EXTENSIONS OF THE BENSON-SOLOMON FUSION SYSTEMS

ELLEN HENKE AND JUSTIN LYND

To Dave Benson on the occasion of his second 60th birthday

Abstract. The Benson-Solomon systems comprise the one currently known family of simple saturated fusion systems at the prime two that do not arise as the fusion system of any finite group. We determine the automorphism groups and the possible almost simple extensions of these systems and of their centric linking systems.

1. Introduction

There is one known family of simple exotic fusion systems at the prime 2, the Benson-Solomon systems. They were first predicted by Dave Benson [Ben98] to exist as finite versions of a 2-local compact group associated to the 2-compact group $DI(4)$ of Dwyer and Wilkerson [DW93]. They were later constructed by Levi and Oliver [LO02] and Aschbacher and Chermak [AC10]. The purpose of this paper is determine the automorphism groups of the Benson-Solomon fusion and centric linking systems, and use that information to determine the fusion systems whose generalized Fitting subsystem is a Benson-Solomon system. This information is needed within certain portions of Aschbacher’s program to classify simple fusion systems of component type at the prime 2. In particular, it is needed within an involution centralizer problem for these systems. Some results on automorphisms of these systems appear in the standard references [LO02,LO05,AC10], and part of our aim is to complete the picture. We now summarize the main results. Slightly more detailed statements are contained in the statements of Theorem 3.10 and Theorem 4.3.

Theorem 1. Fix an odd prime power $q$, and let $l = v_2(q^2 - 1) - 3$ where $v_2$ is the 2-adic valuation. Set $\mathcal{F}_0 = \mathcal{F}_{Sol}(q)$, a Benson-Solomon fusion system over the 2-group $S_0$. Let $\mathcal{F}$ be any almost simple extension of $\mathcal{F}_0$, namely, any saturated fusion system over a 2-group $S$ such that $F^*(F) = F_0$. Then

(a) $Out(\mathcal{F}_0) \cong C_{2^l}$ is cyclic of order $2^l$, induced by field automorphisms, and
(b) $\mathcal{F}_0 = O^2(\mathcal{F})$, $S$ splits over $S_0$, and the induced map $S/S_0 \to Out(\mathcal{F}_0)$ is injective.

Moreover, for each subgroup $A \leq Out(\mathcal{F}_0)$, there is a unique almost simple extension $\mathcal{F}$ of $\mathcal{F}_0$ as above, up to isomorphism, such that the map $S/S_0 \to Out(\mathcal{F}_0)$ has image $A$.

The paper proceeds as follows. In Section 2 we recall the various automorphism groups of fusion and linking systems and the maps between them, following [AOV12]. In Section 3 we look at automorphisms of the fusion and linking systems of $\text{Spin}_7(q)$ and of the Benson-Solomon

Date: October 24, 2018.
2000 Mathematics Subject Classification. Primary 20D20, Secondary 20D05.
Key words and phrases. fusion system, linking system, Benson-Solomon fusion system, group extension.

Justin Lynd was partially supported by NSA Young Investigator Grant H98230-14-1-0312. This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 707758.
systems. We show in Theorem [3.10] that the outer automorphism group of the latter is a cyclic group of field automorphisms of 2-power order. Finally, we show in Theorem [4.3] that the systems having a Benson-Solomon generalized Fitting subsystem are uniquely determined by the outer automorphisms they induce on the fusion system, and that all such extensions are split.

All our maps are written on the left. We would like to thank Jesper Grodal, Ran Levi, and Bob Oliver for helpful conversations.

2. AUTOMORPHISMS OF FUSION AND LINKING SYSTEMS

We refer to [AKO11] for the definition of a saturated fusion system, and also for the definition of a centric subgroup of a fusion system. Let \( \mathcal{F} \) be a saturated fusion system over the finite \( p \)-group \( S \), and write \( \mathcal{F}^c \) for the collection of \( \mathcal{F} \)-centric subgroups. Whenever \( g \) is an element of a finite group, we write \( c_g \) for the conjugation homomorphism \( x \mapsto gxg^{-1} \) and its restrictions.

2.1. Background on linking systems. Whenever \( \Delta \) is an overgroup-closed, \( \mathcal{F} \)-invariant collection of subgroups of \( S \), we have the transporter category \( T_\Delta(S) \) with those objects. This is the full subcategory of the transporter category \( T_S(S) \) where the objects are subgroups of \( S \), and morphisms are the transporter sets: \( N_S(P, Q) = \{ s \in S \mid sP^{-1}sQ^{-1} \leq Q \} \) with composition given by multiplication in \( S \).

A linking system associated to \( \mathcal{F} \) is a nonempty category \( \mathcal{L} \) with object set \( \Delta \), together with functors

\[
\begin{align*}
T_\Delta(S) \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}.
\end{align*}
\]

The functor \( \delta \) is the identity on objects and injective on morphisms, while \( \pi \) is the inclusion on objects and surjective on morphisms. Write \( \delta_{P,Q} \) for the corresponding injection \( N_S(P, Q) \to \text{Mor}_\mathcal{L}(P, Q) \) on morphisms, write \( \delta_P \) for \( \delta_{P,P} \), and use similar notation for \( \pi \).

The category and its structural functors are subject to several axioms which may be found in [AKO11, Definition II.4.1]. In particular, Axiom (B) states that for all objects \( P \) and \( Q \) in \( \mathcal{L} \) and each \( g \in N_S(P, Q) \), we have \( \pi_{P,Q}(\delta_{P,Q}(g)) = c_g \in \text{Hom}_\mathcal{F}(P, Q) \). A centric linking system is a linking system with \( \Delta = \mathcal{F}^c \). Given a finite group \( G \) with Sylow \( p \)-subgroup \( S \), the canonical centric linking system for \( G \) is the category \( \mathcal{L}^c_G(G) \) with objects the \( p \)-centric subgroups \( P \leq S \) (namely those \( P \) whose centralizer satisfies \( C_G(P) = Z(P) \times O_{p'}(C_G(P)) \)), and with morphisms the orbits of the transporter set \( N_G(P, Q) = \{ g \in G \mid gP^{-1}g^{-1} \leq Q \} \) under the right action of \( O_{p'}(C_G(P)) \).

2.1.1. Distinguished subgroups and inclusion morphisms. The subgroups \( \delta_P(P) \leq \text{Aut}_\mathcal{L}(P) \) are called distinguished subgroups. When \( P \leq Q \), the morphism \( \iota_{P,Q} := \delta_{P,Q}(1) \in \text{Mor}_\mathcal{L}(P, Q) \) is the inclusion of \( P \) into \( Q \).

2.1.2. Axiom (C) for a linking system. We will make use of Axiom (C) for a linking system, which says that for each morphism \( \varphi \in \text{Mor}_\mathcal{L}(P, Q) \) and element \( g \in N_S(P) \), the following identity holds between morphisms in \( \text{Mor}_\mathcal{L}(P, Q) \):

\[
\varphi \circ \delta_P(g) = \delta_Q(\pi(\varphi)(g)) \circ \varphi.
\]

2.1.3. Restrictions in linking systems. For each morphism \( \psi \in \text{Mor}_\mathcal{L}(P, Q) \), and each \( P_0, Q_0 \in \text{Ob}(\mathcal{L}) \) such that \( P_0 \leq P, Q_0 \leq Q \), and \( \pi(\psi)(P_0) \leq Q_0 \), there is a unique morphism \( \psi|_{P_0,Q_0} \in \text{Mor}_\mathcal{L}(P_0, Q_0) \) (the restriction of \( \psi \)) such that \( \psi \circ \iota_{P_0,P} = \iota_{Q_0,Q} \circ \psi|_{P_0,Q_0} \). See [Oli10, Proposition 4(b)] or [AKO11, Proposition 4.3].

Note that in case \( \psi = \delta_{P,Q}(s) \) for some \( s \in N_S(P, Q) \), it can be seen from Axioms (B) and (C) that \( \psi|_{P_0,Q_0} = \delta_{P_0,Q_0}(s) \).
2.2. Background on automorphisms.

2.2.1. Automorphisms of fusion systems. An automorphism of $\mathcal{F}$ is, by definition, determined by its effect on $S$: define $\text{Aut}(\mathcal{F})$ to be the subgroup of $\text{Aut}(S)$ consisting of those automorphisms $\alpha$ which preserve fusion in $\mathcal{F}$ in the sense that the homomorphism given by $\alpha(P) \overset{\alpha \varphi \alpha^{-1}}{\longrightarrow} \alpha(Q)$ is in $\mathcal{F}$ for each morphism $P \overset{\varphi}{\longrightarrow} Q$ in $\mathcal{F}$. The automorphisms $\text{Aut}_F(S)$ of $S$ in $\mathcal{F}$ thus form a normal subgroup of $\text{Aut}(\mathcal{F})$, and the quotient $\text{Aut}(\mathcal{F})/\text{Aut}_F(S)$ is denoted by $\text{Out}(\mathcal{F})$.

