Enumeration of Fuss-skew paths

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Abstract

In this paper, we introduce the concept of a Fuss-skew path and then we study the distribution of the semi-perimeter, area, peaks, and corners statistics. We use generating functions to obtain our main results.

Keywords: Skew Dyck path, Fuss-Catalan numbers, generating function

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1. Introduction

A \textit{skew Dyck path} is a lattice path in the first quadrant that starts at the origin, ends on the $x$-axis, and consists of up-steps $U = (1,1)$, down-steps $D = (1,-1)$, and left-steps $L = (-1,-1)$, such that up and left steps do not overlap. The definition of skew Dyck path was introduced by Deutsch, Murnari, and Rinaldi [4]. Some additional results about skew Dyck path can be found in [2, 5, 8, 14].

Let $s_n$ denote the number of skew Dyck path of semilength $n$, where the semilength of a path is defined as the number its up-steps. The sequence $s_n$ is given by the combinatorial sum $s_n = \sum_{k=1}^{n} \binom{n-1}{k-1} c_k$, where $c_n = \frac{1}{n+1} \binom{2n}{n}$ is the $n$-th Catalan number. The sequence $s_n$ appears in OEIS as A002212 [15], and its first few values are

$$1, 1, 3, 10, 36, 137, 543, 2219, 9285, 39587.$$
One way to generalize the classical Dyck paths is to regard the length of an up-step $U$ as a parameter. Given a positive number $\ell$, an $\ell$-Dyck path is a lattice path in the first quadrant from $(0, 0)$ to $((\ell + 1)n, 0)$ where $n \geq 0$ using up-steps $U_\ell = (\ell, \ell)$ and down-steps $U = (1, -1)$. For $\ell = 1$, we recover the classical Dyck path. The total number of $\ell$-Dyck path with length $(\ell + 1)n$ is given by $c_\ell(n) = \frac{1}{(t+1)n} \binom{(t+1)n}{n}$ (cf. [1]). We will refer to $\ell$-Dyck paths here as the “Fuss” case because the sequence $c_\ell(n)$ was first investigated by N. I. Fuss (see, for example, [7, 16] for several combinatorial interpretations for both the Catalan and Fuss-Catalan numbers).

Our focus in this paper is to introduce a Fuss analogue of the skew Dyck path. Given a positive integer $\ell$, an $\ell$-Fuss-skew path is a path in the first quadrant that starts at the origin, ends on the $x$-axis, and consists of up-steps $U_\ell = (\ell, \ell)$, down-steps $D = (1, -1)$, and left steps $L = (-1, -1)$, such that up and left steps do not overlap. Given an $\ell$-Fuss-skew path $P$, we define the semilength of $P$, denote by $|P|$, as the number of up-steps of $P$. For example, Figure 1 shows a 3-Fuss-skew path of semilength 6. It is clear that the 1-Fuss-skew paths coincide with the skew Dyck paths. Let $S_{n,\ell}$ denote the set of all $\ell$-Fuss-skew path of semilength $n$, and $S_\ell = \bigcup_{n \geq 0} S_{n,\ell}$. For example, Figure 4 shows all the paths in $S_{2,2}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure1.png}
\caption{3-Fuss-skew path of semilength 6.}
\end{figure}

2. Counting special steps

For a given path $P \in S_\ell$, we use $u(P)$, $d(P)$, and $t(P)$ to denote the number of up-steps, down-steps, and left-steps of $P$, respectively. In this section, we study the distribution of these parameters over $S_\ell$. Using these parameters, we define the generating function

$$F_\ell(x, p, q) := \sum_{P \in S_\ell} x^{u(P)} p^{d(P)} q^{t(P)}.$$ 

For simplicity, we use $F_\ell$ to denote the generating function $F_\ell(x, p, q)$.

