Lie Symmetries and Similarity transformations for the Generalized Boiti-Leon-Pempinelli equations

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Abstract: We perform a detailed classification of the Lie point symmetries and of the resulting similarity transformations for the Generalized Boiti-Leon-Pempinelli equations. The latter equations for a system of two nonlinear 1+2 partial differential equations of second- and third-order. The nonlinear equations depend of two parameters, namely \( n \) and \( m \), from where we find that for various values of these two parameters the resulting systems admit different number of Lie point symmetries. For every case, we present the complete analysis for the admitted Lie group as also we determine all the possible similarity solutions which follow from the one-dimensional optimal system. Finally, we summarize the results by presenting them in a tabular way.

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1 Introduction

Nonlinear ordinary and partial differential equations are playing very prominent role in field of Mathematics, Physics, Mathematical Physics and Mathematical modeling etc.. Through the study of solutions of such differential equations one can understand the behavior or consequence of a natural phenomena very clearly. Though various methods are available in the literature, Lie’ symmetry method successfully using by researchers due to its systematic and algorithmic way to derive the solution of the differential equations. Actually this method was developed by Sophus Lie, during the period 1872 – 1899 \[1,2,3,4,5,6,7\] and applied it successfully without any ansatz to solve the differential equations.

Since 1960 over the explicit construction of solutions of any sort of problems, even complicated, of mathematical physics, this theory was exploited by the Russian school with L. V. Ovsiannikov \[8,9,10\]. During the last few decades, Lie’s theory continuously enhanced the researchers to solve the differential equations in various form such as generalized symmetries \[10,11\], approximate symmetries \[12,13\], and nonlocal symmetries \[11,14,15,16\] to quote a few. Therefore, researchers using the method
widely on both theoretical and applied point of view [17, 18, 19, 20, 14, 21, 22, 23, 24, 25, 26, 11, 27, 15, 28, 29, 30, 31, 32, 33]. Nowadays researchers are using powerful Computer Algebra Systems (CAS) like Maple and Mathematica (commercial), etc. to do the calculations over the symmetry rapidly. In this work, for the calculation of the symmetries we use the Mathematica add-on Sym [34, 35, 36, 37].

In this work we are interested on the complete classification of the Lie symmetries and of the one-dimensional optimal system for the nonlinear system of 1+2 partial differential equations known as generalized Boiti-Leon-Pempinelli (BLP) equations, the original equations are [38]

\[
\begin{align*}
    u_{ty} - (u^2 - u_x)_{xy} - 2v_{xxx} &= 0, \\
    v_t - v_{xx} - 2uv_x &= 0.
\end{align*}
\]

while in this work we are interesting on the generalization

\[
\begin{align*}
    u_{ty} - (u^n - u_x)_{xy} - \alpha v_{xxx} &= 0, \\
    v_t - v_{xx} - \beta u^m v_x &= 0, \\
\end{align*}
\] (1.1)

where \(nm \neq 0\) and \(\alpha\beta \neq 0\).

The BLP equation describes interactions of two waves with different dispersion relations and are the two-dimensional generalizations of the sine- and sinh-Gordon equations. Some exact and analytic solutions determined in [39, 40, 41, 42, 43], while recently in [44] a complete analysis of the Lie point symmetries and of the Bäcklund transformations performed.

In this work we classify the admitted Lie point symmetries of the generalized BLP equations for various values of the free parameters \(n, m\), while the latter are constrained by the theory of Lie symmetries. Such analysis is important for the detailed study of nonlinear partial differential equations and for the determination of new exact solutions, for other examples we refer the reader in [45, 46, 47, 48] and references therein. The plan of the paper is as follows.

In Section 2, we present the basic properties and definitions for the theory of Lie point symmetries of differential equations. We discuss the main mathematical methods which are applied in this work. Section 3 includes the main analysis of this work where we present the complete classification of the Lie point symmetries for system (1.1). We found four different possible cases for the set of variables \(\{n, m\}\). For every set we determine the Lie point symmetries and we present the commutators and the adjoint representation. The latter are used for the derivation of the one-dimensional optimal system which are applied for the derivation of new similarity solutions for the generalized BLP equations (1.1). Finally, our results are summarized in Section 4, where we draw our conclusions.
2 Lie’s Theory

Consider a system of equations as follows

\[ \Delta_1(t, x, y, u, v, u_t, v_t, u_x, v_x, u_y, v_y, u_{tt}, v_{tt}, u_{tx}, v_{tx}, u_{ty}, v_{ty}, u_{xx}, u_{xy}, \ldots) = 0, \]  
(2.1)

\[ \Delta_2(t, x, y, u, v, u_t, v_t, u_x, v_x, u_y, v_y, u_{tt}, v_{tt}, u_{tx}, v_{tx}, u_{ty}, v_{ty}, u_{xx}, u_{xy}, \ldots) = 0, \]  
(2.2)

where \( t, x, y \) are taken to represent the independent variables and \( u, v \) are taken to represent dependent variables, that is \( u = u(t, x, y) \) and \( v = v(t, x, y) \), while \( u_t = \frac{∂u}{∂t} \).

Infinitesimal point transformation for each variables is defined as in the following manner,

\[ \tilde{t}(t, x, y, \epsilon) = t + \epsilon \xi^1(t, x, y) + o(\epsilon^2), \]

\[ \tilde{x}(t, x, y, \epsilon) = x + \epsilon \xi^2(t, x, y) + o(\epsilon^2), \]

\[ \tilde{y}(t, x, y, \epsilon) = y + \epsilon \xi^3(t, x, y) + o(\epsilon^2), \]

\[ \tilde{u}(t, x, y, \epsilon) = u + \epsilon \eta^1(t, x, y) + o(\epsilon^2), \]

\[ \tilde{v}(t, x, y, \epsilon) = v + \epsilon \eta^2(t, x, y) + o(\epsilon^2), \]

where \( X \) is called infinitesimal generator defined as

\[ X = \frac{∂\tilde{t}}{∂\epsilon} \partial_t + \frac{∂\tilde{x}}{∂\epsilon} \partial_x + \frac{∂\tilde{y}}{∂\epsilon} \partial_y + \frac{∂\tilde{u}}{∂\epsilon} \partial_u + \frac{∂\tilde{v}}{∂\epsilon} \partial_v, \]

or equivalently

\[ X = \xi^1(t, x, y) \partial_t + \xi^2(t, x, y) \partial_x + \xi^3(t, x, y) \partial_y + \eta^1(t, x, y) \partial_u + \eta^2(t, x, y) \partial_v. \]

The invariant condition for the system (2.1) and (2.2), based on the Lie’s theory is given by

\[ \Delta_1(t, x, y, u, v) = \Delta_1(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}) \]

\[ \Delta_2(t, x, y, u, v) = \Delta_2(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}) \]

