Essentially semismall Quasi-Dedekind module relative to a module

Mukdad Q Hussain
Department of Computer Sciences, College of Education for pure science, Diyala University, Diyala, Iraq.

Abstract. Let R be associative ring with identity and M be a unitary R-module. In this paper study the direct summand of essentially semismall quasi-Dedekind module and prove that the direct sum of essentially semismall quasi-Dedekind modules need not be essentially semismall quasi-Dedekind and give the definition of essentially semismall quasi-Dedekind relative to a module with some examples, also give some of their basic properties and some examples that illustrate these properties.

Keywords. Semismall modules, Quasi-Dedekind module.

1. Introduction
In this paper study the direct summand of essentially semismall quasi-Dedekind module and prove that the direct sum of essentially semismall quasi-Dedekind modules need not be essentially semismall quasi-Dedekind. Also, give the definition of essentially semismall Quasi-Dedekind relative to a module. A submodule A of an R-module M is called small in M (A ≪ M) if whenever a submodule B of M with M = A + B implies B = M [1]. An R-submodule N of an R-module M is called essentially small (N ≪_e M), if for every nonzero small submodule K of M, K∩N ≠ {0}. Equivalently, for each 0 ≠ x ∈ M, there exists 0 ≠ r ∈ R such that 0 ≠ rx ∈ N. A proper submodule A of an R-module M is called semismall in M (A ≪_s M) if A = 0 or A/K ≪ M/K for all nonzero submodules K of A [1]. A submodule N of an R-module M is called semismall invertible if N^{-1}N = M, where N^{-1} = \{x ∈ R_T: xN ≪_s M\} and R_T is the localization of R at T in the usual sense, T = \{s ∈ S: sm = 0 for some m ∈ M, then m = 0\}, where S is the set of all nonzero divisors of R. An R-module M is called semismall quasi-Dedekind, if every nonzero R submodule N of M is semismall quasi-invertible; that is Hom (M/N, M) = {0}, for all \{0\} ≠ N ≪_s M. A ring R is semismall quasi-Dedekind if R is a semismall quasi-Dedekind R-module. An R-module M is called essentially semismall quasi-Dedekind if Hom (M/N, M) = \{0\} for all N ≪_se M.
A ring $R$ is essentially semismall quasi-Dedekind if $R$ is an essentially semismall quasi-Dedekind $R$-module. The property of essentially semismall quasi-Dedekind module is inherited by direct summand.

**Proposition 1** A direct summand of an essentially semismall Quasi-Dedekind module is an essentially semismall Quasi-Dedekind module.

**Proof:** Let $J$ be an essentially semismall Quasi-Dedekind $R$-module and let $E \triangleleft J$, then $J = E \oplus V$ for some submodule $V \leq J$. Let $f \in \text{End}_R(E)$, $f \neq 0$, to prove that $\text{Ker}f \triangleleft_{\text{se}} E$. Consider the following:

$$J \xrightarrow{\rho} E \xrightarrow{i} J$$

where $\rho$ is the natural projection, and $i$ is the inclusion mapping. Hence $h = i \circ f \in \text{End}_R(J)$ and $h \neq 0$, so $\text{Ker}h \triangleleft_{\text{se}} J$ and since $\text{Ker}f \subseteq \text{Ker}h$ then $\text{Ker}f \triangleleft_{\text{se}} J$. Now assume that $\text{Ker}f \triangleleft_{\text{se}} E$, this implies $\text{Ker}h \triangleleft_{\text{se}} J$ and so get a contradiction.

