Monodromy of real isolated singularities

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Abstract

Complex conjugation on complex space permutes the level sets of a real polynomial function and induces involutions on level sets corresponding to real values. For isolated complex hypersurface singularities with real defining equation we show the existence of a monodromy vector field such that complex conjugation intertwines the local monodromy diffeomorphism with its inverse. In particular, it follows that the geometric monodromy is the composition of the involution induced by complex conjugation and another involution. This topological property holds for all isolated complex plane curve singularities. Using real morsifications, we compute the action of complex conjugation and of the other involution on the Milnor fiber of real plane curve singularities.

Keywords: fibered knot, monodromy, involution, strong inversion, singularity, real morsification, divide, plane curve, Seifert matrix.

Mathematics Subject Classification: 14D05 (primary), 57M25, 14P25, 14H20, 14H50 (secondary).

Introduction. Let \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) be a map defined by a polynomial. We assume that \( f(0) = 0 \) and that \( 0 \in \mathbb{C}^{n+1} \) is an isolated critical point of \( f \). For \( p \in \mathbb{C}^{n+1} \) let \( ||p|| \) denote the square root of \( |z_0(p)|^2 + |z_1(p)|^2 + \cdots + |z_n(p)|^2 \).

Let \( B_\epsilon := \{ p \in \mathbb{C}^{n+1} \mid ||p|| \leq \epsilon \}, 0 < \epsilon \), be a Milnor ball for the singularity of \( f \) and let \( \text{Tube}_{\epsilon,\delta} := \{ p \in B_\epsilon \mid |f(p)| \leq \delta \}, 0 < \delta << \epsilon \), be a regular tubular neighborhood of \( \{ p \in B_\epsilon \mid f(p) = 0 \} \) in \( B_\epsilon \). A monodromy vector field \( X \) for the singularity is a smooth vector field \( p \in \text{Tube}_{\epsilon,\delta} \mapsto X_p \in \mathbb{C}^{n+1} \) such that we have the following properties for \( p \in \text{Tube}_{\epsilon,\delta} \) (remember \( i = \sqrt{-1} \)):

- \( (df)_p(X_p) = 2\pi i f(p) \),
- \( X_p \) is tangent to \( \partial B_\epsilon \) if \( p \in \partial B_\epsilon \),
- trajectories of $X$ starting at $p \in \partial B_{\varepsilon}$ are periodic with period 1 and are the boundary of a smooth disk in $\partial B_{\varepsilon}$, that is transversal to the function $f$.

Using partition of unity, one can construct monodromy vector fields. The flow at time 1 of a monodromy vector field $X$ defines a monodromy diffeomorphism $T = T_X : F \to F$, where the manifold with boundary $(F, \partial(F)) := \{ p \in B_{\varepsilon} \mid f(p) = \delta \}$ is the Milnor fiber of the singularity. The relative isotopy class of the diffeomorphism $T$ is independent from the chosen monodromy vector field and is the geometric monodromy of the singularity. The geometric monodromy is a topological invariant of the singularity.

From now on we will assume in addition, that the polynomial $f$ is real meaning that its coefficients are real numbers. Let $c : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ denote the involution on complex space given by the complex conjugation of coordinate values. Hence with the above notations, we have $c(\text{Tube}_{\varepsilon, \delta}) = \text{Tube}_{\varepsilon, \delta}$ and $c(F) = F$. We denote by $c_F : F \to F$ the restriction of the involution $c$ to $F$.

Let $X : \text{Tube}_{\varepsilon, \delta} \to \mathbb{C}^{n+1}$ be a monodromy vector field for the isolated singularity of $f$. We may assume that we have constructed the vector field $X$ with more care near the boundary of the Milnor ball in order to achieve that for some $\varepsilon' < \varepsilon$ we have the symmetry $c(X_p) = -c(X_{c(p)}), p \in \text{Tube}_{\varepsilon, \delta}, ||p|| > \varepsilon'$.

