Mean first-passage time of quantum transition processes

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Abstract

In this paper, we consider the problem of mean first-passage time (MFPT) in quantum mechanics; the MFPT is the average time of the transition from a given initial state, passing through some intermediate states, to a given final state for the first time. We apply the method developed in statistical mechanics for calculating the MFPT of random walks to calculate the MFPT of a transition process. As applications, we (1) calculate the MFPT for multiple-state systems, (2) discuss transition processes occurring in an environment background, (3) consider a roundabout transition in a hydrogen atom, and (4) apply the approach to laser theory.

Keywords: mean first-passage time; master equation; lifetime

1 Introduction

In statistical mechanics, for a random-walk process, the mean first-passage time (MFPT) is the average time for the walker to reach some given final site for the first time assuming that the walker started at a given initial site \cite{1,2}.

In quantum mechanics, we can also consider the problem of the MFPT. Like that in statistical mechanics, the MFPT in quantum mechanics is the average time for a quantum system to transit from a given initial state at time $t = 0$ to a given final state for the first time.

Concretely, for a quantum transition process $|i\rangle \rightarrow \text{intermediate states} \rightarrow |f\rangle$, the MFPT represents the average time of the transition from the initial state $|i\rangle$ to the final state $|f\rangle$ for the first time, where $\text{intermediate states}$ denotes the intermediate states that the transition process passing through. Clearly, the MFPT of a transition process contains the information of all states that the transition passes through. Especially, for a two-state system, the MFPT of the transition process $|i\rangle \rightarrow |f\rangle$ is just the lifetime of the state $|i\rangle$.

The difference between the lifetime and the MFPT is as follows. In quantum mechanics, what one considers more frequently is the lifetime of a state. When considering the lifetime of a state, one considers the transition process of a given initial state to all possible final states, while, when considering the MFPT, what one considers is a given transition process whose initial, final, and all intermediate states are assigned.

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The MFPT of a quantum transition can be calculated directly by the master equation. The reason why we use the master equation for the treatment of a quantum transition process is that a transition occurring in a quantum system must need external disturbances. In other words, a system which has no interaction with the environment will not display transitions; or, an isolated system will keep in the initial state forever. It is just the interaction between a quantum system and the environment that causes the quantum transition, e.g., the spontaneous radiation is caused by vacuum fluctuations. That is to say, a quantum system which can display quantum transitions must be an open system. Meanwhile, in fact, no real isolated system exists because of the existence of vacuum fluctuations. Therefore, when we consider a problem of quantum transition, we will face an open system, and, then, the master equation comes in handy.

The method we used in the present paper for the calculation of the MFPT is the master equation method developed in statistical mechanics for the calculation of the MFPT of a random-walk problem [1]. In this paper, we directly apply this method to quantum transition problems. In the following, we first review the method. Then, we calculate the MFPT for multiple-state systems. Concretely, we calculate the MFPT for two-, three-, four-, and five-state systems as examples.

As an application, we consider a transition process occurring in an environment background. In environment backgrounds, the transition will be disturbed. There are many theoretical researches on quantum information and quantum computation [3, 4]. Nevertheless, when realizing quantum information and quantum computation in experiments, one inevitably encounters the problem of decoherence caused by environment backgrounds [5, 6, 7, 8].

Moreover, we also consider a roundabout transition in a hydrogen atom. Concretely, even if a transition is forbidden by selection rules, a roundabout transition can still occur through an indirect way. For example, if the direct transition $|i\rangle \rightarrow |f\rangle$ is forbidden, however, a roundabout transition $|i\rangle \rightarrow |\text{intermediate states}\rangle \rightarrow |f\rangle$ can still occur, so long as the transitions $|i\rangle \rightarrow |\text{intermediate states}\rangle$ and $|\text{intermediate states}\rangle \rightarrow |f\rangle$ are permitted.

Finally, we discuss the application of the MFPT in laser theory.

