Generators of simple modular Lie superalgebras

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Abstract: Let \( X \) be one of the finite-dimensional simple graded Lie superalgebras of Cartan type \( W, S, H, K, HO, KO, SHO \) or \( SKO \) over an algebraically closed field of characteristic \( p > 3 \). In this paper we prove that \( X \) can be generated by one element except the ones of type \( W, HO, KO \) or \( SKO \) in certain exceptional cases, in which \( X \) can be generated by two elements.

As a subsidiary result, we also prove that certain classical Lie superalgebras or their relatives can be generated by one or two elements.

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0. Introduction

Over a field of characteristic \( p > 3 \), the classification problem for finite dimensional simple Lie superalgebras remains open up to now. However, one has known many simple modular Lie superalgebras including the \( \mathbb{Z} \)-graded simple modular Lie superalgebras of Cartan type \( W, S, H, K, HO, KO, SHO \) or \( SKO \) (for example, see \cite{3, 9–11, 13}), which are analogues of the eight series of infinite-dimensional vectorial Lie superalgebras over \( \mathbb{C} \) (for example, see \cite{7}). Roughly speaking, these \( \mathbb{Z} \)-graded Lie superalgebras are subalgebras of the full superderivation algebras of the tensor products of the divided power algebras and the exterior superalgebras and their \( \mathbb{Z} \)-nulls are classical Lie superalgebras or their relatives. As in the modular Lie algebra case, it is conceivable that these Lie superalgebras will play a key role in the final classification of simple modular Lie superalgebras.

In this paper we mainly prove that a simple Lie superalgebra in the eight series mentioned above can be generated by 1 element except the ones of type \( W, HO, KO \) or \( SKO \) in certain exceptional cases, in which \( X \) can be generated by 2 elements. In the process we also prove that the \( \mathbb{Z} \)-nulls of the eight series of Lie superalgebras of Cartan type, which are classical Lie superalgebras or their relatives, can be generated by 1 element except \( gl(m, n) \) with \( m - n \equiv 0 \) (mod \( p \)), \( \hat{P}(m) \) with \( m \neq 0 \) (mod \( p \)) and \( \hat{P}(m) \oplus FI \) with \( m \neq 0 \) (mod \( p \)) and in those exceptional cases, the nulls can be generated by 2 elements.

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This study is originally motivated by two papers of Bois, which provide us with a considerable amount of information for modular Lie algebras. In 2009, Bois [1] proved that any simple Lie algebra in characteristics \( p \neq 2, 3 \) can be generated by 2 elements and moreover, the classical Lie algebras and Zassenhaus algebras can be generated by 1.5 elements, that is, any given nonzero element can be paired a suitable element such that these two elements generate the whole algebra. The results in [1] cover certain classical results in some previous papers: In 1976, Ionescu [5] proved that a simple Lie algebra \( L \) over \( \mathbb{C} \) can be generated by 1 element and in 1951, Kuranashi [8] proved that a semi-simple Lie algebra in characteristic 0 can be generated by 2 elements. Later, Bois [2] proved that the simple graded Lie algebras of Cartan type excluding Zassenhaus algebras are never generated by 1.5 elements. In 2009, we began to consider the problem of generators for simple Lie superalgebras over a field of characteristic 0 and obtained that such a simple Lie superalgebra can be generated by 1 element (see arXiv:1204.2398v1 [math-ph]). We should also mention a well-known result in finite groups: any finite simple group can be generated by 2 elements and only the ones of prime order can be generated by 1 element [4].

Let us point out the main difficulties one will encounter in simple modular Lie superalgebra case, which do not occur in characteristic zero situation. In the characteristic \( p \) case, for a simple \( \mathbb{Z} \)-graded Lie superalgebra of Cartan type \( X = \oplus X_{s}^{i} \), its \( \mathbb{Z} \)-grading is not necessarily consistent with its \( \mathbb{Z}_{2} \)-grading. In particular, its null \( X_{0} \) is not necessarily a Lie algebra. Moreover, a simple \( \mathbb{Z} \)-graded modular Lie superalgebra of Cartan type is not necessarily generated by its local part. In view of certain general properties of simple Lie superalgebras, we overcome these issues by virtue of the weight space decompositions of \( X_{-1}, X_{0} \) and \( X_{s} \) relative to the standard Cartan subalgebra of \( X_{0} \) and find the desirable generators from the odd weight vectors corresponding to different odd weights.

Throughout we write \( \langle Y \rangle \) for the sub-Lie superalgebra generated by a subset \( Y \) in a Lie superalgebra. The ground field \( \mathbb{F} \) is algebraically closed and of characteristic \( p > 3 \). All the vector spaces including algebras and modules are finite dimensional.

1. Classical Lie superalgebras

In this section \( L \) denotes one of the following classical Lie superalgebras and their relatives: \( \mathfrak{gl}(m,n), \mathfrak{sl}(m,n), \mathfrak{osp}(m,n), \mathbb{P}(m), \mathbb{P}(m) \oplus \mathbb{F}I \) or \( \mathfrak{osp}(m,n) \oplus \mathbb{F}I \). The aim of this section is to prove that \( L \) can be generated by 1 element except \( \mathfrak{gl}(m,n) \) with \( m - n \equiv 0 \pmod{p} \), \( \mathbb{P}(m) \) with \( m \not\equiv 0 \pmod{p} \) and \( \mathbb{P}(m) \oplus \mathbb{F}I \) with \( m \not\equiv 0 \pmod{p} \) and in these exceptional cases, \( L \) can be generated by 2 elements. The results will be used in the next section to study generators of the \( \mathbb{Z} \)-graded simple Lie superalgebras of Cartan type. Note that the \( \mathbb{Z} \)-nils of these graded Lie superalgebras are just the classical Lie superalgebras or their relatives mentioned above.

The general linear Lie superalgebra \( \mathfrak{gl}(m,n) \) contains the following subalgebras:

\[
\mathfrak{gl}(m,n) := \{ A \in \mathfrak{gl}(m,n) \mid \text{str}(A) = 0 \};
\]

\[
\mathfrak{osp}(m,n) := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{gl}(m,n) \mid A^{t}G + GA = 0, B^{t}G + MC = 0, D^{t}M + MD = 0 \right\},
\]
Let $\mathfrak{h}$ be the standard Cartan subalgebra of $L$. Consider the weight (root) decompositions relative to $\mathfrak{h}$:

$$L_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0} L^\alpha_0, \quad L_1 = \bigoplus_{\beta \in \Delta_1} L^\beta_1;$$

$$L = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0} L^\alpha_0 \oplus \bigoplus_{\beta \in \Delta_1} L^\beta_1.$$  

Write

$$\Delta = \Delta_0 \cup \Delta_1$$

and

$$L^\gamma = L^\alpha_0 \oplus L^\beta_1$$

for $\gamma \in \Delta$.

