Packing spanning partition-connected subgraphs with small degrees

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Abstract

Let \( G \) be a graph with \( X \subseteq V(G) \) and let \( l \) be an intersecting supermodular subadditive integer-valued function on subsets of \( V(G) \). The graph \( G \) is said to be \( l \)-partition-connected, if for every partition \( P \) of \( V(G) \), \( e_G(P) \geq \sum_{A \in P} l(A) - l(V(G)) \), where \( e_G(P) \) denotes the number of edges of \( G \) joining different parts of \( P \). Let \( \lambda \in [0, 1] \) be a real number and let \( \eta \) be a real function on \( X \). In this paper, we show that if \( G \) is \( l \)-partition-connected and for all \( S \subseteq X \),

\[
\Theta_l(G \setminus S) \leq \sum_{v \in S} (\eta(v) - 2l(v)) + l(V(G)) + l(S) - \lambda l(e_G(S)) + l(S),
\]

then \( G \) has an \( l \)-partition-connected spanning subgraph \( H \) such that for each vertex \( v \in X \), \( d_H(v) \leq \lceil \eta(v) - \lambda l(v) \rceil \), where \( e_G(S) \) denotes the number of edges of \( G \) with both ends in \( S \) and \( \Theta_l(G \setminus S) \) denotes the maximum number of all \( \sum_{A \in P} l(A) - e_G\setminus_S(P) \) taken over all partitions \( P \) of \( V(G) \setminus S \). Finally, we show that if \( H \) is an \((l_1 + \cdots + l_m)\)-partition-connected graph, then it can be decomposed into \( m \) edge-disjoint spanning subgraphs \( H_1, \ldots, H_m \) such that every graph \( H_i \) is \( l_i \)-partition-connected, where \( l_1, l_2, \ldots, l_m \) are \( m \) intersecting supermodular subadditive integer-valued functions on subsets of \( V(H) \).

These results generalize several known results.

Keywords:
Partition-connected; tree-connected; supermodular; edge-decomposition; vertex degree; toughness.

1 Introduction

In this article, all graphs have no loop, but multiple edges are allowed. Let \( G \) be a graph. The vertex set and the edge set of \( G \) are denoted by \( V(G) \) and \( E(G) \), respectively. The degree \( d_G(v) \) of a vertex \( v \) is the number of edges of \( G \) incident to \( v \). We denote by \( d_G(C) \) the number of edges of \( G \) with exactly one end in \( V(C) \), where \( C \) is a subgraph of \( G \). For a set \( X \subseteq V(G) \), we denote by \( G[X] \) the induced subgraph of \( G \) with the vertex set \( X \) containing precisely those edges of \( G \) whose ends lie in \( X \). For a spanning subgraph
$H$ with the integer-valued function $h$ on $V(H)$, the total excess of $H$ from $h$ is defined as follows:

$$te(H, h) = \sum_{v \in V(H)} \max\{0, d_H(v) - h(v)\}.$$ 

According to this definition, $te(H, h) = 0$ if and only if for each vertex $v$, $d_H(v) \leq h(v)$. Let $S \subseteq V(G)$. The vertex set $S$ is called independent, if there is no edge of $G$ connecting vertices in $S$. The graph obtained from $G$ by removing all vertices of $S$ is denoted by $G \setminus S$. Let $F$ be a spanning subgraph of $G$. Denote by $G \setminus [S, F]$ the graph obtained from $G$ by removing all edges incident to the vertices of $S$ except the edges of $F$. Note that while the vertices of $S$ are deleted in $G \setminus S$, no vertices are removed in $G \setminus [S, F]$. Let $A$ and $B$ be two subsets of $V(G)$. This pair is said to be intersecting, if $A \cap B \neq \emptyset$. Let $l$ be a real function on subsets of $V(G)$ with $l(\emptyset) = 0$. For notational simplicity, we write $l(G)$ for $l(V(G))$ and write $l(v)$ for $l(\{v\})$. The function $l$ is said to be supermodular, if for all vertex sets $A$ and $B$,

$$l(A \cap B) + l(A \cup B) \geq l(A) + l(B).$$

Likewise, $l$ is said to be intersecting supermodular, if for all intersecting pairs $A$ and $B$ the above-mentioned inequality holds. The set function $l$ is called (i) nonincreasing, if $l(A) \geq l(B)$, for all nonempty vertex sets $A, B$ with $A \subseteq B$, (ii) subadditive, if $l(A) + l(B) \geq l(A \cup B)$, for any two disjoint vertex sets $A$ and $B$, (iii) element-subadditive, if $l(A) + l(v) \geq l(A \cup \{v\})$, for all vertices $v$ and all vertex sets $A$ excluding $v$, and also is called (iv) weakly subadditive, if $\sum_{v \in A} l(v) \geq l(A)$, for all vertex sets $A$. Note that several results of this paper can be hold for real functions $l$ such that $\sum_{v \in A} l(v) - l(A)$ is integer for every vertex set $A$. For clarity of presentation, we will assume that $l$ is integer-valued. The graph $G$ is said to be $l$-edge-connected, if for all nonempty proper vertex sets $A, d_G(A) \geq l(A)$, where $d_G(A)$ denotes the number of edges of $G$ with exactly one end in $A$. Likewise, the graph $G$ is called $l$-partition-connected, if for every partition $P$ of $V(G)$, $e_G(P) \geq \sum_{A \in P} l(A) - l(V(G))$, where $e_G(P)$ denotes the number of edges of $G$ joining different parts of $P$. An $l$-partition-connected graph $G$ is minimally $l$-partition-connected, if for every edge $e$ of $G$, the resulting $G - e$ is not $l$-partition-connected. We will show that if $l$ is intersecting supermodular, then the vertex set of $G$ can be expressed uniquely (up to order) as a disjoint union of vertex sets of some induced $l$-partition-connected subgraphs. These subgraphs are called the $l$-partition-connected components of $G$. To measure $l$-partition-connectivity of $G$, we define the parameter $\Theta_l(G) = \sum_{A \in P} l(A) - e_G(P)$, where $P$ is the partition of $V(G)$ obtained from $l$-partition-connected components of $G$. The definition implies that for the null graph $K_0$ with no vertices is $l$-partition-connected and $\Theta_l(K_0) = 0$. We will show that $\Theta_l(G)$ is the maximum of all $\sum_{A \in P} l(A) - e_G(P)$ taken over all partitions $P$ of $V(G)$. We say that a spanning subgraph $F$ is $l$-sparse, if for all vertex sets $A$, $e_F(A) \leq \sum_{v \in A} l(v) - l(A)$, where $e_F(A)$ denotes the number of edges of $F$ with both ends in $A$. Clearly, 1-sparse graphs are forests. Note that all maximal $l$-sparse spanning subgraphs of $G$ form the bases of a matroid, when $l$ is an intersecting supermodular weakly subadditive integer-valued function on subsets of $V(G)$, see [4]. Note also that several basic tools in this paper for working with sparse and partition-connected graphs can be obtained using matroid theory. A packing refers to a collection of edge-disjoint subgraphs. A graph is said to be $m$-tree-connected, it has $m$ edge-disjoint spanning trees. It is known that every $m$-partition-connected graph is $m$-tree-connected [17, 21].
denote by $d^+_G(A)$ the number of edges leaving $A$. An orientation of $G$ is called $l$-arc-connected, if for every vertex set $A$, $d^+_G(A) \geq l(A)$. Likewise, an orientation of $G$ is called $r$-rooted $l$-arc-connected, if for every vertex set $A$, $d^+_G(A) \geq l(A) - \sum_{v \in A} r(v)$, where $r$ is a nonnegative integer-valued on $V(G)$ with $l(G) = \sum_{v \in V(G)} r(v)$. Throughout this article, we denote by $e^+_G(A)$ the maximum number of all $e_H(A)$ taken over all minimally $l$-partition-connected spanning subgraphs $H$ of $G$, all set functions are zero on the empty set, and also all variables $k$ and $m$ are integer and positive, unless otherwise stated.

Recently, the present author [15] investigated bounded degree $m$-tree-connected spanning subgraphs and established the following theorem. This result gives a number of new applications on connected factors and generalizes and improves several known results in [5, 6, 11, 14, 16, 23].

**Theorem 1.1.** ([15]) Let $G$ be an $m$-tree-connected graph with $X \subseteq V(G)$. Let $\lambda \in [0, 1]$ be a real number and let $\eta$ be a real function on $X$. If for all $S \subseteq X$,

$$\Theta_m(G \setminus S) \leq \sum_{v \in S} (\eta(v) - 2m) + 2m - \lambda(e_G(S) + m),$$

then $G$ has an $m$-tree-connected spanning subgraph $H$ such that for each $v \in X$, $d_H(v) \leq [\eta(v) - m\lambda]$.

In this paper, we generalize the above-mentioned theorem to the following supermodular version by investigating bounded degree partition-connected spanning subgraphs. Moreover, we generalize several results in [15] toward this concept.

**Theorem 1.2.** Let $G$ be an $l$-partition-connected graph with $X \subseteq V(G)$, where $l$ is an intersecting supermodular subadditive integer-valued function on subsets of $V(G)$. Let $\lambda \in [0, 1]$ be a real number and let $\eta$ be a real function on $X$. If for all $S \subseteq X$,

$$\Theta_l(G \setminus S) \leq \sum_{v \in S} (\eta(v) - 2l(v)) + l(G) + l(S) - \lambda(e_G(S) + l(S)),$$

then $G$ has an $l$-partition-connected spanning subgraph $H$ such that for each $v \in X$, $d_H(v) \leq [\eta(v) - \lambda(v)]$.

In Section 6, we generalize the well-known result of Nash-Williams [17] and Tutte [21] to the following supermodular version. This version can provide an alternative proof for a special case of Theorem 1.2.

**Theorem 1.3.** Let $H$ be a graph and let $l_1, l_2, \ldots, l_m$ be $m$ intersecting supermodular subadditive integer-valued functions on subsets of $V(H)$. Then $H$ is $(l_1 + \cdots + l_m)$-partition-connected, if and only if it can be decomposed into $m$ edge-disjoint spanning subgraphs $H_1, \ldots, H_m$ such that every graph $H_i$ is $l_i$-partition-connected.

## 2 Basic tools

For every vertex $v$ of a graph $G$, consider an induced $l$-partition-connected subgraph of $G$ containing $v$ with the maximal order. The following proposition shows that these subgraphs are unique and decompose the
vertex set of $G$ when $l$ is intersecting supermodular. In fact, these subgraphs are the $l$-partition-connected components of $G$ that already introduced in the Introduction.

**Proposition 2.1.** Let $G$ be a graph with $X, Y \subseteq V(G)$ and let $l$ be an intersecting supermodular real function on subsets of $V(G)$. If $G[X]$ and $G[Y]$ are $l$-partition-connected and $X \cap Y \neq \emptyset$, then $G[X \cup Y]$ is also $l$-partition-connected.

**Proof.** Let $P$ be a partition of $X \cup Y$. Take $A_1, \ldots, A_n$ to be all vertex sets belonging to $P$ such that for each $i$ with $1 \leq i \leq n$, $A_i \cap X \neq \emptyset$ and $A_i \cap Y \neq \emptyset$. Set $A_{n+1} = Y$. Let $P_1$ be the set of all vertex sets $A \in P$ with $A \subseteq X \setminus Y$, and set $P'_1 = P_1 \cup \{A_i \cap X : 1 \leq i \leq n\}$. Let $P_2$ be the set of all vertex sets $A \in P$ with $A \subseteq Y \setminus X$, and set $P'_2 = P_2 \cup \{Z\}$, where $X \cap Y \subseteq Z = (\cup_{1 \leq i \leq n} A_i) \cap Y$. Define $B_i = X$ and for every positive integer $i$ with $1 \leq i \leq n + 1$ recursively define $B_i = B_{i-1} \cup A_{i-1}$. Note that $B_i \cap A_i \neq \emptyset$. It is easy to check that

$$e_{G[X \cup Y]}(P) \geq e_{G[X]}(P'_1) + e_{G[Y]}(P'_2) + \sum_{1 \leq i \leq n+1} d_G(B_i, A_i),$$

where $d_G(B_i, A_i)$ denotes the number of edges of $G$ with one end in $B_i \setminus A_i$ and other one in $A_i \setminus B_i$. Since $G[X]$ and $G[Y]$ are $l$-partition-connected,

$$e_{G[X \cup Y]}(P) \geq \sum_{A \in P_1} l(A) - l(X) + \sum_{A \in P_2} l(A) - l(Y) + \sum_{1 \leq i \leq n+1} d_G(B_i, A_i),$$

which implies that

$$e_{G[X \cup Y]}(P) \geq \sum_{A \in P_1} l(A) + \sum_{1 \leq i \leq n} l(A_i \cap X) - l(X) + \sum_{A \in P_2} l(A) + l(Z) - l(Y) + \sum_{1 \leq i \leq n+1} d_G(B_i, A_i). \quad (1)$$

By the assumption, for each $i$ with $1 \leq i \leq n + 1$, we have

$$l(B_i \cap A_i) + l(B_i \cup A_i) + d_G(B_i, A_i) \geq l(B_i) + l(A_i),$$

which implies that

$$\sum_{1 \leq i \leq n} l(X \cap A_i) + l(Z) + l(X \cup Y) + \sum_{1 \leq i \leq n+1} d_G(B_i, A_i) \geq l(X) + \sum_{1 \leq i \leq n} l(A_i) + l(Y). \quad (2)$$

Therefore, Relations (1) and (2) can conclude that

$$e_{G[X \cup Y]}(P) \geq \sum_{A \in P_1} l(A) + \sum_{1 \leq i \leq n} l(A_i) - l(X \cup Y) + \sum_{A \in P_2} l(A) = \sum_{A \in P} l(A) - l(X \cup Y).$$

Hence the proposition holds. \qed

The next proposition presents a simple way for deducing partition-connectivity of a graph from whose contractions and whose special subgraphs.

**Proposition 2.2.** Let $G$ be a graph with $X \subseteq V(G)$ and let $l$ be an intersecting supermodular real function on subsets of $V(G)$. If $G[X]$ and $G/X$ are $l$-partition-connected, then $G$ itself is $l$-partition-connected.

**Proof.** It is enough to apply the same arguments in the proof of Theorem 2.1, by setting $Y = V(G)$. Note that we still have $e_G(P_2) \geq \sum_{A \in P_2} l(A) - l(Y)$, since $G/X$ is $l$-partition-connected. \qed
2.1 Minimally partition-connected and maximal sparse spanning subgraphs

The following lemma presents a simple way for inducing $l$-partition-connectivity of a graph to whose special subgraphs.

**Lemma 2.3.** Let $G$ be a graph and let $l$ be a real function on subsets of $V(G)$. If $G$ is $l$-partition-connected and $P$ is a partition of $V(G)$ with

$$e_G(P) = \sum_{A \in P} l(A) - l(G),$$

then for any $A \in P$, the graph $G[A]$ is also $l$-partition-connected.

**Proof.** Let $A \in P$ and let $P'$ be an arbitrary partition of $A$. Define $P''$ to be the partition of $V(G)$ with $P'' = P' \cup (P - A)$. Since $G$ is $l$-partition-connected,

$$e_{G[A]}(P') = e_G(P'') - e_G(P) \geq \sum_{A' \in P''} l(A') - l(G) - \left( \sum_{A' \in P} l(A') - l(G) \right) = \sum_{A' \in P'} l(A') - l(A).$$

Hence $G[A]$ is also $l$-partition-connected. \hfill \Box

The following proposition establishes a simple but important property of minimally partition-connected graphs.

**Proposition 2.4.** Let $H$ be a graph and let $l$ be an intersecting supermodular weakly subadditive integer-valued function on subsets of $V(H)$. If $H$ is minimally $l$-partition-connected, then

$$|E(H)| = \sum_{v \in V(H)} l(v) - l(H).$$

**Proof.** By induction on $|V(H)|$. For $|V(H)| = 1$ the proof is clear. So, suppose $|V(H)| \geq 2$. Since $H$ is $l$-partition-connected and $l$ is weakly subadditive, we have $|E(H)| \geq \sum_{v \in V(H)} l(v) - l(H) \geq 0$. If $|E(H)| = 0$, then the theorem holds. So, suppose that $|E(H)| > 0$ and let $e$ be a fixed edge of $H$. Since $H - e$ is not $l$-partition-connected, there is a partition $P$ of $V(H)$ such that $e_H(P) = \sum_{A \in P} l(A) - l(H)$ and $e$ joins different parts of $P$. By Lemma 2.3, for every $A \in P$, the $H[A]$ is $l$-partition-connected. If for an edge $e' \in E(H[A])$, $H[A] - e'$ is still $l$-partition-connected, then by Proposition 2.2, one can conclude that $H \setminus e'$ is still $l$-partition-connected, which is impossible. Thus $H[A]$ is minimally $l$-partition-connected and by induction hypothesis, we therefore have

$$|E(H)| = \sum_{A \in P} e_H(A) + e_H(P) = \sum_{A \in P} \left( \sum_{v \in A} l(v) - l(A) \right) + \sum_{A \in P} l(A) - l(H) = \sum_{v \in V(H)} l(v) - l(H),$$

which completes the proof. \hfill \Box

The following proposition shows that maximal sparse spanning graphs are also partition-connected.
Proposition 2.5. Let $F$ be an $l$-sparse graph with $|E(F)| = \sum_{v \in V(F)} l(v) - l(F)$, where $l$ is a weakly subadditive real function on subsets of $V(F)$. If $P$ is a partition of $V(F)$, then

$$e_F(P) \geq \sum_{A \in P} l(A) - l(F).$$

Furthermore, the equality holds only if for every $A \in P$, the graph $F[A]$ is $l$-partition-connected.

