On the reductive Borel-Serre compactification:
$L^p$-cohomology of arithmetic groups (for large $p$)

Steven Zucker\(^1\)

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218 USA\(^2\)

**Introduction**

The main purpose of this article is to give the proof of the following theorem, as well as some applications of the result.

**Theorem 1.** Let $M$ be the quotient of a non-compact symmetric space by an arithmetically-defined group of isometries, and $M^{\text{RBS}}$ its reductive Borel-Serre compactification. Then for $p$ finite and sufficiently large there is a canonical isomorphism

$$H^\bullet_{(p)}(M) \simeq H^\bullet(M^{\text{RBS}}).$$

Here, the left-hand side is the $L^p$-cohomology of $M$ with respect to a (locally) invariant metric. Though it would be more natural to allow $p = \infty$ in Theorem 1, this is not generally possible (see (3.2.2)). On the other hand, there is a natural mapping $H^\bullet_{(\infty)}(M) \to H^\bullet_{(p)}(M)$ when $p < \infty$, because $M$ has finite volume. The definition of $M^{\text{RBS}}$ is recalled in (1.9).

Theorem 1 can be viewed as an analogue of the so-called Zucker conjecture (in the case of constant coefficients), where $p = 2$:

**Theorem [L],[SS].** Let $M$ be the quotient of a Hermitian symmetric space of non-compact type by an arithmetically-defined group of isometries, i.e., a locally symmetric variety; let $M^{\text{BB}}$ its Baily-Borel Satake compactification. Then there is a canonical isomorphism

$$H^\bullet_{(2)}(M) \simeq IH^\bullet_{\text{m}}(M^{\text{BB}}),$$

where the right-hand side denotes the middle intersection cohomology of $M^{\text{BB}}$.

However, Theorem 1 is not nearly so difficult to prove, once one senses that it is true; it follows without much ado from the methods in [Z3] (the generalization to $L^p$, $p \neq 2$, of those of [Z1] for $L^2$).

As far as I know, the reductive Borel-Serre compactification was first used in [Z1,§4] (where it was called $Y$). This space, a rather direct alteration of the manifold-with-corners constructed in [BS], was introduced there to facilitate the study of the $L^2$-cohomology of $M$. It also plays a central role as the natural setting for the related weighted cohomology of [GHM]. It is a principal theme that $M^{\text{RBS}}$ is an important space when $M$ is an algebraic variety over $\mathbb{C}$, despite the

\(^1\)Support in part by the National Science Foundation, through Grant DMS9820958

\(^2\)e-mail address: zucker@jhu.edu
fact that $M^{RBS}$ is almost never an algebro-geometric, or even complex analytic, compactification of $M$.

This work had its origin in my wanting to understand [GP]. It is convenient to formulate the latter before continuing with the content of this article. Let $Y$ be a Hausdorff topological space. For any complex vector bundle $E$ on $Y$, one has its Chern classes $c_k(E) \in H^{2k}(Y, \mathbb{Z})$. If we further assume that $Y$ is connected, compact, stratified and oriented, then $H_d(Y, \mathbb{Z}) \simeq \mathbb{Z}$, where $d$ is the dimension of $Y$; the orientation picks out a generator $\zeta_Y$ for this homology group, known as the fundamental class of $Y$. We shall henceforth assume that $d$ is even, and we write $d = 2n$. Then, if one has positive integers $k_i$ for $1 \leq i \leq \ell$ such that $\sum_i k_i = n$, one can pair $c_{k_1}(E) \cup \cdots \cup c_{k_\ell}(E)$ with $\zeta_Y$, and obtain what is called a characteristic number, or Chern number, of $E$.

When $Y$ is a $C^\infty$ manifold and $E$ is a $C^\infty$ vector bundle, the Chern classes modulo torsion can be constructed from any connection $\nabla$ in $E$, whereupon they get represented, via the de Rham theorem, by the Chern forms $c_k(E, \nabla)$ in $H^{2k}(Y)$. (For convenience, we will use and understand $\mathbb{C}$-coefficients here and throughout the sequel unless it is specified otherwise.) If $Y$ is compact, the Chern numbers can be computed by integrating $c_{k_1}(E, \nabla) \wedge \cdots \wedge c_{k_\ell}(E, \nabla)$ over $Y$.

For stratified spaces $Y$, there is a lattice of intersection (co)homology theories, with variable perversity $p$ as parameter, as defined by Goresky and MacPherson [GM1]. These range from standard cohomology as minimal object, to standard homology as maximal, and all coincide when $Y$ is a manifold. They can all be defined as cohomology with values in some constructible sheaf whose restriction to the regular locus $Y^{\text{reg}}$ of $Y$ is just $\mathbb{C}_{Y^{\text{reg}}}$. With mappings going in the direction of increasing perversity, we have the basic diagram

\begin{equation}
\begin{array}{cccc}
H^\bullet(Y) & \to & \cdots & \to & IH^\bullet_p(Y) & \to & \cdots & \to & H^\bullet(Y) \\
\downarrow & & & & \uparrow & & & & \uparrow \\
H^\bullet(Y^{\text{reg}}) & \leftarrow & H^\bullet_c(Y^{\text{reg}}) & \simeq & H^\bullet(Y^{\text{reg}}).
\end{array}
\end{equation}

When $Y$ is compact, $\zeta_Y$ lifts to a generator of $IH^\bullet_p(Y)$ for all $p$.

From now on, we write $M$ for $Y^{\text{reg}}$, and start to view the situation in the opposite way, regarding $Y$ as a topological compactification of the manifold $M$. For any vector bundle $E$ on $M$, and bundle extension of $E$ to $\overline{E}$ on $Y$, the functoriality of Chern classes imply that $c_\bullet(\overline{E}) \mapsto c_\bullet(E)$ under the restriction mapping $H^\bullet(Y) \xrightarrow{\rho} H^\bullet(M)$. One might think of this as lifting the Chern class of $E$ to the cohomology of $Y$, but one should be aware that $\rho$ might have non-trivial kernel, so the lift may depend on the choice of $\overline{E}$.

The case where $E = T_M$, the tangent bundle of $M$, is quite fundamental. Finding a vector bundle on $Y$ that extends $T_M$ is not so natural a question when $Y$ has singularities, and one is often inclined to forget about bundles and think instead about just lifting the Chern classes. When $Y$ is a complex algebraic variety, one considers the complex tangent bundle $T'_M$ of $M$. It is shown in [M] that for constructible $\mathbb{Z}$-valued functions $F$ on $Y$, there is a natural assignment of Chern homology classes
$c_\bullet(Y; F) \in H_\bullet(Y)$, such that $c_\bullet(Y, 1)$ recovers the usual Chern classes when $Y$ is smooth. There has been substantial interest in lifting these classes to the lower intersection cohomology (as in the top row of (0.1)), best to cohomology (the most difficult lifting problem) for the reason mentioned earlier.

Next, take for $M$ a locally symmetric variety. For $Y$ we might consider any of the interesting compactifications of $M$, which include: $M^{BB}$, the Baily-Borel Satake compactification of $M$ as an algebraic variety [BB]; $M_\Sigma$, the smooth toroidal compactifications of Mumford (see [Mu]); $M^{BS}$, the Borel-Serre manifold-with-corners [BS]; $M^{RBS}$, the reductive Borel-Serre compactification. These fit into a diagram of compactifications:

\[
\begin{array}{c}
M^{BS} \rightarrow M^{RBS} \\
\downarrow \\
M_\Sigma \rightarrow M^{BB}
\end{array}
\] (0.2)

If one tries to compare $M^{BS}$ and $M_\Sigma$, one sees that there is a mapping (of compactifications of $M$) $M^{BS} \rightarrow M_\Sigma$ only in a few cases (e.g., $G = SU(n, 1)$). However, by a result of Goresky and Tai [GT, 7.3], (if $\Sigma$ is sufficiently fine) there are continuous mappings $M_\Sigma \rightarrow M^{RBS}$ (seldom a morphism of compactifications) such that upon inserting them in (0.2), the obvious triangle commutes in the homotopy category. One thereby gets a diagram of cohomology mappings

\[
\begin{array}{ccc}
H^\bullet(M) & \leftarrow & H^\bullet(M^{RBS}) \\
\uparrow & & \uparrow \\
H^\bullet(M_\Sigma) & \rightleftarrows & H^\bullet(M^{BB});
\end{array}
\] (0.3)

also, the fundamental classes in $H_d(M_\Sigma, \mathbb{Z})$ and $H_d(M^{RBS}, \mathbb{Z})$ are mapped to $\zeta_{M^{BB}}$.

Let $E$ be a (locally) homogeneous vector bundle on $M$ (an example of which is the holomorphic tangent bundle $T'_M$). There always exists an equivariant connection on $E$, whose Chern forms are $L^\infty$ (indeed, of constant length) with respect to the natural metric on $M$. In [Mu], Mumford showed that the bundle $E$ has a so-called canonical extension to a vector bundle $E_\Sigma$ on $M_\Sigma$, such that these Chern forms, beyond representing the Chern classes of $E$ in $H^\bullet(M)$, actually represent the Chern classes of $E_\Sigma$ in $H^\bullet(M_\Sigma)$ (see our (3.2.4)). That served the useful purpose of placing these classes in a ring with Poincaré duality, and implied Hirzebruch proportionality for $M$.

In [GP], Goresky and Pardon lift these classes to the cohomology of $M^{BB}$, minimal in the lattice of interesting compactifications of $M$, so these can be pulled back to the other compactifications in (0.2). (On the other hand, the bundles do not extend to $M^{BB}$ in any obvious way.) They achieve this by constructing another connection in $E$ (see our (5.3.3)), one that has good properties near the singular strata of $M^{BB}$, using features from the work of Harris and Harris-Zucker (see [Z5, App. B]). With this done, the Chern forms lie in the complex of controlled differential forms on $M^{BB}$, whose cohomology groups give $H^\bullet(M^{BB})$. In the case
of the tangent bundle, the classes map to one of the MacPherson Chern homology classes in $H_* (M^{BB})$, viz., $c_* (M^{BB}; \chi_M)$, where $\chi_M$ denotes the characteristic (i.e., indicator) function of $M \subset M^{BB}$ [GP, 15.5].

In [GT, 9.2], an extension of $E$ to a vector bundle $E^{RBS}$ to $M^{RBS}$ is constructed; this does not require $M$ to be Hermitian. There, one finds the following:

**Conjecture A** [GT, 9.5]. Let $M_\Sigma \to M^{RBS}$ be any of the continuous mappings constructed in [GT]. Then the canonical extension $E_\Sigma$ is isomorphic to the pullback of $E^{RBS}$.

In the absence of a proof of Conjecture A, we derive the “topological” analogue of Mumford’s result as a consequence of Theorem 1:

**Theorem 2.** Let $M$ be an arithmetic quotient of a symmetric space of non-compact type. Then the Chern forms of an equivariant connection on $M$ represent $c_* (E^{RBS})$ in $H^* (M^{RBS})$.

We point out that (0.3) and Conjecture A suggest that this is more basic in the Hermitian setting than Mumford’s result.

Goresky and Pardon predict further:

**Conjecture B** [GP]. The Chern classes of $E^{RBS}$ are the pullback of the classes in $H^* (M^{BB})$ constructed in [GP] via the quotient mapping $M^{RBS} \to M^{BB}$.

Our third main result is the proof of Conjecture B.

The material of this article is organized as follows. In §1 we give a canonical construction of the bundle $E^{RBS}$ along the lines of [BS]. We next discuss $L^p$-cohomology, both in general in §2, then on arithmetic quotients of symmetric spaces in §3, achieving a proof of Theorem 1. We make a consequent observation in (3.3) that shows how $L^p$-cohomology can be used to provide definitions of mappings between topological cohomology groups when it is unclear how to define the mappings topologically. In §4, we treat connections and the notion of Chern forms for a natural class of vector bundles on stratified spaces; this allows for the proof in §5 of both Theorem 2 and Conjecture B.

This article was conceived while I was spending Academic Year 1998–99 on sabbatical at the Institute for Advanced Study in Princeton. I wish to thank Mark Goresky and John Mather for helpful discussions.

1. **The Borel-Serre construction for homogeneous vector bundles**

In this section, we make a direct analogue of the Borel-Serre construction for the total space of a homogeneous vector bundle on a symmetric space, and then for any neat arithmetic quotient $M_\Gamma$ thereof. It defines a natural extension of the vector bundle to the Borel-Serre compactification of the space. That the bundle extends is clear, for attaching of a boundary-with-corners does not change homotopy type. Our construction retains at the boundary much of the group-theoretic structure.
The construction is shown to descend to the reductive Borel-Serre compactification \( M^\text{RBS}_F \), reproving [GT, 9.2].

**1.0** *Convention.* Whenever \( H \) is an algebraic group defined over \( \mathbb{Q} \), we also let \( H(\mathbb{R}) \), taken with its topology as a real Lie group, if there is no danger of confusion.

