HOLOMORPHIC BUNDLES ON 2-DIMENSIONAL
NONCOMMUTATIVE TORIC ORBIFOLDS

A. POLISHCHUK

Abstract. We define the notion of a holomorphic bundle on the noncommutative toric
orbifold $\mathcal{O}_{\theta}/G$ associated with an action of a finite cyclic group $G$ on an irrational rotation
algebra. We prove that the category of such holomorphic bundles is abelian and its
derived category is equivalent to the derived category of modules over a finite-dimensional
algebra $\Lambda$. As an application we finish the computation of $K_0$-groups of the crossed
product algebras describing the above orbifolds initiated in [17], [28], [29], [12] and [13].
Also, we describe a torsion pair in the category of $\Lambda$-modules, such that the tilting with
respect to this torsion pair gives the category of holomorphic bundles on $\mathcal{O}_{\theta}/G$.

Introduction

Let $A_{\theta}$ be the algebra of smooth functions on the noncommutative 2-torus $\mathcal{O}_{\theta}$ associated
with an irrational real number $\theta$. Recall that its elements are expressions of the form
$\sum_{m,n} a_{m,n} U_1^m U_2^n$, where the coefficients $(a_{m,n})_{(m,n)\in\mathbb{Z}^2}$ rapidly decrease at infinity, and $U_1$ and $U_2$ satisfy the relation
$$U_1 U_2 = \exp(2\pi i \theta) U_2 U_1.$$ It is convenient to denote $U_v = \exp(-\pi i m n \theta) U_1^m U_2^n$ for $v = (m,n) \in \mathbb{Z}^2$, so that
$$U_v U_w = \exp(\pi i \theta \det(v, w)) U_{v+w}.$$ There is a natural action of $\text{SL}_2(\mathbb{Z})$ on $A_{\theta}$ such that the matrix $g$ acts by the automorphism $U_v \mapsto U_{gv}$. Hence, for a finite subgroup $G \subset \text{SL}_2(\mathbb{Z})$ we can consider the crossed product algebra $B_{\theta} = A_{\theta} \ast G$.

The simplest case is when $G = \mathbb{Z}/2\mathbb{Z}$ generated by $-\text{id} \subset \text{SL}_2(\mathbb{Z})$ acting on $A_{\theta}$ by the so called flip automorphism. In this case the algebra $B_{\theta}$ was studied in the papers [8], [10], [17] and [27]. In particular, it is known that it is simple, has a unique tracial state, and is an AF-algebra. Also its $K$-theory has been computed: one has $K_0(B_{\theta}) = \mathbb{Z}^6$ and $K_1(B_{\theta}) = 0$. However, there are three more examples of finite subgroups $G \subset \text{SL}_2(\mathbb{Z})$ for which the situation is not so well understood. Namely, we can take $G = \mathbb{Z}/3\mathbb{Z}$ generated by $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$; or $G = \mathbb{Z}/4\mathbb{Z}$ generated by the "Fourier" matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; or $G = \mathbb{Z}/6\mathbb{Z}$ generated by $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. In this paper we compute $K_0(B_{\theta})$ in all these cases using holomorphic vector bundles on the corresponding noncommutative orbifolds.

By a vector bundle on the noncommutative toric orbifold $\mathcal{O}_{\theta}/G$ we mean a finitely generated projective right $B_{\theta}$-module. We want to define what is a holomorphic structure

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on such a vector bundle. As in [22], [21], let us consider a complex structure on \( T_\theta \) associated with a complex number \( \tau \in \mathbb{C} \setminus \mathbb{R} \). It is given by a derivation
\[
\delta : A_\theta \to A_\theta : \sum_{m,n} a_{m,n} U_1^m U_2^n \mapsto 2\pi i \sum_{m,n} (m \tau + n) U_1^m U_2^n
\]
of \( A_\theta \) that we view as an analogue of the \( \nabla \)-operator. To descend this structure to the orbifold \( T_\theta / G \) we have to impose some compatibility between the action of \( G \) and \( \delta \). More precisely, we assume that there exists a character \( \varepsilon : G \to \mathbb{C}^* \) such that the following relation holds:
\[
g \delta = \varepsilon(g) \delta g
\]
for all \( g \in G \). For example, for \( G = \mathbb{Z}/2\mathbb{Z} \) acting by the flip automorphism \( \varepsilon \) is the unique nontrivial character of \( \mathbb{Z}/2\mathbb{Z} \). Namely, let us identify \( G = \mathbb{Z}/m\mathbb{Z} \) with the subgroup of \( m \)-th roots of unity in \( \mathbb{C}^* \) and let \( G \) act on \( \mathbb{C} \) by multiplication. Then we can choose \( \tau \) in such a way that the lattice \( \mathbb{Z} \tau + \mathbb{Z} \) is \( G \)-invariant: for \( m = 4 \) we take \( \tau = i \), while for \( m = 3 \) and \( m = 6 \) we take \( \tau = (1 + i\sqrt{3})/2 \). Note that the embedding of \( G \) into \( \text{SL}_2(\mathbb{Z}) \) is induced by its action on the basis \( (\tau, 1) \) of \( \mathbb{Z} \tau + \mathbb{Z} \). Then (0.1) will hold with \( \varepsilon(g) = g^{-1} \in \mathbb{C}^* \).

Recall that in [22], [21] we studied the category \( \text{Hol}(T_{\theta, \tau}) \) of holomorphic bundles on \( T_\theta \). By definition, these are pairs \((P, \nabla)\) consisting of a finitely generated projective right \( A_\theta \)-module \( P \) and an operator \( \nabla : P \to P \) satisfying the Leibnitz identity
\[
\nabla(f \cdot a) = f \cdot \delta(a) + \nabla(f) \cdot a,
\]
where \( f \in P, a \in A_\theta \). Now we extend \( \delta \) to a twisted derivation \( \tilde{\delta} \) of \( B_\theta \) by setting
\[
\tilde{\delta}(\sum_{g \in G} a_g g) = \sum_{g \in G} \varepsilon(g) \delta(a_g) g,
\]
where \( a_g \in A_\theta \) for \( g \in G \). This extended map satisfies the twisted Leibnitz identity
\[
\tilde{\delta}(b_1 b_2) = b_1 \tilde{\delta}(b_2) + \tilde{\delta}(b_1) \kappa(b_2),
\]
where \( \kappa \) is the automorphism of \( B_\theta \) given by \( \kappa(\sum_{g \in G} a_g g) = \sum_{g \in G} \varepsilon(g) a_g g \). We define a holomorphic structure on a vector bundle \( P \) on \( T_\theta / G \) as an operator \( \nabla : P \to P \) satisfying the similar twisted Leibnitz identity
\[
\nabla(f \cdot b) = f \cdot \tilde{\delta}(b) + \nabla(f) \cdot \kappa(b),
\]
where \( f \in P, b \in B_\theta \). By definition, a holomorphic bundle is a pair \((P, \nabla)\) consisting of a vector bundle \( P \) equipped with a holomorphic structure \( \nabla \). One can define morphisms between holomorphic bundles in a natural way, so we obtain the category \( \text{Hol}(T_{\theta, \tau}/G) \) of holomorphic bundles.

Recall that the combined results of [22] and [21] imply that the category \( \text{Hol}(T_{\theta, \tau}) \) is abelian and one has an equivalence of bounded derived categories
\[
D^b(\text{Hol}(T_{\theta, \tau})) \simeq D^b(\text{Coh}(E)),
\]
where \( \text{Coh}(E) \) is the category of coherent sheaves on the elliptic curve \( E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \). Furthermore, the image of the abelian category \( \text{Hol}(T_{\theta, \tau}) \) in the derived category \( D^b(\text{Coh}(E)) \) can be described as the heart of the tilted \( t \)-structure associated with a
certain torsion pair in \( \text{Coh}(E) \) (depending on \( \theta \)). Our main result is a similar explicit description of the category of holomorphic bundles on \( T_{\theta,\tau}/G \), where \( G = \mathbb{Z}/m\mathbb{Z} \subset \text{SL}_2(\mathbb{Z}) \) with \( m \in \{2, 3, 4, 6\} \).

**Theorem 0.1.** The category \( \text{Hol}(T_{\theta,\tau}/G) \) is abelian and one has an equivalence of bounded derived categories

\[
D^b(\text{Hol}(T_{\theta,\tau}/G)) \simeq D^b(\text{mod } -\Lambda),
\]

where \( \text{mod } -\Lambda \) is the category of finite-dimensional right modules over the algebra \( \Lambda = \mathbb{C}Q/(I) \) of paths in a quiver \( Q \) without cycles with quadratic relations \( I \). The number of vertices of \( Q \) is equal to 6, 8, 9 or 10 for \( m = 2, 3, 4 \) or 6, respectively.

The precise description of the algebra \( \Lambda \) will be given in section 1.2. It is derived equivalent to one of canonical tubular algebras considered by Ringel in [23]. Furthermore, the image of \( \text{Hol}(T_{\theta,\tau}/G) \) in \( D^b(\text{mod } -\Lambda) \) corresponds to the tilted \( t \)-structure for a certain explicit torsion pair in \( \text{mod } -\Lambda \) depending on \( \theta \) that we will describe in Theorem 2.8.

