Finite-Length Analysis of BATS Codes

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Abstract

BATS codes are proposed for communication through networks with linear network coding, and can be regarded as a matrix generalization of Raptor codes. In this paper, the performance of finite-length BATS codes is analyzed with respect to both belief propagation (BP) decoding and inactivation decoding. For a fixed number of input symbols and a fixed number of batches, a recursive formula is obtained to calculate the exact probability distribution of the stopping time of the BP decoder. When BP decoding stops before all the inputs symbols are decoded, some input symbols can be inactivated so that the BP procedure can be resumed. Such a decoding approach is called inactivation decoding. Extra computation cost is involved to decode the inactivated symbols by Gaussian elimination. To evaluate the extra computation cost involved in inactivation decoding, a recursive formula is derived to calculate the expected number of inactive symbols. When the number of batches follows a Poisson distribution, recursive formulae with lower computational complexity are derived for the above problems. Since Raptor codes are BATS codes with unit batch size, these results also provide new analytical tools for Raptor codes.

Index Terms

Network coding, fountain codes, BATS codes, finite-length analysis, belief propagation, inactivation decoding

I. INTRODUCTION

BATS (batched sparse) codes [1], [2] are proposed for communication through networks with linear network coding [3]. The outer code of a BATS code is a matrix generalization of a fountain code (e.g. LT code [4] or Raptor code [5]), and encodes the input packets into potentially unlimited batches of coded packets. The inner code of a BATS code comprises linear network codes at the intermediate network nodes which generate new packets using packets of the same batch. BATS codes preserve such desirable properties of fountain codes as ratelessness and low encoding/decoding complexity, and can get the benefit of network coding with low computational and storage requirement for the intermediate network nodes.

BATS code separates the design of the outer code and the inner code, and makes the rigorous performance analysis possible. The asymptotic performance of BATS code under belief propagation (BP) decoding has been analyzed in [2]. A sufficient condition is obtained for the BP decoder to recover a fixed fraction of the input symbols with high probability when the number of input symbols, also known as block length, tends to infinity.
The sufficient condition enables us to design BATS codes with good performance for large block length (e.g., multiple ten thousands). It has been verified theoretically for certain special cases and demonstrated numerically for general cases that BATS codes can achieve rates very close to optimality for a single rank distribution of the transfer matrices. The asymptotic performance of BATS code under BP decoding for multiple rank distributions is studied in [6].

The performance of BATS codes with small block length is of important practical interests. However, the error bound obtained in the asymptotic analysis is rather loose for small numbers of input symbols, and the degree distribution optimized asymptotically does not provide a good performance for small block lengths. Towards designing better codes for a relatively small block length (e.g., hundreds and lower thousand), in this paper, we provide analysis of BATS codes with a fixed finite block length for both the BP decoder and an improved BP decoder called inactivation decoder. Since LT/Raptor codes are BATS codes with unit batch size, our results also provide new analytical tools for LT/Raptor codes.

A. Background and related works

Linear network coding can significantly improve the transmission rate of communication networks compared with routing [3], and can achieve the multicast capacity of networks with packet loss over a wide range of scenarios [7], [8]. However, using linear network coding usually incurs higher computational and storage costs in the network devices and terminals. For example, some network coding schemes require the computational and/or storage capabilities of an intermediate network node to increase linearly with the number of packets for transmission, making them difficult to be implemented in a router-like device that has only constant computational and storage capabilities; and some schemes require the sink node to solve a large order of linear system using Gaussian elimination.

In addition to BATS codes, there is a sequence of works considering how to design practical linear network coding solutions for various scenarios, since the power of linear network coding was recognized. The earliest work towards a practical solution of linear network coding may be Chou, Wu and Jain’s work [9]. We would not review more works here. Readers can find more discussion of related works in [2] and may refer to [10] for the comparison of some schemes.

Finite length analysis of BP decoding is available for LT/Raptor code. Karp et al. provided a recursive formula to compute the error probability of LT codes for a given block length [11]. Maneva and Shokrollahi [12] used a random model of the number of received symbols and obtained a simpler formula. The inactivation decoding has been used for LT/Raptor codes to reduce the coding overhead [13], [14]. However, as far as we know, no analytical results are available for the inactivation decoder of LT/Raptor codes.

B. Summery of results

Specifically, we calculate the exact error probability of BP decoder for given values of block length $K$ and number of batches $n$. We derive the distribution of the stopping time of the BP decoder using a recursive formula,
which can be used to calculate the error probability of the BP decoder (decoding error occurs when the BP decoder stops before a target number of input symbols are recovered). The computational complexity of evaluating the above recursive formula is $O(K^2n^2M)$, where $M$ is the batch size. Our formula applies to LT codes by setting $M = 1$. Though the formula obtained by Karp et al. for LT codes \[11\] has slightly lower complexity than our formula with $M = 1$, directly extending the analysis in \[11\] to $M > 1$ results in much higher order of complexity since more variables should be tracked recursively.

However, using only BP decoding is not sufficient to achieve small coding overhead (or high coding rate) for small block lengths (e.g., hundreds and lower thousands). After BP decoding stops, Gaussian elimination can be used to continue the decoding, which usually involves high computational cost. A better solution is to continue the BP process by assuming certain undecoded input symbols to be known, which is also called inactivation decoding and has been used for LT/Raptor codes to reduce the coding overhead \[13\], \[14\]. Inactivation decoding trades computation cost (decoding inactive input symbols using Gaussian elimination) to get low coding overhead. To understand the tradeoff between computation cost and coding overhead, for given values of $K$ and $n$ we derive a recursive formula to compute the expected number of inactive symbols required for the inactivation decoding. The computational complexity of evaluating the formula is also $O(K^2n^2M)$.

For small values of $K$, e.g., $K \leq 1000$, both of the above iterative formulae can be quickly evaluated, but for larger values of $K$, the computation becomes slow due to the quadratic computation complexity. We show that the complexity can be reduced to linear if we allow certain approximation. When the number of received batches follows a Poisson distribution with mean value $\bar{n}$, we obtain recursive formulae for both BP decoding and inactivation decoding with complexity $O(K\bar{n}M^2D)$, where $D$ is the maximum degree. For large values of $\bar{n}$, numerically results show that evaluation of these formulae provides a good approximation of the case with fixed $\bar{n}$ received number of batches. Our Poisson model is different from the model used to analyze LT codes by Maneva and Shokrollahi \[12\], where the number of received coded symbols is the summation of random variables of binomial distributions. In network communications, the number of received packets in a time interval is usually random, and is typically modelled by Poisson distribution. Therefore, the Poisson model of the number of the received batches is useful to measure the performance of BATS code in such network models.

These finite length analysis results can help us to design BATS codes with short block length. We demonstrate how it works by optimizing the degree distribution using an heuristic algorithm for finite block lengths. Our iterative formulae enable us quickly evaluate the degree distributions obtained from the heuristic algorithm. By tuning the parameters of the heuristic algorithm, we obtain much better degree distributions than the asymptotically optimal ones.

C. Paper organization

After an introduction of BATS codes in Section II, our results are discussed in details in the following of this paper. In Section III we provide a recursive formula that calculates the distribution of the stopping time of the BP decoder for a fixed number of batches. The derivation of the formula is given in Section IV. In Section V we provide another
recursive formula that calculates the distribution of the stopping time of the BP decoder for a Poisson number of batches. After that, inactivation decoding will be introduced in Section VI and the corresponding recursive formulae will be provided. Degree distribution optimization for finite block length is discussed in Section VII. The numerical evaluations of the finite length analysis results can be found in Section VIII.

II. INTRODUCTION OF BATS CODES

In this section, the encoding and decoding processes of BATS codes will be briefly introduced. Readers may refer to [2] for more discussion.

A. Encoding and transmission

Suppose \( K \) input symbols will be transmitted from a source node to a sink node through a network employing linear network coding, where each input symbol is an element \( \mathbb{F}_q \) of the finite field \( \mathbb{F}_q \) of size \( q \). For the purpose of analysis, we describe a random encoding procedure of BATS codes. A batch is a row vector of \( M \) symbols. The encoder of a BATS code generates a potentially unlimited sequence of batches \( X_1, X_2, \ldots \) using the \( K \) input symbols and a degree distribution \( \Psi = \{\Psi_d, d = 1, \ldots, D\} \), where \( D \) is the maximum degree. Each batch \( X_i \) is generated independently using the same procedure as follows.

First, choose a degree \( d_i \) by sampling the distribution \( \Psi \). Second, uniformly at random choose an index set \( A_i \) of \( d_i \) integers ranging from \( 1 \) to \( K \), and form a row vector \( B_i \) by using the input symbols with indices in \( A_i \). Third, set \( X_i = B_iG_i \), where \( G_i \) is a \( d_i \times M \) matrix with all components being uniformly i.i.d. from \( \mathbb{F}_q \). We call the input symbols with indices in \( A_i \) the contributors of the batch \( X_i \).

Symbols in batches are sent out by the source node using certain scheduling scheme. By assuming that the network only applies linear operations on each batch and does not mix different batches together, the received form of batch \( X_i \) is \( Y_i = X_iH_i \), where \( H_i \), called a transfer matrix, is determined by the network operations including linear network coding and packet loss. We assume that \( H_i \), \( i = 1, 2, \ldots \) are mutually independent and following the same distribution of \( H \), and \( H_i \), \( i = 1, 2, \ldots \) are also independent to the encoding process.

The network output \( Y_i \) of a batch can be expressed as

\[
Y_i = B_i \cdot G_i \cdot H_i.
\]

(1)

We call \( Y_i \) a received batch, or a batch for simplicity, and we call (1) the associated linear system of batch \( i \).

Similar to Raptor codes, precoding can be applied to the input symbols before applying the above encoding procedure. With precoding, a BATS code with batch size \( M = 1 \) is just a Raptor code.
B. Belief propagation decoding

Consider $n$ batches $Y_1, Y_2, \ldots, Y_n$ are received. We assume that the sink node knows $G_iH_i$ and $A_i$ for $i = 1, \ldots, n$. The decoding of BATS codes is actually to solve the linear system

\[
\begin{align*}
Y_1 &= B_1 \cdot G_1 \cdot H_1 \\
Y_2 &= B_2 \cdot G_2 \cdot H_2 \\
&\vdots \\
Y_n &= B_n \cdot G_n \cdot H_n.
\end{align*}
\]

(2)

to recover the input symbols. (Note that if the input symbols are precoded by a linear code, $B_i$ will further satisfy a set of parity check constraints.) Gaussian elimination can be used but its complexity is too high to be practical for many applications. Instead, we use the following belief propagation (BP) decoding algorithm illustrated in Fig. 1.

A received batch $Y_i$ is called decodable if $G_iH_i$ has rank $d_i$. If so, then $B_i$ is recovered by solving the linear system (1), which has a unique solution since $\text{rk}(G_iH_i) = |B_i|$. The symbols in $B_i$, once decoded, can be substituted into and hence simplify the associated linear systems of batches $Y_j$ with $A_j \cap A_i \neq \emptyset$. For example in Fig. 1 if $Y_i$ is decoded, $b_k$ can be recovered. Since $b_k$ is a contributor of $Y_j$, the value of $b_k$ can be substituted into the linear system associated with $Y_j$. After the substitution, some previously undecodable batches may become decodable. We repeat the above decoding and substituting procedure until there are no decodable batches.

With precoding, the goal of the BP decoding is to recover a given fraction of the input symbols so that the original input symbols can be recovered using the precode. One objective of this paper is to analyze the number of input symbols that can be recovered by BP decoding. Since directly analyzing the above decoding procedure is difficult, we instead use a decoding process that in each decoding step only one input symbol is decoded, which is described as follows.

The time starts at 0 and increases by 1 after each decoding step. We say an input symbol is decodable if it contributes to a decodable batch. In each decoding step, we first pick a decodable input symbol and mark it as decoded. Then we substitute the input symbol to the associated linear systems of the batches it contributes to. The decoding stops when there is no decodable input symbols.

