On Variants of Network Flow Stability

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Abstract. We present a variant of the stable flow problem. Instead of the traditional flow problem that obeys Kirchhoff’s law, for each vertex, the outflow is monotone and piecewise linear to the inflow. In a directed and capacitated network, each vertex has strict preference over their incident edges. A stable flow assignment does not allow a group of vertices to benefit from privately rerouting along a path. In this paper, we first show the existence of flow stability by reducing this variant of stable flow problem to Scarf’s Lemma, then introduce a path augmenting algorithm that runs in polynomial time to find such a stable flow.

1 Introduction

In the classic stable marriage problem, \(n\) men and \(n\) women with individual preferences order of the opposite gender, are to be matched such that no man-woman pair exists who are inclined to abandon their original partners and marry each other. Gale and Shapley [1] showed the existence of such a stable matching by the deferred acceptance (DA) algorithm. Since then, the stable marriage problem and the DA algorithm have become the cornerstones of market design and have changed the way many centralized markets are organized. (See for example, [2,3].) Motivated by applications in resident matching, school choice, kidney exchange and supply chain networks, numerous extensions of the problem have been studied. Among them a trading network with bilateral contracts is perhaps the most general framework. This problem can be modeled as a directed graph where vertices represent agents and edges represent contracts. The vertex of the outgoing endpoint of an edge is the seller of that contract while the vertex of the incoming endpoint is the buyer. Besides, there may be a source vertex as a representative of a producer who only sells and a sink vertex as a consumer who only buys while other vertices are regarded as intermediate agents. A natural notion of stability is a configuration of trade such that there is no blocking coalition. A blocking coalition is a group of agents who will benefit more by cooperating among themselves.

To model agents’ preference, we assume that agents hold preference lists over individual contracts, and allow capacities over these contracts. One can think of the capacity as the amount of goods traded in this contract.\(^3\) One natural assumption is to require agents to obey Kirchhoff’s law. Namely, the sum of inflow contracts is the same as the sum of outflow contracts. This standard stable flow problem has been well-studied. A preflow-push variant of the Gale-Shapley algorithm can be done in pseudo-polynomial time [7] while a path augmenting variant of the Gale-Shapley algorithm can be done in polynomial time [8]. Besides, a stable flow instance can be reduced to the stable allocation problem [9] and both stable flow and stable allocation inherit the lattice structure [10].

However, many applications do not fit into the traditional flow problem that obeys Kirchhoff’s law. For example, in a supply chain network, one vertex can represent a manufacturing firm that takes raw materials as input and produces certain part-products while another vertex might correspond to an assembly firm whose inputs are the part-products and outputs are finished products. Clearly, the Kirchhoff’s law does not hold for both manufacturing and assembly firms in this example.

\(^3\) Another way to model preference is to use choice functions, that is, an agent evaluates a set of contracts \(C\) by specifying a subset of contracts \(C' \subset C\) that is accepted by the agent. [11,15,16] show that in this framework, a stable solution exists if the choice functions possess some special characteristics. Choice functions are defined over discrete sets, and it is not clear how it can be generalized to continuous capacities as the problem considered in our paper.
Motivated by this, in this paper we assume a more generalized case. Specifically, each agent can sign any amount of outgoing contracts under certain threshold when there are no incoming contracts. When there are incoming contracts, the amount of outgoing contracts is monotone and piecewise linear to the amount of incoming contracts. If all intermediate agents apply this criteria, then this network is called a monotone piecewise linear mapping network (MPLM-network).

Finding a stable solution in this general problem is more challenging. We first show the existence of flow stability by reducing this variant of stable flow problem to Scarf’s Lemma, then introduce a path augmenting algorithm that runs in polynomial time to obtain such a stable solution.

In section 2, we describe the setting and background definition of this problem including how flow and stability are defined when agents utilize some special mapping or functions to their inflow and outflow. Besides, we introduce the concept of monotone piecewise linear mapping (MPLM), convex monotone piecewise linear mapping (CMPLM), and linear mapping (LM).

In section 3, we show the existence of CMPLM-stable-flow by a reduction to Scarf’s Lemma \[\text{[11,12]}\]. Later on, as LM is a subset of CMPLM, by reducing MPLM-stable-flow problem to LM-stable-flow problem, we can show the existence of MPLM-stable-flow.

In section 4, we present a polynomial time path augmenting algorithm that finds an LM-stable-flow for an acyclic LM-network. The main difference of our approach from \[\text{[8]}\] is an augmented path may be a \(\sigma\)-cycle, a path from a source to a vertex that is visited twice. Each iteration in \[\text{[8]}\] augments a path from source to sink or a cycle since when augmenting a cycle, the augmented flow from the source to cycle is always zero. However, this does not apply in our setting because flow conservation property is not guaranteed. An MPLM-network instance can be reduced to an LM-network instance so an MPLM-stable-flow can be found in an acyclic MPLM-network. At the end of this section, we show a reduction from cyclic LM-network to an equivalent acyclic LM-network to enclose the entire problem.

2 Preliminaries

2.1 \(\mathcal{F}\)-agent, \(\mathcal{F}\)-network, \(\mathcal{F}\)-flow, and \(\mathcal{F}\)-stable-flow

A network is a quadruple \((G,s,t,c)\), where \(G = (V,E)\) is a digraph. We abuse a bit of notation, \(V\) does not include \(s\) and \(t\) but \(E\) includes edges with \(s\) or \(t\) as endpoints. \(s\) and \(t\) are the source and sink vertices and \(c : E \rightarrow \mathbb{R}_+\) determines the capacity \(c(e)\) where \(e \in E\). The preference \(\succ_v\) of a vertex \(v \in V\) is defined over edges. \(e_1 \succ_v e_2\) means \(v\) prefers \(e_1\) to \(e_2\). Note that incoming edges and outgoing edges are ranked strictly and separately by \(v\). First, we make assumptions over agents in the network:

**Definition 1.** Let \(\mathcal{F}\) be a set of functions or mappings that maps from \(\mathbb{R}_{\geq 0}\) to \(\mathbb{R}_{\geq 0}\) (or from \(\mathbb{R}_{\geq 0}\) to a subset of \(\mathbb{R}_{\geq 0}\)). For \(v \in V\), \(v\) possesses its own function or mapping \(g_v\). If for \(v \in V\), \(g_v \in \mathcal{F}\), then \(v\) is an \(\mathcal{F}\)-agent. If all agents in \(V\) are \(\mathcal{F}\)-agents, then \((G,s,t,c)\) is an \(\mathcal{F}\)-network.

In a market, each vertex can be regarded as an agent given offers of incoming and outgoing contracts. They evaluate the quantity of desired outgoing contracts to be signed based on how many incoming contracts are accepted. Therefore, the feasibility of contract assignment can be defined as the following:

**Definition 2.** An \(\mathcal{F}\)-flow of an \(\mathcal{F}\)-network is a function \(f : E \rightarrow \mathbb{R}_{\geq 0}\) such that:

1. \(0 \leq f(e) \leq c(e)\).
2. Let \(f_{\text{in}}(v) = \sum_{u \in E} f(uv)\) and \(f_{\text{out}}(v) = \sum_{u \in E} f(vu)\), then \(g_v(f_{\text{in}}(v)) = f_{\text{out}}(v)\) (or \(f_{\text{out}}(v) \in g_v(f_{\text{in}}(v))\) if \(g_v(f_{\text{in}}(v))\) maps to multiple values).

Since agents have preferences over contracts, it is natural to define a scenario when such network flow in a market is stable or not. The flow is not stable when there exist selfish agents who are willing to work together without regarding other agents’ benefit. Although each vertex in \(V\) has their own preference list, \(s\) and \(t\) do not rank their edges, because their preferences are irrelevant with respect to the following definition.
Definition 3. An \( \mathcal{F} \)-stable-flow is an \( \mathcal{F} \)-flow without a blocking path in an \( \mathcal{F} \)-network. Given flow \( f, g \) has a blocking path \( P = (v_1, v_2, ..., v_{k-1}, v_k) \) if all the followings hold:

1. There exists a vector \( V_P = (r_1, r_2, ..., r_{k-1}) > 0 \) such that:
   \( a \) \( r_i \leq c(v_{i+1}) - f(v_{i+1}) \) for \( i = 1, ..., k-1 \).
   \( b \) \( f_{\text{out}}(v_i) + r_i = g_{\text{in}}(f_{\text{in}}(v_i) + r_{i-1}) \) for \( i = 2, ..., k-1 \) (or \( f_{\text{out}}(v_1) + r_1 \) if \( g_{\text{in}}(f_{\text{in}}(v_1) + r_{1-1}) \) if \( g_{\text{in}}(f_{\text{in}}(v)) \) maps to multiple values).

2. \( v_1 = s \) or there is an \( e = v_1u \) such that \( f(e) > 0 \) and \( v_1v_2 > v_1e \).
3. \( v_k = t \) or there is an \( e = v_kv \) such that \( f(e) > 0 \) and \( v_{k-1}v_k > v_k e \).

