Abstract
We define $KK$-theory spectra associated to $C^*$-categories and look at certain instances of the Kasparov product at this level. This machinery is used to give a description of the analytic assembly map as a natural map of spectra.

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1 Introduction
There are a number of different maps referred to as assembly maps, and in some cases several apparently different definitions exist of a given assembly
map. Such difficulties provide a motivation to explore how assembly maps can be characterised. The following result is proved in [28]: an equivariant version of the same result exists in [4].

**Theorem 1.1** Let $F$ be a homotopy-invariant functor from the category of topological spaces to the category of spectra. Then there is a strongly excisive functor, $F^\%$, and a natural transformation, $\alpha: F^\% \to F$ such that the map $\alpha: F^\%(\text{point}) \to F(\text{point})$ is a weak equivalence.

Further, the functor $F^\%$ and the natural transformation $\alpha$ are unique up to weak equivalence. The map $\alpha: F^\%(X) \to F(X)$ is called the assembly map associated to the functor $F$.

Stripped of the technical language, this result says that a given assembly map which can be expressed as a map of spectra rather than simply a group homomorphism is characterised by the fact that it maps something computable (this is what ‘excisive’ means) to the object of interest in a natural way.

It is easy to apply this philosophy to a number of assembly maps, such as the assembly map in algebraic $K$-theory first appearing in [14], the assembly map in Waldhausen $K$-theory used to study the algebraic $K$-theory assembly map in [2] and [3], the assembly map studied in [23], and the topological assembly map used to formulate the Novikov conjecture in [20]. However, there are difficulties with using this approach to attempt to deal with the analytic assembly map $\alpha^\star: K^\star(BG) \to K^\star(C^\star_{\text{max}})\pi_1(X)$ introduced to study the Novikov conjecture in [12].

The problem is that we want to view the analytic assembly map as a special case of an assembly map $\alpha: F^\%(X) \to F(X)$ in the sense of theorem [14] defined for every space $X$. Actually, it is not too hard to formulate the assembly map as a map of $K$-theory spectra

$$\alpha: \mathbb{K}_{\text{hom}}(X) \to \mathbb{K}C^\star_{\text{max}}\pi_1(X)$$

where the spectrum on the left is related to the $K$-homology of the space $X$ and the spectrum on the right is related to the $K$-theory of the group $C^\star_{\text{max}}\pi_1(X)$.

The main difficulty with this approach is that to define the fundamental group $\pi_1(X)$ we need to choose a basepoint for the topological space $X$. The assignment $X \mapsto \mathbb{K}C^\star_{\text{max}}\pi_1(X)$ is therefore not a functor on the category of topological spaces and all continuous maps, and theorem [14] no longer applies.

We can get around this difficulty by replacing the fundamental group $\pi_1(X)$ by the fundamental groupoid $\pi(X)$, and looking at the groupoid $C^\star$-category $C^\star_{\text{max}}\pi(X)$ introduced in [19]. $K$-theory spectra associated to $C^\star$-categories are defined in [15], so we can form the spectrum $\mathbb{K}C^\star_{\text{max}}\pi(X)$, and the assignment $X \mapsto \mathbb{K}C^\star_{\text{max}}\pi(X)$ is a functor on the category of topological spaces and all continuous maps. By theorem [14] we obtain an assembly map

$$\alpha: \mathbb{K}(C^\star_{\text{max}}\pi)^\%(X) \to \mathbb{K}C^\star_{\text{max}}\pi(X)$$

The main purpose of this article is to prove the following result.

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1The idea of using $C^\star$-categories to avoid basepoint issues when defining spectrum-valued assembly maps first appeared in [4].
Theorem 1.2 The map 

$$(\mathbb{K}C_{\text{max}}^* \pi)^\text{an}(X) \to \mathbb{K}C_{\text{max}}^* \pi(X)$$

is in fact the analytic assembly map.

By uniqueness of the assembly map associated to the functor $X \mapsto \mathbb{K}C_{\text{max}}^* \pi(X)$, all we need to do is show that the original assembly map can be written as a natural map of spectra

$$\alpha : \mathbb{K}_{\text{hom}}(X) \to \mathbb{K}C_{\text{max}}^* \pi(X)$$

To write down the assembly map in this way, we need to have some notion of the $KK$-theory, and in particular $KK$-theory spectra, of $C^*$-categories. The next three sections of this paper are devoted to developing the relevant features of such a theory. We conclude in the final section by showing that the analytic assembly map can be written as a natural map of spectra in the way we desire.

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2 Preliminaries

The purpose of this section is to summarise some of the concepts introduced elsewhere that are needed in the rest of this article. In subsection 2.1 we recall the definition of a $C^*$-category and in subsection 2.2 we look at some concepts involved in defining $K$-theory. In the final subsection, we prove a continuity result for the $K$-theory of $C^*$-categories that we will need later on.

2.1 $C^*$-categories

Let $\mathbb{F}$ denote either the field of real numbers or the field of complex numbers. Recall from [16] that a category $\mathcal{A}$ is termed an algebroid if every morphism set is a vector space over the field $\mathbb{F}$, and composition of morphisms

$$\text{Hom}(B,C) \times \text{Hom}(A,B) \to \text{Hom}(A,C)$$

is bilinear. An involution on an algebroid $\mathcal{A}$ is a collection of maps $\text{Hom}(A,B) \to \text{Hom}(B,A)$, written $x \mapsto x^*$, such that

- $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$ for all scalars $\alpha, \beta \in \mathbb{F}$ and morphisms $x, y \in \text{Hom}(A,B)$.
- $(yx)^* = x^* y^*$ for all morphisms $x \in \text{Hom}(A,B)$ and $y \in \text{Hom}(B,C)$
- $(x^*)^* = x$ for all morphisms $x \in \text{Hom}(A,B)$

We call an algebroid with involution a $*$-category. The definition of the objects on which we focus our attention in this article comes from [5] and [19].
Definition 2.1 A unital $C^*$-category is a $\ast$-category in which every morphism set $\text{Hom}(A, B)$ is a Banach space, and

- The inequality $\|xy\| \leq \|x\|\|y\|$ holds for all morphisms $x \in \text{Hom}(B, C)$ and $y \in \text{Hom}(A, B)$.
- The $C^*$-identity, $\|x^*x\| = \|x\|^2$, holds for all morphisms $x \in \text{Hom}(A, B)$.
- For every morphism $x \in \text{Hom}(A, B)$, the product $x^*x$ is a positive element of the $C^*$-algebra $\text{Hom}(A, A)$.

A non-unital $C^*$-category is a collection of objects and morphisms similar to a $C^*$-category except that there need not exist identity morphisms $1 \in \text{Hom}(A, A)$.

Note that a $C^*$-algebra can be considered to be a $C^*$-category with only one object.

We can form the category of all small unital $C^*$-categories. The morphisms are $C^*$-functors, that is to say functors $F: \mathcal{A} \to \mathcal{B}$ between $C^*$-categories such that each map $F: \text{Hom}(A, B) \to \text{Hom}(F(A), F(B))$ is linear, and $F(x^*) = F(x)^*$ for all morphisms $x$ in the category $\mathcal{A}$. We can similarly form the category of all small non-unital $C^*$-categories. Of course, in the definition of a $C^*$-functor it is not necessary to assume that the $C^*$-categories involved are small.

It is proved in [19] that any $C^*$-functor is automatically norm-decreasing, and therefore continuous. Further, the range of any $C^*$-functor is closed.

Definition 2.2 A $C^*$-category $\mathcal{A}$ is said to be graded if we can write each morphism set $\text{Hom}(A, B)$ as a direct sum

$$\text{Hom}(A, B) = \text{Hom}(A, B)_0 \oplus \text{Hom}(A, B)_1$$

of morphisms of degree 0 and degree 1 such that for composable morphisms $x$ and $y$ we have the formula

$$\deg(xy) = \deg(x) + \deg(y)$$

Here addition takes place modulo 2.

A $C^*$-functor $F: \mathcal{A} \to \mathcal{B}$ between graded $C^*$-categories is termed a graded $C^*$-functor if

$$\deg(Fx) = \deg(x)$$

for every morphism $x$ in the category $\mathcal{A}$.

We can consider an ungraded $C^*$-category to be equipped with the trivial grading defined by saying that every morphism is of degree 0. Our attitude is thus to view ungraded $C^*$-categories as special cases of graded $C^*$-categories.

There is a sensible notion of the spatial tensor product, $\mathcal{A} \hat{\otimes} \mathcal{B}$, of graded $C^*$-categories $\mathcal{A}$ and $\mathcal{B}$. The objects are pairs, written $A \otimes B$, for objects $A \in \text{Ob}(\mathcal{A})$ and $B \in \text{Ob}(\mathcal{B})$. The morphism set $\text{Hom}(A \otimes B, A' \otimes B')$ is a completion of the algebraic graded tensor product $\text{Hom}(A, A') \hat{\otimes} \text{Hom}(B, B')$. See section 7 of [19] and definition 2.7 of [18] for details.

Recall that a category is said to be small if the class of all objects forms a set. It is sometimes necessary to restrict our attention to small categories in order to avoid set-theoretic difficulties.
2.2 $K$-theory of $C^*$-categories

Before we are ready to look at the $K$-theory of $C^*$-categories, we need the following definition.

**Definition 2.3** Let $\mathcal{A}$ be a small $C^*$-category. Then we define the *additive completion*, $\mathcal{A}_\oplus$, to be the $*$-category in which the objects are formal sums $A_1 \oplus \cdots \oplus A_m$ where $A_i \in \text{Ob}(\mathcal{A})$. The morphism set $\text{Hom}(A_1 \oplus \cdots \oplus A_m, B_1 \oplus \cdots \oplus B_n)$ is the set of matrices

$$\left\{ \begin{pmatrix} x_{1,1} & \cdots & x_{1,m} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,m} \end{pmatrix} | x_{ij} \in \text{Hom}(A_j, B_i) \right\}$$

Composition of morphisms is defined by matrix multiplication. The involution is defined by the formula

$$\left( \begin{pmatrix} x_{1,1} & \cdots & x_{1,m} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,m} \end{pmatrix} \right)^* = \begin{pmatrix} x^*_{1,1} & \cdots & x^*_{n,1} \\ \vdots & \ddots & \vdots \\ x^*_{1,m} & \cdots & x^*_{n,m} \end{pmatrix}$$

For any small $C^*$-category $\mathcal{A}$ there is an isometric $C^*$-functor into the category of all Hilbert spaces and bounded linear maps; see theorem 1.14 of [3] and theorem 5.2 of [19]. Such a $C^*$-functor extends to a functor on the additive completion, $\mathcal{A}_\oplus$, and can be used to define a $C^*$-norm on the category $\mathcal{A}_\oplus$. If the $C^*$-category $\mathcal{A}$ is graded then the additive completion $\mathcal{A}_\oplus$ is graded in the obvious way.

**Definition 2.4** Let $\mathcal{A}$ be a small graded unital $C^*$-category. Let $A \in \text{Ob}(\mathcal{A}_\oplus)$. Then we define the space of *supersymmetries*

$$\text{SS}(A) = \{ x \in \text{Hom}(A, A)_1 | x = x^*, \ x^2 = 1 \}$$

Given supersymmetries $x \in \text{SS}(A)$ and $y \in \text{SS}(B)$ we can form a supersymmetry

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in \text{SS}(A \oplus B)$$

Suppose that the spaces of supersymmetries $\text{SS}(A)$ are all non-empty. Then we define $V_1(\mathcal{A})$ to be the set

$$\bigsqcup_{A \in \text{Ob}(\mathcal{A}_\oplus)} \pi_0 \text{SS}(A)$$

Here we write $\langle x \rangle$ to denote the path-component of a supersymmetry $x \in \text{SS}(A)$. The set $V_1(\mathcal{A})$ is an Abelian semigroup with operation defined by the formula

$$\langle x \rangle + \langle y \rangle = \langle x \oplus y \rangle$$
**Definition 2.5** Assume that the spaces of supersymmetries $SS(A)$ are all non-empty. Then we define the group $G_1(A)$ to be the Grothendieck completion of the semigroup $V_1(A)$. We define the $K$-theory group $K_1(A)$ to be the set of formal differences

$$\{ (x) - (y) \in G_1(A) \mid x, y \in SS(A), A \in Ob(A_{\bar{}}) \}$$

When $A$ is a graded $C^*$-algebra we recover from the above definition the $K$-theory group $K_1(A)$ defined in [24].

The main construction in [18] is that of a functor, $\mathbb{K}$, from the category of all small graded $C^*$-categories to the category of symmetric spectra. When $A$ is a small unital graded $C^*$-category in which every space $SS(A)$ is non-empty there is a natural isomorphism

$$K_1(A) \cong \pi_1 K(A)$$

For all other integers $n \in \mathbb{Z}$ we define the $K$-theory group $K_n(A)$ to be the stable homotopy group $\pi_n K(A)$.

The main elementary theorem in the $K$-theory of $C^*$-algebras is the Bott periodicity theorem. A version of the Bott periodicity theorem is true for the $K$-theory of graded $C^*$-categories. We introduce it by first considering Clifford algebras, as was done for graded $C^*$-algebras in [23]. See also [4, 10, 11], and [29] for details on this approach to Bott periodicity.

**Definition 2.6** Let $p$ and $q$ be natural numbers. Then we define the $(p,q)$-Clifford algebra, $\mathbb{F}_{p,q}$, to be the algebra over the field $\mathbb{F}$ generated by elements

$$\{ e_1, \ldots, e_p, f_1, \ldots, f_q \}$$

that pairwise anti-commute and satisfy the formulae

$$e_i^2 = 1 \quad f_j^2 = -1$$

The Clifford algebra $\mathbb{F}_{p,q}$ is a graded $C^*$-algebra; the generators themselves are defined to be of degree 1. The following result, proved in [18], is the version of the Bott periodicity theorem that is most useful in the framework of this article.

**Theorem 2.7** Let $A$ be a small graded $C^*$-category. Then there is a natural stable equivalence of spectra

$$\Omega^p \mathbb{K}(A) \simeq \Omega^p \mathbb{K}(A \otimes \mathbb{F}_{p,q})$$

We therefore have isomorphisms

$$K_{1-(p-q)}(A) = K_1(A \otimes \mathbb{F}_{p,q})$$

---

3See [4] for details of the theory of symmetric spectra. The main feature of the theory for our purposes is that there is a well-behaved smash product in the category of symmetric spectra.
The above formulation of the Bott periodicity theorem enables us to define the $K$-theory group $K_1(A)$ without the assumption that the spaces of supersymmetries $SS(A)$ are all non-empty. To do this, observe that the $C^*$-categories $\mathcal{A}$ and $\mathcal{A} \widehat{\otimes} F_{1,1}$ have the same $K$-theory, and the spaces of supersymmetries in the $C^*$-category $\mathcal{A} \widehat{\otimes} F_{1,1}$ are automatically non-empty. In particular, the $K$-theory of ungraded $C^*$-categories can be defined in this way. The usual $K$-theory groups of undgraded $C^*$-algebras are recovered as a special case.

