Identification over Compound Multiple-Input Multiple-Output Broadcast Channels

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Abstract

The identification capacity region of the compound broadcast channel is determined under an average error criterion, where the sender has no channel state information. We give single-letter identification capacity formulas for discrete channels and multiple-input multiple-output Gaussian channels under an average input constraint. The capacity theorems apply to general discrete memoryless broadcast channels. This is in contrast to the transmission setting, where the capacity is only known for special cases, notably the degraded broadcast channel and the multiple-input multiple-output broadcast channel with private messages. Furthermore, the identification capacity region of the compound multiple-input multiple-output broadcast channel can be larger than the transmission capacity region. This is a departure from the single-user behavior of identification, since the identification capacity of a single-user channel equals the transmission capacity.

Index Terms

Identification capacity, compound channels, broadcast communication, multiple-input multiple-output

I. INTRODUCTION

In wireless networks nowadays, massive amounts of data are communicated under challenging conditions [1]. In particular, fading channels are characterized by fluctuating signal level and uncertain gain [2–4]. The compound setting is a worst case scenario that requires reliable communication over every possible channel from an uncertainty set [5]. Thereby, the compound multiple-input multiple-output broadcast channel (MIMO-BC) has raised a lot of attention [6–13], as multiple-antenna transmission allows for substantial performance gains compared to single-antenna transmission [14, 15].

Unlike Shannon’s transmission task [16], where a transmitter sends a message over a noisy channel, and the receiver needs to recover the message that was sent, in some modern event-triggered applications, a receiver performs only a binary hypothesis test to determine whether a particular message of interest was sent or not. This setting is known as identification (ID) via channels [17]. Possible applications for ID include authentication tasks such as watermarking [18–22], private interrogation of devices [23], as well as event-driven applications encountered in sensor communication [24], Industry 4.0 [25] and vehicle-to-X communication [26]. For example, in vehicle-to-X communication, a vehicle may announce information about its future movements to surrounding road users. Every road user is interested in one specific movement that interferes with its plans, and he checks only if this movement is announced or not. This is in contrast to the transmission task, where every road user has to decode every message, regardless of their interest.

In practice, often neither sender nor receiver know the exact channel statistics. The compound channel model is applicable when the channel variation is sufficiently slow, so that the channel is approximately constant throughout the transmission block [2–4, 20]. The capacity of the compound discrete memoryless channel was determined by Blackwell et. al. [5] and Wolfowitz [27]. Subsequently, the result was extended to the Gaussian setting [28]. For compound MIMO channels, the capacity region was optimized under different fading models and input covariance constraints [29–35].

A basic model for multi-user communication is the broadcast channel (BC), where Alice sends messages to Bob and Charly. The discrete memoryless broadcast channel (DMBC) was introduced by Cover [36] in 1972, but in the general case, the transmission capacity region is so far not known. The best known lower bound is due to Marton [37], and the best known upper bound was proven by Nair and El Gamal [38]. The two bounds coincide in special cases such as the more capable, less noisy or degraded DMBC [39]. The MIMO BC [40] is not necessarily degraded, yet the capacity region with private messages equals Marton’s lower bound.

For compound BCs with perfect channel state information (CSI) at the receiver, the transmission capacity region was determined by Weingarten et. al. [6, 7], under a degradedness assumption. Both discrete channels and MIMO Gaussian channels were treated. Chong and Liang [8, 9, 41] extended the result to discrete BCs and MIMO BCs with perfect CSI at the receiver, under a weaker degradedness assumption. Without CSI, Benammar, Piantanida, and Shamai [42] derived lower and upper bounds on the capacity region of compound discrete memoryless and multiple-input single-output BCs, and they determined the capacity region for special cases of hybrid binary symmetric/binary erasure BCs.

The ID capacity region of the discrete memoryless channel was determined by Ahlswede and Dueck [17, 43]. In general, for single-user channels it equals the transmission capacity for known channels [44, 45], and also for compound channels [25] and...
for certain continuous channels \cite{44,46,48} such as MIMO Gaussian channels \cite{49}. However, the ID capacity is a second-order rate, i.e. the ID code size grows doubly exponentially in the block length, provided that the encoder has access to a source of randomness. This is possible by letting the encoding and decoding sets overlap, and thereby accepting a substantial probability that some receiver makes an error, if too many receivers are listening at the same time. For every single receiver, the error probability is still small. General results for ID are surveyed in \cite{44,50}.

The DMBC was also studied for ID, using various error criteria. Under a maximum-error criterion, Verboven and van der Meulen \cite{51} derived an upper bound on the identification capacity region of the DMBC, and a lower bound for the DMBC with feedback. Furthermore, it was shown that the ID capacity region of the deterministic DMBC is the same, whether feedback is available or not. Bilik and Steinberg \cite{52} presented bounds on the ID capacity region of the degraded DMBC. Ahlswede \cite{53} proved that the ID capacity region of a general DMBC with private messages is the same as with degraded message sets. It was further shown in \cite{53} that the ID capacity region of the DMBC is strictly larger than the transmission counterpart with degraded message sets.

Bracher and Lapidoth \cite{54,55} established the ID capacity region of the DMBC for a semi-average error criterion, for which the ID messages that the sender sends to the two receivers are assumed to be uniformly distributed. Since every receiver is interested in one particular message, his error probabilities are maximized over all messages for this receiver, but averaged over the message for the other receiver. In this setting, the ID capacity region of a DMBC \( B \), from \( X \) to \( Y_1, Y_2 \), is given by \cite{54,55}

\[
C_{\text{ID}}(B) = \bigcup_{P_X} \left\{ \left( R_1, R_2 : R_1 \leq I(X; Y_1), \quad R_2 \leq I(X; Y_2) \right) \right\}.
\]

This holds for the general DMBC, without any channel ordering conditions. The achievability proof for ID over the BC is based on more advanced methods than the standard Shannon-theoretic argument. In particular, Bracher and Lapidoth \cite{54} presented a pool-selection code construction with binning, and bounded the error probabilities by analyzing the size of the bin intersections. The converse proof is based on the strong converse for single-user ID over discrete memoryless channels \cite{56}.

In this work, we determine the ID capacity region of the compound BC under a semi-average error criterion, where neither sender nor receiver have access to CSI. We give single-letter ID capacity formulas for discrete channels under an average input constraint defined by an arbitrary positive function, and for MIMO Gaussian channels, under an average input power constraint. CSI at the receiver would result in the same rate region, since the receiver can learn the state with a short training sequence \cite{57, Remark 7.1}. Like Bracher and Lapidoth’s \cite{54,55} result for the DMBC, our results hold for the general compound DMBC and for the compound MIMO BC, without any channel ordering. As for the DMBC, the capacity regions of the compound DMBC and compound MIMO BC can be larger than their transmission counterparts.

As examples, we derive explicit expressions for the ID capacity regions for symmetric channels, the binary erasure channel and scalar Gaussian channels. In those examples, each user can achieve their maximal rate simultaneously. Thereby, the capacity region is rectangular and strictly larger than the corresponding transmission capacity. We also consider the compound broadcast Z-channel with a binary state, and a Gaussian product channel, where the ID capacity region is not rectangular, since different input distributions or transmit power allocations, respectively, are optimal for the two users.

This paper is organized as follows: In Section \( \text{II} \) we introduce the notation, define the communication model, and review basic properties of typicality and laws of large numbers. We make some comments about previous results on ID over single-user compound channels. Section \( \text{III} \) contains our main results. Examples are treated in \( \text{IV} \). Section \( \text{V} \) provides the achievability part of the proof for the capacity region of the compound discrete BC, and Section \( \text{VI} \) provides the converse part. Section \( \text{VII} \) extends the results from the discrete setting to the MIMO Gaussian setting. In Section \( \text{VIII} \) we discuss the implications of our results and further directions of research. Finally, the results are summarized in Section \( \text{IX} \).

II. Preliminaries

A. The Compound Broadcast Channel

A compound broadcast channel (CBC) is specified by a family \( B = \{ B_s \}_{s \in S} \) of discrete memoryless broadcast channels, indexed by a channel state \( s \in S \). Without feedback, the conditional output distribution has a product form, i.e.

\[
B^n_s(y^n_1, y^n_2|x^n) = \prod_{t=1}^n B_s(y_{1,t}, y_{2,t}|x_t),
\]

for \( s \in S \). Note that the channel state \( s \) is chosen once at the beginning of the transmission block, and remains constant throughout the block. Given a compound BC \( B \), the marginal channels of Receiver 1 and Receiver 2 are defined by

\[
W_{1,s}(y_1|x) = \sum_{y_2 \in Y_2} B_s(y_1, y_2|x), \quad \text{for } s \in S.
\]

\[
W_{2,s}(y_2|x) = \sum_{y_1 \in Y_1} B_s(y_1, y_2|x), \quad \text{for } s \in S.
\]
respectively, for \( s \in \mathcal{S} \). The BC is an extension of the single-user channel \( W_s : \mathcal{X} \to \mathcal{P}(\mathcal{Y}) \). We denote a compound single-user channel by \( W = \{ W_s \}_{s \in \mathcal{S}} \).

**Remark 1.** The compound BC is often defined with one state per receiver \([6, 9, 41]\), i.e., \( \mathcal{B} = \{ (W_{1,s_1}, W_{2,s_2}) : s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2 \} \). This is a special case of our definition, where \( \mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \).

A **compound MIMO broadcast channel** is described by a family \( \mathcal{B} = \{ B(G_{1,s}, G_{2,s}) \}_{s \in \mathcal{S}} \) of Gaussian probability density functions (PDFs), where \( \mathcal{S} \) is a finite state set and \( G_{k,s} \in \mathbb{R}^{p_k \times \tau} \), for the number of transmit antennas \( \tau \) and the number of receive antennas \( p_k \), at receiver \( k \in \{ 1, 2 \} \) and state \( s \in \mathcal{S} \). The PDFs are defined such that

\[
Y_{k,s} = G_{k,s}X + Z_{k,s},
\]

where \( X \in \mathbb{R}^\tau \), \( Y \in \mathbb{R}^{p_k \times \tau} \), and the noise vector \( Z_{k,s} \) follows the multivariate Gaussian distribution \( \mathcal{N}(0, I_{p_k}) \), for \( k \in \{ 1, 2 \} \) and \( s \in \mathcal{S} \). In the following, we simply write \( I = I_{p_k} \), since the dimension is determined by the context.

**Remark 2** (see Remark 9.1 in [57]). In general, a MIMO Gaussian channel can always be transformed to have unit noise covariance: Consider the noise \( Z_{k,s} \sim \mathcal{N}(0, \Sigma_{k,s}) \) and \( \Sigma_{k,s} \neq I \). Note that \( \Sigma_{k,s} \) must be strictly positive definite, since \( f_{Z_{k,s}}(z) = |\Sigma_{k,s}|^{-\frac{1}{2}} \exp \left( z^\top \Sigma_{k,s}^{-1} z \right) \). Therefore, \( \Sigma_{k,s}^{-\frac{1}{2}} \) must be invertible. Hence, \( \Sigma_{k,s}^{-\frac{1}{2}} Z_{k,s} \sim \mathcal{N}(0, I) \) and the receiver can postprocess the output to transform the channel into

\[
\tilde{Y}_{k,s} = \Sigma_{k,s}^{-\frac{1}{2}} G_{k,s}X + \Sigma_{k,s}^{-\frac{1}{2}} Z_{k,s}.
\]

### B. Identification Codes

In the following, we define the communication task of identification over compound channels, where the decoder is not required to recover the sender’s message \( i \), but simply determines whether a particular message \( i' \) was sent or not.

**Definition 1.** An \((N_1, N_2, n)\) identification code (ID-code) for the compound BC \( \mathcal{B} \) is a family of pairs of an encoding PMF \( Q_{i_1,i_2} \in \mathcal{P}(\mathcal{X}^n) \) and decoding sets \( D_{k,i,k} \subset \mathcal{Y}^n_k \), for \( i_k \in [N_k] \) and \( k \in \{ 1, 2 \} \). We denote the identification code by \( \mathcal{C} = \{(Q_{i_1,i_2}, D_{1,i_1}, D_{2,i_2}) : i_1 \in [N_1], i_2 \in [N_2]\} \).