An \( \mathcal{F} \)-flow has a blocking path if the first agent prefers to offer contracts to the second agent to other agents she had already offered, while intermediate agents still have space for signing contracts, and the last agent prefers to accept the contracts offered by the penultimate agent to other agents she had already accepted. Note that when \( v_1 = v_k \) and all the conditions are satisfied, \( g_{v_1}(f_{v_1}(v_1) + r_{k-1}) = f_{\text{out}}(v_1) + r_1 \) does not have to hold. The path \( P \) is allowed to have duplicate vertices. When there are duplicate vertices, we just follow Definition 3 regardless of extra incoming or outgoing flow that comes from \( V_P \).

We consider the case that there are no parallel edges since one can always add one vertex between each parallel edge and include the identity function into \( \mathcal{F} \). The vertices between parallel edges apply the identity function. Therefore, whenever we state an edge \( uv \), \( uv \) is unique.

Note that when \( \mathcal{F} \) is just a set of an identity function, this is the standard stable flow problem. We study the case when agents apply monotone piecewise linear mappings.

2.2 MPLM-network, CMPLM-network, and LM-Network

For a vertex \( v \in V \), before defining a monotone piecewise linear mapping (MPLM) \( g_v \) with \( k_v \) segments, we are given the following parameters:

1. \( a_{v,1}, a_{v,2}, ..., a_{v,k_v} \) where \( a_{v,i} > 0 \).
2. \( b_v > 0 \).
3. \( c_{v,0}, ..., c_{v,k_v} \) where \( 0 = c_{v,0} < c_{v,1} < ... < c_{v,k_v} = \infty \).

\( a_{v,i} \) is the slope of segment \( i \), \( b_v \) is the pseudo starting point of \( g_v \), and \( [c_{v,i-1}, c_{v,i}] \) decides the domain of segment \( i \). Now we are able to define \( g_v \):

\[
g_v(x) = \begin{cases} 
[0, b_v] & \text{if } x = 0, \\
\alpha_{v,1}x + b_v & \text{if } x \in (0, c_{v,1}], \\
\alpha_{v,i}x + g_v(c_{v,i-1}) & \text{if } x \in (c_{v,i-1}, c_{v,i}] 
\end{cases}
\]

(1)

Note that when \( x = 0 \), agent \( v \) is willing to sign any amount of outgoing contracts without exceeding the threshold \( b_v \). This mapping also depicts that even a bit of incoming contract is an incentive to sign outgoing contracts beyond certain baseline. If every agent in \( V \) applies MPLM, then \( (G, s, t, c) \) is an MPLM-network.

If \( a_{v,i} < a_{v,j} \) for any \( 1 \leq i < j \leq k_v \), then \( g_v \) is a convex (in \( \mathbb{R}_+ \)) monotone piecewise linear mapping (CMPLM). In this case, we can rewrite \( g_v \):

\[
g_v(x) = \begin{cases} 
[0, b_v] & \text{if } x = 0, \\
\max_{i=1, ..., k_v} \{g_v(x)\} & \text{otherwise}
\end{cases}
\]

(2)

where \( g_{v,i} = a_{v,i}x + b_{v,i} \) and \( b_{v,i} = -a_{v,i}c_{v,i-1} + \sum_{j=1}^{i-1} a_{v,j}(c_{v,j} - c_{v,j-1}) + b_v \). Besides, \( b_{v,1} = b_v \) and

\[
b_{v,i} - b_{v,i-1} = -(a_{v,i}c_{v,i-1} + a_{v,i-1}(c_{v,i-1} - c_{v,i-2}) + a_{v,i-1}c_{v,i-2})
\]

(3)

when \( i > 1 \), so \( b_{v,j} < b_{v,i} \) for any \( 1 \leq i < j \leq k_v \).

If every agent in \( V \) applies CMPLM, then \( (G, s, t, c) \) is a CMPLM-network. If \( k_v = 1 \), then \( g_v \) is a linear function when \( x > 0 \), we say \( g_v \) is a linear mapping (LM). Similarly, if for each \( v \in V, k_v = 1 \), then \( (G, s, t, c) \) is an LM-network.
3 Stability of MPLM-networks

3.1 Scarf’s Lemma

**Definition 4.** Let $A$ be an $m \times n$ nonnegative matrix with at least one positive entry in every column and row, $b \in \mathbb{R}^n_+$ be a positive vector, and $\mathcal{P} = \{x : x \geq 0, Ax \leq b\}$. For each row $i$ of $A$, there is a strict ranking $\succ_i$ over the columns in $\{1 \leq j \leq n : A_{ij} > 0\}$. $k \succ_i j$ means row $i$ prefers column $k$ to column $j$.

We say $x \in \mathcal{P}$ dominates column $j$ if there exists a row $i$ such that:

1. $A_{ij} > 0$ and the constraint $i$ binds, i.e. $(Ax)_i = b_i$.
2. $k \succ_i j$ for any other $k \neq j$ such that $A_{ik} > 0$ and $x_k > 0$.

To simplify our notation, we also say row $i$ dominates column $j$ if the above mentioned conditions hold.

**Lemma 1 (Scarf’s Lemma).** \cite{[12]} For any above mentioned $A, b,$ and $\succ_i$, there exists an $x^* \in \mathcal{P}$ that dominates all columns of $A$.

3.2 The Reductions

Scarf’s Lemma originally appeared as a tool to prove the non-emptiness of the core in an $n$ person game \cite{[11]}. Nowadays, it has been universally used for problems about stability.

In our case, before showing the existence of stable-MPLM-flow in an MPLM-network, we first consider CMPLM-network. MPLM is a more general mapping than CMPLM, so we can reduce MPLM-stable-flow problem to LM-stable-flow problem, a special case of CMPLM-stable-flow. Although LM is a special case of CMPLM and may not be needed in the later part of our proofs, we still prove the existence of stable-CMPLM-flow in CMPLM-networks by a reduction to Scarf’s Lemma.

**Theorem 1.** There exists a CMPLM-stable-flow in a CMPLM-network.

**Proof.** Suppose we are given $(G,s,t,c), G = (V,E), \succ_v,$ and $g_v \in CMPLM$ for any $v \in V$, we would like to reduce this problem to Scarf’s. First construct matrix $A$ as the following:

1. For each $e \in E$, create column $x_e$.
2. For each $v \in V$, create column $x_v^1$ and $x_v^2$.
3. For each $e \in E$, create row $e$ and set $A_{ex_v} = 1$.
4. For each $g_{v,i}$ where $i = 1, \ldots, k_v$, create row $v_i$. For every $e = uv \in E$, set $A_{v^i u x_v} = a_{v,i}$ and $A_{v^i u x_v}^* = A_{v^i u x_v} = 1$.
5. For each $v \in V$, create row $v'$. For every $e = vu \in E$, set $A_{v' u x_v} = 1$. Also set $A_{v' u x_v}^* = A_{v' u x_v} = 1$.
6. Row $v_i$ prefers column $x_v^1$ the most and column $x_v^2$ the least. Preference of $e = uv \in E$ remains the same as $\succ_v$.
7. Row $v'$ prefers column $x_v^1$ the most and column $x_v^2$ the least. Preference of $e = vu \in E$ remains the same as $\succ_v$.

Let $q(v) = \max(\sum_{uv \in E} c(uv), \sum_{vu \in E} c(vu), b_v) + 1$, construct $b$ as the following:

1. Set $b_e = c(e)$.
2. Set $b_{v,i} = q(v) - b_{v,i}$ for $i = 1, \ldots, k_v$.
3. Set $b_{v'} = q(v)$.

Note that all the rows of $A$ and $b$ are labeled either $e, v^i,$ or $v'$. All the columns of $A$ and rows of $x$ are labeled either $x_e, x_v^1,$ or $x_v^2$.

Let $x^*$ be a Scarf’s solution of $A, b, \succ_v, \succ_v', \text{ and } \succ_v$, we construct a CMPLM-stable-flow $f$ of $(G,s,t,c), g_v,$ and $\succ_v$ as the following: $f(e) = x^* e$. Here we abuse a bit of notation by labeling columns or rows by $x_e, x_v^1$, or $x_v^2$ and any superscript or subscript of $x^*$ stands for the value of that entry of $x^*$. The rest of the proof is done in Lemma 2. \[\square\]
Lemma 2. \( f \) is a CMPLM-stable-flow of \((G, s, t, e), G = (V, E), \succ v, \) and \( g_v \) if and only if \( x^* \) is a Scarf’s solution of \( A, b, \succ_v, \) and \( \prec_v \).

