A non-existence result due to small perturbations in an eigenvalue problem

Gelu Paşa,
Simion Stoilow Institute of Mathematics of Romanian Academy
pasa.gelu@gmail.com

Abstract. We consider a well-posed eigenvalue problem on \((a, 0)\), depending on a continuous function \(m\). The boundary conditions in the points \(a, 0\) are depending on the eigenvalues. We divide \((a, 0)\) into small intervals and approximate the function \(m\) by a simple (step) function \(m_S\), constant on each small interval. The eigenfunctions corresponding to \(m_S\) do not exist.

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1. INTRODUCTION.

A well-posed eigenvalue problem is considered on the segment \((a, 0)\), depending on a continuous function \(m\) as parameter. This problem is related with some previous papers concerning the linear stability of displacements in Hele-Shaw cells, see [1] - [10]. The boundary conditions in the points \(a, 0\) are containing the eigenvalues - then we have some compatibility conditions. An existence result can be obtained by using the finite-difference method, see [1]. We divide \((a, 0)\) into small intervals and approximate the function \(m\) by a simple (step) function \(m_S\), constant on each small interval. In this case, the eigenfunctions corresponding to \(m_S\) are known - a combination of exponentials. From this reason, the above mentioned compatibility conditions (related with the boundary conditions in the points \(a\) and 0) can not be verified. As a consequence, we obtain the following result. The functions \(m, m_S\) could be are very “close” (in some suitable functional spaces), but the eigenvalues corresponding to \(m_S\) do not exist. Therefore we get a very strong modification of the eigenvalues, caused by the very small variation of the parameter \(m\). A similar result was given in [12]: the “perturbed” eigenvalues become infinity with increasing wavenumbers.

2. THE ORIGINAL PROBLEM

On the segment \((a, 0)\) we consider the eigenvalue problem (the lower in-
\( -(mf_x)_x + k^2mf = (1/\sigma)k^2fm_x, \quad k \geq 0, \)
\[0 < m_L \leq m(x) \leq m_R, \quad \forall x \in (a, 0). \tag{1} \]

The following conditions for large \(|x|\) are supposed:
\( f(x) = Ae^{kx}, x \leq a; \quad f(x) = Ae^{-kx}, x \geq 0; \)
\( m(x) = m_L, x < a; \quad m(x) = m_R, x > 0. \tag{2} \)

As we specified in Introduction, this problem is related with some previous papers concerning the stability of the flow displacements in Hele-Shaw cells. In these papers, the eigenfunctions \( f \) are the amplitudes of the perturbed velocities. For this reason we consider that \( f \) must decrease to zero at large distances. Moreover, we suppose that \( f \) are continuous in \( a, 0 \) but \( f_x \) could have jumps in these points. On the other hand, the parameter \( m \) can be considered as the viscosity of an intermediate fluid, between a displacing and displaced fluids with constant viscosities \( m_L, m_R \), see the papers [11 - 10]. The function \( m \) is continuous on \( [a, b] \).

We consider the following boundary conditions in the points \( a \) and \( b = 0 \):
\[ m^-(a)f_x^-(a) - m^+(a)f_x^+(a) = [kE(a)/\sigma]f(a), \]
\[ m^-(b)f_x^-(b) - m^+(b)f_x^+(b) = [kE(b)/\sigma]f(b), \]
\[ E(a) := k[m^+(a) - m^-(a)] = k[m^+(a) - m_L], \]
\[ E(b) := k[m^+(b) - m^-(b)] = k[m_R - m^-(b)], \tag{3} \]

where \(-, +\) are denoting the left and right limit values. \( \sigma \) can be considered as the characteristic values of the problem; \( k, f \) are the wavenumbers and the eigenfunctions.

In the relation (3) we have the same eigenvalue(s), therefore \( f \) must verify the compatibility relation
\[ V(a) = V(b), \tag{4} \]
\[ V(a) := \frac{kE(a)f(a)}{m^-(a)f_x^-(a) - m^+(a)f_x^+(a)}, \]
\[ V(b) := \frac{kE(b)f(b)}{m^-(b)f_x^-(b) - m^+(b)f_x^+(b)}. \tag{5} \]
It is important to note that (4) is depending only on \( f \) and is not depending on \( \sigma \). A quite similar problem has been studied in [11], where \( E(a), E(b) \) were polynomials of order 3 in \( k \), depending also on the surface tensions between the displacing fluids.

As in [11], we use the finite difference method to get a classical eigenvalue problem for a matrix, equivalent with (1) - (3). We use the notation
\[
\lambda = 1/\sigma,
\]
thus \( \lambda \) can be considered as eigenvalue(s).

