Quantum Langevin theory of excess noise

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In an earlier work [P. J. Bardroff and S. Stenholm, submitted to Phys. Rev. Lett.], we have derived a fully quantum mechanical description of excess noise in strongly damped lasers. This theory is used here to derive the corresponding quantum Langevin equations. Taking the semi-classical limit of these we are able to regain the starting point of Siegman’s treatment of excess noise [Phys. Rev. A 39, 1253 (1989)]. Our results essentially constitute a quantum derivation of his theory and allow some generalizations.

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I. INTRODUCTION

According to the fluctuation-dissipation theorem, the attenuation or amplification of a signal always adds noise. In optical amplifiers, this fact is usually phrased as “one noise photon” added to the signal from the spontaneous emission processes in the reservoir. This assumption about the noise gives rise to the phase diffusion responsible for the Schawlow-Townes linewidth [1–3] of lasers. However, this is not true generally. In particular cases, the noise can exceed the intensity of one photon by the so-called excess-noise factor or Petermann $K$-factor [4].

Experimentally this phenomenon was first confirmed in a laser cavity with large output coupling leading to an enhancement of a few times [5]. Later, even a factor of a few hundreds was achieved for solid state lasers [6] and gas lasers [7]. Also experiments with a coupling of the polarizations [8] and an inserted small aperture [9] have demonstrated large excess noise. A recent experiment has shown that excess noise can be colored due to saturation effects [10].

After the prediction of excess noise by Petermann for the case of gain-guided semiconductor lasers [4], the first more general theory of excess noise was given by Siegman using a semi-classical description [11]. Until recently, only a few simple systems have been discussed from a quantum mechanical point of view [12].

In a previous paper [13] we introduced a master equation describing a multi-mode field interacting with a reservoir describing the general linear amplifier or attenuator in a strictly quantum mechanical formulation. We find that under certain conditions, the reservoirs create couplings between the undamped modes of the system. Such dissipative couplings lead to a non-Hermitian eigenvalue problem, which introduces non-orthogonal quasi modes in a natural manner. The amplitudes of these modes are then found to display the expected excess noise, which we here ascribe to the reservoir-induced mode-mode coupling.

In this paper we derive the quantum Langevin formalism following from our theory in Ref. [13]. Whereas the dynamic variable of the master equation is the quantum state, the Langevin equations are for the field operators. This allows us a direct comparison of our approach with the well-known semi-classical treatment introduced by Siegman [11]. As this has provided the physical understanding and the mathematical expressions for the excess noise, we are pleased that we can essentially derive his starting equations from our fully quantum mechanical treatment. We are also able to generalize the semi-classical analysis of excess noise to cases beyond the paraxial approximation.

II. MASTER EQUATION

In this section we briefly review the results of the quantum derivation of the excess-noise factor based upon the master equation. We use orthonormal real mode functions $u_n(x)$ of the electromagnetic field with frequency $\omega_n$ which fulfill the boundary conditions for the given configuration in the whole “universe” and satisfy the orthonormality relation

$$\frac{1}{V} \int d^3x u_n(x) u_m(x) = \delta_{nm},$$  \hspace{1cm} (1)$$

where $V$ is the volume of the whole space. Note that the mode function $u_n(x)$ is a vector including the polarization orientation and that we choose them to be real for convenience. The electric field operator then reads
\[ \dot{E}(x) = \sum_n \varepsilon_n u_n(x) \left( \hat{a}_n + \hat{a}_n^\dagger \right), \]  

(2)

where \( \hat{a}_n \) and \( \hat{a}_n^\dagger \) are the usual creation and annihilation operators of the field excitations and the so-called vacuum field amplitude is

\[ \varepsilon_n = \frac{\sqrt{\hbar \omega_n}}{2 \omega V}. \]  

(3)

We start from the multi-mode master equation [13]

\[ \frac{d}{dt} \hat{\rho}(t) = \frac{1}{2} \sum_{n,m} L_{m,n} \{ 2\hat{a}_n^\dagger \hat{a}_m \hat{a}_m^\dagger \hat{a}_n \hat{a}_n \} - \frac{1}{2} \sum_{n,m} \Gamma_{m,n} \{ 2\hat{a}_n \hat{a}_m \hat{a}_m^\dagger \hat{a}_n \} - i \sum_n \omega_n [\hat{a}_n^\dagger \hat{a}_n, \hat{\rho}(t)]. \]  

(4)

with the two symmetric matrices \( \Gamma_{m,n} \) and \( L_{m,n} \) given by

\[ \Gamma_{m,n} = \frac{\tau^2}{\hbar^2} \varepsilon_n \varepsilon_m \frac{1}{V} \int d^3 x r_T(x) |u_n(x)d| |u_m(x)d| \]  

(5a)

and

\[ L_{m,n} = \frac{\tau^2}{\hbar^2} \varepsilon_n \varepsilon_m \frac{1}{V} \int d^3 x r_L(x) |u_n(x)d| |u_m(x)d|. \]  

(5b)

The former describes losses and the latter amplification due to the interaction with the reservoirs.