Proof. Since $F$ is $l$-sparse, $e_F(A) \leq \sum_{v \in A} l(v) - l(A)$, for every $A \in P$, which implies that

$$e_F(P) = |E(F)| - \sum_{A \in P} e_F(A) \geq \sum_{v \in V(F)} l(v) - l(F) - \sum_{A \in P} (\sum_{v \in A} l(v) - l(A)) = \sum_{A \in P} l(A) - l(F).$$

Furthermore, if the equality holds, then for every $A \in P$, we must have $e_F(A) = \sum_{v \in A} l(v) - l(A)$. Since the induced graph $F[A]$ is $l$-sparse, it must be $l$-partition-connected. Hence the proof is completed. □

Proposition 2.6. Let $F$ be an $l$-sparse graph with $x, y \in V(F)$, where $l$ is a weakly subadditive real function on subsets of $V(F)$. Let $Q$ be an $l$-partition-connected subgraph of $F$ with the minimum number of vertices including $x$ and $y$. If $l$ is element-subadditive, then for each $z \in V(Q) \setminus \{x, y\}$, $d_Q(z) \geq 1$. Furthermore, if $l$ is subadditive, then for every vertex set $A$ with $\{x, y\} \subseteq A \subseteq V(Q)$, $d_Q(A) \geq 1$.

Proof. Let $A$ be a vertex set with $\{x, y\} \subseteq A \subseteq V(Q)$ and set $B = V(Q) \setminus A$. According to the minimality of $Q$, the graph $Q[A]$ is not partition-connected and so by Proposition 2.5, we must have $d_Q(B) = d_Q(A) \geq l(A) + l(B) - l(A \cup B) \geq 0$, whether $l$ is element-subadditive and $B = \{z\}$ or $l$ is subadditive.

□

2.2 Exchanging edges and preserving partition-connectivity

The following proposition is a useful tool for finding a pair of edges such that replacing them preserves partition-connectivity of a given spanning subgraph.

Proposition 2.7. Let $G$ be a graph and let $l$ be an intersecting supermodular integer-valued function on subsets of $V(G)$. Let $H$ be an $l$-partition-connected spanning subgraph of $G$ and let $M$ be a nonempty edge subset of $E(H)$. If a given edge $e' \in E(G) \setminus E(H)$ joins different $l$-partition-connected components of $H \setminus M$, then there is an edge $e$ belonging to $M$ such that $H - e + e'$ is still $l$-partition-connected.

Proof. We proceed by induction on $|M|$. Assume first that $M = \{e\}$. Suppose, by way of contradiction, that $H - e + e'$ is not $l$-partition-connected. Consequently, there is a partition $P$ of $V(H')$ such that $e_{H'}(P) < \sum_{A \in P} l(A) - l(G)$, where $H' = H - e + e'$. Since $l$ is integer-valued, $\sum_{A \in P} l(A) - l(G)$ is integer, and so $e_{H'}(P) \leq \sum_{A \in P} l(A) - l(G) - 1$. Since $H$ is $l$-partition-connected, we must have $e_{H'}(P) = e_H(P) - 1$ and $e_H(P) = \sum_{A \in P} l(A) - l(G)$. Therefore, the edge $e$ joins different parts of $P$ and both ends of $e'$ lie in the same part $A$ of $P$. By Lemma 2.3, the graph $H[A]$ is $l$-partition-connected, which is a contradiction.
Now, assume that \(|M| \geq 2\). Pick \(e \in M\). If \(e'\) whose ends lie in different \(l\)-partition-connected components of \(H - (M \setminus e)\), then the proof follows by induction. Suppose that both ends of \(e'\) lies in the same \(l\)-partition-connected component \(C\) of \(H - (M \setminus e)\). By the assumption, both ends of \(e\) must lie in \(C\) and also \(e'\) whose ends lies in different \(l\)-partition-connected components of \(C - e\). By applying induction to \(C\), the graph \(C - e + e'\) must be \(l\)-partition-connected. Thus by Proposition 2.2, the graph \(H - e + e'\) is \(l\)-partition-connected. Hence the proposition holds.

The next proposition is a useful tool for finding a pair of edges such that replacing them preserves sparse property of a given sparse spanning subgraph.

**Proposition 2.8.** Let \(G\) be a graph and let \(l\) be an intersecting supermodular weakly subadditive integer-valued function on subsets of \(V(G)\). Let \(F\) be an \(l\)-sparse spanning subgraph of \(G\). If \(xy \in E(G) \setminus E(F)\) and \(Q\) is an \(l\)-partition-connected subgraph of \(F\) including \(x\) and \(y\) with the minimum number of vertices, then for every \(e \in E(Q)\), the graph \(F - e + xy\) remains \(l\)-sparse.

**Proof.** If \(F - e + xy\) is not \(l\)-sparse, then there is a vertex set \(A\) including \(x\) and \(y\) such that \(e \notin E(F[A])\) and \(e_F(A) = \sum_{v \in A} l(v) - l(A)\). Since \(F\) is \(l\)-sparse,

\[
e_F(A \cap B) \geq e_F(A) + e_F(B) - e_F(A \cup B) \geq \sum_{v \in A} l(v) - l(A) + \sum_{v \in B} l(v) - l(B) - \sum_{v \in A \cup B} l(v) + l(A \cup B).
\]

where \(B = V(Q)\). Since \(l\) is intersecting supermodular, we therefore, \(e_F(A \cap B) \geq \sum_{v \in A \cap B} l(v) + l(A \cap B)\).

By Proposition 2.5, the graph \(F[A \cap B]\) must be \(l\)-partition-connected, which contradicts minimality of \(Q\). Note that \(F[A \cap B]\) includes \(x\) and \(y\). Hence the the proof is completed.

### 2.3 Comparing partition-connectivity measures

The following lemma gives useful information about the existence of non-trivial \(l\)-partition-connected components and develops a result in [24].

**Lemma 2.9.** Let \(G\) be a graph of order at least two and let \(l\) be a real function on subsets of \(V(G)\). If \(G\) contains at least \(\sum_{v \in V(G)} l(v) - l(G)\) edges, then it has an \(l\)-partition-connected subgraph with at least two vertices.

**Proof.** The proof is by induction on \(|V(G)|\). For \(|V(G)| = 2\), the proof is clear. Assume \(|V(G)| \geq 3\). Suppose the lemma is false. Thus there exists a partition \(P\) of \(V(G)\) such that \(e_G(P) < \sum_{A \in P} l(A) - l(G)\).

By induction hypothesis, for every \(A \in P\), we have \(e_G(A) \leq \sum_{v \in A} l(v) - l(A)\), whether \(|A| \geq 2\) or not. Therefore,

\[
\sum_{v \in V(G)} l(v) - l(G) \leq |E(G)| = e_G(P) + \sum_{A \in P} e_G(A) < \sum_{A \in P} l(A) - l(G) + \sum_{A \in P} \left(\sum_{v \in A} l(v) - l(A)\right) = \sum_{v \in V(G)} l(v) - l(G).
\]

This result is a contradiction, as desired.
The following result describes a relationship between partition-connectivity measures of graphs.

**Theorem 2.10.** Let $G$ be a graph and let $l$ an intersecting supermodular real function on subsets of $V(G)$. If $\beta$ is a real number with $\beta \geq 1$, then

$$l(G) \leq \Theta_l(G) \leq \frac{1}{\beta} \Theta_{l\beta}(G).$$

Furthermore, $G$ is $l$-partition-connected if and only if $\Theta_l(G) = l(G)$.

**Proof.** Define $l' = \beta l$. Note that $l'$ is also intersecting supermodular. Let $P$ and $P'$ be the partitions of $V(G)$ obtained from the $l$-partition-connected components and $l'$-partition-connected components of $G$. If $G$ is $l$-partition-connected, then we have $|P| = 1$ and so $e_G(P) = 0$ and $\Theta_l(G) = l(G)$. Oppositely, if $G$ is not $l$-partition-connected, then by applying Lemma 2.9 to the contracted graph $G/P$, $e_G(P) < \sum_{A \in P} l(A) - l(G)$ and hence $\Theta_l(G) > l(G)$. For every $X \in P$, define $P'_X$ to be the partition of $X$ obtained from the vertex sets of $P$. By applying Lemma 2.9 to the graph $G[X]/P'_X$, we have $e_{G[X]}(P'_X) \leq \sum_{A \in P'_X} l'(A) - l'(X)$, whether $|P'_X| = 1$ or not. Therefore,

$$\Theta_l(G) = \sum_{A \in P'} l(A) - e_G(P') = \sum_{X \in P} (\sum_{A \in P'_X} l'(A) - e_{G[X]}(P'_X)) - e_G(P) \geq \sum_{X \in P} l'(X) - e_G(P) \geq \beta \Theta_l(G).$$

This equality completes the proof. \hfill \Box

The following theorem introduces an interesting property of partition-connectivity measures.

**Theorem 2.11.** Let $G$ be a graph and let $l$ be an intersecting supermodular real function on subsets of $V(G)$. Then we have,

$$\Theta_l(G) = \max \{ \sum_{A \in P} l(A) - e_G(P) : P \text{ is a partition of } V(G) \}.$$ 

**Proof.** Consider $P$ with the maximum $\sum_{A \in P} l(A) - e_G(P)$ and with the minimal $|P|$. If for a vertex set $X \in P$, the graph $G[X]$ is not $l$-partition-connected, then there is a partition $P'$ of $X$ such that $e_{G[X]}(P') < \sum_{A \in P'} l(A) - l(X)$. Define $P'' = P' \cup (P - X)$. Then

$$\sum_{A \in P'} l(A) - e_G(P'') = \sum_{A \in P} l(A) - l(X) - e_G(P) + \sum_{A \in P'} l(A) - e_{G[X]}(P') > \sum_{A \in P} l(A) - e_G(P),$$

which contradicts maximality of $\sum_{A \in P} l(A) - e_G(P)$. Hence for every set $X \in P$, the graph $G[X]$ must be $l$-partition-connected. Now, assume that $G[X']$ is $l$-partition-connected, where $X' = \cup_{A \in P'} A$, $P' \subseteq P$, and $|P'| \geq 2$. Thus $e_{G[X']}(P') \geq \sum_{A \in P'} l(A) - l(X')$. Define $P'' = (P \setminus P') \cup \{X'\}$. Then

$$\sum_{A \in P'} l(A) - e_G(P'') = \sum_{A \in P} l(A) + l(X') - \sum_{A \in P'} l(A) - (e_G(P) - e_{G[X']}(P')) \geq \sum_{A \in P} l(A) - e_G(P),$$

which contradicts minimality of $|P|$. It is easy to check that $P$ must be the same partition of $G$ obtained from $l$-partition-connected components of $G$. Hence the theorem holds. \hfill \Box
3 Highly partition-connected spanning subgraphs with small degrees

Here, we state following fundamental theorem, which gives much information about partition-connected spanning subgraphs with the minimum total excess. In Section 4, we present a stronger version for this result with a proof, but we feel that it helpful to state the proof of this special case before the general version.

**Theorem 3.1.** Let $G$ be a graph, let $l$ be an intersecting supermodular element-subadditive integer-valued function on subsets of $V(G)$, and let $h$ be an integer-valued function on $V(G)$. If $H$ is a minimally $l$-partition-connected spanning subgraph of $G$ with the minimum total excess from $h$, then there exists a subset $S$ of $V(G)$ with the following properties:

1. $\Theta_l(G \setminus S) = \Theta_l(H \setminus S)$.
2. $S \supseteq \{v \in V(G) : d_H(v) > h(v)\}$.
3. For each vertex $v$ of $S$, $d_H(v) \geq h(v)$.

**Proof.** Define $V_0 = \emptyset$ and $V_1 = \{v \in V(H) : d_H(v) > h(v)\}$. For any $S \subseteq V(G)$ and $u \in V(G) \setminus S$, let $A(S, u)$ be the set of all minimally $l$-partition-connected spanning subgraphs of $H'$ of $G$ such that $d_{H'}(v) \leq h(v)$ for all $v \in V(G) \setminus V_1$, and $H'$ and $H$ have the same edges, except for some of the edges of $G$ whose ends are in $X$, where $H[X]$ is the $l$-partition-connected component of $H \setminus S$ containing $u$. Note that $H'[X]$ must automatically be $l$-partition-connected. Now, for each integer $n$ with $n \geq 2$, recursively define $V_n$ as follows:

$$V_n = V_{n-1} \cup \{v \in V(G) \setminus V_{n-1} : d_{H'}(v) \geq h(v), \text{ for all } H' \in A(V_{n-1}, v)\}.$$

Now, we prove the following claim.

**Claim.** Let $x$ and $y$ be two vertices in different $l$-partition-connected components of $H \setminus V_{n-1}$. If $xy \in E(G) \setminus E(H)$, then $x \in V_n$ or $y \in V_n$.

**Proof of Claim.** By induction on $n$. For $n = 1$, the proof is clear. Assume that the claim is true for $n-1$. Now we prove it for $n$. Suppose otherwise that vertices $x$ and $y$ are in different $l$-partition-connected components of $H \setminus V_{n-1}$, respectively, with the vertex sets $X$ and $Y$, $xy \in E(G) \setminus E(H)$, and $x, y \notin V_n$. Since $x, y \notin V_n$, there exist $H_x \in A(V_{n-1}, x)$ and $H_y \in A(V_{n-1}, y)$ with $d_{H_x}(x) < h(x)$ and $d_{H_y}(y) < h(y)$. By the induction hypothesis, $x$ and $y$ are in the same $l$-partition-connected components of $H \setminus V_{n-2}$ with the vertex set $Z$ so that $X \cup Y \subseteq Z$. Let $Q$ be the unique $l$-partition-connected subgraph of $H$ with minimum number of vertices including $x$ and $y$. Notice that the vertices of $Q$ lie in $Z$ and also $Q$ includes at least a vertex $z$ of $Z \cap V_{n-1}$ so that $d_H(z) \geq h(z)$. Since $l$ is element-subadditive, $d_Q(z) \geq 1$ which means that there is an
edge $zz'$ of $Q$ incident to $z$. By Proposition 2.7, the graph $H[Z] - zz' + xy$ must be $l$-partition-connected.

Now, let $H'$ be the spanning subgraph of $G$ with

$$E(H') = E(H) - zz' + xy - E(H[X]) + E(H_s[X]) - E(H[Y]) + E(H_g[Y]).$$

By repeatedly applying Proposition 2.2, one can easily check that $H'$ is $l$-partition-connected. For each $v \in V(H')$, we have

$$d_{H'}(v) \leq \begin{cases} d_{H_s}(v) + 1, & \text{if } v \in \{x, y\}; \\ d_H(v), & \text{if } v = z', \end{cases}$$

and

$$d_{H'}(v) = \begin{cases} d_{H_s}(v), & \text{if } v \in X \setminus \{x, z'\}; \\ d_{H_s}(v), & \text{if } v \in Y \setminus \{y, z'\}; \\ d_H(v), & \text{if } v \notin X \cup Y \cup \{z, z'\}. \end{cases}$$

If $n \geq 3$, then it is not hard to see that $d_{H'}(z) < d_H(z) \leq h(z)$ and $H'$ lies in $A(V_{n-1}, z)$. Since $z \in V_{n-1} \setminus V_{n-2}$, we arrive at a contradiction. For the case $n = 2$, since $z \in V_1$, it is easy to see that $h(z) \leq d_{H'}(z) < d_H(z)$ and $te(H', h) < te(H, h)$, which is again a contradiction. Hence the claim holds.

Obviously, there exists a positive integer $n$ such that $V_1 \subseteq \cdots \subseteq V_{n-1} = V_n$. Put $S = V_n$. Since $S \supseteq V_1$, Condition 2 clearly holds. For each $v \in V_1 \setminus V_{i-1}$ with $i \geq 2$, we have $H \in A(V_{i-1}, v)$ and so $d_H(v) \geq h(v)$. This establishes Condition 3. Because $S = V_n$, the previous claim implies Condition 1 and completes the proof. \qed

**Remark 3.2.** The element-subadditive condition of Theorem 3.1 can be replaced by weakly subadditive condition along with the conditions $l(A) + l(v) \geq l(A \cup \{v\})$, where $v$ is a vertex with $h(v) \leq d_G(v)$ and $A \subseteq V(G) \setminus v$. This version allows us to construct another appropriate set function only by increasing $l(G)$.

