Reverse-Order Law of tensors Revisited

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Abstract

The equality \((A \ast_N B)\dagger = B\dagger \ast_N A\dagger\) for any two complex tensors \(A\) and \(B\) of arbitrary order, is called as the reverse-order law of tensors. Panigrahy et al. [Linear Multilinear Algebra; 2018. Doi: 10.1080/03081087.2018.1502252] studied very recently the reverse-order law of the Moore–Penrose inverse of even-order tensors via the Einstein product. This notion is revisited for tensors of any order. In this context, we provide a new expression of the Moore–Penrose inverse of an arbitrary order tensor via the Einstein product. We again obtain some new sufficient conditions for the reverse-order law for the Moore–Penrose inverse of even-order square tensors. We then present several characterizations for tensors of any order to hold the reverse-order law via the same product.

Keywords: Tensor, Moore–Penrose inverse, Einstein product, reverse-order law.
1. Introduction

A tensor is a multidimensional array. Many phenomena are modeled as multilinear systems in engineering and science. Such as, isotropic and anisotropic elasticity are modeled [11] as multilinear systems in continuum physics and engineering. Tensor methods have been practiced to problems in quantum chemistry: the fundamental Hatree-Fock equation is solved by Khoromskij, Khoromskaia, and Flad [8] and the works on multidimensional operators in quantum models were done by Beylkin and Mohlenkamp [2, 3]. Multidimensional boundary and eigenvalue problems are solved by Hackbusch and Khoromskij [6] and Hackbusch, Khoromskij, and Tyrtyshnikov [7] through separated representation and hierarchical Kronecker tensor from the underlying high spatial dimensions using a reduced low-dimensional tensor-product space. An $N$th-order tensor is an element of $\mathbb{C}^{I_1 \times \cdots \times I_N}$ which is the set of order $N$ complex tensors. Here $I_1, I_2, \cdots, I_N$ are dimensions of the first, second, \cdots, $N$th way, respectively. The order of a tensor is the number of dimensions. A first-order tensor is a vector while a second-order tensor is a matrix. Higher-order tensors are tensors of order three or higher. These are denoted by calligraphic letters like $\mathcal{A}$. $a_{ijk}$ denotes an $(i, j, k)$th element of a third order tensor $\mathcal{A}$ and $a_{i_1 \cdots i_N}$ represents an $(i_1, ..., i_N)$th element of an $N$th order tensor $\mathcal{A}$. For tensors $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$ and $\mathcal{B} \in \mathbb{C}^{J_1 \times \cdots \times J_M \times K_1 \times \cdots \times K_L}$, the product $\mathcal{A} \ast_M \mathcal{B} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_L}$ defined via

\[
(\mathcal{A} \ast_M \mathcal{B})_{i_1 \cdots i_N k_1 \cdots k_L} = \sum_{j_1 \cdots j_M} a_{i_1 \cdots i_N j_1 \cdots j_M} b_{j_1 \cdots j_M k_1 \cdots k_L}
\]

by the operation $\ast_M$ is called the Einstein product (5). The associative law of this tensor product holds. In the above formula, if $\mathcal{B} \in \mathbb{C}^{J_1 \times \cdots \times J_M}$, then $\mathcal{A} \ast_M \mathcal{B} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ and

\[
(\mathcal{A} \ast_M \mathcal{B})_{i_1 \cdots i_N} = \sum_{j_1 \cdots j_M} a_{i_1 \cdots i_N j_1 \cdots j_M} b_{j_1 \cdots j_M}.
\]

This product is used in the study of the theory of relativity (5) and in the area of continuum mechanics (11). The Einstein product $\ast_1$ reduces to the standard matrix multiplication as

\[
(A \ast_1 B)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.
\]
for $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times l}$.

There has been active research on tensors for the past four decades. For applications and tensor-based methods, we refer the readers to the survey papers [9, 10, 12] and the references therein. But, research contributions on the theory and applications of generalized inverses of tensors are very little. In fact, the first work in this direction was reported in 2016 (see [15]) where the authors introduced a generalized inverse called the Moore–Penrose inverse of an even-order tensor via the Einstein product. In 2018, Panigrahy and Mishra [14] improved their definition to tensor of any order via the same product, and is recalled below.

**Definition 1.1.** (Definition 1.1, [14]) Let $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$. The tensor $\mathcal{Y} \in \mathbb{C}^{J_1 \times \cdots \times J_M \times I_1 \times \cdots \times I_N}$ satisfying the following four tensor equations:

\begin{align*}
\mathcal{X} *_{M} \mathcal{Y} *_{N} \mathcal{X} & = \mathcal{X}; \quad (1) \\
\mathcal{Y} *_{N} \mathcal{X} *_{M} \mathcal{Y} & = \mathcal{Y}; \quad (2) \\
(\mathcal{X} *_{M} \mathcal{Y})^H & = \mathcal{X} *_{M} \mathcal{Y}; \quad (3) \\
(\mathcal{Y} *_{N} \mathcal{X})^H & = \mathcal{Y} *_{N} \mathcal{X}, \quad (4)
\end{align*}

is defined as the **Moore–Penrose inverse** of $\mathcal{X}$, and is denoted by $\mathcal{X}^\dagger$.

In the case of an even-order invertible tensor, the above definition coincides with the notion of the inverse which was first introduced by Brazell et al. [4]. They also showed that such an inverse can be computed using the singular value decomposition of the same tensor (see the celebrated result in Lemma 3.1, [4]).

This paper is in connection with an open problem stated in the last section of [1] for even-order tensors, and is a continuation of the very recent works done in [13] and [14]. The same open problem is restated next in the setting of tensors of any order.
Problem 1. When does \((A^*N)^\dagger = B^\dagger N A^\dagger\) for any two tensors \(A \in \mathbb{C}^{I_1 \times \ldots \times I_M \times J_1 \times \ldots \times J_N}\) and \(B \in \mathbb{C}^{J_1 \times \ldots \times J_N \times K_1 \times \ldots \times K_L}\)?

Note that Panigrahy et al. [13] attempted first the above problem, very recently. They provided various necessary and sufficient conditions for the same problem, but for even-order tensors only. Nevertheless, our aim is to establish some new necessary and sufficient conditions for the above-stated problem which is well-known as reverse-order law for the Moore–Penrose inverse of tensors via the Einstein product. In this context, the paper is organized as follows. Section 2 collects various useful definitions and results. The next section which contains all our main results is devoted to the above-stated problem and its solution. It has four subsections. The first subsection provides a new expression of the Moore–Penrose inverse of a tensor. The next one presents some results for even-order square tensors. The third one deals with tensors of any order and also contains several results on the reverse-order law. The last one discusses a few properties of different generalized inverses of a tensor and then illustrates a different expression of the Moore-Penrose inverse of the Einstein product of two tensors in terms of the Einstein product of the Moore-Penrose inverse of a tensor and a generalized inverse of a tensor.

