Cosemisimple Hopf algebras are faithfully flat over Hopf subalgebras

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November 1, 2011

Abstract

The question of whether or not a Hopf algebra $H$ is faithfully flat over a Hopf subalgebra $A$ has received positive answers in several particular cases: when $H$ (or more generally, just $A$) is commutative, or cocommutative, or pointed, or when $K$ contains the coradical of $H$. We prove the statement in the title, adding the class of cosemisimple Hopf algebras to those known to be faithfully flat over all Hopf subalgebras. We also show that the third term of the resulting “exact sequence” $A \to H \to C$ is always a cosemisimple coalgebra.

Keywords: cosemisimple Hopf algebra, faithfully flat, right coideal subalgebra, quotient left module coalgebra

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Introduction

The issue of faithful flatness of a Hopf algebra (always over a field) over its Hopf subalgebras arises quite naturally in several ways. One direction is via the so-called Kaplansky conjecture ([Kap75]), which initially asked whether or not Hopf algebras are free over Hopf subalgebras (as an analogue to the Lagrange theorem for finite groups). The answer was known to be negative, with a counterexample having appeared in [OS74], but it is true in certain particular cases: using the notations in the abstract, $H$ is free over $A$ whenever $H$ is finite dimensional (Nichols-Zoeller Theorem, [Mon93, Theorem 3.1.5]), or pointed ([Rad77b]), or $A$ contains the coradical of $H$ ([Rad77a, Corollary 2.3]).

Montgomery then naturally asks whether one can get a positive result by requiring only faithful flatness of a Hopf algebra over an arbitrary Hopf subalgebra ([Mon93, Question 3.5.4]). Again, this turns out not to work in general (see [Sch00] and also [Chi10], where the same problem is considered in the context of whether or not epimorphisms of Hopf algebras are surjective), but
one has positive results in several important cases, such as that when $A$ is commutative ([AG03, Proposition 3.12]), or $H$ is cocommutative ([Tak72, Theorem 3.2], which also takes care of the case when $H$ is commutative). The most recent version of the question, asked in [Sch00], seems to be whether or not a Hopf algebra with bijective antipode is faithfully flat over Hopf subalgebras with bijective antipode.

Another way to get to the faithful flatness issue is via the problem of constructing quotients of affine group schemes. We recall briefly how this goes.

Let $A \rightarrow H$ be an inclusion of commutative Hopf algebras; in scheme language, $A$ and $H$ are affine groups, and the inclusion means that $\text{spec}(A)$ is a quotient group scheme of $\text{spec} H$. The Hopf algebraic analogue of the kernel of this epimorphism is the quotient Hopf algebra $\pi : H \rightarrow C = H/H A^+$, where $A^+$ stands for the kernel of the counit of $A$. $\pi$ is then normal, in the sense of [AD95, Definition 1.1.5]:

$$\text{Lker}(\pi) = \{a \in A \mid (\pi \otimes \text{id}) \circ \Delta(a) = 1_C \otimes a\}$$

equals its counterpart

$$\text{Rker}(\pi) = \{a \in A \mid (\text{id} \otimes \pi) \circ \Delta(a) = a \otimes 1_C\}.$$ 

This means precisely that $\text{spec}(C)$ is a normal affine subgroup scheme of $\text{spec}(A)$ ([Tak72, Lemma 5.1]). This gives a map $A \mapsto C$ from quotient affine group schemes of $H$ to normal subgroup schemes. One naturally suspects that this is probably a bijective correspondence, and this is indeed true (see [Tak72, Theorem 4.3] and also [DG70, III §3 7.2]). In Takeuchi’s paper faithful flatness is crucial in proving half of this result, namely the injectivity of the map $A \mapsto C$: one recovers $A$ as LKer($\pi$).

Many of the technical arguments and constructions appearing in this context go through in the non-commutative setting, so one might naturally be led to the faithful flatness issue by trying to mimic the algebraic group theory in a more general setting, where Hopf algebras are viewed as function algebras on a “quantum” group. This is, for example, the point of view taken in the by now very rich and fruitful theory of compact quantum groups, first introduced and studied by Woronowicz: the main characters are certain $C^*$ algebras $A$ with a comultiplication $A \rightarrow A \otimes A$ (minimal $C^*$ tensor product), imitating the algebras of continuous functions on a compact group (we refer the reader to [KS97, Chapter 11] or Woronowicz’s landmark papers [Wor87, Wor88] for details).

These objects are not quite Hopf algebras, but for any compact quantum group $A$ as above, one can introduce a genuine Hopf algebra $A$, imitating the algebra of representative functions on a compact group (i.e. linear span of matrix coefficients of finite dimensional unitary representations), and which contains all the relevant information on the representation theory of the quantum group in question. The abstract properties of such Hopf $(\ast)$-algebras have been axiomatized, and they are usually referred to as CQG algebras (see [KS97, 11.3] or the original paper [DK94], where the term was coined). They are always cosemisimple (as an analogue of Peter-Weyl theory for representations of compact groups), which is why we hope that despite the seemingly restrictive hypothesis of cosemisimplicity, the results in the present paper might be useful apart from any intrinsic interest, at least in dealing with Hopf algebraic issues arising in the context of compact quantum groups.

We now describe the contents of the paper.

In the first section we introduce the conventions and notations to be used throughout the rest of the paper, and also develop the tools needed to prove the main results. In §1.1 we set up a Galois correspondence between the sets of right coideal subalgebras of a Hopf algebra $H$ and the set of
quotient left module coalgebras of $H$. This correspondence then induces closure operators on both these sets, and the property of being closed under these operators will be very important for us in the proof of the main results. In §1.2 we recall basic results on categories of objects imitating Sweedler’s Hopf modules: these have both a module and a comodule structure, one of them over a Hopf algebra $H$, and the other one over a right coideal subalgebra or a quotient left module coalgebra of $H$. These categories feature prominently in the subsequent discussion.

Section 2 is devoted to the main results: theorem 2.0.3 provides sufficient conditions for a Hopf algebra $H$ to be faithfully flat over a Hopf subalgebra. These are satisfied when $H$ is cosemisimple, and in fact more generally, when its coradical is a Hopf subalgebra; cf. corollaries 2.0.4 and 2.0.5. We also investigate the case of cosemisimple $H$ further, proving in theorem 2.0.8 that for any Hopf subalgebra $A$, the quotient left $H$-module coalgebra $C = H/HA^+$ is always cosemisimple. $H/HA^+$ is the third term of the “exact sequence” which completes the inclusion $A \to H$, and the question of whether or not $C$ is cosemisimple arises naturally in the course of the proof of corollary 2.0.4, which shows immediately that it is true when $HA^+$ happens to be an ideal (both left and right).