**Definition 2.8** Let $F, G : \mathcal{A} \to \mathcal{B}$ be graded $C^*$-functors between unital graded $C^*$-categories. Then a natural isomorphism between $F$ and $G$ consists of a degree 0 unitary morphism $U \in Hom(F(A), G(A))_B$ for each object $A \in Ob(\mathcal{A})$ such that for every morphism $x \in Hom(A, B)_A$ the composites $U_B F(x)$ and $F(x) U_A$ are equal.

A graded $C^*$-functor $F : \mathcal{A} \to \mathcal{B}$ between unital $C^*$-categories is said to be an equivalence of graded $C^*$-categories if there is a graded $C^*$-functor $G : \mathcal{B} \to \mathcal{A}$ such that the composites $FG$ and $GF$ are naturally isomorphic to the identities $1_B$ and $1_A$ respectively.

The following result is proved in [18]

**Proposition 2.9** Let $F : \mathcal{A} \to \mathcal{B}$ be an equivalence of small unital graded $C^*$-categories. Then the induced map $F^* : K_n(\mathcal{A}) \to K_n(\mathcal{B})$ is a stable equivalence of $K$-theory spectra. $\blacksquare$

In particular, if a small graded unital $C^*$-category is equivalent to a $C^*$-algebra, it has the same $K$-theory.

### 2.3 Continuity

In this subsection we investigate ‘continuity’ properties of the $K$-theory groups $K_n(\mathcal{A})$, that is to say we investigate what happens when we look at direct limits.

We begin by looking at a ‘lifting’ result for supersymmetries.

**Lemma 2.10** Let $(A_i, \phi_{ij})$ be a directed family of graded unital $C^*$-algebras, with direct limit the graded $C^*$-algebra $A_\infty$, equipped with morphisms $\phi_i : A_i \to A_\infty$.

Let $x \in SS(A_\infty)$, $\varepsilon > 0$. Then there exists a $C^*$-algebra $A_j$ and a point $y \in SS(A_j)$ such that $\|\phi_j(y) - x\| < \varepsilon$

**Proof:** It is a well-known fact from the theory of direct limits of $C^*$-algebras that the union $\bigcup_{i \in I} \phi_i(A_i)$ must be a dense subset of the $C^*$-algebra $A_\infty$; see for example [26], appendix L for details. Choose a real number $\varepsilon_1 > 0$. Then there exists an element $a \in A_i$ such that $\|\phi_i(a) - x\| < \varepsilon_1$.

Set

$$b = \frac{a + a^*}{2}$$

Then the element $b$ is self-adjoint, and since $\phi_i$ is a morphism of $C^*$-algebras and $x$ is self-adjoint, we can see that $\|\phi_i(b) - x\| < \varepsilon_1$. A similar trick gives us a degree 1 self-adjoint element $c \in A_i$ such that $\|\phi_i(c) - x\| < \varepsilon_1$.
Now, without loss of generality, assume that $\varepsilon_1 < \frac{1}{3}$. Observe
\[
\|\phi_i(c)^2 - 1\| \leq \|\phi_i(c)^2 - \phi_i(c) x\| + \|\phi_i(c) x - x^2\|
\leq \|\phi_i(c)||\phi_i(c) - x\| + \|x||\phi_i(c) - x\| 
\leq (\varepsilon_1 + \varepsilon)\|\phi_i(c) - x\| 
\leq (\varepsilon_1 + 2\varepsilon_1)^2
\]

Define $\varepsilon_2 = \varepsilon_1(2 + \varepsilon_1)$. Then $\varepsilon_1 < \frac{1}{3}$ so $\varepsilon_2 < 1$. Therefore $\|\phi_i(c)^2 - 1\| < 1$. Hence $\|\phi_i(c)^2 - 1\| < 1$. But by definition of the norm on the direct limit $A_\infty$
\[
\|\phi_i(c)^2 - 1\| = \limsup\{\|\phi_j(c)^2 - 1\| : j \geq i}\}
\]

So there exists an element $d = \phi_{ji}(c) \in A_j$ such that $\|\phi_j(d) - x\| < \varepsilon_1$ and $\|d^2 - 1\| < \varepsilon_2$.

Since the element $d$ is self-adjoint, we have spectrum
\[
\sigma(d) \subseteq (-1 - \varepsilon_3, -1 + \varepsilon_3) \cup (1 - \varepsilon_3, 1 + \varepsilon_3)
\]
where $\varepsilon_3 = \varepsilon_2^{1/2}$. Certainly, $\varepsilon_3 < 1$ so we may define a continuous function $f : \sigma(d) \to \mathbb{R}$ by writing
\[
f(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t \leq 0 \end{cases}
\]

Hence, by functional calculus there is an self-adjoint element $y = f(d) \in A_j$ such that $y^2 = 1$. By the formula expressing the norm of a self-adjoint element of a $C^*$-algebra in terms of its spectrum (see for example chapter 12 of [21] for details), we have the inequality
\[
\|y - d\| = \|f(d) - d\| 
\leq \sup_{t \in (-1 - \varepsilon_3, -1 + \varepsilon_3) \cup (1 - \varepsilon_3, 1 + \varepsilon_3)} |f(t) - t| 
< \varepsilon_3
\]

To summarise, $y \in SS(A_j)$ and
\[
\|\phi_j(y) - x\| \leq \|\phi_j\||y - c\| + \|\phi_j(c) - x\| 
< \varepsilon_3 + \varepsilon_1
\]

Recall that $\varepsilon_3 = (\varepsilon_1(2 + \varepsilon_1))^{1/2}$. Given $\varepsilon > 0$, we can easily find a real number $\varepsilon_1 > 0$ such that $\varepsilon_3 + \varepsilon_1 < \varepsilon$. Hence there is a point $y \in SS(A_j)$ such that $\|\phi_j(y) - x\| < \varepsilon$.

It is possible to show that direct limits always exist in the category of small graded $C^*$-categories. The proof very closely follows the proof of the existence of direct limits in the category of $C^*$-algebras. The interested reader can consult [17] for details.

**Theorem 2.11** Let $(A_i, \phi_{ij})$ be a directed family of small unital graded $C^*$-categories with direct limit $A_\infty$. Then the family of groups $(K_1(A_i), \phi_{ij*})$ has direct limit $K_1(A_\infty)$. 

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Proof: Observe that the additive completion \((A_\infty)_\oplus\) is the direct limit of the family of additive completions \(\((A_i)_\oplus, \phi_{ij}\)\). Let the C*-category \(A_\infty\) be equipped with C*-functors \(\phi_i : A_i \rightarrow A_\infty\) such that \(\phi_j \phi_{ij} = \phi_i\) whenever \(i \leq j\). Without loss of generality, assume that for all objects \(A_i \in Ob(A_i)\) the set of supersymmetries \(SS(A_i)\) is not empty.

Consider an object \(A \in Ob((A_\infty)_\oplus)\). By the universal property of direct limits we can find an object \(A_i \in Ob((A_i)_\oplus)\) such that \(\phi_i(A_i) = A\). Whenever \(i \leq j\) define an object \(A_j = \phi_{ij}(A_i)\). Then we have a directed family of C*-algebras \((Hom(A_j, A_j), \phi_{ij})_{j \geq i}\) with direct limit \(Hom(A, A)\).

Recall from proposition 1.11 of [13] that if \(\phi\) is a non-unital C*-category then there are unital C*-categories \(A^+\) and \(\mathbb{F}_A\) together with a natural quotient C*-functor \(\pi : A^+ \rightarrow \mathbb{F}_A\). The K-theory group \(K_1(A)\) can be defined to be the kernel

\[
K_1(A) = \ker(\pi_* : K_1(A^+) \rightarrow K_1(\mathbb{F}_A))
\]

See section 2 of [12] and proposition 2.26 of [13] for further details.

**Corollary 2.12** Let \((A_i, \phi_{ij})\) be a directed family of non-unital graded C*-categories with direct limit \(A_\infty\). Then the directed family of groups \((K_1(A_i), \phi_{ij}^\ast)\) has direct limit \(K_1(A_\infty)\).

Proof: We have short exact sequences of groups

\[
0 \rightarrow K_1(A_i) \rightarrow K_1(A_i^+) \rightarrow K_1(\mathbb{F}_A_i) \rightarrow 0
\]

and the maps \(\phi_{ij}\) induce commutative diagrams

\[
\begin{array}{ccc}
K_1(A_i) & \rightarrow & K_1(A_i^+) \\
\downarrow & & \downarrow \\
K_1(A_j) & \rightarrow & K_1(A_j^+)
\end{array}
\]

\[
\begin{array}{ccc}
& & K_1(\mathbb{F}_A_i) \\
\downarrow & & \downarrow \\
& & K_1(\mathbb{F}_A_j)
\end{array}
\]

By the above theorem we know that the families of groups \((K_1(A_i^+))\) and \((K_1(\mathbb{F}_A_i))\) have direct limits \(K_1(A_\infty^+)\) and \(K_1(\mathbb{F}_A_\infty)\) respectively. Let \(\hat{G}\) be the direct limit of the family of groups \((K_1(A_i))\).

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\(^4\)As we have already mentioned, we can, if necessary, replace each C*-category \(A_i\) by the tensor product \(A_i \otimes \mathbb{F}_{1,1}\) without affecting the K-theory.
It is a well-known result of homological algebra (see for example [22] or [27]) that the direct limit of a family of short exact sequences is a short exact sequence. Thus we have a short exact sequence

\[ 0 \to G \to K_1(A_{\infty}^+) \to K_1(F_{\cdot A_{\infty}}) \to 0 \]

By definition of the group \( K_1(A_{\infty}) \) it follows that \( G \cong K_1(A_{\infty}) \).

\[ \square \]

**Corollary 2.13** Let \((A_i, \phi_{ij})\) be a directed family of graded \(C^*\)-categories with direct limit \( A_{\infty} \). Then the directed family of groups \((K_n(A_i), \phi_{ij}^*)\) has direct limit \( K_n(A_{\infty}) \).

**Proof:** Observe that the family \((A_i \hat{\otimes} F_{p,q}, \phi_{ij} \otimes 1)\) has direct limit \( A_{\infty} \hat{\otimes} F_{p,q} \). The desired conclusion now follows from the above result and the Bott periodicity theorem.

\[ \square \]

As an application of the above continuity theorem we prove a stability result that will be important to us later on.

**Definition 2.14** We define the algebra of compact operators, \( K \), to be the \( C^* \)-algebra direct limit of matrix algebras

\[ K = \lim_k M_k(F) \]

under the inclusions \( M_k(F) \to M_{k+1}(F) \) defined by the formula

\[ x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \]

We equip the \( C^* \)-algebra \( K \) with the trivial grading.

The following result is easy to see.

**Lemma 2.15** Let \( A \) be any \( C^* \)-category. Then

\[ A \hat{\otimes} K = \lim_k (A \hat{\otimes} M_k(F)) \]

\[ \square \]

Let us write \( p \) for the element of the \( C^* \)-algebra \( K \) arising from the element \( 1 \in M_1(F) \). Then our previous machinery gives us the following result.

**Theorem 2.16** The \( C^* \)-functor \( A \to A \hat{\otimes} K \) defined by writing \( a \mapsto a \hat{\otimes} p \) induces an isomorphism

\[ K_n(A) \cong K_n(A \hat{\otimes} K) \]

at the level of \( K \)-theory groups.

**Proof:** It is easy to check that each induced map \( K_n(M_k(A)) \to K_n(M_{k+1}(A)) \) is an isomorphism. The result now follows by corollary 2.13 and lemma 2.15.

\[ \square \]

The \( C^* \)-functor \( A \to A \hat{\otimes} K \) therefore induces a stable equivalence of symmetric spectra \( \mathbb{K}(A) \to \mathbb{K}(A \hat{\otimes} K) \)

\(^5\)Or more generally for any rank 1 projection in the \( C^* \)-algebra \( K \)
3 Hilbert modules over $C^*$-categories

The $KK$-theory of $C^*$-algebras is defined in terms of Hilbert modules. The same is true for the $KK$-theory of $C^*$-categories. We look at the relevant concepts in this section.

3.1 Hilbert Modules

Let $\mathcal{A}$ be a $C^*$-category. Then a right $\mathcal{A}$-module is a linear contravariant functor, $\mathcal{E}$, from the category $\mathcal{A}$ to the category of vector spaces. We use the notation $\eta x = \mathcal{E}(x)\mathcal{E}(\eta)$ to denote the action of a morphism $x \in \text{Hom}(A,B)_\mathcal{A}$ on a vector $\eta \in \mathcal{E}(A)$. It is similarly possible to define left $\mathcal{A}$-modules.

A right $\mathcal{A}$-module, $\mathcal{E}$, is said to be countably generated if there is a countable set $\Omega \subseteq \bigcup_{A \in \text{Ob}(\mathcal{A})} \mathcal{E}(A)$ such that for each object $A \in \text{Ob}(\mathcal{A})$, every element of the vector space $\mathcal{E}(A)$ is a finite linear combination of elements of the form $\eta x$, where $x \in \text{Hom}(A,B)$ and $\eta \in \Omega \cap \mathcal{E}(B)$.

Recall from [19] that a semi-inner product on a right $\mathcal{A}$-module $\mathcal{E}$ is a collection of maps $\langle \cdot, \cdot \rangle : \mathcal{E}(B) \times \mathcal{E}(A) \to \text{Hom}(A,B)_\mathcal{A}$ such that

- For all vectors $\eta \in \mathcal{E}(B)$, $\xi, \zeta \in \mathcal{E}(C)$, and morphisms $x, y \in \text{Hom}(A,C)$ we have the formula $\langle \eta, \xi x + \zeta y \rangle = \langle \eta, \xi \rangle x + \langle \eta, \zeta \rangle y$

- $\langle \eta, \xi \rangle = \langle \xi, \eta \rangle^*$

- For any vector $\eta \in \mathcal{E}(A)$ the product $\langle \eta, \eta \rangle$ is a positive element of the $C^*$-algebra $\text{Hom}(A,A)$.

