GRADED COVERING OF A SUPERMANIFOLD I.
THE CASE OF A LIE SUPERGROUP

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Dedicated to the memory of Arkady Onishchik, 1933–2019

Abstract. We generalize the Donagi and Witten construction of a first obstruction class for splitting of a supermanifold via differential operators using the theory of $n$-fold vector bundles and graded manifolds. Applying the generalized Donagi–Witten construction we obtain a family of embeddings of the category of supermanifolds into the category of $n$-fold vector bundles and into the category of graded manifolds. This leads to a realization of any non-split supermanifold in terms of a collection of vector bundles and some morphism between them. Further we study the images of these embeddings into the category of graded manifolds in the case of a Lie supergroup and a Lie superalgebra. We show that these images satisfy universal property of a graded covering or a graded semicovering.

1. Introduction

This is the first part of a series of papers about $\mathbb{Z}$-graded coverings of a supermanifold. In this paper we investigate the case of a Lie supergroup and its Lie superalgebra. In [DW2, Section 2] Donagi and Witten gave a description of the first obstruction class $\omega_2$ for a supermanifold to be split via differential operators. More precisely, Donagi and Witten constructed an exact sequence of locally free sheaves with Atiyah class $\omega_2$. This is a very interesting and important observation that pure even vector bundles can keep certain information about the original (non-split) supermanifold. In the case a supermanifold has odd dimension $2$, the exact sequence of Donagi and Witten keeps the whole information of this supermanifold.

In this paper we generalize this idea. We show that the Donagi and Witten exact sequence corresponds to a double vector bundle with some additional structure, the whole information of which can be reduced to a graded manifold of degree $2$. Further we suggest to use a dual approach which leads to iterated differential forms, i.e. functions on the iterated antitangent bundle (as it is explained below) instead of differential operators, which leads to a simplification of the Donagi and Witten construction and suggests a natural generalization. For any (non-split) supermanifold $\mathcal{M}$ of odd dimension $\leq n$ we construct a pure even geometric object, an $n$-fold vector bundle $V_{2n}(\mathcal{M})$ which keeps the whole information about the original supermanifold. Moreover the obtained $n$-fold vector bundle possesses additional symmetries. We interpret this in the following way: the $n$-fold vector bundle arising from a supermanifold, can be reduced to a graded manifold of degree $n$.

Summing up we construct a family $F_n$, where $n = 2, 3, \ldots \infty$, of functors from the category of supermanifolds of odd dimension $n$ to the category of $n$-fold vector bundles and to the category of graded manifolds of degree $n$, where the first functor $F_2$ is a modification of the Donagi–Witten construction. Each functor $F_n$
is a composition of four functors: the \((n - 1)\)-iteration of the antitangent functor \(T\), the functor split \(\text{gr}\) (\(\text{gr}M\) denotes the retract of the supermanifold \(M\)), which is defined in the category of supermanifolds, the functor parity change \(\pi\), which is defined in the category of \(n\)-fold vector bundles and another functor \(\iota\), which we call inverse that was discovered independently in \([\text{BCR}]\) (under the name the diagonalization functor) and \([\text{Vi1}]\). We show that the functor \(F_n\) determines an embedding of the category of supermanifolds of odd dimension \(n\) (or smaller) into the category of graded manifolds of degree \(n\). The functor \(F_\infty\) is a limit of functors \(F_n\). This functor defines an embedding of the category of supermanifolds into the category of graded manifolds of any degree. This means that the graded manifold \(F_\infty(M)\), the image of a (non-split) supermanifold \(M\), contains the whole information about the original supermanifold \(M\).

Moreover, for a Lie supergroup \(G\) we prove that \(F_\infty(G)\) satisfies universal properties. These properties of \(F_\infty(G)\) led us to introduce the notion of a graded covering of \(G\). That is: the covering space is unique up to isomorphism and every homomorphism from a graded Lie supergroup to the Lie supergroup \(G\) can be lifted to \(F_\infty(G)\). We also introduce a notion of a graded semicovering of \(G\). This explains the meaning of other functors \(F_n\). Our covering (semicovering) is in some sense a "local diffeomorphism" of Lie supergroups. Further we show that the Lie superalgebra of \(F_\infty(g)\) is an example of a loop algebra construction by Allison, Berman, Faulkner, and Pianzola \([\text{ABFP}]\), see also \([\text{Eld}]\). The loop algebra was used by several authors to investigate graded-simple (Lie) algebras. Therefore results of our paper establish a connection between Donagi–Witten observation and pure algebraic results. Our method leads to a notion of a graded covering and semicovering of any supermanifold as well. However due to technical difficulties of all proofs in this case we present the general theory of covering and semicovering spaces for any supermanifold in the second part of this research \([\text{RV}]\).

Our results are especially interesting in the complex-analytic (and algebraic) category. The reason is the following. According to the Batchelor–Gawedzki Theorem any smooth supermanifold is (non-canonically) split, that is its structure sheaf is isomorphic to the wedge power of a certain vector bundle. Therefore very often we can study geometry of a split supermanifold using geometry of vector bundles. This is not the case in the complex-analytic situation. The study of non-split supermanifolds was initiated in \([\text{Bet, Gr}]\), where the first non-split supermanifold was described. Significant advances in this direction were achieved in the work of A.L. Onishchik. Interest in this problem arose again after Donagi and Witten’s papers \([\text{DW1, DW2}]\), where they proved that the moduli space of super Riemann surfaces is non-split (or more generally not projected). To each (non-split) supermanifold \(M\) our functor \(F_n\) associates a graded manifold \(F_n(M)\) and the image \(F_n(M)\) keeps the whole information about the (non-split) supermanifold \(M\) for sufficiently large \(n\). Therefore we can study a non-split supermanifold in the category of graded manifolds using the tools of classical complex geometry due the fact that geometrically a graded manifold is a family of vector bundles.

Our method studying a supermanifold is interesting in the case of a split supermanifold as well, for instance if a split supermanifold possesses an additional structure. Any Lie supergroup \(G\) is totally split, i.e. as a supermanifold it is a cartesian product of its underlying space \(G_0\) with a fixed odd vector space \(g_1\) — the odd part of the Lie superalgebra of \(G\). However the supermanifold \(G\) has an additional structure which can be non-trivial: the Lie supergroup structure. We show that any graded Lie supergroup \(F_n(G)\) contains the whole information about the original supergroup \(G\) for any \(n\), non only for sufficiently large \(n\). Moreover our
A weight system is a pair $(\Delta, \chi)$, where $\Delta$ is a subset in $\mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_r$, satisfying the following properties:

1. $0 \in \Delta$ and $\alpha_i \in \Delta$, where $i = 1, \ldots, r$;
2. if $\delta \in \Delta$ and $\delta = \sum a_i \alpha_i$, where $a_i \in \mathbb{Z}$, then $a_i \geq 0$,

and $\chi : A \to \mathbb{Z}_2$, $\delta \mapsto |\delta|$, is a group homomorphism. We call $|\delta|$ the parity of $\delta$. A weight system $(\Delta, \chi)$ is called multiplicity free, if $\delta = \sum a_i \alpha_i \in \Delta$ implies $a_i \in \{0, 1\}$.
Let \((\Delta, \chi)\) be a weight system. Then \(\Delta = \Delta_0 \cup \Delta_1\), where \(\Delta_0\) contains all even weights, while \(\Delta_1\) contains all odd ones. Note that 0 is always in \(\Delta_0\). A Lie superalgebra \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) is called a graded Lie superalgebra of type \((\Delta, \chi)\), if \(\mathfrak{g}\) is a \(\mathbb{Z}\)-graded Lie superalgebra of type \(\Delta\) and

\[
\mathfrak{g}_0 = \bigoplus_{\delta \in \Delta_0} \mathfrak{g}_\delta, \quad \mathfrak{g}_1 = \bigoplus_{\delta \in \Delta_1} \mathfrak{g}_\delta.
\]

Very often we will omit \(\chi\) and write a Lie superalgebra of type \(\Delta\) meaning a Lie superalgebra of type \((\Delta, \chi)\).

2.2. Supermanifolds, graded manifolds and \(n\)-fold vector bundles.

2.2.1. Supermanifolds. We consider a complex-analytic supermanifold in the sense of Berezin and Leites [Ber, L], see also [BLMS]. Thus, a supermanifold \(M = (M_0, O_M)\) of dimension \(n|\) is a \(\mathbb{Z}_2\)-graded ringed space that is locally isomorphic to a superdomain in \(\mathbb{C}^{n|m}\). Here the underlying space \(M_0\) is a complex-analytic manifold. The dimension \(n\) of the underlying manifold \(M_0\) is called even dimension of \(M\), while \(m\) is called odd dimension of \(M\). A morphism \(M \to N\) between two supermanifolds is a morphism between \(\mathbb{Z}_2\)-graded ringed spaces, this is, a pair \(F = (F_0, F^*)\), where \(F_0 : M_0 \to N_0\) is a holomorphic mapping and \(F^* : O_N \to (F_0)_*(O_M)\) is a homomorphism of sheaves of \(\mathbb{Z}_2\)-graded ringed spaces. A morphism \(F\) is called an isomorphism if \(F\) is invertible. A supermanifold \(M\) is called split, if its structure sheaf is isomorphic to \(\bigwedge E^*\), where \(E\) is a sheaf of sections of a certain vector bundle \(E[1]\). (Here \(E[1]\) means a usual vector bundle \(E\) additionally assumed that its local sections are odd.) We see that in this case the structure sheaf is \(\mathbb{Z}\)-graded. A morphism of split supermanifolds is called split if it preserves the fixed \(\mathbb{Z}\)-gradings. Split supermanifolds with split morphisms form a category of split supermanifolds.

According to the Batchelor–Gawedzi Theorem any smooth supermanifold is split. This is not true in the complex-analytic case, see [Ber, Gr]. However we can define a functor from the category of supermanifolds to the category of split supermanifolds. Let \(M = (M_0, O)\) be a supermanifold. (Sometimes we will omit \(M\) in \(O_M\) if the meaning of \(O\) is clear from the context.) Then its structure sheaf possesses the following filtration

\[
O = J^0 \supset J \supset J^2 \supset \cdots \supset J^p \supset \cdots,
\]

where \(J\) is the sheaf of ideals generated by odd elements in \(O\). We define

\[
\text{gr}(M) := (M_0, \text{gr}O),
\]

where

\[
\text{gr}O = \bigoplus_{p \geq 0} J^p/J^{p+1}.
\]

The supermanifold \(\text{gr}(M)\) is split, that is its structure sheaf is isomorphic to \(\bigwedge E^*\), where \(\bigwedge E^* = J/J^2\) is a locally free sheaf. Since any morphism \(F\) of supermanifolds preserves the filtration (3), the morphism \(\text{gr}(F)\) is defined. Summing up, the functor \(\text{gr}\) is a functor from the category of supermanifolds to the category of split supermanifolds, see for example [Vi2, Section 3.1] for details. We can apply the functor \(\text{gr}\) to a Lie supergroup \(\mathcal{G}\) and we will get a split Lie supergroup \(\text{gr}(\mathcal{G})\). Later we also will define a corresponding functor \(\text{gr}'\) for Lie superalgebras so that \(\text{Lie}(\text{gr}(\mathcal{G})) = \text{gr}' \circ \text{Lie}(\mathcal{G})\) for any Lie supergroup \(\mathcal{G}\).
2.2.2. Graded manifolds of type $\Delta$, graded manifolds of degree $n$ and r-fold vector bundles. Let us start with a notion of a graded manifold of type $(\Delta, \chi)$, where $(\Delta, \chi)$ is as in Definition 3. Again very often we will omit $\chi$, and write a graded manifold of type $\Delta$. Let
\[
V = \bigoplus_{\delta \in \Delta} V_{\delta}, \quad V_0 = \bigoplus_{\delta \in \Delta_0} V_{\delta}, \quad V_1 = \bigoplus_{\delta \in \Delta_1} V_{\delta}.
\]
Consider a $\mathbb{Z}$-graded ringed space $\mathcal{U} = (\mathcal{U}_0, \mathcal{O}_\mathcal{U})$, where $\mathcal{U}_0 \subset V_0^*$ is an open set, $\mathcal{O}_\mathcal{U} := \mathcal{F}_{\mathcal{U}_0} \otimes_{\mathbb{R}} S^*(V)$ and $\mathcal{F}_{\mathcal{U}_0}$ is the sheaf of smooth or holomorphic functions on $\mathcal{U}_0$.

**Definition 4.** We call the ringed space $\mathcal{U}$ a graded domain of type $(\Delta, \chi)$ and of dimension $\{n_{\delta}\}_{\delta \in \Delta}$, where $n_{\delta} = \dim V_{\delta}$.

Let us choose a basis $(x_i)$ in $V_0$ and a basis $(\xi^j)$ in any $V_\delta$. Then the set $(x_i, \xi^j)_{\delta \in \Delta \setminus \{0\}}$ is a system of local coordinates on $\mathcal{U}$. We assign the weight 0 and the parity 0 to $x_i$ and the weight $\delta$ and the parity $|\delta|$ to $\xi^j$. Such coordinates $(x_i, \xi^j)$ are called graded.

**Definition 5.** A graded manifold of type $(\Delta, \chi)$ and of dimension $\{n_{\delta}\}_{\delta \in \Delta}$ is a $\mathbb{Z}$-graded ringed space $\mathcal{N} = (\mathcal{N}_0, \mathcal{O}_\mathcal{N})$, that is locally isomorphic to a graded domain of type $(\Delta, \chi)$ and of dimension $\{n_{\delta}\}_{\delta \in \Delta}$.

A morphism of graded manifolds of type $(\Delta, \chi)$ is a morphism of the corresponding $\mathbb{Z}$-graded ringed spaces.

If $\Delta = \{0, 1, \ldots, n\}$ and $\chi(1) = \bar{1}$, a graded manifold of type $\Delta$ is also called a graded manifold of degree $n$. (See [R] for more information about graded manifolds of degree $n$.) If $\Delta$ is multiplicity free, this is if $\delta = \sum a_i \alpha_i \in \Delta$ then $a_i \in \{0, 1\}$, a graded manifold of type $\Delta$ is called an r-fold vector bundle of type $\Delta$. This definition of an r-fold vector bundle is equivalent to a classical one as it was shown in [GR] Theorem 4.1.

**Remark 6.** Now let $\Delta$ be as in Definition 3 and $\Delta' \subset \Delta$ satisfies the following property: if $\delta \in \Delta'$ and $\delta' = \sum \delta_i$, for $\delta_i \in \Delta$, then $\delta_i$ are also in $\Delta'$. In this case to any graded manifold $\mathcal{N}$ of type $\Delta$ we can assign a graded manifold $\mathcal{N}'$ of type $\Delta'$. A detailed description of this well-known construction can be found for example in [VI] Section 4.1. For instance to any graded manifold of degree $n$ we can assign a graded manifold of degree $0 \leq n' < n$. In this case we have a natural morphism $\mathcal{N} \to \mathcal{N}'$, which is called projection.

2.3. Lie supergroups and super Harish-Chandra pairs.

2.3.1. Lie supergroups. A Lie supergroup is a group object in the category of supermanifolds. In other words a Lie supergroup is a supermanifold $G = (\mathfrak{g}_0, \mathcal{O}_G)$ with three morphisms: $\mu : G \times G \to G$ (the multiplication morphism), $\kappa : G \to G$ (the inversion morphism), $\varepsilon : (\text{pt}, \mathbb{C}) \to G$ (the identity morphism). Moreover, these morphisms have to satisfy the usual conditions, modeling the group axioms. The underlying manifold $\mathfrak{g}_0$ of $G$ is a smooth or complex-analytic Lie group. As in the theory of Lie groups and Lie algebras, we can define the Lie superalgebra $\mathfrak{g} = \text{Lie}(G)$ of a Lie group $G$. By definition, $\mathfrak{g}$ is the subalgebra of the Lie superalgebra of vector fields on $G$ consisting of all right invariant vector fields on $G$. Any right invariant vector field $Y \in \mathfrak{g}$ has the following form
\[
(4) \quad Y = (Y_c \otimes \text{id}) \circ \mu^*;
\]
where \( Y_e \in (m_e/m_e^2)^* \) and \( m_e \) is the maximal ideal of \((\mathcal{O}_Y)_e\). Note that as the case of classical geometry the vector superspace \((m_e/m_e^2)^*\) may be identified with the vector superspace of all maps \( D : (\mathcal{O}_Y)_e \to \mathbb{C} \) satisfying the Leibniz rule. Further the map \( Y_e \mapsto Y = (Y_e \otimes \text{id}) \circ \mu^* \) is an isomorphism of \((m_e/m_e^2)^*\) onto \( \mathfrak{g} \).

Similarly we can define a graded Lie supergroup of type \( \Delta \) as a group object in the category of graded manifolds of type \( \Delta \) or a graded Lie supergroup of degree \( n \) as a group object in the category of graded manifolds of degree \( n \). Let us consider some examples of Lie supergroups with an additional gradation in the structure sheaf.

**Example 7** (Lie supergroup \( GL(V) \)). The space of endomorphisms of a vector superspace \( V = V_0 \oplus V_1 \) has a natural gradation by integer numbers \(-1,0,1\):

\[
\text{End}(V) = \bigoplus_{i=-1,0,1} \text{End}^i(V),
\]

where

\[
\text{End}(V)^0 = \text{End}(V_0) \oplus \text{End}(V_1), \quad \text{End}(V)^1 = \text{Hom}(V_0, V_1),
\]

\[
\text{End}(V)^{-1} = \text{Hom}(V_1, V_0)
\]

giving rise to a \( \mathbb{Z} \)-graded Lie superalgebra \( \mathfrak{gl}(V) \). It integrates to the \( \mathbb{Z} \)-graded Lie supergroup \( GL(V) \). Consider the case \( \dim V = (1|1) \). The entries of a matrix

\[
A = \begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix} \in GL(\mathbb{K}^{1|1}),
\]

constitute a graded coordinate domain, the subdomain of \( \mathbb{K}^{2|2} \), the coordinates \( a, b \) are of degree 0, and \( \alpha, \beta \) are of degrees \(-1\) and \( 1 \), respectively. The matrix multiplication respects the assigned grading, e.g. \( m^\nu(\alpha) = a'\alpha'' + \alpha'b' \) is of degree \( 0 = 0 + 0 = 1 + (-1) \). Note that \( GL(V) \) with this gradation is not a graded manifold of degree \( n \), since this supermanifold possesses graded coordinates of negative degree.

**Remark 8.** In this paper we denote by \( T \) the antitangent functor. This means that \( T\mathcal{M} \) is the vector bundle obtained from the tangent bundle \( T\mathcal{M} \) by means of the parity functor. If \( (x^A) \) are even/odd coordinates on \( \mathcal{M} \) then the fiber coordinates on \( T\mathcal{M} \) will be denoted by \( (dx^A) \) and the parity of \( dx^A \) is the parity of \( x^A \) plus one modulo two.

A canonical example of graded Lie supergroups of degree \( n \) comes from the tangent lifts.

**Example 9.** Higher order tangent bundles, not to be confused with the related iterated tangent bundles, are the natural geometric home of higher derivative Lagrangian mechanics \([8]\). A higher order (\( k \)-order) tangent bundle \( T^k\mathcal{M} \) of a supermanifold \( \mathcal{M} \) can be defined in a natural way. By applying the functor \( T^k \) to the Lie supergroup \( G \) and the structure morphisms of \( G \) we get a graded Lie supergroup of type \( (\Delta, \chi) \), where \( \Delta = \{0,1,\ldots,k\} \) and \( \chi(1) = 0 \).

