformly over $f$.

We can write
\[ P^\pi[A_n^c | X_n] = \frac{\int_{A_n^c} \varphi_f(X_n) / \varphi_{f_0}(X_n) d\pi(f)}{\int_{\mathbb{R}} \varphi_f(X) / \varphi_{f_0}(X) d\pi(f)} = \frac{N_n}{D_n}. \]
Then the idea is to bound from below the denominator using condition (i) of the Theorem and to bound from above the numerator using a discretization of $A_n$ based on the net $\mathcal{H}_n$ defined in (ii) of the Theorem and on tests.
Let $\varepsilon > \delta > 0$: one has
\[
P_0 \left[ P^\pi \left[ A_\varepsilon^n | X_n \right] \geq e^{-n\delta} \right] 
\leq P_0^n \left[ D_n \leq e^{-n\delta} \right] + P_0^n \left[ N_n \geq e^{-n2\delta} \right]
= p_1 + p_2
\]  
(B.1)

Also, let
\[
\tilde{B}_n(c) = \{ f \in \bar{G}(t, M, m, L, \rho) : nKL_n(f_0, f) \leq nc \}, \quad c > 0.
\]

Using Lemma 2, when $n$ is large enough,\[
\tilde{B}_n(\delta/2) \supset \{ f \in \bar{G}(t, M, m, L, \rho) ; h(f_0; f) \leq \delta/4, \delta(d_0 - d) \leq \rho \wedge 1/2 - t \} = B_{\delta/4}
\]  
so that assumption (i) implies that, for $n$ large enough,
\[
\pi(\tilde{B}_n(\delta/2)) \geq \pi(B_{\delta/4}) \geq e^{-n\delta/2}/2.
\]

Define
\[
\Omega_n = \{ (f, X) : -X^t[T_n(f)^{-1} - T_n(f_0)^{-1}]X + \log (\det(A(f_0, f))) > -n\delta \}.
\]

We then have
\[
p_1 \leq P_0^n \left( \int_{\Omega_n \cap \tilde{B}_n} \frac{\varphi_f(X)}{\varphi_{f_0}(X)} d\pi(f) \leq e^{-n\delta/2} \pi(\tilde{B}_n) \right)
\leq P_0^n \left( \pi(\tilde{B}_n \cap \Omega_n) \leq \frac{\pi(\tilde{B}_n)}{2} \right)
\leq P_0^n \left( \pi(\tilde{B}_n \cap \Omega_n^c) > \frac{\pi(\tilde{B}_n)}{2} \right)
\leq 2 \int_{\tilde{B}_n} P_0^n[\Omega_n^c] d\pi(f) \frac{\pi(\tilde{B}_n)}{\pi(\tilde{B}_n)}
\]

Moreover,
\[
P_0^n[\Omega_n^c] = P_0^n(\Sigma_n T_n(f)^{-1} - T_n(f_0)^{-1})X_n - \log (\det(A(f_0, f))) > n\delta)
= Pr[y^tB(f_0, f)y - \text{tr}(B(f_0, f))]
> n\delta + \log (\det(A(f_0, f))) - \text{tr}(B(f_0, f))\]

where $y \sim N_n(0, \text{id})$. When $f \in \tilde{B}_n$, $n\delta + \log (\det(A(f_0, f))) - \text{tr}(B(f_0, f)) > n\delta/2$.
so that
\[
\begin{align*}
P_n^0[\Omega^c_n] &\leq Pr[y'B(f_0, f)y - \text{tr}(B(f_0, f)) > n\delta/2] \\
&\leq 4E[(y'B(f_0, f)y - \text{tr}(B(f_0, f)))^4]/n^4\delta^4 \\
&\leq \text{tr}(B(f_0, f)^4)C/n^4\delta^4.
\end{align*}
\]

Therefore, for all \( f \in \tilde{\mathcal{B}}_n \),
\[
P_n^0[\Omega^c_n] \leq M'C/n^4,
\]
and
\[
p_1 \leq C_1/n^3,
\]
where \( C_1 \) is a positive constant.

We now consider the second term of (B.1), namely:
\[
p_2 = P_n^0[N_n \geq e^{-2n\delta}] \\
\leq 2e^{2n\delta}P_n[\varphi_f(X_n) > \gamma] + P_n[\varphi_f(X_n) \geq e^{-2n\delta}/2] \\
\leq e^{-n(r-2\delta)} + \tilde{p}_2,
\]
take \( 2\delta < r \) and consider \( \tilde{p}_2 \). Consider the following tests: let \( f_i \in \mathcal{H}_n \),
\[
\phi_i = \mathbb{1}_{X'(T_n^{-1}(f_0) - T_n^{-1}(f_i)) \geq n\rho_i}.
\]

Recall that \( \gamma = \rho \land 1/2 - t \) (or \( \rho \land 1/2 - t, \rho_0 \land 1/2 - t, 1/2 - t \) depending on whether the spectral densities belong to \( \mathcal{G} \) or \( \tilde{\mathcal{L}} \)). We now prove that \( E_n^n[\phi_i] \leq e^{-n\varepsilon|\log \varepsilon|^{-1}} \) and \( E_n^n[1 - \phi_i] \leq e^{-n\varepsilon|\log \varepsilon|^{-1}} \) for \( f \) close to \( f_i \).

