An efficient numerical algorithm for solution of nonlinear delay differential equations

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Abstract. Aim of the paper is to obtain numerical solution of some nonlinear delay differential equations (NDDEs) using iterative scheme. Banach contraction method (BCM) is investigated on some nonlinear delay differential equations to find their solutions. Numerical results and convergence theorem are presented, and error analysis is discussed for some delay differential equations to show that proposed method is suitable for solving NDDEs. The BCM reduces complex calculations, avoids discretization, linearization, perturbation and save calculation time.

Keywords: Delay differential equations; Banach contraction method; Convergence; Error analysis; Mathematica software

1. Introduction
In mathematical formulation of some physical or dynamical processes performance depends not only on the present behavior but on past behavior as well, such systems are categorized as time delay systems. The physical phenomena are represented by delay differential equations (DDEs) with initial and boundary conditions. DDEs find applications in physiological diseases, in control system, in forest harvesting and regeneration, in feedback system control, in tumor growth and cancer therapies, in biochemical reactions etc. The delay terms can be natural, or manmade and not to study time delays involved in physical problems is just to avoid reality.

Many researchers developed and investigated different analytical and numerical methods to obtain solutions of DDEs. Recently various numerical methods have been used for finding solutions of delay differential equations [1, 2, 3, 4]. Authors [5, 6, 7, 8, 9] applied Variational iteration method (VIM) and modified VIM for different types of DDEs.

Biazer and Ghanbargi [10] examined Homotopy perturbation method (HPM) to obtain numerical solution of DDEs with proportional delays. Authors [11, 12, 13, 14] used HPM for different types of differential equations and integro-differential equation. Alomari et al. [15] investigated Homotopy analysis method (HAM) for solution of DDEs. Hu et al. [16] developed strong Milstein approximation scheme for solving numerical solutions for DDEs. The papers cited in [17, 18, 19, 20] used numerical techniques for solving DDEs. Differential transform method (DTM) is used for solving ODEs, PDEs and DDEs [21, 22, 23, 24, 25]. Rebenda et al. [26, 27] combined methods of steps and differential transform method to solve DDEs. Authors [28, 29, 30, 31] examined Adomian decomposition method (ADM) to solve DDEs. Authors [32, 33] discussed existence and uniqueness of the solutions of delay differential equations.
In the present paper, Banach contraction method is investigated on some delay differential equations. The present method is combined with Mathematica software to generate series solutions. The convergence theorems and error analysis are discussed. As per authors knowledge no researcher has used the present method for delay problems. This paper is presented as follows: in Section 2, idea of iterative method is discussed, in Section 3, convergence results are discussed, in Section 4, applications of present method on DDE’s and error analysis are presented. Finally, Conclusion is given in Section 5.

2. Idea of iterative method
To present the idea of iterative method, consider the following general functional equation

\[ A(u(t)) - g(t) = 0, \quad t \in \Omega \tag{1} \]

with boundary conditions

\[ B\left(u, \frac{du}{dt}\right) = 0, \quad t \in T \tag{2} \]

where \( A \) is a general functional operator, \( B \) is the boundary operator, \( g(t) \) is some known function and \( T \) is the boundary of the domain \( \Omega \). Here general operator \( A \) can be divided into linear operator \( L \) and nonlinear operator \( N \).

Applying, inverse operator \( L^{-1} \) on 1 and 2, we obtain the following form

\[ u(t) = f(t) + \int_0^t (t - x) N(u(x)) \, dx \tag{3} \]

where \( f \) represents sum of the finite initial conditions and integration of function \( g \). Now, we define successive approximations

\[ \begin{align*}
\text{initial guess} & \quad u_0 = f \\
\text{first iteration} & \quad u_1 = u_0 + N(u_0) \\
\text{second iteration} & \quad u_2 = u_0 + N(u_1) \\
\vdots & \\
\text{and so on. In general,} & \quad u_n = u_0 + N(u_{n-1}) \, , \, n \in N \tag{4}
\end{align*} \]

Hence, the solution of (4) and (5) is given by

\[ u = \lim_{n \to \infty} u_n \tag{6} \]

3. Convergence analysis
Some convergence steps and theorems are discussed and applied on delay differential equations.

Let the initial guess

\[ v_0 = u_0(t) \]

and first iteration

\[ v_1 = F[v_0] \]
and second iteration

\[ v_2 = F [v_0 + v_1] \]

\[ \ldots \]

and so on. In general,

\[ v_{n+1} = F [v_0 + v_1 + \cdots + v_n] \quad (7) \]

where \( F \) is an operator given by

\[ F [v_k] = S_k - \sum_{i=0}^{k-1} v_i (t) , \quad k \geq 1 \quad (8) \]

where \( S_k \) denotes partial sums up to \( k^{th} \) order. Hence

\[ v_k = v_0 + N \left( \sum_{i=0}^{k-1} v_i (t) \right) , \quad k \geq 1 \quad (9) \]

Therefore, we have series solution given by

\[ u (t) = \sum_{i=0}^{\infty} v_i (t) \quad (10) \]

Following results are used without proof to prove convergence of the problem. The convergence theorem proofs are cited in [34].

