GROUP ACTIONS ON CONTRACTIBLE 2-COMPLEXES I

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ABSTRACT. In this series of two articles, we prove that every action of a finite group $G$ on a finite and contractible 2-complex has a fixed point. The proof goes by constructing a nontrivial representation of the fundamental group of each of the acyclic 2-dimensional $G$-complexes constructed by Oliver and Segev. In the first part we develop the necessary theory and cover the cases where $G = \text{PSL}_2(2^n)$, $G = \text{PSL}_2(q)$ with $q \equiv 3 \pmod{8}$ or $G = \text{Sz}(2^n)$. The cases $G = \text{PSL}_2(q)$ with $q \equiv 5 \pmod{8}$ are addressed in the second part.

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1. INTRODUCTION

A well known result of Jean-Pierre Serre states that an action of a finite group on a tree has a fixed point [Ser80]. A natural attempt to generalize Serre’s result would be to replace “tree” by “contractible $n$-complex”. An example by Edwin E. Floyd and Roger W. Richardson [FR59] implies this generalization does not hold for $n \geq 3$. However, Carles Casacuberta and Warren Dicks conjectured that it holds for $n = 2$ [CD92]. In the compact case and in the form of a question, this was also posed by Michael Aschbacher and Yoav Segev [AS93, Question 3]. In this series of two articles, we give a positive answer to the question of Aschbacher–Segev, settling the compact case of the Casacuberta–Dicks conjecture.

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Theorem A. Every action of a finite group $G$ on a 2-dimensional finite and contractible complex has a fixed point. Moreover, if $G$ is a finite group and $X$ is a 2-dimensional, fixed point free, finite and acyclic $G$-complex, then the fundamental group of $X$ admits a nontrivial unitary representation.

In [CD92] the conjecture is proved for solvable groups. The question of which groups act without fixed points on a finite 2-complex was studied independently by Segev [Seg93], who proved this is not possible for the solvable groups and the alternating groups $A_n$ for $n \geq 6$. Using the classification of the finite simple groups, Aschbacher and Segev proved that for many groups any action on a finite 2-dimensional acyclic complex has a fixed point [AS93]. Then, Bob Oliver and Yoav Segev [OS02] gave the complete classification of the groups that act without fixed points on an acyclic 2-complex (see also the exposition by Alejandro Adem at the Séminaire Bourbaki [Ade03]).

Theorem 1.1 (Oliver–Segev). For any finite group $G$, there is an essential fixed point free 2-dimensional (finite) acyclic $G$-complex if and only if $G$ is isomorphic to one of the simple groups $\text{PSL}_2(2^k)$ for $k \geq 2$, $\text{PSL}_2(q)$ for $q \equiv \pm 3 \pmod{8}$ and $q \geq 5$, or $\text{Sz}(2^k)$ for odd $k \geq 3$. Furthermore, the isotropy subgroups of any such $G$-complex are all solvable.

In [SC20], the author proposed the group $G = A_5 \simeq \text{PSL}_2(2^2) \simeq \text{PSL}_2(5)$ case of Theorem A and proposed a path to prove Theorem A, which consists on representing (in a nontrivial way) the fundamental group of each of the acyclic 2-complexes constructed by Oliver and Segev. Concretely, by the results in [SC20], Theorem A follows from Theorems B and C below.

Theorem B. Let $G$ be one of the groups $\text{PSL}_2(2^n)$ for $n \geq 2$, $\text{PSL}_2(3^n)$ for $n \geq 3$ odd, $\text{PSL}_2(q)$ with $q \equiv 11 \pmod{24}$ or $q \equiv 19 \pmod{24}$, or $\text{Sz}(q)$ for $q = 2^n$ with $n \geq 3$ odd. Then the fundamental group of every 2-dimensional, fixed point free, finite and acyclic $G$-complex admits a nontrivial representation in a unitary group $U(m)$.

Theorem C ([PSC21]). Let $G$ be one of the groups $\text{PSL}_2(q)$ with $q > 5$ and $q \equiv 5 \pmod{24}$ or $q \equiv 13 \pmod{24}$. Then the fundamental group of every 2-dimensional, fixed point free, finite and acyclic $G$-complex admits a nontrivial representation in a unitary group $U(m)$.

The proof of Theorem C appears in the second part of this work [PSC21], which is joint with Kevin Piterman.

To prove Theorems B and C, we use the method of [SC20] but with a more generic approach. If $X_1$ is a $G$-graph we consider the group extension $\Gamma = \pi_1(X_1, x_0) : G$. If $X$ is obtained from $X_1$ by attaching orbits of 2-cells, a result of Kenneth S. Brown [Bro84] gives an extension $\Gamma/\langle\langle w_0, \ldots, w_k \rangle\rangle \simeq \pi_1(X) : G$, where the $w_i \in \ker(\phi: \Gamma \to G) \simeq \pi_1(X_1)$ are words corresponding to the orbits of 2-cells of $X$. Then obtaining a nontrivial representation of $\pi_1(X)$ reduces to obtaining a representation of $\Gamma$ which factors through the quotient $\Gamma \to \Gamma/\langle\langle w_0, \ldots, w_k \rangle\rangle$ and does not factor through $\phi$.

We develop general machinery to obtain a moduli of representations $\mathcal{M}$ of $\Gamma$ from a single representation $\rho_0: G \to G$, where $G$ is a Lie group. Each word $w \in \Gamma$ induces a map $W: \mathcal{M} \to G$ and then the proof reduces to finding a suitable point $\tau \in \mathcal{M}$. With some hypotheses on $\rho_0: G \to G$, there is a single point $\mathcal{I} \in \mathcal{M}$ which gives a representation that factors through $\phi$. Then, by considering $\mathcal{W} = (W_0, \ldots, W_k): \mathcal{M} \to G^{k+1}$, the proof reduces to finding a point $\tau \neq \mathcal{I} \in \mathcal{M}$ such that $\mathcal{W}(\tau) = 1$. When we apply these results to the groups in Theorem 1.1 it turns out that $\mathcal{M}$ and $G^{k+1}$ are orientable manifolds of the same dimension. To complete the proof we show that $\mathcal{I}$ is a regular point of $\mathcal{W}$ and that $\mathcal{W}$ has degree 0.
The groups in Theorem B share a key property: they admit a nontrivial representation which restricts to an irreducible representation of the Borel subgroup. However, the groups in Theorem C lack this property. In [PSC21] some modifications to the approach of the first part are introduced in order to extend the proof to these groups.

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## 2. Preliminaries on Lie groups

Recall that a *Lie group* $G$ is a smooth manifold with a group structure such that the multiplication $\mu: G \times G \to G$, $(x,y) \mapsto xy$ and inversion $i: G \to G$, $x \mapsto x^{-1}$ are differentiable. The group $U(m)$ of $m \times m$ unitary matrices is a compact and connected $m^2$-dimensional Lie group. If $G$ is a Lie group, the *Lie algebra* of $G$ is the tangent space $T_eG$ at the identity element $e \in G$. The *adjoint representation* $\text{Ad}: G \to \text{GL}(T_eG)$ is defined by $g \mapsto d_e\Psi_g$ where $\Psi_g: G \to G$ is the map given by $h \mapsto ghg^{-1}$. Every Lie group is parallelizable and hence orientable.

**Lemma 2.1.** Let $G$ be a Lie group with multiplication $\mu: G \times G \to G$. Then the differential $d_{(p,q)}\mu: T_pG \times T_qG \to T_{pq}G$ is given by $(x,y) \mapsto d_pR_q(x) + d_qL_p(y)$.

**Proof.** The differential $d_{(e,e)}\mu: T_eG \times T_eG \to T_eG$ is given by $(x,y) \mapsto x + y$ (this is [Lee13, Chapter 7, Problem 7-2]). The general case follows by writing $\mu = L_pR_q \circ \mu \circ (L_{p^{-1}} \times R_{q^{-1}})$.

**Proposition 2.2.** Let $M$ be a manifold, $G$ be a Lie group and $f,g: M \to G$ be differentiable maps.

1. We have the product rule $d_p(f \cdot g) = df(p)R_{g(p)} \circ d_p f + dg(p)L_{f(p)} \circ d_pg$.
2. If $f(p) = g(p) = e$, we have $d_p(f \cdot g) = d_p f + d_p g$.
3. If $g(p) = e$, we have $d_p(f \cdot f^{-1}) = df(p)^{-1}L_{f(p)} \circ d_eR_{f(p)}^{-1} \circ d_pg$.
4. If $f(p) = e$, we have $d_p f^{-1} = -d_p f$.
5. If $f(p) = g(p) = e$, we have $d_p[f,g] = 0$.

**Proof.** These properties follow easily from Lemma 2.1. □

**Corollary 2.3.** The adjoint representation is given by $\text{Ad}(g) = dL_g \circ dR_{g^{-1}}$.

We denote the centralizer of $H$ in $G$ by $C_G(H)$ and the center of $G$ by $Z(G)$.

**Proposition 2.4 ([Bou06, Chapter III, §9, no. 3, Proposition 8]).** Let $H$ be a finite subgroup of a Lie group $G$. Then the Lie algebra of the centralizer $C_G(H)$ is obtained by taking the fixed points by $H$ of the adjoint representation of $G$. That is, we have $T_eC_G(H) = (T_eG)^H$.

**Theorem 2.5.** Let $H \leq U(m)$ be a subgroup. Then $C_{U(m)}(H)$ is connected.

**Proof.** A proof using a simultaneous diagonalization argument is given in [Sta05, Proof of Theorem 3.2]. See also [Gra]. □

**Proposition 2.6 ([Lee13, Corollary 21.6]).** Every continuous action by a compact Lie group on a manifold is proper.
Theorem 2.7 (Quotient Manifold Theorem). Suppose $G$ is a Lie group acting smoothly, freely, and properly on a smooth manifold $M$. Then the orbit space $M/G$ is a topological manifold of dimension equal to $\dim M - \dim G$, and has a unique smooth structure with the property that the quotient map $\pi: M \to M/G$ is a smooth submersion.

Moreover, if $M$ is orientable and $G$ is connected, then $M/G$ is orientable.

Proof. The first part is [Lee13, Theorem 21.10]. For the second part we fix an orientation on $M$ and $G$. Since $G$ is connected, the translations $L_g, R_g: G \to G$ and $g: M \to M$ are homotopic to the identity map and thus preserve the orientation. A tedious but straightforward computation with the charts constructed in the proof of [Lee13, Theorem 21.10] allows to extract an oriented atlas, showing that $M/G$ is orientable. \hfill \square

3. A moduli of representations of $\Gamma = \pi_1(X_1, x_0): G$

By $G$-complex we mean a $G$-CW complex. That is, a CW complex with a continuous $G$-action that is admissible (i.e. the action permutes the open cells of $X$, and maps a cell to itself only via the identity). See [OS02, Appendix A] for more details. A graph is a 1-dimensional CW complex. By $G$-graph we always mean a 1-dimensional $G$-complex.

If $X_1$ is a connected $G$-graph, there is a group extension

$$1 \to \pi_1(X_1, v_0) \xrightarrow{i} \Gamma \xrightarrow{\phi} G \to 1$$

which is most easily defined by lifting the action of $G$ to the universal cover $\tilde{X}$ of $X$. In this section we construct a moduli $\mathcal{M}$ of representations of the group extension $\Gamma$ and study its properties (note that we are using the word moduli in a rather informal way, meaning a geometric object whose points correspond to certain representations of $\Gamma$). The starting point to construct $\mathcal{M}$ is a result in Bass–Serre theory due to K.S. Brown [Bro84].

Theorem 3.1 (Brown). Let $X$ be obtained from a $G$-graph $X_1$ by attaching $m$ orbits of 2-cells along (the orbits of) the closed edge paths $\omega_1, \ldots, \omega_m$ based at a vertex $v_0$. Then there is a group extension

$$1 \to \pi_1(X, v_0) \xrightarrow{i} \Gamma/\langle\langle i(\omega_1), \ldots, i(\omega_m)\rangle\rangle \xrightarrow{\phi} G \to 1,$$

where the maps $\bar{i}$ and $\bar{\phi}$ are given by factoring through the quotient.

In order to describe Brown’s construction of $\Gamma$ and the maps $i$ and $\phi$ we need some choices. By admissibility of the action, the group $G$ acts on the set of oriented edges. If $e$ is an oriented edge, the same 1-cell with the opposite orientation is denoted by $e^{-1}$. Each oriented edge $e$ has a source $s(e)$ and a target $t(e)$. For each 1-cell of $X_1$ we choose a preferred orientation in such a way that these orientations are preserved by $G$. This determines a set $P$ of oriented edges. We choose a tree of representatives for $X_1/G$. That is, a tree $T \subset X_1$ such that the vertex set $V$ of $T$ is a set of representatives of $X_1^{(0)}/G$. Such tree always exists and the 1-cells of $T$ are inequivalent modulo $G$. We give an orientation to the 1-cells of $T$ so that they are elements of $P$. We also choose a set of representatives $E$ of $P/G$ in such a way that $s(e) \in V$ for every $e \in E$ and such that each oriented edge of $T$ is in $E$. If $e$ is an oriented edge, the unique element of $V$ that is equivalent to $t(e)$ modulo $G$ will be denoted by $w(e)$. For every $e \in E$ we fix an element $g_e \in G$ such that $t(e) = g_e \cdot w(e)$. If $e \in T$, we specifically choose $g_e = 1$. Then

$$\Gamma = \frac{F(x_e : e \in E) \ast \ast_{v \in V} G_v}{\langle\langle R\rangle\rangle},$$
where $F(x_e : e \in E)$ is the free group with basis $\{x_e : e \in E\}$ and $\langle \langle R \rangle \rangle$ denotes the normal subgroup generated by the set $R$ of relations of the following two types:

(i) $x_e = 1$ if $e \in T$, and

(ii) $x_e^{-1}t_{s(e)}(g)x_e = t_{w(e)}(g^{-1}gg_e)$ for every $e \in E$ and $g \in G_e$,

where $t_v : G_v \rightarrow F(x_e : e \in E) \ast \ast G_v$ denotes the canonical inclusion.

We thus have a moduli of representations of $\Gamma$ in the Lie group $G$. This extends the construction in [SC20, Section 4].

