Thermalization and the chromo-Weibel instability

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Abstract.
Despite the apparent success of ideal hydrodynamics in describing the elliptic flow data which have been produced at Brookhaven National Lab’s Relativistic Heavy Ion Collider, one lingering question remains: is the use of ideal hydrodynamics at times \( t < 1 \text{ fm/c} \) justified? In order to justify its use a method for rapidly producing isotropic thermal matter at RHIC energies is required. One of the chief obstacles to early isotropization/thermalization is the rapid longitudinal expansion of the matter during the earliest times after the initial nuclear impact. As a result of this expansion the parton distribution functions become locally anisotropic in momentum space. In contrast to locally isotropic plasmas anisotropic plasmas have a spectrum of soft unstable modes which are characterized by exponential growth of transverse chromo-magnetic/-electric fields at short times. This instability is the QCD analogue of the Weibel instability of QED. Parametrically the chromo-Weibel instability provides the fastest method for generation of soft background fields and dominates the short-time dynamics of the system.

1. Introduction
With the ongoing ultrarelativistic heavy-ion collision experiments at the Relativistic Heavy-Ion Collider (RHIC) and planned the Large Hadron Collider (LHC) physicists hope to produce and study the properties of a thermalized quark-gluon plasma (QGP) which is expected to be formed when the temperature of nuclear matter is raised above its critical value, \( T_c \sim 200 \text{ MeV} \sim 10^{12} \text{ K} \). Given the small size and short lifetime of the matter created in an ultrarelativistic heavy-ion collision this is not trivially accomplished. One of the chief obstacles to thermalization in ultrarelativistic heavy-ion collisions is the rapid longitudinal expansion of the matter created in the central rapidity region. If the matter expands too quickly then there will not be sufficient time for its constituents to interact and thermalize. During the first 1 fm/c after the nuclear impact the longitudinal expansion causes the created matter to become much colder in the longitudinal direction than in the transverse directions [1], corresponding to \( \langle p_L^2 \rangle \ll \langle p_T^2 \rangle \) in the local rest frame. After this initial period of longitudinal cooling, the expansion slows and one can then ask what are the dominant mechanisms for driving the system towards an isotropic thermal QGP.
The evolution of the partonic matter created during a high-energy nuclear collision was among the questions which the “bottom-up” thermalization scenario [1] attempted to answer. For the first time, it addressed the dynamics of soft modes (fields) with momenta much below \( Q_s \) coupled to the hard modes (particles) with momenta on the order of \( Q_s \) and above [2–4]. However, it has emerged recently that one of the assumptions made in this model was not correct. The debate centers around the fact that the bottom-up scenario implicitly assumed that the underlying soft gauge modes behave the same in an anisotropic plasma as in an isotropic one. However, to be self-consistent one must determine the collective modes which are relevant for an anisotropic plasma and use those. In the case of gauge theories this turns out to be a qualitative rather than quantitative correction since in anisotropic QCD plasmas the most important collective mode corresponds to an instability to transverse chromo-magnetic field fluctuations [5–8]. This instability is the QCD analogue of the QED Weibel instability [5]. There are now a plethora of analytic and numerical studies of the chromo-Weibel instability [8–22].

In particular in the last few years there have been significant advances in the understanding of non-Abelian soft-field dynamics in anisotropic plasmas within the HL framework [13–16]. The HL framework is equivalent to the collisionless Vlasov theory of eikonalized hard particles, i.e. the particle trajectories are assumed to be unaffected (up to small-angle scatterings with \( \theta \sim g \)) by the induced background field. It is strictly applicable only when there is a large scale separation between the soft and hard momentum scales. Even with these simplifying assumptions, HL dynamics for non-Abelian theories is complicated by the presence of non-linear gauge-field interactions. These non-linear interactions become important when the vector potential amplitude is on the order of \( A_{\text{non-Abelian}} \sim p_s/g \sim \sqrt{f_h p_h} \), where \( p_h \) is the characteristic momentum of the hard particles, e.g. \( p_h \sim Q_s \) for CGC initial conditions, \( f_h \) is the angle-averaged occupancy at the hard scale, and \( p_s \) is the characteristic soft momentum of the fields \( (p_s \sim g\sqrt{f_h p_h}) \). In QED there are no gauge field self-interactions and the fields grow exponentially until \( A_{\text{Abelian}} \sim p_h/g \) at which point the hard particles undergo large-angle deflections by the soft background field causing the particles to rapidly isotropize. In fact, in QED the Weibel instability is the fastest process driving plasma isotropization. In QCD, however, the effect of the non-linear gauge self-interactions affects the system’s dynamics primarily slowing down instability-driven particle isotropization.

