HODGE THEORY FOR TWISTED DIFFERENTIALS

DANIELE ANGELLA AND HISASHI KASUYA

Abstract. We study cohomologies and Hodge theory for complex manifolds with twisted differentials. In particular, we get another cohomological obstruction for manifolds in class $\mathcal{C}$ of Fujiki. We give a Hodge-theoretical proof of the characterization of solvmanifolds in class $\mathcal{C}$ of Fujiki, first proven by D. Arapura.

Introduction

One of the possible ways of generalizing Kähler structures is to consider Hermitian metrics being locally conformal to a Kähler metric; see, e.g., [17].

More precisely, consider a complex manifold $(M,J)$. Suppose that it admits a $J$-Hermitian metric $g$ being locally conformal to a Kähler metric. That is, for every point $p \in M$, there exist an open neighbourhood $U \ni p$ in $M$, and a smooth function $f \in C^\infty(U;\mathbb{R})$ such that $g = \exp(-f) \tilde{g}$, where $\tilde{g}$ is Kähler. Consider the associated $(1,1)$-forms $\omega := g(J\cdot,\cdot)$ and $\tilde{\omega} := \tilde{g}(J\cdot,\cdot) = \exp(f) \omega$. Since $\tilde{g}$ is Kähler, it follows that $\omega$ satisfies
\[
d\omega + df \wedge \omega = 0.
\]

In fact, by the Poincaré Lemma, the property of $g$ being locally conformal Kähler is characterized by the existence of a $d$-closed 1-form $\phi \in A^1(M;\mathbb{R})$ such that
\[
d\phi \omega = 0
\]
where $d\phi := d + L\phi$, where $L\phi := \phi \wedge \cdot$.

The cochain complex $(A^\bullet(M;\mathbb{R}), d\phi)$ can be considered as the de Rham complex with values in the topologically trivial flat bundle $M \times \mathbb{R}$ with the connection form $\phi$. Hence it is determined by the character $\rho_\phi : \pi_1(M) \to GL_1(\mathbb{K})$ given by $\rho_\phi(\gamma) = \exp \left( \int_{\gamma} \phi \right)$. In particular, it is determined by the cohomology class $[\phi] \in H^1(M;\mathbb{R})$.

On compact Kähler manifolds, the Hodge theory for local systems was developed by the theory of Higgs bundles, see [31]. In this note, we study cohomologies and Hodge theory for general complex manifolds $(M,J)$ with twisted differentials. More precisely, denote by $H^{\bullet\bullet}_{BC}(M)$ the Bott-Chern cohomology of $(M,J)$; in particular, $H^{1,0}_{BC}(M) = \{ \theta \in A^{1,0}(M)[\partial \theta = \overline{\partial} \theta = 0] \}$. For $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, consider the bi-differential $\mathbb{Z}$-graded complex
\[
(A^\bullet(M)_{\mathbb{C}}, \partial_{(\theta_1,\theta_2)}, \overline{\partial}_{(\theta_1,\theta_2)})
\]
where
\[
\partial_{(\theta_1,\theta_2)} := \partial + L_{\theta_2} + L_{\overline{\theta}_2} \quad \text{and} \quad \overline{\partial}_{(\theta_1,\theta_2)} := \overline{\partial} - L_{\overline{\theta}_2} - L_{\theta_1}.
\]

2010 Mathematics Subject Classification. 32C35; 53C30; 53C56; 58A14.
Key words and phrases. twisted differential, local system, Dolbeault cohomology, Bott-Chern cohomology, Hodge decomposition, solvmanifolds, class $\mathcal{C}$ of Fujiki.

The first author is granted with a research fellowship by Istituto Nazionale di Alta Matematica INdAM, and is supported by the Project PRIN “Varietà reali e complesse: geometria, topologia e analisi armonica”, by the Project FIRB “Geometria Differenziale e Teoria Geometrica delle Funzioni”, and by GNSAGA of INdAM. The second author is supported by the JSPS Grant-in-Aid for Research Activity start-up.
Several cohomologies can be defined, and the identity induces natural maps between them:

\[ H^\bullet (A^\bullet (M)_\mathbb{C}; \partial_{(a_1, a_2)}; \overline{\partial}_{(a_1, a_2)}) \]

Here, \( H^\bullet (A^\bullet (M)_\mathbb{C}; d\phi) \), \( H^\bullet (A^\bullet (M)_\mathbb{C}; \partial_{(a_1, a_2)}) \), and \( H^\bullet (A^\bullet (M)_\mathbb{C}; \overline{\partial}_{(a_1, a_2)}) \) denote the cohomology of the corresponding complex, and, in the notation of \([16]\),

\[ H^\bullet (A^\bullet (M)_\mathbb{C}; \partial_{(a_1, a_2)}; \overline{\partial}_{(a_1, a_2)}) := \frac{\ker \partial_{(a_1, a_2)} \cap \ker \overline{\partial}_{(a_1, a_2)}}{\im \partial_{(a_1, a_2)}} \]

\[ H^\bullet (A^\bullet (M)_\mathbb{C}; \partial_{(a_1, a_2)}; \overline{\partial}_{(a_1, a_2)}) := \frac{\ker \partial_{(a_1, a_2)} \overline{\partial}_{(a_1, a_2)}}{\im \partial_{(a_1, a_2)} + \im \overline{\partial}_{(a_1, a_2)}} \]

are the counterpart of Bott-Chern \([13]\) and Aeppli \([1]\) cohomologies.

The above maps are in general neither injective nor surjective: so they do not allow a direct comparison of the cohomologies. Hence we are especially interested in studying the following properties:
- \((M, J)\) is said to satisfy the \(\partial_{(a_1, a_2)}\overline{\partial}_{(a_1, a_2)}\)-Lemma if the natural map \(\iota_{BC, A}\) is injective;
- \((M, J)\) admits the \((\theta_1, \theta_2)\)-Hodge decomposition if the natural maps \(\iota_{BC, 0}\) and \(\iota_{BC, 4R}\) and \(\iota_{BC, 0}\) are isomorphisms.

Admitting the \((\theta_1, \theta_2)\)-Hodge decomposition is a stronger property than satisfying the \(\partial_{(a_1, a_2)}\overline{\partial}_{(a_1, a_2)}\)-Lemma. For \((\theta_1, \theta_2) = (0, 0)\), the above properties are in fact equivalent. See \([16]\) for a proof. If \((M, J)\) admits Kähler metrics, then it satisfies the two properties for any \(\theta_1, \theta_2 \in \mathbb{H}^{1,0}_{BC}(M)\).

**Corollary 2.3.** Let \((M, J)\) be a compact complex manifold endowed with a Kähler metric. Take \(\theta_1, \theta_2 \in \mathbb{H}^{1,0}_{BC}(M)\). Then \((M, J)\) satisfies the \(\partial_{(a_1, a_2)}\overline{\partial}_{(a_1, a_2)}\)-Lemma and admits the \((\theta_1, \theta_2)\)-Hodge decomposition.

In Example 3.2, we study an explicit example on the completely-solvable Nakamura manifold \(X = \Gamma \backslash G\). It is known that \(\Gamma \backslash G\) satisfies the \(\overline{\partial}\)-Lemma, see \([4, \text{Example 2.17}]\). For \(\theta_1 := \frac{d\xi}{\partial_1} \in \mathbb{H}^{1,0}_{BC}(X)\) and \(\theta_2 := 0\), (see page 17 for notation,) it follows that \(X\) does not admit the \((\theta_1, \theta_2)\)-Hodge decomposition, see \([24, \S 8]\). Furthermore, we show that it does not satisfy the \(\partial_{(a_1, a_2)}\overline{\partial}_{(a_1, a_2)}\)-Lemma.

In particular, we are interested in studying the behaviour of \(\partial_{(a_1, a_2)}\overline{\partial}_{(a_1, a_2)}\)-Lemma and \((\theta_1, \theta_2)\)-Hodge decomposition under modifications. We recall that a modification \(\mu: (\tilde{M}, \tilde{J}) \to (M, J)\) is a holomorphic map between compact complex manifolds of the same dimension that yields a biholomorphism \(\mu|_{\tilde{M}\backslash \mu^{-1}(S)}: \tilde{M}\backslash \mu^{-1}(S) \to M\backslash S\) outside the preimage of an analytic subset \(S \subset M\) of codimension greater than or equal to 1. We prove the following results.

**Theorem 2.5 and Theorem 2.6.** Let \(\mu: (\tilde{M}, \tilde{J}) \to (M, J)\) be a proper modification of a compact complex manifold \((M, J)\). Take \(\theta_1, \theta_2 \in \mathbb{H}^{1,0}_{BC}(M)\).

- If \((M, J)\) satisfies the \(\partial_{(\mu^*\theta_1, \mu^*\theta_2)}\overline{\partial}_{(\mu^*\theta_1, \mu^*\theta_2)}\)-Lemma, then \((M, J)\) satisfies the \(\partial_{(a_1, a_2)}\overline{\partial}_{(a_1, a_2)}\)-Lemma.
- If \((M, J)\) satisfies the \((\mu^*\theta_1, \mu^*\theta_2)\)-Hodge decomposition, then \((M, J)\) satisfies the \((\theta_1, \theta_2)\)-Hodge decomposition.

From this, we get another cohomological obstruction for complex manifolds belonging to class \(\mathcal{C}\) of Fujiki. We recall that a compact complex manifold \((M, J)\) is said to be in class \(\mathcal{C}\) of Fujiki \([19]\) if it admits a proper modification \(\mu: (\tilde{M}, \tilde{J}) \to (M, J)\) with \((\tilde{M}, \tilde{J})\) admitting Kähler metrics.

**Corollary 2.7.** Let \((M, J)\) be a compact complex manifold in class \(\mathcal{C}\) of Fujiki. Take \(\theta_1, \theta_2 \in \mathbb{H}^{1,0}_{BC}(M)\). Then \((M, J)\) satisfies the \(\partial_{(\theta_1, \theta_2)}\overline{\partial}_{(\theta_1, \theta_2)}\)-Lemma and admits the \((\theta_1, \theta_2)\)-Hodge decomposition.
The previous results can be adapted to compact complex orbifolds of global-quotient type, namely, quotients of compact complex manifolds by finite groups of biholomorphisms, see Theorem 2.8 and Corollary 2.9.

The second author studied in [24] the property of satisfying the \((\theta_1, \theta_2)\)-Hodge decomposition for any \(\theta_1, \theta_2 \in H_{BC}^{1,0}(M)\). In [24, Theorem 1.7], he proved that a solvmanifold admitting hyper-strong-Hodge-decomposition admits a Kähler metric. Therefore, we get a more direct proof of the characterization of solvmanifolds in class \(C\) of Fujiki.

**Theorem 3.3** (see also [7, Theorem 9], [10, Theorem 1.1]). Let \((M, J)\) be a solvmanifold endowed with a complex structure. If \((M, J)\) belongs to class \(C\) of Fujiki, then it admits a Kähler metric.

This result was firstly proven by D. Arapura [7], by using the fact that “the classes of fundamental groups of compact manifolds in class \(C\) of Fujiki and compact Kähler manifolds coincide”, which is proven by Hironaka elimination of indeterminacies (see [10, Lemma 2.1]). But our proof does not rely on the Hironaka elimination of indeterminacies.