There are many researches on the calculation and also the application of the MFPT. The MFPT of the Markov process has been considered in Ref. [9], and that of the Non-Markov process has been considered in Refs. [10, 11]. The applications of the MFPT in magnetics have been considered in Ref. [12]. There are also many studies on the MFPT with potentials or noises [13, 14, 15]. The problem of laser with the MFPT has been discussed in Ref. [16]. The MFPT on T-graph has been considered in Refs. [17, 18] and that in random environments has been considered in Refs. [19, 20]. Many other researches consider the application of the MFPT in other fields, e.g., in chemistry [21, 22, 23] and biology [21].

In section 2, we apply the master equation method developed in statistical mechanics to the calculation of the MFPT of quantum transition processes. In section 3, we calculate the MFPT for multiple-state systems. In section 4, we consider the problem of transition processes occurring in environment backgrounds. In section 5, we discuss the problem of roundabout transitions for a hydrogen atom as an example. In section 6, we consider an application in laser theory. In section 7, conclusions and outlook are given.
2 Mean first-passage time of a quantum transition: the master equation

In statistical mechanics, a master equation method is developed for the calculation of the MFPT of random-walk problems [4], and there are many researches about the master equation and its applications in the literature [24, 25, 26]. In this paper we will apply such a method to solve the MFPT in quantum mechanics. For completeness, in this section, we first give a detailed review for the method of the calculation of the MFPT through solving a master equation, following Ref. [1]. It will be seen that this method can be used to solve the problem of MFPT in quantum mechanics without any changes.

Consider a quantum system with $M$ states, $|1\rangle, \cdots, |M\rangle$. The system will transit among these states under the interaction with the environment. In the following, we calculate the MFPT of a transition starting from a given initial state $|n_0\rangle$ at time $t = 0$, going through some intermediate states, to a given final state $|n_p\rangle$.

Let $P(n, t)$ be the probability that the system is at the state $|n\rangle$ at time $t$, and $\Gamma_{mn}$ be the transition probability rate of the transition from $|n\rangle$ to $|m\rangle$. Then, in time interval $\Delta t$, the transition probability of the transition from $|n\rangle$ to $|m\rangle$ is $\Gamma_{mn} \Delta t$. The evolution of such a system obeys the master equation,

$$\frac{\partial}{\partial t} P(n, t) = \sum_{m=1}^{M} \left[ \Gamma_{nm} P(m, t) - \Gamma_{mn} P(n, t) \right].$$

(1)

The first part of the right-hand side of Eq. (1) represents the transition probability from all other states to $|n\rangle$, and the second part represents the transition probability from $|n\rangle$ to other states.

The master equation can be written in a compact form by introducing a transition matrix $W$ with matrix elements $W_{nm} = \Gamma_{nm} - \delta_{nm} \sum_{k=1}^{M} \Gamma_{kn}$ and a column vector $|P(t)\rangle = (P(1, t), P(2, t), \cdots, P(M, t))^T$ denoting the state of the system at time $t$. Here, the vector $|P(t)\rangle$ and the matrix $W$ satisfy $\sum_{n=1}^{M} P(n, t) = 1$, $W_{nm} \geq 0$, and $\sum_{n=1}^{M} W_{nm} = 0$ ($m = 1, 2, \cdots, M$). Then the master equation (1) becomes

$$\frac{d}{dt} |P(t)\rangle = W |P(t)\rangle.$$

(2)

For a system at the initial state $|n_0\rangle$ when $t = 0$, we now calculate the average time for the system transiting to a given final state $|n_p\rangle$ for the first time, i.e., the MFPT.