A semi-inner product is called an inner product if for any vector $\eta \in \mathcal{E}(A)$ the product $\langle \eta, \eta \rangle$ is zero if and only if $\eta = 0$. A Hilbert $\mathcal{A}$-module is a right $\mathcal{A}$-module, $\mathcal{E}$, equipped with an inner product such that each space $\mathcal{E}(A)$ is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$

**Definition 3.1** We call a Hilbert $\mathcal{A}$-module $\mathcal{E}$ countably generated if there is a countably generated right $\mathcal{A}$-module $\mathcal{E}_0$ such that the space $\mathcal{E}_0(A)$ is a dense subset of the space $\mathcal{E}(A)$ for every object $A \in \text{Ob}(\mathcal{B})$.

The countable set $\Omega$ which generates the right $\mathcal{A}$-module $\mathcal{E}_0$ is referred to as a generating set for the Hilbert $\mathcal{A}$-module $\mathcal{E}$.

The above definition of a countably generated differs from the definition given in [19] but agrees with that of [9]. If the $C^*$-category $\mathcal{A}$ is unital and equivalent to a $C^*$-algebra, the above definition is the same as that in [19].
Example 3.2 Let $\mathcal{A}$ be a $C^*$-category. Then for each object $A \in Ob(\mathcal{A})$ we have a Hilbert $\mathcal{A}$-module $\text{Hom}(-, A)_{\mathcal{A}}$ with spaces $\text{Hom}(C, A)_{\mathcal{A}}$. The $\mathcal{A}$-action is defined by composition of morphisms, and the inner product is defined by the formula

$$\langle x, y \rangle = x^* y$$

Example 3.3 Let $\mathcal{A}$ be a $C^*$-category. For each pair of objects $C, A \in Ob(\mathcal{A})$, let us write $\mathcal{H}(C, A)_{\mathcal{A}}$ to denote the space of sequences, $(x_n)$, such that each element $x_n$ belongs to the morphism set $\text{Hom}(C, A)_{\mathcal{A}}$, and the series

$$\sum_n x_n^* x_n$$

converges in norm in the space $\text{Hom}(C, C)_{\mathcal{A}}$.

Then we have a Hilbert $\mathcal{A}$-module $\mathcal{H}(-, A)_{\mathcal{A}}$ defined by associating the space $\mathcal{H}(C, A)_{\mathcal{A}}$ to the object $C$. The action of the $C^*$-category $\mathcal{A}$ is defined by the formula

$$(x_n) y = (x_n y)$$

The inner product is defined by the formula

$$\langle (x_n), (y_n) \rangle = \sum_n x_n^* y_n$$

We can show that the inner product is well-defined by using the Cauchy-Schwarz inequality.

We need to have a notion of the direct sum of a collection of Hilbert $\mathcal{A}$-modules.

Definition 3.4 Let $\{\mathcal{E}_\lambda \mid \lambda \in \Lambda\}$ be a countable collection of Hilbert $\mathcal{A}$-modules. Write

$$\prod_{\lambda \in \Lambda} \mathcal{E}_\lambda(\mathcal{A})$$

to denote the space of sequences of vectors $(\eta_\lambda)$ such that $\eta_\lambda \in \mathcal{E}_\lambda(\mathcal{A})$ and the series

$$\sum_{\lambda \in \Lambda} \langle \eta_\lambda, \eta_\lambda \rangle$$

converges in the $C^*$-algebra $\text{Hom}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$.

Then we define the direct sum

$$\prod_{\lambda \in \Lambda} \mathcal{E}_\lambda$$

to be the Hilbert $\mathcal{A}$-module associating the vector space $\prod_{\lambda \in \Lambda} \mathcal{E}_\lambda(\mathcal{A})$ to the object $A \in Ob(\mathcal{A})$.

The action of the category $\mathcal{A}$ on the direct sum $\prod_{\lambda \in \Lambda} \mathcal{E}_\lambda$ is defined in the obvious way. The inner product is defined by the formula

$$\langle (\eta_\lambda), (\xi_\lambda) \rangle = \sum_{\lambda \in \Lambda} \langle \eta_\lambda, \xi_\lambda \rangle$$
In particular, one can form the direct sum, $E \oplus E'$ of two Hilbert $\mathcal{A}$-modules $E$ and $E'$. If $\{E_\lambda \mid \lambda \in \Lambda\}$ is a countable collection of countably generated Hilbert $\mathcal{A}$-modules, the direct sum $\prod_{\lambda \in \Lambda} E_\lambda(A)$ is countably generated.

Recall that a $C^*$-algebra $\mathcal{A}$ is said to be $\sigma$-unital if there is a countable approximate unit, that is a countable set 
\[ \{e_\lambda \mid \lambda \in \Lambda\} \]
such that
\[ \inf_{\lambda \in \Lambda} \|xe_\lambda - x\| = 0 \quad \inf_{\lambda \in \Lambda} \|e_\lambda x - x\| = 0 \]

**Definition 3.5** A $C^*$-category $\mathcal{A}$ is called $\sigma$-unital if each endomorphism set $\text{Hom}(A, A)_\mathcal{A}$ is a $\sigma$-unital $C^*$-algebra.

If $\mathcal{A}$ is a $\sigma$-unital $C^*$-category, the Hilbert $\mathcal{B}$-modules $\text{Hom}(\cdot, A)_\mathcal{A}$ and $\mathcal{H}(\cdot, A)_\mathcal{A}$ are countably generated.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-categories, that $E$ is a Hilbert $\mathcal{A}$-module, and that $\mathcal{F}$ is a Hilbert $\mathcal{B}$-module. Then we define the outer tensor product, $E \otimes \mathcal{F}$, to be the Hilbert $\mathcal{A} \otimes \mathcal{B}$-module in which the space $(E \otimes \mathcal{F})(A \otimes B)$ is the completion of the algebraic tensor product $E(A) \otimes \mathcal{F}(B)$ with respect to the norm defined by the inner product
\[ \langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle = \langle \eta_1, \eta_2 \rangle \otimes \langle \xi_1, \xi_2 \rangle \]

It is proved in [19] that the outer tensor product $E \otimes \mathcal{F}$ is a Hilbert $\mathcal{A} \otimes \mathcal{B}$-module.

**Definition 3.6** Let $\mathcal{A}$ be a graded $C^*$-category. Then a Hilbert $\mathcal{A}$-module $E$ is called graded if it admits decompositions $E(A) = E(A)_0 \oplus E(A)_1$ into vectors of degree 0 and vectors of degree 1 such that

- $\deg(\eta x) = \deg(\eta) + \deg(x)$ for all vectors $\eta \in E(B)$ and morphisms $x \in \text{Hom}(A, B)_\mathcal{A}$.
- $\deg(\langle \eta, \xi \rangle) = \deg(\eta) + \deg(\xi)$ for all vectors $\eta \in E(B)$ and $\xi \in E(A)$.

Here all addition takes place modulo 2.

The direct sum and outer tensor product of graded Hilbert modules can be graded in the obvious way. If $\mathcal{A}$ is a graded $C^*$-category, the Hilbert $\mathcal{A}$-modules $\text{Hom}(\cdot, A)_\mathcal{A}$ and $\mathcal{H}(\cdot, A)_\mathcal{A}$ inherit a grading from the $C^*$-category $\mathcal{A}$.

Let $\mathcal{A}$ be a graded $C^*$-category, and let $E$ and $\mathcal{F}$ be Hilbert $\mathcal{A}$-modules. Then an operator $T: E \to \mathcal{F}$ is a collection of maps $T_A: E(A) \to \mathcal{F}(A)$ such that there exist maps $T_B: \mathcal{F}(A) \to E(A)$ satisfying the formula
\[ \langle \eta, T_A \xi \rangle = \langle T_B^* \eta, \xi \rangle \]
for all vectors $\eta \in \mathcal{F}(B)$ and $\xi \in E(A)$.

If $T: E \to \mathcal{F}$ is an operator it can be shown (see [19]) that

- $T_A(\eta x + \xi y) = (T_B \eta)x + (T_B \xi)y$ for all vectors $\eta, \xi \in E(B)$ and all morphisms $x, y \in \text{Hom}(A, B)$. In particular, the operator $T$ is natural.
• Each map $T_A : \mathcal{E}(A) \to \mathcal{F}(A)$ is linear and continuous.

• The collection of maps $T_A^*$ is uniquely determined by the operator $T$, and defines an operator $T^*$ such that $(T^*)^* = T$.

Although each map $T_A : \mathcal{E}(A) \to \mathcal{E}'(A)$ is continuous, the norm\[ \|T\| = \sup\{\|T_A\| \mid A \in \text{Ob}(A)\} \]
need not be finite. When the norm $\|T\|$ is finite we call the operator $T$ bounded. We write $L(\mathcal{E}, \mathcal{F})$ to denote the space of all bounded operators $T : \mathcal{E} \to \mathcal{F}$.

Let $L(A)$ be the category of all countably generated graded Hilbert $A$-modules and bounded operators between them. It is proved in [19] that the category $L(A)$ is a $C^*$-category. It can be graded by defining the degree of a bounded operator $T : \mathcal{E} \to \mathcal{E}'$ to be 0 if $\deg(T\eta) = \deg(\eta)$ for all vectors $\eta \in \mathcal{E}(A)$ and to be 1 if $\deg(T\eta) = \deg(\eta) + 1$ for all vectors $\eta \in \mathcal{E}(A)$.

**Definition 3.7** Let $A$ be a $C^*$-category, and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert $A$-modules. Then a rank one operator $T : \mathcal{E} \to \mathcal{F}$ is an operator of the form\[ \zeta \mapsto \eta \langle \xi, \zeta \rangle \]
for elements $\eta \in \mathcal{F}$ and $\xi \in \mathcal{E}$. We write this operator $\eta(\xi, -)$. A compact operator is a norm-limit of finite linear combinations of rank one operators.

The above definition of a compact operator differs from the definition given in [19], but agrees with that of [8]. If the $C^*$-category $A$ is unital and equivalent to a $C^*$-algebra, the above definition is the same as that in [19].

We write $K(\mathcal{E}, \mathcal{E}')$ to denote the space of compact operators from a graded Hilbert $A$-module $\mathcal{E}$ to a graded Hilbert $A$-module $\mathcal{E}'$. We write $K(A)$ to denote the collection of all countably generated graded Hilbert $A$-modules and compact operators. Recall from [3] that a $C^*$-ideal, $\mathcal{J}$, in a $C^*$-category $A$ is a $C^*$-subcategory such that the composite of a morphism in the category $\mathcal{J}$ and a morphism in the category $A$ belongs to the category $\mathcal{J}$. We can form the quotient, $A/\mathcal{J}$, of a $C^*$-category by a $C^*$-ideal.

**Proposition 3.8** The collection $K(A)$ is a $C^*$-ideal in the $C^*$-category $L(A)$.

**Proof:** It is clear that each space $K(\mathcal{E}, \mathcal{E}')$ is a closed subspace of the Banach space $L(\mathcal{E}, \mathcal{E}')$. It is easy to see that the composition of a finite rank operator and a bounded operator is of finite rank. Hence the composition of a compact operator with any other bounded operator is compact. \[\square\]

We write $Q(A)$ to stand for the quotient $L(A)/K(A)$.

The final result we need in this section is a version of the Kasparov stabilization theorem for Hilbert modules over $C^*$-categories. It is more general than the version proved in [19]; such a generalization is needed to perform some of the constructions in $KK$-theory to be found in this article.
Theorem 3.9 Let $A$ be a $\sigma$-unital $C^*$-category, and let $E$ be a countably generated Hilbert $A$-module. Then there is a countable collection, $\Gamma$, of objects of the $C^*$-category such that we have an isomorphism of Hilbert $A$-modules

\[ E \oplus \left( \prod_{A \in \Gamma} \mathcal{H}(-, A)_A \right) \cong \prod_{A \in \Gamma} \mathcal{H}(-, A)_A \]

Before launching ourselves into a proof of this result, we need a lemma. To formulate the lemma, let $E$ and $E'$ be Hilbert $B$-modules. Let us say an operator $T \in \mathcal{L}(E, E')$ has dense range if for each object $A \in \text{Ob}(A)$ the set

\[ \{ T\eta \mid \eta \in \mathcal{E}(A) \} \]

is a dense subset of the space $\mathcal{E}'(A)$.

Lemma 3.10 Let $E$ and $E'$ be Hilbert $A$-modules. Suppose we have an operator $T \in \mathcal{L}(E, E')$ such that the operator $T$ and its adjoint $T^*$ both have dense range. Then the Hilbert $A$-modules $E$ and $E'$ are isomorphic.

Proof: The desired conclusion follows immediately from the corresponding result for Hilbert modules over $C^*$-algebras. For a proof of the result for Hilbert modules over $C^*$-algebras, see for example proposition 3.8 of [13].

Our proof now parallels the proof of the Kasparov stabilisation theorem for Hilbert modules over $C^*$-algebras given in [15].

Proof of theorem 3.9: Let

\[ \Omega \subseteq \bigcup_{A \in \text{Ob}(A)} \mathcal{E}(A) \]

be a countable generating set for the Hilbert $A$-module $E$. Write

\[ \Gamma = \{ A \in \text{Ob}(A) \mid \mathcal{E}(A) \cap \Omega \neq \emptyset \} \]

and let $(\eta_n)$ be a sequence of vectors in the set $\Omega$ such that every element of $\Omega$ occurs infinitely often.

Since the $C^*$-category $A$ is $\sigma$-unital we can construct a sequence, $(e_n)$, of unit vectors in the union $\cup_{A \in \Gamma} \mathcal{H}(A, A)_A$ such that:

- If $\eta_n \in \mathcal{E}(A)$, then $e_n \in \mathcal{H}(A, A)_B$
- The set
  \[ \{ e_n \mid \eta_n \in \mathcal{E}(A) \} \]
  is an orthonormal basis for the space $\mathcal{H}(A, A)$.

