Light-shift-induced quantum gates for ions in thermal motion

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An effective interaction between trapped ions in thermal motion can be generated by illuminating them simultaneously with a single laser resonant with the ion's carrier frequency. The ac Stark-shift induces simultaneous ‘virtual’ two-phonon transitions via several motional modes. Within a certain laser intensity range these transitions can interfere constructively, resulting in a relatively fast, heating-resistant two-qubit logic gate.

Over the past few years, the implementation of a practical quantum information processor has become a major goal for experimentalists across a wide range of disciplines [1]. Among the many candidate scenarios for attaining this goal, one of the most promising and well-known is the system of laser-cooled trapped ions, first suggested by Cirac and Zoller (CZ) [2,3]. Several key features of their seminal proposal have already been experimentally demonstrated [4]. Meanwhile, the inherent difficulty of these experiments has stimulated the development of an array of alternative gate schemes [5–8] making increasingly ingenious use of the ion trap’s physics.

Here we combine the best features of two of these proposals with an extra twist, constructing a two-qubit gate mechanism which is at once (i) robust against environmental heating, (ii) relatively fast, and (iii) requires relatively few experimental resources, namely a single pulse by a single laser and no ground-state-cooling. Our first ingredient is the idea, due to Sørensen and Mølmer [6], of coupling different ions using Raman-like two-phonon exchanges via a “data bus” motional mode. Since the mode is only excited ‘virtually’, this allows the realisation of gates that are relatively insensitive to the ionic vibrational state, and can be implemented even in the presence of moderate motional heating [6]. This method’s largest drawback [6] is the very low switching rate of the gates, which are substantially slower than those obtained with CZ’s method. This is undesirable, since it increases the quantum register’s vulnerability to other sources of decoherence such as technical noise in the experimental apparatus. The problem has been tackled to an extent in ref. [7], where an elegant means of considerably increasing the gate speed is provided. However, this solution requires heavily entangling the internal and motional variables during the gate operation, making it more sensitive to heating.

Our second ingredient is a method we have recently proposed for obtaining relatively fast gates between ground-state-cooled ions [8]. The method relies on the ac Stark-shift (light-shift) induced by driving an ion with a resonant laser. Normally, this driving merely leads to Rabi flopping between the ionic states, realising a one-qubit gate. However, when the induced level-splitting is equivalent to exactly one motional energy quantum, the influence of off-resonant motion-affecting transitions can gradually build up, leading to the exchange of excitations between the ion’s internal and vibrational states. This can then be used to derive a fast CZ-like two-qubit gate between any two ions in a linear chain. As with CZ’s method, however, this scheme requires ground-state cooling and is highly sensitive to motional heating.

In what follows we show a way of combining these ideas and their resulting benefits. Specifically, we propose to use the light-shift effect to drive ‘virtual’ two-phonon transitions via several motional modes simultaneously. Each mode functions as a parallel “data bus”, with an overall gate resulting from the interference of various ‘bus’ paths. The gates have speeds intermediate between those in refs. [6] and [7], but remain insensitive to heating at all times. Moreover, only a single laser beam is needed, instead of the bichromatic illumination used in [6,7]. As in [8], the same beam can also be used to drive single-qubit gates.

Let us begin with an intuitive description of our essential idea (Fig. 1). Consider a chain of $N$ ions, two of which are simultaneously and equally illuminated by a laser resonant with the carrier frequency $\Omega$. At first sight, this simply causes these ions’ internal states to flop periodically at the Rabi frequency $\Omega$. However, when the driving leads to Rabi flopping, exactly one motional energy quantum, the influence of off-resonant motion-affecting transitions can gradually build up, leading to the exchange of excitations between the ion’s internal and vibrational states. This can then be used to derive a fast CZ-like two-qubit gate.

