SELF-COMMUTATOR NORM OF HYPERSONAL TOEPLITZ OPERATORS

TRIEU LE

ABSTRACT. Chu and Khavinson recently obtained a lower bound for the norm of the self-commutator of a certain class of hyponormal Toeplitz operators on the Hardy space. Via a different approach, we offer a generalization of their result.

1. INTRODUCTION

We denote by $\mathbb{D}$ the open unit disk in the complex plane and $\partial \mathbb{D}$ its boundary, the unit circle. Recall that the Hardy space $H^2$ is the closed subspace of $L^2 = L^2(\partial \mathbb{D})$ consisting of all functions whose negative Fourier coefficients vanish. Let $P : L^2 \to H^2$ denote the orthogonal projection. For a bounded function $\varphi \in L^\infty = L^\infty(\partial \mathbb{D})$, the Toeplitz operator $T_\varphi : H^2 \to H^2$ is defined as

$$T_\varphi(u) = P(\varphi u) \quad \text{for all} \quad u \in H^2.$$ 

The study of Toeplitz operators on the Hardy space was initiated by the seminal paper [3] of Brown and Halmos in the sixties.

Define a linear operator $J : L^2 \to L^2$ by $J(u)(z) = \bar{z}u(\bar{z})$ for $u \in L^2$ and $z \in \partial \mathbb{D}$. It is immediate that $J$ is a unitary operator on $L^2$ and it is not hard to verify that $J$ maps $(H^2)^\perp$ onto $H^2$. For $\varphi \in L^\infty$, the Hankel operator $H_\varphi : H^2 \to H^2$ is defined as

$$H_\varphi(u) = J(I - P)(\varphi u) \quad \text{for} \quad u \in H^2.$$ 

We list below a few properties of Toeplitz and Hankel operators that shall be useful for us.

(a) For any $\varphi \in L^\infty$, we have $\|T_\varphi\| = \|\varphi\|_\infty$ and $T_\varphi^* = T_\bar{\varphi}$.

(b) For $f,g \in H^2$,

$$\langle T_\varphi f, g \rangle = \langle \varphi f, g \rangle.$$

(c) If $h$ belongs to $H^\infty$ (which is $H^2 \cap L^\infty$), then $T_h = M_h$, the operator of multiplication by $h$. In addition,

$$T_h(1) = \overline{h(0)}.$$

(d) For any $h_1, h_2 \in H^\infty$,

$$T_{h_1 \varphi h_2} = T_{h_1} T_\varphi T_{h_2}.$$

(e) For $u, f \in H^\infty$, we have $H_\varphi^+(u) = T_{\bar{z}} T_{u(\bar{z})}(f)$. 

2010 Mathematics Subject Classification. Primary 47B35.

Key words and phrases. Toeplitz operator; Hardy space; hyponormality; spectral area.
A bounded linear operator $T$ on a Hilbert space is \textit{hyponormal} if its self-commutator $[T^*, T] = T^*T - TT^*$ is positive. Recall that the spectrum of an operator $T$, denoted by $\text{sp}(T)$, is the set of all complex numbers $\lambda$ for which $T - \lambda I$ is not invertible, where $I$ is the identity operator. The celebrated Putnam’s Inequality \cite{putnam} asserts that for any hyponormal operator $T$,

$$\|[T^*, T]\| \leq \frac{\text{Area}(\text{sp}(T))}{\pi}.$$ 

For a function $\varphi \in H^\infty$, it is well known that the Toeplitz operator $T_\varphi$ is hyponormal and $\text{sp}(T_\varphi) = \varphi(\partial \mathbb{D})$. A lower bound for the norm of the commutator $[T_\varphi^*, T_\varphi]$ was obtained by D. Khavinson \cite{khavinson}. Combining with Putnam inequality, one deduces the isoperimetric inequality (see, e.g. \cite{putnam}). Recently, several papers \cite{putnam1, putnam2, putnam3} have investigated similar problems for analytic Toeplitz operators on the Bergman space.

On the Hardy space, hyponormal Toeplitz operators were characterized by Cowen in \cite{cowen}.

\textbf{Theorem 1} (Cowen). If $\varphi \in L^\infty(\partial \mathbb{D})$, where $\varphi = f + \bar{g}$ for $f, g$ in $H^2$, then $T_\varphi$ is hyponormal if and only if

$$g = c + T_h f$$

for some constant $c$ and some function $h \in H^\infty$ with $\|h\|_\infty \leq 1$.

\textbf{Remark 2}. In general, the function $h$ is not unique. For more details on this, see the discussion following the proof of \cite[Theorem 1]{cowen}.

