Fractional strong matching preclusion of some Cartesian product graphs

Bo Zhu¹, Shumin Zhang², Chenfu Ye³,*

¹Department of Computer, Qinghai Normal University, Xining, Qinghai 810008, China
²Department of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai 810008, China
Emails: zhuboqh@163.com; zhangshumin@qhnu.edu.cn
*Corresponding author’s e-mail: yechf@qhnu.edu.cn

Abstract. The fractional strong matching preclusion number of a graph is the minimum number of edges and vertices whose deletion leaves the resulting graph without a fractional perfect matching. In this paper, we obtain the fractional strong matching preclusion number for the Cartesian product of a graph and a cycle. As an application, the fractional strong matching preclusion number for torus networks is also obtained.

1. Introduction

A matching \( M \) in a graph is a set of pairwise non-adjacent edges. A perfect matching in a graph is a set of edges such that every vertex is incident with exactly one edge in this set. An almost-perfect matching in a graph is a set of edges such that every vertex except one is incident with exactly one edge in this set, and the exceptional vertex is incident to none. A set \( F \) of edges in a graph \( G = (V, E) \) is called a matching preclusion set if \( G - F \) has neither a perfect matching nor an almost-perfect matching. The matching preclusion number of graph \( G \), denoted by \( mp(G) \), is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching or an almost-perfect matching. The concept of matching preclusion was introduced by Brigham et al.[2]. Matching preclusion also has connections to a number of theoretical topics, including conditional connectivity and extremal graph theory, and further studied in [4-6, 9] with special attention given to interconnection networks.

In [13], the concept of strong matching preclusion was introduced. The strong matching preclusion number of a graph \( G \), denoted by \( smp(G) \), is the minimum number of vertices and edges whose deletion leaves the resulting graph without a perfect matching or an almost perfect matching. Let \( F \) be a strong matching preclusion set of a graph \( G = (V, E) \). Suppose \( F = F^V \cup F^E \) where \( F^V \) consists vertices and \( F^E \) consists edges. A strong matching preclusion set is optimal if \( |F| = smp(G) \). According to the definition of \( mp(G) \) and \( smp(G) \), we have that \( smp(G) \leq mp(G) \leq \delta(G) \). We call \( F \) a basic strong matching preclusion set if \( F \) is an optimal strong matching preclusion set of \( G \) and \( G - F \) has an isolated vertex, that is, there exists a vertex \( v \) such that every vertex in \( F^V \) is a neighbor of \( v \) and every edge in \( F^E \) is incident to \( v \). This includes the following scenario: \( F \) is a basic optimal matching preclusion set and \( G - F \) is
odd without almost perfect matchings. We can further restrict this class as follows. If \( G - F \) is even and there is a vertex \( v \) such that every vertex in \( F^V \) is a neighbor of \( v \) and every edge in \( F^E \) is incident to \( v \), then \( F \) is a **trivial strong matching preclusion set**. A strongly maximally matched even graph is **strongly super matched** if every optimal strong matching preclusion set is trivial.

Given a set of edges \( M \) of \( G \), we define \( f^M \) to be the indicator function of \( M \), that is, \( f^M : E(G) \to \{0,1\} \) such that \( f^M(e) = 1 \) if and only if \( e \in M \). Let \( X \) be a set of vertices of \( G \). We define \( \delta'(X) \) to be the set of edges with exactly one end in \( X \). If \( X = \{v\} \), we write \( \delta'(v) \) instead of \( \delta'(\{v\}) \). (We note that although it is common to use \( \delta(X) \) is the literature, we decided to use \( \delta'(X) \) as it is also common to use \( \delta(G) \) to denote the minimum degree of vertices in \( G \).)

With this notation, \( f^M : E(G) \to \{0,1\} \) is the indicator function of the perfect matching \( M \) if \( \sum_{e \in \delta'(v)} f^M(e) = 1 \) for each vertex \( v \) of \( G \). If we replace \( - \) by \( \leq \) in the condition, then \( M \) is a **matching** of \( G \). Similarly, \( f^M : E(G) \to \{0,1\} \) is the indicator function of the almost perfect matching \( M \) if \( \sum_{e \in \delta'(v)} f^M(e) = 1 \) for each vertex \( v \) of \( G \), except one vertex, say \( v' \), and \( \sum_{e \in \delta'(v')} f^M(e) = 0 \). It is also common to use \( f(X) \) to denote \( \sum_{x \in X} f(x) \). It follows from the definition that \( f^M(E(G)) = \sum_{e \in E(G)} f^M(e) = |M| \) for a matching \( M \). In particular, \( f^M(E(G)) = |V(G)|/2 \) if \( M \) is a perfect matching and \( f^M(E(G)) = (|V(G)| - 1)/2 \) if \( M \) is an almost perfect matching.

A standard relaxation from an integer setting to a continuous setting is to a continuous setting to replace the codomain of the indicator function from \( \{0,1\} \) to the interval \([0,1]\). Let \( f : E(G) \to [0,1] \). Then \( f \) is a **fractional matching** if \( \sum_{e \in \delta'(v)} f(e) \leq 1 \) for each vertex \( v \) of \( G \); \( f \) is a **fractional perfect matching** if \( \sum_{e \in \delta'(v)} f(e) = 1 \) for every vertex \( v \) of \( G \); and \( f \) is a **fractional almost perfect matching** if \( \sum_{e \in \delta'(v)} f(e) = 1 \) for every vertex \( v \) of \( G \) except one vertex \( u \), and \( \sum_{e \in \delta'(u)} f(e) = 0 \). Thus, if \( f \) is a fractional perfect matching, then:

\[
 f(E(G)) = \sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \in \delta'(v)} f(e) = \frac{|V(G)|}{2};
\]

and if \( f \) is a fractional almost perfect matching, then:

\[
 f(E(G)) = \sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \in \delta'(v)} f(e) = \frac{|V(G)| - 1}{2}.
\]

We know that although an even graph has no almost perfect matching, an even graph can have a fractional perfect matching. Recently, Liu and Liu in [10] introduced some natural and nice generalizations of the above concepts. An edge subset \( F \) of \( G \) is a **fractional matching preclusion set** (FMP set for short) if \( G - F \) has no fractional perfect matchings. The **fractional matching preclusion number** (FMP number for short) of \( G \), denoted by \( fmp(G) \), is the minimum size of FMP sets of \( G \). That is, \( fmp(G) = \min\{|F| : F \text{ is an FMP set}\} \). A FMP set of minimum cardinality is called optimal. We refer the readers to [10,12] for more details and additional references.

