Gauge-invariant ground state for canonically quantized Yang-Mills theory

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Abstract

We use Hamilton-Jacobi theory to construct a gauge-invariant zero-energy candidate ground state for canonically quantized Yang-Mills theory with a “nonlinear normal” factor ordering, generalizing an analogous ordering introduced by Moncrief and Ryan for problems with finitely many degrees of freedom. Invariance under spatial rotations and translations is immediate; boost invariance remains under investigation. The motivation is to find a model for constructing a candidate ground state in general relativity, canonically quantized a la the Ashtekar variables. We seek to avoid replicating some of the more troublesome features of the Kodama state, inherited from the Chern-Simons state.

1 Introduction

This paper presents a factor ordering of the canonically quantized Yang-Mills Hamiltonian operator, and a corresponding gauge-invariant candidate ground state in what might be called the Schrödinger representation. As usual in a canonically quantized gauge theory, the “position” variable is the vector potential $A$ of the gauge connection. The corresponding momentum $E$ is given by the (negative) Yang-Mills electric field variable. These variables satisfy the Poisson bracket relations

$$\{A_I^i(x), A_J^j(y)\} = 0 = \{E_I^i(x), E_J^j(y)\}, \quad \{E_J^j(y), A_I^i(x)\} = \delta^3(x,y) \delta_I^i \delta_J^j,$$

and can be promoted to quantum operators as

$$\hat{A}_I^i(x) : \psi(A) \rightarrow A_I^i(x)\psi(A), \quad \hat{E}_I^j(x) : \psi(A) \rightarrow -i \frac{\delta}{\delta A_I^i(x)} \psi(A).$$

The commutators of these quantum operators mirror the classical Poisson brackets, as required:

$$[\hat{A}_I^i(x), \hat{A}_J^j(y)] = 0 = [\hat{E}_I^i(x), \hat{E}_J^j(y)], \quad [\hat{E}_J^j(y), \hat{A}_I^i(x)] = -i \delta^3(x,y) \delta_I^i \delta_J^j.$$
where we have set Planck’s constant $\hbar$ equal to 1.

The motivation for choosing a canonical quantization lies in the hope of addressing physical questions relating to ground states, and ultimately measures on the space of field configurations, for quantized gauge theories. In a canonical approach, gauge invariance is implemented at the quantum level by means of Dirac constraints. The ground state presented here is in fact automatically gauge-invariant (as well as spatially rotation and translation invariant) from its construction. Ideally, full Poincare invariance of the ground state is hoped to result by promoting all Poincare generators to quantum conserved quantities and verifying that they annihilate the ground state as well as exhibiting appropriate commutators. Formulated in terms of the Ashtekar variables, general relativity is also a gauge theory, and under a canonical quantization, full diffeomorphism invariance is similarly imposed as a quantum constraint operator (see e.g. [15]).

Yang-Mills theory is a valuable testing ground for ideas to be applied to canonical quantum general relativity. For instance the Kodama state, at present the only known candidate ground state for gravity in the Ashtekar variables (see [15]), arose as a generalization of the Chern-Simons state in Yang-Mills theory. While exhibiting many positive features, the Kodama state as usually constructed seems to inherit unphysical properties of the Chern-Simons state. Alternative candidate ground states for canonical quantum general relativity may aid the ongoing search for a physical inner product. Since the Kodama state is a generalization from the Chern-Simons ground state in Yang-Mills theory, a reasonable effort towards the construction of a normalizable ground state for quantum general relativity would be to search for a well-behaved ground state in Yang-Mills theory.

Such being the aim of the current project, we do not attempt to conform to the usual ideals of quantized Yang-Mills theory per se, as envisioned for instance in the formulation of the Clay Prize Millenium Problem [6]. That is, we do not seek a quantization likely to yield a mass gap, since in quantum gravity a mass gap is not expected.

In the Schrodinger representation [1], a ground state $\Omega (A)$ for the Hamiltonian operator is the first step toward finding a state space of the form $L^2 (A, d\mu)$, for some measure $d\mu$ on the space of connections $A$. Heuristically speaking, the first candidate for the measure $d\mu$ would be something like

$$\langle \Omega (A) \rangle^2 dA,$$

where “$dA$” is a naive “Lebesgue” measure on $\mathcal{A}$. Of course, the usual way (e.g. [2]) to make sense of such expressions will encounter resistance on two fronts: first, $\Omega (A)$ will be non-Gaussian for a nonabelian gauge theory, and secondly the space $\mathcal{A}$ should in fact be composed of equivalence classes of connections modulo gauge transformations $A/\mathcal{G}$, and hence is not a linear space. Similar difficulties are at least partly addressed within the literature on Yang-Mills path integrals (5, 11); however, arriving at a full rigorous understanding of a measure such as (2) is obviously highly nontrivial. In the meantime, however, it would be nice at least to know a well-behaved candidate ground state for
Yang-Mills theory. By well-behaved, we mean that the ground state should
decay rapidly for connections which are “large” in some suitable sense – e.g. of
large $L^2$ norm – so as to be a normalizable ground state with respect to some
measure on $\mathcal{A}$. This is what the Chern-Simons state

$$\Psi_{\text{CS}}(A) = \exp \left[ \int_{\Sigma} tr \left( A \wedge dA - \frac{2}{3} A \wedge A \wedge A \right) d^3x \right]$$

fails to do, since the Chern-Simons form in the exponent changes sign under
parity (for a good discussion of the Chern-Simons state’s problems, see [19]).

For the abelian case of free Maxwell theory, a well-behaved zero-energy ground
state has already been written by Wheeler [18] in closed form as

$$\Omega(A) = N \exp \left( -\frac{1}{4\pi^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left( \nabla \times A(x) \right) \cdot \left( \nabla \times A(y) \right)}{||x-y||^2} d^3x d^3y \right) , \quad (3)$$

and in fact for linearized general relativity, Kuchar [7] derived the strongly
analogous ground state wave functional

$$\Psi(h) = N \exp \left( -\frac{1}{8\pi^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left( h_{TT}^{ik,l}(x) \right) \cdot \left( h_{TT}^{ik,l}(y) \right)}{||x-y||^2} d^3x d^3y \right)$$

in terms of the linearized metric tensor

$$h_{ik} = g_{ik} - \eta_{ik}$$

in transverse traceless gauge (denoted $h_{TT}^{ik}$).

The explicit construction of such ground states, however, relies on integral
kernel methods available only for linear theories. To deal with the nonlinearities
displayed by full nonabelian Yang-Mills theory or general relativity, we need a
new and more indirect means of finding well-behaved ground states, reducing
in free cases to these known examples.

Thus motivated, we generalize a method developed by Moncrief and Ryan
([10], [11], [13]) in nonlinear quantum mechanical settings, using classical Hamilton-
Jacobi theory to derive an expression for an exact quantum state which is a zero-
energy ground state with respect to a particular ordering of the Hamiltonian
operator. Encouragement for the prospect of extending to general relativity
comes from the fact that in [11], Moncrief and Ryan present an explicit solution
for such a ground state in the vacuum Bianchi IX cosmology.

As explained in Sect.3 the ground state we seek is essentially the exponential
of Hamilton’s principal function for the corresponding Euclidean problem, so in
order to find the principal function (or rather functional), we must solve the
Dirichlet problem for Yang-Mills theory. For a compact base manifold, this has
been collectively achieved by Uhlenbeck [16], Sedlacek [14], and Marini [9], using
the direct method in the calculus of variations. We follow a similar technique
but generalize to the case of a noncompact manifold, since we are interested in
Yang-Mills theory on Minkowski space. Some preliminaries necessary to solving the Dirichlet problem on a Riemannian manifold are presented in Sect. 2, and the solution to the Riemannian Yang-Mills Dirichlet problem is presented in Sect. 4. Finally, in Sect. 6, we conclude gauge invariance of the ground state and discuss partial results and ongoing work to test Poincare invariance.

