FRÉCHET MODULES AND DESCENT
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Abstract. We study several aspects of the study of Ind-Banach modules over Banach rings thereby synthesizing some aspects of homological algebra and functional analysis. This includes a study of nuclear modules and of modules which are flat with respect to the projective tensor product. We also study metrizable and Fréchet Ind-Banach modules. We give explicit descriptions of projective limits of Banach rings as ind-objects. We study exactness properties of projective tensor product with respect to kernels and countable products. As applications, we describe a theory of quasi-coherent modules in Banach algebraic geometry. We prove descent theorems for quasi-coherent modules in various analytic and arithmetic contexts.

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1. Introduction

The use of categorical and homological techniques in functional analysis has a long and complicated history which we can not adequately summarize here. This includes work of Helemskii [25], Mayer [32] [31], Cigler, Losert and Michor [16], Paugam [36], Taylor [46], Wengenroth [54] and
others. We follow the approach of using the homological algebra of quasi-abelian categories of Pros-
mans and Schneiders [45], [42] generalized from the functional analysis of Banach and Ind-Banach
spaces over complex numbers to general Banach rings.

Grothendieck developed the theory of nuclearity for topological vector spaces over $\mathbb{C}$. In [42]
these ideas are carried over to the closely related setting of ind-Banach spaces over $\mathbb{C}$. We were
able to prove analogues of these results in the setting of Ind-Banach modules over arbitrary Banach
rings $R$. The definition of nuclearity we use is in Definition 4.9 and an equivalent characterization
in Remark 4.15. Not having Hilbert space techniques available, we were unable to prove that sub-
spaces and quotients of nuclear maps are nuclear. However, we can prove many other standard
“permanence properties” of nuclearity. We discuss countable products and coproducts in Corollary
5.22 and a two out of three rule for strict short exact sequences in Lemma 4.11 and the projective
tensor product of nuclear spaces in Lemma 4.16. A more general theory of nuclearity which works
in both the Archimedean and non-Archimedean settings should probably be inspired by Schnei-
der’s notion (see [44]) of compact morphisms between Banach spaces. Corollary 5.9 proves that
nuclearity also ensures an interesting interaction with products of dual spaces. Following work of
Prosmans and Schneiders we prove that nuclear spaces can be written in certain canonical ways in
Lemmas 4.17 and 4.18. We define metrizability in Definition 5.5. Important examples of metrizable
modules are Banach or Fréchet modules. Notice that as nuclearity of an object implies it is flat for
the projective tensor product (Lemma 4.19), one may ask what condition on an object might ensure
that the projective tensor product with it commutes with countable products. This turns out to
be a complete characterization of metrizability as proven in Lemmas 5.18 and 5.19. Therefore, in
combination the properties of nuclearity and metrizability for an object imply that the projective
tensor product with it commutes with countable limits (Lemma 5.18). Banach algebraic geo-
metry and its derived versions is an approach to analytic geometry which uses geometry relative to
categories of Banach spaces (or modules) in the same way that usual algebraic geometry is based
on categories of abelian groups. In particular, this philosophy applies to rigid analytic geometry
[11], overconvergent rigid geometry [7] and Stein geometry ([8], [39], [5]) and in these articles it was
shown that the homotopy monomorphism topology specializes to conventional ones in special cases.
There are also projects on derived analytic geometry [9] and analytic $\mathbb{F}_1$-geometry [12]. Most of the
constructions in this article are based on an arbitrary Banach ring $R$. If $R$ is a non-archimedean
Banach ring (see Definition 3.23), this entire article can be separately read in two different versions,
depending on whether one considers the categories $\text{Ind}(\text{Ban}_R)$ of all Banach modules or $\text{Ind}(\text{Ban}^{na}_R)$
of non-archimedean Banach modules. Therefore, in this case, notation such as symbols for limits,
colimits, products and coproducts, can sometimes take on two different meanings. We have cho-
sen to write everything with the default version being of the archimedean version. This has the
appealing aspect of being completely the same for any $R$, archimedean or not. In the case that $R$
is non-archimedean, the reader who wants to work in a non-archimedean context should replace all
limits and colimits in the category $\text{Ind}(\text{Ban}_R)$ by those in the category $\text{Ind}(\text{Ban}^{na}_R)$. All the proofs
go through in a similar way. Given a union of subsets one often wants to describe modules on the
union in terms of modules on the components together with gluing data. Our main descent results
can be found in Theorem 7.10. To formulate this we introduce a generalization of a coherent mod-
ule called a quasi-coherent module. This notion was also needed in [11, 7, 8] where some properties
of quasi-coherent modules were studied and in this article we extend that study.

2. Notation

We use the notation $\lim$ instead of $\lim\limits_{\leftarrow}$ and $\text{colim}$ for $\lim\limits_{\rightarrow}$. The letter $R$ denotes a general Banach
ring, defined in Definition 3.21. We denote categorical products by $\prod$ and categorical coproducts
by $[1]$, it should be clear in what category these take place, usually it is sufficient to consider them in the category $\text{Ind}(\text{Ban}_{\mathbb{R}})$. Given an object $A$ in $\text{Comm}(\text{Ind}(\text{Ban}_{\mathbb{R}}))$, we use $\text{spec}(A)$ to just denote the same object in the opposite category. As usual, $\mathbb{Z}_p$ denotes the $p$-adic integers, unless we are scaling the norm on $\mathbb{Z}$ with a real number in the sense of Definition 3.32; this should be clear from the context.

3. Some Category Theory and Its Uses in Functional Analysis and Geometry

3.1. Relative Algebra and Homological Algebra.

**Definition 3.1.** In a category with kernels and cokernels, a morphism $f : E \to F$ is called strict if the induced morphism

$$\text{coim}(f) \to \text{im}(f)$$

is an isomorphism. Here $\text{im}(f)$ is the kernel of the canonical map $F \to \text{coker}(f)$, and $\text{coim}(f)$ is the cokernel of the canonical map $\ker(f) \to E$. An object $P$ is projective if for all strict epimorphisms $E \to F$ the associated map $\text{Hom}(P, E) \to \text{Hom}(P, F)$ is onto. An object $I$ is injective if for all strict monomorphisms $E \to F$ the associated map $\text{Hom}(F, I) \to \text{Hom}(E, I)$ is onto. If the category is equipped with a unital symmetric monoidal structure $\otimes$ then an object $F$ is called flat if the functor $(-) \otimes F$ preserves strict monomorphisms.

Consider a unital, closed, symmetric monoidal category $(C, \otimes, e = \text{id}_C)$ with finiterate limits and colimits (more details in [11]). We will always suppress the commutativity, unitality, and associativity natural transformations from the notation. It is easy to see the following Lemma.

**Lemma 3.2.** The unit of $C$ is flat in $C$. Any colimit of flat objects in $C$ is flat in $C$. A coproduct of objects is flat if and only if each of them is flat.

3.2. Quasi-abelian categories.

**Definition 3.3.** Let $\mathcal{E}$ be an additive category with kernels and cokernels. We say that $\mathcal{E}$ is quasi-abelian if it satisfies the following two conditions:

- In a cartesian square

$$
\begin{array}{ccc}
E' & \xrightarrow{f'} & F' \\
\downarrow & & \downarrow \\
E & \xrightarrow{f} & F
\end{array}
$$

If $f$ is a strict epimorphism then $f'$ is a strict epimorphism.

- In a co-cartesian square

$$
\begin{array}{ccc}
E' & \xrightarrow{f'} & F' \\
\downarrow & & \downarrow \\
E & \xrightarrow{f} & F
\end{array}
$$

If $f$ is a strict monomorphism then $f'$ is a strict monomorphism.

A quasi-abelian category is a category where the strict monomorphisms and strict epimorphisms satisfy the conditions of a Quillen exact category. It may be useful to allow for more general Quillen exact structures (see [12]) for instance using short exact sequences that split over $R$ but in this work we avoid this.
Definition 3.4. Let $E$ be a quasi-abelian category. Let $K(E)$ be its category of complexes up to homotopy. The derived category of $E$ is $D(E) = K(E)/N(E)$ where $N(E)$ is the full subcategory of strictly exact sequences.

A sequence

$$E' \overset{e'}{\rightarrow} E \overset{e''}{\rightarrow} E''$$

in a quasi-abelian category is strictly exact when the image of the first map is isomorphic to the kernel of the second, and $e'$ is strict.

Lemma 3.5. Let $C$ and $D$ be quasi-abelian categories. Let $L : C \to D$ be any functor with right adjoint $R : D \to C$. Then $L$ preserves strict epimorphisms and $R$ preserves strict monomorphisms.

Proof. Let $f : V \to W$ be a strict epimorphism in $C$. Then of course $L(f)$ is an epimorphism. Because $f$ is a strict epimorphism, we have $W = \text{coker}(\ker(f) \to V)$. Therefore, since left adjoints preserve cokernels, $L(f)$ expresses $L(W)$ as the cokernel of the morphism $L(\ker(f)) \to L(V)$. The second statement is proven in the similar way.

\hfill $\square$

Definition 3.6. Let $E$ be a quasi-abelian category. Let $K(E)$ be its homotopy category. A morphism in $K(E)$ is called a strict quasi-isomorphism if its mapping cone is strictly exact.

Definition 3.7. Let $E$ be an additive category with kernels and cokernels. An object $I$ is called injective if the functor $E \mapsto \text{Hom}(E, I)$ is exact, i.e. for any strict monomorphism $u : E \to F$, the induced map $\text{Hom}(F, I) \to \text{Hom}(E, I)$ is surjective. Dually, $P$ is called projective if the functor $E \mapsto \text{Hom}(P, E)$ is exact, i.e. for any strict epimorphism $u : E \to F$, the associated map $\text{Hom}(P, E) \to \text{Hom}(P, F)$ is surjective.

Definition 3.8. A quasi-abelian category $E$ has enough projectives if for any object $E$ there is a strict epimorphism $P \to E$ where $P$ is projective. A quasi-abelian category $E$ has enough injectives if for any object $E$ there is a strict monomorphism $E \to I$ where $I$ is injective.

We will freely use the following proposition which comes from a combination of Propositions 2.1.18 of \cite{15}.

Proposition 3.9. Let $C$ be a small, closed symmetric monoidal quasi-abelian category and $R \in \text{Comm}(C)$. Then $\text{Mod}(R)$ is elementary if $C$ is, $R \otimes P$ is tiny in $\text{Mod}(R)$ whenever $P$ is tiny in $C$, if $G$ is a strictly generating set of $C$ then $\{R \otimes G \mid G \in G\}$ is a strictly generating set of $\text{Mod}(R)$.

3.3. Ind-Categories and Ind-Categories of quasi-abelian categories. Recall that for any category $C$ we can define its ind-completion.

Definition 3.10. Let $C$ be a category. An ind-completion of $C$ is a category $D$ with a functor $i : C \to D$, such that $D$ is closed under filtered colimits, and the functor $i$ is initial with respect to functors into categories closed under filtered colimits.

Lemma 3.11. Let $C$ be a category. Its ind-completion exists and can be realized as the full subcategory of the category $\text{Pr}(C) = \text{Fun}(\text{C}^{\text{op}}, \text{Set})$ whose objects are filtered colimits of representable functors (note that the category of presheaves is cocomplete).

We will denote the ind-completion of $C$ by $\text{Ind}(C)$. Given two presentations of objects $E \cong \colim_{i \in I} E_i$ and $F \cong \colim_{j \in J} F_j$, we have a canonical isomorphism

$$\text{Hom}(E, F) \cong \lim_{i \in I \ j \in J} \text{colim} \text{Hom}(E_i, F_j).$$
Remark 3.12. Therefore, a morphism can be defined as a functor \( \alpha : I \to J \) and for each \( i \in I \) an element of \( \text{Hom}(E_i, F_{\alpha(i)}) \) giving a natural transformation \( E \to F \circ \alpha \).

One way of getting elementary quasi-abelian categories is by looking at ind-completions of quasi-abelian categories (2.1.17 in [45]):

**Theorem 3.13.** Let \( E \) be a small quasi-abelian category with enough projective objects. Then, \( \text{Ind}(E) \) is an elementary quasi-abelian category and there is a canonical equivalence of categories \( \text{LH}(\text{Ind}(E)) \cong \text{Ind}(\text{LH}(E)) \).

The following is 2.1.19 in [45]:

**Proposition 3.14.** The category \( \text{Ind}(E) \) has a canonical closed symmetric monoidal structure extending that on \( E \). Hence, if \( E \) has enough projectives, \( \text{Ind}(E) \) is a closed symmetric monoidal elementary quasi-abelian category.

**Proof.** The extension is given as follows:

\[
\text{Hom}(\text{colim}_{i \in I} E_i, \text{colim}_{j \in J} F_j) = \lim_{i \in I} \text{Hom}(E_i, \text{colim}_{j \in J} F_j).
\]

\[
\text{Hom}(\text{colim}_{i \in I} E_i, F_{(i,j) \in I \times J}) = \text{colim}_{j \in J} \text{Hom}(E_i, F_j).
\]

\[\Box\]

**Definition 3.15.** Let \( E \) be an additive category. An object \( E \) is called:

- small, if
  \[
  \text{Hom}(E, \bigcoplus_{i \in I} F_i) \cong \bigcoplus_{i \in I} \text{Hom}(E, F_i)
  \]
  for any small family \( (F_i)_{i \in I} \) of \( E \) whenever the coproduct on the left exists

- tiny, if
  \[
  \text{Hom}(E, \text{colim}_{i \in I} F_i) \cong \text{colim}_{i \in I} \text{Hom}(E, F_i)
  \]
  for any filtering inductive system \( I \to E \) whenever the colimit on the left exists.

**Definition 3.16.** Let \( E \) be a quasi-abelian category. A strictly generating set of \( E \) is a subset \( G \) of \( \text{Ob}(E) \) such that for any monomorphism

\[
m : S \to E
\]

of \( E \) which is not an isomorphism, there is a morphism

\[
G \to E
\]

with \( G \in G \) which does not factor through \( m \).

The following is 2.1.7 in [45]:

**Lemma 3.17.** Let \( E \) be a cocomplete quasi-abelian category. A small subset \( G \) of objects of \( E \) is a strictly generating set of \( E \) if and only if for any object \( E \) of \( E \), there is a strict epimorphism of the form

\[
\bigcoplus_{j \in J} G_j \to E
\]

where \( (G_j)_{j \in J} \) is a small family of elements of \( G \).

**Definition 3.18.** A quasi-abelian category is quasi-elementary (resp. elementary) if it is cocomplete and has a small strictly generating set of small (resp. tiny) projective objects.
For abelian categories quasi-elementary is equivalent to elementary. We assume the reader is familiar with the notions of a family of injective objects with respect to a functor between in a quasi-abelian categories [45]. In this section we recall how to derive the inverse limit functor in quasi-abelian categories.

Given a functor $V : I \to C$ the Roos complex of $V$ is of the form

$$0 \to \mathcal{R}^0(V) \to \mathcal{R}^1(V) \to \mathcal{R}^2(V) \to \cdots$$

where

$$\mathcal{R}^0(V) = \prod_{i \in I} V_i$$

and

$$\mathcal{R}^n(V) = \prod_{i_0 \to i_1 \to \cdots \to i_{n+1}} V_{i_0}$$

where the product is over all composable sequences of $n$ morphisms in $I$. The differential $\mathcal{R}^n(V) \to \mathcal{R}^{n+1}(V)$ is defined for $\alpha$ the composable sequence $i_0 \to i_1 \to \cdots \to i_{n+1}$

$$(d((v_\beta)_\alpha))_{\alpha} = V(\alpha_1)_{i_1} \to V(\alpha_2)_{i_2} \to \cdots \to V(\alpha_{n+1})_{i_{n+1}}$$

Inverse limits have a derived functor because of the following proposition of Prosmans [41].

**Proposition 3.19.** Let $I$ be a small category and $C$ a quasi-abelian category with exact products. Then the family of objects in $C^I$ which are Roos-acyclic form a $\lim_{i \in I}$-acyclic family. As a result, the functor

$$\lim_{i \in I} : C^I \to C$$

is right derivable and for any object $V \in C^I$, we have an isomorphism

$$\mathcal{R}\lim_{i \in I} V_i \cong \mathcal{R}^\bullet(V)$$

where the right hand side is the Roos complex of $V$. The family of $\lim_{i \in I}$-acyclics for the functor $\lim_{i \in I} : C^I \to C$ form a family of injectives relative to this functor (a concept appearing in [45]).

Because of the explicit formula of the Roos complex, notice that

**Corollary 3.20.** Let $I$ be a small category and $C$ a quasi-abelian closed symmetric monoidal category with exact products. If $W$ is flat in $C$ and $W \otimes (-)$ commutes with products in $C$ then the natural morphism

$$W \otimes (\mathcal{R}\lim_{i \in I} V_i) \to \mathcal{R}\lim_{i \in I} (W \otimes V_i)$$

is an isomorphism. In particular if $V \in C^I$ is $\lim_{i \in I}$-acyclic then so is $W \otimes V$ we recover an isomorphism

$$W \otimes (\lim_{i \in I} V_i) \to \lim_{i \in I} (W \otimes V_i).$$

We will use this material again in Lemma 6.12

### 3.4. Relative Geometry.

Just as algebraic geometry is “built” from the theory of commutative rings and their modules, much work on other kinds of geometry and topology is based on commutative monoids and their modules internal to general symmetric monoidal categories $(C, \otimes, e)$, for instance see [48]. In our approach we also ask that they be equipped with compatible Quillen exact structures. The category most important for us is the quasi-abelian example of Ind-Banach modules over a Banach ring together with its projective tensor product and its applications to analytic and arithmetic geometry. An important class of morphisms between “affine schemes” in...
relative geometry are opposite to those morphisms $A \to B$ in $\text{Comm}(C)$ such that the natural map $B \underline{\otimes}_A B \to B$ is a quasi-isomorphism. Such a morphism will be called a homotopy epimorphism. Because this notion actually is equivalent to the general model or infinity-category notion of a homotopy epimorphism as found in work of Toën and Vezzosi (as used in homotopical or derived algebraic geometry) we use that terminology. However, other sources call this a stably flat morphism \cite{33}, an isocohomological morphism \cite{32} or a homological epimorphism \cite{20,15}. It appears algebraic geometry we use that terminology. However, other sources call this a stably flat morphism. Because this notion actually is equivalent to the general model or infinity-category notion of a

3.5. Banach Rings and Banach Modules.

**Definition 3.21.** By a complete normed (or Banach) ring we mean a commutative ring with identity $R$ equipped with a function, $| \cdot | : R \to \mathbb{R}_{\geq 0}$ such that

- $|a| = 0$ if and only if $a = 0$;
- $|a + b| \leq |a| + |b|$ for all $a, b \in R$;
- there is a $C > 0$ such that $|ab| \leq C|a||b|$ for all $a, b \in R$;
- $R$ is a complete metric space with respect to the metric $(a, b) \mapsto |a - b|$.

The category of Banach rings has as morphisms those ring homomorphisms $R \to S$ such that there exists a constant $C > 0$ such that $|\phi(a)|_S \leq C|a|_R$ for all $a \in R$, in other words bounded ring homomorphisms.

**Definition 3.22.** Let $(R, | \cdot |_R)$ be a Banach ring. A Banach module over $R$ is an $R$-module $M$ equipped with a function $\| \cdot \| : M \to \mathbb{R}_{\geq 0}$ such that for any $m, n \in M$ and $a \in R$:

- $\|0_M\|_M = 0$;
- $\|m + n\|_M \leq \|m\|_M + \|n\|_M$;
- $\|am\|_M \leq C|a|_R \|m\|_M$ for some constant $C > 0$;
- $\|m\|_M = 0$ implies that $m = 0_M$;
- $M$ is complete with respect to the metric $d(m, n) = \|m - n\|$.

**Definition 3.23.** A Banach ring or a Banach module over a Banach ring is called non-archimedean if its semi-norm obeys the strong triangle inequality: for any two elements $v, w$ we have $\|v + w\| \leq \max\{\|v\|, \|w\|\}$.

**Definition 3.24.** If $M$ is a Banach module over a Banach ring $R$ and $r$ is a positive real number then $M_r$ is a Banach module over $R$ defined by $M$ equipped with the Banach structure $r\| \cdot \|_M$.

**Definition 3.25.** Let $(R, | \cdot |_R)$ be a Banach ring. A $R$-linear map between Banach $R$-modules (Definition 3.22), $f : (M, \| \cdot \|_M) \to (N, \| \cdot \|_N)$ is called bounded if there exists a real constant $C > 0$ such that

$$\|f(m)\|_N \leq C\|m\|_M$$

for any $m \in M$. The homomorphism $f$ is called non-expanding if this equation holds for $C = 1$.

The category of Banach modules with bounded morphisms is denoted by $\text{Ban}_R$. If $R$ is non-archimedean $\text{Ban}_R^{na}$ denotes the category of non-archimedean Banach modules with bounded morphisms.

**Lemma 3.26.** Clearly for any Banach ring $R$, $R$ is projective as a Banach $R$-module.
Lemma 3.27. For any projective $R$-module, $P$, and any real number $r > 0$, $P_r$ is also projective.

Definition 3.28. Given $M,N \in \text{Ban}_R$ we define $M \hat{\otimes}_R N$ as the (separated) completion of $M \otimes_R N$ with respect to the semi-norm
\[
\|x\| = \inf \left\{ \sum_{i=1}^{n} \|m_i\|n_i \mid x = \sum_{i=1}^{n} m_i \hat{\otimes}_R n_i \right\}.
\]
Similarly, if $R$ is non-archimedean, given $M,N \in \text{Ban}_R^{na}$ we define $M^{na} \hat{\otimes}_R N$ as the (separated) completion of $M \otimes_R N$ with respect to the semi-norm
\[
\|x\| = \inf \sup_{i=1,\ldots,n} \|m_i\|n_i \mid x = \sum_{i=1}^{n} m_i \hat{\otimes}_R n_i \}.
\]

The internal Hom in this categories is denoted by $\text{Hom}_R(V,W)$ and given by the Banach space whose underlying vector space is just the bounded $R$-linear maps
\[
\{ T \in \text{Lin}_R(V,W) \mid \|T\| < \infty \}
\]
with norm given by $\|T\| = \sup_{v \in V, v \neq 0} \frac{|T(v)|}{\|v\|}$. We write $V'$ for $\text{Hom}_R(V,R) \in \text{Ban}_R$. These categories are both closed symmetric monoidal with unit object given by the multiplicative identity in $R$.

Lemma 3.29. If $P$ and $Q$ are projective in $\text{Ban}_R$ then $P \hat{\otimes}_R Q$ is also projective in $\text{Ban}_R$.

Definition 3.30. The category $\text{Ban}_R^{\leq 1}$ is defined to have the same objects as $\text{Ban}_R$. The morphisms are the linear maps with norm less than or equal to one (they are non-expanding or sometimes just called contracting).

This is closed symmetric monoidal with the same internal hom and tensor product and as before it has two versions (one of which exists only when $R$ is non-archimedean).

Infinite products and coproducts in $\text{Ban}_R^{\leq 1}$ exist even though they do not exist in $\text{Ban}_R$. In the archimedean case (see page 63 of [25]) the product $\prod_{i \in I} V_i$ of a collection $\{V_i\}_{i \in I}$ in $\text{Ban}_R^{\leq 1}$ is given by
\[
\{(v_i)_{i \in I} \in \bigotimes_{i \in I} V_i \mid \sup_{i \in I} \|v_i\| < \infty \}
\]
equipped with the norm
\[
\|(v_i)_{i \in I}\| = \sup_{i \in I} \|v_i\|
\]
while the coproduct $\bigsqcup_{i \in I} V_i$ of a collection $\{V_i\}_{i \in I}$ in $\text{Ban}_R^{\leq 1}$ is given by
\[
\{(v_i)_{i \in I} \in \bigoplus_{i \in I} V_i \mid \sum_{i \in I} \|v_i\| < \infty \}
\]
equipped with the norm
\[
\|(v_i)_{i \in I}\| = \sum_{i \in I} \|v_i\|.
\]
If $R$ is non-archimedean and we choose to work in the the non-archimedean case, they can be computed as in [22]: the product $\prod_{i \in I} V_i$ of a collection $\{V_i\}_{i \in I}$ in $\text{Ban}_R^{\leq 1}$ is given by
\[
\{(v_i)_{i \in I} \in \bigotimes_{i \in I} V_i \mid \sup_{i \in I} \|v_i\| < \infty \}
\]
equipped with the norm
\[
\|(v_i)_{i \in I}\| = \sup_{i \in I} \|v_i\|.
while the coproduct $\bigsqcup_{i \in I} V_i$ of a collection $\{V_i\}_{i \in I}$ in $\text{Ban}_R^{\leq 1}$ is given by
\[
\{(v_i)_{i \in I} \in \prod_{i \in I} V_i \mid \lim_{i \in I} \|v_i\| = 0\}
\]
equipped with the norm
\[
\| (v_i)_{i \in I} \| = \sup_{i \in I} \| v_i \|.
\]

**Lemma 3.31.** Suppose we are given a collection $\{f_i : V_i \to W_i\}_{i \in I}$ in $\text{Ban}_R^{\leq 1}$. Then observe that the natural morphism
\[
\bigsqcup_{i \in I} \ker(f_i) \to \ker\left[\bigsqcup_{i \in I} V_i \to \bigsqcup_{i \in I} W_i\right]
\]
is an isomorphism. Similarly, if $V_i \subset V$ and $W_i \subset W$ are countable increasing unions of complete closed isometric submodules with union $V$ and $W$ respectively then the natural map
\[
\text{colim}_{i \in I} \ker(f_i) \to \ker[V \to W]
\]
is an isomorphism.