Since $f$ is real, we have

$$c((df)_p(X_p)) = (df)_{c(p)}(c(X_p)) = -2\pi i f(c(p))$$

hence, we see (by substituting $q$ for $c(p)$ and accordingly $c(q)$ for $p$) that the vector field $X^c$ defined by:

$$q \in \text{Tube}_{\varepsilon, \delta} \mapsto X^c_q := -c(X_{c(q)}) \in \mathbb{C}^{n+1},$$

is a monodromy vector field too. Let $Y : \text{Tube}_{\varepsilon, \delta} \to \mathbb{C}^{n+1}$ be the vector field $Y := \frac{X + X^c}{2}$, which due to the extra care is also a monodromy vector field. We have $Y^c = Y$. The following is an important symmetry of the geometric monodromy:

**Lemma 1** Let $T_Y$ be a monodromy diffeomorphism, which has been computed with a monodromy vector field $Y$ satisfying $Y^c = Y$. We have the symmetry

$$c_F \circ T_Y \circ c_F = T_Y^{-1}.$$

The geometric monodromy $T$ satisfies (up to relative isotopy) the symmetry

$$c_F \circ T \circ c_F = T^{-1}.$$
**Proof:** The restriction of complex conjugation $c_{\text{Tube}_{\varepsilon,\delta}}' : \text{Tube}_{\varepsilon,\delta} \rightarrow \text{Tube}_{\varepsilon,\delta}$ maps the monodromy vector field $Y'$ to $-Y$ and $F$ to $F$. Hence, since $c_{\text{Tube}_{\varepsilon,\delta}}$ reverses the orientations of the trajectories, we have

$$T_Y^{-1} = (c_F)^{-1} \circ T_Y \circ c_F = c_F \circ T_Y \circ c_F.$$ 

Since the geometric monodromy $T$ is in the relative mapping class group of the Milnor fiber represented by $T_Y$, for $T$ in the relative mapping class group we have the symmetry $c_F \circ T \circ c_F = T^{-1}$. $\blacksquare$

Symmetries of monodromies as in the lemma can occur in the more general context of so-called strongly invertible knots, see for instance [H-To], [T], [K].

The symmetry property $c_F \circ T \circ c_F = T^{-1}$ expresses that the geometric monodromy $T$ of a complex hypersurface with real defining equation is conjugate in the mapping class group by an element of order 2 to its inverse $T^{-1}$. This statement does not refer to any complex conjugation, so it can be stated for any complex hypersurface singularity with complex defining equation. We say that the singularity is strongly invertible if its geometric monodromy diffeomorphism $T$ is conjugate by an element of order 2 in the relative mapping class group of the Milnor fiber to its inverse $T^{-1}$. The property of strong invertibility is a topological property for hypersurface singularities.

We can rewrite the symmetry property as follows: $T_Y \circ c_F \circ T_Y \circ c_F = \text{Id}_F$. We see that $T_Y \circ c_F : F \rightarrow F$ is an involution of $F$. It follows the

**Corollary 1** *The geometric monodromy $T$ of an isolated complex hypersurface singularity, which is defined by a real equation, is the composition of two involutions of the fiber $F_{\delta}, \delta \in \mathbb{R}$, namely: $T = (T \circ c_F) \circ c_F$, where $c_F$ is the restriction of the complex conjugation.*

For $k \in \mathbb{Z}$ we also have the relation $T_Y^k \circ c_F \circ T_Y^k \circ c_F = \text{Id}_F$, which shows that $T_Y^k \circ c_F : F \rightarrow F, k \in \mathbb{Z}$, is a sequence of involutions of $F$.

The above observations can be applied to plane curve singularities in general, since it follows from the Theory of Puiseux Pairs that every plane curve singularity is topologically equivalent to a singularity given by a real equation and hence plane curve singularities are strongly invertible.

For complex hypersurface singularities of higher dimension the situation seems to be opposite. In $\mathbb{C}^{n+1}$, $n > 1$, there exist isolated hypersurface singularities which are not topologically equivalent to a singularity with a real defining
equation [T]. We expect that in general the geometric monodromy $T$ of a complex hypersurface singularity fails to be strongly invertible.

For complex hypersurface singularities the eigenvalues of the monodromy $T_*$ acting on the homology $E := \bigoplus_k H_k(F, \mathbb{Q})$ are roots of unity by the monodromy theorem and the characteristic polynomial of $T_*$ is a product of cyclotomic polynomials. It follows, that the complex homological monodromy is strongly invertible. Mathias Schulze has proved strong invertibility of the homological monodromy with real coefficients [S] and the following is a strengthening of his result.

**Theorem 1** The rational homological monodromy of a complex hypersurface singularity is strongly invertible.

**Proof:** Let $E = \bigoplus_i E_i$ a be finest possible direct sum decomposition in $T_*$-invariant $\mathbb{Q}$-subspaces of $E$. The characteristic polynomial of the restriction $A_i : E_i \to E_i$ of $T_*$ to a summand $E_i$ is a power $\psi(t) = \phi(t)^L$ of a cyclotomic polynomial $\phi(t)$. We have $\psi(A_i) = 0$ by the Hamilton-Cayley theorem and we have $\psi(A_i^{-1}) = 0$ since a power of cyclotomic polynomial satisfies $\psi(1/t) = \pm t^\deg(\psi) \psi(t)$.

