Quantum chromodynamics (QCD) with large number of massless fermion flavors has seen a resurgence of interest due to its connection with technicolor models, originally proposed by Weinberg and Susskind [1], which fall into the category of Beyond the Standard Model Theories. They possess intrinsically attractive features. They do not resort to fundamental scalars to reconcile local gauge symmetry with massive mediators of interaction and have close resemblance with well-studied fundamental strong interactions, i.e., QCD. However, their simple versions do not live up to the experimental electroweak precision constraints, in particular the ones related to flavor changing neutral currents. Walking models containing a conformal window and an infrared fixed point can possibly cure this defect and become phenomenologically viable [2]. This scenario motivates the investigation of QCD for similar characteristics. One looks for such behavior of QCD for large number of light flavors albeit less than the critical value where asymptotic freedom sets in, i.e., $N_f^{c1} = 16.5$, a Nobel prize winning result known since the advent of QCD, [3]. Just as $N_f$ dictates the peculiar behavior of QCD in the ultraviolet, we expect it to determine the onslaught of its emerging phenomena in the infrared, i.e., chiral symmetry breaking and confinement.

Whereas the self interaction of gluons provides anti-screening, the production of virtual quark-antiquark pairs screens and debilitates the strength of this interaction of non abelian origin. For real QCD, light flavors are small in number and hence yield to the gluonic influence which triggers confinement and chiral symmetry breaking. One needs to establish if there is another critical value $N_f^{c2} < N_f^{c1}$ which can sufficiently dilute the gluon-gluon interactions to restore chiral symmetry and deconfine color degrees of freedom. Such a phase transition lies at the non perturbative boundary of the interactions under scrutiny and hence we cannot expect to extract sufficiently reliable information from multiloop calculations of the QCD $\beta$-function. Purely non perturbative techniques are required to solve the problem. Lattice studies in the infrared indicate that just below $N_f^{c1}$, chiral symmetry remains unbroken and color degrees of freedom are unconfined [4]. Below this conformal window, for $8 < N_f^{c2} < 12$, the evolution of the beta function in the infrared is such that QCD enters the phase of dynamical mass generation as well as confinement.

In continuum, Schwinger-Dyson equations (SDEs) of QCD provide an ideal framework to study its infrared properties. [5]. These are the fundamental equations of any quantum field theory, linking all its defining Green functions to each other through intricately coupled nonlinear integral equations. As their formal derivation through variational principle makes no appeal to the weakness of the interaction strength, they naturally connect the perturbative ultraviolet physics with its emerging non perturbative properties in the infrared sector within the same framework. The simplest two-point quark propagator is a basic object to analyze dynamical chiral symmetry breaking and confinement. Within the formalism of the SDEs, the inverse quark propagator can be expressed as

$$ S^{-1}(p) = Z_2(i\gamma \cdot p + m) + \Sigma(p), \quad (1) $$

where $\Sigma(p)$ is the quark self energy:

$$ \Sigma(p) = Z_1 \int \frac{d^4q}{(2\pi)^4} g^2 \Delta_{\mu\nu}(p-q) \frac{\lambda^a}{2} \gamma_{\mu} S(q) \Gamma^a_\nu(q,p), \quad (2) $$

where $Z_1 = Z_1(\mu^2, \Lambda^2)$ and $Z_2 = Z_2(\mu^2, \Lambda^2)$ are the renormalization constants associated respectively with the quark-gluon vertex and the quark propagator. $\Lambda$ is the ultraviolet regulator and $\mu$ is the renormalization point. The solution to this equation is

$$ S^{-1}(p) = \frac{i\gamma \cdot p + M(p^2)}{Z(p^2, \mu^2)}, \quad (3) $$

where $Z(p^2, \mu^2)$ is the quark wavefunction renormalization and the quark mass function $M(p^2)$ is renormalization group invariant. This equation involves the quark-gluon vertex $\Gamma^a_\nu(q,p)$ and the gluon propagator $\Delta_{\mu\nu}(p-q)$.

