Higher-order topological pumping

Wladimir A. Benalcazar,1,* Jiho Noh,1,* Mohan Wang,2,* Sheng Huang,2,* Kevin P. Chen,2 and Mikael C. Rechtsman1

1Department of Physics, The Pennsylvania State University, University Park, PA 16801
2Department of Electrical and Computer Engineering, University of Pittsburgh, Pittsburgh, PA, USA.

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The discovery of the quantization of particle transport in adiabatic pumping cycles of periodic structures by Thouless [Thouless D. J., Phys. Rev. B 27, 6083 (1983)] linked the Chern number, a topological invariant characterizing the quantum Hall effect in two-dimensional electron gases, with the topology of dynamical periodic systems in one dimension. Here, we demonstrate its counterpart for higher-order topology. Specifically, we show that adiabatic cycles in two-dimensional crystals with vanishing dipole moments (and therefore zero ‘particle transport’) can nevertheless be topologically nontrivial. These cycles are associated with higher-order topology and can be diagnosed by their ability to produce corner-to-corner transport in certain metamaterial platforms. We experimentally verify this transport by using an array of photonic waveguides modulated in their separations and refractive indices. By mapping the dynamical phenomenon demonstrated here from two spatial and one temporal to three spatial dimensions, this transport is equivalent to the observation of the chiral nature of the gapless hinge states in a three-dimensional second-order topological insulator.

Topological phases of matter exhibit quantized transport properties and robust unconventional states at their boundaries [1, 2]. Initially, they were classified according to whether they obey time-reversal, chiral, or particle-hole symmetries [3]. More recently, their classification has been refined and enriched by additionally considering crystalline symmetries. In particular, ‘higher-order topological phases’ have emerged as a family of crystalline phases with gapped bulk and boundaries that protect gapless states only at the boundaries of their boundaries, i.e., at their corners or hinges [4–9]. An nth-order topological phase in d dimensions protects gapless states in its d − n dimensional boundaries.

Higher-order topological phases with protected corner states have been realized in several metamaterial platforms [10–22]. In 3D, the hinge-localized states of second-order topological phases are either helical or chiral in nature [6, 7, 9]. Phases with helical hinge states have been observed in Bismuth [23]. Chiral hinge states, however, remain unfound or unrealized, although their existence has been proposed in magnetic axion insulators [24] and superconductors [25].

A second central, though as yet unobserved phenomenon in higher-order topological band theory is the higher-order counterpart of a Thouless pump [26]. According to the hierarchical dimensional structure of topological phases, a 2D Chern insulator is connected to a Thouless pump in 1D via dimensional reduction [26]. The same dimensional reduction leads to an equivalence between the 4D quantum Hall effect [27] and a topological pump that adiabatically connects topological and trivial $Z_2$ time-reversal invariant insulators in 3D [28]. More recently, dimensional reduction has also been used to relate the 4D quantum Hall effect to 2D topological pumps over 2D systems [29, 30], features of which have been realized experimentally in photonic [31] and ultracold atomic [32] systems.

The higher-order counterpart of this hierarchical structure establishes an equivalence between 3D second-order topological phases having chiral hinge states and a topological pump that adiabatically connects the trivial and second-order topological phases in 2D [6, 7, 30].

In this work, we present the first realization of such higher-order topological pump. We do so by studying in detail a 2D lattice that evolves periodically and adiabatically. We show that this pump manifests corner-localized states that cross the bulk bandgap and can be used to adiabatically transport energy from one corner of the structure to its opposite one. We implement this model in an array of photonic waveguides modulated in their separations and refractive indices and use it to experimentally verify that light is indeed transported as predicted. By dimensionally extending our system from two spatial and one temporal to three spatial dimensions, our results are equivalent to the experimental verification of the chiral nature of the hinge states in 3D second-order topological insulators.

