Analytic expressions for the moving infinite line source model

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ABSTRACT

Groundwater flow can have a significant impact on the thermal response of ground heat exchangers. The moving infinite line source model is thus widely used in practice as it considers both conductive and advective heat transport processes. Solution of this model involves a relatively heavy numerical quadrature. Contrarily to the infinite line source model, there is currently no known first-order approximation that could be useful for many practical applications. In this paper, known analytical expressions of the Hantush well function and generalized incomplete gamma function are first revisited. A clear link between these functions and the moving infinite line source model is then established. Then, two new exact and integral-free analytical expressions are proposed, along with two new first-order approximations. The new analytical expressions proposed take the form of convergent power series involving no recursive evaluations. It is shown that relative errors less than 1% can be obtained with only a few summands. The convergence properties of the series, their accuracy and the validity domain of the first-order approximations are also presented and discussed.

1. Introduction

By promoting advective heat transfer, groundwater flow can have a significant impact on ground-source heat pump and borehole thermal energy storage systems. Indeed, research illustrating how groundwater flow can influence mean fluid temperatures (Chiasson et al., 2000), thermal performances (Fan et al., 2007; Zanchini et al., 2012; Nguyen et al., 2017), geometrical layouts (Choi et al., 2013) or operation costs (Capozza et al., 2011; Samson et al., 2018) of these systems are numerous. The result of thermal response tests being also impacted by the advection of groundwater (Signorelli et al., 2007; Chiasson and O’Connell, 2011; Angelotti et al., 2014), much work has been devoted to parameters identification with various interpretation models (Raymond et al., 2011; Wagner et al., 2013; Antelmi et al., 2020; Magraner et al., 2021).

It results that if a significant groundwater flow is present in the aquifer, the design and analysis of ground heat exchangers (GHE) must then be accomplished with simulation models considering both conduction and advection of groundwater. Such a model, known as the moving infinite line source (MILS) model, was first proposed by Carslaw and Jaeger (1959, p. 267) for steady-state. The first known extension to the time-dependent case is due to Zubair and Chaudhry (1996) who also proposed some other interesting solutions and established a clear link with the generalized incomplete gamma function $\Gamma(0, x; b)$. The radial evolution of the temperature around the line source was studied by Zeng et al. (1997) who observed greater temperature variations downstream of the heat source. However, the first applications of the MILS to GHEs are due almost simultaneously to Sutton et al. (2003) and Diao et al. (2004) for design and analysis of ground-source heat pump systems. Nowadays, the general form of the MILS model is:

$$\Delta T = \frac{q}{4\pi k} e^{-r/v_T} \int_{r^2/4k}^{\infty} \frac{1}{\psi} e^{-\frac{1}{\psi}} \left(\frac{C_T}{4\psi}\right)^2 d\psi$$  \hspace{1cm} (1)

with the effective heat transport velocity $v_T$ given by

$$v_T = \frac{C_r}{C} v_D$$ \hspace{1cm} (2)

Solution of Eq. 1 is illustrated in Fig. 1 for various coordinates. One can see the thermal plume, which extends mainly in the flow direction. The mean temperature being useful for design purposes, Sutton et al. (2003) and Diao et al. (2004) used the identity $\int_0^{2\pi} e^{x\cos \theta} d\theta = 2\pi I_0(x)$ to predict it on a circle of radius $r$ centered on the line source. The resulting mean temperature is then given by

$$\overline{\Delta T} = \frac{q I_0(v_T/2\alpha)}{4\pi k} \int_{r^2/4k}^{\infty} \frac{1}{\psi} e^{-\frac{1}{\psi}} \left(\frac{C_T}{4\psi}\right)^2 d\psi$$  \hspace{1cm} (3)

where $I_0$ is the modified Bessel function of the first kind and of order 0.

