PFAFFIAN INTERSECTIONS ANDMULTIPLICITY CYCLES

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Abstract. We consider the problem of estimating the intersection multiplicity between an algebraic variety and a Pfaffian foliation, at every point of the variety. We show that this multiplicity can be majorized at every point \( p \) by the local algebraic multiplicity at \( p \) of a suitably constructed algebraic cycle. The construction is based on Gabrielov’s complex analog of the Rolle-Khovanskii lemma.

We illustrate the main result by deriving similar uniform estimates for the complexity of the Milnor fiber of a deformation (under a smoothness assumption) and for the order of contact between an algebraic hypersurface and an arbitrary non-singular one-dimensional foliation. We also use the main result to give an alternative geometric proof for a classical multiplicity estimate in the context of commutative group varieties.

1. Introduction

1.1. Motivation. Let \( M \) be a complex variety of dimension \( n \). For simplicity of the presentation we assume that \( M = \mathbb{C}^n \). We consider problems of the following type:

I. Given a polynomial vector field \( \xi \) on \( M \) and an algebraic hypersurface \( V \subset M \), estimate for each \( p \in V \) the order of contact between \( V \) and the trajectory of \( \xi \) through \( p \).

II. More generally, given a foliation \( \mathcal{F} \) on \( M \) of codimension \( k \) and an algebraic variety \( V \subset M \) of dimension \( k \), estimate for each \( p \in V \) the intersection multiplicity between \( V \) and the leaf of \( \mathcal{F} \) through \( p \).

III. Given a flat algebraic family \( V \subset M \times \mathbb{C} \), estimate for each point \( p \in M \) the topological complexity (for instance, sum of Betti numbers) of the Milnor fiber of \( V \) at \( (p,0) \).

These question all share a similar nature. Namely, there is some map \( \mu : M \to \mathbb{Z}_{\geq 0} \) defined in algebraic terms, which one would like to estimate from above. Moreover, each of the question involves algebraic degrees (e.g. the degree of the polynomials defining \( \xi, \mathcal{F}, V \)) and one often hopes to obtain an estimate in terms of the degrees of this data. We assume for simplicity that all degrees are bounded by a number \( d \).

Various estimates for each of these problems have been studied (see the appropriate subsection of [14] for references). The most common form for an estimate of this type is

\[
\mu(p) \leq \phi(d)
\]
where $\phi(d)$ is some function depending only on $d$. Motivated by the Bezout theorem, it is reasonable to hope that $\phi$ can be taken to be a polynomial of degree $n$ in $d$, and indeed this is often the case. Moreover, this can easily be shown to be optimal (up to a multiplicative constant) for each of the problems above if one considers only bounds of the form (1).

It is sometimes desirable to have a more detailed understanding of the behavior of $\mu(p)$ as $p$ varies over a set of points. One may hope that in this context more refined estimates can be given. For instance, it is often reasonable to expect that outside a set of codimension $k$ an estimate of order $d^{k-1}$ holds. A result of this nature was proved for problem I in [10], motivated by problems in transcendental number theory (following similar but less refined estimates in [4, 5]).

We propose an algebraic mechanism which is useful in the description of such phenomena. Recall that an algebraic cycle $\Gamma$ is a linear combination of algebraic varieties with integer (for us also positive) coefficients. The multiplicity $\text{mult}_p \Gamma$ for an irreducible variety $\Gamma$ is defined to be the multiplicity of the intersection at $p$ between $\Gamma$ and a generic linear space of complementary dimension. This is extended to the multiplicity $\text{mult}_p \Gamma$ for any cycle. An estimate in terms of a multiplicity cycle is an estimate of the form

$$\mu(p) \leq \text{mult}_p \Gamma$$

for some cycle $\Gamma$. Again motivated by the Bezout theorem, one may hope that the $k$-dimensional part of $\Gamma$, denoted $\Gamma^k$, satisfies $\deg \Gamma^k \leq d^{n-k}$. We prove the existence of an estimate of this form for problem II above, under the assumption that the foliation is Pfaffian (see §4.2.1 for discussion and conjecture regarding the general case). We then use this result to derive similar estimates for problems I in full generality and III under a smoothness assumption.

An estimate of the form (2) has convenient algebraic properties which enable a broader range of applications than the uniform estimate (1). We illustrate this by giving an alternative proof for a classical multiplicity estimate on group varieties due to Masser and Wustholz in [4, 3].

Finally, we make a general remark on the form of the estimate (2). We were first able to obtain an estimate of this form using the topological methods described in this paper. Later, using this expression as a part of an inductive hypothesis, we were also able to obtain a similar result for problem I using an entirely different algebraic approach (see Remark 12 for a brief discussion on the relative advantages of each approach). We view this as an indication that estimates in terms of multiplicities of cycles are fundamentally suitable for the treatment of problems of this type.

1.2. Statement of the main result. Let $M$ denote a complex variety of dimension $n$. Let $\omega_1, \ldots, \omega_k \in \Lambda^1(M)$ be an ordered sequence of one-forms. We will say that $\omega_1, \ldots, \omega_k$ defines an integrable Pfaffian system at a point $p \in M$ if $\omega_1 \wedge \cdots \wedge \omega_k$ does not vanish at $p$ and there exists a (necessarily unique) chain of germs of manifolds $S^k_p \subset \cdots \subset S^1_p \subset M$ containing $p$, where each inclusion is of codimension 1 and such that $\omega_i|_{S^i} \equiv 0$ for $i = 1, \ldots, k$.

Let $\mathbb{C}_e$ denote the germ of $\mathbb{C}$ at the origin, and $e$ the coordinate function on $\mathbb{C}_e$. A function $f : M \times \mathbb{C}_e \to \mathbb{C}$ will be called a family of analytic functions on $M$ depending on the parameter $e$. We adopt the notation $f(x,e) \equiv f(x)$. Recall that a variety $V \subset M \times \mathbb{C}_e$ is said to be flat if each of its components project dominantly on $\mathbb{C}_e$. We denote by $\mathcal{F}(V)$ the flat family obtained from $V$ by
removing any component violating this condition. We denote by $V^\varepsilon := V \cap \{e = \varepsilon\}$ the $\varepsilon$-fiber of $V$.

To simplify the notation we denote $\omega_{1..k} := \omega_1, \ldots, \omega_k$ and similarly for a tuple of families of analytic functions $f_{1..m}$. For brevity, we also denote $\bigwedge_1 \omega_{1..k} := \omega_1 \wedge \cdots \wedge \omega_k$.

**Definition 1.** For a family $V \subset M \times \mathbb{C}_e$ and $p \in M$, we define the deformation multiplicity of $V$ at $p$, denoted $\text{mult}_p^\varepsilon(V)$, to be the number of isolated points in $V^\varepsilon$, $\varepsilon \neq 0$ converging to $p$ as $\varepsilon \to 0$.

Let $\omega_{1..k}$ be a Pfaffian system integrable at $p \in M$ and $f_{1..m}$ be families of meromorphic functions defined near $p$, with $p + m = \dim V - 1$. We define the deformation multiplicity relative to $f_{1..m}$ and $\omega_{1..k}$ to be

$$\text{mult}_p^\varepsilon(f_{1..m}; \omega_{1..k}; V) := \text{mult}_p^\varepsilon(\{f_{1..m} = 0\} \cap S_p^k \cap V).$$

In this case we count each isolated point of intersection (for fixed $\varepsilon$) with its associated multiplicity in the intersection of the cycles $[S_p^k]$ and $[f_{1..m} = 0]$ on $V^\varepsilon$ (which may be negative if $f_{1..m}$ have poles).

We omit $e$ when it is clear from the context, $V$ if it is $M \times \mathbb{C}_e$ and $\omega_{1..k}, f_{1..m}$ if they are empty sequences.

