Two-loop $\mathcal{N} = 4$ Super Yang Mills effective action 
and interaction between D3-branes

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Abstract

We compute the leading low-energy term in the planar part of the 2-loop contribution to the effective action of $\mathcal{N} = 4$ SYM theory in 4 dimensions, assuming that the gauge group $SU(N+1)$ is broken to $SU(N) \times U(1)$ by a constant scalar background $X$. While the leading 1-loop correction is the familiar $c_1 F^4/|X|^4$ term, the 2-loop expression starts with $c_2 F^6/|X|^8$. The 1-loop constant $c_1$ is known to be equal to the coefficient of the $F^4$ term in the Born-Infeld action for a probe D3-brane separated by distance $|X|$ from a large number $N$ of coincident D3-branes. We show that the same is true also for the 2-loop constant $c_2$: it matches the coefficient of the $F^6$ term in the D3-brane probe action. In the context of the AdS/CFT correspondence, this agreement suggests a non-renormalization of the coefficient of the $F^6$ term beyond two loops. Thus the result of hep-th/9706072 about the agreement between the $v^6$ term in the D0-brane supergravity interaction potential and the corresponding 2-loop term in the 1+0 dimensional reduction of $\mathcal{N} = 4$ SYM theory has indeed a direct generalization to 1+3 dimensions, as conjectured earlier in hep-th/9709087. We also discuss the issue of gauge theory – supergravity correspondence for higher order ($F^8$, etc.) terms.

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1 Introduction

The remarkable relation between supersymmetric gauge theories and supergravity implied by existence of D-branes in string theory [1, 2, 3, 4] motives detailed study of quantum corrections in super Yang-Mills theory, and, in particular, their non-renormalization properties. One aspect of this relation which will be of interest to us here is a correspondence between the $\mathcal{N} = 4$ super Yang-Mills theory and type IIB supergravity descriptions of subleading terms in the interaction potential between parallel D3-branes (see, e.g., [5, 6, 7, 8, 9, 10, 11, 12] for related discussions of interactions between D-branes).

Consider the supergravity-implied action for a D3-brane probe in curved background produced by a large number $N$ of coincident D3-branes. Ignoring higher-derivative (“acceleration”) terms, it is given by the Born-Infeld action in the corresponding curved metric and the “electric” part of the R-R 4-form potential,

$$S = -T_3 \int d^4 x \ H^{-1}(X) \left[ \sqrt{-\det(\eta_{mn} + H(X) \partial_m X^i \partial_n X^i + H^{1/2}(X) F_{mn})} - 1 \right]. \quad (1.1)$$

Here $i = 1, ..., 6$, $m, n = 0, 1, 2, 3$, $T_3 = \frac{1}{2\pi g_s}$ and $H = 1 + \frac{Q}{|X|^4}$, $Q \equiv \frac{1}{\pi} N g_s$. In addition, the action contains also the “magnetic” interaction part given by the Chern-Simons term,

$$S_{\text{mag}} = NS_{\text{WZ}} \sim i N f_5 \epsilon_{i_1 ... i_6} \frac{1}{|X|^6} X^{i_1} dX^{i_2} \wedge ... \wedge dX^{i_6}.$$ In what follows we shall consider the case when $X^i = \text{const}$, i.e. ignore all scalar derivative terms. We shall assume that $\frac{Q}{|X|^4} \gg 1$, i.e. that one can drop 1 in the harmonic function $H$, so that (1.1) becomes the same as the action for a D3-brane probe in the $AdS_5 \times S^5$ space (oriented parallel to the boundary of $AdS_5$)

$$S = -T_3 \int d^4 x \frac{|X|^4}{Q} \left[ \sqrt{-\det(\eta_{mn} + \frac{Q^{1/2}}{|X|^2} F_{mn})} - 1 \right]. \quad (1.2)$$

Expanding in powers of $F$, we get

$$S = -T_3 \int d^4 x \left( -\frac{1}{4} F^2 - \frac{1}{8} \frac{Q}{|X|^4} [F^4 - \frac{1}{4} (F^2)^2] - \frac{1}{12} \frac{Q}{|X|^4} [F^6 - \frac{3}{8} F^4 F^2 + \frac{1}{32} (F^2)^3] + ... \right), \quad (1.3)$$

where $F^k$ is the trace of the matrix product in Lorentz indices, i.e. $F^2 = F_{mn} F_{mn}$.

\footnote{We set string tension $T = \frac{1}{2\pi \alpha'}$ to be 1. In general, $T^{-1}$ appears in front of $F_{mn}$ and in the relation between the scale $X$ in the supergravity expressions and the scalar expectation value $\Phi$ in the SYM expressions.}
\[ F^2_{mn}, \quad F^k = F_{m_1 m_2} F_{m_2 m_3} \ldots F_{m_k m_1} \]

The general structure of this expansion is thus

\[ S = \frac{1}{g_s} \int d^4x \sum_{l=0}^{\infty} c_l (g_s N)^l \frac{F^{2l+2}}{|X|^4} . \]  

(1.4)

From the weakly-coupled flat-space string theory point of view, the leading-order interactions between separated D-branes are described by the “disc with holes” diagrams (i.e. annulus, etc). The limit of small separation should be represented by loop corrections in SYM theory, while the limit of large separation – by classical supergravity exchanges. If the coefficient of a particular term in the string interaction potential (like \( v^4 \) term in [6]) happens not to depend on the distance (i.e. on dimensionless ratio of separation and \( \sqrt{\alpha'} \)) then its coefficient should be the same in the quantum SYM and the classical supergravity expressions for the interaction.

In the SYM theory language, computing the interaction potential between a stack of D3-branes and a parallel D3-brane probe carrying constant \( F_{mn} \) background field corresponds to computing the quantum effective action \( \Gamma \) in constant scalar \( \Phi \) background which breaks \( SU(N + 1) \) to \( SU(N) \times U(1) \) and in constant \( U(1) \) gauge field \( F_{mn} \). For the interactions between \( D3 \)-branes, i.e. in the case of finite \( N' = 4, \ D = 4 \) SYM theory, the expansion of \( \Gamma \) in powers of the dimensionless ratio \( F^2/|\Phi|^4 \) has the following general form

\[ \Gamma = \frac{1}{g^2_{YM}} \int d^4x \sum_{l=0}^{\infty} f_l(g^2_{YM}, N) \frac{F^{2l+2}}{|\Phi|^4} . \]  

(1.5)

In the planar (large \( N \), fixed \( \lambda \equiv g^2_{YM} N \)) approximation the functions \( f_l \) should depend only on \( \lambda \).

In more detail, the planar \( l \)-loop diagrams we are interested in are the ones where the background field legs are attached to the “outer” boundary only (in double-line notation). In D-brane interaction picture, this corresponds to one boundary of the \( l \)-loop graph attached to one D-brane (which carries \( F_{mn} \) background) and all other \( l \) boundaries attached to \( N \) coincident “empty” D-branes. This produces the factor of \( N^l \).

The comparison of (1.5) with the supergravity expression (1.4) is done for \( g^2_{YM} = 2\pi g_s \) and \( |\Phi| = T|X| \). Naively, it would work term-by-term if \( f_l(g^2_{YM} N) = c_l(g^2_{YM} N)^l \), i.e. if the \( l \)-th term in (1.5) would receive contribution only from the \( l \)-th loop order.

This is indeed what is known to happen for the leading \( F^4/|X|^4 \) term\(^2\) which ap-

\(^2\)Note that the sum of the \( F^4 \) and \( F^6 \) can be represented as follows: \(-\frac{1}{8|X|^4} [F^4 - \frac{1}{4}(F^2)^2] (1 - \frac{1}{4|X|^4} F_{mn} F_{mn}) \) (in fact, all higher terms in the expansion of the \( D = 4 \) BI action are proportional to the \( F^4 \) term\(^3\)).

\(^3\)Since we set \( T = 1 \) (see footnote 1) in what follows we shall not distinguish between \( \Phi \) and \( X \).
pears only at the first loop order, and not at higher orders due to the existence of a non-renormalization theorem \[14\]. Moreover, the one-loop coefficient of the $F^4$ term in the $\mathcal{N} = 4$ SYM effective action \[18, 19\] is in precise agreement with the supergravity expression (see, e.g., \[5, 12, 9\]).

In the D3-brane case, there is also another viewpoint suggesting a correspondence between the classical supergravity D3-brane probe action \(1.2\) and the quantum SYM effective action \(1.5\) – the AdS/CFT conjecture \[4, 20\]. In the present context it implies that the supergravity action \(1.2\) should agree with the strong 't Hooft coupling limit of the planar (large $N$) part of the SYM action \(1.5\). The AdS/CFT conjecture thus imposes a weaker restriction that $f_\ell(\lambda)_{\lambda \gg 1} \to a_\ell \lambda^\ell$, with $a_\ell$ being directly related to $c_\ell$ in \(1.2\). The simplest possibility to satisfy this condition would be realized if the functions of coupling in front of some of the terms in \(1.5\) would receive contributions only from the particular orders in perturbation theory, and with the right coefficients to match \(1.2\), i.e. if they would not be renormalized by all higher-loop corrections. As we shall see, while this is likely to be the case at the $F^6$ order, the situation for $F^8$, etc., terms is bound to be more complicated.

### 1.1 The $F^6$ term

In this paper we shall study this correspondence for the $F^6$ term by explicitly computing its 2-loop coefficient in the $\mathcal{N} = 4$ SYM theory. The Lorentz structure of this term in the SYM effective action is the same as in the BI action \(1.3\) (the form of the abelian $F^6$ term is, in fact, fixed uniquely by the $\mathcal{N} = 1$ supersymmetry), and the planar ($N \gg 1$) part of its coefficient will turn out to be exactly the same as appearing in \(1.4\).

This precise agreement between the supergravity and the SYM actions at the $F^6$ level we shall establish below was conjectured earlier in \[12\] being motivated by the agreement \[11\] between the $v^6/|X|^{14}$ term in the interaction potential between D0-branes in the supergravity description and the corresponding 2-loop term in the effective action of maximally supersymmetric 1+0 dimensional SYM theory.\[1\]

\[4\]The absence of the 2-loop $F^4$ correction was proved in 1+3 dimensional $\mathcal{N} = 4$ SYM theory in \[15\], and similar result was found in 1+0 dimensional theory \[14\]. For indications of existence of more general non-renormalization theorems see \[17\].

\[5\]It was further conjectured in \[13\] that all terms in the BI action may be reproduced from the SYM effective action (see also \[1, 4, 21\] for similar conjectures). We shall present a more precise version of this conjecture in section 1.2 and section 5.

\[6\]This conjecture was tested in \[12\] (in the general non-abelian case) by demonstrating that there is a
Combined with the known fact that the abelian $F^6$ term does not appear at the one-loop $\mathcal{N} = 4$ SYM effective action \cite{19,10} (see eq.(1.8) below), this suggests that this 2-loop coefficient should be exact, i.e. the abelian $F^6$ term should not receive contributions from all higher ($l \geq 3$) loop orders. Indeed, from the point of view of the AdS/CFT correspondence, higher order $(g_{YM}^2N)^n$ corrections to $f_2$ in (1.5) would dominate over the two-loop one for $\lambda \gg 1$, spoiling the interpretation of (1.4) as the strong-coupling limit of (1.3).

One should thus expect the existence of a new non-renormalization theorem for the abelian $F^6$ term in $\Gamma$ (computed in the planar approximation), analogous to the well-known $F^4$ theorem of \cite{14}. The reasoning used in \cite{14} (based on scale invariance and $\mathcal{N} = 2$ supersymmetry) does not, however, seem to be enough to prove the non-renormalization of the $F^6$ term. It is most likely that one needs to use the full power of 16 supersymmetries of the theory (which are realized in a “deformed” way). One may then expect to show that the $\mathcal{N} = 4$ supersymmetry demands that the coefficient of the $F^6$ term should be rigidly fixed in terms of the $F^4$-coefficient (proportional to its square); then the fact the $F^4$ term appears only at the 1-loop order would imply that the $F^6$ term should be present only at the 2-loop order.