**Theorem 2.1.** The generating function $F_\ell(x, p, q)$ satisfies the functional equation

$$F_\ell = 1 + x(pF_\ell + q)^{\ell-1}(pF_\ell^2 + q(F_\ell - 1)).$$  \hfill (2.1)
Proof. Let $\mathcal{A}_i$ denote the $\ell$-Fuss-skew paths whose last $y$-coordinate is $i$ and let $A_i$ denote the generating function defined by

$$A_i = \sum_{P \in \mathcal{A}_i} x^{u(P)} p^{d(P)} q^{t(P)}.$$ 

A non-empty $\ell$-Fuss-skew path can be uniquely decomposed as either $U_\ell TDP$ or $U_\ell TL$, where $U_\ell T$ is a lattice path in $\mathcal{A}_1$ and $P$ is an $\ell$-Fuss-skew path (see Figure 2 for a graphical representation of this decomposition). From this decomposition, we obtain the functional equation (cf. [6])

$$F_\ell = 1 + x(pA_1 F_\ell + qA_1).$$ (2.2)

\[\begin{array}{c}
\begin{array}{c}
\ell \\
1
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\ell \\
1
\end{array}
\end{array}\]

Figure 2. Decomposition of a $\ell$-Fuss-skew path.

The paths of $\mathcal{A}_i$ can be decomposed as $TDP$ or $TL$, where $T \in \mathcal{A}_{i+1}$ for $i = 1, \ldots, \ell - 2$ and $P \in \mathbb{S}_\ell$ (see Figure 3 for a graphical representation of this decomposition). Moreover, the paths of $\mathcal{A}_{\ell-1}$ are decomposed as $P_1DP_2$ or $P'L$, where $P_1, P_2, P' \in \mathbb{S}_\ell$ and $P'$ is non-empty.

\[\begin{array}{c}
\begin{array}{c}
\ell \\
i+1
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\ell \\
i+1
\end{array}
\end{array}\]

Figure 3. Decomposition of the paths in $\mathcal{A}_i$.

From the above decompositions, we obtain the functional equations

$$A_i = pA_{i+1} F_\ell + qA_{i+1}, \quad \text{for} \quad i = 1, \ldots, \ell - 2, \quad \text{and} \quad A_{\ell-1} = pF^2_\ell + q(F_\ell - 1).$$

Note that in these functional equations we do not consider the first up-step because it was considered in (2.2). Therefore, we have

$$F_\ell = 1 + x(pF_\ell + q)A_1 = 1 + x(pF_\ell + q)^2 A_2 = \cdots = 1 + x(pF_\ell + q)^{\ell-1}(pF^2_\ell + q(F_\ell - 1)).$$
Let \( s_\ell(n, p, q) \) denote the joint distribution over \( S_{n, \ell} \) for the number of down and left steps, that is,

\[
s_\ell(n, p, q) = \sum_{P \in S_{n, \ell}} p^d(P) q^l(P).
\]

It is clear that \( F_\ell = \sum_{n \geq 0} s_\ell(n, p, q) x^n \). From the Lagrange inversion theorem (see for instance [13]), we give a combinatorial expression for the sequence \( s_\ell(n, p, q) \).

**Theorem 2.2.** For \( n \geq 1 \), the sequence \( s_\ell(n, p, q) \) is given by

\[
1 \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} \binom{n(\ell - 1)}{n - 2j + k - 1} p^{2n - 1 - 2j}(2p + q)^k(p + q)^{n(\ell - 2) + 2j - k + 1}.
\]

In particular, the total number of \( \ell \)-Fuss-skew paths of semilength \( n \) is

\[
s_\ell(n) := s_\ell(n, 1, 1) = 1 \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} \binom{n(\ell - 1)}{n - 2j + k - 1} 3^{k} 2^{n(\ell - 2) + 2j - k + 1}.
\]

