Parameter Estimation in Nonlinear Multivariate Stochastic Differential Equations Based on Splitting Schemes

A Preprint

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Abstract

The likelihood functions for discretely observed nonlinear continuous time models based on stochastic differential equations are not available except for a few cases. Various parameter estimation techniques have been proposed, each with advantages, disadvantages, and limitations depending on the application. Most applications still use the Euler-Maruyama discretization, despite many proofs of its bias. More sophisticated methods, such as Kessler’s Gaussian approximation, Ozaki’s Local Linearization, Aït-Sahalia’s Hermite expansions, or MCMC methods, might be complex to implement, do not scale well with increasing model dimension or can be numerically unstable. We propose two efficient and easy-to-implement likelihood-based estimators based on the Lie-Trotter (LT) and the Strang (S) splitting schemes. We prove that S has $L^p$ convergence rate of order 1, a property already known for LT. We show that the estimators are consistent and asymptotically efficient under the less restrictive one-sided Lipschitz assumption. A numerical study on the 3-dimensional stochastic Lorenz system complements our theoretical findings. The simulation shows that the S estimator performs the best when measured on precision and computational speed compared to the state-of-the-art.

Keywords Asymptotic normality · Consistency · $L^p$ convergence · Splitting schemes · Stochastic differential equations · Stochastic Lorenz system

1 Introduction

Stochastic differential equations (SDEs) are popular models for physical, biological, and socio-economic processes. Some recent applications include tipping points in the climate (Ditlevsen and Ditlevsen, 2023), the spread of COVID-19 (Arnst et al., 2022; Kareem and Al-Azzawi, 2021), animal movements (Michelot et al., 2019, 2021) and cryptocurrency rates (Dipple et al., 2020). The advantage of SDEs is their ability to capture and quantify the randomness of the underlying dynamics. They are especially applicable when the dynamics are not entirely understood, and the unknown
We want to estimate the underlying drift parameter $\beta$ and diffusion parameter $\Sigma$ based on discrete observations of $X_t$.

The transition density is necessary for likelihood-based estimators and, thus, a closed-form solution to (1). However, the transition density is only available for a few SDEs, including the Ornstein-Uhlenbeck (OU) process, which has a linear drift function $F$. Extensive literature exists on MCMC methods for the nonlinear case (Fuchs, 2013; Chopin and Papaspiliopoulos, 2020) however, these are often computationally intensive and do not always converge to the correct values for complex models. Thus, we need a valid approximation of the transition density to perform likelihood-based statistical inference.

The most straightforward discretization scheme is the Euler-Maruyama (EM) (Kloeden and Platen, 1992). Its main advantage is the easy-to-implement and intuitive Gaussian transition density. Both frequentist and Bayesian approaches extensively employ EM across theoretical and applied studies. However, the EM-based estimator has many disadvantages. First, it exhibits pronounced bias as the discretization step increases (see Florens-Zmirou (1989) for a theoretical study, or Gloaguen et al. (2018), Gu et al. (2020) for applied studies). Second, Hutzenthaler et al. (2011) showed that it is not mean-square convergent when the drift function $F$ of (1) grows super-linearly. Consequently, we should avoid EM for models with polynomial drift. Third, it often fails to preserve important structural properties, such as hypoellipticity, geometric ergodicity, and amplitudes, frequencies, and phases of oscillatory processes (Buckwar et al., 2022).

Some pioneering papers on likelihood-based SDE estimators are Dacunha-Castelle and Florens-Zmirou (1986); Dohnal (1987); Florens-Zmirou (1989); Genon-Catalot and Jacod (1993); Kessler (1997). The first two only estimate the diffusion parameter. Florens-Zmirou (1989) used EM to estimate both parameters and derived asymptotic properties. Genon-Catalot and Jacod (1993) generalized to higher dimensions, non-equidistant discretization step, and a generic form of the objective function, however only estimating the diffusion parameter. Kessler (1997) proposed an estimator (denoted K) approximating the unknown transition density with a Gaussian density using the true conditional mean and covariance, or approximations thereof using the infinitesimal generator. He proved consistency and asymptotic normality under the commonly used, but too restrictive, global Lipschitz assumption on the drift function $F$.

A competitive likelihood-based approach relies on local linearization (LL), initially proposed by Ozaki (1985) and later extended by Ozaki (1992); Shoji and Ozaki (1998). They approximated the drift between two consecutive observations using a linear function. In the case of additive noise, this corresponds to an OU process with a known Gaussian transition density. Thus, the likelihood approximation is a product of Gaussian densities. Shoji (1998) proved that LL discretization is one-step consistent and $L^2$ convergent with order 1.5. Shoji (2011), Jimenez et al. (2017) extended the theory of LL for SDEs with multiplicative noise. Simulation studies show the superiority of the LL estimator compared to other estimators (Shoji and Ozaki, 1998; Hurn et al., 2007; Gloaguen et al., 2018; Gu et al., 2020). Until recently, the implementation of the LL estimator was numerically ill-conditioned due to the possible singularity of the Jacobian matrix of the drift function $F$. However, Gu et al. (2020) proposed an efficient implementation that overcomes this. The main disadvantage of the LL method is its slow computational speed.

Aït-Sahalia (2002) proposed Hermite expansions (HE) to approximate the transition density, focusing on univariate time-homogeneous diffusions. This method, widely utilized in finance, was later extended to both reducible and irreducible multidiffusive diffusions (Aït-Sahalia, 2008). Chang and Chen (2011) found conditions under which the HE estimator has the same asymptotic distribution as the exact maximum likelihood estimator (MLE). Choi (2013, 2015) further broadened the technique to time-inhomogeneous settings. Picchini and Ditlevsen (2011) used the method for multidimensional diffusions with random effects. When an SDE is irreducible, Aït-Sahalia (2008) applied Kolmogorov’s backward and forward equations to develop a small-time expansion of the diffusion probability densities. Yang et al. (2019) introduced a delta expansion method, using Itô-Taylor expansions to derive analytical approximations of the transition densities of multidiffusive diffusions inspired by Aït-Sahalia (2002). While Aït-Sahalia’s approach allows for a broad class of drift and diffusion functions, the implementation can be complex. To our knowledge, there have not been any applications to models with more than four dimensions. Furthermore, computing coefficients even up to order two can be challenging, while higher-order approximations are often necessary for non-linear models. Hurn et al. (2007) implemented HE up to third order in univariate cases, emphasizing the importance of symbolic computation tools like Mathematica or Maple. Their survey concluded that while LL is the best among discrete maximum likelihood estimators, HE is the preferred overall choice. They highlighted that the HE proposed by Aït-Sahalia (2002) has the best trade-off between speed and accuracy, proving more feasible than LL in most financial applications. Similar results are found in Jensen and Poulsen (2002); López-Pérez et al. (2021). However, LL’s broad applicability contrasts with the limitations of Hermite expansions, particularly for high-dimensional multidiffusive models exceeding three dimensions.

Apart from the above-mentioned general methods, there are some specific setups. Sørensen and Uchida (2003) investigated a small-diffusion estimator. Ditlevsen and Sørensen (2004); Gloter (2006) worked with integrated diffusion, and Uchida and Yoshida (2012) used adaptive maximum likelihood estimation. Bibby and Sørensen (1995) and Forman
and Sørensen (2008) explored martingale estimation functions (EF) in one-dimensional diffusions, but they are difficult to extend to multidimensional SDEs. Ditlevsen and Samson (2019) used the 1.5 scheme to solve the problem of hypoellipticity when the diffusion matrix is not of full rank.

More recently, contributions from Gloter and Yoshida (2020, 2021) have extended the research of Uchida and Yoshida (2012). Gloter and Yoshida (2020) introduced a non-adaptive approach and offered similar analytic asymptotic results as Ditlevsen and Samson (2019) without imposing strict limitations on the model class. Iguchi et al. (2022) proposed sampling schemes for elliptic and hypoelliptic models that often result in conditionally non-Gaussian integrals, distinguishing their approach from prior works. As the transition density of their new scheme is typically complex, Iguchi et al. (2022) created a closed-form density expansion using Malliavin calculus. They recommended a transition density scheme that retained second-order precision through prudent truncation of the expansion. This closed-form expansion aligns with the works of Aït-Sahalia (2002, 2008) and Li (2013) on elliptic SDEs, although with a different approach. Iguchi et al. (2022) deliver asymptotic results with analytically available rates, beneficial for both elliptic and hypoelliptic models.

Table 1 provides a comprehensive overview of estimator properties, finite sample performance, and required model assumptions for the most prominent state-of-the-art methods. While asymptotic properties might be similar in most cases, the finite sample properties are often different. The table also includes the Lie-Trotter (LT) and the Strang (S) splitting estimators, which we propose in this paper. The comparison encompasses four key characteristics: (1) Diffusion coefficient allowed in the model class, distinguishing between additive and general noise; (2) Asymptotic regime, the conditions needed to prove the asymptotic properties; (3) Implementation, assessing the complexity of implementation, dependence on model dimension and parameter optimization time; and (4) Finite sample properties, evaluating performance for fixed sample size $N$ and discretization step size $h$.

An essential aspect of any estimator is the practical execution in real-world applications. Although the previously mentioned research contributes significantly to the theoretical development and broadens our understanding of inference for SDEs, its practical implementations tend not to be user-friendly. Except for precomputed models, applications by non-specialists can be challenging. Our main contribution is proposing estimators that are intuitive, easy to implement, computationally efficient, and scalable with increasing dimensions. These characteristics make the estimators accessible to researchers in various applied sciences while maintaining desirable statistical properties. Moreover, these estimators remain competitive with the best state-of-the-art methods, particularly concerning estimation bias and variance.

We propose to use the LT or the S splitting schemes for statistical inference. These numerical approximations were first suggested for ordinary differential equations (ODEs) (see for example, McLachlan and Quispel (2002); Blanes et al. (2009)), but their extension to SDEs is straightforward. A few studies have investigated numerical properties (Bensoussan et al., 1992; Ableidinger et al., 2017; Ableidinger and Buckwar, 2016; Buckwar et al., 2022). Barbu (1988) applied LT splitting on nonlinear optimal control problems, while Hopkins and Wong (1986) used it for nonlinear filtering. Bou-Rabee and Owhadi (2010); Abdulle et al. (2015) used LT splitting to investigate conditions for preserving the measure of the ergodic nonlinear Langevin equations. Recently, Bréhier et al. (2023) showed that LT splitting successfully preserved positivity for a class of nonlinear stochastic heat equations with multiplicative space-time white noise. Additional studies on the application of splitting schemes to SDEs include those by Misawa (2001); Milstein and Tretyakov (2003); Leimkuhler and Matthews (2015); Alamo and Sanz-Serna (2016); Bréhier and Goudenège (2019). Regarding statistical applications, to the best of our knowledge, only Buckwar et al. (2020); Ditlevsen et al. (2023) used splitting schemes for parametric inference in combination with Approximate Bayesian Computation, and Ditlevsen and Ditlevsen (2023) used it for prediction of a forthcoming collapse in the climate.

This paper presents five main contributions:

1. We introduce two new efficient, easy-to-implement, and computationally fast estimators for multidimensional nonlinear SDEs.
2. We establish $L^p$ convergence of the S splitting scheme.
3. We prove consistency and asymptotic normality of the new estimators under the less restrictive assumption of one-sided Lipschitz. This proof requires innovative approaches.
4. We demonstrate the estimators’ performance in a stochastic version of the chaotic Lorenz system, in contrast to prior studies that primarily addressed the deterministic Lorenz system.
5. We compare the new estimators to four discrete maximum likelihood estimators from the literature in a simulation study, comparing the accuracy and computational speed.

The rest of this paper is structured as follows. In Section 2 we introduce the SDE model class and define the splitting schemes and the estimators. In Section 3, we show that the S splitting has better one-step predictions than the LT, and we prove that the S splitting is $L^p$ consistent with order 1.5 and $L^p$ convergent with order 1. To the best of
| Estimator   | Noise type                  | Asymptotic regime                                                                 | Computational time and implementation                                      | Finite sample properties                                                                                                                                 |
|------------|-----------------------------|----------------------------------------------------------------------------------|-----------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------|
| EM         | General                     | $h \to 0$, $Nh \to \infty$, $Nh^2 \to 0$ (Florens-Zmirou, 1989)                   | Fastest optimization and implementation.                                    | Earliest bias exhibition with increasing $h$.                                                                                                                                                                        |
| K up to order $J$ | General                     | $J$ fixed: $h \to 0$, $Nh \to \infty$, $Nh^p \to 0$, for any $p \in \mathbb{N}$ (Kessler, 1997) | Fastest optimization. Straightforward for any dimension.                    | Unbiased if the exact mean is known.                                                                                                                   |
| EF         | General                     | $h$ fixed: $N \to \infty$ (Bibby and Sørensen, 1995)                             | Fastest optimization. Requires moments of the transition density.           | For larger $h$, a higher order of $J$ is needed.                                                                                                           |
| LL         | Additive (possible generalization) | $h \to 0$, $Nh \to \infty$, $Nh^2 \to 0$ (Ozaki, 1992)                         | Fastest optimization.                                                      | Performance between EM and LL.                                                                                                                                     |
| HE up to order $J$ | General                     | $h$ fixed: $N \to \infty$, $J \to \infty$, $Nh^{2J+2} \to 0$, $J \geq 2$ fixed: $N \to \infty$, $h \to 0$, $Nh^3 \to \infty$, $Nh^{2J+1} \to 0$ (Chang and Chen, 2011) | Slower than LL in the univariate case. Implementation becomes significantly more complex in higher dimensions or for $J \geq 2$. (Hurn et al., 2007) | For larger $h$, a higher order of $J$ is needed. Better than LL in the univariate case. (Hurn et al., 2007) |
| LT (proposed) | Additive (possible generalization) | $h \to 0$, $Nh \to \infty$, $Nh^2 \to 0$                                     | Slower than $K$, but notably faster than LL.                               | Performance relative to EM varies based on splitting strategy and model.                                                                             |
| S (proposed) | Additive (possible generalization) | $h \to 0$, $Nh \to \infty$, $Nh^2 \to 0$                                     | Slower than LT, but notably faster than LL. Straightforward implementation for given nonlinear ODE solution. Scales well with the increasing dimension. | As good as LL.                                                                                                                                              |

Table 1: Comparison of the proposed Lie-Trotter (LT) and Strang (S) splittings (in bold) with five state-of-the-art estimators: Euler-Maruyama (EM), Kessler (K), Estimating functions (EF), Local linearization (LL) and Hermite expansion (HE). The comparison focuses on four key characteristics: (1) Noise type - additive or general, (2) Asymptotic regime – investigating conditions where asymptotic properties align with the exact MLE, (3) Computational time and implementation – evaluating implementation and parameter optimization costs; and (4) Finite sample properties – assessing performance under fixed $N$ and $h$. The finite sample properties of the estimators are likely influenced by specific experiment designs.

*While Kessler (1997) did not explicitly explore the scenario of a fixed $h$, it is a reasonable assumption that the asymptotic results will hold as $N \to \infty$ and $J \to \infty$. 

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our knowledge, this is a new result. Sections 4 and 5 establish the estimator asymptotics under the less restrictive one-sided global Lipschitz assumption. We illustrate in Section 6 the theoretical results in a simulation study on a model that is not globally Lipschitz, the 3-dimensional stochastic Lorenz systems. Since the objective functions based on pseudo-likelihoods are multivariate in both data and parameters, we use automatic differentiation (AD) to get faster and more reliable estimators. We compare the precision and speed of the EM, K, LL, HE, LT, and S estimators. We show that the EM and LT estimators become biased before the others with increasing discretization step (h), HE (of order 2) works only for the smallest h in the simulation study, and the LL and S perform the best. However, S is much faster than LL because LL calculates a new covariance matrix for each combination of data points and parameter values.

Notation. We use capital bold letters for random vectors, vector-valued functions, and matrices, while lowercase bold letters denote deterministic vectors. \(| \cdot |\) denotes both the \(L^2\) vector norm in \(\mathbb{R}^d\) and the matrix norm induced by the \(L^2\) norm, defined as the square root of the largest eigenvalue. Superscript \((i)\) on a vector denotes the \(i\)-th component, while on a matrix it denotes the \(i\)-th row. Double subscript \(ij\) on a matrix denotes the component in the \(i\)-th row and \(j\)-th column. If a matrix is a product of more matrices, square brackets with subscripts denote a component inside the matrix. The transpose is denoted by \(\top\). Operator \(\text{Tr}(\cdot)\) returns the trace of a matrix and \(\det(\cdot)\) the determinant. Sometimes, we denote by \(\left| a_i \right|\) a vector with coordinates \(a_i\), and by \(\left| b_{ij} \right|\) a matrix with coordinates \(b_{ij}\), for \(i, j = 1, \ldots, d\).

We denote with \(\partial_i g(x)\) the partial derivative of a generic function \(g : \mathbb{R}^d \to \mathbb{R}\) with respect to \(x^{(i)}\) and \(\partial^2_{ij} g(x)\) the second partial derivative. The nabla operator \(\nabla\) denotes the gradient vector of a function \(g, \nabla g(x) = [\partial_i g(x)]_{i=1}^d\). The differential operator \(D\) denotes the Jacobian matrix \(DF(x) = [\partial_i F^{(j)}(x)]_{i,j=1}^d\), for a vector-valued function \(F : \mathbb{R}^d \to \mathbb{R}^d\). \(H\) denotes the Hessian matrix of a real-valued function \(g, H_g(x) = [\partial^2_{ij} g(x)]_{i,j=1}^d\). Let \(R\) represent a vector (or a matrix) valued function defined on \((0, 1) \times \mathbb{R}^d\), such that, for some constant \(C\), \(\|R(a, x)\| < aC(1 + \|x\|)^C\) for all \(a, x\). When denoted \(\tilde{R}\), it is a scalar.

The Kronecker delta function is denoted by \(\delta_{ij}\). For an open set \(A\), the bar \(\overline{A}\) indicates closure. We use \(\theta\) to indicate equality up to an additive constant that does not depend on \(\theta\). We write \(x \overset{d}{\to} y\) and \(x \overset{P}{\to} y\) for convergence in probability, distribution, and almost surely, respectively. \(I_d\) denotes the \(d\)-dimensional identity matrix, while \(O_{d \times d}\) is a \(d\)-dimensional zero square matrix. For an event \(E \in \mathcal{F}\), we denote by \(\mathbb{1}_E\) the indicator function.

2 Problem setup

Let \(X\) in (1) be defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P}_\theta)\) with a complete right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\), and let the \(d\)-dimensional Wiener process \(W = (W_t)_{t \geq 0}\) be adapted to \(\mathcal{F}_t\). The probability measure \(\mathbb{P}_\theta\) is parameterized by the parameter \(\theta = (\beta, \Sigma)\). Rewrite equation (1) as follows:

\[
dX_t = A(\beta)(X_t - b(\beta)) \, dt + N(X_t; \beta) \, dt + \Sigma \, dW_t, \quad X_0 = x_0,
\]

such that \(F(x; \beta) = A(\beta)(x - b(\beta)) + N(x; \beta)\). Let \(\Theta = \overline{\Theta}_\beta \times \overline{\Theta}_\Sigma\) be the parameter space with \(\Theta_\beta\) and \(\Theta_\Sigma\) being two open convex bounded subsets of \(\mathbb{R}^r\) and \(\mathbb{R}^{d \times d}\), respectively.

Functions \(F, N : \mathbb{R}^d \times \overline{\Theta}_\beta \to \mathbb{R}^d\) are locally Lipschitz, and \(A, b\) are defined on \(\overline{\Theta}_\beta\) and take values in \(\mathbb{R}^{d \times d}\) and \(\mathbb{R}^d\), respectively. Parameter matrix \(\Sigma\) takes values in \(\mathbb{R}^{d \times d}\). The matrix \(\Sigma \Sigma^\top\) is assumed to be positive definite and determines the variance of the process. Since any square root of \(\Sigma \Sigma^\top\) induces the same distribution, \(\Sigma\) is only identifiable up to equivalence classes. Thus, instead of estimating \(\Sigma\), we estimate \(\Sigma \Sigma^\top\). The drift function \(F\) in (1) is split up into a linear part given by matrix \(A\) and vector \(b\) and a nonlinear part given by \(N\). This decomposition is essential for defining the splitting schemes and the objective functions used for estimating \(\theta\).

We denote the true parameter value by \(\theta_0 = (\beta_0, \Sigma_0)\) and assume that \(\theta_0 \in \Theta\). Sometimes we write \(A_0, \Sigma_0, N_0(x)\) and \(\Sigma \Sigma_0^\top\) instead of \(A(\beta_0), b(\beta_0), N(x; \beta_0)\) and \(\Sigma_0 \Sigma_0^\top\), when referring to the true parameters. We write \(A, b, N(x)\) and \(\Sigma \Sigma^\top\) for any parameter \(\theta\). Sometimes we suppress the parameter to simplify notation, e.g., \(\mathbb{E}\) implicitly refers to \(\mathbb{E}_{\theta_0}\).

Remark. The drift function \(F(x)\) can always be rewritten as \(F(x) = A(x - b) + N(x)\) for any \(A, b\) by setting \(N(x) = F(x) - A(x - b)\), including choosing \(A\) and \(b\) to be zero. The splitting proposed below will then result in a Brownian motion (3) and a nonlinear ODE (4).

Remark. We assume additive noise, sometimes referred to as constant volatility, meaning that the diffusion matrix does not depend on the current state. This assumption can be restrictive and even rejected by the data in some applications. The proposed methodology can be extended if the diffusion is reducible (Definition 1 in (Aït-Sahalia, 2008)) by applying the Lamperti transform to obtain a unit diffusion coefficient. However, if the transform depends on the parameter, estimation is not straightforward. In this paper, we only consider additive noise.
2.1 Assumptions

The main assumption is that (2) has a unique strong solution \( X = (X_t)_{t \in [0, T]} \), adapted to \((\mathcal{F}_t)_{t \in [0, T]}\), which follows from the following first two assumptions (Theorem 2 in Alyushina (1988), Theorem 1 in Krylov (1991), Theorem 3.5 in Mao (2007)). We need the last three assumptions to prove the properties of the estimators.

(A1) Function \( N \) is twice continuously differentiable with respect to \( x \) and \( \theta \), i.e., \( N \in C^2 \). Additionally, it is one-sided globally Lipschitz continuous with respect to \( x \) on \( \mathbb{R}^d \times \mathbb{R}_+ \), i.e., there exists a constant \( C > 0 \) such that:

\[
(x - y)^\top (N(x; \beta) - N(y; \beta)) \leq C\|x - y\|^2, \quad \forall x, y \in \mathbb{R}^d.
\]

(A2) Function \( N \) grows at most polynomially in \( x \), uniformly in \( \theta \), i.e., there exist constants \( C > 0 \) and \( \chi \geq 1 \) such that:

\[
\|N(x; \beta) - N(y; \beta)\|^2 \leq C(1 + \|x\|^{2\chi} + \|y\|^{2\chi}) \|x - y\|^2, \quad \forall x, y \in \mathbb{R}^d.
\]

Additionally, its derivatives are of polynomial growth in \( x \), uniformly in \( \theta \).

(A3) The solution \( X \) of SDE (1) has invariant probability \( \nu_0(dx) \).

(A4) \( \Sigma \Sigma^\top \) is invertible on \( \mathbb{R}^d \).

(A5) Function \( F \) is identifiable in \( \beta \), i.e., if \( F(x; \beta_1) = F(x; \beta_2) \) for all \( x \in \mathbb{R}^d \), then \( \beta_1 = \beta_2 \).

Assumption (A3) is required for the ergodic theorem to ensure convergence in distribution. Assumption (A4) implies that model (1) is elliptic, which is not needed for the S estimator, whereas the EM estimator breaks down in hypoelliptic models. We will treat the hypoelliptic case in a separate paper where the proofs are more involved. Assumption (A5) ensures the identifiability of the parameter.

Assume a sample \( (X_{t_k})_{k=0}^N \equiv X_{0:t_N} \) from (2) at time steps \( 0 = t_0 < t_1 < \cdots < t_N = T \). For notational simplicity, we assume equidistant step size \( h = t_k - t_{k-1} \).

2.2 Moments

Assumption (A1) ensures finiteness of the moments of the solution \( X \) (Tretyakov and Zhang, 2013), i.e.,

\[
\mathbb{E}\left[ \sup_{t \in [0, T]} \|X_t\|^{2p} \right] < C(1 + \|x_0\|^{2p}), \quad \forall p \geq 1.
\]

The infinitesimal generator \( L \) of (1) is defined on sufficiently smooth functions \( g : \mathbb{R}^d \times \Theta \rightarrow \mathbb{R} \) given by:

\[
L_{\theta_0}g(x; \theta) = F(x; \beta_0)^\top \nabla g(x; \theta) + \frac{1}{2} \text{Tr}(\Sigma \Sigma^\top H_{\theta}(x; \theta)).
\]

The moments of (1) are expanded using the following lemma (Lemma 1.10 in Kessler et al. (2012)).

