Vortex Fractionalization in a Josephson Ladder

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We show numerically that, in a Josephson ladder with periodic boundary conditions and subject to a suitable transverse magnetic field, a vortex excitation can spontaneously break up into two or more fractional excitations. If the ladder has \( N \) plaquettes, and \( N \) is divisible by an integer \( q \), then in an applied transverse field of \( 1/q \) flux quanta per plaquette the ground state is a regular pattern of one fluxon every \( q \) plaquettes. When one additional fluxon is added to the ladder, it breaks up into \( q \) fractional fluxons, each carrying \( 1/q \) units of vorticity. The fractional fluxons are basically walls between different domains of the ground state of the underlying \( 1/q \) lattice. The fractional fluxons are all depinned at the same applied current and move as a unit. For certain applied fields and ladder lengths, we show that there are isolated fractional fluxons. It is shown that the fractional fluxons would produce a time-averaged voltage related in a characteristic way to the ac voltage frequency.

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I. INTRODUCTION

Ladder arrays of Josephson junctions have been extensively studied both theoretically and experimentally (see, for example, Refs. 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17). They are of interest, in part, because they show a rich variety of equilibrium and dynamical behavior. In addition, they are an important testing ground for concepts of phase transitions and quantum behavior in lower dimensionality. The interest in such materials is even greater now, because advances in microfabrication allow such ladders to be made almost to specifications, in order to test theoretical predictions or possibly to make new types of Josephson devices.

Recently Chandran and Kulkarni 18 considered the behavior of a hypothetical Josephson ladder array with alternating \( \pi \) and 0 junctions. A \( \pi \) junction is one in which the Josephson supercurrent \( I = I_c \sin(\Delta \phi + \pi) \), where \( \Delta \phi \) is the phase difference across the junction and \( I_c > 0 \). Such junctions can be formed in a variety of ways, e.g., by controlling the energy distribution of the current-carrying states in the normal metal within the junction 19, by preparing a junction between two cuprate superconductors in the presence of bound states in the interface material 20, or by connecting two superconductors across a ferromagnetic layer 21. Thus, an array of alternating 0 and \( \pi \) junctions could, in principle, be made in the laboratory. In Ref. 18, it was shown that when a \( 2\pi \) fluxon is introduced into such a \( 0 - \pi \) ladder, it will break up into two separate \( \pi \) fluxons, each of magnitude \( \Phi_0/2 \). This fractionalized vortex was predicted to have unusual current-voltage characteristics which could readily be detected experimentally.

In this paper, we show that similar fractionalized vortices are produced in conventional ladder Josephson arrays made of 0 junctions, in a transverse magnetic field. Specifically, we consider a ladder having \( N \) square plaquettes, where \( N \) is divisible by an integer \( q \), in a transverse magnetic field. If that field is of magnitude equal to one flux quantum per \( q \) plaquettes, the ground state of the ladder is a periodic array of \( N/q \) equally spaced fluxons. If now one fluxon is added to that ladder, we find that the added fluxon breaks up into \( q \) fractional fluxons, each of magnitude \( \Phi_0/q \). If \( q = 2 \), the fluxon pattern is similar to that found in Ref. 18, but we find analogous patterns for all other values of \( q \) which we have tested. These extra fluxons have IV characteristics and vorticity patterns which should be readily detectable experimentally. Furthermore, the fractionalized fluxons are typically ”confined” that is, they are typically depinned at the same current and move at the same velocity. For an array in which \( N \) is not divisible by an integer \( q \), we show numerically that it is possible to produce isolated fractional fluxons, by applying a transverse magnetic field of suitable magnitude.

The remainder of this paper is arranged as follows. In Section II, we develop the equations of motion from a standard Lagrangian for the Josephson ladder. In Section III, we give our numerical results, which show that fractional vortices are generated in the ladder at suitable applied transverse fields. Finally, in Section IV, we briefly compare our results to other models which give rise to fractional excitations, and give a concluding discussion.

II. FORMALISM

We consider a Josephson ladder with periodic boundary conditions, as shown schematically in Fig. 1. The ladder is assumed to consist of a collection of small junctions, each of critical current \( I_c \), which are inductively coupled together. The edges of the ladder are parallel to the x axis, while the rungs are in the y direction. A dc current \( I \) is injected into each junction on one edge of the ladder and extracted from each junction on the other edge. A magnetic field \( \mathbf{B} = B \hat{z} \) is applied perpendicular to the ladder. The geometry is readily achievable experi-
mentally, in the form of a coplanar ring perpendicular to \( B \).

