Dynamical signature of the edge state in the 1D Aubry–André model

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Abstract

Topological features have become an intensively studied subject in many fields of physics. As a witness of topological phase, the edge states are topologically protected and may be helpful in quantum information processing. In this paper, we define a measure to quantify the dynamical localization of the system and simulate the localization in the one-dimensional Aubry–André model. We find an interesting connection between the edge states and the dynamical localization of the system, this connection may be used as a signature of the edge state and topological phase.

Keywords: dynamical localization, edge state, topological phase

(Some figures may appear in colour only in the online journal)

1. Introduction

The discovery of topological insulators has attracted considerable attention in the field of topological phases of matter [1–8]. Earlier studies in this field mainly focus on three issues: (1) topological properties of the ground state phases; (2) realizations of topological quantum matter [9]; and (3) possible applications of topological matter, e.g., applications of edge states for topological quantum computing and spintronics [10, 11]. Yet, explorations on the dynamical feature of topological quantum matter are scarce. This motivates the study in this paper.

Localization induced by disorder has been recently observed in ultracold bosonic gases in purely random potentials [12] and in bichromatic optical lattice [13]. In both systems, the observations have been interpreted in terms of Anderson localization [14]. Anderson localization is trivial in one-dimensional (1D) systems since the critical disorder at which the system wavefunction changes from being extended to exponentially localized is zero, i.e., all states for any finite disorder are exponentially localized in 1D systems. This makes 1D Anderson localization rather unattractive. However, by using the so-called Aubry–André (A–A) model, Aubry and André [15] predicted a sharp transition from diffusion to localization for a given value of disorder length in 1D systems last century in the 1980s, where the transition arises from the existence of an incommensurate potential of finite strength mimicking disorder in a 1D tight binding model.

The A–A model (also known as the Harper model) can be simulated by trapped fermions on a 1D quasi-periodic optical lattice, which can be generated by superimposing two 1D optical lattices with commensurate or incommensurate wavelengths [12, 16, 17]. One interesting aspect of the 1D A–A model is that it can be mapped into the two-dimensional (2D) Hofstadter model [18, 19], exemplifying the topologically non-trivial 2D quantum Hall system on a 1D lattice.

In this paper, we focus on the manifestation of topological properties in the dynamics of topological states in 1D systems. In the absence of any symmetries, all 1D systems belong to the topologically trivial phases, while in 2D systems there are topological phases of the integer quantum Hall effect [20]. Recently, it has been shown that 1D quasi-periodic optical lattice systems can exhibit edge states [21] and possess the same physical origins of topological phases of 2D quantum
Hall effects on periodic lattices. This makes the study of the 1D A–A model rather interesting, which is adopted as motivation to study the dynamical localization of the 1D A–A model.

To be specific, we connect the dynamical localization with the appearance of edge states in this paper. Because the edge state can be taken as a witness of topological phase, and the appearance of the topologically protected edge states can be witnessed by a measure of localization, the study on dynamical localization is thus the bridge between the dynamical feature and the topological feature of quantum matter.

This paper is organized as follows. In section 2, we introduce the model and present the equation of motion for the system. In section 3, we study the dynamics of the A–A model, a quantity to characterize the dynamical localization, called average inverse participation ratio (AIPR), is introduced and calculated. The dependence of the AIPR on system parameters is given and discussed. In section 4, we study the dynamics of the off-diagonal A–A model. Finally, we conclude our results in section 5.

2. The diagonal A–A model

Let us begin with a specific quasi-crystal, the 1D A–A model [15]. This is a 1D tight-binding model in which the on-site potential is modulated in space. The Hamiltonian of this model takes,

\[ H = -J \sum_{i=1}^{N} (\hat{c}_i^+ \hat{c}_{i+1} + \text{H.c.}) + \sum_{i=1}^{N} V \cos(2\pi \alpha i + \delta) \hat{n}_i, \]

where \( N \) is the number of lattice sites, \( \hat{c}_i^\dagger (\hat{c}_i) \) denotes the creation (annihilation) operator of the fermion, and \( \hat{n}_i = \hat{c}_i^\dagger \hat{c}_i \). \( J \) represents the hopping amplitude, \( V \) is the on-site potential, and \( \alpha \) controls the periodicity of the modulation. Whenever \( \alpha \) is irrational the modulation is incommensurate with the lattice and the on-site term is quasi-periodic. Note that in this model the modulation phase \( \delta \) appears as an additional degree of freedom representing a shift of the origin of the quasi-periodic order. We adopt open boundary conditions with \( n = 1 \) and \( n = N \) being the two edge sites.

