The centralisers of nilpotent elements in the classical Lie algebras

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INTRODUCTION

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$. Consider the coadjoint representation $\text{ad}^\ast(\mathfrak{g})$. The index of $\mathfrak{g}$ is the minimum of dimensions of stabilisers $\mathfrak{g}_\alpha$ over all covectors $\alpha \in \mathfrak{g}^\ast$

$$\text{ind} \mathfrak{g} = \min_{\alpha \in \mathfrak{g}^\ast} \dim \mathfrak{g}_\alpha.$$  

The definition of index goes back to Dixmier [3, 11.1.6]. This notion is important in Representation Theory and also in Invariant Theory. By Rosenlicht’s theorem [12], generic orbits of an arbitrary action of a linear algebraic group on an irreducible algebraic variety are separated by rational invariants; in particular, $\text{ind} \mathfrak{g} = \text{tr.deg} \mathbb{K}(\mathfrak{g}^\ast)^G$.

The index of a reductive algebra equals its rank. Computing the index of an arbitrary Lie algebra seems to be a wild problem. However, there is a number of interesting results for several classes of non-reductive subalgebras of reductive Lie algebras. For instance, parabolic subalgebras and their “relatives” (nilpotent radicals, seaweeds) are considered in [4], [8], [13]. The centralisers of elements form another interesting class of subalgebras. The last topic is closely related to the theory of integrable Hamiltonian systems.

Let $G$ be a semisimple Lie group (complex or real), $\mathfrak{g} = \text{Lie} G$, and $Gx$ an orbit of a covector $x \in \mathfrak{g}^\ast$. Let $\mathfrak{g}_x$ denote the stabiliser of $x$. It is well-known that the orbit $Gx$ possesses a $G$-invariant symplectic structure. There is a family of commuting with respect to a Poisson bracket polynomial functions on $\mathfrak{g}^\ast$ constructed by the argument shift method such that its restriction to $Gx$ contains $\frac{1}{2} \dim(Gx)$ algebraically independent functions if and only if $\text{ind} \mathfrak{g}_x = \text{ind} \mathfrak{g}$.

Conjecture (´Elashvili). Let $\mathfrak{g}$ be a reductive Lie algebra. Then $\text{ind} \mathfrak{g}_x = \text{ind} \mathfrak{g}$ for each covector $x \in \mathfrak{g}^\ast$.

Recall that if $\mathfrak{g}$ is reductive, then the $\mathfrak{g}$-modules $\mathfrak{g}^\ast$ and $\mathfrak{g}$ are isomorphic. In particular, it is enough to prove the “index conjecture” for stabilisers of vectors $x \in \mathfrak{g}$.

Given $x \in \mathfrak{g}$, let $x = x_s + x_n$ be the Jordan decomposition. Then $\mathfrak{g}_x = (\mathfrak{g}_{x_s})x_n$. The subalgebra $\mathfrak{g}_{x_s}$ is reductive and contains a Cartan subalgebra of $\mathfrak{g}$. Hence, $\text{ind} \mathfrak{g}_{x_s} = \text{ind} \mathfrak{g} = \text{rk} \mathfrak{g}$. Thus, a verification of the ”index conjecture” is reduced to the computation of $\text{ind} \mathfrak{g}_{x_n}$ for nilpotent elements $x_n \in \mathfrak{g}$. Clearly, we can restrict ourselves to the case of simple $\mathfrak{g}$.

Note that if $x$ is a regular element, then the stabiliser $\mathfrak{g}_x$ is commutative and of dimension $\text{rk} \mathfrak{g}$. The “index conjecture” was proved for subregular nilpotents and nilpotents of height 2 [2], and also for nilpotents of height 3 [10]. (The height of a nilpotent element $e$ is the maximal number $m$ such that $(\text{ad} e)^m \neq 0$.) Recently, Êlashvili’s conjecture was proved by Charbonnel [2] for $\mathbb{K} = \mathbb{C}$.

In the present article, we prove in an elementary way, that for any nilpotent element $e \in \mathfrak{g}$ of a simple classical Lie algebra the index of $\mathfrak{g}_e$ equals the rank of $\mathfrak{g}$. We assume that the
ground field $\mathbb{K}$ contains at least $k$ elements, where $k$ is the number of Jordan blocks of a nilpotent element $e \in \mathfrak{g}$. For the orthogonal and symplectic algebras, it is also assumed that $\text{char } \mathbb{K} \neq 2$. Note that if a reductive Lie algebra $\mathfrak{g}$ does not contain exceptional ideals, then $\mathfrak{g}_x$ has the same property. Thus, the “index conjecture” is proved for the direct sums of classical algebras.

By Vinberg’s inequality, which is presented in [21 Sect. 1], we have $\text{ind } \mathfrak{g}_x \geq \text{ind } \mathfrak{g}$ for each element $x \in \mathfrak{g}^*$. It remains to prove the opposite inequality. To this end, it suffices to find $\alpha \in (\mathfrak{g}_x)^*$ such that the dimension of its stabiliser in $\mathfrak{g}_x$ is at most $\text{rk } \mathfrak{g}$. For $\mathfrak{g} = \mathfrak{gl}(V)$ and $\mathfrak{g} = \mathfrak{sp}(V)$, we explicitly indicate such a point $\alpha \in \mathfrak{g}_e^*$. In case of the orthogonal algebra, the proof is partially based on induction.

In the last two sections, $\mathbb{K}$ is assumed to be algebraically closed and of characteristic zero. It is shown that the stabilisers $(\mathfrak{g}_e)_\alpha$ constructed for $\mathfrak{g} = \mathfrak{gl}(V)$ and $\mathfrak{g} = \mathfrak{sp}(V)$ are generic stabilisers for the coadjoint representation of $\mathfrak{g}_e$. For the orthogonal case, we give an example of a nilpotent element $e \in \mathfrak{so}_8$ such that the coadjoint action of $\mathfrak{g}_e$ has no generic stabiliser. Similar results for parabolic and seaweed subalgebras of simple Lie algebras were obtained by Panyushev and also by Tauvel and Yu. In [13], there is an example of a parabolic subalgebra of $\mathfrak{so}_8$ having no generic stabilisers for the coadjoint representation. The affirmative answer for series $A$ and $C$ is obtained by Panyushev in [11].

In the last section, we consider the commuting variety of $\mathfrak{g}_e$ and its relationship with the commuting variety of triples of matrices.

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1. Preliminaries

Suppose $\mathfrak{g}$ is a simple classical Lie algebra or a general linear algebra. Let $e \in \mathfrak{g}$ be a nilpotent element and $\mathfrak{z}(e)$ its centraliser in $\mathfrak{g}$. Note that there is no essential difference between $\mathfrak{g} = \mathfrak{gl}(V)$ and $\mathfrak{g} = \mathfrak{sl}(V)$. However, the first case is more suitable for calculations. In case of orthogonal and symplectic algebras, we need some facts from the theory of symmetric spaces.

Let $(\ , \ )_V$ be a non-degenerate symmetric or skew-symmetric form on a finite dimensional vector space $V$ given by a matrix $J$, i.e., $(v, w)_V = v^tJw$, where the symbol $^t$ stands for the transpose. The elements of $\mathfrak{gl}(V)$ preserving $(\ , \ )_V$ are exactly the fixed vectors $\mathfrak{gl}(V)^\sigma$ of the involution $\sigma(\xi) = -J\xi^tJ^{-1}$. There is the $\mathfrak{gl}(V)^\sigma$-invariant decomposition $\mathfrak{gl}(V) = \mathfrak{gl}(V)^\sigma \oplus \mathfrak{g}_1$. The elements of $\mathfrak{g}_1$ multiply the form $(\ , \ )_V$ by $-1$, i.e., $(\xi v, w)_V = (v, \xi w)_V$ for every $v, w \in V$.

