Abstract

We study the phenomenon of revivals for the linear Schrödinger and Airy equations over a finite interval, by considering several types of non-periodic boundary conditions. In contrast with the case of the linear Schrödinger equation examined recently (which we develop further), we prove that, remarkably, the Airy equation does not generally exhibit revivals even for boundary conditions very close to periodic. We also describe a new, weaker form of revival phenomena, present in the case of certain Robin-type boundary conditions for the linear Schrödinger equation. In this weak revival, the dichotomy between the behaviour of the solution at rational and irrational times persists, but in contrast with the classical periodic case, the solution is not given by a finite superposition of copies of the initial condition.

1 Introduction

The phenomenon of revivals in linear dispersive periodic problems, also called in the literature Talbot effect or dispersive quantisation, has been well-studied and is by now well-understood. It was discovered first experimentally in optics, then rediscovered several times by theoretical and experimental investigations. While the term has been used systematically and consistently by many authors, there is no consensus on a rigorous definition. Several have described the phenomenon by stating that a given periodic time-dependent boundary value problem exhibits revival at rational times if the solution evaluated at a certain dense subset of times, see (1.7) below, is given by finite superpositions of translated copies of the initial profile. We will call this the periodic revival property. In particular, when the initial condition has a jump discontinuity at time zero, these discontinuities are propagated and remain present in the solution at each rational time.

This behaviour at rational times should be contrasted with the behaviour at generic time, when the solution is known to be continuous as soon as the initial condition is of bounded variation. Hence generically, while the dispersive propagation has a smoothing
effect on any initial discontinuity, this smoothing does not occur at rational times. Moreover, at generic times and for appropriate initial data, the solution while continuous is nowhere differentiable. In fact its graph has a fractal dimension greater than 1 [1], [2]. There is therefore a dichotomy between generic times and the measure-zero set of rational times, as suggested also by the provocative title of [3].

In this paper we examine the role of boundary conditions in supporting some form of revival phenomenon. In order to illustrate the range of revival behaviour concretely, we focus on two specific linear PDEs of particular significance both from the mathematical point of view and in terms of applications. Namely, we will consider the linear Schrödinger equation with zero potential

\[ iu_t(x,t) + u_{xx}(x,t) = 0, \]

(\text{LS})

and the Airy equation, also known as Stokes problem,

\[ u_t(x,t) - u_{xxx}(x,t) = 0. \]

(AI)

Both these PDEs will be posed on the interval \([0, 2\pi]\) and we set specific boundary conditions either of pseudo-periodic or of Robin type. These represent two essentially different types of boundary conditions. Indeed, in the pseudo-periodic case the boundary conditions couple the ends of the interval, just as in the periodic case, while in the Robin case, the boundary conditions are uncoupled. The type of revival property that we observe in the two cases strongly reflects this difference.

Let \(n\) denotes the order of spatial derivative in the PDE, hence \(n = 2\) for (LS) and \(n = 3\) for (AI). In the first part of the paper, following the work of [4], we will consider specific types of pseudo-periodic boundary conditions of the form

\[ \beta_k \partial_x^k u(0,t) = u(2\pi,t), \quad \beta_k \in \mathbb{C}, \quad k = 0, 1, \ldots, n - 1. \]

(PP)

Of particular interest will be the case when all \(\beta_k\) are equal, that is quasi-periodic boundary conditions of the form

\[ \beta \partial_x^k u(0,t) = u(2\pi,t), \quad \beta \in \mathbb{C}, \quad k = 0, 1, \ldots, n - 1. \]

(QP)

In the second part of the paper, we will consider Schrödinger’s equation (LS) with the specific Robin boundary conditions given by

\[ bu(x_0,t) = (1 - b)\partial_x u(x_0,t), \quad x_0 = 0, \pi, \quad b \in [0, 1]. \]

(R)

The case \(b = 0\) corresponds to Neumann and \(b = 1\) to Dirichlet boundary conditions. For these special cases, the solution of the boundary value problem is obtained by even or odd extensions from the solution of an associated periodic problem. However, for \(0 < b < 1\) the boundary value problem behaves very differently from a periodic one.
It is well established that the periodic problem for any linear dispersive equations exhibits periodic revival (see Theorem 1.4 below). Moreover, subject to consistency conditions on the coefficients $\beta_k$, in [1], it was shown that the periodic revival property holds in general for the equation \((LS)-(PP)\). Below, we give a new proof of the latter. Our arguments elucidate the mathematical reason for the persistence of the periodic revival property for the linear Schrödinger equation \((LS)\), when subject to the fairly general class of boundary conditions \((PP)\). In particular, we show that all pseudo-periodic boundary conditions can be solved in terms of certain associated periodic problems. This is the content of Proposition 3.1 which also enables us to deduce from existing results for the periodic case that, at irrational times, any initial discontinuity is smoothed out. To be precise, even when the initial profile has jump discontinuities, the solution at irrational times becomes a continuous (though nowhere differentiable) function of the space variable.

The spectral properties of pseudo-periodic and other non-periodic boundary value problems for \((AI)\) were first examined in [5], where an explicit general formula for the solution was given. Below we show that, in stark contrast with \((LS)\), the quasi-periodic Airy equation, \((AI)-(QP)\), in general does not exhibit any form of revival at rational times. Indeed, the periodic revival property holds in this case only for values of the quasi-periodicity parameter, $\beta = e^{2\pi i \theta}$ such that $\theta \in \mathbb{Q}$. Remarkably, the latter defies the na"i"ve expectation that the revival property carries onto the case of higher order PDEs, when the boundary conditions support it for the second order case. It also suggests that the general pseudo-periodic case for third-order PDEs, generically, will not exhibit revivals.

In Section 4 we prove the following result.

**Theorem 1.1.** Fix $\theta \in [0,1)$ and consider Airy’s equation \((AI)\) with initial condition $u_0 \in L^2(0,2\pi)$ and quasi-periodic boundary conditions \((QP)\) where $\beta = e^{i2\pi \theta}$. Let $p$ and $q$ be co-prime and let

$$v_0^{(p,q)}(x) = R_3(p,q) \left[ u_0(x) e^{-i\theta x} \right], \quad (1.1)$$

where $R_3(p,q)$ is the third order revival operator defined below in (2.4). Then, the solution at rational time $t_r = \frac{2\pi p}{q}$ is given by

$$u(x,t_r) = e^{-it_r \theta^3} e^{i\theta x} T_{3\theta t_r} v^{(p,q)}(x,3\theta t_r). \quad (1.2)$$

Here $T_s$ is the translation operator (see (2.2)) and $v^{(p,q)}(x,t)$ denotes the solution of the periodic problem for Schrödinger equation with initial condition $v_0^{(p,q)}$.