2. Prerequisites

Here, we collect all those remaining definitions and earlier results which will be used to prove the main results in the next section. We begin with the definition of an identity tensor. A tensor \(I \in \mathbb{C}^{I_1 \times \ldots \times I_N \times I_1 \times \ldots \times I_N}\) with entries \((I)_{i_1 \ldots i_N j_1 \ldots j_N} = \prod_{k=1}^N \delta_{i_k j_k}\) is called an identity tensor if \(\delta_{i_k j_k} = \begin{cases} 1, & \text{if } i_k = j_k \\ 0, & \text{otherwise} \end{cases}\). The conjugate transpose of a tensor \(A \in \mathbb{C}^{I_1 \times \ldots \times I_M \times J_1 \times \ldots \times J_N}\) is denoted by \(A^H\), and is defined as \((A^H)_{j_1 \ldots j_N i_1 \ldots i_M} = \overline{a_{i_1 \ldots i_M j_1 \ldots j_N}}\), where the over-line stands for the conjugate of \(a_{i_1 \ldots i_M j_1 \ldots j_N}\). If the tensor \(A\) is real, then its transpose is denoted by \(A^T\), and is defined as \((A^T)_{j_1 \ldots j_N i_1 \ldots i_M} = a_{i_1 \ldots i_M j_1 \ldots j_N}\). We next present the definition of a unitary and an idempotent tensor. A tensor \(A \in \mathbb{C}^{I_1 \times \ldots \times I_N \times I_1 \times \ldots \times I_N}\) is unitary
if $A^*_N A^H = A^H * N A = I$, and idempotent if $A^*_N A = A$. The following results are useful to prove our main results.

**Lemma 2.1.** (Lemma 2.4, [14]) Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$, $B \in \mathbb{C}^{K_1 \times \cdots \times K_L \times I_1 \times \cdots \times I_N}$ and $C \in \mathbb{C}^{K_1 \times \cdots \times K_L \times I_1 \times \cdots \times I_N}$. If $B^*_N A^* M A^H = C^*_N A^* M A^H$, then $B^*_N A = C^*_N A$.

**Lemma 2.2.** (Lemma 2.6, [14]) Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$, $B \in \mathbb{C}^{J_1 \times \cdots \times J_M \times K_1 \times \cdots \times K_L}$ and $C \in \mathbb{C}^{J_1 \times \cdots \times J_M \times K_1 \times \cdots \times K_L}$. If $A^H * N A^* M B = A^H * N A^* M C$, then $A^* M B = A^* M C$.

If $A = A^H$ for a tensor $A \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$, then $A$ is Hermitian. If $A = -A^H$, then it is skew-Hermitian.

**Lemma 2.3.** (Lemma 2.2, [14]) Let $P \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be Hermitian. Then

(i) If $P^*_N Q = Q$ for $Q \in \mathbb{C}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$, then $Q^*_N P = Q^!$.

(ii) If $Q^*_N P = Q$ for $Q \in \mathbb{C}^{J_1 \times \cdots \times J_M \times I_1 \times \cdots \times I_N}$, then $P^*_N Q^! = Q^!$.

**3. Main Results**

This section is of fourfold. First, we give a new representation of the Moore-Penrose inverse of an arbitrary order tensor. Second, we provide some sufficient conditions to hold reverse-order law for even-order square tensors only. Third, we discuss some necessary and sufficient conditions for the reverse-order law of arbitrary order tensors. Four, we discuss some new properties of $\{1, 2, 3\}$-inverse and $\{1, 2, 4\}$-inverse of a tensor along with the reverse-order law.

**3.1. A new expression of the Moore–Penrose inverse of a tensor**

A new expression of the Moore–Penrose inverse of a tensor is obtained below.

**Theorem 3.1.** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times I_N}$, then $A^! = V^*_M A^* N W$, where $V$ and $W$ are any solutions of

$$V^*_M A^* N A^H = A^H$$

(5)
Proof. Let $\mathcal{X} = \mathcal{V}^* \mathcal{A}^* \mathcal{W}$, where $\mathcal{V}$ and $\mathcal{W}$ are the solutions of Equations (5) and (6), respectively. We then have

$$A^* \mathcal{V}^* \mathcal{A}^* \mathcal{A}^H = A^* \mathcal{A}^H$$

and

$$A^H \mathcal{W}^* \mathcal{A}^* \mathcal{A} = A^H \mathcal{A},$$

which result $A^* \mathcal{V}^* \mathcal{A}^* \mathcal{A}^H = A$ and $A^* \mathcal{W}^* \mathcal{A} = A$ by Lemma 2.1 and Lemma 2.2, respectively. So, $A^* \mathcal{X}^* \mathcal{A} = \mathcal{A}$, $A^* \mathcal{A}^* \mathcal{A} = \mathcal{X}$, $A^* \mathcal{X} = A^* \mathcal{W}$ and $A^* \mathcal{A} = \mathcal{V}^* \mathcal{A}$. Also,

$$\mathcal{V}^* \mathcal{A}^* \mathcal{A}^H \mathcal{V}^* = A^H \mathcal{V}^*$$

and

$$\mathcal{W}^* \mathcal{A}^* \mathcal{A} \mathcal{W} = \mathcal{W}^* \mathcal{A} \mathcal{A}^H,$$

follow from Equations (5) and (6), respectively. Since $\mathcal{V}^* \mathcal{A}^* \mathcal{A}^H \mathcal{V}^*$ and $\mathcal{W}^* \mathcal{A}^* \mathcal{A} \mathcal{W}$ are Hermitian, so we get $(\mathcal{V}^* \mathcal{A})^H = \mathcal{V}^* \mathcal{A}^H \mathcal{V}^*$ and $(\mathcal{A}^* \mathcal{W})^H = \mathcal{A}^* \mathcal{W}^H$. Hence, $A^* \mathcal{X}$ and $A^* \mathcal{A}$ are Hermitian. Thus, $A^\dagger = \mathcal{X} = \mathcal{V}^* \mathcal{A}^* \mathcal{W}$, by Definition 1.1. 

3.2. Reverse-order law of even-order square tensors

Here, we deduce some sufficient conditions to hold the reverse-order law for even-order square tensors. All tensors we assumed to be even-order square tensors in this subsection. The very first result of this subsection shows, how the commutative properties are sufficient to hold the reverse-order law.

**Theorem 3.2.** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ and $B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$. If

$$A^* N (B^* N B^\dagger) = (B^* N B^\dagger)^* N A$$

(7)

$$A^\dagger N (B^* N B^\dagger) = (B^* N B^\dagger)^* N A^\dagger$$

(8)

$$B^* N (A^\dagger N A) = (A^\dagger N A)^* N B$$

(9)

$$B^\dagger N (A^\dagger N A) = (A^\dagger N A)^* N B^\dagger$$

(10)

then $(A^* N B)^\dagger = B^\dagger N A^\dagger$. 

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Proof. Let $X = A^* N B$ and $Y = B^\dagger * N A^\dagger$. Then Equations (9) and (10) confirm that
\[
X^* N Y^* N X = A^* N A^\dagger * N A^* N B^\dagger * N B^\dagger * N B^* N B = A^* N B
\]
and $X = Y$.

