ONE-CYCLES ON GUSHEL-MUKAI FOURFOLDS AND
THE BEAUVILLE-VOISIN FILTERATION

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Abstract. We prove that the invariant locus of the involution associated with a general double EPW sextic is a constant cycle surface and introduce a filtration on $CH_1$ of a Gushel-Mukai fourfold. We verify the sheaf/cycle correspondence for sheaves supported on low-degree rational curves, parallel to the cubic fourfolds case of Shen-Yin’s work.

1. Introduction

A Gushel-Mukai (GM) fourfold is a smooth prime Fano fourfold $X \subset \mathbb{P}^8$ of degree 10 and index 2. Throughout the paper, a GM fourfold is always an ordinary GM fourfold, which can be obtained as a smooth dimensionally transverse intersection

$$X = \text{Gr}(2, 5) \cap \mathbb{P}^8 \cap Q \subset \mathbb{P}^9,$$

where $\text{Gr}(2, 5) \subset \mathbb{P}^9$ is the Plücker embedding, $\mathbb{P}^8$ is a linear subspace and $Q$ is a quadric hypersurface.

GM fourfolds share many properties with cubic fourfolds. One of the distinguished facts is that there is a parallel between GM and cubic fourfolds from a categorical viewpoint. Similarly to the semiorthogonal decomposition of the derived category of a cubic fourfold, Kuznetsov and Perry showed in [15] that the derived category of a GM fourfold admits a semiorthogonal decomposition:

$$D^b(X) = \langle A_X, \mathcal{O}_X, \mathcal{U}_X, \mathcal{O}_X(1), \mathcal{U}_X(1) \rangle,$$

where $\mathcal{U}_X$ is the tautological bundle of $\text{Gr}(2, 5)$ restricted to $X$ and $A_X$ is a K3 category.

Another fact is that a GM fourfold has an associated irreducible holomorphic symplectic (IHS) variety, which is called the dual double EPW sextic, compared with the Fano variety of lines associated with a cubic fourfold, see [3][19][14] for details.

The Fano variety of lines of a cubic fourfold and the dual double EPW sextic of a GM fourfold both admit constructions via moduli spaces of stable objects and Hilbert schemes of rational curves. In [22] Proposition 5.17, it has been shown that at least one of the double EPW sextic and the dual double EPW sextic can be realized as a moduli space of stable objects of $A_X$ and conjecturally both are. In [14], Iliev and Manivel gave a more geometrical description of the dual double EPW sextic. For a general GM
fourfold $X$, let $F(X)$ be the Hilbert scheme of conics on $X$ and $\tilde{Y}^\vee$ be the associated dual double EPW sextic of $X$, there exists a morphism

$$F(X) \to \tilde{Y}^\vee$$

such that a general fiber is $\mathbb{P}^1$.

Quite generally, for a $2n$-dimensional IHS variety $M$, Voisin predicted in [32] that there exists a filtration of the Chow ring $\text{CH}^*(M)$ (called Beauville–Voisin filtration), which can be viewed as an opposite to the conjectural Bloch-Beilinson filtration. For 0-cycles, the filtration $S_i \text{CH}_0(M)$ is defined by

$$S_i \text{CH}_0(M) := \langle x \in M \mid \dim O_y \geq n - i \rangle,$$

where $O_y$ is the orbit of $y$ under the rational equivalence (here we follow the definition in [24], which is opposite to Voisin’s original definition). Let $Z \subset M$ be a subvariety. If any two points on $Z$ have the same class in $\text{CH}_0(M)$, we call that $Z$ is a constant cycle variety on $M$. In particular, for an IHS variety of dimension 4, the filtration is determined by constant cycle surfaces and uniruled divisors. Let $S_i M \subset M$ be the set of points with $\dim O_y \geq n - i$. Voisin conjectured in [32, Conjecture 0.4] that

$$\dim S_i M = n + i.$$

(1.1)

It is closely related to the existence of algebraically coisotropic subvarieties on $M$.

Let $M$ be a nonsingular projective moduli space of stable objects on a K3 category $\mathcal{A}$, then it is an IHS variety in many cases. Shen and Yin predicted that all the Beauville–Voisin filtrations associated with different moduli spaces of stable objects on a fixed K3 category should be controlled by a universal filtration on the Grothendieck group of the K3 category, see [24, Speculation 0.1].

When the K3 category is the derived category of a K3 surface $S$ and $M$ is a $2d$-dimensional nonsingular moduli space of stable objects, the O’Grady’s filtration on $\text{CH}_0(S)$

$$S_i(S) := \bigcup_{\deg([z])=i} \{ [z] + \mathbb{Z} \cdot [o_S] \}$$

serves as the universal filtration for all moduli space of stable objects. It has been shown in [25] that

$$c_2(\mathcal{E}) \in S_d(S)$$

for any $\mathcal{E} \in M$ and that $c_2(\mathcal{E}) \in S_i(S)$ implies $\mathcal{E} \in S_i \text{CH}_0(M)$.

Another example arises when the K3 category is the Kuznetsov component $\mathcal{A}_X$ of a cubic fourfold $X$. Using the Beauville-Voisin filtration on the Fano variety of lines on $X$, Shen-Yin introduced in [25] a filtration $S_•(X)$ on $\text{CH}_1(X)$ induced by the incidence correspondence. They conjectured that the filtration $S_•(X)$ may serve as a universal filtration on $K_0(\mathcal{A}_X)$, i.e., for $\mathcal{E} \in \mathcal{A}_X$, there should be

$$c_3(\mathcal{E}) \in S_d(X),$$

where $d = \frac{1}{2} \dim \text{Ext}^1(\mathcal{E}, \mathcal{E})$. This implies [24] by a standard argument and they verified the prediction when $\mathcal{E}$ is supported on low-degree rational curves.
It is natural to ask whether there exists a universal filtration on $A_X$ for a GM fourfold, similarly to the case of a cubic fourfold. We first study the Beauville–Voisin filtration on $CH_0(\tilde{Y}^\vee)$ by finding a natural constant cycle surface on $\tilde{Y}^\vee$. The invariant locus $Z$ of the involution associated with the dual double EPW sextic is a surface of general type, whose $CH_0$ might be huge. However, it was asked whether $Z$ is a constant cycle surface on $\tilde{Y}^\vee$, see [17, Remark 2.7]. Our first result is (Theorem 4.14):

**Theorem 1.1.** The invariant locus $Z$ of the involution associated with a general dual double EPW sextic $\tilde{Y}^\vee$ is a constant cycle surface. If $\tilde{Y}^\vee$ is very general, any other constant cycle surface $Z'$ meets $Z$.

The $0$-piece of the Beauville–Voisin filtration of $\tilde{Y}^\vee$ is represented by a point on $Z$ and the $1$-piece is represented by a point on a uniruled divisor. Theorem 1.1 allows us to construct a filtration $S_\bullet(X)$ on $CH_1(X)$ by the incidence correspondence of Iliev and Manivel’s construction. For a very general GM fourfold, we predict that the filtration should serve as a universal filtration on $A_X$ in the following way:

The Grothendieck group of $A_X$ is shown to be generated by $pr[O_c(1)]$ and $pr[O_p]$ for a very general $X$, where $c$ is a conic, $p$ is a point on $X$ and $pr$ is the projection $pr: K_0(X) \to K_0(A_X)$.

Unlike the cubic fourfold case, the dual double EPW sextic determines the GM fourfold only up to period partners and the Chern class $c_3(pr[O_p])$ of a skyscraper sheaf might lie in different pieces of the filtration with respect to different GM fourfolds. We define a modification

$$p: K_0(A_X) \to CH_1(X)$$

of $c_3$ by dropping the effect of $pr[O_p]$, see Section 6 for details.

We propose the following conjecture relating the $K3$ category $A_X$ to the filtration $S_\bullet(X)$ as a parallel to the case of a cubic fourfold.

**Conjecture 1.2.** For any object $E \in A_X$, we have

$$p(E) \in S_d(E)(X).$$

Let $i^*$ be the left adjoint functor of the natural inclusion $A_X \hookrightarrow D^b(X)$. We show that the classes of lines and conics on a very general GM fourfold $X$ are in $S_2(X)$, which implies our second main result concerning sheaves supported on low-degree rational curves (Theorem 6.3):

**Theorem 1.3.** We assume that $X$ is very general. If $\mathcal{F}$ is supported on lines or conics on $X$, conjecture 1.2 holds for $E = i^* \mathcal{F}$.

Our approach to deducing Theorem 1.1 has another application, concerning with the relation between $CH_0(\tilde{Y}^\vee)$ and $CH_1(X)$. When $\tilde{Y}^\vee$ is birational to $S[^2]$ for some K3 surface $S$, it has been shown in [17] that $CH_0(\tilde{Y}^\vee)$ with $\mathbb{Q}$-coefficients has a natural decomposition for the Chow group. For $0$-cycles, the decomposition is

$$CH_0(\tilde{Y}^\vee) = \mathbb{Q} \cdot o \oplus CH_0(S)_{hom} \oplus CH_0(\tilde{Y}^\vee)_{hom}^+.$$
where \( CH_0(\tilde{Y}^\vee)_\text{hom}^+ \) is the \( \iota \)-invariant part of \( CH_0(\tilde{Y}^\vee)_\text{hom} \). We show that in the general case, the group \( CH_1(X) \) would play the role of the group \( CH_0(S) \).

**Theorem 1.4.** *(Theorem 5.5)* When the dual double EPW sextic \( \tilde{Y}^\vee \) is general, we have the following isomorphism of groups:

\[
Z \cdot o \oplus CH_0(\tilde{Y}^\vee)_\text{hom}^- \cong CH_1(X),
\]

where \( CH_0(\tilde{Y}^\vee)_\text{hom}^- \) is the \( \iota \)-anti-invariant part of \( CH_0(\tilde{Y}^\vee)_\text{hom} \).

In particular, the group \( CH_1(X) \) is torsion-free and we see that if \( X' \) is another GM fourfold with the same dual double EPW sextic as \( X \), then their groups of 1-cycles are isomorphic.

Our results rely on Iliev and Manivel’s construction and hence all dual double EPW sextics are assumed to be general in the rest of the paper.

2. **Notation**

The following notations appear frequently in the paper, so we gather them together for the sake of convenience.

