Controllability of a Linear System with Nonnegative Sparse Controls

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Abstract—This paper studies controllability of a discrete-time linear dynamical system using nonnegative and sparse inputs. These constraints on the control input arise naturally in many real-life systems where the external influence on the system is unidirectional, and activating each input node adds to the cost of control. We derive the necessary and sufficient conditions for controllability of the system, without imposing any constraints on the system matrices. Unlike the well-known Kalman rank based controllability criteria, the conditions presented in this paper can be verified in polynomial time, and the verification complexity is independent of the sparsity level. The proof of the result is based on the analytical tools concerning the properties of a convex cone. Our results also provide a closed-form expression for the minimum number of control nodes to be activated at every time instant to ensure controllability of the system using positive controls.

Index Terms—Controllability, linear dynamical systems, sparsity, positive control

I. INTRODUCTION

Controllability is one of the most fundamental concepts in control theory, which is related to the ability of a system to maneuver its states. Originally, controllability of a linear system was studied without any constraints on the inputs. These studies lead to the complete characterization of controllability through the classical Kalman rank test and Popov-Belovich-Hautus (PBH) test [1], [2]. Traditional controllability has also been extended to the case where the admissible input set is constrained due to practical limitations. The different types of constraint sets that have been considered in the past are compact, convex, or quasi-convex sets [3]–[6]. In this paper, we deal with nonconvex and noncompact constraints on the input, namely, sparsity and nonnegativity. More precisely, we consider a linear dynamical system whose control input at every time instant has a few number of nonzero entries compared to its dimension, and all nonzero entries are positive. For such a system, we investigate whether it is possible to steer the system from an arbitrary initial state to an arbitrary final state within a finite time duration.

A. Motivation

The sparsity constraint is desired in many real-world applications due to communication bandwidth, cost, or energy constraints [7]. Also, the nonnegativity constraint is frequently encountered in medical, ecological, chemical, and economical applications where the controls have a unidirectional influence [8], [9]. To motivate our setting with both sparsity and nonnegativity constraints, consider the problem of controlling the temperature of a large room with multiple heating elements [10]. The heaters can radiate heat energy into the room, but they cannot extract heat. So the control is in one-way stream, and thus, the inputs are nonnegative. Also, it is desirable to maintain the temperature by operating as few heaters as possible to reduce the overall cost of operation. Hence, the control input at every time instant is sparse.

B. Related work

The two constraints that we consider here, sparsity and nonnegativity, have been studied separately in literature for several decades. We provide a short review of these works below:

1) Sparsity constraint: The study of a linear dynamical system under sparsity constraints dates back to 1972 [11]. Some recent works have addressed the problem of finding the sequence of sparse control inputs, both for the fixed and the time-varying set of control nodes, and other related problems [12]–[14]. However, controllability under sparse inputs was not well-understood, until recently. The widely known condition for sparse-controllability is the extended version of the Kalman rank test. This test is based on the rank of the Gramian matrix, and it is known to have combinatorial complexity. Hence, different quantitative measures of controllability based on the Gramian matrix have been considered: smallest eigenvalue, the trace of the inverse, the inverse of the trace, the determinant, maximum entry in the diagonal, etc [15]–[18]. These strategies make the analysis cumbersome. Recently, a set of algebraically verifiable necessary and sufficient conditions for sparse controllability that are similar to the classical PBH test [2] were presented in [7]. Our work analyzes a more constrained system where the inputs are not only sparse but also nonnegative.

2) Nonnegativity constraint: Controllability of linear systems with nonnegative control inputs was first studied in [3] for continuous-time linear systems. These results were extended to discrete-time systems in [19] for a single input system, and were further extended to multi-input systems in [5]. These papers have triggered a large number of studies dealing with other related problems on controllability like approximate controllability, null controllability, the geometry of the reachability set, etc. [20]–[22]. Other extensions of the controllability result to nonstationary systems and third-order systems have also been investigated [23], [24]. To the best of our knowledge, none of the existing works in the literature has addressed the important problem of controllability of a linear system under both sparsity and nonnegativity constraints.
C. Our contributions

We derive a set of necessary and sufficient conditions for controllability of a linear system under nonnegative sparse inputs. We show that any system is controllable with nonnegative sparse control inputs if and only if it is controllable using nonnegative control inputs and the sparsity level is greater than the dimension of the null space of the state-transition matrix. Using this result, we show that the conditions for verifying controllability are non-combinatorial. Our approach is based on fundamental tools from analysis concerning the properties of positive spanning sets.