By Equation (7), we also get $X H = A^* N A^\dagger * N B^\dagger * N B^* N B = X$, which results in $X^* N Y = Y^* N X$.

The condition (9) then ensures that $X^* N Y = Y^* N X$. Thus, by Definition 1.1, we arrive at the claim.

In an analogous way, one can easily see that Equations (7) and (8) are sufficient to hold the reverse-order law when the tensor $B^\dagger * N A^\dagger * N A^* N B$ is Hermitian.

Theorem 3.3. Let $A \in \mathbb{C}^I_1 \times \cdots \times I_N$ and $B \in \mathbb{C}^I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N$. If
\[
(i) \quad (A^* N B^\dagger * N A^\dagger)^H = A^* N B^\dagger * N B^\dagger * N A^\dagger
(ii) \quad B^\dagger * N (A^\dagger * N A) = (A^\dagger * N A)^* N B
(iii) \quad B^\dagger * N (A^\dagger * N A) = (A^\dagger * N A)^* N B^\dagger,
\]
then $(A* N B)^\dagger = B^\dagger * N A^\dagger$.

Proof. Let $X = A^* N B$ and $Y = B^\dagger * N A^\dagger$. By employing the conditions (ii) and (iii), we have $X^* N Y^* N X = X$, $Y^* N X^* N Y = Y$ and $(Y^* N X)^H = Y^* N X$. The condition (i) then ensures that $(X^* N Y)^H = X^* N Y$. Thus, by Definition 1.1, we arrive at the claim.

In an analogous way, one can easily see that Equations (7) and (8) are sufficient to hold the reverse-order law when the tensor $B^\dagger * N A^\dagger * N A^* N B$ is Hermitian.
Theorem 3.4. Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ and $B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$. If

\begin{align*}
(i) \quad & (B^\dagger \ast_N A^\dagger \ast_N A \ast_N B)^H = B^\dagger \ast_N A^\dagger \ast_N A \ast_N B \\
(ii) \quad & A \ast_N (B \ast_N B^\dagger) = (B \ast_N B^\dagger)^* \ast_N A \\
(iii) \quad & A^\dagger \ast_N (B \ast_N B^\dagger) = (B \ast_N B^\dagger)^* \ast_N A^\dagger,
\end{align*}

then $(A \ast_N B)^\dagger = B^\dagger \ast_N A^\dagger$.

For the class of tensors which satisfies the condition $A^\dagger = A^H$ for any $A$, Equations (9) and (10) are sufficient to hold the reverse-order law.

Theorem 3.5. Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ and $B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$. If Equations (9) and (10) hold and $A^\dagger = A^H$, then $(A \ast_N B)^\dagger = B^\dagger \ast_N A^\dagger$.

Proof. Let $\mathcal{X} = A \ast_N B$ and $\mathcal{Y} = B^\dagger \ast_N A^\dagger$, then using Equations (9) and (10), one can easily obtain $\mathcal{X}^\dagger \ast_N \mathcal{Y}^* \ast_N \mathcal{X} = \mathcal{X}$ and $\mathcal{Y}^\dagger \ast_N \mathcal{X}^\dagger = \mathcal{Y}$, and $\mathcal{Y}^\dagger \ast_N \mathcal{X}$ is Hermitian. On employing $A^\dagger = A^H$, we get $\mathcal{X}^\dagger \ast_N \mathcal{Y}$ is Hermitian. By Definition 1.1 we therefore have $\mathcal{X}^\dagger = \mathcal{Y}$, i.e., $(A \ast_N B)^\dagger = B^\dagger \ast_N A^\dagger$. \hfill $\square$

In an analogous manner, one can easily obtain the following result when the tensor $B$ satisfies $B^\dagger = B^H$.

Theorem 3.6. Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ and $B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$. If Equations (7) and (8) hold and $B^\dagger = B^H$, then $(A \ast_N B)^\dagger = B^\dagger \ast_N A^\dagger$.

Only one equation among Equations (7)-(10) is sufficient to hold the reverse-order law when both the tensors $A$ and $B$ have the property $A^\dagger = A^H$ and $B^\dagger = B^H$.

Theorem 3.7. Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ and $B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$. If at least one of Equations (7) - (10) holds, $A^\dagger = A^H$ and $B^\dagger = B^H$, then $(A \ast_N B)^\dagger = B^\dagger \ast_N A^\dagger$.

Proof. We proceed to the proof by assuming Equation (7). One can approach analogously for the rest of the Equations. Suppose that $A^\dagger = A^H$, $B^\dagger = B^H$ and Equation (7). Let $\mathcal{X} = A \ast_N B$ and $\mathcal{Y} = B^\dagger \ast_N A^\dagger$, then by employing Equation (7) successively twice to $\mathcal{X}^\dagger \ast_N \mathcal{Y}^* \ast_N \mathcal{X}$,
we get $\mathcal{X}_N \mathcal{Y}_N \mathcal{X} = \mathcal{X}$. Equation (7) also yields $\mathcal{Y}_N \mathcal{X}_N \mathcal{Y} = B^\dagger_N A^\dagger_N B^\dagger_N A^\dagger_N A^\dagger_N A$, which on applying $A^\dagger = A^H$ results

$$
\mathcal{Y}_N \mathcal{X}_N \mathcal{Y} = B^\dagger_N A^\dagger_N B^\dagger_N A^\dagger_N A^\dagger_N A \\
= B^\dagger_N (B^\dagger_N A^\dagger_N A) A^\dagger_N A^\dagger_N A \\
= B^\dagger_N A^\dagger_N (A^\dagger_N B^\dagger_N A) A^\dagger_N A^\dagger_N A \\
= B^\dagger_N A^\dagger_N (A^\dagger_N B^\dagger_N A^\dagger_N A^\dagger_N A^\dagger_N A) \\
= B^\dagger_N A^\dagger_N (A^\dagger_N B^\dagger_N A^\dagger_N A^\dagger_N A^\dagger_N A) \\
= B^\dagger_N A^\dagger_N A^\dagger_N A^\dagger_N A^\dagger_N A^\dagger \\
= B^\dagger_N A^\dagger_N A^\dagger_N A^\dagger_N A^\dagger_N A^\dagger \\
= B^\dagger_N A^\dagger_N A^\dagger \\
= \mathcal{Y}.
$$

The conditions $\mathcal{X}_N \mathcal{Y}$ and $\mathcal{Y}_N \mathcal{X}$ are Hermitian, are confirmed using the facts $A^\dagger = A^H$ and $B^\dagger = B^H$, respectively. Therefore, the claim is achieved by Definition 1.1.

\[ \Box \]

3.3. Reverse-order law of arbitrary order tensors

In this subsection, we aim to provide several necessary and sufficient conditions for the reverse-order law of arbitrary order tensors. Here onward a tensor means it is of arbitrary order unless stated otherwise. Below we restate a few results for tensors (arbitrary order) which are proved by Panigrahy et al. [13] very recently for even-order tensors. The very first result stated below extends Theorem 3.25, [13] to any two tensors, and provides necessary and sufficient conditions for the reverse-order law which is mentioned in Problem 1 in the introduction section.