By using infinitesimal transformation and invariant condition one can find the infinitesimal generator which is known as symmetries of the system (2.1) and (2.2). Therefore, if \( X \) is a Lie point symmetry for equation \( \Delta_1 \equiv 0 \) and \( \Delta_2 \equiv 0 \) then the following condition is true

\[ X^{[k]} \Delta_1 = \lambda \Delta_1, \mod \Delta_1 = 0 \]

\[ X^{[k]} \Delta_2 = \phi \Delta_2, \mod \Delta_2 = 0 \]

where \( \lambda \) and \( \phi \) are arbitrary functions, and \( X^{[k]} \) is the k-th extension of \( X \) in the jet-space. These symmetries play a crucial role to perform the reduction of the order of the differential equations as well as number of independent variables. The latter system known as determining system provides the Lie point symmetries for a given system of differential equations.
2.1 Lie invariants

By applying the Lie’s theory one can find the symmetries of a given differential equations. By using the Lie symmetry with some additional regularity assumptions we obtain transformations which is called Lie’s invariant. The invariant reduces the given equation into another which involve fewer independent variables than the original equation. The solution of the reduced equation is called invariant solution to the given equation.

2.2 One-dimensional optimal system

Optimal system forms by a list of $n$–parameter subalgebras if every $n$–parameter subalgebras is equivalent to a unique member of the list under some element of the adjoint representation. P.J. Olver discussed the procedure to find adjoint representation and optimal system which are giving an unique combination of symmetries to perform the reductions of a given differential equation [26]. Later researcher follows slightly different method from Olver which proposed by Ibragimov [49]. The method is given a simple way to find the optimal system [50, 51, 52, 44].

3 Lie symmetry classification

We apply the theory of Lie point symmetries and we have obtained four sets for the set of variables \{n, m\} where we find the symmetries of the system (1.1). In particular we found that of \{n, m\} = (2, 1) which is the original 1+2 BLP equation, for \{n, m\} = \{1, 1\} and for \{n, m\} = (n, n - 1; n \neq 1) the admitted Lie point symmetries are three plus infinity plus infinity but with different admitted Lie algebra. On the other hand for arbitrary \{n, m\} the admitted Lie point symmetries are the common elements, of the common subalgebra for the previous cases.

For each case we present the commutator table and the adjoint representation. From the latter results we determine the one-dimensional optimal system which is applied for the derivation of all the possible similarity transformations and reduction, and when it is feasible of exact solutions.

3.1 Case \( n = 2, m = 1 \)

If \( m = 1 \) and \( n = 2 \) then the system (1.1) yields the following equation

\[
\begin{align*}
    u_{ty} - (u^2 - u_x)_{xy} - \alpha v_{xxx} &= 0, \\
    v_t - v_{xx} - \beta uv_x &= 0.
\end{align*}
\] (3.1)

The admitted Lie symmetries are

\[
\begin{align*}
    X_1 &= \partial_t, \quad X_2 = \partial_x, \quad X_3 = \psi(y)\partial_y - \psi' v\partial_v, \quad X_4^\infty = \phi(y)\partial_v, \\
    X_5 &= 2t\partial_t + x\partial_x - u\partial_u.
\end{align*}
\]
The commutator table and adjoint representation based on the formulas $[X_m, X_n] = X_m X_n - X_n X_m$ and $Ad [e^{X_l}] X_j = X_j - \epsilon [X_l, X_j] + \frac{\epsilon^2}{2} [X_l, [X_l, X_j]] - \ldots$ respectively are given by the following tables (1) and (2).

Table 1: Commutator Table for case 1.1 with $\psi = 1$

| $[X_I, X_J]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
|-------------|-------|-------|-------|-------|-------|
| $X_1$       | 0     | 0     | 0     | 0     | 2$X_1$ |
| $X_2$       | 0     | 0     | 0     | 0     | $X_2$  |
| $X_3$       | 0     | 0     | 0     | ($\psi' \phi + \phi' \psi$)$\partial_v$ | 0     |
| $X_4$       | 0     | 0     | ($\psi' \phi + \phi' \psi$)$\partial_v$ | 0     | 0     |
| $X_5$       | $-2X_1$ | $-X_2$ | 0     | 0     | 0     |

Table 2: Adjoint representation Table for case 1.1 with $\psi = 1$

| $[Ad(e^{X_l})X_J]$ | $X_1$ | $X_2$ | $X_3$ | $X_5$ |
|-------------------|-------|-------|-------|-------|
| $X_1$             | $X_1$ | $X_2$ | $X_3$ | $X_5 - 2\epsilon X_1$ |
| $X_2$             | $X_1$ | $X_2$ | $X_3$ | $X_5 - \epsilon X_2$ |
| $X_3$             | $X_1$ | $X_2$ | $X_3$ | $X_5$ |
| $X_5$             | $e^{2\epsilon} X_1$ | $e^\epsilon X_2$ | $X_3$ | $X_5$ |
Table 3: Reductions for the case $m = 1, n = 2$

| Optimal System | Similarity variable | Reductions |
|----------------|---------------------|------------|
| $X_5 + c_1X_3$ | $X = \frac{x}{\sqrt{t}}, Y = y - \frac{c_1}{2} \log t,$ \[ u = \frac{1}{\sqrt{t}} U[X,Y], v = V[X,Y] \] | (I) $\alpha V_{XXX} - U_{XXY} + 2UU_{XY} + 2U_XU_Y + \frac{1}{2}UU_Y + \frac{c_1}{2}U_{YY} + \frac{1}{2}U_Y = 0$  
$V_{XX} + \beta UV_X + \frac{1}{4} XV_X + \frac{c_1}{2} V_Y = 0$ |
| $X_1 + c_1X_2 + c_2X_3$ | $r = t + c_1x + c_2y, u = U(r), v = V(r)$ | (II) $c_2U'' - c_1c_2 \left(2U'^2 + 2UU''\right) + c_1^2c_2U'' + \alpha c_1^3V'' = 0$  
$(1-c_1\beta U)V'' - c_1^2V'' = 0$ |
| $X_1$ | $u = U[x,y], v = V[x,y]$ | (III) $2(U_xU_y + U_{xy}) - U_{xyy} + \alpha V_{xxx} = 0$  
$\beta UV_x + V_{xx} = 0$ |
| $X_2$ | $u(t,x,y) = U(t,y), v(t,x,y) = V(t,y)$ | (IV) $U_{ty} = 0$,  
$V_t = 0$ |
| $X_3$ | $u(t,x,y) = U(t,x), v(t,x,y) = V(t,x)$. | (V) $\beta UV_x + V_{xx} - V_t = 0$,  
$V_{xxx} = 0$ |
| $X_5$ | $X = x\sqrt{t}, Y = y$,  
$u(t,x,y) = \frac{1}{\sqrt{t}} U(X,Y), v(t,x,y) = V(X,Y)$ | (VI) $2\alpha V_{XXX} + (1 + 4U_X)U_Y + (X + 4U)U_{XY} - 2U_{YXX} = 0$  
$(X + 2\beta U)V_X + 2V_{XX} = 0$ |
As discussed in [51] the optimal system are $X_5 + c_1X_3, X_4 + c_1X_2 + c_2X_3, X_1, X_2, X_3, X_5$. Now we have investigated invariant solution for (3.1) corresponding to each optimal system and tabulated them in table (3). For example the characteristic equation of $X_5 + c_1X_3$ is written as