**Lemma 1.1.** The following statements hold.

1. If $L$ is simple then $[L_1, L_1] = L_0$.

2. If $L \neq P(4)$ then $\dim L^\gamma = 1$ for every $\gamma \in \Delta$.

**Proof.** (1) The proof is standard (see [6, Proposition 1.2.7(1), p.20]).

(2) First, suppose $L = P(m)$ with $m \neq 4$ and $m \not\equiv 0 \pmod{p}$. Then $P(m)$ has a so-called standard basis:

| $L_0$ | $\mathfrak{h}$ | $\epsilon_{i1} - \epsilon_{i+1,m+1} - \epsilon_{m+1,m+1} + \epsilon_{m+i+1,m+i+1}$, $i \in 1, m - 1$ | $\epsilon_{ij} = \epsilon_{m+j,m+i}$, $i, j \in Y_0, i \neq j$ |
|-------|-----------------|-------------------------------------------------|---------------------------------|
| $L_1$ | $\epsilon_{i,m+j} + \epsilon_{j,m+i}$, $\epsilon_{m+i,j} - \epsilon_{m+j,i}$, $i, j \in Y_0$ | | |

Let $\epsilon_i$ be the linear function on $\mathfrak{h}$ such that

$$\epsilon_i(\epsilon_{i1} - \epsilon_{i+1,j+1} - \epsilon_{m+1,m+1} + \epsilon_{m+i+1,m+i+1}) = \delta_{ij}$$

for $i, j \in 1, m - 1$. Write $1 = \sum_{i=1}^{m-1} \epsilon_i$. All the weights and the corresponding weight vectors are as follows:

| even weights | $\delta_{i1} - \delta_{i+1,j-1} + \epsilon_{i1} - \epsilon_{i+1,j}$, $i, j \in Y_0, i \neq j$ |
|--------------|------------------------------------------------------------------|
| weight vectors | $\epsilon_{ij} = \epsilon_{m+j,m+i}$, $i, j \in Y_0, i \neq j$ |
| odd weights | $\delta_{i1} + \delta_{i+1,j-1} - \epsilon_{i1} - \epsilon_{i+1,j}$, $i, j \in Y_0$ |
| weight vectors | $\epsilon_{i,m+j} + \epsilon_{j,m+i}$, $i, j \in Y_0$ |
| odd weights | $\delta_{j1} - \delta_{j+1,i-1} + \epsilon_{j1} - \epsilon_{j+1,i}$, $i, j \in Y_0, i \neq j$ |
| weight vectors | $\epsilon_{m+i,j} - \epsilon_{m+j,i}$, $i, j \in Y_0, i \neq j$ |

According to Table 1.2, the weight spaces of $P(m)$ are all 1-dimensional.
Proposition 1.2. Let $L = \mathfrak{p}(m)$ with $m \neq 4$ and $m \equiv 0 \pmod{p}$. Then, all basis elements of the Cartan subalgebra $\mathfrak{h}$ in Table 1.1 and the element $\sum_{i=1}^{m} e_i - \sum_{j=m+1}^{2m} e_j$ constitute a basis of the standard Cartan subalgebra of $L$. Then the conclusion follows from a case-by-case examination.

Thirdly, for $L \neq \mathfrak{p}(m)$, the proof follows from a computation of weights (see [3, 13]).

For later use we record two basic facts:

F1. Let $V$ be a vector space and $\mathcal{F} = \{f_1, \ldots, f_n\}$ a finite set of linear functions on $V$. Then $\Omega_{\mathcal{F}} := \{v \in V \mid \Pi_{1 \leq i \neq j \leq n}(f_i - f_j)(v) \neq 0\} \neq \emptyset$ (see [1, Lemma 2.2.1]).

F2. Let $\mathfrak{A}$ be an algebra. For $a \in \mathfrak{A}$ write $L_a$ for the left-multiplication operator given by $a$. Suppose $x = x_1 + x_2 + \cdots + x_n$ is a sum of eigenvectors of $L_a$ associated with mutually distinct eigenvalues. Then all $x_i$’s lie in the subalgebra generated by $a$ and $x$.

Proposition 1.2. $L$ can be generated by 1 element except that $L = \mathfrak{gl}(m, n)$ with $m - n \equiv 0 \pmod{p}$, $\mathfrak{p}(m)$ with $m \neq 0 \pmod{p}$ or $\mathfrak{p}(m) \oplus \mathfrak{f}I$ with $m \neq 0 \pmod{p}$.

In the exceptional case, $L$ can be generated by 2 elements.

Proof. We treat two cases separately:

Case 1. Suppose $L = \mathfrak{sl}(m, n)$ with $m - n \neq 0 \pmod{p}$, $\mathfrak{osp}(m, n)$ or $\mathfrak{p}(m)$ with $m \neq 0 \pmod{p}$. If $L \neq \mathfrak{p}(4)$, then all the weights spaces are 1-dimensional. By the basic fact F1, choose any $h \in \Omega_{\Delta_1} \subset H$. Let $x = \sum_{\gamma \in \Delta_1} x^e_{\gamma}$, where $x^e_{\gamma}$ is a weight vector of $\gamma$. By the basic fact F2 and Lemma [1, 1.1] we have $\langle h + x \rangle = L$.

Suppose $L = \mathfrak{p}(4)$. Then the standard basis of $\mathfrak{p}(4)$ is listed above (see Table 1.1 for $m = 4$). Let $x$ be the sum of all standard odd weight vectors (basis elements) (see Table 1.2 for $m = 4$). Choose an element $h \in \Omega_{\Delta_1}$. We want to show that $\langle h + x \rangle = L$.

Recall that $e_i$ is the linear function on $\mathfrak{h}$ given by

$$e_i(e_{11} - e_{1+1,1+j} - e_{55} + e_{5+j,5+j}) = \delta_{ij}$$

for $1 \leq i, j \leq 3$. All the odd weights and the corresponding odd weight vectors are listed below:

| weights | $2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3$ | $-2\varepsilon_1$ | $-2\varepsilon_2$ | $-2\varepsilon_3$ |
|---------|---------------------------------|-----------------|-----------------|-----------------|
| vectors | $e_{15}$                         | $e_{26}$        | $e_{37}$        | $e_{48}$        |
| weights | $\varepsilon_1 + \varepsilon_3$ | $-\varepsilon_1 - \varepsilon_3$ | $-\varepsilon_1 - \varepsilon_3$ | $-\varepsilon_1 - \varepsilon_3$ |
| vectors | $e_{17} + e_{35}, e_{64} - e_{82}$ | $e_{27} + e_{36}, e_{54} - e_{81}$ | $e_{28} + e_{46}, e_{53} - e_{71}$ | $e_{38} + e_{47}, e_{52} - e_{61}$ |
| weights | $\varepsilon_2 + \varepsilon_3$ | $\varepsilon_1 + \varepsilon_2$ | $\varepsilon_2 - \varepsilon_3$ | $\varepsilon_2 - \varepsilon_3$ |
| vectors | $e_{16} + e_{25}, e_{74} - e_{83}$ | $e_{18} + e_{45}, e_{63} - e_{72}$ | $e_{38} + e_{47}, e_{52} - e_{61}$ | $e_{37}, e_{48}, e_{16} + e_{25} + e_{74} - e_{83}, e_{38} + e_{47} + e_{52} - e_{61}, e_{28} + e_{46} + e_{53} - e_{71}, e_{15}, e_{26}, e_{17} + e_{35} + e_{64} - e_{82}, e_{27} + e_{36} + e_{54} - e_{81}, e_{18} + e_{45} + e_{63} - e_{72}$ |