**1.1** *Standard notions.* Let \( G \) be a semi-simple algebraic group over \( \mathbb{Q} \), and \( K \) a maximal compact subgroup of \( G \), and \( X = G/K \). (Note that this implies a choice of basepoint for \( X \), namely the point \( x_0 \) left fixed by \( K \).)

Let \( \mathcal{E} = G \times_K E \) be the homogeneous vector bundle on \( X \) determined by the representation of \( K \) on the vector space \( E \). The natural projection \( \pi : \mathcal{E} \to X = G \times K \{0\} \) is induced by the projection \( E \to \{0\} \), and is \( G \)-equivariant. For \( \Gamma \subset G(\mathbb{Q}) \) a torsion-free arithmetic subgroup, let \( M_\Gamma = \Gamma \backslash X \). (The subscript “\( \Gamma \)” was suppressed in the Introduction.)

If \( P \) is any \( \mathbb{Q} \)-parabolic subgroup of \( G \), the action of \( P \) on \( X \) is transitive. Thus, one can also describe \( \mathcal{E} \to X \) as \( P \times_K E \to P \times_K \{0\} \), where \( K_P = K \cap P \).

**1.2** *Geodesic action.* Let \( U_P \) denote the unipotent radical of \( P \), and \( A_P \) the lift to \( P \) associated to \( x_0 \) of the connected component of the maximal \( \mathbb{Q} \)-split torus \( Z \) of \( P/U_P \). Define the *geodesic action of \( A_P \) on \( \mathcal{E} \) by the formula:

\[
(1.2.1) \quad a \circ (p, e) = (pa, e)
\]

whenever \( p \in P \), \( e \in E \) and \( a \in A_P \); this is well-defined because for \( k \in K_P \),

\[
(1.2.2) \quad a \circ (pk^{-1}, ke) = (pk^{-1}a, ke) = (pak^{-1}, ke),
\]
as \( A_P \) and \( K_P \) commute. The geodesic action of \( A_P \) commutes with the action of \( P \) on \( \mathcal{E} \), and it projects to the geodesic action of \( A_P \) on \( X \) as defined in [BS, §3] (in [BS], the geodesic action is expressed in terms of \( Z \), but the definitions coincide).

**1.3** *Corners.* The simple roots occurring in \( U_P \) set up an isomorphism \( A_P \simeq (0, \infty)^{r(P)} \), where \( r(P) \) denotes the parabolic \( \mathbb{Q} \)-rank of \( P \). Let \( \overline{A}_P \) be the enlargement of \( A_P \) obtained by transport of structure from \( (0, \infty)^{r(P)} \subset (0, \infty)^{r(P)} \). Define the *corner associated to \( P \):* \( \mathcal{E}(P) = \mathcal{E} \times_{A_P} \overline{A}_P \). There is a canonical mapping \( \pi(P) : \mathcal{E}(P) \to X(P) = X \times_{A_P} \overline{A}_P \).

**1.3.1** *Remark.* Though \( X(P) \) is contractible, and hence \( \mathcal{E}(P) \) is trivial, (1.2.1) *does not* yield a canonical trivialization of \( \mathcal{E}(P) \) over \( X(P) \), because of the equivalence relation (1.2.2) determined by \( K_P \).
Let $\infty_P$ denote the zero-dimensional $A_P$-orbit in $\overline{A}_P$, which corresponds to $(\infty, \ldots, \infty) \in (0, \infty)^{|r(P)|}$. The face of $E(P)$ associated to $P$ is

\[(1.3.2) \quad E(P) = E \times_{A_P} \{\infty_P\} \simeq E/A_P.\]

It maps canonically to $X/A_P \simeq e(P) \subset X(P)$ (from [BS, 5.2]). There are geodesic projections implicit in (1.3.2), given by the rows of the commutative diagram

\[(1.3.3) \quad \begin{array}{ccc}
E(P) & \overset{\pi_P}{\longrightarrow} & E(P) \\
\downarrow & & \downarrow \\
X(P) & \overset{\pi_P}{\longrightarrow} & e(P)
\end{array}\]

\[(1.4) \quad \text{Structure of } E(P). \quad \text{There is a natural } P\text{-action on } E(P), \text{ with } A_P \text{ acting trivially, projecting to the action of } P \text{ on } e(P). \text{ We know that } e(P) \text{ is homogeneous under } ^0P \text{ (as in [BS, 1.1]), isomorphic to } P/A_P, \text{ which contains } K_P. \text{ We see that } E(P) \text{ is isomorphic to the homogeneous vector bundle on } e(P) \text{ determined by the representation of } K_P \text{ on } E.\]

\[(1.5) \quad \text{Compatibility. For } Q \subset P, \text{ there is a canonical embedding of } E(P) \text{ in } E(Q), \text{ given as follows. As in [BS, 4.3], write } A_Q = A_P \times A_Q,P, \text{ with } A_Q,P \subset A_Q \text{ denoting the intersection of the kernels of the simple roots for } A_P. \text{ Then there is an embedding}

\[E(P) = E \times_{A_P} \overline{A}_P \simeq (E \times_{A_Q,P} A_Q,P) \times_{A_P} \overline{A}_P \]

\[\subset (E \times_{A_Q,P} \overline{A}_Q,P) \times_{A_P} \overline{A}_P \simeq E \times_{A_Q} \overline{A}_Q = E(Q).\]

Moreover, this projects to $X(P) \subset X(Q)$ via $\pi(Q)$.

\[(1.6) \quad \text{Hereditary property. If } Q \subset P \text{ again, one can view } E(Q) \text{ as part of the boundary of } E(P), \text{ in the same way that } e(Q) \text{ is part of the boundary of } e(P). \text{ This is achieved by considering the geodesic action of } A_Q,P \text{ on } E(P) \text{ (} A_P \text{ acts trivially), and carrying out the analogue of (1.3). Thus, } E(Q) \simeq E(P)/A_Q,P = E(P)/A_Q.\]

\[(1.7) \quad \text{The bundle with corners. Using the identifications given in (1.5), we recall that one puts}

\[(1.7.1) \quad \overline{X} = \bigcup_P X(P) = \bigcup_P e(P),\]

with $P$ ranging over all parabolic subgroups of $G/Q$, including the improper one ($G$ itself). With $\overline{X}$ endowed with the weak topology from the $X(P)$’s, this is the manifold-with-corners construction of Borel-Serre for $X$ (see [BS, §7]). As such, it has a tautological stratification, with the $e(P)$’s as strata.

We likewise put $\overline{E} = \bigcup_P E(P)$, with incidences given by (1.5), and endow it with the weak topology. There is an obvious projection onto $\overline{X}$. Then $\overline{E}$ is a vector
bundle over $\overline{X}$ that is stratified by the homogeneous bundles $E(P)$, given as in (1.4).

(1.8) Quotient by arithmetic groups. We can see that $G(\mathbb{Q})$ acts as vector bundle automorphisms on $\overline{E}$ over its action as homeomorphisms of $\overline{X}$ (given in [BS, 7.6]); also, as it is so for $\overline{X}$, the action on $\overline{E}$ of any neat arithmetic subgroup $\Gamma$ of $G(\mathbb{Q})$ is proper and discontinuous (cf. [BS, 9.3]). Then $\overline{E}_\Gamma = \Gamma \backslash \overline{E}$ is a vector bundle over $M^{BS}_\Gamma = \Gamma \backslash \overline{X}$.

Let $\Gamma_P = \Gamma \cap P$. The action of $\Gamma_P$ (which is contained in $0^P$ of (1.4)) commutes with the geodesic action of $A_P$. The faces of $\overline{E}_\Gamma$ are of the form $E'(P) = \Gamma_P \backslash E(P)$, and are vector bundles over the faces $e'(P) = \Gamma_P \backslash e(P)$ of $M^{BS}_\Gamma$. By reduction theory [BS, §9] (but see also [Z5,(1.3)]), there is a neighborhood of $e'(P)$ in $M^{BS}_\Gamma$ on which geodesic projection $\pi_P$ (from (1.3.3)) descends. The same is true for $\tilde{\pi}_P$ and $E'(P)$ (also from (1.3.3)).

(1.9) The reductive Borel-Serre compactification. We recall the quotient space $X^{RBS}$ of $\overline{X}$. With $\overline{X}$ given as in (1.7) above, one forms the quotient

$$X^{RBS} = \bigsqcup_P X_P, \quad (\text{where } X_P = U_P \backslash e(P)),$$

where $U_P$ is, as in (1.2), the unipotent radical of $P$, and endows it with the quotient topology from $\overline{X}$. Because $U_Q \supset U_P$ whenever $Q \subset P$, $X^{RBS}$ is a Hausdorff space (see [Z1,(4.2)]). There is an induced action of $G(\mathbb{Q})$ on $X^{RBS}$, for which (1.9.1) is a $G(\mathbb{Q})$-equivariant stratification; $G(\mathbb{Q})$ takes the stratum $X_P$ onto that of a conjugate parabolic subgroup, with $P(\mathbb{Q})$ preserving $X_P$. For any arithmetic group $\Gamma \subset G(\mathbb{Q})$, one has a quotient mapping

$$q : M^{BS}_\Gamma \to M^{RBS}_\Gamma = \Gamma \backslash X^{RBS} = \bigsqcup_P \tilde{M}_P,$$

with $\tilde{M}_P = \Gamma_P \backslash X_P$.

(1.10) Descent of $\overline{E}$ to $X^{RBS}$. Analogous to the description of $\overline{X}$ in (1.7), we have

$$\overline{E} = \bigsqcup_P E(P) = \bigsqcup_P E(P),$$

and the corresponding quotient

$$E^{RBS} = \bigsqcup_P (U_P \backslash E(P)).$$

We verify that $E^{RBS}$ is a vector bundle on $X^{RBS}$. Since $\{X(P)\}$ is an open cover of $\overline{X}$ (see (1.7.1)), it suffices to verify this for $E(P) \to X(P)$ for each $P$ separately.

Note that $U_P$ acts on $E(P)$ by the formula: $u \cdot (p,e,a) = (up,e,a)$, and this commutes with the action of $K_P \cdot A_P$. It follows that there is a canonical projection

$$U_P \backslash E(P) \to U_P \backslash X(P).$$
This gives a vector bundle on $U_P \setminus X(P)$ because $U_P \cap (K_P \cdot A_P) = \{1\}$.

Let $X^{RBS}(P)$ be the image of $X(P)$ in $X^{RBS}$, and $E^{RBS}(P)$ be the image of $E(P)$ in $E^{RBS}$. These differ from (1.10.3), for the $U_P$ quotient there is too coarse (for instance, there are no identifications on $X$ or $E$ in $E^{RBS} \to X^{RBS}$). Rather, the pullback of (1.10.3) to $X^{RBS}(P)$ is $E^{RBS}(P)$.

When $\Gamma$ is a neat arithmetic group, $E^{RBS}_\Gamma = \Gamma \setminus E^{RBS}$ is a vector bundle on $M^{RBS}_\Gamma$. This is verified in the same manner as (1.8).

2. $L^p$-cohomology

By now, the notion of $L^p$-cohomology, with $1 \leq p \leq \infty$, is rather well-established. The case of $p = \infty$, though, is visibly different from the case of finite $p$, and was neglected in [Z4]. Morally, Theorem 1 is about $L^\infty$-cohomology, but for technical reasons we will have to settle for $L^p$-cohomology for large finite $p$. It is our first goal to prove Theorem 1.

(2.1) Preliminaries. Let $M$ be a $C^\infty$ Riemannian manifold. For any $C^\infty$ differential form $\phi$ on $M$, its length $|\phi|$ is a non-negative continuous function on $M$. This determines a semi-norm:

\[
\|\phi\|_p = \begin{cases} 
\left( \int_M |\phi(x)|^p dV_M(x) \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty; \\
\sup \{|\phi(x)| : x \in M\} & \text{if } p = \infty,
\end{cases}
\]

where $dV_M(x)$ denotes the Riemannian volume density of $M$. One says that $\phi$ is $L^p$ if $\|\phi\|_p$ is finite.

(2.1.2) Definitions. Let $w$ be a positive continuous real-valued function on the Riemannian manifold $M$.

i) The [smooth] $L^p$ de Rham complex with weight $w$ is the largest subcomplex of the $C^\infty$ de Rham complex of $M$ consisting of forms $\phi$ such that $w\phi$ is $L^p$, viz.

\[
A^\bullet(p; M; w) = \{ \phi \in A^\bullet(M) : w\phi \text{ and } w\alpha\phi \text{ are } L^p \}.
\]

ii) The [smooth] $L^p$-cohomology of $M$ with weight $w$ is the cohomology of $A^\bullet(p; M; w)$. It is denoted $H^\bullet(p; M; w)$.

We note that in the above, there is a difference with the notation used elsewhere: for $p \neq \infty$, $w$ might be replaced with $w^{\frac{1}{p'}}$ in (2.1.2.1).