In the conventional Thouless pump [26, 33], an insulator with discrete translation symmetry adiabatically evolves in a periodic fashion leading to a quantization of the electron transport per cycle. The transport can be tracked by following the dipole moment in a crystal as the cycle progresses [34]. The change of the dipole moment over a cycle is a topologically-protected integer equal to the Chern number of the energy bands calculated in the 2D manifold spanned by the crystal momentum and the adiabatic parameter over that cycle [28].

Figure 1(a-e) illustrates such a process by showing the Wannier centers of a 1D lattice at five stages of a pumping cycle. There is a single Wannier center per unit cell. The
FIG. 1. Evolution of the Wannier centers during one cycle of an adiabatic topological pump. (a-e) A first-order (Thouless) pump. Each unit cell has one Wannier center. (f-j) Our second-order topological pump. Each unit cell has three Wannier centers. In both processes, periodic boundary conditions are adopted. The configurations of Wannier centers at the beginning and end of the cycles are the same. The red arrows indicate the direction of Wannier center flow towards to (first half of cycle) or away from (second half of cycle) their assigned unit cell centers.

Dipole moment is thus proportional to the position of the Wannier centers relative to the centers of their assigned unit cells [34], indicated in Fig. 1 by arrows (up to a sign). The overall effect of the cycle is to transport a Wannier center from left to right by one unit cell, which amounts to the quantization of particle (i.e., Wannier center) transport [34].

The robust quantized transport of Thouless pumps has been observed in lattices of ultra-cold atoms [35, 36]. In systems with open boundaries, Thouless pumps in $d$ spatial dimensions manifest states that cross the energy gap and localize on their $d-1$ dimensional boundaries. These states have been exploited to transport energy from one edge of a sample to the opposite one in several metamaterial platforms [37–39].

Figure 1(f-j) illustrates the cycle of our higher-order topological pump. Each unit cell of the 2D lattice has three Wannier centers. The cycle is $C_3$-symmetric throughout, breaking $C_2$ symmetry at all points except two, at which two topological phases protected by $C_6$ symmetry exist: (i) a second-order topological phase at the beginning of the cycle, with Wannier centers in between the edges of the unit cells [Fig. 1(f,j)], and (ii) a trivial phase, with Wannier centers at the center of the unit cell [Fig. 1(h)] (in Ref. [40] we describe in detail these two topological phases). Notice that the overall dipole moment is zero throughout the cycle and, thus, there is no net particle transport per cycle. This is a necessary property of higher-order pumps. Accordingly, these pumps have vanishing Chern numbers in their spatio-temporal manifolds $(k_1, \theta)$ or $(k_2, \theta)$, where $k_1, k_2$ are crystal momenta along inequivalent non-contractible loops in the 2D Brillouin zone, and $\theta$ is the adiabatically varying parameter. This implies that a system with only edges and no corners will not have protected states that cross the bandgap. In the presence of corners, however, protected in-gap states do cross the bulk bandgap. Such corner states can be used to transport energy among them.

We demonstrate second-order corner-to-corner transport experimentally by employing the two-dimensional photonic waveguide array-based structure demonstrated in Ref. [10]. This structure can host the two phases with Wannier centers shown in Fig. 1(f,j) and 1(h), which we adiabatically connect by modulating its parameters to build the pump.

Specifically, consider the schematic of the three-dimensional photonic structure shown in Fig. 2. At any fixed value of $z$ (the spatial coordinate along the waveguide axis), the photonic structure is a two-dimensional crystalline array of waveguides in the $(x, y)$ plane with six waveguides per unit cell [Fig. 2(a,b)]. Each waveguide binds only the lowest-energy TEM mode, which has an elliptical (almost circular) profile and evanescently couples to its neighbor modes. The couplings are thus real-valued and exponentially decrease with separation between waveguides. We approximate them to be non-vanishing only among nearest neighbors [41]. We slowly vary the separations of the waveguides in the $(x, y)$ plane as well as their refractive indices as a function of $z$, as schematically represented in Fig. 2(c). Here, we exploit the fact that the spatial evolution of the optical beam parallel to the waveguides’ axis directly maps onto the temporal evolution described by the Schrödinger equation of a quantum particle, via the paraxial approximation [40]. Thus, this slow variation of waveguide parameters as a function of $z$ is equivalent to an adiabatic temporal evolution of the system in the $(x, y)$ plane. A tight-binding Bloch Hamiltonian for this process is given by