**Proposition 1.1.** Let \( G \) be a graph. Then \( fmp(G) \leq \delta(G) \), where \( \delta(G) \) is the minimum degree of \( G \). Especially, if the number of vertices in \( G \) is even, then \( \text{mp}(G) \leq fmp(G) \).

If \( G \) is maximally matched, then \( fmp(G) = \delta(G) \). But, if the number of vertices in \( G \) is odd, \( \text{mp}(G) \) and \( fmp(G) \) do not have the same inequality relation. Some examples are given in [10].

A set \( F \) of edges and vertices of \( G \) is a **fractional strong matching preclusion set** (FSMP set for short) if \( G - F \) has no fractional perfect matchings. The **fractional strong matching preclusion number** (FSMP number for short) of \( G \), denoted by \( fspm(G) \), is the minimum size of FSMP sets of \( G \), that is, \( fspm(G) = \min\{||F| : F \text{ is an FSMP set}\} \). A FSMP set of minimum cardinality is called optimal.
Proposition 1.2. Let $G$ be a graph. $fspm(G) \leq fmp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of $G$.

If $fmp(G) = \delta(G)$, then $G$ is fractional maximally matched; if, in addition, $G - F$ has an isolated vertex for every optional fractional matching preclusion set $F$, then $G$ is fractional super matched. If $fspm(G) = \delta(G)$, then $G$ is fractional strongly maximally matched; if, in addition, $G - F$ has isolated vertices for every optimal fractional strong matching preclusion set $F$, then $G$ is fractional strongly super matched.

We refer to the book [1] for graph theoretical notations and terminology that are not defined here. Given a graph $G$, we let $V(G)$, $E(G)$, and $(u, v)$ denote the set of vertices, the set of edges, and the edge whose end vertices are $u$ and $v$, respectively. For a set of $E(G)$, we use $G - F$ to denote the subgraph of $G$ obtained by removing all the vertices or the edges of $F$. For a nonempty subset $S$ of $V(G)$, we use $G[S]$ to denote the subgraph of $G$ induced by $S$. We denote by $K_n$ the complete graph on $n$ vertices. If a cycle contains every vertex of $G$ exactly once, then the cycle is called a Hamiltonian cycle. If there is a Hamiltonian cycle in $G$, then $G$ is called Hamiltonian and if there is a Hamiltonian path between every two distinct vertices of $G$, then $G$ is called Hamiltonian connected.

Let $G$ and $H$ be two simple graphs. The Cartesian Product $G \times H$ is the graph with vertex set $V(G) \times V(H) = \{uv : u \in V(G), v \in V(H)\}$, in which two vertices $u_1v_1$ and $u_2v_2$ are adjacent if and only if $u_1 = u_2$ and $(v_1, v_2) \in E(H)$, or $(u_1, u_2) \in E(G)$ and $v_1 = v_2$. Graph $G_n$ is formed by $n$ copies of $G$, and these copies are “interconnected” by edges of $E(C_n)$. Let these copies be $G_0, G_1, \ldots, G_{n-1}$ labeled along the cycle $C_n$. The edges between different copies of $G_i$ are called cross edges. It follows that each vertex $u$ of $G_i$ has two neighbors that not in $G_i$, called cross neighbors of $u$. Furthermore, the cross neighbors of $u$ are in different copies of $G_i$ (i.e., the one is in $G_{i+1}$ and the other one is in $G_{i-1}$). If $u$ is in $G_0$, then its cross neighbors are in $G_1$ and $G_{n-1}$, respectively. In short, for each vertex in $G_n$, its cross neighbors are in adjacent copies of $G$.

The torus network is one of the most popular interconnection networks for massively parallel computing systems. The torus forms a basic class of interconnection networks. Let $C_k$ be the cycle of length $k$ with the vertex set $\{0, 1, \ldots, k-1\}$. Two vertices $u, v \in V(C_k)$ are adjacent in $C_k$ if and only if $u = v + 1$ (mod $k$). The torus $T(k_1, k_2, \ldots, k_n)$ with $n > 2$ and $k_i > 3$ for all $i$ is defined to be $C_{k_1} \times \ldots \times C_{k_n}$, with the vertex set $\{u_1u_2 \ldots u_n : u_i \in \{0, 1, \ldots, k_i - 1\}, 1 \leq i \leq n\}$. Two vertices $u_1v_1 \ldots u_nv_n$ and $v_1v_2 \ldots v_nv_n$ are adjacent in $T(k_1, k_2, \ldots, k_n)$ if and only if there exists some $j \in \{1, 2, \ldots, n\}$ such that $u_j = v_j + 1$ (mod $k_j$) and $u_i = v_i$ for $i \in \{1, 2, \ldots, n\} - \{j\}$. Clearly, $T(k_1, k_2, \ldots, k_n)$ is a connected $2k \cdot \sum k_i$ regular graph consisting of $k_1k_2 \ldots k_n$ vertices. Let $T(k_1, k_2)$ be a 2-dimensional torus, $k_1 \geq 5$ and $k_2 \geq 5$ are two odd integers. Then $T(k_1, k_2) = C_{k_1} \times C_{k_2}$. We view $C_{k_1}$, $C_{k_2}$ as consisting of $k$ copies of $C_{k_i}$. Let $C_{k_1}^0, C_{k_1}^1, \ldots, C_{k_1}^{k_2-1}$ be these copies labeled along the cycle $C_{k_2}$.

