NOETHERIANITY AND ROOTED TREES

DANIEL BARTER

Abstract. Let $T$ be the category whose objects are rooted trees and morphisms are order embeddings preserving the root. We prove that finitely generated representations of $T$ are Noetherian using techniques developed by Sam and Snowden which generalize classical Gröbner theory. The proof uses a relative version of Kruskal’s tree Theorem.

1. Introduction

Let $C$ be a category and $\text{Vec}$ the category of vector spaces over a field $k$ with arbitrary characteristic. Write $\text{Rep}(C)$ for the category of functors $C \to \text{Vec}$. Such functors are called representations of $C$. Let $T$ be the category whose objects are rooted trees and morphisms are order embeddings preserving the root. In this paper we shall prove

**Theorem 1.** Finitely generated $T$-representations are Noetherian.

Theorem 1 is proved using Gröbner categories, first defined by Richter in [3], and further developed by Sam and Snowden in [4]. Gröbner categories reduce Noetherianity questions to combinatorial questions. In all examples so far, the combinatorial questions reduce to Higman’s lemma, or some variant. For the category $T$, the combinatorial question reduces to Kruskal’s tree Theorem.

1.1. Motivation and previous work. Theorem 1 is a generalization of theorem 2, which was proved independently by Church, Ellenberg and Farb in [7] and by Snowden in [5]:

**Theorem 2.** Let $\text{FI}$ be the category of finite sets with injections. Then finitely generated $\text{FI}$-representations over a field of characteristic 0 are Noetherian.

Theorem 2 has the following Corollary, due to Church, Ellenberg and Farb in [7]:

**Corollary 3.** Let $M$ be a manifold and $S$ a finite set. Then $S \mapsto C_S(M) = \{\text{injections } S \to M\}$ is a functor from $\text{FI}^{\text{op}}$ into the category of manifolds, and $S \mapsto H^d(C_S(M), \mathbb{Q})$ is a finitely generated $\text{FI}$-representation.

We hope that Corollary 3 convinces the reader that Theorems 1 and 2 are interesting. Motivated by Theorem 2, Sam and Snowden developed the theory of Gröbner categories in [4]. They proved

**Theorem 4.** Let $C$ be quasi-Gröbner category. Then every finitely generated $C$-representation is Noetherian.

Sam and Snowden also proved that the categories $\text{FI}_d, \text{FS}^{\text{op}}, \text{VA}, \text{FI}_G, \text{FS}_G^{\text{op}}$ are quasi-Gröbner. In all of these examples, the objects are parameterized by the natural numbers. The category $T$ is the first known example of a quasi-Gröbner category whose objects do not have a natural bijection with $\mathbb{N}^p$. 

\footnote{Date: September 15, 2015.}
1.2. Open problems. This paper raises several questions:

(1) Are there any interesting spaces which are acted upon by tree automorphism groups? If we could find non trivial functors from $T^{op}$ into the category of spaces, then Theorem[1] might imply results like Corollary[3].

(2) If $V$ is a finitely generated $T$-representation, what can one say about the function $T ⟼ \dim V_T$? If $C$ is a quasi-Gröbner category, then it is reasonable to expect that the Hilbert series of finitely generated $C$-representations will be nice. For example, finitely generated $\text{FI}$-representations have rational Hilbert series.

(3) Kruskal’s tree Theorem is an important part of the graph minor Theorem. The category $T$ is quasi-Gröbner because of Kruskal’s tree Theorem. Is there any category which is quasi-Gröbner because of the graph minor Theorem?

1.3. Acknowledgments. I want to thank John Wiltshire-Gordon and Andrew Snowden for many useful discussions. This work would not have been possible without them. I want to thank Steven Sam for expressing interest in the categories $T$ and $PT$ from an early stage, and suggesting ideas for future work.

