

$N = 3$ - extended Supersymmetric Schwarzian and Liouville Theories

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Abstract

$N = 3$ super-Schwarzian and $N = (3, 0)$ super-Liouville theories are formulated by the coadjoint orbit method. We study the coadjoint orbit dependence of the respective theories, represented by a superfield $b$. We show that it is renormalized into the $N = 3$ super-Schwarzian derivative when the $b$ field takes an appropriate configuration at the initial point of the orbit. Then the renormalized actions of the respective theories are invariant under $\text{OSp}(2|3)$ transformations. If the configuration gets further specified, the initial point of the orbit turns out to be stable under one other kind of $\text{OSp}(2|3)$ transformations as well.

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1 Introduction

A discovery of a duality between the SYK model[1, 2] and the $D = 2$ effective gravity raised a vivid interest in the Schwarzian theory. It was considered as playing a role of a mediator between the dual theories. Various interesting generalizations of the Schwarzian theory were undertaken. Supersymmetric generalization is one of them. $N = 1, 2$ and 4 super-Schwarzian theories were discussed in [3, 4, 5]. In [5] the $N = 4$ theory was formulated by the coadjoint orbit method. The differential geometry in the theory then got manifest. Such a formulation was originally given for the non-supersymmetric Schwarzian theory in [6]. The formulation can be applied for the $N = 1$ and 2 theories as well by simply adjusting the arguments in [5].

The coadjoint orbit method was originated by Alekseev and Shatashvili thirty years ago[7]. By the method they studied the Liouville theory which is one of the $D = 2$ effective gravity. Supersymmetric generalization of the arguments was subsequently undertaken. The $N = (1, 0), (0, 1), (1, 1)$ and $(2, 0)$ super-Liouville theories were discussed in the literature [8, 9, 10, 11]. Structural resemblance between these Liouville theories and the Schwarzian theories is notorious. Further extension to the $N = (4, 0)$ super-Liouville theory was discussed only recently in [12].

An $N = 3$ supersymmetric extension is missing for the Schwarzian and Liouville theories both. The aim of this paper is to study it and to fill up the gap in the literature. To this end we need a proper knowledge about the $N = 3$ superconformal algebra. It has been discussed in few occasions in the literature. We begin by giving a brief summary on the algebra in Section 2. The reader familiar with the algebra may skip this section. In Section 3 the coadjoint orbit method is worked out to obtain the Kirillov-Kostant two-form $\widehat{\Omega}_b$ which is closed and invariant under the $N = 3$ superconformal diffeomorphism. It is shown that

$$\widehat{\Omega}_b = dL, \quad i_b\widehat{\Omega}_b = dH.$$ 

Here $L$ is a one-form while $H$ is a zero-form. They give the $N = 3$ super-Liouville and super-Schwarzian theories respectively. (See (3.15).) In Section 4 we study the orbit dependence of the Kirillov-Kostant two-form $\widehat{\Omega}_b$, represented by a superconformal field $b$. It is shown that the $b$ dependence is renormalized into the Kirillov-Kostant two-form $\widehat{\Omega}_b$ with $b = 0$ when the configuration is chosen appropriately at the initial point of the orbit. Then the renormalized Kirillov-Kostant two-form gets invariant under $\text{OSp}(2|3)$ transformation. If the configuration is furthermore specified, the initial point of the orbit gets stable under one other kind of $\text{OSp}(2|3)$ transformation as well. These symmetries are called $\text{OSp}(2|3)_{\text{target}}$ and $\text{OSp}(2|3)_{\text{diff}}$ respectively. In Sections 5 and 6 they are examined for the $N = 3$ super-Schwarzian and $N = (3, 0)$ super-Liouville actions, given in Section 3. The above twofold $\text{OSp}(2|3)$ symmetry of the renormalized Kirillov-Kostant two-form is recovered for these actions. The super-Schwarzian action is worked out by expanding superfields in components. It turns out to be remarkably simple. (See (5.3).) We comment on its quantization in the final section. As for the super-Liouville theory the action contains a non-local term. It can be hardly put in a local form in the supercovariant formulation with superfields. We give it a local expression by expanding the superfields in
components. The expression is rather complicated. (See (3.21).) The similar complication has already appeared in the $N = (4, 0)$ theory, but not in the $N = (0, 0), (1, 0)$ and $(2, 0)$ theories. In this regard we make one more comment on the super-Liouville theories in the final section. Appendices A and B contain helpful formulae for some arguments in the paper. Appendix C is devoted to make clear a subtle difference between the $N = 3$ and 4 super-Schwarzian theories in arguing the twofold OSp(2|3) symmetry.

2 \hspace{1em} N = 3 superconformal symmetry

i) $N = 3$ superconformal diffeomorphism

The $N = 3$ superspace is described by supercoordinates

\[(x, \theta_1, \theta_2, \theta_3) \equiv (x, \theta).\]

Here $x$ is a one-dimensional coordinate and $\theta_a, a = 1, 2, 3$ are fermionic coordinates belonging to the O(3) triplet. The supercovariant derivatives are defined by

\[D_{\theta a} = \frac{\partial}{\partial \theta_a} + \theta_a \partial_x,\]

so as to satisfy

\[\{D_{\theta a}, D_{\theta b}\} = 2\delta_{ab}\partial_x.\]

Under a transformation of the supercoordinates

\[x \longrightarrow f(x, \theta_1, \theta_2, \theta_3) \equiv f(x, \theta), \quad \theta_a \longrightarrow \varphi_a(x, \theta_1, \theta_2, \theta_3) \equiv \varphi_a(x, \theta),\] \hspace{1em} (2.1)

the supercovariant derivative $D_{\theta a}$ changes to $D_{\varphi a}$

\[D_{\theta a} = (D_{\theta a} \varphi_b)D_{\varphi b}.\] \hspace{1em} (2.2)

if the transformation (2.1) satisfies the superconformal condition

\[D_{\theta a} f = \varphi_b D_{\theta a} \varphi_b.\] \hspace{1em} (2.3)

Then (2.1) is called superconformal diffeomorphism. If a superfield $A(x, \theta)$ transforms by (2.1) as

\[A(x, \theta) \longrightarrow \Delta^w A(f(x, \theta), \varphi(x, \theta)),\] \hspace{1em} (2.4)

with a scaling factor $\Delta \equiv \frac{1}{3}(D_{\theta a} \varphi_b)(D_{\theta a} \varphi_b)$, it is called superconformal field with weight $w$. Consider an infinitesimal superconformal diffeomorphism

\[x \longrightarrow f(x, \varphi) = x + \delta x + \cdots, \quad \theta_a \longrightarrow \varphi(x, \varphi) = \theta_a + \delta \theta_a + \cdots.\] \hspace{1em} (2.5)

\[\text{See Appendix A for an alternative expression of } \Delta.\]
It is constrained by (2.3) as
\[
\delta \theta_a = \frac{1}{2} D_{\theta a} (\delta x + \theta_c \delta \theta_c).
\] (2.6)

Calculating the Lie variation we find (2.4) in the infinitesimal form\(^\text{2}\)
\[
\delta A = ((\delta x + \theta_c \delta \theta_c) \partial_x + \delta \theta_c D_{\theta c} + w \partial_x v) A
= (v \partial_x + \frac{1}{2} (D_{\theta c} v) D_{\theta c} + w \partial_x v) A.
\] (2.7)

Here we used the constraint (2.6) and defined a parameter of the Lie variation as \(v \equiv \delta x + \theta_c \delta \theta_c\). From now on this Lie variation is denoted by \(\delta v\). It satisfies
\[
[\delta u, \delta v] = \delta [v, u],
\]
with
\[
[v, u] \equiv v \partial_x u - v \partial_x v + \frac{1}{2} D_{\theta a} v D_{\theta a} u.
\]

The \(N = 3\) super-Schwarzian derivative was given in [13]
\[
S(f, \varphi; x, \theta) = 2 \varepsilon_{abc} \frac{(D_{\theta a} D_{\theta b} \varphi_d)(D_{\theta c} \varphi_d)}{(D_{\theta e} \varphi_f)(D_{\theta e} \varphi_f)}.
\] (2.8)

Consider a further superconformal diffeomorphism
\[
f(x, \theta) \rightarrow F(f(x, \theta), \varphi(x, \theta)), \quad \varphi(x, \theta) \rightarrow \Phi(f(x, \theta), \varphi(x, \theta)),
\]
satisfying the superconformal condition
\[
D_{\varphi a} F = (\Phi_b D_{\varphi a}) \Phi_b.
\]

It obeys the composition rule
\[
S(F(f, \varphi), \Phi(f, \varphi); x, \theta) = \Delta^2 S(F, \Phi; f, \varphi) + S(f, \varphi; x, \theta).
\] (2.9)

Take the infinitesimal transformation (2.5) while fixing \(F\) and \(\Phi\). It then follows that
\[
\delta, S(F, \Phi; x, \theta) = (v \partial_x + \frac{1}{2} \partial_x v + \frac{1}{2} (D_{\theta b} v) D_{\theta b}) S(F, \Phi; x, \theta) + \frac{1}{6} \varepsilon_{abc} D_{\theta a} D_{\theta b} D_{\theta c} v.
\] (2.10)

We note that the transformation law (2.7) can be consistently extended this way, when \(w = \frac{1}{2}\).

