KRULL DIMENSION FOR LIMIT GROUPS I:
BOUNDING STRICT RESOLUTIONS

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ABSTRACT. This is the first paper in a sequence on Krull dimension for limit
groups, answering a question of Z. Sela. In this paper we show that strict reso-
lutions of a fixed limit group have uniformly bounded length. The upper bound
plays two roles in our approach. First, it provides upper bounds for heights of
analysis lattices of limit groups, and second, it enables the construction of JSJ–
respecting sequences in the sequel.

1. INTRODUCTION

We start by drawing the analogy between limit groups and coordinate rings of
algebraic varieties. To an algebraic subset $V$ of $\mathbb{C}^n$, one attaches the coordinate
ring. The points of $V$ are in one-to-one correspondence with ring homomorphisms
$\mathbb{C}[V] \to \mathbb{C}$: If $\mathbb{C}[V]$ is the quotient of $\mathbb{C}[x_i]$ by a radical ideal $I$, then an assignment
$x_i \mapsto z_i \in \mathbb{C}$, $(z_i)$ a point of $V$, vanishes on $I$ and gives a ring homomorphism.
Conversely, one associates to such a ring homomorphism $f$ the tuple $(f(x_i)) \in V$.

The replacement for the notion of a radical ideal in algebraic geometry over the
free group is that of a residually free group. Going further, we need to extend the
notion of prime ideal as well. A zero-divisor in a finitely generated group $G$ is a
tuple of elements of $G$ such that every homomorphism $G \to F$ kills at least one
element of the tuple. A group is residually free if it has no singleton zero divisors.
A group is $\omega$–residually free, or is a limit group, if it has no zero divisors. It will
be convenient to use a slightly different definition of a limit group.

Definition 1.1 (Limit Group [Sel01]). A sequence of homomorphisms $f_n : G \to \Gamma$
is stable if, for all $g \in G$, there exists $n_g$ such that $f_n(g) = 1$ for $n > n_g$ or
$f_n(g) \neq 1$ for $n > n_g$. The stable kernel of a stable sequence of homomorphisms
is the set of elements which have trivial image for large $n$, and is denoted $\text{Ker}(f_n)$.
The quotient of $G$ by the stable kernel of a stable sequence $f_n$ is a $\Gamma$–limit group.

In this paper we study $\Gamma$–limit groups for $\Gamma \cong F$. A sequence $f_n : G \to F$
converges to $G$ if $\text{Ker}(f_n) = \{1\}$. If $G$ is $\omega$–residually free then clearly there
exists a sequence of homomorphisms $f_n : G \to F$ converging to $G$. That a $F$–
limit group is $\omega$–residually free is [Sel01, Theorem 4.6], and follows from finite
presentability.

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The object analogous to the coordinate ring for an algebraic subset $V$ of $\mathbb{F}^n$ is simply a finitely generated residually free group $G$ which requires only $n$ generators. With little work one establishes a one-to-one correspondence between points of $V$ and elements of $\text{Hom}(G, \mathbb{F})$. If an algebraic set $V$ is irreducible then $\mathbb{C}[V]$ has no zero divisors. Likewise, for the set $\text{Hom}(G, \mathbb{F})$ to be irreducible, $G$ must have no zero divisors, hence is a limit group.

The Krull dimension of an algebraic set is the supremum of lengths of chains of irreducible subvarieties, thus it is natural to ask whether any sequence of proper epimorphisms of limit groups, beginning with a fixed free group of rank $n$, terminates in a uniform number of steps. Exhibition of such a bound would establish the finiteness of the Krull dimension of $\mathbb{F}^n$. In this paper we make the first move toward showing this.

The key to constructing limit groups is the notion of a strict resolution. The group $\text{Mod}(L) < \text{Aut}(L)$ is defined in the next section.