2. Rooted Trees

In this section, we explain the terms and notation used throughout the paper. A tree is a connected finite graph with no loops. A rooted tree is a tree equipped with a root vertex. In a rooted tree, we orient every edge towards the root vertex. When drawing rooted trees, the root vertex is at the bottom. Here is an example:

![Rooted Tree Diagram]

If $v$ is a vertex, write $\text{in}(v)$ for the set of incoming edges. When we draw a picture of a rooted tree, we implicitly put an ordering on $\text{in}(v)$ for each vertex $v$. A planar rooted tree is a rooted tree equipped with a total ordering on $\text{in}(v)$ for each vertex $v$. Given a rooted tree $T$ we can build a partially ordered set as follows: The elements are vertices and the relations are generated by the edges pointing towards the root vertex. In other words, given vertices $v, w$ we say that $v \leq w$ if there is a downward path from $v$ to $w$. We call this order the tree order on the vertices of $T$. The root vertex is larger than all other vertices in the tree order. Let $T$ be a planar rooted tree. We can totally order the vertices using a clockwise depth-first tree walk. This total ordering will be called the depth-first ordering on the vertices and
is denoted by $\triangleleft$. When we say order embedding, we mean with respect to the tree order.

$$\mathbf{FT} = \left\{ \begin{array}{l} \text{Objects are rooted trees and morphisms are} \\
\text{order embeddings} \end{array} \right\}$$

$$\mathbf{FPT} = \left\{ \begin{array}{l} \text{Objects are planar rooted trees and morphisms are order embeddings which also preserving the depth-first ordering on vertices} \end{array} \right\}$$

$$\mathbf{T} = \left\{ \begin{array}{l} \text{Objects are rooted trees and morphisms are} \\
\text{order embeddings preserving the root} \end{array} \right\}$$

$$\mathbf{PT} = \left\{ \begin{array}{l} \text{Objects are planar rooted trees and morphisms are order embeddings that preserve the root and the depth-first ordering on vertices} \end{array} \right\}$$

The categories $\mathbf{T}$ and $\mathbf{PT}$ are our main focus, but for many of the proofs, it is useful to work in $\mathbf{FT}$ and $\mathbf{FPT}$. The morphisms in each of the above four categories must be injective on vertices. We can now state our main Theorem, from which Theorem 1 follows.

**Theorem 5.** The category $\mathbf{PT}$ is Gröbner and the forgetful functor $\mathbf{PT} \to \mathbf{T}$ is essentially surjective and has property (F).

Theorem 5 says that $\mathbf{T}$ is quasi-Gröbner. We refer the reader to [4] where the theory of Gröbner categories is developed.

3. A relative version of Kruskal’s tree theorem

We define a sequence of trees $B_1, B_2, B_3, B_4, \ldots$ as follows: $B_n$ is the graph with vertex set $\{\ast\} \cup \{1, \ldots, n\}$ and edges $(i, \ast)$. Diagrammatically, we have

These planar rooted trees form building blocks in the category $\mathbf{FPT}$.

**Lemma 6.** Let $T$ be a planar rooted tree. Let $v$ be a vertex of $T$. Let $T_v$ be the sub tree of $T$ which contains everything above and including $v$. Let $T^v$ be the sub tree of $T$ obtained by removing everything in $T_v$ strictly above $v$. Then we have the following pushout square in $\mathbf{FPT}$:

$$\begin{array}{c}
T_v \\
\downarrow \\
v \\
\downarrow \\
T^v
\end{array} \rightarrow \begin{array}{c}
T \\
\uparrow \\
\uparrow \\
\uparrow \\
 \end{array}$$

Here is an example of such a pushout square:
Proof. To define a morphism $T \rightarrow U$, we need to send edges in $T$ to paths in $U$ so that domains and codomains are preserved. Since every edge in $T$ is contained in either $T_v$ or $T^v$, the lemma follows.