One possible way to demonstrate this would be to apply the component approach, by generalizing to 1+3 dimensions what was done for the $v^6/|X|^{14}$ term in 1+0 dimensions \cite{22} (see also \cite{23}), i.e. by deforming the supersymmetry transformation rules order by order in $1/|X|^2$ and trying to show that the coefficient of the $F^6/|X|^8$ term in $\Gamma$ is completely fixed by the supersymmetry in terms of the coefficient of the $F^4/|X|^4$ term. In effect, this is what was already done in \cite{24} in $D = 10, \mathcal{N} = 1$ SYM theory in a $U(1)$ background. It was shown there that the structure of the abelian $F^4$ and $F^6$ terms in the effective action which starts with the super Maxwell term is completely fixed by the (deformed) $\mathcal{N} = 1, D = 10$ supersymmetry to be the same as in the BI action with the coefficient of the $F^6$ term being related to that of the $F^4$ term in precisely the same way as it comes out of the expansion of the BI action. We thus expect that the arguments

\footnote{In $D = 10$ the role of $1/|X|$ or a fundamental scale is played by an UV cutoff (or $\sqrt{\alpha'}$), powers of which multiply $F^n$.}

\footnote{Let us mention also that the coefficient of the $F^6$ term is fixed uniquely in terms of the coefficient of

universal $\mathcal{N} g_{YM}^2 \left( N \right)^6$ expression on the SYM side which reproduces subleading terms in the supergravity potentials between various bound-state configurations of branes. Since brane systems with different amounts of supersymmetry are described by very different SYM backgrounds, the assumption that all of the corresponding interaction potentials originate from a single universal SYM expression provided highly non-trivial constraints on the structure of the latter.}
may indeed have a direct counterpart in $D = 4$ theory, relating the $F^4$ and $F^6$ coefficients and thus providing a proof of the non-renormalization of the $F^6$ term beyond the 2-loop order.

### 1.2 Comments on higher-order terms

The obvious question then is what happens at the next – $F^8$ order: should one expect that the coefficients of these terms in the SYM effective action are again receiving contributions only from the corresponding – 3-loop – graphs and that they are in agreement with the supergravity expression (1.2)? That the story for the $F^8$ term should be different from the one for the $F^6$ (and $F^4$) term is indicated by the fact that the 1-loop SYM effective action is already containing a non-trivial $F^8$ term (see (1.8)). One could still hope that the $F^8$ term will not receive corrections beyond the 3-loop order, so that the 3-loop contribution will dominate over the 1-loop and 2-loop ones in the supergravity limit ($N g_{YM}^2 \gg 1, N \gg 1$).

However, the situation is more complicated since, in contrast to what was the case for the $F^4$ and $F^6$ terms, the supersymmetry alone does not constrain the structure of the $F^8$ invariant in a unique way – the $F^8$ terms in the BI action and in the 1-loop SYM effective action have, in fact, different Lorentz index structures!

Indeed, considering four Euclidean dimensions and choosing the $U(1)$ gauge field background $F_{mn}$ to have canonical block-diagonal form with non-zero entries $f_1 = F_{12}$ and $f_2 = F_{34}$ one can explicitly compare the expansions of the BI action and the 1-loop SYM effective action. For the BI action we get (we use a constant $s$ instead of $H_{1}^{1/2}$ in (1.2) to facilitate comparison with the SYM expression)

$$\sqrt{\text{det}(\delta_{mn} + sF_{mn})} = 1 + \frac{1}{2}(f_1^2 + f_2^2)s^2 - \frac{1}{8}(f_1^2 - f_2^2)^2s^4 + \frac{1}{16}(f_1^2 - f_2^2)^2(f_1^2 + f_2^2)s^6 - \frac{1}{128}(f_1^2 - f_2^2)^2(5f_1^4 + 6f_1^2f_2^2 + 5f_2^4)s^8 + O(s^{10}).$$

(1.6)

The 1-loop Euclidean Schwinger-type effective action for the $\mathcal{N} = 4$ SYM theory (with gauge group $SU(N+1)$ broken to $SU(N) \times U(1)$ by a scalar field background $X$) depending on a constant $U(1)$ gauge field strength $F_{mn}$ parametrized by $(f_1, f_2)$ has the following form that of $F^4$ term (to be exactly as in the BI action) by the condition of self-duality of the $\mathcal{N} = 4$ SYM effective action written in terms of $\mathcal{N} = 2$ superfields [23, 24].
\[
\Gamma_E^{(1)} = -\frac{4N V_4}{(4\pi)^2} \int_0^\infty \frac{ds}{s^3} e^{-s|X|^2} K(s),
\]

\[
K(s) = \frac{f_1 s}{\sinh f_1 s} \frac{f_2 s}{\sinh f_2 s} (\cosh f_1 s - \cosh f_2 s)^2
\]

\[
= \frac{1}{4} (f_1^2 - f_2^2)^2 s^4 + 0 \times s^6 + \frac{1}{960} (f_1^2 - f_2^2)^4 s^8 + O(s^{10}).
\]

The \( F^6 \) term in (1.8) cancels out, but \( F^8 \sim f^8 \) term is present and has the structure different from that of the \( F^8 \) term in (1.6).

Let \((F^8)_1\) and \((F^8)_2\) denote the bosonic parts of the two abelian super-invariants appearing at the \( F^8 \) order in the expansions of the BI action (1.6) and the SYM 1-loop action (1.8) respectively. We propose the following \textit{conjecture} about the SYM effective action (which replaces the earlier conjecture in [12]): (i) the coefficient of \((F^8)_1\) receives contribution only from the 3-loop order with coefficient which is in precise agreement with the one in the supergravity BI action (1.2); (ii) the coefficient of \((F^8)_2\) receives contributions from all loop orders, but the planar part of the resulting non-trivial function \( f_3(N g_{YM}^2) \) in (1.5) goes to zero in the limit \( N g_{YM}^2 \gg 1 \), so that the requirement of the AdS/CFT correspondence is satisfied.

The possible existence of the \textit{two} independent invariants – one of which has “protected” coefficient and another does not is reminiscent of the situation for the \( R^4 \) invariants in type IIA string theory (see [27] and refs. there). While a “universality” or “BPS saturation” of coefficients of terms with higher than 6 powers of \( F \) may seem less plausible, there are, in fact, string-theory examples of \textit{specific} higher-order terms that receive contributions only from one particular loop order, to all orders in loop expansion [28].

9We shall consider the \( U(1) \) background representing a single D3-brane with gauge field \( F_{mn} \) separated (in one of the 6 transverse directions) by a distance \( X \) from \( N \) coincident D3-branes. The \( U(N+1) \) background matrix in the fundamental representation is then \( \hat{F}_{mn} = \text{diag}(F_{mn},0,...,0) \) and the corresponding \( SU(N+1) \) matrix is \( F = \hat{F} - \frac{1}{N+1} I \), where \( \text{tr} \) is the trace in the fundamental representation. Only this traceless part of the \( U(N+1) \) background couples to quantum fields and thus enters the quantum effective action (the planar part of which will be proportional to a single trace of \( F \) in the fundamental representation). The (Minkowski space) YM Lagrangian will be \( L_{YM} = -\frac{N}{4} \text{tr}(F_{mn} F_{mn}) \), where \( \lambda = g_{YM}^2 N \) and the generators in the fundamental representation are normalized so that \( \text{tr}(T_I T_J) = \delta_{IJ} \). This normalization is convenient for comparison with the kinetic term in the BI action for a collection of D3-branes (\( L_{YM} \) should be evaluated on the diagonal background \( U(N+1) \) matrix in the fundamental representation). In this case \( \lambda = 2\pi g_s N \), \( g_{YM}^2 = 2\pi g_s \). Comparing to the supergravity expression (1.2) note that for \( 2\pi \alpha' = 1 \) one has \( Q = \frac{2N g_s^2}{(2\pi)^2} \); note also that going from Euclidean to Minkowski signature notation one should change the overall sign of the action.
The existence of several independent super-invariants appearing at the same \( F^n \)-order of low-energy expansion of SYM effective action with only one invariant having “protected” coefficient and matching onto the term appearing in the expansion of the BI action may then be a general pattern. Moreover, it should probably apply also in the case of non-abelian backgrounds, here starting already at the \( F^6 \) order.

Indeed, the non-abelian (or multi-\( U(1) \)) \( F^6 \) term is likely to be a combination of the two superinvariants differing by \([F,F] \) “commutator” terms absent in the abelian limit. It is natural to expect that there is, in fact, such commutator term in the 1-loop SYM effective action. Thus the full non-abelian \( F^6 \) term in the SYM effective action will no longer be protected. This may be related to an observation in that supersymmetry does not completely determine the coefficient of the \( O(v^6) \) term in 1+0 dimensions in the case of more general (“\( N > 3 \)-particle”) \( SU(N) \) backgrounds. Particular diagonal \( SU(3) \) (“three D0-brane”) backgrounds in 1+0 dimensional 2-loop SYM effective action were considered in and the agreement with supergravity was demonstrated.

1.3 Superspace form of the \( F^6 \) term

Before describing the content of the technical part of the paper let us discuss the superspace form of the \( F^6 \) term in which it will be computed below.

Using \( \mathcal{N} = 1, D = 4 \) superfield notation, the expansion of the BI action containing the sum of the \( F^4 \) and \( F^6 \) terms in (1.3) can be written as (cf. [38])

\[
S = \frac{1}{4g_{YM}^2} \left[ \left( \int d^6z W^2 + h.c. \right) + \frac{1}{2} \frac{2Ng_{YM}^2}{(2\pi)^2|X|^4} \int d^8z W^2 \bar{W}^2 \left( 1 - \frac{1}{16} \frac{2Ng_{YM}^2}{(2\pi)^2|X|^4} (D^2 W^2 + h.c.) + ... \right) \right],
\]

(1.9)

where \( W \) is the abelian \( \mathcal{N} = 1 \) superfield strength. We assume Minkowski signature choice as in (1.3) and the same superspace conventions as in [39, 26] (in particular, \( D^2 W^2_{\theta=0} = 2F_{mn}^2 + ... \)).

To reproduce this expression by a 2-loop computation on the \( \mathcal{N} = 4 \) SYM side we shall use the \( \mathcal{N} = 2 \) superfield formulation (with the harmonic superspace description for

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\[ \text{\textsuperscript{10}} \text{The precise structure and coefficient of this term can be determined using the general methods of [29, 30, 31], or by computing the non-abelian } O(X^{12}) \text{ 1-loop term in the 1+0 dimensional theory [32] and lifting it to 1+9 or 1+3 dimensions.} \]

\[ \text{\textsuperscript{11}} \text{A possibility of existence, in } SU(N), N > 2 \text{ case, of “unprotected” non-abelian tensor structures already at “} v^4 \text{” order (and that the proof of the non-renormalization theorem of [14] applies in the } SU(2) \text{ or “two brane” case only) was suggested in [34].} \]
the quantum fields in the context of background field method). We shall assume that only one \( \mathcal{N} = 1 \) chiral superfield has a non-zero constant expectation value, namely, the one contained in the \( \mathcal{N} = 2 \) vector superfield, i.e. only the latter and not the hypermultiplet will have a non-vanishing background value. Thus we will be interested in the case when the gauge group \( SU(N + 1) \) (or \( U(N + 1) \), which is the same as we will consider only the leading large \( N \) approximation) is spontaneously broken down to \( SU(N) \times U(1) \) by the abelian constant (in \( x \)-space) \( \mathcal{N} = 2 \) superfield background with non-zero components being (same as in [39])

\[
\mathcal{W}|_{\theta=0} = X = \text{const}, \quad \mathcal{D}^\alpha_{(\alpha} \mathcal{D}_{\beta)} \mathcal{W}|_{\theta=0} = 8 F_{\alpha\beta} = \text{const}.
\]

(1.10)

It is understood that the background function \( \mathcal{W} \) should be multiplied by the diagonal \( su(N + 1) \) matrix (generating the relevant abelian subgroup of \( SU(N + 1) \))

\[
J = \text{diag}(1, 0, ..., 0) - \frac{1}{N + 1} I = \frac{1}{N + 1} \text{diag}(N, -1, ..., -1),
\]

(1.11)

which represents the configuration of \( N \) coincident D3-branes separated by a distance \( X \) from the single D3-brane carrying the background \( F_{mn} \) field.

