Noncommutative geometry of the Moyal plane:
translation isometries,
Connes’ distance on coherent states,
Pythagoras equality.

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\textbf{Abstract}

We study the metric aspect of the Moyal plane from Connes’ noncommutative geometry point of view. First, we compute Connes’ spectral distance associated with the natural isometric action of $\mathbb{R}^2$ on the algebra of the Moyal plane $\mathcal{A}$. We show that the distance between any state of $\mathcal{A}$ and any of its translated is precisely the amplitude of the translation. As a side result, we obtain the spectral distance between coherent states of the quantum harmonic oscillator as the Euclidean distance on the plane, multiplied by the Planck length. Second, we compute the spectral distance in the double Moyal plane, intended as the product - in the sense of spectral triples - of (the minimal unitization of) $\mathcal{A}$ by $\mathbb{C}^2$. On the set of states obtained by translation of an arbitrary state of $\mathcal{A}$, this distance is given by Pythagoras theorem. Applied to the Doplicher-Fredenhagen-Roberts model of quantum spacetime [DFR], these two theorems show that Connes’ spectral distance and the DFR quantum length coincide on the set of states of optimal localization. On the way, we also prove some Pythagoras inequalities for the product of arbitrary unital & non-degenerate spectral triples.

I Introduction

Long after their introduction for the study of quantum mechanics in phase space \cite{25,35}, Moyal spaces are now intensively used in physics and mathematics as a paradigmatic example of noncommutative geometry by deformation (especially, in most recent time, with the aim of developing quantum field theory on noncommutative spacetime). However, their metric aspect has been little studied. The direct approach, consisting in deforming the Riemannian metric tensor by lifting the star product \cite{29}, does not allow to construct a “noncommutative” line element that would be integrated along a “Moyal-geodesic” in order to get a “quantum distance”. Nevertheless, there exist (at least) two alternative proposals for extracting some metric information from Moyal spaces, both starting with an algebraic formulation of the distance: one is Connes’ spectral distance formula \cite{13}, the other is the length operator in the Doplicher-Fredenhagen-Roberts model of quantum spacetime [DFR] \cite{22}. In this paper, we prove two theorems on the spectral distance: the first one gives the distance between any two states obtained from one another by an isometric action of $\mathbb{R}^2$ on the Moyal plane, the second is a Pythagoras equality in the double Moyal plane. Besides their own interest (few such explicit general results are known on the metric aspect of noncommutative geometry), these two results allow us to show \cite{33} how the spectral distance and the DFR quantum length, restricted to the set of physically relevant states, capture the same metric information on a quantum space.
Recall that, given a spectral triple \( [13] \) (or unbounded Fredholm module) \( T = (\mathcal{A}, \mathcal{H}, D) \) where
- \( \mathcal{A} \) is an involutive algebra acting by \( \pi \) on a Hilbert space \( \mathcal{H} \);
- the so called Dirac operator \( D \) is a non-necessarily bounded, densely defined, selfadjoint operator on \( \mathcal{H} \), such that \( \pi(a)(D - \lambda I)^{-1} \) is compact for any \( a \in \mathcal{A} \) and \( \lambda \) in the resolvent set of \( D \) (in case \( \mathcal{A} \) is unital, this means \( D \) has compact resolvent);
- the set \( \{a \in \mathcal{A}, [D, \pi(a)] \in \mathcal{B}(\mathcal{H})\} \) is dense in \( \mathcal{A} \);

Connes has proposed on the state space \( S(\mathcal{A}) \) of \( \mathcal{A} \) the following distance \([12]\),
\[
d_D(\varphi, \tilde{\varphi}) = \sup_{a \in \mathcal{B}_{Lip}(T)} |\varphi(a) - \tilde{\varphi}(a)|, \tag{1.1}
\]
where \( \varphi, \tilde{\varphi} \in S(\mathcal{A}) \) are any two states and
\[
\mathcal{B}_{Lip}(T) \doteq \{ a \in \mathcal{A}, \|[D, \pi(a)]\| \leq 1 \} \tag{1.2}
\]
denotes the \( D \)-Lipschitz ball of \( \mathcal{A} \), that is the unit ball for the Lipschitz semi-norm
\[
L(a) \doteq \|[D, \pi(a)]\|, \tag{1.3}
\]
where \( \|\cdot\| \) is the operator norm coming from the representation \( \pi \),
\[
\|\pi(a)\| = \sup_{0 \neq \psi, \tilde{\psi} \in \mathcal{H}} \left\{ \frac{\|\pi(a)\psi\|_\mathcal{H}}{\|\tilde{\psi}\|_\mathcal{H}} \right\}, \tag{1.4}
\]
with \( \|\psi\|_\mathcal{H} \doteq \sqrt{\langle \psi, \psi \rangle} \) the Hilbert space norm.

In case \( \mathcal{A} = C_0^\infty(\mathcal{M}) \) is the (commutative) algebra of smooth functions vanishing at infinity on a compact Riemannian spin manifold \( \mathcal{M} \), with \( D = \bar{\partial} \doteq -i \sum_\mu \gamma^\mu \partial_\mu \) the Dirac operator of quantum field theory and \( \mathcal{H} \) the Hilbert space of square integrable spinors on \( \mathcal{M} \), the spectral distance \( d_\bar{\partial} \) coincides with the Wasserstein distance of order 1 in the theory of optimal transport \([35]\). This result still holds for locally compact manifolds, as soon as they are geodesically complete \([17]\). For pure states, that is - by Gelfand theorem - evaluation at points \( x \) of \( \mathcal{M} \) - \( \omega_x(f) \doteq f(x) \) for \( f \in C_0^\infty(\mathcal{M}) \) - one retrieves the geodesic distance associated with the Riemannian structure,
\[
d_\bar{\partial}(\omega_x, \omega_y) = d_{geo}(x, y). \tag{1.5}
\]

Therefore, the spectral distance appears as an alternative to the usual definition of the geodesic distance, whose advantage is to make sense also in a noncommutative context. It has been explicitly calculated in several noncommutative spectral triples inspired by high energy physics \([13]\), providing an interpretation to the Higgs field as the component of the metric in a discrete internal dimension \([14, 34]\), and exhibiting intriguing links with other distances, like the Carnot-Carathéodory metric in subriemannian geometry \([31, 32]\). Various examples with finite dimensional algebras have also been investigated \([2, 10, 16, 20]\), as well as for fractals \([9, 10]\) and the noncommutative torus \([7]\).

As often advertised by Connes, formula \((1.1)\) is particularly interesting for it does not rely on any notion ill-defined in a quantum context, such as points or path between points. In this perspective, the spectral distance seems more compatible with a (still unknown) description of spacetime at the Planck scale than the distance viewed as the length of the shortest path. To push this idea further, one investigated in \([6]\) the spectral distance for the simplest spectral triple one may associate to quantum mechanics, namely the isospectral deformation of the Euclidean space based on the noncommutative Moyal product \( \ast \) \([24]\). For technical reasons, in \([6]\) only the stationary states of the quantum harmonic oscillator were taken into account. In the present paper, we extend the analysis to a wider class of states, including coherent states.

We present two main results. The first one is theorem \([III.9]\) in which we show that the spectral distance between any state \( \varphi \) of the 2-dimensional Moyal algebra \( \mathcal{A} \) and any of its translated \( \varphi_\kappa \), \( \kappa \in \mathbb{R}^2 \), is precisely the (geodesic) length of translation,
\[
d_D(\varphi, \varphi_\kappa) = |\kappa|. \tag{1.6}
\]
As an application, we obtain in proposition \([V.3]\) the spectral distance \( d_D \) between coherent states of the one dimensional quantum harmonic oscillator as the Euclidean distance on the plane, multiplied
by the Planck length $\lambda_P$. From the DFR perspective, coherent states are particularly relevant since they are states of optimal localization, that is those which minimize the uncertainty in the simultaneous measurement of the spacetime coordinates (see [20, 21] as well as [30] for a recent review).

The second result is a Pythagoras equality in the double Moyal plane (theorem [IV.5]). By this, we mean the product $T'$ of the unital spectral triple $T^+$ of the Moyal plane with the canonical spectral triple on $\mathbb{C}^2$. We show that for a fixed state $\varphi$ of the Moyal algebra $A$, the spectral distance $d_{T'}$ on the subset of $S(A \otimes \mathbb{C}^2)$ given by

$$\left\{ (\varphi_\kappa, \delta^i), \kappa \in \mathbb{R}^2, i = 1, 2 \right\} \quad \text{with } \delta^1, \delta^2$$

the two pure states of $\mathbb{C}^2$,

(1.7) satisfies Pythagoras theorem, that is

$$d^2_{T'} \left( (\varphi_\kappa, \delta^1), (\varphi_\kappa, \delta^2) \right) = d^2_{T'} \left( (\varphi, \delta^1), (\varphi, \delta^1) \right) + d^2_{T'} \left( (\varphi, \delta^1), (\varphi, \delta^2) \right).$$

Such an equality was known for the product of a manifold by $\mathbb{C}^2$ [34] or - for a very particular class of states - for the product of a manifold by some finite dimensional noncommutative algebra [30] (including the one giving back the gauge group of the standard model of elementary particles [8]). The remarkable point here is that Pythagoras theorem holds true for $A$ an infinite dimensional noncommutative algebra.

We also obtain in proposition [IV.2] some Pythagoras inequalities that hold true in full generality, meaning for any states in the product of any unital and non-degenerate spectral triple with $\mathbb{C}^2$.

Although the paper is self-contained, some of its results can be thought as a continuation of [6], as well as a companion to [33].

The paper is organized as follows. In section III, we recall some basic properties of the Moyal plane and its link with quantum mechanics. We emphasize in particular the unitary implementation of the translations, both in the left-regular and the Schrödinger representations. Section III contains the proof of the first theorem, namely eq. (1.6). We begin by an elementary result on the invariance of the spectral distance under translation. Then, using the characterization of the Lipschitz ball in the Schrödinger representation provided in section III, we show in section III.2 that $d_D(\varphi, \varphi_\kappa) \leq |\kappa|$. The most technical part of the proof consists in exhibiting a sequence of elements in the Lipschitz ball that attains this upper bound. This is done in sections III.3 and III.4. The result is discussed in section III.5 in the light of the commutative case. Section IV deals with the double Moyal space and Pythagoras theorem. We first prove some Pythagoras inequalities for the product of an arbitrary unital non-degenerate spectral triple with $\mathbb{C}^2$ (section IV.1), then Pythagoras equalities for the Moyal plane in section IV.2. Section V deals with the application to the coherent states of the quantum harmonic oscillator and to the states of optimal localization in the DFR model.

Notations and terminology: Formula (1.1) has all the properties of a distance, except it might be infinite. Thus one should call it a pseudo-distance, but for brevity we will omit “pseudo”. Also, for coherence, we keep the terminology used in [32] [17] [33] [6] and called $d_D$ the spectral distance, warning the reader that - e.g. in [3] - formula (1.1) is called Connes distance and is denoted $d_C$.

Recall that a state $\varphi$ of a $C^*$-algebras is a positive ($\varphi(a^*a) \geq 0$) and normalized ($\|\varphi\| = 1$) complex linear form, with

$$\|\varphi\| = \sup_{0 \neq a \in A} \|\varphi(a)\| a^{-1}.$$

A state is pure when it cannot be written as a convex combination of two other states. The set of states of $A$, respectively pure states, is denoted $S(A)$, resp. $P(A)$. In case the algebra $A$ of a spectral triple $T$ is not $C^*$, we call “state” the restriction $\varphi_0$ to $A$ of a state $\varphi$ of the $C^*$-closure of $\pi(A)$. Then $S(A), P(A)$ are shorthand notations for $S(\pi(A)), P(\pi(A))$. By continuity in the $C^*$-norm, $\varphi_0 = \tilde{\varphi}_0$ if and only if $\varphi = \tilde{\varphi}$. So there is no use to distinguish between a state and its restriction and we use the same symbol $\varphi$ for both.

We use Dirac bracket $\langle \cdot, \cdot \rangle$ and parenthesis $\langle \cdot \rangle$ for the inner products on $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$. $1$ and $I_N$, $N \geq 2$, are the identity operators on the infinite and $N$-dimensional separable Hilbert spaces. Gothic letters $q, p, a, n, h, u, f$ denote operators on $L^2(\mathbb{R})$ (i.e. in the Schrödinger representation). $S(\mathbb{R}^3)$ is the space of Schwartz functions on the Euclidean space of dimension $d$.

We use Einstein summation on alternate (up/down) indices.
II Moyal plane

We recall the definition of the spectral triple associated to the Moyal space and stress the interest to switch from the left-regular action $L$ of the Moyal algebra on $\mathbb{R}^{2N}$ to the (integrated) Schrödinger representation $\pi_S$ on $\mathbb{R}^N$, in order to get an easy characterization of the Lipschitz ball (lemma II.7). On our way, we collect various formulas that will be useful for subsequent calculations, including the unitary implementation of the translations in the Moyal plane. Most of this is very well known from von Neumann uniqueness theorem. Nevertheless, it is useful to have all this material, sometimes a bit spread out in the literature, gathered in one single section. The reader familiar with Moyal quantization is invited to jump to section III.

II.1 Spectral triple for the Moyal plane

Hereafter, we call Moyal algebra the noncommutative $\star$-deformation of the algebra of Schwartz functions $S(\mathbb{R}^{2N})$ (with its standard Fréchet topology) by a non-degenerate symplectic form $\sigma$ on $\mathbb{R}^{2N}$ with determinant $\theta_{2N} \in (0,1]$, 

$$ (f \star g)(x) = \frac{1}{(\pi\theta)^{2N}} \int_{\mathbb{R}^{4N}} d^{2N}s d^{2N}t f(x+s) g(x+t) e^{-2i\sigma(s,t)} \quad (2.10) $$

for $f,g \in S(\mathbb{R}^{2N})$, with

$$ \sigma(s,t) = \frac{1}{\theta} \sum_{\mu,\nu=1}^{2N} s^\mu \Theta_{\mu\nu} t^\nu, \quad \Theta = \left( \begin{array}{cc} 0 & -I_M \\ I_N & 0 \end{array} \right). \quad (2.11) $$

A so called isospectral deformation [15, 40] of the Euclidean space is a spectral triple in which the algebra is a noncommutative deformation of some commutative algebra of functions on the space, while the Dirac operator keeps the same spectrum as in the commutative case. For instance

$$ \mathcal{A} = (S(\mathbb{R}^{2N}), \star), \quad \mathcal{H} = L^2(\mathbb{R}^{2N}) \otimes \mathbb{C}^M, \quad D = -i\gamma^\mu \partial_\mu \quad (2.12) $$

satisfy the properties of a spectral triple [24, 41]. Here $M = 2^N$ is the dimension of the spin representation, the $\gamma$'s are the Euclidean Dirac matrices satisfying (with $\delta^{\mu\nu}$ the Euclidean metric)

$$ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu}I_M \quad \forall \mu, \nu = 1, \ldots, 2N; \quad (2.13) $$

and the representation $\pi$ of $\mathcal{A}$ on $\mathcal{H}$ is a multiple of the left regular action

$$ \mathcal{L}(f)\psi = f \star \psi \quad \forall f \in \mathcal{A}, \psi \in L^2(\mathbb{R}^{2N}), \quad (2.14) $$

that is

$$ \pi(f) = \mathcal{L}(f) \otimes 1_M. \quad (2.15) $$

In the following we restrict to the Moyal plane $N = 1$, although the extension of our results to arbitrary $N$ should be straightforward. So, from now on,

$$ \mathcal{A} = (S(\mathbb{R}^2), \star). \quad (2.16) $$

The plane $\mathbb{R}^2$ is parametrized by Cartesian coordinates $x_\mu$ with derivative $\partial_\mu$, $\mu = 1, 2$. We denote

$$ z = \frac{x_1 + ix_2}{\sqrt{2}}, \quad \bar{z} = \frac{x_1 - ix_2}{\sqrt{2}}, \quad (2.17) $$

with corresponding derivatives

$$ \partial \equiv \partial_z = \frac{1}{\sqrt{2}} (\partial_1 - i\partial_2), \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{\sqrt{2}} (\partial_1 + i\partial_2). \quad (2.18) $$

The Dirac operator

$$ D = -i\sigma^\mu \partial_\mu = -i\sqrt{2} \left( \begin{array}{cc} 0 & \bar{\partial} \\ \partial & 0 \end{array} \right), \quad (2.19) $$
with $\sigma^\mu$ the Pauli matrices, acts as a first order differential operator on

$$ H = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2. \quad (2.20) $$

Notice that the spectral triple

$$ T = (\mathcal{A}, \mathcal{H}, D) \quad (2.21) $$

of the Moyal plane is non-unital ($\mathcal{A}$ has no unit) and non-degenerate ($\pi(a)\psi = 0 \forall a \in \mathcal{A}$ implies $\mathcal{H} \ni \psi = 0$). This point will be important when discussing Pythagoras theorem.