Set $\mathfrak{g} = \mathfrak{gl}(V)^\sigma$, and let $e \in \mathfrak{g}$ be a nilpotent element. Denote by $\mathfrak{z}(e)$ and $\mathfrak{z}_{\mathfrak{gl}}(e)$ the centralisers of $e$ in $\mathfrak{g}$ and $\mathfrak{gl}(V)$, respectively. Since $\sigma(e) = e$, $\sigma$ acts on $\mathfrak{z}_{\mathfrak{gl}}(e)$. Clearly, $\mathfrak{z}_{\mathfrak{gl}}(e)^\sigma = \mathfrak{z}(e)$. This yields the $\mathfrak{z}(e)$-invariant decomposition $\mathfrak{z}_{\mathfrak{gl}}(e) = \mathfrak{z}(e) \oplus \mathfrak{z}_1$. Given $\alpha \in \mathfrak{z}_{\mathfrak{gl}}(e)^*$, let $\tilde{\alpha}$ denote its restriction to $\mathfrak{z}(e)$.

**Proposition 1.** Suppose $\alpha \in \mathfrak{z}_{\mathfrak{gl}}(e)^*$ and $\alpha(\mathfrak{z}_1) = 0$. Then $\mathfrak{z}(e)\tilde{\alpha} = \mathfrak{z}_{\mathfrak{gl}}(e)\alpha \cap \mathfrak{z}(e)$. 

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Proof. Take $\xi \in \mathfrak{z}(e)$. Since $[\xi, \mathfrak{z}_1] \subset \mathfrak{z}_1$, $\alpha([\xi, \mathfrak{z}(e)]) = 0$ if and only if $\alpha([\xi, \mathfrak{z}_{gl}(e)]) = 0$. In particular, $\mathfrak{z}(e)_a = \mathfrak{z}(e)_a$. □

Suppose $\mathfrak{h}$ is a Lie algebra and $\tau \in \text{Aut}\mathfrak{h}$ an involution, which defines the decomposition $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$. Each point $\gamma \in \mathfrak{h}^*_0$ determines a skew-symmetric 2-form $\hat{\gamma}$ on $\mathfrak{h}_1$ by $\hat{\gamma}(\xi, \eta) = \gamma([\xi, \eta])$.

Lemma 1. In the above notation, we have $\text{ind}\mathfrak{h} \leq \text{ind}\mathfrak{h}_0 + \min_{\gamma \in \mathfrak{h}^*_0} \dim(Ker \hat{\gamma})$.

Proof. Consider $\gamma$ as a function on $\mathfrak{h}$, which is equal to zero on $\mathfrak{h}_1$. Then $\mathfrak{h}_\gamma = (\mathfrak{h}_0)_\gamma \oplus (\mathfrak{h}_\gamma \cap \mathfrak{h}_1) = (\mathfrak{h}_0)_\gamma \oplus (\text{Ker} \hat{\gamma})$. We have $\dim(\mathfrak{h}_0)_\gamma = \text{dim}\mathfrak{h}_0$ for generic points (= points of some Zariski open subset $U_1 \subset \mathfrak{h}^*_0$). The points of $\mathfrak{h}^*_0$, where $\text{Ker} \hat{\gamma}$ has the minimal possible dimension, form another open subset, say $U_2 \subset \mathfrak{h}^*_0$. For the points of the intersection $U_1 \cap U_2$, the dimension of the stabiliser in $\mathfrak{h}$ equals the required sum. □

2. General linear algebra

Consider a nilpotent element $e \in \mathfrak{gl}(V)$, where $V$ is an $n$-dimensional vector space over $\mathbb{K}$. Denote by $\mathfrak{z}(e)$ the centraliser of $e$. Let us show that the index of $\mathfrak{z}(e)$ equals $n$.

Let $k$ be a number of Jordan blocks of $e$ and $W \subset V$ a $k$-dimensional complement of $\text{Im} e$ in $V$. Denote by $d_i + 1$ the dimension of $i$-th Jordan block. Choose a basis $w_1, w_2, \ldots, w_k$ in $W$, where $w_i$ is a generator of an $i$-th Jordan block, i.e., the vectors $e^{s_i}w_i$ with $1 \leq i \leq k$, $0 \leq s_i \leq d_i$ form a basis of $V$. Let $\varphi \in \mathfrak{z}(e)$. Since $\varphi(e^{s_i}w_i) = e^{s_i}\varphi(w_i)$, the map $\varphi$ is completely determined by its values on $w_i$, $i = 1, \ldots, k$. Each value $\varphi(w_i)$ can be written as

$$\varphi(w_i) = \sum_{j,s} c^{j,s}_i e^{s}w_j,$$

where $c^{j,s}_i \in \mathbb{K}$. (1)

That is, $\varphi$ is completely determined by the coefficients $c^{j,s}_i = c^{j,s}_i(\varphi)$. Note that $\varphi \in \mathfrak{z}(e)$ preserves the space of each Jordan block if and only if $c^{j,s}_i(\varphi) = 0$ for $i \neq j$.

The centraliser $\mathfrak{z}(e)$ has a basis $\{\xi^{i,s}_i\}$, where

$$\begin{align*}
\xi^{i,s}_i(w_i) &= e^{s}w_j, \\
\xi^{i,s}_i(w_i) &= 0 \text{ for } t \neq i, \\
d_j - d_i &\leq s \leq d_j \text{ for } d_j \geq d_i, \\
0 &\leq s \leq d_i \text{ for } d_j
\end{align*}$$

Consider a point $\alpha \in \mathfrak{z}(e)^*$ defined by the formula

$$\alpha(\varphi) = \sum_{i=1}^{k} a_i \cdot c^{i,d_i}_i, \quad a_i \in \mathbb{K},$$

where $c^{j,s}_i$ are the coefficients of $\varphi \in \mathfrak{z}(e)$ and $\{a_i\}$ are non-zero pairwise distinct numbers. We have $\alpha(\xi^{i,s}_i) = a_i$ if $i = j$, $s = d_i$ and zero otherwise.

Theorem 1. The stabiliser $\mathfrak{z}(e)_\alpha$ of $\alpha$ in $\mathfrak{z}(e)$ consist of all maps preserving the Jordan blocks, i.e., $\mathfrak{z}(e)_\alpha$ is the linear span of the vectors $\xi^{i,s}_i$. 

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Proof. Suppose $\varphi \in \mathfrak{z}(e)$ is defined by formula (1). (Some of $c_i^{j,s}$ have to be zeros, but this is immaterial here.) For each basis vector $\xi_i^{j,b}$, we have

$$
\alpha([\varphi, \xi_i^{j,b}]) = \alpha(\sum_{t,s} c_t^{i,s} \xi_t^{s,j,b} - \sum_{t,s} c_j^{t,s} \xi_t^{1,j,b}) = a_j \cdot c_i^{j,-d_j} - a_i \cdot c_i^{j,d_i} - b.
$$

The element $\varphi$ lies in $\mathfrak{z}(e)_\alpha$ if and only if $\alpha([\varphi, \xi_i^{j,b}]) = 0$ for all $\xi_i^{j,b}$.

Note that if $\varphi$ preserves the Jordan blocks, i.e., $c_i^{j,s} = 0$ for $i \neq j$, then $\alpha([\varphi, \mathfrak{z}(e)]) = 0$.