Assuming the \( \mathcal{N} = 2 \) background (1.10), the combination of the superconformal invariants representing the sum of the two unique abelian \( F^4/|X|^4 \) and \( F^6/|X|^8 \) corrections in (1.3), (1.9) may be written in the manifestly \( \mathcal{N} = 2 \) supersymmetric form as follows

\[
S = \frac{1}{8 g_{YM}^2} \left( \int d^8 z \mathcal{W}^2 + \text{h.c.} \right)
+ N \int d^{12} z \left[ c_1 \ln \frac{\mathcal{W}}{\mu} \ln \frac{\bar{\mathcal{W}}}{\mu} + c_2 N g_{YM}^2 \left( \frac{1}{\mathcal{W}^2} \ln \frac{\mathcal{W}}{\mu} \mathcal{D}^4 \ln \frac{\mathcal{W}}{\mu} + \text{h.c.} \right) \right] + ... \quad ,
\]

(1.12)

where (for the relation between the \( \mathcal{N} = 1 \) (1.9) and \( \mathcal{N} = 2 \) (1.12) forms of the \( F^6 \) term in Appendix A)

\[
c_1 = \frac{1}{4(2\pi)^2}, \quad c_2 = \frac{1}{3 \cdot 2^8 (2\pi)^4}.
\]

(1.13)

\( \mu \) is a spurious scale which drops out after one integrates over \( \theta \)'s – it gets replaced by \( \mathcal{W}|_{\theta=0} = X \) in going from the \( \mathcal{N} = 2 \) (1.12) to \( \mathcal{N} = 1 \) (1.9) form.\textsuperscript{12}

The low-energy expansion of the quantum \( \mathcal{N} = 4 \) SYM effective action turns out to have the same structure (1.12). The coefficient of the 1-loop term \( \ln \frac{\mathcal{W}}{\mu} \ln \frac{\bar{\mathcal{W}}}{\mu} \) (computed

\textsuperscript{12}We use that for the given background \( \int d^8 z \frac{1}{|X|^4} W^2 \bar{W}^2 = \int d^{12} z \ln \frac{\mathcal{W}}{\mu} \ln \frac{\bar{\mathcal{W}}}{\mu} \) and \( \int d^8 z \frac{1}{|X|^4} W^2 \bar{W}^2 D^2 W^2 + \text{h.c.} = - \frac{1}{24} \int d^{12} z \ln \frac{\mathcal{W}}{\mu} \mathcal{D}^4 \ln \frac{\mathcal{W}}{\mu} + \text{h.c.} \), etc. These two \( \mathcal{N} = 2 \) structures were discussed in, e.g., [40], [41], and [25], [39], respectively.

\textsuperscript{13}For comparison, the \( \mathcal{N} = 2 \) superfield form of the two abelian \( F^8 \) invariants discussed above is [24], [26]: \( \int d^{12} z \mathcal{W}^{-4} \ln \frac{\mathcal{W}}{\mu} (\mathcal{D}^4 \ln \frac{\mathcal{W}}{\mu})^2 + \text{h.c.} \) and \( \int d^{12} z \mathcal{W}^{-2} \bar{\mathcal{W}}^{-2} \mathcal{D}^4 \ln \frac{\mathcal{W}}{\mu} \mathcal{D}^4 \frac{\bar{\mathcal{W}}}{\mu} \). \( d^{12} z = d^4 x d^4 \theta d^4 \bar{\theta} \).
directly in $\mathcal{N} = 2$ superspace form in [11, 12, 25, 34, 43] coincides indeed with the corresponding coefficient $c_1$ (1.13) in (1.3) [4].

Our aim will be to show that the two-loop correction to the $\mathcal{N} = 4$ SYM effective action in $\mathcal{W}, \bar{\mathcal{W}}$ background has the $\mathcal{N} = 2$ superspace form as the $\int d^{12}z \left( \frac{1}{\mathcal{W}^2} \ln \frac{\mathcal{W}}{\mu} D^4 \ln \frac{\mathcal{W}}{\mu} + \text{h.c.} \right)$ term in (1.12) with the same coefficient $N^2 g_{\text{YM}}^2 c_2$ (in the large $N$ limit). The resulting conclusion will be that both $F^4$ and $F^6$ terms in the SYM effective action coincide exactly with the terms in the second line of (1.9), i.e. with the terms in the BI action in the supergravity background.

The rest of the paper is organized as follows. In section 2 we shall consider the $\mathcal{N} = 4$ super Yang-Mills theory formulated in terms of unconstrained $\mathcal{N} = 2$ superfields in harmonic superspace [44], i.e. represented as the $\mathcal{N} = 2$ SYM theory coupled to hypermultiplet. Then we shall briefly describe the $\mathcal{N} = 2$ superfield background field method allowing one to carry out the calculation of the effective action in a way preserving manifest $\mathcal{N} = 2$ supersymmetry and gauge symmetry. Section 3 will be devoted to the evaluation of the hypermultiplet and ghost corrections. We shall find that their contributions to the leading part of the 2-loop low-energy effective action vanish. In section 4 we shall compute the pure $\mathcal{N} = 2$ SYM contribution to the 2-loop $\mathcal{N} = 4$ SYM effective action. Section 5 will contain a summary and some concluding remarks on possible generalizations. Appendix A will present a relation between $\mathcal{N} = 1$ and $\mathcal{N} = 2$ forms of the $F^6$. Appendix B will describe some details of calculations of integrals over the harmonic superspace.

## 2 $\mathcal{N} = 2$ background superfield expansion

The aim of this work is to calculate the leading two-loop correction to the low-energy $\mathcal{N} = 4$ SYM effective action in the sector where only the $\mathcal{N} = 2$ vector multiplet has a non-trivial background. Our starting point will be the formulation of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 2$ harmonic superspace [44]. In terms of $\mathcal{N} = 2$ superfields $\mathcal{N} = 4$ SYM is simply $\mathcal{N} = 2$ SYM theory interacting with one adjoint hypermultiplet with the action [45, 15]

$$S_{\mathcal{N} = 4 \text{ SYM}} = \frac{1}{4 g_{\text{YM}}^2} \text{tr} \left[ \int d^8 z \mathcal{W}^2 - \int d\zeta (-4) \bar{q}^{++} \nabla^{++} q_i^+ \right].$$

---

14The expression for the 1-loop (Minkowski-space) effective action found in [23, 34, 43] was

$$\Gamma^{(1)} = \frac{1}{16\pi^2} \sum_{k<l} \int d^2 z \ln \frac{\mathcal{W}_k - \mathcal{W}_l}{\mu} \ln \frac{\mathcal{W}_k - \bar{\mathcal{W}}_l}{\mu},$$

where $\mathcal{W}_k (k = 1, ..., N + 1)$ are diagonal values of the background matrix in the fundamental representation of $su(N+1)$. In the case of the present background (1.10), (1.11) the sum produces the factor of $N$ which matches the one in (1.12).
\( \mathcal{W} \) is \( \mathcal{N} = 2 \) superfield strength expressed in terms of \( \mathcal{N} = 2 \) vector superfield \( V^{++} \), and \( q^{\pm i} = (\bar{q}^{+}, -q^{+}) \) is the hypermultiplet field taking values in the adjoint representation of gauge algebra, with \( q^{\pm i} = (q^{+}, \bar{q}^{+}) \). Here and below we use the notation introduced in [14, 15, 16].

The most natural way to calculate loop corrections in this model is to use the \( \mathcal{N} = 2 \) background field method which guarantees manifest \( \mathcal{N} = 2 \) supersymmetry and gauge covariance at each step of the calculation. We shall do background-quantum splitting of the gauge superfield \( V^{++} \) in the form \( V^{++} \rightarrow V^{++} + g v^{++} \) [15, 16] with \( V^{++} \) in the right-hand side being a background and \( v^{++} \) being a quantum superfield. The hypermultiplet will have no background, i.e. will be treated as a quantum superfield.

As explained above, we are interested in the large \( N \) part of the 2-loop effective action in the case of the \( U(1) \) background (1.10), (1.11) corresponding to \( N \) coincident D3-branes separated by a distance \( X \) from one D3-brane carrying \( F_{mn} \) field. The background matrix \( J \) is the traceless part of the \( u(N+1) \) matrix diag\((1, 0, \ldots, 0)\). Since we are after the planar contribution only we may just consider the \( U(N+1) \) theory and ignore the subtraction of traces in the background field expressions and propagators. The relevant planar 2-loop graphs (represented in double-line notation) will have the following structure (see Fig.1): the background fields will be attached only to the “outer” cycle of the diagram (representing, in string theory language, the loop lying on the single separated D3-brane), with two internal cycles (lying on \( N \) coincident D3-branes) each producing a trace factor of \( N \).

\[ \text{Written in adjoint representation it is just a combination of differences of its diagonal elements, } \text{diag}(0, \ldots, 0, 1, -1, \ldots, 1, -1), \text{ i.e. it contains } N^2 + 1 \text{ zeros and } N \text{ pairs } (1, -1). \]

\[ \text{In more general case when the probe is a cluster of } n \text{ D3-branes the planar contribution will be proportional to } N^2 \text{tr(products of background matrices in fundamental representation)}. \]
To develop perturbation theory one needs $\mathcal{N} = 2$ background dependent superfield propagators. They can be found by analogy with [15]

$$
< v_{\tau}^{++}(1) v_{\tau}^{++}(2) > = -\frac{2i}{\Box_1} (\overrightarrow{D_1^+})^4 \left\{ \delta^{12}(z_1 - z_2) \delta^{(-2,2)}(u_1, u_2) \right\},
$$

$$
< q_{\tau}^{++}(1) q_{\tau}^{++}(2) > = \frac{i}{\Box_1} (\overrightarrow{D_1^+})^4 \left\{ \delta^{12}(z_1 - z_2) \frac{1}{(u_1^+ u_2^+)^3} \right\} (\overleftarrow{D_2^+})^4,
$$

$$
< c_{\tau}(1) b_{\tau}(2) > = -\frac{2i}{\Box_1} (\overrightarrow{D_1^+})^4 \left\{ \delta^{12}(z_1 - z_2) \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3} \right\} (\overleftarrow{D_2^+})^4. \tag{2.2}
$$

The first line here defines the $\mathcal{N} = 2$ SYM propagator, the second – hypermultiplet propagator and third – the ghost propagator. The index $\tau$ means that the corresponding superfields are taken in the so called $\tau$-frame [44].