**Acknowledgments.** The first author is greatly indebted to Adriano Tomassini for his constant support and for many useful discussions.

1. Twisted differentials and cohomologies on complex manifolds

1.1. **Twisted differentials.** Let \((M, J)\) be a complex manifold of complex dimension \(n\). For \(K \in \{\mathbb{R}, \mathbb{C}\}\), denote by \(A^*(M)_K\) the space of \(K\)-valued differential forms on \(M\). Consider the cochain complex \((A^*(M)_K, d)\).

For \(\phi \in A^1(M)_K\), we define the operator

\[
\partial_\phi : A^*(M)_K \rightarrow A^{*+1}(M)_K, \quad \partial_\phi(x) := \phi \wedge x.
\]

If \(\phi\) is a \(d\)-closed 1-form, then the operator

\[
d_\phi := d + \partial_\phi
\]

satisfies \(d_\phi \circ d_\phi = 0\). Hence we have the cochain complex

\[
(A^*(M)_K, d_\phi).
\]

Note that \(d_\phi\) satisfies the following Leibniz rule:

\[
\text{for any } \alpha \in A^*(M)_K, \quad [d_\phi, L_\alpha] = L_{d\alpha}.
\]

The cochain complex \((A^*(M)_K, d_\phi)\) is considered as the de Rham complex with values in the topologically trivial flat bundle \(M \times K\) with the connection form \(\phi\). Hence the structure of the cochain complex \((A^*(M)_K, d_\phi)\) is determined by the character \(\rho_\phi : \pi_1(M) \rightarrow GL_1(K)\) given by \(\rho_\phi(\gamma) = \exp \left( \int_\gamma \phi \right)\). In particular, the cochain complex \((A^*(M)_K, d_\phi)\) is determined by the cohomology class \([\phi] \in H^1(M; K)\).

We consider the bi-grading \(A^*(M)_C = A^*(M)\) and the decomposition \(d = \partial + \bar{\partial}\). Take \(\theta_1, \theta_2 \in H_{BC}^{1,0}(M)\). Consider

\[
\partial(\theta_1, \theta_2) := \partial + L_{\theta_2} + L_{\bar{\theta}_1} \quad \text{and} \quad \bar{\partial}(\theta_1, \theta_2) := \bar{\partial} - L_{\theta_2} + L_{\bar{\theta}_1}.
\]

We have

\[
\partial_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} = \bar{\partial}_{(\theta_1, \theta_2)} \bar{\partial}_{(\theta_1, \theta_2)} = \partial_{(\theta_1, \theta_2)} \bar{\partial}_{(\theta_1, \theta_2)} + \bar{\partial}_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} = 0.
\]

Therefore we have the bi-differential \(\mathbb{Z}\)-graded complex

\[
\left( A^*(M)_C, \partial_{(\theta_1, \theta_2)}, \bar{\partial}_{(\theta_1, \theta_2)} \right).
\]

Note that \(\partial_{(\theta_1, \theta_2)}\) and \(\bar{\partial}_{(\theta_1, \theta_2)}\) satisfy the following Leibniz rule:

\[
\text{for any } \alpha \in A^*(M)_C, \quad [\partial_{(\theta_1, \theta_2)}, L_\alpha] = L_{\partial_{\alpha}} \text{ and } [\bar{\partial}_{(\theta_1, \theta_2)}, L_\alpha] = L_{\bar{\partial}_\alpha}.
\]

Note also that the associated total cochain complex is

\[
\left( A^*(M)_C, d_{(\theta_1 + \bar{\theta}_1, \theta_2 - \bar{\theta}_2)} \right).
\]
1.2. **Hodge theory with twisted differentials.** Let \((M, J)\) be a compact complex manifold of complex dimension \(n\). Take a \(J\)-Hermitian metric \(g\) on \(M\). We consider the \((\mathbb{R}\text{-linear, possibly } \mathbb{C}\text{-anti-linear})\) Hodge-\(\ast\)-operator \(\overline{\partial} \colon \Lambda^\ast(M)_K \to \Lambda^{\ast n-\ast}(M)_K\) associated to \(g\). Consider the inner product on \(\Lambda^\ast(M)_K\) given by

\[
(x, y) := \int_M x \wedge \overline{y}.
\]

Consider the adjoint operators \(\partial^\ast\), \(\partial^\ast\), and \(\overline{\partial}^\ast\) of the operators \(\partial\), \(\partial\), and \(\overline{\partial}\), respectively, with respect to \((\cdot, \cdot)\). Then one has

\[
d^\ast = -\overline{\partial} \circ \partial,
\]

\[
\partial^\ast = -\partial \circ \overline{\partial},
\]

\[
\overline{\partial}^\ast = -\overline{\partial} \circ \overline{\partial}.
\]

For \(\phi \in \Lambda^\ast(M)_K\), consider the operator \(L_\phi\) and define its adjoint operator with respect to \((\cdot, \cdot)\):

\[
\Lambda_\phi : \Lambda^\ast(M)_K \to \Lambda^{\ast n-\ast}(M)_K\quad \text{given by }\quad (L_\phi^\ast, \cdot) = (\cdot, \Lambda_\phi \cdot).
\]

For \(x \in \Lambda^{[\ast]}(M)_K\) and \(y \in \Lambda^{[\ast]}(M)_K\), we compute

\[
(L_\phi x, y) = \int_M \phi \wedge x \wedge \overline{y} = (-1)^{\tau([\ast]-\ast)} \int_M x \wedge \overline{\partial} \overline{\phi} \wedge \overline{y} = (-1)^{\tau([\ast]-\ast)} \left( x, \overline{\phi} \overline{y} \right).
\]

Hence we get

\[
\Lambda_\phi |_{\Lambda^{[\ast]}(M)_K} = (-1)^{\tau([\ast]-\ast)} \overline{\partial} \overline{\phi}.
\]

In particular, since the real dimension of \(M\) is even, when \(\phi\) is a 1-form, we have

\[
\Lambda_\phi = \overline{\partial} \overline{\phi}.
\]

For a \(d\)-closed 1-form \(\phi\), by considering the differential \(d_\phi = d + \phi\), the adjoint operator \(d_\phi^\ast\) with respect to \((\cdot, \cdot)\) is given by

\[
d_\phi^\ast = d^\ast + \Lambda_\phi = -\overline{\partial} \circ \partial \phi + \overline{\partial} \overline{\phi} \overline{\partial} = -\overline{\partial} \circ \partial \phi + \overline{\partial} \overline{\phi} \overline{\partial}.
\]

Analogously, for \(\theta_1, \theta_2 \in H^{1,0}_{BC}(M)\), the adjoint operators \(\partial^\ast(\theta_1, \theta_2)\) and \(\overline{\partial}^\ast(\theta_1, \theta_2)\) of the operators \(\partial(\theta_1, \theta_2)\) and \(\overline{\partial}(\theta_1, \theta_2)\), respectively, with respect to \((\cdot, \cdot)\) are

\[
\partial^\ast(\theta_1, \theta_2) = \partial^\ast + \Lambda_{\theta_2} + \Lambda_{\theta_1} = -\partial \circ \overline{\partial} \overline{\theta} + \overline{\partial} \overline{\theta} = -\partial \circ \overline{\partial} \overline{\theta} + \partial \circ \overline{\partial} \overline{\theta}.
\]

and

\[
\overline{\partial}^\ast(\theta_1, \theta_2) = \overline{\partial}^\ast + \Lambda_{\theta_2} + \Lambda_{\theta_1} = -\overline{\partial} \circ \partial \theta + \partial \circ \overline{\partial} \theta = -\overline{\partial} \circ \partial \theta + \partial \circ \overline{\partial} \theta.
\]

Suppose \(g\) is a Kähler metric, with associated Kähler form \(\omega\). Then we have the Kähler identities

\[
\Lambda_\omega \partial - \partial \Lambda_\omega = \sqrt{-1} \overline{\partial}^\ast\quad \text{and} \quad \Lambda_\omega \overline{\partial} - \overline{\partial} \Lambda_\omega = -\sqrt{-1} \partial^\ast.
\]

For a \((1, 0)\)-form \(\theta\), as the local argument for the Kähler identities, see, e.g., [32, Lemma 6.6], we have

\[
\Lambda_\omega L_\theta = L_\theta \Lambda_\omega = \sqrt{-1} \Lambda_{-\theta} = -\sqrt{-1} \Lambda_\theta,
\]

\[
\Lambda_\omega L_\theta = L_\theta \Lambda_\omega = -\sqrt{-1} \Lambda_{-\theta} = \sqrt{-1} \Lambda_\theta.
\]

Hence, for \(\theta_1, \theta_2 \in H^{1,0}_{BC}(M)\), we have

\[
\Lambda_\omega \partial(\theta_1, \theta_2) - \partial(\theta_1, \theta_2) \Lambda_\omega = \sqrt{-1} \overline{\partial}^\ast(\theta_1, \theta_2),
\]

\[
\Lambda_\omega \overline{\partial}(\theta_1, \theta_2) - \overline{\partial}(\theta_1, \theta_2) \Lambda_\omega = -\sqrt{-1} \partial^\ast(\theta_1, \theta_2).
\]
1.3. Cohomologies with twisted differentials. Let $(M, J)$ be a complex manifold; suppose also that $M$ is compact. For $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$ and $\phi := \theta_1 + \overline{\theta}_1 + \theta_2 - \overline{\theta}_2$, we consider the (bi-)differential $\mathbb{Z}$-graded algebras

$$(A^\bullet(M)_{\mathbb{C}}, d_\phi) \quad \text{and} \quad (A^\bullet(M)_{\mathbb{C}}, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2))$$

as above. We consider the following cohomologies:

$$H^\bullet(A^\bullet(M)_{\mathbb{C}}, d_\phi) := \frac{\ker d_\phi}{\operatorname{im} d_\phi},$$

$$H^\bullet(A^\bullet(M)_{\mathbb{C}}, \partial(\theta_1, \theta_2)) := \frac{\ker \partial(\theta_1, \theta_2)}{\operatorname{im} \partial(\theta_1, \theta_2)},$$

$$H^\bullet(A^\bullet(M)_{\mathbb{C}}, \overline{\partial}(\theta_1, \theta_2)) := \frac{\ker \overline{\partial}(\theta_1, \theta_2)}{\operatorname{im} \overline{\partial}(\theta_1, \theta_2)},$$

$$H^\bullet(A^\bullet(M)_{\mathbb{C}}, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) := \frac{\ker \partial(\theta_1, \theta_2) \cap \ker \overline{\partial}(\theta_1, \theta_2)}{\operatorname{im} \partial(\theta_1, \theta_2) \cap \operatorname{im} \overline{\partial}(\theta_1, \theta_2)},$$

$$H^\bullet(A^\bullet(M)_{\mathbb{C}}, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) := \frac{\ker \partial(\theta_1, \theta_2) \cap \ker \overline{\partial}(\theta_1, \theta_2)}{\operatorname{im} \partial(\theta_1, \theta_2) + \operatorname{im} \overline{\partial}(\theta_1, \theta_2)}.$$

The identity induces natural maps

$$H^\bullet(A^\bullet(M)_{\mathbb{C}}, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \quad \rightarrow \quad H^\bullet(A^\bullet(M)_{\mathbb{C}}, d_\phi) \quad \rightarrow \quad H^\bullet(A^\bullet(M)_{\mathbb{C}}, \overline{\partial}(\theta_1, \theta_2))$$

of $\mathbb{Z}$-graded vector spaces.