Let us show that $\mathfrak{z}(e)_\alpha$ contains no other elements. Assume that $c_i^{j,s} \neq 0$ for some $i \neq j$. We have three different possibilities: $d_i < d_j$, $d_i = d_j$ and $d_i > d_j$.

If $d_j \leq d_i$, then put $\xi(w_j) = e^{(d_i-s)} w_i$ and $\xi(w_j) = 0$ for $t \neq j$. It should be noted that $0 \leq s \leq d_j \leq d_i$, hence the expression $e^{(d_i-s)}$ is well defined. One has to check that $e^{d_j+1}(\xi(w_j)) = 0$. Adding the powers of $e$, we get $e^{d_j+1}(\xi(w_j)) = e^{d_j+1+d_i-s} w_i = e^{d_j-s}(e^{d_i+1} w_i) = 0$. We have $\alpha([\varphi, \xi]) = a_i \cdot c_i^{j,s} - a_j \cdot c_j^{i,d_j-d_i+s}$. In case $d_j = d_i$, we obtain $(a_i - a_j) \cdot c_i^{j,s} \neq 0$. If $d_j > d_i$, then $s > d_j - d_i + s$. Choose the minimal $s$ such that $c_i^{j,s} \neq 0$. For this choice, we get $\alpha([\varphi, \xi]) = a_i \cdot c_i^{j,s} \neq 0$.

Suppose now that $d_j > d_i$ and $s$ is the minimal number such that $c_i^{j,s} \neq 0$. Set $\xi(w_j) = e^{(d_j-s)} w_i$ and $\xi(w_i) = 0$ for $t \neq j$. As in the previous case, we have $0 \leq s \leq d_j$. In particular, $d_j - s \geq 0, (d_j + 1 + d_j - s) > d_i + 1$ and, thereby, $e^{d_j+1}(\xi(w_j)) = 0$. We obtain

$$
\alpha([\xi, \varphi]) = a_j \cdot c_i^{j,s} - a_i \cdot c_i^{j,d_j-d_i+s} = a_j \cdot c_i^{j,s} \neq 0.
$$

Here $c_i^{j,d_j-d_i+s} = 0$, since $d_i - d_j + s < s$.

\[ \square \]

Corollary. The index of $\mathfrak{z}(e)$ equals $n$.

Proof. The stabiliser $\mathfrak{z}(e)_\alpha$ consist of all maps preserving Jordan blocks. In particular, it has dimension $n$. Hence, $\text{ind} \mathfrak{z}(e) \leq n$. On the other hand, it follows from Vinberg’s inequality that $\text{ind} \mathfrak{z}(e) \geq n = \text{rk} \mathfrak{g}l(V)$.

\[ \square \]

Let us give another proof of the inequality $\text{ind} \mathfrak{z}(e) \leq n$.

Example 1. Let $e \in \mathfrak{g}l_n$ be a nilpotent element and $\mathfrak{h} = \mathfrak{z}(e)$ the centraliser of $e$. We may assume that the first Jordan block of $e$ is of maximal dimension. Then $V = V_{d_1+1} \oplus V_{\text{oth}}$ and $e = e_1 + e_2$, where $V_{d_1+1}$ is the space of the first Jordan block and $V_{\text{oth}}$ is the space of all other Jordan blocks; $e_1 \in \mathfrak{g}l_{d_1+1}, e_2 \in \mathfrak{g}l_{n-d_1-1}$. Let $\tau \in \mathfrak{g}l(V)$ be the conjugation by a diagonal matrix of order two such that $\mathfrak{g}l(V)^\tau = \mathfrak{g}l_{d_1+1} \oplus \mathfrak{g}l_{n-d_1-1}$. The involution $\tau$ acts on $\mathfrak{h} = \mathfrak{z}(e)$ and induces the decomposition $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$, where $\mathfrak{h}_0 = \mathfrak{z}(e_1) \oplus \mathfrak{z}(e_2)$ (the centralisers are considered in the algebras $\mathfrak{g}l_{d_1+1}$ and $\mathfrak{g}l_{n-d_1-1}$, respectively). Assume that the “index conjecture” is true for all $m < n$; in particular, $\text{ind} \mathfrak{z}(e_2) = n - d_1 - 1$. The subalgebra $\mathfrak{z}(e_1)$ is commutative and its index equals $d_1 + 1$. According to Lemma 1, $\text{ind} \mathfrak{z}(e) \leq \text{ind} (\mathfrak{z}(e_1) \oplus \mathfrak{z}(e_2)) + \min \dim(\text{Ker} \dot{\gamma}) \leq n + \min \dim(\text{Ker} \dot{\gamma})_\gamma$. Now, we make a special choice for $\gamma$. Set $\gamma(\xi_1^{d_1}) = 1$ and $\gamma(\xi_i^{j,s}) = 0$ for all other $\xi_i^{j,s}$. The subspace $\mathfrak{h}_1$ is
generated by the vectors \( \xi_i^{1,s} \) and \( \xi_1^{i,s} \) with \( i \neq 1 \). We have
\[
\begin{cases}
\hat{\gamma}(\xi_1^{i,s}, \xi_i^{1,d_i-s}) = 1, \\
\hat{\gamma}(\xi_1^{i,s}, \xi_i^{1,b}) = 0 \text{ if } s + b \neq d_1, \\
\hat{\gamma}(\xi_i^{i,s}, \xi_i^{1,b}) = \hat{\gamma}(\xi_1^{i,s}, \xi_1^{1,b}) = 0.
\end{cases}
\]
The form \( \hat{\gamma} \) defines a non-degenerate pairing between the spaces \( U_1 := \langle \xi_1^{i,s} | 0 \leq s \leq d_1 \rangle \) and \( U_i^1 := \langle \xi_i^{i,s} | d_1 - d_i \leq s \leq d_1 \rangle \). Hence, \( \hat{\gamma} \) is non-degenerate and \( \text{ind} \, \mathfrak{z}(e) \leq n \).

3. SYMPLECTIC ALGEBRA

In this section \( \mathfrak{g} = \mathfrak{sp}_{2n} = \mathfrak{sp}(V) \), where \( V \) is an \( 2n \)-dimensional vector space over \( \mathbb{K} \). As above, \( e \in \mathfrak{sp}_{2n} \) is a nilpotent element and \( \mathfrak{z}(e) \subset \mathfrak{g} \) is the centraliser of \( e \). Let \( \{w_i\} \) be generators of Jordan blocks associated with \( e \). We may assume that the space of each even-dimensional Jordan block is orthogonal to the space of all other Jordan blocks. If \( d_i \) is even, then the restriction of \( \mathfrak{sp}_{2n} \)-invariant form \( (\ , \,)_V \) on the space of \( i \)-th Jordan block is zero. One can choose generators \( \{w_i\} \) such that the odd-dimensional blocks are partitioned in pairs \( (i, i') \), where \( i' \) is the number of the unique Jordan block which is not orthogonal to the \( i \)-th one. Note that \( d_i = d_i \).

Let \( \mathfrak{z}_0(e) \) be the centraliser of \( e \) in \( \mathfrak{gl}_{2n} \). Recall that \( \mathfrak{z}(e) = \mathfrak{z}_0(e)^\sigma \oplus \mathfrak{z}_1 \), where \( \sigma \) is an involutive automorphism of \( \mathfrak{gl}_{2n} \). For elements of \( \mathfrak{z}_0(e) \) we use notation introduced in the previous section.