2.2.2. Automorphisms of linking systems. A self-equivalence of $\mathcal{L}$ is said to be isotypical if it sends distinguished subgroups to distinguished subgroups, i.e. $\alpha(\delta_P(P)) = \delta_{\alpha(P)}(\alpha(P))$ for each object $P$. It sends inclusions to inclusions provided $\alpha(\iota_{P,Q}) = \iota_{\alpha(P),\alpha(Q)}$ whenever $P \leq Q$. The monoid $\text{Aut}(\mathcal{L})$ of isotypical self-equivalences that send inclusions to inclusions is in fact a group of automorphisms of the category $\mathcal{L}$, and this has been shown to be the most appropriate group of automorphisms to consider. Note that $\text{Aut}(\mathcal{L})$ has been denoted by $\text{Out}_{\text{typ}}(\mathcal{L})$ in [AKO11] and elsewhere. When $\alpha \in \text{Aut}(\mathcal{L})$ and $P$ is an object with $\alpha(P) = P$, we denote by $\alpha_P$ the automorphism of $\text{Aut}_{\mathcal{L}}(P)$ induced by $\alpha$.

The group $\text{Aut}_{\mathcal{L}}(S)$ acts by conjugation on $\mathcal{L}$ in the following way: given $\gamma \in \text{Aut}_{\mathcal{L}}(S)$, consider the functor $c_\gamma \in \text{Aut}(\mathcal{L})$ which is $c_\gamma(P) = \pi(\gamma)(P)$ on objects, and which sends a morphism $P \overset{\varphi}{\longrightarrow} Q$ in $\mathcal{L}$ to the morphism $\gamma \varphi \gamma^{-1}$ from $c_\gamma(P)$ to $c_\gamma(Q)$ after replacing $\gamma$ and $\gamma^{-1}$ by the appropriate restrictions (introduced in [2.1.3]). Note that when $\gamma = \delta_{G}(s)$ for some $s \in S$, then $c_\gamma(P)$ is conjugation by $s$ on objects, and $c_\gamma(\varphi) = \delta_{G,Q}(s) \circ \varphi \circ \delta_{P,P}(s^{-1})$ for each morphism $\varphi \in \text{Mor}_\mathcal{L}(P,Q)$ by the remark on distinguished morphisms in [2.1.3]. In particular, when $\mathcal{L} = \mathcal{L}_{S}(G)$ for some finite group $G$, $c_\gamma$ is truly just conjugation by $s$ on morphisms.

The image of $\text{Aut}_{\mathcal{L}}(S)$ under the map $\gamma \mapsto c_\gamma$ is seen to be a normal subgroup of $\text{Aut}(\mathcal{L})$. The outer automorphism group of $\mathcal{L}$ is

$$\text{Out}(\mathcal{L}) := \text{Aut}(\mathcal{L})/\{c_\gamma \mid \gamma \in \text{Aut}_{\mathcal{L}}(S)\}.$$  

We refer to Lemma 1.14(a) and the surrounding discussion in [AOV12] for more details. This group is denoted by $\text{Out}_{\text{typ}}(\mathcal{L})$ in [AKO11] and elsewhere.

2.2.3. From linking system automorphisms to fusion system automorphisms. There is a group homomorphism

$$\tilde{\mu} : \text{Aut}(\mathcal{L}) \longrightarrow \text{Aut}(\mathcal{F}),$$

given by restriction to $S \cong \delta_S(S) \leq \text{Aut}_{\mathcal{L}}(S)$; see [Oll10, Proposition 6]. The map $\tilde{\mu}$ induces a homomorphism on quotient groups

$$\mu : \text{Out}(\mathcal{L}) \longrightarrow \text{Out}(\mathcal{F}).$$

We write $\mu_{\mathcal{L}}$ (or $\mu_G$ when $\mathcal{L} = \mathcal{L}_{S}^c(G)$) whenever we wish to make clear which linking system we are working with; similar remarks hold for $\tilde{\mu}$. As shown in [AKO11, Proposition II.5.12], $\ker(\mu)$ has an interesting cohomological interpretation as the first cohomology group of the center functor $Z_{\mathcal{F}}$ on the orbit category of $\mathcal{F}$-centric subgroups, and $\ker(\tilde{\mu})$ is correspondingly a certain group of normalized 1-cocycles for this functor.
2.2.4. From group automorphisms to fusion system and linking system automorphisms. We also need to compare automorphisms of groups with the automorphisms of their fusion and linking systems. If $G$ is a finite group with Sylow $p$-subgroup $S$, then each outer automorphism of $G$ is represented by an automorphism that fixes $S$. This is a consequence of the transitive action of $G$ on its Sylow subgroups. More precisely, there is an exact sequence:

$$1 \to Z(G) \xrightarrow{\text{incl}} N_G(S) \xrightarrow{g \to c_g} \text{Aut}(G,S) \to \text{Out}(G) \to 1,$$

where $\text{Aut}(G,S)$ is the subgroup of $\text{Aut}(G)$ consisting of those automorphisms that leave $S$ invariant.

For each pair of $p$-centric subgroups $P,Q \leq S$ and each $\alpha \in \text{Aut}(G,S)$, $\alpha$ induces an isomorphism $O_p'(C_G(P)) \to O_p'(C_G(\alpha(P)))$ and a bijection $N_G(P,Q) \to N_G(\alpha(P), \alpha(Q))$. Thus, there is a group homomorphism

$$\tilde{\kappa}_G: \text{Aut}(G,S) \to \text{Aut}(\mathcal{L}_S^c(G))$$

which sends $\alpha \in \text{Aut}(G,S)$ to the functor which is $\alpha$ on objects, and also $\alpha$ on morphisms in the way just mentioned. This map sends the image of $N_G(S)$ to $\{c_\gamma | \gamma \in \text{Aut}(\mathcal{L}_S^c(G))\}$, and so induces a homomorphism

$$\kappa_G: \text{Out}(G) \to \text{Out}(\mathcal{L}_S^c(G))$$
on outer automorphism groups.

It is straightforward to check that the restriction to $S$ of any member of $\text{Aut}(G,S)$ is an automorphism of the fusion system $\mathcal{F}_S(G)$. Indeed, for every $\alpha \in \text{Aut}(G,S)$, the automorphism $\alpha|_S$ of $\mathcal{F}_S(G)$ is just the image of $\alpha$ under $\tilde{\mu}_G \circ \tilde{\kappa}_G$.

2.2.5. Summary. What we will need in our proofs is summarized in the following commutative diagram, which is an augmented and updated version of the one found in [AKO11, p.186].

\[
\begin{array}{ccccccc}
1 & \to & Z(F) & \xrightarrow{\text{incl}} & Z(S) & \xrightarrow{\delta_S} & \tilde{Z}^1(O(F^c), Z_F) & \xrightarrow{\lambda} & \lim^{\mathbb{Z}}(Z_F) & \to & 1 \\
& & 1 & & 1 & & 1 & & \lambda & & \\
& & \downarrow \delta_S & & \downarrow \tilde{\lambda} & & \downarrow \lambda & & & & \\
& & Z(F) & \xrightarrow{\pi_S} & \text{Aut}_F(S) & \xrightarrow{\tilde{\mu}} & \text{Aut}(F) & \to & \text{Out}(F) & \to & 1 \\
& & 1 & & 1 & & 1 & & & & \\
& & 1 & & 1 & & 1 & & & & \\
\end{array}
\]

All sequences in this diagram are exact. Most of this either is shown in the proof of [AKO11, Proposition II.5.12], or follows from the above definitions. The first and second rows are exact by this reference, except that the diagram was not augmented by the maps out of $Z(F)$ (the center of $F$); exactness at $Z(S)$ and $\text{Aut}_F(S)$ is shown by following the proof there. Given [AKO11, Proposition II.5.12], exactness of the last column is equivalent to the uniqueness of centric linking systems, a result of Chermak. In all the cases needed in this article, it follows from [LO02, Lemma 3.2]. The second-to-last column is then exact by a diagram chase akin to that in a 5-lemma for groups.
3. Automorphisms

The isomorphism type of the fusion systems of the Benson-Solomon systems $F_{\text{Sol}}(q)$, as $q$ ranges over odd prime powers, is dependent only on the 2-share of $q^2 - 1$ by [COS08 Theorem B]. Since the centralizer of the center of the Sylow group is the fusion system of Spin$_7(q)$, the same holds also for the fusion systems of these groups. For this reason, and because of Proposition 3.2 below, it will be convenient to fix a nonnegative integer $l$, and take $q_l = 5^l$ for the sequel. Let $F$ be the algebraic closure of the field with five elements.