We prove Theorem 0.1 in two steps: first, we relate holomorphic bundles on \( T_{\theta,\tau}/G \) to the derived category of \( G \)-equivariant sheaves on the elliptic curve \( E = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) \), and then we construct a derived equivalence with right modules over the algebra \( \Lambda \). The second step is actually well known and works for arbitrary weighted projective curves considered in [15]. We present an alternative derivation working directly with equivariant sheaves. It is based on the semiorthogonal decomposition of the category of \( G \)-equivariant sheaves associated with a ramified \( G \)-covering of smooth curves (see Theorem 1.2).

Combining Theorem 0.1 with the results of [28] and [12] we derive the following result.

**Theorem 0.2.** One has \( K_0(B_{\theta}) \simeq \mathbb{Z}^r \), where \( r = 6, 8, 9 \) or 10 for \( G = \mathbb{Z}/m\mathbb{Z} \) with \( m = 2, 3, 4 \) or 6, respectively.

Note that for \( G = \mathbb{Z}/2\mathbb{Z} \) this was known (see [17]). For \( G = \mathbb{Z}/4\mathbb{Z} \) and \( G = \mathbb{Z}/6\mathbb{Z} \) this was proved for \( \theta \) in a dense \( G \)\text-dρ-set (see [29] and [13]). The case of \( G = \mathbb{Z}/4\mathbb{Z} \) and arbitrary irrational \( \theta \) was done by Lueck, Phillips and Walters in 2003 (unpublished). Our proof shows in addition that the natural forgetful map

\[
K_0(\text{Hol}(T_{\theta,\tau}/G)) \to K_0(B_{\theta})
\]

is, in fact, an isomorphism and identifies the positive cones in these groups.

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1. **Derived categories of \( G \)-sheaves**

1.1. **Generalities on \( G \)-sheaves.** Let \( G \) be a finite group acting on an algebraic variety \( X \) over a field \( k \) of characteristic zero. Then we can consider the category \( \text{Coh}_G(X) \) of \( G \)-equivariant coherent sheaves. We denote its bounded derived category by \( D^b_G(X) \). It is equivalent to the full subcategory in the bounded derived category of \( G \)-equivariant quasicoherent sheaves consisting of complexes with coherent cohomology (see Corollary 1 in [1]). We refer to [11], Section 4, for a more detailed discussion of this category and
restrict ourself to several observations. Below we will use the term $G$-sheaf to denote a $G$-equivariant coherent sheaf.

There is a natural forgetting functor from $D^b_G(X)$ to $D^b(X)$, the usual derived category of coherent sheaves on $X$. For equivariant complexes of sheaves $F_1$ and $F_2$ we denote by $\text{Hom}_G(F_1, F_2)$ (resp., $\text{Hom}(F_1, F_2)$) morphisms between these objects in the former (resp., latter) category. There is a natural action of $G$ on $\text{Hom}_G(F_1, F_2)$ and one has

$$\text{Hom}_G(F_1, F_2) \simeq \text{Hom}(F_1, F_2)^G.$$ 

In particular, the cohomological dimension of $\text{Coh}_G(X)$ is at most that of $\text{Coh}(X)$. Let us also set $\text{Hom}^i_G(F_1, F_2) = \text{Hom}_G(F_1, F_2[\delta])$. If $X$ is a smooth projective variety over $k$ then we can define the bilinear form $\chi_G(\cdot, \cdot)$ on $K_0(D^b_G(X))$ by setting

$$\chi_G(F_1, F_2) = \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Hom}^i_G(F_1, F_2).$$

Many natural constructions with sheaves carry easily to $G$-equivariant setting. For example, the tensor product of $G$-sheaves is defined. Also if $\rho$ is a representation of $G$ then there is a natural tensor product operation $F \mapsto F \otimes \rho$ on $G$-sheaves. If $Y$ is another variety equipped with the action of $G$ and if $f : X \to Y$ is a $G$-equivariant morphism then there are natural functors of push-forward and pull-back:

$$f_* : \text{Coh}_G(X) \to \text{Coh}_G(Y), \quad f^* : \text{Coh}_G(Y) \to \text{Coh}_G(X),$$

We can also consider the derived functor $Rf_* : D^b_G(X) \to D^b_G(Y)$ and if $Y$ is smooth or $f$ is flat, the derived functor $Lf^* : D^b_G(Y) \to D^b_G(X)$ (when $f$ is flat we denote it simply by $f^*$). The pair $(Lf^*, Rf_*)$ satisfies the usual adjunction property.

If $X$ is smooth and projective then the category $D^b_G(X)$ is also equipped with the Serre duality of the form

$$\text{Hom}_G(F_1, F_2)^* \simeq \text{Hom}_G(F_2, F_1 \otimes \omega_X[\dim X]),$$

where the canonical bundle $\omega_X$ is equipped with the natural $G$-action.

The following observation will be useful to us.

**Lemma 1.1.** Let $X$ be a smooth curve equipped with an action of a finite group $G$. Then the category $D^b_G(X)$ is equivalent to the category of $G$-equivariant objects in $D^b(X)$, i.e., the category of data $(F, \phi_g)$, where $F \in D^b(X)$ and $\phi_g : g^*F \to F$, $g \in G$, is a collection of isomorphisms satisfying the natural compatibility condition.

**Proof.** Note that there is a natural functor from $D^b_G(X)$ to the category of $G$-objects in $D^b(X)$. It is easy to see that it is fully faithful, so the only issue is to check that it is essentially surjective. The proof is based on the fact that every object $F \in D^b(X)$ is isomorphic to the direct sum of its cohomology sheaves: $F \simeq \bigoplus_n H^n F[-n]$. A $G$-structure on $F$ is given by a compatible collection of isomorphisms $\phi = (\phi_g)$, where $\phi_g : \bigoplus_n g^* F \to F$. Note that the only nontrivial components of $\phi_g$ with respect to the above direct sum decompositions are maps $g^* H^n F[-n] \to H^n F[-n]$ and $g^* H^{n+1} F[-n] \to H^{n+1} F[-n+1]$. Let $\phi^0 = (\phi_g^0)$ be the $G$-structure on $F$ given by the components of $\phi$ of the first kind (i.e., by the diagonal components). Since the decomposition of $F$ into the direct sum of
cohomology sheaves is compatible with \( \phi^0 \), it suffices to find an isomorphism of \( G \)-objects 
\[
(F, \phi^0) \simeq (F, \phi).
\] (1.1)

Let us write \( \phi_g = a_g \circ \phi^0_g \), where \( a_g \in \text{Aut}(F) \). Note that \( a_g \) belongs to the abelian subgroup
\[
A := \bigoplus_n \text{Hom}(H^nF[-n], H^{n-1}F[-n + 1]) \subset \text{Aut}(F)
\]
of ”upper-triangular” automorphisms with identities as diagonal entries. It is easy to check that the compatibility condition on the data \( \phi \) amounts to the 1-cocycle equation for \( a_g \), where \( G \) acts on \( A \) in a natural way. On the other hand, existence of an isomorphism (1.1) is equivalent to \( g \mapsto a_g \) being a coboundary. It remains to note that \( H^1(G, A) = 0 \) since \( A \) is a vector space over a field of characteristic zero. \( \square \)

1.2. Semiorthogonal decomposition associated with a Galois covering. Recall (see [5], [7]) that a semiorthogonal decomposition of a triangulated category \( \mathcal{A} \) is an ordered collection \( (B_1, \ldots, B_r) \) of full triangulated subcategories in \( \mathcal{A} \) such that \( \text{Hom}(B_i, B_j) = 0 \) whenever \( B_i \in B_i \) and \( B_j \in B_j \), where \( i > j \), and the subcategories \( B_1, \ldots, B_r \) generate \( \mathcal{A} \). In this case we write
\[
\mathcal{A} = \langle B_1, \ldots, B_r \rangle.
\]

Semiorthogonal decompositions are related to admissible triangulated subcategories. For a subcategory \( B \subset \mathcal{A} \) let us denote by \( B^\perp \) the right orthogonal of \( B \), i.e., the full subcategory of \( \mathcal{A} \) consisting of all \( C \) such that \( \text{Hom}(B, C) = 0 \) for all \( B \in B \). A triangulated subcategory \( B \subset \mathcal{A} \) is called right admissible if for every \( X \in \mathcal{A} \) there exists a distinguished triangle \( B \to X \to C \to \ldots \) with \( B \in B \) and \( C \in B^\perp \). Thus, a right admissible subcategory \( B \subset \mathcal{A} \) gives rise to a semiorthogonal decomposition
\[
\mathcal{A} = \langle B^\perp, B \rangle.
\]

Similarly, one can define the left orthogonal and left admissibility of a subcategory.

We are going to use also some results from the theory of exceptional collections (see [3], [23]). Let \( k \) be a field. Recall that an object \( E \) in a \( k \)-linear triangulated subcategory \( \mathcal{A} \) is exceptional if \( \text{Hom}^i(E, E) = 0 \) for \( i \neq 0 \) and \( \text{Hom}(E, E) = k \). An exceptional collection in \( \mathcal{A} \) is a collection of exceptional objects \( (E_1, \ldots, E_n) \) such that \( \text{Hom}^i(E_i, E_j) = 0 \) for \( i > j \). A triangulated subcategory \( (E_1, \ldots, E_n) \) generated by an exceptional collection is known to be left and right admissible (see [3], Theorem 3.2). In the case when \( \langle E_1, \ldots, E_n \rangle = \mathcal{A} \) we will say that \( (E_1, \ldots, E_n) \) is a full exceptional collection. An exceptional collection \( (E_1, \ldots, E_n) \) is strong if \( \text{Hom}^a(E_i, E_j) = 0 \) for \( a \neq 0 \) and all \( i, j \). If an exceptional collection is full and strong then \( E = \bigoplus_{i=1}^n E_i \) is a tilting object in \( \mathcal{A} \), i.e., the functor \( X \mapsto R\text{Hom}(E, X) \) gives an equivalence between \( \mathcal{A} \) and \( D^b(\text{mod} - A) \) (provided \( A \) satisfies some natural finiteness assumptions and is framed, see [6]).