Observe that the set of vectors $e_n$ may be considered a generating set for the Hilbert $A$-module

\[ \prod_{A \in \Gamma} \mathcal{H}(-, A)_A \]
The Hilbert $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{H}(-, A)_\mathcal{A}$ can be regarded as submodules of the direct sum

$$\mathcal{E} \oplus \left( \prod_{A \in \Gamma} \mathcal{H}(-, A)_\mathcal{A} \right)$$

in the obvious way. We can define a compact operator $T \in \mathcal{K}(\prod_{A \in \Gamma} \mathcal{H}(-, A)_\mathcal{A}, \mathcal{E} \oplus (\prod_{A \in \Gamma} \mathcal{H}(-, A)_\mathcal{A}))$ by the formula

$$T\eta = \sum_{n=1}^{\infty} \left( 2^{-n} \eta_n (e_n, \eta) + 4^{-n} e_n (e_n, \eta) \right)$$

In particular:

$$Te_n = 2^{-n} \eta_n + 4^{-n} e_n$$

For every natural number $m$ such that $\eta_m = \eta_n$ we have the formula

$$T(2^m e_m) = \eta_n + 2^{-m} e_m$$

But by construction of the sequence $(\eta_n)$ there are infinitely many such natural numbers $m$. Hence the vector $\eta_n$ belongs to the closure of the image of the operator $T$. Further,

$$T(4^n e_n) = 2^n \eta_n + e_n$$

so the vector $e_n$ must also be in the closure of the range of the operator $T$. But the set of vectors $e_n$ and $\eta_n$ is a generating set for the Hilbert $\mathcal{A}$-module

$$\mathcal{E} \oplus \left( \prod_{A \in \Gamma} \mathcal{H}(-, A)_\mathcal{A} \right)$$

so the operator $T$ has dense range.

Now,

$$T^* \xi = \sum_{n=1}^{\infty} \left( 2^{-n} e_n (\eta_n, \xi) + 4^{-n} e_n (e_n, \xi) \right)$$

Hence $T^*(4^n e_n) = e_n$ and the operator $T^*$ also has dense range. Therefore, by lemma 5.10, the Hilbert $\mathcal{B}$-modules

$$\mathcal{E} \oplus \left( \prod_{A \in \Gamma} \mathcal{H}(-, A)_\mathcal{A} \right)$$

and

$$\prod_{A \in \Gamma} \mathcal{H}(-, A)_\mathcal{A}$$

are isomorphic and we are done. □

There is an obvious version of the Kasparov stabilisation theorem in the graded case.
3.2 Bimodules

The following definition is of great relevance to the $KK$-theory of $C^*$-categories.

**Definition 3.11** Let $\mathcal{A}$ and $\mathcal{B}$ be graded $C^*$-categories. A graded Hilbert $(\mathcal{A}, \mathcal{B})$-bimodule is a graded $C^*$-functor $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{B})$.

For each object $A \in Ob(\mathcal{A})$ we write $\mathcal{F}(\cdot, A)$ to denote the corresponding Hilbert $\mathcal{B}$-module. Given another object $B \in Ob(\mathcal{B})$ we write $\mathcal{F}(A, \cdot)$ to denote the associated vector space. For each morphism $x \in Hom(A, A')_\mathcal{A}$ we write $x: \mathcal{F}(\cdot, A) \rightarrow \mathcal{F}(\cdot, A')$ to denote the induced operator $\mathcal{F}(x)$.

Thus a graded Hilbert $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{F}$ consists of a collection of Hilbert $\mathcal{B}$-modules

$$\{\mathcal{F}(\cdot, A) \mid A \in Ob(\mathcal{A})\}$$

together with a bounded operator $x: \mathcal{F}(\cdot, A) \rightarrow \mathcal{F}(\cdot, A')$ for each morphism $x \in Hom(\mathcal{A}, \mathcal{B})_\mathcal{A}$.

**Example 3.12** Suppose that $\mathcal{B}$ is a $\sigma$-unital graded $C^*$-category. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a graded $C^*$-functor. Recall from example 3.2 that for each object $B \in Ob(\mathcal{B})$ we have a graded Hilbert $\mathcal{B}$-module $Hom(-, B)_\mathcal{B}$ with spaces $Hom(C, B)_\mathcal{B}$. The Hilbert $\mathcal{B}$-module $Hom(-, B)_\mathcal{B}$ is countably generated since the $C^*$-category $\mathcal{B}$ is $\sigma$-unital.

The category $\mathcal{B}$ can thus itself be considered a graded Hilbert $(\mathcal{A}, \mathcal{B})$-bimodule; the $C^*$-functor $F: \mathcal{A} \rightarrow \mathcal{B}$ associates the graded Hilbert $\mathcal{B}$-module $Hom(-, F(A))_\mathcal{B}$ to the object $A \in Ob(\mathcal{A})$. Given a morphism $x \in Hom(\mathcal{A}, \mathcal{A'})_\mathcal{A}$ we have an operator $F(x): Hom(-, F(A))_\mathcal{B} \rightarrow Hom(-, F(A'))_\mathcal{B}$ defined by composition in the category $\mathcal{B}$.

It is possible to form the tensor product of a Hilbert $\mathcal{A}$-module and a Hilbert $(\mathcal{A}, \mathcal{B})$-bimodule. Before we are ready to construct such a product we need some technical results. Our approach here closely follows the analysis of tensor products of Hilbert modules over $C^*$-algebras made in [13].

**Lemma 3.13** Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module, and consider a bounded operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{E})$. Then the operator $T$ is a positive element of the $C^*$-algebra $\mathcal{L}(\mathcal{E}, \mathcal{E})$ if and only if the product $\langle \eta, T \eta \rangle$ is positive for all vectors $\eta \in \mathcal{E}(A)$.

**Proof:** Suppose that the operator $T$ is a positive element of the $C^*$-algebra $\mathcal{L}(\mathcal{E}, \mathcal{E})$. Then we can write $T = S^* S$ for some operator $S \in \mathcal{L}(\mathcal{E}, \mathcal{E})$. Hence for any vector $\eta \in \mathcal{E}(A)$

$$\langle \eta, T \eta \rangle = \langle \eta, S^* S \eta \rangle = \langle S \eta, S \eta \rangle$$

so the product $\langle \eta, T \eta \rangle$ is positive.

Conversely, suppose that the product $\langle \eta, T \eta \rangle$ is positive for all vectors $\eta \in \mathcal{E}(A)$. Then by polarisation the operator $T$ is self-adjoint. We can therefore write $T = R - S$ for positive operators $R, S \in \mathcal{L}(\mathcal{E}, \mathcal{E})$ such that $RS = 0$. Let $\eta \in \mathcal{E}(A)$. By hypothesis the product

$$\langle S \eta, T(S \eta) \rangle = \langle S \eta, -S^2 \eta \rangle = \langle \eta, -S^3 \eta \rangle$$

is positive. But the operator $S$ is positive so we must have $S = 0$. Hence the operator $T$ is positive. \qed
Lemma 3.14 Let $E$ be a Hilbert $A$-module. Suppose that we are given vectors $\eta_1 \in E(A_1), \ldots, \eta_n \in E(A_n)$. Let $A$ be the $C^*$-algebra $\text{Hom}(A_1 \oplus \cdots \oplus A_n, A_1 \oplus \cdots \oplus A_n)$. Form the matrix

$$X = \begin{pmatrix} \langle \eta_1, \eta_1 \rangle & \cdots & \langle \eta_1, \eta_n \rangle \\ \vdots & \ddots & \vdots \\ \langle \eta_n, \eta_1 \rangle & \cdots & \langle \eta_n, \eta_n \rangle \end{pmatrix} \in A$$

Then the matrix $X$ is a positive element of the $C^*$-algebra $A$.

Proof: Consider the Hilbert $A$-module $F = \text{Hom}(A_1, A_1) \oplus \cdots \oplus \text{Hom}(A_n, A_n)$. There is an injective morphism of $C^*$-algebras $\rho: A \to \mathcal{L}(F, F)$ defined by allowing a matrix $T \in A$ to act on a vector $b \in F$ by matrix multiplication.

Form the vector $\eta = (\eta_1, \ldots, \eta_n) \in E(A_1) \oplus \cdots \oplus E(A_n)$. Then for any vector $b = (b_1, \ldots, b_n) \in F$ we know that

$$\langle b, Xb \rangle = \sum_{i,j} b_i^* \langle \eta_i, \eta_j \rangle b_j = \langle \eta b, \eta b \rangle$$

so the product $\langle b, Xb \rangle$ is positive. Hence the matrix $X$ is positive by the previous lemma. \qed

A proof of our final preliminary lemma can be found in [13].

Lemma 3.15 Let $E$ be a Hilbert module over a $C^*$-algebra $A$, and suppose that we are given a vector $\eta \in E$ and a real number $\alpha \in (0, 1)$. Then there is an element $\bar{\eta} \in E$ such that $\eta = \bar{\eta} \langle \eta, \eta \rangle^\alpha$.

Let $E$ be a Hilbert $A$-module and let $F$ be a Hilbert $(A, B)$-bimodule. Then for each object $B \in \text{Ob}(B)$ the functor $F(B, -)$ defined by sending the object $A \in \text{Ob}(A)$ to the vector space $F(B, A)$ is a left $A$-module.

Definition 3.16 We define the algebraic tensor product $E \otimes_A F(B, -)$ to be the vector space

$$\{\lambda_1(\eta_1, \xi_1) + \cdots + \lambda_n(\eta_n, \xi_n) \mid \lambda_i \in F, \eta_i \in E(A), \xi_i \in F(A, B), A \in \text{Ob}(A)\} / \sim$$

where $\sim$ is the equivalence relation defined by writing

- $(\eta_1 + \eta_2, \xi) \sim (\eta_1, \xi) + (\eta_2, \xi)$
- $(\eta, \xi_1 + \xi_2) \sim (\eta, \xi_1) + (\eta, \xi_2)$
- $(\eta, F(x)\xi) \sim (\eta x, \xi)$

We write $\eta \otimes \xi$ for the equivalence class of the pair $(\eta, \xi)$. A more category-theoretic description of the algebraic tensor product of a right module and a left module over a $C^*$-category can be found in [19].

The above algebraic tensor product can be graded in the obvious way. When the grading is taken into account, we write it $E \hat{\otimes}_A F(B, -)$.
Lemma 3.17  The collection of vector spaces

\[ E \odot_A F = \{ E \odot_A F(B, -) \mid B \in Ob(B) \} \]

is a right \( B \)-module. The action of the category \( B \) is defined by the formula

\[(\eta \otimes \xi)(x) = \eta \otimes (\xi x)\]

There is a semi-inner product defined by the formula

\[ \langle \eta \otimes \xi, \eta' \otimes \xi' \rangle = \langle \xi, \langle \eta, \eta' \rangle \xi' \rangle \]

Proof: The only non-trivial part to check is the fact that for any vector

\[ \zeta = \eta_1 \otimes \xi_1 + \cdots + \eta_n \otimes \xi_n \in E \odot_A F(B) \]

where \( \eta_i \in \mathcal{E}(A_i) \) and \( \xi_j \in \mathcal{F}(B, A_i) \) the product \( \langle \zeta, \zeta \rangle \) is positive. Observe that

\[ \langle \zeta, \zeta \rangle = \sum_{i,j} \langle \xi_i, \eta_i \eta_j \xi_j \rangle = \langle \xi, X \xi \rangle \]

where \( \xi = (\xi_1, \ldots, \xi_n) \in \mathcal{E}_{A_1}(B) \oplus \cdots \oplus \mathcal{E}_{A_n}(B) \) and \( X \) is the matrix

\[
X = \begin{pmatrix}
\langle \eta_1, \eta_1 \rangle & \cdots & \langle \eta_1, \eta_n \rangle \\
\vdots & \ddots & \vdots \\
\langle \eta_n, \eta_1 \rangle & \cdots & \langle \eta_n, \eta_n \rangle
\end{pmatrix}
\]

belonging to the \( C^* \)-algebra \( \text{Hom}(A_1 \oplus \cdots \oplus A_n, A_1 \oplus \cdots \oplus A_n) \).

By lemma 3.14, the matrix \( X \) is positive. By lemma 3.13, the product \( \langle \zeta, \zeta \rangle \) is therefore positive. \( \square \)

For each object \( B \in Ob(B) \) form the vector space

\[ N(B) = \{ \eta \in E \odot_A F(B, -) \mid \langle \eta, \eta \rangle = 0 \} \]

Then the collection of quotients \( E \odot_A F(B, -)/N(B) \) is a right \( B \)-module equipped with an inner product. It can be graded if desired.

Definition 3.18 We define the inner tensor product, \( E \hat{\otimes}_A F \), to be the Hilbert \( B \)-module obtained by completing the above inner product \( B \)-module.

If \( T: E \rightarrow E' \) is a bounded operator then we define a bounded operator \( T \otimes 1: E \hat{\otimes}_A F \rightarrow E' \hat{\otimes}_A F \) by the formula

\[(T \otimes 1)(\eta \otimes \xi) = (T\eta) \otimes \xi \]

For example, let \( F: A \rightarrow B \) be a \( C^* \)-functor, and let \( E \) be a Hilbert \( A \)-module. Then by example 3.12 the category \( B \) is itself a Hilbert \( (A, B) \)-bimodule and we can form an inner tensor product which we write \( E \hat{\otimes}_F B \).

The following fact is useful in computations.
Proposition 3.19 Consider a vector $\zeta \in \mathcal{E} \otimes_A \mathcal{F}(B, -)$. Then the vector $\zeta$ belongs to the space $N(B)$ if and only if we can write

$$\zeta = \sum_{i=1}^{n} (\eta_i x_i \otimes \xi_i - \eta_i \otimes x_i \xi_i)$$

for vectors $\eta_i \in \mathcal{E}(A'_i)$, $\xi_i \in \mathcal{F}(B, A_i)$, and morphisms $x_i \in \text{Hom}(A_i, A'_i)_A$.

Proof: Let $\zeta = \eta x \otimes \xi - \eta \otimes x \xi$. Then a straightforward calculation tells us that $\langle \zeta, \zeta \rangle = 0$. Conversely, suppose that $\zeta \in N(B)$. Write

$$\zeta = \sum_{i=1}^{n} (\eta_i \otimes \xi_i)$$

for vectors $\eta_i \in \mathcal{E}(A_i)$ and $\xi_i \in \mathcal{F}(B, A_i)$. Form the matrix

$$X = \begin{pmatrix}
\langle \eta_1, \eta_1 \rangle & \cdots & \langle \eta_1, \eta_n \rangle \\
\vdots & \ddots & \vdots \\
\langle \eta_n, \eta_1 \rangle & \cdots & \langle \eta_n, \eta_n \rangle
\end{pmatrix}$$

and write $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{F}(B, A_1) \oplus \cdots \oplus \mathcal{F}(B, A_n)$. Then

$$\langle \xi, X \xi \rangle = \langle \xi, \zeta \rangle = 0$$

By lemma 3.14 the matrix $X$ is positive, so we can find a positive square root $X^{\frac{1}{2}}$. We have the formula

$$\langle X^{\frac{1}{2}} \xi, X^{\frac{1}{2}} \xi \rangle = \langle \xi, X \xi \rangle = 0$$

and so $X^{\frac{1}{2}} \xi = 0$. Repeating this process we see that $X^{\frac{1}{2}} \xi = 0$.

