On constraint preservation and strong hyperbolicity

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Abstract
We use partial differential equations (PDEs) to describe physical systems. In general, these equations include evolution and constraint equations. One method used to find solutions to these equations is the free-evolution approach, which consists in obtaining the solutions of the entire system by solving only the evolution equations. Certainly, this is valid only when the chosen initial data satisfies the constraints and the constraints are preserved in the evolution. In this paper, we establish the sufficient conditions required for the PDEs of the system to guarantee the constraint preservation. This is achieved by considering quasi-linear first-order PDEs, assuming the sufficient condition and deriving strongly hyperbolic first-order partial differential evolution equations for the constraints. We show that, in general, these constraint evolution equations correspond to a family of equations parametrized by a set of free parameters. We also explain how these parameters fix the propagation velocities of the constraints. As application examples of this framework, we study the constraint conservation of the Maxwell electrodynamics and the wave equations in arbitrary space–times. We conclude that the constraint evolution equations are unique in the Maxwell case and a family in the wave equation case.

Keywords: constraint preservation, strong hyperbolicity, Kronecker decomposition, evolution equations for the constraints, Maxwell equations, wave equation, quasi-linear PDEs

Contents
1. Introduction 2
2. Setting 4
   2.1. \( n + 1 \) foliation 5
1. Introduction

Physical systems are described by sets of partial differential equations (PDEs). In general, these equations can be expressed as first-order partial derivatives, leading us to focus on the study of first-order quasi-linear PDEs. These equations (or systems) usually include gauge freedoms and differential constraints (see [22, 27]). Namely, degrees of freedom (variables) that are not determined by the equations and degrees of freedom that are restricted to a lower-dimensional subspace, respectively. In this paper we do not consider gauge freedoms, we assume that they have already been fixed if the system includes them, and instead we concentrate on the study of the constraints. Furthermore, in our development, we do not assume the presence of a background metric, so the relativistic systems are included in our analysis but we do not restrict exclusively to them.
To study the above mentioned physical systems, the space–time $M$ is foliated as $M = [0, T] \times \bigcup_{0 \leq t \leq T} \Sigma_t$, and the PDEs are divided into two subsets, evolution and constraints equations. The evolution equations determine how the variables change along the different hypersurfaces $\Sigma_t$, with $0 < t \leq T$, while the constraint equations restrict the allowed values of the variables on each $\Sigma_t$. In order for these two subsets to have predictive power, the associated Cauchy problem should be well-posed (see [25, 31, 39]). This means that given an initial data over $\Sigma_t$, the associated solutions of the evolution equations exist, are unique, continuous with respect to the initial data and satisfy the constraints. Although, we emphasize that the evolution equations describing a physical system are not unique, since they can be modified by adding constraint terms to them. Such freedom can lead to well- or ill-posed evolution equations [1, 31]. In this paper, we restrict our attention to one class of equations within the well-posed ones, the strongly hyperbolic (SH). In this class, the recipe for finding SH evolution equations has been given in [3].

Several methods are used to find numerical and analytical solutions to the PDEs. Among the most widely used is the free-evolution approach. In this approach, one begins by verifying that when constraints are initially satisfied they will remain preserved during the evolution, a property commonly referred to as constraint preservation (or conservation). Then, one solves the evolution equations for initial data which satisfy the constraints; and due to the constraints preservation, this method automatically yields solutions of the complete system. Some relevant numerical implementations of this approach can be founded in reviews [18, 33, 36, 39, 40] and references therein.

The present paper addresses the study of the constraint preservation from a PDEs point of view, by establishing sufficient conditions that equations of a physical system have to satisfy to guarantee the constraint preservation. The standard method to verify this conservation is: (1) deriving a set of evolution equations for the constraints, (2) establishing that zero constraints are a solution of this system and (3) checking that this system is SH. When these steps are satisfied, they analytically guarantee that the vanishing solution is unique and therefore that the constraints are preserved. Moreover, the SH is used to find which boundary conditions preserve the constraints (see [16, 16, 39, 43]).

What do we refer to by ‘a set of evolution equations for the constraints’? It is a set of partial differential evolution equations whose variables are the constraints. This set of equations is called subsidiary system (SS). Most well-known physical systems have quasi-linear first-order partial differential SSs. Some examples are: Maxwell [14], and non-linear [2] electrodynamics, different $3+1$ formulations of the Einstein equations [19, 41], Einstein–Christoffel [16], BSSN [7, 47], ADM [28, 46], $N+1$ ADM [42], $f(R)$-gravity [34, 37], bimetric relativity [29] theories, etc. The process for deriving the SS and verifying its SH is conducted separately for each physical theory and usually involves very cumbersome calculations. To assess this problem, we present a theory that simplifies and automatizes the process.

In [38], Reula studies the constant coefficient case, assumes the existence of a first-order SS and explains how the characteristic structure of this system connects with the characteristic structure of the evolution equations of the system. Here, we focus on the quasi-linear case, show which conditions guarantee the existence of a SS and explain how the Reula result arises naturally from our results. Indeed, we show that the principal symbol of the SS is a simple projection of some tensorial objects $C^H_A$ called here Geroch fields (see [22]). We also show that the resulting SSs are not unique. In fact, they are families of equations with free parameters letting to choose the propagation velocities of the constraints.

Finally, we restrict ourselves to the constant-coefficient case to simplify the discussion and give sufficient conditions that the original set of PDEs has to satisfy for having SH evolution
equations and an associated SH SS. To reach this result we give a detailed description of the characteristic structure of both subsystems and explain how they should be chosen to make them SH.

The constraints studied here are called first-class constraints in their Hamiltonian version, see for example [23, 26, 27]. We highlight the work of Hilditch and Richter [27], where a similar problem to the one presented here is studied with a different approach.

There are widely used methods that introduce extra variables to the system and avoid dealing with constraints. Some of the most popular ones are the $\lambda$-systems [13], the divergence cleaning [17, 35] for electrodynamics, the Z4 systems [5, 6, 8–11, 24] and the modified harmonic gauge [30] for Einstein equations. However, for these systems to work properly, the constraints conservation of the original system has to be satisfied. Therefore, this aspect of the problem is crucial.

The outline of the paper is the following. In section 2, we introduce the Geroch fields, used to obtain the expressions for the constraints; the reductions, that select the evolution equations; and the integrability conditions of the system, which allow to obtain first-order partial derivatives SSs. In section 3, we introduce the expressions of the SSs in the first main theorem and the proof to this theorem. In sections 4 and 5, we present the well-posed concepts, the definition of matrix pencils and our second main theorem about the SH of the SS. This section include the subsection 5.1, where we provide the proof of the second main theorem. This proof is splitted in many subsections which include the analysis of the Kronecker decomposition (canonical block structure) of the principal symbol of the system and of the SS (including the constraints of the constraints). In subsection 5.2, we discuss how to suppress one condition of the second main theorem. In section 6, we present two examples of application of the developed formalism: the Maxwell electrodynamics and the wave equation. In section 7, we briefly discuss the results and make some comments on future work. In appendix A, we introduce some lemmas that are used to find the Kronecker decomposition of a pencil. They are used in the proof of theorem 5. Finally, in appendix B, we discuss the introduction of the lapse and shift variables in the $n + 1$ foliations. This is done in the simple system $\nabla_a q^b = 0$, in cases with and without a metric.

2. Setting

Following Geroch’s notation, let $b \xrightarrow{\pi} M$ a fiber bundle over a space–time $M$ with dim $M = n + 1$. We consider the cross-section $\phi^\alpha : M \to b$, which defines the physics fields and satisfies the equation of motion

$$E^\alpha := \Psi_L(x, \phi) \nabla_a \phi^\alpha - J^\alpha(x, \phi) = 0.$$  

Here the Greek indices $\alpha, \beta, \gamma$ represent field indices, lower letters $a, b, c, \ldots$ represent space–time indices and capital letters $A, B, C, \ldots$ represent multi-tensorial indices on the fiber space of equations call $\Psi_L$. We are considering $\dim(A) = e$ and $\dim(\alpha) = u$ such that $e \geq u$. The tensor fields $\Psi_L$ and $J^\alpha$ are the principal symbol and the source term respectively, they do not depend on derivatives of $\phi$. The derivative $\nabla_a$ represent a partial derivative or a Levi Civita connection when the system has a metric.

Solving $E^\alpha = 0$, with a given initial condition for unknown fields $\phi^\alpha$, is called the initial value problem of the system (1). As we commented in the introduction, this can be done numerically or analytically using the free-evolution approach, i.e., separating $E^\alpha = 0$ into two sets of equations, the evolutions and the constraints, finding the SS, showing the constraint conservation and using the evolution equations to find the solutions. Following this path, we present
in the first two sections, and from a PDE perspective, a theory that introduces the sufficient conditions by which a system such as (1) has a SS of first-order partial derivatives. We begin, this section, introducing the main tools of the article, defining the evolution, the constraints and the integrability conditions associated to (1).

2.1. \( n+1 \) foliation

We introduce a foliation of \( M \) given by the level surfaces of a function \( t: M \rightarrow \mathbb{R} \), and call \( \Sigma_{t_0} = \{ p \in M | t(p) = t_0 \} \) to these hypersurfaces. We also introduce coordinates \( x^a = (t, x^i) \), \( i = 1, \ldots, n \), with \( x^i \) adapted to the \( \Sigma_{t_0} \)'s. This is a local \( n+1 \) foliation of \( M \) given by \( [0, T] \times \Sigma_{t_0} \) with \( T \in \mathbb{R} \). We call time coordinate to \( t \) and spatial coordinates to \( x^i \); these names are just names to differentiate one coordinate from the others since we are not assuming the presence of a background metric here. We also consider the vector \( n^a : = (\partial_t, 0, \ldots, 0) \), and co-vector \( n^a : = \nabla_a t = (1, 0, \ldots, 0) \), where the dot in the equal sign means ‘in this coordinates the explicit expressions are’. They satisfy the condition 

\[
\eta^a_b := \delta^a_b - t^a n_b = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

(2)

It has the properties \( \eta^a_b n^b = 0 \) and \( n^a \eta^b_c = \delta^a_c \). We use them to rewrite (1), as follows

\[
E^A = \eta^a_b (n^b \phi^0 - J^A, = \eta^a_b n_c \nabla_b \phi^c + \eta^a_b n_c \nabla_b \phi^c - J^A = 0.
\]

(3)

(4)

In this equation the derivatives are splitting in time and spatial derivatives since the term \( n^a b \nabla_b \) has no derivatives in the \( t \) direction \((t^a n^b \nabla_b = 0)\). Notice that if we use the coordinates \((t, x^i)\) the last expression can be rewritten as

\[
E^A = \eta^a_i \nabla_i \phi^0 + \eta^a_i \nabla_i \phi^0 - J^A = 0,
\]

where \( i = 1, \ldots, n \) and

\[
\eta^a_i \nabla_i := \eta^a_i n_c.
\]

In these expressions the temporal and spatial derivatives are explicit. Without loss of generality, we will assume along this work that we are in these coordinates. Nevertheless, to simplify the notation we will suppress the dot in the symbol \( \dot{=} \) and the use of \((t, x^i)\) will be understood from the context.

2.2. Geroch fields and constraints

We introduce now the main objects of this article, the Geroch fields \( C^A_i(x, \phi) \), which are used to define the constraints as we explain below (see [22] too). These Geroch fields are defined by equation

\[
C^A_i(\eta^A_{\alpha} \phi^\alpha) = 0.
\]

(5)

At each point \( \kappa = (x, \phi) \in b \), they define a vector space\(^1\).

\(^1\) Notice that if \( X^a_\alpha \) and \( Y^a_\alpha \) are Geroch fields (satisfying (5)), any linear combination \( \alpha X^a_\alpha + \beta Y^a_\alpha \) with \( \alpha, \beta \in \mathbb{R} \) is also a Geroch field.
We introduce now a condition about these objects that we will explain after the definition of the constraints. For every open \( U \) of \( M \), we assume that it is possible to choose
\[
e^u \quad (6)
\]
Geroch fields \( C^{T_a}_A \), denoted by the index \( \Gamma \), and such that for all \( x \in U \) and all \( \phi \), the components
\[
C^{T_0}_A := C^{T_a}_A n_a
\]
are linearly independent. This field \( C^{T_0}_A \) can be thought of as a linear transformation from the index \( A \) to the index \( \Gamma \), so the above condition is equivalent to:

**Condition 1.** For each \( \kappa = (x, \phi) \in b \),
\[
C^{T_0}_A \text{ has maximal rank}^2.
\]
These Geroch fields \( C^{T_a}_A \) satisfy the equation
\[
C^{T_0}_A \n^A_{\alpha} = 0 \quad (8)
\]
and define the *constraints* of the system as follows
\[
\psi^{T}_{\alpha} := n_a C^{T_a}_A E^A_{\alpha}.
\]
These \( \psi^{T}_{\alpha} \) are called constraints equations since they have not time derivatives, they only have spatial derivatives as follow
\[
\psi^{T}_{\alpha} = n_a C^{T_a}_A E^A_{\alpha}
\]
\[
= n_a C^{T_a}_A \n^A_{\alpha} n_e \n_i \phi^e + n_a C^{T_a}_A \n^{Ae}_a \phi^e \n_i \phi^e - n_a C^{T_a}_A J^A
\]
\[
= n_a C^{T_a}_A \n^A_{\alpha} n_e \n_i \phi^e - n_a C^{T_a}_A J^A
\]
\[
= C^{T_0}_A \n^A_{\alpha} \n_i \phi^e - C^{T_0}_A J^A = 0.
\]
Where equation (4) was used in the first line and \( n_a C^{T_a}_A \n^A_{\alpha} n_e \n_i \phi^e = 0 \) (follows directly from (8)) in the fourth line. This means that they can be calculated only with the information of \( \phi^e \) pullbacked to the hypersurfaces \( \Sigma_t \).

Notice that these constraints depend on the hypersurfaces considered through \( n_a \). In general and when we have a metric, these hypersurfaces are chosen as spatial, however, there is no restriction on \( n_a \) to be temporal so by changing the choice of \( n_a \) we could obtain constraints associated with any hypersurface.

The motivation of the introduction of condition 7 is that it implies that the \( \psi^{T}_{\alpha} \) are algebraically independents and the number of them is \( c \). This is a natural condition in physical examples as general relativity. Recalling that \( c \) is the difference between the number of equations and fields (see equation (6)), we observe that there remain \( u \) equations in \( E^A_{\alpha} \), these will be the evolution equations.

On the other hand, we notice that assumption (7) is a non-covariant request, this is because, in general, the \( C^{T_a}_A \) are only projections (in certain directions) of a covariant tensor. The full

\[^2\text{There is no } X_{\Gamma} \neq 0 \text{ such that } X_{\Gamma} C^{T_a}_A n_a = 0.\]
covariance can be recovered by including the missing projections, these will be considered as another type of Geroch field in subsection 2.6.

**Remark 2.** A set of PDE’s (as (1)) that includes constraints might not admit Geroch fields. In this case, the system should accept another kind of fields $C_{A}^{\alpha} \phi^{\sigma} \cdot \Phi_{0}$ used to define the constraints. Such systems are not considered here and will be included in future work. However, this does not represent a significant loss of generality since, as Geroch showed in [22], most of the classical physical systems have only Geroch fields.

2.3. Non algebraic constraints

We also assume, in this work, that there is no $X_{A}$ such that

$$X_{A} \Phi_{0} = 0.$$  \hspace{1cm}  (10)

It is equivalent to claim that the system (1) has no algebraic constraints since if such $X_{A}$ exists, then the expression

$$X_{A} E^{A} := - X_{A} J^{A} = 0$$  \hspace{1cm}  (11)

would be an algebraic constraint. In the cases in which the system has an $X_{A}$ satisfying (10), we redefine the system to obtain a new system without algebraic constraints. This is done by suppressing some equations from (1) and adding the derivatives of (11) to the system. We end up with a new system of first order in derivatives with more equations, equivalent to the original system, but without the algebraic constraint. We assume that this new system only admits Geroch fields.

2.4. Evolution equations

We have introduced the constraints, now we introduce the evolution equations. We begin with an additional assumption,

**Condition 3.** For each $\kappa \in b$,

$$\eta_{b}^{\kappa \beta \delta} \Phi_{0}^{\beta \delta} \text{ has maximal rank}^{3}.$$  \hspace{1cm}  (12)

This condition guarantees that we can define evolution equations for each of the unknown variables $\phi^{\alpha}$ as follow.

First, we introduce the reduction tensor $h_{A}^{\alpha}$ such that

$$h_{A}^{\alpha} \eta_{b}^{\kappa \beta \delta} \Phi_{0}^{\beta \delta} = \delta_{A}^{\alpha},$$  \hspace{1cm}  (13)

where $\delta_{A}^{\alpha}$ is the identity map. This reduction is always possible to find when condition (12) holds.

Now, with the aid of the reduction $h_{A}^{\alpha}$, we choose the evolution equations

$$e^{\alpha} := h_{A}^{\alpha} E^{A} = r^{a} \nabla_{a} \phi^{\alpha} + h_{A}^{\alpha} \eta_{b}^{\kappa \beta \delta} \nabla_{b} \phi^{\beta \delta} - h_{A}^{\alpha} J^{A} = 0,$$  \hspace{1cm}  (14)

where expressions (4) and (13) were used. These equations are called evolution equations since they include derivatives in the $r^{a}$ direction for each $\phi^{\alpha}$, telling how the fields $\phi^{\alpha}$ evolve outside

$^{3}$There is not $\delta_{A}^{\alpha} \neq 0$ such that $\eta_{b}^{\kappa \beta \delta} \delta_{A}^{\alpha} = 0$.  

7
of the hypersurfaces $\Sigma_t$. In general, when the system has a background metric, $t^\alpha$ is chosen temporal.

These evolution equations are a linear combination of equation (1) and, of course, they are not unique. We can define other reductions $\tilde{h}_\alpha^A$ as follow

$$\tilde{h}_\alpha^A = h_\alpha^A + p_\alpha^A,$$

where $p_\alpha^A$ satisfies

$$p_\alpha^A \eta_\beta = 0,$$

and produce another set of evolution equations

$$\tilde{e}_\alpha = \tilde{h}_\alpha^A = t^r \nabla_a \phi^r + (h_\alpha^A + p_\alpha^A) \eta_\beta \nabla_b \phi^\beta - (h_\alpha^A + p_\alpha^A) J^A = 0. \quad (15)$$

In general, this freedom is used to obtain well-posed evolution equations (see [3]).

2.5. Evolution and constraints equations

By requiring that system (1) satisfies conditions (7) and (12), we have decomposed (1) into two subsets of equations, the evolution ($e^\alpha$) and constraint equations ($\psi^\Delta$) as follow

$$\begin{bmatrix} e^\alpha \\ \psi^\Delta \end{bmatrix} = \begin{bmatrix} h_\alpha^A \\ C^{\Delta_0}_A \end{bmatrix} E^A. \quad (16)$$

Now, we will show that for each $\kappa \in b$, the matrix $\begin{bmatrix} h_\beta^\alpha \\ C^{\Delta_0}_\beta \end{bmatrix}$ is invertible. This means that $e^\alpha$ and $\psi^\Delta$ are linearly equivalent to the whole system.

**Lemma 4.** For each $\kappa \in b$, the matrix $\begin{bmatrix} h_\beta^\alpha \\ C^{\Delta_0}_\beta \end{bmatrix}$ is invertible, with inverse $\begin{bmatrix} \eta_\beta^0 \\ h_\beta^\gamma \end{bmatrix}$, where

$$C^{\Delta_0}_\beta h_\beta^\gamma = \delta^\Delta_\gamma, \quad h_\beta^\gamma h^\beta_T = 0$$

and

$$\begin{bmatrix} \eta_\alpha^0 \\ h^A \end{bmatrix} \begin{bmatrix} h_\beta^\alpha \\ C^{\Delta_0}_\beta \end{bmatrix} = \eta_\alpha^0 h_\beta^\alpha + h^A C^{\Delta_0}_\beta = \delta^A_B. \quad (17)$$

**Proof.** To show that $\begin{bmatrix} h_\beta^\alpha \\ C^{\Delta_0}_\beta \end{bmatrix}$ is invertible, we will assume that it is not, and conclude an absurd. So, if $\begin{bmatrix} h_\beta^\alpha \\ C^{\Delta_0}_\beta \end{bmatrix}$ is not invertible, there exists $X_\alpha, Y_\Delta \neq 0$ such that

$$X_\alpha h_\beta^\alpha + Y_\Delta C^{\Delta_0}_\beta = 0. \quad (18)$$

Multiplying this expression by $\eta_\beta^0$ and using the equation (13) and $C^{\Delta_0}_\alpha \eta_\alpha^0 = 0$ we conclude

$$X_\alpha = 0.$$

This means that $Y_\Delta C^{\Delta_0}_\beta = 0$, but since $C^{\Delta_0}_\beta$ has only trivial kernel in the $\Delta$ index, it follows $Y_\Delta = 0$. So, the unique solution of equation (18) is $[X_\alpha, Y_\Delta] = 0$, which is a contradiction.

We conclude that $\begin{bmatrix} h_\beta^\alpha \\ C^{\Delta_0}_\beta \end{bmatrix}$ is invertible.
If \[ \begin{bmatrix} \mathcal{N}^{\alpha \beta} \quad h^B_{\Gamma} \\ C^B_{\Gamma} \end{bmatrix} \] is the inverse, the product
\[
\begin{bmatrix} h^B_{\Gamma} \\ C^B_{\Gamma} \end{bmatrix} \begin{bmatrix} \mathcal{N}^{\alpha \beta} \quad h^B_{\Gamma} \\ C^B_{\Gamma} \end{bmatrix} = \begin{bmatrix} h^B_{\Gamma} \mathcal{N}^{\alpha \beta} C^B_{\Gamma} - h^B_{\Gamma} C^B_{\Gamma} \mathcal{N}^{\alpha \beta} \\ C^B_{\Gamma} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]
should be equal to the identity matrix, then \( h^B_{\Gamma} = 0 \).

On the other hand, commuting the matrices we obtain the following equation
\[
\delta^A_B = \begin{bmatrix} \mathcal{N}^{\alpha \beta} \quad h^A_{\Delta} \\ C^A_{\Delta} \end{bmatrix} \begin{bmatrix} h^B_{\Gamma} \\ C^B_{\Gamma} \end{bmatrix},
\]
\[
= \mathcal{N}^{\alpha \beta} h^B_{\Gamma} + h^A_{\Delta} C^B_{\Gamma}.
\]

2.6. Non unicity in the Geroch fields

Frequently, the constraints have certain differential relationships between them on the hypersurfaces \( \Sigma_t \). The canonical example is when a second-order derivative system is reduced to first-order by defining the first derivatives as new variables. This approach reduces the order of the derivatives and at the same time introduces extra constraints to the system. These extra constraints also satisfy certain differential relationships between them, their cross-derivatives should vanish. The way we parameterize this, and any other, differential relationship between the constraints is through another class of Geroch fields that we call \( \mathcal{M}^\Delta_{\tilde{\Delta}}(x, \phi) \), where the \( \tilde{\Delta} \) index numerates them.

