CONTINUOUS-TIME ZERO-SUM STOCHASTIC GAME WITH STOPPING AND CONTROL

CHANDAN PAL AND SUBHAMAY SAHA

Abstract. We consider a zero-sum stochastic game for continuous-time Markov chain with countable state space and unbounded transition and pay-off rates. The additional feature of the game is that the controllers together with taking actions are also allowed to stop the process. Under suitable hypothesis we show that the game has a value and it is the unique solution of certain dynamic programming inequalities with bilateral constraints. In the process we also prescribe a saddle point equilibrium.

Keywords: zero-sum game; stopping time; optimal strategy; dynamic programming inequalities.

1. INTRODUCTION

In this article we consider a zero-sum stochastic game for continuous-time Markov chain. The transition and reward rates are assumed to be unbounded. The additional feature is that players other than taking actions also has the option of stopping the game. We show that the game has a value and it is the unique solution of a set of dynamic programming inequalities with bilateral constrains. The existence of optimal strategies for both players is also establish. These optimal strategies also give optimal stopping rules for both players. Stochastic control problems for continuous time Markov chains, both for one controller and multi-controller setup has been studied by a variety of authors, see [8] and references therein. But to the best of our knowledge there is no work on continuous-time Markov chains with the dual feature of control as well as stopping. Stochastic games with only stopping was introduced by Dynkin [3]. Such games also known in literature as Dynkin games has been investigated for discrete time case, see ([10, 13]), as well as for continuous-time case, see ([2, 11, 14]). Stochastic games with control and stopping has been studied for discrete time case in [6], and for continuous-time non-degenerate diffusion in ([1, 5]). The rest of the paper is organized as follows. In section II, we give the detailed problem formulation and in section III we prove the existence of value of the game and saddle point equilibrium.

2. GAME FORMULATION

The stochastic game model that we are interested in is given by \( \{S, U, V, q, r, \psi_1, \psi_2\} \). The components have the following interpretation. \( S \) is countable set and without any loss
of generality we take $S = \{0, 1, 2, 3, \cdots \}$. $S$ is the state space of the controlled continuous-time Markov chain. $U$ and $V$ are metric spaces representing the action sets of player $I$ and $II$ respectively. The component $q = [q(j|i, a, b)]$ is the controlled transition rate matrix, satisfying the following properties:

(i) $q(j|i, a, b) \geq 0$ for all $i \in S$, $i \neq j$, $a \in U$, $b \in V$.

(ii) It is assumed to be conservative, i.e.,

$$\sum_{j \in S} q(j|i, a, b) = 0 \quad \text{for all } i \in S, a \in U, b \in V.$$

(iii) We also assume it is stable, i.e.,

$$q(i) = \sup_{a \in U, b \in V} \sum_{j \neq i} q(j|i, a, b) < \infty \quad \text{for all } i \in S.$$

The reward rate is given by $r : S \times U \times V \to [0, \infty)$ and $\psi_i : S \to [0, \infty)$, $i = 1, 2$ are the stopped payoff functions for the players.

At time $t = 0$, suppose the process starts from state $i$ and player $I$ and player $II$ independently chooses actions $a$ and $b$ then player $II$ receives a reward at the rate $r(i, a, b)$ until the next jump epoch which occurs after an exponential $(\sum_{j \neq i} q(j|i, a, b))$ amount of time. The next state of the process is $j$ with probability $\frac{q(j|i, a, b)}{\sum_{j \neq i} q(j|i, a, b)}$. The game then repeats from the new state $j$. If at state $i$, player $I$ decides to stop the game then, player $II$ receives a payoff of $\psi_1(i)$, where as if player $II$ decides to stop then she receives a payoff equal to $\psi_2(i)$. Player $II$ tries to maximize her accumulated expected discounted reward, which player $I$ wishes to minimize the same. Here we will consider only randomized stationary control although things go through with randomized Markov control as well.

A randomized stationary control for player $I$ is a measurable function $\Phi : S \rightarrow \mathcal{P}(U)$. similarly $\Psi : S \rightarrow \mathcal{P}(V)$ is a randomized stationary control for player $II$. We denote by $\Pi^I$ and $\Pi^2$ the set of all randomized stationary controls for player $I$ and player $II$ respectively.