Let $x + y$ be any nonzero element of $J$, where $x \in E$, $y \in V$. If $x \neq 0$ and $y \neq 0$, then since $\text{Ker}f \triangleleft_{\text{se}} E$, there exists $0 \neq r \in R$ such that $0 \neq r x \in \text{Ker}f$. Hence $r x + r y = 0$, because if $r x + r y = 0$, then $r x = -r y \in \text{End}_R(V) = \{0\}$ which is a contradiction. Also $h(r x + r y) = 0$; that is $0 \neq r(x+y) \in \text{Ker}h$. If $x = 0$ and $y \neq 0$, then $x + y = y \neq 0$ and $1.y = y$, $h(y) = i \circ f(y) = f(0) = 0$; that is $0 \neq 1(x+y) = y \in \text{Ker}h$. Therefore $\text{Ker}f \triangleleft_{\text{se}} J$ which is a contradiction. Thus the assumption is false and hence $\text{Ker}f \triangleleft_{\text{se}} E$; that is $E$ is an essentially semismall Quasi-Dedekind $R$-module.

The next example show the direct sum of essentially semismall Quasi-Dedekind modules is not necessarily essentially semismall Quasi-Dedekind module.

**Example 2** It is known that $Z$ and $Z_2$ are essentially semismall Quasi-Dedekind as $Z$-modules. But $N = Z \oplus Z_2$ is not essentially semismall Quasi-Dedekind $Z$-module. since if $f : N \longrightarrow N$ define by $f(x, y) = (0, x)$, $x \in Z$, $\overline{y} \in Z_2$, then $f \neq 0$ and $\text{Ker}f = \{(x, y) \in N : f(x, y) = (0, 0)\} = \{(x, y) \in N : \overline{x} = \overline{0}\} = 2Z \oplus Z_2$. Hence $\text{Ker}f \triangleleft_{\text{se}} N$. Thus $N = Z \oplus Z_2$ is not essentially semismall quasi-Dedekind as a $Z$-module.

Let $M$ and $N$ be $R$-modules. $M$ is an essentially semismall quasi-Dedekind $(K$-nonsingular) relative to $N$ if, for all $f \in \text{Hom}(M, N)$, $f \neq 0$, implies $\text{Ker}f \triangleleft_{\text{se}} M$.

**Remarks and Examples 3**

1) Let $J$ be an $R$-module. Then $J$ is essentially semismall Quasi-Dedekind if and only if $J$ is essentially semismall Quasi-Dedekind relative to $J$.

2) Let $J$ be an essentially semismall Quasi-Dedekind $R$-module. Then $J$ is an essentially semismall Quasi-Dedekind relative to $E$, for all $E \leq J$.

**Proof:** Let $E \leq J$. If $E = J$, then $J$ is an essentially semismall Quasi-Dedekind relative to $E$. If $E \subsetneq J$, assume that $f \in \text{Hom}(J, E)$, $f \neq 0$. Hence $i \circ f \in \text{End}_R(J)$, $(i \circ f) \neq 0$, where $i$ is the inclusion mapping. Since $J$ is an essentially semismall Quasi-Dedekind $R$-module, then $\text{Ker}(i \circ f) \triangleleft_{\text{se}} J$. But $\text{Ker}f = \text{Ker}(i \circ f)$, thus $\text{Ker}f \triangleleft_{\text{se}} J$ and so $J$ is an essentially semismall quasi-Dedekind relative to $E$.

3) Every uniform $R$-module $J$ is an essentially semismall Quasi-Dedekind relative to $N$, where $N$ is any $R$-module.

4) Any semisimple $R$-module $M$ is an essentially semismall Quasi-Dedekind relative to $E$, where $E$ is any $R$-module.

5) $Z_{12}$ is not essentially semismall quasi-Dedekind relative to $Z_6$, since there exists $h : Z_{12} \rightarrow Z_6$ defined by $h(x) = 3\bar{x}$ for all $\bar{x} \in Z_{12}$, hence $\text{Ker}h = (\overline{2}) \triangleleft_{\text{se}} Z_{12}$. 


Theorem 4 Let \( \{ O_i \}_{i \in \Lambda} \) be a family of modules. Then \( M = \bigoplus_{i \in \Lambda} O_i \) is essentially semismall quasi-Dedekind if and only if \( O_1 \) is an essentially semismall Quasi-Dedekind relative to \( O_1 \), for all \( i, j \in \Lambda \).

