DERIVED HECKE ALGEBRA AND AUTOMORPHIC L-INVARIANTS

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Abstract. Let \( \pi \) be a cohomological automorphic representation of \( \text{PGL}(2) \) over a number field of arbitrary signature and assume that the local component of \( \pi \) at a prime \( p \) is the Steinberg representation. In this situation one can define an automorphic \( L \)-invariant for each cohomological degree in which the system of Hecke eigenvalues associated to \( \pi \) occurs. We show that these \( L \)-invariants are (essentially) the same if the \( \pi \)-isotypic component of the cohomology is generated by the minimal degree cohomology as a module over Venkatesh’s derived Hecke algebra.

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Introduction

The system of Hecke eigenvalues associated to a Hilbert modular newform of parallel weight \((2, \ldots, 2)\) only shows up in the middle degree cohomology of the corresponding Hilbert modular variety. On the contrary, if one considers an automorphic form \( f \) of parallel weight \((2, \ldots, 2)\) of the group \( \text{PGL}_2 \) over a number field \( F \) which is not totally real, its system of Hecke eigenvalues show up in several degrees. For example, assume that \( F \) is an imaginary quadratic field with class number one and that the field of definition of \( f \) is \( \mathbb{Q} \). Let \( n \subseteq \mathcal{O}_F \) be the level of \( f \) and let \( \Gamma_0(n) \subseteq \text{PSL}_2(\mathcal{O}_F) \) be the subgroup of all matrices which are congruent to an upper triangular matrix modulo \( n \). Then, we have

\[
\dim H^1(\Gamma_0(n), \mathbb{Q})[f] = H^2(\Gamma_0(n), \mathbb{Q})[f] = 1,
\]

where \([f]\) denotes the \( f \)-isotypic component.

In [9] Venkatesh constructs a graded-commutative extension \( \bar{T} = \bigoplus_{i \geq 0} \bar{T}_i \) - called the derived Hecke algebra - of the usual Hecke algebra, which acts in a graded fashion on cohomology with \( p \)-adic coefficients. In particular, it maps the \( f \)-isotypic component to itself. This action should be rich enough to account for the occurrence of the same Hecke eigenvalues in several degrees, i.e. the \( f \)-isotypic part of the cohomology should be generated by its lowest degree as a \( T \)-module. In our example, this simply means that there should exist a derived Hecke operator \( t \in \bar{T}_1 \) such that the corresponding map

\[
H^1(\Gamma_0(n), \mathbb{Q}_p)[f] \xrightarrow{t} H^2(\Gamma_0(n), \mathbb{Q}_p)[f]
\]

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is non-zero. As a consequence, the different cohomology groups attached to \( f \) should carry the same information, e.g. the \( p \)-adic periods one can define using these different cohomology groups should be the same.

The kind of period we study in this article are the so-called automorphic \( \mathcal{L} \)-invariants. Suppose that the automorphic form \( f \) is Steinberg at a prime \( p \), i.e. \( p \) divides the level \( n \) exactly once and the \( U_p \)-eigenvalue of \( f \) is 1. Then, using the lowest degree cohomology one can attach an automorphic \( \mathcal{L} \)-invariant to \( f \) and \( p \), which shows up in the exceptional zero formula of the \( p \)-adic \( \mathcal{L} \)-function attached to \( f \) (see [6] for details.) The study of automorphic \( \mathcal{L} \)-invariants was initiated by Darmon in [5], where he considered elliptic modular forms of weight 2. The construction was generalized to various settings, for example, to Hilbert modular forms of parallel weight \((2,\ldots,2)\) by Spieß (cf. [8]), by Barrera, Dimitrov and Jorza to Hilbert modular forms of arbitrary weight (cf. [1]) and by Barrera and Williams to Bianchi modular forms of arbitrary weight (cf. [2]).

Let us recall the construction of the automorphic \( \mathcal{L} \)-invariant in our example: Let \( \text{St}_p \) be the space of all integer-valued locally constant functions on \( \mathbb{P}^1(F_p) \)-module constant functions. This is canonically a \( PGL_2(F_p) \)-module. Let \( \Gamma_0(p) \subseteq PGL_2(O_p[p^{-1}]) \) be the \( p \)-arithmetic group given by the same congruence conditions as its arithmetic counterpart. Then, the fact that \( f \) is Steinberg at \( p \) implies that evaluation at an Iwahori-fixed vector induces an isomorphism

\[
H^1(\Gamma_0(p), \text{Hom}(\text{St}_p, \mathbb{Q}_p))[f] \xrightarrow{\text{ev}} H^1(\Gamma_0(n), \mathbb{Q}_p)[f]
\]

of one-dimensional vector spaces. \( P \)-adic integration together with Breuil’s construction of extensions of the Steinberg representation yields for every continuous homomorphism \( \ell: F_p^* \to \mathbb{Q}_p \) a map

\[
H^1(\Gamma_0(p), \text{Hom}(\text{St}_p, \mathbb{Q}_p))[f] \xrightarrow{c_\ell} H^2(\Gamma_0(p), \mathbb{Q}_p)[f],
\]

which turns out to be an isomorphism if \( \ell = \text{ord}_p \) is the \( p \)-adic valuation. Then, the automorphic \( \mathcal{L} \)-invariant \( \mathcal{L}_\ell(f, p) \in \mathbb{Q}_p \) is defined as the unique \( p \)-adic number such that

\[
c_\ell = \mathcal{L}_\ell(f, p) \cdot \text{ord}_p.
\]