Suppose that Receiver \( k \) is interested in a particular message \( i_k' \in [N_k] \). He declares that \( i_k' \) was sent if \( Y^n_k \in D_{k,i_k}' \). Otherwise, Receiver \( k \) declares that \( i_k' \) was not sent.

The ID rates of the BC code \( \mathcal{C} \) are defined as \( R_k = \frac{1}{n} \log \log (N_k) \) for \( k \in \{ 1, 2 \} \). In this work, we assume that the ID messages \( i_k \) are uniformly distributed over the set \([N_k]\), for \( k \in \{ 1, 2 \} \). Therefore, the error probabilities are defined on average over the messages for the other receiver. Furthermore, we consider average input constraints of the form \( \frac{1}{n} \sum_{t=1}^{n} \gamma(x_t) \leq \Gamma \), where \( \gamma : \mathcal{X} \to [0, \infty) \) can be any non-negative function. The sender (Alice) makes an error if she transmits a sequence \( X^n \) that doesn’t satisfy the constraint. For \( k \in \{ 1, 2 \} \), Receiver \( k \) (Bob or Charly) makes an error in one of two cases: (1) He decides that \( i_k \) was not sent (missed ID); (2) Receiver \( k \) decides that \( i_k' \) was sent, while in fact \( i_k 
eq i_k' \) was sent (false ID). For every \( s \in \mathcal{S} \), the probabilities of these three kinds of error, averaged over all \( i_k \in [N_k], \ell \neq k \), are defined as

\[
\bar{e}_{k,0}(n, C, i_k) = \sum_{x^n \in \mathcal{X}^n} \frac{1}{N_k} \sum_{i_1=1}^{N_1} Q_{i_1,i_2}(x^n) \mathbb{1} \left( \sum_{t=1}^{n} \gamma(x_t) > n\Gamma \right)
\]

\[
\bar{e}_{k,1}(B_s, n, C, i_k) = \sum_{x^n \in \mathcal{X}^n} \frac{1}{N_k} \sum_{i_1=1}^{N_1} Q_{i_1,i_2}(x^n) W^n_k(D_{k,i_k} | x^n),
\]

\[
\bar{e}_{k,2}(B_s, n, C, i_k', i_k) = \sum_{x^n \in \mathcal{X}^n} \frac{1}{N_k} \sum_{i_1=1}^{N_1} Q_{i_1,i_2}(x^n) W^n_k(D_{k,i_k'} | x^n),
\]

For MIMO Gaussian channels, \( \mathcal{X} \) and \( \mathcal{Y}_{k,s} \) are continuous, \( Q_{i_1,i_2} \) and \( W^n_k \) are PDFs, and the sums are replaced integrals.

An \((N_1, N_2, n, \lambda)\) ID-code \( \mathcal{C} \) for a compound BC \( \mathcal{B} \) and input constraint \( \frac{1}{n} \sum_{t=1}^{n} \gamma(x_t) \leq \Gamma \) satisfies

\[
\max_{i_k \in [N_k]} \bar{e}_{k,0}(n, C, i_k) < \lambda,
\]

\[
\max_{s \in \mathcal{S}} \max_{i_k \in [N_k]} \bar{e}_{k,1}(B_s, n, C, i_k) < \lambda,
\]

\[
\max_{s \in \mathcal{S}} \max_{i_k \in [N_k]} \bar{e}_{k,2}(B_s, n, C, i_k', i_k) < \lambda,
\]

for \( k \in \{ 1, 2 \} \). An ID rate pair \((R_1, R_2)\) is **achievable** if for every \( \lambda > 0 \) and sufficiently large \( n \), there exists an \((e^{nR_1}, e^{nR_2}, n, \lambda)\) ID-code. The ID capacity region \( \mathcal{C}_{ID}(\mathcal{B}, \Gamma) \) of the compound BC \( \mathcal{B} \) under an input constraint
\[
\frac{1}{n} \sum_{t=1}^{n} \gamma(x_t) \leq \Gamma \text{ is defined as the set of all achievable ID rate pairs. The ID capacity without an input constraint is denoted by } C_{\text{ID}}(\mathcal{B}) = C_{\text{ID}}(\mathcal{B}, \infty).
\]

**Remark 3.** In the definition above, \(Q_{i_1,i_2}\) may be a \(0,1\)-distribution, such that we have deterministic encoding. Usually, in identification, one needs randomized encoding to achieve code sizes that grow doubly exponentially in the block length \([17, 25]\). However, under a semi-average error criterion, we can view in identification, one needs randomized encoding to achieve code sizes that grow doubly exponentially in the block length \([17, 25]\). However, under a semi-average error criterion, we can view \(\gamma_{1,1}/\gamma_{1,2}\) as error probabilities of a single-user ID-code \((Q_{i_1,i_2}, P_{1,i_1})\). In the proof of our results, we will use the pool-selection method of \([54, 55]\) to construct \(Q_{i_1,i_2}\) randomly such that it approximates an encoding distribution for a single-user ID-code for the \(k\)-th marginal channel. In this sense, the messages that the sender intends for Receiver 2 are used as randomization for the identification at Receiver 1, and vice versa \([54]\). This is only possible if \(\min \{R_1, R_2\} > 0\). Otherwise, we need stochastic encoding, since we have a single user setting, then.

**C. Laws of Large Numbers**

We use basic concepts of typicality and the method of types, as defined in \([57]\) Section 2.4] and \([58]\) Chapter 2. The definitions and lemmas we use are collected in this section.

The \(n\)-type \(\hat{P}_x^n\) of a sequence \(x^n \in \mathcal{X}^n\) is defined by \(\hat{P}_x^n(x^n) = \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}(x_t = x)\). The set of all \(n\)-types over a set \(\mathcal{X}\) is denoted by \(\mathcal{P}(n, \mathcal{X})\). Joint and conditional types are defined in a similar manner, as in \([57]\). Furthermore, an \(\epsilon\)-typical set is defined as follows. Given a PMF \(P_X \in \mathcal{P}(\mathcal{X})\) over \(\mathcal{X}\), define the \(\epsilon\)-typical set,

\[
T^n_\epsilon(P_X) = \{x^n \in \mathcal{X}^n : |\hat{P}_x^n(a) - P_X(a)| \leq \epsilon P_X(a), \ a \in \mathcal{X}\}.
\]

The set of all jointly typical \(n\)-sequences \(T^n_\epsilon(P_{XY})\) is defined likewise, where \(\mathcal{X}\) is replaced by \(\mathcal{X} \times \mathcal{Y}\). Given a sequence \(x^n \in \mathcal{X}^n\), the **conditionally** \(\epsilon\)-typical set with respect to \(P_{XY}\) is \(T^n_\epsilon(P_{XY}|x^n) = \{y^n \in \mathcal{Y}^n : (x^n, y^n) \in T^n_\epsilon(P_{XY})\}\). Basic law-of-large-numbers type properties are given in the lemmas below. Those are also known as the asymptotic equipartition properties.

**Lemma 1.** For every \(P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})\), \(\epsilon > 0\) and \(x^n \in \mathcal{X}^n\) exists a \(\delta > 0\) such that

\[a) \ \Pr \left( X^n \notin T^n_\epsilon(P_X) \right) < 2|\mathcal{X}| e^{-2n\delta^2} \] \[\text{[59] Theorem 1.1},
\]

\[b) \ \Pr \left( (x^n, y^n) \notin T^n_\epsilon(P_{XY}|x^n) \right) < 2^{-n[H(X) - 2\epsilon H(Y)]} \] \[\text{[59] Theorems 1.3}.
\]

**Theorem 1** (Hoeffding’s inequality \([60]\) Theorem 1]). Let \(X_t, t \in [n]\) be a sequence of i.i.d. random variables \(\sim P_X\), satisfying \(0 < X_t < 1\). Then for all \(\alpha > 0\),

\[
\Pr \left\{ \frac{1}{n} \sum_{t=1}^{n} X_t - \mathbb{E}[X] \geq \alpha \right\} \leq e^{-2n\alpha^2}.
\]

**D. Previous Results**

In the single-user setting, the ID capacity of the single-user compound channel was determined by Boche and Deppe \([25]\).

**Theorem 2** (see \([25]\)). The ID capacity of a compound channel \(\mathcal{W} = \{W_s\}_{s \in \mathcal{S}}\) is given by

\[
C_{\text{ID}}(\mathcal{W}) = \max_{P_X \in \mathcal{P}(\mathcal{X})} \min_{s \in \mathcal{S}} I(X; Y_s),
\]

where \(Y_s \sim W_s(\cdot|X)\).

While the explicit proof in \([25]\) employs a random binning scheme based on transmission codes \([43]\), we will provide an alternative proof by extending the pool-selection method by Bracher and Lapidoth \([54, 55]\). This will enable the same extension to the broadcast setting as in \([54, 55]\).

We note that the capacity is thus upper-bounded by the minimum of the capacities of the channels \(W_s\), i.e.

\[
C_{\text{ID}}(\mathcal{W}) \leq \min_{s \in \mathcal{S}} \max_{P_X \in \mathcal{P}(\mathcal{X})} I(X; Y_s) = \min_{s \in \mathcal{S}} C_{\text{ID}}(W_s),
\]

by the max-min inequality \([61]\) Eq. 5.46]. Equality holds for certain symmetric examples like the binary symmetric and the single-user Gaussian channel.

**Example 1.** Represented as a stochastic matrix, the binary symmetric channel is defined by

\[
\text{BSC}(\delta) = \begin{pmatrix} 1 - \delta & \delta \\ \delta & 1 - \delta \end{pmatrix}.
\]

It is visualized in Figure 7.
Therefore, the crossover probability is \( \delta \), i.e. for all \( x, x' \in \{1, 2\} \) such that \( x \neq x' \), \( P_{Y|X}(x|x) = 1 - \delta \) and \( P_{Y|X}(x'|x) = \delta \). Consider \( W_s = \text{BSC}(\delta_s) \) for every \( s \in S \). The capacity satisfies
\[
C_{ID}(W) = \max_{P_X \in \mathcal{P}(X)} \min_{s \in S} I(X; Y_s) \\
= \max_{P_X \in \mathcal{P}(X)} \min_{s \in S} (H(Y_s) - H(Y_s|X)) \\
\leq 1 - \max_{s \in S} H_2(\delta_s),
\]
where \( H_2(\delta) = -\delta \log \delta - (1 - \delta) \log(1 - \delta) \) is the binary entropy function. Note that the inequality is saturated for the input distribution \( P_X(\cdot) = p = \frac{1}{2} \), for which
\[
P_{Y_s}(y) = p(1 - \delta) + (1 - p)\delta = \frac{1}{2},
\]
for all \( s \in S \). Hence, \( H(Y_s) = H_2(\frac{1}{2}) = 1 \) and
\[
C_{ID}(W) = \min_{s \in S} (1 - H_2(\delta_s)) = \min_{s \in S} C_{ID}(W_s).
\]

This property does not hold in general, as demonstrated in the next example.

**Example 2** (see Example 7.1 in [57]). Consider the following Z-channels,
\[
W_1 = \begin{pmatrix} 1 & 0 \\ \epsilon & 1 - \epsilon \end{pmatrix}, \quad W_2 = \begin{pmatrix} 1 - \epsilon & \epsilon \\ 0 & 1 \end{pmatrix},
\]
where \( \epsilon = 1/2 \). Let \( W = \{W_1, W_2\} \). The channel \( W_1 \) is visualized in Figure 2. The channel \( W_2 \) is the same as \( W_1 \), but with reversed order of the input and output alphabet. The capacity of \( W \) is
\[
C_{ID}(W) = \max_{P_X \in \mathcal{P}(X)} \min_{s \in S} I(X; Y_s) \\
= \max_{P_X \in \mathcal{P}(X)} \min_{s \in S} H_2(\epsilon) = \frac{1}{2} H_2(1/4) - 1/2H_2(1/2) \\
= 0.3113,
\]
and it is achieved by \( P_X(\cdot) = \frac{1}{2} \). Thereby, the ID capacity is strictly lower than the ID capacity of the individual channels, \( W_1, W_2 \), as
\[
C_{ID}(W_1) = C_{ID}(W_2) = H(1/5) - 2/5 = 0.3219.
\]
This is in contrast to Example 1 (see [11]).