Recall that a (mixed) algebraic cycle in $M$ is a linear combination with integer coefficients of subvarieties of $M$. We will only consider cycles with non-negative coefficients. For a mixed cycle $\Gamma$, we denote by $\Gamma^j$ the $j$-codimensional part of $\Gamma$. The (algebraic) multiplicity of a cycle $\Gamma$ of pure dimension $j$ at a point $p$, denoted $\text{mult}_p \Gamma$, is defined to be the multiplicity of the intersection between $\Gamma$ and a generic linear subspace of codimension $j$ passing through $p$. This is extended to arbitrary mixed cycles by linearity.

Our goal is to majorize the multiplicity $\text{mult}_p(f_{1..n-k}; \omega_{1..k})$ as a function of $p$ in terms of algebraic multiplicities. For simplicity of the presentation we assume that the ambient space is given by $M = \mathbb{C}^n$ and $f_{1..n-k}, \omega_{1..k}$ are polynomial (although the construction could clearly be carried out for more general ambient spaces). Our main result is as follows.

**Theorem 1.** There exist a mixed algebraic cycle $\Gamma := \Gamma(f_{1..n-k}; \omega_{1..k})$ of top dimension $k$ in $M$ such that for any point $p$ where $\omega_{1..k}$ is integrable,

$$\text{mult}_p(f_{1..n-k}; \omega_{1..k}) \leq \text{mult}_p \Gamma(f_{1..n-k}; \omega_{1..k}).$$

Moreover, $\Gamma$ is obtained algebraically from $f_{1..n-k}; \omega_{1..k}$ as described in Definition 2. We refer to $\Gamma$ as the multiplicity cycle associated to $f_{1..n-k}; \omega_{1..k}$.

We denote by $\deg(f_{1..n-k}; \omega_{1..k}) := (\beta_1, \ldots, \beta_{n-k})$ where $\deg f_i = \beta_i$ and $\deg \omega_j = \alpha_j$ (where the degree of a one-form is understood to be the maximal degree of any of its coefficients in the standard coordinates).

**Theorem 2.** Let $\deg(f_{1..n-k}; \omega_{1..k}) := (\beta_1, \ldots, \beta_{n-k})$. Then

$$\deg(\Gamma^{n-j}(f_{1..n-k}; \omega_{1..k}) \leq \frac{1}{j! 2^{(k-j)(k-j-1)/2} \beta_1 \cdots \beta_{n-k}} S^{k-j}$$

where

$$S := \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_{n-k}.$$
We sometimes consider the degrees of \( f_1 \ldots n-k, \omega_1 \ldots k \) as growing asymptotically while fixing all other parameters. We give another form of the bound which stresses this particular asymptotic.

**Corollary 2.** Let \( \deg f_1 \ldots n-k, \omega_1 \ldots k \leq d \). Then
\[
\deg \Gamma^{n-j}(f_1 \ldots n-k, \omega_1 \ldots k) \leq C_{n,k,j}d^{n-j}
\]
where
\[
C_{n,k,j} := \frac{k!}{j!} 2^{(k-j)(k-j-1)/2} n^{k-j}.
\]

1.3. **Structure of this paper.** In §2 we survey the Rolle-Khovanskii lemma for real Pfaffian systems and its basic application in the theory of real Fewnomials; and the complex analog of this lemma due to Gabrielov. In §3 we present the construction of the multiplicity cycles for Pfaffian systems and prove the main theorems of the paper. In §4 we discuss various application of the main theorem — to the topological complexity of Milnor fibers in §4.1; to multiplicity estimates for trajectories of non-singular vector fields in §4.2; and to multiplicity estimates on group varieties in §4.3. Finally, in §5 we present an auxiliary compactness result for multiplicity cycles which is used throughout the paper.

2. **Background**

2.1. **Real Pfaffian systems.** Integrable systems of Pfaffian equations were first studied by Khovanskii [11] in the real domain, giving rise to the theory of Fewnomials. We briefly recall the basic elements of this theory in a context suitable for comparison with the present paper. Our presentation is thus restricted in scope, and we refer the reader to [11] for a complete account.

Let \( M \) be a real manifold of dimension \( n \) and \( \omega \in \Lambda^1(M) \) a one-form. We say that \( S \) is a separating solution for \( \omega \) with the coorientation defined by \( \omega \), if:

1. \( S \) has codimension 1 and \( \omega|_S \equiv 0 \).
2. There exists a manifold \( N \subset M \), called the film, such that \( \partial N = S \) and \( \omega \) takes positive values on the vector pointing inside \( N \) at every point of \( S \).

Let \( f_1 \ldots n-1 \) be smooth functions, and \( \Gamma \) their set of common zeros. We assume for simplicity that \( \Gamma \) is a smooth complete intersection curve meeting \( S \) transversally.\(^1\)

The following result forms the basis of the theory of real Fewnomials.

**Lemma 3** (Rolle-Khovanskii, [11] III.4 Corollary 2). Let \( B \) denote the number of non-compact components of \( \Gamma \), and \( N \) denote the number of zeros of \( \omega|_\Gamma \). Then the number of points in \( S \cap \Gamma \) is bounded by \( B + N \).

**Proof.** Choose any orientation for \( \Gamma \), and let \( p, q \in \Gamma \cap S \) be two adjacent points of intersection (i.e. such that the piece of \( \Gamma \) lying between \( p \) and \( q \), denoted \( \Gamma_{p,q} \), does not contain points of \( S \)). It follows that the positive tangent vector to \( \Gamma \) points inside the film \( N \) at \( p \) and outside the film \( N \) at \( q \) or vice versa. By condition (2) above, this means that \( \omega \) changes sign between \( p \) and \( q \), hence by the classical Rolle lemma must have a zero on \( \Gamma \) between the two. A simple counting argument concludes the proof. \( \square \)

\(^1\)this restriction can be relaxed significantly by a perturbation argument which we omit for simplicity.
Assume now that $M = \mathbb{R}^n$. In this case the number of non-compact components $B$ in Lemma 3 can also be algebraically estimated as follows.

**Lemma 4** ([11, Lemma on page 11]). The exists an affine hyperplane $H$ such that the number of non-compact components of $\Gamma$ does not exceed the number of intersections between $\Gamma$ and $H$.

Consider now a sequence of one-forms $\omega_1 \ldots k \in \Lambda^1(M)$ and a chain of submanifolds $S^k \subset \ldots \subset S^1 \subset S^0 = M$ where each inclusion $S^{i+1} \subset S^i$ forms a separating solution for $\omega_i$. Let $f_1 \ldots n - k$ be a sequence of polynomials, which we shall assume to be sufficiently generic. Lemmas 3 and 4 suggest the following estimate for the number of intersections between $S^k$ and $f_1 \ldots n - k = 0$,

$$
\#(S^k \cap f_1 \ldots n - k) \leq \deg(\Gamma(f_1 \ldots n - k, \omega_1 \ldots k)).
$$

(9)

where

$$
g := \frac{\wedge \omega_1 \ldots k \wedge \wedge f_1 \ldots 1 - k \wedge dx_1 \ldots n}{\wedge \wedge dx_1 \ldots n}.
$$

(10)

and $H$ is the hyperplane whose existence is guaranteed by Lemma 4. It is sometimes customary to include $H$ in the definition of $g$ and avoid the second summand in (9). However, for the purposes of comparison with the present paper the form above is more convenient. In particular, a simple inductive argument gives the following.

**Theorem 3.** There exist a mixed algebraic cycle $\Gamma := \Gamma(f_1 \ldots n - k, \omega_1 \ldots k)$ in $M$ such that

$$
\#(S^k \cap f_1 \ldots n - k) \leq \deg(\Gamma(f_1 \ldots n - k, \omega_1 \ldots k)).
$$

(11)

Moreover, $\Gamma$ is obtained algebraically from $f_1 \ldots n - k, \omega_1 \ldots k$.