Since the decomposition has no refinement, we may choose a vector $e_1 \in E_i$, that is cyclic for $A_i$ and for $A_i^{-1}$. The systems

$$e_1, e_2 := A_i(e_1), \ldots, e_{\dim E_i} := A_i^{\dim E_i - 1}(e_1),$$

$$f_1 := e_1, f_2 := A_i^{-1}(e_1), \ldots, f_{\dim E_i} := A_i^{-\dim E_i + 1}(e_1)$$

are bases for the space $E_i$. Let $b_i : E_i \to E_i$ be the linear map defined by $b_i(e_j) = f_j, 1 \leq j \leq \dim E_i$.

We have $A_i^{-1}b_i(e_j) = b_iA_i(e_j), 1 \leq j < \dim E_i$. The polynomial $\psi_1(t) := -\psi(t) + t^{\dim E_i}$ satisfies $\psi_1(A_i^{\pm 1}) = A_i^{\pm \dim E_i}$. Since the degree of the polynomial $\psi_1(t)$ is less than $\dim E_i$, we have

$$A_i^{-1}b_i(e_{\dim E_i}) = A_i^{-\dim E_i}b_i(e_1) = \psi_1(A_i^{-1})b_i(e_1) =$$

$$b_i\psi_1(A_i)(e_1) = b_iA_i^{\dim E_i}(e_1) = b_iA_i(e_{\dim E_i}).$$

We conclude that $A_i^{-1}b_i = b_iA_i$ and $A_i^{-1} = b_iA_i^{-1}$ hold.

The polynomial $\psi_0(t) := \frac{\psi(t) - \psi(0)}{t - \psi(0)}$ satisfies $A_i^{-1} = \psi_0(A_i)$ and $A_i = \psi_0(A_i^{-1})$. We deduce $b_iA_i^{-1} = b_i\psi_0(A_i) = \psi_0(A_i^{-1})b_i = A_i b_i$ and conclude $A_i^{-1} = b_i^{-1}A_i b_i$. 


We observe at this point that both the conjugates of $A_i$ by $b_i$ and by $b_i^{-1}$ are equal to the inverse $A_i^{-1}$.

For $0 \leq j < \dim E_i$ we have (remember $e_1 = f_1 = b_i(e_1) = b_i(f_1)$)

$$b_i(f_{j+1}) = b_i A_i^{-j} (f_1) = A_i^{-j} b_i (f_1) = A_i^{-j} (e_1) = e_{j+1}.$$

Hence $b_i$ is of order two, which shows that $A_i$ is strongly invertible over $\mathbb{Q}$. The sum $b := \bigoplus b_i$ is a rational strong inversion for $T_*$.

**Question:** Is the integral homological monodromy of a complex hypersurface singularity strongly invertible? We would like to know the answer for the isolated surface singularity $f(x, y, z) = L_1 L_2^2 L_3 L_4^4 L_5 L_6^6 L_7 L_8^8 L_9^9 + x^{46} + y^{46} + z^{46}$ where $L_j, 1 \leq j \leq 9$ are linear forms on $\mathbb{C}^3$, such that the 9 lines $\{L_j = 0\}$ in the complex projective plane span the 9_3 configuration of flex tangents to a nonsingular cubic. No real equation for this singularity can exist, since the configuration 9_3 cannot be realized in the real projective plane, see [H-C,T].

In Section 1 we will study in detail the effect of complex conjugation on the topology of plane curve singularities. We show that the fiber of the link of a connected divide carries naturally a cellular decomposition with tri-valent 1-skeleton.

In Section 2 we study the involutions that appear in the decomposition of the geometric monodromy of plane curve singularities. We show that they lift to $\pi$-rotations about an axis in the universal cyclic covering of the complement of the link.

I like to thank Makoto Sakuma for explaining the work of Jeffrey L. Tollefson [To] to me.

I like to thank the referee for having attracted my attention to work of Sabir Gusein-Zade [G-Z2], in which he shows among other results, that the integral homological monodromy of an isolated complex hypersurface singularity with real defining equation $f$ is the composition of two involutions, both conjugated to the action of complex conjugation on the homology of a real regular fiber.

**1. Real plane curve singularities.** Let $f : \mathbb{C}^2 \to \mathbb{C}$ define an isolated plane curve singularity at $0 \in \mathbb{C}^2$ given by a real convergent power series $f \in \mathbb{R}\{x, y\}$. Let $f = f_1 f_2 \cdots f_r$ be the factorization in local branches. A factor $f_i \in \mathbb{C}\{x, y\}$ is a real convergent power series, i.e. $f_i(\mathbb{R}^2) \subset \mathbb{R}$ or the conjugate series $c(f_i)$ is a factor too.