The two reservoirs for amplification and attenuation are assumed to consist of two-level atoms injected in the upper or lower state, respectively [2]. They are completely characterized by the position dependent injection rates \( r_L(x) \) and \( r_T(x) \), the interaction time \( \tau \) of the individual atoms with the field and the orientation of the atomic dipole moment \( d \). In principle, the dipole orientation could be different for damping and attenuation and it may depend on position. This treatment of the damping can describe spatially localized absorption due to an inserted aperture or due to a detector placed outside the cavity. Assuming a perfect absorber (or detector) surrounding our cavity, the reservoir can also model the damping due to output coupling. Taking the limit of a infinitely large “universe” \( V \to \infty \), and hence using a continuum of modes, would be another way of including losses due to output coupling in our model as shown in Ref. [14].

Because of the interaction through the reservoir, the time evolution of the mean values

\[ \frac{d}{dt} \langle \hat{a}_n \rangle = \frac{1}{2} \sum_m \left( L_{m,n} - \Gamma_{m,n} \right) \langle \hat{a}_m \rangle - i \omega_n \langle \hat{a}_n \rangle \]  

(6)

exhibits coupling between different modes. The definition of the quasi modes operator \( \hat{A} \) follows from imposing the condition

\[ \frac{d}{dt} \langle \hat{A} \rangle = \left\{ \frac{1}{2} (\lambda - \gamma) - i \Omega \right\} \langle \hat{A} \rangle, \]  

(7)

where \( \Omega \) is the frequency, \( \lambda \) is the amplification rate and \( \gamma \) is the attenuation rate. For later convenience, we split the net-amplification rate \( \lambda - \gamma \) into the two separate contributions \( \lambda \) and \( \gamma \). Note that \( \Omega, \lambda \) and \( \gamma \) are real. We write this mode operator in terms of the free field mode operators as

\[ \mathcal{E} \hat{A} = \sum_n \varepsilon_n c_n \hat{a}_n \]  

(8)

with the expansion coefficients \( c_n \). This transformation includes the vacuum-field amplitudes \( \varepsilon_n \) and we define \( \mathcal{E} = \sqrt{\frac{\hbar \Omega}{2 \omega V}} \), because then the classical field amplitudes \( \varepsilon_n \langle \hat{a}_n \rangle \) obey the same transformation as the operators. Inserting Eq. (8) into (8) we get an eigenvalue equation

\[ \sum_n \left\{ \frac{1}{2} (L_{m,n} - \Gamma_{m,n}) - i \delta_{n,m} \omega_n \right\} \varepsilon_n \varepsilon_m c_n = \left\{ \frac{1}{2} (\lambda - \gamma) - i \Omega \right\} c_m \]  

(9)
for the non-Hermitian matrix \( \frac{1}{2}(L_{m,n} - \Gamma_{m,n}) - i\delta_{a,m}\omega_n \) \( \frac{c_n}{c_m} \). Here \( c_n^{(\nu)} \) is the right eigenvector; the corresponding left eigenvector is \( \varepsilon_n^{2}c_n^{(\nu)} \). The superscript \( \nu \) distinguishes the different eigenvectors. The detailed properties of the quasi modes are summarized in the Appendix.

We can now calculate the noise of the quadrature operator

\[
\hat{X}_{\nu}(x) = \mathcal{E}_{\nu} \left[ U_{\nu}(x) \hat{A}_{\nu} + U_{\nu}^{*}(x)\hat{A}_{\nu}^{\dagger} \right]
\]

with the definition of the quasi-mode function \( U_{\nu}(x) \) given by Eq. (A7) in the Appendix. Taking the noise averaged over position and comparing to the usual single mode master equation with the same frequency \( \Omega_{\nu} \), damping rate \( \gamma_{\nu} \) and amplification rate \( \lambda_{\nu} \), we find an enhancement by the factor

\[
K_{\nu} = \left| \frac{\sum_{m} \varepsilon_{m}^{2}c_{n}^{(\nu)}c_{m}^{(\nu)}}{\sum_{m} \varepsilon_{m}^{2}} \right|^{2}
\]

for the noise added by the reservoir; cf. Ref. [13]. The excess noise is large when the matrices \( L_{m,n} \) and \( \Gamma_{m,n} \), defined in Eqs. (5), have large off-diagonal terms and when they are not identical. The former follows when the injection rates \( r_L(x) \) and \( r_T(x) \) are not spatially constant whereas the latter when damping and amplification are spatially separated.