**Sketch of proof.** After $k$ inductive applications of (9), one obtains various summands of the form

$$
\#(\{h_1 = \cdots = h_{n-r} = H_1 = \cdots = H_r = 0\})
$$

(12)

where $h_i$ are polynomials and $H_i$ are linear functions. This quantity is certainly bounded by $\deg \Gamma_k$ where $\Gamma_k := \{h_1 = \cdots = h_{n-r} = 0\}$, and collecting all such cycles into $\Gamma$ we obtain the estimate. $\square$

It may appear unclear why one should consider components of different dimensions in $\Gamma(f_1 \ldots n - k, \omega_1 \ldots k)$ when it is equally possible to replace each of the positive dimensional components $\Gamma_h$ in the proof above by its zero-dimensional intersection with $H_1 = \cdots = H_r = 0$. The reader may note the formal analogy between Theorems 1 and 3. We will see that in the local case, the decomposition into components of various dimensions plays a more principal role.

2.2. **Complex Pfaffian systems.** In the complex setting one can no longer expect a global estimate of the type given in Theorem 3. However, a local analog has been developed by Gabrielov [6]. This was later used to establish various estimates on the geometric complexity of sets defined using Pfaffian functions (see [10] for a survey). We recall the fundamental result from this paper, namely a local complex analog of Lemma 3.

Let $M$ be a complex manifold of dimension $n$ and $\omega \in \Lambda^1(M)$ a one-form. Let $p \in M$ and suppose that $\omega$ admits an integral manifold $S_p$ through $p$. Finally, let
\( X \subset M \times \mathbb{C}_\varepsilon \) be a (reduced) flat family, \( \text{dim } X = 2 \). We fix any analytic coordinate system \( x_1, \ldots, x_n \) around \( p \).

**Lemma 5** ([6, Theorem 1.2]). Assume that in a neighborhood of \( p \), \( X^\varepsilon \cap S_p \) consists of isolated points for small \( \varepsilon \neq 0 \). Let \( dH \) be a generic constant one-form. Then
\[
\text{mult}_p(\omega; X) \leq \text{mult}_p(g; X) + \text{mult}_p(dH; X) \tag{13}
\]
where
\[
g = \frac{\omega \wedge (de + c\text{e}dH)|_X}{dH \wedge de|_X} \tag{14}
\]
and \( c \) is a generic complex number.

We remark that the second summand in (13) could be replaced by the equivalent term \( \text{mult}_p(H; X) \), and this is the formulation originally appearing in [6]. With this formulation, one is required to choose a generic \( H \) vanishing at \( p \), whereas in the formulation above we obtain an expression essentially uniform over \( p \). However, the one-form \( dH \) must still satisfy a genericity condition possibly depending on the point \( p \), and we shall have to resolve this technical difficulty in order to obtain truly uniform estimates over \( p \).

The extra generic factor \( c\text{e}dH \) in Lemma 5 is needed in order to avoid certain degeneracies in the case where the intersection \( S_p \cap X^0 \) degenerates into a non-isolated intersection. We sketch the proof for the case where the intersection is isolated and the generic fiber \( X^\varepsilon \) is smooth. We also assume for simplicity that \( \omega \) is a closed form, and hence \( \omega = d\pi \) for some analytic function \( \pi : M \to \mathbb{C} \) around \( p \). In this case case one may take \( c = 0 \), and we will in fact show equality in (13). Our presentation follows that of Gabrielov [6].

**Sketch of proof.** By assumption the fibers \( X^\varepsilon \) are analytic curves. We fix a small positive \( \delta \) and denote \( D_\delta = \{|z| \leq \delta\} \). Consider the fiber
\[
F^\varepsilon := X^\varepsilon \cap \pi^{-1}(D_\delta). \tag{15}
\]
In fact, for \( 0 < \varepsilon \ll \delta \) this fiber has the homotopy type of the Milnor fiber of \( X \) at \( p \).

Let \( \mu := \text{mult}(\omega; X) \). Since the intersection is isolated at \( e = 0 \) by assumption, this means that \( \pi : F^0 \to D_\delta \) is a ramified \( \mu \) to \( 1 \) map. For fixed \( \delta \) and sufficiently small \( \varepsilon \), this remains true for \( \pi : F^\varepsilon \to D_\delta \) as well. By the Riemann-Hurwitz formula,
\[
\chi(F^\varepsilon) = \mu - \text{mult}_p(\omega|_{X^\varepsilon}; X) \tag{16}
\]
where we slightly abuse notation and allow the top-form \( \omega|_{X^\varepsilon} \) in place of a meromorphic function in the second summand, which corresponds to the number of critical points of \( \pi \) on \( F^\varepsilon \) (with their multiplicities).

Arguing similarly with \( H \) in place of \( \pi \), we have
\[
\chi(F^\varepsilon) = \text{mult}_p(H; X) - \text{mult}_p(dH|_{X^\varepsilon}; X). \tag{17}
\]
Note that while \( F^\varepsilon \) appearing in this equation is in fact different (being defined using \( H \) in place of \( \pi \)), in homotopy type both sets agree with the Milnor fiber of \( X \) at \( p \). Thus we may compare the Euler characteristics from (16) and (17) to obtain
\[
\mu = \text{mult}_p(\omega|_{X^\varepsilon}; X) - \text{mult}_p(dH|_{X^\varepsilon}; X) + \text{mult}_p(H; X)
\]
\[
= \text{mult}_p(g; X) + \text{mult}_p(H; X)
\]
and the claim follows since \( \text{mult}_p(dH; X) = \text{mult}_p(H; X). \)

Consider now a sequence of one-forms \( \omega_{1...k} \in \Lambda^1(M) \) defining an integrable Pfaffian system at \( p \in M \), and the corresponding chain of integral submanifolds \( S^k_p \subset \cdots \subset S^1_p \subset S^0_p = M \). Let \( f_{1...n-k} \) be families of polynomials on \( M \). Lemma 5 can be used in a manner analogous to the use of Lemma 8 in the proof of Theorem 2.1 to obtain a bound for \( \text{mult}_p(f_{1...n-k}; \omega_{1...k}) \) in terms of certain algebraic systems of equations, and more specifically the number of solutions for these equations converging to \( p \) (see [6, Theorem 2.1]).

3. The Multiplicity Cycles

Let \( M = \mathbb{C}^n \). Let \( \omega_{1...k} \in \Lambda^1(M) \) be a sequence of one-forms with polynomial coefficients and \( U \subset M \) a set such that \( \omega_{1...k} \) defines an integrable Pfaffian system at \( p \) for each \( p \in U \), and denote the corresponding chain of integral submanifolds by \( S^k_p \subset \cdots \subset S^1_p \subset S^0_p = M \). Let \( f_{1...n-k} \) be families of polynomials on \( M \).

In this section we describe the explicit construction of the multiplicity cycle \( \Gamma(f_{1...n-k}; \omega_{1...k}) \) and prove Theorem 2.1. We begin with a technical result on smoothing deformations which was used in [6, Proof of Theorem 2.1].

Lemma 6. Let \( p \in U \). For any \( N \in \mathbb{N} \) and \( c = c_{1...n-k} \in \mathbb{C} \) define the sequence \( \tilde{f}_{1...n-k} \) by \( \tilde{f}_i = f_i - c_i e^N \). Then for sufficiently large \( N \) and generic \( c \) we have

\[
\text{mult}_p(f_{1...n-k}; \omega_{1...k}) \leq \text{mult}_p(\tilde{f}_{1...n-k}; \omega_{1...k})
\]

and moreover, the intersection \( \{ \tilde{f}_{1...n-k} = 0 \} \cap S^{k-1}_p \) is an effectively non-singular curve intersecting \( S^k_p \) discretely (in a neighborhood of \( p \)) for sufficiently small \( \varepsilon \neq 0 \).

Proof. Standard analytic arguments show that each isolated point of \( f_{1...n-k} = 0 \) on \( S^k \) converging to \( p \) as \( e \to 0 \) survives the perturbation \( f \to \tilde{f} \) as long as \( N \) is large enough (possibly bifurcating into a number of points of the same total multiplicity). This ensures (18).