When $w = 1$, one drops the symbol for the weight. Note that the complex depends on $w$ only through rates of the growth or decay of $w$ at infinity. When $M$ has finite volume, there are inclusions $A^\bullet(p'; M) \hookrightarrow A^\bullet(p; M)$ whenever $1 \leq p < p' \leq \infty$. The preceding extends to metrized local systems (cf. [Z1, §1]). Smooth functions are dense in the Banach space $L^p$ for $1 \leq p < \infty$, but not in $L^\infty$.

We next recall the basic properties of $L^p$-cohomology. Let $\overline{M}$ be a compact Hausdorff topological space that is a compactification of $M$. One defines a presheaf
on $\overline{M}$ by the following rule (cf. [Z4,1.9]): to any open subset $V$ of $\overline{M}$, one assigns $A_{(p)}^*(V \cap M; w)$. Because $\overline{M}$ is compact (see (2.1.5, ii) below), the associated sheaf $A_{(p)}^*(\overline{M}; w)$ satisfies

\begin{equation}
A_{(p)}^*(M; w) \sim \Gamma(\overline{M}, A_{(p)}^*(\overline{M}; w)).
\end{equation}

It follows from the definition that whenever $q : \overline{M} \to \overline{M}$ is a morphism of compactifications of $M$, one has for all $p$

\begin{equation}
q_\ast A_{(p)}^*(\overline{M}; w) \simeq A_{(p)}^*(\overline{M}; w).
\end{equation}

\begin{enumerate}
\item Remarks. i) It is easy to see that the complex $A_{(p)}^*(\overline{M}; w)$ consists of fine sheaves if and only if for every covering of $M^{BS}$ there is a partition of unity subordinate to that covering consisting of functions $f$ whose differential lies in $A^1_\infty(M)$, i.e., $|df|$ is a bounded function on $M$. Thus, (2.1.4) is for $q_\ast$ (as written), not for $Rq_\ast$ in general.

\item Note that in general, the space of global sections of $A_{(p)}^*(M; w)$, defined in the obvious way (or equivalently the restriction of $A_{(p)}^*(\overline{M}; w)$ to $M$) is $A_{(p), loc}^*(M; w) = A^*(M)$. Without a compact boundary, there is no place to store the global boundedness condition.
\end{enumerate}

The following fact makes for a convenient simplification:

\begin{proposition}
Let $M$ be the interior of a Riemannian manifold-with-corners $\overline{M}$ (i.e., the metric is locally extendable across the boundary). Let $\overline{A}_{(p)}^*(\overline{M}; w)$ be the sub-complex of $A_{(p)}^*(\overline{M}; w)$ consisting of forms that are also smooth at the boundary of $\overline{M}$. Then the inclusion

\[ \overline{A}_{(p)}^*(\overline{M}; w) \hookrightarrow A_{(p)}^*(\overline{M}; w) \]

is a quasi-isomorphism. \(\square\)

In other words, one can calculate $H_{(p)}^\bullet(M; w)$ using only forms with the nicest behavior along $\partial M$. Moreover, $\overline{A}_{(p)}^*(\overline{M}; w)$ admits a simpler description; for that and the proof of (2.1.6), see (2.3.7) and (2.3.9) below.

\begin{enumerate}
\item The prototype. We compute a simple case of $L^p$-cohomology, one that will be useful in the sequel.
\end{enumerate}

\begin{proposition}[Z4, 2.1]
Let $R^+$ denote the positive real numbers, and $t$ the linear coordinate from $\mathbb{R}$. For $a \in \mathbb{R}$, let $w_a(t) = e^{at}$. Then

\begin{enumerate}
\item $H_{(p)}^0(\mathbb{R}^+; w_a) \simeq \begin{cases} 0 & \text{if } a > 0, \\ \mathbb{C} & \text{if } a \leq 0. \end{cases}$
\item $H_{(p)}^1(\mathbb{R}^+; w_a) = 0$ for all $a \neq 0$.
\end{enumerate}

\begin{proof}
Again, we carry this out here only for $p = \infty$. First, (i) is obvious: it is just an issue of whether the constant functions satisfy the corresponding $L^\infty$ condition.
\end{proof}

To get started on (ii), proving that a complex is acyclic can be accomplished by finding a cochain homotopy operator $B$ (lowering degrees by one), such that $\phi = dB\phi + Bd\phi$. For the cases at hand (1-forms on $\mathbb{R}^+$), this equation reduces to $\phi = dB\phi$.

When $a < 0$, one takes

\begin{equation}
(2.2.2) \quad B(\phi)(t) = -\int_{t}^{\infty} g(x)dx
\end{equation}

when $\phi = g(t)dt$ (placing the basepoint at $\infty$ is legitimate, as $g$ decays exponentially). We need to check that (2.2.2) lies in the $L^\infty$ complex. By hypothesis,

$$|g(t)| \leq Cw_{-a}(t)$$

for some constant $C$. This implies that

$$|B(\phi)(t)| \leq \int_{t}^{\infty} |g(x)|dx \leq C \int_{t}^{\infty} w_{-a}(x)dx \sim w_{-a}(t)$$

as $t \to \infty$. In other words, $B(\phi)(t)w_{a}(t) \sim 1$, which is what we wanted to show.

When $a > 0$, one takes instead

\begin{equation}
(2.2.3) \quad B(\phi)(t) = \int_{1}^{t} g(x)dx,
\end{equation}

and shows that $|B(\phi)|(t) \sim w_{-a}(t)$, yielding the same conclusion about $B(\phi)$ as before. □

(2.3.4) Remark. One can see that for $a = 0$, one is talking about $H^1_{(p)}(\mathbb{R}^+)$, which is not even finite-dimensional (cf. [Z1, (2.40)]); $H^1_{(\infty)}(\mathbb{R}^+)$ contains the linearly independent cohomology classes of $t^{-\nu}dt$, for all $0 \leq \nu \leq 1$. What was essential in the proof of (2.2.1) was that $w_{a}$ and one of its anti-derivatives had equal rates of growth or decay when $a \neq 0$. That is, of course, false for $a = 0$.

(2.3) Further properties of $L^p$-cohomology. We begin with

(2.3.1) Proposition (A Künneth formula for $L^p$-cohomology). Let $I$ be the unit interval $[0, 1]$, with the usual metric. Then for any Riemannian manifold $N$ and weight $w$, the inclusion $\pi^* : \mathcal{A}_{(p)}^*(N; w) \hookrightarrow \mathcal{A}_{(p)}^*(I \times N; \pi^*w)$ is a quasi-isomorphism; thus

$$H_{(p)}^*(I \times N; \pi^*w) \simeq H_{(p)}^*(N; w).$$

Proof. The argument is fairly standard. The formula (2.2.3) defines an operator on forms on $I$. Because $I$ has finite length, one has now

\begin{equation}
(2.3.2) \quad \phi = H\phi + dB\phi + Bd\phi,
\end{equation}

where $H$ is—well—harmonic projection: zero on 1-forms, mean value on 0-forms. The differential forms on a product of two spaces decompose according to bidegree. On $I \times N$, denote the bidegree by $(e_I,e_N)$ (thus, for a non-zero form, $e_I \in \{0,1\}$).

The exterior derivative on $I \times N$ can be written as $d = d_I + \sigma d_N$, where $\sigma$ is given by $(-1)^{e_I}$. The operators in (2.3.2) make sense for $L^p$ forms on $I \times N$, taking, for each $q$, forms of bidegree $(1,q)$ to forms of bidegree $(0,q)$, and we write them with a subscript “$I$”; thus, we have the identity

\[(2.3.3)\quad \phi = H_I \phi + d_I B_I \phi + B_I d_I \phi.\]

It is clear that $B_I \phi$ is $L^p$ whenever $\phi$ is. Note that $\sigma_I$ anticommutes with $B_I$. We can therefore write (2.3.3) as

\[(2.3.4)\quad \phi = H_I \phi + dB_I \phi - \sigma_I d_N B_I \phi + B_I d\phi - B_I \sigma_I d_N \phi = (H_I \phi + dB_I \phi + B_I d\phi) - (\sigma_I d_N B_I \phi + B_I \sigma_I d_N \phi).\]

Since $\sigma_I$ and $d_N$ commute, the subtracted term equals $(\sigma_I B_I + B_I \sigma_I) d_N \phi = 0$, so (2.3.4) is just $\phi = (H_I \phi + dB_I \phi + B_I d\phi)$. This implies first that $dB_I \phi$ is $L^p$ and then our assertion. \qed

We next use a standard smoothing argument in a neighborhood of $0 \in \mathbb{R}$. To avoid unintended pathology, we consider only monotonic weight functions $w$. Given a smooth function $\psi$ on $\mathbb{R}$ of compact support, let

\[(2.3.5)\quad (\Psi f)(t) = (\psi * f)(t) = \int \psi(x)f(t-x)dx = \int \psi(t-x)f(x)dx,\]

defined for those $t$ for which the integral makes sense. The discussion separates into two cases:

i) $w(t)$ is a bounded non-decreasing function of $t$. In this case, take $\psi$ to be supported in $\mathbb{R}^-$.

ii) Likewise, when $w(t)$ blows up as $t \to 0^+$ take $\psi$ to be supported in $\mathbb{R}^+$, and set $f(x) = 0$ for $x \leq 0$.

**Lemma.** If $f \in L^p(\mathbb{R}^+,w)$ (and $\psi$ is chosen as above), then $\Psi f$ is also in $L^p(\mathbb{R}^+,w)$.

**Proof.** For $p < \infty$, see [Z4,1.5]. When $p = \infty$, we consider each of the above cases. In case (i), we have:

\[w(t)\Psi f(t) = \int \psi(t-x)w(t)f(x)dx = \int \psi(t-x)w(x)f(x)\{w(t)w(x)^{-1}\}dx.\]

By hypothesis, the integral involves only those $x$ for which $t < x$, and there $w(t)w(x)^{-1} \leq 1$. It follows that $w(t)\Psi f(t)$ is uniformly bounded. In case (ii), when $w(t)$ blows up as $t \to 0^+$ the argument is similar and is left to the reader. \qed

We use (2.3.6) to prove:
(2.3.7) Proposition. With \( w \) restricted as above, let \( \overline{A}_{(p)}^*(I;w) \) denote the subcomplex of \( A_{(p)}^*(I;w) \) consisting of forms that are smooth at 0. Then the inclusion

\[
\overline{A}_{(p)}^*(I;w) \hookrightarrow A_{(p)}^*(I;w)
\]

is a quasi-isomorphism, with \( \Psi \) providing a homotopy inverse.

Proof. There is a well-known homotopy smoothing formula, which is at bottom a variant of (2.3.2). We use the version given in [Z4,1.5], valid on the level of germs at 0:

\[
1 - \Psi = dE + Ed, \quad E = (1 - \Psi)B,
\]

with \( B \) as above. Our assertions follow immediately. \( \square \)

The behavior of \( w \) forces the value \( f(0) \) of a function \( f \in L^\infty(I;w) \cap \overline{A}(I) \) to be 0 precisely in case (ii) above. Thus we have:

(2.3.8) Corollary. Write \( \overline{I} \) for the closed interval \([0,1]\). For the two cases preceding (2.3.6),

\[
A_{(\infty)}^*(I;w) \approx \begin{cases} 
A^*(\overline{I}) \text{ in case (i)}, \\
A^*(\overline{I},0) \text{ in case (ii)}.
\end{cases}
\]

There are several standard consequences and variants of (2.3.7) in higher dimension. The simplest to state are (2.1.6) and its corollary; we now give the latter:

(2.3.9) Proposition. Let \( M \) be the interior of a Riemannian manifold-with-corners \( \overline{M} \), and let \( A_{(p)}^*(\overline{M}) \) be the subcomplex of \( A_{(p)}^*(M) \) consisting of forms that are smooth at the boundary. Then the inclusion

\[
A_{(p)}^*(\overline{M}) \hookrightarrow A_{(p)}^*(M)
\]

induces an isomorphism on cohomology. Thus the \( L^p \)-cohomology of \( M \) can be computed as the cohomology of \( A_{(p)}^*(\overline{M}) \), i.e., \( H^*_{(p)}(M) \simeq H^*(\overline{M}) \). \( \square \)

Finally, we will soon need the following generalization of (2.3.1):

(2.3.10) Proposition. Let \( w_M \) and \( w_N \) be positive functions on the Riemannian manifolds \( M \) and \( N \) respectively. Suppose that on the Riemannian product \( M \times N \), one has in the sense of operators on \( L^p \) that \( d = d_M \otimes 1_N + \sigma_M \otimes d_N \), and that \( H^*_{(p)}(N;w_N) \) is finite-dimensional. Then

\[
H^*_{(p)}(M \times N;w_M \times w_N) \simeq H^*_{(p)}(M;w_M) \otimes H^*_{(p)}(N;w_N).
\]

Remarks. i) The condition on \( M \times N \) is asserting that the forms on \( M \times N \) that have separate \( L^p \) exterior derivatives along \( M \) and along \( N \) are dense in the graph norm (cf. [Z1, pp.178–181] for some discussion of when this condition holds.)
When $p = 2$, the above proposition recovers only a special case of what is in [Z1, pp.180–181]; however, the full statement of the latter does generalize to all values of $p$, by a parallel argument.