Definition 1.2 (Strict; Strict resolution). A homomorphism $\pi : L \to L'$ is strict if for every sequence of homomorphisms $f_n : L' \to \mathbb{F}$ converging to $L'$, there exists a sequence of automorphisms $\phi_n \in \text{Mod}(L)$ such that the sequence $f_n \circ \pi \circ \phi_n$ converges to $L$.

A sequence of proper epimorphisms $L \to L_1 \cdots \to \mathbb{F}_k$ is a resolution of $L$. If every homomorphism appearing in a resolution is strict then the resolution is a strict resolution.

By [Sel01, Proposition 5.10] every limit group admits a strict resolution. The first step in our approach to showing that limit groups have finite Krull dimension is to show that any strict resolution of limit groups terminates in a uniformly bounded number of steps.

Theorem 1.3 (Strict resolutions have bounded length). Let $\mathbb{F}_n \to L_0 \to \cdots \to L_k$ be a sequence of proper strict epimorphisms of limit groups. Then $k \leq 3n$.

Let $\mathbb{F}_n \to L_1 \to \cdots \to L_k \cong \mathbb{F}_m$ be a sequence of proper strict epimorphisms of limit groups. Then $k \leq 3(n - m)$.

It is well known that any resolution, strict or not, has finite length [Sel01, Proposition 5.1], [BF03, Corollary 1.9]. The importance of strict resolutions lies in the fact that they witness a group as a limit group. See Definition 1.2. The main use of Theorem 1.3 is along the way to [Lou08a, 7.6], which says, roughly, that if there exist arbitrarily long sequences of proper epimorphisms of rank $n$ limit groups, then there exist arbitrarily long sequences of rank $n$ limit groups such that all groups in the sequence share the same JSJ decomposition.

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2. SPLittings AND Lifting AUtomorphisms

The following definition is somewhat non-standard. Its utility lies in the fact that it streamlines the statements and proofs of the lemmas leading up to Theorem 4.3.
Definition 2.1. Fix a finitely generated group $G$. A splitting $G = G_1 \ast_E G_2$ over a finitely generated abelian subgroup $E$ is an amalgamation. A splitting $G = G' \ast_E$ over a finitely generated abelian subgroup $E$ is an HNN extension. In either case, $E$ is allowed to be trivial, as is $G_1$ in the former.

An important subgroup of the automorphism group of a freely indecomposable finitely generated group is the modular group $\text{Mod}$. To make our exposition as efficient as possible, we define the modular group to be the subgroup of the automorphism group generated by the following elementary automorphisms:

- Automorphisms from amalgamations: $G = G_1 \ast_E G_2$: The Dehn twist in $e \in Z_{G_2}(E)$ is the automorphism of $G$ which is the identity on $G_1$ and which is conjugation by $e$ on $G_2$. This automorphism is denoted $\tau_{\Delta,e}$.
- Automorphisms from HNN extensions: $G' \ast_E$: Let $t$ be the stable letter such that $tEt^{-1} = E_1$ and $e \in Z_{G'}(E)$. Then $\tau_{\Delta,e}$ is the automorphism which is the identity on $G'$ and maps $t$ to $te$.

The modular group as defined above is the group of ‘geometric’ automorphisms arising from one edged splittings. If $G$ is freely indecomposable, then $\text{Mod}(G)$ agrees with the modular group as defined in [BF03] and [Sel01].

Given an HNN extension or amalgamation of $G$, we build a free group $\tilde{G}$ and a homomorphism $\tilde{G} \to G$ so that the automorphism of $G$ engendered by the splitting lifts to an automorphism of $\tilde{G}$.