- \( V_m \) is an \( m \)-dimensional vector space, \( X \) is a smooth ordinary GM fourfold and \( \tilde{Y}^\vee \) is the dual double sextic associated with \( X \).
- \( \iota \) is the involution of \( \tilde{Y}^\vee \) and \( Z \subset \tilde{Y}^\vee \) is the invariant locus of \( \iota \).
- \( c \) is a conic on \( X \), \( F(X) \) is the Hilbert scheme of conics on \( X \) and \( \alpha : F(X) \to \tilde{Y}^\vee \) is the birational \( \mathbb{P}^1 \)-fibration. \( \alpha \) induces an isomorphism \( \alpha^* : CH_0(F(X)) \cong CH_0(\tilde{Y}^\vee) \).
- \( P \) is the universal conic, \( p, q \) are the projections to \( F(X) \) and \( X \).
- \( \Phi = p_*q^* : CH^*(X) \to CH^*(F) \). \( \Psi \) is the composition of \( q_*p^* \) and \( \alpha_*^{-1} \):

\[
\Psi = q_*p^*\alpha_*^{-1} : CH_0(\tilde{Y}^\vee) \to CH_1(X),
\]

sending \( \alpha(c) \) to its class \([c] \in CH_1(X)\).
- \( S_{V_4} = Gr(2, V_4) \cap X \) for some \( V_4 \subset V_5 \), which is a degree 4 Del Pezzo surface.
- \( D_c \) is the subvariety of \( F(X) \) parameterizing conics intersecting the given conic \( c \). \( I \subset F(X) \times F(X) \) is the subvariety of \( F(X) \times F(X) \), parameterizing two intersecting conics.

3. **Preliminaries**

Let \( X = Gr(2, V_5) \cap H \cap Q \) be a very general GM fourfold and \( \tilde{Y}^\vee \to Y^\vee \) be its associated double cover of the dual EPW sextic. Denote by \( \iota : \tilde{Y}^\vee \to \tilde{Y}^\vee \) the associated involution and by \( Z \) the invariant locus of \( \iota \). We recall some facts about GM varieties and EPW sextics in this section. Throughout the paper, \( V_m \) is an \( m \)-dimensional vector space.

3.1. **EPW sextics.** Debarre and Kuznetsov gave a way to associate a GM variety with an EPW sextic in [3] via linear algebra. We will follow [4] to recall some basic facts about EPW sextics. A GM variety \( X_n = Gr(2, V_5) \cap \mathbb{P}(W) \cap Q \) of dimension \( n \) is a smooth intersection of a linear subspace of dimension \( n + 4 \), a quadratic hypersurface and \( Gr(2, V_5) \) inside \( \mathbb{P}(\wedge^2 V_5) \). It has an associated GM data \((W_{n+5}, V_6, V_5, q)\), where
• $V_6$ is a 6-dimensional vector space, consisting of quadrics containing $X$.
• $V_5$ is a hyperplane section of $V_6$, consisting of quadrics containing $\text{Gr}(2, V_5)$.
• $W_{n+5} \subset \bigwedge^2 V_5$ is a subspace of dimension $n + 5$, corresponding to the linear subspace in the definition of GM variety.
• $q : V_6 \rightarrow \text{Sym}^2 W_{n+5}$, is a linear map such that (here we choose an isomorphism $\bigwedge^5 V_5 \cong \mathbb{C}$)
  \[ \forall v \in V_5, q(v)(w, w) = v \wedge w \wedge w. \]
Then $X = \bigcap_{v \in V_6} Q(v) \subset \mathbb{P}(W_{n+5})$, where $Q(v)$ is defined by $q(v)$. For a general GM data, $X$ is a smooth GM variety.

On the other hand, $\bigwedge^3 V_6$ can be endowed with a symplectic form by wedge product. Debarre and Kuznetsov defined the Lagrangian data set $(V_6, V_5, A)$ with $A \subset \bigwedge^3 V_6$ a Lagrangian subspace and $V_5 \subset V_6$ a hyperplane. It gives an EPW data

$Y_A^{\geq l} := \{[v] \in \mathbb{P}(V_6) \mid \dim(A \cap (v \wedge \bigwedge^2 V_6)) \geq l\}$
and an extra $V_5$. Here, $Y_A^{\geq 1}$ is a sextic in $\mathbb{P}(V_6)$ and $Y_A^{\geq 2}$ is a surface. In [20], O’Grady showed that

**Theorem 3.2.** If $A$ contains no decomposable vectors and $Y_A^{\geq 3} = \emptyset$, there is a double cover $\tilde{Y}_A \rightarrow Y_A^{\geq 1}$ branched along the surface $Y_A^{\geq 2}$. $\tilde{Y}_A$ is a smooth IHS variety, called the double EPW sextic.

Iliev and Manivel described in [14] another way to construct the EPW sextic associated with a GM variety $X$. Define $\text{Disc}(X)$ to be the subscheme of $\mathbb{P}(V_6)$ of singular quadrics containing $X$. It is a hypersurface of degree $n + 5$ and the multiplicity of the hyperplane $\mathbb{P}(V_5)$ of Plücker quadrics is at least $n - 1$. Then

$\text{Disc}(X) := \tilde{\text{Disc}}(X) - (n - 1)\mathbb{P}(V_5)$
equals the EPW sextic $Y_A$.

One can use the same construction for the Lagrangian subspace $A^\vee \subset \bigwedge^3 V_6^\vee$, the corresponding $\tilde{Y}_A^\vee$ is called the dual double EPW sextic. We mainly deal with the dual double EPW sextic associated with a GM variety and abbreviate it as $\tilde{Y}^\vee$ in the following sections.

There is a bijection between the GM data sets and the Lagrangian data sets by [4, Theorem 2.2]. However, the GM fourfolds associated with a fixed dual EPW sextic are not unique. Debarre and Kuznetsov proved the following [7, [4, Theorem 2.6]:

**Theorem 3.3.** Let $M^{GM}$ and $M^{EPW}$ be the coarse moduli spaces of ordinary GM $n$-folds and EPW sextics. Then there exists a surjective morphism:

$\pi : M^{GM} \longrightarrow M^{EPW}$

with fibre $\pi^{-1}(A^\vee) = Y_A^{\geq 3-n}$.

If two GM fourfolds are in the same fiber of $\pi$, we say that they are period partners.
3.4. Lines and Conics on $X$. In this subsection, we gather some facts about lines and conics on a general GM fourfold $X$ following [14, Section 3] and [6].

Let $F(X)$ be the Hilbert scheme of conics and $F_1(X)$ the Hilbert scheme of lines on $X$. $F(X)$ is a smooth fivefold. For a conic $c \subset X$, we denote by $(c)$ the plane spanned by $c$. For $V_4 \subset V_5$, we set

$$S_{V_4} = \text{Gr}(2, V_4) \cap H \cap Q \subset \mathbb{P}(\bigwedge^2 V_4) \cap H \cong \mathbb{P}^4,$$

which is a degree 4 Del Pezzo surface.

The hyperplane $H$ can be viewed as a two-form $\omega$ on $V_5$ with a one-dimensional kernel $W_1$. Let $V_i \subset V_5$ be a subspace of dimension $i$. There are three types of conics on $X$:

1. a $\tau$-conic: The plane $(c)$ spanned by the conic $c$ is not contained in $\text{Gr}(2, V_5)$. The normal bundle of a smooth $\tau$-conic $c$ is $N_{c/X} = \mathcal{O}_c \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1)$. A general conic is a $\tau$-conic.

2. a $\rho$-conic: The plane $(c)$ is of the form $\mathbb{P}(V_y^\omega)$ and contained in $H$, or equivalently, $W_1 \subset V_3$ and $V_3$ is isotropic for $\omega$. For a general $X$, the normal bundle of a smooth $\rho$-conic $c$ is $N_{c/X} = \mathcal{O}_c \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1)$. They form a 3-dimensional subvariety in $F(X)$.

3. a $\sigma$-conic: The plane $(c)$ is of the form $\mathbb{P}(V_1 \wedge V_4)$ and contained in $H$, or equivalently, $V_4 \subset V_1^{\perp \omega}$. For $V_1 \neq W_1$, $\dim V_1^{\perp \omega} = 4$. They are parameterized by $Bl_{[W_1]}(\mathbb{P}(V_5))$ and form a 4-dimensional subvariety in $F(X)$.

We see that all the $\rho$-conics (resp. $\sigma$-conics) are represented by the same class in $\text{CH}_1(X)$ and denote it by $\rho$ (resp. $\sigma$) for abuse of notations.

Lines on $X$ are related to the double EPW sextic $\tilde{Y}_A$. The general $\sigma$-conics which are unions of two lines are parameterized by

$$[V_1] \in Y_{\tilde{A}}^{\geq 1} \cap \mathbb{P}(V_5)$$

and a general line is one of the components of a degenerate $\sigma$-conic by [6, Theorem 4.7]. More precisely, the Hilbert scheme of lines $F_1(X)$ is the blow-up of $Y_{\tilde{A}} \times_{\mathbb{P}(V_6)} \mathbb{P}(V_5)$ at a point. There is a rational involution

$$\iota_1 : F_1(X) \dasharrow F_1(X),$$

with $l \cup \iota_1(l)$ a $\sigma$-conic.

The Fano variety of conics $F(X)$ admits birationally a $\mathbb{P}^1$-fibration to $\tilde{Y}^\vee$ in the following way. A conic $c \subset X$ spans a degree 2 surface in $\mathbb{P}(V_5)$, which is degenerate and hence contained in some $\mathbb{P}(V_4)$. Therefore, $c \subset S_{V_4}$. One may check that if $c$ is not of $\rho$-type, the corresponding $V_4$ is unique.