The commutator of $D$ with a Schwartz function $f$ acts by $\ast$-multiplication on

$$ \psi = \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \in \mathcal{H}, \quad (2.22) $$

that is

$$ [D, \pi(f)] \psi = -i\sqrt{2} \left( \begin{array}{cc} 0 & \mathcal{L}(\partial f) \\ \mathcal{L}(\bar{\partial} f) & 0 \end{array} \right) \psi = -i\sqrt{2} \left( \begin{array}{c} \partial f \ast \psi_1 \\ \bar{\partial} f \ast \psi_2 \end{array} \right). \quad (2.23) $$

Easy calculation $[5]$ eq. 3.7 yields

$$ ||[D, \pi(f)]|| = \sqrt{2} \max \{ ||\mathcal{L}(\partial f)||, ||\mathcal{L}(\bar{\partial} f)|| \}. \quad (2.24) $$

There is no easy formula for the operator norm of $\mathcal{L}$: unlike the commutative case, $||\mathcal{L}(f)||$ is not the essential supremum of $f$. That is why $\mathcal{L}(f)$ is not very useful for explicit computations, and one gets a more tractable formula using the Schrödinger representation. To this aim, and to make the link with familiar notions of quantum mechanics, it is convenient to enlarge the algebra.

### II.2 Coordinate operators

Obviously, the (unbounded) Moyal coordinate operators $\psi \to x_\mu \ast \psi$ do not belong to $\mathcal{A}$, so to correctly capture the geometry of the Moyal plane, bigger algebras should be considered.

By continuity on $S(\mathbb{R}^2)$, the Moyal product extends to the dual $S'(\mathbb{R}^2)$ as $(T \ast f, g) = (T, f \ast g)$ for $T \in S'(\mathbb{R}^2)$ (and analogously for $f \ast T$ and the involution $\ast$). One can thus consider the algebra

$$ A = \{T \in S'(\mathbb{R}^2) \mid T \ast g \in L^2(\mathbb{R}^2) \text{ for all } g \in L^2(\mathbb{R}^2) \} \quad (2.25) $$

endowed with the operator norm. We stress $[5]$ that $L(A) \subseteq A$ and, as a $C^*$-algebra, $A$ is isomorphic to $\mathcal{B}(L^2(\mathbb{R}^2))$. Another algebra of interest is the multiplier algebra $\mathcal{M} = \mathcal{M}_L \cap \mathcal{M}_R$, where

$$ \mathcal{M}_L = \{T \in S'(\mathbb{R}^2) \mid T \ast h \in S(\mathbb{R}^2) \text{ for all } h \in S(\mathbb{R}^2) \}, \quad (2.26) $$

$$ \mathcal{M}_R = \{T \in S'(\mathbb{R}^2) \mid h \ast T \in S(\mathbb{R}^2) \text{ for all } h \in S(\mathbb{R}^2) \}. \quad (2.27) $$

$\mathcal{M}$ contains $[5]$ the constant functions, the Dirac $\delta$ distribution together with all its derivatives, all polynomials and plane waves of the form $e^{ik \cdot x} \cdot x \to e^{ik \cdot x}$. The coordinate operators $x_\mu$ do belong to $\mathcal{M}$ and in this space it makes sense to write the fundamental equalities $[23]$ for $f \in S(\mathbb{R}^2)$

$$ x_1 \ast f = \left( x_1 f + i \frac{\theta}{2} \partial_2 f \right) \quad x_2 \ast f = \left( x_2 f - i \frac{\theta}{2} \partial_1 f \right) \quad (2.28) $$

$$ f \ast x_1 = \left( x_1 f - i \frac{\theta}{2} \partial_2 f \right) \quad f \ast x_2 = \left( x_2 f + i \frac{\theta}{2} \partial_1 f \right) \quad (2.29) $$

or, in other terms,

$$ z \ast f = \left( zf + \frac{\theta}{2} \partial f \right) \quad z \ast f = \left( zf - \frac{\theta}{2} \partial f \right) \quad (2.30) $$

$$ f \ast z = \left( zf - \frac{\theta}{2} \partial f \right) \quad f \ast z = \left( zf + \frac{\theta}{2} \partial f \right). \quad (2.31) $$

**Remark II.1** From the very definition above, any element $T \in \mathcal{M}_L$ defines a (possibly unbounded) operator $\mathcal{L}(T)$ on the invariant dense domain $S(\mathbb{R}^2) \subseteq L^2(\mathbb{R}^2)$.
A convenient representation of these various algebras is provided by Wigner transition eigenfunctions \((m,n \in \mathbb{N})\),

\[
h_{mn} = \frac{1}{(\theta^m + n \theta^m n!)^{\frac{1}{2}}} e^{z^*} h_{00} * z^n, \quad h_{00} = \sqrt{\frac{2}{\pi \theta}} e^{-\frac{1}{4 \theta}(z^2 + z^2)}. \tag{2.32}
\]

They form an orthonormal basis of \(L^2(\mathbb{R}^2)\) (see [5], noticing that our \(h_{mn}\) is their \(\frac{f_{m,n}}{\sqrt{2 \pi \theta}}\))

\[
h_{mn} * h_{pq} = \frac{\delta_{mp}}{\sqrt{2 \pi \theta}} h_{mq}, \quad h_{\infty} = h_{mn}, \quad (h_{mn}, h_{kl}) = \delta_{mn} \delta_{kl}. \tag{2.33}
\]

It is easy to see that the linear span \(\mathcal{D}\) of the \(h_{mn}\)'s for \(m,n \in \mathbb{N}\) constitutes an invariant dense domain of analytic vectors for the unbounded operators \(\mathcal{L}(z), \mathcal{L}(\bar{z})\), whose action writes [5 Prop. 5]

\[
\mathcal{L}(z) h_{mn} = \sqrt{\theta} m h_{m-1,n}, \quad \mathcal{L}(\bar{z}) h_{mn} = \sqrt{\theta} (m+1) h_{m+1,n}.
\tag{2.34}
\]

The same is true for the symmetric operators \(\mathcal{L}(x_i), i = 1, 2\), and for the Hamiltonian

\[
\mathcal{L}(z \bar{z}) = \mathcal{L}(z) \mathcal{L}(\bar{z}) - \frac{\theta}{2} = \mathcal{L}(z) \mathcal{L}(\bar{z})^\ast + \frac{\theta}{2} = \mathcal{L}(z) + \frac{\theta}{2} = \mathcal{L}(\bar{z}). \tag{2.35}
\]

By virtue of a theorem of Nelson [37], these operators are essentially self-adjoint on \(\mathcal{D}\) (i.e. \(\mathcal{D}\) is a core for them all). Since \(\mathcal{D} \subset S(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)\), \(S(\mathbb{R}^2)\) is as well a core for all of them. On this domain, we obtain from \((2.28)\) a representation of the Heisenberg algebra [[5], noticing that our \(\mathcal{D}\) and \(\mathcal{D}(\mathbb{R}^2)\)]

\[
[\mathcal{L}(x_1), \mathcal{L}(x_2)] = i \theta \mathbb{I}, \tag{2.36}
\]

which, again by a theorem of Nelson, exponentiates to a representation of the Weyl relations

\[
e^{i k_1 \mathcal{L}(x_1)} e^{i k_2 \mathcal{L}(x_2)} = e^{i \theta k_1 k_2} e^{i k_2 \mathcal{L}(x_2)} e^{i k_1 \mathcal{L}(x_1)}, \tag{2.37}
\]

for \(k_1, k_2 \in \mathbb{R}^2\). Notice that, for \(k \in \mathbb{R}^2\),

\[
e^{i k \mathcal{L}(z)} = \mathcal{L}(e^{i k z}) \tag{2.38}
\]

since, by power series, \(e^{i k \mathcal{L}(z)} \psi = \mathcal{L}(e^{i k z}) \psi\) for \(\psi \in \mathcal{D}\), \(\mathcal{D}\) is dense in \(L^2(\mathbb{R}^2)\) and both operators are bounded.

It may not be useless to stress that defining the exponential of a function \(f\) is potentially ambiguous, regardless convergence problems. Indeed \(\exp f\) may mean \(e^f = 1 + f + \frac{1}{2!} f^2 + ...\) or \(e^f = 1 + f + \frac{1}{2} f f + ...\). For \(f\) a linear combination of \(x_1, x_2\), there is no ambiguity since by \((2.28)\) one checks that \((a x_1 + b x_2) \ast (a x_1 + b x_2) = (a x_1 + b x_2)^2\), and so on for higher degrees. In particular \(e^z = e^1_1 = e^1_2 = e^2_1 = e^2_2\) are unambiguous notations. This is no longer true for the exponential of non-linear functions of the \(x_{p,q}\)'s. For instance, the function \(z \ast e^{\frac{z}{2} z z}\) identically vanishes,

\[
z \ast e^{\frac{z}{2} z z} = \frac{1}{e} z \ast e^{\frac{z}{2} z} = \frac{1}{e} \sqrt{\frac{\pi \theta}{2}} (z \ast h_{00}) = \frac{1}{e} \sqrt{\frac{\pi \theta}{2}} \mathcal{L}(z) h_{00} = 0, \tag{2.39}
\]

as can be checked by direct calculation, or by noticing that \(h_{00} \in \text{Ker} \mathcal{L}(z)\) (as explained in the next section, \(h_{00}\) and \(\mathcal{L}(z)\) are unitarily equivalent - up to tensor product by \(\mathbb{I}\) - to the ground state of the harmonic oscillator and the annihilation operator). On the contrary, \(z \ast e^{\frac{z}{2} z z}\) is non zero since

\[
\mathcal{L}(z \ast e^{\frac{z}{2} z z}) = \mathcal{L}(z) e^{\frac{z}{2} \mathcal{L}(z)^\ast} \mathcal{L}(z) \tag{2.40}
\]

is a non-zero operator, as can be checked from \((2.34)\). We shall not encounter this ambiguity until section \(\text{III.3}\) in which the function \(z \beta\) in lemma \(\text{III.1.7}\) is intended with the Moyal exponential.

Let us conclude this catalog of formulas by a last useful one, namely for all \(g,h \in S(\mathbb{R}^2)\),

\[
\int \left( \hat{f}(k) e^{-i k \ast g} \right) dk = f \ast g \tag{2.41}
\]

where \(\hat{f}\) is Fourier transform. This follows from the linearity of the inner product,

\[
\int \hat{f}(k) \left( e^{-i k \ast g} , h \right) dk = \left( \int \hat{f}(k) e^{-i k} dk , g \ast h \right) = (f , g \ast h) = (f \ast g , h). \tag{2.42}
\]

aIn the literature, formula \((2.36)\) is often written as a Moyal bracket, \(\{x_1, x_2\} = i \theta\), and is the defining property of the so called quantized plane.
II.3 Translations

We collect some notations regarding translations, that is the transformation
\[ \alpha_\kappa f \doteq f \circ \tau_\kappa \]  \hspace{1cm} (2.43)
with \( f \in S(\mathbb{R}^2) \) and, for \( \kappa, x \in \mathbb{R}^2 \), we write
\[ \tau_\kappa(x) \doteq x + \kappa. \]  \hspace{1cm} (2.44)

Obviously \( f_\kappa \doteq \alpha_\kappa f \) is Schwartz and
\[ f_\kappa \ast g_\kappa(x) = \int ds \, dt \, f(x + \kappa + s) \, g(x + \kappa + t) \, e^{-2i\sigma(s,t)} = (f \ast g)(x + \kappa) = (f \ast g)_\kappa, \]  \hspace{1cm} (2.45)
so that \( \alpha_\kappa \) is an \(*\)-automorphism of the Moyal algebra \( \mathcal{A} \).

**Lemma II.2** In the left-regular representation, the \(*\)-automorphism \( \alpha_\kappa, \kappa \in \mathbb{R}^2 \), is obtained as the adjoint action of the plane wave with wave vector \( \frac{i}{\hbar} \Theta \kappa \). Namely, for \( f \in S(\mathbb{R}^2) \),
\[ \mathcal{L}(\alpha_\kappa f) = ad U_\kappa \mathcal{L}(f), \quad \text{where} \quad U_\kappa \doteq \mathcal{L}(e^{\frac{i}{\hbar} \Theta \kappa}). \]  \hspace{1cm} (2.46)

For fixed \( \kappa \in \mathbb{R}^2, \mathbb{R} \ni t \to U_{i\kappa} \) is a one parameter group of unitaries with generator
\[ \mathcal{L} \left( \frac{x \Theta \kappa}{\theta} \right) = \mathcal{L} \left( \frac{\kappa_1 x_2 - \kappa_2 x_1}{\theta} \right) \]  \hspace{1cm} (2.47)
electronically self-adjoint on the domain \( S(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \). Moreover, as operators on \( S(\mathbb{R}^2) \),
\[ \mathcal{L}(\kappa \partial_\mu f) = i \left[ \mathcal{L} \left( \frac{x \Theta \kappa}{\theta} \right), \mathcal{L}(f) \right]. \]  \hspace{1cm} (2.48)

**Proof.** By explicit computation with the definition (2.10) of the star product, one obtains
\[ (e^{i\kappa} \ast f)(x) = e^{i\kappa x} f(x - \frac{\theta}{2} \Theta \kappa), \quad (f \ast e^{ik})(x) = e^{ik y} f(x + \frac{\theta}{2} \Theta \kappa). \]  \hspace{1cm} (2.49)
Hence \( \text{ad} \, \mathcal{L}(e^{i\kappa}) \mathcal{L}(f) = \mathcal{L}(\alpha_{-\theta \kappa} f) \), that is \( \text{ad} \, \mathcal{L}(e^{i\kappa}) \mathcal{L}(f) = \mathcal{L}(\alpha_\kappa f) \) and (2.46) follows. The fact that \( U_\kappa \) defines a one parameter group of unitaries with the required generator is an easy consequence of the discussion leading to (2.37). Equation (2.48) follows by derivation, or from (2.28) and (2.29). \( \square \)

**Remark II.3** \( \text{ad} \, \mathcal{L}(e^{i\kappa}) \) extends naturally to the multiplier algebra \( \mathcal{M} \). In particular, from (2.37) (or equivalently (2.36)) we obtain, as operators on \( S(\mathbb{R}^2) \),
\[ \alpha_\kappa z = ad \, \mathcal{L}(e^{i\kappa}) z = z + \frac{\kappa}{\sqrt{2}}, \quad \alpha_\kappa \bar{z} = ad \, \mathcal{L}(e^{i\kappa}) \bar{z} = \bar{z} + \frac{\bar{\kappa}}{\sqrt{2}} \]  \hspace{1cm} (2.50)
where \( \bar{\kappa} \) is the complex conjugate of \( \kappa = (\kappa_1, \kappa_2) \in \mathbb{R}^2 \) identified to \( \kappa_1 + i\kappa_2 \in \mathbb{C} \).

