Inference from Auction Prices

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Abstract

Econometric inference allows an analyst to back out the values of agents in a mechanism from the rules of the mechanism and bids of the agents. This paper proposes the problem of inferring the values of agents in a mechanism from the social choice function implemented by the mechanism and the per-unit prices paid by the agents (the agent bids are not observed). For single-dimensional agents, this inference problem is a multi-dimensional inversion of the payment identity and is feasible only if the payment identity is uniquely invertible. The inversion is unique for single-unit proportional weights social choice functions (common, for example, in bandwidth allocation); and its inverse can be found efficiently. This inversion is not unique for social choice functions that exhibit complementarities.

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1 Introduction

Traditional econometric inference allows an analyst to determine the values of agents from their equilibrium actions and the rules of a mechanism (Guerre et al., 2000; Haile and Tamer, 2003). This paper studies an inference problem when only the profile of the agents’ per-unit prices is available to the analyst. Such an inference may be applicable when bids are kept private but prices are published; moreover, it is of interest even for incentive compatible mechanisms (where agents truthfully report their preferences). As a motivating example, with the per-unit prices from the incentive compatible mechanism for allocating a divisible item proportionally to agent values (cf. Johari and Tsitsiklis, 2004), we prove that agents’ values are uniquely determined and can be computed efficiently.

Econometric inference is a fundamental topic in a data-driven approach to mechanism design and a number of recent papers have been developing its algorithmic foundations. The following are prominent examples. Chawla et al. (2014, 2016) show that the revenue and welfare of a counterfactual auction can be estimated directly from Bayes-Nash equilibrium bids in an incumbent auction. Nekipelov et al. (2015) develop methods for identifying the rationalizable set of agent values and regret parameters in repeated auctions with learning agents. Hoy et al. (2017) show that the quantities that govern price-of-anarchy analyses can be determined directly from bid data and, thus, empirical price-of-anarchy bounds can be established that improve on the theoretical worst case.

There are two important questions in algorithmic econometrics. First, when are the values uniquely identified? Second, can the values be efficiently computed when the values are identifiable? The first question is studied in depth by the econometrics literature (for inference from actions); the second question is an opportunity for algorithms design and analysis.

We consider inference in single-dimensional environments where a stochastic social choice function maps profiles of agent values to profiles of allocation probabilities. The characterization of incentive compatibility (Myerson, 1981) requires the allocation probability of an agent be monotonically non-decreasing in that agent’s value and that an agent’s expected payments satisfy a payment identity. Per-unit prices – the expected payments conditioned on winning – are easily determined from the total payments in the payment identity by normalizing by the allocation probability. Consequentially, given any social choice function and valuation profile, the allocation probabilities and prices of an incentive compatible mechanism that implements the social choice function are uniquely and easily determined. Our inference problem is the opposite. Given the profile of the agents’ prices, determine the valuation profile that leads to these prices. The social choice function and, thus, the function mapping values to prices is known. The resulting inversion problem is multi-dimensional and this multi-dimensionality leads to a possibility of non-uniqueness (and consequentially, non-identifiability) and computational challenges.

The first goal of this paper is to understand what social choice functions admit inference from prices and which do not. Fundamentally, social choice functions with induced allocation rules that are not strictly increasing do not admit inference. For example, inference possible from the outcome of a second-price auction is only that the winner has value above the winner’s price and the losers

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1Our methods are written assuming that per-unit prices are observed rather than total payments. These prices are more natural for mechanisms usually considered in algorithmic mechanism design as they arise in mechanisms where losers pay nothing, i.e., ex post individually rational mechanism. If instead the realized expected payments and realized allocation probabilities are observed, then these per-unit prices can be easily calculated and our methods applied to the result.
have value below the winner’s price. On the other hand, a “soft max” social choice function like proportional values, where an agent receives a fraction of the item proportional to her value, is strictly continuous and, as we will show, the valuation profile can be uniquely inferred from the outcome. We will characterize social choice functions that admit inference from prices as ones where the Jacobian of the payment identity is positive definite and we will show that, generally, proportional weights social choice functions (with general strictly monotonic weight functions) satisfy this property. In contrast we show that this property does not generally hold for social choice functions that exhibit complementarities.

These identification and non-identification results are complemented by an algorithm for efficiently computing the valuation profile from the prices that corresponds to any proportional weights social choice function for single-item environments.

**Organization.** The rest of this paper is organized as follows. Section 2 gives notation for discussing social choice functions, mechanisms, and agents; reviews the characterization of incentive-compatible single-dimensional mechanisms; and reviews proportional weights allocations. Section 3 then, gives an algorithmic framework for robustly identifying values from prices. It shows that values are identified from payments corresponding to social choice functions given by proportional weights in single-item and multi-unit environments. Section 3.3 shows that values are not identifiable from prices for proportional weights allocations that correspond to environments with complementarities. Section 4 gives an efficient algorithm for inferring values from prices for proportional weights social choice functions in single-item environments.

## 2 Preliminaries

This paper considers general environments for single-dimensional linear agents. Each agent $i$ has value $v_i \in [0, h]$; for allocation probability $x_i$ and expected payment $p_i$, the agent’s utility is $v_i x_i - p_i$. A profile of $n$ agent values is denoted $v = (v_1, \ldots, v_n)$; the profile with agent $i$’s value replaced with $z$ is $(z, v_{-i}) = (v_1, \ldots, v_{i-1}, z, v_{i+1}, \ldots, v_n)$.

A **stochastic social choice function** $x$ maps a profile of values $v$ to a profile of allocation probabilities. A **dominant strategy incentive compatible (DSIC) mechanism** $(x, p)$ maps a profile of values $v$ to profiles of allocations $x(v)$ and payments $p(v)$ so that: for all agents $i$, values $v_i$, and other agent values $v_{-i}$, it is optimal for agent $i$ to bid her value $v_i$. The following theorem of Myerson (1981) characterizes social choice functions that can be implemented by DSIC mechanisms.

**Theorem 1.** (Myerson, 1981) Allocation and payment rules $(x, p)$ are induced by a dominant strategy incentive compatible mechanism if and only if for each agent $i$,

1. (monotonicity) allocation rule $x_i(z, v_{-i})$ is monotone non-decreasing in $z$, and

2. (payment identity) payment rule $p_i(v)$ satisfies

$$ p_i(v) = v_i x_i(v) - \int_0^{v_i} x_i(z, v_{-i}) \, dz + p_i(0, v_{-i}), $$

(1)

where the payment of an agent with value zero is often zero, i.e., $p_i(0, v_{-i}) = 0$. 
Most DSIC mechanisms are implemented to satisfy an ex post individual rationality constraint; specifically, an agent pays nothing when she is not allocated. Her payment when she is allocated, i.e., her per-unit price, is thus her expected payment normalized by her winning probability. Denote the price function by $\pi : \mathbb{R}_+^n \to \mathbb{R}_+^n$, as

$$\pi_i(v) = p_i(v)/x_i(v)$$

$$\pi_i(v) = v_i + \int_0^{v_i} x_i(z, v_{-i}) \, dz / x_i(v)$$

for all agents $i$.

The main objective of this paper is to infer the agents' values from observations of the per-unit prices of the mechanism. A price profile $\rho$ is observed, and it is desired to infer the valuation profile $v$ that generated this price profile by $\rho = \pi(v)$. The key question of this paper is to identify sufficient conditions on the social choice function $x$ such that the price function $\pi$ is invertible.

An important special case is the case where there is $n = 1$ agent and the price function $\pi(\cdot)$ is single-dimensional. When the social choice function $x(\cdot)$ is strictly increasing, the price function $\pi(\cdot)$ is strictly increasing and is uniquely invertible. Thus, the agent’s value can be identified from her observed price $\rho$, e.g., by binary search.

Our goal is to understand families of (multi-agent) social choice functions $x$ that allow values to be inferred from prices. Clearly, as in the single-agent case, if the allocation rule is not strictly increasing in each agent’s value, then the values of the agents cannot be inferred. We assume that the social choice function $x$ is such that it has strictly-increasing allocation functions $x_i$ for any given $v_{-i}$, for all $v_i > 0$. The mechanisms in the literature for welfare and revenue maximization are based on social choice functions that map agents’ values to weights and allocate to maximize the sum of the weights of the agents allocated. In order to satisfy the required strict monotonicity property, our focus is on smoothed versions of these social choice functions under feasibility constraints that correspond to single-item and single-minded combinatorial auctions.

In single-item environments a natural “soft max” is given by proportional weights allocations. A weight function is given for each agent $i$ as a strictly monotone and continuously differentiable function $w_i : \mathbb{R}_+ \to \mathbb{R}_+$ and the proportional weights social choice function maps each agent’s value to a weight and then allocates to an agent with probability proportional to her weight. A canonical example of proportional weights is exponential weights, i.e., $w_i(v_i) = e^{v_i}$ for each agent $i$.

Given the assumptions on functions $w$, they are invertible. Where appropriate we will overload $v_i$ to allow it to be the functional inverse of $w_i$ mapping a weight back to its value. We also overload the notations $x, \pi$ to take weights $w$ as an input, with $x(w) := x(v(w))$ and $\pi(w) := \pi(v(w))$.

### 3 Identification and Non-identification

This section considers sufficient conditions under which values can be inferred from the observed prices $\rho$ of a DSIC mechanism $(x, p)$. We address two theoretical challenges with identifying values from prices. First, values can only possibly be identified from prices if the price function $\pi$ is invertible.

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2 Observe that the derivative of the price function $\pi'(v) = x'(v) / (x(v))^2$ is positive if $x'(v)$ is positive.

3 For simplicity in the main body of the paper, we assume that all weights functions are everywhere strictly positive for all agents, even at $v_i = 0$. The removal of this assumption is handled as a technical extension.
invertible. Second, the image of the price function $\pi$ may not be a product space; for robustness of the inference algorithm, we desire a mapping for any observed prices $\rho$ back to a profile of values.

Our approach to these challenges is to write the problem of inverting the price function $\pi$ at prices $\rho$ as a proxy game between proxy players whose actions are values; and where the payoff function of a proxy player $i$ for a proxy value $\tilde{v}_i$ is the profile of proxy values from the other proxy players $\tilde{v}_{-i}$ -- is a concave function that is optimized where $\pi(\tilde{v})$ on the proxy valuation profile is closest to the observed bid $\rho_i$. For fixed observed prices $\rho$, define the "proxy game." The proxy game is defined with values $\tilde{v}$ as proxy actions, and with utilities for the proxy agents given by the cumulative price-imbalance functions. As desired, when other players select proxy values $\tilde{v}_{-i}$, proxy player $i$ would select proxy value $\tilde{v}_i$ so that agent $i$’s price according to $\pi$ on $\tilde{v}$, i.e., $\pi_i(\tilde{v})$ is close to agent $i$’s given payment $\rho_i$. Based on this proxy game, we define the following inference algorithm.

**Definition 1.** The price-inversion algorithm $A$ on price space $[0, \infty]^n$ for social choice function $x$ on value space $[0, h]^n$ is

1. Observe price profile $\rho$.
2. Select a Nash equilibrium $\tilde{v}$ in the proxy game defined in value space $[0, h]^n$ with utility functions given by the cumulative price-imbalance $\Phi^\rho$ for $\rho$.
3. Return inferred values $\tilde{v}$.

A key property for the proper working of the price-inversion algorithm is whether the proxy game admits a unique pure Nash equilibrium. For example, if there are multiple distinct valuation profiles that map to the same prices via $\pi$, then each of these valuation profiles will be an equilibrium in the proxy game. The computational question of finding a Nash equilibrium of the proxy game is deferred to Section 4.

**Proposition 1.** Any valuation profile $v \in [0, h]^n$ such that observed price profile $\rho = \pi(v)$ is a Nash equilibrium of the proxy game on the social choice function $x$ and prices $\rho$; if this Nash equilibrium $v$ of the proxy game is unique then the inverse $\pi^{-1}(\rho)$ is unique and given by the price inversion algorithm $A$.

**Proof.** The second part follows from the first part. For the first part, assume $\rho = \pi(v)$ for some $v \in [0, h]^n$. Action profile $v$ in the proxy game is a Nash equilibrium as each proxy agent’s first-order condition is satisfied. Specifically, with utilities given by the cumulative imbalances $\Phi^\rho$, the first-order condition is given by $\phi^\rho_i(v_i, v_{-i}) = \rho_i - \pi_i(v_i, v_{-i})$ and is zero by the choice of $v$. Checking first-order conditions is sufficient because $\Phi^\rho$ is strictly concave, i.e., $\frac{d\phi^\rho_i(v_i, v_{-i})}{dv_i} = -\pi_i'(v) < 0$. 

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4Deferring detailed discussion to the full paper, this robustness will allow inference from even "noisy" or "erroneous" price profiles $\tilde{\rho}$ outside the image of the price function $\pi$. 

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Motivated by Proposition 1, the remainder of this section identifies proportional weights as a large natural class of social choice functions for which the proxy game has a unique pure Nash equilibrium for all price profiles. Of course, a necessary condition for the uniqueness of pure Nash in the proxy game is that the price function $\pi$ is one-to-one. In Section 3.1 we show that this condition is implied by a slightly weaker condition than the following: the positive definiteness everywhere of the Jacobian of $\pi$, denoted $J_\pi$. In Section 3.2 we show that all members of the same class of proportional weights social choice functions (for single-unit environments) induce bid functions that satisfy this condition. In contrast, Section 3.3 describes some natural variants of proportional weights social choice functions for environments which resemble single-minded combinatorial auctions, and shows that the bid functions for these social choice functions are not generally invertible, and therefore the proxy game does not have a unique pure Nash equilibrium in this extended setting.

### 3.1 Sufficiency of “Interior” Positive Definiteness

This section shows that a sufficient condition for the uniqueness of a pure Nash equilibrium in the proxy game defined in the price-inversion algorithm (Definition 1) is the “interior” positive definiteness (to be defined shortly) of the Jacobian of the price function $\pi$ for the social choice rule $x$. In this and the subsequent sections we make use of the following facts about positive definite matrices:

**Fact 1.** A positive definite matrix $M$ satisfies $z^\top M z > 0$ for all $z \neq 0$. (Note: $M$ is not required to be symmetric.) Positive (semi-)definite matrices have the following properties:

- For a positive definite matrix $M$, its negation $-M$ is negative definite.
- The product of two symmetric, positive definite matrices is also symmetric, positive definite.
- The sum of a positive definite matrix and a positive semi-definite matrix is positive definite.
- The sum of two positive semi-definite matrices $M_1, M_2$ is positive definite if $\forall z$, they have $z^\top M_1 z > 0$ or $z^\top M_2 z > 0$ (they are both at least 0 by assumption).