**Lemma 2.1** Let Assumptions (A1)-(A2) hold. Let \( X \) be a solution of (1). Let \( g \in C^{(2l+2)} \) be of polynomial growth and \( p \geq 2 \). Then

\[
\mathbb{E}_{\theta_0}[g(X_{t_k}; \theta) \mid \mathcal{F}_{t_{k-1}}] = \sum_{j=0}^l \frac{h^j}{j!} L_{\theta_0}^j g(X_{t_{k-1}}; \theta) + R(h^{l+1}, X_{t_{k-1}}).
\]

We need terms up to order \( R(h^3, X_{t_{k-1}}) \). Applying \( L_{\theta_0} \) on \( g(x) = x^{(i)} \), Lemma 2.1 yields:

\[
\mathbb{E}[X_{t_k}^{(i)} \mid X_{t_{k-1}} = x] = x^{(i)} + hF^{(i)}(x) + \frac{h^2}{2}(F(x)^\top \nabla F^{(i)}(x)) + \frac{1}{2} \text{Tr}(\Sigma \Sigma^\top H_{F^{(i)}}(x)) + R(h^3, x).
\]

2.3 Splitting Schemes

Consider the following splitting of (2):

\[
\begin{align*}
\text{d}X_t^{[1]} &= A(X_t^{[1]} - b) \, \text{d}t + \Sigma \, \text{d}W_t, & X_0^{[1]} &= x_0, \\
\text{d}X_t^{[2]} &= N(X_t^{[2]}) \, \text{d}t, & X_0^{[2]} &= x_0.
\end{align*}
\]

The solution of equation (3) is an OU process given by the following \( h \)-flow:

\[
X_{t_k}^{[1]} = \Phi_h^{[1]}(X_{t_{k-1}}^{[1]}) = e^{Ah}X_{t_{k-1}}^{[1]} + (I - e^{Ah})b + \xi_{h,k}.
\]
where $\xi_{h,k} \overset{i.i.d.}{\sim} N_d(0, \Omega_h)$ for $k = 1, \ldots, N$ (Vatutipong and Phewchean, 2019). The covariance matrix $\Omega_h$ and the conditional mean of the OU process (5) are provided by:

$$\Omega_h = \int_0^h e^{A(h-u)} \Sigma \Sigma^\top e^{A^\top(h-u)} \, du = h^2 \frac{2}{2} (A \Sigma \Sigma^\top + \Sigma \Sigma^\top A^\top) + \mathbf{R}(h, x_0),$$

(6)

$$\mu_h(x; \beta) := e^{A(\beta)h}x + (I - e^{A(\beta)h})b(\beta).$$

(7)

Assumptions (A1) and (A2) ensure the existence and uniqueness of the solution of (4) (Theorem 1.2.17 in Humphries and Stuart (2002)). Thus, there exists a unique function $f_h : \mathbb{R}^d \times \Theta_\beta \to \mathbb{R}^d$, for $h \geq 0$, such that:

$$X^{[2]}_{h,k} = f_h(X^{[2]}_{h,k-1}) = f_h(X^{[2]}_{h,k-1}; \beta).$$

(8)

For all $\beta \in \Theta_\beta$, the time flow $f_h$ fulfills the following semi-group properties:

$$f_0(x; \beta) = x, \quad f_{t+s}(x; \beta) = f_t(f_s(x; \beta); \beta), \quad t, s \geq 0.$$

(9)

Remark Since only one-sided Lipschitz continuity is assumed, the solution to (4) might not exist for all $h < 0$ and all $x_0 \in \mathbb{R}^d$, implying that the inverse $f_h^{-1}$ might not exist. If it exists, then $f_h^{-1} = f_{-h}$. For the S estimator, we need a well-defined inverse. This is not an issue when $\mathbf{N}$ is globally Lipschitz.

We, therefore, introduce the following and last assumption.

(A6) Function $f_h^{-1}(x; \beta)$ is defined asymptotically, for all $x \in \mathbb{R}^d$, $\beta \in \Theta_\beta$, when $h \to 0$.

Before defining the splitting schemes, we present a useful proposition for expanding the nonlinear solution $f_h$ (Section 1.8 in (Hairer et al., 1993)).

**Proposition 2.2** Let Assumptions (A1)-(A2) hold. When $h \to 0$, the $h$-flow of (4) is

$$f_h(x) = x + h\mathbf{N}(x) + \frac{h^2}{2} (D\mathbf{N}(x)) \mathbf{N}(x) + \mathbf{R}(\mathbf{h}^3, x).$$

Now, we introduce the two most common splitting approximations, which serve as the main building blocks for the proposed estimators.

**Definition 2.3** Let Assumptions (A1) and (A2) hold. The Lie-Trotter and Strang splitting approximations of the solution of (2) are given by:

$$X^{[LT]}_{h,k} := \Phi^{[LT]}_h(X^{[LT]}_{h,k-1}) = (\Phi^{[1]}_h \circ \Phi^{[2]}_h)(X^{[LT]}_{h,k-1}) = \mu_h(f_h(X^{[LT]}_{h,k-1})) + \xi_{h,k},$$

(10)

$$X^{[S]}_{h,k} := \Phi^{[S]}_h(X^{[S]}_{h,k-1}) = (\Phi^{[2]}_h \circ \Phi^{[2]}_h)(X^{[S]}_{h,k-1}) = f_{h/2}(\mu_h(f_{h/2}(X^{[S]}_{h,k-1}))) + \xi_{h,k}.$$  

(11)

Remark The order of composition in the splitting schemes is not unique. Changing the order in the S splitting leads to a sum of 2 independent random variables, one Gaussian and one non-Gaussian, whose likelihood is not trivial. Thus, we only use the splitting (11). The reversed order in the LT splitting can be treated the same way as the S splitting.

Remark Splitting the drift $F(x)$ into a linear and a nonlinear part is not unique. However, all theorems and properties, particularly consistency and asymptotic normality of the estimators, hold for any splitting choice. Yet, for fixed step size $h$ and sample size $N$, certain splittings perform better than others. In this paper, we present two general and intuitive strategies. The first applies when the system has a fixed point: here, the linear part of the splitting is the linearization around the fixed point. The linear OU performs accurately near the fixed point, with the nonlinear part correcting for nonlinear deviations. Simulations consistently show this approach to perform best. Another strategy is to linearize around the measured average value for each coordinate. An in-depth analysis of the splitting strategies for a specific example is provided in Section 2.5.

Remark Trajectories of S and LT splittings coincide up to the first $h/2$ and the last $h/2$ steps of the flow $\Phi^{[2]}_{h/2}$. Indeed, when applied $k$ times, the S splitting can be written as:

$$(\Phi^{[S]}_h)^k(x_0) = (\Phi^{[2]}_{h/2} \circ (\Phi^{[LT]}_h)^k \circ \Phi^{[2]}_{-h/2})(x_0).$$

Thus, it is natural that LT and S have the same order of $L^p$ convergence. We prove this in Section 3. However, the LT and S trajectories differ in their output points (10) and (11). Strang splitting outputs the middle points of the smooth steps of the deterministic flow (8), while LT splitting outputs the stochastic increments in the rough steps. We conjecture that this is one of the reasons why the S splitting has superior statistical properties.
2.4 Estimators

In this section, we first introduce two new estimators, LT and S, given a sample \( X_{0:t_N} \). Subsequently, we provide a brief overview of the estimators EM, K, LL, and HE, which will be compared in the simulation study.

2.4.1 Splitting estimators

The LT scheme (10) follows a Gaussian distribution. Consequently, the objective function corresponds to (twice) the negative pseudo-log-likelihood:

\[
\mathcal{L}^{[\text{LT}]}(X_{0:t_N}; \theta) = N \log(\det \Omega_h(\theta)) + \sum_{k=1}^{N} (X_t - \mu_h(f_h(X_{t_{k-1}}; \beta); \beta))^\top \Omega_h(\theta)^{-1}(X_t - \mu_h(f_h(X_{t_{k-1}}; \beta); \beta)). \tag{12}
\]

The S splitting (11) is a nonlinear transformation of the Gaussian random variable \( \mu_h(f_{h/2}(X_{t_{k-1}}; \beta); \beta) + \xi_{h,k} \). We first define:

\[
Z_{t_k}(\beta) := f_{h/2}(X_{t_k}; \beta) - \mu_h(f_{h/2}(X_{t_{k-1}}; \beta); \beta). \tag{13}
\]

Afterwards, we apply a change of variables to derive the following objective function:

\[
\mathcal{L}^{[\text{S}]}(X_{0:t_N}; \theta) = N \log(\det \Omega_h(\theta)) + \sum_{k=1}^{N} Z_{t_k}(\beta) \Omega_h(\theta)^{-1}Z_{t_k}(\beta) - 2 \sum_{k=1}^{N} \log |\det Df_{h/2}(X_{t_k}; \beta)|. \tag{14}
\]

The last term is due to the nonlinear transformation and is an extra term that does not appear in commonly used pseudo-likelihoods.

The inverse function \( f_h^{-1} \) may not exist for all parameters in the search domain of the optimization algorithm. However, this problem can often be solved numerically. When \( f_h^{-1} \) is well defined, we use the identity \(-\log |\det Df_h(x; \beta)| = \log |\det Df_h(x; \beta)|\) in (14) to increase the speed and numerical stability.

Finally, we define the estimators as:

\[
\hat{\theta}_N^{[k]} := \arg\min_{\theta} \mathcal{L}^{[k]}(X_{0:t_N}; \theta), \quad k \in \{\text{LT}, \text{S}\}. \tag{15}
\]

2.4.2 Euler-Maruyama

The EM method uses first-order Taylor expansion of (1):

\[
X_{t_k}^{[\text{EM}]} := X_{t_{k-1}}^{[\text{EM}]} + hF(X_{t_{k-1}}^{[\text{EM}]}; \beta) + \xi_{h,k}^{[\text{EM}]}, \tag{16}
\]

where \( \xi_{h,k}^{[\text{EM}]} \sim N_d(0, h\Sigma \Sigma^\top) \) for \( k = 1, \ldots, N \) (Kloeden and Platen, 1992). The transition density \( p^{[\text{EM}]}(X_{t_k} \mid X_{t_{k-1}}; \theta) \) is Gaussian, so the pseudo-likelihood follows trivially.

2.4.3 Kessler’s Gaussian approximation

The K estimator uses Gaussian transition densities \( p^{[K]}(X_{t_k} \mid X_{t_{k-1}}; \theta) \) with the true mean and covariance of the solution \( X \) (Kessler, 1997). When the moments are unknown, they are approximated using the infinitesimal generator (Lemma 2.1). We implement the estimator K based on the 2nd-order approximation:

\[
X_{t_k}^{[K]} := X_{t_{k-1}}^{[K]} + hF(X_{t_{k-1}}^{[K]}; \beta) + \xi_{h,k}^{[K]}(X_{t_{k-1}}^{[K]}) + \frac{h^2}{2} \left( DF(X_{t_{k-1}}^{[K]}; \beta) F(X_{t_{k-1}}^{[K]}; \beta) + \frac{1}{2} \left[ \text{Tr}(H_F(\xi_{h,k}(X_{t_{k-1}}^{[K]}; \beta))) \right]_{i=1}^d \right), \tag{17}
\]

where \( \xi_{h,k}^{[K]}(X_{t_{k-1}}^{[K]}) \sim N_d(0, \Omega_{h,k}^{[K]}(\theta)) \), and \( \Omega_{h,k}^{[K]}(\theta) = h\Sigma \Sigma^\top + \frac{h^2}{2} \left( DF(X_{t_{k-1}}^{[K]}; \beta) \Sigma \Sigma^\top + \Sigma \Sigma^\top D^\top F(X_{t_{k-1}}^{[K]}; \beta) \right) \). The covariance matrix is not constant which makes the algorithm slower for a larger sample size.

8
2.4.4 Ozaki’s local linearization

Ozaki’s LL method approximates the drift of (1) between consecutive observations by a linear function (Jimenez et al., 1999). The LL method consists of the following steps:

1. Perform LL of the drift $F$ in each time interval $[t, t + h]$ by the Itô-Taylor series;
2. Compute the analytic solution of the resulting linear SDE.

The approximation becomes:

$$X_{t_k}^{[LL]} := X_{t_{k-1}}^{[LL]} + \Phi_{h}^{[LL]}(X_{t_{k-1}}^{[LL]}; \theta) + \xi_{h,k}^{[LL]}(X_{t_{k-1}}^{[LL]}),$$  \hspace{1cm} (18)

where $\xi_{h,k}^{[LL]}(X_{t_{k-1}}^{[LL]}) \sim \mathcal{N}_d(0, \Omega_{h,k}^{[LL]}(\theta))$, and

$$\Omega_{h,k}^{[LL]}(\theta) := \int_0^h e^{DF(X_{t_{k-1}}^{[LL]}; \beta)(h-u)} \Sigma \Sigma^\top e^{DF(X_{t_{k-1}}^{[LL]}; \beta)^\top (h-u)} \, du,$$

$$\Phi_{h}^{[LL]}(x; \theta) := R_{h,0}(DF(x; \beta))F(x; \beta) + (hR_{h,0}(DF(x; \beta)) - R_{h,1}(DF(x; \beta)))M(x; \theta),$$

$$R_{h,i}(DF(x; \beta)) := \int_0^h \exp(DF(x; \beta)u)u^i \, du, \quad i = 0, 1,$$

$$M(x; \theta) := \frac{1}{2} \text{Tr} H_1(x; \theta), \ldots, \text{Tr} H_d(x; \theta))^\top,$$

$$H_k(x; \theta) := \left[ \Sigma \Sigma^\top ight]_{i,j} \frac{\partial^2 F(k)}{\partial x_i \partial x_j}(x)^d_{i,j=1}.$$

We can efficiently compute $R_{h,i}$ and $\Omega_{h,k}^{[LL]}(\theta)$ using formulas from (Van Loan, 1978), see (Gu et al., 2020). For more details, see Supplementary Material S1.

Thus, $p^{[LL]}(X_{t_k} | X_{t_{k-1}}; \theta)$ is Gaussian and standard likelihood inference applies. Similarly to $K$, $\Omega_{h,k}^{[LL]}(\theta)$ depends on the previous state $X_{t_{k-1}}^{[LL]}$, which is a major downside since it is harder to implement and slower to run due to the computation of $N - 1$ covariance matrices. Unlike $K$, LL does not Taylor expand the approximated drift and covariance matrix, so the influence of sample size $N$ on computational times is much larger.

2.4.5 Aït-Sahalia’s Infinite Hermite Expansion

The HE method (Aït-Sahalia, 2002, 2008) approximates the likelihood using two transformations to make data resemble a normal distribution, facilitating corrections for finite samples. First, $X_t$ is transformed to unit diffusion $Y_t$, using the Lamperti transform. Then, $Y_t$ is transformed into a more normal-like $Z_t$. Finally, the objective function is a Hermite expansion in terms of convergent power series in $h$, around this normal density before reverting back to $X_t$. The Lamperti transform can be omitted for non-reducible diffusions (Aït-Sahalia, 2008). For additive noise, the HE objective function of order $J$ is given as:

$$\mathcal{L}^{[HE]}(X_{0:t_N}; \theta) \overset{d}{=} N \log(\det \Sigma \Sigma^\top) - 2 \sum_{k=1}^N \left( \frac{C_{Y}^{(j)}(\gamma(X_{t_k}) | \gamma(X_{t_{k-1}}))}{h} \right) + \sum_{j=0}^{J} \frac{h}{j!} C_{Y}^{(j)}(\gamma(X_{t_k}) | \gamma(X_{t_{k-1}})).$$  \hspace{1cm} (19)

Function $\gamma$ is the Lamperti transform, and functions $C_{Y}^{(j)}$, for $j = -1, 0, 1, \ldots, J$ are calculated recursively according to Theorem 1 in (Aït-Sahalia, 2008).

2.5 An example: the stochastic Lorenz system

The Lorenz system is a 3D system introduced by Lorenz (1963) to model atmospheric convection. The model is originally deterministic exhibiting deterministic chaos, i.e., tiny differences in initial conditions lead to unpredictable and widely diverging trajectories. The Lorenz system evolves around two strange attractors, implying that trajectories remain within some bounded region, while points that start in close proximity may eventually separate by arbitrary distances as time progresses (Hilborn and Hilborn, 2000). We add noise to include unmodelled forces and randomness.
We suggest to split SDE (20) by choosing the OU part (3) as the linearization around one of the two fixed points would still have been there due to the chaotic behavior. However, it has a unique global solution and an invariant probability (Keller, 1996). Lorenz (1963) used the values \( p = 10, r = 28, c = \frac{8}{3} \), \( \sigma_1^2 = 1, \sigma_2^2 = 2 \) and \( \sigma_3^2 = 1.5 \).

The stochastic Lorenz system is given by:

\[
\begin{align*}
\text{d}X_t &= p(Y_t - X_t) \text{d}t + \sigma_1 \text{d}W^{(1)}_t, \\
\text{d}Y_t &= (rX_t - Y_t - X_tZ_t) \text{d}t + \sigma_2 \text{d}W^{(2)}_t, \\
\text{d}Z_t &= (X_tY_t - cZ_t) \text{d}t + \sigma_3 \text{d}W^{(3)}_t.
\end{align*}
\]

The variables \( X_t, Y_t, \) and \( Z_t \) represent convective intensity, and horizontal and vertical temperature differences, respectively. Parameters \( p, r, \) and \( c \) denote the Prandtl number, the Rayleigh number, and a geometric factor, respectively (Tabor, 1989). Lorenz (1963) used the values \( p = 10, r = 28 \) and \( c = \frac{8}{3} \), yielding chaotic behavior.

The system does not fulfill the global or the one-sided Lipschitz condition because it is a second-order polynomial (Humphries and Stuart, 1994). However, it has a unique global solution and an invariant probability (Keller, 1996). Thus, all assumptions (A2)-(A5), except (A1) hold. Even so, we show in Section 6 that the estimators work.

Different approaches for estimating parameters in the Lorenz system have been proposed, mostly in the deterministic case. Zhu et al. (2020) and Lazzús et al. (2016) used sophisticated optimization algorithms to achieve better precision. Dubois et al. (2020) and Ann et al. (2022) used deep neural networks in combination with other machine learning algorithms. Ozaki et al. (2000) used Kalman filtering based on LL on the stochastic Lorenz system.

Figure 1 shows an example trajectory of the stochastic Lorenz system. The trajectory was generated by subsampling from an EM simulation, such that \( N = 10000 \) and \( h = 0.005 \), with parameter values \( p = 10, r = 28, c = \frac{8}{3}, \sigma_1^2 = 1, \sigma_2^2 = 2 \) and \( \sigma_3^2 = 1.5 \). Even if the trajectory had not been stochastic, the unpredictable jumps in the first row of Figure 1 would still have been there due to the chaotic behavior.

We suggest to split SDE (20) by choosing the OU part (3) as the linearization around one of the two fixed points \( (x^*, y^*, z^*) = (\pm\sqrt{c(r-1)}, \pm\sqrt{c(r-1)}, r-1) \). For simplicity, we exclude the fixed point \( (0, 0, 0) \) since \( X \) and \( Y \) spend little time around this point, see Figure 1. Specifically, we apply a mixture of two splittings, linearizing around \( (\sqrt{c(r-1)}, \sqrt{c(r-1)}, r-1) \) when \( X > 0 \) and around \( (-\sqrt{c(r-1)}, -\sqrt{c(r-1)}, r-1) \) when \( X < 0 \). We denote these estimators by \( \text{LT}_{\text{mix}} \) and \( S_{\text{mix}} \). The splitting is given by:

\[
A_{\text{mix}} = \begin{bmatrix} -p & p & 0 \\ 1 & -1 & -x^* \\ y^* & x^* & -c \end{bmatrix}, \quad b_{\text{mix}} = \begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix}, \quad N_{\text{mix}}(x, y, z) = \begin{bmatrix} 0 \\ -(x-x^*)(z-z^*) \\ (x-x^*)(y-y^*) \end{bmatrix}.
\]
The OU process is mean-reverting towards \( b_{\text{mix}} = (x^*, y^*, z^*) \). The nonlinear solution is
\[
\mathbf{f}_{\text{mix}, h}(x, y, z) = \begin{bmatrix} x \\ (y - y^*) \cos(h(x - x^*)) - (z - z^*) \sin(h(x - x^*)) + y^* \\ (y - y^*) \sin(h(x - x^*)) + (z - z^*) \cos(h(x - x^*)) + z^* \end{bmatrix}.
\]
The solution is a composition of a 3D rotation and translation of \((y, z)\) around the fixed point. The inverse always exists, and thus, Assumption (A6) holds. Moreover, \( \det Df_{\text{mix}, h}(\cdot) = 1 \).

The mixing strategy does not increase the complexity of the implementation significantly, and it is straightforward to incorporate into the existing framework. Thus, this splitting strategy is convenient when the model has several fixed points.

An alternative splitting linearizes around the average of the observations. Let \( (\mu_x, \mu_y, \mu_z) \) be the average of the data, where we put \( \mu_x = \mu_y \) since the difference of their averages is small, around \( 10^{-3} \). We denote these estimators by \( \mathbf{L}_{\text{avg}} \) and \( S_{\text{avg}} \). The splitting is given by:
\[
\mathbf{A}_{\text{avg}} = \begin{bmatrix} -p & p & 0 \\ r - \mu_z & -1 & -\mu_x \\ \mu_x & -c & -\mu_x \end{bmatrix}, \quad \mathbf{b}_{\text{avg}} = \begin{bmatrix} \mu_x \\ \mu_x \\ \mu_x \end{bmatrix}, \quad \mathbf{N}_{\text{avg}}(x, y, z) = \begin{bmatrix} -(x - \mu_x)(z - \mu_z) + (r - 1 - \mu_z)\mu_z \\ (x - \mu_x)(y - \mu_x) + \mu_x^2 - c\mu_x \end{bmatrix}.
\]

The nonlinear solution is:
\[
\mathbf{f}_{\text{avg}, h}(x, y, z) = \begin{bmatrix} \mu_x + c\mu_x^2 \frac{\mu_x - \mu_z^2}{x - \mu_x} \\ \mu_z + c\mu_x \frac{\mu_x - \mu_z^2}{x - \mu_x} \\ \frac{x - \mu_x}{\mu_x - \mu_z} \end{bmatrix} + \begin{bmatrix} (y - \mu_x - c\mu_x^2 \frac{\mu_x - \mu_z^2}{x - \mu_x}) \cos(h(x - \mu_x)) - (z - \mu_z - \mu_x \frac{(r - 1 - \mu_z)}{x - \mu_x}) \sin(h(x - \mu_x)) \\ (y - \mu_x - c\mu_x^2 \frac{\mu_x - \mu_z^2}{x - \mu_x}) \sin(h(x - \mu_x)) + (z - \mu_z - \mu_x \frac{(r - 1 - \mu_z)}{x - \mu_x}) \cos(h(x - \mu_x)) \end{bmatrix},
\]
where \( f_{\text{avg}, h}(x, y, z) := (\mu_x, y + h\mu_x (r - 1 - \mu_z), z + h\mu_x^2 - c\mu_x) \) and \( \det Df_{\text{avg}, h}(\cdot) = 1 \).

### 3 Order of one-step predictions and \( L^p \) convergence

In this Section, we investigate \( L^p \) convergence of the splitting schemes and the order of the one-step predictions. Theorem 2.1 in Tretyakov and Zhang (2013) extends Milstein's fundamental theorem on \( L^p \) convergence for global Lipschitz coefficients (Milstein, 1988) to Assumptions (A1) and (A2). This theorem provides the theoretical underpinning for our approach, drawing on the key concepts of \( L^p \) consistency and boundedness of moments.

**Definition 3.1 (\( L^p \) consistency of a numerical scheme)** The one-step approximation \( \tilde{\Phi}_h \) of the solution \( \mathbf{X} \) is \( L^p \) consistent, \( p \geq 1 \), of order \( q_2 - 1/2 \geq 0 \), if for \( k = 1, \ldots, N \), and some \( q_1 \geq q_2 + 1/2 \):
\[
\|E[\mathbf{X}_{t_k} - \tilde{\Phi}_h(\mathbf{X}_{t_{k-1}}) \mid \mathbf{X}_{t_{k-1}} = \mathbf{x}]\| = R(h^{q_1}, \mathbf{x}),
\]
\[
(E[\|\mathbf{X}_{t_k} - \tilde{\Phi}_h(\mathbf{X}_{t_{k-1}})\|^{2p} \mid \mathbf{X}_{t_{k-1}} = \mathbf{x})]^{1/2p} = R(h^{q_2}, \mathbf{x}).
\]

**Definition 3.2 (Bounded moments of a numerical scheme)** A numerical approximation \( \bar{\mathbf{X}} \) of the solution \( \mathbf{X} \) has bounded moments, if for all \( p \geq 1 \), there exists constant \( C > 0 \), such that, for \( k = 1, \ldots, N \):
\[
E[\|\bar{\mathbf{X}}_{t_k}\|^{2p}] \leq C(1 + \|x_0\|^{2p}).
\]

The following theorem (Theorem 2.1 in Tretyakov and Zhang (2013)) gives sufficient conditions for \( L^p \) convergence of a numerical scheme in a one-sided Lipschitz framework.

**Theorem 3.3 (\( L^p \) convergence of a numerical scheme)** Let Assumptions (A1) and (A2) hold, and let \( \bar{\mathbf{X}}_{t_k} \) be a numerical approximation of the solution \( \mathbf{X}_{t_k} \) of (1) at time \( t_k \). If

1. The one-step approximation \( \tilde{\Phi}_h(\bar{\mathbf{X}}_{t_{k-1}}) \) is \( L^p \) consistent of order \( q_2 - 1/2 \); and
2. \( \bar{\mathbf{X}} \) has bounded moments,

then \( \bar{\mathbf{X}} \) is \( L^p \) convergent, \( p \geq 1 \), of order \( q_2 - 1/2 \), i.e., for \( k = 1, \ldots, N \), it holds:
\[
(E[\|\mathbf{X}_{t_k} - \bar{\mathbf{X}}_{t_k}\|^{2p}])^{1/2p} = R(h^{q_2-1/2}, x_0).
\]
3.1 Lie-Trotter splitting

We first show that the one-step LT approximation is of order $R(h^2, x_0)$ in mean. The following proposition is proved in the Supplementary Material S1 for scheme (10), as well as for the reversed order of composition. We demonstrate that the order of one-step prediction can not be improved unless the drift $F$ is linear.