1. If \( |d_0 - d_i| \leq \gamma \), set \( \rho_i = \text{tr}(\text{id} - T_n(f_0)T_n^{-1}(f_i))/n + h_n(f_0, f_i) \), then for all \( 1/4 > s > 0, \)
\[
E_n^n[\phi_i] \leq \exp\{-sn\rho_i\} E_n^n[\exp\{-\frac{1}{2}\log \det[\text{id} + 2sB(f_0, f_i)]\}] \\
= \exp\{-sn\rho_i\} \exp\{-\frac{1}{2}\log \det[\text{id} + 2sB(f_0, f_i)]\} \\
\leq \exp\{-sn\rho_i - s\text{tr}(B(f_0, f_i)) + s^2 \text{tr}((\text{id} + s\tau B(f_0, f_i)^{-1}B(f_0, f_i)^2)\} \\
\leq \exp\{-sn\rho_i - s\text{tr}(B(f_0, f_i)) + 4s^2 \text{tr}(B(f_0, f_i)^2)\},
\]
where \( \tau \in (0, 1) \) and the latter inequality is due to

\[
\mathbf{id} + s\tau B(f_0, f_i) = \mathbf{id}(1 - 2s\tau) + 2s\tau A(f_0, f_i) \geq \frac{1}{2} \mathbf{id}, \quad \text{if } s < 1/4.
\]

Replacing \( \rho_i \) by its above expression and choosing \( s \) to optimize the latter expression, we obtain:

\[
E_0^n[\phi_i] \leq \max \left( \exp \left\{ -n \frac{h_n(f_0, f_i)^2}{16b_n(f_0, f_i)} \right\}, \exp \left\{ -n \frac{h_n(f_0, f_i)}{8} \right\} \right). \tag{B.3}
\]

Uniformly on the support of \( \pi \),

\[
\lim_{n \to \infty} b_n(f_0, f_i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - f_0/f_i)^2(x) dx = b(f_0, f_i),
\]

\[
\lim_{n \to \infty} h_n(f_0, f_i) = h(f_0, f_i).
\]

Therefore, for any \( a > 0 \), if \( n \) is large enough

\[
E_0^n[\phi_i] \leq \max \left( \exp \left\{ -n \frac{h(f_0, f_i)^2 - a}{16(b(f_0, f_i) + a)} \right\}, \exp \left\{ -n \frac{h(f_0, f_i) + a}{8} \right\} \right),
\]

choosing \( a < \varepsilon^2/2 \), since \( f_i \in A^\varepsilon \), we obtain

\[
E_0^n[\phi_i] \leq \max \left( \exp \left\{ -n \frac{h(f_0, f_i)^2}{32(b(f_0, f_i) + \varepsilon^2/2)} \right\}, \exp \left\{ -n \frac{h(f_0, f_i)}{16} \right\} \right). \tag{B.4}
\]

Lemma 4 implies that if \( \varepsilon > 0 \) is small enough, there exists a constant \( C_1 \) such that

\[
E_0^n[\phi_i] \leq \exp \left( -nC_1\varepsilon \right) \left\lvert \log \left( \varepsilon \right) \right\rvert^{-1}.
\]

Moreover, if \( f \) is in the support of \( \Pi \) and satisfies \( f \leq f_i \), and \( 4|d_0 - d| \leq \gamma \), using the same kind of calculations as in the case of \( E_0^n[\phi_i] \) and the fact that

\[
\mathbf{id} - 2sT_{n}^{1/2}(f)(T_{n}^{-1}(f_i) - T_{n}^{-1}(f_0))T_{n}^{1/2} \geq \mathbf{id} + 2sB(f, f_0),
\]

we obtain if \( 0 < s < 1/4, \)

\[
E_f^n[1 - \phi_i] \leq e^{ns\rho_i} \exp\left\{ -s \text{ tr } (B(f, f_0)) + 4s^2 \text{ tr } (B(f, f_0)^2) \right\} \leq \exp\left\{ -ns\rho_i(f_0, f_i) + 4s^2 \text{ tr } (B(f, f_0)^2) + \text{ tr } (A(f_i - f, f_0)) \right\}.
\]

Using Lemma 2 if \( (2\pi)^{-1} \int (f_i - f)f_0^{-1}(x) dx \leq h(f_0, f_i)/4 \), when \( n \) is large enough (uniformly in \( f \)) \( \text{ tr } (A(f_i - f, f_0)) \leq nh_n(f_0, f_i)/2 \) and

\[
E_f^n[1 - \phi_i] \leq \max \left( e^{-n \frac{h(f_0, f_i)^2}{32b(f_0, f_i)}}, e^{-n \frac{h(f_0, f_i)}{4}} \right). \tag{B.5}
\]
Again Lemma 4 implies that if $\varepsilon > 0$ is small enough, there exists a constant $C_1 > 0$ such that
\[
E_f^n [1 - \phi_i] \leq e^{-nC_1\varepsilon \log(\varepsilon)^{-1}}.
\]

2. If $4(d_i - d_0) > \gamma$. Set $\rho_1 = \text{tr} \left( \text{id} - T_n(f_0)T_n^{-1}(f_i) \right) / n + KL_n(f_0; f_i)$, the upper bound of $E_0^n[\phi_i]$ is computed similarly to (B.4) so that
\[
E_0^n[\phi_i] \leq \max \left( \exp \left\{ -nKL_n(f_0, f_i)^2 / 8b_n(f_0, f_i) \right\}, \exp \left\{ -nKL_n(f_0, f_i) / 4 \right\} \right).
\]