II.4 Schrödinger representation and compact operators

We make clear the relation between the left-regular and the Schrödinger representations, that is implicit in (2.36). To make the dependence on \( \theta \) (identified to \( \hbar \)) explicit, we use the standard physicists normalizations for the Schrödinger position and momentum operators,
\[ q : (q \psi)(x) = x \psi(x), \quad p : (p \psi)(x) = -i \theta \partial_x \psi|_x, \quad \psi \in L^2(\mathbb{R}), x \in \mathbb{R}, \]  \hspace{1cm} (2.51)
but we define the annihilation and creation operators as
\[ a \doteq \frac{1}{\sqrt{2}}(q + ip), \quad a^* \doteq \frac{1}{\sqrt{2}}(q - ip). \]  \hspace{1cm} (2.52)
This differs from the usual convention, based on dimensionless operators. In particular we have

\[ [a, a^*] = \theta I. \] (2.53)

The eigenfunctions of the Hamiltonian \( H \) are then \( B V. (35) \) with \( m = \omega = 1 \)

\[ h_n(x) = (\theta \pi)^{-\frac{1}{2}} (2^n n!)^{-\frac{1}{2}} e^{-\frac{x^2}{2\theta}} H_n(x) \sqrt{\theta} \], \quad n \in \mathbb{N} \] (2.54)

where the \( H_n \)'s are the Hermite polynomials. The set \( \{ h_n = \frac{(a^*)^n}{\sqrt{n!}} h_0 \}, n \in \mathbb{N}, \) is an orthonormal basis of \( L^2(\mathbb{R}) \) and spans an invariant dense domain \( D_S \) of analytic vectors for the operators \( q, p. \)

Let \( W \) denote the operator from \( L^2(\mathbb{R}^2) \) to \( L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \) defined as

\[ Wh_{mn} = h_m \otimes h_n \quad m, n \in \mathbb{N}. \] (2.55)

Its main properties are summarized in the following

**Lemma II.4** The operator \( W \) is unitary. Moreover we have \( WD = D_S \otimes D_S \) and

\[ \begin{align*}
W\mathcal{L}(\bar{z})W^* &= a^* \otimes I & W\mathcal{L}(z)W^* &= a \otimes I \tag{2.56} \\
W\mathcal{L}(x_1)W^* &= q \otimes I & W\mathcal{L}(x_2)W^* &= p \otimes I. \tag{2.57}
\end{align*} \]

As a consequence, for \( f \in S(\mathbb{R}^2), \)

\[ W\mathcal{L}(f)W^* = \pi_S(f) \otimes I \tag{2.58} \]

where \( \pi_S \) is the so-called integrated Schrödinger representation (or the Weyl prescription), namely

\[ \pi_S(f) = \int \hat{f}(k_1, k_2) e^{i (\theta q k_1 + \theta p k_2)} dk_1 dk_2. \] (2.59)

**Proof.** Unitarity is evident. The first equality in (2.56) comes from (2.34),

\[ W\mathcal{L}(\bar{z})h_{mn} = W \sqrt{\theta} (m+1) h_{m+1,n} = \sqrt{\theta} (m+1) h_{m+1} \otimes h_n = (a^* \otimes I) h_m \otimes h_n, \] (2.60)

that is \( W\mathcal{L}(\bar{z}) = (a^* \otimes I)W. \) The proof for \( z \) and \( a \) is analogous. Then follows from (2.56).

\[ W\mathcal{L}(e^{\theta k \cdot x}) = (e^{\hat{\theta} ((k_1, k_2) \otimes I)} W \tag{2.61} \]

so that, since \( f \in S(\mathbb{R}^2) \) with its Fourier transform \( \hat{f}, (2.41) \) yields

\[ \begin{align*}
W\mathcal{L}(f)\psi &= W(f \ast \psi) = \int \hat{f}(k_1, k_2) W \left( e^{\hat{\theta} (k_1 q + k_2 p)} \otimes I \right) \psi dk_1 dk_2 \\
&= \int \hat{f}(k_1, k_2) \left( e^{\hat{\theta} (k_1 q + k_2 p)} \otimes I \right) W \psi dk_1 dk_2 \tag{2.62}
\end{align*} \]

where the integral is in the Bochner sense and \( \psi \in L^2(\mathbb{R}^2). \) \qed

In other terms, the representation \( \pi \) of the spectral triple \( T \) given in (2.15) is a multiple of \( \mathcal{L}, \) which in turn is unitary equivalent\(^b\) to a multiple of the integrated Schrödinger representation. Therefore, for any \( f \in \mathcal{A}, \)

\[ \| \mathcal{L}(f) \| = \| \pi(f) \| = \| \pi_S(f) \|, \] (2.64)

and we can denote the corresponding \( C^* \)-closure with the representation-free notation

\[ \mathcal{A} \doteq \overline{\mathcal{L}(\mathcal{A})} \simeq \overline{\pi_S(\mathcal{A})} \simeq \overline{\pi(\mathcal{A})}. \] (2.65)

\(^b\)Our normalization for \( h_{mn}, h_m \) yields the Schrödinger representation without the normalization term \( \sqrt{2} \) of [5].
Remark II.5 This closure is isomorphic to the algebra of compact operators,
\[ \tilde{A} \simeq K. \] (2.66)

Indeed by (2.59) one checks that \( \pi_S(f) \) is a compact for any Schwartz function \( f \). Injectivity of \( \pi : \tilde{A} \to K \) comes from \( \tilde{A} \) being simple and \( \pi_S \) irreducible. Surjectivity can be obtained, for instance, using the isomorphism of Fréchet algebras between \( \tilde{A} \) and the matrices with fast decaying coefficients (cf [33, section 3.1]).

Of course \( L(f) \) is not compact: the left-regular representation is a non-compact representation of the algebra of compact operators. This might sounds as an unnecessary complication, and one could question the interest of defining the spectral triple using \( L \) rather than \( \pi_S \). The point is that the initial motivation is not to build a spectral triple for compact operators, but for the quantum space. From this point of view, the left-regular representation is more suggestive than \( \pi_S \), since the star product (2.10) clearly appears as a deformation of the commutative pointwise product.

By lemma II.4, one easily translates in the Schrödinger representation all the formulas listed in section II.2 and II.3 in particular the unitary implementation of translations.

Lemma II.6 For any \( f \in A \), identifying \( \kappa = (\kappa_1, \kappa_2) \in \mathbb{R}^2 \) to \( \kappa_1 + i\kappa_2 \in \mathbb{C} \), one has
\[ \pi_S(\alpha \kappa f) = ad_{u_\kappa} \pi_S(f) \quad \text{where} \quad u_\kappa = e^{\frac{\kappa \eta}{\sqrt{2}}} \] (2.67)
Moreover, as operators on \( S(\mathbb{R}) \), one has
\[ \pi_S(\kappa \eta \partial f) = \left[ \frac{\kappa \eta}{\sqrt{2}}, \pi_S(f) \right]. \] (2.68)

Proof. Noticing that
\[ i(\kappa_1 p - \kappa_2 q) = \frac{1}{\sqrt{2}} (\kappa \eta - \kappa \eta^*), \] (2.69)
one obtains from (2.61) and the definition (2.46) of \( U_\kappa \),
\[ WU_\kappa W^* = e^{i(\kappa_1 p - \kappa_2 q)} \otimes I = u_\kappa \otimes I. \] (2.70)
The unitary map (2.58) then yields
\[ \pi_S(\alpha \kappa f) \otimes I = W ad U_\kappa L(f) W^* = (ad_{u_\kappa} \pi_S(f)) \otimes I, \] (2.71)
hence (2.67). Similarly, (2.68) follows from (2.48),
\[ \pi_S(\kappa \eta \partial f) \otimes I = W L(\kappa \eta \partial f) W^* = i \left[ \frac{\kappa_1}{\theta} (\kappa \eta - \kappa \eta^*) \otimes I, \pi_S(f) \otimes I \right] = \left[ \frac{\kappa \eta - \kappa \eta^*}{\sqrt{2}}, \pi_S(f) \right] \otimes I. \]

Finally, let us come back to what motivated the introduction of the Schrödinger representation, namely the characterization of the Lipschitz ball.

Lemma II.7 A Schwartz function \( f \in A \) is in the Lipschitz ball \( B_{Lip}(T) \) of the spectral triple (2.21) of the Moyal plane if and only if
\[ \max \{ [\|a\|, \pi_S(f)], [\|a\|, \pi_S(f)] \| \leq \frac{\theta}{\sqrt{2}}. \] (2.72)

Proof. From (2.67) with \( \kappa = 1, i \), one checks that \( \pi_S(\partial f) = \frac{1}{\theta} [p, \pi_S(f)] \) and \( \pi_S(\partial f) = \frac{1}{\theta^2} [q, \pi_S(f)] \). Therefore
\[ \pi_S(\partial f) = -\frac{1}{\theta} [a, \pi_S(f)], \quad \pi_S(\partial f) = \frac{1}{\theta} [a, \pi_S(f)]. \] (2.73)
The result follows from (2.24) together with (2.64).
III Spectral distance between translated states

This section contains the first main result of the paper, namely theorem III.9 where we show that for any state \( \varphi \in \mathcal{S}(A) \) the spectral distance on the set
\[
\mathcal{C}(\varphi) \doteq \{ \varphi_\kappa, \kappa \in \mathbb{R}^2 \}
\]
of translated of \( \varphi \) (see definition III.1 below) is the Euclidean distance on the plane, that is
\[
d_D(\varphi, \varphi_\kappa) = |\kappa|.
\]
We begin by some easy result on isometry by translation, and we show that the Euclidean distance is an upper bound for the spectral distance. We then exhibit a sequence of elements in \( A \) that attains this upper bound, called the optimal element. This proves eq. (3.75). Finally, we discuss the optimal element in the light of the commutative case.

III.1 Translation isometries

Definition III.1 Given any state \( \varphi \in \mathcal{S}(A) \) and \( \kappa \in \mathbb{R}^2 \simeq \mathbb{C} \), the \( \kappa \)-translated of \( \varphi \) is the state
\[
\varphi_\kappa \doteq \varphi \circ \alpha_\kappa
\]
where the translation \( \alpha_\kappa \) is defined in (2.44). We call \( |\kappa| = \sqrt{\kappa_1^2 + \kappa_2^2} \) the translation amplitude.

Notice that \( \varphi_\kappa \) being a state follows from \( \alpha_\kappa \) being a \( \ast \)-automorphism (hence an isometry [39]). We aim at computing the spectral distance on \( \mathcal{C}(\varphi) \) for any \( \varphi \in \mathcal{S}(A) \). Some information comes from the observation that a unitarily implemented automorphism of \( A \) commuting with \( D \) is an isometry of the space of states.

Proposition III.2 Let \((A_1, \mathcal{H}_1, D_1)\) be any spectral triple, and \( \alpha \) a \( \ast \)-automorphism of \( A_1 \) implemented by a unitary \( U \), that is - with \( \pi_1 \) the representation of \( A_1 \) on \( \mathcal{H}_1 \) -
\[
\pi_1(\alpha(a)) = ADU \pi_1(a) \quad \forall a \in A_1.
\]
If \( U \) commutes with \( D_1 \), then for any states \( \varphi, \tilde{\varphi} \) one has
\[
d_{D_1}(\varphi, \tilde{\varphi}) = d_1(\varphi \circ \alpha, \tilde{\varphi} \circ \alpha).
\]

Proof. \( D_1 \) commutes with \( U \), so \([D_1, \pi_1(\alpha^{-1}b)] = (\text{ad} U^\ast)[D_1, \pi_1(b)]\) for any \( b \in A_1 \). Hence
\[
d_{D_1}(\varphi \circ \alpha, \tilde{\varphi} \circ \alpha) = \sup_{b \in \alpha(A_1)} \left\{ \|\varphi(b) - \tilde{\varphi}(b)\|, \|D_1, \pi_1(\alpha^{-1}b)\| \leq 1 \right\},
\]
\[
= \sup_{b \in \alpha(A_1)} \left\{ \|\varphi(b) - \tilde{\varphi}(b)\|, \|D_1, \pi_1(b)\| \leq 1 \right\} = d_{D_1}(\varphi, \tilde{\varphi}).
\]

This proposition has been stated in [30] for inner automorphism, while here (3.77) is less restricting. Also notice that in [3] the authors consider a condition less constraining than \([D_1, U] = 0\). This is not relevant for our purpose since \( D \) does commute with translations, hence the following corollary.

Corollary III.3 Translations are isometries of the Moyal plane, namely for any \( \kappa \in \mathbb{C} \)
\[
d_D(\varphi, \tilde{\varphi}) = d_D(\varphi_\kappa, \tilde{\varphi}_\kappa).
\]

Proof. One has to be careful that the unitary operator \( U_\kappa \) in (2.46) does not commute with \( D \) because of the phase factor appearing in (2.49), that is
\[
U_\kappa \psi = e^{i\kappa_3 x^3} \psi \circ \tau_\kappa
\]
where \( \psi \in L^2(\mathbb{R}) \) and \( \tau_\kappa \) is defined in (2.44). Nevertheless, the Dirac operator commutes with the unitary operator \( V_\kappa \psi = \psi \circ \tau_\kappa \) since
\[
DV_\kappa \psi = -\imath \gamma^\mu \partial_\mu (\psi \circ \tau_\kappa) = -\imath \gamma^\mu ((\partial_\mu \psi) \circ \tau_\kappa) = -\imath (\gamma^\mu \partial_\mu \psi \circ \tau_\kappa) = VD \psi.
\]
The result follows noticing that \( \text{ad} V_\kappa L(f) = L(f \circ \tau_\kappa) \), as can be checked writing
\[
(ad V L(f)) \psi = V L(f) (\psi \circ \tau_\kappa) = (f \ast (\psi \circ \tau_\kappa)) \circ \tau_\kappa = (f \circ \tau_\kappa) \ast \psi.
\]

Corollary III.3 indicates how the spectral distance \( d_D \) transforms under translation, but it gives no information on \( d_D(\varphi, \varphi_\kappa) \). In particular it does not imply (3.75).
\section*{III.2 Upper bound}

We show that $|\kappa|$ is an upper bound for $d_D(\varphi, \varphi_\kappa)$, starting with an easy technical lemma.