**Remark 3.3.** Note that the proof of Theorem 3.1 can be modified to present a polynomial-time algorithm, similar to the algorithm of the proof of Theorem 1 in [5], for producing an appropriate vertex set $S$: by considering calling of subroutines for finding partition-connected components and minimal partition-connected subgraphs as single steps.

### 3.1 Sufficient conditions depending on partition-connectivity measures

The following lemma establishes an important property of minimally $l$-partition-connected graphs.

**Lemma 3.4.** Let $H$ be a graph and let $l$ be an intersecting supermodular weakly subadditive integer-valued function on subsets of $V(H)$. If $H$ is minimally $l$-partition-connected and $S \subseteq V(H)$, then

$$\Theta_l(H \setminus S) = \sum_{v \in S} (d_H(v) - l(v)) + l(H) - e_H(S).$$

**Proof.** Let $P$ be the partition of $V(H) \setminus S$ obtained from the $l$-partition-connected components of $H \setminus S$. Obviously, $e_H(P \cup \{v : v \in S\}) = \sum_{v \in S} d_H(v) - e_H(S) + e_{H \setminus S}(P)$. By Proposition 2.2, one can
conclude that \( H[A] \) is minimally \( l \)-partition-connected, for any \( A \in P \). Hence Proposition 2.4 implies that 
\[
e_H(A) = \sum_{v \in A} l(v) - l(A),
\]
and also \( |E(H)| = \sum_{v \in V(H)} l(v) - l(H) \). Thus 
\[
e_H(P \cup \{v : v \in S\}) = |E(H)| - \sum_{A \in P} e_H(A) = \sum_{v \in S} l(v) + \sum_{A \in P} l(A) - l(H).
\]
Therefore, 
\[
\Theta_l(H \setminus S) = \sum_{A \in P} l(A) - e_{H \setminus S}(P) = \sum_{v \in S} (d_H(v) - l(v)) + l(H) - e_H(S).
\]
Hence the lemma is proved. \( \square \)

The following theorem is essential in this section.

**Theorem 3.5.** Let \( G \) be a graph with \( X \subseteq V(G) \) and let \( l \) be an intersecting supermodular element-subadditive integer-valued function on subsets of \( V(G) \). Let \( \lambda \in [0, 1] \) be a real number and let \( \eta \) be a real function on \( X \). If for all \( S \subseteq X \),
\[
\Theta_l(G \setminus S) < 1 + \sum_{v \in S} (\eta(v) - 2l(v)) + l(G) + l(S) - \lambda(e_G^*(S) + l(S)),
\]
then \( G \) has an \( l \)-partition-connected spanning subgraph \( H \) such that for each \( v \in X \), \( d_H(v) \leq [\eta(v) - \lambda l(v)] \).

**Proof.** For each vertex \( v \), define 
\[
h(v) = \begin{cases} 
s_G(v) + 1, & \text{if } v \notin X; \\
[\eta(v) - \lambda l(v)], & \text{if } v \in X.
\end{cases}
\]
Note that \( G \) is automatically \( l \)-partition-connected, because of \( \Theta_l(G \setminus \emptyset) \leq l(G) \). Let \( H \) be a minimally \( l \)-partition-connected spanning subgraph of \( G \) with the minimum total excess from \( h \). Define \( S \) to be a subset of \( V(G) \) with the properties described in Theorem 3.1. Obviously, \( S \subseteq X \). By Lemma 3.4,
\[
\sum_{v \in S} h(v) + l(H, h) = \sum_{v \in S} d_H(v) = \Theta_l(H \setminus S) + \sum_{v \in S} l(v) - l(H) + e_H(S).
\]
and so
\[
\sum_{v \in S} h(v) + l(H, h) = \Theta_l(G \setminus S) + \sum_{v \in S} l(v) - l(G) + e_H(S). \tag{3}
\]
Also, by the assumption, we have
\[
\Theta_l(G \setminus S) + \sum_{v \in S} l(v) - l(G) + e_H(S) < 1 + \sum_{v \in S} (\eta(v) - l(v)) - \lambda(e_G^*(S) + l(S)) + e_H(S) + l(S). \tag{4}
\]
Since \( e_H(S) \leq e_G^*(S) \) and \( e_H(S) \leq \sum_{v \in S} l(v) - l(S) \),
\[
- \lambda(e_G^*(S) + l(S)) + e_H(S) + l(S) \leq -\lambda(e_H(S) + l(S)) + e_H(S) + l(S) \leq (1 - \lambda) \sum_{v \in S} l(v). \tag{5}
\]
Therefore, Relations (3), (4), and (5) can conclude that
\[ \sum_{v \in S} h(v) + te(H, h) < 1 + \sum_{v \in S} (\eta(v) - \lambda l(v)). \]

On the other hand, by the definition of \( h(v) \),
\[ \sum_{v \in S} (\eta(v) - \lambda l(v) - h(v)) \leq 0. \]

Hence \( te(H, h) = 0 \) and the theorem holds. \( \square \)

When we consider the special cases \( \lambda = 1 \), the theorem becomes simpler as the following result.

**Corollary 3.6.** Let \( G \) be a graph and let \( l \) be an intersecting supermodular element-subadditive integer-valued function on subsets of \( V(G) \). Let \( h \) be an integer-valued function on \( V(G) \). If for all \( S \subseteq V(G) \),
\[ \Theta_l(G \setminus S) \leq \sum_{v \in S} (h(v) - l(v) + l(G) - e^*_G(S)), \]
then \( G \) has an \( l \)-partition-connected spanning subgraph \( H \) such that for each vertex \( v \), \( d_H(v) \leq h(v) \).

**Proof.** Apply Theorem 3.5 with \( \lambda = 1 \) and \( \eta(v) = h(v) + l(v) \). \( \square \)

Note that the above-mentioned corollary is equivalent to Theorem 3.5 and can concludes the next results.

**Corollary 3.7.** Let \( G \) be a graph and let \( l \) be an intersecting supermodular element-subadditive integer-valued function on subsets of \( V(G) \). Let \( h \) be an integer-valued function on \( V(G) \). If for all \( S \subseteq V(G) \),
\[ \Theta_l(G \setminus S) \leq \sum_{v \in S} (h(v) - 2l(v) + l(G) + \Theta_l(G[S])), \]
then \( G \) has an \( l \)-partition-connected spanning subgraph \( H \) such that for each vertex \( v \), \( d_H(v) \leq h(v) \).

**Proof.** Apply Corollary 3.6 along with the inequality \( e^*_G(S) \leq \sum_{v \in S} l(v) - \Theta_l(G[S]) \). \( \square \)

The following corollary provides a necessary and sufficient condition for the existence of a partition-connected spanning subgraph with the described properties.

**Corollary 3.8.** Let \( G \) be a graph with independent set \( X \subseteq V(G) \) and let \( l \) be an intersecting supermodular element-subadditive integer-valued function on subsets of \( V(G) \). Let \( h \) be an integer-valued function on \( X \). Then for all \( S \subseteq X \),
\[ \Theta_l(G \setminus S) \leq \sum_{v \in S} (h(v) - l(v)) + l(G), \]
if and only if \( G \) has an \( l \)-partition-connected spanning subgraph \( H \) such that for each \( v \in X \), \( d_H(v) \leq h(v) \).

**Proof.** It is enough to apply Corollary 3.6 with \( e^*_G(S) = 0 \), and apply Lemma 3.4. \( \square \)
3.2 An alternative proof for a weaker version of Corollary 3.6

In this subsection, we are going to present another proof for the following weaker version of Corollary 3.6. Our proof is based on orientations of partition-connected graphs. In Section 7, we alternatively present a new proof for it based on edge-decompositions with a stronger version on hypergraphs.

**Theorem 3.9.** Let $G$ be a graph and let $\ell$ be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of $V(G)$. Let $h$ be an integer-valued function on $V(G)$. If for all $S \subseteq V(G)$,

$$\Theta_l(G \setminus S) \leq \sum_{v \in S} (h(v) - l(v)) + l(G) - e_G(S),$$

then $G$ has an $l$-partition-connected spanning subgraph $H$ such that for each $v \in X$, $d>H(v) \leq h(v)$.

Before starting the proof, let us state the following two lemmas.

**Lemma 3.10.** (Frank [8]) Let $G$ be a graph and let $\ell$ be an intersecting supermodular nonnegative integer-valued function on subsets of $V(G)$ with $\ell(\emptyset) = \ell(G) = 0$. Then $G$ is $\ell$-partition-connected if and only if it has an $\ell$-arc-connected orientation.

Note that one can apply Theorem 2 in [9] instead of the above-mentioned lemma to obtain further improvement. Hence we state the following lemma in a more general version. This can also be extended to a hypergraph version in the same way, which along with Theorem 3.2 in [12] can provide an alternative proof for a special case of Theorem 7.17.

**Lemma 3.11.** Let $G$ be a directed graph and let $\ell$ be an element-nonincreasing positively intersecting supermodular nonnegative integer-valued function on subsets of $V(G)$ with $\ell(\emptyset) = \ell(G) = 0$. If $H$ is a minimally $\ell$-arc-connected spanning subdigraph of $G$, then for each vertex $v$, we must have $d_H^-(v) = \ell(v)$.

**Proof.** Suppose, by way of contradiction, that $0 \leq \ell(u) < d_H^-(u)$ for a vertex $u$. Let $e = vu$ be a directed edge. Since $|E(H)|$ is minimal, there is a vertex set $A$ including $u$ excluding $v$ such that $\ell(A) = d_H^-(A) > 0$; otherwise, the edge $vu$ can be deleted from $H$. Consider $A$ with minimal $|A|$. Since $\ell$ is element-nonincreasing, $d_H^-(u) > \ell(u) \geq \ell(A) = d_H^-(A)$. Thus there is a directed edge $wu$ with $w \in A$.

Corresponding to $uw$, there is again a vertex set $B$ including $u$ excluding $w$ such that $\ell(B) = d_H^-(B) > 0$. Therefore,

$$\ell(A) + \ell(B) = d_H^-(A) + d_H^-(B) \geq d_H^-(A \cap B) + d_H^-(A \cup B) \geq \ell(A \cap B) + \ell(A \cup B).$$

Since $\ell$ is positively intersecting supermodular and $u \in A \cap B$, we must have $d_H^-(A \cap B) = \ell(A \cap B)$. Since $A \cap B$ includes $u$ and $|A \cap B| < |A|$, we arrive at a contradiction. \hfill \Box

Now, we are ready to state the second proof of the above-mentioned theorem.
The second proof of Theorem 3.9. Let \( r_0 \) be a fixed vertex. For each vertex \( v \), define
\[
\ell'(v) = \max \{ \ell(v), d_G(v) - h(v) + \ell(v) \},
\]
where \( \ell(v) = l(v) - l(G) \) when \( v = r_0 \), and \( \ell(v) = l(v) \) when \( v \neq r_0 \). For all vertex sets \( A \) including \( r_0 \) with \( A \geq 2 \), define \( \ell'(A) = \ell(A) = l(A) - l(G) \), and for all vertex sets \( A \) excluding \( r_0 \) with \( A \geq 2 \), define \( \ell'(A) = \ell(A) = l(A) \). Let \( P \) be a partition of \( V(G) \) and take \( S \) to be the set of all vertices \( v \) with \( \{v\} \in P \) such that \( \ell'(v) = d_G(v) - h(v) + \ell(v) \). Also, define \( P \) to be the set of all vertex sets \( A \in P \) such that
\[
A \neq \{v\}, \text{ when } v \in S.
\]
Note that for every \( A \in \mathcal{P} \), \( \ell'(A) = \ell(A) \). It is not hard to check that \( \ell \) is an intersecting supermodular real function on subsets of \( \Theta_l(G) \).

According to the assumption,
\[
\sum_{A \in \mathcal{P}} l(A) - e_G \setminus S(\mathcal{P}) \leq \Theta_l(G \setminus S) \leq \sum_{v \in S} (h(v) - l(v)) + l(G) - e_G(S).
\]
Since \( e_G(P) = \sum_{v \in S} d_G(v) - e_G(S) + e_G \setminus S(\mathcal{P}) \), we must have
\[
e_G(P) \geq \sum_{A \in \mathcal{P}} l(A) + \sum_{v \in S} (d_G(v) - h(v) + l(v)) - l(G) = \sum_{A \in \mathcal{P}} \ell'(A) + \sum_{v \in S} \ell'(v) = \sum_{A \in \mathcal{P}} \ell'(A).
\]
Thus \( G \) is \( \ell' \)-partition-connected. By Lemma 3.10, the graph \( G \) has an \( \ell' \)-arc-connected orientation so that for every vertex set \( A \), \( d_G^+(A) \geq \ell'(A) \geq \ell(A) \). In particular, for each vertex \( v \), \( d_G^+(v) \geq \ell'(v) \geq d_G(v) - h(v) + \ell(v) \) which implies that \( d_G^+(v) \leq h(v) - \ell(v) \). Let \( H \) be a minimally \( \ell \)-arc-connected spanning subdigraph of \( G \). By Lemma 3.11, for each vertex \( v \), \( d_H^+(v) = \ell(v) \), and so
\[
d_H(v) = d_H^+(v) + d_H^-(v) \leq \ell(v) + d_G^+(v) \leq \ell(v) + (h(v) - \ell(v)) = h(v).
\]
For every partition \( P \) of \( V(H) \), we have
\[
e_H(P) \geq \sum_{A \in \mathcal{P}} d_H^-(A) \geq \sum_{A \in \mathcal{P}} \ell(A) = \sum_{A \in \mathcal{P}} l(A) - l(G).
\]
Hence \( H \) is also \( l \)-partition-connected and the proof is completed.

3.3 Graphs with high edge-connectivity

Highly edge-connected graphs are natural candidates for graphs satisfying the assumptions of Theorem 3.5. We examine them in this subsection, beginning with the following lemma.

Lemma 3.12. Let \( G \) be a graph, let \( l \) be an intersecting supermodular real function on subsets of \( V(G) \), and let \( k \) be a positive real number. If \( S \subseteq V(G) \), then
\[
\Theta_l(G \setminus S) \leq \begin{cases} 
\sum_{v \in S} \frac{d_G(v)}{k} - \frac{2}{k} e_G(S), & \text{when } G \text{ is } k\text{-edge-connected, } k \geq 2, \text{ and } S \neq \emptyset; \\
\sum_{v \in S} \left( \frac{d_G(v)}{k} - l(v) \right) + l(G) - \frac{1}{k} e_G(S), & \text{when } G \text{ is } k\text{-partition-connected and } k \geq 1.
\end{cases}
\]
Proof. Let $P$ be the partition of $V(G) \setminus S$ obtained from the $l$-partition-connected components of $G \setminus S$. Obviously, we have

$$e_G(P \cup \{v : v \in S\}) = \sum_{v \in S} d_G(v) - e_G(S) + e_G(S) = \sum_{v \in S} d_G(v) - e_G(S).$$

If $G$ is $kl$-edge-connected and $S \neq \emptyset$, there are at least $kl(A)$ edges of $G$ with exactly one end in $A$, for any $A \in P$. Thus

$$e_G(P \cup \{v : v \in S\}) \geq \sum_{A \in P} kl(A) - e_{G \setminus S}(P) + e_G(S)$$

and so if $k \geq 2$, then

$$k\Theta_l(G \setminus S) = \sum_{A \in P} kl(A) - k e_{G \setminus S}(P) \leq \sum_{A \in P} kl(A) - 2e_{G \setminus S}(P) \leq \sum_{v \in S} d_G(v) - 2e_G(S).$$

When $G$ is $kl$-partition-connected, we have

$$e_G(P \cup \{v : v \in S\}) \geq \sum_{A \in P} kl(A) + \sum_{v \in S} kl(v) - kl(G)$$

and so if $k \geq 1$, then

$$k\Theta_l(G \setminus S) = \sum_{A \in P} kl(A) - k e_{G \setminus S}(P) \leq \sum_{A \in P} kl(A) - e_{G \setminus S}(P) \leq \sum_{v \in S} (d_G(v) - kl(v)) + kl(G) - e_G(S).$$

These inequalities complete the proof. □

Now, we are ready to generalize a result in [15] as the following theorem.