Theorem 3.8. Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M}$. Then $(A^* B)^\dagger = B^\dagger N A^\dagger$ if and only if

$$
A^\dagger N A^* B^\dagger N A^\dagger = B^\dagger M B^\dagger N A^\dagger (11)
$$

and

$$
B^\dagger N A^\dagger M A^* N B = A^\dagger M A^* N B. (12)
$$
The next two lemmas are again the modified version of Lemma 3.9 and Lemma 3.10. The first one provides a sufficient condition for the commutative property of $A^\dagger *_M A$ and $B^*_M B^H$.

**Lemma 3.9.** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M}$. If $A^\dagger *_M A *^N_B *^N_B^H *^N_A^H = B^*_M B^H *^N_A^H$, then $A^\dagger *_M A$ commutes with $B^*_M B^H$.

Similarly, the next one presents a sufficient condition for the commutative property of $A^H *^N_M A$ and $B^*_M B^\dagger$. These are frequently used to derive some more new reverse-order laws.

**Lemma 3.10.** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M}$. If $B^*_M B^\dagger *^N_B *^N_A^H = A^H *^N_M A *^N_B$, then $A^H *^N_M A$ commutes with $B^*_M B^\dagger$.

Theorem 3.11 together with Lemma 3.9 and Lemma 3.10 yields the following outcome.

**Theorem 3.11.** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M}$. If $(A *^N_B)^\dagger = B^\dagger *^N_A^\dagger$, then

(i) $A^\dagger *^N_M A$ commutes with $B^*_M B^H$.

(ii) $A^H *^N_M A$ commutes with $B^*_M B^\dagger$.

Theorem 3.30, [13] is reproduced here for any two tensors.

**Theorem 3.12.** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M}$. If $(A *^N_B)^\dagger = B^\dagger *^N_A^\dagger$, then $A^\dagger *^N_M A$ and $B^*_M B^\dagger$ commute.

The first main result of this section presents a necessary condition for the reverse-order law.

**Theorem 3.13.** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$. If $(A *^N_B)^\dagger = B^\dagger *^N_A^\dagger$, then

$$(A^\dagger *^N_M A *^L_B L^\dagger)\dagger = B^*_L B^\dagger *^N_A *^N_M A.$$
Proof. Let \( X = A^\dagger_M A^*_N B_L B^\dagger \) and \( Y = B_L B^\dagger_M A^\dagger_M A \). Then
\[
X^* N Y^* N X^* = A^\dagger_M A^*_N B_L B^\dagger_M A^\dagger_M A^*_N B_L B^\dagger
= A^\dagger_M A^*_N B_L B^\dagger
= X,
\]
where we have used the fact \( A^*_N B_L B^\dagger_M A^\dagger_M A = A^*_N B \). Again, since \( B^\dagger_M A^\dagger_M A^*_N B_L B^\dagger_M A^\dagger_M A = B^\dagger_M A^\dagger \), so
\[
Y^* N X^* N Y^* = B_L B^\dagger_M A^\dagger_M A^*_N B_L B^\dagger_M A^\dagger_M A
= B_L B^\dagger_M A^\dagger_M A
= Y.
\]
A simple calculation using Definition 1.1 leads to \( X^* N Y^* = A^\dagger_M A^*_N B_L B^\dagger_M A^\dagger_M A \) and \( Y^* N X^* = B^\dagger_M A^\dagger_M A^*_N B_L B^\dagger_M A \). Since \( A^\dagger_M A^*_N B_L B^\dagger_M A^\dagger_M A \) and \( B^\dagger_M A^\dagger_M A^*_N B_L B^\dagger_M A \) are Hermitian, so are \( X^* N Y \) and \( Y^* N X \), respectively. The claim is thus achieved by Definition 1.1. \( \square \)

We next provide an example which shows that the converse of the above result is not true.

Example 3.14. Let \( A = (a_{ijk}) \in \mathbb{C}^{2 \times 3 \times 4} \) and \( B = (b_{ij}) \in \mathbb{C}^{4 \times 2} \) be such that

\[
A(:, :, 1) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A(:, :, 2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
A(:, :, 3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A(:, :, 4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};
\]

and \( B(:, :) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \). Then \( A^\dagger B = (c_{ijk}) \in \mathbb{C}^{2 \times 3 \times 2} \), where
\[
A^* B(\cdot, 1) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A^* B(\cdot, 2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.
\]

Then \(A^\dagger = (a'_{ij}) \in \mathbb{C}^{4 \times 3}, \ B^\dagger = (b'_{ij}) \in \mathbb{C}^{2 \times 4}\) and \((A^* B)^\dagger = (c'_{ij}) \in \mathbb{C}^{2 \times 3}\) where

\[
A^\dagger(\cdot, 1) = \begin{pmatrix} 1/4 & 0 \\ -1/4 & 0 \\ 1/2 & 0 \\ -1/4 & 0 \end{pmatrix}, \quad A^\dagger(\cdot, 2) = \begin{pmatrix} 3/4 & 0 \\ -1/2 & 0 \\ -1/4 & 1 \end{pmatrix}, \quad A^\dagger(\cdot, 3) = \begin{pmatrix} 3/4 & -1/4 \\ 1/4 & 1/4 \\ -1/2 & 1/2 \\ -3/4 & 1/4 \end{pmatrix}.
\]

\[
B^\dagger(\cdot, :) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}; \quad (A^* B)^\dagger(\cdot, :) = \begin{pmatrix} 6/17 & 0 \\ -4/17 & 0 \end{pmatrix},
\]

\[
(A^* B)^\dagger(\cdot, 2) = \begin{pmatrix} -2/17 & 1/17 \\ 7/17 & 5/17 \end{pmatrix} \quad \text{and} \quad (A^* B)^\dagger(\cdot, 3) = \begin{pmatrix} 3/17 & 1/17 \\ -2/17 & 5/17 \end{pmatrix}.
\]

On pre-multiplying \(B^\dagger\) with \(A^\dagger\), we get

\[
B^\dagger A^\dagger(\cdot, :) = \begin{pmatrix} 3/8 & 0 \\ -1/4 & 0 \end{pmatrix}, \quad B^\dagger A^\dagger(\cdot, 2) = \begin{pmatrix} -1/8 & 0 \\ 1/4 & 1/2 \end{pmatrix}
\]

and

\[
B^\dagger A^\dagger(\cdot, 3) = \begin{pmatrix} 1/8 & 1/8 \\ -1/4 & 1/4 \end{pmatrix}.
\]

Pre-multiplying \(B\) and post-multiplying \(A\) to \(B^\dagger A^\dagger\) gives

\[
B_1 B^\dagger A^\dagger_2 A(\cdot, :) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}.
\]

Also, pre-multiplying \(A^\dagger\) and post-multiplying \(B^\dagger\) to \(A^* B\) yields

\[
A^\dagger A^* B_1 B^\dagger(\cdot, :) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}.
\]
Let $X$ and by Equation (2) we get

\[
B = (A_{1}B_{1}^{\dagger}A_{2})^{\dagger} = B_{1}^{\dagger}A_{2}^{\dagger}A_{1}^{\dagger}.
\]

So, $(A_{1}^{\dagger}A_{2}^{\dagger}B_{1}^{\dagger}B_{2}^{\dagger})^{\dagger} = B_{1}^{\dagger}B_{2}^{\dagger}A_{2}^{\dagger}A_{1}^{\dagger}$, but $(A_{1}B)^{\dagger} \neq B_{1}^{\dagger}A_{1}^{\dagger}$.