Finally, in Section 3 we show that by contrast to the result in the title of the paper, cosemisimple Hopf algebras need not be faithfully coflat over quotient Hopf algebras. This is perhaps surprising, since one might expect that cosemisimple Hopf algebras would, in some vague sense, behave even better ‘coalgebraically’ than they do ‘algebraically’. To construct cosemisimple Hopf algebras, we make use of the free involutive Hopf algebra on a coalgebra, i.e. the left adjoint of the forgetful functor from involutive Hopf algebras to coalgebras. Such universal constructions have proved useful as counterexamples in the past (Takeuchi’s free Hopf algebra on a matrix coalgebra was the first example of Hopf algebra with non-bijective antipode, for example), and we regard this as further evidence of their usefulness, apart from their being rather interesting objects in their own right.

1 Preliminaries

In this section we make the preparations necessary to prove the main results. Throughout, we work over a fixed field $k$, so (co, bi, Hopf)algebra means (co, bi, Hopf)algebra over $k$, etc. We write $\text{Alg}$, $\text{Coalg}$, $\text{Bialg}$ and $\text{Halg}$ for the categories of $k$-algebras, coalgebras, bialgebras and Hopf algebras respectively, and also $\text{Halg}_2$ for the category of involutive Hopf algebras (i.e. those with $S^2 = \text{id}$) in Section 3. We will also refer to the notion of coring over a (not necessarily commutative) $k$-algebra; we refer to [BW03] for basic properties and results.

The reader should feel free to assume $k$ to be algebraically closed whenever convenient, as most results are invariant under scalar extension. In Section 3 we will specialize to characteristic zero. We assume basic familiarity with coalgebra and Hopf algebra theory, for example as presented in [Mon93]. Our notations are standard: $\Delta_C$ and $\varepsilon_C$ stand for comultiplication and counit of the coalgebra $C$ respectively, and we will allow ourselves to drop the subscript when it is clear which coalgebra is being discussed. Similarly, $S_H$ or $S$ stands for the antipode of the Hopf algebra $H$, $1_A$ (or just 1) will be the unit of the algebra $A$, etc. Sweedler notation for comultiplication is used throughout, as in $\Delta(h) = h_1 \otimes h_2$, as well as for left or right coactions: if $\rho : N \to N \otimes C$ ($\rho : N \to C \otimes N$) is a right (left) $C$-comodule structure, we write $n_0 \otimes n_1$ ($\langle n_{-1} \rangle \otimes n_0$) for $\rho(n)$.

One point worth making is that all the categories of (co)algebraic structures that we make use of are presentable (what is usually referred to as ‘locally presentable’ in the literature; cf. [AR94, Chapter 1]). Presentability is proven in [Por08a] for $\text{Alg}$, $\text{Coalg}$ and $\text{Bialg}$ (and many others, such as cocommutative bialgebras, etc.; see diagram 4.3 in that paper, which summarizes the results). For Hopf algebras, presentability follows for example from [Por08b, Proposition 4.3].
and the fact that \( \text{HALG} \to \text{COALG} \) is a right adjoint, due to Takeuchi ([Tak71]), or can be proven directly (as it can for \( \text{HALG}_2 \)). The important consequence for us is that all of these categories are complete and cocomplete, so one can talk about (co)products, pullbacks, pushouts, etc. in the category of algebras, coalgebras, bialgebras, or Hopf algebras.

In fact, in the same context, much more is true: there are various forgetful functors between the categories mentioned above, and one can always rely on the dictum ‘forgetting an algebraic structure is a right adjoint, while forgetting a coalgebraic structure is a left adjoint’. For example, the forgetful functor \( \text{HALG} \to \text{ALG} \) is a left adjoint, \( \text{HALG} \to \text{COALG} \) and \( \text{HALG}_2 \to \text{COALG} \) is a right adjoint, while \( \text{HALG} \to \text{BIALG} \) is both a left and a right adjoint, since the antipode counts as both ‘algebraic’ and ‘coalgebraic’. The recent paper [Por11] treats both interesting situations (i.e. adjoints of the inclusion \( \text{HALG} \to \text{BIALG} \)) in a unified manner, providing both explicit constructions; see also [Man88, §7] and [Par, Theorem 2.6.3] for the left adjoint to and [Ago09, Chi10] for the right adjoint.

It follows from the discussion in the previous paragraph that one never needs to consider (co)limits of objects more complicated than algebras or coalgebras: a limit of a diagram of Hopf algebras is the limit of the underlying diagram of coalgebras, while its colimit is the colimit of the underlying diagram of algebras (because \( \text{HALG} \to \text{ALG} \) is a left adjoint and hence preserves colimits, etc.).

We will also be working extensively with categories of (co)modules over (co)algebras, as well as categories of objects admitting both a module and a comodule structure, compatible in some sense than can be made precise below (cf. §1.2). These categories are always denoted by the letter \( \mathcal{M} \), with left (right) module structures appearing as left (right) subscripts, and left (right) comodule structures appearing as left (right) superscripts. All such categories are abelian (and in fact Grothendieck), and the forgetful functor from each of them to vector spaces is exact. The one exception from this notational convention is the category of \( k \)-vector spaces, which we simply call \( \text{Vec} \).

Recall that the category \( \mathcal{M}^H_f \) of finite dimensional right comodules over a Hopf algebra \( H \) is monoidal left rigid: every object \( V \) has a left dual \( V^* \) (at the level of vector spaces it is just the usual dual vector space), and one has adjunctions \((\otimes V, \otimes V^*)\) and \((V^* \otimes, V \otimes)\) (the left hand member of the pair is the left adjoint) on \( \mathcal{M}^H_f \).

We also use the correspondence between subcoalgebras of a Hopf algebra \( H \) and finite dimensional (right) comodules over \( H \): for such a comodule \( V \), there is a smallest subcoalgebra \( D = \text{COALG}(V) \leq H \) such that the structure map \( V \to V \otimes H \) factors through \( V \to V \otimes D \). Conversely, if \( D \leq H \) is a simple subcoalgebra, then we denote by \( V_D \) the simple right \( D \)-comodule, viewed as a right \( H \)-comodule. Then, for simple subcoalgebras \( D, E \leq H \), the product \( ED \) will be precisely \( \text{COALG}(V_E \otimes V_D) \), while \( S(D) = \text{COALG}(V^*) \).

