A characterization of distance matrices of weighted cubic graphs and Peterson graphs

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Abstract

Given a positive-weighted simple connected graph with $m$ vertices and numbered its vertices by the numbers $1, \ldots, m$, we can construct an $m \times m$ matrix whose entry $(i, j)$ is the minimal weight of a path between $i$ and $j$ for any $i$ and $j$, where the weight of a path is the sum of the weights of its edges. We call this matrix distance matrix of the weighted graph. There is wide literature about distance matrices of weighted graphs. In this paper we characterize distance matrices of positive-weighted $n$-cubic graphs. Moreover we show that a complete bipartite $n$-regular graph with order $2^n$ is not necessarily the $n$-cubic graph. Finally we give a characterization of distance matrices of positive-weighted Peterson graphs.

Introduction

Let $G$ be a simple graph; we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. We can consider a weight function $w : E(G) \rightarrow \mathbb{R}^+$ that assigns to each edge a strictly positive real number, the weight of the edge. A graph $G$ endowed with such a weight function is called a positive-weighted graph; we denote it by $\mathcal{G} = (G, w)$. For any subgraph $H$, we define $w(H)$ to be the sum of the weights of the edges of $H$ and we denote by $e(v, w)$ the edge with endpoints $v$ and $w$ if it exists. Suppose moreover that $G$ is connected; the $k$-weight of a $k$-subset of vertices $\{v_1, \ldots, v_k\}$ is defined to be the minimum among the weights of the subgraphs of $\mathcal{G}$ “connecting” $v_1, \ldots, v_k$, that is of the subgraphs of $\mathcal{G}$ whose vertex set contains $v_1, \ldots, v_k$; we denote it by $D_{v_1, \ldots, v_k}(\mathcal{G})$, regardless of the ordering on the subset $\{v_1, \ldots, v_k\}$. We say that a subgraph $H$ of $G$ realizes $D_{v_1, \ldots, v_k}(\mathcal{G})$ if its vertex set contains $v_1, \ldots, v_k$ and $w(H) = D_{v_1, \ldots, v_k}(\mathcal{G})$. In particular, in the case $k = 2$ we can associate to a positive-weighted graph a symmetric matrix which collects all the informations about the distance between two vertices: if we label, in some way, the vertices by the numbers $1, \ldots, m$, we define the $(i, j)$-entry of this matrix to be $D_{i,j}(\mathcal{G})$. Obviously, the diagonal entries are zero, while the off-diagonal entries are strictly positive. Conversely, the following result characterizes the matrices that are associated to some positive-weighted graph.
Theorem 0.1 (Hakimi-Yau, [5]). A symmetric matrix \((D_{i,j})_{i,j\in\{1,\ldots,m\}}\) with zero diagonal entries and with strictly positive off-diagonal entries is the matrix associated to a positive-weighted graph with vertex set \(\{1,\ldots,m\}\) if and only if the triangle inequalities hold, that is if and only if

\[D_{i,j} \leq D_{i,k} + D_{k,j} \quad \forall i, j, k \in \{1, \ldots, m\}.\]

A square matrix whose diagonal entries are zero and the off-diagonal entries are strictly positive is called a predistance matrix. A predistance matrix satisfying the triangle inequalities is called a distance matrix.

Among the many results on the theory of weighted graphs, we quote also the famous criterion for a distance matrix to be the distance matrix of a positive-weighted tree, see [3], [9], [10]:

Theorem 0.2. (Buneman-Simoes Pereira-Zaretskii) Let \(D\) be a distance matrix. It is the distance matrix of a positive-weighted tree with vertex set \(\{1, \ldots, m\}\) if and only if, for all distinct \(i, j, k, h \in \{1, \ldots, m\}\), the maximum of

\[\{D_{i,j} + D_{k,h}, D_{i,k} + D_{j,h}, D_{i,h} + D_{k,j}\}\]

is attained at least twice.

A wider discussion about the weighted graph theory can be found in [4].

