MULTIPLIER IDEALS AND FILTERED D-MODULES

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Abstract. We give a Hodge-theoretic interpretation of the multiplier ideal of an effective divisor on a smooth complex variety. More precisely, we show that the associated graded coherent sheaf with respect to the jumping-number filtration can be recovered from the smallest piece of M. Saito’s Hodge filtration of the $\mathcal{D}$-module of vanishing cycles.

1. Introduction

Let $D$ be an effective $\mathbb{Q}$-divisor on a nonsingular complex variety $X$ of dimension $n$. The multiplier ideal $\mathcal{J}(D)$ is a subsheaf of ideals of $\mathcal{O}_X$ and measures in a subtle way the singularities of $D$, see [La01]. The singularities of $D$ get ”worse” if $\mathcal{J}(D)$ is smaller. The main goal of this note is to give a Hodge-theoretic interpretation of multiplier ideals. That such an interpretation is possible was hinted by [Bu03] where we proved a local relation at a point $x \in X$ between $\mathcal{J}(D)$ and the mixed Hodge structure on the cohomology the Milnor fiber of an integral divisor $D$ at $x$.

The natural setting for our result is the theory of mixed Hodge modules due to M. Saito ([Sa88], [Sa90]). Since we restrict our attention to the Hodge filtration only and disregard the weight filtration and the rational structure, we end up working with filtered $\mathcal{D}_X$-modules $(M,F)$. Here $\mathcal{D}_X$ is the sheaf of non-commutative rings of linear algebraic differential operators (see [Bo87]). The Hodge filtration $F$ is always assumed here to be increasing. By the Riemann-Hilbert correspondence, $M$ corresponds to a perverse sheaf on $X$. For example, the trivial mixed Hodge module $\mathbb{Q}^H_X[n]$ is represented by the filtered left $\mathcal{D}_X$-module $(\mathcal{O}_X, F)$, where $\text{Gr}_p^F = 0$ for $p \neq 0$. The corresponding perverse sheaf is the shifted trivial complex $\mathbb{Q}_X[n]$.

For a non-constant regular function $f : X \to \mathbb{C}$, one has the vanishing cycles functor $\psi_f$ which can be defined on the abelian category of mixed Hodge modules. By definition, $\psi_f$ corresponds to $^p\psi_f = \psi_f[-1]$ on the category of perverse sheaves on $X$. For $\alpha \in (0,1] \cap \mathbb{Q}$, let $\psi_f^\alpha \mathcal{O}_X$
correspond to the eigenspace of the semisimple part of monodromy for the eigenvalue \( \exp(-2\pi i \alpha) \). M. Saito’s theory provides us with a canonical filtration \( F \) on \( \psi_f^\alpha \mathcal{O}_X \) (for definitions, see the introduction of [Sa88]).

**Theorem.** Let \( X \) be a smooth complex variety of dimension \( n \). Let \( f : X \to \mathbb{C} \) be a non-constant regular function and \( D = f^{-1}(0) \) the corresponding effective divisor. Then for \( \alpha \in (0, 1] \),

\[
\frac{\mathcal{J}((\alpha - \epsilon) \cdot D)}{\mathcal{J}(\alpha \cdot D)} = F_1\psi_f^\alpha \mathcal{O}_X,
\]

where \( 0 < \epsilon \ll 1 \).

Here, \( F_1 \) is the smallest piece of the Hodge filtration of the left \( \mathcal{D}_X \)-module \( \psi_f \mathcal{O}_X \). The values \( \alpha \in (0, 1] \) for which the left-hand side of (1) is nonzero are called jumping numbers (see [Bu03], [La01], [ELSV]). The values \( \alpha \in (0, 1] \) for which the right-hand side of (1) is nonzero were considered in [Sa93]. Thus the Theorem answers a question in [ELSV] regarding the relation between the two sets of values and reproves their theorem that the jumping numbers of \( D \) are roots of the Bernstein-Sato polynomial of \( f \) up to a sign.