**Proposition 3.4 (One-step prediction of LT splitting)** Assume (A1)-(A2), let $X$ be the solution to (1) and let $\Phi^{LT}_h$ be the LT approximation (10). Then, for $k = 1, \ldots, N$, it holds:

$$\|\mathbb{E}[X_{t_k} - \Phi^{LT}_h(X_{t_{k-1}}) | X_{t_{k-1}} = x]\| = R(h^2, X_{t_{k-1}}).$$

$L^p$ convergence of the LT splitting scheme is established in Theorem 2 in Buckwar et al. (2022), which we repeat here for convenience.

**Theorem 3.5 ($L^p$ convergence of the LT splitting)** Assume (A1)-(A2), let $X^{LT}$ be the LT approximation defined in (10), and let $X$ be the solution of (1). Then, there exists $C \geq 1$ such that for all $p \geq 2$ and $k = 1, \ldots, N$, it holds:

$$(\mathbb{E}[(\|X_{t_k} - X^{LT}_t\|_p)^p])^{1/p} = R(h, x_0).$$

Now, we investigate the same properties for the S splitting.

3.2 Strang splitting

The following proposition states that the S splitting (11) has higher order one-step predictions than the LT splitting (10). The proof can be found in Supplementary Material S1.

**Proposition 3.6** Assume (A1)-(A2), let $X$ be the solution to (1), and let $\Phi^{S}_h$ be the S splitting approximation (11). Then, for $k = 1, \ldots, N$, it holds:

$$\|\mathbb{E}[X_{t_k} - \Phi^{S}_h(X_{t_{k-1}}) | X_{t_{k-1}} = x]\| = R(h^3, X_{t_{k-1}}).$$

**Remark** Even though LT and S have the same order of $L^p$ convergence, the crucial difference is in the one-step prediction. The approximated transition density between two consecutive data points depends on the one-step approximation. Thus, the objective function based on pseudo-likelihood from the S splitting is more precise than the one from the LT.

To prove $L^p$ convergence of the S splitting scheme for (1) with one-sided Lipschitz drift, we follow the same procedure as in Buckwar et al. (2022). The proof of the following theorem is in Supplementary Material S1.

**Theorem 3.7 ($L^p$ convergence of S splitting)** Assume (A1), (A2) and (A6), let $X^{S}$ be the S splitting defined in (11), and let $X$ be the solution of (1). Then, there exists $C \geq 1$ such that for all $p \geq 2$ and $k = 1, \ldots, N$, it holds:

$$(\mathbb{E}[(\|X_{t_k} - X^{S}_t\|_p)^p])^{1/p} = R(h, x_0).$$

Before we move to parameter estimation, we prove a useful corollary.

**Corollary 3.8** Let all assumptions from Theorem 3.7 hold. Then, $(\mathbb{E}[\|Z_{t_k} - \xi_{h,k}\|_p])^{1/p} = R(h, x_0)$.

**Proof** From the definition of $Z_{t_k}$ in (13), it is enough to prove that:

$$(\mathbb{E}[\|f^{-1}_{h/2}(X_{t_k}) - \mu_h(f_{h/2}(X_{t_{k-1}})) - \xi_{h,k}\|_p])^{1/p} = R(h, x_0).$$

From (11) we have that $\xi_{h,k} = f^{-1}_{h/2}(X_{t_k}) - \mu_h(f_{h/2}(X_{t_{k-1}})).$ Then, $$\mathbb{E}[\|f^{-1}_{h/2}(X_{t_k}) - \mu_h(f_{h/2}(X_{t_{k-1}})) - \xi_{h,k}\|_p]^{1/p} \leq C(\mathbb{E}[\|f^{-1}_{h/2}(X_{t_k}) - f^{-1}_{h/2}(X^{S}_{t_k})\|_p] + \mathbb{E}[\|f_{h/2}(X_{t_{k-1}}) - f_{h/2}(X^{S}_{t_{k-1}})\|_p]^{1/p}) \leq C(\mathbb{E}[\|X_{t_k} - X^{S}_{t_k}\|_p] + \mathbb{E}[\|X_{t_{k-1}} - X^{S}_{t_{k-1}}\|_p]^{1/p} + R(h, x_0)).$$

We used Proposition 2.2, that $X, X^{S}$ have finite moments and $f_{h/2}, f^{-1}_{h/2}$ grow polynomially. The result follows from $L^p$ convergence of the S splitting scheme, Theorem 3.7.
4 Auxiliary properties

This paper centers around proving the properties of the S estimator. There are two reasons for this. First, most numerical properties in the literature are proved only for LT splitting because proofs for S splitting are more involved. Here, we establish both the numerical properties of the S splitting as well as the properties of the estimator. Second, the S splitting introduces a new pseudo-likelihood that differs from the standard Gaussian pseudo-likelihoods. Consequently, standard tools, like those proposed by Kessler (1997), do not directly apply.

The asymptotic properties of the LT estimator are the same as for the S estimator. However, the following auxiliary properties will be stated and proved only for the S estimator. They can be reformulated for the LT estimator following the same logic.

Before presenting the central results for the estimator, we establish the groundwork with two essential lemmas that rely on the model assumptions. Lemma 4.1 (Lemma 6 in Kessler (1997)) deals with the $p$-th moments of the SDE increments and also provides a moment bound of a polynomial map of the solution. The proof of this lemma in Supplementary Material S1 differs from that in Kessler (1997) due to our relaxation of the global Lipschitz assumption of the drift $F$. Instead, we use a one-sided Lipschitz condition in conjunction with the generalized Grönwall’s inequality (Lemma 2.3 in Tian and Fan (2020) to establish the result, see Supplementary Material S1).

Lemma 4.2 (Lemma 8 in Kessler (1997), Lemma 2 in Sørensen and Uchida (2003)) constitutes a central ergodic property that is essential for establishing the asymptotic behavior of the estimator. The proof when the drift $F$ is one-sided Lipschitz is identical to the one presented in Kessler (1997), particularly when combined with Lemma 4.1.

Lemma 4.1 Assume (A1)-(A2). Let $X$ be the solution of (1). For $t_k \geq t \geq t_{k-1}$, where $h = t_k - t_{k-1} < 1$, the following two statements hold.

1. For $p \geq 1$, there exists $C_p > 0$ that depends on $p$, such that:
   \[ E[\|X_t - X_{t_{k-1}}\|^p | F_{t_{k-1}}] \leq C_p(t - t_{k-1})^{p/2}(1 + \|X_{t_{k-1}}\|)^{Cr}. \]

2. If $g : \mathbb{R}^d \times \Theta \to \mathbb{R}$ is of polynomial growth in $x$ uniformly in $\theta$, then there exist constants $C$ and $C_{t-t_{k-1}}$ that depend on $t - t_{k-1}$, such that:
   \[ E[|g(X_t; \theta)| | F_{t_{k-1}}] \leq C_{t-t_{k-1}}(1 + \|X_{t_{k-1}}\|)^{C}. \]

Lemma 4.2 Assume (A1), (A2), (A3), and let $X$ be the solution to (1). Let $g : \mathbb{R}^d \times \Theta \to \mathbb{R}$ be a differentiable function with respect to $x$ and $\theta$ with derivative of polynomial growth in $x$, uniformly in $\theta$. If $h \to 0$ and $Nh \to \infty$, then,

\[ \frac{1}{N} \sum_{k=1}^{N} g(X_{t_k}, \theta) \xrightarrow{N h \to \infty} \int g(x, \theta) d\nu_0(x), \]

uniformly in $\theta$.

Lastly, we state the moment bounds needed for the estimator asymptotics. The proof is in Supplementary Material S1.

Proposition 4.3 (Moment Bounds) Assume (A1), (A2), (A6). Let $X$ be the solution of (1), and $Z_{t_k}$ as defined in (13). Let $g(x; \beta)$ be a generic function with derivatives of polynomial growth, and $\beta \in \Theta_\beta$. Then, for $k = 1, \ldots, N$, the following moment bounds hold:

1. $E_{\theta_0}[Z_{t_k}(\beta_0) | X_{t_{k-1}} = x] = R(h^3, X_{t_{k-1}})$
2. $E_{\theta_0}[Z_{t_k}(\beta_0)g(X_{t_k}; \beta)^\top | X_{t_k} = x] = \frac{h}{2}(\Sigma \Sigma^\top D^\top g(x; \beta) + Dg(x; \beta) \Sigma \Sigma^\top) + R(h^2, X_{t_{k-1}})$;
3. $E_{\theta_0}[Z_{t_k}(\beta_0)Z_{t_k}(\beta_0)^\top | X_{t_{k-1}} = x] = h^2 \Sigma \Sigma^\top + R(h^2, X_{t_{k-1}})$.

5 Asymptotics

The estimators $\hat{\theta}_N$ are defined in (15). However, the full objective functions (12) and (14) are not needed to prove consistency and asymptotic normality. It is enough to approximate $\Omega_h$ up to the second order by $h \Sigma \Sigma^\top + \frac{h^2}{2}(A \Sigma \Sigma^\top +}$
Theorem 5.1

Let $X$ be the solution of (1) and $\hat{\theta}_N = (\hat{\beta}_N, \hat{\Sigma}\Sigma^T)$ be the estimator that minimizes either (22) or (23). If $h \to 0$ and $Nh \to \infty$, then,

$$\hat{\beta}_N \xrightarrow{p_{\theta_0}} \beta_0,$$

$$\hat{\Sigma}\Sigma^T \xrightarrow{p_{\theta_0}} \Sigma\Sigma^T_0.$$

5.1 Consistency

Now, we state the consistency of $\hat{\beta}_N$ and $\hat{\Sigma}\Sigma^T$. The proof of Theorem 5.1 is in Supplementary Material S1.

Similarly, we approximate the log-determinant as:

$$\log \det \Omega_h(\theta) = \log \det(h \Sigma \Sigma^T + \frac{h^2}{2} (A(\beta) \Sigma \Sigma^T + \Sigma \Sigma^T A(\beta)^T)) + R(h^2, x_0)$$

$$\overset{\theta}{=} \log \det \Sigma \Sigma^T + \log \det(I + \frac{h}{2} (A(\beta) \Sigma \Sigma^T + \Sigma \Sigma^T A(\beta)^T)) + R(h^2, x_0)$$

$$= \log \det \Sigma \Sigma^T + \frac{h}{2} \text{Tr}(A(\beta) + \Sigma \Sigma^T A(\beta)^T) + R(h^2, x_0)$$

Using the same approximation we obtain:

$$2 \log \left| \det Df_{h/2}(x; \beta) \right| = 2 \log \left| \det (I + \frac{h}{2} DN(x; \beta)) \right|$$

$$= 2 \log \left| 1 + \frac{h}{2} \text{Tr} DN(x; \beta) \right| + R(h, x)$$

$$= h \text{Tr} DN(x; \beta) + R(h^2, x_0).$$

Retaining terms up to order $R(Nh^2, x_0)$ from (12) and (14), we establish the approximate objective functions:

$$L_N^{[\theta]}(\theta) := N \log \det \Sigma \Sigma^T + Nh \text{Tr} A(\beta)$$

$$+ \frac{1}{h} \sum_{k=1}^N (X_{t_k} - \mu_h(f_h(X_{t_{k-1}}; \beta); \beta))^T (\Sigma \Sigma^T)^{-1} (X_{t_k} - \mu_h(f_h(X_{t_{k-1}}; \beta); \beta))$$

$$- \sum_{k=1}^N (X_{t_k} - \mu_h(f_h(X_{t_{k-1}}; \beta); \beta)^T (\Sigma \Sigma^T)^{-1} A(\beta)(X_{t_k} - \mu_h(f_h(X_{t_{k-1}}; \beta); \beta))$$

$$L_N^{[S]}(\theta) := N \log \det \Sigma \Sigma^T + Nh \text{Tr} A(\beta) + \frac{1}{h} \sum_{k=1}^N Z_{t_k}(\beta)^T (\Sigma \Sigma^T)^{-1} Z_{t_k}(\beta)$$

$$- \sum_{k=1}^N Z_{t_k}(\beta)^T (\Sigma \Sigma^T)^{-1} A(\beta) Z_{t_k}(\beta) + h \sum_{k=1}^N \text{Tr} DN(X_{t_k}; \beta).$$

Unlike other likelihood-based methods, such as Kessler (1997), Aït-Sahalia (2002, 2008), Choi (2013, 2015), Yang et al. (2019), our estimators do not involve expansions. The objective functions are formulated in simple terms without hyperparameters, such as the order of the expansions. Hence, our approach is robust and user-friendly, as we directly employ (12) and (14). The approximations (22) and (23) are only used for the proofs.

5.1 Consistency

Now, we state the consistency of $\hat{\beta}_N$ and $\hat{\Sigma}\Sigma^T$. The proof of Theorem 5.1 is in Supplementary Material S1.
5.2 Asymptotic normality

First, we need some preliminaries. Let $\rho > 0$ and $B_\rho (\theta_0) = \{ \theta \in \Theta \mid \| \theta - \theta_0 \| \leq \rho \}$ be a ball around $\theta_0$. Since $\theta_0 \in \Theta$, for sufficiently small $\rho > 0$, $B_\rho (\theta_0) \subset \Theta$. Let $\mathcal{L}_N$ be either (22) or (23). For $\theta_N \in B_\rho (\theta_0)$, the mean value theorem yields:

$$
\left( \int_0^1 H_{\mathcal{L}_N}(\theta_0 + t(\hat{\theta}_N - \theta_0)) \, dt \right) (\hat{\theta}_N - \theta_0) = -\nabla \mathcal{L}_N (\theta_0).
$$

(24)

With $\varsigma := \text{vech}(\Sigma \Sigma^T) = ([\Sigma \Sigma^T]_{11}, [\Sigma \Sigma^T]_{12}, [\Sigma \Sigma^T]_{22}, \ldots, [\Sigma \Sigma^T]_{dd_1}, \ldots, [\Sigma \Sigma^T]_{dd_2})$, we half-vectorize $\Sigma \Sigma^T$ to avoid working with tensors when computing derivatives with respect to $\Sigma \Sigma^T$. Since $\Sigma \Sigma^T$ is a symmetric $d \times d$ matrix, $\varsigma$ is of dimension $s = d(d + 1)/2$. For a diagonal matrix, instead of a half-vectorization, we use $\varsigma := \text{diag}(\Sigma \Sigma^T)$.

Define:

$$
C_N(\theta) := \begin{bmatrix}
\frac{1}{N} \partial_{\beta \theta} \mathcal{L}_N(\theta) \\
\frac{1}{N} \partial_{\varsigma \theta} \mathcal{L}_N(\theta)
\end{bmatrix},
$$

(25)

$$
s_N := \begin{bmatrix}
\sqrt{Nh}(\hat{\beta}_N - \beta_0) \\
\sqrt{Nh}(\hat{\varsigma}_N - \varsigma_0)
\end{bmatrix}, \quad \lambda_N := \begin{bmatrix}
-\frac{1}{\sqrt{Nh}} \partial_{\beta \theta} \mathcal{L}_N(\theta_0) \\
-\frac{1}{\sqrt{Nh}} \partial_{\varsigma \theta} \mathcal{L}_N(\theta_0)
\end{bmatrix},
$$

(26)

and $D_N := \int_0^1 C_N(\theta_0 + t(\hat{\theta}_N - \theta_0)) \, dt$. Then, (24) is equivalent to $D_N s_N = \lambda_N$. Let:

$$
C(\theta_0) := \begin{bmatrix}
C_\beta(\theta_0) & 0_{r \times s} \\
0_{s \times r} & C_\varsigma(\theta_0)
\end{bmatrix},
$$

(27)

where:

$$
[C_\beta(\theta_0)]_{i_1,i_2} := \int (\partial_{\beta_{i_1}} F_0(x))^T (\Sigma \Sigma_0^T)^{-1} (\partial_{\beta_{i_2}} F_0(x)) \, d\nu_0(x), \quad 1 \leq i_1, i_2 \leq r,
$$

$$
[C_\varsigma(\theta_0)]_{j_1,j_2} := \frac{1}{2} \text{Tr}((\partial_{\varsigma_{j_1}} \Sigma \Sigma_0^T)(\Sigma \Sigma_0^T)^{-1} (\partial_{\varsigma_{j_2}} \Sigma \Sigma_0^T)(\Sigma \Sigma_0^T)^{-1}), \quad 1 \leq j_1, j_2 \leq s.
$$

Now, we state the theorem for asymptotic normality, the proof is in Supplementary Material S1.

**Theorem 5.2** Assume (A1)-(A6). Let $X$ be the solution of (1), and $\hat{\Theta} = (\hat{\Theta}_N, \hat{\varsigma}_N)$ be the estimator that minimizes either (22) or (23). If $\theta_0 \in \Theta$, $C(\theta_0)$ is positive definite, $h \to 0$, $Nh \to \infty$, and $Nh^2 \to 0$, then, under $\mathbb{P}_{\theta_0}$,

$$
\begin{bmatrix}
\sqrt{Nh}(\hat{\Theta}_N - \Theta_0) \\
\sqrt{Nh}(\hat{\varsigma}_N - \varsigma_0)
\end{bmatrix} \overset{d}{\to} \mathcal{N}(0, C^{-1}(\theta_0)).
$$

(28)

The estimator of the diffusion parameter converges faster than the estimator of the drift parameter. Gobet (2002) showed that for a discretely sampled SDE model, the optimal convergence rates for the drift and diffusion parameters are $1/\sqrt{Nh}$ and $1/\sqrt{Nh}$, respectively. Thus, our estimators reach optimal rates. Moreover, the estimators are asymptotically efficient since $C$ is the Fisher information matrix for the corresponding continuous-time diffusion (see Kessler (1997), Gobet (2002)). Finally, since the asymptotic correlation is zero between the drift and diffusion estimators, they are asymptotically independent.

6 Simulation study

This Section presents the simulation study of the Lorenz system, illustrating the theory and comparing the proposed estimators with other likelihood-based estimators. We briefly recall the estimators, describe the simulation process and the optimization in the programming language R (R Core Team, 2022), and present and analyze the results.

6.1 Estimators used in the study

The EM transition distribution (16) for the Lorenz system (20) is:

$$
\begin{bmatrix}
X_{t_k} \\
Y_{t_k} \\
Z_{t_k}
\end{bmatrix}
| \begin{bmatrix}
X_{t_{k-1}} \\
Y_{t_{k-1}} \\
Z_{t_{k-1}}
\end{bmatrix}
\sim \mathcal{N}
\begin{bmatrix}
\begin{bmatrix}
X + h\rho(y - x) \\
y + h(\rho x - y - z) \\
z + h(\rho xy - cz)
\end{bmatrix},
\begin{bmatrix}
h\sigma_1^2 & 0 & 0 \\
0 & h\sigma_2^2 & 0 \\
0 & 0 & h\sigma_3^2
\end{bmatrix}
\end{bmatrix}.
$$
We do not write the closed-form distributions for K (17), LL (18) and HE (19), but we use the corresponding formulas to implement the likelihoods. We implement the two splitting strategies proposed in Section 2.5, leading to four estimators: \( \text{LT}_{\text{mix}}, \text{LT}_{\text{avg}}, \text{S}_{\text{mix}}, \) and \( \text{S}_{\text{avg}} \). To further speed up computation time, we use the same trick for calculating \( \Omega_h^{[\text{LL}]} \) as for calculating \( \Omega_h^{[\text{mix}]} \), see Supplementary Material S1.

### 6.2 Trajectory simulation

To simulate sample paths, we use the EM discretization with a step size of \( h^{\text{sim}} = 0.0001 \), which is small enough for the EM discretization to perform well. Then, we sub-sample the trajectory to get a larger time step \( h \), decreasing discretization errors. We perform \( M = 1000 \) Monte Carlo repetitions.

### 6.3 Optimization in \( \mathbb{R} \)

To optimize the objective functions we use the \( \mathbb{R} \) package \texttt{torch} (Falbel and Luraschi, 2022), which uses AD instead of the traditional finite differentiation used in \texttt{optim}. The two main advantages of AD are precision and speed. Finite differentiation is subject to floating point precision errors and is slow in high dimensions (Baydin et al., 2017). Conversely, AD is exact and fast and thus used in numerous applications, such as MLE or training neural networks.

We tried all available optimizers in the \texttt{torch} package and chose the resilient backpropagation algorithm \texttt{optim_rprop} based on Riedmiller and Braun (1992). It performed faster than the rest and was more precise in finding the global minimum. We used the default hyperparameters and set the optimization iterations to 200. We chose the precision of \( 10^{-5} \) between the updated and the parameters from the previous iteration as the convergence criteria. For starting values, we used \((0.1, 0.1, 0.1, 0.1, 0.1, 0.1)\). All estimators except HE converged after approximately 80 iterations. The HE estimator only converged with the smallest time step, \( h = 0.005 \), achieving convergence in 43% – 72% of cases across various sample sizes \( N \). This probably occurs due to a polynomial approximation of the likelihood that can be unstable at the boundaries, especially for larger \( h \). Incorporating higher-order approximations and adding constraints in the optimization step might improve performance. For further analysis, see the Supplementary Material S1.

### 6.4 Comparing criteria

We compare eight estimators based on their precision and speed. We compute the absolute relative error (ARE) for each component \( \theta_{N}^{(i)} \) of the estimator \( \hat{\theta}_N \):

\[
\text{ARE}(\hat{\theta}_N^{(i)}) = \frac{1}{M} \sum_{r=1}^{M} \frac{|\hat{\theta}_{N,r}^{(i)} - \theta_{0,r}^{(i)}|}{\theta_{0,r}^{(i)}}.
\]

For S and LL, we compare the distributions of \( \hat{\theta}_N - \theta_0 \) more closely. The running times are calculated using the \texttt{tictoc} package in \( \mathbb{R} \), measured from the start of the optimization step until the convergence criterion is met. To avoid the influence of running time outliers, we compute the median over \( M \) repetitions.

### 6.5 Results

In Figure 2, AREs are shown on log scale as a function of \( h \). While most estimators work well for a step size no greater than 0.01, only \( \text{LL} \), \( \text{S}_{\text{mix}} \), and \( \text{S}_{\text{avg}} \) perform well for \( h = 0.05 \). The \( \text{LT}_{\text{avg}} \) is not competitive even for \( h = 0.005 \). The performance of \( \text{LT}_{\text{mix}} \) varies, sometimes approaching the performance of K, while other times performing similarly to EM. Thus, \( \text{LT}_{\text{mix}} \) is not a good choice for this specific model. The bias of EM starts to show for \( h = 0.01 \) escalating for \( h = 0.05 \). The largest bias appears in the diffusion parameters due to the poor approximation of \( \Omega_h^{[\text{EM}]} \). K is less biased than EM except for \( p \) and \( r \) when \( h = 0.05 \). The HE estimator converged only for \( h = 0.005 \). The ARE is calculated from the 601 simulations out of a total of 1000 in which convergence was achieved. For these, the performance of HE in estimating drift parameters is comparable to the best estimators. However, the diffusion parameters are not well estimated, with the estimation of \( \sigma_b^2 \) being the least accurate. Drift parameters are generally estimated better for larger \( h \) for fixed \( N \) due to a longer observation interval \( T = Nh \), reflecting the \( \sqrt{Nh} \) rate of convergence.

We zoom in on the distributions of \( \text{S}_{\text{mix}}, \text{S}_{\text{avg}}, \text{LL} \) in Figure 3. We also include HE for \( h = 0.005 \), based on the 60% converged estimates. For clarity, we removed some outliers for \( \sigma_b^2 \) and \( \sigma_3^2 \). This did not change the shape of the distributions, it only truncated the tails. Estimators \( \text{S}_{\text{mix}}, \text{S}_{\text{avg}} \) and \( \text{LL} \) perform similarly, especially for the smallest \( h \), where HE performs slightly worse, particularly for \( p, \sigma_2^2 \), and \( \sigma_3^2 \). For \( h = 0.05 \), the drift parameters are underestimated.
SDE Parameter Estimation using Splitting Schemes

Figure 2: Comparing the absolute relative error (ARE) as a function of increasing discretization step $h$ for eight estimators in the stochastic Lorenz system. The sample size is $N = 10000$. The $y$-axis is on log scale. The HE estimator (purple dot) converged only for $h = 0.005$, and only for 60% of the simulated data sets.

by approximately $5 - 10\%$, while the diffusion parameters are overestimated by up to $20\%$. Both S estimators performed better than LL, except for $p$ and $\sigma_1^2$.

While the LL and S estimators perform similarly in terms of precision, Figure 4 shows the superiority of the S estimators over LL in computational costs. The LL becomes increasingly computationally expensive for increasing $N$ because it calculates $N$ covariance matrices for each parameter value. The next slowest estimators are $S_{mix}$ and HE, followed by $LT_{mix}$, $S_{avg}$, $K$, $LT_{avg}$, and, finally, EM is the fastest. The speed of EM is almost constant in $N$. Additionally, it seems that the running times do not depend on $h$. Thus, we recommend using the S estimators, especially for large $N$.

Figures 5 and 6 show that the theoretical results hold for $S_{mix}$ and $LT_{mix}$. We compare how the distributions of $\hat{\theta}_N - \theta_0$ change with sample size $N$ and step size $h$. With increasing $N$, the variance decreases, whereas the mean does not change. For that, we need smaller $h$. To obtain negligible bias for $LT_{mix}$, we need a step size smaller than $h = 0.005$. However, $S_{mix}$ is practically unbiased up to $h = 0.01$. This shows that LT estimators might not be a good choice in practice, while S estimators are.

The solid black lines in Figures 5 and 6 represent the theoretical asymptotic distributions computed from (28). For the Lorenz system (20), the precision matrix (27) is given by:

$$C(\theta_0) = \text{diag} \left( \int \frac{(y - x)^2}{\sigma_1^2} \, d\nu_0(x), \int \frac{x^2}{\sigma_2^2} \, d\nu_0(x), \int \frac{z^2}{\sigma_3^2} \, d\nu_0(x), \frac{1}{2\sigma_1^2}, \frac{1}{2\sigma_2^2}, \frac{1}{2\sigma_3^2} \right).$$

The integrals are approximated by taking the mean over all data points and all Monte Carlo repetitions.