As a consequence of a patient effort, spanning several decades to unravel gluon propagator $\Delta_{\mu\nu}$ in the infrared,
Lattice as well as SDE studies have finally converged on its massive or so-called decoupling solution, see for example [6]. After the gluon propagator solution in the quenched approximation has been chiselled, we now have the first quantitatively reliable glimpses of its quark flavor dependence by incorporating [20] $N_f = 0, 2, 4$ dynamical quark flavors in the QCD action. In Ref. [7], the authors studied the gluon propagator with light quark masses ranging from 20 to 50 [MeV], while the strange quark is roughly set to 95 [MeV] and the heavy charm to 1.51 [GeV] (in $\bar{\text{MS}}$ at $q^2 = 4$ [GeV$^2$]). Very recently also, the authors of Ref. [8] developed a "partially unquenched" approach to incorporate flavor effects in the quenched SDE and yielded results in agreement with those in Ref. [7]. In any case, this two-point function serves as a crucial input to study the quark propagator. The only other ingredient is the three-point quark-gluon vertex $\Gamma^a_{\mu}(q,p)$. Significant advances have been made in pinning it down through its key attributes in the ultraviolet and infrared domains [9]. More recently, the seeds of the most general ansatz for the fermion-boson vertex appeared in [10] and its full blown extension was presented in [11]. Significantly, this ansatz contains nontrivial factors associated with those tensors whose appearance is expressly driven by dynamical chiral symmetry breaking in a perturbatively massless theory. This novel feature enables a direct and positive comparison with the best available symmetry preserving solutions of the inhomogeneous Bethe-Salpeter equation for the vector vertex. The encouraging outcome indicates that this model is likely to provide a much-needed tool for use in Poincaré-symmetric Bethe-Salpeter equation for the vector vertex. Before this is achieved, we restrict ourselves to an effective though efficacious approach. Following the lead of Maris et. al. [12], we employ the following ansatz which has sufficiently integrated strength in the infrared to achieve dynamical mass generation:

$$Z_1 g^2 \Delta_{\mu\nu}(p-q) \Gamma_{\nu}(p,q) \rightarrow g_{\text{eff}}^2 \Delta_{\mu\nu}^N(p-q) \frac{\lambda^a}{2} \gamma_\nu , \quad (4)$$

where

$$\Delta_{\mu\nu}^N(q) = \frac{D(q^2)}{q^2} \left[ \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right] . \quad (5)$$

The effective coupling $g_{\text{eff}}$ is chosen to correctly reproduce the static as well as dynamic properties of mesons below 1 GeV and reproduce perturbation theory in the ultraviolet, see for example review [13] and references therein. Moreover, our modern understanding of the flavor dependence of the gluon propagator provides us with the solid basis to use the following nonperturbative model [14]:

$$D(q^2) = \frac{z(\mu^2)}{q^2} \left( q^2 + M^2 \right)^{-1} \left( M^2 - 15g^2(A^2)/24 + M^2 m_0^2 \right) \quad (6)$$

to describe the gluon dressing renormalized in MOM scheme at $q^2 = \mu^2$. This model is based on the tree-level gluon propagator obtained with the Renormalized Gribov-Zwanziger (RGZ) action which have been shown to describe properly the lattice data in the infrared sector (see refs. [14, 17]). The overall factor $z(\mu^2)$ is introduced to guarantee the multiplicative MOM renormalization prescription, namely, $D(\mu^2) = 1$, and implies no physical consequence as the effective coupling, $g_{\text{eff}}$, is further adjusted to reproduce properly the meson phenomenology. The mass parameter $M^2$ from Eq. (6) is related to the condensate of auxiliary fields only emerging to preserve locality for the RGZ action, i.e., when incorporating the horizon condition to the action. ($A^2$) is the dimension-two gluon condensate that has been elsewhere very much studied [10]. $m_0$ is another mass scale, $m_0^2 = z(\mu^2) \lim_{q^2 \rightarrow 0} q^2/D(q^2)$. It is related to the above mentioned condensates and to the Gribov-Zwanziger parameter determined by the well-known horizon condition. To determine its mass parameters, Eq. (4) can be fitted to the gluon propagator lattice data analyzed in Ref. [7] with the goal of scrutinizing the quark flavor effects on ghost and gluon propagators. These data had been orginally obtained from gauge configurations simulated at several lattices with $N_f=2$ [18] and $N_f=4$ [19] mass-twisted lattice flavors (generated within the framework of ETM collaboration [21, 22]), and with $N_f = 0$ [23]. As can be seen from Fig. 1 we thus obtain the following behavior in terms of the number of fermion flavors for the mass parameters:

$$m_0 = 1.011 (9.161 - N_f)^{-1/2} \text{[GeV]} ,$$

$$g^2(A^2) = 0.474 (16.406 - N_f) \text{[GeV]} ,$$

$$M^2 = 4.85 \text{[GeV]} ; \quad (7)$$

FIG. 1: Parameters $g^2 < A^2 >$ and $1/m_0^2$ in terms of the numbers of flavors and the linear fits yielding the results of Eq. (7).
where $M^2$ is required not to depend on the number of fermion flavors. This is a reasonable approximation, fairly confirmed by a free fit. These parameters, plugged into Eq. (6), result in the gluon propagator prediction, for any flavor number, shown in Fig. 2 to describe properly the lattice data for $N_f = 0, 2, 4$ of Ref. [7]. Thus, we can efficaciously model the dilution of the gluon-gluon interactions with increasing flavor number in order to study the chiral restoration mechanism.