Notation: Boldface lowercase letters denote vectors, boldface uppercase letters denote matrices, and calligraphic letters denote sets. The symbols \( \mathbb{R} \) and \( \mathbb{C} \) denote the set of real numbers and complex numbers, respectively. The notation \( A_{i,j} \) denotes the submatrix of \( A \) formed by the columns indexed by the set \( \mathcal{I} \). The operator \( \| \cdot \|_0 \) represents the \( \ell_0 \) norm of a vector, and \( |\cdot| \) represents the cardinality of a set. The notation \( \geq \) denotes the element-wise inequality, i.e., \( a \geq b \) implies \( a_i \geq b_i \), for all values of \( i \). For any positive integer \( a \), \( [a] \) denotes the set \( \{1, 2, \ldots, a\} \), and \( \text{Span}_+ \{A\} \) denotes the positive span of the columns of \( A \). If \( A \in \mathbb{R}^{n \times m} \)
\[
\text{Span}_+ \{A\} = \left\{ a \in \mathbb{R}^n : a = \sum_{i=1}^{m} \alpha_i A_i, \alpha_i \geq 0 \right\}.
\]
The symbol \( I \) represents the identity matrix and \( 0 \) represents the all zero matrix (or vector).

II. NONNEGATIVE SPARSE CONTROLLABILITY

We consider the discrete-time linear dynamical system \((A, B)\) whose state evolution model is as follows:
\[
x_k = Ax_{k-1} + Bu_k,
\]
where \( x_k \in \mathbb{R}^n \) and \( u_k \in \Omega \subseteq \mathbb{R}^m \) denote the state vector and the control vector at time \( k \), respectively. The set \( \Omega \) is the set of all admissible controls of the system. Also, \( A \in \mathbb{R}^{n \times n} \) is the state-transition matrix, and \( B \in \mathbb{R}^{n \times m} \) is the input matrix of the system. We assume that the control vectors are \( s \)-sparse and its nonzero entries are positive:
\[
\Omega = \Omega_+ \triangleq \{ z \in \mathbb{R}^m : \|z\|_0 \leq s \text{ and } z \geq 0 \} \subset \mathbb{R}^m.
\]
Our goal is to examine controllability of the system, i.e., for any given pair \((x_{\text{initial}}, x_{\text{final}}) \in \mathbb{R}^n \times \mathbb{R}^n\), we test if it is possible to find inputs from \( \Omega \) such that \( x_K = x_{\text{final}} \) when \( x_0 = x_{\text{initial}} \), for some finite positive integer \( K \). This notion of controllability is henceforth referred to as nonnegative sparse controllability. From (2), we know that the state vector at time \( K \) is given by
\[
x_K - A^K x_0 = \sum_{k=1}^{K} A^{K-k} Bu_k.
\]
Therefore, the system is controllable if and only if there exists a positive integer \( K < \infty \) such that
\[
\bigcup_{\{S_k \subseteq [m] : \sum_{i=1}^{K} |S_k| \leq s\}} \text{Span}_+ \{A^{K-1}B_{S_1}, A^{K-2}B_{S_2}, \ldots, B_{S_K}\} = \mathbb{R}^n,
\]
where \( \text{Span}_+ \) is defined in (1). We see that a brute force verification of the above condition is combinatorial, and hence, it is computationally heavy. In the sequel, we present a non-combinatorial verification procedure for testing controllability using nonnegative sparse controls.