In the following proposition we use a morphism $\rho_0 : G \rightarrow G$ to construct a moduli of representations of $\Gamma$ in the Lie group $G$. For $v \in V$, we define $\tau_v = \tau_{e_1, e_2, \ldots, e_s}$ where $(e_1, e_2, \ldots, e_s)$ is the unique path from $v_0$ to $v$ by edges in $T$ (with this definition $\tau_{v_0} = 1$). Then we have a representation $\rho : \Gamma \rightarrow G$ given by

$$\rho_{\tau}(i_v(g)) = \tau_v^{-1}\rho_0(g)\tau_v \quad \text{for } v \in V,$$

$$\rho_{\tau}(x_e) = \tau_s^{-1}\tau_e^{-1}\rho_0(g_e)\tau_w(e) \quad \text{for } e \in E.$$ 

We thus have a moduli of representations

$$\rho : \mathcal{M} \rightarrow \hom(\Gamma, G)$$

$$\tau \mapsto \rho_{\tau}.$$

Moreover, each word $w \in \Gamma$ induces a differentiable map $W : \mathcal{M} \rightarrow G$ given by $\tau \mapsto \rho_{\tau}(w)$.

**Proof.** If $e \in T$ then $\tau_{w(e)} = \tau_{e} = \tau_{s(e)}$ and $g_e = 1$. Therefore $\rho_{\tau}(x_e) = 1$ and relations of type (i) are satisfied. Now if $e \in E$, $g \in G_e$ we have

$$\rho_{\tau}(x_e^{-1}\rho_{\tau}(i_{s(e)}(g))\rho_{\tau}(x_e) = \tau_w^{-1}\rho_0(g^{-1})\tau_e\tau_s\cdot \tau_s^{-1}\rho_0(g)\tau_s\cdot \tau_s^{-1}\tau_e^{-1}\rho_0(g_e)\tau_{w(e)}$$

$$= \tau_w^{-1}\rho_0(g^{-1})\tau_e\rho_0(g)\tau_e^{-1}\rho_0(g_e)\tau_{w(e)}$$

$$= \tau_w^{-1}\rho_0(g^{-1})\rho_0(g)\rho_0(g_e)\tau_{w(e)}$$

$$= \rho_{\tau}(i_{w(e)}(g^{-1}gg_e))$$

and thus the type (ii) relations $x_e^{-1}i_{s(e)}(g)x_e = i_{w(e)}(g^{-1}gg_e)$ also hold. \qed

Different points of $\mathcal{M}$ may correspond to equal representations of $\Gamma$. The quotient $\overline{\mathcal{M}}$ introduced in the following result allows us to deal with this issue.

**Theorem 3.3.** Let $\mathcal{H} = \{(\alpha_v)_{v \in V} : \alpha_{v_0} = 1\} \subseteq \prod_{v \in V} C_G(\rho_0(G_v))$. Assume $\mathcal{H}$ is compact.

(i) There is a free right action $\mathcal{M} \curvearrowright \mathcal{H}$ given by

$$(\tau \cdot \alpha)_e = \rho_0(g_e)\alpha_w^{-1}\rho_0(g_e)^{-1} \cdot \tau_e \cdot \alpha_s(e)$$

(ii) Moreover $\rho_{\tau} = \rho_{\tau'}$ if and only if $\tau, \tau'$ lie in the same orbit of the action of $\mathcal{H}$. 

(iii) The quotient $\overline{M} = M/\mathcal{H}$ is a smooth manifold, the map $p: M \to \overline{M}$ is a smooth submersion and $\dim(\overline{M}) = \dim(M) - \dim(\mathcal{H})$.

(iv) If $\mathcal{H}$ is connected then $\overline{M}$ is orientable.

(v) We have an induced map $\overline{\rho}: \overline{M} \to \text{hom}(\Gamma, G)$. Each word $w \in \Gamma$ induces a differentiable map $\overline{W}: \overline{M} \to G$ such that $\overline{W} = W \circ p$.

Proof. (i) Since $G_{s(e)} \supseteq G_e \subseteq G_{t(e)}$, the good definition follows from $\rho_0(\alpha_e \alpha_w^{-1})^{-1} \in C_G(\rho_0(G_{t(e)}))$ which holds since $t(e) = g_e \cdot w(e)$. If $(\tau \cdot \alpha_e) = \tau_e$ for all $e \in T$, by induction (traversing the tree $T$ starting from the root $v_0$) it follows that $\alpha_v = 1$ for all $v \in V$. Then the action is free.

(ii) Let $\tau \in \mathcal{M}$, $\alpha \in \mathcal{H}$. If $e \in T$ then $(\tau \alpha)_e = \alpha_{t(e)}^{-1} \tau_e \alpha_{s(e)}$. If $v \in V$, $(\tau \alpha)_v = \alpha_v^{-1} \tau_v$. Then

$$\rho_{\tau \alpha}(i_v(g)) = (\tau \alpha)_v^{-1} \rho_0(g)(\tau \alpha)_v$$

$$= \tau_v^{-1} \rho_0(g) \alpha_v^{-1} \tau_v$$

$$= \tau_v^{-1} \rho_0(g) \tau_v$$

$$= \rho_{\tau}(i_v(g)).$$

Moreover, for $e \in E$ we have

$$\rho_{\tau \alpha}(x_e) = (\tau \alpha)_s(e)^{-1} (\tau \alpha)_e^{-1} \rho_0(g_e)(\tau \alpha)_w(e)$$

$$= (\alpha_s(e)^{-1} \tau_s(e))^{-1} (\rho_0(g_e) \alpha_e^{-1} \rho(g_e))^{-1} \rho_0(g_e)(\alpha_e^{-1} \tau_e)$$

$$= \tau_s(e)^{-1} \alpha_v \rho_0(g_e) \tau(w(e))$$

$$= \rho_{\tau}(x_e).$$

Then $\rho_{\tau} = \rho_{\tau \alpha}$. For the other implication, if $\tau, \tau' \in \mathcal{M}$ satisfy $\rho_{\tau} = \rho_{\tau'}$, by defining $\alpha_v = \tau_v(\tau'_{s(v)})^{-1}$ we obtain a point $\alpha = (\alpha_v)_{v \in V} \in \mathcal{H}$ and $\tau \alpha = \tau'$.

(iii) By Proposition 2.6 the action is proper. Then by Theorem 2.7, the quotient $\overline{M} = M/\mathcal{H}$ has a (unique) smooth manifold structure such that $p: M \to \overline{M}$ is a submersion and $\dim(\overline{M}) = \dim(M) - \dim(\mathcal{H})$.

(iv) This follows from the second part of Theorem 2.7.

(v) This follows by passing to the quotient. \qed

Corollary 3.4. If $\mathbb{G} = U(m)$ then $\mathcal{M}$ and $\overline{M}$ are connected and orientable.

Proof. This is immediate from Theorem 2.5. \qed

A representation $\rho: \Gamma \to G$ is said to be universal if $N \subseteq \ker(\rho)$ (or equivalently, if $\rho$ factors through $\phi$). Under suitable hypotheses, $\Gamma = p(1)$ is the only point in $\overline{M}$ which corresponds to a universal representation:

**Proposition 3.5.** Suppose that $G$ is finite and that each element of $G$ fixes a vertex in $X_1$. Let $\mathbb{G} \subseteq \text{GL}_m(\mathbb{C})$ and assume the restriction $\rho_0|_{G_{v_0}}: G_{v_0} \to \mathbb{G}$ is an irreducible representation of $G_{v_0}$. Then $\{\overline{\mathbf{1}}\} = \{\tau \in \overline{M} : \overline{\rho}_\tau \text{ universal}\}$.

Proof. First note that $\overline{\rho}_1 = \rho_0 \circ \phi$ is universal. Now consider $\tau \in \mathcal{M}$ such that $\rho_{\tau}$ is universal. By passing to the quotient we have a representation $\tilde{\rho}_{\tau}: G \to \mathbb{G}$ such that $\rho_{\tau} = \tilde{\rho}_{\tau} \circ \phi$. Now note that, since each element of $G$ fixes a vertex of $X_1$, from the definition of $\rho_{\tau}$ it follows that the representatives $\rho_0$ and $\tilde{\rho}_{\tau}$ have the same character and are therefore isomorphic. Hence, we can take $\alpha \in \text{GL}_m(\mathbb{C})$ such that for all $g \in G$ we have $\alpha \tilde{\rho}_\tau(g)\alpha^{-1} = \rho_0(g)$. Now since for every
\( g \in G_{v_0} \) we have \( \tilde{\rho}_r(g) = \rho_0(g) \), and since \( \rho_0|_{G_{v_0}} \) is irreducible, by Schur’s lemma (Theorem A.1.2) it follows that \( \alpha \) is a scalar matrix and therefore \( \tilde{\rho}_r = \rho_0 \). Then \( \rho_r = \rho_1 \) and therefore by part (ii) of Theorem 3.3, \( p(\tau) = p(1) \) in \( \overline{M} \). \( \square \)

**Remark 3.6.** If \( \rho_0|_{G_{v_0}} : G_{v_0} \to G \) is not irreducible, we could still consider the quotient \( \overline{M} \) of \( M \) by the action of \( C_G(\rho_0(G_{v_0})) \). In this case, the points in \( \overline{M} \) correspond to characters (not representations) of \( G \) and the image of the induced map \( \overline{W} \) is only defined up to conjugation by \( C_G(\rho_0(G_{v_0})) \). Note that the quotient of \( G \) by the conjugation action of \( C_G(\rho_0(G_{v_0})) \) is not, in general, a manifold.

The following result relates a closed edge path \( \omega \in X_1 \) to the differential at 1 of the map \( \mathcal{M} \to G \) induced by the word \( i(\omega) \in \Gamma \).

**Theorem 3.7.** Let \( X_1 \) be a G-graph (with the necessary choices to form \( \mathcal{M} \)). Consider a closed edge path \( \omega = (a_1 e^n_1, \ldots, a_n e^n_n) \) in \( X \), based at \( v_0 \), with \( a_i \in E \), \( a_i \in G \) and \( \varepsilon_i \in \{1, -1\} \).

Let \( w = i(\omega) \in N = \ker(\phi) \). Let \( W : \mathcal{M} \to G \) be the induced differentiable map. Let \( 1 = (1)e \in E \in \mathcal{M} \) and consider the inclusion \( j_e : C_G(\rho_0(G_e)) \hookrightarrow G \). Then, with the identification \( T_1\mathcal{M} \cong \bigoplus_{e \in E} T_1C_G(\rho_0(G_e)) \) we have

\[
d_1 W = - \sum_{i=1}^{n} \varepsilon_i \cdot d_{\rho_0(a_i)}^{-1} L_{\rho_0(a_i)} \circ d_1 R_{\rho_0(a_i)}^{-1} \circ d_1 j_{e_i}.
\]

**Proof.** By the definition of \( i : \pi_1(X_1) \to \Gamma \) (see [Bro84] or [SC20, Section 4]) we can write

\[
w = i_{v_0}(h_1) \cdot x^n_{e_1} \cdot i_{v_0}(h_2) \cdot x^n_{e_2} \cdot \ldots \cdot i_{v_{n-1}}(h_n) \cdot x^n_{e_n} \cdot i_{v_0}(g_1 g_2 \cdots g_n)^{-1}
\]

so that for each \( i \) we have \( g_i = h_i g_{e_i}^{-1} \) and

\[
a_i = \begin{cases} 
g_1 \cdots g_{i-1} h_i & \text{if } \varepsilon_i = 1 \\ 
g_1 \cdots g_{i-1} h_i g_{e_i}^{-1} & \text{if } \varepsilon_i = -1. \end{cases}
\]

Then

\[
W(\tau) = \left( \prod_{i=1}^{n} (\tau_{v_{i-1}} \rho_0(h_i) \tau_{v_i}) (\tau_{s(e_i)}^{-1} \rho_0(g_{e_i}) \tau_{w(e_i)})^{\varepsilon_i} \right) \tau_{v_0} \rho_0(g_1 g_2 \cdots g_n)^{-1} \tau_{v_0}^{-1} \rho_0(g_1 g_2 \cdots g_n)^{-1}.
\]

In the last equality we used that \( \tau_{v_0} = 1 \) and that \( s(e_i) \) and \( w(e_i) \) are (in some order which depends on \( \varepsilon_i \) \( v_{i-1} \)) and \( v_i \). We have \( P_i(1) = \rho_0(a_i) \) where \( P_i \) is the prefix of \( W \) ending just before the occurrence of \( \tau_{e_i}^{-\varepsilon_i} \). Note that, since \( W(1) = 1 \), if \( S_i \) is the suffix of \( W \) starting just after the occurrence of \( \tau_{e_i}^{-\varepsilon_i} \), we have \( S_i(1) = \rho_0(a_i)^{-1} \). To conclude, we apply the product rule Proposition 2.2. \( \square \)

**Lemma 3.8 (cf. [SC20, Lemma 6.7]).** Let \( \Gamma \) be a group, \( G \) be a Lie group, \( M \) be a differentiable manifold, and \( \rho : M \to \text{hom}(\Gamma, G) \) be a function such that for each \( w \in \Gamma \) the mapping \( W : M \to G \) defined by \( W(z) = \rho(z)(w) \) is differentiable. Let \( N \triangleleft \Gamma \) be a normal subgroup and suppose that \( p \in M \) is such that \( \rho(p)(w) = 1 \) for each \( w \in N \). Then for any elements \( w_0, \ldots, w_k \in N \) and \( x_0, \ldots, x_k \in \langle w_0, \ldots, w_k \rangle \Gamma \) \([N, N]\) we have \( \text{rk} d_p W \geq \text{rk} d_p X \), where \( W = (W_0, \ldots, W_k) \) and \( X = (X_0, \ldots, X_k) \) are the induced maps \( M \to G^{k+1} \).
We obtain these homotopies from homotopies $G$
Proof.
For each $I$. SADOFSCHI COSTA
Let $\gamma$ a path with $\gamma(0) = 1$, $\gamma(1) = \rho_0(g)$. The following map

$$H : M \times I \to \mathbb{G}$$

is a homotopy between the maps $M \to \mathbb{G}$ induced by $ww'$ and $wi_v(g)w'$. Moreover, since $\rho_0(g) \in C_G(C_G(\rho_0(G_v)))$, we can take $\gamma(I) \subseteq C_G(C_G(\rho_0(G_v)))$ if the latter is connected and in this case the following computation

$$H(\tau\alpha, t) = W(\tau\alpha)(\tau\alpha)^{-1} \gamma(t)(\tau\alpha)eW'(\tau\alpha)$$

$$= W(\tau)(\tau\alpha)^{-1} \gamma(t)\tau\alpha_1 W'(\tau)$$

$$= W(\tau)(\alpha_1^{-1}\tau\nu)^{-1} \gamma(t)(\alpha_1^{-1}\tau\nu)W'(\tau)$$

$$= W(\tau)\tau\nu^{-1} \gamma(t)\tau\nu W'(\tau)$$

$$= W(\tau)\tau\nu^{-1} \gamma(t)\tau\nu W'(\tau)$$

shows the homotopy $H$ is $H$-equivariant, giving a homotopy between the induced maps $M \to \mathbb{G}$. 