Let \( \pi : X \to Y \) be a ramified Galois covering with Galois group \( G \), where \( X \) and \( Y \) are smooth projective curves over an algebraically closed field \( k \) of characteristic zero. In other words, a finite group \( G \) acts effectively on \( X \) and \( Y = X/G \). We are going to construct a semiorthogonal decomposition of the derived category of \( G \)-sheaves \( D^b_G(X) \) with \( D^b(Y) \) as one of the pieces. Let \( D_1, \ldots, D_r \) be all special fibers of \( \pi \) equipped with the reduced scheme structure and let \( m_1, \ldots, m_r \) be the corresponding multiplicities. Let us also fix a point \( p_i \in D_i \) for each \( i = 1, \ldots, r \), and let \( G_i \subset G \) be the stabilizer subgroup of \( p_i \). Then
we have $G$-equivariant isomorphisms $D_i \simeq G/G_i$. Hence, the category of $G$-sheaves on $D_i$ is equivalent to finite dimensional representations of $G_i$. For every character $\zeta$ of $G_i$ we denote by $\zeta_{D_i}$ the corresponding $G$-sheaf on $D_i$. Note that $G_i$ is a cyclic group of order $m_i$. Moreover, the representation of $G_i$ on $\omega_X|_{p_i}$ allows to identify $G_i$ with the group of $m_i$-th roots of unity in such a way that it acts on $\omega_X|_{p_i}$ via the standard character. Thus, we have an isomorphism of $G$-sheaves
\[
\omega_X|_{D_i} \simeq \zeta(i)_{D_i},
\]
where $\zeta(i)$ is a generator of the character group $G_i$.

**Theorem 1.2.** (i) The natural functor $\pi^*: D^b(Y) \to D^b_G(X)$ is fully faithful.

(ii) For every $i = 1, \ldots, r$, the collection of $G$-sheaves on $X$
\[
(O_{(m_i-1)D_i}, \ldots, O_{2D_i}, O_{D_i})
\]
(1.2)
is exceptional. Let $B_i$ be the triangulated subcategory in $D^b_G(X)$ generated by this exceptional collection. Then $B_i$ and $B_j$ are mutually orthogonal for $i \neq j$, i.e., $\text{Hom}(B_i, B_j) = 0$ for all $B_i \in B_i$ and $B_j \in B_j$.

(iii) One has a semiorthogonal decomposition
\[
D^b_G(X) = \langle \pi^*D^b(Y), B_1, \ldots, B_r \rangle.
\]

**Proof.** (i) For $F_1, F_2 \in D^b(X)$ we have
\[
\text{Hom}_G(\pi^*F_1, \pi^*F_2) \simeq \text{Hom}(F_1, \pi_*\pi^*F_2)^G \simeq \text{Hom}(F_1, F_2 \otimes (\pi_*O_X)^G) \simeq \text{Hom}(F_1, F_2),
\]
since $(\pi_*O_X)^G \simeq O_Y$.

(ii) Let us first prove that the collection of $G$-sheaves on $X$
\[
(O_{D_i}, \zeta(i)_{D_i}[1], \ldots, \zeta(i)^{m_i-2}_{D_i}[m_i-2])
\]
(1.3)
is exceptional. Indeed, it is clear that there are no $G$-morphisms between $\zeta(i)^a_{D_i}$ and $\zeta(i)^b_{D_i}$ for $a \not\equiv b \mod(m_i)$. Also, by Serre duality, for $a, b \in \mathbb{Z}/m_i\mathbb{Z}$ we have
\[
\text{Ext}^1_G(\zeta(i)^b_{D_i}, \zeta(i)^a_{D_i})^* \simeq \text{Hom}_G(\zeta(i)^b_{D_i}, \zeta(i)^{a+1}_{D_i}) \simeq \text{Hom}_G(\zeta(i)^b, \zeta(i)^{a+1}).
\]
The latter space is nonzero only for $b = a + 1$. This proves that (1.3) is exceptional. Now using the exact sequences
\[
0 \to \zeta(i)^a_{D_i} \to O_{(a+1)D_i} \to O_{aD_i} \to 0
\]
for $a = 1, \ldots, m_i - 2$, one can easily show that making a sequence of mutations in (1.3) one gets the sequence (1.2). The fact that $B_i$ and $B_j$ are mutually orthogonal follows from disjointness of $D_i$ and $D_j$.

(iii) Since the subcategory $\langle B_1, \ldots, B_r \rangle$ is admissible it is enough to prove that $\pi^*D^b(Y)$ coincides with its right orthogonal. Since $B_i$ is also generated by the exceptional collection (1.3), the condition $\text{Hom}_G(B_i, F) = 0$ for $F \in D^b_G(X)$ is equivalent
\[
\text{Hom}_G(\zeta(i)^a_{D_i}, F) = 0 \text{ for } a = 0, \ldots, m_i - 2.
\]
Using Serre duality we can rewrite this as
\[
\text{Hom}_G(F, \zeta(i)^a_{D_i}) = 0 \text{ for } a = 1, \ldots, m_i - 1.
\]
Equivalently, $F|_{p_i}$ should have a trivial $G_i$-action. By the main theorem of [26] this implies that $F \in \pi^* D^b(Y)$. □

In the case when $Y = \mathbb{P}^1$ the semiorthogonal decomposition of Theorem 1.2 gives rise to a full exceptional collection in $D^b_G(X)$.

**Corollary 1.3.** Assume $X/G \simeq \mathbb{P}^1$. Then for every $n \in \mathbb{Z}$ we have the following full exceptional collection in $D^b_G(X)$:

$$(\pi^* \mathcal{O}_{\mathbb{P}^1}(n), \pi^* \mathcal{O}_{\mathbb{P}^1}(n + 1), \mathcal{O}_{(m_1 - 1)D_1}, \ldots, \mathcal{O}_{D_1}, \ldots, \mathcal{O}_{(m_r - 1)D_r}, \ldots, \mathcal{O}_{D_r}).$$

In particular, $K^0(D^b_G(X)) \simeq \mathbb{Z}^{2 + \sum_{i=1}^{m_i - 1}}$.

**Definition.** For a collection of $r$ distinct points $\overline{\lambda} = (\lambda_1, \ldots, \lambda_r)$ on $\mathbb{P}^1(k)$ and a sequence of weights $\overline{m} = (m_1, \ldots, m_r)$ let us define the algebra $\Lambda(\overline{\lambda}, \overline{m})$ as the path algebra of a quiver $Q_{\overline{m}}$ modulo relations $I(\overline{\lambda})$, where

(i) $Q_{\overline{m}}$ has $2 + \sum_{i=1}^{r}(m_i - 1)$ vertices named $u$, $v$, and $w_1^1, \ldots, w_{1}^{m_1-1}, \ldots, w_r^1, \ldots, w_r^{m_r-1}$;

(ii) $Q_{\overline{m}}$ has $2 + \sum_{i=1}^{r}(m_i - 1)$ arrows: 2 arrows $u \rightarrow v$, and chains of arrows

$$v \xrightarrow{\epsilon_i} w_i^{m_i-1} \rightarrow w_i^{m_i-2} \rightarrow \ldots \rightarrow w_i^1$$

for every $i = 1, \ldots, r$;

(iii) $I(\overline{\lambda})$ is generated by $r$ quadratic relations: $L_i \cdot e_i = 0$, $i = 1, \ldots, m$, where $L_i \subset kx_1 \oplus kx_1$ is the line corresponding to $\lambda_i \in \mathbb{P}^1 = \mathbb{P}(kx_0 \oplus kx_1)$.

It is easy to see that the endomorphism algebra of the exceptional collection constructed in Corollary 1.3 is isomorphic to $\Lambda(\overline{\lambda}, \overline{m})$, where $D_i = \pi^{-1}(\lambda_i)$. Hence, we obtain the following description of the derived category of $G$-sheaves on a $G$-covering of $\mathbb{P}^1$. (where for a finite-dimensional algebra $A$ we denote by $\text{mod} - A$ the category of finite-dimensional right $A$-modules).