III. PRELIMINARIES

We observe that our system imposes two types of constraints on the set of admissible inputs (as given in (3)): one, nonnegativity and two, sparsity. These two constraints have been separately dealt in the literature and we present the corresponding results below:

**Theorem A** ([5, Theorem 1]). Suppose that the set of admissible vectors \( \Omega \) is a convex cone in \( \mathbb{R}^m \) with nonempty interior. Then, the system \((A, B)\) defined in (2) is controllable if and only if the following conditions hold:
\[(i) \exists z \neq 0 \text{ such that } z^T A = \lambda z^T \text{ and } z^T B = 0, \text{ for any } \lambda \in \mathbb{C}.
\]
\[(ii) \exists z \neq 0 \text{ such that } z^T A = \lambda z^T \text{ and } z^T B u \leq 0, \text{ for any } \lambda \geq 0 \text{ and all } u \in \Omega.
\]

For the special case of nonnegative (non-sparse) vectors, we have the following corollary:

**Corollary A.** Suppose that the set of admissible vectors \( \Omega \) is the set of all nonnegative vectors:
\[
\Omega = \mathbb{R}^m_+ \triangleq \{ z \in \mathbb{R}^m : z \geq 0 \} \subset \mathbb{R}^m.
\]
Then, the system \((A, B)\) defined in (2) is controllable if and only if the following conditions hold:
\[(i) \exists z \neq 0 \text{ such that } z^T A = \lambda z^T \text{ and } z^T B = 0, \text{ for any } \lambda \in \mathbb{C}.
\]
\[(ii) \exists z \neq 0 \text{ such that } z^T A = \lambda z^T \text{ and } z^T B u \leq 0, \text{ for any } \lambda \geq 0 \text{ and all } u \in \Omega.
\]

Next, we present the results for controllability using sparse vectors:

**Theorem B** ([7, Theorem 1]). Suppose that the set of admissible vectors \( \Omega \) is the set of all \( s \)-sparse vectors:
\[
\Omega = \Omega_s \triangleq \{ z \in \mathbb{R}^m : \|z\|_0 \leq s \} \subset \mathbb{R}^m.
\]
Then, the system \((A, B)\) defined in (2) is controllable if and only if the following conditions hold:
\[(i) \exists z \neq 0 \text{ such that } z^T A = \lambda z^T \text{ and } z^T B = 0, \text{ for any } \lambda \in \mathbb{C}.
\]
\[(ii) \exists z \neq 0 \text{ such that } z^T A = \lambda z^T \text{ and } z^T B u \leq 0, \text{ for any } \lambda \geq 0 \text{ and all } u \in \Omega.
\]

In the next section, we present the main result of the paper and the insights that it yields.

IV. NECESSARY AND SUFFICIENT CONDITIONS

The section presents the necessary and sufficient conditions for controllability of the system in (2) under the constraint \( \Omega = \Omega_{s+} \). From (3), (6), and (7) we have
\[
\Omega_{s+} = \mathbb{R}^m_+ \cap \Omega_s.
\]
So the constraint \( \Omega = \Omega_{s+} \) is more restrictive than both the constraints \( \Omega = \mathbb{R}^m_+ \) and \( \Omega = \Omega_s \). Thus, the conditions
The immediate observations from the above result are as follows:

1. Theorem 1 implies that any \( s \)-sparse controllable system is nonnegative \( s \)-sparse controllable if and only if it satisfies Condition (ii) of Theorem 1. This is evident from Theorem B. Similarly, from Corollary A, if a linear system is controllable using nonnegative control inputs, it is nonnegative \( s \)-sparse controllable if and only if \( s \geq N - \text{Rank} \{A\} \). Therefore, the extra condition for ensuring the sparsity of the control inputs is independent of the input matrix \( B \).

2. For the special case when \( s = m \), Theorem 1 reduces to Corollary A, as expected. Also, when \( m = 1 \), the notion of sparse controllability and controllability are same, and hence, Theorem 1 reduces to the well-known result of Evans and Murthy [19, Theorem 1].

3. If the linear system in (2) is nonnegative \( s \)-sparse controllable, it is also nonpositive \( s \)-sparse controllable. This is because if the system given by \((A, B)\) satisfies the conditions of Theorem 1, the system given by \((A, -B)\) also satisfies those conditions. In particular, \( \exists z \neq 0 \) such that \( z^T A = \lambda z^T \) and \( z^T B \geq 0 \), for some \( \lambda \geq 0 \). This follows since for every \( z \) such that \( z^T A = \lambda z^T \), we have \((-z)^T A = \lambda (-z)^T \).