In the absence of dissipation, it is convenient to describe the ladder by the following Lagrangian:

\[
\mathcal{L} = K - V
\]

where

\[
K = \sum_{i=1}^{N} \frac{\hbar^2}{2U} (\dot{\theta}_{i,u} - \dot{\theta}_{i,l})^2
\]

and

\[
V = \sum_{i=1}^{N} \left[ -J_1 \cos(\theta_{i,l} - \theta_{i,u} - A_i) + J_2(\theta_{i,l} - \theta_{i+1,l})^2 + J_3(\theta_{i,u} - \theta_{i+1,u})^2 - J_4(\chi_i - \chi_{i+1})^2 + (\phi_i - \phi_{i+1})^2 \right]
\]

Here \( K \) represents the charging energy of the small junctions, \( U = 4e^2/C \), and \( C \) is the capacitance of the small junction. \( J_1 = hI_c/2e \) is the Josephson coupling energy of the small junction, \( J_3 = hI/(2e) \), and \( J_2 = h^2/(8e^2L) \), where \( L \) is the inductance of the wire segments coupling points \( (i, \ell) \) to \( (i+1, \ell) \) and \( (i, u) \) to \( (i+1, u) \). Finally, if we use the gauge \( A = Bx\hat{y} \), where \( A \) is the vector potential, then \( A_i = (2\pi/\Phi_0) \int_{i,u} \mathbf{A} \cdot d\mathbf{l} = (2\pi/\Phi_0)a^2iB \), where \( \Phi_0 = hc/(2e) \) is the flux quantum, \( a \) is the lattice constant (see Fig. 1), and the subscripts \( \ell \) and \( u \) indicate the lower and upper ends of the junction, as shown in the Figure. The dots indicate time derivatives. Note that we neglect the self-induced magnetic field, by taking \( B \) equal to the externally applied field.

If we now make the substitutions \( \phi_i = \theta_{i,u} - \theta_{i,l} \), \( \chi_i = (\theta_{i,l} + \theta_{i,u})/2 \), we may reexpress the Lagrangian as

\[
\mathcal{L} = \sum_{i=1}^{N} \left[ \frac{E_c}{2} \dot{\phi}_i^2 + J_1 \cos(\phi_i - A_i) + J_3\phi_i + \frac{J_2}{2}(\chi_i - \chi_{i+1})^2 + (\phi_i - \phi_{i+1})^2 \right]
\]

The Lagrange equations of motion are then \((d/dt(\partial\mathcal{L}/\partial\dot{\phi})) = \partial\mathcal{L}/\partial\phi \), with an analogous equation for the variables \( \chi_i \). Since \( \mathcal{L} \) is independent of \( \chi_i \), the equation of motion for \( \chi_i \) leads to the condition that \( \chi_i - \chi_{i+1} \) is independent of \( i \). After some algebra, it is readily shown that the equation of motion for \( \phi_i \) is

\[
\ddot{\phi}_i = -\sin(\phi_i + 2\pi i f) + \lambda_J^2(\phi_{i-1} - 2\phi_i + \phi_{i+1}) + (I/I_c)\phi_i - \dot{\phi}_i/Q_J.
\]

Here, we have introduced the frustration \( f = Ba^2/\Phi_0 \), and a dimensionless time \( \tau = \omega_p t \), where \( \omega_p = [2eI_c/(hC)]^{1/2} \) is the junction plasma frequency, and \( \lambda_J = (J_2/J_1)^{1/2} \) is a dimensionless Josephson penetration depth. The dot now represents a derivative with respect to \( \tau \). Finally, we have added an additional dissipative term \(-\dot{\phi}_i/Q_J\) by hand to the equation of motion, where \( Q_J \) is a dimensionless junction quality factor, assumed to be the same for all junctions. In the resistively and capacitively shunted junction model (RCSJ) \( Q_J = (2eR^2L_c/C^2/(\hbar))^{1/2} \), where \( R \) is the junction shunt resistance.

III. NUMERICAL RESULTS

A. Numerical Method

In most of our calculations, we have studied a ladder with \( N = 120 \) rungs and periodic boundary conditions. This number of rungs was chosen because it is the smallest integer divisible by 2, 3, 4, 5 and 6, and is sufficiently large to allow the fractionalization to be apparent in the simulations. We solved our set of 240 first order, nonlinear, differential equations using a constant-time-step, fourth-order Runge-Kutta method \[23\]. We have generally considered underdamped junctions, with \( Q_J = 10 \), and values of \( \lambda_J^2 \) between 0.01 and 10. We initialized the system in a random phase configuration at \( \tau = 0 \) and \( I/I_c = 0 \). The equations are integrated forward in time up to \( \tau_{eq} = 3000 \), by which time we assume the system has reached a steady state. We then calculate the time-averaged voltage drop across a rung of the ladder (also averaged over the rungs) by averaging over an additional \( \tau_{max} - \tau_{eq} \) dimensionless time units, i.e.,