2 Preliminaries

To solve the Yang-Mills Dirichlet problem for a compact manifold $M$, Marini [9] introduces a terminology for coverings of $M$ by geodesic balls and half-balls; these are described respectively as neighborhoods of type 1 and type 2. Let $M$ be a smooth $n$-dimensional manifold equipped with a Riemannian metric $g$, and let $\partial M$ be its boundary. Then neighborhoods of type 1, in the manifold’s interior, are denoted $U^{(1)} \equiv \{ x = (x^0, ..., x^{n-1}) : |x| < 1 \}$ while neighborhoods of type 2, centered around points in $\partial M$, are of the form $U^{(2)} \equiv \{ x = (x^0, ..., x^{n-1}) : |x| < 1, x^0 \geq 0 \}$, where the coordinate $x^0$ parametrizes unit-speed geodesics orthogonal to $\partial M = \{ x^0 = 0 \}$. The boundary of a type 2 neighborhood divides into $\partial_1 U = \{ x \in \partial U^{(2)} : x^0 = 0 \}$, $\partial_2 U = \{ x \in \partial U^{(2)} : |x| = 1 \}$.

In our problem, the manifold of interest is $\mathbb{R}_+ \times \mathbb{R}^3 = \{ (x^0, x^1, x^2, x^3) : x^0 \geq 0 \}$ with the Euclidean metric; however we solve the Yang-Mills Dirichlet problem for a general smooth 4-dimensional Riemannian manifold with boundary, generalizing Marini’s procedure to the non-compact case (Sect. 4). Certain results used are also valid in general dimension $n$; such distinctions are clearly noted in the statements. We return to consider the importance of dimension more thoroughly in Sect. 4.

The main ingredient in a Yang-Mills theory is the structure group; this is a compact Lie group $G \subset SO(l)$ with Lie algebra $\mathfrak{g}$. For $P$ a principal $G$-bundle over $M$, the Yang-Mills field is a connection $A \in \Lambda^1 P \otimes \mathfrak{g}$. Given a local section $\sigma_\alpha : U_\alpha \to P$ for some neighborhood $U_\alpha \subset M$, the connection 1-form $A$ pulls back to a $\mathfrak{g}$-valued 1-form $A_\alpha = \sigma_\alpha^* A$ on $U_\alpha$; the transformation of $A$ on overlapping neighborhoods $U_\alpha$ and $U_\beta$ is given by the transition function $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ defined by $\sigma_\beta (x) = \sigma_\alpha (x) \tau_{\alpha\beta} (x)$:

$$ A_\alpha (x) = \tau_{\alpha\beta} (x)^{-1} d \tau_{\alpha\beta} (x) + \tau_{\alpha\beta} (x)^{-1} A_\beta (x) \tau_{\alpha\beta} (x), \quad x \in U_\alpha \cap U_\beta. $$

The important quantity for Yang-Mills theory is the curvature $F \in \Lambda^2 P \otimes \mathfrak{g}$ of the connection $A$, given by $F = d_P A + \frac{1}{2} [A, A]$ where the bracket $[\cdot, \cdot]$ denotes
the graded commutator on forms, so that \([A, A] = 2 (A \wedge A)\). In terms of a
local section \(\sigma_\alpha : U_\alpha \to P\), \(F\) pulls back to a \(g\)-valued 2-form \(F_\alpha = \sigma_\alpha^* F\) on \(U_\alpha\),
given by \(F_\alpha = d_M A_\alpha + \frac{1}{2} [A_\alpha, A_\alpha]\), transforming as
\[
F_\alpha (x) = \tau_{\alpha \beta} (x)^{-1} F_\beta (x) \tau_{\alpha \beta} (x), \quad x \in U_\alpha \cap U_\beta
\]
for \(\tau_{\alpha \beta}\) as given above. In local coordinates, \(F\) reduces to
\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],
\]
(here \([\cdot, \cdot]\) is the ordinary commutator in \(g\)).

In order to describe the Yang-Mills action, we use the local expressions of \(F\)
as a \(g\)-valued 2-form on neighborhoods of \(M\); however all definitions are gaugeinvariant and therefore do not depend on the particular section used to pull
back \(F\).

The Yang-Mills action can be conveniently couched in terms of the inner
product for \(g\)-valued \(k\)-forms on the manifold \(M\):
\[
\langle \eta, \theta \rangle_2 = \int_M \text{tr} (\eta \wedge * \theta),
\]
where \(*\) denotes the Hodge dual with respect to the metric \(g\) on \(M\). We
occasionally also write \(\langle \eta, \theta \rangle\) for the pointwise inner product \(\langle \eta, \theta \rangle = \text{tr} (\eta \wedge * \theta)\).
The inner product (4) in turn allows us to define \(L^p\) and Sobolev spaces of
forms, using the norm
\[
\|\eta\|_p = \left( \int_M |\eta|^p \right)^{1/p} = \left( \int_M \langle \eta, \eta \rangle^{p/2} \right)^{1/p} = \left( \int_M \left[ \text{tr} (\eta \wedge * \eta) \right]^{p/2} \right)^{1/p}.
\]
In terms of local coordinates \(\{x^\mu\}\) on \(M\) and a basis \(\{e_I\}\) for the Lie algebra \(g\),
membership of \(\eta\) in the Sobolev space of forms is equivalent to each component
\(\eta_{\mu_1 \ldots \mu_k}\) being Sobolev in the ordinary sense of functions. Using this notation,
the Yang-Mills action can be given as
\[
I (A) = \frac{1}{2} \| F \|_2^2 = \frac{1}{2} \int_M \text{tr} (F \wedge * F) = \frac{1}{4} \int_M \text{tr} F_{\mu \nu} F^{\mu \nu} \sqrt{g} dx^1 \cdots dx^n,
\]
where integration is done using the local form of \(F\) on a neighborhood (gauge
invariance of the form \(\text{tr} (F \wedge * F)\) negates any ambiguity due to choice of local
trivialization).

The above manner of formulating the Yang-Mills action functional also offers
an easy proof of lower semicontinuity, necessary in using the direct method to
find a minimizer:

**Theorem 2.1.** The Yang-Mills functional on a manifold \(M\) of dimension 4 is
lower semicontinuous with respect to the weak topology on \(W^{1,2}_{\text{loc}} (M)\).
Proof. It is good enough to prove that on any open bounded set \( U \subset M \), if \( A_i \rightharpoonup A \) in \( W^{1,2}(U) \), then \( I(A) \leq \lim \inf_{i \to \infty} I(A_i) \). Locally we can write

\[
F_i = dA_i + \frac{1}{2} [A_i, A_i].
\]

Using the same reasoning as Sedlacek’s in Lemma 3.6 of \[14\], weak convergence of \( \{A_i\} \) to \( A \) in \( W^{1,2}(U) \) implies weak convergence of \( dA_i \) to \( dA \) in \( L^2(U) \). The continuity of the imbedding \( W^{1,2} \hookrightarrow L^4 \) and of the multiplication \( L^4 \times L^4 \to L^2 \) along with boundedness of \( \{||A_i||_{2,1}\} \) implies that \( \{||A_i, A_i||_2\} \) is bounded. This together with a.e. pointwise convergence yield \( [A_i, A_i] \rightharpoonup [A, A] \), so that \( F_i \rightharpoonup F \) in \( L^2(U; P) \). Finally, lower semicontinuity of the \( ||\cdot||_2 \) norm concludes lower semicontinuity of the Yang-Mills functional.