Instead of working with the first cohomology group we could work with the second one and get an a priori different \( \mathcal{L} \)-invariant. But let us suppose we have an action of the derived Hecke algebra \( \mathbb{T} \) on \( H^*(\Gamma_0(n), \text{Hom}(\text{St}_p, \mathbb{Q}_p))[f] \) and \( H^*(\Gamma_0(n), \mathbb{Q}_p))[f] \) such that for all \( \ell \in \mathbb{T}_1 \) and all homomorphism \( \ell: F_p^* \to \mathbb{Q}_p \) the diagrams

\[
\begin{array}{ccc}
H^1(\Gamma_0(p), \text{Hom}(\text{St}_p, \mathbb{Q}_p))[f] & \overset{t}{\longrightarrow} & H^2(\Gamma_0(p), \text{Hom}(\text{St}_p, \mathbb{Q}_p))[f] \\
\text{ev} & & \text{ev}
\end{array}
\]

\[
\begin{array}{ccc}
H^1(\Gamma_0(n), \mathbb{Q}_p)[f] & \overset{t}{\longrightarrow} & H^2(\Gamma_0(n), \mathbb{Q}_p)[f] \\
& & \\
\end{array}
\]

and

\[
\begin{array}{ccc}
H^1(\Gamma_0(p), \text{Hom}(\text{St}_p, \mathbb{Q}_p))[f] & \overset{c_\ell}{\longrightarrow} & H^2(\Gamma_0(p), \text{Hom}(\text{St}_p, \mathbb{Q}_p))[f] \\
& & \\
H^2(\Gamma_0(p), \mathbb{Q}_p)[f] & \overset{c_\ell}{\longrightarrow} & H^3(\Gamma_0(p), \mathbb{Q}_p)[f]
\end{array}
\]

commute (up to a sign). Then, the existence of an element \( t \in \mathbb{T}_1 \) as in (0.1) would easily imply that the two \( \mathcal{L} \)-invariants are equal. The main aim of this note is to construct such actions of the derived Hecke algebra and prove that the corresponding diagrams for arbitrary number fields commute.
The paper is structured as follows: The first part is devoted to the construction of automorphic $\mathcal{L}$-invariants in arbitrary cohomological degree. Thankfully, the construction of Spieß can be easily generalized to this setup. A new feature over general number fields, which does not occur when dealing with Hilbert or Bianchi modular forms, is that the cohomology groups in question are not necessarily one-dimensional anymore. Thus, one is forced to define automorphic $\mathcal{L}$-invariants as determinants (see Definition 1.7). We show that it does not matter whether we use cohomology with or without compact support in defining automorphic $\mathcal{L}$-invariants (see Proposition 1.8). Finally, we state our conjecture that automorphic $\mathcal{L}$-invariants coming from different cohomology degrees are essentially the same.

In the second part we review Venkatesh’s derived Hecke algebra. We show that it acts on all cohomology groups in question and that generalizations of the diagrams above commute. Thus, we deduce that our conjecture follows if the $f$-isotypic part of the cohomology of the appropriate locally symmetric space is generated by its lowest degree as a $\mathbb{T}$-module (see Theorem 2.8).

**Notations.** Throughout the article we fix a prime $p$. All rings are assumed to be commutative and unital. The group of invertible elements of a ring $R$ will be denoted by $R^*$.

If $R$ is a ring and $G$ is a group, we write $R[G]$ for the group algebra of $G$ with coefficients in $R$. Given a group $G$ and a group homomorphism $\epsilon : G \to R^*$ we let $R(\epsilon)$ be the $R[G]$-module which underlying $R$-module is $R$ itself and on which $G$ acts via the character $\epsilon$. If $M$ is another $R[G]$-module, we put $M(\epsilon) = M \otimes_R R(\epsilon)$.

If $X$ and $Y$ are topological spaces, we write $C(X, Y)$ for the space of continuous maps from $X$ to $Y$.

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**The setup**

We fix an algebraic number field $F$ of degree $d = r + 2s$, where $r$ (resp. $s$) is the number of real (resp. complex) places of $F$. The ring of integers of $F$ will be denoted by $O$. We denote the set of infinite places of $F$ by $S_\infty$.

If $v$ is a place of $F$, we denote by $F_v$ the completion of $F$ at $v$. If $q$ is a finite place, we let $O_q$ denote the valuation ring of $F_q$ and $\text{ord}_q$ the additive valuation such that $\text{ord}_q(\varpi) = 1$ for any local uniformizer $\varpi \in O_q$. We write $\mathcal{N}(q)$ for the cardinality of the residue field $O/q$.

For a finite set $S$ of places of $F$ we define the ”$S$-truncated adeles” $\mathbb{A}^S$ as the restricted product of all completions $F_v$ with $v \notin S$. In case $S$ is the empty set we drop the superscript $S$ from the notation. We will often write $\mathbb{A}^{S, \infty}$ instead of $\mathbb{A}^{S_\infty}$.

We consider the algebraic group $G = \text{PGL}_2$. If $v$ is a place of $F$, we put $G_v = G(F_v)$ and we write $G_\infty = \prod_{v \notin S_\infty} G_v$. If $q$ is a finite place, we write $K_q = G(O_q)$.

Further, we fix a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$, which is cohomological with respect to the trivial coefficient system. For simplicity, we assume that the field of definition of the finite part of $\pi$ is the field of rationals. We denote the conductor of $\pi$ by $f_\pi$. 

At last, we fix a prime $p$ of $F$ lying above $p$ and we assume that the local representation $\pi_p$ is the Steinberg representation $\text{St}_p(C)$. Hence, the prime $p$ divides the conductor $f_\pi$ exactly once.

**Remark 0.1.** One could also allow non-trivial central characters by working with the group $GL_2$ instead of $PGL_2$. More generally, one could consider automorphic representations of inner forms of $GL_2$. To keep the notation as simple as possible we stick with the special case. But all arguments carry over easily to the more general situation.

1. **Automorphic L-invariants**

1.1. **Extension classes.** We recall a variant of Spieß’ construction of extensions of the continuous Steinberg representation (see [5]).

Let $R$ be a locally pro-finite ring. In our applications $R$ will be one of the following rings: $\mathbb{Z}$, $\mathbb{Z}/p^n\mathbb{Z}$, $\mathbb{Z}_p$ or $\mathbb{Q}_p$. The $R$-valued continuous Steinberg representation $\text{St}_p^{\text{cont}}(R)$ of $G_p$ is given by the set of continuous $R$-valued functions on $\mathbb{P}^1(F_p)$ modulo constant functions.

For a continuous group homomorphism $\ell: F^* \rightarrow R$ we define $\tilde{\mathcal{E}}(\ell)$ as the set of pairs $(\Phi, r) \in C(GL_2(F_p), R) \times R$ with

$$\Phi \left( g \cdot \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} \right) = \Phi(g) + r \cdot \ell(t_1)$$

for all $t_1, t_2 \in F_p^*$, $u \in F$ and $g \in G_p$. The group $GL_2(F_p)$ acts on $\tilde{\mathcal{E}}(\ell)$ via $g.(\Phi(-), r) = (\Phi(g^{-1} \cdot -), r)$. The subspace $\tilde{\mathcal{E}}(\ell)_0$ of tuples of the form $(\Phi, 0)$ with constant $\Phi$ is invariant. We get an induced $G_p$-action on the quotient $\mathcal{E}(\ell) = \tilde{\mathcal{E}}(\ell)/\tilde{\mathcal{E}}(\ell)_0$.