We have suppressed the indices of the fundamental representation, $v = (v)_{kl}$ ($k, l = 1, ..., N + 1$), i.e. the propagators carry the two pairs of indices $kl, k' l'$. In the absence of the background each propagator is proportional to $\delta_{kk'} \delta_{ll'}$ (dropping trace terms subleading at large $N$). In the presence of our $U(1)$ background the propagator $P_{kk', ll'}$ will remain to be diagonal, with $P_{11,11} = P^{(0)} + \text{background-dependent terms}$, $P_{1s,1s'} = \delta_{ss'} P^{(0)} + \text{background-dependent terms}$, $P_{st,s't'} = P^{(0)}_{st,s't'}$, where $s, t = 2, ..., N + 1$ are indices of the fundamental representation of the unbroken $SU(N)$ group and $P_{kl,k'l'}^{(0)} = P^{(0)} \delta_{kl} \delta_{k'l'}$ is the free propagator. Thus all one is to do is to separately take into account the two type of contractions – with the $U(1)$ index “1” (which involve the background-dependent propagator) and with the $SU(N)$ indices $a, b$ taking values $2, ..., N$ (which involve the free propagator). The part $P_{11,11}$ will not contribute to the leading large $N$ part of the diagram. It is obvious also that the only part of $P_{1a,1b'}$ that will be contributing to the relevant diagrams (with all background dependence at outer line only) will be
\( \Pi_{\alpha, \alpha'} = \delta_{\alpha \alpha'} \Pi, \quad \Pi = \Pi^{(0)} + \text{background-dependent terms.} \) It is effectively this non-trivial background-dependent block of the propagator that will be discussed below. The role of the remaining free \( SU(N) \) index contractions is simply to produce the factor of \( N^2 \).

Since the \( \mathcal{N} = 2 \) background superfields \( \mathcal{W}, \bar{\mathcal{W}} \) will be on-shell [13], i.e. will be subject to the equations of motion (at the end, satisfying (1.10))

\[
\mathcal{D}^{\alpha(i} \mathcal{D}^{\beta j)} \mathcal{W} = 0 ,
\]

and are also abelian, one is able to show that \([\mathcal{D}^{+ \alpha}, \mathcal{D}^{- \alpha}] \mathcal{W} = 0, \quad \mathcal{D}^{+ \alpha} \mathcal{D}^{- \alpha} \mathcal{W} = 0\). Then the operator \( \frac{1}{\Box} \) in (2.2) takes the form

\[
\frac{1}{\Box} = \frac{1}{\Box} + \frac{1}{2} (\mathcal{D}^{+ \alpha}\mathcal{W})\mathcal{D}^{- \alpha} + \frac{1}{2} (\bar{\mathcal{D}}^{+ \dot{\alpha}}\bar{\mathcal{W}})\bar{\mathcal{D}}^{- \dot{\alpha}} + \mathcal{W} \bar{\mathcal{W}} .
\]

Within the background field method, the 2-loop corrections to the effective action are given by the “vacuum” diagrams containing only cubic and quartic vertices of quantum superfields. The supergraphs with cubic vertices have the form

\begin{align*}
\text{Fig. 2a} & \quad \text{Fig. 2b} & \quad \text{Fig. 2c}
\end{align*}

Here the wavy, solid and dashed lines stand for the propagators of the \( \mathcal{N} = 2 \) gauge, hypermultiplet and ghost superfields respectively. It is easy to see from (2.2) that the effective action can be represented as a power series in supercovariant derivatives of the background superfields.

To find the leading corrections to the 2-loop effective action \( \Gamma^{(2)} \) coming from these supergraphs, we are to represent the operator \( \frac{1}{\Box} \) as a power series in derivatives of the \( \mathcal{N} = 2 \) background strength. This expansion looks like

\[
\frac{1}{\Box} = \frac{1}{\Box} + \frac{1}{\mathcal{W} \bar{\mathcal{W}}} \sum_{n=0}^{\infty} \left\{ \left( -\frac{i}{2} \right) \left[ \mathcal{D}^{+ \alpha} \mathcal{W} \right] \mathcal{D}^{- \alpha} + \left[ \mathcal{D}^{+ \dot{\alpha}} \bar{\mathcal{W}} \right] \mathcal{D}^{- \dot{\alpha}} \right\} \frac{1}{\Box + \mathcal{W} \bar{\mathcal{W}}} .
\]

Then we are to substitute this expansion into the expressions corresponding to the supergraphs in Figs. 2a – 2c and to carry out (covariant) \( \mathcal{D} \)-algebra transformations (for a description of the \( \mathcal{D} \)-algebra see [16]). We will find that to obtain the leading contribution to the low-energy effective action one should keep only the term which is of fourth order in \( \mathcal{D} \mathcal{W} \) and of second order in \( \bar{\mathcal{D}} \bar{\mathcal{W}} \) (plus the conjugate term). It is this term that, when written in components, contains the \( F^6/|X|^8 \) correction we are interested in computing (cf. (1.12)).
Let us now consider the contribution of the two-loop supergraph in Fig. 2d containing the *quartic* vertex (which is presents only in the $N = 2$ gauge superfield sector).

![Fig. 2d](image)

It is possible to show that, in contrast to the supergraphs in Figs. 2a – 2c, the supergraph in Fig. 2d gives only a sub-leading contribution to $\Gamma$ and thus may be ignored. As usual, the form of the superpropagators implies that the supergraphs contain Grassmann delta-functions allowing one to represent their contributions in the form local in $\theta$-coordinates. Transformations leading to such form may be called “contracting loops into points in Grassmann space”. To contract a single loop into a point we need $8\, D, \bar{D}$ factors. Any propagator carries $4$ manifest $D$-factors and additional $D, \bar{D}$ factors may come from the expansion of $\bar{\square}^{-1}$. To contract two loops in Fig. 2d into points in $\theta$-space we thus need $16\, D, \bar{D}$ factors, with $8$ of then coming from the expansion of $\bar{\square}^{-1}$. Any $D$ or $\bar{D}$ factor from the expansion of $\bar{\square}$ is accompanied by $D\bar{W}$ or $\bar{D}W$ factor. Therefore, a non-zero contribution from the supergraph in Fig. 2d will be proportional to $(D\bar{W})^4(\bar{D}W)^4$. Such term is subleading in the low-energy expansion of the 2-loop effective action since, as we shall see, the leading term coming from the cubic vertex supergraphs in Figs. 2a – 2c is proportional to $(D\bar{W})^4(\bar{D}W)^2 + h.c.$

Let us now discuss the structure of propagators and vertices in our background. The quadratic part of the action of $N = 2$ gauge superfield has the form

$$S_2 = -\frac{1}{4} \text{tr} \int d\xi (-4)v^{++} \bar{\square} v^{++} \tag{2.6}$$

Since the fields take values in the adjoint representation,

$$\bar{\square} v^{++} = \square v^{++} + \frac{i}{2}[D^{+\alpha}W, D^-_{\alpha}v^{++}] + \frac{i}{2}[\bar{D}^{+\dot{\alpha}}\bar{W}, \bar{D}^-_{\dot{\alpha}}v^{++}] + [W\bar{W}, v^{++}] \tag{2.7}$$

The quantum vector field with values in $u(N+1)$ can be written as $v^{++} = v^{++}_{kl}e_{kl}$, where $e_{kl}$ is the Weyl basis of $u(N + 1)$ ($k, l = 1, ..., N + 1$)

$$(e_{kl})_{pq} = \delta_{kp}\delta_{lq} \tag{2.8}$$

The background strength is then $W = W_{kl}e_{kl}$. In the case under consideration (where the background field takes values only in the unbroken $u(1)$ which we label by index “1”) we

\[\begin{align*} &\text{Since we are interested in the planar contribution we may not distinguish between } u(N + 1) \text{ and } su(N + 1). \text{ Note also that since the superfield } v^{++} \text{ is real } v^{++}_{k} = \bar{v}^{++}_{\bar{k}}.\end{align*}\]
have: $\mathcal{W}_{11} \equiv \mathcal{W}_0 \neq 0$, with all other components $\mathcal{W}_{ij}$ equal to zero. Let us denote the matrix $e_{11}$ as $r$, i.e. $r_{ab} = \delta_{1a}\delta_{1b}$, $a, b = 2, ..., N + 1$. Then the quadratic part of the action can be written as

$$S_2 = -\frac{1}{4} \int d\xi^{-4} \left\{ v^{++}_1 \Box v^{++}_1 \text{tr}(e_{kl}e_{mn}) + \right.$$  
$$+ \frac{i}{2} v^{++}_{kl}(\mathcal{D}^{+\alpha}\mathcal{W}_0\mathcal{D}^- - \mathcal{D}^{-\alpha}\mathcal{W}_0\mathcal{D}^+)^{v^{++}_{mn}} \text{tr}(e_{kl}[r, e_{mn}]) +$$  
$$+ v^{++}_{kl}\mathcal{W}_0\mathcal{W}_0 v^{++}_{mn} \text{tr}[e_{kl}[r, e_{mn}]] \right\}. \tag{2.9}$$

Using (2.8) we get

$$S_2 = -\frac{1}{4} \int d\xi^{-4} \left\{ v^{++}_1 \Box v^{++}_1 + 2v^{++}_1 \Box v^{++}_1 + v^{++}_{ab} \Box v^{++}_{ba} + \right.$$  
$$+ 2(v^{++}_1 \mathcal{D}^{+\alpha}\mathcal{W}_0\mathcal{D}^- v^{++}_{1a} + \text{h.c.}) +$$  
$$+ 2v^{++}_{1a} \mathcal{W}_0\mathcal{W}_0 \right\} \tag{2.10}$$

From here we conclude that only the components $v_{1a}, v_{a1}$ ($a \neq 1$) have a non-trivial background-dependent propagator: all other quantum field components $v_{a11}$ and $v_{ab}$ ($a, b \neq 1$) do not interact with the background, i.e. have the free propagator $\Box^{-1}$.

Repeating the same procedure for the ghost and hypermultiplet fields we get the following set of propagators

$$< v^{++}_{1a}(z_1, u_1) \bar{v}^{++}_{1b}(z_2, u_2) > = -2i\delta_{ab} \frac{(\mathcal{D}^+_1)^4}{\Box_w} \delta^{12}_{12} \delta^{(2, -2)}(u_1, u_2)$$

$$< b_{1a}(z_1, u_1)c_{1b}(z_2, u_2) > = -2i\delta_{ab} \frac{(u_1^- u_2^-) \ (\mathcal{D}^+_1)^4 (\mathcal{D}^+_2)^4}{(u_1^+ u_2^+)^3 \Box_w} \delta^{12}_{12}$$

$$< \bar{q}^{+}_{1a}(z_1, u_1) q^{+}_{1b}(z_2, u_2) > = 2i\delta_{ab} \frac{1}{(u_1^+ u_2^+)^3 \Box_w} (\mathcal{D}^+_2)^4 \delta^{12}_{12}$$

$$< \bar{b}_{1a}(z_1, u_1) c_{1b}(z_2, u_2) > = \delta_{ab} \frac{(u_1^- u_2^-) \ (\mathcal{D}^+_1)^4 (\mathcal{D}^+_2)^4}{(u_1^+ u_2^+)^3 \Box_w} \delta^{12}_{12}$$

$$< \bar{q}^{+}_{a1}(z_1, u_1) q^{+}_{b1}(z_2, u_2) > = 2i\delta_{ab} \frac{1}{(u_1^+ u_2^+)^3 \Box_w} (\mathcal{D}^+_1)^4 (\mathcal{D}^+_2)^4 \delta^{12}_{12} \ . \tag{2.11}$$

where

$$\Box_w = \Box + \frac{i}{2} [(\mathcal{D}^{+\alpha}\mathcal{W}_0)\mathcal{D}^- - (\mathcal{D}^{-\alpha}\mathcal{W}_0)\mathcal{D}^+ + \mathcal{W}_0\mathcal{W}_0 ] \ . \tag{2.12}$$

Here the indices $a, b$ are not equal to 1. All other components of the propagators are the same as in free theory (cf. [44]). Note that the propagators (2.11) have extra factor 2 as compared to the ones in [44] because of the extra factor 1/2 in the action (2.1).
The leading $N \to \infty$ contribution from the supergraphs in Figs. 1a – 1c contains the following group-theory factor (i.e. the factor that multiplies the direct product of “singlet” propagators)\[18\] $-\frac{1}{3} \text{tr}(e_{1a}[e_{d1}, e_{be}]) \text{tr}(e_{1a}[e_{d1}, e_{be}])$. Using (2.8) one finds that this factor is equal to $-\frac{1}{3} N^2$.