4. For the linear system in (2), controllability using nonnegative and sparse inputs with a common support (i.e., the positive entries of all control inputs coincide) holds only if \( s \geq N - \text{Rank} \{A\} \). This follows from Condition (iii) of Theorem 1 because here, the control signals are more restricted than the setting in Theorem 1.

We obtain the following interesting corollary from Theorem 1.

Corollary 1. If any system as given in (2) is controllable under the constraint \( \Omega = \mathbb{R}^m_+ \) as given in (6), then it is nonnegative \( s \)-sparse controllable if \( s \geq m - 1 \).

Proof. See Appendix B.

A. Computational complexity

In this subsection, we discuss the computational complexity of the controllability test given in Theorem 1:

- To check Condition (i) of Theorem 1, we solve for all eigenvalues of \( A \) and check if \( \text{Rank} \{[\lambda I - A \ B] \} = N \) for each eigenvalue \( \lambda \). So the complexity of this step is polynomial in \( N \) and \( m \).

- To check Condition (ii) of Theorem 1, for every eigenvalue \( \lambda \geq 0 \) of \( A \), we find a set of linearly independent eigenvectors \( \{z^{(\lambda)}_i\}_{i=1}^{g_{\lambda}} \) corresponding to \( \lambda \), where \( g_{\lambda} \) denotes its geometric multiplicity. Now, Condition (ii) of Theorem 1 can be verified by checking if there exists \( \rho \in \mathbb{R}^p \) such that \( \rho^T B \leq 0 \). Here, \( Z \in \mathbb{R}^{N \times g_{\lambda}} \) is a matrix formed by the vectors \( \{z^{(\lambda)}_i\}_{i=1}^{g_{\lambda}} \). The feasibility of the set of linear inequalities \( B^T Z \rho \leq 0 \) can be verified by solving the following (dummy) linear programming problem:

\[
\max_{\rho \in \mathbb{R}^{p \times g_{\lambda}}} 0 \quad \text{subject to} \quad B^T Z \rho \leq 0. \tag{9}
\]

Thus, the complexity of verification of Condition (ii) of Theorem 1 is also polynomial in \( N \) and \( m \).

- The complexity to verify Condition (iii) of Theorem 1 is independent of the system dimension: \( O(1) \).

Therefore, the overall complexity of our controllability test is non-combinatorial unlike the verification of condition (5). Moreover, the complexity is independent of the sparsity level \( s \) whereas the complexity of the verification of condition (5) is a function of \( \binom{N}{s} \).

B. Comparison with sparse-controllability

From Theorem B and Theorem 1, we see some similarities between sparse-controllability and nonnegative sparse controllability.

- Reversible systems: If the system is reversible, i.e., state-transition matrix \( A \) is invertible, it is \( s \)-sparse-controllable for any \( 0 < s \leq m \) if and only if it is controllable (\( \Omega = \mathbb{R}^m \)). Similarly, it is nonnegative \( s \)-sparse-controllable for any \( 0 < s \leq m \) if and only if it is controllable using nonnegative controls (\( \Omega = \mathbb{R}^m_+ \)).

- Minimal control: Suppose that the system defined by the matrix pair \((A, B_S)\) is controllable under the constraint \( \Omega = \mathbb{R}^N_+ \) as given in (6), for some index set \( S \subseteq [m] \). Then, the system is nonnegative \( s \)-sparse-controllable. In particular, if \( \text{Rank} \{B\} \leq s \), controllability under the constraint \( \Omega = \mathbb{R}^N_+ \) implies nonnegative \( s \)-sparse controllability. Sparse-controllability also posses a similar property.