### III. Results

**A. MIMO Gaussian Channels**

For a compound MIMO Gaussian BC \( B = \{B(G_{1,s}, G_{2,s})\} \), consider an average input power constraint \( \frac{1}{n} \sum_{t=1}^{n} X_t^\top X_t \leq P \), i.e. \( \gamma(X) = X^\top X \). We denote the rate region
\[
\mathcal{R}_P(B) = \bigcup_{P_1, \ldots, P_{n:s} \in \mathcal{P}} \left\{ \{R_1, R_2\} : \text{For all } k \in \{1, 2\} : R_k \leq \min_{s \in S} \frac{1}{2} \log_2(1 + \lambda_{k,s} \delta) \right\},
\]
for all \( s \in S \), where \( \lambda_{k,s} \) is the maximum singular value of the channel matrix.

Figure 1. Binary symmetric channel with crossover probability \( \delta \). Figure 2. Z-channel with parameter \( \epsilon \).
where $\lambda_{k,s}^{(1)}, \ldots, \lambda_{k,s}^{(r)}$ are the eigenvalues of the matrix $G_{k,s}^T G_{k,s}$.

**Theorem 3.** The ID capacity region of the compound MIMO Gaussian broadcast channel $\mathcal{B}$ under an average input power constraint $\frac{1}{n} \sum_{t=1}^{n} X_t^T X_t \leq P$ is given by

$$
C_{ID}(\mathcal{B}, P) = \mathcal{R}(\mathcal{B}).
$$

The proof follows in Section VII. Observe that, for every compound MIMO BC, there exists a compound Gaussian Product BC with the same ID capacity and

$$
Y_{k,s} = \text{diag} \left( \sqrt{\lambda_{k,s}^{(1)}}, \ldots, \sqrt{\lambda_{k,s}^{(r)}} \right) X + Z_{k,s},
$$

for every $k \in \{1, 2\}$, $s \in S$, and $Z_{k,s} \sim \mathcal{N}(0, I)$.

**B. Discrete Channels**

Consider the compound discrete BC $\mathcal{B}$, as defined in Section II-A, and an arbitrary non-negative function $\gamma : \mathcal{X} \rightarrow [0, \infty)$. For this constraint, we denote the rate region

$$
\mathcal{R}_\Gamma(\mathcal{B}) = \bigcup_{P_X \in \mathcal{P}(\mathcal{X}) : \mathbb{E}[\gamma(X)] \leq \Gamma} \left\{ (R_1, R_2) : R_1 \leq \min_{s \in S} I(X; Y_1, s), \quad R_2 \leq \min_{s \in S} I(X; Y_2, s) \right\}.
$$

**Theorem 4.** The ID capacity region of a CBC $\mathcal{B} = \{ B_s : \mathcal{X} \rightarrow \mathcal{Y}_1 \times \mathcal{Y}_2 \}_{s \in S}$ with constraint $\frac{1}{n} \sum_{t=1}^{n} \gamma(x_t) \leq \Gamma$ satisfies

$$
C_{ID}(\mathcal{B}, \Gamma) = \mathcal{R}_\Gamma(\mathcal{B}).
$$

The proof is separated into two parts: In Section VI we show that all rate pairs in the interior of the region $\mathcal{R}_\Gamma(\mathcal{B})$ are achievable. In Section VII we show the converse part, i.e. that no rate pair outside $\mathcal{R}_\Gamma(\mathcal{B})$ can be achieved.

**Remark 4.** By the state approximation of Blackwell, Breiman and Thomasian [5, Lemma 4], the results of Theorem 4 and of Theorem 3 are immediately extended to arbitrarily large or even non-countable state alphabets.

**Remark 5.** In the single-user setting, the ID and transmission capacity characterizations are identical. However, in the broadcast setting, we see a departure from this equivalence [53–55]. The examples in the following section demonstrate this departure in a more explicit manner, showing that the ID capacity region can be strictly larger than the transmission capacity region. We will come back to this in the sequel.

**Remark 6.** In general, one cannot necessarily achieve the full capacity of each marginal channel, i.e.

$$
\mathcal{R}_\Gamma(\mathcal{B}) = \left\{ (R_1, R_2) : R_1 \leq C_{ID}(W_1, \Gamma), \quad R_2 \leq C_{ID}(W_2, \Gamma) \right\},
$$

where $C_{ID}(W_k, \Gamma) = \max_{P_X : \mathbb{E}[\gamma(X)] \leq \Gamma} \min_{s \in S} I(X; Y_k, s)$, since both marginal channels must share the same input distribution. Equality holds if the same input distribution $P_X^*$ maximizes both minimal mutual informations, i.e. if

$$
P_X^* = \arg \max_{P_X : \mathbb{E}[\gamma(X)] \leq \Gamma} \min_{s \in S} I(X; Y_1, s) = \arg \max_{P_X : \mathbb{E}[\gamma(X)] \leq \Gamma} \min_{s \in S} I(X; Y_2, s).
$$

The examples in Sections IV-A and IV-D show that this applies to symmetric channels and to the scalar Gaussian channel, hence the capacity region is rectangular. Sections IV-C and IV-E show that this doesn’t hold in general for broadcast Z-channels and for MIMO channels.

**IV. Examples**

As examples, we consider the compound symmetric broadcast channels, the compound binary erasure channel, the compound binary broadcast Z-channel, the compound scalar Gaussian BC, and a Gaussian product BC.
A. Symmetric Channels

A channel is called weakly symmetric (see [62, p. 190]) if all rows of the transition matrix \( W \) are permutations of each other and all columns \( y \) have the same sum \( \sum_x W(y|x) \). The simplest weakly symmetric channel is the binary symmetric channel

\[
\text{BSC}(\delta) = \begin{pmatrix}
1 - \delta & \delta \\
\delta & 1 - \delta
\end{pmatrix},
\]

where the binary input distribution has the form \( P_X = (p, 1-p) \) for some \( p \in [0, 1] \). By Theorem 7.2.1 in [62], every weakly symmetric channel \( W : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{Y}) \) has

\[
\max_{P_X \in \mathcal{P}(\mathcal{X})} I(X;Y) = \log |\mathcal{Y}| - H(Y|X),
\]

and the maximizing \( P_X \) is the equidistribution over \( \mathcal{X} \) (e.g. \( p = 1/2 \) for the BSC). Since the maximizing \( P_X = P_{X^*} \) is the same for all states \( s \in S \) of a compound channel \( W \), by Theorem 2 the ID capacity of \( W \) is given by

\[
C_{ID}(W) = \max_{P_X \in \mathcal{P}(\mathcal{X})} \min_{s \in S} I(X;Y_s) = \min_{s \in S} I(X^*;Y_s) = \log |\mathcal{Y}| - \max_{s \in S} H(Y_s|X).
\]

Consider a compound broadcast channel \( B \) where the marginal channels \( W_{k,s} \) are weakly symmetric for every receiver \( k \in \{1,2\} \) and state \( s \in S \). Then, we get from Theorem 4 the capacity region

\[
C_{ID}(B) = \left\{ (R_1, R_2) : \begin{array}{l}
R_1 \leq C_{ID}(W_1), \\
R_2 \leq C_{ID}(W_2)
\end{array} \right\}. \tag{20}
\]

This is in accordance with Remark 6 since the same \( P_X(\cdot) = \frac{1}{|\mathcal{X}|} \) maximizes all mutual informations \( I(X;Y_{k,s}) \). Specifically, for \( B \) with marginal binary symmetric channels \( \text{BSC}(\delta_s^{(k)}) \), we have

\[
C_{ID}(B) = \left\{ (R_1, R_2) : \begin{array}{l}
R_1 \leq 1 - \max_{s \in S} H_2(\delta_s^{(1)}), \\
R_2 \leq 1 - \max_{s \in S} H_2(\delta_s^{(2)})
\end{array} \right\}. \tag{21}
\]

By [57], the transmission capacity region of \( B \) is

\[
C_T(B) = \bigcup_{\alpha \in [0,0.5]} \left\{ (R_1, R_2) : \begin{array}{l}
R_1 \leq \min_{s \in S} \left( H_2(\alpha \ast \delta_s^{(1)}) - H_2(\delta_s^{(1)}) \right), \\
R_2 \leq \min_{s \in S} \left( 1 - H_2(\alpha \ast \delta_s^{(2)}) \right)
\end{array} \right\}, \tag{22}
\]

where \( \alpha \ast \delta = \alpha (1 - \delta) + \delta (1 - \alpha) \). The capacity regions \( C_{ID}(B) \) and \( C_T(B) \) are visualized in Figure 3. We see that the \( C_{ID}(B) \) is rectangular and strictly larger than \( C_T(B) \).

B. Channels Composed of Symmetric Channels

Uniform input distribution is not only optimal for weakly symmetric channels, but also for channels composed of those, such as the binary erasure channel

\[
\text{BEC}(\delta) = \frac{1}{2} \begin{pmatrix}
1 & 2 & e \\
1 - \delta & 0 & \delta \\
0 & 1 - \delta & \delta
\end{pmatrix}.
\]
Here, the output alphabet is $\mathcal{Y} = \mathcal{X} \cup \{\epsilon\}$, where $\epsilon$ marks an erasure. Note that both channels $W_1$ and $W_2$ are weakly symmetric. By [62, p. 189], we have $\max_{P_X \in P(\mathcal{X})} I(X; Y) = I(X^*; Y) = 1 - \delta$, where the uniform input distribution $P_X$ is independent of $\delta$.

Consider a compound broadcast channel $B$ where the marginal channels $W_{k,s}$ are binary erasure channels

$$W_{k,s} = \text{BEC}(\delta_{s(k)}),$$

for every $k \in \{1, 2\}$ and $s \in \mathcal{S}$. By Theorem 4, the ID capacity of $W_k$ is given by

$$C_{ID}(W_k) = \max_{P_X \in P(\mathcal{X})} \min_{s \in \mathcal{S}} I(X; Y_{k,s}) = \min_{s \in \mathcal{S}} I(X^*; Y_{k,s}) = 1 - \max_{s \in \mathcal{S}} \delta_s. \quad (24)$$

Then, we get from Theorem 4 the capacity region

$$C_{ID}(B) = \left\{ (R_1, R_2) : R_1 \leq C_{ID}(W_1), R_2 \leq C_{ID}(W_2) \right\}. \quad (25)$$

C. Broadcast Z-Channel

Consider the compound broadcast channel $B$ with the marginal Z-channels

$$W_1(\epsilon) = \begin{pmatrix} 1 - \epsilon & 0 \\ 1 & \epsilon \end{pmatrix}, \quad W_2(\epsilon) = \begin{pmatrix} \epsilon & 1 - \epsilon \\ 0 & 1 \end{pmatrix},$$

for $\epsilon \in \{\epsilon_1, \epsilon_2\}$. The channel $W_2$ is the same as $W_1$, but with reversed order of the input and output alphabet. The resulting broadcast Z-channel $B(\epsilon)$ is visualized in Figure 4.

We derive now the capacity region for the compound BC $B$. Let the input distribution be $P_X = (1 - p, p)$. Then,

$$P_{Y_{k,s}} = (1 - p \epsilon_s, p \epsilon_s). \quad (26)$$

The mutual information for $W_1(\epsilon_s)$ can be calculated as

$$I(X; Y_{1,s}) = H(Y_{1,s}) - H(Y_{1,s}|X) = H(Y_{1,s}) - ((1 - p) \cdot 0 + pH_2(1 - \epsilon_s)) = H_2(p \epsilon_s) - pH_2(\epsilon_s). \quad (27)$$

By [63], the $p$ that maximizes the mutual information for $W_1(\epsilon)$ is given by

$$p_\epsilon = \frac{\gamma}{1 + \epsilon \gamma}, \quad (28)$$

where $\gamma = (1 - \epsilon)(1 - \epsilon)/\epsilon$. By symmetry, the mutual information for $W_2(\epsilon)$ is given by

$$I(X; Y_{2,s}) = H_2((1 - p) \epsilon_s) - (1 - p)H_2(\epsilon_s). \quad (29)$$

This mutual information is maximized for $p = 1 - p_\epsilon$, and it decreases for decreasing $\epsilon$. To see this, consider the derivative

$$\frac{\partial}{\partial \epsilon} (H_2(p \epsilon) - pH_2(\epsilon)) = \frac{\partial}{\partial \epsilon} \left( -p \epsilon \log_2(p \epsilon) - (1 - p) \log_2(1 - p) + p \log_2 \epsilon + p(1 - \epsilon) \log_2(1 - \epsilon) \right) = p \log_2 \left( \frac{1 - p \epsilon}{p - p \epsilon} \right). \quad (30)$$
which is always positive. Hence, the state \( \arg \min_s \epsilon_s \) minimizes the mutual information \( I(X; Y_{k,s}) \), for all \( k \in \{1, 2\} \) and \( P_X \in \mathcal{P}(\mathcal{X}) \), i.e. \( p \in [0,1] \).