![FIG. 1. A laser resonant with the carrier transition illuminates two ions, splitting each dressed level pair $|\pm\rangle$ by $h\Omega$. When this splitting is detuned from all vibrational frequencies $\nu_q$, states $|+\rangle \{n_q\}$ and $|-\rangle \{n_q\}$ are degenerate with each other and non-degenerate with all other levels (for clarity, only levels of a single mode $p$ are shown). Coherent oscillations are then induced between them, at a frequency independent of $\{n_q\}$. This is due to the existence, for each mode, of ‘virtual’ two-phonon transition paths connecting the two states, via levels $|+\rangle |n_p - 1\rangle$ and $|\mp\rangle |n_p + 1\rangle$.](image-url)
allowed depends on the magnitude of the level-splitting. For example, when \( \Omega \) is exactly equal to the frequency \( \nu_p \) of one of the ions’ collective motional modes, then the manifolds \{\ket{--} \ket{n_p + 1}, \ket{+-} \ket{n_p}, \ket{+-} \ket{n_p}, \ket{++} \ket{n_p} \} \) degenerate and one expects transitions altering the phonon number in that mode. This is interesting, but inadequate for obtaining a motion-independent ion-ion gate. If, however, \( \Omega \) is slightly detuned from this value, then transitions involving changes in \( n_p \) become off-resonant and are suppressed. One is left only with oscillations between \ket{+-} \ket{n_p} and \ket{+-} \ket{n_p}, which can be interpreted as occurring via the Raman-like absorption and emission of ‘virtual’ phonons.

Now, just as in ref. [3,10], each of these virtual transitions can occur via two different paths: one ‘via’ level \ket{++} \ket{n_p - 1}, with amplitude proportional to \( n_p \) and positive detuning \( \Delta_p \), and one ‘via’ level \ket{--} \ket{n_p + 1}, with amplitude proportional to \( n_p + 1 \) and negative detuning \(-\Delta_p \). The overall transition amplitude will thus be the difference of these terms; remarkably, the \( n_p \)-dependence should cancel out, and we can expect motion-state-independent transitions between the internal states \ket{+-} and \ket{+-}! Of course, the ion system contains in fact several motional modes, each of which will lead to a separate resonance in the Rabi frequency. However, if \( \Omega \) is sufficiently detuned from all these resonances, the argument above holds simultaneously for all modes. In other words, the overall transition amplitude between \ket{++} and \ket{+-} should result from the interference of multiple Raman-like paths, two for each separate mode. Thus, the totality of modes are used together as a ‘collective data bus’ connecting the two ions.

To see that all this is really the case, consider the Hamiltonian of this system. Within the Lamb-Dicke limit

\[
\eta_p \sqrt{(n_p + 1)} \ll 1 \tag{1}
\]

(where \( \eta_p \) is the Lamb-Dicke (LD) parameter of the \( p^{th} \) collective mode of the ions and \( \bar{n}_p \) is the average number of phonons in that mode), this can be written as [3]

\[
H \simeq \frac{\Omega}{2} \sum_{j=1}^{2} e^{i\phi_j} \sigma_j + \frac{\eta_{jp}}{2} \left[ 1 + \sum_{p=1}^{N} \eta_{jp} \left[ a_p^\dagger e^{i\nu_p t} + h.c. \right] \right] + h.c. \tag{2}
\]

Here \( \Omega \) is the effective Rabi frequency of the laser-ion interaction (assumed to be equal for both ions), \( \phi_j \) is the laser’s phase at the position of the \( j^{th} \) ion, \( \nu_p \) is the frequency of the \( p^{th} \) collective mode (\( p = 1 \) for the centre-of-mass (CM) mode, \( p = 2 \) for the ‘breathing’ mode, etc.) and \( \eta_{jp} \) is an ‘effective’ LD parameter incorporating the relative displacement of the \( j^{th} \) ion in the \( p^{th} \) mode [3,11]. We have also set \( \hbar = 1 \). In what follows we will assume, with no loss of generality, \( \phi_1 = \phi_2 = 0 \) [12].

Following now a reasoning analogous to that of ref. [3], we find that resonance conditions arise for particular values of \( \Omega \). This can be most easily seen by transforming into a dressed-state picture defined by the operator

\[
V(t) \equiv \exp \left( \frac{i\Omega \sigma_{z}}{2} \right) R_1 \otimes \exp \left( \frac{i\Omega \sigma_{z}}{2} \right) R_2 \tag{3}
\]

where \( R_j = \frac{1}{\sqrt{2}} (\ket{1} \ket{1} + \ket{-1} \ket{-1}) \) and in our convention \( \ket{e} = (\ket{1} \ket{0}) \). The Hamiltonian becomes then

\[
H' = \frac{\Omega}{2} \sum_{p=1}^{N} \left[ iJ_{p+}^+ \left[ e^{i\Delta_p \eta_p} a_p + e^{i\nu_p \eta_p} a_p^\dagger \right] + h.c. \right], \tag{4}
\]

where we define \( J_{p+}^+ = \sum_{j=1}^{2} \eta_{jp} \sigma_j^+ \); \( \Delta_p \equiv \Omega - \nu_p \), \( \gamma_p \equiv \Omega + \nu_p \) and primes indicate operators defined in the dressed-state picture. One can see that, when \( \Delta_p = 0 \), the dynamics is dominated by terms of the form \( \sigma_j^+ a_p \), which induce resonant collective excitations of the ions accompanied by the exchange of motional quanta.