Equation 3 gives the mean temperature change around an infinite line-heat source embedded in a homogeneous,
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Fig. 1: Illustration of the thermal plume caused by groundwater advection downstream of an infinite line heat source. Warm temperatures correspond to red colors.

isotropic and infinite media where no hydromechanic dispersion is active. It is important to note that the thermal parameters \( k, C \) and \( \alpha \) represent equivalent values, that is weighted averages of the thermal parameters of the solid and fluid (Nield and Bejan, 2006). Thus, \( \Delta T \) is obtained under the assumption of local thermal equilibrium, which implies that the temperature of the fluid and mineral phases are equal locally. A special attention should also be paid to the fact that \( v_T \) is based on the groundwater flux or Darcy velocity \( v_D \) (Bear, 1979), and not on the actual pore water velocity.

In order to widen its application field, several improvements have been brought to the MILS model. Molina-Giraldo et al. (2011b) extended the model to simulate linesources of finite length. Metzger et al. (2004) added the effect of hydromechanic dispersion in the porous media, a development used later by Molina-Giraldo et al. (2011a) to simulate GHEs response. Rivera et al. (2016) added a Cauchy boundary condition along the ground surface to better reproduce surface conditions. Hu (2017) and Erol and François (2018) proposed models to take into account the multilayer case, thus lifting the assumption of a homogeneous media. Finally, Van de Ven et al. (2021) recently developed a correction function to the MILS model to take into account the impervious grout put in place in GHEs.

So far, the MILS model involves an integral for which no closed-form expression is known in the geoexchange community. Thus, solving this integral requires a numerical quadrature that prevents using widely available spreadsheets and limits the efficiency of optimization methods for GHE design or thermal response test analysis. Lack of simple explicit analytical expressions also limits the physical interpretation of the MILS model. However, several analytical expressions, that went unnoticed to the geoexchange community, can ease and accelerate the use of the MILS model. Indeed, in the field of hydrogeology the integral of Eq. 3 is known as the Hantush well function (Hantush and Jacob, 1955), a well-known integral used to describe the hydraulic head within a leaky confined aquifer. The first analytical solutions to this integral were already given in the seminal paper of Hantush and Jacob (1955) and took the form of a combination of power series and special functions. Since, more practical expressions in the form of analytic expressions, asymptotic expansions and approximations have been proposed. Veling and Maas (2010) realized a comprehensive technical review of the Hantush integral and its various solutions. The expressions proposed so far vary in terms of convergence properties and compactness levels, but the two compact solutions proposed by Hunt (1977) involve the generalized incomplete gamma function \( \Gamma(0, x; b) \) and merits to be known.

Apart from the noticeable work of Capozza et al. (2013) who exploited the properties of \( \Gamma(0, x; b) \) to foster the evaluation of the temperature field in a GHE, the relationship with \( \Gamma(0, x; b) \) has gone quite unnoticed by the geoexchange community. Also, contrarily to the infinite line source model, there is currently no known first-order approximation of the MILS that could be useful in practice. In this paper, we first aim to revisit and make a clear link between the MILS model, the Hantush well function and the generalized incomplete gamma function. Then, based on Hunt’s solutions, two new exact and integral-free analytical expressions to the MILS model are proposed, along with two useful first-order approximations of the MILS model

2. Hantush Well function and Hunt’s solutions

In hydrogeology, the integral of Eq. 3 is known as the Hantush well function (Hantush and Jacob, 1955) and is used to predict the hydraulic head within leaky confined aquifers. Following the work of Simon et al. (2021), doing the change of variable

\[
\psi = \frac{r^2}{4at}
\] (4)

and

\[
\beta = \frac{2\alpha}{v_T}
\] (5)

allows to reconcile the MILS model and the usual form of the Hantush well function given by:

\[
W(u, r/\beta) = \int_{u}^{\infty} \psi^{-1} e^{-\frac{\psi^2 r^2}{4at}} d\psi
\] (6)

Now, by making the change of variable:

\[
b = \frac{1}{4} \left( \frac{r}{\beta} \right)^2 = \frac{1}{4} \left( \frac{r v_T}{2\alpha} \right)^2
\] (7)
the parallel between $W$ and the generalized incomplete gamma function described by Chaudhry and Zubair (1994) becomes clear. The latter reads

$$
\Gamma(a, u; b) = \int_0^\infty \psi^{a-1} e^{-\psi - \frac{b}{\psi}} d\psi
$$

and it is clear that the Hantush well function $W$ corresponds to the case $a = 0$. Note that for $b = 0$, $\Gamma(a, u; b)$ reduces to the incomplete gamma function while the case $a = b = 0$ corresponds to the exponential integral function $E_1$.