Suppose now that \( \epsilon_1 = 0.1 \) and \( \epsilon_2 = 0.5 \). Then, \( \epsilon = \epsilon_1 = 0.1 \) minimizes the mutual informations, and for this state, \( p_{0.1} \approx 0.3727 \) maximizes them. Since different input distributions are optimal for \( W_1(\epsilon) \) and \( W_2(\epsilon) \), the capacity region of \( \mathcal{B} \) is a strict subset of the capacity region for parallel channels \( \mathcal{W}_1, \mathcal{W}_1 \). To see this, consider the input parameter \( p = p_{0.1} \), i.e. \( X \sim P_X = (1 - p_{0.1}, p_{0.1}) \). Let \( R(p) = \min_{s \in \mathcal{S}} (H_2(p_{s}) - pH_2(\epsilon_s)) \). The upper bound on the first rate,

\[
R_1 \leq \min_{s \in \mathcal{S}} I(X; Y_{1,s}) = R(p_{0.1}) \approx 0.0545,
\]

is maximized by this choice of \( p \), but the upper bound on the second rate,

\[
R_2 \leq I(X; Y_{2,s}) = R(1 - p_{0.1}) \approx 0.0418,
\]

is suboptimal for \( \mathcal{W}_2 \), because

\[
\max_{p_X \in \mathcal{P}(\mathcal{X})} \min_{s \in \{1, 2\}} I(X; Y_{2,s}) = \max_{p \in [0,1]} R(1 - p) = R(p_{0.1}) \approx 0.0545.
\]

Therefore, we have that

\[
\mathcal{C}_{ID}(\mathcal{B}) = \mathcal{B} = \bigcup_{P \in [0,1]} \left\{ (R_1, R_2) : \begin{array}{l}
R_1 \leq R(p), \\
R_2 \leq R(1 - p)
\end{array} \right\}
\]

\[
\subseteq \left\{ (R_1, R_2) : \begin{array}{l}
R_1 \leq \max_p R(p), \\
R_2 \leq \max_p R(1 - p)
\end{array} \right\}
\]

\[
= \left\{ (R_1, R_2) : \begin{array}{l}
R_1 \leq \max_{s \in \mathcal{S}} H_2(p_{s}) - pH_2(\epsilon_s) \\
R_2 \leq \max_{s \in \mathcal{S}} H_2(p_{s}) - pH_2(\epsilon_s)
\end{array} \right\}.
\]

The capacity region \( \mathcal{C}_{ID}(\mathcal{B}) \) and the square \([0, \mathcal{C}(\mathcal{W}_1)]^2 \) are visualized in Figure 5 for parameters \( \epsilon_1 = 0.1 \) and \( \epsilon_2 = 0.5 \). Clearly, \( \mathcal{C}_{ID}(\mathcal{B}) \) is not rectangular.

D. Scalar Gaussian Channel

For a scalar Gaussian BC \( \mathcal{B} \), where \( G_{k,s} \in \mathbb{R} \), and input power constraint \( \frac{1}{n} \sum_{i=1}^{n} |X|^2 \leq P \), we obtain from Theorem 3 the closed-form capacity formula,

\[
\mathcal{C}_{ID}(\mathcal{B}, P) = \left\{ (R_1, R_2) : \begin{array}{l}
R_k \leq \min_{s \in \mathcal{S}} \frac{1}{2} \log(G_{k,s}^2 P + 1),
\end{array} \right\}.
\]

In [56], the transmission capacity region of the worst scalar Gaussian BC in \( \mathcal{B} \) with this input constraint is given by

\[
\mathcal{C}_{T}(\mathcal{B}, P) = \bigcup_{\alpha \in [0,1]} \left\{ (R_1, R_2) : \begin{array}{l}
R_1 \leq \min_{s \in \mathcal{S}} \frac{1}{2} \log(\alpha G_{1,s}^2 P + 1), \\
R_2 \leq \min_{s \in \mathcal{S}} \frac{1}{2} \log \left( \frac{(1 - \alpha) G_{2,s}^2 P}{\alpha G_{2,s}^2 P + 1} \right)
\end{array} \right\}.
\]

Figure 6 demonstrates that the rectangular ID capacity region is strictly larger that the transmission capacity region, i.e. \( \mathcal{C}_{T}(\mathcal{B}, P) \subset \mathcal{C}_{ID}(\mathcal{B}, P) \).
E. Gaussian Product Channel

We consider now a Gaussian product channel with \( \tau = \rho_{k,s} \), i.e. \( G_{k,s} \) are diagonal matrices, for \( k \in \{1, 2\} \) and \( s \in S \). Recall from (14) and (15) that for every compound MIMO Gaussian channel \( \mathcal{B}' = \{B(G_{1,s}^1,G_{2,s}^2)\}_{s \in S}' \), we get the Gaussian product channel \( \mathcal{B} = \{B(G_{1,s}^1,G_{2,s}^2)\}_{s \in S} \) with the same capacity region from the eigendecomposition \( G_{k,s}^1G_{k,s}^2 = U_{k,s}G_{k,s}^2U_{k,s}^\top \). Hence, this example yields the capacity region for all channels with eigenvalue matrices \( G_{k,s}^2 \).

For each marginal channel in separate, the optimal rate under a power constraint \( \frac{1}{T} \sum_{t=1}^{T} X_t^\top X_t \leq P \) is achieved by water-filling the transmit powers \( P_1, \ldots, P_T \) (see [14]). However, due to the different optimal water-fillings for the marginal channels, the capacity region is not rectangular, as illustrated in Figure 7.

V. Achievability Proof of Theorem 3

In this section we lower-bound the capacity region in Theorem 3. Specifically, we show that

\[ C_{\text{ID}}(\mathcal{B},\Gamma) \supseteq R_V(\mathcal{B}). \]

In the achievability proof for the DMBC, Bracher and Lapidoth [54, 55] first generate a single-user random code, based on a pool-selection technique, as shown below. Then, a similar pool-selection code is constructed for the BC using a pair of single-user codes, one for each receiver. It is shown in [54, 55] that the corresponding ID error probabilities for the BC can be approximated in terms of the error probabilities of the single-user codes. We use a similar approach, and begin with the single-user compound channel.

A. Single-User Compound Channels

Consider the single-user compound channel \( \mathcal{W} \). We construct an ID-code for the single-user compound channel \( \mathcal{W} = \{W_s\}_{s \in S}, W_s : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{Y}) \), by extending the methods of Bracher and Lapidoth [54, 55] to the compound setting. Later, we will use the results of the derivation in this section to analyze the CBC.

1) Code Construction: Let \( N = \lfloor \exp(n\tilde{R}) \rfloor \) be the code size. We fix some \( \epsilon > 0 \), a PMF \( P_X \) over \( \mathcal{X} \) that satisfies \((1 + \epsilon)\mathbb{E}[\gamma(X)] \leq \Gamma \), a pool rate \( R_{\text{pool}} \), and a binning rate \( \tilde{R} \), such that

\[ R < \tilde{R} < \min_{s \in S} I(X,Y_s) \]  
\[ R_{\text{pool}} > \tilde{R} \]

and

\[ \epsilon < \min_{s' \in S} I(X,Y_{s'}) - \tilde{R} \]

For every index \( v \in \mathcal{V} = [e^{nR_{\text{pool}}}[: V \) perform the following. Choose a code word \( F(v) \sim P^n_X \) at random. Then, for every \( i \in [N] \), decide whether to add \( v \) to the set \( \mathcal{V}_i \) by a binary experiment, with probability \( e^{-n\tilde{R}}/|\mathcal{V}| = e^{-n(R_{\text{pool}}-\tilde{R})} \). That is, decide that \( v \) is included in \( \mathcal{V}_i \) with probability \( e^{-n(R_{\text{pool}}-\tilde{R})} \), and not to include with probability \( 1 - e^{-n(R_{\text{pool}}-\tilde{R})} \). Reveal this construction to all parties. Denote the collection of codewords and index bins by

\[ \Psi = \left\{ \left( F(v) \right)_{v \in \mathcal{V}_i}, \left\{ \mathcal{V}_i \right\}_{i=1}^N \right\}. \]

2) Encoding: To send an ID message \( i \in [N] \), the sender chooses an index \( V \) uniformly at random from \( \mathcal{V}_i \) and transmits the sequence \( F(V) \), if \( \mathcal{V}_i \) is non-empty. Otherwise, the encoder transmits \( F(V^*) \), where \( V^* \in \mathcal{V} \) is any default index. Therefore, the encoding distribution \( Q_V \) is given as follows,

\[ Q_V(x^n) = \begin{cases} \frac{1}{|\mathcal{V}_i|} \sum_v \mathbf{1}(x^n = F(v)) & \text{if } \mathcal{V}_i \neq \emptyset, \\ \mathbf{1}(x^n = F(V^*)) & \text{if } \mathcal{V}_i = \emptyset. \end{cases} \]

3) Decoding: Define the decoding region

\[ \mathcal{D}_i = \bigcup_{v \in \mathcal{V}_i, s' \in S} \mathcal{T}^n(P_X \times W_{s'}|F(v)) \]

\[ = \{ y^n \in \mathcal{Y}^n : (F(v),y^n) \in \mathcal{T}^n(P_X \times W_{s'}) \text{ for some } v \in \mathcal{V}_i \text{ and } s' \in S \}. \]

The receiver receives a sequence \( Y^n \), and he is interested in a particular message \( i' \). If \( Y^n \in \mathcal{D}_{i'} \), the receiver declares that \( i' \) was sent. Otherwise, he declares that \( i' \) was not sent.

We note that our encoder is the same as in the original construction of Bracher and Lapidoth [54, 55], for the discrete memoryless BC. The difference between the constructions, here and in [54, 55], is in the definition of the decoder.

The ID code that is associated with the construction above is denoted by \( \mathcal{C}_\Psi = \{(Q_i,\mathcal{D}_i)\}_{i=1}^N \).
4) **Error Analysis:** We will show that there exists \( \tau > 0 \) such that the error probabilities of the random code \( C_\psi \) satisfy

\[
\lim_{n \to \infty} \Pr \left\{ \max_{s \in S} \max_{v' \in [N]} e_0 (n, C_\psi, i') \right. e_1 (W_s, n, C_\psi, i'), e_2 (W_s, n, C_\psi, i', i) \geq e^{-n\tau} \right\} = 0. \tag{44}
\]

Let \( s \) denote the actual channel state. As mentioned above, the codebook that is used here is the same as \([54, \text{Lemma 5}] \). Therefore, we can use the cardinality bounds for the index bins \( \{V_i\} \) that were established in \([54, 55] \). Denote the collection of index bins by \( L = \{V_i\}_{i \in [N]} \).

**Lemma 2** (see \([54, \text{Lemma 5}] \)). Given \( \mu > 0 \), let \( G_\mu \) be the set of all realizations \( L \) of \( L \) such that

\[
|V_i| > (1 - \delta_n) e^{n\tilde{R}}, \tag{45}
\]

\[
|V_i| < (1 + \delta_n) e^{n\tilde{R}}, \tag{46}
\]

\[
|V_i \cap V_{i'}| < 2\delta_n e^{n\tilde{R}}. \tag{47}
\]

for all \( V_i, V_{i'} \in L, i \neq i' \), where \( \delta_n = e^{-n\mu/2} \). Then, the probability that \( L \in G_\mu \) converges to 1 as \( n \to \infty \), i.e.

\[
\lim_{n \to \infty} \Pr \{ V_i \in G_\mu \} = 1, \tag{48}
\]

for \( \mu < R_{pool} - \tilde{R} \).

