On the abundance theorem in the case $\nu = 0$

Yujiro Kawamata

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Abstract

We present a short proof of the abundance theorem in the case of numerical Kodaira dimension 0 proved by Nakayama and its log generalization.

1 Introduction

Nakayama [5] proved the abundance conjecture for a non-minimal algebraic variety whose numerical Kodaira dimension is equal to 0:

Theorem 1. Let $X$ be a smooth projective variety. Assume that the function $\dim H^0(X, mK_X + A)$ is bounded when $m \to \infty$ for arbitrarily fixed ample divisor $A$. Then there exists a positive integer $m$ such that $H^0(X, mK_X) \neq 0$.

Nakayama’s result is more general in the sense that the theorem holds for KLT pairs. Siu [7] proved the same result by analytic method.

The purpose of this note to present a simplified version of the proof. The main point is to use the numerical version of the Zariski decomposition as in [4] and Simpson’s finiteness result [6].

We shall also prove a logarithmic generalization of Theorem 1 for normal crossing pairs (Theorem 6). We note that the coefficients of the boundary in this case are equal to 1 and the pair is not KLT.

Abundance theorem in the case $\nu = 0$ for minimal algebraic varieties is already proved in [3] as an application of the additivity theorem of the Kodaira dimension for algebraic fiber spaces.

We work over $\mathbb{C}$. We denote by $\equiv$ and $\sim$ the numerical and linear equivalences respectively.
When this paper was posted on the web, the author learned from a message by Frederic Campana that the argument using [6] already appeared in [1].

2 Numerical Zariski decomposition

Let \( X \) be a smooth projective variety. Two \( \mathbb{R} \)-divisors \( D \) and \( D' \) on \( X \) are said to be *numerically equivalent*, and denoted by \( D \equiv D' \), if \( (D \cdot C) = (D' \cdot C) \) for all irreducible curves \( C \). The set of all numerical classes of \( \mathbb{R} \)-divisors form a finite dimensional real vector space \( N^1(X) \). The *pseudo-effective cone* \( \text{Pseff}(X) \) is the smallest closed convex cone in \( N^1(X) \) which contains all the numerical classes of effective divisors. An \( \mathbb{R} \)-divisor is said to be *pseudo-effective* if its numerical class is contained in \( \text{Pseff}(X) \). The *movable cone* \( \text{Mov}(X) \) is the smallest closed convex cone in \( N^1(X) \) which contains all the numerical classes of effective divisors whose complete linear systems do not have fixed components.

**Lemma 2.** Let \( D \) be an \( \mathbb{R} \)-divisor and \( A \) an ample divisor on a smooth projective variety \( X \). The following are equivalent:

1. \( D \) is pseudo-effective.
2. For an arbitrary positive number \( \epsilon \), there exists an effective \( \mathbb{R} \)-divisor \( D' \) such that \( D + \epsilon A \equiv D' \).
3. For an arbitrary positive number \( \epsilon \), there exists a positive integer \( m \) such that \( H^0(X, \mathcal{I}m(D + \epsilon A)) \neq 0 \).

**Proof.** The equivalence of (1) and (2) is clear. Obviously (3) implies (2). If (2) holds, then we can write

\[
D + \frac{1}{3} \epsilon A - D' = \sum_j d_j D_j
\]

for an effective \( \mathbb{R} \)-divisor \( D' \), real numbers \( d_j \) which are linearly independent over \( \mathbb{Q} \) and \( \mathbb{Q} \)-divisors \( D_j \) such that \( D_j \equiv 0 \). Thus we can write

\[
D + \frac{1}{3} \epsilon A \sim_\mathbb{Q} D_1 + D_2 + L
\]

where \( \sim_\mathbb{Q} \) denotes the \( \mathbb{Q} \)-linear equivalence, for an effective \( \mathbb{Q} \)-divisor \( D_1 \), a small \( \mathbb{R} \)-divisor \( D_2 \) and \( L \in \text{Pic}^0(X) \otimes \mathbb{Q} \) such that \( \frac{1}{2} \epsilon A + D_2 \) and \( \frac{1}{2} \epsilon A + L \)
are \( \mathbb{Q} \)-linearly equivalent to effective \( \mathbb{R} \)-divisors. Then there exists a positive integer \( m \) such that \( m(D + \epsilon A) \) is linearly equivalent to an effective \( \mathbb{R} \)-divisor, hence (3).