Then, by Table 1.3, one sees that $\langle h + x \rangle$ contains the following elements:
Lie superbrackets of the odd elements above yield that

\[
P_{57} - e_{31} = [e_{37}, e_{28} + e_{46} + e_{53} - e_{71}], \quad e_{58} - e_{41} = [e_{48}, e_{27} + e_{36} + e_{54} - e_{81}],
\]
\[
P_{56} - e_{21} = [e_{26}, e_{38} + e_{47} + e_{52} - e_{61}], \quad e_{12} - e_{65} = [e_{15}, e_{38} + e_{47} + e_{52} - e_{61}],
\]
\[
P_{13} - e_{75} = [e_{15}, e_{28} + e_{46} + e_{53} - e_{71}], \quad e_{23} - e_{76} = [e_{26}, e_{18} + e_{45} + e_{63} - e_{72}],
\]
\[
P_{24} - e_{86} = [e_{26}, e_{17} + e_{35} + e_{64} - e_{82}], \quad e_{14} - e_{85} = [e_{15}, e_{27} + e_{36} + e_{54} - e_{81}],
\]
\[
P_{34} - e_{87} = [e_{37}, e_{16} + e_{25} + e_{74} - e_{83}], \quad e_{67} - e_{32} = [e_{37}, e_{18} + e_{45} + e_{63} - e_{72}],
\]
\[
P_{68} - e_{12} = [e_{48}, e_{17} + e_{35} + e_{64} - e_{82}], \quad e_{78} - e_{43} = [e_{48}, e_{16} + e_{25} + e_{74} - e_{83}].
\]

So, by Table 1.2 we have \(L_0 \subset (h + x)\). Since

\[
[e_{17} + e_{35} + e_{64} - e_{82}, e_{78} - e_{43}] = e_{18} + e_{45} - e_{63} + e_{72},
\]
\[
[e_{27} + e_{36} + e_{54} - e_{81}, e_{78} - e_{43}] = e_{28} + e_{46} - e_{53} + e_{71},
\]
\[
[e_{18} + e_{45} + e_{63} - e_{72}, e_{86} - e_{24}] = e_{16} + e_{25} - e_{83} + e_{74},
\]
\[
[e_{28} + e_{46} + e_{53} - e_{71}, e_{67} - e_{32}] = e_{38} + e_{47} - e_{52} + e_{61},
\]
\[
[e_{16} + e_{25} + e_{74} - e_{83}, e_{67} - e_{32}] = e_{17} + e_{35} - e_{64} + e_{82},
\]
\[
[e_{38} + e_{47} + e_{52} - e_{61}, e_{86} - e_{24}] = e_{36} + e_{27} - e_{54} + e_{81},
\]

by a direct computation, one sees that all standard odd basis elements (see Table 1.1) lie in \(L\). Therefore, we have \((h + x) = L\).

**Case 2.** Suppose \(L = \mathfrak{gl}(m, n)\) with \(m - n \equiv 0 \pmod{p}\) or \(\mathfrak{osp}(m, n) \oplus \mathbb{F}I\). Let us add a non-zero central element \(I\) of \(L\) to \(h + x\), where \(h \in \Omega_{\Delta_1} \cap [L_1, L_1]\) and \(x\) is the sum of all standard odd weight vectors (basis elements). Assert that \((h + I + x) = L\).

In fact, since \(\text{ad}(h + I)(x) = \text{ad}(x)\), by the basic fact F2 and Lemma 1.11(2), we have \(L_0 \subset (h + I + x)\). Furthermore, \([L_1, L_1] \subset (h + I + x)\) and then \(h \in (h + I + x)\). Thus \(I \in (h + I + x)\) and hence \((h + I + x) = L\).

Suppose \(L = \mathfrak{sl}(m, n)\) with \(m - n \equiv 0 \pmod{p}\). A direct verification shows that \([L_1, L_1] = 0\). Furthermore, by the basic facts F1 and F2 and Lemma 1.11(2), we obtain that \((h + x) = L\), where \(h \in \Omega_{\Delta_1}\) and \(x\) is the sum of all standard odd weight vectors (basis elements) of \(L\).

Suppose \(L = \mathfrak{gl}(m, n)\) with \(m - n \equiv 0 \pmod{p}\). Then \(L_1 = \mathfrak{sl}(m, n)_1\) and \([L_1, L_1] = \mathfrak{sl}(m, n)_0\). Choose an element \(e_{11} \in L\) and let \(x\) be the sum of all standard odd weight vectors (basis elements) and \(h \in \Omega_{\Delta_1}\). Then \((h + x, e_{11}) = L\).

Suppose \(L = \mathfrak{P}(m)\) with \(m \equiv 0 \pmod{p}\). Then, the basis elements of the standard Cartan subalgebra of \(\mathfrak{P}(m)\) with \(m \not\equiv 0 \pmod{p}\) in Table 1.1 and the element \(w := \sum_{i=1}^m e_{ii} - \sum_{j=m+1}^{2m} e_{jj}\) constitute a basis of the standard Cartan subalgebra of \(L\). Let \(x\) be the sum of all standard odd weight vectors (see Table 1.1) and \(h \in \Omega_{\Delta_1}\). Then, by the basic facts F1 and F2 and Lemma 1.11(2), we may deduce that \((h + x) = L\).

Suppose \(L = \tilde{\mathfrak{P}}(m)\) with \(m \equiv 0 \pmod{p}\). By the basic facts F1 and F2 and Lemma 1.11(2), we have \((h + x) = L\), where \(h \in \Omega_{\Delta_1}\) and \(x\) is the sum of all standard odd weight vectors (basis elements).

Suppose \(L = \tilde{\mathfrak{P}}(m) \oplus \mathbb{F}I\) with \(m \equiv 0 \pmod{p}\). We have \((h + x + I) = L\), where \(h \in \Omega_{\Delta_1}\) and \(x\) is the sum of all standard odd weight vectors (basis elements).

Suppose \(L = \tilde{\mathfrak{P}}(m)\) with \(m \not\equiv 0 \pmod{p}\). We have \((h + x, w) = L\), where \(h \in \Omega_{\Delta_1}\), \(w = \sum_{i=1}^m e_{ii} - \sum_{j=m+1}^{2m} e_{jj}\) and \(x\) is the sum of all standard odd weight vectors (basis elements).
Suppose \( L = \tilde{\mathcal{P}}(m) \oplus FI \) with \( m \neq 0 \) (mod \( p \)). Then we have \( \langle h + x + I, w \rangle = L \), where \( h \in \Omega_{\Delta} \cap [L_1, L_1] \), \( w = \sum_{i=1}^{m} e_i - \sum_{j=m+1}^{2m} e_j \) and \( x \) is the sum of all standard odd weight vectors (basis elements). In fact, \( h + I, x \) and \( w \) lie in \( \langle h + x + I, w \rangle \). In view of the basic fact \( F2 \) and Lemma \( \text{(14.2)} \), we obtain that \( L_1 \subset \langle h + x + I, w \rangle \). Then \( h \in [L_1, L_1] \subset \langle h + x + I, w \rangle \). This yields \( I \in \langle h + x + I, w \rangle \) and thereby \( \langle h + x + I, w \rangle = L \).

