KASHIWARA-VERGNE AND DIHEDRAL BIGRADED LIE ALGEBRAS IN MOULD THEORY

HIDEKAZU FURUSHO AND NAO KOMIYAMA

ABSTRACT. We introduce the Kashiwara-Vergne bigraded Lie algebra associated with a finite abelian group and give its mould theoretic reformulation. By using the mould theory, we show that it includes Goncharov’s dihedral Lie algebra, which generalizes the result of Raphael and Schneps.

Contents

0. Introduction 1
1. Preparation on mould theory 3
1.1. Moulds and alternality 3
1.2. Flexions and ari-bracket 8
1.3. Swap and bialternality 10
1.4. Push-invariance and pus-neutrality 12
2. Kashiwara-Vergne Lie algebra 21
2.1. $\Gamma$-variant of the KV condition 21
2.2. Kashiwara-Vergne bigraded Lie algebra 29
3. Dihedral Lie algebra 32
3.1. Dihedral bigraded Lie algebra 32
3.2. Mould theoretic reformulation 34
3.3. Embedding 36
Appendix A. On the ari-bracket of ARI($\Gamma$) 41
A.1. Proof of Proposition 1.14 41
A.2. Proof of Proposition 1.15 50
A.3. Proof of Proposition 1.24 55
Appendix B. Multiple polylogarithms at roots of unity 55
References 57

0. Introduction

The dihedral Lie algebra $D(\Gamma)$ is the bigraded Lie algebra introduced in [G01a] which is associated with a finite abelian group $\Gamma$. It reflects the double shuffle and distribution relations among multiple polylogarithms evaluated at roots of unity. Its relation with a certain bigraded variant of motivic Lie algebra is discussed in loc. cit.
Kashiwara-Vergne Lie algebra $\mathfrak{t}nv_*$ is the filtered graded Lie algebra introduced in [AT] and [AET]. It acts on the set of solutions of ‘a formal version’ of Kashiwara-Vergne conjecture. Related to conjectures on mixed Tate motives, it is expected to be isomorphic to the motivic Lie algebra (cf. [F]).

A bigraded variant $l\mathfrak{t}nv_{**}$ of $\mathfrak{t}nv_*$ is introduced in [RS] where they give its interpretation in terms of Ecalle’s mould theory ([Ec81, Ec03, Ec11]). The results in [M, RS] give an inclusion of bigraded Lie algebras

\begin{equation}
D(\{e\})_* \hookrightarrow \mathfrak{t}nv_*.
\end{equation}

Our objective of this paper is to extend it to any $\Gamma$ by exploiting Ecalle’s mould theory with self-contained proofs. Our results are exhibited as follows:

(i) In Definition 2.1 we introduce the filtered graded $\mathbb{Q}$-linear space $\mathfrak{t}nv(\Gamma)_*$ which generalizes $\mathfrak{t}nv_*$. In Theorem 2.15 we show that $\mathfrak{t}nv(\Gamma)_*$ is identified with the $\mathbb{Q}$-linear space of finite polynomial-valued alternal moulds satisfying Ecalle’s senary relation (2.14) and whose swap’s (1.8) are pus-neutral (1.9), that is, there is an isomorphism of $\mathbb{Q}$-linear spaces

$$\mathfrak{t}nv(\Gamma)_* \simeq \text{ARI}(\Gamma)_{\text{sena/pusnu}} \cap \text{ARI}(\Gamma)_{\text{fin,pol}}.$$  

(ii) In Definition 2.17 we introduce a bigraded $\mathbb{Q}$-linear space $l\mathfrak{t}nv(\Gamma)_{**}$ which is defined by the ‘leading terms’ of the defining equations of $\mathfrak{t}nv(\Gamma)_*$. We also consider its subspace $l\mathfrak{t}nv(\Gamma)_{**}$ by imposing the distribution relation in Definition 2.25. Both of them recover $l\mathfrak{t}nv_{**}$ when $\Gamma = \{e\}$. It is shown that they form Lie algebras in Theorem 2.23 and Corollary 2.26. An inclusion of bigraded $\mathbb{Q}$-linear spaces

$$\text{gr}_D \mathfrak{t}nv(\Gamma)_* \hookrightarrow l\mathfrak{t}nv(\Gamma)_{**}$$

is presented in (2.31), where the first term means the associated bigraded of the filtered graded linear space $\mathfrak{t}nv(\Gamma)_*$.

(iii) In Theorem 2.22 we show that $l\mathfrak{t}nv(\Gamma)_{**}$ is identified with the Lie algebra (cf. Theorem 1.32) of finite polynomial-valued alternal moulds which are push-invariant (1.5) and whose swap’s are pus-neutral (1.9), that is, there is an isomorphism of Lie algebras

$$l\mathfrak{t}nv(\Gamma)_{**} \simeq \text{ARI}(\Gamma)_{\text{push/pusnu}} \cap \text{ARI}(\Gamma)_{\text{fin,pol}}.$$  

(iv) In §3 we consider Goncharov’s dihedral bigraded Lie algebra $D(\Gamma)_{**}$ and its related Lie algebra $D(\Gamma)_*$ with the dihedral symmetry which contains $D(\Gamma)_{**}$. It is explained in Theorem 3.3 that its depth $>1$-part of $D(\Gamma)_{**}$ coincides with the depth $>1$-part of the Lie algebra of finite polynomial-valued part of the set of moulds $\text{ARI}(\Gamma)_{\text{al/alu}}$ (cf. Definition 1.22), namely

$$\text{Fil}_2^D D(\Gamma)_{**} \simeq \text{Fil}_2^D \text{ARI}(\Gamma)_{\text{fin,pol}}.$$  

In Theorem 3.13 we show that there is an inclusion of graded Lie algebras

$$\text{Fil}_2^D \text{ARI}(\Gamma)_{\text{alu}} \hookrightarrow \text{ARI}(\Gamma)_{\text{push/pusnu}}.$$  

In Corollary 3.14 by taking an intersection with $\text{ARI}(\Gamma)_{\text{fin,pol}}$ we obtain the inclusion of bigraded Lie algebras from the depth $>1$ part $\text{Fil}_2^D D(\Gamma)_{**}$ of $D(\Gamma)_{**}$ to $l\mathfrak{t}nv(\Gamma)_{**}$

$$\text{Fil}_2^D D(\Gamma)_{**} \hookrightarrow l\mathfrak{t}nv(\Gamma)_{**}.$$
which extends \([0.1]\). By imposing the distribution relation there the inclusion \(\text{Fil}^1_k D(\Gamma)_{\bullet\bullet} \hookrightarrow \text{tw}(\Gamma)_{\bullet\bullet}\) is similarly obtained in Corollary 3.14.

In Appendix A we give self-contained proofs of several fundamental properties on the \(\text{ari-bracket}\) of the Lie algebra \(\text{ARI}(\Gamma)\) of moulds associated with \(\Gamma\). In Appendix B we discuss moulds arising from the multiple poly logarithms evaluated at roots of unity.

Acknowledgements. We thank for L. Schneps who gave comments on the first version of the paper and informing us [RS]. We are grateful to the referee whose comments helped us to improve the paper in a better form. H.F. and N.K. have been supported by grants JSPS KAKENHI JP18H01110 and JP18J14774 respectively.

1. Preparation on mould theory

We prepare several techniques of moulds which will be employed in our later sections. The notion of moulds, the alternality, flexions and the \(\text{ari-bracket}\) associated with a finite abelian group \(\Gamma\) are explained in §1.1 and §1.2. In §1.3 we explain that the set \(\text{ARI}(\Gamma)\) of bialternal moulds forms a Lie algebra under the \(\text{ari-bracket}\) (whose self-contained proof is given in Appendix A). In §1.4 we introduce and discuss moulds associated with \(\Gamma\) which extends (0.1). By imposing the distribution relation there the inclusion \(\text{Fil}^1_k D(\Gamma)_{\bullet\bullet} \hookrightarrow \text{tw}(\Gamma)_{\bullet\bullet}\) is similarly obtained in Corollary 3.14.

In Appendix A we give self-contained proofs of several fundamental properties on the moulds and alternality.

§4.1

1.1. Moulds and alternality. We introduce and discuss moulds associated with a finite abelian group \(\Gamma\).

The notion of moulds was invented by Ecalle (cf. [Ec81 Tome I, pp.12-13]). For our convenience we employ the following formulation influenced by [Sch15] which is different from the one employed in [C Définition 1 or Définition II.1] and [Sau, §4.1].

Let \(\Gamma\) be a finite abelian group. We set \(\mathcal{F} := \bigcup_{m \geq 1} \mathbb{Q}(x_1,\ldots,x_m)\).

Definition 1.1. A mould on \(\mathbb{Z}_{\geq 0}\) with values in \(\mathcal{F}\) and indexed by \(\Gamma\) in a lower layer is a collection

\[
M = \left( M^m \left( x_{\sigma_1},\ldots,x_{\sigma_m} \right) \right)_{m \in \mathbb{Z}_{\geq 0}, \sigma_i \in \Gamma}
\]

with \(M^m \left( x_{\sigma_1},\ldots,x_{\sigma_m} \right) \in \mathcal{F}(x_1,\ldots,x_m)\) for \(m \geq 0\).\footnote{In this case, we have \(M^0(\emptyset) \in \mathbb{Q}\) for \(m = 0\).} We denote the set of all such moulds with values in \(\mathcal{F}\) by \(\mathcal{M}(\mathcal{F};\Gamma)\). The set \(\mathcal{M}(\mathcal{F};\Gamma)\) forms a \(\mathbb{Q}\)-linear space by

\[
A + B := \left( A^m \left( x_{\sigma_1},\ldots,x_{\sigma_m} \right) + B^m \left( x_{\sigma_1},\ldots,x_{\sigma_m} \right) \right)_{m \in \mathbb{Z}_{\geq 0}, \sigma_i \in \Gamma},
\]

\[
cA := \left( cA^m \left( x_{\sigma_1},\ldots,x_{\sigma_m} \right) \right)_{m \in \mathbb{Z}_{\geq 0}, \sigma_i \in \Gamma},
\]

for \(A,B \in \mathcal{M}(\mathcal{F};\Gamma)\) and \(c \in \mathbb{Q}\), namely the addition and the scalar are taken componentwise. We define a product on \(\mathcal{M}(\mathcal{F};\Gamma)\) by

\[
(A \times B)^m \left( x_{\sigma_1},\ldots,x_{\sigma_m} \right) := \sum_{i=0}^m A^i \left( x_{\sigma_1},\ldots,x_i \right) B^{m-i} \left( x_{i+1},\ldots,x_{\sigma_m} \right),
\]

for \(A,B \in \mathcal{M}(\mathcal{F};\Gamma)\) and for \(m \geq 0\) and for \((\sigma_1,\ldots,\sigma_m) \in \Gamma^\oplus m\). Then the pair \((\mathcal{M}(\mathcal{F};\Gamma), \times)\) is a non-commutative, associative, unital \(\mathbb{Q}\)-algebra. Here, the unit \(I \in \mathcal{M}(\mathcal{F};\Gamma)\) is given by \(I := (1,0,0,\ldots)\).
By the regular action of $\Gamma$ on $\mathbb{Q}[\Gamma]$, $\mathcal{M}(\mathcal{F}; \Gamma)$ admits the action of $\Gamma$. It is described by
$$(\gamma M)^m \left( \frac{x_1, \ldots, x_m}{\sigma_1, \ldots, \sigma_m} \right) = M^m \left( \frac{x_1, \ldots, x_m}{\gamma^{-1} \sigma_1, \ldots, \gamma^{-1} \sigma_m} \right)$$
for $\gamma \in \Gamma$.

The set $\mathcal{M}(\mathcal{F}; \Gamma)$ is encoded with the depth filtration $\{\text{Fil}^m_0 \mathcal{M}(\mathcal{F}; \Gamma)\}_{m \geq 0}$ where $\text{Fil}^m_0 \mathcal{M}(\mathcal{F}; \Gamma)$ is the collection of moulds with $M^r \left( \frac{x_1, \ldots, x_r}{\sigma_1, \ldots, \sigma_r} \right) = 0$ for $r < m$. It is clear that the algebra structure of $\mathcal{M}(\mathcal{F}; \Gamma)$ is compatible with the depth filtration. Put
$$\text{ARI}(\Gamma) := \{ M \in \mathcal{M}(\mathcal{F}; \Gamma) \mid M^0(\emptyset) = 0 \}.$$ It is a filtered (non-unital) subalgebra.

**Definition 1.2.** A mould $M \in \mathcal{M}(\mathcal{F}; \Gamma)$ is called finite when $M^m \left( \frac{x_1, \ldots, x_m}{\sigma_1, \ldots, \sigma_m} \right) = 0$ except for finitely many $m$. It is called polynomial-valued when $M^m \left( \frac{x_1, \ldots, x_m}{\sigma_1, \ldots, \sigma_m} \right) \in \mathbb{Q}[x_1, \ldots, x_m]$ for all $\left( \frac{\sigma_1, \ldots, \sigma_m}{\sigma_1, \ldots, \sigma_m} \right) \in \Gamma^m$ and $m$. We denote $\mathcal{M}(\mathcal{F}; \Gamma)^{\text{fin.pol}}$ (resp. $\text{ARI}(\Gamma)^{\text{fin.pol}}$) to be the subset of all finite polynomial-valued moulds in $\mathcal{M}(\mathcal{F}; \Gamma)$ (resp. $\text{ARI}(\Gamma)$).

We prepare the following algebraic formulation which is useful to present the notion of the alternality of mould: Put $X := \{ \left( \frac{x_i}{\sigma_i} \right) \}_{i \in \mathbb{N}, \sigma \in \Gamma}$. Let $X^\bullet_Z$ be the set such that
$$X^\bullet_Z := \{ \left( \frac{u}{\sigma} \right) \mid u = a_1 x_1 + \cdots + a_k x_k, \ k \in \mathbb{N}, \ a_j \in \mathbb{Z}, \ \sigma \in \Gamma \},$$
and let $X^\bullet_Z^+$ be the non-commutative free monoid generated by all elements of $X^\bullet_Z$ with the empty word $\emptyset$ as the unit. Occasionally we denote each element $\omega = u_1 \cdots u_m \in X^\bullet_Z^+$ with $u_1, \ldots, u_m \in X^\bullet_Z$ by $\omega = (u_1, \ldots, u_m)$ as a sequence. The length of $\omega = u_1 \cdots u_m$ is defined to be $l(\omega) := m$.

For our simplicity we occasionally denote $M \in \mathcal{M}(\mathcal{F}; \Gamma)$ by
$$M = (M^m(x_m))_{m \in \mathbb{Z}_{\geq 0}} \quad \text{or} \quad M = (M(x_m))_{m \in \mathbb{Z}_{\geq 0}},$$
where $x_0 := \emptyset$ and $x_m := \left( \frac{x_1, \ldots, x_m}{\sigma_1, \ldots, \sigma_m} \right)$ for $m \geq 1$. Under the notations, the product of $A, B \in \text{ARI}(\Gamma)$ is expressed as
$$A \times B = \left( \sum_{x_m = \alpha \beta} A^{(\alpha)}(\alpha) B^{(\beta)}(\beta) \right)_{m \in \mathbb{Z}_{\geq 0}},$$
where $\alpha$ and $\beta$ run over $X^\bullet_Z^+$.

We set $A^\mathcal{X} := \mathbb{Q}(X^\bullet_Z)$ to be the non-commutative polynomial algebra generated by $X^\bullet_Z$ (i.e. $A^\mathcal{X}$ is the $\mathbb{Q}$-linear space generated by $X^\bullet_Z$).

We equip $A^\mathcal{X}$ with a product $\mathcal{III} : A^\mathcal{X} \times A^\mathcal{X} \to A^\mathcal{X}$ (called the shuffle product) which is linearly defined by $\emptyset \mathcal{III} \omega := \omega \emptyset \mathcal{III} \emptyset := \omega$ and
$$u \omega \mathcal{III} v \eta := u(\omega \mathcal{III} v \eta) + v(u \omega \mathcal{III} \eta),$$
for $u, v \in X^\bullet_Z$ and $\omega, \eta \in X^\bullet_Z$. Then the pair $(A^\mathcal{X}, \mathcal{III})$ forms a commutative, associative, unital $\mathbb{Q}$-algebra.

We define the family $\{ \text{Sh} \left( \frac{\omega}{\alpha} \right) \}_{\omega, \eta, \alpha \in X^\bullet_Z}$ in $\mathbb{Z}$ by
$$\omega \mathcal{III} \eta = \sum_{\alpha \in X^\bullet_Z} \text{Sh} \left( \frac{\omega \eta}{\alpha} \right) \alpha.$$

---

2We should beware of the inequality $\left( \frac{x^1 + x^2}{x} \right) \neq \left( \frac{x^1}{x} \right) + \left( \frac{x^2}{x} \right)$ and $(\emptyset) \neq 0$. 

---

**References**

4 HIDEKAZU FURUSHO AND NAO KOMIYAMA
Particularly for \( p, q \in \mathbb{N} \) and \( u_1, \ldots, u_{p+q} \in X \), we rewrite the shuffle product by

\[
(u_1, \ldots, u_p) \shuffle (u_{p+1}, \ldots, u_{p+q}) = \sum_{\sigma \in \Pi_{p,q}} (u_{\sigma(1)}, \ldots, u_{\sigma(p)}, u_{\sigma(p+1)}, \ldots, u_{\sigma(p+q)}).
\]

Here the set \( \Pi_{p,q} \) is defined by

\[
\{ \sigma \in S_{p+q} \mid \sigma^{-1}(1) < \cdots < \sigma^{-1}(p), \sigma^{-1}(p+1) < \cdots < \sigma^{-1}(p+q) \},
\]

where \( S_{p+q} \) is the symmetry group with degree \( p+q \).

**Definition 1.3.** A mould \( M \in \ARI(\Gamma) \) is called **alternal** (cf. [Ec81, I–p.118]) if

\[
\sum_{\alpha \in X \cdot Z} \text{Sh}\left( \left(\begin{array}{c} x_1, \ldots, x_p \\ \sigma_1, \ldots, \sigma_p \end{array}\right); \left(\begin{array}{c} x_{p+1}, \ldots, x_{p+q} \\ \sigma_{p+1}, \ldots, \sigma_{p+q} \end{array}\right) \right) M^{p+q}(\alpha) = 0,
\]

for all \( p, q \geq 1 \). The \( \mathbb{Q} \)-linear space \( \ARI(\Gamma)_{al} \) is defined to be the subset of moulds \( M \in \ARI(\Gamma) \) which are alternal (cf. [Ec03, Ec11]).

We encode it with the induced depth filtrations. We exhibit a couple of examples of alternal moulds for \( \Gamma = \{ e \} \) below.

**Example 1.4.** (a). For \( f(x) \in \mathbb{Q}(x) \), we define \( M_f \in \ARI(\Gamma) \) by

\[
M_f(x_m) := \begin{cases} f(x_1) & (m = 1), \\ 0 & (m \neq 1), \end{cases}
\]

that is,

\[
M_f = (0, f(x_1), 0, 0, \ldots).
\]

(b). We define \( A \in \ARI(\Gamma) \) by

\[
A(x_m) := \begin{cases} 0 & (m = 0, 1), \\ \frac{1}{x_2 - x_1} \cdots \frac{1}{x_m - x_{m-1}} & (m \geq 2), \end{cases}
\]

that is,

\[
A = \left( 0, 0, \frac{1}{x_2 - x_1}, \frac{1}{(x_2 - x_1)(x_3 - x_2)}, \frac{1}{(x_2 - x_1)(x_3 - x_2)(x_4 - x_3)}, \ldots \right).
\]

**Proof of their alternalities:** (a). By definition, we have \( M_f(\emptyset) = 0 \). Put \( \omega, \eta \in X^\bullet \) with \( l(\omega), l(\eta) \geq 1 \). Note that for \( \alpha \in X^\bullet \) with \( \text{Sh}(\omega; \eta)_{\alpha} \neq 0 \), we have \( l(\alpha) = l(\omega) + l(\eta) \geq 2 \) and we get

\[
M_f(\alpha) = 0.
\]

Therefore, we obtain

\[
\sum_{\alpha \in X^\bullet} \text{Sh}(\omega; \eta)_{\alpha} M_f(\alpha) = \sum_{\alpha \in X^\bullet, \text{Sh}(\omega; \eta)_{\alpha} \neq 0} \text{Sh}(\omega; \eta)_{\alpha} M_f(\alpha) = 0.
\]

Hence, \( M_f \) is an alternal mould.

(b). The proof is given in [C, Lemme II.5], but we prove the alternality of \( A \) by using the following lemma:

---

3This mould is presented in [Ec81, p. 124] and [C, (II.64)].
Lemma 1.5. For \( r \geq 2 \) and \( s \geq 0 \), we have
\[
\sum_{\alpha \in \mathcal{X}_s^+} \text{Sh}\left( \frac{(\omega_1, \ldots, \omega_{r-1}); (\omega_{r+1}, \ldots, \omega_{r+s})}{\alpha} \right) A(\alpha, \omega_r) = (-1)^s A(\omega_1, \ldots, \omega_{r-1}, \omega_r, \omega_{r+s}, \ldots, \omega_{r+1}),
\]
where \( \omega_i := \langle r_i \rangle \) for \( i \in \mathbb{N} \).

Proof. When \( s = 0 \), the claim is clear. Assume \( s \geq 1 \). We prove by induction on \( r + s \). By the definition of the mould \( A \), we easily see the case of \((r, s) = (2, 1)\). For \( r \geq 2 \) and \( s \geq 1 \), we have
\[
\sum_{\alpha \in \mathcal{X}_s^+} \text{Sh}\left( \frac{(\omega_1, \ldots, \omega_{r-1}); (\omega_{r+1}, \ldots, \omega_{r+s})}{\alpha} \right) A(\alpha, \omega_r)
= \sum_{\alpha \in \mathcal{X}_s^+} \left\{ \text{Sh}\left( \frac{(\omega_1, \ldots, \omega_{r-2}); (\omega_{r+1}, \ldots, \omega_{r+s})}{\alpha} \right) A(\alpha, \omega_{r-1}, \omega_r)
+ \text{Sh}\left( \frac{(\omega_1, \ldots, \omega_{r-1}); (\omega_{r+1}, \ldots, \omega_{r+s-1})}{\alpha} \right) A(\alpha, \omega_{r+s}, \omega_r) \right\}.
\]
Because \( A(\omega_1, \ldots, \omega_{r-1}, \omega_r) = \frac{1}{x_r - x_{r-1}} A(\omega_1, \ldots, \omega_{r-1}) \) for \( r \geq 3 \), we get
\[
= \frac{1}{x_r - x_{r-1}} \sum_{\alpha \in \mathcal{X}_s^+} \text{Sh}\left( \frac{(\omega_1, \ldots, \omega_{r-2}); (\omega_{r+1}, \ldots, \omega_{r+s})}{\alpha} \right) A(\alpha, \omega_{r-1})
+ \frac{1}{x_r - x_{r+s}} \sum_{\alpha \in \mathcal{X}_s^+} \text{Sh}\left( \frac{(\omega_1, \ldots, \omega_{r-1}); (\omega_{r+1}, \ldots, \omega_{r+s-1})}{\alpha} \right) A(\alpha, \omega_{r+s}).
\]
By our induction hypothesis, we calculate
\[
= (-1)^s \frac{1}{x_r - x_{r-1}} A(\omega_1, \ldots, \omega_{r-2}, \omega_{r-1}, \omega_{r+s}, \ldots, \omega_{r+1})
+ (-1)^{s-1} \frac{1}{x_r - x_{r+s}} A(\omega_1, \ldots, \omega_{r-1}, \omega_{r+s}, \omega_{r+s-1}, \ldots, \omega_{r+1})
\]
\[
= (-1)^s \left\{ \frac{1}{x_r - x_{r-1}} - \frac{1}{x_r - x_{r+s}} \right\} A(\omega_1, \ldots, \omega_{r-2}, \omega_{r-1}, \omega_{r+s}, \ldots, \omega_{r+1})
\]
\[
= (-1)^s \frac{x_{r-1} - x_{r+s}}{(x_r - x_{r-1})(x_r - x_{r+s})} A(\omega_1, \ldots, \omega_{r-2}, \omega_{r-1}, \omega_{r+s}, \ldots, \omega_{r+1})
\]
\[
= (-1)^s A(\omega_1, \ldots, \omega_{r-1}) \frac{1}{x_{r-1} - x_{r+s}} \frac{1}{x_{r+s} - x_{r-1}} \frac{1}{x_{r-1} x_{r+s-1} - x_{r+s}} \cdots \frac{1}{x_{r+1} - x_{r+2}}
\]
\[
= (-1)^s A(\omega_1, \ldots, \omega_{r-1}, \omega_r, \omega_{r+s}, \ldots, \omega_{r+1}).
\]
Hence, we obtain the claim. \( \square \)

By using the above lemma, we prove the alternality of \( A \). By the definition, we have \( A(\emptyset) = 0 \), so it is sufficient to prove
\[
\sum_{\alpha \in \mathcal{X}_s^+} \text{Sh}\left( \frac{(\omega_1, \ldots, \omega_r); (\omega_{r+1}, \ldots, \omega_{r+s})}{\alpha} \right) A(\alpha) = 0,
\]
for \( r, s \geq 1 \). Assume \( r \geq s \) without loss of generality. When \( r = s = 1 \), the left hand side of (1.4) is equal to
\[
A(\omega_1, \omega_2) + A(\omega_2, \omega_1) = \frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_2} = 0.
\]

When \( r \geq 2 \) and \( s \geq 1 \), the left hand side of (1.4) is equal to
\[
\sum_{\alpha \in X^*} \text{Sh} \left( (\omega_1, \ldots, \omega_{r-1}); (\omega_{r+1}, \ldots, \omega_{r+s}) \right) A(\alpha, \omega_r)
+ \sum_{\alpha \in X^*} \text{Sh} \left( (\omega_1, \ldots, \omega_r); (\omega_{r+1}, \ldots, \omega_{r+s-1}) \right) A(\alpha, \omega_{r+s}).
\]

By using Lemma 1.5 we get
\[
= (-1)^s A(\omega_1, \ldots, \omega_{r-1}, \omega_r, \omega_{r+s}, \ldots, \omega_{r+1}) + (-1)^{s-1} A(\omega_1, \ldots, \omega_{r}, \omega_{r+s}, \omega_{r+s-1}, \ldots, \omega_{r+1})
= 0.
\]

Hence, the mould \( A \) is alternal. \( \square \)

**Remark 1.6.** Assume that \( u_1, \ldots, u_m \in F \) are algebraically independent over \( Q \). For \( M \in \text{ARI}(\Gamma) \) we denote \( M^m(\bar{x}_1, \ldots, \bar{x}_m) \) to be the image of \( M^m(x_1, \ldots, x_m) \) under the field embedding \( Q(x_1, \ldots, x_m) \hookrightarrow F \) sending \( x_i \mapsto u_i \).

For our later use, we prepare more notations:

**Notation 1.7** ([Ec11 §2.1]). For any mould \( M = (M^m(u_1, \ldots, u_m))_m \in \text{ARI}(\Gamma) \), we define
\[
\text{mantar}(M)^m(\bar{u}_1, \ldots, \bar{u}_m) = (-1)^{m-1} M^m(\bar{u}_1, \ldots, \bar{u}_m),
\]
\[
\text{push}(M)^m(\bar{u}_1, \ldots, \bar{u}_m) = M^m(-u_1, \ldots, -u_m, \bar{u}_1, \ldots, \bar{u}_m),
\]
\[
\text{neg}(M)^m(\bar{u}_1, \ldots, \bar{u}_m) = M^m((-u_1, \ldots, -u_m), \bar{u}_1, \ldots, \bar{u}_m),
\]
\[
\text{teru}(M)^m(\bar{u}_1, \ldots, \bar{u}_m) = M^m(\bar{u}_1, \ldots, \bar{u}_m)
+ \frac{1}{\bar{u}_m} \left\{ M^{m-1}(u_1, \ldots, u_{m-2}, u_{m-1} + u_m) - M^{m-1}(u_1, \ldots, u_{m-1}) \right\}.
\]

Note that they are all \( Q \)-linear endomorphisms on \( \text{ARI}(\Gamma) \). We remark that \( \text{neg} \circ \text{neg} = \text{id} \) and \( \text{mantar} \circ \text{mantar} = \text{id} \).