2. Preliminaries

Many further ideas and results on fractional graph theory can be found in [14] including a fractional analogue of Tutte’s Theorem:

Proposition 2.1. [14] A graph $G$ has a fractional perfect matching if and only if $i(G - S) \leq |S|$ for every set $S \subseteq V(G)$, where $i(G - S)$ is the number of isolated vertices of $G - S$.

In [15], Wang and Feng obtained the following result.

Theorem 2.2. [15] Let $n \geq 3$ be an integer and $C_n$ be a cycle of length $n$. Then $spm(C_n) = 2$.

In [3] E. Cheng and L. Lipták obtained the matching preclusion number of $T(k_1, k_2, \ldots, k_n)$.

Theorem 2.3. [3] Let $T(k_1, k_2, \ldots, k_n)$ be a torus of order even. Then

$$mp(T(k_1, k_2, \ldots, k_n)) = 2n$$
and each of its minimum MP sets is trivial.

In [10], Liu and Liu studied the FMP number and FSMP number and obtained the following result.

**Theorem 2.4**.[10] Let \( n \geq 3 \). Then \( f_{\text{fsmp}}(K_n) = n - 2 \).

Moreover, the existence of a fractional perfect matching in a graph can be inferred from the existence of a particular partition of its vertices:

**Proposition 2.5**.[14] The graph \( G \) has a fractional perfect matching if and only if there is a partition \( \{V_1, V_2, \ldots, V_n\} \) of the vertex set of \( V(G) \) such that, for each \( i \) the graph \( G[V_i] \) is either \( K_2 \) or a Hamiltonian graph on odd number of vertices.

As an immediate application of Proposition 2.5, we have the following proposition.

**Proposition 2.6**.[11] If a graph \( G \) is Hamiltonian, then \( G \) has a fractional perfect matching.

**Lemma 2.7**.[11] Let \( G \) be fractional strongly super matching graph with \( \delta(G) \geq 2 \). If \( F \) is a trivial FSMP set of \( G \) and \( G - F \) has an isolated vertex \( v \), then \( G - F - v \) has a fractional perfect matching.

**Lemma 2.8**.[16] Let \( k \geq 4 \) be an even integer. Then the following statements hold:
- (a) mp\((T(k, k))\) = 4;
- (b) every minimum matching preclusion set in \( T(k, k) \) is trivial.

In [8], Feng and Wang investigated the strong matching preclusion number of a graph.

**Lemma 2.9**.[8] Let \( k_1 > 5 \) and \( k_2 > 5 \) be two odd integers. Then \( T(k_1, k_2) \) is super strong matched.

3. Cartesian product of a graph and a cycle

In this section, we study the FSMP number of the Cartesian product of a graph and a cycle.

**Theorem 3.1**. Let \( n > 5 \) be an odd integer. Let \( G \) be a graph with minimum degree \( \delta(G) \). If \( \delta(G) \geq 4 \) and \( C_n \) is a cycle with \( n \) vertices and \( G \) is fractional strong super matched, then

\[
f_{\text{fsmp}}(G \sq C_n) = \delta(G) + 2
\]

Moreover, every optimal fractional strongly matching preclusion set is trivial.

Proof. Let \( F \subseteq V(G \sq C_n) \cup E(G \sq C_n) \) with \( |F| \leq \delta(G) + 2 \), and \( F_i = F \cap (V(G_i) \cup E(G_i)) \) for \( 0 \leq i \leq n - 1 \). For notational convenience, assume \( |F_{i-1}| \geq |F_i| \) for \( 1 \leq i \leq n - 1 \). The proofs in other cases are similar, and we always start the discussion from the largest \( |F_i| \) set. We use \( f_i \) indicate that there is a fractional perfect matching in \( G \).

We know that if \( G_0 - F_0 \) has fractional perfect matching, then there exists a partition \( \{V_1, V_2, \ldots, V_m\} \) of the vertex set of \( G_0 - F_0 \) such that the graph \( (G_0 - F_0)[V_i] \) is either \( K_2 \) or a Hamiltonian graph with an odd number of vertices by Proposition 2.5. We distinguish the following cases to show this theorem.

**Case 1.** \( |F_0| = \delta(G) + 2 \).

Clearly \( |F_i| = 0 \) for \( 1 < i < n - 1 \). Let \( F'_0 = F_0 - \{\alpha, \beta\} \) for any \( \alpha, \beta \in F_0 \). Since \( G_0 \) is fractional strongly super matched, it follows that either \( G_0 - F'_0 \) has a fractional perfect matching or \( F'_0 \) is a trivial FSMP set of \( G_0 \).

Subcase 1.1. \( G_0 - F'_0 \) has a fractional perfect matching.

From Proposition 2.5, there is a partition \( \{V_1, V_2, \ldots, V_n\} \) of the vertex set of \( G_0 - F'_0 \) such that, for each \( i \) the graph \( (G_0 - F'_0)[V_i] \) is either \( K_2 \) or a Hamiltonian graph on an odd number of vertices.