It is clear from representation (1.2) that we can expect revivals for the Airy quasi-periodic problem only when $\theta \in \mathbb{Q}$, see Corollary 4.1. Indeed, if $\theta \notin \mathbb{Q}$, then the time $3\theta t_r$ is an irrational time for the solution of a periodic problem of the Schrödinger equation, which is therefore a continuous function of $x$. We are not aware of any previous result in the literature concerning the failure of any form of revival to hold for a linear dispersive PDEs with coupling boundary conditions.
We devote the final part of the paper to the linear Schrödinger equation (LS) with the Robin type boundary conditions (R). In this case the boundary conditions do not couple the ends of the interval, in contrast with all other situations considered here. We show that only a weaker form of revival holds, leading us to reconsider what constitutes revival for a linear dispersive evolution equation. Specifically, we show that, while the solution is not given by finitely many translates of the initial condition, the presence of a periodic term in the solution representation guarantees that the dichotomy between the persistence versus regularisation of discontinuities at rational versus generic time still holds. Our main statement can be formulated as follows.

**Theorem 1.2.** Consider the linear Schrödinger equation (LS) with initial condition $u_0 \in L^2(0, \pi)$ and Robin boundary conditions (R) with $b \neq 0, 1$. Let $p$ and $q$ be co-prime and $R_2(p, q)$ be the second order revival operator defined below by (2.4). Let

$$f_1(x) = \sqrt{\frac{\pi}{2}} \frac{b}{(1 - b) e^{-b/(1 - b)}} e^{\frac{b}{1 - b} x}, \quad x \in (0, 2\pi).$$

Then, the solution at rational time $t_r = \frac{2\pi p}{q}$ is given by

$$u(x, t_r) = 2\sqrt{\frac{\pi}{2}} \langle u_0, e^{\frac{b}{1 - b} (\cdot)} \rangle_{L^2(0, \pi)} e^{\frac{b}{1 - b} t_r} f_1(x) + R_2(p, q) \left[ u_0^+ (x) \right] + R_2(p, q) \left[ 2f_1 \ast (u_0^- - u_0^+) (x) \right], \quad x \in (0, \pi).$$

where $u_0^\pm (x)$ are the even/odd extension in $(0, 2\pi)$ of the initial condition, and $\ast$ denotes the $2\pi$-periodic convolution.

We conjecture that this weaker form of revival is generic in the case of boundary conditions that do not couple the interval endpoints. Our observations in this case complement those reported in [6] illustrating a new kind of revival phenomenon.

**Periodic revival**

The original terminology seems to have originated from the experimentally observed phenomenon of *quantum revival* [7]–[9]. This describes how an electron that is initially concentrated near a single location of its orbital shell is found concentrated again, at certain specific times, near a finite number of orbital locations. This led pure mathematicians to pose the question in terms of whether a quantum particle *knows* the time, [3].

A precursor of the phenomenon was observed as far back as 1834 in optical experiments performed by Talbot [10]. This motivated the pioneering work of Berry and collaborators [1], [7], [11], on what they called the *Talbot effect* in the context of the linear free space Schrödinger equation. The concept was extended later to a class of linear dispersive equations that included the linearised Korteweg–deVries equation, first by Oskolkov, [12] and
subsequently rediscovered by Olver [13], who called the effect *dispersive quantisation*. It was later extended by Erdo˘gan and Tzirakis (see the monograph [14] and references therein). An exhaustive introduction to the history and context of this phenomena can be found in the recent survey [15].

Questions have also been addressed on the fractal dimension of the solution profile at irrational times, hence almost everywhere in time, some of them resolved by Rodnianski in [2]. In a different direction, Olver and Chen in [16] and [17] observed and confirmed numerically the revival and fractalisation effect in some non-linear, integrable and non-integrable, evolution problems. A number of their observations have been rigorously confirmed in [18], [19], [20] by Erdo˘gan, Tzirakis, Chousionis and Shakan, and more recently in [6] for linear integro-differential dispersive equations.

The direct link between our present findings and this periodic framework can be put into perspective by following [14, §2.3], as we briefly summarise.

Consider general linear dispersive equations of the form

\[ u_t(x,t) + iP(-i\partial_x)u(x,t) = 0, \quad x \in [0, 2\pi], \quad t > 0. \]  

where \( P(\cdot) \) is a polynomial of degree \( n \) with integer coefficients. Consider purely periodic boundary conditions, i.e. (QP) with \( \beta = 1 \). For initial datum \( u(x,0) = u_0(x) \in L^2(0, 2\pi) = L^2 \), the solution is given in terms of the eigenfunction expansion

\[ u(x,t) = \sum_{m \in \mathbb{Z}} \hat{u}_0(m)e^{-iP(m)t}e_m(x), \quad \hat{u}_0(m) = \langle u_0, e_m \rangle, \]  

where

\[ e_m(x) = \frac{e^{imx}}{\sqrt{2\pi}}, \quad \langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx, \quad f, g \in L^2. \]  

The family \( \{e_m\}_{m \in \mathbb{Z}} \) is the orthonormal family of eigenfunctions of the self-adjoint periodic operator \( P(-i\partial_x) \). Note that the latter has a compact resolvent. If \( u_0 \) is continuous and periodic, the expression (1.5) is also a continuous periodic function of \( x \) and \( t \). The case of equations (LS) and (AI), corresponding to \( P(k) = k^n \) with \( n = 2, 3 \), are among the simplest linear evolution equations, but they are important as they also appear as the linear part of important nonlinear PDEs of mathematical physics, namely the nonlinear Schrödinger and KdV equations respectively.

We focus now on the countable set of rational times, defined as follows.

**Definition 1.3.** We say that \( t > 0 \) is a rational time for the evolution problem (1.4) if there exist co-prime, positive integers \( p, q \in \mathbb{N} \) such that

\[ t = \frac{2\pi p}{q}. \]  

A self-contained proof of the following general result can be found in [14, Theorem 2.14]. This result says that, at these rational times, the solution of the periodic problem for (1.4) has an explicit representation in terms of translates of \( u_0 \).
Theorem 1.4 (Periodic Revival). Consider equation (1.4), with initial condition \( u(x,0) = u_0(x) \in L^2 \) and purely periodic boundary conditions, (PP) with all \( \beta_k = 1 \). At rational time \( t \) given by (1.7) the solution \( u(x,t) \) is given by

\[
 u \left( x, 2\pi \frac{p}{q} \right) = \frac{1}{q} \sum_{k=0}^{q-1} G_{p,q}(k) u_0^* \left( x - 2\pi \frac{k}{q} \right),
\]

where \( u_0^* \) is the \( 2\pi \)-periodic extension of \( u_0 \), see (2.1) below. The coefficients \( G_{p,q}(k) \) are given by

\[
 G_{p,q}(k) = \sum_{m=0}^{q-1} e^{-2\pi i P(m) \frac{k}{q}} e^{2\pi im \frac{k}{q}}.
\]

Note that the functions \( G_{p,q}(k) \) in (1.9) are periodic number-theoretic functions (mod \( q \)), c.f. [21, §27.10] of Gauss type, but they are not Gauss sums, as the coefficients \( e^{-2\pi i P(m) \frac{k}{q}} \) are not Dirichlet characters.

The representation given in Theorem 1.4 describes explicitly the “revival” of the initial condition at rational times, as translated copies of it which are the building blocks of the solution representation. This is in contrast with the behaviour at generic, irrational times. For such times, the solution is continuous and indeed can be shown to have fractal behaviour as soon as the initial condition is sufficiently rough. To be more precise, the following and Theorem 1.4 complement one another for the case of (LS), see [2].

