No-arbitrage implies power-law market impact and rough volatility

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Abstract
Market impact is the link between the volume of a (large) order and the price move during and after the execution of this order. We show that in a quite general framework, under no-arbitrage assumption, the market impact function can only be of power-law type. Furthermore, we prove this implies that the macroscopic price is diffusive with rough volatility, with a one-to-one correspondence between the exponent of the impact function and the Hurst parameter of the volatility. Hence, we simply explain the universal rough behavior of the volatility as a consequence of the no-arbitrage property. From a mathematical viewpoint, our study relies, in particular, on new results about hyper-rough stochastic Volterra equations.

KEYWORDS
Hawkes processes, hyper-rough Heston model, market impact, no-arbitrage property, rough Heston model, rough volatility

1 | INTRODUCTION
It is now well admitted that volatility is rough. This stylized fact first established in Gatheral, Jaisson, and Rosenbaum (2018) and confirmed in Bennedsen, Lunde, and Pakkanen (2016) and Livieri, Mouti, Pallavicini, and Rosenbaum (2018) means that the (log-)volatility process of an asset essentially behaves as a fractional Brownian motion (fBm for short) with Hurst parameter of order 0.1. Recall that a fBm \((W_t^H)_{t\geq0}\) with Hurst parameter \(H \in (0, 1)\) is a Gaussian process that can be written under the Mandelbrot-van Ness representation as

\[
W_t^H = \int_{-\infty}^{0} (t - s)^{H - \frac{1}{2}} - (-s)^{\frac{1}{2}} dB_s + \int_0^t (t - s)^{H - \frac{1}{2}} dB_s,
\]
with $(B_t)_{t \geq 0}$ a classical Brownian motion. For any $\epsilon > 0$, the sample paths of $(W^H_t)_{t \geq 0}$ are almost surely $H - \epsilon$ Hölder (and not $H$ Hölder). Therefore, the trajectories are very rough when $H$ is small.

Various rough volatility models have been recently introduced in the literature, notably for the purpose of risk management of derivatives, such as the rough Heston model of El Euch and Rosenbaum (2019), where the asset price $(P_t)_{t \geq 0}$ satisfies

$$\frac{dP_t}{P_t} = \sqrt{V_t}\left(\rho dB^1_t + \sqrt{1 - \rho^2} dB^2_t\right),$$

with

$$V_t = V_0 + \frac{\lambda}{\Gamma\left(H + \frac{1}{2}\right)} \int_0^t (t - s)^{-H - \frac{1}{2}} (\theta(t) - V_s) ds + \frac{\nu}{\Gamma\left(H + \frac{1}{2}\right)} \int_0^t (t - s)^{-H - \frac{1}{2}} \sqrt{V_s} dB^1_s,$$ (1)

where $B^1$ and $B^2$ are independent Brownian motions, $\lambda$ and $\nu$ two positive constants, $\theta$ a deterministic nonnegative function, and $\rho \in (-1, 1)$ a correlation factor. The particular interest of this model is that, as for the classical Heston model, semiexplicit pricing and hedging formulas can be obtained (see El Euch and Rosenbaum (2018, 2019)).

A puzzling question is the origin of the universal rough volatility property of financial assets. A first explanation is proposed in El Euch, Fukasawa, and Rosenbaum (2018). In this work, the authors place themselves in a highly endogenous market, meaning that most orders are sent in reaction to other orders and without economic motivation. They show that in this context, the widely used trading practice of metaorders splitting (see below for definition of a metaorder) leads to the rough Heston dynamic (1) for the macroscopic price. However, this result is found using a quite specific parametric model for the high frequency price.

In this paper, we wish to obtain a fundamental explanation underlying the rough volatility property. In fact, we prove that in a quite general framework, rough volatility is simply a consequence of the no-arbitrage principle together with the existence of market impact.

Market impact is the fact that on average, a buy order moves the price up and a sell order moves the price down. There are two main viewpoints on market impact. The first one is to consider that agents receive information and reveal their information through transactions, hence impacting the market. The other one is purely mechanical, not referring to any notion of information, and considering that prices move up and down only through volume pressures, following supply and demand. We adopt the mechanical paradigm in this work. The impact of a single order being very difficult to assess, one usually considers large sets of orders split by brokers, so-called metaorders. Empirical studies of market impact have shown that for a buy metaorder (and symmetrically for a sell metaorder) market impact can be decomposed in two phases: a transient phase with a concave rise of the price during the metaorder execution, and a decay phase, where the price decreases toward a long-term level after the execution is completed (see Bacry, Iuga, Lasnier, & Lehalle, 2015; Donier & Bonart, 2015; Lillo, Farmer, & Mantegna, 2003; Tóth et al., 2011).

Let us consider a buy (say) metaorder and let $(q_t)_{t \geq 0}$ be the cumulative volume of this metaorder executed between the initial time 0 and time $t$. The market impact function of this metaorder is defined as

$$MI(t) = \mathbb{E}\left[P_t^{(q_t \leq t)} - P_0\right],$$
where we put the superscript \((q_s)_t\), on \(P\) to insist on the fact that the price dynamic depends on the execution process of the metaorder. Of course the above formula only makes sense in a model where \(P_t^{(q_s)_t}\) is a well-defined stochastic process, as will be the case in the next sections.

The permanent market impact (PMI) of this metaorder is given by the quantity

\[
\text{PMI} = \lim_{t \to +\infty} \text{MI}(t).
\]

Intuitively, it is quite clear that in the long run, the permanent impact of a metaorder with volume \(Q\) should be the same as that of two consecutive orders of volume \(Q/2\). This is formalized in Theorem 1 in Huberman and Stanzl (2004) and further developed in Gatheral (2010), where it is shown that under mild modeling assumptions, the absence of price manipulation on a market implies that the PMI is proportional to the total volume of the metaorder. In particular, it does not depend on the metaorder execution strategy. From now on, we take this linear PMI property as granted. This has consequences for the price dynamics. In particular, we now assume that the price \(P\) is a martingale. We take this martingale hypothesis as a simplifying and convenient version of the classical mathematical finance condition of no almost sure arbitrage, which states that price should be a semimartingale. In this setting, it is shown in Theorem 2.1 in Jaisson (2015) that under the purely mechanical view for market impact,

\[
P_t = \lim_{s \to +\infty} \mathbb{E}[V_s^a - V_s^b | \mathcal{F}_s],
\]

where \(V^a\) (respectively, \(V^b\)) is the cumulated volume of buy (respectively, sell) market orders because the initial time 0 and \((\mathcal{F}_s)_{s \geq 0}\) corresponds to the filtration generated by the order flow process. Hence, the price moves when orders arrive on the market because market participants revise their anticipation about the long-term cumulative imbalance of the order flow. Remark that to derive (2), we only use the set of assumptions that we consider here as our no arbitrage conditions: martingale price and linear PMI.

As for the transient part of the market impact, empirical measurements show that provided the execution rate of the metaorder is relatively constant, the function \(\text{MI}\) is close to a power-law with respect to time, that is \(\text{MI}(t) \sim t^{1-\alpha}\) with \(\alpha \in (0, 1)\) (see Bacry, Iuga, et al., 2015; Bouchaud, 2010; Lillo et al., 2003; Tóth et al., 2011). More precisely, the coefficient \(\alpha\) is found to be about \(1/2\) so that the so-called square root law is approximately satisfied. Actually, it is proved in Pohl, Ristig, Schachermayer, and Tangpi (2017) that under some leverage neutrality assumption, the square root law can be simply derived from dimensional analysis.

We show in this work that under no-arbitrage assumption (represented by the linear permanent impact, the martingale price and thus (2)), the market impact function has indeed to be a power-law of the form \(\text{MI}(t) \sim t^{1-\alpha}\). Then we prove that for any \(\alpha \in (0, 1)\), the scaling limit of the price (2) exists and satisfies

\[
\hat{P}_t = B_{X_t},
\]

with

\[
X_t = \frac{2}{\delta} \int_0^t F^{a,\lambda}(s) ds + \frac{1}{\delta \sqrt{\lambda}} \int_0^t F^{a,\lambda}(t-s) dW_s,
\]

where \(W\) and \(B\) are two Brownian motions, \(\delta\) and \(\lambda\) two positive constants, and \(F^{a,\lambda}\) is the Mittag-Leffler cumulative distribution function (see Appendix A.1 for definition). The correlation between the Brownian motions \(B\) and \(W\) is stochastic and related to the order flow imbalance. The above
equation is a generalization of the rough Heston model (1). Indeed, we can show that when \( \alpha > 1/2 \), after differentiation, Equation (3) can be rewritten under the form of (1) (up to a stochastic correlation factor) with associated Hurst parameter \( H = \alpha - 1/2 \). For \( \alpha \leq 1/2 \), we prove that \( X \) is not continuously differentiable but has Hölder regularity \( 2\alpha - \varepsilon \) for any \( \varepsilon > 0 \). Therefore, we call (3) the \textit{hyper-rough Heston model} when \( \alpha \leq 1/2 \). Hence, we are able to define rough Heston models for Hurst parameter in \( (-1/2, 1/2] \).