**Lemma 1.1.** Let $\text{pr}: G_p \rightarrow \mathbb{P}^1(F_p)$, $g \mapsto g.\infty$, be the canonical projection. The following sequence of $R[G_p]$-modules is exact:

$$0 \rightarrow \text{St}_p^{\text{cont}}(R) \xrightarrow{(\text{pr}_*0)} \mathcal{E}(\ell) \xrightarrow{(0, \text{id}_R)} R \rightarrow 0$$

We write $b_\ell$ for the associated cohomology class in $H^1(G_p, \text{St}_p^{\text{cont}}(R))$.

**Proof.** This is a variant of [8], Lemma 3.11. □

1.2. **Cohomology of $PGL(2)$.** We introduce the cohomology groups which we will use to define automorphic $\ell$-invariants and state their basic properties.

Throughout this section we fix a ring $R$. Let $\text{Div}(\mathbb{P}^1(F))$ be the free abelian group on $\mathbb{P}^1(F)$ with its natural $G(F)$-action. We write $\text{Div}_0(\mathbb{P}^1(F))$ for the kernel of the map

$$\text{Div}(\mathbb{P}^1(F)) \rightarrow \mathbb{Z}, \sum_P m_P P \mapsto \sum_P m_P.$$

For a prime $q$ of $F$ we let

$$\text{St}_q = \text{St}_q^{\text{cont}}(\mathbb{Z}) = C^0(\mathbb{P}^1(F_q), \mathbb{Z})/\mathbb{Z}$$

be the integral Steinberg representation of $G_q$ and for a finite set $S$ of primes of $F$ we put

$$\text{St}_S = \bigotimes_{q \in S} \text{St}_q.$$ 

Given an $R$-module $N$, finite sets $S_0 \subseteq S$ of primes of $F$ and a compact, open subgroup $K \subseteq G(\mathbb{A}_S^{\infty})$ we define $A(K, S_0; N)^S$ to be the space of all functions

$$\Phi: G(\mathbb{A}_S^{\infty}) \rightarrow \text{Hom}_\mathbb{Z}(\text{St}_{S_0}, N)$$
such that $\Phi(kg) = \Phi(g)$ for all $k \in K$ and $g \in G(A^{S,\infty})$. Furthermore, we define

$$\mathcal{A}_c(K, S_0; N)^S = \text{Hom}_\mathbb{Z}(\text{Div}_0(\mathbb{P}^1(F)), \mathcal{A}(K, S_0; N)^S).$$

**Definition 1.2.** For a locally constant character $\epsilon: G_{\infty} \to \{\pm 1\}$ and a set of data as above we define

$$\mathcal{M}_i^e(K, S_0; N)^{S,\epsilon} = \mathcal{H}^{i+r+s}(G(F), \mathcal{A}(K, S_0; N)^S(\epsilon))$$

and

$$\mathcal{M}_i^e(K, S_0; N)^{S,\epsilon} = \mathcal{H}^{i+r+s-1}(G(F), \mathcal{A}_c(K, S_0; N)^S(\epsilon)).$$

Furthermore, we consider

$$\mathcal{M}^\ast(K, S_0; N)^{S,\epsilon} = \bigoplus_{i \in \mathbb{Z}} \mathcal{M}_i^e(K, S_0; N)^{S,\epsilon}.$$

Let $n \subseteq \mathcal{O}$ be a non-zero ideal. If $K = K_0(n)^S \subseteq G(A^{S,\infty})$ is the subgroup of all integral matrices, which are congruent to a upper triangular matrix modulo $n$, we set $\mathcal{M}_i^e(n, S_0; N)^{\epsilon} = \mathcal{M}_i^e(K, S_0; N)^{S,\epsilon}$ and similarly for the version with compact support. If $\epsilon$ is the trivial character, we sometimes drop it from the notation.

The short exact sequence

$$0 \longrightarrow \text{Div}_0(\mathbb{P}^1(F)) \longrightarrow \text{Div}(\mathbb{P}^1(F)) \longrightarrow \mathbb{Z} \longrightarrow 0$$

induces a short exact sequence

$$0 \longrightarrow \mathcal{A}(K, S_0; N)^S \longrightarrow \text{Hom}_\mathbb{Z}(\text{Div}_0(\mathbb{P}^1(F)), \mathcal{A}(K, S_0; N)^S) \longrightarrow \mathcal{A}_c(K, S_0; N)^S \longrightarrow 0$$

and thus a long exact sequence in cohomology. We write

$$\delta: \mathcal{M}_i^e(K, S_0; N)^{S,\epsilon} \longrightarrow \mathcal{M}_i^e(K, S_0; N)^{S,\epsilon}$$

for the connecting homomorphism.

**Proposition 1.3.** Given a set of data as above we have:

(a) The $R$-module $\mathcal{M}^e_i(K, S_0; R)^{S,\epsilon}$ is finitely generated for $? \in \{\emptyset, c\}$ and all $i \in \mathbb{Z}$ if $R$ is Noetherian.

(b) If $N$ is a flat $R$-module, then the canonical map

$$\mathcal{M}^e_i(K, S_0; R)^{S,\epsilon} \otimes_R N \longrightarrow \mathcal{M}^e_i(K, S_0; N)^{S,\epsilon}$$

is an isomorphism for $? \in \{\emptyset, c\}$ and all $i \in \mathbb{Z}$.

**Proof.** See Proposition 4.6 of [8].

Since by Proposition 1.3 (a) the projective system $(\mathcal{M}^e_i(K, S_0; \mathbb{Z}/p^n))_n$ (and also the version with compact support) fulfills the Mittag-Leffler condition we get the following

**Corollary 1.4.** The canonical map

$$\mathcal{M}^e_i(K, S_0; \mathbb{Z}_p) \longrightarrow \lim_n \mathcal{M}^e_i(K, S_0; \mathbb{Z}/p^n)$$

is an isomorphism for $? \in \{\emptyset, c\}$ and all $i \in \mathbb{Z}$.
1.3. Automorphic L-invariants. In the following we will define \( L \)-invariants for our fixed automorphic representation \( \pi \). We show that the definition does not depend on whether we take cohomology with or without compact support. At the end of the section we will state our main conjecture.