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The topology of a plane curve singularity $f$ is completely encoded in a divide for $f$ [AC3,B-K], see also [AC1-2,G-Z1].

We state the results more generally for links of connected divides, since the involution given by complex conjugation on real isolated plane curve singularities corresponds to a natural involution of links of divides as explained below, see [AC3-4].

A divide is the image of a generic relative immersion $P$ of a compact one dimensional manifold in the unit disk $D$ in $\mathbb{R}^2$. The complement in $S^3$ of the link $L(P)$ of $P$ is naturally fibered over $S^1$, if the divide $P$ is connected. Let $\delta_P$ be the number of double points of the divide $P$. We denote the fiber surface above 1 by $F_P$. The local topology of a plane curve singularity is obtained from a divide for the singularity. More precisely, the link $L_f \subset \partial B_\varepsilon$ of an isolated plane curve singularity $\{f = 0\}, f \in \mathbb{R}\{x, y\}$, is equivalent to the link $L_P$ of a divide $P$, see [AC3].

We recall, that for a plane curve singularity $f$ one can obtain a divide $P$ by performing a small real deformation $f_s, 0 \leq s \leq 1$, of the singularity, where for $0 < s \leq 1$ the restriction of $f_s$ to the euclidean disk $D_\varepsilon := B_\varepsilon \cap \mathbb{R}^2$ of radius $\varepsilon$ in $\mathbb{R}^2$ is a Morse function with $\mu(f)$ critical points and such that the 0-level is connected and contains all the saddle points. The divide $P$ for $f$ is the curve $\{p \in D_\varepsilon \ | \ f_1(p) = 0\}$, which we rescale by the factor $\frac{1}{\varepsilon}$ from $D_\varepsilon$ into the unit disk $D$.

![Fig. 1. Divide for the singularity $(x^3 - y^2)((x^3 - y^2)^2 - 4x^8y)$.](image)

The homology $H_1(F_P, \mathbb{Z})$ can be described combinatorially in terms of the
divide $P$ as a direct sum $H_1(F_P, \Z) = E_- \oplus E_0 \oplus E_+$, where $E_-, E_0$ and $E_+$ are the subspaces in $H_1(F, \Z)$, which are freely generated as follows: $E_+ := [\delta_1, \ldots, \delta_{\mu_+}]$, $E_0 := [\delta_{\mu_++1}, \ldots, \delta_{\mu_++\mu_0}]$ and $E_- := [\delta_{\mu_++\mu_0+1}, \ldots, \delta_{\mu_-+\mu_+}]$, where $(\delta_i)_{1 \leq i \leq \mu}$ is the oriented system of vanishing cycles of the divide $P$ with positive upper triangular Seifert form $S : H_1(F, \Z) \to H^1(F, \Z) = \text{Hom}(H_1(F, \Z), \Z)$, see [AC4] Theorem 4. We define $N := S - \text{Id}$, which is upper-triangular nilpotent matrix. The monodromy $T_* : H_1(F, \Z) \to H_1(F, \Z)$ is given by: $T_* = (S^t)^{-1} \circ S$.

We recall briefly the construction of the fiber surface $F_P$ of a divide $P$, see [AC4]. Let $P$ be a connected divide in the unit euclidian disk $D$ consisting of a generic system of immersed copies of $S^1$ and $[0, 1]$. We assume, that $P$ is irreducible, i.e. every differentiable chord of $D$ transversal to $P$, that meets $P$ in at most 1 point, is transversally isotopic to an arc in the boundary of $D$. Let $f : D \to \R$ be a Morse function adapted to $P$. A region of $P$ is a connected component of $D \setminus P$, that does not meet the boundary of $D$. If $P$ has regions, we assume that the sign of $f$ is chosen such that in at least one region the function $f$ is positive. The link $L_P$ of $P$ is the subset in the 3-sphere sitting in the tangent space $S^3 := \{(x, u) \in TR^2 \mid \|x\|^2 + \|u\|^2 = 1\}$ and is given by $L_P := \{(x, u) \in S^3 \mid u \in TP\}$. The number $r = r(P)$ of components of the link $L_P$ is twice the number of immersed circles plus the number immersed intervals in $P$.