Below we shall omit the index $w$ on the background-dependent operator $\mathring{\Box}_w$; we will also rename $\mathcal{W}_0$ and $\mathring{\mathcal{W}}_0$ as simply $\mathcal{W}$ and $\mathring{\mathcal{W}}$, which will stand for the non-zero components of $\mathcal{N} = 2$ superfield strength.

3 Hypermultiplet and ghost contributions to 2-loop low-energy effective action

Let us consider in detail the structure of supergraphs corresponding to Figs. 2a – 2c. We may first contract the gauge, hypermultiplet and ghost loops to a point in $\theta$-space using the rule \[15\]

$$\delta_{12}^8(D^+(u_1))^4(D^+(u_2))^4 \delta_{12}^8 = (u_1^+ u_2^+)^4 \delta_{12}^8.$$  \hspace{1cm} (3.1)

Then the leading $N \to \infty$ contributions of these supergraphs to the effective action can be represented as

\[
I_a = -\frac{1}{3} g_{YM}^2 N^2 \int d^8 \theta_1 d^8 \theta_2 du_1 du_2 dv_1 dv_2 dw_1 \frac{(D^+(u_1))^4}{\Box} \delta_{12}^{12} \delta^{(2,-2)}(u_1, u_2) \\
\times \frac{(v_1^+ v_2^+)^2}{(u_1^+ v_1^+)(u_2^+ v_2^+)(u_1^+ v_2^+)(u_2^+ v_1^+)} \frac{1}{\Box (v_2)} \delta_{12}^{12} \frac{1}{\Box (u_2)} \delta^4(x_1 - x_2) \\
I_b = -\frac{1}{3} g_{YM}^2 N^2 \int d^8 \theta_1 d^8 \theta_2 du_1 du_2 \frac{(D^+(u_1))^4}{\Box} \delta_{12}^{12} \delta^{(2,-2)}(u_1, u_2) \\
\times \frac{1}{(u_1^+ u_2^+)^2} \frac{1}{\Box (u_1)} \delta_{12}^{12} \delta^4(x_1 - x_2) \\
I_c = -\frac{2}{3} g_{YM}^2 N^2 \int d^8 \theta_1 d^8 \theta_2 du_1 du_2 \frac{(D^+(u_1))^4}{\Box} \delta_{12}^{12} \delta^{(2,-2)}(u_1, u_2) \\
\times D_1^{++} D_2^{++} \frac{(u_1^+ v_2^+)^2}{(u_1^+ u_2^+)^2} \frac{1}{\Box (u_1)} \delta_{12}^{12} \delta^4(x_1 - x_2) \]  \hspace{1cm} (3.2)

\[18\] The numerical factor $-\frac{1}{3}$ appears as follows: each vertex carries the factor $\frac{1}{6}$, contracting quantum fields into propagators gives the factor 6, two propagators $<v^+_1 v^+_2>$ contribute factor 4, the propagator $<v_2^+ v_1^+>$ gives factor of 1, and the expansion of $\exp(i S_{int})$ in the path integral to the second order in gauge coupling contributes the factor $-\frac{1}{3}$.

\[19\] We use the notation $\delta_{12}^{12} = \delta_{12}^8 \delta_{12}^8$ for the $\mathcal{N} = 2$ superspace $\delta$-function of argument $z_1 - z_2$, with $\delta_{12}^8$ being its Grassmann part.
Here $D^+$ is the “flat” derivative originating from the free propagator. To obtain these expressions we note that product of the three propagators $\Gamma^{(2)}$ carries factor $i$, and the two-loop correction $\Gamma^{(2)}$ is defined with factor $i$ ($Z = e^{i\Gamma}$). It was already mentioned that due to the structure of the vertices one propagator in each supergraph is the free one.

In $I_a$ we may write $(v_2^+ w_2^+)^2 = D^+_{v_2} D^+_{w_2} \{(v_2^+ w_2^+)(v_2^+ w_2^-)\}$, and integrating by parts, transport the derivatives $D^+_{v_2}, D^+_{w_2}$ to the other part of integrand. We may also do the same operation with harmonic derivatives in $I_c$. These transformations are completely analogous to the ones done in [47] in the case of constant $\mathcal{W}, \bar{\mathcal{W}}$ (where all the derivatives of the background fields $\mathcal{W}, \bar{\mathcal{W}}$ were zero). However, here (cf. (2.10)), unlike the case of the nonholomorphic effective potential considered in [47], the factor $\frac{1}{\Box}$ depends on the harmonic coordinates (since $\mathcal{W}, \bar{\mathcal{W}}$ are not constant in superspace this dependence is implied by (2.3)). Therefore, $I_{a,c}$ contain additional terms where $D^+$’s act on $\frac{1}{\Box}$. We get

$$I_a = \frac{2}{3} g_{YM}^2 N^2 \int d^8 \theta d^8 \theta_2 du_1 du_2 dv_2 dw_2 \frac{(D^+)^4(u_1)}{\Box} \delta_1^{12} \delta^{(2,-2)}(u_1, u_2)(v_2^+ w_2^-)(v_2^- w_2^+)$$

$$\times D^+_{v_2} D^+_{w_2} \left[ \frac{1}{(u_1^+ v_2^+)(u_1^+ w_2^+)(u_2^+ v_2^+)(u_2^+ w_2^+)} \frac{1}{\Box(v_2)} \frac{1}{\Box(w_2)} \delta_1^{12} \delta^{(2,-2)}(u_1, u_2) \right]$$

$$I_c = \frac{4}{3} g_{YM}^2 N^2 \int d^8 \theta d^8 \theta_2 du_1 du_2 \frac{(D^+)^4(u_1)}{\Box} \delta_1^{12} \delta^{(2,-2)}(u_1, u_2)$$

$$\times D^+_{1} D^+_{2} \left[ \frac{(u_1^+ w_2^-)^2}{(u_1^+ w_2^-)^2} \frac{1}{\Box(u_1)} \frac{1}{\Box(u_1)} \delta_1^{12} \delta^{(2,-2)}(u_1, u_2) \right]$$

(3.3)

The contributions of each of these supergraphs $I_{a,c}$ can be separated into two parts, $I = I' + I''$. The first corresponds to the terms in which $D^+$’s do not act on $\frac{1}{\Box}$. The second one contains the terms where $D^+$’s are acting on $\Box$. Note that $I_b$ contains only the contribution of the first type. The sum of contributions in which $D^+$’s do not act on $\Box$ vanishes identically due to $\mathcal{N} = 4$ supersymmetry, as it was pointed out in [47]. That means that diagrams with hypermultiplet propagators do not contribute to the leading 2-loop term in the low-energy effective action.

What remains is to consider the terms in $I_{a,c}$ which contain $D^+$’s acting on $\Box$. Such term in $I_c$ is

$$I''_c = -\frac{4}{3} g_{YM}^2 N^2 \int d^8 \theta \int du_1 du_2 \frac{(D^+)^4(u_1)}{\Box} \delta_1^{12} \delta^{(2,-2)}(u_1, u_2) \left[ \frac{(u_1^- w_2)(u_1^- w_2^-)}{(u_1^+ u_2^+)^2} \frac{1}{\Box(u_1)} \delta_1^{12} \right]$$

$$\times \frac{1}{\Box(u_1)} \left[ (D^{+\alpha}(u_1) W) D^{+\alpha}(u_1) + (\bar{D}^{+\dot{\alpha}}(u_1) \bar{W}) \bar{D}^{+\dot{\alpha}}(u_1) \right]$$

(3.4)

The supercovariant derivatives of $\mathcal{N} = 2$ field strengths arise from the expansion of $\bar{\Box}^{-1}$. In the above expression for $I''_c$ all factors $\bar{\Box}^{-1}$ depend on the same harmonic coordinate.
Hence all derivatives $D^+ W$ which appeared from $\Box^{-1}$ also depend only on $u_1$. Thus, all contributions generated by $I_c$ should be proportional to various powers of $D^+(u_1) W$, $\bar D^+(u_1) \bar W$. However, as one can show, all contributions of second order in $D^+(u_1) W$, $\bar D^+(u_1) \bar W$ are equal to zero. Also, $(D^+_1(u_1) W)^n = 0$ for $n \geq 3$. Hence we find that diagrams with ghost propagators also do not contribute to the leading term in the 2-loop low-energy effective action. This result is similar to the one known in the one-loop approximation where the ghosts and the hypermultiplets also do not contribute to the on-shell low-energy effective action [43].

The term in $I_a$ containing $D^{++}$ acting on $\Box$ takes the form (after integrating over $u_2$ and some transformations)

$$I_a'' = -\frac{2}{3} g_{YM}^2 N^2 \int d^8 \theta \int dudvdw \frac{(D^+)^4(u)}{\Box} \delta_{12}^{12}$$

$$\times \left[ -\frac{1}{4} (v^- w^-)(v^+ w^+) \frac{1}{\Box (v)} ((D^{+\alpha}(v) W) D^+_{\alpha}(v) + (\bar D^+_{\alpha}(v) \bar W) \bar D^{+\alpha}(v)) \frac{1}{\Box (v)} \delta_{12}^{12}$$

$$\times \frac{1}{\Box (w)} ((D^{+\alpha}(w) W) D^+_{\alpha}(w) + (\bar D^+_{\alpha}(w) \bar W) \bar D^{+\alpha}(w)) \frac{1}{\Box (w)} \delta^{4}(x_1 - x_2)$$

$$- i[D_v^{-\delta(1,-1)}(u,v)] \frac{1}{\Box (v)} \delta_{12}^{12} \frac{1}{\Box (u)} \delta_{12}^{12} \frac{1}{\Box (w)} ((D^{+\alpha}(w) W) D^+_{\alpha}(w))$$

$$+ (\bar D^+_{\alpha}(w) \bar W) \bar D^{+\alpha}(w)) \frac{1}{\Box (w)} \delta^{4}(x_1 - x_2) \right] \tag{3.5}$$

The remaining task is to extract from here the leading low-energy contribution to the 2-loop effective action.

4 Leading low-energy term in $\mathcal{N} = 2$ gauge field 2-loop contribution

To develop low-energy expansion of the gauge field contribution $I_a''$ [3.4] we use [2.5] to represent the factors $\frac{1}{\Box}$ in (3.5) as power series in derivatives of $W, \bar W$. As a result, eq. (3.5) will be given by a series of terms containing some number of covariant derivatives acting on $W, \bar W$ and some powers of $(\Box + W \bar W)^{-1}$. The first term in this series has four derivatives and four $W$ factors – it is this term that determines the leading correction to the two-loop effective action. It contains also the factor of $(D W)^2$, so that the leading contribution is proportional to

$$(D W)^4 (D \bar W)^2 + h.c. , \tag{4.1}$$
and, when written in components, contains the required $F^6$ term in the bosonic sector.

The contraction of the remaining loop to a point using the rule (3.1) requires four $\mathcal{D}$-factors and four $\bar{\mathcal{D}}$-factors. Therefore, we must take into account only those terms in the expansion which are at least of fourth order in $\mathcal{D}\mathcal{W}\mathcal{D}$ and of second order in $\mathcal{D}\mathcal{W}\bar{\mathcal{D}}$.

The only non-vanishing supergraph producing such leading correction is given in Fig. 3. All other supergraphs are proportional to either $(u^-u^-) = 0$ or $(\mathcal{D}^+(u)\mathcal{W})^3 = 0$ (this can be shown using the transformations similar to ones which are carried out below). We shall omit also all commutators of the spinor supercovariant derivatives with $\Box + \mathcal{W}\bar{\mathcal{W}}$ since they lead only to subleading corrections.