Some outliers of $\hat{\sigma}_2^2$ are removed from Figures 5 and 6 by truncating the tails.

7 Conclusion

We proposed two new estimators for nonlinear multivariate SDEs. They are based on splitting schemes, a numerical approximation that preserves all important properties of the model. It was known that the LT splitting scheme has $L^p$ convergence rate of order 1. We proved that the same holds for the S splitting. This result was expected because the overall trajectories of the S and LT splittings coincide up to the first $h/2$ and the last $h/2$ move of the flow $\Phi_{h/2}^{[2]}$. 
SDE Parameter Estimation using Splitting Schemes

Figure 3: Comparing the normalized distributions of \((\hat{\theta}_N - \theta_0) \odot \theta_0\) (where \(\odot\) is the element-wise division) of the Lorenz system for the \(S_{\text{mix}}, S_{\text{avg}}, \text{LL}\) and HE estimators for \(N = 10000\). Each column represents one parameter, and each row represents one value of the discretization step \(h\). The black dot with a vertical bar in each violin plot represents the mean and the standard deviation. The HE estimator (purple) converged only for \(h = 0.005\), and only for 60% of the simulated data sets.

Figure 4: Running times as a function of \(N\) for different estimators of the Lorenz system. Each column shows one value of \(h\). On the \(x\)-axis is the sample size \(N\), and on the \(y\)-axis is the running time in seconds. The HE estimator (purple) achieved convergence only for \(h = 0.005\), and only in 43%−72% of cases across various sample sizes \(N\).

Nonetheless, \(S\) splitting is more precise in one-step predictions, which is crucial for the estimators because the objective function consists of densities between consecutive data points. Therefore, the obtained \(S\) estimator is less biased than the LT.

We proved that both estimators have optimal convergence rates for discrete observations of the SDEs. These rates are \(\sqrt{N}\) for the diffusion parameter and \(\sqrt{Nh}\) for the drift parameter. We also showed that the asymptotic variance of the estimators is the inverse of the Fisher information for the continuous time model. Thus, the estimators are efficient.
In the simulation study of the stochastic Lorenz system, we show the superior performance of the S estimators. We compared eight estimators based on different discretization schemes. Estimators based on Ozaki’s LL and the S splitting schemes demonstrated the highest precision. However, the running time of LL is notably influenced by the sample size \( N \), unlike the S estimator, which experiences a more gradual increase in runtime with larger \( N \). This makes the S estimator more appropriate for large sample sizes. The LT, EM, K and HE estimators perform well for small \( h \), but for larger \( h \) the bias increases.

While the proposed estimators are versatile, they come with certain limitations. These include assumptions like additive noise and equidistant observations. However, under specific conditions, the Lamperti transformation can relax the constraint of additive noise. Equidistant observations can easily be relaxed due to the continuous-time formulation. Furthermore, we assumed that the diffusion parameter \( \Sigma \Sigma^T \) is invertible. However, there are applications where models with degenerate noise naturally arise, like second-order differential equations.
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S1 Supplementary Material

Section S2 provides proofs for all propositions, lemmas, and theorems. References to equations and sections that do not begin with "S" refer to the main paper. The properties necessary for subsequent proofs are outlined in Section S3. These properties encompass Grönwall’s and Rosenthal’s inequalities, as well as Central Limit Theorems for a sum of triangular arrays. In Section S4 we discuss in more detail the LL and HE estimators.

If not stated, we assume the parameters are the true ones, and the expectations are taken under the probability measure. Occasionally, we omit explicit parameter notation to enhance clarity. For instance, $\mathbb{E}$ implicitly denotes $\mathbb{E}_\theta$.

S2 Proofs

In Section S2.1, we provide the proof for the Lie-Trotter splitting (LT), while Section S2.2 contains the proofs for the Strang splitting (S). Proof of $L^p$ convergence of the splitting scheme is in Section S2.3. The proof of Lemma 4.1 is in Section S2.4. Additionally, the proofs of moment bounds are detailed in Section S2.5. Sections S2.6 and S2.7 present proofs of consistency and asymptotic normality of the estimators, respectively.

S2.1 Proof for the Lie-Trotter splitting

Proof of Proposition 3.4 To establish the proposition, we compare the actual first moment of the solution to SDE (1), as obtained from Lemma 2.1, with the moment derived through Taylor expansion of the LT approximation. First, we prove the proposition for LT splitting as defined in the paper. By performing the Taylor expansion of $E[\Phi_h^{LT}(X)]=\mu_h(f_{h/2}(x))=e^{Ah}f_h(x)+(1-e^{Ah})b$ around $h=0$, using Proposition 2.2, we arrive at:

$$
\mu_h(f_{h/2}(x)) = x + h(A(x-b)+N(x)) + \frac{h^2}{2}(A^2(x-b)+2AN(x)+(DN(x))N(x)) + R(h^3, x). \quad (S1)
$$

The coefficient of $h$ in (S1) is $F(x)$, which aligns with the coefficient of $h$ in the theoretical moment of the solution to (1) as provided in Lemma 2.1. However, in Lemma 2.1, $\Sigma$ appears in the coefficient of $h^2$, while it does not appear in (S1). Consequently, to achieve the order of convergence $R(h^3, x)$, we need to make the following unrealistic assumption.

$$(SA) \sum_{i=1}^{d} \sum_{j=1}^{d} [\Sigma \Sigma^T]_{ij} \partial^2_{ij} F(i)(x) = 0, \quad \text{for all } k = 1, \ldots, d.$$ 

Upon comparing expression (S1) with the true moments of the SDE solution under Assumption (SA), we arrive at $(DF(x))N(x) = (DN(x))F(x)$ to ensure equality of the coefficient at order $h^2$. However, the last equation holds true for all $x \in \mathbb{R}^d$ only when $N$ is linear. Therefore, achieving the order $R(h^3, x)$ one-step convergence is feasible only if SDE (1) is linear.

We now aim to show that changing the composition order within the LT does not affect the one-step convergence order. To demonstrate this, we define the reversed LT:

$$X_{t_k}^{LT^*} := \Phi_h^{LT^*}(X_{t_{k-1}}^{LT^*}) = (\Phi_h^{[1]} \circ \Phi_h^{[2]})(X_{t_{k-1}}^{LT^*}) = f_h(\mu_h(X_{t_{k-1}}^{LT^*}) + \xi_{h,k}).$$

We compute $E[f_h(\mu_h(X_{t_{k-1}}) + \xi_{h,k}) | X_{t_{k-1}} = x]$, which is equivalent to calculating $E[f_h(X_{t_k}^{[1]}) | X_{t_{k-1}}^{[1]} = x] = E[f_h(\mu_h(X_{t_{k-1}}^{[1]}) + \xi_{h,k}) | X_{t_{k-1}}^{[1]} = x]$. The infinitesimal generator $L_{[1]}$ for SDE (3) is defined on the class of sufficiently smooth functions $g : \mathbb{R}^d \to \mathbb{R}$ by $L_{[1]}g(x) = (A(x-b))^\top \frac{\partial g}{\partial x} + \frac{1}{2} \text{Tr}(\Sigma \Sigma^T H_g(x))$. This yields:

$$E[g(X_{t_k}^{[1]}) | X_{t_{k-1}}^{[1]} = x] = g(x) + hL_{[1]}g(x) + \frac{h^2}{2}L_{[1]}^2g(x) + R(h^3, x). \quad (S2)$$

We apply (S2) to $g(x) = f_h^{(i)}(x)$. For calculating $L_{[1]}f_h^{(i)}(x)$ and $L_{[1]}^2f_h^{(i)}(x)$, we use the Taylor expansion of $f_h(x)$ around $h = 0$, as provided in Proposition 2.2. The partial derivatives are $\partial f_h^{(i)}(x) = \delta^i_j + h\delta^j_i N^{(i)}(x) + R(h^2, x)$ and $\partial^2 f_h^{(i)}(x) = h\delta^j_i N^{(i)}(x) + R(h^2, x)$. Since $L_{[1]}f_h^{(i)}(x)$ is multiplied by $h$ in (S2), we only need to calculate it up to order $R(h, x)$. We have $L_{[1]}f_h^{(i)}(x) = (A(x-b))^\top N^{(i)}(x) + \frac{h}{2} \text{Tr}(\Sigma \Sigma^T H_{N^{(i)}}(x)) + R(h^2, x).$
Similarly, we have $L_{(1)}^2 f_h^{(i)}(x) = (A(x-b))^T \nabla (A(x-b))^{(i)} + R(h, x) = (A(x-b))^T A^{(i)} + R(h, x)$. Thus,

$$
\mathbb{E}[f_h^{(i)}(X_{t_{k-1}}) | X_{t_{k-1}} = x] = x^{(i)} + hN^{(i)}(x) + \frac{h^2}{2} (N(x))^T \nabla N^{(i)}(x) + h(A(x-b))^{(i)} + \frac{h^2}{2} (A(x-b))^T A^{(i)} + R(h^3, x).
$$

Using that $F^{(i)}(x) = (A(x-b))^{(i)} + N^{(i)}(x)$, we have

$$
\frac{\partial F^{(i)}(x)}{\partial x} = (A^{(i)})^T + \nabla N^{(i)}(x) \quad \text{and} \quad H_{F^{(i)}(x)} = H_{N^{(i)}(x)}(x),
$$

the expectation of the true process rewrites as:

$$
\mathbb{E}[X_{t_k} | X_{t_{k-1}} = x] = x^{(i)} + hF^{(i)}(x) + \frac{h^2}{2} ((N(x))^T \nabla F^{(i)}(x) + (A(x-b))^T \nabla F^{(i)}(x) + \frac{1}{2} \text{Tr}(\Sigma \Sigma^T H_{N^{(i)}}(x))) + R(h^3, x).
$$

The final equation coincides with equation (S3) only up to order $R(h, x)$. Despite the reversed LT has the term with $\Sigma \Sigma^T$ at the order $h^2$, the coefficients do not match. Thus, to obtain order $R(h^2, x)$, the condition $(N(x))^T \nabla F^{(i)}(x) = (F(x))^T \nabla N^{(i)}(x)$, must hold for all $i = 1, \ldots, d$. Given Assumption (SA), the condition for achieving a higher one-step convergence order remains equivalent to the case of the original LT.

### S2.2 Proof for the Strang Splitting

We continue employing the Taylor expansion to establish the numerical properties of the $S$ approximation. To begin, we introduce a helpful Lemma S2.1 regarding the approximation of the composition of the mean function $\mu_h$ and the nonlinear solution $f_{h/2}$. Lemma S2.1 expands $\mu_h(f_{h/2}(x))$ around $h = 0$ in various ways, each retaining the crucial terms necessary for the subsequent proofs.

**Lemma S2.1** For the mean function $\mu_h$ and the nonlinear solution $f_{h/2}$ the following three identities hold:

1. $\mu_h(f_{h/2}(x)) = f_{h/2}(x) + hA(x-b) + \frac{h^2}{2} A F(x) + R(h^3, x)$
2. $\mu_h(f_{h/2}(x)) = f_{h/2}^{-1}(x) + hF(x) + \frac{h^2}{2} A F(x) + R(h^3, x)$.
3. $\mu_h(f_{h/2}(x)) = x + hA(x-b) + \frac{h^2}{2} N(x) + \frac{h^2}{2} (A^2(x-b)) + A N(x) + \frac{1}{2} (D N(x)) N(x) + R(h^3, x)$.

**Proof** We prove only the first two identities, as the last one follows the same reasoning. Utilizing the definition of $\mu_h$, its Taylor expansion, and the expansion of $f_{h/2}$, we obtain: $\mu_h(f_{h/2}(x)) = (I + hA + \frac{h^2}{2} A^2)(f_{h/2}(x)-b) + b + R(h^3, x) = f_{h/2}(x) + hA(x-b) + \frac{h^2}{2} A F(x) + R(h^3, x)$, which concludes the first part.

For the second part, Proposition 2.2 gives $f_{h/2}(x) - f_{h/2}^{-1}(x) = hN(x) + R(h^3, x)$. This leads to: $\mu_h(f_{h/2}(x)) = f_{h/2}^{-1}(x) + hF(x) + \frac{h^2}{2} A F(x) + R(h^3, x)$.

**Proof of Proposition 3.6** We begin by introducing a new function of $x$, arising from the third property of Lemma S2.1:

$$
Q_h(x) = \frac{h}{2} ((A(x-b))^T N(x)) + \frac{h^2}{8} (4A^2(x-b) + 4AN(x) + (DN(x))N(x)).
$$

Then, for a generic random vector $X$ we use Proposition 2.2 and Lemma S2.1 to write:

$$
\mu_h(f_{h/2}(X)) + \xi_h = f_{h/2}(X + Q_h(X) + \xi_h + R(h^3, X))
$$

$$
= X + Q_h(X) + \xi_h + \frac{h}{2} N(X + Q_h(X) + \xi_h) + \frac{h^2}{8} (DN(X + Q_h(X) + \xi_h)) N(X + Q_h(X) + \xi_h) + R(h^3, X).
$$

Consequently, we expand:

$$
N(X + Q_h(X) + \xi_h) = N(X) + (DN(X))(Q_h(X) + \xi_h)
$$

$$
+ \frac{1}{2} [(Q_h(X) + \xi_h)^T H_{N^{(i)}}(X)(Q_h(X) + \xi_h)]_{i=1}^d + R(h^2, X).
$$

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The term \([Q_h(X)^T H_{N(o)}(X) Q_h(X)]_{i=1}^d\) is \(R(h^2, X)\), while the terms with only one \(\xi_h\) have zero means. Thus,

\[
\mathbb{E}[N(X + Q_h(X) + \xi_h) \mid X = x] = N(x) + (DN(x)) Q_h(x) + \frac{1}{2} [\mathbb{E}[\xi_h^T H_{N(o)}(X) \xi_h \mid X = x]]_{i=1}^d + R(h^2, x). \quad (S6)
\]

Lastly, we compute:

\[
\mathbb{E}[\xi_h^T H_{N(o)}(X) \xi_h \mid X = x] = \mathbb{E}[\text{tr}(\xi_h^T H_{N(o)}(X) \xi_h) \mid X = x] = \text{tr}(H_{N(o)}(X) \mathbb{E}[\xi_h \xi_h^T])
\]

\[
= \sum_{j,k=1}^d \partial_{j,k}^2 N^{(i)}(x) \text{var}(\xi_h)|_{j,k} = \sum_{j,k=1}^d \partial_{j,k}^2 F^{(i)}(x) \text{tr}(\Omega)|_{j,k}.
\]

We use the approximation of the variance of the random vector \(\xi_h\) to get \(\mathbb{E}[N(X + Q_h(X) + \xi_h) \mid X = x] = N(x) + (DN(x)) Q_h(x) + \frac{1}{2} [\sum_{j,k=1}^d \Sigma \Sigma^T]_{j,k} \partial_{j,k}^2 F^{(i)}(x)]_{i=1}^d + R(h^2, x)\). Taking the expectation of (S4) and incorporating the previous equation completes the proof.

**S2.3 Proof of \(L^p\) convergence of the splitting scheme**

Now, we present the proof of \(L^p\) convergence stated in Theorem 3.7.

**Proof of Theorem 3.7** We use Theorem 3.3 to prove \(L^p\) convergence. It is sufficient to prove the two conditions (1) and (2). To prove condition (1), we need to prove the following property:

\[
(E[||X_{t_k} - \Phi_h^{(1)}(X_{t_{k-1}})||^p \mid X_{t_{k-1}} = x])^{1/p} = R(h^{q_2}, x),
\]

where \(q_2 = 3/2\). We start with \([X_{t_k} - \Phi_h^{(1)}(X_{t_{k-1}})]^p = ||X_{t_k} - X_{t_{k-1}} - hF(X_{t_{k-1}}) - \xi_{h,k} + R(h^{3/2}, X_{t_{k-1}})||^p\).

For more details on the expansion of \(\Phi_h^{(1)}\), see the previous proof. We approximate \(\xi_{h,k} = \int_{t_{k-1}}^{t_k} c_A(t-s) \Sigma \, dW_s\) by:

\[
\xi_{h,k} = \int_{t_{k-1}}^{t_k} (I + (t_k-s)A) \Sigma \, dW_s + h^2, X_{t_{k-1}})
\]

\[
= \Sigma(W_{t_k} - W_{t_{k-1}}) + A \Sigma \int_{t_{k-1}}^{t_k} (t_k-s) \, dW_s + R(h^2, X_{t_{k-1}}).
\]

Using the fact that \(\int_{t_{k-1}}^{t_k} (t_k-s) \, dW_s \sim \mathcal{N}(0, \frac{h^3}{3} I)\), we deduce that \(\xi_{h,k} = \Sigma(W_{t_k} - W_{t_{k-1}}) + R(h^{3/2}, X_{t_{k-1}})\).

Then, Hölder’s inequality yields:

\[
||X_{t_k} - X_{t_{k-1}} - hF(X_{t_{k-1}}) - \Sigma(W_{t_k} - W_{t_{k-1}})||^p
\]

\[
\leq h^{p-1} \int_{t_{k-1}}^{t_k} \||F(X_s) - F(X_{t_{k-1}})||^p \, ds.
\]

Assumption (A2), the integral norm inequality, Cauchy-Schwartz, and Hölder’s inequalities, together with the mean value theorem yield:

\[
E[||X_{t_k} - \Phi_h^{(1)}(X_{t_{k-1}})||^p \mid X_{t_{k-1}} = x]
\]

\[
\leq C(E[h^{p-1} \int_{t_{k-1}}^{t_k} \||F(X_s) - F(X_{t_{k-1}})||^p \, ds \mid X_{t_{k-1}} = x])
\]

\[
\leq C(h^{p-1} \int_{t_{k-1}}^{t_k} E[||X_s - X_{t_{k-1}}||^p \int_0^1 D_x F(X_s - u(X_s - X_{t_{k-1}})) \, du]^p \mid X_{t_{k-1}} = x] \, ds)
\]

\[
\leq C \left( h^{p-1} \int_{t_{k-1}}^{t_k} (E[||X_s - X_{t_{k-1}}||^{2p} \mid X_{t_{k-1}} = x])^{1/2}
\]

\[
\left( E[\int_0^1 D_x F(X_s - u(X_s - X_{t_{k-1}})) \, du]^{2p} \mid X_{t_{k-1}} = x \right)^{1/2} \, ds)
\]

\[
\leq C(h^{p-1} \int_{t_{k-1}}^{t_k} h^{p/2} \, ds) = R(h^{3p/2}, x).
\]
where \( t \)

Now, we apply the generalized Grönwall’s inequality (Lemma 2.3 in Tian and Fan (2020), stated in Section S3) on (S7).

In the second inequality, we used the polynomial growth (A2) of \( \text{moments too. This concludes the proof.} \)

\[
\text{Now, we prove condition (2). We use (5) and (11) to write } X_{t_k}^{[S]} = f_{h/2}(e^{A h/2}(X_{t_k}^{[S]} - X_{t_{k-1}}^{[S]}) + X_{t_k}^{[1]}). \text{ Define } R_{t_k} := e^{A h/2}(X_{t_k}^{[S]} - X_{t_k}^{[1]}), \text{ and use the associativity (9) to get } R_{t_k} = e^{A h/2}(f_h(R_{t_k}^{[1]} + X_{t_k}^{[1]}) - X_{t_k}^{[1]}). \text{ The proof of the boundness of the moments of } R_{t_k} \text{ is the same as in Lemma 2 in Buckwar et al. (2022). Finally, we have } X_{t_k}^{[S]} = f_{h/2}(e^{-A h/2} R_{t_k} + X_{t_k}^{[1]}). \text{ Since } f_{h/2} \text{ grows polynomially and } X_{t_k}^{[1]} \text{ has finite moments, } X_{t_k}^{[S]} \text{ must have finite moments too. This concludes the proof.}

\section{Proof of Lemma 4.1}

**Proof of Lemma 4.1** We first prove (1). In the following, \( C_1 \) and \( C_2 \) denote constants. We use the triangular inequality and Hölder’s inequality to obtain:

\[
\|X_t - X_{t_{k-1}}\|^p \leq 2^{p-1} \left( \int_{t_{k-1}}^t \|F(X_s; \theta)\| ds \right)^p + \|\Sigma(W_t - W_{t_{k-1}})\|^p
\]

\[
\leq 2^{p-1} \left( \int_{t_{k-1}}^t C_1(1 + \|X_s\|) C_1 ds \right)^p + \|\Sigma(W_t - W_{t_{k-1}})\|^p
\]

\[
\leq 2^{p-1} C_1^p \left( \int_{t_{k-1}}^t (1 + \|X_s - X_{t_{k-1}}\| + \|X_{t_{k-1}}\|) ds \right)^p + 2^{p-1} \|\Sigma(W_t - W_{t_{k-1}})\|^p
\]

\[
\leq 2 C_1 + 2^{p-3} C_1^p (t - t_{k-1})^{p-1} \left( \int_{t_{k-1}}^t \|X_s - X_{t_{k-1}}\| ds + (t - t_{k-1})^p \right)^{pC_1}
\]

\[
+ 2^{p-1} \|\Sigma(W_t - W_{t_{k-1}})\|^p.
\]

In the second inequality, we used the polynomial growth (A2) of \( F \). Furthermore, for some constant \( C_2 \) that depends on \( p \), we have \( E[\|\Sigma(W_t - W_{t_{k-1}})\|^p | F_{t_{k-1}}] = (t - t_{k-1})^{p/2} C_2(p) \). Then, for \( h < 1 \), there exists a constant \( C_p \) that depends on \( p \), such that:

\[
C_p (t - t_{k-1})^{2p-1} (1 + \|X_{t_{k-1}}\|)^{pC_p} + C_p (t - t_{k-1})^{p/2} \leq C_p (t - t_{k-1})^{p/2} (1 + \|X_{t_{k-1}}\|)^{pC_p}.
\]

The last inequality holds because the term of order \( p/2 \) is dominating when \( t - t_{k-1} < 1 \). Denote \( m(t) = E[\|X_t - X_{t_{k-1}}\|^p | F_{t_{k-1}}] \). Then, we have:

\[
m(t) \leq C_p (t - t_{k-1})^{p/2} (1 + \|X_{t_{k-1}}\|)^{pC_p} + C_p \int_{t_{k-1}}^t m^{C_1}(s) ds.
\]

\[(S7)\]

Now, we apply the generalized Grönwall’s inequality (Lemma 2.3 in Tian and Fan (2020), stated in Section S3) on (S7). Since we consider a super-linear growth, we can assume that there exist \( C_1 > 1 \) and \( C_p > 0 \), such that:

\[
m(t) \leq C_p (t - t_{k-1})^{p/2} (1 + \|X_{t_{k-1}}\|)^{pC_p} + \left(1 - (C_1 - 1)^2 C_p (t - t_{k-1})^{p/2} \right)^{C_1 - 1} C_p (t - t_{k-1})^p + C \kappa(t),
\]

\[(S8)\]

where \( \kappa(t) = C_p (t - t_{k-1})^{p/2} (1 + \|X_{t_{k-1}}\|)^{pC_p} \). The bound \( C \) in inequality (S8) makes sense, because the term:

\[
(1 - (C_1 - 1)^2 C_p (t - t_{k-1})^p)^{pC_1 - 1}
\]

is positive by Lemma 2.3 from Tian and Fan (2020). Additionally, the same term reaches its maximum value of 1, for \( t = t_{k-1} \). The constant \( C \) in (S8) includes some terms that depend on \( t - t_{k-1} \). However, these terms will not change the dominating term of \( \kappa(t) \) since \( h < 1 \). Finally, the terms in \( \kappa(t) \) are of order \( p/2 \), thus for large enough constant \( C_p \), it holds \( m(t) \leq C_p (t - t_{k-1})^{p/2} (1 + \|X_{t_{k-1}}\|)^{pC_p} \).

To prove (2), we use that \( g \) is of polynomial growth:

\[
E[|g(X_t; \theta)| | F_{t_{k-1}}] \leq C_1 E[|1 + |X_{t_{k-1}}| + |X_t - X_{t_{k-1}}| | C_1 | F_{t_{k-1}}]
\]

\[
\leq C_2 (1 + \|X_{t_{k-1}}\|^{C_1} + E[|X_t - X_{t_{k-1}}| | C_1 | F_{t_{k-1}}]),
\]

Now, we apply the first part of the lemma, to get:

\[
E[|g(X_t; \theta)| | F_{t_{k-1}}] \leq C_2 (1 + \|X_{t_{k-1}}\|^{C_1} + C_{t-t_{k-1}} (1 + \|X_{t_{k-1}}\|)^{C_1})
\]

\[
\leq C_{t-t_{k-1}} (1 + \|X_{t_{k-1}}\|)^C.
\]

That concludes the proof.
S2.5 Proofs of the Moment Bounds

Before proving the moment bounds, we first demonstrate in Lemma S2.2 how the infinitesimal generator $L$ operates on a product of two functions.

**Lemma S2.2** Let $L$ be the infinitesimal generator defined in the main text of SDE (1). For sufficiently smooth functions $\alpha, \beta : \mathbb{R}^d \to \mathbb{R}$, it holds:

$$L(\alpha(x)\beta(x)) = \alpha(x)L\beta(x) + \beta(x)L\alpha(x) + \frac{1}{2} \text{Tr}(\Sigma \Sigma^T (\nabla \alpha(x) \nabla^\top \beta(x) + \nabla \beta(x) \nabla^\top \alpha(x))).$$

**Proof** We use the generator $L$ and the product rule to get:

$$L(\alpha(x)\beta(x)) = F(x)^\top \alpha(x) \nabla \beta(x) + F(x)^\top \beta(x) \nabla \alpha(x) + \frac{1}{2} \text{Tr}(\Sigma \Sigma^T (\alpha(x)H_\beta(x) + \beta(x)H_\alpha(x))).$$

$$= \alpha(x)L\beta(x) + \beta(x)L\alpha(x) + \frac{1}{2} \text{Tr}(\Sigma \Sigma^T (\nabla \alpha(x) \nabla^\top \beta(x) + \nabla \beta(x) \nabla^\top \alpha(x))).$$

This concludes the proof.