To include the effects of gauge self-interactions numerical studies of the time evolution of the gauge-covariant HL equations of motion are required. Recent numerical studies of HL gauge dynamics for SU(2) gauge theory indicate that for moderate anisotropies the gauge field dynamics changes from exponential field growth indicative of a conventional Abelian plasma instability to linear growth when the vector potential amplitude reaches the non-Abelian scale, \( A_{\text{non-Abelian}} \sim p_h \) [14,15]. This linear growth regime is characterized by a turbulent cascade of the energy pumped into the soft modes by the instability to higher-momentum plasmon-like modes [17,18]. These results indicate that there is a fundamental difference between Abelian and non-Abelian plasma...
instabilities in the HL limit.

In addition to numerical studies in the HL limit there have been numerical results from the solution to the full non-linear Vlasov equations for anisotropic plasmas [19,20]. This approach can be shown to reproduce the HL effective action in the weak-field approximation [23–25]; however, when solved fully the approach goes beyond the HL approximation since the full classical transport theory also reproduces some higher $n$-point vertices of the dimensionally reduced effective action for static gluons [26]. Numerical solution of the 3d Vlasov equations show that chromo-instabilities persist beyond the HL limit [20]. Furthermore, the soft field spectrum obtained from full Vlasov simulations shows a cascade or “avalanche” of energy deposited in the soft unstable modes in higher momentum modes similar to HL dynamics.

2. Collective Modes of an Anisotropic Quark-Gluon Plasma

In this section I review the determination of the collective modes of a quark-gluon plasma which has a parton distribution function which is anisotropic in momentum-space. To simplify things we will assume that $f(p)$ can be obtained from an isotropic distribution function by the rescaling of only one direction in momentum space. In practice this means that, given any isotropic parton distribution function $f_{iso}(p)$, we can construct an anisotropic version by changing the argument of the isotropic distribution function, $f(p) = f_{iso}(\sqrt{p^2 + \xi (p \cdot \hat{n})^2})$, where $\hat{n}$ is the direction of the anisotropy, and $\xi > -1$ is an adjustable anisotropy parameter with $\xi = 0$ corresponding to the isotropic case. Here we will concentrate on $\xi > 0$ which corresponds to a contraction of the distribution along the $\hat{n}$ direction since this is the configuration relevant for heavy-ion collisions at early times, namely two hot transverse directions and one cold longitudinal direction. The resulting expression for the gluon polarization tensor is [8]

$$
\Pi^{ij}_{ab}(\omega/k, \theta_n) = m^2_D \delta_{ab} \int \frac{d\Omega}{4\pi} \frac{v^i v^j + \xi (v \cdot \hat{n}) \hat{n}^i \hat{n}^j}{(1 + \xi (v \cdot \hat{n})^2)^2} \left( \delta^{jl} + \frac{v^j k^l}{K \cdot V + i\epsilon} \right),
$$

(1)

where $K = (\omega, k)$, $V = (1, p/p)$, $\cos \theta_n \equiv \hat{k} \cdot \hat{n}$ and $m^2_D > 0$. The isotropic Debye mass, $m_D$, depends on $f_{iso}$ but is parametrically $m_D \sim g p_h$.