In [1] the authors characterized the predistance matrices that are actually distance matrices of some particular graphs, such as paths, caterpillars, cycles, bipartite graphs, complete graphs and planar graphs.

In this work we give a criterion for a distance matrix to be the distance matrix of a positive-weighted \(n\)-cubic graph, that is a positive-weighted graph whose vertices and edges are respectively the vertices and edges of the hypercube in \(\mathbb{R}^n\). In order to show that it was not possible to deduce easily a characterization of distance matrices of \(n\)-cubic graphs from the one for bipartite graphs by adding a condition equivalent to \(n\)-regularity, we exhibit an example of a connected bipartite \(n\)-regular graph with order \(2^n\) that is not an \(n\)-cubic graph.

Finally we give a characterization of the distance matrices of positive-weighted Peterson graphs.

1 Notations and recalls

Notation 1.1. Throughout the paper we use the following notation:

\(\mathbb{N}\) is the set of non-negative integers,
\(\mathbb{N}^+\) is the set of positive integers,
\(#A\) denotes the cardinality of \(A\) for any set \(A\),
\(Z_k^n\) denotes the set of \(z \in \{0,1\}^n\) with exactly \(k\) entries equal to 1, for any \(n, k \in \mathbb{N}^+\).
We recall from [1] the definitions of indecomposable entry of a distance matrix and of useful edge.

**Definition 1.2.** Let $D$ be a distance $m \times m$ matrix for some $m \in \mathbb{N}^+$. We say that an entry $D_{i,j}$ for some $i, j$ with $i \neq j$ is indecomposable if and only if $D_{i,j} < D_{i,k} + D_{k,j}$ for all $k \in \{1, \ldots, m\} \setminus \{i, j\}$.

**Definition 1.3.** In a positive-weighted graph $G$ an edge $e$ is called useful if there exists at least one couple of vertices $i$ and $j$ such that all the paths realizing $D_{i,j}(G)$ contain the edge $e$. Otherwise the edge is called useless.

**Remark 1.4.** We recall from [1] (Remark 2.3) that, if $D$ is the distance matrix of a positive-weighted graph $G = (G, w)$, then $D_{i,j}$ is indecomposable if and only if $E(G)$ contains the edge $e(i,j)$ and $e(i,j)$ is useful; in this case we have that $D_{i,j}(G)$ is realized only by the path given only by the edge $e(i,j)$ and in particular $w(e(i,j)) = D_{i,j}(G)$.

**Notation 1.5.** Let $D$ be a distance $m \times m$ matrix for some $m \in \mathbb{N}^+$. Let us denote the set $\{1, \ldots, m\}$ by $X$ and let us fix an element $x$ in $X$. We can partition the set $X$ as follows:

- let $X_0(x) = \{x\}$;
- let $X_1(x)$ be the set of those elements $y \in X$ such that $D_{x,y}$ is indecomposable.
- let $X_2(x)$ be the set of those elements $y \in X$ for which the minimum $k$ such that $D_{x,y} = D_{x,i_1} + D_{i_1,i_2} + \cdots + D_{i_{k-1},y}$, with every summand indecomposable, is 2;
- in general, for every $t \in \mathbb{N}$, we define $X_t(x)$ to be the set of those elements $y \in X$ for which the minimum $k$ such that $D_{x,y} = D_{x,i_1} + D_{i_1,i_2} + \cdots + D_{i_{k-1},y}$, with every summand indecomposable, is $t$.

Finally, we define $X_{-1}(x) = \emptyset$ and we briefly write $X_t(A)$ in place of $\bigcup_{a \in A} X_t(a)$.

## 2 Distance matrices of weighted cubic graphs

In this section we give a characterization of distance matrices of positive-weighted $n$-cubic graphs.