Under the additional hypothesis $h(0) = 0$, Chu and Khavinson \cite{chu} recently obtained a lower bound for the norm of the self-commutator of $T_\varphi$.

\textbf{Theorem 3} (Chu-Khavinson). If $\varphi = f + T_h f$ for $f, h$ in $H^\infty$, $\|h\|_\infty \leq 1$ and $h(0) = 0$, then

$$\|[T_\varphi^*, T_\varphi]\| \geq \|f - f(0)\|_2^2 = \|P(\varphi) - \varphi(0)\|_2^2. \tag{1}$$

Due to the additional condition that $h(0) = 0$, Theorem \ref{thm3} is not applicable in more general situations. As an example, take $\varphi(z) = z + \bar{z}/2$. A direct calculation shows that

$$[T_\varphi^*, T_\varphi] = \frac{3}{4} e_0 \otimes e_0,$$

where $e_0(z) = 1$ for $z \in \partial \mathbb{D}$. Recall that for elements $u, v$ in $H^2$, we use $u \otimes v$ to denote the operator given by $(u \otimes v)(x) = (x, v)u$ for $x \in H^2$. We then have $\|[T_\varphi^*, T_\varphi]\| = \frac{3}{4}$ while $\|P(\varphi) - \varphi(0)\|_2 = 1$. This shows that inequality \eqref{eq1} is false in this case. More generally, as we shall see later, Theorem \ref{thm3} is not applicable if $\varphi = f + \lambda f$ with $0 < |\lambda| < 1$ and $f$ an inner function vanishing at the origin.

The purpose of this note is twofold. First, we offer an improved version of Theorem \ref{thm3} which is applicable even in the case $h(0) \neq 0$. Second, we present a different and more operator-theoretic proof than that of Chu and Khavinson.

We state here our main result.

\textbf{Theorem 4}. Let $\varphi = f + T_h f$ be a bounded harmonic function on the unit disk, where $f, h \in H^\infty$ with $\|h\|_\infty \leq 1$ and $|h(0)| < 1$. Put $\psi = f - h(0) T_h f$. Then

$$\|[T_\varphi^*, T_\varphi]\| \geq \frac{\|\psi - \psi(0)\|_2^2}{1 - |h(0)|^2}. \tag{2}$$
Remark 5. In the case $h(0) = 0$, we see that (2) reduces to Chu-Khavinson’s result. In the case $|h(0)| = 1$, the function $h$ is a unimodular constant function. It then follows that $[T^*_\varphi, T_\varphi] = 0$, which means that $T_\varphi$ is normal. We assume $|h(0)| < 1$ to avoid such trivial case.

Remark 6. If we fix $f$ and let $h$ vary in the unit ball of $H^\infty$, the norm $||T^*_\varphi, T_\varphi||$ is always bounded by $\|f\|_\infty^2$ (which follows from (3) in Section 2). As a consequence, the right-hand side of (2) remains bounded. We provide here a more direct argument to explain this. From the definition of $\psi$, we have $\psi = T_{1 - h(0)\bar{h}}(f)$. We then compute

$$
\|\psi - \psi(0)\|_2^2 \leq \|\psi\|_2^2 = \|T_{1 - h(0)\bar{h}}f\|_2^2 = \|T_f(1 - h(0)\bar{h})\|_2^2
$$

$$
\leq \|f\|_\infty^2 \|1 - h(0)\bar{h}\|_2^2 = \|f\|_\infty^2 (1 + |h(0)|^2 \|h\|_2^2 - 2|h(0)|^2)
$$

$$
\leq \|f\|_\infty^2 (1 - |h(0)|^2) \quad \text{(since $\|h\|_\infty \leq 1$),}
$$

which implies that the ratio $\|\psi - \psi(0)\|_2^2/(1 - |h(0)|^2)$ is bounded by $\|f\|_\infty^2$.

2. Proof of the main result

In this section we offer a proof of our result and discuss an application. We begin with a simple and probably well-known fact from Functional Analysis. We present here a quick proof.

Lemma 7. Let $T$ be a positive operator on a Hilbert space $\mathcal{H}$. Then for any $v \in \mathcal{H}$, the operator $S = \langle Tv, v \rangle T - \langle Tv \rangle \otimes (Tv)$ is positive as well.