Let \( \alpha \in \mathfrak{z}_0(e)^* \) be a function determined just like in the previous case:
\[
\alpha(\varphi) = a_1 \cdot c_1^{1,d_1} + a_2 \cdot c_2^{2,d_2} + \ldots + a_{2n} \cdot c_k^{k,d_k},
\]
where \( \varphi \) is given by its coefficients \( c_i^{j,s} \), and \( \{a_i\} \) are pairwise distinct non-zero numbers with \( a_{i'} = -a_i \).

**Lemma 2.** In the above notation, we have \( \alpha(\mathfrak{z}_1) = 0 \).

**Proof.** Assume that there is \( \psi \in \mathfrak{z}_1 \) such that \( \alpha(\psi) \neq 0 \). Then there is a non-zero coefficient \( c_i^{j,d_i} \) of \( \psi \). Recall that \( \sigma(\psi) = -\psi \). The element \( \psi \) multiplies the \( \mathfrak{sp}_{2n} \)-invariant skew-symmetric form \( (\ , \,)_V \) by \( -1 \), in particular, \( (\psi(w_i), v)_V = (w_i, \psi(v))_V \) for each vector \( v \in V \). Clearly, \( \psi(w_i) \) and \( w_i \) have to be orthogonal with respect to the skew-symmetric form. If \( d_i \) is odd, then \( (w_i, e^{d_i}w_i)_V \neq 0 \), hence, \( c_i^{j,d_i} = 0 \). If on the contrary \( d_i \) is even, then
\[
c_i^{j,d_i}(e^{d_i}w_i, w_i)V = (\psi(w_i), w_i)_V = (w_i, \psi(w_i))_V = c_i^{j,d_i}(w_i, e^{d_i}w_i)_V = (1)^{d_i}c_i^{j,d_i}(e^{d_i}w_i, w_i)V = c_i^{j,d_i}(e^{d_i}w_i, w_i)V.
\]
Hence, \( c_i^{j,d_i} = c_i^{j',d_i} \). Combining this equality with defining formula of \( \alpha \) we get a sum over pairs of odd-dimensional blocks
\[
\alpha(\psi) = \sum_{(i,i')} (a_i + a_{i'}) c_i^{j,d_i},
\]
which is zero since \( a_i = -a_{i'} \). \( \square \)

Denote by \( \hat{\alpha} \) the restriction of \( \alpha \) to \( \mathfrak{z}(e) \).
Theorem 2. The dimension of the stabiliser \( z(e)_\alpha = \mathfrak{z}_{\mathfrak{gl}}(e)_\alpha \cap \mathfrak{sp}_{2n} \) equals \( n \).

Proof. The stabiliser of \( \alpha \) in \( \mathfrak{z}_{\mathfrak{gl}}(e) \) consist of all maps preserving the spaces of the Jordan blocks. By Proposition 1, \( z(e)_\alpha = \mathfrak{z}_{\mathfrak{gl}}(e)_\alpha \cap z(e) \). Describe the intersection of \( \mathfrak{z}_{\mathfrak{gl}}(e)_\alpha \) with the symplectic subalgebra. If \( w_i \) is a generator of an even-dimensional block, then \( \xi_{i,s}^i \) multiply the skew-symmetric form by \((-1)^{s+1} \), i.e., \((\xi_{i,s}^i(e^b w_i), e^c w_i) = (-1)^{s}(e^b w_i, \xi_{i,s}^i(e^c w_i)) \). Consider a space of a pair \((i, i')\) of odd-dimensional blocks. If \( s_i \) is a generator of an even-dimensional block, then \( \xi_{i,s}^i \) multiply the skew-symmetric form by \((-1)^{s+1} \), i.e., \((\xi_{i,s}^i(e^b w_i), e^c w_i) = (-1)^{s}(e^b w_i, \xi_{i,s}^i(e^c w_i)) \).

4. THE ORTHOGONAL CASE

In this section \( \mathfrak{g} = \mathfrak{so}_n \). As above \( e \in \mathfrak{so}_n \) is a nilpotent element, \( z(e) \) is the centraliser of \( e \) in \( \mathfrak{g} \). Let \( \{w_i\} \) be generators of Jordan blocks associated with \( e \). We may assume that the space of each odd-dimensional Jordan block is orthogonal to the space of all other Jordan blocks. If \( d_i \) is odd, then the restriction of \( \mathfrak{so}_n \)-invariant form \((\cdot, \cdot)_V \) on the space of \( i \)-th Jordan block is zero. One can choose generators \( \{w_i\} \) such that the even-dimensional blocks are partitioned in pairs \((i, i^*)\), where \( i^* \) is the number of the unique Jordan block which is not orthogonal to the \( i \)-th one. Note that \( d_{i^*} = d_i \).

Like the symplectic algebra, the orthogonal algebra is a symmetric subalgebra of \( \mathfrak{gl}_n \). Denote by \( \sigma \) the involution defining it. Since \( \sigma(e) = e \), we have \( \mathfrak{z}_{\mathfrak{gl}}(e) = \mathfrak{z}(e) \oplus \mathfrak{z}_1 \) similarly to the symplectic case. If \( d_i \) is even, set \( i^* = i \). Assume that \((w_{i^*}, e^{d_i} w_i)_V = \pm 1 \) and \((w_i, e^{d_i} w_i)_V = 1 \) for \( i = i^* \). The algebra \( \mathfrak{z}(e) \) is generated (as a vector space) by the vectors \( \xi_{i,j}^{i,j-s} + \varepsilon(i, j, s)\xi_{j,i}^{i,j-s} \), where \( \varepsilon(i, j, s) = \pm 1 \) depending on \( i, j \) and \( s \). In its turn, the subspace \( \mathfrak{z}_1 \) is generated by the vectors \( \xi_{i,j}^{i,j-s} - \varepsilon(i, j, s)\xi_{j,i}^{i,j-s} \). Recall that \((e^s w_i, e^{d_i-s} w_i)_V \neq 0 \) if \( e^s w_i \neq 0 \).

We give some simple examples of linear functions with zero restrictions to \( \mathfrak{z}_1 \). Let \( \varphi \in \mathfrak{z}_{\mathfrak{gl}}(e) \) be a linear map defined by Formula (1). Set \( \beta_i(\varphi) = c_i^{j,d_i-1} \), \( \gamma_{i,j}(\varphi) = c_i^{j,d_j} \).

Lemma 3. If \( i = i^*, j = j^*, t \neq t^* \), then functions \( \beta_i, \beta_j, \gamma_{i,j} - \gamma_{j,i} \) and \( \gamma_{i,t} + \gamma_{t,i} \) are equal to zero on \( \mathfrak{z}_1 \).

Proof. Suppose \( \psi \in \mathfrak{z}_1 \) is defined by Formula (1). Since \( \sigma(\psi) = -\psi \) and \((\psi(w_i), ew_i)_V = c_i^{j,d_i}(e^{d_i-1} w_i, ew_i)_V \), we have

\[
(\psi(w_i), ew_i)_V = (w_i, \psi(ew_i)_V = (w_i, ew_i)_V = -(ew_i, \psi(w_i)_V = -c_i^{j,d_i-1}(ew_i, e^{d_i-1} w_i)_V.
\]

The form \((\cdot, \cdot)_V \) is symmetric and \((ew_i, e^{d_i-1} w_i)_V \neq 0 \), hence \( \beta_i(\psi) = c_i^{j,d_i-1} = 0 \).