Definition 2.2 (Lifts of Automorphisms, $\tilde{G}$). Let $\Delta$ be a splitting of $G$ as $G = G_1 \ast_E G_2$ over a finitely generated abelian group $E$, $e \in Z_{G_2}(E)$. ($G_1$ and $G_2$ are finitely generated.) Let $F_1 = \langle x_1, \ldots, x_{n_1} \rangle$ and $F_2 = \langle y_1, \ldots, y_{n_2} \rangle$ be free groups with homomorphisms $\pi_i: F_i \to G_i$. Define $\tilde{G}$ to be the group $F_1 * F_2$, calling this free splitting $\tilde{\Delta}$, and let $\pi = \pi_1 * \pi_2: \tilde{G} \to G$ be the obvious surjection. This particular free factorization of $\tilde{G}$ is the lift of $\Delta$. Choose $\tilde{e} \in F_2$ such that $\pi_2(\tilde{e}) = e$. Then the automorphism $\tau_{\tilde{\Delta},\tilde{e}}$ of $\tilde{G}$ makes the following diagram commute.

$$
\begin{array}{ccc}
\tilde{G} & \xrightarrow{\tau_{\tilde{\Delta},\tilde{e}}} & \tilde{G} \\
\downarrow{\pi} & & \downarrow{\pi} \\
G & \xrightarrow{\tau_{\Delta,e}} & G \\
\end{array}
$$

If $\Delta$ is an HNN extension $G = G' \ast_E$ we define $\tilde{G}$ similarly. Choose $F' = \langle x_1, \ldots, x_n \rangle$, a homomorphism $\pi': F' \to G'$, and $\tilde{G}$ the HNN extension $F' * \langle \tilde{t} \rangle$. Set $\pi(\tilde{t}) = t$. Choose a lift $\tilde{e} \in F'$ of $e$ and define an automorphism $\tau_{\tilde{\Delta},\tilde{e}}$ which is the identity on $F'$ and maps $\tilde{t}$ to $\tilde{t} \cdot \tilde{e}$.

3. Dehn twisting $\mathcal{R}(F)$

Our approach to Theorem 1.3 is to analyze the action of the modular group of a limit group on its $SL(2, \mathbb{C})$ representation variety.
The set of homomorphisms of a finitely generated group to a complex algebraic Lie group is an affine subvariety of $\mathbb{C}^n$ for some $n$ [CS83]. The $SL(2, \mathbb{C})$ representation variety $\text{Hom}(G, SL(2, \mathbb{C}))$ will be denoted by $\mathcal{R}(G)$. For $g \in G$, $ev_g$ will denote the evaluation map $\mathcal{R}(G) \to SL(2, \mathbb{C})$.

We saw above that if $\tau_{\Delta,e}$ is elementary then it lifts to an elementary automorphism of $\tilde{G}$. The representation variety $\mathcal{R}(G)$ has a natural embedding $\mathcal{R}(G) \hookrightarrow \mathcal{R}(\tilde{G}) = \mathcal{R}(\tilde{F}_n) = SL(2, \mathbb{C})^n \subset \mathbb{C}^{4n}$ where $n$ is the rank of $\tilde{G}$. The lift $\tau_{\Delta,e}$ acts on $\mathcal{R}(\tilde{G})$ and by commutativity of $(\hat{\circ})$ the restriction $\tau_{\Delta}|_{\mathcal{R}(G)}$ agrees with $\tau_{\Delta,e}$.

The exponential map $\exp: M(2, \mathbb{C}) \to GL(2, \mathbb{C})$ is given by the formula

$$\exp(M) = \sum_{i=0}^{\infty} \frac{M^i}{i!}$$

This power series converges everywhere. The following lemma is well known. See [Ros02, Chapter 1, Example 9] for the computation in the real case.

**Lemma 3.1.** The exponential map is biholomorphic in a neighborhood of all points $v \neq 0 \in \mathfrak{sl}_2 \mathbb{C} \subset M(2, \mathbb{C})$ such that $\exp(v) \neq \pm I_2$.

Since all maps we deal with are either polynomials or are power series in matrices of polynomials we suppress mention of the ambient space $\mathbb{C}^k$.