**Corollary 1.4.** Let $\pi : X \rightarrow \mathbb{P}^1$ be a ramified Galois covering, where $X$ is a smooth curve, with Galois group $G$. Let $\overline{\lambda} = (\lambda_1, \ldots, \lambda_r) \subset \mathbb{P}^1$ be the set of ramification points of $\pi$ and let $\overline{m} = (m_1, \ldots, m_r)$ be multiplicities of the corresponding fibers. Then for every $n \in \mathbb{Z}$ one has an exact equivalence of triangulated categories

$$\Phi_n : D^b_G(X) \cong D^b(\text{mod} - \Lambda(\overline{\lambda}, \overline{m})) : F \rightarrow R\text{Hom}_G(V_n, F),$$

where

$$V_n = \pi^* \mathcal{O}_{\mathbb{P}^1}(n) \oplus \pi^* \mathcal{O}_{\mathbb{P}^1}(n + 1) \oplus \bigoplus_{1 \leq i \leq r, 1 \leq j < m_i} \mathcal{O}_{jD_i}.$$

**Remark.** Another natural full exceptional collection in $D^b_G(X)$ is

$$(\mathcal{O}_X, \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X((m_1 - 1)D_1), \ldots, \mathcal{O}_X(D_r), \ldots, \mathcal{O}_X((m_r - 1)D_r), \pi^* \mathcal{O}_{\mathbb{P}^1}(1)).$$

(1.4)

It is obtained from the collection of Corollary 1.3 for $n = 1$ by making the left mutation through $\pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ of the part of the collection following this object.
A right module $M$ over $\Lambda(\overline{\lambda}, \overline{m})$ can be viewed as a representation of the quiver $Q_{\overline{m}}^{\text{op}}$ in which the relations $I(\overline{\lambda})^{\text{op}}$ are satisfied. Thus, $M$ is given by a collection of vector spaces $(U, V, W_i^j)$, $i = 1, \ldots, r$, $j = 1, \ldots, m_i - 1$, equipped with linear maps
\[
W_i^1 \rightarrow \ldots \rightarrow W_i^{m_i - 1} \rightarrow V, \ i = 1, \ldots, r,
\]
and $x_0, x_1 : V \rightarrow U$ satisfying the relations $I(\overline{\lambda})^{\text{op}}$. Let us define the following additive functions of $M$:
\[
\text{deg}_n(M) = |G| \cdot \left( n \dim U - (n - 1) \dim V - \sum_{i=1}^r \sum_{j=1}^{m_i-1} \frac{\dim W_i^j}{m_i} \right),
\]
\[
\text{rk}(M) = \dim U - \dim V.
\]
We extend these functions to additive functions on $D^b(\Lambda(\overline{\lambda}, \overline{m}))$.

**Lemma 1.5.** In the situation of Corollary 1.4 one has $\text{deg}_n(\Phi_n(F)) = \text{deg}(F)$ and $\text{rk}(\Phi_n(F)) = \text{rk}(F)$ for every $F \in D^b_G(X)$.

**Proof.** Let $M = \Phi_n(F) = R\text{Hom}_G(V_n, F)$ for $F \in D^b_G(X)$. Then we have
\[
\dim U = \chi_G(\pi^*\mathcal{O}_{\mathbb{P}^1}(n), F), \ \dim V = \chi_G(\pi^*\mathcal{O}_{\mathbb{P}^1}(n + 1), F), \ \dim W_i^j = \chi_G(\mathcal{O}_{D_i}, F),
\]
and our task is to express $\text{rk}(F)$ and $\text{deg}(F)$ in terms of these numbers. To compute the rank we can use the equality
\[
\text{rk}(F) = -\chi_G(\pi^*\mathcal{O}_q, F),
\]
where $q$ is a generic point of $\mathbb{P}^1$. Since $[\mathcal{O}_q] = [\mathcal{O}_{\mathbb{P}^1}(n + 1)] - [\mathcal{O}_{\mathbb{P}^1}(n)]$, this immediately implies the required formula
\[
\text{rk}(F) = \chi_G(\pi^*\mathcal{O}_{\mathbb{P}^1}(n), F) - \chi_G(\pi^*\mathcal{O}_{\mathbb{P}^1}(n + 1), F).
\]
The formula for $\text{deg}(F)$ should have form
\[
\text{deg}(F) = a\chi_G(\pi^*\mathcal{O}_{\mathbb{P}^1}(n), F) + b\chi_G(\pi^*\mathcal{O}_{\mathbb{P}^1}(n + 1), F) + \sum_{i=1}^r \sum_{j=1}^{m_i-1} c_i^j \chi_G(\mathcal{O}_{D_i}, F)
\]
for some constants $a$, $b$ and $c_i^j$. The constants are determined by substituting in this formula the elements of the dual basis of $K_0(D^b_G(X))$:
\[
([\pi^*\mathcal{O}_{\mathbb{P}^1}(n)], -[\pi^*\mathcal{O}_{\mathbb{P}^1}(n - 1)], -[\zeta(1)_{D_1}], \ldots, -[\zeta(m_i - 1)_{D_i}], \ldots, -[\zeta(r)_{D_r}], \ldots, -[\zeta(r)_{D^{m_r}}]).
\]

\[\square\]

**1.3. Elliptic Galois coverings of $\mathbb{P}^1$.** Now let us specialize to the case of a Galois covering $\pi : E \rightarrow \mathbb{P}^1$, where $E = \mathbb{C}/(\tau \mathbb{Z} + \mathbb{Z})$ is an elliptic curve (so $k = \mathbb{C}$). More precisely, we are interested in the following four cases in which $G$ is a cyclic subgroup in $\mathbb{C}^*$ acting on $E$ in the natural way.

(i) $E$ is arbitrary and $G = \mathbb{Z}/2\mathbb{Z}$. The corresponding double covering $\pi : E \rightarrow \mathbb{P}^1$ is given by the Weierstrass $\wp$-function and is ramified exactly over 4-points
\[
\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{\infty, \wp(\frac{1}{2}), \wp(\frac{\tau}{2}), \wp(\frac{1 + \tau}{2})\}.
\]
(ii) $E = \mathbb{C}/Lt_r$, where $Lt_r = \mathbb{Z}^{1+\sqrt{3}} + \mathbb{Z}$, and $G = \mathbb{Z}/3\mathbb{Z}$. In this case $\pi : E \to \mathbb{P}^1$ is given by $\varphi'(z)$. Note that $E^G$ consists of 3 points: $0 \mod Lt_r$ and $\pm \frac{1+i\sqrt{3}}{6} \mod Lt_r$. Hence, $\pi$ is totally ramified over 3 points.

(iii) $E = \mathbb{C}/Lsq$, where $Lsq = \mathbb{Z}i + \mathbb{Z}$, and $G = \mathbb{Z}/4\mathbb{Z}$. In this case $\pi : E \to \mathbb{P}^1$ is given by $\varphi(z)^2$. We have two points whose stabilizer subgroup is $\mathbb{Z}/2\mathbb{Z}$, namely, $\frac{1}{2} \mod Lsq$ and $\frac{1}{2} \mod Lsq$ (they get exchanged by the generator of $\mathbb{Z}/4\mathbb{Z}$). The two points in $E^G$ are $0 \mod Lsq$ and $\frac{3+1}{2} \mod Lsq$. Hence, $\pi$ is ramified over 3 points and the corresponding multiplicities are $(2, 4, 4)$.

(iv) $E = \mathbb{C}/Lt_r$ (same curve as in (ii)) and $G = \mathbb{Z}/6\mathbb{Z}$. In this case $\pi : E \to \mathbb{P}^1$ is given by $\varphi'(z)^2$. There are 3 points whose stabilizer subgroup is $\mathbb{Z}/2\mathbb{Z}$, namely, all nontrivial points of order 2 on $E$ (they form one $G$-orbit). There is also a $G$-orbit consisting of two points $\pm \frac{3+\sqrt{3}}{6} \mod Lt_r$ with stabilizer subgroup $\mathbb{Z}/3\mathbb{Z}$. Finally, $0 \mod Lt_r$ is the only point in $E^G$. Therefore, $\pi$ is ramified over 3 points with multiplicities $(2, 3, 6)$.

From the above description of the ramification data and from Corollary 1.3 we get

**Corollary 1.6.** One has $K_0(D^b_G(E)) \simeq \mathbb{Z}^r$, where $r = 6, 8, 9$ or 10 in the cases (i)-(iv), respectively.

1.4. Galois coverings of $\mathbb{P}^1$ and weighted projective curves. The results of this section are not used in the rest of the paper. Its purpose is to explain the relation between $G$-sheaves on ramified Galois coverings of $\mathbb{P}^1$ and coherent sheaves on weighted projective curves introduced in [15]. This relation is known to experts, however, our proof seems to be new.

Let us recall the definition of weighted curves \(^1\) $C(\overline{m}, \overline{\lambda})$ of [15] associated with a sequence of positive integers $\overline{m} = (m_1, \ldots, m_r)$ and a sequence of points $\overline{\lambda} = (\lambda_1, \ldots, \lambda_r)$ in $\mathbb{P}^1(\mathbb{k})$. Let $Z(\overline{m})$ be the rank one abelian group with generators $e_1, \ldots, e_r$ and relations $m_1e_1 + \cdots + m_re_r = 0$. Let us also choose for every $i = 1, \ldots, r$ a nonzero section $s_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ such that $s_i(\lambda_i) = 0$. Consider the algebra

$$S(\overline{m}, \overline{\lambda}) = k[x_1, \ldots, x_r]/I(\overline{m}, \overline{\lambda}),$$

where the ideal $I(\overline{m}, \overline{\lambda})$ is generated by all polynomials of the form $a_1x_1^{m_1} + \cdots + a_rx_r^{m_r}$ such that $\sum_{i=1}^r a_i s_i = 0$. Let $Z(\overline{m})_+ \subset Z(\overline{m})$ be the positive submonoid generated by $e_1, \ldots, e_r$. Note that the algebra $S(\overline{m}, \overline{\lambda})$ has a natural $Z(\overline{m})_+$-grading, where $\deg(x_i) = e_i$. The category $\text{coh}(C(\overline{m}, \overline{\lambda}))$ of coherent sheaves on $C(\overline{m}, \overline{\lambda})$ can be defined as the quotient-category of the category of finitely generated $Z(\overline{m})_+$-graded $S(\overline{m}, \overline{\lambda})$-modules by the subcategory of finite length modules.