### III. QUANTUM LANGEVIN EQUATION

Following the usual treatment [3], we replace the time evolution described by the master equation (1) by an equivalent quantum Langevin equation. This contains non-commuting noise forces which are designed such as to give the same moments as those derived from the master equation. The quantum Langevin equation is written

\[
\frac{d}{dt} \hat{a}(t) = \frac{1}{2} \sum_{m} (L_{m,n} - \Gamma_{m,n}) \hat{a}(t) - i\omega_n \hat{a}(t) + \hat{f}_n(t),
\]

where the Langevin noise sources \( \hat{f}_n(t) \) obey the correlation relations

\[
\begin{align*}
\langle \hat{f}_m(t)\hat{f}_m^{\dagger}(t') \rangle &= 2\langle \hat{D}_{\hat{a}_m\hat{a}_m}^{\dagger} \delta(t-t') \rangle, \\
\langle \hat{f}_n^{\dagger}(t)\hat{f}_m(t') \rangle &= 2\langle \hat{D}_{\hat{a}_n\hat{a}_{m}}^{\dagger} \delta(t-t') \rangle
\end{align*}
\]

and

\[
\langle \hat{f}_m(t) \rangle = \langle \hat{f}_m(t)\hat{f}_n(t') \rangle = \langle \hat{f}_n^{\dagger}(t)\hat{f}_n^{\dagger}(t') \rangle = 0.
\]

It then follows from Eq. (12) that the expectation values obey Eq. (1). The diffusion coefficients \( \langle \hat{D}_{\hat{a}_m\hat{a}_m}^{\dagger} \rangle \) and \( \langle \hat{D}_{\hat{a}_n\hat{a}_m} \rangle \) have to be determined to give the correct noise correlations \( \langle \hat{a}_n^{\dagger}\hat{a}_m \rangle \) and \( \langle \hat{a}_m\hat{a}_n^{\dagger} \rangle \). We compare the time evolution of the noise correlations derived from the master equation, given by Eqs. (A12), to the one derived from the Langevin equation (12) to find the relations

\[
\begin{align*}
\langle \hat{f}_n^{\dagger}(t)\hat{a}_m(t) + \hat{a}_m(t)\hat{f}_n(t) \rangle &= \Gamma_{m,n}, \\
\langle \hat{f}_n(t)\hat{a}_m^{\dagger}(t) + \hat{a}_n(t)\hat{f}_m^{\dagger}(t) \rangle &= L_{m,n}.
\end{align*}
\]

With the help of the Einstein relations

\[
\begin{align*}
2\langle \hat{D}_{\hat{a}_m\hat{a}_m}^{\dagger} \rangle &= \frac{d}{dt} \langle \hat{a}_m(t)\hat{a}_m^{\dagger}(t) \rangle - \langle \hat{a}_m(t)\frac{d}{dt} \hat{a}_m^{\dagger}(t) \rangle - \langle (\frac{d}{dt} \hat{a}_m(t) - \hat{f}_m(t))\hat{a}_m^{\dagger}(t) \rangle, \\
2\langle \hat{D}_{\hat{a}_n\hat{a}_m} \rangle &= \frac{d}{dt} \langle \hat{a}_n^{\dagger}(t)\hat{a}_m(t) \rangle - \langle \hat{a}_m(t) - \hat{f}_n^{\dagger}(t)\hat{a}_n^{\dagger}(t) \rangle - \langle \hat{a}_n^{\dagger}(t)(\hat{a}_m(t) - \hat{f}_m(t)) \rangle.
\end{align*}
\]
and Eqs. (14), we determine the diffusion coefficients to be
\[ 2\langle \hat{D}_{\delta,n}\delta_m \rangle = \langle \hat{f}_n(t)\delta_m(t) + \delta_n(t)\hat{f}_m(t) \rangle = \Gamma_{m,n} \quad (17a) \]
and
\[ 2\langle \hat{D}_{\delta,n}\delta_m \rangle = \langle \hat{f}_n(t)\delta_m(t) + \delta_n(t)\hat{f}_m(t) \rangle = L_{m,n}. \quad (17b) \]
These relations clearly show the mode correlations due to the reservoir.