Let \( D \) be a pseudo-effective \( \mathbb{R} \)-divisor, and \( A \) an ample divisor. We define the \textit{numerically fixed part} and the \textit{numerical base locus} of \( D \) by

\[
N(D) = \lim_{\epsilon \to 0} (\inf \{ D' \mid D + \epsilon A \equiv D' \geq 0 \})
\]

\[
NBs(D) = \bigcup_{\epsilon > 0} \left( \bigcap \{ \text{Supp}(D') \mid D + \epsilon A \equiv D' \geq 0 \} \right).
\]

They are independent of \( A \). By setting \( P(D) = D - N(D) \), we obtain a formula \( D = P(D) + N(D) \) called the \textit{numerical Zariski decomposition} of \( D \).

\textbf{Lemma 3.} \begin{enumerate}
\item The irreducible components of \( N(D) \) are numerically independent, i.e., linearly independent in \( N^1(X) \).
\item If \( D \) is a \( \mathbb{Q} \)-divisor and \( D \equiv N(D) \), then \( N(D) \) is also a \( \mathbb{Q} \)-divisor.
\item \( N(D) = 0 \) if and only if the numerical class of \( D \) is contained in \( \text{Mov}(X) \). In this case, \( NBs(D) \) is a countable union of subvarieties of codimension at least 2.
\end{enumerate}

\textit{Proof.} \begin{enumerate}
\item If there is a numerical linear relation, then \( N(D) \) is numerically equivalent to a different effective \( \mathbb{R} \)-divisor, a contradiction.
\item The intersection numbers of \( N(D) \) with curves are rational numbers, hence so are the coefficients of \( N(D) \).
\item We can take the limit \( \epsilon \to 0 \) for only those \( \epsilon \) which are rational numbers.
\end{enumerate}

Let \( D \) be a pseudo-effective \( \mathbb{R} \)-divisor. Then there are two cases:

1. \( \nu(X,D) = 0 \): The function \( \dim H^0(X, \lfloor mD \rfloor + A) \) is bounded when \( m \to \infty \) for any ample divisor \( A \).

2. \( \nu(X,D) > 0 \): There exists an ample divisor \( A \) such that the function \( \dim H^0(X, \lfloor mD \rfloor + A) \) is unbounded when \( m \to \infty \).

The following proposition is [5] Theorem V.1.11. We include a proof for the convenience of the reader.
Proposition 4. Let $X$ be a smooth projective variety, and $D$ a pseudo-effective $\mathbb{R}$-divisor. Assume that $D \not\equiv 0$ and $N(D) = 0$. Then there exist an ample divisor $A$, a positive number $b$ and a positive integer $m_0$ such that

$$\dim H^0(X, \lceil mD \rceil + A) > bm$$

for $m \geq m_0$.

Proof. Since $\text{NBs}(D)$ is a countable union of closed subvarieties of codimension at least 2, a general curve section $C$ does not meet $\text{NBs}(D)$. Since $D \not\equiv 0$, we have $(D \cdot C) > 0$.

We fix an ample divisor $A$ and denote $L_m = \lceil mD \rceil + A$. It is sufficient to prove that the natural homomorphism $H^0(X, L_m) \to H^0(C, L_m|_C)$ is surjective for $m$ large.

Let $\mu : Y \to X$ be the blowing up along $C$ and $E$ the exceptional divisor. We take an effective $\mathbb{R}$-divisor $B_m \equiv mD + \epsilon A$ such that $C \not\subset B_m$ and that $(Y, \mu^*B_m)$ is KLT near $E$.