Hence, it suffices to consider the bin collection realizations \( L \) of \( L \) that satisfy \((46) - (47)\), for \( \mu \in (0, R_{pool} - \tilde{R}) \). Thus, the encoding distribution is given by

\[
Q_i(x^n) = \frac{1}{|V_i|} \sum_{v \in V_i} \mathbb{1} (x^n = F(v)), \tag{49}
\]

for \( i \in [N] \), due to \((42)\) and \( V_i \neq \emptyset \) by \((45)\).

**a) Encoding Error:** First, we bound the probability of an encoding error, i.e.

\[
e_0 (n, C_\psi, i) = \sum_{x^n \in \mathcal{X}} Q_i (x^n) \mathbb{1} \left( \frac{1}{n} \sum_{t=1}^{n} \gamma (X_t) > \Gamma \right) = \frac{1}{|V_i|} \sum_{v \in V_i} \mathbb{1} \left( \sum_{t=1}^{n} \gamma (F(v)_t) > n\Gamma \right), \tag{50}
\]

where \( F(v)_t \) is the \( t \)-th letter of \( F(v) \). For every \( x^n \in T^n (P_X) \) and sufficiently large \( n \), the input constraint is satisfied, since

\[
\sum_{t=1}^{n} \gamma (x_t) \leq n(1 + \epsilon) \mathbb{E}[\gamma (X)] \leq n\Gamma,
\]

by the Typical Average Lemma \([57] \). Therefore,

\[
\mathbb{E}_\Psi \left[ \mathbb{1} \left( \sum_{t=1}^{n} \gamma (F(v)_t) > n\Gamma \right) \right] = \sum_{x^n \in \mathcal{X}^n} P^n_X (x^n) \mathbb{1} \left( \frac{1}{n} \sum_{t=1}^{n} \gamma (x_t) > \Gamma \right) \\
\leq \Pr \left( X^n \notin T^n (P_X) \right) \\
\leq e^{-n\tau_0}, \tag{51}
\]

for some \( \tau_0 > 0 \), where the last inequality follows from Lemma \([14a] \). Now, we show that the encoding error probability is small with high probability. Let \( \alpha \) be such that

\[
0 < \alpha < (\tilde{R} - R)/2. \tag{52}
\]

Then, by the union bound,

\[
\Pr \left\{ \max_{i \in [N]} e_0 (n, C_\psi, i) \geq e^{-n\tau_0} + e^{-n\alpha} \right\} \\
= \Pr \left\{ \exists i \in [N] : \frac{1}{|V_i|} \sum_{v \in V_i} \mathbb{1} \left( \sum_{t=1}^{n} \gamma (F(v)_t) > n\Gamma \right) \geq e^{-n\tau_0} + e^{-n\alpha} \right\} \\
\leq \sum_{i \in [N]} \Pr \left\{ \frac{1}{|V_i|} \sum_{v \in V_i} \mathbb{1} \left( \sum_{t=1}^{n} \gamma (F(v)_t) > n\Gamma \right) \geq e^{-n\tau_0} + e^{-n\alpha} \right\}. \tag{53}
\]
By (45), $|V_i| > e^{nR}/2$, for all $i \in [N]$ and sufficiently large $n$. By symmetry, since the code words $F(v), v \in V$ are i.i.d., the probability terms are bounded by $\exp\left(-2e^{-2\alpha} |V_i|\right) \leq \exp\left(-e^{-2\alpha} e^{nR}\right)$, by Hoeffding’s inequality (Theorem 1). Hence, since $N \leq e^{nR}$, there exists $\theta_0 > 0$ such that

$$
\Pr\left\{ \max_{i \in [N]} e_0(n, C, i) \geq e^{-n\theta_0} \right\} 
\leq N \exp\left(e^{-2\alpha} e^{nR}\right) 
\leq \exp\left(e^{nR} - e^{-2\alpha} e^{nR}\right)
$$

which tends to zero as $n \to \infty$, by (52).

b) Missed ID Error: Next, we bound the probability of the missed-ID error (first kind), given by

$$
e_1(W_s, n, C, i) = \sum_{x^n \in \mathcal{X}^n} Q_i(x^n)W_s^n(\mathcal{D}_i^n|X^n) = \frac{1}{|V_i|} \sum_{v \in V_i} W_s^n(\mathcal{D}_i^n|F(v)) .
$$

(55)

Consider any index bin $V_i \in L$. Let $(X^n, Y^n) \sim P^n_{XY}$, $P^n_X \times W_s^n$, and recall that by (43),

$$
\mathcal{D}_i = \bigcup_{v' \in V_i} \bigcup_{s' \in S} \mathcal{T}_e^n(P_{XY}|F(v')).
$$

Therefore,

$$
W_s^n(\mathcal{D}_i^n|x^n) = \Pr\left(Y^n \notin \bigcup_{s' \in S} \bigcup_{v' \in V_i} \mathcal{T}_e^n(P_{XY}|F(v')) | X^n = x^n\right).
$$

(56)

Observe that in general, the event $Y_s^n \notin \bigcup_{s' \in S} \bigcup_{v' \in V_i} \mathcal{A}(s', v')$ implies that $Y_s^n \notin \mathcal{A}(s, v)$, for $v \in V_i$. Therefore, we have

$$
W_s^n(\mathcal{D}_i^n|x^n) \leq \Pr\left(Y_s^n \notin \mathcal{T}_e^n(P_{XY}|F(v)) | X^n = x^n\right).
$$

(57)

Averaging over the realizations of $F(v)$, we obtain

$$
\mathbb{E}_\psi[ W_s^n(\mathcal{D}_i^n|F(v)) ] = \sum_{x^n \in \mathcal{X}^n} P^n_X(x^n)W_s^n(\mathcal{D}_i^n|x^n)
\leq \Pr\left( (X^n, Y^n) \notin \mathcal{T}_e^n(P_{XY}) \right) 
\leq e^{-n\tau_1},
$$

(58)

for some $\tau_1 > 0$, where the last inequality follows from Lemma 1a. Now, we show that the missed-ID error is small with high probability. As in the proof for the encoding error, it follows from the union bound that

$$
\Pr\left\{ \max_{s \in \mathcal{S}} \max_{i \in [N]} e_1(W_s, n, C, i) \geq e^{-n\tau_1} + e^{-\alpha n} \right\}
\leq \sum_{s \in \mathcal{S}} \sum_{i \in [N]} \Pr\left\{ \frac{1}{|V_i|} \sum_{v \in V_i} W_s^n(\mathcal{D}_i^n|F(v)) \geq e^{-n\tau_1} + e^{-\alpha n} \right\} .
$$

(59)

Hence, by symmetry and Hoeffding’s inequality (Theorem 1), the probability terms are bounded by $\exp\left(-2e^{-2\alpha} |V_i|\right) \leq \exp\left(-e^{-2\alpha} e^{nR}\right)$, for sufficiently large $n$. Therefore, there exists $\theta_1 > 0$ such that

$$
\Pr\left\{ \max_{s \in \mathcal{S}} \max_{i \in [N]} e_1(W_s, n, C, i) \geq e^{-n\theta_1} \right\}
\leq |\mathcal{S}| N \exp\left(e^{-2\alpha} e^{nR}\right)
\leq |\mathcal{S}| \exp\left(e^{nR} - e^{-2\alpha} e^{nR}\right)
$$

(60)

which tends to zero as $n \to \infty$ by (52).
c) False ID Error: Last, we bound the probability of the false-ID error (second kind). Suppose that the sender sends an ID message $i$ and the receiver is interested in $i' \neq i$. Recall that we can restrict our attention to realizations $\mathcal{L} = \{V_i\} \in \mathcal{G}_n$, following Lemma 2. Observe that

$$e_2(W_s, n, C_\Psi, i', i) = \sum_{x^n \in X^n} Q_i(x^n) W_s^n (\mathcal{D}_{i'} | x^n)$$

$$= \frac{1}{|V_i|} \sum_{v \in V_i} W_s^n (\mathcal{D}_{i'} | F(v))$$

$$= \frac{1}{|V_i|} \sum_{v \in V_i \cap V_{i'}} W_s^n (\mathcal{D}_{i'} | F(v)) + \frac{1}{|V_i|} \sum_{v \in V_i \cap V_{i'}} W_s^n (\mathcal{D}_{i'} | F(v))$$

$$\leq \frac{1}{|V_i|} |V_i \cap V_{i'}| + \frac{1}{|V_i|} \sum_{v \in V_i \cap V_{i'}}^{} W_s^n (\mathcal{D}_{i'} | F(v))$$

(61)

where the inequality holds since $|V_i \cap V_{i'}| \leq |V_i|$. By Lemma 2, the first term is bounded by

$$\frac{|V_i \cap V_{i'}|}{|V_i|} < \frac{2 \delta_n}{1 - \delta_n} < \delta_n$$

(62)

(see (45) and (47)), where the second inequality holds as $\delta_n < 1/2$, for sufficiently large $n$.

It remains to bound the second term in the right-hand side of (61), for which $v \in V_i \cap V_{i'}$. Namely, $v \in V_i$ and $v \notin V_{i'}$. Let $X^n = F(v)$ be the transmitted codeword, hence $Y^n_s$ is the corresponding channel output for the actual channel state $s$. Consider a codeword $F(v')$ that is tested by the receiver, where $v' \in V_{i'}$. For every pair of indices $v \notin V_{i'}$ and $v' \in V_{i'}$, we have that the code word $F(v')$ and the channel output $Y^n_s$ are independent, i.e.

$$(F(v'), Y^n_s) \sim P^n_{X,Y}(x, y)$$

(63)

Now, we consider the expectation of $e_2(W_s, n, C_\Psi, i', i)$ By expanding $\mathcal{D}_{i'}$ and applying the bound, we obtain

$$E_\Psi [W_s^n (\mathcal{D}_{i'} | F(v))] = \Pr \left\{ (F(v'), Y^n_s) \in \bigcup_{v' \in V_i} \bigcup_{s' \in S} T^n_c (P_{X,Y,s'}) \right\}$$

$$\leq \sum_{v' \in V_i} \sum_{s' \in S} \Pr \left\{ (F(v'), Y^n_s) \in T^n_c (P_{X,Y,s'}) \right\}$$

$$= \sum_{v' \in V_i} \sum_{s' \in S} \sum_{y \in Y^n} P^n_{X}(X^n, y^n) P^n_{Y}(y^n)$$

(64)

where the last equality follows from (63). By Lemma 1b,

$$P^n_{X}(X^n, y^n) \in T^n_c (P_{X,Y,s'}) \leq e^{-n[I(X;Y_s) - 2 \epsilon H(X) + R]}$$

(65)

for all $y^n \in Y^n$. Therefore, by (64) and (65), there exists $\tau_2 > 0$ such that

$$E_\Psi [W_s^n (\mathcal{D}_{i'} | F(v))] \leq |S| |V_i| \max_{s' \in S} e^{-n[I(X;Y_s) - 2 \epsilon H(X) + R]}$$

(66)

where (a) holds since (45), $|V_i| < (1 + \delta_n)e^{n \tilde{R}}$, and (b) holds since (40), $\min_{s' \in S} I(X;Y_s) - 2 \epsilon H(X) - \tilde{R} > 0$. We show now that the false-ID error is small with high probability. By the union bound,

$$\Pr \left\{ \max_{s' \in S} \max_{i' \notin [N]} \max_{i \notin [N]} e_2(W_s, n, C_\Psi, i', i) \geq \delta_n + e^{-n\tau_2} + e^{-n\alpha} \right\}$$

$$= \Pr \left\{ \exists s \in S, i, i' \in [N], i \neq i' : \frac{1}{|V_i|} \sum_{v \in V_i} W_s^n (\mathcal{D}_{i'} | F(v)) \geq \delta_n + e^{-n\tau_2} + e^{-n\alpha} \right\}$$

$$\leq \sum_{s \in S} \sum_{i' \notin [N]} \sum_{i \notin [N]} \Pr \left\{ \frac{1}{|V_i|} \sum_{v \in V_i} W_s^n (\mathcal{D}_{i'} | F(v)) \geq \delta_n + e^{-n\tau_2} + e^{-n\alpha} \right\}.$$

(67)

Note that by (61) and (62),

$$\frac{1}{|V_i|} \sum_{v \in V_i} W_s^n (\mathcal{D}_{i'} | F(v)) \leq \delta_n + \frac{1}{|V_i \cap V_{i'}|} \sum_{v \in V_i \cap V_{i'}} W_s^n (\mathcal{D}_{i'} | F(v)).$$

(68)
Therefore there exists \( \theta_2 > 0 \) such that

\[
\Pr \left\{ \max_{s \in S} \max_{i \in [N]} \max_{i' \neq i} e_2(W_s, n, C_{\Psi}, i', i) \geq \delta_n + e^{-\theta_2} \right\} 
\leq \sum_{s \in S} \sum_{i \in [N]} \sum_{i' \neq i} \Pr \left\{ \frac{1}{|V_i \cap \nabla_{i'}^c|} \sum_{v \in V_i \cap \nabla_{i'}^c} W_s^n(D_{i'}|F(v)) \geq e^{-n\theta_2} + e^{-n\alpha} \right\}
\leq (a) |S| N^2 \exp \left( -2e^{-2n\alpha} |V_i \cap \nabla_{i'}^c| \right)
\leq (b) |S| \exp \left( 2e^{nR} - e^{-2n\alpha} e^{-nR} \right),
\]

sufficiently large \( n \), where (a) follows from Hoeffding’s inequality (Theorem I), since the codewords \( F(v), v \in \nabla \) are i.i.d., and (b) follows from \( N \leq \exp (e^{nR}) \), and

\[
|V_i \cap \nabla_{i'}^c| = |V_i| - |V_i \cap \nabla_{i'}^c| > (1 - \delta_n)e^{nR} - 2\delta_n e^{nR} \geq e^{nR}/2,
\]

as \( |V_i| \geq (1 - \delta_n)e^{nR} \) and \( |V_i \cap \nabla_{i'}^c| < 2\delta_n e^{nR} \), by Lemma 2 (see (45) and (47), respectively), where the last inequality follows from \( \delta_n < 1/2 \), for sufficiently large \( n \).