Suppose that \( \alpha, \beta \) are two end vertices of \( K_2 \) in some \( (G_0 - F'_0)[V_i] \), then there is a fractional
perfect matching \( f_0 \) in \( G_0 - F_0' - \{ \alpha, \beta \} \), and hence \( \bigcup_{i=0}^{n-1} f_i \) induce a fractional perfect matching of \( G - C_n - F \). Suppose that \( \alpha \) and \( \beta \) are two vertices from two distinct \( K_2 \). Let \( \alpha' \) and \( \beta' \) be two neighbors of \( \alpha \) and \( \beta \) in such two distinct \( K_2 \), respectively. Assume that \( n \) and \( \nu \) are cross neighbors of \( \alpha' \) and \( \beta' \), respectively. By Proposition 2.5, \( G_0 - F_0' - \{ \alpha, \alpha' \} - \{ \beta, \beta' \} \) has a fractional perfect matching \( f_0 \). From the definition of the \( G - C_n \), we know that \( n \) and \( \nu \) are either in \( G_1 \) or in \( G_{n-1} \). Let \( f'_i = \{ u, v \} \cap V(G_i) \) for \( i = 1, n-1 \). It is clear that \( |f'_i| \leq 2 < \delta(G) \) for \( i = 2, n-1 \). Since \( G \) is fractional strong super matched, \( G_i - F_i' \) has a fractional perfect matching \( f_i \) for \( i = 1, n-1 \). Note that \( G_i \) has a fractional perfect matching \( f_i \) for \( 2 < i < n-2 \). Since \( (\alpha', u) \) and \( (\beta', v) \) are two matches, we assign their edges to 1, and both satisfy the condition of the fractional perfect matching. Thus \( \bigcup_{i=0}^{n-1} f_i \) and \( \{(\alpha', u), (\beta', v)\} \) induce a fractional perfect matching of \( G - C_n - F \).

Suppose \( \alpha, \beta \) are vertices of cycle \( C \) on an odd number of vertices in \( (G_0 - F_0' )[V_i] \). It is clear that there is at most one path on an odd number of vertices in \( C - \{ \alpha, \beta \} \). Assume that \( \nu \) is the end vertex of the path on an odd number of vertices. Obviously, there is a fractional perfect matching \( f_0 \) in \( G_0 - \{ \alpha, \beta \} - \nu \). Let \( \nu' \) be cross neighbor of \( \nu \). Let \( f_i' = \{ \nu' \} \cap V(G_i) \) for \( i = 1, n-1 \). It is clear that \( |f_i'| \leq 1 < \delta(G) \) for \( i = 1, n-1 \). Since \( G \) is fractional strong super matched, it follows that \( G_i - F_i' \) has a fractional perfect matching \( f_i \) for \( i = 1, n-1 \). Note that \( G_i \) has a fractional perfect matching \( f_i \) for \( 2 < i < n-2 \). Since \( (\nu, \nu') \) is a match, we assign this edge to 1, which satisfy the condition of the fractional perfect matching. And hence \( \bigcup_{i=0}^{n-1} f_i \) and \( \{(\nu, \nu')\} \) induce a fractional perfect matching of \( G - C_n - F \).

Suppose \( \alpha \) is a vertex of \( K_2 \) and \( \beta \) is a vertex of cycle \( C \) on an odd number of vertices in \( (G_0 - F_0' )[V_i] \). Let \( \alpha' \) be neighbor of \( \alpha \) in \( K_2 \). Then there is a path on an even number of vertices in \( C - \beta \). Assume that \( \nu' \) is a cross neighbor of \( \alpha' \). Let \( f_i' = \{ \alpha' \} \cap V(G_i) \) for \( i = 1, n-1 \). It is clear that \( |f_i'| \leq 1 < \delta(G) \) for \( i = 1, n-1 \). Since \( G \) is fractional strong super matched, it follows that \( G_i - F_i' \) has a fractional perfect matching \( f_i \) for \( i = 1, n-1 \). Note that \( G_i \) has a fractional perfect matching \( f_i \) for \( 2 < i < n-2 \). Therefore, \( \bigcup_{i=0}^{n-1} f_i \) and \( \{(\alpha', \nu')\} \) induce a fractional perfect matching of \( G - C_n - F \).

Suppose \( \alpha = (v, \omega) \) and \( \beta = (x, y) \) are two edges which form two \( K_2 \)’s. Assume that \( \omega', \omega', x' \) and \( y' \) are cross neighbors of \( v, \omega, x \) and \( y \), respectively. It is clear that there is a fractional perfect matching \( f_0 \) in \( G_0 - \{ v, \omega, x, y \} \). Let \( f_i' = \{ \omega', \omega', x', y' \} \cap V(G_i) \) for \( i = 1 \) or \( n-1 \). By the definition of networks \( G - C_n \), we can assume that \( |f_i'| \leq 2 \) for \( i = 1 \) or \( n-1 \). Since \( G \) is fractional strong super matched, it follows that \( G_i - F_i' \) has a fractional perfect matching \( f_i \) for \( 2 < i < n-2 \). Hence \( \bigcup_{i=0}^{n-1} f_i \) and \( \{(\omega', \omega'), (w, \omega'), (x, x'), (y, y')\} \) induce a fractional perfect matching of \( G - C_n - F \).

Suppose \( \alpha, \beta \) are two edges of cycle \( C \) on an odd number vertices in \( (G_0 - F_0') [V_i] \). Then we have the following three cases in \( C - \{ \alpha, \beta \} \):

1. there is an isolate vertex and a path on an even number of vertices;
2. there is a path on odd number of vertices and a path on even number of vertices;
3. there are two paths on an odd number of vertices.

We only need to prove the last case, the other two cases are similar. Let \( e \) be an end vertex of one path on an odd number of vertices and \( w \) be an end vertex of another path on an odd number of vertices. Assume that \( \omega' \) and \( w' \) are cross neighbors of \( v \) and \( w \), respectively. Let \( f_i' = \{ \omega', \omega' \} \cap V(G_i) \) for \( i = 1, n-1 \). It is clear that \( |f_i'| \leq 2 < \delta(G) \) for \( i = 1, n-1 \).
Since $G$ is fractional strong super matched, it follows that $G_i - F_i$ has a fractional perfect matching $f_i$ for $i = 1, n - 1$. Note that $G_i$ has a fractional perfect matching $f_i$ for $2 \leq i \leq n - 2$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v'), (w, w')\}$ induce a fractional perfect matching of $GaC_n - F$.