Two useful solutions to the Hantush integral were proposed by Hunt (1977) a few decades ago. Using the previous definition of $b$ and expressing the factor $e^{-b/\psi}$ in Eq. 6 by his power series, one obtains:

$$
W(u, b) = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} \int_0^\infty \psi^{-n-1} e^{-\psi} d\psi
$$

$$
= \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} u^{-n} \int_1^\infty \psi^{-n-1} e^{-\psi} d\psi'
$$

$$
= \sum_{n=0}^{\infty} \frac{(-b/u)^n}{n!} E_{n+1}(u), \ u > 0
$$

where $E_n$ is the generalized exponential integral of order $n$. Hunt (1977) mentioned initially that the power series is absolutely convergent for $0 < b/u < \infty$. This convergence range was however reduced to $u \geq \sqrt{b}$ by Veling and Maas (2010).

A second expression provided by Hunt (1977) is obtained by substituting $\psi' = b/\psi$ in Eq. 6. One can then derive

$$
W(u, b) = \int_0^{b/u} \psi'^{-1} e^{-\psi'^{-1} - \frac{b}{\psi'}} d\psi'
$$

$$
= 2K_0 \left(\frac{2\sqrt{b}}{u}\right) - \int_{b/u}^{\infty} \psi'^{-1} e^{-\psi'^{-1} - \frac{b}{\psi'}} d\psi'
$$

$$
= 2K_0 \left(\frac{2\sqrt{b}}{u}\right) - \sum_{n=0}^{\infty} \frac{(-u)^n}{n!} E_{n+1}(b/u), \ u \geq 0
$$

where $K_0$ is the modified bessel function of the second kind and of order 0. Hunt (1977) mentioned that the power series is absolutely convergent for $0 \leq u < \infty$. Equation 10 was however deemed appropriate by Veling and Maas (2010) for $0 < u < \sqrt{b}$.

Implementation of formulas 9 and 10 is rather easy since the generalized exponential integral function $E_n(x)$ can be obtained recursively with $E_{n+1}(x) = (e^{-x} - xE_n(x))/n$. Since $E_1(x)$ corresponds to the exponential integral function, an evaluation of $E_1$ is needed for $n = 0$ and the higher-order terms of $E_n(x)$ are then obtained recursively. For most practical applications, $u$ is not a scalar but a vector whose entries correspond to the various evaluation times. To benefit from the parallelism of modern computers, one has then advantage to use formulas 9 and 10 under a vectorized form.

3. New analytical expressions

The recursive evaluation of $E_n$ in Eqs. 9 and 10 does not facilitate their use in spreadsheets and efficiency gains are probably possible by reducing the number of operations achieved on vectors. In this section, two new power series are presented. To ease manipulation and identification of these series, the change of variable $\tau = 4at/r^2$, that is four times the Fourier number, has been made. This leads to:

$$
\tau = \frac{1}{u} = \frac{4at}{r^2}
$$

By setting the characteristic length equal to the radial distance $r$, one can easily establish a link with the Fourier number ($Fo = at/r^2$) and the thermal Péclet number ($Pé = rv_T/a$). Indeed, $\tau$ is related to the Fourier number by $\tau = 4Fo$ and expresses the conduction rate to the rate of heat storage. Similarly, the parameter $b$ is related to the Péclet number by $b = (Pé/r^2)^2$ and expresses the advective heat transfer rate to the conductive heat transfer rate. Although the dimensionless thermal numbers $Fo$ and $Pé$ are widely used, the use of variables $\tau$ and $b$ leads to more compact analytical expressions. For conciseness, the dimensionless numbers $Fo$ and $Pé$ will therefore not be used in this Section.