For each \( \kappa = (x, \phi) \in b \), these new extra fields satisfy equation (8), which means
\[
\mathcal{M}^\Delta_{\tilde{\Delta}} \mathcal{N}^{\alpha (b)} = 0, \tag{19}
\]
and also satisfy the following condition
\[
\mathcal{M}^\Delta_{\tilde{\Delta}} \mathcal{N}^{0 \alpha} = 0. \tag{20}
\]
Notice that, for each \( \kappa \), equations (19) and (20) define a vector subspace within the Geroch field space.

To simplify the analysis, we add an assumption on the \( \mathcal{M}^\Delta_{\tilde{\Delta}} \) which also defines the \( \tilde{\Delta} \) index.

**Condition 5.** In every open \( U \subset M \) and for all \( \kappa = (x, \phi) \in b \) with \( x \in U \), there exist a finite number of fields \( \mathcal{M}^\Delta_{\tilde{\Delta}} \) indexed by \( \tilde{\Delta} \) such that any field satisfying (19) and (20) can be expanded as a linear combination of these \( \mathcal{M}^\Delta_{\tilde{\Delta}} \). This means that any field satisfying (19) and (20) is a linear combination of the \( \mathcal{M}^\Delta_{\tilde{\Delta}} \).

This assumption is usually satisfied in physical systems in such a way that the pair \((\Gamma, \tilde{\Delta})\) define a tensor index. Thus, the covariance of these Geroch tensors is recovered by considering the pair \( \mathcal{C}^\Gamma_{\Delta} \) and \( \mathcal{M}^\Delta_{\tilde{\Delta}} \) together. We will comment more about this at the end of this subsection.

Another result that arises from this assumption is:

**Lemma 6.** If the conditions 1, 3 and 5 hold, then in every open \( U \subset M \) and for all \( \kappa = (x, \phi) \in b \) with \( x \in U \), every Geroch field \( X^a_A \) (satisfying equation (5)) is a linear combination of \( \mathcal{C}^\Gamma_{\Delta} \) and \( \mathcal{M}^\Delta_{\tilde{\Delta}} \), i.e. there exist always \( S_{\Delta} \) and \( S_{\tilde{\Delta}} \) such that
\[
X^a_A = S_{\Delta} \mathcal{C}^\Gamma_{\Delta} + S_{\tilde{\Delta}} \mathcal{M}^\Delta_{\tilde{\Delta}}.
\]
Proof. Let $X^a_A$ be any Geroch field, we have to show that there exist $S_{\Delta}$ and $\tilde{S}_{\Delta}$ such that $X^a_A = S_{\Delta} C_{\Delta}^a + \tilde{S}_{\Delta} M_{\Delta}^a$. We propose $S_{\Delta} = (X^0_A C^A B_{\Delta}) C^\Delta_{\alpha} C_A$ and point out that to conclude the proof, we only need to show that

$$X^a_A = (X^0_A C^A B_{\Delta}) C^\Delta_{\alpha} C_A + \tilde{S}_{\Delta} M_{\Delta}^a.$$  

(21)

satisfies (19) and (20). Since, if this is true, by the condition 5 the field (21) should be expanded by $M_{\Delta}^a$. Showing (19) is trivial from the definitions of Geroch fields, so it only remains to demonstrate (20). Multiplying (21) by $N_{A\alpha}$, we obtain

$$\left( X^a_A - (X^0_A C^A B_{\Delta}) C^\Delta_{\alpha} C_A \right) N_{A\alpha} = - (X^0_A - (X^0_A C^A B_{\Delta}) C^\Delta_{\alpha} C_A) N_{A\alpha},$$

(17)

$$= - X^0_A \left( \delta^B_A - C^A B_{\Delta} C^\Delta_{\alpha} C_A \right) N_{A\alpha},$$

(18)

$$= - X^0_A \left( \delta^B_A - C^A B_{\Delta} C^\Delta_{\alpha} C_A \right) N_{A\alpha},$$

$$= 0.$$

We used $X^a_A N_{A\alpha} = - X^0_A \eta_{\alpha}^A$ and $C_{\Delta}^a X^0_A = - C_{\Delta}^a \eta_{\alpha}^A$ in the first equality, (17) (only valid assuming (7), (12)) in the second one, (17) in the third one and $X^0_A \eta_{\alpha}^A = 0$ in the last one. This concludes the proof.

We said that the $M_{\Delta}^a$ parameterize the differential relationships between the constraints, their explicit expressions can be found in subsection 3.2, equation (38). It is important to mention that to reach those expressions we need the integrability condition (26) which appears in the next subsection. From now on we will refer to (38) as the constraints of the constraints.

Using the identity (17), $M_{\Delta}^a$ can be rewritten as

$$M_{\Delta}^a = M_{\Delta}^a \left[ \eta_{\alpha}^A C^A B_{\Delta} \right] \left[ C^A B_{\Gamma} \right]$$

(22)

where equation (20) has been used. So, by redefining

$$M_{\Delta}^a \equiv M_{\Delta}^a h^A_{\Gamma},$$

(23)

and using equation (8) we derive an equation for $M_{\Delta}^a$

$$M_{\Delta}^a C^A B_{\Gamma} \eta_{\alpha}^A = 0.$$  

(24)

There is an isomorphism between $M_{\Delta}^a$ and $M_{\Delta}^a$, in the sense that one of them completely define the other. For each $M_{\Delta}^a$ satisfying equation (24), $M_{\Delta}^a$ can be defined by equation (22), and for each $M_{\Delta}^a$ satisfying (19) and (20), $M_{\Delta}^a$ can be defined by $M_{\Delta}^a h^A_{\Gamma}$. In addition, considering the following expression,

$$M_{\Delta}^a \left[ \eta_{\alpha}^A h^A_{\Gamma} \right] = \left[ 0 M_{\Delta}^a \right],$$

where we have used that $M_{\Delta}^a \eta_{\alpha}^A = 0$; and recalling that $\left[ \eta_{\alpha}^A h^A_{\Gamma} \right]$ is invertible, the last expression means that any set of $M_{\Delta}^a$’s is linearly independent if and only if the associated set of $M_{\Delta}^a$’s is linearly independent too. From now on, we will regard $M_{\Delta}^a$ and $M_{\Delta}^a$ as equivalent to simplify the discussion.
The $M_i^{\Delta 0}$ has another interesting property. By employing equation (24), we reach to the following lemma.

**Lemma 7.** $M_i^{\Delta 0} = 0$ (or $M_i^{\Delta 0} = 0$).

**Proof.** Considering the $0, b$ components of equation (24) and equation $C_A^{\Gamma^0_0} \gamma_A^a = 0$, we conclude

$$M_i^{\Delta 0} C_A^{\Gamma^0_0} \gamma_A^a = -M_i^{\Delta 0} C_A^{\Gamma^0_0} \gamma_A^a = 0.$$

Since we are working with systems that have not algebraic constraints, there is not $X_A$ such that equation (10) holds, so it should be valid

$$M_i^{\Delta 0} C_A^{\Gamma^0_0} = 0.$$

On the other hand, the $C_A^{\Gamma^0_0}$ has only trivial kernel in the $\Gamma$ index, thus we finally conclude $M_i^{\Delta 0} = 0$. Notice that this result allows us to redefine the Geroch’s fields $C_A^{\Gamma^0_0}$ in the following way

$$\tilde{C}_A^{\Gamma^0_0} = C_A^{\Gamma^0_0} + N_{\Delta} M_i^{\Delta 0},$$

leaving the $C_A^{\Gamma^0_0}$ component unchanged and keeping the same expressions for the constraints equation (9). Here, the $M_i^{\Delta 0}$ fields can be freely chosen and the reader can see that the Geroch fields have significant freedom in their definition. This freedom will also appear in the evolution equations of the constraints.

Finally, we conclude the discussion of the covariance in the Geroch fields. So far, it has been requested that $C_A^{\Delta 0}$ has maximal rank and has shown that $M_i^{\Delta 0} = 0$, but these are not covariant conditions since the $n_a$ direction is privileged among others in these two objects. In general, one can recover the covariance, thinking that we have split the real tensors (the ones which transform good under change of coordinates) into two parts, the $C_A^{\Delta a}$ and $M_i^{\Delta a}$. We will not pay attention to this non-covariant splitting in the future and we will call fields or tensors to $C_A^{\Delta a}$ and $M_i^{\Delta a}$, even if they do not transform as real ones.

We have said nothing about how many derivatives are admitted by the objects considered so far. We will assume that $\eta_{\alpha}^\beta$, $F^\alpha$, $C_A^{\Delta a}$, $M_i^{\Delta a}$, $b^\alpha_\beta$ and $h^\alpha_\Gamma$ admit at least one derivative in any direction for the next conditions to hold.

### 2.7 Integrability conditions

At this point, the Geroch fields $C_A^{\Gamma^0_0}$ have been defined but only the $C_A^{\Gamma^0_0}$ component has been used to define the constraints. We introduce now a set of off-shell identities which include all the components of $C_A^{\Gamma^0_0}$ and $M_i^{\Delta 0}$, they are

$$\nabla_\alpha (C_i^{\Gamma^0_0} E^\alpha) = L_1^i(x, \phi, \nabla \phi) E^\alpha(x, \phi, \nabla \phi),$$

$$\nabla_\alpha (M_i^{\Delta 0} C_A^{\Gamma^0_0} E^\alpha) = L_2^i(x, \phi, \nabla \phi) E^\alpha(x, \phi, \nabla \phi).$$

Equations that hold for any value of $\phi^\alpha$. 

---

[Image and text]
We call integrability conditions to these identities and we assume they hold for the rest of this work. The motivation behind their inclusion will be explained in the following paragraphs. We begin by noticing, the integrability conditions specify that the divergences of certain combinations of the equations of the system are proportional to the system. These proportionality factors may depend on \((x, \phi, \nabla \phi)\). In the case of the Einstein equations, equation (25) follows from the conservation law of the Einstein tensor \(\nabla_a G^a_{\beta} = 0\) (or from the 2nd Bianchi identity). This is obtained by the Noether theorem as a conserved quantity from the Lagrangian coordinate transformation invariance (see [45]). Equation (26) appears when the Einstein equations are cast into first-order form. In the case of the Maxwell equations (or Yang–Mills equations), equation (25) follows from the charge conservation law, this is obtained by the Noether theorem and associated to the Lagrangian gauge invariance symmetry (see [12]). This system does not have an identity as (26). In the case of the wave equations, the identities (25) and (26) appear when the system is reduced into first order in derivative adding extra constraints. The study of the constraints propagation of the Maxwell and wave equations are discussed in section 6, we there give explicit expressions for these identities.

It is likely that at least the identity (25) always comes from the Noether theorem (in its off-shell version) as a conserved quantity and associated to some ‘gauge’ symmetry of the system. As we will show in the next section, these identities are needed to show the constraints conservation since they are the evolutions equations of the constraints. Therefore, in general, we believe that the constraints conservation will be associated to some symmetry of the system.

From the PDE point of view, it has sense that the Geroch fields appear in the expressions (25) and (26). Considering only the left-hand side of (25) and replacing \(E^1\) by its definition (equation (1)), we find that

\[
\nabla_d \left( C^d_A E^1 \right) = \nabla_d \left( C^d_A \left( \eta^{[d] |a|} \right) \right),
\]

\[
= \nabla_d \left( C^d_A \eta^{[d] |a|} \right) \nabla_a \phi^\alpha + \frac{1}{2} \left( C^d_A \eta^{[d] |a|} \right) R^\alpha_{\beta a \phi}.
\]

Where we have used the chain rule, equation (8) and the following definition of the curvature tensor

\[
\nabla_{[d} \nabla_a \phi^\alpha = \frac{1}{2} R^\alpha_{\beta d a \phi}.
\]

We note that the resulting expression (29) does not include second derivatives of the fields \(\phi^\beta\), so it is reasonable that it can be factored as a product of expressions in first derivatives of \(\phi^\beta\), i.e. as \(L^1_A(x, \phi, \nabla \phi) E^1(x, \phi, \nabla \phi)\). The case for (26) is analogous, this is

\[
\nabla_d \left( M^1_A C^A_{\Gamma} E^1 \right) = \nabla_d \left( M^1_A C^A_{\Gamma} \left( \eta^d_\alpha \nabla_a \phi^\alpha - J^1 \right) \right),
\]

\[
= \nabla_d \left( M^1_A C^A_{\Gamma} \eta^{[d] |a|} \right) \nabla_a \phi^\alpha + \frac{1}{2} \left( M^1_A C^A_{\Gamma} \eta^{[d] |a|} \right) R^\alpha_{\beta d a \phi}.
\]

12
standard mechanism to obtain these constraint evolution equations is: take a time derivative of
\( C_{(T0)}^\alpha \) and \( C_{(A)}^\alpha \), and after many calculations) rearrange the resulting terms into expressions that only include

\[ \psi \]

and constraints \( \psi \downarrow (\phi) = C_{(T0)}^\alpha E^\alpha = 0 \) equations. One mechanism to find solutions for these equations is the free-evolution approach. That is, we give an initial data \( \phi|_{t_0} = \phi_0 \) satisfying the constraints \( \psi^T (\phi_0)|_{t_0} = 0 \), and use the evolution equations \( e^\alpha (\phi) = 0 \) to find the solutions of the system over the future hypersurfaces \( \Sigma \) with \( t > t_0 \). However, these found solutions may not satisfy the constraints for \( t > t_0 \), and therefore may not be solutions of the complete system.

For this reason, we need equations that tell us how these constraints evolve for \( t > t_0 \). With these equations, we can determine if the constraint are preserved or not. The standard mechanism to obtain these constraint evolution equations is: take a time derivative of \( \psi^T = n_a C_{(T0)}^\alpha \eta_{\alpha}^b \nabla_b \phi^a - J^A \), which translates into time derivatives of \( \phi^a \), plus other terms; use the evolution equations \( e^\alpha (\phi) = 0 \) to eliminate \( \partial_t \phi^a \) from these expressions and (hopefully and after many calculations) rearrange the resulting terms into expressions that only include spatial derivatives of \( \psi^T \) and lower order terms proportional to \( \psi^T \). As we shall see in the proof of the following theorem, counting with the identities (25) and (26) is equivalent to the mentioned process (since they include the time derivatives of \( \psi^T \)). Actually, if these conditions are not satisfied, the system has extra constraints that should be added to the system and whose conservation has to be studied.

The following theorem, called SS, shows the explicit form of the SS in the quasi-linear case.

**Theorem 8.** Consider the system (1) such that the following conditions hold:

(i) The principal symbol satisfies assumption 3.

(ii) All constraints of the systems come from Geroch fields \( C_{(T)}^\alpha \) as in equation (9) and these \( C_{(T)}^\alpha \) satisfy assumption 1.

(iii) The system can admit (or not) extra Geroch fields \( M_{(c)}^\alpha \) satisfying assumption 5.

(iv) The integrability conditions (25) and (26) are satisfied.

Then, the following off-shell identity is satisfied

\[
\nabla_0 \psi^T + \left( C_{(T)}^A h^{A}_\Delta + N_{(T)}^\Delta M_{(c)}^{\Delta} \right) \nabla_b \psi^\Delta
\]

\[
= (L_{\Delta a}^\alpha \gamma_{\alpha}^0 - \nabla_d (C_{(T)}^{d\alpha} \gamma_{\alpha}^0) ) e^\alpha - C_{(T)}^{d\alpha} \gamma_{\alpha}^0 \nabla_d e^\alpha
\]

\[
+ \left( L_{\Delta a}^\alpha h^{A}_\Delta - \nabla_d (C_{(T)}^{d\alpha} h^{A}_\Delta) + N_{(T)}^\Delta \left( L_{\Delta a}^{\Delta a} - \nabla \left( M_{(c)}^{\Delta a} \right) \right) \right) \psi^\Delta.
\]

(30)

Here the tensors \( N_3^\Delta \) can be freely chosen. If the system does not admit Geroch fields \( M_{(c)}^\alpha \), then \( N_{(T)}^\Delta = 0, L_{\Delta a}^\alpha = 0 \). In this case, the SS is unique.
In the on-shell case i.e. \( e^\alpha(\phi) = 0 \), the above identity reduces to the SS,

\[
\nabla_0 \psi^\Gamma + \left( C^\alpha_A h^\Delta_A + N^\alpha_\Delta M^\Delta_A \right) \nabla_i \psi^\Delta \\
= \left( L^\alpha_{ji} h^\Delta_A - \nabla_d \left( C^\alpha_d h^\Delta_A \right) + N^\alpha \left( L^\alpha_{2d} h^\Delta_A - \nabla_i \left( M^\Delta_i \right) \right) \right) \psi^\Delta 
\]

(31)

where \( i = 1, \ldots, n \).

We notice that if \( \left| \psi^\Gamma_0 \right| = 0 \) then \( \psi^\Gamma = 0 \) is a solution of equation (31). If in addition, equation (31) is well-posed, the solution \( \psi^\Gamma = 0 \) is unique and continuous in the initial data.

It is important to emphasize that the set of constraint evolution equations is not unique when the \( M^\Delta_i \) fields exist. In other words, constraints \( \psi^\Gamma \) are solutions of a family of differential equation (31). This family is parametrized by the field \( N^\alpha_\Delta \) (which can be freely chosen), producing the non-uniqueness. An example of this class of systems is the wave equation, see section 6.2.

On the other hand, the well-posedness of the SS depends on the explicit form of its principal symbol

\[
B^\Gamma_i := C^\Gamma_A h^\Delta_A + N^\Gamma_i M^\Delta_A. 
\]

(32)

Since, this symbol includes the field \( h^\Delta_A \), which is completely determined by the choice of \( h^\alpha_B \) (see equation (17))\(^5\), and the free field \( N^\alpha_\Delta \), the well-posedness of the SSs is given by \( h^\alpha_B \) and \( N^\alpha_\Delta \). We focus on this problem in section 5, where we provide a theorem on how to appropriately chose \( h^\alpha_B \) and \( N^\alpha_\Delta \) to obtain a well-posed SS.

Finally, when there are no \( M^\Delta_i \) fields present in the system, the evolution equations of the constraints are unique, so the analysis of well-posedness is highly simplified. The Maxwell equations are an example of this case, see section 6.1.

We will discuss the hyperbolicity of the SS in the next sections, focusing on the constant-coefficient systems where we can give a closed answer.

3.1. Invariance of the choice of \( e^\alpha \) in subsidiary system

Given a particular physical theory, different reductions \( h^\alpha_B \) and \( \tilde{h}^\alpha_B \) lead to different choices of the evolution equations \( e^\alpha \) and \( \tilde{e}^\alpha \) (see subsection 2.4). Therefore, to study the conservation of the constraints, the SS should be calculated for each choice of \( e^\alpha \) or \( \tilde{e}^\alpha \). In addition, it might happen that the SS does not exist\(^6\) for a particular choice of the evolution equations. In our scheme, this dilemma is solved in a quite elegant way. As we explained before, each reduction \( h^\alpha_B \) (\( \tilde{h}^\alpha_B \)) introduces a unique field \( h^\Delta_A \) (\( \tilde{h}^\Delta_A \)) given by equation (17). This field \( h^\Delta_A \) (\( \tilde{h}^\Delta_A \)) appear in equations of the SS (31), showing that SS always exists and how the information of the reductions \( h^\alpha_B \) (\( \tilde{h}^\alpha_B \)) is propagated to this equation. The reason for this simple answer is that equation (30) follows directly from the integrability conditions (25) and (26), which do not depend on any reduction \( h^\alpha_B \) As it is shown in the proof of the theorem, these reductions appear only as a trick to transform \( C^\tau_A E^A \) into \( \left[ C^\tau_A \psi^\alpha, C^\tau_A h^\Delta_A \right] \left[ e^\alpha \psi^\Delta \right] \), where the explicit form of \( h^\alpha_B \) plays no role in this proof, so that, equation (30) can be obtained for any \( h^\alpha_B \).

\(^5\) Notice that the reverse is true too, \( h^\alpha_B \) is completely defined by choice of \( h^\Delta_A \).

\(^6\) Meaning that it is not possible to write a set of PDE’s where the variables of these PDE’s are the constraints.
It may happen that in some physical theories the integrability conditions (25) and (26) depend on a specific reduction \( h_\alpha \) and consequently on its corresponding evolution equations \( e^\alpha \). In this class of systems, equations (25) and (26) could include two-order or higher partial derivatives of the evolution equations. This should not modify equation (31) since \( e^\alpha = 0 \) in the on-shell case, but this system would be forced to evolve only with \( e^\alpha = 0 \) since any other choice of the evolution equations \( \tilde{e}^\alpha \) would not produce a SS as (31). Therefore, the constraint conservation would not be guaranteed.

### 3.2. Proof of subsidiary system theorem

The proof consists of showing how to go from the sum of (25) and (26),

\[
\nabla_d (C^d_A E^A) + \sum_{\Delta} \nabla_d \left( M_{\Gamma}^{\Delta d} C^{\Gamma 0}_A E^A \right) = L^{\Gamma}_A E^A + N^{\Gamma}_{\Delta} \tilde{L}^{\Delta}_2 A^A,
\]

(33)
to (30), where \( N^{\Gamma}_{\Delta} \) can be freely chosen.