Define a function $f : \mathbb{R}^n \to \mathbb{R}^n$ to be positive definite if its Jacobian is positive definite at all points of the function’s domain. We weaken this functional definition:

**Definition 2.** For product space $\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, a function $f : \Omega \to \mathbb{R}^n$ is interior positive definite if for every point $\omega \in \Omega$:

- the Jacobian of $f$ evaluated at $\omega$ as $J_f(\omega)$ is positive semi-definite, and,
- choosing the minor of $J_f(\omega)$ that removes row/column pairs corresponding to the dimensions in which $\omega$ is on a closed boundary of $\Omega$, this principal minor of $J_f(\omega)$ is positive definite.

\[^5\text{This second requirement of interior positive definite at } \omega \in \Omega \text{ is equivalent to saying that an } \omega \text{ has to pass the } z^\top J_f(\omega) z > 0 \text{ “tests” of strictly positive definiteness on only a weakly smaller set of non-zero vectors } z. \text{ It requires only: } z^\top J_f(\omega) z > 0 \forall z \in \{y | y \neq 0, y_j = 0 \forall j \text{ such that } \omega \in \{j \text{-dimension boundary of product space } \Omega\}\} \]
We show that when the Jacobian of the price function is interior positive definite, the proxy game is related to concave games as defined and studied by Rosen (1965). Rosen defined concave games as games where utility functions are diagonally strictly concave, a property which he also defined. He proves that negative definiteness of the pseudogradient of the Jacobian of the utility functions (for all points in the action space) implies diagonal strict concavity. Notice the structural comparisons which we have already set up: in the proxy game with utilities given by $\Phi^\rho$, the pseudogradient is given by $\phi^\rho$, and the Jacobian of the pseudogradient is the negation of the Jacobian of the function $\pi$. We state two of Rosen’s main results (without needing a formal definition of diagonal strict concavity for our purposes).

**Theorem 2** (Rosen, 1965). A concave game, i.e., with a convex and compact action space and diagonally strictly convex payoffs, has a unique Nash equilibrium, and this equilibrium is pure.

**Lemma 1** (Rosen, 1965). A game where the pseudogradient of the Jacobian of the payoffs is negative definite and the action space is convex and compact has payoffs that are diagonally strictly concave; and therefore the game is a concave game.

Combining Lemma 1 and Theorem 2, a corollary would be that negative definite Jacobian of pseudogradient of utilities is sufficient for existence and uniqueness of pure Nash equilibrium. A direct proof of this corollary- with weaker conditions- was given by Gale and Nikaido (1965).

**Theorem 3** (Gale and Nikaido, 1965). A continuously differentiable function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ with convex $\Omega$ is one-to-one if its Jacobian is everywhere positive (or everywhere negative) definite.

We will need a generalization of these results that relaxes negative definiteness on axis aligned boundary (as in Definition 2). Specifically, below in equation (7), the pseudogradient of the utility function may be only negative semidefinite on the lower boundaries. It will be sufficient to only prove interior negative definiteness as negative semidefiniteness on the boundary will then be implied by continuity. The proof of Theorem 4 here is given in Appendix A.1.

**Theorem 4.** A game with $n$ players and

- a compact and convex product action space $\Omega_n \subset \mathbb{R}^n$;
- a continuous and twice-differentiable utility function $U : \Omega_n \to \mathbb{R}^n$;
- and a pseudogradient of the utility function $U$ that is interior negative definite (Definition 2),

has a unique Nash equilibrium, and this equilibrium is pure.

**Corollary 1.** Given agents with (unknown) values $v \in [0,h]^n$. Consider $\Pi$, the Jacobian of the price function $\pi$ resulting from a strictly increasing, continuous and differentiable proportional weights social choice function $\mathbf{x}$, and dominant-strategy incentive-compatible mechanism implementing $\mathbf{x}$. If $J_{\pi}$ is interior positive definite, then:

\footnote{Though it will not be important for the results of this paper, Uti (2008) shows that the unique Nash equilibrium of a concave game is, in fact, unique among correlated equilibria as well.}

\footnote{Note that for the similar question of whether or not a function gives a bijection between two non-closed subspaces of $\mathbb{R}^n$, negative definiteness of the Jacobian is not sufficient. See, e.g., page 106 of Katok and Climenhaga (2008).}
the inverse function $\pi^{-1}$ given by the unique Nash of the proxy game is well-defined everywhere on a universal domain as $\mathbb{R}^n$;

and in particular on the restricted domain $\rho \in \text{Image}(\pi)$, the price-inversion algorithm $A$ (Definition 1) infers the true values $v$ from the mechanism’s outcome (summarized by prices $\rho = \pi(v)$).

Proof. In the proxy game with payoffs given by $\Phi^\rho$, the pseudogradient of the payoffs is given by $\phi^\rho$, and the Jacobian of the pseudogradient is the negation of the matrix $J_{\pi}$. Interior positive definiteness of $J_{\pi}$ is equivalent to interior negative definiteness of its negation. Thus, the proxy game is concave (Lemma 1) and admits a unique Nash equilibrium which is pure (Theorem 2). Defining the inverse function $\pi^{-1}$ to output the unique Nash of the proxy game is sufficient for its output to be unique.

Finally, Proposition 1 implies that the price-inversion algorithm outputs the true values $v$ when the input is true prices $\rho$. The construction of the proxy game and Theorem 4 proves that interior positive definiteness of $J_{\pi}$ implies that $\pi$ has a unique functional inverse when restricted to its range. \(\square\)

In the subsequent subsections we prove that proportional weights social choice functions for a single-item induces price functions with interior positive definite Jacobian. Thus, the price-inversion algorithm for these social choice functions identifies correct value parameters of the agents from the observed prices.

3.2 Single Item Proportional Weights Social Choice Functions

The goal of this section is to show that every proportional weights social choice function awarding a single item yields price function $\pi$ whose Jacobian $J_{\pi}$ is positive definite (thereby satisfying sufficient conditions from the previous section). We give the main result of the paper:

**Theorem 5.** Given agents with (unknown) values $v \in [0,h]^n$. For a strictly monotone, continuous, differentiable proportional weights social choice rule $x$, the price inversion algorithm of Definition 1 meets the conditions of Proposition 1 and outputs the true values $v$ by inference from the dominant-strategy incentive-compatible mechanism’s outcome (summarized by $\rho$), on every valuation profile $v \in [0,h]^n$.

Proof. The claim follows from Corollary 1 in Section 3.1 and Theorem 7 below. \(\square\)

It will be convenient to restate the inversion problem of Theorem 5 in weight space rather than value space. Recall, weights functions $w_i(v_i)$ are continuously differentiable, positive, strictly increasing functions mapping an agent’s value to weight. Defined as such, they can be inverted as $v_i(w_i) := w_i^{-1}(v_i)$. Recalling equation (3), we can transform the price function $\pi$ to weights-space using calculus-change-of-variables as:

$$\pi_i(w) = v_i(w_i) - \frac{\int_{w_i(0)}^{w_i} x_i(z, w_{-i})v'_i(z)dz}{x_i(w)}.$$  (6)

It is straightforward to calculate the cross derivatives which appear as elements of the Jacobian $J_{\pi}$ (the steps of the calculations are given in Appendix A.2).
Lemma 2. Given the price function $\pi$ for proportional weights, for $j, k \neq i$, the cross derivatives are the same: $\frac{\partial \pi_i}{\partial w_j} = \frac{\partial \pi_i}{\partial w_k}$. Evaluating the Jacobian at $w$, further, all elements of the Jacobian matrix $J_\pi$ are positive, i.e., $\frac{\partial \pi_i}{\partial w_i} > 0$, $\frac{\partial \pi_i}{\partial w_j} > 0$, except at the $w_i(0)$ lower boundary in dimension $i$ where the elements of row $i$ are $\frac{\partial \pi_i}{\partial w_i} = \frac{\partial \pi_i}{\partial w_j} = 0$.

The proof of Lemma 2 is in Appendix A.3.

By Theorem 3, we only need to prove that the Jacobian $J_\pi$ is positive definite, a result which could be of independent interest. We explicitly define the ratio of an agent’s “self-partial” to its “cross-partial” for any $j \neq i$ by $h_i$, which will be needed for analysis throughout the rest of the paper.

$$h_i = \frac{\partial \pi_i}{\partial w_i} \frac{\partial \pi_i}{\partial w_j}$$

Using Lemma 2, we write the Jacobian principal minor as

$$J_{\pi, N} = D \cdot H = \begin{bmatrix} \frac{\partial \pi_1}{\partial w_2} & 0 & 0 & \ldots & 0 \\ 0 & \frac{\partial \pi_2}{\partial w_1} & 0 & \ldots & 0 \\ 0 & 0 & \frac{\partial \pi_3}{\partial w_1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{\partial \pi_N}{\partial w_1} \end{bmatrix} \begin{bmatrix} h_1 & 1 & 1 & \ldots & 1 \\ 1 & h_2 & 1 & \ldots & 1 \\ 1 & 1 & h_3 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & h_N \end{bmatrix}$$

Multiplying by a positive diagonal matrix $D$ is a benign operation with respect to the determination of positive definiteness. We will define $H$ to be the matrix on the right of equation (10) which is composed of $h_i$s on the diagonal and all ones elsewhere. By reduction we need only show that $H$ is positive definite.

We claim the following results, starting with a complete characterization of when an arbitrary matrix $G$ with arbitrary $g_i$ is positive definite, a result which could be of independent interest.

**Theorem 6.** Consider a matrix $G$ with diagonal $g_1, g_2, \ldots, g_n$ and all other entries equal to 1 (and without loss of generality $g_1 \leq g_2 \leq \ldots \leq g_n$). The following is a complete characterization of when $G$ is positive definite.

1. if $g_1 \leq 0$, then the matrix $G$ is not positive definite;
2. if $g_1 \geq 1$ and $g_2 > 1$, then $G$ is positive definite;

*Technically the $h_i$ terms are functions, each of input $w$, but we suppress this in the notation.*
3. if \( g_1, g_2 \leq 1 \), then \( G \) is not positive definite;

4. if \( g_1 < 1 \) and \( g_2 > 1 \), then \( G \) is positive definite if and only if \( \sum_k \frac{1}{1-g_k} > 1 \).

The proof of Theorem 6 is given in Appendix A.4 where the main difficulty is part (4). Theorem 6 is for arbitrary \( G \). We now return to the specific consideration of \( H \) resulting from \( \pi \) and \( J_{\pi,N} \), showing it must be covered by cases (2) or (4) from Theorem 6. The proofs of Lemma 3 and Lemma 4 are given in Appendix A.5.

**Lemma 3.** If \( h_i \leq 1 \), then \( w_i > 0 \). When \( h_1 < 1 \) and \( h_j > 1 \) \( \forall j \neq 1 \), we have \( \sum_k \frac{1}{1-h_k} > 1 \).

**Lemma 4.** When \( h_1 < 1 \) and \( h_j > 1 \) \( \forall j \neq 1 \), we have \( \sum_k \frac{1}{1-h_k} > 1 \).

**Theorem 7.** Let matrix \( J_{\pi} \) be the Jacobian of \( \pi \) at weights \( w \) of a positive, strictly increasing, and differentiable proportional weights social choice functions. \( J_{\pi} \) is interior positive definite.

**Proof.** By the definition of interior positive definiteness (Definition 2), we consider the restriction to the minor \( J_{\pi,N} = D \cdot H \) where coordinates on the boundary of price space have been discarded, see (10). Since weights \( \{w_i\}_{i \in \{1,\ldots,N\}} \) are interior, Lemma 2 implies that \( \{h_i\}_{i \in \{1,\ldots,N\}} \) are strictly positive. By Lemma 3, at most one agent \( i \) has \( h_i \leq 1 \). Without loss of generality, we can set this \( i = 1 \). So there are just two cases:

1. \( h_1 \geq 1 \) and \( h_j > 1 \) \( \forall j \neq 1 \), and

2. \( 0 < h_1 < 1 \) and \( h_j > 1 \) \( \forall j \neq 1 \).

These are respectively cases (2) and (4) of Theorem 6. To satisfy the condition within case (4) of Theorem 6, Lemma 4 is sufficient. Thus, the factor \( H \) of the Jacobian minor \( J_{\pi,N} = D \cdot H \) is positive definite, and so is the product \( J_{\pi,N} = D \cdot H \).

### 3.3 Impossibility Results for Complementarities

In this section we show that natural generalizations of the proportional weights social choice functions to environments with complementarities between agents cannot identify the values of the agents from the outcome of the mechanism.

The impossibility result we present will consider a generalization of exponential weights to environments with complementarities. We will consider the special case where the agents are partitioned and the mechanism can allocate to all agents in any one part, but agents in multiple parts may not be simultaneously served. We prove that a natural extension of exponential weights to partition set systems results in a price function \( \pi \) that is not one-to-one. Thus, it is not invertible, no algorithm can distinguish between the two (or more) valuation profiles that give the same prices.

**Definition 3.** The exponential weights social choice function for an \( n \)-agent partition set system with parts \( S = (S_1, \ldots, S_r) \) is given by:

- \( v_S = \sum_{i \in S} v_i \) for \( S \in S \);
- \( x_S(v) = \frac{e^{v_S}}{\sum_{T \in S} e^{v_T}} \) for \( S \in S \);
- \( x_i(v) = x_S(v) \) for \( i \in S \).
Figure 1: Graphing the “function” \((\pi_1 - \pi_5)(\alpha)\) from the proof of Lemma 5. The zeroes of the function parameterize values for agents in \(S_1\) and \(S_2\) such that all agents across both parts have identical bids, despite the agents of each group having strictly distinct values from each other. (By design, the curve is rotationally symmetric around the point \((5,0)\).)

The resulting price function corresponding to the exponential weights social choice function for partition set systems is

\[
\pi_i(v) = v_i - \frac{\int_0^{v_i} x_i(z, v-\alpha)dz}{x_i(v)}
= v_i - \sum_T e^{v_T} \int_0^{v_i} \frac{e^z e^{v_{S\setminus\{i\}}}}{e^{v_{S\setminus\{i\}}} + \sum_{T \neq S} e^{v_T}} dz
= v_i - \sum_T e^{v_T} \left(\ln \left(\sum_T e^{v_T}\right) - \ln \left(e^{v_{S\setminus\{i\}}} + \sum_{T \neq S} e^{v_T}\right)\right)
\]

The completion of the counter-example is in the following lemma.