**Proof of Proposition 4.3** Proof of (i). Lemma S2.1 yields:

$$\mathbb{E}[f_{-h/2}(X_{t_k}) | X_{t_{k-1}} = x] = \mathbb{E}[f_{-h/2}(X_{t_k}) | X_{t_{k-1}} = x] - \mu_h(f_{-h/2}(x))$$

$$= \mathbb{E}[f_{-h/2}(X_{t_k}) | X_{t_{k-1}} = x] - f_{-h/2}(x) - hF(x)$$

$$- \frac{h^2}{2} A \Sigma \Sigma^T (x; \beta) + R(h^3, x).$$

Now, we use the infinitesimal generator $L$ to evaluate the expectation in the last line where the generator $L$ is applied to a vector-valued function. We have:

$$\mathbb{E}[f_{-h/2}(X_{t_k}) | X_{t_{k-1}} = x] = f_{-h/2}(x) + hL_f_{-h/2}(x) + \frac{h^2}{2} L^2 f_{-h/2}(x) + R(h^3, x).$$

We use $f_{h/2}(x) = f_{-h/2}(x)$ and Proposition 2.2 to get:

$$L_f_{h/2}(x) = Lx - \frac{h}{2} LN(x) + R(h^2, x) = F(x) - \frac{h}{2} LN(x) + R(h^2, x),$$

$$L^2 f_{h/2}(x) = L(A(x-b)) + LN(x) + R(h, x) = A \Sigma \Sigma^T (x; \beta) + LN(x) + R(h, x).$$

It follows that $\mathbb{E}[f_{h/2}(X_{t_k}) | X_{t_{k-1}} = x] = R(h^3, x)$.

Proof of (ii). In this proof, we distinguish the true parameters $\theta_0$ from a generic parameter $\theta$. We start with the expansions of $f_{h/2}$ and $\mu_h$:

$$\mathbb{E}[f_{h/2}(X_{t_k}; \beta_0) - \mu_h(f_{h/2}(X_{t_{k-1}}; \beta_0); \beta_0))g(X_{t_k}; \beta)^\top | X_{t_{k-1}} = x]$$

$$= \mathbb{E}[F_{h/2}(X_{t_k}; \beta_0) | X_{t_{k-1}} = x] - \frac{h}{2} \mathbb{E}[N(X_{t_k}; \beta_0)g(X_{t_k}; \beta)^\top | X_{t_{k-1}} = x]$$

$$- \frac{h}{2} (2A^0(x-b_0) + N_0(x)) \mathbb{E}[g(X_{t_k}; \beta)^\top | X_{t_{k-1}} = x] + R(h^2, x)$$

$$= \mathbb{E}[N_0(x)g(x; \beta)^\top + hL_{\theta_0}(xg(x; \beta)^\top) - \frac{h}{2} N_0(x)g(x; \beta)^\top]$$

$$- \frac{h}{2} N_0(x)g(x; \beta)^\top - hL_{\theta_0}(xg(x; \beta)^\top) - hF_0(x)g(x; \beta)^\top + R(h^2, x).$$

Lastly, Lemma S2.2 and the definition of $L_{\theta_0}$ yield:

$$L_{\theta_0}(xg(x; \beta)^\top) = xL_{\theta_0}(g(x; \beta)^\top + L_{\theta_0}(g(x; \beta)^\top + \frac{1}{2} \Sigma \Sigma^T D^\top g(x; \beta) + D g(x; \beta) \Sigma \Sigma^T)$$

$$= xL_{\theta_0}(g(x; \beta)^\top + F(x; \beta_0)g(x; \beta)^\top + \frac{1}{2} (\Sigma \Sigma^T D^\top g(x; \beta) + D g(x; \beta) \Sigma \Sigma^T).$$

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Proof of (iii). We introduce \( g(X_{t_k}; \beta_0) = f_{h/2}^{-1}(X_{t_k}; \beta_0) \) and use (ii) to show:
\[
\mathbb{E}_{\theta_0}[(f_{h/2}^{-1}(X_{t_k}; \beta_0) - \mu_h(f_{h/2}(X_{t_{k-1}}; \beta_0); \beta_0))(f_{h/2}^{-1}(X_{t_k}; \beta_0) - \mu_h(f_{h/2}(X_{t_{k-1}}; \beta_0); \beta_0))^\top | X_{t_{k-1}} = x]
\]
\[
= \frac{h}{2}(\Sigma \Sigma^\top)^{-1} D^\top g(x; \beta_0) + Dg(x; \beta_0)(\Sigma \Sigma^\top)^{-1}
\]
\[
- \mathbb{E}_{\theta_0}[(f_{h/2}^{-1}(X_{t_k}; \beta_0) - \mu_h(f_{h/2}(X_{t_{k-1}}; \beta_0); \beta_0)) | X_{t_{k-1}} = x] \mu_h(f_{h/2}(x; \beta_0); \beta_0)^\top + R(h^2, x).
\]
The result follows from property (i) and
\[
Dg(x; \beta_0) = I + R(h, x).
\]

S2.6 Proof of consistency of the estimator

The proof of consistency consists in studying the convergence of the objective function that defines the estimators. The objective function \( \mathcal{L}_N(\beta, \zeta) \) (23) can be decomposed into sums of martingale triangular arrays. We thus first state a lemma that proves the convergence of each triangular array involved in the objective function. Then, we will focus on the proof of consistency.

Lemma S2.3 Let Assumptions (A1)-(A6) hold, and \( X \) be the solution of (1). Let \( g, g_1, g_2 : \mathbb{R}^d \times \Theta \times \Theta \to \mathbb{R}^d \) be differentiable functions with respect to \( x \) and \( \theta \), with derivatives of polynomial growth in \( x \), uniformly in \( \theta \). If \( h \to 0 \) and \( Nh \to \infty \), then:

1. \[
\frac{1}{Nh} \sum_{k=1}^N Z_{t_k}(\beta_0)^\top (\Sigma \Sigma^\top)^{-1} Z_{t_k}(\beta_0) \xrightarrow{F_{\theta_0}} \text{Tr}((\Sigma \Sigma^\top)^{-1} \Sigma \Sigma^\top^\top);
\]
2. \[
\frac{h}{N} \sum_{k=1}^N g(X_{t_{k-1}}; \beta_0, \beta)^\top (\Sigma \Sigma^\top)^{-1} g(X_{t_{k-1}}; \beta_0, \beta) \xrightarrow{F_{\theta_0}} 0;
\]
3. \[
\frac{1}{Nh} \sum_{k=1}^N Z_{t_k}(\beta_0)^\top (\Sigma \Sigma^\top)^{-1} g(X_{t_{k-1}}; \beta_0, \beta) \xrightarrow{F_{\theta_0}} 0;
\]
4. \[
\frac{1}{Nh} \sum_{k=1}^N Z_{t_k}(\beta_0)^\top (\Sigma \Sigma^\top)^{-1} g(X_{t_{k-1}}; \beta_0, \beta) \xrightarrow{F_{\theta_0}} 0;
\]
5. \[
\frac{1}{Nh} \sum_{k=1}^N Z_{t_k}(\beta_0)^\top (\Sigma \Sigma^\top)^{-1} g(X_{t_{k-1}}; \beta_0, \beta) \xrightarrow{F_{\theta_0}} 0;
\]
6. \[
\frac{1}{Nh} \sum_{k=1}^N Z_{t_k}(\beta_0)^\top (\Sigma \Sigma^\top)^{-1} g(X_{t_{k-1}}; \beta_0, \beta) \xrightarrow{F_{\theta_0}} 0;
\]
7. \[
\frac{h}{N} \sum_{k=1}^N g_1(X_{t_{k-1}}; \beta_0, \beta)^\top (\Sigma \Sigma^\top)^{-1} g_2(X_{t_{k}}; \beta_0, \beta) \xrightarrow{F_{\theta_0}} 0,
\]
uniformly in \( \theta \).

Lemma S2.3 plays a central role in demonstrating the consistency and asymptotic normality of the proposed estimators. The lemma deals with the uniform convergence of multiple triangular arrays, and proving various aspects of it involves a range of technical tools and methods. Different parts of Lemma S2.3 require distinct strategies to establish appropriate bounds, which can be intricate. Once these bounds are established, we leverage the properties discussed in the preceding section.

For instance, when establishing point-wise convergence, we primarily rely on Lemma S3.2. On the other hand, for proving uniform convergence, we utilize both Lemma S3.3 and Lemma S3.4. Throughout the proof of Lemma S2.3, a recurring theme is to interpret quadratic forms as traces and exploit the cyclic property inherent to them. Additionally, we employ fundamental mathematical tools like the mean value theorem, the Cauchy-Schwartz inequality, and H"older’s inequality in various instances.
Furthermore, there are occasions where we require inequality for norms, particularly the Frobenius norm. To address this, we introduce the Frobenius inner product of matrices $M_1$ and $M_2$ in $\mathbb{R}^{n \times m}$ as $\langle M_1, M_2 \rangle_F := \text{Tr}(M_1^T M_2)$. Leveraging Hölder’s inequality on Frobenius norm provides us with the following bound for the trace of a matrix product: $\| \text{Tr}(M_1 M_2) \| \leq \| \text{Tr}(M_1) \| \| M_2 \|$. 

**Proof of Lemma S2.3** Proof of 1. As previously discussed, we introduce a martingale array that corresponds to the limit outlined in point 1. We then utilize Lemma S3.2 to facilitate our analysis. We denote $Y_k^N(\beta_0, \varsigma) := \frac{1}{Nh}Z_{t_k}(\beta_0)^T (\Sigma \Sigma^T)^{-1}Z_{t_k}(\beta_0)$. We have:

\[
\sum_{k=1}^N \mathbb{E}_{\theta_0}[Y_k^N(\beta_0, \varsigma) | X_{t_k-1}] = \frac{1}{Nh} \sum_{k=1}^N \mathbb{E}_{\theta_0}[\text{Tr}(Z_{t_k}(\beta_0)^T (\Sigma \Sigma^T)^{-1}Z_{t_k}(\beta_0)) | X_{t_k-1}]
\]

\[
= \frac{1}{Nh} \sum_{k=1}^N \text{Tr}(\Sigma \Sigma^T)^{-1} \mathbb{E}_{\theta_0}[Z_{t_k}(\beta_0)Z_{t_k}(\beta_0)^T | X_{t_k-1})]
\]

\[
= \frac{1}{Nh} \sum_{k=1}^N \text{Tr}(\Sigma \Sigma^T)^{-1} h \Sigma \Sigma_0^T + R(h^2, X_{t_k-1}) \xrightarrow{\mathbb{P}_{\theta_0}} \text{Tr}(\Sigma \Sigma^T)^{-1} \Sigma \Sigma_0^T.
\]

To use the result of Lemma S3.2, we need to prove that covariance of $Y_k^N(\beta_0, \varsigma)$ goes to zero. To achieve this, we leverage Corollary 3.8 and recall that if $\rho$ is a Gaussian random vector $\rho \sim \mathcal{N}(0, \Pi)$, then $\mathbb{E}[(\rho^T M \rho)^2] = 2 \text{Tr}((M \Pi)^2) + (\text{Tr}(M \Pi))^2$. This leads to:

\[
\sum_{k=1}^N \mathbb{E}_{\theta_0}[Y_k^N(\beta_0, \varsigma)|X_{t_k}] = \frac{1}{Nh^2} \sum_{k=1}^N \mathbb{E}_{\theta_0}[Z_{t_k}(\beta_0)^T (\Sigma \Sigma^T)^{-1}Z_{t_k}(\beta_0)] + R(h^3/2, X_{t_k-1})
\]

\[
= \frac{1}{Nh} \sum_{k=1}^N (2 \text{Tr}(\Sigma \Sigma^T)^{-1} \Sigma_0 \Sigma_0^T)^2 + (\text{Tr}(\Sigma \Sigma^T)^{-1} \Sigma_0 \Sigma_0^T)^2 + R(h^{3/2}, X_{t_k-1}) \xrightarrow{\mathbb{P}_{\theta_0}} 0,
\]

for $Nh \to \infty$, $h \to 0$. Then, by Lemma S3.2 $\frac{1}{Nh} \sum_{k=1}^N Z_{t_k}(\beta_0)^T (\Sigma \Sigma^T)^{-1}Z_{t_k}(\beta_0) \xrightarrow{\mathbb{P}_{\theta_0}} \text{Tr}(\Sigma \Sigma^T)^{-1} \Sigma_0 \Sigma_0^T$, for $Nh \to \infty$, $h \to 0$. To establish the uniformity of the limits with respect to $\varsigma$, we turn to Lemma S3.3 and introduce sets $\Theta_{\varsigma_j}$ such that $\varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_s) \in \Theta_{\varsigma_1} \times \Theta_{\varsigma_2} \times \cdots \times \Theta_{\varsigma_s} = \Theta_{\varsigma}$. Then it is enough to show that for all $j = 1, \ldots, s$, it holds:

\[
\sup_{N \in \mathbb{N}} \mathbb{E}_{\theta_0}[\sup_{\varsigma_j \in \Theta_{\varsigma_j}} |\partial_{\varsigma_j} \frac{1}{Nh} \sum_{k=1}^N Z_{t_k}(\beta_0)^T (\Sigma \Sigma^T)^{-1}Z_{t_k}(\beta_0)|] < \infty. \quad (S9)
\]

We use the well-known rule of matrix differentiation $\partial_X (a^T X^{-1} a) = -X^{-1} a a^T X^{-1}$, where $a$ is a vector and $X$ is a symmetric matrix, to get:

\[
\partial_{x^{(i)}} \text{Tr}(a^T C^{-1}(x)a) = -\text{Tr}(C^{-1}(x)a a^T C^{-1}(x)) = -\text{Tr}(a a^T C^{-1}(x)(\partial_{x^{(i)}} C(x)) C^{-1}(x)).
\]

We omit writing $\beta_0$ for ease of notation. Then, by using the trace bound, the norm inequality, and Assumption (A4), we can deduce that:

\[
\sup_{N \in \mathbb{N}} \mathbb{E}_{\theta_0}[\sup_{\varsigma_j \in \Theta_{\varsigma_j}} |\partial_{\varsigma_j} \frac{1}{Nh} \sum_{k=1}^N Z_{t_k}(\Sigma \Sigma^T)^{-1}Z_{t_k}|] \leq \sup_{N \in \mathbb{N}} \mathbb{E}_{\theta_0}[\frac{1}{Nh} \sum_{k=1}^N \sup_{\varsigma_j \in \Theta_{\varsigma_j}} |\partial_{\varsigma_j} \text{Tr}(Z_{t_k}(\Sigma \Sigma^T)^{-1}Z_{t_k})|]
\]

\[
\leq \sup_{N \in \mathbb{N}} \mathbb{E}_{\theta_0}[\frac{1}{Nh} \sum_{k=1}^N \text{Tr}(Z_{t_k}^T Z_{t_k}) \sup_{\varsigma_j \in \Theta_{\varsigma_j}} \|\Sigma \Sigma^T\| \|\partial_{\varsigma_j} \Sigma \Sigma^T\|]
\]

\[
\leq C \sup_{N \in \mathbb{N}} \mathbb{E}_{\theta_0}[\frac{1}{Nh} \sum_{k=1}^N \text{Tr}(Z_{t_k}^T Z_{t_k})] = C \sup_{N \in \mathbb{N}} \mathbb{E}_{\theta_0}[\frac{1}{Nh} \sum_{k=1}^N \text{Tr}(h \Sigma \Sigma_0^T + R(h^2, X_{t_k-1}))] < \infty.
\]

**Proof of 2.** We use Lemma 4.2 to deduce:

\[
\frac{1}{N} \sum_{k=1}^N g(X_{t_k-1}; \beta_0, \beta)^T (\Sigma \Sigma^T)^{-1} g(X_{t_k-1}; \beta_0, \beta) \xrightarrow{\mathbb{P}_{\theta_0}} \int g(x; \beta_0, \beta)^T (\Sigma \Sigma^T)^{-1} g(x; \beta_0, \beta) d\nu_0(x),
\]

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uniformly in $\theta$, for $Nh \to \infty$, $h \to 0$. Then we use the bound of $g$ to conclude the proof of 2.

Proof of 3. For $Y_k^N(\beta_0, \theta) := \frac{1}{N} Z_{t_k}(\beta_0)^\top \Sigma^{-1}(X_{t_k}; \beta_0, \beta)$, the limit of $\sum_{k=1}^N \mathbb{E}_0[Y_k^N(\beta_0, \theta) | X_{t_{k-1}}]$ rewrites as:

$$
\sum_{k=1}^N \mathbb{E}_0[Y_k^N(\beta_0, \theta) | X_{t_{k-1}}] = \frac{1}{N} \sum_{k=1}^N \mathbb{E}_0[\text{Tr}(Z_{t_k}(\beta_0)^\top \Sigma^{-1}g(X_{t_{k-1}}; \beta_0, \beta)) | X_{t_{k-1}}]
$$

$$
= \frac{1}{N} \sum_{k=1}^N \text{Tr}((\Sigma^{-1})^{-1}g(X_{t_{k-1}}; \beta_0, \beta) \mathbb{E}_0[Z_{t_k}(\beta_0)^\top | X_{t_{k-1}}])
$$

$$
= \frac{1}{N} \sum_{k=1}^N R(h^2, X_{t_{k-1}}) \xrightarrow{P_{h \theta}} 0,
$$

for $Nh \to \infty$, $h \to 0$. Then, we study the limit of $\sum_{k=1}^N \mathbb{E}_0[Y_k^N(\beta_0, \theta)^2 | X_{t_{k-1}}]$:

$$
\sum_{k=1}^N \mathbb{E}_0[Y_k^N(\beta_0, \theta)^2 | X_{t_{k-1}}]
$$

$$
= \frac{1}{N^2} \sum_{k=1}^N \mathbb{E}_0[g(X_{t_{k-1}}; \beta_0, \beta)^\top \Sigma^{-1}Z_{t_k}(\beta_0) Z_{t_k}(\beta_0)^\top \Sigma^{-1}g(X_{t_{k-1}}; \beta_0, \beta) | X_{t_{k-1}}]
$$

$$
= \frac{1}{N^2} \sum_{k=1}^N g(X_{t_{k-1}}; \beta_0, \beta)^\top \Sigma^{-1} \mathbb{E}_0[Z_{t_k}(\beta_0) Z_{t_k}(\beta_0)^\top | X_{t_{k-1}}](\Sigma^{-1})^{-1}g(X_{t_{k-1}}; \beta_0, \beta)
$$

$$
= \frac{1}{N} \sum_{k=1}^N R(h^2, X_{t_{k-1}}) \xrightarrow{P_{h \theta}} 0,
$$

for $Nh \to \infty$, $h \to 0$. Lemma S3.2 yields that $\frac{1}{N} \sum_{k=1}^N Z_{t_k}(\beta_0)^\top \Sigma^{-1}g(X_{t_{k-1}}; \beta_0, \beta) \xrightarrow{P_{h \theta}} 0$, for $Nh \to \infty$, $h \to 0$. To show the uniformity of the limits with respect to $\theta$, we leverage Lemma S3.4. It is sufficient to demonstrate the existence of constants $p \geq l > r + s$ and $C > 0$ such that for all $\theta, \theta_1$ and $\theta_2$ it holds:

$$
\mathbb{E}_0[|\sum_{k=1}^N Y_k^N(\beta_0, \theta)|^p] \leq C,
$$

(S10)

$$
\mathbb{E}_0[|\sum_{k=1}^N (Y_k^N(\beta_0, \theta_1) - Y_k^N(\beta_0, \theta_2))|^p] \leq C\|\theta_1 - \theta_2\|^l.
$$

(S11)

We begin by considering equation (S10). Based on the definition of $Z_{t_k}(\beta_0)$ and the assumptions made about $N$, as well as the fact that $h < 1$, there exist constants $C_1$ and $C_2$ such that:

$$
\|Z_{t_k}(\beta_0)\|^p \leq \|X_{t_{k-1}} - X_{t_{k-1}}\|^p + C_1 h^p (1 + \|X_{t_{k-1}}\|)^C_1 + C_2 h^p (1 + \|X_{t_{k-1}}\|)^C_2,
$$

(S12)

Then, Lemma 4.1 yields:

$$
\mathbb{E}_0[\|Z_{t_k}(\beta_0)\|^p | X_{t_{k-1}}] \leq Ch^{p/2}(1 + \|X_{t_{k-1}}\|)^C.
$$

(S13)

Subsequently, we use the norm inequality, (S13) and both statements of Lemma 4.1 to get:

$$
\mathbb{E}_0[|\sum_{k=1}^N Y_k^N(\beta_0, \theta)|^p] \leq N^{p-1} \sum_{k=1}^N \mathbb{E}_0[|Y_k^N(\beta_0, \theta)|^p]
$$

$$
= \frac{1}{N} \sum_{k=1}^N \mathbb{E}_0[|\sum_{k=1}^N Z_{t_k}(\beta_0)^\top \Sigma^{-1}g(X_{t_{k-1}}; \beta_0, \beta)\|^p | X_{t_{k-1}}]
$$

$$
\leq \frac{1}{N} \sum_{k=1}^N \mathbb{E}_0[|Z_{t_k}(\beta_0)|^p | X_{t_{k-1}}]|\|\Sigma^{-1}||^p \|g(X_{t_{k-1}}; \beta_0, \beta)\|^p \leq \frac{1}{N} \cdot N \cdot C.
$$

(S14)
This completes the proof of (S10). Now, we focus on (S11). We use the triangular inequality and the Hölder’s inequality to derive:

\[
\mathbb{E}_{\theta_0} \left[ \left| \sum_{k=1}^{N} (Y_k^N(\beta_0, \theta_1) - Y_k^N(\beta_0, \theta_2)) \right|^p \right] \\
\leq \frac{2^{p-1}}{N} \sum_{k=1}^{N} \mathbb{E}_{\theta_0} \left[ \left| \left( \Sigma_1 \Sigma_1^T \right)^{-1}(g(X_{t_{k-1}}; \beta_1, \beta_0) - g(X_{t_{k-1}}; \beta_2, \beta_0)) \right|^p \right] \\
+ \frac{2^{p-1}}{N} \sum_{k=1}^{N} \mathbb{E}_{\theta_0} \left[ \left| \left( \Sigma_1 \Sigma_1^T \right)^{-1} - \left( \Sigma_2 \Sigma_2^T \right)^{-1} \right| g(X_{t_{k-1}}; \beta_2, \beta_0) \right|^p].
\]

(S15)

First, we study sum (S15). We use the mean value theorem and the triangular inequalities to get:

\[
\frac{1}{N} \sum_{k=1}^{N} \mathbb{E}_{\theta_0} \left[ \left| \left( \Sigma_1 \Sigma_1^T \right)^{-1}(g(X_{t_{k-1}}; \beta_1, \beta_0) - g(X_{t_{k-1}}; \beta_2, \beta_0)) \right|^p \right] \\
\leq \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}_{\theta_0} \left[ \left| \left( \Sigma_1 \Sigma_1^T \right)^{-1} \right| \left| g(X_{t_{k-1}}; \beta_1, \beta_0) - g(X_{t_{k-1}}; \beta_2, \beta_0) \right| \right]^p \\
\leq \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}_{\theta_0} \left[ C_p (1 + \left| \left( \Sigma_1 \Sigma_1^T \right)^{-1} \right| ) C_p \left| \beta_1 - \beta_2 \right|^p \int_0^1 D g(X_{t_{k-1}}; \beta_2 + t(\beta_1 - \beta_2), \beta_0) \ dt \right]^p \\
\leq C \left| \beta_1 - \beta_2 \right|^p.
\]

(S17)

To bound sum (S16), we introduce the following multivariate matrix-valued function \( G(\varsigma) := (\Sigma \Sigma^T)^{-1}. \) Then, we use the inequality between the operator 2-norm and Frobenius norm, and the definition of the Frobenius norm to get:

\[
\left\| G(\varsigma_1) - G(\varsigma_2) \right\| \leq \left( \sum_{i,j=1}^{d} \left\| G_{ij}(\varsigma_1) - G_{ij}(\varsigma_2) \right\|^2 \right)^{\frac{1}{2}}.
\]

Now, apply the mean value theorem on each \( G_{ij} \) and Assumption (A4) to get:

\[
\left\| G(\varsigma_1) - G(\varsigma_2) \right\| \leq \left( \sum_{i,j=1}^{d} \left\| \varsigma_1 - \varsigma_2 \right\|^2 \right)^{\frac{1}{2}} \leq C \left\| \varsigma_1 - \varsigma_2 \right\|.
\]

Finally, combining the previous results, we conclude that:

\[
\mathbb{E}_{\theta_0} \left[ \left| \sum_{k=1}^{N} (Y_k^N(\beta_0, \theta_1) - Y_k^N(\beta_0, \theta_2)) \right|^p \right] \\
\leq C \left( \left| \beta_1 - \beta_2 \right|^p + \left\| \varsigma_1 - \varsigma_2 \right\|^p \right) \\
\leq C \left( \left| \beta_1 - \beta_2 \right|^p + \left\| \varsigma_1 - \varsigma_2 \right\|^{p/2} \right) = C \left\| \theta_1 - \theta_2 \right\|^p,
\]

for \( p \geq 2. \) This concludes the proof of 3.