**Definition 3.32.** Let $R$ be a Banach ring and $M$ a Banach $R$-module. $M_r$ is the Banach module which has the same underlying algebraic module as $M$ but the norm on $M_r$ is defined by $\|m\|_{M_r} = r\|m\|_M$.

**Lemma 3.33.** Any projective in $\text{Ban}_R$ is flat in $\text{Ban}_R$.

**Proof.** Let $P$ be a projective in $\text{Ban}_R$. Consider the canonical strict epimorphism $\bigsqcup_{p \in P} \leq 1 R_{\|p\|} \to P$. As usual, it splits and so $\bigsqcup_{p \in P} \leq 1 R_{\|p\|}$ is coproduct of the kernel and $P$. Hence $P$ is flat by Lemma 3.2. \qed

The proof of the following is obvious from the definitions.

**Lemma 3.34.** Any coproduct of projective objects in $\text{Ban}_R$ is projective in $\text{Ban}_R$.

**Lemma 3.35.** [22] A filtered colimit of strict, short exact sequences in $\text{Ban}_R^{\leq 1}$ is a strict short exact sequence.

**Proof.** See proposition 1 on page 69 of [22]. \qed

**Lemma 3.36.** If $V \to W$ is a strict epimorphism and $P$ is projective then the corresponding morphism $\text{Hom}(P,V) \to \text{Hom}(P,W)$ is a strict epimorphism.

**Proof.** By Proposition 1.3.23 of [15] it is enough to show that for any projective $Q$, that
\[
\text{Hom}(Q, \text{Hom}(P,V)) \to \text{Hom}(Q, \text{Hom}(P,W))
\]
is surjective. This follows immediately from adjunction and from Lemma 3.29. \qed

**Lemma 3.37.** For any small set $S$ and projectives $P_s \in \text{Ban}_R$ for each $s \in S$ the object $P = \bigsqcup_{s \in S} \leq 1 P_s$ is projective in $\text{Ban}_R$.

**Proof.** Let $V \to W$ be a strict epimorphism and let $f : P \to W$ be any morphism. Fix $\epsilon > 0$. By Lemma 3.36 for each projective $P_s$ we get a strict epimorphism $\text{Hom}(P_s,V) \to \text{Hom}(P_s,W)$. Let $f_s$ be the restriction of $f$ to $P_s$ so $\|f_s\| = \|f\|$. Using the strict epimorphism property, chose for each $s$ a lift $\tilde{f}_s \in \text{Hom}(P_s,V)$ of $f_s$ such that $\|f_s\| \leq \|f_s\| + \epsilon = \|f\| + \epsilon$. As their norms are bounded independent of $s$, the $\tilde{f}_s$ assemble into a morphism $\tilde{f} : P \to V$ inducing $f$. \qed

**Lemma 3.38.** The category $\text{Ban}_R$ has enough projectives and all projectives in $\text{Ban}_R$ are flat.
Lemma 3.39. Let $R$ be a Banach ring and $M$ a Banach $R$-module. Then for any positive real number $r$ we have $(M_r)^\vee \cong (M^\vee)_r^{-1}$.

Lemma 3.40. Let $R$ be a Banach ring and $M$ a Banach $R$-module. Then for any positive real number $r$, $M_r$ is projective if and only if $M$ is projective.

Lemma 3.41. Given an inductive system $V_i$ in $\text{Ban}^\leq_R$ the canonical morphism

$$(\colim_{i \in I} \leq^1 V_i)^\vee \to \lim_{i \in I} \leq^1 (V_i^\vee)$$

( induced by the duals of the collection of isometric immersions $V_i \to \colim_{i \in I} \leq^1 V_i$) is an isomorphism.

Proof. Since the canonical morphism is strict and non-expanding, it is enough to show that it induces an isomorphism of sets

$$((\colim_{i \in I} \leq^1 V_i)^\vee)^{< r} \to ((\lim_{i \in I} \leq^1 (V_i^\vee))^{< r}$$

for any real number $r \geq 1$. The canonical morphism identifies the left hand side with

$$\text{Hom}^{\leq^1} (R, (\colim_{i \in I} \leq^1 V_i)^\vee) = \text{Hom}^{\leq^1} (R, \text{Hom}_{i \in I} (\colim_{i \in I} \leq^1 V_i, R)) = \text{Hom}^{\leq^1} (R, \widehat{\lim_{i \in I} \leq^1 V_i, R})$$

$$= \text{Hom}^{\leq^1} (\colim_{i \in I} \leq^1 (V_i^\vee), R) = \lim_{i \in I} \text{Hom}^{\leq^1} ((V_i^\vee), R) = \lim_{i \in I} \text{Hom}^{\leq^1} (R, (V_i^\vee)^\vee)$$

$$= \lim_{i \in I} \text{Hom}^{\leq^1} (R, (V_i^\vee)^{-1}) = \lim_{i \in I} \text{Hom}^{\leq^1} (R, V_i^\vee) = \text{Hom}^{\leq^1} (R, \lim_{i \in I} (V_i^\vee))$$

which agrees with the right hand side.

Corollary 3.42. Given a non-expanding morphism $V_i \to W_i$ of inductive systems in $\text{Ban}^\leq_R$ the dual of the corresponding morphism

$$\colim_{i \in I} \leq^1 V_i \to \colim_{i \in I} \leq^1 W_i$$

is the morphism

$$\lim_{i \in I} \leq^1 (V_i^\vee) \to \lim_{i \in I} \leq^1 (W_i^\vee)$$

corresponding to the dual morphisms $V_i^\vee \to W_i^\vee$.

Proof. This is automatic from the definitions and Lemma 3.41.

Conjecture 3.43. Given a non-expanding, nuclear morphism $f_i : V_i \to W_i$ of inductive systems in $\text{Ban}^\leq_R$ the corresponding morphism

$$f : \colim_{i \in I} \leq^1 V_i \to \colim_{i \in I} \leq^1 W_i$$

is nuclear.

Definition 3.44. For any Banach ring $R$ and $n$-tuple of positive real numbers $r = (r_1, \ldots, r_n)$ the poly-disk algebra of poly-radius $r$ is defined by the sub-ring

$$R\left\{ \frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n} \right\} = \left\{ \sum_{J} a_J x^J \in R[[x_1, \ldots, x_n]] \mid \sum_{J} |a_J| r^J < \infty \right\}$$
equipped with the norm $|\sum a_j x^j| = \sum |a_j| r^j$. When $R$ is non-archimedean, one can still use the above if in the archimedean context, or instead if one wants to work in the non-archimedean context one can read this article using the Tate algebra $R(\frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n}) = \{ \sum a_j x^j \in R[[x_1, \ldots, x_n]] \mid \lim_j |a_j| r^j = 0 \}$ equipped with the norm $|\sum a_j x^j| = \sup_j |a_j| r^j$. Similarly, we can define Banach abelian groups $M(\frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n})$ for any Banach abelian group $M$. Notice that these are completions of the normed rings $M(\frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n})$ and subrings of $M[[x_1, \ldots, x_n]]$.

**Remark 3.45.** If $R$ is non-archimedean, all of this subsection goes through for $\text{Ban}_R^a$ in place of $\text{Ban}_R$. Just as finitely presentable rings play an important role in algebraic geometry, in Banach algebraic geometry over $R$, the nice objects of study are quotients of the above disk algebras by ideals, equipped with the quotient Banach structure. As the category of these affinoid algebras is not closed under filtered limits or colimits, it is natural to introduce also Stein and dagger algebras in Section 5 and perhaps even more general limits and colimits like quasi Stein, Stein-dagger, and quasi-Stein-dagger, etc.

**Remark 3.46.** Let $R$ be a non-zero Banach ring with multiplicative norm. It is automatically an integral domain. Let $S$ be a multiplicative subset, and equip the localization $S^{-1}R$ of $R$ with the completion of the norm $|\frac{x}{y}| = \frac{|x|}{|y|}$. Then for any Banach ring $T$, the map $R \to S^{-1}R$ identifies $\text{Hom}(S^{-1}R, T)$ with the bounded ring morphisms $R \to T$ sending $S$ to invertible elements and so categorically, the map $R \to S^{-1}R$ is an epimorphism, equivalently $S^{-1}R \hat{\otimes}_R S^{-1}R \cong S^{-1}R$. Notice also that the morphism $R \to S^{-1}R$ is flat with respect to $\hat{\otimes}_R$ so we can conclude that $R \to S^{-1}R$ is in fact a homotopy epimorphism. For examples of this consider in general inverting an element $r \in R$ using $R \to R(|r|y)/(ry - 1)$ even if the norm on $R$ is not multiplicative. Notice that $R \to Z_p$ is not an example, while $R \to \mathbb{R}$ and $Z \to \mathbb{Z} \{ \frac{p}{q} \} / (pq - 1) = \mathbb{Z}_p$ are.

3.6. **Arithmetic Examples.** In this subsection we discuss several examples of homotopy epimorphisms and derived projective tensor products. These all have a geometric meaning in terms of the Berkovich or Huber spectrum of $Z$. As we are working over $Z$ in this subsection, it takes place entirely in the archimedean context. We return to looking at these in terms of covers of $\text{spec}(Z)$ and descent in Section 7. As we are working over $Z$ in this subsection, it is about the Archimedean context only.

**Observation 3.47.** We have $\mathbb{Q}_p \hat{\otimes}_Z \mathbb{R} = \{ 0 \} = Z_p \hat{\otimes}_Z \mathbb{R}$ for any prime $p$ and for distinct primes $p$ and $q$, we have $\mathbb{Q}_p \hat{\otimes}_Z \mathbb{Q}_q = \{ 0 \} = Z_p \hat{\otimes}_Z \mathbb{Q}_q$. As a consequence, $\mathbb{Q}_p$ and $Z_p$ are not flat with respect to the completed tensor product over $Z$.

**Proof.** In $Z_p \hat{\otimes}_Z \mathbb{R}$ the element $1 \otimes 1$ can be written as $p^n \hat{\otimes} p^{-n}$ which has norm $p^{-2n}$ for any $n$. In $Z_p \hat{\otimes}_Z \mathbb{Q}_q$ choose for each $n$, integers $a_n$ and $b_n$ with $anp^n + bnq^n = 1$. Then $1 \otimes 1$ can be written as $ap^n \hat{\otimes} b_n p^{-n}q^n$ which has norm less than or equal to $p^{-n} + q^{-n}$. Letting $n$ go to infinity we see that in both Banach rings, we must have $1 = 1 \otimes 1$ has norm zero and hence vanishes and so these rings are the zero ring. Applying the functor $Z_p \hat{\otimes}_Z (-)$ to the strict short exact sequence $Z \to \mathbb{R} \to S^1$ gives $Z_p \to \{ 0 \} \to Z_p \hat{\otimes}_Z S^1 = \{ 0 \}$ and so $Z_p$ is not flat. In particular, $S^1 \hat{\otimes}_Z Z_p$ is a non-zero Banach abelian group $K$ sitting in degree $-1$ including for example the element $\frac{1}{p^n} \sum_{i=0}^{\infty} (x/p)^i$ which has norm $\frac{1}{p^n}$ and is in the kernel $K$ of the strict epimorphism $S^1 \{ px \} \to S^1 \{ px \}$. The proofs for the fraction fields with their obvious Banach structures are similar.
Using the resolutions we develop later, its easy to see that these rings are also orthogonal on the derived level. The lack of flatness with respect to the projective tensor product is similar to the known problem in analytic geometry that certain morphisms $A \to B$ of various Banach, Fréchet, or bornological algebras corresponding to the restriction of spaces of functions over various “open” sets do not exhibit $B$ as a flat module with respect to the completed tensor product over $A$.[11][12]. This explains our preference for using homotopy epimorphisms instead of flat epimorphisms. The analogous issue does not arise in algebraic or differential geometry in the standard topologies.

The idea of writing $\mathbb{Q}_p$ and $\mathbb{R}$ in terms of disk algebras over $\mathbb{Z}$ goes back to F. Paugam [37]. We use his idea in the following lemma which uses the disk algebras (Definition 3.44). We show here that one can think of $\mathbb{Z}_p$ and $\mathbb{R}$ as a sort of archimedean type Weirstrass localization of $\mathbb{Z}$ and then in turn $\mathbb{Q}_p$ is a Laurent localization of $\mathbb{Z}_p$ and hence a rational localization of $\mathbb{Z}$. We use the terms Weirstrass, Laurent, and rational localizations because of the analogy of the formulas with those in non-archimedean geometry. The symmetric ring construction in the contracting category [11] works equally well to define infinite dimensional disk algebras.

**Lemma 3.48.** We have

(1) A strict short exact sequence

$$0 \to \prod_{i \in \mathbb{Z}_{22}}^{\leq i+1}(\mathbb{Z}) \to \prod_{i \in \mathbb{Z}_{21}}^{\leq i+1}\mathbb{Z} \to \mathbb{Z}_{\text{triv}} \to 0$$

where here $(\mathbb{Z})_{i+1}$ represents the group $\mathbb{Z}$ with norm scaled by $i+1$ in the sense of Definition 3.32.

(2) A strict short exact sequence

$$0 \to \mathbb{Z}\{2x\}^{2x-1} \to \mathbb{Z}\{2x\} \to \mathbb{R} \to 0$$

in other words

$$\mathbb{Z}\{2x\}/(2x-1) \cong \mathbb{R}.$$ 

As a consequence $\mathbb{R}$ is flat over $\mathbb{Z}$ and hence $\mathbb{Z} \to \mathbb{R}$ is a homotopy epimorphism (Remark 3.46).

(3) A strict short exact sequence

$$0 \to \mathbb{Z}\{px\}^{x-p} \to \mathbb{Z}\{px\} \to \mathbb{Z}_p \to 0$$

in other words

$$\mathbb{Z}\{px\}/(x-p) \cong \mathbb{Z}_p.$$

(4) Strict exact sequences

$$0 \to \mathbb{Z}\{\frac{y}{p}\}^{py-1} \to \mathbb{Z}\{\frac{y}{p}\} \to \mathbb{Q}_p \to 0$$

$$0 \to \mathbb{Z}\{px\}^{x-p} \mathbb{Z}\{\frac{px}{p}\} \to \mathbb{Q}_p \to 0$$

where $\mathbb{Z} \to \mathbb{Z}_{\frac{1}{p}} = \mathbb{Z}\{\frac{y}{p}\}/(py-1)$ is a flat epimorphism and hence a homotopy epimorphism (Remark 3.46). And also there is a strict exact sequence

$$0 \to \mathbb{Z}\{px,\frac{y}{p}\}^{(1-py,x-p)} \to \mathbb{Z}\{px,\frac{y}{p}\}^{2} \to \mathbb{Z}\{px,\frac{y}{p}\}^{x-p,py-1} \to \mathbb{Q}_p \to 0$$

in other words

$$\mathbb{Z}\{px,\frac{y}{p}\}/(x-p,py-1) \cong \mathbb{Q}_p.$$
(5) The morphisms of Banach rings \( \mathbb{Z} \to \mathbb{Z}_p, \mathbb{Z} \to \mathbb{Q}_p \) are homotopy epimorphisms. The morphism \( \mathbb{Z} \to \mathbb{Z}_{\text{triv}} \) is an epimorphism but not a homotopy epimorphism.

**Proof.**

(1) Consider the multiplication by \( n \) maps \( \phi_n : \mathbb{Z} \to \mathbb{Z}_{\text{triv}} \). Since they are non-expanding they induce a morphism

\[
\phi = \sum_{n \in \mathbb{Z}_{\geq 1}} \phi_n : \bigoplus_{Z_{\geq 1}}^{\leq 1} \mathbb{Z} \to \mathbb{Z}_{\text{triv}}
\]

which is easily seen to be a strict epimorphism as any non-zero element can be lifted to an element of norm 1 which is plus or minus an element of the standard basis. Any element of the kernel of \( \phi \) is a linear combination of the elements \( e_i - ie_1 \) for \( i \in \mathbb{Z}_{\geq 1} \). These elements have norm \( i + 1 \) and are independent. Therefore, if we take \( \{ f_i \mid i \geq 2 \} \) as a basis for \( \bigoplus_{Z_{\geq 1}}^{\leq 1}(Z)_{i+1} \) we can define an isometry to \( \ker(\phi) \) just by \( f_i \mapsto e_i - i e_1 \).

(2) There are isomorphisms of normed rings

\[
\mathbb{Z}[x]/(2x - 1) \cong \mathbb{Z}_2[\frac{1}{2}]
\]

where on the left hand side, \( |x| = \frac{1}{2} \) and the norm of \( a_0 + a_1 x + \cdots + a_n x^n \) is \( |a_0| + \frac{1}{2}|a_1| + \cdots + \frac{1}{2^n}|a_n| \). Passing to completions, the kernel of the evaluation map \( \mathbb{Z}\{2x\} \to \mathbb{R} \) is clearly maximal and contains the maximal ideal \( (2x - 1) \) so these must agree. Finally, we apply the right exact completion functor to get the desired isomorphisms. The flatness of \( \mathbb{R} \) follows from the fact that for any Banach abelian group \( M \), the map given by multiplication by \( 2x - 1 \) from \( M\{2x\} \) to itself is a strict monomorphism.

(3) For every prime \( p \) there are isomorphisms of normed rings

\[
\mathbb{Z}[x]/(x - p) \cong \mathbb{Z}_p
\]

where \( |x| = \frac{1}{p} \) and the right hand side has the \( p \)-adic norm. In order to explain this, given a polynomial \( \sum_{i=n}^{m} a_i x^i \) with \( a_n \neq 0 \), it is assigned to a number \( \sum_{i=n}^{m} a_i p^i \) with \( p \)-adic norm bounded by \( \max_{i=n}^{m} (p^{-v(a_i) - 1}) \leq p^{-n} \leq \sum_{i=n}^{m} |a_i| p^{-i} = \| \sum_{i=n}^{m} a_i x^i \|_{\mathbb{Z}(px)} \). This gives a (bounded) morphism \( \mathbb{Z}[x] \to \mathbb{Z} \). It is in fact a strict epimorphism because any integer \( b p^s \) where \( p \) does not divide \( b \) and \( s \geq 0 \) has a \( p \)-adic expansion \( \sum_{i=0}^{m} b_i p^{s+i} \) with \( 0 \leq |b_i| \leq p-1 \), in other words it is the evaluation of \( \sum_{i=0}^{m} a_i x^i \). Therefore, the infimum of the norms of any lift of \( b p^s \) to \( \mathbb{Z}\{px\} \) is bounded by \( \sum_{i=0}^{m} |b_i| p^{-s-i} \leq p^{-s} (p-1) \sum_{i=0}^{\infty} p^{-i} = p^{-1} - s = p \|b p^s\|_{p} \). By computing order by order modulo powers of \( p \) with any polynomial \( \sum_{i=n}^{m} a_i x^i \) where \( \sum_{i=n}^{m} a_i p^i = 0 \), one finds that this element must be in the ideal \( (x - p) \). We apply the completion functor to get the desired isomorphism \( \mathbb{Z}\{px\}/(x - p) \cong \mathbb{Z}_p \).

(4) Consider the morphism \( \mathbb{Z}_p[y] \to \mathbb{Q}_p \) where \( |y| = p \) given by evaluation at \( p^{-1} \). It is bounded because \( |\sum_{i=0}^{n} a_i p^{-i}|_p \leq \max_{i=0}^{n} p^{-v(a_i)} \leq \sum_{i=0}^{n} p^{-v(a)} p^i = \sum_{i=0}^{n} |a_i| p^i = \| \sum_{i=0}^{n} a_i y^i \|_{\mathbb{Z}_p} \). Given \( a y^s \in \mathbb{Z}_p[p^{-1}] \) where \( s \geq 0 \) it lifts to \( a y^s \) which has norm \( p^{s-v(a)} = \|a y^s\|_{\mathbb{Q}_p} \). So \( \mathbb{Z}_p[y]/p \to \mathbb{Q}_p \) is a strict epimorphism with kernel \( p y - 1 \). Similarly, it restricts to a bounded map \( \mathbb{Z}_p[\mathbb{Z}_p]/p \to \mathbb{Q}_p \) which is an isomorphism onto the image \( \mathbb{Z}[p^{-1}] \). Now for every prime \( p \) there are isomorphisms of normed rings

\[
\mathbb{Z}[x,y]/(x - p, py - 1) \cong \mathbb{Z}_p[\frac{1}{p}]
\]

where \( |x| = \frac{1}{p} \) and \( |y| = p \) where the right hand side has the \( p \)-adic norm. We apply the completion functor to get the desired isomorphism.
(5) It is obvious that \( \mathbb{Z}_{\text{triv}} \otimes_{\mathbb{Z}} \mathbb{Z}_{\text{triv}} \cong \mathbb{Z}_{\text{triv}} \) because the norm of an element \( a_1 \otimes b_1 + \cdots + a_n \otimes b_n \) of length \( n \) equates to its minimum length which is 1 because it can be written as \((a_1 b_1 + \cdots + a_n b_n) \otimes 1\).

After applying the functor \((-) \otimes_{\mathbb{Z}} \mathbb{Z}_{\text{triv}}\) we to (3.5) get

\[
0 \to \bigoplus_{i \in \mathbb{Z}_{\geq 2}} \mathbb{Z}_{\text{triv}}^{\leq 1}(Z_{\text{triv}})_{i+1} \to \bigoplus_{i \in \mathbb{Z}_{\geq 1}} \mathbb{Z}_{\text{triv}}^{\leq 1} \to \mathbb{Z}_{\text{triv}} \to 0
\]

The basis elements \( f_i \) of \( \bigoplus_{i \in \mathbb{Z}_{\geq 2}} \mathbb{Z}_{\text{triv}}^{\leq 1}(Z_{\text{triv}})_{i+1} \) still have norm \( i + 1 \) but the elements that they now map to \( e_i + ie_1 \) now have norm 2. Therefore the map \( \bigoplus_{i \in \mathbb{Z}_{\geq 2}} \mathbb{Z}_{\text{triv}}^{\leq 1}(Z_{\text{triv}})_{i+1} \to \bigoplus_{i \in \mathbb{Z}_{\geq 1}} \mathbb{Z}_{\text{triv}}^{\leq 1} \) is not a strict monomorphism. In conclusion, \( Z \to \mathbb{Z}_{\text{triv}} \) is not a homotopy epimorphism. Similarly, neither is \( Z \to \mathbb{Q}_{p,\text{triv}} \). Now notice that \( \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p \) as \( Z \to \mathbb{Z}(p) \) is an algebraic localization which is bounded and hence a bounded epimorphism of rings when \( \mathbb{Z}(p) \) is equipped with the \( p \)-adic norm. Indeed \( \mathbb{Z}(p) \otimes_{\mathbb{Z}} \mathbb{Z}(p) \cong \mathbb{Z}(p) \) where the tensor of the \( p \)-norms on \( \mathbb{Z}(p) \) agrees with the standard \( p \)-norm on \( \mathbb{Z}(p) \). Using the projective resolution discussed above we see that \( \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p \) is represented by

\[
0 \to \mathbb{Z}_p \{px\} \xrightarrow{x-p} \mathbb{Z}_p \{px\}.
\]

Furthermore, the \( x-p \) map remains a strict monomorphism with cokernel \( \mathbb{Z}_p \). This argument clearly can be extended to the needed statement about \( \mathbb{Q}_p \) using the strict exact sequence

\[
0 \to \mathbb{Q}_p \{px\} \xrightarrow{y_p} \mathbb{Q}_p \{px\} \xrightarrow{y_p} \mathbb{Q}_p \{px\} \to 0.
\]

Alternatively, \( \mathbb{Z}_p \to \mathbb{Q}_p \) is a completed localization and hence a flat epimorphism (see Remark 3.49) and hence a homotopy epimorphism. The argument for \( \mathbb{R} \) is similar as \( Z \to \mathbb{Q} \) is an algebraic epimorphism of rings and bounded when \( \mathbb{Q} \) is equipped with the standard norm. Therefore, \( \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R} \) is the completion of \( \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \) with the tensor norm which agrees with the standard one and so \( \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R} \). As far as the derived functor, the statement follows from the strict short exact sequence

\[
0 \to \mathbb{R}\{2x\} \xrightarrow{2x-1} \mathbb{R}\{2x\} \to \mathbb{R} \to 0
\]

given by the completed tensoring of our resolution of \( \mathbb{R} \) with \( \mathbb{R} \).

\[\square\]

**Remark 3.49.** This geometric perspective can be useful, for instance, one could define the \( p \)-adic completion of a Banach ring \( R \) as \( R\{px\}/(x-p) \). There are also interesting new rings to define such as the following Fréchet version of the \( p \)-adic integers. For instance we can consider the functions on the open disk of radius \( 1/p \) over \( \text{spec}(\mathbb{Z}) \), \( \lim_{r \to 1/p} \mathbb{Z}\{x/r\} \) in place of \( \mathbb{Z}\{px\} \) in the role it plays in Lemma 3.48 (3) to get

\[
\overline{\mathbb{Z}}_p = \lim_{r \to 1/p} (\mathbb{Z}\{x/r\}/(x-p)) \cong (\lim_{r \to 1/p} \mathbb{Z}\{x/r\} / (x-p)) \cong \mathcal{O}(D_{x-p}^{-1})/(x-p).
\]

**Definition 3.50.** Consider a ring \( R \), equipped with two different Banach norms \( |.|_1 \) and \( |.|_2 \). We can form another Banach ring \( R_{1,2} \) whose underlying ring is \( R \) and whose norm is defined by \( |r| = \max\{|r|_1,|r|_2\} \). Obviously there are canonical bounded maps \( R_{1,1} \to R_{1,2} \) and \( R_{1,1} \to R_{1,2} \). We will denote for instance, \( \mathbb{Q}_{\text{triv},\infty} \) for the abelian group \( \mathbb{Q} \) equipped with the norm \( |x|_{0,\infty} = \max\{1,|x|\} \)
for $x \neq 0$. We will use $\mathbb{Q}_{p,\infty}$ for the separated completion of the abelian group $\mathbb{Q}$ equipped with the norm $|x|_{p,\infty} = \max\{|x|_p, |x|_\infty\}$ for $x \neq 0$.