Let \( \mathbb{T} \) the \( \mathbb{Z} \)-algebra generated by all good Hecke operators away from \( p \), i.e. the polynomial ring on the variables \( T_q \) with \( q \nmid pf \). In particular, we do not include Hecke operators at our fixed prime \( p \). For \( q \nmid f \) let \( \lambda_q \) be the eigenvalue of \( T_q \) on a spherical vector of \( \pi_q \). Given a \( T \)-module \( M \) we put

\[
M[\pi] = \{ m \in M \mid T_q(m) = \lambda_q \cdot m \ \forall q \}.
\]

Evaluation at a normalized Iwahori-fixed vector yields a Hecke-equivariant map

\[
(1.2) \quad M_i(f, \emptyset; \{ p \}; R)^{[p], \epsilon} \rightarrow M_i(f, \emptyset; R)^{\epsilon}
\]
for any ring \( R \).

**Proposition 1.5.** Let \( \Omega \) be a field of characteristic 0. We have:

(a) For all \( i \geq 0 \) the connecting homomorphism \( (1.1) \) induces an isomorphism

\[
delta : M_i^\epsilon(f, \emptyset; \Omega)[\pi] \rightarrow M_i^\epsilon(f, \emptyset; \Omega)[\pi]
\]
and we have \( \dim M_i^\epsilon(f, \emptyset; \Omega)[\pi] = \binom{i}{\epsilon} \) for \( \emptyset \leq \epsilon \).

(b) The map \( (1.2) \) induces an isomorphism

\[
M_i^\epsilon(f, \emptyset; \Omega)[\pi] \rightarrow M_i^\epsilon(f, \emptyset; \Omega)[\pi]
\]
for \( \emptyset \leq \epsilon \) and all \( i \geq 0 \).

(c) For all \( i \geq 0 \) the connecting homomorphism \( (1.1) \) induces an isomorphism

\[
delta : M_i^\epsilon(f, \emptyset; \Omega)[\pi] \rightarrow M_i^\epsilon(f, \emptyset; \Omega)[\pi]
\]
and we have \( \dim M_i^\epsilon(f, \emptyset; \Omega)[\pi] = \binom{i}{\epsilon} \) for \( \emptyset \leq \epsilon \).

**Proof.** The first claim is a standard fact about the contribution of cuspidal representations to cohomology (see for example [7]). For the second claim see the proof of [8], Proposition 4.8. Here, the fact that \( \pi_p \) is Steinberg is essential. The last claim follows directly from the first two. \( \square \)

Every element in Hom(\( St_p^{cont}(Q_p) \)) can be uniquely extended to a continuous functional on \( St_p^{cont}(Q_p) \). Thus, we get a pairing

\[
(1.3) \quad A(\emptyset, \{ p \}; \mathbb{Z}_p)^{p} \times St_p^{cont}(Q_p) \rightarrow A(\emptyset, \{ p \}; \mathbb{Z}_p)^{p} \otimes Q_p
\]
for every non-zero ideal \( n \subseteq \mathcal{O} \). Hence, using Proposition 1.3 we get a cup product pairing

\[
(1.4) \quad M_i^\epsilon(n, \{ p \}; Q_p)^{[p], \epsilon} \times H^i(G(F), St_p^{cont}(Q_p)) \rightarrow M_i^{\epsilon+j}(n, \emptyset; Q_p)^{[p], \epsilon}.
\]

Let \( \ell : F_p^* \rightarrow \mathbb{Q}_p \) be a continuous homomorphism and \( b_\ell \in H^1(G(F), St_p^{cont}(Q_p)) \) be the (restriction of the) cohomology class associated to it in Section 1.1. Taking the cup product with \( b_\ell \) induces maps

\[
c_{\ell}^{(i)}(\pi)^{\epsilon} : M_i^\epsilon(f, \emptyset; \{ p \}; Q_p)^{[p], \epsilon}[\pi] \xrightarrow{\cup b_\ell} M_i^{\epsilon+1}(f, \emptyset; Q_p)^{[p], \epsilon}[\pi]
\]
and

\[
c_{\ell}^{(s)}(\pi)^{\epsilon} : M^s(f, \emptyset; \{ p \}; Q_p)^{[p], \epsilon}[\pi] \xrightarrow{\cup b_\ell} M^s(f, \emptyset; Q_p)^{[p], \epsilon}[\pi]
\]
In addition, we define

\[
c_{\ell}^{(s)}(\pi)^{\epsilon} = \oplus c_{\ell}^{(s)}(\pi)^{\epsilon}.
\]

**Lemma 1.6.** Let \( ord_p : F_p^+ \rightarrow \mathbb{Z} \subseteq \mathbb{Q}_p \) be the normalized valuation. Then \( c_{\ell}^{(i)}(\pi)^{\epsilon} \) is an isomorphism for \( \emptyset \leq \epsilon \), every sign character \( \epsilon \) and all \( i \).
Proof. The proof of this assertion is completely analogous to that of [8], Lemma 5.2 (b).

Definition 1.7. The \( i \)-th (automorphic) \( \mathcal{L} \)-Invariant \( \mathcal{L}_\ell^{(i)}(\pi, p)^\epsilon \) of \( \pi \) at \( p \) with respect to \( \ell \) and sign \( \epsilon \) is defined by

\[ \mathcal{L}_\ell^{(i)}(\pi, p)^\epsilon = \det((c_{\ord_p}^{(i)}(\pi)^\epsilon)^{-1} \circ c_{\ell}^{(i)}(\pi)^\epsilon). \]

Similarly, there is a variant using cohomology with compact support:

\[ \mathcal{L}_\ell^{(i)}(\pi, p)^c_\epsilon = \det((c_{\ord_p}^{(i)}(\pi)^c_\epsilon)^{-1} \circ c_{\ell}^{(i)}(\pi)^c_\epsilon). \]

The following proposition shows that it does not matter whether we use cohomology with or without compact support for defining \( \mathcal{L} \)-invariants.