To obtain our results, we only need to specify a model for the order flow dynamics. We indeed see from Equation (2) that we do not need to model metaorders individually. Only the aggregated order flow matters in order to derive the price dynamic. More precisely, we consider for buy and sell market order arrivals two independent Hawkes processes \( N^a \) and \( N^b \) and assume that each order is of unit size, see (El Euch et al., 2018; Jaisson & Rosenbaum, 2016). Recall that a Hawkes process \( N \) is a self-exciting point process whose intensity \( \lambda(t) = \mu + \int_0^t \phi(t-s) dN_s \), with \( \mu \) a positive constant and \( \phi \) a nonnegative locally integrable function. Such dynamic is a generalization of the Poisson process that is usually considered when modeling order flows (see, among others, Cont & De Larrard, 2013; Cont, Stoikov, & Talreja, 2010; Smith, Farmer, Gillemot, & Krishnamurthy, 2003). It is nonparametric and very flexible so that it is really reasonable to assume that the actual order flow can be well approximated by a Hawkes based model (see Bacry, Mastromatteo, & Muzy, 2015; Bacry, Jaisson, & Muzy, 2016). Note that we will not put any restriction on the Hawkes parameters \( \mu \) and \( \phi \), except the nonnegativity of \( \mu \), the local integrability of \( \phi \) and the fact that they are the same for the buy and sell flows. In this case, it is shown in Jaisson (2015) that the price process (2) satisfies

\[
P_t = P_0 + \int_0^t \xi(t-s) d(N^a_s - N^b_s),
\]

with

\[
\xi(t) = 1 + (1 + \int_0^{+\infty} \psi(u) du) \int_t^{+\infty} \phi(u) du
\]

and

\[
\psi = \sum_{i \geq 1} (\phi)^{\ast i},
\]

where \((\phi)^{\ast 1} = \phi\) and for \( k \geq 2 \), \((\phi)^{\ast k}\) denotes the convolution product of \((\phi)^{\ast (k-1)}\) with \( \phi \).

Using a rescaling procedure to describe the macroscopic behavior of (4), we show that only one very subtle specification of the Hawkes processes can lead to a nontrivial market impact, which has to be power-law. Furthermore, it implies that the market is highly endogenous. In addition, depending on the market impact shape, the scaling limit of the price is a rough or hyper-rough Heston model (3), with a one-to-one correspondence between the exponent of the impact function and the Hurst parameter of the volatility.

The paper is organized as follows. In Section 2, we show that under the assumption that the market impact function is not degenerate, it can only be a power-law with parameter \( 1 - \alpha \) for some \( \alpha \in (0, 1) \). Then in Section 3, we explain that the macroscopic limit of (4) is a rough or hyper-rough Heston model with Hurst parameter \( H = \alpha - 1/2 \).
2 | MARKET IMPACT IS POWER-LAW

In this section, we show that if there exists a non-degenerate market impact function, it has to be a power-law. Moreover, we will see that it implies a highly endogenous market. By non-degenerate, we essentially mean a market impact function that is ultimately decreasing for buy metaorders (and conversely for sell metaorders), see Assumption 2.1. This is the formalization of the two phases behavior of market impact discussed in Section 1.

2.1 | Asymptotic framework and metaorders modeling

Let $T$ be our final horizon time for the metaorders we will define in the sequel. Recall that the market order flow on $[0, T]$ (and after $T$) is given by two Hawkes processes with the same parameters, $N^a$ for the buy market orders and $N^b$ for the sell orders, with respective intensities $\lambda^a$ and $\lambda^b$. As the time-length of a metaorder is typically large compared to the interarrivals of individual market orders, it is natural to consider that $T$ goes to infinity.

We want to work in a general setting that enables us to be compatible with empirical studies showing that markets are highly endogenous. In the Hawkes process context, the degree of endogeneity of the market is measured by the $L^1$ norm of $\phi$, denoted by $\|\phi\|_1$ (see Filimonov & Sornette, 2015; Hardiman, Bercot, & Bouchaud, 2013; Jaisson & Rosenbaum, 2015, 2016). Therefore, a highly endogenous market corresponds to the case where $\|\phi\|_1$ is close but smaller than unity. So, we allow the model parameters to possibly depend on $T$. Thus, from now on, we use the superscript $T$ for all quantities that could depend on $T$. In particular, $\|\phi_T\|_1$ may go to one as $T$ tends to infinity. We also write $N^{a,T}$, $N^{b,T}$, $\mu_T$, $\phi_T$ to describe the market order flow and model parameters corresponding to the time-horizon $T$, and we set $\phi_T = a_T \phi$ for $\phi$ a nonnegative function such that $\|\phi\|_1 = 1$ and $(a_T)_{T \geq 0}$ a real sequence in $(0,1)$. Note that we do not impose that $a_T$ goes to one. In fact, we will show that one does need to have $a_T$ tends to one for the existence of a non-degenerate market impact function.

We finally need to define a formalism for a sequence of buy (say) metaorders that will be added to the global order flow. We assume that a metaorder is split through market orders of size one over $[0, T]$. In the spirit of Jaisson (2015), we consider that the arrival times of the market orders are given by a nonhomogeneous Poisson process with intensity $\nu^T(t) = I^T f\left(\frac{t}{T}\right)$, where $f$ is a nonnegative continuous function on $[0,1]$ with integral one and $I^T$ a sequence of nonnegative real numbers such that the expected total volume of the metaorder is $I^T T$. The order of magnitude of its duration is $T$. Note that this is slightly smaller than $T$ as the metaorder will end after the last jump time of the Poisson process before $T$. We allow $f$ to be different from a constant to get more realistic splitting schemes than those given by constant rate Poisson processes. Suitable choices for $f$ may be exponentially decaying (arrival price benchmark) or linearly decaying (VWAP benchmark; see, e.g., Almgren & Chriss, 2001; Huang, Lehalle, & Rosenbaum, 2015).

To compute the market impact function in practice, one typically considers the empirical mean of the price movements over many metaorders with similar durations and volumes counted in proportion of the total traded volume. So, in our setting, it is natural to take $I^T \times T$ (the order of magnitude of the total volume of our metaorder) essentially proportional to the total number of other orders executed over $[0, T]$. To do so, we take the intensities proportional

$$I^T = \gamma \beta^T, \quad \text{with} \quad \beta^T = \mu^T (1 - a_T)^{-1},$$
where $\gamma < 1$ and $\beta^T$ is the long-term average intensity of the Hawkes process $N^{a,T}$ ($\beta^T = \lim_{t \to +\infty} (1/t) \int_0^t \lambda^a_s ds$, see Bacry, Delattre, Hoffmann, & Muzy, 2013). Thus, the proportion of the order flow that is due to the considered metaorder is essentially $\gamma/(1 + \gamma)$ and $\gamma$ will be considered reasonably small.

### 2.2 Market impact in the Hawkes setting

In this section, the parameter $T$ is fixed. Assuming that the volume of our metaorder is small enough, the total order flow is not deeply modified by it. Hence, other agents do not observe significant changes in the order flow dynamics. So, the way the market reacts to the incoming orders remains unchanged. Recall that in our model, the market reaction to the order flow (without our metaorder) is given by (4).

We work under the setting of the previous section assuming that the number of shares bought through our metaorder is a nonhomogeneous Poisson process $(n^T_t)_{t \geq 0}$. Therefore, we obtain

$$P^T_t = P_0 + \int_0^t \xi^T(t-s) d(N^{a,T} - N^{b,T} + n^T_s),$$

where $(N^{a,T}_t, N^{b,T}_t)_{t \geq 0}$ corresponds to the aggregated order flows of all other agents. Indeed, all orders being anonymous in the market, our metaorder cannot be distinguished from the global order flow. Thus, the market digests the order flow through the kernel $\xi$, as if it is a bivariate Hawkes process with parameters $\mu$ and $\phi$ ($\gamma$ being small).

Now, we are in the position to properly compute the market impact function of our metaorder. We have

$$MI^T(t) = \mathbb{E}[P^T_t - P_0] = \int_0^t \xi^T(t-s) \mathbb{E}[dn^T_s].$$

This equation together with (5) shows that for any $t \geq 0$, the market impact function can be decomposed into two terms as follows:

$$MI^T(t) = PMI^T(t) + TMI^T(t),$$

where

$$PMI^T(t) = \mathbb{E}[n^T_t]$$

and

$$TMI^T(t) = \int_0^t \Gamma^T(t-s) \mathbb{E}[dn^T_s],$$

with

$$\Gamma^T(s) = (1 - a^T)^{-1} \int_s^{+\infty} \phi^T(u) du,$$

where we have used the fact that

$$\int_0^{+\infty} \psi^T(u) du = \sum_{k \geq 1} \left( \int_0^{+\infty} \phi^T(u) du \right)^k = \frac{a^T}{1 - a^T}.$$
Note that the definition of \( PMI^T(t) \) is compatible with that of the permanent market impact \( PMI \) given in Section 1. Indeed, the order intensity from our metaorder being eventually null and because \( \Gamma^T(t) \) tends to zero as \( t \) goes to infinity, we get

\[
\lim_{t \to +\infty} TMI^T(t) = 0.
\]

The effect of the term \( TMI^T \) is thus only temporary. That is why this term is called transient part of the market impact.

### 2.3 Scaling limit of the market impact

We now rescale the market impact function as the horizon time \( T \) goes to infinity. If the sequence of rescaled market impact functions converges, we call its limit macroscopic market impact function.