The linear system $[c]$ in $S_{V_2}$ is one-dimensional, which defines the birational $\mathbb{P}^1$-fibration to the dual double EPW sextic, see [14, Proposition 4.18]:

$$\alpha : F(X) \to \tilde{Y}^\vee.$$

The fibration maps all $\sigma$ and $\rho$ conics to two points $y_1$ and $y_2 = \iota(y_1)$ on $\tilde{Y}^\vee$ and further to the Plücker hyperplane $[V_5] \in Y \subset \mathbb{P}(V_6^\vee)$, by [14] Lemma
4.12. In conclusion, we have the following diagram.

\[
P \xrightarrow{p} F(X) \xrightarrow{\alpha} \tilde{Y}^\vee,
\]

where \( P \) is the universal conic. The fiber of \( \alpha \) for \( y \neq y_1, y_2 \) is \( \mathbb{P}^1 \) and hence \( \alpha \) induces an isomorphism

\[\alpha_* : \text{CH}_0(F(X)) \cong \text{CH}_0(\tilde{Y}^\vee).\]

We will frequently use some maps between Chow groups: Let \( \Phi = p_* q^* : \text{CH}^*(X) \to \text{CH}^*(F) \) and \( \Psi \) be the composition of \( q_* p^* \) and \( \alpha_1^{-1} \):

\[\Psi = q_* p^* \alpha_1^{-1} : \text{CH}_0(\tilde{Y}^\vee) \to \text{CH}_1(X),\]

sending \( \alpha(c) \) to its class \([c] \in \text{CH}_1(X)\).

The involution \( \iota \) can be described generically as follows. For \( y \in \tilde{Y}^\vee \), we take a conic \( c \) in the fiber \( \alpha_1^{-1}(y) \) and it is contained in \( S_{V_4} \subset \mathbb{P}^4 \). Then taking a hyperplane in \( \mathbb{P}^4 \) containing \( c \), it intersects with \( S_{V_4} \) along another conic \( c' \). Then

\[\iota(y) = \alpha(c').\]

One can check that for a different choice of \( c \) and a hyperplane in \( \mathbb{P}^4 \), the corresponding \( \alpha(c') \) is the same.

Finally, we need to describe the Hilbert scheme of conics \( F(X_3) \) on a linear section \( X_3 \) of \( X \). Let \( B \subset A^3 V_6 \) be the Lagrangian subspace corresponding to \( X_3 \). Then

\[Y_{\geq 2}^B \subset Y_{\geq 1}^A \]

and \( F(X_3) \) is birational to the pull-back of \( Y_{\geq 2}^B \) in \( \tilde{Y}^\vee \), see [14, Section 5.1].

Remark 3.5. In [33, Section 2.4], it was claimed that the locus of \( \sigma \)-conics is contracted to a constant cycle surface on \( \tilde{Y}^\vee \). However, it is not true, because, by Iliev and Manivel’s construction, the locus of \( \sigma \)-conics should be contracted to a point.

3.6. The Beauville-Voisin conjecture. The Beauville-Voisin conjecture implies that the cycle class map of an IHS variety is injective when restricted to the subring generated by divisors in the Chow ring.

For a double EPW sextic, the Beauville-Voisin conjecture for zero cycles was proved in [10] and [17]. The result is slightly stronger, that is, the cycle class map restricted to the subring

\[R^*(\tilde{Y}^\vee) := \langle \text{CH}^1(\tilde{Y}^\vee), \text{CH}^2(\tilde{Y}^\vee)^+, c_j(\tilde{Y}^\vee) \rangle\]

generated by divisors, \( \iota \)-invariant 2-cycles and Chern classes is injective for \( i = 4 \). In particular, there is a distinguished 0-cycle \( o \) on \( \tilde{Y}^\vee \). For a very general double EPW sextic, \( \text{CH}^1(\tilde{Y}^\vee) \) is generated by \( h \), which is the polarization induced by the double cover of the sextic hypersurface \( Y \subset \mathbb{P}^5 \). Let \( Z \) be the invariant locus of the involution. The following relations have been obtained in [10, Proposition 6.1][10, Lemma 9.3]:

\[3Z = 15h^2 - c_2, Z^2 = 192, h^4 = 12, h^2Z = 40.\]

Moreover, the classes of \( Z^2, h^4 \) and \( h^2Z \) are all proportional to the class \( o \).
4. A CONSTANT CYCLE SURFACE ON $\tilde{Y}^\vee$

This section is devoted to proving Theorem 1.1. In the following parts of the paper, we assume that $X$ is general.

4.1. Two facts about $\text{CH}_1(X)$. We first show that $\text{CH}_1(X)$ is rationally generated by conics on $X$ and those conics represented by points on the invariant surface $Z$ have the same class in $\text{CH}_1(X)$.

**Proposition 4.2.** The group $\text{CH}_1(X)_{\mathbb{Q}}$ is generated by conics and $\text{CH}_1(X)$ is generated by lines on $X$.

**Proof.** We first show that $X$ is connected by chains of conics. Every two general points $x_1, x_2$ on $X$ correspond to two 2-dimensional subspaces in $V_5$, hence the two subspaces are in some 4-dimensional $V_4 \subset V_5$. Thus the two points are contained in $S_4$, which is a degree 4 Del Pezzo surface. Choosing two conic fibrations of $S_4$, we see that the $x_1$ and $x_2$ are connected by a chain of 3 conics. Hence by [27, Proposition 3.1], there is a non-zero integer $N$, such that for any 1-cycle $D$ on $X$,

$$N \cdot D \sim \text{sum of conics}.$$ 

Hence $\text{CH}_1(X)_{\mathbb{Q}}$ is generated by conics.

For integral coefficients, it has been shown in [27, Theorem 1.3] that $\text{CH}_1(X)$ is generated by rational curves. By bend and break, a rational curve $C$ of degree $\geq 3$ on $X$ is algebraically equivalent to a sum of lower-degree rational curves. Moreover, a conic on a Del Pezzo surface is linearly equivalent to a sum of lines, therefore $C$ is algebraically equivalent to a sum of lines $\sum_i n_i L_i$. The argument in [27, Lemma 3.4] shows that $\text{CH}_1(X)_{\text{alg}}$ is divisible. Therefore there exists a 1-cycle $C'$ such that

$$N \cdot C' = C - \sum_i n_i L_i \in \text{CH}_1(X)_{\text{alg}},$$

since $N \cdot C'$ is rationally equivalent to a linear combination of lines, we find that $\text{CH}_1(X)$ is generated by lines with $\mathbb{Z}$-coefficients. \qed

**Lemma 4.3.** For $y \in \tilde{Y}^\vee$, we have $\Psi(y + \iota(y))$ is constant in $\text{CH}_1(X)$. In particular, $\Psi(2y)$ is constant in $\text{CH}_1(X)$ if $y$ is on the invariant surface $Z$.

**Proof.** Since any 0-cycle is rationally equivalent to some 0-cycle supported on an open subscheme, we may assume that $y \in Y^\vee$ is a general point such that $y \notin Z$. We recall the description of the map

$$F(X) \overset{\alpha}{\rightarrow} \tilde{Y}^\vee \rightarrow Y^\vee,$$

in [14, Proposition 4.9]. Then the conics in $\alpha^{-1}(y)$ are identical to conics in a linear system in $S_4$ for some $V_4$. The involution $\iota$ sends conics in $S_4$ to their residual conics.

It suffices to show that for a general $V_4$ and any $c, c'$ residual in $S_4$, the sums $c + c'$ are constant in $\text{CH}_1(X)$, which follows from the fact that the hyperplane sections in such $S_4$’s are parameterized by a $\mathbb{P}^3$-bundle over $\mathbb{P}(V_4^\vee)$. \qed
4.4. Relations in $\text{CH}^*(F)$ of incidences. We introduce an incidence $I \in \text{CH}^2(F \times F)$ and compute $I^2 \in \text{CH}^4(F \times F)$, where $F = F(X)$. Denote by $c_s$ the corresponding conic to $s \in F$. The morphisms are the same as in diagram (3.1).

**Definition 4.5.** (1) The incidence subscheme $I \subset F \times F$ is defined to be

$$I := \{(s, t) \in F \times F \mid c_s \cap c_t \neq \emptyset\},$$

which is endowed with the reduced closed subscheme structure and of dimension 8.

(2) $D_c = p(q^{-1}(c)) \subset F$ is the locus in $F$ consisting of conics intersecting the given conic $c$ and $D'_c = q^{-1}(c)$.

(3) $\Sigma_1 \subset F \times F$ is defined to be

$$\Sigma_1 := \{(s, t) \in F \times F \mid c_s \text{ and } c_t \text{ are union of two lines and have a common component}\},$$

which is of dimension 5.

(4) $\Sigma_2 \subset F \times F$ consists of $(s, t) \in F \times F$, where neither $c_s$ nor $c_t$ is of $\tau$-type.

(5) $W \subset F \times F$ is defined to be

$$W := \{(s, t) \mid \exists V_4, c_s \text{ and } c_t \text{ are residual in } S_{V_4}\}.$$

For general $s, t$, that means the two conics meet at two points. Since a conic has a pencil of residual conics in $S_{V_4}$, we have $\dim W = 6$.

**Lemma 4.6.** For a general GM fourfold $X$, two different conics $c_1$ and $c_2$ span different planes. If $c$ is of $\tau$-type, then $\langle c \rangle \cap \text{Gr}(2, V_5) = c$.

**Proof.** A general $X$ contains no planes. Since $X$ is the intersection of quadrics, $\langle c \rangle$ is not contained in some quadric $Q_0 \supset X$. Hence, $\langle c \rangle \cap X \subset Q_0 \cap \langle c \rangle = c$, which means that $\langle c \rangle$ determines $c$.

The second statement follows from the fact that the plane spanned by a $\tau$-conic is not contained in $\text{Gr}(2, V_5)$ and $\text{Gr}(2, V_5)$ is also the intersection of quadrics. $\square$

According to Lemma 4.6, we obtain an injective morphism

$$F \to \text{Gr}(3, 10),$$

(4.1)

sending $t \in F$ to $\langle c_t \rangle \subset \mathbb{P}(\wedge^2 V_5)$. Let $\mathbb{P} = \mathbb{P}(E)$ be the universal plane of $\text{Gr}(3, 10)$ restricted to $F$, we have the following diagram:

$$\begin{array}{ccc}
P & \xrightarrow{q} & X \\
& \searrow & \searrow \\
& & \text{Gr}(3, 10) \\
\text{Gr}(2, V_5) & \xrightarrow{q'} & F \\
\end{array}$$

(4.2)

**Lemma 4.7.** The evaluation morphism $q$ is flat for a general $X$. In particular, the class $I = (p \times p)_*(q \times q)^*\Delta_X \subset \text{CH}^2(F \times F)$. Equivalently, the correspondence $I = \iota P \circ P$. 