Similarly,

\[
c_i^{j,d_i}(e^{d_i} w_j, w_j)_V = (\psi(w_j), w_j)_V = (w_j, \psi(w_j)_V = c_i^{j,d_j}(w_j, e^{d_j} w_j)_V;
\]

\[
c_i^{j,d_j}(w_i, w_i)_V = (w_i, w_i)_V = (w_i, \psi(w_i))_V = c_i^{j,d_j}(w_i, e^{d_i} w_i)_V.
\]
Recall that by our choice \( (e^{d_j}w_j, w_j)_V = (w_i, e^{d_i}w_i)_V = 1, \) \( (e^{d_i}w_i, w_{i^*})_V = -(w_i, e^{d_i}w_{i^*})_V \). Hence \( c_i^{j,d_j} = c_j^{i,d_i}, c_i^{l,d_l} = -c_l^{t,d_t} \).

Let us prove the inequality \( \text{ind } \mathfrak{z}(e) \leq \text{rk } \mathfrak{so}_n \) by the induction on \( n \). In the following two cases, the induction argument does not go through. Therefore we consider them separately.

**The first case.** If \( e \in \mathfrak{so}_{2m+1} \) is a regular nilpotent element, then \( \mathfrak{z}(e) \) is a commutative \( m \)-dimensional algebra.

**The second case.** Let \( e \in \mathfrak{so}_{4d} \) be a nilpotent element with two Jordan blocks of size \( 2d \) each. Set \( \alpha = c_1^{1,2d-2} - c_2^{2,2d-2} \), where \( \varphi \) is defined by Formula (1). One can easily check that \( \mathfrak{z}(e)_\alpha \) has a basis \( \xi_1, -\xi_2 \) with \( 0 \leq s \leq 2d - 1 \) and that \( \dim \mathfrak{z}(e)_\alpha = 2d \).

Order the Jordan blocks of \( e \) according to their dimensions \( d_1 \geq d_2 \geq \ldots \geq d_k \). Here \( d_i + 1 \) stands for the dimension of the \( i \)-th Jordan block, similarly to the case of \( \mathfrak{gl}_n \). Note that the numbers \( n \) and \( k \) have the same parity. Assume that \( k > 1 \) and if \( k = 2 \), then both Jordan blocks are odd dimensional. Then we have the following three possibilities:

1. For some even number \( 2p < k \) the restriction of \( (\ , \ )_V \) to the space of the first \( 2p \) Jordan blocks is non-degenerate;
2. The number \( d_i \) is even for \( i = 1, k \) and odd for all other \( i \);
3. The number \( d_i \) is even if and only if \( i = 1 \).

Each of these three possibilities is considered separately. In the first two cases we make an induction step. In the third one a point \( \alpha \in \mathfrak{z}(e)_\ast \) is given such that \( \dim \mathfrak{z}(e)_\alpha \leq \text{rk } \mathfrak{so}_n \).

1. Suppose the space \( V_{2m} \) of the first \( 2p \) Jordan blocks has dimension \( 2m \) and the restriction of \( (\ , \ )_V \) to \( V_{2m} \) is non-degenerate. Then \( V = V_{2m} \oplus V_{\text{other}} \). Let \( \tau \) be an involution of \( \mathfrak{gl}_n \) corresponding to these direct sum, i.e., \( \mathfrak{gl}_n = \mathfrak{gl}(V_{2m}) \oplus \mathfrak{gl}(V_{\text{other}}) \). Set \( \mathfrak{h} = \mathfrak{z}(e), \mathfrak{h}_0 = \mathfrak{z}(e)^\tau \). Then \( \mathfrak{h}_0 = \mathfrak{z}(e_1) \oplus \mathfrak{z}(e_2) \), where the centralisers of \( e_1 \) and \( e_2 \) are taken in \( \mathfrak{so}_{2m} \) and \( \mathfrak{so}_{n-2m} \), respectively. By the inductive hypothesis, \( \text{ind } \mathfrak{z}(e_1) = m, \text{ind } \mathfrak{z}(e_2) = [n/2] - m \). Hence, \( \text{ind } \mathfrak{z}(e) \leq [n/2] + \min \text{dim}(\text{Ker } \hat{\gamma}) \). To conclude we have to point out a function \( \gamma \in \mathfrak{h}_0^\ast \) such that \( \hat{\gamma} \) is non-degenerate. Recall that the involutions \( \sigma \) and \( \tau \) commute with each other, preserve \( e \) and determine the decomposition \( \mathfrak{z}_\mathfrak{gl}(e) = (\mathfrak{z}(e_1) \oplus \mathfrak{z}(e_2) \oplus \mathfrak{h}_1) \oplus \mathfrak{z}_1 \). If \( \gamma(\mathfrak{z}_1) = \gamma(\mathfrak{h}_1) = 0 \), then \( \text{Ker } \hat{\gamma} = (\mathfrak{h}_1 \cap \mathfrak{z}_\mathfrak{gl}(e))_\gamma \).

Divide odd-dimensional Jordan blocks into pairs \( (i, i') \) (it is assumed that \( i, i' \leq 2p \)). Define a point \( \gamma \) by

\[
\gamma(\varphi) = \sum_{(i,i'), i,i' \leq 2p} (c_i^{i',d_{i'}} - c_i^{j,d_i}) + \sum_{j \leq 2p, (d_j+1) \text{ is even}} c_j^{j,d_j},
\]

where \( \varphi \in \mathfrak{z}_\mathfrak{gl}(e) \) is given by its coefficients \( c_i^{j,d_j} \). The first summand is a sum of \( (\gamma_{i,i'} - \gamma_{i',i}) \) over pairs of odd-dimensional blocks, the second is the sum of \( (\gamma_{j,j} + \gamma_{j',j'}) \) over pairs of even-dimensional blocks. According to Lemma 3, both summands are identical zeros on \( \mathfrak{z}_1 \). Moreover, by the definition \( \gamma(\mathfrak{h}_1) = 0 \).

Set \( j' := j \) for even-dimensional blocks. Assume that an element \( \psi \in \mathfrak{h}_1 \) determined by (1) lies in the kernel of \( \hat{\gamma} \), i.e., \( \gamma([\psi, \mathfrak{h}_1]) = 0 \). Then \( \gamma([\psi, \mathfrak{h}_1]) = \gamma([\psi, \mathfrak{z}_\mathfrak{gl}(e)]) = 0 \). Since \( \psi \in \mathfrak{so}_n \),
and \( \psi \neq 0 \), we may assume that \( c^{j,s}_i \neq 0 \) for some \( j > 2p \geq i \). We have
\[
\gamma([\psi, \eta]) = \pm c^{j,s}_i \neq 0.
\]
Thus we have proved that \( \hat{\gamma} \) is non-degenerate and \( \text{ind} \( e \) \leq \lceil n/2 \rceil \).

(2) Consider a decomposition \( V = V_{\text{oth}} \oplus V_{d_k+1} \), where the second summand is the space of the smallest (odd-dimensional) Jordan block and the first one is the space of all other blocks. As above \( e = e_1 + e_2 \), where nilpotent element \( e_2 \) corresponds to the smallest (odd-dimensional) Jordan block. We define an involution \( \tau \), algebras \( z(e_1, e_2) \), \( z(e_2) \), \( h_0 \) and a subspace \( h_1 \) in the same way as in case (1).

Let \( \gamma \) be the following function
\[
\gamma(\varphi) = c_1^{d_1-1} + \sum_{i=2}^{k-1} c_i^{i,d_i},
\]
where \( \varphi \) is given by formula (1). The first summand is \( \beta_1 \), the second summand is a sum of \( (\gamma_{j,j} + \gamma_{j^*,j^*}) \) over pairs of even-dimensional blocks. Due to Lemma \( \Box \) \( \gamma(\xi_1) = 0 \). Suppose \( \psi \in h_1 \) is given by its coefficients \( c^{j,s}_i \).