Form the vector $\eta = (\eta_1, \ldots, \eta_n) \in \mathcal{E}(A_1) \oplus \cdots \oplus \mathcal{E}(A_n)$. Then by definition of the matrix $X$ we can write $X = \langle \eta, \eta \rangle$. By lemma 3.15 we can find a vector $\eta' = (\eta'_1, \ldots, \eta'_n) \in \mathcal{E}(A_1) \oplus \cdots \oplus \mathcal{E}(A_n)$ such that $\eta' X^{\frac{1}{2}} = \eta$.

Write

$$X^{\frac{1}{2}} = \begin{pmatrix}
x_{1,1} & \cdots & x_{1,n} \\
\vdots & \ddots & \vdots \\
x_{n,1} & \cdots & x_{n,n}
\end{pmatrix}$$

Then we have the formulae

$$\sum_j x_{i,j} \xi_j = 0 \quad \eta_j = \sum_i \eta'_i x_{i,j}$$

and so we conclude that

$$\zeta = \sum_{i,j} (\eta'_i x_{i,j} \otimes \xi_j - \eta'_i \otimes x_{i,j} \xi_j)$$

as required. $\square$

Thus the Hilbert $B$-module $\mathcal{E} \otimes_A \mathcal{F}$ is generated by elementary tensors $\eta \otimes \xi$ satisfying the relation

$$\eta x \otimes \xi = \eta \otimes x \xi$$

Our final lemma is an application of the Kasparov stabilisation theorem.
Lemma 3.20 Suppose that $A$ and $B$ are $\sigma$-unital graded $C^*$-categories. Let $E$ be a countably generated graded Hilbert $A$-module and let $F: A \to B$ be a graded unital $C^*$-functor. Then the inner tensor product $E \hat{\otimes}_F B$ is countably generated.

**Proof:** Let $H$ denote the Hilbert space of all sequences $(\alpha_n)$ in the field $F$ such that the sum $\sum |\alpha_n|^2$ converges. Then the Hilbert $A$-module $H(\cdot, A)_A$ is isomorphic to the outer tensor product $H \hat{\otimes} Hom(\cdot, A)_A$.

Hence, by proposition 3.19, the vector space $H(\cdot, A)_A \hat{\otimes} F Hom(B, \cdot)_B$ is generated by elements of the form

$((\alpha_n) \otimes x) \otimes y = (\alpha_n) \otimes F(x)y$

where $(\alpha_n) \in H$, $x \in Hom(A', A)_A$, and $y \in Hom(B, F(A'))_B$. Since the $C^*$-functor $F$ is unital, every vector of the form $(\alpha_n) \otimes y$, where $y \in Hom(B, F(A))_B$, is obtained in this fashion. Therefore we have isomorphisms

$H(\cdot, A)_A \hat{\otimes} F B \cong H \hat{\otimes} Hom(\cdot, F(A))_B \cong H(\cdot, F(A))_B$

Let $E$ be a countably generated Hilbert $A$-module. By the Kasparov stabilisation theorem the Hilbert module $E$ can be considered to be a submodule of the direct sum of countably many Hilbert $A$-modules of the form $H(\cdot, A)_A$. Hence the above calculation tells us that the inner tensor product $E \hat{\otimes}_F B$ is countably generated.

3.3 Some Computations

We present in this subsection some computations concerning the $C^*$-categories $\mathcal{L}(A)$, $\mathcal{K}(A)$, and $\mathcal{Q}(A)$ which will help us to deduce some of the properties of $KK$-theory.

**Proposition 3.21** The assignments $A \mapsto \mathcal{L}(A)$ and $A \mapsto \mathcal{Q}(A)$ are covariant functors from the category of $\sigma$-unital graded $C^*$-categories to the category of unital graded $C^*$-categories.

**Proof:** Let $F: A \to B$ be a graded $C^*$-functor. Let $E$ be a countably generated graded Hilbert $A$-module. Then by lemma 3.20 we have a countably generated graded Hilbert $B$-module defined by the formula $F_*(E) = E \hat{\otimes}_F B$.

If $T: E \to F$ is an operator between countably generated Hilbert $B$-modules there is an induced operator $T \otimes 1: E \hat{\otimes}_F B \to F \hat{\otimes}_F B$. The operator $T \otimes 1$ is compact if the operator $T$ is compact.

It is now easy to check that the above process defines a $C^*$-functor $F_*: \mathcal{L}(A) \to \mathcal{L}(B)$, that $1_* = 1$, and that $(FG)_* = F_*G_*$. The assignments $A \mapsto \mathcal{L}(A)$ and $A \mapsto \mathcal{Q}(A)$ are therefore covariant functors as claimed.

---

$^6$In fact there are set-theoretic difficulties involved here since for a graded $C^*$-category $A$ the collection of all countably generated graded Hilbert $A$-modules is not a set, even when the $C^*$-category $A$ is itself small. We can evade this problem by insisting that all of our Hilbert modules over a particular $C^*$-category lie within a given universe.
**Definition 3.22** We write \( \mathcal{L}(\mathcal{H}_A) \) to denote the full subcategory of the \( C^* \)-category \( \mathcal{L}(A) \) in which the objects are Hilbert \( A \)-modules of the form \( \mathcal{H}(-, A)_A \) where \( A \in \text{Ob}(A) \). We write \( \mathcal{K}(\mathcal{H}_A) \) to denote the corresponding full subcategory of the \( C^* \)-category \( \mathcal{K}(A) \).

By proposition 3.8 the \( C^* \)-category \( \mathcal{K}(\mathcal{H}_A) \) is a \( C^* \)-ideal in the \( C^* \)-category \( \mathcal{L}(\mathcal{H}_A) \). We may therefore form the quotient \( Q(\mathcal{H}_A) = \mathcal{L}(\mathcal{H}_A)/\mathcal{K}(\mathcal{H}_A) \).

**Proposition 3.23** Let \( A \) be a unital \( C^* \)-category which is equivalent to some \( C^* \)-algebra. Then we have equivalences of \( K \)-theory spectra

\[
\mathbb{K}\mathcal{L}(A) \simeq \mathbb{K}\mathcal{L}(\mathcal{H}_A) \quad \mathbb{K}\mathcal{K}(A) \simeq \mathbb{K}\mathcal{K}(\mathcal{H}_A) \quad \mathbb{K}Q(A) \simeq \mathbb{K}Q(\mathcal{H}_A)
\]

**Proof:** This result is an immediate consequence of the Kasparov stabilisation theorem and the definition of the \( K \)-theory groups.

**Proposition 3.24** The \( K \)-theory groups of the graded \( C^* \)-category \( \mathcal{L}(A) \) are all trivial.

**Proof:** We will use an ‘Eilenberg swindle’ argument to prove this result.

Let \( F_{p,q} \) be any Clifford algebra. Let \( E \) be a countably generated graded Hilbert \( A \)-module, and let \( x \in \text{SS}(\mathcal{L}(E) \otimes F_{p,q}) \) be a supersymmetry.

Consider the direct sum, \( E^\infty \), of (countably) infinitely many copies of the Hilbert \( A \)-module \( E \). We can form a supersymmetry

\[
x \oplus x \oplus \cdots \in \text{SS}(\mathcal{L}(E^\infty) \otimes F_{p,q})
\]

At the level of \( K \)-theory

\[
\langle x \rangle + \langle x \oplus x \oplus \cdots \rangle = \langle x \oplus x \oplus \cdots \rangle
\]

and so \( \langle x \rangle = 0 \). Hence \( K_1(\mathcal{L}(A) \otimes F_{p,q}) = 0 \) for all natural numbers \( p \) and \( q \). By the Bott periodicity theorem we deduce that \( K_n(\mathcal{L}(A)) = 0 \) for all integers \( n \in \mathbb{Z} \).

**Proposition 3.25** The \( C^* \)-categories \( B \otimes \mathcal{K} \) and \( \mathcal{K}(\mathcal{H}_B) \) are isomorphic.

**Proof:** Let \( \mathcal{K}(B^n) \) be the \( C^* \)-category of compact operators between Hilbert \( B \)-modules of the form \( \text{Hom}(-, B)_B^n \). Then the \( C^* \)-category \( \mathcal{K}(\mathcal{H}_B) \) is the direct limit of the \( C^* \)-categories \( \mathcal{K}(B^n) \). Similarly, the \( C^* \)-category \( B \otimes \mathcal{K} \) is the direct limit of the matrix \( C^* \)-categories \( B \otimes M_n(\mathbb{F}) \). It therefore suffices to show that the \( C^* \)-categories \( B \otimes M_n(\mathbb{F}) \) and \( \mathcal{K}(B^n) \) are isomorphic.

We can define a graded \( C^* \)-functor \( F : \mathcal{K}(B^n) \rightarrow B \otimes M_n(\mathbb{F}) \) by mapping the Hilbert \( B \)-module \( \text{Hom}(-, A)_B^n \) to the object \( A \), and the operator

\[
\eta \mapsto (x_1 \oplus \cdots \oplus x_n)/(y_1 \oplus \cdots \oplus y_n, \eta)
\]

to the matrix

\[
\begin{pmatrix}
x_1 y_1^* & \cdots & x_1 y_n^* \\
\vdots & \ddots & \vdots \\
x_n y_1^* & \cdots & x_n y_n^*
\end{pmatrix} \in \text{Hom}(B, C)_B \otimes M_n(\mathbb{F})
\]
where $x_i \in \text{Hom}(A, C)$ and $y_i \in \text{Hom}(A, B)$.

Let $\{e_\lambda \mid \lambda \in \Lambda\}$ be a self-adjoint approximate unit for the $C^*$-algebra $\text{Hom}(B,B)$. A simple calculation shows that

$$\inf_{\lambda \in \Lambda} \{\|xe_\lambda - x\| \mid \lambda \in \Lambda\} = 0$$

for any morphism $x \in \text{Hom}(B,C)$.

It follows that the $C^*$-functor $F$ has dense range. As we mentioned in section 2.1, the image of any $C^*$-functor is closed. Therefore the $C^*$-functor $F$ is surjective.

Now, consider the compact operator

$$T: \eta \mapsto \sum_i (x_1^{(i)} \oplus \cdots \oplus x_n^{(i)}) (y_1^{(i)} \oplus \cdots \oplus y_n^{(i)}, \eta)$$

where $x_j^{(i)} \in \text{Hom}(A, C)$ and $y_j^{(i)} \in \text{Hom}(A, B)$. Let $\eta = \eta_1 \oplus \cdots \oplus \eta_n$. Then

$$T\eta = \sum_{i,j} x_1^{(i)} y_j^{(i)*} \eta_j \oplus \cdots \oplus x_n^{(i)} y_j^{(i)*} \eta_j$$

Thus

$$\|T\| = \sup_{\|\eta\| \leq 1} \left\| \sum_{i,j} x_1^{(i)} y_j^{(i)*} \eta_j \oplus \cdots \oplus x_n^{(i)} y_j^{(i)*} \eta_j \right\|$$

By taking the various vectors $\eta_j$ to be elements of approximate units, we see that the norm, $\|T\|$, is equal to the norm of the matrix

$$\begin{pmatrix}
    x_1 y_1^* & \cdots & x_1 y_n^* \\
    \vdots & \ddots & \vdots \\
    x_n y_1^* & \cdots & x_n y_n^*
\end{pmatrix}$$

Hence the $C^*$-functor $F$ is an isometry, and we are done. \qed

**Theorem 3.26** Let $A$ be a small unital graded $C^*$-category which is equivalent to some $C^*$-algebra. Then we have a natural stable equivalence of spectra

$$\Omega \mathcal{K} \mathcal{Q}(A) \simeq \mathcal{K}(A)$$

**Proof:** We have a natural short exact sequence

$$0 \to \mathcal{K}(A) \to \mathcal{L}(A) \to \mathcal{Q}(A) \to 0$$

and so by proposition 4.7 of a natural fibration

$$\mathbb{K}\mathcal{K}(A) \to \mathbb{K}\mathcal{L}(A) \to \mathbb{K}\mathcal{Q}(A)$$

We therefore have an induced long exact sequence

$$\to K_{n+1}\mathcal{L}(A) \to K_{n+1}\mathcal{Q}(A) \to K_n\mathcal{K}(A) \to K_n\mathcal{L}(A) \to$$
where the boundary map \( \partial_*: K_{n+1}(\mathcal{Q}(A)) \to K_n(\mathcal{K}(A)) \) is induced from a natural map \( \partial: \Omega K(\mathcal{Q}(A)) \to K(\mathcal{K}(A)) \) defined by the homotopy lifting property of a fibration.

By proposition 3.24 the \( K \)-theory groups \( K_{n+1}(\mathcal{L}(A)) \) and \( K_n(\mathcal{K}(A)) \) are both zero. Hence the map \( \partial: \Omega K(\mathcal{Q}(A)) \to K(\mathcal{K}(A)) \) is a stable equivalence of spectra.

But by theorem 2.16 the inclusion \( A \to A \hat{\otimes} K \) induces isomorphisms of \( K \)-theory groups, so by proposition 3.25 there is a natural stable equivalence of spectra \( K(\mathcal{H}_A) \simeq K(A) \). It follows that there is a natural stable equivalence of spectra \( \Omega K(\mathcal{Q}(A)) \simeq K(A) \).

\[ \square \]

4 \( KK \)-theory

4.1 Definition of \( KK \)-theory

We are now ready to generalise the definitions and some results concerning the \( KK \)-theory of \( C^* \)-algebras to the world of small \( C^* \)-categories. For many results concerning \( KK \)-theory to be valid we need to restrict our attention to \( \sigma \)-unital \( C^* \)-categories. We will mention explicitly when the assumption of \( \sigma \)-unitality is needed.

**Definition 4.1** Let \( A \) and \( B \) be small graded \( C^* \)-categories. Then a Kasparov \( (A, B) \)-cycle consists of a Hilbert \( (A, B) \)-bimodule \( \mathcal{E} \) together with a collection of degree 1 operators \( F_A: \mathcal{E}(\cdot, A) \to \mathcal{E}(\cdot, A) \) such that the norm

\[
\| F \| = \sup \{ \| F_A \| \mid A \in \text{Ob}(A) \}
\]

is finite and the operators

\[ x(F_A - F_A^*), \quad x(F_A^2 - 1), \quad x(-1)^{\deg(x)} F_B x \]

are compact for all morphisms \( x \in \text{Hom}(A, B)_A \).