By [6, Theorem 2.4], and using the finite-dimensionality of $H^\bullet(A^\bullet(M)_{\mathbb{C}}, \partial(\theta_1, \theta_2))$ and $H^\bullet(A^\bullet(M)_{\mathbb{C}}, \overline{\partial}(\theta_1, \theta_2))$, see the next subsection or [31, page 22], we have the following inequality à la Frölicher.

**Theorem 1.1.** Let $(M, J)$ be a compact complex manifold of complex dimension $n$. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, and $\phi := \theta_1 + \overline{\theta}_1 + \theta_2 - \overline{\theta}_2$. Then the inequality

$$\dim_{\mathbb{C}} H^\bullet(A^\bullet(M)_{\mathbb{C}}, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) + \dim_{\mathbb{C}} H^\bullet(A^\bullet(M)_{\mathbb{C}}, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2))$$

$$\geq \dim_{\mathbb{C}} H^\bullet(A^\bullet(M)_{\mathbb{C}}, \partial(\theta_1, \theta_2)) + \dim_{\mathbb{C}} H^\bullet(A^\bullet(M)_{\mathbb{C}}, \overline{\partial}(\theta_1, \theta_2))$$

holds.

**Remark 1.2.** Note that, if $\theta_1 = 0$, then $(A^\bullet(M)_{\mathbb{C}}, \partial(0, \theta_2), \overline{\partial}(0, \theta_2))$ has in fact a structure of double complex. Hence we have the Frölicher inequalities, [18, Theorem 2],

$$\dim_{\mathbb{C}} H^\bullet(A^\bullet(M)_{\mathbb{C}}, \partial(0, \theta_2)) = \dim_{\mathbb{C}} H^\bullet(A^\bullet(M)_{\mathbb{C}}, d_\phi)$$

and

$$\dim_{\mathbb{C}} H^\bullet(A^\bullet(M)_{\mathbb{C}}, \overline{\partial}(0, \theta_2)) \geq \dim_{\mathbb{C}} H^\bullet(A^\bullet(M)_{\mathbb{C}}, d_\phi).$$

Hence we get the following inequality à la Frölicher, [6, Corollary 2.6],

$$\dim_{\mathbb{C}} H^\bullet(A^\bullet(M)_{\mathbb{C}}, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) + \dim_{\mathbb{C}} H^\bullet(A^\bullet(M)_{\mathbb{C}}, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2))$$

$$\geq 2 \dim_{\mathbb{C}} H^\bullet(A^\bullet(M)_{\mathbb{C}}, d_\phi).$$
But in general, when \( \theta_1 \neq 0 \), one does not have a double complex structure on \((A^\bullet(M)_C, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2))\). In fact, Example 3.2 shows that the inequality
\[
\dim C H^\bullet (A^\bullet(M)_C; \overline{\partial}(\theta_1, \theta_2)) \geq \dim C H^\bullet(A^\bullet(M)_C; d_\emptyset)
\]
may fail.

1.4. Hodge theory and cohomologies with twisted differentials. Take a Hermitian metric \( g \) on \((M, J)\). We consider the adjoint operators \( d_\emptyset^*, \overline{\partial}(\theta_1, \theta_2)^* \) and \( \overline{\partial}(\theta_1, \theta_2) \) of the operators \( d_\emptyset, \partial(\theta_1, \theta_2), \) and \( \overline{\partial}(\theta_1, \theta_2), \) respectively, with respect to the inner product \((\cdot, \cdot)\) induced by \( g \). We define the Laplacian operators
\[
\Delta d_\emptyset := [d_\emptyset, d_\emptyset^*] := d_\emptyset d_\emptyset^* + d_\emptyset^* d_\emptyset,
\]
\[
\Delta \partial(\theta_1, \theta_2) := \left[ \partial(\theta_1, \theta_2), \partial(\theta_1, \theta_2)^* \right] := \partial(\theta_1, \theta_2) \partial(\theta_1, \theta_2)^* + \partial(\theta_1, \theta_2)^* \partial(\theta_1, \theta_2),
\]
\[
\Delta \overline{\partial}(\theta_1, \theta_2) := \left[ \overline{\partial}(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)^* \right] := \overline{\partial}(\theta_1, \theta_2) \overline{\partial}(\theta_1, \theta_2)^* + \overline{\partial}(\theta_1, \theta_2)^* \overline{\partial}(\theta_1, \theta_2),
\]
\[
\Delta_{BC, \partial(\theta_1, \theta_2); \overline{\partial}(\theta_1, \theta_2)} := \left( \partial(\theta_1, \theta_2) \overline{\partial}(\theta_1, \theta_2) \right) \left( \partial(\theta_1, \theta_2)^* \overline{\partial}(\theta_1, \theta_2)^* \right) + \left( \partial(\theta_1, \theta_2)^* \overline{\partial}(\theta_1, \theta_2) \right) \left( \partial(\theta_1, \theta_2) \overline{\partial}(\theta_1, \theta_2)^* \right) + \left( \partial(\theta_1, \theta_2)^* \overline{\partial}(\theta_1, \theta_2) \right) \left( \partial(\theta_1, \theta_2)^* \overline{\partial}(\theta_1, \theta_2) \right) + \left( \partial(\theta_1, \theta_2) \overline{\partial}(\theta_1, \theta_2)^* \right) \left( \partial(\theta_1, \theta_2)^* \overline{\partial}(\theta_1, \theta_2) \right).
\]

Note that the principal parts of the above operators are equal to the principal parts of the corresponding operators with \( \emptyset = 0 \) and \( (\theta_1, \theta_2) = (0, 0) \). In particular, \( \Delta d_\emptyset, \Delta \partial(\theta_1, \theta_2), \) and \( \Delta \overline{\partial}(\theta_1, \theta_2) \) are 2nd order self-adjoint elliptic differential operators, see [31, page 22]. In particular, one has the orthogonal decompositions
\[
A^\bullet(M)_C = \ker \Delta d_\emptyset \oplus \text{im } \Delta d_\emptyset,
\]
\[
A^\bullet(M)_C = \ker \Delta \partial(\theta_1, \theta_2) \oplus \text{im } \Delta \partial(\theta_1, \theta_2),
\]
\[
A^\bullet(M)_C = \ker \Delta \overline{\partial}(\theta_1, \theta_2) \oplus \text{im } \Delta \overline{\partial}(\theta_1, \theta_2),
\]
with respect to the inner product induced by \( g \), and hence the isomorphisms
\[
H^\bullet (A^\bullet(M)_C; d_\emptyset) \cong \ker \Delta d_\emptyset,
\]
\[
H^\bullet (A^\bullet(M)_C; \partial(\theta_1, \theta_2)) \cong \ker \Delta \partial(\theta_1, \theta_2),
\]
\[
H^\bullet (A^\bullet(M)_C; \overline{\partial}(\theta_1, \theta_2)) \cong \ker \Delta \overline{\partial}(\theta_1, \theta_2)
\]
of vector spaces, depending on the metric, see [31, page 22].

Furthermore, M. Schweitzer proved that \( \Delta_{BC} := \Delta_{BC, \partial(\theta_1, \theta_2); \overline{\partial}(\theta_1, \theta_2)} \) and \( \Delta_A := \Delta_{A, \partial(\theta_1, \theta_2); \overline{\partial}(\theta_1, \theta_2)} \) are 4th order self-adjoint elliptic differential operators, in [30, §2.2, §2.3], see also [26, Proposition 5]. Hence we have the following result.

**Theorem 1.3.** Let \((M, J)\) be a compact complex manifold endowed with a Hermitian metric \( g \). Take \( \theta_1, \theta_2 \in H^{1,0}(M) \). Then the operators \( \Delta_{BC, \partial(\theta_1, \theta_2); \overline{\partial}(\theta_1, \theta_2)} \) and \( \Delta_{A, \partial(\theta_1, \theta_2); \overline{\partial}(\theta_1, \theta_2)} \) are 4th order self-adjoint elliptic differential operators.

For the classical theory of self-adjoint elliptic differential operators, see, e.g., [25, page 450], we get the following corollaries.
Corollary 1.4. Let \((M, J)\) be a compact complex manifold endowed with a Hermitian metric \(g\). Take \(\theta_1, \theta_2 \in H^{1,0}_{BC}(M)\).

- There is an orthogonal decomposition
  \[
  A^\bullet(M)_C = \ker \Delta_{BC, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)} \oplus \img \Delta_{BC, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)}
  \]
  with respect to the inner product induced by \(g\). Hence there is an isomorphism
  \[
  H^\bullet(A^\bullet(M)_C; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \cong \ker \Delta_{BC, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)},
  \]
  depending on the metric.
- There is an orthogonal decomposition
  \[
  A^\bullet(M)_C = \ker \Delta_{A, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)} \oplus \img \Delta_{A, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)}
  \]
  with respect to the inner product induced by \(g\). Hence there is an isomorphism
  \[
  H^\bullet(A^\bullet(M)_C; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \cong \ker \Delta_{A, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)},
  \]
  depending on the metric.

In particular, it follows that the Hodge-\(\ast\)-operator induces isomorphisms between cohomologies. (With abuse of notation, for \(\zeta_1, \zeta_2 \in H^{1,0}_{BC}(X)\), denote \(\partial(\zeta_1, \zeta_2) := \partial + L_{\zeta_1} + L_{\zeta_2}\) and \(\overline{\partial}(\zeta_1, \zeta_2) := \overline{\partial} - L_{\zeta_1} + L_{\zeta_2}\).)

Corollary 1.5. Let \((M, J)\) be a compact complex manifold, of complex dimension \(n\), endowed with a Hermitian metric \(g\). Take \(\theta_1, \theta_2 \in H^{1,0}_{BC}(M)\), and \(\phi := \theta_1 + \overline{\theta}_1 + \theta_2 - \overline{\theta}_2\). Fix a \(J\)-Hermitian metric \(g\), and consider the associated \(C\)-anti-linear Hodge-\(\ast\)-operator \(\tau\): \(A^\bullet(M)_C \to A^{2n-\bullet}(M)_C\). It induces the isomorphisms

\[
\tau: H^\bullet(A^\bullet(M)_C; d_\phi) \cong H^{2n-\bullet}(A^\bullet(M)_C; d_{-\phi}),
\]

\[
\tau: H^\bullet(A^\bullet(M)_C; \partial(\theta_1, \theta_2)) \cong H^{2n-\bullet}(A^\bullet(M)_C; \partial(-\theta_2, -\theta_1)),
\]

\[
\tau: H^\bullet(A^\bullet(M)_C; \overline{\partial}(\theta_1, \theta_2)) \cong H^{2n-\bullet}(A^\bullet(M)_C; \overline{\partial}(-\theta_2, -\theta_1)),
\]

\[
\tau: H^\bullet(A^\bullet(M)_C; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \cong H^{2n-\bullet}(A^\bullet(M)_C; \partial(-\theta_2, -\theta_1), \overline{\partial}(-\theta_2, -\theta_1)).
\]

Proof. Note that

\[
\tau \Delta_{d_\phi} = \Delta_{d_{-\phi}} \tau,
\]

\[
\tau \Delta_{\partial(\theta_1, \theta_2)} = \Delta_{\partial(-\theta_2, -\theta_1)} \tau,
\]

\[
\tau \Delta_{\overline{\partial}(\theta_1, \theta_2)} = \Delta_{\overline{\partial}(-\theta_2, -\theta_1)} \tau,
\]

\[
\tau \Delta_{BC(\theta_1, \theta_2)} = \Delta_{A(-\theta_2, -\theta_1)} \tau.
\]

The statement follows from [31, page 22] and Corollary 1.4.