**Proof.** The functional equation given in Theorem 2.1 can be written as

\[
Q_\ell = x(p(Q_\ell + 1) + q)^{\ell - 1}(p(Q_\ell + 1)^2 + qQ_\ell),
\]

where \( Q_\ell = F_\ell - 1 \). From the Lagrange inversion theorem, we deduce

\[
[x^n] H_\ell = \frac{1}{n} \left[ z^{n-1} \right] (p(z + 1) + q)^{(\ell - 1)n}(p(z + 1)^2 + qz)^n
\]

\[
= \frac{1}{n} \left[ z^{n-1} \right] \sum_{s \geq 0} \binom{\ell - 1}{s} p^s z^s (p + q)^{(\ell - 1)n - s} (p z^2 + (2p + 1) z + p)^n
\]

\[
= \frac{1}{n} \left[ z^{n-1} \right] \sum_{s \geq 0} \binom{\ell - 1}{s} p^s z^s (p + q)^{(\ell - 1)n - s} \times \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} p^{n-j}(2p + q) z^k (p z^2)^j - k
\]

\[
= \frac{1}{n} \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} \binom{n(\ell - 1)}{n - 2j + k - 1} 2^{n-1 - 2j}(2p + q)^k(p + q)^{n(\ell - 2) + 2j - k + 1}.
\]

For example, Figure 4 shows all 2-Fuss-skew paths of semilength 2 counted by the term \( s_\ell(2, p, q) = 3p^4 + 6p^3q + 4p^2q^2 + pq^3 \).
From Theorem 2.2, we obtain that the total number of down-steps over the $\ell$-Fuss-skew paths of semilength $n$ is given by

$$\frac{\partial s_\ell(n, p, 1)}{\partial p} \bigg|_{p=1} = \frac{1}{n} \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} \left( \frac{n(\ell - 1)}{n - 2j + k - 1} \right) 2^{(\ell - 2)n + 2j - k - 3k - 1} (3n(\ell + 2) + k - 6j - 3).$$

Moreover, the total number of left-steps over the $\ell$-Fuss-skew paths of semilength $n$ is

$$\frac{\partial s_\ell(n, 1, q)}{\partial q} \bigg|_{q=1} = \frac{1}{n} \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} \left( \frac{n(\ell - 1)}{n - 2j + k - 1} \right) 2^{(\ell - 2)n + 2j - k - 3k - 1} (3n(\ell - 2) - k + 6j + 3).$$

Equation (2.1) can be explicitly solved for $\ell = 1$. In this case, we obtain the generating function

$$F_1(x, p, q) = \frac{1 - qx - \sqrt{(1 - qx)(1 - (4p + q)x)}}{2px}.$$

Moreover, the generating functions for the total number of down-steps (A026388) and left steps (A026376) over the skew-Dyck paths are respectively

$$\frac{1 - 4x + 3x^2 - \sqrt{1 - 6x + 5x^2(1 - x)}}{2x\sqrt{1 - 6x + 5x^2}}.$$
and
\[
\frac{1 - 3x - \sqrt{1 - 6x + 5x^2}}{2\sqrt{1 - 6x + 5x^2}}.
\]
Notice that we recover some of the results of [5].

Finally, Table 1 shows the first few values of the total number of \( \ell \)-Fuss-skew paths of semilength \( n \).

| \( \ell \backslash n \) | 1  | 2  | 3  | 4  | 5  | 6  | 7   |
|----------------------|----|----|----|----|----|----|-----|
| \( \ell = 1 \)       | 1  | 3  | 10 | 36 | 137| 543| 2219|
| \( \ell = 2 \)       | 2  | 14 | 118| 1114| 11306| 120534| 1331374|
| \( \ell = 3 \)       | 4  | 64 | 1296| 29888| 745856| 19614464| 535394560|
| \( \ell = 4 \)       | 8  | 288| 13568| 734720| 43202560| 2681634816| 172936069120|