Proof. The universal conic $P$ is Cohen-Macaulay since it is a divisor in a smooth variety $\mathbb{P}$, hence it suffices to show that the dimensions of fibers of $q$ are constant. Suppose on the contrary that there is a point $x \in X$ such that $q^{-1}(x)$ has a component $K$ of dimension $\geq 3$. We claim that

Claim. There is a smooth conic $c$ passing through $x$ such that $(c, x) \in P$ belongs to $K$ and $c$ is of $\tau$-type or $\rho$-type.

Proof of Claim. Assume that $x = V_2$. It suffices to show that the dimension of the locus of $\sigma$-conics and singular conics passing through $x$ is less than 3.

In fact, the $\sigma$-conics are parameterized by the planes

$$\mathbb{P}(V_1 \cap V_4) \subset H,$$

where $V_1 \subset V_4$. When $V_1 \neq W_1$, the locus of $\sigma$-conics passing through $x$ are equal to the image of $\mathbb{P}(V_2)$ in $Bl_{W_1}(\mathbb{P}(V_5))$, which is 1-dimensional. When $W_1 = V_1$, a $\sigma$-conic passing through $x$ corresponds to a plane $\langle c \rangle = \mathbb{P}(V_1 \cap V_4)$ with $V_2 \subset V_4 \subset V_5$, and there is a 2-dimensional choice of $V_4$. For singular conics, there are at most 1-dimensional lines passing through a point on $X$ by \cite[Theorem 4.7]{HH}. Therefore the claim follows.

The fiber $p^{-1}(c) = c \subset P$ can be identified with $c \subset X$ via $q$ and

$$N_{c/P} = H^0(N_{c/X}) \otimes \mathcal{O}_c.$$

Considering the differential of $q$ restricted to $c \subset P$, we obtain the following diagram, see also \cite[section 2.3]{HH}:

$$\begin{array}{cccccc}
0 & \longrightarrow & Tc & \longrightarrow & TP|_c & \longrightarrow & H^0(N_{c/X}) \otimes \mathcal{O}_c & \longrightarrow & 0. \\
\downarrow & & \downarrow \cong & & \downarrow dq|_c & & \downarrow ev & & \\
0 & \longrightarrow & Tc & \longrightarrow & TX|_c & \longrightarrow & N_{c/X} & \longrightarrow & 0
\end{array} \tag{4.3}
$$

The normal bundle $N_{c/X} = \mathcal{O}_c \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1)$ for a smooth conic of $\tau$-type or $\rho$-type. Therefore,

$$T_{(c,x)}q^{-1}(x) = \{ s \in H^0(N_{c/X}) | s_x = 0 \},$$

which is 2-dimensional, contradicting to the fact that $\dim K \geq 3$.

Lemma 4.8. Let $c_1$ and $c_2$ be two different conics on a general $X$ and assume that at least one of them is of $\tau$-type. If $c_1$ and $c_2$ have no common component and $\langle c_1 \rangle \cap \langle c_2 \rangle$ is a line $l$, there exists a 4-dimensional subspace $V_4 \subset V_5$, such that $\langle c_1 \rangle \cup \langle c_2 \rangle \subset \mathbb{P}(\wedge^2 V_4)$. In particular, $\iota(\alpha([c_1])) = \alpha([c_2])$.

Proof. A general $X$ contains no planes and hence $l \not\subset X$. The conic $c_1$ is contained in $\mathbb{P}(\wedge^2 V_4)$ for some 4-dimensional vector space $V_4$. If $V_{41} \neq V_{42}$, then $V_3 := V_{41} \cap V_{42}$ is 3-dimensional and

$$l \subset \mathbb{P}(\wedge^2 V_3) \subset \text{Gr}(2, V_5).$$

In particular, $\langle c_i \rangle \cap \text{Gr}(2, V_5) \neq c_i$. Therefore by Lemma 4.6, we have $\langle c_1 \rangle \subset \text{Gr}(2, V_5)$ and it means that the two conics are of $\sigma$-type or $\rho$-type, contradicting to the fact that one of the two conics is of $\tau$-type.
Hence \( V_{41} = V_{42} = V_4 \) and \( c_1, c_2 \) are contained in \( S_{V_4} \). Then \( \iota(\alpha(c_1)) = \alpha(c_2) \) follows from that \( \iota \) sends a conic to its residual conic in \( S_{V_4} \). \( \Box \)

We define \( \tilde{I} \subset P \times P \) and \( I' \subset P \times P \) to be the incidence correspondences, i.e.
\[
\tilde{I} = (q \times q)^{-1}\Delta_X \quad \text{and} \quad I' = (q' \times q')^{-1}\Delta_{\mathbb{P}(\wedge^2 V_5)}.
\]

Let \( I_0 = I \setminus (\Sigma_1 \cup \Sigma_2 \cup W) \), \( \tilde{I}_0 = (p \times p)^{-1}I_0 \) and \( I'_0 = (p' \times p')^{-1}I_0 \). Then Lemma 4.9 says that there is a section from \( I_0 \) to \( I'_0 \), sending two intersecting conics to their common point.

**Lemma 4.9.** The inclusion \( I_0 \subset F \times F \setminus (\Sigma_1 \cup \Sigma_2 \cup W) \) is a regular embedding.

**Proof.** The inclusion \( I' \subset P \times P \) is a regular embedding since it is obtained from \( \Delta_{\mathbb{P}(\wedge^2 V_5)} \subset \mathbb{P}(\wedge^2 V_5) \times \mathbb{P}(\wedge^2 V_5) \) via base change. By the discussion above, there exists a section from \( I_0 \) to \( I'_0 \). We apply [13] B.7.5 and obtain that \( I_0 \subset F \times F \setminus (\Sigma_1 \cup \Sigma_2 \cup W) \) is a regular embedding. \( \Box \)

Now we can compute the self-intersection \( \tilde{I}^2 \), the analogous case of cubic fourfolds can be found in [30] Proposition 2.3 or [12] Appendix A.4.

**Proposition 4.10.** The self-intersection
\[
\tilde{I}^2 = aW + I \cdot A + B + C \in \text{CH}^4(F \times F)
\]
for some constant \( a \). Here, \( A \) and \( B \) are contained in \( \text{pr}_1^* \text{CH}^*(F) \cdot \text{pr}_2^* \text{CH}^*(F) \subset \text{CH}^*(F \times F) \), where \( \text{pr}_i \) is the projection and \( C \) is supported on \( \Sigma_2 \subset F \times F \).

**Proof.** By Lemma 4.9, \( I_0 \subset F \times F \setminus (\Sigma_1 \cup \Sigma_2 \cup W) \) is a regular embedding and the restriction \( p_0 \) of \( p \) from \( \tilde{I}_0 \subset P \times P \) to \( I_0 \subset F \times F \) is an isomorphism. Then,
\[
I_0^2 = c_2(N_{I_0/F \times F})
\]
and
\[
0 \rightarrow \text{pr}_1^* T_{P/F} \oplus \text{pr}_2^* T_{P/F} \rightarrow N_{I_0/P \times P} \rightarrow p_0^* (N_{I_0/F \times F}) \rightarrow 0.
\]

Let \( q_0 \) be the map \( q_0 : \tilde{I}_0 \rightarrow \Delta_X = X \) which maps two conics in \( \tilde{I}_0 \) to their common point. Since \( \tilde{I} = (q \times q)^{-1}\Delta_X \), we have
\[
N_{I_0/P \times P} = q_0^*(TX) = \text{pr}_1^* (TX) \cdot \tilde{I}_0 = \text{pr}_2^* (TX) \cdot \tilde{I}_0
\]
and
\[
c(p_0^*(N_{I_0/F \times F})) = \frac{q_0^* c(TX)}{\text{pr}_1^* c(T_{P/F}) \cdot \text{pr}_2^* c(T_{P/F})}.
\]

Let \( K = \sigma_{1,1}|X \) be the class of \( S_{V_4} \) and \( H = \sigma_{1}|X \) is the hyperplane class. We claim that \( c_2(p_0^*(N_{I_0/F \times F})) \) is a degree 2 polynomial with variables in \( \text{pr}_1^* \text{CH}^*(F) \cdot \text{pr}_2^* \text{CH}^*(F) \) and \( \text{pr}_1^* (H, K) \cdot \text{pr}_2^* (H, K) \) restricting to \( \tilde{I}_0 \). The following computations is standard:

1. The Chern classes of \( X \)
\[
c(T_X) = \frac{c(T_{\text{Gr}(2, V_5)})|X}{(1 + H)(1 + 2H)}.
\]

where \( c(T_{\text{Gr}(2, V_5)}) \) is a polynomial in Schubert cycles. Therefore, the degree 1 and 2 parts of \( q_0^* c(T_X) \) are polynomials in \( H \) and \( \sigma_{1,1} \) via the two projections and restriction to \( \tilde{I}_0 \).
(2) The Chern classes of the relative tangent bundle $T_{P|F}$

$$c(T_{P|F}) = \frac{c(T_{P/F}^P_P)}{c(N/P)} = \frac{c(p^*(E) \otimes O_P(1))_P}{c(O_P(P))},$$

where $c_1(O_P(1))|_X = H$ and $P = 2c_1(O_P(1)) + p^*D \in CH^1(P)$ for some $D \in CH^1(F)$. So the degree 1 and degree 2 parts are polynomials in $H$ and $CH^*(F)$.

The claim follows from the computations and \[(4.4)\]. Next, the terms in $p_0^*c_2(N_{I_0/F \times F})$ are of the following three types:

1. $D \cdot I_0$, where $D \in pr_1^*CH^*(F) \cdot pr_2^*CH^*(F)$.
2. $(q \times q)^*pr_i^*H \cdot D \cdot I_0$ for $i = 1$ or 2, where $D$ is of form $(q \times q)^*pr_j^*H$ or $(p \times p)^*pr_j^*D_1$ for some $D_1 \in CH^1(F)$.
3. $(q \times q)^*pr_i^*K \cdot I_0$ for $i = 1$ or 2.