**Proof of (2.3.10).** The argument is similar to what one finds in [Z1, §2], which is for the case $p = 2$, though we cannot use orthogonal projection here. Let $h^\bullet = h^\bullet_p(N; w_N)$ be any space of cohomology representatives for $H^\bullet_{(p)}(N; w_N)$; by hypothesis, $h^\bullet$ is a finite-dimensional Banach space. It suffices to show that the inclusion

\[(2.3.10.1) \quad A^\bullet_{(p)}(M; w_M) \otimes h^\bullet_p(N; w_N) \subseteq A^\bullet_{(p)}(M \times N; w_M \times w_N)\]

induces an isomorphism on cohomology.

For each $i$, let $Z^i$ denote the closed forms in $A^i = A^i_{(p)}(N; w_N)$. Then $D^i = dA^{i-1}$ is a complement to $h^i$ in $Z^i$; it is automatically closed because of the finite-dimensionality of $h^i$. By the Hausdorff maximal principle, there is a closed linear complement $C^i$ to $Z^i$ in $A^i$ (canonical complements exist when $p = 2$). Then the open mapping theorem of functional analysis (applied for the $L^1$ graph norm on $A^i$), gives that the direct sum of Banach spaces,

\[(2.3.10.2) \quad h^i \oplus D^i \oplus C^i,\]

is boundedly isomorphic to $A^i$. With respect to this decomposition of $A^i$, $d_N$ breaks into the 0-mapping on $Z^i$ and an isomorphism $d^i : C^i \to D^{i+1}$.

We can now obtain a cochain homotopy for $A^\bullet$. Let $B^i$ denote the inverse of $d^{i-1}$, and $B$ and $d$ the respective direct sums of these. One calculates that $dB + Bd$ is equal to $1 - q$, where $q$ denotes projection onto $h^\bullet$ with respect to (2.3.10.2).

Adapting this formula to $M \times N$ runs a standard course. First, $B$ defines an operator $B_N = 1_M \otimes B$ on $M \times N$, and likewise does $q$. We have the identity $1 - q_N = d_NB_N + BNd_N$. Noting that $d_N$ commutes with $\sigma_M$ and that $\sigma^2_M = 1_M$, we obtain

\[ (1 - q_N) = \sigmaMd_N(\sigma_MB_N) + (\sigma_MB_N)\sigmaMd_N, \]

and likewise $d_M(\sigma_MB_N) + (\sigma_MB_N)d_M = 0$. Adding, we get $1 - q_N = d\tilde{B} + \tilde{B}d$, with $\tilde{B} = \sigma_MB_N$, and this gives what we wanted to know about (2.3.10.1), so we are done. \qed

**3. $L^p$ cohomology on the reductive Borel-Serre compactification**

In this section, we determine the cohomology sheaves of $A^\bullet_{(p)}(M^{RBS})$ for large finite values of $p$, and compare the outcome to that of related calculations.

**3.1 Calculations for $M^{RBS}$, and the proof of Theorem 1.** We first observe that $A^\bullet_{(p)}(M^{RBS})$ is a complex of fine sheaves, for the criterion of (2.1.5,i) was verified
Let $y \in U_P \setminus e(P) \subset M^{RBS}$. The issue is local in nature, so it suffices to work with $\tilde{q} : M^{BS}_{U_P} \to X^{RBS}$, and therefore we lift $y$ to $\tilde{y} \in X^{RBS}$. The fiber $\tilde{q}^{-1}(\tilde{y})$ is the compact nilmanifold $N_P = \Gamma_{U_P} \setminus U_P$. Since $N_P$ is compact, neighborhoods of $\tilde{y}$ in $X^{RBS}$ give, via $\tilde{q}^{-1}$, a fundamental system of neighborhoods of $N_P$ in $M^{BS}_{U_P}$.

As in [Z1,(3.6)], the intersection with $M_{\Gamma_{U_P}}$ of such a neighborhood is of the form

$$\tag{3.1.1} \mu_P^+ \times V \times N_P,$$

where $\mu_P^+ \simeq (\mathbb{R}^+)^{\tau(P)}$ and $V$ is a coordinate cell on $\tilde{M}_P$ (notation as in (1.9)). After taking the exponential of the $\mu_P^+$-variable, the metric is given, up to quasi-isometry, as

$$\tag{3.1.2} \sum_i dt_i^2 + dv^2 + \sum_{\alpha} e^{-2\alpha} du_{\alpha},$$

where $\alpha$ runs over the roots in $\mu_P$. By the Künneth formula (2.3.1), we may replace $V$ by a point in (3.1.2); we are reduced to determining $H^{\bullet}_{(p)}(\mu_P^+ \times N_P)$, where the metric is $\sum_i dt_i^2 + \sum_{\alpha} e^{-2\alpha} du_{\alpha}$.

The means of computing this runs parallel to the discussion in [Z1,(4.20)]. We consider the inclusions of complexes

$$\tag{3.1.3} \bigoplus_\beta \left( A^{\bullet}_{(p)}(\mu_P^+; w_\beta) \otimes H^{\bullet}_{(p)}(u_P, \mathbb{C}) \right) \hookrightarrow \bigoplus_\beta \left( A^{\bullet}_{(p)}(\mu_P^+; w_\beta) \otimes \Lambda^{\bullet}_{\beta}(u_P)^* \right) \simeq A^{\bullet}_{(p)}(\mu_P^+ \times N_P)^{U_P} \hookrightarrow A^{\bullet}_{(p)}(\mu_P^+ \times N_P).$$

Here, $u_P$ denotes the Lie algebra of $U_P$, and

$$\tag{3.1.4} w_\beta(a) = a^{p_\beta} a^{-\delta} = a^{p_\beta - \delta} = a^{p(\beta - \frac{\delta}{p})} \quad (a_i = e^{t_i}),$$

where $\delta$ denotes the sum of the positive $\mathbb{Q}$-roots (cf. (3.1.9,ii) below). We can see that the contribution of $\delta$ (which enters because of the weighting of the volume form of $N_P$) is non-zero yet increasingly negligible as $p \to \infty$.

The second inclusion in (3.1.3) is that of the “$U_P$-invariant” forms. Note that this reduces considerations on $N_P$ to a finite-dimensional vector space, viz. $\Lambda^{\bullet}(u_P)^*$. Here, one is invoking the isomorphism

$$\tag{3.1.5} H^{\bullet}(N_P) \simeq H^{\bullet}(u_P, \mathbb{C})$$

for nilmanifolds, which is a theorem of Nomizu [N]. The exterior algebra decomposes into non-positive weight spaces for $a_P$, which we write as

$$\tag{3.1.6} \Lambda^{\bullet}(u_P)^* = \bigoplus_\beta \Lambda^{\bullet}_{\beta}(u_P)^*.$$

The first inclusion in (3.1.3) is given by Kostant’s embedding [K,(5.7.4)] of $H^{\bullet}(u_P, \mathbb{C})$ in $\Lambda^{\bullet}(u_P)^*$ as a set of cohomology representatives, and it respects $a_P$ weights. Our main $L^p$-cohomology computation is based on:
(3.1.7) Proposition. For all \( p \geq 1 \), the inclusions in (3.1.3) are quasi-isomorphisms.

Proof. This is asserted in [Z1,(4.23),(4.25)] for the case \( p = 2 \). The proof given there was presented with \( p = 2 \) in mind, though there is no special role of \( L^2 \) in it (cf. [Z3,(8.6)]). We point out that [Z1,(4.25)] is about the finite-dimensional linear algebra described above, and that the proof (4.23) of [Z1] goes through because the process of averaging a function over a circle (hence a nilmanifold, by iteration) is bounded in \( L^p \)-norm. As such, one sees rather easily that the proof carries over verbatim for general \( p \), and (3.1.7) is thereby proved. \( \square \)

Remark. We wish to point out and rectify a small mistake in the argument in [Z1,§4], one that “corrects itself”. It is asserted that the second terms in (4.37) and (4.41) there vanish by \( U_{j-1} \)-invariance. This is false in general. However, the two expressions actually differ only by a sign, and they cancel, yielding the conclusion of (4.41).

We next show how (3.1.7) yields the determination of \( H^\bullet_\text{p}(A^+_P \times N_P) \). We may use the first complex in (3.1.3) for this purpose. The weights in (3.1.6) are non-positive, and (3.1.4) shows that once \( p \) is sufficiently large, \( w_\beta \) blows up exponentially in some direction whenever \( \beta \neq 0 \), and decays exponentially when \( \beta = 0 \). Applying (2.2.1) and the K"unneth theorem, we obtain:

(3.1.8) Corollary. For sufficiently large \( p < \infty \), \( H^\bullet_\text{p}(A^+_P \times N_P) \simeq H^0(u_P, \mathbb{C}) \simeq \mathbb{C} \).

(3.1.9) Remark. i) We can specify what “sufficiently large” means, using (3.1.4). Write \( \delta \) as a (non-negative) linear combination of the simple \( \mathbb{Q} \)-roots: \( \delta = \sum \beta \alpha_\beta^\beta \). Then we mean to take \( p > \max \{ c_\beta \} \).

ii) When \( p = \infty \), one runs into trouble with the infinite-dimensionality of the unweighted \( H^1_\infty(\mathbb{R}^+) \) (see (2.2.4)). By using instead large finite \( p \), we effect a perturbation away from the trivial weight, thereby circumventing the problem.

There is a straightforward globalization of (3.1.8), which we now state:

(3.1.10) Theorem. For sufficiently large \( p \), the inclusion

\[
\mathbb{C}_{M^{RBS}} \to q_* A^\bullet_\text{p}(M^{BS}) \simeq A^\bullet_\text{p}(M^{RBS})
\]

is a quasi-isomorphism. \( \square \)

From this follows Theorem 1:

(3.1.11) Corollary. For sufficiently large finite \( p \), \( H^\bullet_\text{p}(M) \simeq H^\bullet(M^{RBS}) \).

(3.2) An example (with enhancement). Take first \( G = SL(2) \). Then \( M \) is a modular curve. There are only two distinct interesting compactifications (those in (0.2)): one is \( M^{BS} \), and the other is \( M^{RBS} \) (which is homeomorphic to \( M^{BB} \) and \( M_\Sigma \)). A deleted neighborhood of a boundary point (cusp) of \( M^{BB} \) is a Poincaré
punctured disc \( \Delta_R^* = \{ z \in \mathbb{C} : 0 < |z| < R \} \), with \( R < 1 \), with metric given in polar coordinates by \( ds^2 = (r |\log r|)^{-2}(dr^2 + (rd\theta)^2) \). Because \( R < 1 \), the metric is smooth along the boundary circle \( |z| = R \). Setting \( u = \log |\log r| \) converts the metric to \( ds^2 = du^2 + e^{-2u}d\theta^2 \) (recall (3.1.3)). One obtains from (2.3.1) and (3.1.7):

(3.2.1) Proposition. Write \( \Delta_R^* \) for \( \Delta_R^* \cup \{0\} \). Then for the Poincaré metric on \( \Delta_R^* \)

\[
H^\bullet_{(p)}(\Delta_R^*) \simeq H^\bullet(\Delta_R) \simeq \mathbb{C} \quad \text{whenever } 1 < p < \infty. \quad \square
\]

(3.2.2) Remark. When \( p = 1 \), \( H^2_{(1)}(\Delta_R^*) \) is infinite-dimensional, as is \( H^1_{(\infty)}(\Delta_R^*) \); this follows from (2.2.4) and (3.1.7). By using a Mayer-Vietoris argument, in the same manner as [Z1, §5], we get that when \( M \) is a modular curve, we see that \( H^1_{(\infty)}(M) \) is likewise infinite-dimensional. Thus the assertion in (3.1.11) fails to hold for \( p = \infty \), already when \( G = \text{SL}(2) \).