1.1 Closure operators and (co)dominions

We will be dealing with the kind of situation studied extensively in [Tak79]: \( H \) will be a Hopf algebra, and for most of this section (and in fact the paper), \( \iota : A \to H \) will be a right coideal subalgebra, while \( \pi : H \to C \) will be a quotient left \( H \)-module coalgebra. Recall that this means that \( A \) is a right coideal of \( H \) (i.e. \( \Delta_H(A) \leq A \otimes H \)) as well as a subalgebra, and the induced map \( A \to A \otimes H \) is an algebra map; similarly, \( C \) is the quotient of \( H \) by a left ideal as well as a coalgebra, and the induced map \( H \otimes C \to C \) is supposed to be a coalgebra map.

Given a coalgebra map \( \pi : H \to C \), we write \( \overline{\pi} \) for \( \pi(h), h \in H \). \( H \) will naturally be both a left and a right \( C \)-comodule (via the structure maps \( \pi \otimes \text{id} \circ \Delta_H \) and \( \text{id} \otimes \pi \circ \Delta_H \) respectively),
while \( C \) has a distinguished grouplike element \( \overline{1} \), where \( 1 \in H \) is the unit. Write
\[
\pi H = C H \{ h \in H \mid \overline{h_1} \otimes h_2 = \overline{1} \otimes h \},
\]
\[
H^\pi = H^C = \{ h \in H \mid h_1 \otimes \overline{h_2} = h \otimes \overline{1} \}.
\]
These are what we were calling \( \text{LKER}(\pi) \) and \( \text{RKER}(\pi) \) back in the introduction, following the notation in [AD95]. They are the spaces of \( \overline{1} \)-coinvariants under the left and right coaction of \( C \) on \( H \) respectively, in the sense of [BW03, 28.4].

Dually, let \( \iota : A \to H \) be an algebra map, and set \( A^+ = \iota^{-1}(\text{ker} \varepsilon_H) \). Write \( H_\iota = H_A \) for the left \( H \)-module \( H/\iota(A^+) \), and similarly, \( \iota H =_A H = H/\iota(A^+)H \).

It is now an easy exercise to check that if \( \iota : A \to H \) is a right coideal subalgebra, then \( H_A \) is a quotient left module coalgebra, and vice versa, if \( \pi : H \to C \) is the projection on a quotient left module coalgebra, then \( C H \) is a right coideal subalgebra of \( H \).

\[
\begin{array}{ccc}
A \mapsto H_A & \uparrow \text{set of right coideal} & \downarrow \text{set of quotient left module} \\
\text{subalgebras of } H & \text{coalgebras of } H & C H \leftarrow C
\end{array}
\]

are order-reversing maps with respect to the obvious poset structures on the two sets (whose partial orders we write as \( \preceq \)), and they form a Galois connection between the two posets in the sense of [ML98, IV.5]. \( A \mapsto H^A H \) (henceforth \( A \mapsto \overline{A} \)) and \( C \mapsto H^C_H \) (henceforth \( C \mapsto \overline{C} \)) are the two closure operators associated to the Galois connection, as in [Ore44, Theorem 2] (and the closure operators of the title). Their main properties are that \( A \mapsto \overline{A} \) is order-preserving, idempotent, and \( A \preceq \overline{A} \) (and similarly for \( C \)).

**1.1.1 Definition** Let \( \iota : A \to H \) be a right coideal subalgebra, and \( \pi : H \to C \) a quotient left module coalgebra. If \( A = \overline{A} \ (C = \overline{C}) \) we will say that \( A \) (resp. \( C \)) is closed. We call \( \pi : H \to H_A \) (or \( H_A \) itself) the right reflection of \( \iota : A \to H \) or of \( A \), and \( \iota : C H \to H \) (or \( C H \) itself) the left reflection of \( \pi : H \to C \). We also write \( r(A) \) and \( r(C) \) for \( H_A \) and \( C H \). ♦

Let \( \iota : A \to H \) be an algebra morphism (for us, \( H \) will mostly be a Hopf algebra, while \( A \) will be a right coideal subalgebra). Recall that the dominion of \( \iota \), or of \( A \) in \( H \), is the set of all those \( h \in H \) such that \( h \otimes 1 = 1 \otimes h \) in \( H \otimes_A H \). For the reader’s convenience, we recall alternative characterizations, referring to [Ste75] for the proof and further details:

**1.1.2 Proposition** Let \( \iota : A \to H \) be an algebra morphism. The following properties are equivalent for an element \( h \in H \):

(a) If two algebra morphisms \( \varphi, \psi : H \to K \) agree on \( A \), then they agree on \( h \).

(b) If \( M \in H_MH, \ m \in M \) and \( am = ma \) for all \( a \in A \), then \( hm = mh \).

(c) \( h \otimes 1 = 1 \otimes h \) in \( H \otimes_A H \).

(d) If \( M, N \in H_M \) and \( f : M \to N \) is an \( A \)-module map, then \( f \) commutes with the \( h \)-action.

(e) The images of \( h \) through the two morphisms \( H \mapsto H \coprod_A H \) coincide, where \( \coprod_A \) denotes the pushout over \( A \) in \text{ALG}. 

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Proof (a), (b), (c) and (d) are precisely the corresponding points of [Ste75, Proposition XI.1.1], some of them with very mild rephrasing, while (e) is essentially a reformulation of (a), using the universal property of the pushout to reduce (a) to the corresponding universal statement.

1.1.3 Remark It follows from this proposition (specifically, from (a) $\iff$ (d)) that an algebra map $A \to H$ is an epi in $\text{ALG}$ if and only if the restriction functor $H \mathcal{M} \to A \mathcal{M}$ is full. We will use this observation below, in Section 3.

We will also need the dual notion to that of dominion:

1.1.4 Definition Let $\pi : H \to C$ be a coalgebra morphism. The codominion of $\pi$, or of $C$ in $H$, is the coequalizer of the two maps $H \square_C H \rightrightarrows H$, where $\square_C$ stands for the cotensor product of $C$-comodules, as in [BW03, §10].

1.1.5 Remark There is a characterization of codominions dual to proposition 1.1.2, whose statement and proof we leave to the reader, since it will not be needed in the sequel. We only observe that in definition 1.1.4, we could just as well have used the pullback $H \times_C H$ in $\text{COALG}$ instead of $H \square_C H$; this is the dual analogue of the equivalence (e) $\iff$ (e) in proposition 1.1.2.