Now assume we are given a ramified Galois covering $\pi : X \to \mathbb{P}^1$ with Galois group $G$, where $X$ is a smooth connected curve. Define the associated data $(D_i, \overline{m}, \overline{\lambda})$ as in the previous section. Let $\text{Pic}_G(X)$ be the group of $G$-equivariant line bundles up to $G$-isomorphism. Let us consider the algebra

$$S(X, G) := \bigoplus_{[L] \in \text{Pic}_G(X)} H^0(X, L)^G.$$
\textbf{Theorem 1.7.} One has an isomorphism of algebras

\[ S(\overline{m}, \overline{\lambda}) \simeq S(X, G), \]

compatible with gradings via an isomorphism \( Z(\overline{m}) \simeq \text{Pic}_G(X) \).

\textit{Proof.} We claim that there is a natural homomorphism \( S(\overline{m}, \overline{\lambda}) \rightarrow S(X, G) \) that sends \( x_i \) to a nonzero section \( f_i \) of \( H^0(X, \mathcal{O}_X(D_i)) \). Indeed, note that we have a natural isomorphism \( \mathcal{O}_X(m_i D_i) \simeq \pi^* \mathcal{O}_{\mathbb{P}^1}(1) \) compatible with the action of \( G \), and hence the induced isomorphism

\[ H^0(\mathcal{O}_X(m_i D_i))^G \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)). \]

Let us rescale \( f_i \) in such a way that \( f_i^{m_i} \) corresponds to \( s_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \) under this isomorphism. Then \( f_i^{m_i} \) will satisfy the same linear relations as \( s_i \), hence we get a homomorphism \( \alpha : S(\overline{m}, \overline{\lambda}) \rightarrow S(X, G) \). Note that \( \alpha \) is compatible with gradings via the homomorphism \( Z(\overline{m}) \rightarrow \text{Pic}_G(X) \) sending \( x_i \) to the class of \( D_i \). Let us check that \( \alpha \) is surjective. Assume we are given \( L \in \text{Pic}_G(X) \) and a nonzero \( G \)-invariant section \( f \) of \( L \). If the divisor of zeroes of \( f \) contains \( D_i \) for some \( i \) then \( f \) is divisible by \( f_i \) in the algebra \( S(X, G) \), so we can assume that the divisor of \( f \) is disjoint from all special fibers. Therefore, \( L \simeq \mathcal{O}_{\mathbb{P}^1}(n) \) and \( f \) corresponds to a section of \( \mathcal{O}_{\mathbb{P}^1}(n) \) on \( \mathbb{P}^1 \). Note that \( r \geq 2 \) since \( X \) is connected. Therefore, every section of \( \mathcal{O}_{\mathbb{P}^1}(n) \) can be expressed as a polynomial of \( s_1, \ldots, s_r \). Hence, \( f \) belongs to the image of \( \alpha \). Injectivity of \( \alpha \) follows easily from Proposition 1.3 of [15]. \( \square \)

Using Theorem 1.7 we can derive the following equivalence between the categories of sheaves.

\textbf{Theorem 1.8.} In the above situation one has an equivalence of categories

\[ \text{Coh}_G(X) \simeq \text{Coh}(C(\overline{m}, \overline{\lambda})). \]

\textit{Proof.} This follows from Theorem 1.7 by a version of Serre's theorem. The only nontrivial fact one has to use is that for every \( G \)-sheaf \( F \) on \( X \) there exists a surjection of \( G \)-sheaves \( \bigoplus_{i=1}^n L_i \rightarrow F \), where \( L_i \) are equivariant \( G \)-bundles. Since every \( G \)-sheaf \( F \) is covered by a \( G \)-bundle of the form \( H^0(X, F \otimes L) \otimes L^{-1} \) for sufficiently ample \( G \)-equivariant line bundle \( L \), it suffices to consider the case when \( F \) is locally free. Assume first that the action of \( G_i \) on the fiber \( F|_{D_i} \) is trivial for all \( i = 1, \ldots, r \). Then \( F \) is \( G \)-isomorphic to the pull-back of a vector bundle on \( \mathbb{P}^1 \). In this case the assertion is clear since all vector bundles on \( \mathbb{P}^1 \) are direct sums of line bundles. We are going to reduce to this case using elementary transformations along \( D_i \)'s. Namely, let us decompose a representation of \( G_i \) on the fiber \( F|_{D_i} \) into the direct sum of characters of \( G_i \):

\[ F|_{D_i} \simeq \bigoplus_{j=0}^{m_i-1} V_j \otimes \zeta(i)^j \]

with some multiplicity spaces \( V_j \). Note that if we define the \( G \)-bundle \( F' \) by the short exact triple

\[ 0 \rightarrow F' \rightarrow F \rightarrow V_j \otimes \zeta(i)^j_{D_i} \rightarrow 0 \]

then we have an exact sequence of \( G_i \)-modules

\[ 0 \rightarrow V_j \otimes \zeta(i)^j \otimes \text{Tor}_1(\mathcal{O}_{D_i}, \mathcal{O}_{D_i}) \rightarrow F'|_{D_i} \rightarrow F|_{D_i} \rightarrow V_j \otimes \zeta(i)^j \rightarrow 0. \]
But \( \text{Tor}_1(\mathcal{O}_{D_i}, \mathcal{O}_{p_i}) \simeq \omega_X|_{p_i} \simeq \zeta(i) \), hence,

\[
F'|_{p_i} \simeq \left( \oplus_{j' \neq j} V_{j'} \otimes \zeta(i)_{j'} \right) \oplus V_j \otimes \zeta(i)^{j+1}.
\]

It is clear that using a sequence of transformations of this form we can pass from \( F \) to a vector bundle for which all fibers \( F|_{p_i} \) have trivial \( G_i \)-action. It remains to check that if our claim holds for \( F' \) (i.e., there exists a \( G \)-surjection from a direct sum of \( G \)-equivariant line bundles to \( F' \)) then the same is true for \( F \). To this end we observe that for every \( n \in \mathbb{Z} \) there exists a surjection of \( G \)-sheaves

\[
\omega_j^X \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(n) \to \zeta(i)_D^j.
\]

If \( n \) is sufficiently negative then this map lifts to a morphism \( \omega_j^X \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(n) \to F \). Thus, from a surjection \( \bigoplus_{i=1}^n L_i \to F' \) we obtain a surjection of the form

\[
\bigoplus_{i=1}^n L_i \oplus \omega_j^X \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(n) \to F.
\]

\[\square\]

In the case when \( G = \mathbb{Z}/2\mathbb{Z} \) and \( X \) is an elliptic curve the above equivalence is considered in Example 5.8 of [15].

Note that the tilting bundle on \( C(\overline{m}, \overline{\lambda}) \) constructed in [15] corresponds to the exceptional collection (1.4).

2. Holomorphic bundles on toric orbifolds and derived categories of \( G \)-sheaves

2.1. Remarks on \( B_\theta \)-modules and holomorphic bundles. It is clear that a \( B_\theta \)-module is finitely generates iff it is finitely generated as an \( A_\theta \)-module. We claim that projectivity also can be checked over \( A_\theta \).

**Lemma 2.1.** Let \( M \) be a right \( B_\theta \)-module. Then \( M \) is projective as a \( B_\theta \)-module iff it is projective as an \( A_\theta \)-module.

**Proof.** The “only if” part is clear. Let \( M \) be a right \( B_\theta \)-module, projective over \( A_\theta \). Then we have a natural surjection of \( B_\theta \)-modules \( p : M \otimes_{A_\theta} B_\theta \to M \) given by the action of \( B_\theta \). On the other hand, it is easy to check that the map

\[
s : M \to M \otimes_{A_\theta} B_\theta : m \mapsto \frac{1}{|G|} \sum_{g \in G} mg \otimes g^{-1}
\]

commutes with the right action of \( B_\theta \). Since \( p \circ s = \text{id}_M \), we derive that \( M \) is a direct summand in the projective \( B_\theta \)-module \( M \otimes_{A_\theta} B_\theta \). Hence, \( M \) itself is a projective \( B_\theta \)-module. \( \square \)

Thus, we can identify holomorphic bundles on \( T_{\theta, \tau}/G \) with \( G \)-equivariant holomorphic bundles on \( T_{\theta, \tau} \). Here is a more precise statement. Let us define an automorphism \( g^* \) of the category \( \text{Hol}(T_{\theta, \tau}) \) by setting \( g^*(P, \nabla) = (P^g, \varepsilon(g)\nabla) \), where \( P^g = P \) as a vector space but the \( A_\theta \)-module structure is changed by the automorphism \( g \) of \( A_\theta \). The fact that we again obtain a holomorphic bundle on \( T_{\theta, \tau} \) follows from (0.1).
Lemma 2.2. The category $\text{Hol}(T_{\theta, \tau}/G)$ is equivalent to the category of $G$-equivariant objects of $\text{Hol}(T_{\theta, \tau})$.

Proof. By Lemma 2.1 a holomorphic bundle on $T_{\theta, \tau}/G$ is given by a finitely generated projective right $A_\theta$-modules $P$ equipped with a holomorphic structure $\nabla$ and an action of $G$ such that
\[ g(f \cdot a) = g(f) \cdot g(a), \quad g \circ \nabla = \varepsilon(g) \nabla \circ g, \]
where $g \in G$, $f \in P$, $a \in A_\theta$. This immediately implies the assertion. \qed

Proposition 2.3. Every finitely generated projective $B_\theta$-module admits a holomorphic structure.