Applying $p_0*$ to $p_0^*c_2(N_{I_0/F \times F})$, the terms of the first type are of the form $I \cdot A$ by projection formula. For the other two types, we use $\bar{I} = (q \times q)^*\Delta_X$ and it’s enough to show that $pr_1^*H \cdot pr_2^*H \cdot \Delta_X$ and $pr_i^*K \cdot \Delta_X$ are in $pr_1^*CH^*(X) \cdot pr_2^*CH^*(X)$. Denote by $M = Gr(2, V_5) \cap H$ the Grassmanian hull of $X$ and consider

$$j_1: X \times X \hookrightarrow M \times X.$$

We know that $M$ is a linear variety (Schubert varieties are linear in the sense of \[28\], since they are stratified by Schubert cells) and therefore by \[28\] Proposition 1,

$$CH^*(M \times X) = pr_1^*CH^*(M) \cdot pr_2^*CH^*(X).$$

Hence we have

$$j_1^*j_{1*}\Delta_X \in pr_1^*CH^*(X) \cdot pr_2^*CH^*(X)$$

Since $X$ is of class $2H$ in $M$, we have $j_1^*j_{1*}\Delta_X = 2pr_1^*H \cdot \Delta_X$. Hence $pr_1^*H \cdot \Delta_X \in pr_1^*CH^*(X) \cdot pr_2^*CH^*(X)$.

For the third type, we recall that the class $K \in CH^2(X)$ is represented by a Del Pezzo surface $S = S_{V_4}$ and $pr_1^*K \cdot \Delta_X$ is the push-forward of the diagonal class $\Delta_S$ in $CH^2(S \times S)$. Then by the decomposition of diagonal, we see that

$$\Delta_S = \alpha \times S + \Gamma,$$

where $\alpha \in CH_0(S)$ and $\Gamma$ is supported on $S \times V$ for some proper subvariety $V \subset S$. If $\dim V = 0$, $\Gamma$ is of the form $S \times \beta$ for some $\beta \in CH_0(X)$. If $\dim V = 1$, then $\Gamma \in CH^1(V \times X)$. We see that $CH^1(V \times X) = pr_1^*CH^1(V) \cdot pr_2^*CH^1(S)$ since $H^1(S, O_S) = 0$. In both cases, we obtain:

$$\Delta_S \in pr_1^*CH^*(S) \cdot pr_2^*CH^*(S),$$

which implies that $pr_1^*K \cdot \Delta_X$ are in $pr_1^*CH^*(X) \cdot pr_2^*CH^*(X)$.

Finally, we get the conclusion by the localization exact sequence for Chow groups. □
4.11. **Z is a constant cycle surface.** Now we can prove that Z is a constant cycle surface on $\tilde{Y}$, and a point on Z represents the distinguished 0-cycle o that appeared in (3.2). First, we notice that

$$\Phi([c]) = I_*(e) = D_e$$

since $q$ is flat, where we recall that $\Phi = p_*q^*$. Hence the class of $D_e$ depends only on the class $[e] \in CH_1(X)$.

For a general $V_4$, $SY_4$ is isomorphic to the blow-up of $P^2$ at five points. Let $H$ be the pullback of the hyperplane class on $P^2$ and $E_i$, $i = 1, \ldots, 5$, be the five exceptional curves. There are 10 pencils of conics on $SY_i$, which are $H - E_i$ and $2H - \sum_{j \neq i} E_j$. Denote by $L_i$, $i = 1, \ldots, 10$ the corresponding lines on $F$ and let $L_{i+10} = L_i$. Recall that $L_i$’s are contracted to a point by $\alpha$ and $\alpha(L_i) = \iota(\alpha(L_{i+5}))$. By a straightforward computation in $SY_4$, we have

$$c_i \cdot c_j = 1 \text{ for } j \neq i, i + 5; c_i \cdot c_{i+5} = 2 \text{ and } c_i^2 = 0 \text{ for any } i. \quad (4.5)$$

Here, $c_i$ is a conic in $L_i$. That is, conics in $L_{i+5}$ meet $c_i$ at two points and conics in $L_j, j \neq i, i + 5$ meet $c_i$ at one point. We have the following computation:

**Lemma 4.12.** Let $c_i$ be a conic on a general $SY_4$ and $c_i \in L_i$, then

$$D^2_{c_i} = \sum_{j \neq i, j \neq i+5} L_j + 4L_{i+5} + kL_\sigma \in CH_1(F), \quad (4.6)$$

where $L_\sigma \subset F$ is a curve contained in the locus of $\sigma$-conics.

**Proof.** We may assume that $SY_4$ is general so that $W_1 \not\subset V_4$ and $SY_4$ contains no $\sigma$-conics. Take another conic $c_0 \in [c_i]$ and thus $D_{c_0} = D_{c_i}$ in $CH^2(F)$. The intersection $D_{c_i} \cap D_{c_0}$ consists of conics $c$ which meet both $c_i$ and $c_0$.

We distinguish between the two cases whether $c$ is contained in $SY_4$ or not.

If $c \in D_{c_i} \cap D_{c_0}$ and $c \subset SY_4$, it is contained in another linear system of conics, supported on $L_j$, for some $j \neq i$. Conversely, $L_j \subset Supp D_{c_i} \cap D_{c_0}$ for $j \neq i$ by (4.5).

If $c \in D_{c_i} \cap D_{c_0}$ and $c \not\subset SY_4$, we first show that $c$ must be a $\sigma$-conic. Let $x \in c \cap c_0$, $y \in c \cap c_i$ and $V_x, V_y$ be the corresponding 2-dimensional subspaces in $V_5$. We have $V_x \cup V_y$ is a three-dimensional space $V_3$, otherwise $c$ would be contained in $SY_4$. It follows that $V_x \cap V_y$ is a one-dimensional space $V_1$. Therefore

$$\mathbb{P}(V_1 \cap V_3) \subset Gr(2, V_5) \cap \langle c \rangle.$$ 

By Lemma 4.16 $c$ is a $\sigma$-conic or a $\rho$-conic. Since $W_1 \not\subset V_4$, we see that $W_1 \not\subset V_3$ and $c$ is not a $\rho$-conic. Therefore $c$ is a $\sigma$-conic.

Recall that $\sigma$-conics are parameterized by $Bl_{[W_1]}(\mathbb{P}(V_5))$. Let $S_{c_0}$ and $S_{c_i}$ be the surfaces swept out by lines parameterized by $c_0$ and $c_i$ in $\mathbb{P}(V_4)$. There is a natural morphism

$$S_{c_0} \cap S_{c_i} \rightarrow Bl_{[W_1]}(\mathbb{P}(V_5)),$$

and the image is exactly the locus of $\sigma$-conics meeting $c_i$ and $c_0$ by the previous discussion. The intersection $S_{c_0} \cap S_{c_i}$ is clearly one dimensional. Therefore the conics in $D_{c_i} \cap D_{c_0}$ not contained in $SY_4$ are parameterized by a curve $L_\sigma \subset F$. 

Thus $D_{c_i}$ and $D_{c_0}$ intersect dimensionally transversely and it remains to compute the multiplicity. We consider the differential $dq$ as in (4.3). Let $c$ be a general smooth conic in $D_{c_i} \cap D_{c_0}$ and $c \in L_j, j \neq i, i+5$. Then $c$ meets $c_i$ (resp. $c_0$) at one point $x$ (resp. $y$). We have $T_{c}c \subset T_{c}S_{V_{4}}$, hence the image of $T_{c}c$ in $N_{c/X}$ is $N_{c/S_{V_{4}},x}$. Then by (4.3), we have

$$T_{(c,x)}D'_{c_i} = dq^{-1}(T_{x}c_i) = H^{0}(N_{c/X} \otimes I_{x}) \oplus H^{0}(N_{c/S_{V_{4}},x}),$$

which is 3-dimensional. Thus $D'_{c_i}$ is smooth at $(c,x)$. Since $c_i$ and $c$ meet at a single point, $D'_{c_i}$ maps isomorphically to $D_{c_i}$ via $p$ around $c$. Hence, $D'_{c_i}$ maps the two points $(c_1)_{i}$ and $(c_2)_{i}$ (resp. $(c, y_{1})$ and $(c, y_{2})$ ) to $c$. We obtain that the tangent cones

$$C_{c}D_{c_i} = H^{0}(N_{c/X} \otimes I_{x_{1}}) \oplus H^{0}(N_{c/X} \otimes I_{x_{2}}) \oplus H^{0}(N_{c/S_{V_{4}},x})$$

$$C_{c}D_{c_0} = H^{0}(N_{c/X} \otimes I_{y_{1}}) \oplus H^{0}(N_{c/X} \otimes I_{y_{2}}) \oplus H^{0}(N_{c/S_{V_{4}},x}).$$

(4.7)

Take a general hypersurface $G \subset F$ containing $c$ such that $G$ meets $L_{i+5}$ transversely at $c$ and then the multiplicities

$$\text{mult}_{c}D_{c_i}|_{G} = \text{mult}_{c}D_{c_0}|_{G} = 2.$$

Combining the fact that the direction of $H^{0}(N_{c/S_{V_{4}},x})$ is in the fiber of $\alpha$, we obtain

$$C_{c}D_{c_i}|_{G} \cap C_{c}D_{c_0}|_{G} = 0$$

by (4.7) and the description of the normal bundle of conics. By [8, Proposition 1.29], the intersection multiplicity of $D_{c_i}|_{G}$ and $D_{c_0}|_{G}$ at $c$ is 4. It follows that the multiplicity of the intersection of $D_{c_0}$ and $D_{c_i}$ at $L_{i+5}$ is 4.

Apply this Lemma to every $c_i \subset S_{V_{4}}$ and by a linear combination, we have:

$$6(D^2_{c_i} + D^2_{c_{i+5}}) - \sum_{j=1}^{10} D^2_{c_j} = 12(L_{i} + L_{i+5}) + L',$$

(4.8)

where $L'$ is a 1-cycle on $F$ consisting of $\sigma$-conics.

**Lemma 4.13.** The constant $a$ in Proposition 4.10 is nonzero. If $n[c_1] \sim n[c_2]$ in $CH_{1}(X)$ for some nonzero integer $n$, then $[c_1] \sim [c_2]$ in $CH_{0}(Y')$, where $Y'$ is the pullback of $X$. 