Theorem 1.5. Let \( P(k) = k^2 \) and assume that the hypotheses of Theorem 1.4 hold true. Assume, additionally, that \( u_0(x) \) is of bounded variation. Then, the solution for any value of \( t \) that is not of the form (1.7) is a continuous function of \( x \). Moreover, if \( u_0 \notin \bigcup_{s>1/2} H^s(0,2\pi) \),

where \( H^s \) denotes the standard Sobolev space of order \( s \), then for almost every \( t \), the solution is nowhere differentiable, and the graph of the real part of the solution has fractal dimension \( 3/2 \).

A similar result holds in general for equation (1.4), see [14].

2 Revival operators

This section is devoted to the notion of revival operators, that can be regarded as the basic building blocks of the revival formula (1.8) for solutions of linear dispersive PDEs whose polynomial dispersion is of the form \( P(k) = k^l \), \( l \in \mathbb{N} \). They provide a compact notation, e.g. for the statement of Theorem 1.4. More significantly, while it is straightforward to
compute the corresponding Fourier representation, they give the crucial link for extending
the revival results to more general pseudo-periodic problems from known cases, such as the
linear Schrödinger equation, to higher order case, in particular the Airy equation.

Here and everywhere below, we will denote by $f^\ast$ the $2\pi$-periodic extension to $\mathbb{R}$ of a
function $f$ defined on $[0, 2\pi]$. Explicitly,

$$f^\ast(x) = f(x - 2\pi m), \quad 2\pi m \leq x < 2\pi (m + 1), \quad m \in \mathbb{Z}. \quad (2.1)$$

Because of the role of specific translation operators in what follows, we set our notation
with the following definition.

**Definition 2.1 (Periodic translation operator).** Let $s \in \mathbb{R}$. The periodic translation
operator $T_s : L^2 \to L^2$ is given by

$$T_s f(x) = f^\ast(x - s), \quad x \in [0, 2\pi). \quad (2.2)$$

Note that $T_s$ are isometries. In our scaling, the Fourier coefficients of $T_s f$, $f \in L^2$, turn
out to be

$$\hat{T}_s f(m) = \int_0^{2\pi} T_s f(x) e_{m}(-x) \, dx = e^{-ims} \hat{f}(m). \quad (2.3)$$

Revival operators are formed as finite linear combinations of specific translation operators.

**Definition 2.2 (Periodic revival operator).** Let $p$ and $q$ be integers and co-prime. Let
$\ell \in \mathbb{N}$. The periodic revival operator $R^{\ell}_p(q) : L^2 \to L^2$ of order $\ell$ at $(p, q)$ is given by

$$R^{\ell}_p(q) f = \sqrt{\frac{2\pi}{q}} \sum_{k=0}^{q-1} G^{(\ell)}_{p,q}(k) T^{2\pi k q}_q f, \quad G^{(\ell)}_{p,q}(k) = \sum_{m=0}^{q-1} e^{-im\frac{2\pi k}{q}} e_m \left( \frac{2\pi k}{q} \right), \quad (2.4)$$

where $e_m(x)$ are the normalised eigenfunctions of the $2\pi$-periodic problem given in (1.6).

As we shall see next, from the Fourier representation, it follows that all periodic revival
operators are isometries.

**Lemma 2.3.** Let $p$ and $q$ be integers and co-prime. Let $\ell \in \mathbb{N}$. Then, $R^{\ell}_p(q)$ given by (2.4) is an isometry of $L^2$. Moreover, for all $f \in L^2$ we have

$$\langle R^{\ell}_p(q) f, e_j \rangle = e^{-ij\frac{2\pi k}{q}} \hat{f}(j). \quad (2.5)$$

**Proof.** In order to deduce that $R^{\ell}_p(q)$ is an isometry on $L^2$, it is enough to prove (2.5).

From the left hand side of (2.4) and from (2.3), it follows that

$$\langle R^{\ell}_p(q) f, e_j \rangle = \frac{\sqrt{2\pi}}{q} \sum_{k=0}^{q-1} G^{(\ell)}_{p,q}(k) \langle T^{2\pi k q}_q f, e_j \rangle = \frac{\sqrt{2\pi}}{q} \sum_{k=0}^{q-1} e^{-ij\frac{2\pi k}{q}} G^{(\ell)}_{p,q}(k).$$
Substitute the right hand side of (2.4) to get,
\[
\langle R_\ell(p,q)f,e_j \rangle = \hat{f}(j) = \frac{1}{q} \sum_{m=0}^{q-1} e^{-im^2 \pi p/q} \sum_{k=0}^{q-1} e^{i(m-j)2\pi k/q}.
\]

Now, if \( m \not\equiv j \pmod{q} \), then there exists \( z \in \mathbb{Z} \) not a multiple of \( q \), such that \( m - j = z_1 q + z \) for \( z_1 \in \mathbb{Z} \). Hence,
\[
\sum_{k=0}^{q-1} e^{iz_1 q 2\pi k/q} = \sum_{k=0}^{q-1} e^{iz q 2\pi k/q} = q,
\]
On the other hand, whenever \( m \equiv j \pmod{q} \), we have \( m - j = z_2 q \) for \( z_2 \in \mathbb{Z} \) and so
\[
\sum_{k=0}^{q-1} e^{iz_2 q 2\pi k/q} = 0.
\]
Moreover, in this case we know that, for any \( \ell \in \mathbb{N} \), \( m^\ell \equiv j^\ell \pmod{q} \), and so \( m^\ell = j^\ell + z_3 q \) for some other \( z_3 \in \mathbb{Z} \), hence
\[
e^{-im^\ell 2\pi p/q} = e^{-ij^\ell 2\pi p/q} e^{-iz_3 q 2\pi p/q} = e^{-ij^\ell 2\pi p/q}.
\]
Therefore, as \( m \) runs from 0 to \( q - 1 \), we find that
\[
\langle R_\ell(p,q)f,e_j \rangle = \hat{f}(j)e^{-ij^\ell 2\pi p/q},
\]
as claimed. \( \square \)

The proof of the lemma above relies on elementary arguments and depends on the specific form of the eigenfunctions \( e_m(x) \) and their periodicity. This is in fact at the heart of the periodic revival phenomenon. It suggests strongly that such phenomenon depends crucially on periodicity and will not survive if other boundary conditions are prescribed. The investigation of the validity of this statement is the motivation for this work.

By immediate substitution, Theorem 1.4 applied to the linear Schrödinger and Airy equations can be reformulated in terms of revival operators.