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{dy}{c_1} = \frac{du}{-u} = \frac{dv}{0}.$$  

(3.2)

The solution of the characteristic equations yields four invariant of $X_5 + c_1X_3$:

$$X = \frac{x}{\sqrt{t}}, \quad Y = y - \frac{c_1}{2} \log t, \quad \delta_1 = u\sqrt{t}, \quad \delta_2 = v.$$  

(3.3)

Thus, the solution of (3.1) is given by the invariant form

$$u\sqrt{t} = U\left[\frac{x}{\sqrt{t}}, y - \frac{c_1}{2} \log t\right], \quad v = V\left[\frac{x}{\sqrt{t}}, y - \frac{c_1}{2} \log t\right].$$  

(3.4)

Then

$$u = \frac{1}{\sqrt{t}}U\left[X, Y\right], \quad v = V\left[X, Y\right].$$  

(3.5)

Based on this transformation (3.1) reduced to

$$(I) \quad \alpha V_{XX} - U_{XY} + 2UU_{XY} + 2U_{XY} + \frac{1}{2}XU_{XY} + \frac{c_1}{2}U_{YY} + \frac{1}{2}U_{Y} = 0,$$  

(3.6)

$$V_{XX} + \beta UV_{X} + \frac{1}{2}XV_{X} + \frac{c_1}{2}V_{Y} = 0.$$  

(3.7)

For further reductions and invariant solutions, again we find the symmetries of the system (I) then tabulated our results in table (4). The symmetries of (I) are $\partial_Y$ and $\partial_V$.

### Table 4: Solution for equation (I)

| Symmetry | Similarity variable | Reduction | Solution |
|----------|---------------------|-----------|----------|
| $\partial_Y$ | $U = G(X)$, $V = H(X)$ | $H''' = 0$, $G'' = -\frac{c_1}{2}X + \frac{c_1}{2}V + \frac{c_1}{2}\beta G$ | $H = I_1 + I_2X + I_3X^2$ |
| $\partial_Y + c_3\partial_V$ | $U = G(X)$, $V = c_3Y + H(X)$ | $H''' = 0$, $G'' = -\frac{c_1}{2}X + \frac{c_1}{2}V + \frac{c_1}{2}\beta G$ | $H = I_1 + I_2X + I_3X^2$ |

Next take $X_1 + c_1X_2 + c_2X_3$, then (3.1) is reduced in to the following equation

$$(II) \quad c_2U'' - c_1c_2 \left(2U'' + 2UU''\right) + c_1^2c_2U''' + \alpha c_1^3V''' = 0,$$  

(3.8)

$$\left(1 - c_1\beta U\right)V' - c_1^2V'' = 0.$$  

(3.9)
where $u = U(r)$, $v = V(r)$ and $r = t + c_1x + c_2y$ are invariants obtained from $X_1 + c_1X_2 + c_2X_3$. Here $'$ represents the differentiation with respect to $r$. Integrating (3.8) twice we obtain

$$I_2 + I_1r + c_2U - c_1c_2U^2 + c_1^2c_2U' - c_1^3\alpha V' = 0.$$  

(3.10)

From (3.9), we get

$$V = I_3 \int \exp \left[ \int \left( \frac{1 - c_1\beta U}{c_2} \right)' \right] r.$$  

(3.11)

Substituting (3.11) in (3.10) then we have

$$U' + \frac{1}{c_1^2}U = \frac{1}{c_1}U^2 - \frac{I_2 + I_1r}{c_1^3c_2} + \frac{c_1I_3\alpha}{c_2} \exp \left[ \int \left( \frac{1 - c_1\beta U}{c_1^2} \right)' \right],$$  

(3.12)

where $I_1, I_2$, and $I_3$ are constants of integration.

Similarly we have performed the reduction for all other optimal system then the resultant equations are tabulated in table (3). The follows give the further reductions and invariant solutions for the resultant equation which are tabulated in table (3).

**Reductions and Solutions for (III)**

The equation (III) admits the following symmetries

$$X_{11} = \partial_x, X_{12} = f_1(y)\partial_y - Vf'(y)\partial_V, X_{13} = x\partial_x - U\partial_U, X_{14} = f_2(y)\partial_V$$

Table 5: Adjoint representation Table with $f_1 = 1$ and $f_2 = 0$

| $[\text{Ad}(e^\epsilon X_{1i})X_{1j}]$ | $X_{11}$ | $X_{12}$ | $X_{13}$ |
|---------------------------------|--------|--------|--------|
| $X_{11}$                        | $X_{11}$ | $X_{12}$ | $X_{13} - \epsilon X_{11}$ |
| $X_{12}$                        | $X_{11}$ | $X_{12}$ | $X_{13}$ |
| $X_{13}$                        | $e^\epsilon X_{11}$ | $X_{12}$ | $X_{13}$ |

The table 5 shows the adjoint representation of symmetries of the resultant equation (III) with conditions $f_1 = 1$ and $f_2 = 0$. The corresponding optimal system, similarity variables with respect to the optimal system, reductions and solutions are tabulated in table 6.
Table 6: Solution for equation (III)

| Optimal system | Similarity variable | Reduction | Solution |
|----------------|---------------------|-----------|----------|
| $X_{11}$      | $U(x, y) = G(y)$,  | $H'' = 0$, | $H = I_1 + I_2 x + I_3 x^2$ |
|               | $V(x, y) = H(y)$   | $\beta G' + H'' = 0$ | $G = -\frac{2I_1}{\beta(I_2 + 2I_3 x)}$ |
| $X_{12}$      | $U(x, y) = G(x)$,  | $2(G'' + GG') + c_1 G'' + c_2 \alpha H'' = 0$, | $H = I_1 \int \exp \left[ \frac{\beta}{c_1} \int G dr \right] dr + I_2$ |
|               | $V(x, y) = H(x)$   | $\beta G' - c_1 H'' = 0$ | $c_1 G' = I_4 + I_3 r - G^2 -$ |
| $X_{13} + c_1 X_{12}$ | $r = y - c_1 x$,  | $(1 + 2G)G'' = 0$, | $c_1^2 \alpha I_1 \exp \left[ \frac{\beta}{c_1} \int G dr \right]$ |
|               | $U(x, y) = G(r)$,  | $c_1 G'' + c_1^2 \alpha H'' = 0$, | $(**)$ $c_1^2 G'' + 3c_1 G' + 2G =$ |
|               | $V(x, y) = H(r)$   | $\beta G' - c_1 H'' = 0$ | $I_3 - 2G^2 - 2GG'$ |
|               | $r = y - c_1 x$,  | $(1 - \beta G)H' - c_1 H'' = 0$ | $(G + \beta G^2 + c_1 G')I_1 c_1 \alpha \beta \exp \left[ -\frac{1}{c_1} \int (1 - \beta G) dr \right]$ |
| $X_{13} + c_1 X_{12}$ | $U(x, y) = \frac{G(r)}{x}$ | $2c_1 \alpha H + 3c_1^2 \alpha H'' + c_1^3 \alpha H''' = 0$ | $H = I_1 \int \exp \left[ -\frac{1}{c_1} \int (1 - \beta G) dr \right] dr + I_2$ |