Proof. Let $T^{1/2}$ denote the positive square root of $T$. For any $u \in \mathcal{H}$,

$$
\langle Su, u \rangle = \|T^{1/2}v\|^2 \|T^{1/2}u\|^2 - \langle T^{1/2}v, T^{1/2}u \rangle^2 \geq 0
$$

by Cauchy-Schwarz’s inequality. The conclusion of the lemma now follows. \(\square\)

Lemma 7 provides us with the following immediate consequence for Toeplitz operators with holomorphic symbols.

Proposition 8. Suppose that $h$ is a bounded holomorphic function with $\|h\|_\infty \leq 1$ and $|h(0)| < 1$. Put $\xi = (1 - \bar{h}(0)h)/\sqrt{1 - |h(0)|^2}$. Then

$$
I - T_hT_h \geq \xi \otimes \xi
$$

on the Hardy space $H^2$.

Proof. Because $\|h\|_\infty \leq 1$, the operator $T_h$ has norm at most 1. This implies that $I - T_hT_h \geq 0$. Applying Lemma 7 with $T = I - T_hT_h$ and $v = 1$ gives

$$
(1 - \|h\|_1^2)(I - T_hT_h) - (1 - T_hT_h(1)) \otimes (1 - T_hT_h(1)) \geq 0.
$$

Since $T_h1 = \bar{h}(0)$ and $T_hT_h1 = h(0)h$, we obtain the desired operator inequality. \(\square\)

We are now ready for the proof of our main result.

Proof of Theorem 3. Write $\varphi = f + \bar{g}$ with $g = T_hf$. We have $H_{\bar{g}} = T_{\bar{h}f}H_{\bar{f}}$ (see p. 811), where $h^*(z) = \bar{h}(\bar{z})$. We now compute

$$
[T^*_\varphi, T_\varphi] = [T_{f+g}, T_{f+\bar{g}}] = [T_f, T_f] - [T_{\bar{g}}, T_g]
$$

$$
= H_f^*H_f - H_{\bar{g}}^*H_{\bar{g}} = H_f^*(I - T_hT_{\bar{f}^*})H_{\bar{f}}.
$$
Proposition 8 then implies
\[ [T^\ast_\varphi, T^\ast_\nu] \geq H^\ast_f(\varphi \otimes \nu) H^\ast_f = H^\ast_f(\varphi) \otimes H^\ast_f(\nu), \]
where
\[ \eta = \frac{1 - h^*(0)}{\sqrt{1 - |h^*(0)|^2}} = \frac{1 - h(0)}{\sqrt{1 - |h(0)|^2}}. \]
As a consequence,
\[ \| [T^\ast_\varphi, T^\ast_\nu] \| \geq \| H^\ast_f(\varphi) \otimes H^\ast_f(\nu) \| = \| H^\ast_f(\varphi) \|_2. \]

On the other hand,
\[ H^\ast_f(\varphi) = T^\ast_\eta(T^\ast_f(\varphi)), \]
\[ \frac{1}{\sqrt{1 - |h(0)|^2}} T^\ast_{\eta}(f - h(0)T^\ast_h f) \]
\[ = \frac{1}{\sqrt{1 - |h(0)|^2}} T^\ast_{\eta}(f - h(0)g) \quad \text{(since } g = T^\ast_h f \text{)} \]
\[ = \frac{1}{\sqrt{1 - |h(0)|^2}} T^\ast_{\eta} \psi, \]
since \( \psi = f - h(0)g. \) We then have
\[ \| H^\ast_f(\varphi) \| = \frac{\| T^\ast_{\eta} \psi \|_2^2}{1 - |h(0)|^2} = \frac{\| \psi - \psi(0) \|_2^2}{\sqrt{1 - |h(0)|^2}}, \]
which, together with (4), gives the required inequality (2)
\[ \| [T^\ast_\varphi, T^\ast_\nu] \| \geq \frac{\| \psi - \psi(0) \|_2^2}{1 - |h(0)|^2}. \]

Theorem 4 provides a lower bound for the norm of \([T^\ast_\varphi, T^\ast_\nu]\) in terms of both the holomorphic and anti-holomorphic parts of \( \varphi. \) In the following corollary, we obtain a weaker estimate which only depends on the holomorphic part of \( \varphi. \)

Corollary 9. Let \( \varphi = f + T^\ast_h f \) be a bounded harmonic function on the unit disk, where \( f, h \in H^\infty \) with \( \| h \|_\infty \leq 1 \) and \( |h(0)| < 1. \) Then
\[ \| [T^\ast_\varphi, T^\ast_\nu] \| \geq \frac{(1 - |h(0)| \| h \|_\infty)^2}{1 - |h(0)|^2} \| f - f(0) \|_2. \]