Lemma 7. Assume that $T$ is a planar rooted tree and $v_1, \ldots, v_n$ are the vertices with distance 1 from the root. Then $T$ is a colimit of the following diagram (that we have only drawn for $n = 3$):

\[
\begin{array}{c}
\begin{array}{c}
T_{v_1} \\
v_1
\end{array} & \quad & \begin{array}{c}
T_{v_2} \\
v_2
\end{array} & \quad & \begin{array}{c}
T_{v_3} \\
v_3
\end{array} & \quad & B_3
\end{array}
\]

Proof. This follows by repeated application of Lemma 6.

Lemma 8. We have a natural isomorphism

\[
\text{Mor}_{\text{FPT}}(B_n, T) = \begin{cases}
\text{distinct vertices } v, v_1, \ldots, v_n \in T \text{ such that } v_i \leq v \text{ in the tree order, the } v_i \text{ are pairwise incomparable in the tree order and } v_1 < v_2 < \cdots < v_n \text{ in the depth-first order}
\end{cases}
\]

Let $T$ be a planar rooted tree. Define $\text{PT}_T$ to be the set of morphisms in $\text{PT}$ with domain $T$. If $f, g \in \text{PT}_T$, we define $f \leq g$ if there is a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{f} & U \\
\downarrow{g} & & \downarrow \\
V & & 
\end{array}
\]

in the category $\text{PT}$. Equivalently, $f \leq g$ if there is a morphism $h$ such that $g = hf$. This is called the divisibility quasi-order on $\text{PT}_T$. Now we can state the relative version of Kruskal’s tree Theorem:

Theorem 9. The quasi-order on $\text{PT}_T$ is a well-quasi-order.

The $T = \bullet$ case is very similar to Kruskal’s tree Theorem. Indeed, Lemma 10 is proved by Draisma in [2]. We include a proof to establish notation and demonstrate the main proof technique in the easiest case.

Lemma 10. Theorem 9 is true when $T = \bullet$.

Proof. We use the Nash-Williams theory of good/bad sequences that is explained in [1 Chapter 12]. Suppose that $\text{PT}_{\bullet}$ is not well-quasi-ordered. Given $n \in \mathbb{N}$, assume inductively that we have chosen a sequence $T_0, \ldots, T_{n-1}$ of planar rooted trees such that some bad sequence of planar rooted trees starts with $T_0, \ldots, T_{n-1}$. Choose $T_n$ with a minimal number of vertices such that some bad sequence starts $T_0, T_1, \ldots, T_n$. Then $(T_n)_{n \in \mathbb{N}}$ is a bad sequence. We call $(T_n)$ a minimal bad sequence. Let $v_1, \ldots, v_d$ be the vertices in $T_n$ whose distance from the root is 1, ordered with respect to the depth-first ordering. Let $A_n = T_{n,v_1}T_{n,v_2} \cdots T_{n,v_n}$. If we think of each sequence $A_n$ as a set, we can define $A = \cup_n A_n$. We claim that $A$ is well-quasi-ordered. Let $(U_k)$ be a sequence in $A$. Then $U_k \in A_{n(k)}$, so we have a morphism

\[
\text{Mor}_{\text{FPT}}(B_n, T) = \begin{cases}
\text{distinct vertices } v, v_1, \ldots, v_n \in T \text{ such that } v_i \leq v \text{ in the tree order, the } v_i \text{ are pairwise incomparable in the tree order and } v_1 < v_2 < \cdots < v_n \text{ in the depth-first order}
\end{cases}
\]
$U_k \rightarrow T_{n_k}$ in $\text{FPT}$. This morphism does not preserve the root, but we can modify what the morphism does on the root vertex in the following way:

\[
\begin{array}{c}
\text{node} \\
\rightarrow \\
\text{node}
\end{array}
\]

This allows us to convert $U_k \rightarrow T_{n_k}$ into a morphism which witnesses $U_k \leq T_{n_k}$. Choose $p$ so that $n(p)$ is the smallest element of $\{n(k)\}$. Then we have the following sequence

\[T_0, \ldots, T_{n(p)-1}, U_p, U_{p+1}, \ldots\]