2.1. The width of a path

For a given path \( P \in S_\ell \), we define the width of \( P \), denoted by \( \nu(P) \), as the \( x \)-coordinate of the last point of \( P \). For example, the width of the path given in Figure 1 is 20. We define the generating function
\[
G_\ell(x, y) := G_\ell = \sum_{P \in S_\ell} x^{u(P)} y^{\nu(P)}.
\]
Note that each \( U_\ell \) and \( D \) step of a path increases the width by \( \ell \) units and 1 unit, respectively, while the left-step \( L \) decreases the width by 1 unit. Therefore, we have the functional equation
\[
G_\ell = 1 + xy^\ell (yG_\ell + y^{-1})^{\ell-1} (yG_\ell^2 + y^{-1}(G_\ell - 1))
= 1 + x(y^2G_\ell + 1)^{\ell-1} (y^2G_\ell^2 + (G_\ell - 1)). \tag{2.3}
\]
Let \( g_\ell(n, y) \) denote the distribution over \( S_{n,\ell} \) for the width parameter, i.e.,
\[
g_\ell(n, y) = \sum_{P \in S_{n,\ell}} y^{\nu(P)}.
\]
From the functional equation (2.3) and the Lagrange inversion theorem, we obtain the following theorem.

**Theorem 2.3.** For \( n \geq 1 \), the sequence \( g_\ell(n, y) \) is given by
\[
\frac{1}{n} \sum_{j=0}^n \sum_{k=0}^j \binom{n}{j} \binom{n(\ell-1)}{n-2j+k-1} y^{4(n-j)-2} (y^2 + 1)^{n(\ell-2)+2j-k+1}(2y^2 + 1)^k.
\]
For example, \( g_2(2, y) = y^2 + 4y^4 + 6y^6 + 3y^8 \). This polynomial can be found from the paths in Figure 4. For \( \ell = 1 \), we obtain the explicit generating function with respect to the width of a skew Dyck path.

\[
G_1(x, y) = \frac{1 - x - \sqrt{(1 - x)(1 - x - 4xy^2)}}{2xy^2}.
\]

3. Number of peaks

For a given path \( P \in S_\ell \), we define the peaks of \( P \), denoted by \( \rho(P) \), as the number of subpaths of the form \( U \ell D \) (for counting peaks in a Dyck path, for example, see [9, 11]). For example, the number of peaks of the path given in Figure 1 is 5. We define the generating function

\[
P_\ell(x, y) := P_\ell = \sum_{P \in S_\ell} x^{u(P)} y^{\rho(P)}.
\]

**Theorem 3.1.** The generating function \( P_\ell(x, y) \) satisfies the functional equation

\[
P_\ell = 1 + x(P_\ell + 1)^{\ell - 1}((P_\ell - 1 + y)P_\ell + (P_\ell - 1)).
\]

**Proof.** Let \( C_i \) denote the generating function defined by \( C_i = \sum_{P \in A_i} x^{u(P)} y^{\rho(P)} \). From the decomposition given for the \( \ell \)-Fuss-skew paths, we have the equation

\[
P_\ell = 1 + x(C_1 P_\ell + C_1).
\]

Moreover,

\[
C_i = C_{i+1} P_\ell + C_{i+1}, \quad \text{for } i = 1, \ldots, \ell - 2, \quad \text{and}
\]

\[
C_{\ell-1} = (P_\ell - 1 + y)P_\ell + (P_\ell - 1).
\]

From these relations, we obtain the desired result. \( \square \)

Let \( p_\ell(n, y) \) denote the distribution over \( S_n \) for the peaks statistic, i.e.,

\[
p_\ell(n, y) = \sum_{P \in S_n} y^{\rho(P)}.
\]

From the Lagrange inversion theorem, we deduce the following result.