Using (16) (valid from the assumptions 3 and 1) and (17) the system \( E^A \) can be written as

\[
E^A = [\gamma_{\Delta}^{\alpha 0} h^A_\Delta] \left[ \begin{array}{c}
\frac{h^A_B}{C^B}\end{array} \right] E^B,
\]

(34)

By replacing the latter equation in each of the terms of (33), we obtain

\[
\nabla_d (C^d_A E^A) = \nabla_d \left( [C^d_A \gamma_{\Delta}^{\alpha 0} C^0_A h^A_\Delta] \left[ \begin{array}{c}
e^\alpha \end{array} \right] \right),
\]

\[
= \nabla_d (C^{d 0} A_{\alpha}) e^\alpha + \nabla_d (C^{d 0} h^A_\Delta) e^\alpha + C^d_A \gamma^{\alpha 0}_\alpha \nabla_d e^\alpha
+ C^d_A h^A_\Delta \nabla_d \psi_\Delta,
\]

\[
= \nabla_d (C^{d 0} A_{\alpha}) e^\alpha + \nabla_d (C^{d 0} h^A_\Delta) e^\alpha + C^d_A \gamma^{\alpha 0}_\alpha \nabla_d e^\alpha
+ \nabla_d \psi_\Gamma + C^d_A h^A_\Delta \nabla_d \psi_\Delta.
\]

(35)

Where we used \( C^{d 0}_A h^A_\Delta = \delta^d_\Delta \) and \( i = 1, \ldots, n \) in the last equation. On the other hand,

\[
L^{\Gamma}_A E^A = L^{\Gamma}_A [\gamma_{\Delta}^{\alpha 0} h^A_\Delta] \left[ \begin{array}{c}
e^\alpha \end{array} \right],
\]

\[
= L^{\Gamma}_A \gamma_{\Delta}^{\alpha 0} e^\alpha + L^{\Gamma}_A h^A_\Delta \psi_\Delta.
\]

(36)

We conclude that (25) can be written as

\[
\nabla_d (C^d_A E^A) = L^{\Gamma}_A E^A \nabla_d (C^{d 0} A_{\alpha}) e^\alpha + \nabla_d (C^{d 0} h^A_\Delta) \psi_\Delta + C^d_A \gamma^{\alpha 0}_\alpha \nabla_d e^\alpha
+ \nabla_d \psi_\Gamma + C^d_A h^A_\Delta \nabla_d \psi_\Delta
\]

\[
= L^{\Gamma}_A \gamma_{\Delta}^{\alpha 0} e^\alpha + L^{\Gamma}_A h^A_\Delta \psi_\Delta.
\]

Considering now the divergences which involve \( M_{\Gamma}^{\Delta d} \), we obtain:
\[ \nabla_d \left( M_i^{\tilde{\Delta} d} C_A^{\Gamma 0} E^A \right) = \nabla_d \left( M_i^{\tilde{\Delta} d} \psi^\Gamma \right), \]
\[ = \nabla_d \left( M_i^{\tilde{\Delta} d} \right) \psi^\Gamma + M_i^{\tilde{\Delta} d} \nabla_d (\psi^\Gamma), \]
\[ = \nabla_i \left( M_i^{\tilde{\Delta} i} \right) \psi^\Gamma + M_i^{\tilde{\Delta} i} \nabla_i (\psi^\Gamma). \quad (37) \]

Where in the first line we used the definition of \( \psi^\Gamma = C_A^{\Gamma 0} A^E \) and in the third line that \( M_i^{\tilde{\Delta} 0} \) vanishes (see lemma 7). The expression for \( L_{2A}^{\tilde{\Delta}} E^A \) is
\[ L_{2A}^{\tilde{\Delta}} E^A = L_{2A}^{\tilde{\Delta}} \gamma_i^0 e^\alpha + L_{2A}^{\tilde{\Delta}} h^A \Delta \psi^\Delta, \]
so the expression (26) can be rewritten as
\[ \nabla_d \left( M_i^{\tilde{\Delta} d} C_A^{\Gamma 0} E^A \right) = L_{2A}^{\tilde{\Delta}} E^A \]
\[ \nabla_i \left( M_i^{\tilde{\Delta} i} \right) \psi^\Gamma + M_i^{\tilde{\Delta} i} \nabla_i (\psi^\Gamma) = L_{2A}^{\tilde{\Delta}} \gamma_i^0 e^\alpha + L_{2A}^{\tilde{\Delta}} h^A \Delta \psi^\Delta. \]

Recalling that this expression is an identity that holds for all \( \phi^\alpha \), it should not be possible that the right-hand side of this equation contains time derivatives (on \( e^\alpha \)) while the left-hand side does not. This means that,
\[ L_{2A}^{\tilde{\Delta}} \gamma_i^0 e^\alpha = 0, \]

hence
\[ \nabla_i \left( M_i^{\tilde{\Delta} i} \right) \psi^\Gamma + M_i^{\tilde{\Delta} i} \nabla_i (\psi^\Gamma) = L_{2A}^{\tilde{\Delta}} h^A \Delta \psi^\Delta. \quad (38) \]

Finally, replacing these results in equation (33) gives (30), which concludes the proof.

As a final comment, in section 2.6 we said that the fields \( M_i^{\tilde{\Delta} d} \) parameterize the constraints of the constraint. The explicit expression for these differential relationships between the constraints is given by the last equation (38).

4. Constant coefficient and strong hyperbolicity

In this section we introduce a brief summary of paper [3], we present the definitions and the main result of that work about SH for systems with constraints. This main result gives the necessary and sufficient conditions under which the system
\[ E^A := \gamma_i^0 \partial_\alpha \phi^\alpha, \quad (39) \]
has a SH set of evolution equations.

As in [3], we do not consider the quasi-linear case (1) and focus on the constant coefficient case, where \( \gamma_i^0 \partial_\alpha \) does not depend on \( x, \phi \) (i.e. \( \nabla_i \gamma_i^0 \partial_\alpha = 0 \)). This simplification leads to a closed pseudo-differential theorem about SH for the evolution equations (theorem 11) and will allow us, in the next section, to derive a theorem about the SH of the SS (theorem 12). Note also that the covariant derivatives \( \nabla_i \) have been changed to partial derivatives \( \partial_i \) and that the lower order terms have been suppressed because the SH is not affected by them.

As before, the initial value problem consists on solving
\[ E^A(\phi) = 0 \text{ with initial data } \phi^\alpha|_{\Sigma_0} = \hat{f}^\alpha(x). \quad (40) \]
To discuss its SH, we convert the problem to its Fourier space and present a pseudo-differential analysis. Applying to (40) a Fourier transformation on the spatial variables $x^i$, with $i = 1, \ldots, n$, we obtain

$$\tilde{E}^A := \Omega^A_{\alpha} \partial_\alpha \tilde{\phi}^\alpha + \Omega^A_{\alpha} k_i \tilde{\phi}^\alpha = 0,$$

with

$$\tilde{\phi}^\alpha|_{x_0} = \tilde{f}^\alpha(x).$$

Where $k_a$ is the wave vector such that $k_a t^a = 0$ (i.e. $k_0 = 0$) and $k_a t^a = k_0$ (i.e. $k_a = (0, k_i)$).

Now we introduce the reduction $h^B_A(k_i)$ (satisfying equation (13)) which may depend on the wave vector $k_a$ or not. Applying $h^B_A(k_i)$ to equation (41), we obtain a set of evolution equations for $\tilde{\phi}^\beta$

$$\tilde{\phi}^\beta = \partial_\beta \tilde{\phi}^\beta + i h^B_A(k_i) \Omega^B_{\alpha} k_i \tilde{\phi}^\alpha = 0.$$

As before, the reduction aim is to combine the constraint and time derivatives, to produce systems of evolution equations for each field $\tilde{\phi}^\beta$. The main difference is that now these evolution equations are pseudo-differential and their solutions must be anti-transformed to obtain solutions of the original system.

The set of evolution equations has to be well-posed to be predictive. This is a property that depends on the choice of $h^B_A$ since different reductions may lead to ill-posed or well-posed systems. In particular, we are considering a sub-class within the well-posed equations, namely the SH ones. This leads us to introduce the following definition.

**Definition 9.** Consider $n_\alpha = \nabla_\alpha t$ such that the assumption 3 holds, we say that the system (39) is SH if there exists at least one reduction $h^B_A(k_i)$ satisfying (13), such that for all $k_i$, with $|k_i| = 1$, the principal symbol of the evolution equations $A^B_A k_i := h^B_A \Omega^B_{\alpha} k_i$ is uniformly diagonalizable with real eigenvalues. Namely, for all $k_i$ with $|k_i| = 1$, there exists $(T(k))^\gamma_\beta$ such that $h^B_A \Omega^B_{\alpha} k_i = (T(k))^\gamma_\beta \Lambda^\gamma_\theta (T^{-1}(k))^\theta_\alpha$ with $\Lambda^\gamma_\theta$ diagonal and real; and the diagonalization is uniform, which means that there exists a constant $C > 0$ such that

$$|T(k)| + |T(k)^{-1}| < C.$$ 

The norms $|·|$, used in $|k_i| = 1$ and in equation (43), can be any $k_a$ independent, positive definite norms. We will assume for the following definitions and theorems that the wave vector $k_i$ is normalized to $|k_i| = 1$.

When the reductions $h^B_A$ satisfy the definition, we call them hyperbolizations. Note that when the system has no constraints, the reduction $h^B_A$ is unique and defined by (13).

In the literature, the above definition is presented as a theorem and the original definition of SH is another one, but since this section is only an introduction to the topic, we condense the discussion and present it as a definition. For more details about the theory we suggest [39] and the reference therein.

We introduce now a set of definitions to conclude with the definition of canonical angles and after it, the theorem 11.

We call $\Phi$ and $\Psi$ to the vector fibers that include the vectors $\delta \phi^\alpha$ and $X_A$ respectively. These spaces contain the right and left kernel subspaces of the principal symbol $\Omega^A_{\alpha} w_A$ (for a given $w_A$), whose elements satisfy
increases the dimension to

definition 10. the system (39) is called hyperbolic, if there exists a co-vector 

\( n_a \) such that 

\[ \Theta^{(a)}_{\alpha} u^a \delta \phi^\alpha = 0, \]

\[ X^R_{\lambda} \Theta^{(a)}_{\alpha} u^a = 0 \]

respectively. we will use this notation for any other operator, a vector will belong to the right (left) kernel when it contracts with the down (up) index operator and the result vanishes.

we introduce the set of planes \( S^C_{n_a} = \{ L_\alpha(\lambda) = -\lambda n_a + k_a = (-\lambda, k_a) \} \) for all \( k_a \) not proportional to \( n_a \), with \( |k| = 1 \) and \( \lambda \in \mathbb{C} \). they are complex planes for each fixed \( k_a \) and they reduce to lines when \( \lambda \in \mathbb{R} \). we call \( S_{n_a} \) to the set of these lines.

consider the \( e \times u \) matrices \( A, B \), we call matrix pencil to \( \lambda A + B \), where \( \lambda \in \mathbb{C} \) is a parameter. in this way, the planes \( L_\alpha(\lambda) \in S^C_{n_a} \) transform (for each \( k_a \)) the principal symbol in a \( e \times u \) matrix pencil

\[ \Theta^{(a)}_{\alpha} L_\alpha(\lambda) = \lambda (\Theta^{(a)}_{\alpha} n_a) + \Theta^{(a)}_{\alpha} k_a. \]

we consider now the left and right kernel of this pencil. for each \( k_a \), there exist certain \( \lambda \) called generalized eigenvalues \( \lambda_i(k) \) with \( i \in D(k) \), \( D(k) := \{ 1, 2, \ldots, q(k) \} \) and \( \lambda_1(k) < \cdots < \lambda_{q(k)}(k) \) such that \( \Theta^{(a)}_{\alpha} L_\alpha(\lambda_i(k)) \) has non-trivial right and left kernel. we call \( \Phi^{L(k)}_\lambda \) and \( \Phi^{R(k)}_\lambda \) to these subspaces of \( \Phi \) and \( \Psi \) respectively. notice that the explicit form of these generalized eigenvalues and the \( q(k) \) number of them is \( k \)-dependent. we call \( d_{\lambda_i(k)} \) to the geometric multiplicity of \( \lambda_i(k) \), thus \( \dim \Phi^{L(k)}_\lambda = d_{\lambda_i(k)} \).

the left kernel behaves differently since its dimension is larger, \( \dim \Phi^{R(k)}_\lambda = d_{\lambda_i(k)} + c \). this difference is due to the fact that for all \( k \) and all \( \lambda \), the dimension of the left kernel of \( \Theta^{(a)}_{\alpha} L_\alpha(\lambda) \) is \( c \), except when \( \lambda = \lambda_i(k) \), where the left kernel associated to the generalized eigenvalues increases the dimension to \( d_{\lambda_i(k)} + c \). a better explanation will be given in the proof of theorem sh of the ss.

a necessary condition for the well-posedness of the system is that the generalized eigenvalues be real. when the system satisfies this condition we call it hyperbolic,

definition 10. the system (39) is called hyperbolic, if there exists a co-vector \( n_a = \nabla_a t \) such that

(a) \( \Theta^{(a)}_{\alpha} = \Theta^{(a)}_{\alpha} n_a \) has only trivial right kernel.

(b) for each \( L_\alpha(\lambda) \in S^C_{n_a} \), if \( \Theta^{(a)}_{\alpha} L_\alpha(\lambda) \) has non-trivial right kernel, then \( \lambda \in \mathbb{R} \).

notice that (a) is condition 3.

now, we introduce over \( \Phi \) a positive definite hermitian form \( G^{\alpha\beta} \). this allows us to define the vector subspace \( \Phi^{L(k)}_\lambda \), as the subspace obtained by projecting \( \Psi^{L(k)}_\lambda \) with \( \Theta^{(a)}_{\alpha} G^{\alpha\beta} \). since \( \Phi^{L(k)}_\lambda \) and \( \Phi^{R(k)}_\lambda \) are vector subspaces of \( \Phi \) we can introduce the canonical angles \( \theta^{L(k)}_j \) between them (see [4, 44] for an introduction to the topic), where the index \( j \) runs from 1 to \( d_{\lambda_i(k)} \). these angles measure the ‘separation distance’ of these vector subspaces. as explained below, the sh theorem indicates that these distances should be bounded for all \( k \).

theorem 11 [3]. the constant-coefficient system (39) is sh (admits at least one hyperbolization) if and only if it is hyperbolic for some direction \( n_a = \nabla_a t \) and, for all \( i \in D(k) \) and all normalized \( k_a \) non-proportional to \( n_a \), there is a constant maximum angle \( \vartheta < \frac{\pi}{2} \) between the canonical angles of \( \Phi^{L(k)}_\lambda \) and \( \Phi^{R(k)}_\lambda \).
This last condition is equivalent to: if $\theta_{\lambda}(k)$ are the canonical angles between $\Phi_{\lambda}(k)$ and $\Phi_{\mu}(k)$, then there exists $\vartheta < \frac{\pi}{2}$ such that

$$\cos \theta_{\lambda}(k) \geq \cos \vartheta > 0$$

for all normalized $k_\alpha$ non-proportional to $n_\alpha$, with $i \in D(k)$ and $j = 1, \ldots, d_{\lambda}(k)$.

From the proof of the theorem, it follows how to build all the hyperbolizations that the system admits, pseudo-differential or not (those that do not depend on $k_i$). However, for simplicity, the hyperbolization used in the proof is the one where the eigenvalues, adding by $h^j_A$ to $A^i_A = h^j_A \Omega^i_A k_i$, are simple, different from each other and different from the generalized eigenvalues for all $k_i$. When condition (44) is satisfied and this hyperbolization is chosen, it trivially guarantees the definition for $A^j_A$. We will use this particular hyperbolization to prove our second main theorem.

5. Second main theorem: strong hyperbolicity of the subsidiary system

In this section, we continue considering the constant-coefficient case (39), where $\Omega^i_A$ is constant. This means that $C^i_A$ and $M^\Delta_i A$ are constants too since they are defined by equations $C^i_A \gamma_{(ab)} = 0$ and $M^\Delta_i A \gamma_{(ab)} = 0$. This simplification helps us to present the below closed theorem with simple hypotheses.

Theorem 11 says nothing about the preservation or non-preservation of the constraints during the evolution. For answering this question, we assume valid all the hypotheses of section 2 and show the sufficient conditions for the SH of the evolution equations of the constraints (equation (31)). As already mentioned, these equation (31) may not be unique since $N^\Gamma_A$ can be freely chosen, i.e. $N^\Gamma_A$ plays the role of a reduction. So, we say that the SS is strongly hyperbolic when it is possible to choose at least one reduction $N^\Gamma_A$ (or hyperbolization) such that the set of subsidiary equation (31) is SH. Following definition 9 and considering equation (31), this is equivalent to requiring that there exists a reduction $N^\Gamma_A (k_i)$ such that the principal symbol of the SS

$$B^\Gamma_i k_i := C^i_A h^i_A k_i + N^\Gamma_A M^\Delta_i A k_i$$

is uniformly diagonalizable with real eigenvalues. The following second main theorem, called SH of the SS, explains under which conditions exist such hyperbolization.

**Theorem 12.** Consider the system of constant coefficients (39). This system admits a hyperbolization $h^i_A (k_i)$ and has at least one SH SS associated to the (SH) evolution equations $h^i_A (k_i) \tilde{E}_A = 0$ if the following conditions hold:

(i) The system satisfies the hypotheses of theorem 11.

(ii) All the constraints of the systems come from Geroch fields $C^i_A$ as in equation (9) and these $C^i_A$ satisfy the assumption 1.

(iii) The system can admit (or not) extra Geroch fields $M^\Delta_i A$ which satisfy assumption 5.

(iv) The integrability conditions (25) and (26) are satisfied.

(v) For each $k_i$ with $|k| = 1$, the fields $M^\Delta_i A k_i$ span the left kernel of $C^i_A \gamma_{ab} k_j$.

Notice that (ii), (iii) and (iv) are the same conditions as in the SS theorem, and that we have added one extra condition, the number (v). This has to be included to guarantee the existence of the hyperbolization $N^\Gamma_A (k_i)$, as we will show in the proof of the theorem. However, this is
not a condition that physical systems necessarily satisfy. For example, there could exist $X$ defined only for a particular direction of $k_i$ and such that $X(k_i)C^{(0)}_A N^A k_i = 0$, so $X(k_i)$ could not be expanded by the $M^2_{Aki}$ since the latter and any linear combination of them are defined for all $k_i$. We will explain in subsection 5.2, after the proof of the theorem, how to deal with this class of cases. These results will only be valid in the pseudo-differential version, so we may not be able to extrapolate them so directly to the non-pseudo-differential case.

The proof of this theorem is given in subsection 5.1. This proof uses the fact that the diagonalization bases of the principal symbol of the evolution equations are related in a particular way to the diagonalization bases of the principal symbol of the SS. This relationship was found by Reula [38], assuming that there are subsidiary equations for the constraints and that they are first-order in derivatives. The theorem presented here completes these ideas since Reula’s assumption is obtained as a result when conditions (ii), (iii), (iv) are satisfied. On the other hand, these conditions are the hypotheses of the SS theorem for the quasi-linear case. Therefore, this relationship between the bases is also valid in the quasi-linear case (1), when the reductions $h^B_{\alpha} A$ and $N^A_{\Gamma}$ cannot depend on the wave vector $k_i$. The problem in these non-pseudo differential cases is that the propagation velocities of the SS cannot be freely chosen. This complicates a possible proof of a general theorem. Nevertheless, the steps of the proof presented here can be adapted to each particular theory to conclude similar results. Following all these ideas and to avoid a more complex discussion, we present here a closed theorem with simple hypotheses for the constant-coefficient case.

5.1. Proof of second main theorem

We show the theorem assuming that the system admits non-trivial $M^2_{Aki}$. The proof for the case where the system does not admit $M^2_{Aki}$ will be trivial from the previous case. We comment on this at the end.

By (i) and theorem 11, we know that there exists a family of hyperbolizations $h^B_{\alpha}(k)$ of the system (39). From this family, a particular hyperbolization was used in [3] to prove theorem 11, we call it hyperbolization 1. We shall comment and use it in the following.

The idea of the proof of our theorem is to show that if we choose the hyperbolization 1 $h^B_{\alpha}(k)$, then there exists $N^A_{\Gamma}(k)$ such that the principal symbol of the SS $B^A_{\Gamma} k_i := C^\alpha_{\Delta} h^B_{\alpha} k_i + N^A_{\Gamma} M^2_{Aki}$ satisfies the definition 9 and therefore the subsidiary equations are SH. Recall that $h^B_{\alpha}(k)$ defines the evolution equations $\tilde{e}^{\alpha}(\tilde{\phi}) = h^B_{\alpha}(k) E^B = 0$ whose principal symbol is $A^{\beta}_{\alpha} k_i = h^B_{\alpha} M^A_{\beta} k_i$.

The characteristic structure of $A^{\beta}_{\alpha} k_i$: the eigenvalues of $A^{\beta}_{\alpha} k_i$ define the propagation velocities of the system (see [39]). These eigenvalues are divided into two groups. Those we call the ‘physics’, which are associated to the evolution of the physically relevant fields and do not change no matter the chosen reduction $h^B_{\alpha}$; and the rest, which we call the ‘constraints 1’, that depend on the chosen reduction. As shown in [3], the ‘constraints 1’ can be freely chosen using specific reductions. Particularly, by choosing the ‘hyperbolization 1’, the matrix $A^{\beta}_{\alpha} k_i$ becomes uniformly diagonalizable with real eigenvalues. This hyperbolization satisfies that for each $k_i$ the ‘constraints 1’ are simple, non-degenerate, different from each other and different from the ‘physics’.

The characteristic structure of $B^A_{\Gamma} k_i$: we will show that the eigenvalues of $B^A_{\Gamma} k_i$ are also divided into two groups. The ‘constraints 1’ (inherited from $A^{\beta}_{\alpha} k_i$), which remain unchanged by any choice of $N^A_{\Gamma}$; and the other group which we call the ‘constraints 2’, that depend on $N^F_{\Delta}$. Following these ideas, we will show, for each normalized $k_i$, that:
(a) The set of ‘constrains 1’ are all the generalized eigenvalues of the pencil
\[
\left[ -\delta_{\Delta}^k \lambda + C^\Delta_{\Lambda} h^\Delta_{\Lambda} k_j \right] / M^\Delta_{\Lambda} k_j.
\] (45)

(b) In the Kronecker decomposition (see [20, 21] for its definition) of this pencil, all its Jordan blocks are of dimension 1 for all \(k_i\). In other words, for each \(k_i\), these generalized eigenvalues are simple, non-degenerated and different from each other.

This implies that (45) satisfies the condition (44) for canonical angles. So we can use theorem 11 applied to the following pseudo-differential equations
\[
\left[ \begin{array}{c}
\delta^\Delta_{\Lambda} \partial_t + i C^\Delta_{\Lambda} h^\Delta_{\Lambda} k_j \\
M^\Delta_{\Lambda} k_j
\end{array} \right] \psi^\Delta = 0.
\] (46)

Thus, using the thesis of this theorem, we conclude that there exists (a hyperbolization) \(N^\Gamma_{\Lambda}(k)\) such that \(B^\Delta_{\Lambda}(k)\) and \(M^\Delta_{\Lambda}(k)\) satisfy the definition 9. Here, \(N^\Gamma_{\Lambda}(k)\) is chosen as the ‘hyperbolization 1’ for (46) and such that the eigenvalues ‘constraints 2’ are simple, non-degenerate, different from each other and different from ‘constraints 1’.

This discussion says that we should show (a) and (b) from (i), (ii), (iii), (iv) and (v) to complete the proof of our theorem. Although, we should also justify the use of the pseudo-differential reductions \(h^\alpha_{\Lambda}(k)\), \(h^\Lambda_{\Delta}(k)\) and \(N^\Gamma_{\Lambda}(k)\).

### 5.1.1 Subsidiary system in Fourier’s form.

We first reproduce the calculations of subsection 3.2 until we arrive at the evolution equations for the constraints in their Fourier version. We have to repeat these steps since the quasi-linear system includes non-linearities that complicate the Fourier analysis and these systems do not admit pseudo-differential reductions. The use of these reductions is only valid in the constant coefficients case and after applying the Fourier transform to the system.

Recalling that \(C^\Gamma_{\Lambda}\) and \(M^\Delta_{\Lambda}\) are constants, it is easy to prove the identity
\[
(n_a^\alpha \partial_t + i k_a) \begin{bmatrix} C^\Gamma_{\Lambda} \\ M^\Delta_{\Lambda}\end{bmatrix} \gamma_\alpha^b (n_a^\alpha \partial_t + i k_a) = 0.
\]

Multiplying this expression by \(\bar{\phi}^\alpha\), we obtain
\[
\left[ \begin{array}{c}
C^\Gamma_{\Lambda} \partial_t + i C^\Lambda_{\Lambda} k_j \\
i M^\Delta_{\Lambda} k_j C^\Delta_{\Lambda} \end{array} \right] \bar{E}^\Lambda = 0,
\] (47)

where the expressions \(C^\Gamma_{\Lambda} = C^\Gamma_{\Lambda} n_a^\alpha, k_a = (0, k_i)\) and \(M^\Delta_{\Lambda} = 0\) have been used.