Finally we examine the non-supersymmetric part of \(S(f, \varphi; x, \theta)\). \(f\) and \(\varphi_a\) are expanded in components as (A.2). The non-supersymmetric parts read as
\[
f(x, \theta) = h + \text{superpartners}, \quad \varphi_a(x, \theta) = \theta_a \rho(x) + \text{superpartners}.
\]

\(^2\)Hereinafter we do not write the arguments of superfields explicitly if they are simply \((x, \theta)\).
Put these in (2.3) and (2.8). The superconformal condition becomes\(^3\)
\[
\partial_x h = \rho^2 + \text{superpartners}.
\] (2.11)

Using this we find the \( N = 3 \) super-Schwarzian derivative to tend to the well-known non-supersymmetric limit
\[
S(f, \varphi; x, \theta) = \cdots + \theta_1 \theta_2 \theta_3 \left\{ \frac{\partial^3 h}{\partial_x h} - \frac{3}{2} \left( \frac{\partial^2 h}{\partial_x h} \right)^2 + \text{superpartners} \right\}.
\] (2.12)

(ii) \( N = 3 \) superconformal algebra

The \( N = 3 \) superconformal transformation (2.7) may be represented by means of a commutation relation
\[
(v \partial_x + \frac{1}{2} (D_{\theta c} v) D_{\theta c} + w \partial_x v) A = \int dx d^3 \theta \ v T, A],
\] (2.13)

Here \( T \) is the generator of the transformation, while \( A \) is regarded as an operator representing the superconformal superfield defined by (2.4). Note that \( T \) is a fermionic operator with weight \( \frac{1}{2} \). When \( A \) is taken to be \( T \), the transformation (2.13) may become anomalous like that of the Schwarzian derivative, i.e.,
\[
(v \partial_x + \frac{1}{2} (D_{\theta c} v) D_{\theta c} + \frac{1}{2} \partial_x v) T + \frac{c}{6} \epsilon_{abc} D_{\theta a} D_{\theta b} D_{\theta c} v = \int dx d^3 \theta \ v T, T],
\] (2.14)

with a constant \( c \). It can be written in the standard form of the \( N = 3 \) superconformal algebra by taking the Fourier expansion\(^14\). To see this we begin by taking expansion of \( v \) and \( T \) in \( \theta \) as
\[
v = \alpha(x) + \theta_c \beta_c(x) + \frac{1}{2} \epsilon_{abc} \theta_a \theta_b t_c(x) + \frac{1}{3!} \epsilon_{abc} \theta_a \theta_b \theta_c \beta(x),
\] (2.15)
\[
T = F(x) + \theta_a T_a(x) + \frac{1}{2} \epsilon_{abc} \theta_a \theta_b F_c(x) + \frac{1}{3!} \epsilon_{abc} \theta_a \theta_b \theta_c T(x).
\] (2.16)

The resulting components are expanded in Fourier modes. They take the forms
\[
\alpha(x) = \sum_{n \in \mathbb{Z}} e^{inx} \alpha_n,
\beta_a(x) = \sum_{r \in \mathbb{Z}+\nu} e^{irx} \beta_{ar},
\]
\[
t_a(x) = \sum_{n \in \mathbb{Z}} e^{inx} t_{an},
\beta(x) = \sum_{r \in \mathbb{Z}+\nu} e^{rx} \beta_r,
\] (2.17)

and
\[
F(x) = \sum_{r \in \mathbb{Z}+\nu} e^{irx} F_r,
T_a(x) = \sum_{n \in \mathbb{Z}} e^{inx} T_{an},
\]
\[
F_c(x) = \sum_{r \in \mathbb{Z}+\nu} e^{irx} F_{cr},
T(x) = \sum_{n \in \mathbb{Z}} e^{inx} L_n,
\]

\(^3\)This is one of the constraints following from the superconformal condition (2.8). Other constraints are given in Appendix A.
in which $\nu = 0$ for the R sector or $\frac{1}{2}$ for the NS sector. By these Fourier expansions we find
\[
\int dxd^3\theta \; vT \propto \sum_{n \in \mathbb{Z}} \left( \alpha_n L_n + t_{a-n} T_{an} \right) + \sum_{r \in \mathbb{Z} + \nu} \left( \beta_{a-r} F_{ar} + \beta_{-r} F_r \right).
\]
The r.h.s. of (2.14) can be similarly calculated. Comparing the Fourier modes of both sides we then find the $N = 3$ superconformal algebra\[^{15}\]
\[
[L_m, L_n] = (m - n)L_{m+n} + cm(m^2 - 1)\delta_{m+n,0},
\]
\[
[T_{am}, T_{bn}] = i\varepsilon_{abc}T_{cm+n} + 4cm\delta_{ab}\delta_{m+n,0},
\]
\[
[L_m, T_{an}] = -nT_{am+n},
\]
\[
[L_m, F_{ar}] = \left( \frac{m}{2} - r \right)F_{am+r},
\]
\[
[T_{am}, F_{br}] = -i\varepsilon_{abc}F_{cm+r} + m\delta_{ab}F_{m+r},
\]
\[
\{F_{ar}, F_{bs}\} = -2\delta_{ab}L_{r+s} + (r - s)i\varepsilon_{abc}T_{cr+s} - 4c(r^2 - \frac{1}{4})\delta_{ab}\delta_{r+s,0},
\]
\[
[L_m, F_r] = -(m + r)F_{m+r},
\]
\[
[F_r, T_{am}] = 0,
\]
\[
\{F_{ar}, F_s\} = -T_{ar+s},
\]
\[
\{F_r, F_s\} = -4c\delta_{r+s,0},
\]

after scaling and shifting as\[^{4}\]
\[
F_a \to \frac{1}{2\sqrt{i}}F_a, \quad F \to \frac{1}{2\sqrt{i}}F, \quad T_a \to \frac{1}{2}T_a, \quad L_0 \to L_0 - \frac{c}{2}.
\]

One can eliminate $F_r$ and redefine the remaining generators to find the non-Lie algebraic $O(3)$ superconformal algebra, in the sense that the anti-commutator $\{F_{ar}, F_{bs}\}$ contains a term quadratic in $T_{an}$\[^{17, 18}\].

iii) The superalgebra $\mathfrak{osp}(2|3)$

The following twelve zero-mode generators in the above $N = 3$ superconformal algebra
\[
L_{\pm n}, L_0, F_a \pm \frac{1}{2}n, T_{a0} (= \varepsilon_{abc}T_{b[0]c}) , \quad n \in \mathbb{N} \neq 0,
\]
form the superalgebra $\mathfrak{osp}(2|3)$ By scaling them as
\[
\begin{align*}
\bar{L} &= nL_n, \quad L = nL_{-n}, \quad L_0 = nL_0, \\
\bar{T}_a &= \sqrt{n}F_a \pm \frac{1}{2}n, \quad F_a = \sqrt{n}F_{a-\frac{1}{2}n}, \quad T_a = T_{a0},
\end{align*}
\]

\[^{4}\]Further scaling $F_a \to iF_a$ and $F \to iF$ are needed to get the $N = 3$ superconformal algebra with the normalization in \[^{15} 16\]. See also Footnote 5.
it is given by

\[
[T_a, T_b] = i\varepsilon_{abc} T_c, \quad [T_a, L] = 0, \quad [T_a, L^0] = 0, \\
[T, L] = 2L^0, \quad [L, L^0] = -L, \quad [L, L^0] = T,
\]

\[
[F_a, L^0] = -\frac{1}{2} F_a, \quad [F_a, L^0] = \frac{1}{2} F_a, \\
[F_a, L] = 0, \quad [F_a, T] = F_a, \quad [F_a, L] = -F_a, \quad [F_a, T] = 0, \\
[F_a, T_b] = i\varepsilon_{abc} F_a, \quad [F_a, T_b] = i\varepsilon_{abc} F_a, \\
\{F_a, F_b\} = -2\delta_{ab} L, \quad \{F_a, T\} = -2\delta_{ab} T, \quad \{F_a, F_b\} = -2\delta_{ab} L^0 - i\varepsilon_{abc} T_c.
\] (2.19)

iv) Non-linear realization of \textbf{osp}(2|3)

Let us write the above superalgebra \textbf{osp}(2|3) in a collective form

\[
\delta \epsilon_T^B \equiv [\epsilon_A T^A, T^B] = \epsilon_A f^{AB} C^T,
\]

with \( T^A = \{L, L^0, L, F_a, L_0, T_a\} \). Here \( \epsilon_A \) are infinitesimal parameters, which are graded correspondingly to the grading of \( T^A \). This commutation relation may be realized by non-linear transformations of the \( N = 3 \) superconformal diffeomorphisms \( f \) and \( \varphi_a \)

\[
\delta \epsilon f = \epsilon_A R^A(f, \varphi), \quad \delta \epsilon \varphi_a = \epsilon_A R^A_a(f, \varphi). \tag{2.20}
\]

Here \( R^A(f, \varphi) \) and \( R^A_a(f, \varphi) \) are the Killing vectors realizing the superalgebra \textbf{osp}(2|3) as

\[
[\delta \epsilon, \delta \epsilon'] = \delta [\epsilon, \epsilon'],
\]

with

\[
[\epsilon, \epsilon'] = \epsilon_A \epsilon'_B f^{AB} C^T.
\]