$$h(k, \theta) = c_{\text{ext}}(\theta)h_{\text{ext}}(k) + c_{\text{int}}(\theta)h_{\text{int}} + \delta E(\theta)\Pi,$$  \hspace{1cm} (1)

where $\theta$ is the adiabatic parameter (which varies linearly with $z$); $k = (k_x, k_y)$ is the crystal momentum in the $(x, y)$ plane; $h_{\text{ext}}(k) = \sum_{i=1}^{6} |\cos(k \cdot a_i)| \sigma_x + \sin(k \cdot a_i) \sigma_y|$, and $a_i = (1, 0)$, $a_{2,3} = (\pm 1/2, \sqrt{3}/2)$ are primitive lattice vectors in the $(x, y)$ plane; $h_{\text{int}}$ has entries $[h_{\text{int}}]_{mn} = 1$ for nearest-neighbor waveguides $m$ and $n$ within the same unit cell and 0 otherwise; and and $\Pi = \sigma_z \oplus (-\sigma_z) \oplus \sigma_z$. 

\( c_{\text{ext}} \) (\( c_{\text{int}} \)) are the coupling amplitudes between nearest-neighbor waveguides between (within) unit cells, which exponentially decrease with separation, and \( \delta E \) are the on-site energies, which vary with refractive index.

In the absence of modulation of the refractive index, \( \delta E(\theta) = 0 \), and the Hamiltonian Eq 1 is \( C_6 \) symmetric. Depending on the ratio of the couplings \( c_{\text{ext}} / c_{\text{int}} \), the structure is in one of two phases protected by \( C_6 \) symmetry; when \( c_{\text{ext}} / c_{\text{int}} > 1 \), the structure is in the second-order topological phase with Wannier centers as in Fig. 1(f) which is characterized by the presence of corner-localized zero-energy states [10, 40]. On the other hand, when \( c_{\text{ext}} / c_{\text{int}} < 1 \), the structure is in the trivial phase with Wannier centers as in Fig. 1(h) and hosts no boundary states [40]. These two phases are separated by a gapless transition point at \( c_{\text{ext}} / c_{\text{int}} = 1 \). To avoid closing the bandgap as we modulate the separations, we add a modulation of the refractive index that causes \( \delta E(\theta) \) to be non-zero, thus breaking \( C_6 \) symmetry down to only \( C_3 \) symmetry. Since the added term is proportional to the chiral operator, \( \Pi \), a gap will be guaranteed if \( \delta E(\theta) \neq 0 \) when \( c_{\text{ext}} / c_{\text{int}} = 1 \). A schematic trajectory of the adiabatic pump in Eq. 1 is shown in Fig. 2(d). Topological pumping is protected for any closed trajectory of \( h(k, \theta) \) that encloses the gapless point at \( (E = 0, c_{\text{ext}} = c_{\text{int}}) \). For concreteness, we choose the following modulation of couplings and refractive indices,

\[
\begin{align*}
    c_{\text{ext}}(\theta) &= C e^{-\kappa |L - 2s(\theta)|}, \\
    c_{\text{int}}(\theta) &= C e^{-\kappa s(\theta)}, \\
    \delta E(\theta) &= \epsilon \delta n_0 \sin(\theta),
\end{align*}
\]

where \( s(\theta) = L/3 - A \cos(\theta) \) (for \( A < L/3 \)) is the separation between neighboring waveguides within a unit cell, \( L = 50 \mu m \) is the separation between unit cells [Fig. 2(a)], \( A = 2.2 \mu m \) and \( \delta n_0 = 0.05 \times 10^{-3} \) are the amplitudes of modulation of the separation \( s \) and the refractive index, respectively, and \( \kappa = 0.19 \mu m^{-1}, C = 77 \text{ cm}^{-1}, \) and \( \epsilon = 1.469 \times 10^4 \text{ cm}^{-1} \) are experimental parameters at \( \lambda = 1555 \text{ nm} \).