We introduce the identity \(\delta^\Delta_{\Lambda} = \begin{bmatrix} \gamma_\alpha^0 \\ h^\Lambda_{\Delta}(k) \end{bmatrix} \begin{bmatrix} h^\Gamma_{\Lambda}(k) \\ C^\Delta_{\Lambda} \end{bmatrix}\) in the above equation and separate the terms as follow:
\[
\left[ \begin{array}{c}
C^\Gamma_{\Lambda} \partial_t + i C^\Lambda_{\Lambda} k_j \\
i M^\Delta_{\Lambda} k_j \end{array} \right] \begin{bmatrix} \gamma_\alpha^0 \\ h^\Lambda_{\Delta}(k) \end{bmatrix} = \left[ \begin{array}{c}
-i C^\Gamma_{\Lambda} \gamma_\alpha^0 k_j \\
\delta^\Delta_{\Lambda} \partial_t + i C^\Lambda_{\Lambda} k_j h^\Lambda_{\Delta}(k) \\
0 \\
i M^\Delta_{\Lambda} k_j \end{array} \right],
\] (48)
where we used that $M^{\Delta \beta}_\alpha = 0$, $C^{\Gamma 0}_\beta h^4_\Delta = \delta^{\Gamma}_{\Delta}, -C^{\Gamma 0}_\alpha \Omega^4_{\alpha} = C^{\Gamma 4}_\alpha \Omega^4_{\alpha}$. We conclude

$$
\begin{align*}
\begin{bmatrix}
\tilde{e}^\alpha \\
\tilde{\psi}^\Delta
\end{bmatrix} &= \begin{bmatrix}
h^{\beta}_\alpha(k) \\
C^{\Delta 0}_\beta
\end{bmatrix} \tilde{E}^\beta \\
&= \begin{bmatrix}
\partial_\gamma \tilde{\phi}^\beta + i h^4_{\beta}(k) \Omega^\gamma_{\alpha} k_\delta \tilde{\phi}^\gamma \\
i C^{\Delta 0}_\beta \Omega^4_{\alpha} k_\delta \tilde{\phi}^\gamma
\end{bmatrix},
\end{align*}
$$

(49)

where we used that $h^{\beta}_{\alpha} \Omega^4_{\alpha} = \delta^\beta_\alpha$ and $C^{\Delta 0}_\beta \Omega^4_{\alpha} = 0$. So, by contracting equation (48) with (49), the identity (47) is rewritten as

$$
\begin{align*}
\begin{bmatrix}
-i C^{\Gamma 0}_\alpha \Omega^4_{\alpha} k_j \\
0
\end{bmatrix}
\begin{bmatrix}
\delta^\gamma_\Delta \partial_\delta + i C^{\Gamma 4}_\Delta k_\delta h^4_\Delta \\
i M^{\Delta 4}_\Delta k_j
\end{bmatrix}
\begin{bmatrix}
\tilde{e}^\gamma \\
\tilde{\psi}^\Delta
\end{bmatrix} = 0.
\end{align*}
$$

(50)

In the on-shell case, when the evolution equations are satisfied $\tilde{e}^\gamma = 0$, we obtain the system (46), that is,

$$
\begin{align*}
\begin{bmatrix}
\delta^\gamma_\Delta \partial_\delta + i C^{\Gamma 4}_\Delta k_\delta h^4_\Delta \\
i M^{\Delta 4}_\Delta k_j
\end{bmatrix}
\begin{bmatrix}
\tilde{\psi}^\Delta
\end{bmatrix} = 0.
\end{align*}
$$

The constraints $\tilde{\psi}^\Delta$, satisfy all these pseudo-differential equations, then the final expression for the SS (on-shell case $\tilde{e}^\beta = 0$) is

$$
\begin{align*}
\partial_\beta \psi^\gamma + \left( C^{\Gamma 4}_\Delta h^4_\Delta + N^\gamma_\Delta M^{\Delta 4}_\Delta \right) k_\delta \psi^\Delta = 0.
\end{align*}
$$

(51)

Notice that the principal symbol of the SS is the same as in the quasi-linear case (see equation (32)), the difference here is that $h^{4}_\Delta(k)$ and $N^\Delta_\Delta(k)$ can depend on the wave vector.

We also note that by contracting equation (48) with (50) the following identity is obtained

$$
\begin{align*}
\begin{bmatrix}
-i C^{\Gamma 0}_\alpha \Omega^4_{\alpha} k_j \\
0
\end{bmatrix}
\begin{bmatrix}
\delta^\gamma_\Delta \partial_\delta + i C^{\Gamma 4}_\Delta k_\delta h^4_\Delta \\
i M^{\Delta 4}_\Delta k_j
\end{bmatrix}
\begin{bmatrix}
\tilde{\phi}^\beta \\
i h^4_{\beta}(k) \Omega^\gamma_{\alpha} k_\delta \tilde{\phi}^\gamma
\end{bmatrix} = 0.
\end{align*}
$$

(52)

By condition (iv), we are assuming that $M^{\Delta 4}_\Delta k_j$ expands the entire left kernel of $C^{\Delta 0}_\beta \Omega^4_{\alpha} k_\delta$, otherwise, the above system would be modified as follows

$$
\begin{align*}
\begin{bmatrix}
-i C^{\Gamma 0}_\alpha \Omega^4_{\alpha} k_j \\
0
\end{bmatrix}
\begin{bmatrix}
\delta^\gamma_\Delta \partial_\delta + i C^{\Gamma 4}_\Delta k_\delta h^4_\Delta \\
i M^{\Delta 4}_\Delta k_j
\end{bmatrix}
\begin{bmatrix}
\tilde{e}^\gamma \\
\tilde{\psi}^\Delta
\end{bmatrix} = 0.
\end{align*}
$$

(53)

Where the vectors $X^{\Delta}_\Delta(k)$ satisfy $X^{\Delta}_\Delta(k) C^{\Delta 0}_\beta \Omega^4_{\alpha} k_\delta = 0$ for some $k_\delta$ directions and cannot be obtained from linear combinations of $M^{\Delta 4}_\Delta k_j$. The index $\hat{\delta}$ numbers these vectors such that, for each $k_\delta$, span $\{ X^{\Delta}_\Delta(k) C^{\Delta 0}_\beta \Omega^4_{\alpha} k_\delta \} = \text{left} \ker (C^{\Delta 0}_\beta \Omega^4_{\alpha} k_\delta)$. We will return to this discussion after the proof of this theorem and explain how to obtain the SH of the SS by suppressing condition (v).

5.1.2. Relationship between the principal symbols. As a second step, we study the relationship between the principal symbols of the evolution equations and the SS. Every expression found in this subsubsection is also valid in the quasi-linear case.
For each normalized \( k_a = (0, k_i) \), consider the lines \( I_a(\lambda) = -n_a \lambda + k_a \in S_{n_a} \). Using the equations (5) and (19) we can conclude the following identities

\[
0 = I_a(\lambda) M_{\hat{\lambda}} C^{\tau_0} \gamma^{10}_{\alpha} b_a(\lambda),
\]

\[
= k_i M_{\hat{\lambda}} \left( C^{\tau_0} \gamma^{10}_{\alpha} k_j \right),
\]

and

\[
0 = I_a(\lambda) C^{\tau_2} \left[ \gamma^{10}_{\alpha} \ h^A \left( C_{\hat{\lambda}}^0 b_a(\lambda),
\right.
\]

\[
= -(I_a C^{\tau_0} \gamma^{10}_{\alpha} \ h^A \left( C_{\hat{\lambda}}^0 \gamma^{10}_{\beta} b_b \right) + (I_a C^{\tau_0} h^A \left( C_B^{\Delta_0} \gamma^{10}_{\beta} b_b \right),
\]

\[
= -(k_j C^{\tau_0} \gamma^{10}_{\alpha}) \left( -\lambda \delta^{\beta} + h^A \gamma^{10}_{\beta} k_i \right) + \left( -\lambda \delta^{\beta} + C^{\tau_0} h^A \gamma^{10}_{\beta} k_j \right) \left( C_B^{\Delta_0} \gamma^{10}_{\beta} b_b \right). \tag{57}
\]

Notice that these equations are exactly the rows of (53) by replacing \( \partial \) by \(-i\lambda\) and dividing by \( i \). We conclude the following equation

\[
(k_j C^{\tau_0} \gamma^{10}_{\alpha}) \left( -\lambda \delta^{\beta} + h^A \gamma^{10}_{\beta} k_i \right) = \left( -\lambda \delta^{\beta} + C^{\tau_0} h^A \gamma^{10}_{\beta} k_j \right) \left( C_B^{\Delta_0} \gamma^{10}_{\beta} b_b \right). \tag{58}
\]

This equation was found in [38], assuming that the SS was first order in derivatives. The latter equation is not necessarily valid unless the system has the structure associated with the Geroch fields presented here, i.e. hypotheses (ii), (iii) and (iv) of the theorem.

From equation (56) we know that \( k_i M_{\hat{\lambda}} \) belongs to the left-hand kernel of \( C^{\tau_0} \gamma^{10}_{\alpha} k_j \). Therefore, considering

\[
B^{\tau_0} \gamma^{10}_{\alpha} = \left( C^{\tau_0} h^A \left( N_\Delta^0 \ M_{\hat{\lambda}} \right) \right) k_j
\]

with free \( N_\Delta^0 \), the above equation can be rewritten as

\[
(k_j C^{\tau_0} \gamma^{10}_{\alpha}) \left( -\lambda \delta^{\beta} + A^\alpha_{\beta} k_i \right) = \left( -\delta^{\beta}_A \lambda + B^{\tau_0} k_j \right) \left( C_B^{\Delta_0} \gamma^{10}_{\beta} b_b \right). \tag{58}
\]

This last expression shows how the principal symbols of the evolution equations for \( \phi^\alpha \) are related to the principal symbol of the SS.

### 5.1.3 Left kernel of \( \gamma^{10}_{\alpha} b_a(\lambda) \)

In this subsubsection, we shall choose a basis for the left kernel of \( \gamma^{10}_{\alpha} b_a(\lambda) \). This basis will allow us to find the Kronecker structure of the pencil \( \gamma^{10}_{\alpha} b_a(\lambda) \) in the following subsubsection.

From the previous subsubsections, we know that

\[
0 = \left( -\lambda n_a + k_a \right) \left[ C_{\hat{\lambda}}^{\tau_0} \gamma^{10}_{\alpha} b_a(\lambda) \right].
\]

As before, using equation (17) this expression can be rewritten as

\[
0 = \left[ \begin{array}{cc}
-C^{\tau_0} \gamma^{10}_{\alpha} k_j & -\lambda \delta^{\beta}_A + C^{\tau_0} h^A \gamma^{10}_{\beta} k_i \\
0 & -\lambda \delta^{\beta}_A + C^{\tau_0} h^A \gamma^{10}_{\beta} k_i
\end{array} \right] \left[ \begin{array}{c}
M_{\hat{\lambda}} k_j \\
C_B^{\Delta_0} \gamma^{10}_{\beta} b_b
\end{array} \right].
\]
Therefore, the basis we are looking for will result from the choice of a subset of vectors of
\[
\begin{bmatrix}
-C^\Gamma_A^{\alpha} \Gamma^i_{\alpha kj} & -\lambda \delta^\Gamma_{\alpha} + \left( C^\Gamma_A^{\alpha} h^\Delta_{\alpha} \right) k_j \\
0 & M^\Delta_{\alpha kj}
\end{bmatrix}.
\] (59)

Notice that, for each \(k_j\), the pencil
\[
\lambda f^n_i + K^n_i := \begin{bmatrix}
h^\beta_B & C^0_B \\
C^\Delta_B & 0
\end{bmatrix} \Gamma^b_{\alpha i} I_b(\lambda) = \begin{bmatrix}
-\lambda \delta^\beta_{\alpha} + h^\beta_B \Gamma^b_{\alpha i} k_i \\
0 & C^\Delta_B \Gamma^b_{\alpha i} k_i
\end{bmatrix}
\]
\[
= \lambda \begin{bmatrix}
-\delta^\beta_{\alpha} & 0 \\
0 & C^\Delta_B \Gamma^b_{\alpha i} k_i
\end{bmatrix}
\] (60)

has the form of the pencil (126) in appendix A. So, by making use of the lemma 14 (in this appendix) and noticing that \(\lambda f^n_i + K^n_i\) and \(N^b_{\alpha i} I_b(\lambda)\) are related by an invertible matrix independent of \(\lambda\), we conclude that
\[
\dim(\text{left}_{\ker}(\Gamma^b_{\alpha i} I_b(\lambda))) = c
\]
for any \(\lambda\) different from the generalized eigenvalues \(\lambda_i(k)\). This means that we have to choose \(c\) linearly independent vectors of (59) as the left kernel basis of \(\Gamma^b_{\alpha i} I_b(\lambda)\).

First, since \(C^\Gamma_A^{\alpha} \Gamma^i_{\alpha kj}\) plays a very important role in the rest of the proof, we introduce some definitions associated with this operator. For each \(k_i\), we call
\[
d(k) := \dim(\text{right}_{\ker}(C^\Gamma_A^{\alpha} \Gamma^i_{\alpha kj}))
\] (61)
to the dimension of the right kernel of \(C^\Gamma_A^{\alpha} \Gamma^i_{\alpha kj}\),
\[
r(k) := \text{rank}(C^\Gamma_A^{\alpha} \Gamma^i_{\alpha kj})
\] (62)
to its rank and
\[
s(k) := \dim(\text{left}_{\ker}(C^\Gamma_A^{\alpha} \Gamma^i_{\alpha kj}))
\] (63)
to the dimension of its left kernel. We also recall that, by the rank–nullity theorem,
\[
u = r(k) + d(k),
\] (64)
\[
c = r(k) + s(k).
\] (65)

By hypothesis (v), for each \(k_i\), the vectors \(M^\Delta_{\alpha kj}\) expand the left kernel of \(C^\Gamma_A^{\alpha} \Gamma^i_{\alpha kj}\) (whose dimension is \(s(k)\)). Therefore, we choose the following \(s(k)\) linearly independent vectors
\[
m^i_{\alpha kj} := h^\Delta_{\alpha kj}
\]
as part of our basis. These are obtained from the rows of (59), where \(z = 1, \ldots, s(k)\) and \(h^\Delta_{\alpha kj}\) is the projector representing our choice. These vectors belong to the left kernel of \(\Gamma^b_{\alpha i} I_b(\lambda)\) and have the property that they do not depend on \(\lambda\). Notice also that by the form of (60) any other vector of the left kernel of \(\Gamma^b_{\alpha i} I_b(\lambda)\), linearly independent from the \(m^i_{\alpha kj}\), will depend on \(\lambda\).
On the other hand, since for each \( k_i \), the rank of \( C^\alpha \Gamma^\beta_{\alpha\beta}k_i \) is \( r(\lambda) \) and since \( C^\alpha \Gamma^\beta_{\alpha\beta}k_i \) appears explicitly in the first rows of (59), we complete our basis with the following linearly independent \( r(k) \) vectors

\[
c^\alpha w_{\alpha} l_{\alpha}(\lambda) := h^w \left[ -C^\alpha \Gamma^\beta_{\alpha\beta}k_j - \lambda \delta_\beta^\gamma + \left( C^\gamma_{\alpha\beta}h^\beta \Delta \right) k_j \right].
\]

Where \( w = 1, \ldots, r(\lambda) \), and \( h^w \) is the projector associated with our choice. It only remains to show that \( m^\alpha_w l_{\alpha}(\lambda) \) and \( c^\alpha w_{\alpha} l_{\alpha}(\lambda) \) are linearly independent between them. This is shown by recalling that \( h^w \left( -C^\alpha \Gamma^\beta_{\alpha\beta}k_j \right) \) has no left kernel (in the index \( w \)) and thus the only \([X_w \ X_j]\) which cancels the first \( u \) columns of

\[
\begin{bmatrix}
h^w \left( -C^\alpha \Gamma^\beta_{\alpha\beta}k_j \right) & h^w \left( -\lambda \delta_\beta^\gamma + \left( C^\gamma_{\alpha\beta}h^\beta \Delta \right) k_j \right) \\
0 & h^w \left( M^\beta_{\alpha} k_j \right)
\end{bmatrix}
\]

is the trivial ones.

We conclude by (65) that our chosen basis \( \left\{ m^\alpha_j k_j, c^\alpha w_{\alpha} l_{\alpha}(\lambda) \right\} \) has \( e \) vectors, where the \( m^\alpha_j k_j \) are \( s(k) \) vectors that do not depend on \( \lambda \) and the \( c^\alpha w_{\alpha} l_{\alpha}(\lambda) \) are \( r(k) \) vectors that depend linearly on \( \lambda \). This is a base of the left kernel of \( \mathfrak{R}^\alpha \Gamma^\beta_{\alpha\beta}l_{\alpha}(\lambda) \) valid for \( \lambda \) different to the generalized eigenvalues.

5.1.4. Kronecker decomposition of \( \mathfrak{R}^\alpha \Gamma^\beta_{\alpha\beta}l_{\alpha}(\lambda) \). The aim of this subsubsection is to give the Kronecker decomposition of the principal symbol (pencil matrix)

\[
\mathfrak{R}^\alpha \Gamma^\beta_{\alpha\beta}l_{\alpha}(\lambda) = -\lambda \mathfrak{R}^\alpha \Gamma^\beta_{\alpha\beta} + \mathfrak{R}^\alpha \Gamma^\beta_{\alpha\beta} l_{\alpha}(\lambda)
\]

where \( l_{\alpha}(\lambda) = -\lambda n_\alpha + k_\alpha \in S_\alpha \). This decomposition will guide us in the following subsubsections to complete the proof of the theorem.

This pencil is an \( e \times u \) matrix in \( \mathbb{C} \), although it makes no difference if we think of its components in \( \mathbb{R} \). The Kronecker decomposition consists of rewriting the pencil as

\[
\mathfrak{R}^\alpha \Gamma^\beta_{\alpha\beta}l_{\alpha}(\lambda) = Y^A \Gamma^\beta_{\alpha\beta} K^\beta_{\alpha\beta} (\lambda, k) W^\alpha_{\beta}(k).
\]

Where \( Y^A \Gamma^\beta_{\alpha\beta} \in \mathbb{C}^{e \times e} \) and \( W^\alpha_{\beta}(k) \in \mathbb{C}^{e \times u} \) are two invertible matrices that depend on \( k_i \) and not on \( \lambda \); and \( K^\beta_{\alpha\beta}(\lambda, k) \in \mathbb{C}^{e \times u} \) is a block matrix (see [3, 20, 21] for details), which depends on \( \lambda \) and \( k_i \). The matrices \( Y^A \Gamma^\beta_{\alpha\beta} \) and \( W^\alpha_{\beta}(k) \) can be thought of as change-of-basis matrices, and \( K^\beta_{\alpha\beta}(\lambda, k) \) is the block matrix which we call the Kronecker structure.

We know from lemmas 1 and 2 of [3] (whose hypotheses are valid in this proof), that for each \( k_i \), \( K^\beta_{\alpha\beta}(\lambda, k) \) has all their Jordan blocks of dimension 1. We also know from subsection 3.1 of [3] that since \( \mathfrak{R}^\alpha \Gamma^\beta_{\alpha\beta} \) has only trivial right kernel, the remaining blocks appearing in this decomposition are \( L^T_m \) and zero rows (called \( L^T_0 \) simplifying the notation). However, since \( \mathfrak{R}^\alpha \Gamma^\beta_{\alpha\beta} l_{\alpha}(\lambda) \) is \( k_i \)-dependent, its Kronecker decomposition is \( k_i \)-dependent too. This means that the generalized eigenvalues and block structure \( L^T_m \) of \( K^\beta_{\alpha\beta}(\lambda, k) \) can be different for different values of \( k_i \).

We say that the Kronecker structure of the \( e \times u \) pencil \( \mathfrak{R}^\alpha \Gamma^\beta_{\alpha\beta} l_{\alpha}(\lambda) \) has the following structure:

\[
d(k) \times J_1, \ r(k) \times L^T_1, \ s(k) \times L^T_0.
\]

Where for each \( k_i \), the quantities \( d(k), r(k), s(k) \) are defined by (61), (62) and (63) respectively and they also satisfy the equations (64) and (65). We justify each of the terms of (67) below.
The following analysis is valid for each $k_i$, so we assume $k_i$ is fixed.

- We begin by justifying the block structure $r(k) \times L^T_k$, $s(k) \times L^\omega_k$.

Let us study the left kernel of $L^T_m = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 1 & \cdot & \cdot \\ 0 & 0 & \ldots & \lambda \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{m+1 \times m}$. It has dimension 1 and is expanded by the vector

$$X = [-1, \lambda, \ldots, (-1)^m \lambda^{m-1}, (-1)^{m+1} \lambda^m] \in \mathbb{R}^{1 \times m+1}.$$  

The coefficients of $X$ are polynomials in $\lambda$ whose major degree is $m$. We can increase the degree of these coefficients, for example, by considering the vector $\lambda X$, but we cannot reduce it without obtaining rational functions. This allows us to introduce a method to detect the $L^T_k$ blocks present in a given pencil. We first define the function $gr$ which takes vectors with polynomial coefficients in $\lambda$ and returns the greater polynomial degree between their coefficients (in our example, $gr(X) = m$ and $gr(\lambda X) = m + 1$). If we now consider a pencil with different blocks $L^T_{m_1}, \ldots, L^T_{m_v}$ such that $0 \leq m_1 \leq m_2 \leq \cdots \leq m_v$, there exists a left kernel basis $\{X_i\}$, with $gr(X_i) = z_i$ and ordered such that $z_1 \leq z_2 \leq \cdots \leq z_p$, has at least one $z_i$ such that $m_i = z_i$, and we can increase $gr$ function for their elements. Notice that, if we have a zero row $L^0_k$, its left kernel $F$ can be chosen independent of $\lambda$, then $gr(F) = 0$.

Applying this method to $\Theta^{\beta^h}a_k(\lambda)$, using the basis $\{m^z_k, c^w_k a_k(\lambda)\}$ with $z = 1, \ldots, s(k)$, $w = 1, \ldots, r(k)$ from the previous subsubsection, which satisfies that $gr(m^z_k) = 0$ and $gr(c^w_k a_k(\lambda)) = 1$, we conclude that the Kronecker structure of $\Theta^{\beta^h}a_k(\lambda)$ has the blocks $r(k) \times L^T_k$, $s(k) \times L^\omega_k$ and the rest of the structure are Jordan blocks.

- We have already explained that this pencil only has Jordan blocks of dimension 1, it remains to explain that they are $d(k)$ blocks.

This result can be concluded by counting the number of $\lambda$’s appearing in the columns of $K^\alpha_\nu(\lambda, k)$. Since only one $\lambda$ can be present per column and $K^\alpha_\nu(\lambda, k)$ has $u$ columns, there are $u$ $\lambda$’s in $K^\alpha_\nu(\lambda, k)$. Notice that $L^\omega_k$ has no $\lambda$ since it is a row of zeros and that each $L^T_k$ has only one $\lambda$ per column, then in $r(k) \times L^T_k$, $s(k) \times L^\omega_k$ there are $r(k)$ $\lambda$’s. Finally, the number of Jordan blocks of dimension 1 is $u - r(k) = d(k)$ (see equation (64)).

As a final comment, we note that the sum of the multiplicities of each of the different $q(k)$-generalized eigenvalues has to be equal to $d(k)$, that is,

$$d_{\lambda_1} + d_{\lambda_2} + \cdots + d_{\lambda_{p(k)}} = d(k).$$

### 5.1.5. Basis which diagonalize $A^\nu k_i$

In this subsubsection, we use the obtained information of the Kronecker decomposition of $\Theta^{\beta^h}a_k(\lambda)$ to find the bases that diagonalize $A^\nu k_i$.