Proof. Let $P$ be such a module. Considering $P$ as an $A_\theta$-module we can equip it with a holomorphic structure $\nabla$ making it into a holomorphic bundle on $T_\theta$ (because $P$ is a direct sum of basic modules and every basic module admits a standard holomorphic structure, see [22]). Now replace $\nabla$ with
\[ \frac{1}{|G|} \sum_{g \in G} \varepsilon(g)^{-1} g \nabla g^{-1}. \]
This new structure is compatible with the action of $G$, so that $P$ becomes a holomorphic bundle on $T_{\theta, \tau}/G$. \qed

2.2. Generalities on torsion theory. Recall (see [16]) that a torsion pair in an exact category $C$ is a pair of full subcategories $(T, F)$ in $C$ such that $\text{Hom}(T, F) = 0$ for every $T \in T$, $F \in F$, and every object $C \in C$ fits into a short exact triple
\[ 0 \to T \to C \to F \to 0 \]
with $T \in T$ and $F \in FF$. Note that if $(T, F)$ is a torsion pair then $F$ (resp., $T$) coincides with the right (resp., left) orthogonal of $T$, i.e. with the full subcategory of objects $X$ such that $\text{Hom}(T, X) = 0$ for all $T \in T$ (resp., $\text{Hom}(X, F) = 0$ for all $F \in F$). In particular $T$ and $F$ are stable under extensions and passing to a direct summand.

It will be convenient for us to introduce a slight generalization of the notion of a torsion pair. Given a collection of full subcategories $(C_1, \ldots, C_n)$ in an exact category $C$, let us denote by $[C_1, \ldots, C_n]$ the full subcategory in $C$ consisting of objects $C$ admitting an admissible filtration $0 = F_0 C \subset F_1 C \subset \cdots \subset F_n C = C$ such that $F_i C / F_{i-1} C \in C_i$ for $i = 1, \ldots, n$.

Definition. A torsion $n$-tuple in an exact category $C$ is a collection of full subcategories $(C_1, \ldots, C_n)$ such that $\text{Hom}(C_i, C_j) = 0$ whenever $C_i \in C_i$, $C_j \in C_j$, $i < j$, and $[C_1, \ldots, C_n] = C$.

Sometimes we will write the condition of absence of nontrivial morphisms in the above definition as $\text{Hom}(C_i, C_j) = 0$ for $i < j$. For $n = 2$ we recover the notion of a torsion pair. Moreover, it is clear that if $(C_1, \ldots, C_n)$ is a torsion $n$-tuple then for every $i$ the pair
\[ ([C_1, \ldots, C_i], [C_{i+1}, \ldots, C_n]) \]
is a torsion pair. Note that the subcategories $C_i$ in this definition are automatically stable under extensions. The main reason for introducing torsion $n$-tuples is because it is possible to substitute one such torsion tuple into another. Namely, if $(C_1, \ldots, C_n)$ is a torsion $n$-tuple in $C$, and $(C_{i,1}, \ldots, C_{i,m})$ is a torsion $m$-tuple in $C_i$ then

$$(C_1, \ldots, C_{i-1}, C_{i,1}, \ldots, C_{i,m}, C_{i+1}, \ldots, C_n)$$

is a torsion $(n + m - 1)$-tuple in $C$.

If $C$ is abelian then a torsion pair $(T, F)$ defines a nondegenerate $t$-structure on the derived category $D^b(C)$ with the heart

$$C^p := \{ K \in D^b(C) : H^i(K) = 0 \text{ for } i \neq 0, -1, H^0(K) \in T, H^{-1}(K) \in F\}$$

(see [16]). In other words,

$$C^p = [F[1], T],$$

where for a pair of full subcategories $C_1, C_2$ in a triangulated category $D$ we denote by $[C_1, C_2]$ the full subcategory in $D$ consisting of objects $K$ that fit into an exact triangle

$$C_1 \to K \to C_2 \to C_1[1]$$

with $C_1 \in C_1$, $C_2 \in C_2$. The process of passing from $C$ to $C^p$ is called tilting (also, we will call $C^p$ a tilt of $C$). Note that $(F[1], T)$ is a torsion pair in $C^p$ and applying tilting to this pair we pass back to $C$. If $(C_1, \ldots, C_n)$ is a torsion $n$-tuple in $C$ then we set

$$[C_{i+1}[1], \ldots, C_n[1], C_1, \ldots, C_i] := [[C_{i+1}, \ldots, C_n][1], [C_1, \ldots, C_i]] \subset D^b(C),$$

where $([C_1, \ldots, C_i], [C_{i+1}, \ldots, C_n])$ is the corresponding torsion pair in $C$.

The main example relevant for us is the torsion pair $(\text{Coh}_{>\theta}(X), \text{Coh}_{<\theta}(X))$ in the category $\text{Coh}(X)$ of coherent sheaves on a smooth projective curve $X$, associated with an irrational number $\theta$. Namely, $\text{Coh}^{<\theta}(X) \subset \text{Coh}(X)$ (resp., $\text{Coh}^{>\theta}(X) \subset \text{Coh}(X)$) consists of all coherent sheaves $F$ on $X$ such that all subsequent quotients in the Harder-Narasimhan filtration of $F$ have slope $< \theta$ (resp., $> \theta$), where we consider torsion sheaves as having slope $+\infty$. Note that these torsion pairs arise in connection with stability structures on $D^b(X)$ (see [2]).

2.3. Fourier-Mukai transform for noncommutative two-tori. Let

$$C^\theta(E) = [\text{Coh}^{<\theta}(E)[1], \text{Coh}^{>\theta}(E)] \subset D^b(E)$$

be the tilt of the category of coherent sheaves on the elliptic curve $E = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ associated with $\theta$. We know from [22],[21] that $\text{Hol}(T_{\theta,\tau})$ is equivalent to $C^\theta(E)$. In this section we will show that the construction of this equivalence can be adjusted to be compatible with the action of a finite group $G$.

Recall that the equivalence is given by a version of the Fourier-Mukai transform (see [22], Section 3.3). With a holomorphic vector bundle $(P, \overline{\nabla})$ on $T_{\theta,\tau}$ this transform associates the complex $S(P, \overline{\nabla})$ of $O$-modules on $E$ of the form $d : P_E \to P_E$ concentrated in degrees $[-1,0]$, where $P_E$ is obtained by descending the sheaf of holomorphic $E$-valued functions over $\mathbb{C}$ using an action of $\mathbb{Z}^2$ of the form

$$\rho_v(f)(z) = \exp(\pi i c_v(z))f(z + v)U_v, \quad v \in \mathbb{Z}^2,$$
and the differential $d$ is induced by the operator
\[ f(z) \mapsto \nabla(f(z)) + 2\pi izf(z) \]
. Here $(c_v(z))$ is a collection of holomorphic functions on $\mathbb{C}$ numbered by $\mathbb{Z}^2$ satisfying the condition
\[ c_{v_1}(z) + c_{v_2}(z + v_1) - c_{v_1 + v_2}(z) = \det(v_1, v_2). \quad (2.1) \]

Note that in [22] we made one possible choice of $(c_v(z))$, however, it is not the only choice. In fact, one can easily see that the equivalences corresponding to different choices of $(c_v(z))$ differ by tensoring with a holomorphic line bundle on $E$. One of possible solutions of (2.1) is
\[ c_{m\tau + n}^0(z) = -2mz - m(m\tau + n). \]

It follows from Proposition 3.7 of [22] that $S$ is an equivalence of $\text{Hol}(T_{\theta,\tau})$ with $\mathcal{C}^0(E)$ (note that the definition above differs from that of [22] by the shift of degree).

Now let us assume that a finite group $G$ acts on the elliptic curve $E$ be automorphisms (preserving 0). This means that $G$ is a subgroup in $\mathbb{C}^*$ and multiplication by elements of $G$ preserves the lattice $\mathbb{Z}\tau + \mathbb{Z} \subset \mathbb{C}$. Identifying $\mathbb{Z}^2$ with $\mathbb{Z}\tau + \mathbb{Z}$ by $(m, n) \mapsto m\tau + n$ we can view $G$ also as a subgroup in $\text{SL}_2(\mathbb{Z})$. One can immediately check that the corresponding action of $G$ on $A_\theta$ satisfies (0.1) with $\varepsilon(g) = g^{-1} \in \mathbb{C}^*$. Hence, for every $g \in G$ we have the corresponding automorphism $g^*$ of the category $\text{Hol}(T_{\theta,\tau})$ (see section 2.1). Let us make a $G$-invariant choice of $(c_v(z))$ by setting
\[ c_v(z) = \frac{1}{|G|} \sum_{g \in G} c_{gv}^0 g z, \]
so that $c_{gv}(gz) = c_v(z)$ for all $g \in G$. Then the resulting Fourier-Mukai transform $S$ is compatible with the action of $G$ in the standard way.

**Proposition 2.4.** With the above choice of $(c_v(z))$ one has natural isomorphisms of functors
\[ S \circ g^* \simeq (g^{-1})^* \circ S \]
from $\text{Hol}(T_{\theta,\tau})$ to $\mathcal{C}^0(E)$, where $g \in G$.