**Definition 1.8.** We call a mould \( M \in \text{ARI}(\Gamma) \) push-invariant when we have
\[
\text{push}(M) = M.
\]

We define \( \text{ARI}(\Gamma)_{\text{push}} \) ([Ec11 §2.5]) to be the set of moulds \( M \) in \( \text{ARI}(\Gamma) \) which is push-invariant (1.5).

**Example 1.9.** By direct computations we observe that the mould \( P \in \text{ARI}(\{e\}) \) defined by
\[
P(x_m) := \begin{cases} 0 & (m = 0, 1), \\ \frac{1}{x_1} + \ldots + \frac{1}{x_m} - \frac{1}{x_1 + \ldots + x_m} & (m \geq 2) \end{cases}
\]
is push-invariant, but is not alternal.
1.2. Flexions and ari-bracket. We explain an ari-bracket \( \text{ari}_u \) on \( \text{ARI}(\Gamma) \) by using flexions.

The notion of flexions is introduced by Ecalle in [11] §2.1 for bimoulds (cf. [15] §2.2). Here we consider those for moulds in \( \text{ARI}(\Gamma) \).

**Definition 1.10.** The flexions are the four binary operators \( .[*,*], .[*,*] : X^*_Z \times X^*_Z \rightarrow X^*_Z \) which are defined by

\[
\begin{align*}
\lceil \alpha \rceil := (b_1 + \cdots + b_n + a_1, \sigma_1, \ldots, \sigma_m), \\
\lfloor \alpha \rfloor := (a_1, \ldots, a_m - 1, a_m + b_1 + \cdots + b_n, \sigma_1, \ldots, \sigma_m), \\
\lceil \alpha \rceil := (\sigma_1, \ldots, \sigma_m, a_m), \\
\lfloor \alpha \rfloor := (a_1, \ldots, a_m, \sigma_1, \ldots, \sigma_m - 1), \\
\lceil \gamma \rceil := \gamma, \\
\lfloor \gamma \rfloor := \gamma,
\end{align*}
\]

for \( \alpha = (a_1, \ldots, a_m) \), \( \beta = (b_1, \ldots, b_n) \in X^*_Z (m, n \geq 1) \) and \( \gamma \in X^*_Z \).

Note that we have \( l(\lceil \alpha \rceil) = l(\lfloor \alpha \rfloor) = l(\lceil \alpha \rceil) = l(\alpha) \) and \( l(\alpha, \beta) = l(\alpha) + l(\beta) \) for \( \alpha, \beta \in X^*_Z \).

The derivation ari and bracket ari are introduced for bimoulds in terms of flexions in [11] §2.2 (cf. [15] §2.2) and here we consider those for \( \text{ARI}(\Gamma) \) as follows.

**Definition 1.11.** Let \( B \in \text{ARI}(\Gamma) \). The linear map\(^4\) \( \text{ari}_u(B) : \text{ARI}(\Gamma) \rightarrow \text{ARI}(\Gamma) \) is defined by\(^5\)

\[
(\text{ari}_u(B)(A))^m(x_m) = (\text{ari}_u(B)(A))^m(x_1, \ldots, x_m) := \begin{cases} 
\sum_{x_m = \alpha \beta \gamma} A^{l(\alpha, \gamma)}(\alpha, \gamma)B^{l(\beta)}(\beta), & (m \geq 2), \\
0, & (m = 0, 1),
\end{cases}
\]

for \( A \in \text{ARI}(\Gamma) \).

It is shown in [15] Appendix A.1] that ari forms a derivation on the set of bimoulds. The same holds for \( \text{ARI}(\Gamma) \) as follows, where we present an alternative proof to hers.

**Lemma 1.12.** For any \( A \in \text{ARI}(\Gamma) \), \( \text{ari}_u(A) \) forms a derivation of \( \text{ARI}(\Gamma) \) with respect to the product \( \times \), that is, for any \( B, C \in \text{ARI}(\Gamma) \), we have

\[
\text{ari}_u(A)(B \times C) = \text{ari}_u(A)(B) \times C + B \times \text{ari}_u(A)(C).
\]
Proof. Let \( m \geq 0 \). We have

\[
(\text{arit}(A)(B \times C))(\mathbf{x}_m) = \sum_{\mathbf{x}_m = \text{abc}} (B \times C)(a_i[c]A(b_j)) - \sum_{\mathbf{x}_m = \text{abc}} (B \times C)(a_i,c)A(i|b).
\]

By applying Lemma A.2(1) for \( c \neq 0 \), \( f = 0 \) to the first term and by applying Lemma A.2(2) for \( a \neq 0 \), \( d = 0 \) to the second term, we have

\[
= \sum_{\mathbf{x}_m = \text{abc}} \left\{ \sum_{a = a_1a_2} B(a_1)C(a_2,c)A(b_j) + \sum_{c = c_1c_2} B(a_i,c_1)C(c_2) \right\} A(b_j).
\]

By applying Lemma A.1 to the second term, we get \( A(b_{j,n}) = A(b_{j,n}) \). Similarly, for the third term, we get \( A(a,b) = A(a,b) \). Therefore, we calculate

\[
= \sum_{\mathbf{x}_m = \text{abc}} \left\{ \sum_{d = abc} B(a_i,c_1)A(b_{j,n}) - \sum_{d = abc} B(a_i,c_1)A(i|b) \right\} C(c_2)
\]

\[
+ \sum_{\mathbf{x}_m = \text{a_1d}} B(a_1) \left\{ \sum_{d = abc} C(a_2,c)A(b_{j,n}) - \sum_{d = abc} C(a_2,c)A(i|b) \right\}
\]

\[
= (\text{arit}(A)(B))(C)(\mathbf{x}_m) + (B \times \text{arit}(A)(C))(\mathbf{x}_m).
\]

Hence, we obtain the claim. \( \square \)

We define the following bracket as with the bracket \( \text{ari} \) introduced in \( \text{Ec11} \) (2.40).

**Definition 1.13.** The \( \text{ari}_u \)-bracket means the bilinear map \( \text{ari}_u : \text{ARI}(\Gamma)^{\otimes 2} \rightarrow \text{ARI}(\Gamma) \) which is defined by

\[
\text{ari}_u(A, B) := \text{ari}_u(B)(A) - \text{ari}_u(A)(B) + [A, B]
\]

for \( A, B \in \text{ARI}(\Gamma) \). Here \( ^6 \), we have \([A, B] := A \times B - B \times A.\)

\( ^6 \)In the papers \( \text{Ec11}, \text{SCh14}\) and \( \text{RS}, \) the product \( A \times B \) (resp. the bracket \([A, B]\)) is denoted by \( \text{mu}(A, B) \) (resp. \( \text{lu}(A, B) \)).
We note that the bracket $ari_u(A, B)$ in the case when $\Gamma = \{e\}$ also appears in [R00] (A.3) and is denoted by $[A, B]_{ari}$.

The following is also stated for moulds and bimoulds in [Sch15] Proposition 2.2.2 where her key formula (2.2.10) looks unproven and containing a signature error.

**Proposition 1.14.** The $\mathbb{Q}$-linear space $ARI(\Gamma)$ forms a filtered Lie algebra under the $ari_u$-bracket.

*Proof.* We give a self-contained proof in Appendix [A.1] □

The following proposition for $\Gamma = \{e\}$ is shown in [Sch15] Appendix A).

**Proposition 1.15.** The $\mathbb{Q}$-linear space $ARI(\Gamma)_{al}$ forms a filtered Lie subalgebra of $ARI(\Gamma)$ under the $ari_u$-bracket.

*Proof.* We prove this in Appendix [A.2] □

1.3. **Swap and bialternality.** We encode $ARI(\Gamma)$ with another Lie algebra structure introduced by $ari_u$. We prepare $\overline{ARI}(\Gamma)$, a copy of $ARI(\Gamma)$. We denote $M^m_{\sigma_1, \ldots, \sigma_m}(x_1, \ldots, x_m)$ by $\overline{M}^m_{\sigma_1, \ldots, \sigma_m}(x_1, \ldots, x_m)$ for each element $M$ in $ARI(\Gamma)$ to distinguish it from an element in $ARI(\Gamma)$.

Similarly to our previous sections, we work over the following algebraic formulation: Put $Y := \{(_y^v)^{i, \sigma} \in N, \sigma \in \Gamma \}$, and let $Y^*_Z$ be the set such that

$$Y^*_Z := \{(v_y^u) \mid u = a_1y_1 + \cdots + a_ky_k, k \in \mathbb{N}, a_j \in \mathbb{Z}, \sigma \in \Gamma \},$$

and let $Y^*_Z$ be the non-commutative free monoid generated by all elements of $Y^*_Z$ with the empty word $\emptyset$ as the unit. We set $A_Y := \mathbb{Q}(Y^*_Z)$ to be the non-commutative polynomial algebra generated by $Y^*_Z$. In the same way to $A_X$, it is equipped with a structure of a commutative, associative, unital $\mathbb{Q}$-algebra with the shuffle product $\shuffle : A^*_Y \to A_Y$. The flexions are also introduced in this setting.

**Definition 1.16.** The flexions on $Y^*_Z$ are the four binary operators $,[*, *], , [*, *], : Y^*_Z \times Y^*_Z \to Y^*_Z$ which are defined by

$$\begin{align*}
\beta [\alpha] &:= (\sigma_1, \ldots, \tau_n, a_1, \sigma_2, \ldots, a_2, \ldots, \sigma_m, a_m), \\
\alpha | \beta &:= (\sigma_1, \ldots, a_1, \sigma_2, \ldots, a_2, \ldots, \tau_n, \sigma_{m+1}, \ldots, \sigma_m), \\
\alpha \cdot \beta &:= (\sigma_1, \ldots, \sigma_{m-1}, a_1, \sigma_m, \tau_{n-1}, \ldots, \tau_n), \\
\alpha \cdot \beta &:= (\sigma_1, \ldots, \sigma_{m-1}, \tau_{n-1}, \ldots, \tau_n, \sigma_m, a_1).
\end{align*}$$

for $\alpha = (\sigma_1, \ldots, \sigma_m), \beta = (\tau_1, \ldots, \tau_n) \in Y^*_Z$ (m, n \geq 1) and $\gamma \in Y^*_Z$.

We denote $y_0 := \emptyset$ and $y_m := (y_1, \ldots, y_m)$ for $m \geq 1$.

\footnote{We note that the top and bottom rows are switched.}
Definition 1.17. Let $B \in \overline{\text{ARI}}(\Gamma)$. The linear map $\text{arit}_v(B) : \overline{\text{ARI}}(\Gamma) \to \overline{\text{ARI}}(\Gamma)$ is defined by
\[
(\text{arit}_v(B)(A))^m(y_m) = (\text{arit}_v(B)(A))^m(\sigma_1, \ldots, \sigma_m) \\
:= \sum_{y_m = \sigma_1^\beta \gamma} A^{(\alpha_\gamma)}(\alpha_\gamma \beta) B^{(\beta)\gamma}_\sigma(\gamma) - \sum_{y_m = \sigma_1^\beta \gamma \alpha_\beta \neq \emptyset} A^{(\alpha_\gamma)}(\alpha_\gamma \beta) B^{(\beta)\gamma}_\sigma(\gamma) \quad (m \geq 2),
\]
\[
0 \quad (m = 0, 1),
\]
for $A \in \overline{\text{ARI}}(\Gamma)$.

Similarly to Lemma 1.12, the following holds.

Lemma 1.18. For any $A \in \overline{\text{ARI}}(\Gamma)$, $\text{arit}_v(A)$ forms a derivation of $\overline{\text{ARI}}(\Gamma)$ with respect to the product $\times$.

Proof. The $\text{arit}_v$-bracket can be expressed by the exactly same formula as the $\text{arit}_u$-bracket in terms of flexions. Therefore it can be proved in the same way to the one of Lemma 1.12. □

Definition 1.19. The $\text{ari}_v$-bracket means the bilinear map $\text{ari}_v : \overline{\text{ARI}}(\Gamma) \otimes 2 \to \overline{\text{ARI}}(\Gamma)\text{ which is defined by}$
\[
\text{ari}_v(A, B) := \text{arit}_v(B)(A) - \text{arit}_v(A)(B) + [A, B]
\]
for $A, B \in \overline{\text{ARI}}(\Gamma)$. Here $[A, B] := A \times B - B \times A$.

Similarly to Proposition 1.14, the following holds.

Proposition 1.20. The $\mathbb{Q}$-linear space $\overline{\text{ARI}}(\Gamma)$ forms a filtered Lie algebra under the $\text{ari}_v$-bracket.

Proof. It can be also proved in the same way to the one of Proposition 1.14. □

In the same way to Definition 1.3, alternal moulds in $\overline{\text{ARI}}(\Gamma)$ can be introduced and we denote $\overline{\text{ARI}}(\Gamma)_{al}$ to be its subset consisting of alternal moulds. Similarly to Proposition 1.15, the following holds.

Proposition 1.21. $\overline{\text{ARI}}(\Gamma)_{al}$ forms a Lie algebra under the $\text{ari}_v$-bracket.

Proof. It can be proved in the same way to the one of Proposition 1.15. □

We define the $\mathbb{Q}$-linear map $\text{swap} : \text{ARI}(\Gamma) \to \overline{\text{ARI}}(\Gamma)$ by
\[
\text{swap}(M)^m(\sigma_1, \ldots, \sigma_m) = M^m(\sigma_1 \cdots \sigma_m, \sigma_1 \cdots \sigma_{m-1}, \ldots, \sigma_1 \sigma_2, \sigma_1) \quad \text{for any mould } M = (M^m(\sigma_1, \ldots, \sigma_m)) \in \text{ARI}(\Gamma).
\]

Definition 1.22 (cf. [Ec03], [Ec11]). The subset $\text{ARI}(\Gamma)_{al/al}$ of bialternal moulds is defined to be
\[
\text{ARI}(\Gamma)_{al/al} := \{ M \in \text{ARI}(\Gamma)_{al} \mid \text{swap}(M) \in \overline{\text{ARI}}(\Gamma)_{al}, \ M^1(\sigma_1) = M^1\left(\frac{-x_1}{\sigma_1}\right) \}
\]

Here are observations on Example 1.4.

8 The lower suffix $v$ is reflected by the notion of $v$-moulds in [Sch15].

9 In [RS], the brackets $\text{arit}_v(A, B)$ and $\text{ari}_v(A, B)$ are denoted by $\text{arit}$ and \overline{arit} respectively.
Example 1.23. (a). When \( f(x) \in \mathbb{Q}(x) \) is an even function, the mould \( M_f \) in Example [13] (a) is in \( \text{ARI}(\Gamma)_{\text{al/al}} \).
(b). However, the mould \( A \) in Example [14] (b) is not in \( \text{ARI}(\Gamma)_{\text{al/al}} \). In fact, we have
\[
\text{swap}(A)^2\left( \begin{array}{cc} c_v & e_v \\ v_1 & v_2 \end{array} \right) = A^2\left( \begin{array}{cc} e_v & c_v \\ v_1 - v_2 & e_v \end{array} \right) = \frac{1}{v_1 - 2v_2},
\]
and we get
\[
\sum_{\alpha \in Y_{2 \ast}} \text{Sh}\left( \begin{array}{c} c_v \\ \alpha \end{array} : \begin{array}{c} c_v \\ \alpha \end{array} \right) \text{swap}(A)^2(\alpha) = \text{swap}(A)^2\left( \begin{array}{cc} c_v & e_v \\ v_1 & v_2 \end{array} \right) + \text{swap}(A)^2\left( \begin{array}{cc} e_v & c_v \\ v_1 & v_2 \end{array} \right)
= \frac{1}{v_1 - 2v_2} + \frac{1}{v_2 - 2v_1} = -\frac{v_1 + v_2}{(v_1 - 2v_2)(v_2 - 2v_1)} \neq 0.
\]
Hence, \( \text{swap}(A) \) is not alternal, which says \( A \notin \text{ARI}(\Gamma)_{\text{al/al}} \).

Proposition 1.24. The \( \mathbb{Q} \)-linear space \( \text{ARI}(\Gamma)_{\text{al/al}} \) forms a filtered Lie subalgebra of \( \text{ARI}(\Gamma)_{\text{al}} \) under the \( \text{ari}_\alpha \)-bracket.

Proof. We prove this in Appendix A.3.

1.4. Push-invariance and pus-neutrality. We introduce the set \( \text{ARI}(\Gamma)_{\text{push/pusnu}} \) of moulds which are push-invariant and whose \( \text{swap} \) are pus-neutral and show that it forms a Lie algebra under the \( \text{ari}_\alpha \)-bracket.

By abuse of notation, we introduce the following notations for \( \overline{\text{ARI}}(\Gamma) \) similarly to Notation 1.7.

Notation 1.25 ([Ec11, §2.1]). For any mould \( M = \left( M^m \left( \sigma_1, \ldots, \sigma_m \right) \right) _m \in \overline{\text{ARI}}(\Gamma) \), we define
\[
\text{mantar}(M)^m \left( \sigma_1, \ldots, \sigma_m \right) = (-1)^m-1 M^m \left( \sigma_m, \ldots, \sigma_1 \right),
\]
\[
\text{pus}(M)^m \left( \sigma_1, \ldots, \sigma_m \right) = M^m \left( \sigma_m, \sigma_1, \ldots, \sigma_m \right),
\]
\[
\text{neg}(M)^m \left( \sigma_1, \ldots, \sigma_m \right) = M^m \left( \sigma_1, \ldots, \sigma_m \right).
\]

Definition 1.26. We call a mould \( N \in \overline{\text{ARI}}(\Gamma) \) pus-neutral ([Ec11, (2.73)]) \(^\dagger\) when we have
\[
(1.9) \quad \sum_{i=1}^m \text{pus}^i(N)^m \left( \sigma_1, \ldots, \sigma_m \right) = \sum_{i \in \mathbb{Z}/m\mathbb{Z}} N \left( \sigma_{i+1}, \ldots, \sigma_{i+m} \right) \left( \begin{array}{c} v_{i+1} \\ \vdots \\ v_{i+m} \end{array} \right) = 0
\]
for all \( m \geq 1 \) and \( \sigma_1, \ldots, \sigma_m \in \Gamma \). We define \( \text{ARI}(\Gamma)_{\text{push/pusnu}} \) to be the set of moulds \( M \in \text{ARI}(\Gamma) \) which is push-invariant \(^\ddagger\) and whose \( \text{swap}(M) \) is pus-neutral \(^\ddagger\).

Example 1.27. We consider the mould \( Q \in \overline{\text{ARI}}(\{e\}) \) defined by
\[
Q \left( \begin{array}{c} c_v \\ v_1, \ldots, v_m \end{array} \right) := \left\{ \begin{array}{ll} 0 & (m = 0, 1), \\
\frac{1}{v_2 - v_1} + \cdots + \frac{1}{v_m - v_1} & (m \geq 2). \end{array} \right.
\]

\(^\dagger\) It is not push-neutral but pus-neutral. In [RS] Definition 5\(^{[1]}\), it is called circ-neutral when \(^{[1]}\) holds for \( m > 1 \) when \( \Gamma = \{e\} \).

\(^\ddagger\) We expect that our \( \text{ARI}(\Gamma)_{\text{push/pusnu}} \) might be related to Ecalle’s \( \text{ARI}^{\text{pusnu}} \) ([Ec11, §2.5]), whose precise definition looks missing.
Then this mould $Q$ is pus-neutral. In fact, for $1 \leq i \leq m$, we have

$$Q\left( e, \ldots, e, v, \ldots, v_{i-1} \right) = \sum_{j=i+1}^{m} \frac{1}{v_j - v_i} + \sum_{j=1}^{i-1} \frac{1}{v_j - v_i},$$

and so we calculate

$$\sum_{i=1}^{m} \text{pus}^{i}(Q)\left( e, \ldots, e, v, \ldots, v_{i-1} \right) = \sum_{i=1}^{m} Q\left( e, \ldots, e, v, \ldots, v_{i-1} \right) \sum_{j=i+1}^{m} \frac{1}{v_j - v_i} + \sum_{j=1}^{i-1} \frac{1}{v_j - v_i} = \sum_{i,j \in \{1, \ldots, m\}} \frac{1}{v_j - v_i} + \sum_{i,j \in \{1, \ldots, m\}} \frac{1}{v_j - v_i} = 0.$$

Hence, the mould $Q$ is pus-neutral.

Firstly we show that the set $\text{ARI}_{\text{push}}$ forms a Lie algebra under the $\text{ari}_u$-bracket which was stated in [Ec11] §2.5 without a detailed proof.

**Proposition 1.28.** If $A, B \in \text{ARI}(\Gamma)$ are push-invariant, then $\text{ari}_u(A, B)$ is push-invariant.

**Proof.** Let $m \geq 1$. We put $\omega = (u_1, \ldots, u_m)$. We have

$$\text{push}(A \times B)(\omega) = (A \times B)\left( -u_1, \ldots, -u_m, \sigma_1^{-1}, \ldots, \sigma_m^{-1} \right) = A(\emptyset)B\left( -u_1, \ldots, -u_m, \sigma_1^{-1}, \ldots, \sigma_m^{-1}, \sigma_1 \sigma_m^{-1}, \ldots, \sigma_{i-1} \sigma_m^{-1} \right) + \sum_{i=1}^{m} A\left( -u_1, \ldots, -u_{i-1}, -u_i, \ldots, -u_{m-1} \right)B\left( \sigma_1 \sigma_m^{-1}, \ldots, \sigma_{i-1} \sigma_m^{-1}, \sigma_i \sigma_m^{-1}, \ldots, \sigma_{i-1} \sigma_m^{-1} \right).$$

Because $A, B$ are push-invariant, we get

(1.10)

$$\text{push}(A \times B)(\omega) = A(\emptyset)B(\omega) + A(\omega)B(\emptyset) + \sum_{i=1}^{m-1} A\left( \frac{u_1}{\sigma_1}, \ldots, \frac{u_{i-1}}{\sigma_{i-1}}, \frac{u_i + \cdots + u_m}{\sigma_m} \right)B\left( \frac{\sigma_1 \sigma_m^{-1}}{\sigma_m}, \ldots, \frac{\sigma_{i-1} \sigma_m^{-1}}{\sigma_m}, \frac{\sigma_i \sigma_m^{-1}}{\sigma_m} \right).$$
So by putting $u_0 = u_1 = \cdots = u_m$, and $\tau_0 = \sigma_m^{1}$ and $\tau_i = \sigma_m^{i-1}$ for $1 \leq i \leq m - 1$, we calculate

\[
\text{push}(\text{arit}_u(B)(A))(\omega) = \sum_{0 \leq j < m-1} A\left(\begin{array}{cccc}
-u_j + 1 & \cdots & u_{m-1} & \\
\sigma^{-1}_m & \sigma^{-1}_m & \cdots & \sigma^{-1}_m
\end{array}\right) B\left(\begin{array}{cccc}
u_0 & \cdots & \nu_j & \\
\tau_0 & \tau_j & \cdots & \tau_{j+1}
\end{array}\right)
\]
Because $A, B$ are push-invariant, we get

\[
\begin{align*}
= \sum_{0 \leq j < m-1} A\left( \frac{u_{j+1}}{\sigma_{j+1}}, \ldots, \frac{u_m}{\sigma_{m-1}} \right) B\left( \frac{u_1}{\sigma_1}, \ldots, \frac{u_j}{\sigma_j}, \frac{u_{j+1} + \cdots + u_m}{\sigma_j+1} \right) \\
- \sum_{1 \leq j < m-1} A\left( \frac{u_{j+1}}{\sigma_{j+1}}, \ldots, \frac{u_m}{\sigma_m} \right) B\left( \frac{u_1}{\sigma_1}, \ldots, \frac{u_j}{\sigma_j} \right) \\
+ \sum_{0 < i < j < m-1} A\left( \frac{u_1}{\sigma_1}, \ldots, \frac{u_{i-1}}{\sigma_{i-1}}, \frac{u_i + \cdots + u_j}{\sigma_j+1}, \frac{u_{j+1}}{\sigma_{j+1}}, \ldots, \frac{u_m}{\sigma_m} \right) B\left( \frac{u_1}{\sigma_1}, \ldots, \frac{u_j}{\sigma_j}, \frac{u_{j+1} + \cdots + u_m}{\sigma_j+1} \right) \\
- \sum_{1 < i < j < m-1} A\left( \frac{u_1}{\sigma_1}, \ldots, \frac{u_{i-1}}{\sigma_{i-1}}, \frac{u_i + \cdots + u_j}{\sigma_j+1}, \frac{u_{j+1}}{\sigma_{j+1}}, \ldots, \frac{u_m}{\sigma_m} \right) B\left( \frac{u_1}{\sigma_1}, \ldots, \frac{u_j}{\sigma_j}, \frac{u_{j+1} + \cdots + u_m}{\sigma_j+1} \right)

= \sum_{\omega=\alpha \beta \gamma, \alpha, \beta \neq 0} A(\alpha \beta | \gamma) B(\alpha) - (B \times A)(\alpha \beta | \gamma) - \sum_{\omega=\alpha \beta \gamma, \alpha, \beta \neq 0} A(\alpha | \beta) B(\alpha) - \sum_{\omega=\alpha \beta \gamma, \alpha, \beta \neq 0} A(\alpha \beta | \gamma) B(\alpha \beta | \gamma)
\end{align*}
\]

So we get

\[(1.11) \quad \text{push}(\text{arit}_u(B(A)))(\omega) = (\text{arit}_u(B(A)))(\omega) - (B \times A)(\omega)
- \sum_{i=1}^{m-1} A\left( \frac{u_1}{\sigma_1}, \ldots, \frac{u_{i-1}}{\sigma_{i-1}}, \frac{u_i + \cdots + u_m}{\sigma_m} \right) B\left( \frac{u_1}{\sigma_1}, \ldots, \frac{u_m}{\sigma_m} \right)
+ \sum_{\omega=\alpha \beta \gamma, \alpha, \beta \neq 0} \{ A(\alpha \beta | \gamma) B(\alpha), A(\alpha | \beta) B(\alpha) \}.
\]

Therefore, by using (1.10) and (1.11), we have

\[
\text{push}(\text{arit}_u(A, B))(\omega) = \text{push}(\text{arit}_u(B(A)))(\omega) - \text{push}(\text{arit}_u(A)(B))(\omega)
+ \text{push}(A \times B)(\omega) - \text{push}(B \times A)(\omega)
= (\text{arit}_u(B)(A))(\omega) - (\text{arit}_u(A)(B))(\omega)
+ (A \times B)(\omega) - (B \times A)(\omega)
= \text{arit}_u(A, B)(\omega).
\]

Thus $\text{arit}_u(A, B)$ is push-invariant.

**Remark 1.29.** This proposition improves the results of [RS] §4.1.3, which shows that the intersection $\text{ARI}_{\text{push}}(\Gamma) \cap \text{ARI}_{\text{pol}}(\Gamma)$ forms a Lie algebra under the $\text{ari}_u$-bracket when $\Gamma = \{ e \}$.
Secondly we show that the set of pus-neutral moulds is closed under the arit-\(\Gamma\)-bracket. The following proposition is a generalization of [RS, Lemma 21] which treats the case when \(\Gamma = \{e\}\).

**Proposition 1.30.** If \(A, B \in \overline{\text{ARI}}(\Gamma)\) are pus-neutral, then \(\text{arit}_v(A, B)\) is pus-neutral.

**Proof.** The proof goes in the same way to that of [RS]. Let \(m \geq 1\). Since the algebra \(\overline{\text{ARI}}(\Gamma)\) is graded by depth, it is enough to prove \(A\) and \(B\) are in \(\text{ARI}(\Gamma)\) with depth \(k\) and \(l \in \mathbb{N}\) with \(m = k + l\).