Proposition 1.8. We have:

\[ \mathcal{L}_\ell^{(i)}(\pi, p)^c_\epsilon = \mathcal{L}_\ell^{(i)}(\pi, p)^\epsilon. \]

Proof. Taking cup products and connecting homomorphisms in long exact sequences commute and therefore, the following diagram is commutative:

\[ \begin{array}{ccc}
\mathcal{M}^i(f_\pi, \{p\}; \mathbb{Q}_p)^{\epsilon}[\pi] & \xrightarrow{c_{\ell}^{(i)}(\pi)^c_\epsilon} & \mathcal{M}^{i+1}(f_\pi, \emptyset; \mathbb{Q}_p)^{\epsilon}[\pi] \\
\delta \downarrow & & \delta \downarrow \\
\mathcal{M}^i(f_\pi, \{p\}; \mathbb{Q}_p)^{\epsilon}[\pi] & \xrightarrow{c_{\ell}^{(i)}(\pi)^\epsilon} & \mathcal{M}^{i+1}(f_\pi, \emptyset; \mathbb{Q}_p)^{\epsilon}[\pi]
\end{array} \]

The claim follows immediately since the first vertical arrow is an isomorphism by Proposition 1.5 (c).

Remark 1.9. Previous definitions of automorphic \( \mathcal{L} \)-invariants (see for example [3], [8], [6]) used cohomology with compact support. This is necessary in order to connect these \( \mathcal{L} \)-invariants to derivatives of \( p \)-adic \( \mathcal{L} \)-functions. On the other hand, Venkatesh formulated his conjecture for regular cohomology. Due to the above proposition it does not matter whether we work with cohomology with or without compact support.

Let us now formulate our main conjectures on automorphic \( \mathcal{L} \)-invariants.

Conjecture. For every continuous homomorphism \( \ell: F_p^* \to \mathbb{Q}_p \) we conjecture that

(I) the following equality holds:

\[ \mathcal{L}_\ell^{(i)}(\pi, p)^\epsilon = (\mathcal{L}_\ell^{(0)}(\pi, p)^\epsilon)^{(i)}, \]

(II) more precisely, the following equality holds:

\[ c_{\ell}^{(s)}(\pi)^\epsilon = \mathcal{L}_\ell^{(0)}(\pi, p)^\epsilon \cdot c_{\ord_p}^{(s)}(\pi)^\epsilon. \]

(III) that the \( i \)-th \( \mathcal{L} \)-invariant \( \mathcal{L}_\ell^{(i)}(\pi, p)^\epsilon \) does not depend on the character \( \epsilon \)

(IV) and more generally, that the following equality holds

\[ c_{\ell}^{(s)}(\pi) = \mathcal{L}_\ell^{(0)}(\pi, p)^\epsilon \cdot c_{\ord_p}^{(s)}(\pi) \]

for one (and thus every) sign character \( \epsilon \).

Conjecture (I) follows from Conjecture (II) by Proposition 1.5 (c). Assuming that Conjecture (II) (respectively Conjecture (III)) holds it is enough to check Conjecture (II) for the 0-th \( \mathcal{L} \)-invariant to get the full Conjecture (III) (respectively Conjecture (IV)).
Remark 1.10. The independence of the 0-th automorphic $L$-invariants of the sign character is known in some cases. In the case $F = \mathbb{Q}$ this was shown by Bertolini, Darmon and Iovita if the automorphic representation admits a certain Jacquet-Langlands transfer (see Theorem 6.8 of [3]) and by Breuil in general (see Corollary 5.13 of [3]).

Let us consider the special homomorphism $\ell = \log_p \circ N_{F_p/\mathbb{Q}_p}$. In [6] it is shown that $L^0_\ell(\pi, \mathbb{Q}_p)^\epsilon$, and therefore $L^0_\ell(\pi, \mathbb{Q}_p)$, is independent of the sign character $\epsilon$ as long as a certain simultaneous non-vanishing hypotheses for automorphic $L$-functions holds. If the prime $p$ is of degree 1, this implies the independence of the sign character for the 0-th $L$-invariant for an arbitrary homomorphism $\ell$.

1.4. An approximation lemma. Although automorphic $L$-invariants are constructed using cohomology with characteristic 0 coefficients, all later arguments use finite coefficient rings. We show that the cohomology classes, which we used to define $L$-invariants, can be approximated in an appropriate sense.

Let $\ell : F_p^* \to \mathbb{Z}/p^n$ be a locally constant homomorphism. By taking cup products with the classes by we get maps
\[
\ell^*_n : \mathcal{M}^*(f_\pi, \{\mathbb{Q}_p\})^\epsilon,\mathbb{Z}/p^n \to \mathcal{M}^*(f_\pi, \emptyset, \mathbb{Z}/p^n)^\epsilon.
\]

Now, let $\ell : F_p^* \to \mathbb{Z}_p$ be a continuous homomorphism and write $\ell_n : F_p^* \to \mathbb{Z}/p^n$ for its reduction modulo $p^n$. Then the maps $\ell^*_n$ form a compatible system and, therefore, they induce a map on the projective limit:
\[
\lim_{\longrightarrow} \ell^*_n : \lim_{\longrightarrow} \mathcal{M}^*(f_\pi, \{\mathbb{Q}_p\})^\epsilon,\mathbb{Z}/p^n \to \lim_{\longrightarrow} \mathcal{M}^*(f_\pi, \emptyset, \mathbb{Z}/p^n)^\epsilon.
\]

By Corollary 1.13 the canonical map
\[
\mathcal{M}^*(n, S_0; \mathbb{Z}_p)^\epsilon,\mathbb{Z}/p^n \to \lim_{\longrightarrow} \mathcal{M}^*(n, S_0; \mathbb{Z}/p^n)^\epsilon
\]
for $S_0 = \{\mathbb{Q}_p\}$ or $S_0 = \emptyset$ is an isomorphism. Thus, by inverting $p$ and invoking Proposition 1.3 [11] the maps $\lim_{\longrightarrow} \ell^*_n$ induce a homomorphism
\[
\lim_{\longrightarrow} \ell^*_n(\pi)^\epsilon : \mathcal{M}(\pi, \mathbb{Q}_p)^\epsilon,\mathbb{Z}/p^n[\pi] \to \mathcal{M}(\pi, \emptyset, \mathbb{Q}_p)^\epsilon,\mathbb{Z}/p^n[\pi].
\]

The following lemma follows directly from the definitions.