The eigenvectors which diagonalize $A^\nu k_i$ are divided into two groups:
The generalized eigenvectors $\delta \phi^\beta_{\lambda_i(k)}$, associated to the generalized eigenvalues $\lambda_i(k)$, such that they satisfy
\[
\mathbf{\Omega}^B_{\beta \beta}(-\lambda_i(k)n_a + k_a)\delta \phi^\beta_{\lambda_i(k)} = 0,
\]
with $\lambda_1(k) \leq \lambda_2(k) \leq \cdots \leq \lambda_{d(k)}(k)$. We know from the previous subsubsection that for each $\lambda_i(k)$ there is a generalized eigenvector $\delta \phi^\beta_{\lambda_i(k)}$. Notice that to simplify the notation, we have changed how we denote the generalized eigenvalues with respect to the section 4.

From the above equation, it follows that
\[
\begin{bmatrix}
C_B^\Delta & -\lambda_i(k)n_a + k_a
\end{bmatrix}
\begin{bmatrix}
\delta \phi^\beta_{\lambda_i(k)}
\end{bmatrix} = 0,
\]
therefore, these generalized eigenvectors are eigenvectors of $A^\beta_i k_i$ and belong to the right kernel of $C_B^\Delta \mathbf{\Omega}^B_{\beta \beta} k_i$.

Recalling that the dimension of the right kernel of $C_B^\Delta \mathbf{\Omega}^B_{\beta \beta} k_i$ is $d(k)$ (see equation (61)) and that the $\delta \phi^\beta_{\lambda_i(k)}$ are $d(k)$ linearly independent vectors, we conclude that
\[
\text{right}_\text{ker}(C_B^\Delta \mathbf{\Omega}^B_{\beta \beta} k_i) = \text{span}(\delta \phi^\beta_{\lambda_i(k)}).
\]

We also recall that the set of generalized eigenvalues $\{\lambda_i(k)\}$ are those we called ‘physical’ at the beginning of the proof.

• The eigenvectors $\delta \phi_{\pi_i(k)}$, associated to the eigenvalues $\pi_i(k)$, such that they satisfy
\[
\begin{bmatrix}
C_B^\Delta & \mathbf{\Omega}^B_{\beta \beta} k_i
\end{bmatrix}
\begin{bmatrix}
\delta \phi_{\pi_i(k)}
\end{bmatrix} = 0,
\]

We are considering that the reduction $h_A^\beta(k)$ is chosen such that, for each $k_i$, the $\{\pi_i(k)\}$ are simple, different from each other and different from the $\{\lambda_i(k)\}$.

\[
\text{right}_\text{ker}(C_B^\Delta \mathbf{\Omega}^B_{\beta \beta} k_i) = \text{span}(\delta \phi^\beta_{\lambda_i(k)}),
\]

5.1.6. Kronecker structure of subsidiary system (45). In this subsubsection, we show that the Kronecker structure of the pencil (45) is
\[
J_1(\pi_1(k)), \ldots, J_1(\pi_{d(k)}(k)), s(k) \times L^T_1, y(k) \times L^T_0,
\]
where $y(k) := \dim \left(\text{left}_\text{ker}(C_B^\Delta \mathbf{\Omega}^B_{\beta \beta} k_i)\right)$.

• We begin by showing that $J_1(\pi_1(k)), \ldots, J_1(\pi_{d(k)}(k))$ is part of the Kronecker structure.

For this purpose, we will show that the set of $\pi_i(k)$ are the generalized eigenvalues of (45), with their corresponding linearly independent generalized eigenvectors
\[ \delta \psi_{\tau_i(k)}^\Delta := C_B^{\delta_0} \eta^B j_i \delta \phi_{\tau_i(k)}^\beta, \]  

i.e., for each \( i = 1, \ldots, r(k) \), it holds that

\[ \begin{bmatrix} -\delta \Delta \tau_i(k) + C_A^{T} h_A^k \end{bmatrix} M_{\Delta}^k \delta \psi_{\tau_i(k)}^\Delta = 0. \]  

(72)

First, recalling that \( M_{\Delta}^k j_i \) expands the left kernel of \( C_B^{\delta_0} \eta^B j_i \), we conclude that

\[ M_{\Delta}^k j_i \delta \psi_{\tau_i(k)}^\Delta = M_{\Delta}^k j_i C_B^{\delta_0} \eta^B j_i \delta \phi_{\tau_i(k)}^\beta = 0. \]

Second, evaluating the expression (57) at \( \lambda = \pi_i(k) \), multiplying by \( \delta \phi_{\tau_i(k)}^\beta \) and recalling equation (69), we obtain that

\[
0 = (k_j C_A^{T} \eta^A j_i) (-\pi_i(k) \delta \phi_{\tau_i(k)}^\beta + A^{(\beta)}_k j_i) \delta \phi_{\tau_i(k)}^\beta,
\]

\[
= \left( -\pi_i(k) \delta \Delta \right) + C_A^{T} h_A^k \delta \phi_{\tau_i(k)}^\beta,
\]

\[
= \left( -\pi_i(k) \delta \Delta \right) + C_A^{T} h_A^k \delta \phi_{\tau_i(k)}^\beta
\]

This shows that equation (72) holds. Let us now demonstrate that the \( \delta \psi_{\tau_i(k)}^\Delta \) are linearly independent. For this purpose, we assume that they are not, i.e. that there exists \( U^{\tau_i(k)} \) such that

\[ 0 = U^{\tau_i(k)} \delta \psi_{\tau_i(k)}^\Delta \] (the sum runs into the \( i \) index) and we will conclude that \( U^{\tau_i(k)} = 0 \). We note that

\[ 0 = U^{\tau_i(k)} \delta \psi_{\tau_i(k)}^\Delta = U^{\tau_i(k)} C_B^{\delta_0} \eta^B j_i \delta \phi_{\tau_i(k)}^\beta = C_B^{\delta_0} \eta^B j_i \left( U^{\tau_i(k)} \delta \phi_{\tau_i(k)}^\beta \right), \]

thus \( U^{\tau_i(k)} \delta \phi_{\tau_i(k)}^\beta \) belongs to the right-hand kernel of \( C_B^{\delta_0} \eta^B j_i \) and, by (68), should be a linear combination of the \( \left\{ \delta \phi_{\lambda_i(k)}^\beta \right\} \). Since these \( \left\{ \delta \phi_{\lambda_i(k)}^\beta \right\} \) are linearly independent of the \( \left\{ \delta \phi_{\tau_i(k)}^\beta \right\} \) by construction, it should be that \( U^{\tau_i(k)} = 0 \).

- It remains to show that \( s(k) \times L^T_1, y(k) \times L^T_2 \) is the other part of the Kronecker structure. This can be concluded directly from lemma 15 (in appendix A) if we prove the condition (137). This condition follows from recalling that

\[ \text{right}_\text{ker}(M_{\Delta}^k j_i) = \left( \delta \psi_{\tau_i(k)}^\Delta \right). \]

Furthermore, we conclude that the pairs \( \left\{ \pi_i(k) \right\}, \left\{ \delta \psi_{\tau_i(k)}^\Delta \right\} \) are all the generalized eigenvalues and eigenvectors of the pencil.

Notice that we called the ‘constraints 1’ to the eigenvalues \( \left\{ \pi_i(k) \right\} \), they are at the same time the generalized eigenvalues of the system (45), i.e. they are the ‘constraints 1’ eigenvalues of \( A^{(\beta)}_k j_i \) and the ‘physical’ eigenvalues for the pencil (45).

5.1.7 Basis which diagonalize \( B_{\Delta}^{ij} j_i \). The previous subsubsection is the proof of (a) and (b) (stated at the beginning of this proof). Therefore, there exists \( N^T_{\Delta} \) such that \( B_{\Delta}^{ij} j_i = C_{\Delta}^{ij} h_A^k j_i + h_A^T M_{\Delta}^k j_i \) is uniformly diagonalizable. This concludes the proof of the theorem in the case where \( M_{\Delta}^k j_i \) is non zero.

28
In this subsubsection, we will find the bases that diagonalize $B^\perp k_j$. We will use these bases with the bases that diagonalize $A^\perp k_j$ to give a very simple expression of equation (58). Finally, we will discuss the structure of the latter equation in the cases where $A^\perp k_j$ and/or $B^\perp k_j$ are not diagonalizable.

Analogously to the case of $A^\perp k_j$, the eigenvectors which diagonalize $B^\perp k_j$ are also divided into two groups for each $k_i$:

- The eigenvectors $\{\delta \psi^\perp_{\alpha^\perp}(k_i)\}$ with their corresponding eigenvalues $\pi_i(k)$, with $i = 1, \ldots, r(k)$, found in the previous subsubsection (see (71)) and such that they satisfy

$$
\left(-\delta^\perp \pi_i(k) + C^\perp A^\perp k_j + N^\perp M^\perp_{\alpha^\perp} k_j\right)\delta \psi^\perp_{\alpha^\perp}(k_i) = 0.
$$

- And the eigenvectors $\delta \psi^\perp_{\beta^\perp}(k_i)$ associated to the eigenvalues $\rho_i(k)$, with $i = 1, \ldots, s(k)$, such that

$$
\left(-\delta^\perp \rho_i(k) + C^\perp A^\perp k_j + N^\perp M^\perp_{\beta^\perp} k_j\right)\delta \psi^\perp_{\beta^\perp}(k_i) = 0.
$$

Where the set $\{\rho_i(k)\}$ are simple, different from each other and different from the $\{\pi_i(k)\}$. These $\{\rho_i(k)\}$ was called ‘constraints 2’.

The set $\{\delta \psi^\perp_{\alpha^\perp}(k_i), \delta \psi^\perp_{\beta^\perp}(k_i)\}$ uniformly diagonalizes $B^\perp k_j$ (since, for each $k_i$, all its eigenvalues $\{\pi_i(k), \rho_i(k)\}$ are simple) showing that the system (52) is SH for this choice of $N^\perp$.

Let us now rewrite equation (58) in the found bases.

- For each $k_i$, we consider the co-basis $\{\delta \phi^\perp_{\alpha^\perp}(k_i), \delta \phi^\perp_{\beta^\perp}(k_i)\}$ and $\{\delta \psi^\perp_{\alpha^\perp}(k_i), \delta \psi^\perp_{\beta^\perp}(k_i)\}$ of the bases $\{\delta \phi^\perp_{\gamma^\perp}(k_i), \delta \phi^\perp_{\beta^\perp}(k_i)\}$ and $\{\delta \psi^\perp_{\gamma^\perp}(k_i), \delta \psi^\perp_{\beta^\perp}(k_i)\}$ respectively. They satisfy

$$
\delta \phi^\perp_{\alpha^\perp}(k_i) \delta \phi^\perp_{\beta^\perp}(k_i) = \delta^\perp_i \delta \phi^\perp_{\beta^\perp}(k_i) \delta \phi^\perp_{\alpha^\perp}(k_i) = 0
$$

$$
\delta \phi^\perp_{\alpha^\perp}(k_i) \delta \phi^\perp_{\gamma^\perp}(k_i) = 0
$$

$$
\delta \phi^\perp_{\beta^\perp}(k_i) \delta \phi^\perp_{\gamma^\perp}(k_i) = \delta^\perp_i
$$

$$
\delta \psi^\perp_{\alpha^\perp}(k_i) \delta \psi^\perp_{\beta^\perp}(k_i) = \delta^\perp_i
$$

$$
\delta \psi^\perp_{\alpha^\perp}(k_i) \delta \psi^\perp_{\gamma^\perp}(k_i) = 0
$$

$$
\delta \psi^\perp_{\beta^\perp}(k_i) \delta \psi^\perp_{\gamma^\perp}(k_i) = \delta^\perp_i.
$$

In these bases, equation (58) reduces to

$$
[I \ 0]
[0 \ 0]
[\Pi \ 0]
[0 \ 0]
[0 \ 0 \ \Lambda] =
[\Pi \ 0]
[0 \ 0 \ \Theta]
[I \ 0]
$$

where

$$
I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{r(k) \times r(k)},
$$

$$
\Pi = \begin{bmatrix} \lambda - \pi_i(k) & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & \lambda - \pi_i(k) \end{bmatrix} \in \mathbb{R}^{r(k) \times r(k)}.
$$
\[
\Lambda = \begin{bmatrix}
\lambda - \lambda_1(k) & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \lambda - \lambda_d(k)
\end{bmatrix} \in \mathbb{R}^{d(k) \times d(k)},
\]
and
\[
\Theta = \begin{bmatrix}
\lambda - \rho_1(k) & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \lambda - \rho_d(k)
\end{bmatrix} \in \mathbb{R}^{s(k) \times s(k)}.
\]

Notice that if we consider the not SH case, where \( \Lambda \) has at least one Jordan block \( J_m \), with \( m \geq 2 \), the SS still can be SH. Namely, by equation (73) the matrix \( \begin{bmatrix} \Pi & 0 \\ 0 & \Theta \end{bmatrix} \) could still be diagonalizable.

On the other hand, in the case where \( \Lambda \) shares some eigenvalue with \( \Pi \) forming a Jordan block, for example,

\[
\begin{bmatrix}
\Pi & 0 \\
0 & \Lambda
\end{bmatrix} = \begin{bmatrix}
\lambda - \pi_1(k) & 0 & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 & 0 \\
0 & 0 & \lambda - \lambda_1(k) & 0 & 0 \\
0 & 0 & 1 & \lambda - \lambda_i(k) & 0 \\
0 & 0 & 0 & 0 & \lambda - \lambda_d(k)
\end{bmatrix},
\]

by equation (73), it could still happen that the SS is diagonalizable.

We concluded that by proposing different options for \( \begin{bmatrix} \Pi & 0 \\ 0 & \Lambda \end{bmatrix} \) and \( \begin{bmatrix} \Pi & 0 \\ 0 & \Theta \end{bmatrix} \) such that they satisfy (73) we obtain all possible cases where the evolution equations and the SS are well-posed or not.

5.1.8. Case without \( M^\Delta_{\tilde{\jmath}} \). Let us now study the case where the system (41) does not admit \( M^\Delta_{\tilde{\jmath}} \), i.e., by condition (e), \( C_A^\Delta B^\Delta_{\tilde{\jmath}} k_j \) only has trivial left kernel for any \( k_j \). In this case, \( s(k) = 0 \) and \( c = r(k) \) by equation (65).

The proof of the theorem is the same as presented above, but now we conclude that the set \( \{ \delta \psi_{\delta \pi_{\Delta \Delta}, k} := C_A^\Delta \delta \psi_{\delta \pi_{\Delta \Delta}, k} \} \) uniformly diagonalizes the matrix \( B^\Delta_{\tilde{\jmath}} k_j := C_A^\Delta h^\Delta_{\tilde{\jmath}} k_j \). Therefore, system (52) is SH. This concludes the proof of the theorem.

5.2. Comment about condition (v) in second main theorem

The condition (v) of the theorem can be suppressed without losing the SH of the SS in the case of constant coefficients. For this purpose, we have to change the pencils

\[
\begin{bmatrix}
-M^\Delta_{\tilde{\jmath}} k_j \\
C_A^\Delta h^\Delta_{\tilde{\jmath}} k_j
\end{bmatrix} \rightarrow \begin{bmatrix}
-C_A^\Delta \delta \psi_{\delta \pi_{\Delta \Delta}, k} k_j \\
-M^\Delta_{\tilde{\jmath}} k_j \\
0 \\
M^\Delta_{\tilde{\jmath}} k_j \\
0 \\
X^\Delta_{\tilde{\jmath}} (k)
\end{bmatrix},
\]

as explained in subsubsection 5.1.1 and carry on the same proof as before but with this new
pencil. This change is introduced since $M_j^a k_j$ no longer spans the left_ker($C_B^{\Delta \alpha \beta} k_i$), so it is necessary to add the vectors $X^a_j(k)$ such that, for each $k_i$, it holds that

$$\text{span}(M^a_j k_j, X^a_j(k)) = \text{left_ker}(C_B^{\Delta \alpha \beta} k_i).$$

The cost of this change is reflected in the principal symbol of the SS, whose new form is

$$B_{\Gamma j} = C_{A \Gamma}^{\Delta a} h^a k_j + N^I_{1 \Delta} M^a_j k_j + N^I_{2 \Delta} X^a_j(k). \quad (74)$$

This change the statement of the theorem.

**Theorem 13.** When all the hypotheses of theorem 12 are satisfied except condition (e), it is possible to find $N^I_{1 \Delta}$ and $N^I_{2 \Delta}$ such that the principal symbol of the subsidiary system equation (74) is uniformly diagonalizable and therefore the SS is SH.

Since the $X^a_j(k)$ do not come from Geroch fields, they could be non-zero only for some particular $k_i$ or they could have a non-linear dependence on $k_i$. This latter case would imply that the Fourier anti-transform of the term $N^I_{2 \Delta} X^a_j(k)$ include second or higher derivatives for any $N^I_{2 \Delta}$ (except for $N^I_{2 \Delta} = 0$). This changes the final hyperbolic answer of the SS into a pure pseudo-differential answer, which can not be directly extrapolated to the quasi-linear case. Some known systems admit vectors $X^a_j(k)$, but in general, it is possible to set $N^I_{2 \Delta} = 0$ and find some $N^I_{1 \Delta}$ such that (74) is uniformly diagonalizable.

6. Examples

In this section, we reproduce some known results about the constraint propagations of two specific theories: Maxwell electrodynamics and the wave equation. We use these theories to illustrate the results presented in the previous sections.

We consider the systems on a space–time $M$ of dim $M = 3 + 1$ with a background Lorentzian metric $g_{ab}$ (with signature $-, +, +, +$) and with their equations in first-order derivatives (i.e. equations (79), (80) and (103)).

We show that in the Maxwell case there are no $M^\Delta a$ fields and the $C^\Gamma a$ fields are associated with the standard constraints $\psi_1 := D_a E^a - J^0$ and $\psi_2 = D_a B^a$. We present its subsidiary equations and comment on its characteristic analysis. We also note that it is commonly used in the literature that when equations are coupled to a source $J^a$, this has to satisfy an on-shell integrability condition $\nabla_a J^a = 0$ to preserve the constraints. However, we will show that this condition is relaxed in the off-shell case, by choosing the divergence proportional to the equations of the system (i.e. (75)) and maintaining the preservation of the constraints.

On the other hand, for the wave equation, we find both $M^\Delta a$ and $C^\Gamma a$ fields. They appear as a result of reducing the system from second to a first-order derivative. As we explained in theorem 8, the non-uniqueness of the SS is associated with the presence of $M^\Delta a$. Therefore, we verify this non-uniqueness and comment on its characteristic analysis.

In both cases, we introduce (as in subsection 2.1) a foliation of $M = \bigcup_{t \in \mathbb{R}} \Sigma_t$ associated to the function $t : M \rightarrow \mathbb{R}$, with the spatial (with respect to the metric $g_{ab}$) hypersurfaces $\Sigma_t$. In addition, following appendix B, we consider the definitions
\( n_a := \nabla_a t, \)
\( \tilde{n}_b := -N n_b \) with \( N := \frac{1}{\sqrt{-\nabla_t \nabla_t}} \),
\( p^a = (\partial_t)^a - \beta^a \) with \((\partial_t)^a n_a = 1\) and \( \beta^n n_a = 0 \)
\( \tilde{n}^a := \tilde{n}^a = \frac{1}{N} p^a, \)
\( \tilde{n}^a \tilde{n}_a = -1, \)

and the expression (162) for the metric \( g_{ab} \). We also consider the projection \( \tilde{\eta}_a^b \) to the hypersurfaces \( \Sigma_t \) as
\( \tilde{\eta}_a^b := \delta_a^b - p^a n_b = \delta_a^b + \tilde{n}^a \tilde{n}_b. \)

6.1. Maxwell electrodynamics

We define the fields \( Q^{1,2}_d \) as
\( Q^1_d := \nabla_a F^{ad} - J_d, \)
\( Q^2_d := \nabla_a * F^{ad}, \)
where \( F^{ad} \) is the electromagnetic (antisymmetric) tensor, \( *F^{ad} := \frac{1}{2} \varepsilon^{adq} F_{cq} \) and \( J^d \) is the source of the system. Here, \( J^d = J^d(F^{ad}, *F^{ad}, x^a) \) may depend on \( F^{ad}, *F^{ad} \) and \( x^a \in M \) but it can not depend on derivatives of \( F^{ad} \) or \( *F^{ad} \). In addition, we assume that it satisfies the off-shell identity
\[
\nabla_d J^d = L_{1,d} Q^1_d + L_{2,d} Q^2_d. \quad (75)
\]

As \( J^d \), the fields \( L_{1,2,d}(F^{ad}, *F^{ad}, x^a) \) do not depend on derivatives of \( F^{ad} \) or \( *F^{ad} \).

We notice two extra off-shell identities
\[
\nabla_d \nabla_a F^{ad} = 0 = \nabla_d \nabla_a * F^{ad},
\]
(both easy to verify). These expressions, in addition with (75), give the off-shell identities
\[
\nabla_d Q^1_d = -L_{1,d} Q^1_d - L_{2,d} Q^2_d, \quad (76)
\]
\[
\nabla_d Q^2_d = 0. \quad (77)
\]

Multiplying by \(-N\) and \( N\), these equations can be rewritten as
\[
\nabla_d \begin{bmatrix} -NQ^1_d \\ NQ^2_d \end{bmatrix} + \begin{bmatrix} \nabla_d(N) + NL_{1,d} \\ 0 \end{bmatrix} \begin{bmatrix} Q^1_d \\ Q^2_d \end{bmatrix} = 0. \quad (78)
\]

These latter equations are the integrability conditions (25) from which the evolution equations of the constraints are obtained as we explain below.