3.1. Automorphisms of the fusion system of Spin$_7(q)$. Let $\bar{H} = \text{Spin}_7(F)$. Fix a maximal torus $\bar{T}$ of $\bar{H}$. Thus, $\bar{H}$ is generated by the $\bar{T}$-root groups $X_\alpha = \{x_\alpha(\lambda): \lambda \in F\} \cong (F, +)$, as $\alpha$ ranges over the root system of type $\alpha$ and $\bar{T}$ is subject to the Chevalley relations of [GLS98 Theorem 1.12.1]. For any power $r$ of 5, we let $\psi_r$ denote the standard Frobenius endomorphism of $H$, namely the endomorphism of $\bar{H}$ which acts on the root groups via $\psi_r(x_\alpha(\lambda)) = x_\alpha(\lambda^r)$.

Set $H_l = C_{\bar{H}}(\psi_{ql})$. Thus, $H_l = \text{Spin}_7(q_l)$ since $\bar{H}$ is of universal type (see [GLS98 Theorem 2.2.6(f)]). Also, $T_{\psi_{ql}} := C_{\bar{T}}(\psi_{ql})$ is a maximal torus of $H_l$. For each power $r$ of 5, the Frobenius endomorphism $\psi_r$ of $\bar{H}$ acts on $H_l$ in the way just mentioned, and it also acts on $T_{\psi_{ql}}$ by raising each element to the power $r$. For ease of notation, we denote by $\psi_{ql}$ also the automorphism of $H_l$ induced by $\psi_{ql}$. We next recall some items from [AC10 Lemmas 4.3, 4.8, 4.9]. The normalizer $N_{H_l}(T_{\psi_{ql}})$ contains a Sylow 2-subgroup of $H_l$, and $N_{H_l}(T_{\psi_{ql}})/T_{\psi_{ql}}$ is isomorphic to $C_2 \times S_4$, the Weyl group of $B_3$. We may choose such a Sylow 2-subgroup $S_l$ of $N_{H_l}(T_{\psi_{ql}})$ to be invariant under $\psi_5$; we fix such a choice for the remainder. Set $k := k_l = l + 2$, and denote by

$$T_k := T_{\psi_{ql}} \cap S_l \cong C_2 \times C_2 \times C_2$$

the 2-torsion in the maximal torus $T_{\psi_{ql}}$ of $H_l$.

The automorphism groups of the Chevalley groups were determined by Steinberg [Ste60], and in particular,

$$\text{Out}(H_l) = \text{Outdiag}(H_l) \times \Phi \cong C_2 \times C_2^l,$$

where $\Phi$ is the group of field automorphisms, and where $\text{Outdiag}(H_l)$ is the group of outer automorphisms of $H_l$ induced by $N_{\bar{T}}(H_l)$ [GLS98 Theorem 2.5.1(b)]. We mention that $S_l$ is normalized by some element of $N_{\bar{T}}(H_l) - H_l$ of 2-power order. So we find representatives of the elements of $\Phi$ and of $\text{Outdiag}(H_l)$ in $\text{Aut}(H_l, S_l)$.

We need to be able to compare automorphisms of the group with automorphisms of the fusion and linking systems, and this has been carried out in full generality by Broto, Møller, and Oliver [BMO16] for groups of Lie type. Let $F_{\text{Spin}}(q_l)$ and $L_{\text{Spin}}(q_l)$ be the associated fusion and centric linking systems over $S_l$ of the group $H_l$, and recall the maps $\mu_{H_l}$ and $\kappa_{H_l}$ from §2.2.3.

**Proposition 3.2.** The maps $\mu_{H_l}$ and $\kappa_{H_l}$ are isomorphisms, and hence

$$\text{Out}(L_{\text{Spin}}(q_l)) \cong \text{Out}(F_{\text{Spin}}(q_l)) \cong C_2 \times C_2^l.$$

**Proof.** That $\mu_{H_l}$ is an isomorphism follows from (2.3) and [LO02 Lemma 3.2]. Also, $\kappa_{H_l}$ is an isomorphism by [BMO16 Propositions 5.15, 5.16], using that $\text{Outdiag}(H_l)$ is a 2-group.

3.2. Automorphisms of the Benson-Solomon systems. We keep the notation from the previous subsection. We denote by $\mathcal{F} := F_{\text{Sol}}(q_l)$ a Benson-Solomon fusion system over the 2-group $S_l \in \text{Syl}_2(H_l)$ fixed above, and by $\mathcal{L} := L_{\text{Sol}}(q_l)$ an associated centric linking system with structural functors $\delta$ and $\pi$. 

5
For the remainder of Section 3, we fix \( l \geq 0 \), and we set \( H := H_1 \) and \( S := S_l \).

Observe that \( Z(S) \leq T_k \) is of order 2, and \( N_F(Z(S)) = C_F(Z(S)) = \mathcal{F}_{\text{Spin}}(q_l) \) is a fusion system over \( S \). Since \( Z(S) \) is contained in every \( \mathcal{F} \)-centric subgroup, by Definition 6.1 and Lemma 6.2 of [BLO03], we may take \( N_L(Z(S)) = C_L(Z(S)) \) for the centric linking system of \( \text{Spin}_q(q_l) \). By the items just referenced, \( C_L(Z(S)) \) is a subcategory of \( L \) with the same objects, and with morphisms those morphisms \( \phi \) in \( L \) such that \( \pi(\phi)(z) = z \). Further, \( C_L(Z(S)) \) has the same inclusion functor \( \delta \), and the projection functor for \( C_L(Z(S)) \) is the restriction of \( \pi \). (This was also shown in [LO02, Lemma 3.3(a,b)].)

Write \( F_z \) for \( F_{\text{Spin}}(q_l) \) and \( L_z \) for \( C_L(Z(S)) \) for short. Each member of \( \text{Aut}(F) \) fixes \( Z(S) \) and so \( \text{Aut}(F) \subseteq \text{Aut}(F_z) \). So the inclusion map from \( \text{Aut}(F) \) to \( \text{Aut}(F_z) \) can be thought of as a “restriction map”

\[
(3.3) \quad \rho: \text{Aut}(F) \to \text{Aut}(F_z)
\]

given by remembering only that an automorphism preserves fusion in \( F_z \). We want to make explicit in Lemma 3.5 that the map \( \rho \) of (3.3) comes from a restriction map on the level of centric linking systems. First we need to recall some information about the normalizer of \( T_k \) in \( L \) and \( L_z \).

**Lemma 3.4.** The following hold after identifying \( T_k \) with its image \( \delta_{T_k}(T_k) \leq \text{Aut}_L(T_k) \).

(a) \( \text{Aut}_{L_z}(T_k) \) is an extension of \( T_k \) by \( C_2 \times S_4 \), and \( \text{Aut}_L(T_k) \) is an extension of \( T_k \) by \( C_2 \times \text{GL}_3(2) \) in which the \( \text{GL}_3(2) \) factor acts naturally on \( T_k/\Phi(T_k) \). In each case, a \( C_2 \) factor acts as inversion on \( T_k \). Also, \( T_k \) is equal to its centralizer in each of the above normalizers, \( Z(\text{Aut}_{L_z}(T_k)) = Z(S) \), and \( Z(\text{Aut}_L(T_k)) = 1 \).

(b) \( \text{Aut}_F(S) = \text{Inn}(S) = \text{Aut}_{F_z}(S) \) and \( \text{Aut}_L(S) = \delta_S(S) = \text{Aut}_{L_z}(S) \).

**Proof.** For part (a), see Lemma 4.3 and Proposition 5.4 of [AC10].

Since \( T_k \) is the unique abelian subgroup of its order in \( S \) by [AC10, Lemma 4.9(c)], it is characteristic. By the uniqueness of restrictions (see Section 2.4.3), we may therefore view \( \text{Aut}_L(S) \) as a subgroup of \( \text{Aut}_L(T_k) \). Since \( \text{Aut}_L(T_k) \) has self-normalizing Sylow 2-subgroups by (a), the same holds for \( \text{Aut}_L(S) \). Now (b) follows for \( L_z \) and for \( F_z \) after applying \( \pi \). This also implies the statement for \( L_z \) and \( F_z \), as subcategories. \( \square \)

There is a 3-dimensional commutative diagram related to (2.3) that is the point of the next lemma.