Lemma 1.11. For every continuous character $\ell : F_p^* \to \mathbb{Z}_p$ we have:
\[
\ell^*_n(\pi)^\epsilon = \lim_{\longrightarrow} \ell^*_n(\pi)^\epsilon.
\]

2. Derived Hecke algebra

2.1. A general strategy. In this section we show that the existence of a generalized Hecke algebra fulfilling certain four properties below implies Conjectures [11] and [11]. In subsequent sections we construct a candidate for this algebra using derived Hecke algebras and show that it satisfies the first three properties. Finally, we show how the crucial fourth property is connected to a conjecture of Venkatesh.

Let $\mathbb{T} = \oplus_{j \geq 0} \mathbb{T}_j$ be a graded-commutative $\mathbb{Q}_p$-algebra such that

(DH1) There is a ring homomorphism $\mathbb{T} \to \mathbb{T}_0$.

(DH2) There are graded $\mathbb{T}_j$-actions on $\mathcal{M}^*(f_\pi, \emptyset, \mathbb{Q}_p)^\epsilon,\mathbb{Z}/p^n$ and $\mathcal{M}^*(f_\pi, \emptyset, \mathbb{Q}_p)^\epsilon,\mathbb{Z}/p^n$, which extend the $\mathbb{T}$-action.

Since the image of $\mathbb{T}$ is in the center of $\mathbb{T}$ by property (DH1) the $\mathbb{T}_j$-actions of property (DH2) descend to actions on the $\pi$-typical components $\mathcal{M}^*(f_\pi, \emptyset, \mathbb{Q}_p)^\epsilon,\mathbb{Z}/p^n[\pi]$ and $\mathcal{M}^*(f_\pi, \emptyset, \mathbb{Q}_p)^\epsilon,\mathbb{Z}/p^n[\pi]$. In this case we consider the following two properties:
(DH3) For every continuous homomorphism \( \ell : F^*_{\mathbf{p}} \to \mathbb{Q}_p \) the map
\[
c_{t}(\pi)^{\epsilon} : \mathcal{M}^{*}(f_{\pi}, \emptyset ; \mathbb{Q}_p)^{\mathbb{Z}/p} \cdot [\pi] \to \mathcal{M}^{*}(f_{\pi}, \emptyset ; \mathbb{Q}_p)^{\mathbb{Z}/p} \cdot [\pi]
\]
is a homomorphism of degree 1, i.e. we have
\[
c_{t}(\pi)^{\epsilon}(t \cdot m) = (-1)^{\deg r_{\ell}} c_{t}(\pi)^{\epsilon}(m)[\pi]
\]
for every \( t \in \mathbb{T} \) and \( m \in \mathcal{M}^{*}(f_{\pi}, \emptyset ; \mathbb{Q}_p)^{\mathbb{Z}/p} \cdot \mathbb{Z}/p \cdot [\pi] \).

(DH4) \( \mathcal{M}^{0}(f_{\pi}, \emptyset ; \mathbb{Q}_p)^{\mathbb{Z}/p} \cdot \mathbb{Z}/p \cdot [\pi] \) generates \( \mathcal{M}^{*}(f_{\pi}, \emptyset ; \mathbb{Q}_p)^{\mathbb{Z}/p} \cdot \mathbb{Z}/p \cdot [\pi] \) as a \( \mathbb{T} \)-module.

Now, the fact that \( \mathcal{M}^{0}(f_{\pi}, \emptyset ; \mathbb{Q}_p)^{\mathbb{Z}/p} \cdot \mathbb{Z}/p \cdot [\pi] \) is one-dimensional easily implies our

**Main Lemma 2.1.** Suppose there exists a graded-commutative \( \mathbb{Q}_p \)-algebra \( \mathbb{T} \) which satisfies properties (DH1)-(DH4). Then Conjecture (I) (and therefore also Conjecture (II)) holds.

The subsequent will be used to connect our conjectures to the ones of Venkatesh. If we assume that the following additional property holds

(DH5) There is a graded \( \mathbb{T} \)-action on \( \mathcal{M}^{*}(f_{\pi}, \emptyset ; \mathbb{Q}_p)^{\mathbb{Z}/p} \cdot \mathbb{Z}/p \cdot \mathbb{Z}/p \cdot [\pi] \), which extends the \( \mathbb{T} \)-action and such that the map
\[
\mathcal{M}^{*}(f_{\pi}, \emptyset ; \mathbb{Q}_p)^{\mathbb{Z}/p} \cdot \mathbb{Z}/p \cdot \mathbb{Z}/p \cdot [\pi] \to \mathcal{M}^{*}(f_{\pi}, \emptyset ; \mathbb{Q}_p)^{\mathbb{Z}/p} \cdot \mathbb{Z}/p \cdot \mathbb{Z}/p \cdot [\pi]
\]
is a homomorphism of degree 0, i.e. it is \( \mathbb{T} \)-equivariant,

then by Proposition (I) property (DH4) is equivalent to

(DH4') \( \mathcal{M}^{0}(f_{\pi}, \emptyset ; \mathbb{Q}_p)^{\mathbb{Z}/p} \cdot [\pi] \) generates \( \mathcal{M}^{*}(f_{\pi}, \emptyset ; \mathbb{Q}_p)^{\mathbb{Z}/p} \cdot [\pi] \) as a \( \mathbb{T} \)-module.

**Remark 2.2.** Actually, to ensure that the algebra we construct is as large as possible we will work with slightly different modules (see Section 2.3). But in order to better illustrate the argument we neglected this detail in the above exposition.

2.2. Local derived Hecke algebra. We recall local derived Hecke algebras and their action on various cohomology groups. We follow largely the exposition of Venkatesh (cf. Section 2 of [9]) although we rewrite it in terms of group cohomology.

Let \( q \) be a prime of \( F \), which does not divide \( p \), and \( n \geq 1 \) an integer. We work in the category \( \mathcal{C}(\mathbb{Z}/p^n) \) of smooth \( G_q \)-representations with \( \mathbb{Z}/p^n \)-coefficients. A \( \mathbb{Z}/p^n \)-representation \( M \) is called smooth if the stabilizer of every \( m \in M \) is open in \( G_q \).