- Change of basis: If a system as given in (2) is controllable using inputs that are \( s \)-sparse under the canonical basis, it is controllable using inputs that are \( s \)-sparse under any basis \( \Phi \in \mathbb{R}^{m \times m} \). However, this property does not hold for nonnegative sparse controllability. For example,
consider the following system:
\[
A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
\[
\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

The system \((A, B)\) is nonnegative 1–sparse controllable, but the system \((A, B\Phi)\) is not nonnegative 1–sparse controllable. This is because the eigenvector \(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T\) corresponding the eigenvalue 0 of \(A\) does not satisfy Condition (ii) of Theorem 1.

V. CONCLUSIONS

This paper characterized controllability of discrete-time linear systems subject to sparsity and nonnegativity constraints on the inputs. The characterizations are in terms of algebraic conditions that are similar to the classical results for unconstrained and nonnegative input-constrained linear systems. We showed that the complexity of the controllability test is polynomial in the dimensions of the system. Extending our results to the other notions of controllability like output controllability, approximate controllability, etc., is a related direction of research. Also, this paper dealt with the theoretical question of the existence of a sequence of nonnegative sparse vectors that ensure controllability. Developing computationally efficient algorithms to find the energy-optimal sequence of such nonnegative sparse vectors can be an interesting direction for future work.

APPENDIX A

Proof of Theorem 1

The proof of Theorem 1 relies on the following result on positive spanning sets.

**Lemma 1.** Let \(Z \subseteq \mathbb{R}^N\) be a subspace of dimension \(1 \leq d \leq N\). Also, let \(Z \in \mathbb{R}^{N \times m}\) where \(m > d\) is such that \(Z = \text{Span}_+\{Z\}\). Then, the following relation holds:
\[
Z = \bigcup_{S \subseteq [m], |S| = d} \text{Span}_+\{Z_S\}.
\]

**Proof.** We first note that \(Z = \text{Span}_+\{Z\}\) implies that the columns of \(Z\) belong to \(Z\). Therefore, we get \(\text{Rank}\{Z\} \leq d\).

Also, since \(Z = \text{Span}_+\{Z\}\), the columns of \(Z\) linearly span \(Z\) and this leads to \(\text{Rank}\{Z\} \geq d\). Consequently, we deduce that
\[
\text{Rank}\{Z\} = d.
\]

Further, for any \(z \in Z\), we define
\[
\mathcal{V}_z = \{\alpha \in \mathbb{R}^m : Z\alpha = z \text{ and } \alpha \geq 0\}.
\]

Since \(Z = \text{Span}_+\{Z\}\), the set \(\mathcal{V}_z\) is non-empty. Let \(\bar{Z}\alpha = \bar{z}\) be the reduced-row echelon form of the matrix equation \(Z\alpha = z\), after removing the zero rows. Consequently, we obtain
\[
\mathcal{V}_z = \{\alpha \in \mathbb{R}^m : \bar{Z}\alpha = \bar{z} \text{ and } \alpha \geq 0\}.
\]

By the fundamental theorem in linear programming, the system \(\bar{Z}\alpha = \bar{z}\) has a basic feasible solution \(\alpha \geq 0\) with at most \(\text{Rank}\{\bar{Z}\}\) nonzero elements. However, from (13),
\[
\text{Rank}\{\bar{Z}\} = \text{Rank}\{Z\} = d.
\]

Thus, there exists a \(d\)-sparse vector \(\alpha \geq 0\) such that \(z = Z\alpha\).

So we conclude that
\[
z \in \bigcup_{S \subseteq [m], |S| = d} \text{Span}_+\{Z_S\}.
\]

Since the above result holds for any \(z \in Z\), we deduce that
\[
Z \subseteq \bigcup_{S \subseteq [m], |S| = d} \text{Span}_+\{Z_S\} \subseteq \text{Span}_+\{Z\} = Z.
\]

Hence, the proof is complete.

**Proof of Theorem 1**

The necessity of the three conditions is straightforward from Corollary A and Theorem B. Therefore, we need to show that the conditions of Theorem 1 are sufficient for nonnegative sparse controllability.