\[
\frac{<V>}{RI_c} = \frac{1}{NQ_J(\tau_{max} - \tau_{eq})} \sum_{i=1}^{N} \int_{\tau_{eq}}^{\tau_{max}} \frac{d\phi_i}{d\tau} d\tau,
\]

where \( \tau_{max} = 5000 \). The dc driving current is then increased by 0.01 \( I_c \) and the calculation is repeated. We
FIG. 2: (a) IV curves for \( f = 1/2 \), \( Q_J = 10.0 \), and \( \lambda_J^2 = 1.0 \) for increasing and decreasing current (the directions are indicated by arrows). There are two voltage jumps in the IV curve on the increasing branch: at the first, the fluxon lattice is depinned; at the second, the junctions switch to the voltage state. (b) Schematic of ground state vortex number configuration for the \( f = 1/2 \) ladder at \( I/I_c = 0 \); the +'s and blank squares represent vortex numbers \( n = 1 \) and \( n = 0 \). Only 40 plaquettes are shown.

B. \( f = 1/2 \)

In Fig. 2(a), we show the full IV curve for \( f = 1/2 \), \( Q_J = 10 \), and \( \lambda_J^2 = 1 \). The directions of the arrows indicate whether the current is increasing or decreasing. The IV curve is clearly hysteretic for this value of \( Q_J \), and has two discontinuous jumps on the increasing current branch. At the lower jump, near \( I/I_c = 0.13 \), the system jumps into a state where the fluxon lattice is depinned and starts to move through the ladder as a unit, giving rise to a finite voltage across the ladder. At the upper jump, near \( I/I_c = 0.5 \), all the junctions switch to a finite voltage state and the fluxon excitations are expelled from the ladder.

To represent the fluxon lattice pictorially, we use the concept of a vortex number. The vortex number of the \( \alpha \)th plaquette is defined as

\[
\alpha = f + \frac{1}{2\pi} \sum_{\text{plaquette}} (\theta_i - \theta_j - A_{ij}).
\]

Here the gauge-invariant phase difference for each leg of the plaquette is written \((\theta_i - \theta_j - A_{ij})\) and is defined to lie in the range \([-\pi, \pi]\); the summation is taken around the plaquette in the counterclockwise direction (viewed from the positive \( z \) axis). Thus \( \alpha \) is 0 or a positive or negative integer for each plaquette. In the lowest energy state, we expect that \( \sum_{\alpha} n_\alpha = N_f \), where \( N \) is the number of plaquettes, and also that all the individual \( n_\alpha = 0 \) or 1.

The two values of \( n_\alpha \) can generally be distinguished by the current pattern around the plaquette: \( n_\alpha = 1 \) and 0 correspond to a clockwise and counterclockwise current flow, respectively. The expected ground state for \( f = 1/2 \) at \( I/I_c = 0 \) is shown pictorially in Fig. 2(b). The +\(^\prime\)s and empty squares represent plaquettes of \( n_\alpha = 1 \) and 0, respectively.

In practice, we have not found a way to calculate \( n_\alpha \) directly, because we do not compute the variables \( \chi_i \) mentioned above. Therefore, in order to confirm the vortex pattern at various fields and currents, we calculate the "normalized flux," defined by the relation \( 2\pi \Phi_f/\Phi_0 = \phi_i - \phi_{i+1} + 2\pi f \), in each plaquette. We show a number of examples of this normalized flux below.

When the vortex lattice is depinned near \( I/I_c = 0.13 \), the entire lattice of +\(^\prime\)s moves as a unit through the ladder, generating a voltage. In Fig. 2 we show the space-averaged voltage

\[
\frac{V(\tau)}{RI_c} = \frac{1}{NQ_J} \sum_n \phi_n(\tau).
\]

across the ladder for \( 4500 < \tau < 4520 \), for the parameters of Fig. 2(a) at \( I/I_c = 0.13 \), just above the lower jump in the IV characteristics. We interpret the periodic voltage oscillations as arising from the motion of the fluxon lattice through the bumpy "egg-carton" potential provided by the plaquettes of the Josephson ladder: the velocity of the fluxon lattice varies periodically in time because the fluxon lattice moves alternately faster and slower as it moves through the steeper and less steep part of the egg carton.