The image of an edge group under a representation $\rho$ (usually) lives in a 1-parameter subgroup of $SL(2, \mathbb{C})$. We can therefore twist $\rho$ by elements in the 1-parameter subgroup to produce new representations. It turns out that the twisted representations can be chosen to vary analytically in $\rho$ and a parameter $z$ as long as $z$ is chosen carefully and the representation $\rho$ doesn’t map the edge group to an element whose trace is $-2$.

To get the ball rolling we need to know where we can take logarithms and how to define small pieces of 1-parameter subgroups.

**Lemma 3.2.** Let $P_\epsilon = N_\epsilon(\{x + 0i \mid 0 \leq x \leq 1\}) \subset \mathbb{C}$ be the epsilon neighborhood of $[0, 1] \subset \mathbb{R} \subset \mathbb{C}$.

For all $g \in SL(2, \mathbb{C}), \text{tr}(g) \neq -2$, there is an element $v_g \in \mathfrak{sl}_2 \mathbb{C}$, a neighborhood $U_g \subset SL(2, \mathbb{C}), v_g \in \tilde{U}_g (= \log(U_g))$, and $\epsilon > 0$ such that

- $P = P_\epsilon$
- $\exp(v_g) = g$
- $\exp(P \cdot U_g) \subset SL(2, \mathbb{C}) \setminus \{-I\}$
- $\exp[N_\epsilon(\{0, 1\}, v_g)]$ is biholomorphic onto its image.
- $P \cdot \tilde{U}_g \subset N_\epsilon([0, 1 \cdot v_g])$

**Proof.** The argument is an easy adaptation of the fact that if $N \subset M$ is an embedded smooth, compact, submanifold and $f: M \to M'$ is smooth, injective on $N$, and a local diffeomorphism at every point of $N$, then $f$ is a diffeomorphism on a neighborhood of $N$. \(\square\)
Definition 3.3. A tuple $S = (U_g, \tilde{U}_g, P)$ satisfying Lemma 3.2 is a standard neighborhood of $g$ in $SL(2, \mathbb{C})$. Note that $S$ involves a particular choice of the logarithm $\log : U_g \to sl_2 \mathbb{C}$. If $h \in U_g$ then $\log(h)$ shall be taken to be the element $v_h \in \tilde{U}_g$ such that $\exp(v_h) = h$.

Lemma 3.4. Let $(x_1, \ldots, x_n, y_1, \ldots, y_m) = \mathbb{F}, e \in \langle y_i \rangle, \rho \in \text{Hom}(\mathbb{F}, SL(2, \mathbb{C})) = \mathcal{R}(\mathbb{F})$ such that $\text{tr}(\rho(e)) \neq -2$. Choose a triple $(e, S, V)$ such that $\rho \in V$ and $\text{ev}_e(V) \subset U_{\rho(e)}$. Then the map $\tau_H : V \times P \to \mathcal{R}(\mathbb{F})$ defined by

$$
(\eta, z) \mapsto \begin{cases}
  x_i & \mapsto \eta(x_i) \\
  y_i & \mapsto \eta(y_i) \exp(z \cdot \log(\eta(e)))
\end{cases}
$$

is holomorphic. Similarly, if $F = \langle y_1 \ldots y_{n-1} \rangle * \langle t \rangle = F * \langle t \rangle, e \in F$, then, after choosing an appropriate triple $(e, S, V)$, the map $\tau_H : V \times P \to \mathcal{R}(\mathbb{F})$ defined by

$$
(\eta, z) \mapsto \begin{cases}
  x_i & \mapsto \eta(x_i) \\
  t & \mapsto \eta(t) \exp(z \cdot \log(\eta(e)))
\end{cases}
$$

is holomorphic.