By the minimality of $(T_n)$, it must have a good pair. If $T_i \leq U_j$ then we have $T_i \leq T_{n(j)}$. This is a contradiction because $i < n(p) \leq n(j)$. Therefore there must be a good pair in $(U_k)$. Since our choice of sequence in $A$ was arbitrary, it follows that $A$ is well-quasi-ordered. Consider the following sequence of words in $A$:

\[A_0, A_1, A_2, \ldots\]

By Higman’s lemma, we must have $A_i \leq A_j$ for some $i < j$. What this means is that there is an order preserving injection $\phi : A_i \rightarrow A_j$ such that $U \leq \phi(U)$ for each $U \in A_i$. This gives us $T_i \leq T_j$ which is a contradiction.

**Lemma 11.** Theorem 9 is true when $T = B_n$.

**Proof.** The proof is by induction on $n$. The base case is $n = 1$. Elements in $\text{PT}_{B_1}$ are planar rooted trees with a distinguished non-root vertex and $T \leq U$ if there is a morphism $T \rightarrow U$ preserving the root and the distinguished non-root vertex. Choose a minimal bad sequence $(T_n)$ in $\text{PT}_{B_1}$. Define $A_n$ as in Lemma 10. We can break the sequence $A_n$ up as $B_nU_nC_n$ where $U_n$ is the tree containing the distinguished vertex, $B_n$ is the sequence of trees coming before $U_n$ and $C_n$ is the sequence of trees coming after $U_n$ in the depth first ordering. There are two cases we need to consider:

1. Firstly, suppose that for an infinite subsequence $(U_{n_k})$ of $(U_n)$, the distinguished vertex in $T_{n_k}$ is the root of $U_{n_k}$. Consider the following sequence

\[(B_{n_1}, U_{n_1}, C_{n_1}), (B_{n_2}, U_{n_2}, C_{n_2}), \ldots\]

A product of well-quasi-orders is a well quasi-order. By lemma 10 there must be a good pair $(B_{n_1}, U_{n_1}, C_{n_1}) \leq (B_{n_2}, U_{n_2}, C_{n_2})$ which gives us $T_{n_1} \leq T_{n_2}$ in $\text{PT}_{B_1}$. This is a contradiction.

2. Secondly, suppose that for an infinite subsequence $(U_{n_k})$ of $(U_n)$, the distinguished vertex in $T_{n_k}$ is not the root of $U_{n_k}$. The obvious morphism $U_{n_k} \rightarrow T_{n_k}$ does not preserve roots, but we can use the same trick as in lemma 10 to get $U_{n_k} \leq T_{n_k}$ in $\text{PT}_{B_1}$. Since we started with a minimal bad sequence, $\{U_{n_k}\}$ must be well-quasi-ordered, therefore the sequence

\[(B_{n_1}, U_{n_1}, C_{n_1}), (B_{n_2}, U_{n_2}, C_{n_2}), \ldots\]

must have a good pair $(B_{n_1}, U_{n_1}, C_{n_1}) \leq (B_{n_2}, U_{n_2}, C_{n_2})$ which gives us $T_{n_1} \leq T_{n_2}$ in $\text{PT}_{B_1}$. This is a contradiction.

One of these two cases must occur. Therefore we have proved that $\text{PT}_{B_1}$ is well-quasi-ordered. Now assume that $\text{PT}_{B_i}$ is well-quasi-ordered for $i < n$. We prove that $\text{PT}_{B_n}$ is well-quasi-ordered. Elements of $\text{PT}_{B_n}$ are planar rooted trees with $n$ distinguished non-root vertices $v_1, \ldots, v_n$ that are incomparable in the tree order and ordered in the depth-first order. We have $T \leq U$ if there is a morphism $T \rightarrow U$ in $\text{FPT}$ that preserves the root and
the distinguished non root vertices. Assume that \((T_n)\) is a minimal bad sequence in \(PT_{B_n}\). As usual, form the sequence \((A_n)\). Define \(\omega(A_n)\) as follows: replace each tree in \(A_n\) with the number of distinguished vertices of \(T_n\) it contains, then delete the zeros. By the pigeonhole principle

\[ \omega(A_1), \omega(A_2), \omega(A_3), \ldots \]

must contain some sequence \(m_1, \ldots, m_d\) an infinite number of times. Let \((T_{nk})\) be the corresponding subsequence of \((T_n)\). We must now consider two cases:

1. Suppose \(d = 1\). Write \(A_{nk} = B^1_{nk} U_{nk} B^d_{nk}\) where \(U_{nk}\) has \(m_1\) of the distinguished vertices. Now use the induction hypothesis to get a contradiction.

2. If \(d > 1\) then write \(A_{nk} = B^0_{nk} U^1_{nk} B^1_{nk} \ldots U^d_{nk} B^d_{nk}\) where \(U^i_{nk}\) has \(m_i\) of the distinguished vertices. Now use the induction hypothesis to get a contradiction.

\[\square\]

Proof of Theorem 9. We induct on the number of vertices in \(T\). Lemma 11 is the base case. Choose a non–root vertex \(v\) in \(T\) that has valence \(\geq 2\). Choose a sequence \((\phi_n : T \to U_n)\) in \(PT_T\). Then we get sequences \(\phi_{n,v} : T_v \to U_{n,\phi_n(v)}\) and \(\phi^v_n : T^v \to U_{\phi_n(v)}^n\) in \(PT_T\) and \(PT_{T^v}\) respectively. By induction, there must be a good pair \((\phi_{i,v}, \phi^v_i) \leq (\phi_{j,v}, \phi^v_j)\). This induces \(\phi_i \leq \phi_j\) which completes the proof.

\[\square\]

4. Proof of theorem 5

In this section, we prove that \(PT\) is a Gröbner category and that the forgetful functor \(PT \to T\) has property (F) and is essentially surjective. First, let us recall the definition of a Gröbner category from [4]. Let \(C\) be a small directed category. Write \(C_x = \bigcup_y \text{Mor}_C(x, y)\). If \(f : x \to y\) and \(g : x \to z\) are elements of \(C_x\) then we write \(f \leq g\) if there is a commutative triangle

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow{g} & & \downarrow{} \\
z & & \\
\end{array}
\]

We call this quasi-order on \(C_x\) the **divisibility order**. It is intrinsic to \(C\). An **admissible order** on \(C_x\) is a well-order \(\preceq\) such that if \(f \preceq f'\) then \(gf \preceq gf'\) whenever this makes sense. Admissible orders are not intrinsic to \(C\): they are extra structure.

**Definition 12.** We call \(C\) **Gröbner** if each divisibility order \(C_x\) is a well-quasi-order and each \(C_x\) admits an admissible order.

Theorem 9 says that the divisibility order on \(PT_T\) is a well-quasi-order. Therefore, to prove that \(PT\) is Gröbner, we need to construct admissible orders on each \(PT_T\). Let \(T, U\) be planar rooted trees and choose a morphism \(\phi : T \to U\) in \(PT\). If \(e\) is an edge in \(T\), label every edge in the path \(\phi(e)\) with the distance between target\((e)\) and root\((T)\) in \(T\). (edges point towards
the root. Now we go on a clockwise depth-first tree walk along $U$ (depth-first tree walks are defined in [6, chapter 5]). As we are traveling, record the path as follows:

1. If we travel up an edge marked with an $i$, write (.

2. If we travel down an edge marked with an $i$, write )

3. If we travel up an unmarked edge, write (.

4. If we travel down an unmarked edge, write ).

The resulting string is called the **Catalan word** of $\phi$.

**Example 13.** Consider the map:

![Tree Diagram](image)

Its Catalan word is

(((())())())((())())

If $T = \bullet$ then we recover the the standard bijection between planar rooted trees and strings of balanced brackets which is described in [6, chapter 5].