In what follows we are interested in the complementary regime far from these resonance conditions. Specifically, let us consider the limit where the detunings \( \Delta_p \) of the Rabi frequency are large with respect to the resulting secular frequency of the time evolution, i.e.

\[
\Delta_p \gg \eta_{jp} \Omega = \eta_{jp} (\nu_p + \Delta_p). \tag{5}
\]

In this case, standard time-averaging arguments (see e.g. the Appendix in [3]) can be used to obtain a time-independent effective Hamiltonian, given by

\[
H' \approx \Omega^2 \sum_{p=1}^{N} \left[ \frac{[J_{p+}^+ a_p + a_p^\dagger J_{p+}^-]}{\Delta_p} + \frac{[J_{p+}^- a_p + a_p^\dagger J_{p+}^+]}{\gamma_p} \right]. \tag{6}
\]

Evaluating the commutators, we get after some algebra

\[
\begin{align*}
A_p' &= \eta_{1p} \eta_{2p} \left[ |e'g'\rangle \langle g'e'| + |g'e\rangle \langle e'g'| \right] \tag{7a}
\end{align*}
\]

\[
B_p' = \left. \left( n_p + \frac{1}{2} \right) \left[ (\eta_{1p}^2 + \eta_{2p}^2) \left| |g'e\rangle \langle g'g'| - |e'g\rangle \langle e'g'| \right| + (\eta_{1p}^2 - \eta_{2p}^2) \left| |g'e\rangle \langle g'g'| - |e'g\rangle \langle e'g'| \right| \right) \right) \tag{7b}
\]

and where \( k_1, k_2 \) are constants that may be disregarded by a suitable redefinition of the zero of energy. Thus

\[
H' \approx -\omega \left[ |e'g'\rangle \langle g'e'| + |g'e\rangle \langle e'g'| \right] + \sum_{p=1}^{N} \frac{\Omega^2 B_p}{2(\Omega^2 - \nu_p^2)} \tag{8}
\]

where

\[
\omega = \frac{\Omega^2}{2} \sum_{p=1}^{N} \frac{\eta_{1p} \eta_{2p} \nu_p}{\Omega^2 - \nu_p^2}. \tag{9}
\]

Of course, this Hamiltonian holds only in the dressed picture defined in Eq. (3). In the ‘standard’ picture (i.e., the one corresponding to Eq. (12)), the time evolution operator is \( U(t) \equiv V(t) \exp(-itH')V(0) \).

In order to analyse this result, let us first consider the case \( N = 2 \), where \( \eta_{11} = \eta_{21} = \sqrt{3}\eta_{12} = -\sqrt{3}\eta_{22} \) [3]. The
second term in Eq. (8) then vanishes, and $H'_{\text{eff}}$ splits into one term coupling internal states $|g'g\rangle$ and $|e'g\rangle$, and another affecting only $|g'g\rangle$ and $|e'g\rangle$. Translating back into the “standard” picture, and including also the motional variables, the first term leads to transitions between each pair of states $|\pm + n_1 n_2\rangle$, $|\mp + n_1 n_2\rangle$, as expected from Fig. 1 (note that all time-dependent phases originating in $V(t)$ cancel out). Furthermore, as was also previously suggested, the frequency $\omega$ of the transitions is independent of the phonon numbers $\{n_p\}$. Finally, it is clear from Eqs. (8),(9) that this coupling results from the interference of contributions from both modes. The interference is constructive (both terms contribute positively to $\omega$, leading to faster oscillations) when $\nu_1 \leq \Omega \leq \nu_2 = \sqrt{3}\nu_1$. Note also that in general the further $\Omega$ is to $\nu_q$, the smaller is the amplitude in $\omega$ of the transition path via the corresponding mode. This effect might be useful to screen out “bad” modes with relatively high decoherence rates, such as the centre-of-mass mode in multi-ion traps.