The Maxwell equations are defined by \( Q^{1,2}_d \) as
\[
Q^1_d = \nabla_a F^{ad} - J^d = 0, \quad (79)
\]
\[
Q^2_d = \nabla_a * F^{ad} = 0. \quad (80)
\]
We will use the electric $E^e$ and magnetic $B^e$ fields as the variables of the system. For this purpose, we begin by rewriting $F^e$ in terms of $E^e$ and $B^e$

$$F^e = \tilde{m}^e E^e - \tilde{m}^e E^e + \varepsilon^{qde} \tilde{n}_d B_e.$$ 

These fields are defined by

$$E^e := F^e \tilde{n}_q, \quad B_d := * F_d \tilde{m}^d$$

and they belong to the tangent of $\Sigma_t$, since

$$\tilde{n}_d E^d = 0 = \tilde{m}^e B_e.$$ 

On the other hand, the dual $* F_{lm}$ can be written as

$$* F_{lm} = -\tilde{n}_l B_m + \tilde{n}_m B_l + \varepsilon^{cql} \tilde{m}_q E^l.$$ 

Following the same steps as in appendix B and the definitions (151)–(156), we rewrite $Q^{\ell}_{1,2}$ as

$$Q^1_{\ell} = \tilde{m}^1_{\ell} - \tilde{m}^1_{\ell} \psi_1, \quad (81)$$

$$Q^2_{\ell} = -\tilde{m}^2_{\ell} + \tilde{m}^2_{\ell} \psi_2, \quad (82)$$

where

$$\tilde{J}^l := \tilde{\eta}_l \tilde{f}^l, \quad \tilde{J}^0 = \tilde{n}_l \tilde{f}^l,$$

$$\tilde{e}^1_{\ell} := \tilde{L}_e E^l + \varepsilon^{qde} \tilde{n}_d D_e (N B_e) - N K E^l - N \tilde{J}^l,$$

$$\tilde{e}^2_{\ell} := \tilde{L}_e B^l - \varepsilon^{qde} \tilde{n}_d D_e (NE_e) - N K B^l,$$

$$\psi_1 = D_c E^c - \tilde{J}^0,$$

$$\psi_2 = D_c B^c.$$ 

Here, the expressions (83)–(87) are quantities over $\Sigma_t$ by definition, which means that $\tilde{m}^1_{\ell} \tilde{e}^1_{1,2} = \tilde{e}^1_{1,2}$ and $\tilde{m}^2_{\ell} \tilde{e}^2_{1,2} = \tilde{e}^2_{1,2}$. Notice that $\tilde{e}^1_{1,2}$ and $\psi_{1,2}$ are the evolution and the constraints equations of the system respectively. $\psi_{1,2}$ act as constraints since they have not derivatives in the $\partial_t$ direction. We will see below that these constraints are preserved in the evolutions $\tilde{e}^1_{1,2} = 0$ when the initial data $\phi|_{\Sigma_0}$ is chosen such that $\psi_{1,2}|_{\Sigma_0} = 0$.

Using the expressions (81), (82), (84)–(87), we conclude that the principal symbol of the system is

$$[Q_1^l / Q_2^l] \approx \nabla_{\beta} \phi^\beta,$$

$$= \begin{bmatrix} 2\delta^{\rho}_q \tilde{m}_q \\ \varepsilon^{qde} \tilde{n}_d \end{bmatrix} \begin{bmatrix} 2\delta_{\rho}^q \\ \varepsilon^{qde} \tilde{n}_d \end{bmatrix} \nabla_q \begin{bmatrix} E^e \\ B^e \end{bmatrix}.$$ 

We are considering

$$L_{\rho} E^e \approx \tilde{m}^s_{\rho} \tilde{m}^q_{\rho} \nabla_q E^e,$$
neglecting lower order terms. 

Associated with this system, we have the Geroch fields (see equation (8))

$$C^\Delta_z = \begin{bmatrix} -N\delta^z_i & 0 \\ 0 & N\delta^z_s \end{bmatrix}. \quad (89)$$

Contracting it with $n_z$, we obtain

$$n_z C^\Delta_z = C^\Delta_0 = \begin{bmatrix} \tilde{n}_s & 0 \\ 0 & -\tilde{n}_s \end{bmatrix}. \quad (90)$$

This last expression gives the constraints $\psi_{1,2}$ when it is contracted with $\begin{bmatrix} Q^i_1 \\ Q^i_2 \end{bmatrix}$.

On the other hand, we choose the following reduction

$$h^\alpha_B = \begin{bmatrix} N\tilde{\eta}_l & 0 \\ 0 & -N\tilde{\eta}_s \end{bmatrix}. \quad (92)$$

This reduction leads to the standard symmetric hyperbolic Maxwell evolution equations $\epsilon^l_{1,2}$ when it is contracted with $\begin{bmatrix} Q^i_1 \\ Q^i_2 \end{bmatrix}$. Combining the last two expressions, we conclude that

$$\begin{bmatrix} h^\alpha_B \\ C^\Delta_0 \end{bmatrix} E^\beta = \begin{bmatrix} N\tilde{\eta}_l & 0 \\ 0 & -N\tilde{\eta}_s \\ \tilde{n}_s & 0 \\ 0 & -\tilde{n}_s \end{bmatrix} \begin{bmatrix} Q^i_1 \\ Q^i_2 \end{bmatrix} = \begin{bmatrix} \tilde{\epsilon}_1 \\ \tilde{\epsilon}_2 \\ \psi_1 \\ \psi_2 \end{bmatrix}. \quad (91)$$

The inverse of $\begin{bmatrix} h^\alpha_B \\ C^\Delta_0 \end{bmatrix}$ has the following form

$$[\gamma^A_{\alpha} h^A_\Delta] = \begin{bmatrix} \frac{1}{N} \tilde{\eta}_l & 0 & -\tilde{m}^s & 0 \\ 0 & -\frac{1}{N} \tilde{\eta}_s & 0 & \tilde{m}^s \end{bmatrix}. \quad (92)$$

and, of course, satisfies that

$$[\gamma^A_{\alpha} h^A_\Delta] \begin{bmatrix} h^\alpha_B \\ C^\Delta_0 \end{bmatrix} = \delta^A_B,$$

$$\begin{bmatrix} \frac{1}{N} \tilde{\eta}_l & 0 & -\tilde{m}^s & 0 \\ 0 & -\frac{1}{N} \tilde{\eta}_s & 0 & \tilde{m}^s \\ N\tilde{\eta}_l & 0 & -N\tilde{\eta}_s \\ \tilde{n}_s & 0 & -\tilde{n}_s \end{bmatrix} = \begin{bmatrix} \delta^l_1 & 0 \\ 0 & \delta^l_2 \end{bmatrix}. \quad (93)$$
Using this expression and equations (93) and (91), we conclude that
\[ \nabla_d (C_B^{\alpha \beta} E^\beta) = \nabla_d \left( \left[ C_A^{\Delta d} \left[ \mathfrak{Y}_{110}^{\alpha_0} \ h^\alpha \right] \right] \left( \begin{array}{c} h_B^{\alpha_0} \\ C_B^{\alpha_0} \end{array} \right) E^\beta \right) \]
\[ \times \nabla_d \left( \begin{array}{cc} -N e_2^d & 0 \\ 0 & N e_2^d \end{array} \right) \left[ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right] \]
\[ = \nabla_d \left( \begin{array}{cc} -\tilde{e}_2^d & 0 \\ 0 & -\tilde{e}_2^d \end{array} \right) \left[ \begin{array}{c} \tilde{m}^e \\ \tilde{m}^e \end{array} \right] \left[ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right], \]

which allows to rewrite the identities (78) as
\[ 0 = L_F \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) + \left[ -N - N \tilde{m}^d L_{1d} \ N \tilde{m}^d L_{2d} \ -N K \right] \left[ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right] \]
\[ + \left[ -N \nabla_d \left( \frac{1}{N} e_1^d \right) + L_{1d} \tilde{e}_1^d - L_{2d} \tilde{e}_2^d \right] \left[ \begin{array}{c} \tilde{\eta}_1^d \\ \tilde{\eta}_1^d \end{array} \right] \left[ \begin{array}{c} \tilde{\eta}_2^d \\ \tilde{\eta}_2^d \end{array} \right]. \]

On-shell (i.e., when \( e_{1,2}^d = 0 \)), these equations lead to the SS
\[ 0 = L_F \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) + \left[ -N - N \tilde{m}^d L_{1d} \ N \tilde{m}^d L_{2d} \ -N K \right] \left[ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right] \]
\[ = L_F \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) + \left[ -N - N \tilde{m}^d L_{1d} \ N \tilde{m}^d L_{2d} \ -N K \right] \left[ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right] \]
\[ \text{(94)} \]

Clearly (since there are no spatial derivatives of \( \psi_{1,2} \)) this system is SH, so it has a unique solution for a given initial data. Since the initial data is chosen such that \( \psi_{1,2} |_{\Sigma_0} = 0 \) and \( \psi_{1,2} = 0 \) is a solution of the SS, we conclude (by the uniqueness of the solutions) that the constraints are preserved during the evolutions \( e_{1,2}^d = 0 \).

We now consider the relationship between the principal symbols of the evolution equations and the principal symbols of the constraints equations.

Using equations (88), (89), (90) and (92), we obtain the matrices \( \left[ \begin{array}{c} h_B^{\alpha_0} \\ C_B^{\alpha_0} \end{array} \right] \mathfrak{Y}_{/\beta} \) and \( l_e C_A^{\Delta e} \left[ \begin{array}{c} \mathfrak{Y}_{10}^{\alpha_0} \\ h^\alpha \end{array} \right] \), they are
\[ \left[ \begin{array}{c} h_B^{\alpha_0} \\ C_B^{\alpha_0} \end{array} \right] \mathfrak{Y}_{/\beta} l_e \phi^\beta = \left[ \begin{array}{c} \tilde{\eta}_0^\beta p^\beta \\ -N \tilde{e}_0^{\beta d} \tilde{p}_d \tilde{m}^d \ -\tilde{\eta}_0^\beta p^\beta \ 0 \end{array} \right] \left[ \begin{array}{c} \phi^\beta \end{array} \right], \]
\[ \text{(95)} \]

and
\[ l_e C_A^{\Delta e} \left[ \begin{array}{c} \mathfrak{Y}_{10}^{\alpha_0} \\ h^\alpha \end{array} \right] = l_e \left[ \begin{array}{cc} -\tilde{\eta}_0^\beta & 0 \\ 0 & -\tilde{\eta}_0^\beta \end{array} \right] \left[ \begin{array}{c} \tilde{m}^e \\ \tilde{m}^e \end{array} \right]. \]

Recalling that
\[ l_e C_A^{\Delta e} \left[ \begin{array}{c} \mathfrak{Y}_{10}^{\alpha_0} \\ h^\alpha \end{array} \right] \left[ \begin{array}{c} h_B^{\alpha_0} \\ C_B^{\alpha_0} \end{array} \right] \mathfrak{Y}_{/\beta} l_e \phi^\beta = 0, \]

35
we concluded
\[
L_c \begin{bmatrix}
-\tilde{\eta}_l^\alpha & 0 \\
0 & -\tilde{\eta}_l^\alpha
\end{bmatrix} \begin{bmatrix}
\tilde{\eta}_b^\mu p^\theta \\
-\tilde{N}_e^{qld} b \tilde{\eta}_d^\theta
\end{bmatrix} l_q
\]
\[
= L_c \begin{bmatrix}
p^\phi & 0 \\
0 & p^\phi
\end{bmatrix} \begin{bmatrix}
-\tilde{\eta}_b^\mu \\
0
\end{bmatrix} l_q.
\]

This expression is exactly equation (57), where
\[
\begin{align*}
h_{\alpha}^{\alpha} & = \begin{bmatrix}
\tilde{\eta}_l^\alpha & 0 \\
0 & -\tilde{\eta}_l^\alpha
\end{bmatrix} \\
l_c C_{\lambda}^{\alpha} h_{\lambda}^\alpha & = \begin{bmatrix}
\tilde{\eta}_b^\mu & 0 \\
0 & \tilde{\eta}_b^\mu
\end{bmatrix}
\end{align*}
\]
is the principal symbol of
\[
\begin{bmatrix}
E^\alpha \\
\eta_d^\alpha
\end{bmatrix},
\]

is the principal symbol of the SS and
\[
C_{A}^{\alpha} h_{\lambda}^\alpha I_q = \begin{bmatrix}
\tilde{\eta}_b^\mu & 0 \\
0 & \tilde{\eta}_b^\mu
\end{bmatrix}
\]
is obtained from (97). For each \(k_q\), we have
\[
\dim \left( \ker \left( C_{B}^{\alpha} \eta_d^{\alpha} l_q \right) \right) = 0,
\]

\[
\dim \left( \ker \left( C_{B}^{\alpha} \eta_d^{\alpha} l_q \right) \right) = 2.
\]
These eigenvalues have the associated eigenvectors that can be seen from (94). Its Kronecker decomposition is given by

\[
\text{dim}\left(\text{right}_\ker\left(C_{h\mu}^{\lambda}M_{\alpha}^{\beta}k_{q}\right)\right) = 4.
\]

(101)

This means that there are not Geroch fields \(M_{\Gamma}^{\lambda}\) such that \(l_{\Gamma}M_{\Gamma}^{\lambda}C_{h\mu}^{\lambda}M_{\alpha}^{\beta}k_{q} = 0\) and therefore the condition (v) of theorem 12 is satisfied.

This result allows us to give the Kronecker structure associated with the Maxwell equation, i.e.

\[
2 \times J_{1}\left(N\sqrt{k,k} - \beta k\right), 2 \times J_{1}\left(-N\sqrt{k,k} - \beta k\right), 2 \times L_{I}^{0}.
\]

is the Kronecker structure of the pencil (97).

The \(2 \times L_{I}^{0}\) blocks are justified by (99) and (100) as explained in subsubsection 5.1.4. The Jordan part is justified by giving explicitly the generalized eigenvectors

\[
\left(\delta \phi_{\lambda_{1}}^{1}\right)^{\beta} = \frac{1}{\sqrt{(v,v)}} \frac{1}{1}, \quad \left(\delta \phi_{\lambda_{1}}^{2}\right)^{\beta} = \left[\begin{array}{c}
1 \\
-\sqrt{(v,v)} \v^b
\end{array}\right],
\]

\[
\left(\delta \phi_{\lambda_{2}}^{1}\right)^{\beta} = \left[\begin{array}{c}
1 \\
\v^b
\end{array}\right], \quad \left(\delta \phi_{\lambda_{2}}^{2}\right)^{\beta} = \left[\begin{array}{c}
1 \\
-\sqrt{(v,v)} \v^b
\end{array}\right],
\]

where \(v^d\) and \(w^d\) are linearly independent vectors, defined by

\[
2v^b w^d = \v^b k_d, \quad v.k = v.w = w.k = 0.
\]

They are associated to the generalized eigenvalues \(\lambda_{1} = N\sqrt{k,k} - \beta k\) and \(\lambda_{2} = -N\sqrt{k,k} - \beta k\). These eigenvalues were called the 'physical' eigenvalues in subsection 5.1.

Now, let us study the characteristic structure of \(h_{\alpha}^{\alpha}N_{\alpha}^{\mu}l_q\), whose explicit expression is

\[
h_{\alpha}^{\alpha}N_{\alpha}^{\mu}l_q \phi^\alpha = \left[\begin{array}{c}
\eta_{\mu} - \left(\lambda - \left(\beta k\right)\right) \\
-\v_{\alpha}^{\mu} \\
-\v_{\alpha}^{\mu}
\end{array}\right] E_{\alpha}^{\beta}.
\]

Since \(\left(\delta \phi_{\lambda_{1,2}}\right)^{\beta}\) are the generalized eigenvalues and eigenvectors of \(h_{\alpha}^{\alpha}N_{\alpha}^{\mu}l_q\), they are also the eigenvalues and eigenvectors of \(h_{\alpha}^{\alpha}N_{\alpha}^{\mu}l_q\). Moreover, \(h_{\alpha}^{\alpha}N_{\alpha}^{\mu}l_q\) has two additional eigenvalues \(\pi_{1,2}(k) = -\beta k\), these were called the 'constraints 1' eigenvalues in subsection 5.1. These eigenvalues have the associated eigenvectors

\[
\left(\delta \phi_{\pi_{1}}\right)^{\beta} = \left[\begin{array}{c}
k^0 \\
0
\end{array}\right], \quad \left(\delta \phi_{\pi_{2}}\right)^{\beta} = \left[\begin{array}{c}
0 \\
k^b
\end{array}\right].
\]

To conclude this subsection, let us now study the Kronecker decomposition of the pencil \(C_{h,k}^{\lambda}h_{\gamma}^{\lambda}l_q\) (see equation (98)). This is a square pencil and it is associated to the constraints as can be seen from (94). Its Kronecker decomposition is given by

\[
2 \times J_{1}(-\beta k).
\]
As it was shown in subsubsection 5.1.6, the generalized eigenvectors of this pencil are obtained by projecting \( \{ (\delta \phi_1)^\beta, (\delta \phi_2)^\beta \} \) with \( C_{\alpha}^\beta \Gamma^\beta_{\gamma} k_q \), that is,

\[
[\delta \psi_1^\Gamma, \delta \psi_2^\Gamma] := C_{\alpha}^\beta \Gamma^\beta_{\gamma} l_q \left[ (\delta \phi_1)^\gamma, (\delta \phi_2)^\gamma \right],
\]

\[
= \begin{bmatrix} k_b & 0 \\ 0 & k_b \end{bmatrix} \begin{bmatrix} k_b^\beta & 0 \\ 0 & k_b^\beta \end{bmatrix},
\]

\[
= \begin{bmatrix} (k. k) & 0 \\ 0 & (k. k) \end{bmatrix}.
\]

Where the eigenvectors are the columns of this matrix and they are associated to the 'constraints 1' eigenvalues \( \pi_1,2,3(k) = -\beta \cdot k \). This result follows from equation (57) and can be easily checked, since \( C_{\alpha}^\beta \Delta^\gamma_{\alpha} l_q = 0 \) when \( \lambda = -\beta \cdot k \).

### 6.2. Wave equation

Consider the wave equation

\[
g^{ab} \nabla_a \nabla_b \phi = 0.
\]  

(102)

We lead this equation to first order in derivatives. The wave equation in first-order is

\[
E = 0, \quad E_b = 0, \quad E_{ab} = 0
\]

(103)

where we have defined

\[
u_b := \nabla_b \phi,
\]

and

\[
E := g^{ab} \nabla_a \nu_b,
\]

\[
E_b := \nabla_b \phi - u_b,
\]

\[
E_{ab} := \nabla_{(a} \nu_{b)}.
\]

Notice that \( E_{ab} \) is obtained from \( E_b \) by taking an antisymmetric derivative, i.e. \( \nabla_{(a} E_{b)} = -E_{ab} \). Additionally, there is another identity for \( E_{ab} \), this is \( \nabla_{(a} E_{ab)} = 0 \).\(^8\) Thus, the off-shell identities of the system are

\[
0 = \nabla_{(a} \left( \delta^b_{(a} \delta^d_{b)} E_{b)} \right) + E_{ab},
\]

(104)

\[
0 = \nabla_{(a} \left( \delta^b_{(a} \delta^d_{b)} E_{ab} \right).
\]

(105)

\(^8\)To show this result, you need to use the first Bianchi identity.
The following four projections of these equations

\[
0 = \begin{bmatrix}
2N\tilde{m}^b_i \left( \nabla_j \left( \delta^j_i \delta^{[g} E_{[g} \right) + E_{ab} \right) \\
3N\tilde{m}^b_i \left( \nabla_j \left( \delta^j_i \delta^{[g} E_{[g} \right) + E_{ab} \right) \\
\tilde{m}^b_i \nabla_j \left( \delta^j_i \delta^{[g} E_{[g} \right) + E_{ab} \right) \\
\tilde{m}^b_i \nabla_j \left( \delta^j_i \delta^{[g} E_{[g} \right) + E_{ab} \right)
\end{bmatrix},
\]

can be rewritten as follows

\[
0 = \nabla_z \left( \begin{bmatrix}
0 & 2N\tilde{m}^b_i \\
0 & 3N\tilde{m}^b_i \\
0 & \tilde{m}^b_i \\
0 & \tilde{m}^b_i
\end{bmatrix} \begin{bmatrix}
0 \\
E_g \\
E_{ab}
\end{bmatrix} \right) + \begin{bmatrix}
0 & -\nabla_z (2N\tilde{m}^b_i) \\
0 & -\nabla_z (3N\tilde{m}^b_i) \\
0 & -\nabla_z (\tilde{m}^b_i) \\
0 & -\nabla_z (\tilde{m}^b_i)
\end{bmatrix} \begin{bmatrix}
0 & 2N\tilde{m}^a_i \tilde{e}_b \\
0 & \tilde{e}_b \\
0 & \tilde{e}_b \\
0 & \tilde{e}_b
\end{bmatrix} \begin{bmatrix}
E \\
E_g \\
E_{ab}
\end{bmatrix}.
\] (106)

As we will show, these expressions are exactly the evolution equations of the constraints and the constraints of the constraints of the system. The latter is associated to the Geroch fields $M^\Gamma_\Delta$. We now introduce new variables

\[
\tilde{u}^0 := \tilde{n}_b u^b, \\
\tilde{u}_d := \tilde{n}_b u^b,
\]

where $\tilde{u}_d$ results from projecting $u^b$ onto $\Sigma_t$. Since we are interested in describing the system with the variables $(\tilde{u}^0, \phi, \tilde{u}_a)$, we rewrite the equation (103) as follows

\[
E = -\frac{1}{N} \tilde{e}_1, \\
E_c = -\frac{1}{N} \tilde{n}_b \tilde{e}_2 + \psi_{1c}, \\
E_{ac} = -\frac{1}{N} \tilde{e}_{[a} \tilde{n}_{c]} + \psi_{2ac},
\]

where

\[
\tilde{e}_1 := \mathcal{L}_\mu \tilde{u}^0 - ND_\mu \tilde{u}^d - N(\tilde{u}^a S_a) - N\tilde{u}^0 K, \\
\tilde{e}_2 := \mathcal{L}_\phi - N\tilde{u}^0, \\
\tilde{e}_{3a} := \mathcal{L}_\mu \tilde{u}_a - ND_\mu \tilde{u}^0 - N\left( \tilde{u}^a K_{ac} + \tilde{u}^0 S_a \right) - N\tilde{u}^0 K_{fa}, \\
\psi_{1c} := D_f \phi - \tilde{u}_f, \\
\psi_{2ac} := D_{[a} \tilde{u}_{c]}.
\]

Notice that (110)–(114) are projected onto $\Sigma_t$, where $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_{3a}$ are the evolution equations for our variables and $\psi_{1c}$, $\psi_{2ac}$ are the constraints of the system.
Using the expressions (107), (108) and (109), we obtain the principal symbol of the system
\[
\begin{bmatrix}
E \\
E_c \\
E_{ac}
\end{bmatrix} \approx \tilde{\Omega}_V^{\alpha} \nabla q \phi^3
\]
\[
= \begin{bmatrix}
-\tilde{m}^q & 0 & \tilde{v}^{\beta\nu} \\
0 & \delta^q_k & 0 \\
-\tilde{n}_\nu \tilde{\eta}^{\alpha}_{\nu} & 0 & -\delta^q_k \tilde{\eta}^{\alpha}_{\nu}
\end{bmatrix}
\nabla_q \begin{bmatrix}
\tilde{u}^0 \\
\phi \\
\tilde{u}_\nu
\end{bmatrix}
\]

and the Geroch fields \( C^\Gamma_A \) and \( M_A \)
\[
\begin{bmatrix}
C^\Gamma_A \\
M_A
\end{bmatrix} E^A = \begin{bmatrix}
0 & 2N \tilde{m}^k \tilde{r}_k^\nu & 0 \\
0 & 0 & 3N \tilde{m}^k \tilde{r}_k^\nu \tilde{r}_l^\mu \\
0 & \tilde{r}_k^\nu \tilde{r}_l^\mu & 0 \\
0 & 0 & \tilde{r}_k^\nu \tilde{r}_l^\mu
\end{bmatrix}
\begin{bmatrix}
E \\
E_c \\
E_{ac}
\end{bmatrix}.
\]

(115)

The first two lines are associated with \( C^\Gamma_A \) and the next two with \( M_A \). Notice that
\[
\begin{bmatrix}
C^\Gamma_A \\
M_A
\end{bmatrix} n_A = \begin{bmatrix}
0 & \tilde{r}_k^\nu & 0 \\
0 & 0 & \tilde{r}_k^\nu \tilde{r}_l^\mu \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where \( M_A^{\tilde{A}} := M_A n_A = 0 \) and
\[
C_A^{\tilde{A}} := C_A n_A = \begin{bmatrix}
0 & \tilde{r}_k^\nu & 0 \\
0 & 0 & \tilde{r}_k^\nu \tilde{r}_l^\mu \\
0 & 0 & 0
\end{bmatrix}.
\]

The constraints \( \psi_{1c} \) and \( \psi_{2ac} \) are obtained by contracting this last expression with \( E^A \).