The trivial symmetry of equation (**) is $\partial_r$ and the corresponding canonical variable are $G' = W(G)$. Based on these transformation the equation (**) reduced in to the following

\[ c_1^2 WW' + 3c_1 W + 2G = I_3 - 2G^2 - 2GW - I_1 c_1 \alpha \beta (G + \beta G^2 + c_1 W) \exp \left[ -\frac{1}{c_1} \int (1 - \beta G) dr \right], \]

where $'$ represents the differentiation with respect to $G$. 

9
Reductions and Solutions for (IV)
The solution of system (IV) is given by
\[ U(t, y) = \int f(t) dt + g(y), \quad (3.14) \]
\[ V(t, y) = h(y). \quad (3.15) \]

Reductions and Solutions for (V)
The solution of system (V) is given by
\[ U(t, x) = f'(t) + g'(t)x + h'(t)x^2 - 2h(t) \beta(g(t) + 2xh(t)), \quad (3.16) \]
\[ V(t, y) = f(t) + g(t)x + h(t)x^2. \quad (3.17) \]

Reductions and Solutions for (VI)

The trivial symmetry of the system (VI) is \( \partial_Y \). Based on the symmetry the similarity variables are \( U(X, Y) = G(X) \) and \( V(X, Y) = H(X) \) and then the reduced system is given by
\[ H_{XXX} = 0, \quad (3.18) \]
\[ (X + 2\beta G)H_X + 2H_{XX} = 0. \quad (3.19) \]
The solution of above system are given by
\[ H = I_1 + I_2 X + I_3 X^2, \quad (3.20) \]
\[ G = -\frac{4I_3 + I_2 X + 2I_3 X^2}{2\beta(I_2 + 2I_3 X)}, \quad (3.21) \]
where \( I_1, I_2 \) and \( I_3 \) are constants of integration.

3.2 Case \( m = 1 \) and \( n = 1 \)

If \( m = 1 \) and \( n = 1 \) then the system (1.1) becomes
\[ u_{ty} - (u - u_x)_{xy} - \alpha v_{xxx} = 0, \]
\[ v_t - v_{xx} - \beta uv_x = 0. \quad (3.22) \]
The above system admitted the following Lie symmetries
\[ Y_1 = \partial_t, \quad Y_2 = \partial_x, \quad Y_3 = \psi(y)\partial_y - \psi'y\partial_v, \quad Y_4^\infty = \phi(y)\partial_v, \]
\[ Y_5 = 2t\partial_t + (x - t)\partial_x + \left(\frac{1 - \beta u}{\beta}\right)\partial_u. \]

Table 7 and table 8 give the commutator relation and adjoint representation, respectively, for the symmetries of system (3.22). Indeed, the adjoint representation tabulated with condition \( \psi(y) = 1 \) and \( \phi(y) = 0 \).
Table 7: Commutator Table for case 1.2

| $[Y_i,Y_j]$ | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ |
|-------------|-------|-------|-------|-------|-------|
| $Y_1$       | 0     | 0     | 0     | 0     | $2Y_1 - Y_2$ |
| $Y_2$       | 0     | 0     | 0     | 0     | $Y_2$ |
| $Y_3$       | 0     | 0     | 0     | $(\psi'\phi + \phi'\psi)\partial_v$ | 0 |
| $Y_4$       | 0     | 0     | $-(\psi'\phi + \phi'\psi)\partial_v$ | 0 | 0 |
| $Y_5$       | $-2Y_1 + Y_2$ | $-Y_2$ | 0 | 0 | 0 |

Table 8: Adjoint representation Table for case 1.2

| $[\text{Ad}(e^{\epsilon Y_1})Y_j]$ | $Y_1$ | $Y_2$ | $Y_3$ | $Y_5$ |
|-------------------------------|-------|-------|-------|-------|
| $Y_1$                         | $Y_1$ | $Y_2$ | $Y_3$ | $Y_5 - 2\epsilon Y_1 + \epsilon Y_2$ |
| $Y_2$                         | $Y_1$ | $Y_2$ | $Y_3$ | $Y_5 - \epsilon Y_2$ |
| $Y_3$                         | $Y_1$ | $Y_2$ | $Y_3$ | $Y_5$ |
| $Y_5$                         | $e^{2\epsilon}Y_1 - e^\epsilon(e^\epsilon - 1)Y_2$ | $e^\epsilon Y_2$ | $Y_3$ | $Y_5$ |
| Optimal System | Similarity variable | Reductions |
|----------------|---------------------|------------|
| $Y_1 + c_1 Y_2$ | $X = x - c_1 t, \ Y = y,$ $u = U[X,Y], \ v = V[X,Y]$ | $(VII)$ $U_Y XX - (1 + c_1) U_{YY} - \alpha U_{XXX} = 0$ $V_{XX} + (c_1 + \beta U) V_X = 0$ |
| $Y_5 + c_1 Y_3$ | $X = \frac{1}{\sqrt{t}} (x + t), \ Y = y - \frac{c_1}{2} \log t,$ $u = U[X,Y], \ v = V[X,Y]$ | $(VIII)$ $U_Y + c_1 U_{YY} + X U_{XY} - 2 U_{YXX} + 2 \alpha U_{XXX} = 0$ $c_1 V_Y + (X + 2 \beta U) V_X + 2 V_{XX} = 0$ |
| $Y_1 + c_1 Y_2 + c_2 Y_3$ | $r = t + c_1 x + c_2 y$ $u = U(r), \ v = V(r)$ | $(IX)$ $c_2 U'' - c_1 c_2 U''' + c_1^2 c_2 U'''' - \alpha c_1^3 V''' = 0$ $(1 - c_1 \beta U) V' - c_1^2 V'' = 0$ |
| $Y_1$ | $u(t, x, y) = U(x, y), v(t, x, y) = V(x, y)$ | $(X)$ $U_{xy} - U_{xyy} + \alpha U_{xxx} = 0,$ $\beta U V_x + V_{xx} = 0$ |
| $Y_2$ | $u(t, x, y) = U(t, y), v(t, x, y) = V(t, y)$ | $(XI)$ $U_{ty} = 0,$ $V_t = 0$ |
| $Y_3$ | $u(t, x, y) = U(t, x), v(t, x, y) = V(t, x),$ | $(XII)$ $\beta U V_x + V_{xx} - V_t = 0$ $V_{xxx} = 0$ |
| $Y_5$ | $X = \frac{t + x}{\sqrt{t}}, \ Y = y,$ $u(t, x, y) = \frac{1}{\beta} + \frac{U(X, Y)}{\sqrt{t}}, v(t, x, y) = V(X, Y)$ | $(XIII)$ $2 \alpha V_{XXX} + U_Y + X U_{XY} - 2 U_{YXX} = 0$ $(X + 2 \beta U) V_X + 2 V_{XX} = 0$ |
Now we have examined the optimal system, similarity variables for the optimal system, the resultant equation based on the similarity variables are tabulated in table \[9\]. The follows, giving the reductions and solutions for the resultant equations which are tabulated in table \[9\].