**Definition 2.1.** Let $n \in \mathbb{N}^+$. The $n$-cubic graph is the graph $C_n$ whose vertices and edges are respectively the vertices and the edges of the $n$-hypercube, that is, the graph with vertex set $V(C_n) = \{0, 1\}^n$ and edge set

$$E(C_n) = \{e(v,w) \mid v, w \in \{0, 1\}^n \text{ and } \exists i \text{ such that } v_i \neq w_i\}.$$
Remark 2.2. Let $C_n$ be the $n$-cubic graph. Given $x, y \in V(C_n)$, define the Hamming distance between $x$ and $y$ to be the minimal number of edges of a path connecting $x$ and $y$ or, equivalently, the number $\# \{ i \in \{1, \ldots, n\} \mid x_i \neq y_i \}$. We denote it by $d(x, y)$. For any $x \in V(C_n)$ and any $k \in \{0, \ldots, n\}$, we have obviously the following relation:

$$\# \{ y \in V(C_n) \mid d(x, y) = k \} = \binom{n}{k}. \tag{1}$$

For instance, in the 3-cubic graph, there are one vertex with distance 0 ($x$ itself), three vertices of distance 1 (the adjacent vertices), three vertices of distance 2 and one vertex with distance 3 (the opposite vertex).

Remark 2.3. Let $C_n = (C_n, w)$ be a positive-weighted $n$-cubic graph where each edge is useful and let $X$ denote its vertex set. By (1) and Remark 1.4, we have that

$$\# X_k(x) = \binom{n}{k};$$

in fact, by Remark 1.4, the indecomposable 2-weights correspond to the edges of $C_n$ and so the minimum $k$ such that $D_{x,y} = D_{x,i_1} + D_{i_1,i_2} + \cdots + D_{i_{k-1},y}$, with all the summands indecomposable corresponds to the minimal number of the edges of a path between $x$ and $y$.

Theorem 2.4. Let $n \in \mathbb{N}^+$ and let $D$ be a $2^n \times 2^n$ distance matrix. Let us denote the set $\{1, \ldots, 2^n\}$ by $X$. The matrix $D$ is the distance matrix of a positively weighted $n$-cubic graph $C_n = (C_n, w)$ in which each edge is useful if and only if the following conditions hold:

(a) for any $x, y \in X$ with $y \in X_k(x)$ for some $k \in \mathbb{N}$, we have that $X_1(y)$ is given by $k$ elements of $X_{k-1}(x)$ and $n - k$ elements of $X_{k+1}(x)$;

(b) there exists $\overline{x} \in X$ such that, for any $k \geq 2$ and any $y \in X_k(\overline{x})$, there are exactly $k$ elements $z_1, \ldots, z_k$ of $X_1(\overline{x})$ such that

$$y \in X_{k-1}(z_i) \ \forall i \in \{1, \ldots, k\},$$

and the map from $X_k(\overline{x})$ to $\binom{X_1(\overline{x})}{k}$, defined as $y \mapsto \{z_1, \ldots, z_k\}$, is bijective;

(c) for all $x, y \in X$ such that $y \notin X_1(x)$, we have:

$$D_{x,y} = \min_{i_1, \ldots, i_k \in X \text{ with } k \geq 3, \ i_1 = x, \ i_k = y, \ i_{j+1} \in X_1(i_j) \ \forall j} \{D_{i_1,i_2} + \cdots + D_{i_{k-1},i_k}\}.$$
Proof. ($\Rightarrow$) By Remark 1.3, the indecomposable 2-weights correspond to the edges of $C_n$. So $X_k(x)$ is the set of the $n$-tuples with $n-k$ entries equal to the corresponding entries of $x$ and the others different from the corresponding entries of $x$. So (a) and (b) are obvious.