To satisfy the non-singularity condition it is enough to verify that \( c_{1...n-k} e^N \) is not a critical value for \( f_{1...n-k} \) on \( S^k_{p-1} \), which by the Bertini-Sard theorem is certainly true for generic \( c \) and sufficiently small \( \varepsilon \neq 0 \). Similarly one verifies that the intersection \( \{ \tilde{f}_{1...n-k} = 0 \} \cap S^k_p \) is discrete for generic \( c \) and sufficiently small \( \varepsilon \neq 0 \). \( \square \)

The construction of the multiplicity cycle is based on an inductive process, with the following consequence of Lemma 5 providing the key inductive step.

Lemma 7. Let \( p \in U \) and let \( \partial H \) be a generic constant one-form. Then

\[
\text{mult}_p(f_{1...n-k}; \omega_{1...k}) \leq \text{mult}_p(\tilde{f}_{1...n-k}, g; \omega_{1...k-1})
\]

\[
+ \text{mult}_p(\tilde{f}_{1...n-k}; dH, \omega_{1...k-1})
\]

with

\[
g = \frac{\wedge d\tilde{f}_{1...n-k} \wedge \omega_{1...k} \wedge (de + c\epsilon dH)}{\wedge dx_{1...n} \wedge de}
\]

where \( \tilde{f}_{1...n-k} \) are as given by Lemma 2 and \( c \) is a generic complex number.
Proof. We may apply Lemma \[7\] and assume without loss of generality that \( f_{1...n-k} \) are already in the prescribed form. Since \( dH_1 \) is generic, its integral manifold through \( p \) intersects \( S^1_p, \ldots, S^{k-1}_p \) transversally, and it follows that the sequence \( dH_1, \omega_{1...k-1} \) is an integrable Pfaffian system as well, hence \( \langle 19 \rangle \) is well defined.

Define the flat family \( X \subset M \times \mathbb{C}_\varepsilon \) by

\[
X := \mathcal{F}[\{f_{1...n-k} = 0\} \cap S^{k-1}_p].
\]

We can now apply Lemma \[5\] to \( X \) with the forms \( \omega_k, dH \). By Lemma \[6\] the form

\[
\bigwedge d f_{1...n-k} \bigwedge \omega_{1...k-1}
\]

is non-vanishing on \( X \) (for small \( \varepsilon \neq 0 \)), and it follows that the zeros of the numerator of \( \langle 20 \rangle \) agree with those of \( \langle 14 \rangle \) for such \( \varepsilon \). As the denominator of \( \langle 20 \rangle \) has no zeros, \( \langle 19 \rangle \) now follows from Lemma \[5\].

Applying Lemma \[7\] iteratively \( k \) times gives the following.

**Lemma 8.** Let \( p \in U \) and and let \( dH_1, \ldots, dH_k \) be generic constant one-forms. Then

\[
\text{mult}_p(f_{1...n-k}; \omega_{1...k}) \leq \text{mult}_p \Gamma + \sum_{j=1}^{k-1} \text{mult}_p(f_{1...n-k}, g_j; dH_{1...j-1}, \omega_{1...k-j})
\]

(23)

where \( \Gamma \) is the cycle given by the flat limit of \( \{ f_{1...n-k} = 0 \} \) as \( e \to 0 \),

\[
g_j = \frac{\bigwedge d f_{1...n-k} \bigwedge \omega_{1...k-j-1} \bigwedge d f_{1...j-1} \wedge (\text{de} + ce dH_j)}{dx_1 \wedge \cdots \wedge dx_n \wedge \text{de}}
\]

(24)

and \( f_{1...n-k} \) are as given by Lemma \[7\].

**Proof.** Applying Lemma \[7\] with \( \omega_{1...k} \) and \( dH = dH_1 \) we obtain

\[
\text{mult}_p(f_{1...n-k}; \omega_{1...k}) \leq \text{mult}_p(f_{1...n-k}, g_1; \omega_{1...k}) + \text{mult}_p(f_{1...n-k}; dH_1, \omega_{1...k-1})
\]

(25)

where the first summand corresponds to the \( j = 1 \) summand in \( \langle 23 \rangle \). Applying now Lemma \[7\] with \( dH_1, \omega_{1...k-1} \) and \( dH_2 \) we obtain

\[
\text{mult}_p(f_{1...n-k}; dH_1, \omega_{1...k-1}) \leq \text{mult}_p(f_{1...n-k}, g_2; dH_1, \omega_{1...k-2}) + \text{mult}_p(f_{1...n-k}; dH_1, dH_2, \omega_{1...k-2}).
\]

(26)

where the first summand corresponds to the \( j = 2 \) summand in \( \langle 24 \rangle \). Note that formally one should apply a further smoothing deformation to \( f_{1...n-k} \) when applying Lemma \[7\]. This would not make any difference for the rest of the argument, but to simplify the notation we assume from the start that the deformation \( f_{1...n-k} \) was chosen sufficiently generic to apply Lemma \[7\] with \( dH_{1...j}, \omega_{k-j} \) for \( j = 1, \ldots, k-1 \).

Continuing in this manner one obtains the summands corresponding to \( j = 1, \ldots, k-1 \) in \( \langle 23 \rangle \). In the \( j = k-1 \) step, the second summand is given by \( \text{mult}_p(f_{1...n-k}, dH_{1...k}) \). Since the Pfaffian system \( dH_{1...k} \) defines a generic hyperplane passing through \( p \) and properly intersecting the \( k \)-cycle \( \Gamma \) there, this term is equal to \( \text{mult}_p \Gamma \). \[\square\]
We are now ready to present the construction of the multiplicity cycles. The construction depends on the choice of various parameters, the totality of which we denote by $H$.

**Definition 9.** The multiplicity cycle $\Gamma^H(f_{1\ldots n-k};\omega_{1\ldots k})$ is a mixed cycle in $M$ defined recursively as follows. If $k = 0$, $\Gamma^H(f_{1\ldots n-k};\omega_{1\ldots k})$ is the cycle given by the flat limit of the set $\{\tilde{f}_{1\ldots n} = 0\}$ as $e \to 0$. Otherwise,

$$\Gamma^H(f_{1\ldots n-k};\omega_{1\ldots k}) := \Gamma + \sum_{j=1}^{k-1} \Gamma^H(\tilde{f}_{1\ldots n-k}, g_j; dH_{1\ldots j-1}, \omega_{1\ldots k-j})$$

(27)

where $\Gamma$, $g_{1\ldots k-1}$ and $\tilde{f}_{1\ldots n-k}$ are as given in Lemma 8. In each recursive step we use different generic one-forms $dH_j$ and parameters defining the smoothing deformations of $f_{1\ldots n-k}$, all of which are encoded by $H$.

**Proof of Theorem 4.** Let $\mu : M \to \mathbb{Z}_{\geq 0}$ be defined by

$$\mu(p) := \text{mult}_p(f_{1\ldots n-k};\omega_{1\ldots k})$$

(28)

We denote $\Gamma^H := \Gamma^H(f_{1\ldots n-k};\omega_{1\ldots k})$. If $p_1, \ldots, p_s \in U$ is any finite set of points, then for a sufficiently generic choice of $H$ (and the parameters $N$ for the smoothing deformations sufficiently large) we have

$$\mu(p_i) \leq \text{mult}_{p_i} \Gamma^H \quad i = 1, \ldots, k. \quad (29)$$

Indeed, for such $H$ Lemma 8 applies with $p = p_i$, and (29) follows by reverse induction on $k$. Moreover, one can certainly choose $H$ generic enough (and with large enough $N$) so that this applies to each of the finitely many points under consideration.

To obtain an estimate uniform in $p$ we appeal to the results of Lemma 8. From the construction of $\Gamma^H$ it is clear that it is an algebraic cycle whose total degree is uniformly bounded in terms of the degrees of $f_{1\ldots n-k}$ and $\omega_{1\ldots k}$ (for a more precise statement see Theorem 2). Thus, by Proposition 20 the function $p \to \text{mult}_p \Gamma^H$ is an upper semicontinuous function of complexity bounded by some uniform constant $D$ independent of $H$. Thus, by Proposition 18 there exists some finite set of points $P \subset M$ such that for any $H$,

$$\mu(p)|_P \leq \text{mult}_P \Gamma^H|_P \implies \mu(p) \leq \text{mult}_P \Gamma^H \text{ for any } p \in M. \quad (30)$$

Choosing now $H$ sufficiently generic so that (29) holds for every point of $P$ and setting $\Gamma(f_{1\ldots n-k},\omega_{1\ldots k}) := \Gamma^H$ concludes the proof.