Firstly, we prove that \([A, B]\) is pus-neutral. We calculate

\[
\sum_{i=1}^{m} \text{pus}^i([A, B])^m (\sigma_1, ..., \sigma_m) = \sum_{i \in \mathbb{Z}/m\mathbb{Z}} (A \times B - B \times A)^m (\sigma_{i+1}, ..., \sigma_{m+i})
\]

\[
= \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \left\{ A^k \left( \sigma_{i+1}, ..., \sigma_{k+i} \right) B^l \left( \sigma_{k+i+1}, ..., \sigma_{m+i} \right) - B^l \left( \sigma_{i+1}, ..., \sigma_{k+i} \right) A^k \left( \sigma_{k+i+1}, ..., \sigma_{m+i} \right) \right\}
\]

\[
= \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \left\{ A^k \left( \sigma_{i+1}, ..., \sigma_{k+i} \right) B^l \left( \sigma_{k+i+1}, ..., \sigma_{m+i} \right) - B^l \left( \sigma_{i+1}, ..., \sigma_{k+i} \right) A^k \left( \sigma_{k+i+1}, ..., \sigma_{m+i} \right) \right\}
\]

The first and the second terms are equal, so \([A, B]\) is pus-neutral.

Secondly, we prove that \(\text{arit}_v(B)(A)\) is pus-neutral. Then we have

\[
(\text{arit}_v(B)(A))^m (\sigma_1, ..., \sigma_m) = \sum_{\omega = \alpha \beta \gamma} A^k(\alpha \beta \gamma)B^l(\beta \gamma) - \sum_{\omega = \alpha \beta \gamma} A^k(\alpha \beta \gamma)B^l(\beta \gamma).
\]

Because \(l \geq 1\) and \(B\) is with depth \(l\), we may put \(\alpha = (\sigma_1, ..., \sigma_j)\), \(\beta = (\sigma_{j+1}, ..., \sigma_{j+l})\) and \(\gamma = (\sigma_{j+l+1}, ..., \sigma_m)\) and we get

\[
(\text{arit}_v(B)(A))^m (\sigma_1, ..., \sigma_m) = \sum_{j=0}^{k-1} A^k \left( \sigma_1, ..., \sigma_j, \sigma_{j+1}, ..., \sigma_{j+1+l}, ..., \sigma_m \right) B^l \left( \sigma_{j+1}, ..., \sigma_{j+1+l} \right) - \sum_{j=1}^{k} A^k \left( \sigma_1, ..., \sigma_{j-1}, \sigma_j, ..., \sigma_{j+l}, ..., \sigma_m \right) B^l \left( \sigma_{j+1}, ..., \sigma_{j+l} \right).
\]

\footnote{For the first (resp. second) term, the word \(\alpha\) (resp. \(\gamma\)) can be \(\emptyset\) and the word \(\gamma\) (resp. \(\alpha\)) is not \(\emptyset\). So the index \(j\) must run from 0 (resp. 1) to \(k - 1 = m - l - 1\) (resp. \(k = m - l\)).}
Here, for $0 \leq j \leq k - 1$, we have

$$
\sum_{i \in \mathbb{Z}/m\mathbb{Z}} \sum_{j=0}^{k-1} A^k \left(\sigma_{i+1}, \ldots, \sigma_{i+j}, \sigma_{i+j+1}, \ldots, \sigma_{i+m}\right) \cdot B^i \left(v_{i+j+1} - v_{i+j+l+1}, \ldots, v_{i+m-j}, \ldots, \sigma_{i+j+l+1} \right)
$$

$$
= \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \sum_{j=0}^{k-1} \left(\text{pus}(A)^k \left(\sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{i+m}\right) \cdot B^i \left(v_{i+1} - v_{i+l+1}, \ldots, v_{i+l}, \ldots, \sigma_{i+l+1}\right) \right).
$$

Since $A$ is pus-neutral, this is 0. Similarly by using the pus-neutrality of $A$, we obtain

$$
\sum_{i \in \mathbb{Z}/m\mathbb{Z}} \sum_{j=1}^{k} A^k \left(\sigma_{i+1}, \ldots, \sigma_{i+j-1}, \sigma_{i+j}, \sigma_{i+j+1}, \ldots, \sigma_{i+m}\right) \cdot B^i \left(v_{i+j+1} - v_{i+j}, \ldots, v_{i+j+l} - v_{i+j}\right) = 0
$$

for all $1 \leq j \leq k$. Therefore, we get

$$
\sum_{i=1}^{m} \text{pus}^i (\arit_v(B)(A))^m \left(\sigma_1, \ldots, \sigma_m\right) = \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \left(\arit_v(B)(A)^m \left(\sigma_1, \ldots, \sigma_m\right)\right) = 0.
$$

Whence $\arit_v(B)(A)$ is pus-neutral.

Thirdly, in the same way, we can show that $\arit_v(A)(B)$ is pus-neutral by using the pus-neutrality of $B$.

Thus we complete the proof because $\arit_v(A, B) = [A, B] + \arit_v(B)(A) - \arit_v(A)(B)$. \hfill \qed

We need the following lemma for the proof of Theorem 1.32.

**Lemma 1.31.** If $A, B \in \text{ARI}(\Gamma)$ are push-invariant, then we have

$$\text{swap}(\arit_v(A, B)) = \arit_v(\text{swap}(A), \text{swap}(B)).$$

**Proof.** This follows from the proof of [Sch15, Lemma 2.4.1], which actually works for BARI. \hfill \qed

**Theorem 1.32.** The set $\text{ARI}(\Gamma)_{\text{push/pusnu}}$ forms a Lie subalgebra of $\text{ARI}(\Gamma)_{\text{push}}$ under the $\arit_v$-bracket.

**Proof.** Let $A, B \in \text{ARI}(\Gamma)_{\text{push/pusnu}}$. Then by Proposition 1.28, $\arit_v(A, B) \in \text{ARI}(\Gamma)_{\text{push}}$. Let $m \geq 1$. Because $A$ and $B$ are push-invariant, by Lemma 1.31
we have
\[
\sum_{i \in \mathbb{Z}/m\mathbb{Z}} \text{pus}^i \circ \text{swap}(\text{ari}_u(A, B))^m(\sigma_1, ..., \sigma_m) = \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \text{pus}^i(\text{ari}_u(\text{swap}(A), \text{swap}(B)))^m(\sigma_1, ..., \sigma_m).
\]
By Proposition 1.30 the right hand side is equal to 0. Hence \(\text{swap}(\text{ari}_u(A, B))\) is pus-neutral. Thus we obtain \(\text{ari}_u(A, B) \in \text{ARI}(\Gamma)_{\text{push/pusnu}}\).

**Remark 1.33.** The above theorem improves [RS, Corollary 22] which shows that \(\text{ARI}(\Gamma)\) of \(\text{ARI}(\Gamma)_{\text{push/pusnu}}\)∩\(\text{ARI}_{\text{al}}(\Gamma)\) forms a Lie algebra when \(\Gamma = \{e\}\).

Our Lie algebra \(\text{ARI}(\Gamma)_{\text{push/pusnu}}\) will play an important role in the following sections.

**Definition 1.34.** For \(N \geq 1\), put \(\Gamma^N := \{g^N \in \Gamma \mid g \in \Gamma\}\). We consider the map \(i_N : \text{ARI}(\Gamma) \to \text{ARI}(\Gamma^N)\) which is given by
\[
i_N(M)^m(x_1, ..., x_m) = M^m(x_1, ..., x_m)
\]
and also the map \(m_N : \text{ARI}(\Gamma) \to \text{ARI}(\Gamma^N)\) which is given by
\[
m_N(M)^m(x_1, ..., x_m) = \sum_{\tau^N = \sigma} M^m(N_{x_1}, ..., N_{x_m}).
\]
We define the following \(\mathbb{Q}\)-linear subspaces which are subject to the distribution relations:

\[
\text{ARI}(\Gamma) := \{M \in \text{ARI}(\Gamma) \mid i_N(M) = m_N(M) \text{ for all } N \geq 1 \text{ with } N \mid |\Gamma|\};
\]
\[
\text{ARI}(\Gamma)_{\text{al}} := \text{ARI}(\Gamma) \cap \text{ARI}(\Gamma)_{\text{al/ai}}.
\]

**Proposition 1.35.** The \(\mathbb{Q}\)-linear subspace \(\text{ARI}(\Gamma)\) forms a filtered Lie subalgebra of \(\text{ARI}(\Gamma)\) under the \(\text{ari}_u\)-bracket.

**Proof.** First we prove that \(m_N\) is a Lie algebra homomorphism, that is,
\[
m_N(\text{ari}_u(A, B)) = \text{ari}_u(m_N(A), m_N(B))
\]
for \(A, B \in \text{ARI}(\Gamma)\). Let \(m \geq 0\). We have
\[
m_N(A \times B)^m(x_m) = \sum_{\tau^N = \sigma} (A \times B)^m(N_{x_1}, ..., N_{x_m})
\]
\[
= \sum_{\tau^N = \sigma} \sum_{j=0}^m A^j(N_{x_1}, ..., N_{x_j}) B^{m-i}(N_{x_{j+1}}, ..., N_{x_m})
\]
\[
= \sum_{j=0}^m \left\{ \sum_{\tau^N = \sigma} A^j(N_{x_1}, ..., N_{x_j}) \right\} \left\{ \sum_{\tau^N = \sigma} B^{m-i}(N_{x_{j+1}}, ..., N_{x_m}) \right\}
\]
\[
= \sum_{j=0}^m m_N(A)^j(x_1, ..., x_j) m_N(B)^{m-i}(x_{j+1}, ..., x_m)
\]
\[
= (m_N(A) \times m_N(B))^m(x_m).
\]
Similarly, we have

\[
m_N(\text{arit}_u(B)(A))^m(x_m)
= \sum_{\nu_i^{N}=\sigma_i} \text{arit}(B)(A)^m \left( Nx_1, \ldots, Nx_m \right)
\]

\[
= \sum_{\nu_i^{N}=\sigma_i} \left\{ \sum_{1 \leq k \leq l < m} A \left( Nx_k, \ldots, Nx_{k-1}, \frac{N(x_k + \cdots + x_{l+1})}{\nu_{k+1}}, N_{\nu_{l+2}}, \ldots, N_{\nu_m} \right) \right\}
\cdot B \left( \frac{N_{\nu_k}}{\nu_k\nu_{k+1}}, \ldots, \frac{N_{\nu_l}}{\nu_l\nu_{l+1}} \right) - \sum_{\nu_i^{N}=\sigma_i} \left\{ \sum_{1 \leq k \leq l \leq m} A \left( Nx_k, \ldots, Nx_{k-1}, \frac{N(x_k + \cdots + x_l)}{\nu_{k+1}}, N_{\nu_{k+2}}, \ldots, N_{\nu_m} \right) \right\}
\cdot B \left( \frac{N_{\nu_k}}{\nu_k\nu_{k+1}}, \ldots, \frac{N_{\nu_l}}{\nu_l\nu_{k+1}} \right).
\]

Here, we put \( S_{k,l} := \{ i \in \mathbb{N} | 1 \leq i \leq k-1 \} \cup \{ i \in \mathbb{N} | l+1 \leq i \leq m \} \) for \( 1 \leq k \leq l \leq m \). Then we can divide the set of variables of the summation in the following:

\[
\{ \nu_i | 1 \leq i \leq m \} = \{ \nu_i | i \in S_{k,l} \} \cup \{ \nu_i | k \leq i \leq l \}.
\]

So by substituting \( \nu_i := \tau_i \) for \( i \in S_{k,l} \) and \( \nu_i := \tau_i\tau_{i+1} \) for \( k \leq i \leq l \) for the first summation, and by substituting \( \nu_i := \tau_i \) for \( i \in S_{k,l} \) and \( \nu_i := \tau_i\tau_{k-1} \) for \( k \leq i \leq l \)
for the second summation, we calculate

\[
m_N(\text{arit}_u(B)(A))^m(x_m) = \sum_{1 \leq k \leq l < m} \left\{ \sum_{\substack{\sum_{\tau_i = 1}^{l} = \sigma_l \in S_{k,l} \atop i \in S_{k,l}}} A\left( N_{x_1}, \ldots, N_{\tau_{k-1}}, N_{\tau_k}, \ldots, N_{\tau_{l+1}}, N_{x_{k+1} + \cdots + x_{l+1}}, N_{x_{l+2}}, \ldots, N_{x_m} \right) \right\} \cdots \left\{ \sum_{\sum_{\tau_i = 1}^{l} = \sigma_l \in S_{k,l}} B\left( N_{x_k}, \ldots, N_{x_l} \right) \right\}
\]

\[
- \sum_{1 \leq k \leq l < m} \left\{ \sum_{\substack{\sum_{\tau_i = 1}^{l} = \sigma_l \in S_{k,l} \atop i \in S_{k,l}}} A\left( N_{x_1}, \ldots, N_{\tau_{k-1}}, N_{\tau_k}, \ldots, N_{\tau_{l+1}}, N_{x_{k+1} + \cdots + x_{l+1}}, N_{x_{l+2}}, \ldots, N_{x_m} \right) \right\} \cdots \left\{ \sum_{\sum_{\tau_i = 1}^{l} = \sigma_l \in S_{k,l}} B\left( N_{x_k}, \ldots, N_{x_l} \right) \right\}
\]

\[
= \sum_{1 \leq k \leq l < m} m_N(A)\left( \frac{x_1}{\sigma_1}, \ldots, \frac{x_{k-1}}{\sigma_{k-1}}, \frac{x_k + \cdots + x_{l+1}}{\sigma_{k+1}}, \frac{x_{l+2}}{\sigma_{l+2}}, \ldots, \frac{x_m}{\sigma_m} \right)
\]

\[
\cdot m_N(B)\left( \frac{x_k}{\sigma_k \sigma_{k+1}^{-1}}, \ldots, \frac{x_l}{\sigma_l \sigma_{l+1}^{-1}} \right)
\]

\[
- \sum_{1 \leq k \leq l < m} m_N(A)\left( \frac{x_1}{\sigma_1}, \ldots, \frac{x_{k-1}}{\sigma_{k-1}}, \frac{x_k + \cdots + x_{l+1}}{\sigma_{k+1}}, \frac{x_{l+2}}{\sigma_{l+2}}, \ldots, \frac{x_m}{\sigma_m} \right)
\]

\[
\cdot m_N(B)\left( \frac{x_k}{\sigma_k \sigma_{k+1}^{-1}}, \ldots, \frac{x_l}{\sigma_l \sigma_{l+1}^{-1}} \right).
\]

By putting \( \alpha = (\frac{x_1}{\sigma_1}, \ldots, \frac{x_{k-1}}{\sigma_{k-1}}) \), \( \beta = (\frac{x_k}{\sigma_k}, \ldots, \frac{x_l}{\sigma_l}) \) and \( \gamma = (\frac{x_{k+1}}{\sigma_{k+1}}, \ldots, \frac{x_m}{\sigma_m}) \), we get

\[
= \sum_{x_m = \alpha \beta \gamma, \beta, \gamma \neq \emptyset} m_N(A)(\alpha, \beta) m_N(B)(\beta, \gamma) - \sum_{x_m = \alpha \beta \gamma, \alpha, \beta \neq \emptyset} m_N(A)(\alpha, \beta) m_N(B)(\beta, \gamma).
\]

Hence we get

\[
m_N(\text{arit}_u(A, B)) = m_N(\text{arit}_u(B)(A)) - m_N(\text{arit}_u(A)(B)) + m_N(\alpha \times B) - m_N(\beta \times A)
\]

\[
= \text{arit}_u(m_N(B)(m_N(A))) - \text{arit}_u(m_N(A)(m_N(B)))
\]

\[
+ m_N(A) \times m_N(B) - m_N(B) \times m_N(A)
\]

\[
= \text{arit}_u(m_N(A), m_N(B)).
\]

Therefore, \( m_N \) is a Lie algebra homomorphism.

It is obvious that the map \( i_N \) is a Lie algebra homomorphism. Since both \( m_N \) and \( i_N \) are Lie algebra homomorphisms, \( \ker(i_N - m_N) \) forms Lie algebra.
By the definition of $\text{ARID}(\Gamma)$, we have $\text{ARID}(\Gamma) = \bigcap_{N} \ker(i_{N} - m_{N})$. Therefore $\text{ARID}(\Gamma)$ forms a Lie subalgebra under the $\text{ARI}(\Gamma)$-bracket.

\begin{corollary}
\text{The $Q$-linear subspace $\text{ARID}(\Gamma)_{\text{fil}}$ forms a filtered Lie subalgebra of $\text{ARI}(\Gamma)$ under the $\text{ari}$-bracket.}
\end{corollary}

\begin{proof}
It is a direct consequence of Proposition 1.24 and Proposition 1.35.
\end{proof}

2. Kashiwara-Vergne Lie algebra

We introduce the Kashiwara-Vergne bigraded Lie algebra $\mathfrak{KSV}(\Gamma)_{\bullet\bullet}$ associated with a finite abelian group $\Gamma$ and give its mould theoretical interpretation by using $\text{ARI}(\Gamma)_{\text{push/push}}$.

2.1. $\Gamma$-variant of the KV condition. We investigate a variant of the defining conditions of Kashiwara-Vergne graded Lie algebra associated with a finite abelian group $\Gamma$ (cf. Definition 2.1) and explain its mould theoretical interpretation in Theorem 2.15.

Let $L = \oplus_{w \geq 1} L_{w}$ be the free graded Lie $Q$-algebra generated by $N + 1$ variables $x$ and $y_{\sigma}$ ($\sigma \in \Gamma$) with $\deg x = \deg y_{\sigma} = 1$. Here $L_{w}$ is the $Q$-linear space generated by Lie monomials whose total degree is $w$. Occasionally we regard $L$ as a bigraded Lie algebra $L_{\bullet\bullet} = \oplus_{w,d} L_{w,d}$, where $L_{w,d}$ is the $Q$-linear space generated by Lie monomials whose weight (the total degree) is $w$ and depth (the degree with respect to all $y_{\sigma}$) is $d$. We encode $L$ with a structure of filtered graded Lie algebra by the filtration $\text{Fil}^{d}_{\text{Fil}} L_{w} := \oplus_{N \geq d} L_{w,N}$ for $d > 0$ and denote the associated bigraded Lie algebra by $\text{gr}_{d} L = \oplus_{w,d} \text{gr}_{d} L_{w}$ with $\text{gr}_{d} L_{w} = \text{Fil}^{d}_{\text{Fil}} L_{w}/\text{Fil}^{d+1}_{\text{Fil}} L_{w}$.

The $N + 1$-variable non-commutative polynomial algebra $A = \langle x, y_{\sigma} ; \sigma \in \Gamma \rangle$ is regarded as the universal enveloping algebra of $L$ and is encoded with the induced degree. Similarly $A$ is encoded with a structure of bigraded algebra; $A_{\bullet\bullet} = \oplus_{w,d} A_{w,d}$. By putting $\text{Fil}^{d}_{\text{Fil}} A_{w} := \oplus_{N \geq d} A_{w,N}$ for $d > 0$, we also encode $A$ with a structure of filtered graded algebra. We define the action of $\tau \in \Gamma$ on $A$ (hence on $L$) by

$$
\tau(x) = x \text{ and } \tau(y_{\sigma}) = y_{\tau \sigma}.
$$

For any $h \in A$, we denote

$$
h = h_{x}x + \sum_{\alpha} h_{y_{\alpha}} y_{\alpha} = x h^{x} + \sum_{\alpha} y_{\alpha} h^{y_{\alpha}}.
$$

We denote $\pi_{Y}$ to be the composition of the natural projection and inclusion:

$$
\pi_{Y} : A \to A/ A \cdot x \simeq Q \oplus (\oplus_{\sigma \in \Gamma} A y_{\sigma}) \hookrightarrow A
$$

and the $Q$-linear isomorphism $q$ on $A$ defined by

$$
q(x^{e_0} y_{\sigma_1} x^{e_1} y_{\sigma_2} \cdots x^{e_{r-1}} y_{\sigma_r} x^{e_{r}}) = x^{e_0} y_{\sigma_1} x^{e_1} y_{\sigma_2} y_{\sigma_1}^{-1} \cdots x^{e_{r-1}} y_{\sigma_r} y_{\sigma_{r-1}}^{-1} x^{e_{r}}
$$

(cf. [R02]). The $Q$-linear endomorphism

$$
\text{anti} : A \to A
$$

is the palindrome (backwards-writing) operator in $A_{w}$ (cf. [Sch12] Definition 1.3).

We put $\text{Cyc}(A)$ to be $Q$-linear space generated by cyclic words of $A$ and $\text{tr} : A \to \text{Cyc}(A)$ to be the trace map, the natural projection to $\text{Cyc}(A)$ (cf. [AT]).
Definition 2.1. We define the graded $\mathbb{Q}$-linear space $\text{trv}(\Gamma)_w = \oplus_{w > 1} \text{trv}(\Gamma)_w$, where its degree $w$-part $\text{trv}(\Gamma)_w$ is defined to be the set of Lie elements $F \in L_w$ such that there exists $G = G(F)$ in $L_w$ with

\[(KV1) \quad [x, G] + \sum_{\tau \in \Gamma} [y_{\tau}, \tau(F)] = 0,\]

\[(KV2) \quad \text{tr} \circ q \circ \pi_Y (F(z; (y_\sigma))) = 0\]

with $z = -x - \sum_{\sigma \in \Gamma} y_{\sigma}$.

We note that such $G = G(F)$ uniquely exists when $w > 1$. For $d \geq 1$, we put $\text{Fil}^d_w \text{trv}(\Gamma)_w$ to be the subspace of $\text{trv}(\Gamma)_w$ consisting of $F \in \text{Fil}^d_w L_w$. By $(KV2)$, $\text{trv}(\Gamma)_w = \text{Fil}^1_w \text{trv}(\Gamma)_w$.

Lemma 2.2. Assume that $\Gamma = \{e\}$. Let $F \in L_w$ satisfying $(KV1)$ with $w \geq 2$. Then $(KV2)$ for $F$ is equivalent to

\[(2.1) \quad \text{tr}(G_x x + F_y y) = 0.\]

Proof. By $\text{tr} \circ q \circ \pi_Y (F(z, y)) = \text{tr} \circ \pi_Y (F(z, y)) = \text{tr}(-F_x (z, y)y + F_y (z, y)y)$, the condition $(KV2)$ is equivalent to

\[(2.1) \quad \text{tr}((F_y - F_x)y) = 0.\]

By $(KV1)$, we have $G x - x G = y F - F y$. So $F_y = F^y$ and $G_y = F^x$. By $F \in L_w$, $F_x = (-1)^{w-1} \text{anti}(F^x)$ and $F_y = (-1)^{w-1} \text{anti}(F^y)$. By $G \in L_w$, we have $\text{tr}(G x x + G y y) = 0$. Therefore

\[\text{tr}((F_y - F_x)y) = (-1)^{w-1} \text{tr}((F^y - F^x)y) = (-1)^{w-1} \text{tr}(F_y y - G y y) = (-1)^{w-1} \text{tr}(F_y y + G x x),\]

whence we get the claim. \hfill \qed

Remark 2.3. Since $(2.1)$ agrees with original defining condition $(KV2)$ in [AT], we see that our $\text{trv}(\Gamma)_w$ with $\Gamma = \{e\}$ recovers the original Kashiwara-Vergne Lie algebra denoted by $\text{trv}^0_2$ in [AT] and the depth $>1$-part of $\text{trv}$ in [RS, Definition 3]. We do not know how their Lie algebras $\text{trv}^0_{n+1}$ ($n \geq 0$) in [AT] and also $\text{kv}_{n+1}$ in [AKKN] are related to our $\text{trv}(\Gamma)_w$. It is also not clear if our $\text{trv}(\Gamma)_w$ forms a Lie algebra or not.

Let $h \in A_w$ be a degree $w$ homogeneous polynomial with $h = \sum_{r=0}^w h^r$ and

\[(2.2) \quad h^r = \sum_{(\sigma_1, \ldots, \sigma_r) \in \Gamma^\circ r} \sum_{(e_0, \ldots, e_r) \in \mathcal{E}_w^r} a(h; e_0, \ldots, e_r) x_0^{e_0} y_{\sigma_1} \cdots y_{\sigma_r} x_r \in A_{w,r}\]

where $\mathcal{E}_w^r = \{(e_0, \ldots, e_r) \in \mathbb{N}_0^{r+1} | \sum_{i=0}^r e_i = w - r\}$.

Definition 2.4. By following [Sch12, Appendix A], we associate a mould $\text{ma}_h = (\text{ma}_h^0, \text{ma}_h^1, \text{ma}_h^2, \ldots, \text{ma}_h^w, 0, \ldots) \in \mathcal{M}(\mathbb{F}; \Gamma)$ which is defined by $\text{ma}_h^w = \{\text{ma}_h^w((u_1, \ldots, u_r); (\sigma_1, \ldots, \sigma_r)) \in \Gamma^\circ r \}$ with

\[\text{ma}_h^r((u_1, \ldots, u_r); (\sigma_1, \ldots, \sigma_r)) = \text{vimo}_h^r(0, u_1 + u_2, u_1 + u_2 + u_3, \ldots, u_1 + u_2 + \cdots + u_r),\]

\[\text{vimo}_h^r(0, e_1, e_2, \ldots, e_r) = \sum_{(e_0, \ldots, e_r) \in \mathcal{E}_w^r} a(h; e_0, \ldots, e_r) x_0^{e_0} e_1^{e_1} e_2^{e_2} \cdots e_r^{e_r}.\]

We start with the following technical lemma which is required to our later arguments.
Lemma 2.5. When $h \in \mathbb{L}_{w,r}$, we have

$$\text{(2.3)} \quad \text{vimo}_h^f(z_{1},...,z_{r}) = \text{vimo}_h^f(0,z_{1}−z_{0},...,z_{r}−z_{0}),$$

$$\text{(2.4)} \quad \text{mantar} \circ \text{ma}_h = \text{ma}_h^r.$$ 

Proof. The proof of (2.3) can be done by induction on degree in the same way to the arguments in [Sch12, Lemma A.2]. We give a short proof below. Since $h \in \mathbb{L}_{w,r}$, we have $h = (−1)^{w−1}\text{anti}(h)$. Thus by definition, we have $\text{vimo}_h^f(z_{1},...,z_{r}) = (−1)^{w−1}\text{vimo}_h^f(0,z_{1}−z_{0},...,z_{r}−z_{0})$. Therefore

$$\text{ma}_h^r(z_{1},...,z_{r}) = \text{vimo}_h^f(0,z_{1}−z_{2},...,z_{r}−z_{r}) = (−1)^{w−1}\text{vimo}_h^r(z_{1}−z_{2},...,z_{r}−z_{r}),$$

$$= (−1)^{r−1}\text{vimo}_h^r(z_{1},...,z_{r}) = (−1)^{r−1}\text{vimo}_h^r(0,z_{1},...,z_{r}).$$

The equation (2.4) is a formal generalization of [Sch12, Lemma A.2]. We give a short proof below. Since $h \in \mathbb{L}_{w,r}$, we have $h = (−1)^{w−1}\text{anti}(h)$. Thus by definition, we have $\text{vimo}_h^f(z_{1},...,z_{r}) = (−1)^{w−1}\text{vimo}_h^f(0,z_{1}−z_{0},...,z_{r}−z_{0})$. Therefore

$$\text{ma}_h^r(z_{1},...,z_{r}) = \text{vimo}_h^f(z_{1},...,z_{r}) = (−1)^{w−1}\text{vimo}_h^r(z_{1},...,z_{r}),$$

$$= (−1)^{r−1}\text{vimo}_h^r(z_{1},...,z_{r}) = (−1)^{r−1}\text{vimo}_h^r(0,z_{1},...,z_{r}).$$

Here in the third equality we use that $\text{vimo}_h^r$ is homogeneous with degree $w−r$ and the fourth equality is by (2.3). \[\square\]

Definition 2.6. Let $\mathfrak{d}\mathfrak{t}\mathfrak{e}\mathfrak{r}$ be the set of tangential derivation of $\mathbb{L}$, the derivation $D_{(F_{r},G)}$ of $\mathbb{L}$ defined by $x \mapsto [x,G]$ and $y_{\sigma} \mapsto [y_{\sigma},F_{\sigma}]$ for some $F_{\sigma},G \in \mathbb{L}$. It forms a Lie algebra by the bracket

$$\{D_{(F_{1}^{(1)},G^{(1)})},D_{(F_{2}^{(1)},G^{(1)})}\} = D_{(F_{1}^{(2)},G^{(1)})} \circ D_{(F_{2}^{(2)},G^{(1)})} - D_{(F_{1}^{(2)},G^{(1)})} \circ D_{(F_{2}^{(2)},G^{(1)})}. \text{ (2.5)}$$

The action of $\Gamma$ on $\mathbb{L}$ induces the $\Gamma$-action on $\mathfrak{d}\mathfrak{t}\mathfrak{e}\mathfrak{r}$. We denote its invariant part by $\mathfrak{d}\mathfrak{t}\mathfrak{e}\mathfrak{r}^{\Gamma}$. We mean $\mathfrak{s}\mathfrak{d}\mathfrak{e}\mathfrak{r}$ to be the set of special derivations, tangential derivations such that $D_{(F_{r},G)}(x + \sum_{\sigma}y_{\sigma}) = 0$ and $\mathfrak{s}\mathfrak{d}\mathfrak{e}\mathfrak{r}^{\Gamma}$ to be its intersection with $\mathfrak{d}\mathfrak{t}\mathfrak{e}\mathfrak{r}^{\Gamma}$, both of which forms a Lie subalgebra of $\mathfrak{d}\mathfrak{t}\mathfrak{e}\mathfrak{r}$. We put $\mathbf{m}\mathbf{t} := \oplus_{(w,d)\neq (1,0)}\mathbb{L}_{w,d}$. It forms a Lie algebra by the bracket

$$\{f_{1},f_{2}\} = D_{(\sigma(f_{1}),0)}(f_{2}) - D_{(\sigma(f_{2}),0)}(f_{1}) + [f_{1},f_{2}], \text{ (2.6)}$$

in other words, $D_{(\sigma(f_{1}),0)}(f_{2}) := [D_{(\sigma(f_{1}),0)},D_{(\sigma(f_{2}),0)}].$

We occasionally regard $\mathbf{m}\mathbf{t}$ as a Lie subalgebra of $\mathfrak{d}\mathfrak{t}\mathfrak{e}\mathfrak{r}$ by $f \mapsto D_{(\sigma(f)),0}$. We note that the condition $[\mathbf{K},\mathbf{V}]$ is equivalent to $D_{(\sigma(F)),0} \in \mathfrak{s}\mathfrak{d}\mathfrak{e}\mathfrak{r}^{\Gamma}$.