\footnote{In general, each symbol can be extended to a vector of symbols, but the generalization does not affect the analysis.}

\footnote{How to choose the input symbol does not affect the time when the decoding stops.}
For each batch \( \mathcal{B} \) and time \( t \), let \( \mathcal{A}_i \) be the indices of the contributors of batch \( i \) that have not been decoded. Note that \( \mathcal{A}_i^0 = \mathcal{A}_i \). Let \( j \) be the index of the input symbol decoded in the decoding step right after time \( t \). Then \( \mathcal{A}_i^{t+1} = \mathcal{A}_i^t \setminus \{ j \} \) if \( j \in \mathcal{A}_i^t \), and \( \mathcal{A}_i^{t+1} = \mathcal{A}_i^t \) otherwise. The associated linear system of batch \( i \) at time \( t \) can be denoted by

\[
Y_i^t = B_i^t \cdot G_i^t \cdot H_i,
\]

where \( B_i^t, G_i^t \) and \( Y_i^t \) for each batch \( i \) and time \( t \) are defined as follows. First, \( B_i^0 = B_i, G_i^0 = G_i \) and \( Y_i^0 = Y_i \). For \( t \geq 0 \), \( B_i^{t+1} = B_i^t, G_i^{t+1} = G_i^t \) and \( Y_i^{t+1} = Y_i^t \) if \( A_i^{t+1} = A_i^t \). Otherwise, let \( j \) be the index in \( A_i^t \setminus A_i^{t+1} \). Then \( B_i^{t+1} \) is formed by removing the component of \( B_i^t \) corresponding to \( b_j \), \( G_i^{t+1} \) is formed by removing the row \( g \) of \( G_i^t \) corresponding to \( b_j \), and \( Y_i^{t+1} = Y_i^t - b_j g \).

### C. Solvability of a batch

At time \( t \) of the decoding procedure, the degree of a batch \( i \) is \( |A_i^t| \), and define the rank of the batch to be \( \text{rk}(G_i^t H_i) \). A batch becomes decodable only when its degree equals to its rank. Let us check the probability that a batch is decodable when its degree is \( s, s > 0 \).

Let \( G[s] \) be an \( s \times M \) random matrix with uniformly i.i.d. components in \( \mathbb{F}_q \), and \( G'[s] \) be the submatrix of \( G[s] \) obtained by removing one row. Define

\[
\begin{align*}
\hat{h}_s &:= \Pr\{\text{rk}(G[s+1] H) = \text{rk}(G'[s+1] H) = s\}, \quad (3) \\
\hat{h}'_s &:= \Pr\{\text{rk}(G[s] H) = s\}. \quad (4)
\end{align*}
\]

We can check that \( \hat{h}_s \) is the probability that a batch is decodable for the first time when its degree is \( s \). Once a batch becomes decodable, it keeps to be decodable until all its contributors are decoded. Similarly, we see that \( \hat{h}'_s \) is the probability that a batch is decodable when its degree is \( s \). Note that \( \hat{h}'_s = \sum_{k \geq s} \hat{h}_k \) for \( 0 \leq s \leq M \) and \( \hat{h}_s = 0 \) for \( s > M \).

The explicit forms of \( \hat{h}_s \) and \( \hat{h}'_s \) will not be directly used in the analysis, but are useful in the numerical evaluation. According to [2],

\[
\begin{align*}
\hat{h}_s &= \sum_{i=s}^M \frac{\zeta^i}{q^{i-s}} \hat{h}_i \quad \text{and} \quad \hat{h}'_s = \sum_{k=s}^M \zeta^k \hat{h}_k,
\end{align*}
\]

where

\[
\zeta^m := \begin{cases} 
(1 - q^{-m})(1 - q^{-m+1}) \cdots (1 - q^{-m+r-1}) & r > 0 \\
1 & r = 0
\end{cases}
\]

and \( \hat{h}_r := \Pr\{\text{rk}(H) = r\} \) is the rank distribution of \( H \).

Note that when the field size is large, e.g., \( q = 2^8 \), the difference between \( \hat{h}_r \) and \( \hat{h}'_r \) becomes neglectable. When applying the analysis to LT/Raptor codes, i.e. \( M = 1 \), we can use deterministic generator matrices with all components being the identity of the field, and hence by the definition in [3] and [4], \( \hat{h}_0 = h_0, \hat{h}'_0 = 1 \) and \( \hat{h}_1 = \hat{h}'_1 = h_1 \). In the case of \( M = 1 \), \( h_0 \) can be regarded as the erasure rate.
D. Degree distribution optimization

The asymptotic performance of BATS codes with BP decoding has been analyzed \cite{2}. The result is as follows.

**Theorem 1** (Asymptotic analysis \cite{2}). Let $K$ be the number of input symbols, $n$ be the number of batches received, $\theta = K/n$ and

$$I_{a,b}(x) := \sum_{j=a}^{a+b-1} \binom{a+b-1}{j} x^j (1-x)^{a+b-1-j},$$

be the regularized incomplete beta function. Define

$$\Omega(x) := \sum_{r=1}^{M} \sum_{d=r+1}^{D} d\Psi_d h_r I_{d-r,r}(x) + \sum_{r=1}^{M} h'_r r\Psi_r,$$

where $h_r$ is calculated by (3) and $D$ does not depend on $K$. For any $0 < \eta < 1$, if $\Omega(x) + \theta \ln(1-x) > 0, \forall x \in [0, 1-\eta]$, there exists a positive number $c = c(\theta)$ such that when $K$ is sufficiently large, with probability at least $1 - e^{-cn^{1/8}}$, the BP decoder is able to recover at least $(1 - \eta)K$ input symbols.

The above theorem provides a sufficient condition such that the BP decoding can recover a given fraction of the input symbols, which induces an optimization problem to design the degree distribution. The optimization of degree distribution depends on the rank distribution $\{h_r\}$ of $H_i$ in (1), which is used to calculate $h_i$ and $h'_i$, and a given fraction $\eta \in (0, 1)$ of input symbols to be recovered. The objective of the optimization is to maximize $\theta$, which is related to the design rate of the codes:

$$\max \theta \text{ s.t.}\begin{cases}
\Omega(x) + \theta \ln(1-x) \geq 0, \forall x \in [0, 1-\eta], \\
\sum_{d} \Psi_d = 1, \quad \Psi_d \geq 0 \text{ for } d = 1, \ldots, D.
\end{cases}$$

(6)

It can be shown that using $D$ larger than $\lceil M/\eta \rceil - 1$ does not give better optimal value of (6). We can set $D = \lceil M/\eta \rceil - 1$. Since the maximum degree does not depend on $K$, the per packet encoding/decoding complexity of BATS codes also does not depend on $K$.

Let $\hat{\theta}$ be the optimal value of (6). It can be proved that $(1-\eta)\hat{\theta} \leq \sum_i \imath h_i$. On the other hand, numerical results show that $(1-\eta)\hat{\theta}$ can be very close to $\sum_i \imath h_i$ for all the instances of rank distributions we have tested ($\geq 10^5$). Since $\sum_i \imath h_i$ is the expected rank of the linear system in (2) normalized by $n$, the degree distribution is close to be optimal. For the field sizes of interests, e.g., $q = 2^8$, $\sum_i \imath h_i$ approximately equals to $\sum_i \imath h_i$.

The degree distribution obtained by (6) is guaranteed to have good performance for very large block length. However, simulation results showed that for small block length the true rate has a big gap between design rate. Therefore, we are motivated to understand the finite-length performance of BATS codes and to design better degree distributions.

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3This fact can be shown more easily using the finite-length analysis results in this paper.
III. STOPPING TIME OF BP DECODER

In this section, we fix a number $K$ of input symbols and compute the distribution of the stopping time of the BP decoder. In general, the number of received batches can be random. Let $N$ be the random variable of the number of batches used in the decoder.

We start with the performance of BP decoding for a fixed number $n$ of batches, i.e., the condition $N = n$ is assumed. Let $R^t$ be the number of decodable input symbols at time $t$. The probability that the decoder stops at time $t$ is

$$P_{\text{stop}}(t) := \Pr \{ R^t = 0, R^\tau > 0, \tau < t | N = n \}.$$ 

The stopping time of the BP decoder is equal to the number of packets that can be decoded.

Let $C^t$ be the number of batches with its degree strictly larger than its rank at time $t$, i.e., the number of undecodable batches at time $t$. We are interested in the probabilities

$$\Lambda^t_{c,r}|n := \Pr \{ C^t = c, R^t = r, R^\tau > 0, \tau < t | N = n \},$$

with which, we have $P_{\text{stop}}(t) = \sum_c \Lambda^t_{c,0}|n$. We will express every $\Lambda^t_{c,r}|n$ in terms of $\Lambda^{t-1}_{c',r'}|n$'s, so that we can calculate $\Lambda^t_{c,r}|n$'s recursively.

For $c \leq n$, define

$$\Lambda^t_{c}|n := (\Lambda^t_{c,0}|n, \Lambda^t_{c,1}|n, \ldots, \Lambda^t_{c,K-1}|n),$$

and let $(\Lambda^t_{c}|n)^{\perp 1}$ be the sub-vector of $\Lambda^t_{c}|n$ without the first component. Let

$$\text{Bi}(n, k; p) := \binom{n}{k} (p)^k (1 - p)^{n-k},$$

and let

$$\text{hyge}(n, i, j, k) := \begin{cases} \frac{i}{j} \binom{n-i}{j} & k \leq \min\{i, j\} \\ 0 & \text{o.w.} \end{cases}$$

be the p.m.f. of hyper-geometric distribution. We obtain the following recursive formula of $\Lambda^t_{c}|n$. We postpone the proof to Section IV.

**Theorem 2.** Given the number $K$ of input symbols, the number $n$ of batches, the degree distribution $\{\Psi_d\}$, the rank distribution $\{h_r\}$ of the transfer matrix, the maximum degree $D$ and the batch size $M$, we have

$$\Lambda^0_{c}|n = \text{Bi}(n, c; 1 - \rho^0) \mathbf{e}_1 Q_0^{n-c},$$

and for $t > 0$,

$$\Lambda^t_{c}|n = \sum_{c' \geq c} \text{Bi}(c', c; 1 - \rho^t) (\Lambda^{t-1}_{c'}|n)^{\perp 1} Q_t^{c'-c}$$

where the notations are defined as follows:

1) $\mathbf{e}_1 = (1, 0, \ldots, 0)$.

2) $\rho^0 = \sum_s p^0_s$, where $p^0_s = \Psi_s h'_s$. 

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3) For \( t > 0 \),

\[
\rho^t = \frac{\sum_s p^t_s}{1 - \sum_{t=0}^{t-1} \sum_s p^t_s}.
\]

where

\[
p^t_s = h_s \sum_{d=s+1}^{D} \Psi_{d} d K \text{hyge}(K-1, d-1, t-1, d-s-1).
\]

4) For \( t \geq 0 \), \( Q_t \) is a \((K-t+1) \times (K-t+1)\) matrix with

\[
Q_t(i+1, j+1) = \sum_{s=j}^{j} \frac{p^t_s}{\sum_s p^t_s} \text{hyge}(K-t, i, s, i+s-j)
\]

if \( j-M \leq i \leq j \), and \( Q_t(i+1, j+1) = 0 \) otherwise, where \( Q_t(i+1, j+1) \) is the component of \( Q_t \) on the \((i+1)\)th row and \((j+1)\)th column.

When precoding is applied, the BP decoder only needs to recover a fraction of the input symbols, so the error probability is \( \sum_{t=0}^{K'} \sum_s \Lambda^t_{c,0:n} \), where \( K' \) is the number of input symbols to be recovered.