Explicit expressions of the Killing vectors are found by considering the coset space OSp(2|3)/\( \hat{H} \) with \( \hat{H} \) the subgroup generated by \( L, F_a, L_0, T_a \). Write a coset element

\[
e^{fL+f\varphi_a F_a}
\]

with the \( N = 3 \) superconformal diffeomorphisms \( f \) and \( \varphi_a \). Multiply \( \epsilon^A f^{T_A} \) from the left. Then we follow the procedure elaborated in \[5, 21, 22\] to find the non-linear transformations \( (2.20) \) as

\[
\epsilon_A R^A(f, \varphi) = \epsilon_L + \epsilon_{L_0} f + \epsilon_T f^2 - \epsilon_{F_a} \varphi_a - \epsilon_{T_0} \varphi_c \epsilon_f; \\
\epsilon_A R^A_a(f, \varphi) = \epsilon_{F_b} + \epsilon_{T_b} f - \epsilon_{T_a} \varphi_a \epsilon_{F_a} + \frac{1}{2} \epsilon_{L_0} \varphi_a + i\varepsilon_{abc} \epsilon_b \varphi_c + \epsilon_{T_0} \varphi_c.
\] (2.21)

It is important that they satisfy the superconformal condition \( (2.3) \) as

\[
D_{\theta^a} \delta \epsilon f = \delta \epsilon \varphi_a D_{\theta^b} \varphi^b + \varphi_a D_{\theta^b} \delta \epsilon \varphi^b.
\]
It is also important that the $N = 3$ super-Schwarzian derivative (2.8) is invariant under these non-linear transformations. To see this it is sufficient to show that the numerator and the denominator scale respectively as

$$
\delta \varepsilon (\varepsilon_{abc} D_{\theta a} D_{\theta b} \varphi_d D_{\theta c} \varphi_d) = (\varepsilon_{L} \varphi + 2 \varepsilon_{T} \varphi + 2 \varepsilon_{g} \varphi) (\varepsilon_{abc} D_{\theta a} D_{\theta b} \varphi_d D_{\theta c} \varphi_d),
$$

$$
\delta \varepsilon (D_{\theta c} \varphi_f D_{\theta c} \varphi_f) = (\varepsilon_{L} \varphi + 2 \varepsilon_{T} \varphi + 2 \varepsilon_{g} \varphi) D_{\theta c} \varphi_f D_{\theta c} \varphi_f,
$$

(2.22)

under the non-linear transformations (2.20).

The whole arguments in this section can be similarly done for the $N = 0, 1, 2$ superconformal symmetries. The zero-mode generators of the resulting $N$-extended superconformal algebra form the superalgebra $osp(2|N)$ with $a, b, c = 1, \cdots, N$. But the argument for the $N = 4$ superconformal algebra goes differently. The superalgebra $osp(2|4)$ contains $psu(1,1|2)$. It is this subalgebra which characterized the $N = 4$-extended supersymmetric Schwarzian and Liouville theories\cite{5,12}. The superalgebra $psu(1,1|2)$ was also non-linearly realized by the Killing vectors on an appropriate coset space. But the $N = 4$ super-Schwarzian derivative is no longer invariant under those non-linear transformations. Instead it transforms as

$$
\delta \varepsilon S = \varepsilon_{L} \varphi + 2 f \varepsilon_{T} \varphi + 2 \varphi_c \varepsilon \varphi + 2 \varphi^c \varepsilon_{T} \varphi_c,
$$

(2.23)

with the notation adapted to the $N = 4$ superconformal algebra\cite{5,12}.

### 3 The Kirillov-Kostant two-form

The adjoint and coadjoint actions in the $N = 3$ superconformal algebra are respectively given by a generator $v$ as

$$
\delta_v u = v \partial u - (\partial u) v + \frac{1}{2} (D_{\theta c} v) D_{\theta c} u \equiv [v, u],
$$

$$
\delta_v \tilde{b} = v \partial \tilde{b} + \frac{1}{2} (\partial v) \tilde{b} + \frac{1}{2} (D_{\theta c} v) D_{\theta c} \tilde{b} + c D_{\theta 1} D_{\theta 2} D_{\theta 3} v.
$$

(3.1)

Here $u, v$ are the $N = 3$ superconformal fields with $w = -1$ in the $N = 3$ superspace, discussed in Section 2. But $\tilde{b}$ is the one with $w = \frac{1}{2}$ which transforms anomalously. It is assumed to be fermionic. On top of $u, v, \tilde{b}$ we consider also $f$ and $\varphi_a$, which represented the $N = 3$ superconformal diffeomorphisms in Section 2. Let them to be $N = 3$ superconformal fields with $w = 0$, i.e.,

$$
\delta_v f = v \partial f + \frac{1}{2} (D_{\theta c} v) D_{\theta c} f,
$$

$$
\delta_v \varphi_a = v \partial \varphi_a + \frac{1}{2} (D_{\theta c} v) D_{\theta c} \varphi_a.
$$

\footnote{The subalgebra $osp(2|3)$ has been normalized in Section 2 consistently with the normalization of these subalgebras given in \cite{8,23,5,12}.}
Keep in mind that they satisfy the superconformal condition (2.3). Correspondingly we define the exterior derivative as

\[ du = y\partial_x u - (\partial_x y)u + \frac{1}{2}(D_{\theta c} y)D_{\theta c} u \equiv [y, u] , \]

\[ d\hat{b} = y\partial_x \hat{b} + \frac{1}{2}(\partial_x y)\hat{b} + \frac{1}{2}(D_{\theta c} y)D_{\theta c} \hat{b} + cD_{\theta 1}D_{\theta 2}D_{\theta 3} y , \]  

(3.2)

and

\[ df = y\partial_x f + \frac{1}{2}(D_{\theta c} y)D_{\theta c} f , \]

\[ d\phi_a = y\partial_x \phi_a + \frac{1}{2}(D_{\theta c} y)D_{\theta c} \phi_a . \]  

(3.3)

Here \( y \) is a one-form, while \( u, \hat{b}, f, \phi_a \) are zero-forms. The two derivatives \( \delta_v \) and \( d \) are related by exchanging \( v \) and \( y \). We introduce a third derivative \( i_v \) which replaces \( y \) by \( v \), called skew-derivative. It is assumed to give zero operating on zero-forms. Then they satisfies the well-known identity in the differential geometry[19, 20]

\[ \delta_v = di_v + i_v d . \]  

(3.4)

Solving the equations in (3.3) for \( y \) with the superconformal condition (2.3) we get

\[ y = \frac{df + \phi_c d\phi_c}{\partial_x f + \phi_c \partial_x \phi_c} . \]

Then it is easy to show\(^6\)

\[ dy = y\partial_x y + \frac{1}{4}(D_{\theta c} y)(D_{\theta c} y) = \frac{1}{2}[y, y] , \]

(3.5)

\[ \delta_v y = v\partial_x y - (\partial_x v)y + \frac{1}{2}(D_{\theta c} v)D_{\theta c} y \equiv [v, y] = \frac{1}{2}i_v[y, y] . \]  

(3.6)

It is instructive to see that the identity (3.4) holds for these relations.

The Kirillov-Kostant two-form \( \Omega \) is defined by

\[ \Omega = \int dxd^3\theta\omega \equiv \frac{1}{2} \int dxd^3\theta\hat{b}[y, y] . \]

A smart way to calculate \( d\Omega \) and \( \delta_v \Omega \) is to use the identity relations for the two derivatives

\[ d(\hat{b}u) = (d\hat{b})u + \hat{b}[y, u] = \partial_x (y\hat{b}u) - D_{\theta c} \{ \frac{1}{2}(D_{\theta c} y)\hat{b}u \} + cD_{\theta 1}D_{\theta 2}D_{\theta 3}(y\hat{b}u) , \]

\[ \delta_v(\hat{b}u) = (\delta_v \hat{b})u + \hat{b}[v, u] = \partial_x (v\hat{b}u) - D_{\theta c} \{ \frac{1}{2}(D_{\theta c} v)\hat{b}u \} + cD_{\theta 1}D_{\theta 2}D_{\theta 3}(v\hat{b}u) . \]

\(^6\)To show (3.5) the formula (A.1) in Appendix is useful. To show (3.3) we do not need calculation noting that the nominator of \( y \) has weight \( w = 0 \), while the denominator \( w = 1 \).
They hold even with \( u \) replaced by \([y, y]\), since both have \( w = -1 \). By means of these identity relations we calculate \( d \omega \) and \( \delta_v \omega \) as

\[
d\omega = \frac{1}{2} (d \hat{b})[y, y] \\
= \frac{1}{2} \partial_\nu (\hat{y} \hat{b}[y, y]) - \frac{1}{2} D_{\theta_\nu} \{ \frac{1}{2} (D_{\theta_\nu} \hat{b})[y, y] \} + \frac{1}{2} c (D_{\theta_1} D_{\theta_2} D_{\theta_3} y)[y, y],
\]

(3.7)

\[
\delta_v \omega = \frac{1}{2} (\delta_v \hat{b})[y, y] - \frac{1}{2} \hat{b}[[y, y], v] = \frac{1}{2} (\delta_v \hat{b})[y, y] + \frac{1}{2} \hat{b} \delta_v (|[y, y])
\]

\[
= \frac{1}{2} \partial_\nu (v \hat{b}[y, y]) - \frac{1}{2} D_{\theta_\nu} \{ \frac{1}{2} (D_{\theta_\nu} \hat{b})[y, y] \} + \frac{1}{2} c (D_{\theta_1} D_{\theta_2} D_{\theta_3} v)[y, y].
\]

(3.8)

Here use was made of the Jacobi identity for the commutator. Then we find

\[
d \Omega = \frac{c}{2} \int dx d^3 \theta (D_{\theta_1} D_{\theta_2} D_{\theta_3} y)[y, y] = \frac{c}{2} d \int dx d^3 \theta (D_{\theta_1} D_{\theta_2} D_{\theta_3} y),
\]

\[
\delta_v \Omega = \frac{c}{2} \int dx d^3 \theta (D_{\theta_1} D_{\theta_2} D_{\theta_3} v)[y, y] = cd \int dx d^3 \theta (D_{\theta_1} D_{\theta_2} D_{\theta_3} v),
\]

\[
i_v \Omega = \int dx d^3 \theta \hat{b}[v, y],
\]