In the following, we derive a wave equation for the propagation of the electric field operator including amplification, damping and the corresponding noise source. Starting from Eqs. (12) and (2), we can find the exact equation
\[
\left\{ \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 \right\} \hat{E}(x,t) - \sum \varepsilon_n u_n(x) \left\{ \sum_{k} (L_{k,n} - \_{k,n}) \frac{d}{dt} \hat{a}_k + h.c. \right\} = \\
\sum \varepsilon_n u_n(x) \left\{ - \frac{1}{4} \sum_{k,l} (L_{k,n} - \Cancel{\gamma}_{k,n})(L_{l,k} - \Cancel{\gamma}_{l,k}) \hat{a}_l \\
+ \frac{i}{2} \sum_{k} (L_{k,n} - \gamma_{k,n})(\omega_k - \omega_n) \hat{a}_k \\
- \frac{1}{2} \sum_{k} (L_{k,n} - \gamma_{k,n}) \hat{f}_k - i\omega_n \hat{f}_n + \frac{d}{dt} \hat{f}_n \right\} + h.c. \quad (18)
\]
with the mode functions \( u_n(x) \) fulfilling the Helmholtz equation
\[ (c^2 \nabla^2 + \omega_n^2) u_n(x) = 0 \quad (19) \]
together with the appropriate boundary conditions. At this point we introduce a number of approximations based on the assumption that the average oscillation frequency \( \bar{\omega} \) of the electric field is much higher than the decay or amplification rates, e.g. \( |L_{n,m}| \) or \( |\gamma_{n,m}| \). This is well justified in the optical regime where the former is at least six orders of magnitude larger than the latter. The spectral width \( \Delta \omega \) of the relevant frequencies \( \omega_n \) is assumed to be of the order of the decay or amplification rate. To be more specific, we will neglect terms of the order \( O(\lambda_{\nu}/\bar{\omega}^2) \), \( O(\lambda_{\nu}\Delta \omega/\bar{\omega}^2) \) and \( O(\Delta \omega/\bar{\omega})^2 \) or smaller, and we assume \( \lambda_{\nu} \approx O(\gamma_{\mu}) \).

On the LHS of Eq. (18) the two terms \( \frac{\partial^2}{\partial t^2} \hat{E}(x,t) \) and \( c^2 \nabla^2 \hat{E}(x,t) \) are of the order \( O(\bar{\omega})^2 \). Since the remaining term on the LHS of Eq. (18) is proportional to the damping and amplification rate, we can approximate the frequencies by the mean frequency \( \bar{\omega} \). Hence inserting the definitions of \( L_{n,m} \) and \( \gamma_{n,m} \), Eqs. (3), and of \( \varepsilon_n \), Eq. (3), the remaining term on the LHS of Eq. (18) yields
\[
\sum \varepsilon_n u_n(x) \sum_{k} (L_{k,n} - \gamma_{k,n}) \frac{d}{dt} \hat{a}_k + h.c. = \\
\sum \varepsilon_n u_n(x) \sum_{k} \frac{\tau^2}{\hbar^2 \varepsilon_n \varepsilon_k} \frac{1}{V} \int d^3x'(r_L(x') - r_T(x'))[u_n(x')d][u_L(x')d] \frac{d}{dt} \hat{a}_k + h.c. = \\
\frac{\tau^2}{\hbar^2} \sum_{n} \varepsilon_n u_n(x) \frac{1}{V} \int d^3x'(r_L(x') - r_T(x'))[u_n(x')d][u_L(x')d] \frac{d}{dt} \hat{E}(x', t) \approx \\
\frac{\tau^2 \bar{\omega}}{2\varepsilon_0 \hbar} \int d^3x'(r_L(x') - r_T(x')) \delta_T(x - x') d \otimes d^T \frac{d}{dt} \hat{E}(x', t) = (R_L(x) - R_T(x)) \frac{d}{dt} \hat{E}(x, t). \quad (20)
\]
Here the matrices \( L_{n,m} \) and \( \gamma_{n,m} \) occur in their position representations
\[ \frac{1}{V} \sum_{n,m} u_n(x) \otimes u_m^T(x') L_{n,m} \approx \frac{\tau^2 \bar{\omega}}{2\varepsilon_0 \hbar} R_L(x) \delta_T(x - x') d \otimes d^T \equiv R_L(x) \delta_T(x - x') \quad (21a) \]
and
\[ \frac{1}{V} \sum_{n,m} u_n(x) \otimes u_m^T(x') \gamma_{n,m} \approx \frac{\tau^2 \bar{\omega}}{2\varepsilon_0 \hbar} R_T(x) \delta_T(x - x') d \otimes d^T \equiv R_T(x) \delta_T(x - x'). \quad (21b) \]
Note that $R_L(x)$, $R_T(x)$, $d \otimes d^T$ and the transverse $\delta$-function