Reductions and solutions for (VII)
The resultant system (VII) have the symmetries \(\partial_X, f_1(Y)\partial_Y - V f_1'(Y)\partial_V, f_2(Y)\partial_V\). If \(f_1(Y) = f_2(Y) = 1\) then the optimal system, corresponding similarity variables, reductions and solutions of (VII) tabulated in table \[10\].

Table 10: Solution for equation (VII)

| Optimal system | Similarity variable | Reduction | Solution |
|----------------|---------------------|-----------|----------|
| \(\partial_X + c_2\partial_Y\) | \(r = X - c_2Y, U(X,Y) = G(r), V(X,Y) = H(r)\) | \(c_2(1 + c_1)G'' - c_2G'''' - \alpha H''' = 0\) | \(H = I_1 \int \exp \left[ - \int (c_1 + \beta G)dr \right] dr + (c_1 + \beta G)'H' + H'' = 0\) |
| \(\partial_Y + c_3\partial_V\) | \(U(X,Y) = G(X), V(X,Y) = \frac{Y}{c_3} + H(X)\) | \(H''' = 0, (c_1 + \beta G)'H' + H'' = 0\) | \(H = I_1 + I_2 X + I_3 X^2\) |

Reductions and solutions for (VIII)
The resultant system (VIII) have the symmetries \(\partial_Y, \partial_V\). The optimal system, corresponding similarity variables, reductions and solutions of (VIII) tabulated in table \[11\].

Table 11: Solution for equation (VIII)

| Symmetry | Similarity variable | Reduction | Solution |
|----------|---------------------|-----------|----------|
| \(\partial_Y\) | \(U = G(X), V = H(X)\) | \(H''' = 0, (\frac{X}{2} + \beta G)'H' + H'' = 0\) | \(H = I_1 + I_2 X + I_3 X^2\) |
| \(\partial_Y + c_3\partial_V\) | \(U = G(X), V = \frac{Y}{c_3} + H(X)\) | \(c_1 + 2c_3 + (\frac{X}{2} + \beta G)'H' + H'' = 0\) | \(H = I_1 + I_2 X + I_3 X^2\) |
Integrating the resultant equation (IX) we obtain the following reduced equation

\[ V = I_3 \int \exp \left[ \int \left( \frac{1 - c_1 \beta U}{c_1} \right) r \right] r, \tag{3.23} \]

\[ U' + \frac{1}{c_1^2} U = \frac{1}{c_1} U - \frac{I_2 + I_1 r}{c_1^2 c_2} + \frac{c_1 I_3 \alpha}{c_2} \exp \left[ \int \left( \frac{1 - c_1 \beta U}{c_1^2} \right) r \right], \tag{3.24} \]

where \( I_1, I_2 \) and \( I_3 \) are constants of integration.

**Reductions and solutions for (X)**
The system (X) admits the symmetries \( Y_{11} = \partial_x, Y_{12} = f_1(y) \partial_y - V f'(y) \partial V, Y_{13} = f_2(y) \partial V \). The adjoint representation given by the table \( \PageIndex{12} \) and corresponding optimal system, similarity variables, reductions and solutions are given in table \( \PageIndex{13} \).

**Table 12: Adjoint representation Table with \( f_1 = 1 \) and \( f_2 = 0 \)**

| \([Ad(e^rY_{1i})Y_{1j}]\) | \(Y_{11}\) | \(Y_{12}\) |
|---|---|---|
| \(Y_{11}\) | \(Y_{11}\) | \(Y_{12}\) |
| \(Y_{12}\) | \(Y_{11}\) | \(Y_{12}\) |

**Table 13: Solution for equation (X)**

| Optimal system | Similarity variable | Reduction | Solution |
|---|---|---|---|
| \(Y_{11}\) | \(U(x, y) = G(x)\), \(V(x, y) = H(x)\) | \(H'' = 0, \beta GH' + H'' = 0\) | \(H = I_1 + I_2 x + I_3 x^2\) |
| \(Y_{12}\) | \(U(x, y) = G(x)\), \(V(x, y) = H(x)\) | \(c_1 G'' + G'' + c_1^2 \alpha H'' = 0, (\beta G) H' - c_1 H'' = 0\) | \(H = I_1 \int \exp \left[ \frac{1}{c_1} \int Gdr \right] dr + I_2\) |
| \(Y_{11} + c_1 Y_{12}\) | \(r = y - c_1 x, U(x, y) = \frac{G(r)}{x}, V(x, y) = H(r)\) | | \(c_1 G' + G = I_3 r + I_4 - c_1^2 I_1 \alpha \exp \left[ \frac{1}{c_1} \int Gdr \right]\) |

**Reductions and solutions for (XI)**
The solution of the system \((XI)\) is given by

\[
U(t,y) = \int f(t)dt + g(y), \quad (3.25)
\]

\[
V(t,y) = h(y). \quad (3.26)
\]

**Reductions and solutions for \((XII)\)**

The solution of the system \((XII)\) is given by

\[
U(t,x) = f'(t) + g'(t)x + h'(t)x^2 - 2h(t), \quad (3.27)
\]

\[
V(t,y) = f(t) + g(t)x + h(t)x^2. \quad (3.28)
\]

**Reductions and solutions for \((XIII)\)**

The trivial symmetry of the system \((XIII)\) is \(\partial_Y\). Based on the symmetry the similarity variable is \(U(X,Y) = G(X)\) and \(V(X,Y) = H(X)\) and the reduced equation is given by

\[
H_{XXX} = 0, \quad (3.29)
\]

\[
(X + 2\beta G)H_X + 2H_{XX} = 0. \quad (3.30)
\]

The solution of above system are given by

\[
H = I_1 + I_2X + I_3X^2, \quad (3.31)
\]

\[
G = \frac{-4I_3 + I_2X + 2I_3X^2}{2\beta(I_2 + 2I_3X)}, \quad (3.32)
\]

where \(I_1, I_2\) and \(I_3\) are constants of integration.