Proof. We assume on the contrary that \( a = 0 \). Then
\[
I^2 = I \cdot A + B + C \in \text{CH}^4(F \times F),
\]
where \( A, B, C \) are as in Proposition 4.10. A correspondence \( \Gamma \) of codimension larger than 0 in \( pr_1^* \text{CH}^* (F) \cdot pr_2^* \text{CH}^* (F) \) is of the form \([Z_1 \times Z_2]\), where either \( Z_1 \) or \( Z_2 \) is a proper subvariety of \( F \). In both cases, \( \Gamma \) induces a constant map \( \text{CH}_0(F) \to \text{CH}^* (F) \). Further, by Lemma 4.7, \( I = 4P \circ P \) and hence \( I_* : \text{CH}_0(F) \to \text{CH}_3(F) \) factor through \( \text{CH}_1(X) \). Together with the torsion-freeness of \( \text{CH}_0(F) \), it implies that if \( a = 0 \), then for \( \xi, \zeta \in \text{CH}_0(F) \),
\[
I^2_\xi (\xi) = I^2_\zeta (\zeta)
\]
as long as \( q_p^*(n \xi) = q_p^*(n \zeta) \in \text{CH}_1(X) \), where \( p, q \) are the projections from the universal conic to \( F \) and \( X \).

According to [26, Lemma 17.3], we know that \( I^2_\xi ([c]) = D^2_c \). Then (4.9) together with Lemma 4.3 imply that \( D^2_c + D^2_{t(c)} \) are constant in \( \text{CH}_1(F) \) when varying \( c \in F \). We want to deduce a contradiction between the formula (4.3) and the constantness of \( D^2_c + D^2_{t(c)} \).

Take a very ample divisor \( G \subseteq F \). Intersecting both sides of formula (4.3) with \( G \) and pushing forward to \( \hat{Y}^\vee \), then the constantness of \( D^2_c + D^2_{t(c)} \) implies that \( ([c] + t([c])) \) is constant in \( \text{CH}_0(\hat{Y}^\vee) \), i.e., points on
\[
\Delta' = \{(y, t(y)) \mid y \in \hat{Y}^\vee\} \subseteq \hat{Y}^\vee \times \hat{Y}^\vee
\]
are constant in \( \text{CH}_0(\hat{Y}^\vee) \).

Then for any power \( \sigma^t \) of the two-form \( \sigma \) on \( \hat{Y}^\vee \), we have \( (pr_1^*(\sigma^t) + pr_2^*(\sigma^t))|_{\Delta'} = 0 \) by Mumford’s theorem, see [25, Proposition 10.24]. Then we obtain
\[
2pr_1^*(\sigma^2)\mid_{\Delta'} = 0,
\]
but it is impossible since \( pr_1 \) is an isomorphism between \( \Delta' \) and \( \hat{Y}^\vee \). Therefore, \( a \) is nonzero.

For the second part, assume that \( n[c_1] \sim n[c_2] \) and again take a very ample divisor \( G \subseteq F \). Let \( m \) be the intersection number of \( G \) and a general fiber of \( \alpha : F \to \hat{Y}^\vee \). Then we have \( n^2I^2_\ast ([c_1]) = n^2D^2_{c_1} = n^2D^2_{c_2} = n^2I^2_\ast ([c_2]) \) again by [26, Lemma 17.3] and \( nI_\ast ([c_1]) = nI_\ast ([c_2]) \). Hence due to Proposition 4.10,
\[
0 = G \cdot n^2(I^2_\ast ([c_1]) - I^2_\ast ([c_2]))
= G \cdot n^2(aW + I \cdot A + B + C)([c_1] - [c_2])
= G \cdot an^2W([c_1] - [c_2])
= amn^2 \cdot (\iota([c_1]) - \iota([c_2])),
\]
here we view the cycles in \( \text{CH}^4(F \times F) \) as morphisms from \( \text{CH}_0(F) \) to \( \text{CH}_1(F) \). Then
\[
\iota([c_1]) \sim \iota([c_2])
\]
again by the torsion-freeness of \( \text{CH}_0(\hat{Y}^\vee) \) and the fact that \( a \neq 0 \). Since \( \iota \) is an involution, the result follows.

\[\square\]
Combining Lemma 4.3 and Lemma 4.13, we can get the following result:

**Theorem 4.14.** The invariant locus $Z$ of the involution is a constant cycle surface on $\tilde{Y}^\vee$.

Thus by (3.2), $h^2 \cdot Z = 40\omega$ in $CH_0(\tilde{Y}^\vee)$. That means $\omega$ is represented by a point on the constant cycle surface $Z$. We show that $\omega$ is in fact the class represented by a point on any constant cycle surface:

**Proposition 4.15.** For a very general $\tilde{Y}^\vee$, if $Z' \subset \tilde{Y}^\vee$ is another constant cycle surface, then for any point $z' \in Z'$, $z'$ is rationally equivalent to $\omega$.

**Proof.** It suffices to show that for a very general $\tilde{Y}^\vee$, any two surfaces on $\tilde{Y}^\vee$ intersect. We obtain this by showing that every surface on $\tilde{Y}^\vee$ is strictly nef, i.e., having a positive intersection number with any non-zero effective 2-cycle.

Let $N_2(\tilde{Y}^\vee)$ be the space of 2-cycles modulo numerical equivalence with $\mathbb{R}$-coefficients, which is a 2-dimensional vector space spanned by $h^2$ and $Z$. As in [21], the Lagrangian surface $Z$ is in the boundary of the effective cone $Eff_2(\tilde{Y}^\vee)$, and $c_2(\tilde{Y}^\vee)$ is not contained in the interior of the effective cone. Thus

$$Eff_2(\tilde{Y}^\vee) \subset \langle R_{\geq 0}(Z), R_{\geq 0}(c_2) \rangle.$$ 

Then by [20], we have

$$\langle R_{\geq 0}(Z), R_{\geq 0}(c_2) \rangle \subset \langle R_{\geq 0}(24h^2 - 5Z), R_{\geq 0}(-2h^2 + 5Z) \rangle$$

$$= \langle R_{\geq 0}(Z), R_{\geq 0}(c_2) \rangle^{\vee}$$

$$\subset Neff_2(\tilde{Y}^\vee).$$

The first inclusion is such that the cones have no boundary in common by a straightforward computation. Hence every surface on $\tilde{Y}^\vee$ is strictly nef. □

**5. The filtration on CH$_1(X)$**

**5.1. Basics of the filtration.** A uniruled divisor on a 2n-dimensional IHS variety $M$ is a divisor $D$ which admits a rational map to a $(2n-2)$-dimensional variety $B$:

$$D \hookrightarrow M,$$

$$q:$$

$$\downarrow$$

$$B$$

and the general fibers of $q$ are rational curves. By [2], there exists a uniruled divisor on an IHS variety of $K3^{[n]}$-type with $n \leq 7$, in particular, on the dual double EPW sextic $\tilde{Y}^\vee$.

The Beauville–Voisin filtration on $\tilde{Y}^\vee$ is determined by uniruled divisors and constant cycle surfaces. Points on uniruled divisors serve as the 1st piece and points on constant cycle surfaces serve as the 0th piece. According to [24, Lemma 1.1] and Proposition 4.15, the filtration does not depend on the choice of a uniruled divisor and a constant cycle surface on a very general dual double EPW sextic.
We define a filtration on \( \text{CH}_1(X) \) for a general GM fourfold induced by the Beauville–Voisin filtration on \( \tilde{Y}^\vee \). In the rest of the paper, we study some properties of the filtration.

**Definition 5.2.** The \( i \)th piece of the increasing filtration \( S_i X \subset \text{CH}_1(X) \) consists of \( z \in \text{CH}_1(X) \), such that (i.e., the class of \( az \) can be represented by a sum of multiples of \( \iota \) conics in a uniruled divisor and \( \theta \))

\[
az = a_1 c_1 + a_2 c_2 \ldots + a_i c_i + a_0 \theta
\]

for some integer \( a, a_j \), where \( c_j \)'s are conics with \( \alpha(c_j) \) in a uniruled divisor \( D \subset Y^\vee \) and \( \theta := \Psi(o) \) for \( o \) the class of a point on the constant cycle surface \( Z \).

The EPW sextics associated with period partners \( X \) and \( X' \) are the same. We have an immediate corollary according to the definition:

**Corollary 5.3.** For any \( t \in \text{CH}_0(\tilde{Y}^\vee) \), \( \Psi(t) \in S_i(X) \) iff \( \Psi_i(t) \in S_i(X') \), where \( \Psi_i : \text{CH}_0(\tilde{Y}^\vee) \rightarrow \text{CH}_1(X') \) is similarly defined as \( \Psi \).

By the proof of Proposition 4.2, there exists an integer \( N \) such that \( N \cdot z \) is a linear sum of conics for every \( z \in \text{CH}_1(X) \). At the end of the section, we show in Proposition 5.10 that every conic is rationally equivalent to a sum of conics in \( D \). Hence,

\[
\bigcup_{i=0}^{\infty} S_i(X) = \text{CH}_1(X).
\]

Let \( \text{CH}_0(\tilde{Y}^\vee)_i^- \) be the \( i \)-anti-invariant part of \( \text{CH}_0(\tilde{Y}^\vee)_\text{hom} \) and \( \text{CH}_0(\tilde{Y}^\vee)_i^+ \) be the \( i \)-invariant part. There is a decomposition of \( \text{CH}_0(\tilde{Y}^\vee) \):

\[
\text{CH}_0(\tilde{Y}^\vee) = \mathbb{Z} \cdot o \oplus \text{CH}_0(\tilde{Y}^\vee)_0^- \oplus \text{CH}_0(\tilde{Y}^\vee)_0^+.
\]

We show that the filtration on \( \text{CH}_1(X) \) can actually be defined on the \( i \)-anti-invariant part \( \text{CH}_0(\tilde{Y}^\vee)^- \).

**Proposition 5.4.** The morphism \( \Psi : \text{CH}_0(\tilde{Y}^\vee) \rightarrow \text{CH}_1(X) \) is zero on \( \text{CH}_0(\tilde{Y}^\vee)_\text{hom} \) and an isomorphism onto \( \text{CH}_1(X)_\text{hom} \) when restricting to \( \text{CH}_0(\tilde{Y}^\vee)_\text{hom}^- \).

**Proof.** Due to the divisibility of \( \text{CH}_0(\tilde{Y}^\vee)_\text{hom} \), the elements in \( \text{CH}_0(\tilde{Y}^\vee)_\text{hom}^+ \) are of the form \( \sum (t_i + \iota(t_i) - 2a) \) and elements in \( \text{CH}_0(\tilde{Y}^\vee)^- \) are of the form \( \sum t_i - \iota(t_i) \), where \( t_i \in \tilde{Y}^\vee \). This yields \( \Psi \) is zero on \( \text{CH}_0(\tilde{Y}^\vee)_\text{hom}^+ \) by Lemma 4.3.