Proof. $\tau_H$ is the composition of holomorphic functions. \qed

4. AFTER LIFTING, Restricting

Let $\Delta$ be a one-edged splitting of $G$, $\tau_{\Delta, e}$ an elementary automorphism, $\tilde{G}$ the lifted group and $\tau_{\Delta, \tilde{G}}$ the lift of $\tau_{\Delta, e}$ to $\tilde{G}$. There is a natural inclusion $\mathcal{R}(G) \subset \mathcal{R}(\tilde{G})$. Since $\mathcal{R}(\tilde{G}) \subset SL(2, \mathbb{C})^M \subset \mathbb{C}^{4M}$, if $V \subset \mathcal{R}(G)$ is an open set, then $V = \mathcal{R}(G) \cap W$ for some open subset $W \subset \mathbb{C}^{4M}$. If $\varphi : W \to \mathbb{C}$ is analytic, its restriction to $V$ is analytic by definition.

Lemma 4.1. Let $(\tilde{G}, S, V)$ be as in Lemma 3.4 and choose $\rho \in V \subset \mathcal{R}(G)$. Regard $\rho$ as a representation of $\tilde{G}$ by inclusion, $S = (U_g, \tilde{U}_g, P)$. Then $\tau_H((V \cap \mathcal{R}(G)) \times P) \subset \mathcal{R}(G)$.

Suppose $\mathcal{R}(G)_1, \ldots, \mathcal{R}(G)_k$ are the irreducible components of $\mathcal{R}(G)$ containing $\rho$. If $V_i = V \cap \mathcal{R}(G)_i$ is irreducible as an analytic variety, then $\tau_H(V_i \times P) \subset \mathcal{R}(G)_i$.

Proof. $\tau_H$ is cooked up in such a way that $\tau_H|_{V \cap \mathcal{R}(G) \times P}$ has image in $\mathcal{R}(G)$. We prove the lemma for elementary automorphisms arising from amalgamations. The argument in the case of an HNN extension is identical.

Let $K_i = \text{Ker}(\tilde{G}_i \to G_i)$, and $\{ r_j^i \}_{j=1, \ldots, \infty}$ an enumeration of $K_i$. Since finitely generated rings of polynomials over $\mathbb{C}$ are Noetherian, there exists $k < \infty$ such that

$$
\rho_i \in \mathcal{R}(G_i) \iff \forall j \leq k \left( \text{ev}_{r_j^i}(\rho_i) = I_2 \right)
$$

The inclusions $E \hookrightarrow G_i$ induce restriction maps $\mathcal{R}(G_i) \to \mathcal{R}(E)$. Since $G$ is the pushout of the diagram $\{ E \hookrightarrow G_i \}$, the representation variety $\mathcal{R}(G)$ is the pullback of the diagram $\{ \mathcal{R}(G_i) \to \mathcal{R}(E) \}$, and we identify $\mathcal{R}(G)$ with the set of pairs $(g, h) \in \mathcal{R}(G_1) \times \mathcal{R}(G_2)$ such that the restrictions $g|_E$ and $h|_E$ agree.
Since $E$ is finitely generated there are relations $g^l_i \in F_i$, $l = 1..m$, corresponding to generators of $E$, such that

$$\rho = (\rho_1, \rho_2) \in \mathcal{R}(G) \subset \mathcal{R}(G_1) \times \mathcal{R}(G_2) \iff \forall l \left( ev_{g^l_i}(\rho_1) = ev_{g^l_i}(\rho_2) \right)$$

The Dehn twists $\tau_H$ clearly preserve the relations $ev_{g^l_i}$ and since $e \in Z_{G_2}(E)$,

$$\rho_2(g^{2\cdot\exp(z\cdot\log(\eta(e)))}_i) = \rho_2(g^2_i)$$

for all $g^2_i$. Since $\tau_H$ doesn’t change the values of this finite set of equations, $\tau_H(V \times P) \subset \mathcal{R}(G)$.