Using the Künneth formula (2.3.10), it is easy to obtain the corresponding assertion for \( (\Delta_R^*)^n \):

(3.2.3) Corollary. For the Poincaré metric on \( (\Delta_R^*)^n \),

\[
H^\bullet_{(p)}((\Delta_R^*)^n) \simeq H^\bullet(\Delta_R^n) \simeq \mathbb{C} \quad \text{whenever } 1 < p < \infty. \quad \square
\]

Now, let \( M \) be an arbitrary locally symmetric variety. The smooth toroidal compactifications \( M_\Sigma \) are constructed so that they are complex manifolds and the boundary is a divisor with normal crossings on \( M_\Sigma \). The local pictures of \( M \hookrightarrow M_\Sigma \) are \( (\Delta)^k \times \Delta^{n-k} \hookrightarrow \Delta^n \), for \( 0 \leq k \leq n \). The invariant metric of \( M \) is usually not Poincaré in these coordinates, not even asymptotically. However, it is easy to construct other metrics which are. We will use a subscript “P” to indicate that one is using such a metric instead of the invariant one. We note that such a metric depends on the choice of toroidal compactification. The global version of (3.2.3) follows by standard sheaf theory:

(3.2.4) Proposition. For a metric on \( M \) that is Poincaré with respect to \( M_\Sigma \),

\[
H^\bullet_{(p),P}(M) \simeq H^\bullet(M_\Sigma) \quad \text{whenever } 1 < p < \infty. \quad \square
\]

The above proposition actually gives a reinterpretation of the method in [Mu]. There, Mumford decided to work in the rather large complex of currents that also gives the cohomology of \( M_\Sigma \). However, he shows that the connection and Chern forms involved are “of Poincaré growth”, and that is equivalent to saying that they are \( L^\infty \) with respect to any metric that is asymptotically Poincaré near the boundary of \( M_\Sigma \). One thereby sees that his argument for comparing Chern forms ([Mu, p.243], based on (4.3.4) below) in the complex of currents actually takes place in the subcomplex of Poincaré \( L^\infty \) forms on \( M \).
Remark. For a convenient exposition of the growth estimates in the latter, see [HZ2,(2.6)]. Since there is in general no morphism of compactifications between $M^{RBS}$ and $M_{\Sigma}$, the reader is warned that the comparison of their boundaries is a bit tricky (see [HZ1,(1.5),(2.7)] and [HZ2,(2.5)]).

(3.3) On defining morphisms via $L^p$-cohomology. We give next an interesting consequence of Theorem 1. The space $M$ has finite volume, so there is a canonical morphism (see (2.1))

\[(3.3.1) \quad H^\bullet_{(p)}(M) \to H^\bullet_{(2)}(M)\]

whenever $p > 2$. For $p$ sufficiently large, the left-hand side of (3.3.1) is naturally isomorphic to $H^\bullet(M^{RBS})$. For $p = 2$, there is an analogous assertion: by the Zucker conjecture, proved in [L] and [SS] (see [Z2]), the right-hand side is naturally isomorphic to $IH^\bullet_\mathfrak{m}(M^{BB})$, intersection cohomology with middle perversity $\mathfrak{m}$ of $[GM1]$. These facts transform (3.3.1) into the diagram

\[(3.3.2) \quad \begin{array}{ccc}
H^\bullet(M^{RBS}) & \to & IH^\bullet_\mathfrak{m}(M^{BB}) \\
\uparrow & & \downarrow \approx \\
H^\bullet(M^{BB}) & \to & IH^\bullet_\mathfrak{m}(M^{BB})
\end{array}\]

In other words,

(3.3.3) Proposition. The mapping in (3.3.1) defines a factorization of the canonical mapping

\[H^\bullet(M^{BB}) \to IH^\bullet_\mathfrak{m}(M^{BB})\]

through $H^\bullet(M^{RBS})$.

A related assertion had been conjectured by Goresky-MacPherson and Rapoport, and was proved recently by Saper:

(3.3.4) Proposition. Let $h : M^{RBS} \to M^{BB}$ be the canonical quotient mapping. Then there is a quasi-isomorphism

\[Rh_\ast IC^\bullet_\mathfrak{m}(M^{RBS},\mathbb{Q}) \approx IC^\bullet_\mathfrak{m}(M^{BB},\mathbb{Q}),\]

where $IC$ denotes sheaves of intersection cochains.

This globalizes to an isomorphism $IH^\bullet_\mathfrak{m}(M^{RBS},\mathbb{Q}) \cong IH^\bullet_\mathfrak{m}(M^{BB},\mathbb{Q})$, which underlies (3.3.3), enlarging the triangle into a commutative square defined over $\mathbb{Q}$:

\[
\begin{array}{ccc}
H^\bullet(M^{RBS}) & \to & IH^\bullet_\mathfrak{m}(M^{RBS}) \\
\uparrow & & \downarrow \cong \\
H^\bullet(M^{BB}) & \to & IH^\bullet_\mathfrak{m}(M^{BB})
\end{array}\]

4. Chern forms for vector bundles on stratified spaces

In this section, we will treat the de Rham theory for stratified spaces that will be needed for the proof of Theorem 2. We also develop the associated treatment of Chern classes for vector bundles.

(4.1) **Differential forms on stratified spaces.** Let \( Y \) be a paracompact space with an abstract prestratification (in the sense of Mather) by \( C^\infty \) manifolds. Let \( S \) denote the set of strata of \( Y \). If \( S \) and \( T \) are strata, one writes \( T \prec S \) whenever \( T \neq S \) and \( T \) lies in the closure \( \overline{S} \) of \( S \).

The notion of a prestratification specifies a system \( C \) of Thom-Mather control data (see [GM2, p. 42], [V1],[V2]), and that entails the following. For each stratum \( S \) of \( Y \), there is a neighborhood \( N_S \) of \( S \) in \( Y \), a retraction \( \pi_S : N_S \to S \), and a continuous “distance function” \( \rho_S : N_S \to [0, \infty) \) such that \( \rho_S^{-1}(0) = S \), subject to:

(4.1.1) **Conditions.** Whenever \( T \preceq S \), put \( N_{T,S} = N_T \cap S \), \( \pi_{T,S} = \pi_T|_{N_{T,S}} \), and \( \rho_{T,S} = \rho_T|_{N_{T,S}} \). Then:

i) \( \pi_T(y) = \pi_{T,S}(\pi_S(y)) \) whenever both sides are defined, viz., for \( y \in N_T \cap (\pi_S)^{-1}N_{T,S} \); likewise \( \rho_T(y) = \rho_{T,S}(\pi_S(y)) \).

ii) The restricted mapping \( \pi_{T,S} \times \rho_{T,S} : N_{T,S}^0 \to T \times \mathbb{R}^+ \), where \( N_{T,S}^0 = N_{T,S} - T \), is a \( C^\infty \) submersion.

(The above conditions will be relaxed after (4.1.4) below.) A prestratified space is, thus, the triple \((Y,S,C)\).

Let \( Y^0 \) denote the open stratum of \( Y \). One understands that when \( S = Y^0 \), one has \( N_S = Y^0 \), \( \pi_S = 1_{Y^0} \) and \( \rho_S \equiv 0 \). From (4.1.1), it follows that for all \( S \in S \), \( \pi_{T,S}|_{N_{T,S}} \) is a submersion; moreover, the closure \( \overline{S} \) of \( S \) in \( Y \) is stratified by \( \{T \in S : T \preceq S \} \), and \( C_S = \{(\pi_{T,S}, \rho_{T,S}) : T \prec S \} \) is a system of control data for \( S \).

We also recall the following (see [V2, Def. 1.4]):

(4.1.2) **Definition.** A controlled mapping of prestratified spaces, \( f : (Y,S,C) \to (Y',S',C') \), is a continuous mapping \( f : Y \to Y' \) satisfying:

i) If \( S \in S \), there is \( S' \in S' \) such that \( f(S) \subseteq S' \), and moreover, \( f|_S \) is a smooth mapping of manifolds.

ii) For \( S \) and \( S' \) as above, \( f \circ \pi_S = \pi_{S'} \circ f \) in a neighborhood of \( S \).

iii) For \( S \) and \( S' \) as above, \( \rho_{S'} \circ f = \rho_S \) in a neighborhood of \( S \).

Let \( j : Y^0 \hookrightarrow Y \) denote the inclusion. A subsheaf \( A_{Y,C}^\bullet \) of \( j_* A_{Y^0}^\bullet \), the complex of \( C \)-controlled \( C \)-valued differential forms on \( Y \), is the sheafification of the following presheaf: for \( V \) open in \( Y \), put

\[
A_{Y,C}^\bullet(V) = \{ \varphi \in A^\bullet(V \cap Y^0) : \varphi|_{N_S \cap V \cap Y^0} \in \text{im} \pi_S^* \text{ for all } S \in S \}.
\]

(4.1.4) **Remark.** From (4.1.1, i), one concludes that the condition in (4.1.3) for \( T \) implies the same for \( S \) whenever \( S \succ T \), as \((\pi_T)^* \varphi = (\pi_S)^* (\pi_{T,S})^* \varphi \).
We observe that the definition of $A^\bullet_{Y,C}$ is independent of the distance functions $\rho_S$. Indeed, all that we will need from the control data for most purposes is the collection of germs of $\pi_S$ along $S$. We term this *weak control data* (these are the equivalence classes implicit in [V2, Def. 1.3]). In this spirit, one has the notion of a *weakly controlled mapping*, obtained from (4.1.2) by discarding item (iii); cf. (5.2.2).

The main role that $\rho_S$ plays here is to specify a model for the link of $S$:

$$L_S = \pi_S^{-1}(s_0) \cap \rho_S^{-1}(\varepsilon)$$

for any $s_0 \in V_S$ and sufficiently small $\varepsilon > 0$, but the link is also independent of $C$; besides, we will not need that notion in this paper.

The following is well-known:

**Lemma.** Let $Y$ be a space with prestratification. For any open covering $\mathcal{U}$ of $Y$, there is a partition of unity $\{f_V : V \in \mathcal{U}\}$ subordinate to $\mathcal{U}$ that consists of $C$-controlled functions. $\square$

This is used in [V1, p. 887] to prove the stratified version of the de Rham theorem:

**Proposition.** Let $C$ be a system of (weak) control data on $Y$. Then the complex $A^\bullet_{Y,C}$ is a fine resolution of the constant sheaf $\mathcal{C}_Y$. $\square$

**Corollary.** A closed $C$-controlled differential form on $Y$ determines an element of $H^\bullet(Y)$. $\square$

(4.2) **Controlled vector bundles.** We start with a basic notion.

**Definition.** A $C$-controlled vector bundle on $Y$ is a topological vector bundle $E$, given with local trivializations for all $V$ in some open covering $\mathcal{U}$ of $Y$, such that the entries of the transition matrices are $C$-controlled.

It follows from the definition that a $C$-controlled vector bundle determines a Cech 1-cocycle for $\mathcal{U}$ with coefficients in $GL(r, A^0_{Y,C})$. It thereby yields a cohomology class in $H^1(Y, GL(r, A^0_{Y,C}))$. The latter has a natural interpretation:

**Proposition.** The set $H^1(Y, GL(r, A^0_{Y,C}))$ is in canonical one-to-one correspondence with the set of isomorphism classes of vector bundles $E$ of rank $r$ on $Y$ with $E_S = E|_S$ smooth for all $S \in \mathcal{S}$, together with a system $\{\phi_S : S \in \mathcal{S}\}$, $\{\phi_{T,S} : S,T \in \mathcal{S}\}$ of germs of isomorphisms of vector bundles (total spaces) along each $T \in \mathcal{S}$:

1) $\phi_T : (\pi_T)^* E_T = E_T \times_T N_T \Rightarrow E|_{N_T}$,

2) $\phi_{T,S} : (\pi_{T,S})^* E_T = E_T \times_T N_{T,S} \Rightarrow E|_{N_{T,S}}$

whenever $T \prec S$, satisfying the compatibility conditions $\phi_T = \phi_S \circ \phi_{T,S}$.

**Remark.** Condition (ii) above is, of course, the restriction of (i) along $S$.

If we use $\{E_T : T \in \mathcal{S}\}$, the stratification of $E$ induced by $\mathcal{S}$, then the natural projection $E_T \times_T N_T \rightarrow E_T$ gives weak control data for $E$. Thus we obtain from (4.2.2):
(4.2.3) Corollary. A vector bundle $E$ on $Y$ is $C$-controlled (as in (4.2.1)) if and only if $E$ admits weak control data such that the bundle projection $E \to Y$ is a weakly controlled mapping (as in (4.1)). □

Proof of (4.2.2). Let $\xi$ be a 1-cocycle for the open covering $\mathcal{V}$ of $Y$, with coefficients in $GL(r, \mathcal{A}_{Y,C}^0)$. Since the functions in $\mathcal{A}_{Y,C}^0$ are continuous, $\xi$ determines a vector bundle of rank $r$ in the usual way; putting $E_0$ for $\mathbb{C}^r$, one takes

$$E = E_\xi = \bigsqcup \{(E_0 \times V_\alpha) : V_\alpha \in \mathcal{V}\}$$

modulo the identifications on $V_{\alpha\beta} = V_\alpha \cap V_\beta$:

$$
\begin{align*}
E_0 \times V_{\alpha\beta} & \quad \hookrightarrow \quad E_0 \times V_\alpha \\
1 \times \xi_{\alpha\beta} & \quad \downarrow \\
E_0 \times V_{\alpha\beta} & \quad \hookrightarrow \quad E_0 \times V_\beta
\end{align*}
$$

(4.2.2.3)

It is a tautology that there exist isomorphisms (4.2.2.1) locally on the respective bases ($Y^o$ or $S$), but we want it to be specified globally.