To confirm this picture, we note from Fig. 2 that the dimensionless period of the voltage oscillation, which we denote \( \Delta \tau \), is \( \sim 2.7 \). We expect that this should be the dimensionless time required for the fluxon lattice to advance one lattice constant \( a \). Now if the fluxon lattice moves a distance \( a \) during this period, then the average speed of the fluxon lattice is \( v = (Na)/T \), where \( T \) is the time required for the lattice to complete one circuit around the ladder. During this time, each junction should experience a phase change of \( 2\pi N f \), since \( N_f \) fluxons cross each junction during one circuit. Thus, the time-averaged voltage \( \langle V \rangle \) across each junction should be \( [h/(2e)]2\pi N f/\Delta \tau \). Introducing a dimensionless voltage \( \langle V \rangle/(RI_c) \), and noting that the time required to move one lattice constant \( a \) is \( \omega_c T/N = \Delta \tau \), we obtain

\[
\langle V \rangle/(RI_c) = \frac{2\pi f}{Q_J} \frac{\Delta \tau}{\Delta \tau}.
\]

Using the parameters of Fig. 2(a), together with \( I/I_c = 0.13 \) and \( \Delta \tau = 2.7 \), eq. 3 gives \( \langle V \rangle/(RI_c) = \)
is increased still further, for moving vortex lattice continues to hold. As the current rents slightly higher than the washboard potential provided by the ladder. For current $I/I_c$ slightly larger than the washboard potential provided by the ladder, we show in Figs. 6 (b) and (c) two possible scenarios for how the extra fluxon enters the ladder. However, this fluxon does not enter the ladder as a unit. Instead, it is fractionalized into two 1/2-fluxons, each of which carries 1/2 a unit of flux. These two 1/2-fluxons obviously repel one another, since they prefer to be situated as far away from one another as possible, i.e., about 60 plaquettes apart in the 120-plaquette ladder.

To suggest intuitively why the extra fluxon fractionalizes in the ladder, we show in Figs. 5 and 6 a two possible scenarios for how the extra fluxon enters the ladder. In (b), the fluxon enters as a unit, without distorting the underlying f = 1/2 fluxon lattice. But this scenario is periodic with a period of two unit cells, as suggested by Fig. 2(b). The two ground state fluxes are approximately $2\pi f \pm \Phi_0$ with $f = 1/2$. A similar pattern has been seen previously in Ref. 18 for a ladder with alternating 0 and $\pi$ junctions, and zero applied magnetic field.

C. $f = 1/2 + 1/N$

Next, we consider the case $f = 1/2 + 1/120$, equivalent to adding one fluxon to the $f = 1/2$ ladder. The calculated results are shown in Figs. 4 and 6. Fig. 4 shows the IV characteristic as computed on increasing and decreasing the current in steps of 0.01$I_c$. On the increasing branch, there are now three steps in the IV curve. Two of the steps are easily seen from the Figure, but the lowest jump, which occurs near $I/I_c = 0.01$, is less visible and shown in the inset for $0 < I/I_c < 0.15$. The IV curve is hysteretic in two regions of current, but non-hysteretic at the lowest non-zero voltages. On the decreasing branch there are still three distinct regimes of non-zero voltage. We interpret these as follows: (i) at the lowest currents, a single fluxon is depinned and moves through the fluxon lattice formed by the remaining 60 fluxons; (ii) at higher currents, the fluxon lattice is depinned, as at $f = 1/2$, generating a larger voltage; and finally (iii) at still higher voltage, all the junctions switches to a voltage state and the ladder jumps up to the resistive branch.

Fig. 4(a) shows the flux pattern $2\pi \Phi_0/\Phi_0$ for $f = 61/120$ and $I/I_c = 0$, keeping the rest of the parameters the same as in Fig. 2(a). The $f = 1/2$ pattern is now distorted in two regimes, near $i = 50$ and $i = 110$, while in the rest of the ladder the pattern is similar to that of $f = 1/2$. To compare the two patterns more quantitatively, we note that the sums $\sum \Phi_i/\Phi_0$ for Figs. 4 and 2(a) are 60$(2\pi)$ and 61$(2\pi)$ respectively. This result shows that the effect of the additional magnetic field is to add one additional fluxon to the ladder. However, this fluxon does not enter the ladder as a unit. Instead, it is fractionalized into two 1/2-fluxons, each of which carries 1/2 a unit of flux. These two 1/2-fluxons obviously repel one another, since they prefer to be situated as far away from one another as possible, i.e., about 60 plaquettes apart in the 120-plaquette ladder.

In Fig. 4 we plot the calculated "normalized flux" $\Phi_i/\Phi_0$ for $f = 1/2$ and $I/I_c = 0$. Clearly, $\Phi_i/\Phi_0$ alternates between two values, indicating that the flux lattice...
Thus, at a current increasing and decreasing current (as indicated by arrows). The inset shows the IV curve for $0 < I/I_c < 0.15$. A finite voltage in this regime is produced by the two moving 1/2-fluxons.

The fluxon shown in Fig. 5 (b). We have also found that both halves of the fractionalized fluxon are deinned at the same $I/I_c$, and move together.

