ON ABELIAN FAMILIES AND HOLOMORPHIC NORMAL PROJECTIVE CONNECTIONS

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INTRODUCTION

Our previous results from [JR04] raise the question whether the list of complex projective manifolds admitting a holomorphic normal projective connection is the following (up to étale coverings):

1.) \( \mathbb{P}_m(\mathbb{C}) \),
2.) smooth abelian families,
3.) manifolds with universal covering \( \mathbb{B}_m(\mathbb{C}) \).

Here \( \mathbb{B}_m(\mathbb{C}) \) denotes the ball in \( \mathbb{C}^{m+1} \), the non compact dual of \( \mathbb{P}_m(\mathbb{C}) \) in the sense of hermitian symmetric spaces. The second point includes the flat case of an abelian manifold.

Any compact Riemann surface admits a holomorphic normal projective connection, this is the famous uniformization theorem. Kobayashi and Ochiai showed that the list of projective surfaces with a holomorphic normal projective connection consists of \( \mathbb{P}_2(\mathbb{C}) \), abelian surfaces and ball quotients ([KO80]). The above list was confirmed in the case of projective threefolds in [JR04].

Of particular interest are the manifolds with a holomorphic normal projective connection of intermediate Kodaira dimension \( 0 < \kappa(M) < m \). The type we expect are locally symmetric spaces obtained as quotients of

\[ \mathbb{C}^{m-n} \times \mathbb{B}_n(\mathbb{C}) \]

by some special group of automorphisms as we will see in this article. Concrete examples are given in the last section.

1. HOLOMORPHIC NORMAL PROJECTIVE CONNECTIONS

Cartan’s original definition of projective structures and connections involve the language of principal bundles. We follow Kobayashi and Ochiai ([KO80]).

Let \( M \) be some \( m \)-dimensional projective manifold. Then \( M \) carries a holomorphic normal projective connection if the (normalised) Atiyah class of the holomorphic cotangent bundle has the form

\[
(1.1) \quad a(\Omega^1_M) = \frac{c_1(K_M)}{m+1} \otimes id_{\Omega^1_M} + \frac{c_1(K_M)}{m+1} \otimes id_{\Omega^1_M} \in H^1(M, \Omega^1_M \otimes T_M \otimes \Omega^1_M)
\]

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where we use the identities $\Omega^1_M \otimes T_M \otimes \Omega^1_M \simeq \text{End}(\Omega^1_M) \otimes \Omega^1_M \otimes \text{End}(\Omega^1_M)$.

The following Chern class identities hold, similar to projective space:

\begin{equation}
\tag{1.2} c_r(M) = \frac{1}{(m+1)^r} \left( \frac{m+1}{r} \right) c'_r(M) \quad \text{in } H^r(M, \Omega^r M)
\end{equation}

It was shown in [MM96] that any holomorphic cocycle solution to (1.1) can be thought of as a $C$–bilinear map

$$\Pi : T_M \times T_M \to T_M$$

satisfying certain rules modelled after the Schwarzian derivative

1.) $$\Pi_{fX} Y = f \Pi_X Y - \frac{1}{m+1} X(f)Y, \quad \text{für } f \in C^\infty(M)$$

2.) $$\Pi_X (fY) = f \Pi_X Y + X(f)Y - \frac{1}{m+1} Y(f)X, \quad \text{für } f \in C^\infty(M)$$

3.) $$\Pi_X Y - \Pi_Y X = [X, Y]$$

We shall not use this description in the sequel.

A manifold is said to carry a projective structure if there exists a holomorphic projective atlas, i.e., an atlas with embeddings of the charts into some $\mathbb{P}_m(\mathbb{C})$ such that the coordinate change is given by restrictions of projective automorphisms.

A manifold with a projective structure carries a (flat) projective connection.

