THE AMPLITUDE MODULATION TRANSFORM

IGOR RIVIN

Abstract. Motivated by the study of the local extrema of \( \sin(x)/x \) we define the Amplitude Modulation transform of functions defined on (subsets of) the real line. We discuss certain properties of this transform and invert it in some easy cases.

Introduction

This note has been motivated by the following question:

Let \( 0 = x_0 < x_1 < \cdots < x_n < \cdots \) be the sequence of local maxima of the sinc function \( \text{sinc}(x) = \sin(x)/x \). Is the sequence \( 1 = \text{sinc}(x_0), \text{sinc}(x_1), \ldots, \text{sinc}(x_n), \ldots \) decreasing?

This question is not difficult to answer. Indeed, at a critical point \( x_i \),

\[
\text{sinc}'(x_i) = \frac{x_i \cos(x_i) - \sin(x_i)}{x_i^2} = 0,
\]

which implies that

\[
\cos(x_i) = \frac{\sin(x_i)}{x_i} = \text{sinc}(x_i).
\]

One can also write the above equation as:

\[
x_i = \tan(x_i),
\]

or, equivalently:

\[
\arctan(x_i) = x_i.
\]

Combining equations (0.2) and (0.3), we obtain:

\[
\text{sinc}(x_i) = \cos(\arctan(x_i)) = \frac{1}{\sqrt{1 + x_i^2}},
\]

so the decrease of \( \text{sinc}(x_i) \) is immediate.

1. The Amplitude Modulation Transform

The formula (0.4) suggest the following:

Definition 1.1. The Amplitude Modulation transform \( \text{AM}(f) \) of a function \( f : \mathbb{R} \to \mathbb{R} \) is the set of functions whose values at the critical points of \( f(x) \sin(x) \) agrees with those of \( f(x) \sin(x) \).

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Remark 1.2. In fact, if in the definition of the $\mathcal{AM}$ transform we replace the multiplier $\sin(x)$ by $\sin(x+k)$, we obtain the same function $\mathcal{AM}f$. This is an observation of W. D. Smith, and it allows us to replace the definition above with the more pleasant definition below:

**Definition 1.3.** The *Amplitude Modulation* transform $\mathcal{AM}(f)$ of a function $f : \mathbb{R} \to \mathbb{R}$ is the function whose values at the critical points of $f(x) \sin(x+k)$ agrees with those of $f(x) \sin(x+k)$ for all values of the phase parameter $k$.

The discussion in the Introduction can thus be summarized as follows:

**Theorem 1.4.** The function $\frac{1}{\sqrt{1+x^2}}$ is the $\mathcal{AM}$ transform of $\frac{1}{x}$.

To get an analogous result for a general function $f(x)$, we perform the same sort of computation as in the Introduction:

(We will use the notation $f_x$ for the derivative of $f$ for typographical reasons.)

The critical points of $f(x) \sin(x)$ are the points where:

$$ \frac{df(x) \sin(x)}{dx} = 0. $$

Expanding, we see that $f(x) \cos(x) + f_x(x) \sin(x) = 0$, and so

$$ \cot(x) = -\frac{f_x(x)}{f(x)}, $$

so that

(1.1) $$ x = \arccot \left( -\frac{f_x(x)}{f(x)} \right), $$

while

(1.2) $$ f(x) \sin(x) = -\tan(x) \sin(x) f_x(x). $$

Combining Eq. (1.1) and Eq. (1.2) we see that at the critical points:

(1.3) $$ f(x) \sin(x) = \pm \frac{f^2(x)}{\sqrt{f^2(x) + f^2_x(x)}}, $$

which we can summarize in

**Theorem 1.5 (Theorem-Definition).** The function $\mathcal{AM}(f)$ is defined by

$$ \mathcal{AM}(f)(x) = \frac{f^2(x)}{\sqrt{f^2(x) + f^2_x(x)}}, $$

Here are some examples: As we have seen before, if $f(x) = 1/x$, then

$$ \mathcal{AM}(f)(x) = \frac{1}{\sqrt{1 + x^2}}. $$

If $f(x) = x^\alpha$, then

$$ \mathcal{AM}(f)(x) = \frac{x^{\alpha+1}}{\sqrt{x^2 + \alpha^2}}. $$

If $f(x) = \exp(x)$, then

$$ \mathcal{AM}(f)(x) = \frac{\exp(x)}{\sqrt{2}}, $$
while if \( f(x) = \exp(-x) \), then

\[
\mathcal{AM}(f)(x) = \frac{\exp(-x)}{\sqrt{2}}.
\]

If \( f(x) = \exp(g(x)) \), then

(1.4) \[ \mathcal{AM}(f)(x) = f(x) \frac{1}{\sqrt{1 + g_x^2}}. \]

2. Some algebraic observations

We will need to recall a definition:

**Definition 2.1.** A function \( y = f(x) \) is called *algebraic* if there exists a two-variable polynomial \( P \), such that \( P(x, y) = 0 \).
And some well-known results:

**Theorem 2.2.** If $z$ is an algebraic function of $y$ and $y$ is an algebraic function of $x$, then $z$ is an algebraic function of $x$. Further, if $y_1$ and $y_2$ are algebraic functions of $x$, then so are $y_1 y_2$ and $y_1 + y_2$. Finally, if $y$ is an algebraic function, then so is $y'$.