A plot of the energy bands for this process is shown in Fig. 2(e). Although bulk and edges are gapped (blue lines), there is a pair of gapless states that cross the energy band, which localize at 120° corners (red and green lines) [most of the support of the corner states is in the ‘off-corner’ waveguides marked by green and red colors in Fig. 2(a)]. These states do not hybridize because they localize at opposite corners.

If the adiabatic parameter \( \theta \) is interpreted as a crystal momentum in a third direction, \( k_z \), the spectrum in Fig. 2(d) amounts to that of a 3D second-order topological insulator with open boundaries in the \((x, y)\) plane and periodic boundaries along \( z \). The in-gap protected states then correspond to hinge-localized chiral states with opposite propagation directions, with the left and right hinges having positive and negative group velocities along \( z \), respectively.

This pattern in the localization of the in-gap corner states implies that an initial beam that occupies one of the corner eigenstates in the waveguide array at \( \theta = -\pi \) will delocalize into the bulk as it adiabatically approaches \( \theta = 0 \) but will emerge at the opposite corner at \( \theta = \pi \).

Figure 3(a) shows the instantaneous eigenvalues at each value of \( z \) for a simulation of beam propagation in a system with a modulation adiabatically deformable to that
FIG. 3. Simulation (a) and experimental implementation (b-d) of the second-order topological pump with Bloch Hamiltonian Eq. 1 with the setup in Fig. 2. (a) Spectrum during one pumping cycle. The color map indicates the amplitude of the projection of the beam onto each of the instantaneous eigenstates of the system. (b) Input facet of the pumping segment. The radii of the major and minor axes of the waveguides are 5.35 and 3.5 μm, the separation between unit cells is \( L = 50 \mu m \). The input facet is in the topological phase shown schematically in Fig. 2(a). (c) Beam profile at the output facet of the initial segment, where the topological corner mode is prepared. (d) Beam profile at the output facet of the pumping segment. In (c,d), yellow lines are overlaid to indicate the positions of the waveguides at each facet.

of Eq. (2). The simulation is carried on using the beam propagation method (BPM), which simulates the evolution of the field \( \psi(r, z) \) within the paraxial approximation \([40]\) (with no tight-binding approximation assumed). Here, \( z = 15(\theta + \pi)/2\pi \) cm. The parameters of this simulation are detailed in Ref. [40].

The color in Fig. 3(a) indicates the amplitude of the projection of the wave function into the eigenstates at each value of \( z \). Adiabaticity is evidenced by the fact that the wave function does not occupy other states close in frequency at any point of the pump cycle. As the cycle progresses, we observe that the beam, originally injected at the left corner, appears at the right corner at the end of the cycle. The power transmitted to the right corner state was 99% of the original incident power on the left corner state for a simulated sample of 15 cm.