(3.9)

by using \( dv = 0 \). The consistency of these calculations can be checked by the identity (3.4). \( \Omega \) is not invariant under the superconformal transformation. Hence let us modify it by a counter term

\[
\hat{\Omega} = \Omega + \frac{c}{2} \int dx d^3 \theta y D_{\theta_1} D_{\theta_2} D_{\theta_3} y.
\]

(3.10)

The key point for the modification is that the counter term is closed

\[
d \int dx d^3 \theta y D_{\theta_1} D_{\theta_2} D_{\theta_3} y = 0,
\]

(3.11)

and satisfies

\[
\delta_v \int dx d^3 \theta y D_{\theta_1} D_{\theta_2} D_{\theta_3} y = d \int dx d^3 \theta (2v D_{\theta_1} D_{\theta_2} D_{\theta_3} y).
\]

(3.12)

They are both shown in Appendix B by direct calculations. But it is worth noting that the latter formula follows from the former by means of the identity (3.4). From these formulae it follows that

\[
d \hat{\Omega} = 0, \quad \delta_v \hat{\Omega} = 0, \quad di_v \hat{\Omega} = 0.
\]

(3.13)

Due to (3.11) there exists a one-form \( \gamma \) such as

\[
\int dx d^3 \theta y D_{\theta_1} D_{\theta_2} D_{\theta_3} y = d \int dx d^3 \theta \gamma.
\]

(3.14)
Then we find from (3.13) that
\[
\hat{\Omega} = dL, \quad i_v \hat{\Omega} = dH,
\]
in which
\[
L = -\int dx d^3 \theta \left( \hat{b} y + \frac{c}{2} \gamma \right), \quad H = \int dx d^3 \theta \hat{b} v.
\]

The arguments so far done work for any \( N = 3 \) superconformal field \( \hat{b} \) as long as it transforms as (3.1). We consider one other superconformal field \( b \) obeying the same transformation law but with \( c = 0 \). The whole arguments may be repeated for a sum \( \hat{b} + b \), since it still transforms as (3.1). Let \( \hat{b} \) to be given by \( \hat{b} = cS(f, \varphi; x, \theta) \) with the \( N = 3 \) super-Schwarzian derivative (2.8). Its transformation law reads from (2.10) with a suitable change of the notation as
\[
\delta_v \left( cS(f, \varphi; x, \theta) \right) = v \partial_x \left( cS(f, \varphi; x, \theta) \right) + \frac{1}{2} \left( \partial_x v \right) cS(f, \varphi; x, \theta) + cD_{\theta_1} D_{\theta_2} D_{\theta_3} v,
\]
(3.17)

As for \( b \) we identify it with \( \Delta \frac{1}{2} b(f, \varphi) \). It transforms as
\[
\delta_v \left( \Delta \frac{1}{2} b(f, \varphi) \right) = v \partial_x \left( \Delta \frac{1}{2} b(f, \varphi) \right) + \frac{1}{2} \left( \partial_x v \right) \left( \Delta \frac{1}{2} b(f, \varphi) \right) + \frac{1}{2} \left( D_{\theta_3} v \right) D_{\theta_2} \left( \Delta \frac{1}{2} b(f, \varphi) \right),
\]
(3.18)
as can be checked from the transformation law (2.7) for \( b(x, \theta) \) by a direct calculation. Then the sum \( \hat{b} + b \) can be obtained from \( b(x, \theta) \) as a flow
\[
b(x, \theta) \rightarrow \Delta \frac{1}{2} b(f, \varphi) + cS(f, \varphi; x, \theta),
\]
under the \( N = 3 \) superconformal diffeomorphism
\[
x \rightarrow f(x, \theta) \quad \theta_a \rightarrow \varphi_a(x, \theta).
\]

We think of an orbit connecting both ends of the transformation (3.19). It is a line which starts at \( b \) in a space of all superconformal fields transforming as \( \hat{b} \), called coadjoint orbit \( O_b \). It is important to note that
\[
\lim_{f=x, \varphi=\theta} \delta_v \left( \hat{b} + b \right) \neq \delta_v \left( \lim_{f=x, \varphi=\theta} \left( \hat{b} + b \right) \right) = v \partial_x b + \frac{1}{2} \left( \partial_x v \right) b + \frac{1}{2} \left( D_{\theta_3} v \right) D_{\theta_2} b,
\]
at the initial point of the orbit. With \( \hat{b} \) replaced by \( \hat{b} + b \) we can define the Kirillov-Kostant two-form (3.10) on the coadjoint orbit \( O_b \) as
\[
\hat{\Omega}_b = \int dx d^3 \theta \left\{ \left( \Delta \frac{1}{2} b(f, \varphi) + cS(f, \varphi; x, \theta) \right) [y, y] + \frac{c}{2} y D_{\theta_1} D_{\theta_2} D_{\theta_3} y \right\}.
\]
(3.20)
All the formulae in (3.13) hold for $\hat{\Omega}_b$. So do those in (3.15) with

$$L = -\int dxd^3\theta \left\{ \Delta \frac{1}{2} b(f, \varphi) + cS(f, \varphi; x, \theta) \right\} y + \frac{c}{2} \gamma,$$

$$H = \int dxd^3\theta \left( \Delta \frac{1}{2} b(f, \varphi) + cS(f, \varphi; x, \theta) \right) v.$$

In the next sections they are respectively taken to be the actions of the $N = 3$ super-Liouville and super-Schwarzian theories. (3.13) is the most salient feature of the Kirillov-Kostant two-form in the coadjoint orbit method.

Finally we shall give a concrete expression for the the assumed one-form $\gamma$ in (3.14). However it is hard to find a local expression of $\gamma$ in the supercovariant formulation, i.e., in terms of superconformal fields. Therefore we look for it in the component formulation. To this end the author has done rather massive calculations, using the expansion formulae of $f$ and $\varphi_a$ in Appendix A. The details of the calculations were exposed in [24]. We quote only the result

$$6\gamma = -6 \left\{ \log \rho^2 \partial_x d(\log \rho^2) + \partial_x \left( \frac{1}{\rho^2} \right) \partial_x \left( \frac{1}{\rho^2} \right) dh\rho^2 \right\}$$

$$+ \left\{ -6 \frac{\partial_x \rho^2 \partial_x \rho^2}{\rho^2} \frac{dh}{\rho^2} \left( \frac{\eta}{\rho} \cdot \partial_x \eta \right) + 48 \frac{dh}{\rho^2} \left( \frac{\eta}{\rho} \cdot \partial_x^2 \left( \frac{\eta}{\rho} \right) \right) \right. \right.$$  

$$+ 24 \frac{dh}{\rho^2} \left( \partial_x \left( \frac{\eta}{\rho} \right) \cdot \partial_x^2 \left( \frac{\eta}{\rho} \right) \right) \right\}$$

$$+ \left\{ 24 \frac{d\rho^2}{\rho^2} \left( \partial_x^2 \left( \frac{\eta}{\rho} \right) \right) \right. \right.$$  

$$- 24 \left( \partial_x \left( \frac{\eta}{\rho} \right) \cdot \partial_x \left( \frac{\eta}{\rho} \right) \right) \left( \partial_x \left( \frac{\eta}{\rho} \right) \right) \right.$$  

$$+ 6 \left( \partial_x \left( \frac{\eta}{\rho} \right) \cdot \partial_x \left( \frac{\eta}{\rho} \right) \right)^2 \partial_x \left( \frac{d\rho^2}{\rho^2} \right)$$

$$- 24 \partial_x \left( \frac{\partial_x \rho}{\rho} \right) \left( d \left( \frac{\eta}{\rho} \right) \cdot \frac{\eta}{\rho} \right) \left( \partial_x \left( \frac{\eta}{\rho} \right) \right)$$

$$- 24 \left( \frac{\partial_x \rho}{\rho} \right)^2 \left( d \left( \frac{\eta}{\rho} \right) \cdot \frac{\eta}{\rho} \right) \left( \frac{\eta}{\rho} \cdot \partial_x \left( \frac{\eta}{\rho} \right) \right) \right\}$$

$$+ \left\{ -24 \epsilon_{lmn} \frac{d\eta_l}{\rho} \frac{\partial_x \eta_m}{\rho} \frac{\partial_x \eta_n}{\rho} \frac{\tau}{\rho} \right.$$  

$$+ 24 \left[ \frac{dh}{\rho^2} + \left( \frac{\eta}{\rho} \cdot \frac{d\eta}{\rho} \right) \epsilon_{lmn} \frac{\partial_x \eta_l}{\rho} \frac{\partial_x \eta_m}{\rho} \frac{\partial_x \eta_n}{\rho} \frac{\tau}{\rho} \right] \right.$$  

$$- 24 \left( \frac{dh}{\rho^2} \frac{\partial_x \left( \frac{\tau}{\rho} \right)}{\rho} + \left( \frac{\tau}{\rho} \right) d \left( \frac{\tau}{\rho} \right) - \left( \frac{\eta}{\rho} \cdot \frac{d\eta}{\rho} \right) \frac{\tau}{\rho} \frac{\partial_x \left( \frac{\tau}{\rho} \right)}{\rho} \right) \right\}. \quad (3.21)$$

Such a local expression of $\gamma$ could not be found for the $N = (4, 0)$ case either [5]. But it can be for $N \leq 2$ as commented in the end of this paper.
4 OSp(2|3)$_{\text{target}}$ and OSp(2|3)$_{\text{diff}}$