Fig. 2. Divide and graph of divide.
Let $S_P$ be a system of gradient lines of $f$, which connect saddle points of $f$ with local or relative maxima or minima of $f$. In fact, near the boundary of $D$ the lines of the system $S_P$ are only gradient-like, but end in a relative critical point of $f$. We call the set $S_P$ the graph of the divide $P$. By a slalom construction, see [AC5], one can reconstruct from each of the sets $S^\pm_P := \{x \in S_P \mid \pm f(x) \geq 0\}$ the divide $P$. We assume that at the maxima and minima of $f$, that different arcs of $S_P$ have different unoriented tangent directions. Let $\Sigma_P$ be the following subset of the fiber $F_P$ over 1 of the natural fibration over $S^1$ of the complement of the link of $P$: the subset $\Sigma_P$ is the closure in the tangent space $TD$ of $D$ of

$$\Sigma_P := \{(x,u) \in TD \mid x \in S_P, f(x) > 0, (df)_x(u) = 0, ||x||^2 + ||u||^2 = 1\}.$$ 

We recall that the union of fiber $F_P$ with the link $L_P$ is the closure in $TD$ of

$$F'_P := \{(x,u) \in TD \mid x \in D, f(x) > 0, (df)_x(u) = 0, ||x||^2 + ||u||^2 = 1\},$$

which is a surface of genus $g(P)$ equal to the number of double points in $P$.

![Vanishing cycle δ_2 for a saddle point.](image)

The system of cycles $\delta_i, 1 \leq i \leq \mu$, can be drawn on $\Sigma_P$ as follows. The vanishing cycle $\delta_i, \mu_- + \mu_0 < i \leq \mu$, which corresponds to a local maximum
\( M \) of \( f \) is the circle \( \{(M, u) \mid u \in TD, ||M||^2 + ||u||^2 = 1\} \) oriented counter clockwise. The vanishing cycle \( \delta_i, \mu_+ < i \leq \mu_- + \mu_0 \), which corresponds to a saddle point \( z \) of \( f \) is a curve in the set \( E_z := \{(x, u) \in TD \mid x \in e_z, u \in \text{Kernel}(df)_e_z, ||x||^2 + ||u||^2 = 1\} \), as drawn in Fig-3. The curve \( \delta_i \) is a piecewise smooth embedded copy of \( S^1 \) in \( F_P \) with image in \( E_z \) and with non-constant projection to the disk \( D \). The orientation is chosen such that at both ends the orientation agrees with the oriented vanishing cycle of the maximum. Moreover, the inward tangent vectors \( (x, u) \) to \( e_z \) at end points \( x \in e_z \) do belong to \( \delta_i \). The vanishing cycle \( \delta_i, 1 \leq i \leq \mu_- \), which corresponds to a local minimum of \( f \) projects to an oriented circuit \( e_1, e_2, \cdots, e_k \) of edges of the graph \( S_P \). The circuit surrounds the region in counter clock-sense to which the minimum corresponds. The circuit is a polygon and bounds a cell in \( D \). The curve \( \delta_i \) is a subset of \( \{(x, u) \in TD \mid x \in \cup_{1 \leq j \leq k} e_j, u \in \text{Kernel}(df)_x, ||x||^2 + ||u||^2 = 1\} \) and is the image of a piecewise smooth embedding of \( S^1 \) as drawn in Fig-4. The vectors \( (x, u) \), which belong to \( \delta_i \), point out of the cell of the circuit.

Smooth representatives for the system of vanishing cycles can be obtained using tears as in [AC6].

\[ \text{Fig. 4. Vanishing cycle } \delta_i \text{ for a minimum.} \]

The involution \( c_{TR^2} : (x, u) \in TR^2 \mapsto (x, -u) \in TR^2 \) induces an involution \( c_{S^3} \) on \( S^3 \subset TR^2 \) that preserves the link \( L_P \) of any divide \( P \) and that induces
involutions on the fiber surfaces above $\pm 1$. If the divide $P$ is a divide for a real plane curve singularity $f$, the involutions induced by $c_{\mathbb{TR}^2}$ or by the complex conjugation $c$ on the triples $(S^3, L_P, F_P)$ and on $(\partial B, L_f, F)$ correspond to each other by the homeomorphism of pairs of the main theorem of [AC3].