**Lemma III.4** For any $\varphi \in S(\mathcal{A})$, $f \in B_{L^p}(T)$ and $t \in [0,1]$, let us define

$$F(t) \equiv \varphi_{t\kappa}(f) = \varphi(\alpha_{t\kappa} f),$$

where $\kappa = (\kappa^1, \kappa^2) \in \mathbb{R}^2$ is fixed. Then

$$\frac{dF}{dt}|_t = \kappa^\mu \varphi_{t\kappa}(\partial_\mu f).$$

**Proof.** For $f \in \mathcal{A}$, let us write

$$\hat{f} = \frac{d}{dt}\alpha_{t\kappa} f = \kappa^\mu \alpha_{t\kappa} \partial_\mu f$$

and, for any non-zero real number $h$,

$$f_h = \frac{\alpha(t+h)\kappa f - \alpha_{t\kappa} f}{h}.$$

Notice that $\hat{f}$ and $f_h$ are in $S(\mathbb{R}^2)$. From definition, the result amounts to show that

$$\lim_{h \to 0} \varphi(f_h) = \varphi(\hat{f}).$$

By linearity and continuity of $\varphi$, one has

$$|\varphi(f_h) - \varphi(\hat{f})| \leq \|\varphi\| \left\| \mathcal{L}(f_h) - \mathcal{L}(\hat{f}) \right\| \leq \left\| f_h - \hat{f} \right\|_{L^2(\mathbb{R}^2)}$$

where we used that the operator norm is smaller than the $L^2$ norm \[^{[23]}\text{Lemma 2.12}\]. Observe that $f_h$ tends to $\hat{f}$ in the $S(\mathbb{R}^2)$ topology, meaning that for every $\epsilon > 0$ and integer $i > 0$ we can choose $\delta > 0$ such that for $|h| < \delta$ one has, for instance, $(1 + |x|^i)|f_h(x) - \hat{f}(x)| \leq \epsilon$, that is

$$|f_h(x) - \hat{f}(x)| \leq \frac{\epsilon}{(1 + |x|^i)}.$$  \hspace{1cm} (3.90)

By the dominated convergence theorem, $f_h \to \hat{f}$ in the $L^2$-topology, so (3.88) implies (3.87). $\blacksquare$

**Proposition III.5** For any $\kappa \in \mathbb{C}$ and $\varphi \in S(\mathcal{A})$, $d_D(\varphi, \varphi_\kappa) \leq |\kappa|$.

**Proof.** Let us denote $\bar{\kappa}$ the element of $\mathbb{C}^2$ with component $\bar{\kappa}^1 = \frac{1}{\sqrt{2}} \bar{\kappa}$, $\bar{\kappa}^2 = \frac{1}{\sqrt{2}} \bar{\kappa}$; and write $\bar{\partial}_1 = \partial$, $\bar{\partial}_2 = \bar{\partial}$. Inverting formula \[^{(2.18)}\] yields

$$\kappa^\mu \varphi(\alpha_{t\kappa} \partial_\mu f) = \frac{1}{\sqrt{2}} \left( \kappa^1 \varphi(\alpha_{t\kappa} \partial f) + \bar{\kappa}^2 \varphi(\alpha_{t\kappa} \bar{\partial} f) \right) = \bar{\kappa}^\mu \varphi(\alpha_{t\kappa} \bar{\partial}_\mu f).$$

By Cauchy-Schwartz and the continuity of $\varphi$, at any $\kappa$ one has

$$|\kappa^\mu \varphi(\alpha_{t\kappa} \partial_\mu f)| \leq \|\bar{\kappa}\| \sqrt{\sum_\mu |\varphi(\alpha_{t\kappa} \bar{\partial}_\mu f)|^2} \leq |\kappa| \sqrt{\sum_\mu \|\mathcal{L}(\bar{\partial}_\mu f)\|^2}.$$  \hspace{1cm} (3.91)

For $f$ in the Lipschitz ball, \[^{(2.24)}\] gives $\|\bar{\partial}_\mu f\| \leq \frac{1}{\sqrt{2}}$ for $\mu = 1, 2$. Lemma III.4 together with (3.91) yields

$$\left| \frac{dF}{dt} \right| \leq |\kappa|$$

for any $t$. Hence

$$|\varphi_\kappa(f) - \varphi(f)| = |F(1) - F(0)| \leq \int_0^1 \left| \frac{dF}{dt} \right| dt = |\kappa|. \hspace{1cm} (3.93)$$
III.3 Optimal element & regularization at infinity

Inspired by the analogy (in the commutative case) between the spectral distance and the Wasserstein distance of order 1 [17], let us introduce the following definition, which makes sense whatever algebra (commutative or not).

**Definition III.6** Given a spectral triple $T_1 = (A_1, \mathcal{H}_1, D_1)$, we call optimal element for a pair of states $(\varphi, \tilde{\varphi})$ an element of $\mathcal{B}_{Lip}(T_1)$ that attains the supremum in (1.1) or, in case the supremum is not attained, a sequence of elements $a_n \in \mathcal{B}_{Lip}(T_1)$ such that

$$\lim_{n \to +\infty} |\varphi(a_n) - \tilde{\varphi}(a_n)| = d_{D_1}(\varphi, \tilde{\varphi}).$$

As a first guess, we consider as an optimal element for a pair $(\varphi, \varphi_\kappa)$, $\varphi \in \mathcal{S}(A), \kappa \in \mathbb{C}$, the function

$$f_0(x_1, x_2) = \frac{1}{\sqrt{2}} (ze^{-i\Xi} + \bar{z}e^{i\Xi}),$$

where $\Xi = \text{Arg} \kappa$ and $z, \bar{z}$ are defined in (2.17). Obviously $L(f_0)$ satisfies the commutator norm condition (2.72) since, remembering the commutation relation (2.53), one has

$$\|[a, \pi_S(f_0)]\| = \frac{1}{\sqrt{2}} \|[a, a^*]\| = \frac{\theta}{\sqrt{2}}$$

together with a similar equation for $\|[a^*, \pi_S(f_0)]\|$. Furthermore, with $1$ the constant function $x \to 1$, one obtains

$$\alpha_\kappa f_0 = f_0 + |\kappa| 1$$

coming from

$$(\alpha_\kappa f_0)(x_1, x_2) = f_0(x_1 + \kappa_1, x_2 + \kappa_2) = f_0(x_1, x_2) + \frac{1}{2} (\kappa_1 e^{-i\Xi} + \bar{\kappa} e^{i\Xi}).$$

Therefore, assuming $\varphi(z) < \infty$ (that is, in the Schrödinger representation), assuming that $\varphi$ is in the domain of $a$, and working in the unitization of $A$ one gets, as expected,

$$|\varphi_\kappa(f_0) - \varphi(f_0)| = |\varphi(\alpha_\kappa f_0) - \varphi(f_0)| = \varphi(|\kappa| 1) = |\kappa|.$$

The point is that $f_0$ is not an optimal element, for it is not in the Moyal algebra $A$ but in its multiplier algebra $M$. So we need to regularize it by finding a sequence $\{f_n\}, n \in \mathbb{N}$, in $\mathcal{B}_{Lip}(T)$ with $T$ the spectral triple (2.12), and which converges to $f_0$ in a suitable topology. In the following proposition, inspired by the commutative case, we build from $f_0$ a net of element $f_\beta$ contained in the Lipschitz ball. We will extract from it the required optimal element $\{f_\beta\}$ in the next subsection.

**Proposition III.7** Let $\kappa = |\kappa|e^{i\Xi}$ be a fixed translation. For $\beta \in \mathbb{R}^+$, let us define

$$f_\beta = \frac{1}{\sqrt{2}} (z_\beta + z_\beta^*)$$

where $z_\beta = ze^{-i\Xi} \star e_\kappa - \frac{i}{\theta} z^{\star z}.$

Then there exists $\gamma > 0$ such that $f_\beta \in \mathcal{B}_{Lip}(T)$ for any $\beta \leq \gamma$.

**Proof.** First, let us check that $f_\beta$ is in $A$. As a formal power series of operators, one has

$$W\mathcal{L}(e^{-\frac{i}{\theta} z^{\star z}})W^* = e^{-\frac{i}{\theta} n} \otimes I,$$

where $n \doteq a^*a$ is the number operator. In the Schrödinger representation, $n$ is a diagonal matrix with generic term $n \theta$. Therefore for any $\beta \in (0, \infty)$, the operator $e^{-\frac{i}{\theta} n}$ is a matrix with fast decay coefficient so that - thanks to the isomorphism mentioned in remark 11.5 - the r.h.s. of (3.101) is in $\pi_S(A) \otimes I$ and $e^{-\frac{i}{\theta} z^{\star z}}$ is in $A$. The same is true for $f_\beta$ since $z$ is in the multiplier algebra $M$ of $A$.

From now on we put $\theta = 1$ and assume that $\Xi = 0$. By Lemma 11.7, $f_\beta$ is in the Lipschitz ball if and only if

$$\|[\beta(\beta)]\| \leq 1$$

(3.102)
where we define
\[ c(\beta) = [a, a^*_\beta] = (e^{-\beta} - e^{-\beta}(1 - e^{-\beta})a^2 - (1 - e^{-\beta}n)e^{-\beta n}), \] (3.103)

with \( a_\beta = \pi_S(z_\beta) \). The r.h.s. of (3.103) follows from the Baker-Campbell-Haussdorff formula, \( e^{-\beta a}e^{-\beta a} = e^{-\beta a} \), that is \([a, e^{-\beta a}] = (1 - e^{-\beta})a e^{-\beta a} \). To estimate the norm of \( c(\beta) \), we choose the energy eigenvectors basis and the use Schur’s test, that is \( \|c(\beta)\| \leq \|\epsilon(\beta)\|_1, \|\epsilon(\beta)\|_\infty \)

where \( \|c(\beta)\|_1 \), respectively \( \|c(\beta)\|_\infty \), is the maximum on \( n \in \mathbb{N} \) of the \( L^1 \)-norm of the columns, respectively the raws, of

\[
\epsilon(\beta) = \begin{pmatrix}
\lambda_{00} & 0 & \lambda_{02} & 0 & 0 & \ldots & 0 \\
0 & \lambda_{11} & 0 & \lambda_{13} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 & \lambda_{n-2,n} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \lambda_{n,n} \\
\end{pmatrix} (3.104)
\]

where, recalling that \( a h_n = \sqrt{n} h_{n-1} \),

\[
\lambda_{nn} = (e^{-\beta} - (1 - e^{-\beta})n)e^{-\beta n}, \quad \lambda_{n-2,n} = -e\beta(1 - e^{-\beta})\sqrt{n(n-1)}e^{-\beta n}. \] (3.105) (3.106)

More precisely, we prove that \( \|c(\beta)\| \leq (e^{-\beta}e^\beta)^{\frac{1}{2}} = 1 \) by showing that for \( \beta \) sufficiently small

\[
|\lambda_{n-2,n}| + |\lambda_{nn}| \leq e^{-\beta}, \quad |\lambda_{n+1,n}| + |\lambda_{nn}| \leq e^\beta. \] (3.107) (3.108)

Let us begin with (3.107). For \( n \leq e^{-\beta}/(1 - e^{-\beta}) \equiv n_0 \) one has \( \lambda_{n,n} \geq 0 \), while \( \lambda_{n-2,n} \leq 0 \) for any \( n \in \mathbb{N} \). So

\[
|\lambda_{n-2,n}| + |\lambda_{nn}| = \begin{cases}
(e^{-\beta} + (1 - e^{-\beta})(e^\beta \sqrt{n(n-1)} - n))e^{-\beta n} & \text{for } n \leq n_0, \\
(-e^{-\beta} + (1 - e^{-\beta})(e^\beta \sqrt{n(n-1)} + n))e^{-\beta n} & \text{for } n > n_0.
\end{cases}
\]

Let us assume \( n \leq n_0 \). It is easily verified that \( e^\beta \sqrt{n(n-1)} - n \leq 0 \) for \( n \leq (1 - e^{-2\beta})^{-1} \equiv n_1 \). Since \( n_1 \leq n_0 \) as soon as

\[
\beta \leq \beta_0 \equiv \ln \left(1 + \sqrt{\frac{1}{2}}\right), \] (3.109)

(3.107) is true for \( \beta \leq \beta_0 \) and \( 0 \leq n \leq n_1 \). For \( \beta \leq \beta_0 \) and \( n_1 \leq n \leq n_0 \), we have

\[
|\lambda_{n-2,n}| + |\lambda_{nn}| \leq (e^{-\beta} + n(1 - e^{-\beta})(e^\beta - 1))e^{-\beta n} \leq (e^{-\beta} + e^{-\beta}(e^\beta - 1))e^{-\beta/(1 - e^{-2\beta})} \leq e^{-\beta},
\]

where we simply substitute for \( n_0 \) in the polynomial factor and for \( n_1 \) in the exponential.

Suppose now \( n > n_0 \), so that

\[
|\lambda_{n-2,n}| + |\lambda_{nn}| \leq (-e^{-\beta} + n(1 - e^{-\beta})(e^\beta + 1))e^{-\beta n}. \] (3.110)

The function \((an - b)e^{-\beta n} (a, b > 0)\) reaches its maximum for \( n = \beta^{-1} + b/a \), therefore

\[
|\lambda_{n-2,n}| + |\lambda_{nn}| \leq \frac{(-e^{-\beta} + (1 - e^{-\beta})(e^\beta + 1) \left(1 + \frac{e^{-\beta}}{\beta(1 - e^{-\beta})(e^\beta + 1)}\right))e^{-\beta \left(1 - \frac{\beta^{-1} + b/a}{(1 - e^{-\beta})(e^\beta + 1)}\right)}}{\beta} \leq (e^\beta + 1)e^{-\beta n}.
\]
where we use $1 - e^{-\beta} \leq \beta$ for any $\beta \in \mathbb{R}^+$. One checks that $e^\beta(e^\beta + 1) \leq e$ as soon as

$$\beta \leq \beta_1 \doteq \ln\left(\frac{1}{2}\sqrt{1 + 4e} - 1\right).$$

(3.111)

Consequently, whatever $n$ the inequality (3.107) is true for $\beta \leq \min(\beta_0, \beta_1) = \beta_1$.

We now show (3.108). For any $\beta, n : \lambda_{n,n+2} = -e^{-\beta}(1 - e^{-\beta})\sqrt{(n+1)(n+2)}e^{-\beta}n \leq 0$. So

$$|\lambda_{n,n+2}| + |\lambda_{nn}| = \begin{cases} (e^{-\beta} + (1 - e^{-\beta})(e^{-\beta}\sqrt{(n+1)(n+2)} - n)) e^{-\beta}n & \text{for } n \leq n_0, \\ (-e^{-\beta} + (1 - e^{-\beta})(e^{-\beta}\sqrt{(n+1)(n+2)} + n)) e^{-\beta}n & \text{for } n > n_0. \end{cases}$$

For $n = 0$, this yields

$$|\lambda_{02}| + |\lambda_{00}| = e^{-\beta} + (1 - e^{-\beta})e^{-\beta}\sqrt{2} \leq e^{-\beta}\left(1 + \beta\sqrt{2}\right),$$

(3.112)

which is obviously smaller than $e^\beta$ since $1 + \sqrt{2}\beta \leq e^{2\beta}$ for any $\beta \in \mathbb{R}^+$.