In order to guarantee the existence of a non explosive process (finite jumps in a finite time) we assume the following:

**Assumption (A1):**

There exists $N$ non-negative functions $w_n$ on $S$ and a positive constant $c$ such that for all $i \in S, a \in U, b \in V$,

$$\sum_{j \in S} q(j|i, a, b)w_n(j) \leq w_{n+1}(i), \quad \text{for } n = 1, 2, \cdots, N - 1$$

and furthermore,

$$\sum_{j \in S} q(j|i, a, b)w_N(j) \leq 0$$
and 

\[ q(i) \leq c(w_1(i) + w_2(i) + \cdots + w_N(i)), \quad \text{for all } i \in S. \]

It is well known that under the above assumptions, for randomized stationary controls \((\Phi, \Psi)\) there exists a non explosive continuous time Markov chain, see [7]. We denote the state process by \(X_t\) and let \(U_t\) and \(V_t\) denote the control processes for player I and II respectively.

Let \(\{F_t : t \geq 0\}\) denote the natural filtration of \(\{X_t : t \geq 0\}\). Then a strategy for player I is a pair \((\Phi, \Theta^1)\) where \(\Phi \in \Pi^1\) and \(\Theta^1\) is a \(F_t\)-stopping time. Similarly for player II a strategy is a pair \((\Psi, \Theta^2)\) where \(\Psi \in \Pi^2\) and \(\Theta^2\) is a \(F_t\)-stopping time. The evaluation criterion is given by

\[
J_\alpha(i, \Phi, \Psi, \Theta^1, \Theta^2) = E_{i, \Psi}^\Phi \int_0^{\Theta^1 \wedge \Theta^2} e^{-\alpha t} r(X_t, U_t, V_t) dt + e^{-\alpha (\Theta^1 \wedge \Theta^2)} \{\psi_1(X_{\Theta^1}) 1_{\{\Theta^1 < \Theta^2\}} + \psi_2(X_{\Theta^2}) 1_{\{\Theta^1 \geq \Theta^2\}}\},
\]

where \(\alpha > 0\) is the discount factor and \(1_{\{\cdot\}}\) is the indicator function. Player II wishes to maximize \(J_\alpha(i, \Phi, \Psi, \Theta^1, \Theta^2)\) over her strategies \((\Psi, \Theta^2)\) and player I wishes to minimize the same over all pairs \((\Phi, \Theta^1)\). Define

\[
U(i) = \inf_{(\Phi, \Theta^1)} \sup_{(\Psi, \Theta^2)} J_\alpha(i, \Phi, \Psi, \Theta^1, \Theta^2)
\]

and

\[
L(i) = \sup_{(\Psi, \Theta^2)} \inf_{(\Phi, \Theta^1)} J_\alpha(i, \Phi, \Psi, \Theta^1, \Theta^2).
\]

Then \(U(i)\) is called the upper value of the game and \(L(i)\) is called the lower value of the game. The game is said to have a value if \(U(i) = L(i)\).

A strategy \((\Phi^*, \Theta^{1*})\) is said to be optimal for player I if

\[
L(i) \geq J_\alpha(i, \Phi^*, \Psi, \Theta^{1*}, \Theta^2) \quad \text{for all } i \in S
\]

and for all strategies \((\Psi, \Theta^2)\) of player II. Analogously, A strategy \((\Psi^*, \Theta^{2*})\) is said to be optimal for player II if

\[
U(i) \leq J_\alpha(i, \Phi, \Psi^*, \Theta^1, \Theta^{2*}) \quad \text{for all } i \in S
\]

and for all strategies \((\Phi, \Theta^1)\) of player I. \(((\Phi^*, \Theta^{1*}), (\Psi^*, \Theta^{2*}))\) is called a saddle point equilibrium, if it exists.