We need the following definitions for the proof. Let \(\mathcal{N}\) be the null space of \(A^T\) and \(\mathcal{C}\) be the column space of \(A\). We note that \(\mathcal{N}\) and \(\mathcal{C}\) are orthogonal to each other. The orthogonal projection operator corresponding to \(\mathcal{N}\) and \(\mathcal{C}\) are \(I - AA^\dagger\) and \(AA^\dagger\), respectively, where \(A^\dagger \in \mathbb{R}^{N \times N}\) is the Moore-Penrose pseudo-inverse of \(A\).

From (4), we see that it is sufficient to show that, for any \((x_0 \in \mathbb{R}^N, x_f \in \mathbb{R}^N)\), there exists a positive integer \(K < \infty\) and \(s\)-sparse vectors \(\{u_k \geq 0\}_{k=1}^K\) such that the following holds:
\[
\left(I - AA^\dagger\right)x_f = \left(I - AA^\dagger\right)Bu_K
\]
\[
AA^\dagger x_f - A^Kx_0 = \sum_{k=1}^{K-1} A^{K-k}Bu_k + AA^\dagger Bu_K.
\]

This is because adding (19) and (20) gives (4). We note that the left-hand side term in (19), \(\left(I - AA^\dagger\right)x_K \in \mathcal{N}\), and the term in (20), \(AA^\dagger x_K - A^Kx_0 - AA^\dagger Bu_K \in \mathcal{C}\). As a result, it suffices to prove the following:

(a) For any \(z \in \mathcal{N}\), there exists an \(s\)-sparse vector \(u \geq 0\) such that \(z = (I - AA^\dagger)Bu\).

(b) For any \(z \in \mathcal{C}\), there exists a positive integer \(K < \infty\) and \(\{u_k \in \Omega_{s+1}\}_{k=1}^{K-1}\) such that \(z = \sum_{k=1}^{K-1} A^{K-k}Bu_k\).

We prove this in two steps:

**Step A** We first show that when the conditions of Theorem 1 hold, Statement (a) also holds.

**Step B** Next, we prove that when Statement (a) is true and Statement (b) is false, at least one of the conditions of Theorem 1 does not hold. This claim combined with Step A show that when the conditions of Theorem 1 are satisfied, both Statements (a) and (b) hold.
A. Proof of Statement (a)

Using Corollary A, we know that Conditions (i) and (ii) of Theorem 1 ensure that the system is controllable using nonnegative controls. So there exists a positive integer $K < \infty$ such that

$$\text{Span}_+ \left\{ \left[ A^{K-1} B \ A^{K-2} B \ldots B \right] \right\} = \mathbb{R}^N \supseteq N. \quad (21)$$

So for every $z \in N$, there exists $\{u_k \geq 0\}_{k=1}^K$ such that

$$z = \sum_{k=1}^K A^{K-k} B u_k. \quad (22)$$

Multiplying both sides with $I - AA^\dagger$, we have

$$z = (I - AA^\dagger) z = (I - AA^\dagger) B u_K. \quad (23)$$

As a consequence, we have

$$N \subseteq \text{Span}_+ \left\{ \left( I - AA^\dagger \right) B \right\}. \quad (24)$$

Further, since columns of $(I - AA^\dagger) B$ belong to $N$, we deduce that

$$N = \text{Span}_+ \left\{ \left( I - AA^\dagger \right) B \right\} \quad (25)$$

where (26) follows from Lemma 1 and the fact that the dimension of $N$ is $N - \text{Rank} \{A\}$. Also, (27) is because $s \geq N - \text{Rank} \{A\}$ which is due to Condition (iii) of the theorem. Hence, Step A is completed.

B. Proof of Statement (b)

Let $r \triangleq \text{Rank} \{A\}$ and the Jordan canonical form [25] of $A$ be

$$A = P^{-1}JP$$

$$= P^{-1} \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} 
\in \mathbb{R}^{r \times N-r} \quad || \begin{bmatrix} P^{(1)} \in \mathbb{R}^{r \times N} \\ P^{(2)} \in \mathbb{R}^{N-r \times N} \end{bmatrix}, \quad (29)$$