The computational complexity to evaluate the recursive formula in Theorem 2 is \( O(K^2 n^2 M) \), where \( K \) is the number of input symbols, \( n \) is the number of batches and \( M \) is the batch size. The complexity can be shown by first noting that the quantities \( \{p^t_s, \rho^t, Q_t\}_{t \leq K, s \leq M} \) can be computed in \( O(K^2 MD) \) steps using recursive formulas (e.g., \( \binom{n}{k} = \frac{n}{n-k} \binom{n-1}{k} \)) to compute binomial coefficients. Secondly, the matrix \( Q_t \) has at most \( K+1 \) columns, and each column has most \( M+1 \) non-zero elements. Therefore, the complexity for multiplying a vector to \( Q_t \) is \( O(KM) \). For fixed \( t \leq K, c \leq n \), the vectors \( \{A_{c,n}^{t-1}\} Q_t^{t-c} \) can be computed recursively in \( O(KnM) \) steps. Hence, the total complexity is \( O(K^2 n^2 M) \), where we assume \( D = O(n^2) \). Since \( M \) is small (e.g., 32) and \( n \) is usually linear with \( K \), we can also say that the complexity is \( O(K^4) \).

When \( M = 1 \), Theorem 2 can be simplified as follows. As we have discussed, in this case \( h_0 = h_0, h'_1 = 1 \) and \( h_1 = h'_1 = 1 \). So \( \rho^0 = p^0_1 = \Psi_1 h_1 \) and

\[
\rho^t = \frac{h_1 \sum_{d=2}^{D} \Psi_{d} d K \text{hyge}(K-1, d-1, t-1, d-2) + h_0 \sum_{d=1}^{D} \Psi_{d} d K \text{hyge}(K-1, d-1, t-1, d-1) - \sum_{d=1}^{D} \Psi_{d} \text{hyge}(K, d, t-1, d)}{1 - h_1 \sum_{d=1}^{D} \Psi_{d} \text{hyge}(K, d, t-1, d) - \sum_{d=1}^{D} \Psi_{d} \text{hyge}(K, d, t-1, d)}.
\]

The matrix \( Q_t \) also has a simpler expression: \( Q_t = \frac{p^t_1}{p^t} I_t + \frac{p^t_1}{p^t} S_t \), where \( I_t \) is the \((K-t+1) \times (K-t+1)\) identity matrix, \( S_t \) is a \((K-t+1) \times (K-t+1)\) matrix such that \( S_t(i+1, i+1) = \frac{i}{\kappa-1}, S_t(i+1, i+2) = 1 - \frac{i}{\kappa-1} \) and \( S_t(i, j) = 0 \) otherwise.

Karp et al. have given a finite length analysis of LT codes [11]. However, directly extending their approach for BATS codes results in a formula that has much higher evaluation complexity. We outline the extension of the Karp et al.’s approach in Appendix. In their approach the number of decodable received symbols (called output ripple) is calculated recursively. Directly extending their approach for \( M > 1 \) requires us to calculate the number of decodable batches. But since decodable batches can have \( M \) different degrees, \( M \) recursive formulas must be provided for each degree value of decodable batches so that the evaluation complexity increases exponentially with \( M \).
approach in Theorem 2, instead, tracks the number of decodable input symbols. Though our formula has slightly higher complexity ($O(K^4)$) when $M = 1$ than the formula of LT codes in [11] ($O(K^3 \log^2(K) \log(K))$), our approach is simpler to apply for cases with $M > 1$.

IV. PROOF OF THEOREM 2

Let $\tilde{\Theta}_s^t$ be the set of indices of batches that both the degree and the rank at time $t$ equal to $s$. In other words, a batch with index in $\tilde{\Theta}_s^t$, $s > 0$, is solvable and can decode $s$ symbols. Let $\Theta_s^t$ be the set of indices of batches that are not in $\tilde{\Theta}_s^t := \cup_{s=0}^M \tilde{\Theta}_s^t$. We see that $R_t = |\cup_{i \in \Theta_s^t} A_i^t|$, which is valid since $A_i^t = \emptyset$ for $i \in \tilde{\Theta}_0^t$. Also, we see that $C_t = |\Theta_s^t|$.

A. Initial status

We first calculate $\Lambda_{c,r|n}^0 = \Pr\{C^0 = c, R^0 = r|N = n\}$. The condition $N = n$ will be implied in the following of the proof.

When $t = 0$, the probability that a batch has degree $s$ is $\Psi_s$ and for a batch with degree $s$, it is decodable with probability $h_s'(s)$ (see (4)). Therefore, the probability that a batch is in $\tilde{\Theta}_0^s$ is $\Psi_s h_s'$, i.e., for $i \leq n$ and $s \leq M$,

$$\Pr\{i \in \tilde{\Theta}_s^0\} = p_s^0 := \Psi_s h_s' .$$

Hence,

$$\Pr\{i \in \Theta_s^0\} = \sum_{s=0}^M p_s^0 := \rho^0 .$$

Since all batches are independent, we obtain that

$$\Pr\{C^0 = k\} = \Pr\{|\Theta^0| = k\} = \text{Bi}(n,k;1-\rho^0) .$$

(8)

Recall that $Q_0$ is a $(K+1) \times (K+1)$ matrix defined as

$$Q_0(i+1,j+1) = \sum_{k=j-i}^{j} \text{hyge}(K,i,k,i+k-j) \frac{p_s^0}{\rho^0}$$

if $j-M \leq i \leq j$, and $Q_0(i+1,j+1) = 0$ otherwise.

Lemma 1. We have

$$(\Pr\{R^0 = j|C^0 = n-k\} : j = 0, \ldots, K) = e_1 Q_0^k ,$$

where $e_1 = (1,0,\ldots,0)$.

Proof: Fix $n$ and $k$. If $k = 0$, then $\tilde{\Theta}_0^0 = \emptyset$, and hence $\Pr\{R^0 = 0|C^0 = n\} = 1$, i.e., the lemma with $k = 0$ is proved. In the following, we assume $k > 0$. We have

$$\Pr\{R^0 = j|C^0 = n-k\}$$

$$= \Pr\{|\cup_{i \in \tilde{\Theta}_0^s} A_i^0| = j|\tilde{\Theta}_0^0| = k\}$$

$$= \Pr\{|\cup_{i=1}^k A_i^0| = j|\tilde{\Theta}_0^0 = \{1,\ldots, k\}\}$$
where the second equality follows that since all batches are i.i.d., Pr \( \{ \cup_{i \in B} A^0_i \} = j \mid \Theta^0 = B \} \) is the same for any \( B \subset \{1, 2, \ldots, n\} \) with \( |B| = k \).

Let

\[
Q_{r|k}(j) = \Pr \{ \cup_{i=1}^r A^0_i = j \} \mid \Theta^0 = \{1, \ldots, k\} \}.
\]

We give a recursive formula to compute \( Q_{r|k}(\cdot) \) for \( r = 1, \ldots, k \). Note that \( Q_{k|k}(j) = \Pr \{ R^0 = j \} C^0 = n - k \} \). First,

\[
Q_{1|k}(s) = \Pr \{ |A^0_1| = s \} \mid \Theta^0 = \{1, \ldots, k\} \}
= \Pr \{ 1 \in \Theta^0, 1 \in \Theta^0 \}
= \frac{p^0_1}{\rho^0}.
\]

Second, for \( r > 1 \),

\[
Q_{r|k}(s) = \Pr \{ |\cup_{i=1}^r A^0_i| = s \} \mid \Theta^0 = \{1, \ldots, k\} \}
= \sum_{i=0}^s Q_{i,s} Q_{r-1|k}(i),
\]

where

\[
Q_{i,s} = \Pr \{ |\cup_{i=1}^r A^0_i| = s \} \mid |\cup_{i=1}^{r-1} A^0_i| = i, \Theta^0 = \{1, \ldots, k\} \}
= \sum_{j=s-i}^s \Pr \{ |\cup_{i=1}^r A^0_i| = s \} \mid |\cup_{i=1}^{r-1} A^0_i| = i, |A^0_r| = j \} \times
\times \Pr \{ |A^0_r| = j \} \mid \cup_{i=1}^{r-1} A^0_i = i, \Theta^0 = \{1, \ldots, k\} \}
= \sum_{j=s-i}^s \text{hyge}(K, i, j, i + j - s) \frac{\rho^{0-j}}{\rho^0}.
\]

where term \((a)\) is a hypergeometric distribution; and term \((b)\) is equal to \( \Pr \{ |A^0_0| = j \} r \in \Theta^0 \} \) and can be obtained similar to \( Q_{1|k} \).

Note that \( Q_{i,s} \) does not depend on \( r \). Let \( \pi_{r|k} = (Q_{r|k}(0), Q_{r|k}(1), \ldots, Q_{r|k}(K)) \). We have

\[
Q_{r|k} = \pi_{r-1|k} Q_0 = \pi_{1|k} Q_0^{r-1}.
\]

Noting that \( \pi_{1|k} = \frac{1}{\rho^0}(p_0^0, p_1^0, \ldots, p_K^0) \) is the same as the first row of \( Q_0 \), the proof is completed. \( \blacksquare \)

By \((8)\) and Lemma 1, we have

\[
\Lambda^0_{c|n} = (\Pr \{ C^0 = c, R^0 = j \} : j = 0, \ldots, K) \]
= \( \Pr \{ C^0 = c \} (\Pr \{ R^0 = j \} C^0 = c \} : j = 0, \ldots, K) \)
= \( \text{Bi}(n, c; 1 - \rho^0) e_1 Q_0^{n-c} \).
B. Recursive formula

We have computed $A_{c,r|n}^0$ in the last part. Now we compute $A_{c,r|n}^t = \Pr \{ C^t = c, R^t = r, R^{\tau} > 0, \tau < t | N = n \}$ for $t > 0$. Define event $E_t$, $t \geq 0$, as $\{ R^{\tau} > 0, \tau < t, N = n \}$ and $E_{t-1} = \{ N = n \}$. For $t > 0$, we have

\[
A_{c,r|n}^t = \Pr \{ C^t = c, R^t = r, R^{\tau} > 0, \tau < t | N = n \} = \sum_{c', r'} \Pr \{ C^t = c, C^{t-1} = c', R^t = r, R^{t-1} = r', R^{\tau} > 0, \tau < t | N = n \} A_{c',r'|n}^{t-1}.
\]

We characterize (c) and (d) in the above equation as follows.

Recall that for $\tau \geq 1$

\[
p_s^\tau := h_s \sum_{d=s+1}^D \Psi_d \frac{d}{K} \text{hyge}(K-1, d-1, \tau-1, d-s-1).
\]

**Lemma 2.** Consider BATS codes with BP decoding. We have for $r^t > 0$,

\[
\Pr \{ C^t = c | C^{t-1} = c', R^{t-1} = r', E_{t-1} \} = \text{Bi}(c', c; 1 - \rho^t),
\]

where

\[
\rho^t := \frac{\sum_s p_s^t}{1 - \sum_{\tau=0}^{t-1} \sum_s p_s^\tau}.
\]

**Proof:** Since $r^t > 0$, the decoding does not stop at time $t-1$. We calculate the distribution of $C^t$ under a special instance of the condition $\{ R^{t-1} = r', E_{t-1} \}$ that the input symbol decoded from time $\tau - 1$ to $\tau$ has index $\tau - 1$ for $1 \leq \tau \leq t$. Since the distribution we will obtain does not depend on the instance, the distribution is equal to the conditional distribution of the lemma. To simplify the notation, the condition is implied in the notations in the following proof of the lemma.

We first compute $\Pr \{ j \in \Theta_s^\tau \cap \Theta^{t-1} \}$ for an arbitrary batch $j$. Note that $\tau - 1$ is the index of the input symbol chosen to decode in the decoding step from $\tau - 1$ to $\tau$, and there are totally $\tau$ input symbols decoded at time $\tau$. Given the initial degree of batch $j$ being $d$, $j \in \Theta_s^\tau \cap \Theta^{t-1}$ is equivalent to

1) $\tau - 1 \in A_j$,
2) $|A_j| = s$, and
3) $\text{rk}(G_j^{d-1}H_j) = \text{rk}(G_j^dH_j) = s$.