Now, using the same calculations as in Dahlhaus (1989, p. 1754), there exists a constant $C \geq 1$ such that
\[
KL_n(f_0, f_i) \geq \frac{b_n(f_0, f_i)}{C}
\]
so that, for large $n$ (independently of $f_i$),
\[
KL_n(f_0, f_i) \geq \frac{b(f_0, f_i)}{2C}
\]
We finally obtain that there exists a constant $c > 0$ such that
\[
E_0^n[\phi_i] \leq \exp \left\{ -ncb(f_0, f_i) \right\}. \tag{B.6}
\]

Since $b(f_0, f_i)$ is uniformly bounded from below on the set
\[
\{ f \in \mathcal{G}(t, m, M, L, \rho); d \geq d_0 + 1/4(\rho \wedge \rho_0 \wedge 1/2) - t/4 \}
\]
(or $\mathcal{L}(t, m, M, L)$), if $\varepsilon$ is small enough
\[
E_0^n[\phi_i] \leq \exp \left\{ -n\varepsilon \right\}.
\]

Consider $f \leq f_i$, such that $4(d_i - d) \leq \rho \wedge 1/2 - t$. Similarly to before, let $h \in (0, 1)$:
\[
E_f^n [1 - \phi_i] \leq e^{(1-h)n\rho_i / 2 - \frac{1}{2} \log \det [\text{id} - (1-h)T_n^{-1/2}(f_i)](T_n^{-1}(f_i) - T_n^{-1}(f_0))T_n^{-1/2}(f_i)}
\]
\[
\leq e^{(1-h)n\rho_i / 2 - \frac{1}{2} \log \det [\text{id} - (1-h)B(f, f_0)]}
\]
\[
= e^{(1-h)n\rho_i / 2 - \frac{1}{2} \log \det [\text{id} - T_n^{1/2}(f_i)(T_n^{-1}(f_i) - T_n^{-1}(f_0))T_n^{1/2}(f_i)]}
\]
then using the same kind of expansions as in (B.5), we obtain
\[
E_f^n [1 - \phi_i] \leq \text{det}[A(f_i, f)]^{1/2}
\]
\[
\times \max \left\{ \exp \left( -nKL_n(f_0, f_i)^2 / 32 \text{tr} (B(f_0, f)^2) / n \right), \exp \left( -nKL_n(f_0, f_i) / 8 \right) \right\}.
\]
Since $\log \det [A(f, f_i)] = - \log \det [\mathbf{id} + T_n(f_i - f)T_n(f)^{-1}]$, using a Taylor expansion of $\log \det$ around $\mathbf{id}$, we obtain that for $n$ large enough

$$- \log \det [A(f, f_i)] \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} (f_i - f) / f(x) dx + a$$

where $a$ can be chosen as small as necessary. Also

$$\frac{KL_n(f_0, f_i)^2}{32 \text{ tr } (B(f_0, f)^2)/n} \geq \frac{cb(f_0, f_i)^2 - a}{b(f_0, f) + a}.$$ 

Since

$$b(f_0, f) \leq 2b(f_0, f_i) + \frac{1}{\pi} \left( \int_{-\pi}^{\pi} f_0^2 (f^{-1} - f_i^{-1})^2(x) dx \right),$$

if

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f_i - f) / f(x) dx \leq cb(f_0, f_i)/4,$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f_0^2 (f^{-1} - f_i^{-1})^2(x) dx \leq b(f_0, f_i),$$

there exists a constant $c_1 > 0$ such that when $n$ is large enough

$$E_f^n [1 - \phi_i] \leq \exp \{-nc_1 b(f_0, f_i)\} \leq e^{-n\varepsilon}$$

for $\varepsilon$ small enough.

3. If $4(d_0 - d_i) > \gamma$. Set $\rho_0 = \log \det [T_n(f_i)T_n(f_0)^{-1}]/n$, then if $0 < h < 1$

$$E_0^n [\phi_i] \leq e^{-(1-h)n\rho_0}/2 - \log \det [A(f_0, f_i)]/2 - \log \det [\mathbf{id}(1-h) + hT_n^{-1/2}(f_0)T_n(f_i)T_n^{-1/2}(f_0)]$$

$$\leq e^{-nKL_n(f_i, f_0)h^2} \text{ tr } (B(f, f_0)^2) \leq e^{-n\varepsilon}$$

where the last inequality can be obtained by following the same lines as for (B.6).

Moreover, for all $f \leq f_i$, satisfying $4(d_i - d) \leq \rho \wedge 1/2 - t$, if

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f_i - f) / f(x) dx \leq cb(f_i, f_0)/4, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f_0^2 (f - f_i)^2(x) dx \leq b(f_i, f_0),$$

where the constant $c$ is defined such that, for $n$ large enough, for all $f$ in $\tilde{G}(t, m, L, \rho)$ (resp. $\tilde{L}(t, m, M, L)$) such that $4(d_0 - d) > \gamma$, $KL_n(f_i, f_0) \geq cb(f_i, f_0)$ (see the calculations presented in the case $4(d - d_0) > \gamma$),

$$E_f^n [1 - \phi_i] \leq \exp \{-2snKL_n(f_i, f_0) - \text{ tr } (A(f_i - f, f)) / n\} + 4s^2 nb_n(f, f_0)$$

$$\leq e^{-nc_1 b(f_i, f_0)} \leq e^{-n\varepsilon}$$

for $\varepsilon$ small enough.
Then, in each case, we have, for large $n$ (independently of $f_i$),

$$E^n_0 [\phi_i] \leq e^{-nc|\log \varepsilon|^{-1}}$$

for all $\varepsilon < \varepsilon_0$,

$$E^n_f [1 - \phi_i] \leq e^{-nc|\log \varepsilon|^{-1}}.$$ 

Let $\phi^{(n)} = \max_i \phi_i$; then, using Markov inequality,

$$\bar{p}_2 \leq E^n_0 [\phi^{(n)}] + 2e^{2n\delta} \int A_n \cap \mathcal{F}_n E_f [1 - \phi^{(n)}] d\pi(f) \leq e^{N_n} e^{-nc|\log \varepsilon|^{-1}} + 2e^{2n\delta} e^{-nc|\log \varepsilon|^{-1}} \leq e^{-nc|\log \varepsilon|^{-1}/2},$$