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2. **Proof of the Theorem**

We used left \( \mathcal{D} \)-modules only for the introduction. We will work, as our references do, with right \( \mathcal{D}_X \)-modules. The trivial right \( \mathcal{D}_X \)-module is \( \omega_X = \bigwedge^n \Omega^1_X \), the sheaf of regular \( n \)-forms. Locally, the action of a vector field \( \xi \) on \( w \in \omega_X \) is given by \( w \xi = -\text{Lie}_\xi w \), the Lie derivative. \( \mathbb{Q}[n] \) is represented by the filtered right \( \mathcal{D}_X \)-module \( (\omega_X, F) \), where \( \text{Gr}_p F = 0 \) for \( p \neq -n \). In general, the transformation from left to right \( \mathcal{D}_X \)-modules is given by

\[
(M, F) \mapsto (\omega_X \otimes M, F_p = \omega_X \otimes F_{p+n}),
\]

\[
(w \otimes u) \xi = w \xi \otimes u - w \otimes \xi u.
\]

Hence (1) is equivalent to

\[
\omega_X \otimes \mathcal{O}_X \frac{\mathcal{J}((\alpha - \epsilon) \cdot D)}{\mathcal{J}(\alpha \cdot D)} = F_{1-n} \psi_f^\alpha \omega_X.
\]
Let \( \mu : Y \to X \) be a log resolution of \((X, D)\). Recall that the multiplier ideal \( \mathcal{J}(\alpha \cdot D) \) is defined for all \( \alpha > 0 \) by
\[
\mathcal{J}(\alpha \cdot D) = \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor \alpha \cdot D \rfloor),
\]
where \( K_{Y/X} = K_Y - \mu^* K_X \) and \( \lfloor \cdot \rfloor \) means rounding down the coefficients in a \( \mathbb{Q} \)-divisor. Put
\[
\mathcal{K}_\alpha(X, D) = \frac{\mathcal{J}((\alpha - \epsilon) \cdot D)}{\mathcal{J}(\alpha \cdot D)},
\]
where \( 0 < \epsilon \ll 1 \). Then \( \mathcal{J}(\alpha \cdot D) \) and \( \mathcal{K}_\alpha(X, D) \) are independent of the choice of \( \mu \) and
\[
(3) \quad \mathcal{K}_\alpha(X, D) = \mu_* \left( \mathcal{O}_Y(K_{Y/X}) \otimes \mathcal{K}_\alpha(Y, \mu^* D) \right).
\]

Put \( g = f \circ \mu \). Let \( \mathcal{M} = (\omega_Y, F) \) be the filtered \( \mathcal{D}_Y \)-module with \( Gr^F_p = 0 \) for \( p \neq -n \). By Theorem 2.14 of [Sa90], for \( \alpha \in (0, 1] \),
\[
\psi^\alpha_j \mathcal{H}^j \mu_* \mathcal{M} = \mathcal{H}^j \mu_* \psi^\alpha_g \mathcal{M},
\]
for all \( j \in \mathbb{Z} \). Here \( \mu_* : D^b \text{MHM}(Y) \to D^b \text{MHM}(X) \) is the direct image functor on the bounded derived categories of mixed Hodge modules (we care only about the complexes of filtered \( \mathcal{D} \)-modules), and \( \mathcal{H}^j \) is the \( j \)-th cohomology of a complex. In particular, we have an equality of filtered \( \mathcal{D}_X \)-modules
\[
(4) \quad \psi^\alpha_j \mathcal{H}^0 \mu_* \mathcal{M} = \mathcal{H}^0 \mu_* \psi^\alpha_g \mathcal{M}.
\]
We will show that (2), hence the Theorem, follows by taking \( F_{1-n} \) of both sides of (4).

**Lemma 2.1.** \( F_{1-n} \psi^\alpha_j \mathcal{H}^0 \mu_* \mathcal{M} = F_{1-n} \psi^\alpha_j (\omega_X, F) \), for \( \alpha \in (0, 1] \).