3.1. Analytical expression for $0 < \tau \leq 1/b$

Using the dimensionless variables $\tau$ and $b$ in Eq. 9 and expanding the sum gives:

$$
W(\tau, b) = \sum_{n=0}^{\infty} \frac{(-b\tau)^n}{n!} E_{n+1}(1/\tau)
$$

and

$$
W(\tau, b) = + \frac{E_1(1/\tau)}{1} - \frac{b\tau E_1(1/\tau)}{1} + \frac{b\tau^2 E_2(1/\tau)}{2} - \frac{(b\tau)^3 E_3(1/\tau)}{6} + \frac{(b\tau)^4 E_4(1/\tau)}{24} - \ldots
$$

The order of the generalized exponential integral $E_n$ can be lowered using the recurrence relation $(e^{-x} - x E_n(x))/x = E_{n+1}(x)$. Thus, using this relation iteratively to get rid
of the generalized exponential integral of order \( n > 1 \)
leads to the following continued fraction:
\[
W(\tau, b) = E_1(1/\tau) + \frac{b^2}{4} + \frac{b^4}{36} + \frac{b^4}{576} + \ldots
\]
Expanding the previous expression and collecting the terms separately involving \( E_1(1/\tau) \) and \( e^{-1/\tau} \) now gives:
\[
W(\tau, b) = E_1(1/\tau) \left( 1 + b + \frac{b^2}{4} + \frac{b^3}{36} + \frac{b^4}{576} + \ldots \right)
+ e^{-1/\tau} \left( -b + \frac{b^2}{4} - \frac{b^3}{36} + \frac{b^4}{576} + \frac{b^5}{36} + \frac{b^7}{576} - \frac{b^9}{288} + \frac{b^{10}}{96} - \ldots \right)
\]
The polynomial that multiplies \( E_1(1/\tau) \) in Eq. 15 can be expressed easily with a power series
\[
W_1 = E_1(1/\tau) \sum_{n=0}^{\infty} \frac{b^n}{n!^2}
\]
For the special case \( k = 0 \), the modified bessel function of the first kind \( I_k(z) \) comes down to the power series
\[
I_0(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!^2}
\]
Thus, \( W_1 \) can be simplified further by setting \( z = 2\sqrt{b} \), which gives
\[
W_1 = E_1(1/\tau) I_0(2\sqrt{b})
\]
The second polynomial in Eq. 15 is an alternating power series that can be expressed by a double summation. The second term then simplifies to:
\[
W_2 = e^{-1/\tau} \left( \sum_{m=0}^{\infty} (-\tau)^{m+1} m! \left( \sum_{n=m+1}^{\infty} \frac{b^n}{n!^2} \right) \right)
\]
It is worthy noting that the summation will converge to a negative value. \( W_2 \) will however tends towards zero for small evaluation times \( \tau \) since \( e^{-1/\tau} \rightarrow 0 \). On the contrary, for long evaluation times, \( e^{-1/\tau} \rightarrow 1 \) and \( W_2 \) will take a negative value.

Finally, a first analytical expression for the MILS model involving special functions \( E_1 \) and \( I_0 \), as well as a double infinite summation is thus provided by:
\[
\Delta T = \frac{q I_0(2\sqrt{b})}{4\pi k} \left( E_1 \left( \frac{1}{\tau} \right) I_0(2\sqrt{b}) \right)
+ e^{-1/\tau} \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} (-\tau)^{m+1} m! \frac{b^n}{n!^2}
\]
It is interesting to note that without groundwater flow, \( b = 0 \), \( I_0(0) = 1 \) and Eq. 20 corresponds exactly to the infinite line source model of Ingersoll et al. (1954).

For \( b > 0 \), \( I_0(2b^{1/2}) > 1 \) and the temperature plateau typically observed at the long-term (see Fig. 2) will be controlled mostly by the negative values taken by \( W_2 \). We will show in Section 4 that this equation is appropriate for \( 0 < \tau \leq 1/b \).