The Yang-Mills field equations are

\[
d_D \star F = 0
\]

where \( d_D = d + [A, \cdot] \); solutions to this system correspond exactly to critical points of the Yang-Mills action \( I(A) \). To see this, vary \( I(A) \) by varying \( A \) as \( A + \lambda h \), where \( h \) vanishes at \( t = 0 \) and is supported on some compact subset \( N \) (dependent on \( h \)) of \( M \):

\[
\delta_h (I)(A) = \int_N \langle d_D h, F \rangle = \int_N \text{tr} (d_D h \wedge \star F)
\]

\[
= \int_{\partial N} \text{tr} (h \wedge \star F) - \int_N \text{tr} (h \wedge d_D \star F).
\]

It is evident that \( \delta_h (I)(A) \) vanishes for all variations \( h \) precisely when \( d_D \star F \) is identically 0.

As described in Sect. 1, this paper will deal with the canonical quantization ansatz; therefore we must derive the canonical variables. To make the transformation from a Lagrangian to a Hamiltonian formulation, we specialize to the case of interest \( M = \mathbb{R}_+ \times \mathbb{R}^3 \). Since \( M \) is contractible, every bundle over \( M \) is trivial and therefore admits a global section. We can then drop the distinction between \( A \) and its local coordinate representation. Because the Lagrangian is independent of \( A_0 \), the Legendre transformation breaks down for an arbitrary gauge (see e.g. \[5\]), and we must choose to work in the Weyl gauge \( A_0 = 0 \). Thus our canonical position variable is \( A_i^I \), where \( i \) runs over the three spatial parameters and \( I \) over the basis of the Lie algebra \( \mathfrak{g} \), and the canonical momentum is \( E_i^I = \dot{A}_i^I \) (this is the negative of the “electric field” variable).

With respect to these variables we obtain the Hamiltonian

\[
H = \frac{1}{2} \int_{\mathbb{R}^3} \text{tr} \left( E^2 + B^2 \right),
\]

(5)

where \( B_i^I = \frac{1}{2} \varepsilon^{ijk} F_{jk} \). Hamilton’s equations follow by writing the integral \( J = \int_0^\infty H \, dt = \int_0^\infty \int_{\mathbb{R}^3} \mathcal{H} \, d^3x \, dt = -I + \int_0^\infty \int_{\mathbb{R}^3} E A \, d^3x \, dt \). Varying both
expressions with respect to a one-parameter family $A_{\lambda}$ (where the variation has compact support in $\mathbb{R}_+ \times \mathbb{R}^3$ and vanishes for $t = 0$), we arrive at the equality

$$\frac{dJ}{d\lambda} = \int_0^\infty \int_{\mathbb{R}^3} \frac{\delta H}{\delta A} \delta A + \frac{\delta H}{\delta E} \delta E = \int_0^\infty \int_{\mathbb{R}^3} \left[ E \delta A - \dot{A} \delta E \right] - \frac{dI}{d\lambda}$$

using integration by parts. In order for equality to hold between the first and last lines for all variations, Hamilton’s equations

$$\dot{E} = -\frac{\delta H}{\delta A}, \quad \dot{A} = \frac{\delta H}{\delta E}$$

must be equivalent to the vanishing of the variation $\frac{\delta I}{\delta A}$. Notice that in taking the Weyl gauge $A_0 = 0$, we have lost the field equations describing gauge transformations, and therefore in the quantized theory, the Gauss law constraint

$$D_i E^i = 0$$

must be dealt with separately, either by promoting to a quantum operator and verifying that it annihilates the ground state, or by other means. For the ground state we construct here, gauge invariance in fact turns out the be directly verifiable (see Sect. 6).

### 3 Nonlinear normal ordering

For nonlinear quantum mechanical situations, Moncrief [10] and Ryan [13] present a “normal” ordering scheme for the Hamiltonian operator, yielding a well-behaved associated ground state. Consider a nonlinear quantum mechanical Hamiltonian of the form

$$H = \frac{1}{2} |p|^2 + V(x).$$

In a linear system, the function $V(x)$ would be a quadratic form $\langle x, Mx \rangle$, $M$ a positive self-adjoint operator. The normal ordering would proceed by factoring $M$ as $T^2$, in terms of its unique positive self-adjoint square root $T$, and defining creation and annihilation operators $a^* = \frac{1}{\sqrt{2}} (T \hat{x} - i \hat{p})$ and $a = \frac{1}{\sqrt{2}} (T \hat{x} + i \hat{p})$. Under usual assignment of canonical quantum operators $\hat{x} : \psi(x) \rightarrow x^i \psi(x)$, $\hat{p}^i : \psi(x) \rightarrow -i \frac{\partial}{\partial x^i} \psi(x)$, this immediately yields the ground state $\psi(x) = \mathcal{N} \exp \left( -\frac{1}{2} \langle x, T^2 x \rangle \right)$ with energy $E = \frac{1}{2} \frac{\langle T^2 x \rangle}{2}$, for $H$ expressed as a quantum operator $\hat{H} = a^* a + \frac{\langle T^2 x \rangle}{2} I$. 

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The idea of the nonlinear normal ordering is to factorize the function \( V(x) \geq 0 \) by solving the imaginary-time zero-energy Hamilton-Jacobi equation

\[
\frac{1}{2} \sum_i \left( \frac{\partial S}{\partial x^i} \right)^2 - V(x) = 0.
\]

We can then order the quantum Hamiltonian operator as

\[
\hat{H} = \frac{1}{2} \sum_i \left( \frac{\delta S}{\delta x^i} - i\hat{p}^i \right) \left( \frac{\delta S}{\delta x^i} + i\hat{p}^i \right),
\]

admitting the zero-energy ground state

\[
\mathcal{N} \exp (-S(x)).
\]

This factorization can be illustrated with the anharmonic oscillator Hamiltonian

\[
H = \frac{1}{2} \hat{p}^2 + \frac{1}{2} x^2 + \frac{1}{4} \lambda x^4,
\]

in which case the Hamilton-Jacobi equation \( \frac{1}{2} \left( \frac{dS}{dx} \right)^2 = \frac{1}{2} x^2 + \frac{1}{4} \lambda x^4 \) is easily integrated to yield the solution \( S(x) = \frac{2}{3\lambda} \left( 1 + \frac{1}{2} x^2 \right)^{3/2} - \frac{2}{3\lambda} \). While the resulting ground state is not the usual anharmonic oscillator ground state obtained from the factor ordering \( \hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{m \omega^2}{2} \hat{q}^2 + \frac{1}{4} \lambda \hat{q}^4 \), it is the correct zero-energy ground state for nonlinear normal ordering \( \hat{7} \). Finding a ground state with zero energy is not a priority for an ordinary quantum mechanical system like the anharmonic oscillator, but in the realm of relativistic field theories, a quantum ground state must have zero energy to be invariant under infinitesimal time-translations. Hence full Poincare invariance in a canonically quantized relativistic field theory requires a zero energy ground state, suggesting that the nonlinear normal ordering approach may yield well-behaved candidate ground states for a canonically quantized nonlinear field theory.