Finally, we consider the expected time-dependent voltage for the two moving 1/2-fluxons in the 61/120 ladder. As explained above, each 1/2 fluxon is really a kind of domain wall, consisting of two adjacent + vortex charges. The expected mechanism by which a 1/2-fluxon moves is that one of the two adjacent + charges moves to the nearest empty plaquette. This jump causes the domain wall to move by two plaquettes. Thus, if the ladder has length $N$ plaquettes, a 1/2-fluxon will circulate around the ladder twice in $N$ moves. This causes a phase slip of $2\pi$, since there is a $\pi$ phase slip each time the 1/2-fluxon circulates once around the ladder. Thus, if $T$ is the time required for the 1/2-fluxon to circulate once around the ladder, the time-averaged voltage is $\langle V \rangle = (\hbar/2e)\pi/T$. On the other hand, $V(t)$ for a single 1/2-fluxon should have an ac component with a period of $1/\nu = 2T/N$, where $\nu$ is the ac frequency, since $2T/N$ is the time required for a 1/2-fluxon to make a single move of two plaquettes.

Combining these two relations, we get $\langle V \rangle = [\hbar/(2eN)]\nu$ for a single 1/2-fluxon. This is precisely the relation between $\langle V \rangle$ and $\nu$ which we find for a ladder containing a single 1/2-fluxon, as is discussed later in this paper.

For the present case of a 61/120 ladder, there are, as already mentioned, two 1/2-fluxons, as shown in Fig. 6. Thus, at a current $I$ in the regime where the two 1/2-fluxons are moving, we expect that $\langle V \rangle$ will equal twice this value, or

$$\langle V \rangle = 2[\hbar/(2eN)]\nu,$$  \hspace{1cm} (10)

where $\nu$ represents the frequency of the ac voltage generated by the motion of a single 1/2-fluxon at the same current. We can rewrite this equation in terms of $Q_J$ and $\Delta \tau \equiv \omega_p/\nu$ with the result

$$\frac{\langle V \rangle}{RI_c} = \frac{2 \cdot (2\pi)}{NQ_J\Delta \tau}.$$  \hspace{1cm} (11)

In the present case $N = 120$ so the right-hand side equals $2\pi/(60Q_J\Delta \tau)$. The calculated $\langle V \rangle$, in the regime where the two 1/2-fluxons are moving, agrees very well with the prediction of eq. (11). Thus, the IV characteristics in this regime are produced by two 1/2-fluxons moving with the same average velocity.

### D. $f = 1/3$

If an added fluxon fractionalizes when added to a ladder at $f = 1/2$, what happens at other values of $f$? To answer this question, we have carried out similar calculations for $1/3, 1/4, 1/5$ and $1/6$ in an $N = 120$ ladder, as well as for each of these values of $f$ with one fluxon added.
or subtracted. The calculated IV characteristics and normalized flux for \( f = 1/3 \) are shown in Figs. 4 and 4(a). The behavior of the IV curves shown in Fig. 4 is most easily understood on the decreasing current branch. For the highest currents, the junctions are all in the voltage state, and the fluxon lattice has disappeared. At intermediate currents, the \( f = 1/3 \) fluxon lattice is moving through the washboard potential provided by the ladder. At the lowest currents, the fluxon lattice is pinned and the time-averaged voltage vanishes. On the increasing current branch, the fluxon lattice appears to depin in two stages, corresponding to the two voltage jumps visible in the Figure.

Fig. 8(a) shows the normalized flux \( \Phi_i/\Phi_0 \) versus plaquette number \( i \) at \( I/I_c = 0 \) and figure 8(b) shows the vortex charge pattern configuration in the equilibrium state. The flux pattern is again periodic, as at \( f = 1/2 \), but now with a period of three plaquettes, with fluxes \( 2\pi f + f, 2\pi f + f, \) and \( 2\pi f - 2f \).

As a further check that the fluxon lattice really moves as a unit, we have used eq. 9 to compare the time-averaged voltage \( \langle V \rangle \) at \( I/I_c = 0.04 \), on the upward current sweep just above the first voltage jump, to the dimensionless period \( \Delta \tau \) of the a. c. component. The prediction for \( \langle V \rangle/(RL_c) \) from eq. 9 (namely \( \langle V \rangle/(RL_c) = 0.0381 \)) is in excellent agreement with the value calculated by a direct time-average of \( \langle V \rangle/(RL_c) \) (0.0372).

**E. \( f = 1/3 + 1/N \)**

Next, we consider the behavior of the same system with one additional fluxon. Fig. 9 shows the IV curve, and Fig. 10(a) displays the normalized flux vs. plaquette number at \( I/I_c = 0 \), for \( f = 1/3 + 1/120 \). In this case, the additional fluxon fractionizes into three excitations. These are visible in Fig. 10(a) as regions which are distorted, in comparison to the flux plot at \( f = 1/3 \).