The transformation (3.19) defines the finite form of the coadjoint action on $\hat{b} + b$ along the orbit $O_b$. Let it be denoted by

$$\text{Ad}_{f,\varphi}^* b \equiv \Delta_{b}^{+} b(f, \varphi) + c S(f, \varphi; x, \theta).$$  \hspace{1cm} (4.1)

In this section we discover that it can have OSp(2|3) symmetry twofold. One is the OSp(2|3) symmetry of the $N = 3$ super-Schwarzian derivative $S(f, \varphi; x, \theta)$ under the non-linear transformations by the Killing vectors (2.21). It was proved in the end of Section 2. We show that it becomes a symmetry of the coadjoint action (4.1) as well when the initial point $b$ of the coadjoint orbit $O_b$ is chosen appropriately. We call the symmetry OSp(2/3)$_{\text{target}}$. The other is a symmetry under the superalgebra osp(2|3) given by (2.19), which is a subalgebra of the the $N = 3$ superconformal algebra. It stabilizes the coadjoint action (4.1). That is, we can show that the initial point $b$ of the orbit $O_b$ is stable under the superalgebra osp(2|3), if $b$ is tuned to be a specific configuraton. Therefore the coadjoint action (4.1) is invariant under osp(2|3) as well. We call this subsymmetry of the coadjoint action OSp(2/3)$_{\text{diff}}$. We shall discuss the two OSp(2|3) symmetries in detail.

4.1 OSp(2|3)$_{\text{target}}$

It seems that the the coadjoint action (4.1) no longer keeps the OSp(2|3)$_{\text{target}}$ invariance of the Schwarzian derivative, due to the $b$-dependence. We show that the invariance remains if the initial point $b$ of the coadjoint orbit $O_b$ is appropriately chosen. The idea is that for any configuration of $b$ we may find certain functions $F_0(x, \theta)$ and $\Phi_0(x, \theta)$ such that

$$b(x, \theta) = c S(F_0, \Phi_0; x, \theta).$$  \hspace{1cm} (4.2)

Here the Schwarzian derivative is regarded as a background configuration for $b(x, \theta)$. The initial point $b(x, \theta)$ flows to $\Delta_{b}^{+} b(f, \varphi)$ along the coadjoint orbit under the $N = 3$ superconformal diffeomorphism. Putting (4.2) into (4.1) and using the composition law of the Schwarzian derivative (2.9) we get

$$\text{Ad}_{f,\varphi}^* b = \Delta_{b}^{+} b(f, \varphi) + c S(f, \varphi; x, \theta) = S(F_0(f, \varphi), \Phi_0(f, \varphi); x, \theta).$$  \hspace{1cm} (4.3)

We can say that the $b$-dependence of the coadjoint action (4.1) has been renormalized into the original Schwarzian derivative. Clearly this renormalized Schwarzian derivative has the same OSp(2|3)$_{\text{target}}$ invariance as the original one because the non-linear transformations (2.21) flow as

$$\delta_{f} F_0 = \lim_{f = F_0, \varphi = \Phi_0} R^A(f, \varphi), \quad \delta_{\varphi} \Phi_0 = \lim_{f = F_0, \varphi = \Phi_0} R^A_{a}(f, \varphi).$$  \hspace{1cm} (4.4)

Thus the coadjoint action (4.1) can keep OSp(2|3)$_{\text{target}}$ invariance.

The background (4.2) should obey the superconformal condition

$$D_{\theta a} F_0 = \Phi_{0b} D_{\theta a} \Phi_{0b}. \hspace{1cm} (4.5)$$

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Then it follows that
\[
y = \frac{df + \varphi_c d\varphi_c}{\partial_x f + \varphi_c \partial_x \varphi_c} = \frac{dF_0(f, \varphi) + \Phi_{0c}(f, \varphi)d\Phi_{0c}(f, \varphi)}{\partial_x F_0(f, \varphi) + \Phi_{0c}(f, \varphi)\partial_x \Phi_{0c}(f, \varphi)} \equiv \lim_{f = F_0, \varphi = \Phi_0} y. \tag{4.6}
\]

This implies that the one-form \( y \) in the Kirillov-Kostant two-form is invariant under the renormalization of the \( b \)-dependence. Another important property of \( y \) is that it has also \( \text{OSp}(2|3)_{\text{target}} \) invariance similarly to the Schwarzian derivative. That is, the denominator and the numerator respectively transform as
\[
\delta_c(df + \varphi_c d\varphi_c) = (\epsilon L^9 + 2\epsilon T^z + 2\epsilon T_a \varphi_a)(df + \varphi_c d\varphi_c),
\]
\[
\delta_c(\partial_x f + \varphi_c \partial_x \varphi_c) = (\epsilon L^9 + 2\epsilon T^z + 2\epsilon T_a \varphi_a)(\partial_x f + \varphi_c \partial_x \varphi_c), \tag{4.7}
\]
under the non-linear transformations (2.21).

Owing to (4.3) and (4.6), the Kirillov-Kostant two-form \( \hat{\Omega}_b \) (3.20) gets the \( b \) dependence renormalized. The renormalized Kirillov-Kostant two-form has \( \text{OSp}(2|3)_{\text{target}} \) invariance under the renormalized non-linear transformations (4.4).

### 4.2 \( \text{OSp}(2|3)_{\text{diff}} \)

The initial point \( b(x, \theta) \) generically flows along the orbit \( O_b \) by the \( N = 3 \) superconformal transformation. But it could be stable under its subalgebra \( \mathfrak{osp}(2|3) \) if \( F_0 \) and \( \Phi_0 \) in the initial configuration (4.2) are constrained appropriately. That is, we could have
\[
\lim_{f = F_0, \varphi = \Phi_0} \left( \delta_v \text{Ad}^*_f \varphi b(x, \theta) \right) = v \partial_x b(x, \theta) + \frac{1}{2}(\partial_x v)b(x, \theta) + \frac{1}{2}(D_{\theta a} v)D_{\theta b} b(x, \theta) + cD_{\theta 1}D_{\theta 2}D_{\theta 3} v = 0, \tag{4.8}
\]
while \( v \) is restricted to the submodes generated by \( \mathfrak{osp}(2|3) \) in (2.17). This stability implies the \( \text{OSp}(2|3)_{\text{diff}} \) symmetry of the finite coadjoint action (4.11) which was mentioned in the introduction of the section. It is different from \( \text{OSp}(2|3)_{\text{target}} \) discussed previously.

We shall show that such an \( \text{OSp}(2|3)_{\text{diff}} \) symmetry indeed comes true by assuming \( b(x, \theta) \) to take a specific expansion form such that
\[
b(x, \theta) = \theta_1 \theta_2 \theta_3 d(x). \tag{4.9}
\]
By expanding \( v \) as (2.15) we find (4.8) in components to be
\[
c\beta(x) + \theta_a[c\partial_x t_a(x)] + \frac{1}{2}\varepsilon_{abc}\theta_a\theta_b[\frac{1}{4}\varepsilon_{abc}\beta_c d(x) + \frac{c}{2}\varepsilon_{abc}\partial_x^2 \beta_c(x)] + \theta_a\theta_2\theta_3[\alpha(x)\partial_x d(x) + 2\partial_x \alpha(x)d + c\partial_x^3 \alpha(x)] = 0. \tag{4.10}
\]
This gives as many differential equations as the components of \( v \). They can be solved by
\[
\alpha(x) = e^{\pm inx} \alpha_\pm, \quad \alpha_0, \\
\beta_a(x) = e^{\pm \frac{1}{2}inx} \beta_a \pm \frac{1}{2}, \\
t_a(x) = t_a 0, \\
\beta(x) = 0, \tag{4.11}
\]

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when the configuration of $b$ is specified by

$$d(x) = \frac{1}{2}cn^2,$$  \hspace{1cm} (4.12)

with a fixed integer $n \in \mathbb{Z}_{>0}$. With $v$ restricted by (4.11) the $N = 3$ superconformal diffeomorphism becomes exactly the superalgebra $\mathfrak{osp}(2|3)$ discussed in Section 2. Thus $b(x, \theta)$ given by (4.9) with (4.12) is a right configuration for the constraint (4.8) to be satisfied.

Putting (4.12) into (4.2) we have

$$\theta_1 \theta_2 \theta_3 \left[ \frac{1}{2} cn^2 \right] = cS(F_0, \Phi_0; x, \theta).$$

Finally we give a solution of this equation for $F_0(x, \theta)$ and $\Phi_{0a}(x, \theta)$. To this end it is sufficient to know the non-supersymmetric part of the r.h.s. of the equation, i.e., the non-supersymmetric Schwarzian derivative. By using the expansion of the $N = 3$ super-Schwarzian derivative in components (2.12) and taking into account the superconformal condition (4.5) it turns out that

$$F_0(x, \theta) = \frac{2\lambda^2}{n} \tan\left(\frac{nx}{2}\right), \quad \Phi_{0a}(x, \theta) = \theta_a \lambda \sec\left(\frac{nx}{2}\right),$$  \hspace{1cm} (4.13)

with a constant $\lambda$.

With this solution for $F_0$ and $\Phi_0$ the finite coadjoint action (4.1) is invariant under $\text{OSp}(2|3)_{\text{diff}}$. On the other hand $y$ is also stable at the initial point of the orbit $O_b$ as

$$\lim_{f=x, \varphi=\theta} \delta_v y = 0.$$  \hspace{1cm} (4.14)

Therefore with the above solution for $F_0$ and $\Phi_0$ the Kirillov-Kostant two-form (3.20) has $\text{OSp}(2|3)_{\text{diff}}$ invariance.