Following this line of reasoning for Yang-Mills theory, we set up the imaginary-time zero-energy Hamilton-Jacobi equation for the Yang-Mills Hamiltonian \( \hat{5} \):

\[
\int_{\mathbb{R}^3} tr \left( \frac{\delta S}{\delta A} \right)^2 = \int_{\mathbb{R}^3} tr B^2,
\]

whose solution \( S(A) \) will yield the nonlinear normal ordering

\[
\hat{H} = \frac{1}{2} \int_{\mathbb{R}^3} tr \left( \frac{\delta S}{\delta A} - i\hat{E} \right) \left( \frac{\delta S}{\delta A} + i\hat{E} \right)
\]

and associated zero-energy ground state

\[
\mathcal{N} \exp (-S(A)).
\]
However, the Hamilton-Jacobi equation (9) is not as easily solved as in the anharmonic oscillator problem. Fortunately, the classical Hamilton-Jacobi theory provides us with a means of constructing the solution, essentially as Hamilton’s principal function for the imaginary-time problem. With the transformation to imaginary time $t \rightarrow it$, the chain rule yields $A \rightarrow A$, $\dot{A} \rightarrow -i \dot{A}$, and $E \rightarrow -iE$, so that the imaginary-time Lagrangian and Hamiltonian are

$$\tilde{L} = \frac{1}{2} \int_{\mathbb{R}^3} \left(-\dot{A}^2 - B^2\right) \, d^3x$$

$$\tilde{H} = \frac{1}{2} \int_{\mathbb{R}^3} \left(-E^2 + B^2\right) \, d^3x.$$

The full Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + \tilde{H} \left(A, \frac{\delta S}{\delta A}, t\right) = 0; \quad (10)$$

a time-independent solution $S(A)$ to this equation will be the solution we seek for (9). In fact, the solution we construct will be time-independent, but for the moment we assume that an explicit time dependence is possible, using the definition of functional derivatives to write

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \int_{\mathbb{R}^3} \frac{\delta S}{\delta A} \frac{\partial A_t}{\partial t} \, d^3x.$$

Subsituting from (10) we obtain

$$\frac{dS}{dt} = -\tilde{H} \left(A_t, \frac{\delta S}{\delta A_t}\right) + \int_{\mathbb{R}^3} \frac{\delta S}{\delta A_t} \frac{\partial A_t}{\partial t} \, d^3x$$

$$= \int_{\mathbb{R}^3} \delta S \frac{\partial A_t}{\partial t} \, d^3x - \tilde{H} \left(A_t, \frac{\delta S}{\delta A_t}\right) \, d^3x.$$  

For $A_t$ the solution to

$$\frac{\partial A_t}{\partial t} = \frac{\delta \tilde{H}}{\delta E}\bigg|_{E=\frac{\delta S}{\delta A_t}}$$  

(11)

with initial data $A_{t=0} = A$, we get

$$\int_{\mathbb{R}^3} \frac{\delta S}{\delta A_t} \frac{\partial A_t}{\partial t} \, d^3x - \tilde{H} \left(A_t, \frac{\delta S}{\delta A_t}\right) \, d^3x = \int_{\mathbb{R}^3} E \dot{A}_t - \tilde{H} \left(A_t, E\right) \, d^3x$$

$$= L \left(A_t, DA_t\right)$$

$$\Rightarrow S \left(A_{t_0}\right) - S \left(A\right) = \int_0^{t_0} \tilde{L} \left(A_t, \dot{A}_t\right) \, dt$$

Taking $S \left(A\right) = -\int_0^\infty \tilde{L} \left(A_t, \dot{A}_t\right) \, dt$ clearly satisfies this relation, since we will then have $S \left(A_{t_0}\right) = -\int_0^\infty \tilde{L} \left(A_t, \dot{A}_t\right) \, dt$. The exponential $\exp(-S(A))$ will peak about the field configuration $A = 0$. To prove that the functional $S(A)$ exists, we need only prove the existence of a solution $A_t$ of the Euclidean Yang-Mills equations, with initial data $A$. 

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4 Solving the Yang-Mills Dirichlet problem

For a compact manifold $M$ with boundary $\partial M$, the method of Uhlenbeck [16], Sedlacek [14], and Marini [9] begins with a localizing theorem, proving that given a sequence of connections with a uniform global bound on the Yang-Mills action, there exists a cover for $M$ (possibly missing a finite collection of points) such that on neighborhoods of the cover, the Yang-Mills action for connections in the sequence eventually becomes lower than an arbitrary pre-set bound $\varepsilon$. This result depends on compactness as proved in [14] (see Proposition 3.3, or in [9] Theorem 3.1). We reprove the result here in a manner independent of compactness (Theorem 4.2), so that the overall argument now applies to noncompact manifolds with boundary as well.

Note that another possible solution to the problem would be to transform $M = \mathbb{R}_+ \times \mathbb{R}^3$ into a compact manifold with boundary, using inversion in the sphere (this suggestion is due to T. Damour). In this approach, one considers the unit sphere centered at the origin of $\mathbb{R}^4$, imbedding $\mathbb{R}_+ \times \mathbb{R}^3$ into $\mathbb{R}^4$ as the set $\{x : x^0 \geq 2\}$. The inversion mapping is

$$y^i = \frac{x^i}{r^2},$$

where $r^2 = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$. Under this transformation, the hyperplane $x^0 = 2$ maps to a sphere $S_{1/4}$ of radius $\frac{1}{4}$, with its south pole (the image of all points at infinity) at the origin. The half-space $\{x : x^0 > 2\}$ maps to the interior of $S_{1/4}$.

Since the mapping is conformal, the Yang-Mills action remains invariant, and the problem of interest on $\mathbb{R}_+ \times \mathbb{R}^3$ has been effectively mapped to a compact problem, to which the arguments of [16], [14], and [9] should apply directly. We do not pursue this approach here, since the crucial result using compactness can be shown to generalize (Theorem 4.2); however we note its potential usefulness to future work, such as the investigation of uniqueness of solution to the Yang-Mills Dirichlet problem (see Sect. 5). An issue pertinent to the conformal mapping approach is the behavior of initial data at the south pole of $S_{1/4}$, the image of points at infinity.

Returning to our sketch of the Yang-Mills Dirichlet problem’s solution, local control over the Yang-Mills action is used to prove existence and regularity of a minimizer. From this point on, the proofs in [16], [14], and [9] are purely local and hold unchanged in the noncompact case; proofs are thus not repeated here. Locally, the argument for existence of a Yang-Mills minimizer consists in finding a Sobolev-bounded minimizing sequence satisfying the boundary conditions; this sequence then has a weakly convergent subsequence, which proves to be a solution to the original Dirichlet problem. Local solutions are related by transition functions on overlapping neighborhoods.

Gauge freedom turns out to be a help as well as a hindrance. Of course it forces the necessity of working locally and proving compatibility on overlaps, but at the same time gauge freedom offers an elegant solution to the regularity
problem. A judicious choice of gauge – the "Hodge gauge" – complements the Yang-Mills equation in such a way as to yield an elliptic system. In the Yang-Mills equation

\[ d^* dA + [A, dA] + [A, [A, A]], \]

the highest order term is related to the first term of the Laplace-de Rham operator \( \Delta = \delta d + d\delta \), where \( \delta \) is the codifferential \( \delta = (-1)^{(k+1)+1} \ast d^* \). Choosing the Hodge gauge, in which \( d^* A = 0 \), ensures that every solution of our system in this gauge is also a solution of the elliptic system \( \Delta A + *([A, dA] + [A, [A, A]]) = 0 \), and therefore enjoys the regularity properties of such solutions. (Additional work is needed to establish boundary regularity; Marini accomplishes this in [9] using the technique of local doubling.)