An element of \( KK \)-theory is defined to be a certain equivalence class of Kasparov cycles. We will abuse notation slightly, and write \( F \) to denote both the collection of operators \( F_A: \mathcal{E}(\cdot, A) \to \mathcal{E}(\cdot, A) \) and an individual operator of the form \( F_A \).

**Definition 4.2** Let \( (\mathcal{E}, F) \) and \( (\mathcal{E}', F') \) be Kasparov \( (A, B) \)-cycles. Then the direct sum is the Kasparov cycle

\[
(\mathcal{E}, F) \oplus (\mathcal{E}', F') = (\mathcal{E} \oplus \mathcal{E}', F \oplus F')
\]

**Definition 4.3**

- A Kasparov \( (A, B) \)-cycle \( (\mathcal{E}, F) \) is called degenerate if the operators

\[ x(F - F^*), \quad x(F^2 - 1), \quad x(-1)^{\deg(x)} F x \]

are equal to zero for all morphisms \( x \in \text{Hom}(A, B)_A \).
• An operator homotopy between Kasparov \((\mathcal{A}, \mathcal{B})\)-cycles \((\mathcal{E}, F)\) and \((\mathcal{E}, F')\) is a norm-continuous path \((\mathcal{E}, F_\theta)\) of Kasparov \((\mathcal{A}, \mathcal{B})\)-cycles such that \(F_0 = F\) and \(F_1 = F'\).

• Kasparov \((\mathcal{A}, \mathcal{B})\)-cycles \((\mathcal{E}_1, F_1)\) and \((\mathcal{E}_2, F_2)\) are called equivalent if there are degenerate Kasparov \((\mathcal{A}, \mathcal{B})\)-cycles \((\mathcal{E}'_1, F'_1)\) and \((\mathcal{E}'_2, F'_2)\) such that the direct sums \((\mathcal{E}_1, F_1) \oplus (\mathcal{E}'_1, F'_1)\) and \((\mathcal{E}_2, F_2) \oplus (\mathcal{E}'_2, F'_2)\) are operator homotopic.

We write \([[\mathcal{E}, F]]\) to denote the equivalence class of a Kasparov \((\mathcal{A}, \mathcal{B})\)-cycle \((\mathcal{E}, F)\), and \(KK(\mathcal{A}, \mathcal{B})\) to denote the set of equivalence classes.

**Theorem 4.4** The set \(KK(\mathcal{A}, \mathcal{B})\) is an Abelian group with an operation defined by taking the direct sum of Kasparov cycles.

**Proof:** It is easy to check that the set \(KK(\mathcal{A}, \mathcal{B})\) is an Abelian semigroup, with identity element \([[\mathcal{E}, F]]\)) where \((\mathcal{E}, F)\) is any degenerate Kasparov \((\mathcal{A}, \mathcal{B})\)-cycle.

If \(\mathcal{E}\) is a graded Hilbert \(\mathcal{B}\)-module, with grading \(\mathcal{E}(A) = \mathcal{E}(A)_0 \oplus \mathcal{E}(A)_1\), define \(\mathcal{E}^{op}\) to be the Hilbert \(\mathcal{B}\)-module with the opposite grading. Given a Kasparov \((\mathcal{A}, \mathcal{B})\) cycle \((\mathcal{E}, F)\) define \((\hat{\mathcal{E}}, -F)\) to be the Kasparov cycle in which the bimodule \(\hat{\mathcal{E}}\) is defined by writing \(\hat{\mathcal{E}}(-, A) = \mathcal{E}(-, A)^{op}\) for all objects \(A \in Ob(\mathcal{A})\) and \(\hat{\mathcal{E}}(x_0 + x_1) = \mathcal{E}(x_0 - x_1)\) for all morphisms \(x_0, x_1 \in Hom(A, \mathcal{A}'_1)\) of degrees 0 and 1 respectively.

We can define an operator homotopy between the Kasparov cycle \((\hat{\mathcal{E}}, F) \oplus (\hat{\mathcal{E}}, -F)\) and the cycle \(\left(\mathcal{E} \oplus \hat{\mathcal{E}}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\) by the formula

\[
G_\theta = \begin{pmatrix}
F \cos \theta & \sin \theta \\
\sin \theta & -F \cos \theta
\end{pmatrix}, \quad \theta \in [0, \frac{\pi}{2}]
\]

But the cycle \(\left(\mathcal{E} \oplus \hat{\mathcal{E}}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\) is degenerate so we have proved that

\([[\mathcal{E}, F]] + [\hat{\mathcal{E}}, -F]] = 0\]

Therefore the set \(KK(\mathcal{A}, \mathcal{B})\) is an abelian group. \(\square\)

It is easy to see that when \(A\) and \(B\) are graded \(C^*\)-algebras the group \(KK(A, B)\) defined above is the usual \(KK\)-theory group associated to graded \(C^*\)-algebras.

**Proposition 4.5** Let \(\mathcal{A}\) and \(\mathcal{B}\) be small \(\sigma\)-unital graded \(C^*\)-categories. Then the abelian group \(KK(\mathcal{A}, \mathcal{B})\) is contravariantly functorial in the variable \(\mathcal{A}\) and covariantly functorial in the variable \(\mathcal{B}\).

**Proof:** Let \((\mathcal{E}, F)\) be a Kasparov \((\mathcal{A}, \mathcal{B})\)-cycle and let \(G: \mathcal{B} \to \mathcal{B}'\) be a \(C^*\)-functor. Then we can define a Hilbert \((\mathcal{A}, \mathcal{B})\)-bimodule \(\mathcal{E} \otimes_G \mathcal{B}'\) and a collection of operators \(F_A \otimes 1: \mathcal{E} \otimes_G \mathcal{B}' \to \mathcal{E} \otimes_G \mathcal{B}'\). It is straightforward to check that we have defined a Kasparov \((\mathcal{A}, \mathcal{B}')\)-cycle\(^7\)

\[
G_*(\mathcal{E}, F) = (\mathcal{E} \otimes \mathcal{B}', F \otimes 1)
\]

\(^7\)Each Hilbert \(\mathcal{B}\)-module \(\mathcal{E}_A \otimes_G \mathcal{B}'\) is countably generated by lemma 3.20. Here we need the assumption that the \(C^*\)-category \(\mathcal{B}'\) is \(\sigma\)-unital.
and that with such induced cycles we have a functorially induced map $G_* : KK(A, B) \to KK(A, B')$.

Now suppose we have a $C^*$-functor $H : A' \to A$. Then the composite $C^*$-functor $E \circ H : A' \to L(B)$ is a Hilbert $(A', B)$-bimodule. We have a functorially induced map of $KK$-theory groups defined by the formula

$$H^*(E, F) = (E \circ H, F)$$

Note that the above proof does not work if we allow the $C^*$-category $B$ to be non-$\sigma$-unital.

4.2 Duality

In [6] the $K$-homology of a $C^*$-algebra $A$ is defined in terms of the ordinary $K$-theory of a ‘dual algebra’ constructed from $A$. We can in fact extend this approach to define the groups $KK^{-n}(A, B)$ for graded $C^*$-categories $A$ and $B$ in terms of the ordinary $K$-theory of some ‘dual category’. Having done this we will be able to deduce information concerning $KK$-theory from information concerning $K$-theory.

**Definition 4.6** An operator, $T$, between graded Hilbert $(A, B)$-bimodules $E$ and $F$ is a collection of bounded operators $T_A : E(-, A) \to F(-, A)$ such that the norm

$$\|T\| = \sup\{\|T_A\| \mid A \in Ob(A)\}$$

exists. The operator $T$ is said to be of degree 1 if each operator $T_A$ is of degree 1, and of degree 0 if each operator $T_A$ is of degree 0.

Note that we do not require an operator between Hilbert $(A, B)$-bimodules to be a natural transformation.

**Definition 4.7** Let $A$ and $B$ be $\sigma$-unital small graded $C^*$-categories. We define the dual $C^*$-category, $D(A, B)$, of the categories $A$ and $B$ to be the $C^*$-category in which the objects are graded $(A, B)$-bimodules and the morphisms between $(A, B)$-bimodules $E$ and $F$ are operators $T$ such that the graded commutator $xT_A - (-1)^{\deg(x)\deg(T)}T_Bx$ is compact for all morphisms $x \in Hom(A, B)_A$.

We define the subcategory of locally compact operators, $KD(A, B)$, to be the subcategory of the category $D(A, B)$ in which the morphisms are operators $T : E \to F$ such that the operators $T_Bx$ and $xT_A$ are compact for all morphisms $x \in Hom(A, B)_A$.

It is easy to check that the collection of objects and morphisms $D(A, B)$ is a unital graded $C^*$-category with direct sum, and that the collection $KD(A, B)$ is a $C^*$-ideal. We can form the quotient $QD(A, B) = D(A, B)/KD(A, B)$.

The quotient $QD(A, B)$ is a graded unital $C^*$-category. The following result is proved similarly to proposition 4.5.

**Proposition 4.8** The $C^*$-category $QD(A, B)$ is covariantly functorial in the variable $B$, and covariantly functorial in the variable $A$. 

□
There is a natural isomorphism \( K_1 \mathcal{QD}(\mathcal{A}, \mathcal{B}) \cong KK(\mathcal{A}, \mathcal{B}) \).

**Theorem 4.9**

**Proof:** The additive completion \( \mathcal{QD}(\mathcal{A}, \mathcal{B})_\oplus \) is equivalent to the \( C^* \)-category \( \mathcal{QD}(\mathcal{A}, \mathcal{B}) \) in the sense of definition 2.9, so the group \( K_1 \mathcal{QD}(\mathcal{A}, \mathcal{B}) \) is defined to be the set of formal differences

\[
\{ (x) - (y) \mid x, y \in SS(\mathcal{E}), \mathcal{E} \in Ob(\mathcal{QD}(\mathcal{A}, \mathcal{B})) \}
\]

It is easy to check that we can define a natural homomorphism \( \alpha: K_1 \mathcal{QD}(\mathcal{A}, \mathcal{B}) \rightarrow KK(\mathcal{A}, \mathcal{B}) \) by writing

\[
\alpha((x) - (y)) = [(\mathcal{E}, x)] - [(\mathcal{E}, y)]
\]

for supersymmetries \( x, y \in SS(\mathcal{E}) \).

Let \( (\mathcal{E}, F) \) be any Kasparov cycle. Then we can find degenerate cycles \( (\mathcal{E}', F') \) and \( (\mathcal{E} \oplus \mathcal{E}', G) \) for some Hilbert \( (\mathcal{A}, \mathcal{B}) \)-bimodule \( \mathcal{E}' \). Hence

\[
[(\mathcal{E}, F)] = [(\mathcal{E} \oplus \mathcal{E}', F \oplus F')] - [(\mathcal{E} \oplus \mathcal{E}', G)] = \alpha((F \oplus F') - (G))
\]

and we have proved that the homomorphism \( \alpha \) is surjective.

Now suppose that \( (\mathcal{E}, F) \) and \( (\mathcal{E}, F') \) are Kasparov cycles and that \( [(\mathcal{E}, F)] = [(\mathcal{E}, F')] \). Then we can find a degenerate cycle \( (F, G) \) such that the cycles \( (\mathcal{E} \oplus \mathcal{F}, F \oplus G) \) and \( (\mathcal{E} \oplus \mathcal{F}, F' \oplus G) \) are operator-homotopic. Hence the supersymmetries \( F \oplus G \) and \( F' \oplus G \) lie in the same path-component of the space \( SS(\mathcal{E} \oplus \mathcal{F}) \).

Thus

\[
\langle F \oplus G \rangle = \langle F' \oplus G \rangle \\
\Rightarrow \quad \langle F \rangle = \langle F' \rangle
\]

and the homomorphism \( \alpha \) is also injective. \( \square \)

**Lemma 4.10** Suppose that \( \mathcal{A} \) and \( \mathcal{B} \) are unital graded \( C^* \)-categories that are equivalent to \( C^* \)-algebras. Let \( p, q \in \mathbb{N} \). Then there is a natural isomorphism

\[
K_1 \mathcal{QD}(\mathcal{A}, \mathcal{B} \hat{\otimes} \mathcal{F}_{p,q}) \cong K_1(\mathcal{QD}(\mathcal{A}, \mathcal{B}) \hat{\otimes} \mathcal{F}_{p,q})
\]

**Proof:** Consider the Clifford algebra \( \mathcal{F}_{1,0} \) generated by an element \( e \) such that \( e^2 = 1 \). We can define a natural \( C^* \)-functor \( \alpha: \mathcal{QD}(\mathcal{A}, \mathcal{B}) \hat{\otimes} \mathcal{F}_{1,0} \rightarrow \mathcal{QD}(\mathcal{A}, \mathcal{B} \hat{\otimes} \mathcal{F}_{1,0}) \) by mapping the Hilbert \( (\mathcal{A}, \mathcal{B}) \)-bimodule \( \mathcal{E} \) to the Hilbert \( (\mathcal{A}, \mathcal{B} \hat{\otimes} \mathcal{F}_{1,0}) \)-bimodule \( \mathcal{E} \hat{\otimes} \mathcal{F}_{1,0} \) and by mapping the morphism \( x \otimes 1 + y \otimes e \), where \( x, y \in Hom(\mathcal{E}, \mathcal{E}')_{\mathcal{QD}(\mathcal{A}, \mathcal{B})} \), to the operator \( T \) defined by the formula

\[
T_A(\eta \otimes 1 + \xi \otimes e) = x(\eta) \otimes 1 + x(\xi) \otimes e + (-1)^{\text{deg}(\eta)} y(\eta) \otimes e + (-1)^{\text{deg}(\xi)} y(\xi) \otimes 1
\]

It is clear that the \( C^* \)-functor \( \alpha \) is faithful, with image the full subcategory of the \( C^* \)-category \( \mathcal{QD}(\mathcal{A}, \mathcal{B} \hat{\otimes} \mathcal{F}_{1,0}) \) in which the set of objects is the set of Hilbert \( (\mathcal{A}, \mathcal{B}) \)-bimodules of the form \( \mathcal{E} \hat{\otimes} \mathcal{F}_{1,0} \), where \( \mathcal{E} \) is a Hilbert \( (\mathcal{A}, \mathcal{B}) \)-bimodule. By the Kasparov stabilisation theorem we obtain a natural isomorphism

\[
K_1 \mathcal{QD}(\mathcal{A}, \mathcal{B} \hat{\otimes} \mathcal{F}_{1,0}) \cong K_1(\mathcal{QD}(\mathcal{A}, \mathcal{B}) \hat{\otimes} \mathcal{F}_{1,0})
\]
A similar calculation yields a natural isomorphism

\[ K_1 QD(A, B \hat{\otimes} F_{0,1}) \cong K_1 (QD(A, B) \hat{\otimes} F_{0,1}) \]

The desired result now follows by induction since we have isomorphisms

\[ F_{p,q} \hat{\otimes} F_{r,s} \cong F_{p+r,q+s} \]

Because of the above lemma, theorem 4.9, and the Bott periodicity theorem it makes sense to define further KK-theory groups by the formula

\[ KK^{p-q}(A, B) = K(K(A, B \hat{\otimes} F_{p,q})) \]

We have natural isomorphisms

\[ KK^{p-q}(A, B) \cong K_1 (QD(A, B) \hat{\otimes} F_{p,q}) \cong K_{1-(p-q)} QD(A, B) \]

We can use the above notion of ‘duality’ to define a spectrum for KK-theory. This idea comes from [7].