In the following two propositions we use the notation $\prod_{i=\ell}^{1} b_i = b_{\ell}b_{\ell-1}b_{\ell-2} \cdots b_2b_1$. 

\qed
Proposition 3.10. Let \( \eta \in E - T \) and let \((e_1, \ldots, e_k)\) and \((e'_1, \ldots, e'_{\ell})\) be the unique paths in \( T \) from \( v_0 \) to \( s(\eta) \) and \( w(\eta) \) respectively (see Figure 1). Suppose that \( \gamma_0, \ldots, \gamma_k, \beta_0, \ldots, \beta_{\ell} : I \to \mathbb{G} \) are paths such that:

- For \( i = 1, \ldots, k \) and for every \( t \in I \), \( \gamma_i(t) \) commutes with \( C_G(\rho_0(G_t(e_i))) \).
- For \( i = 1, \ldots, \ell \) and for every \( t \in I \), \( \beta_t(t) \) commutes with \( C_G(\rho_0(G_t(e'_i))) \).

Then there is an \( \mathcal{H} \)-equivariant homotopy \( F : \mathcal{M} \times I \to \mathbb{G} \) defined by

\[
F(\tau, t) = \gamma_0(t) \left( \prod_{i=1}^{k} \tau_{e_i}^{-1} \gamma_i(t) \right) (\tau \alpha)^{-1} \rho_0(g_\eta) \left( \prod_{i=\ell}^{1} \beta_t(t) \right) \beta_0(t)
\]

Moreover, if \( \gamma_i(0) = 1 \) for \( i = 0, \ldots, k \) and \( \beta_i(0) = 1 \) for \( i = 0, \ldots, \ell \) then \( F_0 = X_\eta \) where \( X_\eta \) is the map induced by \( x_\eta \).

![Figure 1](image-url) The paths in Propositions 3.10 and 3.11. Note that \( t(e'_i) = w(\eta) = g_\eta^{-1} \cdot t(\eta) \). Also note that we may have \( k = 0 \) or \( \ell = 0 \).

**Proof.** The following computation shows that \( F \) is \( \mathcal{H} \)-equivariant.

\[
F(\tau \alpha, t) = \gamma_0(t) \left( \prod_{i=1}^{k} (\tau \alpha)_{e_i}^{-1} \gamma_i(t) \right) (\tau \alpha)^{-1} \rho_0(g_\eta) \left( \prod_{i=\ell}^{1} \beta_t(t) (\tau \alpha)_{e'_i} \right) \beta_0(t)
\]

\[
= \gamma_0(t) \left( \prod_{i=1}^{k} \alpha_{s(e_i)}^{-1} \tau_{e_i}^{-1} \alpha_{t(e_i)} \gamma_i(t) \right) \alpha_{s(\eta)}^{-1} \tau_{\eta}^{-1} \rho_0(g_\eta) \alpha_{w(\eta)} \rho_0(g_\eta)^{-1} \rho_0(g_\eta) \left( \prod_{i=\ell}^{1} \beta_t(t) \right) \beta_0(t)
\]

\[
= \gamma_0(t) \left( \prod_{i=1}^{k} \tau_{e_i}^{-1} \gamma_i(t) \right) \tau_{\eta}^{-1} \rho_0(g_\eta) \left( \prod_{i=\ell}^{1} \beta_t(t) \right) \beta_0(t)
\]

\[
= F(\tau, t).
\]

For the second part, note that

\[
X_\eta(\tau) = \rho_? (x_\eta)
\]

\[
= \tau_{\eta}^{-1} \tau_{s(\eta)}^{-1} \rho_0(g_\eta) \tau_{w(\eta)}
\]

\[
= \left( \prod_{i=1}^{k} \tau_{e_i}^{-1} \right) \tau_{\eta}^{-1} \rho_0(g_\eta) \prod_{i=\ell}^{1} \tau_{e'_i}.
\]

\[ \square \]
Proposition 3.11. Suppose that $C_G(C_G(\rho_0(G_v)))$ is connected for each $v \in V$. Let $\eta \in E - T$ and let $(e_1, \ldots, e_k)$ and $(e'_1, \ldots, e'_\ell)$ be the unique paths in $T$ from $v_0$ to $s(\eta)$ and $w(\eta)$ respectively. Let $A_e \in G$ be elements defined for every $e \in E$. Suppose that $C_1, \ldots, C_k, B_1, \ldots, B_\ell \in G$ satisfy:

- $A^{-1}_{e_i} C_i A_{e_{i+1}}$ commutes with $C_G(\rho_0(G_{t(e_i)}))$ for $i = 1, \ldots, k - 1$.
- $A^{-1}_{e_k} C_k A_\eta$ commutes with $C_G(\rho_0(G_{t(e_k)}))$.
- $A^{-1}_{e'_i} B_i A_{e'_i}$ commutes with $C_G(\rho_0(G_{t(e'_i)}))$ for $i = 1, \ldots, \ell - 1$.
- $\rho_0(g_\eta)^{-1} A^{-1}_\eta g_\eta B_{i_\ell} A_{e'_i}$ commutes with $C_G(\rho_0(G_{w(\eta)}))$.

Then there is an $H$-equivariant homotopy between the map $X_\eta : M \to G$ induced by $x_\eta$ and the map $Z : M \to G$ defined by

$$Z(\tau) = \left( \prod_{i=1}^{k} A_{e_i} \tau_{e_i}^{-1} A^{-1}_{e_i} C_i \right) A_\eta \tau_\eta^{-1} A^{-1}_\eta \rho_0(g_\eta) \left( \prod_{i=\ell}^{\ell} B_i A_{e'_i} \tau_{e'_i}^{-1} A^{-1}_{e'_i} \right).$$

Proof. Since the centralizers $C_G(C_G(\rho_0(G_v)))$ are connected, we can take paths:

- $1 \stackrel{\gamma_i}{\to} A_{e_i}$, in $G$.
- $1 \stackrel{\gamma_i}{\to} A^{-1}_{e_i} C_i A_{e_{i+1}}$ such that $\gamma_i(I)$ commutes with $C_G(\rho_0(G_{t(e_i)}))$ for $i = 1, \ldots, k - 1$.
- $1 \stackrel{\gamma_k}{\to} A^{-1}_{e_k} C_k A_\eta$ such that $\gamma_k(I)$ commutes with $C_G(\rho_0(G_{t(e_k)}))$.
- $1 \stackrel{\beta_i}{\to} \rho_0(g_\eta)^{-1} A^{-1}_\eta \rho_0(g_\eta) B_{i_\ell} A_{e'_i}$ such that $\beta_i(I)$ commutes with $C_G(\rho_0(G_{w(\eta)}))$.
- $1 \stackrel{\beta_i}{\to} A^{-1}_{e'_i} B_i A_{e'_i}$ such that $\beta_i(I)$ commutes with $C_G(\rho_0(G_{t(e'_i)}))$ for $i = 1, \ldots, \ell - 1$.
- $1 \stackrel{\beta_i}{\to} A^{-1}_{e'_i}$, in $G$.

The result now follows from Proposition 3.10. \hfill \Box

4. SOME REPRESENTATION THEORY

In this section we prove Lemma 4.7, which is useful to apply Proposition 3.11. We also prove Lemma 4.5, a consequence of Schur’s lemma which is used later to compute the dimension of centralizers in $U(n)$. We start by recalling the following classical results

Theorem 4.1 ([EGH+11, Theorem 4.6.2]). Every representation $\rho : G \to GL_n(\mathbb{C})$ of a finite group $G$ is isomorphic to a unitary representation $\tilde{\rho} : G \to U(n)$.

Theorem 4.2. Let $G$ be a finite group. If two unitary representations of $G$ are isomorphic then there is a unitary isomorphism between them.

Proof. When the representations are irreducible this is [Dor71, Lemma 33.1]. For a proof in the general case see [Was]. \hfill \Box

If $A, A'$ are matrices then $A \oplus A'$ denotes the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}$. If $\rho, \rho'$ are representations of a group $G$ then $\rho \oplus \rho'$ denotes the representation such that $(\rho \oplus \rho')(g) = \rho(g) \oplus \rho'(g)$ for all $g \in G$. We denote the $n \times n$ identity matrix by $I_n$.

It is easy to verify that block scalar matrices commute with scalar block matrices:

Proposition 4.3. Let $X \in M_n(\mathbb{C})$ and $\lambda \in M_k(\mathbb{C})$ be two matrices. Let $A = X \oplus \ldots \oplus X \in M_{kn}(\mathbb{C})$ and let $B \in M_{kn}(\mathbb{C}) = M_k(M_n(\mathbb{C}))$ be the matrix defined by $B_{i,j} = \lambda_{i,j} I_n$. Then $A$ and $B$ commute.
Remark 4.4. Let \( \rho_1, \ldots, \rho_k \) be pairwise non-isomorphic irreducible representations of a finite group \( G \) and let \( n_1, \ldots, n_k \) be natural numbers. Consider the representation \( \rho = \bigoplus_{i=1}^k \rho_i^{n_i} \), where \( \rho_i^{n_i} \) denotes the sum \( \rho_i \oplus \ldots \oplus \rho_i \) of \( n_i \) copies of \( \rho_i \). Let \( d_i \) be the degree of \( \rho_i \) and let \( n = \sum_{i=1}^k d_in_i \) be the degree of \( \rho \). Then, by Schur’s lemma (Theorem A.1.2), we have

\[
C_{U(n)}(\rho(G)) = \prod_{i=1}^k C_{U(d_in_i)}(\rho_i^{n_i}),
\]

where the product on the right is included in \( U(n) \) as block diagonal matrices. Again by Schur’s lemma, we have an isomorphism \( U(n_i) \xrightarrow{\sim} \prod_{i=1}^k C_{U(d_in_i)}(\rho_i^{n_i}) \) which is given by \( A \mapsto \tilde{A} \), where \( \tilde{A} = \bigoplus_{i=1}^k \tilde{A}_i \), which is in fact unitary. Then \( C_{U(n)}(\rho(G)) \simeq \prod_{i=1}^k U(n_i) \) and in particular we have \( \dim C_{U(n)}(\rho(G)) = \sum_{i=1}^k n_i^2 \).

Lemma 4.5. Let \( G \) be a finite group and let \( \rho : G \to U(n) \) be a unitary representation with character \( \chi \). Then \( \dim C_{U(n)}(\rho(G)) = \langle \chi, \chi \rangle_G \).

Proof. If \( \rho \) is isomorphic to \( \bigoplus_{i=1}^k \rho_i^{n_i} \), where \( \rho_1, \ldots, \rho_k \) are pairwise non-isomorphic irreducible representations of \( G \), from the orthogonality relations and Remark 4.4 we obtain \( \langle \chi, \chi \rangle_G = \sum_{i=1}^k n_i^2 = \dim C_{U(n)}(\rho(G)) \).

Recall that \( \text{Res}^G_H \chi \) denotes the restriction of a character \( \chi \) of \( G \) to a subgroup \( H \leq G \).

Definition 4.6. Let \( G \) be a finite group and let \( H \leq G \) be a subgroup. Let \( \Theta \) be a subset of the irreducible characters of \( H \). If \( \rho \) is a representation of \( G \) with character \( \chi \) we define

\[
d(\rho, \Theta) = \left\langle \text{Res}^G_H \chi, \sum_{\theta \in \Theta} \theta(1) \theta \right\rangle_H.
\]

Note that \( d(\rho, \Theta) \) is the sum of the dimensions of the irreducible factors of \( \rho|_H \) whose characters belong to \( \Theta \).

Lemma 4.7. Let \( H_1, H_2 \) be two subgroups of a finite group \( G \) and let \( \rho : G \to U(n) \) be a representation. Suppose that \( n = n' + n'' \) and \( A_1, A_2 \in U(n) \) are matrices such that, for \( i = 1, 2 \), the representations \( \tilde{\rho}_i : H_i \to U(n) \) given by \( \tilde{\rho}_i(g) = A_i\rho(g)A_i^{-1} \) satisfy

- There is a number \( k_i \in \mathbb{N} \) and irreducible representations \( \tilde{\rho}_{i,s} : H_i \to U(m_{i,s}) \) with \( s = 1, \ldots, k_i \) such that we have a block diagonal decomposition

  \[
  \tilde{\rho}_i = \tilde{\rho}_{i,1} \oplus \ldots \oplus \tilde{\rho}_{i,k_i}.
  \]

- If \( \tilde{\rho}_{i,s} \) and \( \tilde{\rho}_{i,s'} \) are isomorphic then \( \tilde{\rho}_{i,s} = \tilde{\rho}_{i,s'} \).

- There exists \( l_i \in \mathbb{N} \) such that \( n' = m_{i,1} + \ldots + m_{i,l_i} \) and \( n'' = m_{i,l_i+1} + \ldots + m_{i,k_i} \).

- If \( s, s' \) are numbers such that \( 1 \leq s \leq l_i \) and \( l_i + 1 \leq s' \leq k_i \) then \( \tilde{\rho}_{i,s} \) and \( \tilde{\rho}_{i,s'} \) are not isomorphic.

Suppose that for each irreducible factor \( \rho' \) of \( \rho \) we have \( d(\rho', \Theta_1) = d(\rho', \Theta_2) \), where \( \Theta_i \) is the set given by the characters of the representations \( \tilde{\rho}_{i,s} \) with \( 1 \leq s \leq l_i \).

Then there is a matrix \( C \in U(n') \times U(n'') \) such that \( A_2^{-1}CA_1 \) commutes with \( C_{U(n)}(\rho(G)) \).

Proof. By Theorem 4.1 and Theorem 4.2, we can take \( T \in U(n) \) and irreducible representations \( \rho_j : G \to U(m_j) \) with \( j = 1, \ldots, k \) such that \( TpT^{-1} = \rho_1 \oplus \ldots \oplus \rho_k \). Moreover, we can do this
so that whenever \( \rho_j \) and \( \rho_{j'} \) are isomorphic we have \( \rho_j = \rho_{j'} \). For each \( i = 1, 2 \) we take matrices \( D_{i,1}, \ldots, D_{i,k} \) with \( D_{i,j} \in \mathbf{U}(m_j) \) such that

\[
D_{i,j}\rho_j D_{i,j}^{-1} : H_i \to \mathbf{U}(m_j)
\]
is block diagonal, with each block equal to some \( \tilde{\rho}_{i,s} \) and such that the irreducible blocks with characters in \( \Theta_i \) appear consecutively at the start of the diagonal. These blocks have a total size of \( d(\rho_j, \Theta_i) = d(\rho_j, \Theta_2) \) (see Figure 2). We choose the \( D_{i,j} \) so that \( \rho_j = \rho_{j'} \) implies \( D_{i,j} = D_{i,j'} \).