2. Lie superalgebras of Cartan type

Fix two \( m \)-tuples of positive integers \( \underline{t} \) := \((t_1, t_2, \ldots, t_m) \) and \( \pi \) := \((\pi_1, \pi_2, \ldots, \pi_m) \), where \( \pi_i := \binom{p}{\pi_i} - 1 \). Let \( \mathcal{O}(m; \underline{t}) \) be the divided power superalgebra over \( \mathbb{F} \) with basis \( \{ x^{(a)} | \alpha \in A \} \), where \( A := \{ \alpha \in \mathbb{N}^m | \alpha_i \leq \pi_i \} \). Let \( \Lambda(n) \) be the exterior superalgebra over \( \mathbb{F} \) of \( u \) variables \( x_{m+1}, x_{m+2}, \ldots, x_{m+n} \). Hereafter \( s, t \) is the set of integers \( s + 1, \ldots, t \). Let

\[
B = \{ (i_1, i_2, \ldots, i_k) | m + 1 \leq i_1 < i_2 < \cdots < i_k \leq m + n, k \in [1, n] \}.
\]

For \( u := (i_1, i_2, \ldots, i_k) \in B \), set \( |u| = k \) and write \( x^u = x_{i_1} x_{i_2} \cdots x_{i_k} \). The tensor product \( \mathcal{O}(m, n; \underline{t}) := \mathcal{O}(m; \underline{t}) \otimes \Lambda(n) \) is an associative superalgebra. The tensor \( \mathcal{O}(m, n; \underline{t}) \) has a standard \( \mathbb{F} \)-basis \( \{ x^{(a)} x^u | (a, u) \in A \times B \} \). For convenience, put

\[
Y_0 = \underline{1}, m, Y_1 = \underline{m + 1}, m + n \quad \text{and} \quad Y = Y_0 \cup Y_1.
\]

Let \( \partial_r \) be the superderivation of \( \mathcal{O}(m, n; \underline{t}) \) such that

\[
\partial_r (x^{(a)}) = x^{(a - e_r)} \quad \text{for} \quad r \in Y_0 \quad \text{and} \quad \partial_r (x_s) = \delta_{rs} \quad \text{for} \quad r, s \in Y.
\]

For \( g \in \mathcal{O}(m; \underline{t}), f \in \Lambda(n) \), write \( gf \) for \( g \otimes f \). For \( e_i := (\delta_{i_1}, \ldots, \delta_{i_m}) \), we abbreviate \( x^{(e_i)} \) by \( x_i \). For a \( \mathbb{Z}_2 \)-graded vector space \( V \), we denote by \( |x| = \theta \) the parity of a homogeneous element \( x \in V_\theta, \theta \in \mathbb{Z}_2 \). Now we list the eight series of modular graded Lie superalgebras of Cartan type \( \mathbb{F} [\mathbb{E}, \mathbb{I}, \mathbb{H}, \mathbb{J}, \mathbb{G}, \mathbb{F}, \mathbb{D}, \mathbb{S}] \), which are subalgebras of the Lie superalgebra \( \text{Der} \mathcal{O}(m, n; \underline{t}) \):

\section*{W} The generalized Witt superalgebra

\[
W(m, n; \underline{t}) := \text{span}_\mathbb{F} \{ f \partial_k | f \in \mathcal{O}(m, n; \underline{t}), r \in Y \}.
\]

\section*{S} The special superalgebra

\[
S(m, n; \underline{t}) := \text{span}_\mathbb{F} \{ D_{ij} (a) | a \in \mathcal{O}(m, n; \underline{t}), i, j \in Y \}.
\]

Let \( m = 2r \) or \( 2r + 1 \), where \( r \in \mathbb{N} \). Put

\[
i' = \begin{cases} 
i + r, & \text{if } i \in [1, r]; \\ 
i - r, & \text{if } i \in [r + 1, 2r]; \\ 
i, & \text{if } i \in Y_1. \end{cases}
\]

\[
\sigma(i) = \begin{cases} 1, & \text{if } i \in [1, r]; \\ 
-1, & \text{if } i \in [r + 1, 2r]; \\ 
1, & \text{if } i \in Y_1. \end{cases}
\]

\[
\sigma(i') = \begin{cases} \sigma(i), & \text{if } i \in [1, r]; \\ 
-\sigma(i), & \text{if } i \in [r + 1, 2r]; \\ 
\sigma(i), & \text{if } i \in Y_1. \end{cases}
\]
**H:** Suppose $m = 2r$. Let $D_H(a) = \sum_{i\in Y} Y_{\sigma(i)}(-1)^{|\partial_i||a|}\partial_i(a)\partial_i$. The Hamiltonian superalgebra $H(m, n; t)$ is the derived algebra of the Lie superalgebra $H(m, n; t) := \text{span}_F \{D_H(a) \mid a \in \mathcal{O}(m, n; t)\}$.

**K:** Suppose $m = 2r + 1$. Then $(\mathcal{O}(m, n; t), [\ , \ ]_K)$ is a Lie superalgebra with respect to multiplication

$$[a, b]_K = D_K(a) (b) - 2\partial_m(a)(b)$$

for $a, b \in \mathcal{O}(m, n; t)$, where

$$D_K(a) := \sum_{i\in Y\setminus\{m\}} (-1)^{|\partial_i||a|}(x_i\partial_m(a) + \sigma(i')\partial_i(a))\partial_i + \left(2a - \sum_{i\in Y\setminus\{m\}} x_i\partial_i(a)\right)\partial_m.$$  

The derived algebra of this Lie superalgebra is simple, called the contact superalgebra, denoted by $K(m, n; t)$.

In the below, suppose $m > 2$ and $n = m$ or $m + 1$. Put

$$i'' = \begin{cases} i + m, & \text{if } i \in Y_0; \\ i - m, & \text{if } i \in m + 1, 2m. \end{cases}$$

We have the following simple Lie superalgebras of Cartan type $[8, 11]$.

**HO:** Let $T_H(a) = \sum_{i\in Y} Y_{\sigma(i)}(-1)^{|\partial_i||a|}\partial_i(a)\partial_i$ for $a \in \mathcal{O}(m, m; t)$. Then $T_H$ is an odd linear mapping with kernel $F \cdot 1$ and Z-degree $-2$. The odd Hamiltonian superalgebra is

$$HO(m, m; \mathfrak{L}) := \text{span}_F \{T_H(a) \mid a \in \mathcal{O}(m, m; \mathfrak{L})\}.$$  

**SHO:** The special odd Hamiltonian Lie superalgebra $SHO(m, m; \mathfrak{L})$ is the second derived superalgebra of the Lie superalgebra $S(m, m; \mathfrak{L}) \cap HO(m, m; \mathfrak{L})$.

From $[3, 10]$ we have the following simple Lie superalgebras.

**KO:** The odd contact Lie superalgebra is

$$KO(m, m + 1; \mathfrak{L}) := (\mathcal{O}(m, m + 1; \mathfrak{L}), [\ , \ ]_{KO})$$  

with multiplication

$$[a, b]_{KO} = D_{KO}(a) (b) - (-1)^{|a|}2\partial_{2m+1}(a) b \quad \text{for } a, b \in \mathcal{O}(m, m + 1; \mathfrak{L}),$$

where

$$D_{KO}(a) := T_H(a) + (-1)^{|\partial_i|}\partial_{2m+1}(a)D + (D(a) - 2a)\partial_{2m+1}$$

and $D := \sum_{i=1}^{2m} x_i\partial_i$. Note that $D$ is just the degree superderivation of $\mathcal{O}(m, m; \mathfrak{L})$. Here, both $T_H$ and $D$ are naturally extended to $\mathcal{O}(m, m + 1; \mathfrak{L})$. 