*Proof Sketch and lightning introduction to elimination theory.* The proofs of all the assertions follow from the following basic fact: two univariate polynomials $P_1$ and $P_2$ over a domain $R$ with unity have no common zeros if and only if their greatest common divisor is 1, or, equivalently, there exist polynomials $Q_1$ and $Q_2$, such that

$$Q_1 P_1 + Q_2 P_2 = 1.$$  

Since Eq. (2.1) is a system of linear equations for the coefficients of $Q_1$ and $Q_2$, the existence of $Q_1$ and $Q_2$ as above is easily seen to be equivalent to the non-vanishing of the determinant of the linear system. This determinant is the so-called resultant of the polynomials $P_1$ and $P_2$. Now, if we have two polynomial equations $P(x, y) = 0$ and $Q(x, y) = 0$, they can regarded as two polynomials in $y$ whose coefficients are polynomials in $x$, and so the set of $x$-coordinates of the points in the common zero-set of $P$ and $Q$ all have the property that at those points, the resultant of the two equations vanishes. The resultant is a polynomial in $x$, and so $y$ has been eliminated from consideration, hence the name “elimination theory.” Now, to proceed with the proof of the Theorem 2.2. If $z$ is an algebraic function of $y$ and $y$ is an algebraic function of $x$, then there are equations $P(x, y) = 0$ and $Q(y, z) = 0$. Eliminating $y$ from the two equations, we see that $z$ is algebraic. If $y_1$ and $y_2$ are algebraic, and $y = y_1 y_2$, then we have the the three equations (the ones satisfied by $y_1$ and $y_2$ and $y = y_1 y_2$. We can eliminate first $y_1$ and then $y_2$, to show that $y$ is algebraic, similarly with $y_1 + y_2$.

Finally, to show that the derivative is algebraic, we differentiate $P(x, y) = 0$ implicitly, to obtain $Q(x, y, y') = 0$. Eliminating $y$ from the two equations we obtain the algebricity of $y'$.

\[\square\]

**Figure 3.** $f(x) = \exp(-x)$. 

![Figure 3. $f(x) = \exp(-x)$.](image-url)
Remark 2.3. For considerably more detail on the subject of elimination, please see 2.

As corollaries of the above Theorem, we see that

Corollary 2.4. If $f(x)$ is an algebraic function, then so is $S(f)$. If $f, g$ are algebraic functions, then if $h(x) = f(x) \exp(g(x))$, it follows that $\mathcal{A}\mathcal{M}(h)(x) = k(x)h(x)$, where $k$ is algebraic.

3. INVERSE PROBLEMS

The first obvious inverse problem is the following:

Which functions $g(x)$ are $\mathcal{A}\mathcal{M}$ transforms?

Construction of the inverse transform is equivalent to the solution of the ODE

$$f'(x) = \pm f(x) \sqrt{\frac{f^2(x)}{g^2(x)} - 1}.$$  

(3.1)

The choice of plus or minus is already troubling, as is the fact that the right hand side is frequently not Lipschitz, so the usual Picard existence theorem for ODE does not apply everywhere, and uniqueness fails spectacularly: the functions $1/\sin x$ and 1 have the same $\mathcal{A}\mathcal{M}$ transform. This example also demonstrates that the initial value problem can develop singularities in finite time. Nevertheless, some things can be said. First:

Lemma 3.1. The transformed function $\mathcal{A}\mathcal{M}(f)$ has a critical point whenever $f$ has a critical point. Furthermore, at such a critical point $x$, $\mathcal{A}\mathcal{M}(f)(x) = f(x)$, and the last equality only holds at a critical point of $f$.

Proof. A simple computation. □

We can thus simplify our life by attempting to solve (3.1) on an interval $[a, b]$ where $g$ is monotone, and in addition, $0 < g(x) < M$. We can pick between the two equations:

$$f'(x) = f(x) \sqrt{\frac{f^2(x)}{g^2(x)} - 1}$$  

(3.2)

$$f'(x) = -f(x) \sqrt{\frac{f^2(x)}{g^2(x)} - 1}$$  

(3.3)

Now, by the Lemma 3.1 we know that we have local existence and uniqueness of solutions, and so the only thing we need check is that singularities do not develop in finite time. To do this we analyze two separate cases:

- Case 1. $g(x)$ is decreasing on $[a, b]$. In this case we take Eq. (3.3). Local existence and uniqueness is assured by the Picard theorem (see [1, Chapter 1]). We pick the initial value $f(a)$ at will (as long as it is bigger than $g(a)$.) Since $f'(x)$ is always negative we know that $f(x) < f(a)$, and since we know that $g(x) > 0$ on $[a, b]$ we know that $f(x) > g(x) > 0$, so it follows that we have a solution on $[a, b]$.

- Case 2. $g(x)$ is increasing. In this case we start at the right endpoint $b$, and use Eq. (3.2), and then construct the solution going right to left. The reasoning in Case 1 goes through verbatim.
What happens if \( g(x) \) has critical points on \([a, b]\)? In that case, it is fairly obvious that we can construct a “weak inverse,” but anything more seems to require much more work. In case the reader is dissatisfied with the nonexplicit nature of our construction, (s)he will perhaps be mollified by the observation that the following problem can be solved explicitly:

Given a function \( r(x) > 1 \) on \([a, b]\), construct a positive \( f(x) \), such that \( f(x)/\mathcal{A}M(f)(x) = r(x) \).

Using Eq. (1.4), it is easy to see that

\[
(3.4) \quad f(x) = \exp \left( \pm C \int_a^x dt \sqrt{r^2(t) - 1} \right)
\]

is the desired solution.

4. Questions

The most natural question is:

**Question 4.1.** Given a smooth \( g(x) \), is there a natural way to construct an \( f(x) \) such that \( g(x) = \mathcal{A}M(f) \)?

Changing categories:

**Question 4.2.** Is \( \mathcal{A}M \) invertible on the set of algebraic functions?

or

**Question 4.3.** Suppose \( f \) satisfies a first order linear differential equation with algebraic coefficients. Is the same always true of \( \mathcal{A}M(f) \)?

References

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