Let \( D_i = D_{i,1} \oplus \ldots \oplus D_{i,k} \). Note that by Proposition 4.3 and Remark 4.4, \( D_i \) commutes with \( \mathbf{C}_{\mathbf{U}(n)}(T\rho(G)T^{-1}) \).

![Figure 2](image.png)

**Figure 2.** The matrices \( D_{i,j}\rho_j D_{i,j}^{-1} \) for \( i = 1, 2 \). Each shaded block is equal to some \( \tilde{\rho}_{i,s} \). Later in the proof of Lemma 4.7, the condition \( d(\rho_j, \Theta_i) = d(\rho_j, \Theta_2) \) allows to take the same permutation matrix \( \sigma \) for \( i = 1, 2 \).

Now note that we can take a permutation matrix \( \sigma \in \mathbf{U}(n) \) such that for \( i = 1, 2 \) the representation

\[
(\sigma D_i T) \rho |_{H_i} (\sigma D_i T)^{-1} : H_i \to \mathbf{U}(n)
\]
is block diagonal with irreducible blocks and such that the blocks with characters in \( \Theta_i \) appear consecutively at the start of the diagonal.

Now for \( i = 1, 2 \) we take a permutation matrix \( \sigma_i \in \mathbf{U}(n') \times \mathbf{U}(n'') \) such that

\[
A_i |_{H_i} A_i^{-1} = (\sigma_i \sigma D_i T) \rho |_{H_i} (\sigma_i \sigma D_i T)^{-1}.
\]

Let \( C_i = A_i (\sigma_i \sigma D_i T)^{-1} \). Then \( C_i \in \mathbf{C}_{\mathbf{U}(n)}(A_i \rho(H_i)A_i^{-1}) \subseteq \mathbf{U}(n') \times \mathbf{U}(n'') \) and we have \( A_i \rho(H_i)A_i^{-1} = C_i \sigma_i \sigma D_i T \). Finally, letting \( C = C_2 \sigma_2 \sigma_1^{-1} C_1^{-1} \in \mathbf{U}(n') \times \mathbf{U}(n'') \) the matrix

\[
A_2^{-1} C A_1 = T^{-1} D_2^{-1} D_1 T
\]
commutes with \( \mathbf{C}_{\mathbf{U}(n)}(\rho(G)) \), because \( D_i \) commutes with \( \mathbf{C}_{\mathbf{U}(n)}(T \rho(G) T^{-1}) \) for \( i = 1, 2 \). \( \square \)

In what follows, \( C_n \) denotes a cyclic group of order \( n \) and \( D_n \) denotes a dihedral group of order \( n \).

**Proposition 4.8.** Let \( n \) be an odd number and consider subgroups \( H_1 = \langle g \rangle \cong C_n \) and \( H_2 \cong C_2 \rangle \) of a group \( G \cong D_{2n} \). Let \( \mu_k : H_i \to \mathbb{C} \) be the character given by \( \mu_k(g^i) = \xi^k \), where \( \xi = e^{2\pi i / n} \) and let \( 1_{H_2} \) be the trivial character of \( H_2 \).

(i) Letting \( \Theta_1 = \{ \mu_1, \ldots, \mu_{(n-1)/2} \} \) and \( \Theta_2 = \{ 1_{H_2} \} \), we have \( d(\rho, \Theta_1) = d(\rho, \Theta_2) \) for any nontrivial irreducible representation \( \rho \) of \( G \).

(ii) Letting \( \Theta_1 = \{ \mu_0, \mu_1, \ldots, \mu_{(n-1)/2} \} \) and \( \Theta_2 = \{ 1_{H_2} \} \), we have \( d(\rho, \Theta_1) = d(\rho, \Theta_2) \) for any irreducible representation \( \rho \) of \( G \) other than the nontrivial degree 1 representation.
Proof. The irreducible characters of $D_{2n}$ are given in Proposition A.3.1. For a representation $\rho$ with character $\psi_1$ we have $\text{Res}_{H_1}^G \psi_1 = \mu_0$ and $\text{Res}_{H_2}^G \psi_1 = 1_{H_2}$. For a representation $\rho$ with character $\psi_2$ we have $\text{Res}_{H_1}^G \psi_2 = \mu_0$ and $\text{Res}_{H_2}^G \psi_2 = \nu$, where $\nu : H_2 \to \mathbb{C}$ denotes the nontrivial irreducible character of $H_2$. For a representation $\rho$ with character $\chi_i$ we have $\text{Res}_{H_1}^G \chi_i = \mu_i + \mu_{-i}$ and $\text{Res}_{H_2}^G \chi_i = 1_{H_2} + \nu$. \hfill $\square$

5. THE GRAPH $X_1^{OS}(G)$

We recall here the examples obtained by Oliver and Segev for each of the groups in Theorem 1.1.

Proposition 5.1 ([OS02, Example 3.4]). Set $G = \text{PSL}_2(q)$, where $q = 2^k$ and $k \geq 2$. Then there is a 2-dimensional acyclic fixed point free $G$-complex $X$, all of whose isotropy subgroups are solvable. More precisely, $X$ can be constructed to have three orbits of vertices with isotropy subgroups isomorphic to $B = \mathbb{F}_q \rtimes C_{q-1}$, $D_{2(q-1)}$, and $D_{2(q+1)}$; three orbits of edges with isotropy subgroups isomorphic to $C_{q-1}$, $C_2$ and $C_2$; and one free orbit of 2-cells.

Proposition 5.2 ([OS02, Example 3.5]). Assume that $G = \text{PSL}_2(q)$, where $q = p^k \geq 5$ and $q \equiv \pm 3 \pmod{8}$. Then there is a 2-dimensional acyclic fixed point free $G$-complex $X$, all of whose isotropy subgroups are solvable. More precisely, $X$ can be constructed to have four orbits of vertices with isotropy subgroups isomorphic to $B = \mathbb{F}_q \rtimes C_{(q-1)/2}$, $D_{q-1}$, $D_{q+1}$, and $A_4$; four orbits of edges with isotropy subgroups isomorphic to $C_{(q-1)/2}$, $C_2$, $C_3$ and $C_2$; and one free orbit of 2-cells.

Proposition 5.3 ([OS02, Example 3.7]). Set $q = 2^{2k+1}$ for any $k \geq 1$. Then there is a 2-dimensional acyclic fixed point free $\text{Sz}(q)$-complex $X$, all of whose isotropy subgroups are solvable. More precisely, $X$ can be constructed to have four orbits of vertices with isotropy subgroups isomorphic to $M(q, \theta)$, $D_{2(q-1)}$, $C_{q+\sqrt{q+1}} \rtimes C_4$, $C_{q-\sqrt{q+1}} \rtimes C_4$; four orbits of edges with isotropy subgroups isomorphic to $C_{q-1}$, $C_4$, $C_4$ and $C_2$; and one free orbit of 2-cells.

The Oliver–Segev graph $X_1^{OS}(G)$ is the 1-skeleton of any 2-dimensional fixed point free acyclic $G$-complex of the type constructed in Propositions 5.1 to 5.3. The graph $X_1^{OS}(G)$ is unique up to $G$-homotopy equivalence (see [SC20, Proposition 3.10]). For any $k \geq 0$, we also consider the $G$-graph $X_1^{OS+k}(G)$ obtained from $X_1^{OS}(G)$ by attaching $k$ free orbits of 1-cells. The $G$-homotopy type of $X_1^{OS+k}(G)$ does not depend on the particular way these free orbits are attached (again by [SC20, Proposition 3.10]).

Lemma 5.4. Let $G$ be a finite group and let $X_1$ be a $G$-graph. Let $u, v, w$ be vertices of $X_1$ and let $e, e'$ be edges such that $e$ has endpoints $\{u, v\}$ and $e'$ has endpoints $\{v, w\}$. Suppose that $G_{e'} \subseteq G_e$. Consider the $G$-graph $\tilde{X}_1$ obtained from $X_1$ by removing the orbit of $e'$ and attaching an orbit of edges of type $G/G_{e'}$ with endpoints $\{u, w\}$. Then $X_1$ and $\tilde{X}_1$ are $G$-homotopy equivalent.
Proof. We can obtain \( \overline{X} \) from \( X_1 \) by doing an equivariant elementary expansion of type \( G/G_e \) followed by an equivariant elementary collapse of type \( G/G_{e'} \) (see [Ill74] for the precise definition of equivariant expansion and equivariant collapse). These modifications are \( G \)-homotopy equivalences.

For each of the groups \( G \) in Theorem 1.1, we describe a feasible way to connect the orbits in the graph \( X_1^{OS}(G) \). Recall that, by [OS02, Theorem 4.1], the graph \( X_1^{OS}(G)^H \) is a tree for each solvable subgroup \( H \leq G \). When considering the groups \( \text{Sz}(q) \) we use the notation \( r = \sqrt{2q} \).

**Proposition 5.5.** For each of the groups \( G \) in Theorem 1.1, we can construct \( X_1^{OS}(G) \) so that the orbits are connected as in Figure 3.

Proof. In all cases, the orbit types are given by Propositions 5.1 to 5.3. For the groups \( G = \text{PSL}_2(q) \), a possible way to connect the orbits is described in [OS02, Section 3]. For each of these groups, the structure in Figure 3 coincides with this one up to an application of Lemma 5.4.

For \( G = \text{Sz}(q) \) we give more detail here. First note that, since \( q - 1 \nmid 4(q \pm r + 1) \), the orbit of type \( C_{q-1} \) has to connect \( B \) to \( D_{2(q-1)} \). Now the two orbits of type \( C_1 \) must connect \( B, C_{q+r+1} \times C_4 \) and \( C_{q-r+1} \times C_4 \) (in some way). The orbit \( C_2 \) must connect \( D_{2(q-1)} \) to one of the other three orbits of vertices. Note that, in any case, we can repeatedly use Lemma 5.4 to obtain the desired structure.

\[
\begin{array}{|l|c|c|c|c|c|}
\hline
G & q & G_{v_0} & G_{v_1} & G_{v_2} & G_{v_3} \\
\hline
\text{PSL}_2(q) & 2^n & B = \mathbb{F}_q \times C_{q-1} & D_{2(q-1)} & D_{2(q+1)} & - \\
\text{PSL}_2(q) & q \equiv 3 \pmod{8} & B = \mathbb{F}_q \times C_{(q-1)/2} & D_{q-1} & D_{q+1} & A_4 \\
\text{Sz}(q) & 2^n & B = M(q, \theta) & D_{2(q-1)} & C_{q+r+1} \times C_4 & C_{q-r+1} \times C_4 \\
\hline
\end{array}
\]

**Table 1.** Stabilizers of vertices for the graph \( X_1^{OS}(G) \)

\[
\begin{array}{|l|c|c|c|c|c|}
\hline
G & q & G_{v_0} & G_{v_1} & G_{v_2} & G_{v_3} \text{ or } G_{v'_3} \\
\hline
\text{PSL}_2(q) & 2^n & C_{q-1} & C_2 & C_2 & - \\
\text{PSL}_2(q) & q \equiv 3 \pmod{8} & C_{(q-1)/2} & C_2 \times C_2 & C_3 & 1 \\
\text{Sz}(q) & 2^n & C_{q-1} & C_2 & C_4 & 1 \\
\hline
\end{array}
\]

**Table 2.** Stabilizers of edges for the graph \( X_1^{OS}(G) \)

Now, for each of the groups \( G \) in Theorem B, we fix our choices regarding \( X_1^{OS+k}(G) \) in order to apply Brown’s result to it. In each case the stabilizers are given in Tables 1 and 2. Our choices in each case are the following:

- For \( G = \text{PSL}_2(2^n) \) we take \( V = \{v_0, v_1, v_2\}, E = \{\eta_0, \eta_1, \eta_2, \eta_1', \ldots, \eta_k\}, T = \{\eta_0, \eta_1\}, \) with \( v_0 \overset{\eta_0}{\to} v_1, v_1 \overset{\eta_1}{\to} v_2, v_2 \overset{\eta_2}{\to} g_{v_2}v_0 \) and \( v_0 \overset{\eta_i}{\to} v_0 \) for \( i = 1, \ldots, k \).

- For \( G = \text{PSL}_2(q) \) with \( q = 3^n \) or \( q \equiv 19 \pmod{24} \) we take \( V = \{v_0, v_1, v_2, v_3\}, E = \{\eta_0, \eta_1, \eta_2, \eta_3, \eta_1', \ldots, \eta_k'\}, T = \{\eta_0, \eta_1, \eta_2\}, \) with \( v_0 \overset{\eta_0}{\to} v_1, v_1 \overset{\eta_1}{\to} v_2, v_2 \overset{\eta_2}{\to} v_3, v_3 \overset{\eta_3}{\to} g_{v_3}v_0 \) and \( v_0 \overset{\eta_i}{\to} v_0 \) for \( i = 1, \ldots, k \).

- For \( G = \text{PSL}_2(q) \) with \( q \equiv 11 \pmod{24} \) we take \( V = \{v_0, v_1, v_2, v_3\}, E = \{\eta_0, \eta_1, \eta_2, \eta_3, \eta_1', \ldots, \eta_k'\}, T = \{\eta_0, \eta_1, \eta_2\}, \) with \( v_0 \overset{\eta_0}{\to} v_1, v_1 \overset{\eta_1}{\to} v_2, v_2 \overset{\eta_2}{\to} v_3, v_3 \overset{\eta_3}{\to} g_{v_3}v_2 \) and \( v_0 \overset{\eta_i}{\to} v_0 \) for \( i = 1, \ldots, k \).
We have a free orbit of orbits of edges in Lemma 5.6. Note that in all cases the stabilizer of \( i \in x_i \) is only needed in this proof. If \( i \in x_i \) is a Borel subgroup of \( G \), then we define \( \tilde{g} \) for \( i = 1, \ldots, k \). Let \( \eta \) be the unique edge of \( X_1^{OS}(G) \) which lies in \( E - T \). We define \( y_0 = x_\eta \) and \( y_i = x_{y_i} \) for \( i = 1, \ldots, k \).