Proof. Scarf’s \( \rightarrow \) CMPLM-stable-flow:

Suppose \( x^* \) is a solution of Scarf’s, then \( x^* \) dominates every column of \( A \). Assume there is a row \( v^i \) that dominates column \( x^i_1 \), then \( i = 1 \) and row \( v^i \) must bind since \( q(v) - b_{v,1} \) is less than \( q(v) - b_{v,i} \), for \( i > 1 \) by equation (3). Row \( v^i \) prefers other nonzero entries (column \( x_e \), where \( e = uv \in E \) and column \( x^i_2 \)) to column \( x^i_0 \). Row \( v^i \) prefers column \( x^i_1 \) the most, so other entries including \( x^i_2 \) must be zero and \( x^i_1 = q(v) - b_v > 0 \).

In this case, \( f_{in}(v) = \sum_{v \in \omega \in E} x^e_v = 0 \) and from constraint row \( v^i \), \( f_{out}(v) = \sum_{e \in \omega \in E} x^e_v \leq b_v \).

If there is no row \( v^i \) that dominates column \( x^i_1 \), then row \( v^i \) must dominate column \( x^i_0 \), so row \( v^i \) binds.

If at the same time row \( v^i \) dominates column \( x^i_1 \), then \( v^i \) prefers other nonzero entries (column \( x_e \), where \( e = uu \in E \) and column \( x^i_1 \)) to column \( x^i_2 \). \( v^i \) prefers \( x^i_2 \) the most, so other entries including \( x^i_1 \) must be zero and \( x^i_2 = q(v) \). However, every row \( v^i \) prefers \( x^i_1 \) to \( x^i_2 \) and \( x^i_2 > 0 \), so row \( v^i \) cannot dominate \( x^i_1 \). \( x^i \) is not dominated by \( x^i \), a contradiction. Thus, there are some row \( v^i \) that dominates column \( x^i_2 \) and binds. Since row \( v^i \) also binds, \( q(v) = a_{v,i} \sum_{e \in \omega \in E} x^e_v + b_{v,1} + x^i_1 + x^i_2 = \sum_{e \in \omega \in E} x^e_v + x^i_1 + x^i_2 \).

Besides, row \( v^j \) may not bind for \( j \neq i \), so \( a_{v,j} \sum_{e \in \omega \in E} x^e_v + b_{v,1} + x^j_1 + x^j_2 = \sum_{e \in \omega \in E} x^e_v + x^j_1 + x^j_2 \). Besides, row \( v^j \) and \( v^i \) are no constraints for these two vertices. For the other case \( v^i \) prefers \( v^j \) to some other nonzero edges. For an unsaturated edge \( e \in A \), this means for some \( i_1 \) row \( v^j \) does not dominate column \( x_{e_i} \), since \( v^i \) prefers some other nonzero edges to \( e_{i_1} \). Besides, if \( f(e_{i_1}) > 0 \), then \( v_j > 0 \) since otherwise \( V_P = 0 \) (the path corresponds to a series of functions in \( \mathbb{R}_+ \) and the upcoming \( r_i \)’s remain unchanged). That is, \( c(e_{i_1}) > f(e_{i_1}) \) whenever \( f(e_{i_1}) > 0 \) so row \( e_{i_1} \) cannot dominate \( x_{e_i} \). This means for some \( i_1 \), row \( v^j \) must dominate column \( x_{e_i} \). Since row \( v^j \) dominates column \( x_{e_i} \) and column \( x^j_2 \) is the least preferred, we have \( x^j_2 = 0 \). \( v^j \) must bind and the maximum sum of either incoming or outgoing flow cannot reach \( q(v^j) \). \( x^j_2 > 0 \) indicates that \( v^j \) cannot dominate \( x_{e_i} \) because column \( x^j_2 \) is row \( v^j \)’s least preferred. Similarly, for some \( i_2 \), row \( v^j \) must dominate column \( x_{e_{i_2}} \) where \( e_{i_2} = v_{k-1}v_k \). If we keep on repeating the similar argument, there must be some \( i_{k-1} \) such that row \( v^k \) must dominate column \( x_{e_{i_{k-1}}} \), \( e_{k-1} = v_{k-1}v_k \). However, from the third condition in Definition 3, \( v_k \) must prefer \( e_{k-1} \) to some other nonzero edges. This means row \( v^k \) must prefer \( e_{k-1} \) to some other nonzero edges so \( x^* \) does not dominate column \( x_{e_{k-1}} \), a contradiction.

CMPLM-stable-flow \( \rightarrow \) Scarf’s:

Suppose we have a CMPLM-stable-flow \( f \), first set \( x^* = f(e) \) for each \( e \in E \). If \( x^* = c(e) \) then row \( e \) dominates column \( x_e \). For an unsaturated edge \( e = uv, \) exactly one of the following cases holds:

1. \( u \) prefers \( e \) to some other nonzero edges or \( u = s \).
2. \( v \) prefers \( e \) to some other nonzero edges or \( u = t \).
3. None of the above.

The first and second cannot hold at the same time or \( uv \) is a blocking path.

In the first case, \( v \) must prefer all other nonzero edges to \( e \) so some row \( v^i \) dominates column \( x_{e_i} \). This leads to \( x^i_0 = 0 \) because \( x_{e_i} \succ_v x^i_0 \) for any \( i = 1, \ldots, k \). Suppose \( g_v(f_{in}(v)) = g_v, f_{in}(v) \succ_{v,i} \sum_{e \in \omega \in E} x^e_v - b_{v,i} \).

Similarly in the second case, \( u \) must prefer all other nonzero edges to \( e \) so row \( u^j \) dominates column \( x_{e_i} \). This leads to \( x^j_0 = 0 \) because \( x_{e_i} \succ_u x^j_0 \). In order to let row \( u^j_B \), \( x^j_0 = q(u) = \sum_{e \in \omega \in E} x^e_v \).

In the third case, column \( x_{e_i} \) is already dominated. The next step is to move on and examine how column \( x^i_0 \) and column \( x^i_2 \) can be dominated.

We can classify vertex \( v \) into 3 types:

1. \( v \) has an incoming unsaturated edge \( uw \) and \( u \) prefers \( uv \) to some other nonzero edges.
2. \( v \) has an outgoing unsaturated edge \( vw \) and \( w \) prefers \( uv \) to some other nonzero edges.
3. None of the above.
The first and the second case cannot happen at the same time by our construction, since otherwise \( u \) prefers \( uv \) to some other nonzero edges and \( w \) prefers \( vw \) to some other nonzero edges, \((u,v,w)\) forms a blocking path, a contradiction. Besides, the first case implies \( x^{s^2}_{v} = 0 \) and the second case implies \( x^{s^1}_{v} = 0 \). By our selection of \( q(v) \), \( x^{s^1}_{v} = x^{s^2}_{v} = 0 \) cannot happen because the sum of incoming or outgoing flow cannot reach \( q(v) \).

In the third case, if \( f_{in}(v) = 0 \) and \( f_{out}(v) < b_v \), row \( v' \) cannot bind and row \( v^1 \) binds. In order to dominate column \( x^{s^2}_{v} \), set \( x^{s^2}_{v} = q(v) - b_v \). Otherwise, arbitrarily set \( x^{s^1}_{v} \) and \( x^{s^2}_{v} \) such that \( x^{s^1}_{v} \geq 0 \), \( x^{s^2}_{v} \geq 0 \), and \( x^{s^1}_{v} + x^{s^2}_{v} = q(v) - a_{v,i} \sum_{uv \in E} f(uv) - b_{v,i} = q(v) - \sum_{uv \in E} f(uv) \) where \( i \) is the one that maximizes \( g_{v,i}(f_{in}(v)) \). In this case, both row \( v' \) and \( v^1 \) bind. Column \( x^{s^2}_{v} \) as the least preferred by row \( v^i \) is dominated while column \( x^{s^1}_{v} \) as the least preferred by row \( v^j \) is also dominated. \( \square \)

The existence of CMPLM-stable-flow also implies the existence of LM-stable-flow. One can have an analogous proof for LM-stable-flow by setting \( k_v = 1 \) for each \( v \in V \). In the following proof, we show that MPLM-stable-flow can be reduced to LM-stable-flow and each MPLM-agent is equivalent to a subnetwork that consists of LM-agents.

Theorem 2. There exists a MPLM-stable-flow in an MPLM-network.