However, the converse of Theorem 3.13 holds under the assumption of two additional conditions, and is shown next.

**Theorem 3.15.** Let $A \in \mathbb{C}^{l_{1} \times \cdots \times l_{M} \times \cdots \times l_{N}}$ and $B \in \mathbb{C}^{j_{1} \times \cdots \times j_{N} \times k_{1} \times \cdots \times k_{L}}$. If $(A_{1}^{\dagger}A_{1}B_{1}^{\dagger}B_{2}^{\dagger})^{\dagger} = B_{1}^{\dagger}B_{2}^{\dagger}A_{1}^{\dagger}A_{2}^{\dagger}$, then $(A_{1}B_{1}B_{2}^{\dagger})^{\dagger} = B_{1}^{\dagger}B_{2}^{\dagger}A_{1}^{\dagger}A_{2}^{\dagger}$.

**Proof.** Since $(A_{1}^{\dagger}A_{1}B_{1}B_{2}^{\dagger})^{\dagger} = B_{1}^{\dagger}B_{2}^{\dagger}A_{1}^{\dagger}A_{2}^{\dagger}$, so by Equation (11) we get

\[
(A_{1}^{\dagger}A_{1}B_{1}B_{2}^{\dagger})^{\dagger} = A_{1}^{\dagger}A_{1}B_{1}B_{2}^{\dagger},
\]

and by Equation (2) we get

\[
B_{1}B_{2}^{\dagger}A_{1}^{\dagger}A_{2}^{\dagger} = B_{1}B_{2}^{\dagger}A_{1}^{\dagger}A_{2}^{\dagger}.
\]

Let $X = A_{1}B_{1}$ and $Y = B_{1}^{\dagger}A_{1}^{\dagger}$. Pre-multiplying $A$ and post-multiplying $B$ to Equation (13) yields $X_{L}Y_{M}X = X$, and on pre-multiplication of $B_{1}^{\dagger}$ and post-multiplication of $A_{1}^{\dagger}$ with Equation (1) results $Y_{M}X_{L}Y = Y$. The Hermitian property of $X_{L}Y$ and $Y_{M}X$ are confirmed by the second and third assumptions. The claim is thus attained by Definition 1.11.

The following example illustrate the above theorem.

**Example 3.16.** Let $A = (a_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2}$ and $B = (b_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2}$ such that

\[
A(\cdot, \cdot, 1, 1) = \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix},
\]

\[
A(\cdot, \cdot, 2, 1) = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]

and

\[
A(\cdot, \cdot, 1, 2) = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix},
\]

\[
A(\cdot, \cdot, 2, 2) = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]
\begin{align*}
A(:,;1,1) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
A(:,;1,2) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
A(:,;2,1) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A(:,;2,2) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}

The multiplication \( A*_{2}B = (c_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2 \times 2} \) such that
\begin{align*}
A*_{2}B(:,;1,1) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
A*_{2}B(:,;1,2) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
A*_{2}B(:,;2,1) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A*_{2}B(:,;2,2) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}

The Moore-Penrose inverses \( A^\dagger = (a'_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2 \times 2} \) and \( B^\dagger = (b'_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2 \times 2} \) such that
\begin{align*}
A^\dagger(:,;1,1) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
A^\dagger(:,;1,2) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
A^\dagger(:,;2,1) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A^\dagger(:,;2,2) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};
\end{align*}
\begin{align*}
B^\dagger(:,;1,1) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
B^\dagger(:,;1,2) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
B^\dagger(:,;2,1) &= \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix} \quad \text{and} \quad B^\dagger(:,;2,2) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\end{align*}

Pre-multiplying \( A^\dagger \) and post-multiplying \( B^\dagger \) to \( A*_{2}B \) yields \( A^\dagger*_{2}A*_{2}B*_{2}B^\dagger = (t_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2 \times 2} \) such that
\begin{align*}
A^\dagger*_{2}A*_{2}B*_{2}B^\dagger(:,;1,1) &= \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}, \\
A^\dagger*_{2}A*_{2}B*_{2}B^\dagger(:,;1,2) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
A^\dagger*_{2}A*_{2}B*_{2}B^\dagger(:,;2,1) &= \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix} \quad \text{and} \quad A^\dagger*_{2}A*_{2}B*_{2}B^\dagger(:,;2,2) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}
The Moore-Penrose inverse \((A^\dagger^1 A^\dagger^2 A^\dagger^2 B^\dagger^2 B^\dagger^2 B^\dagger^1)^\dagger = (u'_{ijkl}) \in \mathbb{C}^{2\times 2\times 2\times 2}\) such that

\[
(A^\dagger^1 A^\dagger^2 A^\dagger^2 B^\dagger^2 B^\dagger^2 B^\dagger^1)^\dagger(:, :, 1, 1) = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}, \quad (A^\dagger^1 A^\dagger^2 A^\dagger^2 B^\dagger^2 B^\dagger^2 B^\dagger^1)^\dagger(:, :, 1, 2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

\[
(A^\dagger^1 A^\dagger^2 A^\dagger^2 B^\dagger^2 B^\dagger^2 B^\dagger^1)^\dagger(:, :, 2, 1) = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix} \quad \text{and} \quad (A^\dagger^1 A^\dagger^2 A^\dagger^2 B^\dagger^2 B^\dagger^2 B^\dagger^1)^\dagger(:, :, 2, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