Lemma 3.51. The evaluation at $p^{-1}$ map $\mathbb{Z}[y] \to \mathbb{Q}$ induces bounded inclusions $\mathbb{Z}_{p}^{1} = \mathbb{Z}\langle \frac{y}{p} \rangle / (py - 1) \to \mathbb{Q}_{p,\infty}$ and $\mathbb{Z}_{p}^{1} = \mathbb{Z}\langle \frac{y}{p} \rangle / (py - 1) \to \mathbb{Q}_{\operatorname{triv},\infty}$.

Proof. For $p$ not dividing the integer $a$ and $n \geq 0$ suppose that $\sum_{i=s}^{t} a_{i}p^{-i} = ap^{-n}$ with $a_{s}$ and $a_{t}$ non-zero we get $a = \sum_{i=s}^{t} a_{i}p^{-i}$ and so $t = n$ we have $|ap^{-n}|_{\mathbb{Q},\infty} = \max\{p^{n}, |ap^{-n}|\} = \max\{p^{n}, |\sum_{i=s}^{t} a_{i}p^{-i}|\} \leq \max\{p^{n}, |\sum_{i=s}^{t} a_{i}p^{-i}|\} \leq \sum_{i=s}^{t} |a_{i}|p^{i} = \sum_{i=s}^{t} |a_{i}y^{i}|_{\mathbb{Z}(\frac{y}{p})}$ showing that the evaluation map is bounded. The case for $\mathbb{Q}_{\operatorname{triv},\infty}$ is similar. \hfill \square

Lemma 3.52. We have $\mathbb{Z}_{p}^{1} \otimes_{\mathbb{Z}} \mathbb{Q}_{\operatorname{triv},\infty} \cong \mathbb{Q}_{\operatorname{triv},\infty}$.

Proof. The map

$$\mathbb{Q}_{\operatorname{triv},\infty} \begin{pmatrix} \frac{y}{p} \end{pmatrix} \to \mathbb{Q}_{\operatorname{triv},\infty} \begin{pmatrix} \frac{y}{p} \end{pmatrix}$$

is an isomorphism with two sided inverse given by multiplication by $-\sum_{i=0}^{\infty} (py)^{i}$. \hfill \square

Lemma 3.53. The morphism $\mathbb{Z} \to \mathbb{Q}_{\operatorname{triv},\infty}$ is a homotopy epimorphism. The morphism $\mathbb{Q}_{\operatorname{triv},\infty} \to \mathbb{R}$ is a homotopy epimorphism. The morphism $\mathbb{Z} \to \mathbb{Q}_{p,\infty}$ is a homotopy epimorphism. The morphisms $\mathbb{Q}_{p,\infty} \to \mathbb{Q}_{p}$ and $\mathbb{Q}_{p,\infty} \to \mathbb{R}$ are homotopy epimorphisms.

Proof. Consider the tensor semi-norm on $\mathbb{Q}_{\operatorname{triv},\infty} \otimes_{\mathbb{Z}} \mathbb{Q}_{\operatorname{triv},\infty}$. We have for $x = a_{1}b_{1} + \cdots + a_{k}b_{k}$, $|a_{1}||b_{1}| + \cdots + |a_{k}||b_{k}| \geq |a_{1}b_{1} + \cdots + a_{k}b_{k}| = |x \otimes 1|$ so the product map is an isometry. Therefore, $\mathbb{Q}_{\operatorname{triv},\infty} \otimes_{\mathbb{Z}} \mathbb{Q}_{\operatorname{triv},\infty} \cong \mathbb{Q}_{\operatorname{triv},\infty}$. Consider the resolutions

Equation 3.54.

$$0 \to \prod_{r \in \mathbb{Q}_{>0}, r \neq 1} \leq 1(\mathbb{Q}_{\operatorname{triv},\infty})_{|u|_{r} + |v|_{r}|_{\operatorname{triv},\infty}} \to \prod_{r \in \mathbb{Q}_{>0}} \leq 1(\mathbb{Q}_{\operatorname{triv},\infty})_{|r|_{\operatorname{triv},\infty}} \to \mathbb{Q}_{\operatorname{triv},\infty} \to 0$$

and the projective resolution

Equation 3.55.

$$0 \to \prod_{r \in \mathbb{Q}_{>0}, r \neq 1} \leq 1\mathbb{Z}_{|u|_{r} + |v|_{r}|_{\operatorname{triv},\infty}} \to \prod_{r \in \mathbb{Q}_{>0}} \leq 1\mathbb{Z}_{|r|_{\operatorname{triv},\infty}} \to \mathbb{Q}_{\operatorname{triv},\infty} \to 0$$

In these equations, the right morphism is a strict epimorphism given by sum weighted by index:

$$(a_{r})_{r \in \mathbb{Q}_{>0}} \to \sum_{r \in \mathbb{Q}_{>0}} ra_{r}.$$
In both cases,
\[
\|\psi(x)\| = \sum_{r \in \mathbb{Q}_{0, r} \setminus 1} |x_r(u_r e_1 - v_r e_r)| = \sum_{r \in \mathbb{Q}_{0, r} \setminus 1} |x_r u_r|_{triv, \infty} + |x_r v_r|_{triv, \infty} + \sum_{r \in \mathbb{Q}_{0, r} \setminus 1} |x_r| \|u_r\|_{triv, \infty} = \|x\|
\]
(3.7)
so \(\psi\) is an isometry and a strict monomorphism. If we apply the functor \(\mathbb{Q}_{triv, \infty} \tilde{\otimes}_Z (-)\) to (3.55) we get (3.54). Therefore \(\mathbb{Q}_{triv, \infty} \tilde{\otimes}_Z \mathbb{Q}_{triv, \infty} \simeq \mathbb{Q}_{triv, \infty}\). As we already showed that \(\mathbb{Z} \to \mathbb{R}\) is a homotopy epimorphism we can conclude that \(\mathbb{Q}_{triv, \infty} \to \mathbb{R}\) is as well.

**Lemma 3.56.** The morphism \(\mathbb{Z} \to \mathbb{Q}_{p, \infty}\) is a homotopy epimorphism. The morphisms \(\mathbb{Q}_{p, \infty} \to \mathbb{Q}_p\) and \(\mathbb{Q}_{p, \infty} \to \mathbb{R}\) are homotopy epimorphisms.

**Proof.** Consider the tensor semi-norm on \(\mathbb{Q}_{p, \infty} \otimes_Z \mathbb{Q}_{p, \infty}\). We have for \(x = a_1 b_1 + \cdots + a_k b_k\), \(|a_1| \cdots + |a_k|\|b_k\| \geq |a_1 b_1 + \cdots + a_k b_k| = |x \otimes 1|\) so the product map is an isometry. Therefore, \(\mathbb{Q}_{p, \infty} \otimes_Z \mathbb{Q}_{p, \infty} \simeq \mathbb{Q}_{p, \infty}\). Consider the resolutions

**Equation 3.57.**
\[
0 \to \prod_{r \in \mathbb{Q}_{0, r} \setminus 1} \mathbb{Q}_{p, \infty}^{\leq 1}|u_r| + |v_r| |r|_{p, \infty} \to \prod_{r \in \mathbb{Q}_{0}} \mathbb{Q}_{p, \infty}^{\leq 1}|r|_{p, \infty} \to \mathbb{Q}_{p, \infty} \to 0
\]

and the projective resolution

**Equation 3.58.**
\[
0 \to \prod_{r \in \mathbb{Q}_{0, r} \setminus 1} \mathbb{Q}_{p, \infty}^{\leq 1}|u_r| + |v_r| |r|_{p, \infty} \to \prod_{r \in \mathbb{Q}_{0}} \mathbb{Q}_{p, \infty}^{\leq 1}|r|_{p, \infty} \to \mathbb{Q}_{p, \infty} \to 0
\]

In these equations, the right morphism is a strict epimorphism given by sum weighted by index:
\[
(a_r)_{r \in \mathbb{Q}_{0}} \to \sum_{r \in \mathbb{Q}_{0}} ra_r.
\]
The kernel is freely generated by the independent elements \(ue_1 - ve_s\) for \(s = \frac{n}{2} \in \mathbb{Q}_{0}\), for \(u, v \in \mathbb{Z}\), \(s \neq 1\). In the first case, these elements have norm \(|u|_{p, \infty} + |v|_{p, \infty}|s|_{p, \infty}\) while in the second, they have norm \(|u|_{\infty} + |v|_{\infty}|s|_{p, \infty}\). The left morphism \(\psi\) is induced by the morphisms defined for any \(s \in \mathbb{Q}_{0}, s \neq 1\):
\[
\psi_s : x_s \mapsto (c_r)_{r \in \mathbb{Q}_{0}}
\]
where where \(c_r = ux_s\) for \(r = 1\) and \(c_r = -vx_s\) for \(r = s\) and \(c_r = 0\) otherwise. Notice that for any \(s \in \mathbb{Q}_{0}, s \neq 1\) we have \(|u|_{\infty} + |v|_{\infty}|s|_{p, \infty} = |u|_{p, \infty} + |v|_{p, \infty}|s|_{p, \infty}\). Thus in the both cases, it is clear that in the morphisms \(\psi_s\) are strict and bounded.

In both cases,
\[
\|\psi(x)\| = \sum_{r \in \mathbb{Q}_{0, r} \setminus 1} |x_r(u_r e_1 - v_r e_r)| = \sum_{r \in \mathbb{Q}_{0, r} \setminus 1} |x_r u_r|_{p, \infty} + |x_r v_r|_{p, \infty} = \sum_{r \in \mathbb{Q}_{0, r} \setminus 1} |x_r| \|u_r\|_{p, \infty} = \|x\|
\]
so \(\psi\) is an isometry and a strict monomorphism. If we apply the functor \(\mathbb{Q}_{p, \infty} \tilde{\otimes}_Z (-)\) to (3.55) we get (3.54). Therefore \(\mathbb{Q}_{p, \infty} \tilde{\otimes}_Z \mathbb{Q}_{p, \infty} \simeq \mathbb{Q}_{p, \infty}\). As we already showed that \(\mathbb{Z} \to \mathbb{Q}_p\) and \(\mathbb{Z} \to \mathbb{R}\) is a homotopy epimorphism we can conclude that \(\mathbb{Q}_{p, \infty} \to \mathbb{R}\) and \(\mathbb{Q}_{p, \infty} \to \mathbb{Q}_p\) is as well.

**Lemma 3.59.** We have
\[
\mathbb{Q}_{triv, \infty} \tilde{\otimes}_Z \mathbb{Q}_p^p \simeq \{1\}.
\]
Proof. For every integer \( n > 0 \) we have
\[
|1 \otimes 1| \leq |p^{-n}|_{\text{triv}} |p^n|_p = |p^n|_p = p^{-n}
\]
and hence \(|1 \otimes 1| = 0\) and so when we pass from \( \mathbb{Q} \otimes_p \mathbb{Z} \) with the tensor semi-norm where \( \mathbb{Z} \) is equipped with the \( p \)-norm to the to completion we have \( \mathbb{Q}_{\text{triv, } \infty} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \{1\} \). As the map \( \mathbb{Q}_{\text{triv, } \infty}(px) \to \mathbb{Q}_{\text{triv, } \infty}(px) \) given by multiplication by \( x - p = -p(1 - \frac{x}{p}) \) is an isomorphism with inverse \( \frac{1}{p} \sum_{i=0}^{\infty} (\frac{x}{p})^i \) which has norm \( \sum_{i=0}^{\infty} \frac{1}{p^i} |\cdot|_{\text{triv, } \infty} = \sum_{i=0}^{\infty} \frac{1}{p^i} = \frac{p}{p-1} \), the fact that the derived projective tensor product is trivial follows from the resolution of \( Z_p \) given in Lemma 3.48. \( \square \)

3.7. Bornological and Ind-Banach Modules. Let \( R \) be a Banach ring as defined in Definition 3.21. A bornological module over \( R \) is a pair consisting of an \( R \)-module \( M \) together with a bornology on the set \( M \) as in Definition 3.36 of [7] such that the structure morphisms for addition and action of \( R \) are bounded. The morphisms of bornological modules are bounded \( R \)-linear maps. The homological properties of bornological spaces over \( \mathbb{C} \) was studied by Houzel [24]. Given an \( \prod \) subsets, each of which has the structure of an object of \( \text{Ban} \) full subcategory \( \text{CBorn} \) of a diagram in \( \text{Ban} \) has enough projectives, it has exact products by Proposition 1.4.5 of [45].

By Proposition 3.14

Lemma 3.60. The category \( \text{CBorn} \) is closed, symmetric monoidal, quasi-abelian, complete and co-complete. It has enough flat projectives.

Lemma 3.61. Direct products in \( \text{CBorn} \) commute with cokernels.

Proof. Suppose we have \( f_i : V_i \to W_i \) with cokernels \( C_i \). Then we have a natural map \( \text{coker}(\prod f_i) \to \prod C_i \). Since \( \text{CBorn} \) has enough projectives, it has exact products by Proposition 1.4.5 of [45]. Therefore \( \prod W_i \to \prod C_i \) is a strict epimorphism as it is the product of strict epimorphisms. The kernel is \( \prod V_i \) so we are done. \( \square \)

By Proposition 3.14

Lemma 3.62. If \( R \) is a Banach ring the categories \( \text{Ind}(\text{Ban}_R) \) (or \( \text{Ind}(\text{Ban}_R) \) for \( R \) non-archimedean) is a closed, symmetric monoidal, complete and co-complete elementary quasi-abelian category.

Definition 3.63. A Fréchet module over \( R \) is an object of \( \text{Ind}(\text{Ban}_R) \) which is a countable limit of a diagram in \( \text{Ban}_R \). We consider Fréchet modules as a full subcategory of \( \text{CBorn} \).

Note that many function spaces in complex analytic geometry carry natural Fréchet structures or more generally locally convex structures. We would like to relate these to category \( \text{Ind}(\text{Ban}_C) \). Let \( \text{Tc} \) be the category of locally convex topological vector spaces over \( \mathbb{C} \) and \( \text{Fr} \) the sub-category of Fréchet spaces. Note that both of these categories are quasi-abelian but they don’t share all of the nice properties of \( \text{Ind}(\text{Ban}_C) \) such as having enough projectives and having a closed symmetric monoidal structure. The following definition is [42] definition 1.1:

Definition 3.64. For any object \( E \) of \( \text{Tc} \), let \( B_E \) be the set of absolutely convex bounded subsets of \( E \). Given \( B \in B_E \), let \( B_E \) be the linear span of \( B \) with its gauge semi-norm \( p_B \). Let
\[
\text{IB} : \text{Tc} \to \text{Ind}(\text{Ban}_C)
\]
be defined as
\[ \text{IB}(E) = \colim E_B \]
where the colimit is taken over the directed system \( B_E \). Given \( f : E \to F \) in \( \mathcal{T}_c \) and \( B \in B_E \), then \( f(B) \in B_F \). Hence we get a natural map \( \colim E_B \to \colim f(B) \). Composing this with the canonical map \( \colim_{B \in B_E} f(B) \to \colim_{B' \in B_F} f(B) \) we get the functoriality of IB.

Note that if \( E \) is a Banach space then \( \text{IB}(E) = E \). The functor IB is in many cases fully faithful (\[42\] proposition 1.5):

**Proposition 3.65.** Let \( E, F \) be objects of \( \mathcal{T}_c \). Assume that \( E \) is bornological and that \( F \) is complete. Then
\[ \text{Hom}_{\mathcal{T}_c}(E, F) = \text{Hom}_{\text{ind (Ban)}}(\text{IB}(E), \text{IB}(F)). \]

We also have that IB has a left adjoint when restricted to the category of complete locally convex vector spaces (\[42\] proposition 1.6).

### 4. Nuclear Modules

#### 4.1. Nuclear Banach Modules.

Let \( C \) be a closed symmetric monoidal category. This subsection will only be applied to the \( C \) being one of the categories \( \text{Ban}_R, \text{Ban}_R^{\leq 1} \) and if \( R \) is non-archimedean, we could still consider those but also in this case we could consider \( \text{Ban}_R^{na}, \text{Ban}_R^{na, \leq 1} \). These definitions and results will not be applied to the Ind-categories we consider, on the other hand, in subsection 4.2 we will separately define nuclear objects of Ind-categories and work with objects in Ind-categories in a way that extends the definitions given in this subsection.

**Definition 4.1.** Let \( V \) and \( W \) be Banach modules. A element of \( \text{Hom}(V, W) \) is called nuclear if it lies in the image of the composition
\[ \text{Hom}(e, V^\vee \otimes W) \to \text{Hom}(e, \text{Hom}(V, W)) = \text{Hom}(V, W). \]
An object is called nuclear if the identity morphism of this object is nuclear.

From Higgs and Rowe [26]

**Lemma 4.2.** If a morphism is nuclear then so is its dual. If a morphism is nuclear then so is any pre or post composition with it. The monoidal product of two nuclear morphisms is nuclear.

**Proof.** The first two statements can be found in Proposition 2.2 of [26]. The statement about the monoidal product can be found in Proposition 2.3 of [26].

**Lemma 4.3.** The following are equivalent [26]:

1. The object \( V \) is nuclear.
2. The natural morphism \( V^\vee \otimes V \to \text{Hom}(V, V) \) is an isomorphism.
3. For every object \( W, W \otimes V^\vee \to \text{Hom}(V, W) \) is an isomorphism.
4. For every object \( W, V \otimes W^\vee \to \text{Hom}(W, V) \) is an isomorphism.

**Lemma 4.4.** [26] If an object \( V \) is nuclear then its dual is also nuclear. Any nuclear object is reflexive.

**Lemma 4.5.** Let \( V \) be a nuclear object of \( C \).
Proof. (1) Since $V$ is nuclear, we can consider the naturally isomorphic functor

$$W \mapsto V^\vee \otimes W$$

By Lemma 3.5, this functor preserves strict epimorphisms so we are done.

(2) Because $V$ is reflexive, we have $W \otimes V \cong W \otimes (V^\vee) \cong \text{Hom}(V^\vee, W)$. Therefore, we can consider the naturally isomorphic functor

$$W \mapsto \text{Hom}(V^\vee, W).$$

This preserves strict monomorphisms by Lemma 3.5.

The following Lemma was shown over $\mathbb{C}$ in [12]. The following is our version over a Banach ring $R$.

Lemma 4.6. Let $V$ and $W$ be two Banach modules over a Banach ring $R$ and let $f : V \to W$ be a nuclear morphism in $\text{Ban}_R$. Then there exists a countable set $S$ and a map $m : S \to R_{>0}$, a nuclear morphism $p : V \to \bigsqcup_{s \in S} R_m(s)$ and a non-expanding morphism $c : \bigsqcup_{s \in S} R_m(s) \to W$ such that $f = c \circ p$.

Proof. Let $P$ be the element of $W \otimes_R V^\vee$ corresponding to $f$. We have a countable set $S$ so that $P$ is a sum $\sum_{s \in S} w_s \otimes \alpha_s$ where $L = \sum_{s \in S} ||w_s|| ||\alpha_s|| < \infty$. Let $m(s) = ||w_s||$. We define

$$p : V \to \bigsqcup_{s \in S} R_m(s)$$

by $p(v) = (\alpha_s(v))_{s \in S}$ where

$$||p(v)|| \leq \sum_{s \in S} m(s)||\alpha_s|| ||v|| = L ||v||.$$

The morphism $p$ is actually nuclear as it can be written as $\sum_{s \in S} \delta_s \otimes \alpha_s$ where $\delta_s$ is the vector with $1 \in R_m(s)$ in position $s$ and 0 elsewhere. Define $c$ by $c((\mu_s)_{s \in S}) = \sum_{s \in S} \mu_s w_s$. We have

$$||c(u)|| \leq \sum_{s \in S} ||w_s|| m(s) m(s)^{-1} \leq \left( \sum_{t \in S} ||\mu_t|| m(t) \right) \left( \sup_{s \in S} ||w_s|| m(s)^{-1} \right) = ||u||$$

which shows that $c$ is non-expanding. For any element $v \in V$ we have $p(v) = (\alpha_s(v))_{s \in S}$ so $c(p(v)) = \sum_{s \in S} \alpha_s(v) w_s = f(v)$. \qed
Lemma 4.7. Let $V$ and $W$ be two Banach modules over a Banach ring $R$ and let $f : V \to W$ be a nuclear morphism in $\text{Ban}_R$. Then there exists a countable set $S$ and a map $m : S \to \mathbb{R}_{\geq 0}$, a non-expanding morphism $c : V \to \prod_{s \in S}^{\leq 1} R_{m(s)}$ and a nuclear morphism $p : \prod_{s \in S}^{\leq 1} R_{m(s)} \to W$ such that $f = p \circ c$.

Proof. Let $P$ be the element of $W \hat{\otimes}_R V^*$ corresponding to $f$. We have a countable set $S$ so that $P$ is a sum $\sum_{s \in S} w_s \hat{\otimes} \alpha_s$ where $L = \sum_{s \in S} ||w_s|| ||\alpha_s|| < \infty$. Let $m(s) = L^{-1} ||w_s||$. We define

$$c : V \to \prod_{s \in S}^{\leq 1} R_{m(s)}$$

by $c(v) = (\alpha_s(v))_{s \in S}$ where

$$||c(v)|| \leq \sup_{s \in S} m(s)||\alpha_s|| ||v|| \leq ||v||.$$

Define $p$ by $p((\mu_s)_{s \in S}) = \sum_{s \in S} \mu_s w_s$. We have

$$||p(u)|| \leq \sum_{s \in S} ||\mu_s|| ||w_s|| m(s)m(s)^{-1} \leq (\sum_{s \in S} ||w_s|| m(s)^{-1})(\sup_{t \in S} ||\mu_t|| m(t)) = (\sum_{s \in S} ||w_s|| m(s)^{-1}) ||u||.$$

The morphism $p$ is actually nuclear as it can be written as $\sum_{s \in S} \delta_s \hat{\otimes} w_s$ where $\delta_s$ is the vector with $1 \in R_{m(s)}$ in position $s$ and $0$ elsewhere. For any element $v \in V$ we have $c(v) = (\alpha_s(v))_{s \in S}$ so $p(c(v)) = \sum_{s \in S} \alpha_s(v) w_s = f(v).$ \hfill \Box

Remark 4.8. If $R$ is non-archimedean, all of this subsection goes through for $\text{Ban}_R^{\text{na}}$ in place of $\text{Ban}_R$.