**Lemma 5.** The price function \(\pi\) corresponding to the exponential weights social choice function for partition set systems (with at least one partition containing two or more agents) is not one-to-one.

**Proof.** We prove that the price function is not one-to-one (consequently by the contrapositive of Corollary 1 its Jacobian is not positive definite). We first set up a parameterized analysis and then choose the parameters later.

Let there be \(k\) agents in set \(S_1\) who all have the same valuation \(\alpha/k\), and another \(k\) agents in set \(S_2\) who all have the same valuation \((\beta - \alpha)/k\). Note \(\beta = v_{S_1} + v_{S_2}\). Players in all other sets \(S_r, r > 2\) have a constant value \(v_{\text{others}}\) and can be summarized by a single parameter \(\delta\) by letting \(e^\delta = \sum_{r \geq 2} e^{v_{S_r}}\). Parameters \(k, \alpha, \beta\) and \(v_{\text{others}}\) will be determined later.

Then the price for agent 1 in part \(S_1\) is

\[
\pi_1 = \frac{\alpha}{k} - \frac{e^\alpha + e^{\beta-\alpha} + e^\delta}{e^\alpha} \left[ \ln(e^\alpha + e^{\beta-\alpha} + e^\delta) - \ln(e^{(1-1/k)\alpha} + e^{\beta-\alpha} + e^\delta) \right]
\]
The price for agent $k + 1$ in part $S_2$ is
\[
\pi_{k+1} = \frac{\beta - \alpha}{k} - \frac{e^\alpha + e^{\beta - \alpha} + e^\delta}{e^{\beta - \alpha}} \left[ \ln(e^\alpha + e^{\beta - \alpha} + e^\delta) - \ln(e^{(1-1/k)(\beta-\alpha)} + e^\alpha + e^\delta) \right]
\]

It is possible that player 1 and player $k + 1$ have different valuations but are charged the same prices. Consider the case $k = 4, \beta = 10$, and there is one additional part $S_3$ with a single agent $9$ with $v_9 = v_{\text{others}} = 4$ inducing $\delta = 4$. Then we can consider the quantity $(\pi_1 - \pi_5)$ as a function of parameter $\alpha$ with $v_1 = ... = v_4 = \alpha/4$ and $v_5 = ... = v_8 = (10 - \alpha)/4$.

This function $(\pi_1 - \pi_5)(\alpha)$ is graphed in Figure 1 where we can see that there are three solutions for $\pi_1 = \pi_5$. Then (without showing the calculation), $\pi_1 = \pi_5$ holds for a value profile where $v_1 = ... v_4 = \alpha/4 \approx 0.375$ and $v_5 = ... v_8 = (10 - \alpha)/4 \approx 2.125$, and $v_9 = 4$. In this case, the seller cannot tell between $S_1$ and $S_2$ which part has agents with identical values $\approx 0.375$ versus the other part whose agents all have values $\approx 2.125$.

This lemma can be generalized as follows. A set system is downward-closed if all subsets of feasible sets are feasible. Agents are substitutes if the set system satisfies the matroid augmentation property, i.e., for any pair of feasible sets with distinct cardinalities, an element from the smaller set that is not in the larger set can be added to the larger set and the resulting set remains feasible. A set system exhibits complementarities if agents are substitutes (i.e., there exists sets that fail the augmentation property). Exponential weights can be generalized to any set system by choosing a maximal set with probability proportional to its exponentiated weight. The impossibility result above can then be easily generalized to any set system that exhibits complementarities by identifying the sets and taking $S_1$ and $S_2$ to be the agents uniquely in each set (i.e., and not in their intersection) and setting all other agent values to zero.

4 Computational Methods for Inverting the Price Function

In Section 3, we gave a well-defined, continuous function that inverts the payment identity $\pi$ to map prices $\rho$ back to weights $w$ (equivalently, values $v$). The price-inversion algorithm (Definition 1) is straightforward except for Step 2 which requires the computation of a Nash equilibrium in the defined proxy game. In this section we give a simple algorithm for identifying a Nash equilibrium of the proxy game and thus show that the inverse function can be easily computed.

The algorithm for solving for solving the proxy game is enabled by two observations. First, for player $i$, the sum of weights $s = \sum_k w_k$ summarizes everything that needs to be known about the other players and this observation leads to a many-to-one reduction in the dimension of search space. Consequently, the price function can be rewritten as a function $\bar{\pi}_i(s, w_i)$.

Second, because the price function $\pi$ is invertible, the sum $s$ is uniquely determined from the prices.

Obviously at most one agent can have strictly more than half the total weight $s$. For the rest of this section, without loss of generality we fix agent $i^*$ to mean that $w_{i^*}$ is not restricted and $w_i \leq s/2$ for all $i \neq i^*$.

Fix observed input prices $\rho$. For any agent $i$, consider the set of points $(s, w_i)$ for which $\bar{\pi}_i$ outputs $\rho_i$. Our first key Lemma 6 will show that, restricting to the space $w_i \leq s/2$, this set of points can be interpreted as a real-valued, monotone decreasing function of $s$, denoted $w_{i^*}(\cdot)$. With

\footnote{See equation (27) in Appendix B}
this property holding for all agents other than $i^*$, we can express the price function for agent $i^*$ with dependence on prices $\rho_{-i^*}$ and sum $s$.

$$\pi_{i^*}^\rho(s) := \pi_{i^*}(\max\{s - \sum_{i \neq i^*} w_i^\rho(s), w_{i^*}(0)\}, w_{-i^*}^\rho(s))$$  \hfill (11)

where, for guess $s$, the quantity $s - \sum_{i \neq i^*} w_i^\rho(s)$ assigns an intermediate guess of $w_{i^*}$ as the “balance” of the quantity $s$ having subtracted the implied weights of the “small” agents for guess $s$. Our second key Lemma 7 (below) shows that, on the range of $s$ for which it is well-defined, the function $\pi_{i^*}^\rho$ is strictly monotonically increasing.

This setup suggests a natural binary search procedure. For some agent $i^*$ and small initial guess of $s$, the implied price for $i^*$ is smaller than the observed input, i.e., $\pi_{i^*}^\rho(s) < \rho_{i^*}$. A large guess of $s$ implying too big of a price and monotonicity will then guarantee a crossing. The algorithm has the following steps:

1. Find an agent $i^*$ by iteratively running the following for each fixed assignment of agent $i \in \{1, \ldots, n\}$:
   (a) temporarily set $i^* = i$;
   (b) determine the range of $s$ on which $\pi_{i^*}^\rho$ is well-defined and searching is appropriate;\footnote{A note for functions $w_{i^*}^\rho$ and $\pi_{i^*}^\rho$. We write them both indexed by vector $\rho$ to demark them in a common, simple way because their usage is always related. However $\rho$ implies an over-dependence on parameters. $w_{i^*}^\rho$ only uses $\rho_{i^*}$ and $\pi_{i^*}^\rho$ uses all of $\rho_{-i^*}$ but not $\rho_{i^*}$.}
   (c) if this range of $s$ is non-empty, permanently fix $i^* = i$ and break the for-loop;
2. use the monotonicity of $\pi_{i^*}^\rho$ to binary search on $s$ for the true $s^*$, converging $\pi_{i^*}^\rho(s)$ to $\rho_{i^*}$;\footnote{When the guess $w_{i^*} = s - \sum_{i \neq i^*} w_i^\rho(s)$ is irrationally small or even negative, the structure of the problem allows}
3. when the binary search has been run to satisfactory precision and reached a final estimate $\tilde{s}$, output weights $\tilde{w} = (\tilde{s} - \sum_{i \neq i^*} w_i^\rho(\tilde{s}), w_{-i^*}^\rho(\tilde{s}))$ which invert to values $\tilde{v}$ via respective $v_i(\cdot)$ functions.

The rest of this section formalizes our key results.

### 4.1 Computation through Total Sum Weights

The following theorem claims correctness of the algorithm, and is the main result of this section.

**Theorem 8.** Given weights $w$ and payments $\rho = \pi(w)$ according to a proportional weights social choice function, the algorithm identifies weights $\tilde{w}$ within $\varepsilon$ of the true weights $w$ in time polynomial in the number of agents $n$, the logarithm of the ratio of high to low weights $\max_i \ln(w_i(h)/w_i(0))$, and the logarithm of the desired precision $\ln 1/\varepsilon$.

A major object of interest for this sequence of results is the price level set defined by $Q^\rho = \{(s, w_i) \mid \pi_i(s, w_i) = \rho_i\}$, i.e., all of the $(s, w_i)$ pairs that result in the input price $\rho_i$, and also in particular its subset $P^\rho = \{(s, w_i) \mid \pi_i(s, w_i) = \rho_i \text{ and } w_i \leq s/2\} \subseteq Q^\rho$ which restricts the set to the region where $w_i$ is at most half the total weight $s$. Define $r^\rho = \min\{s : (s, w_i) \in P^\rho\}$ as the lower bound on the sum $s$ on which the set $P^\rho$ is supported. These quantities are depicted in Figure 2.
Figure 2: The price level set curve \( \mathcal{Q}_i^\rho = \{(s, w_i) : \bar{\pi}_i(s, w_i) = \rho_i\} \) (thick, gray, dashed), is decreasing below the \( w_i = s/2 \) line (Lemma 6) where it is defined by its subset \( \mathcal{P}_i^\rho \) (thin, black, solid). It is bounded above by the \( w_i = s \) line (trivially as \( s \) sums over all weights) and the \( w_i = w_i(h) \) line (the maximum weight in the support of the values), and below by the \( w_i = w_i(0) \) line which we have assumed to be strictly positive. \( r_i^\rho \) is the minimum weight-sum consistent with observed price \( \rho_i \) and weights \( w_i \leq s/2 \).

**Lemma 6.** The price level set \( \mathcal{Q}_i^\rho \) is a curve; further, restricting \( \mathcal{Q}_i^\rho \) to the region \( w_i \leq s/2 \), the resulting subset \( \mathcal{P}_i^\rho \) can be written as \( \{(s, w_i^\rho(s)) : s \in [r_i^\rho, \infty)\} \) for a real-valued decreasing function \( w_i^\rho \) mapping sum \( s \) to a weight \( w_i \) that is parameterized by the observed price \( \rho_i \).

**Lemma 7.** For any agent \( i^* \) and \( s \in [\max_j \rho_j, \infty) \), function \( \pi_i^\rho \) is strictly increasing.

A key step in the proof of Lemma 7 will depend on Lemma 4, so we will see that the correctness of our algorithm’s technique is critically related to the proof of existence of a unique inverse in the first place. The \( \frac{1}{1-n_k} \) terms in the statement of Lemma 4 are realized to be the derivatives of the \( w_i^\rho \) functions.

We give the proofs of Theorem 8, Lemma 6, and Lemma 7 in Appendix B.3. Preceding these proofs within Appendix B.3 is a more detailed analysis of the structure of the search space, with a more explicit description of the binary search algorithm in Appendix B.2.

**References**

Chawla, S., Hartline, J., and Nekipelov, D. (2014). Mechanism design for data science. In *Proceedings of the fifteenth ACM conference on Economics and computation*, pages 711–712. ACM.

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us to round it up to constant \( w_i^\rho(0) \) while still preserving monotonicity. See the proof of Lemma 6 in Appendix B.3.

\[^{12}\text{In the proper algorithm and proof, we will give better bounds on the range of the search; for now, as a simple indication that bounds exist, note that there exists a solution for some appropriate } i^* \text{ within the general bounds on } s \text{ as } \sum_k w_k(\rho_k) \leq \sum_k w_k(v_k) = s \leq \sum_k w_k(h) \text{ for known } \rho_i \text{ and max value } h, \text{ because } \rho_i \leq v_i \leq h.\]
Supporting Material for Section 3

A.1 Proof of Theorem 4

First, existence is from Rosen.

**Theorem 9 (Rosen, 1965).** A game with a convex and compact action space in which all the utilities functions are individually continuous and continuously differentiable and weakly concave has a Nash equilibrium in pure strategies. In particular, the utility function having Jacobian of the pseudogradient which is negative semi-definite is sufficient.

The following re-statement and proof of Theorem 4 subsumes Theorem 9 (as well as extending Theorem 2). Also note that the statement of Theorem 4 here has been appended to unpack the definition of *interior negative definite* and define this assumption in terms of the notation of the proxy game and new notation as needed in the proof.

**Theorem 4.** A game with n players and

1. a compact product action space $\Omega_n \subset \mathbb{R}^n$;
2. a continuous and twice-differentiable utility function $U : \Omega_n \to \mathbb{R}^n$;
3. and a pseudogradient of the utility function $U$ that is interior negative definite (Definition 2),
has a unique Nash equilibrium, and this equilibrium is pure.

Proof. Re-writing (3) explicitly in the notation of the theorem statement, (3) is equivalent to assuming the following condition on the Jacobian of the pseudogradient of \( U \) denoted \( J_{pgU} \):

- for all points \( a \) strictly interior to the action space, \( J_{pgU}(a) \) is negative definite;
- for points \( a \) on the boundary of the action space, specifically on the boundary with respect to all dimensions in a set \( C(a) \), \( J_{pgU}(a) \) can be either negative definite, or negative semi-definite, however: where \( J_{pgU}(a) \) is negative semi-definite, it must be that its principal minor removing rows/columns corresponding to the set \( C(a) \) as \( J_{pgU} - C(a) \) is negative definite.

The following proof is adapted from Rosen’s proof of the sufficiency of a diagonal strictly concave \( J_{pgU} \) matrix, and the Gale-Nikkaio proof of Theorem 3. First, existence of Nash in pure strategies follows quite simply from Rosen and Theorem 9.

For uniqueness, by contradiction, assume there are two action vectors \( a^0 \) and \( a^1 \) that are both Nash equilibria in the game.