Let $Q_n(t)$ denote the conditional probability for the system being at state $|n\rangle$ at time $t$ given the condition that this system is at state $|n_0\rangle$ when $t = 0$. $Q_n(t)$ satisfies the same master equation as $P(n, t)$, Eq. (1), with the initial condition $Q_n(0) = \delta_{n_0 n}$. Take the absorbing boundary condition $Q_{n_p}(t) = 0$, then

$$\frac{\partial}{\partial t} Q_n(t) = \sum_{m=1}^{M} \left[ \Gamma_{nm} Q_m(t) - \Gamma_{mn} Q_n(t) \right], \quad n \neq n_p.$$

(3)

Also, we can write this equation in the matrix form

$$\frac{d}{dt} Q(t) = MQ(t),$$

(4)

where the matrix element $Q_n(t) = \langle n | Q(t) | n_0 \rangle$ ($n \neq n_p$) and $M_{nm} = \Gamma_{nm} - \delta_{nm} \sum_{k=1}^{M} \Gamma_{kn}$ ($n \neq n_p$).
and \( m \neq n_p \). Here, \( M \) and \( Q(t) \) are obtained by removing the \( n_p \)-th component from the matrix \( W \) and the vector \( |P(t)\rangle \).

Using the method of the matrix spectral decomposition, we can solve \( Q(t) \):

\[
Q(t) = Q(0) \sum_{i=1}^{M-1} e^{\lambda_i t} |\psi_i\rangle \langle \chi_i|,
\]

where \( \lambda_i \) is the \( i \)-th eigenvalue of the matrix \( M \), \( \langle \chi_i | \rangle \) and \( |\psi_i\rangle \) are the normalized orthogonal left eigenvector and right eigenvector belonging to \( \lambda_i \). These eigenvectors satisfy

\[
\sum_{i=1}^{M-1} |\psi_i\rangle \langle \chi_i| = I.
\]

Then, \( Q_n(t) \) reads

\[
Q_n(t) = \langle n | Q(t) | n_0 \rangle = \sum_{i=1}^{M-1} e^{\lambda_i t} \langle n | \psi_j \rangle \langle \chi_i | n_0 \rangle, \quad n \neq n_p.
\]

Let \( P(t) \) denote the probability that the system has not transited to the state \( |n_p\rangle \) at the time \( t \); in other words, \( P(t) \) is the sum of the probability that the system is at other states, i.e.,

\[
P(t) = \sum_{n=1, n \neq n_p}^{M} Q_n(t).
\]

Introducing \( f_{n_p}(t) dt \) to represent the probability that the system transits to the state \( |n_p\rangle \) during the time interval \( t \rightarrow t + dt \), we have

\[
f_{n_p}(t) = -\frac{d}{dt} P(t).
\]

Therefore, the MFPT from \( |n_0\rangle \) to \( |n_p\rangle \) is

\[
\langle t \rangle = \int_{0}^{\infty} t f_{n_p}(t) dt = \int_{0}^{\infty} P(t) dt.
\]

### 3 Mean first-passage time of multiple-state systems

In this section, we will calculate the mean first-passage time of a transition from the highest state to the lowest state through spontaneous transitions for three-, four-, and five-state systems, respectively.

**Two-state systems:** Obviously, the MFPT of a transition from the higher state to the lower state of a two-state system is just the lifetime of the excited state, i.e.,

\[
\langle t \rangle = \frac{1}{\Gamma_{01}},
\]

where \( \Gamma_{01} \) is the spontaneous transition probability rate of the transition from the excited state \( |1\rangle \) to the ground state \( |0\rangle \).

**Three-state systems:** Consider a three-state system with the ground state \( |0\rangle \), the first excited state \( |1\rangle \), and the second excited state \( |2\rangle \). Let \( \Gamma_{02}, \Gamma_{01} \), and \( \Gamma_{12} \) be the spontaneous transition probability rates among these three states.
Suppose that at time $t = 0$ the system is at state $|2\rangle$. Now, we calculate the MFPT for the transition $|2\rangle \rightarrow |0\rangle$.