Since all batches are formed independently, we know that 1) holds with probability $d/K$; given 1) the probability that 2) holds is the hypergeometric distribution $\text{hyge}(K - 1, d - 1, \tau - 1, d - s - 1)$; given both 1) and 2) the
probability that 3) holds is $h_s$ (see (3)). Therefore, the probability for 1), 2) and 3) holding given $|A_j| = d$ is

$$\frac{d}{K} h_s \text{hyge}(K - 1, d - 1, \tau - 1, d - s - 1).$$

Hence

$$\Pr\{j \in \tilde{\Theta}_s^\tau \cap \Theta^{\tau-1}\} = p_s^\tau. \quad (10)$$

Now we compute $\Pr\{j \in \Theta^t\}$. Since the following collection of sets is a partition of all batches:

$$\tilde{\Theta}^0, \{\Theta^\tau \cap \Theta^{\tau-1} : \tau = 1, \ldots, t\}, \Theta^t,$$

we have

$$\Pr\{j \in \Theta^t\}$$

$$= 1 - \Pr\{j \in \tilde{\Theta}^0\} - \sum_{\tau=1}^t \Pr\{j \in \tilde{\Theta}^\tau \cap \Theta^{\tau-1}\}$$

$$= 1 - \sum_s p_s^0 - \sum_{\tau=1}^t \sum_s p_s^\tau$$

$$= \frac{\sum p_s^{t+1}}{\rho^{t+1}},$$

where the last equality follows from the definition of $\rho^t$ in (9). Hence

$$\Pr\{j \in \tilde{\Theta}^t | j \in \Theta^{t-1}\} = \frac{\Pr\{j \in \tilde{\Theta}^t \cap \Theta^{t-1}\}}{\Pr\{j \in \Theta^{t-1}\}} = \rho^t. \quad (11)$$

Finally, since batches in $\Theta^{t-1}$ stay in $\Theta^t$ independently, for $B \subset \{1, \ldots, n\}$ with $|B| = c'$,

$$\Pr \{ C^t = c | \Theta^{t-1} = B \}$$

$$= \Pr \{ |\Theta^t| = c | \Theta^{t-1} = B \}$$

$$= \text{Bi}(c', c; 1 - \rho^t).$$

Since the above distribution depends on $B$ only through its cardinality, we have

$$\Pr \{ C^t = c | C^{t-1} = c' \}$$

$$= \sum_{B \subset \{1, \ldots, n\} : |B| = c'} \Pr \{ C^t = c | \Theta^{t-1} = B \} \times \Pr \{ \Theta^{t-1} = B | C^{t-1} = c' \}$$

$$= \text{Bi}(c', c; 1 - \rho^t).$$

Recall that $Q_t$ is a $(K - t + 1) \times (K - t + 1)$ matrix defined as:

$$Q_t(i + 1, j + 1) = \sum_{s=j+1}^j \frac{p_s^t}{\sum_s p_s^t} \text{hyge}(K - t, i, s, i + s - j)$$

if $j - M \leq i \leq j$, and $Q_t(i + 1, j + 1) = 0$ otherwise.
Lemma 3. Consider BATS codes with BP decoding. For \( r' > 0 \),

\[
\Pr \{ R^t = r | C^t = c, C^{t-1} = c', R^{t-1} = r', E_{t-1} \}
\]
is equal to the \( r' \)th row and \( (r+1) \)th column component of \((Q_i)^{c'-c}\).

Proof: Consider an instance of \( \{ C^t = c, C^{t-1} = c', R^{t-1} = r', E_{t-1} \} \) with \( \Theta^{t-1} = B', \Theta^t = B \) and let \( \mathcal{A} \) be the set of indices of decodable input symbols at time \( t - 1 \), excluding the input symbol decoded from time \( t - 1 \) to \( t \). We have \(|B| = c, \ |B'| = c', B \subset B'\), and \(|\mathcal{A}| = r' - 1\), which is valid since \( r' > 0 \). We will compute the distribution of \( R^t \) by assuming this instance. Since the distribution we will obtain only depends on the instance through \( c, c' \) and \( r' \), the distribution of \( R^t \) under the condition \( \{ C^t = c, C^{t-1} = c', R^{t-1} = r', E_{t-1} \} \) is the same.

For convenience, let \( B' \setminus B = \{1, \ldots, \delta := c' - c\} \). Since batches with index in \( B' \setminus B \) becomes decodable only starting at time \( t \), we have

\[
\Pr \{ R^t = j \} = \Pr \{ |\mathcal{A} \cup (\cup_{i=1}^{\delta} A_i^t)\} = j \}.
\]

We use the similar method as in Lemma 1 to compute the above distribution. For \( 0 \leq j \leq K - t \), let

\[
Q_{0|\delta}(j) = \Pr \{ |\mathcal{A}| = j \},
\]

and

\[
Q_{r|\delta}(j) = \Pr \{ |\mathcal{A} \cup (\cup_{i=1}^{r-1} A_i^t)\} = j \}.
\]

For \( r \geq 1 \),

\[
Q_{r|\delta}(j) = \sum_{i=0}^{j} Q_{i|\delta} Q_{r-i|\delta}(i),
\]

where

\[
Q_{i,j}
\]

\[
= \Pr \{ |\mathcal{A} \cup (\cup_{i=1}^{j} A_i^t)\} = j \} \Pr \{ |\mathcal{A} \cup (\cup_{i=1}^{r-1} A_i^t)\} = i \}
\]

\[
= \sum_{s=j-i}^{j} \Pr \{ |A_s^t| = s \} \Pr \{ |\mathcal{A} \cup (\cup_{i=1}^{r-1} A_i^t)\} = i \} \times
\]

\[
\times \Pr \{ |\mathcal{A} \cup (\cup_{i=1}^{r-1} A_i^t)\} = j \} \Pr \{ |\mathcal{A} \cup (\cup_{i=1}^{j-1} A_i^t)\} = i \}
\]

\[
= \sum_{s=j-i}^{j} \frac{p_{s}^{t}}{\sum_{s' \leq j-i} p_{s'}^{t}} \text{hyge}(K - t, i, s, i + s - j),
\]

where term \((e)\) is equal to \( \Pr \{ r \in \tilde{\Theta}_s | r \in \tilde{\Theta}^{t-1} \cap \tilde{\Theta}^t \} \), which can be obtained using \((10)\); \((f)\) is a hypergeometric distribution.

Let \( Q_{0|\delta} = (Q_{0|\delta}(0), \ldots, Q_{0|\delta}(K - t)) \). We have

\[
(\Pr \{ R^t = 0 \}, \ldots, \Pr \{ R^t = K - t \}) = Q_{0|\delta}(Q_i)^{\delta}.
\]

The proof is completed by noting that the only non-zero component of \( Q_{0|\delta} \) is the \( r' \)th component.
Proof of Theorem 2: With the above two lemmas, we can write
\[
\Lambda_t^{c,n} = \sum_{c', r' > 0} (Q_t)^{c' - c}(r', \cdot)B_i(c', c; 1 - \rho^t)\Lambda_{c', r'|n}^{t-1}
\]
\[
= \sum_{c' > c} B_i(c', c; 1 - \rho^t)\Lambda_{c'|n}^{t-1}(Q_t)^{c' - c},
\]
where \((Q_t)^{c' - c}(r', \cdot)\) is the \(r'\)th row of \((Q_t)^{c' - c}\). This completes the proof of Theorem 2.

V. POISSON NUMBER OF BATCHES

When \(N\) follows a general distribution, we can calculate the error probability using the above formulas by first calculating the error probabilities for all possible instances \(n\) of \(N\), and then combining these error probabilities according the distribution of \(N\), i.e., the error probability is
\[
\sum_n \Pr\{N = n\} \sum_{c \leq n} \Lambda_{c|n}^t.
\]
But this approach is not usable when the support of \(N\) is large. For example, when \(N\) has a Poisson distribution, the support of \(N\) is non-negative integers, and hence accurately computing the error probability using (12) directly is not possible.

In this subsection, we show that there exists a simpler recursive formula to compute the exact distribution of the stopping time of BP decoder when \(N\) has a Poisson distribution. We consider that the number \(N\) of batches is Poisson distributed with mean \(\bar{n}\), that is,
\[
\Pr\{N = n\} = \frac{\bar{n}^n}{n!}e^{-\bar{n}}.
\]
For \(0 \leq t \leq K\), define
\[
\Lambda_t^t := \Pr\{R_t^t = r, R_t^t > 0, \tau < t\},
\]
which is the probability that the BP decoder successfully decodes \(t\) input symbols and \(R_t^t = r\). Define a vector of size \(K - t + 1\):
\[
\Lambda^t := (\Lambda_0^t, \Lambda_1^t, \ldots, \Lambda_{K-t}^t).
\]
Then, we have
\[
\Lambda^t = \sum_n \Pr\{N = n\} \sum_{c \leq n} \Lambda_{c|n}^t.
\]
Before giving the general result, we show \(\Lambda^0\) as an example. Substituting \(\Pr\{N = n\}\) and \(\Lambda_{c|n}^{0}\) given in Theorem 2
\[
\Lambda^0 = \sum_n \frac{\bar{n}^n}{n!}e^{-\bar{n}} \sum_{c \leq n} B_i(n, c; 1 - \rho^0)e_1Q_0^{n-c}
\]
\[
= \sum_{c, n: c \leq n} \frac{\bar{n}^n}{n!}e^{-\bar{n}} \binom{n}{c}(1 - \rho^0)^c(\rho^0)^{n-c}e_1Q_0^{n-c}
\]
\[
= e^{-\bar{n}}e_1 \sum_{c, n: c \leq n} \frac{(\bar{n}(1 - \rho^0))^c (\bar{n}\rho^0Q_0)^{n-c}}{c! (n-c)!}.
\]
By defining \( m = n - c \) and using matrix exponential defined for a square matrix \( \mathbf{A} \) as
\[
\exp(\mathbf{A}) := \sum_{i=0}^{\infty} \frac{\mathbf{A}^i}{i!},
\]
we can further simplify the above formula as
\[
\mathbf{A}^0 = e^{-\hat{n}} \mathbf{e}_1 \sum_c \left( \frac{\hat{n}(1 - \rho^0)}{c!} \right)^c \sum_m \left( \hat{n}\rho^0 \mathbf{Q}_0 \right)^m
= e^{-\hat{n}} \mathbf{e}_1 \exp(\hat{n}(1 - \rho^0)) \exp(\hat{n}\rho^0 \mathbf{Q}_0)
= \mathbf{e}_1 \exp(-\hat{n}\rho^0) \exp(\hat{n}\rho^0 \mathbf{Q}_0).
\]

The general result is as follows.

**Theorem 3.** Given the number \( K \) of input symbols, the degree distribution \( \{\Psi_d\} \), the rank distribution \( \{h_r\} \) of the transfer matrix, the maximum degree \( D \), the batch size \( M \), and the number of batches being Poisson distributed with the mean equal to \( \hat{n} \), we have for \( t \geq 0 \),
\[
\mathbf{A}^t = \exp(-\hat{n} \sum_s p_s^t) (\mathbf{A}^{t-1})^{1 \backslash 1} \exp(\hat{n} \sum_s p_s^t \mathbf{Q}_t),
\]
where \( (\mathbf{A}^{-1})^{1 \backslash 1} := \mathbf{e}_1 \), while \( p_s^t \) and \( \mathbf{Q}_t \) are defined as in **Theorem 2**.