Another interesting example of \( \mathbb{Z} \)-gradation on a Lie supergroup was noticed by V. Kac:

**Example 10.** Let \( V = V_0 \oplus V_1 \) be a vector superspace equipped with a non-degenerate even symmetric bilinear form \( Q \), i.e. \( Q(x,y) = (-1)^{|x||y|}Q(y,x) \) for homogeneous \( x, y \in V \) and \( Q : V \times V \to \mathbb{K} \) is an even map. It follows that \( V_0 \perp V_1 \)
and $Q$ restricted to $V_0$ (resp. $V_1$) is symmetric (resp. skew-symmetric). Thus the dimension of $V_1$ is even and one can find lagrangian subspaces $L, L' \subset V_1$ such that $V_1 = L \oplus L'$, where $L'$ is naturally identified with the dual to $L$ and $Q|_{V_1}$ has the following natural form with respect to this decomposition:

$$Q((l_1, l_1'), (l_2, l_2')) = l_1'(l_2) - l_2'(l_1),$$

where $l_i \in L$ and $l_1' \in L'$. The orthosymplectic Lie superalgebra $\mathfrak{osp}(V, Q)$ is spanned by homogeneous endomorphism $T : V \to V$ such that

$$Q(Tx, y) + (-1)^{|T||x|}Q(x, Ty) = 0$$

for any homogeneous $x, y \in V$. It follows that $\mathfrak{osp}(V, Q)_0 = \mathfrak{so}(V_0) \oplus \mathfrak{sp}(V_1)$ and $\mathfrak{osp}(V, Q)_1 = V_0 \otimes V_1$. There are well known isomorphisms $\mathfrak{so}(W) = W \wedge W$ and $\mathfrak{sp}(W) = S^2W$. Assigning the weight 1 to $L$ and the weight $-1$ to $L'$ one find a canonical $Z$-grading on $\mathfrak{osp}(V)$ with non-zero parts in degrees $\pm 2$, $\pm 1$ and 0:

$$\mathfrak{osp}(V) = S^2(L') \oplus (V_0 \otimes L') \oplus (L' \otimes L \oplus V_0 \wedge V_0) \oplus (V_0 \otimes L) \oplus S^2L.$$  

Thus the corresponding Lie supergroup, the orthosymplectic Lie supergroup, is also naturally $Z$-graded.

2.3.2. Super Harish-Chandra pairs. To study a Lie supergroup one uses the theory of super Harish-Chandra pairs, see [Ber] and also [BCC [CCF] [V83].

**Definition 11.** A super Harish-Chandra pair $(G, \mathfrak{g})$ consists of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, a Lie group $G$ such that $\mathfrak{g}_0 = \text{Lie}(G)$, and an action $\alpha$ of $G$ on $\mathfrak{g}$ by automorphisms such that

- for any $g \in G$ the action $\alpha(g)$ restricted to $\mathfrak{g}_0$ coincides with the adjoint action $Ad(g)$;

- the differential $(d\alpha)_e : \mathfrak{g}_0 \to T_e \text{Aut}(\mathfrak{g}) = \text{Der}(\mathfrak{g})$ at the identity $e \in G$ coincides with the adjoint representation $\text{ad}$ of $\mathfrak{g}_0$ in $\mathfrak{g}$.

Super Harish-Chandra pairs form a category. The following theorem was proved in [V83] for complex-analytic Lie supergroups.

**Theorem 12.** The category of complex-analytic Lie supergroups is equivalent to the category of complex-analytic super Harish-Chandra pairs.

A similar result in the real case was obtained in [Kos], the algebraic case was treated by several authors, see for example [Gav], [Mas], [MS] and references therein. Theorem 12 implies that any Lie supergroup is globally split, that is, its structure sheaf is isomorphic to a wedge product of a certain trivial vector bundle. More explanation of this fact will be given later.

The notion of super Harish-Chandra pair has an obvious $A$-graded generalization, where $A$ is an abelian group. Namely, we assume that $\mathfrak{g}$ is an $A$-graded Lie superalgebra, the Lie algebra of $G$ is $\mathfrak{g}_0$, where $\mathfrak{g}_0$ is the part of $\mathfrak{g}$ in degree 0, $\alpha(g) : \mathfrak{g} \to \mathfrak{g}$, $g \in G$, is an automorphism of $\mathfrak{g}$, in particular it preserves the $A$-gradation and $d_\alpha(X) = [X, \cdot]_\alpha$ for any $X \in \mathfrak{g}_0$. In [JKPS, Theorem 5.6] it was noticed that an analogue of Theorem 12 holds true for graded Lie supergroups of degree $n$. We will also work with super Harish-Chandra pairs of type $\Delta$. A super Harish-Chandra pairs of type $\Delta$ is a $Z^r$-graded super Harish-Chandra pair $(G, \mathfrak{g})$, where $\mathfrak{g}$ is a $Z^r$-graded Lie superalgebra of type $\Delta$. Recall that in this case $Z^r$ is generated by the set $\Delta$, with additional agreement about parities of generators of $\Delta$, see Definition 3. In this case repeating argument [JKPS, Theorem 5.6], we get an analogue of Theorem 12 as well.
Example 13. Let $V = \bigoplus_{k=0}^{\infty} V_k$ be a non-negatively $\mathbb{Z}$-graded vector space with $\dim V < \infty$. Denote by $\text{End}_q(V)$, where $q \in \mathbb{Z}$, the space of endomorphisms $T : V \to V$ increasing the degree by $q$, i.e. $T : V \to V$ such that $T|_{V_i} : V_i \to V_{i+q}$. Thus $\text{End}(V) = \bigoplus_{q=-\infty}^{\infty} \text{End}_q(V)$ is a $\mathbb{Z}$-graded vector space. It will be considered as a $\mathbb{Z}$-graded Lie superalgebra with the bracket given by the graded commutator. The corresponding $\mathbb{Z}$-graded Lie supergroup $GL(V)$ has the body $G_0 = \bigtimes_{i=0}^{\infty} GL(V_i)$ and its sheaf of functions is given by the super symmetric algebra over $F = \bigoplus_{0 \leq i \neq j \leq n} V_i \otimes V_j^*$, as $V_i \otimes V_j^* = \text{Hom}(V_i,V_j)^*$, i.e. $\mathcal{O}(U) = \mathcal{C}^\infty(U) \otimes \text{Sym} F$. The gradation in $F$ is obvious, $|V_i \otimes V_j^*| = i-j$ and the group multiplication preserve this gradation. The graded Harish-Chandra pair corresponding to $GL(V)$ is $(G_0, \text{End}(V))$ equipped with the adjoint action of $G_0$ on $\text{End}(V)$. The language of Harish-Chandra pairs allows us to study some infinite dimensional generalizations of Lie supergroups in a convenient way. We simply drop the assumption that $\mathfrak{g}$ has finite dimension. For example, if $V = \bigoplus_{k=0}^{\infty} V_k$ is a $\mathbb{Z}$-graded vector space such that $\dim V_k < \infty$, but the dimension of $V$ can be infinite, then $GL(V)$ can be understood as a pair consisting of the direct sum of $GL(V_i)$, $G_0 = \bigoplus_{i=0}^{\infty} GL(V_i)$ and the natural action of $G_0$ on $\text{End}(V) = \bigoplus_{i,j \geq 0} \text{Hom}(V_i,V_j)$.

2.3.3. Construction of the Lie supergroup corresponding to a super Harish-Chandra pair. Let us remind a reader how to construct a Lie supergroup $\mathcal{G}$ (or a graded Lie supergroup of type $\Delta$) using a given super (or of type $\Delta$) Harish-Chandra pair $(G, \mathfrak{g})$. Denote by $\mathcal{U}(\mathfrak{h})$ the universal enveloping algebra of Lie superalgebra $\mathfrak{h}$. We need to define a structure sheaf $\mathcal{O}$ of $\mathcal{G}$. In the super and graded cases we, respectively, put

\begin{equation}
\mathcal{O} = \text{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), \mathcal{F}_G), \quad \mathcal{O} = \text{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), \mathcal{F}_G).
\end{equation}

Here $\mathcal{F}_G$ is the sheaf of (holomorphic) functions on $G$. Using the Hopf algebra structure on $\mathcal{U}(\mathfrak{g})$ we can define explicitly the multiplication morphism $\mu$, the inversion morphism $\kappa$ and the identity $\varepsilon$, see for instance [V6]. Indeed, assume that a super or graded of type $\Delta$ Harish-Chandra pair $(G, \mathfrak{g})$ is given. Let us define the supergroup structure of the corresponding Lie supergroup or graded Lie supergroup $\mathcal{G}$. Let $X \cdot Y \in \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$, where $X$ is from the first copy of $\mathcal{U}(\mathfrak{g})$ and $Y$ from the second one, $f \in \mathcal{O}$, see [5], and $g, h \in G$. The following formulas define a multiplication morphism, an inverse morphism and an identity morphism respectively:

\begin{equation}
\mu^*(f)(X \cdot Y)(g,h) = f(\alpha(h^{-1})(X) \cdot Y)(gh);
\end{equation}
\begin{equation}
\kappa^*(f)(X)(g^{-1}) = f(\alpha(g^{-1})(S(X)))(g);
\end{equation}
\begin{equation}
\varepsilon^*(f) = f(1)(e).
\end{equation}

Here $S$ is the antipode map in $\mathcal{U}(\mathfrak{g})$ considered as a Hopf algebra and $\alpha$ is as in the definition of a super Harish-Chandra pair.

2.4. (Skew-symmetric) double vector bundles. A double vector bundle (DVB, in short) is a graded manifold of type $\Delta = \{0, \alpha, \beta, \alpha + \beta\}$. In this subsection we assume that the weights $\alpha$ and $\beta$ are even, however later on we shall drop this assumption. Geometrically we can see a double vector bundle as a quadruple $(D; A, B; M)$ with the following diagram of morphisms

\begin{equation}
\begin{array}{ccc}
D & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & M
\end{array}
\end{equation}
where all maps are bundle projections and there are imposed some natural compatibility conditions, see [P, M].

In particular, $D$ has two vector bundle structures, one over the manifold $A$ and the second over $B$. Moreover, $A$, $B$ are vector bundles over $M$ called the side bundles of $D$. The compatibility condition can be easily expressed using Euler vector fields: $[E_A, E_B] = 0$ [GR]. Essentially, a DVB can be consider as a manifold equipped with two Euler vector fields, $D = (D; E_A, E_B)$, describing the vector bundle structure on the legs $D \to A$, $D \to B$ of $D$. This lead to many simplifications in the theory of DVBs.

The core $C$ of $DVB$ is defined as intersection of the kernels of two bundle projections $D \to A$ and $D \to B$. Moreover, $A, B$ are vector bundles over $M$ called the side bundles of $D$. The compatibility condition can be easily expressed using Euler vector fields: $[\Gamma_B \cdot A, \Gamma_B \cdot B] = 0$ [GR]. Essentially, a DVB can be considered as a manifold equipped with two Euler vector fields, $D = (D; E_A, E_B)$, describing the vector bundle structure on the legs $D \to A$, $D \to B$ of $D$. This lead to many simplifications in the theory of DVBs.

The flip of a DVB is obtained by interchanging the legs of $D$, i.e. the flip of $D = (D; E_A, E_B)$ is $(D; E_B, E_A)$.

Symmetric and skew-symmetric DVBs are DVBs $(D; A, B; M)$ with an involution $\sigma: D \to D$ satisfying a certain addition condition. They were introduced in [BGR] in order to recognize the image of the linearization functor which associates with a graded manifold of type $\Delta = \{0, \alpha, 2\alpha\}$ a DVB. Under this association graded manifolds of type $\Delta$ with even $\alpha$ (i.e., purely even graded bundles of degree 2) are in one-to-one correspondence with symmetric DVBs, while graded manifolds of type $\Delta$ with odd $\alpha$ (i.e., $N$-manifolds of degree 2) — with skew-symmetric DVBs. The results are extended to any order. For the purposes of this manuscript it is enough to recall a definition of skew-symmetric DVBs.

**Definition 14.** A skew-symmetric DVB is a pair $(D, \sigma)$ consisting of a DVB $D$ and an involution $\sigma: D \to D$ (i.e. $\sigma \circ \sigma = \text{id}$) such that

(i) $\sigma$ exchanges the legs of $D$, i.e. $\sigma$ is a DVB morphism from $D$ to the flip of $D$,

(ii) the restriction of $\sigma$ to the core is minus the identity.

A morphism between skew-symmetric DVBs is assumed to intertwines with their involutions.

It follows immediately that the side bundles of a skew-symmetric DVB are isomorphic and that any skew-symmetric DVB admits an atlas with graded coordinates $(x^a; y^i; \alpha, \beta; z_{ij}^\mu \alpha + \beta)$ such that

- the transition functions for $y^i$ and $Y^i$ are the same and
  $$z_{ij}^\mu = Q_{ij}^\mu z^{\nu} + \frac{1}{2} Q_{ij}^{\mu\nu} y^i Y^j$$
  with $Q_{ij}^\mu = -Q_{ji}^\mu$,

- the involution $\sigma$ has a form: $\sigma^*(y^i) = Y^i$ (hence $\sigma^*(Y^i) = y^i$) and $\sigma^*(z^\mu) = -z^\mu + \sigma_{ij}^\mu (x) y^i Y^j$ with $\sigma_{ij}^\mu = \sigma_{ji}^\mu$.

To any double vector bundle $D$ (with the core $C$ and side bundles $A$, $B$) we can assign a short exact sequence

$$0 \to C \to \hat{D} \to A \otimes B \to 0,$$

which is obtained by dualizing the short exact sequence

$$0 \to \ker \text{pr}_C \to A^{\alpha+\beta}(D) \xrightarrow{\text{pr}_C^*} C^* \to 0$$
where $A^{\alpha+\beta}(D)$ is the $F(M)$-module of homogeneous functions on $D$ of weight $\alpha+\beta$ and $pr_i$ denotes the restriction of a function $f$ to the core $C$, i.e. $pr_i(f) = f|_C$. Locally, $A^{\alpha+\beta}(D)$ is generated by the functions $y_i^\alpha$, $Y_j^\beta$ and $z_{\alpha+\beta}$. In other words, the dual to $D$ is the vector bundle whose space of sections is $A^{\alpha+\beta}(D)$.

If $(D, \sigma)$ is skew-symmetric then we can decompose
\[ A^{\alpha+\beta}(D) = A_+^{\alpha+\beta}(D) \oplus A_-^{\alpha+\beta}(D), \]
where $A^{\alpha+\beta}_-(D)$ consists of those $f \in A^{\alpha+\beta}(D)$ such that $\sigma^*(f) = \varepsilon f$ and $\varepsilon$ is plus or minus.

**Theorem 15.** Let $(D, \sigma)$ be a skew-symmetric DVB with side bundle $E \to M$. Then $A^{\alpha+\beta}_-(D)$ coincides with the space of sections of $S^2E^*$ while $A^{\alpha+\beta}_+(D)$ gives rise to a short exact sequence
\[ 0 \to \Gamma(\bigwedge^2 E^*) \to A^{\alpha+\beta}_-(D) \to \Gamma(C^*) \to 0. \]
The skew-symmetric DVB $(D, \sigma)$ can be reconstructed completely from the above sequence.

**Proof.** Let $(x^a, y^i, Y^j, z^\mu)$ be adopted coordinates for a skew-symmetric DVB $(D, \sigma)$. In these coordinates the decomposition of $A^{\alpha+\beta}(D)$ is clear: $A^{\alpha+\beta}_+(D)$ is locally generated by functions $y^i Y^j + y^j Y^i$ and $A^{\alpha+\beta}_-(D)$ by $z^\mu$ and $y^i Y^j - y^j Y^i$. The projection on $\Gamma(C^*)$ is the same, it is given as the restriction to the core bundle of $D$. □

### 2.5. Inverse limit of supermanifolds and Lie supergroups

We shall work with some types of infinite dimensional supermanifolds and Lie supergroups. There is no need to present a general theory as all examples we shall work with have a form of an inverse limit of supermanifolds (even with the same body $M_0$):

\[ M_1 \stackrel{pr_1}{\leftarrow} M_2 \stackrel{pr_2}{\leftarrow} M_3 \leftarrow \ldots \]

where $M_k = (M_0, O_{M_k})$ and $pr_{k+1}^k : M_{k+1} \to M_k$ is the projection of the graded manifold $M_{k+1}$ of degree $k+1$ to the graded manifold $M_k$ of degree $k$, see Remark 10. We define
\[ M_\infty = (M_0, O_{M_\infty}), \quad \text{where } O_{M_\infty} = \bigcup_{k=1}^\infty O_{M_k}, \]
i.e. $f \in O_{M_\infty}$ if and only if there exist $k$ such that $f \in O_{M_k}$. A morphism $f : M_\infty \to N_\infty$ is a family of morphisms $f = (f_k : M_k \to N_{a_k})$ where $(a_k)$ is a non-decreasing sequence of positive integers and the family $(f_k)$ is compatible with the projections:

\[ M_{k+1} \xrightarrow{f_{k+1}} N_{a_{k+1}} \]
\[ M_k \xrightarrow{pr_{k+1}^k} N_{a_k} \]
\[ M_k \xrightarrow{pr_{a_k}} N_{a_k} \]

where $pr_{a_k}$ is the composition of projections $pr_{i+1}^i$ for $i = a_{k+1} - 1$ to $a_k$.

The inverse limit of Lie supergroups has a Lie supergroup structure. We additionally have to assume that the projections $pr_{i+1}^i$ are Lie supergroup homomorphisms.

Also the inverse limit of Lie (super)algebras has a Lie (super)algebra structure thus
a Lie functor makes sense for inverse limit of Lie supergroups. For example a \(Z\)-graded Lie superalgebra \(g = \bigoplus_{k=0}^{\infty} g_k\) is an inverse limit of Lie algebras \(g/I_k\) where \(I_k = g_{k+1} \oplus g_{k+2} \oplus \ldots\). There is one-to-one correspondence of the inverse limit of graded Lie supergroups and the inverse limit of their graded Harish-Chandra pairs.

3. The Donagi–Witten construction, supermanifolds, double vector bundles and graded manifolds of degree 2

In [DW2] Section 2,1 Donagi and Witten gave a description of the first obstruction class \(\omega = \omega_2\) via differential operators. In this section we remind the definition of \(\omega_2\) and the Donagi–Witten construction. Further we give an interpretation of the Donagi–Witten construction using the language of double vector bundles and graded manifolds of degree 2. At the end using differential forms instead of differential operators we simplify this construction, which allows us to find its higher analogue.

3.1. First obstruction class \(\omega_2\). Let us describe the first obstruction class to splitting a supermanifold using results [Ber, Gr, Oni, R]. We follow the exposition of [Oni]. First of all consider a split supermanifold \(M = (M_0, \mathcal{O})\), where \(\mathcal{O} = \Lambda \mathcal{E}^*\) and \(\mathcal{E}\) is the sheaf of sections of \(E[1]\). Denote by \(\text{Der}\mathcal{O}\) the sheaf of vector fields on \(M\) and by \(\text{Der}F\) the sheaf of vector fields on the underlying space \(M_0\). The sheaf \(\text{Der}\mathcal{O} = \bigoplus_{p \geq -1} \text{Der}_p\mathcal{O}\) is naturally \(Z\)-graded since \(\Lambda \mathcal{E}^*\) is \(Z\)-graded. We have the following exact sequence

\[
(10) \quad 0 \to \bigwedge^3 \mathcal{E}^* \otimes \mathcal{E} \to \text{Der}_2\mathcal{O} \to \bigwedge^2 \mathcal{E}^* \otimes \text{Der}F \to 0,
\]

see [Oni] Formula (5)].

According Green [Gr] we can describe all non-split supermanifolds \(\mathcal{N}\) such that \(\text{gr}(\mathcal{N}) \simeq M\) using the sheaf of automorphisms \(\text{Aut}\mathcal{O}\) of \(\mathcal{O}\). More precisely consider the following subsheaf of \(\text{Aut}\mathcal{O}\):

\[
\text{Aut}_{(2)}\mathcal{O} = \{a \in \text{Aut}\mathcal{O} \mid a(u) = u \in J^2 \text{ for any } u \in \mathcal{O}\},
\]

see also [Oni] Formula (17)]. Recall that \(J\) is the sheaf of ideals generated by odd elements in \(\mathcal{O}\). Denote by \(\text{Aut}(\mathcal{E}^*)\) the group of automorphisms of \(\mathcal{E}^*\). There is a natural action of \(\text{Aut}(\mathcal{E}^*)\) on the sheaf \(\text{Aut}_{(2)}\mathcal{O}\), see [Oni] Section 1.4. This action induces an action of \(\text{Aut}(\mathcal{E}^*)\) on the set \(H^1(M_0, \text{Aut}_{(2)}\mathcal{O})\). By Green [Gr] points of the set of orbits \(H^1(M_0, \text{Aut}_{(2)}\mathcal{O})/\text{Aut}(\mathcal{E}^*)\) are in one-to-one correspondence with isomorphism classes of supermanifolds \(\mathcal{N}\) such that \(\text{gr}(\mathcal{N}) \simeq M\). More precisely to any supermanifold \(\mathcal{N}\) such that \(\text{gr}(\mathcal{N}) \simeq M\) we can assign a class \(\gamma \in H^1(M_0, \text{Aut}_{(2)}\mathcal{O})\). If \(\gamma_i\) is the class corresponding to a supermanifold \(\mathcal{N}_i\) such that \(\text{gr}(\mathcal{N}_i) \simeq M_i\), where \(i = 1, 2\). Then \(\mathcal{N}_1 \simeq \mathcal{N}_2\) if and only if \(\gamma_1\) and \(\gamma_2\) are in the same orbit of \(\text{Aut}(\mathcal{E}^*)\).