1.5. Hodge theory on Kähler manifolds with twisted differentials. Consider the case of a compact complex manifold endowed with a Kähler metric. Thanks to the Kähler identities for twisted differentials, we have the following analogue of the classical Hodge decomposition theorem.

Proposition 1.6. Let \((M, J)\) be a compact complex manifold endowed with a Kähler metric \(g\). Take \(|\phi| \in H^1(M; \mathbb{C})\). Consider \(\theta_1, \theta_2 \in H^{1,0}_{BC}(M)\) such that \(\phi = \theta_1 + \overline{\theta}_1 + \theta_2 - \overline{\theta}_2\). Then

\[
\Delta_{d_\phi} = 2 \Delta_{\partial(\theta_1, \theta_2)} = 2 \Delta_{\overline{\partial}(\theta_1, \theta_2)}
\]

and

\[
\Delta_{BC, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)} = \Delta_{\partial(\theta_1, \theta_2)}^2 + \partial(\theta_1, \theta_2) \partial(\theta_1, \theta_2) + \overline{\partial}(\theta_1, \theta_2) \overline{\partial}(\theta_1, \theta_2)
\]

and

\[
\Delta_{A, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)} = \Delta_{\partial(\theta_1, \theta_2)}^2 + \partial(\theta_1, \theta_2) \partial(\theta_1, \theta_2) + \overline{\partial}(\theta_1, \theta_2) \overline{\partial}(\theta_1, \theta_2).
\]
Proof. For the sake of completeness, we detail the proof.

Take $[\phi] \in H^1(M; \mathbb{C})$. Then $[\phi] = [\text{Re}\phi] + \sqrt{-1}[\text{Im}\phi]$ where $[\text{Re}\phi] \in H^1(M; \mathbb{R}) \subset H^1(M; \mathbb{C})$ and $[\text{Im}\phi] \in H^1(M; \mathbb{R}) \subset H^1(M; \mathbb{C})$. By the Hodge decomposition theorem for compact Kähler manifolds, one has that the identity induces the isomorphism $H^{1,0}_{BC}(M) \oplus H^{0,1}_{BC}(M) \cong H^1(M; \mathbb{C})$. Hence there exists $\theta_1 \in H^{1,0}_{BC}(M)$ such that $\text{Re}\phi = \theta_1 + \overline{\theta}_1$, and there exists $\theta_2 \in H^{0,1}_{BC}(M)$ such that $\sqrt{-1}\text{Im}\phi = \theta_2 - \overline{\theta}_2$.

Note that
\[
\frac{d\phi}{\theta} = d + L_\phi = \partial + L_{\theta_2} + L_{\overline{\theta}_1} = \partial_{(\theta_1, \theta_2)} + \overline{\partial}_{(\theta_1, \theta_2)}.
\]

Firstly, note that, by the Kähler identities for twisted differentials, we have:
\[
\begin{align*}
\left[\partial_{(\theta_1, \theta_2)}, \overline{\partial}_{(\theta_1, \theta_2)}\right] &= -\sqrt{-1} \left[\partial_{(\theta_1, \theta_2)}, \Lambda, \partial_{(\theta_1, \theta_2)} - \partial_{(\theta_1, \theta_2)} \Lambda, \right] \\
&= -\sqrt{-1} \left[\partial_{(\theta_1, \theta_2)} \Lambda, \partial_{(\theta_1, \theta_2)} - \partial^2_{(\theta_1, \theta_2)} + \Lambda, \partial^2_{(\theta_1, \theta_2)} \Lambda, \right] \\
&= 0
\end{align*}
\]
and, by conjugation,
\[
\left[\overline{\partial}_{(\theta_1, \theta_2)}, \partial^*_{(\theta_1, \theta_2)}\right] = 0,
\]
where $\omega$ denotes the $(1,1)$-form associated to $g$.

Therefore
\[
\Delta_{\theta_1} = \left[\frac{d\phi}{\theta}, \frac{d^*\phi}{\theta}\right] = \left[\partial_{(\theta_1, \theta_2)}, \partial^*_{(\theta_1, \theta_2)} + \overline{\partial}_{(\theta_1, \theta_2)}\right]
\]
\[
= \left[\partial_{(\theta_1, \theta_2)}, \partial^*_{(\theta_1, \theta_2)}\right] + \left[\partial_{(\theta_1, \theta_2)}, \overline{\partial}_{(\theta_1, \theta_2)}\right] + \overline{\partial}_{(\theta_1, \theta_2)}
\]
\[
= \Delta_{\partial_{(\theta_1, \theta_2)}} + \Delta_{\overline{\partial}_{(\theta_1, \theta_2)}}.
\]

Hence we have to show that
\[
\Delta_{\partial_{(\theta_1, \theta_2)}} = \Delta_{\overline{\partial}_{(\theta_1, \theta_2)}}.
\]

Indeed, by using the Kähler identities, we have
\[
\Delta_{\overline{\partial}_{(\theta_1, \theta_2)}} = \left[\partial_{(\theta_1, \theta_2)}, \partial^*_{(\theta_1, \theta_2)}\right] = \sqrt{-1} \left[\Lambda, \partial_{(\theta_1, \theta_2)}\right]
\]
\[
= \sqrt{-1} \left[\Lambda, \left[\partial_{(\theta_1, \theta_2)}, \overline{\partial}_{(\theta_1, \theta_2)}\right]\right] + \sqrt{-1} \left[\overline{\partial}_{(\theta_1, \theta_2)}, \left[\partial_{(\theta_1, \theta_2)}, \Lambda, \right]\right]
\]
\[
= \left[\overline{\partial}_{(\theta_1, \theta_2)}, \overline{\partial}_{(\theta_1, \theta_2)}\right] = \Delta_{\overline{\partial}_{(\theta_1, \theta_2)}}.
\]

Again by the Kähler identities, we have (compare [26, Proposition 6], [30, Proposition 2.4])
\[
\Delta^2_{\overline{\partial}_{(\theta_1, \theta_2)}} = \Delta_{\overline{\partial}_{(\theta_1, \theta_2)}} \Delta_{\overline{\partial}_{(\theta_1, \theta_2)}}
\]
\[
= \partial_{(\theta_1, \theta_2)} \overline{\partial}_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} - \partial_{(\theta_1, \theta_2)} \overline{\partial}_{(\theta_1, \theta_2)} \partial^*_{(\theta_1, \theta_2)} + \overline{\partial}_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} \partial^*_{(\theta_1, \theta_2)} - \overline{\partial}_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} \overline{\partial}_{(\theta_1, \theta_2)}
\]
\[
= - \partial_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} \overline{\partial}_{(\theta_1, \theta_2)} - \overline{\partial}_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} \partial^*_{(\theta_1, \theta_2)} - \overline{\partial}_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} \overline{\partial}_{(\theta_1, \theta_2)}
\]
\[
= \partial_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} \partial^*_{(\theta_1, \theta_2)} + \overline{\partial}_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} \partial^*_{(\theta_1, \theta_2)} - \partial_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} \overline{\partial}_{(\theta_1, \theta_2)}
\]
\[
= \Delta_{BC,\partial_{(\theta_1, \theta_2)}} = \Delta_{\partial_{(\theta_1, \theta_2)}}.
\]
Theorem 1.8. Consider analogously, isomorphisms in Corollary 1.7.

Corollary 1.7. Let \((M, J)\) be a compact complex manifold endowed with a Kähler metric \(g\). Take \([\phi] \in H^1(M; \mathbb{C})\). Consider \(\theta_1, \theta_2 \in H^{1,0}_B(M)\) such that \(\phi = \theta_1 + \theta_2 - \overline{\theta}_2\). Then there are isomorphisms

\[
\begin{align*}
\text{H}^\bullet (A^\bullet(M)_C; \partial (\theta_1, \theta_2)) & \xrightarrow{\simeq} \text{H}^\bullet (A^\bullet(M)_C; \overline{\partial}(\theta_1, \theta_2)) \\
\text{H}^\bullet (A^\bullet(M)_C; \overline{\partial}(\theta_1, \theta_2)) & \xrightarrow{\simeq} \text{H}^\bullet (A^\bullet(M)_C; \partial (\theta_1, \theta_2))
\end{align*}
\]

of \(\mathbb{Z}\)-graded vector spaces.

Since the isomorphisms in [31, page 22] and Corollary 1.4 depend on the Kähler metric, also the isomorphisms in Corollary 1.7, a priori, depend on the Kähler metric. In fact, the following result holds, analogous to the \(\partial \overline{\partial}\)-Lemma.

Theorem 1.8. Let \((M, J)\) be a compact complex manifold endowed with a Kähler metric \(g\). Take \(\theta_1, \theta_2 \in H^{1,0}_B(M)\). Then the natural map

\[
\text{H}^\bullet (A^\bullet(M)_C; \partial (\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2), \partial (\partial_1, \partial_2), \overline{\partial}(\partial_1, \partial_2)) \to \text{H}^\bullet (A^\bullet(M)_C; \partial (\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2), \partial (\partial_1, \partial_2), \overline{\partial}(\partial_1, \partial_2))
\]

induced by the identity is injective.

Proof. We detail the proof, which follows the argument in [16, pages 266–267].

We prove that

\[
\ker \partial (\theta_1, \theta_2) \cap \ker \overline{\partial}(\theta_1, \theta_2) \cap \left( \text{im} \partial (\theta_1, \theta_2) + \text{im} \overline{\partial}(\theta_1, \theta_2) \right) = \text{im} \partial (\partial_1, \partial_2) \overline{\partial}(\theta_1, \theta_2).
\]

Consider \(\alpha = \partial (\theta_1, \theta_2) \beta + \overline{\partial}(\theta_1, \theta_2) \gamma \in A^k(M)_C\) such that \(\partial (\theta_1, \theta_2) \alpha = \overline{\partial}(\theta_1, \theta_2) \alpha = 0\), where \(\beta \in A^{k-1}(M)_C\) and \(\gamma \in A^{k-1}(M)_C\).