**Theorem 3.2.** For \( n \geq 1 \), we have

\[
p_\ell(n, y) = \frac{1}{n} \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} \binom{n(\ell - 1)}{n - 2j + k - 1} 2^{n(\ell - 2) + 2j - k + 1} y^{n-j} (y + 2)^k.
\]

In particular, the total number of peaks in all \( \ell \)-Fuss-skew paths of semilength \( n \) is

\[
\frac{\partial p_\ell(n, y)}{\partial y} \bigg|_{y=1} = \frac{1}{n} \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} \binom{n(\ell - 1)}{n - 2j + k - 1} 2^{n(\ell - 2) + 2j - k + 1} 3^{k-1} (3(n - j) + k).
\]
For example, $p_2(2, y) = 8y + 6y^2$. This polynomial can be found from the paths in Figure 4. For $\ell = 1$ we obtain the generating function

$$P_1(x, y) = \frac{1 - xy - \sqrt{(1 - xy)^2 - 4(1 - x)x}}{2x}.$$

Moreover, the generating function for the total number of peaks is

$$\frac{1 - x - \sqrt{1 - 6x + 5x^2}}{2\sqrt{1 - 6x + 5x^2}}.$$

Table 2 shows the first few values of the number of peaks in $\ell$-Fuss-skew paths of semilength $n$.

| $\ell \setminus n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|---------------------|-----|-----|-----|-----|-----|-----|-----|
| $\ell = 1$          | 1   | 4   | 17  | 75  | 339 | 1558| 7247|
| $\ell = 2$          | 2   | 20  | 226 | 2696| 33138| 415164| 5270850|
| $\ell = 3$          | 4   | 96  | 2672| 78848| 2400896| 74568704| 2347934464|
| $\ell = 4$          | 8   | 448 | 29440| 2054144| 147986432| 10878189568| 810813030400|

### 4. Number of corners

For a given path $P \in S_\ell$, we define a **corner** of $P$ as a right angle caused by two consecutive steps in the graph of $P$. For example, the path given in Figure 5 has 4 corners, depicted in red. This statistic has been studied in other combinatorial structures as integer partitions [3], compositions [10], and bargraphs [12].

![Figure 5. Corners of a path.](image)

Let $\tau(P)$ denote the number of corners of $P$. We define the bivariate generating function

$$W_\ell(x, y) := W_\ell = \sum_{P \in S_\ell} x^{u(P)} y^{\tau(P)}.$$

In this section, we analyze the cases $\ell = 1$ and $\ell = 2$. We leave as an open question the case $\ell \geq 3$. 
Theorem 4.1. The generating function $W_1(x, y)$ satisfies the functional equation

$$xy(1 + y)W_1^3 - (2 - x(2 - y^2))W_1^2 + 3(1 - x)W_1 + x - 1 = 0.$$ 

Proof. Let $\mathcal{D}$ and $\mathcal{L}$ denote the skew Dyck paths whose last step is a down-step or a left-step, respectively. Let $D$ and $L$ denote the generating functions defined by

$$D = \sum_{P \in \mathcal{D}} x^{u(P)} y^{\tau(P)} \quad \text{and} \quad L = \sum_{P \in \mathcal{L}} x^{u(P)} y^{\tau(P)}.$$ 

A non-empty skew Dyck path can be uniquely decomposed as either $UT_1L$ or $UT_2DT_3$, where $T_1, T_2,$ and $T_3$ are lattice paths in $S_1$ with $T_1$ non-empty. In the first case, $T_1$ has two options: the last step is a down-step or a left step, see Figure 6. Then, this case contributes to the generating function the term $x(yD + L)$.