(Co)Dominions are important for us because of the following simple observation:

1.1.6 Proposition Let $\iota : A \to H$ be a right coideal subalgebra of the Hopf algebra $H$, and $\pi : H \to C$ a quotient left module coalgebra. Then, the closure $\overline{A}$ coincides with the domain of $A$ in $H$, and dually, the closure $\overline{C}$ coincides with the codominion of $C$ in $H$.

Proof Recall (e.g. [Tak79, page 456, line 4]) that the isomorphism $H \otimes H \cong H \otimes H$ defined by $h \otimes k \mapsto h_1 \otimes h_2 k$ descends to an isomorphism $H \otimes_A H \cong H_A \otimes H$. Transporting the equality $h \otimes 1 = 1 \otimes h$ in $H \otimes_A H$ to $H_A \otimes H$ by means of this isomorphism yields $h_1 \otimes h_2 = 1 \otimes h$, which is exactly the characterization of $\overline{A} = H^\natural H$. This takes care of the statement about $A$.

Now dualize everything in the previous paragraph to prove the second statement: by, say, [Tak79, line 4 in the proof of Theorem 2], the same isomorphism $H \otimes H \cong H \otimes H$ we used above induces $H \otimes C H \cong H \square_C H$. After using this isomorphism to substitute $H \otimes C H$ for $H \square_C H$, the two maps appearing in definition 1.1.4 are $h \otimes k \mapsto hk$ and $h \otimes k \mapsto h\varepsilon(k)$. Their coequalizer is precisely $\overline{C} = HC_H$.

1.1.7 Remark One way to regard the closedness property $A = \overline{A}$ for a right coideal subalgebra $\iota : A \to H$ is as a weaker version of faithful flatness: certainly, if $H$ is either left or right faithfully flat over $A$, then $A$ is closed. Similar remarks apply to the dual situation: if $\pi : H \to C$ is a quotient left module coalgebra making $H$ either left or right faithfully coflat over $C$ (dual analogue of faithful flatness, with $\square_C$ replacing $\otimes_A$; cf. [BW03, 10.8]), then $C = \overline{C}$.

1.1.8 Remark Using the notation in definition 1.1.1, the case when $A$ and $C$ are Hopf algebras (and $\iota$ and $\pi$ are Hopf algebra morphisms) and they are each other’s reflections is precisely what the authors of [AD95] call a short exact sequence of Hopf algebras. In view of proposition 1.1.6, $A$ and $r(A)$ are each other’s reflections as soon as $A$ is closed. In conclusion, the hypotheses of [AD95, Proposition 1.2.4] can be weakened quite a bit: using that paper’s terminology, if the Hopf subalgebra $A \to H$ is conormal in the sense of [AD95, Definition 1.1.9], then the sequence $k \to A \to H \to r(A) \to k$ is short exact if and only if $A = \overline{A}$. A dual discussion can be carried out to strengthen [AD95, Proposition 1.2.13] in the appropriate way.
1.1.9 Definition In view of this, we will say that $k \to A \to H \to C \to k$ is exact if $A$ is a right coideal subalgebra of $H$, $C$ is a quotient left module coalgebra, and they are each other’s reflections (so in particular, they are closed). Since we only deal with short sequences, we will usually drop the $k$’s.

1.2 Descent data and adjunctions

If $H$ is a Hopf algebra and $C$ is a left $H$-module coalgebra, then $C_H^H M$ will be the category of left $H$-modules endowed with a left $C$-comodule structure which is a left $H$-module map from $M$ to $C \otimes M$ (where the latter has the left $H$-module structure induced by the comultiplication on $H$). Similarly, if $A$ is a right $H$-comodule algebra, then $M^A_H$ is the category of vector spaces right $H$-comodules with a right $A$-module structure such that $M \otimes A \to M$ is a map of right $H$-comodules. The morphisms in each of these categories are required to preserve both structures.

Let $\xi : A \to H$ be a right coideal subalgebra and $\pi : H \to C$ a quotient left module coalgebra such that $\pi \circ \xi$ factors through $A \ni a \mapsto \varepsilon(a) 1 \in C$ (this is equivalent to saying that $A \preceq r(C)$, or $C \preceq r(A)$, in the two posets discussed before definition 1.1.1). Then, there is an adjunction between the categories $A M$ and $C_H^H M$, and dually, an adjunction between $M^H_A$ and $M^C$. We will recall briefly these how these are defined, omitting most of the proofs, which are routine.

Let $M \in A M$. $H \otimes_A M$ then has a left $H$-module structure, as well as a left $C$-comodule structure inherited from the left $C$-coaction on $H$ (checking this is where the condition $A \preceq r(C)$ is needed). This defines a functor $L : A M \to C^H_H M$. To go in the other direction, for $N \in C^H_H M$, let

$$R(N) = \{ n \in N \mid n_{(-1)} \otimes n_{(0)} = \mathbb{1} \otimes n \}. $$

This defines a functor, and as the notation suggests, $L$ is a left adjoint to $R$.

For the other adjunction, given $M \in M^H_A$, define $L'(M) = M/MA^+$. This is a functor (with the obvious definition on morphisms), and it is left adjoint to $R' : M^C \to M^H_A$ defined by $R'(N) = N \sqcup C H$; the latter has a right $H$-comodule structure obtained by making $H$ coact on itself, as well as a right $A$-module structure obtained from the right $A$-action on $H$.

Let us now focus on the adjunction $A M \leftrightarrow C_H^H M$. In [Tak79], the same discussion is carried out in a slightly less general situation: the adjunction described above is considered in the case $A = r(C)$. On the other hand, we remark that when $C = r(A)$, the category $C_H^H M$ introduced above is nothing but the category of descent data for the ring extension $A \to H$. Recall ([BW03, Proposition 25.4]) that in our case, this would be the category $H \otimes_A H^H M$ of left comodules over the canonical $H$-coring $H \otimes_A H$ associated to the algebra extension $A \to H$. This means left $H$-modules $M$ with an appropriately coassociative and counital left $H$-module map $\rho : M \to (H \otimes_A H) \otimes_H M \cong H \otimes_A M$.

Using the identification $H \otimes_A H \cong r(A) \otimes H$ from the proof of proposition 1.1.6, we see that a map $\rho$ as above is the same thing as a map $\psi : M \to r(A) \otimes M$. The other properties of $\rho$, namely being a coassociative, counital, left $H$-module map, precisely translate to $\psi$ being coassociative, counital, and a left $H$-module map respectively. Taking into account this equivalence $r(A) \otimes H^H M$, the adjunction $(L, R) : A M \leftrightarrow r(A) \otimes H^H M$ is an equivalence as soon as $H$ is right faithfully flat over $A$ (this is the faithfully flat descent theorem; cf. [Nus97, Theorem 3.8]).