In [R] the following map of sheaves was defined

\[
(11) \quad \text{Aut}_{(2)}\mathcal{O} \to \text{Der}_2\mathcal{O},
\]

see also [Oni] Formula (19)]. Combining the map (11) and the map \(\text{Der}_2\mathcal{O} \to \bigwedge^3 \mathcal{E}^* \otimes \text{Der}F\) from (10) we get the following map of cohomology sets

\[
H^1(M_0, \text{Aut}_{(2)}\mathcal{O}) \to H^1(M_0, \bigwedge^2 \mathcal{E}^* \otimes \text{Der}F)
\]
and the corresponding map of $\text{Aut}(E^*)$-orbits
\[ H^1(M_0, \text{Aut}(2)\mathcal{O})/\text{Aut}(E^*) \to H^1(M_0, \bigwedge^2 E^* \otimes \text{Der}\mathcal{F})/\text{Aut}(E^*). \]

If $\gamma \in H^1(M_0, \text{Aut}(2)\mathcal{O})$ corresponds to a non-split supermanifold $N$, the image of $\gamma$ in $H^1(M_0, \bigwedge^2 E^* \otimes \text{Der}\mathcal{F})$ is called the first obstruction class to splitting of $N$ and following [DW2] Section 2.1 we denote this class by $\omega_2$.

Consider the case when $M$ has odd dimension 2 in details. Since $\bigwedge^3 E^* = \{0\}$, from (10) it follows that $\text{Der}_2 \mathcal{O} \simeq \bigwedge^2 E^* \otimes \text{Der}\mathcal{F}$. In this case the map (11) is an isomorphism. Therefore we have the following set bijection
\[ H^1(M_0, \text{Aut}(2)\mathcal{O}) \simeq H^1(M_0, \bigwedge^2 E^* \otimes \text{Der}\mathcal{F}) \]
and the corresponding bijection of the sets of orbits. Now Green’s result [Gr] implies that $\omega_2$ is the only obstruction for a supermanifold to be split in this case. In other words a supermanifold $N$ of odd dimension 2 such that $\text{gr}(N) \simeq M$ is split if and only if $\omega_2 = 0$. Note that the notion of a split and a projectable, see [DW1] for details, supermanifold coincide in this case.

3.2. The Donagi and Witten construction. In [DW2] Section 2.1 the first obstruction class $\omega_2$ to splitting a supermanifold $M = (M_0, \mathcal{O})$ was interpreted in terms of a certain sheaf of differential operators on $M$. Namely, the obstruction class $\omega$ defined above is the Atiyah class of the extension
\[ 0 \to TM \to D_\omega \to \bigwedge E \to 0 \]
where sections of $D_\omega$ are identified with some factor of $\mathcal{D}_2|_{M_0}$ where the meaning of the latter is explained below.

Let us remind this construction using charts and local coordinates. Consider two charts $U_1$ and $U_2$ on $M$ with non-empty intersection and with local coordinates $(x_i, \xi_a)$ and $(y_j, \eta_b)$, where $i, j = 1, \ldots, n$ and $a, b = 1, \ldots, m$. Let in $U_1 \cap U_2$ we have the following transition functions
\begin{equation}
\begin{aligned}
y_j &= F_j + \frac{1}{2} G_j^{\alpha_1 \alpha_2} \xi_{\alpha_1} \xi_{\alpha_2} + \cdots + j = 1, \ldots, n; \\
\eta_b &= H_b^a \xi_a + \cdots, \quad b = 1, \ldots, m,
\end{aligned}
\end{equation}
where $F_j, G_j^{\alpha_1 \alpha_2}, H_b^a$ are (holomorphic) functions depending only on even coordinates $(x_i)$.

If $D : \mathcal{O} \to \mathcal{O}$ is a differential operator let $[D] : \mathcal{O} \to \mathcal{F}$ denotes the composition of $D$ with the projection $\mathcal{O} \to \mathcal{O}/\mathcal{J} = \mathcal{F}$.

Following Donagi and Witten, see [DW2] Section 2.1, we define the sheaf $\mathcal{D}_2|_{M_0}$ on the underlying manifold $M_0$ as a sheaf locally generated over $\mathcal{F}$ by
\[ \langle 1, \left[ \frac{\partial}{\partial x_i} \right], \left[ \frac{\partial}{\partial \xi_j} \right], \left[ \frac{\partial^2}{\partial \xi_a \partial \xi_b} \right] \rangle. \]

In [DW2] Theorem 2.5 it was shown that this definition does not depend on local coordinates. Note that the operators $\frac{\partial}{\partial \xi_a} \frac{\partial}{\partial \xi_b}$ anticommute, i.e. $\frac{\partial^2}{\partial \xi_a \partial \xi_b} = - \frac{\partial^2}{\partial \xi_b \partial \xi_a}$. We
can get transition function for generators of $D^2|_{\mathcal{M}_0}$ writing down the transition function for $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial \xi^j}$, $\frac{\partial^2}{\partial \xi^a \partial \xi^b}$ and then factorizing them by $J$. Indeed, 

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} + \frac{\partial \eta^\alpha}{\partial x^i} \frac{\partial}{\partial \eta^\alpha}; \quad \frac{\partial}{\partial \xi^k} = \frac{\partial y^j}{\partial \xi^k} \frac{\partial}{\partial y^j} + \frac{\partial \eta^\alpha}{\partial \xi^k} \frac{\partial}{\partial \eta^\alpha}.$$ 

Therefore modulo $J$ we have:

$$\left[ \frac{\partial^2}{\partial \xi^a \partial \xi^b} \right] = \left( \frac{\partial \eta^\alpha}{\partial \xi^a} \frac{\partial^2(y_j)}{\partial y^j} \right) \text{red} \left[ \frac{\partial}{\partial y^j} \right] - \left( \frac{\partial \eta^\alpha}{\partial \xi^a} \frac{\partial \eta^\beta}{\partial \xi^b} \right) \text{red} \left[ \frac{\partial^2}{\partial \eta^\alpha \partial \eta^\beta} \right].$$

Compare with [DW2, Formula 2.13]. Using (12) we can write the transition functions for $D^2|_{\mathcal{M}_0}$ explicitly.

$$x_i = F_i^{-1}(y_j);$$

$$\left[ \frac{\partial}{\partial x^i} \right] = \frac{\partial F_i}{\partial x^j} \left[ \frac{\partial}{\partial y^j} \right]; \quad \left[ \frac{\partial}{\partial \xi^a} \right] = H_a^b \left[ \frac{\partial}{\partial \eta^b} \right];$$

$$\left[ \frac{\partial^2}{\partial \xi^a \partial \xi^b} \right] = -G^a_b \left[ \frac{\partial}{\partial y^j} \right] - H_a^b H^c_d \left[ \frac{\partial^2}{\partial \eta^a \partial \eta^c} \right].$$

Note that in (13) it is more correct to use the index red for even coordinates, for example $(x_i)_{\text{red}}$. However we omit red for simplicity of notations. We conclude this subsection with the following important remark.

**Remark 16.** Comparing Formulas (12) and (13), we see that Formulas (13) contain the whole information about Formulas (12) modulo $J^3$. In other words using Formulas (13) we can reconstruct Formulas (12) modulo $J^3$.

One of purposes of this paper is to develop this observation. This leads to an idea to use the theory of $n$-fold vector bundles and the theory of graded manifolds of degree $n$ to recover a supermanifolds of odd dimension greater than 2.

### 3.3. A geometric interpretation of the Donagi and Witten construction.

In this subsection we give a description of the geometric object with transition functions (13). We use the theory of double vector bundles and graded manifolds of degree 2.

**Lemma 17.** Let $\mathcal{D}$ be a vector bundle over $M$ characterized by the space of sections, $\Gamma(\mathcal{D}) = D^2|_{\mathcal{M}_0}$. Then the dual bundle $\mathcal{D}^*$ has transition functions of a skew-symmetric DVB. More precisely, there exist a skew-symmetric DVB $(\mathcal{D}, \sigma)$ such that the space $A^{a+\beta}(D)$ coincides with the space of sections of the vector bundle $\mathcal{D}^*$.

**Proof.** We simply transpose and reverse the formulas (13). \[\Box\]

The skew-symmetric DVB arising from the above lemma will be denoted by $\mathcal{V}b_2(M)$.

The Atiyah class of the exact sequence (8), that is the obstruction class of splitting of this sequence, is an element in $H^1(M; A^* \otimes B^* \otimes C)$. It will be also called the Atiyah class associated with the DVB $D$ and denoted by $At(D)$. Note that the Atiyah class associated with any dual (vertical or horizontal) of $D$ coincides with $At(D)$. Conversely if we have the sequence (8) we can reconstruct the double vector bundle $D$. 


Now let $\mathcal{M}$ be a supermanifold with transition function (12) and let $E = Vb_1(\mathcal{M})$. The Atiyah sequence associated with $D = Vb_2(\mathcal{M})$ (which is a double vector bundle with the same side bundles $E$ and the core $TM_0$) is

$$0 \to TM_0 \to \hat{D} \to E \otimes E \to 0.$$ 

The obstruction for splitting of this exact sequence is

$$At(Vb_2(\mathcal{M})) \in H^1(\mathcal{M}_0, TM(\mathcal{M}_0) \otimes E^* \otimes E^*).$$

Due to the decomposition $E \otimes E = \text{Sym}^2 E \oplus \Lambda^2 E$, the Atiyah class $At(D)$, where $D = Vb_2(\mathcal{M})$, decomposes to $At_+ (D) + At_-(D)$, where $At_+(D) \in H^1(\mathcal{M}_0, TM(\mathcal{M}_0) \otimes \text{Sym}^2 E^*)$ and $At_-(D) \in H^1(\mathcal{M}; TM \otimes \Lambda^2 E^*)$. We are only interested with $At_-(D)$ because our $DVB$ $Vb_2(\mathcal{M})$ is a skew-symmetric $DVB$, hence $At_+(D) = 0$.

The class $At_-(\mathcal{M})$ is here the same as the Atiyah class of the sequence (11) associated with the skew-symmetric $DVB$ $(D, \sigma)$. It also coincides with the first obstruction class to splitting of the supermanifold $\mathcal{M}$ as in [DW2, Section 2].

Later we will give a geometric interpretation of this fact.

### 3.4. A modification of the Donagi–Witten construction.

In this subsection we suggest a different way how to obtain a double vector bundle with obstruction class $\omega_2$. First of all to write Formulas (14) we need the transition functions (12) and their inverse. To avoid this inconvenience we can use differential forms instead of differential operators.

Let $\mathcal{M}$ be a supermanifold as above. Consider as above two charts $\mathcal{U}_1$ and $\mathcal{U}_2$ with non-empty intersection on $\mathcal{M}$ with local coordinates $(x_i, \xi_a)$ and $(y_j, \eta_b)$ and transition functions (12). Further consider the antitangent bundles $T(\mathcal{M})$ of $\mathcal{M}$ and two charts $T(\mathcal{U}_1)$ and $T(\mathcal{U}_2)$ with standard coordinates $(x_i, \xi_a, dx_i, d\xi_a)$ and $(y_j, \eta_b, dy_j, d\eta_b)$, respectively. Thus $x_i$, $d\xi_a$ are even local coordinates in $T(\mathcal{U}_1)$, while $\xi_a, dx_i$ are odd ones. In $T(\mathcal{U}_1) \cap T(\mathcal{U}_2)$ we get the following transition functions.

$$y_j = F_j + G_j^{a_1 a_2} \xi_{a_1} \xi_{a_2} + \cdots, \quad \eta_a = H_a^b \xi_b + \cdots,$$

$$dy_j = \sum_{b=1}^n (F_j)_x b dx_b + \sum_{b=1}^n (G_j^{a_1 a_2})_x b dx_b \xi_{a_1} \xi_{a_2} + G_j^{a_1 a_2} (d(\xi_{a_1}) \xi_{a_2} - \xi_{a_1} d(\xi_{a_2})) + \cdots;$$

$$d\eta_a = \sum_{b=1}^n (H_a^b)_x b dx_b \xi_j + H_a^b d\xi_j + \cdots.$$

Here we denote by $(F_j)_x b$ the derivation of $F_j$ by $x_b$. Now we apply the functor $gr$ to $T(\mathcal{M})$. In local coordinates this means that we factorize our transition functions by all terms that contain more than one odd variable. Then $gr T(\mathcal{M})$ has the following transition functions

$$y_j = F_j(x);$$

$$\eta_a = H_a^b \xi_b;$$

(14)

$$dy_j = \sum_{b=1}^n (F_j)_x b dx_b + G_j^{a_1 a_2} (d(\xi_{a_1}) \xi_{a_2} - \xi_{a_1} d(\xi_{a_2}));$$

$$d\eta_a = H_a^b d\xi_b.$$

If we compare Formulas (14) with [V2, Section 2.2, Formulas (9)-(12)], we see that a manifold with such transition functions is a double vector bundle, which we denote...
by D. We can see $\text{gr}T(M)$ as a graded manifold of type $\Delta = \{0, \alpha, \beta, \alpha + \beta\}$, where $\alpha$ is odd (the weight of $\xi_a$) and $\beta$ is even (the weight of $d\xi$). Formulas (14) shows that all weights are well-defined. The double vector bundle

$$
\begin{align*}
D' & \longrightarrow E[1] \\
\downarrow & \\
E & \longrightarrow M_0
\end{align*}
$$

is only slightly different from the double vector bundle $Vb_2(M)$. In our case one side bundle is pure even.

**Remark 18.** By definition the composition of functors $\text{gr} \circ T$ is a functor from the category of supermanifolds to the category of (split) supermanifolds. However above implies that we can see the image of $\text{gr} \circ T$ as the category of double vector bundles (with some additional structure). For simplicity we will use the same notation $\text{gr} \circ T$ meaning a functor from the category of supermanifolds to the category of double vector bundles. Similarly in next sections we will consider the functor $\text{gr} \circ T^{(n)}$ as a functor from the category of supermanifolds to the category of $n$-fold vector bundles.

Now we can go further and give an interpretation of $\omega_2$ as the obstruction class of splitting of a graded manifold of degree 2. First of all let us change that parity of the side bundle $E$. (In other words we apply the functor parity change, see [Vo].) To do this we need to rewrite (14) in the following form

$$
y_j = F_j;
\eta_a = H_a^b \xi_b;
$$

$$
dy_j = \sum_{b=1}^n (F_j)_a x_a dx_b + G_{j}^{a_1 a_2} (\xi_{a_2} d(\xi_{a_1}) - \xi_{a_1} d(\xi_{a_2}));
\eta_a = H^b_a d\xi_b.
$$

and to change the parity of the weight $\beta$.

Now we use a result obtained by [JL, CM] for double vector bundles and by [BGR, VII] for $n$-fold vector bundles. Later we will call this step "to apply the functor inverse". In more details in [JL, CM] a functor was constructed from the category of graded manifolds of degree 2 to the category of double vector bundles with some additional structures. In [BGR, VII] an analogue of this result was obtained for graded manifolds of degree $n$ and for graded manifolds of type $\Delta$. (Note that in all these papers [JL, CM, BGR, VII] the categories of double vector bundles with additional structures are different. In this paper we follow approaches of [BGR, VII].) In [JL, CM] it was shown that this functor is an equivalence of the category of double vector bundles with some additional structures and the category of graded manifolds of degree 2. In [BGR, VII] it was shown that this functor is an equivalence of the category of $n$-fold vector bundles with some additional structures and the category of graded manifolds of degree $n$ or of type $\Delta$. The inversion of this functor, that is the functor from the category of $n$-fold vector bundles with some additional structures to the category of graded manifolds of degree $n$, we call the functor inverse. We will denote the functor inverse by $\iota$. 
In terms of local coordinates "to apply the functor inverse" means that we identify $\zeta_i$ with $d\zeta_i$ in (16). We get

$$
y_j = F_j; \quad \theta_a = H_a^i \zeta_i; \quad t_j = \sum_{b=1}^n (F_j)_x z_b + 2G^a_{j}^{a^2} \zeta_{a^1} \zeta_{a^2}.
$$

We obtained transition functions of a graded manifold $N$ of degree 2, which we will denote also by $F_2(M)$. We assign the following weights to our local coordinates: $x_i$ (weight 0); $\zeta_j$ (weight $\alpha$); $z_a$ (weight $2\alpha$). In other words, the transition functions (17) defines a graded manifold of type $\Delta = \{0, \alpha, 2\alpha\}$, where $\alpha$ is odd. Note that we can remove the coefficient 2 in (17). Indeed, it is enough to replace $\zeta_i$ by $\frac{1}{\sqrt{2}} \zeta_i$ in any chart.

**Remark 19.** Let us give another explanation of the procedure "to apply the functor inverse". Consider two graded domain $V_1$ and $V_2$ with graded coordinates $(x_i, \zeta_j, x_i)$ and $(y_j, \theta_a, t_i)$, respectively, and with weights as above. Define transition functions $V_1 \to V_2$ by (17). (Let us omit the coefficient 2.) Now we apply the tangent functor $T$ to $V_1$ and $V_2$ and to the morphism (17) and factorize the result by the sheaf of ideals locally generated by $dx_i$ (or by $dy_j$). We get

$$
y_j = F_j; \quad \theta_a = H_a^i \zeta_i; \quad dt_j = \sum_{b=1}^n (F_j)_x z_b + G^a_{j}^{a^2} (\zeta_{a^1} d(\zeta_{a^2}) = \zeta_{a^1} d(\zeta_{a^2})).
$$

We obtain Formulas (16) up to appropriate change of variables. The inversion of this procedure is called "to apply the functor inverse".

The structure sheaf $\mathcal{O}_N = \bigoplus_{q \geq 0} (\mathcal{O}_N)_q$ of $N$ is $\mathbb{Z}$-graded. Clearly $(\mathcal{O}_N)_1 \cdot (\mathcal{O}_N)_1 \subset (\mathcal{O}_N)_2$. Therefore we can assign to $N^*$ the following exact sequence

$$
0 \to (\mathcal{O}_N)_1 \cdot (\mathcal{O}_N)_1 \to (\mathcal{O}_N)_2 \to (\mathcal{O}_N)_2/(\mathcal{O}_N)_1 \cdot (\mathcal{O}_N)_1 \to 0.
$$

The Atiyah class of this sequence is represented by a cocycle ($\mathcal{G}^a_{j}^{a^2}$) and coincides with the obstruction class of the double vector bundle (16). It is called the obstruction class of the graded manifold $F_2(M)$ of degree 2.

Let us summarize our results in the following theorem.

**Theorem 20** (Main theorem about the first obstruction class). The first obstruction class $w_2$ for a supermanifold $M$ coincides with the obstruction class $At(\text{Vb}_2(M))$ of the skew-symmetric double vector bundle $\text{Vb}_2(M)$ and with the obstruction class of the graded manifold $F_2(M)$ of degree 2.