Suppose there is only one excitation in the 1D lattice, the wavefunction of the system at time \( t \) can be written as \( |\Psi(t)\rangle = \sum_n \psi_n(t) |c_n\rangle |0\rangle \). Substituting this wavefunction into the Schrödinger equation leads to the following equation:

\[ i\hbar \frac{\partial}{\partial t} \psi_n(t) = -J(\psi_{n+1}(t) + \psi_{n-1}(t)) + V \cos(2\pi \alpha n + \delta) \psi_n(t). \]

Here, \( \psi_n(t) \) is the probability amplitude of finding the excitation at site \( n \). For irrational \( \alpha \), it is shown that a localization transition appears in the A–A model as \( V \) is increased beyond the critical value \( (V = 2J) \) with all states being extended (localized) for \( V < 2J \) (\( V > 2J \)). For rational \( \alpha \), the A–A model can be mapped into a 2D Hofstadter lattice by treating \( \delta \) as another momentum of another spatial dimension [18, 19]. For \( \alpha \neq \frac{1}{2} \), the Hofstadter lattice has gapped energy bands with non-trivial topology, characterized by non-zero Chern numbers.

The time-reversal symmetry protected edge states were predicted to occur in quantum wells (2D system) of mercury telluride sandwiched between cadmium telluride in 1986 and were observed in 2007 [22]. Since then it is widely accepted that the appearance of edge states in a finite system with an open boundary condition is a signature for a state of matter with topological properties distinct from the conventional one. Thus, localized edge states are expected for a finite-size system with a boundary that possesses non-zero Chern numbers.

3. Results

To simulate numerically the quantum evolution of the system, we choose two different initial states. In the first, the system is initially prepared to occupy site 1, while in the second, the system is initialized at the centre of the lattice. To quantify the dynamical localization/extension of the system, we define an AIPR,

\[ L_0 = \frac{1}{T} \int_0^T \sum_n |p_n(t)|^2, \]

where \( p_n(t) \) denotes the probability of finding the excitation at site \( n \), therefore, \( \sum_n p_n(1) = 1 \). \( T \) denotes the evolution time. This definition can be understood as an extension of the inverse participation ratio [23] averaged over the evolution time \( T \). Therefore, the AIPR depends on the initial state of the system. In the following numerical simulation, we initialize the system to occupy one of the edge sites, say site 1, at the beginning of evolution. Due to the exchange symmetry of the 1D system, the edge eigenstates must appear in pairs or occupy the two edge sites with equal probability. In other words, when we find an edge state located at edge site 1, there must be another one at edge site \( N \). Otherwise, the edge eigenstate distributes equally at both edge sites. Assume the edge state is \( \psi_1(0) \), i.e., the edge eigenstate is exactly the excited state of site 1, it is easy to show that \( L_0 = 1 \). For edge states that not only locate exactly at the edge sites, the probability of finding the system at site 1 after an evolution time \( t \) is \( P_1(t) = \langle |1\rangle |\Phi(t)\rangle^2 \), where \( |\Phi(t)\rangle \) is the wavefunction of the system at time \( t \) with initial state \( |1\rangle \). Straightforward calculation yields,

\[ P_1(t) = \sum_n |a_n|^2 + 2 \sum_{n \neq a} |a_n|^2 |a_m|^2 \cos(E_m - E_n)t, \]

where \( a_n \) denotes a coefficient defined by \( |1\rangle = \sum_n a_n |E_n\rangle \), and \( |E_n\rangle \) is the \( n \)th eigenstate of the system with the eigenvalue denoted by \( E_n \). Therefore,

\[ L_0 = \frac{1}{T} \int_0^T P_1(t) dt \sim |a_1|^4 + 4 |a_1|^4 \sum_{m \neq 1} |a_m|^2 |a_n|^2, \]

where \( |E_1\rangle \) being the edge state located at the edge site 1 is assumed. Thus, for a system having edge eigenstates, \( L_0 \rightarrow 1 \), whereas \( L_0 \rightarrow 0 \), for a system where all eigenstates are extended states. Thus the AIPR can be taken as a measure to quantify the localization of the dynamics. It is worth addressing that the evolution time \( T \) affects the value of localization measure \( L_0 \), but it does not change the peak position of \( L_0 \). In this sense, it qualifies for quantifying localization with respect to a reference dynamics.
Figure 1. The average inverse participation ratio as a function of $\alpha$ with a given $V = 1.5J$ (top panel), and as a function of $\alpha$ and $V$ (bottom panel). The total number of sites is $N = 99$ and $T = 600$ (in units of $1/J$). The other parameters chosen are $J = 1$, $\delta = \frac{\pi}{2}$. The system is initially at site 1. $V$ is chosen in units of $J$.