For anisotropic systems ($\xi \neq 0$) one can solve for the collective modes of the plasma and the result is that there is one additional stable mode compared to the isotropic case and, more importantly, that there are now purely imaginary solutions in the lower- and upper-half of the complex plane. These new solutions correspond to damped and unstable modes, respectively. We can determine the growth rate for these unstable modes by taking $\omega \to i\Gamma$ and then solving the resulting dispersion relations for $\Gamma(k)$ [8]. Typical dispersion relations are shown in Fig. 1. As can been seen from this figure for $\xi > 0$ there are two types of unstable modes corresponding to magnetic and electric field instabilities. The magnetic instability has a slightly higher growth rate than the electric one and so will dominate the dynamics of the system at short times. Additionally, since the unstable mode growth rate has a maximum at a wave number $k^*$ this means that modes with this wavenumber will be dominant.
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Figure 1. Instability growth rates as a function of wave number for $\xi = 10$ and $\theta_n = \pi/8$. Note that both growth rates vanish at $k = 0$ and have a maximum $\Gamma^* \sim m_D/10$ at $k^* \sim m_D/3$.

3. Numerical Solution of Hard-Loop Dynamics

It is possible to go beyond an analysis of gluon polarization tensor to a full effective field theory for the soft modes and then solve this numerically. The effective field theory for the soft modes that is generated by integrating out the hard plasma modes at one-loop order and in the approximation that the amplitudes of the soft gauge fields obey $A \ll |p|/g$ is that of the gauge-covariant collisionless Boltzmann-Vlasov equations [27]. In equilibrium, the corresponding (nonlocal) effective action is the so-called hard-thermal-loop effective action which has a simple generalization to plasmas with anisotropic momentum distributions [11]. For the general non-equilibrium situation the resulting equations of motion are

\[ D_\nu(A) F^{\nu\mu} = -g^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2|p|} \rho^\mu \frac{\partial f(p)}{\partial p^\beta} W^\beta(x; \mathbf{v}), \]

\[ F_{\mu\nu}(A) v^\nu = [v \cdot D(A)] W_\mu(x; \mathbf{v}), \]

where $f$ is a weighted sum of the quark and gluon distribution functions and $v^\mu \equiv p^\mu/|p| = (1, \mathbf{v})$.

These equations include all hard-loop resummed propagators and vertices and are implicitly gauge covariant. At the expense of introducing a continuous set of auxiliary fields $W_\beta(x; \mathbf{v})$ the effective field equations are also local. These equations of motion are then discretized in space-time and $\mathbf{v}$, and solved numerically. The discretization in $\mathbf{v}$-space corresponds to including only a finite set of the auxiliary fields $W_\beta(x; \mathbf{v}_i)$ with $1 \leq i \leq N_W$. For details on the precise discretizations used see Refs. [14, 15].
3.1. Discussion of Numerical Hard-Loop Simulation Results

During the process of instability growth the soft gauge fields get the energy for their growth from the hard particles and, of course, the total energy is conserved. In an Abelian plasma the energy deposited in soft fields grows exponentially until the energy in the soft fields is of the same order as the energy remaining in the hard particles at which point the back-reaction of the fields on the particle motion causes rapid isotropization. As mentioned above in a non-Abelian plasma the situation is quite different and one must rely on numerical simulations due to the presence of strong gauge field self-interactions.

In Fig. 2 I have plotted the time dependence of the chromo-magnetic/-electric energy densities obtained from a 3+1 dimensional from a typical HL simulation run initialized with “weak” random color noise with $\xi = 10$ [15]. As can be seen from this figure at $m_\infty t \simeq 30$ there is a change from exponential to linear growth. Another interesting feature of the isotropic linear growth phase is that it exhibits a cascade of energy pumped into the unstable soft modes to higher energy plasmon like modes. This is demonstrated in Fig. 3 which shows the soft gauge field spectrum as a function of momentum at different simulation times along with the estimated scaling coefficient of the spectrum [17]. From Fig. 2 we can conclude that the chromo-Weibel instability will be less efficient at isotropizing a QCD plasma than the analogous Weibel instability seen in Abelian plasmas due to the slower than exponential growth at late times. On the positive side, from a theoretical perspective “saturation” at the soft scale implies that one can still apply the hard-loop effective theory self-consistently to understand the behavior of the system in the linear growth phase.