**Lemma 3.5.** There is a restriction map \( \tilde{\rho}: \text{Aut}(L) \to \text{Aut}(L_z) \), with kernel the automorphisms induced by conjugation by \( \delta_S(Z(S)) \leq \text{Aut}_L(S) \), which makes the diagram

\[
\begin{array}{ccc}
\text{Aut}(L) & \xrightarrow{\tilde{\rho}} & \text{Aut}(L_z) \\
\mu_L & \downarrow & \mu_{L_z} \\
\text{Aut}(F) & \xrightarrow{\rho} & \text{Aut}(F_z),
\end{array}
\]

commutative, which commutes with the conjugation maps out of

\[
\begin{array}{ccc}
\text{Aut}_L(S) & \xrightarrow{id} & \text{Aut}_{L_z}(S) \\
\pi_S & \downarrow & \pi_S \\
\text{Aut}_F(S) & \xrightarrow{id} & \text{Aut}_{F_z}(S),
\end{array}
\]

\[6\]
and which therefore induces a commutative diagram

\[
\begin{array}{ccc}
\text{Out}(L) & \stackrel{[\tilde{\rho}]}{\longrightarrow} & \text{Out}(L_z) \\
\mu_L & | & \mu_{L_z} \\
\text{Out}(F) & \stackrel{[\rho]}{\longrightarrow} & \text{Out}(F_z).
\end{array}
\]

**Proof.** Recall that we have arranged $L_z \subseteq L$. Thus, the horizontal maps in the second diagram are the identity maps by Lemma 3.4 and so the lemma amounts to checking that an element of $\text{Aut}(L)$ sends morphisms in $L_z$ to morphisms in $L_z$. For then, we can define its image under $\tilde{\rho}$ to have the same effect on objects, and to be the restriction to $L_z$ on morphisms.

Now fix an arbitrary $\alpha \in \text{Aut}(L)$, objects $P,Q \in F^c = F^c_z$, and a morphism $\varphi \in \text{Mor}_L(P,Q)$. Let $Z(S) = \langle z \rangle$. By two applications of Axiom (C) for a linking system (§2.1.2),

(3.6) \[ \iota_{P,S} \circ \delta_P(z) = \delta_S(z) \circ \iota_{P,S} \quad \text{and} \quad \iota_{\alpha(P),S} \circ \delta_{\alpha(P)}(z) = \delta_S(z) \circ \iota_{\alpha(P),S}, \]

because $\pi(\iota_{P,S})(z) = \pi(\iota_{\alpha(P),S})(z) = z$. Since $\alpha_S$ is an automorphism of $\delta_S(S) \cong S$, it sends $\delta_S(z)$ to itself. Thus, $\alpha$ sends the right side of the first equation of (3.6) to the right side of the second, since it sends inclusions to inclusions. Thus

\[ \iota_{\alpha(P),S} \circ \alpha(\delta_P(z)) = \iota_{\alpha(P),S} \circ \delta_{\alpha(P)}(z). \]

However, each morphism in $L$ is a monomorphism [Oli10 Proposition 4], so we obtain

(3.7) \[ \alpha(\delta_P(z)) = \delta_{\alpha(P)}(z), \]

and the same holds for $Q$ in place of $P$.

Since $\varphi \in \text{Mor}(L_z)$, we have $\pi(\varphi)(z) = z$, so by two more applications of Axiom (C),

(3.8) \[ \varphi \circ \delta_P(z) = \delta_Q(z) \circ \varphi \quad \text{and} \quad \alpha(\varphi) \circ \delta_{\alpha(P)}(z) = \delta_{\alpha(Q)}(\pi(\alpha(\varphi))(z)) \circ \alpha(\varphi). \]

After applying $\alpha$ to the left side of the first equation of (3.8), we obtain the left side of the second by (3.7). Thus, considering right sides, we obtain

\[ \delta_{\alpha(Q)}(z) \circ \alpha(\varphi) = \delta_{\alpha(Q)}(\pi(\alpha(\varphi))(z)) \circ \alpha(\varphi) \]

Since each morphism in $L$ is an epimorphism [Oli10 Proposition 4], it follows that

\[ \delta_{\alpha(Q)}(z) = \delta_{\alpha(Q)}(\pi(\alpha(\varphi))(z)). \]

Hence, $\pi(\alpha(\varphi))(z) = z$ because $\delta_{\alpha(Q)}$ is injective (Axiom (A2)). That is, $\alpha(\varphi) \in \text{Mor}(L_z)$ as required.

The kernel of $\tilde{\rho}$ is described via a diagram chase in (2.3). Suppose $\tilde{\rho}(\alpha)$ is the identity. Then, $\alpha$ is sent to the identity automorphism of $S$ by $\tilde{\mu}_L$, since $\rho$ is injective. Thus, $\alpha$ comes from a normalized 1-cocycle by (2.3) and these are in turn induced by elements of $Z(S)$ since $\varprojlim 1(Z_F)$ is trivial [LO02 Lemma 3.2].

**Lemma 3.9.** Let $G$ be a finite group and let $V$ be an abelian normal 2-subgroup of $G$ such that $C_G(V) \subseteq V$. Let $\alpha$ be an automorphism of $G$ such that $[V, \alpha] = 1$. Then $[G, \alpha] \subseteq V$, and if $G$ acts fixed point freely on $V/\Phi(V)$ and $\alpha^2 \in \text{Inn}(G)$, then the order of $\alpha$ is at most the exponent of $V$.

**Proof.** As $[V, \alpha] = 1$, we have $[V, G, \alpha] \subseteq [V, \alpha] = 1$ and $[\alpha, V, G] = [1, G] = 1$. Hence, by the Three subgroups lemma, it follows that $[G, \alpha, V] = 1$. As $C_G(V) \subseteq V$, this means

\[ [G, \alpha] \subseteq V. \]
Assume from now on that $G$ acts fixed point freely on $V/\Phi(V)$. Write $G^* := G \rtimes \langle \alpha \rangle$ for the semidirect product of $G$ by $\langle \alpha \rangle$. As $[V, \alpha] = 1$ and $[G, \alpha] \subseteq V$, the subgroup $W := V\langle \alpha \rangle$ is an abelian normal subgroup of $G^*$ with $[W, G^*] \subseteq V$.

As $[V, \alpha] = 1$, it follows that $[V, \alpha^2] = 1$. So $\alpha^2 \in \text{Inn}(G)$ is realized by conjugation with an element of $C_G(V) = V$. Pick $u \in V$ with $\alpha^2 = c_u|G$. This means that, for any $g \in G$, we have $u^{-1}\alpha^2 g = g$ in $G^*$. So $Z := \langle u^{-1}\alpha^2 \rangle$ centralizes $G$ in $G^*$. Since $W$ is abelian and contains $Z$, it follows that $Z$ lies in the centre of $G^* = WG$. Set

$$G^* = G^*/Z.$$ 

Because $C_G(V) \subseteq V$, the order of $u$ equals the order of $c_u|G = \alpha^2$. Hence, $Z \cap G = 1 = Z \cap \langle \alpha \rangle$. So $|\alpha| = |\bar{\alpha}|$ and $G \cong \bar{G}$. In particular, we have $\bar{V} \cong V$ and $\bar{G}$ acts fixed point freely on $\bar{V}/\Phi(\bar{V})$. Note also that $\bar{\alpha}^2 = \bar{u}$. Hence $|\bar{W}/\bar{V}| = 2 = \Phi(\bar{W}) \subseteq \bar{V}$. Moreover, letting $n \in \mathbb{N}$ such that $2^n$ is the exponent of $\bar{V}$, we have $|\alpha| = |\bar{\alpha}| \leq 2 \cdot 2^n = 2^{n+1}$. Assume $|\bar{\alpha}| = 2^{n+1}$. Then $\bar{u} = \bar{\alpha}^2$ has order $2^n$ and is thus not a square in $\bar{V}$. Note that $\Phi(\bar{V}) = \{v^2 : v \in \bar{V}\}$ and $\bar{\Phi}(\bar{W}) = \{w^2 : w \in \bar{W}\} = \langle \bar{\alpha}^2 \rangle \Phi(\bar{V}) \subseteq \bar{V}$. Hence, $\Phi(\bar{W})/\Phi(\bar{V})$ has order 2. As $\bar{G}$ normalizes $\Phi(\bar{W})/\Phi(\bar{V})$, it thus centralizes $\Phi(\bar{W})/\Phi(\bar{V})$ contradicting the assumption that $\bar{G}$ acts fixed point freely on $\bar{V}/\Phi(\bar{V})$. Thus $|\alpha| = |\bar{\alpha}| \leq 2^n$ which shows the assertion. \hfill $\square$

We are now in a position to determine the automorphisms of $\mathcal{F} = \mathcal{F}_{\text{Sol}}(q_l)$ and $\mathcal{L} = \mathcal{L}_{\text{Sol}}^c(q_l)$. It is known that the field automorphisms induce automorphisms of these systems as we will make precise next. Recall that the field automorphism $\psi_{\mathcal{F}}$ of $H$ of order $2^l$ normalizes $S$ and so $\psi_{\mathcal{F}}|_S$ is an automorphism of $\mathcal{F}_2 = \mathcal{F}_S(H)$. By $[\text{ACT10}]$ Lemma 5.7], the automorphism $\psi_{\mathcal{F}}|_S$ is actually also an automorphism of $\mathcal{F}$. We thus denote it by $\psi_{\mathcal{F}}$ and refer to it as the field automorphism of $\mathcal{F}$ induced by $\psi_{\mathcal{F}}$. By Proposition 3.2 this automorphism has order $2^l$.