Suppose $\alpha = (v, w)$ is an edge of $K_2$ and $\beta$ is an edge of cycle $C$ on an odd number of vertices in $(G_0 - F_0)[V_i]$. There exists a path on an odd number of vertices in $C - \beta$. Let $x$ be an end vertex of the path on an odd number of vertices. Assume that $v', w'$ and $x'$ are cross neighbors of $v, w$ and $x$, respectively. Let $F_i = \{v', w', x'\} \cap V(G_i)$ for $i = 1, n - 1$. It is clear that $|F_i| \leq 3 < \delta(G)$ for $i = 1, n - 1$. Since $G$ is fractional strong super matched, it follows that $G_i - F_i$ has a fractional perfect matching $f_i$ for $i = 1, n - 1$. Note that $G_i$ has a fractional perfect matching $f_i$ for $2 < i < n - 2$. Then $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v'), (w, w'), (x, x')\}$ induce a fractional perfect matching of $GaC_n - F$.

Suppose $\alpha$ is a vertex of $K_2$ and $\beta$ is an edge of cycle $C$ on an odd number of vertices in $(G_0 - F_0)[V_i]$. Let $\alpha'$ be neighbor of $\alpha$. There exists a path on an odd number of vertices in $C - \beta$. Let $x$ be an end vertex of the path on an odd number of vertices. Assume that $x'$ and $\alpha''$ are cross neighbors of $x$ and $\alpha'$, respectively. There is a fractional perfect matching $f_0$ in $G_0 - F_0 - \{\alpha, \alpha', \beta, x\}$. Let $F_i = \{\alpha'', x'\} \cap V(G_i)$ for $i = 1, n - 1$. It is clear that $|F_i| \leq 2 < \delta(G)$ for $i = 1, n - 1$. Since $G$ is fractional strong super matched, it follows that $G_i - F_i$ has a fractional perfect matching $f_i$ for $i = 1, n - 1$. Since $G_i$ has a fractional perfect matching $f_i$ for $2 < i < n - 2$, it follows that $\bigcup_{i=0}^{n-1} f_i$ and $\{(\alpha', \alpha''), (x, x')\}$ induce a fractional perfect matching of $GaC_n - F$.

Suppose $\alpha = (v, w)$ is an edge of $K_2$ and $\beta$ is a vertex of cycle $C$ on an odd number of vertices in $(G_0 - F_0)[V_i]$. Obviously, there is a fractional perfect matching in $C - \beta$. Assume that $v'$ and $w''$ are cross neighbors of $v$ and $w$, respectively. Let $F_i = \{v', w''\} \cap V(G_i)$ for $i = 1, n - 1$. It is clear that $|F_i| \leq 2 < \delta(G)$ for $i = 1, n - 1$. Since $G$ is fractional strong super matched, $G_i - F_i$ has a fractional perfect matching $f_i$ for $i = 1, n - 1$. Note that $G_i$ has a fractional perfect matching $f_i$ for $2 \leq i \leq n - 2$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v'), (w, w'')\}$ induce a fractional perfect matching of $GaC_n - F$.

Subcase 1.2. $F_0' = \emptyset$ is a trivial FSMP set.

Let $G_0' = G_0 - F_0' = v$ where $v$ is an isolated vertex in $G_0 - F_0'$. By Lemma 2.7, we know that $G_0'$ has a fractional perfect matching. Thus, the proof in Subcase 1.2 only needs to consider an isolated $\{v\}$ more than the proof in Subcase 1.1. From the proof in Subcase 1.1, it is clear that $GaC_n - F - v$ has a fractional perfect matching. In Subcase 1.1, there are at most four vertices not matched in $G_0 - F$. Now, there are at most five vertices that not matched in $G_0 - F$, let us say $\{w_1, w_2, w_3, w_4, v\}$. So there is a fractional perfect matching $f_0$ in $G_0 - F = \{w_1, w_2, w_3, w_4, v\}$. Let $w_1', w_2', w_3', w_4'$ and $v'$ be cross neighbors of $w_1, w_2, w_3, w_4$ and $v$, respectively. By the definition of $GaC_n$, we can assume that $F_i = \{w_1', w_2', w_3', w_4', v'\} \cap V(G_i)$ such that $|F_i| \leq 3 < \delta(G)$ for $i = 1, n - 1$. Since $G$ is fractional strong super matched, $G_i - F_i$ has a fractional perfect matching $f_i$ for $i = 1, n - 1$. Note that $G_i$ has a fractional perfect matching $f_i$ for $2 < i < n - 2$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(w_1, w_1'), (w_2, w_2'), (w_3, w_3'), (w_4, w_4'), (v, v')\}$ induce a fractional perfect matching of $GaC_n - F$.

Case 2. $|F_0| = \delta(G) + 1$.

Clearly $|F_0| \leq 1$ for $1 \leq i \leq n - 1$. Let $F_0' = F_0 - \{\alpha\}$ for every $\alpha \in F_0$. Since $G_0$ is fractional strongly super matched, it follows that either $G_0 - F_0'$ has fractional perfect
matching or $F''_0$ is a trivial FSMP set of $G_0$. Then we have the following two subcases to consider.

Subcase 2.1. $G_0 - F'_0$ has fractional perfect matching.

Suppose $\alpha$ is a vertex of $K_2$. Let $\alpha'$ be neighbor of $\alpha$. Assume that $\alpha''$ is a cross neighbor of $\alpha'$. It is clear that $G_0 - F''_0 - \{\alpha, \alpha''\}$ has a fractional perfect matching $f_0$.

Let $F'_i = \{(\alpha'') \cap V(G_i)\} \cup F_i$ for $1 < i < n - 1$. Obviously, $|F'_i| \leq 2 \delta(G)$, $(1 \leq i \leq n - 1)$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(\alpha', \alpha'')\}$ induce a fractional perfect matching of $G_0 C_n - F$. Let $\alpha$ be a vertex of cycle $C$ on an odd number of vertices in $(G_0 - F''_0)[V_i]$. It is obvious that $G_0 - F''_0 - \{\alpha\}$ has a fractional perfect matching $f_0$. Then, $|F'_i| \leq 2 \delta(G)$, $(1 \leq i \leq n - 1)$, thus $G_i - F'_i$ has a fractional perfect matching $f_i$. Thus $\bigcup_{i=0}^{n-1} f_i$ induce a fractional perfect matching of $G_0 C_n - F$.