Conversely, we want to conclude that if $(L, R)$ is an equivalence, then $H$ is right $A$-faithfully flat; indeed, $A \otimes_B$ is then exact on $B M$. Note that we are using the fact that $r(A) \otimes H$ is abelian, with the same exact sequences as $VEC$. All in all, this proves

1.2.1 Proposition Let $\iota A \to H$ be a right coideal subalgebra. Then, the adjunction $(L, R) : A M \leftrightarrow r(A) \otimes H^H M$ is an equivalence iff $H$ is right $A$-faithfully flat.
1.2.2 Remark This result is very similar in spirit to the equivalence (5) ⇐⇒ (3) in [Sch90, Theorem I], or to (1) ⇐⇒ (2) in [SS05, Lemma 1.7]. These can all be deduced from much more general, coring-flavored descent theorems that are now available, such as, say, [CDGV07, Theorem 2.7].

2 Main results

We will prove a statement somewhat more general than the one announced in the title of the paper:

2.0.3 Theorem Let $H$ be a Hopf algebra, and $\iota : A \to H$ a Hopf subalgebra. Then, if $\iota$ splits as an $A$-bimodule right $H$-coideal map, $H$ is right $A$-faithfully flat.

Before going into the proof, let us record the consequence we are after.

2.0.4 Corollary A cosemisimple Hopf algebra is faithfully flat over all its Hopf subalgebras.

Proof By theorem 2.0.3 above, we only need to show that an inclusion $\iota : A \to H$ of cosemisimple Hopf algebras (as $A$ is automatically cosemisimple) splits as an $A$-bimodule right $H$-coideal map. In fact, one can even find a subcoalgebra $B \leq H$ with $H = A \oplus B$ as $A$-bimodules.

Let $I$ be the set of simple subcoalgebras of $H$, and $J$ the subset of $I$ consisting of subcoalgebras contained in $A$. One then has $H = \bigoplus_I D$, and $A = \bigoplus_J D$. Define $B = \bigoplus_{I \setminus J} D$; in other words, $B$ is the direct sum of those simple subcoalgebras of $H$ which are not in $A$. Clearly, $B$ is a subcoalgebra, and $H = A \oplus B$, and we now only need to check that $B$ is invariant under (either left or right) multiplication by $A$.

Let $D \in J$ and $E \in I \setminus J$ be simple subcoalgebras of $A$ and $B$ respectively. $ED$ (product inside $H$) is then $\text{coalg}(V_E \otimes V_D)$ (cf. last paragraph above §1.1). Now assume $F \in J$ is a summand of $ED$. This means that $V_F \leq V_E \otimes V_D$, so $V_F^* \leq V_D \otimes V_F^*$. This is absurd: $V_F^*$ is a $B$-comodule, while $V_D \otimes V_F^*$ is an $A$-comodule. ■

In fact, this result can be slightly strengthened. Recall that the coradical $C_0$ of a coalgebra $C$ is the sum of all its simple subcoalgebras.

2.0.5 Corollary A Hopf algebra $H$ whose coradical $H_0$ is a Hopf subalgebra is faithfully flat over its cosemisimple Hopf subalgebras.

Proof Any cosemisimple Hopf subalgebra $A \leq H$ will automatically be contained in the coradical $H_0$. By the previous corollary, $H_0$ is faithfully flat over $A$. On the other hand, Hopf algebras are faithfully flat (and indeed free) over sub-bialgebras which contain the coradical ([Rad77b, Corollary 1]); in particular, in this case, $H$ is faithfully flat over $H_0$. The conclusion follows. ■

We will need the following observation.

2.0.6 Lemma Under the hypotheses of theorem 2.0.3, the right adjoint functor $R : r^H(M) \to A^H \otimes \text{forget} = C \square C - \cong (k \square C -) \oplus ((B/BA^+) \square C -)$.
Since it is naturally a direct summand of the exact functor forget, $k \square_C$ must be exact, proving the first statement.

To prove the second statement, note that $R(N)$ can be realized as the following equalizer in $AM$:

$$R(N) \xrightarrow{e} N \xrightarrow{\partial_1 \partial_0} C \otimes N.$$ 

Here, $\partial_1$ is the comodule structure map, while $\partial_0$ is $m \mapsto T \otimes m$. Now add two arrows going in the opposite direction:

$$R(N) \xleftarrow{e} N \xrightarrow{\partial_1 \partial_0} C \otimes N,$$

where $u$ is the projection induced by the splitting

$$R(N) \cong k \square_C N \leq (k \oplus (B/BA^+)) \square_C N = C \square_C N \cong N,$$

and similarly, $v$ is the projection coming from

$$N \cong k \otimes N \leq (k \oplus (B/BA^+)) \otimes N = C \otimes N$$

(cf. the proof of lemma 2.0.6). It is now easy to check that these extra arrows make our equalizer split in the sense of [ML98, VI.6], i.e. one has

$$\partial_0 e = \partial_1 e, \quad u e = \text{id}, \quad v \partial_0 = \text{id}, \quad v \partial_1 = eu.$$ 

But this then means that the equalizer is universal, in the sense that it is preserved by any functor ([ML98, dual analogue of Corollary VI.6]). In particular, applying $L$ to the initial diagram yields an equalizer diagram. Since the obvious monomorphism $N \rightarrow L(N) = H \otimes_A N$ defined by $n \mapsto 1 \otimes n$ equalizes $L(\partial_0)$ and $L(\partial_1)$, it must factor through $LR(N)$; this implies that there is a monomorphism $N \rightarrow LR(N)$, which finishes the proof of the lemma. 

**Proof of theorem 2.0.3** As in the proof of lemma 2.0.6, let $C = r(A)$. We are going to show that the adjunction $(L, R) : AM \leftarrow \leftarrow CHM$ is an equivalence, and then conclude via proposition 1.2.1.

Write $H = A \oplus B$, direct sum of $A$-bimodule right $H$-coideals. It is already clear from the splitting of $\iota$ as an $A$-bimodule map that $A = \overline{A}$: one then has

$$H \otimes_A H \cong A \oplus B \oplus (B \oplus B \otimes_A B),$$

and the two maps $H \rightarrow H \otimes_A H$ send elements $b \in B$ to the two different copies of $B$ in the above direct sum. This means that, as claimed, no non-zero $b \in B$ belong to $\overline{A}$.