All the arguments about the twofold $\text{OSp}(2|3)$ symmetry in this section can be straightforwardly applied to the cases of $N \leq 2$ as well. But there is some difference in the application to the $N = 4$ case. It is explained in Appendix C.

5  \hspace{1cm} $N = 3$ super-Schwarzian theory

In the end of Section 2 we have shown that the Kirillov-Kostant two-form $\hat{\Omega}_b$ on the coadjoint orbit satisfies $i_c \hat{\Omega}_b = dH$ with a zero-form. The $N = 3$ super-Schwarzian theory may be defined by using this zero-form $H$ as

$$H|_{v=1} = \int dx d^3\theta \text{Ad}^*_f, \varphi b \equiv \int dx d^3\theta \left( \Delta^+ b(f, \varphi) + cS(f, \varphi; x, \theta) \right).$$  \hspace{1cm} (5.1)
The finite coadjoint action (4.1) is renormalized as (4.3) when \( b \) is given by (4.2). As shown in Subsection 4.1 it has OSp(2\( |3 \)) target invariance at any point of the renormalization flow. So does the action (5.1) being written as

\[
H|_{v=1} = c \int dx d^3 \theta \, S(F_0(f, \varphi), \Phi_0(f, \varphi); x, \theta). \tag{5.2}
\]

This action with \( b \neq 0 \) is not invariant under the superconformal diffeomorphism. This is because the renormalized Schwarzian derivative transforms as (3.17). Note here that the anomalous part of the transformation disappears as a boundery term in the integration (5.2), but the other part does not. When \( b \) is further specified as (4.9) with (4.12), this configuration of \( b \) is stable under OSp(2\( |3 \))\(^{\text{diff}} \) satisfying the constraint (4.8). Therefore the action (5.2) has the OSp(2\( |3 \))\(^{\text{diff}} \) symmetry as has been shown in Section 4.2.

The action (5.2) with this \( b \) dependence deserves to be studied by expanding \( f \) and \( \varphi \) in components as in Appendix A. We find that

\[
H|_{v=1} = c \int dx \left[ \frac{n^2}{2} \left\{ - \frac{1}{3} \epsilon_{ijk} \eta_i \eta_j \eta_k \frac{\partial_x^2 \rho}{\rho} + \rho^4 - 3 \rho^2 (\eta \cdot \partial_x \eta) + 2 (\eta \cdot \partial_x \eta)^2 \right\} 
+ 2 \left\{ \frac{1}{\rho} \partial_x^2 \rho - 2 \left( \frac{\partial_x \rho}{\rho} \right)^2 + \frac{1}{\rho^2} \tau \partial_x \tau + \frac{1}{\rho^2} (\partial_x \eta \cdot \partial_x^2 \eta) \right\} \right], \tag{5.3}
\]

with the constraints also given in Appendix A. The first bracket is the top component of \( \Delta \frac{1}{2} \epsilon_{ijk} \varphi_i \varphi_j \varphi_k \), i.e., the \( b \) dependent term with (4.9), while the second bracket is the one of \( S(F_0; x, \theta) \). They are exact results with no approximation. The reader may refer to [24] for details of the calculation. To the quadratic order of fermionic fields the action turns out to take a fairly compact form

\[
H_{\text{quad}}|_{v=1} = c \int dx \left[ S(F_0, x) - S(F_0, x)(\zeta \cdot \partial_x \zeta) + 2(\partial_x \zeta \cdot \partial_x^2 \zeta) + \frac{2}{\partial_x h} \tau \partial_x \tau \right]. \tag{5.4}
\]

Here we have used the constraint \( \rho^2 = \partial_x h + (\eta \cdot \partial_x \eta) = \partial_x h[1 + (\zeta \cdot \partial_x \zeta)] \) with \( n_{\mu} = \sqrt{\partial_x h} \zeta_{\mu} \). \( S(F_0; x) \) is the non-supersymmetric Schwarzian action with \( F_0 = \frac{2 \lambda^2}{n} \tan \frac{n h}{2} \), given by (4.13). It has resulted from the use of the composition law (2.9) at the non-supersymmetric limit. Now we are in a position to evaluate this quadratic action at the stable point \( h = x \). Then \( S(F_0, x) \) becomes \( \frac{n^2}{2} \), which is the value at the stable point of the \( N = 3 \) supersymmetric action (5.2). (5.4) becomes

\[
H_{\text{quad}}|_{v=1} = c \int dx \left[ \frac{n^2}{2} \left( \zeta \cdot \partial_x \zeta \right) + 2(\partial_x \zeta \cdot \partial_x^2 \zeta) + \frac{2}{\partial_x h} \tau \partial_x \tau \right].
\]

It is worth comparing this result with the one for the case of \( N = 1 \) or \( N = 2 \), discussed in [3]. They can be obtained by supersymmetry truncation \( \tau = 0 \) and \( (\eta \cdot \partial_x \eta) = \eta \partial_x \eta \) or \( n_+ \partial_x n_- + n_- \partial_x n_+ \) for the respective case. Firstly we comment on the \( N = 1 \) case. With the \( b \) dependence turned off the truncated action of (5.4) reduces to the \( N = 1 \) action
which was given by (4.13) in [3]. To see this, set \(n = 0\) in (5.2). Then \(F_0\) becomes \(\lambda^2 h\). As the result \(S(F_0; x) = S(h; x)\). We find the truncated action of (5.3) to become the \(N = 1\) action given in [3]. Secondly we comment on the \(N = 2\) case. The truncated action of (5.4) is needed to be modified by a kind of U(1) anomaly. In [6] they gave an interesting argument for its quantum effect. It is due to chiral splitting by the chirality condition.

The chiral condition allows us to U(1)-rotate the fermionic components \(\eta_{\pm}\) independently of each other. Therefore supersymmetric truncation of the action of (5.4) should be done again by using

\[
\eta_{\pm}(x) = \eta_{\pm}(x) + \theta_{\pm}(x)e^{\pm i\sigma(x)} + \theta_{1}\theta_{2}\partial_{x}\eta_{\pm}(x).
\]

They are independent of each other.

---

6 \(N = (3, 0)\) super-Liouville theory

In the end of Section 2 we have also shown that the Kirillov-Kostant two-form \(\hat{\Omega}_b\) on the coadjoint orbit can be written in an exact form \(\hat{\Omega}_b = dL\). Remember that \(L\) is a one-form in the space of the adjoint orbit. The \(N = (3, 0)\) super-Liouville theory is given by integrating \(L\) along the orbit as

\[
L = -\int_{O_b} \int dx d^3\theta \left\{ y \left( \Delta \frac{1}{2} b(f, \varphi) + cS(f, \varphi; x, \theta) \right) + \frac{c}{2} \gamma \right\}.
\]

The left-moving sector is described by the \(N = 3\) superspace, while the right-moving sector by the extra space for the orbit \(O_b\). It is worth checking the non-supersymmetric limit of the theory. The purely bosonic part of \(S(f, \varphi; x, \theta)\) has been already given by (2.12), while the one of \(\gamma\) can be found in the first line in the formula (3.21). \(\Delta\) and \(y\) take the familiar forms in the non-supersymmetric limit. By the assumption (4.9) it follows that

\[
\Delta \frac{1}{2} b(f, \varphi) = \cdots + \rho \varphi_1 \varphi_2 \varphi_3 d(f) = \rho_1 \rho_2 \rho_3, d(h) + \text{superpartners}.
\]

---

9The \(N = 2\) chiral superfields \(\varphi_{\pm}\) are expanded in \(\theta\) as

\[
\varphi_{\pm}(x, \theta) = \eta_{\pm}(x) + \theta_{\pm} \rho(x)e^{\pm i\sigma(x)} + \theta_{1}\theta_{2}\partial_{x}\eta_{\pm}(x).
\]
Putting these results together and using the constraint (2.11) we get
\[
L = - \int_{O_b} \int dx \frac{dh}{\partial_x h} \left\{ (\partial_x h)^2 d(h) + \frac{c}{2} \left( \frac{\partial^3 h}{\partial_x h} - 2 \left( \frac{\partial^2 h}{\partial_x h} \right)^2 \right) + \text{superpartners} \right\},
\]
which is indeed the non-supersymmetric Liouville action given in [7].

Let us study symmetries of the action. The \( N = 3 \) superconformal invariance is evident from the original form of the action (6.1) given by (3.16)
\[
L = - \int dxd^3 \theta (\tilde{b}y + \frac{c}{2} \gamma).
\]

We may be interested in finding the energy-momentum tensor of the theory. The usual recipe for this is to recalculate \( \delta_v L \) by assuming \( dv \neq 0 \). All the calculations in Section 3 have been done with \( dv = 0 \). With \( dv \neq 0 \) the formula (3.6) for \( \delta_v y \) is modified as
\[
\delta_v y = dv \theta_j y - (\partial_y v) + \frac{1}{2} (D_{\theta_j} v) D_{\theta_j} y \equiv [v, y] = \frac{1}{2} i_v [y, y].
\]

In calculating \( \delta_v \gamma \) the formula (3.12) should be used in a modified form
\[
\delta_v \int dxd^3 \theta y D_{\theta_1} D_{\theta_2} D_{\theta_3} y = \int dxd^3 \theta (2 dv D_{\theta_1} D_{\theta_2} D_{\theta_3} y)
+ d \int dxd^3 \theta (2v D_{\theta_1} D_{\theta_2} D_{\theta_3} dyy).
\]

Keeping in mind these contributions from \( dv \neq 0 \) we find
\[
\delta_v L = - \int_{O_b} \int dxd^3 \theta dv \left( \Delta^\frac{1}{2} b(f, \varphi) + cS(f, \varphi; x, \theta) \right).
\]

It may be written in the form
\[
\delta_v I = \int_{O_b} dt \int dxd^4 \theta \frac{d}{dt} \left( \Delta^\frac{1}{2} b(f, \varphi) + cS(f, \varphi; x, \theta) \right),
\]
by parametrizing the extra space of the orbit with \( t \). Therefore the energy-momentum tensor of the theory in the left-moving sector is given by the finite coadjoint action (4.1). When \( dv/dt = 0 \) it is conserved. The appearance of the Schwarzian derivative in energy-momentum tensor is the hallmark of the Liouville theory. So far we have discussed the symmetry in the left-moving sector. It is obvious that the right-moving sector is invariant under reparametrization of the orbit. Thus the Liouville theory given by (6.1) is invariant under the \( N = (3, 0) \) superconformal diffeomorphism. It contrasts with the Schwarzian theory given by (5.1).