One can see that the kernel of \( \hat{\gamma} \) is one-dimensional and generated by \( (\xi_1^{k,b} - \xi_1^{k,d_k}) \). Hence, \( \text{ind} \( e \) \leq \lceil n/2 \rceil - 1 + 1 = \lceil n/2 \rceil \).

(3) In this case \( n \) and \( k \) are odd, and \( e \) has a unique odd-dimensional Jordan block whose size is maximal. Assume that \( k = 2m+1 \). Enumerate the Jordan blocks by integers ranging from \(-m\) to \( m \). Let the unique odd-dimensional block has number zero. Suppose that pairs of blocks \((-i,i)\) and \((-j,j)\) are orthogonal to each other if \( i \neq \pm j \), and dimensions of Jordan blocks are increasing from \(-m\) to \( 0 \) and decreasing from \( 0 \) to \( m \), i.e., if \( |i| \leq |j| \), then \( d_i \geq d_j \).

Note that \( d_i = d_{-i} \). Such enumeration is shown on Picture 1. Choose the generators \( w_i \) of Jordan blocks such that \( i(w_i,e^{d_i}w_{-i})_V = |i| \) for \( i \neq 0 \) and \( (w_0,e^{d_0}w_0)_V = 1 \).

\begin{center}
\text{Picture 1.}
\end{center}
Suppose \( \varphi \in \mathfrak{z}_{gl}(e) \) is given by Formula (1). Consider the following point \( \alpha \in \mathfrak{z}_{gl}(e)^* \):

\[
\alpha(\varphi) = \sum_{i=-m+1}^{m} c_{i,d_i}^i.
\]

One can check by direct computation that \( c_{i,d_i}^i(\psi) = -c_{-i}^{1-i,d_{i-1}}(\psi) \) for each \( \psi \in \mathfrak{z}_1 \) and, hence, \( \alpha(\mathfrak{z}_1) = 0 \). Let \( \mathfrak{a} \in \mathfrak{z}(e)^* \) be the restriction of \( \alpha \). Let us describe the stabiliser \( \mathfrak{z}(e)_{\mathfrak{a}} = \mathfrak{z}_{gl}(e)_{\mathfrak{a}} \cap \mathfrak{z}(e) \). Note that \( \alpha([\varphi, \xi_i^q]) = c_{i,d_i-1}^i(\varphi) = c_{i}^{j+1,d_{i+1}^j-1} \).

**Lemma 4.** Suppose \( \varphi \in \mathfrak{z}(e) \) and \( \text{ad}^*(\varphi) \alpha = 0 \). Then \( c_{i}^{j+1} = c_{i}^{j+1} \) for \( i < j \).

**Proof.** Assume that the statement is wrong and take a maximal \( i \) for which there are \( j > i \) and \( s \) such that \( c_{i}^{j,s} = 0 \). Because \( \varphi \) preserves \( (\ , )_V \), \( c_{-i}^{-i,d_i-1} = \pm c_{j}^{j,s} \neq 0 \). Hence, \( -j \leq i < j, j > 0, |i| \leq j \) and \( d_i \geq d_j \). Moreover, \( -j < (i+1) \leq j \) and \( d_{i+1} \geq d_j \). Evidently, \( d_{i+1} - s \geq d_j - s \geq 0 \) and there is an element \( \xi_j^{i+1,d_{i+1}-s} \in \mathfrak{z}_{gl}(e) \). We have

\[
0 = \alpha([\varphi, \xi_j^{i+1,d_{i+1}-s}]) = c_{i}^{j,s} - c_{i+1}^{j,\delta} = c_{i}^{j,s}.
\]

Here we do not give a precise value of \( \delta \). Anyway all coefficients \( c_{i}^{j+1} \) are zero, because \( j + 1 > i + 1 > i \). We get a contradiction. Thus the lemma is proved. \( \square \)

Let us say that \( \varphi \in \mathfrak{z}_{gl}(e) \) has a step \( l \) whenever \( c_{i}^{j,s} = 0 \) for \( j \neq i + l \). Each vector \( \varphi \in \mathfrak{z}_{gl}(e) \) can be represented as a sum \( \varphi = \varphi_{-2m} + \varphi_{-2m+1} + \ldots + \varphi_{2m-1} + \varphi_{2m} \), where the step of \( \varphi \) equals \( l \). The notion of the step is well-defined on \( \mathfrak{z}(e) \), due to an equality \((-i) - (-j) = j - i \). From the definition of \( \alpha \), one can deduce that \( \alpha(\varphi_l, \varphi_l) = 0 \) only if \( l + t = 1 \). The stabiliser \( \mathfrak{z}(e)_{\alpha} \) is a direct sum of its subspaces \( \Phi_l \), consisting of elements having step \( l \). As we have seen, \( \Phi_l = \varnothing \) if \( l > 0 \). It is remains to describe elements with non-positive steps.

**Example 2.** Let us show that \( \dim \Phi_l \leq d_l/2 \). Suppose \( \varphi \in \Phi_0, \varphi \neq 0 \) and \( \varphi(w_0) = 0 \). Take a minimal by the absolute value \( i \) such that \( \varphi(w_i) \neq 0 \). Since \( \varphi \in \mathfrak{so}_n \), we have also \( \varphi(w_{-i}) \neq 0 \). Assume that \( i > 0 \) and a coefficient \( c_{i}^{j,s} \) of \( \varphi \) is non-zero. Then \( |i - 1| < i \), \( d_{i-1} \geq d_i \), there is an element \( \xi_i^{1,d_i-1} \in \mathfrak{z}_{gl}(e) \) and \( 0 = \alpha([\xi, \varphi]) = c_{i}^{j,s} - c_{i-1}^{j,s} = c_{i}^{j,s} \). Hence, if \( \varphi(w_0) = 0 \), then also \( \varphi = 0 \). Thus, a vector \( \varphi \in \Phi_0 \) is entirely determined by its value on \( w_0 \). In its turn \( \varphi(w_0) = c_1 w_0 + c_3 e^3 w_0 + \ldots + c_{d_0-1} e^{d_0-1} w_0 \).

**Lemma 5.** If \( q = 2l \) or \( q = 2l - 1 \), where \( 0 < l \leq m \), then \( \dim \Phi_{-q} \leq (d_l + 1)/2 \).

**Proof.** Similarly to the previous example, we show that if \( \varphi \in \Phi_{-q} \) and \( \varphi(w_l) = 0 \), then also \( \varphi = 0 \). Since \( \varphi \in \mathfrak{so}_n \), if \( \varphi(w_l) \neq 0 \), then also \( \varphi(w_{q-l}) \neq 0 \). Suppose \( \varphi(w_j) \neq 0 \) for some \( j \). If \( j < l \), then \( j - q \geq l \), but \( \varphi(w_l) = 0 \), hence \( j > l \). Find the minimal \( j > l \) such that \( \varphi(w_j) \neq 0 \). Suppose \( c_{j-1}^{l,q,s} = c_{j-1}^{l,q,s}(\varphi) \neq 0 \). We have \( -j < -l \leq j - q - 1 < j \), \( d_j \leq d_{j-q}-1, d_j - s \geq 0 \). Hence, there is an element \( \xi := c_{j-1}^{l,q,s} \in \mathfrak{z}_{gl}(e) \). As above

\[
0 = \alpha([\xi, \varphi]) = c_{j-1}^{l,-q} - c_{j-1}^{l,-q,q} = c_{j-1}^{l,-q} \text{ (we do not give a precise value of } \delta, \text{ anyway, } \varphi(w_{j-1}) = 0, \text{ since } l \leq j - 1 < j \).
\]