In the physical problem, we are interested in Yang-Mills theory over a 4-dimensional manifold with boundary. However many theorems which follow are also valid over any smooth \( n \)-dimensional Riemannian manifold with boundary, and we retain this level of generality in stating and proving results. The caveat lies in stringing together the individual theorems into a complete argument for existence and regularity of a solution to the Yang-Mills Dirichlet problem; to accomplish this, the dimension must be 4 (see the remarks in [9] following Theorem 3.1). The “good cover” theorem (Theorem 4.2 here, or Theorem 3.1 in [9]) guarantees a cover of \( M \setminus \{ x_1, ..., x_k \} \) on whose neighborhoods the local Yang-Mills action for the connections in the sequence is eventually bounded by an arbitrary pre-set bound \( \varepsilon \). However, the condition for existence of a Hodge gauge solution to the Yang-Mills Dirichlet problem is a bound on the local \( L^{n/2} \) norm of the Yang-Mills field strength \( F \), which except in dimension 4 is not the same as a local bound on the Yang-Mills action.

Without further ado, we give the precise statements of all theorems needed for the existence and regularity of a Yang-Mills minimizer on a 4-dimensional manifold with boundary. On a neighborhood \( U \) of type 1 or 2, the condition for local existence of a gauge satisfying \( d^* A = 0 \) is an \( L^{n/2} \) bound on Yang-Mills field strength. Consider the sets

\[
\mathfrak{A}^{1,p}_K(U) = \left\{ D = d + A : A \in W^{1,p}(U), \| F_A \|_{L^{n/2}(U)} < K \right\}
\]

\[
\mathfrak{A}^{1,p}_K(U) = \left\{ D = d + A : A \in W^{1,p}(U), A_\tau \in W^{1,p}(\partial_1 U) \right\}
\]

\[
\| F_A \|_{L^{n/2}(\partial_1 U)} < K \}
\]

describing connections with field strength locally \( L^{n/2} \)-bounded on a neighborhood \( U \) of type 1 and type 2, respectively. (All norms are defined on \( U \), unless otherwise specified.) As proven in [16] (Thm 2.1) for interior neighborhoods and in [9] (Thms 3.2 and 3.3) for boundary neighborhoods, a good choice of gauge exists for connections belonging to \( \mathfrak{A}^{1,p}_K(U) \) or \( \mathfrak{A}^{1,p}_K(U) \). More precisely,

**Theorem 4.1.** For \( \frac{n}{2} \leq p < n \), there exists \( K \equiv K(n) > 0 \) and \( c \equiv c(n) \) such that every connection \( D = d + A \in \mathfrak{A}^{1,p}_K(U) \) (\( \mathfrak{A}^{1,p}_K(U) \)) is gauge equivalent to a
connection $d + \mathring{A}$, $\mathring{A} \in W^{1,p}(U)$, satisfying
\begin{align*}
(a) \quad d * \mathring{A} &= 0 & (a') \quad d * \mathring{A} &= 0 \\
(b) \quad \mathring{A}_v &= 0 \text{ on } \partial U & (b') \quad \mathring{A}_v &= 0 \text{ on } \partial_2 U \\
(c = c') \quad \|\mathring{A}\|_{1,n/2}^{1/2} &< c(n) \|F_{\mathring{A}}\|_{n/2} \\
(d = d') \quad \|\mathring{A}\|_1^{1/2} &< c(n) \|F_{\mathring{A}}\|_p
\end{align*}

(Unprimed conditions (a)-(d) refer to $\mathfrak{B}^{1,p}_K(U)$; primed conditions (a')-(d') to $\mathfrak{B}^{1,p}_K(U)$). Moreover, the gauge transformation $s$ satisfying $\mathring{A} = s^{-1}ds + s^{-1}A$ can be taken in $W^{2,n/2}(U)$ (so will in fact always be one degree smoother than $A$; see Lemma 1.2 in [10]).

Proof. See [10], [9]. \hfill \Box

As noted in [11], the condition $\|F_A\|_{n/2} < K$ is conformally invariant, while the norm $\|F_{A_\epsilon}\|_{L^{n/2}(\partial_1 U)}$ picks up a factor of $\epsilon$ under the dilations $x' = \epsilon x$, so that the simultaneous conditions $\|F_A\|_{n/2} < K$, $\|F_{A_\epsilon}\|_{L^{n/2}(\partial_1 U)} < K$ on a neighborhood $U$ of type 2 can always be achieved by applying an appropriate dilation (the Dirichlet boundary data is prescribed to be smooth, so $\|F_{A_\epsilon}\|_{L^{n/2}(\partial_1 U)}$ already satisfies some bound).

To find a regular minimizer of the Yang-Mills action on a 4-dimensional manifold $M$, we must find a cover $\{U_\alpha\}$ of $M$ and a minimizing sequence $\{A_i\}$ whose members satisfy
\[ SY_M(A_i|_{U_\alpha}) = \int_{U_\alpha} |F_{A_i}|^2 dx < K \quad \forall \alpha, i, \]

where $K \equiv K(4)$ is as given in Theorem [11] for a compact manifold this is proved in [14] using a counting argument. Here we use dilations of the neighborhoods in a cover to construct a proof valid for any smooth Riemannian manifold.

**Theorem 4.2.** Let $\{A(i)\}$ be a sequence of connections in $G$-bundles $P_i$ over $M$, with uniformly bounded action $\int_M |F(i)|^2 dx < B \quad \forall i$. For any $\varepsilon > 0$, there exists a countable collection $\{U_\alpha\}$ of neighborhoods of type 1 and 2, a collection of indices $I_\alpha$, a subsequence $\{A(i)\}_{\mathcal{T}} \subset \{A(i)\}_{\mathcal{T}}$, and at most a finite number of points $\{x_1, ..., x_k\} \in M$ such that
\[ \bigcup U_\alpha \supset M \setminus \{x_1, ..., x_k\} \]
\[ \int_{U_\alpha} |F(i)|^2 dx < \varepsilon \quad \forall i \in \mathcal{T}, \quad i > I_\alpha. \]

Proof. For each $n \in \mathbb{N}$, consider the cover $\{B_n(x) : x \in M\}$ of $M$ given by geodesic balls of radius $\frac{1}{n}$ centered at each point $x \in M$ (for $x \in \partial M$, the geodesic “ball” $B_n(x)$ will actually be a half-ball, a fact which makes no difference in the proof). By separability, each such cover has a countable subcover $C_n = \{B_n(x_{n,m}) : m \in \mathbb{N}\}$. 

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On any ball $B_n(x_{n,m})$, we have the uniform bound $\int_{B_n(x_{n,m})} |F(i)|^2 \, dx < B \forall \ i$. Therefore for the ball $B_n(x_{n,1})$ in a given cover $C_n$, there exists a subsequence of the set $\{A(i)\}$ for which the corresponding subsequence of $\left\{ \int_{B_n(x_{n,1})} |F(i)|^2 \, dx \right\}$ converges. Of this subsequence, there exists a further subsequence such that the corresponding subsequence of $\left\{ \int_{B_n(x_{n,2})} |F(i)|^2 \, dx \right\}$ converges, and so on, for every $m$. Diagonalizing over these nested subsequences, we obtain a subsequence of $\{A(i)\}$ such that the corresponding subsequence of $\left\{ \int_{B_n(x_{n,m})} |F(i)|^2 \, dx \right\}$ converges for every $m \in \mathbb{N}$.