**Theorem: 1** Let \( F \) defined in 8, be an operator from a Hilbert space \( H \) to \( H \). The series solution \( u_n (x) = \sum_{i=0}^{n} v_i (x) \) converges if \( \exists 0 < \gamma < 1 \) such that

\[ F [v_0 + v_1 + v_2 + \cdots + v_i + 1] \leq F [v_0 + v_1 + v_2 + \cdots + v_i] \] such that \( v_{i+1} \leq v_i \ \forall \ i = 0, 1, 2, \ldots \)

**Theorem: 2** If the series solution \( u (t) = \sum_{i=0}^{\infty} v_i (t) \) is convergent then series will give the solution of problem.

From above theorems, we draw following, for each \( i \) if we define parameters

\[ \beta_i = \begin{cases} \frac{\|v_{i+1}\|}{\|v_i\|}, & \|v_i\| \neq 0 \\ 0, & \|v_i\| = 0 \end{cases} \]

then, the series solution \( \sum_{i=0}^{\infty} v_i (t) \) for problem presented in 1 is convergent when \( 0 \leq \beta_i < 1, \ \forall \ i=0,1,2, \ldots \)

4. Applications

Two examples are discussed to show effectiveness and robustness of the present method. The error analysis is also discussed. Numerical computations are performed using Mathematica software version 11.1.1.
4.1. Example 1
Consider the second order proportional delay differential equation [5]

\[ y''(t) = -y(t) + 5y^2 \left( \frac{t}{2} \right) \]  

(11)

initial condition

\[ y(0) = 1, \quad y'(0) = -2 \]  

(12)

The exact solution is given by

\[ y(t) = e^{-2t} \]  

(13)

Integrating equation 11 from 0 to \( t \) and using 12, we obtain recurrence relation given by

\[ y_{n+1}(t) = 1 - 2t + \int_0^t (t-x) \left( -y_n(x) + 5y^2_n \left( \frac{x}{2} \right) \right) dx \]  

(14)

The various series terms are given by

\[ y_0(t) = 1 - 2t \]

\[ N(y_{n-1}) = \int_0^t (t-x) \left( -y_{n-1}(x) + 5y^2_{n-1} \left( \frac{x}{2} \right) \right) dx \]

\[ y_1(t) = 1 - 2t + 2t^2 - \frac{4t^3}{3} + \frac{5t^4}{12} \]

\[ y_2(t) = 1 - 2t + 2t^2 - \frac{4t^3}{3} + \frac{2t^4}{3} - \frac{4t^5}{15} + \frac{53t^6}{576} - \frac{5t^7}{192} + \frac{155t^8}{32256} - \frac{25t^9}{41472} + \frac{25t^{10}}{663552} \]  

(15)

We continue the process and compute more terms but due to space constraints other terms are not listed here. However, seven terms are determined using Mathematica software version 11.1.1.

The following terms are generated to examine convergence condition

\[ v_0 = y_0 = 1 - 2t \]

\[ v_1 = 2t^2 - \frac{4t^3}{3} + \frac{5t^4}{12} \]

\[ v_2 = \frac{t^4}{4} - \frac{4t^5}{15} + \frac{53t^6}{576} - \frac{5t^7}{192} + \frac{155t^8}{32256} - \frac{25t^9}{41472} + \frac{25t^{10}}{663552} \]  

(16)

and so on.

Further, \( \beta_i \) values for example 1 are given by

\[ \beta_0 = \frac{v_1}{v_0} = 0.023385, \]

\[ \beta_1 = \frac{v_2}{v_1} = 0.001198, \]

\[ \beta_2 = \frac{v_3}{v_2} = 0.000135 \]

(17)

and so on. Clearly, \( \beta_i \) values satisfy the convergence criterion, for \( 0 < t \leq 1 \).