Suppose $\alpha$ is an edge and $\alpha = (v, w)$. Let $\alpha$ be an edge of $K_2$. There still exists a fractional perfect matching $f_0$ in $G_0 - F'_0 - \{v, w\}$. Assume that $v'$ and $w'$ are cross neighbors of $v$ and $w$, respectively. Let $F'_i = \{(v', w') \cap V(G_i)\} \cup F_i$ for $1 < i < n - 1$. Obviously, $|F'_i| \leq 3 \delta(G)$, $(1 \leq i \leq n - 1)$, thus $G_i - F'_i$ has a fractional perfect matching $f_i$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v'), (v', w')\}$ induce a fractional perfect matching of $G_0 C_n - F$. Suppose $\alpha$ is an edge of cycle $C$ on an odd number of vertices in $(G_0 - F''_0)[V_i]$. It is clear that there is a path on an even number of vertices in $C - \{v\}$ or $C - \{w\}$. We only need to prove the case of $C - \{v\}$. By Proposition 2.5, we can derive that $G_0 - F''_0 - v$ has a fractional perfect matching $f_0$. We assume that $v'$ is a cross neighbor of $v$. Let $F'_i = \{(v') \cap V(G_i)\} \cup F_i$ for $1 \leq i \leq n - 1$. Thus $|F'_i| \leq 2 \delta(G)$, $G_i - F'_i$ has a fractional perfect matching $f_i$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v')\}$ induce a fractional perfect matching of $G_0 C_n - F$.

Subcase 2.2. $F'_0$ is a trivial FSMP set.

This proof is similar to the proof of Subcase 1.2.

Case 3. $|F'_0| = \delta(G)$.

Because $G_0$ is fractional strongly super matched, either $G_0 - F_0$ has a fractional perfect matching or $F'_0$ is a trivial FSMP set. Suppose that $G_0 - F_0$ has a fractional perfect matching $f_0$. It is clear that $|F'_i| \leq 2 \delta(G)$ for $1 < i < n - 1$, so $G_i - F_i$ has a fractional perfect matching $f_i$. Thus $\bigcup_{i=0}^{n-1} f_i$ induces a fractional perfect matching of $G_0 C_n - F$. Suppose that $F'_0$ is trivial FSMP set, let $v$ be an isolated vertex in $G_0 - F_0$. There is a fractional perfect matching $f_0$ in $G_0 - F_0 - v$ by Lemma 2. Assume that $v'$ is a cross neighbor of $v$. Let $F'_i = \{(v') \cap V(G_i)\} \cup F_i$ for $1 \leq i \leq n - 1$. Thus $|F'_i| \leq 3 \delta(G)$, $G_i - F'_i$ has a fractional perfect matching $f_i$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v')\}$ induce a fractional perfect matching of $G_0 C_n - F$.

Case 4. $|F'_0| \leq \delta(G) - 1$.

Since $G_i$ is fractional strong super matching, $G_i - F_i$ has a fractional perfect matching $f_i$. Thus $\bigcup_{i=0}^{n-1} f_i$ induce a fractional perfect matching of $G_0 C_n - F$.

Now that we have completed the proof to show that $G_0 C_n$ is fractional strongly super matched. Let us look at some applications of this result to torus networks.

4. Results for torus

In this section, first we obtain $fmp(T(k_1, k_2, \ldots, k_n)) = 2n$ when $T(k_1, k_2, \ldots, k_n)$ is a torus with an even number of vertices. Secondly we prove that $fsmmp(T(k_1, k_2)) = 4$, which $k_1 > 5$ and $k_2 \geq 5$ are both odd integers and every optional fractional matching preclusion set is trivial. Then we can infer that $fsmmp(T(k_1, k_2, \ldots, k_n)) = 2n$ for $n \geq 2$, $k_i \geq 5$ ($k_i$ is odd).

Theorem 4.1. Let $T(k_1, k_2, \ldots, k_n)$ be a torus with an even number of vertices. Then
Moreover, every optional fractional matching preclusion set is trivial.

Proof. From the definition of $T(k_1, k_2, \ldots, k_n)$, it is a connected $2n$-regular graph. By Theorem 2.3, we know that $mp(T(n_1, n_2, \ldots, n_k)) = 2n$. It is clear that $mp(T(k_1, k_2, \ldots, k_n)) \leq mp(T(n_1, n_2, \ldots, n_k))$ and $mp(T(k_1, k_2, \ldots, k_n)) = \delta(T(k_1, k_2, \ldots, k_n)) = 2n$. By Theorem 2.3, each of minimum $MP$ sets of $T(k_1, k_2, \ldots, k_n)$ is trivial, so every optimal fractional matching preclusion set is trivial. The proof is complete.

**Theorem 4.2.** Let $k_1 > 5$ and $k_2 > 5$ be two odd integers. Then $f_{sm}(T(k_1, k_2)) = 4$ and every optional fractional matching preclusion set is trivial.

Proof. In this part, first we introduce some notations. Let $F \subseteq V(T(k_1, k_2)) \cup E(T(k_1, k_2))$, $F_i = F \cap (V(C_{k_1}^i) \cup E(C_{k_1}^i))$ for $0 \leq i \leq k_2 - 1$. Let $F^V = F \cap V(T(k_1, k_2))$, $F^E = F \cap E(T(k_1, k_2))$ and $|F| = |F^V| + |F^E| = 4$. It suffices to prove that $T(k_1, k_2) - F$ has a fractional perfect matching or $F$ is a trivial fractional matching preclusion set. We may assume that $T(k_1, k_2) - F$ has a fractional perfect matching. If $|F^V|$ is even, then $T(k_1, k_2) - F$ has a perfect matching by Lemma 2.9. So we only consider that $|F^V|$ is odd. Without loss of generality, we assume $|F_{i-1}| \geq |F_i|$ for $1 < i < k_2 - 1$. The proofs in other cases are similar, and we always start the discussion from the largest $|F_i|$ set. If a graph contains an isolated vertex or a path with an odd number of vertices, then the graph has no fractional perfect matching. So in this proof, we consider an isolated vertex as a path with an odd number of vertices. We use $f_i$ indicate that there is a fractional perfect matching in $C_{k_1}^i$.