**Proof.** Follows from \( \mathcal{H}^0 \mu_* \mathcal{Q}_X^H[n] = \mathcal{Q}_X^H[n] \). \( \square \)

**Lemma 2.2.** \( F_{1-n} \mathcal{H}^0 \mu_* \psi^\alpha_g \mathcal{M} = \omega_X \otimes_{\mathcal{O}_X} \mathcal{K}_\alpha(X, D) \), for \( \alpha \in (0, 1] \).

**Proof.** Recall the definition of \( \mu_*(M, F) \). Let \( i_\mu : Y \to Y \times X \) be the graph of \( \mu \). Let \( p : Y \times X \to X \) be the natural projection. Then
\[
\mu_*(M, F) = Rp \ DR_{X \times Y/X}(i_\mu)_*(M, F),
\]
where \( Rp \) is the usual derived direct image for sheaves. We put from now \( p_* = H^0(Rp_*) \) for the usual direct image of sheaves. Recall that \( DR_{X \times Y/X}(M', F) \) is defined by
\[
F_p DR_{X \times Y/X}(M', F) = \left[ F_{p-n} M' \otimes \mathcal{O}_Y \longrightarrow \ldots \longrightarrow F_p M' \right],
\]
where \( F_p M' \) sits in degree zero in the last complex, and \( \mathcal{O}_Y = (\mathcal{O}_Y^\vee)\).

The definition of \( (i_\mu)_*(M, F) = (M', F) \) is the same as for \( \mathcal{D}_Y \)-modules,
and all we need to know about the Hodge filtration is that $F_q M^\prime = (i_\mu)_p F_p M$ if $q = \min\{ p \mid F_p M \neq 0 \}$. In this case, also $q = \min\{ p \mid F_p M^\prime \neq 0 \}$.

If $(M, F) = \psi_Y^\alpha(\omega_Y, F)$ and $\alpha \in (0, 1]$ is such that $M \neq 0$, then $q = 1 - n$. Hence,

$$F_{1-n} H^0 \mu_* \psi_Y^\alpha (M, F) = p. (i_\mu)_p F_{1-n} M$$

$$= \mu. F_{1-n} M,$$

where the last $\mu$ is the usual sheaf direct image. By Lemma 2.3, $F_{1-n} M = \omega_Y \otimes K_\alpha(Y, \mu^* D)$. By (3), this proves the claim.

### Lemma 2.3

The Theorem is true if $D$ is a simple normal crossing divisor.

**Proof.** By definition, for $\alpha \in (0, 1],$

$$\psi_f^\alpha \omega_X = \text{Gr}_V^\alpha (i_f)_* \omega_X,$$

where $i_f : X \to X \times \mathbb{C}$ is the graph of $f$, $(i_f)_*$ is the direct image for (filtered) $\mathcal{D}_X$-modules, and $V$ is the decreasing filtration of Malgrange and Kashiwara. The Hodge filtration on $\psi_f^\alpha$ is the filtration $F[1]$, where $F$ is induced by $(i_f)_*$. Here $F[i]p = F_{p-i}$. In particular,

$$F_{1-n} \psi_f^\alpha \omega_X = F_{-n} \text{Gr}_V^\alpha (i_f)_* \omega_X.$$

By Proposition 3.5-(3.5.1) of [Sa90] applied to $(\omega_X, F[n])$, one has that

$$F_{-n} V^\alpha (i_f)_* \omega_X = \omega_X \otimes \mathcal{O}_X (-\downarrow (\alpha - \epsilon) \cdot D),$$

where $0 < \epsilon \ll 1$. Indeed, to apply that Proposition one only has to check locally, where $X$ has coordinates $x_1, \ldots, x_n$, that $\omega(x_i \partial_i + 1) = 0$, for $\omega = dx_1 \wedge \ldots \wedge dx_n$, and $\partial_i = \partial_{x_i}$. This follows from $\text{Lie}_f (x_i \omega) = \omega$. Hence it gives $\mu = (-1, \ldots, -1)$ in the above-mentioned Proposition.

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