The error between two consecutive solutions is presented in table 1 and figure 1 shows difference of consecutive terms versus number of iterations for example 1, where \( D_i \) denotes the difference between two consecutive solutions.
Table 1. Error between two consecutive solutions

| i | $D_i$   | $\log \frac{D_{i+1}}{D_i}$ |
|---|---------|-----------------|
| 1 | 7.81290e-1 | -3.73318        |
| 2 | 1.86859e-2 | -6.72531        |
| 3 | 2.24258e-5 | -7.38861        |
| 4 | 1.38648e-8 | -45.8235        |
| 5 | 1.74193e-12| -66.5493        |
| 6 | 2.18294e-17|                |

Figure 1. Difference of consecutive terms versus number of iterations for example 1.

4.2. Example 2
Consider the nonlinear proportional delay differential equation

$$y''(t) = y(t) - \frac{8}{t^2} y^2 \left(\frac{t}{2}\right)$$

(18)

initial condition

$$y(0) = 0, y'(0) = 1$$

(19)

The exact solution is given by

$$y(t) = te^{-t}$$

(20)

Integrating equation 18 from 0 to $t$ and using 19, we obtain recurrence relation given by

$$y_{n+1}(t) = t + \int_0^t (t - x) \left( y_n(x) - \frac{8}{x^2} y^2_n \left(\frac{x}{2}\right) \right) dx$$

(21)

The various series terms are given by

$$y_0(t) = t$$

$$N\left(y_{n-1}\right) = \int_0^t (t - x) \left( y_{n-1}(x) - \frac{8}{x^2} u^2_{n-1} \left(\frac{x}{2}\right) \right) dx$$
\[ y_1(t) = t - t^2 + \frac{t^3}{6} \]
\[ y_2(t) = t - t^2 + \frac{t^3}{2} - \frac{5t^4}{36} + \frac{t^5}{80} - \frac{t^6}{8640} \]  

We continue the process and compute more terms but due to space constraints other terms are not listed here. However, seven terms are determined using Mathematica software version 11.1.1. The following terms are generated to examine convergence condition

\[ v_0 = y_0 = t \]
\[ v_1 = -t^2 + \frac{t^3}{6} \]
\[ v_2 = \frac{t^3}{3} - \frac{5t^4}{36} + \frac{t^5}{80} - \frac{t^6}{8640} \]  

and so on. Further, \( \beta_i \) values for example 2 are given by

\[ \beta_0 = \frac{v_1}{v_0} = 0.101667, \]
\[ \beta_1 = \frac{v_2}{v_1} = 0.031433, \]
\[ \beta_2 = \frac{v_3}{v_2} = 0.007822 \]  

and so on. Clearly, \( \beta_i \) values satisfy the convergence criterion, for \( 0 < t \leq 1 \).

The error between two consecutive solutions is presented in table 1 and figure 2 shows difference of consecutive terms versus number of iterations for example 2.

| \( i \) | \( D_i \) | \( \log \frac{D_{i+1}}{D_i} \) |
|------|------|-----------------|
| 1    | 8.98333e-2 | -2.21078 |
| 2    | 9.8471e-3  | -3.42017 |
| 3    | 3.22069e-4 | -4.8563  |
| 4    | 2.50545e-6 | -35.0889 |
| 5    | 1.44532e-9 | -49.3495 |
| 6    | 5.34253e-13| -5.01667 |