3. Existence of Value and Saddle Point Equilibrium

In order to characterize the value of the game and to establish the existence of a saddle point equilibrium we will need the following assumption:

Assumption (A2):
(i) $U$ and $V$ are compact sets;
(ii) $r(i, a, b)$ and $q(j|i, a, b)$ are continuous in $(a, b) \in U \times V$;
(iii) Let $W(i) = w_1(i) + w_2(i) + \cdots + w_N(i)$, for all $i \in S$. The function $\sum_{j \in S} q(j|i, a, b)W(j)$ is continuous in $(a, b) \in U \times V$;
(iv) there is a constant $M$ such that
$$r(i, a, b) \leq MW(i), \text{ for all } i \in S \text{ and } (a, b) \in U \times V,$$
and
$$\psi_1(i) \leq MW(i), \text{ for all } i \in S$$
and
$$\psi_2(i) < \psi_1(i), \text{ for all } i \in S;$$
(v) there exists a non-negative function $\tilde{W}$ on $S$ and positive constants $c$ and $\tilde{c}$ such that
$$q(i)W(i) \leq M\tilde{W}(i), \text{ for all } i \in S$$
and
$$\sum_{j \in S} q(j|i, a, b)\tilde{W}(j) \leq c\tilde{W}(i) + \tilde{c} \text{ for all } i \in S \text{ and } (a, b) \in U \times V.$$

Set
$$B_W(S) = \left\{ f : S \to [0, \infty) \mid \sup_{i \in S} \frac{f(i)}{W(i)} < \infty \right\},$$
where $W$ is as in (A2). Define for $f \in B_W(S)$,
$$\|f\|_W = \sup_{i \in S} \frac{f(i)}{W(i)}.$$

Then $B_W(S)$ is a Banach space with the norm $\| \cdot \|_W$.

For any two states $i, j \in S$, any two probability measures $\mu \in \mathcal{P}(U)$ and $\nu \in \mathcal{P}(V)$ define
$$\tilde{r}(i, \mu, \nu) = \int_V \int_U r(i, a, b)\mu(da)\nu(db)$$
and
$$\tilde{q}(j|i, \mu, \nu) = \int_V \int_U q(j|i, a, b)\mu(da)\nu(db).$$

For $\phi \in B_W(S)$ define
$$H^+_\alpha(i, \phi) = \inf_{\mu \in \mathcal{P}(U)} \sup_{\nu \in \mathcal{P}(V)} \left[ \tilde{r}(i, \mu, \nu) + \sum_{j \in S} \tilde{q}(j|i, \mu, \nu)\phi(j) \right]$$
and
$$H^-_\alpha(i, \phi) = \sup_{\nu \in \mathcal{P}(V)} \inf_{\mu \in \mathcal{P}(U)} \left[ \tilde{r}(i, \mu, \nu) + \sum_{j \in S} \tilde{q}(j|i, \mu, \nu)\phi(j) \right].$$

Further define
$$I^+_\alpha(i, \phi) = \inf_{\mu \in \mathcal{P}(U)} \sup_{\nu \in \mathcal{P}(V)} \left[ \tilde{r}(i, \mu, \nu) + \frac{q(i) + 1}{\alpha + q(i) + 1} + \frac{q(i) + 1}{\alpha + q(i) + 1} \sum_{j \in S} \tilde{p}(j|i, \mu, \nu)\phi(j) \right],$$
where
\[ \bar{p}(j|i, \mu, \nu) = \frac{q(j|i, \mu, \nu) + \delta_{ij}}{q(i) + 1} \]
(\(\delta_{ij}\) is the Kronecker delta). Similarly define,
\[
I_\alpha^-(i, \phi) = \sup_{\nu \in \mathcal{P}(V)} \inf_{\mu \in \mathcal{P}(V)} \left[ \frac{\bar{r}(i, \mu, \nu)}{\alpha + q(i) + 1} + \frac{q(i) + 1}{\alpha + q(i) + 1} \sum_{j \in S} \bar{p}(j|i, \mu, \nu)\phi(j) \right].
\]
Note that by Fan’s minimax theorem \[\ref{fan},\] \(H_\alpha^+ = H_\alpha^- := H_\alpha\) and \(I_\alpha^+ = I_\alpha^- := I_\alpha\). Now consider the following dynamic programming inequalities with bilateral constraints:
\[
\psi_2(i) \leq \phi(i) \leq \psi_1(i)
\]
\[
\alpha \phi(i) - H_\alpha(i, \phi) = 0 \quad \text{if} \quad \psi_2(i) < \phi(i) < \psi_1(i)
\]
\[
\alpha \phi(i) - H_\alpha(i, \phi) \geq 0 \quad \text{if} \quad \psi_2(i) = \phi(i) \quad \text{(3.1)}
\]
\[
\alpha \phi(i) - H_\alpha(i, \phi) \leq 0 \quad \text{if} \quad \psi_1(i) = \phi(i)
\]
or equivalently
\[
\psi_2(i) \leq \phi(i) \leq \psi_1(i)
\]
\[
\phi(i) - I_\alpha(i, \phi) = 0 \quad \text{if} \quad \psi_2(i) < \phi(i) < \psi_1(i)
\]
\[
\phi(i) - I_\alpha(i, \phi) \geq 0 \quad \text{if} \quad \psi_2(i) = \phi(i) \quad \text{(3.2)}
\]
\[
\phi(i) - I_\alpha(i, \phi) \leq 0 \quad \text{if} \quad \psi_1(i) = \phi(i).
\]