Because the action space is a compact product space, and resulting from the assumption of negative semi-definite \( J_{pgU} \) everywhere, at any equilibrium \( a^{eq} \) the following first-order conditions are necessary (and sufficient) in each dimension \( i \):

- if \( a^{eq}_i \) is the smallest action type possible in player \( i \)’s closed and bounded action space in \( \mathbb{R} \), then the partial \( \frac{\partial U_i}{\partial a_i}(a^{eq}) \leq 0 \);
- if \( a^{eq}_i \) is the largest action type possible, then \( \frac{\partial U_i}{\partial a_i}(a^{eq}) \geq 0 \);
- if \( a^{eq}_i \) is strictly interior to player \( i \)’s action space, then \( \frac{\partial U_i}{\partial a_i}(a^{eq}) = 0 \).

Let vector \( a^\lambda \) be a linear combination as

\[
a^\lambda = a^0 + \lambda(a^1 - a^0)
\]

Define the pseudogradient vector \( U' = \left[ \frac{\partial U_i}{\partial a_i} \right]_{i} \) and the function

\[
f(\lambda) = \left[ U'(a^\lambda) \right]^\top \cdot (a^1 - a^0)
\]

and then simple evaluation gives:

\[
f(0) = \left[ U'(a^0) \right]^\top \cdot (a^1 - a^0)
\]
\[
f(1) = \left[ U'(a^1) \right]^\top \cdot (a^1 - a^0)
\]

Consider the change in the value of the function \( f \) between inputs 0 and 1 as

\[
0 = f(1) - f(0) = \left[ U'(a^1) \right]^\top \cdot (a^1 - a^0) - \left[ U'(a^0) \right]^\top \cdot (a^1 - a^0)
\]
\[
= \left[ U'(a^1) \right]^\top \cdot (a^1 - a^0) + \left[ U'(a^0) \right]^\top \cdot (a^0 - a^1)
\]

\[13\] Let the notation \((C.)\) within \( a_{-C} \) denote the reminder that \( C \) is itself a function of the original \( a \) vector.
(See footnote for the connection from this point to Rosen’s definition of a diagonally strictly concave function \( f \).\(^\text{14}\))

Working from equation \((13)\), we note that its two terms each have the following general form: 
\[ [U'(a^{eq})] \cdot (a^{eq} - a^{other}) \geq 0. \]
Within the dot product over the dimensions, each term in the sum at coordinate \( j \) must be individually non-negative, by the first order conditions at Nash described above. Recall the action space is a closed product space. Specifically we have:

- if an individual coordinate \( a^{eq}_j \) is an interior point, then \( U'_j(a^{eq}) = 0 \) from first order conditions at Nash, and \( [U'_j(a^{eq})] \cdot (a^{eq}_j - a^{other}_j) = 0 \) as well;
- if \( a^{eq}_j \) is the smallest action type for player \( j \), then the partial \( \frac{\partial U'_j}{\partial a_j}(a^{eq}) \leq 0 \) is non-positive; but also \( (a^{eq} - a^{other}) \) is a vector coming into \( a^{eq} \), and in the dimension \( j \) the ending point has the smallest type such that regardless of the starting point we get \( (a^{eq}_j - a^{other}_j) \leq 0 \), and we conclude \([U'_j(a^{eq})] \cdot (a^{eq}_j - a^{other}_j) \geq 0\);
- by symmetry, the argument for the smallest type holds for the largest action types as well, with both inequalities flipped.

From this analysis, we get that \( f(1) \geq f(0) \). Function \( f \) is continuously differentiable, so by fundamental theorem of calculus there exists \( \lambda^* \in (0, 1) \) (choosing strictly an interior point on the line segment from \( a^0 \) to \( a^1 \)) such that

\[
0 \leq f'(\lambda)\bigg|_{\lambda^*} \tag{14}
\]

Then we get the following:

\[
0 \leq f'(\lambda)\bigg|_{\lambda^*} = \left[ \frac{\partial U'}{\partial \lambda}(a^{\lambda^*}) \right]^\top \cdot (a^1 - a^0) \\
= \left[ \frac{\partial U'_1}{\partial \lambda}(a^{\lambda^*}), \frac{\partial U'_2}{\partial \lambda}(a^{\lambda^*}), \ldots, \frac{\partial U'_n}{\partial \lambda}(a^{\lambda^*}) \right]^\top \cdot (a^1 - a^0) \\
= \left[ \sum_j \frac{\partial U'_j}{\partial a^\lambda_j}(a^{\lambda^*}), \ldots, \sum_j \frac{\partial U'_j}{\partial a^\lambda_j}(a^{\lambda^*}) \right]^\top \cdot (a^1 - a^0) \\
= (a^1 - a^0)^\top \cdot M \cdot (a^1 - a^0)
\]

\(^{14}\) Combining terms in equation line \((13)\) would give us

\[
0 = f(1) - f(0) = ([U'(a^1)] - [U'(a^0)])^\top \cdot (a^1 - a^0)
\]

Rosen defines the following property which here is trivially sufficient to guarantee uniqueness of pure Nash equilibrium. (If Definition \((3)\) holds everywhere, then in particular it holds for \( a^0, a^1 \) and it would give an immediate contradiction at this point of the analysis.) Also see Theorem \((2)\)

**Definition 4.** Given domain \( \Omega \subseteq \mathbb{R}^n \), a function \( f : \Omega \to \mathbb{R}^n \) with pseudogradient \( f'(\omega) = \left[ \frac{\partial f}{\partial \omega} \right]_i \) is diagonally strictly concave if for every distinct \( \omega_0, \omega_1 \in \Omega \), it has

\[ [f'(\omega_1) - f'(\omega_0)]^\top \cdot (\omega_1 - \omega_0) < 0 \]
by $\frac{\partial a_j^\lambda}{\partial x}(a^\lambda) = (a^1_j - a^0_j)$ and letting $\tilde{M}$ be the matrix where the $(i,j)$ entry is

$$\frac{\partial U'_i}{\partial a_j^\lambda}(a^\lambda)$$

But then $\tilde{M}$ is observed to be exactly $J_{pgU}$ the Jacobian matrix of the pseudogradient of $U$ evaluated at $a^\lambda$. Substituting back and summarizing we get

$$0 \leq (a^1 - a^0) \cdot J_{pgU}(a^\lambda) \cdot (a^1 - a^0) \leq 0$$

with the left hand side from the previous block of analysis, and the right hand side by assumption of interior negative definiteness (in the theorem statement). Clearly the middle term must be identically 0. The final step of our proof will be to use the technical definition of interior negative definiteness to show that it actually implies that the right hand inequality is strict (contradicting that it must hold with equality from line (15)).

By our starting assertion, $a^0$ and $a^1$ are distinct such that $z = (a^1 - a^0)$ is a non-zero vector, and now equation (15) can be interpreted as a test of negative definiteness of $U'$ at $a^\lambda$, by the definition that for negative definite matrix $M$, it has $z^\top M z < 0, \forall z \neq 0$.

For clarity we rewrite equation (15) (incorporating that it holds with equality from above):

$$f'_\lambda |_{a^\lambda} = (a^1 - a^0) \cdot J_{pgU}(a^\lambda) \cdot (a^1 - a^0) = 0$$

If our assumption was that we had strict negative definiteness everywhere, this would already be a contradiction.\footnote{At this point we have proved Rosen’s Lemma which claimed that strict negative definiteness of $U'$ was sufficient for diagonal strict concavity, and hence uniqueness of Nash.} Within the assumption of interior negative definiteness, the only possible way that we do not have strict negative definiteness is if the point $a^\lambda$ is on one or more of the action boundaries. It is a strict convex combination of $a^0$ and $a^1$, so it can only be on the boundary for some coordinate $j$ if both $a^0_j$ and $a^1_j$ are both originally in the same $j$-dimension boundary.

For all dimensions $j$ for which $a^\lambda$ is in a boundary (recall the set $C(a)$ defined from the theorem statement, let it again represent the set of such $j$), it must be that $z_j = (a^1_j - a^0_j) = 0$.

Removing quantities that must multiply out to 0 from equation (16), because of the 0-vector-coordinate and regardless of the entries of the matrix, effectively taking a minor, we can re-write it as a reduced equation

$$(a^1_{-C} - a^0_{-C}) \cdot J_{pgU_{-C}}(a^\lambda_{-C}) \cdot (a^1_{-C} - a^0_{-C}) = 0$$

where a vector with subscript $(-C)$ removes the dimensions represented in set $C(a)$, and where the matrix $J_{pgU_{-C}}$ (note the dependence on $a^\lambda$) was analogously defined in the theorem statement where it was further assumed to be strictly negative definite.

But then the non-zero vector $z_{-C} = (a^1_{-C} - a^0_{-C})$ gives us a final contradiction of the assumption of $J_{pgU_{-C}}$, as strictly negative definite, as seen in equation (17). We conclude that distinct pure Nash equilibria $a^0$ and $a^1$ can not exist with the conditions on $U$ as given in the theorem statement.

\footnote{At this point we have proved Rosen’s Lemma which claimed that strict negative definiteness of $U'$ was sufficient for diagonal strict concavity, and hence uniqueness of Nash.}
A.2 Derivative Calculations for Section 3.2

Allocation rule sub-calculations:

\[
x_i(w) = \frac{w_i}{\sum_k w_k}
\]

\[
\frac{\partial x_i}{\partial w_i}(w) = \frac{(\sum_k w_k) - w_i}{(\sum_k w_k)^2}
\]

\[
\frac{\partial x_i}{\partial w_j}(w) = -\frac{w_i}{(\sum_k w_k)^2} = -\frac{x_i(w)}{\sum_k w_k}
\]

Re-stating the bid function:

\[
\pi_i(w) = v_i(w) - \int_{w_i(0)}^{w_i} x_i(z, w_{-i})v'_i(z)dz
\]

Self-partial:

\[
\frac{\partial \pi_i}{\partial w_i}(w) = v'_i(w_i) - \frac{x_i(w)v'_i(w_i)}{x_i(w)} + \int_{w_i(0)}^{w_i} x_i(z, w_{-i})v'_i(z)dz \cdot \frac{\partial x_i}{\partial w_i}(w)
\]

\[
= \int_{w_i(0)}^{w_i} x_i(z, w_{-i})v'_i(z)dz \cdot \frac{(\sum_k w_k) - w_i}{(\sum_k w_k)^2}
\]

\[
= \int_{w_i(0)}^{w_i} x_i(z, w_{-i})v'_i(z)dz \left[ \frac{\sum_k w_k}{w_i} - \frac{1}{w_i(0)} \right] dz
\]

\[
= \int_{w_i(0)}^{w_i} v'_i(z) \frac{1}{w_i} \cdot \frac{z}{(\sum_k w_k) - w_i + z} \cdot \left[ \frac{\sum_k w_k}{w_i} - 1 \right] dz
\]
Cross-partial:

\[
\frac{\partial \pi_i}{\partial w_j}(w) = -\int_{w_i(0)}^{w_i} \frac{\partial x_j}{\partial w_i}(z, w_-) v_i'(z) dz + \frac{\partial x_i}{\partial w_j}(w) \int_{w_i(0)}^{w_i} x_i(z, w_-) v_i'(z) dz
\]

\[
= \int_{w_i(0)}^{w_i} v_i'(z) \cdot \left[ x_i(z, w_-) \frac{\partial x_j}{\partial w_i}(w) - \frac{\partial x_i}{\partial w_j}(z, w_-) x_i(w) \right] dz
\]

\[
= \int_{w_i(0)}^{w_i} v_i'(z) \cdot \left[ x_i(z, w_-) - x_i(w) \frac{z}{\sum_k w_k} - x_i(w) \frac{z^2}{\sum_k w_k} - w_i + z \right] dz
\]

\[
= \int_{w_i(0)}^{w_i} v_i'(z) \cdot \left[ \frac{-z}{\sum_k w_k} - w_i + z \right] dz
\]

A.3 Proof of Lemma \textbf{2}

Lemma 2. Given the price function $\pi$ for proportional weights, for $j, k \neq i$, the cross derivatives are the same: $\frac{\partial \pi_i}{\partial w_j} = \frac{\partial \pi_i}{\partial w_k}$. Evaluating the Jacobian at $w$, further, all elements of the Jacobian matrix $J_\pi$ are positive, i.e., $\frac{\partial \pi_i}{\partial w_i} > 0$, $\frac{\partial \pi_i}{\partial w_j} > 0$, except at the $w_i(0)$ lower boundary in dimension $i$ where the elements of row $i$ are $\frac{\partial \pi_i}{\partial w_i} = \frac{\partial \pi_i}{\partial w_j} = 0$.

Proof. All cross-derivatives $\frac{\partial \pi_i}{\partial w_j}$ for fixed $i$ and $j \neq i$ are equal because a $dw_j$ increase in the weight of any other agent $j$ “looks the same” mathematically to the proportional weights allocation rule of agent $i$, which is $x_i(w) = \frac{w_i}{w_i + \sum_{j \neq i} w_j}$.

We continue by recalling our assumption that weights are strictly positive and strictly increasing in value. Then all terms in the derivative equations (7) and (8) within the integrals are non-negative everywhere by inspection. All denominator terms are strictly positive elsewhere.

For any dimension $i$, consider $w_i > w_i(0)$. For integrand $z$ strictly interior to the endpoints in $(w_i(0), w_i)$, all terms in the derivative equations are strictly positive everywhere. With non-negativity everywhere and positivity somewhere, all derivatives evaluate to be strictly positive.

For $w_i = w_i(0)$, the integrals start and end at $w_i(0)$ and trivially evaluate to 0.

A.4 Proof of Theorem \textbf{6} in Section \textbf{3.2}

Theorem 6. Consider a matrix $G$ with diagonal $g_1, g_2, \ldots, g_n$ and all other entries equal to 1 (and without loss of generality $g_1 \leq g_2 \leq \ldots \leq g_n$). The following is a complete characterization of when $G$ is positive definite.

1. if $g_1 \leq 0$, then the matrix $G$ is not positive definite;

2. if $g_1 \geq 1$ and $g_2 > 1$, then $G$ is positive definite;
3. if \( g_1, g_2 \leq 1 \), then \( G \) is not positive definite;

4. if \( 0 < g_1 < 1 \) and \( g_2 > 1 \), then \( G \) is positive definite if and only if \( \sum_k \frac{1}{1-g_k} > 1 \).

Proof. To prove positive definiteness in cases (2) and (4), we will show that for any non-zero vector \( z \), it must be true that \( z^\top G z > 0 \). For cases (1) and (3) we give counterexamples of \( z \) for which \( z^\top G z \leq 0 \). Given the structure of \( G \), we have

\[
z^\top G z = \left( \sum_i z_i \right)^2 + \sum_i (g_i - 1)z_i^2.
\]  

(18)

We recall for use throughout this proof the assumption that, without loss of generality, the diagonal elements are such that \( g_1 \leq g_2 \leq \ldots \leq g_n \). We prove each case of the characterization in turn.