We conclude this section with a lemma needed in Section 8. We explain here some notation which is only needed in this proof. If \( x = \sum_{g \in G} x_g g \in \mathbb{Z}[G] \) then we define \( \overline{x} = \sum_{g \in G} x_g g^{-1} \). We have \( x + \overline{y} = \overline{x + y} \) and \( x \cdot \overline{y} = \overline{y} \cdot \overline{x} \). If \( H \leq G \) is a subgroup we define \( N(H) = \sum_{h \in H} h \).

**Lemma 5.6.** Let \( G \) be one of the groups in Theorem 1.1. Let \( E \) be a set of representatives of the orbits of edges in \( X_1^{OS}(G) \). Let \( X \) be an acyclic 2-complex obtained from \( X_1^{OS}(G) \) by attaching a free orbit of 2-cells along (the orbit of) a closed edge path \( \xi = (a_1 e_1^i, \ldots, a_n e_n^i) \) with \( e_i \in E \), \( a_i \in G \) and \( \varepsilon_i \in \{-1, 1\} \). Let \( G_e \) be the stabilizer of \( e \). Then it is possible to choose, for each

\[
\begin{array}{c}
| B \quad C_{q-1} \quad C_2 \\
D_{2(q-1)} \quad C_2 \quad D_{2(q+1)}
\end{array}
\]

\[
G = PSL_2(2^n).
\]

\[
\begin{array}{c}
| B \quad C_3 \quad A_4 \\
C_{q+1} \quad C_2 \quad C_2 \quad C_2
\end{array}
\]

\[
D_{q-1} \quad C_2 \quad D_{q+1}
\]

\[
G = PSL_2(q), \ q \equiv 19 \pmod{24}.
\]

\[
\begin{array}{c}
| B \quad C_3 \quad D_{q+1} \\
C_{q+1} \quad C_2 \quad C_3
\end{array}
\]

\[
D_{q-1} \quad C_2 \quad A_4
\]

\[
G = PSL_2(q), \ q \equiv 5 \pmod{24}.
\]

\[
\begin{array}{c}
| B \quad C_3 \quad C_{q-r+1} \times C_4 \\
C_{q-1} \quad C_4
\end{array}
\]

\[
D_{2(q-1)} \quad C_2 \quad C_{q+r+1} \times C_4
\]

\[
G = Sz(q), \ q = 2^n.
\]

\[
\begin{array}{c}
| B \quad C_3 \quad A_4 \\
C_{q+1} \quad C_2 \quad C_2 \quad C_2
\end{array}
\]

\[
D_{q-1} \quad C_2 \quad D_{q+1}
\]

\[
G = PSL_2(3^n), \ odd \ n.
\]

\[
\begin{array}{c}
| B \quad C_3 \quad D_{q+1} \\
C_{q+1} \quad C_2 \quad C_3
\end{array}
\]

\[
D_{q-1} \quad C_2 \quad A_4
\]

\[
G = PSL_2(q), \ q \equiv 11 \pmod{24}.
\]

\[
\begin{array}{c}
| B \quad C_3 \quad C_{q-r+1} \times C_4 \\
C_{q-1} \quad C_4
\end{array}
\]

\[
D_{2(q-1)} \quad C_2 \quad C_{q+r+1} \times C_4
\]

\[
G = Sz(q), \ q = 2^n.
\]
e ∈ E an element \( x_e \in \mathbb{Z}[G] \) such that
\[
1 = \sum_{i=1}^{n} \varepsilon_i a_i N(G_{e_i}) x_{e_i}.
\]
Therefore for any representation \( V \) of \( G \) we have \( V = \sum_{e \in E} s_e V^{G_e} \), where \( s_e = \sum_{i \in I_e} \varepsilon_i a_i \) and
\[I_e = \{ i : e_i = e \}.\]

**Proof.** We consider the cellular chain complex of \( X \) (which is a complex of left \( \mathbb{Z}[G] \)-modules). Let \( \alpha \) be the 2-cell attached along \( \xi \). We have isomorphisms \( C_2(X) \cong \mathbb{Z}[G] \) and \( C_1(X) \cong \bigoplus_{e \in E} \mathbb{Z}[G/G_e] \) given by \( \alpha \mapsto 1 \) and \( e \mapsto 1 \cdot G_e \) respectively. With these identifications, the differential \( d_2 : C_2(X) \to C_1(X) \) is given by \( d_2(1) = \sum_{i=1}^{n} \varepsilon_i a_i G_{e_i} = \sum_{e \in E} s_e G_e \). Now the differential \( d^2 : C^1(X; \mathbb{Z}) \to C^2(X; \mathbb{Z}) \) identifies with the map
\[
d^2 : \bigoplus_{e \in E} \mathbb{Z}[G/G_e] \to \mathbb{Z}[G]
\]
\[1 \cdot G_e \mapsto N(G_e)\overline{e}.
\]

Since \( X \) is acyclic, the differential \( d^2 \) is surjective and there are elements \( y_e \in \mathbb{Z}[G] \) such that
\[
1 = \sum_{e \in E} s_e N(G_e)\overline{e}.
\]
Finally, since \( \overline{N(H)} = N(H) \) and letting \( x_e = \overline{e} \) we have
\[
1 = \sum_{e \in E} s_e N(G_e) x_e.
\]

### 6. Representations and centralizers

The results in this section provide, for each of the groups \( G \) in Theorem B, a suitable irreducible representation \( \rho_0 \) of \( G \) in \( \mathbb{G} = \mathbb{U}(m) \). The values of \( m \) are recorded in Table 3.

| \( G \)       | \( q \)          | \( m \)          |
|--------------|------------------|------------------|
| \( \text{PSL}_2(q) \) | \( 2^n \)        | \( q - 1 \)      |
| \( \text{PSL}_2(q) \) | \( 3^n \) with \( n \) odd | \( \frac{q-1}{2} \) |
| \( \text{PSL}_2(q) \) | \( q \equiv 11 \) or \( 19 \) \( \text{mod} \) \( 24 \) | \( \frac{q-1}{2} \) |
| \( \text{Sz}(q) \)  | \( 2^n \) with \( n \) odd | \( \frac{1}{2}r(q-1) \) |

**Table 3.** The degree \( m \) of \( \rho_0 \) in each case.

For \( i = 0, 1 \) and each of the groups \( G \) in Theorem B, we fix a generator \( \hat{g}_i \) of \( G_n \). In the following propositions \( I_n \) denotes the \( n \times n \) identity matrix.

**Proposition 6.1.** Let \( G = \text{PSL}_2(q) \) with \( q = 2^n \) and \( n \geq 2 \). Let \( \mathbb{G} = \mathbb{U}(q-1) \). There is an irreducible representation \( \rho_0 : G \to \mathbb{G} \) satisfying the following properties:

(i) There is a matrix \( A_{\rho_0} \in \mathbb{G} \) such that
\[
A_{\rho_0} \rho_0(\hat{g}_0) A_{\rho_0}^{-1} = \text{diag}(\xi, \xi^2, \ldots, \xi^{q-1})
\]
where \( \xi = e^{\frac{2\pi i}{q-1}} \). Then
\[
C_{\mathbb{G}}(\rho_0(G_{\rho_0})) = A_{\rho_0}^{-1} \mathbb{U}(1)^{q-1} A_{\rho_0}
\]
has dimension \( q - 1 \).
(ii) There is a matrix $A_{n_0} \in \mathbb{G}$ such that we have
\[ A_{n_0} \rho_0(\hat{g}_1) A_{n_0}^{-1} = I_{\frac{n_0}{2}} \oplus -I_{\frac{n_0}{2}} \]
and therefore
\[ \mathbb{C}_G(\rho_0(G_{n_0})) = A_{n_0}^{-1} \left( U \left( \frac{q}{2} - 1 \right) \times U \left( \frac{q}{2} \right) \right) A_{n_0} \]
has dimension $\left( \frac{q}{2} - 1 \right)^2 + \left( \frac{q}{2} \right)^2$.
(iii) The centralizer $\mathbb{C}_G(\rho_0(G_{n_0}))$ has dimension $\left( \frac{q}{2} - 1 \right)^2 + \left( \frac{q}{2} \right)^2$.
(iv) The restriction of $\rho_0$ to the Borel subgroup $G_{n_0}$ is irreducible.
(v) The centralizer $\mathbb{C}_G(\rho_0(G_{n_1}))$ has dimension $\frac{q}{2}$.
(vi) The centralizer $\mathbb{C}_G(\rho_0(G_{n_1}))$ has dimension $\frac{q}{2}$.
(vii) The trivial representation $1_{G_{n_1}}$ does not occur in the restriction $\rho_0|_{G_{n_1}}$.

Proof. For $G = \text{PSL}_2(q)$ and $q = 2^n$ we take $\rho_0$ realizing the degree $\frac{q+1}{2}$ character $\theta_3$ of Theorem A.4.1. By Theorem 4.1, we can take $\rho_0$ to be unitary. By Lemma A.1.1 and Lemma 4.5 we can prove parts (i) to (vii) by computing inner products of the restrictions of $\theta_3$. These restrictions are computed using Proposition A.4.2. \qed

Proposition 6.2. Let $G = \text{PSL}_2(q)$ where $q \equiv 3 \pmod{8}$ and $q > 3$. Let $r = \sqrt{q/p}$ and let $G = U \left( \frac{2-1}{2} \right)$. There is an irreducible representation $\rho_0: G \to \mathbb{G}$ satisfying the following properties:

(i) There is a matrix $A_{n_0} \in \mathbb{G}$ such that
\[ A_{n_0} \rho_0(\hat{g}_1) A_{n_0}^{-1} = \text{diag}(1, \xi, \xi^2, \ldots, \xi^{\frac{q-1}{2}}) \]
where $\xi = e^{\frac{2\pi i}{q+1}}$. Then
\[ \mathbb{C}_G(\rho_0(G_{n_0})) = A_{n_0}^{-1} U \left( \frac{q+1}{2} \right) A_{n_0} \]
has dimension $\frac{q+1}{2}$.
(ii) There is a matrix $A_{n_1} \in \mathbb{G}$ such that
\[ A_{n_1} \rho_0(\hat{g}_1) A_{n_1}^{-1} = I_{\frac{n_1}{4}} \oplus -I_{\frac{n_1}{4}} \]
and therefore
\[ \mathbb{C}_G(\rho_0(G_{n_1})) = A_{n_1}^{-1} \left( U \left( \frac{q+1}{4} \right) \times U \left( \frac{q-3}{4} \right) \right) A_{n_1} \]
has dimension $\left( \frac{q+1}{4} \right)^2 + \left( \frac{q-3}{4} \right)^2$.
(iii) The centralizer $\mathbb{C}_G(\rho_0(G_{n_1}))$ has dimension $\left( \frac{q+1}{4} \right)^2 + 3\left( \frac{q-3}{8} \right)^2$.
(iv) The dimension of $\mathbb{C}_G(\rho_0(G_{n_1}))$ is given by
\[ \dim \mathbb{C}_G(\rho_0(G_{n_1})) = \begin{cases} (\frac{q+3}{6})^2 + (\frac{q+3}{6})^2 + (\frac{q-3}{6})^2, & \text{if } q \equiv 0 \pmod{3}, \\ 3\left(\frac{q+1}{6}\right)^2, & \text{if } q \equiv 1 \pmod{3}, \\ (\frac{q-5}{6})^2 + 2\left(\frac{q+1}{6}\right)^2, & \text{if } q \equiv 2 \pmod{3}. \end{cases} \]
(v) The restriction of $\rho_0$ to the Borel subgroup $G_{n_1}$ is irreducible.
(vi) The centralizer $\mathbb{C}_G(\rho_0(G_{n_1}))$ has dimension $\frac{q+1}{4}$.
(vii) The centralizer $\mathbb{C}_G(\rho_0(G_{n_1}))$ has dimension $\frac{q+1}{4}$.
(viii) The dimension of $C_G(\rho_0(G_{v_i}))$ is given by

$$\dim C_G(\rho_0(G_{v_i})) = \begin{cases} \frac{q^2 + 6q + 21}{48} & \text{if } q \equiv 0 \pmod{3} \\ \frac{q^2 - 2q + 13}{48} & \text{if } q \equiv 1 \pmod{3} \\ \frac{q^2 - 2q + 45}{48} & \text{if } q \equiv 2 \pmod{3}. \end{cases}$$

(ix) The nontrivial degree 1 representation of $G_{v_1} \simeq D_{q-1}$ does not occur in the restriction $\rho_0|_{G_{v_1}}$.

Proof. For $G = \text{PSL}_2(q)$ with $q \equiv 3 \pmod{8}$ we take $\rho_0$ by factoring a representation realizing the degree $\frac{q-1}{2}$ character $\eta_1$ of Theorem A.5.1 through the quotient $\text{SL}_2(q) \to \text{PSL}_2(q)$. By Theorem 4.1, we can take $\rho_0$ to be unitary. By Lemma A.1.1 and Lemma 4.5 we can prove parts (i) to (ix) by computing inner products of the restrictions of the restrictions of $\eta_1$. These restrictions are computed using Proposition A.5.3.

Proposition 6.3. Let $G = \text{Sz}(q)$ with $q = 2^n$ and $n \geq 3$ odd. Let $r = \sqrt{2q}$ and let $\mathbb{G} = U\left(\frac{r(q-1)}{2}\right)$. There is an irreducible representation $\rho_0 : G \to \mathbb{G}$ satisfying the following properties:

(i) There is a matrix $A_{n_0} \in \mathbb{G}$ such that

$$A_{n_0} \rho_0(\hat{g}_0) A_{n_0}^{-1} = \xi I_{r/2} \oplus \xi^2 I_{r/2} \oplus \ldots \oplus \xi^{q-2} I_{r/2} \oplus \xi^{q-1} I_{r/2},$$

where $\xi = e^{\frac{2\pi i}{q-1}}$. Then

$$C_G(\rho_0(G_{n_0})) = A_{n_0}^{-1} U(r/2)^{q-1} A_{n_0}$$

has dimension $\frac{2(q-1)}{2}$.

(ii) There is a matrix $A_{n_1} \in \mathbb{G}$ such that

$$A_{n_1} \rho_0(\hat{g}_1) A_{n_1}^{-1} = I_{r(q-2)/4} \oplus -I_{r_q/4}$$

and then

$$C_G(\rho_0(G_{n_1})) = A_{n_1}^{-1} (U(r(q-2)/4) \times U(rq/4)) A_{n_1}$$

has dimension $\frac{q(q^2 - 2q + 2)}{4}$. (viii) The trivial representation $1_{G_{v_1}}$ does not occur in the restriction $\rho_0|_{G_{v_1}}$.