By $[\text{LO02}]$ Proposition 3.3(d)], there is a unique lift $\psi$ of $\psi_{\mathcal{F}}$ under $\bar{\mu}_L$ that is the identity on $\pi^{-1}(\mathcal{F}_{\text{Sol}}(5))$ and restricts to $\bar{\kappa}_H(\psi_5)$ on $\mathcal{L}_z$. We refer to $\psi$ as the field automorphism of $\mathcal{L}$ induced by $\psi_5$.

**Theorem 3.10.** Fix $l \geq 0$, and set $q_l = 5^{2^l}$ as before. The map $\mu_{\mathcal{L}} : \text{Out}(\mathcal{L}_{\text{Sol}}^c(q_l)) \to \text{Out}(\mathcal{F}_{\text{Sol}}(q_l))$ is an isomorphism, and

$$\text{Out}(\mathcal{L}_{\text{Sol}}^c(q_l)) \cong \text{Out}(\mathcal{F}_{\text{Sol}}(q_l)) \cong C_{2^l}$$

is induced by field automorphisms. Also, the automorphism group $\text{Aut}(\mathcal{L}_{\text{Sol}}^c(q_l))$ is a split extension of $S$ by $\text{Out}(\mathcal{L}_{\text{Sol}}(q_l))$; in particular, it is a 2-group.

More precisely, if $\psi$ is the field automorphism of $\mathcal{L}_{\text{Sol}}^c(q_l)$ induced by $\psi_5$, then $\psi$ has order $2^l$ and $\text{Aut}(\mathcal{L}_{\text{Sol}}^c(q_l)) = \text{Out}(\mathcal{L}_{\text{Sol}}(S)) \cong S$ with the cyclic group generated by $\psi$.

**Proof.** We continue to write $\mathcal{L} = \mathcal{L}_{\text{Sol}}^c(q_l)$, $\mathcal{F} = \mathcal{F}_{\text{Sol}}(q_l)$, $\mathcal{L}_z = \mathcal{L}_{\text{Spin}}^c(q_l)$, and $\mathcal{F}_z = \mathcal{F}_{\text{Spin}}(q_l)$, and we continue to assume that $\mathcal{L}$ has been chosen so as to contain $\mathcal{L}_z$ as a linking subsystem. Recall that $T_k \subseteq S$ is homocyclic of rank 3 and exponent $2^k = 2^{k+2}$.

We first check whether the outer automorphism of $\mathcal{L}_z$ induced by a diagonal automorphism of $H$ extends to $\mathcal{L}$, and we claim that it doesn’t. A non-inner diagonal automorphism of $H$ is induced by conjugation by an element $t \in \tilde{T}$ by $[\text{GLS98}]$ Theorem 2.5.1(b)]. Its class as an outer automorphism has order 2, so if necessary we replace $t$ by an odd power and assume that $t^2 \in T_k$. Now $T_k$ consists of the elements of $\tilde{T}$ of order dividing $2^{k}$, so $t$ has order $2^{k+1}$ and induces an automorphism of $H$ of order at least $2^k$. For ease of notation, we identify $T_k$ with $\delta_{T_k}(T_k) \subseteq \text{Aut}_{\mathcal{L}}(T_k)$, and we identify $s \in S$ with $\delta_s(s) \in \text{Aut}_{\mathcal{L}}(S)$.

Let $\tau = \bar{\kappa}_H([c_l]) \in \text{Aut}(\mathcal{L}_z)$, and assume that $\tau$ lifts to an element $\bar{\tau} \in \text{Aut}(\mathcal{L})$ under the map $\bar{\rho}$ of Lemma 3.3. As $\bar{\rho}(\bar{\tau}) = \tau = \bar{\kappa}_H(c_l)$, we have $\bar{\rho}(\bar{\tau}^2) = \tau^2 = \bar{\kappa}_H(c_{2^l})$, i.e. $\bar{\rho}(\bar{\tau}^2)$ acts on
every object and every morphism of $L_z = L^c_S(H)$ as conjugation by $t^2$. Similarly, if we take the conjugation automorphism $c_{t^2}$ of $L$ by $t^2$ (or more precisely the conjugation automorphism $c_{\delta_S(t^2)}$ of $L$ by $\delta_S(t^2)$), then $\hat{\rho}(c_{t^2})$ is just the conjugation automorphism of $L_z$ by $t^2$. So according to the remark at the end of §2.2.2, the automorphism $\hat{\rho}(c_{t^2})$ acts also on $L_z$ via conjugation by $t^2$, which shows that $\hat{\rho}(\tau^2) = \hat{\rho}(c_{t^2})$. By the description of the kernel in Lemma 3.5 we have $\tau^2 = c_{t^2}$ or $\tau^2 = c_{t^2}$.

Now set $\alpha := \tau T_k \in \text{Aut}(\text{Aut}_L(T_k))$. From what we have shown, it follows that $\alpha$ equals the conjugation automorphism $c_{t^2}$ or $c_{t^2}$ of $\text{Aut}_L(T_k)$. Note that $|c_{t^2}| = |t^2 z| = |t^2| = |c_{t^2}|$, since $Z(\text{Aut}_L(T_k)) = 1$ by Lemma 3.4(a). Hence,

$$\alpha = 2|\alpha^2| = 2|t^2| = 2^{k+1},$$

(3.11)

On the other hand, $\alpha$ centralizes $T_k$, and we have seen that $\alpha^2$ is an inner automorphism of $\text{Aut}_L(T_k)$. Moreover, by Lemma 3.3(a), $C_{\text{Aut}_L(T_k)}(T_k) = T_k$, and $\text{Aut}_L(T_k)$ acts fixed point freely on $T_k/\Phi(T_k)$. The hypotheses of Lemma 3.3 thus hold for $G = \text{Aut}_L(T_k)$ and $\alpha \in \text{Aut}(G)$. By the description of the kernel in Lemma 3.6 and the exactness of the third column of (2.3), the maps $\mu_G$ and $\mu_{L_z}$ are isomorphisms. Thus,

$$\text{Out}(L) \cong \text{Out}(F) \cong C_{2^l}.$$　

Let $\psi$ be the field automorphism of $L$ induced by $\psi_0$ as above. Then $\psi$ is the identity on $\pi^{-1}(F)_{\text{Sol}(5)}(5))$ by definition. It remains to show that $\psi$ has order $2^l$, since this will imply that $\text{Aut}(L)$ is a split extension of $\text{Aut}_L(S) \cong S$ by $\langle \psi \rangle \cong \text{Out}(L) \cong \text{Out}(F)$.

The automorphism $\psi^{2^l}$ maps to the trivial automorphism of $F$, and so is conjugation by an element of $Z(S)$ by (2.2). Now $\psi^{2^l}$ is trivial on $\text{Aut}_{L^c_{\text{Sol}(5)}}(\Omega_2(T_k))$, whereas $z \notin Z(\text{Aut}_{L^c_{\text{Sol}(5)}}(\Omega_2(T_k)))$ by Lemma 3.3(a) as $T_2 = \Omega_2(T_k)$ is the torus of $L^c_{\text{Sol}(5)}$. Thus, since a morphism $\varphi$ is fixed by $c_z$ if and only if $\pi(\varphi)(z) = z$ (Axiom (C)), we conclude that $\psi^{2^l}$ is the identity automorphism of $L$, and this completes the proof.

4. Extensions

In this section, we recall a result of Linckelmann on the Schur multipliers of the Benson-Solomon systems, and we prove that each saturated fusion system $F$ with $F^+(F) = F_{\text{Sol}(q)}^+(q)$ is a split extension of $F^+(F)$ by a group of outer automorphisms.