We now have an exact sequence $A \rightarrow H \rightarrow C$, as in definition 1.1.9. The unit $\text{id} \rightarrow RL$ is an isomorphism on the object $A \in AM$ (this is precisely the statement $A = \overline{A}$), and hence on every projective object. Now let $M \in AM$ be an arbitrary module, and

$$P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

an exact sequence, with projective $P_i$'s. On then has a diagram

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P_1 \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
P_0 \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
M \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
0 \\
\downarrow
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
RL(P_1) \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
RL(P_0) \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
RL(M) \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
0
\end{array}
\end{array}
$$

9
in $\mathcal{A}M$. Applying the right exact functor $L$ to the bottom row $E$ of this diagram yield an exact row $L(E)$. But we know from lemma 2.0.6 that $R$ is exact, so the top row $RL(E)$ is also exact. The vertical arrows, except maybe the third one, are isomorphisms. The five lemma then says that all vertical arrows are isomorphisms, and hence $L$ is fully faithful.

Now let $N \in \mathcal{C}_H$ be an arbitrary object, and consider the exact sequence

$$0 \to \bullet \to LR(N) \to N \to \bullet \to 0,$$

obtained of the unit $LR(N) \to N$ of our adjunction. Applying the exact functor $R$ (lemma 2.0.6), we get an exact sequence

$$0 \to R(\bullet) \to RLR(N) \to R(N) \to R(\bullet) \to 0.$$

But $id \cong RL$ implies that the map in the middle of this sequence is an isomorphism, and hence $R(\bullet)$ are both zero. By lemma 2.0.6, we conclude that $\bullet$ must both have been zero to begin with, and hence $LR \cong id$.

2.0.7 Remark We have presented the full proof for the convenience of the reader; however, it is a consequence of [CDGV07, Theorem 2.7, (2) $\iff$ (4)] that proving $L$ to be fully faithful suffices.

Now let us place ourselves in the setting of corollary 2.0.4, assuming in addition that the Hopf subalgebra $A \to H$ is conormal in the language of [AD95]. This simply means that $HA^+ = A^+H$, and it is equivalent to $C = r(A)$ being a quotient Hopf algebra of $H$, rather than just a quotient coalgebra (cf. [AD95, Definition 1.1.9]). We observed in the proof of lemma 2.0.6 that $C$ breaks up as the direct sum of $k = k^+$ and $B/BA^+$, both of which, in this particular case, are coalgebras. In other words, the coalgebra spanned by the unit of the Hopf algebra $C$ has a coalgebra complement in $C$. It follows (cf. [Swe69, Theorem 14.0.3, (c) $\iff$ (f)]) that $C$ is a cosemisimple Hopf algebra. Our aim, in the rest of this section, is to extend this result to the general case covered by corollary 2.0.4:

2.0.8 Theorem If $\iota : A \to H$ is a Hopf subalgebra of a cosemisimple Hopf algebra $H$, then the coalgebra $C = r(A)$ is cosemisimple.

Proof We know from corollary 2.0.4 that $H$ is right $A$-faithfully flat, and hence also left faithfully flat (just flip everything by means of the bijective antipode). This then implies, for example by [Tak71, Theorem 1], that the second adjunction we introduced above, $(L', R') : \mathcal{M}_A^H \leftrightarrow \mathcal{M}_C$ is an equivalence. It is then enough to show that all objects of the category $\mathcal{M}_A^H$ are projective, and this is precisely what the next two results do.

2.0.9 Definition An object of $\mathcal{M}_A^H$ is said to be $A$-projective if it is projective as an $A$-module.

2.0.10 Proposition Under the hypotheses of theorem 2.0.8, every object of $\mathcal{M}_A^H$ embeds as a retract of some $A$-projective object.

Proof Let $M \in \mathcal{M}_A^H$ be an arbitrary object. Endow $M \otimes H$ with a right $H$-comodule structure by making $H$ coact on itself, and also a right $A$-module structure by the diagonal right action (i.e. $M \otimes H$ is the tensor product in the monoidal category $\mathcal{M}_A$). It is easy to check that these are compatible in the sense that they make $M \otimes H$ into an object of $\mathcal{M}_A^H$, and the map $\rho : m \mapsto m_{(0)} \otimes m_{(1)} \in M \otimes H$ giving $M$ its right $H$-comodule structure is actually a morphism
in $\mathcal{M}_{A}^{H}$. Similarly, id $\otimes \varepsilon_{H} : M \otimes H \to M$ is a morphism in $\mathcal{M}_{A}^{H}$, and it splits the inclusion $\rho$. It follows that it is enough to show that the object $M \otimes H \in \mathcal{M}_{A}^{H}$ described above is $A$-projective.

Corollary 2.0.4 says that $H$ is $A$-faithfully flat, and it follows from [MW94, Corollary 2.9] that it is then (left and right) $A$-projective. This means that $M \otimes H$ can be split embedded (in the category $\mathcal{M}_{A}$) into a direct sum of copies of $M \otimes A$, with the diagonal right action of $A$. But

$$M \otimes A \longrightarrow M \otimes A, \quad m \otimes a \mapsto ma_{1} \otimes a_{2}$$

exhibits an isomorphism from $M \otimes A$ with the right $A$-action on the right tensorand to $M \otimes A$ with the diagonal $A$-action (its inverse is $m \otimes a \mapsto mS(a_{1}) \otimes a_{2}$). This means that in $\mathcal{M}_{A}$, $M \otimes H$ is a direct summand of a direct sum of copies of $A$, i.e. projective.

**2.0.11 Proposition** Under the hypotheses of theorem 2.0.8, $A$-projective objects of $\mathcal{M}_{A}^{H}$ are projective.

Before going into the proof, we need some preparation, including additional notation to keep track of the several $A$-module or $H$-comodule structures that might exist on the same object.

As in the proof of corollary 2.0.4, denote by $I$ and $J \subseteq J$ the sets of simple right comodules over $H$ and $A$, respectively. Recall that these are also in one-to-one correspondence with the simple subcoalgebras of $H$ and $A$, respectively. We will henceforth denote by $\varphi : H \to A$ the map which is the identity on $A$, and sends every simple subcoalgebra $D \in I \setminus J$ to 0.