Performing a similar diagonalization over all covers $C_n$, there exists a subsequence $\{A(i)\}_{I^*} \subset \{A(i)\}_I$ such that for every ball in every cover, the sequence $\left\{ \int_{B_n(x_{n,m})} |F(i)|^2 \, dx \right\}_{I^*}$ converges. For each $C_n$, consider the collection of balls $\{B_n(y_{n,m})\}$, $\{y_{n,m}\} \subset \{x_{n,m}\}$, for which $\left\{ \int_{B_n(y_{n,m})} |F(i)|^2 \, dx \right\}_{I^*}$ converges to a value greater than or equal to $\varepsilon$. Note that for any $i \in I$, there is an upper bound on the number $N_{i,n}$ of disjoint balls of radius $\frac{1}{n}$ for which $\int_{B_n(y_{n,m})} |F(i)|^2 \, dx \geq \varepsilon$:

$$B \geq N_{i,n}\varepsilon.$$ 

Thus the upper bound $\frac{B}{\varepsilon}$ limits the number of disjoint balls in the set $\{B_n(y_{n,m})\}$.

Choose a maximal disjoint set $\{B_n(y_{n,m_j})\}_{j=1}^J$ of balls in $\{B_n(y_{n,m})\}$, and consider the set $\{B_n(y_{n,m_j})\}_{j=1}^J$ of balls centered at the points $y_{n,m_j}$ but having radius $\frac{1}{n}$. Then we have $\bigcup_{j=1}^J B_n(y_{n,m_j}) \subset \bigcup_{j=1}^J B_n^*(y_{n,m_j})$. This shows that if we discard the balls $\{B_n(y_{n,m})\}$ from the cover $C_n$, we will only have discarded a set which was contained in $J \leq \frac{B}{\varepsilon}$ balls of radius $\frac{1}{n}$. We can then safely discard the balls $\{B_n(y_{n,m})\}$ from each cover $C_n$, and form the union $C = \bigcup_{n \in \mathbb{N}} C_n \setminus \{B_n(y_{n,m})\}$ to obtain a cover $C$ of $M \setminus \{x_1, \ldots, x_k\}$, where $k \leq \frac{B}{\varepsilon}$, and each ball $B_n(x_{n,m}) \subset C$ satisfies $\int_{B_n(x_{n,m})} |F(i)|^2 \, dx < \varepsilon$.

Since a minimizing sequence $\{A(i)\}_{i \in I}$ by definition admits a uniform bound on the action, we can use Theorem 4.2 to select a subsequence $\{A(i)\}_{i \in I'}$ of the minimizing sequence and a cover $\{U_n\}$ satisfying $\int_{U_n} |F(i)|^2 \, dx < K(4) \forall \ i \in I'$, $i > I$. On any neighborhood $U$ of the cover, Theorem 4.1 implies that each member $A_n(i)$ of the subsequence is gauge-equivalent to a connection $A_n(i)$ in the Hodge gauge, satisfying a uniform $W^{1,2}(U)$ bound on $A_n(i)$. Weak compactness of Sobolev spaces now yields a further subsequence of $\{A_n(i)\}$.

1Diagonalizing over a list of sequences $\{a_j(i)\}$ such as $\{a_1(1), a_1(2), a_1(3), \ldots\}$, $\{a_2(1), a_2(2), a_2(3), \ldots\}$, and so on, extracts the new sequence $\{a_i(i)\}$. In the case important for this proof, each row represents a subsequence of the previous row, so that for any $j$, the diagonalized sequence $\{a_k(k)\}$ is a subsequence of $\{a_j(i)\}$ for $k \geq j$. 

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weakly convergent in $W^{1,2}$ to some $\hat{A}_\alpha$. It only remains to show that $\hat{A}_\alpha$ retains the desired regularity properties and boundary data, and that the set $\{\hat{A}_\alpha\}$ can be patched to a global connection on $M$. These objectives are accomplished by Theorem 3.4 in [9] (generalizing Theorem 3.6 in [10] and Theorem 3.1 in [14]). Their results are paraphrased below; the proof is by weak compactness of Sobolev spaces:

**Theorem 4.3.** Let $\{A(i)\}_{i \in I}$ be a sequence of $G$-connections with uniformly bounded action as described in Theorem 4.2 and with prescribed smooth tangential boundary components $(A(i))_\partial = a_\tau$ on $\partial M$. Let $\varepsilon = K(4)$, where $K(4)$ is the constant from Theorem 4.4. Then, for the subsequence $\{A(i)\}_{i \in I'}$, found in Theorem 4.2 and cover $\{U_\alpha\}$, there exists a further subsequence $\{A(i)\}_{i \in I''}$ and connections $A_\alpha$ on $U_\alpha$ such that

(c) $\sigma^\alpha_\beta(i) (A_i) = A_\alpha - A_\beta$ in $W^{1,2}(U_\alpha)$

(f) $F(A_\alpha(i)) = F_\alpha(i) \to F_\alpha$ in $L^2(U_\alpha)$

(g) $s_{\alpha\beta}(i) \to s_{\alpha\beta}$ in $W^{2,2}(U_\alpha \cap U_\beta)$

(h) $(A_\alpha)_{|\partial U_\alpha} \sim a_\tau_{|\partial U_\alpha}$ by a smooth gauge transformation

(i) $d * A_\alpha = 0$ on $U_\alpha$

(j) $d_\tau * (A_\alpha)_\partial = 0$ on $\partial U$

Here $s_{\alpha\beta}(i)$ is the transition function $A_\beta(i) \to A_\alpha(i)$; i.e,

$$A_\alpha(i) = s_{\alpha\beta}^{-1}(i) A_\beta(i) s_{\alpha\beta}(i) + s_{\alpha\beta}^{-1}(i) d s_{\alpha\beta}(i).$$

**Proof.** See [14], [9]. Note that the result follows by weak compactness of Sobolev spaces, after applying diagonalization (as in the proof of Theorem 4.2) over the countable cover $\{U_\alpha\}$.

Lower semicontinuity of the Yang-Mills functional now implies that the value of the action on the limiting connection $A$ of the sequence described in Theorem 4.3 is in fact $m(a_\tau) = \min I(A)$ where $A$ is the set of connections on $G$-bundles on $M$ such that $A_\tau_{|\partial M}$. In Theorem 3.5 in [9] and 4.1 in [14], it is proved by contradiction that $A$ in fact satisfies the Yang-Mills equations. These proofs are completely local, and hold unchanged in our case.

The proofs of regularity of the connection in Hodge gauge are also local and hold unchanged. Regularity except for at the points $\{x_1, ..., x_k\}$ from Theorem 4.2 is a consequence of the ellipticity of the Yang-Mills equations in Hodge gauge. At the points $\{x_1, ..., x_k\}$, the limiting connection may not be defined, so removable singularity theorems are needed to extend $A$ to these points. The case of interior points is covered by Theorem 4.1 in [17], and that of boundary points by Theorem 4.6 in [9], so that the connection $A$ extends to a smooth connection (provided the Dirichlet boundary data is smooth). More precisely, we have

**Theorem 4.4.** Let $U^{(1)}$ ($U^{(2)}$) be a neighborhood of type 1 (2); let $U^{(1)}_\alpha = U^{(1)} \setminus \{0\}$. Let $A$ be a connection in a bundle $P$ over $U^{(1)}_\alpha$, $\|F_A\|_{L^2(U)} < B < \infty$. 
Then

(Type 1) If \( A \) is Yang-Mills on \( P|U_*^{(1)} \), there exists a \( C^\infty \) connection \( A_0 \) defined on \( U^{(1)} \) such that \( A_0 \) is gauge-equivalent to \( A \) on \( U_*^{(1)} \).