We finally obtain that for some $\delta > 0$, if $n$ is large enough

$$P^n_0 [P^n [A^c_\varepsilon \mid \mathcal{X}_n] > e^{-n\delta}] \leq \frac{C_0}{n^3}$$

for some positive constant $C_0$, so that $\pi[A^c_\varepsilon \mid \mathcal{X}_n] \rightarrow 0$ $P_0^\infty$ a.s.

**Appendix C: Lemma 3**

**Lemma 3.** Let $f_j$, $j \in \{1, 2\}$ be such that $f_j(\lambda) = |\lambda|^{-2d_j} \tilde{f}_j(\lambda)$, where $d_j < 1/2$ and $\tilde{f}_j \in \mathcal{S}(L, \beta)$, for some constant $L > 0$ and consider $b$ a bounded function on $[-\pi, \pi]$. Assume that $h_n(f_1, f_2) < \varepsilon$ where $\varepsilon > 0$. Then $\forall \delta > 0$, there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, there exists $C > 0$ such that

$$\frac{1}{n} \text{tr} (T_n(f_1)^{-1} T_n(f_1 b) T_n(f_2)^{-1} T_n(f_1 b)) \leq C (\log n)^3 [\|b\|_2^2 + \|b\|_\infty^2 n^{\delta - 1} + n^{-1/2 + \delta}],$$

(C.1)

$$\frac{1}{n} \text{tr} (T_n(f_1^{-1}) T_n(f_1 - f_2) T_n(f_2^{-1}) T_n(f_1 - f_2)) \leq Ch_n(f_1, f_2).$$

(C.2)

Let $g_j = (1 - \cos \lambda)^{d_j}$ and $f_j = g_j^{-1} \tilde{f}_j$, where $\tilde{f}_1 \in \mathcal{S}(L, \beta) \cap \mathcal{L}$ and $\tilde{f}_2 \in \mathcal{S}(L, \beta)$, written in the form $\log \tilde{f}_2(\lambda) = \sum_{l=0}^K \theta_l \cos(l\lambda)$; then

$$\left\| \frac{1}{n} \text{tr} (T_n(g_1(f_1 - f_2)) T_n(g_2(f_1 - f_2))) - \text{tr} (T_n(g_1 g_2(f_1 - f_2)^2)) \right\| \leq C n^{-1+\delta} + n^{-1} \log n \sum_{l=0}^K |\theta_l| \left( \int_{[-\pi, \pi]} g_1 g_2(f_1 - f_2)^2(\lambda) d\lambda \right)^{1/2},$$

(C.3)

for any $\delta > 0$. 
Proof. Throughout the proof $C$ denotes a generic constant. We first prove (C.1).
To do so, we obtain an upper bound on another quantity, namely
\[
\gamma(b) = \frac{1}{n} \text{tr} \left( T_n(f_1^{-1})T_n(f_1 b)T_n(f_2^{-1})T_n(f_1 b) \right). \tag{C.4}
\]
First note that $b$ can be replaced by $|b|$ so that we can assume that it is positive. Let $\Delta_n(\lambda) = \sum_{j=1}^{n} \exp(-i j \lambda)$ and $L_n$ be the $2\pi$-periodic function defined by $L_n(\lambda) = n$ if $|\lambda| \leq 1/n$ and $L_n(\lambda) = |\lambda|^{-1}$ if $1/n \leq |\lambda| \leq \pi$. Then $|\Delta_n(\lambda)| \leq C L_n(\lambda)$ and we can express traces of products of Toeplitz matrices in the following way. Let the symbol $d\lambda$ denote the quantity $d\lambda_1 d\lambda_2 d\lambda_3 d\lambda_4$:
\[
\gamma(b) = \frac{C}{n} \int_{[-\pi,\pi]^4} \frac{f_1(\lambda_1) b(\lambda_1) f_1(\lambda_3) b(\lambda_3)}{f_0(\lambda_2) f_0(\lambda_4)} \times
\Delta_n(\lambda_1 - \lambda_2) \Delta_n(\lambda_2 - \lambda_3) \Delta_n(\lambda_3 - \lambda_4) \Delta_n(\lambda_4 - \lambda_1) d\lambda
\leq \frac{C}{n} \int_{[-\pi,\pi]^4} \frac{f_1(\lambda_1) f_1(\lambda_3) b(\lambda_1)}{f_0(\lambda_2) f_0(\lambda_4)} (b(\lambda_3) - b(\lambda_1)) \times
\Delta_n(\lambda_1 - \lambda_2) \Delta_n(\lambda_2 - \lambda_3) \Delta_n(\lambda_3 - \lambda_4) \Delta_n(\lambda_4 - \lambda_1) d\lambda
\]
On the set $b(\lambda_1) > b(\lambda_3)$, $0 < b(\lambda_1) - b(\lambda_3) < b(\lambda_1)$ and on the set $b(\lambda_3) > b(\lambda_1)$, $0 < b(\lambda_3) - b(\lambda_1) < b(\lambda_3)$, therefore the second term of the r.h.s. of the above inequality is bounded by (in absolute value)
\[
\gamma(b) \leq \frac{C}{n} \int_{[-\pi,\pi]^4} \frac{f_1(\lambda_1) f_1(\lambda_3) b(\lambda_1)^2}{f_1(\lambda_2) f_2(\lambda_4)} L_n(\lambda_1 - \lambda_2) L_n(\lambda_2 - \lambda_3) \times
L_n(\lambda_3 - \lambda_4) d\lambda
\leq \frac{C}{n} \int_{[-\pi,\pi]^4} b(\lambda_1)^2 \frac{\lambda_1^{-2d_1} \lambda_3^{-2d_1}}{\lambda_2^{-2d_1} \lambda_4^{-2d_1}} L_n(\lambda_1 - \lambda_2) L_n(\lambda_2 - \lambda_3) \times
L_n(\lambda_3 - \lambda_4) d\lambda
\]
Note that
\[
\int_{[-\pi,\pi]} L_n(\lambda_1 - \lambda_2) L_n(\lambda_2 - \lambda_3) d\lambda \leq C \log n L_n(\lambda_1 - \lambda_3), \tag{C.5}
\]
therefore