Case (1) is correct by counter-example, setting \( z = (-1,0,\ldots,0) \).

Case (2) is correct by inspection of equation (18) in which all terms are non-negative. The vector \( z \) is non-zero, so either a \((g_j - 1)z_j^2\) term for \( j \neq 1 \) in the second sum is strictly larger than 0, or all such \( z_j \) are 0 but then \( z_1 \neq 0 \) and the first sum-squared is strictly larger than 0.

Case (3) is correct by counter-example, setting \( z = (1,-1,0,\ldots,0) \).

For case (4), we need to prove that when \( 0 < g_1 < 1 \) and \( g_2 > 1 \), then the matrix \( G \) is positive definite if and only if \( \sum_k \frac{1}{1-g_k} > 1 \).

For this last case, given the assumptions on the \( g_i \) elements, only the \((g_1 - 1)z_1^2\) term from equation (18) is negative, all other terms are non-negative. Therefore, from this point on, we can ignore any sub-case where \( z_1 = 0 \), as some \((g_j - 1)z_j^2\) term for \( j \neq 1 \) must be strictly positive.

Now consider fixing the value \( z_1 \) to any real number \( \bar{z}_1 \neq 0 \). We will show that equation (18) is strictly positive for any \( z_{-1} \in \mathbb{R}_{n-1} \). Specifically, for any \( \bar{z}_1 \neq 0 \), equation (18) has a global minimum in variables \( z_{-1} \) that is strictly positive. This global minimum \( z_{-1}^* \) satisfies

\[
z_{-1}^* = \operatorname{argmin}_{z_{-1}} (\bar{z}_1, z_{-1})^\top G (\bar{z}_1, z_{-1}) \tag{19}
\]

\[
= \operatorname{argmin}_{z_{-1}} \left( \bar{z}_1 + \sum_{j \geq 2} z_j \right)^2 + \sum_{j \geq 2} (g_j - 1)z_j^2 \tag{20}
\]

where the second line substitutes equation (18) and drops the constant \( \bar{z}_1 \) term from the right hand sum. It will be convenient to denote the sum of the variables as \( S(\bar{z}_1) = \bar{z}_1 + \sum_{i > 2} z_i^* \). After the brief argument that the minimizer \( z_{-1}^* \) exists and is characterized by its first-order conditions, we will use first-order conditions on \( z_{-1}^* \) to write all variables in terms of \( S(\bar{z}_1) \) which we substitute into (18) to analyze.

To show that \( z_{-1}^* \) exists and is characterized by its first-order conditions, observe that the polynomial \((\bar{z}_1, z_{-1})^\top G (\bar{z}_1, z_{-1})\) is a quadratic form with Hessian \( 2 \cdot G_{[2:n,2:n]} \), i.e., twice the matrix \( G \) without the first row and column:

\[
\text{Hessian}((\bar{z}_1, z_{-1})^\top G (\bar{z}_1, z_{-1})) = G_{[2:n,2:n]} = \begin{bmatrix}
g_2 & 1 & \cdots & 1 \\
1 & g_3 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & g_n
\end{bmatrix}.
\]

\textsuperscript{16}Of course, it is a well-known property of positive definite matrices \( G \) that all diagonal elements must be strictly positive, otherwise they have \( z^\top G z \leq 0 \) with a simple counter-example \( z \) described by all zeroes except \(-1\) in the index of the matrix’s non-positive diagonal element.

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Matrix $G_{[2:n,2:n]}$ is ones except by assumption we have $g_j > 1$ for $j \geq 2$ in the diagonal; thus, by case (2) of the theorem, it is positive definite. A quadratic form with strictly positive definite Hessian has a unique local minimum which is characterized by its first-order conditions.

We now use the first-order conditions to write optimizer $z^*_{j-1}$ of equation (20) in terms of $S(\tilde{z}_1)$.

\begin{align*}
0 &= 2 \left( \tilde{z}_1 + \left( \sum_{k \geq 2, k \neq j} z^*_k \right) + (g_j - 1) \cdot z^*_j \right) \quad \text{for each } j \geq 2 \tag{21} \\
z^*_j &= \frac{1}{1 - g_j} S(\tilde{z}_1) \quad \text{for each } j \geq 2 \tag{22}
\end{align*}

We now similarly identify a substitution of $\tilde{z}_1$ in terms of $S(\tilde{z}_1)$. Starting from equation (22), sum the $z^*_j$ first-order condition equalities over all $j \geq 2$:

\begin{equation}
\sum_{j \geq 2} z^*_j = \sum_{j \geq 2} \left( \frac{1}{1 - g_j} S(\tilde{z}_1) \right) \tag{23}
\end{equation}

Add $\tilde{z}_1$ to both sides of the equation:

\begin{equation}
1 \cdot \left( \tilde{z}_1 + \sum_{j \geq 2} z^*_j \right) = \tilde{z}_1 + \left( \sum_{j \geq 2} \frac{1}{1 - g_j} \right) S(\tilde{z}_1) \tag{24}
\end{equation}

Substitute $S(\tilde{z}_1)$ on the left and solve for the right-hand side $\tilde{z}_1$ term:

\begin{equation}
\tilde{z}_1 = \left( 1 - \sum_{j \geq 2} \frac{1}{1 - g_j} \right) \cdot S(\tilde{z}_1). \tag{25}
\end{equation}

Notice that equation (25) and the definition of $\tilde{z}_1 \neq 0$ excludes the possibility that $S(\tilde{z}_1) = 0$.

In the analysis below, the first line re-writes the objective function in (18). The second line substitutes equations (22) and (25). Subsequent lines are elementary manipulations.

\begin{align*}
(\tilde{z}_1, z^*_{-1})^T \cdot G \cdot (\tilde{z}_1, z^*_{-1}) &= (g_1 - 1) \tilde{z}_1 + S(\tilde{z}_1)^2 + \sum_{j \geq 2} \left( g_j - 1 \right) z^*_j \\
&= (g_1 - 1) \left( 1 - \sum_{j \geq 2} \frac{1}{1 - g_j} \right) S(\tilde{z}_1)^2 + \left( 1 - \sum_{j \geq 2} \frac{1}{1 - g_j} \right) S(\tilde{z}_1)^2 \\
&= S(\tilde{z}_1)^2 \left[ (g_1 - 1) \left( 1 - \sum_{j \geq 2} \frac{1}{1 - g_j} \right) + \sum_{j \geq 2} \frac{1}{1 - g_j} \right] \\
&= S(\tilde{z}_1)^2 \left[ \sum_{j \geq 2} \frac{1}{1 - g_j} \right] \left( 1 - g_1 \right) \left[ \sum_{j \geq 2} \frac{1}{1 - g_j} - 1 \right] + \frac{1 - g_1}{1 - g_1} \\
&= S(\tilde{z}_1)^2 \left[ \sum_{j \geq 2} \frac{1}{1 - g_j} \right] \left( 1 - g_1 \right) \left[ \sum_{k \geq 2} \frac{1}{1 - g_k} - 1 \right].
\end{align*}

Given the assumptions on the $g_i$ for current case (4), the first three terms of this product are strictly positive (recalling $\tilde{z}_1 \neq 0$ and $S(\tilde{z}_1) \neq 0$, so $(S(\tilde{z}_1))^2 > 0$). To finish, we observe that the
exact dependence of positive definiteness of the matrix $G$ is on the bracketed fourth term (where the first term $k = 1$ of the sum is positive and all of the other terms are negative):

For $0 < g_1 < 1$ and $g_j > 1 \forall j \geq 2$, $G$ is positive definite iff
$$\left[ \sum_k \frac{1}{1-g_k} - 1 \right] > 0.$$  

A.5 Lemmas Supporting Theorem 7 in Section 3.2

Lemma 3. If $h_i \leq 1$, then $w_i > 0.5 \sum w_k$, and all other weights must have $w_j < 0.5 \sum w_k$, and all other $h_j > 1$.

Proof. Writing out $h_i$ from its definition as the ratio of partial derivatives,

$$h_i = \frac{\int u_i v'(z) \frac{w_i}{w_k} \cdot \frac{z}{w_i} \cdot \sum_k w_z - w_i + z}{\int u_i v'(z) \frac{w_i}{w_k} \cdot \frac{z}{w_i} \cdot \sum_k w_z - w_i + z}$$

If $h_i \leq 1$, by implication it is well-defined so the denominator can not disappear and $w_i > w_i(0)$. There must exist $z \in (0, w_i]$, such that

$$\sum_k w_k \sum_k w_z - w_i + z \geq \sum_k w_k w_i$$

which implies $w_i > 0.5 \sum w_k$ by noting equal numerators and comparison of denominators. The rest of the claim follows as $w_i$ is obviously the only weight more than half the total, and claiming $h_j > 1$ for other $j$ is simply an explicit statement of the contrapositive.

We give the necessary and sufficient lower bound for each term in the final equation [A.4] that will appear in the proof of Theorem 7.

Lemma 4. When $h_1 < 1$ and $h_j > 1 \forall j \neq 1$, we have $\sum_k \frac{1}{1-h_k} > 1$.

Proof. With $h_1 < 1$ by assumption, then $w_1 > 0.5 \sum w_k$ by Lemma 3 and $x_1 > 0.5$. Thus $x_j < 0.5$ for $j \neq 1$ and we can apply Lemma 3 (below), to get the first inequality in the following analysis:

$$\sum_k \frac{1}{1-h_k} > \frac{x_1^2}{2x_1 - 1} + \sum_{k>1} \frac{x_k^2}{2x_k - 1}$$

$$\geq \frac{x_1^2}{2x_1 - 1} + \frac{(1-x_1)^2}{2(1-x_1) - 1}$$

$$= 1$$

and with the second step following because $\frac{x_k^2}{2x_k - 1} \Big|_0 = 0$ and is a concave function when $0 < x_k < 0.5$ and $\sum_{k>1} x_k = (1-x_1)$ (its second derivative is $\frac{2}{(2x_k-1)^2}$ and it acts submodular).

Lemma 8. When $h_1 < 1$ and $h_j > 1 \forall j \neq 1$, then $\forall i \in \{1, \ldots, n\}$, we have $\frac{1}{1-h_i} > \frac{x_i^2}{2x_i - 1}$. 

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Proof. By subtracting 1 from both sides, it is equivalent to prove the inequality on the right:

\[
\frac{1}{1-h_i} > \frac{x_i^2}{2x_i - 1} \iff \frac{h_i}{1-h_i} > \frac{x_i^2 - 2x_i + 1}{2x_i - 1} = \frac{(1-x_i)^2}{2x_i - 1}
\]

Working from the definition of \( h_i \):

\[
\frac{h_i}{1-h_i} = \frac{\int_{w_i(0)}^{w_i} v'_i(z) \frac{1}{w_i} \cdot \frac{z}{\sum_k w_k - w_i + z} \cdot \left[ \frac{\sum_k w_k}{w_i} - 1 \right] dz}{\int_{w_i(0)}^{w_i} v'_i(z) \frac{1}{w_i} \cdot \frac{z}{\sum_k w_k - w_i + z} \cdot \left[ \frac{\sum_k w_k}{w_i} - \frac{\sum_k w_k}{w_i} \right] dz}
\]

The numerator is always positive.

For the denominator, we would like to get a less complex upper bound on it by dropping the \( z \) term within the brackets. Generally we can do this but we have to be careful that the overall sign of the denominator does not change.

For \( i \neq 1 \) and \( h_i > 1 \), then the denominator is negative by simple inspection of the left hand side. For \( i = 1 \), \( h_1 < 1 \), then the denominator is positive. We relax the denominator and increase it, arguing after the calculations that doing this does not change the sign of the expression.

\[
\int_{w_i(0)}^{w_i} v'_i(z) \frac{1}{w_i} \cdot \frac{z}{\sum_k w_k - w_i + z} \cdot \left[ \frac{\sum_k w_k}{w_i} - \frac{\sum_k w_k}{w_i} \right] dz
\]

The important term is \( (2w_i - \sum_k w_k) \). For \( i = 1 \), \( w_1 > 0.5 \sum_k w_k \) by Lemma \( \text{[3]} \) and also for \( j \neq 1 \), \( w_j < 0.5 \sum_k w_k \) by Lemma \( \text{[3]} \). Then clearly the denominator is still positive for \( i = 1 \); and still negative for agents \( i \neq 1 \). So we give a lower bound on the fraction using the proved upper bound on the denominator.

\[
\frac{h_i}{1-h_i} > \frac{\int_{w_i(0)}^{w_i} v'_i(z) \frac{1}{w_i} \cdot \frac{z}{\sum_k w_k - w_i + z} \cdot \left[ \frac{\sum_k w_k}{w_i} - 1 \right] dz}{\int_{w_i(0)}^{w_i} v'_i(z) \frac{1}{w_i} \cdot \frac{z}{\sum_k w_k - w_i + z} \cdot \left[ \frac{\sum_k w_k}{w_i} - \frac{\sum_k w_k}{w_i} \right] dz}
\]

\[
= \frac{\sum_k w_k}{w_i} - \frac{1}{\sum_k w_k - w_i + z} \cdot \left[ \frac{\sum_k w_k}{w_i} - \frac{\sum_k w_k}{w_i} \right] dz
\]

\[
= \frac{(1-x_i)^2}{2x_i - 1}
\]

\[
\square
\]

B Supporting Material for Section \( \text{[4]} \)

The goal of this section is to show in detail how to reduce the price inversion question to binary search. We do this by showing that the analysis is largely many-to-one separable: we can make meaningful observations about each agent individually, in particular by treating the (initially unknown) sum total of all weights \( s = \sum_k w_k \) as an independent variable used as input to the analysis of each agent.
Before getting to the key results, we use a more measured pace than is possible in the main body of the paper to give some preliminary analysis of the problem regarding price functions and structure of search spaces, in particular for “small” agents with weight at most half the total. We do this in Appendix B.1 and then the rest of this section is laid out as follows: Appendix B.2 gives both the intuition and fully detailed version of the algorithm; Appendix B.3 gives the proofs of the critical lemmas and Theorem 8 from Section 4; and finally technical Appendix B.4 is used to describe within the algorithm how we set up the “oracle checks” to find the correct sub-space of value space to search for a solution, and the endpoints of binary search.