For $1 \leq n \leq n_0$, either $(e^{-\beta}\sqrt{(n+1)(n+2)} - n) \leq 0$ and we are done; or

$$\left(e^{-\beta} + (1 - e^{-\beta})(e^{-\beta}\sqrt{(n+1)(n+2)} - n)) e^{-\beta}n \leq \left(e^{-\beta} + \beta(\sqrt{(n+1)(n+2)} - n)\right)e^{-\beta}.$$

(3.113)

Observing that

$$\sqrt{(n+1)(n+2)} - n = \frac{3n + 2}{\sqrt{(n+1)(n+2)} + n} \leq \frac{3n + 2}{2n + 1} \leq \frac{3}{2} + \frac{1}{n} \leq \frac{5}{2}$$

(3.114)

one obtains that (3.113) is smaller than $e^\beta$ as soon as $e^{-\beta} + 5\beta/2 \leq e^{2\beta}$. Noticing that $e^{2\beta} \geq 1 + 2\beta$, this is true as soon as $e^{-\beta} + 5\beta/2 \leq 1 + 2\beta$, or equivalently as soon as $(\beta - 2)e^{\beta - 2} \leq -2e^{-2}$, that is - denoting $W$ the Lambert function - for

$$\beta \leq \beta_2 = 2 + W(-2e^{-2}).$$

(3.115)

We are left with the case $n \geq n_0$. Then

$$|\lambda_{n,n+2}| + |\lambda_{nn}| \leq (-e^{-\beta} + (1 - e^{-\beta})(e^{-\beta}(n + 2) + n)) e^{-\beta}n$$

(3.116)

$$= ((1 - e^{-2\beta})n - (2e^{-2\beta} - e^{-\beta})) e^{-\beta}n.$$  

(3.117)

By the same reasoning as before, this is maximum for $n = \beta^{-1} + (2e^{-2\beta} - e^{-\beta})/(1 - e^{-2\beta})$, so that

$$|\lambda_{n,n+2}| + |\lambda_{nn}| \leq \frac{1 - e^{-2\beta}}{\beta} e^{-1}e^{-\beta}\frac{2e^{-2\beta} - e^{-\beta}}{1 - e^{-2\beta}} \leq \frac{2\beta}{\beta} e^{-1} = \frac{2}{e} < e^\beta \quad \forall \beta \in \mathbb{R}^+.$$  

(3.118)

To summarize, (3.102) holds true for any $n \in \mathbb{N}$ as soon as $\beta \leq \gamma \doteq \min(\beta_1, \beta_2) = \beta_1$. To conclude, we notice that restoring $\theta$ and $\Xi$ simply amounts to multiply $\lambda_{nn}$ by $\theta e^{-\Xi}$, and $\lambda_{n,n+2}$ by $\theta e^{-\Xi}$, so that the proof is unchanged.

III.4 Main result

At this point it might be useful to recall some well known facts regarding the state space of $\hat{A}$. By (2.66) and a classical result of operator algebras (see for example [39]), in every representation of $\hat{A}$ all states are normal, while all pure states are actually vector states. When the representation is irreducible (like the integrated Schrödinger representation), the correspondence between pure and vector states becomes one to one. In addition, normality has the following important consequence.
Remark III.8 [27] Theo. 7.1.12] Any non-pure state \( \varphi \in S(A) \) is a convex combination of pure states,

\[
\varphi(a) = \sum_{n=1}^{\infty} \lambda_n \langle \psi_i, \pi_S(a) \psi_i \rangle \quad \forall a \in A,
\]

(3.119)

where \( \psi_i \) are unit vectors in \( L^2(\mathbb{R}) \) and the \( \lambda_i \)'s are positive real numbers with \( \sum_{n=1}^{\infty} \lambda_i = 1 \). Consequently, the restriction of \( \varphi \) to the closed ball of radius \( r \in \mathbb{R}^+ \), \( B_r(A) \doteq \{ a \in A, ||a|| \leq r \} \), can be approximated by a finite combination of pure states. Indeed, denoting \( n_\epsilon \) the smallest integer such that \( \sum_{i=n_\epsilon+1}^{\infty} \lambda_i \leq \epsilon \) for some arbitrary fixed \( \epsilon \), one has

\[
|\varphi(a) - \sum_{n=1}^{n_\epsilon} \lambda_n \langle \psi_i, \pi_S(a) \psi_i \rangle| \leq r \epsilon \quad \forall a \in B_r(A).
\]

(3.120)

We can now prove the first main result of this paper, namely eq. (3.75) for any state \( \varphi \) in \( S(A) \) and any translation \( \kappa \in \mathbb{R}^2 \simeq \mathbb{C} \).

Theorem III.9 The spectral distance between a state and its translated is the Euclidean distance,

\[
d_{D}(\varphi, \varphi_{\kappa}) = |\kappa| \quad \forall \varphi \in S(A), \kappa \in \mathbb{C}.
\]

(3.121)

Proof. We split the proof in three parts: first we show that the result follows if

\[
\lim_{n \rightarrow \infty} \varphi(A(\beta_n)_{\kappa}) = 0
\]

(3.122)

where \( \{ \beta_n \} \) is a sequence of positive numbers tending to 0, satisfying \( \beta_n \leq \gamma \forall n \in \mathbb{N} \) for \( \gamma \) introduced in lemma III.7 \( A(\beta_n)_{\kappa} \) is defined below. It is an element of the Moyal algebra and, as such, sends Schwartz functions into Schwartz functions. Then we show that (3.122) actually holds for pure states. Finally we extend the result to arbitrary states.

i) Let us fix \( \beta > 0 \) and consider the net \( f_\beta, 0 < \beta \leq \gamma \) in the Lipschitz ball defined in (3.100). To lighten notation, we incorporate \( \theta \) into \( \beta \), i.e. \( \frac{\beta}{\theta} \rightarrow \beta \). The theorem amounts to show that, for any any state \( \varphi \in S(A) \) and any \( \kappa \in \mathbb{C} \), one has

\[
\lim_{\beta \rightarrow 0} |\varphi_{\kappa}(f_\beta) - \varphi(f_\beta)| = |\kappa|.
\]

(3.123)

Defining, as in lemma III.4, \( F(t) \doteq \varphi_{\tau_{\kappa}}(f_\beta) = \varphi(\alpha_{\kappa}, f_\beta) \), we will be done as soon as we show that

\[
\lim_{\beta \rightarrow 0} \frac{dF}{dt} = |\kappa|.
\]

(3.124)

To this aim, we fix \( \kappa \in \mathbb{C} \) and use the explicit form of the differential given by lemma III.4, namely

\[
\frac{dF}{dt} \big|_t = \kappa^\mu \varphi_{\tau_{\kappa}}(\partial_{\mu} f_\beta) = \kappa^\mu \varphi((\partial_{\mu} f_\beta) \circ \tau_{\kappa}) = \kappa^\mu \varphi(\partial_{\mu}(\alpha_{\kappa}, f_\beta)),
\]

(3.125)

where \( \tau_{\kappa} \) is defined in (2.44). By (2.67), using that \( \kappa a - \kappa a^* \) commutes with \( u_{\kappa} \), this gives

\[
\kappa^\mu \pi_S(\partial_{\mu}(\alpha_{\kappa}, f_\beta)) = \left[ \frac{\kappa a - \kappa a^*}{\theta \sqrt{2}}, \pi_S(\alpha_{\kappa}, f_\beta) \right] = \text{ad} u_{\kappa} \left[ \frac{\kappa a - \kappa a^*}{\theta \sqrt{2}}, \pi_S(f_\beta) \right].
\]

(3.126)

Denoting \( f_\beta \doteq \pi_S(f_\beta) \), one obtains

\[
\frac{dF}{dt} \big|_t = \frac{1}{\theta \sqrt{2}} \varphi(\text{ad} u_{\kappa} [\kappa a - \kappa a^*, f_\beta]),
\]

(3.127)

Notice that the pointwise limit is sufficient: substituting in (3.123) \( |\varphi_{\kappa}(f_\beta) - \varphi(f_\beta)| \) with its integral form (3.93), \( f_\beta \) being in the Lipschitz ball allows to exchange the limit and the integral thanks to the dominated convergence theorem.
where, with a slight abuse of notation, we write the evaluation of a state as \( \varphi(\pi_S(f)) \) instead of \( \varphi(f) \). By easy computations, one has

\[
[\kappa a - \kappa a^*, f_\beta] = \frac{1}{\sqrt{2}} (\{[\kappa a, a_\beta] + [\kappa a^*, a_\beta^*]\} + \text{adjoint}),
\]

and the result.

Let us denote the sum of the commutators in the equation above as a single operator \( A(\beta) \), which is in \( \mathcal{A} \) since both \( e^{-\beta a} \) and \( [\kappa a - \kappa a^*, f_\beta] = \theta \sqrt{2\kappa^2} \partial_\beta f_\beta \) are in \( \mathcal{A} \). Define similarly

\[
A(\beta)_{\kappa} \doteq \text{ad} u_{\kappa} A(\beta) = \text{ad} u_{\kappa} [\kappa a - \kappa a^*, f_\beta] - \sqrt{2\kappa} |e^{-\beta n_{\kappa}}|,
\]

with

\[
a_{\kappa} \doteq (\text{ad} u_{\kappa}) a = a + \frac{t_{\kappa}}{\sqrt{2}} r_{\beta}, \quad a_{\beta}^* \doteq (\text{ad} u_{\kappa}) a^* = a^* + \frac{t_{\kappa}}{\sqrt{2}} r_{\beta}, \quad u_{\kappa} \doteq (a^* a)_{\kappa} = a_{\kappa}^* a_{\kappa}.
\]

Again \( A(\beta)_{\kappa} \) is in \( \mathcal{A} \), for the latter is invariant by \( \text{ad} u_{\kappa} \). This allows to write \( \text{(3.127)} \) as

\[
\frac{dF_{\beta}}{dt} = \frac{1}{\theta} \varphi(e^{-\beta n_{\kappa}}) + \frac{1}{\theta \sqrt{2}} \varphi(A(\beta)_{\kappa}).
\]

The operator \( n_{\kappa} \) is positive and selfadjoint, so by the Hille-Yosida theorem \([37]\) the application \( (0, +\infty) \ni \beta \to e^{-\beta n_{\kappa}} \) defines a contraction semi-group. In particular one has for \( \beta \geq 0 \) and any \( \psi \in L^2(\mathbb{R}) \),

\[
\|e^{-\beta n_{\kappa}}\| \leq 1 \quad \text{and} \quad \lim_{\beta \to 0} e^{-\beta n_{\kappa}} \psi = \psi,
\]

so that remark [III.8] yields

\[
\lim_{\beta \to 0} \varphi(e^{-\beta n_{\kappa}}) = 1.
\]

So as soon as the limit \( \text{(3.122)} \) holds true for some sequence \( 0 < \beta_n \leq \gamma \), \( \text{(3.132)} \) reduces to \( \text{(3.124)} \) and the theorem follows.

\[\text{ii)}\] To prove the limit \( \text{(3.122)} \), we need to evaluate the various terms of \( \varphi(A(\beta)_{\kappa}) \). Let us first do it assuming \( \varphi \) is a pure state \( \langle \psi, \psi \rangle \) with \( \psi \in S(\mathbb{R}) \). Developing the commutator in \( \text{(3.130)} \), one obtains

\[
A(\beta)_{\kappa} = \frac{1}{\sqrt{2}} (\kappa e^{-i\kappa} a_{\kappa} [a, e^{-\beta n_{\kappa}}] + \kappa e^{i\kappa} [a, e^{-\beta n_{\kappa}}] a_{\kappa}^*) + \text{adjoint}.
\]

Let us consider the first term of this equation, disregarding the constant coefficients. One has

\[
\|a_{\kappa} [a, e^{-\beta n_{\kappa}}] \psi\| = \|a_{\kappa} [a + \frac{t_{\kappa}}{\sqrt{2}} r_{\beta}, I - e^{-\beta n_{\kappa}}] \psi\|
\]

\[
\leq \|a_{\kappa}^2 (I - e^{-\beta n_{\kappa}}) \psi\| + \|a_{\kappa} (I - e^{-\beta n_{\kappa}}) a_{\kappa} \psi\|.
\]

Calculating explicitly the first norm in \( \text{(3.137)} \), one finds

\[
\|a_{\kappa}^2 (I - e^{-\beta n_{\kappa}}) \psi\|^2 = \langle a_{\kappa}^2 (I - e^{-\beta n_{\kappa}}) \psi, a_{\kappa}^2 (I - e^{-\beta n_{\kappa}}) \psi \rangle
\]

\[
= \langle a_{\kappa}^2 e^{-\beta n_{\kappa}} \psi, a_{\kappa}^2 e^{-\beta n_{\kappa}} \psi \rangle + \langle a_{\kappa}^2 \psi, a_{\kappa}^2 \psi \rangle - 2 \text{Re} \langle a_{\kappa}^2 e^{-\beta n_{\kappa}} \psi, a_{\kappa}^2 \psi \rangle
\]

\[
= \langle e^{-\beta n_{\kappa}} \psi, a_{\kappa}^2 a_{\kappa}^2 \psi \rangle + \langle \psi, a_{\kappa}^2 a_{\kappa}^2 \psi \rangle - 2 \text{Re} \langle e^{-\beta n_{\kappa}} \psi, a_{\kappa}^2 a_{\kappa}^2 \psi \rangle.
\]

The three terms in \( \text{(3.140)} \) are finite, for \( \psi \) is Schwartz. Moreover, by \( \text{(3.133)} \) they cancel each other as \( \beta \to 0 \). The same argument applies to \( \|a_{\kappa} [a, e^{-\beta n_{\kappa}}] a_{\kappa} \psi\| \). Repeating the procedure for \( [a, e^{-\beta n_{\kappa}}] a_{\kappa}^* \) and the adjoints, one gets

\[
\lim_{\beta \to 0} \|A(\beta)_{\kappa} \psi\| = 0,
\]

so that, by Cauchy-Schwartz, \( \lim_{\beta \to 0} |\varphi(A(\beta)_{\kappa})| \leq \lim_{\beta \to 0} \|A(\beta)_{\kappa} \psi\| = 0 \). This implies \( \text{(3.122)} \) and the result.