**Definition 4.11** Let \( A \) and \( B \) be small graded \( C^* \)-categories. Then we define the \( KK \)-theory spectrum of the \( C^* \)-categories \( A \) and \( B \) to be the spectrum

\[ \mathbb{K}K(A, B) = \Omega K QD(A, B) \]

We can similarly define \( K \)-homology spectra.

**Definition 4.12** Let \( A \) be a small graded \( C^* \)-category. Then we define the \( K \)-homology spectrum of the \( C^* \)-category \( A \) to be the spectrum

\[ K_{\text{hom}}(A) = \Omega K QD(A, F) \]

Suppose that \( X \) is a locally compact Hausdorff space. Then the above definition gives us a spectrum for locally-finite \( K \)-homology

\[ K_{\text{hom}}^l(X) = \Omega K QD(C_0(X), F) \]

A slight adjustment delivers a spectrum for ordinary rather than locally finite \( K \)-homology

\[ K_{\text{hom}}(X) = \lim_{K \text{compact } \subseteq X} K^l_{\text{hom}}(K) \]

Finally, let us note that we can recover the definition of \( K \)-theory from that of \( KK \)-theory.

**Proposition 4.13** Let \( B \) be a small \( \sigma \)-unital graded \( C^* \)-category. Then we have a natural stable equivalence of spectra

\[ \mathbb{K}K(F, B) \simeq K(B) \]

**Proof:** Observe that

\[ \mathbb{K}K(F, B) = \Omega K Q(B) \]

The result now follows from theorem 3.26. \( \square \)
4.3 Products

The purpose of this section is to define a few special cases of the Kasparov product for the $KK$-theory of $C^*$-categories. The products we consider in this section can be expressed as natural morphisms in the stable category of symmetric spectra.

We begin with an aside on trivially graded $C^*$-categories.

**Definition 4.14** Let $A$ be a small unital trivially graded $C^*$-category. For each object $A \in \text{Ob}(A_{\oplus})$ we form the set of projections

$$P(A) = \{ p \in \text{Hom}(A, A) \mid p = p^*, p^2 = p \}$$

We write $p \sim_h q$ when the projections $p \oplus 0_B \oplus 0_C$ and $0_A \oplus q \oplus 0_C$ lie in the same path-component of the space $P(A \oplus B \oplus C)$ for some object $C$. We write $[p]$ for the equivalence class containing the projection $p$.

Form the set $V_0(A)$ of all equivalence classes of projections in the $C^*$-category $A_{\oplus}$. Then the set $V_0(A)$ is an abelian semigroup with group operation induced by taking the direct sum, $p \oplus q$, of projections $p$ and $q$. The following result comes from [18]. It is a consequence of the Bott periodicity theorem.

**Theorem 4.15** The $K$-theory group $K_0(A)$ is naturally isomorphic to the Grothendieck completion of the semigroup $V_0(A)$.

The first of the products we need to consider also comes from [18].

**Theorem 4.16** There is a natural product $K(A) \wedge K(B) \to K(A \hat{\otimes} B)$ in the stable category of symmetric spectra. When the $C^*$-categories $A$ and $B$ are trivially graded and unital, the induced map $K_0(A) \otimes K_0(B) \to K_0(A \otimes B)$ is defined by the formula

$$([p], [q]) \mapsto [p \otimes q]$$

The above product is called the exterior product of $K$-theory spectra.

**Proposition 4.17** Let $A$ and $B$ be small $\sigma$-unital graded $C^*$-categories. Then there is a canonical $C^*$-functor

$$P: A \hat{\otimes} QD(A, B) \to Q(B)$$

The $C^*$-functor $P$ is natural in the variable $B$ in the obvious sense and natural in the variable $A$ in the sense that if we have a $C^*$-functor $F: A \to A'$ then there is an induced commutative diagram

$$
\begin{array}{ccc}
A \hat{\otimes} QD(A, B) & \xrightarrow{P} & Q(B) \\
\uparrow & & \uparrow \\
A \hat{\otimes} QD(A', B) & \xrightarrow{P} & Q(B) \\
\downarrow & & \downarrow \\
A' \hat{\otimes} QD(A', B) & \xrightarrow{P} & Q(B)
\end{array}
$$
Proof: We can define such a $C^*$-functor by writing

\[ P(A \otimes E) = E(-, A) \quad P(x \otimes T) = x T_A \]

for all objects $A \otimes E \in \text{Ob}(A \hat{\otimes} \mathcal{QD}(A, B))$ and morphisms $x \otimes T \in \text{Hom}(A \otimes E, B \otimes E')_{A \hat{\otimes} \mathcal{QD}(A, B)}$. \qed

**Corollary 4.18** There is a natural morphism

\[ \mathbb{K}(A) \land \mathbb{K}(A, B) \to \mathbb{K}(B) \]

in the stable category of symmetric spectra

**Proof:** By the previous proposition and theorem 3.26 we have a natural morphism

\[ \Omega \mathbb{K}(A \hat{\otimes} \mathcal{QD}(A, B)) \to \Omega \mathbb{K}(Q(B)) \to \mathbb{K}(B) \]

Composition with the exterior product gives us a natural morphism

\[ \mathbb{K}(A) \land \mathbb{K}(A, B) \to \Omega \mathbb{K}(A \hat{\otimes} \mathcal{QD}(A, B)) \to \mathbb{K}(B) \]

as required. \qed

**Proposition 4.19** Let $A$, $B$, and $C$ be small $\sigma$-unital graded $C^*$-categories. Then there is a canonical $C^*$-functor

\[ D : \mathcal{QD}(A, B) \to \mathcal{QD}(A \hat{\otimes} C, B \hat{\otimes} C) \]

The $C^*$-functor $D$ is natural in the variables $A$ and $B$ and natural in the variable $C$ in the sense that for any $C^*$-functor $F : C \to C'$ there is an induced commutative diagram

\[
\begin{array}{ccc}
\mathcal{QD}(A, B) & \xrightarrow{D} & \mathcal{QD}(A \hat{\otimes} C, B \hat{\otimes} C) \\
\| & & \downarrow \\
\mathcal{QD}(A, B) & \xrightarrow{D} & \mathcal{QD}(A \hat{\otimes} C', B \hat{\otimes} C')
\end{array}
\]

**Proof:** We can define such a $C^*$-functor by writing

\[ D(E) = E \hat{\otimes} C \quad D(T) = T \otimes 1 \]

for all objects $E \in \text{Ob}(\mathcal{QD}(A, B))$ and morphisms $T \in \text{Hom}(E, E')_{\mathcal{QD}(A, B)}$. \qed

We can use the above construction to define a ‘slant product’ between $K$-theory and $K$-homology.

**Theorem 4.20** There is a canonical morphism

\[ S : \mathbb{K}(A \hat{\otimes} B) \land \mathbb{K}_\text{hom}(A) \to \mathbb{K}(B) \]
in the stable category of symmetric spectra. The morphism $S$ is natural in the variable $B$ in the obvious sense and natural in the variable $A$ in the sense that a $C^*$-functor $F: A \to A'$ induces a commutative diagram

$$
\begin{array}{ccc}
\mathbb{K}(A \hat{\otimes} B) \wedge \mathbb{K}_{\text{hom}}(A) & \xrightarrow{S} & \mathbb{K}(B) \\
\uparrow & & \uparrow \\
\mathbb{K}(A \hat{\otimes} B) \wedge \mathbb{K}_{\text{hom}}(A') & \xrightarrow{S} & \mathbb{K}(B) \\
\downarrow & & \downarrow \\
\mathbb{K}(A' \hat{\otimes} B) \wedge \mathbb{K}_{\text{hom}}(A') & \xrightarrow{S} & \mathbb{K}(B) \\
\end{array}
$$

**Proof:** As usual we write $\mathbb{K}_{\text{hom}}(A) = \mathbb{K}\mathbb{K}(A, F)$. By the above proposition and corollary 4.18 we can form a morphism

$$
\mathbb{K}(A \hat{\otimes} B) \wedge \mathbb{K}_{\text{hom}}(B) \to \mathbb{K}(A \hat{\otimes} B) \wedge \mathbb{K}\mathbb{K}(A \hat{\otimes} B, B) \to \mathbb{K}(B)
$$

$$\square$$

5 Assembly

5.1 Finitely Generated Projective Hilbert Modules

Let $A$ be a (trivially graded) unital $C^*$-algebra. Recall that a Hilbert $A$-module, $E$, is called finitely generated and projective if we can find a Hilbert $A$-module $E'$ and an isomorphism of Hilbert modules $\phi: E \oplus E' \to A^n$ for some $n$. A finitely generated Hilbert module defines an element of the $K$-theory group $K_0(A)$.

We can generalise this idea to see how certain Hilbert modules over a (trivially graded) unital $C^*$-category $A$ satisfying similar conditions defines elements of the $K$-theory group $K_0(A)$.

**Definition 5.1** Let $A$ be a small unital $C^*$-category. Then a Hilbert $A$-module, $E$, is called finitely generated and projective if there is a Hilbert $A$-module $E'$ such that the direct sum $E \oplus E'$ is isomorphic to the direct sum of finitely many Hilbert $A$-modules of the form $\text{Hom}(\cdot, A)_A$.

We write $\mathcal{L}(A_{\text{fgp}})$ to denote the category of finitely generated projective Hilbert $A$-modules and bounded operators.

**Lemma 5.2** The natural inclusion $i: A \hookrightarrow \mathcal{L}(A_{\text{fgp}})$ defined by mapping an object $A \in \text{Ob}(A)$ to the Hilbert $A$-module $\text{Hom}(\cdot, A)_A$ induces a stable equivalence of $K$-theory spectra

$$
i_*: \mathbb{K}(A) \to \mathbb{K}\mathcal{L}(A_{\text{fgp}})
$$

**Proof:** Consider supersymmetries $x, y \in SS(E \hat{\otimes} F_{p,q})$ for some finitely generated projective Hilbert $A$-module $E$. Since the module $E$ is finitely generated and projective, there is a Hilbert $A$-module $E'$ and objects $A_1, \ldots, A_n \in \text{Ob}(A)$ such that

$$E \oplus E' \cong \text{Hom}(\cdot, A_1) \oplus \cdots \oplus \text{Hom}(\cdot, A_n)$$
Choose a supersymmetry \( E \in SS(E' \hat{\otimes} F_{p,q}) \). Then, looking at \( K \)-theory classes:
\[
\langle x \rangle - \langle y \rangle = \langle x + E \rangle - \langle y + E \rangle
\]
which by the above isomorphism is the image of a \( K \)-theory class under the map induced by the \( C^* \)-functor \( i \). Hence the induced map
\[
i_* : K_1(A \hat{\otimes} F_{p,q}) \rightarrow K_1(L(A_{fgp}) \hat{\otimes} F_{p,q})
\]
is surjective. Injectivity of the induced map \( i_* \) can be proved similarly.

By the Bott periodicity theorem it follows that the induced maps \( i_* : K_n(A) \rightarrow K_n(L(A_{fgp})) \) are all isomorphisms. Hence the induced map of \( K \)-theory spectra
\[
i_* : \mathbb{K}(A) \rightarrow \mathbb{K}(L(A_{fgp}))
\]
is a stable equivalence. \( \Box \)

Now, let \( A \) be a trivially graded unital \( C^* \)-category. Then it is shown in [18] that a projection, \( p \), in the additive completion \( A \oplus \) defines an element of the zeroth space in the \( K \)-theory spectrum:
\[
[p] \in \mathbb{K}(A)_0
\]
If \( E \) is a finitely generated projective Hilbert \( A \)-module, there is an associated projection \( 1_E : E \rightarrow E \) in the category \( L(A_{fgp}) \). By the above theorem we have an associated \( K \)-theory class
\[
[E] = i_*^{-1}[1_E] \in \mathbb{K}(A)_0
\]
By theorem [4.20] we have the following result.

**Proposition 5.3** Let \( A \) and \( B \) be trivially graded unital \( C^* \)-categories. Let \( E \) is a finitely generated projective Hilbert \( A \otimes B \)-module. Then there is an induced morphism of spectra:
\[
[E] \otimes : \mathbb{K}_{\text{hom}}(A) \rightarrow \mathbb{K}(B)
\]
For all \( C^* \)-functors \( F : A \rightarrow A' \) and \( G : B \rightarrow B' \) there are commutative diagrams
\[
\begin{array}{ccc}
\mathbb{K}_{\text{hom}}(A) & \xrightarrow{[E] \otimes} & \mathbb{K}(B) \\
\uparrow & & \uparrow \\
\mathbb{K}_{\text{hom}}(A') & \xrightarrow{F_*[E] \otimes} & \mathbb{K}(B)
\end{array}
\]
and
\[
\begin{array}{ccc}
\mathbb{K}_{\text{hom}}(A) & \xrightarrow{[E] \otimes} & \mathbb{K}(B) \\
\| & & \downarrow \\
\mathbb{K}_{\text{hom}}(A) & \xrightarrow{G_*[E] \otimes} & \mathbb{K}(B')
\end{array}
\]
\( \Box \)
5.2 The Analytic Assembly Map

We are now ready to apply our machinery in order to give a natural description of the analytic assembly map at the level of spectra. Such a description will enable us to use results from [28] to characterise the assembly map.