We calculate

$$\mu^*L_m - E - (K_Y + \mu^*B_m)$$

$$= \mu^*(\lceil mD \rceil + A - (K_X + B_m)) - (n - 1)E$$

$$\equiv \mu^*((1 - \epsilon)A - \langle mD \rangle - K_X) - (n - 1)E$$

where $n = \dim X$ and $\langle mD \rangle = mD - \lceil mD \rceil$. It is ample for any $m > 0$ if $A$ is sufficiently large compared to the irreducible components of $D$, $K_X$ and $E$.

Let $I$ be the multiplier ideal sheaf for the pair $(Y, \mu^*B_m)$. We have $E \cap \text{Supp}(\mathcal{O}_Y/I) = \emptyset$. By the Nadel vanishing theorem, we have $H^1(Y, I(\mu^*L_m - E)) = 0$. It follows that the homomorphism $H^0(Y, \mu^*L_m) \to H^0(E, \mu^*L_m|_E)$ is surjective, and our assertion is proved.

Corollary 5. Let $X$ be a smooth projective variety, and $D$ a pseudo-effective $\mathbb{R}$-divisor. Assume that $\dim H^0(X, \lceil mD \rceil + A)$ is bounded. Then $D$ is numerically equivalent to an effective $\mathbb{R}$-divisor $N(D)$.

When $D = K_X$, we denote $\nu(X) = \nu(X, K_X)$. The Kodaira dimension $\kappa(X)$ is a birational invariant. Its numerical version $\nu(X)$ is also a birational invariant: if $X$ and $X'$ are birationally equivalent smooth projective varieties, then $\nu(X) = 0$ if and only if $\nu(X') = 0$. For more precise definition of the numerical Kodaira dimension $\nu(X)$, we refer the reader to [5].
3 Proof of the theorem

Proof of Theorem 1. By assumption, $K_X$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. We take the smallest possible positive integer $m$ such that $m(K_X + L)$ for some $L \in \text{Pic}^r(X)$ is linearly equivalent to an effective divisor $N$. We shall prove that $L$ is a torsion in $\text{Pic}^r(X)$.

By blowing up $X$ further, we may assume that the support of $N$ is a normal crossing divisor. We take a holomorphic section $h$ of $O_X(m(K_X + L))$ such that div$(h) = N$. By taking the $m$-th root of $h$, we construct a finite and surjective morphism $\pi : Y' \to X$ from a normal variety with only rational singularities. Let $\mu : Y \to Y'$ be a desingularization. We have

$$\mu_* K_Y \sim \pi^*(K_X + (m - 1)(K_X + L) - N')$$

for some $\mathbb{Q}$-divisor $N'$ such that $0 \leq N' \leq N$. Then

$$\mu_* K_Y + \pi^* L \sim \pi^*(m(K_X + L) - N') \sim \pi^*(N - N')$$

Since $Y'$ has only rational singularities, it follows that $H^0(Y, K_Y + \mu^* \pi^* L) \neq 0$.

By [6], it follows that there exists a torsion element $L' \in \text{Pic}^r(Y)$ such that $H^0(Y, K_Y + L') \neq 0$. If $L' \not\sim \mu^* \pi^* L$, then $\pi^*(N - N')$ is numerically equivalent to a different effective divisor. Then $N$ must be numerically equivalent to a different effective $\mathbb{Q}$-divisor, a contradiction. Therefore $L' \sim \mu^* \pi^* L$, and $L$ is a torsion.

We prove a logarithmic version:

**Theorem 6.** Let $X$ be a smooth projective variety and $D = \sum_i D_i$ a simple normal crossing divisor. Assume that $\dim H^0(X, m(K_X + D) + A)$ is bounded when $m \to \infty$ for any fixed ample divisor $A$. Then there exists a positive integer $m$ such that $H^0(X, m(K_X + D)) \neq 0$.

*Proof.* The proof is parallel to the non-log case. We have $m(K_X + D + L) \sim N$ as before. We make the union of $N$ and $D$ to be normal crossing by blowing up $X$, and take a ramified covering $\pi : Y' \to X$ branching along $N$. By resolution, we obtain a smooth projective variety $Y$ with a simple normal crossing divisor $E$ which is the union of the preimage of $D$ and the exceptional divisors of the resolution. We note that common irreducible components of $N$ and $D$ do not cause any trouble for the formula

$$\mu_*(K_Y + E) \sim \pi^*(K_X + D + (m - 1)(K_X + D + L) - N')$$
though we have to modify $N'$. We have $\mu_\ast(K_Y + E) + \pi^\ast L \sim \pi^\ast(N - N')$ as before. We have to prove that $\pi^\ast L$ is torsion.