**Proof:** Define
\[
\tilde{\mathbf{A}}^t_{c|n} = [\mathbf{0}_t \ \mathbf{A}^t_{c|n}],
\]
where \( \mathbf{0}_t \) is a zero vector of length \( t \). Let \( \tilde{\mathbf{Q}}_t \) be a \((K+1) \times (K+1)\) matrix formed as follows: The submatrix formed by the last \( K - t + 1 \) rows and the last \( K - t + 1 \) columns is \( \mathbf{Q}_t \), and all other components of \( \tilde{\mathbf{Q}}_t \) are zero.

We see that \( \tilde{\mathbf{A}}^0_{c|n} = \mathbf{A}^0_{c|n} \), and by **Theorem 2** for \( t \geq 1 \)
\[
\tilde{\mathbf{A}}^t_{c|n} = \sum_{c' \geq c} \text{Bi}(c', c; 1 - \rho^t) \tilde{\mathbf{A}}^{t-1}_{c'|n} \tilde{\mathbf{Q}}^{c'-c}_t.
\]

Expanding the recursive formula, we have
\[
\tilde{\mathbf{A}}^t_{c|n} = \mathbf{e}_1 \sum \text{Bi}(n, c_0; 1 - \rho^0) \tilde{\mathbf{Q}}^{n-c_0}_0 \times
\]
\[
\times \text{Bi}(c_0, c_1; 1 - \rho^1) \tilde{\mathbf{Q}}^{c_0-c_1}_1 \times \cdots \times
\]
\[
\times \text{Bi}(c_{t-1}, c; 1 - \rho^t) \tilde{\mathbf{Q}}^{c_{t-1}-c}_t
\]
\[
= \mathbf{e}_1 \sum \begin{pmatrix} n \\ c_0 \end{pmatrix} (1 - \rho^0)^{c_0} (\rho^0 \tilde{\mathbf{Q}}_0)^{n-c_0} \times
\]
\[
\times \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} (1 - \rho^1)^{c_1} (\rho^1 \tilde{\mathbf{Q}}_1)^{c_0-c_1} \times \cdots \times
\]
\[
\times \begin{pmatrix} c_{t-1} \\ c \end{pmatrix} (1 - \rho^t)^{c} (\rho^t \tilde{\mathbf{Q}}_t)^{c_{t-1}-c}
\]
where the summation is over all \( \{c_0, \ldots, c_{t-1}\} \) such that \( n \geq c_0 \geq c_1 \geq \cdots \geq c_{t-1} \geq c \).
Note that $1 - \rho^0 = \Pr\{j \in \Theta^0\}$, and for $t > 0$, $1 - \rho^t = \Pr\{j \in \Theta^t|j \in \Theta^{t-1}\}$ (see (11)). Since $\Theta^{t-1} \subset \Theta^t$,

$$
\Pr\{j \in \Theta^t\} = \Pr\{j \in \Theta^t \cap \Theta^{t-1}\} = \prod_{\tau=1}^{t} \Pr\{j \in \Theta^\tau|j \in \Theta^{\tau-1}\} \Pr\{j \in \Theta^0\}.
$$

On the other hand, by (9),

$$
\Pr\{j \in \Theta^t\} = \sum_{s} p_{s}^{t+1} \rho_{s}^{t+1}.
$$

So we have for $t > 0$

$$
\prod_{\tau=0}^{t} (1 - \rho^{\tau}) = \sum_{s} p_{s}^{t+1} \rho_{s}^{t+1}.
$$

Reorganizing (16) using the above equality, we obtain

$$
\tilde{\Lambda}^t = \sum_{s} \bar{n}^{n} \frac{p_{s}^{t+1}}{n!} \frac{n!}{(n - s)! (n - s - 1)!} \cdots \frac{n!}{(n - s - c)!} e^{\bar{n} \sum_{s} p_{s}^{t+1} \tilde{Q}_{s}} \sum_{e} \tilde{\Lambda}_{e|n}^t.
$$

Define

$$
\tilde{\Lambda}^t := [0_t \; \Lambda^t].
$$

We have

$$
\tilde{\Lambda}^t = \sum_{e} \bar{n}^{n} \frac{p_{s}^{t+1}}{n!} \frac{n!}{(n - s)! (n - s - 1)!} \cdots \frac{n!}{(n - s - c)!} e^{\bar{n} \sum_{s} p_{s}^{t+1} \tilde{Q}_{s}} \sum_{e} \tilde{\Lambda}_{e|n}^t.
$$

Substituting the expression of $\tilde{\Lambda}_{e|n}^t$ and using the fact that

$$
\binom{n}{e_0} \binom{e_1}{c_1} \cdots \binom{e_{t-1}}{c_{t-1}} = \frac{n!}{(n - e_0)! (n - e_0 - e_1)! \cdots (n - e_0 - e_1 - \cdots - e_{t-1})!},
$$

we have

$$
\tilde{\Lambda}^t = e_1 \sum_{s} e^{-\bar{n}} \frac{\bar{n} \sum_{s} p_{s}^{t+1} \tilde{Q}_{s}}{e_s!} \binom{n - s}{c_0} \binom{s - c_0}{c_0} \cdots \binom{s - c_{t-1}}{c_{t-1} - c} e^{\bar{n} \sum_{s} p_{s}^{t} \tilde{Q}_{s}} (c_0 - 1)! \cdots (c_{t-1} - c)!.
$$

Let $x_{t+1} = c$, $x_0 = n - c_0$, $x_t = c_{t-1} - c$ and $x_{\tau} = c_{\tau-1} - c_{\tau}$ for $1 \leq \tau \leq t - 1$. We can rewrite the above
expression as

\[
\hat{A}^t = \mathbf{e}_1 \sum_{x_1:t=0,\ldots,t+1} e^{-\hat{n}} \left( \frac{\hat{n} \sum_s p_s^{t+1}}{\rho^{t+1}} \right) x_{t+1}! \left( \frac{\hat{n} \sum_s p_s^0 \tilde{Q}_0}{x_0!} \right) \times \left( \frac{\hat{n} \sum_s p_s^1 \tilde{Q}_1}{x_1!} \right) \ldots \left( \frac{\hat{n} \sum_s p_s^t \tilde{Q}_t}{x_t!} \right)
\]

\[
= \mathbf{e}_1 e^{-\hat{n}} \sum_{x_{t+1}} \left( \frac{\hat{n} \sum_s p_s^{t+1}}{\rho^{t+1}} \right) x_{t+1}! \sum_{x_0} \left( \frac{\hat{n} \sum_s p_s^0 \tilde{Q}_0}{x_0!} \right) \times \sum_{x_1} \left( \frac{\hat{n} \sum_s p_s^1 \tilde{Q}_1}{x_1!} \right) \ldots \sum_{x_t} \left( \frac{\hat{n} \sum_s p_s^t \tilde{Q}_t}{x_t!} \right)
\]

\[
= \mathbf{e}_1 e^{-\hat{n}} \exp \left( \sum_{s} \left( \frac{\hat{n} \sum_s p_s^{t+1}}{\rho^{t+1}} \right) \right) \exp \left( \sum_{s} \left( \frac{\hat{n} \sum_s p_s^0 \tilde{Q}_0}{x_0!} \right) \right) \times \exp \left( \sum_{s} \left( \frac{\hat{n} \sum_s p_s^1 \tilde{Q}_1}{x_1!} \right) \right) \ldots \exp \left( \sum_{s} \left( \frac{\hat{n} \sum_s p_s^t \tilde{Q}_t}{x_t!} \right) \right)
\]

\[
= \mathbf{e}_1 \exp \left( -\hat{n} \sum_{s} \left( \frac{\sum_{s} \sum_{x_0} p_s^0 \tilde{Q}_0}{x_0!} \right) \right) \times \exp \left( \sum_{s} \left( \frac{\hat{n} \sum_s p_s^1 \tilde{Q}_1}{x_1!} \right) \right) \ldots \exp \left( \sum_{s} \left( \frac{\hat{n} \sum_s p_s^t \tilde{Q}_t}{x_t!} \right) \right)
\]

(18)

where (18) is obtained using the definition of matrix exponential, and (19) follows from (9).

Thus, we have

\[
\hat{A}^t = \exp(-\hat{n} \sum_s p_s^t) \hat{A}^{t-1} \exp(\hat{n} \sum_s p_s^t \tilde{Q}_t)
\]

with \( \hat{A}^0 = \Lambda^0 \) given in (13). The proof is complete by noting the definition of \( \hat{A}^t \) in (17), the submatrix of \( \exp(\hat{n} \sum_s p_s^t \tilde{Q}_t) \) formed by the last \( K-t+1 \) rows and the last \( K-t+1 \) columns is \( \exp(\hat{n} \sum_s p_s^t \tilde{Q}_t) \), and all other components of \( \exp(\hat{n} \sum_s p_s^t \tilde{Q}_t) \) are zero.

The computational complexity of evaluating the recursive formula in Theorem 3 is \( O(K\hat{n}M^2D/tol) \), where \( tol \) is the tolerable error in the computation. Assume the average number of symbols received is larger than the number of input symbols, i.e., \( M\hat{n} \geq K \). Since the complexity of computing the quantities \( \{p_s^t, \tilde{Q}_t\}_{s \leq K, s \leq M} \) is \( O(K^2MD) \), to show the overall complexity it suffices to consider the cost of computing the action of matrix exponential \( (\Lambda^{t-1})^{1/\lambda} \exp(\hat{np}^t \tilde{Q}_t) \), which is usually faster than computing \( \exp(\hat{np}^t \tilde{Q}_t) \), where \( p^t = \sum_s p_s^t \).

Using the algorithm in [15], the cost for computing an action of an matrix exponential \( e^A \) is \( O(||A||_1 Mul(A)/tol) \), where \( Mul(A) \) is the cost for multiplying \( A \) with a vector. From the expression of \( \hat{np}^t \tilde{Q}_t \), we have \( ||\hat{np}^t \tilde{Q}_t||_1 \leq \frac{\hat{np}}{\lambda} \) for \( t > 0 \) and \( ||\hat{np}^0 \tilde{Q}_0||_1 \leq \hat{n} \). Also, as in the discussion following Theorem 2 \( Mul(\tilde{Q}_t) = O(KM) \). Therefore the complexity of calculating every \( \hat{A}^t \) is

\[
O(K \times \hat{n}MD/tol + \hat{n}KM/tol) = O(K\hat{n}MD/tol).
\]

(20)
When precoding is applied, we know that \( D = \mathcal{O}(M) \) is sufficient to achieve the maximum rate \([2]\). So the complexity in \([20]\) becomes \( \mathcal{O}(KnM^3/tol) \). When there is no precoding, we know the average degree must be of the order \( \mathcal{O}(\log K) \) \([5]\). So the maximum degree \( D \) must also increase with \( K \) to get the optimal performance.

### VI. Inactivation Decoding

BATS codes demonstrate nearly optimal asymptotic performance \([2]\), but the achievable rates of BATS codes with finite input symbols are lower than the asymptotic optimal value since extra number of batches is required to guarantee the success of BP decoding. An effective method to improve the rate for small block length is to use inactivation decoding.

In the following of this section, we first introduce inactivation decoding, then we calculate an important parameter of inactivation decoding – the expected number of inactive symbols.

#### A. Introduction of inactivation decoding

Inactivation decoding can be regarded as an efficient way to solve sparse linear systems \([16]\), \([17]\). This approach has been proposed for LT/Raptor codes \([13]\), \([14]\), and similar algorithm \([18]\) has been used for efficient encoding of LDPC codes. Here we introduce how to use inactivation for the decoding of BATS codes.

In the BP decoding algorithm introduced in Section II-B, the decoding stops when there are no decodable input symbols. Though BP decoding stops, Gaussian elimination can still be used to decode the remaining input symbols (by combining the linear systems associated with the undecoded batches to a single linear system involving all undecoded input symbols). But the decoding complexity using Gaussian elimination is higher than BP decoding. Inactivation decoding combines BP decoding with Gaussian elimination in a more efficient way.