$$\delta_T(x-x') = \frac{1}{V} \sum_n u_n(x) \otimes u_n^T(x')$$  \hspace{1cm} (22)

are tensors. We can neglect the terms on the RHS of Eq. (13) containing the field operator since they are of the order $\mathcal{O}(\lambda_\nu)^2$ or $\mathcal{O}(\lambda_\nu \Delta \omega)$, respectively. For the noise we only take terms of lowest order. Therefore we may neglect the first of the noise terms in Eq. (18) and we approximate $\frac{d}{dt} f_n \approx -i\bar{\omega}_n f_n$. Introducing the position representation

$$\hat{f}(x,t) = \sum_n \epsilon_n u_n(x) f_n(t)$$ \hspace{1cm} (23)

of the noise source we find

$$\left\{ \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 - (R_L(x) - R_T(x)) \frac{\partial}{\partial t} \right\} \hat{E}(x,t) = -2i\bar{\omega} \hat{f}(x,t) + h.c. \hspace{1cm} (24)$$

The correlations of the noise operators are

$$\langle \hat{f}(x,t) \hat{f}^\dagger(x',t') \rangle = \frac{\hbar \bar{\omega}}{2\epsilon_0} R_T(x) \delta_T(x-x') \delta(t-t')$$ \hspace{1cm} (25a)$$

and

$$\langle \hat{f}^\dagger(x,t) \hat{f}(x',t') \rangle = \frac{\hbar \bar{\omega}}{2\epsilon_0} R_L(x) \delta_T(x-x') \delta(t-t').$$ \hspace{1cm} (25b)$$

Consequently the total noise on the RHS of Eq. (24) obeys

$$\langle (-2i\bar{\omega} \hat{f}(x,t) + h.c.)^2 \rangle = \frac{\hbar \bar{\omega}^3}{\epsilon_0} (R_L(x) + R_T(x)) \delta_T(x-x') \delta(t-t'). \hspace{1cm} (26)$$

As expected, the effects of amplification and damping add for the noise whereas they subtract for the amplification.

**IV. SEMI-CLASSICAL TREATMENT**

Starting from Eq. (24) we can now perform the transition to the semi-classical treatment replacing operators with $\epsilon$-numbers. The solution of Eq. (24) can conveniently be written using the positive frequency part $E^{(+)}(x)$ of the electromagnetic field. The real part is the electric field and the imaginary part relates to the magnetic field. With the help of the Green function

$$G^{(+)}(x,x',t) = \sum_\nu U_\nu(x) \bar{U}_\nu(x') e^{\frac{i}{\hbar} (\lambda_\nu - \gamma_\nu) t - i\Omega_\nu t} \hspace{1cm} (27)$$

and the accumulated noise

$$N^{(+)}(x,t) = -2i\bar{\omega} \int_0^t dt' \sum_\nu e^{\frac{i}{\hbar} (\lambda_\nu - \gamma_\nu) (t-t') - i\Omega_\nu (t-t')} U_\nu(x) \frac{1}{V} \int d^3x' \bar{U}_\nu(x') f(x',t'), \hspace{1cm} (28)$$

we find the field to be given by

$$E^{(+)}(x,t) = \frac{1}{V} \int d^3x' G^{(+)}(x,x',t) E^{(+)}(x',0) + N^{(+)}(x,t) \hspace{1cm} (29)$$

starting from the initial field $E^{(+)}(x',t=0)$. Within the approximations made, the quasi-mode functions $U_\nu(x)$ and $\bar{U}_\nu(x)$ and their eigenvalues $\lambda_\nu$, $\gamma_\nu$ and $\Omega_\nu$ are the same as defined using the master equation, Eqs. (A7) and (A8). When we now calculate the variance of the electric field $E(x,t)$ averaged over position and compare with damping and amplification processes described by the usual single mode master equation, we recover the same $K$-factor, Eq. (11), as before. We find for the noise term
with the enhancement factor in the commonly used form

\[ K_\nu = \frac{\int d^3x U_\nu(x)U^*_\mu(x)}{\int d^3x U_\nu(x)} \frac{\int d^3x \tilde{U}_\nu(x)(R_L(x) + R_T(x))\tilde{U}^*_\mu(x)}{[\int d^3x U_\nu(x)]^2}. \]