**Proof of Theorem 2.** Suppose that $(f_{1\ldots n-k};\omega_{1\ldots k})$ have degrees $(\beta_{1\ldots n-k};\alpha_{1\ldots k})$. If $j = n - k$ then $\Gamma^H(f_{1\ldots n-k},\omega_{1\ldots k})$ is equal to the flat limit of a family of cycles defined by equations of degrees $\beta_{1\ldots n-k}$ and the claim follows.

Otherwise, Definition 9 implies that the $j$-dimensional piece of $\Gamma^H(f_{1\ldots n-k},\omega_{1\ldots k})$ is a sum of the corresponding $j$-dimensional pieces of $k$ multiplicity cycles, each having degrees bounded by $(\beta_{1\ldots n-k};S;\alpha_{1\ldots k-1})$. Continuing inductively, each such cycle gives rise to $k - 1$ multiplicity cycles, each having degrees bounded by $(\beta_{1\ldots n-k};S,2S;\alpha_{1\ldots k-2})$ and so on. The result follows by induction.

Theorem 1 deals particularly with complete intersections, i.e. with the situation where the number of equations $f_{1\ldots n-k}$ is complementary to the number of one-forms $\omega_{1\ldots k}$. However, this can easily be extended to more general intersections.
Corollary 10. Let $V ⊂ M × \mathbb{C}_e$ be a family given by the common zero locus of families of polynomials $f_{1,...,m}$ (not necessarily a complete intersection). Then there exist a mixed algebraic cycle $Γ := Γ(f_{1,...,m}; ω_{1,...,k})$ of dimension at most $k$ in $M$ such that for every $p ∈ U$,
\[ \text{mult}_p(ω_{1,...,k}; V) ≤ \text{mult}_p Γ(f_{1,...,m}; ω_{1,...,k}). \]  
Moreover, the cycle is obtained algebraically from $f_{1,...,m}; ω_{1,...,k}$ and satisfies the same degree bounds as in Theorem 11.

Proof. Let $μ : M → \mathbb{Z}_{≥ 0}$ be defined by
\[ μ(p) := \text{mult}_p(ω_{1,...,k}; V). \]
Let $p ∈ M$ be any point, and denote by $V_p$ the germ of $V$ at $p$. Then $V_p \cap S^k$ is a union of curves $γ^1_p, ..., γ^r_p$ projecting dominantly on $\mathbb{C}_e$ (and possibly some other components), and only these curves contribute to the multiplicity defining $μ(p)$. If we let $g^L_{1,...,n−k}$ denote $n−k$ generic linear combinations of $f_{1,...,m}$ (with $L$ denoting the collection of coefficients defining these combinations), then by standard arguments $γ^1_p, ..., γ^r_p$ are also irreducible components of the intersection $\{g^L_{1,...,n−k} = 0\} \cap S^k_p$, and therefore
\[ μ(p) ≤ μ^L(p) := \text{mult}_p(\{g^L_{1,...,n−k} = 0\}; ω_{1,...,k}). \]  
We pick generic $H$ and denote
\[ Γ^{H,L} := Γ^H(g^L_{1,...,n−k}; ω_{1,...,k}). \]
Then by Theorem 11 we have
\[ μ(p) ≤ μ^L(p) ≤ \text{mult}_p Γ^{H,L}. \]
Moreover, for sufficiently generic choices of $H,L$ the same holds for any finite collection of points $P ⊂ M$. The proof can now be concluded in the same way as in the proof of Theorem 11. □

4. Applications

4.1. Critical points and Euler characteristics of Milnor fibers. Let $ω_{1,...,k} ∈ Λ^1(M)$. We suppose for simplicity that they define an integrable Pfaffian system at every point $p ∈ M$ with the corresponding chain of integral submanifolds $S^k_p \subset ⋯ \subset S^1_p \subset S^0_p = M$ (although one could easily relax this requirement).

For a family of polynomials $f_{1,...,m}$ on $M$ we define
\[ Σ(f_{1,...,m}; ω_{1,...,k}) := \{ f_{1,...,m} = 0 \} \mathbf{d} f_{1,...,m} \mathbf{d} ω_{1,...,k} \mathbf{d} \mathbf{d} e = 0 \}. \]
In other words, for each fiber $e = ε$ we define $Σ(f_{1,...,m}; ω_{1,...,k})$ to be the set of points $p$ where $\{ f_{1,...,m} = 0 \} \cap S^k_p$ is not effectively smooth. Note that (36) is not in general a complete intersection: it involves the vanishing of an $m + k + 1$ form in $n + 1$ variables, and is thus a determinantal variety. However, its deformation multiplicity at any point $p ∈ M$ may still be estimated by Corollary 10. We will apply this result to estimate the topological complexity of Milnor fibers of deformations.

Consider the flat family defined by
\[ X ⊂ M × \mathbb{C}_e, \quad X := Σ(\{ f_{1,...,m} = 0 \}). \]
Recall that the Milnor fiber of $X$ at a point $p$ is defined to be
\[ F_p := X^e \cap D_δ(p), \quad 0 ≤ ε ≤ δ ≤ 1 \]
where $D_\delta(p)$ denotes the disc of radius $\delta$ around $p$ in any analytic coordinate system.

The homotopy type of the set $F_p$ is independent on the choice of coordinates and $\varepsilon, \delta$.

We call a point $p \in M$ good if the Milnor fiber at $p$ is effectively smooth. In this case a local complex analog of real Morse theory allow one to study the topological structure of the Milnor fiber in terms of critical points of linear functionals $\ell^2$. This result is known as Le’s attaching formula (see for instance [21]; cf. [5, Proposition 2] and [2, Proposition 12]). In particular, this implies the following.

**Proposition 11.** Let $\ell := \ell_1, \ldots, n-m+1$ be generic linear functionals on $M = \mathbb{C}^n$, and $p \in M$ a good point. Then for $k = 0, \ldots, n-m$,

$$b_{n-m-k}(F_p) \leq \text{mult}_p(d\ell_1, \ldots, d\ell_{k+1}; \Sigma(f_1, \ldots, f_{n-m}, d\ell_1, \ldots, k))$$

(39)

**Sketch of proof.** Denote $X_0 := X$. According to Le’s attaching formula, the Milnor fiber of $X$ is obtained from the Milnor fiber of $X_1 := X \cap \{\ell_1 = \ell_1(p)\}$ by attaching $c_{n-m}$ cells, where $c_{n-m}$ is given by the number of critical points of $\ell_1$ on $X_0$ converging to $p$ as $\varepsilon \to 0$. Thus

$$c_{n-m} = \text{mult}_p(\Sigma(f_1, \ldots, f_{n-m}; d\ell_1)).$$

(40)

Similarly, the Milnor fiber of $X_1$ is obtained from the Milnor fiber of $X_2 := X_1 \cap \{\ell_2 = \ell_2(p)\}$ by attaching $c_{n-m-1}$ cells, where $c_{n-m-1}$ is given by the number of critical points of $\ell_2$ on $X_1^\varepsilon$ converging to $p$ as $\varepsilon \to 0$. Thus

$$c_{n-m-1} = \text{mult}_p(d\ell_1; \Sigma(f_1, \ldots, f_{n-m}; d\ell_1, \ldots, 2)).$$

(41)

Continuing inductively and noting that $b_r \leq c_r$ concludes the proof. □

In combination with Corollary 10 we have the following multiplicity cycle estimate for the Betti numbers of the Milnor fiber of a deformation over a general point.