Proof. For $G = \text{Sz}(q)$ we take $\rho_0$ realizing the degree $\frac{r(q-1)}{2}$ character $W_1$ of Proposition A.6.3. By Theorem 4.1, we can take $\rho_0$ to be unitary. By Lemma A.1.1 and Lemma 4.5 we can prove parts (i) to (viii) by computing inner products of the restrictions of $W_1$. These restrictions are computed using Proposition A.6.4.

7. The Dimension of $\overline{M}_k$

From now on, let $\mathcal{M}_k$ be the moduli of representations of $\Gamma_k$ obtained from the representation $\rho_0$ given by Propositions 6.1 to 6.3 using Theorem 3.2. Let $\overline{M}_k$ be the corresponding quotient obtained using Theorem 3.3. Note that $\mathcal{M}_k = \mathcal{M}_0 \times \mathbb{G}^k$ and that $\overline{M}_k = \overline{M}_0 \times \mathbb{G}^k$. From Corollary 3.4 we know that $\mathcal{M}_k$ and $\overline{M}_k$ are connected and orientable.
Proposition 7.1. For each of the groups $G$ in Theorem $B$, the dimension of $\overline{M}_k$ is equal to the dimension of $\mathcal{G}^{k+1}$.

Proof. This reduces to the case of $k = 0$ and by part (iii) of Theorem 3.3 we only have to verify that
\[
\sum_{e \in E} \dim C_G(\rho_0(G_e)) - \sum_{v \in V, v \neq v_0} \dim C_G(\rho_0(G_v)) = \dim \mathcal{G}.
\]
In each case, this follows from a computation with Propositions 6.1 to 6.3.

The following result, whose proof is due to Piterman, gives an alternative proof of Proposition 7.1 which sheds light on why $\dim \overline{M}_k = \dim \mathcal{G}^{k+1}$.

Lemma 7.2 (Piterman). Let $X$ be an acyclic 2-dimensional $G$-complex and let $\varphi, \psi$ be two characters of $G$. Let $V, E, F$ be representatives of the orbits of vertices, edges and 2-cells in $X$. Then
\[
\langle \varphi, \psi \rangle_G + \sum_{e \in E} \langle \text{Res}^G_{G_e} \varphi, \text{Res}^G_{G_e} \psi \rangle_{G_e} = \sum_{v \in V} \langle \text{Res}^G_{G_v} \varphi, \text{Res}^G_{G_v} \psi \rangle_{G_v} + \sum_{f \in F} \langle \text{Res}^G_{G_f} \varphi, \text{Res}^G_{G_f} \psi \rangle_{G_f}.
\]

Proof. Since $X$ is acyclic, $\tilde{C}_1(X; \mathbb{C}) \oplus \tilde{C}_0(X; \mathbb{C}) \simeq \tilde{C}_0(X; \mathbb{C}) \oplus \tilde{C}_2(X; \mathbb{C})$ as $G$-modules. Then, letting $\alpha_H$ be the character of the $G$-module $\mathbb{C}[G/H]$ we have
\[
\alpha_G + \sum_{e \in E} \alpha_{G_e} = \sum_{v \in V} \alpha_{G_v} + \sum_{f \in F} \alpha_{G_f}
\]
and now the result follows from Frobenius reciprocity:
\[
\langle \text{Res}^G_H \varphi, \text{Res}^G_H \psi \rangle_H = \langle \varphi, \text{Ind}_H^G \text{Res}^G_H \psi \rangle_G = \langle \varphi, \alpha_H \psi \rangle_G.
\]

Let $\chi$ be the character of $\rho_0$. Since $\langle \chi, \chi \rangle_G = 1$ and $\langle \text{Res}^G_{G_{v_0}} \chi, \text{Res}^G_{G_{v_0}} \chi \rangle_{G_{v_0}} = 1$, applying Lemma 7.2 to $\chi$ (and using Lemma 4.5) we obtain another proof of Proposition 7.1.

8. The differential of $\overline{W}$ at $\mathbf{1}$

For $i = 0, \ldots, k$ we consider the map $X_i: M \to \mathcal{G}$ induced by $x_i$.

Proposition 8.1. Let $X = (X_0, \ldots, X_k): M_k \to \mathcal{G}^{k+1}$. Then $1$ is a regular point of $X$.

Proof. The proof reduces to the case of $k = 0$. Consider the representation
\[
\text{Ad} \circ \rho_0: G \to \text{GL}(T_1 G)
\]
which (by Corollary 2.3) is given by $g \cdot v = d_{\rho_0(g)^{-1}}L_{\rho_0(g)} \circ d_1 R_{\rho_0(g)^{-1}}(v)$. By Proposition 2.4 we have $T_1 C_G(\rho_0(H)) = (T_1 G)^H$. Then by Lemma 5.6 we have $T_1 \mathcal{G} = \sum_{e \in E} s_e \cdot T_1 C_G(\rho_0(G_e))$. Now the result follows from Theorem 3.7.

Proposition 8.2. If $w_0, \ldots, w_k \in N$ satisfy $N = \langle [w_0, \ldots, w_k] \rangle^G [N, N]$, then $1$ is a regular point of $W = (W_0, \ldots, W_k): M_k \to \mathcal{G}^{k+1}$.

Proof. This is follows from Lemma 3.8 and Proposition 8.1.

Now since $\overline{W} \circ p = W$ we have:

Corollary 8.3. If $w_0, \ldots, w_k \in N$ satisfy $N = \langle [w_0, \ldots, w_k] \rangle^G [N, N]$, then $\mathbf{1}$ is a regular point of $W = (\overline{W}_0, \ldots, \overline{W}_k): \overline{M}_k \to \mathcal{G}^{k+1}$. □
9. The Degree of $\overline{W}$

In this section we prove the degree of $\overline{W}$ is 0. We start by recalling the definition and some properties of the degree (see e.g. [Lee13, Chapter 17] for a detailed exposition). Let $M$ and $M'$ be oriented $m$-manifolds. The degree $\deg(f)$ of a smooth map $f: M \to M'$ is the unique integer $k$ such that

$$\int_{M} f^* (\omega) = k \int_{M'} \omega$$

for every smooth $m$-form $\omega$ on $M'$. If $x \in M$ is a regular point of $f$, then $d_x f: T_x M \to T_{f(x)} M'$ is an isomorphism between oriented vector spaces and we can consider its sign $\sgn d_x f$. If $y \in M'$ is a regular value of $f$ we have

$$\deg(f) = \sum_{x \in f^{-1}(y)} \sgn d_x f.$$

In particular if $f$ is not surjective then $\deg(f) = 0$. Homotopic maps have the same degree. If $N$ and $N'$ are oriented $n$-manifolds and $g: N \to N'$ is a smooth map then $\deg(f \times g) = \deg(f) \deg(g)$). If $M''$ is an oriented $m$-manifold and $h: M' \to M''$ is smooth then $\deg(h \circ f) = \deg(h) \deg(f)$.

For $i = 0, \ldots, k$ we consider the map $Y_i: \mathcal{M} \to G$ induced by $y_i$. Table 4 gives the value of $Y_0$ in the different cases that we consider.

$$
\begin{array}{|c|c|c|}
\hline
G & q & Y_0(\tau) \\
\hline
\text{PSL}_2(q) & 2^n & \tau_{n_0}^{-1} \tau_{n_1}^{-1} \tau_{n_2} \rho_0(y_{n_2}) \\
\text{PSL}_2(q) & 3^n & \tau_{n_0}^{-1} \tau_{n_1}^{-1} \tau_{n_2} \rho_0(y_{n_2}) \\
\text{PSL}_2(q) & q \equiv 19 \pmod{24} & \tau_{n_0}^{-1} \tau_{n_1}^{-1} \tau_{n_2} \rho_0(y_{n_2}) \\
\text{PSL}_2(q) & q \equiv 11 \pmod{24} & \tau_{n_0}^{-1} \tau_{n_1}^{-1} \tau_{n_2} \rho_0(y_{n_2}) \\
\text{Sz}(q) & 2^n & \tau_{n_0}^{-1} \tau_{n_1}^{-1} \tau_{n_2} \rho_0(y_{n_2}) \\
\hline
\end{array}
$$

Table 4. The map $Y_0$, for each of the groups $G$ in Theorem B.

The proof goes by proving that $\overline{Y}_0: \mathcal{M}_0 \to G$ has degree 0. We do this by homotoping the map $Y_0$ to a map $Z: \mathcal{M}_0 \to G$ which is not surjective. This homotopy needs to be $\mathcal{H}$-equivariant. In order to define the map $Z$, we need to choose a matrix $C \in U(m') \times U(m'')$ with certain property. The numbers $m'$ and $m''$ satisfy $m = m' + m''$ and are defined in Table 5. Let $A_{n_0}$ and $A_{n_1}$ be the matrices in Propositions 6.1 to 6.3. By Lemma 4.7 and Proposition 4.8, in each case we can take a matrix $\tilde{C} \in U(m_1) \times U(m_2)$ such that $A_{n_0}^{-1} \tilde{C} A_{n_1}$ commutes with $C_G(G_{v_1})$. We set $C = A_{n_1}^{-1} \tilde{C} A_{n_1} \in C_G(G_{v_1})$.

$$
\begin{array}{|c|c|c|c|}
\hline
G & q & m' & m'' \\
\hline
\text{PSL}_2(q) & 2^n & q/2 + 1 & q/2 \\
\text{PSL}_2(q) & 3^n & (q + 1)/4 & (q - 3)/4 \\
\text{PSL}_2(q) & q \equiv 19 \pmod{24} & (q + 1)/4 & (q - 3)/4 \\
\text{PSL}_2(q) & q \equiv 11 \pmod{24} & (q + 1)/4 & (q - 3)/4 \\
\text{Sz}(q) & 2^n & r(q - 2)/4 & r q/4 \\
\hline
\end{array}
$$

Table 5. The numbers $m'$ and $m''$.

Proposition 9.1. In all cases the map $Z: \mathcal{M}_0 \to G$ of Table 6 is not surjective.
\[
\begin{array}{|c|c|c|c|}
\hline
G & q & Z(\tau) & T(\tau) \\
\hline
\text{PSL}_2(q) & 2^n & (A^{-1}_{0}A_{0})\tau_0^{-1}(A_{0}^{-1}A_{0})^{-1}C & \tau_1^{-1}\tau_2^{-1}\rho_1(g_{r_2}) \\
\text{PSL}_2(q) & 3^n & (A^{-1}_{0}A_{0})\tau_0^{-1}(A_{0}^{-1}A_{0})^{-1}C & \tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\rho_1(g_{r_3}) \\
\text{PSL}_2(q) & q \equiv 19 \text{ (mod 24)} & (A^{-1}_{0}A_{0})\tau_0^{-1}(A_{0}^{-1}A_{0})^{-1}C & \tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\rho_1(g_{r_3}) \\
\text{PSL}_2(q) & q \equiv 11 \text{ (mod 24)} & (A^{-1}_{0}A_{0})\tau_0^{-1}(A_{0}^{-1}A_{0})^{-1}C & \tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\rho_1(g_{r_3}) \\
\text{Sz}(q) & 2^n & (A^{-1}_{0}A_{0})\tau_0^{-1}(A_{0}^{-1}A_{0})^{-1}C & \tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\rho_1(g_{r_3}) \\
\hline
\end{array}
\]

Table 6. The definition of the maps \( Z \) and \( T \) for each of the groups \( G \) in Theorem B.

\textbf{Proof.} In each case, we consider the manifold \( M = \prod_{i>0} C_G(\rho_0(G_{i})) \) and the Lie group \( H = \prod_{i>1} C_G(\rho_0(G_{i})) \). The action \( M_0 \curvearrowright \mathcal{H} \) restricts to a free action of \( H \leq \mathcal{H} \) on the factor \( M \) of \( M_0 \). Now consider the map \( T : M \to G \) defined in Table 6. Note that \( T \) factors through the quotient \( M \to M/H \) giving a map \( \tilde{T} : M/H \to \mathbb{G} \). By Theorem 2.7, \( M/H \) is a manifold and we have

\[
\dim M/H = \dim M - \dim H \\
= \dim M_0 - \dim \mathcal{H} + \dim C_G(\rho_0(G_{v})) - \dim C_G(\rho_0(G_{v})) \\
< \dim M_0 - \dim \mathcal{H} \\
= \dim \mathbb{G}.
\]

Therefore \( T \) is not surjective (this is a corollary to Sard’s Theorem, see [Lee13, Corollary 6.11]). Now note that

\[
A_{m} C_G(\rho_0(G_{v})) A_{m}^{-1} \subseteq U(m_1) \times U(m_2) = A_{n} C_G(\rho_0(G_{i})) A_{n}^{-1}
\]

and then \( A_{m}^{-1} A_{m} C_G(\rho_0(G_{v}))(A_{m}^{-1} A_{m})^{-1} \subseteq C_G(\rho_0(G_{v})) \). Moreover since \( \tilde{C} \in U(m') \times U(m'') \) we have \( C = A_{m}^{-1} \tilde{C} A_{n} \in C_G(\rho_0(G_{v})) \). To conclude, note that the image of \( Z \) equals the image of \( T \).

\textbf{Proposition 9.2.} For each of the groups \( G \) in Theorem B, the degree of \( \overline{Y}_0 : \overline{M}_0 \to \mathbb{G} \) is 0.

\textbf{Proof.} By Proposition 3.11 (and Theorem 2.5) the map \( Y_0 : M_0 \to \mathbb{G} \) is \( \mathcal{H} \)-equivariantly homotopic to the map \( Z \) defined in Table 6. Passing to the quotient we see that there is a homotopy between the maps \( \overline{Y}_0, Z : \overline{M}_0 \to \mathbb{G} \). By Proposition 9.1, \( Z \) is not surjective and therefore \( \overline{Z} \) is not surjective. We conclude the degree of \( \overline{Y}_0 \) is 0.

\textbf{Remark 9.3.} In the power of 2 and the Suzuki cases, with some more work we can use Proposition 3.11 to deform \( \overline{Y}_0 \) into a map \( \overline{Z} \) with image \( (U(m') \times U(m'')) \cdot \rho_0(g_{r_3}) \). This can be achieved by extending Propositions 6.1 and 6.3 and inserting additional matrices in the definition of \( Z \) (which again can be obtained using Lemma 4.7).