**Theorem 2** Let $P \subset D$ be a connected divide consisting of the image of a relative generic immersion of a compact one dimensional manifold in the unit disk $D \subset \mathbb{R}^2$. Let $E_-, E_0$, and $E_+$ be the summands in $H_1(F, \mathbb{Z})$ as above. The involution $c_{F*} : H_1(F, \mathbb{Z}) \to H_1(F, \mathbb{Z})$ fixes pointwise the summand $E_+$, in particular for $\delta_i \in E_+$ we have

$$c_{F*}(\delta_i) = \delta_i.$$

For $\delta_i \in E_0$ we have

$$c_{F*}(\delta_i) = -\delta_i + \sum_{1 \leq j \leq \mu_+} \langle N(\delta_j), \delta_i \rangle.$$

For $\delta_i \in E_-$ we have

$$c_{F*}(\delta_i) = \delta_i + \sum_{1 \leq j \leq \mu_+} \langle N(\delta_j), \delta_i \rangle - \sum_{\mu_+ < j \leq \mu_0 + \mu_0} \langle N(\delta_j), \delta_i \rangle \delta_j.$$

The trace of the involution $c_{F*} : H_1(F, \mathbb{Z}) \to H_1(F, \mathbb{Z})$ is given by:

$$\text{Trace}(c_{F*}) = \mu_- - \mu_0 + \mu_+.$$

**Proof:** A vanishing cycle $\delta_i \in E_+$ corresponds to a maximum $M \in D$ of $f$, hence as set we have $\delta_i = \{(M, u) \in TD \mid ||M||^2 + ||u||^2 = 1\}$. The involution $(x, u) \mapsto (x, -u)$ induces on $\delta_i$ the antipodal map, which is orientation preserving. It follows $c_{F*}(\delta_i) = \delta_i$ in homology. A vanishing cycle $\delta_i \in E_0$ corresponds to a saddle point $z \in D$ of $f$. Working with the tear model of [AC6], see also Section 2, we see that the involution reverses the orientation and that at the endpoints, which are maxima of $f$, we have outward instead of inward vectors. Hence,

$$c_{F*}(\delta_i) = -\delta_i + \sum m_{i,j} \delta_j,$$

where in the sum $j$ runs through the maxima of $f$ in the interior of $D$ which are connected by gradient lines of $S_P$ to the saddle point $i$. The coefficient $m_{i,j}$ equals 1 or 2 depending on whether the connection by gradient lines is simple or double. Finally one gets $c_{F*}(\delta_i) = -\delta_i + \sum_{1 \leq j \leq \mu_+} \langle N(\delta_j), \delta_i \rangle$. For a vanishing cycle $\delta_i \in E_-$ one can work with the model of [AC6] and
get \( c_{F*}(\delta_i) = \delta_i + \sum_{1 \leq j \leq \mu_+} \langle N(\delta_j), \delta_i \rangle - \sum_{\mu_+ < j \leq \mu_+ + \mu_0} \langle N(\delta_j), \delta_i \rangle \). Since \( N \) is strictly upper-triangular, the matrix of \( c_{F*} \) is upper-triangular with \( \pm 1 \) on the diagonal. We get \( \text{Trace}(c_{F*}) = \mu_- - \mu_0 + \mu_+ \).

We can present the Seifert form and homological monodromy with block-matrices \( S, T \) as in [G-Z2]. On \( H_1(F, \mathbb{Z}) \), \( E_-, E_0, E_+ \) and \( H^1(F, \mathbb{Z}) \) we work with the basis or dual basis given by the system \( \delta_i, 1 \leq i \leq \mu \). The matrix in block form of the Seifert form is

\[
S = \begin{pmatrix}
\text{Id}_{\mu_+} & A & G \\
O & \text{Id}_{\mu_0} & B \\
O & O & \text{Id}_{\mu_-}
\end{pmatrix}
\]

where the block \( G \) equals the block matrix product \( \frac{1}{2}(A \circ B) \). The matrix coefficients of \( A \circ B \) and \( G \) have interpretations in terms of the divide \( P \) or in terms of the Morse function \( f_P \) on the disk \( D \). The matrix coefficient \( (A \circ B)_{(i,j)}, 1 \leq i \leq \mu_+, \mu_+ + \mu_0 < j \leq \mu \), counts the number of sector adjacencies, that has the + region \( i \) of \( P \) with the - region \( j \), while the coefficient \( G_{(i,j)} \) counts the number of common boundary segments of the regions \( i \) and \( j \). The coefficient \( (A \circ B)_{(i,j)} \) also counts the number of saddle connections via gradient lines of \( f_P \) in between the minimum \( j \) and maximum \( i \), while the coefficient \( G_{(i,j)} \) counts the number of components of regular gradient line connections from the minimum \( j \) to the maximum \( i \). This explains the above factor \( \frac{1}{2} \), since a segment of \( P \) is twice incident with a saddle point of \( f_P \).