Next, let

$$
(4.2.2.4) \quad \mathcal{V}_S = \{V \in \mathcal{V} : V \cap S \neq \emptyset\}, \quad \mathcal{V}(S) = \{V \cap S : V \in \mathcal{V}_S\}.
$$

Then $\mathcal{V}(S)$ is an open cover of $S$. By refining $\mathcal{V}$, we may assume without loss of generality that $\xi_{\alpha\beta} \in \text{im}(\pi_S)^*$ on $V_{\alpha\beta} \cap N_S$ whenever $V_\alpha, V_\beta \in \mathcal{V}_S$, and write $\xi_{\alpha\beta} = (\pi_S)^* \xi^S_{\alpha\beta}$. The bundle $E_S = E|_S$ is constructed from the 1-cocycle $\xi^S$. Let

$$N'_S = N_S \cap \bigcup \{V : V \in \mathcal{V}_S\}.$$

The relation $\xi = (\pi_S)^* \xi^S$ on $N'_S$ determines a canonical isomorphism $\phi_S : E|_{N'_S} \overset{\sim}{\longrightarrow} (\pi_S)^* E_S$, for the local ones patch together; it is smooth on each stratum $R \rhd S$. One produces $\phi_{S,T}$ by doing the above for the restriction of $E$ to $\mathfrak{S}$, along its stratum $T$.

The consistency condition, $\phi_T = \phi_S \circ \phi_{T,S}$ whenever $T \rhd S$, holds because of (4.1.4). Replacing $\mathcal{V}$ by any refinement of it, only serves to make $N'_S$ smaller, so the germs of the pullback relations do not change. Also, we must check that the isomorphisms above remain unchanged when we replace $\xi$ by an equivalent cocycle. Let $\xi'_{\alpha\beta} = \psi_{\beta} \psi_{\alpha\beta} \psi_{\alpha}^{-1}$, where $\psi$ is a 0-cochain for $\mathcal{V}$ with coefficients in $GL(r, \mathcal{A}_{Y,C}^0)$. Without loss of generality again, we assume that $\psi$ is of the form $(\pi_S)^* \psi^S$ on $N'_S$. The isomorphism $E(\xi'_S) \simeq E(\xi_S)$ induced by $\psi^S$ then pulls back to the same for the restrictions of $E(\xi')$ and $E(\xi)$ to $N'_S$, respecting the compatibilities.

Thus, we have constructed a well-defined mapping from $H^1(Y, GL(r, \mathcal{A}_{Y,C}^0))$ to isomorphism classes of bundles on $Y$ with pullback data along the strata. We wish to show that it is a bijection.

Actually, we can invert the above construction explicitly. Given $E$, $\phi_T$, etc., as in (4.2.2.1), let, for each $T \in \mathcal{S}$, $\mathcal{V}_T$ be a covering of $T$ that gives a 1-cocycle $\xi^T$ for
$E_T$ (as a smooth vector bundle on $T$); $N'_T$, a neighborhood of $T$, contained in $N_T$, on which the isomorphisms $\phi_T$ and $\phi_{T,S}$ (for all $S \succ T$) are defined; $\Psi(T) = \pi_T^{-1} N_T$ the corresponding covering of $N'_T$, on which $(\pi_T)^* \xi_T$ is a cocycle giving $E|_{N'_T}$. Then

$$\Psi = \bigcup \{ \Psi_T : T \in S \}$$

is a covering of $Y$, such that for all $V \in \Psi$, $E|_V$ has been trivialized.

We claim that the 1-cocycle for $E$, with respect to these trivializations, has coefficients in $GL(r, A^0_Y, \mathbb{C})$. For $V_\alpha$ and $V_\beta$ in the same $\Psi_T$, we have seen already that $\xi_{\alpha\beta}$ is in $im(\pi_T)^*$. Suppose, then, that $T \prec S$, and that $V_\alpha \in \Psi_T$ and $V_\beta \in \Psi_S$ have non-empty intersection. Then $\xi_{\alpha\beta}$ is actually in $im(\pi_S)^*$, which one sees is a consequence of the compatibility conditions for (4.2.2.1), and our claim is verified.

That we have described the inverse construction is easy to verify. □

(4.3) Controlled connections on vector bundles. When we speak of a connection on a smooth vector bundle over a manifold, and write the symbol $\nabla$ for it, we mean foremost the covariant derivative. Then, the difference of two connections is a 0-th order operator, given by the difference of their connection matrices with respect to any one frame.

We can define the notion of a connection on a $\mathcal{C}$-controlled vector bundle:

(4.3.1) Definition. Let $E$ be a $\mathcal{C}$-controlled vector bundle on $Y$. A $\mathcal{C}$-controlled connection on $E$ is a connection $\nabla$ on $E|_{Y^\circ}$ for which there is a covering $\Psi$ of $Y$ such that for each $V \in \Psi$, there is a frame of $E|_V$ such that the connection forms lie in $A^1_Y \otimes \operatorname{End}(E)$.

Remark [added]. It is more graceful to define a controlled connection so as to be in accordance with (4.2.2.1): it is a system of connections $\{(E_T, \nabla_T)\}$, with germs of isomorphisms

$$(\nabla_S)|_{N_T,S} = (\pi_T,S)^* \nabla_T \quad \text{whenever} \ T \prec S.$$ 

One sees that (4.1.1) and (4.3.1) imply that a $\mathcal{C}$-controlled connection on $E$ defines a usual connection on $E|_S$ for every $S \in \mathcal{S}$. The next observation is evident from the definition:

(4.3.2) Lemma. The curvature form $\Theta \in j_*(A^2_Y \otimes \operatorname{End}(E|_{Y^\circ}))$ of a $\mathcal{C}$-controlled connection $\nabla$ lies in $A^2_Y \otimes \operatorname{End}(E)$. □

It is also obvious that $A^*_Y \otimes \operatorname{End}(E)$ is closed under exterior multiplication. One can thus define for each $k$ the Chern form $c_k(E, \nabla)$, a closed $\mathcal{C}$-controlled $2k$-form on $Y$, by the usual formula:

$$c_k(E, \nabla) = P_k(\Theta, \ldots, \Theta),$$

where $P_k$ is the appropriate invariant polynomial of degree $k$. By (4.1.8), $c_k(E, \nabla)$ defines a cohomology class in $H^{2k}(Y)$. 

(4.3.3) Proposition. i) Every C-controlled vector bundle $E$ on $Y$ admits a C-controlled connection.

ii) The cohomology class of $c_k(E, \nabla)$ in $H^{2k}(Y)$ is independent of the C-controlled connection $\nabla$ on $E$.

Proof. Let $\{\phi_T, \phi_{T:S} : T \prec S\}$ be the data defining a C-controlled vector bundle, as in (4.2.2.1), and $N_T' \subset N_T$ a domain for the isomorphisms involving $E_T$. For each $T$, let $\nabla^T$ be any smooth connection on $E_T$, and $(\pi_T)^* \nabla^T$ the pullback connection on $E|_{N_T'}$. Then $\mathcal{U} = \{N_T' : T \in \mathcal{S}\}$ is an open covering of $Y$. Apply (4.1.6) to get a C-controlled partition of unity $\{f_T\}$ subordinate to $\mathcal{U}$. Then $\nabla = \sum_V f_V \nabla^V$ is a C-controlled connection on $E$. This proves (i).

The argument for proving (ii) is the standard one. For two connections on a smooth manifold, such as $Y^\circ$, there is an identity:

\begin{equation}
(4.3.4) \quad c_k(E, \nabla_1) - c_k(E, \nabla_0) = d\eta_k,
\end{equation}

where

\begin{equation}
(4.3.4.1) \quad \eta_k = k \int_0^1 P_k(\omega_1, \Theta_t, \ldots, \Theta_t) dt,
\end{equation}

$\omega = \nabla_1 - \nabla_0$, $\nabla_t = (1-t)\nabla_0 + t\nabla_1$, and $\Theta_t$ denotes the curvature of $\nabla_t$. Now, if $\nabla_0$ and $\nabla_1$ are both C-controlled, one sees easily that $\omega$ and $\nabla_t$ are likewise, and then so is $\eta_k$. It follows that (4.3.4) is an identity in $A^*_{Y, C}$, giving (ii).

We have been leading up to the following:

(4.3.5) Theorem. Let $E$ be a C-controlled vector bundle on the stratified space $Y$. Then the cohomology class in (4.3.3, ii) gives the topological Chern class of $E$ in $H^{2k}(Y)$; in particular, it is independent of the choice of $C$.

Proof. This argument, too, follows standard lines. We start by proving the assertion when $E$ is a line bundle $L$. On $Y$, there is the short exact exponential sequence (of sheaves):

\begin{equation}
(4.3.5.1) \quad 0 \to \mathbb{Z}_Y \to A^0_{Y,C} \to (A^0_{Y,C})^* \to 1.
\end{equation}

The Chern class of $L$, $c_1(L)$, is then the image of any controlled Cech cocycle that determines $L$, under the connecting homomorphism

\begin{equation}
(4.3.5.2) \quad H^1(Y, (A^0_Y)^*) \to H^2(Y, \mathbb{Z}).
\end{equation}

To prove the theorem for line bundles, it is convenient to work in the double complex $\mathfrak{c}^*(A^*_{Y,C})$, where $\mathfrak{c}^*$ denotes Cech cochains. It has differential $D = \delta + \sigma d$ (i.e., Cech differential plus a sign $\sigma = (-1)^a$ times exterior derivative, where $a$ is the Cech degree). On a sufficiently fine covering of $Y$ we have a cochain giving $L$,
Construction, we have a retraction induced by the weak control data for $E$.

Denote the restriction of $p$ to $(E/F)^*$ by $\pi : (E/F)^* → (E/F)^*$. Then, we have the change-of-frame formula for connections gives $d\omega + d\lambda = 0$. Finally, the curvature (for a line bundle) is $\Theta = d\omega$, so we wish to show that $d\omega$ and $\delta\lambda$ are cohomologous in the double complex. By definition, $D\lambda = d\lambda - d\lambda$, and $D\omega = d\omega + d\omega = d\omega - d\lambda$. This gives $\delta\lambda - d\omega = D(\lambda - \omega)$, and we are done.

To get at higher-rank bundles, we invoke a version of the splitting principle. Let $p : \mathbf{F}(E) → Y$ be the bundle of total flags for $E$. As $E$ is locally the product of $Y$ and a vector space, $\mathbf{F}(E)$ is locally on $Y$ just $F_r × Y$, where $F_r$ is a (smooth compact) flag manifold. As such, $\mathbf{F}(E)$ is a stratified space that is locally no more complicated than $Y$; we take as the set of strata $\tilde{S} = p^{-1}(S) = \{p^{-1}S : S ∈ \mathcal{S}\}$. For weak control data, we deduce it from $C$ in the same way it is done for $E$ (see (4.2.2.2)); we take $N_{F_T} = \mathbf{F}(E|N_T)$, and use the natural projection $\mathbf{F}(E|N_T) → \mathbf{F}(E_T)$ induced by (4.2.2.1).

It is standard that the vector bundle $p^*E$ on $\mathbf{F}(E)$ decomposes (non-canonically) into a direct sum of line bundles: $p^*E = \bigoplus_{1≤j≤r} \Lambda_j$. (4.2.2.1) is canonically filtered: $\Lambda_1 = F_1 ⊂ F_2 ... F_r = p^*E$, with $\Lambda_j ≃ F_j/F_{j-1}$.) To obtain this, one starts by taking $\Lambda_1$ to be the line bundle given at each point of $\mathbf{F}(E)$ by the one-dimensional subspace from the corresponding flag. Then, one splits the exact sequence

(4.3.5.3) $0 → \Lambda_1 → p^*E → p^*E/\Lambda_1 → 0$

using a controlled metric on $E$. By that, we mean a metric that is a pullback via the isomorphisms (4.2.2.1); these can be constructed by the usual patching argument, using controlled partitions of unity (4.1.6). One obtains $\Lambda_j$, for $j > 1$, by recursion. We need a little more than that:

(4.3.6) Lemma. i) The vector bundle $p^*E$ is, in a tautological way, a controlled vector bundle on $\mathbf{F}(E)$.

ii) The line bundles $\Lambda_j$ are controlled subbundles of $p^*E$.

Proof. We have that $p^*E = E × Y \mathbf{F}$, and its strata are $(p^*E)_{F_T} = E_T × F_T$, for all $T ∈ \mathcal{S}$. There is natural weak control data for $p^*E$ that we now specify. By construction, we have a retraction

(4.3.6.1) $\pi_{(p^*E)} : (p^*E)|_{N_{F_T}} = (p^*E)|_{F|N_T} ≃ E|_{N_T} × (E|_{N_T}) → E_T × F_T = (p^*E)_{F_T}$,

induced by the weak control data for $E$ (and thus also $\mathbf{F}$), and likewise for the restriction to $(p^*E)|_{N_{F_T,F_S}}$, when $S ⋵ T$. These provide $\phi_{F_T}$ and $\phi_{F_T,F_S}$ (from (4.2.2.1)) respectively for $p^*E$, and (i) is proved.