On the other hand, we choose the following reduction \( h_A^{\alpha} \), such that
\[
h_A^{\alpha} E^A = \begin{bmatrix}
-N & 0 & 0 \\
0 & N \tilde{m}^k & 0 \\
0 & 0 & -2N \tilde{r}_k^\nu \tilde{r}_l^\mu
\end{bmatrix}
\begin{bmatrix}
E \\
E_c \\
E_{ac}
\end{bmatrix}.
\]

which gives the (symmetric hyperbolic) evolution equations \( \tilde{e}_{1,2,3} \).

Combining the last two results, we have that
\[
\begin{bmatrix}
h_A^{\alpha} \\
C_A^{\tilde{A}}
\end{bmatrix} E^A = \begin{bmatrix}
-N & 0 & 0 \\
0 & N \tilde{m}^k & 0 \\
0 & 0 & -2N \tilde{r}_k^\nu \tilde{r}_l^\mu
\end{bmatrix}
\begin{bmatrix}
E \\
E_c \\
E_{ac}
\end{bmatrix} = \begin{bmatrix}
\tilde{e}_1 \\
\tilde{e}_2 \\
\tilde{e}_3
\end{bmatrix}.
\]

(116)

The principal symbol \( \begin{bmatrix}
h_A^{\alpha} \\
C_A^{\tilde{A}}
\end{bmatrix} \tilde{\Omega}_V^{\alpha} \nabla_q \phi \) of the latter expression is
\[
\begin{bmatrix}
h_A^{\alpha} \\
C_A^{\tilde{A}}
\end{bmatrix} \tilde{\Omega}_V^{\alpha} \nabla_q \begin{bmatrix}
\tilde{u}^0 \\
\phi \\
\tilde{u}_\nu
\end{bmatrix} = \begin{bmatrix}
N \tilde{m}^k & 0 & -N \tilde{r}_k^\nu \\
0 & N \tilde{m}^k & 0 \\
-\tilde{n}_\nu \tilde{\eta}^{\alpha}_{\nu} & 0 & N \tilde{r}_k^\nu \tilde{r}_l^\mu \\
0 & \tilde{r}_k^\nu & 0 \\
0 & 0 & \tilde{r}_k^\nu \tilde{r}_l^\mu
\end{bmatrix}
\nabla_q \begin{bmatrix}
\tilde{u}^0 \\
\phi \\
\tilde{u}_\nu
\end{bmatrix}.
\]

(117)
On the other hand, the inverse of \( \left[ \begin{array}{c} h^0_\beta \\ C_{\beta 0} \end{array} \right] \) has the following form

\[
\left[ \begin{array}{c} h^0_\alpha \\ C_{\alpha 0} \end{array} \right]^{-1} = \begin{bmatrix}
\frac{-1}{N} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{N} \eta_c & 0 & \tilde{\eta}_c^w & 0 \\
0 & 0 & \frac{1}{N} \eta^\beta_\mu \tilde{a}_\lambda & 0 & \tilde{\eta}^\nu_\beta \tilde{\eta}_\nu^{\lambda} 
\end{bmatrix},
\]

and satisfy that

\[
\left[ \begin{array}{c} \tilde{\gamma}^\mu_0 \\ \tilde{C}_{\mu 0} \end{array} \right] \left[ \begin{array}{c} h^0_\beta \\ C_{\beta 0} \end{array} \right] = \delta^\beta_\gamma,
\]
i.e.

\[
\begin{bmatrix}
\frac{-1}{N} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{N} \tilde{\eta}_c & 0 & \tilde{\eta}_c^w & 0 \\
0 & 0 & \frac{1}{N} \eta^\beta_\mu \tilde{a}_\lambda & 0 & \tilde{\eta}_c^w \tilde{\eta}_\nu^{\lambda} 
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \delta^c_c & 0 \\
0 & 0 & \delta^c_\gamma 
\end{bmatrix}.
\]

Using the above expressions, we obtain

\[
\begin{bmatrix}
C_{\alpha 0}^{\Delta d} \\
M_{\alpha 0}^{\Gamma} \end{bmatrix} E^A = \left( \begin{bmatrix}
C_{\alpha 0}^{\Delta d} \\
M_{\alpha 0}^{\Gamma} \end{bmatrix} \left[ \begin{array}{c} h^0_\beta \\ C_{\beta 0} \end{array} \right] \left( \begin{bmatrix}
\tilde{h}^0_\beta \\
\tilde{C}_{\beta 0} \end{bmatrix} E^B \right) \right),
\]

\[
= \begin{bmatrix}
0 & -\tilde{\eta}_c & 0 & N\tilde{m}^c \tilde{\eta}_c^w & 0 \\
0 & 0 & \tilde{\eta}_c^w \tilde{\eta}_\nu^{\lambda} & 0 & N\tilde{m}^c \tilde{\eta}_c^w \tilde{\eta}_\nu^{\lambda} \\
0 & 0 & 0 & \tilde{\eta}_c^w \tilde{\eta}_\nu^{\lambda} & 0 \\
0 & 0 & 0 & 0 & \tilde{\eta}_c^w \tilde{\eta}_\nu^{\lambda} 
\end{bmatrix} \begin{bmatrix}
\tilde{e}^1_c \\
\tilde{e}_2 \\
\tilde{c}_{3\gamma} \\
\tilde{\psi}_{1\nu} \\
\tilde{\psi}_{2\gamma}
\end{bmatrix}.
\]

Furthermore, \( \begin{bmatrix}
C_{\alpha 0}^{\Delta d} \\
M_{\alpha 0}^{\Gamma} \end{bmatrix} E^A \) can be written as (115) and its divergence as (106). This leads to the following identity

\[
0 = \nabla^2 \left( \begin{bmatrix}
0 & -\tilde{\eta}_c & 0 & N\tilde{m}^c \tilde{\eta}_c^w & 0 \\
0 & 0 & \tilde{\eta}_c^w \tilde{\eta}_\nu^{\lambda} & 0 & N\tilde{m}^c \tilde{\eta}_c^w \tilde{\eta}_\nu^{\lambda} \\
0 & 0 & 0 & \tilde{\eta}_c^w \tilde{\eta}_\nu^{\lambda} & 0 \\
0 & 0 & 0 & 0 & \tilde{\eta}_c^w \tilde{\eta}_\nu^{\lambda} 
\end{bmatrix} \begin{bmatrix}
\tilde{e}^1_c \\
\tilde{e}_2 \\
\tilde{c}_{3\gamma} \\
\tilde{\psi}_{1\nu} \\
\tilde{\psi}_{2\gamma}
\end{bmatrix} \right)
\]

\[
+ \begin{bmatrix}
0 & -\nabla^2 \left( 2N\tilde{m}^c \tilde{\eta}_c^w \right) \\
0 & -\nabla^2 \left( 2N\tilde{m}^c \tilde{\eta}_c^w \right) \\
0 & -\nabla^2 \left( 2N\tilde{m}^c \tilde{\eta}_c^w \right) \\
0 & -\nabla^2 \left( 2N\tilde{m}^c \tilde{\eta}_c^w \right) 
\end{bmatrix} \times \begin{bmatrix}
-\frac{1}{N} \tilde{e}^1_c \\
-\frac{1}{N} \tilde{e}_2 \\
\frac{1}{N} \tilde{c}_{3\gamma} \\
\frac{1}{N} \tilde{\psi}_{1\nu} \\
\frac{1}{N} \tilde{\psi}_{2\gamma}
\end{bmatrix}.
\]
where equations (107), (108) and (109) have been used. Finally, this expression can be rewritten as

\[
0 = \begin{bmatrix}
\tilde{\eta}^q_{t^p} & 0 & 0 \\
0 & \tilde{\eta}^q_{\eta^p} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \mathcal{L}_p \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} D_f \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \mathcal{L}_p \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} D_f \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \mathcal{L}_p \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} D_f \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix}
\]

In the on-shell case \((e_1, e_2) = 0\), these equations are the SS (1st and 2nd line) and the constraints of the constraints (3rd and 4th line) which are trivially satisfied in the evolution. Namely,

\[
0 = \begin{bmatrix}
\tilde{\eta}^q_{t^p} & 0 & 0 \\
0 & \tilde{\eta}^q_{\eta^p} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \mathcal{L}_p \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} D_f \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \mathcal{L}_p \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} D_f \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \mathcal{L}_p \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} D_f \begin{bmatrix}
\psi_{1q} \\
\psi_{2q} \\
\end{bmatrix}
\]

(120)

where \(\mathcal{L}_p = \partial_t - \mathcal{L}_\beta\), since we are considering the coordinates \((t, x^i)\). We note that the evolution equations obtained are quite simple. However, by adding to these evolution equations terms proportional to the constraints, one can modify this system and obtain a family of SSs. This family is obtained by contracting the expression (120) with the following reduction

\[
N^\Gamma_{\Delta} = \begin{bmatrix}
\tilde{\eta}_w & 0 & N^w_{1w} & N^w_{2w} \\
\tilde{\eta}_w & 0 & N^w_{1w} & N^w_{2w} \\
\tilde{\eta}_m & 0 & N^w_{1m} & N^w_{2m} \\
\tilde{\eta}_m & 0 & N^w_{1m} & N^w_{2m} \\
\end{bmatrix},
\]

where \(N^w_{1w}, N^w_{2w}, N^w_{1m},\) and \(N^w_{2m}\) can be freely chosen. Of course, different choices of \(N^w_{\Delta}\) give rise to ill/well-posed evolution equations.

Let us now study the characteristic structures of the system and the SS. In other words, we will study the Kronecker decomposition of the pencils \(\begin{bmatrix} h^\alpha \mu^\alpha q_{\alpha} \\
C^\alpha \gamma^\alpha q_{\alpha} \\
\end{bmatrix} l_q \) and \(\begin{bmatrix} C^\alpha \gamma^\alpha q_{\alpha} \\
M^\alpha \gamma^\alpha q_{\alpha} \\
\end{bmatrix} l_d \) with \(l_q = -\lambda n_a + k_a\) and \(k_a(\partial_j)^a = 0\). In their pullback version to \(\Sigma_t\), these pencils are

\[
\begin{bmatrix}
h^\alpha \gamma^\alpha q_{\alpha} \\
C^\alpha \gamma^\alpha q_{\alpha} \\
\end{bmatrix} l_q \phi^a = \begin{bmatrix}
(-\lambda - (\beta, k)) & 0 & -Nk^w \\
0 & (-\lambda - (\beta, k)) & 0 \\
-Nk_s & 0 & (-\lambda - (\beta, k))\tilde{\eta}_w \\
0 & k_s & 0 \\
0 & 0 & k_s\tilde{\eta}_w \\
\end{bmatrix} \begin{bmatrix}
\phi^a \\
\eta^a \\
\end{bmatrix},
\]

(121)
The Jordan part is justified by giving explicitly the generalized eigenvectors from 1 to 3. We also note that

\[
\begin{bmatrix}
\eta^w
\end{bmatrix}
\]

This result allows us to give the Kronecker structure associated to the wave equation, i.e., the Kronecker structure of the pencil (121). It is

\[
J_1 \left( N \sqrt{k} \cdot k - \beta \cdot k \right), J_1 \left( -N \sqrt{k} \cdot k - \beta \cdot k \right), 3 \times L_1^T, 3 \times L_0^T.
\]

The 3 \times L_1^T, 3 \times L_0^T blocks are justified by (123) and (124) as explained in subsubsection 5.1.4.

The Jordan part is justified by giving explicitly the generalized eigenvectors

\[
\begin{bmatrix}
1 & 0 \\
0 & k_w
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
0 & k_w
\end{bmatrix}
\]

They are associated to the generalized eigenvalues \(\lambda_1 = N \sqrt{k} \cdot k - \beta \cdot k\) and \(\lambda_2 = -N \sqrt{k} \cdot k - \beta \cdot k\). These eigenvalues were called the ‘physical’ eigenvalues in subsection 5.1.
Now, let us study the characteristic structure of $h^\alpha A N Aq\phi^\alpha$, whose explicit expression is

$$h^\alpha A N Aq\phi^\alpha = \begin{bmatrix} -\lambda - (\beta, k) & 0 & -Nk^w \\ 0 & -\lambda - (\beta, k) & 0 \\ -Nk_s & 0 & -\lambda - (\beta, k) \end{bmatrix} \begin{bmatrix} \tilde{\eta}_w \\ \phi \\ \tilde{\mu}^q \end{bmatrix}.$$ 

Since $(\lambda_1, 2, (\delta\phi^1, 2, \lambda_2))$ are the generalized eigenvalues and eigenvectors of $[h^\alpha A N Aq\phi^\alpha]_{lq}$, they are also the eigenvalues and eigenvectors of $h^\alpha A N Aq\phi^\alpha$. Moreover, $h^\alpha A N Aq\phi^\alpha$ has three more eigenvalues $\pi_{1,2,3}(k) = -\beta, k$, these were called the 'constraints 1' eigenvalues in subsection 5.1. These eigenvalues have the associated eigenvectors

$$\delta\phi_1, 2, \lambda = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \delta\phi_2, 3, \lambda = \begin{bmatrix} 0 \\ 0 \\ v_1 \end{bmatrix}, \delta\phi_3, 3, \lambda = \begin{bmatrix} 0 \\ 0 \\ v_2 \end{bmatrix}$$

where $v_1, v_2$ are linearly independent vectors, and they are defined by the following conditions

$$(v_1, v_2) = (v_{1,2}, k) = (v_{1,2}, \tilde{\mu}) = 0.$$ 

Let us now study the Kronecker decomposition of the pencil $[C^\lambda C^\Gamma A hA a \Delta q]_{lq}$ (see equation (122)). This pencil is associated to the constraints as can be seen from (119) and (120); and its Kronecker decomposition is given by

$$3 \times J_1(-\beta, k), 3 \times L_1^T, 1 \times L_0^T,$$

as we will show below.

We begin by studying the Jordan blocks. As it was shown in subsection 5.1.6, the generalized eigenvectors of this pencil are obtained by projecting $\{\delta\phi_1, 2, \lambda, \delta\phi_2, 3, \lambda, \delta\phi_3, 3, \lambda\}$ with $C^\Gamma B^\beta k_q$, that is,

$$[\delta\psi_1, 2, \delta\psi_2, \delta\psi_3] := C^\Gamma B^\beta k_q [\delta\phi_1, 2, \lambda, \delta\phi_2, 3, \lambda, \delta\phi_3, 3, \lambda],$$

$$= \begin{bmatrix} 0 & k_s & 0 & 0 & 0 \\ 0 & 0 & k_1 \tilde{\eta}_1 & 1 & 0 & 0 \\ 0 & k_1 v_1 & 0 & v_1 & v_2 \end{bmatrix},$$

$$= k_s \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ v_1 \end{bmatrix} = k_s \begin{bmatrix} v_1 \end{bmatrix}.$$ 

Where the eigenvectors are the columns of this matrix and they are associated to the eigenvalues the 'constraints 1' $\pi_{1,2,3}(k) = -\beta, k$.

To complete the Kronecker decomposition, we need to find the

$$\dim \left( \ker \left( M^\Delta h^\Lambda A k_q \right) \right),$$

where $M^\Delta h^\Lambda A k_q$ is given by

$$M^\Delta h^\Lambda A k_q \psi^\Delta = \begin{bmatrix} \tilde{\eta}_w \tilde{\eta}_r \psi^w \\ 0 \end{bmatrix} = k \begin{bmatrix} \psi^w \\ \psi^w \end{bmatrix}$$.
Notice that
\[ \text{span}(\tilde{\eta}^f, f, g) = \text{left}_\Delta \ker \left( M_A^{\Delta^2} h^A \right) \]
therefore
\[ \dim \left( \text{left}_\Delta \ker \left( M_A^{\Delta^2} h^A \right) \right) = 1. \]

Using this result and the expressions (123) and (70), we conclude that the rest of the Kronecker decomposition is \( 3 \times L^T_1, 1 \times L^T_0 \).

Finally, we give the basis that diagonalizes matrix \( C^\Gamma_A h^A \psi = \psi^{1w} \).

This matrix
\[ C^T_A h^A \psi = \begin{bmatrix} \eta^w_{\psi}(-\lambda - (\beta, k)) & 0 & \eta^w_{\psi}(-\lambda - (\beta, k)) \\ 0 & \psi^{1w} \end{bmatrix} \]
can be read from (122). By construction, we know that \( \{ \delta \psi^T_{\pi^1}, \delta \psi^T_{\pi^2}, \delta \psi^T_{\pi^3} \} \) are eigenvectors associated to the eigenvalues \( \pi_{1,2,3}(k) = -\beta, k \). We still need to find three more eigenvectors associated to the ‘constraints 2’ eigenvalues \( \rho_{1,2,3}(k) \). These eigenvectors \( \{ \delta \psi^T_{\rho^1}, \delta \psi^T_{\rho^2}, \delta \psi^T_{\rho^3} \} \) are any three vectors, linearly independent from \( \{ \delta \psi^T_{\pi^1}, \delta \psi^T_{\pi^2}, \delta \psi^T_{\pi^3} \} \).

7. Conclusions and discussion

In this paper, we have considered generic systems of first-order PDEs that include differential constraints. We have shown sufficient conditions for these systems to have first-order partial differential SS with a SH evolution. This guarantees the constraint preservation.

We have shown that if the constraints of the system are defined by the Geroch fields \( C^\Gamma_A \) and the system admits the integrability conditions (25) and (26), then the SS exists and it is a set of first-order PDEs. Furthermore, we have shown that when the system only admits Geroch fields \( C^\Gamma_A \) and \( M_A^{\Delta^2} \), it implies that the Kronecker structures of the principal symbol, of the system and of the SS, does not include \( L^T_m \) blocks with \( m \geq 2 \). Since most-known physical systems have these kind of Geroch fields, we conclude that they only have \( L^T_1 \)-blocks and vanishing rows in their Kronecker decomposition. On the other hand, this connection between Geroch fields and the Kronecker structure indicates a possible extension to first-order PDEs, of the classifications presented in the cases of ordinary differential equations [20] and algebraic differential equations [32]. In these latter classifications, the Kronecker structure of the principal symbol is used to find the integrability conditions (as we have done here) and the solutions of the system. Currently, we are working on extending the results presented here to systems that admit other kinds of Geroch fields.

The study of SH of the SS is performed in the case of constant coefficients and presented as a continuation of the work [3]. We have given a complete analysis of the characteristic structure of the system, showing how the propagation velocities of the physical fields and of the constraints can be chosen. As in [3], the analysis is algebraic and pseudo-differential, so the possible non analyticity in the evolution equations as well as in the SSs may introduce causality issues. However, the steps of the proofs presented here can be readapted to each physical system (including the quasi-linear ones) avoiding the non analytical pseudo-differential reductions. This procedure will/may come at the cost of less freedom in the choice of the propagation velocities of the constraints.

With the new tools developed here, it seems natural to continue with the study of boundary conditions that guarantee the constraint preservation. We have made great progress in
this direction and we are currently writing an article where we generalize the ideas of [15, 16, 39, 43] to the variable and quasi-linear coefficient cases.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Lemmas

In this appendix, we introduce some lemmas about the Kronecker decomposition of a pencil. These results are used in the proof of theorem SH of the SS.

Consider the following matrix pencil

\[
\lambda I^A_\beta + K^B_\beta = \lambda \begin{bmatrix} -\delta^\alpha_\beta \\ 0 \end{bmatrix} + \begin{bmatrix} A^\alpha_\beta \\ C^\Gamma_\beta \end{bmatrix},
\]

such that \(\delta^\alpha_\beta, A^\alpha_\beta \in \mathbb{R}^{u \times u}, C^\Gamma_\beta \in \mathbb{R}^{c \times u}, \delta^\alpha_\beta\) is the identity matrix, \(I^A_\beta = \begin{bmatrix} -\delta^\alpha_\beta \\ 0 \end{bmatrix}\) and \(K^B_\beta = \begin{bmatrix} A^\alpha_\beta \\ C^\Gamma_\beta \end{bmatrix}\).

Since \(I^A_\beta\) has only trivial right kernel, the Kronecker decomposition of this pencil may only include Jordan blocks (with \(\lambda_i\) as the generalized eigenvalues)

\[
J_m(\lambda_i) = \begin{bmatrix} (\lambda - \lambda_i) & 1 & 0 & 0 \\ 0 & (\lambda - \lambda_i) & 1 & 0 \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & (\lambda - \lambda_i) \end{bmatrix} \in \mathbb{C}^{m \times m},
\]

\(L_m^T\) blocks

\[
L_m^T = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & \lambda \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{C}^{(m+1) \times m},
\]

and zero rows (called here \(L_0^T\)).

Lemma 14.

(a) Let

\[
J, m_0 \times L_0^T, m_1 \times L_1^T, m_2 \times L_2^T, \ldots, m_n \times L_n^T
\]
be the Kronecker structure of $\lambda I_\beta^A + K_\beta^A$, where $J \in \mathbb{C}^{a \times a}$ includes only the Jordan blocks, then

$$m_0 + m_1 + m_2 + \cdots + m_n = c,$$

$$a + m_1 + 2m_2 + \cdots + nm_n = u.$$  \hspace{1cm} (129)

(b) For any $\lambda$ other than the generalized eigenvalues, it holds that

$$\dim(\text{left}_\lambda \ker(\lambda I_\beta^A + K_\beta^A)) = c.$$  \hspace{1cm} (131)

(c) Let (128) with

$$J = d_1 \times J_1(\lambda_1), d_2 \times J_1(\lambda_2), \ldots, d_q \times J_1(\lambda_q),$$

and $\lambda_1 < \cdots < \lambda_q$ be the Kronecker structure of $\lambda I_\beta^A + K_\beta^A$, then

$$\dim(\text{left}_\lambda \ker(\lambda I_\beta^A + K_\beta^A)) = c \forall \lambda \neq \lambda_i \text{ with } i = 1, \ldots, q.$$  \hspace{1cm} (132)

$$\dim(\text{left}_\lambda \ker(\lambda I_\beta^A + K_\beta^A|_{\lambda=\lambda_i})) = c + d_i.$$  \hspace{1cm} (133)

**Proof.**

(a) The pencil $\lambda I_\beta^A + K_\beta^A$ is a $(u + c) \times u$ matrix so its Kronecker matrix is of the same size. Therefore, the number of columns of $J \in \mathbb{C}^{a \times a}$ plus that of the $L_m \in \mathbb{C}^{m+1 \times m}$ blocks in (128) has to be equal to $u$, i.e.,

$$a + m_1 + 2m_2 + \cdots + nm_n = u.$$  \hspace{1cm} (130)

This concludes the proof of equation (130).

The same analysis on the rows shows that

$$a + m_0 + 2m_1 + 3m_2 + \cdots + (n + 1)m_n = u + c.$$  \hspace{1cm}

Subtracting these last two expressions, we obtain equation (129) and conclude the proof of (a). Notice that (129) is independent of the size of $J$.