Proof. Let us estimate the numerator of the right hand side of (2) in Theorem 4
\[ \| \psi - \psi(0) \|_2 \geq \| f - f(0) \|_2 - |h(0)| \cdot \| g - g(0) \|_2 \]
\[ = \| f - f(0) \|_2 - |h(0)| \cdot \| T^\ast_{\eta} g \|_2 \]
\[ = \| f - f(0) \|_2 - |h(0)| \cdot \| T^\ast_{\eta} T^\ast_h(f) \|_2 \]
\[ = \| f - f(0) \|_2 - |h(0)| \cdot \| T^\ast_{\eta} T^\ast_h(f - f(0)) \|_2 \quad \text{(since } T^\ast_{\eta}(f(0)) = 0 \text{)} \]
\[ \geq \| f - f(0) \|_2 - |h(0)| \| h \|_\infty \cdot \| f - f(0) \|_2 \quad \text{(since } \| T^\ast_{\eta} T^\ast_h \| \leq \| h \|_\infty \text{).} \]

Combining with (2) gives
\[ \frac{\| \psi - \psi(0) \|_2^2}{1 - |h(0)|^2} \geq \frac{(1 - |h(0)| \| h \|_\infty)^2}{1 - |h(0)|^2} \| f - f(0) \|_2^2 \]
as desired. \( \square \)
Example 10. Consider \( \varphi(z) = z + \bar{z}/2 \) so that \( f(z) = z \) and \( h(z) = \frac{1}{2} \). We have seen in Introduction that \( \| [T_{\varphi}^*, T_{\varphi}] \| = \frac{3}{4} \). On the other hand, the right-hand side of (5) is also equal to \( \frac{3}{4} \). Consequently, (5) is in fact an equality in this case.

Using Putnam’s Inequality and Corollary 9, we obtain

\[
\text{Corollary 11. If } \varphi = f + T_h^* f \text{ for } f, h \in H^\infty \text{ with } \|h\|_\infty \leq 1 \text{ and } |h(0)| < 1, \text{ then }
\text{Area}(\text{sp}(T_{\varphi})) \geq \pi \frac{(1 - |h(0)| \|h\|_\infty)^2}{1 - |h(0)|^2} \|f - f(0)\|^2.
\]

Given a bounded function \( \varphi \) for which \( T_{\varphi} \) is hyponormal, the existence of the function \( h \) in the representation \( \varphi = f + T_h^* f \) is not unique. Our lower estimate of the norm of \([T_{\varphi}^*, T_{\varphi}]\) in Theorem 4 depends on the value of \( h(0) \). In some cases, it turns out that \( h(0) \) is independent of the choice of \( h \). We illustrate this in the following example.

Example 12. Let \( \chi \) be an inner function with \( \chi(0) = 0 \). Suppose \( f \) is a non-constant polynomial of \( \chi \) and \( g \) belongs to \( H^\infty \) such that \( T_{f+g} \) is hyponormal. Then for any function \( h \in H^\infty \) satisfying \( \|h\|_\infty \leq 1 \) and \( g = c + T_h f \) for some constant \( c \), the value \( h(0) \) is independent of \( h \). In the case \( f = \chi \), we have \( h(0) = \langle f, g \rangle \).

**Proof.** Since \( f \) is a non-constant polynomial of \( \chi \), there exist \( M \geq 1 \) and complex numbers \( c_1, \ldots, c_M \) such that \( c_M \neq 0 \) and

\[
f = c_0 + \cdots + c_M \chi^M.
\]

We then compute

\[
\langle \chi^M, g \rangle = \langle \chi^M, T_h f + c \rangle = \langle \chi^M, T_h f \rangle \quad \text{(since } \chi^M(0) = 0) = \langle \chi^M, h, f \rangle = \sum_{j=0}^M \bar{c}_j \langle \chi^M, \chi^j \rangle = \bar{c}_M h(0) = \langle \chi^M, f \rangle h(0).
\]

It follows that

\[
h(0) = \frac{\langle \chi^M, g \rangle}{\langle \chi^M, f \rangle},
\]

which is independent of the choice of \( h \). In the case \( f = \chi \), we have \( M = 1 \) and hence \( h(0) = \langle f, g \rangle \). \( \Box \)