Note that the map $M \mapsto F_2(M)$ is a functor from the category of supermanifolds of odd dimension 2 to the category of graded manifolds of degree 2. Indeed, $F_2$ is a composition of four functors: $T$, gr, $\pi$ and $\iota$. This functor defines an equivalence of the category of supermanifolds of odd dimension 2 and the category of graded manifolds of degree 2 with the following additional condition. If $N = (N_0, \mathcal{O}_N)$ is a graded manifold of degree 2, we additionally assume that the locally free sheaf $E := (\mathcal{O}_N)_2/(\mathcal{O}_N)_1 \cdot (\mathcal{O}_N)_1$ is isomorphic to the sheaf of sections of $T[1]|N_0$. In


Denote by $\Delta$ the maximal multiplicity free weight system generated by an odd supermanifold. Let $\text{Smf}_2$ denote the category of supermanifolds of odd dimension $2$. By Theorem 21, the functor $\text{F}_n$ splits $\text{Smf}_2$ for $(n - 1)$-times (or infinity many times for $n = \infty$) iterated antitangent functor $T^{(n-1)} := T \circ \ldots \circ T$, the functor split $\text{gr}$, the functor parity change $\pi$ and the functor inverse $\iota$. If $\mathcal{M}$ is a supermanifold, then $T^{(n-1)}(\mathcal{M})$, $\text{gr} \circ T^{(n-1)}(\mathcal{M})$ are already defined. However, for any dimension the functor $\text{F}_2$ is not an embedding. In this case from a graded manifold of degree 2 we can recover a supermanifold module $\mathcal{J}^3$. Therefore, $R = \text{F}_2(\mathcal{M})$. Since $\text{F}_2$ is a functor, we get

$$\text{F}_2(T_{31}) \circ \text{F}_2(T_{23}) \circ \text{F}_2(T_{12}) = \text{F}_2(R).$$

The composition $\text{F}_2(T_{31}) \circ \text{F}_2(T_{23}) \circ \text{F}_2(T_{12})$ is equal to $\text{id}$, since the cocycle condition for the graded manifold $\mathcal{N}$ holds true. Therefore $\text{F}_2(R) = \text{id}$. The morphism $R : U_i \rightarrow U_i$ is completely defined by its image $F_2(R) = \text{id}$. Therefore, $R = \text{id}$.

Denote the category of supermanifolds of odd dimension $2$ by $\text{Smf}_2$ and the category of graded manifolds of degree 2 and of odd dimension $2$ with the additional condition for $(\mathcal{O}_N)_2/(\mathcal{O}_N)_1 \cdot (\mathcal{O}_N)_1$ as above, by $\text{GrMan}_2T$. Now we can summarize our results in the following theorem.

**Theorem 21.** The categories $\text{Smf}_2$ and $\text{GrMan}_2T$ are equivalent.

In [RV] we generalize this theorem to the category of supermanifolds of any odd dimension.

**Remark 22.** If our supermanifold $\mathcal{M}$ has the odd dimension $m > 2$, we still can repeat the procedure above. Therefore the functor $\text{F}_2$ is a functor from the category of supermanifolds to the category of graded manifold of degree 2. However for any dimension the functor $\text{F}_2$ is not an embedding. In this case from a graded manifold of degree 2 we can recover a supermanifold module $\mathcal{J}^3$.

Summing up, in this section we showed that the first obstruction class to splitting of the supermanifold in the sense of [DW2, Section 2] coincides with the obstruction class of the splitting of the double vector bundle $\text{gr}T(\mathcal{M})$ and with the obstruction class of the splitting of the graded manifold $F_2(\mathcal{M})$. A general splitting theory for supermanifolds of odd dimension $m$ based on splitting of the corresponding $m$-fold vector bundles and graded manifolds of degree $m$ will be developed in our oncoming paper.

4. A generalization of the Donagi–Witten construction

In this section we give a construction of functors $F_n$, where $n = 2, 3, \ldots, \infty$, from the category of supermanifolds to the category of graded manifolds of degree $n$. Due to the size of this paper and technical difficulty of the proof of the main result, the existence of the functor inverse $\iota$ for any supermanifold of the form $\text{gr} \circ T(\mathcal{M})$, where $\mathcal{M}$ is a supermanifold, we leave details of this proof to Part II of this paper. Here we give only the idea of the proof.

The functor $F_n$ is again a composition of four functors: the $(n - 1)$-times (or infinity many times for $n = \infty$) iterated antitangent functor $T^{(n-1)} := T \circ \ldots \circ T$, the functor split $\text{gr}$, the functor parity change $\pi$ and the functor inverse $\iota$. If $\mathcal{M}$ is a supermanifold, then $T^{(n-1)}(\mathcal{M})$, $\text{gr} \circ T^{(n-1)}(\mathcal{M})$ are already defined. Denote by $\Delta$ the maximal multiplicity free weight system generated by an odd
weight $\alpha$ and by even weights $\beta_1, \ldots, \beta_{n-1}$, see Definition 3. We need the following propositions.

**Proposition 23.** The supermanifold $\text{gr} \circ T^{(n-1)}(\mathcal{M})$ is an $n$-fold vector bundle of type $\Delta$, where $\Delta$ is the maximal multiplicity free system generated by an odd weight $\alpha$ and by even weights $\beta_1, \ldots, \beta_{n-1}$.

**Proof.** Let $d_i$, where $i = 1, \ldots, n-1$, be the iterated de Rham differentials. They are vector fields in the structure sheaf of $T^{(n-1)}(\mathcal{M})$. Denote by $d_{I}$, where $I = (i_1, \ldots, i_k)$, the composition $d_{i_1} \circ \ldots \circ d_{i_k}$. Let $\mathcal{I}$ be the ideal generated by odd elements in $\mathcal{O}_{\text{gr} T^{(n-1)}(\mathcal{M})}$ and let us choose a chart $\mathcal{U}$ on $\mathcal{M}$ with local coordinates $(x_a, \xi_i)$. Then the corresponding standard local coordinates in the chart $T^{(n-1)}(\mathcal{U})$ are $(d_I(x_a), d_I(\xi_i))_{C(I), C(J) \leq n-1}$, where $C(I)$ is the cardinality of $I$. Hence the corresponding local coordinates in the chart $\text{gr} \circ T^{(n-1)}(\mathcal{U})$ are $(d_I(x_a) + \mathcal{I}^2, d_I(\xi_i) + \mathcal{I}^2)_{C(I), C(J) \leq n-1}$. We assign the weight $\alpha + \beta_i + \cdots + \beta_i$ to $d_I(Z) + \mathcal{I}^2$, where $Z \in \{x_a, \xi_i\}$, if $d_I(Z)$ is odd and the weight $\beta_i + \cdots + \beta_i$ to $d_I(Z) + \mathcal{I}^2$ if $d_I(Z)$ is even.

It remains to prove that transition functions in $\text{gr} \circ T^{(n-1)}(\mathcal{M})$ preserve our weights. Let $\mathcal{U}'$ be another chart on $\mathcal{M}$ with coordinates $(x'_a, \xi'_i)$. Let

$$x'_a = \sum_{C(K) = 2k} F_K(x) \xi_K$$

be the expression of $x'_a$ in coordinates of $\mathcal{U}$. Then $d_I(x'_a) + \mathcal{I}^2$ in coordinates of $\text{gr} \circ T^{(n-1)}(\mathcal{U})$ is a sum of monomials in the following form.

$$(18) \quad H_{I,K}(x)d_{I_1}(x_{a_1}) \cdots d_{I_q}(x_{a_q})d_{I_{p+1}}(\xi_{b_1}) \cdots d_{I_q}(\xi_{b_q}) + \mathcal{I}^2,$$

where $H_{I,K}(x)$ is a certain derivative of $F_K(x)$, which has weight 0, and $(I_1, \ldots, I_q)$ is a decomposition of the sequence $I$ into $q$ parts. We see that the weights of $d_I(x'_a) + \mathcal{I}^2$ and of $(18)$ coincide. Similarly for $d_I(\xi'_i) + \mathcal{I}^2$. This completes the proof. \qed

**Proposition 24.** The functor $\text{gr} \circ T^{(n-1)}$ is an embedding of the category of supermanifolds of odd dimension $n$ into the category of $n$-fold vector bundles of type $\Delta$.

**Proof.** Let $\mathcal{N}$ be an $n$-fold vector bundle. Let us show that if a preimage $\mathcal{M}$ of $\mathcal{N}$ exists, it is unique. The idea of the proof of similar to the idea of the proof of Theorem 21. Assume that there exists another supermanifold $\mathcal{M}'$ with $\text{gr} \circ T^{(n-1)}(\mathcal{M}) = \text{gr} \circ T^{(n-1)}(\mathcal{M}')$. By construction we have $\mathcal{M}_0 = \mathcal{M}'_0 = N_0$. Let us choose an atlas $\{\mathcal{U}_I\}$ on $\mathcal{M}$ and an atlas $\{\mathcal{U}'_{I}\}$ on $\mathcal{M}'$ such that $(U_I)_0 = (U'_I)_0$. Clearly such atlases there exist, since it is sufficient to choose any atlas on $\mathcal{M}_0$ of Stein domains. Now consider two charts $\mathcal{U}'_1$ and $\mathcal{U}'_2$ with $\mathcal{U}'_1 \cap \mathcal{U}'_2 \neq \emptyset$ on $\mathcal{M}'$. Denote by $T'$ the transition function $T' : \mathcal{U}'_1 \rightarrow \mathcal{U}'_2$ and by $T$ the transition function $T : U_1 \rightarrow U_2$. By our assumption $\text{gr} \circ T^{(n-1)}(T) = \text{gr} \circ T^{(n-1)}(T')$. But this implies that $T = T'$. The proof is complete. \qed

**Remark 25.** The question when the functor $\text{gr} \circ T^{(n-1)}$ possesses a preimage is treated in [RV].

Now we can use results of [BGR] and [VII] to define the functor inverse. Note that these two results lead to two different approaches to the problem. Let us start with a description of results [BGR] or [VII]. In [BGR] a functor $G_n$ was...
constructed from the category of graded manifolds of degree $n$ to the category of $n$-fold symmetric vector bundles. In [VII] a functor $V_n$ was constructed from the category of graded manifolds of degree $n$ to the category of $n$-fold vector bundles with $n-1$ odd commutative homological vector fields. In both cases it was proven that these functors are equivalence of categories. In Section 3 we saw that due to the fact that the first obstruction class $\omega_2$ is an element in

$$H^1(\mathcal{M}_0, T[1](\mathcal{M}_0) \otimes \bigwedge^2 \mathcal{E}^*)$$

we can construct a graded manifold of degree 2 with transition functions $\{\mathcal{E}_i\}$ from a double vector bundle given by [16]. In terms of [BGR] this means that the double vector bundle with the obstruction class $\omega_2$ is symmetric and $\pi \circ \text{gr} \circ T(\mathcal{M})$ is in the image of the functor $G_2$ constructed in [BGR]. Our functor inverse $\iota$ is the inversion of the equivalence $G_2$. More generally we have

**Theorem 26.** For a supermanifold $\mathcal{M}$ of odd dimension $n$ the image $\pi \circ \text{gr} \circ T^{(n-1)}(\mathcal{M})$ is a symmetric $n$-fold vector bundle. Therefore there exists a graded manifold $\mathcal{N}$ of degree $n$ such that $G_n(\mathcal{N}) \simeq \pi \circ \text{gr} \circ T^{(n-1)}(\mathcal{M})$. In other words the functor inverse $\iota = G_n^{-1}$ is defined on the image of the functor $\pi \circ \text{gr} \circ T^{(n-1)}$.

Let us describe another approach based on results of [VII]. The graded supermanifold $\text{gr} \circ T(\mathcal{M})$ possesses a natural odd homological vector field. Indeed, let $d_R$ be the exterior derivative (de Rham differential) of the supermanifold $\mathcal{M}$. Clearly $d_R$ is an odd homological vector field in the structure sheaf $\mathcal{O}_{T(\mathcal{M})}$ of $T(\mathcal{M})$. Denote by $\mathcal{I}$ the sheaf of ideals locally generated by all odd variables of $\mathcal{O}_{T(\mathcal{M})}$. Then we have

$$d_R(\mathcal{I}) \subset \mathcal{O}_{T(\mathcal{M})}, \quad d_R(\mathcal{I}^2) \subset \mathcal{I}.$$ 

Therefore we have an induced operator in the structure sheaf of $\text{gr} \circ T(\mathcal{M})$

$$\mathcal{O}_{\text{gr} \circ T(\mathcal{M})} = \text{gr} \mathcal{O}_{T(\mathcal{M})} = \bigoplus_{p \geq 0} \mathcal{I}^p / \mathcal{I}^{p+1}.$$

Clearly this induced operator is odd and homological. Hence the double vector bundle $\text{gr} \circ T(\mathcal{M})$ is a double vector bundle with a homological vector field. Therefore by [VII] there exists a graded manifold $V^{-1}_2(\text{gr} \circ T(\mathcal{M}))$ of degree 2. This procedure can be generalized. Now we can formulate a general result.

**Theorem 27.** For a supermanifold $\mathcal{M}$ of odd dimension $n$ the image $\text{gr} \circ T^{(n-1)}(\mathcal{M})$ is an $n$-fold vector bundle with $n-1$ commuting odd homological vector fields. Therefore there exists a graded manifold $\mathcal{N}$ of degree $n$ such that $V_n(\mathcal{N}) \simeq \text{gr} \circ T^{(n-1)}(\mathcal{M})$. In other words the functor $\iota' = V_n^{-1}$ is defined on the image of $\text{gr} \circ T^{(n-1)}$.

The idea of the proof is simple: the $(n - 1)$-times iterated tangent functor leads to $n-1$ commuting de Rham differentials, which induce $(n - 1)$ commuting odd homological vector fields on $\text{gr} \circ T^{(n-1)}(\mathcal{M})$. We leave details of the proof to Part II of this paper. Note that in this case the functor parity change $\pi$ is included in the functor $V_n$, therefore we do not need to apply it explicitly. We conclude this section with the following proposition

**Proposition 28.** The functors $\iota \circ \pi \circ \text{gr} \circ T^{(n-1)}$ and $\iota' \circ \text{gr} \circ T^{(n-1)}$ are embeddings of the category of supermanifolds of odd dimension $n$ to the category of graded manifolds of degree $n$. 
Proof. The proposition follows from Proposition 24, from the fact that \( \pi \) is an equivalence of categories of \( n \)-fold vector bundles and from Theorems 20 and 21. \( \square \)

The inverse limit of functors \( F_n \) is denoted by \( F_\infty \). That is if \( \mathcal{M} \) is a supermanifold, we put \( F_\infty(\mathcal{M}) := \varprojlim F_n(\mathcal{M}) \) and the same for morphisms.

5. DONAGI–WITTEN FUNCTOR FOR LIE SUPERALGEBRAS AND LIE SUPERGROUPS

In this section, we adapt the results of Section 3 to the case of Lie superalgebras and Lie supergroups. We show that there is an analogue of the functor \( F_2 \) in the category of Lie superalgebras. We denote by \( \text{sLieAlg} \) the category of Lie superalgebras. We denote by \( \text{grLieAlg}_\infty \) the category of non-negatively \( \mathbb{Z} \)-graded Lie superalgebras of type \( \{0, 1, \ldots, n\} \) with \( |1| = 1 \) and by \( \text{grLieAlg}_\infty \) the category of non-negatively \( \mathbb{Z} \)-graded Lie superalgebras.

5.1. DONAGI AND WITTEN CONSTRUCTION FOR LIE SUPERALGEBRAS. In this section we will construct a functor \( F_2' : \text{sLieAlg} \to \text{grLieAlg}_2 \). In next sections we will use this construction to obtain a functor from the category of Lie superalgebras \( \text{sLieAlg} \) to the category \( \text{grLieAlg}_\infty \). As in [3] the functor \( F_2' \) is a composition of four functors, which we denote by \( T' \) (tangent), \( \text{gr}' \) (split), \( \pi' \) (parity change) and \( \iota' \) (inverse). (To distinguish the category of Lie superalgebras in the case we will use superscript. That is \( T' \) instead of \( T \) and so on.) Let \( g = g_0 \oplus g_1 \) be a Lie superalgebra. Recall that to define a Lie superalgebra we need to define a Lie algebra \( g_0 \), a \( g_0 \)-module \( g_1 \) and a symmetric \( g_0 \)-module map \([\cdot, \cdot] : g_1 \otimes g_1 \to g_0\) such that \([[Y, Y], Y] = 0\), for any \( Y \in g_1 \).

Tangent functor \( T' \). Since a Lie superalgebra \( g = g_0 \oplus g_1 \) is a vector superspace, its antitangent bundle is the following linear superspace

\[ T'(g) = g \oplus d(g). \]

Here \( dg \) denote a copy of \( g \) with reversed parity. In more details, \( V = g \oplus d(g) \) is a linear superspace with the underlying space \( V_0 = g_0 \oplus dg_1 \) and with the structure sheaf \( F_{V_0} \otimes S^*(g_1^0 \oplus dg_0^0) \). If \( X \in g \), sometimes we will denote by \( d(X) \) the corresponding element in \( d(g) \). Further the Lie superalgebra structure on \( T'(g) \) is defined as follows. The Lie bracket on the vector subspace \( g \oplus \{0\} \) coincides with the Lie bracket of \( g \), further \( d(g) \) is a \( g \)-module, where the action of \( g \) coincide with the adjoint action up to sign

\[ [X, dY] := (-1)^{|X|}d([X, Y]). \]

and the product \([d(g), d(g)] = 0\) is trivial.

Remark 29. The Lie superalgebra \( T'(g) \) sometimes is called in the literature a Takiff superalgebra, named after the author of [1]. This Lie superalgebra is a Lie superalgebra in the form \( g \otimes_{K} K[\tau] \), where \( \tau \) is an odd element and \( K[\tau] \) are all polynomials in \( \tau \). Summing up, in our paper \( T' \) is derived from both “Takiff” and “antitangent”.

Functor split \( \text{gr}' \). The functor \( \text{gr}' \) is defined as follows. For a Lie superalgebra \( g \) we put \( gr'(g) := g' \), where \( g' \) is obtained from the Lie superalgebra \( g \) putting \( g_1' = 0 \). In [12, Theorem 3], see also Theorem 39 below, it was shown that \( \text{Lie}(grG) = gr'(Lie G) \) for any Lie supergroup \( G \). In other words \( g' \) is the Lie superalgebra of \( gr(G) \), where \( G \) is a Lie supergroup with the Lie superalgebra \( g \). Clearly \( gr' \) is defined on morphisms as well. Let us compute the composition of the functors \( gr' \circ T' \).
Lemma 30. Let \( \mathfrak{g} \) be a Lie superalgebra. Then the Lie superalgebra \( \text{gr}'(T'(\mathfrak{g})) \) is equal to \( T'(\mathfrak{g}) \) as \( \mathfrak{g}_0 \)-modules. And the following holds true for the Lie superalgebra multiplication
\[
[g_1, g_1] = 0, \quad [g_1, d(g_0)] = 0, \quad [d(g), d(g)] = 0.
\]
and \( [Y_1, d(Y_2)] = -d([Y_1, Y_2]) \) for \( Y_1, Y_2 \in g_1 \).

Remark 31. Note that the Lie algebra \( \text{gr}T(\mathfrak{g}) \) keeps all the information on the bracket on \( \mathfrak{g} \). We can see the Lie superalgebra \( \mathfrak{a} = \text{gr}'(T'(\mathfrak{g})) \) as a \( \mathbb{Z} \times \mathbb{Z} \)-graded Lie superalgebra of type \( \{0, \alpha, \beta, \alpha + \beta\} \), where \( \alpha \) is odd and \( \beta \) is even. Indeed, we put
\[
a_0 := g_0, \quad a_\alpha := g_1, \quad a_\beta := dg_1, \quad a_{\alpha+\beta} := dg_0.
\]
Lemma 30 implies that the multiplication in \( \mathfrak{a} \) is \( \mathbb{Z} \times \mathbb{Z} \)-graded.

Functor parity change \( \pi' \). In previous sections we reminded how to define the functor \( \pi \) for double vector bundles. In this section we show that on the parity reversed double vector bundle \( \pi \circ \text{gr} \circ T(G) \), where \( G \) is a Lie supergroup, a Lie supergroup structure can be defined. We start with the case of Lie superalgebras.

We define the functor \( \pi' \) on the image of the composition of functors \( \text{gr}' \circ T' \) as follows. The Lie superalgebra \( \text{gr}'(T'(\mathfrak{g})) \) possesses a parity reversion of the weight \( \beta \). Indeed, by definition the Lie superalgebra \( \pi'(\text{gr}'(T'(\mathfrak{g}))) \) is equal to \( \text{gr}'(T'(\mathfrak{g})) \) as \( \mathfrak{g}_0 \)-modules, but now we assume that elements \( X \) and \( d(X) \) have the same parities. In other words we assume that \( d \) is even. More precisely, let \( \mathfrak{a} = \text{gr}'(T'(\mathfrak{g})) \). Denote by \( \mathfrak{h} \) the vector superspace \( \pi'(\text{gr}'(T'(\mathfrak{g}))) \). We have
\[
\mathfrak{h} = \bigoplus_{\beta \in \Delta} \mathfrak{h}_\beta,
\]
where \( \Delta = \{0, \alpha, \beta, \alpha + \beta\} \) with \( |\alpha| = |\beta| = 1 \). Let us define a Lie superalgebra structure on \( \mathfrak{h} \) of type \( \Delta \). We denote by \( [.,. \rceil_a \) the Lie bracket on \( \mathfrak{a} \) and by \( [.,. \rceil_\beta \) the Lie bracket on \( \mathfrak{h} \).