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+ e^{-1/\tau} \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} (-\tau)^{m+1} m! \frac{b^n}{n!^2}
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### 3.2. Analytical expression for \( \tau \geq 1 \)

A second expression is obtained by developing the summation in Eq. 10 and reducing the order of the generalized exponential integral until only \( E_1 \) remains. Finally, collecting separately the terms involving \( E_1(br) \) and \( e^{-br} \) leads to:
\[
\sum_{n=0}^{\infty} (-\tau)^n n! E_{n+1}(br) =
E_1(br) \left( 1 + b + \frac{b^2}{4} + \frac{b^3}{36} + \frac{b^4}{576} + \ldots \right)
+ e^{-br} \left( \frac{1}{18} + \frac{b^2}{35} + \frac{b^3}{288} + \ldots \right)
\]
Using again the identity of Eq. 17 allows simplifying the first polynomial of Eq. 21 to
\[
W_1 = E_1(br) I_0(2\sqrt{b})
\]
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The second polynomial can be expressed by a double summation and the resulting expression is

\[ W_2 = e^{-br} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{b^n (m - 1)!}{(-r)^m (m + n)!} \]  \hspace{1cm} (23)

which can be simplified further using the generalized hypergeometric function \( _1F_2 \):

\[ W_2 = e^{-br} \sum_{m=1}^{\infty} \frac{1}{m!} _1F_2 \left( 1; m + 1, m + 1; b \right) (-r)^m \] \hspace{1cm} (24)

Numerical tests indicated that Eq. 23 was 1 to 2 orders of magnitude faster than Eq. 24. For performance purposes, a second expression for \( \overline{T} \) is thus provided by:

\[ \Delta \overline{T} = \frac{q I_0 \left( 2 \sqrt{b} \right)}{4\pi k} \left( 2K_0 \left( 2\sqrt{b} \right) - E_1 (br) I_0 \left( 2\sqrt{b} \right) \right) - e^{-br} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{b^n (m - 1)!}{(-r)^m (m + n)!} \]  \hspace{1cm} (25)

As \( E_1 (0) = \infty \), it is this time impossible to identify the infinite line source model for the special case \( b = 0 \). The physical interpretation of Eq. 25 is however instructive. For long evaluation times \( r \to \infty \), \( W_1 \) and \( W_2 \) will be nil and the steady-state temperature plateau will be equal to \( \overline{T} = q I_0 (2b^{1/2}) K_0 (2b^{1/2}) / 2\pi k \), a result already obtained by Sutton et al. (2003) and Diao et al. (2004). Since \( _1F_2 \left( 1; m + 1, m + 1; b \right) \to 1 \) as \( m \to \infty \),
the alternating series test indicates again convergence of the power series for \( r \geq 1 \) and for any \( b \) value. We will show in Section 4 that Eq. 25 is appropriate for \( r \geq 1 \).

### 3.3. First-order approximations

First-order approximations are often useful for practical applications. In particular, they allow analysis of thermal response tests by simplifying the temperature (Mogensen, 1983) or time derivative response (Pasquier, 2018). Such approximations are obtained for the MILS model using a single summand in the power series and the relations \( I_0(x) \approx 1 + x \), \( K_0(x) \approx -2 \ln(x) \) and \( E_1(x) \approx -\ln(x) - \gamma \). Using these relations, one can derive the following first-order approximation easily from Eqs. 18 and 19:

\[
\tilde{W} = (\ln (r) - \gamma)(1 + b) - b e^{-\frac{1}{r}} \tag{26}
\]

Similarly, a second approximation is derived from Eqs. 22 and 23:

\[
\tilde{W} = (\ln (br) + \gamma)(1+b) + \frac{1}{r}e^{-br} - 2 \ln \left( 2\sqrt{b} \right) \tag{27}
\]

Note that the same formulas would have been obtained using Eqs. 9 and 10 with two summands, and by expressing \( E_2 \) as a function of \( E_1 \) with the recursion formula presented earlier. The accuracy of these approximations will be presented in the next Section.

### 4. Numerical evaluation and efficiency

The mean temperature change \( \Delta \bar{T}(b, r) \) obtained by Eqs. 20 and 25 for a finite number of summands are illustrated in Fig. 2 for various values of \( b \) and for \( 10^{-2} \leq r \leq 10^{6} \). This range covers a few minutes to several years for radial distances \( r \) and thermal diffusivities \( \alpha \) typical of geothermal applications. Both the inner and outer summations were evaluated with the same number of terms. In addition, the relative error \( \varepsilon \) made by Eqs. 20 and 25 with respect to quadrature of Eq. 3 is shown in Fig. 4 for partial sums made of 1, 2, 10 and 50 summands. Note that both figures contain a vertical line at \( r = 1/b \) or \( r = 1 \) that represents the convergence criteria of the power series mentioned previously. These values were chosen to ensure an error less than 1% when only 10 summands are used to evaluate the infinite power series. Recall that \( r = 4 Fo \) and \( b = (P \dot{e}/4)^2 \). The results presented hereinafter can therefore also be expressed as a function of the dimensionless numbers \( Fo \) and \( P \dot{e} \).