Notice now that $A$ acts on $H$ (as well as on itself) not just by the usual right regular action, but also by the right adjoint action: $h \triangleleft a = S(a_{1})ha_{2}$ ($h \in H$; $a \in A$). This gives $H$ and $A$ a second structure as objects in $\mathcal{M}_{A}^{H}$. When working with this structure rather than the obvious one, we denote these objects by $H_{ad}$ and $A_{ad}$.

**2.0.12 Lemma** (a) For any object $M \in \mathcal{M}_{A}^{H}$, $M \otimes H_{ad}$ becomes an object of $\mathcal{M}_{A}^{H}$ when endowed with the diagonal $A$-action (where $A$ acts on $M \in \mathcal{M}_{A}^{H}$ and on $H$ by the right adjoint action) and the diagonal $H$-coaction.

(b) Similarly, $M \otimes A_{ad} \in \mathcal{M}_{A}^{H}$.

(c) $\text{id} \otimes \varphi : M \otimes H_{ad} \to M \otimes A_{ad}$ respects the structures from (a) and (b), and hence is a morphism in $\mathcal{M}_{A}^{H}$.

**Proof** We will only prove (a); (b) is entirely analogous, while (c) follows immediately, since $\varphi$ clearly preserves both the right $H$-coaction and the adjoint $A$-action.

Proving (a) amounts to checking that the diagram

$$\begin{array}{ccc}
M \otimes H_{ad} \otimes A & \longrightarrow & M \otimes H_{ad} \\
\downarrow & & \downarrow \\
M \otimes H_{ad} \otimes H \otimes A & \longrightarrow & M \otimes H_{ad} \otimes H
\end{array}$$

is commutative. The path passing through the upper horizontal line is

$$m \otimes h \otimes a \longrightarrow ma_{1} \otimes S(a_{2})ha_{3} \longrightarrow m_{0}a_{1} \otimes S(a_{4})h_{1}a_{5} \otimes m_{1}a_{2}S(a_{3})h_{2}a_{6},$$

while the other composition is

$$m \otimes h \otimes a \longrightarrow m_{0} \otimes h_{1} \otimes m_{1}h_{2} \otimes a \longrightarrow m_{0}a_{1} \otimes S(a_{2})h_{1}a_{3} \otimes m_{1}h_{2}a_{4}.$$
Using the properties of the antipode and counit in a Hopf algebra, we have

\[ m_0a_1 \otimes S(a_4)h_1a_5 \otimes m_1a_2S(a_3)h_2a_6 = m_0a_1 \otimes S(\varepsilon(a_2)a_3)h_1a_4 \otimes m_1h_2a_5 \]

\[ = m_0a_1 \otimes S(a_2)h_1a_3 \otimes m_1h_2a_4, \]

concluding the proof.

Now denote by \((M \otimes H)^r \in \mathcal{M}_A^H\) the the object from the proof of proposition 2.0.10: the \(A\)-action is diagonal, while \(H\) coacts on the right tensorand alone. The upper \(r\) is meant to remind the reader of this.

2.0.13 Lemma For \(M \in \mathcal{M}_A^H\), \(\psi_M : M \otimes H \rightarrow M \otimes H\) defined by

\[ m \otimes h \mapsto m_0 \otimes S(m_1)h. \]

is a morphism in \(\mathcal{M}_A^H\) from \((M \otimes H)^r\) to \(M \otimes H_{ad}\).

Proof We only check compatibility with the \(A\)-actions, leaving the task of doing the same for \(H\)-coactions to the reader. The composition \((M \otimes H)^r \otimes A \rightarrow (M \otimes H)^r \rightarrow M \otimes H_{ad}\) is

\[ m \otimes h \otimes a \xrightarrow{\psi_M} ma_1 \otimes ha_2 \xrightarrow{\psi_M} m_0a_1 \otimes S(m_1a_2)ha_3, \]

while the other relevant composition is

\[ m \otimes h \otimes a \xrightarrow{\psi_M \otimes \text{id}} m_0 \otimes S(m_1)h \otimes a \xrightarrow{\psi_M} m_0a_1 \otimes S(a_2)S(m_1)ha_3. \]

Since \(S\) is an algebra anti-morphism, they are equal.

Finally, we have

2.0.14 Lemma Let \(M \in \mathcal{M}_A^H\). The map \(M \otimes A \rightarrow M\) giving \(M\) its \(A\)-module structure is a morphism \(M \otimes A_{ad} \rightarrow M\) in \(\mathcal{M}_A^H\).

Proof Compatibility with the \(H\)-coactions is built into the definition of the category \(\mathcal{M}_A^H\), so one only needs to check that the map is a morphism of \(A\)-modules. In other words, we must show that the diagram

\[
\begin{array}{ccc}
M \otimes A_{ad} \otimes A & \rightarrow & M \otimes A_{ad} \\
\downarrow & & \downarrow \\
M \otimes A & \rightarrow & M
\end{array}
\]

is commutative. The right-down composition is

\[ m \otimes a \otimes b \xrightarrow{\psi_M} mb_1 \otimes S(b_2)ab_3 \xrightarrow{\psi_M} mb_1S(b_2)ab_3, \]

while the other composition is

\[ m \otimes a \otimes b \xrightarrow{\psi_M} ma \otimes b \xrightarrow{\psi_M} mab; \]

they are thus equal.
2.0.15 Lemma For \( M \in \mathcal{M}_A^H \), the composition

\[
t_M : (M \otimes H)^r \xrightarrow{\psi_M} M \otimes H_{ad} \xrightarrow{\text{id} \otimes \varphi} M \otimes A_{ad} \longrightarrow M
\]

where the last arrow gives \( M \) its \( A \)-module structure is a natural transformation from the \( \mathcal{M}_A^H \)-endofunctor \((\bullet \otimes H)^r\) to the identity functor, and it exhibits the latter as a direct summand of the former.

Proof The fact that \( t_M \) is a map in \( \mathcal{M}_A^H \) follows from lemmas 2.0.12 to 2.0.14. Naturality is immediate (one simply checks that it holds for each of the three maps), as is the fact that \( t_M \) is a left inverse of the map \( M \to (M \otimes H)^r \) giving \( M \) its \( H \)-comodule structure.

We are now ready to prove the result we were after.

Proof of proposition 2.0.11 Let \( P \in \mathcal{M}_A^H \) be an \( A \)-projective object. We must show that \( \mathcal{M}_A^H(P, \bullet) \) is an exact functor. Embedding the identity functor as a direct summand into \((\bullet \otimes H)^r\) (lemma 2.0.15), it suffices to show that \( \mathcal{M}_A^H(P, (\bullet \otimes H)^r) \) is exact.