The state vector and the transition matrix are then

$$\langle P(t) \rangle = \begin{pmatrix} P(0, t) \\ P(1, t) \\ P(2, t) \end{pmatrix}$$

and

$$W = \begin{pmatrix} 0 & \Gamma_{01} & \Gamma_{02} \\ 0 & -\Gamma_{01} & \Gamma_{12} \\ 0 & 0 & -\Gamma_{02} - \Gamma_{12} \end{pmatrix},$$

respectively. Removing the zeroth component, we obtain the conditional probabilities,

$$Q_1(t) = \langle 1 | Q(t) | 2 \rangle,$$

$$Q_2(t) = \langle 2 | Q(t) | 2 \rangle,$$

and the matrix

$$M = \begin{pmatrix} -\Gamma_{01} & \Gamma_{12} \\ 0 & -\Gamma_{02} - \Gamma_{12} \end{pmatrix}.$$

$Q(t)$ is determined by Eq. (4), i.e.,

$$\frac{d}{dt} Q(t) = MQ(t),$$

with the initial condition

$$\langle n | Q(0) | 2 \rangle = \delta_{n2}.$$

Solving this equation directly, we achieve

$$Q_1(t) = \frac{\Gamma_{12}}{\Gamma_{02} + \Gamma_{12} - \Gamma_{01}} \left[ 1 - e^{-(\Gamma_{02} + \Gamma_{12} - \Gamma_{01})t} \right] e^{-\Gamma_{01}t},$$

$$Q_2(t) = e^{-(\Gamma_{02} + \Gamma_{12})t}.$$

Thus the MFPT of such a transition process is

$$\langle t \rangle = \int_0^\infty [Q_1(t) + Q_2(t)] dt$$

$$= \frac{\Gamma_{01} + \Gamma_{12}}{\Gamma_{01} (\Gamma_{02} + \Gamma_{12})}. \tag{18}$$

**Four-state systems:** By the same approach, we can calculate the MFPT for a four-state system with the ground state $|0\rangle$ and the excited states $|1\rangle$, $|2\rangle$, and $|3\rangle$ with the spontaneous transition rates $\Gamma_{03}$, $\Gamma_{02}$, $\Gamma_{01}$, $\Gamma_{13}$, $\Gamma_{12}$, and $\Gamma_{23}$:

$$\langle t \rangle_{|3\rangle \rightarrow |0\rangle} = \frac{(\Gamma_{12} + \Gamma_{02})(\Gamma_{01} + \Gamma_{13}) + \Gamma_{24}(\Gamma_{01} + \Gamma_{12})}{\Gamma_{01}(\Gamma_{02} + \Gamma_{12})(\Gamma_{03} + \Gamma_{13} + \Gamma_{23})}. \tag{19}$$

**Five-state systems:** Similarly, for a five-state system, the MFPT is

$$\langle t \rangle_{|4\rangle \rightarrow |0\rangle} = \frac{\Gamma_{01} + \Gamma_{12}}{\Gamma_{01}(\Gamma_{02} + \Gamma_{12})(\Gamma_{03} + \Gamma_{13} + \Gamma_{23})(\Gamma_{04} + \Gamma_{14} + \Gamma_{24} + \Gamma_{34})} \left[ \Gamma_{23}\Gamma_{34} + \Gamma_{24}(\Gamma_{03} + \Gamma_{13} + \Gamma_{23}) \right]$$

$$+ \frac{\Gamma_{34}(\Gamma_{01} + \Gamma_{13})(\Gamma_{12} + \Gamma_{02}) + (\Gamma_{01} + \Gamma_{14})(\Gamma_{12} + \Gamma_{02})(\Gamma_{03} + \Gamma_{13} + \Gamma_{23})}{\Gamma_{01}(\Gamma_{02} + \Gamma_{12})(\Gamma_{03} + \Gamma_{13} + \Gamma_{23})(\Gamma_{04} + \Gamma_{14} + \Gamma_{24} + \Gamma_{34})}. \tag{20}$$

Once $\Gamma_{ij} = \Gamma = \text{const } (i < j)$, we can obtain the MFPT of an $n$-state system: $\langle t \rangle_{|n-1\rangle \rightarrow |0\rangle} = 1/\Gamma$. This is just like a two-state system with the spontaneous transition rate $\Gamma$. 

5
4 Transitions with background interferences

In this section, we consider a transition with background interferences.

A transition occurring in an environment background is an important problem in practice. For example, in realistic quantum information and quantum computation processes, the background interference cannot be ignored [5, 6, 27, 28, 29]. In the following, we will discuss the influence of environment backgrounds on a transition process by examples.