Recall that the hyperfocal subgroup of a saturated $p$-fusion system $F$ over $S$ is defined to be the subgroup of $S$ given by

$$\text{hfp}(F) = \langle [\varphi, s] := \varphi(s)s^{-1} | s \in P \leq S \text{ and } \varphi \in O^p(\text{Aut}_F(P)) \rangle.$$　

A subsystem $F_0$ over $S_0 \leq S$ is said to be of $p$-power index in $F$ if $\text{hfp}(F) \leq S_0$ and $O^p(\text{Aut}_F(P)) \leq \text{Aut}_{F_0}(P)$ for each $P \leq S_0$. There is always a unique normal saturated subsystem on $\text{hfp}(F)$ of $p$-power index in $F$, which is denoted by $O^p(F)$ [AKO11]. We will use the next lemma in §4.2.

\begin{lemma} 4.1. \textit{Let $F$ be a saturated fusion system over $S$, and let $F_0$ be a weakly normal subsystem of $F$ over $S_0 \leq S$. Assume that $O^p(\text{Aut}_F(S_0)) \leq \text{Aut}_{F_0}(S_0)$. Then $O^p(\text{Aut}_F(P)) \leq \text{Aut}_{F_0}(P)$ for every $P \leq S_0$. Thus, if in addition $\text{hfp}(F) \leq S_0$, then $F_0$ has $p$-power index in $F$.}
\end{lemma}
Proof. Note that \( \operatorname{Aut}_{\mathcal{F}_0}(P) \) is normal in \( \operatorname{Aut}_{\mathcal{F}}(P) \) for every \( P \leq S_0 \), since \( \mathcal{F}_0 \) is weakly normal in \( \mathcal{F} \). We need to show that \( \operatorname{Aut}_{\mathcal{F}}(P)/\operatorname{Aut}_{\mathcal{F}_0}(P) \) is a \( p \)-group for every \( P \leq S_0 \). Suppose this is false and let \( P \) be a counterexample of maximal order. Our assumption gives \( P < S_0 \). Hence, \( P < Q := N_{S_0}(P) \), and the maximality of \( P \) implies that
\[
\operatorname{Aut}_{\mathcal{F}}(Q)/\operatorname{Aut}_{\mathcal{F}_0}(Q)
\]
is a \( p \)-group. Notice that
\[
N_{\operatorname{Aut}_{\mathcal{F}}(Q)}(P)/N_{\operatorname{Aut}_{\mathcal{F}_0}(Q)}(P) \cong N_{\operatorname{Aut}_{\mathcal{F}}(Q)}(P)\operatorname{Aut}_{\mathcal{F}_0}(Q)/\operatorname{Aut}_{\mathcal{F}_0}(Q)
\leq \operatorname{Aut}_{\mathcal{F}}(Q)/\operatorname{Aut}_{\mathcal{F}_0}(Q)
\]
and thus \( N_{\operatorname{Aut}_{\mathcal{F}}(Q)}(P)/N_{\operatorname{Aut}_{\mathcal{F}_0}(Q)}(P) \) is a \( p \)-group.

If \( \alpha \in \operatorname{Hom}_{\mathcal{F}}(P,S) \) with \( \alpha(P) \in \mathcal{F}^f \) then conjugation by \( \alpha \) induces a group isomorphism from \( \operatorname{Aut}_{\mathcal{F}}(P) \) to \( \operatorname{Aut}_{\mathcal{F}}(\alpha(P)) \). As \( \mathcal{F}_0 \) is weakly normal, we have \( \alpha(P) \leq S_0 \) and conjugation by \( \alpha \) takes \( \operatorname{Aut}_{\mathcal{F}_0}(P) \) to \( \operatorname{Aut}_{\mathcal{F}_0}(\alpha(P)) \). So upon replacing \( P \) by \( \alpha(P) \), we may assume without loss of generality that \( P \) is fully \( \mathcal{F} \)-normalized. Then \( P \) is also fully \( \mathcal{F}_0 \)-normalized by [Asc08, Lemma 3.4(5)]. By the Sylow axiom, \( \operatorname{Aut}_{S_0}(P) \) is a Sylow \( p \)-subgroup of \( \operatorname{Aut}_{\mathcal{F}_0}(P) \). So the Frattini argument yields
\[
\operatorname{Aut}_{\mathcal{F}}(P) = \operatorname{Aut}_{\mathcal{F}_0}(P)\operatorname{Aut}_{\mathcal{F}}(P)(\operatorname{Aut}_{S_0}(P))
\]
and thus
\[
\operatorname{Aut}_{\mathcal{F}}(P)/\operatorname{Aut}_{\mathcal{F}_0}(P) \cong N_{\operatorname{Aut}_{\mathcal{F}}(P)}(\operatorname{Aut}_{S_0}(P))/N_{\operatorname{Aut}_{\mathcal{F}_0}(P)}(\operatorname{Aut}_{S_0}(P)).
\]
By the extension axiom for \( \mathcal{F} \) and \( \mathcal{F}_0 \), each element of \( N_{\operatorname{Aut}_{\mathcal{F}}(P)}(\operatorname{Aut}_{S_0}(P)) \) extends to an automorphism of \( \operatorname{Aut}_{\mathcal{F}}(Q) \), and each element of \( N_{\operatorname{Aut}_{\mathcal{F}_0}(P)}(\operatorname{Aut}_{S_0}(P)) \) extends to an automorphism of \( \operatorname{Aut}_{\mathcal{F}_0}(Q) \). Therefore, the map
\[
\Phi: N_{\operatorname{Aut}_{\mathcal{F}}(Q)}(P) \to N_{\operatorname{Aut}_{\mathcal{F}}(P)}(\operatorname{Aut}_{S_0}(P)), \varphi \mapsto \varphi|_P
\]
is an epimorphism which maps \( N_{\operatorname{Aut}_{\mathcal{F}_0}(Q)}(P) \) onto \( N_{\operatorname{Aut}_{\mathcal{F}_0}(P)}(\operatorname{Aut}_{S_0}(P)) \). Hence,
\[
\operatorname{Aut}_{\mathcal{F}}(P)/\operatorname{Aut}_{\mathcal{F}_0}(P) \cong N_{\operatorname{Aut}_{\mathcal{F}}(P)}(\operatorname{Aut}_{S_0}(P))/N_{\operatorname{Aut}_{\mathcal{F}_0}(P)}(\operatorname{Aut}_{S_0}(P))
\cong N_{\operatorname{Aut}_{\mathcal{F}}(Q)}(P)/N_{\operatorname{Aut}_{\mathcal{F}_0}(Q)}(P)\ker(\Phi).
\]
We have seen above that \( N_{\operatorname{Aut}_{\mathcal{F}}(Q)}(P)/N_{\operatorname{Aut}_{\mathcal{F}_0}(Q)}(P) \) is a \( p \)-group, and therefore also
\[
N_{\operatorname{Aut}_{\mathcal{F}}(Q)}(P)/N_{\operatorname{Aut}_{\mathcal{F}_0}(Q)}(P)\ker(\Phi)
\]
is a \( p \)-group. Hence, \( \operatorname{Aut}_{\mathcal{F}}(P)/\operatorname{Aut}_{\mathcal{F}_0}(P) \) is a \( p \)-group, and this contradicts our assumption that \( P \) is a counterexample. \( \square \)

4.1. **Extensions to the bottom.** A **central extension** of a fusion system \( \mathcal{F}_0 \) is a fusion system \( \mathcal{F} \) such that \( \mathcal{F}/Z \cong \mathcal{F}_0 \) for some subgroup \( Z \leq Z(\mathcal{F}) \). The central extension is said to be **perfect** if \( \mathcal{F} = \mathcal{O}^p(\mathcal{F}) \). Linckelmann has shown that the Schur multiplier of a Benson-Solomon system is trivial.

**Theorem 4.2** (Linckelmann). Let \( \mathcal{F} \) be a perfect central extension of a Benson-Solomon fusion system \( \mathcal{F}_0 \). Then \( \mathcal{F} = \mathcal{F}_0 \).

*Proof.* This follows from Corollary 4.4 of [Lin06a] together with the fact that \( \text{Spin}_7(q) \) has Schur multiplier of odd order when \( q \) is odd [GLS98, Tables 6.1.2, 6.1.3]. \( \square \)
4.2. Extensions to the top. The next theorem describes the possible extensions \((S,F)\) of a Benson-Solomon system \((S_0,F_0)\). The particular hypotheses are best stated in terms of the generalized Fitting subsystem of Aschbacher \([Asc11]\), but they are equivalent to requiring that \(F_0 \leq F\) and \(C_S(F_0) \leq S_0\), where \(C_S(F_0)\) is the centralizer constructed in \([Asc11] \S 6\). This latter formulation is sometimes expressed by saying that \(F_0\) is \textit{centric normal} in \(F\).

**Theorem 4.3.** Let \(l\) be any nonnegative integer, and let \(F_0 = F_{Sol}(5^{2^l})\) be a Benson-Solomon system over the 2-group \(S_0\).