**3.3 Case \(n \neq 1, m = n - 1\)**

If \(m = n - 1\) and \(n \neq 1\) then the system \((LI)\) yields the following equation

\[
\begin{align*}
\frac{u_{ty}}{(u^n - u_x)_{xy} - \alpha v_{xxx}} &= 0, \quad (3.33) \\
v_t - v_{xx} - \beta u^{n-1}v_x &= 0.
\end{align*}
\]

The above system admitted Lie symmetries as we are given below:

\[
\begin{align*}
Z_1 &= \partial_t, \quad Z_2 = \partial_x, \quad Z_3 = \psi(y)\partial_y - \psi'v\partial_v, \quad Z_4^\infty = \phi(y)\partial_v, \\
Z_5 &= 2t\partial_t + x\partial_x + \left(\frac{1}{1 - n}\right)u\partial_u + \left(\frac{2 - n}{1 - n}\right)v\partial_v.
\end{align*}
\]

The commutator table for the symmetries are displayed in table 14. Table 15 gives adjoint representation of the symmetries with condition \(\psi(y) = 1\) and \(\phi(y) = 0\). Optimal system, similarity variables and corresponding resultant equations are given in table 16.
### Table 14: Commutator Table for case 1.3

| $[Z_1, Z_j]$ | $Z_1$ | $Z_2$ | $Z_3$ | $Z_4$ | $Z_5$ |
|--------------|-------|-------|-------|-------|-------|
| $Z_1$        | 0     | 0     | 0     | 0     | $2Z_1$|
| $Z_2$        | 0     | 0     | 0     | 0     | $Z_2$ |
| $Z_3$        | 0     | 0     | 0     | $(\psi' \phi + \phi' \psi)\partial v$ | 0 |
| $Z_4$        | 0     | 0     | $-(\psi' \phi + \phi' \psi)\partial v$ | 0 | $(\frac{n-2}{n-1}) Z_4$ |
| $Z_5$        | $-2Z_1$ | $-Z_2$ | 0 | $-\left(\frac{n-2}{n-1}\right) Z_4$ | 0 |

### Table 15: Adjoint representation Table for case 1.3

| $[\text{Ad}(e^{\epsilon Z_1})Z_j]$ | $Z_1$ | $Z_2$ | $Z_3$ | $Z_5$ |
|---------------------------------|-------|-------|-------|-------|
| $Z_1$                           | $Z_1$ | $Z_2$ | $Z_3$ | $Z_5 - 2\epsilon Z_1$ |
| $Z_2$                           | $Z_1$ | $Z_2$ | $Z_3$ | $Z_5 - \epsilon Z_2$ |
| $Z_3$                           | $Z_1$ | $Z_2$ | $Z_3$ | $Z_5$ |
| $Z_5$                           | $e^{2\epsilon Z_1}$ | $e^{\epsilon Z_2}$ | $Z_3$ | $Z_5$ |
Table 16: Reductions for the case $m = n - 1, n \neq 1$

| Optimal System | Similarity variable | Reductions |
|----------------|---------------------|------------|
| $Z_5 + c_1Z_3$ | $X = \frac{x}{\sqrt{t}}, Y = y - \frac{c_1}{2} \log t,$ $u = t^{\frac{1}{n-1}} U[X,Y], v = t^{\frac{2-n}{2(n-1)}} V[X,Y]$ | (XIV) $\alpha V_{XXX} - U_{XY} + nU_{X}U_{XY} + \frac{1}{2} XU_{XY} + n(n-1)U_{X}U_{XY} + \frac{c_1}{2} U_{YY} + \frac{1}{2(n-1)} U_{Y} = 0$ |
| $Z_1 + c_1Z_2 + c_2Z_3$ | $r = t + c_1 x + c_2 y,$ $u = U(r), v = V(r)$ | (XV) $c_2 U'' - c_1 c_2 \left( n(n-1)U^{n-2}U'' + nU^{n-1}U'' \right) + c_1^2 c_2 U'' + \alpha c_1^3 V''' = 0$ |
| $Z_1$ | $u = U[x, y], v = V[x, y]$ | (XVI) $n(n-1)U^{n-2}U_x U_y + nU^{n-1}U_{xy} - U_{xxy} + \alpha V_{xxx} = 0$ $\beta U^{n-1}V_x + V_{xx} = 0$ |
| $Z_2$ | $u(t, x, y) = U(t, y), v(t, x, y) = V(t, y)$ | (XVII) $U_t = 0,$ $V_t = 0$ |
| $Z_3$ | $u(t, x, y) = U(t, x), v(t, x, y) = V(t, x)$ | (XVIII) $\beta U^{n-1}V_x + V_{xx} - V_t = 0$ $V_{xxx} = 0$ |
| $Z_5$ | $x = X\sqrt{t}, y = Y,$ $u(t, x, y) = (2(1-n) t^{\frac{1}{(n-1)}} U(X, Y), v(t, x, y) = (2(n-1) t^{\frac{2-n}{2(n-1)}} V(X, Y)$ | (XIX) $A^\frac{1}{2} B^\frac{1}{2} U_y + n A^\frac{n}{2} B^\frac{n}{2}((n^2 + 1)A - nA^n)U^{n-2}U_X U_Y + (n-1)A^\frac{n}{2} B^\frac{n}{2} XU_{XY} + n(n-1)A^\frac{n}{2} B^\frac{n}{2} U^{n-1}U_{XY} + A^\frac{1}{2} B^\frac{1}{2} \alpha V_{XXX} + A^\frac{1}{2} B^\frac{1}{2} \alpha V_{XXX} = 0$ $(n-2) V - (n-1) XV + (2(1-n))^{\frac{1}{2}} \beta U^{n-1}V_X - 2(n-1) V_{XX} = 0$ |

where $A = 2^{\frac{1}{n-1}}$ and $B = (1-n)^{\frac{1}{(n-1)}}$. 
Table 17: Solution for equation (XIV)

| Symmetry                         | Similarity variable | Reduction                                      | Solution                                      |
|----------------------------------|---------------------|------------------------------------------------|------------------------------------------------|
| \(\partial_Y\)                  | \(U = G(X),\)       | \(H''' = 0,\)                                   | \(H = I_1 + I_2X + I_3X^2\)                  |
|                                  | \(V = H(X)\)        | \(\frac{(n-2)}{(n-1)} H - (X + 2\beta G^{n-1})H' - 2H'' = 0\) | \(G = \left[ \frac{2(n-1)\beta I_2 + 2I_3X}{(2-n)I_1 + I_2X + 4(n-1)I_3 + nI_4X^2} \right]^{\frac{1}{n-1}}\) |
| \(\partial_Y + c_3e^{(n-1)c_1} \partial_Y\) | \(U = G(X),\)       | \(H''' = 0,\)                                   | \(H = I_1 + I_2X + I_3X^2\)                  |
|                                  | \(V = \frac{(n-1)c_1}{c_3(n-2)}e^{(n-1)c_1} + H(X)\) | \(\frac{(n-2)}{(n-1)} H - (X + 2\beta G^{n-1})H' - 2H'' = 0\) | \(G = \left[ \frac{2(n-1)\beta I_2 + 2I_3X}{(2-n)I_1 + I_2X + 4(n-1)I_3 + nI_4X^2} \right]^{\frac{1}{n-1}}\) |
Reductions and solutions for (XIV)

The system (XIV) has two symmetries which are given as \( \partial_Y \) and \( e^{\frac{(n-2)Y}{c_1^2}} \partial_Y \). Corresponding reductions and solutions are tabulated in table 17.