To conclude we describe possible values \( \varphi(w_l) \). If \( q = 2l \), then \( \varphi(w_l) = c_0 w_1 + c_2 e^2 w_1 + \ldots + c_{d_l} e^{d_l} w_1 \). In case \( q = 2l - 1 \) we get an equation on...
coefficients of $\varphi$: 
$$0 = \alpha[(\xi_{-1}^{t,b}, \varphi)] = c_{l}^{-l+1,d_{l-1}-b} - c_{l-1}^{-l,d_{l}-b},$$
i.e., $c_{l}^{-l+1,d_{l-1}-b} = c_{l-1}^{-l,d_{l}-b}$. This is possible only for odd $b$. \hfill \Box

**Theorem 3.** Suppose $e \in \mathfrak{so}_n$ is a nilpotent element. Then $\text{ind} \ z(e) = \text{rk} \mathfrak{so}_n = [n/2]$.

**Proof.** If possibility (3) takes place, i.e., only one Jordan block of $e$ is odd-dimensional and it is also maximal, then, as we have seen, $\mathfrak{z}(e)_\alpha = \bigoplus_{q=0}^{m} \Phi_{-q}$. Moreover, $\dim \Phi_q$ is at most half of dimension of the Jordan block with number $[(q+1)/2]$. Thereby, $\dim \mathfrak{z}(e)_\alpha \leq ((d_0+1)/2 + (\sum_{l=1}^{m} d_l)) = [n/2]$. On the other hand, according to Vinberg’s inequality, $\text{ind} \ z(e) \geq [n/2]$.

In cases (1) and (2) the inequality $\text{ind} \ z(e) \leq \text{rk} \mathfrak{so}_n$ was proved by induction.

If none of these three possibilities takes place, then either $k = 1$ and $e$ is a regular nilpotent element, or $k = 2$ and both Jordan blocks of $e$ are even-dimensional. These two cases have been considered separately. \hfill \Box

## 5. Generic points

In this section we assume that $\mathbb{K}$ is algebraically closed and of characteristic zero. Suppose we have a linear action of a Lie algebra $\mathfrak{g}$ on a vector space $V$.

**Definition.** A vector $x \in V$ (a subalgebra $\mathfrak{g}_x$) is called a *generic point* (a *generic stabiliser*), if for every point $y \in U \subset V$ of some open in Zariski topology subset $U$ algebras $\mathfrak{g}_y$ and $\mathfrak{g}_x$ are conjugated in $\mathfrak{g}$.

It is well known that generic points exist for any linear action of a reductive Lie algebra.

It is proved in [3, §1] that a subalgebra $\mathfrak{g}_x$ is a generic stabiliser if and only if $V = V^\mathfrak{g}_x + \mathfrak{g}_x$, where $V^\mathfrak{g}_x$ is the subspace of all vectors of $V$ invariant under $\mathfrak{g}_x$.

Tauvel and Yu have noticed that in case of a coadjoint representation $\mathfrak{g}_x = (\mathfrak{g}/\mathfrak{g}_x)^* = \text{Ann}(\mathfrak{g}_x)$, $(\mathfrak{g}^*)^\mathfrak{g}_x = \text{Ann}(x_{\mathfrak{g}_x, \mathfrak{g}})$. From this observation they have deduced a simple and useful criterion.

**Theorem 4.** [13, Corollaire 1.8.] Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}^*$. The subalgebra $\mathfrak{g}_x$ is a generic stabiliser of the coadjoint representation of $\mathfrak{g}$ if and only if $[\mathfrak{g}_x, \mathfrak{g}] \cap \mathfrak{g}_x = \{0\}$.

Unfortunately, the authors of [13] were not aware of the aforementioned Élashvili’s result and have proved it anew.

Let $e \in \mathfrak{gl}_n$ be a nilpotent element and $\mathfrak{z}(e)$ the centraliser of $e$. Set $\mathfrak{h} = \mathfrak{z}(e)_\alpha$, where $\alpha \in \mathfrak{z}(e)^*$ is the same as in Section 2.

**Proposition 2.** There is an $\mathfrak{h}$-invariant decomposition $\mathfrak{z}(e) = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is generated by the vectors $\xi_i^{i,s}$ with $i \neq j$.

**Proof.** Recall that $\mathfrak{h}$ is generated by the vectors $\xi_i^{i,s}$. The inclusion $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ follows immediately from the equality

$$[\xi_i^{i,s}, \xi_j^{l,b}] = \begin{cases} 
\xi_i^{i,s+b} & \text{if } i = j, i \neq t; \\
-\xi_j^{i,s+b} & \text{if } i = t, i \neq j; \\
0 & \text{otherwise.}
\end{cases}$$

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There is a similar decomposition in the case of symplectic algebras. Let \( e \in \mathfrak{sp}(V) \subset \mathfrak{gl}(V) \). Denote by \( \mathfrak{z}_{\mathfrak{gl}}(e) \) and \( \mathfrak{z}_{\mathfrak{sp}}(e) \) the centralisers of \( e \) in \( \mathfrak{gl}(V) \) and \( \mathfrak{sp}(V) \), respectively. We use notation of Section 3. Suppose \( \mathfrak{z}_{\mathfrak{gl}}(e) = \mathfrak{h} \oplus \mathfrak{m} \). Evidently, this decomposition is \( \sigma \)-invariant and \( \mathfrak{z}_{\mathfrak{sp}}(e) = \mathfrak{h}^\sigma \oplus \mathfrak{m}^\sigma \), where \( \mathfrak{h}^\sigma = \mathfrak{z}_{\mathfrak{sp}}(e)_{\tilde{\alpha}} \).

**Theorem 5.** The Lie algebras \( \mathfrak{z}_{\mathfrak{gl}}(e)_\alpha \) and \( \mathfrak{z}_{\mathfrak{sp}}(e)_{\tilde{\alpha}} \) constructed in Sections 2 and 3 in cases of general linear and symplectic algebras are generic stabilisers of the coadjoint actions of \( \mathfrak{z}_{\mathfrak{gl}}(e) \) and \( \mathfrak{z}_{\mathfrak{sp}}(e) \).

**Proof.** Let us verify the condition of Theorem 4. Since \( [\mathfrak{h}, \mathfrak{z}_{\mathfrak{gl}}(e)] = [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \), we have \( [\mathfrak{h}, \mathfrak{z}_{\mathfrak{gl}}(e)] \cap \mathfrak{h} = 0 \). Similarly, \( [\mathfrak{h}^\sigma, \mathfrak{z}_{\mathfrak{sp}}(e)] \subset \mathfrak{m}^\sigma \). □

In case of orthogonal algebras it can happen that a generic stabiliser of the coadjoint action of \( \mathfrak{z}(e) \) does not exist.

**Example 3.** Let \( e \in \mathfrak{so}_8 \) be a subregular nilpotent element. Then it has two Jordan blocks of dimensions 3 and 5. Choose the generators \( w_1, w_2 \) of Jordan blocks such that \( (w_1, e^2w_1)_V = (w_2, e^4w_2)_V = 1 \). The dimension of \( \mathfrak{z}(e) \) is 6 and \( \mathfrak{z}(e) \) has a three-dimensional center, generated by the vectors \( e, e^3 = \xi_2^{2,3} \) and \( \varphi_3 = \xi_1^{2,4} - \xi_1^{1,2} \). Since \( \text{ind} \mathfrak{z}(e) = 4 \), we have \( \dim \mathfrak{z}(e)_\alpha = 4 \) for points of some open subset \( U \subset \mathfrak{z}(e)^\ast \).