Fix a Hermitian metric \(g\). By Corollary 1.4 and Proposition 1.6, one has

\[
\alpha \in \text{im} \partial (\theta_1, \theta_2) + \text{im} \overline{\partial}(\theta_1, \theta_2) \subseteq \text{im} \Delta, \partial (\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)
\]

\[
\implies \ker \Delta, \partial (\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2) = \ker \Delta, \partial (\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)
\]

therefore, again by Corollary 1.4, one has

\[
\alpha \in \text{im} \Delta, \partial (\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2) = \text{im} \partial (\theta_1, \theta_2) \overline{\partial}(\theta_1, \theta_2) \oplus \left( \text{im} \partial (\theta_1, \theta_2) + \text{im} \overline{\partial}(\theta_1, \theta_2) \right).
\]
Since $\partial(\theta_1, \theta_2)\alpha = 0$, then $\alpha \perp \im\partial(\theta_1, \theta_2)$. Since $\overline{\partial}(\theta_1, \theta_2)\alpha = 0$, then $\alpha \perp \im\overline{\partial}(\theta_1, \theta_2)$. Hence $\alpha \in \im(\partial(\theta_1, \theta_2)\overline{\partial}(\theta_1, \theta_2))$. This concludes the proof. □

**Corollary 1.9.** Let $(M, J)$ be a compact complex manifold endowed with a Kähler metric. Take $[\phi] \in H^1(M; \mathbb{C})$. Consider $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$ such that $\phi = \theta_1 + \overline{\theta}_1 + \theta_2 - \overline{\theta}_2$. Then the natural maps

$$H^* (A^* (M)_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2); \partial(\theta_1, \theta_2)\overline{\partial}(\theta_1, \theta_2))$$

$$H^* (A^* (M)_\mathbb{C}; d\phi)$$

$$H^* (A^* (M)_\mathbb{C}; \overline{\partial}(\theta_1, \theta_2))$$

induced by the identity are isomorphisms.

**Proof.** By [16, Lemma 5.15], see also [6, Lemma 1.4], the maps $\iota_{BC,A}$, $\iota_{BC,B}$, $\iota_{BC,\overline{A}}$, and $\iota_{BC,A}$ are injective, and the maps $\iota_{BC,A}$, $\iota_{A,A}$, $\iota_{\overline{A},A}$, and $\iota_{A,A}$ are surjective. By Corollary 1.7, they are in fact isomorphisms, being either injective or surjective maps between finite-dimensional vector spaces. □

**1.6. Homologies with twisted differentials.** Consider the space $D^* (M)_\mathbb{C} = D^{*+}(M)$ of currents, where we denote $D^p (M)_\mathbb{C}$ the space of complex $(\dim M - p)$-dimensional currents. Then we also have the bi-differential ($\mathbb{Z}$-graded) algebra

$$(D^* (M)_\mathbb{C}, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) .$$

We consider the inclusion

$$T: A^* (M)_\mathbb{C} \to D^* (M)_\mathbb{C} , \quad T_\eta := \int_M \eta \wedge .$$

Then we have the following result.

**Theorem 1.10.** Let $(M, J)$ be a compact complex manifold endowed with a Hermitian metric $g$. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, and $\phi := \theta_1 + \overline{\theta}_1 + \theta_2 - \overline{\theta}_2$. The inclusion $T: A^* (M)_\mathbb{C} \to D^* (M)_\mathbb{C}$ induces the cohomology isomorphisms

$$H^* (A^* (M)_\mathbb{C}; d\phi) \cong H^* (D^* (M)_\mathbb{C}; d\phi) ,$$

$$H^* (A^* (M)_\mathbb{C}; \partial(\theta_1, \theta_2)) \cong H^* (D^* (M)_\mathbb{C}; \partial(\theta_1, \theta_2)) ,$$

$$H^* (A^* (M)_\mathbb{C}; \overline{\partial}(\theta_1, \theta_2)) \cong H^* (D^* (M)_\mathbb{C}; \overline{\partial}(\theta_1, \theta_2)) ,$$

$$H^* (A^* (M)_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2); \partial(\theta_1, \theta_2)\overline{\partial}(\theta_1, \theta_2)) \cong H^* (D^* (M)_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2); \partial(\theta_1, \theta_2)\overline{\partial}(\theta_1, \theta_2)) ,$$

$$H^* (A^* (M)_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2); \partial(\theta_1, \theta_2)\overline{\partial}(\theta_1, \theta_2)) \cong H^* (D^* (M)_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2); \partial(\theta_1, \theta_2)\overline{\partial}(\theta_1, \theta_2)) .$$

**Proof.** Consider each case. For a fixed Hermitian metric, consider the corresponding Laplacian operator $\Delta$. Then, by [34, Theorem 4.12], we have the operators

$$G: A^* (M)_\mathbb{C} \to A^* (M)_\mathbb{C} \quad \text{and} \quad H: A^* (M)_\mathbb{C} \to A^* (M)_\mathbb{C} ,$$

where $H$ is given by the projection $A^* (M)_\mathbb{C} \to \ker \Delta$ and $G$ is given by the inverse of the restriction of $\Delta$ on $A^* (M)_\mathbb{C} \cap (\ker \Delta)^\perp$, such that

$$\Delta \circ G + H = G \circ \Delta + H = \text{id} .$$

Since $\Delta$ is self-adjoint, $G$ and $H$ are also self-adjoint. We can define the operators $\Delta, G,$ and $H$ on $D^* (M)_\mathbb{C}$. They still satisfy $\Delta \circ G + H = G \circ \Delta + H = \text{id}$. By the regularity of the kernel of elliptic differential operators in Sobolev spaces, see, e.g., [34, Theorem 4.8], we have

$$\ker \Delta |_{A^* (M)_\mathbb{C}} = \ker \Delta |_{D^* (M)_\mathbb{C}} .$$
Hence we have

$$D^\bullet(M)_C = \ker \Delta_{(A^\bullet(M)_C)} + \Delta (D^\bullet(M)_C) .$$

For $T \in D^\bullet(M)_C$ and $h \in \ker \Delta_{(A^\bullet(M)_C)}$, suppose that $\Delta_{D^\bullet(M)_C} T = h$. Then, by [34, Theorem 4.9], we have $T \in A^\bullet(M)_C$, and hence $h = 0$. Thus we have

$$D^\bullet(M)_C = \ker \Delta_{(A^\bullet(M)_C)} \oplus \Delta (D^\bullet(M)_C) .$$

This completes the proof. \qed

### 2. Hodge decomposition with twisted differentials and modifications

Let $f: M_1 \to M_2$ be a holomorphic map between compact complex manifolds $M_1$ and $M_2$. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M_2)$, and $\phi := \theta_1 + \theta_1 + \theta_2 - \theta_2$.

#### 2.1. Modifications and cohomologies with twisted differentials. Consider the pull-back $f^*: A^{**}(M_2) \to A^{**}(M_1)$. We have $f^*\theta_1, f^*\theta_2 \in H^{1,0}_{BC}(M_1)$. Since $f^*$ commutes with $\partial$ and $\overline{\partial}$, then

$$f^*: (A^\bullet(M_2)_C, d_\phi) \to (A^\bullet(M_1)_C, d_{f^*\phi})$$

is a morphism of differential $Z$-graded complexes, and

$$f^*: (A^\bullet(M_2)_C, \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \to (A^\bullet(M_1)_C, \partial(f^*\theta_1, f^*\theta_2), \overline{\partial}(f^*\theta_1, f^*\theta_2))$$

is a morphism of bi-differential $Z$-graded complexes. In particular, $f$ induces the maps

$$f_{\partial, (\theta_1, \theta_2)}: H^\bullet(A^\bullet(M_2)_C; d_\phi) \to H^\bullet(A^\bullet(M_1)_C; d_{f^*\phi}) ,$$

$$f_{\overline{\partial}, (\theta_1, \theta_2)}: H^\bullet(A^\bullet(M_2)_C; \overline{\partial}(\theta_1, \theta_2)) \to H^\bullet(A^\bullet(M_1)_C; \overline{\partial}(f^*\theta_1, f^*\theta_2)) ,$$

$$f_{{\partial}, (\theta_1, \theta_2)}: H^\bullet(A^\bullet(M_2)_C; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \to H^\bullet(A^\bullet(M_1)_C; \partial(f^*\theta_1, f^*\theta_2), \overline{\partial}(f^*\theta_1, f^*\theta_2)) .$$

#### 2.2. Modifications and homologies with twisted differentials. Suppose that $f$ is proper. Then we have the map $f_\ast: D^{**}(M_1) \to D^{**}(M_2)$, called the direct image map, such that $f_\ast$ commutes with $\partial$ and $\overline{\partial}$, and $f_\ast(f^*\alpha \land C) = \alpha \land f_\ast C$ for any $\alpha \in A^{**}(M_2)$ and $C \in D^{**}(M_1)$. Hence the map

$$f_\ast: (D^\bullet(M_1)_C, d_{f^*\phi}) \to (D^\bullet(M_2)_C, d_\phi)$$

is a morphism of differential $Z$-graded complexes, and the map

$$f_\ast: (D^\bullet(M_1)_C; \partial(f^*\theta_1, f^*\theta_2), \overline{\partial}(f^*\theta_1, f^*\theta_2)) \to (D^\bullet(M_2)_C; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2))$$

is a morphism of bi-differential $Z$-graded complexes. In particular, $f$ induces the maps

$$f_{dR^\bullet, \phi}: H^\bullet(D^\bullet(M_1)_C; d_{f^*\phi}) \to H^\bullet(D^\bullet(M_2)_C; d_\phi) ,$$

$$f_{\overline{\partial}, (\theta_1, \theta_2)}: H^\bullet(D^\bullet(M_1)_C; \overline{\partial}(f^*\theta_1, f^*\theta_2)) \to H^\bullet(D^\bullet(M_2)_C; \overline{\partial}(\theta_1, \theta_2)) ,$$

$$f_{\overline{\partial}, (\theta_1, \theta_2)}: H^\bullet(D^\bullet(M_1)_C; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \to H^\bullet(D^\bullet(M_2)_C; \partial(f^*\theta_1, f^*\theta_2), \overline{\partial}(f^*\theta_1, f^*\theta_2)) .$$
2.3. Hodge decomposition and $\partial\bar{\partial}$-Lemma with twisted differentials. As in [16], we consider the following definitions in the case of twisted differentials.

**Definition 2.1.** Let $(M, J)$ be a compact complex manifold of complex dimension $n$. For $\theta_1, \theta_2 \in H_{BC}^{1,0}(M)$, and $\phi := \theta_1 + \bar{\theta}_1 + \theta_2 - \bar{\theta}_2$, consider the bi-differential $\mathbb{Z}$-graded complex

$$(A^\bullet(M)_{\mathbb{C}}; \partial_{(\theta_1, \theta_2)}, \bar{\partial}_{(\theta_1, \theta_2)}) .$$

We say that $(M, J)$:

(i) satisfies the $\partial(\theta_1, \theta_2)\bar{\partial}(\theta_1, \theta_2)$-Lemma if

$$\ker \partial(\theta_1, \theta_2) \cap \ker \bar{\partial}(\theta_1, \theta_2) \cap (\text{im} \partial(\theta_1, \theta_2) + \text{im} \bar{\partial}(\theta_1, \theta_2)) = \text{im} \partial(\theta_1, \theta_2) \bar{\partial}(\theta_1, \theta_2) ;$$

i.e., if the natural map

$$H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2)) \to H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2))$$

induced by the identity is injective;

(ii) admits the $(\theta_1, \theta_2)$-Hodge decomposition if the natural maps

$$H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2)) \to H^\bullet (A^\bullet(M)_{\mathbb{C}}; d_\phi)$$

and

$$H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2)) \to H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2))$$

and

$$H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2)) \to H^\bullet (A^\bullet(M)_{\mathbb{C}}; \bar{\partial}(\theta_1, \theta_2))$$

induced by the identity are isomorphisms.