**Lemma 2.4.** Let \( u_0 \in L^2 \) and assume periodic boundary conditions, \((QP)\) with \( \beta = 1 \). At rational time \( t = 2\pi \frac{p}{q} \), the solution to the periodic problem for equation \((LS)\) starting at \( u_0 \) is given by
\[
u \left( x, 2\pi \frac{p}{q} \right) = R_2(p,q)u_0(x)
\]
and the solution to the periodic problem for equation \((AI)\) starting at \( u_0 \) is given by
\[
u \left( x, 2\pi \frac{p}{q} \right) = R_3(p,q)u_0(x).
\]
3 Pseudo-periodic problems for the linear Schrödinger equation

In this section we give an alternative proof of the results reported in [4], by deriving a new representation of the solution of the problem (LS)-(PP), namely

\[ iu_t + u_{xx} = 0, \quad u(x,0) = u_0(x) \in L^2, \]

\[ \beta_0 u(0,t) = u(2\pi,t), \quad \beta_1 u_x(0,t) = u_x(2\pi,t), \] \tag{3.1}

where \( \beta_0, \beta_1 \in \mathbb{C} \) satisfy

\[ \arccos \left( \frac{1 + \beta_0 \beta_1}{\beta_0 + \beta_1} \right) \in \mathbb{R}. \]

The latter condition ensures that all the eigenvalues of the underlying (closed) spatial operator are real and that this operator has a family of eigenfunctions which is complete in \( L^2 \), forming a bi-orthogonal basis. Moreover, this family reduces to an orthonormal basis, i.e. the operator is self-adjoint, if and only if \( \beta_0 \beta_1 = 1 \). For details, see [4].

Our goal is to show that the solution of (3.1) can be written as the sum of four terms, each obtained as the solution of a periodic problem. These four periodic problems start from an initial condition obtained by a suitable transformation of the given initial \( u_0(x) \).

In order to construct a solution of (3.1), we consider the bi-orthogonal basis \( \{ \phi_j, \psi_\ell \}_{j,\ell \in \mathbb{Z}} \) formed by the eigenfunctions of the spatial operator and their adjoint pairs. The spectral problem is given by

\[ -\phi''(x) = \lambda \phi(x), \quad \beta_0 \phi(0) = \phi(2\pi), \quad \beta_1 \phi'(0) = \phi'(2\pi). \]

As shown in [4], the eigenvalues \( \{ \lambda_j \}_{j \in \mathbb{Z}} \) are given by

\[ \lambda_j = k_j^2, \quad k_j = (j + k_0), \quad k_0 = \frac{1}{2\pi} \arccos \left( \frac{1 + \beta_0 \beta_1}{\beta_0 + \beta_1} \right). \] \tag{3.2}

and the corresponding eigenfunctions are

\[ \phi_j(x) = \frac{1}{\sqrt{2\pi \tau}} (e^{ik_j x} + \Lambda_0 e^{-ik_j x}), \]

where

\[ \tau = \frac{(\gamma^2 + 1)(\beta_0 \beta_1 + 1) - 2\gamma(\beta_0 + \beta_1)}{(\beta_0 \gamma - 1)(\beta_1 \gamma - 1)}, \quad \Lambda_0 = \frac{\gamma - \beta_0}{\beta_0 - \gamma^{-1}} = \frac{\gamma - \beta_1}{\gamma^{-1} - \beta_1}, \] \tag{3.3}

and

\[ \gamma = e^{ik_j 2\pi} = e^{i2\pi k_0} = \frac{1 + \beta_0 \beta_1}{\beta_0 + \beta_1} + i \sqrt{1 - \left( \frac{1 + \beta_0 \beta_1}{\beta_0 + \beta_1} \right)^2} \cdot \]

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We require also the eigenfunctions of the adjoint spectral problem

\[-\psi''(x) = \lambda \psi(x), \quad \psi(0) = \bar{\beta}_1 \psi(a), \quad \psi'(0) = \bar{\beta}_0 \psi'(a).\]

These are given by

\[\psi_j(x) = \frac{1}{\sqrt{2\pi \tau}}(e^{ik_jx} + I_0 e^{-ik_jx})\]

where \(\tau\) is as in (3.3) and

\[I_0 = \frac{\gamma - 1/\bar{\beta}_1}{1/\bar{\beta}_1 - \gamma^{-1}}.\] (3.4)

The family \(\{\phi_j\}_{j \in \mathbb{Z}}\) is a complete system of \(L^2\). Then, for any fixed time \(t \geq 0\) and initial \(v_0 \in L^2\), the solution to (3.1) is given by the spectral expansion

\[u(x,t) = \sum_{j \in \mathbb{Z}} \langle v_0, \psi_j \rangle e^{-ik_j^2 t} \phi_j(x)\]

\[= \frac{1}{2\pi \tau} \sum_{j \in \mathbb{Z}} \left( \int_0^{2\pi} u_0(y)e^{-ik_jy} dy + \bar{I}_0 \int_0^{2\pi} u_0(y)e^{ik_jy} dy \right) e^{-ik_jt^2} \left( e^{ik_jx} + \Lambda_0 e^{-ik_jx} \right).\] (3.5)

Our alternative proof that this problem exhibits the periodic revival phenomenon will rely on the existence of revivals for suitable periodic problems. Given \(u_0 \in L^2\), we define \(v_0, w_0 \in L^2\) as

\[v_0(x) = u_0(x)e^{-ik_0x}, \quad w_0(x) = u_0(x)e^{ik_0x},\] (3.6)

where \(k_0 \in \mathbb{R}\) is defined in (3.2). For any \(f \in L^2\), we will denote by the symbol \(f^\natural(x)\) the reflection of \(f(x)\) with respect to \(x = \pi\), namely

\[f^\natural(x) = f(2\pi - x).\] (3.7)

**Proposition 3.1.** The solution \(u(x,t)\) of (3.1) admits the following representation,

\[u(x,t) = \frac{e^{-ik_0^2 t}}{\tau} \left\{ e^{ik_0x} \mathcal{T}_{2k_0t}v(x,t) + \Lambda_0 e^{-ik_0x} \mathcal{T}_{-2k_0t}v^\natural(x,t) + \bar{I}_0 e^{ik_0x} \mathcal{T}_{2k_0t}w^\natural(x,t) + \Lambda_0 \bar{I}_0 e^{-ik_0x} \mathcal{T}_{-2k_0t}w(x,t) \right\},\] (3.8)

where \(\mathcal{T}\) is the translation operator defined by (2.2), the constants \(\tau, \Lambda_0\) are given in (3.3) and \(I_0\) by (3.4). Here \(v, w, v^\natural, w^\natural\) are the solutions of the periodic problem, i.e. \(\beta_0 = \beta_1 = 1\), with initial conditions as follows,

- \(v(x)\) denotes the solution corresponding to initial condition \(v_0(x)\)
- \(w(x)\) denotes the solution corresponding to initial condition \(w_0(x)\)
- \(v^\natural(x)\) denotes the solution corresponding to initial condition \(v_0^\natural(x)\)
• \( w^\pm(x) \) denotes the solution corresponding to initial condition \( w^\pm_0(x) \).

Before giving a proof, we highlight the important consequence of this proposition. Substituting the expression \([2.6]\) for the solution of the periodic problem in \([3.8]\), one obtains revival for the pseudo-periodic linear Schrödinger equation.