**Case 1.** $|F^V| = 3$.

Suppose that $|F_0| = 4$. Clearly $|F_i| = 0$ for $1 < i < k_2 - 1$. Since $|F^V| = 3$, it follows that $|F^E| = 1$. It is clear that there are at most four paths with an odd number of vertices in $C_{k_1}^0 - F_0$. Suppose that $v_1, v_2, v_3$ and $v_4$ are end vertices of the four paths, respectively. Obviously, $C_{k_1}^0 - F_0 - \{v_1, v_2, v_3, v_4\}$ has a fractional perfect matching $f_0$. By definition of $T(k_1, k_2)$, each vertex of $C_{k_1}^0$ has two cross neighbors, one is in the $C_{k_1}^1$ and other one is in the $C_{k_1}^{k_2-1}$. Assume that $v_1'$ and $v_2'$ in $C_{k_1}^1$ are cross neighbors of $v_1$ and $v_2$, $v_3'$ and $v_4'$ in $C_{k_1}^{k_2-1}$ are cross neighbors of $v_3$ and $v_4$. It is clear that there is at most one path with an odd number of vertices in $C_{k_1}^1 - \{v_1', v_2'\}$ and there is at most one path with an odd number of vertices in $C_{k_1}^{k_2-1} - \{v_3', v_4'\}$. Suppose that $w_1$ is an end vertex of the path with an odd number of vertices in $C_{k_1}^1 - \{v_1', v_2'\}$; $w_2$ is an end vertex of the path with an odd number of vertices in $C_{k_1}^{k_2-1} - \{v_3', v_4'\}$. Obviously, $C_{k_1}^0 - \{v_1', v_2'\} - w_1$ has a fractional perfect matching $f_1$ and hence $C_{k_1}^{k_2-1} - \{v_3', v_4'\} - w_2$ has a fractional perfect matching $f_{k_2-1}$. Assume that $w_1'$ in $C_{k_1}^0$ is a cross neighbor of $w_1$; $w_2'$ in $C_{k_1}^{k_2-2}$ is a cross neighbor of $w_2$. Obviously, $C_{k_1}^0 - W_1$ has a fractional perfect matching $f_2$ and $C_{k_1}^{k_2-2} - w_2'$ has a fractional perfect matching $f_{k_2-2}$. We can conclude that $\bigcup_{i=0}^{k_2-1} f_i$ and $\{(v_1, v_1'), (v_2, v_2'), (v_3, v_3'), (v_4, v_4'), (w_1, w_1'), (w_2, w_2')\}$ induce a fractional perfect matching of $T(k_1, k_2) - F$.

Suppose that $|F_0| = 3$, $|F_1| = 1$. Then $|F_i| = 0$ for $2 \leq i \leq k_2 - 1$. If $F_0$ contains three vertices, then $F_1$ contains an edge. So there are at most two paths with an odd number of vertices in $C_{k_1}^0 - F_0$ and there is one path on an odd number of vertices in $C_{k_1}^1 - F_1$. Suppose that $v_1$ and $v_2$ are end vertices of the two paths on odd number of vertices in $C_{k_1}^0 - v_1, v_2$ is end vertex of the path with an odd number of vertices in $C_{k_1}^1$. Then $C_{k_1}^0 - F_0 - \{v_1, v_2\}$ has a fractional perfect matching $f_0$ and hence $C_{k_1}^1 - F_1 - v_3$ has a fractional perfect matching $f_1$. Assume that $v_1'$ and $v_2'$ in $C_{k_1}^1$ are cross neighbors of $v_1$ and $v_2$; $v_3'$ in $C_{k_1}^2$ is cross neighbor of $v_3$. It is clear that
has a fractional perfect matching $f_2$ and hence there is at most one path with an odd number of vertices in $C_{k_i}^{2} - \{v'_i, v'_j\}$. Let $w$ be an end vertex of the path with an odd number of vertices. Suppose $w'$ in $C_{k_i}^{2} - w$ has a fractional perfect matching $f_{k_i-2}$ and hence $\bigcup_{i=0}^{k_i-1} f_i$ and $\{(v_1, v'_1), (v_2, v'_2), (v_3, v'_3), (w, w')\}$ induce a fractional perfect matching of $T(k_1, k_2) - F$. If $F_0$ contains two vertices and an edge, then $F_1$ contains a vertex. The proof of this case is similar to the one above. We can always find a fractional perfect matching in $T(k_1, k_2) - F$.

Suppose $|F_0|=2$. Clearly, $|F_1|=2$ or $|F_1|=1$ and $|F_2|=1$. It is easy to show that there is at most one path with an odd number of vertices in $C_{k_i}^{2} - F_0$. Let $w$ be an end vertex of the path with an odd number of vertices and $w'$ be a cross neighbor of $w$ in $C_{k_i}^{2} - v$. When $|F_1|=2$, suppose $F_1$ contain two vertices. Moreover, there is at most one path with an odd number of vertices in $C_{k_i}^{2} - F_1$. Assume that $v$ is an end vertex of the path with an odd number of vertices. So $C_{k_i}^{2} - F_1 - v$ has a fractional perfect matching $f_1$. Let $v'$ in $C_{k_i}^{2} - v$ be the cross neighbor of $v$. Obviously, $C_{k_i}^{2} - v'$ also has a fractional perfect matching $f_2$. At last, we can conclude that $\bigcup_{i=0}^{k_i-1} f_i$ and $\{(v, v'), (w, w')\}$ induce a fractional perfect matching of $T(k_1, k_2) - F$. When $|F_1|=1$ and $|F_2|=1$, let $F_1$ contain an edge. The proof is similar to the edge in $F_0$ or $F_0$, so we only need to prove that the edge is in $F_1$. There is a path with an odd number of vertices in $C_{k_i}^{2} - F_0$ and let $v_1$ be an end vertex of the path with an odd number of vertices; there is a path with an odd number of vertices in $C_{k_i}^{2} - F_1$ and let $v_2$ be an end vertex of the path with an odd number of vertices. Suppose that $v'_i$ in $C_{k_i}^{2} - v_i$ is a cross neighbor of $v_i$; $v'_j$ in $C_{k_i}^{2} - v_j$ is a cross neighbor of $v_j$. Moreover, there is at most one path on an odd number of vertices in $C_{k_i}^{2} - F_2 - v_2$ and let $v_3$ be the end vertex of the path. Assume that $v'_3$ in $C_{k_i}^{2} - v_3$ is a cross neighbor of $v_3$. At last, we can conclude that $\bigcup_{i=0}^{k_i-1} f_i$ and $\{(v_1, v'_1), (v_2, v'_2), (v_3, v'_3), (w, w')\}$ induce a fractional perfect matching of $T(k_1, k_2) - F$.