The matrix of the action of complex conjugation on \( H_1(F, \mathbb{Z}) \) is

\[
C = \begin{pmatrix}
\text{Id}_{\mu_+} & A & G \\
O & -\text{Id}_{\mu_0} & -B \\
O & O & \text{Id}_{\mu_-}
\end{pmatrix}
\]

and is obtained from the matrix \( S \) by multiplying the middle row of blocks by \(-1\). It is interesting to compute the matrix of the involution \( T \circ c_F \) on \( H_1(F, \mathbb{Z}) \)

\[
TC = \begin{pmatrix}
\text{Id}_{\mu_+} & O & O \\
-tA & -\text{Id}_{\mu_0} & O \\
tG & tB & \text{Id}_{\mu_-}
\end{pmatrix}
\]

The matrix \( T \circ C \) is the transgradient of the matrix of \( C \).

It turns out that the combinatorial property \( G = \frac{1}{2}(A \circ B) \) for divides is equivalent to \( C \circ C = \text{Id}_\mu \) or \( T \circ C \circ T \circ C = \text{Id}_\mu \).
For an isolated plane curve singularity at $0 \in \mathbb{C}^2$, which is given by a real equation $\{f = 0\}, f \in \mathbb{R}\{x, y\}, f(0) = 0$, we will denote by $\delta_R(f)$ the number of double points of a divide $P$ for the singularity. Hence $\delta_R(f)$ is the maximal number of local real saddle points in some level, that can occur for a small real deformation of $f$. Observe, that one has $\delta_R(f) \leq \delta(f)$, where $\delta(f)$ is the maximal number of local critical points in some level, that can occur for a small deformation of $f$. We recall the formula $\mu(f) = 2\delta(f) - r + 1$ of Milnor [M]. As example for $f = x^4 + K x^2 y^2 + y^4, -2 \neq K \neq 2$, one has $\delta_R(f) = 4, \delta(f) = 6, r = 4$ and very surprisingly, Callagan shows that for $-2 < K < 2$ there exists a small real deformation with 5 local minima in the same level [C].

**Theorem 3** For an isolated plane curve singularity at $0 \in \mathbb{C}^2$, which is given by a real equation $\{f = 0\}, f \in \mathbb{R}\{x, y\}, f(0) = 0$, we have

$$\mu(f) = 2\delta_R(f) + \text{Trace}(c_{F_P})$$

**Proof:** We have $\delta_R = \dim E_0 = \mu_0$. Hence $\mu(f) = \mu_+ + \mu_0 + \mu_+ = \text{Trace}(c_{F_P}) + 2\mu_0 = 2\delta_R(f) + \text{Trace}(c_{F_P})$. 

Curves $\delta_i$ that correspond to maxima of $f_P$ are invariant by the involution $c$. A curve $\delta_i$ that corresponds to a saddle point or minimum of $f_P$ is in general not invariant by $c$. The union $\Sigma_P$ of the curves $\delta_i$ is invariant by $c$. We get

**Theorem 4** Let $P$ be a connected divide. The pair $(F_P, \Sigma_P)$ defines a tri-valent cellular decomposition with $r(P)$ punctured cells of the fiber surface with boundary $F_P$. The involution $c : (x, u) \rightarrow (x, -u)$ acts on the map $(F_P, \Sigma_P)$.

**Proof:** The union $\Sigma_P$ of the vanishing cycles $\delta_i$ is a tri-valent graph $\Sigma_P$ that is invariant by $c$. The inclusion $\Sigma_P \subset F_P$ induces an isomorphism $H_1(\Sigma_P, \mathbb{Z}) \rightarrow H_1(F_P, \mathbb{Z})$, hence, $\Sigma_P$ is a spine for the surface $F_P$. 

In particular if $r(P) = 1$ the triple $(F_P, \Sigma_P, c)$ is a so-called maximal unicell map of genus $g(P)$ with orientation reversing involution $c$ with $r(P) = 1$ fixed points on the graph of the map. The involution $c_F$ has a unique fixed point on $\Sigma_P$, which corresponds to the intersection of the folding curve $F_P \cap \partial D$ of $c_F$ with the graph $\Sigma_P$. Maximal here means that the number of edges of the graph of the map is maximal, i.e. $3g$. 