First we reparameterize in time and consider \( (MI^T(f,tT))_{t \in \mathbb{R}^+} \) (we put the function \( f \) as parameter of \( MI^T \) to insist on the fact that the market impact function depends on the metaorder strategy). Thus, \( t = 1 \) corresponds essentially to the end of the metaorder. Regarding the scaling in space, because in our framework the size of a metaorder is measured relatively to the total volume, which is of order \( T^\beta \) on \([0,T]\), we finally define our rescaled market impact function \( \overline{MI}^T \) on \( \mathbb{R}^+ \) by

\[
\overline{MI}^T(f,t) = \frac{MI^T(f,tT)}{T^\beta} = \overline{PMI}^T(f,t) + \overline{TMI}^T(f,t),
\]

with

\[
\overline{PMI}^T(f,t) = \gamma \int_0^t f(x)dx
\]

the rescaled permanent impact and

\[
\overline{TMI}^T(f,t) = \gamma \frac{a^T(1-a^T)^{-1}}{T} \int_0^{Tt} f(t-x/T) \int_x^{+\infty} \phi(u)du dx
\]

the rescaled transient impact. Remark that the permanent impact term does not depend on \( T \). Thus, there always exists a macroscopic PMI function and the convergence of the sequence \( (\overline{MI}^T(f,\cdot))_{T \geq 0} \) is equivalent to that of \( (\overline{TMI}^T(f,\cdot))_{T \geq 0} \). Motivated by the empirical results on market impact (Bacry, Iuga, et al., 2015; Bouchaud, 2010; Gomes & Waelbroeck, 2015; Lillo et al., 2003; Potters & Bouchaud, 2003) discussed in Section 1, we make the following natural assumption.

**Assumption 2.1.** For constant execution rate, that is \( f = \mathbf{1}_{[0,s]} \) for some \( s \in (0,1] \), the scaling limit of the market impact function exists pointwise and is nonincreasing after time \( s \). Furthermore, there exists \( t > s \) such that the value of this limiting function at time \( t \) is smaller than that at time \( s \).

There do exist some model parameters such that Assumption 2.1 is satisfied. For example, any kernel \( \phi \) such that \( \phi(t) \sim_{+\infty} ct^{-\alpha-1} \) with \( c > 0 \). Assumption 2.1 implies that for \( f = \mathbf{1}_{[0,s]} \) with \( s \in (0,1] \), we can define the macroscopic market impact function \( \overline{MI}(f,t) \) and its transient and permanent components \( \overline{TMI}(f,t) \) and \( \overline{PMI}(f,t) \) as

\[
\overline{MI}(f,t) = \lim_{T \to +\infty} \overline{MI}^T(f,t), \quad \overline{TMI}(f,t) = \lim_{T \to +\infty} \overline{TMI}^T(f,t), \quad \overline{PMI}(f,t) = \lim_{T \to +\infty} \overline{PMI}^T(f,t).
\]

Using Tauberian theorems, see Appendix A.2, we obtain the following result.
Theorem 2.2. Under Assumption 2.1, for any nonnegative function \( f \) defined on \( \mathbb{R}^+ \), continuous on \([0,1]\) and supported on \([0,1]\), the macroscopic market impact function and its transient part exist. More precisely, there exists a parameter \( \alpha \in (0, 1) \) such that for any \( t > 0 \), when \( \alpha < 1 \),

\[
\lim_{T \to +\infty} \frac{T}{T} TMI_T(f, t) = \gamma K(1 - \alpha) \int_0^t f(t - u)u^{-\alpha} du,
\]

for some \( K > 0 \), and when \( \alpha = 1 \)

\[
\lim_{T \to +\infty} TMI_T(f, t) = \gamma K f(t).
\]

Furthermore, the Hawkes kernel \( \phi \) necessarily satisfies

\[
\int_0^t \int_s^{+\infty} \phi(u)du \, ds = t^{1-\alpha} L(t),
\]

where \( L \) is a slowly varying function (see definition in Appendix A.2). Finally, we necessarily have

\[
(1 - a_T)^{-1}T^{-\alpha} L(T) \to K,
\]

and consequently \( a_T \to 1 \) (see Proposition A.2 in Appendix A.2).

Considering, for example, \( f = 1_{[0,1]} \), Theorem 2.2 shows that under no-arbitrage together with the assumption of the existence of the macroscopic market impact function, the transient part of the market impact is power-law while the permanent part is linear. Moreover, Equation (6) gives that the decay of the market impact is essentially a power-law with exponent \(-\alpha\), see Figure 1 for illustration.

We see that the connection between \( a_T \) and \( T \) is completely specified in Theorem 2.2. For given \( \alpha \), there is only one asymptotic regime leading to a nontrivial limit. Note that the fact that \( a_T \) goes to one implies that the nonlinear transient part of the market impact (case \( \alpha < 1 \)) can arise only in a highly endogenous market. This is actually very natural for the following reason. A nonlinear transient impact means that the market reacts differently to a child order that is in the end of the metaorder compared to a child order in the beginning of the metaorder. For this to be possible, one needs that correlations in the order flow to survive all over the time-length of our metaorder. In our probabilistic setting, using the population approach to Hawkes processes (see Bacry & Muzy, 2016; Filimonov & Sornette, 2015; Jaisson & Rosenbaum, 2015, 2016), it is easily seen that such property can hold only provided \( a_T \) goes to one.

In this regard, the case \( \alpha = 1 \) is quite degenerate because the market has somehow no memory and reacts the same way to market orders, independently of their position within the metaorder. Even more, the price instantaneously decreases to its permanent level when the metaorder is completed. This means that the market is able to detect instantaneously the end of a metaorder, which seems unrealistic and incompatible with empirical measurements.

3 | MACROSCOPIC LIMIT OF THE PRICE

We finally show in this section that under Assumption 2.1, the macroscopic price, that is the limit as time goes to infinity of the properly rescaled microscopic price (4), is diffusive with rough or hyper-rough volatility. Moreover, we make explicit the link between the market impact shape exponent and the Hurst parameter of the volatility.
3.1 Scaling limit of the price process

We start with an assumption that is necessary to get a nontrivial long-term limit for the price (4).

**Assumption 3.1.** For some $\delta > 0$, we have

$$\lim_{T \to +\infty} (1 - a^T)\mu^T T = \delta.$$  

Assumption 3.1 is classical in the context of Hawkes processes with kernel whose $L^1$ norm tends to one (see Jaisson & Rosenbaum, 2016). Indeed, it ensures that the number of events does not explode asymptotically.

According to Equation (2), price and volume are homogeneous. Therefore, we rescale the price the same way as the metaorders. Taking for simplicity and without loss of generality $P_0 = 0$, we define for $t \in [0, 1]$

$$\bar{P}_t^T = \frac{1}{T\beta^T}P_t^T = \frac{1 - a^T}{T\mu^T} \int_0^t \xi^T(T(t-s))d\left(N_{tT}^a - N_{tT}^h\right),$$

where

$$\xi^T(t) = \left(1 + \int_0^{+\infty} \psi^T(u)du\right)\left(1 - \int_0^t \phi^T(u)du\right).$$

Let $\alpha$ be the parameter of the market impact function in Theorem 2.2, $K$ the constant introduced in Equations (6) and (7) and $\lambda = (K(2 - \alpha))^{-1}$. We have the following result for the macroscopic limit of the price process, whose proof is given in Section 4.2.
Theorem 3.2. Under Assumptions 2.1 and 3.1, the sequence of rescaled price processes \((\hat{P}^T)_{T \geq 0}\) converges in law for the Skorokhod topology toward a process \(\hat{P}\) such that for \(t \in [0, 1]\)

\[
\hat{P}_t = \frac{1}{\sqrt{\delta}} \left( B^a_{X^a_t} - B^b_{X^b_t} \right),
\]

where \(B^a\) and \(B^b\) are two independent Brownian motions such that \(B^a_{X^a_t}\) and \(B^b_{X^b_t}\) are two martingales, \(X^a\) is increasing and satisfies

\[
X^a_t = \int_0^t F^{\alpha, \lambda}(s) ds + \frac{1}{\sqrt{\delta \lambda}} \int_0^t F^{\alpha, \lambda}(t - s) dB^a_{X^a_s},
\]

and \(X^b\) is increasing and solution of the same equation as above replacing the superscript \(a\) by \(b\).

In particular, there exists a Brownian motion \(W\) such that the integrated variance \(X = (X^a + X^b)/\delta\) of \(\hat{P}\) is solution of the stochastic rough Volterra equation

\[
X_t = \frac{2}{\delta} \int_0^t F^{\alpha, \lambda}(s) ds + \frac{1}{\delta \sqrt{\lambda}} \int_0^t F^{\alpha, \lambda}(t - s) dW_{X_s}.
\] (8)

Moreover, for any \(\varepsilon > 0\), the process \(X\) has Hölder regularity \(1 \wedge (2\alpha - \varepsilon)\). It is continuously differentiable for \(\alpha > 1/2\) and not continuously differentiable for \(\alpha \leq 1/2\).

Theorem 3.2 shows that the no-arbitrage principle together with the existence of market impact imply that the macroscopic price\(^2\) is a diffusive process whose cumulative variance is solution of a stochastic rough Volterra equation (except when \(\alpha = 1\) that corresponds to the classical Heston model, see Corollary 3.3). Note that \(X\) plays the role of an integrated variance and that when \(\alpha \leq 1/2\) it is not continuously differentiable. Thus, in that case, the spot variance is not well defined and only its integrated version makes sense. This is why for \(\alpha \leq 1/2\), we call this model hyper-rough volatility model (more precisely, hyper-rough Heston model).

From Theorem 3.2 in Jaisson and Rosenbaum (2016), we have that for \(\alpha > 1/2\), the process \(X^a\) is almost surely differentiable and its derivative \(Y^a\) is the unique solution of

\[
Y^a_t = \frac{\lambda}{\Gamma(\alpha)} \left( \int_0^t (t - s)^{\alpha - 1} \left( 1 - Y^a_s \right) ds + \frac{1}{\sqrt{\delta \lambda}} \int_0^t (t - s)^{\alpha - 1} \sqrt{Y^a_s} dB^a_s \right).
\]

The same result holds for \(X^b\) replacing the superscript \(a\) by \(b\). We deduce that when \(\alpha > 1/2\), the integrated volatility admits a derivative and the macroscopic limit of the price follows a rough Heston model. More precisely, we have the following corollary.