One caveat is that the latest published HL simulation results [14, 15] are for distributions with a finite $O(1 \rightarrow 10)$ anisotropy due to computational limitations and the
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Figure 3. Field mode spectrum for an SU(2) run showing saturation of soft field growth at \( f \sim 1/g^2 \) and an associated cascade of energy to the UV as the simulation time increases.

observed saturation seems to imply that for weak anisotropies field instabilities will not rapidly isotropize the hard particles. This means, however, that due to the continued expansion of the system the anisotropy will increase. It is therefore necessary to study the hard-loop dynamics in an expanding system. Naively, one expects this to change the growth from \( \exp(\tau) \) to \( \exp(\sqrt{\tau}) \) at short times but there is no clear expectation of what will happen in the linear regime. A significant advance in this regard has occurred recently for a U(1) gauge theory [16]. Work is underway to do the same for non-Abelian gauge theories.

4. Beyond Hard-Loops

It is also possible to go beyond the hard-loop approximation and solve instead the full classical transport equations in three dimensions [20]. The Vlasov transport equation for hard gluons with non-Abelian color charge \( q^a \) in the collisionless approximation are [28, 29],

\[
p^\mu [\partial_\mu - g q^a F^a_{\mu \nu} \partial_\nu - g f_{abc} A^b_\mu q^c \partial_{q^c}] f(x, p, q) = 0 ,
\]

were \( f(t, x, p, p_q) \) denotes the single-particle phase space distribution function.

The Vlasov equation is coupled self-consistently to the Yang-Mills equation for the soft gluon fields,

\[
D_\mu F^{\mu \nu} = J^\nu = g \int \frac{d^3 p}{(2\pi)^3} dq q \nu f(t, x, p, q) ,
\]

where again \( \nu^\mu \equiv (1, p/p) \). These equations reproduce the hard-loop effective action near equilibrium [23–25]. However, the full classical transport theory (3,4) also reproduces some higher \( n \)-point vertices of the dimensionally reduced effective action for static
Figure 4. Time evolution of the field energy densities for $SU(2)$ gauge group resulting from a highly anisotropic initial particle momentum distribution. Simulation parameters are $L = 5$ fm, $p_h = 16$ GeV, $g^2 n_g = 10/\text{fm}^3$, $m_\infty = 0.1$ GeV.

The back-reaction of the long-wavelength fields on the hard particles (“bending” of their trajectories) is, of course, taken into account, which is important for understanding particle dynamics in strong fields. For details of the numerical implementation used see Ref. [20].

In Fig. 4 I present the results of a three-dimensional Wong-Yang-Mills (WYM) simulation published in Ref. [20]. The figure shows the time evolution of the field energy densities for $SU(2)$ gauge group resulting from a highly anisotropic initial particle momentum distribution. The behavior shown in Fig. 4 indicates that the results obtained from the hard-loop simulations and direct numerical solution of the WYM equations are qualitatively similar in that both show that for non-Abelian gauge theories there is a saturation of the energy transferred to the soft modes by the gauge instability. Although I don’t show it here the corresponding Coulomb gauge fixed field spectra show that the field saturation is accompanied by an “avalanche” of energy transferred to soft field modes to higher frequency field modes with saturation occurring when the hardest lattice modes are filled [20]. A thorough analytic understanding of this ultraviolet avalanche is lacking at this point in time although some advances in this regard have been made recently [30].

5. Outlook

An important open question is whether quark-gluon plasma instabilities and/or the physics of anisotropic plasmas in general play an important phenomenological role at RHIC or LHC energies. In this regard the recent papers of Refs. [21, 31–33] provide theoretical frameworks which can be used to calculate the impact of anisotropic momentum-space distributions on observables such as jet shapes and the rapidity...
dependence of medium-produced photons. A concrete example of work in this direction is the recent calculation of photon production from an anisotropic QGP [34]. The results of that work suggest that it may be able possible to determine the time-dependent anisotropy of a QGP by measuring the rapidity dependence of high-energy medium photon production.

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