\(\begin{align*}
(a) & \quad \text{If } F \text{ is a saturated fusion system over } S \text{ such that } F^*(F) = F_0 \text{, then } F_0 = O^2(F), \text{ } S \text{ splits over } S_0, \text{ and the map } S/S_0 \to \text{Out}(F_0) \text{ induced by conjugation is injective.} \\
(b) & \quad \text{Conversely, given a subgroup of } A \leq \text{Out}(F_0) \cong C_2, \text{ there is a saturated fusion system } F \text{ over a 2-group } S \text{ such that } F^*(F) = F_0 \text{ and the map } S/S_0 \to \text{Out}(F_0) \text{ induced by conjugation on } S_0 \text{ has image } A. \text{ Moreover, the pair } (S,F) \text{ with these properties is uniquely determined up to isomorphism.}
\end{align*}\)

\[
\text{If } L_0 \text{ is a centric linking system associated to } F_0, \text{ then } \text{Aut}_{L_0}(S_0) = S_0, \text{ and the } p\text{-group } S \text{ can be chosen to be the preimage of } A \text{ in } \text{Aut}(L_0) \text{ under the quotient map from } \text{Aut}(L_0) \text{ to } \text{Out}(L_0) \cong \text{Out}(F_0).
\]

**Proof.** Let \(F\) be a saturated fusion system over \(S\) such that \(F^*(F) = F_0\). Set \(F_1 = F_0 S\), the internal extension of \(F_0\) by \(S\), as in \([Hen13]\) or \([Asc11] \S 8\). According to \([AOV12]\) Proposition 1.31, there is a normal pair of linking systems \(L_0 \leq L_1\), associated to the normal pair \(F_0 \leq F_1\). Furthermore, \(L_0 \leq L_1\) can be chosen such that \(L_0\) is a centric linking system. There is a natural map from \(\text{Aut}_{L_0}(S_0)\) to \(\text{Aut}(L_0)\) which sends a morphism \(\varphi \in \text{Aut}_{L_0}(S_0)\) to conjugation by \(\varphi\). (So the restriction of this map to \(\text{Aut}_{L_0}(S_0)\)) is the conjugation map described in \([Sem15] \S 2.2\).

The centralizer \(C_S(F_0)\) depends a priori on the fusion system \(F\), but it is shown in \([Lyn15]\) Lemma 1.13 that it does not actually matter whether we form \(C_S(F_0)\) inside of \(F\) or inside of \(F_1\). Moreover, since \(F^*(F) = F_0\), it follows from \([Asc11] \text{Theorem } 6\) that \(C_S(F_0) = Z(F_0) = 1\). Thus, by a result of Semeraro \([Sem15] \text{Theorem A}\), the conjugation map \(\text{Aut}_{L_1}(S_0) \to \text{Aut}(L_0)\) is injective. By Lemma 3.3, we have \(S_0 = \text{Aut}_{L_0}(S_0)\) via the inclusion functor \(\delta_1\) for \(L_1\). By Theorem 3.10, \(\text{Aut}(L_0)\) is a 2-group which splits over \(S_0\). Moreover, by the same theorem, we have that \(\text{Out}_{L_0}(S_0) \leq S_0\) and \(\text{Out}(L_0) \cong \text{Out}(F_0)\) is cyclic. Since \((\delta_1)_S(S) \cong S\) is a Sylow 2-subgroup of \(\text{Aut}_{L_0}(S_0)\) by \([Oli10] \text{Proposition 4(d)}\), we can conclude that

\[
S_0 = \text{Aut}_{L_0}(S_0) \leq \text{Aut}_{L_1}(S_0) = S,
\]

via the inclusion functor \(\delta_1\) for \(L_1\). Moreover, it follows that \(S\) splits over \(S_0\), and \(C_S(S_0) \leq S_0\). The latter property means that the map

\[
S/S_0 \to \text{Out}(F_0)
\]

is injective. In particular, \(S/S_0\) is cyclic as \(\text{Out}(F_0)\) is cyclic.

Next, we show that \(O^2(F) = F_0\). Fix a subgroup \(P \leq S\), and let \(\alpha \in \text{Aut}_F(P)\) be an automorphism of odd order. Then \(\alpha\) induces an odd-order automorphism of the cyclic 2-group \(P/(P \cap S_0) \cong PS_0/S_0 \leq S/S_0\). This automorphism must be trivial, and so \([P,\alpha] \leq S_0\). Hence, \([P, O^2(\text{Aut}_F(P))] \leq S_0\) for all \(P \leq S\). Since \(\text{hp}(F_0) = S_0\), we have \(\text{hp}(F) = S_0\). Note that \(\text{Aut}_F(S_0)\) is a 2-group as \(\text{Aut}_F(S_0) \leq \text{Aut}(F_0)\) and \(\text{Aut}(F_0)\) is a 2-group by Theorem 3.10. Therefore \(O^2(F) = F_0\) by Lemma 4.1. We conclude that \(F_1 = F\) by the uniqueness statement in \([Hen13] \text{Theorem } 1\). This completes the proof of (a). Moreover, we have seen that the following property holds for any normal pair \(L_0 \leq L\) attached to \(F_0 \leq F\):

\[
(4.4) \quad S_0 = \text{Aut}_{L_0}(S_0) \leq \text{Aut}_{L_1}(S_0) = S \quad \text{and} \quad S \cong \text{Aut}(L_0) \to \text{Aut}(L_0) \text{ is injective.}
\]

\[\text{11}\]
Finally, we prove (b). Fix a centric linking system $\mathcal{L}_0$ associated to $F_0$ with inclusion functor $\delta_0$. Let $S \subseteq \text{Aut}(\mathcal{L}_0)$ be the preimage of $A$ under the quotient map to $\text{Out}(F_0)$. We will identify $S_0$ with $\delta_0(S_0)$ so that $S_0 = \text{Aut}_{\mathcal{L}_0}(S_0)$ by Lemma 3.4. Write $\iota : S_0 \to \text{Aut}(\mathcal{L}_0)$, $s \mapsto c_s$ for map sending $s \in S_0$ to the automorphism of $\mathcal{L}_0$ induced by conjugation with $s$ in $\mathcal{L}_0$. Then $\iota(S_0)$ is normal in $S$. Let $\chi : S \to \text{Aut}(S_0)$ be the map defined by $\alpha \mapsto \iota^{-1} \circ c_{\iota(S_0)} \circ \iota$; i.e. $\chi$ corresponds to conjugation in $S$ if we identify $S_0$ with $\iota(S_0)$. We argue next that the following diagram commutes:

\[
\begin{array}{ccc}
S_0 & \xrightarrow{\iota} & \text{Aut}(\mathcal{L}_0) \\
\downarrow{\chi} & & \downarrow{\alpha \mapsto \alpha_{S_0}} \\
S & \xrightarrow{\iota} & \text{Aut}(S_0)
\end{array}
\]

(4.5)

The upper triangle clearly commutes. Observe that $\alpha \circ \iota(s) \circ \alpha^{-1} = \alpha \circ c_s \circ \alpha^{-1} = c_{\iota(S_0)}(s) = \iota(\alpha_{S_0}(s))$ for every $s \in S_0$ and $\alpha \in S$. Hence, for every $\alpha \in S$ and $s \in S_0$, we have $(\iota^{-1} \circ c_{\iota(S_0)} \circ \iota)(s) = \iota^{-1}(\alpha \circ \iota(s) \circ \alpha^{-1}) = \alpha_{S_0}(s)$ and so the lower triangle commutes.

We will now identify $S_0$ with its image in $S$ under $\iota$, so that $\iota$ becomes the inclusion map and $\chi$ corresponds to the map $S \to \text{Aut}(S_0)$ induced by conjugation in $S$. As the above diagram commutes, it follows then that the diagram in [Oli10, Theorem 9] commutes when we take $\Gamma = S$ and $\tau : S \to \text{Aut}(\mathcal{L}_0)$ to be the inclusion. Thus, by that theorem, there is a saturated fusion system $\mathcal{F}$ over $S$ in which $F_0$ is a weakly normal, and there is a corresponding normal pair of linking systems $\mathcal{L}_0 \leq \mathcal{L}$ (in the sense of [AOV12, §1.5]) such that $S = \text{Aut}_{\mathcal{L}}(S_0)$ has the given action on $\mathcal{L}_0$ (i.e. the automorphism of $\mathcal{L}_0$ induced by conjugation with $s \in S$ in $\mathcal{L}$ equals the automorphism $s$ of $\mathcal{L}_0$). By the same theorem, the pair $(\mathcal{F}, \mathcal{L})$ is unique up to isomorphism of fusion systems and linking systems with these properties. Since $F_0$ is simple [Lin06b], $F_0$ is in fact normal in $\mathcal{F}$ by a result of Craven [Cra11, Theorem A]. Thus, since $C_S(F_0) \leq C_S(S_0) \leq S_0$, it is a consequence of [Asc11, (9.1)(2), (9.6)] that $\mathcal{F}^*(\mathcal{F}) = F_0$.