(32)

Note that our choice of the normalization for the quasi-mode functions is given as in Eq. (A8). We have shown in Ref. [13] that Eq. (32) agrees with Eq. (11) up to the order \( O(\Delta \omega/\bar{\omega})^2 \).

Siegman [11] used an equation analogous to Eq. (24) as the starting point for his derivation of the excess-noise factor. However, there are two interesting differences in the details of the noise source correlations, Eq. (26).

The first difference in Ref. [11] is that the spatial transverse \( K \)-factor is replaced by the Hertzian bandwidth \( \Delta \omega/(2\pi) \) of the reservoir. The latter circumstance is explained by using the Fourier representation of our noise correlation in Eq. (26); this leads to the same equation with \( \delta(t - t') \) replaced by \( \delta(\omega - \omega')/2\pi \). We then integrate with respect to \( \omega \) and \( \omega' \) over the frequency bandwidth \( \Delta \omega \) to obtain

\[
\frac{1}{2} \int \int \frac{d\omega d\omega'}{2\pi} \delta(\omega - \omega')e^{-i(\omega t - \omega't')} + c.c. \approx \frac{\sin(\Delta \omega(t - t')/2)}{2\pi(t - t')} e^{-i\omega(t - t') + c.c.} \approx \frac{1}{2} \frac{\Delta \omega}{2\pi} e^{-i\omega(t - t') + c.c.}
\]

(33)

for \( |t - t'| < (\Delta \omega)^{-1} \). This approximation is reasonable when the mean frequency \( \bar{\omega} \) of the noise is much larger then the bandwidth \( \Delta \omega \).

The second difference in Ref. [11] is that \( (R_L(x) + R_T(x)) \) is replaced by \( 2 \left( \frac{R_L}{R_L + R_T} \right) (R_L - R_T) = 2R_L \) with spatially constant \( R_L \) and \( R_T \). This simplification is justified only when averaging over the whole volume \( V \) and when amplification and damping are balanced. For the derivation of the \( K \)-factor which involves an average over position, this is a valid replacement. However, one has to be aware of the subtlety that only non-constant \( R_L(x) \) and \( R_T(x) \) with \( R_L(x) \neq R_T(x) \) lead to non-orthogonal quasi modes and hence can give \( K > 1 \).

It is interesting to note that within the paraxial approximation, we obtain an equation analogous to the position representation of Eq. (12)—the starting point of our semi-classical analysis. Making the ansatz

\[ E^{(+)}(x, t) = e^{i\bar{\omega}(z/c - t)} \tilde{E}^{(+)}(x) \]

with \( \tilde{E}^{(+)}(x) \) slowly varying with respect to the longitudinal coordinate \( z \), we get from Eq. (24)

\[ c \frac{\partial}{\partial z} \tilde{E}^{(+)}(x) = \left\{ \frac{1}{2} (R_L(x) - R_T(x)) + \frac{i c^2}{2 \bar{\omega}} \nabla^2 \right\} \tilde{E}^{(+)}(x) + \tilde{f}(x) \]

(35)

where \( f(x, t) \approx e^{i\bar{\omega}(z/c - t)} \hat{f}(x) \). The time derivative \( d/dt \) of Eq. (12) is replaced by the derivative with respect the longitudinal coordinate \( c \partial / \partial z \) which is equivalent in a frame moving with the electromagnetic wave. The frequency part of Eq. (12) is replaced by the transverse Laplacian. Frequently, Eq. (35) is solved with mode functions of the transverse Laplace equation, depending only parametrically on the longitudinal coordinate. This distinction between longitudinal and transverse coordinates leads to a factorization of the \( K \)-factor into a longitudinal and a transverse part.
V. DISCUSSION

In an earlier paper [13], we derived the master equation for a set of modes coupled to amplifying and attenuating reservoirs. This introduces couplings between the undamped modes of the total “universe” and leads directly to the introduction of quasi modes, which are found to exhibit the excess noise described originally by Petermann [4].