![FIG. 2. Time required for the creation of a Bell state starting from the initial state $|\pm\pm\rangle$, calculated from Eq. (11)]. Thin solid curves represent the limits imposed by Eq. (11). The method should hold well in the region above both curves.

Meanwhile, states $|\pm + n_1 n_2\rangle$ and $|\mp - n_1 n_2\rangle$ acquire time-dependent phases, originating both from $V(t)$ and from $H'_{\text{eff}}$. The latter do depend on the phonon numbers and can lead to unwanted correlations between internal and motional variables, and thus to decoherence of the internal state. These results are analogous to those obtained in ref. [3] for the case where each ion is separately illuminated by a different laser, one slightly detuned from the first red sideband frequency and the other from the first blue one. As in that case, here it turns out that the motion-state-dependent component of the evolution can be cancelled by altering the Hamiltonian in the course of the gate operation itself (see below).

As a result, maximally entangled states of the two ions can be produced after a time $\tau_1 = |\pi/4\omega|$. To see this, note that the evolution operator $U(\tau_1)$ is essentially a $\sqrt{\text{SWAP}}$ gate with respect to the $|\pm\rangle$ basis: it turns the disentangled states $|\pm\pm\rangle, |\mp\mp\rangle$ into the maximally entangled (or “Bell”) states $|\beta_\pm\rangle \equiv (|\pm\pm\rangle \pm i|\mp\mp\rangle)/\sqrt{2}$, while leaving $|\mp\pm\rangle$ and $|\pm\mp\rangle$ unaffected (up to motion-state-independent phases $\phi_\pm(\tau_1)$ due to $V^\dagger(t)$). We stress again that this evolution occurs regardless of the initial motional state, and even regardless of whether it might be changing (due to heating effects) during the gate operation itself. Note further that, after a pulse of duration $2\tau_1$, $|\pm\mp\rangle \rightarrow i|\mp\pm\rangle, |\mp\pm\rangle \rightarrow \exp(\pm i\Omega \tau_1)|\mp\pm\rangle$, and so all four standard basis states $|g'(e')g(e)\rangle$ become maximally entangled. In other words, a gate locally equivalent to a CNOT is realised. Our method therefore allows the implementation of universal quantum logic on the ion chain, even in the presence of heating. The speed with which this is accomplished can be seen in Fig. 2, where we use Eq. (11) to plot $\tau_1$ as a function of $\Omega, \eta_{11}$. The thin solid lines indicate the validity limits imposed by choosing a factor of 10 in Eq. (11). We can conclude that the method should allow $\tau_1$ to be as low as a few hundred trap periods, a performance intermediate between that in [3] and that of the enhanced but heating-sensitive method in [10]. A significant difference with regard to these proposals is that our method should work in a different parameter range, including a comparatively high Rabi frequency and a comparatively small LD parameter. This may give it a substantial speed advantage in situations where $\eta_{11}$ is forcibly small, such as a tight trap containing many ions.

The trick which cancels the motion-state-dependent part of the evolution is to use a “photon-echo”-like procedure [4], whereby midway along the time evolution one inverts the sign of the motion-dependent term in the Hamiltonian (in our case $B_p$ in Eq. (8)). As a result, phases acquired in the first half of the evolution are cancelled out exactly during the second one.

In the present setup, this idea can be implemented in an experimentally simple way, by suddenly shifting the laser phase by $\pi$ (using, for instance, an electro-optic modulator). To see this, let us return to eq. (8). Shifting $\phi_j \rightarrow \phi_j + \pi$ is equivalent to changing the sign of $M$, which corresponds to exchanging the signs of the light-shifts suffered by $|\pm\rangle$ and $|\mp\rangle$. A development analogous to eqs. (3)-(9) results then in an effective Hamiltonian identical to $H'_{\text{eff}}$ but with $|e'\rangle$ and $|g'\rangle$ everywhere interchanged. It can be easily seen that this leaves the motion-independent term $A_p$ intact, while changing the sign of $B_p$, as required. In fact, following again ref. [3], in order to suppress motional heating it is desirable to perform the sign inversion several times during the course of the system’s evolution (specifically, at frequency $F = M/\tau$, where $\tau$ is the total time during which the laser is applied and $M \gg 1$ is an integer). If $F \gg \omega, \Gamma$, where $\Gamma$ is the typical heating rate, then the effects of the motion-dependent terms are cancelled before heating can affect them [3].