For the second part, there exists an integer \( N \), such that for any \( z \in \text{CH}_1(X) \), \( N \cdot z \) is a linear combination of conics. Then \( \text{CH}_1(X)_\text{hom} \) is generated by conics by the divisibility of \( \text{CH}_1(X)_\text{hom} \). Therefore, it suffices to show that \( \Psi \) is injective on \( \text{CH}_0(\tilde{Y}^\vee)^- \).

Let \( G \subset F \) be an ample divisor and \( G \cdot L = m \), where \( L \) is a general fibre of \( \alpha : F \rightarrow \tilde{Y}^\vee \). Denote by \( c_i \) and \( c_i' \) the conics representing the class \( \Psi(t_i) \) and \( \Psi(t_i') \). If we have the relation

\[
\Psi(\sum t_i - \iota(t_i)) = \Psi(\sum t_i' - \iota(t_i')) \in \text{CH}_1(X),
\]

it implies that:

\[
\sum (D_{c_i} - D_{\iota(c_i)}) = \sum (D_{c_i'} - D_{\iota(c_i')}).
\text{(5.1)}
\]
By Lemma 4.12 we have:

\[ G \cdot (D_{c_i} - D_{i(c_i)}) \cdot (D_{c_i} + D_{i(c_i)}) = G \cdot (D_{c_i}^2 - D_{i(c_i)^2}) = 4m(\iota(t_i) - t_i). \]

Here, we identify \( \text{CH}_0(F) \) and \( \text{CH}_0(\tilde{Y}^\vee) \) via the isomorphism \( \alpha^* \). By Lemma 4.3, we know that \( D_c + D_{i(c)} \) is constant in \( \text{CH}^2(F) \), we denote it by \( D_{0} \).

Therefore we obtain

\[ G \cdot D_{0} \cdot \sum (D_{c_i} - D_{i(c_i)}) = 4m \sum (\iota(t_i) - t_i) \]

and

\[ G \cdot D_{0} \cdot \sum (D_{c_i} - D_{i(c_i)}) = 4m \sum (\iota(t_i) - t_i). \]

The left-hand sides are equal by (5.1). Therefore by torsion-freeness we obtain:

\[ \sum t_i - \iota(t_i) = \sum t'_i - \iota(t'_i). \]

Hence \( \Psi \) is injective on \( \text{CH}_0(\tilde{Y}^\vee)^- \). \( \square \)

Consequently, we immediately deduce that

**Theorem 5.5.** We have an isomorphism between groups

\[ Z \cdot o \oplus \text{CH}_0(\tilde{Y}^\vee)_{\text{hom}} \cong Z \cdot \theta \oplus \text{CH}_1(X)_{\text{hom}} \cong \text{CH}_1(X). \]  

(5.2)

In particular, the group \( \text{CH}_1(X) \) is torsion-free.

When \( \tilde{Y}^\vee \) is birational to \( S^{[2]} \) for some K3 surface \( S \), it has been shown in [17] that \( \text{CH}_0(\tilde{Y}^\vee)_Q \) with \( Q \)-coefficient has a natural decomposition for the Chow group. For 0-cycles, the decomposition is

\[ \text{CH}_0(\tilde{Y}^\vee)_Q = Q \cdot o \oplus \text{CH}_0(S)_{\text{hom}} \oplus \text{CH}_0(\tilde{Y}^\vee)^+_\text{hom}. \]

Theorem [5.5] can be viewed as a generalization of this decomposition, where the group \( \text{CH}_1(X) \) takes the place of \( \text{CH}_0(S) \).

**5.6. A result for conics.** In this subsection, we prove that the class of a conic on \( X \) is in \( S_2(X) \). The proof is similar to the case of cubic fourfolds by showing any conic is contained in a certain singular cubic threefold. Here, we need the flexibility to change the GM fourfold to its period partners, guaranteed by Corollary 5.3.

We consider the following incidence relation

\[ \Omega := \{(A, B) \mid \dim(A \cap B) \geq 9\} \subset L\mathbb{G}(\bigwedge^3 V_6) \times L\mathbb{G}(\bigwedge^3 V_6). \]

Denote the fibre of the projection over \( A \) by \( \Omega_A \) and \( F_y = y \wedge \bigwedge^2 V_6 \subset \bigwedge^3 V_6 \). Clearly, we have

\[ Y_{\bigwedge^2 B} \subset Y_{\bigwedge^1 A} \]

for \( (A, B) \in \Omega \). We let

\[ \Sigma = \{ A \in L\mathbb{G}(\bigwedge^3 V_6) \mid A \text{ contains a decomposable vector} \}, \]

and \( \Sigma_k \) be the closure of the locus of \( A \) that contains exactly \( k \) decomposable vectors.

We can deduce the following lemmas.
Lemma 5.7. For any $A, B \subset \bigwedge^3 V_6$ with $(A, B) \in \Omega$ and $A \neq B$, we can find $V_5 \subset V_6$, such that the corresponding GM varieties satisfy $X_B \subset X_A$ and are of dimension 3 and 4.

Proof. It’s enough to take $V_5 \in Y_{B^1} \cap Y_{B^2}^{-2}$ by Theorem 3.3.

Lemma 5.8. For a general $B \in \Sigma_8$ and $V_5 \in Y_{B^2}^{-2}$, let $F(X_B)$ be the surface of conics on the corresponding GM threefold $X_B$, then $\dim \text{Alb}(F(X_B)) = 2$.

Proof. For a general choice of $B \in \Sigma_8$ and $V_5$, we may assume that the 8 decomposable vectors are

$$\bigwedge^3 V_{31} \ldots \bigwedge^3 V_{38} \in B \cap \bigwedge^3 V_6$$

and $V_{31}, \ldots, V_{38} \not\subset \bigwedge^3 V_5$. Then, by [3] Proposition 2.24, the Grassmanian hull $\text{Gr}(2, V_5) \cap W_B$ is smooth. $\bigwedge^3 V_{31}$ can be written as $v \wedge v_1 \wedge v_2$ with $v \in V_6 \setminus V_5$, and $O = v_1 \wedge v_2$ is the kernel of $q(v)$, by [3] Theorem 3.16. Then by [5] Lemma 4.1, $X_B$ has a node at $O$. Then we can project $X_B \subset \mathbb{P}^7$ to $X_O \subset \mathbb{P}^6$ from $O$ as in [5].

The quadrics containing $X_O$ form a net $P$. Let $D$ be the discriminant curve parameterizing singular quadrics. Then there is a line $L \subset D$ corresponding to quadrics containing the projection of $\text{Gr}(2, V_5)$. Therefore,

$$D = C \cup L$$

for some degree 6 curve $C \subset P$.

We see that the curve $\mathbb{P}(V_{31}) \cap Y_{B^2}^{-2}$ parameterizing quadrics in $Y_{B^2}^{-2}$ of corank $\geq 2$ whose vertices contain $O$, thus it equals $C$. Then by [3] Proposition 2.20, $C$ has 7 nodes corresponding to the extra decomposable vectors. For a general $B$,

$$Y_{B^2}^{-3} = \emptyset,$$

hence the quadrics in $C$ are all of rank 6. There is a étale double cover $\tilde{C} \rightarrow C$, corresponding to the choice of a family of 3-planes contained in a quadric in $C$, see [5] section 4.2. Let $p' : \tilde{N} \rightarrow N$ be the normalization of $p : C \rightarrow C$.

There is a morphism $P' \rightarrow C^{(6)}$, sending a line in $P$ to its intersection with $C$. Denote $S'$ the pull back of $P'$ in $\tilde{C}^{(6)}$ and $S''$ the further pull back in $\tilde{N}^{(6)}$. By [18] Proposition 5.8, $S'$ has two irreducible components and $F(X_B)$ is birational to one of the components. It follows that $F(X_B)$ is birational to a component of $S''$. Then by [34] Theorem 8.19,

$$\text{Alb}(F(X_B)) \cong \text{Pr}(\tilde{N}/N),$$

which is of dimension 2 since $g(N) = 3$.

Lemma 5.9. For a general point $y \in Y_A$, we can find $B \in \Omega_A \cap \Sigma_8$, such that $y \in Y_B^{-2}$.

Proof. The proof is similar to [11] Proposition 5.1. Let

$$\Gamma = \Omega \cap \Lambda G(\bigwedge^3 V_6) \times \Sigma_8 \cap \Lambda G(\bigwedge^3 V_6) \times \Omega_y,$$

where $\Omega_y = \{ B \mid \dim(B \cap F_y) \geq 2 \}$ and $F_y = y \wedge \bigwedge^2 V_6$. We have the two projections:
The fiber of $\rho$ is a codimension-2 subvariety of $\Omega_B$, which is 8-dimensional, see [11 Lemma 5.3]. It yields that $\dim \Gamma = \dim \mathbb{L}G(\wedge^3 V_6)$. Therefore it suffices to show that $d\pi$ at a general point $(A, B) \in \Gamma$ is an isomorphism.

Let $(A, B) \in \Gamma$ be general such that the 8 decomposable forms $\alpha_1, \ldots, \alpha_8$ in $B$ are linearly independent and that $\alpha_1, \ldots, \alpha_8$ and $F_y \cap B$ are not contained in $A$. Let $U = A \cap B$. According to [11 Lemma 5.4], we have the description of tangent spaces:

$$T_{(A, B)} \Omega = \{(q_A, q_B) \in \text{Sym}^2(A^\vee) \times \text{Sym}^2(B^\vee) \mid q_A|_U = q_B|_U\}$$

$$T_B \Sigma_8 = \{q_B \in \text{Sym}^2(B^\vee) \mid q_B(\alpha_1) = \ldots = q_B(\alpha_8) = 0\}$$

$$T_B \Omega_y = \{q_B : B \to B^\vee \mid q_B(F_y) \subset B + F_y/B\}.$$

We assume on the contrary that there is a nonzero $t = (q_A, q_B) \in \text{Ker} d\pi$. It implies that $q_A = 0$ and thus $\text{Ker} q_B = U \cup U'$ for some hyperplane $U' \subset B$. According to the assumptions and the description of tangent spaces, the 8 decomposable forms and $F_y \cap B$ are contained in $\text{Ker} q_B$ hence contained in $U'$. Then for a general $B$, $\dim U' = 10$, which is a contradiction. □

Now we can deduce the main result of this subsection:

**Proposition 5.10.** For any conic $c \subset X$, we have $c \in S_2(X)$.

**Proof.** We may assume that $c$ is a general conic. Combining the above Lemmas, there exist a GM 3-fold $X_B$ with 8 nodes containing $c$ and a period partner $X'$ of $X$ such that $X_B \subset X'$. Let $S'$ be the image of

$$F(X_B) \subset F(X) \xrightarrow{\alpha} \hat{Y}^\vee$$

and $\hat{S}$ be the resolution of $S'$. Then a conic lying in $R'$ is in $S_1(X)$, where $R'$ is the normalization of $S' \cap D$.