Proof. We can reduce MPLM-stable-flow problem to LM-stable-flow problem to show the existence. First construct \((G', s', t', c')\), \( G' = (V', E'), \succ_v' \), and \( g'_v \in \text{LM} \) for each \( v' \in V' \) as the following:

1. For each \( v \in V \), create \( v_{in}, v_{out}, \) and \( v_1, ..., v_{k_v} \) in \( V' \).
2. For each \( uv \in E \), create \( u_{out}v_{in} \) in \( E' \) and let \( c'(u_{out}v_{in}) = c(uv) \) (assume \( s_{out} = s' \) and \( t_{in} = t' \)).
3. For each \( v_i \), create edges \( v_{in}v_i \) and \( v_i v_{out} \) in \( E' \). If \( i < k_v \), set \( c'(v_{in}v_i) = c_{v,i} - c_{v,i-1} \) and \( c'(v_i v_{out}) = a_{v,i}c'(v_{in}v_i) \). If \( i = k_v \), set \( c'(v_{in}v_i) = \max\{0, \sum_{uv \in E} c(uv) - c_{v,i-1}\} \) and \( c'(v_i v_{out}) = a_{v,i}c'(v_{in}v_i) \).
4. The preference of \( v_{in} \) over \( u_{out}v_{in} \) is the same as \( v \) over \( uv \). Similarly, the preference of \( v_{out} \) over \( v_{out}u_{in} \) is the same as \( v \) over \( vu \).
5. \( v_{in} \) prefers \( v_{in}v_i \) with smaller \( i \) and \( v_{out} \) prefers \( v_i v_{out} \) with smaller \( i \).
6. \( g'_{v_{in}}(x) = x \), \( g'_{v_{out}}(x) = a_{v,i}x \), \( g'_{v_{in}}(0) \in [0, b_v] \) and \( g'_{v_{out}}(x) = x + b_v \) if \( x \in \mathbb{R}_+ \).

Let \( f' \) be the LM-stable-flow, it is clear that \( g'_{v_{out}}(f'_{v_{in}}(v_{out})) = g_v(f'_{v_{in}}(v_{in})) = f'_{v_{in}}(v_{out}) \) since the sum of outgoing flow from \( v_{in} \) is the same as the sum of the incoming flow, both \( v_{in} \) and \( v_{out} \) prioritize flow that passes through smaller \( v_i \)’s and \( v_i \) output flow weighted by \( a_{v,i} \), \( v_i \) does not have any flow unless \( v_{in}v_{i-1} \) and \( v_{i-1}v_{out} \) are saturated, and eventually, \( v_{out} \) just merely output a flow by adding \( b_v \) when \( f'_{v_{in}}(v_{out}) > 0 \).

Therefore, assigning \( f(uv) = f'(u_{out}v_{in}) \) makes \( f \) an MPLM-stable-flow. \( \square \)

Note that there are alternative ways to prove the existence of CMPLM-stable-flow and MPLM-stable-flow. One can first prove the existence of CMPLM-stable-flow and reduce a MPLM-network to a CMPLM-network via decomposing MPLM into CMPLM segments instead of LM segments in our proof. Another way is to first prove the existence of LM-stable-flow then reduce both CMPLM-stable-flow and MPLM-stable-flow to LM-stable-flow.

4 Path Augmenting Algorithms for MPLM-networks

4.1 Acyclic Networks

From the proof of Theorem 2, each MPLM-agent can be transformed as a subnetwork of several LM-agents. As a consequence, it suffices to design algorithms for LM-stable-flow problem in order to solve the MPLM-stable-flow problem. We present a path augmenting algorithm for LM-networks. This algorithm is a slight modification of the one in \cite{3} which considers the case when network does not have cycles.
The Algorithm for Acyclic LM-networks

Suppose we are given \((G, s, t, c)\), \(G = (V, E), \succ_v, g_v \in LM\) for any \(v \in V\), and \(G\) does not have any cycles. In the algorithm, each vertex \(v \in V \cup \{s\}\) is associated with a state and an edge. The states for vertices are \{\text{PROPOSE}, \text{REJECT}, \text{DONE}\}. If \(v\) is at the \text{PROPOSE} state, then the associated edge is the outgoing edge that \(v\) currently prefers the most. \(v\) reaches the \text{REJECT} state when \(v\) is running out of outgoing edges for proposing. The associated edge at this state is the incoming edge with positive flow value that \(v\) currently prefers the least. \(v\) reaches the \text{DONE} state when its most preferred edge is also rejected and there is no associated edge.

Initially, for each \(v \in V \cup \{s\}\), \(v.\text{state} = \text{PROPOSE}\), \(v.\text{edge}\) is set to the most preferred outgoing edge of \(v\).

For \(s\), the order can be arbitrary. We can construct the auxiliary graph \(H = (V_H, E_H)\) as the following:

1. \(V_H = V\).
2. \(E_H = \{v.\text{edge} : v \in V \cup \{s\}, v.\text{state} = \text{PROPOSE}\} \cup \{\text{rev}(v.\text{edge}) : v \in V \cup \{s\}, v.\text{state} = \text{REJECT}\}\) where \(\text{rev}(uv) = vu\).

For each \(uv \in E_H\), the residual capacity \(c_f(uv)\) based on the current flow \(f\) is:

\[
c_f(uv) = \begin{cases} c(uv) - f(uv) & \text{if } u.\text{state} = \text{PROPOSE} \text{ and } u.\text{edge} = uv \\ f(vu) & \text{if } u.\text{state} = \text{REJECT} \text{ and } u.\text{edge} = uv \end{cases}
\]

Before showing how the algorithm works, we introduce a different path to augment from traditional path augmenting algorithms for flows.

\textbf{Definition 5.} A \(\sigma\)-cycle is a path \(P = (s, v_1, v_2, ..., v_k)\) where all vertices are distinct except \(v_k = s\) or \(v_k = v_j\) for some \(1 \leq j < k\).

The algorithm iteratively augments the flow \(f\) in \(H\). In each round, one can always augment an \(s\)-\(t\) path or a \(\sigma\)-cycle \(P\) such that \(c_e(f) = 0\) for some \(e\) in \(P\).

After the augmentation, update either the associated edge or the state for each outgoing vertex \(u\) of the saturated edges \(uv\). If \(u\) just finished proposing to \(v\), then move on to the next preferred edge. If \(u\) is running out of vertices to propose, then move to the \text{REJECT} state, set the associated edge to the least preferred edge that is currently accepted, and update the vertices that are going to propose to \(u\). If \(u\) is running out of vertices to reject, then \(u\) has rejected all the edges, set \(u.\text{state}\) to \text{DONE}.

The algorithm stops when \(s.\text{state} = \text{REJECT}\). Namely, there are no vertices for \(s\) to propose to.

\textbf{Algorithm 1: Path Augmenting}

1. Initialize the state and associated edge for \(v \in V \setminus \{t\}\) and set \(f\) as zero flow.
2. \textbf{while} \(s.\text{state} = \text{PROPOSE}\) \textbf{do} \hfill \triangleright\ assume \(H\) and \(f\) as global variables
3. \hspace{1em} let \(P\) be an \(s\)-\(t\)-path or a \(\sigma\)-cycle in \(H\)
4. \hspace{1em} augment \(f\) by \(P\) such that the capacity of at least one edge in \(H\) drops to 0
5. \hspace{1em} \textbf{for} \(uv\) in \(P\) \textbf{do}
6. \hspace{2em} \textbf{if} \(c_f(uv) = 0\) \textbf{then}
7. \hspace{3em} \text{Update}(u, P)
8. \hspace{2em} return \(f\)
Details of Path Augmentation Suppose after running some iterations, the current flow is $f$ and the auxiliary graph is $H$. Let $P$ be an $s$-$t$-path or a $\sigma$-cycle (there must be one, see Lemma 3) and $\Delta$ be a list of edge values that we are about to augment along $P$.

If we find an $s$-$t$-path $P = (s, v_1, ..., v_k, t)$ to augment, let $\Delta = (\Delta_0, \Delta_1, ..., \Delta_k)$. We start from $sv_1$ and set $\Delta_0 = cf(sv_1), v_0 = s$, and $v_{k+1} = t$, then traverse through $P$, there are three cases in each step:

1. Push flow: If $v_{i-1}$ and $v_i$ are at PROPOSE state, set $\Delta_i = \min\{g_v(f_{in}(v_i) + \Delta_{i-1}) - f_{out}(v_i), cf(v_i, v_{i+1})\}$.
2. Redirect flow: If $v_{i-1}$ and $v_i$ are at different state, set $\Delta_i = \min\{\Delta_{i-1}, cf(v_i, v_{i+1})\}$.
3. Remove flow: If $v_{i-1}$ and $v_i$ are at REJECT state, set $\Delta_i = \min\{g_v^{-1}(f_{out}(v_i) - \Delta_{i-1}) - f_{in}(v_i), cf(v_i, v_{i+1})\}$.

Once $t$ is reached, we fix the flow by traversing backwards along $P$:

1. If $v_{i-1}$ and $v_i$ are at PROPOSE state, set $\Delta_{i-1} = g_v^{-1}(f_{out}(v_i) + \Delta_i) - f_{in}(v_i)$.
2. If $v_{i-1}$ and $v_i$ are at different state, set $\Delta_{i-1} = \Delta_i$.
3. If $v_{i-1}$ and $v_i$ are at REJECT state, set $\Delta_{i-1} = g_v(f_{in}(v_i) - \Delta_i) - f_{out}(v_i)$.

In the first case, when $f_{out}(v_i) + \Delta_i < b_v, g_v^{-1}(f_{out}(v_i) + \Delta_i) = 0$ and $f_{in}(v_i) = 0$, $\Delta_{i-1}$ is going to be 0. Similarly, in the third case, when $f_{in}(v_i) - \Delta_i = 0$, it must be the case that $g_v(f_{in}(v_i) - \Delta_i) = f_{out}(v_i)$, so $\Delta_{i-1}$ is going to be 0.