In the moduli space of local systems $V$ of rank 1 on $X$, we consider the closed subvarieties where the dimensions of the cohomology groups

$$H^p_B(Y \setminus E, V) = H^p(Y \setminus E, \mu^\ast \pi^\ast V)$$

jump. Let $i: Y \setminus E \to Y$ be an open immersion, and denote by $E^{[p]}$ the disjoint union of all the $p$-fold intersections of the irreducible components of $E$ as in [2]. We have $E^{[0]} = Y$ by convention. Then the canonical filtration on the complex $Ri_\ast \mathbb{Q}_{Y \setminus E}$ induces a spectral sequence among Betti cohomologies

$$E_{1}^{p,q} = H^{2p+q}_B(E^{[p] \setminus E}, V) \Rightarrow H^{p+q}_B(Y \setminus E, V)$$

where we denote $H^{q}_B(E^{[p]}, V) = H^q(E^{[p]}, \mu^\ast \pi^\ast V \otimes \Omega^\bullet_{E^{[p]}})$. We denote furthermore

$$H^{q}_{DR}(E^{[p]}, V) = H^q(E^{[p]}, \mu^\ast \pi^\ast V \otimes \Omega^\bullet_{E^{[p]}})$$
$$H^{q}_{DR}(Y \setminus E, V) = H^q(Y, \mu^\ast \pi^\ast V \otimes \Omega^\bullet_{Y \setminus E})$$
$$H^{q}_{Dol}(E^{[p]}, V) = H^q(E^{[p]}, (\mu^\ast \pi^\ast V \otimes \Omega^\bullet_{E^{[p]}}, \phi))$$
$$H^{q}_{Dol}(Y \setminus E, V) = H^q(Y, (\mu^\ast \pi^\ast V \otimes \Omega^\bullet_{Y \setminus E}, \phi))$$

where $\phi \in H^0(X, \Omega^1_X)$ is the Higgs field corresponding to the flat connection on $V$. The filtration with respect to the orders of log poles on the complex $\Omega^\bullet_Y(\log E)$ induces spectral sequences

$$E_{1}^{p,q} = H^{2p+q}_{DR}(E^{[p] \setminus E}, V) \Rightarrow H^{p+q}_{DR}(Y \setminus E, V)$$
$$E_{1}^{p,q} = H^{2p+q}_{Dol}(E^{[p] \setminus E}, V) \Rightarrow H^{p+q}_{Dol}(Y \setminus E, V)$$

By [2], these spectral sequences are compatible with the isomorphisms

$$H^q_B(E^{[p]}, V) \cong H^q_{DR}(E^{[p]}, V) \cong H^q_{Dol}(E^{[p]}, V)$$

so that we have isomorphisms

$$H^q_B(Y \setminus E, V) \cong H^q_{DR}(Y \setminus E, V) \cong H^q_{Dol}(Y \setminus E, V).$$

Thus our jumping loci are canonically defined in the sense of Simpson [6]. We apply Simpson’s result, and deduce that there is a torsion element $L' \in \text{Pic}^+(X)$ such that $H^0(Y, L' \otimes \mathcal{O}_Y(K_Y + E)) \neq 0$. The rest of the proof is the same as in Theorem [4].
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1 Introduction

Nakayama [6] proved the abundance conjecture for a non-minimal algebraic variety whose numerical Kodaira dimension is equal to 0:

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Nakayama’s result is more general in the sense that the theorem holds for KLT pairs. Siu [8] proved the same result by analytic method.

The purpose of this note to present a simplified version of the proof. The main point is to use the numerical version of the Zariski decomposition as in [5] and Simpson’s finiteness result [7].