Let us prove (c). Let $x, y$ with $D_{x,y}$ decomposable. By definition of 2-weights, we have that $D_{x,y}(C_n)$ is equal to

$$\min_{i_1, \ldots, i_k \in X \text{ with } k \geq 2, i_1 = x, i_k = y \text{ and } i_j, i_{j+1} \text{ adjacent } \forall j} \{w(e(i_1, i_2)) + \cdots + w(e(i_{k-1}, i_k))\},$$

but, by assumption, every edge of $C_n$ is useful, so, by Remark 1.3, for any adjacent vertices $r, s$, we have that $w(e(r, s)) = D_{r,s}(C_n)$, which is equal to $D_{r,s}$ by assumption; thus $D_{x,y}(C_n)$ is equal to

$$\min_{i_1, \ldots, i_k \in X \text{ with } k \geq 2, i_1 = x, i_k = y \text{ and } i_j, i_{j+1} \text{ adjacent } \forall j} \{D_{i_1,i_2} + \cdots + D_{i_{k-1},i_k}\}. \tag{3}$$

Since, by Remark 1.3, two vertices $r, s$ are adjacent if and only if $r \in X_1(s)$ and $D_{x,y}$ is decomposable, we get (c).

($\Leftarrow$) First observe that assumption (a) implies that $\#X_1(y) = n$ for any $y \in X$; in particular $\#X_1(\overline{x}) = n$ and so, by (b), we get also that $X_k(\overline{x}) = \emptyset$ for any $k > n$. Moreover, by (a), for any $z \in X$, we have that $X_1(X_s(z)) \subseteq X_{s+1}(z) \cup X_{s-1}(z)$, so

$$X_1(X_s(z)) \cap X_t(z) = \emptyset \text{ if } s + 1 < t. \tag{4}$$

We define $G_n$ to be the graph whose vertex set is $X$ and, for any $i, j \in X$, we have that $e(i, j) \in E(G_n)$ if and only if $i \in X_1(j)$.

We want to show that $G_n$ is isomorphic to the $n$-cubic graph. Let us consider the following map $\varphi : X \rightarrow \{0, 1\}^n$: define $\varphi(\overline{x})$ to be $(0, \ldots, 0)$ and send the $n$ elements of $X_1(\overline{x})$ to the elements of $Z_1^n$ in any injective way; let $k \geq 2$ and $y \in X_k(\overline{x})$, and let $z_1, \ldots, z_k$ be as in (b); define $\varphi(y)$ to be $\sum_{i=1}^{k} \varphi(z_i)$. Obviously $\varphi$ is a bijection between $X$ and $\{0, 1\}^n$; in fact $\varphi$ restricted to $X_k(\overline{x})$ is the composition of the following bijective maps:

$$X_k(\overline{x}) \rightarrow \left(\frac{X_1(\overline{x})}{k}\right) \rightarrow \left(\frac{Z_1^n}{k}\right) \rightarrow Z_k^n,$$

where the first map is the map $y \mapsto (z_1, \ldots, z_k)$ described in (b), the second map is the map induced by $\varphi$ and the last map is given by the sum.

In order to show that $G_n$ is isomorphic to the $n$-cubic graph, we have to show that, for any $k \geq 1$ and any $y \in X_k(\overline{x})$, if $X_1(y) \cap X_{k-1}(\overline{x}) = \{v_1, \ldots, v_k\}$, then $\varphi(v_1), \ldots, \varphi(v_k)$ are the $n$-uples obtained from $\varphi(y)$ by changing an entry equal to 1 into 0. Let $z_1, \ldots, z_k$ be the elements of $X_1(\overline{x})$ such that