**Corollary 3.2** (Pseudo-periodic revival property). The solution of the pseudo-periodic problem \([3.1]\) at rational times, is given by

\[
\begin{align*}
    u(x, 2\pi \frac{p}{q}) &= e^{-\frac{2\pi i k_0 p}{q}} \left\{ e^{i k_0 x} \left[ T_{\frac{4\pi k_0 p}{q}} R_2(p, q) \right] e^{-i k_0 x} u_0(x) \\
    &+ e^{-i k_0 x} \left[ \Lambda_0 e^{-i k_0 2\pi x} T_{\frac{4\pi k_0 p}{q}} R_2(p, q) \right] e^{i k_0 x} u_0^\pm(x) \\
    &+ e^{i k_0 x} \left[ \bar{I}_0 e^{i k_0 2\pi x} T_{\frac{4\pi k_0 p}{q}} R_2(p, q) \right] e^{-i k_0 x} u_0^\pm(x) \\
    &+ e^{-i k_0 x} \left[ \Lambda_0 \bar{I}_0 e^{i k_0 2\pi x} T_{\frac{4\pi k_0 p}{q}} R_2(p, q) \right] e^{i k_0 x} u_0(x) \right\}.
\end{align*}
\]

**Remark 3.3.** In expression \([3.2]\), the solution is given explicitly in terms of a finite number of translated copies of \( u_0(x) e^{\pm i k_0 x} \). Note that the final result is then multiplied by \( e^{\mp i k_0 x} \), and hence the solution is indeed given in terms of a finite linear combination of translated copies of \( u_0(x) \). This can be verified by substituting the expression for \( R_2(p, q) \) in the first part of formula \([3.2]\), to obtain

\[
e^{i k_0 x} T_{\frac{4\pi k_0 p}{q}} R_2(p, q) e^{-i k_0 x} u_0(x) = e^{i k_0 x} T_{\frac{4\pi k_0 p}{q}} \tilde{R}_2(p, q) u_0(x),
\]

where \( \tilde{R}_2(p, q) \) differs from \( R_2(p, q) \) only in that each term \( e_m(\frac{2\pi k}{q}) \) is replaced by \( e_m(\frac{2\pi k}{q} (1 + \frac{k_0}{m})) \). The other three terms in the expression \([3.2]\) for the solution can be handled similarly.

**Proof of Proposition 3.1.** Consider each of the terms in the series \([3.5]\). Using the definition \([3.6]\) of \( v_0 \) and \( w_0 \), we have

\[
\int_0^{2\pi} u_0(y) \frac{e^{-i k_0 y}}{\sqrt{2\pi}} dy + \bar{I}_0 \int_0^{2\pi} u_0(y) \frac{e^{i k_0 y}}{\sqrt{2\pi}} dy = \hat{v}_0(j) + \bar{I}_0 \hat{w}_0(-j). \tag{3.9}
\]

Recall that \( k_j = k_0 + j \). Moreover, we have the elementary but key relation,

\[
e^{-i k_j^2 t} = e^{-i k_0^2 t} e^{-2 k_0 j t} e^{-i j^2 t} \tag{3.10}
\]

and for the eigenfunctions

\[
\frac{e^{i k_j x}}{\sqrt{2\pi}} + \Lambda_0 \frac{e^{-i k_j x}}{\sqrt{2\pi}} = e^{i k_0 x} e_j(x) + \Lambda_0 e^{-i k_0 x} e_{-j}(x). \tag{3.11}
\]
Here the $e_j(x)$ are the periodic eigenfunctions.

By substituting (3.9), (3.10) and (3.11) in (3.5) we obtain

$$u(x, t) = \frac{e^{-ik_0^2t}}{\tau} \sum_{j \in \mathbb{Z}} e^{-ik_0jt} e^{-ij^2t} \left( e^{ik_0x} \bar{v}_0(j) e_j(x) + \Lambda_0 e^{-ik_0x} \bar{w}_0(-j) e_{-j}(x) \right)$$

(3.12)

Each term in (3.12) is the solution of a periodic problem. Indeed, from (2.3) it follows that for $f \in L^2$, $T_s f(x) = \sum_{j \in \mathbb{Z}} e^{-ij^2x} \hat{f}(j) e_j(x)$, hence we have

$$\sum_{j \in \mathbb{Z}} e^{-ik_0jt} e^{-ij^2t} e^{ik_0x} \bar{v}_0(j) e_j(x) = e^{ik_0x} T_{2k_0t} \left( \sum_{j \in \mathbb{Z}} \hat{v}_0(j) e^{-ij^2t} e_j(x) \right) = e^{ik_0x} T_{2k_0t} v(x, t),$$

where $v(x, t)$ solves the periodic equation with initial condition $v_0(x)$. Similar calculation for the remaining terms yields the representation (3.8).

Note that for the self-adjoint case, $\beta_0 \beta_1 = 1$, the following reduction of (3.8) is valid,

$$u(x, t) = \frac{e^{-ik_0^2t}}{1 + |\Lambda_0|^2} \left\{ e^{ik_0x} T_{2k_0t} v(x, t) + \Lambda_0 e^{-ik_0x} T_{-2k_0t} w(x, t) \right\} + |\Lambda_0|^2 e^{-ik_0x} T_{-2k_0t} w(x, t),$$

(3.13)

with all notation as in Proposition 3.1.

### 3.1 The quasi-periodic case

We now describe the specific form of the solution of the quasi-periodic boundary value problem for (LS), corresponding to $\beta_0 = \beta_1 = \beta$ in (3.1). This specific case appears to be of importance for the study of the vortex filament equation with non-zero torsion [22].

The self-adjoint case corresponds to $|\beta|^2 = 1$ and it has been studied in the context of quantum revivals, as well as experimentally, in [23].

Set $\beta = e^{2\pi i \theta}$ for $\theta \in (0, 1)$ in (3.1). For $k_0$ and $\Lambda_0$ as in (3.3), we have

$$\cos(2\pi k_0) = \frac{1 + \beta^2}{2\beta} = \frac{1 + e^{4\pi i \theta}}{2e^{2\pi i \theta}} = \cos(2\pi \theta).$$

So we pick

$$k_0 = \theta, \quad \gamma = e^{2\pi i \theta} = \beta \quad \text{and} \quad \Lambda_0 = \frac{\gamma - \beta}{\beta - \gamma^{-1}} = 0.$$

Substituting these values into (3.13), yields the significantly reduced expression,

$$u(x, t) = e^{-i\theta^2t} e^{i\theta x} T_{2\theta t} v(x, t).$$
where \(v(x,t)\) the solution of the periodic problem with initial condition \(v_0(x)\) as in (3.6). In particular, at rational times we obtain the representation formula

\[
\begin{align*}
\v(x,2\pi p/q) &= e^{-i\theta^22\pi^2/4}e^{i\theta x}T_{4\pi\theta^2/q}\mathcal{R}_2(p,q)e^{-i\theta x}u_0(x) .
\end{align*}
\]

(3.14)

Remark 3.4. The comment made in Remark 3.3 applies also to the revival expression (3.14). The latter can also be obtained directly, by expanding the solution in terms of the eigenfunctions of the associated spatial operator and their adjoint pair.

4 Quasi-periodic problems for the Airy equation

We now turn to the time evolution problem for the Airy equation with quasi-periodic boundary conditions, defined by (AI)-(QP) with \(\beta = e^{i2\pi\theta}\) for \(\theta \in [0,1)\), namely

\[
\begin{align*}
\v_t(x,t) - \v_{xxx}(x,t) &= 0, \quad \v(x,0) = \v_0(x), \\
e^{i2\pi\theta} \partial_x^m \v(0,t) &= \partial_x^m \v(2\pi,t), \quad m = 0,1,2.
\end{align*}
\]

(4.1)

We give the proof of Theorem 1.1 which describes the solution of (4.1) in terms of the solution of a periodic problem for the linear Schrödinger equation.