4.2. Nuclear Ind-Banach Modules. This subsection is about nuclear objects in $\text{Ind}(\text{Ban}_R)$ or if $R$ is non-archimedean, about nuclear objects in $\text{Ind}(\text{Ban}_R^{\text{na}})$. For readability, we suppress the non-archimedean versions, all the statements and proofs in the non-archimedean case are the same, up to the obvious substitutions. Of course in the non-archimedean version, all categorical constructions in $\text{Ban}_R^{\leq 1}$ are replaced by those in $\text{Ban}_R^{\leq 1,\text{na}}$ and $\hat{\otimes}_R$ is replaced by $\hat{\otimes}_R^{\text{na}}$. As the beginning is more general, we work with a closed symmetric monoidal category $C$ with monoidal structure $\hat{\otimes}$ and unit $e$ with finite limits and colimits, but the reader is invited to take $C = \text{Ban}_R$, $\hat{\otimes} = \hat{\otimes}_R$ and $e = R$ or if $R$ is non-archimedean there is also the option $C = \text{Ban}_R^{\text{na}}$, $\hat{\otimes} = \hat{\otimes}_R^{\text{na}}$ and $e = R$.

Definition 4.9. An object $V$ in $\text{Ind}(C)$ is called nuclear if for every object $W$ of $C$ the natural morphism

$$W^\vee \hat{\otimes} V \to \text{Hom}(W,V)$$

is an isomorphism.

Remark 4.10. In general, these objects are definitely not nuclear in the sense of subsection 4.1 applied to the category $\text{Ind}(C)$. If we consider an object $V$ of $\text{Ind}(C)$ which happens to be in $C \subset \text{Ind}(C)$ itself then the definitions agree by Lemma 4.3. Later, we find another situation when the two definitions agree in Lemma 5.8.

Lemma 4.11. If $0 \to V_1 \to V_2 \to V_3 \to 0$ is a strict short exact sequence in $\text{Ind}(C)$ then any two out of three of the $V_i$ are nuclear; the third is as well.

Proof. This follows immediately from the fact that if any two out of three of the $V_i$ are nuclear we can identify the strict sequences

$$W^\vee \hat{\otimes} V_1 \to W^\vee \hat{\otimes} V_2 \to W^\vee \hat{\otimes} V_3 \to 0$$
and

0 \rightarrow \text{Hom}(W, V_1) \rightarrow \text{Hom}(W, V_2) \rightarrow \text{Hom}(W, V_3).

Hence both can be completed to strict short exact sequences. Now because two of the three terms are identified by assumption, the needed isomorphism also holds for the missing $V_i$ and hence it is nuclear.

\textbf{Lemma 4.12.} For any nuclear object $V$ of $\text{Ind}(C)$ and an arbitrary object $W$ of $\text{Ind}(C)$ represented as $V = \text{colim}_{i \in I} V_i$ and $W = \text{colim}_{j \in J} W_j$ any morphism in $\text{Hom}(W, V)$ can be represented in terms of a system of nuclear maps in $C, W_j \rightarrow V_i$.

\textit{Proof.} Notice that

$\text{Hom}(W, V) = \text{Hom}(e, \text{Hom}(W, V)) = \text{Hom}(e, \lim_{j \in J} \text{Hom}(W_j, V)) \cong \text{Hom}(e, \lim_{j \in J} (V \otimes W_j^\vee))$

\begin{equation}
\text{(4.1)}
= \lim_{j \in J} \text{Hom}(e, V \otimes W_j^\vee) = \lim_{j \in J} \text{Hom}(e, (\text{colim}_{i \in I} V_i) \otimes W_j^\vee)
\end{equation}

\begin{equation}
= \lim_{j \in J} \text{Hom}(e, \text{colim}_{i \in I} V_i \otimes W_j^\vee) = \lim_{j \in J} \text{colim}_{i \in I} \text{Hom}(e, V_i \otimes W_j^\vee).
\end{equation}

On the other hand, by definition

$\text{Hom}(W, V) = \lim_{j \in J} \text{colim}_{i \in I} \text{Hom}(W_j, V_i) = \lim_{j \in J} \text{colim}_{i \in I} \text{Hom}(e, \text{Hom}(W_j, V_i)).$

Therefore, the canonical map

$\text{Hom}(W, V) \cong \lim_{j \in J} \text{colim}_{i \in I} \text{Hom}(e, V_i \otimes W_j^\vee) \rightarrow \lim_{j \in J} \text{colim}_{i \in I} \text{Hom}(e, \text{Hom}(W_j, V_i)) = \lim_{j \in J} \text{colim}_{i \in I} \text{Hom}(W_j, V_i)$

is an isomorphism and so any element $\phi$ in $\text{Hom}(W, V)$ and for any $j \in J$ there exists $i \in I$ and a nuclear map $\phi_{i,j} : W_j \rightarrow V_i$ assigned to it. The collection of the $\phi_{i,j}$ define a morphism of inductive systems representing $\phi$. \hfill \Box

\textbf{Lemma 4.13.} Given an filtered inductive system $W_i$ of $C$ where all the maps in the system are nuclear, then the object of $\text{Ind}(C)$ given by $\text{colim}_{i \in I} W_i$ is nuclear in the sense of Definition 4.9.

\textit{Proof.} Let $V$ be any object of $C$. Consider the canonical morphism

$f : (\text{colim}_{i \in I} W_i) \otimes V^\vee \rightarrow \text{Hom}(V, \text{colim}_{i \in I} W_i)$.

Since $V$ is in $C$ (hence a compact object in $\text{Ind}(C)$) we can equivalently write this as the colimit of the morphisms

$W_i \otimes V^\vee \rightarrow \text{Hom}(V, W_i)$.

Consider any $W_i \rightarrow W_j$ in the system corresponding to a non-identity arrow $i \rightarrow j$. Since they are nuclear, the precomposition $V \rightarrow W_i \rightarrow W_j$ is also nuclear by Lemma 4.12. Therefore it lies in the image of $\text{Hom}(e, V \otimes W_j)$. This constructs a two-sided inverse

$(\text{colim}_{i \in I} W_i) \otimes V^\vee \cong (\text{colim}_{i \in I} (W_i \otimes V^\vee)) \leftarrow (\text{colim}_{i \in I} \text{Hom}(V, W_i)) \cong \text{Hom}(V, \text{colim}_{i \in I} W_i)$

to $f$. \hfill \Box

\textbf{Lemma 4.14.} If an object $W$ is nuclear in $\text{Ind}(C)$ and presented as $\text{colim}_{i \in I} W_i$ for $I$ a filtering ordered set with transition maps $\phi_{ij} : W_i \rightarrow W_j$, then for each $i \in I$ there exists a $j \in I$ with $j > i$ such that morphism $\phi_{ij}$ is nuclear.
Proof. Consider Lemma 4.12 in the case $V = W = \text{“colim”}_i W_i$ in the case of identical inductive systems applied to the element $\text{id} \in \text{Hom}(W, W)$. A representative of the identity is given by a cofinal choice of transition maps. The lemma provides the nuclear maps $\phi_{ij}$ representing the identity which are therefore transition maps in the given presentation of $W$. \hfill $\square$

Remark 4.15. Because of Lemmas 4.13 and 4.14 we can conclude that nuclear objects are just those representable by an ind-system with nuclear transition maps.

Lemma 4.16. If $X$ and $Y$ are nuclear in $\text{Ind}(C)$ then so is $X \Box Y$.

Proof. If we consider Remark 4.15 and choose presentations of $X$ and $Y$ with nuclear transition maps, then the induced monoidal structure presentation of $X \Box Y$ also has nuclear transition maps by Lemma 4.2 \hfill $\square$

Lemma 4.17. Given $I$, an infinite filtering ordered set and a functor $I \to \text{Ban}_R$ such that the corresponding object $W = \text{“colim”}_i W_i$ is nuclear in $\text{Ind}(\text{Ban}_R)$, there is a filtered category $K$ with the same cardinality of objects and morphisms as $I$, a functor $K \to \text{Ban}_R$ with corresponding object $P = \text{“colim”}_k P_k$ and an isomorphism $W \cong P$ such that each Banach space $P_k$ is a countable coproduct in $\text{Ban}^\text{op}_R$ of weighted copies of $R$ and the transition functions are nuclear.

Proof. Using Lemma 4.14 we may assume that for each $i$ there is a $j > i$ such that $\phi_{ij} : W_i \to W_j$ is nuclear. Define

$$K = \{(i, j) \in I \times I | j \geq i, \phi_{ij} : W_i \to W_j \text{ is nuclear}\}.$$ 

Using Lemma 4.6 we can decompose each such morphism $\phi_{ij}$ into $c_{ij} \circ p_{ij} : W_i \to P_{ij} \to W_j$ where $P_{ij}$ is a countable non-expanding coproduct of weighted copies of $R$ and $p_{ij}$ is nuclear. Given two pairs $k = (i, j)$ with $j \geq i$ and $k' = (i', j')$ with $j' \geq i'$ of $K$ we define the nuclear (see Lemma 4.2) morphism $n_{kk'} : P_k \to P_{k'}$ by $p_{k'} \circ c_k$ for any pair $k, k'$ such that $j = i'$. As in the proof of Lemma 2.3 of [42] we have a filtering inductive system $(K, \{P_k\}, \{n_{kk'}\})$ which defines an object $P$ of $\text{Ind}(\text{Ban}_R)$ isomorphic to $W$ which has the desired properties. \hfill $\square$

Lemma 4.18. Given $I$, an infinite filtering ordered set and a functor $I \to \text{Ban}_R$ such that the corresponding object $W = \text{“colim”}_i W_i$ is nuclear in $\text{Ind}(\text{Ban}_R)$, there is a filtered category $K$ with the same cardinality of objects and morphisms as $I$, a functor $K \to \text{Ban}_R$ with corresponding object $L = \text{“colim”}_k L_k$ and an isomorphism $W \cong L$ such that each Banach space $L_k$ is a countable product in $\text{Ban}^\text{op}_R$ of weighted copies of $R$ and the transition functions are nuclear.

Proof. Using Lemma 4.14 we may assume that for each $i$ there is a $j > i$ such that $\phi_{ij} : W_i \to W_j$ is nuclear. Define

$$K = \{(i, j) \in I \times I | j \geq i, \phi_{ij} : W_i \to W_j \text{ is nuclear}\}.$$ 

Using Lemma 4.7 we can decompose each such morphism $\phi_{ij}$ into $p_{ij} \circ c_{ij} : W_i \to L_{ij} \to W_j$ where $L_{ij}$ is a countable non-expanding product of weighted copies of $R$ and $p_{ij}$ is nuclear. Given two pairs $k = (i, j)$ with $j \geq i$ and $k' = (i', j')$ with $j' \geq i'$ of $K$ we define the nuclear (see Lemma 4.2) morphism $n_{kk'} : L_k \to L_{k'}$ by $c_{k'} \circ p_k$ for any pair $k, k'$ such that $j = i'$. As in the proof of Lemma 2.3 of [42] we have a filtering inductive system $(K, \{L_k\}, \{n_{kk'}\})$ which defines an object $L$ of $\text{Ind}(\text{Ban}_R)$ isomorphic to $W$ which has the desired properties. \hfill $\square$

Lemma 4.19. Any nuclear object of $\text{Ind}(\text{Ban}_R)$ is flat in $\text{Ind}(\text{Ban}_R)$.
Proof. Using Lemma 4.17 we can write a nuclear object in a certain nice form as the formal filtered colimit of countable coproducts in $\text{Ban}^≤_R$ of weighted copies of $R$. Each weighted copy of $R$ is projective by Lemma 3.40. Therefore their coproduct in $\text{Ban}^≤_R$ is projective by Lemma 3.34 and hence flat by Lemma 3.33. By Lemma 3.34 this colimit of flat objects is flat. □

Remark 4.20. If $R$ is non-archimedean, all of this subsection goes through for $\text{Ban}^na_R$ in place of $\text{Ban}_R$.

5. THE INTERACTION OF PRODUCTS AND TENSOR PRODUCTS

Just as flatness in our context is about commuting kernels and completed tensor products, we need to investigate the interaction of the other type of limit (products) with the completed tensor product. Many of the results in this section are either taken from or inspired by the book [31] by R. Meyer and discussions with him. In this section we define metrizable modules and examine how the tensor product with them interacts with products. We work in as general as possible context and as usual, for simplicity we look at the archimedean setting, even if $R$ is non-archimedean. In that setting, all the proofs in this section go through with the obvious modifications for $\text{Ban}^na_R$ in place of $\text{Ban}_R$.

Definition 5.1. Let $\lambda$ be a cardinal. A poset $J$ is called $\lambda$-filtered if any subset $S$ of $J$ with $|S| < \lambda$ has an upper bound.

Remark 5.2. Let $\lambda$ be a cardinal. A finite product of $\lambda$-filtered posets is $\lambda$-filtered.

Lemma 5.3. Let $\lambda$ be a cardinal. Suppose that $I$ is a poset and we are given a functor $F : I \times J \to \text{Set}$ where $J$ has cardinality less than $\lambda$ and $I$ is $\lambda$-filtered. Then the natural morphism

$$\text{colim}_{i \in I} \lim_{j \in J} F(i, j) \to \lim_{j \in J} \text{colim}_{i \in I} F(i, j)$$

is an isomorphism.

Proof. This is well known in set theory. For example when $\lambda = \aleph_0$, one can consider sets $X_1 \subset X_2 \subset X_3 \subset \ldots$, $Y_1 \subset Y_2 \subset Y_3 \subset \ldots$, and $Z_1 \subset Z_2 \subset Z_3 \subset \ldots$, along with maps $X_i \to Z_i \leftarrow Y_i$ compatible with inclusions and then the claim is that $(\bigcup_i X_i) \times \bigcup_i Z_i = \bigcup_i (X_i \times Z_i)$ as can be shown by hand.

By considering objects in $\text{Ind}(C)$ as functors from $C^{\text{op}}$ to sets, Lemma 5.3 immediately implies the following.

Lemma 5.4. Let $\lambda$ be a cardinal. Suppose that $I$ is a poset and we are given a functor $F : I \times J \to \text{Ind}(C)$ where $J$ has cardinality less than $\lambda$ and $I$ is $\lambda$-filtered. Then the natural morphism

$$\text{colim}_{i \in I} \lim_{j \in J} F(i, j) \to \lim_{j \in J} \text{colim}_{i \in I} F(i, j)$$

is an isomorphism.

Suppose now that $C$ is closed symmetric monoidal category.

Definition 5.5. An object $V$ of $\text{Ind}(C)$ will be called metrizable if the category whose objects consist of objects of $C$ along with morphisms to $V$ and whose morphisms are commuting triangles is $\aleph_1$-filtered.

Lemma 5.6. An object $V$ of $\text{Ind}(C)$ is metrizable if and only if there is an $\aleph_1$-filtered category $I$, a functor $F : I \to C$ and an isomorphism $V \cong \text{colim}_I F$.
Proof. If there exists a functor as in the statement of the Lemma, any morphism \( W \to V \) would factor via \( F(i) \) for some object \( i \in I \).

\[ \text{Corollary 5.7. Let } V = \text{"colim"} \bigoplus_{i \in I} V_i \in \text{Ind}(C) \text{ where } V_i \in C \text{ and } I \text{ has cardinality less than } \lambda. \text{ Let } W = \text{"colim"} \bigoplus_{j \in J} W_j \text{ where } J \text{ is } \lambda\text{-filtered. Then there is an isomorphism } \]

\[ \text{Hom}(V, W) \cong \text{colim}_{j \in J} \text{Hom}(V, W_j). \]

Proof. By Lemma 5.4 we have

\[ \text{Hom}(V, W) \cong \text{colim}_{i \in I} \text{colim}_{j \in J} \text{Hom}(V_i, W_j) \cong \text{colim}_{i \in I} \text{colim}_{j \in J} \text{Hom}(V_i, W_j) \cong \text{colim}_{i \in I} \text{Hom}(V, W_i). \]

\[ \text{Lemma 5.8. Let } V = \text{"colim"} \bigoplus_{i \in I} V_i \in \text{Ind}(C) \text{ where } V_i \in C \text{ and } I \text{ has cardinality less than } \lambda. \text{ Let } W = \text{"colim"} \bigoplus_{j \in J} W_j \text{ where } J \text{ is } \lambda\text{-filtered. Assume that } W \text{ is nuclear in } \text{Ind}(C), \text{ then } \]

\[ \text{Hom}(V, W) \cong V^\text{\bigoplus} W \]

Proof. Consider the morphism

\[ \left( \text{"colim"} \bigoplus_{i \in I} V_i \right)^\text{\bigoplus} W \to \text{Hom}(V, \text{"colim"} \bigoplus_{i \in I} V_i). \]

By corollary 5.7 we can equivalently write this as the colimit of the morphisms

\[ W_i^\text{\bigoplus} W \to \text{Hom}(V, W_i). \]

Consider any \( W_i \to W_j \) in the system corresponding to a non-identity arrow \( i \to j \). Since they are nuclear, the precomposition \( V \to W_i \to W_j \) is also nuclear by Lemma 4.2. Therefore it lies in the image of \( \text{Hom}(e, V^\text{\bigoplus} W_j) \). This constructs an inverse

\[ \left( \text{"colim"} \bigoplus_{i \in I} V_i \right)^\text{\bigoplus} W \leftarrow \text{Hom}(V, \text{"colim"} \bigoplus_{i \in I} V_i). \]

\[ \text{Corollary 5.9. Let } V = \text{"colim"} \bigoplus_{i \in I} V_i \in \text{Ind}(C) \text{ where } V_i \in C \text{ and } I \text{ has cardinality less than } \lambda. \text{ Let } W = \text{"colim"} \bigoplus_{j \in J} W_j \text{ where } J \text{ is } \lambda\text{-filtered. Assume that } W \text{ is nuclear in } \text{Ind}(C). \text{ Then the natural morphism } \]

\[ \left( \text{lim}_{i \in I} (V_i) \right)^\text{\bigoplus} W \to \text{lim}_{i \in I} (V_i^\text{\bigoplus} W) \]

is an isomorphism. Also, if \( \lambda \approx \aleph_1 \) and countable list of objects \( V_1, V_2, V_3, \ldots \) in \( C \) the natural morphism

\[ \left( \prod_{i \in I} (V_i) \right)^\text{\bigoplus} W \to \prod_{i \in I} (V_i^\text{\bigoplus} W) \]

is an isomorphism.

Proof. The left hand side is \( V^\text{\bigoplus} W \) which is isomorphic to

\[ \text{Hom}(V, W) \cong \text{lim}_{i \in I} \text{Hom}(V_i, W) \cong \text{lim}_{i \in I} (V_i^\text{\bigoplus} W). \]

For the second statement let \( I = \mathbb{Z}_{>0} \) and just consider the system

\[ V_1 \to V_1 \oplus V_2 \to V_1 \oplus V_2 \oplus V_3 \to \ldots \]

and apply the statement already proven.
Definition 5.10. Let $\Psi$ be the poset consisting of functions $\psi : I \to \mathbb{Z}_{\geq 1}$ with the order $\psi_1 \leq \psi_2$ if $\psi_1(i) \leq \psi_2(i)$ for all $i \in I$. Let $\Upsilon$ be the poset consisting of functions $\psi : I \to \mathbb{Z}_{\geq 1}$ with the order $\psi_1 < \psi_2$ if $\psi_1(i) < \psi_2(i)$ for all $i \in I - J$ where $J$ is a finite subset of $I$. The category $\Psi = \prod_{i \in I} \mathbb{Z}_{>0}$ can be thought of as the category of maps $I \to \mathbb{Z}_{>0}$.

At this point in the subsection, we need to reduce the generality and take $C = \text{Ban}_R$ for a Banach ring $R$. Of course, as usual, if $R$ is non-archimedean, one can use $C = \text{Ban}^{\text{sa}}_R$ instead with the obvious modifications which we suppress to save space.

Lemma 5.11. Suppose we are given a set $V_i$ in $\text{Ban}_R$ (n.b. not in $\text{Ind}(\text{Ban}_R)$) indexed by a set $I$. Then the natural morphism in $\text{Ind}(\text{Ban}_R)$

$$\text{"colim"}_{\psi \in \Psi} \prod_{i \in I}^{\leq 1} ((V_i)_{\psi(i)-1}) \to \prod_{i \in I} V_i$$

is an isomorphism in $\text{Ind}(\text{Ban}_R)$ where the product on the right is taken in $\text{Ind}(\text{Ban}_R)$, the product on the left is taken in $\text{Ban}_R^{\leq 1}$ and the notation $(V_i)_{\psi(i)-1}$ uses Definition 3.32.

Proof. It is enough to show that the morphisms

$$\text{Hom}(M, \text{"colim"}_{\psi \in \Psi} \prod_{i \in I}^{\leq 1} ((V_i)_{\psi(i)-1})) \to \text{Hom}(M, \prod_{i \in I} V_i)$$

are isomorphisms of abelian groups for any $M \in \text{Ban}_R$. We have

$$\text{Hom}(M, \text{"colim"}_{\psi \in \Psi} \prod_{i \in I}^{\leq 1} ((V_i)_{\psi(i)-1})) \cong \text{colim}_{\psi \in \Psi} \text{Hom}(M, \prod_{i \in I}^{\leq 1} ((V_i)_{\psi(i)-1}))$$

$$\cong \prod_{\psi \in \Psi} \text{colim}_{j \in \mathbb{Z}_{>0}} \text{Hom}(M, \prod_{i \in I}^{\leq 1} ((V_i)_{\psi(i)-1}))$$

$$\cong \prod_{\psi \in \Psi} \text{colim}_{j \in \mathbb{Z}_{>0}} \text{Hom}(M, (\prod_{i \in I}^{\leq 1} ((V_i)_{\psi(i)-1})))_{j-1}$$

$$\cong \prod_{\psi \in \Psi} \text{colim}_{j \in \mathbb{Z}_{>0}} \text{Hom}(M, \prod_{i \in I}^{\leq 1} ((V_i)_{j\psi(i)-1})))$$

$$\cong \prod_{\psi \in \Psi} \text{colim}_{j \in \mathbb{Z}_{>0}} \text{Hom}(M, (V_i)_{\psi(i)-1}))$$

(5.1)

$$\cong \prod_{i \in I} \text{colim}_{j \in \mathbb{Z}_{>0}} \text{Hom}(M, (V_i)_{j-1})$$

$$\cong \prod_{i \in I} \text{colim}_{j \in \mathbb{Z}_{>0}} \text{Hom}(M, (V_i)_{j-1})$$

$$\cong \prod_{i \in I} \text{Hom}(M, V_i)$$

$$= \text{Hom}(M, \prod_{i \in I} V_i)$$

Notice here that in order to pass from colimits over $\Psi$ to colimits over $\mathbb{Z}_{>0}$, in the isomorphism

$$\text{colim}_{\psi \in \Psi} \prod_{i \in I} \text{Hom}^{\leq 1}(M, (V_i)_{\psi(i)-1}) \cong \prod_{i \in I} \text{colim}_{j \in \mathbb{Z}_{>0}} \text{Hom}^{\leq 1}(M, (V_i)_{j-1})$$

we have used that in the category of sets, products and filtered colimits distribute (not commute!). This means that for a set indexed by $I$ of filtered sets $\{S_{i,j}\}_{j \in J}$ we have

$$\prod_{i \in I} \text{colim}_{j \in \mathbb{Z}_{>0}} \text{Hom}^{\leq 1}(M, (V_i)_{j-1}) \cong \prod_{i \in I} \text{colim}_{j \in \mathbb{Z}_{>0}} S_{i,j} \psi(i) \cdot$$
Corollary 5.13. Suppose we are given a system 
\[ \cdots \to V_4 \to V_3 \to V_2 \to V_1 \]
in \text{Ban}_R (n.b. not in \text{Ind}(\text{Ban}_R)) such that all morphisms in the system are in \text{Ban}_R^{\leq 1}. Let \( \Psi \) be the poset consisting of non-decreasing functions \( \psi: \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1} \) with the order \( \psi_1 \leq \psi_2 \) if \( \psi_1(i) \leq \psi_2(i) \) for all \( i \in \mathbb{Z}_{\geq 1} \). Then the natural morphism in \text{Ind}(\text{Ban}_R)
\[ \text{“colim”} \prod_{i \in \mathbb{Z}_{\geq 1}}^{\leq 1} (V_i)_{\psi(i)-1} \to \text{“colim”} \prod_{i \in \mathbb{Z}_{\geq 1}}^{\leq 1} (V_i)_{\psi(i)-1} \]
is an isomorphism in \text{Ind}(\text{Ban}_R). Furthermore, \( \lim_{i \in \mathbb{Z}_{\geq 1}} V_i \) is metrizable and so any Fréchet module is metrizable.