We regard $\mathfrak{s}\mathfrak{d}\mathfrak{e}\mathfrak{r}^{\Gamma}$ and $\mathbf{m}\mathbf{t}$ as filtered graded Lie algebras by encoding them with $\text{Fil}_{\mathfrak{d}\mathfrak{t}\mathfrak{e}\mathfrak{r}^{\Gamma}} = \{D_{(\sigma(F)),G} \in \mathfrak{s}\mathfrak{d}\mathfrak{e}\mathfrak{r}^{\Gamma} \mid F \in \text{Fil}_{\mathfrak{d}\mathfrak{t}\mathfrak{e}\mathfrak{r}^{\Gamma}}\mathbb{L}_{w,d}\}$ and $\text{Fil}_{\mathfrak{d}\mathfrak{t}\mathfrak{e}\mathfrak{r}^{\Gamma}} = \{D_{(\sigma(F)),0} \in \mathbf{m}\mathbf{t} \mid F \in \text{Fil}_{\mathfrak{d}\mathfrak{t}\mathfrak{e}\mathfrak{r}^{\Gamma}}\mathbb{L}_{w,d}\}$. The following is required in the next section.
Lemma 2.7. We have a natural graded Lie algebra homomorphism
\[ \text{res} : \text{gr}_D(\mathfrak{der}^\Gamma) \to \text{mt} \]
sending \( D_{(\tau(F)),G} \mapsto D_{(\tau(F)),0} \).

Proof. Let \( D_i = D_{(\tau(F)),G_i} \in \text{Fil}^{d_i}_D\mathfrak{der}^\Gamma_{w_i} \) for \( i = 1,2 \). Put \( D_3 = [D_1,D_2] \). Since it belongs to \( \mathfrak{der}^\Gamma \), \( D_3 \) is expressed as \( D_{(\tau(F)),G_3} \) for some \( F_3 \in L \) and \( G_3 \in L \).

By [KVI], we have \( \tau(F_i) \in \text{Fil}^{d_i+1}_D L_{w_i}, G_i \in \text{Fil}^{d_i+1}_D L_{w_i} \) (\( i = 1,2 \)). Since \( F_3 \) is calculated to be
\[ F_3 = D_{(\tau(F_1)),G_1}(F_2) - D_{(\tau(F_2)),G_2}(F_1) + [F_1,F_2] \]
by (2.5), we see that \( F_3 \in \text{Fil}^{d_1+d_2}_D L_{w_1+w_2} \). The residue class \( \bar{F}_3 \in \text{gr}^{d_1+d_2}_D L_{w_1+w_2} \) is calculated as
\[ \bar{F}_3 = D_{(\tau(F_1)),0}(\bar{F}_2) - D_{(\tau(F_2)),0}(\bar{F}_1) + [\bar{F}_1,\bar{F}_2], \]
where \( \bar{F}_1 \in \text{gr}^{d_1}_D L_{w_1} \) and \( \bar{F}_2 \in \text{gr}^{d_2}_D L_{w_2} \) are the residue classes of \( F_1 \) and \( F_2 \). Therefore our map is a Lie algebra homomorphism. \( \square \)

Proposition 2.8. The map \( ma \) sending \( h \to ma_h \) induces a filtered graded Lie algebra isomorphism
\[ ma : \text{mt} \simeq \text{ARI}(\Gamma)_{\text{fin.pol}} \]
where \( \text{ARI}(\Gamma)_{\text{fin.pol}} \) (cf. Definition 1.3) is the Lie algebra (cf. Proposition 1.15) equipped with the \( \text{ari}_n \)-bracket.

Proof. It can be proved completely in a same way to that of [Sch12] Theorem 3.4.2. \( \square \)

We prepare the following technical lemma which is required to the proof of a reformulation of [KVI] in Lemma 2.10

Lemma 2.9. Let \( H \in L_w \) with \( w \geq 2 \). Assume that \( H \) has no words starting with any \( y_\sigma \) and ending in any \( y_\tau \). Then there exists \( G \in L_{w-1} \) such that \( H = [x,G] \).

Proof. The proof goes on the same way to the proof of [Sch12] Proposition 2.2]. Define the derivation \( \partial_x \) of \( A \) sending \( x \mapsto 1 \) and \( y_\sigma \mapsto 0 \) and the \( \mathbb{Q} \)-linear endomorphism \( \text{sec} \) of \( A \) by \( \text{sec}(h) := \sum_{i \geq 0} \frac{(-1)^i}{i!} \partial_x^i(h)x^i \) for \( h \in A \). Let us write \( H = H_x x + \sum_{\sigma} H_{y_\sigma} y_\sigma \). Then by our assumption, we have \( P \in A \) such that \( xP = \sum_{\sigma} H_{y_\sigma} y_\sigma \). Then by [R02] Proposition 4.2.2 and \( \partial_x^i(xP) = i \partial_x^{i-1}(P) + x \partial_x^i(P) \), we have
\[ H = \text{sec}(\sum_{\sigma} H_{y_\sigma} y_\sigma) = \text{sec}(xP) = i \partial_x^i(xP)x^i = \sum_{i \geq 1} \frac{(-1)^i}{(i-1)!} \partial_x^{i-1}(P)x^i + x \sum_{i \geq 0} \frac{(-1)^i}{i!} \partial_x^i(P)x^i = - \text{sec}(P)x + x \text{sec}(P) = [x,\text{sec}(P)]. \]

It remains to show that \( G = \text{sec}(P) \) is in \( L_w \), which follows from exactly the same arguments to the last half of the proof of [Sch12] Proposition 2.2]. \( \square \)
Let \( F = F(x; (y_\sigma)) \in L \). As in \[\text{Sch12} \], we put
\[
(2.10) \quad f(x; (y_\sigma)) = F(z; (y_\sigma)) \quad \text{and} \quad \tilde{f}(x; (y_\sigma)) = f(x; (-y_\sigma))
\]
with \( z = -x - \sum_\sigma y_\sigma \).

The following generalizes the equivalence between (i) and (v) in \[\text{Sch12} \, \text{Theorem 2.1}]\.

**Lemma 2.10.** Let \( F \in L_w \) with \( w \geq 1 \). Then saying \( [\text{KV1}] \) for \( F \) is equivalent to saying that
\[
(2.11) \quad \{ \tilde{f}_{y_\gamma} + \tilde{f}_x \} = (-1)^{w-1} \gamma \cdot \text{anti} \{ \tilde{f}_{y_{-1}} + \tilde{f}_x \}
\]
for all \( \gamma \in \Gamma \).

**Proof.** Firstly, we show that \( [\text{KV1}] \) for \( F \) is equivalent to \( \alpha(F)_{y_\beta} = \beta(F)^{y_\alpha} \) for any \( \alpha, \beta \in \Gamma \). Set \( H = \sum_{\alpha, \beta} [y_\sigma, \sigma(F)] \). Then we have
\[
(2.12) \quad H = \sum_{\alpha, \beta} y_\alpha \{ \alpha(F)_{y_\beta} - \beta(F)^{y_\alpha} \} y_\beta + \sum_\alpha y_\alpha \alpha(F)_x x - \sum_\beta x \beta(F)^x y_\beta.
\]

- Assume \( [\text{KV1}] \) for \( F \). Then
\[
(2.13) \quad H = Gx - xG.
\]

So \( H \) has no words starting with any \( y_\sigma \) and ending in any \( y_\tau \). By (2.12), we have \( \alpha(F)_{y_\beta} = \beta(F)^{y_\alpha} \).

- Conversely assume \( \alpha(F)_{y_\beta} = \beta(F)^{y_\alpha} \). Then \( H \) has no words starting with any \( y_\sigma \) and ending in any \( y_\tau \). By Lemma 2.9, there is a \( H = G \in L_w \) such that \( H = [G, x] \). Whence we get \( [\text{KV1}] \).

Secondly, by \( \alpha(F) \in L_w \), we have \( \alpha(F)_{y_0} = (-1)^{w-1} \text{anti} (\alpha(F)_{y_0}) \). Therefore \( \alpha(F)_{y_0} = \beta(F)^{y_0} \) is equivalent to \( \alpha(F)_{y_0} = (-1)^{w-1} \text{anti} (\beta(F)_{y_0}) \).

Lastly, by (2.10) we have \( f_{y_\alpha}(z; (y_\sigma)) - f_x(z; (y_\sigma)) = F_{y_\alpha} \). Hence
\[
\alpha(F)_{y_\beta} = \alpha(f_{y_{-1, \beta}}(z; (y_\sigma)) - f_x(z; (y_\sigma))) \text{.}
\]
Whence \( \alpha(F)_{y_\beta} = (-1)^{w-1} \text{anti} (\beta(F)_{y_\alpha}) \) is equivalent to
\[
\alpha \{ f_{y_{-1, \beta}}(z; (y_\sigma)) - f_x(z; (y_\sigma)) \} = (-1)^{w-1} \text{anti} \{ \beta \{ f_{y_{-1, \beta}}(z; (y_\sigma)) - f_x(z; (y_\sigma)) \} \}
\]
from which our claim follows. \( \square \)

The following reformulation is suggested by the arguments in \[\text{Sch12} \, \text{Appendix A}]\.

**Lemma 2.11.** Let \( \tilde{f} \in L_w \) with \( w > 1 \). Then \((\text{2.11})\) for all \( \gamma \in \Gamma \) is equivalent to the following senary relation \[\text{3.64}(\text{cf. } \text{Ec11})]\)
\[
(2.14) \quad \text{teru}(M)^r = \text{push} \circ \text{mantar} \circ \text{teru} \circ \text{mantar}(M)^r
\]
for \( 1 \leq r \leq w \) with \( M = ma_{\tilde{f}} \).

**Proof.** Because \( M = ma_{\tilde{f}} \) is mantar invariant for \( \tilde{f} \in L_w \) by (2.1), the senary relation \( (2.14) \) is equivalent to
\[
(2.15) \quad \text{swap} \circ \text{teru}(M)^r = \text{swap} \circ \text{push} \circ \text{mantar} \circ \text{teru}(M)^r.
\]

\[\text{It is because it consists of 6 terms, 3 terms on each hand sides.}\]
For simplicity, we denote $\sigma_i, \ldots, \sigma_j$ by $\sigma_{i,j}$ for $1 \leq i \leq j \leq r$. By definition, its left hand side is calculated to be

\begin{equation}
\vimo f_r \left( \begin{array}{cccc}
0, & \nu_r, & \ldots, & \nu_2, \\
\sigma_{1,r}, & \ldots, & \sigma_{1,2}, & \sigma_1
\end{array} \right)
+ \frac{1}{v_1 - v_2} \left\{ \vimo f_r^{-1} \left( \begin{array}{cccc}
0, & \nu_r, & \ldots, & \nu_2, \\
\sigma_{1,r}, & \ldots, & \sigma_{1,3}, & \sigma_{1,2}
\end{array} \right) - \vimo f_r^{-1} \left( \begin{array}{cccc}
0, & \nu_r, & \ldots, & \nu_2, \\
\sigma_{1,r}, & \ldots, & \sigma_{1,3}, & \sigma_1
\end{array} \right) \right\}.
\end{equation}

By $\vimo f_r \left( \begin{array}{cccc}
\tau_0, & \ldots, & \tau_r, \\
\sigma_1, & \ldots, & \sigma_r
\end{array} \right) = (-1)^{w-r} \vimo f_r \left( \begin{array}{cccc}
-\tau_0, & \ldots, & -\tau_r, \\
\sigma_1, & \ldots, & \sigma_r
\end{array} \right)$ and definition, its right hand side is calculated to be

\begin{equation}
(-1)^{w-1} \left\{ \vimo f_r \left( \begin{array}{cccc}
0, & \nu_r - v_2, & \ldots, & \nu_r - \nu_2, \\
\sigma_{2,2}, & \ldots, & \sigma_{2,r}, & \sigma_2, \sigma_{1,2}
\end{array} \right)
+ \frac{1}{v_1} \left\{ \vimo f_r^{-1} \left( \begin{array}{cccc}
0, & \nu_r - v_2, & \ldots, & \nu_r - \nu_2, \\
\sigma_{2,2}, & \ldots, & \sigma_{2,r}, & \sigma_2, \sigma_{1,2}
\end{array} \right) - \vimo f_r^{-1} \left( \begin{array}{cccc}
0, & \nu_r - v_2, & \ldots, & \nu_r - \nu_2, \\
\sigma_{2,2}, & \ldots, & \sigma_{2,r}, & \sigma_2, \sigma_{1,2}
\end{array} \right) \right\} \right\}.
\end{equation}

By $\vimo f_r \left( \begin{array}{cccc}
\tau_0, & \ldots, & \tau_r, \\
\sigma_1, & \ldots, & \sigma_r
\end{array} \right) = \vimo f_r \left( \begin{array}{cccc}
\tau_0, & \ldots, & \tau_r, \\
\sigma_1, & \ldots, & \sigma_r
\end{array} \right)$ in (2.3) and the change of variables

$\nu_0 = \nu_1, \nu_1 = \nu_r - \nu_1, \ldots, \nu_{r-1} = \nu_2 - \nu_1$ and $\gamma = \sigma_1, \tau_1 = \sigma_1, \ldots, \tau_{r-1} = \sigma_1, 2$, the formula (2.10) is equal to

\begin{equation}
\vimo f_r \left( \begin{array}{cccc}
\tau_0, & \ldots, & \tau_{r-1}, 0 \\
\sigma_1, & \ldots, & \sigma_r
\end{array} \right) + \frac{1}{\nu_1 - \nu_r} \left\{ \vimo f_r^{-1} \left( \begin{array}{cccc}
\tau_0, & \ldots, & \tau_{r-1}, 1 \\
\sigma_1, & \ldots, & \sigma_r
\end{array} \right) - \vimo f_r^{-1} \left( \begin{array}{cccc}
\tau_0, & \ldots, & \tau_{r-1}, 0 \\
\sigma_1, & \ldots, & \sigma_r
\end{array} \right) \right\},
\end{equation}

and (2.14) is equal to

\begin{equation}
(-1)^{w-1} \left\{ \vimo f_r \left( \begin{array}{cccc}
\nu_{r-1}, & \ldots, & \nu_0, \\
\gamma - 1, & \gamma - 1, & \ldots, & \gamma - 1, \nu_{r-1}, \nu_1, \gamma \end{array} \right)
+ \frac{1}{\nu_1} \left\{ \vimo f_r^{-1} \left( \begin{array}{cccc}
\nu_{r-1}, & \ldots, & \nu_0, \\
\gamma - 1, & \gamma - 1, & \ldots, & \gamma - 1, \nu_{r-1}, \nu_1, \gamma \end{array} \right) - \vimo f_r^{-1} \left( \begin{array}{cccc}
\nu_{r-1}, & \ldots, & \nu_0, \\
\gamma - 1, & \gamma - 1, & \ldots, & \gamma - 1, \nu_{r-1}, \nu_1, \gamma \end{array} \right) \right\} \right\}.
\end{equation}

Whence (2.14) is equivalent to (2.18) = (2.19) for $1 \leq r \leq w$ and $\tau_1, \ldots, \tau_{r-1}, \gamma \in \Gamma$.

On the other hand, (2.11) for all $\gamma \in \Gamma$ is equivalent to \footnote{Here $\tilde{f}_x$ and $\tilde{f}_y$ mean $(\tilde{f}_x)^r$ and $(\tilde{f}_y)^r$.}
Hence, by [2.21] and [2.22], the left hand side of [2.20] is calculated to be

\[(2.23) \quad \text{vimo}_{f+1}^{\tau_r} \left( \frac{z_0, \ldots, z_0, \ldots, z_0, \ldots, z_0}{\gamma_0, \gamma_1, \ldots, \gamma_0} \right) + \frac{1}{z_0} \left\{ \text{vimo}_{f_r}^{\tau_r} \left( \frac{z_0, \ldots, z_0, \ldots, z_0, \ldots, z_0}{\gamma_0, \gamma_1, \ldots, \gamma_0} \right) - \text{vimo}_{f_r}^{\tau_r} \left( \frac{z_0, \ldots, z_0, \ldots, z_0, \ldots, z_0}{\gamma_0, \gamma_1, \ldots, \gamma_0} \right) \right\}, \]

and the left hand side of [2.20] is calculated to be

\[(2.24) \quad (-1)^w \left[ \text{vimo}_{f+1}^{\tau_r} \left( \frac{z_0, \ldots, z_0, \ldots, z_0, \ldots, z_0}{\gamma_0, \gamma_1, \ldots, \gamma_0} \right) + \frac{1}{z_0} \left\{ \text{vimo}_{f_r}^{\tau_r} \left( \frac{z_0, \ldots, z_0, \ldots, z_0, \ldots, z_0}{\gamma_0, \gamma_1, \ldots, \gamma_0} \right) - \text{vimo}_{f_r}^{\tau_r} \left( \frac{z_0, \ldots, z_0, \ldots, z_0, \ldots, z_0}{\gamma_0, \gamma_1, \ldots, \gamma_0} \right) \right\} \right]. \]

So (2.11) for all \( r \in \Gamma \) is equivalent to (2.23)=(2.24) for 1 \( r \leq w \) and \( \gamma \in \Gamma \). This is nothing but (2.13)=(2.14) for \( r-1 \) instead of \( r \). Therefore we get the equivalence between (2.11) and (2.14).

Next we consider the condition [KV2].

**Lemma 2.12.** Let \( F \in \mathbb{L}_w \). Then (KV2) for \( F \) is equivalent to

\[(2.25) \quad \sum_{i \in \mathbb{Z}/w} a(q \circ \pi_Y(f)) : \frac{c_i, c_{i+1}, \ldots, c_{i+r-1}, 0}{\sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{i+r}} = 0 \]

for each 1 \( r \leq w \), \( (\sigma_1, \ldots, \sigma_r) \in \Gamma^r \) and \( (e_0, \ldots, e_r) \in E^r_w \) with \( f \) as in (2.10).

**Proof.** It is immediate that (KV2) for \( F \) is equivalent to

\[(2.26) \quad \text{tr}(q \circ \pi_Y(f)) = 0. \]

For \( W = u_1 \cdots u_w \in \mathcal{A}_w \) with \( u_i = x \) or \( y_{\sigma} \) (\( \sigma \in \Gamma \)), we define its cyclic permutation by \( \text{cy}(W) := u_2 \cdots u_w u_1 \in \mathcal{A}_w \). We put \( c(W) \) to be the number of its cycles. It divides \( w \). Then the natural pairing in \( \text{Cyc}(A) \) is calculated by the one in \( A \) as follows: When \( W = x^{e_0} y_{\sigma_1} \cdots x^{e_{r-1}} y_{\sigma_r} \), we have

\[(\text{tr}(q \circ \pi_Y(f)) \mid \text{tr}(W))_{\text{Cyc}(A)} = \langle q \circ \pi_Y(f) \mid \sum_{i=0}^{c(W)-1} \text{cy}^i(W) \rangle_{\mathcal{A}} = \frac{c(W)}{w} \langle q \circ \pi_Y(f) \mid \sum_{i=0}^{w-1} \text{cy}^i(W) \rangle_{\mathcal{A}} = \frac{c(W)}{w} \sum_{i \in \mathbb{Z}/w} a(q \circ \pi_Y(f), \frac{e_i, \ldots, e_{i+r-1}, 0}{\sigma_{i+1}, \ldots, \sigma_{i+r}}). \]

So we get the claim.

**Lemma 2.13.** Let \( f \in \mathbb{L}_w \). Then (2.26) for \( f \) is equivalent to the pus-neutrality (1.3) for \( \text{swap}(ma_f) \), i.e.

\[(2.27) \quad \sum_{i \in \mathbb{Z}/w} \text{pus}^i \circ \text{swap}(M)^r \left( \frac{e_1, \ldots, e_r}{\sigma_1, \ldots, \sigma_r} \right) = 0 \]

with \( M = ma_f \) for all 1 \( r \leq w \) and \( \sigma_i \in \Gamma \) (1 \( i \leq r \)).

**Proof.** We decompose \( f = \sum_r \hat{f}^r \) and describe \( \hat{f}^r \in \mathbb{L}_{w,r} \) as in (2.2). Then

\[ma_f^{(u_1, \ldots, u_r)} = \sum_{(e_0, \ldots, e_r) \in E^r_w} \langle a(\hat{f} : \frac{e_0, \ldots, e_{r-1}}{\sigma_1, \ldots, \sigma_r})u_1, (u_2)^{e_2} \cdots (u_1 + \cdots + u_r)^{e_r} \rangle. \]

By \( \hat{f}^r \in \mathbb{L}_{w,r} \), we have

\[a(\hat{f} : \frac{e_r}{\sigma_r}) = (-1)^{w+1} a(\hat{f} : \frac{e_r}{\sigma_r}). \]
because \( \text{anti}(\tilde{f}^r) = (-1)^{w+1} \tilde{f}^r \). So
\[
\text{swap} \circ \text{ma}_f^r(\sigma_1,\ldots,\sigma_r) = \sum_{e_0=0}^{w+1} a(\tilde{f} : (\sigma_1,\ldots,\sigma_r) \in \mathcal{E}_w^r)
\]
\[
= (-1)^{w+1} \sum_{(e_0,\ldots,e_r) \in \mathcal{E}_w^r} a(\tilde{f} : (\sigma_1,\ldots,\sigma_r) \in \mathcal{E}_w^r)
\]
\[
= (-1)^{w+1} \sum_{(e_0,\ldots,e_r) \in \mathcal{E}_w^r} a(\tilde{f} : (\sigma_1,\ldots,\sigma_r) \in \mathcal{E}_w^r)
\]
\[
= (-1)^{w+r+1} \sum_{(e_0,\ldots,e_r) \in \mathcal{E}_w^r} a(\tilde{f} : (\sigma_1,\ldots,\sigma_r) \in \mathcal{E}_w^r)
\]
\[
= (-1)^{w+r+1} \sum_{(e_0,\ldots,e_r) \in \mathcal{E}_w^r} a(\tilde{f} : (\sigma_1,\ldots,\sigma_r) \in \mathcal{E}_w^r)
\]
\[
= (-1)^{w+r+1} \sum_{(e_0,\ldots,e_r) \in \mathcal{E}_w^r} a(\tilde{f} : (\sigma_1,\ldots,\sigma_r) \in \mathcal{E}_w^r)
\]

Here in the last equality we use
\[
a(\text{swap} \circ \text{ma}_f^r(\sigma_1,\ldots,\sigma_r)) = a(\text{swap} \circ \text{ma}_f^r(\sigma_1,\ldots,\sigma_r))
\]
for any \( h \in \pi(A) \). Therefore we obtain
\[
\sum_{i \in \mathbb{Z}/r\mathbb{Z}} \text{push}^i \circ \text{swap} \circ \text{ma}_f^r(\sigma_1,\ldots,\sigma_r) = (-1)^{w+r} \sum_{(e_0,\ldots,e_r) \in \mathcal{E}_w^r} a(\text{swap} \circ \text{ma}_f^r(\sigma_1,\ldots,\sigma_r))
\]
It is immediate to see that it is equivalent to (2.24).

The following definition of the mould version of \( \text{fur} \) is suggested by our previous lemmas.

**Definition 2.14.** We define the \( \mathbb{Q} \)-linear space \( \text{ARI}(\Gamma)_{\text{senu}} \) to be the subset of moulds \( M \) in \( \text{ARI} \) which satisfy the senary relation (2.14) and whose swap satisfy the pus-neutrality (2.27).

The \( \mathbb{Q} \)-linear space \( \text{ARI}(\Gamma)_{\text{senu}} \) is filtered by the depth filtration \( \{ \text{Fil}^m_{\text{D}} \}_{m} \).

We note that \( \text{Fil}^m_{\text{D}} \text{ARI}(\Gamma)_{\text{senu}} \cap \text{ARI}(\Gamma)_{\text{al}} \) is identified with the finite and depth >1-part of \( \text{ARI}(\Gamma)_{\text{al}} \) in [RS, Proposition 28] when \( \Gamma = \{ e \} \). It is because their formula (79), actually (81), agrees with (2.14) for a mould in \( \text{ARI}(\Gamma)_{\text{al}} \) by Lemma 3.7.

There is an embedding of \( \mathbb{Q} \)-linear space
\[
(2.28) \quad \text{gr}_{\text{D}} \text{ARI}(\Gamma)_{\text{senu}} \hookrightarrow \text{ARI}(\Gamma)_{\text{push}}
\]
that is, the associated graded quotient \( \text{gr}_{\text{D}} \text{ARI}(\Gamma)_{\text{senu}} \) of the filtered \( \mathbb{Q} \)-linear space \( \text{ARI}(\Gamma)_{\text{senu}} \) is embedded to \( \text{ARI}(\Gamma)_{\text{push}} \) introduced in Definition 1.26.
Theorem 2.15. The map sending $F \in \mathcal{A} \mapsto m_a \tilde{\psi} f \in M(\mathcal{F}; \Gamma)$ induces an isomorphism of filtered $\mathbb{Q}$-linear spaces
\begin{equation}
\text{trv}(\Gamma)_{\bullet} \simeq \text{ARI}(\Gamma)_{\text{sena/pusnu}} \cap \text{ARI}(\Gamma)_{\text{fin,pol}}^{\text{fin,pol}}.
\end{equation}

Proof. The restriction of our map decomposes as
\begin{equation}
F \in \text{trv}(\Gamma)_{\bullet} \mapsto \tilde{f} \in \mathfrak{m} \mapsto m_a \tilde{\psi} f \in \text{ARI}(\Gamma)_{\text{fin,pol}}^{\text{fin,pol}}.
\end{equation}
By Proposition 2.8 we see that it gives an isomorphism $\mathfrak{m} \simeq \text{ARI}(\Gamma)_{\text{al}}^{\text{fin,pol}}$ as (actually filtered) $\mathbb{Q}$-linear spaces. Thus we get the claim by our previous lemmas. \hfill \Box

Remark 2.16. When $\Gamma = \{e\}$, the above isomorphism (2.29) can be recovered by the composition of the isomorphisms (29), (71) and Proposition 28 in [RS].