We show that $\Lambda_1$ is preserved by $\phi_{F_T}$ and $\phi_{F_T,F_S}$. (As before, we explain this only for the former, the other being its restriction to the strata.) Let $p_T : F_T → T$ denote the restriction of $p$ to $F_T$. Since $p_T$ gives the flag manifold bundle associated
to $E_T$, $(p^*_T E_T)$ contains a tautological line bundle, which we call $\Lambda_{1,T}$. We have from the control data that $(\Lambda_1)|_{N_{p_T}} \simeq (\phi_{p_T})^*\Lambda_{1,T}$.

We claim further that (4.3.6.1) takes $\Lambda_1|_{N_{p_T}}$ to $\Lambda_{1,T}$, as desired. The explicit formula for (4.3.6.1), obtained by unwinding the fiber products, is as follows. Let $e$ be in the vector space $E_T,t$, the fiber of $E_T$ over $t \in T$, and $f$ a point the flag manifold of $E_T,t$. Also, let $n \in (\pi_T)^{-1}(t)$. Then $\pi(p^*_{E_T})(e,f,n) = (e,f)$, which implies (ii) for $j = 1$. The assertion for $j > 1$ is obtained recursively. □

We return to the proof of (4.3.5). Let $\nabla_0$ be the direct sum of $\tilde{\mathcal{C}}$-controlled connections on each $\Lambda_j$; and take $\nabla_1 = p^*\nabla$, where $\nabla$ is a $\mathcal{C}$-controlled connection on $E$. Both $\nabla_0$ and $\nabla_1$ are $\tilde{\mathcal{C}}$-controlled connections on $F(E)$. By construction, $c_k(p^*E, \nabla_0)$ represents the $k$-th Chern class of $p^*E$ in $H^{2k}(F(E))$. We then apply (4.3.3, ii) to obtain that $c_k(p^*E, \nabla_1) = p^*c_k(E, \nabla)$ represents $p^*c_k(E) \in H^{2k}(F(E))$. Since $p^*: H^{2k}(Y) \rightarrow H^{2k}(F(E))$ is injective, it follows that $c_k(E, \nabla)$ represents $c_k(E)$ in $H^{2k}(Y)$, and (4.3.5) is proved. □

5. Proofs of Theorem 2 and Conjecture B

In this section, we apply the methods of §4 in the case $Y = M_\Gamma^{RBS}$.

(5.1) Control data for a manifold-with-corners. Let $Y$ be a manifold-with-corners, with its open faces as strata. For each codimension one boundary stratum $S$, let

$$\overline{\phi}_S : [0, 1] \times S \rightarrow Y$$

define the collar $\overline{N}_S$ of $S$ in $Y$, so that $\{0\} \times S$ is mapped identically onto $S$. This determines partial control data (that is, without distance functions) for $Y$ as follows.

As $N_S$, one takes $\overline{\phi}([0, 1) \times S)$, and as $\pi_S$ projection onto $S$. For a general boundary stratum $T$, write

$$T = \bigcap\{S : S \text{ of codimension one, } T \prec S\}.$$ 

Let $\overline{N}_T = \bigcap\{\overline{N}_S : S \text{ of codimension one, } T \prec S\}$; given the $\overline{\phi}_S$’s above, this set is canonically diffeomorphic to $[0, 1)^r \times T$, where $r$ is the codimension of $T$. Then $N_T$ is the subset of $\overline{N}_T$ corresponding to $[0, 1)^r \times T$, in which terms $\pi_T$ is simply projection onto $T$.

(5.2) Compatible control data. The existence of natural (partial) control data for $M_\Gamma^{RBS}$ is, in essence, well-known, as is compatible control data for $M_\Gamma^{BB}$ in the Hermitian case. We give a brief presentation of that here. This will enable us to determine that Conjecture B is true.

The relevant notions are variants of (4.1.2).
(5.2.1) Definition (see [GM2, 1.6]). Let $Y$ and $Y'$ be stratified spaces. A proper smooth mapping $f : Y \to Y'$ is said to be stratified when the following two conditions are satisfied:

i) If $S'$ is a stratum of $Y'$, then $f^{-1}(S')$ is a union of connected components of strata of $Y$;

ii) Let $T \subset Y$ be a stratum component as in (i) above. Then $f|_T : T \to S'$ is a submersion.

It follows that a stratified mapping $f$ is, in particular, open. We assume henceforth, and without loss of generality, that all strata are connected.

(5.2.2) Definition (cf. [V1: 1.4]). Let $f : Y \to Y'$ be a stratified mapping, with (weak) control data $C$ for $Y$, and $C'$ for $Y'$. We say that $f$ is weakly controlled if for each stratum $S$ of $Y$, the equation $\pi_{S'} \circ f = f \circ \pi_S$ holds in some neighborhood of $S$ (here $f$ maps $S$ to $S'$).

(5.2.3) Remark. Note that there is no mention of distance functions in (5.2.2). This is intentional, and is consistent with our stance in (4.1).

(5.2.4) Lemma. Let $f : Y \to Y'$ be a stratified mapping. Given partial control data $C$ for $Y$, there is at most one system of germs of partial control data $C'$ for $Y'$ such that $f$ becomes weakly controlled. Such $C'$ exists if and only if for all strata $S$ of $Y$, there is a neighborhood of $S$ (contained in $N_S$) in which $f(y) = f(z)$ implies $f(\pi_S(y)) = f(\pi_S(z))$. $\square$

When the condition in (5.2.4) is satisfied, one uses the formula

$$\pi_{S'}(f(y)) = f(\pi_S(y))$$

to define $C'$, and we then write $C' = f_* C$. In the usual manner, the mapping $f$ determines an equivalence relation on $Y$, viz., $y \sim z$ if and only if $f(y) = f(z)$. The condition on $C$ thereby becomes:

$$y \sim z \implies \pi_S(y) \sim \pi_S(z) \quad \text{(near $S$).}$$

We will use the preceding for the stratified mappings $M^B_\Gamma \to M^{RBS}_\Gamma$ in general, and $M^{RBS}_\Gamma \to M^{BB}_\Gamma$ in the Hermitian case. The reason for bringing in $M^B_\Gamma$ is that it is a manifold-with-corners, and it also has natural partial control data.

The boundary strata of $\overline{X}$, the universal cover of $M^B_\Gamma$, are the sets $e(P)$ of (1.3), as $P$ ranges over all rational parabolic subgroups of $G$. Those of $M^{RBS}_\Gamma$ itself are the arithmetic quotients $e'(P)$ of $e(P)$, with $P$ ranging over the finite set of $\Gamma$-conjugacy classes of such $P$. There are projections $X \to X/A_P = e(P)$, defined by collapsing the orbits of the geodesic action of $A_P$ to points. This extends to a $P$-equivariant smooth retraction (geodesic projection), given in (1.3.3):

$$X(P) \xrightarrow{\pi_P} X(P)/A_P = e(P).$$
Recall from (1.8) that there is a neighborhood of $e'(P)$ in $M^{BS}_\Gamma$ on which geodesic projection onto $e'(P)$, induced by (5.2.6), is defined. We take the restriction of this geodesic projection over $e'(P)$ as the definition of $\pi_P$ in our partial control data $C$ for $M^{BS}_\Gamma$. We have been leading up to:

\textbf{(5.2.7) Proposition.} The quotient mapping $M^{BS}_\Gamma \to M^{RBS}_\Gamma$ satisfies (5.2.5).

\textit{Proof.} The mappings $e(P) \to e(P)/U_P$, as $P$ varies, induce the mapping $M^{BS}_\Gamma \to M^{RBS}_\Gamma$. It is a basic fact (\cite[4.3]{BS}) that for $Q \subset P$, $A_Q \supset A_P$ and $\pi_Q \circ \pi_P = \pi_Q$. This gives $e(P) \to e(P)/U_P$. Since it is also the case that $Q \subset P$ implies $U_Q \supset U_P$, we see that (5.2.5) is satisfied. $\square$

Only a little more complicated is:

\textbf{(5.2.8) Proposition.} In the Hermitian case, the quotient mapping $M^{RBS}_\Gamma \to M^{BB}_\Gamma$ satisfies (5.2.5).

\textit{Proof.} When the symmetric space $X$ is Hermitian, the $P$-stratum of $X^{RBS}$, for each $P$, decomposes as a product:

\begin{equation}
 e(P)/U_P \simeq X_{\ell,P} \times X_{h,P}.
 \end{equation}

This is induced by a decomposition of reductive algebraic groups over $\mathbb{Q}$:

\begin{equation}
 P/U_P = G_{\ell,P} \cdot G_{h,P}
 \end{equation}

(cf. \cite[p. 254]{Mu}). Fixing $G_h$, one sees that the set of $Q$ with $G_{h,Q} = G_h$ (if non-empty) is a lattice, whose greatest element is a maximal parabolic subgroup $P$ of $G$. The lattice is then canonically isomorphic to the lattice of parabolic subgroups $R$ of $G_{\ell,P}$, whereby $Q/U_Q \simeq (R/U_R) \times G_{h,P}$. (Thus $G_{h,Q} = G_{h,P}$. In the language of \cite[(2.2)]{HZ1} such $Q$ are said to be \textit{subordinate to} $P$.)

The mapping $M^{RBS}_\Gamma \to M^{BB}_\Gamma$ is induced, in terms of (5.2.8.1), by

\begin{equation}
 e(Q)/U_Q \to X_{h,Q},
 \end{equation}

for all $Q$; perhaps more to the point, the terms can be grouped by lattice, yielding

\begin{equation}
 \overline{X}_{\ell,P} \times X_{h,P} \to X_{h,P} \hookrightarrow (X_{h,P})^{BB}
 \end{equation}

for $P$ maximal (see \cite[2.6.3]{GT}). One sees that (5.2.5) is satisfied. $\square$

We have thereby reached the conclusion:

\textbf{(5.2.9) Corollary.} The natural partial control data for $M^{BS}_\Gamma$ induces compatible partial control data for $M^{RBS}_\Gamma$ and $M^{BB}_\Gamma$. $\square$

\textbf{(5.3) Conjecture B.} Let $\mathcal{E}_\Gamma$ be a homogeneous vector bundle on $M_\Gamma$, and $\mathcal{E}^{RBS}_\Gamma$ its extension to $M^{RBS}_\Gamma$ from \cite{GT} that was reconstructed in our §1. We select as partial control data $C$ for $M^{RBS}_\Gamma$ that given in (5.2.9). It is essential that the following hold:
(5.3.1) Proposition. $\mathcal{E}_Γ^{RBS}$ is a controlled vector bundle on $M_Γ^{RBS}$, with the $\tilde{π}_P$'s of (1.3.3) providing the weak control data.

Proof. This is almost immediate from the construction in §1. Recall that the weak control data for $M_Γ^{RBS}$ consists of the geodesic projections $π_P$, defined in a neighborhood of $\tilde{M}_P$. The vector bundle $\mathcal{E}_Γ^{RBS}$ also gets local geodesic projections $\tilde{π}_P$, induced from those of $\mathcal{E}^{BS}$, that are compatible with those of $M_Γ^{RBS}$ because of (1.2). The same holds within the strata of these spaces, by (1.6). We see that the criterion of (4.2.3) is satisfied. □

We proceed with a treatment of $∇^{GP}$, the connection on $\mathcal{E}_Γ$ constructed in [GP]. For each maximal $Q$-parabolic subgroup of $G$, let $M_P$ be the corresponding stratum of $M^{BB}$; it is a locally symmetric variety for the group $G_{h,P}$. We also use “$P$” to label the strata: thus, we have for (4.1.2), $π_P : N_P → M_P$, etc. Then $∇^{GP}$ can be defined recursively, starting from the strata of lowest dimension ($Q$-rank zero), and then increasing the $Q$-rank by one at each step.

There is, first, the equivariant Nomizu connection for homogeneous vector bundles, whose definition we recall. Homogeneous vector bundles are associated bundles of the principal $K$-bundle:

\[(5.3.2) \quad κ : Γ \backslash G → M_Γ.\]

When we write the Cartan decomposition $g = \mathfrak{k} ⊕ \mathfrak{p}$, we note that (5.3.2) has a natural equivariant connection whose connection form lies in the vector space $\text{Hom}(g, \mathfrak{k})$; it is given by the projection of $g$ onto $\mathfrak{k}$ (with kernel $\mathfrak{p}$). This is known as the Nomizu connection. The homogeneous vector bundle $\mathcal{E}_Γ$ on $M_Γ$ is associated to the principal bundle (5.3.2) via the representation $K → GL(\mathcal{E})$. The connection induced on $\mathcal{E}_Γ$ via $\mathfrak{k} → \mathfrak{gl}(\mathcal{E}) = \text{End}(\mathcal{E})$ is also called the Nomizu connection (of $\mathcal{E}_Γ$), and will be denoted $∇^{No}$; its connection form is denoted $θ ∈ g^* ⊗ \text{End}(\mathcal{E})$.