Note that each vertex cannot be in the DONE state (see Lemma 3). The usage of $g_v$ in the push flow case is well-defined since in the forward stage of traversal along $P$, $\Delta_0$ is positive and at each step, $\Delta_{i-1}$ was previously set to a positive value and by induction $\Delta_i$ will also be positive. There exists a bottleneck edge $v_jv_{j+1}$ that is saturated. The backward traversal keeps the same edge values as in the forward traversal stage from $t$ to $v_j$ and fixes the edge values from $v_j$ to $s$.

If we find a $\sigma$-cycle $P = (s, v_1, ..., v_j, ..., v_k)$, then let $v_0 = s, v_k = v_j$ for some $0 \leq j < k$, and set $\Delta = (\Delta_0, ..., \Delta_{k-1})$. There are two cases:

1. The saturated edge $e = v_{i-1}v_i$ and $i > j$, i.e. $e$ belongs to a cycle.
2. The saturated edge $e = v_{i-1}v_i$ and $i \leq j$, i.e. $e$ belongs to a path from $s$ to $v_j$.

In order to figure out which is the case, first assume that it is the first case. Regardless of the path from $s$ to $v_j$, compute only the cycle part of $\Delta$. Next check whether it is possible to augment along the path from $s$ to $v_j$ such that it matches the cycle part of $\Delta$ we computed. If not, then it may be the second case.

To calculate the cycle part of $\Delta$, one can start from $v_j$ and do exactly the same as in the $P$ is an $s$-$t$-path case until $v_k$ is reached. By traversing the cycle part of $P$ forth and back, the cycle part of $\Delta$ is computed. However, this may not be a valid augmentation since $a_{v_j, \Delta_{k-1}}$ may not be the same as $\Delta_{j}$ if one of the following conditions is met then we are done:
1. If \( j = 0 \) then we are done since \( s \) has no incoming flow and is not an LM-agent.
2. If \( \Delta_j = a_{v,1}\Delta_{k-1} \), we don’t have to fix anything from \( s \) to \( v_j \).

If \( f_{\text{out}}(v_j) + \Delta_j \leq b_v \), then \( \Delta_k = 0 \) and \( \Delta_j = 0 \). This cannot happen since after augmenting a positive value on \( v_j \) to \( v_j \), the augmented value from \( v_j \) to \( v_k \) must be positive. Therefore, \( f_{\text{out}}(v_j) + \Delta_j - \Delta_{k-1} > b_v \), so the outgoing flow of \( v_j \) so far at least meets the threshold and \( f(v_j v_j) > 0 \), we have to fix the \( s \) to \( v_j \) part of \( \Delta \). To do this, first apply a similar method as in the \( s-t \) path case and traverse forward from \( s \) to \( v_j \) to get \( \Delta_j \). Let the required change of \( v_j \)'s outflow be \( A = g_{v_j}^{-1}(f_{\text{out}}(v_j) + \Delta_j - a_{v,1}\Delta_{k-1}) - f_{\text{in}}(v_j) \). By comparing \( \Delta_j - 1 \) and \( A \), there are two cases:

1. If \( \Delta_j - 1 > A \), then set \( \Delta_j = 1 \), traverse backwards from \( v_j \) to \( s \) and apply the similar method as in the \( s-t \) path case to fix the \( v_j \) to \( s \) part of \( \Delta \). Note that \( \Delta_j - 1 \) will be negative if \( \Delta_j - a_{v,1}\Delta_{k-1} < 0 \).
2. If \( \Delta_j - 1 < A \), we have to use the property of LM. If \( \Delta_j = 0 \), then by the property of \( g_{v_j}, \Delta_j - 1 = 0 \) and \( \Delta_{k-1} = 0 \). This cannot happen because we are not augmenting anything along the \( \sigma \)-cycle.

The remaining case is both \( \Delta_j - 1 \) and \( \Delta_j \) are positive. \( v_j \) applies a series of linear functions (in \( \mathbb{R}_+ \)) to reach \( v_k \). For a vertex \( v_m \) where \( j + 1 \leq m < k \), it either applies \( g_{v_m} \) for the push flow case, identity function for the redirect flow case, or \( g_{v_m}^{-1} \) for the remove flow case. As a consequence, \( \Delta_j = \alpha\Delta_{k-1} \) for some \( \alpha > 0 \). If \( \alpha = a_{v,1} \), we are in the former case \( \Delta_j = a_{v,1}\Delta_{k-1} \). If \( \alpha < a_{v,1} \), then \( \Delta_j - a_{v,1}\Delta_{k-1} < 0 \) which was also covered earlier. As a result, \( \alpha > a_{v,1} \). \( A = (\frac{1}{\alpha a_{v,1}} - \frac{1}{\alpha}) \Delta_j \) since \( g_{v_j}^{-1} \) is linear and \( f_{\text{in}}(v_j) = g_{v_j}^{-1}(f_{\text{out}}(v_j)) \). \( \Delta_j - 1 \) cannot reach the required change, so we have to set \( \Delta_j = \frac{a_{v,1}}{\alpha - a_{v,1}} \Delta_j - 1 \) and traverse the cycle part of \( \Delta \) as the forward traversal in the \( s-t \) path case. Next, traverse backwards from \( v_j \) to \( s \) and fix \( \Delta \) by the same method in the \( s-t \) path case. The following is an example:

**Example 1.** Suppose each node is using identity function except for \( d \) the outflow is half of the inflow. The left graph is before augmentation and the right is after augmentation.

The \( \sigma \)-cycle is \((s, v_1, v_2, v_3, v_1)\). Before augmenting, \( v_3.\text{state} = \text{REJECT}, v_3.\text{edge} = v_1 v_3, v_2.\text{state} = \text{PROPOSE}, \) and \( v_2.\text{edge} = v_2 v_3 \) so \( v_3 \) is the redirecting vertex and \( v_3 v_1 \in E_H \). \( v_1 v_2 \) is directed and \( v_1 v_2 \) to \( v_1 \) proposed to \( v_2 \) earlier. For the cycle part, we would like to augment \((4, 2, 2)\) along \((v_1, v_2, v_3, v_1)\). However, this cannot be done because the net outgoing flow of \( v_1 \) is 2 and \( s \) can only push 1 flow value to \( v_1 \). We know that \( \alpha = 2 \) and \( a_{v,1} = 1 \), so the augmenting value for \( v_1 v_2 \) must be twice of \( s v_1 \). We augment \((1, 2, 1, 1)\) along \((s, v_1, v_2, v_3, v_1)\) and obtain the right graph.

**Analysis** We start from proving the correctness of Algorithm 1. To ensure that Algorithm 1 can be executed properly in each iteration, we start with the following lemma:

**Lemma 3.** At the beginning of any iteration in Algorithm 1 line 2, \( H \) always contains an \( s-t \)-path or a \( \sigma \)-cycle.

**Proof.** Consider any \( v \in V \), \( v \) has an outgoing edge if \( v \) is at \text{PROPOSE} or \text{REJECT} state. When \( v.\text{state} = \text{DONE} \), \( v \) rejected its last incoming edge. Therefore, it must be the removing flow case (see details of path augmentation) otherwise \( v \) will still be in \text{REJECT} state. To remove the last flow, \( v \) was rejected by some vertex \( w \) then \( v \) rejected some vertex \( u \) with \( f(uv) > 0 \) such that \( f_{\text{in}}(v) \) drops to 0. Once \( f \) is updated after a path augmentation that involved the removal of \( f(uv) \), there are no incoming edges of \( v \) in \( E_H \). Hence \( v \) does not have incoming and outgoing edges if and only if \( v.\text{state} = \text{DONE} \).

Before finding an augmenting path, \( s.\text{state} = \text{PROPOSE} \) so \( s \) has an outgoing edge in \( H \). \( s \) always reaches vertices that is in either the \text{PROPOSE} or \text{REJECT} state and each such vertex has an outgoing edge. One can always traverse from \( s \) until a vertex is visited twice which forms a \( \sigma \)-cycle or until \( t \) is reached. \( \square \)
Theorem 3. Algorithm 1 computes an LM-stable-flow in polynomial time.

Proof. Suppose there is a blocking path \( P = (v_1, \ldots, v_k) \) in \( G \). Edges in this blocking path must be unsaturated. From the second condition in Definition 3, either \( v_1 = s \) or there is an edge \( v_1 u \) such that \( v_1 v_2 \succ_v v_1 u \) and \( f(v_1 u) > 0 \).