B.1 First Computations and Analysis of the Search Space

This section exhibits the fundamentals of a reduced, separated, one-agent analysis of the price inversion question, starting with some analysis. Note the following explicit conversion of the function $\pi_i(\cdot)$ to accept sum $s = \sum_k w_k$ as an input variable in place of $w_{-i}$. We recall equation (6):

$$
\pi_i(w) = v_i(w_i) - \frac{\int_{w_i(0)}^{w_i} x_i(z, w_{-i})v'_i(z)dz}{x_i(w)}
$$

where we also recall $v_i(\cdot)$ is overloaded to be the function that maps from buyer $i$’s weight back to buyer $i$’s valuation (well-defined by the assumption that $w_i(\cdot)$ is strictly increasing). Re-arranging we have:

$$
\bar{\pi}_i(s, w_i) = v_i(w_i) - \frac{s}{w_i} \int_{w_i(0)}^{w_i} \frac{z}{s - w_i + z} v'_i(z)dz
$$

The form of equation (27) illustrates the critical relationships between $\bar{\pi}_i$, $s$, and $w_i$. Our high level goal will be to understand the behavior of the function $\bar{\pi}_i$ in the space ranging over feasible $s$ and $w_i$, starting with the technical computations of the partials on $\bar{\pi}_i$. Recall from Lemma 2 that functions $\pi_i$ have the same cross-partials with respect to $w_j \forall j \neq i$. This property extends to $\bar{\pi}_i$:

$$
\bar{\pi}_i(s + dw_i, w_i + dw_i) = \pi_i(w_i + dw_i, w_{-i})
\Rightarrow \frac{\partial \bar{\pi}_i}{\partial s} dw_i + \frac{\partial \bar{\pi}_i}{\partial w_i} dw_i = \frac{\partial \pi_i}{\partial w_i} dw_i
$$

$$
\bar{\pi}_i(s, w_i + dw_i) = \pi_i(w_i + dw_i, w_{-i, j})
\Rightarrow \frac{\partial \bar{\pi}_i}{\partial w_i} dw_i = \frac{\partial \pi_i}{\partial w_i} dw_i - \frac{\partial \pi_i}{\partial w_j} dw_i
$$

Combining the above equations together, and any $j \neq i$ we get

$$
\frac{\partial \bar{\pi}_i}{\partial w_i} = \frac{\partial \pi_i}{\partial w_i} - \frac{\partial \pi_i}{\partial w_j}
$$

(28)

$$
\frac{\partial \bar{\pi}_i}{\partial s} = \frac{\partial \pi_i}{\partial w_j}
$$

(29)

We give the intuition for these calculations. If $w_i$ increases unilaterally without a change in $s$, then it must be that some other $w_j$ decreases by an equal amount. If we increase $s$ without an
observed change in \( w_i \), then it must be some other \( w_j \) that increased. The result is the symbolic identities as given in equations (28) and (29) above. We will evaluate them in more detail in Lemma 9 below.

We formally identify three objects of interest (initially discussed in Section 4, see Figure 3). These quantities are defined for each agent \( i \), weight function \( w_i \), and the observed price \( \rho_i \) of this agent. Importantly, though the notation includes the whole profile of observed prices \( w \), these objects only depend on its \( i \)th coordinate \( \rho_i \).

- First, the price level set \( Q_i^\rho \) is defined as \( \{(s, w_i) \mid \bar{\pi}_i(s, w_i) = \rho_i \} \), i.e., these are the \( \rho_i \) level-sets of \( \bar{\pi}_i(s, w_i) \). Its pertinent subset is \( P_i^\rho = \{(s, w_i) \mid \bar{\pi}_i(s, w_i) = \rho_i \text{ and } w_i \leq s/2 \} \subseteq Q_i^\rho \), i.e., the subset which restricts the set \( Q_i^\rho \) to the region where \( w_i \) is at most half the total weight \( s \). These sets are illustrated respectively by the dashed and solid lines in Figure 3.

- Second, the elements of the price level-set \( P_i^\rho \) each have unique \( s \) coordinate (see Lemma 9). It will be convenient to describe it as a function mapping sum \( s \) to weight \( w_i \) of agent \( i \), parameterized by the price \( \rho_i \). Denote this function \( w_i^\rho(s) \). This function is illustrated in Figure 3 where below the dotted line \( w_i = s/2 \), the curve is a function in \( s \). Qualitatively, it is monotone decreasing and not necessarily convex.

- Third, \( P_i^\rho \) is non-empty and possesses a smallest total weights coordinate \( s \) which we define as \( r_i^\rho = \min\{s \mid (s, w_i) \in P_i^\rho \} \). In the example of Figure 3, \( r_i^\rho \) is the \( s \)-coordinate of the point where the level-set \( P_i^\rho \) intersects the \( w_i = s/2 \) line. In the case that the entire set \( Q_i^\rho \) is below the \( w_i = s/2 \) line, \( P_i^\rho = Q_i^\rho \) and \( r_i^\rho \) is the sum \( s \) that uniquely satisfies \( \bar{\pi}_i(s, w_i(h)) = \rho_i \).

Continuing, consider price level set \( Q_i^\rho \). We note again that \( \bar{\pi}_i(\cdot) \) can be used to map a domain of \( (s, w_i) \) to price level sets (as depicted in Figure 3). In this context we return to analyzing the partial derivatives of \( \bar{\pi}_i(\cdot) \), formally with Lemma 9 (immediately to follow). Intuitively, the statement of Lemma 9 claims the following, with relation to Figure 3.

- Part 1 of Lemma 9 below the \( w_i = s/2 \) line, starting at any point \((\hat{s}, \hat{w}_i)\), we strictly “move up” fixed-price level sets as we move up to \((\hat{s}, \hat{w}_i + \delta)\), or to the right to \((\hat{s} + \delta, \hat{w}_i)\).

- Part 2 of Lemma 9 below the \( w_i = s/2 \) line, price level sets are necessarily decreasing curves; further they are defined for arbitrarily large \( s \), which reflects the many-to-one nature of this analysis: other than the summary statistic \( s \), nothing specific is known about the other agents, for example we do not need to know the number of other agents or their weights functions or bounds on their weights.

- Additionally, above the \( w_i = s/2 \) line, we “move up” fixed-price level sets with an increase in \( s \) but not necessarily with an increase in \( w_i \).

For use in Lemma 9 and the rest of this section, we overload the notation \( h_i \) as defined in equation (9) to be a function of \( w_i \) and \( s \) rather than \( w \), with the obvious substitution in its definition to replace \( \sum_k w_k \) with \( s \).

---

17 Note that because all the cross-derivatives are the same, it is without loss of generality that we assume that changes \( \partial s \) are entirely attributable to one other particular agent \( j \neq i \) as \( \partial w_j \).

18 We can not assume that the set \( P_i^\rho \) is non-empty without proof. We prove that it is non-empty in Lemma 12.

19 Further discussion will be given in Appendix B.4 where we show that \( r_i^\rho \) can be computed via a binary search between easy to identify upper and lower bounds.
Lemma 9. Assume \( w_i \leq s/2 \) and fix the price of agent \( i \) to be \( \rho_i > 0 \). Let \( Q_i^\rho, P_i^\rho, w_i^\rho(s) \) and \( r_i^\rho \) be defined as above, and \( h_i = \frac{\partial \pi_i}{\partial w_i} / \frac{\partial \pi_i}{\partial w_j} \) as defined in equation (9). Then restricting analysis to the cone described by \( w_i \leq s/2 \) and non-negative weight \( w_i \):

1. \( \pi_i(s, w_i) \) is a continuous and strictly increasing function in both variables \( s \) and \( w_i \), with specifically \( \frac{\partial \pi_i(s, w_i)}{\partial s} = \frac{\partial \pi_i}{\partial w_j} \) and \( \frac{\partial \pi_i(s, w_i)}{\partial w_i} = \frac{\partial \pi_i}{\partial w_j} \cdot (h_i - 1) \);

2. \( w_i^\rho(s) \) is a well-defined and strictly decreasing function on \( s \in [r_i^\rho, \infty) \) with \( \frac{dw_i^\rho(s)}{ds} = \frac{1}{1-h_i} \); in particular the function is well-defined for arbitrarily large \( s \) independent of the number of other agents or their weight functions;

3. \( w_i^\rho(s) \) can be computed to arbitrary precision using binary search.

Further, (1) partially extends such that \( \pi_i(s, w_i) \) is increasing in \( s \) with \( \frac{\partial \pi_i(s, w_i)}{\partial s} = \frac{\partial \pi_i}{\partial w_j} \) holding everywhere, (so including above the line \( w_i = s/2 \)).

Proof. For (1), as in Section 3.2 we set \( h_i = \frac{\partial \pi_i}{\partial w_i} / \frac{\partial \pi_i}{\partial w_j} \), i.e., the diagonal entry in the Jacobian matrix after the normalization (divide each row by its common cross-partial term), and with the substitution \( s = \sum_k w_k \).

From equation (28), \( \frac{\partial h_i}{\partial w_i} = \frac{\partial \pi_i}{\partial w_i} - \frac{\partial \pi_i}{\partial w_j} = \frac{\partial \pi_i}{\partial w_j} \cdot (h_i - 1) \). By Lemma 3, \( h_i \) is larger than 1 when \( w_i \leq s/2 \). By Lemma 2, \( \frac{\partial h_i}{\partial w_i} > 0 \). Hence \( \frac{\partial h_i}{\partial s} \cdot (h_i - 1) > 0 \) when \( w_i \leq s/2 \). The \( \frac{\partial h_i}{\partial s} \) direction follows directly from equation (29) with Lemma 2 applying to \( \frac{\partial \pi_i}{\partial w_j} \). This argument is also sufficient to prove the last claim of the lemma statement extending (1).

For (2), we first observe that the function \( w_i^\rho(s) \) is well-defined (on an appropriate domain) because \( w_i^\rho(s) \) uses fixed \( \rho_i \), otherwise it would contradict the monotonicity properties proved in (1) which requires we “move up” price level sets whenever we unilaterally increase \( w_i \). Therefore
we can take the derivative with respect to $s$. We get $\frac{dw_i^\rho(s)}{ds}$ is negative for $w_i \leq s/2$ by the following calculation (from first-order conditions as we move along the fixed curve resulting from $\bar{\pi}_i(\cdot)$ having constant output $\rho_i$):

$$0 = \frac{\partial \bar{\pi}_i}{\partial w_i^\rho(s)} dw_i^\rho(s) + \frac{\partial \bar{\pi}_i}{\partial s} ds$$

$$\Rightarrow \frac{dw_i^\rho(s)}{ds} = -\frac{\frac{\partial \bar{\pi}_i}{\partial s}}{\frac{\partial \bar{\pi}_i}{\partial w_i^\rho(s)} - \frac{\partial \bar{\pi}_i}{\partial w_j^\rho}} = \frac{1}{1 - h_i} < 0$$

The last inequality uses Lemma 3 from which $w_i \leq s/2$ implies $h_i > 1$. We next prove for (2) that $w_i^\rho(\cdot)$ and its domain are well-defined.

Technical Lemma 12 (below) will show that $P_i^\rho$ is non-empty. Consider starting at any of its elements. We can theoretically use its continuous derivative to “trace out” the curve of the function $w_i^\rho(s)$. As $s$ increases from the starting point, we note that positive prices can never be consistent with non-positive weights, then the continuous and negative derivative implies that the function converges to some positive infimum as $s \to \infty$. As $s$ decreases, the function increases until either we reach a maximum feasible point with $(s, w_i^\rho(s) = w_i(h))$ from the maximum value type $h$, or otherwise the input-output pair $(s, w_i^\rho(s))$ intersects the line $w_i = s/2$, and minimum total weight $r_i^\rho$ is realized at the point of intersection.

This shows that “reals at least $r_i^P$” is a valid domain for $w_i^\rho(s)$, and this completes the first statement in (2). The second statement of (2) follows because the construction of the set $Q_i^\rho$ is independent of other agents: for any realization of the set of other agents, their effect is summarized with the variable $s$.

For (3), we note that the output of function $w_i^\rho(s)$ has constant lower-bound $w_i(0)$ and is upper-bounded by $s/2$, so we can indeed run binary search.

Within Lemma 9 we make a final note of the significance of the $h_i$ terms in derivative calculations. As the last part of the statement shows, these derivative calculations also hold for the space $w_i > s/2$ (with a carefully extended interpretation of the $w_i^\rho$ function to be sure to apply the mapping from $s$ at the correct $w_i$); but we do not get the contrapositive of Lemma 3 in this region to guarantee the sign of $(1 - h_i)$, and do not get the a monotonicity property of (2) everywhere.

Further, recall the statement of Lemma 4 (its proof was on page 22):

**Lemma 4.** When $h_1 < 1$ and $h_j > 1 \forall j \neq 1$, we have $\sum_k \frac{1}{1 - h_k} > 1$.

As economic intuition for this result, we now see that the terms in the sum are exactly the derivatives $\frac{\partial \bar{\pi}_i(s, w_i)}{\partial s}$, i.e., the derivatives of the respective agents’ $\rho_i$ level set curves. We will see the importance of Lemma 4 below as the key final step in the proof of Lemma 7.

**B.2 The Full Algorithm**

Because the observed profile of prices $\rho$ is invertible to a unique profile of weights (from Section 3.2), the quantity $s = \sum_k w_k$ is uniquely determined by observed prices. The intuitive description of the algorithmic strategy to compute the inversion from prices to weights is as follows.

Motivated by Appendix B.1, we intend to split the search space for the unique $s$. Clearly at most one of the agents can have strictly more than half the weight. We cover the valuation space...
by considering \( n \) subspaces, representing the \( n \) possibilities that any one agent \( i^* \) is allowed but not required to have strictly more than half the weight. (The region where all agents have at most half the weight is covered by all subspaces, without introducing a conflict.) Explicitly, define

\[
\text{Space-}i = \left\{ w \mid w_i \text{ unrestricted } \land w_j \leq \sum_k w_k / 2, \forall j \neq i \right\} \quad \text{for } i \in \{1, \ldots, n\} \tag{30}
\]

A specific monotonicity property within each \text{Space-}i (see Lemma \text{[7]} and its proof) will allow the algorithm to use a natural binary search for the solution. Considering such a search in each of \( n \) spaces will deterministically find \( \hat{s} \) to yield a vector of weights \( \hat{w} \) as \((\hat{s} - \sum_{i \neq i^*} w_i(\hat{s})), w_{i^*}(\hat{s}))\), which are arbitrarily close to the true \( s^* \) and true \( w^* \) consistent with \( \rho \) (i.e., \( \pi \) maps \( w^* \) to \( \rho \)).