Corollary 3.3. When \(\alpha > 1/2\), the process \(X\) is differentiable almost surely and its derivative \(Y\) is the unique solution of the stochastic rough Volterra equation

\[
Y_t = \frac{1}{\delta} \left( \int_0^t (t - s)^{\alpha - 1} \left( \frac{2}{\delta} - Y_s \right) ds + \frac{1}{\sqrt{\delta \lambda}} \int_0^t (t - s)^{\alpha - 1} \sqrt{Y_s} dW_s \right).
\]

with \(W\) a Brownian motion. Furthermore, the dynamic of the price \(\hat{P}\) is

\[
d\hat{P}_t = \frac{1}{\sqrt{\delta}} \left( \sqrt{Y^a_t} dB^a_t - \sqrt{Y^b_t} dB^b_t \right).
\]
This result highlights the fact that at the macroscopic limit, the correlation \( \rho_t \) between the two Brownian motions driving price and volatility is stochastic. More precisely, we have

\[
\rho_t = \frac{Y_t^a - Y_t^b}{Y_t^a + Y_t^b}.
\]

Hence, the correlation sign depends on that of \( Y_t^a - Y_t^b \). The process \( Y^a \) (respectively, \( Y^b \)) corresponding to the volatility of the ask (respectively, bid) side of the market (see Step 4 in Section 4.2), this can be interpreted in terms of order flow dynamics. Indeed, suppose that \( Y_t^a \gg Y_t^b \) and that price is increasing. Then the instantaneous imbalance has the same sign as price returns. Thus, the volatility increases as the order flow excites the price dynamic. Conversely, if the price increases and \( Y_t^a \ll Y_t^b \), the volatility decreases as the order flow tends to compensate the upward price variation.

To prove the convergence in law in Theorem 3.2, we show that \( (\bar{P}_t^T)_{T \geq 0} \) is tight and that all limit points have the same law. This is done using the characteristic function of Hawkes processes in the spirit of El Euch and Rosenbaum (2019). A direct proof would consist in obtaining uniqueness in law for solutions of Equation (8) as done in Jaber, Larsson, and Pulido (2019) for \( \alpha > 1/2 \). However, such approach seems quite intricate to adapt for \( \alpha \leq 1/2 \). We have the following result whose proof is given in Section 4.3.

**Theorem 3.4.** Let \( X \) be the cumulated variance process given in Theorem 3.2 and \( h \) a continuously differentiable function from \( \mathbb{R}^+ \) to \( \mathbb{R} \) such that \( h(0) = 0 \). The function

\[
\psi(h, t) = \mathbb{E} \left[ \exp \left( \int_0^t i h(t - s)dx_s \right) \right]
\]

satisfies

\[
\psi(h, t) = \exp \left( \int_0^t g(s)ds \right),
\]

with \( g \) the unique continuous solution of the Volterra Riccati equation

\[
g(t) = \int_0^t f^{a,\lambda}(t - s)(\delta^{-1} \frac{1}{4} g(s)^2 + \delta^{-1} 2ih(s))ds, \tag{9}
\]

where \( f^{a,\lambda} \) is the Mittag-Leffler density function (see Appendix A.1).

Theorem 3.4 extends some already known results about characteristic functions related to rough Heston models for \( \alpha > 1/2 \) (see El Euch, Gatheral, & Rosenbaum, 2019; El Euch & Rosenbaum, 2019; Jaber et al., 2019). Note that the characteristic function of the macroscopic price process \( \hat{P}_t \) can also be obtained using the same type of proof as that for Theorem 3.4.

### 3.2 Conclusion

We have considered three main assumptions in this work:

- No arbitrage, in the sense that the price process is a martingale and PMI is linear.
- Existence of market impact with a transient component.
- The order flow can be fitted by a Hawkes process (with no restriction on the Hawkes parameters).
We have shown that in this quite general framework, the market impact function can only be a power-law with exponent \( 1 - \alpha \) for some \( \alpha \in (0, 1) \) (we drop here the case \( \alpha = 1 \) that leads to a somehow degenerate market impact function). The parameter \( \alpha \) also appears necessarily in the tail of the kernel of the Hawkes process driving the order flow: \( \phi(x) \sim x^{-(1+\alpha)} \) as \( x \) goes to infinity. Furthermore, this also implies that the market is highly endogenous. Even more interestingly, we obtain that the macroscopic behavior of the price is that of a rough or hyper-rough Heston model with Hurst parameter \( H = \alpha - 1/2 \).

The relationship between market impact, tail of Hawkes kernel and volatility Hurst parameter enables us to confront our results to empirical measurements. In Bacry, Iuga, et al. (2015), it is found that the market impact function fits a power-law with exponent 0.45. In Gatheral et al. (2018), it is shown that volatility is rough with a Hurst parameter of order 0.1. Finally, in Hardiman et al. (2013), the authors calibrate a Hawkes process on market orders arrival and obtain that the kernel decays as a power-law function with exponent around \(-1.45\). All these measurements are compatible with our results (and suggest that market impact is close to square root).

4 | PROOFS

4.1 | Proof of Theorem 2.2

Let \( f = 1_{[0,s]} \), \( s \in (0, 1] \). From Assumption 2.1, we have the pointwise convergence of \( \left( \overline{MI}_T(f, \cdot) \right)_{T \geq 0} \). As previously explained, this is equivalent to the convergence of \( \left( TMI_T(f, \cdot) \right)_{T \geq 0} \). Moreover, \( \left( \overline{PMI}_T(f, \cdot) \right)_{T \geq 0} \) being independent of \( T \), Assumption 2.1 implies that the sequence of functions

\[
\overline{TMI}_T(f, t) = \gamma \int_0^t a_T^{-1} \int_{y_T}^{+\infty} \phi(u)dudf(t - y)dy
\]

converges pointwise. The function \( \phi \) being nonnegative and integrable, \( \overline{TMI}_T(f, \cdot) \) is nonnegative, nondecreasing and concave on \([0, s]\) and then nonincreasing. Hence, \( \overline{TMI}_T(f, \cdot) \) reaches its maximum in \( s \). By pointwise convergence, \( \overline{TMI}(f, \cdot) \) has the same properties. Because we have assumed that \( \overline{MI}(f, t) < \overline{MI}(f, s) \) for some \( t > s \) and \( \overline{PMI}(f, \cdot) \) is nondecreasing, we deduce that \( \overline{MI}(f, s) > 0 \).

Let \( g(t) = \gamma^{-1} \overline{MI}(1_{[0,t]}, t) \) for \( t \in (0, 1] \) and consider

\[
R(t) = \int_0^t \int_y^{+\infty} \phi(u)dudy > 0.
\]

According to Equation (10), we have for \( t \in (0, 1] \)

\[
\frac{R(T_t)}{R(T)} \to \frac{g(t)}{g(1)} > 0.
\]

By the characterization theorem, see Theorem A.3 in Appendix A.2, we deduce that the previous limit holds for all \( t > 0 \) with some suitable extension of the function \( g \). Moreover, there exist some \( \beta \in \mathbb{R} \), \( K > 0 \) and \( L \) a slowly varying function such that for \( t > 0 \)

\[
g(t) = K t^\beta, \quad R(t) = L(t) t^\beta.
\]
Remark that for \( t \in (0, 1] \), we have \( g(t) = \overline{TMI}(I_{\{0,1\}}T) \), which is concave. Thus, \( \beta \in [0, 1] \). Taking \( s = t = 1 \) in the pointwise convergence (10), we get

\[
T(1 - a^T)^{-1} \int_0^T \int_y^{+\infty} \phi(u)du \, dy = a^T(1 - a^T)^{-1}T^{\beta - 1}L(T) \rightarrow_{T \rightarrow +\infty} g(1) = K > 0.
\]  

(12)

Consider now the sequence of functions

\[
\Gamma_T(y) = a^T(1 - a^T)^{-1} \int_{y}^{+\infty} \phi(u)du.
\]

We get from (11), (12) and property of slowly varying function that for any \( t > 0 \)

\[
\lim_{T \rightarrow +\infty} \int_0^t \Gamma_T(y)dy = Kt^\beta.
\]

Suppose that \( \beta \neq 0 \). Let \( 0 \leq a < b \) and \( y \in [a, b] \). We have

\[
\lim_{T \rightarrow +\infty} \int_a^b \frac{\Gamma_T(u)}{\Gamma_T(v)} du = \frac{y^\beta - a^\beta}{b^\beta - a^\beta}.
\]

The right-hand side is the cumulative distribution function of a random variable with support on \([a, b]\) whose law is denoted by \( m_{a,b}^\beta \). Hence, we have the convergence in law

\[
I_{[a,b]} \frac{\Gamma_T(u)du}{\int_{a}^{b} \Gamma_T(v)dv} \rightarrow_{T \rightarrow +\infty} m_{a,b}^\beta(du).
\]

So, for any bounded continuous function \( g \) on \([a, b]\), we get

\[
\lim_{T \rightarrow +\infty} \int_a^b \frac{\Gamma_T(u)}{\Gamma_T(v)} g(u)du = \int_a^b g(u)m_{a,b}^\beta(du).
\]