If \( v_1 \neq s \) and \( v_1 v_2 \) is unsaturated and \( f(v_1 u) > 0 \) either because \( v_2 \text{state} = \text{REJECT} \) and \( v_2 \text{edge} = v_1 v_2 \) or \( v_1 \text{edge} \) was updated once \( v_2 \text{state} \) changed to \text{REJECT}. In the former case, \( v_1 v_2 \) was once saturated but \( f(v_1 v_2) \) decreased later on. In the latter case, \( v_1 \) did not get the chance to make \( v_1 v_2 \) saturated and \( v_2 \text{edge} = v_1 v_2 \). If \( v_1 = s \), the termination of Algorithm 1 implies that \( v_2 \text{state} = \text{REJECT} \). In either case, \( v_2 \text{state} = \text{REJECT} \) and \( v_2 \text{edge} = v_1 v_2 \).

From \( v_2 \text{state} = \text{REJECT} \) and \( v_2 \text{edge} = v_1 v_2 \) unsaturated, \( v_2 \) proposed to all available vertices and either \( v_2 \) has proposed to \( v_3 \) earlier and \( f(v_2 v_3) \) decreased later on or \( v_2 \text{edge} \) was updated once \( v_3 \text{state} \) changed to \text{REJECT} and \( v_2 \) did not get the chance to make \( v_2 v_3 \) saturated. In either case \( v_3 \text{state} = \text{REJECT} \) and \( v_2 \text{edge} = v_2 v_3 \).

By continuing analogous arguments, \( v_k \text{state} = \text{REJECT} \) and \( v_k \text{edge} = v_{k-1} v_k \). From the third condition of Definition 3, either \( v_k = t \) or there is an edge \( w v_k \) such that \( v_{k-1} v_k \succ_{v_k} w v_k \) and \( f(w v_k) > 0 \). For the former case, \( t \) never rejects vertices or forces vertices to update so this cannot happen. For the latter case, \( v_k \) rejected \( w \) earlier and \( f(w v_k) = 0 \) since \( v_k \text{edge} = v_{k-1} v_k \) and \( v_{k-1} v_k \succ_{v_k} w v_k \), a contradiction.

For time complexity, in each iteration, the residual capacity of at least one edge drops to 0. There are at most \( 2|E| \) iterations since each edge can first be fully saturated then be fully removed. It takes \( O(|V|) \) time to find an \( s-t \) path or a \( \sigma \)-cycle in each iteration and deciding the edge augmenting values of the path takes \( O(|V|) \). Therefore the running time of Algorithm 1 is \( O(|V||E|) \). \( \square \)

Theorem 4. An MPLM-stable-flow of an acyclic MPLM-network can be computed in polynomial time where the polynomial involves the number of segments of each vertex.

Proof. First apply the reduction from MPLM-stable-flow to LM-stable-flow in Theorem 2, then run Algorithm 1 on the LM-network. For each \( v \in V \) in the old graph, \( k_v + 2 \) vertices and \( 2k_v \) extra edges are created in the new graph. Let \( K = \sum_{v \in V} k_v \), a rough bound for time complexity will be \( O((|V| + K)(|E| + K)) \).

The new graph has a special structure. Although the number of iterations is still bounded by \( |E| + K \), the length of path in each iteration is bounded by \( 3|V| \). The time complexity is therefore \( O(|V||E| + K|V|) \). \( \square \)

4.2 A Reduction from LM-Scarf’s to Acyclic LM-Network

Previously, we designed Algorithm 1 to find MPLM-stable-flow in an acyclic network. With cycles, we cannot apply Algorithm 1 in certain scenarios. Sometimes there is no proper augmenting path that starts from \( s \) as the following example:

Example 2. In this graph, the outflow is twice of inflow for \( v_1 \) and \( v_2 \). For the preference, \( v_2 v_1 \succ_v v_1 \) and \( v_1 v_2 \succ_v v_1 \). Algorithm 1 does not work properly for this graph. The first \( \sigma \)-cycle found is \((s, v_1, v_2, v_1)\). We would like to augment \( (\Delta_0, \Delta_1, \Delta_2) \) along this \( \sigma \)-cycle as the left graph shows. The following must be satisfied: \( \Delta_1 = 2(\Delta_0 + \Delta_2) \) and \( \Delta_2 = 2\Delta_1 \). This implies \( \Delta_0 = -\frac{1}{2} \Delta_1 \) and flows cannot be negative, so it is impossible to augment this \( \sigma \)-cycle such that one edge is saturated. Besides, it is possible that there is no outflow from \( s \) or there is no inflow to \( t \) in a stable flow. The right graph is a stable assignment.

From Theorem 2, we know that an MPLM-stable-flow always exists. We can first reduce MPLM-stable-flow to LM-stable-flow then reduce LM-stable-flow to Scarf’s no matter whether there are cycles or not. If a Scarf’s instance corresponds to an LM-stable-flow instance, it is an LM-Scarf’s instance. It turns out that LM-Scarf’s can be reduced to LM-stable-flow where the LM-network does not contain any cycles.
Theorem 5. Any LM-Scarf’s instance can be reduced to a LM-network that is solvable by Algorithm 1.

Proof. We will use the same notation as in the proof of Theorem 1 except $k_v = 1$ for $v \in V$ since we are only given a network with LM-agents. Construct an LM-network as the following:

1. For each column $x^1_v$, create vertex $v_{in}$.
2. For each column $x^2_v$, create vertex $v_{out}$.
3. Create $s$ and $s_{out}$, set $c(sv_{out}) = \infty$, and set $c(sv_{in}) = b_v$.
4. Create $t$ and $t_{in}$, set $c(t_{out}t) = \infty$, and set $c(t_{in}t) = b_v$.
5. For each $e = uv$, create a vertex $m_e$, set $c(uv) = b_v$ and $c(m_e) = A_{uv}b_v$.
6. For each $v$, create vertices $m^1_v$ and $m^2_v$, set $c(v_{out}m^1_v) = c(v_{out}m^2_v) = c(m^1_v) = c(m^2_v) = b_v$.
7. For each $v$, $v_{out}$ prefers $v_{out}m^2_v$ the most and $v_{out}m^1_v$ the least, the preference of $v_{in}m_e$ for some $e = vu$ is the same as in row $v$.
8. For each $v$, $v_{in}$ prefers $m^1_v$ the most and $m^2_v$ the least, the preference of $m_e v_{in}$ for some $e = uv$ is the same as in row $v$.
9. The preference of $s_{out}$ and $t_{in}$ is arbitrary.
10. The outflow of $s_{out}$, $t_{in}$, $v_{in}$'s, $v_{out}$'s, $m^1_v$'s and $m^2_v$'s are the same as each of their inflow, they apply identity functions.
11. The outflow of $m_e$ where $e = uv$ and $w \neq t$ is $A_{uv}x^v$ times of its inflow. If $w = t$ then $m_e$ applies identity function.

By setting the capacity of an edge in the network, the edge is automatically included in the edge set. Note that this network has three layers. Starting from $s$, the first layer has $s_{out}$ and $v_{out}$'s, the second layer has $m^1_v$'s, $m^2_v$'s, and $m_e$'s, and the third layer has $t_{in}$ and $v_{in}$'s that eventually merge to $t$.

Suppose we have an LM-stable-flow $f$ of this network, we show that by setting $x^e_v = f(uv)$ for $e = uv$, $x^1_v = f(m^1_v)$, and $x^2_v = f(m^2_v)$, we have a solution for FM-Scarf’s. Conversely, the solution of Scarf’s also corresponds to a LM-stable-flow. The rest of the proof is done in Lemma 4.

Lemma 4. $f$ is a LM-stable-flow of the above mentioned network if and only if $x^*$ is a solution of LM-Scarf’s.

Proof. Scarf’s $\Rightarrow$ LM-stable-flow:

Suppose $x^*$ is a solution of Scarf’s, then $x^*$ dominates every column of $A$. We set $f$ as the following:

1. For $e = uv$, set $f(uv) = x^e_v$, $f(m_e v_{in}) = A_{uv}x^v_x x^e_v$.
2. For each $v$, set $f(v_{out} m^1_v) = f(m^1_v v_{in}) = x^1_v$ and $f(v_{out} m^2_v) = f(m^2_v v_{in}) = x^2_v$.
3. Set $f(s_{out}) = \sum_{e = vu} A_{uv}x^e_v x^e_u + x^1_v + x^2_v = \sum_{e = vu} x^e_v + x^1_v + x^2_v$.
4. Set $f(t_{in}) = \sum_{e = st} x^e_v$.
5. Set $f(s_{out}) = \sum_{e = ss} A_{su}x^e_s = \sum_{e = ss} x^e_s$.
6. Set $f(t_{in}) = \sum_{e = st} x^e_v$.