We recall the definition of \( \pi_{i^*}^\rho \), given previously in the main body of the paper, and extend it in terms of \( \tilde{\pi}_{i^*} \):

\[
\pi_{i^*}^\rho(s) := \tilde{\pi}_{i^*}(\max\{s - \sum_{i \neq i^*} w_i^\rho(s), w_{i^*}(0)\}, w_{i^*}^\rho(s)) = \tilde{\pi}_{i^*}(s, \max\{s - \sum_{i \neq i^*} w_i^\rho(s), w_{i^*}(0)\}) \tag{31}
\]

Intuitively the definition here is: given \( s \), we assign guesses of weights \( w_j^\rho(s) \) to other agents \( j \) and agent \( i^* \) gets weight as the balance \( s - \sum_{j \neq i^*} w_j^\rho(s) \), then we calculate the price charged to agent \( i^* \) (simply from Myerson's characterization).

The goal of the algorithm is to find the agent \( i^* \) and unique \( s \) such that \( \pi_{i^*}^\rho(s) \) outputs \( \rho_{i^*} \), the true payment. I.e., we search for the equality of \( \pi_{i^*}^\rho(s) = \rho_{i^*} \).

We now give the full version of the algorithm. Beyond the outline in the main body of the paper, the new key technical piece in the expanded description is the use of \( r_j^\rho \) variables to lower bound the search for \( s \) in any given candidate \text{Space-}i. The \( r_j^\rho \) variables were described as the third item of interest in Appendix \text{[B.1]}. They are used in the expanded descriptions of new pre-process step 0, and steps 1(a)(b)(c). We also newly use \( s(h) = \sum_k w_k(h) \) to denote the maximum sum of weights possible.

The full algorithm (with intuitive remarks):

0. \text{Pre-process:} For each \( i \), compute \( r_i^\rho \):

\[
\text{(a) (general case: } \mathcal{P}_i^\rho \neq \mathcal{Q}_i^\rho \text{) if } \pi_i(2w_i(h), w_i(h)) \geq \rho_i \text{, run binary search “diagonally” on the line segment of } w_i = s/2 \text{ between } (0, 0) \text{ and } (2w_i(h), w_i(h)) \text{ to find an element of } \mathcal{Q}_i^\rho \text{ and use its } s \text{ coordinate as } r_i^\rho \text{ (which we can do because } \pi_i(\cdot) \text{ is strictly increasing on this domain);}
\]

\[
\text{(b) (edge case: } \mathcal{P}_i^\rho = \mathcal{Q}_i^\rho \text{) otherwise, fix } w_i \text{ coordinate to its maximum } w_i(h) \text{ and run binary search “horizontally” to find } \hat{s} \in [2w_i(h), s(h)] \text{ representing } (\hat{s}, w_i(h)) \in \mathcal{Q}_i^\rho \text{ (which we can do because } \pi_i(\cdot) \text{ is strictly increasing in } s \text{ for constant } w_i); \text{ set minimum total weight } r_i^\rho = \hat{s}.
\]

1. find an agent \( i^* \) and search a range \([s_L, s_U]\) over possible \( s \) by iteratively running the following for each fixed assignment of agent \( i \in \{1, \ldots, n\} \):

\[
\text{(a) temporarily set } i^* = i;
\]

\footnote{See Appendix \text{[B.4]} for further explanation.}
(b) determine the range \([s_L, s_H]\) on which \(\pi^\rho_i\) is well-defined and searching is appropriate:
- identify a candidate lower bound \(s_L = \max_{j \neq i^*} \pi^\rho_j\) (because any smaller \(s \in [0, s_L]\) is outside the domain of \(w_j^\rho\), for some \(j\));
- run an "oracle check" on the lower bound, specifically, exit this iteration of the for-loop if we do not observe:
  \[
  \pi^\rho_i(s_L) = \bar{\pi}_i^*(\max\{s_L - \sum_{k \neq i^*} w_k^\rho(s_L), w_{i^*}(0)\}, s_L) \leq \rho_{i^*}
  \]
  (because recall the goal of the algorithm, to search for equality of \(\pi^\rho_i(s) = \rho_{i^*}\); but by Lemma 6, \(\pi^\rho_i\) is increasing, then if this does not hold at the lower bound, the left hand side is already too big and will never decrease);
- identify a candidate upper bound \(s_H\) using binary search to find \(s_H\) as the largest total weight consistent with the maximum weight of agent \(i^*\), i.e., such that \(s_H - \sum_{k \neq i^*} w_k^\rho(s_H) = w_{i^*}(h)\):
  1. search for \(s_H \in [s_L, s(h)]\) \((s > s_H\) will “guess” impossible weights \(w_{i^*} > w_{i^*}(h)\) as input to \(\bar{\pi}_{i^*}\), because \(w_{i^*}\) gets the balance of \(s\) after subtracting the decreasing functions in \(\sum_{k \neq i^*} w_k^\rho(s)\), see Lemma 13;
  2. run an "oracle check" on the upper bound, specifically, exit this iteration of the for-loop immediately after either of the following fail (in order):
    \[
    w_{i^*}(0) \leq s_H - \sum_{k \neq i^*} \pi^\rho_k(s_H)
    \]
    \[
    \rho_{i^*} \leq \pi^\rho_i(s_H) = \bar{\pi}_i^*(s_H - \sum_{k \neq i^*} w_k^\rho(s_H), s_H)
    \]
    (with the first checking the rationality of the interim guess of weight \(w_{i^*}\) and the second applying reasoning symmetric to the justification of the oracle on the lower bound);
  3. permanently fix \(i^* = i, s_L, s_H\) and break the for-loop (if this step is reached, then the range \([s_L, s_H]\) over \(s\) is non-empty and contains a solution by passing both oracles);

2. use the monotonicity of \(\pi^\rho_i\) to binary search on \(s\) for the true \(s^*\), converging \(\pi^\rho_i(s)\) to \(\rho_{i^*}\);
3. when the binary search has been run to satisfactory precision and reached a final estimate \(\tilde{s}\), output weights \(\tilde{\mathbf{w}} = (\tilde{s} - \sum_{i \neq i^*} w_i^\rho(\tilde{s}), w_{i^*}^\rho(\tilde{s}))\) which invert to values \(\tilde{\mathbf{v}}\) via respective \(v_i(\cdot)\) functions.

### B.3 Proofs of Lemma 6, Lemma 7, and Theorem 8 (Algorithm Correctness)

We now prove the key lemmas claimed in the main body of the paper. The purpose of Lemma 6 is to show that if we fix the “large weight candidate agent” \(i^*\) putting us in \(\text{Space-}i^*\), then all other agents have weights that are a precise, monotonically decreasing function of \(s\). Critically, recall that we can set \(i^*\) to be any agent, it is not restricted to be the agent (if any) who actually has more than half the weight (according to the true weights of any specific problem instance).

**Lemma 6.** The price level set \(Q^\rho_i\) is a curve; further, restricting \(Q^\rho_i\) to the region \(w_i \leq s/2\), the resulting subset \(P^\rho_i\) can be written as \(\{(s, w_i^\rho(s)) : s \in [r_i^\rho, \infty)\}\) for a real-valued decreasing function \(w_i^\rho\) mapping sum \(s\) to a weight \(w_i\) that is parameterized by the observed price \(r_i\).
Proof. This lemma follows as a special case of Lemma 9.

The purpose of Lemma 7 is to prove that function \( \pi^\rho_i \) is monotone increasing in \( s \) within Space-\( i^* \); setting up our ability to identify end points \( s_L \) and \( s_H \) where we run oracle checks, and our ability to run binary search for the unique solution \( s^* \) in a correct space.

Lemma 7. For any agent \( i^* \) and \( s \in [\max_{j \neq i^*} \rho_j^*, \infty) \), function \( \pi^\rho_i \) is strictly increasing.

Proof. The quantity \( s - \sum_{j \neq i^*} \rho_j^*(s) \) is monotone increasing in \( s \) as every term in the negated sum is decreasing in \( s \) (Lemma 9). Therefore, the guess of weight \( w_i^* = \max\{s - \sum_{j \neq i^*} \rho_j^*(s), w_i^*(0)\} \) lies in one of two ranges that are delineated by the threshold where the increasing quantity \( s - \sum_{j \neq i^*} \rho_j^*(s) \) crosses the constant \( w_i^*(0) \).

For small weight sums \( s \), the guess \( w_i^* \) evaluates to \( w_i^*(0) \). In this region \( \pi_i^\rho(s) = \pi_i^*(s, w_i^*(0)) \) from equation (31), and we know that \( \pi_i^\rho(s) \) is indeed strictly increasing when holding \( w_i^* = w_i^*(0) \) constant because \( \frac{\partial \pi_i^*(s, w_i^*)}{\partial s} \) is strictly positive (Lemma 9).

The remainder of this proof is devoted to showing for large weight sums \( s \) where the guess \( w_i^* \) evaluates to \( s - \sum_{j \neq i^*} \rho_j^*(s) \), that the function \( \pi_i^\rho(s) \) is strictly increasing. For the following, we use the result of Lemma 3 and the definition of \( h_i \) in equation (9). Note that when we are in Space-\( i^* \), we have \( h_k > 1 \) for \( k \neq i^* \).

\[
\frac{d\pi_i^\rho(s)}{ds} = \frac{d\pi_i((s - \sum_{k \neq i^*} \rho_k^*(s)), \rho_i^*(s))}{ds}
\]

\[
= \frac{\partial \pi_i}{\partial w_i} \left( 1 - \sum_{k \neq i} \frac{dw_k^\rho(s)}{ds} \right) + \frac{\partial \pi_i}{\partial w_i} \sum_{k \neq i} \frac{dw_k^\rho(s)}{ds}
\]

\[
= \frac{\partial \pi_i}{\partial w_i} \left( 1 - \sum_{k \neq i} \frac{1}{1 - h_k} + \frac{1}{h_i} \sum_{k \neq i} \frac{1}{1 - h_k} \right)
\]

\[
= \frac{\partial \pi_i}{\partial w_i} \left[ 1 + \left( \frac{1}{h_i} - 1 \right) \sum_{k \neq i} \frac{1}{1 - h_k} \right]
\]

In the second line here, the notation \( \frac{\partial \pi_i}{\partial w_j \neq i} \) recalls that all cross-partial are the same; moving from the second line to the third line, we replaced \( \frac{dw_k^\rho(s)}{ds} = 1/(1 - h_k) \) from Part 2 of Lemma 9, which also guarantees that each of these terms is strictly negative. When \( h_i \geq 1 \), the total bracketed term is positive, and \( \frac{d\pi_i((s - \sum_{k \neq i} \rho_k^*(s)), \rho_i^*(s))}{ds} > 0 \).

Alternatively to make an argument when \( h_i < 1 \), we further rearrange the algebra of the partial.
Continuing from the last line:

\[
\frac{d\pi_i^0(s)}{ds} = \frac{\partial \pi_i}{\partial w_i} \left[ 1 + \left(\frac{1}{h_i} - 1\right) \sum_{k \neq i} \frac{1}{1 - h_k} \right]
\]

\[
= \frac{\partial \pi_i}{\partial w_i} \left[ \frac{h_i - 1}{h_i - 1} + \left(\frac{1}{h_i} - 1\right) \sum_{k \neq i} \frac{1}{1 - h_k} \right]
\]

\[
= \frac{\partial \pi_i}{\partial w_i} \left[ \left(\frac{1}{h_i} - 1\right) \left(\sum_{k \neq i} \frac{1}{1 - h_k}\right) + \frac{h_i}{1 - h_i} + \frac{1}{1 - h_i} - \frac{1}{1 - h_i} \right]
\]

\[
= \frac{\partial \pi_i}{\partial w_i} \left[ \left(\frac{1}{h_i} - 1\right) \left(\sum_k \frac{1}{1 - h_k}\right) - \frac{h_i}{1 - h_i} + \frac{1}{1 - h_i} \right]
\]

When \(h_i < 1\), this quantity is again necessarily positive (with the last term positive by Lemma 4).

So again \(\frac{d\pi_i^0(s - \sum_{k \neq i} w_k(s), w_i^0(s))}{ds} > 0\). We conclude that \(\pi_i^0(s) := \pi_i((s - \sum_{k \neq i} w_k(s)), w_i^0(s))\) is everywhere strictly increasing in \(s\).

Finally we argue the correctness of the algorithm. However, correctness of the technical computations in pre-processing step 0 will be delayed to Appendix 3.4.

**Theorem 8.** Given weights \(w\) and payments \(\rho = \pi(w)\) according to a proportional weights social choice function, the algorithm identifies weights \(\tilde{w}\) within \(\varepsilon\) of the true weights \(w\) in time polynomial in the number of agents \(n\), the logarithm of the ratio of high to low weights \(\max_i \ln(w_i(h)/w_i(0))\), and the logarithm of the desired precision \(\ln 1/\varepsilon\).

**Proof.** Fix observed prices \(\rho\) that correspond to true weights \(w\) with sum \(s = \sum_i w_i\). Fix an agent \(i^*\) with \(w_{i^*} > s/2\) if one exists or \(i^* = 1\) if none exists. Set \(s_L = \max_{i \neq i^*} r_i^0\), and \(s_H\) as calculated in the algorithm for Space-i*. It must be that \(\pi_i^0(s_L) \leq \rho_{i^*} \leq \pi_i^0(s_H)\). The bounds follow by \(w_i \leq s/2\) for all \(i \neq i^*\), and Lemma 12 and Lemma 13 (stated and proved in the next section). Monotonicity of \(\pi_i^0(\cdot)\) then implies binary search will identify a sum \(\tilde{s}\) arbitrarily close to satisfying \(\pi_{i^*}^0(\tilde{s}) = \rho_{i^*}\). By the definition of \(\pi_i^0(\cdot)\), the weights \(\tilde{w} = w^0(\tilde{s})\) satisfy \(\pi(\tilde{w}) \approx \rho\); by uniqueness of the inverse \(\pi^{-1}\), these weights are approximately the original weights, i.e., \(\tilde{w} \approx w\).