1.3. Example. Projective space $\mathbb{P}_m(\mathbb{C})$ carries a projective structure. Any manifold whose universal covering space admits an embedding into $\mathbb{P}_m(\mathbb{C})$ such that its fundamental group acts by restrictions of projective transformations admits a projective structure. In particular $\mathbb{B}_m(\mathbb{C})$, the non compact dual of $\mathbb{P}_m(\mathbb{C})$, carries a projective structure. Any abelian manifold carries a projective structure. Any Riemann surface carries a projective structure.

We call $\mathbb{P}_m(\mathbb{C})$, ball quotients and étale quotients of abelian manifolds the standard examples of manifolds with a projective structure.

Let $M$ be a any projective manifold with a holomorphic normal projective connection. If $K_M$ is not nef, then $M \simeq \mathbb{P}_m(\mathbb{C})$ ([JR04]). If $K_M$ is nef, then it is expected that some multiple of $|K_M|$ is spanned defining a map

$$f : M \to N.$$ 

This is the famous abundance conjecture. Our results from [JR04] suggest that for $M$ as above $f$ should be a smooth abelian fibration, perhaps after some étale covering.

We should mention that there are more examples if one drops the Kähler condition. Twistor spaces over conformally flat Riemannian fourfolds are complex threefolds with a projective structure. The only Kählerian twistor space is $\mathbb{P}_3(\mathbb{C})$ by a result of Hitchin. There are of course even more non–compact examples.
2. Families of Abelian Varieties

We are dealing with the following situation. Let \( M \) be some projective manifold of dimension \( m \) admitting a smooth holomorphic map
\[
f : M_m \rightarrow N_n
\]
on to some smooth projective manifold \( N \) of positive dimension \( n \). The fibers are assumed to be \( (m - n) \)-dimensional abelian varieties.

We assume the existence of a smooth section. Projectivity of \( M \) is not always necessary, but this is the case we are interested in. The holomorphic one forms on \( M \) and \( N \) give rise to the exact sequence
\[
(2.1) \quad 0 \rightarrow f^* \Omega^1_N \xrightarrow{df} \Omega^1_M \rightarrow \Omega^1_{M/N} \rightarrow 0
\]
where \( \Omega^1_{M/N} \) is the sheaf of relative one forms. Here in this case
\[
E = f^* \Omega^1_{M/N}
\]
is a vector bundle on \( N \) of rank \( m - n \) and \( f^* E \simeq \Omega^1_{M/N} \) via the canonical map \( f^* f_* \Omega^1_{M/N} \rightarrow \Omega^1_{M/N} \), as \( \Omega^1_{M/N} \) is relatively spanned.

2.2. Proposition. In the above situation, assume that \( M \) admits a holomorphic normal projective connection. Then \( N \) has a holomorphic normal projective connection, and
\[
(2.3) \quad a(E(-K^N_n/n+1)) = 0 \quad \text{in} \quad H^1(N, \text{End}(E) \otimes \Omega^1_N),
\]
where \( a \) denotes the Atiyah class of a vector bundle.

2.4. Remark. The formula is in terms of classes, we do not assume the existence of a theta characteristic on \( N \).

Proposition 2.2 will be proved below, we will first derive some consequences. The trace of the Atiyah class gives the first Chern class, hence
\[
(2.5) \quad c_1(E) = \frac{m-n}{n+1} c_1(K^N) \quad \text{in} \quad H^1(N, \Omega^1_N).
\]
Let as usual \( K_{M/N} \) denote a divisor representing the determinant of \( \Omega^1_{M/N} \). We have
\[
K_M = K_{M/N} + f^* K^N.
\]
The divisor \( K_{M/N} \) also represents \( f^* \det E \) and \( 2.5 \) gives

2.6. Corollary. In the situation of the proposition the following identities hold in \( H^1(M, \Omega^1_M) \):
\[
c_1(K_{M/N}) = \frac{m-n}{n+1} c_1(f^* K^N) \quad \text{and} \quad c_1(K_M) = \frac{m+1}{n+1} c_1(f^* K^N)
\]
In particular, \( c_1(K_M) \) and \( c_1(f^* K^N) \) are proportional.