In the below, suppose $m > 2$ and $n = m$ or $m + 1$. Put

$$i'' = \begin{cases} i + m, & \text{if } i \in Y_0; \\ i - m, & \text{if } i \in m + 1, 2m. \end{cases}$$

We have the following simple Lie superalgebras of Cartan type $[8, 11]$.

**HO:** Let $T_H(a) = \sum_{i\in Y} Y_{\sigma(i)}(-1)^{|\partial_i||a|}\partial_i(a)\partial_i$ for $a \in \mathcal{O}(m, m; t)$. Then $T_H$ is an odd linear mapping with kernel $F \cdot 1$ and Z-degree $-2$. The odd Hamiltonian superalgebra is

$$HO(m, m; \mathfrak{L}) := \text{span}_F \{T_H(a) \mid a \in \mathcal{O}(m, m; \mathfrak{L})\}.$$  

**SHO:** The special odd Hamiltonian Lie superalgebra $SHO(m, m; \mathfrak{L})$ is the second derived superalgebra of the Lie superalgebra $S(m, m; \mathfrak{L}) \cap HO(m, m; \mathfrak{L})$.

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**KO:** The odd contact Lie superalgebra is

$$KO(m, m + 1; \mathfrak{L}) := (\mathcal{O}(m, m + 1; \mathfrak{L}), [\ , \ ]_{KO})$$  

with multiplication

$$[a, b]_{KO} = D_{KO}(a) (b) - (-1)^{|a|}2\partial_{2m+1}(a) b \quad \text{for } a, b \in \mathcal{O}(m, m + 1; \mathfrak{L}),$$

where

$$D_{KO}(a) := T_H(a) + (-1)^{|\partial_i|}\partial_{2m+1}(a)D + (D(a) - 2a)\partial_{2m+1}$$

and $D := \sum_{i=1}^{2m} x_i\partial_i$. Note that $D$ is just the degree superderivation of $\mathcal{O}(m, m; \mathfrak{L})$. Here, both $T_H$ and $D$ are naturally extended to $\mathcal{O}(m, m + 1; \mathfrak{L})$. 

In the below, suppose $m > 2$ and $n = m$ or $m + 1$. Put

$$i'' = \begin{cases} i + m, & \text{if } i \in Y_0; \\ i - m, & \text{if } i \in m + 1, 2m. \end{cases}$$

We have the following simple Lie superalgebras of Cartan type $[8, 11]$.
Generators of simple modular Lie superalgebras

**SKO**: Given \( \lambda \in \mathbb{F} \), for \( a \in \mathcal{O}(m, m + 1; \mathbb{L}) \), let

\[
\text{div}_\lambda(a) = (-1)^{|a|} 2 \left( \sum_{i=1}^{m} \partial_i \partial_{\nu^i} (a) + (\mathfrak{D} - m \lambda \text{id}_{\mathcal{O}(m, m + 1; \mathbb{L})}) \partial_{2m+1} (a) \right).
\]

The kernel of \( \text{div}_\lambda \) is a subalgebra of \( KO(m, m + 1; \mathbb{L}) \). Its second derived algebra is simple, called the *special odd contact Lie superalgebra*, denoted by \( \text{SKO}(m, m + 1; \mathbb{L}) \).

If \( V = \bigoplus_{r \in \mathbb{Z}} V_r \) is a \( \mathbb{Z} \)-graded vector space and \( x \in V \) is a \( \mathbb{Z} \)-homogeneous element, write \( zd(x) \) for the \( \mathbb{Z} \)-degree of \( x \). From now on \( X \) denotes one of the simple graded Lie superalgebras \( W, S, H, K, HO, SHO, KO \) or \( SKO \). The \( \mathbb{Z} \)-grading of \( X \) given by \( zd(x_i) = -zd(\partial_i) = a_i \), where \( a_i = 1 + \delta_{X=K} \delta_i = m + \delta_{X=K \mathcal{O}} \delta_i = 2m+1 + \delta_{X=SHO} \delta_i = 2m+1 \), induces a \( \mathbb{Z} \)-grading \( X = X_{-1} \oplus \cdots \oplus X_1 \). Note that \( l = 1 \) for \( X = W, S, H, HO \) or \( SHO \) and \( l = 2 \) for \( X = K, KO \) or \( SKO \). Put \( \theta = \sum_{i=1}^{m} p^i - m + n \). Then

\[
s = \begin{cases} 
\theta - 1, & X = W, KO; \\
\theta - 2, & X = S, HO or SKO with m\lambda + 1 \not\equiv 0 \pmod{p}; \\
\theta - 3, & X = H, SKO with m\lambda + 1 \equiv 0 \pmod{p}; \\
\theta - 5, & X = SHO; \\
\theta + \pi_m - 2, & X = K with n - m - 3 \not\equiv 0 \pmod{p}; \\
\theta + \pi_m - 3, & X = K with n - m - 3 \equiv 0 \pmod{p}. 
\end{cases}
\]

For \( X = W, S, HO, SHO, KO \) or \( SKO \), the null \( X_0 \) is isomorphic to a classical modular Lie superalgebra or its relatives under the canonical isomorphism

\[
\phi : W(m, n; \mathbb{L})_0 \rightarrow \mathfrak{gl}(m, n)
\]
given by \( x_i D_j \mapsto e_{ij} \), where \( i, j \in Y \). For \( X = H \), as in [14, p.164], put

\[
\mathfrak{L}(m, n) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{gl}(m, n) \mid A^t G + GA = 0, B^t G + C = 0, D^t + D = 0 \right\}.
\]

There is an isomorphism of algebras:

\[
\varphi : H(m, n; \mathbb{L})_0 \rightarrow \mathfrak{L}(m, n)
\]
given by

\[
D_H(x_i x_j) \mapsto \sigma(j)(-1)^{|\partial_j|} e_{ij'} + \sigma(i) \sigma(j)(-1)^{|\partial_i| + |\partial_j| + |\partial_j'|} e_{ij'} \text{ for } i, j \in Y.
\]

For \( K \), there is an isomorphism of algebras:

\[
\psi : K(m, n; \mathbb{L})_0 \rightarrow \mathfrak{L}(m - 1, n) \oplus \mathbb{F} I
\]
given by \( D_K(x_{im}) \mapsto I \) and for \( i, j \in \{1, m - 1 \cup Y \} \),

\[
D_K(x_i x_j) \mapsto \sigma(j)(-1)^{|\partial_j|} e_{ij'} + \sigma(i) \sigma(j)(-1)^{|\partial_i| + |\partial_j| + |\partial_j'|} e_{ij'}.
\]

Suppose \( \mu \in \mathbb{F} \) and \( \mu^2 = -1 \). Put

\[
P_n = \begin{cases} I_q & \text{if } n = 2q \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_q \end{bmatrix} & \text{if } n = 2q + 1. 
\end{cases}
\]
Generators of simple modular Lie superalgebras