One can see easily in Fig. 2 and 4 that both analytical expressions converge rapidly with an error inferior to 1% when only 10 terms are used to approximate the infinite power series. As illustrated in Fig. 3, the summands decrease and reach machine precision rapidly. A summand is then evaluated numerically equal to zero and adding the subsequent terms of the series is then useless. This confirms that in practice just a few terms are required to obtain a good numerical accuracy. However, this result is only valid if the convergence criteria are met. Otherwise, the series increase rapidly and diverge with relative errors that could easily exceed \( 10^{10}\% \). For the special case \( b = 0 \), Eq. 25 will return an infinite value due to the evaluation of \( E_1(0) \) and \( K_0(0) \). Formula 20, which simplifies to the infinite line source model for \( b = 0 \), should then be used. We mention that we observed similar convergence properties and relative errors with Hunt’s original solutions (Eqs. 9 and 10). One can then use both approaches. It is interesting to note that the series are valid for \( r \leq 1/b \) or \( r \geq 1 \) and overlap over a significant validity range for most values of \( b \). In practice, one can then use the convergence criteria to create two smaller \( r \) sub-vectors and thus accelerate the evaluation of equations 20 and 25. Combination of the individual solutions allows easily to cover the full simulation span. A Matlab implementation of Eqs. 20 and 25 has been developed (see Appendix). The mean computation time for a vector \( r \) composed of 10 000 time steps with \( n_y = 10 \) was approximately 13 ms, which is 200 times faster than the numerical quadrature of the Hantush well function. The implementation Eqs. 20 and 25 was 15 to 20% faster than Hunt’s solutions, a relatively small but significant real gain.

Finally, a comparison between the first-order approximation of the Hantush well function and its reference value is illustrated in Fig. 4 for several values of \( b \). If we except the case \( b = 10 \), the approximations
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Fig. 4: Relative absolute error $\varepsilon$ made on $W$ when using Eqs. 20 (left) and 25 (right) with respect to the integral of Eq. 3, for various values of $b$ and number of summands $ns$. The vertical black dashed lines correspond to $\tau = 1/b$ (left) or $\tau = 1$ (right). The thin horizontal black lines correspond to an error of 1%. Dimensionless parameter $b = (P\acute{e}/4)^2$ is proportional to the ratio of advective to conductive heat transfer, while $\tau = 4Fo$ is proportional to the ratio of conductive heat transfer to the rate of heat storage.

provided by Eqs. 26 and 27 appear valid during the conduction-dominant period, that is during the rising part of $W$ and before the plateau is reached. For early times and advective-dominant periods, the approximations clearly failed to reproduce $W$. As shown in Fig. 6, a relative error of less than 10% is however achievable and covers two wide isosceles triangles. A vertical narrow valley of small errors around $\tau = 1$ is also observed for Eq. 27. To guide the use of the approximations, the triangular areas are limited for error levels of 0.1%, 1% and 10% by $\tau \geq \tau_{\text{min}} \cap \tau \leq \tau_{\text{max}}$, where $\log_{10}(\tau_{\text{min}})$ and $\log_{10}(\tau_{\text{max}})$ are given by the parameters listed in Table 1. Using these simple relations directly provides the maximum error made by the first-order approximations, which can be useful for practical or pedagogical applications.