Assume that a generic stabiliser of the coadjoint action of \( \mathfrak{z}(e) \) exists and denote it by \( \mathfrak{f} \). Evidently, \( \mathfrak{f} \) contains the center of \( \mathfrak{z}(e) \). Consider an element \( \varphi_2 = \xi_1^{2,3} + \xi_2^{1,1} \in \mathfrak{z}(e) \). Clearly, the subspace \( [\varphi_2, \mathfrak{z}(e)] \) is a linear span of \( e^3 \) and \( \varphi_3 \). In particular, it is contained in the center of \( \mathfrak{z}(e) \), and, hence, in \( \mathfrak{f} \). Hence, \( [\varphi_2, \mathfrak{f}] \subset \mathfrak{f} \), and, by Theorem 4 \( \mathfrak{f} \subset \mathfrak{z}(e)_{\varphi_2} \). Since \( \dim \mathfrak{z}(e)_{\varphi_2} = 4 \), we have \( \mathfrak{f} = \mathfrak{z}(e)_{\varphi_2} \). On the other hand, \( \mathfrak{z}(e)_{\varphi_2} = \langle e, e^3, \varphi_3, \varphi_2 \rangle_{\mathfrak{g}} \) is a normal, but not a central subalgebra of \( \mathfrak{z}(e) \).

Consider the embedding \( \mathfrak{so}_8 \subset \mathfrak{so}_9 \) as the stabiliser of the first basis vector in \( \mathbb{K}^9 \). By a similar argument one can show that a generic stabiliser does not exist for the coadjoint action of \( \mathfrak{z}_{\mathfrak{so}_9}(e) \) either.

6. **Commuting varieties**

Let \( \mathfrak{g} \) be a Lie algebra over an algebraically closed field \( \mathbb{K} \) of characteristic zero. A closed subset \( Y = \{(x, y)|x, y \in \mathfrak{g}, [x, y] = 0\} \subset (\mathfrak{g} \times \mathfrak{g}) \) is called the *commuting variety* of the algebra \( \mathfrak{g} \). The question of whether \( Y \) is irreducible or not is of a great interest. In case of a reductive algebra \( \mathfrak{g} \) the commuting variety \( Y \) is irreducible and coincides with the closure of \( G(\mathfrak{a}, \mathfrak{a}) \), where \( \mathfrak{a} \subset \mathfrak{g} \) is a Cartan subalgebra and \( G \) is a connected algebraic group with \( \text{Lie} G = \mathfrak{g} \).

Let \( e \in \mathfrak{gl}_n \) be a nilpotent element and \( \mathfrak{z}(e) \) the centraliser of \( e \). We use notation introduced in Section 2. Set \( \mathfrak{h} = \mathfrak{z}(e)_\alpha \). Consider a subalgebra \( \mathfrak{t} \subset \mathfrak{z}(e) \) generated by the vectors \( \xi_i^{0,0} \). Evidently, \( \mathfrak{t} \subset \mathfrak{h} \). Moreover, since \( [\xi_i^{0,s}, t_i \xi_i^{0,0} + t_j \xi_j^{0,0}] = (t_j - t_i) \xi_i^{s,0} \), the algebra \( \mathfrak{h} \) coincides with the normaliser (= centraliser) of \( \mathfrak{t} \) in \( \mathfrak{z}(e) \). Hence, \( \mathfrak{h} \) coincides with its normaliser in \( \mathfrak{z}(e) \).
Let $Z(e)$ be the identity component of the centraliser of $e$ in $\text{GL}_n$. Then $Y_0 = \overline{Z(e)(\mathfrak{h}, \mathfrak{h})}$ is an irreducible component of $Y$ of maximal dimension. As in the reductive case, $Y$ is irreducible if and only if $Y_0 = Y$. It is known that if a nilpotent element $e$ has at most two Jordan blocks, then $Y$ is irreducible \[7\]. In the general case, the statement is not true, since it would lead to the irreducibility of the commuting varieties of triples of matrices.

**Example 4.** Assume that $Y_0 = Y$ for all nilpotent elements $e \in \mathfrak{gl}_m$ with $m \leq n$. Consider the set of triples of commuting matrices

$$C_3 = \{(A, B, C)|A, B, C \in \mathfrak{gl}_n, [A, B] = [A, C] = [B, C] = 0\}.$$ 

Let $\mathfrak{a} \subset \mathfrak{gl}_n$ be a subalgebra of diagonal matrices. Clearly, $\overline{\text{GL}_n(\mathfrak{a}, \mathfrak{a}, \mathfrak{a})}$ is an irreducible component of $C_3$. Let us prove by induction that it coincides with $C_3$. There is nothing to prove for $n = 1$. Let $n > 1$. We show that each triple $(A, B, C)$ of commuting matrices is contained in the closure $\overline{\text{GL}_n(\mathfrak{a}, \mathfrak{a}, \mathfrak{a})}$. Without loss of generality, we may assume that $A, B, C \in \mathfrak{sl}_n$. Let $A = A_s + A_n$ be the Jordan decomposition of $A$. If $A_s \neq 0$, consider the centraliser $\mathfrak{z}(A_s)$ of $A_s$ in $\mathfrak{gl}_n$. Clearly, $A, B, C \in \mathfrak{z}(A_s)$ and $\mathfrak{z}(A_s)$ is a sum of several algebras $\mathfrak{gl}_n$ with strictly smaller dimension. We may assume that $\mathfrak{a} \subset \mathfrak{z}(A_s)$. Then, by the inductive hypothesis

$$(A, B, C) \in \overline{Z(A_s)(\mathfrak{a}, \mathfrak{a}, \mathfrak{a})} \subset \overline{\text{GL}_n(\mathfrak{a}, \mathfrak{a}, \mathfrak{a})}.$$ 

Suppose now that all three elements $A, B, C$ are nilpotent and at least one of them, say $A$, is not regular. Consider the centraliser $\mathfrak{z}(A) \subset \mathfrak{gl}_n$. We have assumed that $Y_0 = Y$, i.e., the pair $(B, C)$ lies in the closure of $Z(A)(\mathfrak{h}, \mathfrak{h})$. It will be enough to show that $(A, \mathfrak{h}, \mathfrak{h}) \subset \overline{\text{GL}_n(\mathfrak{a}, \mathfrak{a}, \mathfrak{a})}$. Let $x \in \mathfrak{t} \subset \mathfrak{h}$ be a non-central semisimple element. Then $A \in (\mathfrak{gl}_n)_x$ and $\mathfrak{h} \subset (\mathfrak{gl}_n)_x$. Once again we can make an induction step, passing to a subalgebra $(\mathfrak{gl}_n)_x$.

If all three elements $A, B, C$ are regular nilpotent, then there is a non-trivial linear combination $A'$ of them, which is non-regular. In particular, the triple $(A, B, C)$ is equivalent under the action of $\text{GL}_n$ to some other triple $(A', B', C')$ of commuting nilpotent matrices.

It is known that for $n > 31$ the variety $C_3$ is reducible, see \[6\]. Hence, the commuting variety $Y$ is certainly reducible for some nilpotent elements. It will be interesting to find minimal (in some sense) nilpotent elements for which $Y$ is reducible and/or describe some classes of nilpotent elements for which $Y$ is irreducible.

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