Then
\[ \mathfrak{osp}(m, n) = \{ P^{-1}EP \mid E \in \mathfrak{L}(m, n) \}, \ P := \begin{bmatrix} I_m & 0 \\ 0 & P_n \end{bmatrix}. \]

It follows that \( \mathfrak{L}(m, n) \simeq \mathfrak{osp}(m, n) \). We write down the following simple facts:

\[ W(m, n; \mathfrak{L}_0) \simeq \mathfrak{gl}(m, n), \ S(m, n; \mathfrak{L}_0) \simeq \mathfrak{sl}(m, n), \]
\[ HO(m, m; \mathfrak{L}_0) \simeq \tilde{P}(m), \ K_{0}(m, m; \mathfrak{L}_0) \simeq \tilde{P}(m) \oplus FI, \]
\[ SHO(m, m; \mathfrak{L}_0) \simeq P(m), \ SKO(m, m + 1; \mathfrak{L}_0) \simeq \tilde{P}(m), \]
\[ H(m, n; \mathfrak{L}_0) \simeq \mathfrak{osp}(m, n), \ K(m, n; \mathfrak{L}_0) \simeq \mathfrak{osp}(m - 1, n) \oplus \mathfrak{FI}. \]

For convenience, we list the so-called standard basis of the Cartan subalgebra \( \mathfrak{h}_{X_0} \) of \( X_0 \):

| \( X \) | basis of \( \mathfrak{h}_{X_0} \) |
|--------|------------------|
| \( W(m, n; \mathfrak{L}) \) | \( x_i \partial_i, \ i \in \mathbf{Y} \) |
| \( S(m, n; \mathfrak{L}) \) | \( -D_{11}(x_i x_j), \ D_{11}(x_i x_j), \ i \in 2, m, j \in \mathbf{Y} \) |
| \( H(2r, n; \mathfrak{L}) \) | \( D_{10}(x_i x_{n + j + \frac{r}{2}}), \ i \in 1, r, j \in 2r + 1, 2r + \frac{r}{2} \) |
| \( K(2r + 1, n; \mathfrak{L}) \) | \( x_i x_{r + j + \frac{1}{2}}, x_m, \ i \in 1, r, j \in 2r + 2, 2r + 1 + \frac{r}{2} \) |
| \( HO(m, m; \mathfrak{L}) \) | \( x_{0r} \), \( r \in \mathbf{Y}_0 \) |
| \( SHO(m, m; \mathfrak{L}) \) | \( x_{0r} \), \( r \in 2, m \) |
| \( KO(m, m + 1; \mathfrak{L}) \) | \( x_{2m + 1} + m \lambda x_i x_{r + j}, \ i \in \mathbf{Y}_0 \) |

The weight decomposition of the component \( X_k \) relative to the standard Cartan subalgebra \( \mathfrak{h}_{X_0} \) is:

\[ X_k = \delta_{k, 0} \mathfrak{h}_{X_0} \oplus \alpha \in \Delta_m X_k^\alpha, \text{ where } k \in -1, s. \]

Write \( (\Delta_i)_1 \) for the subset of odd weights in \( \Delta_i \), \( i = -1, 0, s - 1 \) or \( s \).

**Lemma 2.1.** Let \( X \) be one of eight series of Lie superalgebras of Cartan type. If

\[ X = HO(m, m; \mathfrak{L}) \text{ or } KO(m, m + 1; \mathfrak{L}), \]

where \( m \) is odd, then there exist odd weights \( \alpha_{-1} \in (\Delta_{-1})_1 \) and \( \alpha_{s-1} \in (\Delta_{s-1})_1 \) such that \( \alpha_{-1}, \alpha_{s-1} \) and any \( \alpha \in (\Delta_0)_1 \) are pairwise distinct.

Otherwise, there exist odd weights \( \alpha_{-1} \in (\Delta_{-1})_1 \) and \( \alpha_{s} \in (\Delta_{s})_1 \) such that \( \alpha_{-1}, \alpha_{s} \) and any \( \alpha \in (\Delta_0)_1 \) are pairwise distinct.

**Proof.** For \( W(m, n; \mathfrak{L}) \), let \( \zeta_i \) be the linear function on \( \mathfrak{h}_{W_0} \) given by

\[ \zeta_i(x_j \partial_j) = \delta_{ij}, \ i, j \in \mathbf{Y}. \]

We have

\[ (\Delta_{-1})_1 = \{-\zeta_k \mid k \in \mathbf{Y}_1\}; \]
\[ (\Delta_0)_1 = \{ \pm(\zeta_k - \zeta_l) \mid k \in \mathbf{Y}_0, l \in \mathbf{Y}_1\}; \]
\[ (\Delta_s)_1 = \left\{ \begin{array}{ll}
-\sum_{l \in \mathbf{Y}_0} \zeta_l + \sum_{k \in \mathbf{Y}_1} \zeta_k - \zeta_m & \text{if } m \in \mathbf{Y}_0 \\
-\sum_{l \in \mathbf{Y}_0} \zeta_l + \sum_{k \in \mathbf{Y}_1} \zeta_k - \zeta_n & \text{if } n \in \mathbf{Y}_1
\end{array} \right\} \]

if \( n \) is odd,

if \( n \) is even.
Generators of simple modular Lie superalgebras

For $S(m, n; \mathfrak{U})$, let $\eta_i$ be the linear function on $\mathfrak{h}_{S_n}$ given by

$$
\eta_i(x_1 \partial_1 - x_j \partial_j) = \delta_{ij}, \; i, j \in \mathbb{Z}, m + n
$$

and write $\eta_i = \sum_{l=2}^{m+n} \eta_i$. We have

$$(\Delta_{-1})_1 = \{ \eta_k \mid k \in \mathbf{Y}_1 \};$$

$$(\Delta_0)_1 = \{ \pm (\eta_k - \eta_l) \mid k \in \mathbf{Y}_0, l \in \mathbf{Y}_1 \};$$

$$(\Delta_1)_1 = \left\{ - \sum_{j \in \mathbf{Y}_0} \eta_j + \sum_{l \in \mathbf{Y}_1} \eta_l - \eta_i \mid i + k + n \text{ is an odd number} \right\}.$$

For $H(2r, 2q; \mathfrak{L})$, let $\vartheta_i$ be the linear function on $\mathfrak{h}_{H_{2q}}$ given by

$$
\vartheta_i(D_H(x_j x_{j'} + x_k x_{k+q})) = \delta_{ij} + \delta_{ik}, i, j \in \mathbb{Z}; i, k \in 2r + 1, 2r + q.
$$

We have

$$(\Delta_{-1})_1 = \{ \pm \vartheta_i \mid l \in 2r + 1, 2r + q \};$$

$$(\Delta_0)_1 = \{ \pm (\vartheta_i - \vartheta_j) \mid i \in \mathbb{Z}, j \in 2r + 1, 2r + q \};$$

$$(\Delta_1)_1 = \{ \pm \vartheta_j \mid j \in 1, r \}.$$

For $H(2r, 2q + 1; \mathfrak{L})$, let $\iota_i$ be the linear function on $\mathfrak{h}_{H_{2q}}$ given by

$$
\iota_i(D_H(x_j x_{j'} + x_k x_{k+q})) = \delta_{ij} + \delta_{ik}, i, j \in \mathbb{Z}; i, k \in 2r + 1, 2r + q.
$$