By [16, Lemma 5.15], see also [6, Lemma 1.4], we have the following equivalent characterizations of $\partial(\theta_1, \theta_2)\bar{\partial}(\theta_1, \theta_2)$-Lemma. (In case of double complex, a further characterization is proven in [5].)

**Lemma 2.2.** Let $(M, J)$ be a compact complex manifold of complex dimension $n$. Take $\theta_1, \theta_2 \in H_{BC}^{1,0}(M)$, and $\phi := \theta_1 + \bar{\theta}_1 + \theta_2 - \bar{\theta}_2$. The following conditions are equivalent:

- $(M, J)$ satisfies the $\partial(\theta_1, \theta_2)\bar{\partial}(\theta_1, \theta_2)$-Lemma, i.e., the natural map

  $$H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2)) \to H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2))$$

induced by the identity is injective;

- the natural map

  $$H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2)) \to H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2))$$

induced by the identity is an isomorphism;

- the natural maps

  $$H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2)) \to H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2))$$

and

  $$H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2)) \to H^\bullet (A^\bullet(M)_{\mathbb{C}}; \bar{\partial}(\theta_1, \theta_2))$$

induced by the identity are injective;

- the natural maps

  $$H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2)) \to H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2))$$

and

  $$H^\bullet (A^\bullet(M)_{\mathbb{C}}; \bar{\partial}(\theta_1, \theta_2)) \to H^\bullet (A^\bullet(M)_{\mathbb{C}}; \partial(\theta_1, \theta_2), \bar{\partial}(\theta_1, \theta_2))$$

induced by the identity are surjective.

Furthermore, they imply the following conditions:
• the natural map

\[ H^\bullet(A^\bullet(M)_\Bbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2); \partial(\theta_1, \theta_2)\overline{\partial}(\theta_1, \theta_2)) \rightarrow H^\bullet(A^\bullet(M)_\Bbb{C}; d_\phi) \]

induced by the identity in injective;

• the natural map

\[ H^\bullet(A^\bullet(M)_\Bbb{C}; d_\phi) \rightarrow H^\bullet(A^\bullet(M)_\Bbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2); \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \]

induced by the identity in surjective.

In particular, we have that admitting the \((\theta_1, \theta_2)\)-Hodge decomposition is a stronger condition than satisfying the \(\partial(\theta_1, \theta_2)\overline{\partial}(\theta_1, \theta_2)\)-Lemma. We wonder whether it is strictly stronger, namely, whether there exists an example of a compact complex manifold \(M\) satisfying the \(\partial(\theta_1, \theta_2)\overline{\partial}(\theta_1, \theta_2)\)-Lemma but not admitting the \((\theta_1, \theta_2)\)-Hodge decomposition, for some \(\theta_1, \theta_2 \in H^0_{\text{BC}}(M)\).

In the Kähler case, we can summarize Theorem 1.8 and Theorem 1.9 in the following.

**Corollary 2.3.** Let \((M, J)\) be a compact complex manifold endowed with a Kähler metric. Take \(\theta_1, \theta_2 \in H^1_{\text{BC}}(M)\). Then \((M, J)\) satisfies the \(\partial(\theta_1, \theta_2)\overline{\partial}(\theta_1, \theta_2)\)-Lemma and admits the \((\theta_1, \theta_2)\)-Hodge decomposition.

### 2.4. Modifications and cohomologies with twisted differentials

We recall that a **modification** of a compact complex manifold \((M, J)\) of complex dimension \(n\) is a holomorphic map

\[ \mu: (\tilde{M}, \tilde{J}) \rightarrow (M, J) \]

such that:

- \((\tilde{M}, \tilde{J})\) is a compact \(n\)-dimensional complex manifold;
- there exists an analytic subset \(S \subset M\) of codimension greater than or equal to 1 such that \(\mu|_{\tilde{M}\setminus\mu^{-1}(S)}: \tilde{M}\setminus\mu^{-1}(S) \rightarrow M\setminus S\) is a biholomorphism.

For the following, compare [33, Theorem 3.1].

**Theorem 2.4.** Let \(\mu: (\tilde{M}, \tilde{J}) \rightarrow (M, J)\) be a proper modification of a compact complex manifold \((M, J)\). Take \(\theta_1, \theta_2 \in H^1_{\text{BC}}(M)\), and \(\phi := \theta_1 + \overline{\theta_1} + \theta_2 - \overline{\theta_2}\). Then the map \(\mu\) induces the injective maps

\[
\begin{align*}
\mu_{\text{AR}, \phi}: & H^\bullet (A^\bullet(M)_\Bbb{C}; d_\phi) \rightarrow H^\bullet (A^\bullet(\tilde{M})_\Bbb{C}; d_\mu \circ \phi), \\
\mu_{\text{AR}, \phi}(\theta_1, \theta_2): & H^\bullet (A^\bullet(M)_\Bbb{C}; \partial(\theta_1, \theta_2)) \rightarrow H^\bullet (A^\bullet(\tilde{M})_\Bbb{C}; \partial(\mu \circ \theta_1, \mu \circ \theta_2)), \\
\mu_{\text{AR}, \phi}(\theta_1, \theta_2): & H^\bullet (A^\bullet(M)_\Bbb{C}; \overline{\partial}(\theta_1, \theta_2)) \rightarrow H^\bullet (A^\bullet(\tilde{M})_\Bbb{C}; \overline{\partial}(\mu \circ \theta_1, \mu \circ \theta_2)), \\
\mu_{\text{AR}, \phi}(\theta_1, \theta_2): & H^\bullet (A^\bullet(M)_\Bbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2); \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \rightarrow H^\bullet (A^\bullet(\tilde{M})_\Bbb{C}; \partial(\mu \circ \theta_1, \mu \circ \theta_2), \overline{\partial}(\mu \circ \theta_1, \mu \circ \theta_2); \partial(\mu \circ \theta_1, \mu \circ \theta_2), \overline{\partial}(\mu \circ \theta_1, \mu \circ \theta_2)), \\
\mu_{\text{AR}, \phi}(\theta_1, \theta_2): & H^\bullet (A^\bullet(M)_\Bbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2); \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \rightarrow H^\bullet (A^\bullet(\tilde{M})_\Bbb{C}; \partial(\mu \circ \theta_1, \mu \circ \theta_2), \overline{\partial}(\mu \circ \theta_1, \mu \circ \theta_2); \partial(\mu \circ \theta_1, \mu \circ \theta_2), \overline{\partial}(\mu \circ \theta_1, \mu \circ \theta_2)).
\end{align*}
\]
and the surjective maps
\[ \mu^\ast_{\phi} : H^\ast (M; \mathcal{L}) \to H^\ast (M; \mathcal{L}) , \]
\[ \mu^\ast_{\phi} : H^\ast (M; \mathcal{L}) \to H^\ast (M; \mathcal{L}) , \]
\[ \mu^\ast_{\phi} : H^\ast (M; \mathcal{L}) \to H^\ast (M; \mathcal{L}) \]
\[ \mu^\ast_{\phi} : H^\ast (M; \mathcal{L}) \to H^\ast (M; \mathcal{L}) \]
\[ \mu^\ast_{\phi} : H^\ast (M; \mathcal{L}) \to H^\ast (M; \mathcal{L}) \]

**Proof.** We follow closely the proof by R. O. Wells in [33, Theorem 3.1].

Consider the diagram
\[ \begin{array}{ccc}
A^\ast (\tilde{M})_C & \xrightarrow{T} & D^\ast (\tilde{M})_C \\
\mu^\ast & \downarrow & \mu^\ast \\
A^\ast (M)_C & \xrightarrow{T} & D^\ast (M)_C 
\end{array} \]

There exists a proper analytic subset \( \tilde{S} \subset \tilde{M} \) such that
\[ \mu|_{\tilde{M}\setminus\tilde{S}} : \tilde{M} \setminus \tilde{S} \to M \setminus \mu(S) \]
is a finitely-sheeted covering map of sheeting number \( \ell \in \mathbb{N} \setminus \{0\} \). Let \( U := \{ U_a \}_{a \in J} \) be an open covering of \( M \setminus \mu(S) \), and let \( \{ \rho_a \}_{a \in J} \) be an associated partition of unity. For every \( \alpha, \beta \in A^\ast (M)_C \), one has that
\[ \langle \mu_* T^\ast \alpha, \beta \rangle = \langle T^\ast \mu_* \alpha, \beta \rangle = \int_{M} \mu^\ast \alpha \wedge \mu^\ast \beta = \int_{\tilde{M}\setminus\tilde{S}} \mu^\ast (\alpha \wedge \beta) = \sum_{j \in J} \int_{\mu^{-1}(U_j)} \mu^\ast (\rho_j \cdot (\alpha \wedge \beta)) = \sum_{j \in J} \sum_{\mathcal{U} \in \mu^{-1}(U_j)} \int_{U_j} \rho_j \cdot (\alpha \wedge \beta) \]
\[ = \ell \cdot \sum_{j \in J} \int_{U_j} \rho_j \cdot (\alpha \wedge \beta) = \ell \cdot \int_{M \setminus \mu(S)} (\alpha \wedge \beta) = \ell \cdot \int_{M \setminus \mu(S)} (\alpha \wedge \beta) = \langle \ell \cdot T^\ast \alpha, \beta \rangle , \]
and hence one gets that
\[ \mu_* T^\ast \mu^\ast = \ell \cdot T . \]

In particular, one gets, for \( \xi \in \{ \partial, \nabla, BC, A \} \),
\[ \mu^\ast_{\phi} = \ell \cdot T \quad \text{and} \quad \mu^\ast_{\phi} = \ell \cdot T . \]
Hence, in particular, for \( \xi \in \{ \partial, \nabla, BC, A \} \), the maps \( \mu^\ast_{\phi} \) and \( \mu^\ast_{\phi} \) are injective, and the maps \( \mu^\ast_{\phi} \) and \( \mu^\ast_{\phi} \) are surjective.

2.5. **Modifications and \( \partial \nabla \)-Lemma with twisted differentials.** As a consequence, we get the following two results. The first one concerns the behaviour of \( \partial_{(\theta_1, \theta_2)} \nabla_{(\theta_1, \theta_2)} \)-Lemma under proper modifications.