The multiplication \(B^\dagger^1 A^\dagger^2 = (d_{ijkl}) \in \mathbb{C}^{2\times 2\times 2\times 2}\) such that

\[
B^\dagger^1 A^\dagger^2(:, :, 1, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B^\dagger^1 A^\dagger^2(:, :, 1, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
B^\dagger^1 A^\dagger^2(:, :, 2, 1) = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix} \quad \text{and} \quad B^\dagger^1 A^\dagger^2(:, :, 2, 2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Pre-multiplying \(B\) and post-multiplying \(A\) to \(B^\dagger^1 A^\dagger^2\) yields \(B^\dagger^1 A^\dagger^2 A^\dagger^2 = (u_{ijkl}) \in \mathbb{C}^{2\times 2\times 2\times 2}\) such that

\[
B^\dagger^1 A^\dagger^2 A^\dagger^2(:, :, 1, 1) = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}, \quad B^\dagger^1 A^\dagger^2 A^\dagger^2(:, :, 1, 2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

\[
B^\dagger^1 A^\dagger^2 A^\dagger^2(:, :, 2, 1) = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix} \quad \text{and} \quad B^\dagger^1 A^\dagger^2 A^\dagger^2(:, :, 2, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Thus, \((A^\dagger^1 A^\dagger^2 A^\dagger^2 B^\dagger^2)^\dagger = B^\dagger^1 A^\dagger^2 A^\dagger^2 A^\dagger^2.\) Post-multiplication of \(B^\dagger^1 A^\dagger^2\) to \(A^\dagger^2 B\) yields \(A^\dagger^2 B^\dagger^1 A^\dagger^2 A^\dagger^2 = (r_{ijkl}) \in \mathbb{C}^{2\times 2\times 2\times 2}\) such that

\[
A^\dagger^2 B^\dagger^1 A^\dagger^2 A^\dagger^2(:, :, 1, 1) = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}, \quad A^\dagger^2 B^\dagger^1 A^\dagger^2 A^\dagger^2(:, :, 1, 2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

\[
A^\dagger^2 B^\dagger^1 A^\dagger^2 A^\dagger^2(:, :, 2, 1) = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix} \quad \text{and} \quad A^\dagger^2 B^\dagger^1 A^\dagger^2 A^\dagger^2(:, :, 2, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Clearly, \(A^\dagger^2 B^\dagger^1 A^\dagger^2\) is Hermitian. Pre-multiplication of \(B^\dagger^1 A^\dagger^2\) to \(A^\dagger^2 B\) yields \(B^\dagger^1 A^\dagger^2 A^\dagger^2 A^\dagger^2 = (r_{ijkl}) \in \mathbb{C}^{2\times 2\times 2\times 2}\).
\((s_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2} \) such that
\[
B^{{\dagger}_2} A^{{\dagger}_2} A^*_2 B(:, ;, 1, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B^{{\dagger}_2} A^{{\dagger}_2} A^*_2 B(:, ;, 1, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
B^{{\dagger}_2} A^{{\dagger}_2} A^*_2 B(:, ;, 2, 1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B^{{\dagger}_2} A^{{\dagger}_2} A^*_2 B(:, ;, 2, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Clearly, \(B^{{\dagger}_2} A^{{\dagger}_2} A^*_2 B\) is also Hermitian. The Moore-Penrose inverse \((A^*_2 B)^\dagger = (c'_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2}\) such that
\[
(A^*_2 B)^\dagger(:, ;, 1, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (A^*_2 B)^\dagger(:, ;, 1, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
(A^*_2 B)^\dagger(:, ;, 2, 1) = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix} \quad \text{and} \quad (A^*_2 B)^\dagger(:, ;, 2, 2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Thus, \((A^*_2 B)^\dagger = B^{{\dagger}_2} A^\dagger\).

Theorem 3.13 and 3.15 can be together stated as:

**Theorem 3.17.** Let \(A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\) and \(B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}\). Then \((A^*_N B)^\dagger = B^{{\dagger}_N} A^\dagger\) if and only if

\[(i) \quad (A^*_M A^*_N B^*_L B^\dagger)^\dagger = B^*_M B^*_N A^*_M A^\dagger,\]

\[(ii) \quad (A^*_N B^*_L B^*_N A^* A^\dagger)^H = A^*_N B^*_L B^*_N A^\dagger,\]

\[(iii) \quad (B^*_N A^*_M A^*_N B^* B^\dagger)^H = B^*_N A^*_M A^*_N B^\dagger.\]

To prove Theorem 3.21, the following two results are useful. We reproduce the same here from [13] where the authors proved for even-order tensors. These results show that the reverse-order law holds if at least one of the tensors is unitary.

**Theorem 3.18.** (a) Let \(A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\) and \(B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}\). If \(B\) is unitary, then \((A^*_N B)^\dagger = B^H * N A^\dagger\).
(b) Let $A \in \mathbb{C}^{I_1 \times \ldots \times I_N}$ and $B \in \mathbb{C}^{I_1 \times \ldots \times J_1 \times \ldots \times J_M}$. If $A$ is unitary, then \((A^*b)B^*A^H\).

The example given below shows that the converse of the above result is not true in general.

**Example 3.19.** Let $A = (a_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2 \times 2}$ and $B = (b_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2 \times 2}$ be such that

\[
A(\cdot, 1, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A(\cdot, 1, 2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A(\cdot, 2, 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A(\cdot, 2, 2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};
\]

and

\[
B(\cdot, 1, 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B(\cdot, 1, 2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B(\cdot, 2, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B(\cdot, 2, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then, $A^*B = (c_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2 \times 2}$ where

\[
A^*B(\cdot, 1, 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^*B(\cdot, 1, 2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
A^*B(\cdot, 2, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^*B(\cdot, 2, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

We then have $A^\dagger = (a'_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2 \times 2}$, $B^H = (b'_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2 \times 2}$ and $(A^*B)^\dagger = (c'_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2 \times 2}$ where

\[
A^\dagger(\cdot, 1, 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^\dagger(\cdot, 1, 2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
A^\dagger(\cdot, 2, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A^\dagger(\cdot, 2, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};
\]

\[
B^H(\cdot, 1, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B^H(\cdot, 1, 2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
B^H(\cdot, 2, 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B^H(\cdot, 2, 2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};
\]

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Thus, \( \textbf{Theorem 3.22.} \)

Also, \( \mathcal{B}^H *_2 \mathcal{A}^\dagger = (d_{ijkl}) \in \mathbb{C}^{2 \times 2 \times 2 \times 2} \) such that

\[
(\mathcal{B}^H *_2 \mathcal{A}^\dagger)(::1,1) = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \\
(\mathcal{B}^H *_2 \mathcal{A}^\dagger)(::1,2) = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \\
(\mathcal{B}^H *_2 \mathcal{A}^\dagger)(::2,1) = \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}, \\
(\mathcal{B}^H *_2 \mathcal{A}^\dagger)(::2,2) = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}.
\]

Thus, \( (\mathcal{A} *_2 \mathcal{B})^\dagger = \mathcal{B}^H *_2 \mathcal{A}^\dagger \) but \( \mathcal{B} \) is not unitary.

The result below shows that \( (\mathcal{A} *_M \mathcal{B} *_N \mathcal{C})^\dagger = \mathcal{C}^\dagger *_N \mathcal{B}^\dagger *_M \mathcal{A}^\dagger \) holds when the tensors \( \mathcal{A} \) and \( \mathcal{C} \) are unitary.

**Theorem 3.20.** Let \( \mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M} \), \( \mathcal{B} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( \mathcal{C} \in \mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N} \). If \( \mathcal{A} \) and \( \mathcal{C} \) are unitary, then \( (\mathcal{A} *_M \mathcal{B} *_N \mathcal{C})^\dagger = \mathcal{C}^\dagger *_N \mathcal{B}^\dagger *_M \mathcal{A}^\dagger \).