Proof. First notice that
\[ \lim_{i \in \mathbb{Z}_{\geq 1}} V_i = \ker[ \prod_{i \in \mathbb{Z}_{\geq 1}} V_i^{d_s} \to \prod_{i \in \mathbb{Z}_{\geq 1}} V_i]. \]
Using Lemma 5.11 we can write \( \prod_{i \in \mathbb{Z}_2} V_i \xrightarrow{id-s} \prod_{i \in \mathbb{Z}_2} V_i \) as a map
\[
(5.2) \quad \text{“colim”}_{\psi \in \Psi} \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{\psi(i)-1}) \to \text{“colim”}_{\psi \in \Psi} \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{\psi(i)-1})
\]
however, the shift of an element \((v_i)_{i \in \mathbb{Z}_2}\) of \( \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{\psi(i)-1}) \) lands in \( \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{\phi(i)-1}) \) whenever \( \sup_{i \in \mathbb{Z}_2} \|v_i\|_{i-1} \phi(i-1)^{-1} < \infty \). This happens as long as \( \sup_{i \in \mathbb{Z}_2} \|v_i\|_{i-1} \phi(i-1)^{-1} < \infty \) since the maps are non-expanding. For \( \phi = \psi \), \( \psi(i-1)^{-1} \geq \psi(i)^{-1} \) so there is no reason why this should be true. However, if we define \( \phi = s\psi \) by \( (s\psi)(i) = \psi(i+1) \) then we do have the map
\[
s : \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{(s\psi)(i)-1}) \to \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{(s\psi)(i)-1})
\]
induced by the obvious maps \( s_i : (V_i)_{(s\psi)(i)-1} \to (V_{i-1})_{\psi(i)^{-1}} = (V_{i-1})_{(s\psi)(i-1)^{-1}} \). Luckily, there is also the map
\[
id : \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{\psi(i)-1}) \to \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{(s\psi)(i)-1})
\]
induced by \( id_i : (V_i)_{\psi(i)-1} \to (V_i)_{(s\psi)(i)-1} \) since \( s\psi \geq \psi \). Consider the morphisms
\[
(id-s)_j : \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{\psi(i)-1}) \to (V_j)_{(2s\psi)(j)-1}
\]
defined by
\[
(\alpha_1, \alpha_2, \ldots) \mapsto \alpha_j - \alpha_{j+1}
\]
which is non-expanding because of the inequalities
\[
(5.3) \quad \|\alpha_j - \alpha_{j+1}\|_{(V_j)_{(2s\psi)(j)-1}} = 2^{-1}\psi(j+1)^{-1}\|\alpha_j - \alpha_{j+1}\|_{V_j} \leq 2^{-1}\psi(j+1)^{-1}\|\alpha_{j+1}\| + 2^{-1}\psi(j+1)^{-1}\|\alpha_j\|
\]
\[
\leq 2^{-1}\psi(j+1)^{-1}\|\alpha_{j+1}\| + 2^{-1}\psi(j)^{-1}\|\alpha_j\| \leq \sup_{i \in \mathbb{Z}_2} \psi(i)^{-1}\|\alpha_i\|_{V_i}
\]
Therefore, we can rewrite (5.2) as
\[
\text{“colim”}_{\psi \in \Psi} \left( \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{\psi(i)-1}) \to \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{(2s\psi)(i)-1}) \right).
\]
Because the functor “colim” is exact, we are done. As in the proof of Lemma 5.12 we can replace \( \Psi \) with the \( \mathbb{N}_1 \)-filtered category \( \Psi \) and conclude that
\[
\text{“colim”}_{\psi \in \Psi} \ker \left[ \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{\psi(i)-1}) \xrightarrow{id-s} \prod_{i \in \mathbb{Z}_2} \leq 1((V_i)_{2^{-1}\psi(i+1)^{-1}}) \right] \to \lim_{i \in \mathbb{Z}_2} V_i
\]
is an isomorphism in \( \text{Ind}(\text{Ban}_R) \). Therefore, \( \lim_{i \in \mathbb{Z}_2} V_i \) is metrizable. \( \square \)

**Remark 5.14.** It is completely fine to take some or all of the maps \( V_{i+1} \to V_i \) to be the identity. In particular, taking them all to be the identity we see that any Banach module is metrizable.

**Lemma 5.15.** For each \( k \in K \) suppose we are given an inductive system \( I_k \to \text{Ban}_R \) given by the system of Banach modules \( W_i^{(k)} \). Let \( W^{(k)} = \text{“colim”}_{i \in I_k} W_i^{(k)} \). Assume that for \( i_1 < i_2 \) the morphisms \( W_i^{(k)} \to W_i^{(k)} \) are non-expanding. Let \( \Phi \) be the poset whose objects are pairs \( (\phi_1, \phi_2) \) where \( \phi_1 : K \to \coprod_{k \in K} I_k \) such that \( \phi_1(k) \in I_k \) for all \( k \) and \( \phi_2 : K \to \mathbb{N}_{\geq 1} \). This has a partial order
defined by \((\phi_1, \phi_2) \leq (\phi'_1, \phi'_2)\) if and only if \(\phi_1(k) \leq \phi'_1(k)\) and \(\phi_2(k) \leq \phi'_2(k)\) for all \(k \in K\). Then the natural morphism

\[
\text{colim}_{(\phi_1, \phi_2) \in \Phi} \prod_{k \in K} \phi_1(k) \phi_2(k) \to \prod_{k \in K} W(k)
\]

is an isomorphism. To explain the structure maps in the formal filtered colimit, for \((\phi_1, \phi_2) \leq (\phi'_1, \phi'_2)\) the morphism

\[
\prod_{k \in K} \phi_1(k) \phi_2(k) \to \prod_{k \in K} \phi'_1(k) \phi'_2(k)
\]

is the non-expanding product over \(k \in K\) of the obvious morphisms \((W(k)_{\phi_1(k)}) \phi_2(k) \to (W(k)_{\phi'_1(k)}) \phi'_2(k)\). If each \(I_k\) is \(\aleph_1\)-filtered then if \(K\) is countable then \(\prod_{k \in K} W(i)\) is metrizable.

Proof. It is enough to show that the morphisms

\[
\Hom(M, \text{colim}_{(\phi_1, \phi_2) \in \Phi} \prod_{k \in K} \phi_1(k) \phi_2(k)) \to \Hom(M, \prod_{k \in K} \phi_1(k) \phi_2(k))
\]

are isomorphisms of abelian groups for any \(M \in \Ban_R\). We have

\[
\Hom(M, \text{colim}_{(\phi_1, \phi_2) \in \Phi} \prod_{k \in K} \phi_1(k) \phi_2(k)) = \text{colim}_{(\phi_1, \phi_2) \in \Phi} \Hom(M, \prod_{k \in K} \phi_1(k) \phi_2(k))
\]

As in the proof of Lemma 5.11, we have used in the isomorphism

\[
\text{colim}_{(\phi_1, \phi_2) \in \Phi} \prod_{k \in K} \Hom(M, (W(k)_{\phi_1(k)}) \phi_2(k)) \cong \prod_{k \in K} \text{colim}_{j \in \mathbb{Z}_+} \Hom(M, (W_i(k))_{j^{-1}})
\]

that in the category of sets, products and filtered colimits distribute (not commutate!) as explained in [1]. Let us now assume that each \(I_k\) is \(\aleph_1\)-filtered and \(K\) is countable (so we can assume that \(K = \mathbb{N}\)). Let \(\Lambda\) be the set whose objects are pairs \(\lambda = (\phi_1, \phi_2)\) where \(\phi_1 : K \to \prod_{k \in K} I_k\) such that
φ₁(k) ∈ I_k for all k and φ₂ : K → N_{≥1}. This has a partial order defined by (φ₁, φ₂) < (φ₁', φ₂') when φ₁(k) < φ₁'(k) and φ₂(k) < φ₂'(k) for all but a finite number of k ∈ K. Say that we are given a collection λ⁽¹⁾ = (φ₁⁽¹⁾, φ₂⁽¹⁾), λ⁽²⁾ = (φ₁⁽²⁾, φ₂⁽²⁾), · · · ∈ Λ. Define β = (β₁, β₂) : K → ( ∐ k∈K I_k ) × N_{≥1} by choosing for each k an element β₁(k) ∈ I_k such that β₁(k) > φ₁⁽ᵐ⁾(k) for all m and β₂(k) = 1 + ∑ᵢ≤k φ₂⁽ⁱ⁾(k) ∈ N_{≥1}. Then for any fixed m, β₂(k) > λ₂⁽ᵐ⁾(k) for all k ≥ m and β₁(k) > λ₁⁽ᵐ⁾(k) for all m. Therefore β(k) > λ⁽ᵐ⁾(k) for all k ≥ m and so β > λ⁽ᵐ⁾ for all m.

By comparing Lemma 5.11 and Lemma 5.15 we get:

**Corollary 5.16.** Consider the set of functions φ₁ : K → ∐ k∈K I_k such that φ₁(k) ∈ I_k for all k. It has a partial order defined by φ₁ < φ₁' when φ₁(k) < φ₁'(k) for all but a finite number of k ∈ K. Denote this poset by Φ₁. The natural morphism

\[
\text{colim}\limits_{φ₁∈Φ₁} \prod_k W_i(k) → \prod_k \text{"colim"} W_i(k)
\]

is an isomorphism.

**Lemma 5.17.** Let K be a countable set. For each k ∈ K suppose we are given an inductive system I_k → Ban_R given by the system of Banach modules W_i(k). Let W(k) = "colim" W_i(k) ∈ Ind(Ban_R) for each k. Then for any U ∈ Ban_R the natural morphism

\[
U ⊗_R \left( \prod_{k∈K} W(k) \right) → \prod_{k∈K} \left( U ⊗_R W(k) \right)
\]

is an isomorphism. If U is flat over R and K → Ind(Ban_R) is any functor written as k → W(k) then the natural morphism

\[
U ⊗_R \left( \lim\limits_{k∈K} W(k) \right) → \lim\limits_{k∈K} \left( U ⊗_R W(k) \right)
\]

is an isomorphism.

**Proof.** Let P = ∐₁ ≤₁ R_₁ with r_₁ > 0. Notice that

\[
P ⊗_R \left( \prod_{k∈K} W(k) \right) ≅ P ⊗_R \left( \text{"colim"}_{(φ₁, φ₂)∈Φ} \prod_{k∈K} ≤₁ (W(k))_{φ₁(k)φ₂(k)}^{-1} \right) ≅ \text{"colim"}_{(φ₁, φ₂)∈Φ} \prod_{s∈S} ≤₁ \prod_{k∈K} ≤₁ (W(k))_{φ₁(k)φ₂(k)}^{-1}
\]

while we can rewrite \( P ⊗_R W(k) \) as

\[
\prod_{k∈K} \left( P ⊗_R \text{"colim"}_{φ₁(k)} W_i(k) \right) ≅ \prod_{k∈K} \left( \text{"colim"}_{s∈I_k} ≤₁ (W_i(k))_{r_sφ₂(k)}^{-1} \right)
\]

Let \( f_{(φ₁, φ₂)}^{(φ₁', φ₂')} \) denote the morphisms

\[
\prod_{s∈S} ≤₁ \prod_{k∈K} ≤₁ (W(k))_{φ₁(k)φ₂(k)}^{-1} → \prod_{s∈S} ≤₁ \prod_{k∈K} ≤₁ (W(k))_{φ₁(k)φ₂(k)}^{-1}
\]

for \( (φ₁, φ₂) ≤ (φ₁', φ₂') \) and similarly let \( g_{(φ₁, φ₂)}^{(φ₁', φ₂')} \) denote the morphisms

\[
\prod_{k∈K} ≤₁ \prod_{s∈S} ≤₁ (W(k))_{φ₁(k)φ₂(k)}^{-1} → \prod_{k∈K} ≤₁ \prod_{s∈S} ≤₁ (W(k))_{φ₁(k)φ₂(k)}^{-1}
\]
Now clearly for each \((\phi_1, \phi_2)\) we have that \(\bigcup_{s \in S} \prod_{k \in K}^{\leq 1} (W_{\phi_1(k)})_{r,s,\phi_2(k)}^{-1}\) is a Banach submodule of \(\prod_{k \in K}^{\leq 1} \bigcup_{s \in S}^{\leq 1} (W_{\phi_1(k)})_{r,s,\phi_2(k)}^{-1}\), denote the bounded inclusion by
\[
l(\phi_1, \phi_2) : \bigcup_{s \in S}^{\leq 1} \prod_{k \in K}^{\leq 1} (W_{\phi_1(k)})_{r,s,\phi_2(k)}^{-1} \to \prod_{k \in K}^{\leq 1} \bigcup_{s \in S}^{\leq 1} (W_{\phi_1(k)})_{r,s,\phi_2(k)}^{-1}.
\]

Notice that
\[
(5.5) 
\]
\[
g(\phi_1', \phi_2') \circ l(\phi_1, \phi_2) = l(\phi_1', \phi_2') \circ f(\phi_1, \phi_2).
\]

We now want maps in the other direction but this will not work without increasing \((\phi_1, \phi_2)\). Suppose that we are given an element \((w_{k,s})_{k \in K, s \in S} \in \prod_{k \in K}^{\leq 1} \bigcup_{s \in S}^{\leq 1} (W_{\phi_1(k)})_{r,s,\phi_2(k)}^{-1}\). By definition this means that
\[
\sum_{s \in S} ||w_{k,s}||_{\phi_1(k)}^{(k)} r_s \phi_2(k)^{-1} < \infty
\]
and
\[
\sup_{k \in K} \sum_{s \in S} ||w_{k,s}||_{\phi_1(k)}^{(k)} r_s \phi_2(k)^{-1} < \infty.
\]

For each \(k\) this implies that \(\{s \in S | w_{k,s} \neq 0\}\) is countable. Let \(S_c\) be the subset of \(S\) defined by \(S_c = S - \{s \in S | w_{k,s} = 0 \text{ for all } k\}\). Notice that \(S_c\) is countable since it is a countable union of countable subsets: \(S_c = \bigcup_{k \in K} (S - \{s \in S | w_{k,s} = 0\})\). Because of this countability, we can choose a collection of positive real numbers \(p_s\) for \(s \in S_c\) such that \(p = \sum_{s \in S_c} p_s\) is finite. Chose \(\phi_2'\) so that \(\phi_2'(k) = 2^k \phi_2(k)\) for all \(k\). Then for any \(s \in S_c\) there exists a \(k_s \in K\) such that we have
\[
\sup_{k \in K} ||w_{k,s}||_{\phi_1(k)}^{(k)} r_s \phi_2'(k)^{-1} \leq p_s + ||w_{k,s}||_{\phi_1(k)}^{(k)} r_s \phi_2'(k)^{-1}
\]
Now
\[
\sum_{s \in S \cap K} \sup_{k \in K} ||w_{k,s}||_{\phi_1(k)}^{(k)} r_s \phi_2'(k)^{-1} \leq p + \sum_{s \in S} ||w_{k,s}||_{\phi_1(k)}^{(k)} r_s \phi_2'(k)^{-1}
\]
\[
\leq p + \sum_{k \in K} \sum_{s \in S} ||w_{k,s}||_{\phi_1(k)}^{(k)} r_s \phi_2'(k)^{-1}
\]
\[
\leq p + \sum_{k \in K} 2^{-k} \sum_{s \in S} ||w_{k,s}||_{\phi_1(k)}^{(k)} r_s \phi_2'(k)^{-1}
\]
\[
\leq p + \sum_{k \in K} 2^{-k} \left( \sup_{k \in K} \sum_{s \in S} ||w_{k,s}||_{\phi_1(k)}^{(k)} r_s \phi_2'(k)^{-1}\right)
\]
and therefore \(\sum_{s \in S \cap K} ||w_{k,s}||_{\phi_1(k)}^{(k)} r_s \phi_2'(k)^{-1}\) is finite. This gives whenever \(\phi_2'(k) = 2^k \phi_2(k)\) for all \(k\) bounded morphisms
\[
\pi(\phi_1', \phi_2') : \prod_{k \in K}^{\leq 1} \bigcup_{s \in S}^{\leq 1} (W_{\phi_1(k)})_{r,s,\phi_2'(k)}^{-1} \to \prod_{k \in K}^{\leq 1} \bigcup_{s \in S}^{\leq 1} (W_{\phi_1(k)})_{r,s,\phi_2'(k)}^{-1}.
\]

Clearly we also have
\[
\pi(\phi_1', \phi_2') \circ g(\phi_1, \phi_2) = f(\phi_1, \phi_2') \circ \pi(\alpha_1, \alpha_2')
\]
Finally, notice also that
\[
l(\phi_1, \phi_2') \circ \pi(\phi_1', \phi_2') = g(\phi_1, \phi_2')
\]
and

\[ \eta(\phi_1, \phi_2) \circ \lambda(\phi_1, \phi_2) = f(\phi_1, \phi_2). \]

These three identities combined with Equation (5.5) imply that

\[ \text{colim}_{(\phi_1, \phi_2) \in \Phi} \prod_{k \in K}^{\leq 1} \prod_{s \in S}^{\leq 1} (W_{\phi_1(k)}^k)_{r_s \phi_2(k)}^{-1} \cong \text{colim}_{(\phi_1, \phi_2) \in \Phi} \prod_{k \in K}^{\leq 1} \prod_{s \in S}^{\leq 1} (W_{\phi_1(k)}^k)_{r_s \phi_2(k)}^{-1} \]

and therefore

\[ P\wedge_R \left( \prod_{k \in K} W^k \right) \cong \prod_{k \in K} \left( P\wedge_R W^k \right). \]

Now let \( U \in \text{Ban}_R \) be arbitrary. Similarly to Lemma A.39 of \cite{[citation]} we can find a projective resolution

\[ K \to P \to U \to 0 \]

where all morphisms are strict epimorphisms and \( K = \bigcup_{t \in T}^{\leq 1} R_t \), while \( P = \bigcup_{s \in S}^{\leq 1} R_s \). The fact that products are right exact immediately implies that \( U\wedge_R \left( \prod_{k \in K} W^k \right) \cong \prod_{k \in K} \left( U\wedge_R W^k \right) \).

\[ \square \]

**Lemma 5.18.** Let \( K \) be a countable set, For each \( k \in K \) suppose we are given an inductive system \( I_k \to \text{Ban}_R \) given by the system of Banach modules \( W_i^k \). Let \( W^k = \text{colim}_{i \in I_k} W_i^k \). Let \( V \in \text{Ind}(\text{Ban}_R) \) be metrizable. Then the natural morphism

\[ V\wedge_R \left( \prod_{k \in K} W^k \right) \to \prod_{k \in K} \left( V\wedge_R W^k \right) \]

is an isomorphism. Now let \( K \) be a category with a countable set of objects and morphisms. If \( V \) is metrizable and flat over \( R \) (or metrizable and nuclear) and \( K \to \text{Ind}(\text{Ban}_R) \) is any functor then the natural morphism

\[ V\wedge_R \left( \text{lim}_{k \in K} W^k \right) \to \text{lim}_{k \in K} \left( V\wedge_R W^k \right) \]

is an isomorphism.

**Proof.** Let \( V = \text{colim}_{j \in J} V_j \) where \( J \) is \( \aleph_1 \)-filtered. Using Lemma 5.4 we have

\[ \prod_{k \in K} \left( V\wedge_R W^k \right) \cong \text{colim}_{j \in J} \prod_{k \in K} \left( V_j\wedge_R W^k \right) \cong \text{colim}_{j \in J} \prod_{k \in K} \left( V_j\wedge_R W^k \right) \]

because colimits over an \( \aleph_1 \)-filtered category commute with countable products by Lemma 5.4. Also

\[ V\wedge_R \left( \prod_{k \in K} W^k \right) \cong \text{colim}_{j \in J} \left( V_j\wedge_R \left( \prod_{k \in K} W^k \right) \right) \cong \text{colim}_{j \in J} \prod_{k \in K} \left( V_j\wedge_R W^k \right) \]

by Lemma 5.17. \( \square \)

**Lemma 5.19.** The converse to Lemma 5.18 holds in the sense that we can conclude that an object \( V \in \text{Ind}(\text{Ban}_R) \) is metrizable if an only the functor \( V\wedge_R (-) \) commutes with countable products.

**Proof.** We will prove that if for some \( V \in \text{Ind}(\text{Ban}_R) \), that the natural morphism

\[ V\wedge_R \left( \sum_{z} R \right) \to \sum_{z} V \]
is an isomorphism, then \( V \) is metrizable. Consider the category \( J \) of all objects of \( \text{Ban}_R \) mapping to \( V \). Then of course \( V \cong \text{colim}_{j \in J} V_j \). The above isomorphism combined with Lemma 5.17, which tells us
\[
V_j \otimes_R (\prod_{Z} R) \cong \prod_{Z} V_j
\]

immediately implies that the natural morphism
\[
\text{colim}_{j \in J} (\prod_{Z} V_j) \to \prod_{Z} \text{colim}_{j \in J} V_j
\]
is an isomorphism. Suppose we are given a chain \( V_{j_1} \to V_{j_2} \to V_{j_3} \to V_{j_4} \to \cdots \) in \( J \). We can lift the natural morphism \( \prod_{k \in Z} V_{j_k} \to \prod_{k \in Z} \text{colim}_{j \in J} V_j \) to a morphism \( \prod_{k \in Z} V_{j_k} \to \text{colim} \prod_{j \in J} V_j \). Therefore there exists some \( j \in J \) such that all morphisms \( V_{j_k} \to V \) factor through some \( V_j \to V \). Therefore \( J \) is \( \aleph_1 \)-filtered and so \( V \) is metrizable.

\[\Box\]

**Remark 5.20.** This result is surprising since the analogous result in the purely algebraic case is not true. In fact, in the algebraic case the tensor product of a module will commute with all products of other modules if and only if the first module is finitely presented \([30]\). However, there is no contradiction here because if we take a ring, endow it with the discrete Banach structure, then we can consider the category of discrete modules over the ring but this category is not closed under the operation \( \otimes_R \).

**Lemma 5.21.** Say we fix \( A \in \text{Comm}(\text{Ind}(\text{Ban}_R)) \). Let \( K \) be a countable set. For each \( k \in K \) suppose we are given an inductive system \( I_k \to \text{Ban}_R \) given by the system of Banach modules \( W_i^{(k)} \). Suppose we are given objects \( W^{(k)} \in \text{Mod}(A) \) with underlying object \( \text{colim}_{i \in I_k} W_i^{(k)} \in \text{Ind}(\text{Ban}_R) \). Let \( V \in \text{Mod}(A) \). Suppose the objects underlying \( A \) and \( V \) in \( \text{Ind}(\text{Ban}_R) \) are metrizable. Then the natural morphism
\[
V \otimes_A \left( \prod_{k \in K} W^{(k)} \right) \to \prod_{k \in K} \left( V \otimes_A W^{(k)} \right)
\]
is an isomorphism.

**Proof.** This follows from the case of \( A = R \) which was proven in Lemma 5.18 and the description of \( \otimes_A \) as a coequalizer in Equation 3.3 together with the fact that \( \otimes_R \) is right exact in each variable as discussed in subsection 3.3.

\[\Box\]

**Corollary 5.22.** As a corollary of Remark 5.14 (or of Lemma 5.17) and Lemma 5.19, we see that if we have a countable collection \( V_i \) of nuclear objects of \( \text{Ind}(\text{Ban}_R) \), their product is nuclear. A countable coproduct of any collection of nuclear objects is also nuclear.