We have

$$(\Delta_{-1})_1 = \{ \pm \iota_i \mid l \in 2r + 1, 2r + q \} \cup \{ 0 \};$$

$$(\Delta_0)_1 = \{ \pm (\iota_i - \iota_j) \pm \iota_k \mid i \in 1, r, j \in 2r + 1, 2r + q \};$$

$$(\Delta_1)_1 = \{ \pm \iota_j \mid j \in 1, r \} \cup \{ 0 \}.$$

For $K(2r + 1, 2q; \mathfrak{L})$, let $\kappa_i$ be the linear function on $\mathfrak{h}_{K_{2q}}$ given by

$$
\kappa_i(x_j x_{j'} + x_k x_{k+q}) = \delta_{ij} + \delta_{ik}, i, j \in \mathbb{Z}; i, k \in 2r + 1, 2r + q
$$

and $\kappa_m(x_m) = 1$. We have

$$(\Delta_{-1})_1 = \{ \pm \kappa_i \mid l \in 2r + 2, 2r + q + 1 \};$$

$$(\Delta_0)_1 = \{ \pm (\kappa_i - \kappa_j) \mid i \in 1, r, j \in 2r + 2, 2r + q + 1 \};$$

$$(\Delta_1)_1 = \left\{ \begin{array}{ll}
{-4 \kappa_m} & \text{if } n - m - 3 \equiv 0 \pmod{p}, \\
{-2 \kappa_j - 5 \kappa_m \mid j \in 1, r \cup 2r + 2, 2r + q + 1} & \text{otherwise.}
\end{array} \right\}$$

For $K(2r + 1, 2q + 1; \mathfrak{L})$, let $\lambda_i$ be the linear function on $\mathfrak{h}_{K_{2q}}$ given by

$$
\lambda_i(D_H(x_j x_{j'} + x_k x_{k+q})) = \delta_{ij} + \delta_{ik}, i, j \in \mathbb{Z}; i, k \in 2r + 1, 2r + q
$$

and $\lambda_m(x_m) = 1$. We have

$$(\Delta_{-1})_1 = \{ \pm \lambda_i \mid l \in 2r + 2, 2r + q + 1 \} \cup \{ 0 \};$$

$$(\Delta_0)_1 = \{ \pm (\lambda_i - \lambda_j) \lambda_k \mid i \in 1, r, j \in 2r + 2, 2r + q + 1 \};$$

$$(\Delta_1)_1 = \left\{ \begin{array}{ll}
{-4 \lambda_m} & \text{if } n - m - 3 \equiv 0 \pmod{p}, \\
{-2 \lambda_j - 5 \lambda_m \mid j \in 1, r \cup 2r + 2, 2r + q + 1} & \text{otherwise.}
\end{array} \right\}$$
For $HO(m, m; \mathbb{L})$, let $\mu_i$ be the linear function on $\mathfrak{h}_{HO_o}$ given by
$$\mu_i(T_H(x_j x'_j)) = \delta_{ij}, \ i, j \in \mathcal{Y}_0.$$We have
$$\begin{align*}
(\Delta_{-1})_1 &= \{ \mu_l \mid l \in \mathcal{Y}_0 \}; \\
(\Delta_0)_1 &= \{ 2\mu_i, \pm (\mu_i + \mu_j) \mid i, j, l \in \mathcal{Y}_0, i \neq j \}; \\
(\Delta_{-1})_1 &= \{ -\mu_l \mid l \in \mathcal{Y}_0 \} \cup \{ 0 \} \text{ if } m \text{ is odd,} \\
(\Delta_s)_1 &= \left\{ -2 \sum_{i=1}^{m} \mu_i \right\} \text{ if } m \text{ is even.}
\end{align*}$$

For $SHO(m, m; \mathbb{L})$, let $\nu_i$ be the linear function on $\mathfrak{h}_{SHO_o}$ given by
$$\nu_i(T_H(x_1 x'_1 - x_j x'_j)) = \delta_{ij}, \ i, j \in \mathcal{Y}_0$$and write $\nu_1 = \sum_{i=2}^{m} \nu_i$. We have
$$\begin{align*}
(\Delta_{-1})_1 &= \{ \nu_i \mid i \in \mathcal{Y}_0 \}; \\
(\Delta_0)_1 &= \{ 2\nu_i, \pm (\nu_i + \nu_j) \mid i, j, l \in \mathcal{Y}_0, i \neq j \}; \\
(\Delta_s)_1 &= \{ \nu_i \mid i \in \mathcal{Y}_0 \}.
\end{align*}$$

For $KO(m, m+1; \mathbb{L})$, let $\xi_i$ be the linear function on $\mathfrak{h}_{KO_o}$ given by
$$\xi_i(x_1 x'_1 - x_j x'_j) = \delta_{ij}, \ i, j \in \mathcal{Y}_0$$and $\xi_{2m+1}(x_{2m+1}) = 1$. We have
$$\begin{align*}
(\Delta_{-1})_1 &= \{ \xi_j + \xi_{2m+1} \mid j \in \mathcal{Y}_0 \}; \\
(\Delta_0)_1 &= \{ 2\xi_i, \pm (\xi_i + \xi_j) \mid i, j, l \in \mathcal{Y}_0, i \neq j \}; \\
(\Delta_{-1})_1 &= \{ -\xi_j - 2\xi_{2m+1} \mid j \in \mathcal{Y}_0 \} \text{ if } m \text{ is odd,} \\
(\Delta_s)_1 &= \left\{ -2 \sum_{i=2}^{m} \xi_i - \xi_{2m+1} - 2\xi_{2m+1} \right\} \text{ if } m \text{ is even.}
\end{align*}$$

For $SKO(m, m+1; \mathbb{L})$, let $\omega_i$ be the linear function on $\mathfrak{h}_{SKO_o}$ given by
$$\omega_i(x_1 x'_1 - x_j x'_j) = \delta_{ij}, \ i, j \in \mathcal{Y}_0$$and $\omega_{2m+1}(x_{2m+1}) = 1$. We have
$$\begin{align*}
(\Delta_{-1})_1 &= \left\{ \sum_{i=2}^{m} \omega_i + \omega_{2m+1} + \omega_k \mid l \in \mathcal{Y}_0 \right\}; \\
(\Delta_0)_1 &= \{ 2\omega_i, \pm (\omega_i + \omega_j) \mid i, j, l \in \mathcal{Y}_0, i \neq j \}; \\
(\Delta_s)_1 &= \left\{ \begin{array}{ll}
2 \left( \sum_{i=2}^{m} \omega_i + \omega_{2m+1} \right) & \text{if } m\lambda + 1 \neq 0 \pmod{p}, \\
2 \left( \sum_{i=2}^{m} \omega_i + \omega_{2m+1} \right) - \omega_k & \text{if } k \in \mathcal{Y}_0
\end{array} \right\} \text{ otherwise.}
\end{align*}$$

Summarizing, if $X = HO(m, m; \mathbb{L})$ with $m$ odd or $KO(m, m+1; \mathbb{L})$ with $m$ odd, we may easily find the desired weights $\alpha_{-1} \in (\Delta_{-1})_1$ and $\alpha_{s-1} \in (\Delta_s)_1$ such that $\alpha_{-1}, \alpha_{s-1}$ and any $\alpha \in (\Delta_0)_1$ are pairwise distinct. Otherwise, we may easily find the desired weights $\alpha_{-1} \in (\Delta_{-1})_1$ and $\alpha_s \in (\Delta_s)_1$ such that $\alpha_{-1}, \alpha_s$ and any $\alpha \in (\Delta_0)_1$ are pairwise distinct. \[ \square \]
Lemma 2.2. Let $\mathfrak{g} = \oplus_{i=-r}^{s} \mathfrak{g}_i$ be a finite-dimensional simple Lie superalgebra. Then the following statements hold.