Next result assumes the reverse-order law for tensors \( \mathcal{A} \) and \( \mathcal{B} \), and provides the reverse-order law for another two tensors \( \mathcal{X} \) and \( \mathcal{Y} \) which are obtained from \( \mathcal{A} \) and \( \mathcal{B} \) by pre-multiplication and post-multiplication two unitary tensors.

**Theorem 3.21.** Let \( (\mathcal{A} *_N \mathcal{B})^\dagger = \mathcal{B}^\dagger *_N \mathcal{A}^\dagger \) for \( \mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( \mathcal{B} \in \mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M} \). If \( \mathcal{X} = \mathcal{U} *_M \mathcal{A} \) and \( \mathcal{Y} = \mathcal{B} *_M \mathcal{U}^H \) where \( \mathcal{U} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M} \) is any unitary tensor, then

\[
(\mathcal{X} *_N \mathcal{Y})^\dagger = \mathcal{Y}^\dagger *_N \mathcal{X}^\dagger.
\]

**Proof.** By Theorem \ref{3.20} we have \( (\mathcal{X} *_N \mathcal{Y})^\dagger = \mathcal{U} *_M \mathcal{B}^\dagger *_N \mathcal{A}^\dagger *_M \mathcal{U}^H \). But, by Theorem \ref{3.18}, we obtain \( \mathcal{X}^\dagger = \mathcal{A}^\dagger *_M \mathcal{U}^H \) and \( \mathcal{Y}^\dagger = \mathcal{U} *_M \mathcal{B}^\dagger \). The claim is therefore established. \( \square \)

**Theorem 3.22.** Let \( \mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( \mathcal{B} \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_T} \). Then \( (\mathcal{A} *_N \mathcal{B})^\dagger = \mathcal{B}^\dagger *_N \mathcal{A}^\dagger \) is equivalent to

\[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}, \\
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}.
\]
(i) \((A^\dagger_M A^* N B)^\dagger = B^\dagger_N A^\dagger_M\).

(ii) \((A^* N B)^\dagger = (A^\dagger_M A^* N B)^\dagger_N A^\dagger_M\).

Similarly, we obtain the following results.

**Theorem 3.23.** Let \(A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\) and \(B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}\). Then \((A^* N B)^\dagger = B^\dagger_N A^\dagger\) is equivalent to

(i) \((A^* N B^* L B^\dagger)^\dagger = B^\dagger_L B^\dagger_N A^\dagger\),

(ii) \((A^* N B)^\dagger = B^\dagger_N (A^* N B^* L B^\dagger)^\dagger\).

We next extended Theorem 3.1 [13], for arbitrary tensor which can be proved by adopting similar steps as in [13]. In Problem 1, if \(B = A^H\), then the reverse-order law holds and is stated in the next result.

**Theorem 3.24.** Let \(A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\). Then

(a) \((A^H \ast_M A)^\dagger = A^\dagger_M (A^H)^\dagger_M\),

(b) \((A^* N A^H)^\dagger = (A^H)^\dagger_N A^\dagger\).

The above result helps to obtain two more equivalent condition of reverse-order law.

**Theorem 3.25.** Let \(A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\) and \(B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}\). Then \((A^* N B)^\dagger = B^\dagger_N A^\dagger\) is equivalent to

(i) \((A^H \ast_M A^* N B)^\dagger = B^\dagger_N (A^H \ast_M A)^\dagger\),

(ii) \((A^* N B)^\dagger = (A^H \ast_M A^* N B)^\dagger_N A^H\).

The following result can be proved in an analogous manner.

**Theorem 3.26.** Let \(A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\) and \(B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}\). Then \((A^* N B)^\dagger = B^\dagger_N A^\dagger\) is equivalent to

(i) \((A^* N B^L B^H)^\dagger = (B^* L B^H)^\dagger_N A^\dagger\).
Another equivalent condition for reverse-order law of tensor is obtained next.

**Theorem 3.27.** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$. Then \((A^* N B)^\dagger = B^\dagger N A^\dagger\) is equivalent to

\begin{enumerate}[(i)]
\item \((A^H M A^* N B^* L B^H)^\dagger = (B^* L B^H)^\dagger N (A^H M A)^\dagger,
\item \((A^* N B)^\dagger = B^H N (A^H M A^* N B^* L B^H)^\dagger N A^H.
\end{enumerate}

It can be easily shown that for any tensor $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, we have

\[
(A^* N A^H M A)^\dagger = A^\dagger M A^H N A^\dagger.
\]

The above equality helps us to obtain the next equality.

\[
A^* N A^H M A^* N (A^* N A^H M A)^\dagger = A^* N A^H M A^* N A\dagger M A^H N A^\dagger
= A^* N A^H M A^\dagger H N A^\dagger
= A^* N A^\dagger M A^* N A^\dagger
= A^* N A^\dagger.
\]  

Similarly, one can easily prove that

\[
(A^* N A^H M A)^\dagger M A^* N A^H M A = A^\dagger M A,
\]

for any tensor $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$. Another representation of the reverse-order law is obtained next using Equations (15) and (16).

**Theorem 3.28.** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$. Then \((A^* N B)^\dagger = B^\dagger N A^\dagger\) is equivalent to

\begin{enumerate}[(i)]
\item \((A^* N A^H M A^* N B^* L B^H)^\dagger = (B^* L B^H)^\dagger N (A^* N A^H M A)^\dagger,
\item \((A^* N B)^\dagger = B^H N B^* L (A^* N A^H M A^* N B^* L B^H)^\dagger M A^* N A^H.
\end{enumerate}
We have \((A^H * M A)^\dagger = A^{\dagger M} A^{\dagger H}\) and \((A^* N A^H)^\dagger = A^{\dagger H^* N} A^\dagger\) by Theorem 3.24. Then
\[
[(A^H * M A)^2]^{\dagger} = [(A^H * M A)]^{2}\nonumber
= A^{\dagger M} A^{\dagger H^* N} A^\dagger * M A^{\dagger H},
\] (17)
and
\[
[(A^* N A^H)^2]^{\dagger} = [(A^* N A^H)]^{2}\nonumber
= A^{\dagger H^* N} A^\dagger * M A^{\dagger H^* N} A^\dagger.
\] (18)

Equations (17) and (18) lead the following simpler equivalences.
\[
(A^* N A^H)^2 * N [(A^* N A^H)^2]^{\dagger} = A^* N A^H * M A^* N A^{H^* N} A^\dagger * M A^{\dagger H^* N} A^\dagger
= A^* N A^H * M A^{\dagger M} A^{\dagger H^* N} A^\dagger
= A^* N A^H * M A^{\dagger H^* N} A^\dagger
= A^* N A^\dagger * M A^{\dagger M} A^\dagger
= A^* N A^\dagger.
\] (19)

Analogously, one can show that
\[
(A^H * M A)^2 * N [(A^H * M A)^2]^{\dagger} = A^{\dagger M} A, \tag{20}
\]
\[
[(A^H * M A)^2]^{\dagger} * N (A^H * M A)^2 = A^{\dagger M} A, \tag{21}
\]
\[
[(A^* N A^H)^2]^{\dagger} * N (A^* N A^H)^2 = A^* N A^\dagger. \tag{22}
\]

We next produce a result which provides another two equivalent conditions for the reverse-order law using Equations (17) to (22).