Our treatment has been carried out only in the linear regime so far. This describes an amplifier or an attenuator, where the treatment is most straightforward and the results display the most transparent physical insight. However, the saturation in an operating laser will need to be considered, and we are for the moment carrying out such calculations, which show the influence of the excess noise in the strong field situation.

The best physical picture of this noise was provided by Siegman [11], who also supplied the quasi-mode expression for the excess-noise factor. This has then been used successfully to describe the experimental findings [5–9]. In [13] we showed that our quantum mechanical approach naturally provides an expression which is essentially identical with Siegman’s results.

Siegman, however, utilized a semi-classical Langevin approach, where the noise forces were added ad hoc to the classical equations for the amplitudes; the noise forces were then supplied with properly chosen correlation properties, which was shown to imply the presence of excess noise. Because this approach has been found to give both a physically attractive and theoretically justified description of the situation, we find it interesting to connect that treatment to our quantum approach in some detail.

In this paper we derive the quantum Langevin equations following from our general master equation. Here we utilize techniques known from quantum noise theory, and obtain results that can be directly compared with the treatment of Siegman’s, when the semi-classical limit is taken. Except for some minor differences, our resulting equations are identical with those used by Siegman. We thus claim that we have justified his formulation of the problem from a more fundamental quantum mechanical point of view. The differences found are either based on natural approximations or obvious qualifications of the results as e.g. the introduction of the transverse delta function in the noise correlations. In addition, we have been able to generalize the theory to situations outside the paraxial approximation.

The results of our treatment, however, have bearings beyond the problem of excess noise in highly lossy cavities. The approach is quite general, and in addition to the Markov approximation we only need the rotating wave approximation for the interaction with the reservoirs. The master equation is then derived from first principles, and the nonorthogonal quasi modes emerge in a natural manner. The theory is fully general and may well be applicable to other high loss physical systems as well. For the moment we know of no observation that would show the equivalent of the laser excess noise, but novel situations may soon turn up. The lively research activity in quantum information processing, atom optics and novel measurement situations may provide potential applications of the present theory.

The physics of our approach resides in the coupling of the undamped modes through the reservoirs. In such a situation, the only essential assumption in our derivation is the Markovian approximation. In highly damped systems, this may not necessarily hold, and the introduction of memory effects in our theory has not been considered so far. Some features are, however, expected to survive, but also unexpected complications may appear. These questions remain to be investigated.

APPENDIX A: PROPERTIES OF THE QUASI MODES

In this Appendix we recall from Ref. [13] those properties of the quasi modes which are relevant for the derivation of the excess noise. The only properties of the left and right eigenvectors of non-Hermitian matrices which we need for our analysis are their mutual orthogonality and completeness [16]: The eigenvectors fulfill the orthogonality condition

$$\sum_n \varepsilon_n^2 \langle \nu | c_n^\dagger (\mu) \rangle = \delta_{\nu,\mu} \sum_n \varepsilon_n^2 \langle \nu | c_n \rangle^2 \quad (A1)$$

and the completeness relation

$$\sum_\nu \left( \frac{\varepsilon_n^2 \langle \nu | c_n \rangle \langle \nu | c_m \rangle}{\sum_{n'} \varepsilon_{n'}^2 \langle \nu | c_{n'} \rangle^2} \right) = \delta_{n,m} \quad (A2)$$

with $\sum_{n'} \varepsilon_{n'}^2 \langle \nu | c_{n'} \rangle^2 \neq 0$. Therefore, we can uniquely define the set of quasi-mode operators as
\[ \hat{A}_\nu = \frac{1}{\mathcal{E}_\nu} \sum_n c_n^{(\nu)} \varepsilon_n \hat{a}_n \]  

(A3)

with the vacuum field amplitude

\[ \mathcal{E}_\nu = \sqrt{\frac{\hbar \Omega_\nu}{2\epsilon_0 V}}. \]  

(A4)

The inverse transformation is

\[ \hat{a}_n = \varepsilon_n \sum_\nu \frac{c_n^{(\nu)}}{\sum_m \varepsilon_m^{(\nu)^2}} \mathcal{E}_\nu \hat{A}_\nu. \]  

(A5)

Consequently the positive frequency part of the electric field operator is given by

\[ \hat{E}^{(+)}(x) = \sum_n \varepsilon_n u_n(x) \hat{a}_n = \sum_\nu \mathcal{E}_\nu U_\nu(x) \hat{A}_\nu. \]  

(A6)