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It would be interesting to compute the generating series

\[ \text{divide}(t) := \sum_{g \in \mathbb{N}} d(g) t^g = 1 + t + 2t^2 + 8t^3 + 36t^4 + \cdots, \]

where \( d(g) \) denotes the number of simple, relative and generic immersions the interval \([0, 1]\) with \( g \) double points in the disk \( D \), counted up to homeomorphism in the source and image. We have taken the coefficients up to the term \( t^4 \) of \( \text{divide}(t) \) from the listings of simple, relative, free or oriented divides by Masaharu Ishikawa [I]. We ask to compare the numbers \( d(g) \) with the numbers \( m^+(g) \) of maximal maps of genus \( g \) with orientation reversing involution having 1 fixed point on its graph.

2. Involutions induced by \( \pi \)-rotations.

Our next goal is to visualize the two involutions \( c_F \) and \( T \circ c_F \). We assume that the monodromy diffeomorphism \( T \) is chosen in its isotopy class such that \( T \circ c_F \) is an involution of the fiber surface \( F_P \). In fact, we assume that the monodromy \( T \) was given by a monodromy vector field \( X \), which satisfies \( X = X^c \).

We assume for simplicity, that the divide \( P \) meets the boundary of the disk \( D \). The involution \( c_F \) is an orientation reversing diffeomorphism of \( F_1 = F_P \), which fixes pointwise the system of arcs \( a := F_P \cap \partial D \). The Lefschetz number of the orientation reversing involution \( T \circ c_F \) is equal to the Lefschetz number of \( c_F \) by Th. 1. Hence the involution \( T \circ c_F \) also fixes pointwise exactly a system of arcs \( b \) on the fiber surface \( F_P \) with \( \chi(a) = \chi(b) \), i.e. the systems \( a \) and \( b \) consist of the same number of arcs.

Let \( z : Z \rightarrow S^3 \setminus K_P \) be the infinite cyclic covering of the knot complement. Let \( X^Z \) be the lift of the vector field \( X \). Let \( T_{\ast} : Z \rightarrow Z \) be the flow diffeo-
morphism of the vector field $X^Z$ with stopping time $\frac{1}{2}$. Let $F'_1 \subset Z$ be a lift of the fiber surface $F_1$ and let $F'_{-1}$ be the lift $T_{\frac{1}{2}}(F'_1)$ of the fiber surface $F_{-1}$. The involution $c_F$ can first be lifted unambiguously to an involution of $F'_1$ and then extended unambiguously to an involution $A$ of $Z$ such that $A$ maps the vector field $X^Z$ to its opposite. The involution $A$ is a $\pi$-rotation about a lift of the system of arcs $a$ into $F'_1$. The involution $(x, u) \mapsto (x, -u)$ induces an involution $c_F$ of the fiber surface $F_{-1}$ above $-1$, and, as we have done for the involution $c_F$, the involution $c_{F_{-1}}$ lifts unambiguously to a $\pi$-rotation $B : Z \to Z$ about a system of arcs $b'' \subset F'_{-1}$ that satisfies $z(b'') = b' := \partial D \cap F_{-1}$. We have $B \circ A = T_{\frac{1}{2}} \circ T_{\frac{1}{2}} = T_1$. Since $T_1$ is a lift of the monodromy, it follows from $(T \circ c_F) \circ c_F = T$, that the involution $T \circ c_F$ fixes the system of arcs $b := z(T_{-1}^{-1}(b'))$.

In [AC4] we have computed the monodromy diffeomorphism $T_+ : F_1 \to F_{-1}$ for which $T_{\frac{1}{2}}$ is a lift to $Z$ as a product of half Dehn twists for a divide. From the above we deduce, that the involution $T \circ c_F : F_1 \to F_1$ is the composition $T_{-1}^{-1} \circ c_{F_{-1}} \circ T_+$. We see that $T \circ c_F$ fixes the arc $b = T_{-1}^{-1}(\partial D \cap F_{-1})$. We also see that the involution $c_F \circ T$ of $F_1$ fixes the system of arcs $c_F(b) = T_{+}(\partial D \cap F_{-1})$. Both arc systems $b$ and $c_F(b)$ have equal projections $\beta$ into $D$. See Fig. 5, 6 for examples.

The composition of the orientation reversing involutions, that fix pointwise the arc systems $a, b$ of Fig. 5 on the cylinder surface, is indeed a Dehn twist. To see this, think of the cylinder as $[0, \pi] \times S^1$; the two involutions induce on the circle $\{\theta\} \times S^1$ reflections about diameters that make an angle $\theta$, so the composition is a rotation of angle $2\theta$ of the circle $\{\theta\} \times S^1$. 

![Fig. 6. Divide for the $E_8$ singularity and the curve $\beta$.](image-url)
In Fig. 6 the curve \( \beta \) is drawn for a more complicated divide for the plane curve singularity \( E_8 \) with equation \( y^3 - x^5 = 0 \).

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