**Theorem 4.** For $r = 0, \ldots, n-m$ there exists a mixed algebraic cycle $\Gamma_r := \Gamma_r(X)$ such that for any good point $p \in M$,

$$b_r(F_p) \leq \text{mult}_p(\Gamma_r(X)).$$

(42)

Moreover, if $\deg f_1, \ldots, m \leq d$ then

$$\deg \Gamma_r^{n-j}(X) \leq D_{n,n-m-r,j}d^{m-j}, \quad D_{n,k,j} := mn-jC_{n,k,j}$$

(43)

for $C_{n,k,j}$ given in Corollary [10]

**Proof.** For any finite collection of points $P \subset M$ the claim is proved by choosing sufficiently generic $\ell$ and applying Proposition 11 and Corollary 10 noting that the equations defining $\Sigma(f_1, \ldots, f_{n-m}, d\ell_{1, \ldots, k+1})$ have degrees bounded by $md$ (for any $k$). To obtain a bound uniform over all good point $p \in M$ one can argue in the same way as the proof of Theorem 1. □

\[\text{[1]}\]

\[\text{[2]}\]
4.2. Multiplicity estimates for non-singular vector fields. Let $\xi$ be a polynomial vector field,

$$\xi = \xi_1 \frac{\partial}{\partial x_1} + \cdots + \xi_n \frac{\partial}{\partial x_n}, \quad \xi_1 \ldots \eta \in \mathbb{C}[x_1 \ldots n]$$

and $P$ be a polynomial. Let $p \in M$ be a non-singular point of $\xi$. The multiplicity of $P$ along $\xi$ at $p$ is defined to be

$$\text{mult}_\xi^p(P) := \text{ord}_p P \big|_{\gamma_p}$$

where $\gamma_p$ denotes the trajectory of $\xi$ through $p$. If $\xi$ is singular at $p$, or if $P$ vanishes identically on $\gamma_p$ we define $\text{mult}_\xi^p(P) = \infty$.

The problem of estimating $\text{mult}_\xi^p(P)$ in terms of $d := \deg P$ and $\delta := \deg \xi$ has been considered by various authors motivated by applications in transcendental number theory [4, 5, 16, 17], control theory [20, 7, 8] and dynamical systems [18, 23]. We present below the strongest estimate known for $\text{mult}_\xi^p(P)$ as a function of $p$, for arbitrary non-singular vector fields. This result (in a slightly different formulation) was first obtained using algebraic methods in [16] and later using topological methods in [2]. We give the formulation of [2] as it is more suitable for comparison with the present paper.

For a (possibly mixed) cycle $\Gamma$ we define a function $\mathcal{D}_p(\Gamma) : M \to \mathbb{Z}_{\geq 0}$ as follows. For an irreducible variety $V \subset M$ we define

$$\mathcal{D}_p(V) := \begin{cases} \deg V & p \in V \\ 0 & \text{otherwise,} \end{cases}$$

and extend this to arbitrary cycles $\Gamma$ by linearity. Then the following holds.

**Theorem 5.** There exists a mixed algebraic cycle $\Gamma \subset M$ such that

$$\text{mult}_\xi^p(P) \leq \mathcal{D}_p(\Gamma)$$

whenever the left-hand side is finite, and moreover,

$$\deg \Gamma^j \leq C_{n,\xi} d^j$$

where $C_{n,\xi}$ is some constant depending only on $n$ and $\xi$ (see [2] for precise expressions for the constants).

The function $\mathcal{D}_p(\Gamma)$ is somewhat artificial, involving global properties of the cycle $\Gamma$ when evaluated at a particular point $p$. The multiplicity function $\text{mult}_p \Gamma$ is a more natural local analog, and moreover it is clear that $\text{mult}_p \Gamma \leq \mathcal{D}_p \Gamma$. We will prove the following result.

**Theorem 6.** There exists a mixed algebraic cycle $\Gamma(P; \xi) \subset M$ such that

$$\text{mult}_\xi^p(P) \leq \text{mult}_p \Gamma(P; \xi)$$

whenever the left hand side is finite, and moreover,

$$\deg \Gamma^j(P; \xi) \leq C_{n,\xi} d^{n-j}$$

where the constant $C_{n,\xi}$ can be easily recovered from the proof.

**Remark 12.** During the preparation of this manuscript, the author has found an additional, algebraic proof for Theorem 6 cf. [3]. The two results are not comparable: the approach of [3] gives estimates valid also for singular vector fields under an additional assumption known as the D-property; on the other hand, with
this approach the D-property is necessary even in the non-singular case in order to produce estimates with explicit constants.

We note that the expression for $\text{mult}_p^\xi(P)$ in terms of local multiplicities of cycles was first obtained using the topological methods of the present paper. Having this expression in mind, the author was later able to produce an inductive algebraic construction in [3].

The proof of Theorem 6 is based on a beautiful topological characterization of the multiplicity $\text{mult}_p^\xi(P)$ discovered by Gabrielov [8], which we now recall. Let $Q \in \mathbb{C}[x_1...n]$ be a polynomial of degree $n - 1$, and denote $P^Q = P + eQ$. For $r = 1,...,n$ we define the family $X_r \subset M \times \mathbb{C}e$ by

$$X_r := \mathcal{F} \left[ \{ P^Q = \cdots = \xi^{r-1}P^Q = 0 \} \right].$$

Denote by $F^r_p$ the Milnor fiber of $X_r$ at the point $p$. Note that for simplicity of the notation we suppress the dependence of $X_r, F^r_p$ on $Q$.

**Proposition 13.** Let $p \in M$ be such that $\text{mult}_p^\xi(P) < \infty$. Then for a sufficiently generic choice of $Q$,

$$\text{mult}_p^\xi(P) \leq \sum_{r=1}^n \chi(F^r_p).$$

Moreover, $p$ is a good point of each of the families $X_r$ (i.e., the Milnor fiber $F^r_p$ is effectively smooth).

The proof of Theorem 6 is now a simple exercise using Theorem 4.

**Proof of Theorem 6.** Let $p \in M$ be such that $\text{mult}_p^\xi(P) < \infty$. Choose $Q$ sufficiently generic so that Proposition 13 applies. Let $r = 1,...,n$. Applying Theorem 4 to $X_r$ we obtain for every $q = 0,...,n - r$ a cycle $\Gamma_{r,q}$ such that

$$b_q(F^r_p) \leq \text{mult}_p \Gamma_{r,q}.$$  

Noting that the Euler characteristic is bounded by the sum of Betti numbers, and setting $\Gamma_r := \sum_q \Gamma_{r,q}$ gives

$$\chi(F^r_p) \leq \text{mult}_p \Gamma_r.$$  

Finally setting $\Gamma(P;\xi) := \sum_r \Gamma_r$ and using Proposition 13 gives

$$\text{mult}_p^\xi(P) \leq \text{mult}_p \Gamma(P;\xi).$$

The same arguments show that one can find a cycle $\Gamma(P;\xi)$ satisfying the conditions of the theorem for any finite collection of points $S \subset M$. One can then conclude that (for sufficiently generic $Q$) the bound is in fact uniform over all $p \in M$ in the same way as in the proof of Theorem 4.

**4.2.1. Multiplicity estimates for non-singular foliations.** Let $\xi_1...m$ be $m$ commuting polynomial vector fields defining a foliation $\mathcal{F}$ on $M$ and $P_1...m$ be $m$ polynomial functions. We define the multiplicity $\text{mult}_p(P_1...m;\mathcal{F})$ to be the intersection multiplicity between the leaf of $\mathcal{F}$ at $p$ and $\{ P_1...m = 0 \}$ (or infinity if the foliation is singular at $p$, or the intersection is not proper). In other words, we extend our usual definition for Pfaffian foliations to the case of arbitrary foliations. Note however that we intentionally introduce this notion for proper intersections rather than general deformations as we did in the Pfaffian case.