**Theorem 3.1.** Under assumptions (A1) and (A2), the following are equivalent.

(i) \(\phi\) satisfies \[\ref{assumptions}.\]

(ii) \(\phi(i) = \min\{\max\{I_\alpha(i, \phi); \psi_2(i)\}; \psi_1(i)\}\).

(iii) \(\phi(i) = \max\{\min\{I_\alpha(i, \phi); \psi_1(i)\}; \psi_2(i)\}\).

**Proof.** Here we prove only the equivalence of (i) and (ii), others can be proved similarly. Suppose (i) is true and \(i \in S\) is such that \(\psi_2(i) < \Phi(i) < \psi_1(i)\). Then
\[
\phi(i) = I_\alpha(i, \phi)
\]
\[
= \max\{I_\alpha(i, \phi); \psi_2(i)\}
\]
\[
= \min\{\max\{I_\alpha(i, \phi); \psi_2(i)\}; \psi_1(i)\}\]
If \(\psi_2(i) = \phi(i)\). Then \(\phi(i) \geq I_\alpha(i, \phi)\). Therefore
\[
\phi(i) = \max\{I_\alpha(i, \phi); \psi_2(i)\}
\]
\[
= \min\{\max\{I_\alpha(i, \phi); \psi_2(i)\}; \psi_1(i)\},
\]
since \(\psi_2(i) \leq \phi(i) \leq \psi_1(i)\).

If \(\psi_1(i) = \phi(i)\). Then \(\phi(i) \leq I_\alpha(i, \phi)\) and \(\psi_2(i) \leq \phi(i) \leq \psi_1(i)\). Hence
\[
I_\alpha(i, \phi) = \max\{I_\alpha(i, \phi); \psi_2(i)\}.
\]
Therefore
\[ \phi(i) = \psi_1(i) = \min \{ \max \{ I_\alpha(i, \phi); \psi_2(i) \}; \psi_1(i) \} . \]

Now assume that (ii) is true, i.e.,
\[ \phi(i) = \min \{ \max \{ I_\alpha(i, \phi); \psi_2(i) \}; \psi_1(i) \} . \]

Suppose \( i \in S \) is such that \( \psi_2(i) < \phi(i) < \psi_1(i) \). Then
\[ \phi(i) = \max \{ I_\alpha(i, \phi); \psi_2(i) \} = I_\alpha(i, \phi) \]

If \( \psi_2(i) = \phi(i) \). Then by assumption (A2), \( \phi(i) < \psi_1(i) \). Therefore
\[ \phi(i) = \max \{ I_\alpha(i, \Phi); \psi_2(i) \} \geq I_\alpha(i, \phi) . \]

If \( \psi_1(i) = \phi(i) \). Then by assumption (A2), \( \phi(i) > \psi_2(i) \). Therefore \( \phi(i) = \psi_1(i) \leq \max \{ I_\alpha(i, \phi); \psi_2(i) \} \), which implies that \( \phi(i) \leq I_\alpha(i, \phi) \) (since \( \phi(i) > \psi_2(i) \) ). It is easy to see that \( \psi_2(i) \leq \phi(i) \leq \psi_1(i) \). Hence \( \phi \) satisfies (3.2).

Now define the operator \( T : B_W(S) \to B_W(S) \) by
\[ T\phi(i) := \min \{ \max \{ I_\alpha(i, \phi); \psi_2(i) \}; \psi_1(i) \} . \]

Let \( u_0(i) = \psi_2(i) \) and \( u_n = Tu_{n-1}, \ n \geq 1 \). Then the following is true.