In inactivation decoding, when there are no decodable input symbols at time \( t \), we instead randomly pick an undecoded symbol \( b_k \) and mark it as inactive. We substitute the inactive symbol \( b_k \) into the batches like a decoded symbol, except that \( b_k \) is an indeterminate. For example, if \( k \in A_t^t \), each element of \( Y_{i+1}^t = Y_i^t - b_kg \) will be expressed as a linear polynomial of \( b_k \). The decoding process is repeated until all input symbols are either decoded or inactive.

The inactive input symbols can be recovered by solving a linear system of equations using Gaussian elimination, and then the values of inactive symbols are substituted into the decoded input symbols. The inactive symbols are uniquely solvable if and only if the linear system \([2]\) formed by the linear systems associated with all the batches is uniquely solvable.

Inactivation decoding incurs extra computation cost that includes solving the inactive symbols using Gaussian elimination and substituting the values of the inactive symbols. Since both terms depend on the number of inactive symbols, knowing this number can help us to understand the tradeoff between computation cost and coding rate. In the following, we provide methods to compute the expected number of inactive symbols.
B. Recursive formulas

When the number of received batches is \( n \), the expectation of the number of inactive symbols is given by

\[
\sum_{t=0}^{K-1} \Pr\{ R^t = 0 | N = n \}.
\]

(21)

Theorem 2 can be modified to compute (21). For inactivation decoding, we define

\[
\Gamma_{c,r|n}^t := \Pr\{ C^t = c, R^t = r | N = n \}.
\]

Let

\[
\Gamma_{c|n}^t := (\Gamma_{c,0|n}, \Gamma_{c,1|n}, \ldots, \Gamma_{c,K-t|n}),
\]

\[(\Gamma_{c|n}^t)^{1+2} := (\Gamma_{c,0|n} + \Gamma_{c,1|n}, \Gamma_{c,2|n}, \ldots, \Gamma_{c,K-t|n}),\]

i.e., \((\Gamma_{c|n}^t)^{1+2}\) is obtained by combining the first two components of \(\Gamma_{c|n}^t\). The expected number of inactive symbols for \(K\) input symbols and \(n\) batches can be expressed as \(\sum_{t=0}^{K-1} \sum_c \Gamma_{c,0|n}\).

Theorem 4. Under the notations and assumption of Theorem 2, we have for inactivation decoding

\[
\Gamma_{c|n}^0 = \text{Bi}(n, c ; 1 - \rho^0) e_1 Q_0^n - c,
\]

and for \(t > 0\),

\[
\Gamma_{c|n}^t = \sum_{c' \geq c} \text{Bi}(c', c ; 1 - \rho^t) (\Gamma_{c'|n}^{t-1})^{1+2} Q_t^{c' - c}.
\]

Proof: First we have \(\Lambda_{c,r|n}^0 = \Gamma_{c,r|n}^0\) by their definitions. For every \(t > 0\), let

\[
\begin{align*}
\Gamma_{c,r|n}^{t(1)} &= \Pr\{ C^t = c, R^t = r, R^{t-1} > 0 | N = n \} \\
\Gamma_{c,r|n}^{t(2)} &= \Pr\{ C^t = c, R^t = r, R^{t-1} = 0 | N = n \}.
\end{align*}
\]

Then we have

\[
\Gamma_{c,r|n}^t = \Gamma_{c,r|n}^{t(1)} + \Gamma_{c,r|n}^{t(2)}.
\]

We can write

\[
\begin{align*}
\Gamma_{c,r|n}^{t(1)} &= \sum_{c', r' > 0} \Pr\{ R^t = r | C^t = c, C^{t-1} = c', R^{t-1} = r', N = n \} \times \\
&\quad \times \Pr\{ C^t = c | C^{t-1} = c', R^{t-1} = r', N = n \} (\Gamma_{c',r'|n}^{t-1}) \Gamma_{c',r'|n}^{t-1},
\end{align*}
\]

where term \((a)\) and \((b)\) can be obtained using Lemma 3 and Lemma 2 respectively, since only normal BP decoding is applied from time \(t - 1\) to \(t\). So similar to the procedure to obtain (12), we have

\[
\Gamma_{c|n}^t = \sum_{c' \geq c} \text{Bi}(c', c ; 1 - \rho^t) (\Gamma_{c'|n}^{t-1})^{1} Q_t^{c' - c},
\]

(22)
where
\[ \Gamma_{c|n}^{(1)} := \left( \Gamma_{c,0|n}, \Gamma_{c,1|n}, \ldots, \Gamma_{c,K-1|n} \right). \]

The term \( \Gamma_{c,r|n}^{(2)} \) corresponds to the case that inactivation has occurred at time \( t \). In such case, an undecoded input symbol is marked as inactive and is treated as decoded in the decoding step from time \( t-1 \) to \( t \). We can write
\[
\Gamma_{c,r|n}^{(2)} = \sum_{c'} \Pr\{ R^t = r | C^t = c, C^{t-1} = c', R^{t-1} = 0, N = n \} \times \Pr\{ C^t = c | C^{t-1} = c', R^{t-1} = 0, N = n \} \Gamma_{c',0|n}^{(t-1)}. 
\]

Since with inactivation decoding, the inactive symbol in the decoding step from time \( t-1 \) to \( t \) can be regarded as the only decodable input symbol in time \( t-1 \), we can obtain \((c)\) and \((d)\) using Lemma 3 with \( r' = 1 \) and Lemma 2 with \( r' = 1 \), respectively. Thus, we have
\[
\Gamma_{c|n}^{(2)} = \sum_{c' \geq c} \text{Bi}(c', c; 1 - \rho') (\Gamma_{c',0|n}^{(t-1)} e_1) Q_{c'}^{n-c}, \tag{23}
\]
where
\[ \Gamma_{c|n}^{(2)} := \left( \Gamma_{c,0|n}^{(2)}, \Gamma_{c,1|n}^{(2)}, \ldots, \Gamma_{c,K-1|n}^{(2)} \right). \]

Combining (22) and (23), the recursive formula of Theorem 4 follows.

When the number of received batches follows a Poisson distribution, the expected number of inactive symbols is given by
\[
\sum_{t=0}^{K-1} \Pr\{ R^t = 0 \}. \tag{24}
\]

We can modify Theorem 3 to compute (24). Define
\[ \Gamma^t := (\Gamma_0^t, \Gamma_1^t, \ldots, \Gamma_{K-1}^t), \]
where
\[ \Gamma^t = \Pr\{ R^t = r \}. \]

The expected number of inactive symbols for Poisson distributed number of received batches can be expressed as
\[ \sum_{t=0}^{K-1} \Gamma_t^t. \]

**Theorem 5.** Under the notations and assumption of Theorem 3 we have the following for inactivation decoding: For \( t \geq 0 \)
\[
\Gamma^t = \exp(-\bar{n} \sum_s p_s^t) (\Gamma^{t-1})^{1+2} \exp(\bar{n} \sum_s p_s^t Q_t),
\]
where \((\Gamma^{-1})^{1+2} := e_1.\]
Though the objective functions are linear, both optimization problems are difficult to solve since they are not convex due to the non-linear equality constraints.

For inactivation decoding, we can optimize the degree distribution by minimizing the expected number of inactivation symbols of inactivation decoding. For BP decoding, we can optimize the degree distribution by minimizing the error probability as follows:

VII. Finite-length degree distribution optimization

The degree distribution obtained using the optimization \((6)\) is guaranteed to have good performance for sufficiently large block length by Theorem 1. However, this degree distribution does not perform well for relative small block length (see the numerical results in the next section). In this section, we discuss how to design good degree distributions for a finite block length.

Consider the degree distribution of a BATS code with fixed block length \(K\) and batch size \(M\). To find an optimal degree distribution, one may consider to minimize the error probability of BP decoding, or to minimize the expected number of inactive symbols of inactivation decoding. For BP decoding, we can optimize the degree distribution by minimizing the error probability as follows:

\[
\min \sum_{t=0}^{K^\prime-1} \sum_{c} \Lambda_c^{t \mid n} \text{s.t.} \begin{cases} 
\Lambda_0^{t \mid n} = \text{Bi}(n, c; 1 - \rho^0) e_1 Q_0^{n-c} \\
\Lambda_t^{c \mid n} = \sum_{c' \geq c} \text{Bi}(c', c; 1 - \rho^t) (\Lambda_{c'}^{t-1 \mid n})^{c'} Q_t^{c'-c}, \quad t > 0 \\
\sum_d \Psi_d = 1, \quad \Psi_d \geq 0 \quad \text{for} \quad d = 1, \ldots, D.
\end{cases}
\]

For inactivation decoding, we can optimize the degree distribution by minimizing the expected number of inactivation as follows:

\[
\min \sum_{t=0}^{K-1} \sum_{c} \Gamma_c^{t \mid n} \text{s.t.} \begin{cases} 
\Gamma_0^{c \mid n} = \text{Bi}(n, c; 1 - \rho^0) e_1 Q_0^{n-c} \\
\Gamma_t^{c \mid n} = \sum_{c' \geq c} \text{Bi}(c', c; 1 - \rho^t) (\Gamma_{c'}^{t-1 \mid n})^{c'} Q_t^{c'-c}, \quad t > 0 \\
\sum_d \Psi_d = 1, \quad \Psi_d \geq 0 \quad \text{for} \quad d = 1, \ldots, D.
\end{cases}
\]

Though the objective functions are linear, both optimization problems are difficult to solve since they are not convex due to the non-linear equality constraints.

Proof: Theorem 5 can be proved similarly to Theorem 3. The recursive formula in Theorem 4 can be rewritten as

\[ \Gamma_c^{t \mid n} = \sum_{c' \geq c} \text{Bi}(c', c; 1 - \rho^t) (\Gamma_{c'}^{t-1 \mid n}) Q_t^{c'-c}, \]

where \(N_t\) is a \((K - t + 2) \times (K - t + 1)\) matrix such that the first row is \(e_1\) and the latter rows forms the \((K - t + 1) \times (K - t + 1)\) identity matrix. The above equation can be further modified to the form similar to (15):

\[ \Gamma_c^{t \mid n} = \sum_{c' \geq c} \text{Bi}(c', c; 1 - \rho^t) \tilde{N}_t Q_t^{c'-c}, \]

where \(\tilde{N}_t\) is a \((K + 1) \times (K + 1)\) matrix such that the submatrix formed by the last \(K - t + 2\) rows and the last \(K - t + 1\) columns is \(N_t\) and other components are zeros.

The proof can be completed by following the steps after (15).
Instead of solving (25) and (26), we use an heuristic method to design the degree distribution of a BATS code with finite block length. Our method is to modify (6) for finite block length as follows:

\[
\max \theta \text{ s.t. } \begin{cases} 
\Omega(x) + \theta \left[ \ln(1 - x) - \frac{c}{K} (1 - x)^c' \right] \geq 0 \text{ for } 0 \leq x \leq 1 - \eta \\
\sum_d \Psi_d = 1, \, \Psi_d \geq 0 \text{ for } d = 1, \ldots, D,
\end{cases}
\]  

(27)

where \(c\) and \(c'\) are parameters that we can tune. For given values of \(c\) and \(c'\), optimization (27) provides us candidate degree distributions that could have better performance. We then evaluate these degree distributions using our finite-length analysis results to find the best one.

Let us explain the intuition behind optimization (27). From the asymptotic analysis in [2], we know that when \(K\) is large the number of decodable input symbols converges to \(K \theta (1 - x) \Omega(x) + \theta \ln(1 - x)\) where \(x\) is the fraction of the input symbols has been decoded, and \(\theta\) is a parameter related to the design coding rate. For the finite block length case, we hope this number is decreasing linearly with \(x\) and is proportional with the expected number of input symbols that can be recovered by a batch. Therefore, we modify the constraint to

\[
\frac{K}{\theta} (1 - x) [\Omega(x) + \theta \ln(1 - x)] \geq c(1 - x)^{c + 1}.
\]

We see that \(c\) should be proportional to \(\sum_i i h_i\).