The contribution of the supergraph in Fig. 3 (after contraction of one of the loops to a point in $\theta$-space as in section 3) is given by

$$
\Gamma^{(2)} = -4g^2_{YM}N^2 \int d^{12}z_1d^{12}z_2 \int du dv dw A(u, v, w) \frac{1}{(\Box + \mathcal{W}\bar{\mathcal{W}})^3}\delta_{12}^{12}(\mathcal{D}^{+\alpha}(v)\mathcal{W})\mathcal{D}^{-\alpha}(v) \\
\times (\mathcal{D}^{+\gamma}(v)\mathcal{W})\mathcal{D}^{-\gamma}(v)(\mathcal{D}^{+\beta}(w)\mathcal{W})\mathcal{D}^{-\beta}(w)(\mathcal{D}^{+\delta}(w)\mathcal{W})\mathcal{D}^{-\delta}(w) \\
\times \frac{1}{(\Box + \mathcal{W}\bar{\mathcal{W}})^3}\delta_{12}^{12}(x_1 - x_2) + h.c. \quad (4.2)
$$

Here $A(u, v, w)$ is a function of harmonics

$$
A(u, v, w) = -\frac{1}{4} \frac{(v^-w^-)(v^+w^+)}{(u^+v^+)^2(u^+w^+)^2} . \quad (4.3)
$$

As above, $\mathcal{D}$ is the background dependent derivative, and $D$ is the flat one. We took into account that there are 6 supergraphs of the form similar to that one given in Fig. 2 with
difference being only in position of external superfield strength insertions. Each of these supergraphs gives the same contribution. The derivatives in $(DW)$ and $(D\bar{W})$ act only on $\mathcal{W}, \bar{\mathcal{W}}$. All other derivatives act on all terms to the right. Since we are interested in the background we may use the property $\partial_m \mathcal{W} = \partial_m \bar{\mathcal{W}} = 0$. The equations of motion $(2.3)$ lead to $(D^\pm)^2(v)\mathcal{W} = 0$ (here $v$ is an arbitrary harmonic coordinate). Using this identity, we conclude that for on-shell $\partial_m$-constant background field strengths no more than one spinor derivative can act on the background field. As a result, we can transport all “excess” spinor derivatives (additional to the ones incorporated in $(DW)$ and $(D\bar{W})$) to act on the same Grassmann $\delta$-function. This allows us to rewrite $(4.2)$ in the form

$$\Gamma^{(2)} = -4g_{YM}^2N^2 \int d^{12}z_1d^{12}z_2 \int dudvdw A(u,v,w) \frac{1}{(\Box + \mathcal{W}\bar{\mathcal{W}})^5} \delta^{12}_{12} \times (D^+_{\alpha}(v)\mathcal{W})(D^+_{\alpha}(v)\mathcal{W})(D^+_{\beta}(v)\mathcal{W})(D^+_{1\alpha}(v)\mathcal{W})(D^+_{1\alpha}(v)\mathcal{W})$$

$$\times (D^+_{1\gamma}(v)\mathcal{W}D^\pm_{\gamma1}(v)\mathcal{W}D^+_{\delta}(v)\mathcal{W}D^\pm_{\delta1}(w)\mathcal{W}D^\mp_{\delta1}(w)\mathcal{W}D^\pm_{\delta1}(w)\mathcal{W}D^\pm_{\delta1}(w)\mathcal{W}) \times \frac{1}{(\Box + \mathcal{W}\bar{\mathcal{W}})^3} \delta^4(x_1 - x_2) + h.c. \quad (4.4)$$

Now we can integrate over $\theta_2$ using the rule $\int d^4x_2d^8\theta_2f(\theta_2, \ldots) = \int d^4x_2D^+_2\bar{D}^+_2f(\theta_2, \ldots)|_{\theta_2 = 0}$. Here dots stand for all arguments except $\theta_2$, and $D^+_2 = D^+_2\bar{D}^+_2D^+_2\bar{D}^+_2D^+_{2j\beta}$, where $i, j$ are the indices numerating $\mathcal{N} = 2$ spinor derivatives. The equations of motion $(2.3)$ lead to the following on-shell identity $\bar{D}^+_2(w)\mathcal{W}D^+_{\delta1}(w)\mathcal{W} = \frac{1}{2}D^+_2(w)\mathcal{W}^2$. Using this identity we rewrite $(4.4)$ as

$$\Gamma^{(2)} = -2g_{YM}^2N^2 \int d^4x_1d^4x_2d^8\theta_1 \int dudvdw A(u,v,w)(D^+(v)\mathcal{W})^2(D^+_1(v)\mathcal{W})^2$$

$$\times (D^+_1(v))^{2}\mathcal{W}^2D^+_2\bar{D}^+_2\frac{1}{(\Box + \mathcal{W}\bar{\mathcal{W}})^5} \delta^{12}_{12} (D^+_{1\gamma}(v)\mathcal{W}D^\pm_{\gamma1}(v)\mathcal{W}D^+_{\delta}(v)\mathcal{W}D^\pm_{\delta1}(w)\mathcal{W}D^\mp_{\delta1}(w)\mathcal{W}D^\pm_{\delta1}(w)\mathcal{W}D^\pm_{\delta1}(w)\mathcal{W})$$

$$\times (D^+_2(u))^{2}(D^+_2(u)^2) \frac{1}{(\Box + \mathcal{W}\bar{\mathcal{W}})^3} \delta^4(x_1 - x_2) + h.c. \quad (4.5)$$

The integrand in $(4.5)$ is local in Grassmann coordinates, as usual in superfield theories (see, e.g., [10]).

The integrand here is evaluated at $\theta_2 = 0$, i.e. it depends only on $\theta_1$. To simplify $(4.5)$ it is convenient to transport $\mathcal{D}^2$ from $\mathcal{W}^2$ to the rest of the factors – this allows us to get an expression in which the numbers of chiral and anti-chiral derivatives acting on delta functions are equal to each other. Since the background is abelian, the gauge covariant derivatives acting on the background strengths are equivalent to the “flat” ones, $D^+_\alpha(v)\mathcal{W} = D^+_\alpha(v)\mathcal{W}$. We integrate by parts using the rule $\int d^8\theta_1(\bar{D}^2_1\mathcal{W}^2)Y = \int d^8\theta_1\mathcal{W}^2\bar{D}^2_1Y$. The action of $\bar{D}$ on $D\mathcal{W} = DW$ leads to the space-time derivatives of the
background field strength, i.e. to the terms which we are ignoring here. As a result, we arrive at

$$
\Gamma^{(2)} = -2g_Y^2N^2 \int d^4x d^4y d^8\theta_1 \int dudvdw A(u,v,w)(D^+(v)\mathcal{W})^2(D^+(w)\mathcal{W})^2\bar{\mathcal{W}}^2
\times \frac{1}{4+3WW}\left[\delta^4_\theta\gamma^\gamma(v)v\mathcal{D}^{-\gamma}_\gamma(w)\mathcal{D}^{+\delta}_\delta(w)\mathcal{D}^{-\beta}_\beta(w)\mathcal{D}^{+\beta}_\beta(w)\right]^{2}\delta^4(x_1 - x_2) + h.c. \quad (4.6)
$$

Now we are to evaluate the result of applying ten covariant derivatives to the factor in the square brackets in \((4.6)\). Fortunately, only a few terms in the final complicated expression are actually contributing to the term \((4.7)\) in the low-energy effective action we are interested in. To extract them let us consider the component form of this expression. The component content of the term \((\mathcal{D}_1\mathcal{W})^4\mathcal{W}^2\) in pure gauge sector is \(F^a_{\theta_1}\theta_1^a + \ldots\). Therefore, all dependence of the integrand in \((4.6)\) on \(\theta_1\) is concentrated in \((\mathcal{D}_1^+(v)\mathcal{W})^2(\mathcal{D}_1^+(w)\mathcal{W})^2\bar{\mathcal{W}}^2\), and to find the contribution of \((4.6)\) it is enough to study the component

$$
R = \bar{\mathcal{D}}^{+\gamma}_\gamma(w)\mathcal{D}^{+\delta}_\delta(w)\mathcal{D}^{-\beta}_\beta(w)\mathcal{D}^{+\beta}_\beta(w)
\times \frac{1}{4+3WW}\left[\delta^4_\theta\gamma^\gamma(v)v\mathcal{D}^{-\gamma}_\gamma(w)\mathcal{D}^{+\delta}_\delta(w)\mathcal{D}^{-\beta}_\beta(w)\mathcal{D}^{+\beta}_\beta(w)\right]^{2}\delta^4(x_1 - x_2) + h.c. \quad (4.7)
$$

i.e. the term which is of zeroth order in both \(\theta_1\) and \(\theta_2\). Then eq. \((4.6)\) can be written as

$$
\Gamma^{(2)} = -2g_Y^2N^2 \int d^4x d^4y d^8\theta_1 \int dudvdw A(u,v,w)(\mathcal{D}_1^+(v)\mathcal{W})^2(\mathcal{D}_1^+(w)\mathcal{W})^2\bar{\mathcal{W}}^2
\times \frac{1}{4+3WW}\left[R^{2+\gamma}_\gamma(v)v\mathcal{D}^{+\delta}_\delta(w)\mathcal{D}^{-\beta}_\beta(w)\mathcal{D}^{+\beta}_\beta(w)\right]^{2}\delta^4(x_1 - x_2) + h.c. \quad (4.8)
$$

To obtain a non-trivial contribution from \((4.7)\) we should act with at least four \(\mathcal{D}\) and four \(\bar{\mathcal{D}}\) derivatives on each of Grassmann \(\delta\)-function (otherwise we get terms of first and higher orders in \(\theta,\bar{\theta}\)’s which vanish at \(\theta_1 = \theta_2 = 0\)). Since all derivatives now act on \(\delta\)-functions which are symmetric with respect to the indices 1 and 2 we may arrange all of them acting on the first argument \(z_1\). We then find that the zeroth order in \(\theta_1\) in \((4.7)\) corresponds to acting by at least eight spinor derivatives on each of the \(\delta\)-functions. The only non-zero term in \(R\) \((4.7)\) is then

$$
\delta^4(x_1 - x_2) \left[\bar{\mathcal{D}}^{+\gamma}_\gamma(v)v\mathcal{D}^{+\delta}_\delta(w)\mathcal{D}^{-\beta}_\beta(w)\mathcal{D}^{+\beta}_\beta(w)\left[R^{2+\gamma}_\gamma(v)v\mathcal{D}^{+\delta}_\delta(w)\mathcal{D}^{-\beta}_\beta(w)\mathcal{D}^{+\beta}_\beta(w)\right]^{2}\delta^4(x_1 - x_2) + h.c. \quad (4.9)
$$

Other terms are either of odd order in Grassmann coordinates (and hence vanish at \(\theta_1 = \theta_2 = 0\)) or proportional to \((w^+w^+) = 0\).
Next, we are to contract the remaining loop into a point in \( \theta \)-space. To do this we commute \( \mathcal{D} \) and \( \bar{\mathcal{D}} \) factors in (4.9) using the identity \( \{ \mathcal{D}^\pm_\alpha(v), \bar{\mathcal{D}}^\pm(w)_\dot{\alpha}\} = 2i(v^+ w^+)\mathcal{D}^\alpha \dot{\alpha} \).

As a result, we obtain a sum of terms in which two \( \mathcal{D} \)- and two \( \bar{\mathcal{D}} \)-factors form \( \Box \). As a result, we arrive at the following expression for (4.9):

\[
R = \delta^4(x_1 - x_2)\left[ (u^+ v^-)(u^+ v^+)(u^+ w^+)(u^+ w^-) + (u^+ v^-)(u^+ v^+)(u^+ w^+)^2(u^+ w^-)(v^- w^-) \\
+ (u^+ v^-)(u^+ v^+)(u^+ w^+)^2(v^- w^-) + (u^+ v^-)(u^+ v^+)(u^+ w^+)^3(w^- v^-) \\
+ (u^+ v^+)(u^+ w^+)^2(u^+ w^-)(v^- w^-) + (u^+ v^-)(u^+ v^+)(u^+ w^+)(u^+ w^-)^2(v^- w^+) \right] \\
\times \sqrt{21} \delta^4(x_1 - x_2). \tag{4.10}
\]

The factor \( \Box \) thus cancels against the same factor in the denominator.