When \( b \) is given by (1.2), the Liouville action can be also rewritten in the renormalized form
\[
L = - \int_{O_b} \int dxd^3 \theta \left\{ cyS(F_0(f, \varphi), \Phi_0(f, \varphi); x, \theta) + \frac{c}{2} \gamma \right\}.
\]
As has been shown in Subsection 4.1 $y$ is invariant under the renormalization. Therefore the integrand of $L$ has $\text{OSp}(2|3)_{\text{target}}$ invariance at any point of the renormalization flow.

At this stage the action with (4.12) is still invariant under the superconformal diffeomorphism on the contrary to the Schwarzian action. When $b$ is further specified as (4.9) with (4.12), the action is no longer invariant under superconformal diffeomorphism. However we have (4.14) as well as (4.8). Hence $b$ is stable under $\text{OSp}(2|3)_{\text{diff}}$ and the action remains invariant under $\text{OSp}(2|3)_{\text{diff}}$. The reader may remember a similar discussion for the Kirillov-Kostant two-form in the end of Subsection 4.2.

7 Conclusions

The respective actions (5.1) and (6.1) are valid not only for the $N = 3$ super-Schwarzian and Liouville theories, but also for the cases of $N \leq 2$, if $\mathcal{S}(f, \psi; x, \theta)$, $y$ and the anomalous term therein are replaced appropriately for the supersymmetries. For those quantities see Appendix A in [12]. We can argue the actions for those cases similarly to the $N = 3$ case. We obtain the same conclusions as in Sections 5 and 6. The twofold symmetry is given by $\text{OSp}(2|N)_{\text{diff}}$ and $\text{OSp}(2|N)_{\text{target}}$. For the $N = 4$ case the similar arguments go through and end up with almost the same conclusions. But there is a difference. It will be argued in Appendix C.

On top of the feature summarized above there is one more point worthy to be remarked. For the Liouville theories with $N \leq 2$ the $\text{OSp}(2|N)_{\text{target}}$ symmetry becomes local in the right-moving sector. For those theories the anomalous terms can be expressed in exact forms in terms of superfields. They are given by

$$\int dxdy \partial^2_y y = -d \int dx \left[ y \left( \mathcal{S} + \frac{1}{2} \left( \partial^2 h \right)^2 \right) \right],$$

$$\int dxd\theta \frac{1}{2} y D_{\theta} \partial^2_{x} y = -d \int dxd\theta \left[ y \left( \mathcal{S} + \left( \frac{D_{\theta} \varphi}{D_{\theta} \varphi} \right) \right) \right],$$

$$\int dxd^2 \theta \frac{1}{2} y \partial_x \left[ D_{\theta+}, D_{\theta-} \right] y = -d \int dxd^2 \theta \left[ y \left( \mathcal{S} - 2 \frac{\partial_{\theta+} \varphi^+}{D_{\theta+} \varphi^+} \frac{\partial_{\theta-} \varphi^-}{D_{\theta-} \varphi^-} \right) \right],$$

for $N = (0,0), (1,0), (2,0)$ respectively. Owing to these formulae we can show that the respective Liouville actions are invariant locally in the right-moving sector under nonlinear transformations by the Killing vectors like (2.21). That is, they are invariant even when the infinitesimal parameters are local as $\epsilon^A(t)$ [7,8,23]. On the contrary the anomalous term in (6.2) for the case of $N = 3$ did not admit such a supercovariant expression. It did not for the case of $N = 4$ either. The anomalous term $\gamma$ in (6.2) can get a local expression only in terms of components as (3.21). For the $N = 4$ case the reader may refer to [12]. Without a supercovariant expression of $\gamma$ we can hardly study local invariance under $\text{OSp}(2|N)_{\text{target}}$ or $\text{PSU}(1,1|2)_{\text{target}}$ in the right-moving sector.

In this paper the arguments have been limited at the classical level. Obtaining a simple form of the action (5.4) one may be interested in a saddle-point calculation of
the partition function of the $N = 3$ super-Schwarzian theory. One may ask about one-loop exactness of the calculation. Here the measure of the path-integration does matter. In [6] it was worked out for the non-supersymmetric case and one-loop exactness of the partition function was shown by means of the Duistermaat-Heckman formula. It was also pointed out that such an argument is applicable in supersymmetric cases as long as it is formulated by the coadjoint orbit method. We have all the materials at hand to discuss the issue for the $N = 3$ case as well. In Section 4 we have discussed $\text{OSp}(2|3)_{\text{diff}}$ invariance of the Kirillov-Kostant 2-form $\hat{\Omega}_b$ as a residual symmetry after the gauge-fixing. Hence the relevant symplectic supermanifold is $\text{diff}(S^{1|3})/\text{OSp}(2|3)$. It is described by the component fields $f, \eta_1, \eta_2, \eta_3$ of the $N = 3$ superconformal symmetry and the ones with opposite grading, which are denoted by $df, d\eta_1, d\eta_2, d\eta_3$ according to [6]. (Here $d$ no longer means an exterior derivative, but it is used as a convention to denote partner fields avoiding new naming.) The Duistermaat-Heckman formula for the partition function reads as

$$Z = \int \frac{d\mu(f, \eta_1, \eta_2, \eta_3, df, d\eta_1, d\eta_2, d\eta_3)}{\text{Osp}(2|3)} \exp\{\hat{\Omega}_b + H\}_{v=1},$$

with the Kirillov-Kostant 2-form and the $N = 3$ super-Schwarzian action given by given by (3.20) and (5.3) respectively. However we may immediately note that both actions in the exponent contain the $\text{O}(3)$-singlet fermionic field $\tau$ and its odd-graded partner $d\tau$. We then ask how to path-integrate them. $\tau$ is related to the generator $F(x)$ in (2.16) of the $N = 3$ superconformal algebra discussed in ii) of Section 2. As known in the literature this generator may be eliminated from the superconformal algebra to get the algebra in a non-Lie algebraic form[17, 18]. Correspondingly we think of eliminating the dependence on $\tau$ and $d\tau$ from the action $\hat{\Omega}_b + H|_{v=1}$ in favour of $\eta$ and $d\eta$. But it can be hardly done even by using the constraints in Appendix A. $\tau$ and $d\tau$ are not constituents of the symplectic supermanifold $\text{diff}(S^{1|3})/\text{Osp}(2|3)$. In the presence of them it is a point at issue whether the Duistermaat-Heckman formula is still applicable for the $N = 3$ super-Schwarzian theory. The partition function might loose the reason for one-loop exactness for the $N = 3$ theory. The author will study the issue furthermore.

### A The superconformal condition

From the superconformal condition (2.3) it follows that

$$D_{\theta_a}\varphi_c D_{\theta_b}\varphi_c = D_{\theta_b}\varphi_a D_{\theta_c}\varphi_b = \delta_{ab}(\partial_x f + \varphi_c \partial_x \varphi_c).$$

Hence the scaling factor in (2.9) can be written as

$$\Delta = \partial_x f + \varphi_c \partial_x \varphi_c.$$

The similar relations can be found also for the case of $N = 1, 2$ and 4 superconformal diffeomorphisms. Keep in mind that $\Delta$ is a superconformal field with weight 1.
We can write the superconformal condition (2.3) in components. \( f \) and \( \theta_a \) are expanded in \( \theta \) as

\[
\begin{align*}
f(x, \theta) &= h(x) + \theta_c \psi_c(x) + \frac{1}{2} \varepsilon_{abc} \theta_a \theta_b t_c(x) + \theta_a \theta_2 \theta_3 \omega(x), \\
\varphi_a(x, \theta) &= \eta_a(x) + \theta_a \rho(x) + \frac{1}{2} \varepsilon_{abc} \theta_b \tau_c(x) + \theta_a \theta_2 \theta_3 \tau_c(c) + \theta_1 \theta_2 \theta_3 r_a(x). 
\end{align*}
\]

Putting these into the superconformal condition (2.3) gives the constraints

\[
\begin{align*}
\partial_x h &= -\eta_b \partial_x \eta_c + \rho^2, \quad \psi_a = \eta_a \rho, \\
t_a &= -\eta_b \tau - \varepsilon_{abc} \eta_b \tau_c, \quad \omega = \eta_c \tau_c - \tau \rho, \\
\tau_a &= \partial_x \eta_a, \quad 0 = \rho \tau_a + \tau \tau_a + \frac{1}{2} \varepsilon_{abc} \tau_b \tau_c.
\end{align*}
\]