We can see that for each vertex, the inflow and outflow satisfy the condition of an LM-flow. The remaining is to show that $f$ is stable. Assume there is a blocking path $P$ in this network. Observing the structure of this three-layer network, each vertex $m_e$, $m^1_v$, or $m^2_v$ in the second layer only have one incoming and one outgoing edge. This indicates they cannot be the starting or ending vertex of the blocking path. Besides, $P$ cannot start from $s$ and end at a vertex in the first layer since each vertex in that layer has only one incoming edge. Similarly, $P$ cannot start from a vertex in the third layer and end at $t$ as each vertex in the third layer has only one outgoing edge. The remaining cases of a blocking path are:

1. $P$ starts from $s$ and ends at a vertex $v_{in}$ in the third layer:

   This means there exists a vertex $v_{out}$ in the first layer such that row $u'$ does not bind since $su_{out}$ is not saturated. For $e = uv$, $m_e v_{in}$ and $u_{out} m_e$ are not saturated and $v_{in}$ prefers $m_e v_{in}$ to some other nonzero incoming edges indicate column $x_e$ is dominated by neither row $e$ nor row $v$. Column $x_e$ is not dominated by $x^*$, a contradiction.
2. $P$ means starts from a vertex $u_{out}$ in the first layer and ends at $t$:

This means there exists a vertex $v_{in}$ in the third layer such that row $v^1$ does not bind since $v_{in} t$ is not saturated. For $e = uv, m_e v_{in}$ and $u_{out} m_e$ are not saturated and $u_{out}$ prefers $u_{out} m_e$ to some other nonzero incoming edges indicate column $x_e$ is dominated by neither row $e$ nor row $u'$. Column $x_e$ is not dominated by $x^*$, a contradiction.

3. $P$ starts from a vertex $u_{out}$ in the first layer and ends at a vertex $v_{in}$ in the third layer:

This means for $e = uv, m_e v_{in}$ and $u_{out} m_e$ are not saturated so column $x_e$ is not dominated by row $e$. $u_{out}$ prefers $u_{out} m_e$ to some other nonzero incoming edges indicates column $x_e$ is not dominated by row $u'$. $v_{in}$ prefers $m_e v_{in}$ to some other nonzero incoming edges indicates column $x_e$ is not dominated by row $v^1$. Column $x_e$ is not dominated by $x^*$, a contradiction.

4. $P$ starts from $s$ to $t$:

This means row $v^1$ and $u'$ do not bind and for $e = uv, m_e v_{in}$ and $u_{out} m_e$ are not saturated which indicates column $x_e$ is not dominated by row $e$. Column $x_e$ is not dominated by $x^*$, a contradiction.

LM-stable-flow $\rightarrow$ Scarf’s:

Suppose $f$ is a LM-stable-flow of the network. For $e = uv$, if $m_e v_{in}$ and $u_{out} m_e$ are saturated, then $x^* = c(u_{out} m_e) = b_e$. Column $x_e$ is dominated by row $e$. Otherwise, the following two cases cannot happen at the same time or there is a blocking path from $u_{out}$ to $v_{in}$:

1. $u_{out}$ prefers $u_{out} m_e$ to some other nonzero incoming edges.
2. $v_{in}$ prefers $m_e v_{in}$ to some other nonzero incoming edges.

If the first case happens, $v_{in}$ must prefer all nonzero incoming edges to $m_e v_{in}$ and $v_{in} t$ must be saturated or the path from $u_{out}$ to $t$ is blocking. This forces $f(m^1_e v_{in}) = c(v_{in} t) - \sum_{e' = uv_{in}} f(e')$ which corresponds to row $v^1$ binds and dominates column $x_e$.

If the second case happens, $u_{out}$ must prefer all nonzero outgoing edges to $u_{out} m_e$ and $s v_{out}$ must be saturated or the path from $s$ to $v_{in}$ is blocking. This forces $f(u_{out} m^2_e) = c(s v_{out}) - \sum_{e' = u_{out} w} f(e')$ which corresponds to row $u'$ binds and dominates column $x_e$.

The remaining is to show how $f$ makes column $x^1_e$ and $x^2_e$ dominated. If $s v_{out}$ is not saturated, then either $m^1_e v_{in}$ and $u_{out} m^1_e$ are saturated or the flow of incoming edges of $v_{in}$ except $m^1_e v_{in}$ have to be zero. Otherwise, $s$ to $v_{in}$ forms a blocking path since $v_{in}$ prefers $m^1_e v_{in}$ the most. However, $m^1_e v_{in}$ and $u_{out} m^1_e$ cannot be saturated since $b_v = c(v_{in} t) \leq c(u_{out} m^1_e) = c(m^1_e v_{in}) = c(s v_{out}) = b_{uv}$ and now $e = f(u_{out} m^1_e) = f(m^1_e v_{in}) \leq f(s v_{out}) < c(s v_{out})$ so $f(u_{out} m^1_e) < c(u_{out} m^1_e)$ and $f(m^1_e v_{in}) < c(m^1_e v_{in})$. This also forces $v_{in} t$ saturated otherwise path $s$ to $t$ is blocking. Consequently, $f(u_{out} m^1_e) = f(m^1_e v_{in}) = c(v_{in} t)$ corresponds to row $v^1$ binds and dominates column $x^1_e$ and $x^2_e$.

If $s v_{out}$ is saturated, then either $m^2_e v_{in}$ and $u_{out} m^2_e$ are saturated or $v_{in} t$ is saturated or $v_{out}$ to $t$ forms a blocking path since $u_{out}$ prefers $u_{out} m^2_e$ the most. If $m^2_e v_{in}$ and $u_{out} m^2_e$ are saturated, then $v_{in} t$ is also saturated since $f(v_{in} t) \geq f(m^2_e v_{in}) = c(m^2_e v_{in})$ and $f(v_{in} t) \leq b_v = c(v_{in} t) \leq c(m^2_e v_{in}) = c(s v_{out}) = b_{uv}$ implies $f(v_{in} t) = c(v_{in} t)$. In either case, $v_{in} t$ must be saturated. This corresponds to row $v^1$ binds and dominates $x^2_e$ and row $v'$ binds and dominates $x^1_e$. \hfill \Box

### 4.3 Cyclic Networks

The standard stable flow problem can be reduced to the stable allocation problem \cite{9} according to \cite{10}. Conversely, the stable allocation problem can also be reduced to the stable flow problem. For cyclic networks, one can always reduce stable flow to stable allocation, then reduce the obtained stable allocation instance to stable flow. It turns out that the new network is acyclic. The number of vertices is still $O(|V|)$ and the number of edges is still $O(|E|)$. Applying the path augmentation algorithm in \cite{8} to the new flow still takes $O(|V| |E|)$ time.

Inspired by this observation, we transform a cyclic LM-network to a new acyclic LM-network then apply Algorithm 1. MPLM-stable-flow can be reduced to LM-stable-flow by Theorem 2 so one can always transform a cyclic MPLM-network to an acyclic LM-network.

First we simplify the transformation from cyclic LM-networks to acyclic LM-networks.
Theorem 6. For an LM-network \((G, s, t, c), G = (V, E), \succ_v\), and \(g_v \in LM\) for any \(v \in V\) with cycles, there is an equivalent LM-network \((G', s', t', c'), G' = (V', E'), \succ_{v'}\), and \(g_{v'} \in LM\) for any \(v' \in V'\) without cycles.

Proof. This is a direct result by combining Theorem 1 and Theorem 5. To wrap everything up, for each \(v \in V\), let \(q(v) = \max(\sum_{uv \in E} c(uv), \sum_{vu \in E} c(vu), b_v) + 1\), we construct the new LM-network as the following:

1. Create \(s'_\text{out} \in V'\) and \(t'_\text{in} \in V'\).
2. For each \(v \in V\), create \(v_{\text{in}}, v_{\text{out}}, m^1_v, \text{and } m^2_v \) in \(V'\).
3. For each \(v_{\text{out}} \in V'\), set \(c(s'v_{\text{out}}) = q(v)\) (by setting capacity of \(c', c'\) is automatically added to \(E'\)).
4. For each \(v_{\text{in}} \in V'\), set \(c(v_{\text{in}}t') = q(v) - b_v\).
5. Set \(c(s's'_{\text{out}}) = c(t'_\text{in}t') = \infty\).
6. For each \(e = uv \in E\), set \(c(u_{\text{out}}m_e) = c(uv)\) and \(c(m_{v_{\text{in}}}a_v) = a_v, c(uv)\).
7. Set \(c(v_{\text{out}}m^1_v) = c(v_{\text{out}}m^2_v) = c(m^1_{v_{\text{in}}}a_v) = c(m^2_{v_{\text{in}}}a_v) = q(v)\).
8. For each \(v_{\text{out}} \in V'\), \(v_{\text{out}}\) prefers \(v_{\text{out}}m^2_v\) the most and \(v_{\text{out}}m^1_v\) the least, the preference of \(v_{\text{out}}m_e\) for some \(e = uv\) is the same as in row \(v'\).
9. For each \(v_{\text{in}} \in V'\), \(v_{\text{in}}\) prefers \(m^1_{v_{\text{in}}}v_{\text{in}}\) the most and \(m^2_{v_{\text{in}}}v_{\text{in}}\) the least, the preference of \(m_{v_{\text{in}}}v_{\text{in}}\) for some \(e = uv\) is the same as in row \(v^1\).
10. The preference of \(s_{\text{out}}\) and \(t_{\text{in}}\) is arbitrary.
11. Set \(g_{s_{\text{out}}}(x) = x, g_{t_{\text{in}}}(x) = x\). For each \(v \in V\), set \(g'_{v_{\text{in}}}(x) = x, g'_{v_{\text{out}}}(x) = x, g'_{m^1_v}(x) = x, \text{and } g'_{m^2_v}(x) = x\).
12. For each \(e = uv \in E'\), set \(g_{m_e}(x) = a_v, x\) if \(v \neq t'\). If \(v = t'\), set \(g_{m_e}(x) = x\).