**Proof:** give the details of proof of this theorem for \( i \in \Lambda = \{1, 2\} \), and the proof for any \( \Lambda \) is similarly.

\( \Rightarrow \) Since \( M = M_1 \oplus M_2 \) is an essentially semismall quasi-Dedekind \( R \)-module, then by Prop 1, \( M_1 \) and \( M_2 \) are essentially semismall quasi-Dedekind \( R \)-modules. So \( M_1 \) is an essentially semismall quasi-Dedekind relative to \( M_1 \) and \( M_2 \) is an essentially semismall quasi-Dedekind relative to \( M_2 \). Now, to prove that \( M_1 \) is an essentially semismall quasi-Dedekind relative to \( M_2 \). Let \( f: M_1 \rightarrow M_2, f \neq 0 \). Consider the following: \( M \xrightarrow{\rho} M_1 \xrightarrow{f} M_2 \xrightarrow{i} M \), where \( \rho \) is the natural mapping. Then \( h = iof \circ \rho \in \text{End}_R(M) \) and \( h \neq 0 \), thus \( \ker h \leq \ker f \leq \ker M \), but \( \ker f \leq \ker M \) which implies \( \ker f \leq \ker h \). Now, to prove that \( \ker f \leq \ker M \). Suppose that \( \ker f \leq \ker M \), then \( \ker f \leq \ker M \leq \ker f \oplus M_2 = M_2 \), to show that \( \ker f = \ker f \oplus M_2 \). Let \( x \in \ker f \), \( y \in M_2 \), \( h(x + y) = iof_{\rho}(x + y) = iof(x) = 0 \), thus \( \ker f \oplus M_2 \leq \ker h \), and let \( x + y \in \ker h \leq M_1 \oplus M_2 \), so \( x \in M_1, y \in M_2 \), since \( h(x + y) = 0 \) implies (iofo\( \rho(x + y) = 0 \), so \( iof(x) = 0 \) then \( f(x) = 0 \); that is \( x \in \ker f \), thus \( \ker h \leq \ker f \oplus M_2 \). Hence \( \ker h = \ker f \oplus M_2 \leq \ker M_2 \leq M_2 = M \), which is a contradiction. Therefore \( \ker f \leq \ker M_1 \) and hence \( M_1 \) is an essentially semismall quasi-Dedekind relative to \( M_2 \).

Similarly, \( M_2 \) is an essentially semismall quasi-Dedekind relative to \( M_1 \).

\( \Leftarrow \) Assume \( \psi : M \rightarrow M \) such that \( \ker \psi \leq \ker M \), so \( \ker \psi \cap M_1 \leq \ker M_1 \). Let \( \psi \big|_{M_1} : M_1 \rightarrow M \) such that \( \psi \big|_{M_1}(x) = \psi(x + 0), \forall x \in M_1 \), then \( \ker \psi \big|_{M_1} = \ker \psi \cap M_1 \), to see this: let \( x \in \ker \psi \big|_{M_1} \) implies \( 0 = \psi \big|_{M_1}(x) = \psi(x + 0) = \psi(x) \). It follows that \( x \in \ker \psi \cap M_1 \). Also, let \( x \in \ker \psi \cap M_1 \), so \( x \in M_1 \) and \( 0 = \psi(x) = \psi(x + 0) = \psi \big|_{M_1}(x) \), so \( x \in \ker \psi \big|_{M_1} \). Consider the following: \( M_1 \xrightarrow{\psi \big|_{M_1}} M \xrightarrow{\rho_1} M_1 \) and