Proposition 32. The Lie superalgebra structure on \( \mathfrak{h} \) is defined by the following data.

- We set \( [X, Y]_a = [X, Y]_a \) for all homogeneous \( X, Y \), except for the case \( X \in \mathfrak{h}_a \) and \( Y \in \mathfrak{h}_\beta \).
- For \( X \in \mathfrak{h}_a \) and \( Y \in \mathfrak{h}_\beta \) we set \( [X, Y]_a = [X, Y]_a = -[Y, X]_a =: [Y, X]_\beta \).

Proof. Step 1. Clearly \( \mathfrak{h}_0 = \mathfrak{h}_0 \oplus \mathfrak{h}_{\alpha+\beta} \) is a Lie algebra, namely it is a semidirect product of the Lie algebra \( a_0 \) and \( a_0 \)-module \( a_{\alpha+\beta} \).

Step 2. \( \mathfrak{h}_1 \) is a \( \mathfrak{h}_0 \)-module. Clearly, \( \mathfrak{h}_1 \) is a \( \mathfrak{h}_0 \)-module and \( \mathfrak{h}_{\alpha+\beta} \) acts trivially on \( \mathfrak{h}_1 \).

Step 3. Note that the map \( \mathfrak{h}_1 \otimes \mathfrak{h}_1 \rightarrow \mathfrak{h}_0 \) induced by \( [,]_b \) is an \( \mathfrak{h}_0 \)-module map. It remains only to check Jacobi identity for homogeneous \( X, Y, Z \in \mathfrak{h}_1 = \mathfrak{h}_a \oplus \mathfrak{h}_\beta \). However the product of any three elements of this form is 0.

Later we will give another proof of this result. It is easy to check that \( \pi' \) is defined on the morphisms of the form \( \text{gr}' \circ T'(\phi) \), where \( \phi : \mathfrak{g} \rightarrow \mathfrak{g}_1 \) is a morphism of Lie superalgebras.

Functor inverse \( \iota' \). Above we explained the meaning of the functor inverse \( \iota \) for supermanifolds. Now we show that an analogue of this functor can be defined in the category of Lie superalgebras. Let \( \mathfrak{g} \) be a Lie superalgebra. Denote by
\[ p := \iota' \circ \pi' \circ \text{gr}' \circ T'(g) \] a \( \mathbb{Z} \)-graded subsuperspace of \( \mathfrak{h} := \pi'(\text{gr}'(T'(g))) \) with support \( \{0, 1, 2\} \), where \( |1| = 1 \), given by
\[
p_0 := h_0, \quad p_1 = \{ Y + dY \mid Y \in \mathfrak{g}_1 \} \subset h_\alpha \oplus h_\beta, \quad p_2 := h_{\alpha+\beta}.
\]

**Proposition 33.** The superspace \( p \) is a \( \mathbb{Z} \)-graded Lie superalgebra with support \( \{0, 1, 2\} \) and with \( |1| = 1 \).

**Proof.** The proposition follows from a direct calculation. For example let us show that \( [p_1, p_1] \subset p_2 \). For \( Y_1, Y_2 \in \mathfrak{g}_1 \) we have
\[
|Y_1 + dY_1, Y_2 + dY_2| = |Y_1, Y_2| + [dY_1, Y_2] + |Y_1, dY_2| + [dY_1, dY_2] = d|Y_1, Y_2| + d|Y_1, Y_2| = 2d|Y_1, Y_2| \in p_2.
\]
\[ \square \]

Note that the functor \( \iota' \) is defined only for Lie superalgebras of the form \( \pi' \circ \text{gr}' \circ T'(g) \). Further, if \( f : g \to \mathfrak{g}_1 \) is a morphism of Lie superalgebras, then the morphism \( \pi' \circ \text{gr}' \circ T'(f) \) can be restricted to the subalgebras \( \iota' \circ \pi' \circ \text{gr}' \circ T'(g) \to \iota' \circ \pi' \circ \text{gr}' \circ T'(\mathfrak{g}_1) \). Summing up, we constructed the following functor
\[
F'_2 := \iota' \circ \pi' \circ \text{gr}' \circ T' : \text{sLieAlg} \to \text{grLieAlg}_2.
\]
This functor is an embedding of category \( \text{sLieAlg} \) into \( \text{grLieAlg}_2 \).

**Remark 34.** (1) It is necessary to apply the functor \( \pi' \) before the functor inverse. Indeed, if \( Y \) and \( dY \), where \( Y \in p_1 \), have different parities, we will get
\[
|Y_1 + dY_1, Y_2 + dY_2| = |Y_1, Y_2| + [dY_1, Y_2] + |Y_1, dY_2| + [dY_1, dY_2] = d|Y_1, Y_2| - d|Y_1, Y_2| = 0.
\]

(2) Another observation is the following. Lie superalgebras of the form \( \text{gr}'(\mathfrak{k}) \), where \( \mathfrak{k} \) is a Lie superalgebra possesses another parity reversion
\[
\mathfrak{e}_0 \oplus \mathfrak{e}_1 \longmapsto \mathfrak{e}_0 \oplus \mathfrak{e}_1[\bar{1}],
\]
where \( \bar{1} \) is the shift of parity be 1. In other words we assume that all odd elements are even. Clearly \( \mathfrak{e}_0 \oplus \mathfrak{e}_1[\bar{1}] \) is a Lie algebra (not superalgebra). In this case the same argument, see (14), shows that the functor inverse loses information about original Lie superalgebra.

5.2. Donagi and Witten construction for Lie supergroups. In this section we develop the construction of Section 3 for Lie supergroups. In details, we apply the functor \( F_2 \) to a Lie supergroup \( G \). Again \( F_2 \) is a composition of four functors: \( T \) (tangent), \( \text{gr} \) (split), \( \pi \) (parity change) and \( \iota \) (inverse).

**Tangent functor** \( T \). Let \( G \) be a Lie supergroup with multiplication \( \mu \), inversion \( \kappa \) and identity \( e \). Clearly, \( T(G) \) is a Lie supergroup as the functor \( T \) preserves products. Indeed, since \( T \) is a functor the morphisms \( T(\mu) \), \( T(\kappa) \) and \( T(e) \) satisfy the Lie supergroup axioms.

**Proposition 35.** The Lie superalgebra of the Lie supergroup \( T(G) \) is equal to \( T'(g) \), where \( g = \text{Lie}(G) \).

**Proof.** We shall follow the definition of the Lie functor of a Lie supergroup given in [BLMS]. Let \( X_1, X_2 \in T_e G \) be homogeneous. Any homogeneous vector \( X \in T_e G \)
can be represented by a curve $g : \mathcal{R} \to \mathcal{G}$ where $\mathcal{R}$ is $\mathbb{R}^{1|0}$ or $\mathbb{R}^{0|1}$ depending on the parity of $X$, so that
\[
g^*(f) = f(e) + tX(f) \mod I_t^2,
\]
for any $f \in O_q(U)$ defined in a neighbourhood $U \subset \mathcal{G}_0$ of $e$, where $t$ is the distinguished coordinate on $\mathcal{R}$ of parity $0$ or $1$, and $I_t = (t)$ is the ideal generated by $t$. Say $X_1, X_2 \in T_e\mathcal{G}$ are represented by curves $g_i : \mathcal{R}_i \to \mathcal{G}$, where $i = 1, 2$ and $s, t$ be the distinguished coordinates on $\mathbb{R}_1, \mathbb{R}_2$, respectively. Then there exist $Z \in T_e\mathcal{G}$ such that for $\Phi := g_1g_2(g_1)^{-1}(g_2)^{-1} : \mathcal{R}_1 \times \mathcal{R}_2 \to \mathcal{G}$ we have
\[
\Phi^*(f) = f(e) + (-1)^{|X||Y|}tsZ(f) \mod I_t^2 + I_s^2.
\]
The bracket $[X_1, X_2]$ is defined as $Z$.

We are ready to describe the Lie superalgebra $\text{Lie}(T\mathcal{G})$. Let $(U, x^A)$ be a chart around $e \in U \subset \mathcal{G}_0$ with local coordinates $(x^A)$. Without loss of generality we may assume that $x^4(e) = 0$. The group law assures that the Taylor expansion of the multiplication map $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ has the form
\[
m((x^A), (\dot{x}^A)) = (x^A + xx^A + \frac{1}{2}Q_{BC}x^C\dot{x}^B \mod I),
\]
where $I = (\dot{x}^2, \dot{x}^2)$. Moreover, $Q_{BC}^2 = -(-1)^{\dot{B}\dot{C}}Q_{BC}$. The inverse $(x^A)^{-1}$ is given by $(-x^A)(\mod I)$. The multiplication on $T(\mathcal{G})$ is given by $T(m)$, i.e.
\[
(x^A, \dot{x}^A)(\dot{x}^A, \dot{x}^A) = \left( x^A + xx^A + \frac{1}{2}Q_{BC}x^C\dot{x}^B, \dot{x}^A + \dot{x}^A + \frac{1}{2}Q_{BC}(\dot{x}^C\dot{x}^B + x^C\dot{x}^B) \right).
\]
The inverse of $(x^A, \dot{x}^A)$ is $(-x^A, -\dot{x}^A)$ modulo $I$, hence
\[
(x^A, \dot{x}^A) \cdot (\dot{x}^A, \dot{x}^A) \cdot (\dot{x}^A, \dot{x}^A) \cdot (\dot{x}^A, \dot{x}^A) \equiv \left( x^A + xx^A + \frac{1}{2}Q_{BC}x^C\dot{x}^B, \dot{x}^A + \dot{x}^A + \frac{1}{2}Q_{BC}(\dot{x}^C\dot{x}^B + x^C\dot{x}^B) \right),
\]
\[
\equiv (Q_{BC}x^C\dot{x}^B, Q_{BC}(\dot{x}^C\dot{x}^B + x^C\dot{x}^B)) \pmod {I}.
\]
We read off from the last formula that $[\partial_x^B, \partial_x^B] = Q_{BC}^1\partial_x^A$, $[\partial_x^C, \partial_x^A] = Q_{BC}^A\partial_x^A$, and $[\partial_x^A, \partial_x^A] = 0$. This completes the proof. \( \square \)

The following theorem was proved \[36\].

**Theorem 36.** \[35\] Let $\mathcal{G}$ be a Lie supergroup corresponding to the super Harish-Chandra pair $(\mathcal{G}_0, \mathfrak{g})$, where $\mathcal{G}_0$ is the underlying space of $\mathcal{G}$ and $\mathfrak{g} = \text{Lie}(\mathcal{G})$. Then $\text{gr}(\mathcal{G})$ is a Lie supergroup corresponding to the following super Harish-Chandra pair $(\mathcal{G}_0, \mathfrak{g}')$, where $\mathfrak{g}' = \text{gr}'(\mathfrak{g})$.

Proposition \[35\] and Theorem \[36\] imply the following corollary.

**Corollary 37.** Let $\mathcal{G}$ be a Lie supergroup, then
\[
\text{gr}' \circ T'(\text{Lie} \mathcal{G}) = \text{Lie}(\text{gr} \circ T(\mathcal{G})).
\]

As a consequence we get that the Lie supergroup $\text{gr} \circ T(\mathcal{G})$ is a graded Lie supergroup of type $\{0, \alpha, \beta, \alpha + \beta\}$, where $|\alpha| = 1$ and $|\beta| = 0$, with the graded Harish-Chandra pair $(\mathcal{G}_0, \text{gr}' \circ T'(\mathfrak{g}))$ of the same type $\{0, \alpha, \beta, \alpha + \beta\}$. (Compare also with Proposition \[23\])
Lemma 38. Verify the following lemma.

The multiplication in $T\varepsilon T(G)$ is a group object in the category of double vector bundles. As we have seen in Section 5.1, the double vector bundle $gr\circ T(G)$ possesses the following parity reversion: we change the parity of the weight $\beta$ from even to odd. We denote this new vector bundle by $\pi \circ gr \circ T(G)$. Further we can define a Lie supergroup structure on $\pi \circ gr \circ T(G)$ using the graded Harish-Chandra pair $(g_0, \pi' \circ gr' \circ T(g))$ of type $\Delta = \{0, \alpha, \beta, \alpha + \beta\}$, where $|\alpha| = |\beta| = 1$. Now the Lie supergroup $\pi \circ gr \circ T'(G)$ is defined.

Let us describe the Lie supergroup morphisms of $\pi \circ gr \circ T'(G)$ using the language of double vector bundles. If $\mu, \kappa$ and $e$ are group morphisms of the Lie supergroup $G$, then $gr \circ T(G)$ is a group object in the category of double vector bundles with the structure morphisms $gr \circ T(\mu)$, $gr \circ T(\kappa)$ and $gr \circ T(e)$. Since the category of double vector bundles possesses the parity change: $\beta$ even to $\beta$ odd, we denote by $\pi \circ gr \circ T(G)$ with $\pi \circ gr \circ T(\mu)$, $\pi \circ gr \circ T(\kappa)$ and $\pi \circ gr \circ T(e)$ the result of this parity change. Formulas (6) tells us that this definition coincides with the definition in terms of graded Harish-Chandra pairs.

Functor inverse $\iota$. In Proposition 33 we saw that the Lie superalgebra $\pi' \circ gr' \circ T'(g)$ possesses a $\mathbb{Z}$-graded Lie subsuperalgebra $\mathfrak{p} = \iota' \circ \pi' \circ gr' \circ T'(g)$. We denote by $\iota \circ \pi \circ gr \circ T(G)$ the corresponding Lie subsupergroup. More precisely, we define $\iota \circ \pi \circ gr \circ T(G)$ using the graded Harish-Chandra pairs $(g_0, \iota' \circ \pi' \circ gr' \circ T'(g))$.

6. Generalized Donagi–Witten construction for Lie superalgebras

In Section 4 we constructed an injective functor $F := F_\infty$ from the category of supermanifolds to the category of graded manifolds. In this section we show that this functor can be defined in the category of Lie supergroups. We start with Lie superalgebras. In more details we will construct a functor $F'$ from the category of Lie superalgebras $\mathfrak{g}$ to the category of non-negatively $\mathbb{Z}$-graded Lie algebras $gr\mathfrak{g}_{\mathbb{Z}_{\geq 0}}$. Further we will use these results for the category of Lie supergroups. Again the functor $F'$ is a composition of four functors: the iterated antitangent functor $T^\infty$, the functor split $\iota'$ for the parity change $\pi'$ and the functor inverse $\iota'$.

6.1. Iterated tangent functor $T^\infty$. Let $g$ be a Lie superalgebra. Let us describe the superalgebra $T^\infty(g)$, where $T^\infty := T' \circ T' \circ \cdots$ is the infinitely many times iterated antitangent functor. Let us consider first twice iterated tangent functor $T^{2}(g)$. By definition

$$T^{2}(g) = T'(T'(g)) = T'(g \oplus d_1(g)) = g \oplus d_1(g) \oplus d_2(g \oplus d_1(g)).$$

We replaced $d$ from Section 5.1 by $d_1$ and $d_2$ is the second de Rham differential. The multiplication in $T' \circ T'(g)$ is defined in a natural way. Moreover we can easily verify the following lemma.

Lemma 38. To obtain the multiplication in $T^{2}(g)$ we can use the following rule: $[d_1X, d_1Y] = (-1)^{|X|}d_1d_2([X, Y])$ for any $X, Y \in T^{2}(g)$.

Corollary 39. Since the operators $d_1$ assumed to be odd, we have $d_1d_1([X, Y]) = 0$ for any $X, Y \in T^{2}(g)$. 
To define the functor $T^{\infty}$ we use de Rham differentials $d_1, d_2, \ldots, d_n, \ldots$. Lemma 39 and Corollary 39 imply the following lemma.

**Lemma 40.** The Lie superalgebra $T^{\infty}(g)$ is an infinite dimensional Lie superalgebra with the underlying vector space
\[
\bigoplus_{p \geq 0} \bigoplus_{i_1 < \cdots < i_p} d_{i_1} \cdots d_{i_p}(g)
\]
and multiplication defined by the following formula
\[
[d_i X, d_j Y] = (-1)^{|X|} d_i d_j ([X, Y])
\]
for any $X, Y \in T^{\infty}(g)$.

### 6.2. A connection with the functor of points for Lie superalgebras

For details about the functor of points for Lie superalgebras we refer for example to [Gav Section 2.2.4]. Let us recall this construction. To any Lie superalgebra $g$ we can associate a functor $L_g$ from the category of supercommutative algebras to the category of Lie algebras. It is defined as follows
\[
L_g(A) = (A \otimes g)_0,
\]
where $A$ is a super-commutative algebra. The product $A \otimes g$ is a Lie superalgebra with the following multiplication
\[
[a \otimes X, a' \otimes X'] := (-1)^{|X'||a|} a' \otimes [X, X'].
\]
Comparing with Lemma 39 we see that the Lie superalgebra $T^{\infty}(g)$ is isomorphic to the Lie superalgebra $\bigwedge(\xi_1, \ldots, \xi_k) \otimes g$. Clearly, $T^{\infty}(g)$ is isomorphic to $\bigwedge(\xi_1, \xi_2, \ldots) \otimes g$, where $\bigwedge(\xi_1, \xi_2, \ldots)$ is the Grassmann algebra with infinitely many variables $\xi_1, \xi_2, \ldots$. Now we see that
\[
L_g(\bigwedge(\xi_1, \xi_2, \ldots)) = T^{\infty}(g)_0.
\]

### 6.3. Functors split $\text{gr}'$ and parity change $\pi'$. The functor split is defined as above.

**Remark 41.** The Lie superalgebra $\text{gr}' \circ T^{\infty}(g)$ is a graded Lie superalgebra with support $\Delta$, where $\Delta$ is the maximal multiplicity free weight system generated by $\alpha, \beta_1, \ldots, \beta_k, \ldots$. Here $\alpha$ is odd, while $\beta_1, \ldots, \beta_k, \ldots$ are even. The grading is defined as follows. We assign the weight $\alpha + \beta_i + \cdots + \beta_p$ to $d_{i_1} \cdots d_{i_p}(Z)$, if $d_{i_1} \cdots d_{i_p}(Z)$ is odd, and we assign the weight $\beta_i + \cdots + \beta_p$ to $d_{i_1} \cdots d_{i_p}(Z)$ if $d_{i_1} \cdots d_{i_p}(Z)$ is even.

For example elements of the subspace $g_0$ have weight 0 and elements of $g_1$ have weight $\alpha$.

Let us define the parity change functor $\pi'$. Let us take the Lie superalgebra $\text{gr}'(T^{\infty}(g))$, where $g$ is a Lie superalgebra. The functor parity change $\pi'$ is defined as follows
\[
\mathfrak{h} := \pi'(\text{gr}'(T^{\infty}(g))) = \text{gr}'(T^{\infty}(g))
\]
as $g_0$-modules. Further we assume that all operators $d_i$ are even and again $d_i \circ d_i = 0$. In other words, this means that
\[
\mathfrak{h}_0 := \bigoplus_{p \geq 0} \bigoplus_{i_1 < \cdots < i_p} d_{i_1} \cdots d_{i_p}(g_0), \quad \mathfrak{h}_1 := \bigoplus_{p \geq 0} \bigoplus_{i_1 < \cdots < i_p} d_{i_1} \cdots d_{i_p}(g_1).
\]
To simplify our presentation we will use the following notations
\[ I = \{i_1, \ldots, i_p\}, \quad J = \{j_1, \ldots, j_q\}, \quad K = \{k_1, \ldots, k_r\}, \]
and \( d_i \) for \( d_{i_1} \cdots d_{i_p} \). We denote by \( \sharp I := C(I) \mod 2 = \bar{p} \) the cardinality of \( I \) modulo 2. The multiplication in \( \mathfrak{h} = \pi'(\text{gr}^r(T^{\infty}_{\infty}(\mathfrak{g}))) \) is defined by the following rules

(Rule 1) If \( I \cap J \neq \emptyset \), we have \([d_I(\mathfrak{g}), d_J(\mathfrak{g})] = \{0\}\).