Consequently,

\[
\lim_{T \rightarrow +\infty} \int_a^b \Gamma_T(u)g(u)du = K \int_a^b g(u)m_{a,b}^\beta(du)(b^\beta - a^\beta) = K\beta \int_a^b g(u)u^{\beta - 1}du.
\]

Now let \( h \) be a nonnegative measurable function defined on \( \mathbb{R}^+ \), continuous on \([0,1]\) and supported on \([0,1]\). For \( t \leq 1 \), we have

\[
\int_0^t h(t - u)\Gamma_T(u)du \rightarrow_{T \rightarrow +\infty} K\beta \int_0^t h(t - u)u^{\beta - 1}du
\]

and for \( t > 1 \)

\[
\int_0^t h(t - u)\Gamma_T(u)du = \int_{t - 1}^t h(t - u)\Gamma_T(u)du \rightarrow_{T \rightarrow +\infty} K\beta \int_{t - 1}^t h(t - u)u^{\beta - 1}du.
\]
Finally, for any $t \geq 0$
\[
\hat{TMI}(h, t) = \lim_{T \to +\infty} \frac{TMI^T(h, t)}{T} = \gamma K \beta \int_0^t h(t-u)u^{\beta-1} du.
\]
Thus, when $\beta \in (0, 1]$, we have the existence of a macroscopic limit for the transient part of the market impact function (and therefore for the market impact function). Remark that for $\beta = 1$
\[
\hat{TMI}(f, t) = \gamma K \int_0^t f(u) du.
\]
Consequently, in that case, $\hat{TMI}(1_{[0,1]}, \cdot)$ is a nondecreasing function. This is in contradiction with Assumption 2.1, hence $\beta$ cannot be equal to 1.

Suppose that $\beta = 0$. For any $t > 0$, we have
\[
1_{[0,t]} \frac{\Gamma^T(u) du}{\int_0^t \Gamma^T(v) dv} \xrightarrow{T \to +\infty} \delta_0(du),
\]
where $\delta_0$ is the Dirac measure in 0. Then for any bounded continuous function $g$
\[
\lim_{T \to +\infty} \int_0^t \frac{\Gamma^T(u)}{\int_0^t \Gamma^T(v) dv} g(u) du = g(0).
\]
Now let $f$ be a nonnegative measurable function defined on $\mathbb{R}^+$, continuous on $[0,1]$ and supported on $[0,1]$. For $t \leq 1$, we have
\[
\int_0^t f(t-s)\tilde{\Gamma}^T(s) ds \xrightarrow{T \to +\infty} K f(t)
\]
and for $t > 1$
\[
0 \leq \int_0^t f(t-s)\tilde{\Gamma}^T(s) ds \leq \int_0^t \tilde{f}(t-s)\tilde{\Gamma}^T(s) ds \xrightarrow{T \to +\infty} 0,
\]
where $\tilde{f}$ is a nonnegative continuous extension of $f 1_{[0,1]}$ on $\mathbb{R}^+$ supported on $[0, 1 + \frac{t-1}{2}]$. Finally, for any $t \geq 0$
\[
\hat{TMI}(f, t) = \gamma K f(t).
\]
Consequently, for $\beta = 0$, we also have the existence of a macroscopic limit for the transient part of the market impact function (and therefore of the market impact function). We obtain the result letting $\alpha = 1 - \beta$.

4.2 | Proof of Theorem 3.2

We proceed in five steps:

1. Step 1: We first prove a preliminary result on the characteristic function of Hawkes processes that we use later in Step 3.
2. Step 2: We rewrite the sequence $(P^T)_{T \geq 0}$ in a convenient way.
3. Step 3: We adapt results from El Euch and Rosenbaum (2019) and Jaisson and Rosenbaum (2016) on scaling limits of nearly unstable heavy-tailed Hawkes processes to our more general framework.

4. Step 4: We deduce from the previous steps the convergence in law for the Skorokhod topology of the sequence \( \overline{P}_T \) for \( T \geq 0 \) and make explicit the equation satisfied by the limit.

5. Step 5: We prove the results on the regularity of solutions of Equation (8).

For simplicity and without loss of generality, we take \( P_0 = 0 \).

### 4.2.1 Step 1

We derive a result on the characteristic function of Hawkes processes using similar arguments as those introduced in El Euch and Rosenbaum (2019). Recall that the notation \( * \) stands for the convolution product on \( \mathbb{R}^+ \). More precisely, for \( f \) and \( g \) suitable measurable functions and \( m \) a measure,

\[
(f \ast g)(t) = \int_0^t f(t - s)g(s)ds
\]

and

\[
(f \ast dm)(t) = \int_0^t f(t - s)m(ds).
\]

We have the following proposition.

**Proposition 4.1.** Let \( N \) be a Hawkes process with parameters \( (\nu, \phi) \), with \( \nu \) a locally integrable non-negative function and \( \phi \) a nonnegative measurable function such that \( \|\phi\|_1 < 1 \). For any continuous function \( h \) from \( \mathbb{R}^+ \) into \( \mathbb{R} \),

\[
L(h, t) = \mathbb{E}[\exp((ih \ast dN)(t))]
\]

satisfies

\[
L(h, t) = \exp\left(\int_0^t (C(h, s) - 1)(t - s)ds\right),
\]

where \( C \) is solution of the equation

\[
C(h, \cdot) = \exp(ih + (C(h, \cdot) - 1) \ast \phi).
\]

**Proof.** Let \( \tilde{N} \) be a Hawkes process with parameters \( (\phi, \phi) \) and \( N^0 \) a Poisson process with intensity \( \nu \). Let \( (\tilde{N}^j)_{j \in \mathbb{N}^*} \) be independent copies of \( \tilde{N} \), also independent of \( N^0 \). Using the population interpretation of Hawkes processes, see appendix C.1 in El Euch and Rosenbaum (2018), we deduce the following equality in law

\[
N_t \overset{\mathcal{D}}{=} N_t^0 + \sum_{j=1}^{N_t^0} \tilde{N}_t^{j},
\]
where \((T_j)_{j \in \mathbb{N}^*}\) are the jump times of the process \(N^0\). Consequently,

\[
(ih * dN)(t) = (ih * dN^0)(t) + \sum_{j=1}^{N_t^0} (ih * d\tilde{N}^j)(t - T_j).
\]

Then taking the exponential and conditional expectation with respect to \(N^0\), we get

\[
\mathbb{E}[\exp((ih * dN)(t))|N^0] = \exp((ih * dN^0)_t) \prod_{j=1}^{N_t^0} \tilde{L}(h, t - T_j)
\]

\[
= \exp\left(\left((ih + \log(\tilde{L}(h, \cdot))) * dN^0)(t)\right)\right),
\]

where \(\tilde{L}\) is defined as \(L\) with \(\tilde{N}\) instead of \(N\). Remark that

\[
\left((ih + \log(\tilde{L}(h, \cdot))) * dN^0)(t) = \sum_{j=1}^{N_t^0} ih(t - T_j) + \log(\tilde{L}(h, t - T_j))
\]

and that \(\text{Re}(\log(\tilde{L}(h, \cdot))) \leq 0\) as \(|\tilde{L}(h, \cdot)| \leq 1\). Thus, using Proposition A.5 in Appendix A.4, we get

\[
L(h, t) = \exp\left(\int_0^t \left(e^{ih(t-s)} \tilde{L}(h, t - s) - 1\right) \nu(s)ds\right).
\]

In the same way, we have

\[
\tilde{L}(h, t) = \exp\left(\int_0^t \left(e^{ih(t-s)} \tilde{L}(h, t - s) - 1\right) \phi(s)ds\right).
\]

Thus, setting

\[
C(h, t) = e^{ih(t)} \tilde{L}(h, t),
\]

we obtain

\[
L(h, t) = \exp\left(\int_0^t \left(C(h, s) - 1\right) \nu(t - s)ds\right)
\]

and

\[
C(h, \cdot) = \exp(ih + (C(h, \cdot) - 1) * \phi).
\]

\[
\square
\]

### 4.2.2 Step 2

We consider the price model (4). Let \(M^{a,T}\) be defined by

\[
M_t^{a,T} = N_t^{a,T} - \int_0^t \lambda_s^{a,T} ds.
\]

We define \(M^{b,T}\) the same way replacing the superscript \(a\) by \(b\) in the above equation. We have the following result.
Lemma 4.2. The price process (4) can be written as

\[ P_t^T = \left( 1 + \int_0^{+\infty} \psi^T(v)dv \right) \left( M_{t}^{a,T} - M_{t}^{b,T} \right). \]

Proof. We have

\[ P_t^T = \int_0^t \left( 1 + \int_0^{+\infty} \psi^T(v)dv \right) \left( 1 - \int_0^{t-u} \phi^T(v)dv \right) d \left( N_{a,T}^a - N_{b,T}^b \right)_u. \]

We first deal with the term \( T_1 \) defined by

\[ T_1 = \int_0^t \left( 1 + \int_0^{+\infty} \psi^T(v)dv \right) \int_0^{t-u} \phi^T(v)dv \left( N_{a,T}^a - N_{b,T}^b \right)_u. \]

Using Fubini–Tonelli theorem, we get

\[ T_1 = \left( 1 + \int_0^{+\infty} \psi^T(v)dv \right) \int_0^t \int_0^v \phi^T(v-u)dv \left( N_{a,T}^a - N_{b,T}^b \right)_u dv. \]

Thus, we deduce

\[ T_1 = \left( 1 + \int_0^{+\infty} \psi^T(v)dv \right) \int_0^t \left( \lambda_{v,T}^a - \mu - \lambda_{v,T}^b + \mu \right) dv. \]

Finally,

\[ P_t^T = \left( 1 + \int_0^{+\infty} \psi^T(v)dv \right) \int_0^t \left( dN_{v,T}^a - \lambda_{v,T}^a dv - dN_{v,T}^b + \lambda_{v,T}^b dv \right) \]

\[ = \left( 1 + \int_0^{+\infty} \psi^T(v)dv \right) \left( M_{t}^{a,T} - M_{t}^{b,T} \right). \]

Lemma 4.2 leads to

\[ \frac{1}{P_t^T} = \frac{1 - a^T}{T \mu^T} \left( 1 + \int_0^{+\infty} \psi^T(v)dv \right) \left( M_{t}^{a,T} - M_{t}^{b,T} \right). \]

4.2.3 | Step 3

We temporarily drop the superscripts \( a \) and \( b \). Indeed, the results are valid both for buy and sell order flows. Consider the sequences

\[ X_t^T = \frac{1 - a^T}{T \mu^T} N_{t,T}^T, \quad \Lambda_t^T = \frac{1 - a^T}{T \mu^T} \int_0^{t,T} \lambda_{s,T}^T ds, \quad Z_t^T = \sqrt{\frac{T \mu^T}{1 - a^T}} (X_t^T - \Lambda_t^T) \]

defined for \( t \in [0, 1] \). The following result is borrowed from Jaisson and Rosenbaum (2016).