Theorem 3.13. Let $G$ be a graph with $X \subseteq V(G)$, let $l$ be an intersecting supermodular element-subadditive nonnegative integer-valued function on subsets of $V(G)$, and let $k$ be a positive real number. Then $G$ has an $l$-partition-connected spanning subgraph $H$ such that for each $v \in X$,

$$d_H(v) \leq \begin{cases} \left\lceil \frac{1}{k}(d_G(v) - 2l(v)) \right\rceil + 2l(v), & \text{if } G \text{ is } kl\text{-edge-connected and } k \geq 2; \\ \left\lceil \frac{1}{k}(d_G(v) - l(v)) \right\rceil + l(v), & \text{if } G \text{ is } kl\text{-partition-connected and } k \geq 1; \\ \left\lfloor \frac{1}{k}d_G(v) \right\rfloor + l(v), & \text{if } G \text{ is } kl\text{-edge-connected, } k \geq 2, \text{ and } X \text{ is independent}; \\ \left\lceil \frac{1}{k}d_G(v) \right\rceil, & \text{if } G \text{ is } kl\text{-partition-connected, } k \geq 1, \text{ and } X \text{ is independent}. \end{cases}$$

Proof. Let $S \subseteq V(G)$. If $G$ is $kl$-edge-connected, $k \geq 2$, and $S \neq \emptyset$, then by Lemma 3.12, we have

$$\Theta_l(G \setminus S) \leq \sum_{v \in S} \frac{d_G(v)}{k} - \frac{2}{k} e_G(S) \leq \sum_{v \in S} (\eta(v) - 2l(v)) + l(G) + l(S) - \frac{2}{k}(e_G(S) + l(S)),$$

where for each vertex $v$, $\eta(v) = \frac{d_G(v)}{k} + 2l(v)$. Note that when $G$ is $kl$-edge-connected, $k \geq 2$, and $S = \emptyset$, we must have $\Theta_l(G \setminus S) = l(G)$. If $G$ is $kl$-partition-connected and $k \geq 1$, then by Lemma 3.12, we also have

$$\Theta_l(G \setminus S) \leq \sum_{v \in S} \left(\frac{d_G(v)}{k} - l(v)\right) + l(G) - \frac{1}{k} e_G(S) \leq \sum_{v \in S} (\eta(v) - 2l(v)) + l(G) + l(S) - \frac{1}{k}(e_G(S) + l(S)),$$

where for each vertex $v$, $\eta(v) = \frac{d_G(v)}{k} + l(v)$. Thus the first two assertions follow from Theorem 3.5 for $\lambda \in \{2/k, 1/k\}$. The second two assertions can similarly be proved. □

The following corollary can improve a result in [1] by replacing minimum degree condition. We denote below by $\delta^+(G)$ the minimum out-degree of a directed graph $G$. 

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Corollary 3.14. Let $G$ be a graph with $X \subseteq V(G)$ and let $k$ be a real number with $k \geq 1$. If $G$ has an orientation with $\delta^+(G) \geq km$, then it has a spanning subgraph $H$ with a new orientation such that $\delta^+(H) \geq m$ and for each $v \in X$,

$$d_H(v) \leq \begin{cases} \left\lceil \frac{1}{k}d_G(v) \right\rceil, & \text{when } X \text{ is independent;} \\ \left\lceil \frac{1}{k}(d_G(v) - m) \right\rceil + m, & \text{otherwise.} \end{cases}$$

Proof. Since $\delta^+(G) \geq km$, the graph $G$ is $kl$-partition-connected, where $l(v) = m$ for each vertex $v$ and $l(A) = 0$ for every vertex set $A$ with $|A| \geq 2$. Let $H$ be an $l$-partition-connected spanning subgraph of $G$ with the properties described in Theorem 3.13. Since $H$ is $l$-partition-connected, by Lemma 3.10, it has an orientation such that $\delta^+(H) \geq m$.

4 Highly partition-connected spanning subgraphs with bounded degrees

In this section, we shall strengthen Theorem 3.5 for finding partition-connected spanning graphs with bounded degrees, when $l$ is nonincreasing. Before doing so, we establish the following promised generalization of Theorems 3.1. Note that $\Theta(G \setminus [S, F]) = \Theta(G \setminus S) + \sum_{v \in S} l(v)$, when $F$ is the trivial spanning subgraph and $l$ is element-subadditive.

Theorem 4.1. Let $G$ be an $l$-partition-connected graph with the spanning $l$-sparse subgraph $F$, where $l$ is a intersecting supermodular weakly subadditive integer-valued function on subsets of $V(G)$. Let $h$ be an integer-valued function on $V(G)$. If $H$ is a minimally $l$-partition-connected spanning subgraph of $G$ containing $F$ with the minimum total excess from $h + d_F$, then there exists a subset $S$ of $V(G)$ with the following properties:

1. $\Theta_l(G \setminus [S, F]) = \Theta_l(H \setminus [S, F])$.
2. $S \supseteq \{v \in V(G) : d_H(v) > h(v) + d_F(v)\}$.
3. For each vertex $v$ of $S$, $d_H(v) \geq h(v) + d_F(v)$.

Proof. Define $V_0 = \emptyset$ and $V_1 = \{v \in V(H) : d_H(v) > h(v) + d_F(v)\}$. For any $S \subseteq V(G)$ and $u \in V(G) \setminus S$, let $A(S, u)$ be the set of all minimally $l$-partition-connected spanning subgraphs $H'$ of $G$ containing $F$ such that $d_{H'}(v) \leq h(v) + d_F(v)$ for all $v \in V(G) \setminus V_1$, and $H'$ and $H$ have the same edges, except for some of the edges of $G$ whose ends are in $X$, where $H[X]$ is the $l$-partition-connected component of $H \setminus [S, F]$ containing $u$. Note that $H'[X]$ must automatically be $l$-partition-connected. Now, for each integer $n$ with $n \geq 2$, recursively define $V_n$ as follows:

$$V_n = V_{n-1} \cup \{v \in V(G) \setminus V_{n-1} : d_{H'}(v) = h(v) + d_F(v), \text{ for all } H' \in A(V_{n-1}, v)\}.$$
Now, we prove the following claim.

**Claim.** Let \( x \) and \( y \) be two vertices in different \( l \)-partition-connected components of \( H \setminus [V_{n-1}, F] \). If \( xy \in E(G) \setminus E(H) \), then \( x \in V_n \) or \( y \in V_n \).

**Proof of Claim.** By induction on \( n \). For \( n = 1 \), the proof is clear. Assume that the claim is true for \( n - 1 \). Now we prove it for \( n \). Suppose otherwise that vertices \( x \) and \( y \) are in different \( l \)-partition-connected components of \( H \setminus [V_{n-1}, F] \), respectively, with the vertex sets \( X \) and \( Y \), \( xy \in E(G) \setminus E(H) \), and \( x, y \notin V_n \). Since \( x, y \notin V_n \), there exist \( H_x \in \mathcal{A}(V_{n-1}, x) \) and \( H_y \in \mathcal{A}(V_{n-1}, y) \) with \( d_{H_x}(x) < h(x) + d_F(x) \) and \( d_{H_y}(y) < h(y) + d_F(y) \). By the induction hypothesis, \( x \) and \( y \) are in the same \( l \)-partition-connected component of \( H \setminus [V_{n-2}, F] \) with the vertex set \( Z \) so that \( X \cup Y \subseteq Z \). Let \( M \) be the nonempty set of edges of \( H[Z] \setminus E(F) \) incident to the vertices in \( V_{n-1} \setminus V_{n-2} \) whose ends lie in different \( l \)-partition-connected components of \( H[Z] \setminus [Z \cap V_{n-1}, F] \). By Proposition 2.7, there exists an edge \( zz' \notin M \) with \( z \in Z \cap V_{n-1} \) such that \( H[Z] - zz' + xy \) is \( l \)-partition-connected. Now, let \( H' \) be the spanning subgraph of \( G \) containing \( F \) with

\[
E(H') = E(H) - zz' + xy - E(H[X]) + E(H_x[X]) - E(H[Y]) + E(H_y[Y]).
\]

By repeatedly applying Proposition 2.2, one can easily check that \( H' \) is \( l \)-partition-connected. For each \( v \in V(H') \), we have

\[
d_{H'}(v) \leq \begin{cases} 
    d_{H_x}(v) + 1, & \text{if } v \in \{x, y\}; \\
    d_H(v), & \text{if } v = z'; \\
    d_H(v), & \text{if } v \in X \setminus \{x, z'\}; \\
    d_H(v), & \text{if } v \notin X \cup Y \cup \{z, z'\}.
\end{cases}
\]

If \( n \geq 3 \), then it is not hard to see that \( d_{H'}(z) < d_H(z) \leq h(z) + d_F(z) \) and \( H' \) lies in \( \mathcal{A}(V_{n-2}, z) \). Since \( z \in V_{n-1} \setminus V_{n-2} \), we arrive at a contradiction. For the case \( n = 2 \), since \( z \in V_1 \), it is easy to see that \( h(z) + d_F(z) \leq d_{H'}(z) < d_H(z) \) and \( te(H', h + d_F) < te(H, h + d_F) \), which is again a contradiction. Hence the claim holds.

Obviously, there exists a positive integer \( n \) such that \( V_1 \subseteq \cdots \subseteq V_{n-1} = V_n \). Put \( S = V_n \). Since \( S \supseteq V_1 \), Condition 2 clearly holds. For each \( v \in V_i \setminus V_{i-1} \) with \( i \geq 2 \), we have \( H \in \mathcal{A}(V_{i-1}, v) \) and so \( d_H(v) \geq h(v) + d_F(v) \). This establishes Condition 3. Because \( S = V_n \), the previous claim implies Condition 1 and completes the proof. \( \square \)

In the above-mentioned theorem, we could assume that \( \Theta_l(H) = \Theta_l(G) \) and choose \( H \) with the minimum \( te(H, h + d_F) \), whether \( G \) is \( l \)-partition-connected or not. Conversely, if we assume that \( te(H, h + d_F) = 0 \) and choose \( H \) with the minimum \( \Theta_l(H) \), the next theorem can be derived. However, the above-mentioned theorem works remarkably well, we shall use this result to get further improvement in the last subsection.

**Theorem 4.2.** Let \( G \) be a graph with the spanning \( l \)-sparse subgraph \( F \), where \( l \) is an intersecting supermodular weakly subadditive integer-valued function on subsets of \( V(G) \). Let \( h \) be an integer-valued function on \( V(G) \). If \( H \) is a spanning subgraph of \( G \) containing \( F \) with \( te(H, h + d_F) = 0 \) and with the minimum \( \Theta_l(H) \), then there exists a subset \( S \) of \( V(G) \) with the following properties:

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1. \( \Theta_l(G \setminus [S, F]) = \Theta_l(H \setminus [S, F]) \).

2. For each vertex \( v \) of \( S \), \( d_H(v) = h(v) + d_F(v) \).

**Proof.** Define \( V_0 = \emptyset \). For any \( S \subseteq V(G) \) and \( u \in V(G) \setminus S \), let \( \mathcal{A}(S, u) \) be the set of all spanning subgraphs \( H' \) of \( G \) containing \( F \) with \( te(H', h + d_F) = 0 \) such that \( \Theta_l(H') = \Theta_l(H) \), \( H'|X] \) is \( l \)-partition-connected, \( H' \) and \( H \) have the same edges, except for some of the edges of \( G \) whose ends are in \( X \), where \( H[X] \) is the \( l \)-partition-connected component of \( H \setminus [S, F] \) containing \( u \). Now, for each integer \( n \) with \( n \geq 2 \), recursively define \( V_n \) as follows:

\[
V_n = V_{n-1} \cup \{ v \in V(G) \setminus V_{n-1} : d_{H'}(v) = h(v) + d_F(v), \text{ for all } H' \in \mathcal{A}(V_{n-1}, v) \}.
\]

Now, we prove the following claim.

**Claim.** Let \( x \) and \( y \) be two vertices in different \( l \)-partition-connected components of \( H \setminus [V_{n-1}, F] \). If \( xy \in E(G) \setminus E(H) \), then \( x \in V_n \) or \( y \in V_n \).

**Proof of Claim.** By induction on \( n \). Suppose otherwise that vertices \( x \) and \( y \) are in different \( l \)-partition-connected components of \( H \setminus [V_{n-1}, F] \), respectively, with the vertex sets \( X \) and \( Y \), \( xy \in E(G) \setminus E(H) \), and \( x, y \notin V_n \). Since \( x, y \notin V_n \), there exist \( H_x \in \mathcal{A}(V_{n-1}, x) \) and \( H_y \in \mathcal{A}(V_{n-1}, y) \) with \( d_{H_x}(x) < h(x) + d_F(x) \) and \( d_{H_y}(y) < h(y) + d_F(y) \). For \( n = 1 \), define \( H' \) to be the spanning subgraph of \( G \) containing \( F \) with

\[
E(H') = E(H) + xy - E(H[X]) + E(H_y[X]) - E(H[Y]) + E(H_y[Y]).
\]

Since the edge \( xy \) joins different \( l \)-partition-connected components of \( H \), we must have \( \Theta_l(H') < \Theta_l(H) \). Since \( te(H', h + d_F) = 0 \), we arrive at a contradiction. Now, suppose \( n \geq 2 \). By the induction hypothesis, \( x \) and \( y \) are in the same \( l \)-partition-connected component of \( H \setminus [V_{n-2}, F] \) with the vertex set \( Z \) so that \( X \cup Y \subseteq Z \). Let \( M \) be the nonempty set of edges of \( H[Z] \setminus E(F) \) incident to the vertices in \( V_{n-1} \setminus V_{n-2} \) whose ends lie in different \( l \)-partition-connected components of \( H[Z] \setminus [Z \cap V_{n-1}, F] \). By Proposition 2.7, there exists an edge \( zz' \in M \) with \( z \in Z \cap V_{n-1} \) such that \( H[Z] - zz' + xy \) is \( l \)-partition-connected. Now, let \( H' \) be the spanning subgraph of \( G \) containing \( F \) with

\[
E(H') = E(H) - zz' + xy - E(H[X]) + E(H_z[X]) - E(H[Y]) + E(H_y[Y]).
\]

It is easy to see that the \( l \)-partition-connected components of \( H' \) and \( H \) have the same vertex sets. Since \( H \) and \( H' \) have the same edges joining these \( l \)-partition-connected components, \( \Theta_l(H') = \Theta_l(H) \). For each \( v \in V(H') \), we have

\[
d_{H'}(v) \leq \begin{cases} 
  d_{H_x}(v) + 1, & \text{if } v \in \{ x, y \}; \\
  d_{H}(v), & \text{if } v = z'.
\end{cases}
\]

and

\[
d_{H'}(v) = \begin{cases} 
  d_{H_x}(v), & \text{if } v \in X \setminus \{ x, z' \}; \\
  d_{H_y}(v), & \text{if } v \in Y \setminus \{ y, z' \}; \\
  d_{H}(v), & \text{if } v \notin X \cup Y \cup \{ z, z' \}.
\end{cases}
\]

It is not hard to check that \( d_{H'}(z) < d_{H}(z) \leq h(z) + d_F(z) \) and \( H' \) lies in \( \mathcal{A}(V_{n-2}, z) \). Since \( z \in V_{n-1} \setminus V_{n-2} \), we arrive at a contradiction. Hence the claim holds.
Obviously, there exists a positive integer \( n \) such that \( V_1 \subseteq \cdots \subseteq V_{n-1} = V_n \). Put \( S = V_n \). For each \( v \in V_i \setminus V_{i-1} \), we have \( H \in \mathcal{A}(V_{i-1}, v) \) and so \( d_H(v) = h(v) + d_F(v) \). This establishes Condition 2. Because \( S = V_n \), the previous claim implies Condition 1 and completes the proof. \( \square \)

4.1 Prerequisites

The following lemma provides a generalization for Lemma 3.4. Recall that \( \Theta_l(G \setminus [S, F]) = \Theta_l(G \setminus S) + \sum_{v \in S} l(v) \) when \( F \) is the trivial spanning subgraph and \( l \) is element-subadditive.

Lemma 4.3. Let \( H \) be an \( l \)-sparse graph with the spanning subgraph \( F \), where \( l \) is an intersecting supermodular weakly subadditive integer-valued function on subsets of \( V(H) \). If \( S \subseteq V(H) \) and \( F = H \setminus E(F) \), then

\[
\sum_{v \in S} d_F(v) = \Theta_l(H \setminus [S, F]) - \Theta_l(H) + e_F(S).
\]

Proof. By induction on the number of edges of \( F \) which are incident to the vertices in \( S \). If there is no edge of \( F \) incident to a vertex in \( S \), then the proof is clear. Now, suppose that there exists an edge \( e = uu' \in E(F) \) with \( |S \cap \{u, u'\}| \geq 1 \). Hence

1. \( \Theta_l(H) = \Theta_l(H \setminus e) - 1 \),
2. \( \Theta_l(H \setminus [S, F]) = \Theta_l((H \setminus e) \setminus [S, F]) \),
3. \( e_F(S) = e_{F \setminus e}(S) + |S \cap \{u, u'\}| - 1 \),
4. \( \sum_{v \in S} d_F(v) = \sum_{v \in S} d_{F \setminus e}(v) + |S \cap \{u, u'\}|. \)

Therefore, by the induction hypothesis on \( H \setminus e \) with the spanning subgraph \( F \) the lemma holds. \( \square \)

The following lemma provides a useful relationship between two parameters \( \Theta_l(G \setminus S) \) and \( \Theta_l(G \setminus [S, F]) \), when \( l \) is nonincreasing. We shall apply it in the subsequent subsections.