So it remains only to prove that $(S, \mathcal{F})$ is uniquely determined up to an isomorphism of fusion systems. Let $\mathcal{F}'$ be a saturated fusion system over a 2-group $S'$ such that $\mathcal{F}^*(\mathcal{F}') = F_0$, and such that the map $S'/S_0 \to \text{Out}(F_0)$ induced by conjugation has image $A$. Then by (a), $F_0 = O^2(\mathcal{F}')$. So by [AOV12, Proposition 1.31], there is a normal pair of linking systems $\mathcal{L}_0' \leq \mathcal{L}'$ associated to the normal pair $\mathcal{F}_0 \leq \mathcal{F}'$. Moreover, we can choose $\mathcal{L}_0'$ to be a centric linking system. Since a centric linking system attached to $F_0$ is unique, there is an isomorphism $\theta : \mathcal{L}_0' \to \mathcal{L}_0$ of linking systems. We may assume that the set of morphisms which lie in $\mathcal{L}'$ but not in $\mathcal{L}_0'$ is disjoint from the set of morphisms in $\mathcal{L}_0$. Then we can construct a new linking system from $\mathcal{L}'$ by keeping every morphism of $\mathcal{L}'$ which is not in $\mathcal{L}_0'$ and replacing every morphism $\psi$ in $\mathcal{L}_0'$ by $\theta(\psi)$, and then carrying over the structure of $\mathcal{L}'$ in the natural way. Thereby we may assume $\mathcal{L}_0' = \mathcal{L}_0$. So we are given a normal pair $\mathcal{L}_0 \leq \mathcal{L}'$ attached to $\mathcal{F}_0 \leq \mathcal{F}'$. By (4.4) applied with $\mathcal{L}'$ and $\mathcal{F}'$ in place of $\mathcal{F}$ and $\mathcal{L}$, we have $S_0 = \text{Aut}_{\mathcal{L}_0}(S_0) \leq \text{Aut}_{\mathcal{L}'}(S_0) = S'$ via the inclusion functor $\delta'$ of $\mathcal{L}'$. Let $\tau : S' \to \text{Aut}(\mathcal{L}_0)$ be the map taking $s \in S'$ to the automorphism of $\mathcal{L}_0$ induced by conjugation with $s$ in $\mathcal{L}'$. Again using (4.3), we see that $\tau$ is injective. Note also that $\tau$ restricts to the identity on $S_0$ if we identify $S_0$ with $\iota(S_0)$ as above. Recall that the map $S'/S_0 \to \text{Out}(F_0)$ induced by conjugation has image $A$. So Theorem 3.10 implies $\tau(S') = S$, i.e. we can regard $\tau$ as an isomorphism $\tau : S' \to S$. So replacing $(S', \mathcal{F}')$ by $(S, \mathcal{F})$ and then choosing $\mathcal{L}_0 \leq \mathcal{L}'$ as before, we may assume $S = S'$. So $\mathcal{F}'$ is a fusion system over $S$ with $\mathcal{F}_0 \leq \mathcal{F}'$, and $\mathcal{L}_0 \leq \mathcal{L}'$ is a normal pair of linking systems associated to $\mathcal{F}_0 \leq \mathcal{F}'$ such that $\text{Aut}_{\mathcal{L}'}(S_0) = S$ via $\delta'$. Let $s \in S$. Recall that $\tau(s)$ is
the automorphism of $\mathcal{L}_0$ induced by conjugation with $s$ in $\mathcal{L}$. Observe that the automorphism of $S_0 = \text{Aut}_{\mathcal{L}_0}(S_0)$ induced by $\tau(s)$ equals just the automorphism of $S_0$ induced by conjugation with $s$ in $S$. Similarly, the automorphism $s$ of $\mathcal{L}_0$ equals the automorphism of $\mathcal{L}_0$ given by conjugation with $s$ in $\mathcal{L}$, and so induces on $S_0 = \text{Aut}_{\mathcal{L}_0}(S_0)$ just the automorphism given by conjugation with $s$ in $S$. Theorem 3.10 gives $C_{\text{Aut}(\mathcal{L}_0)}(S_0) \leq S_0$ and this implies that any two automorphisms of $\mathcal{L}_0$, which induce the same automorphism on $S_0$, are equal. Hence, $\tau(s) = s$ for any $s \in S$. In other words, $S = \text{Aut}_{\mathcal{L}}(S_0)$ induces by conjugation in $\mathcal{L}'$ the canonical action of $S$ on $\mathcal{L}_0$. The uniqueness of the pair $(\mathcal{F}, \mathcal{L})$ implies now $\mathcal{F}' \cong \mathcal{F}$ and $\mathcal{L}' \cong \mathcal{L}$. This shows that $(S, \mathcal{F})$ is uniquely determined up to isomorphism.

\[ \square \]

References

[AC10] Michael Aschbacher and Andrew Chermak, A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver, Ann. of Math. (2) 171 (2010), no. 2, 881–978.

[AKO11] Michael Aschbacher, Radha Kessar, and Bob Oliver, Fusion systems in algebra and topology, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR 2848834

[AOV12] Kasper K. S. Andersen, Bob Oliver, and Joana Ventura, Reduced, tame and exotic fusion systems, Proc. Lond. Math. Soc. (3) 105 (2012), no. 1, 87–152. MR 2948790

[Asc08] Michael Aschbacher, Normal subsystems of fusion systems, Proc. Lond. Math. Soc. (3) 97 (2008), no. 1, 239–271. MR 2434097 (2009e:20044)

[Asc11] , The generalized Fitting subsystem of a fusion system, Mem. Amer. Math. Soc. 209 (2011), no. 986, vi+110. MR 2752788

[Ben98] David J. Benson, Cohomology of sporadic groups, finite loop spaces, and the Dickson invariants, Geometry and cohomology in group theory (Durham, 1994), London Math. Soc. Lecture Note Ser., vol. 252, Cambridge Univ. Press, Cambridge, 1998, pp. 10–23. MR 1709949 (2001i:55017)

[BLO03] Carles Broto, Ran Levi, and Bob Oliver, The homotopy theory of fusion systems, J. Amer. Math. Soc. 16 (2003), no. 4, 779–856 (electronic).

[BMO16] Carles Broto, Jesper M. Møller, and Bob Oliver, Automorphisms of fusion systems of finite simple groups of lie type, preprint (2016), arXiv:1601.04566.

[COS08] Andrew Chermak, Bob Oliver, and Sergey Shpectorov, The linking systems of the Solomon 2-local finite groups are simply connected, Proc. Lond. Math. Soc. (3) 97 (2008), no. 1, 209–238. MR 2434096 (2009g:55018)

[Cra11] David A. Craven, Normal subsystems of fusion systems, J. Lond. Math. Soc. (2) 84 (2011), no. 1, 137–158. MR 2819694

[DW93] W. G. Dwyer and C. W. Wilkerson, A new finite loop space at the prime two, J. Amer. Math. Soc. 6 (1993), no. 1, 37–64. MR 1161306

[GL83] Daniel Gorenstein and Richard Lyons, The local structure of finite groups of characteristic 2 type, Mem. Amer. Math. Soc. 42 (1983), no. 276, vii+731. MR 690900

[GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, The classification of the finite simple groups. Number 3. Part I. Chapter A, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998, Almost simple $K$-groups. MR 1490581 (98j:20011)

[Hen13] Ellen Henke, Products in fusion systems, J. Algebra 376 (2013), 300–319. MR 3003728

[Lin06a] Markus Linckelmann, A note on the Schur multiplier of a fusion system, J. Algebra 296 (2006), no. 2, 402–408.

[Lin06b] , Simple fusion systems and the Solomon 2-local groups, J. Algebra 296 (2006), no. 2, 385–401.

[LO02] Ran Levi and Bob Oliver, Construction of 2-local finite groups of a type studied by Solomon and Benson, Geom. Topol. 6 (2002), 917–990 (electronic).

[LO05] Correction to: “Construction of 2-local finite groups of a type studied by Solomon and Benson” [Geom. Topol. 6 (2002), 917–990 (electronic); mr1943386], Geom. Topol. 9 (2005), 2395–2415 (electronic).

[Lyn15] Justin Lynd, A characterization of the 2-fusion system of $L_4(q)$, J. Algebra 428 (2015), 315–356. MR 3314296
Bob Oliver, *Extensions of linking systems and fusion systems*, Trans. Amer. Math. Soc. **362** (2010), no. 10, 5483–5500. MR 2657688 (2011f:55032)

Bob Oliver, *Reductions to simple fusion systems*, Bulletin of the London Mathematical Society **48** (2016), no. 6, 923–934.

Jason Semeraro, *Centralizers of subsystems of fusion systems*, J. Group Theory **18** (2015), no. 3, 393–405. MR 3341522

Robert Steinberg, *Automorphisms of finite linear groups*, Canad. J. Math. **12** (1960), 606–615. MR 0121427

E-mail address: ellen.henke@abdn.ac.uk

Institute of Mathematics, University of Aberdeen, Fraser Noble Building, Aberdeen AB15 5LY, United Kingdom

E-mail address: lynd@louisiana.edu

Department of Mathematics, University of Louisiana at Lafayette, Maxim Doucet Hall, Lafayette, LA 70504