Proof of 4. For \( Y_k^N(\beta_0, \theta) := \frac{1}{N h} Z_{t_k} (\beta_0)^\top (\Sigma \Sigma^T)^{-1} g(X_{t_{k-1}}; \beta_0, \beta), \) we repeat the same derivations as in the proof of 3. to show that the limit of \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0} \left[ Y_k^N(\beta_0, \theta) \mid X_{t_{k-1}} \right] \) satisfies:

\[
\sum_{k=1}^{N} \mathbb{E}_{\theta_0} \left[ Y_k^N(\beta_0, \theta) \mid X_{t_{k-1}} \right] \\
= \frac{1}{N h} \sum_{k=1}^{N} \text{Tr} \left( (\Sigma \Sigma^T)^{-1} g(X_{t_{k-1}}; \beta_0, \beta) \mathbb{E}_{\theta_0} \left[ Z_{t_k} (\beta_0)^\top \mid X_{t_{k-1}} \right] \right) \\
= \frac{1}{N} \sum_{k=1}^{N} R(h^2, X_{t_{k-1}}) \xrightarrow{\mathbb{P}_{\theta_0}} 0,
\]

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for \( h \to 0 \). Similarly we deduce that:

\[
\sum_{k=1}^{N} \mathbb{E}_{\theta_0}[Y_k^{N}(\beta_0, \theta)^2 \mid X_{t_{k-1}}] = \frac{1}{N^2 h^2} \sum_{k=1}^{N} g(X_{t_{k-1}}; \beta_0, \beta)^\top (\Sigma \Sigma^\top)^{-1} \mathbb{E}_{\theta_0}[Z_{t_k}(\beta_0)Z_{t_k}(\beta_0)^\top \mid X_{t_{k-1}}] (\Sigma \Sigma^\top)^{-1} g(X_{t_{k-1}}; \beta_0, \beta) = \frac{1}{N} \sum_{k=1}^{N} R\left(\frac{1}{Nh}, X_{t_{k-1}}\right) \xrightarrow{p_{\theta_0}} 0,
\]

for \( Nh \to \infty \). To prove uniform convergence, we use Lemma S3.4 along with Rosenthal’s inequality from Theorem S3.5, resulting in:

\[
\mathbb{E}_{\theta_0}[\| \sum_{k=1}^{N} Y_k^{N}(\beta_0, \theta) \|^p] \leq C(\mathbb{E}(\sum_{k=1}^{N} \mathbb{E}[Y_k^{N}(\beta_0, \theta)^2 \mid X_{t_{k-1}}])^{p/2}) + \sum_{k=1}^{N} \mathbb{E}[\| Y_k^{N}(\beta_0, \theta) \|^p]).
\]

The first term is bounded because of (S18). To bound the second term on the right-hand side, we use (S14). Then, for \( Nh \to \infty \) and \( h \to 0 \) and \( p > 2 \) it holds:

\[
\sum_{k=1}^{N} \mathbb{E}[\| Y_k^{N}(\beta_0, \theta) \|^p] \leq \frac{1}{(Nh)^p} \cdot Nh^{p/2} \cdot C = \frac{1}{(Nh)^p} \cdot h^{p/2-1} \cdot C \leq C.
\]

To conclude the proof of uniform convergence, we once again apply Rosenthal’s inequality to get:

\[
\mathbb{E}_{\theta_0}[\| \sum_{k=1}^{N} (Y_k^{N}(\beta_0, \theta_1) - Y_k^{N}(\beta_0, \theta_2)) \|^p] 
\leq CE\left(\sum_{k=1}^{N} \mathbb{E}[\| Y_k^{N}(\beta_0, \theta_1) - Y_k^{N}(\beta_0, \theta_2)^2 \mid X_{t_{k-1}}]\right)^{p/2} + C \sum_{k=1}^{N} \mathbb{E}[\| (Y_k^{N}(\beta_0, \theta_1) - Y_k^{N}(\beta_0, \theta_2)) \|^p].
\]

To bound the first term in (S19), we follow the reasoning from (S17) and start with:

\[
\mathbb{E}[\| Y_k^{N}(\beta_0, \theta_1) - Y_k^{N}(\beta_0, \theta_2)^2 \mid X_{t_{k-1}}] 
\leq 2 \mathbb{E}_{\theta_0}[\| (\Sigma_1 \Sigma_1^\top)^{-1} (g(X_{t_{k-1}}; \beta_1, \beta_0) - g(X_{t_{k-1}}; \beta_2, \beta_0))^2 \mid X_{t_{k-1}}] 
+ 2 \mathbb{E}_{\theta_0}[\| (\Sigma_2 \Sigma_2^\top)^{-1} (\Sigma_1 \Sigma_1^\top)^{-1} g(X_{t_{k-1}}; \beta_2, \beta_0))^2 \mid X_{t_{k-1}}].
\]

Then, the rest is the same. Similarly, to bound the second term in (S19), we repeat derivations from (S17) to get:

\[
\sum_{k=1}^{N} \mathbb{E}[\| (Y_k^{N}(\beta_0, \theta_1) - Y_k^{N}(\beta_0, \theta_2)) \|^p] \leq \frac{1}{(Nh)^p} \cdot Nh^{p/2} \cdot C \cdot \| \theta_1 - \theta_2 \|^p \leq C \cdot \| \theta_1 - \theta_2 \|^p.
\]

Finally, (S18) and conclusions after (S17) complete the proof of 4.

Proof of 5. We introduce \( Y_k^{N}(\beta_0, \theta) := \frac{1}{N} Z_{t_k}(\beta_0)^\top (\Sigma \Sigma^\top)^{-1} g(X_{t_k}; \beta_0, \beta) \). Proposition 4.3 yields that \( \mathbb{E}[Z_{t_k}(\beta_0)g(X_{t_k}; \beta_0, \beta)^\top \mid X_{t_{k-1}}] = R(h, X_{t_{k-1}}) \). Then, we conclude that \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[Y_k^{N}(\beta_0, \theta) \mid X_{t_{k-1}}] \to 0 \in p_{\theta_0}, \) for \( Nh \to \infty, \) \( h \to 0 \). Moreover, to prove the convergence of \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[Y_k^{N}(\beta_0, \theta)^2 \mid X_{t_{k-1}}] \), it is enough to bound \( \frac{1}{N^2} \sum_{k=1}^{N} \mathbb{E}[\| (\Sigma_1 \Sigma_1^\top)^{-1} g(X_{t_k}; \beta_0, \beta))^2 \mid X_{t_{k-1}}] \), Hölder’s inequality, together with Cauchy-Schwartz inequality, Lemma 4.1 and (S13), yield:

\[
\frac{1}{N^2} \sum_{k=1}^{N} \mathbb{E}[\| (\Sigma_1 \Sigma_1^\top)^{-1} g(X_{t_k}; \beta_0, \beta))^2 \mid X_{t_{k-1}}] 
\leq \frac{1}{N^2} \sum_{k=1}^{N} \mathbb{E}[\| Z_{t_k}(\beta_0)^\top \mathbb{E}[\| (\Sigma_1 \Sigma_1^\top)^{-1} (\Sigma \Sigma^\top)^{-1} \|^2]
\leq C \frac{1}{N^2} \sum_{k=1}^{N} \mathbb{E}[\| Z_{t_k}(\beta_0)^\top \mathbb{E}[\| (\Sigma_1 \Sigma_1^\top)^{-1} (\Sigma \Sigma^\top)^{-1} \|^2]
\leq \frac{1}{N} \sum_{k=1}^{N} R(h/N, X_{t_{k-1}}) \xrightarrow{p_{\theta_0}} 0,
\]

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Again, the proofs of (S10) and (S11) are the same as in property 3, with a distinction of rewriting: 

$$
\sum_{k=1}^{N} \mathbb{E}_{\theta_{0}}[Y_{k}^{N}(\beta_{0}, \theta) | X_{t_{k-1}}] = \frac{1}{N h} \sum_{k=1}^{N} \text{Tr}(\Sigma_{\Sigma}^{T})^{-1} \mathbb{E}_{\theta_{0}}[Z_{t_{k}}(\beta_{0})g(X_{t_{k}}; \beta_{0}, \beta)^{T} | X_{t_{k-1}}]) 
$$

$$
= \frac{1}{2N} \sum_{k=1}^{N} \text{Tr}((\Sigma_{\Sigma}^{T})^{-1}(\Sigma_{0}^{T}D^{T}g(X_{t_{k-1}}; \beta_{0}, \beta) + Dg(X_{t_{k-1}}; \beta_{0}, \beta)\Sigma_{0}^{T} + R(h, X_{t_{k-1}}))) 
$$

$$
\xrightarrow{P_{\theta_{0}}} \int \text{Tr}(Dg(x; \beta_{0}, \beta)\Sigma_{\Sigma}^{T})^{-1} d\nu_{0}(x), 
$$

for $N h \to \infty$, $h \to 0$. On the other hand, $\sum_{k=1}^{N} \mathbb{E}_{\theta_{0}}[Y_{k}^{N}(\beta_{0}, \theta)^{2} | X_{t_{k-1}}] = \frac{1}{N} \sum_{k=1}^{N} R(\frac{1}{N h}, X_{t_{k-1}}) \to 0$, in $P_{\theta_{0}}$, for $N h \to \infty$, $h \to 0$, which follows from derivations in (S20). To prove uniform convergence, we repeat the same approach as in the previous two proofs.

Proof of 7. First, we use the fact that $\mathbb{E}[g(X_{t_{k}}; \beta_{0}, \beta) | X_{t_{k-1}} = x] = g(x; \beta_{0}, \beta) + R(h, x)$, for a generic function $g$. Then, for $Y_{k}^{N}(\beta_{0}, \theta) := \frac{1}{N} g_{1}(X_{t_{k-1}}; \beta_{0}, \beta)(\Sigma_{\Sigma}^{T})^{-1}g_{2}(X_{t_{k}}; \beta_{0}, \beta)$ it follows

$$
\sum_{k=1}^{N} \mathbb{E}_{\theta_{0}}[Y_{k}^{N}(\beta_{0}, \theta) | X_{t_{k-1}}] \xrightarrow{\mathbb{P}_{\theta_{0}}} 0, \quad \sum_{k=1}^{N} \mathbb{E}_{\theta_{0}}[Y_{k}^{N}(\beta_{0}, \theta)^{2} | X_{t_{k-1}}] \xrightarrow{\mathbb{P}_{\theta_{0}}} 0. 
$$

Again, the proofs of (S10) and (S11) are the same as in property 3, with a distinction of rewriting:

$$
g_{1}(\beta_{1})^{T}(\Sigma_{1}^{T})^{-1}g_{2}(\beta_{1}) - g_{1}(\beta_{2})^{T}(\Sigma_{2}^{T})^{-1}g_{2}(\beta_{2}) 
$$

$$
= (g_{1}(\beta_{1}) - g_{1}(\beta_{2}))^{T}(\Sigma_{1}^{T})^{-1}g_{2}(\beta_{1}) + g_{1}(\beta_{2})^{T}(\Sigma_{1}^{T})^{-1}(g_{2}(\beta_{1}) - g_{2}(\beta_{2})) 
$$

$$
+ g_{1}(\beta_{2})^{T}(\Sigma_{2}^{T})^{-1} - (\Sigma_{2}^{T})^{-1}g_{2}(\beta_{2}). 
$$

Proof of Theorem 5.1 To establish consistency, we follow the proof of Theorem 1 in Kessler (1997) and study the limit of $L_{N}^{[S]}(\beta, \varsigma)$ from (23), rescaled by the correct rate of convergence. More precisely, the consistency of the diffusion parameter is proved by studying the limit of $\frac{1}{N}L_{N}^{[S]}(\beta, \varsigma)$, while the consistency of the drift parameter is proved by studying the limit of $\frac{1}{N}L_{N}^{[S]}(\beta, \varsigma) - L_{N}^{[S]}(\beta_{0}, \varsigma)$. We start with the consistency of the diffusion parameter $\varsigma$. We need to prove that:

$$
\frac{1}{N}L_{N}^{[S]}(\beta, \varsigma) \to \log(\text{det}(\Sigma_{\Sigma}^{T})) + \text{Tr}((\Sigma_{\Sigma}^{T})^{-1}\Sigma_{0}^{T}) =: G_{1}(\varsigma, \varsigma_{0}), 
$$

in $P_{\theta_{0}}$, for $N h \to \infty$, $h \to 0$, uniformly in $\theta$. To study the limit, we first decompose $\frac{1}{N}L_{N}^{[S]}(\beta, \varsigma)$ as follows:

$$
\frac{1}{N}L_{N}^{[S]}(\beta, \varsigma) = \log \text{det} \Sigma_{\Sigma}^{T} + T_{1} + T_{2} + T_{3} + 2(T_{4} + T_{5} + T_{6}) + R(h, x_{0}). 
$$
The terms $T_1, \ldots, T_6$ are derived from the quadratic form in (23) by adding and subtracting the corresponding terms with $\beta_0$, followed by rearrangements, resulting in the following expressions:

\[
T_1 := \frac{1}{N h} \sum_{k=1}^{N} Z_{tk} (\beta_0)^T (\Sigma \Sigma^T)^{-1} Z_{tk} (\beta_0),
\]

\[
T_2 := \frac{1}{N h} \sum_{k=1}^{N} (f_{h/2,k}(\beta) - f_{h/2,k}(\beta_0))^T (\Sigma \Sigma^T)^{-1} (f_{h/2,k}(\beta) - f_{h/2,k}(\beta_0)),
\]

\[
T_3 := \frac{1}{N h} \sum_{k=1}^{N} (\mu_{h,k-1}(\beta_0) - \mu_{h,k-1}(\beta))^T (\Sigma \Sigma^T)^{-1} (\mu_{h,k-1}(\beta) - \mu_{h,k-1}(\beta)),
\]

\[
T_4 := \frac{1}{N h} \sum_{k=1}^{N} Z_{tk} (\beta_0)^T (\Sigma \Sigma^T)^{-1} (\mu_{h,k-1}(\beta) - \mu_{h,k-1}(\beta)),
\]

\[
T_5 := \frac{1}{N h} \sum_{k=1}^{N} (f_{h/2,k}(\beta) - f_{h/2,k}(\beta_0))^T (\Sigma \Sigma^T)^{-1} (\mu_{h,k-1}(\beta) - \mu_{h,k-1}(\beta)),
\]

\[
T_6 := \frac{1}{N h} \sum_{k=1}^{N} (f_{h/2,k}(\beta) - f_{h/2,k}(\beta_0))^T (\Sigma \Sigma^T)^{-1} Z_{tk} (\beta_0).
\]

Previously, we defined $f_{h/2,k}(\beta) := f_{h/2}(X_{tk}; \beta)$ and $\mu_{h,k-1}(\beta) := \mu_h(f_{h/2}(X_{tk-1}; \beta); \beta)$. These terms will also play a significant role in proving the asymptotic normality.

The first term of (S22) is a constant. Properties 1, 2, 3, 5, and 7 from Lemma S2.3 give the following limits $T_1 \rightarrow \text{Tr}((\Sigma \Sigma^T)^{-1} \Sigma \Sigma_0^T)$ and for $l = 2, 3, \ldots, 6$, $T_l \rightarrow 0$, uniformly in $\theta$. The convergence in probability is equivalent to the existence of a subsequence converging almost surely. Thus, the convergence in (S21) is almost sure for a subsequence $(\tilde{\beta}_{N_l}, \tilde{\xi}_{N_l})$. This implies:

\[
\tilde{\xi}_{N_l} \overset{P_{\theta_0} \text{ a.s.}}{\underset{N h \rightarrow \infty, h \rightarrow 0}{\rightarrow}} \xi_\infty.
\]

The compactness of $\mathbb{G}$ implies that $(\tilde{\beta}_{N_l}, \tilde{\xi}_{N_l})$ converges to a limit $(\beta_\infty, \xi_\infty)$ almost surely. By continuity of the mapping $\xi \mapsto G_1(\xi, \varsigma_0)$ we have $\frac{1}{N h} L_{N}^{[S]}(\tilde{\beta}_{N_l}, \tilde{\xi}_{N_l}) \rightarrow G_1(\xi_\infty, \varsigma_0)$, in $P_{\theta_0}$, for $N h \rightarrow \infty, h \rightarrow 0$, uniformly in $\theta$. By the definition of the estimator, $G_1(\xi_\infty, \varsigma_0) \leq G_1(\varsigma_0, \varsigma_0)$. We also have:

\[
G_1(\xi_\infty, \varsigma_0) \geq G_1(\varsigma_0, \varsigma_0)
\]

\[
\Leftrightarrow \log(\det(\Sigma \Sigma_\infty^T)) + \text{Tr}((\Sigma \Sigma_\infty^T)^{-1} \Sigma \Sigma_0^T) \geq \log(\det(\Sigma \Sigma_0^T)) + \text{Tr}(I_d)
\]

\[
\Leftrightarrow \text{Tr}((\Sigma \Sigma_\infty^T)^{-1} \Sigma \Sigma_0^T) \geq \log(\det(\Sigma \Sigma_\infty^T)) - \log(\det(\Sigma \Sigma_0^T)) \geq d
\]

\[
\Leftrightarrow \sum_{i=1}^{d} \lambda_i - \log \prod_{i=1}^{d} \lambda_i \geq \sum_{i=1}^{d} \lambda_i - \log \lambda_i \geq 0,
\]

where $\lambda_i$ are the eigenvalues of $(\Sigma \Sigma_\infty^T)^{-1} \Sigma \Sigma_0^T$, which is a positive definite matrix. The last inequality follows since for any positive $x$, $\log x \leq -x - 1$. Thus, $G_1(\xi_\infty, \varsigma_0) \geq G_1(\varsigma_0, \varsigma_0)$. Then, all the eigenvalues $\lambda_i$ must be equal to 1, hence, $\Sigma \Sigma_\infty^T = \Sigma \Sigma_0^T$. We proved that a convergent subsequence of $\tilde{\xi}_{N_l}$ tends to $\varsigma_0$ almost surely. From there, the consistency of the estimator of the diffusion coefficient follows.

We now focus on the consistency of the drift parameter. The objective is to prove that the following limit in $P_{\theta_0}$, for $N h \rightarrow \infty, h \rightarrow 0$, uniformly with respect to $\theta$:

\[
\frac{1}{N h} (L_N^{[S]}(\beta, \varsigma) - L_N^{[S]}(\beta_0, \varsigma)) \rightarrow G_2(\beta_0, \varsigma_0, \beta, \varsigma), \tag{S23}
\]

where:

\[
G_2(\beta_0, \varsigma_0, \beta, \varsigma) := \int (F_0(x) - F(x))^T (\Sigma \Sigma^T)^{-1} (F_0(x) - F(x)) d\nu_0(x)
\]

\[
+ \int \text{Tr}(D(F_0(x) - F(x))(\Sigma \Sigma_0^T (\Sigma \Sigma^T)^{-1} - I)) d\nu_0(x).
\]
To prove the same statement for the LT estimator, the representation of the objective function \((S22)\) has to be adapted.

Then,

The limit of \(\frac{1}{N} \sum_{k=1}^{N} (Z_{tk}(\beta_0)^T (\Sigma \Sigma^T)^{-1} A(\beta_0) Z_{tk}(\beta_0) - Z_{tk}(\beta)^T (\Sigma \Sigma^T)^{-1} A(\beta) Z_{tk}(\beta))\) converges to \(\text{Tr}(A(\beta_0) - A(\beta))\), which thus cancels out with the first term in \((34)\). Lemma 4.2 provides the uniform convergence of \(\frac{1}{T_2}\) with respect to \(\theta\):

The limit of \(\frac{1}{N} T_3\) computes analogously. To prove \(\frac{1}{N} T_4\) → 0, we use Lemma 9 in Genon-Catalot and Jacod (1993) and Property 4 from Lemma S2.3. Lemma 4.2 yields:

Finally, \(\frac{1}{N} T_4\) → \(\frac{1}{2} \int \text{Tr}(D(N_0(x) - N(x))^T \Sigma \Sigma^T)^{-1} (N_0(x) - N(x)) d\nu_0(x)\) uniformly in \(\theta\), by Property 6 of Lemma S2.3. Lemma 4.2 gives:

uniformly in \(\theta\). This proves \((S23)\). Then, there exists a subsequence \(N_l\) such that \((\hat{\beta}_{N_l}, \hat{\varsigma}_{N_l})\) converges to a limit \((\beta_\infty, \varsigma_\infty)\), almost surely. By continuity of the mapping \((\beta, \varsigma) \mapsto G_2(\beta_0, \varsigma_0, \beta, \varsigma)\), for \(N_l h \to \infty, h \to 0\), we have the following convergence in \(P_{\theta_0}\):

Then, \(G_2(\beta_0, \varsigma_0, \beta_\infty, \varsigma_\infty) \geq 0\) since \(\Sigma \Sigma^T = \Sigma \Sigma^T\). On the other hand, by the definition of the estimator \(L_{N_l}^N(\hat{\beta}_{N_l}, \hat{\varsigma}_{N_l}) - L_{N_l}^N(\beta_0, \hat{\varsigma}_{N_l}) \leq 0\). Thus, the identifiability assumption \((A5)\) concludes the proof for the \(S\) estimator.

To prove the same statement for the LT estimator, the representation of the objective function \((S22)\) has to be adapted. In the LT case, this representation is straightforward. There is no extra logarithmic term and only three instead of six auxiliary \(T\) terms are used. This is due to the Gaussian transition density in the LT approximation.

**S2.7 Proof of asymptotic normality of the estimator**

In this section, we distinguish between the true parameter \(\theta_0\) and a generic parameter \(\theta\).

**Proof of Theorem 5.2** According to Theorem 1 in Kessler (1997) or Theorem 1 in Sørensen and Uchida (2003), Lemmas S2.4 and S2.5 below are enough for establishing the asymptotic normality of \(\theta_N\). Here, we only present the outline of the proof. For more details, see proof of Theorem 1 in Sørensen and Uchida (2003).

**Lemma S2.4** Let \(C_N(\theta_0)\) and \(C(\theta_0)\) be defined in \((25)\) and \((27)\), respectively. If \(h \to 0, N h \to \infty, \rho_N \to 0,\) then:

\[
C_N(\theta_0) \xrightarrow{\rho_N} C(\theta_0), \quad \sup_{\|\theta\| \leq \rho_N} \|C_N(\theta_0 + \theta) - C_N(\theta_0)\| \xrightarrow{\rho_N} 0.
\]
Lemma S2.5 Let $\lambda_N$ be as defined (26). If $h \to 0$, $Nh \to \infty$ and $Nh^2 \to 0$, then:

$$\lambda_N \overset{d}{\to} \mathcal{N}(0, 4C(\theta_0)),$$

under $\mathbb{P}_{\theta_0}$.

Lemma S2.4 states that $C_N(\theta_0)$ approaches $2C(\theta_0)$ as $h \to 0$ and $Nh \to \infty$. Moreover, the difference between $C_N(\theta_0 + \theta)$ and $C_N(\theta_0)$ approaches zero when $\theta$ approaches $\theta_0$, within a distance specified by balls $B_{\rho_N}(\theta_0)$, where $\rho_N \to 0$. To ensure the asymptotic normality of $\hat{\theta}_N$, Lemma S2.4 is employed to restrict the term $||D_N - C_N(\theta_0)||$ when $\hat{\theta}_N \in \Theta \cap B_{\rho_N}(\theta_0)$ as follows:

$$||D_N - C_N(\theta_0)||_1 \to_{\theta \in \Theta \cap B_{\rho_N}(\theta_0)} \sup_{\theta \in B_{\rho_N}(\theta_0)} ||C_N(\theta) - C_N(\theta_0)||_1 \to 0 \quad \text{as} \quad Nh \to \infty.$$

Applying again Lemma S2.4 on the previous line, we get $D_N \to 2(C(\theta_0))$ in $\mathbb{P}_{\theta_0}$, as $h \to 0$ and $Nh \to \infty$.

Lemma S2.5 establishes the convergence in distribution of $\lambda_N$ to $\mathcal{N}(0, 4C(\theta_0))$, under $\mathbb{P}_{\theta_0}$, as $h \to 0$ and $Nh \to \infty$. This result provides the groundwork for the asymptotic normality of $\hat{\theta}_N$. Indeed, consider the set $D_N$ composed of instances where $D_N$ is invertible. The probability, under $\theta_0$, of $D_N$ occurring approaches 1, as $h \to 0$ and $Nh \to \infty$. This implies that $D_N$ is almost surely invertible in this limit. Furthermore, we define $\mathcal{E}_N$ as the intersection of $\{\theta_N \in \Theta\}$ and $D_N$. Then, it can be shown that $\mathcal{E}_N \to 1$ in $\mathbb{P}_{\theta_0}$, when $h \to 0$ and $Nh \to \infty$. For $E_N := D_N$ on $\mathcal{E}_N$, we have $E_N \to 2(C(\theta_0))$ in $\mathbb{P}_{\theta_0}$ as $h \to 0$ and $Nh \to \infty$. Given that $s_N \mathbb{P}_{\theta_0} = E_N^{-1} D_N s_N \mathbb{P}_{\theta_0} = E_N^{-1} \lambda_N \mathbb{P}_{\theta_0}$ and according to Lemma S2.5, $s_N \mathbb{P}_{\theta_0} \to \mathcal{N}(0, C(\theta_0)^{-1})$ in distribution as $h \to 0$, $Nh \to \infty$ and $Nh^2 \to 0$.

In conclusion, under $\mathbb{P}_{\theta_0}$, as $h \to 0$, $Nh \to \infty$ and $Nh^2 \to 0$, $s_N \mathbb{P}_{\theta_0}$ is shown to converge in distribution to $\mathcal{N}(0, C(\theta_0)^{-1})$. The asymptotic normality for $\hat{\theta}_N$ is, thus, confirmed due to the convergence of $s_N \mathbb{P}_{\theta_0} \to 1$.