\textbf{Corollary 9.4.} The degree of \( \overline{Y} = (\overline{Y}_0, \ldots, \overline{Y}_k) : \overline{M}_k \to \mathbb{G}^{k+1} \) is 0.
Proof. We have $\overline{M}_k = \overline{M}_0 \times G^k$. Now, by Proposition 9.2, the map $\overline{Y}: \overline{M}_0 \times G^k \to G^{k+1}$ has degree 0 since it is the product of the maps $\overline{Y}_0: \overline{M}_0 \to G$ and the identity maps $\overline{Y}_i: G \to G$ for $i = 1, \ldots, k$. □

Proposition 9.5. Let $w_0, \ldots, w_k \in \Gamma_k$ and let $\overline{W} = (\overline{W}_0, \ldots, \overline{W}_k): \overline{M}_k \to G^{k+1}$. Then $\deg(\overline{W}) = 0$ ∈ $\mathbb{Z}$.

Proof. First note that, by Lemma 3.9 (and Theorem 2.5), we only need to address the case when the $w_i$ are words in the generators $y_0, \ldots, y_k$. Now consider the map $\overline{Y} = (\overline{Y}_0, \ldots, \overline{Y}_k)$ and consider the map $\overline{W}: G^{k+1} \to G^{k+1}$ induced by the words $w_0, \ldots, w_k \in F(y_0, \ldots, y_k)$, which makes the following diagram commute

$\overline{M}_k \xrightarrow{\overline{Y}} G^{k+1} \xrightarrow{\overline{W}} G^{k+1}$

By Corollary 9.4 $\overline{Y}$ has degree 0 and since $\deg(\overline{W}) = \deg(\overline{W}) \cdot \deg(\overline{Y})$ we are done. □

10. Group actions on contractible 2-complexes

We now prove the main results of this article.

Theorem 10.1. Let $G$ be one of the groups in Theorem B. Let $w_0, \ldots, w_k \in N$. If $N = \langle w_0, \ldots, w_k \rangle^{\Gamma_k}[N, N]$ then there is a point $\tau \in \overline{M}_k$ such that

(i) $p_\tau(w_i) = 1$ for $i = 0, \ldots, k$; and
(ii) $p_\tau$ is not universal.

Proof. By Proposition 9.5 the degree of $\overline{W}$ is 0. By Corollary 8.3, $\mathbf{1}$ is a regular point of $\overline{W}$. Therefore, there must exist a point $\tau \in \overline{W}^{-1}(1)$ with $\tau \neq \mathbf{1}$. To conclude note that by Proposition 3.5, $\tau$ is not universal. □

Theorem 10.2 ([SC20, Theorem 3.6]). Let $G$ be one of the groups in Theorem 1.1. Let $X$ be a fixed point free 2-dimensional finite acyclic $G$-complex. Then there is a fixed point free 2-dimensional finite acyclic $G'$-complex $X'$ obtained from the $G$-graph $X_1^{OS}(G)$ by attaching $k \geq 0$ free orbits of 1-cells and $k + 1$ free orbits of 2-cells and an epimorphism $\pi_1(X) \to \pi_1(X')$.

Proof of Theorem B. By Theorem 10.2 it is enough to prove the result when $X$ is obtained from $X_1^{OS}(G)$ by attaching $k \geq 0$ free orbits of 1-cells and $k + 1$ free orbits of 2-cells. By Theorem 3.1, there are words $w_0, \ldots, w_k \in N$ such that $\pi_1(X) \simeq \frac{N}{\langle w_0, \ldots, w_k \rangle^{\Gamma_k}}$ and since $H_1(X) = 0$ we have $N = \langle w_0, \ldots, w_k \rangle^{\Gamma_k}[N, N]$. Now passing to the quotient the representation $p_\tau$ given by Theorem 10.1 we obtain a nontrivial representation $\pi_1(X) \to \mathbf{U}(m)$. □

Theorem 10.3 ([SC20, Theorem 3.8]). Let $P(G)$ be the following proposition: “there is a nontrivial representation in $\mathbf{U}(m)$ of the fundamental group of every acyclic $G$-complex obtained from $X_1^{OS}(G)$ by attaching $k \geq 0$ free orbits of 1-cells and $(k + 1)$ free orbits of 2-cells”. To prove Theorem A it is enough to prove $P(G)$ for each of the following groups $G$:

- $\text{PSL}_2(2^p)$ for $p$ prime;
- $\text{PSL}_2(3^p)$ for an odd prime $p$;
- $\text{PSL}_2(q)$ for a prime $q > 3$ such that $q \equiv \pm 3 \pmod{5}$ and $q \equiv \pm 3 \pmod{8}$;
- $\text{Sz}(2^p)$ for $p$ an odd prime.
Remark 10.4. Note that in [SC20], Theorem 10.3 is proved for representations in SO(m). However the same proof gives the unitary version: since the Gerstenhaber–Rothaus theorem holds for any compact connected Lie group [GR62], we have that [SC20, Proposition A.3] also holds for unitary representations. Also note that the second part of Theorem A is not addressed in the original statement of Theorem 10.3. Nevertheless, this also is immediate from the original proof of Theorem 10.3.

Proof of Theorem A. By Theorem 10.3, the result now follows from Theorem B and Theorem C.

Appendix A. Representation theory of PSL_2(q) and Sz(q)

by Kevin Piterman

A.1. Summary on representation theory for finite groups. In this subsection, we give a brief summary on representation theory for finite groups. The main references are [Dor71] and [Ser77].

Let $G$ be a finite group. We will work with the field $\mathbb{C}$ of complex numbers. Denote by $\overline{x}$ the complex conjugate of an element $x \in \mathbb{C}$.

By Maschke’s theorem, the group algebra $\mathbb{C}[G]$ is semisimple. Hence every $\mathbb{C}[G]$-module splits as a direct sum of irreducible (or simple) $\mathbb{C}[G]$-modules (that is, with no nontrivial and proper $\mathbb{C}[G]$-submodule). Recall also that a $\mathbb{C}[G]$-module $V$ is the same as a representation of $G$ on a $\mathbb{C}$-vector space $V$, which is group homomorphism $\rho: G \to \text{Aut}_{\mathbb{C}}(V)$. Here $\text{Aut}_{\mathbb{C}}(V)$ denotes the group of $\mathbb{C}$-linear automorphisms of $V$. We make no distinction and we denote such a representation by $\rho$ or $V$. The degree of the representation $\rho$ is the dimension of $V$ as $\mathbb{C}$-vector space. The character of $\rho$ is the function $\chi_{\rho}: G \to \mathbb{C}$ such that $\chi_{\rho}(g)$ equals the trace of $\rho(g)$, for $g \in G$. Note that characters are constant on conjugacy classes, and that the degree of $\rho$ equals the value of $\chi_{\rho}$ at $1 \in G$. From the theory of characters, the representation $\rho$ is completely determined by $\chi_{\rho}$. This means that two $\mathbb{C}[G]$-modules $V, W$ give rise to the same character if and only if $V$ and $W$ are isomorphic. A character $\chi$ of $G$ is irreducible if it is the character of an irreducible $\mathbb{C}[G]$-module.

Let $\text{Irr}(G)$ denote the set of irreducible characters of $G$ and let $\mathcal{C}(G)$ denote the set of functions $\alpha: G \to \mathbb{C}$ which are constant on the conjugacy classes of $G$. Then $\mathcal{C}(G)$ is a $\mathbb{C}$-vector space whose dimension is equal to the number of conjugacy classes of $G$. There is an inner product in $\mathcal{C}(G)$ defined as follow. If $\alpha, \beta \in \mathcal{C}(G)$, then

$$\langle \alpha, \beta \rangle_G := \frac{1}{|G|} \sum_{g \in G} \alpha(g) \overline{\beta(g)}.$$ 

By Schur’s orthogonality relations, $\text{Irr}(G)$ is an orthonormal basis for this inner product, from which it follows that $|\text{Irr}(G)|$ equals the number of conjugacy classes of $G$. In particular, the character of every representation $\rho: G \to \text{Aut}_{\mathbb{C}}(V)$ can be uniquely written as a linear combination of irreducible characters, with non-negative integer coefficients. Let $\text{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$ and denote by $V_i$ the irreducible module associated to $\chi_i$.

If $\rho$ is a representation that splits as a direct sum $V = \bigoplus_{i=1}^{r} V_i^{n_i}$ with $n_i \geq 0$, then the character of $V$ is the linear combination $\chi = \sum_{i=1}^{r} n_i \chi_i$. Since $\text{Irr}(G)$ forms an orthonormal basis, the coefficients $n_i$ for an arbitrary representation $\rho$ can be computed by taking the inner
products \( n_i = \langle \chi_{r^i}, \chi_i \rangle \). In particular, we have the following useful lemma to detect irreducible characters.

**Lemma A.1.1.** Let \( G \) be a finite group and \( \chi: G \to \mathbb{C} \) be a character. Then \( \langle \chi, \chi \rangle = 1 \) if and only if \( \chi \) is irreducible.

Below we state Schur’s lemma. Recall from the Frobenius theorem that \( \mathbb{C} \) is the only division algebra (of finite dimension) over \( \mathbb{C} \).

**Theorem A.1.2** (Schur’s Lemma). Let \( V, W \) be two irreducible \( \mathbb{C}[G] \)-modules. Then every \( G \)-linear map between \( V \) and \( W \) is either the zero map or an isomorphism. Therefore the following assertions hold:

- If \( V \) and \( W \) are not isomorphic, then the zero map is the unique \( G \)-linear map between them.
- The endomorphism algebra \( \text{End}_{\mathbb{C}[G]}(V) \) is a division algebra over \( \mathbb{C} \). Hence, the elements of \( \text{End}_{\mathbb{C}[G]}(V) \) are the scalar multiples of the identity map.

Note that an element \( x \in \text{Aut}_\mathbb{C}(V) \) satisfies \( x\rho(g) = \rho(g)x \) for all \( g \in G \) if and only if \( x \) preserves the action of \( G \). That is, \( \mathbb{C}_{\text{Aut}_\mathbb{C}(V)}(G) \) is exactly the group of \( \mathbb{C}[G] \)-automorphisms of \( V \). This centralizer can be computed by using Schur’s lemma.

A.2. **Characters of cyclic groups.** The following proposition computes the irreducible characters of cyclic groups.

**Proposition A.2.1.** Let \( C_n \) be a cyclic group of order \( n \) generated by an element \( g \). The irreducible characters for \( C_n \) are \( \mu_k \) for \( k = 1, \ldots, n \), and they are defined as follows. Let \( \mu = e^{2\pi i/n} \). Then the irreducible character \( \mu_k \) has degree 1 and is given by \( g^i \mapsto \mu^{ki} \).

A.3. **Characters of dihedral group \( D_{2n} \) (\( n \) odd).** Recall that the dihedral group of order \( 2n \), denoted by \( D_{2n} \), is the group generated by an involution \( s \) and an element \( r \) of order \( n \) such that \( srs = r^{-1} \).

**Proposition A.3.1** ([Ser77, Section 5.3]). Let \( n \) be an odd number and let \( \mu = e^{\frac{2\pi i}{n}} \). Then the character of table of \( D_{2n} \) is

|     | 1   | \( \mu^k \) | \( sp^k \) |
|-----|-----|------------|------------|
| \( \psi_1 \) | 1   | 1          | 1          |
| \( \psi_2 \) | 1   | 1          | -1         |
| \( \chi_i \) | 2   | \( \mu^{ik} + \mu^{-ik} \) | 0          |

Here, \( i = 1, \ldots, \frac{n-1}{2} \).

A.4. **Linear groups in even characteristic.** We review the conjugacy classes and irreducible characters of the groups \( SL_2(q) \) and \( PSL_2(q) \).

Recall that \( PSL_2(q) \) is the quotient of \( SL_2(q) \) by its center, which is \( \mathbb{Z}(SL_2(q)) = \{ 1, -1 \} \) if \( q \) is odd, and \( \mathbb{Z}(SL_2(q)) = 1 \) if \( q \) is even. By composing with the quotient map \( SL_2(q) \to PSL_2(q) \), we see that every character of \( PSL_2(q) \) arises from a character of \( SL_2(q) \) whose kernel contains \( \mathbb{Z}(SL_2(q)) \) (recall that the kernel of a character of a group \( G \) is the kernel of the associated group homomorphism \( G \to \text{Aut}_\mathbb{C}(V) \)). Furthermore, it is known that the irreducible characters \( \chi \) of \( PSL_2(q) \) are exactly those arising from the irreducible characters \( \tilde{\chi} \) of \( SL_2(q) \) containing \( \mathbb{Z}(SL_2(q)) \) in their kernel.
We recall below the conjugacy classes and character tables for linear groups defined over a field of even characteristic. For any \( x \in G \), let \( (x) \) denote the conjugacy class of \( x \).

**Theorem A.4.1** ([Dor71, Theorem 38.2]). Let \( F \) be a finite field of \( q = 2^f \) elements, and let \( \nu \) be a generator of the cyclic group \( F^* = F - 0 \). Denote
\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}
\]
in \( G = \text{SL}_2(F) \).

Then \( |a| = q - 1, |c| = 2, G \) contains an element \( b \) of order \( q + 1 \) and there are exactly \( q + 1 \) conjugacy classes
\[
(1), (c), (a), (a^2), \ldots, (a^{q^{-2}}), (b), (b^2), \ldots, (b^{q^{-2}}),
\]
such that
\[
\begin{array}{c|cccc}
 x & 1 & c & a^l & b^m \\
\hline
|\langle x \rangle| & 1 & q^2 - 1 & q(q + 1) & q(q - 1) \\
\end{array}
\]
for \( 1 \leq l \leq q^{-2}, 1 \leq m \leq \frac{q}{2} \).

Let \( \rho \in \mathbb{C} \) be a primitive \((q-1)\)-th root of unity, \( \sigma \in \mathbb{C} \) a primitive \((q+1)\)-th root of unity. Then the character table of \( G \) over \( \mathbb{C} \) is
\[
\begin{array}{ccccc}
 & 1 & c & a^l & b^m \\
1_G & 1 & 1 & 1 & 1 \\
\psi & q & 0 & 1 & -1 \\
\chi_i & q + 1 & 1 & \rho^i + \rho^{-i} & 0 \\
\theta_j & q - 1 & -1 & 0 & -(\sigma^{jm} + \sigma^{-jm}) \\
\end{array}
\]
for \( 1 \leq i \leq \frac{q-2}{2}, 1 \leq j \leq \frac{q}{2}, 1 \leq l \leq \frac{q-2}{2}, 1 \leq m \leq \frac{q}{2} \).