(Rule 2) If \( \sharp I + \bar{p} \) and \( \sharp J + \bar{q} \) are odd, we have \([d_I(\mathfrak{g}), d_J(\mathfrak{g})] = \{0\}\).

(Rule 3) In other cases we have \([d_I(\mathfrak{g}), d_J(\mathfrak{g})] = d_{I \cup J}(\{[d_I, d_J]\})\).

**Theorem 42.** The superspace \( \mathfrak{h} = \pi'(\text{gr}^r(T^{\infty}_{\infty}(\mathfrak{g}))) \) is a Lie superalgebra.

**Proof.** **Step 1.** Let us prove first that \( \mathfrak{h}_0 \) is a Lie algebra. Consider the following subspace
\[ \mathfrak{r} := \bigoplus_{p \geq 0} \bigoplus_{i_1 < \ldots < i_p} d_{i_1} \cdots d_{i_p}(\mathfrak{g}_0) \subseteq \text{gr}^r(T^{\infty}_{\infty}(\mathfrak{g})). \]
Clearly, \( \mathfrak{r} \) is a Lie subsuperalgebra in \( \text{gr}^r(T^{\infty}_{\infty}(\mathfrak{g})) \) and this Lie superalgebra is split. This is \( \mathfrak{r} = \mathfrak{r}_0 \oplus \mathfrak{r}_1 \) with \([\mathfrak{r}_0, \mathfrak{r}_1] = \{0\}\). Such Lie superalgebras possesses a parity reversion, see Remark [24] part (2). We can see that in notations of Remark [24] we have \( \mathfrak{r}_0 \oplus \mathfrak{r}_1[1] = \mathfrak{h}_0 \).

**Step 2.** Let us prove that \( \mathfrak{h}_1 \) is an \( \mathfrak{h}_0 \)-module. Let us take \( d_I(X) \in d_I(\mathfrak{g}_0), \) \( d_J(Y) \in d_J(\mathfrak{g}_0) \) and \( d_K(Z) \in d_K(\mathfrak{g}_1) \) with \( I \cap J = \emptyset, I \cap K = \emptyset \) and \( J \cap K = \emptyset \). We consider the following cases

(1) Let \( \sharp I = \sharp J = 0 \), any \( K \). Then
\[ [d_I(X), [d_J(Y), d_K(Z)]] + [d_J(Y), [d_K(Z), d_I(X)]] + [d_K(Z), [d_I(X), d_J(Y)]] = d_I \circ d_J \circ d_K([X, [Y, Z]]) + d_J \circ d_K \circ d_I([Y, [Z, X]]) + d_K \circ d_I \circ d_J([Z, [X, Y]]) = d_I \circ d_J \circ d_K([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]) = 0. \]

(2) Let \( \sharp I = 1, \sharp J = 0 \) and \( \sharp K = 0 \). Then
\[ [d_I(X), [d_J(Y), d_K(Z)]] = [d_J(Y), [d_K(Z), d_I(X)]] = [d_K(Z), [d_I(X), d_J(Y)]] = 0. \]

(3) Let \( \sharp I = \bar{1}, \sharp J = 0 \) and \( \sharp K = \bar{1} \). Then
\[ [d_I(X), [d_J(Y), d_K(Z)]] + [d_J(Y), [d_K(Z), d_I(X)]] + [d_K(Z), [d_I(X), d_J(Y)]] = d_I \circ d_J \circ d_K([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]) = 0. \]

(4) Let \( \sharp I = \sharp J = \bar{1} \) and \( \sharp K = 0 \). Then
\[ [d_I(X), [d_J(Y), d_K(Z)]] = [d_J(Y), [d_K(Z), d_I(X)]] = [d_K(Z), [d_I(X), d_J(Y)]] = 0. \]

(5) Let \( \sharp I = \sharp J = \sharp K = 1 \). Then
\[ [d_I(X), [d_J(Y), d_K(Z)]] = [d_J(Y), [d_K(Z), d_I(X)]] = [d_K(Z), [d_I(X), d_J(Y)]] = 0. \]

**Step 3.** Let us check Jacobi identity for elements from \( \mathfrak{h}_1 \). Let us take \( d_I(X) \in d_I(\mathfrak{g}_1), d_J(Y) \in d_J(\mathfrak{g}_1) \) and \( d_K(Z) \in d_K(\mathfrak{g}_1) \) with \( I \cap J = \emptyset, I \cap K = \emptyset \) and \( J \cap K = \emptyset \) and consider the following cases

(1) Let \( \sharp I = \sharp J = 0 \), any \( K \). Then
\[ [d_I(X), [d_J(Y), d_K(Z)]] = [d_J(Y), [d_K(Z), d_I(X)]] = [d_K(Z), [d_I(X), d_J(Y)]] = 0. \]
(2) Let $\sharp I = 0$, $\sharp J = 1$ and $\sharp K = 1$. Then
\[
[d_I(X), d_J(Y), d_K(Z)] + [d_J(Y), d_K(Z), d_I(X)] + [d_K(Z), d_I(X), d_J(Y)] =
\]
\[
d_I \circ d_J \circ d_K([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]) = 0.
\]

(3) Let $\sharp I = \sharp J = \sharp K = 1$. Then again
\[
[d_I(X), d_J(Y), d_K(Z)] + [d_J(Y), d_K(Z), d_I(X)] + [d_K(Z), d_I(X), d_J(Y)] =
\]
\[
d_I \circ d_J \circ d_K([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]) = 0.
\]
The proof is complete. □

**Remark 43.** It is unexpected that such a parity change can give a well-defined Lie superalgebra.

6.4. **Functor inverse** $\iota'$. Our goal now is to define a $\mathbb{Z}$-graded subsuperalgebra $p := \iota' \circ \pi' \circ \text{gr}^\infty(\mathfrak{g})$ in $\pi' \circ \text{gr}^\infty(\mathfrak{g})$. We put
\[
\begin{align*}
p_0 & := \mathfrak{g}_0; \\
p_1 & := \text{diag}(\mathfrak{g}_1 \oplus \bigoplus_i d_i(\mathfrak{g}_1)); \\
p_2 & := \text{diag}(\bigoplus_i d_i(\mathfrak{g}_0) \oplus \bigoplus_{i<j} d_i d_j(\mathfrak{g}_0)); \\
p_3 & := \text{diag}(\bigoplus_{i<j} d_i d_j(\mathfrak{g}_1) \oplus \bigoplus_{i<j<k} d_i d_j d_k(\mathfrak{g}_1)); \\
& \quad \vdots \\
p_n & := \text{diag}(\bigoplus_{C(I) = n-1} d_I(\mathfrak{g}_n) \oplus \bigoplus_{C(J) = n} d_J(\mathfrak{g}_n)); \\
& \quad \vdots
\end{align*}
\]
Here $C(I)$ is the cardinality of $I$.

**Proposition 44.** The subsuperspace $p$ is a $\mathbb{Z}$-graded Lie subsuperalgebra.

**Proof.** Let us prove that $[p_i, p_j] \subset p_{i+j}$. Let us take $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$. Consider
\[
\begin{bmatrix}
\sum_{C(I) = i-1} d_I(X) + \sum_{C(J) = i} d_J(X), & \sum_{C(I) = j-1} d_I(Y) + \sum_{C(J) = j} d_J(Y)
\end{bmatrix}.
\]
Using Rules (2) we get
\[
\begin{bmatrix}
\sum_{C(I) = i-1} d_I(X), & \sum_{C(I) = j-1} d_I(Y)
\end{bmatrix} = 0.
\]
Further,

\[
\begin{align*}
\left[ \sum_{C(J)=i-1} d_{I}(X), \sum_{C(J')}=j} d_{J}(Y) \right] + \left[ \sum_{C(J)=i} d_{I}(X), \sum_{C(J')}=j-1} d_{J}(Y) \right] + \\
\left[ \sum_{C(J)=i} d_{I}(X), \sum_{C(J')}=j} d_{J}(Y) \right] = \\
\underbrace{\left( \binom{i+j-1}{i} + \binom{i+j-1}{j-1} \right)}_{(\ast)} \sum_{C(K)=i+j-1} d_{K}([X,Y]) + \\
\left( \binom{i+j}{i} \right) \sum_{C(K)=i+j} d_{K}([X,Y]).
\end{align*}
\]

The proof is complete. \(\square\)

If we have a morphism \(\psi\) of Lie superalgebras, then \(\pi' \circ \text{gr}' \circ T'^{\infty}(\psi)\) preserves subalgebras \(p\)'s. Therefore we can define \(\iota'\) on morphisms. Therefore the functor \(\iota'\) is defined on the image of the functor \(\pi' \circ \text{gr}' \circ T'^{\infty}\). Summing up, we constructed the following functor

\[
F':=\iota' \circ \pi' \circ \text{gr}' \circ T'^{\infty}: \text{sLieAlg} \to \text{grLieAlg}.
\]

**Remark 45.** The Lie superalgebra \(p\) is “locally isomorphic” to \(g\) in the following sense. We have \(p_0 \approx \bar{g}_0\) as Lie superalgebras and \(p_n \approx \bar{g}_n\) as \(\bar{g}_0\)-modules for any \(n\).

### 6.5. The functor \(F_{\infty}\) as a inverse limit

We can define the Lie superalgebras \(T'^{\infty}(g)\), \(\text{gr}'(T'^{\infty}(g))\), \(\pi'(\text{gr}'(T'^{\infty}(g)))\) and \(\iota'(\pi'(\text{gr}'(T'^{\infty}(g))))\) using the inverse limit. Indeed,

\[
\begin{align*}
T'^{\infty}(g) &= \lim_{\leftarrow} T'^{n}(g); \\
\text{gr}'(T'^{\infty}(g)) &= \lim_{\leftarrow} \text{gr}'(T'^{n}(g)); \\
\pi'(\text{gr}'(T'^{\infty}(g))) &= \lim_{\leftarrow} \pi'(\text{gr}'(T'^{n}(g))); \\
\iota'(\pi'(\text{gr}'(T'^{\infty}(g)))) &= \lim_{\leftarrow} \iota'(\pi'(\text{gr}'(T'^{n}(g)))).
\end{align*}
\]

Lemma 40 implies (21). Further, Equality (22) follows from definition of the functor \(\text{gr}'\). To obtain (23) we can repeat Theorem 42 assuming that we have only \(n\) de Rham differentials \(d_i\). Finally (24) follows from Proposition 44 again assuming that we have only \(n\) de Rham differentials \(d_i\). Summing up,

\[
F'(g) = F_{\infty}'(g) = \lim_{\leftarrow} F_{n}'(g).
\]

### 7. Generalized Donagi–Witten construction for Lie supergroups

Again the functor \(F\) is a composition of four functors: the iterated tangent functor \(T^{\infty}\), the functor split \(\text{gr}\), the functor parity change \(\pi\) and the functor inverse \(\iota\). Let \(G\) be a Lie supergroup with the supergroup morphisms \(\mu\), \(\kappa\) and \(e\).
7.1. Iterated antitangent functor $T^\infty$. Above we considered the tangent functor $T$ applying to a Lie supergroup $G$. We saw that $T(G)$ is a Lie supergroup again. Therefore we can iterate this procedure and get the Lie supergroup

$$T^2(G) := T(T(G)).$$

By definition we put

$$T^n(G) := T \circ \cdots \circ T(G),$$

where on the left hand side the tangent functor $T$ is iterated $n$ times. Now we define

$$T^\infty(G) = \lim_{n \to \infty} T^n(G),$$

$$T^\infty(\mu) = \lim_{n \to \infty} T^n(\mu), \quad T^\infty(\kappa) = \lim_{n \to \infty} T^n(\kappa), \quad T^\infty(e) = \lim_{n \to \infty} T^n(e),$$

see Section 2.2. Clearly the infinite dimensional supermanifold $T^\infty(G)$ is a Lie supergroup, since the morphisms $T^\infty(\mu)$, $T^\infty(\kappa)$ and $T^\infty(e)$ satisfy the group axioms.

Let us describe $T^\infty(G)$ in terms of graded Harish-Chandra pairs. Above we saw that

$$T^n(n) \simeq \bigwedge (\xi_1, \xi_2, \ldots, \xi_n) \otimes g$$

and

$$T^\infty(n) \simeq \bigwedge (\xi_1, \xi_2, \ldots) \otimes g,$$

where $\bigwedge (\xi_1, \xi_2, \ldots)$ is the Grassmann algebra with infinitely many variables $\xi_1, \xi_2, \ldots$ labeled by natural numbers. We can identify the structure sheaf $\mathcal{O}_{T^\infty(G)}$ of $T^\infty(G)$ with

$$\text{Hom}'_{U(\mathfrak{g}_0)}(U(T^\infty(g)), F_{\mathfrak{g}_0}) \subset \text{Hom}_{U(\mathfrak{g}_0)}(U(T^\infty(g)), F_{\mathfrak{g}_0}),$$

where $\text{Hom}'$ are all $U(\mathfrak{g}_0)$-homomorphisms that are zero on an ideal generated by

$$\bigoplus_{q > 0} \bigwedge (\xi_n, \xi_{n+1}, \ldots) \otimes g \subset U(T^\infty(g))$$

for some $n \geq 1$.

7.2. Functor split $\text{gr}$, functor parity change $\pi$ and functor inverse $\iota$. The functor $\text{gr}$ for $T^\infty(G)$ is defined as above. More precisely, let $J$ be the sheaf of ideals in $\mathcal{O}_{T^\infty(G)}$ generated by odd elements. We get a filtration in $\mathcal{O}_{T^\infty(G)}$ by the subsheaves $J^p$, where $p \geq 0$. The corresponded graded sheaf we denote by $\text{gr}\mathcal{O}_{T^\infty(G)}$. By definition $\text{gr}\mathcal{O}_{T^\infty(G)}$ is the structure sheaf of $\text{gr}(T^\infty(G))$. In terms of graded Harish-Chandra pairs we have

$$\mathcal{O}_{\text{gr}(T^\infty(G))} = \text{Hom}'_{U(\mathfrak{g}_0)}(U(\text{gr}' \circ T^\infty(g)), F_{\mathfrak{g}_0}),$$

where $\text{Hom}'$ are all $U(\mathfrak{g}_0)$-homomorphisms that are zero on some ideal generated by

$$\text{gr}' \left( \bigoplus_{q > 0} \bigwedge (\xi_n, \xi_{n+1}, \ldots) \otimes g \right) \subset U(\text{gr}' \circ T^\infty(g)), \quad n \geq 1.$$

The supermanifold $\text{gr}(T^\infty(G))$ is an $\infty$-fold vector bundle of type $\Delta$, where $\Delta$ is generated by an odd weight $\alpha$ and even weights $\beta_1, \beta_2, \ldots$, see Proposition 23. Since $\text{gr}' \circ T^\infty(g))$ is a graded Lie superalgebra of type $\Delta$, the universal enveloping algebra

$$U(\text{gr}' \circ T^\infty(g)) = \bigoplus_{\delta \in \mathbb{Z} \times \mathbb{Z} \times \cdots} U(\text{gr}' \circ T^\infty(g))_{\delta}$$

is $\mathbb{Z} \times \mathbb{Z} \times \cdots$-graded. We have

$$\mathcal{O}_{\text{gr}(T^\infty(G))} = \bigoplus_{\delta \in \mathbb{Z} \times \mathbb{Z} \times \cdots} (\mathcal{O}_{\text{gr}(T^\infty(G)))_{\delta}.$$


where

\[(\mathcal{O}_{gr(T^n(G))})_\delta = \text{Hom}_{U(g_0)}(U(gr' \circ T^{\infty}(g))_\delta, F_{\bar{\nu}_0}).\]

Note that in this formula we can omit \(\prime\) and write simply Hom. We can use the equality (22) as a definition of the structure sheaf \(\mathcal{O}_{gr(T^{\infty}(G))}\). Since gr is a functor, the structure morphisms in \(\text{gr}(T^{\infty}(G))\) are graded as well. In terms of inverse limit we have \(\text{gr}(T^{\infty}(G)) = \varprojlim \text{gr}(T^n(G)).\)

To define the functor \(\pi\) we change parities in any \(gr \circ T^n(G)\). In terms of graded Harish-Chandra pairs we get

\[\pi_{\text{gr}g}(T^{\infty}(G)) = \text{Hom}_{U(g_0)}(U(\pi' \circ gr' \circ T^{\infty}(g)), F_{\bar{\nu}_0}),\]

where \(\pi'\) are all \(U(g_0)\)-homomorphisms that are zero on some ideal generated by \(n\geq 1\).

In terms of inverse limit again we have \(\pi \circ gr(T^{\infty}(G)) = \varprojlim \pi \circ gr(T^n(G)).\) The Lie supergroup \(\pi \circ gr \circ T^{\infty}(G)\) is a graded Lie supergroup of type \(\pi(\Delta)\), where \(\pi(\Delta)\) is the maximal multiplicity free system generated by odd weights \(\alpha, \beta, \ldots\).

A similar idea we use for functor \(\iota\). Recall that \(F := \iota \circ \pi \circ gr \circ T^{\infty}\) and we denote \(F_{\bar{\nu}} := \iota \circ \pi \circ gr \circ T^{n-1}\). First of all we define the graded Lie supergroup \(F_{\bar{\nu}}(G)\) of degree \(n\) (that is of type \(\{0, 1, \ldots, n\}\) with \(|l| = 1\) as the \(\mathbb{Z}\)-graded Lie supergroup corresponding to the \(\mathbb{Z}\)-graded Harish-Chandra pair \((G_0, F'_{\bar{\nu}}(g))\). For any \(n\) we have a natural homomorphism \(F_{n+1}(g) \rightarrow F'_{n}(g)\). This induces a homomorphism of enveloping algebras

\[\Phi_n : U(F'_{n+1}(g)) \rightarrow U(F'_n(g)).\]

Therefore we have a natural map of sheaves

\[\mathcal{O}_{F_{n}(G)} = \text{Hom}_{U(g_0)}(U(F'_n(g)), F_{\bar{\nu}_0}) \leftarrow \mathcal{O}_{F_{n+1}(G)} = \text{Hom}_{U(g_0)}(U(F'_{n+1}(g)), F_{\bar{\nu}_0}).\]

Now we can identify the structure sheaf of \(F(G)\) with

\[\mathcal{O}_{F(G)} = \text{Hom}_{U(g_0)}(U(F'(g)), F_{\bar{\nu}_0}),\]

where \(\text{Hom}'\) are all \(U(g_0)\)-homomorphisms that are zero on some \(\text{Ker}(\Phi_n)\). Further we see that \(\mathcal{O}_{F(G)}\) is \(\mathbb{Z}\)-graded. Indeed,

\[\mathcal{O}_{F(G)} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{O}_{F(G)})_n = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{U(g_0)}(U(F'(g))_n, F_{\bar{\nu}_0}).\]

The graded Lie supergroup morphisms can be defined by formulas (9) or using inverse limit. In terms of inverse limit again we have \(F(G) = \varprojlim F_n(G)\).

8. Coverings and semicoverings of a Lie superalgebra and a Lie supergroup

In this section we give a definition of a covering and a semicovering of a Lie superalgebra and a Lie supergroup. Further we show that the generalized Donagi–Witten construction leads to a construction of a covering and semicovering spaces of a Lie superalgebra and a Lie supergroup. The case of any supermanifold will be considered in [RV].

8.1. Coverings and semicoverings of a Lie superalgebra. We start with Lie superalgebras. Throughout this subsection we fix a surjective homomorphism \(\phi : A \rightarrow B\) of abelian groups.
8.1.1. A $\phi$-covering of a $B$-graded Lie superalgebra along a homomorphism $\phi : A \to B$.

**Definition 46.** A $\phi$-covering of a $B$-graded Lie superalgebra $g$ along a surjective homomorphism $\phi : A \to B$ of abelian groups is an $A$-graded superalgebra $p = \bigoplus_{\alpha \in A} p_{\alpha}$ together with a homomorphism $\Pi' : p \to g$ such that $\Pi'|_{p_{\alpha}} : p_{\alpha} \to g_{\phi(\alpha)}$ is a linear bijection for any $\alpha \in A$.