**Theorem 3.29.** Let \(A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\) and \(B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}\). Then \((A^* N B)^\dagger = B^{\dagger * N} A^\dagger\) is equivalent to

(i) \([(A^H * M A)^2 * N (B * L B^H)^2]^{\dagger} = [(B * L B^H)^2]^{\dagger * N} [(A^H * M A)^2]^{\dagger},\]

(ii) \((A^* N B)^\dagger = B^H * N B * L B^H * N [(A^H * M A)^2 * N (B * L B^H)^2]^{\dagger * N} A^{H^* M} A^* N A^H.\)
3.4. Reverse-order law using \{1, 2, 3\}-inverse and \{1, 2, 4\}-inverse

Let us first recall the definitions of \{1, 2, 3\} and \{1, 2, 4\}-inverse of a tensor \(A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\).

A tensor \(X \in \mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M}\) is called \{1, 2, 3\}-inverse of \(A\) if \(X\) satisfies Equations (1) to (3), and is called \{1, 2, 4\}-inverse if it satisfies Equations (1), (2) and (4). Here we first obtain some interesting properties of \{1, 2, 3\}-inverse and \{1, 2, 4\}-inverse of a tensor which are helpful to deduce some reverse-order law.

**Lemma 3.30.** If \(A^{(1, 2, 3)} \in A\{1, 2, 3\}\) for any tensor \(A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\), then \(A^* N A^{(1, 2, 3)} = A^* N A^\dagger\).

**Proof.** Since \(A^{(1, 2, 3)} \in A\{1, 2, 3\}\), then \(A^* N A^{(1, 2, 3)} = A^* N A^\dagger M A = A^* N A^\dagger A^* N A^{(1, 2, 3)}\) by post-multiplying \(A^{(1, 2, 3)}\). The proof completes by taking the conjugate transpose of both sides of \(A^* N A^{(1, 2, 3)} = A^* N A^\dagger M A^* N A^{(1, 2, 3)}\).

The following result is the analogous version of the above result for the \{1, 2, 4\}-inverse of a tensor.

**Lemma 3.31.** If \(A^{(1, 2, 4)} \in A\{1, 2, 4\}\) for any tensor \(A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\), then \(A^{(1, 2, 4)} = A^\dagger M A^* N A\).

We next present a characterization of \{1, 2, 3\}-inverse of a tensor.

**Theorem 3.32.** Let \(A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\). Then \(A^{(1, 2, 3)}\) is a \{1, 2, 3\}-inverse for \(A\) if and only if there exists a tensor \(X \in \mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M}\) such that \(A^* N A = 0\), \(X^* M A^* N A^\dagger = X\), and \(A^{(1, 2, 3)} = A^\dagger + X\).

**Proof.** Let \(A^{(1, 2, 3)}\) be an \{1, 2, 3\}-inverse and \(X = A^{(1, 2, 3)} - A^\dagger\). By Lemma 3.30, we
then have $A^*_N X = 0$ and

$$X^*_M A^*_N A^\dagger = (A^{(1, 2, 3)} - A^\dagger)_M A^*_N A^\dagger$$

$$= A^{(1, 2, 3)}_M A^*_N A^\dagger - A^\dagger M A^*_N A^\dagger$$

$$= A^{(1, 2, 3)}_M A^*_N A^{(1, 2, 3)} - A^\dagger$$

$$= A^{(1, 2, 3)} - A^\dagger$$

$$= X.$$

Conversely, assume that there exists a $X$ such that $A^*_N X = 0$ and $X^*_M A^*_N A^\dagger = X$. Let $Z = A^\dagger + X$. Then

$$A^*_N Z^*_N A = A^*_N (A^\dagger + X)^*_M A$$

$$= A^*_N A^\dagger N A + A^*_N X^*_M A$$

$$= A$$

and

$$Z^*_M A^*_N Z = (A^\dagger + X)^*_M A^*_N (A^\dagger + X)$$

$$= (A^\dagger + X)^*_M A^*_N A^\dagger$$

$$= A^\dagger + X$$

$$= Z.$$

Also, we have $A^*_N Z = A^*_N (A^\dagger + X) = A^*_N A^\dagger$. Since $A^*_N A^\dagger$ is Hermitian, so $A^*_N Z$ is also Hermitian. Thus, $A^{(1, 2, 3)} = Z = A^\dagger + X$.

We now ready to present equivalent conditions for reverse-order law using $\{1, 2, 3\}$-inverse and $\{1, 2, 4\}$-inverse and is presented next.

**Theorem 3.33.** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$. Then $(A^*_N B)^\dagger = B^\dagger N A^\dagger$ if and only if there exists a $\{1, 2, 3\}$-inverse, $B^{(1, 2, 3)}$, of $B$ satisfying $(A^*_N B)^\dagger = B^{(1, 2, 3)} N A^\dagger$. 

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Proof. Suppose that there exists a $\mathcal{B}^{(1, 2, 3)}$ such that $(\mathcal{A} \ast_N \mathcal{B})^\dagger = \mathcal{B}^{(1, 2, 3)} \ast_N \mathcal{A}^\dagger$. By Lemma 2.3, we have $(\mathcal{A} \ast_N \mathcal{B})^\dagger = \mathcal{B}^\dagger \ast_N \mathcal{B} \ast_L \mathcal{B}^{(1, 2, 3)} \ast_N \mathcal{A}^\dagger$, whose right-hand side reduces to $\mathcal{B}^\dagger \ast_N \mathcal{A}^\dagger$ using Lemma 3.30.

Conversely, assume that $(\mathcal{A} \ast_N \mathcal{B})^\dagger = \mathcal{B}^\dagger \ast_N \mathcal{A}^\dagger$. Since $\mathcal{B}^\dagger$ is also a $\{1, 2, 3\}$-inverse of $\mathcal{B}$, so the claim is justified.

As a consequence, we have the following result which is the last result of this article.

Theorem 3.34. Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $\mathcal{B} \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$. Then $(\mathcal{A} \ast_N \mathcal{B})^\dagger = \mathcal{B}^\dagger \ast_N \mathcal{A}^\dagger$ if and only if there exists a $\{1, 2, 4\}$-inverse, $\mathcal{A}^{(1, 2, 4)}$, of $\mathcal{A}$ satisfying $(\mathcal{A} \ast_N \mathcal{B})^\dagger = \mathcal{B}^\dagger \ast_N \mathcal{A}^{(1, 2, 4)}$.

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