**Proposition 3.1.** Under assumptions (A1) and (A2), the sequence of functions \( \{ u_n \}_{n \geq 0} \) is a non-decreasing and there exists \( u^* \in B_W(S) \) such that \( \lim_{n \to \infty} u_n = u^* \). Further \( u^* \) is a fixed point of \( T \), i.e., \( Tu^* = u^* \).

**Proof.** Clearly \( u_1(i) = Tu_0(i) \geq \psi_2(i) = u_0(i) \) for all \( i \in S \). Now suppose \( u_n \geq u_{n-1} \). It is easy to see that \( I_\alpha(i, \Phi) \) is monotone in \( \Phi \). Thus we have
\[ u_{n+1}(i) = Tu_n(i) = \min \{ \max \{ I_\alpha(i, u_n); \psi_2(i) \}; \psi_1(i) \} \geq \min \{ \max \{ I_\alpha(i, u_{n-1}); \psi_2(i) \}; \psi_1(i) \} = Tu_{n-1}(i) = u_n(i) . \]

Thus by induction we have that \( \{ u_n \}_{n \geq 0} \) is a non-decreasing. Therefore there exists \( u^* \in B_W(S) \) such that \( u^*(i) = \lim_{n \to \infty} u_n(i) \) for all \( i \in S \). Now clearly
\[ Tu^*(i) \geq Tu_n(i) = u_{n+1}(i) . \]

Taking limit \( n \to \infty \) on both sides we get,
\[ Tu^*(i) \geq u^*(i) \text{ for all } i \in S . \]
Now for the reverse inequality,

\[ u_{n+1}(i) = T u_n(i) \]

\[ = \min \left\{ \max \{ I_\alpha(i, u_n); \psi_2(i) \}; \psi_1(i) \right\} \]

\[ = \min \left\{ \max \left\{ \sup_{\nu \in \mathcal{P}(V)} \inf_{\mu \in \mathcal{P}(U)} \left[ \frac{\tilde{r}(i, \mu, \nu)}{\alpha + q(i) + 1} + \frac{q(i) + 1}{\alpha + q(i) + 1} \sum_{j \in S} \tilde{p}(j|i, \mu, \nu) u_n(j) \right] ; \psi_2(i) \right\}; \psi_1(i) \right\} \]

\[ \geq \min \left\{ \max \left\{ \left[ \frac{\tilde{r}(i, \mu^*, \nu)}{\alpha + q(i) + 1} + \frac{q(i) + 1}{\alpha + q(i) + 1} \sum_{j \in S} \tilde{p}(j|i, \mu^*, \nu) u_n(j) \right] ; \psi_2(i) \right\}; \psi_1(i) \right\}, \]

where \( \nu \in \mathcal{P}(V) \) is arbitrary. The existence of \( \mu^*_n \in \mathcal{P}(U) \) is ensured by assumption (A2).

Now since \( \mathcal{P}(U) \) is compact. There exists \( \mu^* \in \mathcal{P}(U) \) such that \( \mu_n \to \mu^* \) in \( \mathcal{P}(U) \). Thus by extended Fatou's [Lemma 8.3.7(b) in [9]] we have by letting \( n \to \infty \) on both sides,

\[ u^*(i) \geq \min \left\{ \max \left\{ \frac{\tilde{r}(i, \mu^*, \nu)}{\alpha + q(i) + 1} + \frac{q(i) + 1}{\alpha + q(i) + 1} \sum_{j \in S} \tilde{p}(j|i, \mu^*, \nu) u^*(j) \right\}; \psi_1(i) \right\} \]

\[ \geq \min \left\{ \max \left\{ \inf_{\mu \in \mathcal{P}(U)} \left[ \frac{\tilde{r}(i, \mu, \nu)}{\alpha + q(i) + 1} + \frac{q(i) + 1}{\alpha + q(i) + 1} \sum_{j \in S} \tilde{p}(j|i, \mu, \nu) u^*(j) \right] ; \psi_1(i) \right\} \right\}. \]

Since the above is true for any \( \nu \in \mathcal{P}(V) \). Hence we have

\[ u^*(i) \geq \min \{ \max \{ I_\alpha(i, u^*); \psi_2(i) \}; \psi_1(i) \} \]

\[ = Tu^*(i). \]

Thus we are done. \( \square \)