Lemma 4.4. Let \( G \) be a graph with the spanning subgraph \( F \) and let \( l \) be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of \( V(G) \). If \( S \subseteq V(G) \) then

\[
\Theta_l(G \setminus [S, F]) \leq \Theta_l(G \setminus S) + \sum_{v \in S} \max \{0, l(v) - d_F(v)\} + e_F(S).
\]

Furthermore, \( \Theta_l(G \setminus [S, F]) \leq \Theta_l(G \setminus S) + \frac{1}{(c-1)} e_F(S) \), when every \( l \)-partition-connected component \( C \) of \( F \) we have \( \sum_{v \in C} l(v) \geq cl(C) - \frac{1}{2} d_F(C) \) and \( c \geq 2 \).
Proof. Define $P$ and $P'$ to be the partitions of $V(G)$ and $V(G) \setminus S$ obtained from the $l$-partition-connected components of $G \setminus [S,F]$ and $G \setminus S$. Let $R = \{A \in P : A \subseteq S\}$, $R_1 = \{A \in R : |A| = 1\}$, and $R_2 = \{A \in R : |A| \geq 2\}$. It is not difficult to check that
\[
e_{G \setminus [S,F]}(P) \geq e_{G \setminus S}(P') - \sum_{A \in P'R} e_{G[A \setminus S]}(P'_{A \setminus S}) + D_F(R),
\]
where $P'_{A \setminus S}$ denotes the partition of $A \setminus S$ obtained from vertex sets of $P'$, and $D_F(R)$ denotes the number of edges of $F$ incident to vertex sets in $R$. Thus
\[
\Theta_l(G \setminus [S,F]) - \sum_{A \in P} l(A) \leq \Theta_l(G \setminus S) - \sum_{A \in P'R} \Theta_l(G[A \setminus S]) - D_F(R).
\]
Since $\Theta_l(G[A \setminus S]) \geq l(A \setminus S)$, for any $A \in P \setminus R$, we have
\[
\Theta_l(G \setminus [S,F]) \leq \Theta_l(G \setminus S) + \sum_{A \in P'R} (l(A) - l(A \setminus S)) + \sum_{A \in R} l(A) - D_F(R).
\]
Since $l$ is nonincreasing,
\[
\Theta_l(G \setminus [S,F]) \leq \Theta_l(G \setminus S) + \sum_{A \in R} l(A) - D_F(R).
\]
In the first statement, $e_F(A) \geq \sum_{v \in A} l(v) - l(A) \geq l(A)$, for any $A \in R_2$, and so
\[
\sum_{A \in R} l(A) - D_F(R) \leq \sum_{\{v\} \in R_1} l(v) + \sum_{A \in R_2} e_F(A) - \sum_{\{v\} \in R_1} d_F(v) + e_F(R_1) \leq \sum_{\{v\} \in R_1} (l(v) - d_F(v)) + e_F(R).
\]
Therefore,
\[
\Theta_l(G \setminus [S,F]) \leq \Theta_l(G \setminus S) + e_F(S) + \sum_{v \in S} \max\{0, l(v) - d_F(v)\}.
\]
In the second statement, $\sum_{v \in A} l(v) \geq c l(A) - \frac{c-1}{2} d_F(A)$ for any $A \in R$, and so
\[
\sum_{A \in R} \left( l(A) - \frac{1}{c-1} \left( \sum_{v \in A} l(v) - l(A) \right) \right) \leq \sum_{A \in R} \frac{1}{2} d_F(A) \leq D_F(R).
\]
Since $e_F(A) \geq \sum_{v \in A} l(v) - l(A)$, for any $A \in R$, it is easy to check that
\[
\Theta_l(G \setminus [S,F]) \leq \Theta_l(G \setminus S) + \frac{1}{(c-1)} e_F(S).
\]
Hence the lemma holds. \qed

4.2 A strengthened version of a special case of Theorem 3.5

A strengthened version of Theorems 3.5 is given in the following theorem, when $l$ is nonincreasing.

**Theorem 4.5.** Let $G$ be a graph with $X \subseteq V(G)$ and with the spanning subgraph $F$, and let $l$ be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of $V(G)$. Let $\lambda \in [0,1]$ be a real number and let $\eta$ be a real function on $X$. If for all $S \subseteq X$,
\[
\Theta_l(G \setminus S) < 1 + \sum_{v \in S} (\eta(v) - 2l(v)) + l(G) + l(S) - \lambda(e^*_G(S) + l(S)),
\]

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then it has an $l$-partition-connected spanning subgraph $H$ containing $F$ such that for each $v \in X$,
\[
d_H(v) \leq \left\lceil \eta(v) - \lambda l(v) \right\rceil + \max\{0, d_F(v) - l(v)\}.
\]

**Proof.** For each vertex $v$, define
\[
h(v) = \begin{cases} d_G(v) + 1, & \text{if } v \notin X; \\ \left\lceil \eta(v) - \lambda l(v) \right\rceil - \min\{l(v), d_F(v)\}, & \text{if } v \in X. \end{cases}
\]
First, suppose that $F$ is $l$-sparse. Note that $G$ is automatically $l$-partition-connected, because of $\Theta_l(G \setminus \emptyset) \leq l(G)$. Let $H$ be a minimally $l$-partition-connected spanning subgraph of $G$ containing $F$ with the minimum total excess from $h + d_F$. Define $S$ to be a subset of $V(G)$ with the properties described in Theorem 4.1. Obviously, $S \subseteq X$. Put $F = H \setminus E(F)$. By Lemma 4.3,
\[
\sum_{v \in S} h(v) + te(H, h + d_F) = \sum_{v \in S} d_F(v) = \Theta_l(H \setminus [S,F]) - l(G) + e_F(S),
\]
and so
\[
\sum_{v \in S} h(v) + te(H, h + d_F) = \Theta_l(G \setminus [S,F]) - l(G) + e_F(S).
\]
Since $e_F(S) + e_F(S) = e_H(S)$, Lemma 4.4 implies that
\[
\sum_{v \in S} (h(v) - \max\{0, l(v) - d_F(v)\}) + te(H, h + d_F) \leq \Theta_l(G \setminus S) - l(G) + e_H(S). \tag{6}
\]
Also, by the assumption,
\[
\Theta_l(G \setminus S) - l(G) + e_H(S) < 1 + \sum_{v \in S} (\eta(v) - 2l(v)) - \lambda(e_G(S) + l(S)) + e_H(S) + l(S). \tag{7}
\]
Since $e_H(S) \leq e^*_G(S)$ and $e_H(S) \leq \sum_{v \in S} l(v) - l(S)$,
\[
- \lambda(e^*_G(S) + l(S)) + e_H(S) + l(S) \leq -\lambda(e_H(S) + l(S)) + e_H(S) + l(S) \leq (1 - \lambda) \sum_{v \in S} l(v). \tag{8}
\]
Therefore, Relations (6), (7), and (8) can conclude that
\[
\sum_{v \in S} (h(v) - \max\{0, l(v) - d_F(v)\}) + te(H, h + d_F) < 1 + \sum_{v \in S} (\eta(v) - \lambda l(v) - l(v)).
\]
On the other hand, by the definition of $h(v)$,
\[
\sum_{v \in S} (\eta(v) - \lambda l(v) - l(v) - h(v) + \max\{0, l(v) - d_F(v)\}) \leq 0.
\]
Hence $te(H, h + d_F) = 0$ and the theorem holds. Now, suppose that $F$ is not $l$-sparse. Remove some of the edges of the $l$-partition-connected components of $F$ until the resulting $l$-sparse graph $F'$ have the same $l$-partition-connected components. For each vertex $v$ with $d_F(v) < d_F(v)$, we have $d_F(v) \geq d_F(v) \geq l(v)$, since $v$ must lie in a non-trivial $l$-partition-connected component of $F'$ and $l$ is nonincreasing. It is enough, now, to apply the theorem on $F'$ and finally add the edges of $E(F) \setminus E(F')$ to that explored $l$-partition-connected spanning subgraph. \qed
4.3 Tough enough graphs

In this subsection, we improve below Theorems 3.5 for graphs that the values of \( \Theta_l(G \setminus S) \) are small enough compared to \( |S| \), which enables us to choose \( \eta(v) \) small enough, in compensation we require that the given spanning subgraph \( F \) approximately have large \( l \)-partition-connected components.

**Theorem 4.6.** Let \( G \) be a graph and \( l \) be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of \( V(G) \). Let \( h \) be an integer-valued function on \( V(G) \). Let \( F \) be a spanning subgraph of \( G \) in which for every \( l \)-partition-connected component \( C \) of \( F \), we have \( \sum_{v \in V(C)} l(v) \geq cl(C) - \frac{d_F(C)}{2}l(C) \) and \( c \geq 2 \). If for all \( S \subseteq V(G) \),

\[
\Theta_l(G \setminus S) < 1 + \sum_{v \in S} \left( \frac{c}{2c-2} h(v) - \frac{1}{c-1} l(v) \right) + l(S) + \frac{1}{c-1} l(S),
\]

then \( G \) has an \( l \)-partition-connected spanning subgraph \( H \) containing \( F \) such that for each vertex \( v \), \( d_H(v) \leq h(v) + d_F(v) \).

**Proof.** First, suppose that \( F \) is \( l \)-sparse. Note that \( G \) is automatically \( l \)-partition-connected, because of \( \Theta_l(G \setminus \emptyset) \leq l(G) \). Let \( H \) be an \( l \)-sparse spanning subgraph of \( G \) containing \( F \) with \( te(H, h + d_F) = 0 \) and with the minimum \( \Theta_l(H) \). Define \( S \) to be a subset of \( V(G) \) with the properties described in Theorem 4.2. Put \( F = H \setminus E(F) \). By Lemma 4.3,

\[
\sum_{v \in S} h(v) = \sum_{v \in S} d_F(v) = \Theta_l(H \setminus [S, F]) - \Theta_l(H) + e_F(S),
\]

and so

\[
\Theta_l(H) = \Theta_l(G \setminus [S, F]) + e_F(S) - \sum_{v \in S} h(v). \tag{9}
\]

Since \( e_F(S) + e_F(S) = e_H(S) \leq \sum_{v \in S} l(v) - l(S) \) and \( e_F(S) \leq \frac{1}{2} \sum_{v \in S} d_F(v) = \frac{1}{2} \sum_{v \in S} h(v) \), we have

\[
e_F(S) + \frac{1}{c-1} e_F(S) \leq \frac{1}{2} \sum_{v \in S} h(v) + \frac{1}{c-1} \left( \sum_{v \in S} l(v) - l(S) - \frac{1}{2} \sum_{v \in S} h(v) \right). \tag{10}
\]

Also, by Lemma 4.4,

\[
\Theta_l(G \setminus [S, F]) + e_F(S) \leq \Theta_l(G \setminus S) + e_F(S) + \frac{1}{c-1} e_F(S). \tag{11}
\]

Therefore, Relations (9), (10), and (11) can conclude that

\[
\Theta_l(H) \leq \Theta_l(G \setminus S) - \frac{c}{2c-2} \sum_{v \in S} h(v) + \frac{\sum_{v \in S} l(v) - l(S)}{c-1} < l(G) + 1.
\]

Hence \( \Theta_l(H) = l(H) \) and the theorem holds. Now, suppose that \( F \) is not \( l \)-sparse. Remove some of the edges of the \( l \)-partition-connected components of \( F \) until the resulting \( l \)-sparse graph \( F' \) have the same \( l \)-partition-connected components. For every \( l \)-partition-connected component \( C \) of \( F' \), we still have \( d_{F'}(C) = d_F(C) \).

It is enough, now, to apply the theorem on \( F' \) and finally add the edges of \( E(F) \setminus E(F') \) to that explored \( l \)-partition-connected spanning subgraph.
When we consider the special cases \( h(v) = 1 \), the theorem becomes simpler as the following result.

**Corollary 4.7.** Let \( G \) be a graph and \( l \) be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of \( V(G) \). Let \( F \) be a spanning subgraph of \( G \) in which for every \( l \)-partition-connected component \( C \) of \( F \), we have \( \sum_{v \in C} l(v) \geq c_l(C) - \frac{1}{c-1}d_F(C) \) and \( c \geq 2 \). If for all \( S \subseteq V(G) \),

\[
\Theta_l(G \setminus S) \leq \sum_{v \in S} \frac{e - 2ml(v)}{2m(c - 1)} + l(G) + \frac{1}{c - 1}l(S)
\]

then \( G \) has an \( ml \)-partition-connected spanning subgraph \( H \) containing \( F \) such that for each vertex \( v \), \( d_H(v) \leq d_F(v) + 1 \).

**Proof.** Let \( G' \) be the union of \( m \) copies of \( G \) with the same vertex set and define \( l' = ml \). It is easy to check that \( \Theta_l(G' \setminus S) = m\Theta_l(G \setminus S) \), for every \( S \subseteq V(G) \). Define \( h(v) = 1 \) for each vertex \( v \). By Theorem 4.6, the graph \( G' \) has an \( l' \)-partition-connected spanning subgraph \( H \) containing \( F \) such that for each vertex \( v \), \( d_H(v) \leq h(v) + d_F(v) \leq 1 + d_F(v) \). According to the construction, the graph \( H \) must have no multiple edges of \( E(G') \setminus E(F) \). Hence \( H \) itself is a spanning subgraph of \( G \) and the proof is completed. \( \square \)

## 5 Total excesses from comparable functions

In this section, we formulate the following strengthened versions of the main results of this paper which are motivated by Ozeki-type condition [19]. As their proofs require only minor modifications, we shall only state the strategy of the proof in the subsequent subsection.

**Theorem 5.1.** Let \( G \) be an \( l \)-partition-connected graph, where \( l \) is an intersecting supermodular elementsubadditive integer-valued function on subsets of \( V(G) \). Let \( p \) be a positive integer. For each integer \( i \) with \( 1 \leq i \leq p \), let \( t_i \) be a nonnegative integer, let \( \lambda_i \in [0, 1] \) be a real number, and let \( \eta_i \) be a real function on \( V(G) \) with \( \eta_1 - \lambda_1 l \geq \cdots \geq \eta_p - \lambda_p l \). If for all \( S \subseteq V(G) \) and \( i \in \{1, \ldots, p\} \),

\[
\Theta_l(G \setminus S) < 1 + \sum_{v \in S} (\eta_i(v) - 2l(v)) + l(G) + l(S) - \lambda_i(c_G^l(S) + l(S)) + t_i,
\]

then \( G \) has an \( l \)-partition-connected spanning subgraph \( H \) satisfying \( t e(H, h_i) \leq t_i \) for all \( i \) with \( 1 \leq i \leq p \), where \( h_i(v) = [\eta_i(v) - \lambda_i l(v)] \) for all vertices \( v \).

**Theorem 5.2.** Let \( G \) be an \( l \)-partition-connected graph with the spanning subgraph \( F \), where \( l \) is a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of \( V(G) \). Let \( p \) be a positive integer. For each integer \( i \) with \( 1 \leq i \leq p \), let \( t_i \) be a nonnegative integer, let \( \lambda_i \in [0, 1] \) be a real number, and let \( \eta_i \) be a real function on \( V(G) \) with \( \eta_1 - \lambda_1 l \geq \cdots \geq \eta_p - \lambda_p l \). If for all \( S \subseteq V(G) \) and \( i \in \{1, \ldots, p\} \),

\[
\Theta_l(G \setminus S) < 1 + \sum_{v \in S} (\eta_i(v) - 2l(v)) + l(G) + l(S) - \lambda_i(c_G^l(S) + l(S)) + t_i,
\]
then $F$ can be extended to an $l$-partition-connected spanning subgraph $H$ satisfying $te(H, h_i) \leq t_i$ for all $i$ with $1 \leq i \leq p$, where $h_i(v) = [\eta_i(v) - \lambda_i l(v)] + \max\{0, d_F(v) - l(v)\}$ for all vertices $v$.

5.1 Strategy of the proof

Let $G$ be a graph with the spanning subgraph $H$ and take $xy \in E(G) \setminus E(H)$. Let $h$ be an integer-valued function on $V(G)$. It is easy to check that if $d_H(x) < h(x)$ and $d_H(y) < h(y)$, then $te(H + xy, h) = te(H, h)$, and also this equality holds for any other integer-valued function $h'$ on $V(G)$ with $h' \geq h$. This observation was used by Ozeki (2015) to prove Theorem 6 in [19] with a method that decreases total excesses from comparable functions, step by step, by starting from the largest function to the smallest function. Inspired by Ozeki’s method, we now formulate the following strengthened version of Theorem 4.1.

Theorem 5.3. Let $G$ be an $l$-partition-connected graph with the $l$-sparse spanning subgraph $F$, where $l$ is an intersecting supermodular weakly subadditive integer-valued function on subsets of $V(G)$. Let $h_1, \ldots, h_q$ be $q$ integer-valued functions on $V(G)$ with $h_1 \geq \cdots \geq h_q$. Define $\Gamma_0$ to be the set of all $l$-partition-connected spanning subgraphs $H$ of $G$ containing $F$. For each positive integer $n$ with $n \leq q$, recursively define $\Gamma_n$ to be the set of all graphs $H$ belonging to $\Gamma_{n-1}$ with the smallest $te(H, h_n + d_F)$. If $H \in \Gamma_q$, then there exists subset $S$ of $V(G)$ with the following properties:

1. $\Theta_l(G \setminus [S, F]) = \Theta_l(H \setminus [S, F])$.
2. $S \supseteq \{v \in V(G) : d_H(v) > h_q(v) + d_F(v)\}$.
3. For each vertex $v$ of $S$, $d_H(v) \geq h_q(v) + d_F(v)$.