$$\gamma(b) \leq \frac{C(\log n)^3}{n} \int_{[-\pi, \pi]} b(\lambda)^2 d\lambda$$

$$+ C \int_{[-\pi, \pi]^4} b(\lambda_1)^2 |\lambda_1|^{-2(d_1-d_2)} \left( \frac{|\lambda_3|^{-2d_1}}{|\lambda_2|^{-2d_1}} - 1 \right) \left( \frac{|\lambda_1|^{-2d_2}}{|\lambda_4|^{-2d_2}} - 1 \right)$$

$$\times L_n(\lambda_1 - \lambda_2)L_n(\lambda_2 - \lambda_3)L_n(\lambda_3 - \lambda_4)L_n(\lambda_4 - \lambda_1) d\lambda$$

$$+ 2C \int_{[-\pi, \pi]^4} b(\lambda_1)^2 |\lambda_1|^{-2(d_1-d_2)} \left( \frac{|\lambda_3|^{-2d_1}}{|\lambda_2|^{-2d_1}} + \frac{|\lambda_1|^{-2d_2}}{|\lambda_4|^{-2d_2}} - 2 \right)$$

$$\times L_n(\lambda_1 - \lambda_2)L_n(\lambda_2 - \lambda_3)L_n(\lambda_3 - \lambda_4)L_n(\lambda_4 - \lambda_1) d\lambda$$

Since

$$\frac{|\lambda_1|^{-2d_1}}{|\lambda_2|^{-2d_1}} - 1 \leq C \frac{|\lambda_1 - \lambda_2|^{1-\delta}}{|\lambda_1|^{1-\delta}}, \text{ for } j = \{1, 2\},$$

using Dahlhaus’ (1989) calculations as in his proof of Lemma 5.2, we obtain that, if $d_1 - d_2 < \delta/4$,

$$\int_{[-\pi, \pi]^4} b(\lambda_1)^2 |\lambda_1|^{-2(d_1-d_2)} \left( \frac{|\lambda_3|^{-2d_1}}{|\lambda_2|^{-2d_1}} - 1 \right) \left( \frac{|\lambda_1|^{-2d_2}}{|\lambda_4|^{-2d_2}} - 1 \right)$$

$$\times L_n(\lambda_1 - \lambda_2)L_n(\lambda_2 - \lambda_3)L_n(\lambda_3 - \lambda_4)L_n(\lambda_4 - \lambda_1) d\lambda$$

$$\leq |b|^2 \int_{[-\pi, \pi]^4} \frac{L_n(\lambda_1 - \lambda_2)L_n(\lambda_2 - \lambda_3)^\delta L_n(\lambda_3 - \lambda_4)^\delta L_n(\lambda_4 - \lambda_1)^\delta}{|\lambda_1|^{1-\delta/2}|\lambda_4|^{1-\delta}} d\lambda$$

$$\leq Cn^{2\delta} |b|^2 \infty (\log n)^2,$$

as long as $|d_1 - d_2| < \delta/2$. By considering $h_\infty (f, f_0) < \epsilon$ with $\epsilon > 0$ small enough, we can impose that $|d_1 - d_2| < \delta/2$, and we finally obtain that

$$\gamma(b) \leq C |b|^2 (\log n)^3 + C |b|^2 \infty n^{2\delta-1}(\log n)^2.$$ (C.7)

We now prove that, for large $n$ and $\forall \delta > 0$,

$$\frac{1}{n} \text{tr} (T_n(f_1)T_n^{-1}(f_1 b)T_n(f_2)T_n^{-1}(f_1 b)) \leq C \frac{1}{n} \text{tr} (T_n(f_1)T_n(f_1 b)T_n(f_2)T_n(f_1 b)) + C n^{-1+\delta}.$$ (C.8)