Note that (27), when \(M = 1\), is slightly different from heuristic approach of designing degree distribution for Raptor codes [5]. We will show in the next section that if we chose proper parameters, the heuristic method provides much better degree distributions for small block lengths.

VIII. NUMERICAL EVALUATION

In this section, we use an example to demonstrate how to use the finite length analysis and the heuristic approach to optimize the degree distribution. Let us first describe the settings of the example. Let batch size \(M = 16\) and field size \(q = 2^8\). We use a rank distribution\( h = (0, 0, 0, 0, 0, 0, 0.0004, 0.0025, 0.0110, 0.0387, 0.1040, 0.2062, 0.2797, 0.2339, 0.1038, 0.0190, 0.0008)\). (28)

We can calculate that normalized maximum achievable rate \(C = \sum_i i h_i / M\) of BATS codes is 0.7442. The design coding rate of a BATS code is defined as \(R = \frac{(1 - \eta)K}{nM}\), where \(n\) is the number of batches. Another parameter used in evaluating BATS code is overhead, which is the difference between the number of packets used in decoding and the number of input packets before precoding. Define the design (relative) overhead as

\[
\frac{C}{R} - 1 = \frac{\sum_i i h_i}{(1 - \eta)K/n} - 1.
\]

For a good code, the design overhead should be close to small. We set \(\eta = 0.04\) in the following numerical evaluation.
TABLE I
Degree distributions for the rank distribution in (25) with $M = 16$, $q = 2^8$ and $\eta = 0.04$. Degree distribution $\Psi^\infty$ is obtained by solving optimization (6). Degree distribution $\Psi^{196A}$, $\Psi^{392A}$ and $\Psi^{784A}$ are obtained by solving (27) with $c = 15$, $c' = 0.5$. Degree distribution $\Psi^{196B}$, $\Psi^{392B}$ and $\Psi^{784B}$ are obtained by solving (27) with $c = 30$, $c' = 0.25$.

| degree | $\Psi^\infty$ | $\Psi^{196A}$ | $\Psi^{196B}$ | $\Psi^{392A}$ | $\Psi^{392B}$ | $\Psi^{784A}$ | $\Psi^{784B}$ |
|--------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 12     | 0              | 0              | 0.0604         | 0              | 0              | 0              | 0              |
| 13     | 0              | 0.1509         | 0.2597         | 0              | 0.1466         | 0              | 0              |
| 14     | 0.0478         | 0.1134         | 0              | 0.2695         | 0.1196         | 0.1471         | 0.2676         |
| 15     | 0.2665         | 0              | 0              | 0              | 0              | 0.1583         | 0.0051         |
| 17     | 0              | 0.1486         | 0              | 0.0072         | 0.1475         | 0              | 0              |
| 18     | 0              | 0              | 0.2022         | 0.1537         | 0              | 0              | 0.1599         |
| 19     | 0              | 0              | 0              | 0              | 0              | 0.0627         | 0              |
| 20     | 0.1012         | 0              | 0              | 0              | 0              | 0.1251         | 0              |
| 21     | 0.0977         | 0.0909         | 0              | 0              | 0.0981         | 0              | 0              |
| 22     | 0              | 0.0461         | 0              | 0.0377         | 0.0386         | 0              | 0.0333         |
| 23     | 0              | 0              | 0              | 0.0951         | 0              | 0              | 0.0989         |
| 25     | 0              | 0              | 0              | 0.1425         | 0              | 0              | 0              |
| 26     | 0              | 0              | 0              | 0.0031         | 0              | 0              | 0.0813         |
| 27     | 0              | 0.0496         | 0              | 0              | 0.0584         | 0.0624         | 0              |
| 28     | 0.141          | 0.0654         | 0              | 0              | 0.0575         | 0              | 0              |
| 29     | 0              | 0              | 0              | 0.0947         | 0              | 0              | 0.0932         |
| 30     | 0              | 0              | 0              | 0.0121         | 0              | 0              | 0.013          |
| 35     | 0              | 0              | 0              | 0.1054         | 0              | 0              | 0              |
| 36     | 0              | 0.0417         | 0              | 0              | 0.0463         | 0.1092         | 0              |
| 37     | 0              | 0.0539         | 0              | 0.0105         | 0.0496         | 0              | 0.0114         |
| 38     | 0.0899         | 0              | 0              | 0.0785         | 0              | 0              | 0.0774         |
| 39     | 0.0122         | 0              | 0              | 0              | 0              | 0              | 0              |
| 49     | 0              | 0              | 0              | 0.038          | 0              | 0              | 0              |
| 50     | 0              | 0.0681         | 0.04           | 0.005          | 0.0698         | 0.0842         | 0.0049         |
| 51     | 0.0034         | 0.0104         | 0              | 0.0712         | 0.0083         | 0.0008         | 0.0712         |
| 52     | 0.0734         | 0              | 0              | 0              | 0              | 0              | 0              |
| 71     | 0              | 0              | 0              | 0.0002         | 0              | 0              | 0              |
| 72     | 0              | 0.0623         | 0.0592         | 0.0394         | 0.0633         | 0.0581         | 0.0383         |
| 73     | 0.0579         | 0.0014         | 0              | 0.0252         | 0              | 0.0091         | 0.0261         |
| 74     | 0.0061         | 0              | 0              | 0              | 0              | 0              | 0              |
| 111    | 0.0251         | 0.0418         | 0.0319         | 0.0335         | 0.0428         | 0.0357         | 0.0327         |
| 112    | 0.0286         | 0.0093         | 0.0151         | 0.019          | 0.0079         | 0.0178         | 0.0196         |
| 198    | 0              | 0              | 0              | 0.0039         | 0              | 0              | 0.0055         |
| 199    | 0.049          | 0.0423         | 0.0407         | 0.0476         | 0.0402         | 0.0483         | 0.0473         |
| 200    | 0              | 0              | 0              | 0.0018         | 0              | 0              | 0              |
TABLE II

For $K = 196$, $n = 16, 17, \ldots, 20$, three degree distributions $\Phi^\infty$, $\Phi^{196A}$ and $\Phi^{196B}$ are evaluated for both BP decoding and inactivation decoding.

| $n$  | 16   | 17       | 18       | 19       | 20       |
|------|------|----------|----------|----------|----------|
| $P_{\text{error}}(\Psi^\infty)$ | 1.0000 | 0.9997   | 0.9975   | 0.9901   | 0.9754   |
| $P_{\text{error}}(\Psi^{196A})$ | 1.0000 | 0.9994   | 0.9926   | 0.9656   | 0.9064   |
| $P_{\text{error}}(\Psi^{196B})$ | 1.0000 | 0.9995   | 0.9930   | 0.9631   | 0.8920   |
| Expected inact. ($\Psi^\infty$) | 42.70  | 36.88    | 31.79    | 27.45    | 23.82    |
| Expected inact. ($\Psi^{196A}$) | 39.56  | 33.15    | 27.46    | 22.53    | 18.37    |
| Expected inact. ($\Psi^{196B}$) | 38.60  | 32.01    | 26.13    | 21.02    | 16.73    |
| Design overhead | 0.0125 | 0.0758   | 0.1391   | 0.2024   | 0.2656   |

TABLE III

For $K = 392$, $n = 32, 34, \ldots, 40$ three degree distributions $\Phi^\infty$, $\Phi^{392A}$ and $\Phi^{392B}$ are evaluated for both BP decoding and inactivation decoding.

| $n$  | 32   | 34       | 36       | 38       | 40       |
|------|------|----------|----------|----------|----------|
| $P_{\text{error}}(\Psi^\infty)$ | 1.0000 | 0.9997   | 0.9934   | 0.9675   | 0.9228   |
| $P_{\text{error}}(\Psi^{392A})$ | 1.0000 | 0.9994   | 0.9844   | 0.9148   | 0.7859   |
| $P_{\text{error}}(\Psi^{392B})$ | 1.0000 | 0.9994   | 0.9816   | 0.8877   | 0.7078   |
| Expected inact. ($\Psi^\infty$) | 61.19  | 49.06    | 39.09    | 31.32    | 25.53    |
| Expected inact. ($\Psi^{392A}$) | 57.77  | 44.77    | 33.88    | 25.28    | 18.88    |
| Expected inact. ($\Psi^{392B}$) | 56.15  | 42.69    | 31.33    | 22.31    | 15.65    |
| Design overhead | 0.0125 | 0.0758   | 0.1391   | 0.2024   | 0.2656   |

A. Performance of the asymptotically optimal degree distribution

Solving (6) w.r.t. the rank distribution, we obtain degree distribution $\Phi^\infty$ (see the 2nd column of Table I). Though $\Phi^\infty$ is optimal for large $K$, its BP decoding performance for small block length is poor. We evaluate the decoding error probability of BP decoding with $\Phi^\infty$ for $K = 196, 392$ or $784$ (see the second row in Table II, III and IV). Note for $\Phi^\infty$, the maximum possible degree is 199, which is larger than the number of input symbols when $K = 196$. To handle this case, we treat all the degree larger than 196 as 196.

We plot the accumulative distribution functions (CDFs) of the stopping time for several $(K, n)$ pairs w.r.t. $\Phi^\infty$ in Fig. 2. For each value of $K$, we pick two values of $n$ corresponding to a low rate code and a high rate code, respectively. From these figures, we can see that for the high rate codes, the CDF curve increases quickly at the early stage of the decoding process; while for low rate codes, the CDF curve increases quickly at both the early and the late stages. In the place of the CDF curve increases faster, the decoding stops with higher probability.

Though the error probability of BP decoding is large, using inactivation can effectively improve the coding rate. We evaluate the expected number of inactivation for inactivation decoding with $\Phi^\infty$ for $K = 196, 392$ and $784$ (see the 5th row in Table II, III and IV). For $K = 196$, by choosing $n = 16$ we obtain a BATS code with design rate very close to the maximum achievable rate and the extra computation induced by inactivation is still small.
Fig. 2: The cumulative distribution functions (CDF) of the stopping time for degree distribution $\Phi^\infty$, $\Phi^K_B$ and $\Phi^K_A$. 
TABLE IV
For \( K = 784, n = 64, 68, \ldots, 80 \), three degree distributions \( \Phi^\infty, \Phi^{784B} \) and \( \Phi^{784A} \) are evaluated for both BP decoding and inactivation decoding.

| \( n \) | 64  | 68  | 72  | 76  | 80  |
|-------|-----|-----|-----|-----|-----|
| \( P_{\text{err}}(\Phi^\infty) \) | 1.0000 | 0.9998 | 0.9758 | 0.9002 | 0.8138 |
| \( P_{\text{err}}(\Phi^{784A}) \) | 1.0000 | 0.9988 | 0.9540 | 0.8028 | 0.6346 |
| \( P_{\text{err}}(\Phi^{784B}) \) | 1.0000 | 0.9986 | 0.9376 | 0.7217 | 0.4857 |
| Expected inact. (\( \Psi^\infty \)) | 97.21 | 72.31 | 53.48 | 40.50 | 31.86 |
| Expected inact. (\( \Psi^{784A} \)) | 93.73 | 67.63 | 47.57 | 33.76 | 24.87 |
| Expected inact. (\( \Psi^{784B} \)) | 91.35 | 64.49 | 43.69 | 29.47 | 20.66 |
| Design overhead | 0.0125 | 0.0758 | 0.1391 | 0.2024 | 0.2656 |

TABLE V
Numerical results for \( K = 1600 \). When the number of batches follows a Poisson distribution, the mean value of the number of batches is \( n \).

| \( n \) | 136 | 144 | 152 | 160 |
|-------|-----|-----|-----|-----|
| \( P_{\text{err}} \) | 0.9997 | 0.9533 | 0.8085 | 0.6897 |
| \( P_{\text{err}} \) (Poisson) | 0.9561 | 0.8952 | 0.8092 | 0.7152 |
| Expected inact. (Poisson) | 122.67 | 84.39 | 61.35 | 47.23 |
| Expected inact. (Poisson) | 139.14 | 100.38 | 73.17 | 54.81 |

B. Performance of degree distributions optimized for finite block lengths
To further improve the performance, we obtain degree distributions using optimization (27). We use two configurations: In the first configuration \( c = 1 \) and \( c' = 0.5 \). We generate degree distributions for \( K = 196, 392 \) and 784, which are listed in Table I, denoted by \( \Phi^{196A}, \Phi^{392A} \) and \( \Phi^{784A} \), respectively. In the second configuration \( c = 30 \) and \( c' = 0.25 \). We generate degree distributions for \( K = 196, 392 \) and 784, which are listed in Table I, denoted by \( \Phi^{KB}, \Phi^{392B} \) and \( \Phi^{784B} \), respectively.