We shall also prove a logarithmic generalization of Theorem [1] for normal crossing pairs (Theorem [6]). We note that the coefficients of the boundary in this case are equal to 1 and the pair is not KLT.

Abundance theorem in the case $\nu = 0$ for minimal algebraic varieties is already proved in [4] as an application of the additivity theorem of the Kodaira dimension for algebraic fiber spaces.

We work over $\mathbb{C}$. We denote by $\equiv$ and $\sim$ the numerical and linear equivalences respectively.
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2 Numerical Zariski decomposition

Let $X$ be a smooth projective variety. Two $\mathbb{R}$-divisors $D$ and $D'$ on $X$ are said to be numerically equivalent, and denoted by $D \equiv D'$, if $(D \cdot C) = (D' \cdot C)$ for all irreducible curves $C$. The set of all numerical classes of $\mathbb{R}$-divisors form a finite dimensional real vector space $N_1(X)$. The pseudo-effective cone $Pseff(X)$ is the smallest closed convex cone in $N_1(X)$ which contains all the numerical classes of effective divisors. An $\mathbb{R}$-divisors is said to be pseudo-effective if its numerical class is contained in $Pseff(X)$. The movable cone $Mov(X)$ is the smallest closed convex cone in $N_1(X)$ which contains all the numerical classes of effective divisors whose complete linear systems do not have fixed components.

Lemma 2. Let $D$ be an $\mathbb{R}$-divisor and $A$ an ample divisor on a smooth projective variety $X$. The following are equivalent:

1. $D$ is pseudo-effective.
2. For an arbitrary positive number $\epsilon$, there exists an effective $\mathbb{R}$-divisor $D'$ such that $D + \epsilon A \equiv D'$.
3. For an arbitrary positive number $\epsilon$, there exists a positive integer $m$ such that $H^0(X, \mathcal{I}_m(D + \epsilon A)) \neq 0$.

Proof. The equivalence of (1) and (2) is clear. Obviously (3) implies (2). If (2) holds, then we can write

$$D + \frac{1}{3} \epsilon A - D' = \sum_j d_j D_j$$

for an effective $\mathbb{R}$-divisor $D'$, real numbers $d_j$ which are linearly independent over $\mathbb{Q}$ and $\mathbb{Q}$-divisors $D_j$ such that $D_j \equiv 0$. Thus we can write

$$D + \frac{1}{3} \epsilon A \sim \mathbb{Q} D_1 + D_2 + L$$

where $\sim \mathbb{Q}$ denotes the $\mathbb{Q}$-linear equivalence, for an effective $\mathbb{Q}$-divisor $D_1$, a small $\mathbb{R}$-divisor $D_2$ and $L \in \text{Pic}^g(X) \otimes \mathbb{Q}$ such that $\frac{1}{3} \epsilon A + D_2$ and $\frac{1}{3} \epsilon A + L$
are $\mathbb{Q}$-linearly equivalent to effective $\mathbb{R}$-divisors. Then there exists a positive integer $m$ such that $m(D + \epsilon A)$ is linearly equivalent to an effective $\mathbb{R}$-divisor, hence (3). \hfill \Box

Let $D$ be a pseudo-effective $\mathbb{R}$-divisor, and $A$ an ample divisor. We define the **numerically fixed part** and the **numerical base locus** of $D$ by

$$N(D) = \lim_{\epsilon \to 0^-}(\inf\{D' | D + \epsilon A \equiv D' \geq 0\})$$

$$\text{NBs}(D) = \bigcup_{\epsilon > 0}(\bigcap\{\text{Supp}(D') | D + \epsilon A \equiv D' \geq 0\}).$$

They are independent of $A$. By setting $P(D) = D - N(D)$, we obtain a formula $D = P(D) + N(D)$ called the **numerical Zariski decomposition** of $D$.

**Lemma 3.** (1) The irreducible components of $N(D)$ are numerically independent, i.e., linearly independent in $N^1(X)$.

(2) If $D$ is a $\mathbb{Q}$-divisor and $D \equiv N(D)$, then $N(D)$ is also a $\mathbb{Q}$-divisor.