The authors are not sure if the bigger space $\text{ARI}(\Gamma)_{\text{sena/pusnu}}$ is equipped with a structure of Lie algebra under the $\text{ari}_{\text{al}}$-bracket or not although we show that a related space $\text{ARI}(\Gamma)_{\text{push/pusnu}}$ forms a Lie algebra in Theorem 1.32.

2.2. Kashiwara-Vergne bigraded Lie algebra. Based on our arguments in the previous subsection, we introduce a $\Gamma$-variant $\text{ltkv}(\Gamma)_{\bullet \bullet}$ of the Kashiwara-Vergne bigraded Lie algebra $\text{ltkv}(\Gamma)_{\bullet}$ in Definition 2.17 and give a mould theoretical interpretation in Theorem 2.22. As a corollary we show that it forms a Lie algebra in Theorem 2.23.

Definition 2.17. Kashiwara-Vergne bigraded Lie algebra is defined to be the bigraded $\mathbb{Q}$-linear space $\text{ltkv}(\Gamma)_{\bullet \bullet} = \bigoplus_{w > 1, d > 0} \text{ltkv}(\Gamma)_{w,d}$, where $\text{ltkv}(\Gamma)_{w,d}$ is the $\mathbb{Q}$-linear space consisting of $\bar{F} \in \text{gr}^d_{\mathcal{D}} \mathbb{L}_w$ whose lift $F \in \text{Fil}^2_{\mathcal{D}} \mathbb{L}_w$ satisfies the following relations, ‘the leading-terms’ of (KV1) and (KV2):
\begin{align}
(LKV1) \quad [x, G] + \sum_{\tau \in \Gamma} [y_{\tau}, \tau(F)] & \equiv 0 \mod \text{Fil}^{d+2}_{\mathcal{D}} \mathbb{L}_{w+1}, \\
(LKV2) \quad \text{tr} \circ \psi \circ \pi_Y(F) & \equiv 0 \mod \text{tr} \text{Fil}^{d+1}_{\mathcal{D}} \mathbb{A}_w.
\end{align}
We note that such $G \in \mathbb{L}_w$ is in $\text{Fil}^{d+1}_{\mathcal{D}} \mathbb{L}_w$ and is uniquely determined modulo $\text{Fil}^{d+2}_{\mathcal{D}} \mathbb{L}_w$ by (LKVI). We note that, by (LKVI) and dim $\mathbb{L}_{w,1} = 1$, we have $\text{ltkv}_{w,d} = \{0\}$ for $d = 1$.

Remark 2.18. When $\Gamma = \{e\}$, our definition of $\text{ltkv}(\Gamma)_{\bullet \bullet}$ does not agree with that of $\text{ltkv}$ in [RS Definition 5 and 10], which is constructed as a Lie polynomial version of $\text{ARI}(\Gamma)_{\text{push/pusnu}}$. However their depth>1-parts are eventually isomorphic to our depth>1-parts via Theorem 2.22.

By definition, there is an inclusion of bigraded $\mathbb{Q}$-linear spaces
\begin{equation}
\text{gr}_{\mathcal{D}} \text{trv}(\Gamma)_{\bullet} \hookrightarrow \text{ltkv}(\Gamma)_{\bullet \bullet},
\end{equation}
which generalizes [RS Proposition 2], that is, the associated graded $\mathbb{Q}$-linear space $\text{gr}_{\mathcal{D}} \text{trv}(\Gamma)_{\bullet}$ of the filtered Lie algebra $\text{trv}(\Gamma)_{\bullet}$ is embedded to $\text{ltkv}(\Gamma)_{\bullet \bullet}$. We do not know if it is an isomorphism.

\footnote{It looks that there is a slip on the isomorphism on [RS Proposition 28] because its right hand side is completed by depth while its left hand side is not.}
For $F \in \text{Fil}^d L_w$, we put $f$ to be the element in $L_{w,d}$ corresponding to $(-1)^{w-d}F \in \text{gr}^{w}_{d} L_w$ under the natural identification $\text{gr}^{w}_{d} L_w \simeq L_{w,d}$. By abuse of notation, 
\[(2.32) \quad f = (-1)^{w-d}F, \]

We write $f = f_\sigma x + \sum_{\sigma} f_{\gamma, y_\sigma} y_\sigma$. We also put 
\[\tilde{f} = f(x, (-y_\sigma)_\sigma)\]

and write $\tilde{f} = \tilde{f}_\sigma x + \sum_{\sigma} \tilde{f}_{\gamma, y_\sigma}$.

The following is a bigraded variant of Lemma 2.10.

**Lemma 2.19.** Let $F \in \text{Fil}^d L_w$ for $w \geq 1$. Then \((LKV1)\) for $F$ is equivalent to 
\[(2.33) \quad \tilde{f}_{y_\gamma} = (-1)^{w-1}\gamma(\text{anti}(\tilde{f}_{y_{\gamma - 1}})) \]

for any $\gamma \in \Gamma$.

**Proof.** The proof goes similarly to Lemma 2.10.

Firstly, we show that \((LKV1)\) for $F$ is equivalent to $\alpha(F)_{y_\beta} \equiv \beta(F)^{y_\alpha}$ mod $\text{Fil}^d L_{w-1}$. Set $H = \sum_{\sigma} [y_\sigma, \sigma(F)] \in \text{Fil}^{d+1} L_{w+1}$.

Assume \((LKV1)\) for $F$. Then $H \equiv GX - xG$ mod $\text{Fil}^{d+2} L_{w+1}$. So the depth $(d + 1)$-part of $H$ has no words starting and ending in any $y_\sigma$. By (2.32), we have $\alpha(F)_{y_\beta} \equiv \beta(F)^{y_\alpha}$ mod $\text{Fil}^d L_{w-1}$.

Conversely assume $\alpha(F)_{y_\beta} \equiv \beta(F)^{y_\alpha}$ mod $\text{Fil}^d L_{w-1}$. Then the depth $(d + 1)$-part of $H$ has no words starting and ending in any $y_\sigma$. By Lemma 2.9, there is a $G \in L_w$ which express this part as $[G, x]$, i.e. $H \equiv [G, x]$ mod $\text{Fil}^{d+2} L_{w+1}$. Whence we get \((LKV1)\).

Secondly, by $\alpha(F) \in L_1$, we have $\alpha(F)_{y_\beta} = (-1)^{w-1}\gamma(\text{anti}(\alpha(F)_{y_\beta}))$. Therefore $\alpha(F)_{y_\beta} \equiv \beta(F)^{y_\alpha}$ mod $\text{Fil}^d L_{w-1}$ is equivalent to 
\[\alpha(F)_{y_\beta} \equiv (-1)^{w-1}\gamma(\text{anti}(\beta(F)_{y_\alpha})) \quad \text{mod} \quad \text{Fil}^d L_{w-1}.\]

Lastly, by (2.32) and $\alpha(F)_{y_\beta} = \alpha(F)_{y_{\beta - 1}}$, it is equivalent to $\alpha(\tilde{f}_{y_{\gamma - 1}}) = (-1)^{w-1}\gamma(\text{anti}(\beta(f_{y_{\gamma - 1}})))$, so for $\tilde{f}_{y_\gamma}$.

The following might be regarded as a bigraded variant of Lemma 2.11.

**Lemma 2.20.** Let $\tilde{f} \in L_{w,d}$ with $w > 1$. Then (2.33) for any $\gamma \in \Gamma$ is equivalent to push-invariance \((1.3)\) for $M = ma_{\tilde{f}}$.

**Proof.** The proof goes similarly to Lemma 2.11. The condition (2.33) for $\gamma \in \Gamma$ is reformulated to 
\[\text{vimo}_{\tilde{f}}^{\tau+1} (\tau_{1, \ldots, \tau_{r}}, \tau_{r}, \ldots, \tau_{r}) = (-1)^{w-1}\gamma(\text{anti}(\tilde{f}_{y_{\gamma - 1}}), \tau_{1, \ldots, \tau_{r}})\]

for $1 \leq r \leq w$, which is equivalent to 
\[\text{vimo}_{\tilde{f}}^{r+1} (\tau_{1, \ldots, \tau_{r}}, \tau_{r}, \ldots, \tau_{r}) = (-1)^{w-1}\gamma(\text{anti}(\tilde{f}_{y_{\gamma - 1}}), \tau_{1, \ldots, \tau_{r}, \tau_{r}}, \gamma_{1, \ldots, \gamma_{r-1}})\]

for $1 \leq r + 1 \leq w$.

On the other hand, (1.3) can be shown to be equivalent to 
\[\text{vimo}_{\tilde{f}}^{r} (\tau_{1, \ldots, \tau_{r}}, \tau_{r}, \ldots, \tau_{r}) = (-1)^{w-1}\gamma(\text{anti}(\tilde{f}_{y_{\gamma - 1}}), \tau_{1, \ldots, \tau_{r}}, \gamma_{1, \ldots, \gamma_{r}}, \gamma_{1, \ldots, \gamma_{r-1}})\]

for $1 \leq r \leq w$, which turns to be equivalent to (2.34).
The following is a bigraded variant of Lemma 2.13.

**Lemma 2.21.** Then (LK2) for \( F \in \operatorname{Fil}^D D_L \) is equivalent to the pus-neutrality (1.9) for \( M = \text{swap} (\text{maj}) \), i.e. (2.27) for all \( r \geq 1 \).

**Proof.** We note that actually only the terms for \( r = d \) contributes in the above equation. Decompose \( \tilde{f} \) as in (2.2). Then by the arguments in the proof of Lemma 2.13 (2.27) is equivalent to

\[
\sum_{i \in \mathbb{Z}/r\mathbb{Z}} a(q \circ \pi_Y (f) : e_{r+1} \ldots e_{r+1}, e_{r} \ldots e_{r}) = 0
\]

for all \( (\sigma_1, \ldots, \sigma_r) \in \Gamma^r \) and \( (e_0, \ldots, e_{r-1}, 0) \in \mathcal{E}_c^r \). It is nothing but

\[
\text{tr} \circ q \circ \pi_Y (f) = 0,
\]

which is equivalent to (LK2). \( \square \)

**Theorem 2.22.** The map sending \( \bar{F} \in \operatorname{gr} D \mapsto \text{maj} \) induces an isomorphism of bigraded \( \mathbb{Q} \)-linear spaces

\[
\mathfrak{tr} (\Gamma) \simeq \operatorname{ARI}(\Gamma)_{\text{push/pusnu}} \cap \operatorname{ARI}(\Gamma)_{\text{fin, pol}}.
\]

**Proof.** Our map decomposes as

\[
(2.35) \quad \bar{F} \in \mathfrak{tr} (\Gamma) \mapsto \tilde{f} \in \mathfrak{mt} \mapsto \text{maj} \in \operatorname{ARI}(\Gamma)_{\text{fin, pol}}.
\]

The first map is injective and the second one is isomorphic by Proposition 2.8. Our claim follows by our previous lemmas. \( \square \)

As a generalization of [RS Proposition 1], we obtain the following.

**Theorem 2.23.** The space \( \mathfrak{tr} (\Gamma) \) forms a bigraded Lie algebra under the bracket (2.6).

**Proof.** As for the morphism (2.35), the second map is Lie algebra homomorphism by Proposition 2.8. It is easy to see that the first map forms a Lie algebra homomorphism when we encode \( \mathfrak{tr} (\Gamma) \) with the bracket (2.3). Thus our claim follows because it is shown that \( \operatorname{ARI}(\Gamma)_{\text{push/pusnu}} \cap \operatorname{ARI}(\Gamma)_{\text{fin, pol}} \) forms a Lie algebra by Theorem 1.32 and so \( \operatorname{ARI}(\Gamma)_{\text{al}} \) does by Lemma 1.15. \( \square \)

**Remark 2.24.** By (2.28), (2.31), Theorem 2.15 and Theorem 2.22 we obtain the following commutative diagram of Lie algebras.

\[
\begin{array}{ccc}
\text{gr}_D \mathfrak{tr} (\Gamma) & \overset{\simeq}{\longrightarrow} & \text{gr}_D (\operatorname{ARI}(\Gamma)_{\text{push/pusnu}} \cap \operatorname{ARI}(\Gamma)_{\text{fin, pol}}) \\
\downarrow & & \downarrow \\
\mathfrak{tr} (\Gamma) & \overset{\simeq}{\longrightarrow} & \operatorname{ARI}(\Gamma)_{\text{push/pusnu}} \cap \operatorname{ARI}(\Gamma)_{\text{fin, pol}}
\end{array}
\]

The diagram is commutative because (2.35) is associated graded with (2.20).

Similarly to Definition 1.34, we impose a distribution relation on \( \mathfrak{tr} (\Gamma) \).

**Definition 2.25.** For \( N \geq 1 \) and \( \Gamma^N = \{ g^N \in \Gamma \mid g \in \Gamma \} \), we consider the map \( i_N : \mathfrak{tr} (\Gamma) \to \mathfrak{tr} (\Gamma^N) \) which is induced by

\[
i_N (x) = x, \quad i_N (y_\tau) = \begin{cases} y_\tau & \text{when } \tau \in \Gamma^N, \\ 0 & \text{when } \tau \not\in \Gamma^N \end{cases}
\]
and also the map $m_N : \frak{l}_{tr}(\Gamma)_{\bullet\bullet} \to \frak{l}_{tr}(\Gamma^N)_{\bullet\bullet}$ which is induced by

$$m_N(x) = Nx, \quad m_N(y) = y_{\sigma N}$$

for $\sigma \in \Gamma$. We define the following $\mathbb{Q}$-linear subspace

$$\frak{l}_{tr}(\Gamma)_{\bullet\bullet} := \{ F \in \frak{l}_{tr}(\Gamma)_{\bullet\bullet} \mid i_N(F) = m_N(F) \text{ for all } N \geq 1 \}.$$

As a corollary of Theorems 2.22 and 2.23 we obtain the following corollary.

**Corollary 2.26.** The space $\frak{l}_{tr}(\Gamma)$ forms a bigraded Lie algebra and we have the following isomorphism of bigraded Lie algebras

$$\frak{l}_{tr}(\Gamma)_{\bullet\bullet} \simeq \ARI(\Gamma)_{\text{push/push}} \cap \ARI(\Gamma)_{\text{fin, pol}}.$$

**Proof.** By definition we see that both $i_N$ and $m_N$ form Lie algebra homomorphisms and by Proposition 1.3.5 $\ARI(\Gamma)$ forms a Lie algebra. Therefore our claim follows from the commutativity of the following diagrams and Proposition 1.3.5. 

\[ \begin{array}{ccc} \frak{l}_{tr}(\Gamma)_{\bullet\bullet} & \longrightarrow & \ARI(\Gamma) \\
i_N \downarrow & & \downarrow m_N \\
\frak{l}_{tr}(\Gamma^N)_{\bullet\bullet} & \longrightarrow & \ARI(\Gamma^N), \end{array} \]

\[ \begin{array}{ccc} \frak{l}_{tr}(\Gamma)_{\bullet\bullet} & \longrightarrow & \ARI(\Gamma) \\
m_N \downarrow & & \downarrow m_N \\
\frak{l}_{tr}(\Gamma^N)_{\bullet\bullet} & \longrightarrow & \ARI(\Gamma^N). \end{array} \]

\[ \square \]

3. Dihedral Lie algebra

By using mould theoretic interpretations of the bigraded Lie algebra $\mathbb{D}(\Gamma)_{\bullet\bullet}$ (a finite abelian group) with a dihedral symmetry and of the Kashiwara-Vergne bigraded Lie algebra $\frak{l}_{tr}(\Gamma)_{\bullet\bullet}$, we realize an embedding $\text{Fil}_2^D \mathbb{D}(\Gamma)_{\bullet\bullet} \hookrightarrow \frak{l}_{tr}(\Gamma)_{\bullet\bullet}$ which extends the result of [RS].

**3.1. Dihedral bigraded Lie algebra.** We recall the definition of the dihedral bigraded Lie algebra $D(\Gamma)_{\bullet\bullet}$ introduced by Goncharov [G01a] and introduce a related Lie algebra $\mathbb{D}(\Gamma)_{\bullet\bullet}$ which contains $D(\Gamma)_{\bullet\bullet}$ in Definitions 3.1.4 and 3.1.5.

We call an element $f = f(t_1, \ldots, t_{m+1})$ in $\mathbb{Q}[t_1, \ldots, t_{m+1}]$ translation invariant when $f(t_1, \ldots, t_{m+1}) = f(t_1 + c, \ldots, t_{m+1} + c)$ for any $c \in \mathbb{Q}$. We often denote this by $f(t_1 : \ldots : t_{m+1})$. We consider a set of collections

$$Z = \{ Z(g_1, \ldots, g_m) \mid t_1 : \ldots : t_{m+1} \} g_1, \ldots, g_m \in \Gamma$$

with

$$g_{m+1} = (g_1 \cdots g_m)^{-1}$$

of translation invariant element in $\mathbb{Q}[t_1, \ldots, t_{m+1}]$. For such $Z$, we associate

$$Z(g_1 : \cdots : g_{m+1} | t_1 : \cdots : t_{m+1}) := Z(g_1^{-1} g_2, \ldots, g_m^{-1} g_{m+1}, g_1 | t_1 : t_2 : \cdots : t_1 \cdots : t_{m+1} = 0)$$

with any $g_1, \ldots, g_{m+1} \in \Gamma$ and $t_1 + \cdots + t_{m+1} = 0$ and also

$$Z(g_1 : \cdots : g_{m+1} | t_1 : \cdots : t_{m+1}) := Z(g_1 : \cdots : g_{m+1} | t_1 : t_{m+1} = 0, t_2 = t_{m+1} - t_{m+1}, \ldots, t_{m} = t_{m+1} - t_{m+1})$$

(3.3)

$$Z(g_1^{-1} g_2, \ldots, g_m^{-1} g_{m+1}, g_1 | t_1 : \cdots : t_{m+1})$$

(3.4)

with any $g_1, \ldots, g_{m+1} \in \Gamma$ and any $t_1, \ldots, t_{m+1}$. 

**Definition 3.1 (G01a).** Set-theoretically the dihedral bigraded Lie algebra means the \( \mathbb{Q} \)-linear bigraded space

\[
D(\Gamma)_{\ast \ast} = \bigoplus_{w, m} D(\Gamma)_{w, m},
\]

where the bidegree \((w, m)\)-part \(D(\Gamma)_{w, m}\) is defined to be the set \(Z\) in (3.1) of translation invariant elements in \(\mathbb{Q}[t_1, \ldots, t_{m+1}]\) with total degree \(w - m\) satisfying (a). *the double shuffle relation*, that is,

\begin{equation}
(a-i).
\sum_{\sigma \in \Pi_{p,q}} Z(g_{\sigma(1)} \cdots g_{\sigma(m)} \cdot g_{m+1} | t_{\sigma(1)} \cdots t_{\sigma(m)} : t_{m+1}) = 0
\end{equation}

\begin{equation}
(a-ii).
\sum_{\sigma \in \Pi_{p,q}} Z(g_{\sigma(1)} \cdots g_{\sigma(m)} : g_{m+1} | t_{\sigma(1)} \cdots t_{\sigma(m)} : t_{m+1}) = 0
\end{equation}

for any \(p, q \geq 1\) with \(p + q = m\).

(b). *the distribution relation* for \(N \in \mathbb{Z}\) such that \(|N|\) divides \(|\Gamma|\),

\begin{equation}
Z(g_1 \cdots g_{m+1} | t_1 \cdots t_{m+1}) = \frac{1}{|\Gamma_N|} \sum_{h_i \equiv g_i} Z(h_1 \cdots h_{m+1} | N t_1 \cdots N t_{m+1})
\end{equation}

except that a constant in \(t\) is allowed when \(m = 1\) and \(g_1 = g_2\). Here \(|\Gamma_N|\) is the order of the \(N\)-torsion subgroup of \(\Gamma\) and \(\Pi_{p,q}\) is defined by (1.2).

(c). Additionally we put

\begin{equation}
Z(e : e | 0 : 0) = 0
\end{equation}

for a technical reason.

Its Lie algebra structure is explained in G01a §§4-5. In G01a Theorem 4.1, it is shown that the double shuffle relation implies the dihedral symmetry relations:

**Theorem 3.2 (G01a Theorem 4.1).** The double shuffle relation implies the dihedral symmetry relations, which consist of the cyclic symmetry relation

\[Z(g_1 \cdot g_2 \cdots g_{m+1} | t_1 : t_2 \cdot \cdots : t_{m+1}) = Z(g_2 \cdots g_{m+1} | g_1 | t_2 : \cdots : t_{m+1} : t_1),\]

the inversion relation (*the distribution relation for \(N = -1\)*)

\[Z(g_1 \cdot g_2 \cdots g_{m+1} | t_1 : \cdots : t_{m+1}) = Z(g_1^{-1} \cdots g_{m+1}^{-1} | -t_1 : \cdots : -t_{m+1}).\]

the reflection relation

\[Z(g_1 \cdot g_2 \cdots g_{m+1} | t_1 : \cdots : t_m : t_{m+1}) = (-1)^{m+1} Z(g_{m+1} \cdots g_1 | -t_m : \cdots : -t_1 : -t_{m+1}),\]

for \(m \geq 2\).

The following reformulation of the above dihedral symmetry relations is useful in our later arguments.

**Remark 3.3.** The transformation G01 allows us to rewrite the above dihedral symmetry relations as follows:

the cyclic symmetry relation

\begin{equation}
Z(g_1, g_2, \ldots, g_{m+1} | t_1 : t_2 : \cdots : t_{m+1}) = Z(g_2, \ldots, g_{m+1}, g_1 | t_2 : \cdots : t_{m+1} : t_1),
\end{equation}
the inversion relation
(3.10)
\[ Z(g_1, g_2, \ldots, g_{m+1}; t_1 : t_2 : \cdots : t_{m+1}) = Z(g_1^{-1}, g_2^{-1}, \ldots, g_{m+1}^{-1}; -t_1 : -t_2 : \cdots : -t_{m+1}), \]
the reflection relation
(3.11)
\[ Z(g_1, \ldots, g_m, g_{m+1}; t_1 : \cdots : t_m : t_{m+1}) = (-1)^{m+1} Z(g_m^{-1}, \ldots, g_1^{-1}, g_{m+1}^{-1}; -t_m : \cdots : -t_1 : -t_{m+1}) \]
with \( g_1 g_2 \cdots g_{m+1} = 1 \).

**Definition 3.4.** Goncharov \([\text{G01a}, \S 4.5 \text{ and } \S 5.2]\) also introduced a related Lie algebra
\[ D'(\Gamma)^{\bullet \bullet} = \oplus_{w, m} D'(\Gamma)_{w, m} \]
which consists of the collections \( Z \) satisfying the shuffle product \( (3.6) \), the cyclic symmetry relation \( (3.2) \) and the additional condition \( (3.8) \) (cf. \([\text{G01a}, \text{Proposition 4.6}]\)). For our purpose, we consider its \( \mathbb{Q} \)-linear subspace
\[ \mathcal{D}(\Gamma)^{\bullet \bullet} = \oplus_{w, m} \mathcal{D}(\Gamma)_{w, m} \]
which consists of elements in \( D'(\Gamma)^{\bullet \bullet} \) satisfying the double shuffle relations (a).

By Theorem 3.2, we have
\[ D'(\Gamma)^{\bullet \bullet} \supset \mathcal{D}(\Gamma)^{\bullet \bullet} \supset D(\Gamma)^{\bullet \bullet}. \]

3.2. Mould theoretic reformulation. We explain a reformulation of \( \mathcal{D}(\Gamma)^{\bullet \bullet} \) and \( D(\Gamma)^{\bullet \bullet} \) in terms of moulds in Theorem 3.5 and Corollary 3.6 respectively.

Let \( Z \) be a collection \( (3.1) \) of translation invariant elements. We associate a mould
\[ M^w_Z = (M^w_Z)^{i \in \mathbb{Z}_{\geq 0}} \in \mathcal{M}(\mathcal{F}; \Gamma) \]
by \( M^w_Z = 0 \) for \( i \neq m \) and
\[ M^m_Z(g_1^{u_1}, \ldots, g_m^{u_m}) = Z(g_1 : \cdots : g_m : 1| u_1, \ldots, u_{m+1}). \]

The following is a generalization of the results in \([\text{M}, \text{Sch15}]\) which treat the case of \( \Gamma = \{ \epsilon \} \).

**Theorem 3.5.** The map sending \( M : Z \in D'(\Gamma)^{\bullet \bullet} \mapsto M^w_Z \in \text{ARI}(\Gamma) \) forms a Lie algebra homomorphism and it induces an isomorphism between
\[ \text{Fil}^2_{\mathcal{D}} D(\Gamma)^{\bullet \bullet} \cong \text{Fil}^2_{\mathcal{D}} \text{ARI}(\Gamma)_{\text{fin, pol}}^{\ast / \mathbb{Q}}. \]

Here the left hand side means the depth\( >1 \)-part of \( D(\Gamma)^{\bullet \bullet} \) and the right hand side means the finite polynomial-valued part of the subset of \( \text{ARI}(\Gamma)_{\text{fin, pol}}^{\ast / \mathbb{Q}} \) (cf. Definition 1.22) consisting of \( M \) with depth\( >1 \).

**Proof.** It is immediate that that \( (3.6) \) is equivalent to the condition for \( M^w_Z \) being in \( \text{ARI}(\Gamma)_{\text{fin}} \). By (3.2), we have
\[ \text{swap}(M^w_Z(g_1^{v_1}, \ldots, g_m^{v_m})) \]
\[ = Z(g_1 \cdots g_m : g_1 \cdots g_{m-1} : \cdots : g_1 : 1| v_m, v_{m-1} - v_m, \ldots, v_2 - v_3, v_1 - v_2, -v_1) \]
\[ = Z(g_m^{-1}, g_{m-1}^{-1}, g_1^{-1}, g_1 \cdots g_m | v_m : \cdots : v_2 : v_1 : 0). \]
Thus we see that (3.5) is equivalent to the condition for \( \text{swap}(M_Z) \) being in \( \overline{\text{ARI}}(\Gamma)_{al} \). Therefore our map forms a \( \mathbb{Q} \)-linear isomorphism (3.112) by Theorem 3.72.