First, consider a model that a transition process of a two-state system occurs in an environment background. The two-state system consists of a ground state $|g\rangle$ and an excited state $|e\rangle$. The transition probability rate of the spontaneous transition from $|e\rangle$ to $|g\rangle$ is $\Gamma$. Suppose that the environment consists of $n$ states $|1\rangle$, $|2\rangle$, ..., $|n\rangle$ satisfying $E_e > E_n > E_{n-1} > \cdots > E_1 > E_g$.

The transition probability rate between the background states $|i\rangle$ and $|j\rangle$ is denoted as $\Gamma_{ij}$. For convenience, we denote the transition probability rate of the spontaneous transition from $|e\rangle$ to $|i\rangle$ by $\Gamma_{i,n+1}$ and from $|i\rangle$ to $|g\rangle$ by $\Gamma_{0i}$.

If there is no influence of background interferences, the MFPT from $|e\rangle$ to $|g\rangle$ is just the lifetime of the excited state $|e\rangle$, i.e., $\tau = 1/\Gamma$. If there exist background interferences, however, the MFPT from $|e\rangle$ to $|g\rangle$ will be influenced. In the following, we will consider two simplified models of background interferences.

Case (1): $|e\rangle \rightarrow |n\rangle \rightarrow |n-1\rangle \rightarrow \cdots \rightarrow |1\rangle \rightarrow |g\rangle$.

For a transition process $|e\rangle \rightarrow |n\rangle \rightarrow |n-1\rangle \rightarrow \cdots \rightarrow |1\rangle \rightarrow |g\rangle$, we can directly obtain the MFPT:

$$\langle t \rangle_{|e\rangle \rightarrow |g\rangle} = \frac{\Gamma_{n,n+1}}{\Gamma_{n,n+1} + \Gamma} \left( \frac{1}{\Gamma_{01}} + \frac{1}{\Gamma_{12}} + \cdots + \frac{1}{\Gamma_{n,n+1}} \right)$$

(21)

Consider a simple case: $\Gamma_{i,i+1} = \gamma = \text{const.}$ In such a case, we have

$$\langle t \rangle_{|e\rangle \rightarrow |g\rangle} = \frac{n+1}{\gamma + \Gamma}.$$  

(22)

Obviously, the MFPT is proportional to the total number of the background states.

When $\gamma \gg \Gamma$, i.e., the transition probability rate of the transition involving background states is much greater than the transition probability rate between the two system states, the MFPT is

$$\langle t \rangle_{|e\rangle \rightarrow |g\rangle} \sim \frac{n+1}{\gamma}.$$  

(23)

In this case, when the influence of the background dominates, the MFPT is almost determined by the transition probability among background states and the total number of the background states.

When $\gamma \ll \Gamma$, the transition probability, Eq. (22), becomes

$$\langle t \rangle_{|e\rangle \rightarrow |g\rangle} = \frac{n+1}{\Gamma}.$$  

(24)

In this case, the MFPT is determined by the transition probability between two system states and the total number of the background states.

Case (2): $|e\rangle \rightarrow |\text{background states}\rangle \rightarrow |g\rangle$ with $|\text{background states}\rangle$ denoting $n$ intermediate states.
In this case, the background consists of \( n \) background states. Suppose that the transition probability rates from the excited state \( |e\rangle \) to every background states are the same, denoted by \( \Gamma_1 \), and the transition probability rates from every background states to the ground state \( |g\rangle \) are the same, denoted by \( \Gamma_2 \). Moreover, we also suppose that the transition probability among the background states is zero. We then can obtain the MFPT from \( |e\rangle \) to \( |g\rangle \):

\[
\langle t \rangle_{|e\rangle \to |g\rangle} = \frac{n \Gamma_1 + \Gamma_2}{\Gamma_2 (n \Gamma_1 + \Gamma)}. \tag{25}
\]

Comparing with Eq. (18), we can see that such a transition is just the transition among three states with \( \Gamma_{12} \) replaced by \( n \Gamma_1 \).