Since $f_i(\lambda) \leq C|\lambda|^{-2d_1} \propto g_i(\lambda), \ i = 1, 2$. This implies that $T_n^{-1}(f_i) \geq C^{-1} T_n^{-1}(g_i)$ so that we can replace $T_n^{-1}(f_i)$ by $T_n^{-1}(g_i)$ in the above term. Then

$$\delta_n = \text{tr} (T_n(f_1 b)T_n^{-1}(g_1)T_n(f_1 b)T_n^{-1}(g_2))$$

$$= \text{tr} (T_n(f_1 b)T_n(g_1^{-1}T_n^{-1}(g_1)T_n(g_2^{-1}T_n^{-1}(g_2)))$$

$$+ \text{tr} (T_n(f_1 b)T_n^{-1}(g_1)T_n(f_1 b)T_n^{-1/2}(g_2)T_n^{-1/2}(g_2))$$

$$+ \text{tr} (T_n(f_1 b)T_n(g_1^{-1/2}T_n^{-1/2}(g_1)T_n(f_1 b)T_n(g_2^{-1/2}T_n^{-1/2}(g_2))).$$
where \( R_i = T_n(g_i)^{1/2}T_n(g_i^{-1}/(4\pi^2))T_n(g_i)^{1/2} - \text{id}, \) \( i = 1, 2. \) Using (C.7) we obtain that
\[
\text{tr} \left( T_n(f_1b)T_n(g_1^{-1})T_n(f_1b)T_n(g_2^{-1}) \right) \leq Cn(\log n)^3|b_n|^2 + O(|b|_\infty n^\delta) = n\gamma.
\]
Moreover
\[
\left| \text{tr} \left( T_n(f_1b)T_n^{-1}(g_1)T_n(f_1b)T_n^{-1/2}(g_2)R_2T_n^{-1/2}(g_2) \right) \right| \\
\leq |R_2||T_n^{-1/2}(g_2)T_n(f_1b)T_n^{-1}(g_1)T_n(f_1b)T_n^{-1/2}(g_2)| \\
\leq \delta_n^{1/2}|R_2||T_n^{-1/2}(g_2)T_n(f_1b)^{1/2}||T_n(f_1b)^{1/2}T_n^{-1/2}(g_1)|
\]
Lemmas 5.2 and 5.3 in Dahlhaus (1989) lead to: \( \forall \delta > 0 \)
\[
\left| \text{tr} \left( T_n(f_1b)T_n^{-1}(g_1)T_n(f_1b)T_n^{-1/2}(g_2)R_2T_n^{-1/2}(g_2) \right) \right| \leq Cn^{\delta+2(d_0-d)}|b_n|\infty \delta_n^{1/2} \leq C\delta_n^{1/2}
\]
Similarly,
\[
\left| \text{tr} \left( T_n(f_1b)T_n(g_1)^{-1/2}R_1T_n(g_1)^{-1/2}T_n(f_1b)T_n(g_1^{-1}/(4\pi^2)) \right) \right| \\
\leq |R_1|\delta_n^{1/2}|R_1| + 1|T_n(g_1)^{-1/2}T_n(f_1b)^{1/2}||T_n(f_1b)^{1/2}T_n(g_1)^{-1/2}|
\]
Since \( |T_n(g_1)^{1/2}T_n(g_1^{-1}/(4\pi^2))^{1/2}| \leq 1 + \|T_n(g_1)^{1/2}T_n(g_1^{-1}/(4\pi^2))^{1/2} - \text{id} \| \leq n^{\delta} \)
for all \( \delta > 0 \) and using Lemma 5.3 of Dahlhaus (1989)
\[
\left| \text{tr} \left( T_n(f_1b)T_n(g_1)^{-1/2}R_1T_n(g_1)^{-1/2}T_n(f_1b)T_n(g_2^{-1}/(4\pi^2)) \right) \right| \leq Cn^{\delta}\delta_n^{1/2}
\]
Finally we obtain for all \( \delta > 0 \), when \( n \) is large enough
\[
\delta_n/n \leq Cn^{-1/2+\delta}\sqrt{\delta_n/n} + \gamma/n \leq 2\gamma/n + 0(n^{-1+\delta}),
\]
and (C.1) is proved. We now prove (C.2). since \( f_j \geq m|\lambda|^{-2d_j} = g_j \) where
\( m = e^{-L}, T_n^{-1}(f_j) < T_n^{-1}(g_j) \), i.e. \( T_n^{-1}(g_j) - T_n^{-1}(f_j) \) is positive semidefinite, and
\[
h_n(f_1, f_2) = \frac{1}{2n} \text{tr} \left( T_n(f_1 - f_2)T_n^{-1}(f_2)T_n(f_1 - f_2)T_n^{-1}(f_1) \right) \\
\geq \frac{1}{2n} \text{tr} \left( T_n(f_1 - f_2)T_n^{-1}(f_2)T_n(f_1 - f_2)T_n^{-1}(g_1) \right) \\
\geq \frac{1}{2n} \text{tr} \left( T_n(f_1 - f_2)T_n^{-1}(f_2)T_n(f_1 - f_2)T_n^{-1/2}(g_1)R_1T_n^{-1/2}(g_1) \right)
\]
(C.8)
+ \frac{1}{2n} \text{tr} \left( T_n(f_1 - f_2)T_n^{-1}(g_2)T_n(f_1 - f_2)T_n \left( \frac{g_1^{-1}}{4\pi^2} \right) \right) \\
= \frac{1}{2n(16\pi^4)} \text{tr} \left( T_n(f_1 - f_2)T_n(g_2^{-1})T_n(f_1 - f_2)T_n \left( g_1^{-1} \right) \right) \\
+ \frac{1}{2n} \text{tr} \left( T_n(f_1 - f_2)T_n^{-1}(f_2)T_n(f_1 - f_2)T_n^{-1/2}(g_1)R_1T_n^{-1/2}(g_1) \right) \quad \text{(C.9)} \\
+ \frac{1}{2n(4\pi^2)} \text{tr} \left( T_n(f_1 - f_2)T_n^{-1/2}(g_2)R_2T_n^{-1/2}(g_2)T_n(f_1 - f_2)T_n \left( g_1^{-1} \right) \right)