Note that the bracket on $p$ is fully determined by the bracket on $g$. Indeed, for $X \in p_{\alpha}$, $X' \in p_{\alpha'}$ we have $\Pi'(\,[X,X']_{p}) = \,[\Pi'(X),\Pi'(X')]_{g} \in g_{\phi(\alpha+\alpha')}$, hence

$$[X,Y]_{p} = (\Pi'_{\alpha+\alpha'}|_{p_{\alpha+\alpha'}})^{-1}(\,[\Pi'(X),\Pi'(X')]_{g}) \,.$$

Also $\Pi'_0 = \Pi'|_{p_0} : p_0 \to g_{0}$ is a $g_0$-module map, where we identify the Lie superalgebras $p_0$ and $g_0$ via $\Pi_0$.

**Proposition 47.** Let $a$, $g$ be an $A$- and a $B$-graded Lie superalgebras, respectively. Let $\psi : a \to g$ be a $B$-graded homomorphism of Lie superalgebrasootnote{Any $A$-graded Lie superalgebra is automatically $B$-graded.}, and let $p$ be a covering of $g$ along $\phi : A \to B$. Then there exists a unique $A$-graded homomorphism $\Psi : a \to p$ such that the following diagram is commutative

\[
\begin{array}{ccc}
p & \xrightarrow{\Psi} & p' \\
\downarrow & & \downarrow \\
a & \xrightarrow{\psi} & g
\end{array}
\]

*Proof.* We define $\Psi$ as a linear map such that $\Psi(a_s) \subset p_s$ for any $s \in A$ and such that $\Pi' \circ \Psi = \psi$. Let us check that $\Psi$ is a homomorphism. Indeed, let us take $X \in a_s$ and $Y \in a_t$. Then

$$\Pi'([\Psi(X),\Psi(Y)]) = [\Pi' \circ \Psi(X),\Pi' \circ \Psi(Y)] = [\psi(X),\psi(Y)] = \psi([X,Y]) = \Pi' \circ \Psi([X,Y]).$$

Since by definition of $\Psi$ both $\Psi([X,Y])$ and $[\Psi(X),\Psi(Y)]$ are in $p_{s+t}$ and $\Pi'$ is locally bijective, we get the equality $[\Psi(X),\Psi(Y)] = \Psi([X,Y])$. □

**Proposition 48.** Let $f : g \to \tilde{g}$ be a homomorphism of $B$-graded Lie superalgebras. Then there exists unique homomorphism $F$ of $\phi$-coverings $p$ and $\tilde{p}$ such that the following diagram is commutative

\[
\begin{array}{ccc}
p & \xrightarrow{F} & \tilde{p} \\
\downarrow & & \downarrow \\
g & \xrightarrow{f} & \tilde{g}
\end{array}
\]

*Proof.* It follows immediately from Proposition 47, just take $\psi = f \circ \Pi'$. □

From Proposition 48 it follows that $\phi$-coverings are unique up to isomorphism.
8.1.2. A \(\phi\)-covering and \(\phi\)-semicovering with support \(C\). Let \(A\) and \(B\) be abelian groups, \(\phi : A \to B\) be a surjective homomorphism and \(C \subset A\) be a subset.

**Definition 51.** A \(\phi\)-covering with support \(C\) of a \(B\)-graded Lie superalgebra \(g\) along a surjective homomorphism \(\phi : A \to B\) is an \(A\)-graded superalgebra \(p = \oplus_{\alpha \in A} p_{\alpha}\) with \(\text{supp}(p) = C\) such that \(\phi(C) = B\) together with a surjective homomorphism \(\Pi' : p \to g\) such that \(\Pi'|_{p_{\alpha}} : p_{\alpha} \to g_{\phi(\alpha)}\) is a linear bijection for any \(\alpha \in C\).

Let \(A\), \(B\), \(C\) and \(\phi\) be as above, \(g\) and \(g'\) be a \(A\)-graded and \(B\)-graded Lie superalgebra, respectively.

**Definition 50.** A map \(\Psi : g \to g'\) is called a partial homomorphism if \(\Psi([X,Y]) = [\Psi(Y), \Psi(Y)]\) for any \(X \in g_\alpha\), \(Y \in g_\beta\) such that \(\alpha\), \(\beta\) and \(\alpha + \beta\) are in \(C\).

**Definition 51.** A \(\phi\)-semicovering with support \(C\) of a \(B\)-graded Lie superalgebra \(g\) along a surjective homomorphism \(\phi : A \to B\) is an \(A\)-graded Lie superalgebra \(p = \oplus_{\alpha \in C} p_{\alpha}\) with \(\text{supp}(p) = C\) such that \(\phi(C) = B\) together with a surjective partial homomorphism \(\Pi' : p \to g\) such that \(\Pi'|_{p_{\alpha}} : p_{\alpha} \to g_{\phi(\alpha)}\) is a linear bijection for any \(\alpha \in C\).

For a \(\phi\)-covering and \(\phi\)-semicovering with support \(C\) of a \(B\)-graded Lie superalgebra \(g\) we can prove analogues of Propositions \([47] \text{and} [48]\).

**Proposition 52.** (1) Let \(a\), \(g\) be an \(A\)- and \(B\)-graded Lie superalgebras, respectively, and \(\text{supp}(a) = C\). Let \(\psi : a \to g\) be a \(B\)-graded homomorphism of Lie superalgebras, and let \(p\) be a \(\phi\)-covering (or \(\phi\)-semicovering) with support \(C\) of \(g\). Then there exists a unique \(A\)-graded homomorphism \(\Psi : a \to p\) such that \(\psi = \Pi' \circ \Psi\).

(2) Let \(f : g \to g'\) be a homomorphism of \(B\)-graded Lie superalgebras. Then there exists unique homomorphism \(F\) of \(\phi\)-coverings (or \(\phi\)-semicoverings) \(p\) and \(p'\) with support \(C\) such that \(f \circ \Pi' = \Pi' \circ F\).

*Proof.* We define \(\Psi\) as a linear map such that \(\Psi(a_s) \subset p_s\) for any \(s \in C\) and such that \(\Pi' \circ \Psi = \psi\). Now we just repeat arguments of the proofs of Propositions \([47]\) and \([48]\). One non-trivial point is the proof that \(\Psi\) is a homomorphism in the case of a semicovering. We have for \(X \in a_s\) and \(Y \in a_t\), where \(s, t, s + t \in C\),

\[
\Pi'([\Psi(X), \Psi(Y)]) = [\Pi' \circ \Psi(X), \Pi' \circ \Psi(Y)] = [\psi(X), \psi(Y)] = \psi([X,Y]) = \Pi' \circ \Psi([X,Y]).
\]

Since by definition of \(\Psi\) both \(\Psi([X,Y])\) and \(\Pi'([X,Y])\) are in \(p_{s+t}\) and \(\Pi'\) is locally bijective, we get the equality \([\Psi(X), \Psi(Y)] = \Psi([X,Y])\). In the case \(s, t \in C\), but \(s + t \notin C\), we have

\[
[\Psi(X), \Psi(Y)] = \Psi([X,Y]) = \Psi([X,Y]) = 0.
\]

\(\square\)

From Proposition \([52]\) it follows that \(\phi\)-coverings (and \(\phi\)-semicoverings) with support \(C\) are unique up to isomorphism.
8.2. A covering and a semicovering of a Lie supergroup. Unlike the notion of a covering of a Lie superalgebra, as far as we know, the notion of a covering of a Lie supergroup was never considered in the literature before. One possible way to give a definition of a covering of a Lie supergroup is to use the graded covering of the corresponding Lie superalgebra. However we suggest a different way that is closer to the notion of a topological covering space and this approach can be used to define a covering for any supermanifold, see [RV].

We start with a definition of a semicovering spaces for a Lie supergroup. In Section 8.2 we will show that for any Lie supergroup \(G\) we will give a simple explicit construction of a covering of any matrix Lie supergroup. Note that in general such a simple construction for a Lie supergroup is not applicable for the case of a supermanifold. Recall that if \(M\) is a supermanifold, we denote by \(O_M\) its structure sheaf and by \(\mathcal{F}_M\) the structure sheaf of the underlying space \(M_0\).

8.2.1. A semicovering of a Lie supergroup. Let \(P\) be a graded Lie supergroup of degree \(n\) (or equivalently of type \(C = \{0, 1, \ldots, n\}\) with \(|1| = 1\)) with multiplication morphism \(\nu\) and \(G\) be a Lie supergroup with multiplication morphism \(\mu\). Assume in addition that \(P_0 = g_0\).

**Definition 53.** A sum of \(n\) morphisms \(\Pi^n = \bigoplus_{c=0}^n \Pi_c\), where
\[
\Pi_c = (id, \Pi^*_c) : (G_0, (O_P)_c) \rightarrow (G_0, O_G), \ c \in C,
\]
and \(\Pi^*_c\) is a morphism of sheaves of vector spaces, is called a partial homomorphism of \(P\) to \(G\) with support \(C\) if
\[
\Pi^*_c(f \circ g) = \sum_{c_1 + c_2 = c} \Pi^*_c(f) \cdot \Pi^*_c(g), \ f, g \in O_G, \ c, c_i \in C,
\]
and
\[
\nu^* \circ \Pi^*_c = \sum_{c_1 + c_2 = c} (\Pi^*_c \times \Pi^*_c) \circ \mu^* \text{ for any } c, c_i \in C.
\]

Sometimes we will write a partial homomorphism \(\Pi^n\) of \(P\) to \(G\) in the form \(\Pi^n : P \rightarrow G\). Note that the sheaf morphism \((\Pi^n)^* : O_G \rightarrow O_P\) is in general not defined.

**Definition 54.** A \(\phi : Z \rightarrow Z_2\)-semicovering with support \(C = \{0, 1, 2, \ldots, n\}\) of a Lie supergroup \(G\) is a graded Lie supergroup \(P\) of degree \(n\) with \(P_0 = g_0\) together with a partial homomorphism \(\Pi^n : P \rightarrow G\) such that we can choose atlases \(\{U_i\}\) and \(\{V_i\}\) on \(G\) and \(P\), respectively, with the same base space \((U_i)_0 = (V_i)_0\), with even and odd coordinates \((x_a, \xi_s)\) in \(U_i\) and with graded coordinates \((y^*_a, \eta^*_s)\), where \(s \in C\) is an even integer and \(t \in C\) is an odd integer, in \(V_i\) such that
\[
\Pi^*_c(x_a) = y^*_a, \quad \Pi^*_c(\xi_s) = \eta^*_s.
\]

8.2.2. A covering of a Lie supergroup. Denote by \(P = \varinjlim P_n\) the inverse limit of graded Lie supergroups \(P_n\) of degree \(n\) with the same underlying space \((P_n)_0 = G_0\).

**Definition 55.** A \(\phi : Z \rightarrow Z_2\)-covering with support \(C = \{0, 1, 2, \ldots\}\) of a Lie supergroup \(G\) is a Lie supergroup \(P = \varinjlim P_n\) together with a Lie supergroup homomorphism \(\Pi = \varinjlim (\Pi^n) : P \rightarrow G\) such that \(\Pi^n : P_n \rightarrow G\) is a semicovering with support \(C = \{0, 1, 2, \ldots, k\}\) for any \(k \geq 0\).
Remark 56. Definition 54 implies that in the case of a $\phi : \mathbb{Z} \to \mathbb{Z}_2$-covering $\Pi : \mathcal{P} \to \mathcal{G}$ with support $C = \{0, 1, 2, \ldots, \}$ we can choose "atlases" $\{\mathcal{U}_i\}$ and $\{\mathcal{V}_i\}$ on $\mathcal{G}$ and $\mathcal{P}$, respectively, with the same base space $(\mathcal{U}_i)_0 = (\mathcal{V}_i)_0$, with even and odd coordinates $(x_a, \xi_b)$ in $\mathcal{U}_i$ and with graded coordinates $(y_a^s, \eta_b^t)$, where $s$ is an even integer and $t$ is an odd integer, in $\mathcal{V}_i$ such that

$$pr_s \circ \Pi^*(x_a) = y_a^s, \quad pr_t \circ \Pi^*(\xi_b) = \eta_b^t,$$

where $pr_q : \mathcal{O}_\mathcal{P} \to (\mathcal{O}_\mathcal{P})_q$, $q \in \mathbb{Z}$, is the natural projection. In this case each $\mathcal{V}_i$ is an "infinite graded domain", which we understand as inverse limit of the corresponding finite graded domains.

Sometimes we will call a $\phi : \mathbb{Z} \to \mathbb{Z}_2$-covering with support $C = \{0, 1, 2, \ldots, \}$ of a Lie supergroup $\mathcal{G}$ simply a $\mathbb{Z}^{\geq 0}$-covering of $\mathcal{G}$.

Remark 57. It looks more natural to give a definition of a covering space with support $\mathbb{Z}$. The study of $\mathbb{Z}$-graded Lie supergroups (not necessary non-negatively $\mathbb{Z}$-graded) was announced in [KPS]. In the case of any $\mathbb{Z}$-graduation graded coordinates may be nilpotent, this leads to significant difficulties, see [CGP]. Therefore a definition of a covering space with support $\mathbb{Z}$ we leave for a future research.

Remark 58. Our definition implies that in some sense $\mathcal{P}$ is locally diffeomorphic to $\mathcal{G}$. Indeed, let us choose $q \in \mathbb{Z}$ assuming for example that $q$ is even. Then the sheaf morphism $pr_s \circ \Pi^*$ determines a sheaf isomorphism of $S^*(x_a)_{(\mathcal{U}_i)_0} \subset \mathcal{F}_{\mathbb{Z}^q}$ and $S^*(y_a^s)_{(\mathcal{U}_i)_0}$. Here $S^*(x_a)$ is the supersymmetric algebra generated by $x_a$. And similarly in the case of odd coordinates. In other words we have the following isomorphisms of superdomains

$$(\mathcal{U}_i)_0, S^*(x_a)) \simeq (\mathcal{V}_i)_0, S^*(y_a^s)), \quad (\mathcal{U}_i)_0, S^*(\xi_b) \simeq (\mathcal{V}_i)_0, S^*(\eta_b^t)).$$

We say that a Lie supergroup $\mathcal{G}$ possesses an additional non-negative $\mathbb{Z}$-grading if its structure sheaf possesses a $\mathbb{Z}$-grading $\mathcal{O}_\mathcal{G} = \bigoplus_{q \geq 0} (\mathcal{O}_\mathcal{G})_q$ and all Lie supergroup morphisms are $\mathbb{Z}$-graded. In this paper we assume in addition that $(\mathcal{O}_\mathcal{G})_q \subset (\mathcal{O}_\mathcal{G})_q$. Note that a Lie supergroup with a $\mathbb{Z}$-grading is not the same as a graded Lie supergroup of degree $n$. Now we can prove that our covering satisfies the same universal properties as a covering for a Lie superalgebra.

Theorem 59. Let $\mathcal{G}$ be a Lie supergroup with an additional non-negative $\mathbb{Z}$-grading or a graded Lie supergroup of degree $n$ and $\mathcal{G}'$ be a Lie supergroup. Let $\psi : \mathcal{G} \to \mathcal{G}'$ be a Lie supergroup homomorphism and let $\mathcal{P}'$ be a $\mathbb{Z}^{\geq 0}$-covering of $\mathcal{G}'$. Then there exists a unique homomorphism of Lie supergroups $\Psi : \mathcal{G} \to \mathcal{P}'$, which preserves the $\mathbb{Z}$-gradings, such that the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{P}' & \xrightarrow{\Pi} & \mathcal{G}' \\
\downarrow^{\exists! \Psi} & & \\
\mathcal{G} & \xrightarrow{\psi} & \mathcal{G}'
\end{array}$$

Proof. We split the proof into steps.

Step 1. We put $W_i := \psi^{-1}(\mathcal{U}_i)_0$. Let us define first $(\Psi_{W_i})^*$ for any $i$ using coordinates from Definition 54. We put

$$(\Psi_{W_i})^*(y_a^s) := pr_s \circ \psi^*(x_a) \in (\mathcal{O}_\mathcal{G})_s, \quad (\Psi_{W_i})^*(\eta_b^t) := pr_t \circ \psi^*(\xi_b) \in (\mathcal{O}_\mathcal{G})_t,$$

where $pr_q : \mathcal{O}_\mathcal{G} \to (\mathcal{O}_\mathcal{G})_q$. By [L] Section 2.1.7, Theorema we defined a morphism $\Psi_{W_i} : (W_i, \mathcal{O}_{\mathcal{P}'}) \to \mathcal{V}_i$ of superdomains.
Step 2. The morphism $\Psi_{W_i}$ satisfies the following equality
\begin{equation}
(\Psi_{W_i})^*(pr_s \circ \Pi^*(F)) = pr_s \circ \psi^*(F),
\end{equation}
where $s \in \mathbb{Z}$ and $F \in \mathcal{O}_{G'}(U_i)$. First of all assume that $F$ is a polynomial in $(x_a, \xi_b)$. It is sufficient to assume that
\[ F = x_{a_1} \cdots x_{a_k} \xi_{b_1} \cdots \xi_{b_l} \]
is a monomial. By Definition 55 we have
\[ \Pi^*(x_a) = \sum_{s=2q} y_a^s, \quad \Pi^*(\xi_b) = \sum_{t=2q+1} \eta_b^t. \]
(Here for simplicity of notations we write infinite sums. We understand this sum as an inverse limit.) Further, we get
\[ pr_s \circ \Pi^*(x_{a_1} \cdots x_{a_k} \xi_{b_1} \cdots \xi_{b_l}) = \sum_{s_1 + \sum_1 t_j = s} y_{a_1}^{s_1} \cdots y_{a_k}^{s_k} \eta_{b_1}^{t_1} \cdots \eta_{b_l}^{t_l}. \]
Therefore,
\begin{align*}
(\Psi_{W_i})^*(pr_s \circ \Pi^*(F)) &= \sum_{s_1 + \sum_1 t_j = s} (\Psi_{W_i})^*(y_{a_1}^{s_1}) \cdots (\Psi_{W_i})^*(y_{a_k}^{s_k})(\Psi_{W_i})^*(\eta_{b_1}^{t_1}) \cdots (\Psi_{W_i})^*(\eta_{b_l}^{t_l}).
\end{align*}
On the other hand,
\begin{align*}
pr_s \circ \psi^*(F) &= pr_s \circ \psi^*(x_{a_1} \cdots x_{a_k} \xi_{b_1} \cdots \xi_{b_l}) = \sum_{s_1 + \sum_1 t_j = s} pr_{s_1} \circ \psi^*(x_{a_1}) \cdots pr_s \circ \psi^*(x_{a_k}) pr_{t_1} \circ \psi^*(\xi_{b_1}) \cdots pr_{t_l} \circ \psi^*(\xi_{b_l}).
\end{align*}
Now the result follows from the definition of $\Psi_{W_i}$. If morphisms coincide on all polynomials, a standard argument, see [4], implies the result for any functions $F \in \mathcal{O}_{G'}$.

Step 3. We have to show that $\Psi_{W_i} = \Psi_{W_j}$ in $W_i \cap W_j$. Let $\tilde{x}_a = H(x_a, \xi_a)$ in $U_i \cap U_j$ and $\tilde{y}_a = pr_s \circ \psi^*(x_a)$. We have by Step 2
\begin{align*}
(\Psi_{W_i})^*(\tilde{y}_a) = (\Psi_{W_j})^*(pr_s \circ \Pi^*(\tilde{x}_a)) = (\Psi_{W_j})^*(pr_s \circ \Pi^*(H(x_a, \xi_a))) = pr_s \circ \psi^*(H(x_a, \xi_a)) = (\Psi_{W_i})^*(pr_s \circ \Pi^*(H(x_a, \xi_a))).
\end{align*}
Now we can define the morphism $\Psi$ by $\Psi|_{W_i} := \Psi_{W_i}$.