Since \( M_Z \) is in \( \text{ARI}(\Gamma)_{\text{fin}, \text{pol}} \) when \( Z \in D'(\Gamma)_{\text{even}} \), by Proposition 2.28 there is an \( h \in L \) with depth \( m \) such that
\[
\text{ma}(h) = M_Z,
\]
that is,
\[
\text{ma}_h^m(u_1, \ldots, u_m) = \tilde{Z}(g_1 : \cdots : g_m : 1|u_1, \ldots, u_{m+1})
\]
with \( u_1 + \cdots + u_m + u_{m+1} = 0 \). By (3.3) and (3.3), we have
\[
\text{vimo}_h^m(u_1, \ldots, u_m) = Z(g_1 : \cdots : g_m : 1|u_1 : \cdots : u_m : u_0).
\]

In [G01a, Theorem 5.2], \( D'(\Gamma)_{\text{even}} \) is realized as a Lie subalgebra of \( \mathfrak{sl}_2 \mathfrak{det}^{\Gamma} \) under the morphism
\[
\xi^c_{\Gamma} : D'(\Gamma)_{\text{even}} \to \mathfrak{sl}_2 \mathfrak{det}^{\Gamma},
\]
sending each \( Z = \{ Z(g_1, \ldots, g_m | t_1 : \cdots : t_{m+1}) \}_{g_1, \ldots, g_m \in \Gamma} \in D_{w,m}(\Gamma) \) to the residue class of \( D_{\sigma(F), G(F)} \in \mathfrak{sl}_2 \mathfrak{det}^{\Gamma} \) in \( \mathfrak{gr}_H^G \mathfrak{det}^{\Gamma} \) with
\[
F = |\Gamma|^{-1} \sum_{n_1, \ldots, n_{m+1} > 0 \atop g_1 \ldots, g_{m+1} \in \Gamma} I_{n_1, \ldots, n_{m+1}}(g_1 : \cdots : g_{m+1}) X^{n_1-1} Y_{g_1} \cdots X^{n_{m-1}} Y_{g_m} X^{n_m-1} = \sum_{n_0, \ldots, n_m > 0 \atop g_1 \ldots, g_m \in \Gamma} I_{n_0, \ldots, n_m}(1 : g_1 : \cdots : g_m) X^{n_0-1} Y_{g_1} \cdots X^{n_m-1} Y_{g_m} X^{n_m-1} \in \mathbb{L}_{w,m}
\]
when the associated element \( Z = \{ Z(g_1 : \cdots : g_{m+1} | t_1 : \cdots : t_{m+1}) \}_{g_1, \ldots, g_{m+1} \in \Gamma} \) given by (3.31) is expressed as
\[
Z(g_1 : \cdots : g_{m+1} | t_1 : \cdots : t_{m+1}) := \sum_{n_1, \ldots, n_{m+1} > 0} I_{n_1, \ldots, n_{m+1}}(g_1 : \cdots : g_{m+1}) t_1^{n_1-1} \cdots t_{m+1}^{n_{m+1}-1}
\]
in \( \mathbb{Q}[t_1, \ldots, t_{m+1}] \). We note that \( I_{n_1, \ldots, n_{m+1}}(g_1 : \cdots : g_{m+1}) \) is uniquely determined.

By combining the Lie algebra homomorphisms \( \xi^c_{\Gamma} \) with (2.24) and (2.29), we obtain
\[
\text{ma} \circ \text{res} \circ \xi^c_{\Gamma}(Z) = \text{ma}(F).
\]

By definition, we have
\[
\text{vimo}_F^m(u_1, \ldots, u_m) = \sum_{n_1, \ldots, n_{m+1} > 0} I_{n_0, \ldots, n_m}(1 : g_1 : \cdots : g_m) u_0^{n_0-1} \cdots u_m^{n_m-1}
\]
\[
= Z(1 : g_1 : \cdots : g_m | u_0 : \cdots : u_m).
\]

By the cyclic symmetry relation (3.2),
\[
= Z(g_1 : \cdots : g_m : 1|u_1 : \cdots : u_m : u_0) = \text{vimo}_h^m(u_1, \ldots, u_m).
\]

Therefore we have \( F(x, (y_\sigma)_{\sigma}) = h(x, (y_{\sigma^{-1}})_{\sigma}) \). So
\[
M(Z) = \text{ma}(h) = \iota \circ \text{ma} \circ \text{res} \circ \xi^c_{\Gamma}(Z)
\]
where \( \iota \) is the map defined by \( (\iota M)^m(u_1, \ldots, u_m) = M^m(u_1, \ldots, u_m) \) which forms a Lie algebra homomorphism. Since \( \iota, \text{ma} \) in (2.29), \( \text{res} \) in (2.27) and \( \xi^c_{\Gamma} \) in (3.13) are all Lie algebra homomorphisms, we see that \( M \) is so. Whence we get our claims.
Since $\text{ARI}(\Gamma)_{al/\text{al}}$, and hence the right hand side of (3.12), forms a Lie algebra by Proposition 1.24, we learn that $\text{Fil}_2^2 D(\Gamma)_{\text{••}}$ forms a Lie subalgebra of $D'(\Gamma)_{\text{••}}$.

**Corollary 3.6.** The map $M$ in Theorem 3.5 induces a Lie algebra isomorphism between

$$\text{(3.14) } \text{Fil}_2^2 D(\Gamma)_{\text{••}} \simeq \text{Fil}_2^2 \text{ARID}(\Gamma)^{\text{fin, pol}}_{\text{al/\text{al}}}. $$

Here the left hand side means the depth $>1$-part of $D(\Gamma)_{\text{••}}$ and the right hand side means the finite polynomial-valued part of the subset of $\text{ARID}(\Gamma)_{\text{al/\text{al}}}$ (cf. Definition 1.34) consisting of $M$ with depth $>1$.

**Proof.** Since the distribution relation (3.7) is equivalent to

$$Z(g_1, \ldots, g_{m+1}|t_1 : \cdots : t_{m+1}) = \sum_{h_i^N = g_i} Z(h_1, \ldots, h_{m+1}|Nt_1 : \cdots : Nt_{m+1}),$$

which corresponds to $i_N(M_Z) = m_N(M_Z)$, that is, $M_Z \in \text{ARID}(\Gamma)$. Since $\text{ARID}(\Gamma)_{\text{al/\text{al}}}$ and hence the right hand side of (3.14), forms a Lie algebra by Corollary 1.36 and $\text{Fil}_2^2 D(\Gamma)_{\text{••}}$ forms a Lie subalgebra of $D'(\Gamma)_{\text{••}}$ by [G01a, Theorem 5.2], our claim follows. \qed

### 3.3. Embedding

In this subsection, we construct an embedding $\text{Fil}_2^2 \text{ARID}(\Gamma)_{\text{al/\text{al}}} \hookrightarrow \text{ARI}(\Gamma)_{\text{push/pushu}}$ in Theorem 3.13. As a corollary, we get an embedding $\text{Fil}_2^2 D(\Gamma)_{\text{••}} \hookrightarrow \text{lkrv}(\Gamma)_{\text{••}}$ in Corollary 3.14.

The following generalizes [Sch15, Lemma 2.5.3].

**Lemma 3.7.** Any mould $M \in \text{ARI}(\Gamma)_{\text{al}}$ is mantar-invariant (cf. Notation [LA]), namely, for $m \geq 1$ and $\sigma_1, \ldots, \sigma_m \in \Gamma$, we have

$$\text{(3.15) } M^m \left( \sigma_1, \ldots, \sigma_m \right) = (-1)^{m-1} M^m \left( \sigma_m, \ldots, \sigma_1 \right).$$

**Proof.** For simplicity, we denote $\omega_i := \left( \frac{x_i}{\sigma_i} \right)$. By using alternality of $M$, we have

$$\sum_{i=1}^{m-1} (-1)^{i-1} \left\{ \sum_{\alpha \in \mathcal{X}_2^*} \text{Sh} \left( (\omega_i, \ldots, \omega_1); (\omega_{i+1}, \ldots, \omega_m) \right) M^m(\alpha) \right\} = 0.$$
Here, we calculate the left hand side as follows:

\[
\begin{align*}
\sum_{i=1}^{m-1} (-1)^{i-1} \left\{ \sum_{\alpha \in \mathcal{X}_2^\bullet} \text{Sh} \left( (\omega_i, \ldots, \omega_1); (\omega_{i+1}, \ldots, \omega_m) \right) M^m(\alpha) \right\} \\
= \sum_{i=1}^{m-1} (-1)^{i-1} \left\{ \sum_{\alpha \in \mathcal{X}_2^\bullet} \text{Sh} \left( (\omega_i, \ldots, \omega_1); (\omega_{i+1}, \ldots, \omega_m) \right) M^m(\omega_i, \alpha) \right\} \\
+ \sum_{\alpha \in \mathcal{X}_2^\bullet} \text{Sh} \left( (\omega_i, \ldots, \omega_1); (\omega_{i+2}, \ldots, \omega_m) \right) M^m(\omega_{i+1}, \alpha) \\
= \sum_{i=1}^{m-1} (-1)^{i-1} \sum_{\alpha \in \mathcal{X}_2^\bullet} \text{Sh} \left( (\omega_i, \ldots, \omega_1); (\omega_{i+1}, \ldots, \omega_m) \right) M^m(\omega_i, \alpha) \\
- \sum_{i=2}^{m} (-1)^{i-1} \sum_{\alpha \in \mathcal{X}_2^\bullet} \text{Sh} \left( (\omega_i, \ldots, \omega_1); (\omega_{i+1}, \ldots, \omega_m) \right) M^m(\omega_i, \alpha) \\
= M^m(\omega_1, \omega_2, \ldots, \omega_m) - (-1)^{m-1} M^m(\omega_m, \omega_{m-1}, \ldots, \omega_1).
\end{align*}
\]

So we obtain \ref{eq:3.15}.

\[\Box\]

We define the parallel translation map \(\tilde{t} : \overline{ARI}(\Gamma) \to \overline{ARI}(\Gamma)\) by

\[
\tilde{t}(M)^m(y_m) := \begin{cases} 
M^m(y_m) & (m = 0, 1), \\
M^{m-1}(y_2 - y_1, \ldots, y_m - y_1) & (m \geq 2),
\end{cases}
\]

for \(M \in \overline{ARI}(\Gamma)\). For our simplicity, we put

\[u := \tilde{t} \circ \text{swap}.\]

\begin{lemma}
Let \(M \in \text{ARI}(\Gamma)_{\mathcal{W}_{m \overline{\mathcal{W}}}^\mathcal{W}}\) and \(m \geq 2\) and \(\sigma_1, \ldots, \sigma_m \in \Gamma\) with \(\sigma_1 \cdots \sigma_m = 1\). Then we have

\begin{equation}
(3.16) \quad u(M)^m(\sigma_{x_1}, \ldots, x_m) = u(M)^m(\sigma_{\overline{x}_2}, \ldots, \overline{x}_m, \overline{x}_1).
\end{equation}

\end{lemma}

\begin{proof}
We have

\[
\begin{align*}
u(M)^m(\sigma_{x_1}, \ldots, x_m) &= \text{swap}(M)^{m-1}(\sigma_{x_2}, \sigma_3, \ldots, \sigma_{m-1}, \sigma_m) \\
&= M^{m-1}(x_{m-x_1}, x_{m-1-x_m}, \ldots, x_2-x_1) \\
&= (-1)^{m-2}M^{m-1}(x_2-x_3, \ldots, x_{m-1}-x_m, x_{m-x_1}) \\
&= (-1)^{m-2}\text{swap}(M)^{m-1}(\sigma_2, \ldots, \sigma_{m-1}, \sigma_{m-x_1}).
\end{align*}
\]

By using \ref{eq:3.15}, we get

\[
= (-1)^{m-2}M^{m-1}(x_2-x_3, \ldots, x_{m-1}-x_m, x_{m-x_1}) \\
= (-1)^{m-2}M^{m-1}(x_2-x_3, \ldots, x_{m-1}-x_m, x_{m-x_1}),
\]

By \ref{eq:3.15}, \(\sigma_1 \cdots \sigma_m = 1\) and \(M \in \text{ARI}(\Gamma)_{\mathcal{W}_{m \overline{\mathcal{W}}}^\mathcal{W}}\), we calculate

\[
\begin{align*}
\text{swap}(M)^{m-1}(\sigma_{x_1}, \ldots, x_m) &= \text{swap}(M)^{m-1}(\sigma_{x_2}, \ldots, x_{m-1}, x_m) \\
&= u(M)^m(\sigma_2, \ldots, x_{m-1}, x_m) \\
&= u(M)^m(\sigma_2, \ldots, x_m, x_1).
\end{align*}
\]

\end{proof}
Therefore, we obtain the claim. □

The following three relations should be called as the cyclic symmetry relation, the inversion relation and the reflection relation respectively (compare them with (3.9), (3.10) and (3.11) with \( m + 1 \) replaced with \( m \)).

**Lemma 3.9.** Let \( m \geq 3 \) and \( M \in \text{ARI}(\Gamma)_{\mathbb{K}/\mathbb{A}} \). Then, for \( \sigma_1, \ldots, \sigma_m \in \Gamma \) with \( \sigma_1 \cdots \sigma_m = 1 \), we have the following:

\[
\begin{align*}
(3.17) & \quad u(M)^m \frac{\sigma_1, \sigma_2, \ldots, \sigma_m}{x_1, x_2, \ldots, x_m} = u(M)^m \frac{\sigma_2, \ldots, \sigma_m, \sigma_1}{x_2, \ldots, x_m, x_1}, \\
(3.18) & \quad u(M)^m \frac{\sigma_1, \sigma_2, \ldots, \sigma_m}{x_1, x_2, \ldots, x_m} = u(M)^m \frac{\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_m^{-1}}{-x_1, -x_2, \ldots, -x_m}, \\
(3.19) & \quad u(M)^m \frac{\sigma_1, \ldots, \sigma_m^{-1}, \sigma_m}{x_1, \ldots, x_m-x_1, \ldots, x_m} = (-1)^m u(M)^m \frac{\sigma_m^{-1}, \ldots, \sigma_1^{-1}, \sigma_1}{-x_m, \ldots, -x_1, -x_m}.
\end{align*}
\]

**Proof.** It is easy to get (3.18) from (3.16) and (3.17) and to get (3.19) from (3.15), (3.17) and (3.18). So it is enough to prove (3.17).

Firstly, we have

\[
\begin{align*}
\sum_{\alpha \in Y_2^*} \text{Sh} \left( \frac{(\sigma_1)}{(\sigma_2, \ldots, \sigma_m)} \right) u(M)^m(\alpha) \\
&= u(M)^m \frac{\sigma_1, \sigma_2, \ldots, \sigma_m}{x_1, x_2, \ldots, x_m} + \sum_{\alpha \in Y_2^*} \text{Sh} \left( \frac{(\sigma_1)}{(\sigma_2, \ldots, \sigma_m)} \right) u(M)^m(\frac{\sigma_2}{x_2}, \alpha) \\
&= u(M)^m \frac{\sigma_1, \ldots, \sigma_m}{x_1, \ldots, x_m} \\
&\quad + \sum_{\alpha \in Y_2^*} \text{Sh} \left( \frac{(x_1-x_2)}{(x_3-x_2, \ldots, x_m-x_2)} \right) \text{swap}(M)^{m-1}(\alpha).
\end{align*}
\]

So by using alternality of \( \text{swap}(M) \), we obtain

\[
(3.20) \quad \sum_{\alpha \in Y_2^*} \text{Sh} \left( \frac{(\sigma_1)}{(\sigma_2, \ldots, \sigma_m)} \right) u(M)^m(\alpha) = u(M)^m \frac{\sigma_1, \ldots, \sigma_m}{x_1, \ldots, x_m}.
\]

Secondly, we have

\[
\begin{align*}
\sum_{\alpha \in Y_2^*} \text{Sh} \left( \frac{(\sigma_1)}{(x_2, \ldots, x_m, x_1)} \right) u(M)^m(\alpha) \\
&= u(M)^m \frac{\sigma_2, \ldots, \sigma_m, \sigma_1}{x_2, \ldots, x_m, x_1} + \sum_{\alpha \in Y_2^*} \text{Sh} \left( \frac{(\sigma_1)}{(x_2, \ldots, x_m-1)} \right) u(M)^m(\alpha, (\frac{\sigma_m}{x_m})).
\end{align*}
\]

By applying Lemma 3.8 to the second term, we get

\[
\begin{align*}
&= u(M)^m \frac{\sigma_2, \ldots, \sigma_m, \sigma_1}{x_2, \ldots, x_m, x_1} \\
&\quad + \sum_{\alpha \in Y_2^*} \text{Sh} \left( \frac{(\sigma_1)}{(x_1-x_m)} \right) u(M)^m(\frac{\sigma_m^{-1}}{-x_m}, \alpha_-) \\
&= u(M)^m \frac{\sigma_2, \ldots, \sigma_m, \sigma_1}{x_2, \ldots, x_m, x_1} \\
&\quad + \sum_{\alpha \in Y_2^*} \text{Sh} \left( \frac{(x_1-x_m)}{(x_2-x_m, \ldots, x_m-1-x_m)} \right) \text{swap}(M)^{m-1}(\alpha_-).
\end{align*}
\]
Here, the symbol $\alpha_-$ is $(\tau_1, \ldots, \tau_{m-1})$ for $\alpha = (\tau_1, \ldots, \tau_m)$ and for $(\tau_1), \ldots, (\tau_m) \in Y_2$.

So by using alternality of $\text{swap}(M)$, we obtain

$$
\sum_{\alpha \in Y_2^*} \text{Sh} \left( \left( \sigma_1 \right); \left( \sigma_2, \ldots, \sigma_m \right) \right) y(M)^m(\alpha) = y(M)^m(\sigma_1, \sigma_2, \ldots, \sigma_m, \sigma_1).
$$

Therefore, by combining (3.20) and (3.21), we obtain (3.17). □

**Lemma 3.10.** For any mould $M \in \text{ARI}(\Gamma)|_{\mathbb{A}/\mathbb{A}^1}$, its $\text{swap}(M)$ is neg-invariant and mantar-invariant (cf. Notation 1.7) namely we have

$$
\text{swap}(M)^m(\sigma_1, \ldots, \sigma_m) = \text{swap}(M)^m(-\sigma_1, \ldots, -\sigma_m),
$$

$$
\text{swap}(M)^m(\sigma_1, \ldots, \sigma_m) = (-1)^{m-1}\text{swap}(M)^m(-\sigma_m, \ldots, -\sigma_1),
$$

for $m \geq 0$.

**Proof.** For $m \geq 2$, the first equation follows from (3.18) and the second one follows from (3.17) and (3.19). For $m = 0, 1$, they follow from the definition of $\text{ARI}(\Gamma)|_{\mathbb{A}/\mathbb{A}^1}$.

The following can be also found in [Sch15, Lemma 2.5.5] but we give a different proof below.

**Proposition 3.11.** Any mould $M \in \text{ARI}(\Gamma)|_{\mathbb{A}/\mathbb{A}^1}$ is push-invariant (1.3).

**Proof.** For $m = 0, 1$, it is obvious because we have $M^1(x_{\sigma_1}) = M^1(-x_{\sigma_1})$. Assume $m \geq 2$. By using (3.17), we have

$$
y(M)^{m+1}(\tau_1, \tau_2, \ldots, \tau_{m+1}) = y(M)^{m+1}(\tau_2, \ldots, \tau_{m+1}, \tau_1),
$$

for $(\tau_1), \ldots, (\tau_{m+1}) \in Y_2$ with $\tau_1 \cdots \tau_{m+1} = 1$. We calculate the left hand side

$$
y(M)^{m+1}(\tau_1, \tau_2, \ldots, \tau_{m+1}) = \text{swap}(M)^m(y_{2^{-1}y_1}, y_2, \ldots, y_{m+1}y_1),
$$

$$
= y(M)^m(y_{m+1}y_1, y_m y_{m-1}y_1, \ldots, y_2 y_3 y_2).
$$

Similarly, we calculate the right hand side

$$
y(M)^{m+1}(\tau_2, \ldots, \tau_{m+1}, \tau_1) = y(M)^m((y_1 y_2, y_1 y_{m+1}y_1), \ldots, y_3 y_4 y_3).
$$

By substituting $y_1 := 0$ and $y_i := x_1 + \cdots + x_{m+2-i}$ ($2 \leq i \leq m+1$) and $\tau_1 := \sigma_1^{-1}$ and $\tau_2 := \sigma_m$ and $\tau_j := \sigma_{m+2-j}^{-1}$ ($3 \leq j \leq m+1$) for any $\sigma_1, \ldots, \sigma_m \in \Gamma$, we have

$$
M^m(\sigma_1, \sigma_2, \ldots, \sigma_m) = M^m(-x_1, \ldots, -x_{m+2-j}^{-1}, \ldots, -x_1, \ldots, -x_m^{-1})
$$

$$
= \text{push}(M)^m(\sigma_1, \sigma_2, \ldots, \sigma_m).
$$

Whence we obtain the claim. □

The following generalizes [RS] Theorem 14.

**Proposition 3.12.** For any mould $M \in \text{Fil}^2_D\text{ARI}(\Gamma)|_{\mathbb{A}/\mathbb{A}^1}$, its $\text{swap}(M)$ is push-neutral (1.3), that is, for all $m \geq 1$ and $\sigma_1, \ldots, \sigma_m \in \Gamma$,

$$
\sum_{i \in \mathbb{Z}/m\mathbb{Z}} \text{push}^i \circ \text{swap}(M)^m(\sigma_1, \ldots, \sigma_m, x_{\sigma_1}, \ldots, x_m) = 0.
$$
Proof. For $m = 0, 1$, it is obvious because we have $M^1(x_1) = 0$. Assume $m \geq 2$. Let $(\tau^{(0)}_1, \tau^{(1)}_1, \ldots, \tau^{(m)}_1) \in Y_2$ with $\tau_0 \tau_1 \cdots \tau_m = 1$. By using alternality of swap($M$), we have

$$
\sum_{\alpha \in Y_2^*} \text{Sh} \left( \left( \begin{array}{cccc}
\tau_2, & \cdots & \tau_m \\
u_2 - u_1, & \cdots & u_m - u_1
\end{array} \right) \cdot \left( \begin{array}{c}
\tau_0 \\
u_0 - u_1
\end{array} \right) \right) \text{swap}(M)^m(\alpha) = 0.
$$

On the other hand, we calculate

$$
\sum_{\alpha \in Y_2^*} \text{Sh} \left( \left( \begin{array}{cccc}
\tau_2, & \cdots & \tau_m \\
u_2 - u_1, & \cdots & u_m - u_1
\end{array} \right) \cdot \left( \begin{array}{c}
\tau_0 \\
u_0 - u_1
\end{array} \right) \right) \text{swap}(M)^m(\alpha)
= u(M)^{m+1} \left( \tau_1, \tau_2, \ldots, \tau_m, \tau_0 \right) + u(M)^{m+1} \left( \tau_1, \tau_2, \ldots, \tau_{m-1}, \tau_0, \tau_m \right)
+ \cdots + u(M)^{m+1} \left( \tau_1, \tau_2, \tau_3, \ldots, \tau_m \right) + u(M)^{m+1} \left( \tau_1, \tau_2, \ldots, \tau_m \right).
$$

Repeated applications of (3.17) yield

$$
u(M)^{m+1} \left( \tau_0, \tau_1, \tau_2, \ldots, \tau_m \right) + u(M)^{m+1} \left( \tau_0, \tau_1, \tau_2, \ldots, \tau_{m-1} \right)
+ \cdots + u(M)^{m+1} \left( \tau_0, \tau_1, \tau_2, \ldots, \tau_m \right) + u(M)^{m+1} \left( \tau_0, \tau_1, \tau_2, \ldots, \tau_m \right)
= \sum_{i \in Z/mZ} u(M)^{m+1} \left( \tau_0, \tau_{i+1}, \tau_{i+2}, \ldots, \tau_{i+m} \right).
$$

Therefore, by substituting $u_0 = 0$ and $u_i = x_i$ ($1 \leq i \leq m$) and $\tau_0 = (\sigma_1 \cdots \sigma_m)^{-1}$ and $\tau_i = \sigma_i$ ($1 \leq i \leq m$), we get the claim.

\begin{theorem}
There is an embedding
$$\text{Fil}_2^2 \text{ARI}(\Gamma)_{\text{al/alg}} \hookrightarrow \text{ARI}(\Gamma)_{\text{push/pushnu}}$$
of graded Lie algebras.
\end{theorem}

\begin{proof}
Let $M \in \text{Fil}_2^2 \text{ARI}(\Gamma)_{\text{al/alg}}$. Then we have $M \in \text{ARI}(\Gamma)_{\text{al/alg}}$. So by Proposition 3.11 and Proposition 3.12, $M$ is push-invariant and swap($M$) is pus-neutral. Therefore, we obtain $M \in \text{ARI}(\Gamma)_{\text{push/pushnu}}$. To see that it is a Lie algebra homomorphism is immediate.
\end{proof}

As a corollary, by taking an intersection with $\text{ARI}(\Gamma)_{\text{fin/pol}}$ in the embedding of the above theorem, we obtain the following inclusion which generalizes [RS, Theorem 3].

\begin{corollary}
There is an embedding
$$\text{Fil}_2^2 \text{D}(\Gamma)_{\ast \ast} \hookrightarrow \text{ltrv}(\Gamma)_{\ast \ast}$$
of bigraded Lie algebras.
\end{corollary}

\begin{proof}
It follows from Theorem 2.22 and Theorem 3.13.
\end{proof}

By imposing the distribution relation, we also obtain the following.

\begin{corollary}
There is an embedding
$$\text{Fil}_2^2 \text{D}(\Gamma)_{\ast \ast} \hookrightarrow \text{ltrvd}(\Gamma)_{\ast \ast}$$
of bigraded Lie algebras.
\end{corollary}

\begin{proof}
It follows from Corollary 2.26, Corollary 3.6 and Theorem 3.13.
\end{proof}
It looks interesting to see if the map is an isomorphism.

**Remark 3.16.** By (3.12), Theorem 2.22, Theorem 3.13 and Corollary 3.14 we obtain the commutative diagram (3.24) of Lie algebras:

\[
\begin{array}{ccc}
\text{Fil}^2_D D(\Gamma)_{\text{fil}} & \simeq & \text{Fil}^2_D \text{ARI}(\Gamma)_{\text{al}} \\
\downarrow & & \downarrow \\
\text{ARI}(\Gamma)_{\text{push/push}} & \simeq & \text{ARI}(\Gamma)_{\text{al}}
\end{array}
\]

**APPENDIX A. ON THE ari-BRACKET OF ARI(\Gamma)**

In this appendix, we give self-contained proofs of fundamental properties of the ari-bracket of ARI(\Gamma), that is, Proposition 1.14, which are required in this paper.

**A.1. Proof of Proposition 1.14.** We give a proof Proposition 1.14 which claims that ARI(\Gamma) forms a Lie algebra, by showing that it actually forms a pre-Lie algebra.

We start with three fundamental lemmas which can be proved directly by simple computations.