(3) $N(D) = 0$ if and only if the numerical class of $D$ is contained in $\text{Mov}(X)$. In this case, $\text{NBs}(D)$ is a countable union of subvarieties of codimension at least 2.

**Proof.** (1) If there is a numerical linear relation, then $N(D)$ is numerically equivalent to a different effective $\mathbb{R}$-divisor, a contradiction.

(2) The intersection numbers of $N(D)$ with curves are rational numbers, hence so are the coefficients of $N(D)$.

(3) We can take the limit $\epsilon \to 0$ for only those $\epsilon$ which are rational numbers. \hfill \Box

Let $D$ be a pseudo-effective $\mathbb{R}$-divisor. Then there are two cases:

1. $\nu(X, D) = 0$: The function $\dim H^0(X, \lceil mD \rceil + A)$ is bounded when $m \to \infty$ for any ample divisor $A$.

2. $\nu(X, D) > 0$: There exists an ample divisor $A$ such that the function $\dim H^0(X, \lceil mD \rceil + A)$ is unbounded when $m \to \infty$.

The following proposition is [6] Theorem V.1.11. We include a proof for the convenience of the reader.
Proposition 4. Let $X$ be a smooth projective variety, and $D$ a pseudo-effective $\mathbb{R}$-divisor. Assume that $D \not\equiv 0$ and $N(D) = 0$. Then there exist an ample divisor $A$, a positive number $b$ and a positive integer $m_0$ such that

$$\dim H^0(X, mD + A) > bm$$

for $m \geq m_0$.

Proof. Since $\text{NBs}(D)$ is a countable union of closed subvarieties of codimension at least 2, a general curve section $C$ does not meet $\text{NBs}(D)$. Since $D \not\equiv 0$, we have $(D \cdot C) > 0$.

We fix an ample divisor $A$ and denote $L_m = mD + A$. It is sufficient to prove that the natural homomorphism $H^0(X, L_m) \to H^0(C, L_m|_C)$ is surjective for $m$ large.

Let $\mu : Y \to X$ be the blowing up along $C$ and $E$ the exceptional divisor. We take an effective $\mathbb{R}$-divisor $B_m \equiv mD + \epsilon A$ such that $C \not\subset B_m$ and that $(Y, \mu^*B_m)$ is KLT near $E$. We calculate

$$\mu^*L_m - E - (K_Y + \mu^*B_m)$$

$$= \mu^*(mD + A - (K_X + B_m)) - (n - 1)E$$

$$\equiv \mu^*((1 - \epsilon)A - \langle mD \rangle - K_X) - (n - 1)E$$

where $n = \dim X$ and $\langle mD \rangle = mD - mD \cdot$. It is ample for any $m > 0$ if $A$ is sufficiently large compared to the irreducible components of $D, K_X$ and $E$.

Let $I$ be the multiplier ideal sheaf for the pair $(Y, \mu^*B_m)$. We have $E \cap \text{Supp}(\mathcal{O}_Y/I) = \emptyset$. By the Nadel vanishing theorem, we have $H^1(Y, I(\mu^*L_m - E)) = 0$. It follows that the homomorphism $H^0(Y, \mu^*L_m) \to H^0(E, \mu^*L_m|_E)$ is surjective, and our assertion is proved.

Corollary 5. Let $X$ be a smooth projective variety, and $D$ a pseudo-effective $\mathbb{R}$-divisor. Assume that $\dim H^0(X, mD + A)$ is bounded. Then $D$ is numerically equivalent to an effective $\mathbb{R}$-divisor $N(D)$.

When $D = K_X$, we denote $\nu(X) = \nu(X, K_X)$. The Kodaira dimension $\kappa(X)$ is a birational invariant. Its numerical version $\nu(X)$ is also a birational invariant: if $X$ and $X'$ are birationally equivalent smooth projective varieties, then $\nu(X) = 0$ if and only if $\nu(X') = 0$. For more precise definition of the numerical Kodaira dimension $\nu(X)$, we refer the reader to [6].
3 Proof of the theorem

Proof of Theorem 1. By assumption, $K_X$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. We take the smallest possible positive integer $m$ such that $m(K_X + L)$ for some $L \in \text{Pic}^\tau(X)$ is linearly equivalent to an effective divisor $N$. We shall prove that $L$ is a torsion in $\text{Pic}^\tau(X)$.