Based on (54), (60) and (69), we have established that (44) holds for \( \tau = \min \{ \theta_0, \theta_1, \theta_2 \} \).

**B. Broadcast Channels**

In this section, we show the direct part for the ID capacity region of the CBC. That is, we show that \( C_{\text{ID}}(B) \supseteq \mathcal{R}(B) \). The analysis makes use of the our single-user derivation above.

1) **Code Construction:** We extend Bracher and Lapidoth’s [54, 55] idea to combine two BL code books \( \Psi_1, \Psi_2 \) that share the same pool. Fix a PMF \( P_X \) over \( \mathcal{X} \) and rates \( R_k, \bar{R}_k \), for \( k \in \{1, 2\} \), that satisfy

\[
\begin{align*}
R_1 &< \bar{R}_1 < \min_{s \in S} I(X; Y_{1,s}) \\
R_2 &< \bar{R}_2 < \min_{s \in S} I(X; Y_{2,s}) \\
\max \left\{ \bar{R}_1, \bar{R}_2 \right\} &< R_{\text{pool}} \\
R_{\text{pool}} &< \bar{R}_1 + \bar{R}_2.
\end{align*}
\]

(71a)

(71b)

(71c)

(71d)

Let \( N_k = e^{nR_k} \). For every index \( v \in \nabla = [e^{nR_{\text{pool}}}] \), perform the following. Choose a code word \( F(v) \sim P_X^n \) at random, as in the single-user case. Then, for every \( i_k \), decide whether to add \( v \) to the set \( \nabla_{k,i_k} \) by a binary experiment, with probability \( e^{-nR_k} / |\nabla| = e^{-n(R_{\text{pool}}-R_k)} \). That is, decide that \( v \) is included in \( \nabla_{k,i_k} \) with probability \( e^{-n(R_{\text{pool}}-R_k)} \), and not to include with probability \( 1 - e^{-n(R_{\text{pool}}-R_k)} \). Finally, for every pair \((i_1, i_2) \in [N_1] \times [N_2] \), select a common index \( V_{i_1,i_2} \) uniformly at random from \( \nabla_{1,i_1} \cap \nabla_{2,i_2} \), if this intersection is non-empty. Otherwise, if \( \nabla_{1,i_1} \cap \nabla_{2,i_2} = \emptyset \), then draw \( V_{i_1,i_2} \) uniformly from \( \nabla \). Reveal this construction to all parties.

Denote the collection of codewords and index bins for the compound BC by

\[
\Psi_B = \left( F, \{ \nabla_{1,i_1}, i_1 \in [N_1] \}, \{ \nabla_{2,i_2}, i_2 \in [N_2] \}, \{ V_{i_1,i_2}, (i_1,i_2) \in [N_1] \times [N_2] \} \right).
\]

(72)

Note that, for \( k \in \{1, 2\} \), \( \Psi_k \) includes all elements of

\[
\Psi_k = \left( F, \{ \nabla_{k,i_k}, i_k \in [N_k] \} \right),
\]

defined for the single-user marginal channels \( \nabla_k \) as in Section V-A. We denote the corresponding single-user code by

\[
C_{\Psi_k} = \left\{ \left( \hat{Q}_{k,i_k}, \hat{D}_{k,i_k} \right) \right\}_{i_k=1}^{N_k}.
\]

2) **Encoding:** To send an ID message pair \((i_1, i_2) \in [N_1] \times [N_2] \), the sender transmits the sequence \( F(V_{i_1,i_2}) \). Therefore, given \( \Psi_B \), the encoding distribution \( Q_{i_1,i_2} \) is given by the following 0-1-rule:

\[
Q_{i_1,i_2}(x^n) = \mathbb{1} (x^n = F(V_{i_1,i_2})),
\]

(73)

for \( x^n \in \mathcal{X}^n \). We note that for every realization \( C_{\Psi_B} \) of a BL code, this encoding function is deterministic.

3) **Decoding:** Receiver \( k \), for \( k = 1, 2 \), employs the decoder of the single-user code \( C_{\Psi_k} \). Specifically, suppose that Receiver \( k \) is interested in an ID message \( i_k' \in [N_k] \). Then, he uses the decoding set \( D_{k,i_k'} \) to decide whether \( i_k' \) was sent or not.

We denote the ID code for the CBC that is associated with the construction above by

\[
C_{\Psi_B} = \{ (Q_{i_1,i_2}, D_{1,i_1}, D_{2,i_2}) : i_1 \in [N_1], i_2 \in [N_2] \}.
\]

(74)
4) Error Analysis: Based on the idea of Bracher and Lapidoth [54, 55], we will show that the semi-average error probabilities of the ID code defined above can be approximately upper-bounded by the respective error probabilities of the single-user ID-codes $C_{\Psi_1}$ and $C_{\Psi_2}$ for the respective receivers. Denote the total variation distance between two PMFs $P$ and $Q$ over a given finite set $A$. It is defined as
\[
d(P, Q) = \max_{A' \subseteq A} \left( P(A') - Q(A') \right) = \frac{1}{2} \sum_{a \in A} |P(a) - Q(a)|.
\]
For the second equality see (11.137) in [62].

We consider now only Receiver 1 and his marginal channel $\mathcal{W}_1$. Since the code construction is completely symmetric between the two receivers, the same arguments hold for Receiver 2 and $\mathcal{W}_2$. The encoding distribution of $C_{\Psi_B}$ can be expressed as
\[
Q_{i_1, i_2}(x^n) = \sum_{v \in V} 1\{x^n = F(v)\} Q_{i_1, i_2}(v),
\]
where $Q_{i_1, i_2}(v) = 1(V_{i_1, i_2} = v)$. Similarly, the encoding distribution of the single-user code
\[
C_{\Psi_k} = \left\{ (\tilde{Q}_{k, i_k}, D_{k, i_k}) \right\}_{i_k \in \mathcal{N}_k}
\]
can be rewritten as
\[
\tilde{Q}_{k, i_k}(x^n) = \sum_{v \in V} 1\{x^n = F(v)\} \tilde{Q}_{k, i_k}(v)
\]
for some family of PMFs $\tilde{Q}_{k, i_k} \in \mathcal{P}(V)$, $i_k \in \mathcal{N}_k$. Thus, for every BC $B_s$, $s \in \mathcal{S}$, the error probabilities are bounded by
\[
\bar{e}_{1, 0}(n, C_{\Psi_B}, i_1) = \frac{1}{N_2} \sum_{i_2 \in \mathcal{N}_2} \sum_{x^n \in X^n} Q_{i_1, i_2}(x^n) 1\left( \sum_{t=1}^{n} \gamma(X_t) > n \Gamma \right)
\leq \sum_{x^n \in X^n} \tilde{Q}_{1, i_1}(x^n) 1\left( \sum_{t=1}^{n} \gamma(X_t) > n \Gamma \right) + d\left( \frac{1}{N_2} \sum_{i_2 \in \mathcal{N}_2} Q_{i_1, i_2}, \tilde{Q}_{1, i_1} \right)
\leq e_0(n, C_{\Psi_1}, i_1) + d\left( \frac{1}{N_2} \sum_{i_2 \in \mathcal{N}_2} Q_{i_1, i_2}, \tilde{Q}_{1, i_1} \right), \quad (75a)
\]
\[
\bar{e}_{1, 1}(B_s, n, C_{\Psi_B}, i_1) = \frac{1}{N_2} \sum_{i_2 \in \mathcal{N}_2} Q_{i_1, i_2} W_{1, i_2}^n(D_{1, i_1})
\leq \tilde{Q}_{1, i_1} W_{1, i_2}^n(D_{1, i_1}) + \delta
\leq e_1(B_s, n, C_{\Psi_1}, i_1) + \delta \quad (75b)
\]
\[
\bar{e}_{1, 2}(B_s, n, C_{\Psi_B}, i_1', i_1) = \frac{1}{N_2} \sum_{i_2 \in \mathcal{N}_2} Q_{i_1, i_2} W_{1, i_2}^n(D_{1, i_1'})
\leq \tilde{Q}_{1, i_1} W_{1, i_2}^n(D_{1, i_1'}) + \delta
\leq e_2(B_s, n, C_{\Psi_1}, i_1', i_1) + \delta, \quad (75c)
\]
where $\delta = d\left( \frac{1}{N_2} \sum_{i_2 \in \mathcal{N}_2} Q_{i_1, i_2} W_{1, i_2}^n, \tilde{Q}_{1, i_1} W_{1, i_2}^n \right)$. The error probabilities $\bar{e}_{2, 0}, \bar{e}_{2, 1}, \bar{e}_{2, 2}$ for Receiver 2 are bounded analogously, in terms of the error probabilities for $C_{\Psi_2}$. By the data-processing inequality for the total variation distance [64, Lemma 1],
\[
\delta \leq d\left( \frac{1}{N_2} \sum_{i_2 \in \mathcal{N}_2} Q_{i_1, i_2}, \tilde{Q}_{1, i_1} \right), \quad (76)
\]
which is independent of the channel.

Now, let $Q_{2, i_2}$ be a single-user encoding distribution for $\mathcal{W}_2$, analogous to $\tilde{Q}_{1, i_1}$. The next lemma bounds the right-hand side of (76) to zero in probability as $n \to \infty$.

**Lemma 3** (see [54] Equations (98, 113–156)). For some $\tau > 0$
\[
\lim_{n \to \infty} \Pr\left\{ \max_{i_1 \in \mathcal{N}_1} d\left( \frac{1}{N_2} \sum_{i_2 \in \mathcal{N}_2} Q_{i_1, i_2}, \tilde{Q}_{1, i_1} \right) \geq e^{-\tau} \right\} = 0 \quad (77)
\]
Thus, we get single-user codes with \( C_{\Psi, B} \) intervals
\[ \Pr \left\{ \max_{i_2 \in [N_2]} d \left( \frac{1}{N_1} \sum_{i_1 \in [N_1]} Q_{i_1, i_2}, \tilde{Q}_{2, i_2} \right) \geq e^{-n\tau} \right\} = 0. \] (78)

By this lemma and (76), the total variation distances in (75) are upper-bounded with high probability, for sufficiently large \( n \). Hence, the error probabilities for the BC code \( C_{\Psi, B} \) are approximately upper-bounded by the corresponding error probabilities for the single-user marginal codes \( C_{\Psi_1}, C_{\Psi_2} \).