Reductions and solutions for (XV)

The resultant equation further reduced to the following equations

\[
V = I_3 \int \exp \left[ \int \left( \frac{1 - c_1 \beta U^{n-1}}{c_1^2} \right) r \right] r, \tag{3.34}
\]

\[
U' + \frac{1}{c_1^2} U = \frac{1}{c_1} U^n - \frac{I_2 + I_1 r}{c_1^2 c_2} + \frac{c_1 I_3 \alpha}{c_2} \exp \left[ \int \left( \frac{1 - c_1 \beta U^{n-1}}{c_1^2} \right) r \right], \tag{3.35}
\]

where \( I_1, I_2 \) and \( I_3 \) are constants of integration.

Reductions and solutions for (XVI)

The equation (XVI) admits the following symmetries

\[
Z_11 = \partial_x, Z_{12} = f_1(y) \partial_y - V f'(y) \partial_V, Z_{13} = x \partial_x - \left( \frac{1}{n-1} \right) U \partial_U + \left( \frac{n-2}{n-1} \right) V \partial_V, Z_{14} = f_2(y) \partial_V.
\]

The adjoint representation with conditions \( f_1(y) = 1 \) and \( f_2(y) = 0 \) is tabulated in table 18 and corresponding optimal system, similarity variables, reduced system and solutions are given by table 19.

| Table 18: Adjoint representation Table with \( f_1 = 1 \) and \( f_2 = 0 \) |
|---------------------------------|-----------------|-----------------|-----------------|
| \([\text{Ad}(e^r Z_U)] Z_{1j}\) | \(Z_{11}\) | \(Z_{12}\) | \(Z_{13}\) |
| \(Z_{11}\) | \(Z_{11}\) | \(Z_{12}\) | \(Z_{13} - \epsilon Z_{11}\) |
| \(Z_{12}\) | \(Z_{11}\) | \(Z_{12}\) | \(Z_{13}\) |
| \(Z_{13}\) | \(e^r Z_{11}\) | \(Z_{12}\) | \(Z_{13}\) |
| Optimal system | Similarity variable | Reduction | Solution |
|---------------|----------------------|-----------|----------|
| $Z_{11}$      | $U(x, y) = G(y)$, $V(x, y) = H(y)$ | $H'' = 0$, $\beta G^{n-1} H' + H'' = 0$ | $U(x, y) = G(y)$ $V(x, y) = H(y)$ |
| $Z_{12}$      | $U(x, y) = G(x)$, $V(x, y) = x \frac{n}{n-2} H(y)$ | $H'' = 0$, $\beta G^{n-1} H' + H'' = 0$ | $H = I_1 + I_2 x + I_3 x^2$ $G = \left[ -\frac{2I_3}{\beta(I_2 + 2I_3 x)} \right] \frac{1}{n-1}$ |
| $Z_{13}$      | $U(x, y) = (x - nx) \frac{1}{1-n} G(y)$, $V(x, y) = H(y)$ | $G + \beta G'' = 0$ | $G(y) = (-\beta)^{\frac{1}{1-n}}$ |
| $Z_{11} + c_1 Z_{12}$ | $r = y - c_1 x$, $U(x, y) = G(r)$, $V(x, y) = H(r)$ | $n(n-1)(G^{n-2} G'^2) + nG^{n-1} G'' + c_1 G''' + c_1^2 \alpha H''' = 0$, $\beta G^{n-1} H' - c_1 H'' = 0$ | $H(y) = 0$ $H = I_1 \int \exp \left[ \frac{\beta}{c_1} \int G^{n-1} dr \right] dr + I_2$ $c_1 G' = I_4 + I_3 r - G^n$ $c_1^2 \alpha I_1 \exp \left[ \frac{\beta}{c_1} \int G^{n-1} dr \right]$ |
| $Z_{13} + c_1 Z_{12}$ | $r = y - c_1 x$, $U(x, y) = x^{\frac{1-n}{n-2}}(1-n)^{\frac{1}{1-n}} G(r)$, $V(x, y) = x^{\frac{n-2}{n-1}} H(r)$ | $n \left( \frac{n-2}{n-1} \right) \alpha H + n^2(1-n)^{\frac{1}{1-n}} G^{n-1} G' + c_1 n(1-n)^{\frac{2-n}{1-n}} G^{n-2} G'^2 + c_1 (n^2 - 2n - 2) \alpha H' + c_1 (1+n)(1-n)^{\frac{1}{1-n}} G'' - c_1 n(1-n)^{\frac{2-n}{1-n}} G^{n-1} G'' - 3c_1^2(n-1) \alpha H'' - c_1^2(1-n)^{\frac{2-n}{1-n}} G''' + n(1-n)^{\frac{1}{1-n}} G' - c_1^3 (n-1)^{2} \alpha H''' = 0$ $\left( \frac{n-2}{n-1} \right) (1 + \beta G^{n-1}) H + c_1 (n-3) H' - c_1 \beta G^{n-1} H' - c_1^2 (n-1) H'' = 0$ | | |
Reductions and solutions for (XVII)

The solution of the system (XVII) is given by

\[
U(t, y) = \int f(t) dt + g(y), \quad (3.36)
\]

\[
V(t, y) = h(y). \quad (3.37)
\]

Reductions and solutions for (XVIII)

The solution of the system (XVIII) is given by

\[
U(t, x) = \left[ \frac{f'(t) + g'(t)x + h'(t)x^2 - 2h(t)}{\beta(g(t) + 2xh(t))} \right]^{\frac{1}{n-1}}, \quad (3.38)
\]

\[
V(t, y) = f(t) + g(t)x + h(t)x^2. \quad (3.39)
\]

Reductions and solutions for (XIX)

The trivial symmetry of the system (XIX) is \( \partial_Y \). Based on this symmetry the system is reduced to

\[
H''' = 0, \quad (3.40)
\]

\[
(n - 2)H - (n - 1)XH' + (2(1 - n))^{\frac{1}{2}} \beta G^{n-1}H' - 2(n - 1)H'' = 0. \quad (3.41)
\]