**Proof.** Suppose that we have a countable collection of nuclear objects \( V_i \) indexed by a countable set \( I \). For any Banach module \( W \) the map
\[
(\prod_{i \in I} V_i) \otimes_R W^\vee \to \text{Hom}(W, \prod_{i \in I} V_i)
\]
breaks up as a product of maps \( V_i \otimes_R W^\vee \to \text{Hom}(W, V_i) \). Write the coproduct of nuclear objects \( V_i \) over a countable set \( I \) as a filtered colimit of coproducts over finite subsets. The finite coproducts of \( V_i \) are clearly nuclear then both sides of the needed equation \( V \otimes_R W^\vee = \text{Hom}(W, V) \) filtered colimits of true equations.

\[\Box\]
6. Spaces of Functions

In this section, we use the previous results to study rings of analytic functions and their modules on Stein spaces over Banach rings \( R \). As usual, “affine” spaces are considered as the opposite category of commutative, associative, unital ring objects over \( \text{Ind(} \text{Ban}_R \text{)} \). As these form a huge category, one often wants to do geometry with a more manageable class of objects. Although we will not be precise here, one could define Stein algebras as limits of a sequence \( \cdots \to A_3 \to A_2 \to A_1 \) where the \( A_i \) are quotients of Banach disk algebras by finitely generated ideals which are flat over \( R \), and the morphisms are ring homomorphisms over \( R \) in \( \text{Ind(} \text{Ban}_R \text{)} \) which are injective, homotopy epimorphisms, nuclear, non-expanding, and dense. Similarly, dagger algebras over \( R \) are defined as colimits of systems made up of the same type of morphisms \( A_1 \to A_2 \to A_3 \to \cdots \). These limits and colimits take place in \( \text{Ind(} \text{Ban}_R \text{)} \). In this section we focus on limits of colimits of canonical maps of disk algebras, without quotienting by any ideals. It may be interesting to extend these results to more general contexts to develop a complete theory.

**Definition 6.1.** The analytic functions on \( n \)-dimensional affine space over \( R \) are defined by

\[
\mathcal{O}(\mathbb{A}^n_R) = \lim_{r \in \mathbb{Z}_{>0}} R\{\frac{x_1}{r}, \ldots, \frac{x_n}{r}\}.
\]

Similarly, given an \( n \)-tuple of positive real numbers \( r = (r_1, \ldots, r_n) \) the \( n \)-dimensional open disk with multi-radius \( r \) is defined by

\[
\mathcal{O}(D^{n}_{<r,R}) = \lim_{\rho \leq r} R\{\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n}\}.
\]

The ring \( \mathcal{O}(\mathbb{A}^n_R) \) is actually a bornological (in fact Fréchet) ring over \( R \) isomorphic to

\[
\{ \sum_{I \subseteq \mathbb{Z}_{\geq 0}} a_I x^I \in R[[x_1, \ldots, x_n]] \mid \text{for each } r \in \mathbb{Z}_{>0}, \sum_{I \subseteq \mathbb{Z}_{\geq 0}} |a_I| r^I < \infty \}.
\]

The bornology is induced by the family of semi-norms \( ||f||_r = \sum_{I \subseteq \mathbb{Z}_{\geq 0}} |a_I| r^I \) in the sense that a subset is bounded if it is simultaneously bounded for all the metrics induced by this collection of semi-norms. It is easy to see that the limit of algebras gives the standard algebra of global analytic functions on Stein spaces over Banach rings \( R \). These limits and colimits take place in \( \text{Ind(} \text{Ban}_R \text{)} \). In the case (with the standard non-archimedean adaptations of using the Tate algebras instead of the \( \ell^1 \) disk algebras). The ring \( \mathcal{O}(D^{n}_{<r,R}) \) thought of as functions on the open disk of multi-radius \( r \) is isomorphic to

\[
\{ \sum_{I \subseteq \mathbb{Z}_{\geq 0}} a_I x^I \in R[[x_1, \ldots, x_n]] \mid \text{for each } \rho \leq r, \sum_{I \subseteq \mathbb{Z}_{\geq 0}} |a_I| \rho^I < \infty \}.
\]

Here, the bornology is induced by the family of seminorms \( ||f||_m = \sum_{I \subseteq \mathbb{Z}_{\geq 0}} |a_I| r^I (m) \) where \( r(m) = (r_1 - \frac{1}{m}, \ldots, r_n - \frac{1}{m}) \).

**Observation 6.2.** Recall the definition of the poset \( \Upsilon = \prod_{i \in I} \mathbb{Z}_{\geq 0} \) from Definition 5.10. We can give an explicit description of the algebras of functions on the affine line or on an open disk. For example, in the one dimensional cases Lemma 5.13 applied in the case \( V_n = R\{x^n\} \subset R[[x]] \) we have

\[
\mathcal{O}(\mathbb{A}^1_R) = \text{“colim”} \{ f = \sum_{i=0}^{\infty} a_i x^i \in R[[x]] \mid \sup_{n \in \mathbb{N}} \left( \psi(n)^{-1} \sum_{i=0}^{\infty} |a_i| n^i \right) < \infty \}
\]
Lemma 6.3. For each $\psi \in \mathcal{Y}$, every finite subset of the non-singular elements $f ∈ \mathcal{Y}$ such that $\sum_{i=0}^{\infty} a_i f^i ∈ R[[x]]$ are the elements in $V_n$ such that $\lim_{n→\infty} \sum_{i=0}^{\infty} a_i f^i ∈ R[[x]]$. In fact, the same description can be done for any type of categorical limit $\lim_{i→\infty} V_i$ of Banach modules $V_i ⊂ V_{i+1}$ as submodules of a given algebraic module where $V_n$ are the elements $f$ in $V_1$ with $\|f\|_{V_n} < ∞$:

$$\lim_{\psi ∈ \mathcal{Y}} \sum_{i=0}^{\infty} a_i f^i ∈ R[[x]] \mid \sup_{n∈\mathbb{N}}(\psi(n)−1∥f∥_{V_n}) < ∞.$$  

Lemma 6.3. For each $ρ < τ$ the restriction map

$$R\left(\frac{x_1}{τ_1}, \ldots, \frac{x_n}{τ_n}\right) → R\left(\frac{x_1}{ρ_1}, \ldots, \frac{x_n}{ρ_n}\right)$$

is nuclear.

Proof. This map clearly decomposes into the completed tensor product of its factors $R\left(\frac{x_i}{τ_i}\right) → R\left(\frac{x_i}{ρ_i}\right)$. By the compatibility of nuclearity with the completed tensor product discussed in Lemma 4.2, it is enough to to the one dimensional case and to show that for $ρ < τ$ that the map $R\left(\frac{x}{τ}\right) → R\left(\frac{x}{ρ}\right)$ is nuclear. For any $τ$ and $ρ$ the Banach module $R\left(\frac{x}{τ}\right)\otimes_R R\left(\frac{x}{ρ}\right)$ can be described as

$$\left(\prod_{j∈\mathbb{Z}_{20}} \leq 1 R_{τ−j}\right) \otimes_R \left(\prod_{i∈\mathbb{Z}_{20}} \leq 1 R_{ρ^i}\right) \approx \prod_{j∈\mathbb{Z}_{20}} \leq 1 R_{ρ^j τ−j}$$

The right hand side consists of elements $(a_{ij})_{i,j∈\mathbb{Z}_{20}}$ such that first of all

$$\sup_{j∈\mathbb{Z}_{20}} |a_{ij}|ρ^j τ−j < ∞$$

for any $i∈\mathbb{Z}_{20}$ and that furthermore,

$$\sum_{i∈\mathbb{Z}_{20}} |a_{ij}|ρ^j τ−j < ∞.$$  

In the case that $a_{ij} = δ_{ij}$ which gives the restriction map, the first condition is vacuous because it just says that $(\frac{x}{τ})^i < ∞$ for any $i∈\mathbb{Z}_{20}$. The second condition gives $\sum_{i∈\mathbb{Z}_{20}} (\frac{x}{τ})^i < ∞$ which holds precisely when $ρ < τ$. □

Let $ψ$ be a non-decreasing sequence $\mathbb{Z}_{20} → \mathbb{Z}_{21}$. Define a Banach ring over $R$ by

$$R\left(\frac{x}{τ}\right)ψ = \{ \sum_{j∈\mathbb{Z}_{20}} a_j x^j ∈ R[[x]] \mid \sum_{j∈\mathbb{Z}_{20}} |a_j| r^j ψ(j) < ∞ \}.$$  

equipped with the norm

$$\| \sum_{j∈\mathbb{Z}_{20}} a_j x^j \| = \sum_{j∈\mathbb{Z}_{20}} |a_j| r^j ψ(j).$$

Consider the morphisms

$$\prod_{i∈\mathbb{Z}_{21}} \leq 1 R\left(\frac{x}{r + i−1}\right)_{2ψ(i+1)} \rightarrow \prod_{i∈\mathbb{Z}_{21}} \leq 1 R\left(\frac{x}{r + i−1}\right)ψ(i) \rightarrow R\left(\frac{x}{r}\right)ψ$$

(6.1)
given by

\[(id - s)(f_1, f_2, f_3, \ldots) = (f_1, f_2 - f_1, f_3 - f_2, \ldots)\]

and where the map \(\sigma\) is defined by summation. Here, \(s(f)_i = f_{i-1}\) for \(i \geq 1\) and \(s(f)_0 = 0\). The map \(s\) is contracting because it comes from the contracting maps \(R\{\frac{x}{r+t-1}\}|_{\psi(i)} \to R\{\frac{x}{r+t}\}|_{\psi(i)}\) and similarly the map \(id\) is contracting because it comes from the contracting maps \(R\{\frac{x}{r+t}\}|_{\psi(i+1)} \to R\{\frac{x}{r+t-1}\}|_{\psi(i)}\) For \(f \in R\{\frac{x}{r+t}\}_2\psi(i+1)\)

\[\| (id - s)(f) \| = \| f \| R\{\frac{x}{r+t-1}\}|_{\psi(i)} + \| f \| R\{\frac{x}{r+t}\}|_{\psi(i+1)} \leq 2\| f \| R\{\frac{x}{r+t-1}\}|_{\psi(i+1)} = \| f \| R\{\frac{x}{r+t}\}_2\psi(i+1)\]

The map \(\sigma\) is non-expanding as it is induced by the obvious non-expanding morphisms

\[R\{\frac{x}{r+t-1}\}|_{\psi(i)} \to R\{\frac{x}{r}\}_i\psi(i)\]

Define \(\delta : R\{\frac{x}{r}\}_i\psi \to \bigsqcup_{i \in \mathbb{Z}_2} \leq 1 R\{\frac{x}{r+t}\}|_{\psi(i)}\) by

\[\delta(\sum_j a_j x^j) = (a_i x^i)_i.\]

The fact that \(\delta\) is bounded follows from the estimate:

\[\sum_{i \in \mathbb{Z}_2} |a_i|(r + i^{-1})^j \psi(i) \leq (\sup_{j \in \mathbb{Z}_2} (1 + \frac{1}{jr})^j) \sum_{i \in \mathbb{Z}_2} |a_i| r^i \psi(i) = e^{(r^{-1})} \sum_{i \in \mathbb{Z}_2} |a_i| r^i \psi(i).\]

Then we have \(\sigma \circ (id - s) = 0\). Indeed the norm of \(\sum_i (id - s)f\) is smaller than or equal to \(e^{(r^{-1})} \| id - s \| \| f_N \|\) where \(f_N\) is the component of \(f\) in \(R\{\frac{x}{r+N}\}_N\psi(N+1)\). Because the terms of \(f\) are summable, \(\| f_N \| \to 0\) as \(N\) goes to infinity and so the norm of \(\sigma \circ (id - s)f\) is zero for any \(f\). Also notice that \(\sigma \circ \delta = id_{R\{\frac{x}{r}\}_\psi}\). We conclude that (6.1) is a strict short exact sequence. By dualizing it and using Lemma 3.41 we get the strict exact sequence

\[(6.2) \quad 0 \to (R\{\frac{x}{r}\}_i\psi)^\vee \to \bigsqcup_{i \in \mathbb{Z}_2} \leq 1 (R\{\frac{x}{r+t}\}|_{\psi(i)}^\vee)^{\psi(i)-1} \to \bigsqcup_{i \in \mathbb{Z}_2} \leq 1 (R\{\frac{x}{r+t}\}|_{\psi(i)}^{\vee})^{2 \psi(i+1)-1}\]

where the second morphism is given by

\[(f_1, f_2, \ldots) \mapsto (f_1 - f_2, f_3 - f_2, \ldots).\]

**Lemma 6.4.** For two sequences \(\phi, \psi\) such that \(\psi(i) > 2^i \phi(i)\) for all \(i\) the natural morphism

\[R\{\frac{x}{r}\}_i\psi \to R\{\frac{x}{r}\}_i\phi\]

is nuclear.

**Proof.** As a module, we can identify \(R\{\frac{x}{r}\}_i\phi = \bigsqcup_{j \in \mathbb{Z}_2} \leq 1 R_{(r+j^{-1})}\phi(j)\) and so

\[(R\{\frac{x}{r}\}_i\psi)^\vee = \bigsqcup_{j \in \mathbb{Z}_2} \leq 1 R_{(r+j^{-1})}\psi(j)^{-1}.\]

We have

\[R\{\frac{x}{r}\}_i\phi \otimes_{R} (R\{\frac{x}{r}\}_i\psi)^\vee = \bigsqcup_{j \in \mathbb{Z}_2} \leq 1 R_{(r+j^{-1})} (\psi(j)^{-1} \phi(j)).\]

An element of this space is just a collection \((a_{j,l})_{j,l}\) such that

\[\sup_{l \in \mathbb{Z}_2} |a_{j,l}| (r + l^{-1})^{-1} \psi(l)^{-1} (r + j^{-1})^i \phi(j) < \infty\]
for each \( j \) and
\[
\sum_{j \in \mathbb{Z}_{\geq 0}} \sup_{l \in \mathbb{Z}_{\geq 0}} |a_{j,l}| (r + t^{-1})^{-1} \psi(l)^{-1} (r + j^{-1})^{-1} \phi(j) < \infty.
\]
The morphism we care about is nuclear if and only if \( a_{j,l} = \delta_{j,l} \) satisfies these conditions. The first condition is obvious and the second reduces to checking that \( \sum_{j \in \mathbb{Z}_{\geq 0}} \psi(j)^{-1} \phi(j) \) is finite, but this finiteness follows from our assumptions.

Corollary 6.5. \( \mathcal{O}(D^{n}_{\mathbb{R}}) \) is nuclear (and hence flat over \( R \)) for any \( r = (r_{1}, \ldots, r_{n}) \) with \( r_{i} \geq 0 \).

Proof. We have by Equation 6.6 and the description of limits from Corollary 5.13 that \( \mathcal{O}(D^{n}_{\mathbb{R}}) \) is isomorphic to
\[
\text{colim}_{\psi \in \Psi} \ker \left[ \prod_{i} \left( R \left\{ \frac{x_{1}}{r_{1} + t^{-1}}, \ldots, \frac{x_{n}}{r_{n} + t^{-1}} \right\} \right)^{\psi(i)-1} \right]
\]
which is isomorphic to \( \text{colim} \left( R \{ x_{1}, \ldots, x_{n} \} \right)^{\psi} \) by Equation 6.2. Lemma 6.4 implies that we can find a final indexing set so that all the morphisms in the system are nuclear. Therefore, by Lemma 4.19, \( \mathcal{O}(D^{n}_{\mathbb{R}}) \) is nuclear and hence flat by Lemma 4.19.

Let \( R \) be a Banach ring. Let \( V \) be finite rank free Banach module over \( R \). Let \( S_{R}(V) \) be the symmetric algebra, a free object in \( \text{Comm}(\text{Ind}(\text{Ban}_{R})) \). Note that as a bornological ring, \( A \) is a polynomial algebra over \( R \) with number of generators equal to the rank of \( V \). Consider the algebra \( S^{\leq 1}_{R}(V) \), a free object in \( \text{Comm}(\text{Ban}^{\leq 1}_{R}) \). This is a Banach ring which can be explicitly described as the subring \( R \{ x_{1}, \ldots, x_{n} \} \) of \( R \{ [x_{1}, \ldots, x_{n}] \} \) where \( n \) is the rank of \( V \) consisting of elements \( \sum a_{I}x^{I} \) such that \( \sum_{I \in \mathbb{Z}_{\geq 0}^{n}} |a_{I}| < \infty \) and equipped with the norm \( \| \sum_{I \in \mathbb{Z}_{\geq 0}^{n}} a_{I}x^{I} \| = \sum_{I \in \mathbb{Z}_{\geq 0}^{n}} |a_{I}| \).

Note that even when \( R \) is a non-archimedean valued field, we are not restricting our attention to non-archimedean modules. The following theorem and proof closely follows [46].

Theorem 6.6. For any Banach ring \( R \), the natural morphism \( S_{R}(V) \rightarrow S^{\leq 1}_{R}(V) \) is a homotopy epimorphism in \( \text{Comm}(\text{Ind}(\text{Ban}_{R})) \). Similarly, if \( R \) is non-archimedean, the natural morphism \( S^{\text{na}}_{R}(V) \rightarrow S^{\text{na},\leq 1}_{R}(V) \) is a homotopy epimorphism in \( \text{Comm}(\text{Ind}(\text{Ban}^{\text{na}}_{R})) \).

Proof. Let \( A = S_{R}(V) \) and \( B = S^{\leq 1}_{R}(V) \). We first do the case where \( V \) has rank one, so \( V = R \). In order to show that the multiplication induces a weak equivalence \( B \otimes_{A} B \rightarrow B \) in \( \text{Comm}(\text{Ind}(\text{Ban}_{R})) \). It is enough to show it is a weak equivalence in \( \text{Mod}(B) \). Notice that \( A \) is just the ring \( R \{ x \} \) equipped with the bornology coming from the coproduct definition of the symmetric algebra construction. Consider the Koszul resolution \( K_{A} \rightarrow A \) of \( A \) in \( \text{Mod}(A \otimes_{R} A) \) where \( K_{A} \) defined by the complex of free objects of \( \text{Mod}(A \otimes_{R} A) \):
\[
R[x,y] \xrightarrow{\delta} R[y,z]
\]
where the morphism is given by multiplication by \((y - z)\). Since \( A \) is flat over \( R \) we have
\[
B \otimes_{A} B \cong B \otimes_{A} A \otimes_{A} B \cong B \otimes_{A} (K_{A}) \otimes_{A} B = [B \otimes_{R} B \rightarrow B \otimes_{R} B]
\]
where the morphism is given by multiplication by \((y - z)\) splits over \( R \) over its image. Notice that because this is true in the world of formal power series, the set theoretical image is the kernel of the multiplication map \( B \otimes_{R} B \rightarrow B \) given by setting \( y \) to be \( z \). What remains to be seen is that
\( \delta \) is strict. Such a splitting \( s \) can be given by sending an \( f \) which satisfies \( f(x,x) = 0 \) defined by
\[
(f(y,z) - f(z,z)) = \sum_{i,j \geq 0, t \geq 0} a_{i,j} y^i z^j y^{-i-t}.
\]
The composition \( s \circ \delta \) is the identity since \( s \) sends \( (y-z)g(y, z) \) to \( g(y, z) \). Notice that \( s \) is bounded because if we consider the isometric isomorphism \( \mathbb{R}\{2\xi, \eta\} \to R\{y, z\} \) given by \( \xi \mapsto y - z \) and \( \eta \mapsto z \) and let \( h(\xi, \eta) = g(\xi + \eta, \eta) \) then
\[
\frac{|g|}{|(y-z)g|} = \frac{|h|}{|\xi h|} \leq \frac{1}{2}.
\]
So \( \delta \) is a strict monomorphism onto the kernel of the multiplication map and hence \( \delta \) is a strict monomorphism into \( R\{y, z\} \). Since the derived monoidal product of homotopy epimorphisms is a homotopy epimorphism and since \( A \) and \( B \) are both flat over \( R \), the \( n \)-fold tensor product morphism
\[
S_R(V) = A \overline{\otimes}_R A \overline{\otimes}_R \cdots \overline{\otimes}_R A \to B \overline{\otimes}_R B \overline{\otimes}_R \cdots \overline{\otimes}_R B = S_R^1(V)
\]
is also a homotopy epimorphism. \( \square \)

**Remark 6.7.** Let \( V_r = R_{r_1} \oplus \cdots \oplus R_{r_n} \) where the \( r_i \) are real numbers greater than zero. The previous discussion can be repeated for \( A = S_R(V) \) and the (archimedean version of the) Tate algebra \( B = S_R^1(V_r) = R_{(\frac{a_1}{r_1}, \ldots, \frac{a_n}{r_n})} \). When \( R \) is non-archimedean, this whole discussion can be repeated using the categories \( \text{Ind}(\text{Ban}^{\text{na}}_R) \) and \( \text{Ban}^{\text{na}, \leq 1}_R \) and the symmetric algebras in those categories.

**Definition 6.8.** The dagger algebra \([3]\) of overconvergent functions on the polydisk of polyradius \((r_1, \ldots, r_n) \in \mathbb{R}_{>0}^n\) is the colimit of the monomorphic restrictions of the functions on closed polydisks in \( \text{Comm}(\text{Ind}(\text{Ban}_R)) \):
\[
R_{(\frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n})}^1 = \text{"colim"}_{\rho \geq r} R_{(\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n})}.
\]
This also makes sense when \( r = (0, \ldots, 0) \). This bornological ring can be realized as the subring of elements \( f = \sum_{I \subset \mathbb{Z}_{>0}^n} a_I x^I \) of \( R[[x_1, \ldots, x_n]] \) such that for some \( \rho \geq r \) we have that \( \sum_{I \subset \mathbb{Z}_{>0}^n} |a_I| \rho^I < \infty \). A subset is bounded precisely when it is bounded in one of the Banach rings \( R_{(\frac{a_1}{r_1}, \ldots, \frac{a_n}{r_n})} \).

**Lemma 6.9.** There are canonical injective morphisms
\[
\left( R_{(\frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n})}^1 \right)^\vee \to \mathcal{O}(D_{<r^{-1}}^n R)
\]
\[
\mathcal{O}(D_{<r^{-1}}^n R)^\vee \to R_{(\frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n})}^1
\]
where \( r^{-1} = (r_1^{-1}, \ldots, r_n^{-1}) \).

**Proof.** For the second, we first define a bounded pairing between \( \mathcal{O}(D_{<r^{-1}}^n R) \) and \( R_{(\frac{a_1}{r_1}, \ldots, \frac{a_n}{r_n})}^1 \) which is \( R \)-linear and non-degenerate in each variable. Consider the partially defined map
\[
f : R[[x_1, \ldots, x_n]] \times R[[x_1, \ldots, x_n]] \to R
\]
given by
\[
(\sum_{I \subset \mathbb{Z}_{>0}^n} a_I x^I, \sum_{J \subset \mathbb{Z}_{>0}^n} b_J x^J) \mapsto \sum_{I \subset \mathbb{Z}_{>0}^n} a_I b_I.
\]
It suffices to show that it restricts to a well defined, bounded non-degenerate morphism on the product
\[ R\left(\frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n}\right) \times \mathcal{O}(D^n_{<r^{-1}, R}). \]
The non-degeneracy is obvious. The fact that it is well defined follows from the estimate
\[ (6.3) \quad |\sum_{I \in \mathbb{Z}^n_{>0}} a_I b_I| \leq \sum_{I \in \mathbb{Z}^n_{>0}} |a_I b_I| = \sum_{I \in \mathbb{Z}^n_{>0}} |a_I \rho^{-I}||b_I \rho^I| \leq \left( \sum_{I \in \mathbb{Z}^n_{>0}} |a_I \rho^{-I}| \right) \left( \sum_{J \in \mathbb{Z}^n_{>0}} |b_J \rho^J| \right) \]
assuming that \( \sum_{J \in \mathbb{Z}^n_{>0}} b_J x^J \in R\left(\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n}\right) \) and \( \rho > r \). In order to show that it is bounded we need to take a bounded subset \( B_1 \subset R\left(\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n}\right) \) for \( \rho > r \) and another bounded subset \( B_2 \subset \mathcal{O}(D^n_{<r^{-1}, R}) \) and show that \( f(B_1 \times B_2) \) is bounded. Let \( \mathcal{O}(D^n_{<r^{-1}, R})_{\rho^{-1}} \) denote the space \( \mathcal{O}(D^n_{<r^{-1}, R}) \) equipped with the norm coming from \( R\left(\rho_1 x_1, \ldots, \rho_n x_n\right) \). The map \( f : R\left(\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n}\right) \times \mathcal{O}(D^n_{<r^{-1}, R})_{\rho^{-1}} \to R \) factorizes as
\[ R\left(\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n}\right) \times \mathcal{O}(D^n_{<r^{-1}, R}) \to R\left(\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n}\right) \times \mathcal{O}(D^n_{<r^{-1}, R})_{\rho^{-1}} \to R. \]
The first map is clearly bounded and so \( B_1 \times B_2 \) is still bounded in \( R\left(\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n}\right) \times \mathcal{O}(D^n_{<r^{-1}, R})_{\rho} \) and so lands inside the product of disks \( D_1 \times D_2 \) where \( D_i \) consists of elements of norm less than \( d_i \). The estimate (6.3) again shows the image in \( R \) is bounded.