We fabricated a sample that reproduced the modulation scheme of the BPM simulations to observe the corner-to-corner transport experimentally. We do this by fabricating two waveguide arrays, an ‘initial segment’, that populates the corner topological state, and a ‘pumping segment’, in which the waveguides are modulated to produce the pump. Figure 3(b-d) shows microscope images of the sample’s input facet of the pumping segment, as well as of the optical beams at the beginning and end of the pumping cycle. At the start of the pumping cycle, the beam must occupy the topological corner state. In the initial segment of the sample, we indirectly excite the topological corner mode by using an auxiliary waveguide in close proximity to the left corner of the waveguide array. For this initial segment, of length 17 cm, we set a refractive index of \( \Delta n = 3.00 \times 10^{-3} \) for both the waveguide array and the auxiliary waveguide, and a ratio of \( L/s = 2.64 \) for the topological waveguide array. The topological corner mode, being degenerate with the mode in the auxiliary waveguide, weakly couples to it. We then input the beam into the auxiliary waveguide and let it leak into the topological corner mode of the array during the entire 17 cm of this initial segment. Figure 3(c) shows the diffracted light measured at the output facet of the initial segment. From this output facet, only the two off-corner waveguides that comprise the vast majority of the left corner topological state are directly butt-coupled to the off-corner waveguides at the left corner of the input facet of the pumping segment (a separate waveguide array).

In the pumping segment of the sample, the positions and the refractive indices of the waveguides were modulated according to the parameters of the full-wave simulation [40]. The \( L/s \) ratio at the input and output facets was set to 2.64, while half way through propagation (\( z = 7.5 \) cm) it was set to 3.32. \( \delta n_0 \) was set to \( 0.5 \times 10^{-4} \). The output facet of the pumping cycle shows that the beam was transported to the opposite topological corner mode [Fig. 3(d)], as expected.

It is worth noting that higher-order pumps should not cause edge-to-edge transport via in-gap states (i.e., its edges should remain gapped). In our system, this is the case as the dipole moment identically vanishes at each instant of the pump. If edge-to-edge transport were to occur, the dimensionally-extended 3D second-order phase would have gapless states on its 2D surfaces into which the hinge states could scatter in the presence of disorder. This distinguishes the present result from that of the 2D pump (dimensionally reduced from 4D) of Ref. [31]: while the corner states observed in that work correspond to localized states on the hinges of the three-dimensional cube, they are not protected against scattering into the degenerate surface states.

Higher-order topological phases with chiral hinge
states remain to be found in condensed matter systems. In photonics, generating chiral hinge states is particularly difficult due to the necessity to break time-reversal symmetry in a 3D bulk. We have circumvented this difficulty via dimensional reduction, by which chiral hinge states in a 3D second-order topological insulator map to the topological pump on a 2D second-order TI, as the one we probe in the present experiment. Thus our experiment effectively provides experimental access to the anomalous chiral hinge states in 3D second-order TIs.

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* these authors contributed equally

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[41] Including additional couplings to waveguides farther away will not preclude topological pumping because the Wannier centers are still pinned to either the center of the unit cell (for the trivial configuration) or at the three middle of the edges of the unit cell (for the topological configuration) as long as $C_6$ symmetry is preserved.
Supplementary Information:
Higher-order topological pumping

Wladimir A. Benalcazar,1,* Jiho Noh,1,* Mohan Wang,2,* Sheng Huang,2,* Kevin P. Chen,2 and Mikael C. Rechtsman1

1Department of Physics, The Pennsylvania State University, University Park, PA 16801
2Department of Electrical and Computer Engineering,
   University of Pittsburgh, Pittsburgh, PA, USA.

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In this Supplementary Information, we describe the steps that justify using a tight-binding approximation for this photonic system. We also describe the two $C_6$ symmetry-protected topological phases in the waveguide array, following Ref. [1], which are connected in the higher-order topological pump. Finally, we describe details of the simulation and the experiment to which the Main Text refers.