The solution of above system are given by

\[
H = I_1 + I_2X + I_3X^2, \quad (3.42)
\]

\[
G = \left[ \frac{(2(1 - n))^{\frac{1}{2}} \beta (I_2 + 2I_3X)}{(2-n)I_1 + I_2X + 4(n - 1)I_3 + nI_3X^2} \right]^{\frac{1}{n-1}}. \quad (3.43)
\]

3.4 Arbitrary \( n, m \)

For arbitrary \( m \) and \( n \):

The system (1.1) admitted Lie symmetries as follows

\[
\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = \partial_y, \quad \Gamma_4 = \psi(y) \partial_v
\]

The commutator table for the symmetries are displayed in table 20. Table 21 gives adjoint representation of the symmetries with condition \( \psi(y) = 0 \). Optimal system, similarity variables and corresponding resultant equations are given in table 22.
Table 20: Commutator Table for case 1.4

| $[\Gamma_I, \Gamma_J]$ | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ | $\Gamma_4$ |
|------------------------|----------|----------|----------|----------|
| $\Gamma_1$            | 0        | 0        | 0        | 0        |
| $\Gamma_2$            | 0        | 0        | 0        | 0        |
| $\Gamma_3$            | 0        | 0        | 0        | $\Gamma_4$ |
| $\Gamma_4$            | 0        | 0        | $-\Gamma_4$ | 0        |

Table 21: Adjoint representation table for case 1.4

| $[Ad(e^{i\Gamma_I})\Gamma_J]$ | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ |
|-------------------------------|----------|----------|----------|
| $\Gamma_1$                   | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ |
| $\Gamma_2$                   | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ |
| $\Gamma_3$                   | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ |
Table 22: Reductions for the arbitrary case $m$ and $n$

| Optimal System | Similarity variable | Reductions |
|----------------|---------------------|------------|
| $\Gamma_1 + c_1\Gamma_2 + c_2\Gamma_3$ | $r = t + c_1x + c_2y,$ $u = U(r), v = V(r)$ | (XX) $c_2U'' - c_1c_2 \left( n(n-1)U^{n-2}U'' + nU^{n-1}U'' \right) + c_1^2c_2U''' + \alpha c_1^3V''' = 0$  
| | | $\left( 1 - c_1\beta U^m \right) V' - c_1^2V'' = 0$ |
| $\Gamma_1$ | $u(t, x, y) = U(x, y), v(t, x, y) = V(x, y)$ | (XXI) $n(n-1)U^{n-2}U_xU_y + nU^{n-1}U_{xy} - U_{xxy} + \alpha V_{xxx} = 0$  
| | | $\beta U^m V_x + V_{xx} = 0$ |
| $\Gamma_2$ | $u(t, x, y) = U(t, y), v(t, x, y) = V(t, y)$ | (XXII) $U_{ty} = 0,$  
| | | $V_t = 0$ |
| $\Gamma_3$ | $u(t, x, y) = U(t, x), v(t, x, y) = V(t, x)$ | (XXIII) $\beta U^m V_x + V_{xx} - V_t = 0$  
| | | $V_{xxx} = 0$ |
Reductions and solutions for \((X X)\)

\[
V = I_3 \int \exp \left[ \int \left( \frac{1 - c_1 \beta U'^m}{c_1^2} \right) r. \right] r., \quad (3.44)
\]

\[
U'' + \frac{1}{c_1^2} U = \frac{1}{c_1} U'' + \frac{1}{c_1} U'' + \frac{I_2 + I_3 r}{c_1^2 c_2} + \frac{c_1 I_3 \alpha}{c_2} \exp \left[ \int \left( \frac{1 - c_1 \beta U'^m}{c_1^2} \right) r. \right], \quad (3.45)
\]

where \(I_1, I_2\) and \(I_3\) are constants of integration.

Reductions and solutions for \((X XI)\)

The system \((X XI)\) admits the following symmetries \(\Gamma_{11} = \partial_x, \Gamma_{12} = \partial_y\). The corresponding optimal system, similarity variables, reductions and solutions are given in table 23.

Table 23: Solution for equation \((X XI)\)

| Optimal system | Similarity variable | Reduction | Solution |
|----------------|---------------------|-----------|----------|
| \(\Gamma_{11}\) | \(U(x, y) = G(y),\) \(V(x, y) = H(y)\) | \(H''' = 0, \beta G^m H' + H'' = 0\) | \(U(x, y) = G(y)\) \(V(x, y) = H(y)\) |
| \(\Gamma_{12}\) | \(U(x, y) = G(x),\) \(V(x, y) = H(x)\) | \(H''' = 0, \beta G^m H' + H'' = 0\) | \(H = I_1 + I_2 x + I_3 x^2\) \(G = \left[ \frac{2 I_4}{3 (I_2 + 2 I_3 x)} \right]^{\frac{1}{m}}\) |
| \(\Gamma_{11} + c_1 \Gamma_{12}\) | \(r = y - c_1 x, \) \(U(x, y) = \frac{G(r)}{x},\) \(V(x, y) = H(r)\) | \(n(n-1)G'^{n-2}G'' + nG^{n-1}G'' + c_1 G'''' + c_1^2 \alpha H'''' = 0\) \((\beta G) H' - c_1 H'' = 0\) | \(H = I_1 \int \exp \left[ \frac{\beta}{c_1} \int (G^m) dr \right] dr + I_2\) \(c_1 G' + G^m = I_3 r + I_4\) \(c_1^2 I_1 \alpha \exp \left[ \frac{\beta}{c_1} \int (G^m) dr \right]\) |

Reductions and solutions for \((X XII)\)

The solution of the system \((X XII)\) is given by

\[
U(t, y) = \int f(t) dt + g(y), \quad (3.46)
\]

\[
V(t, y) = h(y). \quad (3.47)
\]

Reductions and solutions for \((X XIII)\)

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The solution of the system (XXIII) is given by

\begin{align*}
U(t, x) &= \left[ f'(t) + g'(t)x + h'(t)x^2 - 2h(t) \right] \frac{1}{\beta(g(t) + 2xh(t))} \\
V(t, y) &= f(t) + g(t)x + h(t)x^2.
\end{align*} 

(3.48) (3.49)

4 Conclusion

In this work, we studied the generalized BLP system by using the theory of Lie symmetries. Specifically, we performed a detailed classification of the admitted Lie point symmetries for the generalized BLP system by constraint the free parameters of the system with the Lie conditions.

We found four different sets for the unknown parameters in which the resulting systems admits different Lie point symmetries. For each system we determined the commutators and the Adjoint representation for the admitted Lie point symmetries. The latter applied to determine the one-dimensional optimal system, an important information in order to determine all the possible and independent similarity transformations which lead to invariant solutions. Finally invariant solutions have been presented for all the cases of study.

This work contributes on the application of Lie’s theory on nonlinear differential equations. In a future work we want to investigate the existence of conservation laws which follow from the admitted group properties for the generalized BLP system.

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