Lemma 6.10.
\[ \left(R\left(\frac{x_1}{r}\right)^\tau\right)^{\nu} \cong \mathcal{O}(D^1_{<r^{-1}, R}) \]

Proof. We have
\[ \left(R\left(\frac{x}{r}\right)^\mu\right) \cong \lim_{\rho \to r} R\left(\frac{x}{\rho}\right)^{\nu} = \lim_{\rho \to r} \left( \prod_{j \in \mathbb{Z}_{>0}} \left( \sum_{j \in \mathbb{Z}_{>0}} a_j x^j \sup_{j \in \mathbb{Z}_{>0}} |a_j| \rho^{-j} < \infty \right) \right)^{\nu} \]
\[ = \lim_{\rho \to r} \left( \prod_{j \in \mathbb{Z}_{>0}} \left( \sum_{j \in \mathbb{Z}_{>0}} \left( R(\rho)^{\nu} \right) \right) \right) \]
\[ \cong \lim_{\rho \to r} \left( \prod_{j \in \mathbb{Z}_{>0}} \left( R(\rho)^{\nu} \right) \right). \]
On the other hand,
\[ \mathcal{O}(D^1_{<r^{-1}, R}) = \lim_{\tau \to r} R\left(\frac{x}{\tau}\right) \cong \lim_{\rho \to r} R(\rho x). \]
Now notice that \( \prod_{j \in \mathbb{Z}_{>0}} \left( R(\rho)^{\nu} \right) = \left( \sum_{j \in \mathbb{Z}_{>0}} a_j x^j \sup_{j \in \mathbb{Z}_{>0}} |a_j| \rho^{-j} < \infty \right) \) and
\[ R(\rho x) = \left\{ \sum_{j \in \mathbb{Z}_{>0}} a_j x^j \sum_{j \in \mathbb{Z}_{>0}} |a_j| \rho^{-j} < \infty \right\} \]
and
\[ R\left(\frac{x}{\tau}\right) = \left\{ \sum_{j \in \mathbb{Z}_{>0}} a_j x^j \sum_{j \in \mathbb{Z}_{>0}} |a_j| \tau^{-j} < \infty \right\} \]
and that the inclusion map for \( \tau = \rho^{-1} \)
\[ R\left(\frac{x}{\tau}\right) \to \prod_{j \in \mathbb{Z}_{>0}} \left( R(\rho)^{\nu} \right) \]
or in other words
\[ R\{p\} \to \prod_{j \in \mathbb{Z}_2} \leq 1 (R_{\rho,j}) \]
is bounded. Also if \( \eta \) satisfies \( \eta < \rho^{-1} < r^{-1} \) we have a bounded inclusion \( \prod_{j \in \mathbb{Z}_2} \leq 1 (R_{\rho,j}) \to R\{\eta\} \) because of the inequality
\[ \sum_{j \in \mathbb{Z}_2} |a_j| \eta^j = \sum_{j \in \mathbb{Z}_2} |a_j| \rho^{-j}(\eta \rho)^j \leq (\sup_{k \in \mathbb{Z}_2} |a_k| \rho^{-k})(\sum_{k \in \mathbb{Z}_2} (\eta \rho)^k). \]
Together these inclusions give the desired isomorphisms. \( \square \)

**Lemma 6.11.**
\[ \left( R\{\frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n}\} \right)^\vee \cong \mathcal{O}(D_{\leq r^{-1}}) \]

**Proof.** Let \( V_\rho = R_{\rho_1} \oplus \cdots \oplus R_{\rho_n} \). We have
\[ \left( R\{\frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n}\} \right)^\vee \cong \lim_{\rho \to r} \left( R\{\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n}\} \right)^\vee \cong \lim_{\rho \to r} \left( \prod_{j \in \mathbb{Z}_2} \leq 1 (V_{\rho,j}/\Sigma_j) \right)^\vee = \lim_{\rho \to r} \left( \prod_{j \in \mathbb{Z}_2} \leq 1 (V_{\rho^{-1},j}/\Sigma_j) \right)^\vee \]
\[ (6.5) \]
On the other hand,
\[ \mathcal{O}(D_{\leq r^{-1}}) = \lim_{\tau \to r^{-1}} R\{\frac{x_1}{\tau_1}, \ldots, \frac{x_n}{\tau_n}\} \cong \lim_{\rho \to r} R\{\rho_1 x_1, \ldots, \rho_n x_n\} = \lim_{\rho \to r} \left( \prod_{j \in \mathbb{Z}_2} \leq 1 (V_{\rho^{-1},j}/\Sigma_j) \right) \]
Notice that for each \( \rho > r \) we have injective bounded maps
\[ f_\rho : R\{\rho_1 x_1, \ldots, \rho_n x_n\} \to R\{\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n}\}^\vee \]
sending \( \sum_I a_I x^I \) to the map sending \( \sum_I b_I x^I \) to \( \sum_I a_I b_I \). The later is bounded by \( \sum_J ||a_J|| \rho^{-J} \) because
\[ ||\sum_I a_I b_I|| \leq \sum_I ||a_I|| ||b_I|| = \sum_I ||a_I|| ||\rho^I|| ||b_I|| \rho^{-I} \leq (\sum_J ||a_J|| \rho^{-J})(\sum_K ||b_K|| \rho^K). \]
This then shows that \( f_\rho \) has norm less than or equal to one. For any \( \eta > \rho > r \) we have an injective bounded map
\[ g_{\rho,\eta} : R\{\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n}\} \to R\{\eta_1 x_1, \ldots, \eta_n x_n\} \]
given by sending any \( \alpha \) to \( \sum_I \alpha(x^I) x^I \) which can be seen to be well defined and bounded by the estimate
\[ ||\sum_I \alpha(x^I) x^I|| = \sum_I ||\alpha(x^I)|| \eta^{-I} \leq \sum_I ||\alpha|| ||x^I|| \eta^{-I} = (\sum_I (\rho/\eta)^I) ||\alpha||. \]
Notice that the composition \( g_{\rho, \eta} \circ f_\rho \) is simply restriction from the disk of radius \( \rho^{-1} \) to \( \eta^{-1} \). The composition \( f_\eta \circ g_{\rho, \eta} \) is the identity. Therefore these maps give maps of systems which give the required isomorphisms:

\[
O(D^n_{< r^{-1}, R}) = \lim_{\rho > r} R\{\rho_1 x_1, \ldots, \rho_n x_n\} \cong \lim_{\rho > r} \left( R\left( \frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n} \right) \right)^{\vee} \cong \left( \lim_{\rho > r} \left( \frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n} \right) \right)^{\vee}.
\]

\[\square\]

**Lemma 6.12.** Given a system \( \cdots \to V_2 \to V_1 \to V_0 \) in \( \text{Ban}_R \) where all structure morphisms are dense, the canonical map

\[\lim_i V_i \to \text{Rlim}_i V_i\]

is a quasi-isomorphism.

**Proof.** Because the object \( \text{Rlim}_i V_i \) has bounded cohomological dimension (in fact with amplitude \([0, 1]\) by [11]), it is enough to show that for any projective \( P \) that the natural morphism \( R\text{Hom}(P, \lim_i V_i) \to R\text{Hom}(P, \text{Rlim}_i V_i) \) is a quasi-isomorphism of complexes of abelian groups. We have

\[R\text{Hom}(P, \text{Rlim}_i V_i) \cong R\text{lim}_i \text{RHom}(P, V_i)\]

and also (since \( P \) is projective) \( R\text{lim}_i \text{RHom}(P, V_i) \cong \text{RHom}(P, V_i) \). Similarly, \( \text{RHom}(P, \lim_i V_i) \cong \text{lim}_i \text{Hom}(P, V_i) \).

Therefore, we need to show that the morphism

\[\lim_i \text{Hom}(P, V_i) \to \text{Rlim}_i \text{Hom}(P, V_i)\]

is a quasi-isomorphism. In the system defined by the \( \text{Hom}(P, V_i) \) the structure maps are bounded and dense. By the Mittag-Leffler lemma for abelian groups equipped with compatible metrics given in work of Palamadov [35], Vogt, and Retakh, we get a quasi-isomorphism

\[\lim_i \text{Hom}(P, V_i) \to \text{Rlim}_i \text{Hom}(P, V_i)\]

in the derived category of (the quasi-abelian category of) topological abelian groups. This implies by that we in fact have the needed quasi-isomorphism for the underlying abelian groups in equation (6.7).

\[\square\]

**Corollary 6.13.** The natural morphism \( O(K^n_R) \to \text{Rlim}_r R(\frac{x_1}{r}, \ldots, \frac{x_m}{r}) \) is a quasi-isomorphism and therefore \( O(K^n_R) \cong \lim_{r \in \mathbb{Z}_{>0}} R(\frac{x_1}{r}, \ldots, \frac{x_m}{r}) \).

**Lemma 6.14.** For any \( \tau_1, \ldots, \tau_n \in (0, \infty) \) and \( \rho_1, \ldots, \rho_m \in (0, \infty) \), we have

\[O(D^n_{< \tau, \rho}) \otimes_R O(D^m_{< \rho}) \cong O(D^{n+m}_{< (\tau, \rho)})\]

In particular \( O(K^n_R) \otimes_R O(K^m_R) \cong O(K^{n+m}_R) \).

**Proof.** This follows immediately from the facts that both \( R\{\rho_1 x_1, \ldots, \rho_n x_n\} \) and \( O(D^m_{< \rho}) \) are metrizable by Corollary 5.13 and \( O(D^n_{< \rho}) \) is nuclear by Corollary 6.5 and hence flat by Lemma 4.19 that the second statement in Lemma 6.17 as all the limits can be pulled outside.

\[\square\]

**Theorem 6.15.** Let \( V = \prod_{i=1}^n R \). The natural morphism \( S_R(V) \to O(K^n_R) \) is a homotopy epimorphism in \( \text{Comm}(\text{Ind}(\text{Ban}_R)) \).
Proof. Let $A = S_R(V)$ and $B = \mathcal{O}(\mathbb{A}^n_R)$. We first do the case where $V$ has rank one and $n = 1$ so $V = R$. In order to show that the multiplication induces a weak equivalence $B \otimes^L_A B \to B$ in $\text{Comm}(\text{slnd}(\text{Ban}_R))$. It is enough to show it is a weak equivalence in $\text{Mod}(B)$. Consider the Koszul resolution $K_A \to A$ of $A$ in $\text{Mod}(A \otimes_R A)$ where $K_A$ defined by the complex of free objects of $\text{Mod}(A \otimes_R A)$:

$$R[y, z] \to R[y, z]$$

where the morphism is given by multiplication by $(y - z)$. Since $A$ is flat over $R$ we have

$$B \otimes^L_A B \cong B \otimes^L_A A \otimes^L_A B \cong B \otimes_A (K_A) \otimes_A B = [B \otimes_R B \to B \otimes_R B].$$

Therefore, it is enough to show that the last complex, which is by Lemma 6.14

$$\mathcal{O}(\mathbb{A}^2_R) \xrightarrow{\delta} \mathcal{O}(\mathbb{A}^2_R),$$

where the morphism is given by multiplication by $(y - z)$, splits over $R$. Such a splitting $s$ can be given by sending $f$ defined by $f(y, z) = \sum_{(i, j) \in \mathbb{Z}^2} a_{i, j} y^i z^j$ to $sf$ defined by

$$(sf)(y, z) = \frac{f(y, z) - f(z, z)}{y - z} = \sum_{i > 0, j \geq 0} a_{i, j} (y - z)^{i-1}.$$

The splitting $s : R[[y, z]] \to R[[y, z]]$ preserves each $R \{\frac{y}{r}, \frac{z}{r}\}$. It is clear that $s$ is bounded and that $s \circ \delta$ is the identity. Since the derived monoidal product of homotopy epimorphisms is a homotopy epimorphism and since $A$ and $B$ are both flat over $R$, the $n$-fold tensor product morphism

$$S_R(V) = A \otimes_R A \otimes_R \cdots \otimes_R A \to B \otimes_R B \otimes_R \cdots \otimes_R B = S^1_R(V)$$

is also a homotopy epimorphism. $\square$

Lemma 6.16. For any $\rho < r$ the natural morphism

$$\mathcal{O}(D^n_{cr,R}) \to R\{\frac{x_1}{\rho_1}, \ldots, \frac{x_n}{\rho_n}\}$$

is a homotopy epimorphism.

Proof. The main fact we use in this proof is the flatness over $R$ of the two types of bornological rings which appear in the statement of the Lemma in the sense of $[3, 1]$. Therefore we can pass back and forth between $\otimes^L_R$ and $\otimes_R$ in our expressions. We first consider the case when $n = 1$. Because by Corollary $6.5$, $\mathcal{O}(D^1_{cr,R})$ is nuclear, it is flat over $R$ by $[4, 19]$ Consider the short exact sequence

$$0 \to \mathcal{O}(D^2 \subset (r, r), R) \xrightarrow{y - z} \mathcal{O}(D^2 \subset (r, r), R) \to \mathcal{O}(D^1_{cr,R}) \to 0$$

where we have used variables corresponding to the presentation

$$\mathcal{O}(D^2 \subset (r, r), R) = \operatorname{lim}_{\rho < r} R\{\frac{y}{\rho}, \frac{z}{\rho}\}.$$
The term on the right hand side is isomorphic to \( R(\xi_R^L)\hat{\otimes}_{\mathcal{O}(D^1_{c,r},R)} R(\xi_R^L) \). Because \( \mathcal{O}(D^2_{c,(r,r),R}) = A\hat{\otimes}_R A \) by Lemma 6.14 and of course \( R(\xi_R^L) = R(\xi_R^L)\hat{\otimes}_R R(\xi_R^L) \) the other part of the triangle simplifies to
\[
R\left(\frac{y}{\rho}, \frac{z}{\rho}\right) \rightarrow R\left(\frac{x}{\rho}, \frac{z}{\rho}\right).
\]
Because this fits into a short exact sequence
\[
0 \rightarrow R\left(\frac{y}{\rho}, \frac{z}{\rho}\right) \rightarrow R\left(\frac{y}{\rho}, \frac{z}{\rho}\right) \rightarrow R\left(\frac{x}{\rho}\right) \rightarrow 0
\]
we conclude that \( R(\xi_R^L)\hat{\otimes}_{\mathcal{O}(D^1_{c,r},R)} R(\xi_R^L) \cong R(\xi_R^L) \).

**Remark 6.17.** If we denote \( A^n_{R,\text{ind}} = \text{spec}(S_R(V)) \) then the methods of this section can be used to show that any natural restriction of functions on pairs of objects included in one another given by dagger disks (including radius 0), or affinoid disks, Stein disks, \( A^n_{R,\text{ind}} \) or \( A^n_R \) is a homotopy epimorphism. This should similarly hold for the map given by quotienting the pair and their restriction morphism by the same ideal.

### 7. Topologies and Descent

Different considerations of abstract topologies and descent that we know of have appeared for example in works of Orlov [34] and Kontsevich/Rosenberg [29]. We consider descent in the infinity-category version of the homotopy monomorphism, flat, and other topologies in our project on derived analytic geometry [12]. On the other hand, in this article we try to focus on non-derived categories of modules (i.e. homotopicly discrete modules or complexes in degree 0) and the underved pullback functors of restriction. As this would not work for arbitrary modules concentrated in degree zero, we need to specialize to quasi-coherent modules. The Grothendieck pre-topology that we work is not quasi-compact, it has covers consisting of a countable collection of homotopy monomorphisms \( \text{spec}(A_i) \rightarrow \text{spec}(A) \) such that given a morphism \( f : M \rightarrow N \) in \( \text{Mod}(A) \) with \( \mathcal{O}_A A_i \rightarrow \mathcal{N} \mathcal{O}_A A_i \) an isomorphism for all \( i \), then \( f \) is an isomorphism. This property is called being conservative. It is expected to correspond to surjectivity of the cover for the topos-theoretic notion of points. For instance, it is known that the Huber points correspond with the topos-theoretic notion of points in the rigid-analytic context with the \( G \)-topology (see [11] where more explanation and citations are given).

#### 7.1. Quasi-coherent modules

**Definition 7.1.** Let \( A \) be an object of \( \text{Comm}(\text{Ind}(\text{Ban}_R)) \) flat over \( R \). An object \( M \) of \( \text{Mod}(A) \) is called quasi-coherent if it is flat over \( R \) and for all homotopy epimorphisms \( A \rightarrow B \) where \( B \) is metrizable that \( M \) is transverse to \( B \) over \( A \). The full subcategory of quasi-coherent modules is denoted by \( \text{Mod}^{RR}(A) \).

**Definition 7.2.** Let \( A \) be an object of \( \text{Comm}(\text{Ind}(\text{Ban}_R)) \) which is metrizable. An object \( M \) of \( \text{Mod}_F(A) \) is called quasi-coherent if for all homotopy epimorphisms \( A \rightarrow B \) in \( \text{Comm}(\text{Ind}(\text{Ban}_R)) \) where \( B \) is metrizable that \( M \) is transverse to \( B \) over \( A \). The full subcategory of quasi-coherent modules is denoted by \( \text{Mod}^{FRR}(A) \).

Our notation “RR” is in credit to Ramis and Ruget who introduced a similar notion in the context of complex analysis in [43].

**Example 7.3.** Let \( A \in \text{Comm}(\text{Ind}(\text{Ban}_R)) \) and \( V \in \text{Ind}(\text{Ban}_R) \). Assume that both \( A \) and \( V \) are flat over \( R \) and metrizable. Then \( A\hat{\otimes}_R V \in \text{Mod}^{FRR}(A) \).
Lemma 7.4. Say that $C$ is a category with countably many objects and morphisms. Then given any functor $F : C \to \text{Mod}_{R}^{\mathbb{N}}(A)$ the limit computed in $\text{Mod}(A)$ lives in $\text{Mod}_{R}^{\mathbb{N}}(A)$.

7.2. General Results on Descent.

Lemma 7.5. Let $A \in \text{Comm}(\text{Ind}(\text{Ban}(R)))$, and let $\{E_{i}\}_{i \in I}$ be projective system in $\text{Mod}(A)$ indexed by the countable poset $I$. Let $F$ be an object in $\text{Mod}(A)$. Suppose that the underlying objects of $A$ and $F$ in $\text{Ind}(\text{Ban}(R))$ are metrizable and flat over $R$. Suppose in addition $F$ is transverse to $E_{i}$ over $A$ for each $i$ and the system $\{E_{i}\}_{i \in I}$ is lim-acyclic. Then $\{F \hat{\otimes}_{A} E_{i}\}_{i \in I}$ is a lim-acyclic projective system, $F$ is transverse to $\lim_{i \in I} E_{i}$ over $A$ and the natural morphism

$$F \hat{\otimes}_{A} (\lim_{i \in I} E_{i}) \to \lim_{i \in I} (F \hat{\otimes}_{A} E_{i})$$

is an isomorphism. If instead of the condition that $F$ is flat over $R$ we instead have that both the $E_{i}$ and $\lim_{i \in I} E_{i}$ are flat over $R$ then the same conclusion holds.

Proof. Recall that the Bar complex $\mathcal{L}^{\bullet}_{A}(F)$ is strictly quasi-isomorphic to $F$ and that $F \hat{\otimes}_{A} (-)$ is commuted by $\mathcal{L}^{\bullet}_{A}(F) \otimes_{R} (-)$. Using the explicit form of this complex together with the fact that $F \hat{\otimes}_{R} (-)$ and $A \hat{\otimes}_{R} (-)$ commute with products by Lemma 5.18 we can see that it interacts well with the Roos complex of $\{E_{i}\}_{i \in I}$ in the sense that there is a strict quasi-isomorphism

$$\text{Tot}(\mathcal{L}^{\bullet}_{A}(F) \otimes_{R} \mathcal{R}^{\bullet}((\{E_{i}\}_{i \in I}) \approx \text{Tot}(\mathcal{R}^{\bullet}((\{\mathcal{L}^{\bullet}_{A}(F) \otimes_{R} E_{i}\}_{i \in I})).$$

The left hand side computes $F \hat{\otimes}_{A} (\mathcal{R}\lim_{i \in I} E_{i})$ and the right hand side commutes $\mathcal{R}\lim_{i \in I} (F \hat{\otimes}_{A} E_{i})$. Using that $F$ is transverse to $E_{i}$ over $A$ for each $i$ and the system $\{E_{i}\}_{i \in I}$ is lim-acyclic the above equation simplifies to a quasi-isomorphism

$$F \hat{\otimes}_{A} (\lim_{i \in I} E_{i}) \approx \mathcal{R}\lim_{i \in I} (F \hat{\otimes}_{A} E_{i}).$$

From which the rest of the claims follow immediately as one side is in non-negative degrees and the other is in non-positive degrees and so both are concentrated in degree zero.

Recall that for a countable collection $A \to A_{i}$ and $M \in \text{Mod}(A)$ we can form the usual complex

$$\mathcal{C}^{\bullet}(M, \{A_{i}\}) = \prod_{i} (M \hat{\otimes}_{A} A_{i} \to \prod_{i,j} (M \hat{\otimes}_{A} A_{i} \hat{\otimes}_{A} A_{j} \to \cdots)$$

Lemma 7.6. Suppose that the underlying objects of $A$ and $M$ in $\text{Ind}(\text{Ban}(R))$ are metrizable and flat over $R$. Suppose $M$ is transverse to all $A_{i_{1}} \hat{\otimes}_{A} A_{i_{2}} \hat{\otimes}_{A} \cdots \hat{\otimes}_{A} A_{i_{n}}$ in $\text{Mod}(A)$ and the natural morphism $A \to \mathcal{C}^{\bullet}(A, \{A_{i}\})$ is a quasi-isomorphism, then the natural morphism $M \to \mathcal{C}^{\bullet}(M, \{A_{i}\})$ is a quasi-isomorphism. If instead of the condition that $M$ is flat over $R$ we have that all the $A_{i_{1}} \hat{\otimes}_{A} A_{i_{2}} \hat{\otimes}_{A} \cdots \hat{\otimes}_{A} A_{i_{n}}$ are flat over $R$ then the same conclusion holds.

Proof. $M$ is quasi-isomorphic to $M \hat{\otimes}_{A} \mathcal{C}^{\bullet}(A, \{A_{i}\})$. Using Lemma 5.18 each term

$$\prod_{i_{1}, \ldots, i_{n}} A_{i_{1}} \hat{\otimes}_{A} A_{i_{2}} \hat{\otimes}_{A} \cdots \hat{\otimes}_{A} A_{i_{n}}$$

is transverse to $M$ over $A$. Therefore, there is a quasi-isomorphism

$$\text{Tot}(\mathcal{L}^{\bullet}_{A}(M) \otimes_{R} \mathcal{C}^{\bullet}(A, \{A_{i}\})) \approx \text{Tot}(\mathcal{C}^{\bullet}(M, \{A_{i}\}).$$

As our conditions guarantee that $\mathcal{C}^{\bullet}(-, \{A_{i}\})$ is an exact functor the right hand side is quasi-isomorphic to $\mathcal{C}^{\bullet}(M, \{A_{i}\})$ and $M = M \hat{\otimes}_{A} A = M \hat{\otimes}_{A} \mathcal{C}^{\bullet}(A, \{A_{i}\})$ is computed by the left hand side so we are done. □
Remark 7.7. The non-archimdean version of this (the proof is the same) can give new settings for Tate’s acyclicity theorem.

Corollary 7.8. Let \( A \in \text{Comm}(\text{Ind}(\text{Ban}_R)) \) such and \( A_i \) indexed by \( i \) in a countable poset \( I \) in \( \text{Comm}(\text{Ind}(\text{Ban}_R)) \). Suppose that the system \( \{ A_i \}_{i \in I} \) is lim-acyclic and that the objects of \( \text{Ind}(\text{Ban}_R) \) underlying \( A_i \) are metrizable and flat over \( R \) and transverse to one another over \( A \) and in \( \text{Mod}^{RR}_F(A) \). The natural functor \( \text{Mod}(A) \to \text{limMod}(A_i) \) induces a functor \( \text{Mod}^{RR}_F(A) \to \text{limMod}^{RR}_{i \in I}(A_i) \). Then \( A = \text{lim}A_i \) if and only if the collection of functors \( (\text{-}) \otimes A_i \) is conservative. When this holds the natural functor

\[
\left\{ N_i \right\}_{i \in I} \to \text{lim}N_i
\]

is essentially surjective.

Proof. Given an object \( M \in \text{Mod}^{RR}_F(A) \) and \( A_i \to B \) is a homotopy epimorphism we have

\[
(M \otimes A_i) \otimes A_i B \cong (M \otimes A_i) \otimes A_i B \cong M \otimes A_i B \cong M \otimes A B \cong (M \otimes A_i) \otimes A_i B
\]

so \( M \otimes A_i \in \text{Mod}^{RR}_F(A) \) and each \( M \otimes A_i \) is metrizable.