1. $\mathfrak{g}_0$-modules $\mathfrak{g}_{-1}$ and $\mathfrak{g}_s$ are irreducible.
2. $\mathfrak{g}$ can be generated by $\mathfrak{g}_{-1}$ and $\mathfrak{g}_s$.

Theorem 2.3. Let $X$ be one of eight series of Lie superalgebras of Cartan type. If

$$X = \begin{cases} 
W(m, n; \mathbb{L}) & \text{with } m - n \equiv 0 \pmod{p}, \\
HO(m, m; \mathbb{L}) & \text{with } m \not\equiv 0 \pmod{p}, \\
KO(m, m + 1; \mathbb{L}) & \text{with } m \not\equiv 0 \pmod{p}, \\
SKO(m, m + 1; \mathbb{L}) & \text{with } m \not\equiv 0 \pmod{p},
\end{cases}$$

then $X$ can be generated by 2 elements. Otherwise, $X$ can be generated by 1 element.

Proof. Recall that the null module $X_0$ is isomorphic to one of the Lie superalgebras

$$\mathfrak{gl}(m, n), \mathfrak{so}(m, n), \mathfrak{osp}(m, n), \mathfrak{osp}(m - 1, n) \oplus \mathbb{F}I, \tilde{\mathfrak{P}}(m), \tilde{\mathfrak{P}}(m) \oplus \mathbb{F}I \text{ or } \mathfrak{P}(m).$$

Let us prove the first conclusion. From the proof of Proposition 1.2, $X_0$ can be generated by 2 elements. Write $h_X$ for one of two generators which is associated with $h$ (see Proposition 1.2). Let $\Phi_X$ be one of eight series of Lie superalgebras of Cartan type. If

$$X = \begin{cases} 
W(m, n; \mathbb{L}) & \text{with } m - n \equiv 0 \pmod{p}, \\
HO(m, m; \mathbb{L}) & \text{with } m \not\equiv 0 \pmod{p}, \\
KO(m, m + 1; \mathbb{L}) & \text{with } m \not\equiv 0 \pmod{p}, \\
SKO(m, m + 1; \mathbb{L}) & \text{with } m \not\equiv 0 \pmod{p},
\end{cases}$$

then $X$ can be generated by 2 elements. Otherwise, $X$ can be generated by 1 element.

Proof. Recall that the null module $X_0$ is isomorphic to one of the Lie superalgebras

$$\mathfrak{gl}(m, n), \mathfrak{so}(m, n), \mathfrak{osp}(m, n), \mathfrak{osp}(m - 1, n) \oplus \mathbb{F}I, \tilde{\mathfrak{P}}(m), \tilde{\mathfrak{P}}(m) \oplus \mathbb{F}I \text{ or } \mathfrak{P}(m).$$

Let us prove the first conclusion. From the proof of Proposition 1.2, $X_0$ can be generated by 2 elements. Write $h_X$ for one of two generators which is associated with $h$ (see Proposition 1.2). Let $\Phi_X$ be one of eight series of Lie superalgebras of Cartan type. If

$$X = \begin{cases} 
W(m, n; \mathbb{L}) & \text{with } m - n \equiv 0 \pmod{p}, \\
HO(m, m; \mathbb{L}) & \text{with } m \not\equiv 0 \pmod{p}, \\
KO(m, m + 1; \mathbb{L}) & \text{with } m \not\equiv 0 \pmod{p}, \\
SKO(m, m + 1; \mathbb{L}) & \text{with } m \not\equiv 0 \pmod{p},
\end{cases}$$

then $X$ can be generated by 2 elements. Otherwise, $X$ can be generated by 1 element.
For $X \neq HO(m,m;\mathbb{Q})$ and $KO(m,m+1;\mathbb{Q})$ with $m$ odd, by Lemma 2.1, we choose weights $\alpha_{-1} \in (\Delta_{-1})_1$ and $\alpha_s \in (\Delta_s)_1$ such that $\alpha_{-1}, \alpha_s$ and any $\alpha \in (\Delta_0)_1$ are pairwise distinct. Put $x = x_{-1} + x_0 + x_s$ for some odd weight vectors $x_{-1} \in X_{\alpha_{-1}}$ and $x_s \in X_s^{\alpha_s}$. Let $\Phi = \{\alpha_{-1}\} \cup \{\alpha_s\} \cup (\Delta_0)_1$. Choose any $h \in \Omega_\Phi \subset \Omega_{(\Delta_0)_1}$. Assert that $\langle h + x \rangle = X$. In fact, we have $x_{-1} + x_0 + x_s \in \langle h + x \rangle$ and $h \in \langle h + x \rangle$. Furthermore, according to the basic fact F2, we have $x_{-1}, x_0, x_s \in \langle h + x \rangle$. Then $\langle h + x \rangle = X_0 \subset \langle h + x \rangle$. By Lemma 2.1, the irreducibility of $X_{-1}$ and $X_s$ as $X_0$-modules ensures that $X_{-1} \subset X$ and $X_s \subset X$. According to Lemma 2.2(2), we have $X = \langle h + x \rangle$.

For $X = HO(m,m;\mathbb{Q})$ with $m$ odd or $KO(m,m+1;\mathbb{Q})$ with $m$ odd, by Lemma 2.1, we choose weights $\alpha_{-1} \in (\Delta_{-1})_1$ and $\alpha_{s-1} \in (\Delta_{s-1})_1$ such that $\alpha_{-1}, \alpha_{s-1}$ and any $\alpha \in (\Delta_0)_1$ are pairwise distinct. Let $\Phi = \{\alpha_{-1}\} \cup \{\alpha_{s-1}\} \cup (\Delta_0)_1$. Put $y = x_{-1} + x_0 + x_{s-1}$ and choose $h \in \Omega_\Phi \subset \Omega_{(\Delta_0)_1}$. Then $x_{-1}, x_0, x_{s-1} \in \langle h + y \rangle$. From the proof of Proposition 1.2, we have $X_0 = \langle h + x \rangle \subset \langle h + y \rangle$. The irreducibility of $X_{-1}$ as $X_0$-module ensures $X_{-1} \subset X$. Since $x_{s-1} \in \langle h + y \rangle$ and transitivity of $X$, we conclude that there exists an element $x_1$ of $X_1$ in $\langle h + y \rangle$ such that $X_s = \mathbb{F}[x_1, x_{s-1}] \subset \langle h + y \rangle$. Then by Lemma 2.2 we have $X = \langle h + y \rangle$. 

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