By (44) for the single-user compound channel, given any receiver \( k \in \{1, 2\} \), state \( s \in S \), and message pair \( i_k, i'_k \in [N_k] \) such that \( i_k \neq i'_k \), the error probabilities \( e_0(n, C_{\Psi, k}, i_k), e_1(W_{k,s}, n, C_{\Psi, k}, i_k) \) and \( e_2(W_{k,s}, n, C_{\Psi, k}, i'_k, i_k) \) converge with high probability to zero exponentially in \( n \). This completes the proof of the direct part.

VI. Converse Proof of Theorem 4

Next, we upper-bound the capacity region in Theorem 4, i.e. we prove that
\[ C_{id}(B, \Gamma) \subseteq \mathcal{R}_k(B). \] (79)

Brach and Lapidoth [54] Claim 16 modified the single-user converse from [56] so that the bounds for the single-user marginal channels can be combined for the DMBC. We additionally combine the bounds for all states of the CBC.

We denote \( I(P_X; P_Y | X) = I(X; Y) \). The key lemma we use is the following.

**Lemma 4** (see Lemma 21 in [54]). For every discrete memoryless channel \( W : X \to \mathcal{Y} \), every ID rate \( R \), all positive constants \( \lambda, \epsilon, \kappa \) satisfying \( 2\lambda + \lambda < \kappa < 1 \), \( N = \exp e^{nR} \) and sufficiently large \( n \), if \( \{ (Q_i, D_i) \}_{i=1}^N \) is an \( \mathcal{N}(n, n, \lambda) \) ID-code for \( W \), then
\[ \frac{1}{N} \sum_{i=1}^{N} Q_i \left( X^n \in \left\{ x^n \in X^n : I(\tilde{P}_{x^n}, W) \leq R - \epsilon \right\} \right) < \kappa - \exp \left( -e^{nR/2} \right). \] (80)

Consider now an \( (N_1, N_2, n, \lambda) \) ID code, \( C = \{(Q_{i_1, i_2}, D_{1,i_1}, D_{2,i_2}) : i_1 \in [N_1], i_2 \in [N_2]\} \), for the CBC \( B \), and the PMFs
\[ Q_{1,i_1} = \frac{1}{N_2} \sum_{i_2=1}^{N_2} Q_{1,i_2}, \]
\[ Q_{2,i_2} = \frac{1}{N_1} \sum_{i_1=1}^{N_1} Q_{i_1,i_2}, \]
\[ Q' = \frac{1}{N_1N_2} \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} Q_{i_1,i_2}. \]

Thus, we get single-user codes \( C_k = \{Q_{k,i_k}, D_{k,i_k}\}_{i_k=1}^{N_k} \), for \( k \in \{1, 2\} \). By the definition of the error probabilities (8),
\[ \lambda > \bar{e}_{1,2}(B_s, n, C, i_k) \]
\[ = \sum_{x^n \in X^n} \frac{1}{N_2} \sum_{i_2=1}^{N_2} Q_{i_1,i_2}(x^n) W^n_{1,i_1}(D_{1,i_1}^{c} | x^n), \]
\[ = \sum_{x^n \in X^n} Q_{1,i_1}(x^n) W^n_{1,i_1}(D_{1,i_1}^{c} | x^n), \]
\[ = e_1(W_{1,s}, n, C_1, i_1) \] (81a)

for all \( s \in S \), and similarly we get
\[ \lambda > \bar{e}_{2,1}(B_s, n, C, i'_k) \]
\[ = e_1(W_{1,s}, n, C_1, i'_1, i_2), \] (81b)
\[ \lambda > \bar{e}_{1,2}(B_s, n, C, i'_2) \]
\[ = e_1(W_{2,s}, n, C_2, i'_2, i_2), \] (81c)
\[ \lambda > \bar{e}_{2,1}(B_s, n, C, i'_1) \]
\[ = e_1(W_{2,s}, n, C_2, i'_1, i_2), \] (81d)

for all \( s \in S \). Hence, for every \( k \in \{1, 2\} \) and \( s \in S \), \( C_k \) is an \( \mathcal{N}(N_k, n, \lambda) \) ID-code for \( W_{k,s} \). Let \( R_k = \frac{1}{n} \log \log N_k \). Since for \( s \in S \) and \( k \in \{1, 2\} \) the marginal channel \( W_{k,s} \) is a discrete memoryless channel, Lemma 4 proves that for all \( s \in S \), all constants \( \epsilon > 0 \), \( \lambda \in [0, \frac{1}{n} |\log|], \kappa \in (2\lambda, \frac{1}{n} \log \log N_k), \) and sufficiently large \( n \),
\[ Q' \left( X^n \in \left\{ x^n \in X^n : I(\tilde{P}_{x^n}, W_{k,s}) \leq R_k - \epsilon \right\} \right) < \kappa + \exp \left( -e^{nR_k/2} \right). \] (82)
Let $X^n \sim Q'$. Then, by the union bound,

$$
\Pr \left( X^n \in \bigcap_{s \in S} \left\{ x^n \in X^n : \begin{array}{l}
I(\hat{P}_x^n, W_{1,s}) > R_1 - \epsilon, \\
I(\hat{P}_x^n, W_{2,s}) > R_2 - \epsilon
\end{array} \right\} \right) = 1 - \Pr \left( X^n \in \bigcup_{s \in S} \bigcup_{k \in \{1,2\}} \left\{ x^n \in X^n : I(\hat{P}_x^n, W_{k,s}) \leq R_k - \epsilon \right\} \right) \\
\geq 1 - \sum_{s \in S} \sum_{k \in \{1,2\}} \Pr \left( X^n \in \{ x^n \in X^n : I(\hat{P}_x^n, W_{k,s}) \leq R_k - \epsilon \} \right) \\
> 1 - \sum_{s \in S} \sum_{k \in \{1,2\}} \left( \kappa + \exp \left( e^{-nR/2} \right) \right) \\
= 1 - 2 |S| \left( \kappa + \exp \left( -e^{nR/2} \right) \right).
$$

(83)

Since $\kappa < 1/(2 |S|)$, the right-hand side is positive for sufficiently large $n$. We deduce that there exists a sequence $x^n \in X^n$ such that $I(P_x^n, W_{k,s}) > R_k - \epsilon$, for every receiver $k \in \{1, 2\}$ and every state $s \in S$, for $\lambda < 1/(4 |S|)$. Therefore, let $X$ be distributed according to the type of this sequence, i.e. $X \sim \hat{P}_x^n$, and let $Y_{1,s}, Y_{2,s}$ denote the corresponding outputs. Then, the rates satisfy

$$
R_1 \leq \min_{s \in S} I(\hat{X}; Y_{1,s}),
$$

(84a)

and

$$
R_2 \leq \min_{s \in S} I(\hat{X}; Y_{2,s}).
$$

(84b)

Furthermore, if the input constraint is satisfied, then $\mathbb{E}[\gamma(\hat{X})] = \frac{1}{n} \sum_{i=1}^{n} \gamma(x_i) \leq \Gamma$. Therefore, we have

$$
C_{ID}(B) \subseteq \mathcal{R}(B).
$$

(85)

VII. PROOF OF THEOREM 3

We obtain the ID capacity region for MIMO Gaussian channels by approximating them by discrete channels. This has been done for continuous single-user channels with certain properties in [44, 46, 48]. Based on these results, Labidi, Deppe and Boche [49] determined the ID capacity of a single-user MIMO Gaussian channel. We combine the discretization technique by Han [44] with a continuity argument discussed for a single-user channel in [65, Section VI] and apply them to the capacity region of the CBC in Theorem 3. Specifically, we consider a discrete channel $B^0$ that converges to $B$ for $\delta \to 0$. We show that the rate regions $C_{ID}(B^0, \Gamma)$ and $\mathcal{R}_I(B^0)$ are functions of $\delta$, which are continuous in $\delta = 0$. To this end, we generalize $\mathcal{R}_I(B)$ to the continuous case by taking the union over all PDFs that satisfy the input constraint in [16]. We obtain Theorem 3 as the limit in $\delta \to 0$ of Theorem 2 by evaluating this expression for the MIMO Gaussian channel, following Telatar [14]. Thus, we obtain the exact capacity region of the MIMO Gaussian BC.

A. Continuity of $C_{ID}(B^0, \Gamma)$

Let $C = \{ (Q_{i_1,i_2}, D_{1,i_1}, D_{2,i_2}) : i_1 \in [N_1], i_2 \in [N_2] \}$ be an $(N_1, N_2, n, \lambda)$ ID-code for a compound MIMO channel $B = \{ B_s \}_{s \in S}$. By [44, Lemma 6.7.1], there exists a PMF $Q^0_{i_1,i_2}$, for every $i_1 \in [N_1]$, $i_2 \in [N_2]$, and $\delta > 0$, such that the total variation distance $d(Q_{i_1,i_2}W_{k,n}, Q^0_{i_1,i_2}W_{k,n}) \leq \delta$, for all $k \in \{1, 2\}$ and sufficiently large $n$. Then, since $B_s$ is continuous and smooth, there exists an ID code $B^0 = \{ B^0_s \}_{s \in S}$ of discretized measures $B^0_s$, for every $s \in S$, such that

$$
\lim_{\delta \to 0} B^0_s = B_s.
$$

(86)

Furthermore, we define discrete decoding sets $D_{k,i_k}^0$, such that $\lim_{\delta \to 0} D_{k,i_k}^0 = D_{k,i_k}$. The error probabilities are approximated in terms of the resulting discrete code $C^0 = \{ (Q^0_{i_1,i_2}, D_{1,i_1}, D_{2,i_2}) : i_1 \in [N_1], i_2 \in [N_2] \}$, i.e.

$$
|\tilde{e}_{k,1}(B, n, C, i_k) - \tilde{e}_{k,1}(B^0, n, C^0, i_k)| < |\tilde{e}_{k,1}(B, n, C, i_k) - \tilde{e}_{k,1}(B, n, C^0, i_k)| + \delta < 2\delta,
$$

(87)

and similarly for $\tilde{e}_{k,2}$, for $k \in \{1, 2\}$ and sufficiently large $n$. Hence, by letting $B^0 = B$ and $C^0 = C$, the capacity region $C_{ID}(B^0, \Gamma)$ is continuous in $\delta \geq 0$.

We denote the largest achievable rate $R_1$ for a given $R_2$ and input constraint $\Gamma$ by

$$
C_1(B^0, \Gamma|R_2) = \max \left\{ R_1 : (R_1, R_2) \in C_{ID}(B^0, \Gamma) \right\}.
$$

The equality $C_{ID}(B^0, \Gamma) = \mathcal{R}_I(B^0)$ implies that

$$
C_1(B^0, \Gamma|R_2) = R^*_1(B^0, \Gamma|R_2),
$$

(88)

for every $R_2 \in [0,1]$ and $\delta > 0$. 

B. Continuity of $\mathcal{R}_\Gamma(B^\delta)$

Next, we show that $\mathcal{R}_\Gamma(B^\delta)$ is also continuous in $\delta = 0$. Note that $\mathcal{R}_\Gamma(B)$ is convex, for every $B$. To see this, define $P_X = aP_X + (1 - a)P_{X'}$, for any $a \in [0, 1]$. If $P_X$ and $P_{X'}$ satisfy $\max\{\mathbb{E}[\gamma(X)], \mathbb{E}[\gamma(X')]\} \leq \Gamma$, then $\mathbb{E}[\gamma(X)] = a\mathbb{E}[X] + (1 - a)\mathbb{E}[X'] \leq \Gamma$. Furthermore, by the concavity of $\min$ and the mutual information,

$$a \min_\delta I(X; Y_{k,s}) + (1 - a) \min_\delta I(X'; Y_{k,s}) \leq \min_\delta I(X; Y_{k,s}).$$

Therefore, by Slater’s condition or, equivalently, by representing $\mathcal{R}_\Gamma(B)$ by its supporting hyperplanes (see [66, Theorem 6.20]), the optimal $R_1$, given some rate $R_2$, is the optimum of the Lagrangian function

$$R^*_1(B^\delta, \Gamma | R_2) = \min_{\lambda \in \mathbb{R}} \max_{\mathbf{X}^s} I(X; Y_{1,s}) + \lambda \min_{\mathbf{Y}^s} I(X; Y_{2,s}) - \lambda R_2,$$

and thus $\mathcal{R}_\Gamma(B^\delta) = \{ (R_1, R_2) : R_1 \leq R^*_1(B^\delta, \Gamma | R_2) \}$.