It is reasonable to ask whether Theorems 1 and 6 can be generalized to a result about general foliations, as follows.
Conjecture 14. There exists a mixed algebraic cycle $\Gamma(P_1...m; F) \subset M$ such that
\[
\text{mult}_p^{\mathcal{F}}(P_1...m) \leq \text{mult}_p \Gamma(P_1...m; F)
\]
whenever the left hand side is finite, and moreover,
\[
\text{deg} \Gamma^j(P_1...m; F) \leq C_n,\mathcal{F} d^{n-j}
\]
where $C_n,\mathcal{F}$ is a constant depending only on $n, \mathcal{F}$.

In [9] this problem was studied for the case of a single point $p$ (i.e., with estimates not depending on $p$). More specifically, it was proven that the multiplicity in this case is bounded by $C_n,\mathcal{F} d^n$ where $C_n,\mathcal{F}$ is some constant. It appears that in combination with the ideas of [2] the approach of [9] can be sharpened to give an estimate $C_n,\mathcal{F} d^n$, and moreover to prove Corollary 14 in full generality (albeit with worse constants). However, this result falls outside the scope of this paper and will appear separately.

We note that the formulation of Corollary 14 applies only to proper intersections, rather than to multiplicities of deformations as in the case of Theorem 1. It is in fact natural to conjecture that the same result would hold for arbitrary deformations. However, this result may require new ideas. A conjecture of this type was made in [9] for the case of a single point $p$ (with a less accurate asymptotic), and proved under a minor technical condition in [1]. But it is unclear whether this approach can be extended to produce multiplicity cycles and the precise asymptotic of Conjecture 14.

4.3. Multiplicity estimates on group varieties. Suppose that our ambient space is a commutative algebraic group variety $G$ with the associated Lie algebra $g$ of invariant vector fields. Let $P$ be a regular function on $G$. Results imposing restrictions on the geometry of the set of points where $P$ vanishes to a given order, with respect to one or several vector field, are known as multiplicity estimates on group varieties. Several authors have studied such multiplicity estimates, with important applications in transcendental number theory.

The area was initiated by Masser and Wustholz in the two important papers [12, 13]. In [12], the authors consider the zeros of $P$ without prescribing the order along a vector field. In [13], the authors consider zeros of $P$ of a prescribed order $T$ in the direction of an invariant vector field $\xi \in g$. Both papers rely on an algebraic technique involving manipulation of ideals, which was later elaborated to account for zeros of a prescribed order in the direction of several vector fields [22] and strengthened in [19].

In [14, 15], Moreau gave a simple geometric proof of the main result of [12]. However, the treatment of multiplicities still required the more general algebraic approach of [13]. Our goal in this section is to present a simple extension of Moreau’s geometric approach to the case involving multiplicities, relying on the notion of a multiplicity cycle. We begin by presenting the result of [12] in the formulation of [15].

For simplicity of the presentation we restrict ourselves to the case $G = (\mathbb{C}^*)^n$, and present estimates in terms of a pure degree $d$ rather than a sequence of mixed degrees (although the results of this section extend to arbitrary $G$ and more general notions of degree with minor modifications). To maintain the conventional notation in our presentation, we use additive notation for the addition rule in $G$. Following
Moreau, for a set \( \Sigma \subset G \) we denote
\[
\Sigma^{(p)} := \{ \gamma_1 + \cdots + \gamma_p : \gamma_i \in \Sigma \}
\]
\[
\Sigma^{(0)} := \{ 0 \}.
\]
For a subgroup \( H \) of \( G \), we denote by \( \#(\Sigma + H)/H \) the size of the coset space \( (\Sigma + H)/H \). With these notations, we have

**Theorem 7.** Let \( \Sigma \subset G \) be a finite subset containing \( 0 \), and \( P \in \mathbb{C}[x_1, \ldots, n] \) be a polynomial of degree \( d \). Suppose that for any proper algebraic subgroup \( H \subset G \) we have
\[
\#(\Sigma + H)/H > d \text{codim}_G H.
\]
Then
\[
\bigcap_{\gamma \in \Sigma^{(n)}} (-\gamma + \{ P = 0 \}) = \emptyset
\]

If \( V \subset G \) is an irreducible variety, we define its \( d \)-weight to be \( \text{wt}_d V := \text{deg} V / d \text{codim}_G V \). We extend this by linearity to arbitrary varieties and cycles in \( G \). For any (effective) cycle \( \Gamma \), we denote by \( \circ \Gamma \) the variety underlying \( \Gamma \) (i.e. without associated multiplicities). With this notation, Theorem 7 follows from the following more general assertion.

**Lemma 15.** Let \( \Sigma \subset G \) be a finite subset containing \( 0 \). Let \( V \subset G \) be a variety of top dimension \( m \) which is cut out set-theoretically by polynomials of degree at most \( d \). Suppose that for any proper algebraic subgroup \( H \subset G \) we have
\[
\#(\Sigma + H)/H > d \text{codim}_G H \text{wt}_d V.
\]
Then
\[
\bigcap_{\gamma \in \Sigma^{(m+1)}} (-\gamma + V) = \emptyset
\]

**Proof.** We proceed by induction on \( m \). For \( m = 0 \), it follows from (60) (with \( H = \{ 0 \} \)) that \( \# \Sigma > \text{deg} V = \# V \). Thus for any \( g \in G \) there exists \( \gamma \in \Sigma \) with \( g + \gamma \not\in V \), proving (61).

Assume now that \( m > 0 \) and write \( V^m = V_1 \cup \cdots \cup V_s \). Let \( i = 1, \ldots, s \) and let \( H_i \) denote the stabilizer of \( V_i \) in \( G \). Certainly \( \text{codim}_G H_i \geq m \), so by (60) we have
\[
\#(\Sigma + H_i)/H_i > d^m \text{wt}_d V \geq \text{deg} V^m \geq s
\]
It follows that there exists an element \( \gamma_i \in \Sigma \) such that \( \gamma_i + V_i \neq V_j \) for \( j = 1, \ldots, s \), and hence \( \gamma_i + V_i \not\subset V \). Since \( V \) is cut out by polynomials of degree bounded by \( d \), we can find such a polynomial \( P_i \) which vanishes on \( V \) but not on \( \gamma_i + V_i \). Set
\[
V'_i = V_i \cap (-\gamma_i + \{ P_i = 0 \}).
\]
By the Bezout theorem, \( \text{wt}_d V'_i \leq \text{wt}_d V_i \).

We now set
\[
V' := \bigcup_{i=1}^s V'_i \cup V^{m-1} \cup \cdots \cup V^0.
\]
Then the top-dimension of \( V' \) is at most \( m-1 \) and \( \text{wt}_d V' \leq \text{wt}_d V \). By construction we have
\[
\bigcap_{\gamma \in \Sigma} (-\gamma + V) \subset V'.
\]
Finally, by induction we have
\[ \bigcap_{\gamma \in \Sigma^{(m+1)}} (-\gamma + V) \subseteq \bigcap_{\gamma \in \Sigma^{(m)}} (-\gamma + V') = \emptyset \] (66)
as claimed. \hfill \Box

Let \( V \subseteq G \) be an irreducible variety and \( H \) a divisor. If \( V \not\subseteq H \) we denote by \( [V] * H \) the intersection of \([V]\) and \( H \) as cycles. Otherwise, we denote \([V] * H := V\). We extend this by linearity to define \( \Gamma * H \) for an arbitrary cycle \( \Gamma \). Note that \( * \) is not associative, though it is associative at the level of the underlying sets. If \( \deg H \leq d \) then by the Bezout theorem, \( \text{wt}_d \Gamma * H \leq \text{wt}_d \Gamma \).