Substituting the above \( R \) into (4.8), doing Fourier transform, and integrating over \( x_2 \) one obtains \( \Gamma^{(2)} \) (4.8) in the form:

\[
\Gamma^{(2)} = -2g^2_N N^2 \int d^4 x d^8 \theta \int dv dw A(u, v, w)\left[ 4(u^+ v^+)(u^+ w^+)(u^+ w^-)^2(v^+ v^-) \\
+ 4(u^+ w^+)(u^+ w^-)^2(v^+ v^-)(w^+ v^+) \\
+ 4(u^+ w^+)(u^+ w^-)^2(u^+ v^-)(w^+ v^+) - 4(u^+ w^+)^2(u^+ w^-)(u^+ v^-)(w^+ v^-) \\
+ 4(u^+ w^+)^3(w^+ v^-)(w^- v^-) - 4(u^+ w^+)^2(u^+ w^-)(u^+ v^-)(w^- v^-)^2 \\
- 4(u^+ v^+)(u^+ w^+)^2(u^+ w^-)(u^+ v^-)(w^- v^+) + 4(u^+ w^+)^2(u^+ w^-)^2(w^+ v^-)(w^- v^+) \\
+ 4(u^+ w^+)^2(u^+ w^-)^2(w^+ v^-)(w^+ v^-) + 4(u^+ v^+)(u^+ w^+)^2(u^+ w^-)(w^- v^-) \\
+ 4(u^+ w^+)^2(u^+ w^-)^2(u^+ v^-)(w^- v^-) - 4(u^+ w^+)(u^+ w^-)^2(u^+ v^-)(w^- v^-) \\
- 4(u^+ v^-)(u^+ w^+)(u^+ w^-)^2(v^+ v^+) + 6(u^+ w^+)(u^+ w^-)^2(u^+ v^-)(w^- v^-) \\
+ 6(u^+ w^-)^2(u^+ v^-)(u^+ w^+)^2(v^+ v^+) + 3(u^+ w^+)(u^+ w^-)^3(w^+ v^-)(w^+ v^+) \\
+ 3(u^+ w^+)(u^+ w^-)(u^+ v^-)(u^+ v^+) + 3(u^+ w^+)^2(u^+ w^-)^2(v^- v^-)(w^- v^+) \\
+ 3(u^+ w^+)^2(u^+ w^-)^2(v^+ v^-) \right] \\
\times (\mathcal{D}^+(v)\mathcal{W})^2(\mathcal{D}^+(w)\mathcal{W})^2\bar{\mathcal{W}}^2 \int \frac{d^4 k d^4 l}{(2\pi)^8 (k^2 + \mathcal{W} \mathcal{W})^5 (l^2 + \mathcal{W} \mathcal{W})^5} \frac{1}{\Gamma(1) \Gamma(3)} + h.c. \tag{4.11}
\]

Here we renamed the coordinates as \( x_1 = x, \theta_1 = \theta \). The momentum integrals can be easily calculated:

\[
\int \frac{d^4 k d^4 l}{(2\pi)^8 (l^2 + \mathcal{W} \mathcal{W})^3 (k^2 + \mathcal{W} \mathcal{W})^5} = \frac{1}{(4\pi)^4} \frac{1}{(\mathcal{W} \mathcal{W})^4} \Gamma(1) \Gamma(3) \frac{1}{24(4\pi)^4(\mathcal{W} \mathcal{W})^4}. \tag{4.12}
\]

Schematically, these transformations can be summarized as follows. We consider (4.3), choose two derivatives \( \mathcal{D} \) and two derivatives \( \bar{\mathcal{D}} \) and replace them by the factor \( \Box \) multiplied by a product of harmonic arguments of chosen derivatives. Then we add together all such terms corresponding to all possible choices of pairs \( \mathcal{D} \) and \( \bar{\mathcal{D}} \).
It remains to integrate over the harmonics. We use the identities like \((u^+w^+) = D_u^{++}(u^-w^+)\) and transport the harmonic derivatives \(D^{++}\) to other factors in the integrand. The identity \(D_u^{++}1/(u^+w^+)=\delta^{(0,0)}(u,v)\) produces harmonic \(\delta\)-functions which allow to do the integral over \(u\). As a result, after substituting of integral (4.12) into (4.11) we get

\[
\Gamma^{(2)} = \frac{1}{48(4\pi)^4} g_{YM}^2 \int d^{12}z \int dv dw (D_+^+(v)W)(D_+^+(w)W) \frac{1}{(WW)^4} \\
\times (v^-w^-)^2[2+2(v^-w^-)(v^+w^+)-4(v^-w^-)(v^+w^+)(v^-w^+) \\
- 4(v^+w^-)(v^-w^+)-6(v^-w^-)^2(v^+w^-)^2-3(v^-w^-)^2(v^+w^+)^2]
\] (4.13)

Calculating the integrals over the harmonics using the rules of [44] (see Appendix B) gives (here 1, 2 are indices of the two \(\mathcal{N}=1\) Grassmann coordinates)

\[
\Gamma^{(2)} = \frac{1}{48(4\pi)^4} g_{YM}^2 N^2 \int d^{12}z \ \tilde{W}^2 (D_1 W)^2 (D_2 W)^2 \frac{1}{(WW)^4} + h.c. \ .
\] (4.14)

Using the equations of motion (2.3) for \(W, \bar{W}\) we obtain the following final \(\mathcal{N}=2\) supersymmetric expression for the leading part of the 2-loop \(\mathcal{N}=4\) SYM effective action

\[
\Gamma^{(2)} = \frac{1}{48(4\pi)^4} g_{YM}^2 N^2 \int d^{12}z \ \frac{1}{W^2} \ln \frac{W}{\mu} D^4 \ln \frac{W}{\mu} + h.c. \ .
\] (4.15)

This expression matches the one in (1.12) – it reproduces exactly the value \(\frac{1}{32^8(2\pi)^4}\) of the coefficient \(c_2\) in (1.12). We conclude that the coefficient of the \(F^6\) term in the quantum 2-loop SYM effective action is the same as in the BI action (1.3).

5 Summary and concluding remarks

In this paper we have calculated the leading part of the planar two-loop contribution to the low-energy \(\mathcal{N} = 4\) \(SU(N+1)\) SYM effective action in the abelian \(\mathcal{N}=2\) gauge superfield background. We used the formulation of \(\mathcal{N}=4\) SYM theory in terms of \(\mathcal{N}=2\) superfields in harmonic superspace and the background field method. We have found that the relevant leading two-loop term does not appear from the 2-loop diagrams with hypermultiplet and ghost propagators, so that the result is given entirely by the \(\mathcal{N}=2\) gauge superfield contribution.\(^{21}\)

\(^{21}\)While the \(F^6\) term thus appears to be generated only by the \(\mathcal{N}=2\) SYM sector, the role of \(\mathcal{N}=4\) supersymmetry is still important: it ensures the cancellation of additional contributions to \(F^6\) in which the harmonic derivative \(D^{++}\) does not act on the “box” operator (2.4). Thus the result for the \(F^6\) term in pure \(\mathcal{N}=2\) SYM theory is expected to be different from the \(\mathcal{N}=4\) SYM one.
The calculation of the two-loop low-energy correction we have described is a good illustration of the efficiency of the harmonic superspace approach to computing the effective action in $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetric gauge theories.

The correction we have calculated is the $\mathcal{N} = 2$ supersymmetrization of the $F^6/|X|^8$ term depending on a constant $U(1)$ gauge field strength and constant scalar field background. This term does not appear in the 1-loop approximation. We have found that the large $N$ part of its 2-loop coefficient coincides exactly with that of the corresponding term in the expansion of the Born-Infeld action describing the supergravity interaction between a stack of $N$ D3-branes and a parallel D3-brane probe carrying constant $F_{mn}$ background. Since, in general, the coefficient of this term in the SYM effective action could be a non-trivial function of 't Hooft coupling (receiving corrections also from third and higher loop orders), the agreement of the 2-loop coefficient with the supergravity expression (which, according to the AdS/CFT correspondence, should be describing the large-coupling limit of the SYM theory) is a strong indication of the existence of a non-renormalization theorem for this term (at least in the planar limit).

In section 1.2 we proposed a conjecture on how the correspondence between the low-energy SYM effective action and the D3-brane supergravity interaction potential can be extended to higher-order terms. Higher-dimensional $F^{2l+2}/|X|^{4l}$ terms in the SYM effective action should be combinations of bosonic parts of several $\mathcal{N} = 1$ (or $\mathcal{N} = 2$) super-invariants. Only one of them (for each $l$) should have “protected” coefficient which receives contribution only from the $l$-th loop order. This term (its planar part) should thus be the only one at given order that survives in the strong coupling limit. It is this term that should then match onto the corresponding structure in the expansion of the Born-Infeld action, in agreement with the expectation based on AdS/CFT philosophy.

There are several possible extensions of the present work that may help to clarify the situation with the higher-order terms and may provide some evidence supporting the validity of the above conjecture:

(i) Compute the abelian $F^8$ term in the 2-loop SYM effective action to demonstrate that it indeed contains only the same “unprotected” invariant $(F^8)_2$ as the 1-loop action (1.8) – the second “protected” invariant $(F^8)_1$ should appear only at the 3-loop order.

(ii) More generally, compute the full non-linear expression for the 2-loop SYM effective action in constant abelian $F_{mn}$ background, i.e. the $D = 4, \mathcal{N} = 4$ SYM analog of the 2-loop Schwinger action in quantum electrodynamics [48], or the 1+3 dimensional counterpart of the 1+0 dimensional SYM result of [49]. A comparison of the corresponding
2-loop function of the $F_{mn}$ eigen-values $f_1, f_2$ with the 1-loop function (1.7), (1.8) should be useful for identifying $F^n$ invariants that have “unprotected” coefficients, i.e. that appear in both 1-loop and 2-loop effective actions.

(iii) Consider the SYM effective action in a non-abelian $F_{mn}$ background and compute the 1-loop and 2-loop coefficients of the “second” non-abelian $F^6$ invariant ("tr$F^4[F,F]"$, see [12]) to confirm that its coefficient indeed gets renormalized.

(iv) It is well known that the full non-abelian 1-loop $F^4$ term in the $\mathcal{N} = 4$ SYM effective action $-\text{Str}[F^4 - \frac{1}{4}(F^2)^2]$ can be obtained by taking the $\alpha' \to 0$ limit in the superstring partition function on the annulus [50]; it would be of interest to give a similar string-theoretic derivation for the two-loop $F^6$ term (see in this connection [51]).

(v) Another useful generalization would be to repeat the calculation of the $F^6$ term in other superconformal $\mathcal{N} = 2$ theories, corresponding, e.g., to orbifold versions of the AdS/CFT correspondence [72]. This would allow one to check whether the relation between the subleading $F^6$ interactions on the supergravity and the SYM sides is still holding in the less supersymmetric situation (i.e. whether the conjectured non-renormalization of the abelian $F^6$ term is indeed depending on the full $\mathcal{N} = 4$ supersymmetry).