**Theorem 2.5.** Let \( \mu : (\tilde{M}, \tilde{J}) \to (M, J) \) be a proper modification of a compact complex manifold \((M, J)\). Take \( \theta_1, \theta_2 \in H^{1,0}_{BC}(M) \). If \((\tilde{M}, \tilde{J})\) satisfies the \( \partial_{(\theta_1, \theta_2)} \nabla_{(\theta_1, \theta_2)} \)-Lemma, then \((M, J)\) satisfies the \( \partial_{(\theta_1, \theta_2)} \nabla_{(\theta_1, \theta_2)} \)-Lemma.
Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
H^*_B(M; (\theta_1, \theta_2)) & \xrightarrow{\mu_{BC, (\theta_1, \theta_2)}} & H^*_B(\tilde{M}; (\mu^* \theta_1, \mu^* \theta_2)) \\
\downarrow{\text{id}_M^*} & & \downarrow{id_{\tilde{M}}^*} \\
H^*_A(M; (\theta_1, \theta_2)) & \xrightarrow{\mu_{A, (\theta_1, \theta_2)}} & H^*_A(\tilde{M}; (\mu^* \theta_1, \mu^* \theta_2))
\end{array}
\]

where, for simplicity, we have denoted, e.g.,

\[
H^*_B(\tilde{M}; (\mu^* \theta_1, \mu^* \theta_2)) := H^*((A^*(\tilde{M}))_C; \partial(\mu^* \theta_1, \mu^* \theta_2), \partial((\mu^* \theta_1, \mu^* \theta_2)_C) \tilde{\partial}(\mu^* \theta_1, \mu^* \theta_2)
\]

and

\[
H^*_A(\tilde{M}; (\mu^* \theta_1, \mu^* \theta_2)) := H^*((A^*(\tilde{M}))_C; \partial(\mu^* \theta_1, \mu^* \theta_2), \partial((\mu^* \theta_1, \mu^* \theta_2)_C) \tilde{\partial}(\mu^* \theta_1, \mu^* \theta_2)
\]

Suppose that \((\tilde{M}, \tilde{J})\) satisfies the \(\partial(\mu^* \theta_1, \mu^* \theta_2)\tilde{\partial}(\mu^* \theta_1, \mu^* \theta_2)\)-Lemma. Then, by definition, the map \(\text{id}_{\tilde{M}}^*\) is injective. Furthermore, the map \(\mu_{BC, (\theta_1, \theta_2)}^*\) is injective by Theorem 2.4. Hence the map \(\text{id}_M^*\) is injective, that is, \((M, J)\) satisfies the \((\theta_1, \theta_2)\)-Hodge decomposition. \(\square\)

2.6. Modifications and Hodge decomposition with twisted differentials. The second result concerns Hodge decomposition with twisted differential.

**Theorem 2.6.** Let \(\mu: (\tilde{M}, \tilde{J}) \to (M, J)\) be a proper modification of a compact complex manifold \((M, J)\). Take \(\theta_1, \theta_2 \in H^0_{BC}(M)\). If \((\tilde{M}, \tilde{J})\) satisfies the \((\mu^* \theta_1, \mu^* \theta_2)\)-Hodge decomposition, then \((M, J)\) satisfies the \((\theta_1, \theta_2)\)-Hodge decomposition.

**Proof.** For simplicity, we have denoted, e.g.,

\[
H^*_B(\tilde{M}; (\mu^* \theta_1, \mu^* \theta_2)) := H^*((A^*(\tilde{M}))_C; \partial(\mu^* \theta_1, \mu^* \theta_2), \partial((\mu^* \theta_1, \mu^* \theta_2)_C) \tilde{\partial}(\mu^* \theta_1, \mu^* \theta_2)
\]

and

\[
H^*_A(\tilde{M}; (\mu^* \theta_1, \mu^* \theta_2)) := H^*((A^*(\tilde{M}))_C; \partial(\mu^* \theta_1, \mu^* \theta_2), \partial((\mu^* \theta_1, \mu^* \theta_2)_C) \tilde{\partial}(\mu^* \theta_1, \mu^* \theta_2)
\]

Consider the commutative diagram

\[
\begin{array}{ccc}
H^*_B(M; (\theta_1, \theta_2)) & \xrightarrow{\mu_{BC, (\theta_1, \theta_2)}} & H^*_B(\tilde{M}; (\mu^* \theta_1, \mu^* \theta_2)) \\
\downarrow{id_M^*} & & \downarrow{id_{\tilde{M}}^*} \\
H^*_A(M; \phi) & \xrightarrow{\mu_{A, \phi}} & H^*_A(\tilde{M}; \mu^* \phi)
\end{array}
\]

Then by Theorem 2.4, \(\mu_{BC, (\theta_1, \theta_2)}^*\) and \(\mu_{A, \phi}^*\) are injective. Hence by the injectivity of \(\text{id}_{\tilde{M}}^*\), the map \(\text{id}_M^*: H^*_B(M; (\theta_1, \theta_2)) \to H^*_B(\tilde{M}; (\mu^* \theta_1, \mu^* \theta_2))\) is injective.

Considering the direct image maps, we have the commutative diagram

\[
\begin{array}{ccc}
H^*_B(M; (\theta_1, \theta_2)) & \xrightarrow{\mu_{BC, (\theta_1, \theta_2)}} & H^*_B(\tilde{M}; (\mu^* \theta_1, \mu^* \theta_2)) \\
\downarrow{id_M^*} & & \downarrow{id_{\tilde{M}}^*} \\
H^*_A(M; \phi) & \xrightarrow{\mu_{A, \phi}} & H^*_A(\tilde{M}; \mu^* \phi)
\end{array}
\]

By Theorem 2.4, \(\mu_{BC, (\theta_1, \theta_2)}^*\) and \(\mu_{A, \phi}^*\) are surjective. By the surjectivity of \(\text{id}_M^*\), the map \(\text{id}_M^*: H^*_B(M; (\theta_1, \theta_2)) \to H^*_B(\tilde{M}; (\mu^* \theta_1, \mu^* \theta_2))\) is surjective.

Arguing in the same way with the Dolbeault cohomologies, we get that \(M\) satisfies the \((\theta_1, \theta_2)\)-Hodge decomposition. \(\square\)
We recall that a compact complex manifold \((M, J)\) is said to be in class \(\mathcal{C}\) of Fujiki, \([19]\), if it admits a proper modification \(\mu: \left(\tilde{M}, \tilde{J}\right) \to (M, J)\) with \((\tilde{M}, \tilde{J})\) admitting Kähler metrics. In particular, a Moishezon manifold, \([27]\), (that is, a compact complex manifold of complex dimension \(n\) such that the degree of transcendence over \(\mathbb{C}\) of the field of meromorphic functions is equal to \(n\)) admits a proper modification from a projective manifold, \([27, \text{Theorem 1}]\), and therefore belongs to class \(\mathcal{C}\) of Fujiki.

Corollary 2.7. Let \((M, J)\) be a compact complex manifold in class \(\mathcal{C}\) of Fujiki. Take \(\theta_1, \theta_2 \in H^{1,0}_{BC}(M)\). Then \((\tilde{M}, \tilde{J})\) satisfies the \(\partial(\theta_1, \theta_2)\)-Lemma and admits the \((\theta_1, \theta_2)\)-Hodge decomposition.

Proof. Consider a proper modification \(\mu: \left(\tilde{M}, \tilde{J}\right) \to (M, J)\) with \((\tilde{M}, \tilde{J})\) admitting Kähler metrics. From Corollary 2.3, (see also Theorem 1.8 and Theorem 1.9,) the compact Kähler manifold \((\tilde{M}, \tilde{J})\) satisfies the \(\partial(\mu^*\theta_1, \mu^*\theta_2)\)-Lemma, and admits the \((\mu^*\theta_1, \mu^*\theta_2)\)-Hodge decomposition. Therefore, from Theorem 2.5 and Theorem 2.6, the compact complex manifold \((M, J)\) satisfies the \(\partial(\theta_1, \theta_2)\)-Lemma and admits the \((\theta_1, \theta_2)\)-Hodge decomposition. \(\square\)

2.7. Complex orbifolds of global-quotient type. I. Satake introduce in \([29]\) the notion of orbifold, also called \(V\)-manifold; see also \([8, 9]\). It is a singular complex space whose singularities are locally isomorphic to quotient singularities \(\mathbb{C}^n/G\), for finite subgroups \(G \subset \text{GL}_n(\mathbb{C})\). In particular, we are interested in compact complex orbifolds of global-quotient type, namely, compact complex orbifolds given by \(\hat{M} = M/G\) where \(M\) is a compact complex manifold and \(G\) is a finite group of biholomorphisms of \(M\). See \([3]\) and the references therein for motivations.

From the cohomological point of view, one can adapt both the sheaf-theoretic and the analytic tools to complex orbifolds, see \([29, 8, 9]\). In particular, let \(M = M/G\) be a compact complex orbifold of global-quotient type. Consider the double-complex \(\left(\Lambda^{*,*} \hat{M}, \partial, \bar{\partial}\right)\), where the space \(\Lambda^{*,*} \hat{M}\) of differential forms on \(\hat{M}\) is defined as the space of \(G\)-invariant differential forms on \(M\). Consider the associated cohomologies. Fix a Hermitian metric on \(M\), namely, a \(G\)-invariant Hermitian metric on \(M\). Consider the Laplacian operators defined as in the smooth case. Then Hodge theory applies, \([8, \text{Theorem H, Theorem K}]\), \([3, \text{Theorem 1}]\).

By considering objects on \(\hat{M}\) as \(G\)-invariant objects on \(M\), one can adapt all the definitions and results in the previous sections in a straightforward way. In particular, as an analogue of \([3, \text{Theorem 2}]\), we can restate Theorem 2.5 and Theorem 2.6 as follows.

Theorem 2.8. Let \(\mu: \hat{N} \to \hat{M}\) be a proper modification between compact complex orbifolds of global-quotient type. Take \(\theta_1, \theta_2 \in H^{1,0}_{BC}(N)\).

- If \(\hat{N}\) satisfies the \(\partial(\mu^*\theta_1, \mu^*\theta_2)\)-Lemma, then \(\hat{M}\) satisfies the \(\partial(\theta_1, \theta_2)\)-Lemma.
- If \(\hat{N}\) satisfies both the \((\mu^*\theta_1, \mu^*\theta_2)\)-Hodge decomposition, then \(\hat{M}\) satisfies the \((\theta_1, \theta_2)\)-Hodge decomposition.

Therefore, we have the following corollary. (As usual, by compact complex orbifold \(\hat{M}\) of global-quotient type in class \(\mathcal{C}\) of Fujiki, we mean that there exists a proper modification \(\mu: \hat{N} \to \hat{M}\) where \(\hat{N}\) is a compact complex orbifold of global-quotient type admitting Kähler metrics.)

Corollary 2.9. Let \(\hat{M}\) be a compact complex orbifold of global-quotient type in class \(\mathcal{C}\) of Fujiki. Take \(\theta_1, \theta_2 \in H^{1,0}_{BC}(\hat{M})\). Then \(\hat{M}\) satisfies the \(\partial(\theta_1, \theta_2)\)-Lemma and admits the \((\theta_1, \theta_2)\)-Hodge decomposition.

3. Solvmanifolds

In this section, we consider solvmanifolds, i.e., compact quotients \(\Gamma \backslash G\) where \(G\) is a connected simply-connected solvable Lie group and \(\Gamma\) is a co-compact discrete subgroup.

3.1. Cohomology computations for solvmanifolds. Let \(G\) be a connected simply-connected solvable Lie group endowed with a left-invariant complex structure \(\mathcal{J}\) and admitting a lattice \(\Gamma\). Its associated Lie algebra is denoted by \(\mathfrak{g}\), and its complexification by \(\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes \mathbb{C}\). Then we consider the sub-double complex

\[
\left(\Lambda^{*,*} \mathfrak{g}_\mathbb{C}, \partial, \bar{\partial}\right) \to \left(\Lambda^{*,*}(\Gamma \backslash G)_\mathbb{C}, \theta, \overline{\theta}\right).
\]
Take \( \theta_1, \theta_2 \in H^{1,0}(\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial, \overline{\partial}; \overline{\partial} \mathcal{D}) \leftrightarrow H^{1,0}(\mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; \partial, \overline{\partial}; \overline{\partial} \mathcal{D}) \). (For the injectivity, see [2, Lemma 3.6]; see also [14, Lemma 9].) Then we have the bi-differential \( \mathbb{Z} \)-graded sub-complex

\[
\left( \mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2) \right) \leftrightarrow \left( \mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2) \right)
\]

We firstly prove the following result, which generalizes [14, Lemma 9] and [2, Lemma 3.6] to the case of twisted differentials. (Consider also the F. A. Belgun symmetrization trick, [11, Theorem 7], as a different argument.)