Proof. Apply the same arguments of Theorems 4.1 with replacing $h_q(v)$ instead of $h(v)$. □

Proof of Theorem 5.2. First, define $h_q(v)$ and $\lambda_q$ as with $h(v)$ and $\lambda$ in the proof of Theorem 4.5 by replacing $\eta_q$ instead of $\eta$, where $1 \leq q \leq p$. Next, for a fixed graph $H \in \Gamma_p \subseteq \cdots \subseteq \Gamma_1$, show that $te(H, h_q + d_F) \leq t_q$, for any $q$ with $1 \leq q \leq p$, by repeatedly applying Theorem 5.3 and using the same arguments in the proof of Theorem 4.5. □

6 Packing spanning partition-connected subgraphs

In this section, we investigate edge-decomposition of highly partition-connected graphs into partition-connected spanning subgraphs. For this purpose, we first form the following lemma, which provides a generalization for Lemma 3.5.3 in [3].

Theorem 6.1. Let $G$ be a graph and let $l_1, l_2, \ldots, l_m$ be $m$ intersecting supermodular subadditive integer-valued functions on subsets of $V(G)$. If $F_1, \ldots, F_m$ is a family of edge-disjoint spanning subgraphs of $G$ with
the maximum $|E(F_1 \cup \cdots \cup F_m)|$ such that every graph $F_i$ is $l_i$-sparse, then there is a partition $P$ of $V(G)$ such that there is no edges in $E(G) \setminus E(F_1 \cup \cdots \cup F_m)$ joining different parts of $P$, and also for each $i$ with $1 \leq i \leq m$ and every $A \in P$, the graph $F_i[A]$ is $l_i$-partition-connected.

**Proof.** Define $F = (F_1, \ldots, F_m)$. Let $A$ be the set of all $m$-tuples $F = (F_1, \ldots, F_m)$ with the maximum $|E(F)|$ such that $F_1, \ldots, F_m$ are edge-disjoint spanning subgraphs of $G$ and every $F_i$ is $l_i$-sparse, where $E(F) = E(F_1 \cup \cdots \cup F_m)$. Note that if $e \in E(G) \setminus E(F)$, then every graph $F_i + e$ is not $l_i$-sparse; otherwise, we replace $F_i$ by $F_i + e$ in $F$, which contradicts maximality of $|E(F)|$. Thus both ends of $e$ lie an $l_i$-partition-connected subgraph of $F_i$. Let $Q_i$ be the $l_i$-partition-connected subgraph of $F_i$ including both ends of $e$ with minimum number of vertices. Let $e' \in Q_i$. Define $F'_i = F_i - e' + e$, and $F'_j = F_j$ for all $j$ with $j \neq i$.

According to Proposition 2.8, the graph $F'_i$ is again $l_i$-sparse and so $F' = (F'_1, \ldots, F'_m) \in A$. We say that $F'$ is obtained from $F$ by replacing a pair of edges. Let $A_0$ be the set of all $m$-tuples $F$ in $A$ which can be obtained from $F$ by a series of edge replacements. Let $G_0$ be the spanning subgraph of $G$ with

$$E(G_0) = \bigcup_{F \in A_0} (E(G) \setminus E(F)).$$

Now, we prove the following claim.

**Claim.** Let $F = (F_1, \ldots, F_m) \in A_0$ and assume that $F' = (F'_1, \ldots, F'_m)$ is obtained from $F$ by replacing a pair of edges. If $x$ and $y$ are two vertices in an $l_i$-partition-connected subgraph of $F'_i \cap G_0$, then $x$ and $y$ are also in an $l_i$-partition-connected subgraph of $F_i \cap G_0$, where $1 \leq i \leq m$.

**Proof of Claim.** Let $e'$ be the new edge in $E(F') \setminus E(F)$. Define $Q'_i$ to be the minimal $l_i$-partition-connected subgraph of $F'_i \cap G_0$ including $x$ and $y$. We may assume that $e' \in E(Q'_i)$; otherwise, $E(Q'_i) \subseteq E(F_i) \cap E(G_0)$ and the proof can easily be completed. Since $e' \in E(F') \setminus E(F)$, both ends of $e'$ must lie in an $l_i$-partition-connected subgraph of $F_i$. Define $Q_i$ to be the minimal $l_i$-partition-connected subgraph of $F_i$ including both ends of $e'$. By Proposition 2.8, for every edge $e \in E(Q_i)$, the graph $F_i - e + e'$ remains $l_i$-sparse, which can imply that $E(Q_i) \subseteq E(G_0)$. Define $Q = (Q_1 \cup Q'_1) - e'$. Note that $Q$ includes $x$ and $y$, and also $E(Q) \subseteq E(G_0) \cap E(F_i)$. Since $Q \cup V(Q_i)$ and $Q[V(Q_i)]$ are $l_i$-partition-connected, by Proposition 2.2, the graph $Q$ itself must be $l_i$-partition-connected. Hence the claim holds.

Define $P$ to be the partition of $V(G)$ obtained from the components of $G_0$. Let $i \in \{1, \ldots, m\}$, let $C_0$ be a component of $G_0$, and let $xy \in E(C_0)$. By the definition of $G_0$, there is no edges in $E(G) \setminus E(F_1 \cup \cdots \cup F_m)$ joining different parts of $P$, and also there are some $m$-tuples $F^1, \ldots, F^n$ in $A_0$ such that $xy \in E(G) \setminus E(F_n)$, $F = F^1$, and every $F^k$ can be obtained from $F^{k-1}$ by replacing a pair of edges, where $1 \leq k \leq n$. As we stated above, $x$ and $y$ must lie in an $l_i$-partition-connected subgraph of $F^n_i$. Let $Q'_i$ be the minimal $l_i$-partition-connected subgraph of $F^n_i$ including $x$ and $y$. By Proposition 2.8, for every edge $e \in E(Q'_i)$, the graph $F^n_i - e + xy$ remains $l_i$-sparse, which can imply $E(Q'_i) \subseteq E(G_0)$. Thus $x$ and $y$ must also lie in an $l_i$-partition-connected subgraph of $F^i_1 \cap G_0$. By repeatedly applying the above-mentioned claim, one can conclude that $x$ and $y$ lie in an $l_i$-partition-connected subgraph of $F_i \cap G_0$. Let $Q_i$ be the minimal $l_i$-partition-connected subgraph of $F_i$ including $x$ and $y$ so that $E(Q_i) \subseteq E(G_0)$. Since $l$ is subadditive, Proposition 2.6 implies that $d_{Q_i}(A) \geq 1$, for every vertex set $A$ with $\{x, y\} \subseteq A \subseteq V(Q_i)$. Since $C_0$ is
connected, we must have \( V(Q_i) \subseteq V(C_0) \). In other words, for every \( xy \in E(C_0) \), there is an \( l_i \)-partition-connected subgraph of \( F_i \cap C_0 \) including \( x \) and \( y \). Since \( C_0 \) is connected, all vertices of \( C_0 \) must lie in an \( l_i \)-partition-connected subgraph of \( F_i \cap C_0 \). Thus the graph \( F_i[V(C_0)] \) itself must be \( l_i \)-partition-connected. Hence the proof is completed.

The following theorem generalizes the well-known result of Nash-Williams [17] and Tutte [21].

**Theorem 6.2.** Let \( G \) be a graph and let \( l_1, l_2, \ldots, l_m \) be \( m \) intersecting supermodular subadditive integer-valued functions on subsets of \( V(G) \). If \( G \) is \( (l_1 + \cdots + l_m) \)-partition-connected, then it can be decomposed into \( m \) edge-disjoint spanning subgraphs \( H_1, \ldots, H_m \) such that every graph \( H_i \) is \( l_i \)-partition-connected.

**Proof.** Let \( F_1, \ldots, F_m \) be a family of edge-disjoint spanning subgraphs of \( G \) with the maximum \( |E(F)| \) such that every graph \( F_i \) is \( l_i \)-sparse, where \( F = F_1 \cup \cdots \cup F_m \). Let \( P \) be a partition of \( V(G) \) with the properties described in Theorem 6.1. Since for every \( A \in P \), the induced subgraph \( F_i[A] \) is \( l_i \)-partition-connected, we must have \( e_{F_i}(A) = \sum_{v \in A} l_i(v) - l_i(A) \). Define \( l = l_1 + \cdots + l_m \). By the assumption, \( e_G(P) \geq \sum_{A \in P} l(A) - l(G) \). Since \( e_F(P) = e_G(P) \), we have

\[
|E(F)| = e_F(P) + \sum_{A \in P} e_F(A) \geq \sum_{A \in P} l(A) - l(G) + \sum_{A \in P} \left( \sum_{v \in A} l(v) - l(A) \right) = \sum_{v \in V(G)} l(v) - l(G).
\]

On the other hand,

\[
|E(F)| = \sum_{1 \leq i \leq m} |E(F_i)| \leq \sum_{1 \leq i \leq m} \left( \sum_{v \in V(G)} l_i(v) - l_i(G) \right) = \sum_{v \in V(G)} l(v) - l(G).
\]

Therefore, for every graph \( F_i \), the equality \( |E(F_i)| = \sum_{v \in V(G)} l_i(v) - l_i(G) \) must be hold, which implies that \( F_i \) is \( l_i \)-partition-connected. This can complete the proof.

**Corollary 6.3.** ([22], see [10, Theorem 10.5.9]) Every \( l_{p+m,m} \)-partition-connected graph has a packing of \( m \) spanning trees and \( p \) spanning \( l_{1,0} \)-partition-connected subgraphs, where \( l_{n,m} \) denotes the set function that is \( n \) on vertices and is \( m \) on vertex sets with at least two vertices.

In the following, we give an alternative proof for a special case of Theorem 3.13.

**Corollary 6.4.** Let \( G \) be a graph and let \( l \) be an intersecting supermodular subadditive nonnegative integer-valued function on subsets of \( V(G) \). If \( G \) is \( 2l \)-edge-connected, then it has a spanning \( l \)-partition-connected subgraph \( H \) such that for each vertex \( v \),

\[
d_H(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor + l(v).
\]

Furthermore, for a given arbitrary vertex \( u \) the upper bound can be reduced to \( \left\lfloor \frac{d_G(u)}{2} \right\rfloor + l(u) - l(G) \).
Proof. Define \( \ell(u) = [d_G(u)/2] - l(u) + l(G) \), and \( \ell(v) = [d_G(v)/2] - l(v) \), for each vertex \( v \) with \( v \neq u \) so that \( d_G(u) \geq 2l(u) + 2\ell(u) - 2l(G) - 1 \) and \( d_G(v) \geq 2l(v) + 2\ell(v) \). Define \( \ell(A) = 0 \) for every vertex set \( A \) with \( |A| \geq 2 \). Let \( P \) be a partition of \( V(G) \). By the assumption,

\[
\sum_{A \in P} d_G(A) \geq \sum_{A \in P, |A| \geq 2} 2\ell(A) + \sum_{A \in P, |A| = 1} d_G(A) \geq \sum_{A \in P} (2\ell(A) + 2\ell(A)) - 2l(G) - 1.
\]

which implies that

\[
\epsilon_G(P) = \frac{1}{2} \sum_{A \in P} d_G(A) \geq \sum_{A \in P} (\ell(A) + \ell(A)) - l(G) - \ell(G).
\]

Thus \( G \) is \((l + \ell)\)-partition-connected. By Theorem 6.2, the graph \( G \) can be decomposed into an \( l \)-partition-connected spanning subgraph \( H \) and an \( \ell \)-partition-connected spanning subgraph \( H' \). For each vertex \( v \), we must have \( d_{H'}(v) \geq \ell(G) - \ell(v) = \ell(v) \). This implies that \( d_H(v) = d_G(v) - d_{H'}(v) \leq [d_G(v)/2] + l(v) \). Likewise, \( d_H(u) = d_G(u) - d_{H'}(u) \leq [d_G(u)/2] + l(u) - l(G) \). Hence the corollary is proved.

Corollary 6.5. Let \( G \) be a graph and let \( \ell_1, \ell_2, \ldots, \ell_m \) be \( m \) nonincreasing intersecting supermodular nonnegative integer-valued functions on subsets of \( V(G) \) with \( \ell_1(G) = \cdots = \ell_m(G) = 0 \). If \( G \) is \((2\ell_1 + \cdots + 2\ell_m)\)-edge-connected, then it has an orientation and \( m \) edge-disjoint spanning subdigraphs \( H_1, \ldots, H_m \) such that every digraph \( H_i \) is \( \ell_i \)-arc-connected and for each vertex \( v \),

\[
d^+_G(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor.
\]

Furthermore, for a given arbitrary vertex \( u \) the upper bound can be reduced to \( \left\lfloor \frac{d_G(u)}{2} \right\rfloor \).

Proof. Define \( \ell_0(u) = [d_G(u)/2] - l(u) \), and \( \ell_0(v) = [d_G(v)/2] - l(v) \) for each vertex \( v \) with \( v \neq u \), where \( l = \ell_1 + \cdots + \ell_m \). Define \( \ell_0(A) = 0 \) for every vertex set \( A \) with \( |A| \geq 2 \). Let \( P \) be a partition of \( V(G) \). By the assumption,

\[
\sum_{A \in P} d_G(A) \geq \sum_{A \in P, |A| \geq 2} 2\ell(A) + \sum_{A \in P, |A| = 1} d_G(A) \geq \sum_{A \in P} (2\ell(A) + 2\ell_0(A)) - 1.
\]

which implies that

\[
\epsilon_G(P) = \frac{1}{2} \sum_{A \in P} d_G(A) \geq \sum_{A \in P} (\ell(A) + \ell_0(A)) - l(G) - \ell_0(G).
\]

Thus \( G \) is \((l + \ell_0)\)-partition-connected. By Theorem 6.2, the graph \( G \) can be decomposed into \( m + 1 \) edge-disjoint spanning subgraphs \( H_0, \ldots, H_m \) such that every \( H_i \) is \( \ell_i \)-partition-connected. By Lemma 3.10, every \( H_i \) has an \( \ell_i \)-arc-connected orientation. Consider the orientation of \( G \) obtained from these orientations. For each vertex \( v \), we must have \( d^+_G(v) \leq d_G(v) - \sum_{0 \leq i \leq k} d^-_{H_i}(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor \). Likewise, \( d^+_G(u) \leq d_G(u) - \sum_{0 \leq i \leq k} d^-_{H_i}(u) \leq [d_G(u)/2] \). Hence the corollary is proved.

Corollary 6.6. Let \( G \) be a graph and let \( l_1, l_2, \ldots, l_m \) be \( m \) nonincreasing intersecting supermodular nonnegative integer-valued functions on subsets of \( V(G) \) and let \( r_1, \ldots, r_m \) be \( m \) nonnegative integer-valued
functions on \(V(G)\) with \(l_i(G) = \sum_{v \in V(G)} r_i(v)\). If \(G\) is \((2l_1 + \cdots + 2l_m)\)-edge-connected, then it has an orientation and \(m\) edge-disjoint spanning subdigraphs \(H_1, \ldots, H_m\) such that every digraph \(H_i\) is \(r_i\)-rooted \(l_i\)-arc-connected and for each vertex \(v\),
\[
d^+_G(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor.
\]
Furthermore, for a given arbitrary vertex \(u\) the upper bound can be reduced to \(\left\lfloor \frac{d_G(u)}{2} \right\rfloor\).

**Proof.** Apply Corollary 6.5 with \(l_i = l_i - r_i\), where \(r_i(A) = \sum_{v \in A} r_i(v)\) for every vertex set \(A\). \(\square\)

## 7 Packing spanning partition-connected sub-hypergraphs

In this subsection, we shall develop several results in this paper to hypergraphs in the same way. Before doing so, we introduce the needed definitions and notations for hypergraphs.