**Proof.** By definition $g^*(P, \nabla) = (P^g, \varepsilon(g)\nabla)$. Hence, $Sg^*(P, \nabla) = [d_1 : P_E^g \to P_E^g]$ where $P_E^g$ is obtained from the action of $\mathbb{Z}^2$ on $P_C$ given by
\[ f(z) \mapsto \exp(\pi i \theta c_v(z))f(z + v)U_{gv}, \]
and the differential $d_1$ is induced by the operator
\[ f(z) \mapsto \varepsilon(g)\nabla(f(z)) + 2\pi izf(z). \]

On the other hand, $(g^{-1})^*S(P, \nabla)$ is given by the complex $[d_2 : (g^{-1})^*P_E \to (g^{-1})^*P_E]$, where $(g^{-1})^*P_E$ is obtained from the action of $\mathbb{Z}^2$ on $P_C$ given by
\[ f(z) \mapsto \exp(\pi i \theta c_v(gz))f(z + g^{-1}v)U_v, \]
and $d_2$ is induced by the operator
\[ f(z) \mapsto \nabla(f(z)) + 2\pi igzf(z), \]
where we view $g$ as an element of $\mathbb{C}^\ast$. Making a change of variables $v \mapsto gv$ we can identify two $\mathbb{Z}^2$-actions above, and hence we can identify with $P^g_E$ with $(g^{-1})^* P_E$. Since $\varepsilon(g) = g^{-1}$, under this identification $d_1 = \varepsilon(g)d_2$, so we get the required isomorphism. \[\Box\]

2.4. Proof of Theorem 0.1. From Lemma 2.2 and Proposition 2.4 we obtain that the category $\text{Hol}(T_{\theta, \tau}/G)$ is equivalent to the category of $G$-equivariant objects of $\mathcal{C}^\theta(E)$.

Note that the Harder-Narasimhan filtration of a $G$-sheaf is stable under the action of $G$ and hence, can be considered as a filtration in $\text{Coh}_G(E)$. Therefore, we can define a torsion theory $((\text{Coh}_{\theta}^\theta(E), \text{Coh}_{G}^\theta(E)))$ in $\text{Coh}_G(E)$, where $\text{Coh}_{G}^\theta(E)$ consists of $G$-sheaves $F$ such that after forgetting the $G$-structure we have $F \in \text{Coh}^\ast(E)$. Let

$$\mathcal{C}_G^\theta(E) = [\text{Coh}_G^\theta(E)[1], \text{Coh}_{\theta}^\theta(E)] \subset D_G^b(E)$$

be the corresponding tilted abelian subcategory. By Lemma 1.1 the category $\mathcal{C}_G^\theta(E)$ is equivalent to the category of $G$-equivariant objects on $\mathcal{C}^\theta(E)$, and hence, to $\text{Hol}(T_{\theta, \tau}/G)$.

Let us show that $D_G^b(E)$ is equivalent to $D(G^\theta(E))$. By Proposition 5.4.3 of [4] it is enough to check that that our torsion pair in $\text{Coh}_G(E)$ is cotilting, i.e., for every $G$-sheaf $F$ on $E$ there exists a $G$-equivariant vector bundle $V \in \text{Coh}^\theta(E)$ and a $G$-equivariant surjection $V \to F$. However, this is clear since for every $G$-sheaf $F$ there is a surjection

$$H^0(X, F \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(n)) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(-n) \to F,$$

where $n$ is large enough (and the space of global sections is equipped with the natural $G$-action).

Thus, we showed that $\text{Hol}(T_{\theta, \tau})/G$ is abelian and its derived category is equivalent to $D_G^b(E)$. It remains to apply Corollary 1.4 to the covering $\pi : E \to E/G \simeq \mathbb{P}^1$. The statement about the number of vertices follows from the explicit description of these coverings in section 1.3. \[\Box\]

2.5. Proof of Theorem 0.2. Using Theorem 0.1 and Corollary 1.6 we obtain an isomorphism

$$K_0(\text{Hol}(T_{\theta, \tau}/G)) \simeq K_0(D_G^b(E)) \simeq \mathbb{Z}^r,$$

where $r = 6, 8, 9, 10$ for $G = \mathbb{Z}/m\mathbb{Z}$ with $m = 2, 3, 4, 6$, respectively. Now we observe that by Proposition 2.3 the natural homomorphism

$$K_0(\text{Hol}(T_{\theta, \tau}/G) \to K_0(B_\theta)$$

is surjective. To prove that this map is an isomorphism, it is enough to check that the rank of $K_0(B_\theta)$ is at least $r$. This was done in [8], [28] and [12] for $m = 2$, $m = 4$, and $m = 3, 6$, respectively (by explicitly constructing $r$ elements in $K_0(B_\theta)$ and using unbounded traces to check their linear independence). \[\Box\]

This result was known for $G = \mathbb{Z}/2\mathbb{Z}$ (see [17]), however, with a different proof. For $G = \mathbb{Z}/4\mathbb{Z}$ and $G = \mathbb{Z}/6\mathbb{Z}$ it was known for $\theta$ in a dense $G_\delta$-set (see [29] and [13]). The case of $G = \mathbb{Z}/4\mathbb{Z}$ and general $\theta$ was done by Lueck, Walters and Phillips (unpublished).

Note that from the above proof we also get the following

**Corollary 2.5.** The natural homomorphism $K_0(\text{Hol}(T_{\theta, \tau}/G)) \to K_0(B_\theta)$ is an isomorphism. Moreover, the positive cones are the same.
Remark. In [27] it was shown that for $G = \mathbb{Z}/2\mathbb{Z}$ the positive cone in $K_0(B_\theta)$ coincides with the preimage of the positive cone in $K_0(A_\theta)$ under the natural homomorphism $K_0(B_\theta) \to K_0(A_\theta)$ (in other words, it consists of all elements $x \in K_0(B_\theta)$ such that $\text{tr}_+(x) > 0$, where $\text{tr}_+: K_0(B_\theta) \to \mathbb{R}$ is the homomorphism induced by the trace). As was pointed to us by Chris Phillips, similar statement is also known to hold for other groups $G$. Namely, it follows from the fact that the corresponding crossed products are simple AH algebras with slow dimension growth and real rank zero (see Theorems 8.11 and 9.10 of [20]).

2.6. Tiltings associated with $\theta$. Let $\pi : X \to \mathbb{P}^1$ be a ramified Galois covering with the Galois group $G$, and let $D^b_G(X)$ be the derived category of $G$-sheaves on $X$. As in the above proof of Theorem 0.1 we can define the torsion pair $(\text{Coh}^{>\theta}_G(X), \text{Coh}^{\leq \theta}_G(X))$ in $\text{Coh}_G(X)$ associated with an irrational number $\theta$. Our goal in this section is to describe the image of the corresponding tilted abelian subcategory

$$\mathcal{C}^\theta_G(X) := [\text{Coh}^{>\theta}_G(X)[1], \text{Coh}^{\leq \theta}_G(X)] \subset D^b_G(X)$$

under the equivalence $\Phi_n$ of Corollary 1.4 (for suitable $n$).

Let us start by describing the torsion pair in $\text{Coh}_G(X)$ giving rise to the $t$-structure on $D^b_G(X)$ associated with $\Phi_n$. By definition, the heart $\mathcal{M}_n$ of this $t$-structure consists of objects $F$ such that $\text{Hom}^i_G(V_n, F) = 0$ for $i \neq 0$.

**Proposition 2.6.** Let $\mathcal{T}_0 \subset \text{Coh}_G(X)$ denote the full subcategory consisting of all $G$-sheaves isomorphic to a direct sum of $G$-sheaves from the collection

$$(\mathcal{O}_{(m_1-1)D_1}, \ldots, \mathcal{O}_{D_1}, \ldots, \mathcal{O}_{(m_r-1)D_r}, \ldots, \mathcal{O}_{D_r}).$$

Let also $\mathcal{T}_1 \subset \text{Coh}_G(X)$ be the full subcategory of torsion $G$-sheaves obtained by successive extensions from simple $G$-sheaves of the form $\zeta^{a}(i)_{D_i}$, where $i = 1, \ldots, r$, $a = 1, \ldots, m_i-1$ (so $\mathcal{O}_{D_0}$ are not included). Then for every $n \in \mathbb{Z}$ we have a torsion quadruple

$$(\mathcal{T}_0, \pi^* \text{Coh}^{\leq n}(\mathbb{P}^1), \pi^* \text{Coh}^{\leq n-1}(\mathbb{P}^1), \mathcal{T}_1)$$

in $\text{Coh}_G(X)$. Furthermore, we have the equality of abelian subcategories in $D^b_G(X)$

$$\mathcal{M}_n = [\pi^* \text{Coh}^{\leq n-1}(\mathbb{P}^1)[1], \mathcal{T}_1[1], \mathcal{T}_0, \pi^* \text{Coh}^{\leq n}(\mathbb{P}^1)].$$