**Light propagation in the waveguide array and its tight-binding approximation**

The diffraction of light through the waveguide array of Fig. 2 of the Main Text is governed by the paraxial wave equation

$$i\partial_z \psi(r, z) = \left[ -\frac{1}{2k_0} \nabla^2_r - k_0 \frac{\Delta n(r, z)}{n_0} \right] \psi(r, z), \quad (S1)$$

where $\psi(r, z)$ is the envelope function of the electric field $E(r, z) = \psi(r, z)e^{i(k_0 z - \omega t)}\hat{x}$, $k_0 = 2\pi n_0/\lambda$ is the wavenumber within the medium, $\lambda$ is the wavelength of light, $\nabla^2_r$ is the Laplacian in the transverse $(x,y)$ plane, $\omega = 2\pi c/\lambda$, and $\Delta n$ is the refractive index relative to $n_0$. Assuming that only the lowest TEM mode is bound to each waveguide, we may employ the tight-binding approximation

$$i\partial_z \psi_i(z) = -\sum_j c_{ij}(\lambda)\psi_j(z), \quad (S2)$$

where $\psi_n$ is amplitude of the optical field in the $n$-th waveguide, and $c_{ij}(\lambda)$ is the coupling constant between waveguides $i$ and $j$ at wavelength $\lambda$. For a tight-binding description in the adiabatic regime, we assume the propagation direction $z$ to be constant. Thus, we can explicitly write the $z$-dependence of the propagating modes in equation (S2) as $\psi_n(z) = \psi_n e^{iEz}$. This leads to

$$E\psi_i = \sum_j c_{ij}(\lambda)\psi_j, \quad (S3)$$

where $E$ plays the role of energy in the analogous Schrödinger equation $H\psi = E\psi$, where $H_{ij} \equiv c_{ij}$. 

2
As described in the Main Text, the higher-order topological pump of Eq. 1 and Eq. 2 adiabatically connects two symmetry protected topological phases. A detailed account of these two phases can be found in Ref. [1]. For the sake of completeness, however, here we summarize the essential aspects of these phases.

These two phases are characterized by the Hamiltonian in Eq. 1 with the parameters in Eq. 2 at values of $\theta = 0$ and $\pi$ for the trivial and topological phases, respectively. At these two values of the adiabatic parameter, $\theta$, this model has $C_6$ symmetry,

$$\hat{r}_6 h(k) \hat{r}_6^\dagger = h(R_6 k), \quad \hat{r}_6 = \begin{pmatrix} 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \\ \sigma_x & 0 & 0 \end{pmatrix}, \quad R_6 = \begin{pmatrix} \cos(2\pi/6) & \sin(2\pi/6) \\ -\sin(2\pi/6) & \cos(2\pi/6) \end{pmatrix}, \quad (S4)$$

where $\hat{r}_6$ is the rotation operator acting on the internal degrees of freedom of the unit cell, which obeys $\hat{r}_6^6 = 1$, and $R_6$ is the matrix that rotates the crystal momentum by $2\pi/6$ radians. The $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices and $\sigma_0$ is the $2 \times 2$ identity matrix.

As long as $C_6$ symmetry is preserved, the crystalline structure can transition from a topological phase to the trivial phase as we vary the ratio $c_{\text{int}}/c_{\text{ext}}$ from $c_{\text{int}}/c_{\text{ext}} < 1$ (when $\theta = \pi$) to $c_{\text{int}}/c_{\text{ext}} > 1$ (when $\theta = 0$). To diagnose the topology of these phases, it is necessary to look at the $C_6$ symmetry representations of the bands below the energy gap at the high symmetry point of the Brillouin zone. Based on the $C_n$ representations, a classification of the topological phases has been constructed in Ref. [2]. In $C_6$ symmetric crystalline phases, the topological phase is specified by the topological invariant

$$\chi^{(6)} = ([M], [K]), \quad (S5)$$

where $[M], [K] \in \mathbb{Z}$ are topological invariants calculated as follows,

$$[M] = \#M_1 - \#\Gamma_1^{(2)} \quad (S6)$$

$$[K] = \#K_1 - \#\Gamma_1^{(3)} \quad (S7)$$
where \( \#M_1 (\#\Gamma_1^{(2)}) \) is the number of states below the energy gap with \( C_2 \) rotation eigenvalue \( M_1 = +1 (\Gamma_1^{(2)} = +1) \) at the \( M (\Gamma) \) point of the BZ, and \( \#K_1 (\#\Gamma_1^{(3)}) \) is the number of states below the energy gap with \( C_3 \) rotation eigenvalues \( K_1 = 1 (\Gamma_1^{(3)} = 1) \) at the \( K (\Gamma) \) point of the BZ.