Proposition 4.3. The sequence \( (\Lambda^T, X^T, Z^T) \) is tight. Furthermore, for any limit point \( (\Lambda, X, Z) \) of \( (\Lambda^T, X^T, Z^T) \), \( Z \) is a continuous martingale, \([Z, Z] = X\) and \( \Lambda = X \).
In addition, we have the following proposition that extends Theorem 3.1 in Jaisson and Rosenbaum (2016).

**Proposition 4.4.** Under Assumptions 2.1 and 3.1, for any limit point \((X, Z)\) of \((X^T, Z^T)\), there exists a Brownian motion \(B\) on \((\Omega, \mathcal{A}, \mathbb{P})\) (up to extension of the space) such that

\[ Z_t = B_{X_t}, \]

and \(X\) is a solution of the stochastic rough Volterra equation

\[ X_t = \int_0^t F^{\alpha, \lambda}(t - s)ds + \frac{1}{\sqrt{\delta \lambda}} \int_0^t F^{\alpha, \lambda}(t - s)dB_{X_s}. \] (14)

Moreover, for any \(\epsilon > 0\), the process \(X\) has Hölder regularity \(1 \wedge (2\alpha - \epsilon)\).

Note that we are here under more general assumptions than in Theorem 3.1 in Jaisson and Rosenbaum (2016). Indeed, in Jaisson and Rosenbaum (2016), we have

\[ \int_0^t \phi(s)ds = K t^{-\alpha}, \]

while we only know that

\[ \int_0^t \int_0^{+\infty} \phi(u)duds = L(t) t^{1-\alpha}, \]

with \(L\) a slowly varying function. To prove Proposition 4.4, it is enough to get the following lemma. The rest of the proof is similar to that in Jaisson and Rosenbaum (2016).

**Lemma 4.5.** The sequence of functions \(\rho^T(t) = \frac{1 - a^T}{a^T} \psi^T(Tt)T\) converges weakly toward \(F^{\alpha, \lambda}\). Furthermore, \(\int_0^t \rho^T(s)ds\) converges uniformly toward \(F^{\alpha, \lambda}\).

**Proof.** The function \(\rho^T\) is nonnegative with integral equal to 1. So, it can be interpreted as the density of a random variable. Hence, it is enough to show that its Laplace transform converges pointwise to get weak convergence. We have for \(z > 0\)

\[ \hat{\rho}^T(z) = \frac{1 - a^T}{a^T} \hat{\psi}^T \left( \frac{z}{T} \right) = \frac{\hat{\phi}(\frac{z}{T})}{1 - a^T (1 - a^T)^{-1} (\hat{\phi}(\frac{z}{T}) - 1)}. \]

Let

\[ R(t) = \int_0^t \int_0^{+\infty} \phi(u)duds. \]

Recall that from Theorem 2.2, \(R(t) = t^{1-\alpha} L(t)\). By Karamata’s Tauberian theorem, see Theorem A.4 in Appendix A.2, we have

\[ \hat{R}(z) \sim_{0^+} \frac{1}{z} \Gamma(2 - \alpha). \]

Integrating by parts twice, we obtain

\[ \hat{R}(z) = \int_0^{+\infty} e^{-zs} R(s)ds = \frac{1}{z} (1 - \hat{\phi}(z)). \]
So, we get

\[ a^T (1 - a^T)^{-1} \left( 1 - \hat{\phi} \left( \frac{Z}{T} \right) \right) \sim_{T \to +\infty} a^T (1 - a^T)^{-1} T^{-\alpha} L(T) \frac{L(T)}{L(T)} z^\alpha \Gamma(2 - \alpha). \]

We have shown in Theorem 2.2 that

\[ a^T (1 - a^T)^{-1} T^{-\alpha} L(T) \to K. \]

The function \( L \) being slowly varying, see Appendix A.2, we deduce

\[ \lim_{T \to +\infty} a^T (1 - a^T)^{-1} \left( 1 - \hat{\phi} \left( \frac{Z}{T} \right) \right) = z^\alpha \Gamma(2 - \alpha) K, \]

and finally

\[ \lim_{T \to +\infty} \hat{\rho}^T(z) = \frac{1}{1 + K \Gamma(2 - \alpha) z^\alpha} = \frac{\lambda}{\lambda + z^\alpha} = \hat{f}^{a,\lambda}(z), \]

with \( \lambda = (K \Gamma(2 - \alpha))^{-1}. \) The uniform convergence in Lemma 4.5 is obviously deduced from Dini’s theorem.

We finally show that the sequence \((X^T, Z^T)_{T \geq 0}\) converges in law for the Skorokhod topology. We already know that it is tight, so it is enough to prove that all the limit points have the same law.

Let \((X, Z)\) be a limit point of \((X^T, Z^T)_{T \geq 0}\). Using Proposition 4.4 together with the stochastic Fubini theorem, see Veraar (2012), we have

\[ X_t = \int_0^t f^{a,\lambda}(t - s) \left( s + \frac{1}{\sqrt{\delta \lambda}} Z_s \right) ds. \]

Example 42.2 in Samko, Kilbas, and Marichev (1993) leads to

\[ D^\alpha X_t + \lambda X_t - \lambda t = \sqrt{\frac{\lambda}{\delta}} Z_t, \]

where \( D^\alpha \) is the fractional derivative operator defined in Appendix A.3. Thus, the law of \((X, Z)\) is uniquely determined by the law of \( X \). Consequently, it is enough to prove uniqueness in law for limit points of \((X^T, Z^T)_{T \geq 0}\) to get convergence in law of \((X^T, Z^T)_{T \geq 0}\). For this, we prove that the characteristic function of any limit point \( X \) of the sequence \((X^T)_{T \geq 0}\) is a functional of the solution of a fractional Riccati equation. Uniqueness in law is then a consequence from the uniqueness of the solution of this equation.

**Proposition 4.6.** Let \( X \) be a limit point of \((X^T)_{T \geq 0}\) and \( h \) a continuously differentiable function from \( \mathbb{R}^+ \) to \( \mathbb{R} \) such that \( h(0) = 0 \). The function

\[ \psi(h, t) = \mathbb{E}[\exp(ih \ast dX_t)] \]

satisfies

\[ \psi(h, t) = \exp \left( \int_0^t g(s) ds \right), \]
with $g$ the unique continuous solution of the rough Volterra Riccati equation

$$g = f^{a,\lambda} * \left( \delta^{-1} \frac{1}{2} g^2 + ih \right). \quad (15)$$

To show this result, we are inspired by the methodology of El Euch and Rosenbaum (2019). However, note again that we are in a more general setting.

**Proof.** Recall that

$$X_t^T = \frac{1 - a^T}{T \mu^T} N_t^T.$$ 

We introduce the following quantities:

$$h^T(t) = \frac{1 - a^T}{T \mu^T} h\left( \frac{t}{T} \right), \quad L^T(h^T, t) = \mathbb{E}[\exp(i h^T \ast dN_t^T)] \text{ and } \psi^T = L^T(h^T, tT).$$

For every $T$, according to Proposition 4.1, there exists a function $C^T$ solution of

$$C^T = \exp(i h^T + (C^T - 1) \ast \phi^T)$$

such that

$$L^T(h^T, t) = \exp\left( \int_0^t (C^T(s) - 1) \mu^T ds \right).$$

Now define the sequence $g^T$

$$g^T(s) = C^T(s T) - 1.$$ 

We have

$$\psi^T = \exp\left( \frac{g^T}{1 - a^T} \ast (T(1 - a^T)\mu^T 1_{\mathbb{R}^+}) \right) \quad \text{and} \quad g^T + 1 = \exp\left( \frac{1 - a^T}{T \mu^T} ih + g^T \ast (T \phi^T(\cdot)) \right). \quad (16)$$

An immediate adaptation of Proposition 6.4. in El Euch and Rosenbaum (2019) gives that for any $s \in [0, t]$

$$|g^T(s)| \leq c(h)(1 - a^T), \quad (17)$$

with $c(h)$ a positive constant depending only on $h$. Hence, for $T$ large enough, we have

$$\log(1 + g^T) = g^T - \frac{1}{2} (g^T)^2 - \epsilon^T(h, \cdot), \quad (18)$$

with $|\epsilon^T(h, \cdot)| \leq c(h)(1 - a^T)^3$. According to Equations (16) and (18), we get for every $s \in [0, t]$

$$g^T(s) = \frac{1}{2} g^T(s)^2 + \epsilon^T(h, s) + \frac{1 - a^T}{T \mu^T} ih(s) + g^T \ast (T \phi^T(\cdot))(s).$$