2.7. Remark. 1.) The formulas hold in the case \( m = n \).

2.) If \( c_1(K_M) = 0 \) or \( c_1(K^N) = 0 \), then \( c_1(K_M) = c_1(K^N) = 0 \). By 1.3 all Chern classes of \( M \) and \( N \) vanish. Then \( M \) and \( N \) are abelian.

3.) If \( K_M \) or \( K^N \) is not nef, then \( K_M \) and \( K^N \) are not nef. Then \( M \simeq \mathbb{P}_M(\mathbb{C}) \) and \( N \simeq \mathbb{P}_N(\mathbb{C}) \) (JR14). As \( n > 0 \) by assumption, \( m = n \) and \( f \) is an automorphism of projective space.
4.) In the case $\dim N = 1$ the formula reads

$$2c_1(E) = 2c_1(f_\ast \Omega^1_{M/N}) = (m - 1)c_1(K_N).$$

If $c_1(K_N) \neq 0$, i.e., in the non–abelian case, we get a family reaching the Arakelov bound in the sense of Viehweg and Zuo in [VZ02].

Because of (2.3) and $c_1(K_N) \neq 0$, the bundle $E$ does not admit a flat direct summand of positive rank. The result of [VZ02] in this case seems to be that $M$ is isogeneous to some fiber product

$$Z \times_Y \cdots \times_Y Z,$$

where $Z \longrightarrow Y$ is a modular family of abelian varieties with special Hodge group built from quaternion algebras.

Proof of Proposition 2.4. The arguments can essentially be found in [JR04]. By assumption we have a section $s : N \longrightarrow M$.

Consider the pull back to $N$ by $s$ of (2.1)

$$(2.8) \quad 0 \longrightarrow \Omega^1_N \longrightarrow s^\ast \Omega^1_M \longrightarrow s^\ast \Omega^1_{M/N} \simeq E \longrightarrow 0.$$

We have the map $ds : s^\ast \Omega^1_M \longrightarrow \Omega^1_N$. As $$(ds)(s^\ast df) = d(f \circ s) = id_{\Omega^1_N},$$ sequence (2.8) splits holomorphically.

The Atiyah class of $s^\ast \Omega^1_M$ is obtained from the Atiyah class of $\Omega^1_M$ by applying $ds$ to the last $\Omega^1_M$ factor in (1.1). What we get is

$$a(s^\ast \Omega^1_M) = \frac{s^\ast c_1(K_M)}{m + 1} \otimes ds + id_{s^\ast \Omega^1_M} \otimes \frac{c_1(s^\ast K_M)}{m + 1} \in H^1(N, s^\ast \Omega^1_M \otimes s^\ast T_M \otimes \Omega^1_N),$$

where we carefully distinguish between $s^\ast c_1(K_M) \in H^1(N, s^\ast \Omega^1_M)$ and the class $c_1(s^\ast K_M) = ds(c_1(K_M)) \in H^1(N, \Omega^1_N)$.

The Atiyah class of a direct sum is the direct sum of the Atiyah classes. As the pull back of (2.1) splits holomorphically, we get the Atiyah classes of $\Omega^1_N$ and $E$ by projecting (2.8) onto the corresponding summands.

We begin with $E$. The class $c_1(K_M) \in H^1(M, \Omega^1_M)$ is the pull back of some class in $H^1(N, \Omega^1_N)$; it therefore vanishes under $H^1$ of $\Omega^1_M \longrightarrow \Omega^1_{M/N}$. This means the first summand in (2.9) vanishes if we project, while the second summand becomes

$$id_E \otimes \frac{c_1(s^\ast K_M)}{m + 1} \in H^1(N, E \otimes E^{\ast} \otimes \Omega^1_N).$$

This is the Atiyah class of $E$. The trace gives

$$c_1(E) = rk(E) \frac{c_1(s^\ast K_M)}{m + 1} \in H^1(N, \Omega^1_N)$$

The determinant of (2.8) gives the following identities of classes in $H^1(N, \Omega^1_N)$:

$$(2.11) \quad c_1(K_N) = c_1(s^\ast K_M) - c_1(E) = \frac{m + 1 - (m - n)}{m + 1}c_1(s^\ast K_M)$$

Now (2.5) follows from (2.10).
We compute the Atiyah class of $\Omega^1_N$. We have to apply $ds$ to the first factor of the first summand in (2.9). This gives $c_1(s^*K_M)_{m+1}$. As the splitting maps compose to the identity we get

$$c_1(K_N)_{n+1} = c_1(s^*K_M)_{m+1}. $$

Replacing this in the above formula we see that $N$ has a holomorphic normal projective connection. The proposition is proved.