\( M_1 \xrightarrow{\psi \big|_{M_1}} M \xrightarrow{\rho_2} M_2 \), where \( \rho_1, \rho_2 \) are the natural projections. Claim that \( \ker(\rho_1 \circ \psi \big|_{M_1}) \cap \ker(\rho_2 \circ \psi \big|_{M_1}) \supseteq \ker \psi \big|_{M_1} \). To prove our assertion: Let \( x \in \ker(\psi \big|_{M_1}) \) then \( \psi \big|_{M_1}(x) = 0 \), hence \( \rho_1 \circ \psi \big|_{M_1}(x) = \rho_1(\psi \big|_{M_1}(x)) = 0 \) and \( \rho_2 \circ \psi \big|_{M_1}(x) = 0 \). Thus \( x \in \ker(\rho_1 \circ \psi \big|_{M_1}) \cap \ker(\rho_2 \circ \psi \big|_{M_1}) \); that is \( \ker(\rho_1 \circ \psi \big|_{M_1}) \cap \ker(\rho_2 \circ \psi \big|_{M_1}) \supseteq \ker \psi \big|_{M_1} \) But \( \ker(\psi \big|_{M_1}) = \ker \psi \cap M_1 \leq \ker M_1 \), so \( \ker(\rho_1 \circ \psi \big|_{M_1}) \rangle \ker(\rho_2 \circ \psi \big|_{M_1}) \leq \ker \psi \big|_{M_1} \) and hence \( \ker(\rho_1 \circ \psi \big|_{M_1}) \langle \ker M_1 \) and \( \ker(\rho_2 \circ \psi \big|_{M_1}) \langle \ker M_1 \). But \( M_1 \) is an essentially semismall quasi-Dedekind relative to \( M_1 \) and \( M_1 \).
is essentially semismall quasi-Dedekind relative to $M_2$, through hypothesis. So that $\rho_1 o \psi \big|_{M_1} = 0$, $\rho_2 o \psi \big|_{M_1} = 0$ ... (1), by a similar way, obtain $\rho_1 o \psi \big|_{M_2} = 0$, $\rho_2 o \psi \big|_{M_2} = 0$ ... (2). Hence by (1) and (2), have $\psi = 0$.

**Proposition 5** Let $J$ be an essentially semismall Quasi-Dedekind ($K$-nonsingular) module, and let $N \leq j$ Whether $E \ll E_1 \leq \oplus M$, for all $I = 1, 2$, then $E_1 = E_2$.

**Proof** Consider the endomorphism $(I - \rho_1) \rho_2$, $\rho_1$ is the natural projections of $J$ onto $N_i$, $i = 1, 2$; that is $\rho_1: J \to E_1, \rho_2: J \to E_2$. Since $E \subseteq E_1$ and $E \subseteq E_2$, so $d_1(e) = e, d_2(e) = e$, for all $e \in E$. Hence for each $e \in E ((I - d_1)(d_2)(e)) = I - d_1(d_2(e)) = I - d_1(e) = I(e) - d_1(e) = 0$, so $E \subseteq \ker([I - d_1]d_2)$ ... (1). Since $E_2 \leq \oplus J$, so there exists $K_2 \subseteq J$ such that $E_2 \oplus K_2 = J$, and since for each $k \in K_2$, $([I - d_1]d_2)k = (I-d_1)(d_2)k = (I-d_1)(0) = 0$ implies $K_2 \subseteq \ker([I - d_1]d_2)$ ... (2). Now, from (1) and (2) $E \oplus K_2 \subseteq \ker([I - \rho_1]\rho_2)$, but $E \ll E_2, K_2 \ll E_2, E \oplus K_2 \ll E_2 \oplus K_2 = J$. Hence $\ker([I - \rho_1]\rho_2) \ll J$, so $(I - \rho_1)\rho_2 = 0$ (since $J$ is an Essentially semismall Quasi-Dedekind module). It follows that $\rho_2 = \rho_1 o \rho_2$. Now, to prove that $E_2 \subseteq E_1$. Let $u \in E_2$, then $\rho_2(u) = u$. Hence $d_1(d_2(u)) = d_1(u)$, then $d_1(u) = d_2(u) = u$. Hence $x \in E_1$, thus $E_2 \subseteq E_1$. Similarly by taking $(I - \rho_2)\rho_1$ and showing it is zero, then obtain $E_1 \subseteq E_2$. Thus $E_1 = E_2$.

**References**

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