As in Fig. 6(a), these regions show how the \( f = 1/3 \) flux pattern changes to accommodate the additional flux quantum. To understand why this extra fluxon fractionalizes into three parts, we show in Figs. 10(b) and (c) two possible patterns of vortex number when the additional fluxon is added. The \( f = 1/3 \) ground state consists of a +1 vortex every three plaquettes. If the additional fluxon is un-fractionalized, we expect that it will enter the ladder as in Fig. 10(b). This pattern must have two neighboring plaquettes with a +1 vortex number, and hence is
a relatively high-energy state. It is much more plausible that the lowest-energy pattern would never have +1 vortices in adjacent plaquettes. Such a pattern is shown in Fig. (c). The 1/3 fluxon shows up as a region where two +1 vortices are separated by only one empty plaquette, rather than two as in the unperturbed $f = 1/3$ pattern. To conserve the number of +1 plaquettes, there must be three such regimes; hence, the added fluxon is expected to fractionalize into three 1/3-fluxons, as we find numerically. These 1/3-fluxons are equidistant in order to minimize the repulsive energy between these added excitations.

When the extra fractionalized fluxon moves through the lattice, it generates voltage. Once again, we can use the analog of eq. (14) to connect the time-averaged voltage to the oscillation period of $V(\tau)$. This analog is

$$\frac{\langle V \rangle}{RI_c} = \frac{3 \cdot (2\pi)}{NQf \Delta \tau}.$$  \hspace{1cm} (12)

Here $\Delta \tau$ is now the period of the a. c. voltage in the regime where three 1/3-fluxons are moving, measured in units of $1/\omega_p$. When we use this relation to calculate $\langle V \rangle$ from our calculated $\Delta \tau$, at $I/I_c = 0.02$ (in the regime of three moving 1/3-fluxons), we find that the value of $\langle V \rangle$ obtained from eq. (12) is in excellent agreement with the value of $\langle V \rangle$ calculated directly.

F. $f = 1/q$ and $f = 1/q + 1/N$, with $q > 3$

We have also carried out similar calculations for $1/4$, $1/5$ and $1/6$. The ground state (at $I/I_c = 0$) consists, as expected, of a periodic array of +1 vortices separated by $q - 1$ zero vortices ($q = 4$, 5 or 6). When we carry out the corresponding calculations for $f = 1/4 + 1/N$, $f = 1/5 + 1/N$ and $1/6 + 1/N$, we again find that the added vortex fractionalizes into $q$ equally spaced pieces, as suggested by our results for $q = 2, 3$.

G. Effects of Varying Ladder Length, Inductive Coupling, and Damping Coefficient. Effects of Removing One Fluxon.

We have also studied $f = 1/q + 1/N$ ($q > 1$) for a ladder of double the length, i.e., 240 plaquettes. The flux patterns are very similar to those of the shorter ladders, except that, of course, the fractionalized fluxons are twice as far apart as for $N = 120$; the IV curves are also qualitatively similar.

As another important test case, we have computed both the ground state configurations and IV characteristics for $f \rightarrow f - 1/N$, in order to compare the effects of removing rather than adding one fluxon. (We do not display these results graphically.) In general, the results are similar to those found for adding one fluxon. The results for $f = 1/2 - 1/N$ are nearly identical to those for $f = 1/2 + 1/N$: the missing fluxon is fractionalized into two half-fluxons. For $f = 1/q - 1/N$, with $q > 1$ an integer, the IV and flux characteristics are similar to those for $f = 1/q + 1/N$; however, the flux curves are not identical, except for $q = 2$. It is not surprising that one obtains different results for adding and subtracting a fluxon for $q > 2$: only at $q = 2$ would one expect, on symmetry grounds, that adding and subtracting a vortex would give the same response.

We have also tried varying the strength of the inductive coupling constant $\lambda^2$, considering $\lambda^2 = 0.01, 0.1$ and 1.0 as well as 1 for $f = 1/2$. For $\lambda^2 = 0.01$, the system behaves like a collection of independent junctions: $\langle V \rangle = 0$ for all $I/I_c < 0.9$, and the flux pattern is very similar to that of Fig. 4. For $\lambda^2 = 0.1$ also gives results very similar to those of Fig. 4. But for $\lambda^2 = 10$, the fluxons are depinned even at $I/I_c = 0$, and the IV curve is very similar to that seen in Ref. 20 for a long junction in the continuum limit. We have also carried out calculations (not shown here) for values of $Q_J = 1.0, 5.0$ and 50.0 for $f = 1/2$, $N = 120$ and $\lambda^2 = 1.0$. The position of the first jump in the IV curve remains unchanged even for the critically damped case ($Q_J = 1.0$). We found that, for the larger
FIG. 11: Normalized flux vs. plaquette number, plotted for $N = 121$ and $f = 61/121$.

FIG. 12: IV characteristics for $N = 121$ and $f = 61/121$, for the parameters of the preceding figure. Inset: enlargement of IV characteristics at small currents.