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Now, fix any pure state \( \hat{\varphi} = (\tilde{\psi}, \tilde{\psi}) \) for some unit vector \( \tilde{\psi} \in L^2(\mathbb{R}) \), and take a Schwartz-pure state \( \varphi \) as before such that
\[
\|\varphi - \hat{\varphi}\| < \frac{\epsilon}{r}
\] (3.142)
for arbitrary real positive numbers \( r \) and \( \epsilon \). This is always possible for \( S(\mathbb{R}) \) is dense in \( L^2(\mathbb{R}) \) (by Cauchy-Schwartz one has \( |(\varphi - \hat{\varphi})(a)| \leq 2\|\psi\|_{L^2(\mathbb{R})} \|\delta\psi\|_{L^2(\mathbb{R})} + \|\delta\psi\|^2_{L^2(\mathbb{R})} \) for any \( a \) of norm 1, where \( \delta\psi \equiv \tilde{\psi} - \psi \) has arbitrary small norm). Then
\[
|\hat{\varphi}(A(\beta)_{t\kappa})| \leq \|\varphi - \hat{\varphi}\| \|A(\beta)_{t\kappa}\| + |\varphi(A(\beta)_{t\kappa})| \leq \frac{\epsilon}{r} \|A(\beta)_{t\kappa}\| + |\varphi(A(\beta)_{t\kappa})|.
\] (3.143)

From the definition \( (3.130) \) of \( A(\beta)_{t\kappa} \), the explicit form \( (3.126) \) of the derivative and the strong continuity \( (3.133) \), using moreover that \( \beta \) is in the Lipschitz ball so that - by \( (2.24) \) - \( \|\partial_\mu f_\beta\| \leq 2^{-\frac{\epsilon}{r}} \), one obtains
\[
\|A(\beta)_{t\kappa}\| \leq \theta \sqrt{2\kappa} \|\partial_\mu f_\beta\| + \sqrt{2\theta}|\kappa| \leq \theta \sum_{\mu} |\kappa^\mu| + \sqrt{2\theta}|\kappa|,
\] (3.144)

Taking as a parameter \( r \) the r.h.s. of the equation above,
\[
r = \theta \sum_{\mu} |\kappa^\mu| + \sqrt{2\theta}|\kappa|,
\] (3.145)
and remembering, as shown above, that eq.\( (3.122) \) holds true for \( \varphi \), eq.\( (3.143) \) yields
\[
\lim_{\beta \to 0} |\hat{\varphi}(A(\beta)_{t\kappa})| = 0,
\]
hence the result.

iii) The argument for an arbitrary state in \( S(A) \) is now straightforward. For any \( t \in [0, 1] \), the net \( A(\beta)_{t\kappa} \), \( 0 < \beta \leq \gamma \), is contained within the closed ball \( B_r(A) \subset B(L^2(\mathbb{R})) \) with radius \( r \) given by \( (3.145) \). Since \( B_r(A) \) is compact (and metrizable) in the \( \sigma \)-weak topology of \( B(L^2(\mathbb{R})) \) (as any closed ball, see \( [39] \)), from any sequence \( \{A(\beta_n)_{t\kappa}\}_{n=1}^{\infty} \) with \( \beta_n \to 0 \), one can extract a sub-sequence \( \{A(\beta_n)_{t\kappa}\}_{n=1}^{\infty} \) such that, for every (normal) state \( \varphi \) in the predual \( B(H)_* \),
\[
\lim_{j \to \infty} \varphi(A(\beta_n)_{t\kappa}) = \varphi(A(0))
\] (3.146)
for some \( A(0) \in B_r(A) \). Fixing \( \epsilon > 0 \), the same is true for the finite convex combination
\[
\sigma_\epsilon \equiv \sum_{n=1}^{n_\epsilon} \lambda_i(\psi_i, \cdot \psi_i)
\] (3.147)
developed in remark \( III.8 \) that is
\[
\lim_{j \to \infty} \sigma_\epsilon(A(\beta_n)_{t\kappa}) = \sigma_\epsilon(A(0)).
\] (3.148)
But by the result of ii), each of the terms of \( \sigma_\epsilon(A(\beta_n)_{t\kappa}) \) tends to zero, that is \( \sigma_\epsilon(A(0)) = 0 \). Therefore
\[
|\varphi(A_0)| \leq r\epsilon,
\] (3.149)
that is
\[
\lim_{j \to \infty} |\varphi(A(\beta_n)_{t\kappa})| \leq r\epsilon
\]
which again is \( (3.122) \).
III.5 Discussion on the optimal element

In the commutative case, the optimal element for two pure states \( \omega_x, \omega_y \in \mathcal{P}(C_0^\infty(\mathbb{R}^2)) \) such that \( y \) belongs to the segment \([0, x]\) is the function

\[
l(z) \doteq \sqrt{2}|z|,
\]

regularized at infinity, for instance considering the sequence

\[
l_n(z) = \sqrt{2}|z|e^{-\frac{|z|^2}{n}} \in C_0^\infty(\mathbb{R}^2).
\]

Indeed, one has

\[
\omega_n(l) - \omega_y(l) = |x - y|
\]

(the \( \sqrt{2} \) factor is compensated by the one in the definition (2.17)) and \( l \) is in the (commutative) Lipschitz ball, defined similarly as in (2.24) with the supremum norm instead of \( \|\mathcal{L}(\cdot)\| \).

\[
\left\| \frac{\partial}{\partial L} |z| \right\| = \frac{1}{\sqrt{2}} \left\| \frac{\bar{z}}{|z|} \right\| = \frac{1}{\sqrt{2}}.
\]

In the Moyal plane (meaning in the left regular representation), a first natural candidate as an optimal element for the pair \( \varphi, \varphi_\kappa \) could be a suitable regularization of either

\[
\mathcal{L}(l) = \sqrt{2}|\mathcal{L}(|z|)| \quad \text{or} \quad \sqrt{2}|\mathcal{L}(z)|.
\]

The first of these operators is not very tractable since, to our knowledge, there is no easy formula for \(|z| \ast z \) or \(|z| \ast \bar{z} \). On the contrary, the second operator written as \( \sqrt{2} \sqrt{\mathcal{L}(z^2)\mathcal{L}(\bar{z})} \) is easy to deal with: its Schrödinger representation is \( \sqrt{2a^\ast a} \). However, one easily checks by (2.72) that \( \sqrt{2} |\mathcal{L}(z)| \) does not satisfy the commutator norm condition, since \([a^\ast, \sqrt{a^\ast a}] \) is unbounded \( ([a^\ast, \sqrt{a^\ast a}^\ast][a^\ast, \sqrt{a^\ast a}] \) is diagonal with spectrum \( 2\sqrt{n(n-1)} \), \( n \in \mathbb{N} \). Consequently, building a regularized sequence \( f_n \in \mathcal{B}_{Lip}(T) \) such that

\[
(\varphi - \varphi_\kappa)(f_n) \xrightarrow{n \to \infty} (\varphi - \varphi_\kappa)(|\mathcal{L}(z)|)
\]

is hopeless. Said differently, the optimal element for the Moyal plane is not the left-regular representation of the optimal element of the commutative case. Furthermore, one can show that the supremum of \( \varphi(f) - \varphi_\kappa(f) \) on \( f \in \mathcal{B}_{Lip}(T) \) such that \( \pi_S(f) \) is diagonal yields a quantity lower than the spectral distance [28].

Interestingly, the optimal (up to regularization) element \( f_0 \) [3.35] is similar to the optimal element \( l_0 \) used in [17] to compute the distance between non-pure states \( \varphi \) of \( C_0^\infty(\mathbb{R}^2) \),

\[
\varphi(f) = \int_{\mathbb{R}^2} f d\mu, \quad \alpha_n \varphi(f) = \int_{\mathbb{R}^2} \alpha_n f d\mu,
\]

where \( d\mu \) is the measure of probability characterizing the state \( \varphi \); that is [17, eq. 3.5]

\[
l_0(z) \doteq \sqrt{2}ze^{i\varphi}.
\]

This is not surprising, since pure states of \( \mathcal{A} \) are integrations against a probability distribution and therefore are similar to non-pure states of \( C_0^\infty(\mathbb{R}^2) \).

We conclude by noticing that in proposition III.7, we regularize by \( e^{-\frac{p}{n} |\mathcal{L}(z)|^2} \), instead of \( e^{-\frac{2}{n} |z|^2} \) in (3.151). The constant coefficients are not relevant, what is interesting is the exponent of the argument of the exponential. In the commutative case, one could choose as well \( |z|^2 \) or \( |z|^3 \), for \( \left\| \text{grad} \left(|z|e^{-\frac{|z|^2}{n}}\right) \right\| \) is never greater than 1 for \( p = 1, 2, 3 \). In proposition III.7 we chose the exponential \( p = 2 \) rather than \( p = 1 \), for the commutation relations of \( |\mathcal{L}(z)|^2 = n \) are easier to deal with than those of \( |\mathcal{L}(z)| = \sqrt{n} = |a| \).
IV Pythagoras relations

The product of an (even) spectral triple \( T_1 = (A_1, \mathcal{H}_1, D_1) \) with the canonical spectral triple on \( \mathbb{C}^2 \) given by

\[
A_2 = \mathbb{C}^2, \quad \mathcal{H}_2 = \mathbb{C}^2, \quad D_2 \doteq \begin{pmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{pmatrix} \quad (4.158)
\]

where \( \Lambda \) is a constant complex parameter and \( \mathbb{C}^2 \) acts on itself as

\[
\pi_2(z^1, z^2) \doteq \begin{pmatrix} z^1 & 0 \\ 0 & z^2 \end{pmatrix} \quad z^1, z^2 \in \mathbb{C}, \quad (4.159)
\]

is (see [14]) the spectral triple \( T' = (\mathcal{A}', \mathcal{H}', D') \) with

\[
\mathcal{A}' \doteq A_1 \otimes A_2, \quad \mathcal{H}' \doteq \mathcal{H}_1 \otimes \mathcal{H}_2, \quad D' \doteq D_1 \otimes I_2 + \Gamma_1 \otimes D_2, \quad (4.160)
\]

where \( \Gamma_1 \) denotes the chirality of \( T_1 \), that is a graduation of \( H_1 \) such that

\[
\Gamma_1^2 = I_1 \text{ is the identity of } \mathcal{H}_1 \text{ and } [\Gamma_1, \pi_1(A_1)] = 0. \quad (4.161)
\]

We consider the subspace \( Q(\mathcal{A}') \) of \( S(\mathcal{A}') \) consisting in pairs

\[
\varphi^i \doteq (\varphi, \delta^i) \quad (4.162)
\]

where \( \varphi \in S(A_1) \) and \( \delta^i \in P(A_1) \) is one of the two pure states of \( \mathbb{C}^2 \),

\[
\delta^i(z^1, z^2) = z^i \quad i = 1, 2. \quad (4.163)
\]

\( Q(\mathcal{A}') \) is the disjoint union of two copies of \( S(A_1) \). Since \( A_2 \) is commutative, pure states of \( \mathcal{A}' \) are pairs 39 so that

\[
\mathcal{P}(\mathcal{A}') \subset Q(\mathcal{A}') \subset S(\mathcal{A}'). \quad (4.165)
\]

A generic element of \( \mathcal{A}' = A_1 \otimes \mathbb{C}^2 \simeq A_1 \oplus A_1 \) is \( a' = (f, g) \) with \( f, g \in A_1 \). Its evaluation on a state \( \varphi^i \in Q(\mathcal{A}') \) reads

\[
\varphi^i(a') = \varphi(f), \quad \varphi^2(a') = \varphi(g). \quad (4.166)
\]

We make the extra-assumption that the spectral triple \( T_1 \) is unital and non-degenerate. These conditions are discussed in remark IV.6.

Considering either two states \( \varphi^1, \varphi^2 \) on the same copy of \( S(A_1) \), or two copies \( \varphi^1, \varphi^2 \) of the same state \( \varphi \in S(A_1) \), one finds that the spectral distance \( d_{P'} \) on \( T' \) reduces either to the distance \( d_{D_1} \) on \( T_1 \) or \( d_{D_2} \) on \( T_2 \), namely \( 34, 30 \)

\[
d_{D'}(\varphi^i, \varphi^j) = d_{D_1}(\varphi, \varphi), \quad (4.167)
\]

\[
d_{D'}(\varphi^i, \varphi^j) = d_{D_2}(\delta^i, \delta^j) = |\Lambda|^{-1}. \quad (4.168)
\]

Furthermore, for \( T_1 \) the canonical (commutative) spectral triple of a compact spin manifold \( \mathcal{M} \), Pythagoras equality holds true for pure states \( \omega_x^i = (\omega_x \in \mathcal{P}(C_0^\infty(\mathcal{M}), \delta^i)) \), that is \( 34, 30 \)

\[
d_{D'}(\omega_x^1, \omega_y^1) = \sqrt{d_{D'}^2(\omega_x^1, \omega_y^1) + d_{D'}^2(\omega_x^1, \omega_y^2)} \quad (4.169)
\]

We prove below a Pythagoras inequality,

\[
\sqrt{d_{D'}^2(\varphi^1, \tilde{\varphi}^1) + d_{D'}^2(\varphi^2, \tilde{\varphi}^2)} \leq d_{D'}(\varphi^1, \tilde{\varphi}^1) \leq \sqrt{2 \sqrt{d_{D'}^2(\varphi^1, \tilde{\varphi}^1) + d_{D'}^2(\varphi^1, \tilde{\varphi}^2)}}, \quad (4.170)
\]

for an arbitrary unital, non-degenerate spectral triple \( T_1 \). Then we show that on the Moyal plane and for arbitrary \( \varphi \in S(\mathcal{A}) \), Pythagoras equality holds true on the subset of \( Q(\mathcal{A}') \) given by two disjoint copies of the set \( C(\varphi) \) introduced in (3.74), namely (see figure 1)

\[
Q(\varphi) = C(\varphi) \times C(\varphi) = \{ (C(\varphi), \delta^i) \, i = 1, 2 \}. \quad (4.171)
\]
IV.1 Pythagoras inequalities

Let us begin by an elementary lemma.

**Lemma IV.1** Let \((A_1, \mathcal{H}_1, D_1)\) be any spectral triple. For any states \(\varphi, \tilde{\varphi} \in \mathcal{S}(A_1)\) and any \(a\) in \(B_{\text{Lip}}(T_1)\) that does not commute with \(D_1\), one has \(|\varphi(a) - \tilde{\varphi}(a)| \leq \|[D_1, \pi_1(a)]\| d_{D_1}(\varphi, \tilde{\varphi})\).