**Lemma A.1.** For \(\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \in X^*_Z\), we have

\[
\begin{align*}
\alpha_n[\beta] &= \alpha([\beta]_n, \beta)_n = ([\beta]_n)_n, \\
\alpha_n[\beta_1, \beta_2] &= \alpha_n[\beta_1, \beta_2]_n = ([\beta_1, \beta_2]_n)_n.
\end{align*}
\]

Especially, if \(\beta_1 \neq \emptyset\), we have

\[
\alpha_n[\beta_1, \beta_2] = \alpha([\beta_1]_n, \beta_2)_n = \beta_2([\beta_1]_n),
\]

and if \(\alpha_2 \neq \emptyset\), we have

\[
\alpha_n[\beta_1, \beta_2] = \alpha_1[\beta_1, \beta_2]_n = \beta_1([\beta_2]_n).
\]

**Proof.** Let \(\alpha_1 = (a_1, \ldots, a_i, \ldots, a_l)\), \(\alpha_2 = (a_{i+1}, \ldots, a_{i+m})\), \(\beta = (b_1, \ldots, b_n)\). By \(\alpha_1 \alpha_2 = (a_1, \ldots, a_{i+m})\), we have

\[
\alpha_n[\beta] = (a_1 + \cdots + a_{i+m} + b_1, b_2, \ldots, b_n) = \alpha([a_1 + \cdots + a_{i+m} + b_1, b_2, \ldots, b_n])\]

All the other cases can be shown similarly. \(\Box\)

**Lemma A.2.** Let \(a, b, c, d, e, f \in X^*_Z\).

(1) If we have \(a|c = def\), one of the following holds:

(a) \(a_1, a_2, a_3 \in X^*_Z\) such that \(a = a_1 a_2 a_3\) and \((d, e, f) = (a_1, a_2, a_3|c)\),

(b) \(a_1, a_2, c_1, c_2 \in X^*_Z\) such that \(a = a_1 a_2\) and \(c = c_1 c_2\) and \((d, e, f) = (a_1, a_2|c_1, c_2)\) and \(c_1 \neq \emptyset\),

(c) \(c_1, c_2, c_3 \in X^*_Z\) such that \(c = c_1 c_2 c_3\) and \((d, e, f) = (a_1|c_1, c_2, c_3)\) and \(c_1 \neq \emptyset\).
(2) If we have \( a \cdot c = def \), one of the followings holds:

(I) : There exists \( a_1, a_2, a_3 \in X^*_2 \) such that
\[ a = a_1 a_2 a_3 \text{ and } (d, e, f) = (a_1, a_2, a_3) \text{ and } a_3 \neq \emptyset, \]

(II) : There exists \( a_1, a_2, c_1, c_2 \in X^*_2 \) such that
\[ a = a_1 a_2 \text{ and } c = c_1 c_2 \text{ and } (d, e, f) = (a_1, a_2, c_1, c_2) \text{ and } a_2 \neq \emptyset, \]

(III) : There exists \( c_1, c_2, c_3 \in X^*_2 \) such that
\[ c = c_1 c_2 c_3 \text{ and } (d, e, f) = (a_1, c_1, c_2, c_3). \]

Proof. We present a proof for (1). When \( a \cdot c = def \), the following depict all the possible cases of the positions of \( a, i[c, d, e \text{ and } f]. \)

The first, the second and the third cases correspond (I), (II) and (III) in (1) respectively. The claim for (2) can be proved in the same way. □

Lemma A.3. For \( a, b, c \in X^*_2 \), the following hold:

(1) (commutativity)
\[
\begin{align*}
& a \cdot i[c, b] = i[c, a], \quad (c \cdot i)[b] = (i[c]) \cdot b, \\
& a \cdot i(c) = i(c), \quad (c \cdot i) = i, \quad (a \cdot i) = i.
\end{align*}
\]

(2) (composition)
\[
\begin{align*}
& (c \cdot i)[b] = c \cdot i[b], \quad (c \cdot i)[b] = (c \cdot i)[b], \\
& a \cdot c = c \cdot a, \quad (c \cdot i)[b] = (c \cdot i)[b].
\end{align*}
\]

(3) (independence)
\[
\begin{align*}
& a \cdot i[c] = i[c], \quad (c \cdot i) = i[c], \\
& c \cdot a = a \cdot c, \quad (c \cdot i)[b] = (c \cdot i)[b].
\end{align*}
\]

Proof. Let \( a = (a_1, \ldots, a_i), b = (b_1, \ldots, b_m), c = (c_1, \ldots, c_n) \). We give proofs for specific cases because all the other cases can be proved in a similar way.

(1). We calculate
\[
i[c] = \left( a_1 + \cdots + a_i + b_1 + \cdots + b_m + c_1 + \cdots + c_n \right)_{\mu_1, \mu_2, \ldots, \mu_n} = \left( a_1 + \cdots + a_i + b_1 + \cdots + b_m + c_1 + c_2 + \cdots + c_n \right)_{\mu_1, \mu_2, \ldots, \mu_n} = i[c].
\]

(2). By \( b \cdot a = (b_1, \ldots, b_m, a_1, \ldots, a_i) \) and \( ab = (a_1, \ldots, a_i, b_1, \ldots, b_m) \), we get
\[
(b \cdot a)[c] = \left( a_1 + \cdots + a_i + b_1 + \cdots + b_m + c_1 + c_2 + \cdots + c_n \right)_{\mu_1, \mu_2, \ldots, \mu_n} = ab[c].
\]

On the other hand, by \( b \cdot a = \left( b_1, \ldots, b_m \right)_{\mu_1, \mu_2, \ldots, \mu_n} \) and \( c \cdot a = \left( c_1, \ldots, c_n \right)_{\mu_1, \mu_2, \ldots, \mu_n} \), we get
\[
(b \cdot a)[c] = \left( c_1, \ldots, c_n \right)_{\mu_1, \mu_2, \ldots, \mu_n} = c \cdot a.
\]

\(^{16}\)Note that, in order to decompose \( i[c] \), we need the condition \( c_1 \neq \emptyset \) for the second and third cases.
(3). By the above expression of $b_{l.}$ we get
\[
_i c = \left( b_1 + \cdots + b_m + c_1, c_2, \ldots, c_n \right) = v_i c. \\
\]

The following formula is essential to prove Proposition 1.14. It is stated in [Sch15, (2.2.10)] (where it looks that there is an error on the signature) without a proof.

**Proposition A.4.** For any $A, B \in \text{ARI}(\Gamma)$, we have
\[
\text{arit}_{u}(B) \circ \text{arit}_{u}(A) - \text{arit}_{u}(A) \circ \text{arit}_{u}(B) = \text{arit}_{u}(\text{arit}_{u}(A, B)).
\]

**Proof.** Let $m \geq 0$. Then we have
\[
\text{arit}_{u}(B) \circ \text{arit}_{u}(A) - \text{arit}_{u}(A) \circ \text{arit}_{u}(B) = \sum_{x_m = abc \atop b, c \neq 0} \text{arit}_{u}(A)(a_i [c] B(b_\cdot)) - \sum_{x_m = abc \atop a, b \neq 0} \text{arit}_{u}(A)(a_i [c] B(b_\cdot))
\]
\[
- \sum_{x_m = abc \atop b, c \neq 0} \text{arit}_{u}(B)(a_i [c] A(b_\cdot)) + \sum_{x_m = abc \atop a, b \neq 0} \text{arit}_{u}(B)(a_i [c] A(b_\cdot))
\]
\[
= \sum_{x_m = abc \atop b, c \neq 0} \left\{ \sum_{a_i [c] = def \atop c, f \neq 0} \text{C}(d_\cdot [f] A(e_\cdot)) - \sum_{a_i [c] = def \atop d, c \neq 0} \text{C}(d_\cdot [f] A(e_\cdot)) \right\} B(b_{l.})
\]
\[
- \sum_{x_m = abc \atop a, b \neq 0} \left\{ \sum_{a_i [c] = def \atop c, f \neq 0} \text{C}(d_\cdot [f] A(e_\cdot)) - \sum_{a_i [c] = def \atop d, c \neq 0} \text{C}(d_\cdot [f] A(e_\cdot)) \right\} B(b_{l.})
\]
\[
- \sum_{x_m = abc \atop b, c \neq 0} \left\{ \sum_{a_i [c] = def \atop c, f \neq 0} \text{C}(d_\cdot [f] B(e_\cdot)) - \sum_{a_i [c] = def \atop d, c \neq 0} \text{C}(d_\cdot [f] B(e_\cdot)) \right\} A(b_{l.})
\]
\[
+ \sum_{x_m = abc \atop a, b \neq 0} \left\{ \sum_{a_i [c] = def \atop c, f \neq 0} \text{C}(d_\cdot [f] B(e_\cdot)) - \sum_{a_i [c] = def \atop d, c \neq 0} \text{C}(d_\cdot [f] B(e_\cdot)) \right\} A(b_{l.}).
\]
By using Lemma [A.2] we have

\[
\begin{align*}
&= \sum_{x = abc} \left\{ \sum_{a, b, c \neq 0} C(a_1, \lfloor a_2, [c]\rfloor) A((a_2)_m) + \sum_{a, b, c \neq 0} C(a_1, \lfloor c_2\rfloor) A((a_2, [c_1])_m) + \sum_{a, b, c \neq 0} C(a_1, \lfloor c_2\rfloor) A((a_2)_m) \right\} - B(b_1, c_2) \\
&- \sum_{a, b, c \neq 0} \left\{ \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) A((a_2)_m) + \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) A((a_2, [c_1])_m) + \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) A((a_2)_m) \right\} - B(b_1, c_2) \\
&- \sum_{a, b, c \neq 0} \left\{ \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) B((a_2)_m) + \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) B((a_2, [c_1])_m) + \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) B((a_2)_m) \right\} - A(b_1, c_2) \\
&- \sum_{a, b, c \neq 0} \left\{ \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) B((a_2)_m) + \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) B((a_2, [c_1])_m) + \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) B((a_2)_m) \right\} - A(b_1, c_2) \\
&+ \sum_{a, b, c \neq 0} \left\{ \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) B((a_2)_m) + \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) B((a_2, [c_1])_m) + \sum_{a, b, c \neq 0} C((a_1, [c_1]), c_2) B((a_2)_m) \right\} - A(b_1, c_2).
\end{align*}
\]
By using Lemma A.1 and Lemma A.3 (2), (3) and changing variables, we calculate

\[\sum_{m \neq \emptyset} C(a_m [c_e] A(b_m) B(d_m)) + \sum_{m \neq \emptyset} C(a_m [e] A(b_m[d_m]) B(c_m)) + \sum_{m \neq \emptyset} C(a_m [c_e] A(d_m) B(b_m)) - \sum_{m \neq \emptyset} C(a_m [c_e] A(b_m) B(c_m)) - \sum_{m \neq \emptyset} C(a_m [e] A(b_m[d_m]) B(b_m)) - \sum_{m \neq \emptyset} C(a_m [c_e] A(d_m)) B(b_m) = 0\]

\[\sum_{m \neq \emptyset} C(a_m [c_e] A(b_m) B(d_m)) - \sum_{m \neq \emptyset} C(a_m [e] A(b_m[d_m]) B(c_m)) - \sum_{m \neq \emptyset} C(a_m [c_e] A(d_m) B(b_m)) + \sum_{m \neq \emptyset} C(a_m [e] A(b_m[d_m]) B(c_m)) + \sum_{m \neq \emptyset} C(a_m [c_e] A(d_m)) B(b_m) = 0\]

Cancellation yields

\[\sum_{m \neq \emptyset} C(a_m [c_e] A(b_m) B(d_m)) + \sum_{m \neq \emptyset} C(a_m [e] A(b_m[d_m]) B(c_m)) + \sum_{m \neq \emptyset} C(a_m [c_e] A(d_m) B(b_m)) - C(a_m [c_e] A(b_m) B(c_m)) - C(a_m [e] A(b_m[d_m]) B(b_m)) - C(a_m [c_e] A(d_m)) B(b_m) = 0\]

\[\sum_{m \neq \emptyset} C(a_m [c_e] A(b_m) B(d_m)) - \sum_{m \neq \emptyset} C(a_m [e] A(b_m[d_m]) B(c_m)) - \sum_{m \neq \emptyset} C(a_m [c_e] A(d_m) B(b_m)) + C(a_m [e] A(b_m[d_m]) B(c_m)) + C(a_m [c_e] A(d_m)) B(b_m) = 0\]

\[\sum_{m \neq \emptyset} C(a_m [c_e] A(b_m) B(d_m)) - \sum_{m \neq \emptyset} C(a_m [e] A(b_m[d_m]) B(c_m)) - \sum_{m \neq \emptyset} C(a_m [c_e] A(d_m) B(b_m)) + \sum_{m \neq \emptyset} C(a_m [e] A(b_m[d_m]) B(c_m)) + \sum_{m \neq \emptyset} C(a_m [c_e] A(d_m)) B(b_m) = 0\]

17 We apply Lemma A.1 to the 4th, 7th, 16th and 19th terms.

18 Especially, we apply Lemma A.3 (2) to the middle terms of each lines, and apply Lemma A.3 (3) to the first terms of each lines.

19 The cancellations occur on the four pairs: 4th and 19th, 6th and 21th, 7th and 16th, 9th and 18th.
Therefore, cancellation \(^{20}\) yields

\[
\sum_{x_m = abedc, b, d, e \neq \emptyset} C(a_i[(c_i[e])A(b_j]_m)B(d]_m)
= \sum_{x_m = abedc, b, d, e \neq \emptyset} C(a_i[(c_i[e])A(b_j]_m)B(d]_m) + \sum_{x_m = abedc, b, d, e \neq \emptyset} C(a_i[(c_i[e])A(b_j]_m)B(d]_m).
\]

By using Lemma \text{A.1} we get

\[
\sum_{x_m = abedc, b, d, e \neq \emptyset} C(a_i[(c_i[e])A(b_j]_m)B(d]_m) + \sum_{x_m = abedc, b, d, e \neq \emptyset} C(a_i[(c_i[e])A(b_j]_m)B(d]_m).
\]

Therefore, cancellation \(^{20}\) yields

\[
(arit_u(B) \circ arit_u(A) - arit_u(A) \circ arit_u(B))(C(x_m)
= \sum_{x_m = abedc, b, d, e \neq \emptyset} C(a_i[(c_i[e])A(b_j]_m)B(d]_m) + \sum_{x_m = abedc, c, d, e \neq \emptyset} C(a_i[(c_i[e])A(b_j]_m)B(c]_m)
- \sum_{x_m = abedc, a, c, d \neq \emptyset} C(a_i[c]_m)A((b_i][d]_m)B(c]_m)
- \sum_{x_m = abedc, b, c, d \neq \emptyset} C(a_i[c]_m)A((b_i][d]_m)B(c]_m)
+ \sum_{x_m = abedc, a, b, d \neq \emptyset} C((a_i][c_i][e])A(\bar{[}b_i]_m)B(\bar{[}d_i]_m) + \sum_{x_m = abedc, a, b, e \neq \emptyset} C((a_i][c_i][e])A(\bar{[}b_i]_m)B(\bar{[}c_i]_m)
- \sum_{x_m = abedc, b, d, e \neq \emptyset} C(a_i[(c_i[e])B(b_j]_m)A(d]_m) + \sum_{x_m = abedc, c, d, e \neq \emptyset} C(a_i[(c_i[e])B(b_j]_m)A(c]_m)
+ \sum_{x_m = abedc, a, c, d \neq \emptyset} C(a_i[c]_m)B((\bar{[}b_i]_m)[d]_m)A(c]_m)
+ \sum_{x_m = abedc, b, c, d \neq \emptyset} C(a_i[c]_m)B((\bar{[}b_i]_m)[d]_m)A(c]_m)
- \sum_{x_m = abedc, a, b, d \neq \emptyset} C((a_i][c_i][e])B(\bar{[}d_i]_m)A(\bar{[}b_i]_m) - \sum_{x_m = abedc, a, b, e \neq \emptyset} C((a_i][c_i][e])B(\bar{[}d_i]_m)A(\bar{[}c_i]_m).
\]

\(^{20}\)The cancellations occur on the four pairs when \(c \neq \emptyset\): 1st and 11th, 3rd and 9th, 6th and 16th, 8th and 14th.
By rearranging each terms and by calculating the terms with $c = \emptyset$, we have

\[
\begin{align*}
&= \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b), B(d), A) - \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b), A(d), A) \\
&\quad + \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b), B(d), A) - \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b), B(d), A) \\
&\quad + \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b, [d], c), B(c), d) - \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b, [d], c), B(c), d) \\
&\quad - \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), d) + \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), d) \\
&\quad + \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), b) - \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), b) \\
&\quad - \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), b) + \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), b).
\end{align*}
\]

Here, the first term is calculated such that

\[
\sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b), B(d), A) = \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b), B(c), c) = \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b, [d], c), B(c), d).
\]

The second, third and fourth terms are also calculated respectively as

\[
\begin{align*}
&- \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b), A(c), d) = - \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), d), \\
&\quad \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b, [d], c), B(c), d) = \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b, [d], c), B(c), d), \\
&\quad \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), b) = \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), b).
\end{align*}
\]

These computations yield

\[
(\text{arit}_u(B) \circ \text{arit}_u(A) - \text{arit}_u(A) \circ \text{arit}_u(B))(C)(x_m)
\]

\[
= \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b, [d], c), B(c), d) - \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b, [d], c), B(c), d) \\
- \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), d) + \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), d) \\
+ \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b, [d], c), B(c), d) - \sum_{x_m=\text{abde}} C(a_{\omega} [e] A(b, [d], c), B(c), d) \\
- \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), b) + \sum_{x_m=\text{abde}} C(a_{\omega} [e] B(b, [d], c), A(c), b).
\]
Therefore, we obtain the claim.

By using Lemma A.3(2), (3) and Lemma A.1, we have

\[
= \sum_{x_m = a} C(a)[e] \left\{ \sum_{f \in b} A((b, d) | c) B(c, e) - \sum_{f = b, c \neq \emptyset} A((b, d) | c) B(c, e) \right\}
\]

\[
- \sum_{x_m = a} C(a)[e] \left\{ \sum_{f \in b} B((b, d) | c) A(c, e) - \sum_{f = b, c \neq \emptyset} B((b, d) | c) A(c, e) \right\}
\]

\[
+ \sum_{x_m = a} C(a)[e] \left\{ \sum_{f \in b} A((b, d) | c) B(c, e) - \sum_{f = b, c \neq \emptyset} A((b, d) | c) B(c, e) \right\}
\]

\[
- \sum_{x_m = a} C(a)[e] \left\{ \sum_{f \in b} B((b, d) | c) A(c, e) - \sum_{f = b, c \neq \emptyset} B((b, d) | c) A(c, e) \right\}
\]

By using Lemma A.3(2), (3) and Lemma A.1, we have

\[
= \sum_{x_m = a} C(a)[e] \left\{ (\text{arit}_u(B)(A))(f, c) + (A \times B)(f, c) - (\text{arit}_u(A)(B))(f, c) - (B \times A)(f, c) \right\}
\]

\[
+ \sum_{x_m = a} C(a)[e] \left\{ -(\text{arit}_u(B)(A))(f, c) + (B \times A)(f, c) + (\text{arit}_u(A)(B))(f, c) - (A \times B)(f, c) \right\}
\]

\[
= \sum_{x_m = a} C(a)[e] \text{arit}_u(A, B)(f, c) - \sum_{x_m = a} C(a)[e] \text{arit}_u(A, B)(f, c)
\]

\[
= (\text{arit}_u(\text{arit}_u(A, B))(C))(x_m)
\]

Therefore, we obtain the claim.
**Definition A.5** ([EcL] (2.46)). We consider a binary operation $\text{preari}_u : \text{ARI}(\Gamma)^{\otimes 2} \to \text{ARI}(\Gamma)$ which is defined by

$$
\text{preari}_u(A, B) := \text{arit}_u(B)(A) + A \times B
$$

for $A, B \in \text{ARI}(\Gamma)$.

Then we have $\text{ari}_u(A, B) = \text{preari}_u(A, B) - \text{preari}_u(B, A)$.

**Proposition A.6.** The pair $(\text{ARI}(\Gamma), \text{preari}_u)$ forms a pre-Lie algebra, i.e., the following formula holds:

$$
\text{preari}_u(A, \text{preari}_u(B, C)) - \text{preari}_u(\text{preari}_u(A, B), C) = \text{preari}_u(A, \text{preari}_u(C, B)) - \text{preari}_u(\text{preari}_u(A, C), B)
$$

for $A, B, C \in \text{ARI}(\Gamma)$.

**Proof.** We have

$$
\begin{align*}
\{&\text{preari}_u(A, \text{preari}_u(B, C)) - \text{preari}_u(\text{preari}_u(A, B), C) \\
&- \{\text{preari}_u(A, \text{preari}_u(C, B)) - \text{preari}_u(\text{preari}_u(A, C), B)\} \\
= &\text{ari}_u(\text{preari}_u(B, C))(A) + A \times \text{ari}_u(B, C) - \text{ari}_u(C)(\text{preari}_u(A, B)) - \text{preari}_u(A, B) \times C \\
&- \text{ari}_u(\text{preari}_u(C, B))(A) - A \times \text{preari}_u(C, B) + \text{ari}_u(B)(\text{preari}_u(A, C)) + \text{preari}_u(A, C) \times B
\end{align*}
$$

By using associativity of $(\text{ARI}(\Gamma), \times)$ and using Lemma [1.12] we get

$$
\begin{align*}
= &\text{ari}_u(\text{ari}_u(C)(B))(A) + \text{ari}_u(B \times C)(A) + \text{ari}_u(C)(\text{ari}_u(B)(A)) \\
&- \text{ari}_u(\text{ari}_u(B)(C))(A) - \text{ari}_u(C \times B)(A) + \text{ari}_u(B)(\text{ari}_u(C)(A))
\end{align*}
$$

Therefore, by using Proposition [A.4] we obtain the claim. \hfill \square

**Proof of Proposition I.13** It is sufficient to prove the Jacobi identity

$$
\text{ari}_u(\text{ari}_u(A, B), C) + \text{ari}_u(\text{ari}_u(B, C), A) + \text{ari}_u(\text{ari}_u(C, A), B) = 0
$$

for $A, B, C \in \text{ARI}(\Gamma)$. By the relationship between $\text{ari}_u$ and $\text{preari}_u$, we calculate

$$
\begin{align*}
\text{ari}_u(\text{ari}_u(A, B), C) + \text{ari}_u(\text{ari}_u(B, C), A) + \text{ari}_u(\text{ari}_u(C, A), B)
= &\text{preari}_u(\text{preari}_u(A, B), C) - \text{preari}_u(\text{preari}_u(C, A), B) \\
&+ \text{preari}_u(\text{preari}_u(A, B), C) - \text{preari}_u(\text{preari}_u(A, C), B) \\
&+ \text{preari}_u(\text{preari}_u(A, B), C) - \text{preari}_u(\text{preari}_u(B, A), C) \\
&+ \text{preari}_u(\text{preari}_u(B, C), A) - \text{preari}_u(\text{preari}_u(A, B), C) \\
&+ \text{preari}_u(\text{preari}_u(B, C), A) - \text{preari}_u(\text{preari}_u(A, C), B) \\
&+ \text{preari}_u(\text{preari}_u(B, C), A) - \text{preari}_u(\text{preari}_u(C, A), B) \\
&+ \text{preari}_u(\text{preari}_u(B, C), A) - \text{preari}_u(\text{preari}_u(A, C), B) \\
&+ \text{preari}_u(\text{preari}_u(B, C), A) - \text{preari}_u(\text{preari}_u(C, A), B)
\end{align*}
$$

By Proposition [A.6] it is equal to 0. So we obtain the Jacobi identity. \hfill \square
A.2. Proof of Proposition 1.15. We give a proof of Proposition 1.15 which claims that $\text{ARI}(\Gamma)_\text{al}$ forms a Lie algebra.

We show the following key lemma in this section.

Lemma A.7. For $\omega, \eta, \alpha_1, \ldots, \alpha_r \in X^*_Z$, we have

$$\text{Sh}\left(\frac{\omega; \eta}{\alpha_1 \cdots \alpha_r}\right) = \sum_{\substack{\omega = \omega_1 \cdots \omega_r \eta = \eta_1 \cdots \eta_r}} \text{Sh}\left(\frac{\omega_1; \eta_1}{\alpha_1}\right) \cdots \text{Sh}\left(\frac{\omega_r; \eta_r}{\alpha_r}\right).$$

where $\omega_1, \ldots, \omega_r, \eta_1, \ldots, \eta_r$ run over $X^*_Z$.

Proof. We consider the deconcatenation coproduct $\Delta : A_X \to A_X^\otimes 2$ defined by

$$\Delta(\omega) := \sum_{\omega = \omega_1 \otimes \omega_2} \omega_1 \otimes \omega_2$$

for $\omega \in X^*_Z$. We recursively define $\mathbb{Q}$-linear maps $\Delta_r : A_X \to A_X^\otimes r$ by $\Delta_2 := \Delta$ and for $r \geq 3$

$$\Delta_r := (\text{Id} \otimes \cdots \otimes \text{Id} \otimes \Delta) \circ \Delta_{r-1}.$$

It is clear that $\Delta_r$ is an algebra homomorphism, i.e., for $\omega, \eta \in X^*_Z$, we have

$$\Delta_r(\omega \triangledown \eta) = \Delta_r(\omega) \triangleleft \Delta_r(\eta).$$

By expanding the above both sides and taking the coefficient of $\alpha_1 \otimes \cdots \otimes \alpha_r$ for $\alpha_1, \ldots, \alpha_r \in X^*_Z$, we obtain the claim.

Proof of Proposition 1.15. Because we have

$$\text{ari}(A, B) = \text{ari}(B)(A) - \text{ari}(A)(B) + [A, B],$$

it is sufficient to prove the following two formulae for $A, B \in \text{ARI}(\Gamma)_\text{al}$:

(A.2) \[ [A, B] \in \text{ARI}(\Gamma)_\text{al}, \]

(A.3) \[ \text{ari}(B)(A) \in \text{ARI}(\Gamma)_\text{al}. \]

Firstly, we prove (A.2). Let $p, q \geq 1$ and put $\omega = (x^1, \ldots, x^p)$ and $\eta = (x^{p+1}, \ldots, x^{p+q})$. For our simplicity, we denote

$$\text{Sh}(M)(\omega; \eta) := \sum_{\alpha \in X^*_Z} \text{Sh}\left(\frac{\omega; \eta}{\alpha}\right)M^{p+q}(\alpha).$$

Then we have

$$\text{Sh}(A \times B)(\omega; \eta) = \sum_{\alpha \in X^*_Z} \text{Sh}\left(\frac{\omega; \eta}{\alpha}\right) \sum_{\alpha = \alpha_1 \alpha_2} A(\alpha_1)B(\alpha_2) = \sum_{\alpha_1, \alpha_2 \in X^*_Z} \text{Sh}\left(\frac{\omega; \eta}{\alpha_1 \alpha_2}\right)A(\alpha_1)B(\alpha_2).$$

Note that the product $\triangledown$ of $A_X$ induces the product of $A_X^\otimes r$ (we also denote this product to the same symbol $\triangledown$) as

$$(\omega_1 \otimes \cdots \otimes \omega_r) \triangledown (\eta_1 \otimes \cdots \otimes \eta_r) := (\omega_1 \triangledown \eta_1) \otimes \cdots \otimes (\omega_r \triangledown \eta_r)$$

for any $\omega_1, \eta_1 \in X^*_Z$. 

\[\text{□}\]
By using Lemma \[A.7\] for \( r = 2 \), we get

\[
\begin{align*}
&= \sum_{\alpha_1, \alpha_2 \in X_2^*} \sum_{\omega = \omega_1 \omega_2 \atop \eta = \eta_1 \eta_2} \text{Sh} \left( \frac{\omega_1; \eta_1}{\alpha_1} \right) \text{Sh} \left( \frac{\omega_2; \eta_2}{\alpha_2} \right) A(\alpha_1) B(\alpha_2) \\
&= \sum_{\omega = \omega_1 \omega_2 \atop \eta = \eta_1 \eta_2} \left\{ \sum_{\alpha_1 \in X_2^*} \text{Sh} \left( \frac{\omega_1; \eta_1}{\alpha_1} \right) A(\alpha_1) \right\} \left\{ \sum_{\alpha_2 \in X_2^*} \text{Sh} \left( \frac{\omega_2; \eta_2}{\alpha_2} \right) B(\alpha_2) \right\}.
\end{align*}
\]

By using alternality of \( A \), we calculate

\[
\begin{align*}
&= \sum_{\omega = \omega_1 \omega_2 \atop \omega_1 \neq \emptyset} A(\omega_1) \left\{ \sum_{\alpha_2 \in X_2^*} \text{Sh} \left( \frac{\omega_2; \eta}{\alpha_2} \right) B(\alpha_2) \right\} + \sum_{\eta = \eta_1 \eta_2 \atop \eta_1 \neq \emptyset} A(\eta_1) \left\{ \sum_{\alpha_2 \in X_2^*} \text{Sh} \left( \frac{\omega; \eta_2}{\alpha_2} \right) B(\alpha_2) \right\}.
\end{align*}
\]

By using alternality of \( B \), we obtain

\[
= A(\omega) B(\eta) + A(\eta) B(\omega).
\]

Because we have \([A, B] = A \times B - B \times A\), we get \( \mathcal{S}h([A, B])(\omega; \eta) = 0 \), that is, we obtain \[A.2\].