### 7.1 Definitions

Let \(\mathcal{H}\) be a hypergraph (possibly with repetition of hyperedges). The rank of \(\mathcal{H}\) is the maximum size of its hyperedges. The vertex set and the hyperedge set of \(\mathcal{H}\) are denoted by \(V(\mathcal{H})\) and \(E(\mathcal{H})\), respectively. The degree \(d_{\mathcal{H}}(v)\) of a vertex \(v\) is the number of hyperedges of \(\mathcal{H}\) including \(v\). For a set \(X \subseteq V(\mathcal{H})\), we denote by \(\mathcal{H}[X]\) the induced sub-hypergraph of \(\mathcal{H}\) with the vertex set \(X\) containing precisely those hyperedges \(Z\) of \(\mathcal{H}\) with \(Z \subseteq X\). We also denote by \(\mathcal{H}/X\) the hypergraph obtained from \(\mathcal{H}\) by contracting \(X\) into a single vertex \(u\) and replacing each hyperedge \(Z\) with \(Z \cap X \neq \emptyset\) by \((Z \setminus X) \cup \{u\}\). A spanning sub-hypergraph \(F\) is called \(l\)-**sparse**, if for all vertex sets \(A\), \(e_F(A) \leq \sum_{v \in A} l(v) - l(A)\), where \(e_F(A)\) denotes the number of hyperedges of \(F\) with \(Z \subseteq A\). Likewise, the hypergraph \(\mathcal{H}\) is called \(l\)**-partition-connected**, if for every partition \(P\) of \(V(\mathcal{H})\), \(e_{\mathcal{H}}(P) \geq \sum_{A \in P} l(A) - l(\mathcal{H})\), where \(e_{\mathcal{H}}(P)\) denotes the number of hyperedges of \(\mathcal{H}\) joining different parts of \(P\). Note that if \(l\) is intersecting supermodular, then the vertex set of \(\mathcal{H}\) can be expressed uniquely (up to order) as a disjoint union of vertex sets of some induced \(l\)-partition-connected sub-hypergraphs. These sub-hypergraphs are called the \(l\)-partition-connected components of \(\mathcal{H}\). To measure \(l\)-partition-connectivity of \(\mathcal{H}\), we define the parameter \(\Theta_l(\mathcal{H}) = \sum_{A \in P} l(A) - e_{\mathcal{H}}(P)\), where \(P\) is the partition of \(V(\mathcal{H})\) obtained from \(l\)-partition-connected components of \(\mathcal{H}\). It is not difficult to show that \(\Theta_l(\mathcal{H})\) is the maximum of all \(\sum_{A \subseteq P} l(A) - e_{\mathcal{H}}(P)\) taken over all partitions \(P\) of \(V(\mathcal{H})\). The hypergraph \(\mathcal{H}\) is said to be \(l\)**-edge-connected**, if for all nonempty proper vertex sets \(A\) of \(V(\mathcal{H})\), \(d_{\mathcal{H}}(A) \geq l(A)\), where \(d_{\mathcal{H}}(A)\) denotes the number of hyperedges \(Z\) of \(\mathcal{H}\) with \(Z \cap A \neq \emptyset\) and \(Z \setminus A \neq \emptyset\). For a vertex set \(S\), we denote by \(\sigma_{\mathcal{H}}(S)\) the sum of all \(|Z \cap S| - 1\) taken over all hyperedges \(Z\) of \(\mathcal{H}\) with \(Z \cap S \neq \emptyset\). Note that for graphs we have \(\sigma_G(S) = e_G(S)\). We call a hypergraph \(\mathcal{H}\) directed, if for every hyperedge \(Z\), a head vertex \(u\) in \(Z\) is specified; other vertices of \(Z - u\) are called the tails of \(Z\). For a vertex \(v\), we denoted by \(d_{\mathcal{H}}(v)\) the number of hyperedges with head \(v\) and denote by \(d_{\mathcal{H}}^+(v)\) and the number of hyperedges with tail \(v\). We say that a directed hypergraph \(\mathcal{H}\) is \(l\)**-arc-connected**, if for every vertex set \(A\), \(d_{\mathcal{H}}^+(A) \geq l(A)\), where
$d^-(A)$ denotes the number of hyperedges $Z$ with head vertex in $A$ and $Z \setminus A \neq \emptyset$. Likewise, $H$ is called \textit{$r$-rooted $l$-arc-connected}, if for every vertex set $A$, $d^-(A) \geq l(A) - \sum_{v \in A} r(v)$, where $r$ is a nonnegative integer-valued on $V(H)$ with $l(H) = \sum_{v \in V(H)} r(v)$. \textbf{Trimming} a hyperedge $Z$ of size at least three is the operation that $Z$ is replaced by a subset of it with size at least two, see \cite{10}. Trimming a directed hyperedge $Z$ of size at least three with head $u$ is the operation that $Z$ is replaced by a subset of it including $u$ with size at least two. A trimmed (directed) hypergraph refers to a (directed) hypergraph which is obtained by a series of trimming operations. Throughout this article, all hypergraphs have hyperedges with size at least two.

\section*{7.2 Basic tools}

For every vertex $v$ of a hypergraph $H$, consider an induced $l$-partition-connected sub-hypergraph of $H$ containing $v$ with the maximal order. The following proposition shows that these sub-hypergraphs are unique and decompose the vertex set of $H$ when $l$ is intersecting supermodular. The proofs of the results in this subsection are similar to whose graph versions.

\textbf{Proposition 7.1.} Let $H$ be a hypergraph with $X, Y \subseteq V(H)$ and let $l$ be an intersecting supermodular real function on subsets of $V(H)$. If $H[X]$ and $H[Y]$ are $l$-partition-connected and $X \cap Y \neq \emptyset$, then $H[X \cup Y]$ is also $l$-partition-connected.

\textbf{Proposition 7.2.} Let $H$ be a hypergraph with $X \subseteq V(H)$ and let $l$ be an intersecting supermodular real function on subsets of $V(H)$. If $H[X]$ and $H/X$ are $l$-partition-connected, then $H$ is also $l$-partition-connected.

\textbf{Lemma 7.3.} Let $H$ be a hypergraph and let $l$ be a real function on subsets of $V(H)$. If $G$ is $l$-partition-connected and $P$ is a partition of $V(H)$ with

$$e_H(P) = \sum_{A \in P} l(A) - l(H),$$

then for any $A \in P$, the hypergraph $H[A]$ is also $l$-partition-connected.

\textbf{Proposition 7.4.} Let $H$ be a hypergraph and let $l$ be an intersecting supermodular weakly subadditive integer-valued function on subsets of $V(H)$. If $H$ is minimally $l$-partition-connected, then

$$|E(H)| = \sum_{v \in V(H)} l(v) - l(H).$$

\textbf{Proposition 7.5.} Let $F$ be an $l$-sparse hypergraph with $|E(F)| = \sum_{v \in V(F)} l(v) - l(F)$, where $l$ is a weakly subadditive real function on subsets of $V(F)$. If $P$ is a partition of $V(F)$, then

$$e_F(P) \geq \sum_{A \in P} l(A) - l(F).$$
Furthermore, the equality holds only if for every $A \in P$, the hypergraph $F[A]$ is $l$-partition-connected.

**Proposition 7.6.** Let $F$ be an $l$-sparse hypergraph with $Y \subseteq V(F)$, where $l$ is a weakly subadditive real function on subsets of $V(F)$. Let $Q$ be an $l$-partition-connected sub-hypergraph of $F$ with the minimum number of vertices including all vertices of $Y$. If $l$ is element-subadditive, then for each $z \in V(Q) \setminus Y$, $d_Q(z) \geq 1$. Furthermore, if $l$ is subadditive, then for every vertex set $A$ with $Y \subseteq A \subseteq V(Q)$, $d_Q(A) \geq 1$.

**Proposition 7.7.** Let $\mathcal{H}$ be a hypergraph and let $l$ be an intersecting supermodular weakly subadditive integer-valued function on subsets of $V(\mathcal{H})$. Let $F$ be an $l$-sparse spanning sub-hypergraph of $\mathcal{H}$. If $Z'$ is an $l$-sparse supermodular subadditive intersecting supermodular weakly subadditive hypergraph with the minimum number of vertices, then for every $Z \in E(Q)$, the hypergraph $F - Z + Z'$ remains $l$-sparse.

### 7.3 Packing spanning partition-connected sub-hypergraphs

The following theorem provides an extension for Theorem 6.1 on hypergraphs.

**Theorem 7.8.** Let $\mathcal{H}$ be a hypergraph and let $l_1, l_2, \ldots, l_m$ be $m$ intersecting supermodular subadditive integer-valued functions on subsets of $V(\mathcal{H})$. If $F_1, \ldots, F_m$ is a family of edge-disjoint spanning sub-hypergraphs of $\mathcal{H}$ with the maximum $|E(F_1 \cup \cdots \cup F_m)|$ such that every hypergraph $F_i$ is $l_i$-sparse, then there is a partition $P$ of $V(\mathcal{H})$ such that there is no hyperedges in $E(\mathcal{H}) \setminus E(F_1 \cup \cdots \cup F_m)$ joining different parts of $P$, and also for each $i$ with $1 \leq i \leq m$ and every $A \in P$, the hypergraph $F_i[A]$ is $l_i$-partition-connected.

**Proof.** Define $F = (F_1, \ldots, F_m)$. Let $A$ be the set of all $m$-tuples $\mathcal{F} = (F_1, \ldots, F_m)$ with the maximum $|E(\mathcal{F})|$ such that $F_1, \ldots, F_m$ are edge-disjoint spanning sub-hypergraphs of $\mathcal{H}$ and every $F_i$ is $l_i$-sparse, where $E(\mathcal{F}) = E(F_1 \cup \cdots \cup F_m)$. Note that if $Z \in E(\mathcal{H}) \setminus E(\mathcal{F})$, then every hypergraph $F_i + Z$ is not $l_i$-sparse; otherwise, we replace $F_i$ by $F_i + Z$ in $\mathcal{F}$, which contradicts maximality of $|E(\mathcal{F})|$. Thus all vertices of $Z$ lie in an $l_i$-partition-connected subgraph of $F_i$. Let $Q_i$ be the $l_i$-partition-connected sub-hypergraph of $F_i$ including all vertices of $Z$ with minimum number of vertices. Let $Z' \in Q_i$. Define $F'_i = F_i - Z' + Z$, and $F'_j = F_j$ for all $j$ with $j \neq i$. According to Proposition 7.7, the hypergraph $F'_i$ is again $l_i$-sparse and so $\mathcal{F}' = (F'_1, \ldots, F'_m) \in A$. We say that $\mathcal{F}'$ is obtained from $\mathcal{F}$ by replacing a pair of hyperedges. Let $\mathcal{A}_0$ be the set of all $m$-tuples $\mathcal{F}$ in $A$ which can be obtained from $\mathcal{F}$ by a series of hyperedge replacements. Let $\mathcal{H}_0$ be the spanning sub-hypergraph of $\mathcal{H}$ with

$$E(\mathcal{H}_0) = \bigcup_{\mathcal{F} \in \mathcal{A}_0} (E(\mathcal{H}) \setminus E(\mathcal{F})).$$

Now, we prove the following claim.

**Claim.** Let $\mathcal{F} = (F_1, \ldots, F_m) \in \mathcal{A}_0$ and assume that $\mathcal{F}' = (F'_1, \ldots, F'_m)$ is obtained from $\mathcal{F}$ by replacing a pair of hyperedges. If all vertices of a given vertex set $Y$ lie in an $l_i$-partition-connected sub-hypergraph of
Proposition \( \in \langle k \rangle \) where \( 1 \leq i \leq m \).

**Proof of Claim.** Let \( Z' \) be the new hyperedge in \( E(F') \setminus E(F) \). Define \( Q_i' \) to be the minimal \( l_i \)-partition-connected sub-hypergraph of \( F_i' \cap \mathcal{H}_0 \) including all vertices of \( Y \). We may assume that \( Z' \in E(Q_i') \); otherwise, \( E(Q_i') \subseteq E(F_i) \cap E(\mathcal{H}_0) \) and the proof can easily be completed. Since \( Z' \in E(F') \setminus E(F) \), all vertices of \( Z' \) must lie in an \( l_i \)-partition-connected sub-hypergraph of \( F_i \). Define \( Q_i \) to be the minimal \( l_i \)-partition-connected sub-hypergraph of \( F_i \) including all vertices of \( Z' \). By Proposition 7.7, for every hyperedge \( Z \in E(Q_i) \), the hypergraph \( F_i - Z + Z' \) remains \( l_i \)-sparse, which can imply that \( E(Q_i) \subseteq E(\mathcal{H}_0) \). Define \( Q = (Q_i \cup Q_i') - Z' \). Note that \( Q \) includes all vertices of \( Y \), and also \( E(Q) \subseteq E(\mathcal{H}_0) \cap E(F_i) \).

Since \( Q/V(Q_i) \) and \( Q[V(Q_i)] \) are \( l_i \)-partition-connected, by Proposition 7.2, the hypergraph \( Q \) itself must be \( l_i \)-partition-connected. Hence the claim holds.

Define \( P \) to be the partition of \( V(\mathcal{H}) \) obtained from the components of \( \mathcal{H}_0 \). Let \( i \in \{1, \ldots, m\} \), let \( C_0 \) be a component of \( \mathcal{H}_0 \), and let \( Y \in E(C_0) \). By the definition of \( \mathcal{H}_0 \), there is no hyperedges in \( E(\mathcal{H}) \setminus E(F_1 \cup \cdots \cup F_m) \) joining different parts of \( P \), and also there are some \( m \)-tuples \( F^1, \ldots, F^n \) in \( A_0 \) such that \( Y \in E(\mathcal{H}) \setminus E(F^n) \), \( F = F^1 \), and every \( F^k \) can be obtained from \( F^{k-1} \) by replacing a pair of hyperedges, where \( 1 < k \leq n \). As we stated above, all vertices of \( Y \) must lie in an \( l_i \)-partition-connected sub-hypergraph of \( F^n \). Let \( Q'_i \) be the minimal \( l_i \)-partition-connected sub-hypergraph of \( F^n \) including all vertices of \( Y \). By Proposition 7.7, for every hyperedge \( Z \in E(Q_i') \), the hypergraph \( F^n - Z + Y \) remains \( l_i \)-sparse, which can imply \( E(Q_i') \subseteq E(\mathcal{H}_0) \). Thus all vertices of \( Y \) must also lie in an \( l_i \)-partition-connected subgraph of \( F_i' \cap \mathcal{H}_0 \). By repeatedly applying the above-mentioned claim, one can conclude that all vertices of \( Y \) lie in an \( l_i \)-partition-connected sub-hypergraph of \( F_i \cap \mathcal{H}_0 \). Let \( Q_i \) be the minimal \( l_i \)-partition-connected sub-hypergraph of \( F_i \) including all vertices of \( Y \) so that \( E(Q_i) \subseteq E(\mathcal{H}_0) \). Since \( l \) is subadditive, Proposition 7.6 implies that \( d_{Q_i}(A) \geq 1 \), for every vertex set \( A \) with \( Y \subseteq A \subseteq V(Q_i) \). Since \( C_0 \) is connected, we must have \( V(Q_i) \subseteq V(C_0) \). In other words, for every \( Y \in E(C_0) \), there is an \( l_i \)-partition-connected sub-hypergraph of \( F_i \cap C_0 \) including all vertices of \( Y \). Since \( C_0 \) is connected, all vertices of \( C_0 \) must lie in an \( l_i \)-partition-connected sub-hypergraph of \( F_i \cap C_0 \). Thus the hypergraph \( F_i[V(C_0)] \) itself must be \( l_i \)-partition-connected. Hence the proof is completed.

The following theorem generalizes a result in [13, Theorem 2.8] due to Frank, Király, and Kriesell (2003).

**Theorem 7.9.** Let \( \mathcal{H} \) be a hypergraph and let \( l_1, l_2, \ldots, l_m \) be \( m \) intersecting supermodular subadditive integer-valued functions on subsets of \( V(\mathcal{H}) \). If \( \mathcal{H} \) is \((l_1 + \cdots + l_m)\)-partition-connected, then it can be decomposed into \( m \) edge-disjoint spanning sub-hypergraphs \( H_1, \ldots, H_m \) such that every hypergraph \( H_i \) is \( l_i \)-partition-connected.

**Proof.** Let \( F_1, \ldots, F_m \) be a family of edge-disjoint spanning sub-hypergraphs of \( \mathcal{H} \) with the maximum \( |E(F)| \) such that every hypergraph \( F_i \) is \( l_i \)-sparse, where \( F = F_1 \cup \cdots \cup F_m \). Let \( P \) be a partition of \( V(\mathcal{H}) \) with the properties described in Theorem 7.8. Since for every \( A \in P \), the induced sub-hypergraph \( F_i[A] \)
is \( l_i \)-partition-connected, we must have \( e_{F_i}(A) = \sum_{v \in A} l_i(v) - l_i(A) \). Define \( l = l_1 + \cdots + l_m \). By the assumption, \( e_\mathcal{H}(P) \geq \sum_{A \in P} l(A) - l(\mathcal{H}) \). Since \( e_\mathcal{H}(P) = e_\mathcal{H}(P) \), we have
\[
|E(F)| = e_\mathcal{H}(P) + \sum_{A \in P} e_\mathcal{H}(P) \geq \sum_{A \in P} l(A) - l(\mathcal{H}) + \sum_{A \in P} (\sum_{v \in A} l(v) - l(A)) = \sum_{v \in V(\mathcal{H})} l(v) - l(\mathcal{H}).
\]
On the other hand,
\[
|E(F)| = \sum_{1 \leq i \leq m} |E(F_i)| \leq \sum_{1 \leq i \leq m} (\sum_{v \in V(\mathcal{H})} l_i(v) - l_i(\mathcal{H})) = \sum_{v \in V(\mathcal{H})} l(v) - l(\mathcal{H}).
\]
Therefore, for every hypergraph \( F_i \), the equality \( |E(F_i)| = \sum_{v \in V(\mathcal{H})} l_i(v) - l_i(\mathcal{H}) \) must hold, which implies that \( F_i \) is \( l_i \)-partition-connected. This can complete the proof. \( \square \)

The following result provides an improvement for Corollary 2.9 in [13].