The values \( f_x^-(a), f_x^+(b) \) are given by the above relations (2), because we know the values of \( f \) outside the segment \((a, b)\):
\[
f_x^-(a) = kf(a), \quad f_x^+(b) = -kf(b).
\]
From the relations (3) we can obtain the limit values of \( f_x \) in the interior ends of the segment \((a, b)\):
\[
m^+(a)f_x^+(a) = mLkf(a) - \lambda kE(a)f(a),
m^-(b)f_x^-(b) = -mRkf(b) + \lambda kE(b)f(b).
\]
We consider \((M + 1)\) equidistant points \( x_i \) s.t.
\[
a = x_M < x_{M-1} < \ldots < x_1 < x_0 = b = 0, \quad x_{i-1} - x_i = d,
\]
and the following approximations for \( f_x^+(a), f_x^-(b), f_{xx}(z) \) for \( z \in (a, b) \):
\[
\frac{f_0 - f_1}{d} = f_x^+(a), \quad \frac{f_{M-1} - f_M}{d} = f_x^-(b),
\]
\[
f_{xx}(z) = \frac{f(z + d) - 2f(z) + f(z - d)}{d^2}.
\]
The last formula is obtained by using the symmetric approximation of \( f_x \) with the approximation step \( d/2 \). Therefore, the boundary conditions become
\[
-m^+(a) \frac{k}{kE(a)} f_{M-1} + \left[ m^+(a) \frac{k}{kE(a)} + m^+(a) \right] f_M = \lambda f_M,
\]
\[
\left[ m_R^k \frac{k}{kE(b)} + \frac{m^-(b)}{kE(b)} \right] f_0 - \frac{m^-(b)}{kE(b)} f_1 = \lambda f_0.
\]
The discrete form of our problem is

\[ Pf = Qf, \quad f = (f_0, f_1, f_2, \ldots, f_M). \]  

(9)

In the case of 4 interior points we have

\[
P = \begin{pmatrix}
    a_{00} & a_{01} & 0 & 0 & 0 \\
    a_{10} & a_{11} & a_{12} & 0 & 0 \\
    0 & a_{21} & a_{22} & a_{23} & 0 \\
    0 & 0 & a_{32} & a_{33} & a_{34} \\
    0 & 0 & 0 & a_{43} & a_{44} & a_{45} \\
    0 & 0 & 0 & 0 & a_{54} & a_{55}
\end{pmatrix},
\]

where

\[
a_{00} = \left[ \frac{m_R k}{k E(b)} + \frac{m^-(b)}{kd E(b)} \right], \quad a_{01} = -\frac{m^-(b)}{kd E(b)}
\]

\[
a_{54} = -\frac{m^+(a)}{kd E(a)}, \quad a_{55} = \left[ \frac{m_L k}{k E(a)} + \frac{m^+(a)}{kd E(a)} \right]
\]

The matrix \( P \) is tridiagonal. The first and the last lines contain the boundary conditions (8). The other lines contain only 3 elements different from zero related to the approximation of \((m f_x)_x\) in \(x_1, \ldots, x_{M-1}\).

The matrix \( Q \) is diagonal: \( N_{00} = N_{MM} = 1 \) and the other elements of the diagonal contain the values of \(m_x\) in the interior points \(x_1, \ldots, x_{M-1}\).

If \(m_x \neq 0\) in all interior points, then \(Q^{-1}\) exist and (9) is equivalent with

\[ Q^{-1} Pf = \lambda f. \]  

(10)

This is an eigenvalue problem for the matrix \((Q^{-1} P)\) and the eigenvalues \(\lambda\) are obtained by using the classical method.

**Remark 1.** If the parameter \(m\) is constant, we prove that the eigenvalues of the problem (1) - (5) do not exist. This fact is due to the particular form of the boundary conditions. When \(m\) is constant, the right hand side of (1) is zero, thus the eigenfunctions \(f\) are known:

\[ f(x) = Je^{kx} + Le^{-kx}, \quad J, L \text{ constant.} \]

(11)

We consider large values of the wavenumbers \(k\). In order to avoid large values of \(f\), it follows

\[ f(x) = Je^{kx}. \]
We need continuity of $f$ in $a$, then $J = A$ where $A$ appears in the formula $(2)$. We also need the continuity of $f$ in $b = 0$. Thus we are led to the eigenfunctions

$$f(x) = Ae^{kx}, x \leq 0; \quad f(x) = Ae^{-kx}, x \geq 0. \quad (12)$$

Just now we prove that the eigenvalue(s) corresponding to the eigenfunctions $(12)$ do not exist. To this end, we use the equivalent form of the relations $(8)$:

$$\lambda = m^+(a) \left( -\frac{f_{M-1}}{f_M} \right) + \left[ \frac{m_Lk}{kE(a)} + \frac{m^+(a)}{kdE(a)} \right], \quad (13)$$

$$\lambda = m^-(b) \left( -\frac{f_1}{f_0} \right) + \left[ \frac{m_Rk}{kE(b)} + \frac{m^-(b)}{kdE(b)} \right]. \quad (14)$$