When \( \Gamma_i \gg \Gamma \), the MFPT, Eq. (25), becomes

\[
\langle t \rangle_{|e\rangle \to |g\rangle} \sim \frac{1}{n \Gamma_1} + \frac{1}{\Gamma_2}. \tag{26}
\]

The MFPT is almost independent of the transition probability between the system states. Moreover, if \( \Gamma_i \gg \Gamma \) and \( n \gg 1 \), we arrive at

\[
\langle t \rangle_{|e\rangle \to |g\rangle} \sim \frac{1}{\Gamma_2}, \tag{27}
\]

i.e., the MFPT is only determined by the transition probability from the background states to the ground state \( |g\rangle \).

In another case, when \( n \) is not large enough so that \( n \Gamma_1 \ll \Gamma \), \( \langle t \rangle \sim (1 + n \Gamma_1 / \Gamma_2) / \Gamma \). That is different from the intuitive result \( 1 / \Gamma \), but relies on \( n \Gamma_1 / \Gamma_2 \).

### 5 Roundabout transitions: hydrogen atoms

In this section, we consider a roundabout transition process in a hydrogen atom.

Selection rules forbid certain transitions in a hydrogen atom. Nevertheless, what the selection rule forbids is the direct transition, a transition between two states. If a transition between two states \( |i\rangle \) and \( |f\rangle \) is forbidden, however, a roundabout transition from \( |i\rangle \) to \( |f\rangle \) can also occur.

Concretely, even the transition \( |i\rangle \to |f\rangle \) is forbidden, a roundabout transition \( |i\rangle \to |\text{intermediate states}\rangle \to |f\rangle \) can in principle occur through some intermediate states, \( |n\rangle, |n - 1\rangle, \cdots, |1\rangle \), between \( |i\rangle \) and \( |f\rangle \).

In this paper, taking hydrogen atoms as an example, we calculate the MFPT for such a roundabout transition.

For hydrogen atoms, the selection rule of electric dipole transition between states \( |nlm\rangle \) read

\[
\Delta l = \pm 1, \quad \Delta m = 0, \pm 1, \tag{28}
\]

where \( n, l, \) and \( m \) are principal quantum number, angular quantum number, and magnetic quantum number, respectively.

In the following, as an example, we calculate the MFPT of a roundabout transition from \( |300\rangle \) to \( |100\rangle \); the direct transition from \( |300\rangle \) to \( |100\rangle \) is forbidden by the selection rule of electric dipole transition. Nevertheless, there still some roundabout transitions from \( |300\rangle \) to \( |100\rangle \) can occur:

\[
|300\rangle \to |21m\rangle \to |100\rangle, \quad m = 1, 0, -1. \tag{29}
\]
In this case, the corresponding vector $|P(t)\rangle$ and transition matrix $W$ are

$$|P(t)\rangle = \begin{pmatrix} P([100], t) \\ P([21, -1], t) \\ P([210], t) \\ P([211], t) \\ P([300], t) \end{pmatrix},$$  \hspace{1cm} (30)$$

and

$$W = \begin{pmatrix} 0 & \Gamma_{[21,-1] \rightarrow [100]} & \Gamma_{[210] \rightarrow [100]} & \Gamma_{[211] \rightarrow [100]} & 0 \\ 0 & -\Gamma_{[21,-1] \rightarrow [100]} & 0 & 0 & \Gamma_{[300] \rightarrow [21,-1]} \\ 0 & 0 & -\Gamma_{[210] \rightarrow [100]} & 0 & \Gamma_{[300] \rightarrow [210]} \\ 0 & 0 & 0 & -\Gamma_{[211] \rightarrow [100]} & \Gamma_{[300] \rightarrow [211]} \\ 0 & 0 & 0 & 0 & -\Gamma_{[300] \rightarrow [21,-1]} - \Gamma_{[300] \rightarrow [210]} - \Gamma_{[300] \rightarrow [211]} \end{pmatrix}. \hspace{1cm} (31)$$

The master equation given by Eq. (2) is $d\frac{d}{dt}|P(t)\rangle = W|P(t)\rangle$.