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\footnote{It would also be useful to understand how to organize the calculation of the 2-loop $F^6$ term directly in 10-dimensional terms, making it possible to carry out the calculation in any $1+n$ dimensional reduction of the $D = 10, \mathcal{N} = 1$ SYM theory. One would then need to address the issues of the UV and IR cutoff dependence which were absent in the $D = 4$ case.}
Appendix A

Here we discuss the relation between the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetric forms of $F^6$ term. Let us start with the $\mathcal{N} = 2$ form of the $F^6$ correction (see (1.12))

$$S_6 = \int d^{12}z \frac{1}{W^2} \log \frac{W}{\mu} D^4 \log \frac{W}{\mu} + h.c. . \quad (A.1)$$

Using that

$$\int d^{12}z = \int d^4x d^4\theta \frac{1}{16} \bar{D}^2 D^2$$

we get

$$S_6 = \frac{1}{16} \int d^8z \bar{D}^2 D^2 (\frac{1}{W^2} \log \frac{W}{\mu} D^2 D^2 D^2 \log \frac{W}{\mu}) + h.c. . \quad (A.2)$$

Since $D^2 \bar{D}^2 = 0$ this is equal to

$$S_6 = \int d^8z \bar{D}^2 D^2 (\frac{1}{W^2} \log \frac{W}{\mu}) D^2 D^2 D^2 \log \frac{W}{\mu} + h.c. . \quad (A.3)$$

Here the symbol $|$ denotes the value at $\theta_2 = 0$. The chirality of $W$ implies

$$D^2 \bar{D}^2 (\frac{1}{W^2} \log \frac{W}{\mu}) = D^2 (\frac{1}{W^2} \bar{D}^2 \log \frac{W}{\mu}) . \quad (A.4)$$

To express this in $\mathcal{N} = 1$ form we use (see eq.(3.22) in [39]; by definition, $D_\alpha^2 W = 2i W_\alpha$)

$$D^2 D^2 \log \frac{W}{\mu} = 4 \frac{D^2 W^2}{\Phi^2} , \quad \bar{D}^2 \frac{1}{W^2} = -24 \frac{\bar{W}^2}{\Phi^4} , \quad D_2^2 \log \frac{W}{\mu} = 4 \frac{W^2}{\Phi^2} . \quad (A.5)$$

We thus finish with the following expression for the $\mathcal{N} = 1$ form of $S_6$

$$S_6 = -24 \int d^8z W^2 \bar{W}^2 D^2 W^2 \frac{1}{|\Phi|^8} + h.c. . \quad (A.6)$$

Appendix B

Here we shall discuss the calculation of the integral over the harmonics in the expression for two-loop low-energy effective action (1.13). Any integral of a function of the harmonics can be decomposed into a sum of integrals of products, so let us first describe the calculation of the integral of an arbitrary product of harmonics, e.g.,

$$H = \int du u_1^+ \ldots u_n^+ u_1^- \ldots u_n^- . \quad (B.1)$$
Here \( i_1, \ldots, i_n, j_1, \ldots, j_n = 1, 2 \) are the \( SU(2) \) harmonic indices taking values 1, 2. This expression contains equal number of \( u^+ \) and \( u^- \) harmonics, since otherwise the integral is zero. Any such integral can be calculated exactly using the general formalism of [44], in particular, the properties

\[
\int du = 1, \quad (B.2)
\]

\[
u^+ u^- = u^+ u^- = \epsilon_{ij}, \quad (B.3)
\]

\[
\int du (u^+ u^-)^{i_1} \cdots u^+ u^-^{i_n} u^-^{j_1} \cdots u^-^{j_m} = 0. \quad (B.4)
\]

The first relation defines the normalization of the integral, the second expresses the fact that the determinant of a matrix of the harmonics

\[
\begin{bmatrix}
u^+ & \nu^- \\ \nu^- & \nu^+
\end{bmatrix}
\]

is equal to one, and the third means that the integral from the symmetrized product of harmonics is equal to zero.

We first expand the product of harmonics into a sum of terms symmetric and antisymmetric in any pair of indices, and then use the property (B.3) which allows to replace the term \( u^+ u^- = \epsilon_{ij} \) by 1. We finish with the integral which contains the sum of products of epsilon symbols with all possible \((n!)\) permutations of indices \( i_1, \ldots, i_n, j_1, \ldots, j_n \), and the sum of various symmetrized products of harmonics. Due to (B.4) the integral of any symmetrized product of harmonics gives zero. As a result, a non-vanishing contribution comes from the sum of products of epsilon symbols with different orders of the indices. Since the expression (B.1) is symmetric with respect to permutations of the indices \( i_a \) as well as the indices \( j_b \) \((a, b = 1, \ldots, n)\), after (anti)symmetrization it should have the factor \( \frac{1}{n!} \). Then, the number of epsilon symbols arising from antisymmetrization is \( n \) (the terms with less number of epsilon symbols contain symmetric products of harmonics, and their integral is zero (B.4)), and there are \( n! \) equivalent arrangements of these epsilon symbols. Therefore, we get the combinatoric factor \( \frac{1}{n!} \). Also, the (anti)symmetrization of any pair of indices is accompanied by \( \frac{1}{2} \), so that \( n \) pairs of them require the factor \( \frac{1}{2^n} \). As a result, the integral (B.1) is equal to

\[
H = \frac{1}{2^n n!} \sum_{i_1, \ldots, i_n} \prod_{m=1}^n \epsilon_{imj_l}. \quad (B.5)
\]

To compute the integral in (4.13) we note that \( (D^+(v)W)^2(D^+(w)W)^2 \) can be represented as \( v_i^+ w_j^+ w_k^+ w_l^+ D^{i\alpha} D^{j\beta} D^{k\alpha} D^{l\beta} \). Then the only non-zero term in

\[
D^{i\alpha} D^{j\alpha} D^{k\beta} D^{l\beta} W
\]

is due to anticommutativity of \( D^{i\alpha} W \)-factors

\[
D^{i\alpha} D^{j\alpha} D^{k\beta} D^{l\beta} W = (D^1 W)^2 (D^2 W)^2 (v^+ w^+ + w^+ v^+)^2.
\]
It is convenient to introduce the notation

\[ v^{+1}w^{+2} + v^{+2}w^{+1} = d_{ij}v^{+i}w^{+j} , \quad d_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Using the relation (4.14):

\[ (v^+w^+)(v^-w^-) = 1 + (v^+w^-)(v^-w^+) \] (B.6)

we can express (4.13) in the form

\[
\Gamma^{(2)} = \frac{1}{48(4\pi)^4} g_{YM}^2 N^2 [ \int d^{12}z \frac{1}{(\mathcal{W}^2)^4} \mathcal{W}^2(D^1\mathcal{W})^2(D^2\mathcal{W})^2 + h.c.] \\
\times \int dwdv(d_{ij}v^{+i}w^{+j})^2 (v^-w^-)^2 (1 - 4(v^+w^-)(v^-w^+) + (v^+w^-)^2(v^-w^+)^2) \] (B.7)

Then the products of the harmonics in (4.13) can be represented as:

\[
(v^+w^-)(v^-w^+) = \epsilon_{ij} \epsilon_{kl} v^{+i} w^{-j} v^{-k} w^{+l} \\
((v^+w^-)(v^-w^+))^2 = \epsilon_{ij} \epsilon_{kl} \epsilon_{mn} \epsilon_{pq} v^{+i} w^{-j} v^{-k} w^{-l} v^{-m} w^{+n} v^{-p} w^{+q} \\
((v^+w^-)(v^-w^+))^3 = \epsilon_{ij} \epsilon_{kl} \epsilon_{mn} \epsilon_{pq} \epsilon_{ab} \epsilon_{cd} v^{+i} w^{-j} v^{-k} w^{-l} v^{-a} w^{+b} v^{-m} w^{+n} v^{-p} w^{+q} v^{-c} w^{+d} .
\] (B.8)

Substituting these expression into (B.7) we get

\[
\Gamma^{(2)} = \frac{1}{48(4\pi)^4} g_{YM}^2 N^2 [ \int d^{12}z \frac{1}{(\mathcal{W}^2)^4} \mathcal{W}^2(D^1\mathcal{W})^2(D^2\mathcal{W})^2 + h.c.] \\
\times \int dwdv(d_{ij}v^{+i}w^{+j})(d_{kl}v^{+k}w^{+l})(\epsilon_{mn} v^{-m} w^{-n})(\epsilon_{pq} v^{-p} w^{-q}) \\
\times \left[ 1 - 4\epsilon_{ab} \epsilon_{cd} v^{+a} w^{-b} v^{-c} w^{+d} + \epsilon_{ab} \epsilon_{cd} \epsilon_{rs} \epsilon_{tu} v^{+a} w^{+c} v^{-r} w^{-b} w^{-d} w^{+s} w^{+u} \right] \] (B.9)

The integrand here is the sum of three terms. Using (B.3) we find for the first term

\[
\int dwdv(d_{ij}v^{+i}w^{+j})(d_{kl}v^{+k}w^{+l})(\epsilon_{mn} v^{-m} w^{-n})(\epsilon_{pq} v^{-p} w^{-q}) \\
= d_{ij}d_{kl}\epsilon_{mn}\epsilon_{pq} \left( \frac{1}{2^{222!}} \right)^2 \\
\times (\epsilon^{im} \epsilon^{kp} + \epsilon^{km} \epsilon^{ip})(\epsilon^{jn} \epsilon^{lq} + \epsilon^{jq} \epsilon^{ln}) = \frac{1}{16} , \] (B.10)

for second term

\[
-4 \int dwdv(d_{ij}v^{+i}w^{+j})(d_{kl}v^{+k}w^{+l})(\epsilon_{mn} v^{-m} w^{-n})(\epsilon_{pq} v^{-p} w^{-q})\epsilon_{ab} \epsilon_{cd} v^{+a} w^{-b} v^{-c} w^{+d} \\
= -4d_{ij}d_{kl}\epsilon_{mn}\epsilon_{pq} \epsilon_{ab} \epsilon_{cd} \left( \frac{1}{2^{233!}} \right)^2 \\
\times (\epsilon^{im} \epsilon^{kp} \epsilon^{ac} + \epsilon^{im} \epsilon^{kc} \epsilon^{ap} + \epsilon^{km} \epsilon^{ip} \epsilon^{ac} + \epsilon^{km} \epsilon^{ic} \epsilon^{ap} + \epsilon^{am} \epsilon^{kp} \epsilon^{ic} + \epsilon^{am} \epsilon^{ip} \epsilon^{kc}) \\
\times (\epsilon^{jn} \epsilon^{lq} \epsilon^{db} + \epsilon^{jn} \epsilon^{lb} \epsilon^{dq} + \epsilon^{ln} \epsilon^{jq} \epsilon^{db} + \epsilon^{ln} \epsilon^{jb} \epsilon^{dq} + \epsilon^{dn} \epsilon^{jz} \epsilon^{db} + \epsilon^{dn} \epsilon^{jb} \epsilon^{lq}) = \frac{15}{32} \] (B.11)
and for the third term
\[
\int d\nu d\omega (d_{ij}\nu^{i}\omega^{j})(d_{kl}\nu^{k}\omega^{l})(\epsilon_{mn}\nu^{m}\omega^{n})(\epsilon_{pq}\nu^{p}\omega^{q})\epsilon_{ab}\epsilon_{cd}\epsilon_{ef}\epsilon_{tu}
\]
\[
\times \epsilon_{rs}\epsilon_{tu}\nu^{r}\nu^{t}\omega^{s}\omega^{u}
\]
\[
= d_{ij}d_{kl}\epsilon_{mn}\epsilon_{pq}\epsilon_{ab}\epsilon_{cd}\epsilon_{ef}\epsilon_{tu}\left(\frac{1}{24!}\right)^{2}
\]
\[
\times \left(\epsilon^{im}\epsilon^{kp}\epsilon^{ar}\epsilon^{ct} + \epsilon^{im}\epsilon^{kp}\epsilon^{at}\epsilon^{er} + \epsilon^{im}\epsilon^{kt}\epsilon^{ar}\epsilon^{cp} + \ldots\right)
\]
\[
\times \left(\epsilon^{jn}\epsilon^{ls}\epsilon^{sb}\epsilon^{ud} + \epsilon^{jn}\epsilon^{ls}\epsilon^{sd}\epsilon^{ub} + \epsilon^{jn}\epsilon^{ld}\epsilon^{sb}\epsilon^{uq} + \ldots\right)
\]
\[
= \frac{15}{32}
\]
(B.12)

The sum of (B.10), (B.11), (B.12) is equal to 1 and thus we finish with the expression (4.14).

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