In the case where \(w_{i^*} > s/2\), the iterative searches of Space-i for \(i \neq i^*\) will fail as these searches only consider points \((s, w_{i^*})\) where \(w_{i^*} < s/2\), but the weights \(w\) that corresponds to \(\rho\) are unique (by Theorem 3) and do not satisfy \(w_{i^*} < s/2\). When \(w_{i^*} \leq s\) then all searches, in particular \(i^* = 1\), will give the same result of \(w\).

Lastly, we show that binary search over \(s\)-coordinates within Space-i* is sufficient to converge the algorithm’s approximate \(\tilde{w}\) to \(w\) (measured by \(L_1\)-norm distance) at the same asymptotic rate of the binary search on \(s\), a rate which has only polynomial dependence on \(n, \max_i \ln(w_i(h)/w_i(0))\), and \(\ln 1/\varepsilon\).

---

21The significance of Lemma 3 here was discussed after the original statement of Lemma 7 at the end of Section 4.1 and after Lemma 9 at the end of Appendix 3.1.
By Lemma [10] below, for each agent $k \neq i^*$ there is a bound $B_k$ on the magnitude of the slope of $\frac{\partial \pi^\rho}{\partial s}$ as a function of the value space and weight functions inputs to the problem. $B_k$ depends on the factor $w_i/w_i(0) \leq w_i(h)/w_i(0)$ leading to the running time dependence.

Given a binary-search-step range on $s$ with size $S$, for every agent $k \neq i^*$, the size of the range containing $w_k$ can not be larger than $S \cdot B_k$. Every time the range of $s$ gets cut in half, the upper bound on the range of $w_k$ also gets cut in half. The convergence of $\tilde{w}_{k^*}$ to $w_{i^*}$ follows from the convergence in coordinates $s, w_{-i^*}$ and Lemma [11].

We conclude this section with the lemmas supporting the convergence rate claims of Theorem 8. Within the statement of Lemma [10] recall that the definition of the derivative was proved by Lemma [8].

**Lemma 10.** Given agent $i$ with $w_i \leq s/2$ and function $\pi^\rho_i$, the slope $\frac{\partial \pi^\rho}{\partial s} = \frac{1}{1-h_i} < 0$ has magnitude bounded by $\frac{w_i(h)}{2w_i(0)}$. 

**Proof.** We will show $\frac{1}{1-h_i} \leq \frac{w_i(h)}{w_i(0)}$. To upper bound $\frac{1}{1-h_i}$, we lower bound $h_i > 1$. Note that a lower bound on $h_i$ will only be useful for us if it strictly separates $h_i$ above 1. Substitute $s = \sum_k w_k$ into the definition of $h_i$ in equation (9) and bound, with justification to follow, as:

$$h_i = \frac{\int_{w_i(0)}^{w_i} v'^i_1(z) \frac{\frac{s}{w_i} - 1}{s - w_i + z} \cdot \frac{\frac{s}{w_i} - 1}{s - w_i + z}}{\int_{w_i(0)}^{w_i} v'^i_1(z) \frac{\frac{s}{w_i} - 1}{s - w_i + z} \cdot \frac{\frac{s}{w_i} - 1}{s - w_i + z}}$$

$$\geq \frac{\left[\frac{s}{w_i} - 1\right]}{\left[\frac{s}{s - w_i + w_i(0)} - 1\right]} \frac{\int_{w_i(0)}^{w_i} v'^i_1(z) \frac{\frac{s}{w_i} - 1}{s - w_i + z} \cdot \frac{\frac{s}{w_i} - 1}{s - w_i + z} \cdot \frac{s}{w_i(0)} \cdot \frac{z}{s - w_i + z} \cdot \frac{s}{w_i(0)} \cdot \frac{z}{s - w_i + z} \cdot \frac{s}{w_i(0)} \cdot \frac{z}{s - w_i + z}}{\int_{w_i(0)}^{w_i} v'^i_1(z) \frac{\frac{s}{w_i} - 1}{s - w_i + z} \cdot \frac{\frac{s}{w_i} - 1}{s - w_i + z} \cdot \frac{s}{w_i(0)} \cdot \frac{z}{s - w_i + z} \cdot \frac{s}{w_i(0)} \cdot \frac{z}{s - w_i + z} \cdot \frac{s}{w_i(0)} \cdot \frac{z}{s - w_i + z} \cdot \frac{s}{w_i(0)} \cdot \frac{z}{s - w_i + z}}$$

$$\geq \frac{s - w_i + w_i(0)}{w_i - w_i(0)} > 1 + \frac{2w_i(0)}{w_i} > 1$$

The first inequality replaces the integrand $z$ in the bracketed term in the denominator with its constant lower bound $w_i(0)$ (which only decreases a denominator, in the denominator); thereafter both bracketed terms can be brought outside of their respective integrals. The second inequality replaces the numerator with 1 because $w_i \leq s/2$ by statement assumption. The third (strict) inequality both replaces $s$ with $2w_i$ by the same reason, and adds $w_i(0)$ to both numerator and denominator, which makes the fraction smaller because it was originally larger than 1.

Using this bound we get:

$$\left|\frac{1}{1-h_i}\right| \leq \left|\frac{1}{1 - \left(1 + \frac{2w_i(0)}{w_i}\right)}\right| = \frac{w_i}{2w_i(0)} \leq \frac{w_i(h)}{2w_i(0)}$$

**Lemma 11.** Given agent $i^*$, true $s^* \in [s^-, s^+]$, and true weights $w_k \in [w_k^-, w_k^+]$ for agents $k \neq i^*$, which induce the range for $i^*$’s weight of $w_{i^*} \in [s^+ - \sum_{k \neq i^*} w_k^+, s^+ - \sum_{k \neq i^*} w_k^-]$. If the sizes of the ranges $[s^-, s^+]$ and $[w_k^-, w_k^+]$ are each individually reduced by (at least) a constant factor $\alpha$, then the size of the range of $w_{i^*}$ is also reduced by (at least) $\alpha$. 

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Proof. The statement follows immediately from the induced range of $w_i^*$. Its size is exactly equal to the sum of the $n$ other ranges, i.e.,

$$\left| s^- - \sum_{k \neq i^*} w_k^+ , s^+ - \sum_{k \neq i^*} w_k^- \right| = \left| s^- , s^+ \right| + \sum_{k \neq i^*} \left| w_k^- , w_k^+ \right| \square$$

B.4 Correctness of Algorithm Search End Points as Oracle Checks

This section has four purposes:

- analyze the structure of $r_i^p$ corresponding to level set $Q_i^p$;
- prove the correctness and run-time of the pre-process step 0 of the algorithm, which pre-computes $r_i^p$ for all $i$;
- conclude that the lower bounds $s_L$ of search in any given Space-$i$, determined within each iteration of step 1 of the algorithm, are the correct lower bounds of feasibility;
- conclude that the upper bounds $s_H$ calculated within each iteration of step 1 are the correct upper bounds of feasibility;

For strictly positive observed payment $\rho_i > 0$, the level set $Q_i^p$ takes on the full range of weights $w_i \in (w_i(\rho_i), w_i(h)]$ (the lower bound of $w_i(\rho_i)$ will not play an important role, our algorithms will use the less restrictive bound of $w_i(0)$ instead). Our search for the minimum $s$-coordinate of $P_i^p$, i.e., $r_i^p$, which is the intersection of $Q_i^p$ with the points below the $w_i = s/2$ line is either on the $w_i = s/2$ boundary or on the $w_i = w_i(h)$ boundary. This follows because constrained to $w_i \leq s/2$ the level set is given by a decreasing function (Lemma 6) and all level sets extend to $s = \infty$ (this second fact is true, but will not need to be explicitly proven). The two cases are depicted in Figure 4. For convenience, we restate the preprocessing step of the algorithm:

0. Pre-process: For each $i$, compute $r_i^p$:

- (general case: $P_i^p \neq Q_i^p$) if $\bar{\pi}_i(2w_i(h), w_i(h)) \geq \rho_i$, run binary search “diagonally” on the line segment of $w_i = s/2$ between $(0, 0)$ and $(2w_i(h), w_i(h))$ to find an element of $Q_i^p$ and use its $s$ coordinate as $r_i^p$ (which we can do because $\bar{\pi}_i(\cdot)$ is strictly increasing on this domain);
- (edge case: $P_i^p = Q_i^p$) otherwise, fix $w_i$ coordinate to its maximum $w_i(h)$ and run binary search “horizontally” to find $\bar{s} \in [2w_i(h), s(h)]$ representing $(\bar{s}, w_i(h)) \in Q_i^p$ (which we can do because $\bar{\pi}_i(\cdot)$ is strictly increasing in $s$ for constant $w_i$); set minimum total weight $r_i^p = \bar{s}$.

There is an intuitive explanation to the order of operations in the pre-processing step 0. First we check if we are in the general case. We can do this because price level-sets are strictly increasing on the line $w_i = s/2$ (see Lemma 12 below, extending Lemma 9). So we can check the largest the price at the largest possible point as $\bar{\pi}_i(2w_i(h), w_i(h))$; if it is too big, we can run binary search down to $\bar{\pi}_i(2w_i(0), w_i(0)) = 0$; otherwise we are in the edge case where $r_i^p$ corresponds to $w_i(h)$. In this case, we can binary search the line $w_i = w_i(h)$ for the point with payment $\rho_i$ as, again, price level-sets are strictly increasing (Lemma 9). The formal proof is given as Lemma 12.
As mentioned previously, denote the maximum sum of weights possible by \( \rho \). Proof. For the lower bound on the range, an agent wins with certainty and makes no payment \( \rho \leq s \) when the sum of the other agent weights is zero; the upper bound is from the natural upper bound \( \rho \geq \bar{s} \) corresponding price level sets \( P \) and \( Q \) are depicted. When \( \rho = 0 \) both of the corresponding price level sets \( P_i \) and \( Q_i \) are on the line \( w_i = w_i(0) \) (depicted, but not labeled). For observed price \( \rho_i \leq \bar{s}_i(2w_i(h), w_i(h)) \), and \( r_i^p \) corresponds to the \( s \)-coordinate at the intersection with the \( w_i = s/2 \) line. For observed price \( \rho_i \geq \bar{s}_i(2w_i(h), w_i(h)) \) the high level sets look like the depicted \( P_i^p = Q_i^p \), and \( r_i^p' \) corresponds to the \( s \)-coordinate at the intersection with the \( w_i = \hat{s}(h) \). line.

**Lemma 12.** For any realizable payment \( \rho_i \), price level set \( P_i^p \) is non-empty and its \( s \)-coordinates are lower bounded by \( r_i^p \) which can be computed to arbitrary precision by a binary search.

**Proof.** As mentioned previously, denote the maximum sum of weights possible by \( s(h) = \sum_i w_i(h) \). To find \( r_i^p \), we first focus attention on the horizontal line with constant weight \( w_i(h) \).

A point \((\hat{s}, w_i(h))\) on price level set \( Q_i^p \), i.e., with \( \bar{s}_i(\hat{s}, w_i(h)) = \rho_i \), can be found to arbitrary precision with binary search over \( s \in (w_i(h), s(h)) \). Correctness of this binary search follows because a realizable payment \( \rho_i \) must satisfy \( 0 = \bar{s}_i(w_i(h), w_i(h)) \) and because increasing \( s \)-coordinate corresponds to increasing price-level set on any line with fixed weight \( w_i \) by Lemma 9. For the lower bound on the range, an agent wins with certainty and makes no payment when the sum of the other agent weights is zero; the upper bound is from the natural upper bound \( s \leq s(h) \).

There are now two cases depending on whether this point \((\hat{s}, w_i(h))\) is above or below the \( w_i = s/2 \) line. If below, then \( r_i^p = \hat{s} \) because this point is tight to the maximum weight \( w_i(h) \) (see Figure 4), and (again by Lemma 9) the slope of curve \( P_i^p \) is strictly negative and all smaller \( s \) are infeasible.

Alternatively suppose \((w_i(h), \hat{s})\) is above the \( w_i = s/2 \) line, then \( r_i^p \) can be found by searching the \( w_i = s/2 \) line. Part (1) of Lemma 9 guarantees that points on this line are consistent with unique and increasing observed prices (partials of the price function are strictly positive in both dimensions \( w_i \) and \( s \), we can first move right \( ds \), and then move up \( dw_i \), with the price function strictly increasing as a result of both “moves”). On this line we have \( 0 = \bar{s}_i(2w_i(0), w_i(0)) \leq \rho_i \leq \bar{s}_i(2w_i(h), w_i(h)) \) where the lower bound observes an agent with value 0 to always pay 0, and the upper bound follows from the supposition \( w_i(h) \geq \hat{s}/2 \) of this case. Thus, a binary search of the \( w_i = s/2 \) line with \( w_i \in [w_i(0), w_i(h)] \) is guaranteed to find a point with price arbitrarily close to \( \rho_i \). Since \( P_i^p \) as a curve is decreasing in \( s \), the identified point, which is in \( P_i^p \), has the minimum \( s \)-coordinate.

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\(^{22}\) If on the line, the cases are equal and either suffices.
The two cases are exhaustive and so $r_i^P$ is identified and $\mathcal{P}_i^P$ is non-empty.

We finish the section with the lemma showing the correctness of the search range of sum $s$ within $[s_L, s_H]$.

**Lemma 13.** For true weights $w$, true weight sum $s = \sum w_i$, and $i^*$ with $w_i \leq s/2$ for $i \neq i^*$, sum $s$ is contained in interval $[s_L, s_H]$ (defined in step 1 of the algorithm for $i^*$).

**Proof.** First for the lower bound $s_L$, the assumption of the lemma requires $i \neq i^*$ satisfy $w_i \leq s/2$. Therefore the true pair $(s, w_i)$ must be a point in $\mathcal{P}_i^P$, and the true sum $s$ must be at least the lower bound $r_i^P$ for each $i \neq i^*$.

Second for the upper bound $s_H$, recall the definition

$$\pi_i^P(s) := \pi_i^*(\max\{s - \sum_{i \neq i^*} w_i^P(s), w_{i^*}(0)\}, w_{i^*}^P(s))$$

which uses a guess at the total weights $s'$ to guess the corresponding the weight of agent $i^*$ as $(s' - \sum_{i \neq i^*} w_i^P(s'))$. In fact, this guessed weight is strictly increasing in $s'$ as each term in the negated sum is strictly decreasing (Lemma 9). Our choice of $s_H$ equates this guessed weight with its highest possible value $w_{i^*}(h)$. By monotonicity of the guessed weight the true $s$ must be at most $s_H$. 

$\square$