The intersection of an irreducible algebraic variety with an open subset of $\mathbb{C}^{4n}$ is an analytic variety. If the intersection $V \cap \mathcal{R}(G_i)$ is irreducible then $V \times P$ is also irreducible. The image of an irreducible complex analytic variety under a holomorphic map has irreducible closure (preimages of closed sets are closed), hence must have image in an irreducible component of the range, in this case $\mathcal{R}(G)$.

Thus, since $V_i \notin \bigcup_{j \neq i} \mathcal{R}(G)_j$, $\tau_H(V_i \times P) \subset \mathcal{R}(G)_i$. □

**Definition 4.2.** Let $\mathcal{R}_2(G)$ be the union of the irreducible components of $\mathcal{R}(G)$ such that for all $g \in G$, $tr(ev_g(\cdot)) + 2$ doesn’t vanish. A component $V$ of $\mathcal{R}(G)$ is a component of $\mathcal{R}_2(G)$ if, for every $g \in G$, there is $\rho \in V$ such that $tr(\rho(g)) \neq -2$.

**Theorem 4.3.** Suppose $\tau_{\Delta, e}$ is an elementary automorphism of $G$, $\rho \in \mathcal{R}(G)_i$, and $tr(\rho(e)) \neq -2$. Then $\tau_{\Delta, e}(\mathcal{R}(G)_i) = \mathcal{R}(G)_i$.

The modular group acts trivially on the set of irreducible components of $\mathcal{R}_2(G)$.

**Proof.** By Lemma 4.1 for an appropriate triple $(\ell, S, V)$, the 1-parameter family of Dehn twists $\tau_H$ maps $V_i \times P$ to $\mathcal{R}(G)_i$ as long as $V_i$ is irreducible. The restrictions $\tau_H|_{V_i \times \{0\}}$ and $\tau_H|_{V_i \times \{1\}}$ agree with the restrictions of $id_G$ and $\tau_{\Delta, e}$, respectively. Thus $\tau_{\Delta, e}$ maps $V_i$ to $\mathcal{R}(G)_i$, and since $\tau_{\Delta, e}$ is an automorphism, $\tau(V_i)$ cannot lie in the intersection $\mathcal{R}(G)_i \cap (\bigcup_{j \neq i} \mathcal{R}(G)_j)$. Thus $\tau(\mathcal{R}(G)_i)$ shares a point with $\mathcal{R}(G)_i \backslash (\bigcup_{j \neq i} \mathcal{R}(G)_j)$, hence $\tau(\mathcal{R}(G)_i) = \mathcal{R}(G)_i$.

If $\tau_{\Delta, e}$ is an elementary automorphism of $G$ and $V$ is a component of $\mathcal{R}_2(G)$, then there is a representation $\rho \in V$ such that $tr(\rho(e)) \neq -2$. By the above, $\tau(V) = V$. Since the modular group is generated by elementary automorphisms, the claim holds. □

If $\phi: G \to H$ is a homomorphism then $\phi^{-1}(\mathcal{R}_2(H)) \subset \mathcal{R}_2(G)$. If $V$ is a component of $\mathcal{R}_2(H)$ then $\phi^{-1}(V)$ is contained in some irreducible component $W$ of $\mathcal{R}(G)$. If $g \in G$ and $\phi(g) \neq 1$ then $tr(ev_g(\cdot)) + 2$ doesn’t vanish on $W$ since it doesn’t vanish on $V$. If $\phi(g) = 1$ the same holds since $tr(I_2) = 2$.

5. **APPLICATION TO STRICT RESOLUTIONS OF LIMIT GROUPS**

In this section we give our main application of Theorem 4.3: a bound on the length of a strict resolution of a limit group which depends only on its rank. Limit groups possess two qualities which make the theory developed so far useful: many
maps to free groups, which have large representation varieties, and large automorphism groups generated by Dehn twists in one-edged splittings.

We give only enough definitions to make sense of the statement of the theorem and its proof. They will be very economical. For more information on limit groups see the exposition by Bestvina and Feighn [BF03] or Sela’s original work [Sel01].