$$y \in X_{k-1}(z_i) \quad \forall i \in \{1, \ldots, k\}$$
and, for any $j = 1, \ldots, k$, let $z_1^j, \ldots, z_{k-1}^j$ be the $k - 1$ elements of $X_1(\pi)$ such that
\[ v_j \in X_{k-2}(z_i^j), \quad \forall i \in \{1, \ldots, k-1\} \] (they exist by (b)). We have that
\[ y \in X_{k-1}(z_i^j) \quad \forall i \in \{1, \ldots, k-1\}, \quad \forall j \in \{1, \ldots, k\}, \] in fact: if we had $y \in X_t(z_i^j)$ with $t > k - 1$, we would get that
\[ y \in X_1(X_{k-2}(z_i^j)) \cap X_t(z_i^j) \] (by (5) and by the fact that $y \in X_1(v_j)$), which would contradict (4); if we had that $y \in X_t(z_i^j)$ with $t < k - 1$, we would get that
\[ \pi \in X_1(X_t(y)) \cap X_k(y) \] (because $\pi \in X_1(z_i^j)$ and $z_i^j \in X_t(y)$), which would contradict again (4). By (b), we must have that $z_i^j \in \{z_1, \ldots, z_k\}$ for any $i, j$. By (b) we have also that, for any distinct $j, j' \in \{1, \ldots, k\}$, the set $\{z_1^j, \ldots, z_{k-1}^j\}$ is different from the set $\{z_1^{j'}, \ldots, z_{k-1}^{j'}\}$, so we must have that the sets $\{z_1^j, \ldots, z_{k-1}^j\}$ for $j = 1, \ldots, k$ are exactly the sets $\{z_1, \ldots, z_k\} \setminus \{z_s\}$ for $s = 1, \ldots, k$. Hence \( \varphi(v_1), \ldots, \varphi(v_k) \), which, by definition of \( \varphi \), are equal respectively to
\[ \sum_{l=1}^{k-1} \varphi(z_1^l), \ldots, \sum_{l=1}^{k-1} \varphi(z_k^l), \] are the $n$-uples obtained from $\varphi(y) = \sum_{l=1}^{k} \varphi(z_l)$ by changing an entry equal to 1 into 0, as we wanted to prove.

Finally consider the weighted graph $G_n = (G_n, w)$, where $w(e(i, j))$ is defined to be $D_{i, j}$ for every $e(i, j) \in E(G_n)$. For any $x, y \in X$ with $y \in X_1(x)$ we have obviously that $D_{x,y}(G_n) = D_{x,y}$ because by construction $e(x, y) \in E(G_n)$ and its weight is $D_{x,y}$, so $D_{x,y}(G_n) = D_{x,y}$ by triangle inequalities. For any $x, y \in X$ with $y \notin X_1(x)$, by definition of 2-weights, we have that $D_{x,y}(G_n)$ is equal to the number in (2); thus, by definition of $w$, it is equal to the number in (3). By the fact that two vertices $r, s$ are adjacent if and only if $r \in X_1(s)$ (by definition of $G_n$), by the decomposability of $D_{x,y}$ and by condition (c), we get that $D_{x,y}(G_n) = D_{x,y}$. \( \square \)

As we have already said, the distance matrices of positive-weighted bipartite graphs were characterized in [1]. Obviously an $n$-cubic graph is a $n$-regular bipartite graph with order $2^n$. In order to show that it was not possible to deduce a characterization of distance matrices of $n$-cubic graphs from the one for bipartite graphs simply by adding the condition that, for every vertex $x$, there are exactly $n$ other vertices $y_1, \ldots, y_n$ such that $D_{x,y_i}$ is indecomposable, we show an example of a connected bipartite $n$-regular
graph with order $2^n$ that is not an $n$-cubic graph. We exhibit here the case $n = 4$, being the general case completely analogous.

To construct our example, we start by partitioning the set of vertices $X$ (of cardinality equal to $16 = 2^4$), into two equipotent subsets of cardinality 8, say $Y = \{y_1, \ldots, y_8\}$ and $Z = \{z_1, \ldots, z_8\}$.

Now we build the complete bipartite graph $K_{2,4}$ with vertex set $\{y_1, y_2, z_1, z_2, z_3, z_4\}$, connecting $y_1$ and $y_2$ to each $z_i$; now $y_1$ and $y_2$ have degree 4, while each $z_i$ has degree 2: see Figure 1.