(Type 2) If \( A \) is Yang-Mills and \( C^\infty \) on \( P|U_*^{(2)} \), there exists a \( C^\infty \) connection \( A_0 \) defined on \( U^{(2)} \) such that \( A_0 \) is gauge-equivalent to \( A \) on \( U_*^{(2)} \), by a gauge transformation in \( C^\infty (U_*^*) \).

Proof. See [17], [9].

5 The Euclidean Yang-Mills Hamilton-Jacobi functional

Having shown the existence of an absolute minimizer \( A_t \) for the Euclidean Yang-Mills action given prescribed smooth initial tangential components \( A = a_\tau \), we can now define the Hamilton-Jacobi functional\(^2\)

\[
S(A) = \int_{\mathbb{R}_+ \times \mathbb{R}^3} tr (F_{A_t} \wedge * F_{A_t}) \, dt,
\]

where \(*\) indicates the Hodge star operator in the Euclidean metric.

The values of this functional are well-defined even allowing for the possible existence of more than one gauge-equivalence class of minimizers for the given initial data; in principle we can simply choose a minimizer starting from the field configuration \( A = a_\tau \). However, while still an open question, there exist partial results toward establishing uniqueness of a minimizer for given initial data in the compact case. In [9], Isobe has shown that for flat boundary values, the Dirichlet problem on a star-shaped bounded domain in \( \mathbb{R}^n \) can only have a flat solution. Non-uniqueness results are proven by Isobe and Marini [4] for Yang-Mills connections in bundles over \( B^4 \), but the solutions are topologically distinct, belonging to differing Chern classes. On the domain \( M = \mathbb{R}_+ \times \mathbb{R}^3 \), it seems likely that given initial data determines a minimizer unique up to gauge transformation. Future work will aim to settle this question; one possible means of approach is a conformal transformation to the compact case, as described in Sect. [3].

In order to make the claim that \( S(A) \) solves the imaginary-time zero-energy Yang-Mills Hamilton-Jacobi equation, we must also verify its functional differentiability. This can be done using the same integration by parts argument as in the derivation of the Euler-Lagrange equation. However, we must first write the solution to the Euclidean Dirichlet problem in a global gauge which is smooth and decays sufficiently rapidly at spatial and temporal infinity.

\(^2\)Note that for \( S(A) \) to be finite, we are implicitly making the assumption that for all initial data \( A \) of physical interest, there exists at least one trajectory \( A_s (A_{s=0} = A) \) such that \(-I(A_s) < \infty\). This constraint defines the set of physical fields, since for any \( A \) on \( \mathbb{R}^3 \) for which no such \( A_s \) can be found, allowing \( S(A) \) to take an infinite value implies that evaluated on this \( A \), the ground state \( \Omega(A) \) is zero.
First, Theorem 4.4 implies that the solution $A$ to the Yang-Mills Dirichlet problem extends to a smooth connection on a smooth bundle over all of $M = \mathbb{R}_+ \times \mathbb{R}^3$. Since the only bundle over a contractible base manifold is the trivial one (see e.g. [14]), $A$ is also a connection on the trivial bundle $P \cong M \times G$. Therefore we can write $A$ in terms of a smooth global section $\sigma : M \to G$. Using this trivialization, $D = d + A$ is smoothly defined over all of $M$.

The following lemma controls the growth of $A$ and $F$, for a good choice of gauge. Part (a) is a version of Uhlenbeck’s Corollary 4.2 [17] for our base manifold $M = \mathbb{R}_+ \times \mathbb{R}^3$; part (b) extends the same principle to bound the growth of the connection 1-form $A$.

**Lemma 5.1.** Let $D = d + A$ be a connection in a bundle $P$ over an exterior region $\mathcal{V} = \{ y \in \mathbb{R}_+ \times \mathbb{R}^3 : |y| \geq N \}$ satisfying $\int_\mathcal{V} |F|^2 < \infty$. Then

(a) $|F| \leq C |y|^{-4}$ for some constant $C$ (not uniform);

(b) There exists a gauge in which $D = d + \tilde{A}$ satisfies $|\tilde{A}| \leq K |y|^{-2}$.

**Proof.** (a) Following the reasoning in [17], we define the conformal mapping

$$f : U_\ast \to \mathcal{V},$$

$$y = f(x) = N \frac{x}{|x|^2},$$

where $U_\ast = \{ x \in \mathbb{R}_+ \times \mathbb{R}^3 : 0 < |x| \leq 1 \}$. By conformal invariance of the Yang-Mills action, we have

$$\int_{U_\ast} |f^* F|^2 = \int_{\mathcal{V}} |F(f^* D)|^2 = \int_{\mathcal{V}} |F|^2.$$

Applying part (b) of Theorem 4.4 to the pullback $f^* D$ of $D$ under $f$, there exists a gauge transformation $\sigma : U_\ast \to G$ in which $f^* D$ extends smoothly to $U$. Thus using the transformation law for 2-forms, we have the following

$$|F(y)| = |f^* F(x)| |df(x)|^{-2} \leq \max_{x \in U} |f^* F(x)| \left( N/|x|^2 \right)^{-2} = C' N^2 |y|^{-4}.$$

(b) Define the gauge transformation $s = \sigma \circ f^{-1} : \mathcal{V} \to G$. Denoting $A^* = s^{-1} ds + s^{-1} A s$ by $\tilde{A}$ and $(f^* A)^\sigma = \sigma^{-1} d\sigma + \sigma^{-1} (f^* A) \sigma$ by $\tilde{f^* A}$, we have $f^* \tilde{A} = \tilde{f^* A}$. Thus again applying Theorem 11(b) and using the transformation law for 1-forms,

$$|\tilde{A}(y)| = |f^* \tilde{A}(x)| |df(x)|^{-1} \leq \max_{x \in U} |f^* \tilde{A}(x)| \left( N/|x|^2 \right)^{-1} = C'' N |y|^{-2}. $$

\[\square\]
We are now ready to prove differentiability of our Hamilton-Jacobi functional. Thanks are due to V. Moncrief for suggesting the form of this argument.

**Theorem 5.2.** The functional

\[ S(A) = -\tilde{I}(A) = \int_{\mathbb{R}_+ \times \mathbb{R}^3} \text{tr} \left( F_{A_t} \wedge F_{A_t} \right) dt \]

is functionally differentiable, and \( \frac{dS}{dA} = E = \dot{A}_{t=0} \).