**Proposition 3.1.** Let \( \Gamma\backslash G \) be a solvmanifold endowed with a \( G \)-left-invariant complex structure, and with associated Lie algebra \( \mathfrak{g} \). Take \( \theta_1, \theta_2 \in H^{1,0}(\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial, \overline{\partial} \mathcal{D}) \), and \( \phi := \theta_1 + \overline{\theta}_1 + \theta_2 - \overline{\theta}_2 \).

The maps

\[
H^* (\mathcal{A}^*\mathfrak{g}_\mathbb{C}; d_\phi) \rightarrow H^* (\mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; d_\phi),
\]

\[
H^* (\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial(\theta_1, \theta_2)) \rightarrow H^* (\mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; \partial(\theta_1, \theta_2)),
\]

\[
H^* (\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \overline{\partial}(\theta_1, \theta_2)) \rightarrow H^* (\mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; \overline{\partial}(\theta_1, \theta_2)),
\]

\[
H^* (\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2); \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \rightarrow H^* (\mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)),
\]

induced by the inclusion \( (\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \leftrightarrow (\mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \) are injective.

**Proof.** Consider each case. Fix \( g \) a \( G \)-left-invariant Hermitian metric on \( \Gamma\backslash G \). The metric \( g \) and the forms \( \theta_1 \) and \( \theta_2 \) being left-invariant, the associated Laplacian \( \Delta^2 \) satisfies \( \Delta^2 \left. \right|_{\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)} \rightarrow \mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2) \). In particular, Hodge theory applies both to \( (\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \) and to \( (\mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \).

Hence we have the commutative diagram

\[
\Delta^2 \left. \right|_{\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)} \longrightarrow \Delta^2 \left. \right|_{\mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)} \]

\[
\cong \quad \cong
\]

\[
H^* (\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \longrightarrow H^* (\mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)),
\]

where \( H^* (\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \) and \( H^* (\mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \) denote the corresponding cohomologies. It yields the injectivity of the map \( H^* (\mathcal{A}^*\mathfrak{g}_\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \rightarrow H^* (\mathcal{A}^*\mathcal{W}(\Gamma\backslash G)\mathbb{C}; \partial(\theta_1, \theta_2), \overline{\partial}(\theta_1, \theta_2)) \). Compare [4, Proposition 2.2]. \( \square \)

**Example 3.2.** Take \( G := \mathbb{C} \ltimes \phi \mathbb{C}^2 \) where

\[
\phi(z_1) := \begin{pmatrix}
e^{\frac{z_1+z_2}{2}} & 0 \\
0 & e^{-\frac{z_1+z_2}{2}}
\end{pmatrix} \in \text{GL}(\mathbb{C}^2).
\]

Then for some \( a \in \mathbb{R} \) the matrix \( \begin{pmatrix}e^a & 0 \\
0 & e^{-a}\end{pmatrix} \) is conjugate to an element of \( \text{SL}(2; \mathbb{Z}) \). Hence, for any \( b \in \mathbb{R} \setminus \{0\} \), we have a lattice \( \Gamma := \langle a \mathbb{Z} + b \sqrt{-1} \mathbb{Z} \rangle \ltimes \Gamma'' \) of \( G \), where \( \Gamma'' \) is a lattice of \( \mathbb{C}^2 \). The solvmanifold \( \Gamma\backslash G \) is called completely-solvable Nakamura manifold, [28, page 90]; see also, e.g., [15, §3], [22, Example 1], [4, Example 2.17]. If \( b \not\in \pi \mathbb{Z} \), then \( \Gamma\backslash G \) satisfies the \( \partial \overline{\partial} \)-Lemma, see [4, Example 2.17] (see also [23]).

Consider local holomorphic coordinates \( (z_1, z_2, z_3) \) for \( \mathbb{C} \ltimes \phi \mathbb{C}^2 \). We have

\[
\wedge^\bullet \mathfrak{g}_\mathbb{C} = \wedge^\bullet \left( \langle d z_1, e^{-\frac{z_1+z_2}{2}} \cdot d z_2, e^{\frac{z_1+z_2}{2}} \cdot d z_3 \rangle \otimes \langle d \bar{z}_1, e^{-\frac{z_1+z_2}{2}} \cdot d \bar{z}_2, e^{\frac{z_1+z_2}{2}} \cdot d \bar{z}_3 \rangle \right)
\]

Take

\[
\theta_1 := \frac{d z_1}{2} \quad \text{and} \quad \theta_2 := 0,
\]

and set

\[
\phi := \theta_1 + \overline{\theta}_1 + \theta_2 - \overline{\theta}_2 = \frac{d z_1 + d \bar{z}_1}{2}.
\]
In [24, §8], the second author computed

$$H^*(\Gamma \backslash G; \mathfrak{d}_0) \neq \{0\}$$

and

$$H^*(\Gamma \backslash G; \overline{\mathfrak{d}}_{(\theta_1, \theta_2)}) = \{0\}.$$  

Hence $\Gamma \backslash G$ does not admit the $(\theta_1, \theta_2)$-Hodge decomposition.

We show now that also the $\mathfrak{d}_{(\theta_1, \theta_2)}\overline{\mathfrak{d}}_{(\theta_1, \theta_2)}$-Lemma does not hold on $\Gamma \backslash G$. Consider

$$\frac{1}{2} e^{\frac{1}{2}(\theta_1 + \overline{\theta}_1)} (d z_1 + d \overline{z}_1) \wedge d \overline{z}_3 \in \Lambda^2 g^*_C.$$  

We have

$$0 \neq \left[ \frac{1}{2} e^{\frac{1}{2}(\theta_1 + \overline{\theta}_1)} (d z_1 + d \overline{z}_1) \wedge d \overline{z}_3 \right] \in H^2(\Lambda^* g^*_C; \partial_{(\theta_1, \theta_2)}, \operatorname{\overline{\partial}}_{(\theta_1, \theta_2)}).$$  

On the other hand, we have

$$\frac{1}{2} e^{\frac{1}{2}(\theta_1 + \overline{\theta}_1)} (d z_1 + d \overline{z}_1) \wedge d \overline{z}_3 = \overline{\partial}_{(\theta_1, \theta_2)} \left(e^{\frac{1}{2}(\theta_1 + \overline{\theta}_1)} d \overline{z}_3\right).$$  

Therefore the map

$$H^2(\Lambda^* g^*_C; \partial_{(\theta_1, \theta_2)}, \overline{\partial}_{(\theta_1, \theta_2)}; \partial_{(\theta_1, \theta_2)} \overline{\partial}_{(\theta_1, \theta_2)}) \to H^2(\Lambda^*(\Gamma \backslash G) \cap; \partial_{(\theta_1, \theta_2)}, \overline{\partial}_{(\theta_1, \theta_2)}; \partial_{(\theta_1, \theta_2)} \overline{\partial}_{(\theta_1, \theta_2)})$$

$$\to H^2(\Lambda^*(\Gamma \backslash G) \cap; \partial_{(\theta_1, \theta_2)}, \overline{\partial}_{(\theta_1, \theta_2)}; \partial_{(\theta_1, \theta_2)} \overline{\partial}_{(\theta_1, \theta_2)})$$

is not injective. Since the first map is injective by Proposition 3.1, it follows that the natural map

$$H^2(\Lambda^*(\Gamma \backslash G) \cap; \partial_{(\theta_1, \theta_2)}, \overline{\partial}_{(\theta_1, \theta_2)}; \partial_{(\theta_1, \theta_2)} \overline{\partial}_{(\theta_1, \theta_2)}) \to H^2(\Lambda^*(\Gamma \backslash G) \cap; \partial_{(\theta_1, \theta_2)}, \overline{\partial}_{(\theta_1, \theta_2)}; \partial_{(\theta_1, \theta_2)} \overline{\partial}_{(\theta_1, \theta_2)})$$

is not injective. It follows that $\Gamma \backslash G$ does not satisfy the $\partial_{(\theta_1, \theta_2)}\overline{\partial}_{(\theta_1, \theta_2)}$-Lemma.

### 3.2. Solvmanifolds and $\partial\overline{\partial}$-Lemma with twisted differentials

The Weinstein and Thurston problem, concerning the characterization of nilmanifolds admitting Kähler structures, was solved by Ch. Benson and C. S. Gordon, [12, Theorem A]. In fact, in [20, Theorem 1, Corollary], K. Hasegawa proved that a solvmanifold admitting hyper-strong-Hodge-decomposition admits a Kähler metric.

As regards the characterization of solvmanifolds admitting Kähler structure, K. Hasegawa proved the following in [21, Main Theorem]. Let $X$ be a compact homogeneous space of solvable Lie group, that is, a compact differentialable manifold on which a connected solvable Lie group acts transitively. Then $X$ admits a Kähler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus-bundle over a complex torus. In particular, a completely-solvable solvmanifold has a Kähler structure if and only if it is a complex torus.

As regards solvmanifolds in class $C$ of Fujiki, they are characterized in [7, Theorem 9] by D. Arapura. More precisely, it follows from [7, Theorem 3, Theorem 9] that, for solvmanifolds endowed with complex structures, the properties of admitting Kähler metrics and of belonging to class $C$ of Fujiki are equivalent. The proof is sketched at [7, page 136], and is based on the fact that a finitely-presented group is a Fujiki group if and only if it is a Kähler group, see also [10, Theorem 1.1] by G. Bharali, I. Biswas, and M. Mj. In fact, their result founds on the Hironaka elimination of indeterminacies, [10, §2]. By using the results by the second author in [24] and the above results, we can provide a different and more direct proof, of cohomological flavour.

**Theorem 3.3.** Let $(M, J)$ be a solvmanifold endowed with a complex structure. If $(M, J)$ belongs to class $C$ of Fujiki, then it admits a Kähler metric.

**Proof.** Take any $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$. By Corollary 2.7, the manifold $(M, J)$ admits the $(\theta_1, \theta_2)$-Hodge decomposition. In [24], the property of satisfying the Hodge-decomposition with respect to any $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$ is called hyper-strong-Hodge-decomposition. The second author proved in [24, Theorem 1.7] that a solvmanifold admitting hyper-strong-Hodge-decomposition admits a Kähler metric. \qed
(Daniele Angella) ISTITUTO NAZIONALE DI ALTA MATEMATICA
Current address: Dipartimento di Matematica e Informatica, Università di Parma, Parco Area delle Scienze 53/A, 43124, Parma, Italy
E-mail address: daniele.angella@math.unipr.it

(Hisashi Kasuya) DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, 1-12-1 H-7, O-okayama, Meguro, Tokyo 152-8551, Japan
E-mail address: khs@ms.u-tokyo.ac.jp
E-mail address: kasuya@math.titech.ac.jp