Secondly, we prove \([A.3]\). We remark that, for \( \alpha \in X_2^* \) with \( l(\alpha) \geq 2 \), we have

\[
(A.5)\quad \arit(A)(B)(\alpha)
\]

\[
\begin{align*}
&= \sum_{\alpha_3 \neq \emptyset} A(\alpha_1, \alpha_2, \alpha_3 \neq \emptyset) B(\alpha_2, \alpha_3) - \sum_{\alpha_3 \neq \emptyset} A(\alpha_1, \alpha_2, \alpha_3 \neq \emptyset) B(\alpha_1, \alpha_3)
\end{align*}
\]

Then, by the definition \[A.4\] of the map \( \mathcal{S}h \), we calculate

\[
\mathcal{S}h(\arit(A)(B))(\omega; \eta) = \sum_{\alpha \in X_2^*} \text{Sh} \left( \frac{\omega; \eta}{\alpha} \right) \arit(A)(B)(\alpha).
\]

Then by \[A.5\], we have

\[
\begin{align*}
&= \sum_{\alpha_1 \in X_2^*, \alpha_2, \alpha_3 \neq \emptyset} \text{Sh} \left( \frac{\omega; \eta}{\alpha_1 \alpha_2} \right) A(\alpha_1, \alpha_2, \alpha_3 \neq \emptyset) B(\alpha_2, \alpha_3) - \sum_{\alpha_1 \in X_2^*, \alpha_2, \alpha_3 \neq \emptyset} A(\alpha_1, \alpha_2, \alpha_3 \neq \emptyset) B(\alpha_1, \alpha_3)
\end{align*}
\]

\[
\begin{align*}
&= \sum_{\alpha_1 \alpha_2, \alpha_3 \neq \emptyset, x \in X_2} \text{Sh} \left( \frac{\omega; \eta}{\alpha_1 \alpha_2 \alpha_3} \right) A(\alpha_1 \alpha_2, \alpha_3 \neq \emptyset) B(\alpha_2, \alpha_3) - \sum_{\alpha_1 \alpha_2, \alpha_3 \neq \emptyset, x \in X_2} A(\alpha_1 \alpha_2, \alpha_3 \neq \emptyset) B(\alpha_1, \alpha_3)
\end{align*}
\]
By using Lemma [A.7], we have

\[
= \sum_{\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}^*_{X_2}} \left\{ \sum_{\omega = \omega_1 \omega_2 \omega_3, \eta = \eta_1 \eta_2 \eta_3} \text{Sh} \left( \frac{\omega_1; \eta_1}{\alpha_1} \right) \text{Sh} \left( \frac{\omega_2; \eta_2}{\alpha_2} \right) \text{Sh} \left( \frac{\omega_3; \eta_3}{\alpha_3} \right) A(\alpha_1 x, [x \alpha_3] B(\alpha_2 x, \).) \right.
\]

\[
- \sum_{\omega = \omega_1 \omega_2 \omega_3, \eta = \eta_1 \eta_2 \eta_3} \text{Sh} \left( \frac{\omega_1; \eta_1}{\alpha_1} \right) \text{Sh} \left( \frac{\omega_2; \eta_2'}{x} \right) \text{Sh} \left( \frac{\omega_3; \eta_3}{\alpha_3} \right) A(\alpha_1 x, [x \alpha_3] B(., [\alpha_2) \right) .
\]

Here, if \( \text{Sh} \left( \frac{\omega_2; \eta_2}{\alpha_2} \right) \neq 0 \) holds for \( \alpha_2 \in \mathbb{Z}^*_{X_2} \), then all letters appearing in \( \alpha_2 \) match with all ones appearing in \( \omega_2 \) and \( \eta_2 \). So we have \( \alpha_2 [x = (\omega, \eta_2), x \text{ and } x]_n = x]_2 \) for \( \alpha_2 \in \mathbb{Z}^*_{X_2} \) and we continue

\[
= \sum_{\alpha_1, \alpha_3 \in \mathbb{Z}^*_{X_2}} \left\{ \sum_{\omega = \omega_1 \omega_2 \omega_3, \eta = \eta_1 \eta_2 \eta_3} \text{Sh} \left( \frac{\omega_1; \eta_1}{\alpha_1} \right) \text{Sh} \left( \frac{\omega_2; \eta_2'}{x} \right) \text{Sh} \left( \frac{\omega_3; \eta_3}{\alpha_3} \right) A(\alpha_1 x, [x \alpha_3] \text{Sh}(B)(\omega_2 \), \eta_2),) \right.
\]

\[
- \sum_{\omega = \omega_1 \omega_2 \omega_3, \eta = \eta_1 \eta_2 \eta_3} \text{Sh} \left( \frac{\omega_1; \eta_1}{\alpha_1} \right) \text{Sh} \left( \frac{\omega_2; \eta_2'}{x} \right) \text{Sh} \left( \frac{\omega_3; \eta_3}{\alpha_3} \right) A(\alpha_1 x, [x \alpha_3] \text{Sh}(B)(\omega_2, [\eta_2) \right) .
\]

Because \( B \in \text{ARI}(\Gamma)_a \), we have \( \text{Sh}(B)(\emptyset; \emptyset) = 0 \) and \( \text{Sh}(B)(\omega_2; \eta_2),) = \text{Sh}(B)(\emptyset; \emptyset, \eta_2) = 0 \) for \( \omega_2, \eta_2 \neq \emptyset \). So we have

\[
= \sum_{\alpha_1, \alpha_3 \in \mathbb{Z}^*_{X_2}} \left\{ \sum_{\omega = \omega_1 \omega_2 \omega_3, \eta = \eta_1 \eta_2 \eta_3} \text{Sh} \left( \frac{\omega_1; \eta_1}{\alpha_1} \right) \text{Sh} \left( \frac{\omega_2; \eta_2'}{x} \right) \text{Sh} \left( \frac{\omega_3; \eta_3}{\alpha_3} \right) A(\alpha_1 x, [x \alpha_3] B(\omega_2),) \right.
\]

\[
- \sum_{\omega = \omega_1 \omega_2 \omega_3, \eta = \eta_1 \eta_2 \eta_3} \text{Sh} \left( \frac{\omega_1; \eta_1}{\alpha_1} \right) \text{Sh} \left( \frac{\omega_2; \eta_2'}{x} \right) \text{Sh} \left( \frac{\omega_3; \eta_3}{\alpha_3} \right) A(\alpha_1 x, [x \alpha_3] B(\omega_2), \right)
\]
Here, \( \text{Sh}(\omega_2'; \eta_2') \neq 0 \) holds for \( x \in X_z \) if and only if \((\omega_2', \eta_2') = (x, \emptyset) \) or \((\emptyset, x) \). So we get

\[
\begin{align*}
\sum_{\omega_2 \neq \emptyset} \left[ \sum_{\omega = \omega_2, \eta = \eta_2, \omega_2 \neq \emptyset} \text{Sh} \left( \frac{\omega_2}{\omega_3}; \frac{\eta_2}{\eta_3} \right) \text{Sh} \left( \frac{\omega_3}{\alpha_3}; \frac{\eta_3}{\alpha_3} \right) A(\alpha_1, x) B(\omega_2, x) \
- \sum_{\omega_2 \neq \emptyset} \text{Sh} \left( \frac{\omega_1}{\alpha_1}; \frac{\eta_1}{\eta_2} \right) \text{Sh} \left( \frac{\omega_3}{\alpha_3}; \frac{\eta_3}{\alpha_3} \right) A(\alpha_1, x) B(\omega_2, x) \right]
\end{align*}
\]

\[
+ \sum_{\omega_2 \neq \emptyset} \left[ \sum_{\omega = \omega_2, \eta = \eta_2, \omega_2 \neq \emptyset} \text{Sh} \left( \frac{\omega_1}{\alpha_1}; \frac{\eta_1}{\eta_2} \right) \text{Sh} \left( \frac{\omega_3}{\alpha_3}; \frac{\eta_3}{\alpha_3} \right) \left\{ A(\alpha_1, x) B(\omega_2, x) - A(\alpha_1, x) B(\omega_2, x) \right\} \right]
\]

\[
+ \sum_{\omega_2 \neq \emptyset} \left[ \sum_{\omega = \omega_2, \eta = \eta_2, \omega_2 \neq \emptyset} \text{Sh} \left( \frac{\omega_1}{\alpha_1}; \frac{\eta_1}{\eta_2} \right) \text{Sh} \left( \frac{\omega_3}{\alpha_3}; \frac{\eta_3}{\alpha_3} \right) A(\alpha_1, x) B(\omega_2, x) \right]
\]

\[
- \sum_{\omega_2 \neq \emptyset} \left[ \sum_{\omega = \omega_2, \eta = \eta_2, \omega_2 \neq \emptyset} \text{Sh} \left( \frac{\omega_1}{\alpha_1}; \frac{\eta_1}{\eta_2} \right) \text{Sh} \left( \frac{\omega_3}{\alpha_3}; \frac{\eta_3}{\alpha_3} \right) A(\alpha_1, x) B(\omega_2, x) \right].
\]
Because \( x \) runs over \( X_Z \), we get \( \omega \| x = x \| \) and \( \eta \| x = x \| \) and \( \eta_2 \| = \| \eta_2 \) and \( \omega_2 \| = \| \omega_2 \). Hence we calculate

\[
\begin{align*}
&= \sum_{\alpha_1, \alpha_3 \in X^*_Z} \left\{ \sum_{\omega = \omega_1 \omega_2 \omega_3, \eta = \eta_1 \eta_3 \omega_2 \neq \emptyset} \text{Sh} \left( \omega_1; \eta_1 \right) \text{Sh} \left( \omega_3; \eta_3 \right) A(\alpha_1 \| x \alpha_3) B(\omega) \right\} \\
&\quad - \sum_{\omega = \omega_1 \omega_2 \omega_3, \eta = \eta_1 \eta_3 \omega_2 \neq \emptyset} \text{Sh} \left( \omega_1; \eta_1 \right) \text{Sh} \left( \omega_3; \eta_3 \right) A(\alpha_1 \| x \alpha_3) B(\omega) \right\} \\
&+ \sum_{\alpha_1, \alpha_3 \in X^*_Z} \left\{ \sum_{\omega = \omega_1 \omega_2 \omega_3, \eta = \eta_1 \eta_2 \eta_3 \omega_2 \neq \emptyset} \text{Sh} \left( \omega_1; \eta_1 \right) \text{Sh} \left( \omega_3; \eta_3 \right) A(\alpha_1 \| x \alpha_3) B(\eta_2) \right\} \\
&\quad - \sum_{\omega = \omega_1 \omega_2 \omega_3, \eta = \eta_1 \eta_2 \eta_3 \omega_2 \neq \emptyset} \text{Sh} \left( \omega_1; \eta_1 \right) \text{Sh} \left( \omega_3; \eta_3 \right) A(\alpha_1 \| x \alpha_3) B(\eta_2) \right\} .
\end{align*}
\]

For \( \omega_1, \omega_2, \omega_3, x \) with \( \omega = \omega_1 \omega_2 \omega_3 x \omega_3 \), by using Lemma A.7 with \( r = 3 \) and \( \omega = \omega_1 \| x \omega_3 \) and \( \alpha_2 = \omega \| x \), we have

\[
\text{Sh} \left( \omega_1 \| x \omega_3; \eta \right) = \sum_{\omega = \omega_1 \| x \omega_3 = \omega_1 \| x \omega_3 \omega_2 \neq \emptyset} \text{Sh} \left( \omega_1; \eta_1 \right) \text{Sh} \left( \omega_3; \eta_2 \right) \text{Sh} \left( \omega_3; \eta_3 \right).
\]

Because \( \eta = (\sigma_{p_1}, \ldots, \sigma_{p_q}) \) and \( \omega_2 \neq \emptyset \), the letter \( \omega_1 \| x \in X_Z \) does not appear in \( \eta \). So for any word \( \eta_2 \) such that \( \eta = \eta_1 \eta_2 \eta_3 \), we get \( \eta_2 \neq \omega \| x \). Hence, \( \text{Sh} \left( \omega_1 \| x \omega_2 \right) \neq \emptyset \) holds if and only if \( \omega_2 \| = \omega \| x \) and \( \eta_2 = \emptyset \). So we have

\[
= \sum_{\eta = \eta_1 \eta_3} \text{Sh} \left( \omega_1; \eta_1 \right) \text{Sh} \left( \omega_3; \eta_3 \right).
\]

Similarly, we get

\[
\sum_{\omega = \omega_1 \omega_3} \text{Sh} \left( \omega_1; \eta_1 \right) \text{Sh} \left( \omega_3; \eta_3 \right) = \text{Sh} \left( \omega_1 \| x \omega_3 \right).
\]
forms a filtered Lie subalgebra of $ari$ and $ARI(\Gamma)$ have the definition of $\text{swap}(ari)$ for $ari$ by the following power series

\[
\begin{align*}
\text{Sh}(ari, A, B) = \sum_{\omega, \eta} \text{Sh}(A)(\omega_1, \ldots, \omega_3; \eta) B(\eta_2, \eta_3) = \sum_{\omega_1, \ldots, \omega_3} \text{Sh}(A)(\omega_1, \ldots, \omega_3; \eta) B(\eta_2, \eta_3).
\end{align*}
\]

Lastly, similarly to footnote 5 in Definition 1.11, all letters appearing in two words $\omega_1, \omega_2, \ldots, \omega_3$ and $\eta$ (resp. $\omega$ and $\eta$) are algebraically independent over $\mathbb{Q}$, so by Remark 1.6, the component $\text{Sh}(A)(\omega_1, \ldots, \omega_3; \eta)$ (resp. $\text{Sh}(A)(\omega; \eta)$) is well-defined. Hence, by using alternality of $A$, we obtain (A.3). 

\[\square\]

A.3. Proof of Proposition 1.24. We give a proof Proposition 1.24 which claims that $ARI(\Gamma)_{al}$ forms a filtered Lie subalgebra of $ARI(\Gamma)_{al}$ under the $ari$-bracket.

For $A, B \in ARI(\Gamma)_{al}$, it is enough to prove $ari_u(A, B) \in ARI(\Gamma)_{al}$, that is,

\[
\begin{align*}
&ari_u(A, B) \in ARI(\Gamma)_{al}, \\
&\text{swap}(ari_u(A, B)) \in ARI(\Gamma)_{al}, \\
&ari_u(A, B)^{1} (\sigma_{1}) = arir(A, B)^{1} (\sigma_{1}).
\end{align*}
\]

Because $ARI(\Gamma)_{al}$ forms a Lie algebra under the $ari$-bracket, (A.6) is obvious. By the definition of $ari_u$, (A.8) is also clear. By Lemma 1.31 and Proposition 3.11, we have

\[\text{swap}(ari_u(A, B)) = arir(u_u(A), swap(B)).\]

Since we have $\text{swap}(A), \text{swap}(B) \in ARI(\Gamma)_{al}$ for $A, B \in ARI(\Gamma)_{al}$, we get $\text{swap}(ari_u(A, B)) \in ARI(\Gamma)_{al}$ by Proposition 1.21. Thus we obtain (A.7). 

\[\square\]

APPENDIX B. MULTIPLE POLYLOGARITHMS AT ROOTS OF UNITY

In this appendix we recall how $ARI(\Gamma)_{al}$ and $D(\Gamma)_{\text{••}}$ are related to multiple polylogarithms at roots of unity.

Multiple polylogarithm is the several variable complex function which is defined by the following power series

\[\text{Li}_{n_1, \ldots, n_r}(z_1, \ldots, z_r) := \sum_{0 < k_1 < \cdots < k_r} \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1^{n_1} \cdots k_r^{n_r}}\]

for $n_1, \ldots, n_r, r \in \mathbb{N}$. For $N \in \mathbb{N}$, we denote $\mu_N$ to be the group of $N$-th roots of unity in $\mathbb{C}$. Its limit value at $r$-tuple $(\epsilon_1, \ldots, \epsilon_r) \in \mu_N^{\otimes r}$ makes sense if and only if $(n_r, \epsilon_r) \neq (1, 1)$. They are called multiple $L$-values (particularly multiple zeta values when $N = 1$).

In [Ec03] §8 and [Ec11] §1.2 (it is also recalled in [R00] §A.3] in the case when $\Gamma = \{\epsilon\}$), the following moulds in $M(F; \Gamma)$ with $F = \mathbb{C}[u_1, \ldots, u_m]$ and $\Gamma = \mu_N$

\[\text{Zag} = \{\text{Zag}(u_1, \ldots, u_m)\}_m \quad \text{and} \quad \text{Zig} = \{\text{Zig}(v_1, \ldots, v_m)\}_m\]

for $n_1, \ldots, n_r, r \in \mathbb{N}$. For $N \in \mathbb{N}$, we denote $\mu_N$ to be the group of $N$-th roots of unity in $\mathbb{C}$. Its limit value at $r$-tuple $(\epsilon_1, \ldots, \epsilon_r) \in \mu_N^{\otimes r}$ makes sense if and only if $(n_r, \epsilon_r) \neq (1, 1)$. They are called multiple $L$-values (particularly multiple zeta values when $N = 1$).
are introduced and defined by
\[ \text{Zag}(\ell_1, \ldots, \ell_m) = \sum_{n_1, \ldots, n_m > 0} \text{Li}^i_{\ell_1, \ldots, \ell_m} \frac{\epsilon_1}{\epsilon_2}, \ldots, \frac{\epsilon_m-1}{\epsilon_m} u_1^{n_1-1} (u_1 + u_2)^{n_2-1} \cdots (u_1 + \cdots + u_m)^{n_m-1} \]
\[ \text{Zig}(s_1, \ldots, s_m) = \sum_{n_1, \ldots, n_m > 0} \text{Li}^s_{n_m, \ldots, n_1} \epsilon_1^{n_1-1} \cdots v_m^{n_m-1} \]

where \( \text{Li}^i_{\ell_1, \ldots, \ell_m}(\epsilon_1, \ldots, \epsilon_m) \) and \( \text{Li}^s_{n_m, \ldots, n_1}(\epsilon_1, \ldots, \epsilon_m) \) mean the shuffle regularization and the harmonic (stuffle) regularization of \( \text{Li}_{n_1, \ldots, n_m}(\epsilon_1, \ldots, \epsilon_m) \) respectively (cf. [AK]). Particularly, \( \text{Li}^i_{\ell_1, \ldots, \ell_m}(\epsilon_1, \ldots, \epsilon_m) = \text{Li}^s_{n_m, \ldots, n_1}(\epsilon_1, \ldots, \epsilon_m) = \text{Li}_{n_1, \ldots, n_m}(\epsilon_1, \ldots, \epsilon_m) \) when \( (n_m, \epsilon_m) \neq (1, 1) \). In [Ec03] (37) and [Ec11] (1.27), it is explained that they are related as follows

\[ \text{Zig} = \text{Mini} \times \text{swap}(\text{Zag}) \]

(see (1.8) for swap). Here \( \text{Mini}(s_1, \ldots, s_m) \) is the mould defined by

\[ \text{Mini}(s_1, \ldots, s_m) = \begin{cases} \text{Mono}_m & \text{when } \epsilon_1, \ldots, \epsilon_m = (1, \ldots, 1), \\ 0 & \text{otherwise} \end{cases} \]

with \( 1 + \sum_{r=2}^{\infty} \text{Mon}_r \cdot t^r := \exp \{ \sum_{k=2}^{\infty} (-1)^{k-1} \frac{\zeta_k}{k} t^k \} \) (cf. [Ec11] (1.30)).

Goncharov’s arguments in [G01b] suggest us to express them as

\[ \text{Zag}(\ell_1, \ldots, \ell_m) = \text{reg}^i \int_{0 < s_1 < \cdots < s_m < 1} \frac{\epsilon_1 s_1^{-k_1}}{1 - \epsilon_1 s_1} d s_1 \wedge \cdots \wedge \frac{\epsilon_m s_m^{-k_m}}{1 - \epsilon_m s_m} d s_m, \]
\[ \text{Zig}(s_1, \ldots, s_m) = \text{reg}^s \sum_{0 < k_m < \cdots < k_1} \frac{\epsilon_1 \cdots \epsilon_m}{(k_m - v_m) \cdots (k_1 - v_1)} \]

by using its regularizations. These might also help to understand [Ec11] (9.5) saying that Zag belongs to \( \text{GARI}(\Gamma)_{\text{as/ls}} \). The authors are not aware of its definition but expect that it means the symmetrality for Zag and the symmetrality for Zig ([Ec11] §§1.1–1.2), which looks corresponding to the shuffle and the harmonic product among multiple \( L \)-values. In [Ec11] §4.7 it is directed to combine a related group \( \text{GARI}(\Gamma)_{\text{as/al}} \) and its Lie algebra \( \text{ARI}(\Gamma)_{\text{al/ls}} \) with a bigraded variant \( \text{GARI}(\Gamma)_{\text{as/al}} \) and \( \text{ARI}(\Gamma)_{\text{al/ls}} \) (cf. Definition 1.22) under maps \( \text{adgar}(\text{pal}) \) and \( \text{adari}(\text{pal}) \) (cf. [Ec11] (2.54), (2.55), and §4.2)).

\[ \text{GARI}(\Gamma)_{\text{as/ls}} \xrightarrow{\text{adgar}(\text{pal})} \text{GARI}(\Gamma)_{\text{as/al}} \]
\[ \text{ARI}(\Gamma)_{\text{al/ls}} \xrightarrow{\text{adari}(\text{pal})} \text{ARI}(\Gamma)_{\text{al/al}}. \]

While the cyclotomic analogue of Drinfeld’s KZ-associator ([D]) which is constructed from the KZ-like differential equation in \( \hat{A}_C \) over \( \mathbb{P}^1(\mathbb{C}) \setminus \{0, \mu_N, \infty\} \) with a complex variable \( s \)

\[ \frac{d}{ds} H(s) = \left( \frac{x}{s} + \sum_{\xi \in \mu_N} \frac{y_\xi}{\xi s - 1} \right) H(s) \]

is introduced and discussed in [R02] [En]. It is a non-commutative formal power series \( \Phi_N^{KZ} \in \hat{A}_C \) with \( \Gamma = \mu_N \) whose coefficients are given by multiple \( L \)-values. In
particular, the coefficient of
\[ x^{n_r-1}y_{e_r}x^{n_{r-1}-1}y_{e_{r-1}} \cdots x^{n_1-1}y_{e_1} \]
is \((-1)^r\text{Li}_{n_1, \ldots, n_r}(\epsilon_1, \ldots, \epsilon_r)\). Definition \((B.4)\) enables us to calculate a relation between the associator \(\Phi_{KZ}^N\) and the mould \(\text{Zag}\) as follows
\[ \text{max}_{\Phi_{KZ}^N} = \{ \text{Zag} \left( \frac{-u_1, \ldots, -u_m}{\epsilon_1, \ldots, \epsilon_m} \right) \} \in \mathcal{M}(F; \mu N). \]

Put \(\Phi_{KZ}^N = \Phi_{KZ}(x, (-y))\) and \(\Phi_{KZ,corr} = \exp \left\{ \left( \sum_{n \geq 2} \frac{(-1)^{n-1}}{n} \text{Li}_n(1) y^1 \right) \right\} \in \mathcal{A}_C\) and define
\[ \Phi_{KZ,\ast}^N := \Phi_{KZ,corr} \cdot q(\pi_y (\Phi_{KZ}^N)) \]
(for \(\pi_y\) and \(q\), see \((2.4)\)). It is shown in \((R.02)\) that \(\Phi_{KZ}^N\) is group-like with respect to the shuffle (deconcatenation) coproduct of \(\mathcal{A}_C\) and \(\Phi_{KZ,\ast}^N\) is so with respect to the harmonic coproduct of \(\text{Im} \pi_y\). They correspond to the shuffle and the harmonic product among multiple \(L\)-values respectively. The regularized double shuffle relations (the shuffle and the harmonic products and the regularization relations \((B.4)\)) are the defining equations of his torsor DMR\(\mu\) for \(\mu \in \mathbb{C}^\times\) which contains \(\Phi_{KZ}^N\) as a specific point when \(\mu = 2\pi \sqrt{-1}\). It is equipped with a free and transitive action of the group DMR\(\mu\). Its associated Lie algebra, which he denotes by \(\mathfrak{dmr}\), is a filtered graded Lie algebra defined by the regularized double shuffle relations modulo products. Our dihedral Lie algebra \(D(\mu N)\) should be called its bigraded variant, defined by ‘their highest depth parts’ of the relations. By definition, it contains the associated graded \(\mathfrak{grdmr}\). By translating Ecalle’s pictures including the diagram \((B.2)\) to this setting, we might learn more enriched structures on these Lie algebras.

References

[AET] Alekseev, A., Enriquez, B. and Torossian, C., Drinfeld associators, braid groups and explicit solutions of the Kashiwara-Vergne equations, Publ. Math. Inst. Hautes Études Sci. No. 112 (2010), 143–189.

[AKKN] Alekseev, A., Kawazumi, N, Kuno, Y. and Naef, F., The Goldman-Turaev Lie bialgebra in genus zero and the Kashiwara-Vergne problem, Advances in Mathematics, 326, 1-53 (2018).

[AT] Alekseev, A. and Torossian, C., The Kashiwara-Vergne conjecture and Drinfeld’s associators, Ann. of Math. (2) 175 (2012), no. 2, 415–463.

[AK] Arakawa, T., Kaneko, M., On multiple \(L\)-values, J. Math. Soc. Japan 56 (2004), no.4, 967–991.

[C] Cresson, J., Calcul moulien, Ann. Fac. Sci. Toulouse Math. (6) 18 (2009), no. 2, 307–395.

[D] V. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\), Leningrad Math. J. 2 (1991), no. 4, 829–860.

[Ec81] Ecalle, J., Les fonctions résurgentes. Tome I et II, Publications Mathématiques d’Orsay 81.6, Université de Paris-Sud, Département de Mathématique, Orsay, 1981.

[Ec03] Ecalle, J., ARI/GARI, la dimorphie et l’arithmétique des multizêtas: un premier bilan, J. Théor. Nombres Bordeaux 15 (2003), no. 2, 411–478.

[Ec11] Ecalle, J., The flexion structure and dimorphy: flexion units, singulators, generators, and the enumeration of multizeta irreducibles, With computational assistance from S. Carr. CRM Series, 12, Asymptotics in dynamics, geometry and PDEs; generalized Borel summation. Vol. II, 27–211, Ed. Norm., Pisa, 2011.

[En] Enriquez, B., Quasi-reflection algebras and cyclotomic associators, Selecta Math.(N.S.) 13 (2007), no. 3, 391–463.

\(22\) The word DMR stands for the French ‘double mélange et régularisation’.
[F] Furusho, H., *Around associators*, Automorphic forms and Galois representations. Vol. 2, 105–117, London Math. Soc. Lecture Note Ser., 415, Cambridge Univ. Press, Cambridge, 2014.

[G01a] Goncharov, A. B., *The dihedral Lie algebras and Galois symmetries of \( \pi_1(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N)) \)*, Duke Math. J. 110 (2001), no. 3, 397–487.

[G01b] Goncharov, A. B., *Multiple polylogarithms and mixed Tate motives*, preprint, arXiv:math/0103059.

[M] Maassarani, M., *Bigraded Lie algebras related to MZVs*, preprint, arXiv:1907.07200.

[R00] Racinet, G., * Séries génératrices non-commutatives de polyzétales et associateurs de Drinfeld*, Ph.D. dissertation, Paris, France, 2000.

[R02] Racinet, G., *Doubles mélange des polylogarithmes multiples aux racines de l’unité*, Publ. Math. Inst. Hautes Études Sci. 95 (2002), 185–231.

[RS] Raphael, E., Schneps, L., *On linearised and elliptic versions of the Kashiwara-Vergne Lie algebra*, arXiv:1706.08299v1, preprint.

[SuSch] Salerno, A., Schneps, L., *Mould theory and the double shuffle Lie algebra structure*, Periods in Quantum Field Theory and Arithmetic, Springer Proceedings in Mathematics & Statistics, vol 314 (2020), Springer, 399–430.

[Sau] Sauzin, D., *Mould expansions for the saddle-node and resurgence monomials*, Renormalization and Galois theories, 83–163, IRMA Lect. Math. Theor. Phys., 15, Eur. Math. Soc., Zürich, 2009.

[Sch12] Schneps, L., *Double shuffle and Kashiwara-Vergne Lie algebras*, J. Algebra 367 (2012), 54–74.

[Sch15] Schneps, L., *ARI, GARI, ZIG and ZAG: An introduction to Ecalle’s theory of multiple zeta values*, arXiv:1507.01534, preprint.