**Proof.** Let \(\bar{a} \doteq \frac{a}{\|[D_1, \pi_1(a)]\|}\). Then \(\|[D_1, \pi_1(\bar{a})]\| = 1\). Hence the result by definition of \(d_{D_1}\). \(\square\)

**Proposition IV.2** Let \((A', \mathcal{H}', D')\) be the product \((4.161)\) of an arbitrary even, unital and non-degenerate spectral triple \(T\) with the spectral triple \(T_2\) \((4.158)\) of \(\mathbb{C}^2\). For any \(\varphi^1, \tilde{\varphi}^2 \in \mathcal{Q}(A')\), one has

\[
\sqrt{d^2_{D'}(\varphi^1, \tilde{\varphi}^1) + d^2_{D'}(\varphi^1, \tilde{\varphi}^2)} \leq d_{D'}(\varphi^1, \tilde{\varphi}^2) \leq \sqrt{2} \sqrt{d^2_{D'}(\varphi^1, \tilde{\varphi}^1) + d^2_{D'}(\varphi^1, \tilde{\varphi}^2)}. \tag{4.172}
\]

**Proof.** \(a' = (f, g)\) is represented as

\[
\pi'(a') = \begin{pmatrix} \pi_1(f) & 0 \\ 0 & \pi_1(g) \end{pmatrix}. \tag{4.173}
\]

The Dirac operator \(D'\) acts as

\[
D' = \begin{pmatrix} D_1 & \Lambda \Gamma_1 \\ \Lambda \Gamma_1 & D_1 \end{pmatrix}, \tag{4.174}
\]

so that, by \((4.161)\),

\[
[D', \pi'(a)] = \begin{pmatrix} [D_1, \pi_1(f)] & \Lambda \Gamma_1 \pi_1(f - g) \\ \Lambda \Gamma_1 \pi_1(g - f) & [D_1, \pi_1(g)] \end{pmatrix}. \tag{4.175}
\]

Define the subset

\[
A' \supset B \doteq \{(f, g) = (f^*, g^*) \in A', f - g = \lambda \mathbf{1} \text{ for some } \lambda \in \mathbb{R}^+\} \tag{4.176}
\]

with \(\mathbf{1}\) the unit of \(A_1\), and let

\[
d_B(\varphi^1, \tilde{\varphi}^2) \doteq \sup_{b \in B} \{\|\varphi^1(b) - \tilde{\varphi}^2(b)\|, \|[D', \pi'(b)]\| \leq 1\}. \tag{4.177}
\]

Obviously \(d_{D'}(\varphi^1, \tilde{\varphi}^2) \geq d_B(\varphi^1, \tilde{\varphi}^2)\), so the l.h.s. of \((4.172)\) follows if we show that

\[
d_B(\varphi^1, \tilde{\varphi}^2) = \sqrt{d^2_{D'}(\varphi^1, \tilde{\varphi}^1) + d^2_{D'}(\varphi^1, \tilde{\varphi}^2)}. \tag{4.178}
\]

To this aim, let us fix \(\varphi, \tilde{\varphi}\) in \(\mathcal{S}(A_1)\), and consider

\[
b = (f, f + \lambda \mathbf{1}) \in B \cap B_{\text{Lip}}(T'). \tag{4.179}
\]

Noticing that \([D_1, \pi_1(f)]^* = -[D_1, \pi_1(f)]\), \((4.175)\) yields

\[
[D', \pi'(b)][D', \pi'(b)]^* = ([D_1, \pi_1(f)][D_1, \pi_1(f)]^* + \lambda^2|\Lambda|^2 \mathbb{1}) \otimes \mathbb{1}. \tag{4.180}
\]

For any positive element in a unital \(C^*\)-algebra and \(\lambda \in \mathbb{R}^+, \|a + \lambda \mathbf{1}\| = \|a\| + \lambda\), hence

\[
1 \geq \|[D', \pi'(b)]\| = \sqrt{\|[D_1, \pi_1(f)]\|^2 + \lambda^2|\Lambda|^2}. \tag{4.181}
\]

Furthermore, for any \(b \in B\) lemma \((IV.1)\) yields

\[
|\varphi^1(b) - \tilde{\varphi}^2(b)| = |\varphi(f) - \tilde{\varphi}(f) + \lambda| \leq d_{D_1}(\varphi, \tilde{\varphi}) \|[D_1, \pi_1(f)]\| + \lambda. \tag{4.182}
\]

Therefore, using \((4.181)\) as \(\|[D_1, \pi_1(f)]\| \leq \sqrt{1 - \lambda^2|\Lambda|^2} d_1 + \lambda\), one obtains

\[
d_B(\varphi^1, \tilde{\varphi}^2) \leq \sup_{\lambda \in \mathbb{R}^+} F(\lambda) \quad \text{with} \quad F(\lambda) \doteq \sqrt{1 - \lambda^2|\Lambda|^2} d_1 + \lambda \tag{4.183}
\]
where we write $d_1$ instead of $d_{D_1}(\varphi, \varphi')$. The function $F$ reaches its maximum on $\mathbb{R}^+$,

$$F(\lambda_{\text{max}}) = \sqrt{\frac{1}{|\Lambda|^2} + d_1^2},$$

(4.184)

for

$$\lambda_{\text{max}} = \frac{1}{|\Lambda|\sqrt{d_1^2 + 1}}.$$  

(4.185)

This upper bound in the spectral distance formula is attained by $b = (f, f + \lambda_{\text{max}}1)$ with

$$f = \frac{|\Lambda|d_1}{\sqrt{|\Lambda|^2d_1^2 + 1}}f_1,$$

(4.186)

where $f_1$ is the element that attains the supremum in the computation of $d_1$. Therefore

$$d_B(\varphi^1, \varphi^2) = \sqrt{\frac{1}{|\Lambda|^2} + d_1^2},$$

(4.187)

which, by (4.167), (4.168) is nothing but (4.178). Hence the l.h.s. of (4.172).

The r.h.s. follows from (4.167) and (4.168) by the triangle inequality,

$$d_{D'}(\varphi^1, \varphi^2) \leq d_{D'}(\varphi^1, \tilde{\varphi}^1) + d_{D'}(\tilde{\varphi}^1, \varphi^2),$$

(4.188)

together with $(a + b)^2 \leq 2a^2 + 2b^2$.

Figure 1: The spectral distance on $Q(\varphi)$.

### IV.2 Pythagoras equality

Since the Moyal algebra $\mathcal{A}$ has no unit, in order to apply proposition IV.2 above we should work with the (minimal) unitization $\mathcal{A}^+$ of $\mathcal{A}$, that is

$$\mathcal{A}^+ = S(\mathbb{R}^2, \ast) \otimes \mathbb{C}$$

(4.189)

as a vector space, with product $(f, \lambda), (g, \lambda) = (fg + \lambda g + \bar{\lambda} f, \lambda \bar{\lambda})$ and unit $1 = (0, 1)$. The left-regular representation extends to $\mathcal{A}^+$ as

$$\mathcal{L}(f_\lambda) \doteq \mathcal{L}(f) + \lambda \mathbb{1} \quad \text{where } f_\lambda \doteq (f, \lambda),$$

(4.190)

while the representation $\pi$ of the spectral triple (2.21) extends to $\mathcal{A}^+$ as

$$\pi(f_\lambda) = \mathcal{L}(f_\lambda) \otimes \mathbb{1}_2 = \pi(f) + \lambda \mathbb{1} \otimes \mathbb{1}_2.$$  

(4.191)

We denote $T^+ = (\mathcal{A}^+, \mathcal{H}, D)$ the unital spectral triple of the Moyal plane, with $\mathcal{H}$ and $D$ still given by (2.20) and (2.19). Notice that any state $\varphi \in S(\mathcal{A})$ linearly extends to a state in $S(\mathcal{A}^+)$ by setting $\varphi(0,1) = 1$.

Switching from $T$ to $T^+$ has no incidence on the spectral distance on $S(\mathcal{A})$.  

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Lemma IV.3 Let \((A_1, \mathcal{H}_1, D_1)\) be a non-unital, non-degenerate spectral triple. For any \(\varphi, \tilde{\varphi} \in \mathcal{S}(A_1)\),
\[d_{D_1}(\varphi, \tilde{\varphi}) = \sup_{a \in A_1^+} \{||\varphi(a) - \tilde{\varphi}(a)||, ||[D_1, \pi_1(a)]|| \leq 1\}\] (4.192)
where \(\pi_1\) is extended to \(A_1^+\) by setting \(\pi_1(1) = 1\).

Proof. One has \(\varphi((a, \lambda)) = \varphi(a) + \lambda\), so that \(\varphi((a, \lambda)) - \tilde{\varphi}((a, \lambda)) = \varphi(a) - \tilde{\varphi}(a)\) does not depend on \(\lambda\). Similarly \(\|D_1, \pi_1(a, \lambda)\| = [D_1, \pi_1(a)] + \lambda [D_1, \pi_1(\lambda)] = [D_1, \pi_1(a)]\) does not depend on \(\lambda\). Hence it is equivalent to look for the supremum on \(A_1\) or on \(A_1^+\).

We now consider the product \(T^\prime\) of the unital, non-degenerate, spectral triple \(T^+\) of the Moyal plane with the spectral triple \(T_2\) on \(\mathbb{C}^2\) given in (4.158), namely
\[A' = A^+ \otimes \mathbb{C}^2, \quad \mathcal{H}' = \mathcal{H} \otimes \mathbb{C}^2, \quad D' = D \otimes 1_{2} + \Gamma \otimes D_2\] (4.193)
The graduation \(\Gamma\) is the third Pauli matrix \(\sigma_3\).

We begin with a technical lemma regarding the Lipschitz ball.

Lemma IV.4 For any \(a' = (f_\lambda, g_\lambda)\) in \(B_{Lip}(T^\prime)\), with \(f_\lambda, g_\lambda \in A^+\), one has
\[\left\|\mathcal{L}(\partial f)^* \mathcal{L}(\partial f) + \frac{\|\Lambda\|^2}{2} \mathcal{L}(g_\lambda - f_\lambda)^* \mathcal{L}(g_\lambda - f_\lambda)\right\| \leq \frac{1}{2}\] (4.194)
\[\left\|\mathcal{L}(\partial g)^* \mathcal{L}(\partial g) + \frac{\|\Lambda\|^2}{2} \mathcal{L}(g_\lambda - f_\lambda)^* \mathcal{L}(g_\lambda - f_\lambda)\right\| \leq \frac{1}{2}\] (4.195)
and similarly for \(\partial\).

Proof. From (4.173), one gets
\[\left[\mathcal{L}(\partial f)^* \mathcal{L}(\partial f) + \frac{\|\Lambda\|^2}{2} \mathcal{L}(g_\lambda - f_\lambda)^* \mathcal{L}(g_\lambda - f_\lambda)\right]\] (4.196)
Multiplying on the left and right by the diagonal matrices of norm 1
\[\Delta_L \doteq \text{diag}(1, 0), \quad \Delta_R \doteq \text{diag}(1, \Gamma),\] (4.197)
one obtains
\[\|\Delta_L [D', \pi'(a')]_{\Delta_R}\| \leq \|[D', \pi'(a')]\| \leq 1\] (4.198)
where, using the properties (4.161) of \(\Gamma\),
\[\Delta_L [D', \pi'(a')]_{\Delta_R} = \left( \begin{array}{cc} [D, \pi(f_\lambda)] & \Lambda \pi(g_\lambda - f_\lambda) \\ \Gamma \Lambda \pi(g_\lambda - f_\lambda) & [D, \pi(g_\lambda)] \end{array} \right)\] (4.199)
has norm
\[\|\Delta_L [D', \pi'(a')]_{\Delta_R}\|^2 = \|[D, \pi(f_\lambda)]^* [D, \pi(f_\lambda)] + \|\Lambda\|^2 \pi(g_\lambda - f_\lambda)^* \pi(g_\lambda - f_\lambda)\|\] (4.200)
Let us evaluate the various terms of the equation above. On the one hand,
\[\left[D, \pi(f_\lambda)\right] = [D, \pi(f) + \lambda [1_2 \otimes 1]] = [D, \pi(f)],\] (4.201)
so that by (2.23)
\[\left[D, \pi(f_\lambda)\right]^* [D, \pi(f_\lambda)] = 2 \left( \begin{array}{cc} \mathcal{L}(\partial f)^* \mathcal{L}(\partial f) & 0 \\ 0 & \mathcal{L}(\partial f)^* \mathcal{(\partial f)} \end{array} \right)\] (4.202)
On the other hand,
\[\pi(g_\lambda - f_\lambda) = \mathcal{L}(g_\lambda - f_\lambda) \otimes 1_2,\] (4.203)
Consequently \(\|\Delta_L [D', \pi'(a')]_{\Delta_R}\|^2 = \max\{\|A_\theta\|, \|A_\theta \|\}\) where
\[A_\theta \doteq 2 \mathcal{L}(\partial f)^* \mathcal{L}(\partial f) + \|\Lambda\|^2 \mathcal{L}(g_\lambda - f_\lambda)^* \mathcal{L}(g_\lambda - f_\lambda),\] (4.204)
By Cauchy-Schwartz, one thus obtains
\[ d_D'(\varphi^1, \varphi^2) = \sqrt{d_D'(\varphi^1, \varphi^1) + d_D'(\varphi^2, \varphi^2)} = \sqrt{|\kappa|^2 + |\Lambda|^{-2}}. \] (4.205)

Proof. We first show that
\[ d_D'(\varphi^1, \varphi^2) \leq \sqrt{|\Lambda|^{-2} + |\kappa|^2}, \] (4.206)
using a similar procedure as in proposition [III.5]. Let us fix \( \varphi \in S(\mathcal{A}), \kappa \in \mathbb{R}^2 \). For any \( \mathcal{A} \ni a' = (f_\lambda, g_\lambda) \) define
\[ F'(u, v) = \varphi_{u \kappa} \left( (1 - v|\Lambda|) f_\lambda + v|\Lambda| g_\lambda \right) \] (4.207)
with \( u \in [0, 1], v \in [0, |\Lambda|^{-1}] \) and \( \varphi_{u \kappa} = \varphi \circ \alpha_{u \kappa} \) the \( u \kappa \)-translated of \( \varphi \) defined in (3.76). One has
\[ F'(0, 0) = \varphi(f_\lambda) = \varphi^1(a'), \] (4.208)
\[ F'(u, |\Lambda|^{-1}) = \varphi_{u \kappa}(g_\lambda) = \varphi^2(a'). \] (4.209)

Viewing \( F' \) as a real function \( F'(c(t)) \) on \( \mathbb{R}^2 \), where \( c(t) = (u(t), v(t)) \) denote a curve in \( \mathbb{R}^2 \) such that \( u(0) = v(0) = 0, u(1) = 1, v(1) = |\Lambda|^{-1} \), one obtains
\[ |\varphi^2(a') - \varphi^1(a')| = |F'(c(1)) - F'(c(0))| \leq \int_0^1 \left| \frac{dF'}{dt} \right| dt. \] (4.210)

Now fix \( c(t) = (u(t) = t, v(t) = |\Lambda|^{-1}t) \). The derivative of \( F' \) along it is
\[ \frac{dF'}{dt} = \frac{\partial F'}{\partial u} |t| + \frac{\partial F'}{\partial v} |t| = \tilde{a} \tilde{\partial}_a F'_t, \] (4.211)
where for \( a \in [0, 2] \), we define \( \tilde{a} \equiv |\Lambda|^{-1} \) and
\[ \partial_a F' \equiv \frac{\partial F'}{\partial v} = \varphi_{u \kappa} \left(|\Lambda| \left( g_\lambda - f_\lambda \right) \right), \] (4.212)
while for \( a = \mu = 1, 2, \tilde{\partial}_a \) and \( \tilde{\partial}_a \) are as in proposition [III.5] that is, using \( \tilde{\partial}_\mu f_\lambda = \tilde{\partial}_\mu f, \tilde{\partial}_\mu g_\lambda = \tilde{\partial}_\mu g \),
\[ \partial_\mu F' = \varphi_{u \kappa} \left( (1 - v|\Lambda|^{-1}) \tilde{\partial}_\mu f + v|\Lambda|^{-1} \tilde{\partial}_\mu g \right). \] (4.213)

By Cauchy-Schwartz, one thus obtains
\[ \left| \frac{dF'}{dt} \right| \leq \sqrt{|\Lambda|^{-2} + |\kappa|^2} \sqrt{\sum_{a=0}^2 |\partial_a F'_t|^2}. \] (4.214)

To lighten notations, let us write \( \tilde{\varphi}_\kappa(t) \equiv \varphi_{u \kappa} (\tilde{\partial}_\mu f) \) and similarly for \( g \). Eq. (4.213) yields
\[ |\partial_\mu F'_t|^2 \leq v^2 |\Lambda|^{-2} \left( \tilde{g}_\mu(t) - \tilde{f}_\mu(t) \right)^2 + 2v|\Lambda|^{-1} \tilde{f}_\mu(t) \left( \tilde{g}_\mu(t) - \tilde{f}_\mu(t) \right) + \tilde{f}_\mu(t). \] (4.215)

As a function of \( v \in [0, |\Lambda|] \), this is a parabola with positive leading coefficient, hence it is maximum either at \( v = 0 \) or \( v = |\Lambda| \). Therefore
\[ |\partial_\mu F'_t| \leq \max \left\{ |\tilde{f}_\mu(t)|, |\tilde{g}_\mu(t)| \right\} \equiv |\varphi_{u \kappa} (h_{\mu})|, \] (4.216)
where \( h_{\mu} = \tilde{\varphi}_\kappa f \) or \( \tilde{\varphi}_\kappa g \) is a blind notation to denote the maximum. It is important to stress that, at fixed \( t \), nothing guarantees that \( h_{\mu} \) and \( h_{\nu} \) should be given by the same function. One may
have \( \hat{h}_{t_2} = \hat{\delta}_1 f \) while \( \hat{h}_{t_2} = \hat{\delta}_2 g \). As well, for the same index \( \mu \), nothing forbids the maximums at \( t_1 \) and \( t_2 \neq t_1 \) to be given by distinct functions: \( \hat{h}_{t_1\mu} = \hat{\delta}_1 f \) and \( \hat{h}_{t_2\mu} = \hat{\delta}_2 g \). In any case, again by Cauchy-Schwartz one gets

\[
|\partial_\alpha F'_\alpha|^2 \leq \varphi_\mu(h_{t_\mu}^* h_{t_\mu}), \quad (4.217)
\]

\[
|\partial_\alpha F'_\alpha|^2 \leq |A|^2 \varphi_\mu \left((g_\mu - f_\mu)^* (g_\mu - f_\mu)\right). \quad (4.218)
\]

Therefore

\[
\sum_{\alpha=0}^2 |\partial_\alpha F'_\alpha|^2 \leq \sum_{\mu=1}^2 \varphi_\mu \left(h_{t_\mu}^* h_{t_\mu} + \frac{|A|^2}{2} (g_\mu - f_\mu)^* (g_\mu - f_\mu)\right), \quad (4.219)
\]

\[
\leq \sum_{\mu=1}^2 \left| \mathcal{L}(h_{t_\mu})^* \mathcal{L}(h_{t_\mu}) + \frac{|A|^2}{2} \mathcal{L}(g_\mu - f_\mu)^* \mathcal{L}(g_\mu - f_\mu) \right| \leq 1 \quad (4.220)
\]

by lemma 4.4. Hence 4.206.