Using that

$$\sum_{i \geq 1} (T \phi^T(\cdot))^i = T \psi^T(\cdot),$$
we deduce from Lemma 4.1 in Jaisson and Rosenbaum (2015) that
\[ g^T(s) = (T \psi^T(T)) \ast \left( \frac{1}{2} (g^T)^2 + e^T(h, \cdot) + \frac{1 - a^T}{T \mu^T} i h \right)(s) + \frac{1}{2} g^T(s)^2 + e^T(h, s) + \frac{1 - a^T}{T \mu^T} i h(s). \]

Consequently, letting \( \theta_T = (1 - a^T)^{-1} g^T \)
\[ \theta_T(s) = (T(1 - a^T)\psi^T(T)) \ast \left( \frac{1}{2} \theta_T^2 + \frac{1}{\delta} i h \right)(s) + r_1^T(s), \]
with
\[
\begin{align*}
r_1^T(s) &= (T(1 - a^T)\psi^T(T)) \ast \left( e^T(h, \cdot)(1 - a^T)^{-2} + \left( \frac{1}{T(1 - a^T)\mu^T} - \delta^{-1} \right) i h \right)(s) \\
&\quad + (1 - a^T)^{-1} \frac{1}{2} (g^T(s))^2 + (1 - a^T)^{-1} e^T(h, s) + \frac{1}{T \mu^T} i h(s).
\end{align*}
\]

Because \( a^T \) goes to 1, we know from Lemma 4.5 that in the sense of weak convergence
\[ T(1 - a^T)\psi^T(T) \to_{T \to +\infty} f^{a, \lambda}. \]
Finally, we have
\[ \theta_T = f^{a, \lambda} \ast \left( \frac{1}{2} \theta_T^2 + \frac{1}{\delta} i h \right) + r_1^T + r_2^T, \]
where
\[ r_2^T = (T(1 - a^T)\psi^T(T) - f^{a, \lambda}) \ast \left( \frac{1}{2} \theta_T^2 + \frac{1}{\delta} i h \right). \]

We now prove that \( (r_1^T)_{T \geq 0} \) and \( (r_2^T)_{T \geq 0} \) goes to 0 in \( C^0([0, t], \mathbb{R}) \) for the sup-norm.

Using Assumption 3.1, the second part of Lemma 4.5 and Equation (17), we get that \( (r_1^T)_{T \geq 0} \) goes to zero in \( C^0([0, t], \mathbb{R}) \). The sequence \( (\theta_T)_{T \geq 0} \) is bounded for the sup-norm according to Equation (17). Moreover, according to Lemma 4.7 (see after the proof), \( \theta_T \) is differentiable, and \( (\theta_T')_{T \geq 0} \) is bounded for the sup-norm. By integration by parts, we have
\[ r_2^T(t) = \left( \int_0^t T(1 - a^T)\psi^T(sT)ds - F^{a, \lambda} \right) \ast \left( \theta_T' + \frac{1}{\delta} i h' \right)(t), \]
where we have used the fact that \( \theta_T(0) = 0 \) and \( h(0) = 0 \). We then conclude that \( (r_2^T)_{T \geq 0} \) converges toward 0 in \( C^0([0, t], \mathbb{R}) \) using dominated convergence. Lemma 4.7 together with the Ascoli theorem gives that the sequence \( (\theta_T)_{T \geq 0} \) is relatively compact in \( (C^0([0, t], \mathbb{R}), \| \cdot \|_\infty) \). Moreover, for any limit point \( \theta \) of the sequence \( (\theta_T)_{T \geq 0} \), we have that \( \theta \) is solution of
\[ \theta = f^{a, \lambda} \ast \left( \frac{1}{2} \theta^2 + \frac{1}{\delta} i h \right). \]
The above equation has a unique continuous solution in \( C^0([0, t], \mathbb{R}) \) (see Section 6.2.4 in El Euch & Rosenbaum, 2019). Thus, the sequence \( (\theta_T)_{T \geq 0} \) converges toward this solution.

Finally, remark that
\[ \psi^T(t) = \mathbb{E}[\exp(ih \ast dX^T)(t)]. \]
Thus, convergence in law of \((X^T)_{T \geq 0}\) toward \(X\) implies that \((\psi^T)_{T \geq 0}\) converges pointwise toward the function \(\psi\). Passing to the limit in (16), we get

\[
\psi(t) = \exp((\theta \ast (\delta 1_{\mathbb{R}^+}))_t) = \exp\left(\delta \int_0^t \theta(s)ds\right).
\]

Letting \(g = \delta \theta\), we have the result.

\[\square\]

It is enough to characterize the law of \(X\) to know \(\psi(h, t)\) for any \(t \in \mathbb{R}^+\) and \(h \in C^1_0([0, t], \mathbb{R})\). Therefore, uniqueness in law for the limit points of \((X^T)_{T \geq 0}\) is a corollary from the uniqueness of continuous solution for the Volterra Riccati Equation (15) (see Section 6.2.4 in El Euch & Rosenbaum, 2019).

We now give the lemma we used in the proof of Proposition 4.6.

**Lemma 4.7.** The functions \((\theta^T)_{T \geq 0}\) are continuously differentiable and \((\theta^T_t)'_{T \geq 0}\) is bounded in \(C^0([0, t], \mathbb{R})\).

**Proof.** Using the proof of Proposition 4.1, we have

\[
\theta_T = (1 - a^T)^{-1}\left[\mathbb{E}\left[\exp\left((ih + ih \ast d\tilde{N}^T_T)\frac{1 - a^T}{T\mu^T}\right)\right] - 1\right],
\]

with \(\tilde{N}\) a Hawkes processes with parameters \((\phi^T, \phi^T)\), where \(\phi^T = a^T \phi\). Because \(h(0) = 0\), \(h \ast d\tilde{N}^T_T\) admits a derivative and for any \(s \in [0, t]\)

\[
(h \ast d\tilde{N}^T_T)'(s) = (h' \ast d\tilde{N}^T_T)(s).
\]

Furthermore, we have

\[
|(h' \ast d\tilde{N}^T_T)(s)| \leq \|h'\|_{\infty} \tilde{N}^T_T.
\]

Using that

\[
\tilde{\lambda}^T_s = \psi^T(s) + \int_0^t \psi^T(t - s)d\tilde{M}^T_s,
\]

we get

\[
(1 - a^T)\mathbb{E}[\tilde{N}^T_T] \leq (1 - a^T)\mathbb{E}\left[\int_0^t \tilde{\lambda}^T_s ds\right] \leq \int_0^T (1 - a^T)\psi^T(s)ds \leq 1.
\]

Consequently, using derivation for integral with parameters, \(\theta_T\) is differentiable and

\[
\theta_T' = (1 - a^T)^{-1}\mathbb{E}\left[(ih' + ih' \ast d\tilde{N}^T_T)\frac{1 - a^T}{T\mu^T}\exp\left((ih + ih \ast d\tilde{N}^T_T)\frac{1 - a^T}{T\mu^T}\right)\right].
\]

Thus, we have for all \(s \in [0, t]\)

\[
|\theta_T'(s)| \leq \frac{1}{T\mu^T(1 - a^T)}(1 - a^T)\mathbb{E}\left[\|h'\|_{\infty} + \|h'\|_{\infty} \tilde{N}^T_T\right].
\]

The right-hand side is finite and independent of \(s\), consequently the sequence \((\theta_T')_{T \geq 0}\) is bounded in \(C^0([0, t], \mathbb{R})\).

\[\square\]

Finally, we have proved that the sequence \((X^T, Z^T)_{T \geq 0}\) converges in law for the Skorokhod topology.
4.2.4 Step 4

Consider the sequence \((X^a, T, Z^a, T)_{T \geq 0}\) (respectively, \((X^b, T, Z^b, T)_{T \geq 0}\)) defined the same way as in Equation (13) with \((N^a, T)_{T \geq 0}\) (respectively, \((N^b, T)_{T \geq 0}\)) instead of \((N^T)_{T \geq 0}\). According to Lemma 4.2, we have

\[
\overline{P}_t^T = \frac{1 - a_T^T}{T \mu^T} \left(1 + \int_0^{+\infty} \psi^T(u)du\right) \left(M^a_{iT} - M^b_{iT}\right) = \frac{1}{T \mu^T} \left(M^a_{iT} - M^b_{iT}\right).
\]

Thus,

\[
\overline{P}_t^T = \frac{1}{\sqrt{T \mu^T (1 - a_T)}} \left(Z^a_t - Z^b_t\right).
\]

Using Step 3, we get that \((Z^a, T)_{T \geq 0}\) and \((Z^b, T)_{T \geq 0}\) converge for the Skorohod topology. These sequences being independent, \((\overline{P}^T)_{T \geq 0}\) converges toward a process \(\hat{P}\) in the Skorokhod topology. Furthermore, we deduce from Proposition 4.4 together with Assumption 3.1 that there exist two independent Brownian motions \(B^a\) and \(B^b\) such that

\[
\hat{P}_t = \frac{1}{\sqrt{\delta}} \left(B^a_{X^a_t} - B^b_{X^b_t}\right),
\]

where \(X^a\) (respectively, \(X^b\)) is the limit of the sequence \((X^a, T)_{T \geq 0}\) (respectively, \((X^a)_{T \geq 0}\)) and is solution of Equation (14) with Brownian motion \(B^a\) (respectively, \(B^b\)). Hence, \(X = \frac{X^a + X^b}{\delta}\) is solution of

\[
X_t = \frac{2}{\delta} \int_0^t F^{a, \lambda}(t - s)ds + \frac{1}{\delta \sqrt{\lambda}} \int_0^t F^{a, \lambda}(t - s) \frac{1}{\sqrt{\delta}} d\left(B^a_{X^a_t} + B^b_{X^b_t}\right).
\]

Moreover, there exists a Brownian motion \(W\) such that \(W_{X_t} = \frac{1}{\sqrt{\delta}} (B^a_{X^a_t} + B^b_{X^b_t})\). Consequently,

\[
X_t = \frac{2}{\delta} \int_0^t F^{a, \lambda}(t - s)ds + \frac{1}{\delta \sqrt{\lambda}} \int_0^t F^{a, \lambda}(t - s)dW_{X_t}.
\]

4.2.5 Step 5

We first recall a result from Jaisson and Rosenbaum (2016).

