A SIMPLE CLASS OF INFINITELY MANY ABSOLUTELY EXOTIC MANIFOLDS

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Abstract. We show that the smooth 4-manifold $M$ obtained by attaching a 2-handle to $B^4$ along a certain knot $K \subset \partial B^4$ admits infinitely many absolutely exotic copies $M_n$, $n = 0, 1, 2, \ldots$, such that each copy $M_n$ is obtained by attaching 2-handle to a fixed compact smooth contractible manifold $W$ along the iterates $f^n(c)$ of a knot $c \subset \partial W$ by a diffeomorphism $f : \partial W \to \partial W$. This generalizes the example in author’s 1991 paper, which corresponds to $n = 1$ case.

0. Construction

A relative exotic structure on a compact smooth 4-manifold $M$ with boundary, is a self diffeomorphism $f : \partial M \to \partial M$, which extends to a self homeomorphism of $M$, but does not extend to a self diffeomorphism of $M$. If $F : M \to M$ a homeomorphism extending $f$, then the pullback smooth structure $M_F$ provides a relative exotic copy of $M$. We say that $N$ is an absolutely exotic copy of $M$, if it is homeomorphic but not diffeomorphic to $M$ (no condition on the boundary). The technique introduced in [AR] turns relative exotic structures to absolute exotic structures. This is done by choosing an invertable cobordism $H$ with $\partial H = H_- \sqcup H_+$ and $H_- \approx \partial M$, and then gluing $H$ it to the boundary of $M$ in two different ways. Then the manifolds $M' = M \cup_{f_1} H$ and $M'' = M \cup_{f_2} H$ become absolutely exotic copies of each other. Applying this construction to a cork $W$ produces an absolutely exotic copy of $W$; and when $W$ is an infinite order loose-cork ([A2]) then we get infinitely many absolutely exotic copies of $W$. This construction results a boundary $H_+$, which consists of hyperbolic manifolds glued along tori.

Ideally we want to produce small 4-manifolds with simple boundaries, admitting absolutely exotic copies (with the hope of capping boundaries to get small closed exotic manifolds). One such example is the cusp $C^4$ ([A3]) with a Seifert fibered space boundary, which is obtained by attaching a 2-handle to $B^4$ along the trefoil knot with 0-framing. Performing knot surgeries $C \sim C_K$ along the torus inside (by using different knots $K$) provides infinitely many absolutely exotic

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copies of $C$. An interesting open problem here is to find corks inside of $C$, such that twisting $C$ along them induce the exotic copies $C_K$ of $C$. Here we produce another example $M^4$, similar to $C^4$, which also has a Seifert fibered space boundary, and is obtained by attaching a 2-handle to $B^4$ along a slice knot with $-2$ framing. But from this we can construct infinitely many different exotic copies of $M$, each obtained by cork-twisting along an infinite order loose-cork $W \subset M$ ([A2]), rather than a knot surgery to $M$, as in the case of $C$ above.

**Theorem 1.** The manifold $M$ of Figure 1, which is obtained by attaching 2-handle to $B^4$ along the knot $K$ of Figure 1 with $-2$ framing, admits infinitely many distinct absolutely exotic copies, and they can be detected twisting an infinite order loose-cork inside of $M$.

**Proof.** First of all by blowing up and down as in Figure 2, we see that $\partial M$ can also be identified by $+2$ surgery to $(-4, 2)$ twist knot (stevedores knot). $\partial M$ is the small Seifert fibered space $M(-2; 1/2, 3/4, 7/9)$ (e.g. [BW], [S], [T]), therefore its mapping class group is finite order.

Now, recall the infinite order loose-cork $W$ of [A2], which is shown in Figure 3 (where two of its alternative handlebody pictures are given).
The order \( n \) diffeomorphism \( f_n : \partial W \to \partial W \) is obtained by a delta move along the curve \( \delta \subset \partial W \) (the dotted curve in the figure).

![Figure 3. W](image)

Now check that the handlebody pictures of Figure 4 describe the manifold \( M \) above (to see this cancel 1/2- handle pairs). The second picture of Figure 4 shows an imbedding \( W \subset M \). That is, \( M \) is obtained from \( W \) by attaching a 2-handle along the knot \( c \) with 0-framing.

![Figure 4. M](image)

Now apply \( \delta \)-moves to \( W \) inside \( M \), \( n \)-times (where \( \delta \) is chosen as in Figure 3), and call the resulting manifold \( M_n \) (which is the first picture of Figure 5). We claim \( \{M_n\} \) are exotic copies of \( M \) rel boundary. To see this attach 2-handles to \( M_n \) along the knots \( a \) and \( b \) of the picture.

Call the manifold obtained from \( M_n \) by attaching 2-handles along \( a \) and \( b \) with \(-1\) framings by \( S_n = M_n + a^{-1} + b^{-1} \). Now we proceed as in [A2] by handle slides, to show that \( S_n \) is the manifold obtained from the Stein manifold \( S \) of Figure 7 by the knot surgery using the twist knot \((-2, -n)\). Furthermore we can compactify \( S \) into some closed symplectic manifold \( Z \) with \( \delta_2^+(Z) > 1 \) (by [LM], [AO], or [A4] p.108).
This shows that manifolds \{M_n\} are exotic copies of \(M\) rel boundary, and they are obtained by iterating \(\delta\)-moves to \(f: \partial W \to \partial W\) inside \(W \subset M\). Since the mapping class group of the Seifert fibered space \(\partial M \cong M_n\) is finite [BO], by going to a subsequence we can assume that all \{M_n\} are absolutely exotic copies of each other.

Now we analyze what an \(n\)-iterate of the \(\delta\)-move does to \(M\), well it turns it into \(M_n\), and a close inspection shows that \(M_n\) is obtained from \(M\) by attaching 2-handle to \(W\) along the loop \(f^n(c)\) with 0-framing, as shown in Figure 8. Recall also, performing the \(\delta\)-move to \(W\) inside of \(M\), has the affect of attaching a cancelling pair of 2/3-handles to \(M\) and performing the diffeomorphism described in [AI] resulting \(M_n\).
Remark 1. Note the new features of Theorem [1] which can not be reached by the techniques of [AR], they are: (1) We don’t need to modify the boundary of $M$ (by a homology cobordism) in order to construct its absolutely exotic copy. (2) The construction here produces infinitely many absolutely exotic copies of $M$. (3) $M$ contains the tangent disc bundle of $S^2$ (an imbedded $-2$ sphere) and vice versa, so every smooth manifold which contains a $-2$ sphere contains a copy of $M$ inside.

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