As a final calculation, we have considered a ladder with an odd number of plaquettes, namely $N = 121$, and $f = 61/121$. Intuitively, one might expect that this ladder would correspond to a single added $1/2$-fluxon superimposed on an $f = 1/2$ ground state. The latter corresponds to $f = 1/2 = 60.5/121$, while the extra $1/2$-fluxon is needed in order to bring the total flux up to $f = 61/121$. Indeed, the flux plot, shown in Fig. 11, is consistent with this picture: an alternating flux pattern of the $f = 1/2$ ground state on which is superimposed the flux pattern of a single $1/2$-fluxon, i.e., one of the two excitations distinguishable in Fig. 6.

H. $f = 1/2 + 1/(2N)$: One Added Half-Fluxon

As a final calculation, we have considered a ladder with an odd number of plaquettes, namely $N = 121$, and $f = 61/121$. Intuitively, one might expect that this ladder would correspond to a single added $1/2$-fluxon superimposed on an $f = 1/2$ ground state. The latter corresponds to $f = 1/2 = 60.5/121$, while the extra $1/2$-fluxon is needed in order to bring the total flux up to $f = 61/121$. Indeed, the flux plot, shown in Fig. 11, is consistent with this picture: an alternating flux pattern of the $f = 1/2$ ground state on which is superimposed the flux pattern of a single $1/2$-fluxon, i.e., one of the two excitations distinguishable in Fig. 6.

The IV characteristics of this ladder (shown in Fig. 12), are consistent with this picture. There are three regimes of non-zero voltage (seen most clearly on the decreasing current branch), namely (i) motion of the $1/2$-fluxon through the background of the pinned $f = 1/2$ lattice; (ii) motion of the depinned $f = 1/2$ lattice; and (iii) purely resistive behavior, with all small junctions in the voltage state. The motion of the half-fluxon [in regime (i)] is particularly intriguing. At a given current, the voltage generated by this $1/2$-fluxon is $\sim 1/2$ that produced by the two moving $1/2$-fluxons at the corresponding current in Fig. 6. This result is consistent with the picture that the two $1/2$-fluxons in Fig. 6 move at the same average velocity at any given applied current. In addition, as mentioned earlier, at all currents in this regime, the time-averaged voltage $\langle V \rangle$ generated by the $1/2$-fluxon is related to the ac frequency $\nu$ of that voltage by

$$\langle V \rangle = \frac{2 \pi}{N Q J \Delta \tau},$$

where $\Delta \tau = \omega_p/\nu$. As an illustration, we show in Fig. 13 $V(\tau)/(R I_c)$ for $f = 61/121$ and a ladder of length 121 plaquettes, at a current $I/I_c = 0.03$, in the regime where the voltage is produced by a single $1/2$-fluxon.

IV. DISCUSSION

We have shown that a single fluxon introduced into a Josephson ladder will fractionalize if the ladder is initially in an $f = 1/q$ ground state with $q = 2, 3, 4, 5, \text{or} 6$. This fractionalization leads to a characteristic behavior of the (suitably defined) flux in the ground state at $I/I_c = 0,$
and even for $I/I_c \neq 0$, provided that $I$ is smaller than the critical current for onset of voltage. Once a measurable voltage is obtained across the ladder, for these fields, there is a characteristic series of regimes with increasing current: first, the fractionalized fluxon moves, generating a voltage; then, the underlying fluxon lattice is depinned; and finally, the ladder as a whole switches to the voltage state and the fluxons are expelled from the ladder.

The type of fractional excitation found here are basically walls between different domains of the underlying 1/q ground state. For an applied field of 1/q, there are q discrete flux lattice ground states, each shifted by one plaquette from the next ground state. If two different ground state domains, shifted by one plaquette, are placed next to one another, a 1/q fractional fluxon is formed. Such walls, with fractional vortex charge, are the analogs of fractional excitations well known in many other physics contexts, especially in one dimension. For example, solitons in polyacetylene are fractionally charged excitations which can form between two domains of alternating singly and doubly bonded carbons in this one-dimensional chain compound[28]. The fractional fluxons in the Josephson ladder are unusual because it appears that a fluxon of any rational fraction can form in a suitable applied magnetic field.

The fractional vortices described in this paper generally produce a non-hysteretic IV characteristic, even if the individual junctions which make up the ladder are hysteretic. This behavior is similar to that previously observed for integer vortices in two-dimensional arrays[28], which also move in a non-hysteretic fashion even when the junctions in the array are hysteretic. In the case of the 2D arrays, the explanation for this behavior is that the effective charge factor describing the voltage generated by the moving vortex is $Q_{eff} = (2eR^2I_cC/\hbar)^{1/2}$, where $C$ is the junction capacitance, $R$ the junction shunt resistance, and $I_c$ is the current required to depin a vortex from its equilibrium position within a plaquette of the 2D lattice. Since $I_d \sim 0.1I_c$ in the 2D case, where $I_c$ is the critical current of one junction, $Q_{eff}$ is much smaller than the junction quality factor $(2eR^2I_cC/\hbar)^{1/2}$, and hence the motion of the vortex is overdamped even when the individual junctions are underdamped and hysteretic.