**Proposition 4.8.** Let \(X\) be a solution of the stochastic Volterra equation (8). Then for any \(\varepsilon > 0\), almost surely, \(X\) has Hölder regularity \(1 \wedge (2\alpha - \varepsilon)\). And if \(\alpha > 1/2\), \(X\) is almost surely differentiable.

We now give a new result on the regularity of the solution of Equation (8).

**Proposition 4.9.** Let \(\alpha \leq \frac{1}{2}\). Let \(X\) be a solution of the stochastic Volterra Equation (8). Then, almost surely, \(X\) is not continuously differentiable.

**Proof.** As already seen in Step 3, \(X\) satisfies

\[
D^\alpha X_t = -\lambda X_t + \frac{2\lambda}{\delta} t + \frac{\sqrt{\lambda}}{\delta} W_{X_t},
\]

(19)
Applying the law of iterated logarithm, we get for \(0 \leq t \leq 1\)

\[
\limsup_{s \to t^{-}} \frac{D^\alpha X_t - D^\alpha X_s - \frac{2\lambda}{\delta} (t - s)}{\sqrt{2(X_t - X_s) \log \log ((X_t - X_s)^{-1})}} = \frac{\sqrt{\lambda}}{\delta}.
\]

Assume that \(X\) is continuously differentiable. According to Appendix A.3, we have

\[
D^\alpha X_t = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} X'_s ds.
\]

Let \(t\) be such that \(X'_t \neq 0\). Such a point almost surely exists because \(X\) is not constant. Indeed, suppose it is constant, as \(X_0 = 0\) it implies that \(X = 0\). But obviously the null function is not solution of Equation (19). For such \(t\) using that

\[
X_t - X_s \sim_{s \to t} (t - s)X'_t,
\]

we have

\[
\lim_{s \to t^{-}} \frac{t - s}{\sqrt{2(X_t - X_s) \log \log ((X_t - X_s)^{-1})}} = 0.
\]

Hence,

\[
\limsup_{s \to t^{-}} \frac{D^\alpha X_t - D^\alpha X_s}{\sqrt{2(X_t - X_s) \log \log ((X_t - X_s)^{-1})}} = \frac{\sqrt{\lambda}}{\delta}. \quad (20)
\]

We now give a bound on \(|D^\alpha X_t - D^\alpha X_s|\), for \(s < t\), where \(\|X'\|_\infty\) denotes the supremum norm of \(X'\)

\[
|D^\alpha X_t - D^\alpha X_s| = \left| \int_0^s ((t - u)^{-\alpha} - (s - u)^{-\alpha}) X'_s du + \int_s^t (t - u)^{-\alpha} X'_u du \right|
\]

\[
\leq \int_0^s |(t - u)^{-\alpha} - (s - u)^{-\alpha}| \|X'\|_\infty du + \frac{\|X'\|_\infty}{1 - \alpha} (t - s)^{1 - \alpha}
\]

\[
\leq \left( \int_0^s u^{-\alpha} du + \int_s^t u^{-\alpha} du \right) \|X'\|_\infty + \frac{\|X'\|_\infty}{1 - \alpha} (t - s)^{1 - \alpha}
\]

\[
\leq \left( \frac{1}{1 - \alpha} (t - s)^{1 - \alpha} + (t - s)s^{-\alpha} \right) \|X'\|_\infty + \frac{\|X'\|_\infty}{1 - \alpha} (t - s)^{1 - \alpha}.
\]

We get

\[
\lim_{s \to t^{-}} \frac{D^\alpha X_t - D^\alpha X_s}{\sqrt{2(X_t - X_s) \log \log ((X_t - X_s)^{-1})}} = 0.
\]

This is in contradiction with Equation (20), hence \(X\) cannot be continuously differentiable. \(\square\)
4.3 Proof of Theorem 3.4

We have seen in Section 4.2.4 that $X = (X^a + X^b)/\delta$, with $X^a$ and $X^b$ being independent copies of the limit of the sequence $(X^T)_{T \geq 0}$. From Proposition 4.6, we immediately obtain Theorem 3.4.

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ENDNOTES

1 A price manipulation is a round-trip (strategy starting and finishing with null inventory), whose expected cost is negative.

2 Remark that under our completely symmetric setting, price can become negative. This is, however, obviously not very important for our purpose here.

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APPENDIX A

A.1 Mittag-Leffler functions

Let \((\alpha, \beta) \in (\mathbb{R}^+_0)^2\). The Mittag-Leffler function \(E_{\alpha, \beta}(z)\) is defined for \(z \in \mathbb{C}\) by

\[
E_{\alpha, \beta}(z) = \sum_{n \geq 0} \frac{z^n}{\Gamma(an + \beta)}.
\]

For \((\alpha, \lambda) \in (0, 1] \times \mathbb{R}^+\), we also define

\[
f^{\alpha, \lambda}(t) = \lambda t^{\alpha-1} E_{\alpha, \alpha}(-\lambda t^\alpha), \quad t > 0,
\]

\[
F^{\alpha, \lambda}(t) = \int_0^t f^{\alpha, \lambda}(s)ds, \quad t \geq 0.
\]

The function \(f^{\alpha, \lambda}\) is a density function on \(\mathbb{R}_+\) called the Mittag-Leffler density function. Its Laplace transform is

\[
\hat{f}^{\alpha, \lambda}(z) = \frac{\lambda}{\lambda + z^\alpha}.
\]

When \(\alpha = 1\), the Mittag-Leffler density simply corresponds to the exponential law with parameter \(\lambda\).

A.2 Tauberian theorems

The following results can be found in Bingham, Goldie, and Teugels (1989).

**Definition A.1.** A measurable function \(L : \mathbb{R}^+ \rightarrow \mathbb{R}\) is slowly varying if for all \(s > 0\)

\[
\frac{L(st)}{L(t)} \rightarrow 1 \quad \text{as} \quad t \rightarrow +\infty.
\]

**Proposition A.2.** Let \(L\) be a slowly varying function and \(\alpha > 0\), then

\[
t^{-\alpha} L(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.
\]

**Theorem A.3.** (Characterization theorem). Let \(U\) be a positive measurable function on \(\mathbb{R}_+\) such that for all \(s \in C\), with \(C\) a set with positive Lebesgue measure

\[
\frac{U(ts)}{U(t)} \rightarrow g(s) > 0 \quad \text{as} \quad t \rightarrow +\infty,
\]

for some function \(g\). Then the previous limit can be extended for all \(s > 0\). Let \(\hat{g}\) be this limiting function extending \(g\). There exist \(\alpha \in \mathbb{R}\) such that \(g(t) = t^\alpha\) and a slowly varying function \(L\) such that \(U(t) = t^\alpha L(t)\).

**Theorem A.4.** (Karamata’s Tauberian theorem). Let \(U\) be a measurable nonnegative function, \(c \geq 0\), \(\rho > -1\) and assume \(\hat{U}(z) = \int_0^{+\infty} e^{-zs} U(s)ds\) is finite for any \(z > 0\). Then

\[
U(t) \sim_{+\infty} c t^\rho \frac{L(t)}{\Gamma(1 + \rho)}
\]
for $L$ a slowly varying function implies

$$
\hat{U}(z) \sim_{0^+} c z^{-\rho-1} L \left( \frac{1}{z} \right).
$$

### A.3 Fractional derivative

For $\alpha \in [0, 1)$, the fractional derivative operator $D^\alpha$ is defined for $h \lambda$-Hölder function (with $\lambda > \alpha$) by

$$
D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s)ds.
$$

Note that if the function $h$ is continuously differentiable and $h(0) = 0$. The derivation for integral with parameters gives

$$
D^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} h'(s)ds.
$$

More information on fractional differential operator can be found in Samko et al. (1993).

### A.4 A result on inhomogeneous Poisson process

We recall the following well-known result.

**Proposition A.5.** (Exponential formula). Let $N$ be an inhomogeneous Poisson process with intensity $\nu$ and $f$ be a complex measurable function defined on $\mathbb{R}^+$ such that $\text{Re}(f) \leq 0$. Consider the function

$$
N_f(t) = \sum_{i=1}^{N_f} f(T_i),
$$

where $(T_i)_{i \in \mathbb{N}}$ are the jump times of $N$. For any $t \geq 0$, we have

$$
\mathbb{E}[\exp(N_f(t))] = \exp \left( \int_0^t (e^{f(s)} - 1) \nu(s)ds \right).
$$