Suppose \(f'\) is the LM-stable-flow of the new LM-network, to obtain the LM-stable-flow \(f\) for the old LM-network with cycles, for each \(e = uv \in E\), set \(f(uv) = f'(u_{\text{out}}m_e)\).

Theorem 7. An MPLM-stable-flow of a cyclic network can be computed in polynomial time where the polynomial involves the number of segments of each vertex.

Proof. We follow a series of reductions:

1. Reduce MPLM-network to LM-network by Theorem 2.
2. Reduce cyclic LM-network to the equivalent acyclic LM-network by Theorem 6.
3. Run Algorithm 1 on the new LM-network.

Suppose the starting MPLM-network has \(|V|\) vertices and \(|E|\) edges. In the first step, for each \(v \in V\) in the MPLM-network, \(k_v\) extra vertices and \(2k_v\) extra edges are created in the LM-network.

Let \(K = \sum_{v \in V} k_v\). In the second step, the new acyclic LM-network \(O(|E| + K)\) vertices and \(O(|E| + K)\) edges.

In the last step, the rough bound for running time of Algorithm 1 is \(O((|E| + K)^2)\). By utilizing the special structure of the network, the length of the augmenting path is \(O(|V|)\) so the running time is \(O(|V||E| + K|V|)\), which is asymptotically the same as the running time on acyclic MPLM-networks.

5 Conclusion

In this paper, we first state a variant of the stable flow problem where agents apply MPLM instead of identity functions as in traditional flow problems. For a CMPLM-network, a subclass of MPLM-network, the existence of stable flow can be proved by the reduction to Scarf’s Lemma. For an MPLM-network, each agent can be transformed into an LM-subnetwork, and the entire network is equivalent to a large LM-network. The existence of CMPLM-stable-flow also implies the existence of LM-stable-flow, hence there always exists a stable flow for an MPLM-network.

An acyclic MPLM-network can be reduced to an acyclic LM-network and the stable flow of an acyclic network can be found by a path augmenting algorithm Algorithm 1. This algorithm augments either an \(s-t\)-path or a \(\sigma\)-cycle in each iteration and runs in polynomial time. For MPLM-networks, the polynomial involves the number of segments. However, Algorithm 1 is not applicable for cyclic MPLM-networks or LM-networks. To circumvent this issue, we find an equivalent acyclic LM-network and solve the problem by Algorithm 1.
A monotone continuous mapping (MCM) can output any value below a threshold when inflow is zero and applies a monotone continuous function when inflow is positive. MPLM can approximate any MCM by setting a bunch of infinitesimal segments. Therefore, it is natural to assume that there always exists an MCM-stable-flow in an MCM-network. One can assume that for a vertex \( v \) and \( g_v \in \text{MCM} \), if we are given \( x \), \( g_v(x) \) can be computed in constant time. However, similar approach as Algorithm 1 may not be applicable to an acyclic MCM-network. This is because it may not be possible to augment a \( \sigma \)-cycle such that there is one edge saturated in the auxiliary graph. For acyclic LM-networks, we can do so because of the linear relation between the change of inflow and outflow. An engrossing open problem is: How to design an algorithm to find an MCM-stable-flow in an MCM-network? Are there any other special set of functions or mappings \( F \) that makes the augmentation along an \( \sigma \)-cycle computable in an \( F \)-network?

The standard stable flow problem can be solved in \( O(|E| \log |V|) \) time \[8\] if one utilizes a more sophisticated data structure, the dynamic trees \[13\]. This technique was also used in \[14\]. Can we modify dynamic trees and design faster algorithms for finding stable flow in LM-networks? Another open problem is the regular flow problem for graphs only depicts contracts with two agent involved. How about contracts with more agents involved? Contracts with multiple agents can be interpreted by introducing hypergraphs and there are studies of flow problems for directed hypergraphs \[15\]. How will this problem be reduced to Scarf’s Lemma? Is there always a stable assignment for agents?

Based on the properties of the standard stable flow problem such as the lattice structure \[10\], perhaps another more accomplishable research direction for MPLM-networks and LM-networks is to examine the structure of stable solutions. An MPLM-network can be described as a special form of LM-network. It will be interesting to see how structures of MPLM-stable-flow differs from LM-stable-flow.

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A An Example for Theorem 7

Let the following be an input instance of MPLM-stable-flow:

\[ g_{v_1}(x) = \begin{cases} [0, 2] & \text{if } x = 0, \\ 2x + 2 & \text{if } x \in (0, 2] \\ x + 4 & \text{if } x > 2 \end{cases} \]

\[ g_{v_2}(x) = \begin{cases} [0, 1] & \text{if } x = 0, \\ x + 1 & \text{if } x \in (0, 3] \\ 2x - 2 & \text{if } x > 3 \end{cases} \]

First reduce this MPLM-network to LM-network by Theorem 2. In this network, the subnetwork of LM-agents \( a, b, c, \) and \( d \) is equivalent to MPLM-agent \( v_1 \), and the subnetwork of LM-agents \( e, f, g, \) and \( h \) is equivalent to MPLM-agent \( v_2 \).

Next, reduce this LM-network to an equivalent LM-network without cycle by Theorem 6 and solve it by Algorithm 1. Note that \( g_{m_{ae}}(x) = 2x \) and \( g_{m_{cg}}(x) = 2x \), any \( m_e \) that applies identity function is not in the graph. Because of the space limit, \( m_1 v_v, m_2 v_v, v_{out} m_1 v_v, v_{out} m_2 v_v, m_1 v_{in}, \) and \( m_2 v_{in} \) are not shown in the graph.

Preference list (from most preferred to least preferred):

| vertex | a | d | e | h |
|--------|---|---|---|---|
| in     | ha \succ_a sa | bd \succ_a cd | fh \succ_a gh |
| out    | ab \succ_a ac | cd \succ_a de | ef \succ_e eg |

\[ g_v(x) = x, 2x, x + 2, x, 2x, x + 1 \]

| vertex | \( a \) | \( b \) | \( c \) | \( d \) | \( e \) | \( f \) | \( g \) | \( h \) |
|--------|------|------|------|------|------|------|------|------|
| in     |      |      |      |      |      |      |      |      |
| out    |      |      |      |      |      |      |      |      |

| vertex | \( a \) | \( b \) | \( c \) | \( d \) | \( e \) | \( f \) | \( g \) | \( h \) |
|--------|------|------|------|------|------|------|------|------|
| in     |      |      |      |      |      |      |      |      |
| out    |      |      |      |      |      |      |      |      |

| vertex | \( a \) | \( b \) | \( c \) | \( d \) | \( e \) | \( f \) | \( g \) | \( h \) |
|--------|------|------|------|------|------|------|------|------|
| in     |      |      |      |      |      |      |      |      |
| out    |      |      |      |      |      |      |      |      |

| vertex | \( a \) | \( b \) | \( c \) | \( d \) | \( e \) | \( f \) | \( g \) | \( h \) |
|--------|------|------|------|------|------|------|------|------|
| in     |      |      |      |      |      |      |      |      |
| out    |      |      |      |      |      |      |      |      |
Rest of the flow:

| vertex | $a_{out}$ | $b_{out}$ | $c_{out}$ | $d_{out}$ | $e_{out}$ | $f_{out}$ | $g_{out}$ | $h_{out}$ |
|--------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| to     | $m_a^1$   | $m^b_1$   | $m^c_1$   | $m^d_1$   | $m^e_1$   | $m^f_1$   | $m^g_1$   | $m^h_1$   |
| value  | 9         | 1         | 9         | 7         | 5         | 1         | 9         | 8         |

Note that by only changing $c(f_{out}m^1_f) = c(m^1_f, e_{in}) = 1$ and $c(f_{out}m^2_f) = c(m^2_f, e_{in}) = 0$ also returns a stable solution but we list the one returned by Algorithm 1.
The corresponding instance of the old LM-network is:

The corresponding instance of the original MPLM-network is:

\[
\begin{align*}
g_{v_1}(x) &= \begin{cases} 
[0, 2] & \text{if } x = 0, \\
2x + 2 & \text{if } x \in (0, 2] \\
x + 4 & \text{if } x > 2 
\end{cases} \\
g_{v_2}(x) &= \begin{cases} 
[0, 1] & \text{if } x = 0, \\
x + 1 & \text{if } x \in (0, 3] \\
2x - 2 & \text{if } x > 3 
\end{cases}
\end{align*}
\]