The spatial operator \(i\partial_x^3\) with the given boundary conditions is self-adjoint. Moreover, unlike the general quasi-periodic boundary conditions, we can find the eigenpairs of this operator explicitly. Because of this, it is possible to argue in similar fashion as in Section 3. This leads to the conclusion that, in contrast to the linear Schrödinger equation, it is not possible to establish a direct correspondence between the solution of (4.1) and the solution of one or more periodic problems evaluated at the same time. The correspondence that we establish in Theorem 1.1 connects the solution of the Airy equation at a rational time \(t_r\) to the solution of an associated problem for the linear Schrödinger equation evaluated at a time \(t_\theta\) that depends on \(t_r\) and on \(\theta\). As a consequence, we show below that revivals for problem (4.1) arise if and only if \(\theta \in \mathbb{Q}\).

The eigenvalue problem is now given by

\[
\begin{align*}
-\phi'''(x) &= i\lambda \phi(x), \quad e^{i2\pi\theta} \phi(0) = \phi(2\pi), \\
e^{i2\pi\theta} \phi'(0) &= \phi'(2\pi), \\
e^{i2\pi\theta} \phi''(0) &= \phi''(2\pi).
\end{align*}
\]

Hence, it is straightforward to compute that the eigenvalues are given by

\[
\lambda_m = k_m^3, \quad k_m = m + \theta, \quad m \in \mathbb{Z}
\]

and the corresponding normalized eigenfunctions by

\[
\phi_m(x) = e^{ik_m x}/\sqrt{2\pi} = e^{i\theta x}e_m(x), \quad m \in \mathbb{Z}.
\]

(4.2)
Thus, for any fixed time $t \geq 0$ and initial $u_0 \in L^2$, the solution to (4.1) is

$$u(x,t) = \sum_{m \in \mathbb{Z}} \langle u_0, \phi_m \rangle e^{-ik_3^m t} \phi_m(x).$$

(4.3)

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** According to (4.2),

$$\langle u_0, \phi_j \rangle = \int_0^{2\pi} u_0(x)e^{-i\theta x} e_j(x) dx = \hat{u}_0(j), \quad w_0(x) = u_0(x)e^{-i\theta x}.$$

The exponential term $e^{-ik_3^j t}$ can be written as

$$e^{-ik_3^j t} = e^{-i(j+\theta^j) t} = e^{-i\theta^j t} e^{-ij^3 t} e^{ij^2 3 \theta t}.$$

Substituting all this into (4.3) for the solution of (4.1), we find

$$u(x,t) = \sum_{j \in \mathbb{Z}} \langle u_0, \phi_j \rangle e^{-i(j+\theta^j) t} \phi_j(x)
= \sum_{j \in \mathbb{Z}} \hat{u}_0(j) e^{-i\theta^j t} e^{-ij^3 t} e^{-ij^2 3 \theta t} e^{i\theta x} e_j(x)
= e^{-i\theta^j t} e^{i\theta x} \sum_{j \in \mathbb{Z}} \hat{u}_0(j) e^{-ij^3 t} e^{-ij^2 3 \theta t} e_j(x) \quad (4.4)
= e^{-i\theta^j t} e^{i\theta x} T_{3\theta^j t} \left( \sum_{j \in \mathbb{Z}} \hat{u}_0(j) e^{-ij^3 t} e^{-ij^2 3 \theta t} e_j(x) \right).$$

For the last equality we have used the Fourier representation (2.3) of the translation operator $T_s$.

Now, by virtue of Lemma 2.3,

$$\hat{w}_0(j) e^{-ij^3 t} = \langle \mathcal{R}_{3}(p,q) w_0, e_j \rangle = \langle v_0^{(p,q)}, e_j \rangle = \hat{v}_0^{(p,q)}(j),$$

where the function $v_0^{(p,q)}(x)$ is given by (1.1). Substituting this final identity into (4.4), gives

$$u(x,t) = e^{-i\theta^j t} e^{i\theta x} T_{3\theta^j t} \left( \sum_{j \in \mathbb{Z}} \hat{v}_0^{(p,q)}(j) e^{-ij^2 3 \theta t} e_j(x) \right) = e^{-i\theta^j t} e^{i\theta x} T_{3\theta^j t} v^{(p,q)}(x, 3 \theta t),$$

as claimed. \qed
The fundamental difference with the case of the linear Schrödinger equation analysed in the previous section, lies in the fact that the solution of the quasi-periodic problem for the Airy equation corresponds to the solution of a suitable periodic problem but \textit{evaluated at a different time}. Indeed, Theorem 1.1 states that the solution of (4.1) at time $t = t_r$ is obtained via the solution of a periodic problem for the Schrödinger equation evaluated at time $t = 3\theta t_r$. If $\theta \notin \mathbb{Q}$, this is an irrational time, for which the fractalisation result of Theorem 1.5 applies. From this it follows that, the quasi-periodic Airy problem exhibits revivals at rational times if and only if $\theta \in \mathbb{Q}$. To be more precise, we have the following two possibilities.

1. Case $\theta \in \mathbb{Q}$. The time $t = 3\theta t_r$ is a rational time for Schrödinger’s periodic problem. Hence Airy’s quasi-periodic problem will exhibit revivals at any rational time $t_r$.

2. Case $\theta \notin \mathbb{Q}$. The time $t = 3\theta t_r$ is irrational for Schrödinger’s periodic problem. It follows that the solution of Airy’s quasi-periodic problem at rational times $t_r$ is a continuous but nowhere differentiable function, and there is no revival at rational times in this case.

We now establish a representation formula for the solution at rational times, which implies the validity of the revival phenomenon observed in (4.1) in the case $\theta \in \mathbb{Q}$. The proof of the next statement is a direct consequence of combining Theorem 1.1 with Lemma 2.4.

**Corollary 4.1 (Quasi-Periodic Revival).** Let $(p, q), (c, d)$ be pairs of co-prime positive integers, with $c < d$. Set $\theta = c/d < 1$. Let $u_0 \in L^2$. For $\theta = \theta_r$, the solution $u(x, t)$ of the linear Airy equation with initial condition given by $u_0$ and quasi-periodic boundary conditions (4.1), at rational time $t_r = 2\pi p/q$ is given by,

$$u(x, t_r) = e^{i\theta_r x} \left[ e^{-i\theta_r (3t_r)_{pq}} R_3(p, q) R_2(p, dq) \right] e^{-i\theta_r x} u_0(x). \quad (4.5)$$

The comment made in Remark 3.3 applies also to the revival expression (4.5). Indeed, the latter has an alternative representation in terms of the eigenfunctions $\phi_m(x)$ of the spatial quasi-periodic operator given by (4.2). This alternative representation is the direct analogue of the representation in Theorem 1.4 for the periodic case, with a modified revival operator $\tilde{R}_3$ defined in terms of the eigenfunctions of the quasi-periodic problem directly. We state this representation, without proof. It can be obtained from algebraic manipulations of the expression (4.5), or directly following the lines of the proof of Lemma 2.3.