\text{where } R_j = \text{id} - T_n^{1/2}(g_j)T_n(g_j^{-1}/(4\pi^2))T_n^{-1/2}(g_j). \text{ We first bound the first term of the r.h.s. of (C.9). Let } \delta > 0 \text{ and } \epsilon < \epsilon_0 \text{ such that } |d - d_0| \leq \delta \text{ (Corollary 1 implies that there exists such a value } \epsilon_0). \text{ Then using Lemmas 5.2 and 5.3 of (8)}

\left| \text{tr} \left( T_n(f_1 - f_2)T_n^{-1}(f_2)T_n(f_1 - f_2)T_n^{-1/2}(g_1)R_1T_n^{-1/2}(g_1) \right) \right| \\
\leq 2|R_1|T_n^{-1/2}(g_1)T_n(f_1-f_2)T_n^{-1/2}(f_2)||T_n(|f_1-f_2|)^{1/2}T_n^{-1/2}(f_2)|| \\
\times |T_n(|f_1-f_2|)^{1/2}T_n^{-1/2}(g_1)| \\
\leq Cn^{\delta}|T_n^{-1/2}(g_1)T_n(f_1-f_2)T_n^{-1/2}(f_2)|.

\text{Since } g_1 \leq Cf_1,

|T_n^{-1/2}(g_1)T_n(f_1-f_2)T_n^{-1/2}(f_2)|^2 = \text{tr} \left( T_n^{-1}(g_1)T_n(f_1-f_2)T_n^{-1}(f_2)T_n(f_1-f_2) \right) \\
\leq C \text{tr} \left( T_n^{-1}(f_1)T_n(f_1-f_2)T_n^{-1}(f_2)T_n(f_1-f_2) \right) \\
= Cnh_n(f_1, f_2),

\text{and}

\frac{1}{n} \left| \text{tr} \left( T_n(f_1-f_2)T_n^{-1}(f_2)T_n(f_1-f_2)T_n^{-1/2}(g_1)R_1T_n^{-1/2}(g_1) \right) \right| \leq Cn^{2\delta-1/2}h_n(f_1, f_2).

\text{We now bound the second term of the r.h.s. of (C.9).}

\left| \frac{1}{n} \text{tr} \left( T_n(f_1-f_2)T_n^{-1/2}(g_2)R_2T_n^{-1/2}(g_2)T_n(f_1-f_2)T_n \left( g_1^{-1} \right) \right) \right| \\
\leq \frac{1}{n} |R_2||T_n^{-1/2}(g_2)T_n(f_1-f_2)T_n(g_1^{-1})||T_n(g_1)^{1/2}T_n(g_1^{-1})T_n(|f_1-f_2|)T_n^{-1/2}(f_2)| \\
\leq Cn^{\delta} \sqrt{nh_n(f_2, f_1)} |T_n(g_1)^{1/2}T_n(g_1^{-1})T_n(|f_1-f_2|)T_n^{-1/2}(f_2)| \\
\leq Cn^{\delta+1/2} \sqrt{nh_n(f_2, f_1)} |T_n(g_1)^{1/2}T_n(g_1^{-1})|^{1/2} \times |T_n(g_1)^{-1/2}T_n(|f_1-f_2|)^{1/2}||T_n(|f_1-f_2|)^{1/2}T_n^{-1/2}(f_2)|| \\
\leq Cn^{3\delta-1/2}h_n(f_1, f_2).
The first 2 terms of the right hand side are of order $O(n^{24} \log n)$. We now study the last term, here the problem is due to the fact that $\tilde{f}_2$ does not necessarily belong to $\mathcal{L}$. We have:

$$\int_{[-\pi,\pi]^2} g_1 g_2 (f_1 - f_2)(\lambda_1)[f_2(\lambda_2) - f_2(\lambda_1)] \Delta_n(\lambda_1 - \lambda_2) \Delta_n(\lambda_2 - \lambda_1) d\lambda$$

$$= \int_{[-\pi,\pi]^2} g_1 g_2 (f_1 - f_2)(\lambda_1)[\tilde{f}_2(\lambda_2) - g_2^{-1}(\lambda_2)][g_2^{-1}(\lambda_1) - g_2^{-1}(\lambda_1)] \Delta_n(\lambda_1 - \lambda_2) \Delta_n(\lambda_2 - \lambda_1) d\lambda$$

$$+ \int_{[-\pi,\pi]^2} g_1 (f_1 - f_2)(\lambda_1)[\tilde{f}_2(\lambda_2) - \tilde{f}_2(\lambda_1)] \Delta_n(\lambda_1 - \lambda_2) \Delta_n(\lambda_2 - \lambda_1) d\lambda.$$