A homomorphism $\mathbb{F} \to SL(2, \mathbb{C})$ is nondegenerate if it is injective and every element has image with trace not equal to $-2$.

**Definition 5.1.** The essential subvariety, $\mathcal{R}_e(L)$, comprises the irreducible components $V$ of $\mathcal{R}(L)$ such that there is a nondegenerate $i \in \mathcal{R}(\mathbb{F})$ and a sequence $f_n : L \to \mathbb{F}$ converging to $L$, such that $i \circ f_n \in V$.

The important feature the essential subvariety has is that for all $g \in L$ the evaluation map $ev_g$ takes non-identity values on every component. The essential subvariety for a limit group is non-empty since $\mathbb{F}$ has a nondegenerate embedding in $SL(2, \mathbb{C})$, limit groups are $\omega$-residually free, and $\mathcal{R}(L)$ has finitely many components. Since there exist nondegenerate elements of $\mathcal{R}(\mathbb{F})$, $\mathcal{R}_e(L) \neq \emptyset$. By definition $\mathcal{R}_e(L) \subset \mathcal{R}_2(L)$.

**Lemma 5.2.** If $\pi : L \to L'$ is strict then $\mathcal{R}_e(L') \subset \mathcal{R}_e(L)$ and if $V' \subset V$ are irreducible components of $\mathcal{R}_e(L')$ and $\mathcal{R}_e(L)$, respectively, then $\dim V' < \dim V$. Thus $\dim \mathcal{R}_e(L') < \dim \mathcal{R}_e(L)$.

**Proof.** To prove the first part of the claim, choose an irreducible component $V'$ of $\mathcal{R}_e(L')$ and an irreducible component $V$ of $\mathcal{R}(L)$ containing $V'$. We show that $V \subset \mathcal{R}_e(L)$.

Choose a sequence $f_n \in \text{Hom}(L', \mathbb{F})$ converging to $L'$, and a nondegenerate $i \in \mathcal{R}(\mathbb{F})$ such that $i \circ f_n$ is contained $V'$ for all $n$. Choose $\phi_n \in \text{Mod}(L)$ such that $g_n = f_n \circ \pi \circ \phi_n$ converges to $L$. Since $\mathcal{R}_2(L') \subset \mathcal{R}_2(L)$ we have $\mathcal{R}_e(L') \subset \mathcal{R}_2(L)$. By Theorem 4.3, the $\phi_n$ fix $V$ setwise and we have $i g_n \in V$ for all $n$, hence $V$ is a component of $\mathcal{R}_e(L)$ and we have the claim.

To prove the second claim choose some $g \neq 1 \in \text{Ker}(L \to L')$. By the definition of the essential subvariety, the evaluation map $ev_g$ doesn’t vanish on any irreducible component of $\mathcal{R}_e(L)$ and, since $g \in \text{Ker}(\pi)$, $ev_g$ vanishes on $\mathcal{R}_e(L')$, hence the inequality on dimensions of irreducible components is strict.

The last statement follows immediately from the first two. □

**Proof of Theorem 1.3.** We prove only the second assertion. Let $\mathbb{F}_n \to L_1 \to \ldots \to L_k \cong \mathbb{F}_m$ be a strict resolution and consider the sequence of essential subvarieties

$$\mathcal{R}_e(L_k) \cong SL(2, \mathbb{C})^m \subset \cdots \subset \mathcal{R}_e(L_1) \subset SL(2, \mathbb{C})^n$$

Let $d_i = \dim(\mathcal{R}_e(L_i))$. Since $L_i \to L_{i+1}$ is strict and proper, by Lemma 5.2, $d_i > d_{i-1}$. Since $d_k = 3m$ and $d_1 \leq 3n$, $k$ must be at most $3(n - m)$. □

6. ACCESSIBILITY FOR LIMIT GROUPS

Sela proves [Sel01, Theorem 4.1 and Proposition 4.3] that the height of the cyclic analysis lattice of a limit group is quadratic in the first Betti number. One
application of Theorem 1.3 is the following companion to Sela’s result. This kind of argument is used in [Lou08a, Theorem 2.11] and [Lou08b, Theorem 2.5].