**Proposition 4.2.** Let $p, q, c, d, u_0(x)$ and $\theta_r$ be as in the previous statement. Let $u(x, t)$ denote the solution of Airy’s quasi-periodic problem (4.1), with $\theta = \theta_r$. The solution $u(x, t_r)$ at rational time $t_r = 2\pi p/q$ admits the representation

$$u(x, t_r) = \sqrt{2\pi} \frac{d^q e^{-i\frac{\pi}{2} d q^{-1} t_r} e^{-i(m + \xi)^3 t_r} \phi_m \left( \frac{\pi k}{dq} \right) u_0 \left( x - \frac{\pi k}{2d q} \right).}$$
Here $\phi_m(x)$ are the eigenfunctions of the spatial operator given by (4.2) and $\tilde{u}_0(x)$ is the quasi-periodic extension of $u_0$,

$$\tilde{u}_0(x) = e^{2\pi i \frac{m}{d} x} u_0(x - 2\pi m), \quad 2\pi m \leq x < 2\pi(m + 1).$$

In appendix A, we illustrate with several numerical examples the revival behaviour described by the results in this section.

Remark 4.3. By an induction argument, the above results can generalise to higher order equations with a monic dispersion relation $P(k) = k^p$, $p \geq 4$.

5 The Linear Schrödinger equation with Robin boundary conditions

In this final section we consider the linear Schrödinger equation (LS) posed on $(0, \pi)$, but now we impose the Robin boundary conditions (R). Namely, the problem we consider is

$$
\begin{align*}
&iu_t + u_{xx} = 0, \quad u(x, 0) = u_0(x) \in L^2(0, \pi), \\
&bu(x_0, t) = (1 - b) \partial_x u(x_0, t), \quad x_0 = 0, \pi, \quad b \in [0, 1],
\end{align*}
$$

and we give the proof of Theorem 1.2, whose results describes the behaviour of the solution of (5.1) at rational times.

A routine calculation shows that the eigenvalues and the normalised eigenfunctions of the spatial operator are as follows. When $0 < b < 1$, there is one negative eigenvalue, which depends on the parameter $b$, given by

$$\lambda_b = -m_b^2 < 0, \quad m_b = \frac{b}{1 - b},$$

with associated normalised eigenfunction

$$\phi_b(x) = A_b e^{m_b x}, \quad A_b = \sqrt{\frac{2m_b}{e^{2\pi m_b} - 1}}.$$ 

The rest of the spectrum is the sequence of eigenvalues, independent of $b$, given by $\lambda_j = j^2 > 0$, $j \in \mathbb{N}$ with associated normalised eigenfunctions,

$$\phi_j(x) = \frac{1}{\sqrt{2\pi}} \left[ e^{ijx} - \Lambda_j e^{-ijx} \right], \quad \Lambda_j = \frac{b - (1 - b)ij}{b + (1 - b)ij}.$$ 

Note that the cases $b \to 1$ and $b \to 0$ correspond to Dirichlet and Neumann boundary conditions respectively. It is a routine calculation to verify that, by taking the even or odd extension, these can be treated as periodic problems posed on the double-length interval $(0, 2\pi)$.
In order to simplify the presentation we set the following notation. For \( f \in L^2(0, \pi) \), the even and odd extensions of \( f \) to the segment \([0, 2\pi]\) are denoted by

\[
f^\pm(x) = \begin{cases} f(x), & 0 \leq x < \pi, \\ \pm f(2\pi - x), & \pi \leq x < 2\pi, \end{cases}
\]

and we write the \( 2\pi \)-periodic convolution of \( f, g \in L^2(0, 2\pi) \), as

\[
f \ast g(x) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f^*(x - y)g^*(y)dy, \quad x \in (0, 2\pi),
\]

where the symbol \( \ast \) on top of functions denotes the \( 2\pi \)-periodic extension as in (2.1).

Finally, as in the statement of Theorem 1.2, we define

\[
f_1(x) = \sqrt{\frac{\pi}{2}} \frac{m_b}{e^{2\pi m_b} - 1} e^{im_b x}, \quad x \in (0, 2\pi).
\]

We first state a representation of the solution of (5.1) in terms of the solutions of five periodic problems for (LS), each with an initial condition specified by an explicit transformation of \( u_0 \). Four of these initial conditions are obtained as the \( 2\pi \)-periodic convolution of an explicit exponential \( 2\pi \)-periodic function with corresponding odd or even \( 2\pi \)-periodic extensions of the initial data.

**Proposition 5.1.** Let \( u_0 \in L^2(0, \pi) \), and consider the following solutions to the \( 2\pi \)-periodic problem for equation (LS):

- \( n(x,t) \) denotes the solution corresponding to initial condition \( n_0(x) = u_0^+(x) \)
- \( h(x,t) \) denotes the solution corresponding to initial condition \( h_0(x) = (f_1 + f_1^\dagger) \ast u_0^+(x) \)
- \( v(x,t) \) denotes the solution corresponding to initial condition \( v_0(x) = (f_1^\dagger - f_1) \ast u_0^+(x) \)
- \( z(x,t) \) denotes the solution corresponding to initial condition \( z_0(x) = (f_1 - f_1^\dagger) \ast u_0^-(x) \)
- \( w(x,t) \) denotes the solution corresponding to initial condition \( w_0(x) = (f_1 + f_1^\dagger) \ast u_0^-(x) \)

where \( f_1(x) \) is defined by (5.2) and \( \dagger \) denotes reflection as given in (3.7). Then, at each \( t \geq 0 \) the solution \( u(x,t) \) to the Robin problem (5.1) is given by

\[
u(x,t) = \langle u_0, \phi_b \rangle_{L^2(0,\pi)} e^{im_b^2 t} \phi_b(x) + n(x,t) - h(x,t) + v(x,t) + z(x,t) + w(x,t).
\]

We omit the proof of this proposition, which is entirely analogous to the proof of Proposition 3.1. Various numerical examples which illustrate revival and non-revival for (5.1) are given in Appendix B.
The proof of Theorem 1.2 is an immediate consequence of Proposition 5.1, which expresses the solution of this problem, in (1.3), as the sum of three terms:

\[
u(x, t_r) = 2\sqrt{\frac{2}{\pi}} \left( u_0, e^{\frac{1}{4i\beta^2}t_r} e^{i\frac{\beta^2}{4}t_r} f_1(x) + \mathcal{R}_2(p, q) \left[ u_0^+(x) \right] \right. \\
+ \left. \mathcal{R}_2(p, q) \left[ 2f_1 \ast (u_0^- - u_0^+)(x) \right] \right), \quad x \in (0, \pi).
\]

The three components on the right hand side of the equation correspond to the following:

* The first term is a rank one perturbation and represents the contribution of the negative eigenvalue \( \lambda_b \).
* The second terms is the periodic revival of the (even extension of the) given initial condition.
* The last term is the periodic revival of a continuous function.