By blowing up $X$ further, we may assume that the support of $N$ is a normal crossing divisor. We take a holomorphic section $h$ of $O_X(m(K_X + L))$ such that $\text{div}(h) = N$. By taking the $m$-th root of $h$, we construct a finite and surjective morphism $\pi : Y' \to X$ from a normal variety with only rational singularities. Let $\mu : Y \to Y'$ be a desingularization. We have

$$\mu_*K_Y \sim \pi^*(K_X + (m - 1)(K_X + L) - N')$$

for some $\mathbb{Q}$-divisor $N'$ such that $0 \leq N' \leq N$. Then

$$\mu_*K_Y + \pi^*L \sim \pi^*(m(K_X + L) - N') \sim \pi^*(N - N').$$

Since $Y'$ has only rational singularities, it follows that $H^0(Y, K_Y + \mu^*\pi^*L) \neq 0$.

By [7], it follows that there exists a torsion element $L' \in \text{Pic}^\tau(Y)$ such that $H^0(Y, K_Y + L') \neq 0$. If $L' \not\sim \mu^*\pi^*L$, then $\pi^*(N - N')$ is numerically equivalent to a different effective divisor. Then $N$ must be numerically equivalent to a different effective $\mathbb{Q}$-divisor, a contradiction. Therefore $L' \sim \mu^*\pi^*L$, and $L$ is a torsion.

We prove a logarithmic version:

Theorem 6. Let $X$ be a smooth projective variety and $D = \sum_i D_i$ a simple normal crossing divisor. Assume that $\dim H^0(X, m(K_X + D) + A)$ is bounded when $m \to \infty$ for any fixed ample divisor $A$. Then there exists a positive integer $m$ such that $H^0(X, m(K_X + D)) \neq 0$.

Proof. The proof is parallel to the non-log case. We have $m(K_X + D + L) \sim N$ as before. We make the union of $N$ and $D$ to be normal crossing by blowing up $X$, and take a ramified covering $\pi : Y' \to X$ branching along $N$. By resolution, we obtain a smooth projective variety $Y$ with a simple normal crossing divisor $E$ which is the union of the preimage of $D$ and the exceptional divisors of the resolution. We note that common irreducible components of $N$ and $D$ do not cause any trouble for the formula

$$\mu_*(K_Y + E) \sim \pi^*(K_X + D + (m - 1)(K_X + D + L) - N').$$
though we have to modify $N'$. We have $\mu_\ast(K_Y + E) + \pi^\ast L \sim \pi^\ast(N - N')$ as before. We have to prove that $\pi^\ast L$ is torsion.

In the moduli space of local systems $V$ of rank 1 on $X$, we consider the closed subvarieties where the dimensions of the cohomology groups

$$H_B^p(Y \setminus E, V) = H^p(Y \setminus E, \mu^\ast \pi^\ast V)$$

jump. Let $i : Y \setminus E \to Y$ be an open immersion, and denote by $E^{[p]}$ the disjoint union of all the $p$-fold intersections of the irreducible components of $E$ as in [3]. We have $E^{[0]} = Y$ by convention. Then the canonical filtration on the complex $Ri_\ast Q_{Y \setminus E}$ induces a spectral sequence among Betti cohomologies

$$E_1^{p,q} = H^{2p+q}_B(Y^{[-p]}, V) \Rightarrow H^{p+q}(Y \setminus E, V)$$

where we denote $H^q_B(E^{[p]}, V) = H^q(E^{[p]}, \mu^\ast \pi^\ast V \otimes \Omega_{E^{[p]}}^\bullet)$. We denote furthermore