The two phases have topological invariants

\[
\chi^{(6)} = \begin{cases} 
(0, 0) & \text{for } c_{\text{int}}/c_{\text{ext}} > 1 \\
(2, 0) & \text{for } c_{\text{int}}/c_{\text{ext}} < 1
\end{cases}.
\]

(S8)

The transition at \( c_{\text{int}}/c_{\text{ext}} = 1 \) occurs by closing the bulk energy gap at the \( \Gamma \) point of the BZ. This transition point corresponds to the usual honeycomb lattice, which is well known in the context of graphene to have two Dirac cones. The difference in our formulation resides exclusively in our unit cell definition having six instead of two degrees of freedom (see Fig. 2 of the Main Text). The energy bands are shown in Fig. S1 for the trivial and topological phases, as well as at the transition point.

When the lattice is in the topological phase, \( \chi^{(6)} = (2, 0) \), it has Wannier centers at the edges of the unit cells (Fig. S2, right). Notice that it impossible to deform the Wannier centers away from that position in a way that preserves \( C_6 \) symmetry. This obstruction to symmetry-preserving deformations is a real-space manifestation of the symmetry protection of the phase. For this reason, this phase is said to be in an ‘obstructed’ atomic limit [3]. It was recently shown that obstructed atomic limits with no dipole moments (such this lattice), can present a ‘filling anomaly’ when corners are introduced. In insulators, a corner-induced
filling anomaly results in the fractionalization of corner charge [2]. In our system, the filling anomaly manifests a fractional local density of states at 120° corners (Fig. S2). In the additional presence of chiral symmetry, the filling anomaly is compensated by the existence of zero-energy corner states. These states, shown in Fig. S3, are the states that need to be excited at the beginning of the pumping cycle. These states are robust. Their energies are protected at a value of zero by chiral symmetry; their degeneracy is further protected by $C_6$ symmetry. Breaking either of these symmetries softly (i.e., without causing a bulk phase transition) will generally preserve these corner states. Indeed, adding terms that break chiral symmetry (e.g., by including next nearest neighbor couplings) will not compromise the pump, as the Wannier centers will remain fixed by $C_6$ symmetry to the edges of the unit cells. The pumping procedure described in the Main Text may consequently not be symmetric around $\theta = 0$ or $E = 0$, but the in-gap corner states will necessarily cross the gap, and therefore traverse the structure from one corner to the opposite one.
FIG. S3. Real part of the two topological zero-energy corner states in the topological phase protected by $C_6$ symmetry (imaginary part is zero. There is one state per corner, each associated with a fractional filling anomaly of $1/2$ (see Fig. S2, right). Notice that the two ‘off-corner’ sites have have opposite phases.

Detailed description of the simulations

The simulations presented in this work are based on the beam propagation method (BPM), which is a full-wave simulation of the evolution of $\psi(r,z)$ in the propagation direction, $z$, using the paraxial equation Eq. S1. The waveguide in the simulation is modeled as having a Gaussian profile for the variation in the waveguide refractive index: $\Delta n(x,y) = (3.00 \times 10^{-3} \pm \delta n_0) \exp(-x^2/\sigma_x^2 - y^2/\sigma_y^2)$, with $\sigma_x = 3.5 \, \mu m$ and $\sigma_y = 5.35 \, \mu m$. Using BPM simulations, we optimized the parameters of modulation of the waveguide’s separations and refractive indices to maximize the adiabaticity and the efficiency of the pump. For that purpose, we fixed the sample length to $z = 15 \, cm$, which is the maximum length attainable in our experimental setup. The modulation pattern we obtained makes the relation between the adiabatic parameter, $\theta$, and the direction of propagation of light, $z$, piecewise linear. Additionally, the amplitude of modulation of the waveguides’ separations, $A$, varies. Specifically, $\theta$ and $A$ vary according to