**Corollary 7.10.** Let \( \mathcal{H} \) be a hypergraph with the rank \( r \) and let \( l \) be an intersecting supermodular subadditive nonnegative integer-valued function on subsets of \( V(\mathcal{H}) \). If \( \mathcal{H} \) is \( rl \)-edge-connected, then it has an \( l \)-partition-connected spanning sub-hypergraph \( H \) such that for each vertex \( v \),
\[
d_H(v) \leq \left\lfloor \frac{r-1}{r} d_\mathcal{H}(v) \right\rfloor + l(v).
\]
Furthermore, for a given arbitrary vertex \( u \) the upper bound can be reduced to \( \left\lfloor \frac{r-1}{r} d_\mathcal{H}(u) \right\rfloor + l(u) - l(\mathcal{H}) \).

**Proof.** Define \( \ell(u) = \lfloor d_\mathcal{H}(u)/r \rfloor - l(u) + l(\mathcal{H}) \), and \( \ell(v) = \lfloor d_\mathcal{H}(v)/r \rfloor - l(v) \), for each vertex \( v \) with \( v \neq u \) so that \( d_\mathcal{H}(u) \geq rl(u) + r\ell(u) + rl(\mathcal{H}) - (r-1) \) and \( d_\mathcal{H}(v) \geq rl(v) + r\ell(v) \). Define \( \ell(A) = 0 \) for every vertex set \( A \) with \( |A| \geq 2 \). Let \( P \) be a partition of \( V(\mathcal{H}) \). By the assumption,
\[
\sum_{A \in P} d_\mathcal{H}(A) \geq \sum_{A \in P, |A| \geq 2} rl(A) + \sum_{A \in P, |A| = 1} d_\mathcal{H}(A) \geq \sum_{A \in P} (rl(A) + r\ell(A)) - rl(\mathcal{H}) - (r-1),
\]
which implies that
\[
e_\mathcal{H}(P) \geq \frac{1}{r} \sum_{A \in P} d_\mathcal{H}(A) \geq \sum_{A \in P} (l(A) + \ell(A)) - l(\mathcal{H}) - l(\mathcal{H}).
\]
Thus \( \mathcal{H} \) is \((l + \ell)\)-partition-connected. By Theorem 7.9, the hypergraph \( \mathcal{H} \) can be decomposed into an \( l \)-partition-connected spanning sub-hypergraph \( H \) and an \( \ell \)-partition-connected spanning sub-hypergraph \( H' \). For each vertex \( v \), we must have \( d_{H'}(v) \geq \ell(\mathcal{H} - v) + \ell(v) - \ell(\mathcal{H}) = \ell(v) \). This implies that \( d_{H'}(v) = d_\mathcal{H}(v) - d_{H'}(v) \leq \left\lfloor \frac{r-1}{r} d_\mathcal{H}(v) \right\rfloor + l(v) \). Likewise, \( d_{H'}(u) = d_\mathcal{H}(u) - d_{H'}(u) \leq \left\lfloor \frac{r-1}{r} d_\mathcal{H}(u) \right\rfloor + l(u) - l(\mathcal{H}) \). Hence the theorem holds. \( \square \)

For every hypergraph \( \mathcal{H} \), one may associate a nonnegative set function \( r \) such that for every vertex set \( A \), \( r(A) \) is the maximum of all \( |A \setminus Z| + 1 \) taken over all hyperedges \( Z \) with \( |Z \cap A| \neq \emptyset \). We call \( r(A) \) the local rank of \( \mathcal{H} \) on \( A \). According this definition, the above-mentioned corollary could be refined to the following version.
Corollary 7.11. Let $\mathcal{H}$ be a hypergraph with the local rank function $r$ and let $l$ be an intersecting supermodular subadditive nonnegative integer-valued function on subsets of $V(\mathcal{H})$. If $\mathcal{H}$ is $rl$-edge-connected, then it has an $l$-partition-connected spanning sub-hypergraph $H$ such that for each vertex $v$, $d_H(v) \leq \left\lceil \frac{r(v) - 1}{r(v)} d_H(v) \right\rceil + l(v)$.

Proof. Apply the same arguments of Corollary 7.10 by replacing the inequality $e_H(P) \geq \sum_{A \in P} \frac{1}{r(A)} d_H(A)$.

The following result can be proved similarly to whose graph version and can also be formulated in a rooted arc-connected version.

Corollary 7.12. Let $\mathcal{H}$ be a hypergraph with the rank $r$ and let $\ell_1, \ell_2, \ldots, \ell_m$ be $m$ nonincreasing intersecting supermodular nonnegative integer-valued functions on subsets of $V(\mathcal{H})$ with $\ell_1(\mathcal{H}) = \cdots = \ell_m(\mathcal{H}) = 0$. If $\mathcal{H}$ is $(r\ell_1 + \cdots + r\ell_m)$-edge-connected, then it has an orientation and $m$ edge-disjoint spanning sub-hypergraphs $H_1, \ldots, H_m$ such that every hypergraph $H_i$ is $\ell_i$-arc-connected and for each vertex $v$,

$$d_H^+(v) \leq \left\lfloor \frac{r - 1}{r} d_H(v) \right\rfloor.$$

Furthermore, for a given arbitrary vertex $u$ the upper bound can be reduced to $\left\lfloor \frac{r - 1}{r} d_H(u) \right\rfloor$.

7.4 Trimming hypergraphs and preserving partition-connectivity

As we observed in the previous subsection, the proof of Theorem 7.9 follows from the same arguments of whose graph version. In fact, Theorem 7.9 can easily be derived from whose graph version, using the following generalization of Theorem 9.4.5 in [10].

Theorem 7.13. Let $\mathcal{H}$ be a hypergraph and let $l$ be an intersecting supermodular weakly subadditive integer-valued function on subsets of $V(\mathcal{H})$. If $\mathcal{H}$ is $l$-partition-connected, then it can be trimmed to an $l$-partition-connected graph.

We show below that the operations can be done without removing specified vertices from hyperedges.

Theorem 7.14. Let $\mathcal{H}$ be a directed hypergraph and let $l$ be an intersecting supermodular weakly subadditive integer-valued function on subsets of $V(\mathcal{H})$. If $\mathcal{H}$ is $l$-partition-connected, then it can be trimmed to an $l$-partition-connected directed graph.

Proof. By induction on the sum of all $|Z| - 2$ taken over all hyperedges $Z$. If this sum is zero, then $\mathcal{H}$ itself is a graph. So assume that a directed hyperedge $Z$ with head $u$ has size at least three. Let $x$ a vertex of $Z \setminus \{u\}$. If replacing $Z$ by $Z - x$ preserves partition-connectivity, then the proof follows by induction. Otherwise, there is a partition $P$ of $V(\mathcal{H})$ such that $e_H(P) = \sum_{A \in P} l(A) - l(\mathcal{H})$ and $Z \setminus X = \{x\}$, for a
vertex set \( X \in P \). Let \( y \) be a vertex of \( Z \cap X \setminus \{u\} \). Now, replace \( Z \) by \( Z - y \) and call the resulting hypergraph \( H' \). According to this construction, we must have \( H'/X = H/X \) and \( H[X] = H'[X] \). Since \( H'/X \) and \( H'[X] \) are \( l \)-partition-connected, by Proposition 7.2, the hypergraph \( H' \) itself must be \( l \)-partition-connected. Thus by the induction hypothesis the theorem can be hold.

The following theorem is a counterpart of Theorem 7.14.

**Theorem 7.15.** Let \( H \) be a directed hypergraph and let \( l \) be an intersecting supermodular weakly subadditive integer-valued function on subsets of \( V(H) \). If \( H \) is \( l \)-sparse, then it can be trimmed to an \( l \)-sparse directed graph.

**Proof.** By induction on the sum of all \(|Z| - 2\) taken over all hyperedges \( Z \). If this sum is zero, then \( H \) itself is a graph. So assume that a directed hyperedge \( Z \) with head \( u \) has size at least three. Let \( x \) and \( y \) be two vertices of \( Z \setminus \{u\} \). If replacing \( Z \) by \( Z - x \) preserves sparse property, then the proof follows by induction.

Otherwise, there is a vertex set \( X \) including \( u \) such that \( Z \setminus X = \{x\} \) and \( e_H(X) = \sum_{v \in X} l(v) - l(X) \). Corresponding to \( y \), there is a vertex set \( Y \) including \( u \) such that \( Z \setminus Y = \{y\} \) and \( e_H(Y) = \sum_{v \in Y} l(v) - l(Y) \).

Note that \( Z \) is neither a subset of \( X \) nor a subset of \( Y \). Thus
\[
e_H(X \cup Y) \geq e_H(X) + e_H(Y) - e_H(X \cap Y) + 1.
\]

Since \( l \) is intersecting supermodular, we must have
\[
e_H(X \cup Y) \geq \sum_{v \in X} l(v) - l(X) + \sum_{v \in Y} l(v) - l(Y) - \sum_{v \in X \cup Y} l(v) + l(X \cap Y) + 1 > \sum_{v \in X \cup Y} l(v) + l(X \cup Y).
\]

This is a contradiction, as desired.

The following theorem generalizes Theorem 7.4.9 in [10].

**Theorem 7.16.** Let \( H \) be a directed hypergraph and let \( l \) be a positively intersecting supermodular integer-valued function on subsets of \( V(H) \) with \( \ell(\emptyset) = \ell(H) = 0 \). If \( H \) is \( \ell \)-arc-connected, then it can be trimmed to an \( \ell \)-arc-connected directed graph.

**Proof.** By induction on the sum of all \(|Z| - 2\) taken over all hyperedges \( Z \). If this sum is zero, then \( H \) itself is a graph. So assume that a directed hyperedge \( Z \) with head \( u \) has size at least three. Let \( x \) and \( y \) be two vertices of \( Z \setminus \{u\} \). If replacing \( Z \) by \( Z - x \) preserves arc-connectivity, then the proof follows by induction.

Otherwise, there is a vertex set \( X \) including \( u \) such that \( Z \setminus X = \{x\} \) and \( \ell(X) = d^+_H(X) > 0 \). Corresponding to \( y \), there is a vertex set \( Y \) including \( u \) such that \( Z \setminus Y = \{y\} \) and \( \ell(Y) = d^+_H(Y) > 0 \). Note that \( Z \) is neither a subset of \( X \) nor a subset of \( Y \). Thus
\[
\ell(X) + \ell(Y) = d^+_H(X) + d^+_H(Y) \geq d^+_H(X \cup Y) + d^+_H(X \cap Y) + 1 > \ell(X \cup Y) + \ell(X \cap Y).
\]

Since \( \ell \) is intersecting supermodular, we arrive at a contradiction.
7.5 Spanning partition-connected sub-hypergraphs with restricted degrees

The following theorem provides a generalization for Theorem 3.9 with a new proof.

**Theorem 7.17.** Let \( H \) be a hypergraph and let \( l \) be an intersecting supermodular subadditive integer-valued function on subsets of \( V(H) \). Let \( h \) be an integer-valued function on \( V(H) \). If for all \( S \subseteq V(H) \),

\[
\Theta_l(H \setminus S) \leq \sum_{v \in S} (h(v) - l(v)) + l(H) - \sigma_H(S),
\]

then \( H \) has an \( l \)-partition-connected spanning sub-hypergraph \( H \) such that for each vertex \( v \), \( d_H(v) \leq h(v) \).

**Proof.** Define \( \ell(v) = \max\{0, d_H(v) - h(v)\} \) for each vertex \( v \), and define \( \ell(A) = 0 \) for every vertex set \( A \) with \( |A| \geq 2 \). Let \( P \) be a partition of \( V(H) \). Define \( S \) to be the set of all vertices \( v \) such that \( \{v\} \in P \) and \( \ell(v) = d_H(v) - h(v) \). Also, define \( P \) to be set of all vertex sets \( A \in P \) such that \( A \neq \{v\} \), when \( v \in S \). Note that for every \( A \in P \), \( \ell(A) = 0 \). By the assumption,

\[
\sum_{A \in P} \ell(A) - e_{H \setminus S}(P) \leq \Theta_l(H \setminus S) \leq \sum_{v \in S} (h(v) - l(v)) + l(H) - \sigma_H(S).
\]

Since \( e_H(P) = \sum_{v \in S} d_H(v) - \sigma_H(S) + e_H \setminus S(P) \), we must have

\[
e_H(P) \geq \sum_{A \in P} \ell(A) + \sum_{v \in S} (d_H(v) - h(v) + l(v)) - l(H),
\]

which implies that

\[
e_H(P) \geq \sum_{A \in P} \ell(A) + \sum_{v \in S} (\ell(v) + l(v)) - l(H) = \sum_{A \in P} (\ell(A) + l(A) + l(H)) - l(H) - l(H).
\]

Thus \( H \) is \((l + \ell)\)-partition-connected. By Theorem 7.9, the hypergraph \( H \) can be decomposed into an \( l \)-partition-connected spanning sub-hypergraph \( H \) and an \( \ell \)-partition-connected spanning sub-hypergraph \( H' \). For each vertex \( v \), we must have \( d_{H'}(v) \geq l(H - v) + \ell(v) - l(H) = \ell(v) \). This implies that \( d_H(v) \leq d_{H'}(v) \leq h(v) \). Hence the theorem is proved. \( \Box \)

The following theorem provides two upper bounds on \( \Theta_l(H \setminus S) \) depending on two parameters of connectivity of \( H \) and \( d_H(v) \) of the vertices \( v \) in \( S \), which generalizes Lemma 3.12.

**Theorem 7.18.** Let \( H \) be a hypergraph with the rank \( r \), let \( l \) be an intersecting supermodular real function on subsets of \( V(H) \), and let \( k \) be a positive real number. If \( S \subseteq V(H) \), then

\[
\Theta_l(H \setminus S) \leq \begin{cases} \sum_{v \in S} \frac{d}{k}d_H(v) - \frac{1}{k}\sigma_H(S), & \text{when } H \text{ is }kl\text{-edge-connected, } k \geq r, \text{ and } S \neq \emptyset; \\ \sum_{v \in S} \left(\frac{d_H(v)}{k} - l(v)\right) + l(H) - \frac{1}{k}\sigma_H(S), & \text{when } H \text{ is }kl\text{-partition-connected and } k \geq 1. \end{cases}
\]

**Proof.** Let \( P \) be the partition of \( V(H) \setminus S \) obtained from \( l \)-partition-connected components of \( H \setminus S \). For every integer \( i \) with \( 0 \leq i \leq r \), denote by \( c_i \) the number of hyperedges \( Z \) with \( |Z \cap S| = i \). If \( H \) is
kl-edge-connected and $S \neq \emptyset$, for any $A \in P$, there are at least $kl(A)$ hyperedges $Z$ with $Z \cap A \neq \emptyset$ and $Z \setminus A \neq \emptyset$. Thus

$$\sum_{A \in P} kl(A) - rc_0 \leq \sum_{1 \leq i \leq r} (r - i)c_i = \sum_{1 \leq i \leq k} (r - 1)i - \sum_{v \in S} r(i - 1)c_i = \sum_{v \in S} (r - 1)d_H(v) - r\sigma_H(S),$$

which implies that

$$\Theta_l(H \setminus S) \leq \sum_{A \in P} l(A) - \frac{r}{k}e_{H \setminus S}(P) \leq \sum_{v \in S} \frac{(r - 1)}{k}d_H(v) - \frac{r}{k}\sigma_H(S).$$

When $H$ is $kl$-partition-connected and $k \geq 1$, we have

$$\sum_{A \in P} kl(A) + \sum_{v \in S} kl(v) - kl(H) \leq e_H(P \cup \{v : v \in S\}) = \sum_{v \in S} d_H(v) - \sigma_H(S) + e_{H \setminus S}(P),$$

and so

$$k\Theta_l(H \setminus S) = \sum_{A \in P} kl(A) - ke_{H \setminus S}(P) \leq \sum_{A \in P} kl(A) - e_{H \setminus S}(P) \leq \sum_{v \in S} (d_H(v) - kl(v)) + kl(H) - \sigma_H(S).$$

These inequalities can complete the proof. \qed

7.6 An application to packing Steiner trees with restricted degrees

The following theorem is a strengthened version of Theorem 3.1 in [13] and can be proved in the same way, by replacing the new improved version of Corollary 2.9 in [13].

**Theorem 7.19.** Let $G$ be a graph with $S \subseteq V(G)$, where $V(G) \setminus S$ is an independent set. If $G$ is $3m$-edge-connected in $S$, then it has a spanning subgraph $H$ containing $m$ edge-disjoint Steiner trees spanning $S$ such that for each $v \in S$, $d_H(v) \leq \lceil \frac{2}{3}d_G(v) \rceil + m$.

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