**Proof.** First, let us check that

$$(\mathcal{T}_0, \pi^* \text{Coh}(\mathbb{P}^1), \mathcal{T}_1)$$

is a torsion triple in $\text{Coh}_G(X)$. The conditions

$$\text{Hom}_G(\mathcal{T}_0, \mathcal{T}_1) = 0, \text{Hom}_G(\mathcal{T}_0, \pi^* \text{Coh}(\mathbb{P}^1)) = 0, \text{and } \text{Hom}(\pi^* \text{Coh}(\mathbb{P}^1), \mathcal{T}_1) = 0$$

easily follow from the vanishings

$\text{Hom}_G(\mathcal{O}_{bD_1}, \zeta^{a}(i)_{D_1}) = 0, \text{Hom}_G(\mathcal{O}_{bD_1}, \pi^* \mathcal{O}_{\pi(bD_1)}) = 0,$

where $a = 1, \ldots, m_i-1, b = 1, \ldots, m_i-1$. Using elementary transformations along $D_i$’s as in the proof of Theorem 1.8 one can easily see that every $G$-bundle on $X$ belongs to $[\pi^* \text{Coh}(\mathbb{P}^1), \mathcal{T}_1]$. Now let $F$ be an indecomposable torsion $G$-sheaf on $X$ supported on $D_i$. Then there exists a filtration

$$0 = F_0 \subset F_1 \subset \ldots \subset F_n = F$$
by $G$-subsheaves such that $F_a / F_{a-1} \simeq \zeta(i)^{c-a}$ for $a = 1, \ldots, n$ (where $c \in \mathbb{Z}/m_i \mathbb{Z}$). If $c - a \not\equiv 0 \mod(m_i)$ for all $a = 1, \ldots, n$ then $F \in \mathcal{T}_1$. Otherwise, let $a_1$ (resp., $a_2$) be the minimal (resp., maximal) $a$ such that $c - a \equiv 0 \mod(m_i)$. Then it is easy to see that

$$F_{a_1} \simeq \mathcal{O}_{a_1 D_1} \in \mathcal{T}_0, \quad F_{a_2} / F_{a_1} \in \pi^* \text{Coh}(\mathbb{P}^1), \quad F / F_{a_2} \in \mathcal{T}_1,$$

and hence, $F \in [\mathcal{T}_0, \pi^* \text{Coh}(\mathbb{P}^1), \mathcal{T}_1]$.

Substituting the torsion pair

$$(\pi^* \text{Coh}^{\geq n}(\mathbb{P}^1), \pi^* \text{Coh}^{\leq n-1}(\mathbb{P}^1))$$

into $\pi^* \text{Coh}(\mathbb{P}^1)$ we obtain the required torsion quadruple. One can immediately check that each of the subcategories $\pi^* \text{Coh}^{\leq n-1}(\mathbb{P}^1)[1]$, $\mathcal{T}_1[1]$, $\mathcal{T}_0$ and $\pi^* \text{Coh}^{\geq n}(\mathbb{P}^1)$ belongs to $\mathcal{M}_n$. Hence, the RHS of (2.2) is contained in $\mathcal{M}_n$. Since both subcategories are hearts of nondegenerate $t$-structures, this implies the required equality.

Note that $\pi^* \text{Coh}^{\geq n}(\mathbb{P}^1) \subset \text{Coh}_G^{2n}(X)$ and $\pi^* \text{Coh}^{\leq n-1}(\mathbb{P}^1) \subset \text{Coh}_G^{2n-2}(X)$. Hence, the subcategories $\mathcal{M}_n$ and $\mathcal{C}_G^\theta(X)$ have a large intersection provided $2n - 2 < \theta < 2n$, i.e. $n = [\theta/2] + 1$. Let us show that in this case these categories are related by tilting.

**Proposition 2.7.** Set $n = [\theta/2] + 1$. Then we have the following torsion quadruple in $\mathcal{C}_G^\theta(X)$:

$$(\text{Coh}^{\leq \theta}(X)[1], \mathcal{T}_0, \pi^* \text{Coh}^{\geq n}(\mathbb{P}^1), [\pi^* \text{Coh}^{\leq n-1}(\mathbb{P}^1), \mathcal{T}_1] \cap \text{Coh}_G^\theta(X)).$$

(2.3)

Furthermore, we have

$$\mathcal{M}_n = [[\pi^* \text{Coh}^{\leq n-1}(\mathbb{P}^1)[1], \mathcal{T}_1[1]] \cap \text{Coh}_G^\theta(X)[1], \text{Coh}_G^\theta(X)[1], \mathcal{T}_0, \pi^* \text{Coh}^{\geq n}(\mathbb{P}^1)].$$

(2.4)

**Proof.** First, we observe that

$$(\mathcal{T}_0, \pi^* \text{Coh}^{\geq n}(\mathbb{P}^1), [\pi^* \text{Coh}^{\leq n-1}(\mathbb{P}^1), \mathcal{T}_1] \cap \text{Coh}_G^\theta(X))$$

is a torsion triple in $\text{Coh}_G^\theta(X)$. Indeed, this follows immediately from Proposition 2.6 and from the fact that the subcategory $\text{Coh}_G^\theta(X) \subset \text{Coh}_G(X)$ is stable under passing to quotients. Substituting this triple into the standard torsion pair $(\text{Coh}_G^\theta(X)[1], \text{Coh}_G^\theta(X))$ we obtain the torsion quadruple (2.3). It remains to check that all the constituents in the RHS of (2.4) belong to $\mathcal{M}_n$. For most of them this follows from (2.2). The remaining inclusion $\text{Coh}_G^\theta(X)[1] \subset \mathcal{M}_n$ is implied by the fact that $\pi^* \mathcal{O}_{\mathbb{P}^1}(n)$ and $\pi^* \mathcal{O}_{\mathbb{P}^1}(n + 1)$ have slope $\geq 2n > \theta$. 

In conclusion we are going to interpret the torsion pair in $\mathcal{M}_n$ arising in the above proposition in terms of right modules over the algebra $\Lambda(\overline{X}, \overline{m})$ (see Corollary 1.4) assuming that $X$ is an elliptic curve.

**Theorem 2.8.** Assume that $\pi : E \rightarrow \mathbb{P}^1$ is a ramified Galois covering with the Galois group $G$, where $E$ is an elliptic curve, and let $\overline{m}, \overline{X}$ be the associated ramification data. Fix an irrational number $\theta$ and set $n = [\theta/2] + 1$. Let us define full subcategories $\mathcal{T}^\theta, \mathcal{F}^\theta \subset \text{mod} -\Lambda(\overline{X}, \overline{m})$ as follows: $\mathcal{T}^\theta$ (resp., $\mathcal{F}^\theta$) consists of all modules $M \simeq \bigoplus_{i=1}^k M_i$, where $M_i$
are indecomposable and \(\deg_{\pi}(M) - \theta \mathrm{rk}(M) < 0\) (resp., \(\deg_{\pi}(M) - \theta \mathrm{rk}(M) > 0\)). Then \((\mathcal{T}^\theta, \mathcal{F}^\theta)\) is a torsion pair in \(\mod -\Lambda(\overline{X}, \overline{M})\) and one has
\[
\Phi_n(\mathcal{C}^\theta_G(E)) = [\mathcal{F}^\theta, \mathcal{T}^\theta[-1]] \subset D^b(\Lambda(\overline{X}, \overline{M})).
\]

**Proof.** From Proposition 2.7 we know that \(\mathcal{C}^\theta_G(E) = [\mathcal{F}, \mathcal{T}[-1]]\) for the torsion pair \((\mathcal{T}, \mathcal{F})\) in \(\mathcal{M}_n = \Phi_n^{-1}(\mod -\Lambda(\overline{X}, \overline{M}))\) given by
\[
\mathcal{T} = [\pi^* \text{Coh}^{\leq n-1}(\mathbb{P}^1)[1], \mathcal{T}_1[1]] \cap \text{Coh}^{>\theta}_G(E)[1], \quad \mathcal{F} = [\text{Coh}^{<\theta}_G(E)[1], \mathcal{T}_0, \pi^* \text{Coh}^{\geq n}(\mathbb{P}^1)].
\]

We claim that one has \(\operatorname{Ext}^1_{\mathcal{M}_n}(F, T) = 0\) for every \(F \in \mathcal{F}\) and \(T \in \mathcal{T}\). It suffices to check that \(\operatorname{Hom}_G(F, T[1]) = 0\) for \(T \in \mathcal{T}\) in the following three cases: (i) \(F \in \text{Coh}^{<\theta}_G(E)[1]\); (ii) \(F \in \mathcal{T}_0\); (iii) \(F \in \pi^* \text{Coh}^{\geq n}(\mathbb{P}^1)\). Note that in cases (ii) and (iii) this is clear since cohomological dimension of \(\text{Coh}_G(E)\) is equal to 1. In case (i) we obtain by Serre duality (using triviality of \(\omega_E\))
\[
\operatorname{Hom}(F, T[1])^* \simeq \operatorname{Hom}(T, F) = 0,
\]
since \(T \in \text{Coh}^{>\theta}(E)[1]\) and \(F \in \text{Coh}^{<\theta}(E)[1]\).

It follows that every indecomposable object of \(\mathcal{C}^\theta_G(E)\) is contained either in \(\mathcal{T}\) or in \(\mathcal{F}\). Therefore, \(\mathcal{T}\) (resp., \(\mathcal{F}\)) coincides with the full subcategory of objects \(F\) such that \(F \simeq \bigoplus_{i=1}^n F_i\), where \(F_i\) are indecomposable objects and \(F_i \in \mathcal{T}\) (resp., \(F_i \in \mathcal{F}\)). Since \(\deg(C) - \theta \mathrm{rk}(C) > 0\) for \(C \in \mathcal{C}^\theta(E)\), it follows that \(\deg(F) - \theta \mathrm{rk}(F) > 0\) for \(F \in \mathcal{F}\) and \(\deg(T) - \theta \mathrm{rk}(T) < 0\) for \(T \in \mathcal{T}\). Taking into account Lemma 1.5 we derive that \(\mathcal{T}^\theta = \Phi_n(\mathcal{T})\) and \(\mathcal{F}^\theta = \Phi_n(\mathcal{F})\). \(\square\)

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