Figure 1 displays the AIPR $Lo$ as a function of $V$ and $\alpha$ (lower panel). To show clearly the dependence of $Lo$ on $\alpha$, we present the $Lo$ versus $\alpha$ with a fixed $V = 1.5$ in the upper panel. We find that $Lo$ arrives at its maximum at about ($\alpha = \text{integer} + \frac{1}{2}$). As $V$ is increased, the system becomes more dynamically localized at the edge sites. $Lo$ is a periodic function of $\alpha$ and $\delta$ with periods 1 and $2\pi$, respectively. This is a reflection of symmetry in the Hamiltonian, i.e., the Hamiltonian remains unchanged by substitution, $\alpha \rightarrow (1+\alpha), \delta \rightarrow (\delta+2\pi)$. From figure 1 we can also observe that $Lo$ is very close to zero at $\alpha = m$ and $\alpha = \frac{(2m+1)\pi}{2}$ where $m$ is an integer. This can be explained as a direct consequence of the space-independent on-site potential, $V_i = V \cos(2\pi\alpha i + \delta)|_{\alpha=m} = V \cos\delta$ and $V_i|_{\alpha=(2m+1)\pi/2} = -\cos\delta$.

Figure 2 shows the AIPR as a function of $V$ and $\delta$. As $\alpha$ changes, $Lo$ may have one peak or many peaks within one period of $\delta$, each peak corresponds to an eigenstate well localized at the edge site 1. The observed dynamical localization quantified by AIPR depends on the initial state of the system, for example, the system is well dynamically localized with the excitation being initially at site 1, while it is extended with the central site being occupied, see figure 3. Further numerical results show that $Lo$ is sharply suppressed when the sites other than site 1 are occupied initially.

To understand the physics behind the dynamical localization, we calculate the largest probability of finding the system at site 1 when the system is in one of its eigenstates. This calculation would show the overlap between site 1 and the eigenstate which exhibits largest probability to occupy site 1. The eigenvalue equation $H|\alpha\rangle = E|\alpha\rangle$, the other parameters chosen are $J = 1$. The total number of sites is $N = 99$ and $T = 600$ (in units of $1/J$), the other parameters chosen are $J = 1$. For the top panel, $\alpha = \frac{1}{2}$, and for the bottom panel $\alpha = (\sqrt{3} - 1)/4$. The system is initially prepared at site 1. $V$ is in units of $J$, $\delta$ is in units of $\pi$.

To shed light on the effect of the boundary, we now turn to discuss the situation with a periodic boundary condition. The A–A model in this situation can be solved analytically as follows. Suppose that the $n$th eigenstate of the Hamiltonian restricted to a single particle in the 1D lattice is given by $|E_n\rangle = \sum_i u_{i,n} |\psi_i\rangle$, the eigenvalue equation $H|E_n\rangle = E_n|E_n\rangle$ leads to the Harper equation,

$$E_n u_{i,n} = -J(u_{i+1,n} + u_{i-1,n}) + V \cos(2\pi\alpha i + \delta) u_{i,n}. \quad (5)$$

One may wonder how the dynamical localization connects with the topological feature, i.e., the edge states. The AIPR is calculated with the system initially at one of the edges, e.g., site 1 in this paper. The appearance of edge states indicates that the system initially occupies one of the eigenstates of the system with large probability (nearly 100% if the edge state is well localized). As the eigenstate acquires only a phase in the time evolution, the appearance of the edge state leads to dynamical localization. The is the reason why the AIPR can grasp the appearance of the edge states.
where $V_1 = V \cos(2\pi \alpha_1 + \delta)$ with a rational $\alpha_1$ given by $\alpha_1 = p/q$ with $p$ and $q$ being integers which are prime to each other. Since the potential $V_1$ is periodic with a period $q$, the wave functions take the Bloch form, $u_{k+i} = e^{i\delta_k}u_k$, for the lattice with the periodic boundary condition. Omitting the eigenstate index $n$, where not confused, and taking $u_j = e^{i\delta_k}\psi_j(k)$ for $|k| < \pi / q$, we have $\psi_{j+q}(k) = \psi_j(k)$, equation (5) then follows

$$E(k)\psi_j = -J(e^{i\delta}\psi_{j+1} + e^{-i\delta}\psi_{j-1}) + V \cos(2\pi j p/q + \delta)\psi_j,$$