As a consequence of this representation, we conclude that 5.1 exhibits a weaker form of revivals. While the solution is not simply obtained as a linear combination of translated copies of the initial condition, the second term in 1.3 ensures that the functional class of the initial condition is preserved at rational times. In particular, whenever \( u_0 \) has a finite number of jump discontinuities, then the same will be true for the solution at rational times, and the dichotomy between the solution behaviour at rational or irrational times is present. We may say that the quantum particle that solves the linear Schrödinger equation with Robin boundary conditions still knows the time.

**Conclusions**

The main goal of this work was to examine a variety of boundary conditions for the linear Schrödinger and Airy equations, and identify how the revival phenomenon depends on these boundary conditions. The starting point was the periodic case, for which it is known that the solution at rational times can be obtained as a finite linear combination of translated copies of the initial condition, and the dichotomy between revival at rational times and fractalisation at irrational times is well established.

We analysed pseudo-periodic conditions, which couple the two ends of the interval of definition, and Robin-type boundary conditions imposed separately at the two ends. We derived two main new results. One that establishes the constraints on the validity of the revival property for the third-order Airy equation. The other that describes a new, weaker form of revival for the case of Robin conditions.

More specifically, we confirmed that in the second-order case of the linear Schrödinger equation, every pseudo-periodic problem admits revival, by expressing its solution in terms of a purely periodic problem. We then show, by virtue of this new expression, that the revival property is more delicate for the third-order case of the Airy equation. In fact, it does
not even hold in general for quasi-periodic boundary conditions. The rational/irrational
time dichotomy, typical of the revival phenomenon, holds in this case only for rational
value of the quasi-periodicity parameter.

The particular case of Robin boundary conditions that we have chosen, revealed a new
weaker form of revival phenomenon, which is worth further investigation. In this case, while
the rational/irrational time dichotomy still holds, it is not true anymore that the solution
at rational times is simply obtained by a finite linear combinations of copies of the initial
profile. It is worth highlighting that the validity of a form of revival in this case is due to the
presence of one term in the solution representation that is due to a purely periodic problem.
This new manifestation of revival complements the one recently reported in [6] for the case
of periodic linear integro-differential equations. The latter displays a rational/irrational
time dichotomy similar to the present one, but the representation of the solution is more
involved.

Our analysis strongly support the conjecture that periodicity, and the number-theoretic
properties of the purely exponential series that represent periodic solutions, are essential
to any revival phenomenon. Future work will aim to confirm this conjecture, by extend-
ing consideration to general linear, constant coefficients boundary conditions for the both
Schrödinger and Airy equation. In the latter case, there exists boundary conditions for
which the associate spatial operator does not admit a complete basis of eigenfunctions
- an example of such conditions are the pseudo-Dirichlet conditions $u(0, t) = u(2\pi, t) =
ux(2\pi, t) = 0$, see [3], [24]. While preliminary numerical evidence suggests that at rational
and irrational times the solution of this boundary value problem behaves fundamentally
differently, the analysis for these types of boundary conditions requires a different approach.

The equations we have considered are the linear part of important nonlinear equations of
mathematical physics, the nonlinear Schrödinger and KdV equations respectively. In work
of Erdoğan, Tzirakis, Chousionis and Shakan, see [18]–[20], [25], the dichotomy between the
behaviour at rational and irrational times has been established rigorously for the periodic
problem for these nonlinear equations. We expect that our result for the pseudo-periodic
case would extend to the nonlinear case in an analogous manner. This would also provide
theoretical foundation for recent results on the vortex filament equation with non-zero
torsion [22], a problem that can be represented in terms of the solution of a quasi-periodic
problem for the Schrödinger equation.

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A Numerical examples for Airy’s equation

In this first appendix, we display the numerical solutions of the quasi-periodic problem for Airy’s equation \((\text{4.1})\). We illustrate the phenomenon of revivals and fractalisation for two choices of the quasi-periodicity parameter, one rational and one irrational. The initial condition is the piecewise constant function: \(u_0(x) = 0\) in \((0, \pi)\) and \(u_0(x) = 1\) in \((\pi, 2\pi)\).

In figures 1 and 2 we plot the profile of the solution in space variable, with quasi-periodic boundary conditions determined by \(\beta = e^{2\pi i \theta}\) with \(\theta \in \mathbb{Q}\). In the first figure, the
time is set to be rational. The re-appearance of the initial jump discontinuity is clearly seen.
In the second figure, the time is irrational. The solution has both its real and imaginary parts continuous. Indeed, the discontinuity has been smoothed out. This is consistent with Oskolkov’s results [12], stating that for the periodic problem for linear Schrödinger and Airy equations, at irrational times, the solution is a continuous functions of $x$ provided the initial condition is of bounded variation.

In figures 3 and 4 we plot the solution with quasi-periodic boundary conditions determined by $\beta = e^{2\pi i \theta}$ with $\theta \notin \mathbb{Q}$. In this case, no discontinuities appear in the solution at any time, either rational or irrational. This is consistent with the representation (1.2) of the solution of this problem in terms of the solution of a periodic problem for the Schrödinger equation at an irrational time.

Figure 1: Real (blue) and imaginary (red) parts of the solution of Airy’s problem (4.1) with $\theta = 1/4$ at rational times $t = 2\pi p/q$. 

\begin{enumerate}
\item \begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Real (blue) and imaginary (red) parts of the solution of Airy’s problem $p=0$, $q=1$; $p=1$, $q=4$; $p=3$, $q=7$; $p=5$, $q=8$; $p=3$, $q=11$; $p=5$, $q=16$.}
\end{figure}
\end{enumerate}
Figure 2: Real (blue) and imaginary (red) parts of the solution of Airy’s problem (4.1) with $\theta = 1/4$ at generic times.

Figure 3: Real (blue) and imaginary (red) parts of the solution of Airy’s problem (4.1) with $\theta = \sqrt{2}/3$ at rational times $t = 2\pi p/q$. 

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Figure 4: Real (blue) and imaginary (red) parts of the solution of Airy’s problem (4.1) with $\theta = \sqrt{2}/3$ at generic times.

B Numerical Examples for the Robin linear Schrödinger problem

These final numerical experiments correspond to the equation (5.1). We take as initial condition the piecewise constant function: $u_0(x) = 0$ in $(0, \pi/2)$ and $u_0(x) = 1$ in $(\pi/2, \pi)$. Picking different values of the parameter $b \in [0, 1]$, we plot the real and imaginary part of the solution $u(x,t)$, in space, at generic and rational times.

At rational times, in figures (5) and (7), we notice that the solution evolves to, not exactly, only translations and/or reflections of the initial profile. However, the revival of the discontinuities is preserved, as predicted by Theorem 1.2. On the other hand, see figures 6 and 8, at generic times the solution profile is clear of discontinuities.
Figure 5: Real (blue) and imaginary (red) part of the solution of Robin’s problem (5.1) with $b = 0.35$ at rational times $t = 2\pi p/q$.

Figure 6: Real (blue) and imaginary (red) parts of the solution of Robin’s problem (5.1) with $b = 0.35$ at generic times.
Figure 7: Real (blue) and imaginary (red) parts of the solution of Robin’s problem (5.1) with $b = 0.6$ at rational times $t = 2\pi p/q$.

Figure 8: Real (blue) and imaginary (red) parts of the solution of Robin’s problem (5.1) with $b = 0.6$ at generic times.