For the ladder arrays, the depinning current $I_d$ for a fractional fluxon is less than 0.01$I_c$ for all cases we have considered in this paper. Hence, even though $Q_J = 10$, the analogous effective quality factor for the fractional fluxon will be less than 0.1. The resulting IV characteristics should thus be non-hysteretic and similar to those of an overdamped Josephson junction, as we have observed numerically (see Fig. 13). However, we have not succeeded in deriving a driven-pendulum-like equation of motion for the fractional vortex, similar to that given in Refs. [28] and [29].

One striking feature of these fractionalized fluxons is that they do not appear to form in two dimensions. When a fluxon is added to an $f = 1/2$ ground state of a square lattice in two dimensions, the fluxon is depinned at a current much lower than that of the $f = 1/2$ lattice, just as in one dimension. However, numerical studies suggest that the fluxon is not fractionalized, but remains a compact object[28]. Fractionalization is apparently easier in 1D because the domain walls are zero-dimensional, and hence require much lower energy to form.

It is of interest to compare this prediction to what we would expect from a single un-fractionalized fluxon in an otherwise empty ladder. Here, the expected mechanism of fluxon motion is that the fluxon moves by one plaquette. An argument similar to that just given yields $\langle V \rangle = \langle h/(2eN) \rangle \nu$.

We can also consider the motion of an un-fractionalized vortex through the background of a pinned $f = 1/q$ fluxon lattice. Since the motion of the single vortex would have a period of $q$ lattice constants, and since there would be a phase slip of $2\pi$ each time the vortex circulates once around the ladder, the equation relating the time-averaged voltage to the period would be $\langle V \rangle/(RL_c) = 2\pi q/(Q_J \Delta \tau)$, where $\Delta \tau$ is the period of the ac voltage, in units of $1/\omega_p$. This relation is the same as would hold for a vortex fractionalized into $q$ parts and moving through the same background.

Thus, the clearest experimental way to see a signature of the fractionalization would be to compare the period of the ac voltages with the time-averaged voltage in a ladder with isolated fractional vortices. Such ladders were discussed earlier - an example is the case of $N = 121$ and $f = 61/121$, which contains a single 1/2-fluxon in a background of an $f = 1/2$ fluxon lattice. Besides searching for a signal of the fractionalized vortices in the IV characteristics, perhaps the flux profile of the fractionalized vortices could also be directly imaged.

As mentioned earlier, the fractional vortices discussed here were previously suggested[18] for a ladder of alternating 0 and $\pi$ junctions. Since fractional vortices can thus appear in both types of ladders, one might ask if any information about the nature of the junctions (i.e., whether they are 0 or $\pi$ junctions) could be gleaned from a simple inspection of the IV characteristics. If all the rungs of the ladder are $\pi$ junctions, they would have the same IV characteristics as a ladder made entirely of 0 junctions, since the presence of $\pi$ junctions would produce no additional frustration in this case. Thus, the IV characteristics would not distinguish between all 0 and all $\pi$ junctions. If the ladder contained alternating 0 and $\pi$ junctions, the IV characteristics, in the presence of a magnetic field, would differ from that for all 0 (or all $\pi$) junctions by being shifted in magnetic field by 1/2 a flux quantum per plaquette - that is, the IV characteristics of the alternating ladder at field $f$ would be the same as that of the 0 or $\pi$ ladder at $f + 1/2$. For a more complicated distribution of 0 and $\pi$ junctions, the IV characteristics of the ladder would be more complicated, and the presence of fractional vortices might be difficult to detect. Thus, the existence of fractional vortices, of itself, might not be sufficient to determine if the ladder were partly...
composed of $\pi$ junctions.

Finally, we note that the present results are all obtained from a classical set of governing equations. If the individual junctions are sufficiently small, we expect that the fractional fluxons would behave like quantum-mechanical particles. Specifically, the phase configuration of the ladder would be described by an N-dimensional wave function $\Psi(\phi_1, ..., \phi_N)$, which would be an eigenfunction of the ladder Hamiltonian (expressed as an operator). It would be of interest to calculate the spectrum of such eigenstates, and to study the behavior of a lattice, in the small junction regime, containing two or more such quantum-mechanical fractional fluxons. One important question would be the statistics (bose, fermi, or fractional) obeyed by such particles. Such questions, though fascinating, are, however, beyond the scope of the present paper. In addition, there should be a broad range of experimental parameters where the classical equations are applicable. Thus, the present predictions should be readily testable experimentally.

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