Step 4. Let us prove that $\Psi$ is a homomorphism of Lie supergroups. Denote by $\mu_G$, $\mu_P$ and by $\mu_P$ the multiplication morphism in $G$, $G$ and $P$, respectively. We have to show that
\begin{equation}
(\Psi^* \times \Psi^*) \circ \mu_P^* = \mu_G^* \circ \Psi^*.
\end{equation}
Clearly it is sufficient to show (27) only for coordinates $(y_a^s, \eta_b^t)$. Step 2 implies that
\[ \Psi^* \circ pr_s \circ \Pi^* = pr_s \circ \psi^*. \]
We have
\begin{align*}
\mu_G^* \circ \Psi^*(y_a^s) &= \mu_G^* \circ \Psi^* \circ pr_s \circ \Pi^*(x_a) = \mu_G^* \circ pr_s \circ \psi^*(x_a) = pr_s \circ \mu_G^* \circ \psi^*(x_a) = pr_s \circ (\psi^* \times \psi^*) \circ \mu_G^*(x_a) = (\Psi^* \times \Psi^*) \circ pr_s \circ \Pi^*(x_a) = (\Psi^* \times \Psi^*) \circ \mu_G^* \circ \Pi^*(x_a) = (\Psi^* \times \Psi^*) \circ \mu_G^*(y_a^s).
\end{align*}
For $\eta_b^t$ the proof is similar. The proof is complete. \hfill $\square$

Corollary 60. If a $\mathbb{Z}_{\geq 0}$-covering of a Lie supergroup $G$ exists, it is unique up to isomorphism.
Proof. In Theorem 59 we assume that \(G\) is finite dimensional. However the same argument as in the proof of Theorem 59 implies that we can replace \(G\) by a \(\mathbb{Z}_{\geq 0}\)-covering of \(G\). Therefore we get the result. □

If there exists a \(\mathbb{Z}_{\geq 0}\)-covering \(\Pi : P \to G\) of a Lie supergroup \(G\), then from Definition 55 it follows that there exists a \(\phi : Z \to \mathbb{Z}_{\geq 0}\)-semicovering of \(G\) with support \(C = \{0, 1, 2, \ldots, n\}\) for any \(n\). Indeed, let \(P_n\) be the graded Lie supergroup of degree \(n\) as in Definition 55. Then by Definition 55 \(\Pi^*\) is a partial homomorphism. Further, for any Lie supergroups homomorphism \(\psi : G' \to G\), where \(G'\) is a graded Lie supergroup of degree \(n\), there exists unique Lie supergroup homomorphism (not a partial homomorphism!) \(\Psi_n : G' \to P_n\) such that \(pr_s \circ \psi^* = \Psi^* \circ \Pi_s\).

We finalize this section with the following proposition.

**Proposition 61.** Let \(G, \tilde{G}\) be Lie supergroups and \(P, \tilde{P}\) be their \(\mathbb{Z}_{\geq 0}\)-coverings, respectively. Let \(\psi : G \to \tilde{G}\) be a homomorphism of Lie supergroups. Then there exists unique Lie supergroup homomorphism \(\Psi\) of \(\mathbb{Z}_{\geq 0}\)-coverings \(P\) to \(\tilde{P}\) such that the following diagram is commutative

![Diagram]

\[P \xrightarrow{\Psi} \tilde{P} \quad \Pi \downarrow \quad \Pi \]

\[G \xrightarrow{\psi} \tilde{G}\]

*Proof.* Locally we put

\[\Psi^*(\tilde{g}_a) := pr_s \circ \Pi^* \circ \psi^* (\tilde{x}_a), \quad \Psi^*(\tilde{\eta}_b) := pr_t \circ \Pi^* \circ \psi^* (\tilde{\xi}_b).\]

The rest of the proof is similar to the proof of Theorem 59. □

Later we will prove that such a covering exists for any Lie supergroup. More precisely we will show that \(F(G)\) is a covering of a Lie supergroup \(G\).

**9. Existence of a \(\mathbb{Z}_{\geq 0}\)-covering of a Lie superalgebra and a Lie supergroup**

In this section we will show that the \(F^*\)-image of a Lie superalgebra \(g\) and the \(F\)-image a Lie supergroup \(G\) are \(\mathbb{Z}_{\geq 0}\)-coverings of \(g\) and \(G\), respectively. We start with a Lie superalgebra.

**9.1. A \(\mathbb{Z}_{\geq 0}\)-covering of a Lie superalgebra.** In this section we will show that the Lie superalgebra \(p := F^*(g)\) is a \(\mathbb{Z}_{\geq 0}\)-covering of \(g\), see Definition 49. We are ready to prove the following theorem.

**Theorem 62.** For any Lie superalgebra \(g\) the Lie superalgebra \(p := F^*(g)\) is a \(\mathbb{Z}_{\geq 0}\)-covering of \(g\).

*Proof.* In notations of Proposition 44 the homomorphism \(\Pi' : p \to g\) is defined as follows

\[\Pi' \left( \sum_{i=1}^{\infty} d_i (X) + \sum_{i,j=1}^{\infty} d_j (X) \right) = \frac{1}{i!} X \in g_i,\]

Denote \(Z' := \sum_{i=1}^{\infty} d_i (Z) + \sum_{i,j=1}^{\infty} d_j (Z) \in p_i\) for any \(Z \in g_i\). Using arguments from the proof of Proposition 44 we see that

\[\Pi'([X', Y']) = \frac{1}{i!j!} [X, Y] = [\Pi'(X'), \Pi'(Y')]\]
Lemma 64. We need the following two lemmas.

Choose global coordinates of degree 0.

Coordinates of a (graded) Lie supergroup standard the coordinate \( b_q \).

By definition of standard coordinates, we have the following sheaf isomorphisms

\[
G \quad \text{where} \quad \xi = b_q \cdot \zeta.
\]

Theorem 66. Now we are ready to prove the main result of this section.

We conclude this section with the following observation.

Proposition 63. The Lie superalgebra \( F^p_\theta(g) \) is a semicovering with support \( \{0, \ldots, n\} \) of the Lie superalgebra \( g \) along the homomorphism \( \mathbb{Z} \to \mathbb{Z}_2 \).

9.2. A \( \mathbb{Z}^{\geq 0} \)-covering of a Lie supergroup. Let \( G \) be a Lie supergroup with the Lie superalgebra \( g \) and \( \mathcal{P} \) be a graded Lie supergroup with the graded Lie superalgebra \( p \) and with \( P_0 = G_0 \). We put \( p_{(1)} := p_1 \oplus p_2 \oplus \cdots \). By the Poincaré–Birkhoff–Witt theorem we have the following sheaf isomorphisms

\[
\tag{28}
\begin{align*}
\mathcal{O}_\mathcal{G} & = \text{Hom}_{\text{U}}(U(g), F_{G_0}) \cong \text{Hom}_{\mathbb{C}}(S^*(g_1), F_{G_0}) \cong S^*(g_1) \otimes_{\mathbb{C}} F_{G_0}, \\
\mathcal{O}_{P_q} & = \text{Hom}_{\text{U}(p)}(U(p), F_{P_0}) \cong \text{Hom}_{\mathbb{C}}(S^*(p_{(1)}), F_{P_0}) \cong (S^*(p_{(1)}))^* \otimes_{\mathbb{C}} F_{G_0},
\end{align*}
\]

where \( q \in \mathbb{Z}^{\geq 0} \). The isomorphism (28) leads to a certain choice of global odd and global graded coordinates on \( G \) and \( \mathcal{P} \), respectively. Indeed, let \( (\xi_i) \) be a basis in \( g_1 \) and let \( (\xi^q_{i}) \) be a basis in \( p^q \), where \( q \geq 1 \). Since \( g_1 \) is a direct term in \( S^*(g_1) \), we assume that \( \xi_i \in S^*(g_1) \), and similarly \( \xi^q_i \in S^q(p_{(1)}) \). Then the system \( (\xi_i) \) is a system of global odd coordinates on \( G \), and \( (\xi^q_i) \), where \( q \geq 1 \) is the degree of the coordinate \( \xi^q_i \), is a system of global graded coordinates of \( \mathcal{P} \). We will call such coordinates of a (graded) Lie supergroup standard. Note that in general we cannot choose global coordinates of degree 0.

We need the following two lemmas.

Lemma 64. Let us fix a graded chart \( U \) on a graded Lie supergroup \( \mathcal{P} \) with standard graded coordinates \( (\xi^q_i) \), where \( q \geq 1 \). Denote by \( \theta^q_\xi \) the basis in \( p^q \), dual to the basis \( (\xi^q_i) \) of \( p^q \). Let \( f \in \mathcal{O}_{U_\mathcal{P}} \) be a graded function of degree \( p \). If \( f(\theta^q_\xi) = 0 \) for any \( c \), then \( f = \sum f^{i_1 \ldots i_k} \theta^{q_1}_{i_1} \cdots \theta^q_{i_k} \) is a sum of products of coordinates of degree \( p' < p \) with respect to the coordinates \( (\xi^q_i) \). Therefore the same holds for any other coordinates.

Proof. By definition of standard coordinates, \( f \) is a sum of products of coordinates of degree \( p' < p \) with respect to the coordinates \( (\xi^q_i) \). Therefore the same holds for any other coordinates.

Lemma 65. Let \( \mathcal{U} \) be a graded superdomain with graded coordinates \( (\theta^q_\xi) \), where \( q \in \mathbb{Z} \) is the degree of a coordinate. The system \( (\tilde{\theta}^q_\xi) \), where

\[
\tilde{\theta}^q_\xi = \theta^q_\xi + \sum_{q_1 + \ldots + q_k = q} f_{i_1 \ldots i_k} \theta^{q_1}_{i_1} \cdots \theta^q_{i_k}, \quad f_{i_1 \ldots i_k} = f_{i_1 \ldots i_k}(\theta^q_\xi),
\]

is a new graded coordinate system in \( \mathcal{U} \).

Proof. By induction.

Now we are ready to prove the main result of this section.

Theorem 66. For any Lie supergroup \( G \) the image \( F(G) \) is a \( \mathbb{Z}^{\geq 0} \)-covering of \( G \).
Proof. Denote $\mathcal{P} := F(\mathcal{G})$. We have a natural homomorphism of Lie supergroups $\Pi : \mathcal{P} \to \mathcal{G}$ given by

$$
\Pi^* : \text{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{F}_{\mathcal{G}_0}) \to \text{Hom}_{\mathcal{U}(\mathfrak{p}_0)}(\mathcal{U}(\mathfrak{p}), \mathcal{F}_{\mathcal{G}_0}),
$$

$$
\Pi^*(f)(X) = f(\Pi^*(X)), \quad f \in \text{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{F}_{\mathcal{G}_0}), \quad X \in \mathcal{U}(\mathfrak{p}),
$$

where $\Pi^* : \mathfrak{p} \to \mathfrak{g}$ is the differential of $\Pi$. (With the same letter we denote the corresponding homomorphism of the universal enveloping algebras.) Let us choose an atlas $\mathcal{A}$ on $\mathcal{G}$ and let us fix $\mathcal{U} \in \mathcal{A}$ with coordinates $(x_a, \xi_b)$, where $\xi_b$ are standard global odd coordinates.

Let us choose a basis $Z_1, \ldots, Z_n$ in $\mathfrak{g}_0$ and a basis $T_1, \ldots, T_m$ in $\mathfrak{g}_1$, dual to the basis $(\xi_b)$. Let $Z_1^1, \ldots, Z_n^s$ be the corresponding basis in $\mathfrak{p}_s$, where $s$ is even, and $T_1^t, \ldots, T_m^t$ is the corresponding basis in $\mathfrak{p}_s$, where $t$ is odd. Further let us identify $x_a$ with elements in $X_a \in \mathcal{O}_G$, where

$$
X_a(1) = x_a, \quad X_a(T_j) = 0, \quad X_a(Z_j) = \tilde{Z}_j(x_a)_{\text{red}}.
$$

Here $\tilde{Z}_j$ is the fundamental vector field on $\mathcal{G}$ corresponding to $Z_j$ and $\tilde{Z}_j(x_a)_{\text{red}}$ is the image of the function $\tilde{Z}_j(x_a)$ via the natural projection $\mathcal{O}_G \to \mathcal{F}_{\mathcal{G}_0}$.

Let us define an atlas $\{V_i\}$ as in Definition 55 and 54. We put

$$(29) \quad y^s_a := pr_s \circ \Pi^*(x_a), \quad \eta^s_b := pr_t \circ \Pi^*(\xi_b)$$

for $s$ even and $t$ odd. Let us check that $(x_a, y^s_a, \eta^s_b)$ are graded coordinates in $\mathcal{P}$. According above we may choose standard graded coordinates $(x_a, \tilde{y}^s_a, \tilde{\eta}^s_b)$, where $x_a \in \mathcal{F}_{\mathcal{G}_0}$ are as above, and

$$
\tilde{y}^s_a(Z_j) = \delta_{ij}, \quad \tilde{y}^s_a(T_j^s) = 0, \quad \tilde{\eta}^s_b(T_j^s) = \delta_{ij}, \quad \tilde{\eta}^s_b(Z_j^s) = 0.
$$

And we denote by the same letter $\tilde{y}^s_a$ the corresponding elements in $(S^s(\mathfrak{p}/\mathfrak{p}_0))^*$ (or by $\tilde{\eta}^s_b$ the corresponding elements in $(S^t(\mathfrak{p}/\mathfrak{p}_0))^*$).

Let us show that the coordinates $(x_a, \tilde{y}^s_a, \tilde{\eta}^s_b)$ may be expressed via $(x_a, y^s_a, \eta^s_b)$. This will imply that the system $(x_a, \tilde{y}^s_a, \tilde{\eta}^s_b)$ is a system of local coordinates as well. We have

$$
\eta^s_b(T_j^s) = \Pi^*(\xi_b)(T_j^s) = \xi_b(\Pi^*(T_j^s)) = \xi_b(T_j) = \delta_{ij}.
$$

Similarly, checking other values of $\eta^s_b(A)$ and $\tilde{\eta}^s_b(A)$, where $A \in \mathcal{U}(\mathfrak{p})$, we get $\eta^s_b = \tilde{\eta}^s_b$. Further, the systems of vector fields $(\partial x_a)$ and $(\tilde{Z}_a)$ are both local bases of the sheaf of vector fields on $\mathcal{G}$. Therefore the matrix $A = (\tilde{Z}_a(x_a))_{\text{red}}$ is invertible. Let $B = A^{-1}$. Consider $\sum_a B^s_a y^s_a$. Then

$$
\sum_a B^s_a y^s_a = \sum_a B^s_a \tilde{Z}_j(x_a) = \delta_{ij} \tilde{y}^s_a(Z_j^s).
$$

By Lemma 54 the difference $\sum_a B^s_a y^s_a - \tilde{y}^s_a$ is a sum of products of coordinates of degrees smaller than $s$. By Lemma 55 and induction of $s$ we conclude that $\tilde{y}^s_a$ may be expressed via $y^s_a$. The proof is complete. \qed

9.3. A semicovering of a Lie supergroup with support $\{0, 1, \ldots, n\}$. As we have seen after Definition 54 we can construct a semicovering of a Lie supergroup $\mathcal{G}$ with support $\{0, 1, \ldots, n\}$ using a $\mathbb{Z}^{\geq 0}$-covering of $\mathcal{G}$. The definition of the functor $F$ implies that $F_n(\mathcal{G})$ is a semicovering of $\mathcal{G}$ with support $\{0, 1, \ldots, n\}$.

Remark 67. The functor $F_n$ is an embedding of the category of Lie supergroups to the category of $\mathbb{Z}$-graded Lie supergroups with support $\{0, 1, \ldots, n\}$. Clearly we can recover the Lie supergroup structure on the supermanifold $\mathcal{G}$ using the Lie supergroup structure of $F_n(\mathcal{G})$. 
10. Loop superalgebras

In this section we give an explicit matrix realization of a $\mathbb{Z}^{\geq 0}$-covering for a finite dimensional Lie superalgebra and for a matrix Lie supergroup. Let $\mathfrak{g}$ be a Lie superalgebra and $\mathfrak{p} := F'(\mathfrak{g})$. We can easily see that $\mathfrak{p}$ is a subalgebra of a loop algebra in the sense [ABFP, Definition 3.1.1], see also [Eld]. Indeed, clearly we have

$$\mathfrak{p} \simeq \bigoplus_{i \geq 0} \mathfrak{g}_i \otimes t^i,$$

where $t$ is a formal even variable. (We will get a definition of a loop algebra if we replace $i \geq 0$ by $i \in \mathbb{Z}$.)

Now consider the case $\mathfrak{g} = \mathfrak{gl}_{m|n}(\mathbb{K})$. The Lie superalgebra $\mathfrak{p} = F'(\mathfrak{g})$ possesses a simple matrix realization. This implies by Ado’s Theorem that we have such a matrix realization for any finite dimensional Lie superalgebra $\mathfrak{h} \subset \mathfrak{gl}_{m|n}(\mathbb{K})$. Recall that the general linear Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{K})$ contains all matrices in the following form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A$ is $m \times m$-matrix and $D$ is $n \times n$-matrix over $\mathbb{K}$. Then the Lie superalgebra $\mathfrak{p}$ contains all matrices in the following form

$$\begin{pmatrix} A_1 & 0 & 0 & 0 & \cdots \\ C_1 & D_1 & 0 & 0 & \cdots \\ A_2 & B_1 & A_1 & 0 & \cdots \\ C_2 & D_2 & C_1 & D_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Here $A_i$ are $m \times m$-matrices, $D_i$ are $n \times n$-matrices, $B_i$ are $m \times n$-matrices, $C_i$ are $n \times m$-matrices over $\mathbb{K}$ and so on. We have

$$\mathfrak{p} = \bigoplus_{n \geq 0} \mathfrak{p}_n, \quad \mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1,$$

where $\mathfrak{p}_0$ contains all matrices with $B_i = 0$ and $C_i = 0$, $\mathfrak{p}_1$ contains all matrices with $A_i = 0$ and $D_j = 0$. The $\mathbb{Z}$-grading of $\mathfrak{p}$ has natural meaning: $\mathfrak{p}_0$ contains all matrices with $B_i = 0$, $C_i = 0$ for any $i$, $A_j = 0$ and $D_j = 0$ for $j > 1$; $\mathfrak{p}_1$ contains all matrices with $A_i = 0$, $D_i = 0$ for any $i$, $C_j = 0$ and $B_j = 0$ for $j > 1$.

Let $\mathcal{G}$ be a Lie subsupergroup in the general linear Lie supergroup $\text{GL}_{m|n}(\mathbb{K})$. Using our results for the Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{K})$ we can explicitly construct $F(\text{GL}_{m|n}(\mathbb{K}))$. Indeed, let $\text{GL}_{m|n}(\mathbb{K})$ has the following coordinate superdomain

$$U = \begin{pmatrix} X & \Xi \\ H & Y \end{pmatrix},$$

where $X \in \text{GL}_m(\mathbb{K})$, $Y \in \text{GL}_n(\mathbb{K})$ are matrices of even coordinates of $U$ and $\Xi$, $H$ are matrices of odd coordinates. Now $F(\text{GL}_{m|n}(\mathbb{K}))$ has the following coordinate superdomain

$$\begin{pmatrix} X_1 & 0 & 0 & 0 & \cdots \\ H_1 & Y_1 & 0 & 0 & \cdots \\ X_2 & \Xi_1 & X_1 & 0 & \cdots \\ \Xi_2 & Y_2 & H_1 & Y_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$X_i \in \text{GL}_m(\mathbb{K})$, $Y_j \in \text{GL}_n(\mathbb{K})$ are matrices of even coordinates of $F(\text{GL}_{m|n}(\mathbb{K}))$ and $\Xi_k$, $H_l$ are matrices of odd coordinates. The Lie supergroup multiplication is given
by usual matrix multiplication. A similar idea can be used to construct explicitly $F(G)$ for any matrix Lie supergroup $G$.

We can give a matrix realization of the Lie superalgebras $F_n' (g)$, $n \geq 2$. The Lie superalgebra $F_n' (g)$ contains all matrices in the following form

$$
\begin{pmatrix}
A_1 & 0 & 0 & \cdots & 0 \\
C_1 & D_1 & 0 & \cdots & 0 \\
A_2 & B_1 & A_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_l & D_l & C_{l-1} & \cdots & D_1
\end{pmatrix},
\begin{pmatrix}
A_1 & 0 & 0 & \cdots & 0 \\
C_1 & D_1 & 0 & \cdots & 0 \\
A_2 & B_1 & A_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{r+1} & B_r & A_r & \cdots & A_1
\end{pmatrix},
$$

for $n = 2l - 1$ and for $n = 2r$, respectively. Similarly for matrix Lie supergroups.

Note that in general these constructions are not applicable for supermanifolds.

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