The quasi-mode eigenfunctions

\[ U_\nu(x) = \sum_n \frac{\varepsilon_n^{(\nu)}}{\sum_m \varepsilon_m^{(\nu)^2}} u_n(x) \]  

(A7)

satisfy an orthogonality relation

\[ \frac{1}{V} \int d^3x U_\nu(x) \bar{U}_\mu(x) = \delta_{\nu,\mu}. \]  

(A8)

with their adjoint quasi-mode functions

\[ \bar{U}_\nu(x) = \sum_n \varepsilon_n^{(\nu)^*} u_n(x). \]  

(A9)

The properties

\[ \Omega_\nu = \frac{\sum_n \varepsilon_n^{(\nu)^2} \omega_n |c_n^{(\nu)}|^2}{\sum_n \varepsilon_n^{(\nu)^2} |c_n^{(\nu)}|^2} = \frac{2 \epsilon_0 V \sum_n \varepsilon_n^{(\nu)^2} |c_n^{(\nu)}|^2}{\hbar \sum_n \varepsilon_n^{(\nu)^2} |c_n^{(\nu)}|^2}. \]  

(A10)

\[ \lambda_\nu = \frac{\sum_{n,m} L_{m,n} \varepsilon_n \varepsilon_m c_n^{(\nu)*} c_m^{(\nu)}}{\sum_n \varepsilon_n^{(\nu)^2} |c_n^{(\nu)}|^2} \]  

(A11a)

and

\[ \gamma_\nu = \frac{\sum_{n,m} \Gamma_{m,n} \varepsilon_n \varepsilon_m c_n^{(\nu)*} c_m^{(\nu)}}{\sum_n \varepsilon_n^{(\nu)^2} |c_n^{(\nu)}|^2} \]  

(A11b)

can be obtained from the real and imaginary parts of Eq. (9) after taking the scalar product with the vector \( \varepsilon_m^{(\nu)*} c_m^{(\nu)} \).

From the master equation (4) follows the time evolution of the noise correlations

\[ \frac{d}{dt} \langle \hat{a}_n^{\dagger} \hat{a}_m \rangle = \frac{1}{2} \sum_k (L_{k,n} - \Gamma_{k,n}) \langle \hat{a}_k^{\dagger} \hat{a}_m \rangle + \frac{1}{2} \sum_k (L_{m,k} - \Gamma_{m,k}) \langle \hat{a}_n^{\dagger} \hat{a}_k \rangle + i(\omega_n - \omega_m) \langle \hat{a}_n^{\dagger} \hat{a}_m \rangle + L_{m,n} \]  

(A12a)

and

\[ \frac{d}{dt} \langle \hat{a}_m^{\dagger} \hat{a}_n \rangle = \frac{1}{2} \sum_k (L_{k,m} - \Gamma_{k,m}) \langle \hat{a}_k^{\dagger} \hat{a}_n \rangle + \frac{1}{2} \sum_k (L_{n,k} - \Gamma_{n,k}) \langle \hat{a}_m^{\dagger} \hat{a}_k \rangle + i(\omega_n - \omega_m) \langle \hat{a}_m^{\dagger} \hat{a}_n \rangle + \Gamma_{m,n}. \]  

(A12b)
For the quasi-mode operators, we find the correlations to be

\[
\frac{d}{dt} \langle \hat{A}_\nu^\dagger \hat{A}_\mu \rangle = \left\{ \frac{1}{2} (\lambda_\nu + \lambda_\mu - \gamma_\nu - \gamma_\mu) + i (\Omega_\nu - \Omega_\mu) \right\} \langle \hat{A}_\nu^\dagger \hat{A}_\mu \rangle + \frac{1}{\epsilon_\nu \epsilon_\mu} \sum_{n,m} L_{n,m} \epsilon_n \epsilon_m c_\nu^{(n)} c_\mu^{(m)}
\]  
(A13a)

and

\[
\frac{d}{dt} \langle \hat{A}_\mu^\dagger \hat{A}_\nu \rangle = \left\{ \frac{1}{2} (\lambda_\nu + \lambda_\mu - \gamma_\nu - \gamma_\mu) + i (\Omega_\nu - \Omega_\mu) \right\} \langle \hat{A}_\mu^\dagger \hat{A}_\nu \rangle + \frac{1}{\epsilon_\nu \epsilon_\mu} \sum_{n,m} \Gamma_{n,m} \epsilon_n \epsilon_m c_\nu^{(n)} c_\mu^{(m)}
\]  
(A13b)

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