\((\bullet \otimes H)^r : \mathcal{M}_A \to \mathcal{M}_A^H \) is right adjoint to forget : \( \mathcal{M}_A^H \to \mathcal{M}_A \) (as \( \mathcal{M}_A^H \) is the category coalgebras for the comonad \( \bullet \otimes H \) on \( \mathcal{M}_A \); cf. [ML98, Theorem VI.2.1]), so \( \mathcal{M}_A^H(P, (\bullet \otimes H)^r) \) is naturally isomorphic to \( \mathcal{M}_A(P, \bullet) \), which is exact by our assumption that \( P \) is \( A \)-projective.

2.0.16 Remark In the above proof, the forgetful functor forget : \( \mathcal{M}_A^H \to \mathcal{M}_A \) has been suppressed in several places, in order to streamline the notation; we trust that this has not caused any confusion.

2.0.17 Remark The proof of proposition 2.0.10 is essentially a rephrasing of the usual proof that Hopf algebras \( H \) with a (right, say) integral sending \( 1_H \) to 1 are cosemisimple (cf. [Swe69, §14.0]; we will call such integrals unital). The map \( \varphi : H \to A \) introduced in lemma 2.0.12 might be referred to as an \( A \)-valued right integral (by which we mean a map preserving both the right \( H \)-comodule structure and the right adjoint action of \( A \)), and specializes to a unital integral when \( A = k \). In conclusion, one way of stating proposition 2.0.11 would be:

If the inclusion \( \iota : A \to H \) of a right coideal subalgebra is split by an \( A \)-valued right integral, then the forgetful functor \( \mathcal{M}_A^H \to \mathcal{M}_A \) reflects projectives.

3 A counterexample

In this section we provide an example which shows that, perhaps surprisingly, a cosemisimple Hopf algebra need not be coflat over quotient Hopf algebras. For a coalgebra \( C \) we will denote by \( H(C) \) the free involutive Hopf algebra on \( C \); it is the image of \( C \) through the left adjoint to the forgetful functor \( \text{HALG}_2 \to \text{COALG} \) (cf. the discussion at the beginning of Section 1).

The \( C \) component of the unit of the adjunction between \( \text{COALG} \) and \( \text{HALG}_2 \) is a coalgebra map \( C \to H(C) \), which is always an injection. The proof provided of [Tak71, Corollary 9], which is the analogous statement about free Hopf algebras, works in the involutive case as well. Note also that just from the relevant universal property of \( H(C) \) it follows that it must be generated, as an algebra, by \( C \) and \( S(C) \).

Now let \( C \) be the matrix coalgebra \( \mathcal{M}_3^k \), i.e. the dual of the \( 3 \times 3 \) matrix algebra \( M_3 = M_3(k) \) over \( k \). Similarly, we consider the dual \( D \) of the algebra \( T_3 \) of \( 3 \times 3 \) upper triangular matrices. The inclusion of the latter into \( M_3(k) \) dualizes to a surjection \( C \to D \). The claim is that these provide the announced counterexample:
3.0.18 Proposition If the base field $k$ has characteristic zero, then $H(C)$ is cosemisimple, and moreover, it is not faithfully coflat over the quotient Hopf algebra $H(D)$.

Proof The fact that $H(C)$ is cosemisimple in characteristic zero follows for example from [Bic07, Theorem 1.1]; in the notation of that paper, $H(C)$ would be $H(F)$, where $F$ is the $3 \times 3$ identity matrix. In other words, $H(C)$ is an example of what Bichon calls a universal cosovereign Hopf algebra.

The surjection $C \to D$ certainly induces a Hopf algebra map $H(C) \to H(D)$, which is easily seen to be surjective again using the remark made above that $H(C)$ and $H(D)$ are generated as algebras by $C$, $S(C)$ and $D$, $S(D)$ respectively.

Finally, assume $H(C)$ is (right, say) faithfully coflat over $H(D)$. Since $C$ is a subcoalgebra of $H(C)$ and hence a direct summand right subcomodule, it must also be faithfully coflat over $H(D)$. But since it maps onto $D$, this implies that our original coalgebra map $C \to D$ must have been faithfully coflat to begin with. Dualizing, this would mean that the inclusion $T_3 \to M_3$ is faithfully flat. This, however, is not true: faithfully flat epimorphisms of algebras are always isomorphisms (this is well known; cf. [Chi10, Proposition 2.3] for a short proof), and the next lemma states that our inclusion $T_3 \to M_3$ is in fact an epimorphism in the category Alg.

3.0.19 Lemma For any field $k$ (regardless of the characteristic) and any positive integer $n$, the inclusion $T_n \to M_n$ of the algebra of $n \times n$ upper triangular matrices into that of all $n \times n$ matrices is an epimorphism in Alg.

Proof To keep the notation simple, we write $T$ and $M$ for $T_n$ and $M_n$ respectively.

We want to show that the scalar restriction functor from $M$ to $T$-modules is full, or, in other words, that any $T$-module map between $M$-modules automatically respects the $M$-module structures (cf. remark 1.1.3). Since $M$ is Morita equivalent to $k$ with the usual column vectors module $k^n$ as unique simple module, it is enough to show that the $T$-endomorphism ring of $k^n$ is just $k$. Since extending scalars only makes this statement stronger, we may (and do) assume that $k$ is algebraically closed.

Denote by $e_{ij}$, $i, j = 1, \ldots, n$ the $n \times n$ matrix units, and set $e_i = e_{ii}$. Let $S_i$, $i = 1, \ldots, n$ be the $n$ simple (1-dimensional) modules of $T$, and $P_i = Te_i$ their respective projective covers. We have $k^n \cong P_n$, and there is a composition series $0 < P_1 < P_2 < \ldots < P_n$ with $P_i/P_{i-1} \cong S_i$. On the one hand, this means that any non-zero $T$-endomorphism of $P_n$ must act as an automorphism on $S_1$ and hence cannot be nilpotent, while on the other hand, since $P_n$ is indecomposable and has finite length, Fitting’s lemma says that every endomorphism is either nilpotent or an automorphism.

The conclusion of the above argument is that $\text{End}_T(k^n)$ is a division ring (finite-dimensional over $k$). By our assumption that $k$ is algebraically closed, it must be $k$ itself.

3.0.20 Remark See also [NT94, example following Theorem 3.6] for an alternative approach to the same result. In that paper, the authors show directly that $(M/T) \otimes_T M = 0$ (in the case $n = 2$), which turns out to be equivalent to $T \to M$ being an epimorphism.

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