![Figure 6. Decomposition of a skew Dyck path.](image1)

On the other hand, $T_2$ can be an empty path or a path in $\mathcal{D}$ or $\mathcal{L}$. If $T_3$ is empty, then this case contributes to the generating function the term $x(y + D + Ly)$. On the other hand, if the path $T_3$ is non-empty, then this case contributes to the generating function the term $x(y + D + yL)y(W_1 - 1)$, see Figure 7. Summarizing these cases, we obtain the functional equation

$$W_1 = 1 + x(yD + L) + x(y + D + yL)(1 + y(W_1 - 1)).$$

From a similar argument, we obtain the equations

$$D = x(y + D + yL)(1 + yD) \quad \text{and} \quad L = x(yD + L) + x(y + D + yL)(yL).$$

![Figure 7. Decomposition of a skew Dyck path.](image2)

Using the Gröbner basis on the polynomial equations for $W_1, D,$ and $L$, we obtain the desired result. 

\qed
We can use a symbolic software computation to obtain the first few terms of the formal power series of $W_1(x, y)$ as follows:

\[
W_1(x, y) = 1 + xy + x^2(y + y^2 + y^3) + x^3(y + 2y^2 + 4y^3 + 2y^4 + y^5) + x^4(y + 3y^2 + 9y^3 + 9y^4 + 10y^5 + 3y^6 + y^7) + \cdots.
\]

From the equation given in Theorem 4.1, we obtain

\[
3xS^3(x) + 6xS^2(x)K(x) - 2xS^2(x) - 2(2 - x)S(x)K(x) + 3(1 - x)K(x) = 0,
\]

where $K(x)$ is the generating function for the total number of corners in skew Dyck paths and $S(x) = (1 - x - \sqrt{1 - 6x + 5x^2})/(2x)$ is the generating function for the number of the skew Dyck paths. Solving the above equation, we obtain the generating function

\[
K(x) = \frac{2(1 - x)(3 + x)x}{(1 - x)(3 - 2x)(1 - 5x) + (3 - 11x + 4x^2)\sqrt{1 - 6x + 5x^2}} = x + 6x^2 + 30x^3 + 145x^4 + 695x^5 + 3327x^6 + 15945x^7 + \cdots.
\]

**Theorem 4.2.** The generating function $W_2(x, y)$ satisfies the functional equation

\[
x^2y^4(1 + y)^3W_2^6 - xy^2(1 + y)^2(1 - x(1 + 6y + y^2 - 3y^3))W_2^5 + xy(-4 - 7y + 3y^3 + x(1 + y)^2(4 + 9y - 11y^2 - 6y^3 + 3y^4))W_2^4 + (4 - 2x(1 + y)^2(4 - 7y + y^2) - x^2(1 + y)^2(-4 + 2y + 21y^2 - 8y^3 - 5y^4 + y^5))W_2^3 + (-12 - x^2(1 + y)^2(8 + 4y - 18y^2 + 4y^3 + y^4) - 2x(-10 - 9y + 6y^2 + 6y^3 + y^4))W_2^2 + (12 + x^2(1 + y)^2(5 + 4y - 7y^2 + y^3) + x(-17 - 16y + 2y^2 + 4y^3 + 3y^4))W_2 + (-4 + x^2(1 + y)^2(-1 - y + y^2) + x(5 + 4y - y^4)) = 0.
\]

**Proof.** Let $D_2$ and $L_2$ denote the 2-Fuss-skew paths whose last step is a down-step or a left-step, respectively. Let $D_2$ and $L_2$ denote the generating functions defined by

\[
D_2 = \sum_{P \in D_2} x^u(P)y^\tau(P) \quad \text{and} \quad L_2 = \sum_{P \in L_2} x^u(P)y^\tau(P).
\]

From a similar argument as in the proof of Theorem 4.1, we obtain the system of polynomial equations

\[
W_2 = 1 + x((y + yD_2 + L_2)(1 + y^2D_2 + yL_2)(1 + y(W_2 - 1)) + (D_2 + yL_2)) + (D_2 + yL_2)y(1 + y(W_2 - 1)) + (y + yD_2 + L_2)(y + yD_2 + y^2L_2)),
\]

\[
D_2 = x((y + yD_2 + L_2)(1 + y^2D_2 + yL_2)yD_2 + (D_2 + yL_2) + (D_2 + yL_2)y(yD_2) + (y + yD_2 + L_2)(y + yD_2 + y^2L_2)),
\]

\[
L_2 = x((y + yD_2 + L_2)(1 + y^2D_2 + yL_2)(1 + yL_2) + (D_2 + yL_2)y(1 + yL_2)).
\]