To be specific, we take $\alpha_1 = 1/3$, then $q = 3$ and $-1/3 < k/\pi < 1/3$. By $\psi_{j+q}(k) = \psi_j(k)$, equation (6) reduces to

$$M\Phi_E = E\Phi_E,$$

where

$$M = \begin{pmatrix}
V \cos\left(\delta + \frac{2\pi}{3}\right) & -J e^{i\delta} & -J e^{-i\delta} \\
-J e^{-i\delta} & V \cos\left(\delta - \frac{2\pi}{3}\right) & -J e^{i\delta} \\
-J e^{i\delta} & -J e^{-i\delta} & V \cos \delta
\end{pmatrix},$$

and $\Phi_E = (\psi_1, \psi_2, \psi_3)^T$. Solving the eigenvalue equation (7), we can obtain three eigenvalues for a given $k$,

$$E_1 = \sqrt{4J^2 + V^2 \cos \theta},$$
$$E_2 = \sqrt{4J^2 + V^2 \cos \left(\theta + \frac{2\pi}{3}\right)},$$
$$E_2 = \sqrt{4J^2 + V^2 \cos \left(\theta - \frac{2\pi}{3}\right)},$$

where $\theta = \frac{1}{2} \arccos \left(-\frac{d}{2\sqrt{J^2 + 4V^2}}\right)$, $d = 2J^3 \cos(3k) - \frac{V^2}{4} \cos(3\delta)$. The corresponding three eigenstates are,

$$\Phi_E = \begin{pmatrix}
\psi_1(E) \\
\psi_2(E) \\
\psi_3(E)
\end{pmatrix}, \quad E = E_1, E_2, E_3,$$

where

$$\psi_1(E) = g(E)\psi_2,$$
$$\psi_2(E) = \frac{1}{\sqrt{1 + |g(E)|^2 + |e(E)|^2}},$$
$$\psi_3(E) = e(E)\psi_2,$$
$$g(E) = \frac{e^{ik} + B(E) e^{-2ik}}{A(E) - B(E)},$$
$$e(E) = B(E) \cdot \frac{e^{2ik} + A(E) e^{-ik}}{A(E) - B(E)},$$
$$A(E) = V \cos\left(\frac{2\pi}{3} + \delta\right) - E,$$
$$B(E) = \frac{J}{V \cos \delta - E}.$$

Collecting these equations, we calculate the AIPR and exhibit the results in figure 5 (bottom panel). The AIPR under the periodic condition is very small, by contrast, the open boundary condition helps dynamical localization, as shown in the top panel of figure 5 shows. This again can be explained as a consequence of small overlap between site 1 and the eigenstates of the system. In fact, under the periodic condition, all eigenstates of the system are extended, this is enforced by the Bloch theorem.

4. Off-diagonal $A$–$A$ model

Now we extend the study to the generalized 1D $A$–$A$ model, which is described by the following Hamiltonian,

$$H = -J \sum_{i=1}^{N} \left[1 + \alpha \cos(2\pi \alpha_1 + \delta)\delta_{1i}\right] (\hat{c}_i^\dagger \hat{c}_{i+1} + \text{H.c.})$$
$$+ \sum_{i=1}^{N} V \cos(2\pi \alpha_1 + \delta)\hat{n}_i.$$
This Hamiltonian is different from the model in equation (1) at the inhomogeneity in hopping strength described by cosine modulations. The modulations have the same periodicity as in the on-site potential energy and its amplitude is characterized by $\lambda$. The special case with $\lambda = 0$ corresponds to the diagonal A–A model, and the generalized A–A model can be derived starting from an ancestor 2D Hofstadter model with next-nearest-neighbour hopping terms. It has been shown recently [24] that the commensurate off-diagonal A–A model is topologically non-trivial in the gapless regime and supports zero-energy edge modes. Unlike the incommensurate case, the non-trivial topology in the off-diagonal A–A model is attributed to the topological properties of the 1D Majorana chain.