In both above relations we have the same eigenvalue(s) $\lambda$. However we get

$$(-\frac{f_{M-1}}{f_M}) = -\frac{e^{k(a+d)}}{e^{ka}} = -e^{kd}, \quad (-\frac{f_1}{f_0}) = -e^{-kd}. \quad (15)$$

For large $k$, from $(13)$ we get $\lambda \to -\infty$ and $(14)$ gives us $\lambda \to 0$. As we obtain different limits for $\lambda$, the eigenvalue(s) do not exist.

\[ \square. \]

3. THE PERTURBED PROBLEM

Consider the step function

$$m_S(x) = m_i, \quad x \in (x_{i-1}, x_i), \quad (16)$$

where $m_i$ is constant in each small interval. Thus we have

$$-(f_S)_{xx} + k^2f_S = 0, \quad x \neq x_i, \quad x \in (a, 0). \quad (17)$$

Both $m_S, f_S$ verify the relations $(2)$. The boundary conditions $(3)$ exist only in $x = a, x = b = 0$:

$$m_S^-(a)(f_S)_x^- (a) - m_S^+(a)(f_S)_x^+ (a) = \frac{kE_S(a)f_S(a)}{\sigma_S},$$

$$m_S^-(b)(f_S)_x^- (b) - m_S^+(b)(f_S)_x^+ (b) = \frac{kE_S(b)f_S(b)}{\sigma_S}. \quad (18)$$
where $E_S(a), E_S(b)$ are given by (3) with $m_S$ instead of $m$ and $\sigma_S$ are the perturbed characteristic values. As in Remark 1, (18) contain the same $\sigma_S$, thus we need

$$V_S(a) = V_S(b);$$

$$V_S(a) = \frac{kE_S(a)f_S(a)}{m_S^-a(f_S)_x^-(a) - m_S^+(a)(f_S)_x^+(a)},$$

$$V_S(b) = \frac{kE_S(b)f_S(b)}{m^-b_S(f_S)_x^-(b) - m^+(b)_S(f_S)_x^+(b)}.$$ (20)

**Remark 2.** The problem (16) - (18) has no solution As in Remark 1, from (17) we get the following perturbed eigenfunctions

$$f_S(x) = A_i e^{kx} + B_i e^{-kx}, \ x \in (x_{i-1}, x_i).$$

Thus for large $k$ it follows

$$f_S(x) = A_i e^{kx}, \ x \in (x_{i-1}, x_i).$$ (21)

As $f$ is continuous, there exists a constant $A$ s.t.

$$A = A_1 = A_2 = ... = A_N.$$

Then in fact we have the same solution (12), which is continuous also in $a, 0$:

$$(f_S)_-(a) = f_S(a) = Ae^{ka}; \ (f_S)_+(b) = f_S(b) = Ae^{kb} = A.$$  

We can use the same properties of $\lambda$, obtained in Remark 1 and get the nonexistence of the eigenvalues, because the boundary conditions in $a, 0$ are the same.

However, we give here a different proof, based on the relation (20). A jump of $f_x$ exists only in $x_0 = b = 0$:

$$(f_S)_x^-(a) = (f_S)_x^+(a) = kAe^{ka};$$

$$(f_S)_x^-(b) = kAe^{kb}, \ (f_S)_x^+(b) = -kAe^{kb}.$$  

We insert the above derivatives in the relation (20) and it follows

$$V_S(a) = \frac{k(m_1 - m_L)}{m_L - m_1}; \ V_S(b) = \frac{k(m_R - m_N)}{m_N + m_R}.$$
As a consequence, we get

\[ V_S(a) = V_S(b) \Rightarrow m_R = 0, \quad (22) \]

which is not possible, because \( m_R \) must be positive. Thus \( V_S(a) \neq V_S(b) \).

\[ \square \]

**Remark 3.** As the relation (19) is not fulfilled, the problem (16) - (20) has no solution. Thus \( \sigma_S \) do not exist. Moreover, the problem (1) - (5) can not be approximated by the problem (16) - (20).

\[ \square \]

4. CONCLUSION

The functions \( m, m_S \) could be very “close” (in a suitable functional space) if the number of the jump points \( x_i \) (for \( m_S \)) is very large. A very small perturbation of a parameter can give a strong variation of the perturbed eigenvalues - see [13]. In our case, the perturbed eigenvalues do not exist, even if the initial problem is well-posed and the “perturbations” are very small.

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