If \( A \cong \text{lim}A_i \), give a morphism \( f : M \to N \), we can rewrite \( f \) using Lemma 7.5 as \( \text{lim}f_i \) where the \( f_i : M \otimes A_i \to N \otimes A_i \). Therefore, the collection is conservative. Conversely, if the collection is conservative let \( \pi : A \to \text{lim}A_i \) be the canonical morphism. To show it is an isomorphism it is enough to know that it becomes so after applying the \( A_i \otimes (-) \). But after doing this we get using Lemma 7.5

\[
A_j \to A_j \otimes A (\text{lim}A_i) \cong \text{lim}(A_j \otimes A_i) \cong A_j
\]

which is an isomorphism. The essential surjectivity holds because again using Lemma 7.5 we have

\[
M = M \otimes A \cong M \otimes A (\text{lim}A_i) \cong \text{lim}(M \otimes A_i)
\]

for any \( M \in \text{Mod}^{RR}_F(A) \).

Lemma 7.9. Let \( A \in \text{Comm}(\text{Ind}(\text{Ban}_R)) \) and say we are given a countable poset \( A \to A_i \) of epimorphisms of \( \text{Comm}(\text{Ind}(\text{Ban}_R))_A \). The functor

\[
\text{limMod}(A_i) \to \text{Mod}(A)
\]

is fully faithful.

Proof. The natural “pushforward” functors \( \text{Mod}(A_i) \to \text{Mod}(A) \) are fully faithful. The limit of these functors is therefore fully faithful.

By combining Corollary 7.8 and Lemma 7.9 we have

Theorem 7.10. Let \( A \in \text{Comm}(\text{Ind}(\text{Ban}_R)) \) and \( A \to A_i \) homopopy epimorphisms indexed by \( i \) in a countable poset \( I \) in \( \text{Comm}(\text{Ind}(\text{Ban}_R))_A \) whose underlying modules are in \( \text{Mod}^{RR}_F(A) \). Suppose that the system \( \{ A_i \}_{i \in I} \) is lim-acyclic and that the objects of \( \text{Ind}(\text{Ban}_R) \) underlying \( A_i \) are metrizable
and flat over $R$. Assume the collection of functors $(-)\otimes_A A_i$ is conservative. When this holds the natural functor
\[ \lim_{i \in I} \text{Mod}^{RR}_F(A_i) \to \text{Mod}^{RR}_F(A) \]
is an equivalence of categories.

This theorem can be used in the case of hypercovers. We now give a more explicit proof in the case of covers which can be easily adapted to general posets.

**Theorem 7.11.** Let $A \in \text{Comm}(\text{Ind}(\text{Ban}_R))$ and say we are given a countable collection $A \to A_i$ of objects of $\text{Comm}(\text{Ind}(\text{Ban}_R))_A$ indexed by $i \in S$. Suppose that $A$ and $A_i$ are metrizable objects which are flat over $R$, in $\text{Mod}^{RR}_F(A)$. Suppose that each morphism $A \to A_i$ is a homotopy epimorphism and the collection of functors $\text{Mod}^{RR}_F(A) \to \text{Mod}^{RR}_F(A_i)$ is conservative. Suppose that the corresponding system $A_w = A_i \otimes_A \cdots \otimes_A A_{i_m}$ for words $w$ in $S$ is a lim-acyclic projective system as above. Then the canonical functor
\[ D : \text{Mod}^{RR}_F(A) \to \lim_{w \in P} \text{Mod}^{RR}_F(A_w) \]
is an equivalence of categories.

**Proof.** Given $N_w \in \text{Mod}^{RR}_F(A_w)$ and suppose that $A \to B$ is a homotopy epimorphism.
\[ B \otimes_A N_w \approx B \otimes_A (A_w \otimes_A N_w) \approx (B \otimes_A A_w) \otimes_A N_w \approx (B \otimes_A A_w \otimes_A N_w) \approx B \otimes_A N_w \]
since $A_w$ is quasi-coherent and $A_w \to B \otimes_A A_w$ is a homotopy epimorphism and $N_w$ is a quasi-coherent $A_w$-module. Since $B$ is transverse to $N_w$ for each $w$ over $A$ we have that $\lim_{w \in P} N_w$ is transverse to $B$ over $A$ by Lemma 7.5. Hence using Lemma 5.15 $\lim_{w \in P} N_w \in \text{Mod}^{RR}_F(A)$.

Consider the functor
\[ \text{Mod}^{RR}_F(A) \leftarrow \lim_{w \in P} \text{Mod}^{RR}_F(A_w) : R \]
in the other direction defined by taking the limit. We have by Lemma 7.5
\[ A_i \otimes_A (\lim_{w \in P} N_w) \approx \lim_{w \in P} (A_i \otimes_A N_w) \approx N_v \]
showing that $D \circ R$ is naturally equivalent to the identity. Using Lemma 7.5 and Corollary 7.8 we have
\[ \lim_{w \in P} (A_w \otimes_A M) \approx (\lim_{w \in P} A_w) \otimes_A M \approx A \otimes_A M \approx M \]
showing that $R \circ D$ is naturally equivalent to the identity. 

### 7.3. Examples of Descent

This article was originally motivated by a desire to make “more categorical” the results on descent for Stein algebras from [8] (see also [5]). We believe that we have succeeded in a large aspect in terms of the issues surrounding infinite products and completed tensor products and their interaction. Unfortunately, we have not been able to make categorical the Mittag-Leffler aspects which involve dense maps of algebras in the projective system and lim-acyclicity. In standard complex analysis one often exhausts a Stein open subset by an increasing union compact, convex subsets with the Noether property. Then one would like to understand how certain quasi-abelian categories of quasi-coherent modules on the Stein open are constructed as categorical limits of the similar categories on the compact subsets. More precisely, can a nice enough module over the algebra of holomorphic functions on the Stein open be determined in terms of gluing data for modules on the compact subsets. These questions also have a rich history in the non-archimedean literature, for example see work of Ardakov and Wadsley [3], where one uses affinoids in place of compact convex subsets. In our desire for a unified approach to the
archimedean and non-archimedean case, we can restate Theorem 7.10 to this case. Let $A_i$ be Banach modules, flat over $\mathbb{R}$ together with a sequence of dense, nuclear, homotopy epimorphisms $\cdots \rightarrow A_2 \rightarrow A_1$ any quasi-coherent, metrizable, ind-Banach module $M$ flat over $R$ over $A = \lim A_i$ can be expressed as a limit in $\text{Mod}(A)$ of a sequence $\cdots \rightarrow M_2 \rightarrow M_1$ where each $M_i$ is a nuclear, metrizable ind-Banach object of $\text{Mod}(A_i)$ flat over $R$ and the morphisms are consistent with this action in the sense that there are isomorphisms $A_{i+1} \otimes_A M_i \simeq A_{i+1} \otimes_A M_i \simeq M_{i+1}$ compatible with one another and with the maps in the sequence. Any element of $\text{Hom}(M,N)$ in the category of ind-Banach $A$-modules is a consistent limit of elements of $\text{Hom}(M_i,N_i)$ in the category of ind-Banach $A_i$-modules. This should have applications in non-archimedean geometry for instance in the case of analytic differential operators as appear in work of Ardakov and Wadsley (see also [4]) which are Frechet (and as we have shown therefore metrizable) modules which are not coherent over the functions, but which can be shown to be quasi-coherent in our definition. A version of this theorem was stated in [8] over a complete valuation field but there we needed to separately prove the theorem in the archimedean and non-archimedean cases whereas in this article we provide a single proof over a Banach ring that works in the archimedean or non-archimedean case.

In the arithmetic setting, for any prime $p$ one can cover $\text{spec}(\mathbb{Z})$ by $\text{spec}(\mathbb{R})$, $\text{spec}(\mathbb{Z}_p)$, and $\text{spec}(\mathbb{Q}_p)$ with these last two elements of the cover having intersection $\text{spec}(\mathbb{Q})$ and we can obtain descent results for modules on this cover applying Theorem 7.10. There could also be an adelic cover. We postpone this to future work.

8. The Fargues-Fontaine Curve

A thorough treatment of the Fargues-Fontaine Curve from the point of view of Banach algebraic geometry appears in [9]. Therefore, we only focus on the aspects here which are relevant to the current article. Let

$$\mathbb{Z}\{\left(\frac{x}{r}\right)^{\frac{1}{n}}\} = \mathbb{Z}\{\frac{x}{r}, \frac{y}{r}\} / (y^n - x) \cong \mathbb{Z}\{\frac{x}{r}\} \oplus \mathbb{Z}\{\frac{x}{r}\}.\quad \text{for } r_2 < r_1 < 1,$$

For $r_2 < r_1 < 1$, the non-expanding morphisms $\mathbb{Z}\{\left(\frac{x}{r_1}\right)^{\frac{1}{n}}\} \rightarrow \mathbb{Z}\{\left(\frac{x}{r_2}\right)^{\frac{1}{n}}\}$ is nuclear being a sum of nuclear morphisms. Using the non-expanding morphisms

$$\alpha_{n,m} : \mathbb{Z}\{\left(\frac{x}{r_1}\right)^{\frac{1}{n}}\} \rightarrow \mathbb{Z}\{\left(\frac{x}{r_m}\right)^{\frac{1}{m}}\}$$

we have the Banach ring $\colim_n^{\leq 1} \mathbb{Z}\{\left(\frac{x}{r}\right)^{\frac{1}{n}}\}$.

**Conjecture 8.1.** The induced morphisms

$$\colim_n^{\leq 1} \mathbb{Z}\{\left(\frac{x}{r_1}\right)^{\frac{1}{n}}\} \rightarrow \colim_n^{\leq 1} \mathbb{Z}\{\left(\frac{x}{r_2}\right)^{\frac{1}{n}}\}$$

are nuclear for all $r_2 < r_1 < 1$ and more generally a countable contracting colimit of nuclear morphisms is nuclear.

Let $E$ be the field

$$E = \mathbb{F}_p((\mathbb{Q})) = \{ \sum_{\gamma \in \mathbb{Q}} a_{\gamma} x^\gamma | a_{\gamma} \in \mathbb{F}_p, \ \text{support}(a_{\gamma}) \ \text{well ordered} \}.$$

It is equipped with the valuation given by

$$v\left(\sum_{\gamma \in \mathbb{Q}} a_{\gamma} x^\gamma\right) = \min\{\gamma : a_{\gamma} \neq 0\}.$$
The associated valuation ring is
\[ \mathcal{O}_E = \mathbb{F}_p(((\mathbb{Q}_{\geq 0}))) = \left\{ \sum_{\gamma \in \mathbb{Q}_{\geq 0}} a_{\gamma} x^\gamma | a_{\gamma} \in \mathbb{F}_p, \text{ support}(a_{\gamma}) \text{ well ordered} \right\}. \]

In Fargues-Fontaine theory one encounters a scheme \( \mathbb{Y}_E \) whose set of closed points \( |\mathbb{Y}_E| \) parametrize un-tilts of \( E \). An un-tilt of \( E \) is an isomorphism class of pairs \((F, t)\) where \( F \) is a perfectoid field of characteristic \( 0 \), \( t : E \to F^h \) is an embedding of topological fields and the quotient is a finite extension. Here \( F^h = \text{Frac}(\lim_{\to} \mathcal{O}_F/p) \) where \( \mathcal{O}_F \) is the ring of integers of \( F \). Let \( W \) denote the Witt vectors construction. Let \( \mathbb{Z}_r \) be the Banach \( \mathbb{Z} \)-module which is \( \mathbb{Z} \) with norm \( r|\cdot| \) where \( |\cdot| \) is the usual absolute value. For any \( M \in \text{Ban}_{\mathbb{Z}} \), let \( S^{\leq 1}(M) \) be the symmetric ring construction in the category \( \text{Ban}_{\mathbb{Z}}^{\leq 1} \) consisting of Banach modules with non-expanding morphisms, i.e., \( S^{\leq 1}(M) = \text{Lim}_{n=0}^\infty (M \otimes \mathbb{Z}/\mathbb{Z}_n) \) where the coproduct is taken in \( \text{Ban}_{\mathbb{Z}}^{\leq 1} \). Consider the colimit in \( \text{Ban}_{\mathbb{Z}}^{\leq 1} \) of the \( l \)-th power morphisms \( x \mapsto x^l \) in the ring of functions on the “closed 1-dimensional disk of radius \( r \)” given by the contracting coproduct \( S^{\leq 1}(\mathbb{Z}_r) \). One then has that for each prime \( p \),

\[ \left( \text{colim}_{t \in \mathbb{N}} \mathbb{Z}^{\leq 1}\left( \left( \frac{X}{r} \right)^t \right) \right) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \lim_{t \in \mathbb{N}} \mathbb{Z}_p^{\leq 1} \left( \left( \frac{X}{r} \right)^t \right) \]

and this question is addressed more carefully in [9] using results from this article. Consider the Fréchet completion of \( W(\mathcal{O}_E) \) with respect to the semi-norms

\[ | \sum_{n \gg -\infty} [f_n]p^n |_r = \sup_{n \gg -\infty} |f_n|p^{-rn}. \]

The importance of this completion is that the closed maximal ideals of the localization at \( p \) are in bijection with \( \mathbb{Y}_E/\mathbb{Z} \) where \( n \in \mathbb{Z} \) acts by \((F, \iota) \mapsto (F, \iota \circ \phi^n)\) where \( \phi \) the \( p \)-th power Frobenius automorphism of \( E \). In [14], Cuntz and Deninger found a nice description of the additive group structure on the ring of \( p \)-typical Witt vectors of a perfect \( \mathbb{F}_p \)-algebra with basis \( b \). They found it to be simply the \( p \)-adic completion of the free \( \mathbb{Z} \)-algebra with basis \( b \).

**Lemma 8.2.** The natural functor

\[ F : \text{Ind}(\text{Ban}_{\mathbb{R}}^{K_{\mathbb{R}}}) \to \left( \text{Ind}(\text{Ban}_{\mathbb{R}}) \right)^{K_{\mathbb{R}}} \]

is fully-faithful for any poset \( K \) with cardinality less than \( \aleph_1 \).

**Proof.** Given objects \( X : k \mapsto \text{colim}_{t \in T} X_t^k \) and \( Y : k \mapsto \text{colim}_{s \in S} Y_s^k \) of \( \text{Ind}(\text{Ban}_{\mathbb{R}}^{K_{\mathbb{R}}}) \), where \( T \) and more importantly \( S \) is a \( \aleph_1 \)-filtered poset, we have

\[ \text{Hom}(X,Y) = \lim_{t \in T} \text{colim}_{s \in S} \int_{k \in K} \text{Hom}(X_t^k, Y_s^k) \]

where \( \int_{k \in K} \) is a limit over the usual diagram used to define morphisms in diagram categories. This is a limit over a diagram with cardinality less than \( \aleph_1 \) since \( K \) itself is such a diagram. It is a limit in the category of sets of a diagram of sets whose vertices are of the form \( \text{Hom}(X_t^k, Y_s^l) \). On the other hand,

\[ \text{Hom}(FX, FY) = \int_{k \in K} \lim_{t \in T} \text{colim}_{s \in S} \text{Hom}(X_t^k, Y_s^k) \cong \lim_{t \in T} \int_{k \in K} \text{colim}_{s \in S} \text{Hom}(X_t^k, Y_s^k). \]

The term \( \int_{k \in K} \text{colim}_{s \in S} \text{Hom}(X_t^k, Y_s^k) \) is a limit in the category of sets over the same diagram whose vertices are of the form \( \text{colim}_{s \in S} \text{Hom}(X_t^k, Y_s^l) \) where the functor \( \text{colim} \) has been applied to the previous diagram. By Lemma 5.3 we can interchange \( \int_{k \in K} \) and \( \text{colim} \) so these different Hom-sets agree, finishing the proof. \( \square \)
Definition 8.3. Let $F: \mathbb{N} \to \text{Ind}(\text{Ban}_R)$ be a functor such that there exists an $\aleph_1$-filtered category $L$ and a functor $\tilde{F}: \mathbb{N} \times L \to \text{Ban}_R^{\leq 1}$ such that using the composition $\mathbb{N} \to (\text{Ban}_R^{\leq 1})^L \to \text{Ind}(\text{Ban}_R)$ agrees with $F$. Define $\operatorname{colim}^{\leq 1} \tilde{F}$ by the composition $L \to (\text{Ban}_R^{\leq 1})^N \to \text{Ban}_R^{\leq 1}$. Define $\operatorname{colim}^{\leq 1} L = \operatorname{colim}^{\leq 1} \tilde{F}$.

This is well defined because the full subcategory of $\text{Ind}(\text{Ban}_R)^N$ admitting such lifts is by Lemma 8.2 actually equivalent to $\text{Ind}((\text{Ban}_R^{\leq 1})^N)$. This equivalence can be realized by sending $F$ to the equivalence class $[\tilde{F}]$ in $\text{Ind}((\text{Ban}_R^{\leq 1})^N)$ of a lift $\tilde{F}$ and then we have

$$\operatorname{colim}^{\leq 1} F = \text{Ind}(\operatorname{colim}^{\leq 1} \tilde{F}).$$

Therefore, under this equivalence, we simply have

$$\operatorname{colim}^{\leq 1} F = \text{Ind}(\operatorname{colim}^{\leq 1}) : \text{Ind}((\text{Ban}_R^{\leq 1})^N) \to \text{Ind}(\text{Ban}_R)).$$

This functor $\operatorname{colim}^{\leq 1} F$ an exact functor from a full subcategory of $\text{Ind}(\text{Ban}_R)^N$ to $\text{Ind}(\text{Ban}_R)$ (takes kernels to kernels) because both the ordinary non-expanding colimit and the formal filtered colimit are exact functors. The functor we have described commutes with $V \otimes_R (-)$ for any $V \in \text{Ban}_R$.

Lemma 8.4. Consider a functor $K \times \mathbb{N} \to \text{Ban}_R^{\leq 1}$ where $K$ is a countable category. There exists a chain of isomorphisms:

$$\begin{align*}
\operatorname{colim}^{\leq 1} \lim_{i \in \mathbb{N}} V_{(k)} 
\text{ker} \big[ \big( \prod_{k \in K} \operatorname{colim}^{\leq 1} (V_{(k)}) \big)_{(k)} \big] 
\to \big( \prod_{k \in K} \lim_{i \in \mathbb{N}} (V_{(k)})_{(k)} \big)_{(k+1)} 
\end{align*}$$

(8.1)

Proof. The first and last morpisms are determined by the description of limits found in Corollary 8.13 in which they are shown to be isomorphisms. The second morpism is isomorphism is a consequence of Definition 8.3. The natural third morpism is an isomorphism because the non-expanding colimit functor is exact by Lemma 8.31 (see also Lemma 8.35).

Lemma 8.5. The natural morpism

$$\begin{align*}
\operatorname{colim}^{\leq 1} \lim_{r \in \mathbb{N}} \mathbb{Z}\{ \langle \frac{r}{x} \rangle \} 
\text{ker} \big[ \big( \prod_{r \in \mathbb{N}} \operatorname{colim}^{\leq 1} (\mathbb{Z}\{ \langle \frac{r}{x} \rangle \}) \big) \big] 
\to \big( \prod_{r \in \mathbb{N}} \lim_{i \in \mathbb{N}} (\mathbb{Z}\{ \langle \frac{r}{x} \rangle \}) \big) \big) 
\end{align*}$$

(8.2)

is an isomorphism and this object of $\text{Comm}(\text{Ind}(\text{Ban}_Z)))$ is flat over $\mathbb{Z}$.

Proof. The first statement follows immediately from Lemma 8.4 because taking a cofinal system with $r$ within a countable set, $\lim_{r \in \mathbb{N}} \mathbb{Z}\{ \langle \frac{r}{x} \rangle \}$ is $\aleph_1$-filtered by Corollary 8.13. Given any $V \in \text{Ban}_Z$ and any $F$ as in Definition 8.3 admitting a suitable lift $\tilde{F}$, then $V \otimes_Z F$ admits $V \otimes_Z \tilde{F}$ as a suitable lift and therefore, the exact functor $\operatorname{colim}^{\leq 1}$ commutes with $V \otimes Z (-)$ and hence commutes with
\[ V \otimes^L_Z (\cdot) \] as well. Hence it preserves flatness. \( \lim_{r < 1} Z\{ (\frac{1}{r})^\perp \} \) is flat because it is isomorphic to 
\( \left( \lim_{r < 1} Z\{ (\frac{1}{r})^\perp \} \right) \otimes Z \mathbb{Z} \) which is flat since \( \lim_{r < 1} Z\{ (\frac{1}{r})^\perp \} \) is flat by Corollary 6.5. So \( \colim_{l \in \mathbb{N}} \lim_{r < 1} Z\{ (\frac{1}{r})^\perp \} \) is flat. Therefore, using the isomorphism of Equation (8.2) we get that \( \lim_{r < 1} \colim_{l \in \mathbb{N}} Z\{ (\frac{1}{r})^\perp \} \) is flat as well.

**Lemma 8.6.** The object
\[ \left( \lim_{r < 1} \colim_{l \in \mathbb{N}} Z\{ (\frac{1}{r})^\perp \} \right) \otimes Z R \]
is isomorphic to
\[ \lim_{r < 1} \colim_{l \in \mathbb{N}} R\{ (\frac{1}{r})^\perp \} \]
for any Banach ring \( R \).

**Proof.** Since \( \colim_{l \in \mathbb{N}} R\{ (\frac{1}{r})^\perp \} \) and \( \lim_{r < 1} \colim_{l \in \mathbb{N}} Z\{ (\frac{1}{r})^\perp \} \) are flat over \( Z \), Lemma 7.3 gives this result immediately. \( \square \)

Notice that
\[ \left( \colim_{l \in \mathbb{N}} Z_p\{ (\frac{1}{r})^\perp \} \right) / p \left( \colim_{l \in \mathbb{N}} Z_p\{ (\frac{1}{r})^\perp \} \right) \cong \colim_{l \in \mathbb{N}} F_p\{ (\frac{1}{r})^\perp \} . \]
and \( \colim_{l \in \mathbb{N}} Z_p\{ (\frac{1}{r})^\perp \} \) is a strict \( p \)-ring. So we should show that the natural morphism
\[ \colim_{l \in \mathbb{N}} F_p\{ (\frac{1}{r})^\perp \} \rightarrow O_E \]
is an isomorphism where \( F_p \) carries the residue norm from \( Z \). This question is addressed in [\textbf{?}].

## 9. Appendix

As remarked above, most of this article has a non-archimedean version in the case that \( R \) is non-archimedean field, the standard Tate algebra representing an affinoid disk is \( k\{ \frac{1}{r_1}, \ldots, \frac{1}{r_n} \} \). In order to compare this with the “archimedean” disk algebra we used in this article which we denote the non-archimedean version by \( R\{ \frac{1}{r_1}, \ldots, \frac{1}{r_n} \}_na \). Interestingly, Stein and Dagger algebras as defined in the introduction to Section 6 constructed from these two versions of disk algebras actually agree as we show in this informal Appendix. For any \( r > 0 \) there is an injective map \( \text{Comm}(\text{CBorn}_R) \)
\[ R\{ \frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n} \} \rightarrow R\{ \frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n} \}_na \]
which by density is an epimorphism. Let \( A_i \) and \( C_i \) respectively be for any \( s < t \) we have
\[ \sum_{t} a_i s^t \leq \sum_{t} (s/t)^t \leq (\sum_{t} (s/t)^t)(\sup_{t}(a_i t^t)) \).
Therefore, we get isomorphisms \( A = \lim A_i \cong \lim C_i = C \). Similarly, this should hold for general Stein or dagger algebras as defined in the introduction to Section 6 described in the two different ways (using archimedean or non-archimedean disk algebras or their quotients) for a non-archimedean Banach ring \( R \).

Consider the category \( D \) whose objects are pairs consisting of a sequence of objects \( M_i \in \text{Mod}(A_i) \) and a collection of compatible isomorphisms \( \pi_i \) and \( M_i \rightarrow M_{i-1} \) where \( M_{i-1} \rightarrow M_i \) where morphisms are the obvious thing. Similarly there is the category \( D_{na} \) whose objects are pairs consisting of a sequence of
objects $N_i \in \text{Mod}^{\alpha}(C_i)$ and a collection of compatible isomorphisms $N_i \otimes_{C_i}^{\alpha} C_{i-1} \to N_{i-1}$ where morphisms are the obvious thing. These categories are isomorphic and if we specialize to the nuclear metrizable modules and algebras flat over $\mathbb{Z}$ we get by descent (Theorem 7.10) an equivalence of categories for these modules on $A$ and $C$.

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