Since minima and maxima over continuous functions $D \to \mathbb{R}$ with the same domain $D$ are continuous, $R^*_1(B^\delta, \Gamma | R_2)$ is continuous in $B^\delta$. The composition of two continuous functions is also continuous. Therefore, $R^*_1(B^\delta, \Gamma | R_2)$ is a continuous function in $\delta$.

C. Equality of $C_{ID}(B, \Gamma)$ and $\mathcal{R}_\Gamma(B)$

Theorem 4 proves for all $\delta > 0$ that $C_{ID}(B^\delta, \Gamma) = \mathcal{R}_\Gamma(B^\delta)$. Equality holds also in the limit $\delta = 0$, since

$$C_1(B, \Gamma | R_2) \overset{(a)}{=} C_1(\lim_{\delta \to 0} B^\delta, \Gamma | R_2) \overset{(b)}{=} \lim_{\delta \to 0} C_1(B^\delta, \Gamma | R_2) \overset{(c)}{=} \lim_{\delta \to 0} R^*_1(B^\delta, \Gamma | R_2) \overset{(d)}{=} R^*_1(\lim_{\delta \to 0} B^\delta, \Gamma | R_2) \overset{(e)}{=} R^*_1(B, \Gamma | R_2),$$

(90)

for every $R_2 \in [0, 1]$, where (a) follows from the convergence property in [86], (b) holds by the continuity of $C_1(\cdot, \Gamma | R_2)$ and the Gaussian channel measure, (c) follows from the ID capacity theorem for the compound discrete BC, Theorem 4, and (d) holds by the continuity of $R^*_1(\cdot, \Gamma | R_2)$ and the Gaussian channel measure. Hence, (90) implies

$$C_{ID}(B, \Gamma) = \mathcal{R}_\Gamma(B).$$

(91)

D. Evaluation of the Mutual Information $I(X; Y_{k,s})$

It remains to show that, for an average power constraint and $\Gamma = P$, $\mathcal{R}_\Gamma(B)$ evaluates to the expression [14] for $\mathcal{R}_P(B)$. The matrix $G^T_{k,s}G_{k,s}$ is symmetric and thus has an eigendecomposition $G^T_{k,s}G_{k,s} = U_{k,s} \text{diag}(\lambda_{k,s}^{(1)}, \ldots, \lambda_{k,s}^{(\tau)}) U^T_{k,s}$ with real eigenvalues $\lambda_{k,s}^{(j)} \in \mathbb{R}$ and an orthogonal matrix $U_{k,s}$. By Telatar [14], we have

$$I(X; Y_{k,s}) = \frac{1}{2} \log_2 \left| G_{k,s} K_X G^T_{k,s} + I \right| \leq \sum_{j=1}^\tau \frac{1}{2} \log_2 (1 + \lambda_{k,s}^{(j)} P_j),$$

(92)

for $P_j = (Q_{k,s})_{jj} \geq 0$, $1 \leq j \leq \tau$, and $Q_{k,s} = U^T_{k,s} K_X U_{k,s}$, with equality if $Q_{k,s}$ is diagonal. Furthermore, since $\text{trace}(AB) = \text{trace}(BA)$, and $U_{k,s}$ is orthogonal,

$$\sum_{j=1}^\tau P_j = \text{trace}(U^T_{k,s} K_X U_{k,s}) = \text{trace}(K_X) = \mathbb{E}[X'^T X],$$

(93)

for $k \in \{1, 2\}$ and $s \in S$. Hence,

$$C_{ID}(B, P) = \mathcal{R}_P(B) = \bigcup_{P_1, \ldots, P_{\tau} : \sum_{j=1}^\tau P_j \leq P} \left\{ (R_1, R_2) : \text{ For all } k \in \{1, 2\}: \right.$$

$$R_k \leq \min_{s \in S} \left( \sum_{j=1}^\tau \frac{1}{2} \log_2 (1 + \lambda_{k,s}^{(j)} P_j) \right) \big\}.$$
A. Channel State Information at the Receiver

In the literature (e.g., [6, 9, 11]) it is often assumed that the receivers have perfect knowledge of the channel, i.e., they know which \( s \in S \) was selected. However, CSI at the receiver does not improve the capacity for compound channels, since the receiver can estimate the state from a small training sequence [57, Remark 7.1].

B. Pool-Selection vs. Binning

The pool-selection construction by Bracher and Lapidoth [54, 55] (see Section V) has similarities with traditional binning, as used in [37], and with the binning scheme for identification in [17].

In traditional binning, Receiver \( k \) assigns randomly a code word \( u_k^n(v_k) \) to every index from an index set, \( v_k \in \mathcal{V}_k \), and the index set \( \mathcal{V}_k \) is partitioned into disjoint index bins \( \mathcal{V}_{k,i} \subset \mathcal{V}_k \) of a specified size. The set \( \{ u_k^n(v_k) : v_k \in \mathcal{V}_k \} \) must be a reliable code book for the marginal channel \( W_k \). To simultaneously bin for two receivers, one transmits a sequence \( x^n(u_1^n, u_2^n) \).

In contrast, in the pool-selection construction, there is only one set \( \mathcal{V} \) of indices \( v \in \mathcal{V} \), and a pool \( \{ x^n(v) : v \in \mathcal{V} \} \). It may be larger than any reliable code book. Then, the \( v \in \mathcal{V} \) are assigned to index bins \( \mathcal{V}_{k,i} \) at random, for Receiver \( k \), and the bin sizes are bounded stochastically (see Lemma 2) such that every bin \( \{ x^n(v) : v \in \mathcal{V}_{k,i} \} \) is effectively a reliable transmission code book, with high probability. The sizes of the index sets are chosen such that every bin for Receiver 1 overlaps every bin for Receiver 2 with high probability, and therefore we can simultaneously encode for two receivers without needing intermediate code words \( u_1^n, u_2^n \) that would require auxiliary variables in the random code construction. However, this construction yields no improvement for transmission, as a receiver can only reliably distinguish pairwise between bins, while for transmission one needs to distinguish between all bins simultaneously, and therefore the union of the bins, the pool, must be a reliable code book. Then, to transmit over two different channels, one needs two pools of different sizes, which leads to the traditional binning scheme.

Next, we compare the pool-selection construction to the ID code construction in [17]. There, a transmission code book is divided into overlapping bins, and each bin corresponds to one ID message. Hence, the pool is a transmission code book. Thus, in a broadcast setting, the size of the pool is constrained by the capacity of the worse marginal channel, since we have one pool that must be a reliable transmission code book for both marginal channels. As the number of possible ID messages is approximately \( 2^M \) for bins of size \( M \) [17] and the bins must be smaller than the pool, this construction cannot achieve the ID capacity for semi-average errors, if the marginal channels admit different rates for the same input distribution.

C. Identification with Maximal Errors

It is well-known that in order to identify one establishes common randomness between a sender and a receiver [45, 67]. In this work, we considered a semi-average error criterion. As pointed out in Remark 3, one doesn’t need local randomness at the sender to establish common randomness with one receiver, because the messages for the other receiver can be used as local randomness, i.e., they randomize the channel input.

Under a maximum-error criterion, it is impossible to extract randomness from the message, since one has to identify reliably, regardless of the message that was sent to the other receiver. Hence the error probabilities are maximized over all message pairs \((i, j) \in [N_1] \times [N_2]\). In this setting, Ahlswede [53] claimed to have proven the exact ID capacity region, yet Bracher and Lapidoth [54, 55] pointed out a gap in the converse proof. Nevertheless, Ahlswede [53] showed that the ID capacity region is the same for two private messages as for degraded message sets, where one receiver must identify two messages with rates \( R_0, R_1 \), while the other must identify only one of them with rate \( R_0 \). Furthermore, the ID capacity region is strictly larger than the transmission capacity region with degraded message sets [53], because the encoder randomly selects two messages from sets of cardinalities \( 2^{nR_0} \) and \( 2^{nR_1} \) and sends both of them over the better channel. Hence, the receiver can decode a random message from a set of cardinality \( 2^{nR_0 + nR_1} \), and thus, the ID rate is \( R_0 + R_1 \).

Therefore, using maximal errors, one can achieve a rate region [53, Theorem 11]

\[
\mathcal{R}_{\text{max}}(\mathcal{B}) = \mathcal{R}_1(\mathcal{B}) \cap \mathcal{R}_2(\mathcal{B}),
\]

where

\[
\mathcal{R}_1(\mathcal{B}) = \left\{ (R_1, R_2) : \begin{array}{l}
R_1 \leq \min \left\{ I(X; Y_1), I(X; Y_1|U) + I(U; Y_2) \right\}, \\
R_2 \leq I(U; Y_2), \\
\|U\| \leq |X| + 2
\end{array} \right\}
\]

and

\[
\mathcal{R}_2(\mathcal{B}) = \left\{ (R_1, R_2) : \begin{array}{l}
R_1 \leq I(U; Y_1), \\
R_2 \leq \min \left\{ I(X; Y_2), I(X; Y_2|U) + I(U; Y_1) \right\}, \\
\|U\| \leq |X| + 2
\end{array} \right\}.
\]

The converse proof is still open, as described above.
D. Open Problems

We mention a few related open problems. First, it is of high practical importance to determine optimal coding strategies for the MIMO BC as is done for transmission \[6, 7, 29, 35, 40, 42, 68]. Furthermore, different models of channel uncertainty can be considered. The authors are working on a follow-up paper extending the results of this work to the arbitrarily varying channel, where the state may change during one transmission/identification. For transmission, such extensions use Ahlswede’s Robustification [69, 71] and Elimination [72] techniques. However, they cannot be applied directly to ID, since robustification extends the bound for one error probability to the arbitrarily varying setting, whereas for identification, one error probability per ID message pair \(i \neq i'\) has to be bounded. Furthermore, elimination does not apply in the general form of transmission, where one can prove that for every random code there exists a deterministic code with randomized encoding that achieves the same rate. This does not apply, since common randomness between the sender and the receiver, such as a random code, increases the ID capacity [45, 67]. In practice, one often encounters stochastic channel uncertainty. Appropriate performance measures should also be considered for ID, such as ergodic capacities for fast fading and outage capacities for slow fading. A lower bound on the outage ID capacity has been determined for single-user channels [73]. For deterministic identification over single-user channels, without randomized encoding, lower and upper bounds on the ergodic capacity are known [74, 75]. The latter setting is quite different from the randomized setting considered in this work. The capacity regions for ID with maximal errors over BCs with fading can be lower-bounded by the respective transmission capacities, as was done for single-user channels [25, 67]. But in contrast to those results, ID converse proofs for BCs seem to be difficult, especially for maximal errors (see also Section VIII-C).

IX. Summary

We determined the ID capacity region of the compound BC under a semi-average error criterion in the MIMO Gaussian setting (Theorem 3) and in the discrete setting (Theorem 4). To this end, we extended the proofs for the DMBC in [54, 55] to the compound setting. In the achievability proof in Section V we extended the pool-selection construction of [54, 55] by modified the decoding sets, specifically by taking the union over all possible instances of the channel state. In the converse proof in Section VI we extended the union bound arguments that were used by Bracher and Lapidoth [54, 55] to combined the rate constraints for the marginal channels. We applied them to combine bounds for the different states as well, and thereby showed the existence an input distribution that satisfies the mutual information constraints on the rates, for every state. Our capacity theorem holds for the general compound BC, as channel ordering conditions, such as degradedness, are not required in the ID setting.

As examples, we derived explicit expressions for the ID capacity regions for symmetric channels in Section IV-A for the binary erasure channel in Section IV-B and for the scalar Gaussian channel in Section IV-D where the sender has no channel state information. In those examples, each user can achieve the capacity of the respective marginal channel. Thereby, the capacity region is rectangular and strictly larger than the transmission capacity region. However, the ID capacity region is not rectangular in general, as demonstrated by the example of a compound broadcast Z-channel with a binary state in Section IV-C and by the example of a Gaussian product BC in Section IV-E.

The shape of the ID capacity regions emphasizes a different behavior than in the single-user setting (see Section II-D). There, the ID capacity equals the transmission capacity [45]. In the broadcast setting, however, the ID capacity can be strictly larger, since interference between receivers can be seen as part of the randomization of the coding scheme, as discussed in Remark 3 and Section VIII-C.

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