Then, we build the complete bipartite graph $K_{4,2}$ with vertex set $\{y_5, y_6, y_7, y_8, z_7, z_8\}$, connecting $z_7$ and $z_8$ to each $y_j$; now $z_1$ and $z_2$ have degree 4, while each $y_j$ for $j = 5, \ldots, 8$ has degree 2: see Figure 2

At this point $y_3$, $y_4$, $z_5$ and $z_6$ are still “isolated”; since we are building a 4-regular graph, $z_1$, $z_2$, $z_3$, $z_4$, $y_5$, $y_6$, $y_7$ and $y_8$ can be linked to two more vertices each. So we connect $y_3$ to $z_1$, $z_2$, $z_5$ and $z_6$, then we connect $y_4$ to $z_3$, $z_4$, $z_5$ and $z_6$; moreover we build an edge from $z_5$ to $y_5$ and $y_6$ and from $z_6$ to $y_7$ and $y_8$, as in Figure 3. In this situation, the vertices $y_5$, $y_6$, $y_7$, $y_8$, $z_1$, $z_2$, $z_3$ and $z_4$ have degree 3, while the others have degree 4; so we simply connect $z_i$ to $y_{i+4}$ for each $i \in \{1, 2, 3, 4\}$, having a connected bipartite $n$-regular graph with order $2^n$, as desired (Figure 4).

But this graph is not a 4-cubic graph, since there exist at least two different vertices (for example $y_1$ and $y_2$) connected to the same four vertices, while this does not happen
in an 4-cubic graph. In fact, given two different vertices \( u = (u_1, u_2, u_3, u_4) \in \{0, 1\}^4 \) and \( v = (v_1, v_2, v_3, v_4) \in \{0, 1\}^4 \) of a 4-cubic graph, they have at least one different coordinate, say \( 0 = u_1 \neq v_1 = 1 \); we can suppose \( u = (0, 0, 0, 0) \); if \( u \) and \( v \) were adjacent to the same vertices \( x, y, z, w \), then each of them would have exactly one coordinate different from \( u \), say \( 0 = u_1 \neq x_1 = 1, 0 = u_2 \neq y_2 = 1, 0 = u_3 \neq z_3 = 1, 0 = u_4 \neq w_4 = 1 \), so \( x = (1, 0, 0, 0), y = (0, 1, 0, 0), z = (0, 0, 1, 0), w = (0, 0, 0, 1) \); but also \( v \) must have the same property, and since \( v \neq u \), we have necessarily \( v = (1, 1, 0, 0) \) or \( v = (1, 0, 1, 0) \) or \( v = (1, 0, 0, 1) \), using the adjacency of \( v \) and \( x \); in each case, we see that \( v \) and at least one of \( y, z \) and \( w \) differ in two coordinates, so they cannot be adjacent and this is a contradiction.

### 3 Distance matrices of weighted Peterson graphs

In this section we characterize distance matrices of positive-weighted Peterson graphs.

**Theorem 3.1.** Let \( D \) be a 10 \times 10 distance matrix. Let us denote the set \( \{1, \ldots, 10\} \) by \( X \). The matrix \( D \) is the distance matrix of a positively weighted Peterson graph in which each edge is useful if and only if the following conditions hold:

(a) for any \( x \in X \), we have \( \#X_1(x) = 3 \);
for instance, if \( D \) is indecomposable by assumption (c), and, finally, statement (d) follows from the definition of 2-weights, Remark 1.4 and the implication of Theorem 2.4.

Finally, statement (d) follows from the definition of 2-weights, Remark 1.4 and the assumption that all the edges are useful as in the proof of statement (c) in the right implication of Theorem 2.4.

(\( \Rightarrow \)) We point out that all the edges are useful by assumption and an edge \( e(i, j) \) is useful if and only if \( D_{i,j}(G) \) (which is equal to \( D_{i,j} \) by assumption) is indecomposable (see Remark 1.4); so the edges correspond to the indecomposable 2-weights. Hence statement (a) follows from the fact that all the vertices of the Peterson graph have degree 3 and statement (b) follows from the fact that in the Peterson graph there are not cycles of length 3 or 4. Statement (c) is obvious (take \( v_1, \ldots, v_5 \) as in Figure 5).

Finally, statement (d) follows from the definition of 2-weights, Remark 1.4 and the assumption that all the edges are useful as in the proof of statement (c) in the right implication of Theorem 2.4.