Definition 5.4 Let $X$ be a path-connected locally compact Hausdorff topological space that is locally path-connected. Let $K \subseteq X$ be a compact subspace. Choose a basepoint $x_0 \in K$. Then we form the space

$$(\tilde{K}, x_0) = \{ \tilde{x} \in C([0, 1] \to X) \mid \tilde{x}(0) \in K, \tilde{x}(1) = x_0 \}$$

where the equivalence relation $\sim$ is that of paths with fixed endpoints being homotopic in the space $X$.

We write $p: (\tilde{K}, x_0) \to X$ to denote the natural map defined by sending a path $\tilde{x}$ to the initial point $\tilde{x}(0)$. The following result is easy to see.

Proposition 5.5 Let $x \in K$. Then there is an open subset $U \subseteq K$ such that $x \in U$ and we have a homeomorphism $p^{-1}[U] \approx U \times \pi_1(X, x_0)$.

Let $\tilde{x}_1, \tilde{x}_2: [0, 1] \to X$ be paths such that $\tilde{x}_1(0) = \tilde{x}_2(1)$. Then we write $\tilde{x}_1 \tilde{x}_2$ to denote the path obtained by moving along the path $\tilde{x}_2$ followed by the path $\tilde{x}_1$. We can define an action of the fundamental group $\pi_1(X, x_0)$ on the space $(\tilde{K}, x_0)$ by the formula $(g, \tilde{x}) \mapsto g \tilde{x}$.

Following [12], there is a canonical Hilbert $C(K) \otimes C^*_{\max} \pi_1(X, x_0)$-module based on this action.

Proposition 5.6 Define $F_K(x_0)$ to be the space of all continuous maps $\mu: (\tilde{K}, x_0) \to C^*_{\max} \pi_1(X, x_0)$ such that $\mu(g \tilde{x}) = g \mu(\tilde{x})$ for all group elements $g \in \pi_1(X, x_0)$ and paths $\tilde{x} \in (\tilde{K}, x_0)$. Then the space $F_K(x_0)$ is a finitely generated projective Hilbert $C(K) \otimes C^*_{\max} \pi_1(X, x_0)$-module with inner product

$$\langle \mu, \mu' \rangle(p(\tilde{x})) = \mu^*(\tilde{x}) \mu'(\tilde{x})$$

and $C(K) \otimes C^*_{\max} \pi_1(X, x_0)$-action

$$\mu(f \otimes g)(\tilde{x}) = f(p(\tilde{x})) \mu(\tilde{x}) g$$

By theorem 5.3 we can define a mapping

$$[F_K(x_0)] \otimes: K_*(K) \to K_*(C^*_{\max} \pi_1(X, x_0))$$

Recall that we can define the $K$-homology groups of the space $X$ to be the direct limit of the $K$-homology groups $K_*(K)$ where $K$ is a compact subspace of $X$. We therefore have a map

$$\beta_*: K_*(X) \to K_* C^*_{\max} \pi_1(X, x_0)$$

\[\text{This notation may appear 'backwards', but is consistent with other conventions in this article such as writing xy for the composition of a morphism y with a morphism x in some category.}\]
Definition 5.7 The map $\beta_*$ is called the analytic assembly map.

We can generalise the above definition to form a basepoint-free map at the level of spectra. The key idea is the following construction from $[19]$.

Proposition 5.8 There is a functor $G \mapsto C^*_{\text{max}} G$ from the category of discrete groupoids to the category of unital $C^*$-categories such that

- There is a natural inclusion $G \hookrightarrow C^*_{\text{max}} G$

- Let $f : G \to H$ be an equivalence of groupoids. Then the induced map $f_* : C^*_{\text{max}} G \to C^*_{\text{max}} H$ is an equivalence of $C^*$-categories

- If $G$ is a group then the $C^*$-category $C^*_{\text{max}} G$ is the maximal $C^*$-algebra of the group $G$

The groupoid $C^*_{\text{max}}(G)$ is called the maximal $C^*$-category of the groupoid $G$. It is constructed similarly to the reduced $C^*$-category of a groupoid that is considered in $[4]$.

Now, let $\pi(X)|_K$ be the full subgroupoid of the fundamental groupoid $\pi(X)$ in which the objects are the points of the subspace $K$.

Choose a point $x_0 \in K$. For each point $x \in K$ choose a path, $\gamma_x$, from the point $x_0$ to the point $x$. Then there is a unitary element $u_x = 1 \otimes |\gamma_x| \in \text{Hom}(x_0, x)_{C(K) \otimes C^*_{\text{max}}(\pi(X)|_K)}$ for each point $x \in K$. The unital $C^*$-category $C(K) \otimes C^*_{\text{max}}(\pi(X)|_K)$ is therefore equivalent to the $C^*$-algebra $C(K) \otimes C^*_{\text{max}}(X, x_0)$. Similarly the $C^*$-category $C^*_{\text{max}}(\pi(X)|_K)$ is equivalent to the $C^*$-algebra $C^*_{\text{max}}(X, x_0)$.

Definition 5.9 For a point $x \in K$, define $\mathcal{F}_K(x)$ to be the set of maps

$$\{\mu_y : (\tilde{K}, y) \to \text{Hom}(x, y)_{C^*_{\text{max}}(\pi(X)|_K)} \mid y \in K\}$$

such that

$$\mu_y(g\tilde{y}) = g\mu_y(\tilde{y})$$

for all groupoid elements $g \in \text{Hom}(y, y')_{\pi(X)|_K}$ and paths $\tilde{y} \in (\tilde{K}, y)$.

The collection, $\mathcal{F}_K$, of spaces of the form $\mathcal{F}_K(x)$ is a Hilbert $C(K) \otimes C^*_{\text{max}}(\pi(X)|_K)$-module with inner product

$$\langle \{\mu_y\}, \{\mu_y'\} \rangle(p(\tilde{y})) = \mu_y^*(\tilde{y})\mu_y'^*(\tilde{y})$$

and $C(K) \otimes C^*_{\text{max}}(\pi(X)|_K)$-action

$$\mu(f \otimes g)(\tilde{y}) = f(p(\tilde{y}))\mu(\tilde{y})g$$

By proposition $3.3$ we obtain a morphism

$$\mathcal{F}_K : \mathbb{K}_{\text{hom}}(K) \to \mathbb{K}(C^*_{\text{max}}(\pi(X)|_K))$$

At the level of homotopy groups we have an induced map

$$[\mathcal{F}_K] \otimes : K^*(K) \to K^*(C^*_{\text{max}}(\pi(X)|_K))$$

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Lemma 5.10 We have a commutative diagram

\[
\begin{array}{ccc}
K_s(K) & \xrightarrow{[F_K(x_0)]\otimes} & K_s C_{\text{max}}^* \pi_1(X, x_0) \\
\|

\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
K_s(K) & \xrightarrow{[F]\otimes} & K_s C_{\text{max}}^* \pi(X)|_K \\
\end{array}
\]

such that the vertical map on the right is an isomorphism.

Proof: The result follows immediately from the fact that the \(C^*\)-category \(C(K) \otimes C_{\text{max}}^* \pi(X)|_K\) is equivalent to the \(C^*\)-algebra \(C(K) \otimes C_{\text{max}}^* (X, x_0)\) and the \(C^*\)-category \(C_{\text{max}}^* \pi(X)|_K\) is equivalent to the \(C^*\)-algebra \(C_{\text{max}}^* \pi(X, x_0)\). \(\square\)

Lemma 5.11 Let \(s: (X, K) \to (Y, L)\) be a map of pairs of spaces, where the spaces \(X\) and \(Y\) are path-connected Hausdorff spaces, and the subspaces \(K \subseteq X\) and \(L \subseteq Y\) are compact. Then we have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{K}_{\text{hom}}(K) & \xrightarrow{[F_K]\otimes} & \mathbb{K}C_{\text{max}}^* \pi(X)|_K \\
\downarrow & & \downarrow \\
\mathbb{K}_{\text{hom}}(L) & \xrightarrow{[F_L]\otimes} & \mathbb{K}C_{\text{max}}^* \pi(Y)|_L \\
\end{array}
\]

Proof: Consider the induced \(C^*\)-functors

\[
1 \otimes s^* : C(K) \otimes C_{\text{max}}^* \pi(X)|_K \to C(K) \otimes C_{\text{max}}^* \pi(Y)|_L
\]

\[
s^* \otimes 1 : C(L) \otimes C_{\text{max}}^* \pi(Y)|_L \to C(K) \otimes C_{\text{max}}^* \pi(Y)|_L
\]

Then by proposition 5.13 we have commutative diagrams

\[
\begin{array}{ccc}
\mathbb{K}_{\text{hom}}(K) & \xrightarrow{[F_K]\otimes} & \mathbb{K}C_{\text{max}}^* \pi(X)|_K \\
\|

\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
\mathbb{K}_{\text{hom}}(K) & \xrightarrow{(1 \otimes s^*)[F_K]\otimes} & \mathbb{K}C_{\text{max}}^* \pi(Y)|_L \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathbb{K}_{\text{hom}}(K) & \xrightarrow{[s^* \otimes 1][F_L]\otimes} & \mathbb{K}C_{\text{max}}^* \pi(Y)|_L \\
\downarrow & & \downarrow \\
\mathbb{K}_{\text{hom}}(L) & \xrightarrow{[F_L]\otimes} & \mathbb{K}C_{\text{max}}^* \pi(Y)|_L \\
\end{array}
\]

Consider the Hilbert \(C(K) \otimes C_{\text{max}}^* (\pi(Y)|_L)\)-module \(\mathcal{E}\), where the space \(\mathcal{E}(x)\) consists of all sets of maps

\[
\{\mu_y: (\tilde{K}, y) \to Hom(s(x), s(y))C_{\text{max}}^* \pi(Y)|_L \mid y \in K\}
\]

such that

\[
\mu_y'(g\tilde{y}) = s^*(g)\mu_y(\tilde{y})
\]

for all paths \(\tilde{y} \in (\tilde{K}, y)\) and groupoid elements \(g \in Hom(y, y')\pi(X)|_K\).

There are obvious inclusions \(i: (1 \otimes s^*)[F_K] \to \mathcal{E}\) and \(j: (s^* \otimes 1)[F_L] \to \mathcal{E}\).

By proposition 5.13, for each point \(x \in K\) we can find open sets \(U \subseteq K\) and \(V \subseteq L\) such that \(x \in U, s(x) \in V\), and

\[
p^{-1}[U] \approx U \times C_{\text{max}}^* \pi_1(X, x)
\]

\[
p^{-1}[V] \approx V \times C_{\text{max}}^* \pi_1(X, x)
\]

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Define modules

\[ E|_U(x) = \{ \phi |_{p^{-1}[U]} \mid \phi \in E(x) \} \]
\[ \mathcal{F}_K|_U(x) = \{ \phi |_{p^{-1}[U]} \mid \phi \in \mathcal{F}_K(x) \} \]
\[ \mathcal{F}_L|_V(x) = \{ \phi |_{p^{-1}[V]} \mid \phi \in \mathcal{F}_L(x) \} \]

Then for all compact sets \( A \subseteq U \) and \( B \subseteq V \) there are isomorphisms

\[ E|_A \cong C(A) \otimes C_{\max}^* \pi(Y)|_L \]
\[ \mathcal{F}_K|_A \cong C(A) \otimes C_{\max}^* \pi(X)|_K \]
\[ \mathcal{F}_L|_B \cong C(B) \otimes C_{\max}^* \pi(Y)|_L \]

Therefore the inclusions \( i: (1 \otimes s_*)[\mathcal{F}_K] \hookrightarrow \mathcal{E} \) and \( j: (s^* \otimes 1)[\mathcal{F}_L] \hookrightarrow \mathcal{E} \) are isomorphisms. Hence the Hilbert modules \( (1 \otimes s_*)[\mathcal{F}_K] \) and \( (s^* \otimes 1)[\mathcal{F}_L] \) are actually equal so we have a commutative diagram

\[
\begin{array}{ccc}
K_{\text{hom}}(K) & \xrightarrow{\mathcal{F}_K \otimes} & K C_{\max}^* \pi(X)|_K \\
\downarrow & & \downarrow \\
K_{\text{hom}}(L) & \xrightarrow{\mathcal{F}_L \otimes} & K C_{\max}^* \pi(Y)|_L
\end{array}
\]

Forming direct limits we obtain a morphism

\[ \beta : K_{\text{hom}}(X) \rightarrow K C_{\max}^* \pi(X) \]

**Theorem 5.12** The map \( \beta \) is natural, and is a weak equivalence when the space \( X \) is a single point. The induced map between stable homotopy groups

\[ \beta_* : K_*(X) \rightarrow K_* C_{\max}^* \pi_1(X) \]

is the analytic assembly map. \( \square \)

**Proof:** Naturality follows immediately from lemma 5.11. The fact that the induced map between stable homotopy groups is the usual analytic assembly map is a consequence of lemma 5.10. The fact that the map \( \beta \) is a weak equivalence when the space \( X \) is a single point now follows from the fact that the analytic assembly map \( \beta_* \) is an isomorphism when the space \( X \) is a single point. \( \square \)

The above theorem can be used to give a characterisation of the analytic assembly map. The main tool is the following result of Weiss and Williams from [28].

**Theorem 5.13** Let \( F \) be a homotopy-invariant functor from spaces to spectra. Then there exists a homotopy-invariant functor \( F^\% \) from spaces to spectra and a natural transformation \( \alpha : F^\% \rightarrow F \) such that

- The functors \( X \mapsto \pi_* F^\%(X) \) form a generalised homology theory.

\[ 9 \]In this theorem all spaces have the homotopy type of \( CW \)-complexes

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The map $\alpha: F(\text{point}) \to F(\text{point})$ is a stable equivalence.

Further, the functor $F$ and map $\alpha$ are unique up to stable equivalence.

The transformation $\alpha$ is called the assembly map associated to the functor $F$. We can use the above theorem on the functor $X \mapsto K_{C^*_\text{max}}^* \pi(X)$ together with theorem 5.12 to obtain the main result of this article.

**Theorem 5.14** Let $F$ be a homotopy-invariant functor from spaces to spectra such that the functors $X \mapsto \pi_* F(X)$ form a generalised homology theory. Let $\alpha: F(X) \to K_{C^*_\text{max}}^* \pi(X)$ be a natural morphism of spectra which is a stable equivalence when the space $X$ is a single point. Then the map $\alpha$ is equivalent to the analytic assembly map.

Roughly speaking this result says that anything which resembles the analytic assembly map at the level of spectra actually is the analytic assembly map.

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