Proof of Lemma S2.4 To prove the first part of the lemma, we aim to represent $C_N(\theta_0)$ from the objective function (14). In doing so, we again employ the approximation (23), focusing solely on the terms that do not converge to zero as $Nh \to \infty$ and $h \to 0$. We start as in the approximation (34) and compute the corresponding derivatives to obtain the first block matrix of $C_N$ (25). We begin with $\partial_{\beta_1 \beta_2} L^S_N(\beta, \varsigma)$:

$$\frac{1}{Nh} \partial_{\beta_1 \beta_2} L^S_N(\beta, \varsigma) = \partial_{\beta_1 \beta_2} \text{Tr} A(\beta) + \frac{1}{N} \sum_{k=1}^{N} \partial_{\beta_1 \beta_2} \text{Tr} DN(X_t \beta) + T_2(\beta_0, \beta, \varsigma) + T_3(\beta_0, \beta, \varsigma) + T_4(\beta_0, \beta, \varsigma) + T_5(\beta_0, \beta, \varsigma) + T_6(\beta_0, \beta, \varsigma))$$

$$- \frac{1}{Nh} \sum_{k=1}^{N} \partial_{\beta_1 \beta_2} (Z_{\beta_1}^{\beta_2}(\beta)^T (\Sigma \Sigma^T)^{-1} A(\beta) Z_{\beta_1}(\beta)) + R(h, x_0).$$

To determine the convergence of each of the previous terms, we use the definitions of the sums $T_i$s and approximate each $T_i$ using Proposition 2.2 and the Taylor expansion of the function $\mu_k$. As we apply the derivatives $\partial_{\beta_1 \beta_2}$, the order of $h$ in each sum increases since terms of order $R(1, x_0)$ are constant with respect to $\beta$. Finally, when evaluating $\frac{1}{Nh} \partial_{\beta_1 \beta_2} L^S_N(\beta, \varsigma)$ at $\beta = \theta_0$, numerous terms will cancel out due to differences of the type $g_1(\beta_0; X_{t_k}, X_{t_{k-1}}) - g_1(\beta; X_{t_k}, X_{t_{k-1}})$. Using the results from Lemma S2.3 and the proof of Theorem 5.1, we get the following limits:

$$\partial_{\beta_1 \beta_2} \frac{1}{Nh} T_2(\beta_0, \beta, \varsigma_0) \bigg|_{\beta = \beta_0} \overset{P_{\theta_0}}{\to} \frac{1}{2} \int (\partial_{\beta_1} N_0(x))^T (\Sigma \Sigma^T)^{-1} \partial_{\beta_2} N_0(x) \, dv_0(x),$$

$$\partial_{\beta_1 \beta_2} \frac{1}{Nh} T_3(\beta_0, \beta, \varsigma_0) \bigg|_{\beta = \beta_0} \overset{P_{\theta_0}}{\to} \frac{1}{2} \int (\partial_{\beta_1} A_0(x-b_0))^T (\Sigma \Sigma^T)^{-1} (\partial_{\beta_2} A_0(x-b_0)) dv_0(x),$$

$$\partial_{\beta_1 \beta_2} \frac{1}{Nh} T_4(\beta_0, \beta, \varsigma_0) \bigg|_{\beta = \beta_0} \overset{P_{\theta_0}}{\to} \frac{1}{2} \int (\partial_{\beta_1} F_0(x))^T (\Sigma \Sigma^T)^{-1} (\partial_{\beta_2} N_0(x)) \, dv_0(x) + \frac{1}{2} \int (\partial_{\beta_2} A_0(x-b_0))^T (\Sigma \Sigma^T)^{-1} \partial_{\beta_1} N_0(x) \, dv_0(x),$$

$$\partial_{\beta_1 \beta_2} \frac{1}{Nh} T_6(\beta_0, \beta, \varsigma_0) \bigg|_{\beta = \beta_0} \overset{P_{\theta_0}}{\to} -\frac{1}{2} \text{Tr}(D\partial_{\beta_1 \beta_2} N_0(x)) \, dv_0(x).$$
for $Nh \to \infty$, $h \to 0$. Since $\frac{1}{h}T_4 \to 0$, the partial derivatives go to zero too. From Lemma 4.2, for $Nh \to \infty$, $h \to 0$, we have:

$$
\frac{1}{N} \sum_{k=1}^{N} \partial_{\beta_1, \beta_2} \left( \frac{1}{N} \sum_{k=1}^{N} \partial_{\beta_1, \beta_2} \left( Z_{t_k}(\beta)^{\top} (\Sigma \Sigma^\top)^{-1} A(\beta) Z_{t_k}(\beta) \right) \right) \xrightarrow{P_{\theta_0}} \int \text{Tr} \left( D\partial_{\beta_1, \beta_2} N_0(x) \right) \, d\nu_0(x).
$$

Term $\frac{1}{N} \sum_{k=1}^{N} \partial_{\beta_1, \beta_2} \left( Z_{t_k}(\beta)^{\top} (\Sigma \Sigma^\top)^{-1} A(\beta) Z_{t_k}(\beta) \right)$, evaluated in $\theta = \theta_0$, has only one term of order $h$:

$$
\frac{1}{N} \sum_{k=1}^{N} Z_{t_k}(\beta_0)^{\top} (\Sigma \Sigma^\top)^{-1} A(\beta_0) Z_{t_k}(\beta_0),
$$

which converges to $\partial_{\beta_1, \beta_2} \text{Tr} A(\beta_0)$ (Property 1 Lemma S2.3).

Thus, $\frac{1}{N} \text{Tr} \partial_{\beta_1, \beta_2} L^{[S]}_N(\beta, \varsigma_0) \mid_{\beta=\beta_0, \varsigma=\varsigma_0} \to 2 \int \text{Tr} \left( \partial_{\beta_2} F_0(\mathbf{x}) \right) (\Sigma \Sigma^\top)^{-1} \partial_{\beta_2} F_0(\mathbf{x}) \, d\nu_0(\mathbf{x})$, in $P_{\theta_0}$ for $Nh \to \infty$, $h \to 0$.

Now, we prove $\frac{1}{N} \text{Tr} \partial_{\beta_1} L^{[S]}_N(\beta, \varsigma_0) \mid_{\beta=\beta_0, \varsigma=\varsigma_0} \to 0$, in $P_{\theta_0}$ for $Nh \to \infty$, $h \to 0$. For a constant $C_h$, depending on $h$, $l = 2, 3, \ldots, 6$, and generic functions $g, g_1$, the following term is at most of order $R(h, x_0)$:

$$
\partial_{\beta_1} T_l(\beta, \varsigma) = C_h \sum_{k=1}^{N} \left( g(\beta_0; X_{t_k}, X_{t_k-1}) - g(\beta; X_{t_k}, X_{t_k-1}) \right)^\top (\Sigma \Sigma^\top)^{-1} g_1(\beta; X_{t_k}, X_{t_k-1}).
$$

Then, term $\partial_{\beta_1} L^{[S]}_N(\beta, \varsigma)$ still contains $g(\beta_0; X_{t_k}, X_{t_k-1}) - g(\beta; X_{t_k}, X_{t_k-1})$ which is 0 for $\beta = \beta_0$. Moreover, the term $\frac{1}{N} \sum_{k=1}^{N} \partial_{\beta_1} \left( Z_{t_k}(\beta)^{\top} (\Sigma \Sigma^\top)^{-1} A(\beta) Z_{t_k}(\beta) \right)$ is at most of order $R(h, x_0)$. Thus, $\frac{1}{N} \text{Tr} \partial_{\beta_1} L^{[S]}_N(\beta, \varsigma) \mid_{\beta=\beta_0, \varsigma=\varsigma_0} \to 0$.

Finally, we compute $\frac{1}{N} \text{Tr} \partial_{\varsigma_1, \varsigma_2} L^{[S]}_N(\beta, \varsigma)$. As before, it holds $\frac{1}{N} \text{Tr} \partial_{\varsigma_1, \varsigma_2} T_l(\beta, \varsigma) \mid_{\beta=\beta_0, \varsigma=\varsigma_0} \to 0$, for $l = 2, 3, \ldots, 6$. Similarly, we see $\frac{1}{N} \sum_{k=1}^{N} Z_{t_k}(\beta_0)^{\top} \partial_{\varsigma_1, \varsigma_2} Z_{t_k}(\beta_0)$ is at most of order $R(h, x_0)$. So, we need to compute the following second derivatives $\partial_{\varsigma_1, \varsigma_2} \log(\det(\Sigma \Sigma^\top))$ and $\partial_{\varsigma_1, \varsigma_2} \frac{1}{N} \sum_{k=1}^{N} Z_{t_k}(\beta_0)^{\top} (\Sigma \Sigma^\top)^{-1} Z_{t_k}(\beta_0)$.

The first one yields:

$$
\partial_{\varsigma_1, \varsigma_2} \log(\det(\Sigma \Sigma^\top)) = \text{Tr}((\Sigma \Sigma^\top)^{-1}(\partial_{\varsigma_1} \Sigma \Sigma^\top)(\Sigma \Sigma^\top)^{-1}(\partial_{\varsigma_2} \Sigma \Sigma^\top)).
$$

On the other hand, we have:

$$
\partial_{\varsigma_1, \varsigma_2} \frac{1}{N} \sum_{k=1}^{N} Z_{t_k}(\beta_0)^{\top} Z_{t_k}(\beta_0) \xrightarrow{P_{\theta_0}} \frac{1}{N} \sum_{k=1}^{N} \text{Tr}(Z_{t_k}(\beta_0)^{\top} Z_{t_k}(\beta_0) (\Sigma \Sigma^\top)^{-1}(\partial_{\varsigma_1} \Sigma \Sigma^\top)(\Sigma \Sigma^\top)^{-1}).
$$

Thus, from Property 1 of Lemma S2.3, we get:

$$
\partial_{\varsigma_1, \varsigma_2} \left( \frac{1}{N} \sum_{k=1}^{N} Z_{t_k}(\beta_0)^{\top} Z_{t_k}(\beta_0) \right) \bigg|_{\varsigma=\varsigma_0} \xrightarrow{P_{\theta_0}} \frac{1}{2N} \sum_{k=1}^{N} \text{Tr}((\Sigma \Sigma^\top)^{-1}(\partial_{\varsigma_1} \Sigma \Sigma^\top)(\Sigma \Sigma^\top)^{-1}(\partial_{\varsigma_2} \Sigma \Sigma^\top)(\Sigma \Sigma^\top)^{-1}).
$$

Thus, $\frac{1}{N} \text{Tr} \partial_{\varsigma_1, \varsigma_2} L^{[S]}_N(\beta, \varsigma) \mid_{\beta=\beta_0, \varsigma=\varsigma_0} \to \text{Tr}((\Sigma \Sigma^\top)^{-1}(\partial_{\varsigma_1} \Sigma \Sigma^\top)(\Sigma \Sigma^\top)^{-1}(\partial_{\varsigma_2} \Sigma \Sigma^\top)(\Sigma \Sigma^\top)^{-1})$. Since all the limits used in this proof are uniform in $\theta$, the first part of the lemma is proved. The second part is trivial, because all limits are continuous in $\theta$. 

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Proof of Lemma S2.5 First, we compute the first derivatives. We start with:

\[
\partial_{\beta_i} L_N^S(\beta, \varsigma) = -2 \sum_{k=1}^{N} \text{Tr}(Df_{h/2,k}(\beta)D\beta_i f_{h/2,k}^{-1}(\beta)) \\
+ \frac{2}{h} \sum_{k=1}^{N} (f_{h/2,k}(\beta) - \mu_{h,k-1}(\beta))^{\top}(\Sigma \Sigma^{\top})^{-1}(\partial_{\beta_i} f_{h/2,k}(\beta) - \partial_{\beta_i} \mu_{h,k-1}(\beta)).
\]

The first derivative with respect to \( \varsigma \) is:

\[
\partial_{\varsigma} L_N^S(\beta, \varsigma) = N \partial_{\varsigma} \log \det(\Sigma \Sigma^{\top}) \\
+ \frac{1}{h} \sum_{k=1}^{N} \left( \text{Tr} \left( (f_{h/2,k}(\beta) - \mu_{h,k-1}(\beta))(f_{h/2,k}(\beta) - \mu_{h,k-1}(\beta))^{\top} \right) \\
(\Sigma \Sigma^{\top})^{-1}(\partial_{\varsigma} \Sigma \Sigma^{\top})(\Sigma \Sigma^{\top})^{-1}ight).
\]

Define:

\[
\eta^{(i)}_{N,k}(\theta) := \frac{2}{\sqrt{Nh}} \text{Tr}(Df_{h/2,k}(\beta)D\beta_i f_{h/2,k}^{-1}(\beta))
\]

\[
\zeta^{(j)}_{N,k}(\theta) := \frac{1}{\sqrt{Nh}} \text{Tr}(Z_{t_k}^{1}(\beta)Z_{t_k}^{1}(\beta)^{\top}(\Sigma \Sigma^{\top})^{-1}(\partial_{\varsigma} \Sigma \Sigma^{\top})(\Sigma \Sigma^{\top})^{-1})
\]

and rewrite \( \lambda_N \) as \( \lambda_N = \sum_{k=1}^{N}[\eta^{(i)}_{N,k}(\theta), \ldots, \eta^{(r)}_{N,k}(\theta), \zeta^{(j)}_{N,k}(\theta), \ldots, \zeta^{(s)}_{N,k}(\theta)]^{\top} \). Now, by Proposition 3.1 from Crimalli and Pratelli (2005), it is sufficient to prove Lemma S2.6.

Lemma S2.6 Let \( \eta^{(i)}_{N,k}(\theta) \) and \( \zeta^{(j)}_{N,k}(\theta) \) be defined as in (S25) and (S26), respectively. If \( h \to 0, Nh \to \infty, \) and \( Nh^2 \to 0 \), then for and all \( i, i_1, i_2 = 1, 2, \ldots, r, \) and \( j, j_1, j_2 = 1, 2, \ldots, s, \) it holds:

(i) \( \mathbb{E}_{\theta_0}[\sup_{1 \leq k \leq N}[\eta^{(i)}_{N,k}(\theta_0)]] \to 0 \), and \( \mathbb{E}_{\theta_0}[\sup_{1 \leq k \leq N}|\zeta^{(j)}_{N,k}(\theta_0)|] \to 0 \);

(ii) \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[\eta^{(i)}_{N,k}(\theta_0) | X_{t_{k-1}}] \xrightarrow{\mathbb{P}_{\theta_0}} 0 \), and \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[\zeta^{(j)}_{N,k}(\theta_0) | X_{t_{k-1}}] \xrightarrow{\mathbb{P}_{\theta_0}} 0 \);

(iii) \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[\eta^{(i)}_{N,k}(\theta_0) | X_{t_{k-1}}, \eta^{(i2)}_{N,k}(\theta_0), X_{t_{k-1}}] \xrightarrow{\mathbb{P}_{\theta_0}} 0 \);

(iv) \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[\zeta^{(j)}_{N,k}(\theta_0) | X_{t_{k-1}}, \zeta^{(j2)}_{N,k}(\theta_0), X_{t_{k-1}}] \xrightarrow{\mathbb{P}_{\theta_0}} 0 \);

(v) \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[\eta^{(i)}_{N,k}(\theta_0) | X_{t_{k-1}}, \zeta^{(j)}_{N,k}(\theta_0), X_{t_{k-1}}] \xrightarrow{\mathbb{P}_{\theta_0}} 0 \);

(vi) \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[\eta^{(i)}_{N,k}(\theta_0) | X_{t_{k-1}}, \eta^{(i2)}_{N,k}(\theta_0), X_{t_{k-1}}] \xrightarrow{\mathbb{P}_{\theta_0}} 4[C_{\beta}(\theta_0)]_{i_1 i_2} \);

(vii) \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[\zeta^{(j)}_{N,k}(\theta_0) | X_{t_{k-1}}, \zeta^{(j2)}_{N,k}(\theta_0), X_{t_{k-1}}] \xrightarrow{\mathbb{P}_{\theta_0}} 4[C_{\varsigma}(\theta_0)]_{j_1 j_2} \);

(viii) \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[\eta^{(i)}_{N,k}(\theta_0) | X_{t_{k-1}}, \eta^{(i2)}_{N,k}(\theta_0)] \xrightarrow{\mathbb{P}_{\theta_0}} 0 \);

(ix) \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[\eta^{(i)}_{N,k}(\theta_0) | X_{t_{k-1}}]^2 \xrightarrow{\mathbb{P}_{\theta_0}} 0 \);
(x) \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[(\zeta_{N,k}^{(i,j)}(\theta_0))\zeta_{N,k}^{(i,j)}(\theta_0))^2 | X_{t_{k-1}}] \stackrel{P_{\theta_0}}{\longrightarrow} 0; \)

(xi) \( \sum_{k=1}^{N} \mathbb{E}_{\theta_0}[(\eta_{N,k}^{(i)}(\theta_0)\zeta_{N,k}^{(i)}(\theta_0))^2 | X_{t_{k-1}}] \stackrel{P_{\theta_0}}{\longrightarrow} 0. \)

Proof of Lemma S2.6 The proof of Lemma S2.6 is technical and involves bounding the sums of triangular arrays in such a way that the bound converges to zero in probability \( P_{\theta_0} \) as \( h \to 0, Nh \to \infty, \) and \( Nh^2 \to 0. \) Unlike in the previous proof, this time we do not require uniform convergence.

We begin by expanding \( \eta_{N,k}^{(i)} \) to differentiate between terms that vanish and those that do not in the limits:

\[
\eta_{N,k}^{(i)}(\theta_0) = \frac{2}{\sqrt{Nh}} \text{Tr}(I + \frac{h}{2} DN_0(X_{t_k}))( - \frac{h}{2} \partial_{\beta_i} N_0(X_{t_k})) \\
- \frac{2}{\sqrt{Nh}} Z_{t_k}(\beta_0)^T (\Sigma \Sigma_0^T)^{-1}( - \frac{h}{8} \partial_{\beta_i} (DN_0(X_{t_k})) N_0(X_{t_k})) \\
+ \frac{2}{\sqrt{Nh}} Z_{t_k}(\beta_0)^T (\Sigma \Sigma_0^T)^{-1} \partial_{\beta_i} \mu_h(f_{h/2}(X_{t_{k-1}}; \beta_0); \beta_0) + R(\sqrt{h^3/N}, X_{t_{k-1}}) \\
- \frac{1}{\sqrt{Nh}} Z_{t_k}(\beta_0)^T (\Sigma \Sigma_0^T)^{-1} \partial_{\beta_i} (DN_0(X_{t_k})) N_0(X_{t_k}) \\
+ \frac{2}{\sqrt{Nh}} Z_{t_k}(\beta_0)^T (\Sigma \Sigma_0^T)^{-1} \partial_{\beta_i} \mu_h(f_{h/2}(X_{t_{k-1}}; \beta_0); \beta_0) + R(\sqrt{h^3/N}, X_{t_{k-1}}). \tag{S27}
\]

Proof of (i). Let us begin by examining the limit of the expectation of \( \sup_{1 \leq k \leq N} |\zeta_{N,k}^{(i,j)}(\theta_0)| \). In equation (S27), all the involved functions are bounded, and the term with the largest order is \( R(\sqrt{Nh}, X_{t_{k-1}}) \) because \( \partial_{\beta_i} \mu_h(f_{h/2}(X_{t_{k-1}}; \beta_0); \beta_0) \) is \( R(h, X_{t_{k-1}}) \). The remaining terms converge to zero. Moreover, terms with coefficients \( \frac{1}{\sqrt{Nh}} \) take the form \( Z_{t_k}(\beta_0)^T (\Sigma \Sigma_0^T)^{-1} g \), where \( g \) is a vector-valued function of either \( X_{t_{k-1}} \) or \( X_{t_k} \). Their expected values are bounded by \( R(h, X_{t_{k-1}}) \) at most. Thus, the dominant order becomes \( R(\sqrt{h^3/N}, X_{t_{k-1}}) \), which indeed converges to zero.

We proceed to analyze the limit of the expectation of \( \sup_{1 \leq k \leq N} |\zeta_{N,k}^{(i,j)}(\theta_0)| \). The leading term in \( \zeta_{N,k}^{(i,j)}(\theta_0) \), as defined in the paper, has an order \( R(1/\sqrt{Nh^2}, X_{t_{k-1}}) \). Upon calculating its expected value, we obtain an order of \( R(h, X_{t_{k-1}}) \). This concludes the proof of (i).

To establish limits (ii)-(v), we need to calculate the expectations of \( \eta_{N,k}^{(i)} \) and \( \zeta_{N,k}^{(i,j)} \). By analyzing (S27), we can deduce that \( \mathbb{E}_{\theta_0}[\eta_{N,k}^{(i)}(\theta_0) | X_{t_{k-1}}] = R(\sqrt{h^3/N}, X_{t_{k-1}}) \), since Proposition 4.3 gives:

\[
\mathbb{E}_{\theta_0}[\frac{1}{\sqrt{Nh}} Z_{t_k}(\beta_0)^T (\Sigma \Sigma_0^T)^{-1} \partial_{\beta_i} N_0(X_{t_k}) | X_{t_{k-1}}] = \sqrt{\frac{h}{N}} \text{Tr}(D_\gamma \partial_{\beta_i} N_0(X_{t_k})) + R(\sqrt{h^3/N}, X_{t_{k-1}}),
\]

Similarly, from:

\[
\mathbb{E}_{\theta_0}[\text{Tr}(Z_{t_k} Z_{t_k}^T (\Sigma \Sigma_0^T)^{-1}(\partial_{\beta_i} \Sigma \Sigma_0^T)(\Sigma \Sigma_0^T)^{-1}) | X_{t_{k-1}}] = h \text{Tr}((\Sigma \Sigma_0^T)^{-1} \partial_{\beta_i} \Sigma \Sigma_0^T) + R(h^2, X_{t_{k-1}})
\]
we conclude that \(\mathbb{E}_0[\zeta_{N,k}^{(i)}(\theta_0) \mid X_{t_{k-1}}] = R(h/\sqrt{N}, X_{t_{k-1}})\). Then, combining the previous, we get:

\[
\sum_{k=1}^{N} \mathbb{E}_0[\eta_{N,k}^{(i)}(\theta_0) \mid X_{t_{k-1}}] = R(\sqrt{Nh^3}, X_{t_{k-1}}) \xrightarrow{P_{\theta_0}} 0,
\]
\[
\sum_{k=1}^{N} \mathbb{E}_0[\eta_{N,k}^{(j)}(\theta_0) \mid X_{t_{k-1}}] = R(\sqrt{Nh^2}, X_{t_{k-1}}) \xrightarrow{P_{\theta_0}} 0,
\]
\[
\sum_{k=1}^{N} \mathbb{E}_0[\eta_{N,k}^{(i)}(\theta_0) \mid X_{t_{k-1}}] = R(\sqrt{Nh}, X_{t_{k-1}}) \xrightarrow{P_{\theta_0}} 0,
\]
\[
\sum_{k=1}^{N} \mathbb{E}_0[\eta_{N,k}^{(j)}(\theta_0) \mid X_{t_{k-1}}] = R(\sqrt{Nh^5/2}, X_{t_{k-1}}) \xrightarrow{P_{\theta_0}} 0.
\]

Now, we prove limit (vi). Here, we focus on the terms of order \(1/\sqrt{Nh}\) in \(\eta_{N,k}^{(i)}\), which are the only ones that will not converge to zero when multiplying \(\eta_{N,k}^{(i)}\) and \(\eta_{N,k}^{(j)}\):

\[
\eta_{N,k}^{(i)}(\theta_0) = \frac{1}{\sqrt{Nh}} Z_{t_k}^\top (\Sigma \Sigma_0^\top)^{-1} \partial_{\beta_i} N_0(X_{t_k})
+ \frac{2}{h\sqrt{Nh}} Z_{t_k}^\top (\Sigma \Sigma_0^\top)^{-1} \partial_{\beta_i} \mu_h(f_{h/2}(X_{t_{k-1}}; \beta_0)) + R(\sqrt{h/N}, X_{t_{k-1}})
\]
\[
= \frac{1}{\sqrt{Nh}} Z_{t_k}^\top (\Sigma \Sigma_0^\top)^{-1} \partial_{\beta_i} N_0(X_{t_k}) + \frac{1}{\sqrt{Nh}} Z_{t_k}^\top (\Sigma \Sigma_0^\top)^{-1} \partial_{\beta_i} (N_0(X_{t_{k-1}}))
+ 2A_0(X_{t_{k-1}} - b_0) + R(\sqrt{h/N}, X_{t_{k-1}})
\]
\[
= \frac{2}{\sqrt{Nh}} Z_{t_k}^\top (\Sigma \Sigma_0^\top)^{-1} \partial_{\beta_i} F_0(X_{t_{k-1}}) + \frac{1}{\sqrt{Nh}} Z_{t_k}^\top (\Sigma \Sigma_0^\top)^{-1} \psi_{k,k-1}^{(i)}(\beta_0) + R(\sqrt{h/N}, X_{t_{k-1}}),
\]