Figure 6 shows the AIPR as a function of $\lambda$, $J$ and $\delta$ with the system initially at site 1. We find that $V$, the modulation amplitude of the on-site potential, does not shift the peak, but the larger the $V$, the bigger the AIPR is (figure 6(a)). Randomness of $V$ increases the AIPR, see figure 6(b), which is reminiscent of Anderson localization in a disordered medium. The phases $\delta$ and $\delta_{\text{off}}$ can be tuned independently in experiment, so $\delta$ and $\delta_{\text{off}}$ can be treated as independent variables. The phase $\delta_{\text{off}}$ can alter the AIPR, see figure 5(c), the AIPR arrives at a maximum when $\delta_{\text{off}} = 0$, and it is very close to 0 when $\delta_{\text{off}}$ in an interval of $[\pi/2, 3\pi/2]$.

5. Conclusion

To conclude, we have found a signature for the existence of edge states in terms of dynamical localization of the system. The physics behind the connection between the edge state and the dynamical localization is that the eigenstate of the system never evolves up to a phase. A quantity to measure the dynamical localization called AIPR is introduced and discussed. We have calculated the AIPR for the diagonal and off-diagonal A–A model. Our findings suggest that the AIPR can be taken as a measure to quantify the dynamical localization of the quantum system, in particular, it can be chosen as a witness of the edge state. Our strategy is to make use of the definition of edge states and it is applicable for both gapped and gapless systems in one dimension.

The time evolution of an isolated macroscopic quantum system initially prepared in an out-of-equilibrium state is currently turning from an abstract concept to a real phenomenon that can be observed and studied experimentally.
This striking change has been mainly driven by experiments on cold atoms [25, 26], but it will be surely given further impulse in the near future by the fast progresses in time-resolved spectroscopy on condensed-matter systems.

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References

[1] Bernevig B A, Hughes T L and Zhang S-C 2006 Science 314 1757
[2] Fu L, Kane C L and Mele E J 2007 Phys. Rev. Lett. 98 106803
[3] König M, Wiedmann S, Brüne C, Roth A, Buhmann H, Molenkamp L W, Qi X-L and Zhang S-C 2007 Science 318 766
[4] Hsieh D, Qian D, Wray L, Xia Y, Hor Y, Cava R J and Hasan M Z 2008 Nature 452 970
[5] Chen Y L et al 2009 Science 325 178
[6] Xia Y et al 2009 Nature Phys. 5 398
[7] Hsieh D et al 2009 Phys. Rev. Lett. 103 146401
[8] Kuroda K et al 2012 Phys. Rev. Lett. 108 206803
[9] Yao N Y, Gorshkov A V, Laumann C R, Läuchli A M, Ye J and Lukin M D 2013 Phys. Rev. Lett. 110 185302
[10] Nayak C, Simon S H, Stern A, Freedman M and Das Sarma S 2008 Rev. Mod. Phys. 80 1083
[11] Moore J E 2010 Nature 464 194
[12] Roati G, D’Errico C, Fallani L, Fattori M, Fort C, Zaccanti M, Modugno G, Modugno M and Inguscio M 2008 Nature 453 895
[13] Billy J, Josse V, Zuo Z, Bernard A, Hambrecht B, Lugan P, Cemert D, Sanchez-Palencia L, Bouyer P and Aspect A 2008 Nature 453 891
[14] Anderson P W 1958 Phys. Rev. 109 1492
[15] Aubry S and Andre G 1980 Ann. Isr. Phys. Soc. 3 133
[16] Fallani L, Lye J E, Guerrera V, Fort C and Inguscio M 2007 Phys. Rev. Lett. 98 130404
[17] Deissler B et al 2010 Nature Phys. 6 354
[18] Harper P G 1955 Proc. Phys. Soc. A 68 874
[19] Hofstadter D R 1976 Phys. Rev. B 14 2239
[20] Avron Y, Seiler R and Shapira B 1986 Nucl. Phys. B 265 364
[21] Lang L-J, Cai X and Chen S 2012 Phys. Rev. Lett. 108 220401
[22] Pankratov O A, Pahkhomov S V and Volkov B A 1986 Solid State Commun. 61 93
[23] Haake F 1991 Quantum Signatures of Chaos (Berlin: Springer)
[24] Brown W G, Santos L F, Starling D J and Viola L 2008 Phys. Rev. E 77 021106
[25] Dukesz F, Zilbergerts M and Santos L F 2009 New J. Phys. 11 043026
[26] Ganesan S, Sun K and Das Sarma S 2013 Phys. Rev. Lett. 110 180403
[27] Bloch I, Dalibard J and Zwerger W 2008 Rev. Mod. Phys. 80 885
[28] Polkovnikov A, Sengupta K, Silva A and Vengalattore M 2011 Rev. Mod. Phys. 83 863