In this problem, the initial state is $|300\rangle$ and the final state is $|100\rangle$, so $Q(t)$ and $M$ read

$$Q_{[21,-1]}(t) = (21, -1)|Q(t)|300\rangle,$$
$$Q_{[210]}(t) = (210)|Q(t)|300\rangle,$$
$$Q_{[211]}(t) = (211)|Q(t)|300\rangle,$$
$$Q_{[300]}(t) = (300)|Q(t)|300\rangle$$ \hspace{1cm} (32)

and

$$M = \begin{pmatrix} -\Gamma_{[21,-1] \rightarrow [100]} & 0 & 0 & \Gamma_{[300] \rightarrow [21,-1]} \\ 0 & -\Gamma_{[210] \rightarrow [100]} & 0 & \Gamma_{[300] \rightarrow [210]} \\ 0 & 0 & -\Gamma_{[211] \rightarrow [100]} & \Gamma_{[300] \rightarrow [211]} \\ 0 & 0 & 0 & -\Gamma_{[300] \rightarrow [21,-1]} - \Gamma_{[300] \rightarrow [210]} - \Gamma_{[300] \rightarrow [211]} \end{pmatrix}. \hspace{1cm} (33)$$

Here, $Q(t)$ is determined by $\frac{d}{dt}Q(t) = MQ(t)$ with the initial condition

$$\langle 300 | Q(0) | 300 \rangle = 1 \text{ and } \langle 21m | Q(0) | 300 \rangle = 0.$$ \hspace{1cm} (34)

Then, by solving $Q(t)$, we achieve the MFPT of the roundabout transition (29):

$$\langle t \rangle = \int_0^\infty [Q_{[21,-1]}(t) + Q_{[210]}(t) + Q_{[211]}(t) + Q_{[300]}(t)] \, dt.$$ \hspace{1cm} (35)

Here, $\Gamma_{[300] \rightarrow [21m]} = 2.1046 \times 10^6 s^{-1}$ and $\Gamma_{[21m] \rightarrow [100]} = 6.2649 \times 10^8 s^{-1}$ ($m = 0, \pm 1$) \hspace{1cm} (36), we then have

$$\langle t \rangle |_{[300] \rightarrow [21m] \rightarrow [100]} = 1.5998 \times 10^{-7} \text{ s.}$$ \hspace{1cm} (36)

It should be noted here that the MFPT, $\langle t \rangle$, is different from the lifetime of a state. The lifetime does not associate a certain final state. The MFPT, however, associates a certain initial state and a certain final state. In this case, the lifetime of $|300\rangle$ is $1.5838 \times 10^{-7} \text{ s}$, which is very close to the value of the MFPT, $\langle t \rangle$; while the lifetimes of $|21,-1\rangle$, $|210\rangle$ and $|211\rangle$ are all $1.5962 \times 10^{-9} \text{ s},$
which is so small compared to the lifetime of \(|300\rangle\) and \(\langle t\rangle\). Thus, the transition from \(|300\rangle\) to \(|21, -1\), |210\rangle\) and \(|211\rangle\) is the dominant contribution to the transition \(|300\rangle \rightarrow |21m\rangle \rightarrow |100\rangle\).

Note that in this case, by chance, \(\Gamma_{300 \rightarrow 21 - 1} = \Gamma_{300 \rightarrow 210} = \Gamma_{300 \rightarrow 211}\), so we can also use the result \(25\) to achieve Eq. \(36\) by only regarding the three states \(|21, -1\), |210\rangle\) and \(|211\rangle\) as background states and \(|300\rangle\) and \(|100\rangle\) as two system states.

6 Applications to lasers

The above result of the MFPT of a three-state system can be directly applied to the problem of laser.