We can now employ the gap equation to provide quantitative details of chiral symmetry breaking in terms of the quark mass function for an increasing number of light quarks. We take the effective coupling $g_{eff}$ to be independent of $N_f$ which is justified by the results of [2] (see Eq.(5.2)) which suggest that an effective coupling can be constructed such that there is an absence of any flavor dependence in the infrared region, more precisely starting from $q^2 \lesssim 1 \text{ GeV}^2$. Note that we have not considered the flavor dependence which would arise from the quark-gluon vertex. No explicit handle on this dependence is available at the moment. Within the abelian theory of QED, restrictions imposed by the all order multiplicative renormalizability of the photon propagator may provide a handle on the transverse part of the electron-photon vertex, see the last reference in [9]. A consequent non perturbative construction of such a vertex with imprints of the massless charged fermion flavors and its subsequent extension to QCD is still not available. Once the quark mass function is available for varying light quark flavors, one can investigate any of the interrelated order parameters, namely, the Euclidean pole mass defined as $m^2_{\text{dy}} + M^2(p^2 = m^2_{\text{dy}}) = 0$, the quark-antiquark condensate which is obtained from the trace of the quark propagator or the pion leptonic decay constant $f_\pi$ defined through the Pagel-Stokar equation [24], or through considering the residue at the pion pole of the meson propagator. Each of these quantities involves the quark wave-function renormalization, the mass function and/or its derivatives and is hence calculable from the solution for the full quark propagator. Moreover, these order parameters can help locate the critical number of flavors above which chiral symmetry is restored. We choose to present the Euclidean pole mass of the quark, shown in Fig. 4. At a critical value of about $N_f \approx 8$, chiral symmetry breaking ceases to exist. The phase transition appears second order, described by the following mean field behavior (shown by the solid line in Fig. 4) :

$$m_{\text{dy}} \sim \sqrt{N_f^c - N_f}. \quad (8)$$

It has been established that confinement is related to the analytic properties of QCD Schwinger functions which are the Euclidean space Green functions, namely, propagators and vertices. One deduces from the reconstruction theorem [25] that the only Schwinger functions which can be associated with expectation values in the Hilbert space of observables; namely, the set of measurable expectation values, are those that satisfy the axiom of reflection positivity. When that happens, the real-axis mass-pole splits,

FIG. 2: Gluon propagator in terms of momenta, as given from Eqs. (6), for different number of fermion flavors. The points correspond to the same "unquenched" ($N_f = 2,4$), and quenched ($N_f = 0$) lattice data analyzed in Ref. [7].

FIG. 3: The quark mass function diminishes height for increasing light quark flavors. Above $N_f = 8.33685$, only the chirally symmetric Wigner solution exists.

FIG. 4: Quark pole mass in the Euclidean space clearly demonstrates that chiral symmetry is restored above a critical number of quark flavors. The solid line is the mean-field scaling fit described by Eq. (8).
moving into pairs of complex conjugate singularities. No mass-shell can be associated with a particle whose propagator exhibits such singularity structure. We define the following Schwinger function:

\[
\Delta(t) = \int d^3x \int \frac{d^4p}{(2\pi)^4} e^{i(p \cdot x - t)} \sigma_s(p^2),
\]

(9)
to study the analytic properties of the quark propagator; where \(\sigma_s(p^2)\) is the scalar term for the quark propagator in Eq. (3), that can be written in terms of the quark wavefunction renormalization and mass function as \(Z(p^2, \mu^2) M(p^2)/(p^2 + M(p^2))\). One can show that if there is a stable asymptotic state associated with this propagator, with a mass \(m\), then \(\Delta(t) \sim e^{-mt}\), whereas two complex conjugate mass-like singularities, with complex masses \(\mu = a \pm ib\) lead to an oscillating behavior of the sort \(\Delta(t) \sim e^{-\delta t} \cos(bt + \delta)\) for large \(t\). [26]. Fig. 5 analyzes this function for varying \(N_f\). The existence of oscillations clearly demonstrates that the quarks correspond to a confined excitation for small \(N_f\). With increasing \(N_f\), the onslaught of oscillations moves towards higher values of \(t\) and eventually never takes place above a critical \(N_f\) when quarks deconfine and correspond to a stable asymptotic state.

As an order parameter of confinement, we therefore employ \(\nu(N_f) = 1/\gamma_1(N_f)\), where \(\gamma_1(N_f)\) is the location of the first zero, [27]. This order parameter vanishes when confinement is lost. It is notable that when the dynamically generated mass approaches zero, \(\nu(N_f)\) diminishes rapidly, as can be seen in Fig. 6. This highlights the intimate connection between chiral symmetry restoration and deconfinement. In fact, within the numerical accuracy of our work, Fig. 6 shows \(N_f^c\) to be the same for both the transitions.

Therefore, the SDE analysis of the latest lattice results for the quark flavor dependence of the gluon propagator in the infrared hint towards chiral symmetry restoration and deconfinement in QCD when the number of light quark flavors exceeds a critical value of \(N_f^c \approx 8\). Noticeably, roughly the same number is also an indication of modern lattice QCD analysis [28].

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