In the previous calculations, we introduced a new notation \(\psi_{k,k-1}^{(i)}(\beta_0) := \partial_{\beta_i} (N_0(X_{t_k}) - N_0(X_{t_{k-1}}))\). Now, we consider the product \(\eta_{N,k}^{(i)}(\theta_0) \eta_{N,k}^{(j)}(\theta_0)\) and again focus only on the terms with coefficient \(1/Nh\):

\[
\eta_{N,k}^{(i)}(\theta_0) \eta_{N,k}^{(j)}(\theta_0) = \frac{4}{Nh} Z_{t_k}^\top (\Sigma \Sigma_0^\top)^{-1} \partial_{\beta_i} F_0(X_{t_{k-1}}) \partial_{\beta_j} F_0(X_{t_{k-1}})^\top (\Sigma \Sigma_0^\top)^{-1} Z_{t_k}
+ \frac{2}{Nh} Z_{t_k}^\top (\Sigma \Sigma_0^\top)^{-1} \psi_{k,k-1}^{(i)}(\beta_0) \partial_{\beta_j} F_0(X_{t_{k-1}})^\top (\Sigma \Sigma_0^\top)^{-1} Z_{t_k}
+ \frac{2}{Nh} Z_{t_k}^\top (\Sigma \Sigma_0^\top)^{-1} \partial_{\beta_i} F_0(X_{t_{k-1}}) \psi_{k,k-1}^{(j)}(\beta_0)^\top (\Sigma \Sigma_0^\top)^{-1} Z_{t_k}
+ \frac{1}{Nh} Z_{t_k}^\top (\Sigma \Sigma_0^\top)^{-1} \psi_{k,k-1}^{(i)}(\beta_0) \psi_{k,k-1}^{(j)}(\beta_0)^\top (\Sigma \Sigma_0^\top)^{-1} Z_{t_k} + R(1/N, X_{t_{k-1}}).
\]

In the previous equation, we must show that the sum of expectations of all the terms except the first converges to zero. We only prove this for the second row; the rest follows analogously. Due to the definition of \(\psi^{(i)}\), it is clear that \(\mathbb{E}_0[||\psi_{k,k-1}^{(i)}(\beta_0)||^p \mid X_{t_{k-1}}] = R(h, X_{t_{k-1}})\), for all \(p \geq 1\). Then, we use property (S13) to obtain:

\[
\frac{1}{Nh} \mathbb{E}_0[||Z_{t_k}^\top (\Sigma \Sigma_0^\top)^{-1} \psi_{k,k-1}^{(i)}(\beta_0) \partial_{\beta_j} F_0(X_{t_{k-1}})^\top (\Sigma \Sigma_0^\top)^{-1} Z_{t_k} \mid X_{t_{k-1}}]|]
\leq \frac{1}{Nh} \mathbb{E}_0[||\psi_{k,k-1}^{(i)}(\beta_0)||^p \mid X_{t_{k-1}}] \mathbb{E}_0[||Z_{t_k}^\top Z_{t_k}^\top \mid X_{t_{k-1}}|]
\leq C \frac{1}{Nh} \mathbb{E}_0[||Z_{t_k}^\top Z_{t_k}^\top \mid X_{t_{k-1}}|] \mathbb{E}_0[||\psi_{k,k-1}^{(i)}(\beta_0)||^p \mid X_{t_{k-1}}]
\leq \frac{1}{Nh} (R(h^2, X_{t_{k-1}}) R(h, X_{t_{k-1}}))^\frac{p}{2} = R(\sqrt{Nh}/N, X_{t_{k-1}}).
\]
Finally, we use Lemma 4.2 to get:

\[
\sum_{k=1}^{N} \mathbb{E}_{\theta_0}[\tilde{u}(\bar{\theta}_0)_{i_1(\bar{\theta}_0)}^T_{i_2(\bar{\theta}_0)}|X_{t_{k-1}}] = \frac{4}{N} \sum_{k=1}^{N} \left( \mathbb{E}_{\theta_0}[Z_{t_k}^T(\Sigma \Sigma_0^T)^{-1}(\partial_{\bar{\theta}_1} \Sigma \Sigma_0^T)^{-1}Z_{t_k} | X_{t_{k-1}}] + R(h^{3/2}, X_{t_{k-1}}) \right)
\]

\[
= \frac{4}{N} \sum_{k=1}^{N} \left( \text{Tr}(\partial_{\bar{\theta}_1} \Sigma \Sigma_0^T)^{-1} \mathbb{E}_{\theta_0}[Z_{t_k}^T(\Sigma \Sigma_0^T)^{-1}Z_{t_k} | X_{t_{k-1}}] + R(\sqrt{h}/N, X_{t_{k-1}}) \right)
\]

To prove (vii) we use Corollary 3.8:

\[
\mathbb{E}_{\theta_0}[\xi_{N,h,k}(\theta_0)_{i_1(\theta_0)}^T_{i_2(\theta_0)}|X_{t_{k-1}}] = \frac{1}{h^2 N} \mathbb{E}_{\theta_0}[Z_{t_k}^T(\Sigma \Sigma_0^T)^{-1}(\partial_{\bar{\theta}_2} \Sigma \Sigma_0^T)^{-1}Z_{t_k} | X_{t_{k-1}}] = \frac{1}{h^2 N} \text{Tr}((\Sigma \Sigma_0^T)^{-1} \partial_{\bar{\theta}_1} \Sigma \Sigma_0^T)(\Sigma \Sigma_0^T)^{-1} \mathbb{E}_{\theta_0}[\xi_{N,h,k}(\theta_0)_{i_1(\theta_0)}^T_{i_2(\theta_0)}|X_{t_{k-1}}] + R(\sqrt{h}/N, X_{t_{k-1}})
\]

Now, we use the expectation of a product of two quadratic forms of normally distributed random vectors (see for example Section 2 in Kumar (1973)) to get:

\[
\frac{1}{h^2 N} \mathbb{E}_{\theta_0}[\xi_{N,h,k}(\theta_0)_{i_1(\theta_0)}^T_{i_2(\theta_0)}|X_{t_{k-1}}] = \frac{2}{N} \text{Tr}((\Sigma \Sigma_0^T)^{-1} \partial_{\bar{\theta}_1} \Sigma \Sigma_0^T)(\Sigma \Sigma_0^T)^{-1} \xi_{N,h,k} | X_{t_{k-1}}]
\]

This proves (vii). We omit the proofs of (viii)-(xi) since they follow the same pattern. Namely, we find the leading term and ensure it goes to zero. For the expectations of squares, we can apply the same approach with a product of two quadratic forms.

### S3 Auxiliary properties

In this section, we revisit crucial properties essential for establishing the consistency and asymptotic normality of the proposed estimators. To begin, we invoke Lemma 2.3 from Tian and Fan (2020) as Lemma S3.1, which was used in proving Lemma 4.1. This lemma offers a generalization of the Grönwall’s inequality.

Furthermore, Lemma 9 in Genon-Catalot and Jacod (1993) provides conditions for the convergence of a sum of a triangular array and is recalled as Lemma S3.2.

Lemmas S3.3 and S3.4 give sufficient conditions for uniform convergence. The former is sourced from Proposition A1 in Gloter (2006), while the latter comes from Lemma 3.1 from Yoshida (1990). On occasions, Lemma S3.3 might not suffice, warranting the use of Lemma S3.4. Theorem S3.5 is a helpful tool for assessing the conditions of these two lemmas is the Rosenthal’s inequality for martingales (Theorem 2.12 in Hall and Heyde (1980)).

Lastly, Theorem S3.6 presents a special case of the central limit theorem for multivariate martingale triangular arrays (Proposition 3.1 from Crimaldi and Pratelli (2005)). This theorem is pivotal for proving the asymptotic normality of the proposed estimators.

**Lemma S3.1 (Generalized Grönwall’s inequality, Lemma 2.3 in Tian and Fan (2020))** Let \( p > 1 \) and \( b > 0 \) be constants, and let \( a : (0, +\infty) \to (0, +\infty) \) be a continuous function. If

\[
|u(t)| \leq a(t) + b \int_0^t u^p(s) \, ds,
\]

then \( u(t) \leq a(t) + \left( \kappa^{1-p}(t) - (p - 1)2^{p-1}bt \right)^{\frac{1}{p-1}} \) and \( \kappa^{1-p}(t) > (p - 1)2^{p-1}bt, \) where

\[
\kappa(t) := 2^{p-1}b \int_0^t a^p(s) \, ds.
\]
Lemma S3.2 (Lemma 9 in Genon-Catalot and Jacod (1993)) Let \((X^N_k)_{N \in \mathbb{N}, 1 \leq k \leq N}\) be a triangular array with each row \(N\) adapted to a filtration \((\mathcal{G}^N_k)_{1 \leq k \leq N}\), and let \(U\) be a random variable. If
\[
\sum_{k=1}^{N} E[X^N_k \mid \mathcal{G}^N_{k-1}] \xrightarrow{P} U,
\]
then \(\sum_{k=1}^{N} X^N_k \xrightarrow{P} U\).

Lemma S3.3 (Proposition A1 in Gloter (2006)) Let \(S_N(\omega, \theta)\) be a sequence of measurable real-valued functions defined on \(\Omega \times \Theta\), where \((\Omega, \mathcal{F}, P)\) is a probability space, and \(\Theta\) is product of compact intervals of \(\mathbb{R}\). We assume that \(S_N(\cdot, \theta)\) converges to a constant \(C\) in probability for all \(\theta \in \Theta\); and that there exists an open neighbourhood of \(\Theta\) on which \(S_N(\omega, \cdot)\) is continuously differentiable for all \(\omega \in \Omega\). Furthermore, we suppose that:
\[
\sup_{N \in \mathbb{N}} \mathbb{E}[\sup_{\theta \in \Theta} ||\nabla \theta S_N(\theta)||] < \infty.
\]
Then, \(S_N(\theta) \xrightarrow{P} C\) uniformly in \(\theta\).

Lemma S3.4 (Lemma 3.1 in Yoshida (1990)) Let \(F \subset \mathbb{R}^d\) be a convex compact set, and let \(\{\xi_N(\theta); \theta \in F\}\), be a family of real-valued random processes for \(N \in \mathbb{N}\). If there exist constants \(p \geq l > d\) and \(C > 0\) such that for all \(\theta, \theta_1\) and \(\theta_2\), it holds:
(1) \(\mathbb{E}[|\xi_N(\theta_1) - \xi_N(\theta_2)|^p] \leq C||\theta_1 - \theta_2||^l\);
(2) \(\mathbb{E}[|\xi_N(\theta)|^p] \leq C\);
(3) \(\xi_N(\theta) \xrightarrow{P} 0\),
then \(\sup_{\theta \in F} |\xi_N(\theta)| \xrightarrow{P} 0\).

Theorem S3.5 (Rosenthal’s inequality, Theorem 2.12 in Hall and Heyde (1980)) Let \((X^N_k)_{N \in \mathbb{N}, 1 \leq k \leq N}\) be a triangular array with each row \(N\) adapted to a filtration \((\mathcal{G}^N_k)_{1 \leq k \leq N}\) and let:
\[
S_N = \sum_{k=1}^{N} X^N_k, \quad N \in \mathbb{N}
\]
be a martingale array. Then, for all \(p \in [2, \infty)\) there exist constants \(C_1, C_2\) such that:
\[
C_1(\mathbb{E}(\sum_{k=1}^{N} [E[(X^N_k)^2 \mid \mathcal{G}^N_{k-1}]]^{\frac{p}{2}}) + \sum_{k=1}^{N} E[[|X^N_k|^p]] \leq E[||S_N|^p] \leq C_2(\mathbb{E}(\sum_{k=1}^{N} [E[(X^N_k)^2 \mid \mathcal{G}^N_{k-1}]]^{\frac{p}{2}}) + \sum_{k=1}^{N} E[[|X^N_k|^p]])
\]

Theorem S3.6 (Proposition 3.1. in Crimaldi and Pratelli (2005)) Let \((X_{N,k})_{N \in \mathbb{N}, 1 \leq k \leq N}\) be a triangular array of \(d\)-dimensional random vectors, such that, for each \(N\), the finite sequence \((X_{N,k})_{1 \leq k \leq N}\) is a martingale difference array with respect to a given filtration \((\mathcal{G}^N_k)_{1 \leq k \leq N}\) such that:
\[
S^*_N = \sum_{k=1}^{N} X_{N,k}, \quad N \in \mathbb{N}
\]
If
(1) \(\mathbb{E}[\sup_{1 \leq k \leq N} \|X_{N,k}\|_1] \xrightarrow{N \to \infty} 0\);
(2) \(\sum_{k=1}^{N} X_{N,k}X_{N,k}^\top \xrightarrow{P} U\), for some non-random positive semi-definite matrix \(U\),
then, \(S^*_N \xrightarrow{d} N_d(0, U)\).
Remark Instead of using the second condition of Theorem S3.6, Lemma S3.4 yields that it is sufficient to prove that, for all \(i, j = 1, \ldots, d\), it holds:

\[
\sum_{k=1}^{N} \mathbb{E}[X_{N,k}^{(i)} X_{N,k}^{(j)} | G_{k-1}^{N}] \xrightarrow{P} U_{ij}, \quad \sum_{k=1}^{N} \mathbb{E}[(X_{N,k}^{(i)} X_{N,k}^{(j)})^2 | G_{k-1}^{N}] \xrightarrow{P} 0.
\]

Remark For a martingale difference array the conditional expectations need to be zero almost surely, i.e:

\[
\mathbb{E}[X_{N,k} | G_{k-1}^{N}] = 0, \text{ a.s. for all } N \in \mathbb{N}, \ 1 \leq k \leq N.
\]

In our case, \((X_{N,k})_{N \in \mathbb{N}, 1 \leq k \leq N}\) does not fulfill the previous condition. Hence, similar to the approach in Corollary 2.6 of McLeish (1974), we need the following two additional conditions on \((X_{N,k})_{N \in \mathbb{N}, 1 \leq k \leq N}\):

\[
\sum_{k=1}^{N} \mathbb{E}[X_{N,k}^{(i)} | G_{k-1}^{N}] \xrightarrow{P} 0, \quad \sum_{k=1}^{N} \mathbb{E}[X_{N,k}^{(i)} | G_{k-1}^{N}][X_{N,k}^{(j)} | G_{k-1}^{N}] \xrightarrow{P} 0.
\]

Indeed, martingale difference array \(Y_{N,k} = X_{N,k} - \mathbb{E}[X_{N,k} | G_{k-1}^{N}]\) satisfies conditions of the previous theorem. To prove that the first condition is satisfied, we write:

\[
\mathbb{E}[\sup_{1 \leq k \leq N} \|Y_{N,k}\|_1] \leq \mathbb{E}[\sup_{1 \leq k \leq N} \|X_{N,k}\|_1] + \mathbb{E}[\sup_{1 \leq k \leq N} \mathbb{E}[\|X_{N,k}\|_1 | G_{k-1}^{N}]]
\]

\[
\leq \mathbb{E}[\sup_{1 \leq k \leq N} \|X_{N,k}\|_1] + \mathbb{E}[\sup_{1 \leq j \leq N} \mathbb{E}[\|X_{N,j}\|_1 | G_{k-1}^{N}]] \leq 3\mathbb{E}[\sup_{1 \leq k \leq N} \|X_{N,k}\|_1] \xrightarrow{N \to \infty} 0.
\]

We used the Doob’s inequality for the last submartingale. To demonstrate the second condition we fix \(i, j\) to get:

\[
\sum_{k=1}^{N} Y_{N,k}^{(i)} Y_{N,k}^{(j)} = \sum_{k=1}^{N} X_{N,k}^{(i)} X_{N,k}^{(j)} - \sum_{k=1}^{N} X_{N,k}^{(i)} \mathbb{E}[X_{N,k}^{(j)} | G_{k-1}^{N}]
\]

\[
- \sum_{k=1}^{N} X_{N,k}^{(i)} \mathbb{E}[\mathbb{E}[X_{N,k}^{(j)} | G_{k-1}^{N}]] + \sum_{k=1}^{N} \mathbb{E}[X_{N,k}^{(i)} | G_{k-1}^{N}] \mathbb{E}[X_{N,k}^{(j)} | G_{k-1}^{N}].
\]

The first term goes to \(U_{ij}\), and the last term goes to zero. To prove that middle terms also vanish, we use the following inequalities:

\[
\left| \sum_{k=1}^{N} X_{N,k}^{(i)} \mathbb{E}[X_{N,k}^{(j)} | G_{k-1}^{N}] \right| \leq \sum_{k=1}^{N} |X_{N,k}^{(i)}| \mathbb{E}[|X_{N,k}^{(j)}| | G_{k-1}^{N}]
\]

\[
\leq \left( \sum_{k=1}^{N} (X_{N,k}^{(i)})^2 \right) \sum_{k=1}^{N} \mathbb{E}[|X_{N,k}^{(j)}| | G_{k-1}^{N}] \xrightarrow{N \to \infty} 0.
\]

Theorem S3.6 yields that \(\sum_{k=1}^{N} Y_{N,k} \xrightarrow{d} N_{d}(0, \mathbb{I})\), which together with (S29), gives \(\mathcal{S}_N \xrightarrow{d} N_{d}(0, \mathbb{I})\).

### S4 Estimators

In this section, we treat the computation of integrals involving matrix exponentials, using formulas from (Van Loan, 1978) and apply it to the LL estimator, following (Gu et al., 2020). In the main paper, we extend this approach to calculate \(\Omega_h\) for the splitting schemes.

Additionally, we present the coefficients for the HE log-likelihood expansion up to order \(J = 2\) for the Lorenz system, with our gratitude to the third reviewer for providing these formulas. The section concludes with a detailed analysis of the simulation results for the HE method.

#### S4.1 Ozaki’s local linearization

Building on the approach by Gu et al. (2020), we can efficiently compute \(R_{h,i}\) and \(\Omega_{h,k}^{[LL]}(\theta)\) using the following procedure. To begin, define the three block matrices:

\[
P_1(x) = \begin{bmatrix} 0_{d \times d} & I_d \\ F^{\top}(x; \beta) \end{bmatrix}, \quad P_2(x) = \begin{bmatrix} -DF(x; \beta) \\ 0_{d \times d} \end{bmatrix}, \quad P_3(x) = \begin{bmatrix} DF(x; \beta) \\ 0_{d \times d} \end{bmatrix}, \quad P_4(x) = \begin{bmatrix} 0_{d \times d} \\ DF(x; \beta) \end{bmatrix}.
\]

(S30)
Then, we compute the matrix exponential of matrices $h\mathbf{P}_1(x)$ and $h\mathbf{P}_2(x)$:

$$
\exp(h\mathbf{P}_1(x)) = \begin{bmatrix}
\mathbf{R}_{h,0}(DF(x;\beta)) & \mathbf{B}_{h,0}(DF(x;\beta)) \\
0_{d \times d} & \mathbf{R}_{h,1}(DF(x;\beta)) \\
\end{bmatrix},
\quad
\exp(h\mathbf{P}_2(x)) = \begin{bmatrix}
\mathbf{R}_{h,0}(DF(x;\beta)) & \mathbf{B}_{h,1}(DF(x;\beta)) \\
0_{d \times d} & \mathbf{R}_{h,1}(DF(x;\beta)) \\
\end{bmatrix}.
$$

The terms marked with * symbols can be disregarded. Starting with the first matrix, we derive $\mathbf{R}_{h,0}(DF(x;\beta))$. Then, we compute $\mathbf{R}_{h,1}(DF(x;\beta))$ using the formula $\mathbf{R}_{h,1}(DF(x;\beta)) = \exp(hDF(x;\beta))\mathbf{B}_{h,1}(DF(x;\beta))$. Finally, we obtain $\Omega^{[\text{LL}]}_{h,k}(\theta)$ from the matrix exponential:

$$
\exp(h\mathbf{P}_3(x)) = \begin{bmatrix}
\mathbf{B}_{\Omega_{h,k}}(DF(x;\beta);\theta) & \mathbf{C}_{\Omega_{h,k}}(DF(x;\beta);\theta) \\
0_{d \times d} & \mathbf{R}_{h,1}(DF(x;\beta)) \\
\end{bmatrix},
\quad
\Omega^{[\text{LL}]}_{h,k}(\theta) = \mathbf{C}_{\Omega_{h,k}}(DF(x;\beta);\theta)\mathbf{B}_{\Omega_{h,k}}(DF(x;\beta;\theta))^\top.
$$

### S4.2 Aït-Sahalia’s Infinite Hermite Expansion

Polynomial coefficients $C^{(j)}_Y(\gamma(X_{i,j}) \mid \gamma(X_{i,j-1}))$, for $j = -1, 0, 1, \ldots, J$ are calculated recursively according to Theorem 1 in (Aït-Sahalia, 2008). In the following, we present $C^{(j)}_Y$ for the Lorenz system up to order $J = 2$ (provided by the third reviewer):

\[
C^{(-1)}_Y(\gamma(x, y, z) \mid \gamma(x_0, y_0, z_0)) = \frac{1}{2} \left( \frac{(x - x_0)^2}{\sigma_1^2} + \frac{(y - y_0)^2}{\sigma_2^2} + \frac{(z - z_0)^2}{\sigma_3^2} \right);
\]

\[
C^{(0)}_Y(\gamma(x, y, z) \mid \gamma(x_0, y_0, z_0)) = \frac{1}{3} (x - x_0)(y - y_0)(z - z_0) \left( -\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2} \right) + \frac{1}{2} \sigma_x^2 (y - y_0)(z - z_0) \left( -\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2} \right) + \frac{1}{2} \sigma_y^2 (x - x_0)(z - z_0) \left( -\frac{1}{\sigma_1^2} + \frac{1}{\sigma_3^2} \right) + \frac{1}{2} \sigma_z^2 (x - x_0)(y - y_0) \left( -\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right);
\]

\[
C^{(1)}_Y(\gamma(x, y, z) \mid \gamma(x_0, y_0, z_0)) = \frac{1}{24} (x - x_0)^2 \left( \frac{p^2\sigma_2^2 - 4p^2 + 2p(r - z_0)^2}{\sigma_1^2} - \frac{3(r - z_0)^2}{\sigma_2^2} - \frac{3y_0^2}{\sigma_3^2} \right) + \frac{1}{24} (y - y_0)^2 \left( \frac{\sigma_x^2 (r - z_0)^2 + \sigma_y^2 x_0^2}{\sigma_1^2} - \frac{3p^2}{\sigma_1^2} + \frac{2(x_0 - p(r - z_0) - 2)}{\sigma_2^2} - \frac{3x_0^2}{\sigma_3^2} \right) + \frac{1}{24} (z - z_0)^2 \left( \frac{\sigma_x^2 y_0^2 + \sigma_y^2 x_0^2}{\sigma_1^2} - \frac{3x_0^2}{\sigma_2^2} + \frac{2(x_0 - 2c_2)}{\sigma_3^2} \right) + \frac{1}{12} (x - x_0)(y - y_0) \left( \frac{4p^2 + x_0 y_0 + 4(r - z_0)}{\sigma_1^2} - \frac{7x_0 y_0 - 4c z_0}{\sigma_2^2} \right) + \frac{1}{12} (y - y_0)(z - z_0) \left( \frac{\sigma_x^2 y_0 (r - z_0)}{\sigma_1^2} - \frac{4x_0}{\sigma_2^2} + \frac{py_0 - 4c x_0}{\sigma_3^2} \right) + \frac{1}{12} (x - x_0)(z - z_0) \left( \frac{px_0 \sigma_2^2}{\sigma_1^2 \sigma_3^2} + \frac{px_0 \sigma_1^2}{\sigma_2^2 \sigma_3^2} - \frac{4y_0 + 7x_0 (r - z_0)}{\sigma_2^2} + \frac{4c y_0 - x_0 (r - z_0)}{\sigma_3^2} \right) + \frac{1}{2} (x - x_0) \left( \frac{p^2(x_0 + y_0)}{\sigma_1^2} + \frac{z_0 (r - z_0) - y_0 (r - z_0) - y_0 (x_0 y_0 + c z_0)}{\sigma_2^2} - \frac{y_0 (x_0 y_0 + c z_0)}{\sigma_3^2} \right) + \frac{1}{2} (y - y_0) \left( \frac{p^2(x_0 - y_0)}{\sigma_1^2} + \frac{x_0 (r - z_0) - y_0 (r - z_0) + x_0 (x_0 y_0 + c z_0)}{\sigma_2^2} - \frac{y_0 (x_0 y_0 + c z_0)}{\sigma_3^2} \right) + \frac{1}{2} (z - z_0) \left( \frac{x_0 (y_0 + x_0 (r - z_0))}{\sigma_1^2} + c (x_0 y_0 - c z_0) \right) + \frac{1}{2} \left( 1 + p + \frac{p^2(x_0 - y_0)^2}{\sigma_1^2} - \frac{(x_0 y_0 - c z_0)^2}{\sigma_2^2} - \frac{(-x_0 (r - z_0) + y_0)^2}{\sigma_3^2} \right);\]
Figure 7: Comparing S (red) and HE (blue) objective functions of a data set generated from the Lorenz system where all parameters except $\sigma_3^2$ are fixed to the true values. The sample size is fixed to $N = 10000$. Each row represents one value of the discretization step $h$. The black vertical dashed line is the true value of $\sigma_3^2$.

The poor performance of the HE estimator (no convergence for larger discretization step sizes $h$, and only $\approx 43 – 72\%$ convergence for small $h$) in the simulation study can probably be attributed to the polynomial approximation of the likelihood function, which can become unstable, particularly for larger $h$, as illustrated in Figure 7. Additional coefficients $C^{(2)}_y$ in the approximation might mitigate this problem.

Figure 7 shows the objective functions of HE and S for a fixed trajectory, $h$, and $N$, with all parameters fixed to their true values except for $\sigma_3^2$. Consequently, the objective functions are presented as functions of $\sigma_3^2$. The HE function tends towards $-\infty$ as $\sigma_3^2$ approaches zero. This is also the case for the smallest $h$, although it is not evident in the figure due to the $x$-scale used. However, in this case the objective function do possess a local minimum close to the true value. As a result, the global minimum of the HE objective function is always at $-\infty$. For sufficiently small $h$, this issue can be mitigated by imposing constraints on $\sigma_3^2$. However, as $h$ increases, the local minimum vanishes. In contrast, the objective functions of other estimators like S tend towards $+\infty$, when $\sigma_3^2$ goes to zero, ensuring that the minimum around the true value of $\sigma_3^2$ is also the global minimum of their objective functions.