For a three-state laser scheme, let \(|0\rangle\), \(|1\rangle\), and \(|2\rangle\) represent the ground state, the upper laser state, and the pumping state, respectively. The laser procedure is realized as follows. Some pumping processes take atoms from \(|0\rangle\) to \(|2\rangle\). The atoms at \(|2\rangle\) drop very rapidly to the upper laser state \(|1\rangle\), and the transition from \(|1\rangle\) to \(|0\rangle\) produces a photon we needed. As long as the pumping process is effective enough, the lifetime of state \(|2\rangle\) is short enough, and the lifetime of state \(|1\rangle\) is long enough, the number of atoms in \(|1\rangle\) will exceed the number of atoms in \(|0\rangle\), i.e., the population inversion will be achieved. The transition between \(|1\rangle\) to \(|0\rangle\) will yield a laser.

According to the above discussion, if we want to obtain a stable output of laser, a necessary condition is that the probability of an atom pumping from \(|0\rangle\) to \(|2\rangle\) must exceed the probability of an atom transiting from \(|2\rangle\) to \(|0\rangle\).

As the above discussion, the MFPT of the transition \(|2\rangle\) to \(|0\rangle\) can describe the probability of the transition from \(|2\rangle\) to \(|0\rangle\) exactly. If we introduce the pumping rate \(P\) to describe the probability of an atom pumping from \(|0\rangle\) to \(|2\rangle\), then the above necessary condition can be expressed as

\[ P \geq \frac{1}{\langle t \rangle}, \]  

or

\[ P \geq \frac{\Gamma_{01} \Gamma_{02} + \Gamma_{12}}{\Gamma_{01} + \Gamma_{12}}. \]  

As a comparison, in the usual treatment of laser, the transition from \(|2\rangle\) to \(|1\rangle\) is assumed as instantaneous and the transition from \(|2\rangle\) to \(|0\rangle\) is neglected, which means

\[ \Gamma_{12} \rightarrow \infty \text{ and } \Gamma_{02} \rightarrow 0. \]  

Under these assumptions, the usual result of the pumping rate satisfies \[31\]

\[ P \geq \Gamma_{01}. \]  

From Eq. \(38\), we can see that our result based on the MFPT will reduce to the usual result \(40\) under the condition Eq. \(39\); Eq. \(38\) is a more accurate result. In fact, the power of a laser is usually expressed as \[31, 32\]

\[ P_{\text{laser}} = A (P - \Gamma_{01}), \]  

where \(A\) is a parameter determined by the character of the specific laser medium and laser facility. However, a more accurate expression of the power is expressed by the MFPT:

\[ P_{\text{laser}} = A \left( P - \frac{1}{\langle t \rangle} \right) = A \left( P - \Gamma_{01} \frac{\Gamma_{02} + \Gamma_{12}}{\Gamma_{12} + \Gamma_{01}} \right). \]
7 Conclusions and outlook

In this paper, we discuss the problem of the MFPT in quantum mechanics. In the problem of the MFPT, we concentrate on a given transition process: $|i\rangle \rightarrow |\text{intermediate states}\rangle \rightarrow |f\rangle$.

We apply the method developed in statistical mechanics for calculating the MFPT of the problem of random walks to calculate the MFPT of transition processes in quantum mechanics. Such a method is based on the master equation. Furthermore, we calculate the MFPT in multiple-state systems. Especially, we consider a transition process occurring in an environment background by examples. Taking a hydrogen atom as an example, we calculate a roundabout transition process $|300\rangle \rightarrow |100\rangle$. Finally, we discuss the application to laser theory.

When realizing a quantum information or a quantum computation process in experiments, one has to face the influence of environments. The environmental background interferences will cause the problem of quantum decoherence\cite{5,6,27,28,29}. That is to say, in a realistic quantum information or a quantum computation process, the environmental background interferences cannot be ignored. Although many researches are devoted to suppress the influence of environment\cite{7,8,33,34,35}, decoherence is still one of the most important obstacles in quantum information processes. Based on the result of the present paper, we can view the background interference as some intermediate states and consider the quantum decoherence by analyzing the MFPT. We will discuss the application to the problem of quantum decoherence elsewhere.

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