**Proof.** To find the functional derivative of \( S(A) = -\tilde{I}(A) \) at a given connection \( A_0 \) on the slice \( x^0 = 0 \), consider the 1-parameter family \( A_0 + \lambda h \), constructing

\[
\left. \frac{d}{d\lambda} [S(A_0 + \lambda h)] \right|_{\lambda=0} = \lim_{\lambda \to 0} \frac{S(A_\lambda) - S(A_0)}{\lambda} = \lim_{\lambda \to 0} \frac{-\tilde{I}(A_{\lambda,t}) - \left( -\tilde{I}(A_{0,t}) \right)}{\lambda} = -\lim_{\lambda \to 0} \frac{1}{\lambda} \left[ \tilde{I}(A_{\lambda,t}) - \tilde{I}(A_{0,t}) \right]
\]

where for each \( A_\lambda = A_0 + \lambda h \), \( A_{\lambda,t} \) denotes the absolute minimizer of \( -\tilde{I} \) given initial data \( A_\lambda \). For any given value \( \lambda_0 \), the difference \( \tilde{I}(A_{\lambda_0,t}) - \tilde{I}(A_{0,t}) \) can be expressed in terms of a Taylor series, as follows. First, use the parameter \( \lambda \) to interpolate between \( A_{0,t} \) and \( A_{\lambda_0,t} \), describing a 1-parameter family \( X_{\lambda,t} \),

\[
X_{\lambda,t} \equiv \frac{\lambda}{\lambda_0} A_{\lambda_0,t} + \left( 1 - \frac{\lambda}{\lambda_0} \right) A_{0,t},
\]

so that \( X_{\lambda,0} = A_\lambda \). The standard Taylor series expansion of \( \tilde{I}(X_{\lambda,t}) \) as a function of \( \lambda \) then gives

\[
\tilde{I}(A_{\lambda_0,t}) - \tilde{I}(A_{0,t}) = \lambda_0 \left( \frac{\partial \tilde{I}}{\partial \lambda} \right)_{\lambda=0} + O(\lambda_0^2).
\tag{12}
\]

Let \( h_t = \frac{1}{\lambda_0} (A_{\lambda_0,t} - A_{0,t}) \), so that \( X_{\lambda,t} = A_{0,t} + \lambda h_t \). Then

\[
\left. \frac{\partial \tilde{I}}{\partial \lambda} \right|_{\lambda=0} = \frac{\partial}{\partial \lambda} \left[ \int_{\mathbb{R}_+ \times \mathbb{R}^3} \langle F_{X_{\lambda,t}}, F_{X_{\lambda,t}} \rangle \right]_{\lambda=0} = 2 \int_{\mathbb{R}_+ \times \mathbb{R}^3} \langle dh_t + [A_{0,t}, h], F_{A_{0,t}} \rangle = 2 \lim_{R \to \infty} \left( \int_{\partial_1} \langle h, F_{A_0} \rangle + \int_{\partial_2} \langle h_t, F_{A_{0,t}} \rangle - \int_{0 \leq |x| < R} \langle h_t, D^* F_{A_{0,t}} \rangle \right)
\]

where \( \partial_1 = \{|x| < R, x^0 = 0\} \), \( \partial_2 = \{|x| = R, x^0 > 0\} \).
The last term on the right-hand side vanishes due to the fact that $F_{A_0,t}$ is a solution to the Yang-Mills equations. Working with $A_{\lambda_0,t}$ and $A_{0,t}$ both in the gauge guaranteed by Lemma 5.1 (for some fixed $N$ which $R$ eventually surpasses), the middle term also approaches zero as $R$ approaches infinity, since

$$\langle h_t, F_{A_0,t} \rangle \leq |h_t| |F_{A_0,t}| \leq \frac{1}{\lambda_0} (|A_{\lambda_0,t}| + |A_{0,t}|) |F_{A_0,t}| \leq \frac{1}{\lambda_0} (K_{\lambda_0} + K_0) C_0 \cdot R^{-6}.$$ 

Since the area element on $\partial_2$ contributes only a factor of $R^2$, the middle term is easily seen to vanish. Thus we are left with only the first term, so that

$$\left. \frac{\partial I}{\partial \lambda} \right|_{\lambda=0} = \int_{\mathbb{R}^3} \langle h, F_{A_0} \rangle,$$

and the definition of functional derivative implies that

$$\frac{\delta S}{\delta A} = E = \dot{A}_t = 0.$$

### 6 Gauge and Poincare invariance

In order for the candidate ground state wave functional

$$\Omega(A) = N \exp (-S(A))$$

to be physical, it must remain invariant under the action of gauge transformations $g(x), x \in \mathbb{R}^3$, on the connection $A(x)$:

$$S (g^{-1}dg + g^{-1}Ag) = S(A),$$

so that $S$ is in fact a functional on the physical configuration space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations, rather than the kinematical configuration space $\mathcal{A}$. Gauge invariance of $S$ follows immediately from its form

$$S(A) = - \int_0^\infty \tilde{L} \left( A_t, \dot{A}_t \right) dt = \int_{\mathbb{R}^+ \times \mathbb{R}^3} \text{tr} (F_{A_t} \wedge * F_{A_t})$$

where $*$ denotes the Hodge star operator in the Euclidean metric on $\mathbb{R}^+ \times \mathbb{R}^3$. The gauge transformation $g(x), x \in \mathbb{R}^3$, can simply be extended to $\mathbb{R}^+ \times \mathbb{R}^3$ by taking $g(t,x) = g(x)$ constant over $\mathbb{R}^+$, and the cyclic property of the trace implies

$$S (g^{-1}dg + g^{-1}Ag) = \int_{\mathbb{R}^+ \times \mathbb{R}^3} \text{tr} (F_{gA_t} \wedge * F_{gA_t})$$

$$= \int_{\mathbb{R}^+ \times \mathbb{R}^3} \text{tr} (F_{A_t} \wedge * F_{A_t}) = S(A).$$
Similarly, rotations and translations applied to \( \mathbb{R}^3 \) do not affect the value of \( S(A) \), because we can extend them constantly through time over \( \mathbb{R}_+ \times \mathbb{R}^3 \), and by a change of coordinates the value of the integral defining \( S(A) \) is unchanged. The only remaining Poincare transformations are boosts, which cannot be verified directly in our canonical framework. The conserved quantity generating an infinitesimal boost in the \( x^i \) direction is

\[
C_{B(i)} = \int_{\mathbb{R}^3} (x^0 \delta^i_\mu + x^i \delta^0_\mu) T^\mu_0 d^3x,
\]

where \( T^{\mu\nu} = -\frac{1}{16\pi} tr \left\{ F_\alpha^\mu F_\alpha^\nu - \frac{1}{4} \eta^{\mu\nu} F_\alpha F_\alpha^\beta \right\} \) is the stress-energy tensor of Yang-Mills theory. This infinitesimal generator must be promoted to a quantum operator which annihilates our candidate ground state. A test case in which this can be done is the abelian case of \( U(1) \) gauge theory (free Maxwell theory), using Wheeler’s zero-energy ground state \( \Omega \) as in Sect. 1. This is alternatively attainable as

\[
\Omega(A) = \mathcal{N} \exp \left( -\langle A, (\ast d) \Delta^{-1/2} (\ast d) A \rangle_2 \right)
\]

by using the normal ordering as described in Sect. 3 to write the unique positive square root of the operator \( \ast d \ast d \) in the form \( (\ast d) \Delta^{-1/2} (\ast d) \). Writing the operator \( \Delta^{-1/2} \) in terms of its integral kernel shows the equality of this state with \( \Omega \). Invariance under infinitesimal boosts follows by expressing \( \Omega \) as the sum of a translation generator \( \frac{x}{8\pi} \int \epsilon^{ijk} E_j B_k d^3x \) plus the term \( \frac{x}{8\pi} \int x^i \left( |E|^2 + |B|^2 \right) d^3x \). Translation invariance being already established, it only remains to verify that \( \Omega \) is annihilated by the remaining term under our ordering. Indeed, the functional \( S(A) \) in the exponent of \( \Omega \) can be directly shown to satisfy

\[
\int_{\mathbb{R}^3} x^i \left| \frac{\delta S}{\delta A} \right|^2 d^3x = \int_{\mathbb{R}^3} x^i |B|^2 d^3x,
\]

allowing the extra term to be ordered in the same way as the Hamiltonian. Using the abelian case as a model, we hope to extend invariance under boosts to the nonabelian case in future work.

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