By using the Gröbner basis, we obtain the desired result. \qed
Expanding with Mathematica the functional equation for $W_2$, we find
\[
W_2(x, y) = 1 + (y + y^2)x + (y + 3y^2 + 5y^3 + 4y^4 + y^5)x^2 \\
+ (y + 5y^2 + 16y^3 + 27y^4 + 33y^5 + 25y^6 + 9y^7 + 2y^8)x^3 + \cdots.
\]
Moreover, the first few terms of the total number of corners in $S_2$ are
\[
3x + 43x^2 + 561x^3 + 92703x^5 + 1197151x^6 + 15532917x^7 + 202428373x^8 + \cdots.
\]
From Figure 4 one can verify that there are 43 corners over all paths in $S_{2, 2}$.

5. Other generalization

Let $H_\ell$ denote the skew Dyck paths where left steps are below the line $y = \ell$. In particular, $H_0$ are the Dyck path and $H_\infty$ are the skew Dyck path. We define the generating function
\[
H_\ell(x, p, q) := \sum_{P \in H_\ell} x^{u(P)} p^{d(P)} q^{t(P)}.
\]
For simplicity, we use $H_\ell$ to denote the generating function $H_\ell(x, p, q)$.

**Theorem 5.1.** For $\ell \geq 1$, we have
\[
H_\ell = 1 + qx(H_{\ell-1} - 1) + pxH_{\ell-1}H_\ell, \tag{5.1}
\]
with the initial value $H_0 = \frac{1 - \sqrt{1 - 4px}}{2px}$.

**Proof.** A non-empty skew Dyck path in $H_\ell$ can be decomposed as $UT_1L$ or $UT_2DT_3$, where $T_1, T_2 \in H_{\ell-1}$ with $T_1$ a non-empty path, and $T_3 \in H_\ell$. From this decomposition follows the functional equation.

Recall that the $m$th Chebyshev polynomial of the second kind satisfies the recurrence relation $U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t)$ with $U_0(t) = 1$ and $U_1(t) = 2t$. Thus by induction on $\ell$ and Theorem 5.1, we obtain the following result.

**Theorem 5.2.** Let $t = \frac{1 + qx}{2\sqrt{x(p + q - pqx)}}$ and $r = \sqrt{x(p + q - pqx)}$. The generating function $H_\ell$ is given by
\[
\frac{(qxU_{n-1}(t) - rU_{n-2}(t))C(px) + (1 - qx)U_{n-1}(t)}{U_{n-1}(t) - rU_{n-2}(t) - pxU_{n-1}(t)C(px)},
\]
where $U_m$ is the $m$th Chebyshev polynomial of the second kind and $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$ the generating function for the Catalan numbers $\frac{1}{n+1} \binom{2n}{n}$. 
The generating functions for the total number of skew Dyck path in $\mathbb{H}_\ell$ for $\ell = 1, 2, 3$ are

$$H_1(x, 1, 1) = \frac{3 - 2x - \sqrt{1 - 4x}}{1 + \sqrt{1 - 4x}},$$

$$H_2(x, 1, 1) = \frac{1 + 2x - 2x^2 - (1 - 2x)\sqrt{1 - 4x}}{1 - x - 2(1 - x)x + (1 + x)\sqrt{1 - 4x}},$$

$$H_3(x, 1, 1) = \frac{1 - 3x + 7x^2 - 4x^3 + (1 + x - 3x^2)\sqrt{1 - 4x}}{1 - 4x + 2x^3 + (1 + 2x^2)\sqrt{1 - 4x}}.$$

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