The following proposition can be deduced by using Theorem A.4.1 and a structure description of the groups listed in the first column.

**Proposition A.4.2.** Let \( G = \text{SL}_2(2^n) \). The following table shows the size of \( (x) \cap L \), for each conjugacy class \( (x) \) of \( G \).

| \( L \) | \( B \) | \( D_{2(q-1)} \) | \( D_{2(q+1)} \) | \( C_{q-1} \) | \( C_2 \) |
|---|---|---|---|---|---|
| \( B \) | 1 & \( q - 1 \) & 2q & 0 |
| \( D_{2(q-1)} \) | 1 & \( q - 1 \) & 2 & 0 |
| \( D_{2(q+1)} \) | 1 & \( q + 1 \) & 0 & 2 |
| \( C_{q-1} \) | 1 & 0 & 2 & 0 |
| \( C_2 \) | 1 & 1 & 0 & 0 |

Here \( 1 \leq l \leq \frac{q-2}{2} \) and \( 1 \leq m \leq \frac{q}{2} \).

A.5. **Linear groups in odd characteristic.** We provide here similar results for the linear groups in odd characteristic.

**Theorem A.5.1** ([Dor71, Theorem 38.1]). Let \( F \) be the finite field of \( q = p^n \) elements, \( p \) odd prime, and let \( \nu \) be a generator of the cyclic group \( F^* = F - 0 \). Denote
\[
\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
in $G = \text{SL}_2(F)$.

Then $|a| = q - 1$, $G$ has an element $b$ of order $q + 1$ and there are exactly $q + 4$ conjugacy classes:

$$(1), (z), (c), (d), (zc), (a), (a^2), \ldots, (a^{q-3}), (b), (b^2), \ldots, (b^{q-1})$$

such that

$$
\begin{array}{ccccccc}
 x & |(x)| & z & c & d & zc & zd & a^l & b^m \\
1_G & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\psi & q & q & 0 & 0 & 1 & -1 & -1 \\
\chi_i & q + 1 & (-1)^i(q + 1) & 1 & 1 & \rho^{il} + \rho^{-il} & 0 & 0 \\
\theta_j & q - 1 & (-1)^j(q - 1) & -1 & -1 & 0 & - (\sigma^{jm} + \sigma^{-jm}) & 0 \\
\xi_1 & \frac{1}{2}(q + 1) & \frac{1}{2}(q + 1) & \frac{1}{2}(1 + \sqrt{eq}) & \frac{1}{2}(1 - \sqrt{eq}) & (-1)^{j} & 0 & 0 \\
\xi_2 & \frac{1}{2}(q + 1) & \frac{1}{2}(q + 1) & \frac{1}{2}(1 - \sqrt{eq}) & \frac{1}{2}(1 + \sqrt{eq}) & (-1)^{j} & 0 & 0 \\
\eta_1 & \frac{1}{2}(q - 1) & -\frac{1}{2}(q - 1) & \frac{1}{2}(-1 + \sqrt{eq}) & \frac{1}{2}(-1 - \sqrt{eq}) & 0 & (-1)^{m+1} & 0 \\
\eta_2 & \frac{1}{2}(q - 1) & -\frac{1}{2}(q - 1) & \frac{1}{2}(-1 - \sqrt{eq}) & \frac{1}{2}(-1 + \sqrt{eq}) & 0 & (-1)^{m+1} & 0 \\
\end{array}
$$

with $1 \leq l \leq 2^{q-3}/2$, $1 \leq m \leq 2^{q-1}/2$.

Denote $\epsilon = (-1)^{2^{q-3}/2}$. Let $\rho \in \mathbb{C}$ be a primitive $(q - 1)$-th root of unity, $\sigma \in \mathbb{C}$ a primitive $(q + 1)$-th root of unity. Then the complex character table of $G$ is

$$
\begin{array}{ccccccc}
 \chi(zc) & \chi(1) & \chi(c) & \chi(d) & \chi(zd) & \chi(1) & \chi(d) \\
\end{array}
$$

for all irreducible character $\chi$ of $G$.

**Remark A.5.2.** By [Dor71, p.234], if $q \equiv 3 \pmod{4}$, then $c^{-1} \in (d)$, $(zc)^{-1} \in (zd)$, $d^{-1} \in (c)$ and $(zd)^{-1} \in (zc)$.

Note also that the unique element of order 2 in $\text{SL}_2(q)$ is $z$, and that every element of order 4 in $\text{SL}_2(q)$ is conjugate to $b^{2^{q+1}}$ if $q \equiv 3 \pmod{4}$. By the above assertion on the inverse of the elements, we see that there is a unique class of involutions in the quotient $\text{PSL}_2(q)$, corresponding to the involution $b^{2^{q+1}}$.

The following proposition can be deduced from Theorem A.5.1, Remark A.5.2 and a structure description of the groups listed in the first column.

We denote by $\overline{x}$ the image of an element $x \in \text{SL}_2(q)$ in the quotient $\text{PSL}_2(q)$. Recall that if $q = 3^n$, $n$ odd, then $q \equiv 3 \pmod{4}$. Denote by $\delta_{x,y}$ the Kronecker delta function, which is 1 if $x = y$ and 0 otherwise.

**Proposition A.5.3.** Let $G = \text{PSL}_2(q)$ with $q \equiv 3 \pmod{4}$. The following table shows the size of $(x) \cap L$, for each conjugacy class $(x)$ of $G$. 

| $\overline{x}$ | 1 | $z$ | $c$ | $d$ | $zd$ | $a^l$ | $b^m$ |
|---------------|---|-----|-----|-----|------|-------|-------|
| $1_G$         | 1 | 1   | 1   | 1   | 1    | 1     | 1     |
| $\psi$        | $q$ | $q$ | 0   | 0   | 1    | -1    | -1    |
| $\chi_i$      | $q + 1$ | $(-1)^i(q + 1)$ | 1 | 1 | $\rho^{il} + \rho^{-il}$ | 0 |
| $\theta_j$    | $q - 1$ | $(-1)^j(q - 1)$ | -1 | -1 | 0 | $-(\sigma^{jm} + \sigma^{-jm})$ |
| $\xi_1$       | $\frac{1}{2}(q + 1)$ | $\frac{1}{2}(q + 1)$ | $\frac{1}{2}(1 + \sqrt{eq})$ | $\frac{1}{2}(1 - \sqrt{eq})$ | $(-1)^{j}$ | 0 |
| $\xi_2$       | $\frac{1}{2}(q + 1)$ | $\frac{1}{2}(q + 1)$ | $\frac{1}{2}(1 - \sqrt{eq})$ | $\frac{1}{2}(1 + \sqrt{eq})$ | $(-1)^{j}$ | 0 |
| $\eta_1$      | $\frac{1}{2}(q - 1)$ | $-\frac{1}{2}(q - 1)$ | $\frac{1}{2}(-1 + \sqrt{eq})$ | $\frac{1}{2}(-1 - \sqrt{eq})$ | 0 | $(-1)^{m+1}$ |
| $\eta_2$      | $\frac{1}{2}(q - 1)$ | $-\frac{1}{2}(q - 1)$ | $\frac{1}{2}(-1 - \sqrt{eq})$ | $\frac{1}{2}(-1 + \sqrt{eq})$ | 0 | $(-1)^{m+1}$ |
The following table gives the irreducible characters for $q$-powers of a primitive element of order $q-1$.

| $L$ | $B$ | $C_2$ | $C_2 \times C_2$ | $C_{q-1}$ | $C_{q+1}$ | $C_{q-1}$ | $C_{q+1}$ |
|-----|-----|-----|----------------|-----------|-----------|-----------|-----------|
| $A_1 : q \equiv 0 \pmod{3}$ | 1 | 0 | 1 | 1 | 1 | 1 | 0 |
| $q \equiv 1 \pmod{3}$ | 1 | 0 | 0 | 8 \delta_l \frac{q-1}{2} | 0 | 0 | 8 \delta_l \frac{q+1}{2} |
| $q \equiv 2 \pmod{3}$ | 1 | 0 | 0 | 0 | 8 \delta_l \frac{q-1}{2} | 3 | 3 |

Here $1 \leq l \leq \frac{q^3-3}{4}$ and $1 \leq m \leq \frac{q^3-3}{4}$.

A.6. The Suzuki groups. In this section we recall the basic facts on the conjugacy classes and irreducible characters of the Suzuki groups $G = Sz(q)$, where $q$ is an odd power of 2. Let $r := \sqrt{2q}$. The following results are due to Suzuki [Suz62].

Proposition A.6.1 (Maximal subgroups). The solvable maximal subgroups of $Sz(q)$ are:

1. $B$, the Borel subgroup, of order $q^2(q-1)$ and normalizer of a Sylow 2-subgroup.
2. $D_{2(q-1)}$, dihedral of order $2(q-1)$ and normalizer of a cyclic subgroup of order $q-1$.
3. $C_+ \simeq C_{q+r+1} \rtimes C_4$, normalizer of a cyclic subgroup of order $q+r+1$.
4. $C_- \simeq C_{q-r+1} \rtimes C_4$, normalizer of a cyclic subgroup of order $q-r+1$.

Proposition A.6.2 (Conjugacy classes). Let $G = Sz(q)$, with $q = 2^n$ and $n$ odd. There are $q+3$ conjugacy classes of elements in $G$, and they are given as follows. Let $\sigma, \rho, \pi_0, \pi_1, \pi_2 \in G$ be elements of order 2, 4, $q-1$, $q+r+1$ and $q-r+1$ respectively. Then the conjugacy classes are (1), ($\sigma$), ($\rho$), ($\rho^{-1}$), ($\pi_0^a$) ($1 \leq a \leq \frac{q-2}{2}$), ($\pi_1^b$) ($1 \leq b \leq \frac{q+r}{2}$) and ($\pi_2^c$) ($1 \leq c \leq \frac{q-r}{2}$).

In Proposition A.6.3 we reproduce the character table of the Suzuki group $Sz(q)$, following the notation and results from [Suz62, Section 17].

Proposition A.6.3 (Character table). Let $G = Sz(q)$, with $q = 2^n$ and $n$ odd. Let $\epsilon_0$ be a primitive $(q-1)$-th root of unity. For $1 \leq i \leq \frac{q-2}{2}$, define the function $\epsilon_i^0$ in the powers of $\pi_0$ as follows:

$$\epsilon_i^0(\pi_0^a) = \epsilon_0^{ia} + \epsilon_0^{-ia}.$$  

Let $\epsilon_1$ be a primitive $(q+r+1)$-th root of unity. For $1 \leq j \leq \frac{q+r}{2}$, define the function $\epsilon_j^1$ in the powers of $\pi_1$ as follows:

$$\epsilon_j^1(\pi_1^b) = \epsilon_1^{jb} + \epsilon_1^{-jb} + \epsilon_1^{-jbq}.$$  

Let $\epsilon_2$ be a primitive $(q-r+1)$-th root of unity. For $1 \leq k \leq \frac{q-r}{2}$, define the function $\epsilon_k^2$ in the powers of $\pi_2$ as follows:

$$\epsilon_k^2(\pi_2^c) = \epsilon_2^k + \epsilon_2^{-kc} + \epsilon_2^{-kc} + \epsilon_2^{-kcq}.$$  

The following table gives the irreducible characters for $G$. 

| $L$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_0^a$ | $\pi_1^b$ | $\pi_2^c$ |
|-----|-------|-------|-------|-------|-------|-------|
| $A_1 : q \equiv 0 \pmod{3}$ | 1 | 0 | 0 | 1 | 0 | 0 |
| $q \equiv 1 \pmod{3}$ | 1 | 0 | 0 | 8 \delta_l \frac{q-1}{2} | 0 | 0 |
| $q \equiv 2 \pmod{3}$ | 1 | 0 | 0 | 0 | 8 \delta_l \frac{q-1}{2} | 3 | 3 |

Here $1 \leq l \leq \frac{q^3-3}{4}$ and $1 \leq m \leq \frac{q^3-3}{4}$. 

| $L$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_0^a$ | $\pi_1^b$ | $\pi_2^c$ |
|-----|-------|-------|-------|-------|-------|-------|
| $A_1 : q \equiv 0 \pmod{3}$ | 1 | 0 | 0 | 1 | 0 | 0 |
| $q \equiv 1 \pmod{3}$ | 1 | 0 | 0 | 8 \delta_l \frac{q-1}{2} | 0 | 0 |
| $q \equiv 2 \pmod{3}$ | 1 | 0 | 0 | 0 | 8 \delta_l \frac{q-1}{2} | 3 | 3 |
Here \(1 \leq a, i \leq \frac{q-2}{2}, 1 \leq b, j \leq \frac{q+r}{4}\) and \(1 \leq c, k \leq \frac{q-r}{4}\).

In the following proposition, we compute the size of \((x) \cap L\) for each conjugacy class \((x)\) of the Suzuki group \(G = Sz(q)\). The proof of this proposition follows from the structure description of each one of the groups appearing in the first column. See [Suz62] for more details on these descriptions.

**Proposition A.6.4.** Let \(G = Sz(q)\). The following table shows the size of \((x) \cap L\), for each conjugacy class \((x)\) of \(G\).

| \(L\) | 1 \(\sigma\) \(\rho\) \(\rho^{-1}\) \(\pi_0^a\) \(\pi_1^b\) \(\pi_2^c\) |
|--------|--------|--------|--------|--------|--------|--------|
| \(B\)  | 1 \(q-1\) \(\frac{q}{2}(q-1)\) \(\frac{q}{2}(q-1)\) \(2q^2\) \(0\) \(0\) |
| \(D_{2(q-1)}\) | 1 \(q-1\) \(0\) \(0\) \(2\) \(0\) \(0\) |
| \(C_+\) | 1 \(q+r+1\) \(q+r+1\) \(q+r+1\) \(0\) \(4\) \(0\) |
| \(C_-\) | 1 \(q-r+1\) \(q-r+1\) \(q-r+1\) \(0\) \(4\) \(0\) |
| \(C_{q-1}\) | 1 \(0\) \(0\) \(0\) \(2\) \(0\) \(0\) |
| \(C_4\) | 1 \(1\) \(1\) \(1\) \(0\) \(0\) \(0\) |
| \(C_2\) | 1 \(1\) \(0\) \(0\) \(0\) \(0\) \(0\) |

Here \(1 \leq a \leq \frac{q-2}{2}, 1 \leq b \leq \frac{q+r}{4}\) and \(1 \leq c \leq \frac{q-r}{4}\).

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