Let \( \Phi^* \in \Pi^1 \) and \( \Psi^* \in \Pi^2 \) be such that

\[ H_\alpha(i, u^*) = \sup_{\nu \in \mathcal{P}(V)} \left[ \tilde{r}(i, \Phi^*(i), \nu) + \sum_{j \in S} \tilde{q}(j|i, \Phi^*(i), \nu) u^*(j) \right] \]

and

\[ H_\alpha(i, u^*) = \inf_{\mu \in \mathcal{P}(U)} \left[ \tilde{r}(i, \mu, \Psi^*(i)) + \sum_{j \in S} \tilde{q}(j|i, \mu, \Psi^*(i)) u^*(j) \right]. \]

The existence of \( \Phi^* \in \Pi^1 \) and \( \Psi^* \in \Pi^2 \) follows from assumption (A2) and measurable selection theorem [12].

Define

\[ A_1 = \{ i \in S | u^*(i) = \psi_1(i) \} \]

and

\[ A_2 = \{ i \in S | u^*(i) = \psi_2(i) \}. \]

Let \( \{ X_t; t \geq 0 \} \) be the state process governed by the stationary controls \( \Phi^* \in \Pi^1 \) and \( \Psi^* \in \Pi^2 \). Let

\[ \Theta^{1*} = \inf \{ t \geq 0 | X_t \in A_1 \} \]

and

\[ \Theta^{2*} = \inf \{ t \geq 0 | X_t \in A_2 \}. \]
Then we have our main theorem.

**Theorem 3.2.** Under assumptions \((A1)\) and \((A2)\), the stochastic game with stopping and control has a value and \(u^*(i) = U(i) = L(i)\). Thus \(U^*\) is the unique fixed point of \(T\). Further \(((\Phi^*, \Theta^1*), (\Psi^*, \Theta^2*))\) is a saddle point equilibrium.

**Proof.** Let \(i \in S\) be such that \(u^*(i) < \psi_1(i)\). Let \(\Psi\) be any stationary control of player \(\Pi\). Let \(\{\tilde{X}_t; t \geq 0\}\) be the process governed by the stationary controls \(\Phi^*\) and \(\Psi\) and let \(\tilde{\Theta}^1 = \inf\{ t \geq 0 | \tilde{X}_t \in A_1 \}\), \(\Theta^2\) be any stopping time of player \(\Pi\). Then by Dynkin’s formula we get for \(T \geq 0\),

\[
E_i^{\Phi^*, \Psi} \left[ e^{-\alpha (T \wedge \tilde{\Theta}^1 \wedge \Theta^2)} u^* (\tilde{X}_{T \wedge \tilde{\Theta}^1 \wedge \Theta^2}) \right] - u^*(i) = E_i^{\Phi^*, \Psi} \int_0^{T \wedge \tilde{\Theta}^1 \wedge \Theta^2} e^{-\alpha t} \left[ -\alpha u^* (\tilde{X}_t) + \sum_{j \in S} \tilde{q}(j | \tilde{X}_t, \Phi^*(\tilde{X}_t), \Psi(\tilde{X}_t)) u^*(j) \right] dt 
\]

\[
\leq -E_i^{\Phi^*, \Psi} \int_0^{T \wedge \tilde{\Theta}^1 \wedge \Theta^2} e^{-\alpha t} r(\tilde{X}_t, U_t, V_t) dt.
\]

Now letting \(T \to \infty\) we get,

\[
u^*(i) \geq E_i^{\Phi^*, \Psi} \int_0^{\tilde{\Theta}^1 \wedge \Theta^2} e^{-\alpha t} r(\tilde{X}_t, U_t, V_t) dt \]

\[
+ E_i^{\Phi^*, \Psi} \left[ e^{-\alpha (\tilde{\Theta}^1 \wedge \Theta^2)} u^* (\tilde{X}_{\tilde{\Theta}^1 \wedge \Theta^2}) \right] \geq J_\alpha (i, \Phi^*, \Psi, \tilde{\Theta}^1, \Theta^2).
\]

Since the above is true for any strategy \((\Psi, \Theta^2)\) of player \(\Pi\), we obtain \(u^*(i) \geq U(i)\).

Analogously it can be shown that \(u^*(i) \leq L(i)\). Hence we are done. \(\square\)

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Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, Assam, India
E-mail address: cpal@iitg.ernet.in

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, Assam, India
E-mail address: saha.subhamay@iitg.ernet.in