(b) Studying the left kernel of $\lambda I_\beta^A + K_\beta^A$ is equivalent to studying the left kernel of its Kronecker structure, so we assume that $\lambda I_\beta^A + K_\beta^A$ is already in its Kronecker form.

We begin by noting that the Jordan blocks only have non-trivial left kernel when $\lambda = \lambda_i$. Since we are not considering these values of $\lambda$, we conclude that to show (131), it is sufficient to study the left kernel of the blocks $L_m^u$ and the zero rows $L_0^u$.

The left kernel of each block $L_m^u$ (equation (127)) has dimension 1 and is expanded by the vector

$$X = [-1, \lambda, \ldots, (-1)^m \lambda^{m-1}, (-1)^{m+1} \lambda^m] \in \mathbb{R}^{1 \times m+1}.$$  \hspace{1cm}

Therefore, if we assume without loss of generality, that the Kronecker structure of the pencil is given by (128) we conclude that the left kernel of the pencil has dimension

$$m_0 + m_1 + m_2 + \cdots + m_n.$$  \hspace{1cm}

Using equation (129) of (a) we conclude the proof of (b).
(c) The proof of equation (132) is the same as the proof of (b), since (b) is independent of the form of $J$.

The proof of equation (133) is followed by noting that all Jordan blocks in $J$ are of $1 \times 1$ and that each generalized eigenvalue $\lambda_i$ has degeneracy $d_i$; therefore, when $\lambda = \lambda_i$ there are $d_i$ extra vectors added to the left kernel of the pencil justifying the expression (133).

The following lemma reveals the connection between $C^\Gamma_\beta$ and the Kronecker structure of the pencil (126).

Lemma 15. Let the pencil (126) such that part of its Kronecker structure is

\[ d_1 \times J_1(\lambda_1), d_2 \times J_1(\lambda_2), \ldots, d_q \times J_1(\lambda_q), \]

with

\[ d_1 + d_2 + \cdots + d_q = \dim(\text{right}_\ker(C^\Gamma_\beta)). \] (137)

Then the complete Kronecker structure of this pencil is

\[ d_1 \times J_1(\lambda_1), d_2 \times J_1(\lambda_2), \ldots, d_q \times J_1(\lambda_q), s \times L^T_0, r \times L^T_1. \] (138)

Proof. We begin by showing that if the conditions (136) and (137) are satisfied, the system does not admit any other Jordan blocks than those present in (136). Consider the set of generalized eigenvectors $\delta \phi^{\beta \lambda_i}_{j}$ associated to the structure (136), they compound the set $\{ \delta \phi^{\beta \lambda_i}_{j} \}$ of $d$ linearly independent vectors with $j = 1, \ldots, d_i$ and $i = 1, \ldots, q$ such that

\[ \left( \lambda_i \begin{bmatrix} -\delta^\alpha \beta \\ 0 \end{bmatrix} + \begin{bmatrix} A^\beta \\ C^\alpha_\beta \end{bmatrix} \right) \delta \phi^{\beta \lambda_i}_{j} = 0. \] (139)

This means that

\[ C^\Gamma_\beta \delta \phi^{\beta \lambda_i}_{j} = 0 \quad \forall j = 1, \ldots, d_i \quad \text{and} \quad \forall i = 1, \ldots, q. \]

Notice that any other possible generalized eigenvector satisfying (139) should also belong to the right kernel of $C^\Gamma_\beta$. But since $d = \dim(\text{right}_\ker(C^\Gamma_\beta))$, the entire right kernel of $C^\Gamma_\beta$ must be expanded by the $d$ vectors $\{ \delta \phi^{\beta \lambda_i}_{j} \}$. In other words, the pencil cannot admit any extra Jordan block, otherwise, we will reach a contradiction.
It only remains to show that the complete pencil structure is given by (138). We call $J$ the part of the Jordan blocks $J = d_1 \times J_1(\lambda_1), d_2 \times J_1(\lambda_2), \ldots, d_q \times J_1(\lambda_q)$ and we notice that $J$ is a $d \times d$ matrix, so using equation (130) of the previous lemma, we obtain

$$d + m_1 + 2m_2 + \cdots + nm_n = u.$$  

Since by (134), $u = r + d$, we conclude that

$$m_1 + 2m_2 + \cdots + nm_n = r. \tag{140}$$

On the other hand, since $\dim(\text{left}_\ker(C^{\beta}_\Gamma)) = s$, the pencil $\lambda I^{\beta}_j + K^{\beta}_j$ has $s$ linearly independent vectors $[0 \ v_i^\beta]$, with $i = 1, \ldots, s$, belonging to the left kernel of the pencil. These vectors do not depend on $\lambda$ since $C^{\beta}_\Gamma$ does not. Moreover, due to the explicit form of $\lambda I^{A}_j + K^{A}_j$, every other vector of the left kernel, independent of $\lambda$ should be a linear combination of the previous ones. Therefore, the set of vectors $[0 \ v_i^\beta]$ are associated to $s$ null rows in the Kronecker decomposition of $\lambda I^{\beta}_j + K^{\beta}_j$, i.e.,

$$m_0 = s.$$  

Replacing this expression in (140) and (135) we obtain

$$c = r + s = m_1 + 2m_2 + \cdots + nm_n + m_0.$$  

Using now equation (129), we conclude

$$m_0 + m_1 + m_2 + \cdots + m_n = c = m_1 + 2m_2 + \cdots + nm_n + m_0$$

and, therefore,

$$0 = m_2 + 2m_3 + \cdots + (n - 1)m_n.$$  

Since $m_i \geq 0$ the unique solution for this equation is

$$m_i = 0 \text{ para } i \geq 2.$$  

Therefore,

$$c = r + s = m_1 + m_0 = m_1 + s,$$

from which we conclude that $m_1 = r$, ending the proof.

\section*{Appendix B. Lapse and shift}

In this appendix, we consider a space–time $M$, with the foliation $\Sigma_\tau$ described in subsection 2.1, and we study the equation

$$\nabla_a q^b = 0, \tag{141}$$
where $\nabla_a$ is any covariant derivative without torsion. We present this simple system as an example of the process of rewriting the equations in their $n+1$ version. Any more complex system can be handled in the same way by repeating the steps presented here. We perform this $n+1$ decomposition by using a different projector $\tilde{\eta}_{ab}$ to the one $\eta_{ab}$ used in section 2 and assuming that the system does not have a background metric. This new projector is parametrized by the lapse function $N$ and the shift vector $\beta^a$. Of course, we recover the standard results in the cases with a background metric.

We remark that theorems 8 and 12 can be re-adapted to the type of projection used here. This is showing in the examples of section 6.

We begin by considering the definitions introduced in subsection 2.1, and defining the shift vector $\beta^a$ such that

$$\beta^a n_a = 0.$$ 

It allows us to define the projector

$$\tilde{\eta}_{ab} := \delta_{ab} - p^a n_b,$$

with

$$p^a = t^a - \beta^a.$$ 

Notice that this projector reduces to the $\eta_{ab}$ projector when $\beta^a = 0$.

Using the coordinates $(t, x^i)$ adapted to the foliation introduced in subsection 2.1 we obtain

$$\tilde{\eta}_{ib} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\beta^1 & 1 & 0 & 0 \\
\beta^2 & 0 & 1 & 0 \\
\beta^3 & 0 & 0 & 1
\end{pmatrix},$$

where the indices $a$ and $b$ correspond to the columns and the rows of the matrix respectively.

On the other hand, since

$$p^a n_a = t^a n_a - \beta^a n_a = t^a n_a = 1,$$

it is easy to check that $\tilde{\eta}_{ib}$ satisfies similar projector properties as $\eta_{ib}$, that is,

$$\tilde{\eta}_{ib} \tilde{\eta}_{ic} = \tilde{\eta}_{ib}, \quad \tilde{\eta}_{ib} p^b = 0, \quad \tilde{\eta}_{ib} n_a = 0.$$

We introduce now the lapse function $N$ in the following way, consider the projector

$$\tilde{\eta}_{ib} := \delta_{ib} + \frac{1}{N} p^a (-N n_b),$$

$$\tilde{\eta}_{ib} = \delta_{ib} + \tilde{m}^a n_b,$$

where we have defined

$$\tilde{m}^a := \frac{1}{N} p^a,$$

$$\tilde{n}_b := - N n_b.$$
such that

\[ \tilde{m}^d \tilde{n}_a = -1. \]  

(144)

With these definitions, we can project equation (141) and its variables as follows.

**Lemma 16.**

\[
\nabla_a \tilde{q}^b = -\tilde{m}^b n_a (\mathcal{L}_p \tilde{q}^0 - N((\tilde{q}^w S_w) + \tilde{q}^0 (\tilde{m}^d Z_d))) 
+ \tilde{\eta}_w^a n_a (\mathcal{L}_p \tilde{q}^w - N(\tilde{q}^w K^w + \tilde{q}^0 S^w)) 
+ \tilde{\eta}_w^b \tilde{\eta}_a^d (D_a \tilde{q}^w + \tilde{q}^0 K^w) 
- \tilde{m}^b \tilde{\eta}_a^d (D_a \tilde{q}^0 + \tilde{q}^0 K_{wd} - \tilde{q}^0 Z_d)
\]

(145)

with \(\tilde{q}^w, S^r, S_r, K_{ba} \) and \(K^0_a\) tangents to \(\Sigma_t\) and such that

\[
\tilde{q}^w := \tilde{\eta}_{w}^{r} q^r, \quad (149)
\]

\[
\tilde{q}^0 := \tilde{n}_{a} q^a, \quad (150)
\]

\[
Z_d := \tilde{n}_w \nabla_d \tilde{m}_w, \quad (151)
\]

\[
S_r := \tilde{\eta}_w^a \tilde{m}_d \nabla_d \tilde{n}_w, \quad (152)
\]

\[
S' := \tilde{\eta}_w^a \tilde{m}_d \nabla_d \tilde{m}_w, \quad (153)
\]

\[
-K^b_a := \tilde{\eta}_w^b \tilde{\eta}_a^d \nabla_d \tilde{m}_w, \quad (154)
\]

\[
-K_{ba} := \tilde{\eta}_w^b \tilde{\eta}_a^d \nabla_d \tilde{n}_r. \quad (155)
\]

The covariant derivative \(D_a\) is defined over \(\Sigma_t\) in the standard form

\[
D_a \tilde{q}^w := \tilde{\eta}_w^r \tilde{\eta}_a^s \nabla_s \tilde{q}^r, \quad (156)
\]

and the Lie derivative \(\mathcal{L}_p\) is

\[
\mathcal{L}_p = \mathcal{L}_{\partial_t} - \mathcal{L}_\beta, \quad (157)
\]

\[
= \partial_t - \mathcal{L}_\beta, \quad (158)
\]

where the last equation holds since we are considering the coordinates \((t, x^i)\).

Moreover, it holds

\[
S_r = \tilde{\eta}_w^w (D_w (\ln N) - Z_w) \quad (159)
\]

and

\[
\mathcal{L}_w \tilde{\eta}_r^a = \tilde{m}^d D_a (\ln N), \quad (160)
\]

\[
\mathcal{L}_p \tilde{\eta}_r^d = 0. \quad (161)
\]
Proof. We start by isolating $\delta_q^b$ from equations (142) and (143) and rewriting (141) as follows

$$\nabla_a q^b = \delta_q^b \nabla_a (\delta_q^b) \quad \Rightarrow \quad \nabla_a q^b = (\tilde{\eta}_a^b + p^b n_a) (\tilde{\eta}_a^b + p^b n_a) \nabla_a ((\tilde{\eta}_a^b - \tilde{m}^b \tilde{n}_a) q^b).$$

Defining

$$\tilde{q}^0 := \tilde{n}_a q^a,$$
$$\tilde{q}^b := \tilde{\eta}_a^b q^a$$

and introducing these definitions in the last expression we obtain

$$\nabla_a q^b = -\tilde{m}^b n_a (p^f \tilde{n}_w (\nabla_a \tilde{q}^w - \tilde{q}^b \nabla_a \tilde{m}^w - \tilde{m}^w \nabla_a \tilde{q}^0)) + \tilde{\eta}_a^b \tilde{n}_a (p^f (\nabla_a \tilde{q}^w - \tilde{q}^b \nabla_a \tilde{m}^w - \tilde{m}^w \nabla_a \tilde{q}^0)) + \tilde{\eta}_a^b \tilde{\eta}_a^d (\nabla_a \tilde{q}^w - \tilde{q}^b \nabla_a \tilde{m}^w - \tilde{m}^w \nabla_a \tilde{q}^0) - \tilde{m}^b \tilde{\eta}_a^d (\tilde{n}_w (\nabla_a \tilde{q}^w - \tilde{q}^b \nabla_a \tilde{m}^w - \tilde{m}^w \nabla_a \tilde{q}^0)).$$

Using now that

$$\tilde{\eta}_a^b \tilde{m}^w = 0,$$
$$\tilde{n}_a \tilde{q}^w = 0,$$
$$\tilde{n}_w \nabla_a \tilde{q}^w = -\tilde{q}^w \nabla_a \tilde{n}_w,$$
$$\tilde{n}_a \tilde{m}^w = -1,$$
$$\tilde{n}_w \nabla_a \tilde{m}^w = -\tilde{m}^w \nabla_a \tilde{n}_w,$$

we conclude

$$\nabla_a q^b = -\tilde{m}^b n_a (-p^f ((\tilde{q}^w - \tilde{q}^0 \tilde{m}^w) \nabla_a \tilde{n}_w) + p^f \nabla_a \tilde{q}^0) + \tilde{\eta}_a^b \tilde{n}_a (p^f (\nabla_a \tilde{q}^w - \tilde{q}^b \nabla_a \tilde{m}^w)) + \tilde{\eta}_a^b \tilde{\eta}_a^d (\nabla_a \tilde{q}^w - \tilde{q}^b \nabla_a \tilde{m}^w) - \tilde{m}^b \tilde{\eta}_a^d (-(\tilde{q}^w - \tilde{q}^b \tilde{m}^w) \nabla_a \tilde{n}_w + \nabla_a \tilde{q}^0).$$

Using also that

$$D_{a} \tilde{q}^b := \tilde{\eta}_a^b \tilde{n}_a \nabla_a \tilde{q}^w = \tilde{n}_w \tilde{\eta}_a^d D_{d} \tilde{q}^w,$$
$$D_{a} \tilde{q}^0 := \tilde{\eta}_a^b \nabla_a \tilde{q}^b = \tilde{\eta}_a^d D_{a} \tilde{q}^0,$$
$$L_{p} \tilde{q}^0 := p^f \nabla_a \tilde{q}^0,$$
$$L_{p} \tilde{q}^w := p^f \nabla_a \tilde{q}^w - \tilde{q}^0 \nabla_a p^w,$$

we conclude

$$\nabla_a q^b = -\tilde{m}^b n_a (-p^f ((\tilde{q}^w - \tilde{q}^0 \tilde{m}^w) \nabla_a \tilde{n}_w) + L_{p} \tilde{q}^0)$$
\[+ \tilde{\eta}^b_a n_a (\mathcal{L}_p \tilde{q}^w + \tilde{q}^w \nabla_p p^w - \tilde{q}^0 \nabla_d \tilde{m}^w)\]
\[+ \tilde{\eta}^b_a \tilde{\eta}^d_a (D_a \tilde{q}^w - \tilde{q}^0 \nabla_d \tilde{m}^w)\]
\[- \tilde{m}^b \tilde{\eta}^d_a (-(\tilde{q}^w - \tilde{q}^0 \tilde{m}^w) \nabla_d \tilde{n}_w + D_d \tilde{q}^0).\]

Recalling that \[\tilde{m}^w = \frac{1}{N} p^w,\]
we rewrite \(\nabla_p p^w\) as follows
\[\nabla_p p^w = \tilde{m}^w \nabla_p N + N \nabla_p \tilde{m}^w.\]

Replacing this expression in the previous development we arrive at the following result
\[\nabla_a q^b = -\tilde{m}^b_a n_a (-(\tilde{q}^w - \tilde{q}^0 \tilde{m}^w) p^d \nabla_d \tilde{n}_w + \mathcal{L}_p \tilde{q}^0)\]
\[+ \tilde{\eta}^b_a n_a (\mathcal{L}_p \tilde{q}^w + \tilde{q}^w \nabla_p \tilde{m}^w - \tilde{q}^0 \nabla_d \tilde{m}^w)\]
\[+ \tilde{\eta}^b_a \tilde{\eta}^d_a (D_a \tilde{q}^w - \tilde{q}^0 \nabla_d \tilde{m}^w)\]
\[- \tilde{m}^b \tilde{\eta}^d_a (-(\tilde{q}^w - \tilde{q}^0 \tilde{m}^w) \nabla_d \tilde{n}_w + D_d \tilde{q}^0).\]

Finally, introducing the definitions
\[-K^b_w := \tilde{\eta}^b \tilde{\eta}^d \nabla_d \tilde{n}_w,\]
\[-K_{ew} := \tilde{\eta}^b \tilde{\eta}^d \nabla_d \tilde{n}_r,\]
\[Z_d := \tilde{n}_w \nabla_d \tilde{m}^w = -\tilde{m}^w \nabla_d \tilde{n}_w,\]
\[S_r := \tilde{n}_r \tilde{m}^r \nabla_d \tilde{n}_w,\]
\[S^r := \tilde{n}_r \tilde{m}^r \nabla_d \tilde{m}^w,\]
we obtain
\[\nabla_a q^b = -\tilde{m}^b_a n_a (\mathcal{L}_p \tilde{q}^w - N(\tilde{q}^w S_w + \tilde{q}^0 (\tilde{m}^d Z_d)))\]
\[+ \tilde{\eta}^b_a n_a (\mathcal{L}_p \tilde{q}^w - \tilde{q}^w NK^e_w - N \tilde{q}^0 S^w)\]
\[+ \tilde{\eta}^b_a \tilde{\eta}^d_a (D_a \tilde{q}^w + \tilde{q}^0 K^w_d)\]
\[- \tilde{m}^b \tilde{\eta}^d_a (D_a \tilde{q}^0 + \tilde{q}^0 K^w_d - \tilde{q}^0 Z_d).\]

In addition, we show that
\[\tilde{m}^d \nabla_d \tilde{n}_b = -\tilde{m}^d \nabla_d (N \nabla_b t) = -\tilde{m}^d \nabla_d \nabla_b t - \tilde{m}^d N \nabla_d (\nabla_b t),\]
\[= \frac{1}{N} \tilde{n}_b \tilde{m}^d \nabla_d N - \tilde{m}^d N \nabla_b (\nabla_d t),\]
\[= \frac{1}{N} \tilde{n}_b \tilde{m}^d \nabla_d N + \tilde{m}^d N \nabla_b \left( \frac{1}{N} \tilde{n}_d \right),\]
\[= \frac{1}{N} \tilde{n}_b \tilde{m}^d \nabla_d N - N \nabla_b \left( \frac{1}{N} \right) + \tilde{m}^d \nabla_b \tilde{n}_d.\]
\[ \frac{1}{N} \tilde{n}_p \tilde{m}_q \nabla_p N + \frac{1}{N} \tilde{m}_b \nabla_b N + \tilde{m}_d \nabla_d \tilde{n}_d, \]
\[ \frac{1}{N} \tilde{n}_b \tilde{m}_d \nabla_d N + \tilde{m}_d \nabla_d \tilde{n}_d, \]
\[ = D_b (\ln N) - Z_b, \]

and

\[ \mathcal{L}_p \tilde{n}_d = \tilde{m}_d \nabla_d \tilde{n}_d - \tilde{m}_d \nabla_d \tilde{m}_d + \tilde{m}_d \nabla_d \tilde{m}_d, \]
\[ = \tilde{m}_d \nabla_d (\tilde{m}_d \tilde{n}_d) - (\delta^d_q + \tilde{m}_d \tilde{n}_q) \nabla_q \tilde{m}_d + (\delta^d_q + \tilde{m}_d \tilde{n}_q) \nabla_q \tilde{m}_d, \]
\[ = \tilde{n}_d \tilde{m}_d \nabla_q \tilde{n}_d + \tilde{m}_d \tilde{n}_q \nabla_q \tilde{m}_d - \tilde{n}_d \tilde{m}_q \nabla_q \tilde{n}_d + \tilde{m}_d \tilde{n}_q \nabla_q \tilde{m}_d, \]
\[ = \tilde{m}_d (\tilde{m}_d \nabla_q \tilde{n}_q + Z_r), \]
\[ = \tilde{m}_d D_r (\ln N). \]

Notice that \( \mathcal{L}_p \tilde{n}_d \) can be obtained from the latter expression by taking \( N = 1 \), so that

\[ \mathcal{L}_p \tilde{n}_d = 0. \]

Let us now consider the case when we have a Lorentzian metric \( g_{ab} \), such that in coordinates \( (t, x^i) \) it has the following form

\[ ds^2 = (-N^2 + \beta_k \beta^k) dt^2 + 2 \beta_i \, dt \, dx^i + \gamma_{ij} \, dx^i \, dx^j. \]

Here \( N \) and \( \beta^k \) have been defined as before. In its matrix mode \( g_{ab} \) and its inverse \( g^{bc} \) are given by

\[ g_{ab} = \begin{bmatrix}
-N^2 + \beta_k \beta^k & \beta_j \\
\beta_j & \gamma_{ij}
\end{bmatrix}, \]
\[ g^{bc} = \begin{bmatrix}
-\frac{1}{N^2} & \frac{\beta^j}{N^2} \\
\frac{\beta^j}{N^2} & \frac{1}{N^2} - \frac{\beta^j \beta_j}{N^2}
\end{bmatrix}. \]

With this ansatz, it is easy to check that

\[ N = \frac{1}{\sqrt{-\nabla t \cdot \nabla t}}, \]

and if we use the metric \( g_{ab} \) (and its inverse) to raise and lower indices we obtain that

\[ \tilde{m}_b = g^{bc} \tilde{n}_c. \]

Notice that the latter expression is equivalent to

\[ \tilde{n}_b = \tilde{m}_b \]

and furthermore \( \tilde{n}_c \) is temporal due to equation (144), i.e.,

\[ \tilde{n}_b \tilde{n}_b = \tilde{m}_b \tilde{m}_b = -1. \]
Moreover, we see that the projector \( \tilde{\eta}^a_b \) (equation (142)), takes the form
\[
\tilde{\eta}^a_b := \delta^a_b + \tilde{n}^a \tilde{n}_b. \tag{163}
\]
When we lower the \( a \) component with \( g_{ac} \), the latter expression is a Riemannian metric on \( \Sigma_t \), it is
\[
\tilde{\eta}_{ab} = g_{ab} + \tilde{n}^a \tilde{n}_b. \tag{164}
\]
Notice also that when we do not consider a metric we have two unrelated types of 'extrinsic curvatures' \( K^b_a \) and \( K_{ba} \). However, when we have a metric like (162) and we consider a Levi-Civita connection \( \nabla_d \) then
\[
K_{ba} = g_{bc} K^c_a, \\
S_r = g_r^c S^c, 
\]
furthermore
\[
Z_d := \tilde{n}_w \nabla_d \tilde{n}^w = \tilde{n}_w \nabla_d \tilde{n}^w = \frac{1}{2} \nabla_d (\tilde{n} \tilde{n}) = 0. 
\]

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