$$\theta(z) = \begin{cases} 
-\pi + \frac{\theta_c}{z_c} z & 0 < z < z_c \\
(\theta_c - \pi) + \frac{2(\pi - \theta_c)}{z_L - 2z_c} (z - z_c) & z_c < z < z_L - z_c, \\
(\pi - \theta_c) + \frac{\theta_c}{z_c} (z - z_L + z_c) & z_L - z_c < z
\end{cases}$$

$$A = \begin{cases} 
1.6 \, \mu m & |\theta| < 0.5\pi \\
2.3 \, \mu m & |\theta| > 0.5\pi
\end{cases}$$

(S9)
FIG. S4. Adiabatic parameter $\theta$ as a function of $z$ in the optimized modulation scheme (blue line) described by Eq. S9. The green-dashed horizontal lines indicate the points at which the value of $A$ change. As a comparison, the red line indicates the original function of $\theta$ before the optimization.

where $z_L = 15$ cm is the total sample length, $\theta_c = 0.32 \pi$, $z_c = 5.6$ cm and $s(\theta) = L/3 - A \cos(\theta)$. These relations are shown in the plot of Fig. S4. During fabrication of the sample, the transitions in the values of $\theta$ were smoothened.

Figures S5 shows the simulation of a pump cycle for this modulation. Fig. S5(a) plots the instantaneous energies at each depth in the array. The color indicates the projection of the beam into the instantaneous eigenstates of the system in the $(x,y)$plane. Figs. S5(b-d) show the intensities of the wave function at the cross-sections of the system at the beginning, middle, and end of the cycle, respectively. The initial wavefunction occupies the left topological corner state of the second-order topological phase [Fig. S5(b)]. As the wave function adiabatically propagates, it delocalizes into the bulk. Such delocalization is maximal in the middle of the cycle [Fig. S5(c)]. In the second half of the cycle, the beam increasingly localizes on the right corner. At the output facet, the wavefunction largely occupies the right topological corner state [Fig. S5(d)].

**Waveguide fabrication**

We fabricated the waveguides using femtosecond direct laser writing technique. Using the optimized waveguide parameters found by BPM simulation, we wrote the waveguides using a 800 nm Titanium:sapphire laser and amplifier system (Coherent:RegA 9000 with pulse duration 270 fs, repetition rate 250 kHz, and pulse energy 820 nJ) in borosilicate glass (Corning Eagle XG borosilicate glass) with refractive index $n_0 = 1.473$ at $\lambda = 1550$ nm. The shape and size of the focal volume were controlled by first sending the laser writing
FIG. S5. Simulation of beam propagation during the second-order pumping process. (a) Spectrum during one pumping cycle. The color map indicates the amplitude of the projection of the beam onto each of the instantaneous eigenstates of the system [same as Fig. 3(a) in the Main Text]. (b) Beam profile at the input facet, \( z = 0 \), which occupies the left topological corner state. (c) Beam at half cycle, occupying the lowest bulk state above the gap due to state-level adiabaticity. (d) Beam profile at output facet, \( z = 15 \) cm, which occupies the right topological corner state.

beam through a beam-shaping cylindrical telescope and then focusing it inside the glass chip using a \( \times50 \), aberration-corrected microscope objective (NA = 0.55). The waveguides were fabricated by translating the glass chip through the focal volume of the laser beam using a high-precision three-axis Aerotech motion stage (model ABL20020). The refractive index modulation of the sample was achieved by speed variation of the laser writing beam.

* these authors contributed equally

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