**Definition 6.1 (Analysis lattice).** The abelian (cyclic) analysis lattice of a finitely generated group $G$ is the following tree of groups. Free and abelian groups have no children.

- Level 0 consists of $G$.
- Level 0.1 consists of the freely indecomposable free factors of $G$ and the free group of some Grushko free factorization of $G$. The groups in level 0.1 are the children of the node labeled $G$.
- Level 1 consists of the rigid, abelian, and quadratically hanging subgroups in the abelian (cyclic) JSJ’s of the freely indecomposable free factors of $G$ at level 0.1. The parent of a group at level 1 is the freely indecomposable free factor it is a subgroup of.
- Level 1.1 is constructed exactly as level 0.1 was.

**Theorem 6.2.** The height of the abelian analysis lattice of a limit group is bounded by three times its rank.

**Remark 6.3.** Before we begin, note if that $L_v$ is a vertex group of the abelian JSJ decomposition of $L$ then the restriction of every modular automorphism to $L_v$ agrees with the restriction of an inner automorphism since $L_v$ is elliptic in every splitting of $L$ over an abelian subgroup. If $\pi: L \to L'$ is strict and $g \in \ker(L_v \to L')$, then $g$ is in the kernel of every element of $\pi \circ \text{Mod}(L)$. Since $\pi$ is strict $g$ must be the identity element, therefore $L_v$ embeds in $L'$.

**Proof.** Let $L$ be a limit group generated by $n$ elements. We prove something slightly more general: that the height of the abelian analysis lattice is bounded linearly by the length of the shortest strict resolution $L$ admits. Observe that if $L' < L$ then any strict resolution of $L$ restricts to a strict resolution of $L'$.

Let $L = L_1 \to L_2 \to \cdots \to L_k$ be a shortest strict resolution of $L$. If $k = 1$ then $L$ has height 0 and we may stop. Otherwise, let $L''$ be a group at level 1 of the analysis lattice. If $L''$ is free or abelian it has height 0 so we may stop the procedure. If not, then by the previous paragraph $L''$ embeds in $L_2$, and by induction the height of the abelian analysis lattice of $L_2$ is at most $k - 1$, the height of the analysis lattice of $L''$ is at most $k - 1$. Since the analysis lattice of $L$ is obtained by grafting the analysis lattices of the vertex groups of the freely indecomposable free factors of $L$ (the groups $L''$) to the leaves at level 0.1 in the analysis lattice of $L$, its height is at most $k$. Since $k \leq 3n$ the theorem is proven. □

7. Remarks

There is an easier proof of Theorem 1.3 which does not use modular group and generalizes to limit groups over linear groups. One uses equational Noetherianness, the fact that the representation variety has only finitely many components, and a diagonal argument to achieve the same end.
The preprint [Hou08] of Ould-Houcine is written in more model theoretic language than this paper, and uses an argument like the above. Theorem 1.3 follows from his bound on the Cantor-Bendixon rank of the closure of the space of marked free groups plus finite presentability. To clarify, if a sequence of marked limit groups \((G, S_i)\) converges to a marked limit group \((H, S)\), then \(H\) has a strict homomorphism onto \(G\). Conversely, let \(S\) be a generating set for \(H\), and fix a strict homomorphism \(\pi: H \to G\). Suppose \(f_i: G \to \mathbb{R}\) converges to \(G\). If \(f_i \circ \pi \circ \phi_i, \phi_i \in \text{Mod}(H)\), converges to \(H\), then the sequence of marked limit groups \((G, \pi \circ \phi_i(S))\) converges to the marked limit group \((H, S)\).

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