The first term of the above inequality is of order $O(n^{28} \log n)$ because $g_2$ belongs to $\mathcal{L}$. Since

$$\tilde{f}(\lambda) = \exp \left( \sum_{t=0}^{K_n} \theta_t \cos(t\lambda) \right),$$

since $|T_n(f_1)^{1/2}T_n(f_1^{-1})T_n(f_1)^{1/2}| \leq |\text{id}| + |T_n(f_1)^{1/2}T_n(f_1^{-1})T_n(f_1)^{1/2} - \text{id}| \leq Cn^6$.
one gets
\[
I = \int_{[-\pi, \pi]^2} g_1(f_1 - f_2)(\lambda_1)[\tilde{f}_2(\lambda_2) - \tilde{f}_2(\lambda_1)]\Delta_n(\lambda_1 - \lambda_2)\Delta_n(\lambda_2 - \lambda_1)d\lambda
\]
\[
\leq C \int_{[-\pi, \pi]^2} g_1|f_1 - f_2|(\lambda_1)\left|\sum_{j=0}^{K_n} \theta_i(\cos(j\lambda_2) - \cos(j\lambda_1))\right| L_n(\lambda_1 - \lambda_2)L_n(\lambda_2 - \lambda_1)d\lambda
\]
\[
\leq C \log n \left(\sum_{l=0}^{K_n} \theta_i|l|\right) \int_{[-\pi, \pi]} g_1|f_1 - f_2|(\lambda)d\lambda
\]
\[
\leq C \log n \sum_{l=0}^{K_n} \theta_i|l| \left(\int_{[-\pi, \pi]} g_1 g_2(f_1 - f_2)^2(\lambda)d\lambda\right)^{1/2},
\]
where the latter inequality holds because \(\int g_1/g_2(\lambda)d\lambda\) can be proved to be bounded by an application of an application of Hölder inequality. \(\square\)

**Appendix D: Relations between \(b(f_0, f)\) and \(h(f_0, f)\)**

**Lemma 4.** Let \(m, M, L > 0\) and \(\rho \leq 1\). There exists \(\tau > 0\) and \(C > 0\) such that for any \(f, f_0 \in \tilde{G}(t, m, M, \rho) \cup \tilde{L}(t, m, M, L)\), if \(h(f, f_0) < \tau\),
\[
b(f, f_0) \leq h(f, f_0) \log h(f, f_0).
\]

We need to bound \(b(f, f_0)\) in terms of \(h(f_0, f)\) when \(|d - d_0|\) is small. Assume that \(f_0 = |x|^{-2d_0} \tilde{f}_0\) and \(f = |x|^{-2d} \tilde{f}\) with \(d \geq d_0\) (otherwise the bound is straightforward) we have
\[
b(f, f_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f/f_0 - 1)^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(f - f_0)^2}{f_0^2} dx,
\]
\[
h(f, f_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f/f_0 - 1)^2 f_0 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(f - f_0)^2}{f f_0} dx.
\]
Let \(A > 0\) be large enough; using the fact that \(m \leq \tilde{f}, \tilde{f}_0 \leq M\) we obtain, if \(A > 2\), that
\[
b(f, f_0) \leq Ah(f_0, f) + \int_{f/f_0 > A} \frac{1}{2\pi} \int_{-\pi}^{\pi} (f/f_0 - 1)^2 dx
\]
\[
\leq Ah(f_0, f) + \int_{f/f_0 > A} \frac{2M^2}{2m^2\pi} |x|^{-4(d-d_0)} dx.
\]
Let $A > M/m$ then if $f/f_0 > A$, $|x|^{-2(d-d_0)} > Am/M$, so that

$$b(f, f_0) \leq Ah(f_0, f) + C \int_{|x|^{-2(d-d_0)} > KA} |x|^{-4(d-d_0)} \, dx$$

Now assume that $h(f_0, f) \leq \tau$ where $\tau > 0$ is fixed and small. Consider $t > 0$ small enough so that

$$h(f_0, f) \gtrsim \int_{x^{-2(d-d_0)} \geq t^{-1}} x^{-2(d-d_0)} \, dx$$

where $\gtrsim$ means that the inequality is up to a multiplicative constant whose value does not depend on $f$ and $f_0$ (but it does depend on $M$ and $m$). It implies that

$$\frac{1}{1 - 2(d - d_0)} t^{-1 + 1/2(d-d_0)} \leq Ch(f, f_0),$$

so that, if $t^{-1} = \log 1/h(f, f_0)$,

$$\log(\log 1/h(f, f_0)) \frac{1}{2(d - d_0)} \geq \log 1/(\rho h(f, f_0)), \quad \rho > 0 \text{ fixed}.$$ 

Hence, if $h(f, f_0)$ is small enough,

$$2(d - d_0) \leq \frac{2\log(\log 1/h(f, f_0))}{\log 1/h(f_0, f)}.$$

Now using the fact that there exists $C > 0$ such that

$$b(f, f_0) \leq Ah(f_0, f) + CA^{2-1/(2(d-d_0))}, \quad h(f_0, f) \geq C' \frac{A^{1-2(d-d_0)}}{1 - 2(d - d_0)}$$

and considering $A = \log 1/h(f, f_0)$ we finally obtain

$$b(f, f_0) \leq \log 1/h(f, f_0)h(f_0, f) + C'h(f, f_0) \log (1/h(f, f_0)).$$

Hence, there exists $\tau > 0$ (depending only on $m, M$ and $C > 0$) such that if $h(f, f_0) < \tau$, $b(f, f_0) \leq h(f, f_0) |\log h(f, f_0)|$.

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