Suppose that $|F_i|=1$ for $0 < i < 3$ and $|F_i|=0$ for $4 \leq i \leq k_2 - 1$. Let $F_i$ contain an edge, the proof is similar to the other case. Obviously, $C_{k_i}^{2} - F_0$ has a fractional perfect matching $f_0$ and $C_{k_i}^{2} - F_1$ has a fractional perfect matching $f_1$. There is a path with an odd number of vertices in $C_{k_i}^{2} - F_2$ and let $v_1$ be an end vertex of the path on an odd number of vertices. Suppose $v'_i$ in $C_{k_i}^{2} - v_i$ is a cross neighbor of $v_i$. Thus there is at most a path with an odd number of vertices in $C_{k_i}^{2} - F_3 - v'_i$ and let $v_2$ be an end vertex of the path with an odd number of vertices. Suppose that $v'_2$ in $C_{k_i}^{2} - v_2$ is a cross neighbor of $v_2$. At last, we can conclude that $\bigcup_{i=0}^{k_i-1} f_i$ and $\{(v_1, v'_1), (v_2, v'_2)\}$ induce a fractional perfect matching of $T(k_1, k_2) - F$.

Case 2. $|F^V|=1$.

We can prove Case 2 in the same way as Case 1. It is easy to show that this case is identical to the corresponding Case 1 in the proof of Theorem 4.2, thus completing the proof.

**Corollary 4.3.** Let $k_i > 5$ be an odd integer for $1 < i < n$. Then $f_{smp}(T(k_1, k_2, \ldots, k_n)) = 2n$. Moreover, $T(k_1, k_2, \ldots, k_n)$ is fractional super strong matched.

**5. Conclusion**

The topic of fractional matching preclusion is getting more and more attention and we already know different research groups who are interested in this topic. The concept of fractional matching preclusion introduced in [10] is very useful in networks. In this paper, we obtain the result for Cartesian product graphs and apply this result to torus networks which is an important class of interconnection networks.
Acknowledgments
This work is supported by the National Science Foundation of China (No. 11661068), the Science Found of Qinghai Province (No. 2021 - ZJ - 703).

References
[1] J.A. Bondy,(2008) Graph Theory. GTM244, Springer U.S.R. Murty.
[2] R.C. Brigham, F. Harary, E.C. Violin, J. Yellen. (2005) Perfect matching preclusion, *Congr. Numer.* 174 185–192.
[3] E. Cheng, L. Lipták. (2012) Matching preclusion and conditional matching preclusion problems for tori and related Cartesian products, *Discrete Appl. Math.* 160 (12) 1699–1716.
[4] E. Cheng, L. Lipták. (2007) Matching preclusion for some interconnection networks, *Networks* 50 173–180.
[5] E. Cheng, L. Lesniak. M.J. Lipman, L. Lipták. (2008) Matching preclusion for alternating group graphs and their generalizations, *Int. J. Found. Comp. Sci.* 19 1413–1437.
[6] E. Cheng, L. Lesniak. M.J. Lipman, L. Lipták. (2009) Conditional Matching preclusion sets, *Inf. Sci.* 179 1092–1101.
[7] E. Cheng, S. Shah, V. Shah, D.E. Steffy. (2013) Strong matching preclusion for augmented cubes, *Theor. Comput. Sci.* 491 71–77.
[8] K. Feng, S. - Y. Wang. (2015) Strong matching preclusion for two - dimensional torus networks, *Inter. J. Comput. Math.* 473 - 485.
[9] J. - S. Jwo, S. Lakshmivarahan, S.K. Dhall. (1993) A new class of interconnection networks based on the alternating group, *Networks* 23 315–326.
[10] Y. Liu, W. Liu. (2017) Fractional matching preclusion of graphs, *J. Comb. Optim.* 34 522–533.
[11] T. Ma, Y. Mao, E. Cheng, C. Melekian. (2018) Fractional matching preclusion for (burnt) pancake graphs, *J - SPAN.* 00030 133 - 141.
[12] T. Ma, Y. Mao, E. Cheng, J. Wang. (2019) Fractional matching preclusion for arrangement graphs, *Discrete Appl. Math.*, 270 181 - 189.
[13] J. - H. Park, I. Ihm. (2011) Strong matching preclusion, *Theor. Comput. Sci.* 412 6409–6419.
[14] E.R. Scheinerman, D.H. Ullman. (1997) Fractional Graph Theory: A Rational Approach to the Theory of Graphs, *John Wiley,* New York.
[15] S. Wang, K. Feng. (2014) Strong matching preclusion for torus networks, *Theor. Comput. Sci.* 520 97 - 110.
[16] S. Wang, R. Wang, S. Lin, J. Li. (2010) Matching preclusion for $k$ - ary $n$ - cubes, *Discrete Appl. Math.* 158 2066 - 2070.