**Lemma 14.** The mapping $\phi \mapsto$ Catalan word is injective.

*Proof.* We can reproduce $\phi$ from its Catalan word as follows. The top row of parentheses gives the target. The bottom row of numbers tells us how the domain is mapped in, and also gives the domain since all tree maps are fully faithful.

We use Catalan words to equip each set $PT_T$ with an admissible order. Given a Catalan word, build the tuple $(p, n)$ where $p$ is the top row and $n$ is the second row. We order the alphabets in the following way:

\[ ) \prec ( \prec - \prec 0 \prec 1 \prec 2 \prec 3 \prec \ldots \]

Order words in the parentheses alphabet using the length lexicographic ordering. Order words in $\{-, 0, 1, 2, \ldots\}$ using lexicographic ordering. Given two Catalan words $(p, n), (p', n')$, define $(p, n) \prec (p', n')$ if $p \prec p'$ or $p = p'$ and $n \prec n'$.

**Lemma 15.** Let $f, g : T \to U$ be morphisms in $PT$ such that $f \prec g$ with respect to the above Catalan word ordering. Let $h : U \to V$ be a morphism. Then $hf \prec hg$.

*Proof.* First we interpret $f \prec g$. When we go on a clockwise depth-first tree walk along $U$, the first time we notice a difference in the edge labeling, the label for $g$ is larger than the label for $f$. Now go on a clockwise depth-first tree walk along $V$ labeled by $hf$ and $hg$. The first difference that we notice is going to be induced by the difference we noticed on our walk along $U$ and the label for $hg$ will be bigger than the label for $hf$ because the labels are mapped from $U$. \(\square\)
This completes the construction of admissible orders on each $\mathbf{PT}_T$. Therefore we have proved that $\mathbf{PT}$ is Gröbner. To conclude the proof of Theorem 5, we need to prove that the forgetful functor $i : \mathbf{PT} \to \mathbf{T}$ is essentially surjective and has property (F). First we recall the definition of property (F).

**Definition 16.** Let $i : \mathcal{C}' \to \mathcal{C}$ be a functor. We say that $i$ has **property (F)** if for each principal projective $P_x = \mathbb{C}C(x, -)$ in $\text{Rep}(\mathcal{C})$, the $\mathcal{C}'$-representation $i^* P_x$ is finitely generated.

Let $J : \mathbf{PT} \to \mathbf{T}$ be the functor which forgets the plane ordering. Since every rooted tree can be drawn on the plane it follows that $J$ is essentially surjective. Let $U$ be a rooted tree and $V$ a planar rooted tree. Then we have

$$T(U, J(V)) = \mathbf{PT}(U_1, V) \sqcup \mathbf{PT}(U_2, V) \sqcup \cdots \sqcup \mathbf{PT}(U_e, V)$$

where $U_1, \ldots, U_e$ are all the planar representations of $U$. This implies that

$$J^* P_U = \bigoplus_{i=1}^{e} P_{U_i}$$

which proves that $J$ has property (F).

**References**

[1] Reinhard Diestel. *Graph theory*. Springer Verlag, 2005.
[2] Jan Draisma. *Noetherianity up to symmetry*. Springer, 2014.
[3] Günther Richter. *Noetherian semigroup rings with several objects*. Elsevier Science Publishers, 1986.
[4] Steven V Sam and Andrew Snowden. Grobner methods for representations of combinatorial categories. 2014, 1409.1670.
[5] Andrew Snowden. Syzygies of Segre embeddings and $\Delta$-module. *Duke Mathematics Journal*, 162, 2013.
[6] Richard Stanley. *Enumerative combinatorics volume 2*. Cambridge University Press, 1999.
[7] Jordan Ellenberg Thomas Church and Benson Farb. FI-modules and stability for representations of symmetric groups. 2012, 1204.4533.