3. Examples

The examples we give are well known PEL type Shimura families. We follow the classical description of Shimura ([Sh59]) which makes the projective structure clearly visible.

The examples are quotients of $C^{m-1} \times B_1$. We identify $B_1(\mathbb{C})$ and the upper half plane $H = \{ \tau \in \mathbb{C} | \Im \tau > 0 \}$. For any Fuchsian group $\Gamma \subset PGl_2(\mathbb{R})$ acting on $H$ as a group of M"obius transformations we denote by $H/\Gamma = Y_\Gamma$ the corresponding quotient.

3.1. Elliptic curves. A non–compact example and the case of a split algebra: let $B = M_{2\times 2}(\mathbb{Q})$ and let $\Gamma$ be a torsion free congruence subgroup of the group of positive units, i.e., of $Sl_2(\mathbb{Z})$. We let $\Gamma$ act on $H$ in the usual way.

The elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ where $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$. If $\tau' = \gamma(\tau) = \frac{a\tau + b}{c\tau + d}$ for some $\gamma \in \Gamma$, then $E_\tau \simeq E_{\tau'}$. We will do the computation below in the analogous case of false elliptic curves. Consider the subgroup $H_\Gamma$ of $Sl_3(\mathbb{C})$ of matrices

$$\begin{pmatrix} \gamma & 0 & 0 \\ m & n & 1 \end{pmatrix}$$

with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $m, n \in \mathbb{Z}$.

The subgroup of maps where $\gamma = id$ is normal and isomorphic to $\Lambda \simeq \mathbb{Z} \oplus \mathbb{Z}$, and we obtain the exact sequence

$$0 \rightarrow \Lambda \rightarrow H_\Gamma \simeq \Gamma \times \mathbb{Z}^2 \rightarrow \Gamma \rightarrow 1.$$ (3.1)

Consider $\mathbb{P}_2$ with homogeneous coordinates $\tau, u, z$. Think of $H \times \mathbb{C}$ with coordinates $\tau, z$ as an open subset of the standard chart $u = 1$. The group $H_\Gamma$ induces a group of projective automorphisms stabilizing $H \times \mathbb{C}$. The above matrix acts as

$$(\tau, z) \mapsto \left( \frac{a\tau + b}{c\tau + d}, \frac{z + m\tau + n}{c\tau + d} \right).$$

If we choose for $\Gamma$ some group such that $S_\Gamma = (H \times \mathbb{C})/H_\Gamma$ becomes a smooth surface, then this surfaces has a projective structure (example 1.3). By construction $S_\Gamma$ comes with an elliptic fibration over $H/\Gamma$; the fiber over $[\tau]$ is isomorphic to the above curve $E_\tau$.

Examples for $\Gamma$ are the well known congruence groups $\Gamma(N), \Gamma_0(N), \Gamma_1(N)$ for certain level $N$. The surface $S_\Gamma$ is not compact; any smooth compactification destroys the projective structure ([KOS80]).
3.2. **False elliptic Curves.** A compact example and the case of a non split algebra: let $B$ be a total indefinite quaternion algebra over $\mathbb{Q}$. We fix an isomorphism $B \otimes \mathbb{R} \simeq M_2(\mathbb{R})$. A false elliptic curve is an abelian surface $F$ with

$$\text{End}(F) \otimes \mathbb{Q} \simeq B.$$ 

The representing lattice is $\mathfrak{o}_B \simeq \mathbb{Z}^4$, a maximal order in $B$. The orbit in $\mathbb{C}^2$ of the vector $(\tau, 1)^t$, $\tau \in \mathbb{H}$ under the matrices in $\mathfrak{o}_B$ is a complete lattice $\Lambda_\tau \simeq \mathfrak{o}_B$. The quotient $F = \mathbb{C}^2/\Lambda_\tau$, for general $\tau$, is an example of a false elliptic curve ([Sh59]).