We compare the performance of these degree distributions in terms of both BP decoding and inactivation decoding (see Fig. 2 and Table II, III and IV). It is clearly that for all the values of \( K \) we test, both degree distributions obtained from (27) outperform \( \Phi^\infty \). Further, the section configuration shows clearly better performance than the first one.

C. Fixed vs Poisson number of batches
For larger block lengths, e.g., \( K = 1600 \), the evaluation of the iterative formulas for a given number of chunks becomes time consuming. For a given number \( n \) of batches, instead of evaluating this code directly, we can assume that the number of batches follows a Poisson distribution with mean \( n \) and evaluate the performance using the iterative formulas of the Poisson case. In Table V and Fig. 3, we compare the results of the formula for fixed number of batches and the formula for Poisson number of batches.
IX. CONCLUDING REMARKS

The recursive formulas in this paper can be easily evaluated numerically using matrix operations. So without heavy simulation, we can directly calculate the error probability of BP decoder and the expected number of inactive symbols. Numerical results show that an asymptotically optimal degree distribution may not work well for small number of input symbols. For a given degree distribution, those recursive formulas can help us to determine the number of batches to received, and the distribution of stopping time also provides hints on how to tune the degree distribution to improve the performance.

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To extend Karp et al.’s approach for BATS codes, we have to calculate the distribution of the number of decodable batches throughout the decoding process. Since decodable batches of different degrees behave differently when the BP decoder steps forward, it is difficult to directly work on the total number of decodable batches. Instead, we will keep track the numbers of decodable batches of different degrees separately. More precisely, given an integer $c$ and a vector of integers $\mathbf{o} = (o_1, \ldots, o_M)$, we are interested in the probabilities

$$P_{c, \mathbf{o}/n} := \Pr\{C^t = c, \mathbf{O}^t = \mathbf{o}, O^t > 0, \tau < t | N = n\},$$

where $O^t := \sum_{s=1}^{M} O_s^t$, $\mathbf{O}^t := (O_1^t, \ldots, O_M^t)$ and $O_s^t := |\bar{\Theta}_s^t|$ is the number of decodable batches of degree $s$ at time $t$. The condition $\{O^t > 0, \tau < t\}$ means there are decodable batches of non-zero degrees, therefore the BP decoder does not stop for $\tau < t$. This condition is equivalent to $\{R^t > 0, \tau < t\}$ in Section III.

In analogous to Section III we will express $P_{c, \mathbf{o}/n}^{t'}$ in terms of $P_{c', \mathbf{o}'/n'}$’s for $t' < t$, so that we can calculate $P_{c, \mathbf{o}/n}^{t}$ recursively. Before moving to the details, we have to consider how to choose the input symbol to be decoded in each decoding step. The order of the input symbols being decoded would not affect the error probabilities. However, to facilitate the derivation of the formulae, we always decode an input symbol which contributes to a decodable batches of the least degree. That is, if $s'$ is the integer such that $\bigcup_{s=1}^{s'-1} \bar{\Theta}_s = \emptyset$ and $\bar{\Theta}_{s'} \neq \emptyset$, then the input symbol decoded in the decoding step from $t-1$ to $t$ must belong to the set $\bigcup_{j \in \bar{\Theta}_{s'}} A_j^{t-1}$.

Under the assumption above, we have the following theorem:

**Theorem 6.** Given the number $K$ of input symbols, the number $n$ of batches, the degree distribution $\{\Psi_d\}$, the rank distribution $\{h_r\}$ of the transfer matrix, the maximum degree $D$ and the batch size $M$. 

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When \( t = 0 \),

\[
\Pr\{C^t = c, O^t = o | N = n\} = \text{Bi}(n, c; 1 - \rho^t) \frac{(n - c)!}{(n - c - \sum s o_s)!} \left( \frac{\rho^t}{p^t} \right)^{n-c-\sum s o_s} \prod_{s=1}^{M} \left[ \frac{1}{o_s!} \left( \frac{p^s}{p^t} \right)^{o_s} \right].
\]

if \( n - c - \sum s o_s \geq 0 \). Otherwise the probability equals to 0.

For any \( t > 0 \), if \( \sum s o'_s > 0 \), we have

\[
\Pr\{C^t = c, O^t = o | C^{t-1} = c', O^{t-1} = o', \tau < t - 1, N = n\} = \sum_{(a, b) \in S(c, o, c', o')} \text{Bi}(c', c; 1 - \rho^t) \frac{(c' - c)!}{b_0!} \left( \frac{\rho^t}{p^t} \right)^{b_0} \prod_{s=1}^{M} \left[ \frac{1}{b_s!} \left( \frac{p^s}{p^t} \right)^{b_s} \text{Bi}(o'_s - o'_s, a_s; 1 - \frac{s}{K - t + 1}) \right],
\]

where \( p^t := \sum_{s=0}^{M} p^t_s \), \( s = 1 \) if \( s = \min_s (o'_s \neq 0) \) and \( \delta_s = 0 \) otherwise. \( S(c, o, c', o') \) is the set of non-negative integer vectors \( (a_1, \ldots, a_M, b_0, \ldots, b_M) \) that satisfy the equations:

\[
\begin{align*}
c &= c' - (b_0 + b_1 + \cdots + b_M) \\
o_s &= (a_s - \delta_s) + b_s + (o'_{s+1} - a_{s+1} + \delta_{s+1}), \quad s = 1, \ldots, M - 1 \\
o_M &= (a_M - \delta_M) + b_M
\end{align*}
\]

If \( \sum_s o'_s = 0 \), then the probability equals to 0.

Outline of Proof: We will only proof the case for \( t > 0 \). Observe that we have the following partitions for \( \Theta^{t-1} \) and \( \Theta^t_s \):

\[
\begin{align*}
\Theta^{t-1} &= \Theta^t \cup (\cup_{s=0}^{M}(\Theta^t_s \cap \Theta^{t-1})) \\
\Theta^t_s &= (\Theta^t_{s-1} \cap \Theta^t_s) \cup (\Theta^t_s \cap \Theta^{t-1}) \cup (\Theta^t_{s+1} \setminus \Theta^t_s), \quad s = 0, \ldots, M - 1 \\
\Theta^t_M &= (\Theta^t_{M-1} \cap \Theta^t_M) \cup (\Theta^t_M \cap \Theta^{t-1})
\end{align*}
\]

Hence it is sufficient to compute the probability distribution for \( |\Theta^t|, |\Theta^t_s \cap \Theta^{t-1}| \) and \( |\Theta^t_{s-1} \cap \Theta^t_s|, s = 0, \ldots, M \).

First, from the proof in Subsection [IV-B] we have already known that

\[
\Pr\{j \in \Theta^t | j \in \Theta^{t-1}\} = 1 - \rho^t
\]

and for \( s = 0, 1, \ldots, M \),

\[
\Pr\{j \in \Theta^t_s | j \in \Theta^{t-1}\} = \frac{p^t_j}{\sum_{s} p^t_s} \rho^t.
\]

Therefore we can compute the joint distribution of \( |\Theta^t| \) and \( |\Theta^t_s \cap \Theta^{t-1}| \)'s as a multinomial distribution.

Secondly, let \( j \in \Theta^s_s \). If the input symbol decoded in the decoding step from \( t - 1 \) to \( t \) belongs to \( A_j \), then \( j \in \Theta^s_{s-1} \). Otherwise, \( j \in \Theta^s_s \). Using this fact, it is not difficult to show that, for \( s = 1, \ldots, M \),

\[
\Pr\{j \in \Theta^s_{s-1} | j \in \Theta^t_s\} = \frac{s}{K - t + 1} \quad \text{and} \quad \Pr\{j \in \Theta^t_s | j \in \Theta^t_s\} = 1 - \frac{s}{K - t + 1}.
\]

From these two equalities, and also note that \( \Theta^t_{s'} \cap \Theta^t_{s'-1} \neq \emptyset \) when \( s' = \min_s (o'_s \neq 0) \), we can express the distribution of \( |\Theta^t_{s'-1} \cap \Theta^t_s| \) and \( |\Theta^t_{s'-1} \setminus \Theta^t_s| \) in terms of binomial distributions for each \( s \).

Finally, the steps above will provide us an expression for the conditional probability

\[
\Pr\{|\Theta^t| = c, |\Theta^t_s \cap \Theta^{t-1}| = b_s, 0 \leq s \leq M, |\Theta^t_{s'} \cap \Theta^t_{s'-1}| = a_{s'}, 1 \leq s' \leq M \ | C^{t-1} = c', O^{t-1} = o', O^t > 0, \tau < t - 1, N = n\}.
\]
Summing over the conditional probabilities which the numbers \((a_1, \ldots, a_M, b_0, \ldots, b_M)\) are a possible combination for \(\{C^t = c, O^t = o\}\), i.e. \((a, b) \in S(c, o, c', o')\), we get the desired result.

**Corollary 1.**

\[
P_{c,o|n}^t = \sum_{c',o'} P_{c',o'|n}^{t-1} \sum_{(a, b) \in S(c, o, c', o')} Bi(c', c; 1 - \rho^t) \left( \frac{c' - c}{b_0} \right)! \left( \frac{p_0}{p^t} \right) b_0 \prod_{s=1}^{M} \left[ \frac{1}{b_s} \left( \frac{p_s^t}{p^t} \right)^{b_s} Bi(o_s' - \delta_s, a_s; 1 - \frac{s}{K-t+1}) \right].
\]

We can also express the recursive formula in terms of polynomials.

**Corollary 2.** Let

\[
P_n^t(x_1, \ldots, x_M, y) = \sum_{c,o} P_{c,o|n}^t (x_1)^{o_1} \cdots (x_M)^{o_M} y^{c'},
\]

then we have

\[
P_n^0(x_1, \ldots, x_M, y) = \left[ (1 - \rho^0) y + \rho^0 \left( \frac{p_0}{p^0} x_0 + \frac{p_1}{p^0} x_1 + \cdots + \frac{p_M}{p^0} x_M \right) \right]^n,
\]

and for \(t > 0\),

\[
P_n^t(x_1, \ldots, x_M, y) = \sum_{s=1}^{M} x_{s-1} \frac{P_n^{t-1}(0, \ldots, 0, \alpha_s^t, \ldots, \alpha_{M-s}^t, \beta^t) - P_n^{t-1}(0, \ldots, 0, \alpha_s^t, \ldots, \alpha_{M-s}^t, \beta^t)}{x_s},
\]

where \(x_0 = 1\), \(\alpha_s^t = \left(1 - \frac{s}{K-t+1}\right) x_s + \frac{s}{K-t+1} x_{s-1}\), and \(\beta^t = (1 - \rho^t) y + \rho^t \left( \frac{p_0^t}{p^t} x_0 + \frac{p_1^t}{p^t} x_1 + \cdots + \frac{p_M^t}{p^t} x_M \right)\).

To implement the above approach for the finite length analysis of BATS code, we will have to compute the \(O(n^{M+1})\) quantities \(P_{c,o|n}^t\) for each \(t\), where \(n\) is the number of batches. Hence the evaluation complexity for this approach is at least \(O(n^{M+1})\).