Remark 13. The lower estimate in Theorem 4 makes use of the value of \( h \) at the origin. For any \( a \in \mathbb{D} \), we briefly discuss here how an estimate involving \( h(a) \) may be obtained. However, the formula is a little more complicated. Recall that \( k_a(z) = \sqrt{1 - |a|^2}/(1 - \bar{a}z) \) is the normalized reproducing kernel of the Hardy space at \( a \). We shall write

\[
\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}
\]

for the Mobius automorphism of the unit disk that interchanges \( a \) and the origin. Note that \( \varphi_a \circ \varphi_a(z) = z \) for all \( z \in \mathbb{D} \). Define the operator \( W_a \) by

\[
W_a(u) = k_a \cdot (u \circ \varphi_a), \quad u \in L^2(\partial \mathbb{D}).
\]

A change-of-variables on the unit circle shows that \( W_a \) is a unitary operator on \( L^2(\partial \mathbb{D}) \). It is well known that \( H^2 \) is a reducing subspace of \( W_a \) and

\[
W_a^* T_{\varphi} W_a = T_{\varphi \circ \varphi_a}.
\]
for any bounded \( \varphi \). As a consequence, \( T_\varphi \) is hyponormal if and only if \( T_{\varphi \circ \varphi_a} \) is hyponormal and their self-commutators have the same norm. Note that if \( g = T_\overline{h} f \) then it can be checked that \( g \circ \varphi_a = T_\overline{h} \circ \varphi_a (f \circ \varphi_a) + c \) for some constant \( c \). Applying Theorem 2 for \( \varphi \circ \varphi_a \) gives

\[
\|[T_{\varphi \circ \varphi_a}^*, T_{\varphi \circ \varphi_a}]\| \geq \frac{\| f \circ \varphi_a - (h \circ \varphi_a(0))g \circ \varphi_a \|^2 - |f \circ \varphi_a(0) - (h \circ \varphi_a(0))g \circ \varphi_a(0)|^2}{1 - |h \circ \varphi_a(0)|^2}.
\]

(6)

Since \( W_a \) is a unitary operator, the first term in the numerator of (6) is equal to

\[
\|W_a(f \circ \varphi_a - h(a)g \circ \varphi_a)\|^2_2 = \|(f - h(a)g)k_a\|^2_2.
\]

We then have

\[
\|[T_\varphi, T_\varphi]\| = \|[T_{\varphi \circ \varphi_a}^*, T_{\varphi \circ \varphi_a}]\| \geq \frac{\|(f - h(a)g)k_a\|^2_2 - |f(a) - h(a)g(a)|^2}{1 - |h(a)|^2}.
\]

(7)

It is possible to obtain estimate (7) by following the proof of Theorem 4. One needs to modify Proposition 8 by setting \( v = k_a \) instead of \( v = 1 \). However, some parts of calculation are a bit more complicated. We leave this for the interested reader.

Acknowledgments. The author would like to thank D. Khavinson and C. Chu for insightful comments and suggestions on the first version of the paper.

References

[1] S. R. Bell, T. Ferguson, and E. Lundberg, Self-commutators of Toeplitz operators and isoperimetric inequalities, Math. Proc. R. Ir. Acad. 114A (2014), no. 2, 115–133. MR 3353499

[2] C. Béntéteau and D. Khavinson, The isoperimetric inequality via approximation theory and free boundary problems, Comput. Methods Funct. Theory 6 (2006), no. 2, 253–274. MR 2291136

[3] A. Brown and P. R. Halmos, Algebraic properties of Toeplitz operators, J. Reine Angew. Math. 213 (1963/1964), 89–102. MR 0160136 (28 #3350)

[4] C. Chu and D. Khavinson, A note on the spectral area of Toeplitz operators, Proc. Amer. Math. Soc. 144 (2016), no. 6, 2533–2537. MR 3477969

[5] C. C. Cowen, Hyponormality of Toeplitz operators, Proc. Amer. Math. Soc. 103 (1988), no. 3, 809–812. MR 947663

[6] M. Fleeman and D. Khavinson, Extremal domains for self-commutators in the Bergman space, Complex Anal. Oper. Theory 9 (2015), no. 1, 99–111. MR 330527

[7] D. Khavinson, A note on Toeplitz operators, Banach spaces (Columbia, Mo., 1984), Lecture Notes in Math., vol. 1166, Springer, Berlin, 1985, pp. 89–94. MR 827763

[8] J. Olsen and M. C. Reguera, On a sharp estimate for Hankel operators and Putnam’s inequality, Rev. Mat. Iberoam. 32 (2016), no. 2, 495–510. MR 3512424

[9] C. R. Putnam, An inequality for the area of hyponormal spectra, Math. Z. 116 (1970), 323–330. MR 0270193

Department of Mathematics and Statistics, Mail Stop 942, University of Toledo, Toledo, OH 43606

Email address: Trieu.Le2@utoledo.edu