Let $\Gamma$ be a torsion free congruence subgroup of the group of positive units in $\mathfrak{o}_B$. Using $B \otimes \mathbb{R} \simeq M_2(\mathbb{R})$ we consider $\Gamma$ as a group of matrices and study its fixed point free action on $\mathbb{H}$.

The quotient $Y_\Gamma$ is compact here ([Sh59]). If $\tau' = \gamma(\tau) = \frac{a \tau + b}{c \tau + d}$ for some $\gamma \in \Gamma$, then

$$\Lambda_{\tau'} = \mathfrak{o}_B \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \\ 1 \end{array} \right) = \frac{1}{c \tau + d} \mathfrak{o}_B \left( \begin{array}{c} a \tau + b \\ c \tau + d \end{array} \right) = \frac{1}{c \tau + d} \mathfrak{o}_B \gamma \left( \begin{array}{c} \tau \\ 1 \end{array} \right) = \frac{1}{c \tau + d} \Lambda_\tau$$

implying that $F_{\tau'}$ and $F_\tau$ are isomorphic.

As above we obtain a subgroup $H_B$ of $SL_4(\mathbb{C})$ and an exact sequence like (3.1) if we consider the following matrices:

$$\left( \begin{array}{cc} \gamma_1 \\ \gamma_2 \\ 0 \\ 0 \end{array} \right)$$

with $\gamma_1, \gamma_2 \in \mathfrak{o}_B$.

Consider $\mathbb{P}_3$ with homogeneous coordinates $\tau, u, z_1, z_2$. We think of $\mathbb{H} \times \mathbb{C}^2$ with coordinates $\tau, z_1, z_2$ as an open subset of the standard chart $u = 1$. The group $H_B$ induces a group of projective automorphisms acting on $\mathbb{H} \times \mathbb{C}^2$. In coordinates

$$\gamma = \left( \begin{array}{cc} a_k & b_k \\ c_k & d_k \end{array} \right), \quad k = 1, 2$$

act as

$$(\tau, z_1, z_2) \mapsto \left( \frac{a_1 \tau + b_1}{c_1 \tau + d_1}, \frac{z_1 + a_2 \tau + b_2}{c_1 \tau + d_1}, \frac{z_2 + c_2 \tau + d_2}{c_1 \tau + d_1} \right).$$

We see that $H_B$ acts on the first factor by $\gamma_1 \in \Gamma$. The quotient

$$M_{B, \Gamma} = (\mathbb{H} \times \mathbb{C}^2)/H_B$$

carries the structure of a smooth projective threefold and has a projective structure (example 1.3). The fiber of $M_{B, \Gamma} \rightarrow Y_\Gamma$ over $[\tau]$ is isomorphic to the above false elliptic curve $F_\tau$.

The latter construction also works if we consider the split algebra $M_{2 \times 2}(\mathbb{Q})$ instead of a quaternion algebra. The quotient in this case, taking for $\Gamma$ the same group as above, is just the fiber product $S_\Gamma \times_{Y_\Gamma} S_\Gamma$ from above. Likewise, if we consider

$$\left( \begin{array}{cc|cc} \gamma_1 & 0 & 0 \\ \gamma_2 & 1 & 0 \\ \gamma_3 & 0 & 1 \end{array} \right), \quad \gamma_1, \gamma_2, \gamma_3 \in \mathfrak{o}_B.$$
acting on \( \mathbb{H} \times \mathbb{C}^2 \times \mathbb{C}^2 \) we get \( M_{B, \Gamma} \times \chi \), \( M_{B, \Gamma} \), a compact example of a smooth projective manifold with a projective structure.

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