Lemma 16. Let \( \Gamma \) by a mixed cycle in \( G \) and \( H \) a divisor. For any \( p \in H \) we have
\[ \text{mult}_p \Gamma * H \geq \text{mult}_p \Gamma. \] (67)

Proof. It suffices to prove the claim for the case \( \Gamma = [V] \) for an irreducible variety \( V \subseteq G \). If \( V \not\subseteq H \) then the claim is obvious. Otherwise, the right hand side is given by the intersection at \( p \) of \( \dim V \) generic divisors passing through \( p \) and the right hand side is given by the intersection at \( p \) of \( H \) and \( \dim V - 1 \) generic divisors passing through \( p \). The result follows by the semicontinuity of the intersection multiplicity. \hfill \Box

We now consider multiplicities in the direction of an invariant vector field. Let \( \xi \in g \) be an invariant vector field on \( G \) and \( P \) a polynomial of degree \( d \). In particular, \( \xi \) is a linear vector field, and derivation with respect to \( \xi \) does not increase the degree of a polynomial. The trajectories of \( \xi \) can be written as the solution of a Pfaffian chain. Indeed, \( \xi \) spans the kernel of \( n - 1 \) invariant one-forms \( \omega_1 \ldots \omega_{n-1} \in g^* \), and the integrability condition for these forms follows from the commutativity of \( g \). By Theorem 1 there exists a cycle \( \Gamma(P; \xi) \) in \( G \) with
\[ \text{mult}_p \xi P = \text{mult}_p (P; \omega_1 \ldots \omega_{n-1}) \leq \text{mult}_p \Gamma(P; \xi) \] (68)
and
\[ \text{wt}_d \Gamma(P; \xi) \leq C_G \] (69)
for some constant \( C_G \) depending only on \( G \). Alternatively, one can apply Theorem 6 to \( \xi \) directly (giving a somewhat worse constant).

Proposition 17. Let \( P \) be a polynomial of degree \( d \). Let \( T \in \mathbb{N} \) and let \( V_T \) denote the variety of points where \( P \) vanishes to order at least \( T \) along \( \xi \),
\[ V_T := \{ P = \xi P = \cdots = \xi^{T-1} P = 0 \} \] (70)
Then \( \text{wt}_d V_T \leq C_G / T \).

Proof. Let \( \Gamma \) be the cycle given by
\[ \Gamma := (\cdots (\Gamma(P; \xi) * \{ P = 0 \}) \cdots) * \{ \xi^{T-1} P = 0 \} \] (71)
By Lemma 16 and the remark preceding it we have \( \text{wt}_d \Gamma \leq C_G \) and for every \( p \in V_T \),
\[ \text{mult}_p \Gamma \geq \text{mult}_p \Gamma(P; \xi) \geq T. \] (72)
Moreover, \( \Gamma \) is clearly supported on \( V_T \). It follows that each irreducible component \( W \) of \( V_T \) must appear as a summand \( m_W[W] \) of \( \Gamma \) with the \( m_W \geq T \) (because the
\[ \text{mult}_p \Gamma \geq \text{mult}_p \Gamma(P; \xi) \geq T. \] (72)
Moreover, \( \Gamma \) is clearly supported on \( V_T \). It follows that each irreducible component \( W \) of \( V_T \) must appear as a summand \( m_W[W] \) of \( \Gamma \) with the \( m_W \geq T \) (because the
\[ \text{mult}_p \Gamma \geq \text{mult}_p \Gamma(P; \xi) \geq T. \] (72)
Moreover, \( \Gamma \) is clearly supported on \( V_T \). It follows that each irreducible component \( W \) of \( V_T \) must appear as a summand \( m_W[W] \) of \( \Gamma \) with the \( m_W \geq T \) (because the
multiplicity of \( \Gamma \) at generic points of \( W \) where \( W \) is smooth is equal to \( m_W \). Thus \( \text{wt}_d V_T \leq C_G/T \) as claimed.

An application of Lemma \( \ref{lem:mult} \) now gives the following multiplicity version of Moreau's result, essentially agreeing with the multiplicity estimate of [13].

**Theorem 8.** Let \( \Sigma \subset G \) be a finite subset containing 0, and \( P \in \mathbb{C}[x_1, \ldots, n] \) be a polynomial of degree \( d \). Let \( T \in \mathbb{N} \) and let \( V_T \) denote the variety of points where \( P \) vanishes to order at least \( T \) along \( \xi \),

\[
V_T := \{ P = \xi P = \cdots = \xi^{T-1} P = 0 \}.
\]

Suppose that for any proper algebraic subgroup \( H \subset G \) we have

\[
\#(\Sigma + H)/H > C_G d^{\text{codim}_H} H / T.
\]

Then

\[
\bigcap_{\gamma \in \Sigma^{(n)}} (-\gamma + V_T) = \emptyset
\]

5. A compactness property for semicontinuous bounds

In this section we assume for simplicity of the formulation that the ambient variety \( M \) is given by \( M = \mathbb{C}^n \) (although similar results would hold, nearly verbatim, in a much more general context).

Recall that a function \( F : M \to \mathbb{N} \) is said to be (algebraic) upper semicontinuous if the sets

\[
F \geq n := F^{-1}([n, \infty))
\]

are closed varieties for each \( n \in \mathbb{N} \). We will say that \( F \) has complexity bounded by \( D \) if moreover, all of these sets can be defined by equations of degree at most \( D \).

The following proposition appeared in [2]. We include the proof for the convenience of the reader.

**Proposition 18.** Let \( D \in \mathbb{N} \) and \( f : M \to \mathbb{N} \) an arbitrary bounded function. Then there exists a finite set of points \( P \subset M \) such that for any upper semicontinuous function \( F \) of complexity bounded by \( D \),

\[
f|_P \leq F|_P \implies f \leq F.
\]

**Proof.** Denote by \( N \) an upper bound for \( f \). Then \( f \leq F \) if and only if \( f_{\geq i} \subset F_{\geq i} \) for \( i = 1, \ldots, N \). Thus it will suffice to construct a finite set \( P_i \subset f_{\geq i} \) such that for any set \( S \) of complexity bounded by \( D \),

\[
P_i \subset S \implies f_{\geq i} \subset S
\]

and take \( P = \bigcup_{i=1}^N P_i \).

Let \( L \) denote the linear space of polynomials of degree bounded by \( D \) on \( M \). For any \( p \in M \) let \( \phi_p : L \to \mathbb{C} \) denote the functional of evaluation at \( p \). Finally, for any set \( P \subset M \) denote by \( L_P \subset L \) the linear subspace of polynomials which vanish at every point of \( P \).

We need to construct a finite set \( P_i \subset f_{\geq i} \) with \( L_{P_i} = L_{f_{\geq i}} \). This is clearly possible. Indeed, \( L_{f_{\geq i}} \) is the kernel of the set of functionals \( \{ \phi_p : p \in f_{\geq i} \} \). Since \( L_{f_{\geq i}} \) has finite codimension in \( L \), one can choose a finite subset \( P_i \) (in fact, of size equal to this codimension) of functionals whose kernel, \( L_P \), agrees with \( L_{f_{\geq i}} \). This concludes the proof.

The following simple exercises is left for the reader.
Lemma 19. Let $F_i, i = 1, \ldots, N$ be an upper semicontinuous functions with complexity $D_i$ and bounded by $B_i$. Then $\sum_{i=1}^{N} F_i$ is an upper semicontinuous function with complexity bounded by $D' = D'(D_1 \ldots N, B_1 \ldots N)$.

Finally, the following proposition allows the application of Proposition 18 to multiplicity cycles.

Proposition 20. Let $\Gamma$ be an algebraic cycle (possibly of mixed dimension) of total degree bounded by $d$. Then $p \mapsto \mult_p \Gamma$ is an upper semicontinuous function of complexity bounded by a constant $D$ depending only on $d$.

Proof. By Lemma 19 it is enough to establish the result for a cycle of pure dimension $k$. The family of all such cycles is parametrized by the projective Chow variety $\mathcal{C}_{k,d}$. Moreover, the correspondence

$$M_\mu \subset \mathcal{C}_{k,d} \times M \quad M_\mu = \{ (\Gamma', p) : \mult_p \Gamma' \geq \mu \}$$

(78)

is algebraic for $\mu = 1, \ldots, d$ and empty for $\mu > d$. In particular, the fibers of $M_\mu$ under the projection to $\mathcal{C}_{k,d}$ have uniformly bounded degrees (and hence are also set-theoretically cut out by equations of uniformly bounded degrees). This concludes the proof. \(\square\)

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