**B Proofs of (3.11) and (3.12)**

The integrand of (3.11) can be rewritten as

\[
3 \varepsilon_{abc} y(D_{\theta a} y) D_{\theta b} D_{\theta c} \partial_x y = -\varepsilon_{abc} (y \partial_x y) D_{\theta a} D_{\theta b} D_{\theta c} y + \text{boundary terms.} \quad (B.1)
\]

We shall show the formula in this form. The l.h.s. can be calculated in two ways

\[
\begin{align*}
\varepsilon_{abc} y(D_{\theta a} y) D_{\theta b} D_{\theta c} \partial_x y &= A + D_{\theta a} (\cdots), \quad (B.2) \\
\varepsilon_{abc} y(D_{\theta a} y) D_{\theta b} D_{\theta c} \partial_x y &= -B - A + \partial_x (\cdots), \quad (B.3)
\end{align*}
\]

in which

\[
\begin{align*}
A &= \varepsilon_{abc} y(D_{\theta a} \partial_x y) D_{\theta b} D_{\theta c} y, \\
B &= \varepsilon_{abc} \partial_x y(D_{\theta a} y) D_{\theta b} D_{\theta c} y.
\end{align*}
\]

(B.3) can be put in one other form by calculating the r.h.s. as

\[
\varepsilon_{abc} y(D_{\theta a} y) D_{\theta b} D_{\theta c} \partial_x y = -2A - \varepsilon_{abc} (y \partial_x y) D_{\theta a} D_{\theta b} D_{\theta c} y + D_{\theta a} (\cdots). \quad (B.4)
\]

The three equations cannot be independent. Eliminating \( A \) and \( B \) we find (B.1).

Next we show (3.12), i.e.,

\[
\delta_v \int dx^3 \theta \varepsilon_{abc} y D_{\theta a} D_{\theta b} D_{\theta c} y = 2d \int dx^3 \theta \varepsilon_{abc} v D_{\theta a} D_{\theta b} D_{\theta c} y. \quad (B.5)
\]

By using (3.5) and (3.6) we calculate both sides of the equation as

\[
\begin{align*}
\text{l.h.s.} &= 2 \int dx^3 \theta \left( \varepsilon_{abc} [v, y] D_{\theta a} D_{\theta b} D_{\theta c} y \right) \\
&= 2 \int dx^3 \theta \varepsilon_{abc} \left( -\frac{1}{2} v \partial_x y - (\partial_x v) y \right) D_{\theta a} D_{\theta b} D_{\theta c} y + \frac{3}{2} v(D_{\theta a} y) D_{\theta b} D_{\theta c} \partial_x y, \\
\text{r.h.s.} &= 2 \int dx^3 \theta \varepsilon_{abc} v D_{\theta a} D_{\theta b} D_{\theta c} dy = -2 \int dx^3 \theta \varepsilon_{abc} dy D_{\theta a} D_{\theta b} D_{\theta c} v \\
&= -2 \int dx^3 \theta \varepsilon_{abc} (y \partial_x y + \frac{1}{4} D_{\theta a} y D_{\theta b} y) D_{\theta a} D_{\theta b} D_{\theta c} v.
\end{align*}
\]
By repeating integration by parts in the r.h.s. the integrand becomes
\[ \epsilon_{abc} \left( -y D_{\theta a} \partial_x y + \frac{1}{2} (D_{\theta a} y) \partial_x y \right) D_{\theta b} D_{\theta c} v + D_{\theta a} (\cdots) \]
\[ = \epsilon_{abc} \left( -\frac{1}{2} (D_{\theta b} y) D_{\theta a} \partial_x y - y D_{\theta b} D_{\theta a} \partial_x y + \frac{1}{2} (D_{\theta b} D_{\theta a} y) \partial_x y \right) D_{\theta c} v + D_{\theta b} (\cdots) \]
\[ = \epsilon_{abc} \left( -\frac{3}{2} (D_{\theta b} y) D_{\theta c} D_{\theta a} \partial_x y + y D_{\theta c} D_{\theta b} D_{\theta a} \partial_x y + \frac{1}{2} (\partial_x y) D_{\theta c} D_{\theta b} D_{\theta a} y \right) v + D_{\theta c} (\cdots). \]

Integrating this over the superspace we find the l.h.s.

\section{The $N = 4$ super-Schwarzian theory revisited}

The issue for the $N = 3$ Schwarzian theory in Section 5 were studied also for the $N = 4$ super-Schwarzian action in [5]. However the $b$ field renormalization was not discussed. In this Appendix we complete the arguments for $b \neq 0$ giving the missing part. We show that the $N = 4$ super-Schwarzian action can also have the two symmetries similarly to the $N = 3$ action. They are $\text{PSU}(1,1|2)_{\text{target}}$ and $\text{PSU}(1,1|2)_{\text{diff}}$, but do not appear simultaneously. It contrasts with the $N \leq 3$ case. Moreover we show that they are not symmetries of the action density. This feature also makes the $N = 4$ super-Schwarzian action different from the $N \leq 3$ case.

To explain these features let us remember the $N = 4$ super-Schwarzian action, which corresponds to (5.1), i.e.,
\[ H = \int dxd^4\theta \ Ad_{f,\varphi}^* b = \int dxd^4\theta \left( b(f, \varphi) + c\mathcal{S}(f, \varphi; x, \theta) \right). \]  
(See (5.14) in [5].) When $b$ is given by $b(x, \theta) = c\mathcal{S}(F_0, \Phi_0; x, \theta)$, the finite coadjoint action $Ad_{f,\varphi}^* b$ is renormalized similarly to the $N = 3$ case, i.e.,
\[ Ad_{f,\varphi}^* b(x, \theta) = b(f, \varphi) + c\mathcal{S}(f, \varphi; x, \theta) \]
\[ = c\mathcal{S}(F_0(f, \varphi), \Phi_0(f, \varphi); x, \theta). \]  
(C.2)

The action $H$ has $\text{PSU}(1,1|2)_{\text{target}}$ symmetry at any point of the renormalization flow. But it is not a symmetry of the action density. It is because the $N = 4$ super-Schwarzian derivative $\mathcal{S}(f, \varphi; x, \theta)$ transforms as (2.23) under non-linear transformations realized by the Killing vectors like (2.21). In [5] it was shown that the breaking terms (2.23) disappear by the integration in (C.1) as boundary terms.

So far the action $H$ with $b(x, \theta) = c\mathcal{S}(F_0, \Phi_0; x, \theta)$ is not invariant under the superconformal diffeomorphism, since the integrand transforms as a superconformal field with $w = 0$. Note here that the anomalous part of the transformation disappears as a boundary term in the integration (C.1), but the other part does not. This is also a feature we have seen for the $N = 3$ super-Schwarzian theory.
Now we may ask whether the action (C.1) can have the $\text{PSU}(1,1|2)_{\text{diff}}$ symmetry as the $N = 3$ theory had the $\text{OSp}(2|3)_{\text{diff}}$ symmetry. The question is whether we could have for the action (C.1)

$$
\delta vH = \int dxd^4\theta \lim_{f=\varphi;=\theta} \left( \delta v \text{Ad}^* (f, \varphi) b(x, \theta) \right)
$$

$$
= \int dxd^4\theta \left\{ v \partial_x b(x, \theta) + \frac{1}{2} D_{\theta a} v D_\theta^a b(x, \theta) + \frac{1}{2} D_\theta^a v D_{\theta a} b(x, \theta) + c \partial_x v \right\},
$$

$$
= 0 \quad (C.3)
$$

while restricting the parameter $v$ to the modes of the subalgebra $\mathfrak{usp}(1,1|2)$. It was shown in [5] that this constraint is satisfied when $b(x, \theta)$ takes the specific configuration

$$
b(x, \theta) = (\theta_a \theta^a) \left( -\frac{1}{4} cn^2 \right) + \text{superpartners}, \quad (C.4)
$$
given by (5.19) therein. Therefore with this configuration of $b$ the action has the $\text{PSU}(1,1|2)_{\text{diff}}$ symmetry. But we then loose the $\text{PSU}(1,1|2)_{\text{target}}$ symmetry. This is because with the configuration (C.4) we can not find functions $F_0(x, \theta)$ and $\Phi_0(x, \theta)$ satisfying $b(x, \theta) = cS(F_0, \Phi_0; x, \theta)$. For this it is enough to note that the $N = 4$ super-Schwarzian derivative takes the form in the non-supersymmetric limit

$$
S(f, \varphi; x, \theta) = \log \partial_x h + \frac{1}{2} (\theta_a \theta^a)^2 \left[ -\frac{\partial^2 h}{\partial_x h} + 2 \left( \frac{\partial^2 h}{\partial_x h} \right)^2 \right] + \text{superpartners}. \quad (C.5)
$$

(See (3.17) in [5].) The term $\log \partial_x h$ hinders from equating (C.4) and (C.5). Its appearance is consistent with the transformation law (2.23). It is characteristic for the $N = 4$ super-Schwarzian derivative in contrast with the case of $N \leq 3$.

Even though we have lost the $\text{PSU}(1,1|2)_{\text{target}}$ symmetry we may be interested in the $\text{PSU}(1,1|2)_{\text{diff}}$ symmetry by itself. We then wonder if it might be a symmetry of the action density (C.2). As shown in [5] it is too strong to require the integrand in (C.3) to be vanishing while restricting $v$ to the modes of the subalgebra $\mathfrak{usp}(1,1|2)$. There is no way to gauge-fix the $b$ field so that it is stable at the initial point of the coadjoint orbit under $\mathfrak{usp}(1,1|2)$. This point is also different from the $N \leq 3$ case.

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