This robustness is illustrated in Fig. 3, where we plot time evolution curves simulated using the full trap Hamiltonian (i.e., including all orders of the LD parameter) and allowing also for heating in the form of quantum jumps, described by jump operators $\sqrt{\Gamma_p} \delta_p a$ and $\sqrt{\Gamma_p} (\delta_p + 1) a^\dagger$ [14]. (The heating rate $\Gamma_1$ for the CM mode is assumed to be an order of magnitude greater than $\Gamma_2$ [13]). The curves are averages obtained over 25 Monte Carlo runs, each containing an average of 18.4 jumps. In the bottom curve, the ions’ internal state is initially $|\psi_0\rangle = |\pm\rangle$, and we plot its squared over-
lap over time with the Bell state $|\beta_\circ\rangle$. A fidelity of 98% is achieved at the time $\nu_1 \tau_1 \approx 515$ expected from Fig. 2. In the top curve, $|\psi_0\rangle = (|++\rangle + |--\rangle)/\sqrt{2}$, and we plot the squared overlap $V^\dagger(t)V(0)|\psi_0\rangle$, the expected evolution in the absence of the motion-state-dependent term in Eq. (8). The small deviation from 1 shows that this term has been effectively suppressed by the switching of the laser phase.

![Graph](image)

Fig. 3. Time evolution of ions in thermal motion. In the bottom curve the initial internal state is $|\psi_0\rangle = |++\rangle$, and we plot the squared overlap over time with the Bell state $|\beta_\circ\rangle$. In the top curve, $|\psi_0\rangle = (|++\rangle + |--\rangle)/\sqrt{2}$, and we plot the squared overlap over time with $V^\dagger(t)V(0)|\psi_0\rangle$. In both cases, the motional modes are initially in thermal states with mean phonon numbers $\bar{n}_1 = 1, \bar{n}_2 = 0.1$, undergoing heating at rates $\Gamma_1 = 10^{-3}\nu_1, \Gamma_2 = 10^{-4}\nu_1$. Other parameter values are $\eta_{11} = 0.025, \Omega = 1.5\nu_1, F = \nu_1/50$.

At first sight, it may appear that ever greater suppression of the motion-dependent evolution can be achieved by increasing the switching rate $F$. One must be careful, however, not to invalidate the approximations leading to Eq. (8). For instance, $F$ must be limited by the Rabi frequency $\Omega$, in order to allow the rotating-wave_average implicit in the derivation of Eq. (3) to hold within each interval between phase inversions. In fact, our numerical simulations indicate that this averaging only occurs fully for particular “resonant” values of $F$. For most other choices, the time evolution of the system gradually becomes degraded. Determining analytically these “good” values is an open question; in practice, one would likely “tune” $F$ until an appropriate value was found.

In multi-ion chains with $N > 2$ our conclusions still hold, but high-frequency phase shifts become a necessity even in the absence of external heating. To see this, consider first that the LD parameters $\eta_{ijp}$ will generally differ from ion to ion [3], and so $[A_p, B_p] \neq 0$. In this case, inverting the sign of $B_p$ at a low frequency (e.g., comparable to $\omega$) is not sufficient to undo its effect on states $|++\rangle$ and $|--\rangle$. However, if the frequency is much larger than $\omega$ then this cancellation does occur, since for very short timescales the effect of the commutator above becomes negligible [15]. As a result, once again $B_p$ can be neglected, and the Hamiltonian reduces effectively to the first term in Eq. (8). A further generalisation to the case where $m > 2$ ions are illuminated together is also possible, as is the counter-intuitive possibility of coupling different pairs of ions simultaneously (cf the concluding remark in [1]). The latter case is achievable by illuminating each pair with a separate beam of sufficiently different intensity. In this case, following the argument in Eqs. (3)-(9) shows that the Hamiltonian effectively decouples into a sum of terms like in Eq. (8), one for each pair. Thus, adding a further mind-boggling twist to the situation in [3], every single vibrational mode can be used collectively and simultaneously for different tasks, despite only ever being ‘virtually’ excited!

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