The theorem then follows from proposition 4.2 together with 4.167 and 4.168. \( \blacksquare \)

**Remark IV.6** In proposition 4.2, the hypothesis that \( A_1 \) is unital is crucial to define the subset \( B \) of \( A^1 \) in 4.166, and the non-degeneracy condition is used as \( \pi(1) = 1_1 \) to obtain 4.186. On the contrary, the proof of eq. 4.206 does not require unitarity. So in the non-unital double Moyal space, one has

\[
d_{D^2}(\varphi^1, \varphi^2) \leq \sqrt{d_{D^2}(\varphi^1, \varphi^1) + d_{D^2}(\varphi^2, \varphi^2)}. \quad (4.221)
\]

Whether the full theorem may hold without the unitarity condition will be investigated in [12].

**V Applications**

**V.1 Coherent states**

Coherent - or semi-classical - states of the quantum harmonic oscillator are, by definition, quantum states that reproduce the behaviour of a classical harmonic oscillator. We recall their basic properties in the Schrödinger representation, taking the material from e.g. [11], and give their quantum states that reproduce the behaviour of a classical harmonic oscillator. We recall their

V Applications

Whether the full theorem may hold without the unitarity condition will be investigated in [12].

\[
|\kappa| \leq |\kappa(0)|, \quad \Xi \leq \text{Arg} \kappa(0). \quad (5.223)
\]

Notice that the energy \( \frac{\hbar \omega}{2} |\kappa|^2 \) is constant in time. In other terms, a state of a classical oscillator is fully characterized by one complex number \( \kappa = |\kappa|e^{i\Xi} \). The same is true for a quantum coherent state. Indeed, such a state is defined (in the Schrödinger representation) by a vector \( \psi(t) \in L^2(\mathbb{R}) \) such that, at any time \( t \), the mean value of the observables \( q, p \) and \( h \) coincide with their classical counterpart; that is

\[
\omega_{\psi(t)}(q) \equiv \langle \psi(t), \hat{q}\psi(t) \rangle = x(t), \quad \omega_{\psi(t)}(p) = p(t), \quad \omega_{\psi(t)}(h) \equiv \frac{\hbar \omega}{2} |\kappa|^2. \quad (5.224)
\]

From now on we make the identification \( \theta = \hbar \) and assume that \( \omega = m = 1 \) so that \( \beta = \theta^{-\frac{1}{2}} \). Solving the classical evolution equation for \( \kappa \), one gets from the first two requirements of 5.224

\[
\omega_{\psi(0)}(a) = \sqrt{\theta} \kappa(0). \quad (5.225)
\]
Assuming that $|\kappa| >> 1$ (i.e. the energy of a classical oscillator is much greater than the quantum), the last requirement of (5.224) implies

$$\omega_{\psi(0)}(a^*a) = \theta|\kappa|^2.$$  

(5.226)

Easy calculations show that (5.225) and (5.226) are equivalent to $\psi(0)$ being an eigenstate of $a$ with eigenvalue $\sqrt{\theta|\kappa|^2}$. Notice that, by the Schrödinger equation, $\psi(t)$ remains an eigenstate of $a$, with eigenvalue $\sqrt{\theta|\kappa|^2}e^{-i\omega t}$.

Definition V.1 A coherent state of the Moyal algebra $A$ is a linear form

$$\omega_\kappa(f) = \langle \kappa, \pi_S(f)\kappa \rangle \quad \forall f \in A$$  

(5.227)

where $|\kappa\rangle \in L^2(\mathbb{R})$, $||\kappa||_{L^2(\mathbb{R})} = 1$, is a solution of

$$a|\kappa\rangle = \sqrt{\theta|\kappa|^2}|\kappa\rangle \quad \kappa \in \mathbb{C}.$$  

(5.228)

Although formula (5.227) is often used in quantum mechanics, for our purposes it is not very helpful: in [6] one computed the distance between stationary states $\omega_n$ of the Hamiltonian $H$, that is vector state defined by a vector $\psi_n$ with only one non-zero component $c_m = \frac{1}{\sqrt{2\pi}}\delta_{mn}$. In $[33]$ we partially extended the computation to states with two non-zero components. It seems out of reach to obtain a formula for arbitrary states, especially those with an infinite number of non-zero components. However, coherent states can also be characterized by a simple geometrical property.

Proposition V.2 The coherent state $\omega_\kappa$ is the translated of the ground state of the quantum harmonic oscillator, with translation $\sqrt{\theta|\kappa|^2}$. That is to say

$$\omega_\kappa(f) = \omega_0 \circ a_\sqrt{\theta|\kappa|^2}(f)$$  

(5.230)

where $\omega_0(\cdot) = \langle h_0, \pi_S(\cdot)h_0 \rangle$, with $h_0$ the ground state vector of the harmonic oscillator.

Proof. Remembering the unitary implementation (2.67) of translations, let us define

$$v_\kappa \equiv u_{\sqrt{\theta|\kappa|^2}} = e^{\frac{\alpha^* a - a \alpha}{\sqrt{\theta|\kappa|^2}}}.$$  

(5.231)

One checks that

$$v_\kappa h_0 = \sum_{m \in \mathbb{N}} c_m^\kappa h_m = |\kappa\rangle.$$  

(5.232)

Therefore

$$\omega_\kappa(f) = \langle h_0, \text{ad} v_\kappa^* \pi_S(f) h_0 \rangle = \langle h_0, \pi_S(\alpha_{-\sqrt{\theta|\kappa|^2}}f) h_0 \rangle,$$  

(5.233)

and the result by lemma II.6. ■

By theorem III.9 one immediately obtains that the distance between coherent states is the Euclidean distance on the plane, multiplied by $\sqrt{\theta|\kappa|^2}$.

Proposition V.3 Let $\omega_\kappa^\xi, \omega_\kappa^\zeta$ be any two coherent states of the Moyal algebra, then

$$d_D(\omega_\kappa^\xi, \omega_\kappa^\zeta) = \sqrt{2\theta|\kappa| - \kappa}.$$  

(5.234)
V.2 Quantum length in the DFR model

We summarize in this section the analysis developed at length in [33]. The 2N-dimensional DFR model of quantum spacetime is described by coordinate operators $q_μ, μ = 1, 2N$, that satisfy the commutation relations [22]

$$[q_μ, q_ν] = iλ_P Θ_{μν} \mathbb{I},$$

(5.235)

with $Θ$ the matrix given in (2.11) and $λ_P$ the Planck length. It carries a representation of the Poincaré group $G$ under which (5.235) is covariant (the left-hand side transforms under $\text{ad} G$). We shall not take into account this action here, since we are interested in the Euclidean length operator,

$$L = \sqrt{\sum_{μ=1}^{2N} dq_μ^2}, \quad dq_μ ≃ q_μ \otimes \mathbb{I} - \mathbb{I} \otimes q_μ,$$

(5.236)

whose spectrum is obviously not Poincaré invariant. Said differently, we fix once and for all the matrix $Θ$ in (5.235). Incidentally, this means that our analysis also applies to the so-called canonical noncommutative spacetime (or $θ$-Minkowski), characterized by the invariance (opposed to covariance) of the commutators (5.235) under the action of the quantum group $θ$-Poincaré. In both models, the length operator $L$ is promoted to a quantum observable [1, 2], and

$$l_p ≃ \min{λ \in \text{Sp} L}$$

(5.237)

is interpreted as the minimal value that may come out from a length measurement. The link with the spectral distance is obtained by identifying $q_μ$ with the Moyal coordinate $x_μ$, viewed as an unbounded operator affiliated to $\mathbb{K}$. The choice of the representation, left-regular on $\mathcal{H} = L^2(\mathbb{R}^{2N})$, that is $q_μ = \mathcal{L}(x_μ)$; or integrated Schrödinger on $L^2(\mathbb{R}^N), q_μ = π_Σ(x_μ)$; is not relevant for the following discussion. In both cases, $q_μ$ is affiliated to $\mathbb{K}$. To any pair of states $(ϕ, \tilde{ϕ}) \in S(\mathbb{K}) \times S(\mathbb{K})$ in the domain of the $q_μ$’s, one associates the quantum length [33]

$$d_L(ϕ, \tilde{ϕ}) ≃ (ϕ \otimes \tilde{ϕ})(L).$$

(5.238)

Obviously $d_L$ is not a distance: for $N = 1$, an explicit computation yields

$$l_p = \sqrt{2}λ_P,$$

(5.239)

so that $d_L(ϕ, \tilde{ϕ}) ≥ l_p$ never vanishes. Consequently, there is a priori little sense to compare the quantum length with the spectral distance.

Nevertheless, we have shown in [33] that it does make sense to compare the quantum square-length,

$$d_{L^2}(ϕ, \tilde{ϕ}) ≃ (ϕ \otimes \tilde{ϕ})(L^2),$$

(5.240)

with the (square of the) spectral distance $d_{L^2}$ in the double unital Moyal space of section IV. The doubling procedure allows to implement the notion of minimal length between a state and itself into the spectral distance framework, by identifying $d_{L^2}(ϕ, ϕ)$ to $d_{L^2}^D(ϕ^1, ϕ^2)$. Technically, this simply amounts to fix the parameter $Λ$ in the Dirac operator $D_2$ to the required value since, by (4.168), one has

$$d_{L^2}(ϕ, ϕ) = d_{L^2}^D(ϕ^1, ϕ^2) \quad \text{if and only if} \quad Λ = d_{L^2}(ϕ, ϕ)^{-\frac{1}{2}}.$$  

(5.241)

Since $d_{L^2}^D(ϕ, ϕ)$, as a function of $ϕ$, is constantly equal to $Λ^{-1}$ on $S(\mathcal{A})$ (in fact, one could make it non-constant by introducing a Higgs field; this point is discussed in [33]), once the free parameter $Λ$ is fixed, the identification of $d_{L^2}^D$ with $d_{L^2}$ may make sense only for those states $\tilde{ϕ}$ such that $d_{L^2}(ϕ, ϕ) = d_{L^2}(ϕ, ϕ)$. This is indeed the case for the states in the set $\mathcal{C}(ϕ)$ defined in (3.74). One then gets by theorem IV.5 that the identification $d_{L^2} ↔ d_{L^2}^D$ extends to any pair of states $(ϕ, \tilde{ϕ})$ with $C(ϕ) \ni ϕ \neq \tilde{ϕ}$ if and only if the spectral distance on a single copy of the Moyal plane is

$$d_{L^2}^D(ϕ, ϕ) = \sqrt{d_{L^2}(ϕ, ϕ) - d_{L^2}(ϕ, ϕ)}.$$  

(5.242)

Eq. (5.242) is the true condition guaranteeing that, once the obvious discrepancy due to the non-vanishing of $d_{L^2}(ϕ, ϕ)$ is solved, the spectral distance and the quantum length capture the same metric information on the Moyal plane. Remarkably, this conditions holds true for the states that
are of particular physical importance from the DFR point of view, namely the states $\varphi$ of optimal localization, for which

$$d_{L^2}(\varphi, \varphi) = l_P.$$  \hfill (5.243)

Indeed, these states are nothing but the coherent states discussed in the previous subsection, so that by proposition V.3 one obtains a link between a priori two distinct ways of defining a length and a distance on a quantum space.

**Proposition V.4** On the states of optimal localization, the DFR quantum length coincides with Connes spectral distance in that, for any two coherent states $\omega^c_\kappa, \omega^c_{\tilde{\kappa}}$ with $\kappa, \tilde{\kappa} \in \mathbb{C}$, one has

$$d_D(\omega^c_\kappa, \omega^c_{\tilde{\kappa}}) = \sqrt{d_{L^2}(\omega^c_\kappa, \omega^c_{\tilde{\kappa}}) - 2\lambda_P^2}.$$  \hfill (5.244)

**Proof.** See [33].

Let us stress that, while the identification of $d_{L^2}(\varphi, \varphi)$ with $d_{L^2}(\varphi, \varphi)$ in (5.241) is nothing but fixing the free parameter $\Lambda$ to the desired value $l_P^{-1}$, the identification of $d_{L^2}$ with $d_{L^2}$ on the set of states of optimal localization is possible only because both theorems III.9 and IV.5 hold true.

## VI Conclusion

We have proved two theorems which allow to identify Connes’ spectral distance on the Moyal plane with the DFR quantum length between states of optimal localization: first, the spectral distance between any state of the unital Moyal algebra and any of its translated is precisely the amplitude of the translation; second the product of the unital Moyal spectral triple by $\mathbb{C}^2$ is an orthogonal product in the sense of Pythagoras theorem, restricted to the classes of states $\mathcal{C}(\varphi)$.

Both results heavily rely on the fact that the states of optimal localization are obtained by the isometric action of a group (here $\mathbb{R}^2$). This strongly suggests that similar results should hold true in the context of Rieffel deformation. This will be the object of a future work.

Finally, we would like to mention the very recent work of Wallet [41], who shows that the spectral distance on the Moyal plane, with the harmonic Dirac operator used to study the renormalisability of quantum field theories on noncommutative space instead of $\partial \mathcal{C}$, is homothetic to the spectral distance calculated in [8] and in this paper.

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