A $K$-frame for $\mathcal{E}_Γ$ on an open subset $O ⊂ M_Γ$ is given by a smooth cross-section $σ : O → κ^{-1}(O)$ of $κ$; the resulting connection matrix is the pullback of $θ$ via $σ^*$, an element of $A^1(O, \text{End}(\mathcal{E}))$.

With that stated, we can start to describe $∇^{GP}$. For any maximal $Q$-parabolic $P$, one will be taking expressions of the form

\[(5.3.3) \quad ∇^P = ψ^P_P ∇^{P,No} + \sum_{Q < P} ψ^Q_P Φ^*_Q, (∇^Q)\]

[GP, 11.2]. Here, $∇^{P,No}$ is the Nomizu connection for the homogeneous vector bundle on $M_P$ determined by the restriction $K_{h,P} → K → GL(\mathcal{E})$, and the functions $\{ψ^Q_P : Q ≤ P\}$ form a partition of unity on $M_P$ of a selected type, given in [GP, 3.5, 11.1.1]; the function $ψ^Q_P$ is a cut-off function for a large relatively compact open subset $V_{Q,P}$ of $M_Q$ in $M_P$, with

\[\bigcup_{Q ≤ P} V_{Q,P} = M^*_P,\]
and can be taken to be supported inside the neighborhood $N_{Q,P}$ of the partial control data when $Q \neq P$.

Next, $\Phi_{Q,P}^*$ indicates the process of parabolic induction from $M_Q$ to $N_{Q,P}$, by means of $\pi_{Q,P}$. It is defined as follows. Fix a maximal parabolic $Q$ and a representation $K \to \text{GL}(E)$. The latter restricts, of course, to $K_Q = K_{h,Q} \times K_{\ell,Q}$, but through the Cayley transform, this actually extends to a representation $\lambda$ of $K_{h,Q} \times G_{\ell,Q}$. That allows one to define an action of all of $Q$ on $\mathcal{E}_{h,Q}$ [GP,10.1], which induces a $Q$-equivariant mapping

$$\mathcal{E} = Q \times_{K_Q} E \longrightarrow G_{h,Q} \times_{K_{h,Q}} E = \mathcal{E}_{h,Q},$$

given by $(q,e) \mapsto (g_h, \lambda(g_\ell)e)$ for $q = u g h g_\ell \in U_Q G_{h,Q} G_{\ell,Q} = Q$. That in turn defines a $U_Q$-invariant isomorphism of vector bundles homogeneous under $Q$:

$$\mathcal{E} \simeq \pi_Q^* (\mathcal{E}_{h,Q}).$$

One then takes $\nabla^{GP}$ to be $\nabla^G$ in (5.3.3). Given any connection $\nabla^{h,Q}$ on $\mathcal{E}_{h,Q}$, the pullback connection $\nabla = \pi_Q^* (\nabla^{h,Q})$ satisfies the same relation for its curvature form, viz.,

$$\Theta(\nabla) = \pi_Q^* \Theta(\nabla^{h,Q}).$$

It follows that the Chern forms of $\nabla^{GP}$ are controlled on $M^{BB}_\Gamma$ [GP,11.6]. On the other hand, the connection itself is not. To proceed, weaker information about $\nabla^{GP}$ suffices:

**Proposition.** With an appropriate choice of the functions $\psi_{Q,P}^*$, the connection $\nabla^{GP}$ is a controlled connection when viewed on $M^{RBS}_\Gamma$.

*Proof.* This is not difficult. Recall from (4.3.1) that the issue is the existence of local frames at each point of $M^{RBS}_\Gamma$, with respect to which the connection matrix is controlled. For each rational parabolic subgroup $Q$ of $G$, we work in the corner $X(Q)$. By (5.3.4), one gets local frames for $\mathcal{E}(Q)$, the restriction of $\mathcal{E}$ to $X(Q) \subset \overline{X}$, from local frames for $\mathcal{E}_{h,Q}$. We can write $\mathcal{E}(Q)$ as:

$$\mathcal{E}(Q) \simeq U_Q \times A_Q \times M_Q \times_{K_Q} E.$$  

This also provides good variables for calculations. We note that $\Phi_{Q,P}^*$ is independent of the $U_Q$-variable. Likewise, $\psi_{Q,P}^*$ can be chosen to be a function of only $(a, mK_Q)$, constant on the compact nilmanifold fibers $N_P$ (i.e., the image of the $U_P$-orbits). It follows by induction that $\nabla^{GP}$ is controlled on $M^{RBS}_\Gamma$. □

As we said, the Chern forms of $\nabla^{GP}$ are controlled differential forms for $M^{BB}_\Gamma$, so are *a fortiori* controlled for $M^{RBS}_\Gamma$. It follows from (4.3.5) that
(5.3.7) Proposition. \( c_\bullet(\mathcal{E}_{\Gamma}^{RBS}, \nabla^{GP}) \) represents \( c_\bullet(\mathcal{E}_{\Gamma}^{RBS}) \in H^\bullet(M_{\Gamma}^{RBS}). \) □

Thereby, Conjecture B is proved.

(5.4) Theorem 2. Let \( \nabla^{ctrl} \) be any \( C \)-controlled connection on \( \mathcal{E}_{\Gamma}^{RBS} \), and \( \nabla^{No} \) the equivariant Nomizu connection on \( \mathcal{E}_{\Gamma} \). By (4.3.5) we know that \( c_k(\nabla^{ctrl}) \) represents \( c_k(\mathcal{E}_{\Gamma}^{RBS}) \); we want to conclude the same for \( c_k(\nabla^{No}) \). Toward that, we recall the standard identity on \( M \) satisfied by the Chern forms:

(5.4.1)
\[
c_k(\nabla^{No}) - c_k(\nabla^{ctrl}) = d\eta_k,
\]

which is a case of (4.3.4). The following is straightforward:

(5.4.2) Lemma. i) \( A_{M_{\Gamma}^{RBS}, C}^\bullet \) is contained in \( A_{(\infty)}^\bullet(M_{\Gamma}^{RBS}) \).

ii) A \( G \)-invariant form on \( M_{\Gamma} \) is \( L^\infty \).

Proof. In terms of (3.1.1), a controlled differential form on \( M_{\Gamma}^{RBS} \) is one that is, for each given \( P \), pulled back from \( V \subset \hat{M}_P \). Such forms are trivially weighted by \( A_{\hat{M}_P}^\bullet \) in the metric (3.1.2). It follows that a controlled form is locally \( L^\infty \) on \( M_{\Gamma}^{RBS} \).

This proves (i). As for (ii), an invariant form has constant length, so is in particular \( L^\infty \). □

(5.4.3) Proposition. The closed forms \( c_k(\nabla^{No}) \) and \( c_k(\nabla^{ctrl}) \) represent the same class in \( H^{2k}_{(\infty)}(M_{\Gamma}) \).

Proof. Since \( M_{\Gamma}^{RBS} \) is compact, a global controlled form on \( M_{\Gamma}^{RBS} \) is globally \( L^\infty \).

As such, (5.4.2) gives that the Chern forms for both \( \nabla^{ctrl} \) and \( \nabla^{No} \) are in the complex \( L_{(\infty)}^\bullet(M_{\Gamma}) \). It remains to verify that \( \eta_k \) in (5.4.1) is likewise \( L^\infty \), for then the relation (5.4.1) holds in the \( L^\infty \) de Rham complex \( A_{(\infty)}^\bullet(M_{\Gamma}) \), so \( c_k(\nabla^{No}) \) and \( c_k(\nabla^{ctrl}) \) are cohomologous in the \( L^\infty \) complex.

By (4.3.4.1), it suffices to check that the difference \( \omega = \nabla^{No} - \nabla^{ctrl} \) is \( L^\infty \). That can be accomplished by taking the difference of connection matrices with respect to the same local frame of \( \mathcal{E}^{RBS} \), and for that purpose we use, for each \( Q \), frames pulled back from \( \hat{M}_Q \). For that, it is enough to verify the boundedness for \( \omega \) in a neighborhood of every point of the boundary of \( M_{\Gamma}^{RBS} \), and we may as well calculate on \( X^{RBS} \).

Consider a point in the \( Q \)-stratum \( X_Q \) of \( X^{RBS} \). As in (3.1.1), we can take as neighborhood base, intersected with \( X \), sets that decompose with respect to \( Q \) as

(5.4.3.1)
\[
N_Q \times A^+_Q \times V,
\]

with \( V \) open in \( X_Q \). In these terms, \( \pi_Q \) is just projection onto \( V \). As in (3.1.2) and (3.1.4), we use as coordinates \((u_\alpha,a,v)\). We also decompose (see the end of (1.1)),

(5.4.3.2)
\[
\mathcal{E} \simeq Q \times_{K_Q} E \simeq U_Q \times A^+_Q \times X_Q \times_{K_Q} E.
\]
We obtain a canonical isomorphism $\mathcal{E} \simeq \pi_Q^*\mathcal{E}_Q$, with $\mathcal{E}_Q$ a homogeneous vector bundle on $X_Q$. By (5.4.2, ii), the connection matrix of a connection that is pulled back from $X_Q$, with respect to a local frame pulled back from $X_Q$ is $L^\infty$, so we wish to do the same for the Nomizu connection.

First, we have:

(5.4.3.3) **Lemma.** Let $\hat{Q} = Q/A_QU_Q$ and consider the diagram

$$
\begin{array}{ccc}
Q & \longrightarrow & \hat{Q} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi_Q} & X_Q.
\end{array}
$$

Then $Q \simeq \hat{Q} \times_{X_Q} X$, the pullback of $\hat{Q}$ with respect to $\pi_Q$.

**Proof.** We note that both $\hat{Q}$ and $Q$ are exhibited as principal $K_Q$-bundles. To prove our assertion, it is simplest use the Langlands decomposition (of manifolds) $Q \simeq \hat{Q} \times A_Q \times U_Q$ to yield the decomposition $X \simeq X_Q \times A_Q \times U_Q$ (cf. (5.4.3.1)). Then

$$
\hat{Q} \times_{X_Q} X \simeq \hat{Q} \times A_Q \times U_Q \simeq Q. \quad \square
$$

It follows that if $\sigma : O \subseteq X_Q \to \hat{Q}$ gives a local $K_Q$-frame, then $\bar{\sigma} : \pi_Q^{-1}(O) \subseteq X \to Q \simeq \hat{Q} \times_{X_Q} X$, defined by $\bar{\sigma}(x) = (\sigma(\pi_Q(x)), x)$, gives the pullback frame $\pi_Q^*\sigma$. In other words, $\pi_Q^*\sigma$ takes values in the principal $K_Q$-bundle $Q \to X$ that is the restriction of structure group of (5.3.2) from $K$ to $K_Q$.

Let $\nabla^{No}$ be the Nomizu connection on $\mathcal{E}$. Recall that this is determined by

$$(5.4.3.4) \quad T_X \xrightarrow{\bar{\sigma}^*} \mathfrak{g} \to \mathfrak{k} \to \text{End}(E).$$

As such, $\nabla^{No}$ is not a $K_Q$-connection. However, a frame for the restriction to $X$ of the canonical extension $\mathcal{E}$ can be taken to be of the form $\bar{\sigma}$ as above (cf. (1.10)). It follows that for $x \in \pi_Q^{-1}(O)$, the Nomizu connection is given by

$$(5.4.3.5) \quad T_{X,x} \xrightarrow{\bar{\sigma}^*} \mathfrak{q} \to \mathfrak{k} \to \text{End}(E),$$

where $\mathfrak{q}$ denotes the Lie algebra of $Q$. This is a mapping that is of constant norm along the fibers of $\pi_Q$. It follows that the connection matrix is $L^\infty$. Therefore, we have:

(5.4.3.6) **Proposition.** The connection difference $\omega$ is $L^\infty$. \quad \square

This finishes the proof of (5.4.3).

(5.4.4) **Remark.** The reader may find it instructive to compare, in the case of $G = SL(2)$, the above argument to the one used in [Mu, pp. 259–260]. The two discussions, seemingly quite different, are effectively the same.

We now finish the proof of Theorem 2 by demonstrating:
(5.4.5) Proposition. $c_k(\nabla^{No})$ and $c_k(\nabla^{ctrl})$ represent the same class in $H^{2k}(M^{RBS}_\Gamma)$.

Proof. Because $M_\Gamma$ has finite volume, there is a canonical mapping

$$H^\bullet_{(\infty)}(M_\Gamma) \to H^\bullet_{(p)}(M_\Gamma)$$

for all $p$ (see (3.3.1)). It follows from (5.4.3) that $c_k(\nabla^{No})$ and $c_k(\nabla^{ctrl})$ represent the same class in $H^{2k}_{(p)}(M_\Gamma)$ for all $p$. Taking $p$ sufficiently large, we apply Theorem 1 (i.e., (3.1.11)) to see that $c_k(\nabla^{No})$ and $c_k(\nabla^{ctrl})$ represent the same class in $H^{2k}(M^{RBS}_\Gamma)$. □

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