**Definition 2.3.** The (spherical) derived Hecke algebra at \( q \) with \( \mathbb{Z}/p^n \)-coefficients is the graded commutative algebra
\[
\mathcal{H}_{q;\mathbb{Z}/p^n} = \mathcal{H}(G_q, K_q; \mathbb{Z}/p^n) = \text{Ext}_{\mathcal{C}(\mathbb{Z}/p^n)}(\mathbb{Z}/p^n[G_q/K_q], \mathbb{Z}/p^n[G_q/K_q]).
\]

Note that the degree 0 subalgebra of \( \mathcal{H}_{q;\mathbb{Z}/p^n} \) is the classical spherical Hecke algebra of \( G_q \) with \( \mathbb{Z}/p^n \)-coefficients. The derived Satake isomorphism (see Theorem 3.3 of [9]) implies that the (spherical) derived Hecke algebra is graded-commutative in a lot of cases. More precisely, we have the following

**Proposition 2.4.** Assume that \( p \neq 2 \) and that \( N(q) \equiv 1 \mod p^n \). Then the algebra \( \mathcal{H}_{q;\mathbb{Z}/p^n} \) is graded commutative.

Note that in this case the classical spherical Hecke algebra lies in the center of the derived one.

Although the above definition of the derived Hecke algebra is the conceptional most satisfying one it is easier to define its action on various cohomology groups with a description in terms of double cosets: For a coset \( x \in G_q/K_q \) with representative \( g_x \in G_q \) we put \( K_{q;G_q} = K_q \cap g_x K_q g_x^{-1} \). We fix a set of representatives \( [K_q;G_q] \).

of the $K_q$-orbits of its left action on $G_q/K_q$. There there is an isomorphism of graded $\mathbb{Z}/p^n$-modules

\begin{equation}
\bigoplus_{x \in [G_q/K_q]} \mathcal{H}^*(G_q, \mathbb{Z}/p^n) \to \mathcal{H}_{q, \mathbb{Z}/p^n},
\end{equation}

where on the left hand side we are considering continuous group cohomology of the profinite group $G_q$. For a detailed discussion of this isomorphism see Section 2.3 and 2.4 of [9]. Thus, we can view an element of the derived Hecke algebra as a tuple $(x, \alpha)$ with $x \in G_q/K_q$ and $\alpha \in H^*(G_q, \mathbb{Z}/p^n)$.

Let $S_0 \subseteq S$ be finite sets of finite places, which do not contain the prime $q$, and let $n \subseteq \mathcal{O}$ be a non-zero ideal such that $q$ does not divide $n$. In the following we are going to construct the Hecke operator

$$h_{x, \alpha} : \mathcal{M}^*(n, S_0; \mathbb{Z}/p^n)^{S, x} \to \mathcal{M}^*(n, S_0; \mathbb{Z}/p^n)^{S, x},$$

associated to a tuple $(x, \alpha)$ as above.

Let $K_0(n)_x^S \subseteq G(S^{\infty})$ be the compact, open subgroup given by the product

$$K_0(n)_x^S = K_{q, x} \times K_0(n)^{S, x}.$$

We have the following chain of maps

$$H^*(G_q, \mathbb{Z}/p^n) \to H^*(K_0(n)^S_x, \mathbb{Z}/p^n) \xrightarrow{\alpha} H^*(G(S^{\infty}), A(K_0(n)_x^S, S_0; \mathbb{Z}/p^n)^S) \to \mathcal{M}^*(K_0(n)_x^S, S_0; \mathbb{Z}/p^n)^S,$$

where the first arrow is given by inflation, the second by Shapiro’s Lemma and the third by restriction. By abuse of notation we denote the image of $\alpha$ under the above chain of maps also by $\alpha$.

The operator $h_{x, \alpha}$ is defined via the composition of the following three homomorphisms: Firstly, the natural inclusion

$$A(K_0(n)_x^S, S_0; \mathbb{Z}/p^n)^S \to A(K_0(n)_x^S, S_0; \mathbb{Z}/p^n)^S$$

yields a map

$$\mathcal{M}^*(n, S_0; \mathbb{Z}/p^n)^{S, x} \to \mathcal{M}^*(K_0(n)_x^S, n, S_0; \mathbb{Z}/p^n)^{S, x}.$$  

Secondly, the bilinear pairing

$$\mathcal{A}(K_0(n)_x^S, S_0; \mathbb{Z}/p^n)^S \times \mathcal{A}(K_0(n)_x^S, S_0; \mathbb{Z}/p^n)^S \to \mathcal{A}(K_0(n)_x^S, S_0; \mathbb{Z}/p^n)^S$$

given by pointwise multiplication yields a cup product pairing. Therefore, taking the cup product with $\alpha$ yields

$$\mathcal{M}^*(n, S_0; \mathbb{Z}/p^n)^{S, x} \to \mathcal{M}^*(K_0(n)_x^S, n, S_0; \mathbb{Z}/p^n)^{S, x}.$$  

Finally, summing over right translations by coset representatives of $K_0(n)_x^S \setminus K_0(n)^S$ gives a map

$$\mathcal{A}(K_0(n)_x^S, S_0; \mathbb{Z}/p^n)^S \to \mathcal{A}(K_0(n)^S, S_0; \mathbb{Z}/p^n)^S,$$

which in turn induces a homomorphism

$$\mathcal{M}^*(K_0(n)_x^S, S_0; \mathbb{Z}/p^n)^{S, x} \to \mathcal{M}^*(n, S_0; \mathbb{Z}/p^n)^{S, x}.$$  

Note that this action agrees with the usual Hecke action on elements of degree 0. We do not check that this indeed defines an action of the local derived Hecke algebra, i.e. that the action is compatible with multiplication. This has been done for the cohomology of arithmetic groups with trivial coefficients (the case $S_0 = S = \emptyset$) in [3]. The arguments carry over to the more general case easily. Alternatively, we could simply work with the algebra generated by all endomorphisms $h_{x, \alpha}$.
Standard properties of the cup product immediately implies the following proposition, which is essential for proving that the global derived Hecke algebra (see Section 2.3) fulfills most of the sought-after properties of Section 2.1.