(\( \Leftarrow \)) Let \( v_1, \ldots, v_5, \overline{v}_1, \ldots, \overline{v}_5 \) as in (c). Let \( G \) be the graph in Figure 5 and for any adjacent vertices \( i, j \), define \( w(e(i, j)) = D_{i,j} \). Let \( G = (G, w) \). We want to show that \( D_{x,y}(G) = D_{x,y} \) for any \( x, y \in X \).

Case 1: \( D_{x,y} \) is indecomposable.

First observe that, by (b), we have that \( D_{v_i,v_j} \) is indecomposable if and only if \( \{i, j\} \) is one of the following: \( \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\} \). So we observe that \( D_{v_i,v_j} \) is indecomposable if and only if in the graph \( G \) we have constructed there is an edge with endpoints \( v_i \) and \( v_j \).

Observe also that \( D_{\overline{v}_i,\overline{v}_j} \) is indecomposable if and only if in the graph \( G \) there is an edge with endpoints \( \overline{v}_i \) and \( \overline{v}_j \); otherwise we would have a contradiction with (b): for instance, if \( D_{\overline{v}_1,\overline{v}_2} \) were indecomposable, we would have that \( D_{\overline{v}_1,\overline{v}_2} \), \( D_{v_1,v_2}, D_{\overline{v}_1,\overline{v}_2}, D_{\overline{v}_2,\overline{v}_1} \) are indecomposable (in fact \( D_{\overline{v}_1,\overline{v}_2} \) is indecomposable because \( \overline{v}_1 \in X_1(v_1), D_{v_1,v_2} \) is indecomposable by assumption (c), and, finally, \( D_{\overline{v}_2,\overline{v}_1} \) is indecomposable because \( \overline{v}_2 \in X_1(v_2) \)) and this would contradict assumption (b).
Finally observe that $D_{v_i,v_j}$ is indecomposable if and only if in $G$ there is an edge with endpoints $v_i$ and $v_j$ (otherwise we would have again a contradiction with (b)).

Thus we can conclude that $D_{x,y}$ is indecomposable if and only if in the graph $G$ we have constructed there is an edge with endpoints $x$ and $y$. In this case we have that $D_{x,y}(G) = D_{x,y}$ by the triangle inequalities.

**Case 2:** $D_{x,y}$ is decomposable. By definition of 2-weights we have that $D_{x,y}(G)$ is equal to the number in (2). So, by the definition of $w$, it is equal to the number in (3). By the fact, we have proved before, that $D_{i,j}$ is indecomposable if and only if in $G$ there is an edge with endpoints $i$ and $j$, by the decomposability of $D_{x,y}$ and, finally, by assumption (d), we get that $D_{x,y}(G) = D_{x,y}$.

![Peterson graph](image)

**Figure 5:** Peterson graph

\[\square\]

## 4 Open problems

We list here some possible open problems.

(1) We could try to generalize the result for positive-weighted Peterson graphs in Section 3 to positive-weighted Kneser graphs.

(2) Let $n$ be a natural number with $n \geq 2$ and let $\{m_I\}_{I \in \binom{\{1,\ldots,n\}}{2}}$ and $\{M_I\}_{I \in \binom{\{1,\ldots,n\}}{2}}$ be two families of positive real numbers with $m_I \leq M_I$ for any $I$; in the paper [3], the author studied when there exist a positive-weighted graph $G$ and an $n$-subset $\{1,\ldots,n\}$ of the set of its vertices such that $D_I(G) \in [m_I, M_I]$ for any $I \in \binom{\{1,\ldots,n\}}{2}$ and the analogous problem for trees. It would be interesting to study when there exist a positive-weighted graph of a particular kind (for instance a hypercube, a cycle, a bipartite graph...) and an $n$-subset $\{1,\ldots,n\}$ of the set of its vertices such that $D_I(G) \in [m_I, M_I]$ for any $I$.

(3) In the last years $k$-weights of weighted graphs for $k \geq 3$ have been investigated, see for instance [2], [6], [7]. One could try to characterize families of $k$-weights of some
particular graphs for \( k \geq 3 \).

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