$$H^q_{DR}(E^{[p]}, V) = H^q(E^{[p]}, \mu^\ast \pi^\ast V \otimes \Omega_{E^{[p]}}^\bullet)$$
$$H^q_{DR}(Y \setminus E, V) = H^q(Y, \mu^\ast \pi^\ast V \otimes \Omega_Y^\bullet (\log E))$$
$$H^q_{Dol}(E^{[p]}, V) = H^q(E^{[p]}, (\mu^\ast \pi^\ast V \otimes \Omega_{E^{[p]}}^\bullet, \phi))$$
$$H^q_{Dol}(Y \setminus E, V) = H^q(Y, (\mu^\ast \pi^\ast V \otimes \Omega_Y^\bullet (\log E), \phi))$$

where $\phi \in H^0(X, \Omega_X^1)$ is the Higgs field corresponding to the flat connection on $V$. The filtration with respect to the orders of log poles on the complex $\Omega_Y^\bullet (\log E)$ induces spectral sequences

$$E_1^{p,q} = H^{2p+q}_{DR}(Y^{[-p]}, V) \Rightarrow H^{p+q}_{DR}(Y \setminus E, V)$$
$$E_1^{p,q} = H^{2p+q}_{Dol}(Y^{[-p]}, V) \Rightarrow H^{p+q}_{Dol}(Y \setminus E, V)$$

By [3], these spectral sequences are compatible with the isomorphisms

$$H^q_B(E^{[p]}, V) \cong H^q_{DR}(E^{[p]}, V) \cong H^q_{Dol}(E^{[p]}, V)$$

so that we have isomorphisms

$$H^q_B(Y \setminus E, V) \cong H^q_{DR}(Y \setminus E, V) \cong H^q_{Dol}(Y \setminus E, V).$$

Thus our jumping loci are canonically defined in the sense of Simpson [4]. We apply Simpson’s result, and deduce that there is a torsion element $L' \in \text{Pic}^+(X)$ such that $H^0(Y, L' \otimes \mathcal{O}_Y(K_Y + E)) \neq 0$. The rest of the proof is the same as in Theorem [4]
4 Addendum

After the first version of this paper is written, the author received a paper [1]. Then the author realized that a more precise calculation on the construction of this paper yields a more general result for LC pairs as follows. The proof is added for the sake of completeness though it is basically the same.

**Theorem 7.** Let \((X, B)\) be a projective pair with log canonical (LC) singularities. Assume that \(H^0(X, m(K_X + B + L)) \neq 0\) for a positive integer \(m\) and \(L \in \text{Pic}^\tau(X)\). Then there exists a positive integer \(m'\) such that \(H^0(X, m'(K_X + B)) \neq 0\).

**Proof.** We have \(m(K_X + B + L) \sim N\) for an effective divisor \(N\) as before. By a resolution of singularities, we may assume that the support of \(B + N\) is a normal crossing divisor. Moreover we may assume that \(B\) and \(N\) have no common irreducible components by subtracting the overlap and multiplying \(m\) if necessary. We denote \(B = D + B'\) for \(D = \cup B_i\). Let \(\pi : Y' \to X\) be the ramified covering obtained by taking the \(m\)-th root of \(N - mB\), and \(\mu : Y \to Y'\) a log resolution as before. Let \(f = \pi \mu\).

Let \(E = (f^*D)_{\text{red}}\). Then we have \(R\mu_* \mathcal{O}_Y(K_Y + E) \cong \mathcal{O}_{Y'}(K_{Y'} + \pi^*D)\) by the vanishing theorem. We have the following more precise formula:

\[
R_f_* \mathcal{O}_Y(K_Y + E) \cong \bigoplus_{i=0}^{m-1} \mathcal{O}_X(K_X + D + i(K_X + B + L - N/m)\gamma).
\]

Since \(H^0(X, \bullet)\) of each term on the right hand side is upper semicontinuous, the jumping locus on the moduli space of flat line bundles on \(X\) for each term is a union of torsion translations of triple tori by Simpson’s result as before.

If we set \(i = m - 1\), then we have

\[
K_X + D + (m-1)(K_X + B + L - N/m)\gamma = m(K_X + B) + (m-1)L - \cup N/m_i.
\]

Therefore we conclude that there exists a torsion line bundle \(L'\) such that \(H^0(X, m(K_X + B) + L' - \cup N/m_i) \neq 0\). 

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