where $P \in \mathbb{R}^N$ is an invertible matrix, and $J \in \mathbb{R}^N$ is an upper triangular matrix. Also, $J \in \mathbb{R}^{r \times N}$ is an invertible upper triangular matrix with the nonzero eigenvalues of $A$ along its diagonal. Then, Statement (b) simplifies as follows: For any $z = A \bar{z}$, there exists a positive integer $K < \infty$ and $\{u_k \in \Omega_s\}_{k=1}^K$ such that

$$\sum_{k=1}^K J^{K-1-k} B u_k, \quad (30)$$

where we define $\bar{B} \in \mathbb{R}^{r \times m}$ as follows:

$$\bar{B} = P^{(1)} B. \quad (31)$$

If Statement (b) is false, using the Kalman rank type condition, we know that

$$\bigcup_{s_k \in [m], |s_k| \leq K-1} \text{Span}_+ \left\{ \left[ J^{K-1} B_{S_1} \ldots B_{S_K} \right] \right\} \subseteq \mathbb{R}^r, \quad (32)$$

for any positive integer $K < \infty$.

Let the sets $\{S_i \subseteq [m]\}_{i=1}^K$, each with cardinality $s_i$, be such that they partition the set $[m]$ as follows:

$$|S_i| \leq s$$

Therefore, the linear dynamical system $(J, B^*)$ is not controllable using nonnegative controls. Applying Corollary A to the system $(J, B^*)$, we see that one of the following conditions hold:

**C1:** There exists $(\lambda, y \neq 0)$ such that

$$y^T J = \lambda y^T \text{ and } y^T B^* = 0. \quad (37)$$

**C2:** There exists $(\bar{u}, \bar{v} \neq 0)$ with $\bar{v}^T \neq 0$ and $\bar{y}^T \neq 0$ such that

$$\lambda \bar{v}^T J = \bar{v}^T P A P^{-1}. \quad (38)$$

Further, from (36) and (37), we obtain

$$y^T B = y^T P^{(1)} B = v^T P A P^{-1} = \lambda z^T. \quad (39)$$

Therefore, $z^T B = 0$, and combining this claim with (39), we get that Condition (i) of Theorem 1 is violated.

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1This condition is similar to (5) with $A$ replaced with $J$ and $B$ replaced with $B$.

2For example, $\bar{K} = [m/s]$ and $S_i = \{(i-1)s+1,(i-1)s+2, \ldots, \min(is,m)\}$.
C2: There exists $\lambda$ such that $y^T J = \lambda y^T$ and $y^T B^* \leq 0$. From (40) and (41), we have $z^T B \leq 0$ because $\lambda > 0$. Combining this condition with (39), we observe that Condition (ii) is violated.

Thus, Step B is completed, and hence, the theorem is proved.

APPENDIX B
PROOF OF COROLLARY 1

The proof relies on the following result:

Lemma 2. Suppose $Z \subseteq \mathbb{R}^{N \times m}$ is that $Z \subseteq \text{Span}_+ \{Z\}$, for some subspace $Z$ with dimension $d$. Then, $m \geq d + 1$.

Proof. The proof is straightforward from [26, Corollary 5.5].

Proof of Corollary 1

Since the system is controllable using nonnegative control inputs, from Corollary A, we know that Conditions (i) and (ii) of Theorem 1 hold. Therefore, it suffices to check if Condition (iii) holds. Also, controllability of the system using nonnegative inputs yields that there exists a positive integer $K < \infty$ such that

$$\text{Span}_+ \{A^{K-1} B, A^{K-2} B, \ldots, B\} = \mathbb{R}^N \supseteq \mathcal{N}, \quad (42)$$

where $\mathcal{N}$ denotes the null space of $A$. Then, the arguments similar to (21) to (25) guarantee that

$$\mathcal{N} = \text{Span}_+ \{I - AA^\dagger\} B\right\}. \quad (43)$$

Further, from Lemma 2, we have $m \geq N - \text{Rank} \{A\} + 1$, since the dimension of $\mathcal{N}$ is $N - \text{Rank} \{A\}$. Thus, we conclude that

$$s \geq m - 1 \geq N - \text{Rank} \{A\}. \quad (44)$$

Hence, Condition (iii) of Theorem 1 also holds, and the proof is complete.

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