5. Conclusion
In the present paper, Banach contraction method is applied on some delay differential equations. Numerical solutions of DDE’s are determined. Error analysis are presented resulting in efficiency of BCM. On increasing number of iterations error becomes very small. Also, BCM can be applied to constant and time dependent delay differential equations.
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References
[1] Saeed R K and Rahman B M 2010 Adomian decomposition method for solving system of delay differential equation Australian Journal of Basic and Applied Sciences 108(2) 4(8) 3613—3621
[2] Rebenda J, Smarda Z and Khan Y 2015 A new semi-analytical approach for numerical solving of Cauchy problem for functional differential equations Filomat 31(15) 1–14
[3] Edeki S O, Akinlabi G O and Hinov N 2017 Zhou Method for the Solutions of System of Proportional Delay Differential Equations MATEC Web of Conferences 125 1–5
[4] Ahmad J and Rubab S 2017 Efficient Homotopy Analysis Method to System of Delay Differential Equations Annals of the Faculty of Engineering Hunedoara-International Journal of Engineering 15(2) 133—140
[5] Chen X and Wang L 2010 The variational iteration method for solving a neutral functional-differential equation with proportional delays Comput. Math. Appl. 59(8) 2696—2702
[6] Liu H, Xiao A and Su L 2013 Convergence of variational iteration method for second order delay differential equations J. Appl. Math. 2013 1–9
[7] Yu Z H 2008 Variational iteration method for solving the multi-pantograph delay equation Phys. Lett. A 372(43) 6475—6479
[8] Saadatmandi A and Dehghan M 2009 Variational iteration method for solving a generalized pantograph equation Computers & Mathematics with Applications 58(11) 2190–2196
[9] Alipour M, Karimi K and Rostamy D 2011 Modified variational iteration method for the multi-pantograph equation with convergence analysis Australian Journal of Basic and Applied Sciences 5(5) 886—893
[10] Biazar J and Ghanbari B 2012 The homotopy perturbation method for solving neutral functional differential equations with proportional delays Journal of King Saud University Science 24(1) 33–37
[11] Olvera D, Zuniga A E, Lacalle L N and Rodriguez C A 2015 Approximate solutions of delay differential equations with constant and variable coefficients by the enhanced multistage homotopy perturbation method Abstr. Appl. Anal. 2015 1–12
[12] Soltanian F, Dehghan M and Karbassi S M 2010 Solution of the differential algebraic equations via homotopy perturbation method and their engineering applications International Journal of Computer Mathematics 87(9) 1950—1974
[13] Biazar J and Eslami M 2011 A new homotopy perturbation method for solving systems of partial differential equations Computers & Mathematics with Applications 62(1) 225–234
[14] Araghi M F and Sotoodeh M 2012 An enhanced modified homotopy perturbation method for solving nonlinear volterra and fredholm integro-differential equation World Applied Sciences Journal 20(12) 1646—1655
[15] Alomari A K, Noorani M S and Nazar R 2009 Solution of delay differential equation by means of homotopy analysis method Acta Applicandae Mathematicae 108(2) 395–412
[16] Hu Y, Mohammed S E A and Yan F 2004 Discrete-time approximations of stochastic delay equations: the Milstein scheme Ann. Probab. 32 265–314
[17] Wang W S 2015 High order stable Runge-Kutta methods for nonlinear generalized pantograph equations on the geometric mesh Appl. Math. Model. 39(11) 270—283
[18] Li D and Liu M Z 2005 Runge–Kutta methods for the multi-pantograph delay equation Appl. Math. Comput. 163(1) 383—395
[19] Liu M Z and Spijker M 1990 The stability of the θ methods in the numerical solution of delay differential equations IMA J. Numer. Anal. 10 31—48
[20] Baker C T H 2000 Retarded differential equations. J. Comput. Appl. Math. 125 309—335
[21] Gokdogan A, Merdan M and Yildirim A 2012 The modified algorithm for the differential transform method to solution of genosio systems Commun Nonlinear Sci Numer Simul 17(1) 45—51
[22] Kurnaz A and Oturanc G 2005 The differential transform approximation for the system ordinary differential equations International Journal of Computer Mathematics 82 709—719
[23] Kangalgil F and Ayaz F 2009 Solitary wave solutions for the kdv and mkdv equations by differential transform method Chaos Solitons Fractals 41(1) 464—472
[24] Yu J, Jing J, Sun Y and Wu S 2016 (n + 1)-dimensional reduced differential transform method for solving partial differential equations Appl. Math. Comput. 273 697—707
[25] Mohammed G J and Fadhel F S 2011 Extend differential transform methods for solving differential equations with multiple delay Ibn Al- Haitham J. for Pure and Appl. Sci. 24(3) 1—5
[26] Rebenda J and Smarda Z 2017 A differential transformation approach for solving functional differential equations with multiple delays Commun. Nonlinear Sci. Numer. Simul. 48 246—257
[27] Rebenda J, Smarda Z and Khan Y 2017 A new semi-analytical approach for numerical solving of cauchy problem for differential equations with delay Filomat 31(15) 4725—4733
[28] Cakir M and Arslan D 2015 The Adomian decomposition method and the differential transform method for numerical solution of multi-pantograph delay differential equations Appl. Math. 6 1332—1343
[29] Evans D J and Raslan K R 2005 The Adomian decomposition method for solving delay differential equation Int. J. Comput. Math. 82(1) 49—54
[30] Widatalla S and Koroma M A 2012 Approximation algorithm for a system of pantograph equations J. Appl. Math. 2012 1—9
[31] Cakir M and Arslan D 2015 The Adomian Decomposition Method and the Differential Transform Method for Numerical Solution of Multi-Pantograph Delay Differential Equations, Applied Mathematics 6 1332—1343
[32] Ding L, Li X and Li Z 2010 Fixed points and stability in nonlinear equations with variable delays Fixed Point Theory Appl. 195—216
[33] Eloe P W, Raffoul Y and Tisdell C C 2005 Existence, uniqueness and constructible results for delay differential equations Ex. Electron J Differ Equ 121 1—11
[34] Odibat Z M and Mamani S 2006 Applications of variational iteration method to nonlinear differential equations of fractional order International Journal of Nonlinear Science and Numerical simulation 7(1) 27—34