Table 1
Numerical values for $a_0$, $a_1$ and $a_2$ used with $\log_{10}(\tau_{\text{min or max}}) = (a_0 + (\log_{10}(b) - a_1)/a_2)$.

| $\varepsilon$ | Bound | $a_0$ | $a_1$ | $a_2$ |
|---------------|-------|-------|-------|-------|
| 0.1%          | $\tau_{\text{min}}$ | 2.32  | 0     | $\infty$ |
| $\tau_{\text{max}}$ | 0    | -1.04 |       | -0.94 |
| 1%            | $\tau_{\text{min}}$ | 1.52  | 0     | $\infty$ |
| $\tau_{\text{max}}$ | 0    | -0.46 |       | -0.94 |
| 10%           | $\tau_{\text{min}}$ | 0.81  | 0     | 0.08  |
| $\tau_{\text{max}}$ | 0    |       |       | $\infty$ |
| Eq. 26        | 1%   | $\tau_{\text{min}}$ | 0    | -0.53 | -1.07 |
| $\tau_{\text{max}}$ | 0    | -0.55 |       | -0.97 |
| 10%           | $\tau_{\text{min}}$ | 1.22  | 0     | $\infty$ |
| $\tau_{\text{max}}$ | 0    | -0.36 |       | -0.88 |

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5. Conclusions

We presented analytical expressions and approximations for the moving infinite line source model, a widely used thermal model that takes into account groundwater flow. All expressions were derived from the Han-tush well function and generalized incomplete gamma function that are commonly used in hydrogeology to model leaky confined aquifers.

The proposed new analytical expressions are exact, integral-free and take the form of convergent power series involving no recursive evaluations. We have shown that for most cases, providing the convergence domain is respected, evaluation of the series with only a few summands provides efficiently a high accuracy.

New first-order approximations were also presented along with their validity domain. It was shown that a relative error less than 10% can be obtained easily with these approximations which can be useful for many professional or academic activities.

To conclude, we stress that the approach used in this work is not limited to the moving infinite line source model and could probably lead to new analytical expressions for other models that currently require a heavy numerical quadrature. Future work could then lead to new series expansion useful to the design of ground heat exchangers.

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Matlab Implementation of the MILS model

function [dT]=MILS(t,k,Cs,Cw,vD,r,q,ns)
% t : Column vector of evaluation times [s]
% k : Thermal conductivity [W/(mK)]
% Cs: Soil volumetric capacity [J/(m³ K)]
% Cw: Water volumetric capacity [J/(m³ K)]
% vD: Darcy velocity/groundwater flux [m/s]
% q: Normalized heat flux [W/m²]
% r: Radial distance [m]
% n: Number of summands [-]
% T: Mean temperature change [K]. The columns correspond to Eq. 20 and 24 respect.
% rv: Radial distance [m]
% I: Fourier number (at/r²) [-]
% Péclet number (rvF/α) [-]
% α: Ground thermal conductivity [W/(mK)]
% K₀: Modified bessel function of the second kind and of order 0 [-]
% m: Summation index [-]
% n: Summation index [-]
% P: Péclet number (rvF/α) [-]
% q: Normalized heat load [W/m²]
% r: Radial distance [m]
% u: r²/4αt [-]
% vD: Darcy velocity or groundwater flux [m/s]
% vT: Effective heat transport velocity [m/s]
% W: Hantush well function [-]

% Implementation of Eq. 20
S1=sum((repmat(b.^n./factorial(n),.^2,ns,1).*id,1));
S2=sum((-tau).^((m+1).*factorial(m)).*S2,2);
T(:,1)=a0.*exp(int(1./tau))+10.*exp(-1./tau).*S1;

% Implementation of Eq. 25
S2=b.^((n-1)./factorial ((m+1)+(n-1)).);^2;
S1=sum(factorial(m)./-tau).^((m+1).+sum(S2,1,2);
T(:,2)=a0*2*besseli(0,2.*sqrt(b))...
exp(int(b*tau)+10.*exp(-b*tau).*S1);

end

% end of function

Nomenclature

Acronyms

GHE Ground heat exchanger
MILS Moving infinite line source

Variables

α Thermal diffusivity [m²/s]
β 2α/vT [m]
ΔT Mean temperature change[°C]
ΔT Temperature change [°C]
ψ Integration variable [-]
τ Adimensional time (4αt/r²) [-]
θ Angle with the x-axis [rad]
b (rvF/4α)² [-]
C Ground volumetric heat capacity [J/(m³-K)]
Cw Water volumetric heat capacity [J/(m³-K)]
Eₙ Generalized exponential integral function [-]
F₀ Fourier number (αt/r²) [-]
I₀ Modified bessel function of the first kind and of order 0 [-]

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