Lemma 2.5. Suppose that \( q \) does not divide the conductor \( f_\pi \) of \( \pi \) and let \((x, \alpha) \in \mathcal{M}^*\) be an element of degree \( i \).

(a) Let \( \tilde{\ell} : F^*_p \rightarrow \mathbb{Z}/p^a \) be a locally constant character. Then, we have
\[
c^\ell_h(x, \alpha)(m) = (-1)^i h(x, \alpha)(c^\ell_h(m))
\]
for all \( m \in \mathcal{M}(f_\pi, \{p\}; \mathbb{Z}/p^n)^{p} \).

(b) The evaluation maps \((1.2)\) with \( R = \mathbb{Z}/p^n \) commute with the action of \( h \).

2.3. Global derived Hecke algebra. In the following we will patch together the actions of various local Hecke algebras to construct a candidate for the algebra \( \tilde{T}\).

Since we make use of the derived Satake isomorphism, or rather its corollary in the form of Proposition 2.4, we will assume from now on that \( p \neq 2 \). We first introduce the modules on which the global derived Hecke algebra acts.

Remember that for a prime \( q \), that does not divide the conductor of \( \pi \), we write \( \lambda_q \in \mathbb{Z} \) for the eigenvalue of \( T_q \) on a spherical vector of \( \pi_q \). Let \( m \subseteq T \) be the kernel of the homomorphism
\[
T \longrightarrow \mathbb{Z}/p, \quad T_q \longmapsto \lambda_q \bmod p.
\]
Firstly, we put
\[
A_n = \mathcal{M}^*(f_\pi, \emptyset; \mathbb{Z}/p^n)_n.
\]
Here, the subscript \( m \) denotes localization. Secondly, we define
\[
B'_n = \mathcal{M}^*(f_\pi, \{p\} ; \mathbb{Q}_p/p^n(r),
\]
and
\[
B_n = B'_n / \text{ker}(\text{ev}_n),
\]
where \( \text{ev}_n \) denotes the map \( B_n \overset{\text{def}}{\longrightarrow} A_n \) induced by evaluation at an Iwahori-fixed vector. We put
\[
C_n = \frac{1}{n} \sum_{\ell} c^\ell_B(B_n)
\]
where the sum runs over all locally constant homomorphism \( \ell : F^*_p \rightarrow \mathbb{Z}/p^n \) and \( c^\ell_B \) is the (localization at \( m \) of the) map constructed in Section 1.3. Finally, we define
\[
C_n = \frac{1}{n} \sum_{\ell} c^\ell_B(\text{ker}\text{ev}_n).
\]
Thus, we have that \( \text{ev}_n \) induces an injective map
\[
\text{ev}_n : B_n \longrightarrow A_n
\]
and every locally constant homomorphism \( \ell : F^*_p \rightarrow \mathbb{Z}/p^n \) induces a map
\[
c^\ell_B : B_n \longrightarrow C_n.
\]
Let \( q \) be a prime of \( F \), which is co-prime to \( p \) and the conductor of \( \pi \) and which fulfills \( N(q) \equiv 1 \bmod p \). Thus, the local derived Hecke algebra \( \mathcal{H}_{q, \mathbb{Z}/p^n} \) is graded-commutative. This together with Lemma 2.5 implies that the actions of \( \mathcal{H}_{q, \mathbb{Z}/p^n} \) constructed in the previous section descends to actions on the modules \( A_n, B_n \) and \( C_n \). Furthermore, the map \( \text{ev}_n \) is a \( \mathcal{H}_{q, \mathbb{Z}/p^n} \)-homomorphism of degree 0 and the maps \( c^\ell_B \) are \( \mathcal{H}_{q, \mathbb{Z}/p^n} \)-homomorphism of degree 1.

Let \( \mathbb{H}^*_n \subseteq \text{End}(A_n) \) be the graded-commutative algebra generated by all Hecke operators \( T_q, q \nmid pF \) and the actions of \( \mathcal{H}_{q, \mathbb{Z}/p^n} \) for all primes \( q \) of \( F \) which are co-prime to \( p \) and the conductor of \( \pi \) and which fulfill \( N(q) \equiv 1 \bmod p \). There are natural reduction maps \( \mathbb{H}^*_n \longrightarrow \mathbb{T}_m^* \) for \( n \geq m \).
Definition 2.6. The global derived Hecke algebra is the graded-commutative $\mathbb{Q}_p$-algebra

$$\mathring{T} = \lim_{\leftarrow} \mathring{T}^* \otimes \mathbb{Q}_p.$$  

Remark 2.7. Note that the algebra defined above is smaller than the one considered by Venkatesh in [9] since we are only using Hecke operators at primes for which derived Satake holds. But the arguments in loc.cit. suggest that it should be enough to work with this smaller algebra.

We put $A_\infty = \lim_{\leftarrow} A_n \otimes \mathbb{Q}_p$ and similarly for $B_\infty$ and $C_\infty$. These are the modified modules mentioned in Remark 2.2. For example, there is a canonical isomorphism $A_\infty[\pi] \cong M^*(f_\pi, \emptyset; \mathbb{Q}_p)^{\mathfrak{m}}[\pi]$. By construction the derived Hecke algebra acts on the modules $A_\infty$, $B_\infty$ and $C_\infty$. By Lemma 1.11 and Lemma 2.5, $\mathring{T}$ fulfills (the analogues of) properties (DH1) - (DH5) and (DH5). Therefore, the following theorem follows by applying (a variant of) our Main Lemma 2.1.

Theorem 2.8. Let $p \neq 2$. If $M^0(f_\pi, \emptyset; \mathbb{Q}_p)^{\mathfrak{m}}[\pi]$ generates $M^*(f_\pi, \emptyset; \mathbb{Q}_p)^{\mathfrak{m}}[\pi]$ as a $\mathring{T}$-module, then Conjecture (II) (and therefore also Conjecture (I)) holds.

Remark 2.9. The assumption in the theorem was formulated as a question by Venkatesh in [9] although he uses a slightly larger algebra (see Remark 2.7) and only considers primes $p$, which are co-prime to the conductor of $\pi$. He answers the question affirmatively in a lot of cases which are close to ours, i.e. he considers inner forms of $SL_n$ over $\mathbb{Q}$ such that the corresponding locally symmetric spaces are compact.

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