The argument of [24 Section 2.4] implies that there is a surjection

$$R^{(2)} \to \text{Alb}(\hat{S}).$$

For a resolution $\hat{X}_B \to X_B$, we have $\text{CH}_1(\hat{X}_B)_{\text{hom}} \cong J(\hat{X}_B)$ by [31 Theorem 0.3]. Hence, $\text{CH}_0(\hat{S})_{\text{hom}} \to \text{CH}_1(\hat{X}_B)_{\text{hom}}$ factors through $\text{Alb}(\hat{S})$. By the surjectivity of $R^{(2)} \to \text{Alb}(\hat{S})$, there exists two conics $c_1$ and $c_2 \in S_1(X')$ such that $c = c_1 + c_2 - \theta \in \text{CH}_1(\hat{X}_B)$. Then the result follows from Corollary [30]. □

6. **Sheaves supported on conics and lines**

In this section, we introduce a link between the filtration on $\text{CH}_1(X)$ and the Kuznetsov component $A_X$ of the derived category of $X$ and prove Theorem [13]. We assume that $X$ is very general.

Let $i^* : D^b(X) \to A_X$ be the left adjoint of the inclusion $i_* : A_X \to D^b(X)$ and $\text{pr} : K_0(X) \to K_0(A_X)$ be the projection of Grothendieck groups.
Let $\operatorname{pr}$ be the further projection to $K_{\text{num}}(X)$. For a very general $X$, the numerical Grothendieck group $K_{\text{num}}(X) \cong \mathbb{Z}^{\oplus 2}$ and under the basis $\lambda_1, \lambda_2$ the Euler form is given by (cf. [23, Lemma 2.4]):

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

where $\lambda_1 = \operatorname{pr}[\mathcal{O}_c(1)]$ and $\operatorname{pr}[\mathcal{O}_p] = -\lambda_1 - 2\lambda_2$ for a conic $c \subset X$ and a point $p \in X$.

**Lemma 6.1.** $K_0(A_X) \otimes \mathbb{Q}$ is generated by $\operatorname{pr}[\mathcal{O}_c(1)]$ and $\operatorname{pr}[\mathcal{O}_p]$.

**Proof.** Since $CH_0(X) \cong \mathbb{Z}$, the class $[\mathcal{O}_p]$ in $K_0(X)$ is independent of the choice of $p \in X$ and the cycle class map $\operatorname{cl} : CH^i(X) \to H^i(X, \mathbb{Z})$ is injective for $i \neq 3$, by [1, Theorem 1.1].

We have the following commutative diagram:

$$
\begin{array}{ccc}
K_0(A_X) \otimes \mathbb{Q} & \xrightarrow{\operatorname{ch}} & CH^*(X) \otimes \mathbb{Q} \\
\downarrow{\operatorname{pr}} & & \downarrow{\operatorname{cl}} \\
K_{\text{num}}(A_X) \otimes \mathbb{Q} & \xrightarrow{\overline{\operatorname{ch}}} & H^*(X, \mathbb{Q}),
\end{array}
$$

It’s enough to show that $\operatorname{cl}^{-1}(\overline{\operatorname{ch}}(K_{\text{num}}(A_X)))$ is generated by $\operatorname{ch}(\mathcal{O}_c(1))$ and $\operatorname{ch}(\mathcal{O}_p)$ and this follows from the injectivity of the cycle class map for $i \neq 3$ and the fact that $CH_1(X) \otimes \mathbb{Q}$ is generated by conics. $\square$

For a cubic fourfold, the link between the Kuznetsov component and 1-cycles is by taking the Chern class $c_3$. However, the situation is slightly different for GM fourfolds:

If $X$ and $X'$ are period partners, their associated dual double EPW sextics are the same by Theorem 3.3. Therefore, their Fano varieties of conics both admit a birational $\mathbb{P}^1$-fibration to $\tilde{Y}^\vee$. Then by Lemma 6.1 there is a group isomorphism

$$K_0(A_X)_Q \cong K_0(A'_X)_Q,$$

which maps $\operatorname{pr}[\mathcal{O}_p]$ to $\operatorname{pr}[\mathcal{O}_p']$ and $\operatorname{pr}[\mathcal{O}_c(1)]$ to $\operatorname{pr}[\mathcal{O}_c'(1)]$. Here $c'$ is a conic mapping to the same point with $c$ in $\tilde{Y}^\vee$ via the fibration.

However, the classes $c_3(\operatorname{pr}[\mathcal{O}_p])$ and $c_3(\operatorname{pr}[\mathcal{O}_p'])$ do not coincide in $CH_0(\tilde{Y}^\vee)$ via the isomorphism (5.5) even modulo the class $o$. We want the link between the Kuznetsov component and 1-cycles intrinsic for the EPW sextic, so we modify the $c_3$ to be compatible with (5.5) by dropping the term of $\operatorname{pr}[\mathcal{O}_p]$:

If $[E] = \sum a_i \operatorname{pr}[\mathcal{O}_{c_i}(1)] + b \operatorname{pr}[\mathcal{O}_p] \in K_0(A_X)_Q$, we set

$$p(E) = \sum a_i c_i \in CH_1(X) \otimes \mathbb{Q}.$$ 

It is independent of the choice of representations, since $b$ relies only on the numerical class of $[E]$. Let $\mathcal{F}$ be a sheaf supported on a nonsingular connected rational curve $C \subset X$ of degree $e > 0$. Then $[\mathcal{F}]$ is numerically equivalent to

$$re[\mathcal{O}(1)] + m[\mathcal{O}_p]$$
in $K_{\text{num}}(X)$ for some integers $r > 0$ and $m$, where $l$ is a line on $X$. Let
\[
d(i^*F) := \frac{1}{2} \dim \text{Ext}^1_{\mathcal{A}_X}(i^*F, i^*F)
\]
and we have:

**Lemma 6.2.** If $\mathcal{F}$ is supported on a line or a conic, then $d(i^*\mathcal{F}) \geq 2$.

*Proof.* By straightforward computation, we have $\mathcal{P}(\mathcal{O}_p) = -\lambda_1 - 2\lambda_2$ and $\mathcal{P}(\mathcal{O}_1(1)) = -\lambda_1$. Hence $\mathcal{P}(\mathcal{E}) = -m\lambda_1 - (2m + re)\lambda_2$. Then
\[
d(i^*\mathcal{F}) \geq \frac{1}{2}(i^*\mathcal{F}, i^*\mathcal{F}) + 1 = 5m^2 + 4mre + r^2e^2 + 1.
\]
For $e = 1$ or $2$, we can easily deduce that $d(i^*\mathcal{F}) \geq 2$. In fact, there is a lower bound by a quadric polynomial in $e$, but we only deal with $e = 1$ or $2$.

Now we can prove the main theorem:

**Theorem 6.3.** If $\mathcal{F}$ is supported on a line or a conic and $\mathcal{E} = i^*\mathcal{F}$, then $p(\mathcal{E}) \in S_d(\mathcal{E})(X)$.

*Proof.* By the definition of $p$, it is enough to show that every line or conic belongs to $S_2(X)$. For conics, it is proved in Proposition 5.10. For lines, the strategy is to degenerate the residual curve of a line on a certain surface to a sum of conics and lines.

The locus in $F(X)$ and its image in $\widetilde{\Psi}$ parameterizing double lines is a surface by [13, Lemma 3.8]. Since $X$ is very general, this surface meets $Z$ by Proposition 4.15. Hence, there exists a line $l_0 \subset X$ such that $2l_0 = \theta$. Take $\mathbb{P}(V'_2 \bigcap V'_2) = l' = \epsilon_1(l_0)$, we obtain
\[
2l' = 2\sigma - \theta,
\]
where $\epsilon_1$ is the rational involution of the Hilbert scheme of lines $F_1(X)$.

For $l = \mathbb{P}(V'_1 \bigcap V'_3) \subset X$ a general line, let $V_2 = V_1 \oplus V'_1$ and $V_4 = V_1 \oplus V_3$ and $S_{V_i}$ be the surface parameterizing points $x \in X$ with $V_x \cap V_2 \neq \emptyset$, where $V_x$ is the two dimensional vector space corresponding to $x$. We see that $S_{V_4}$ is a degree 6 surface in $\mathbb{P}(V_2 \bigcap V_3) \cap H = \mathbb{P}^1$. By construction, $l$ and $l'$ are contained in $S_{V_3}$ and $S_{V_2} \cap S_{V_4}$ is a hyperplane section of $S_{V_4} \subset \mathbb{P}(\bigwedge^2 V_4) \cap H$, which is the union of $l$ and a degree 3 curve $C_3$. Hence
\[
l + C_3 = 2\theta.
\]
Since $V_3' \cap V_4$ is 2-dimensional, $l'$ meets $S_{V_4}$ at the point corresponding to $V_3' \cap V_4$. For a general choice of $l$, it does not meet $l'$. It follows that $l'$ and $C_3$ meet at a point, and therefore they span a $\mathbb{P}^4 \subset \mathbb{P}(V_2 \bigcap V_3) \cap H$, which cuts $S_{V_4}$ into a degree 6 curve $l' \cup C_3 \cup c$ for some conic $c$.

We claim that
\[
l' + C_3 + c = 2\theta + \sigma.
\]
In fact, we can choose the hyperplane section $\mathbb{P}^4 \subset \mathbb{P}(V_2 \bigcap V_3) \cap H$ to be the span of $\mathbb{P}(V_2 \bigcap V_3) \cap H$ and $\mathbb{P}(V_1 \bigcap V_1^4)$ for some $V_1 \subset V_2$ with $V_1 \neq V_4$. Then the class of a section is $l + C_3 + c'$, which equals $2\theta + \sigma$. Here $c'$ is the $\sigma$-type conic associated with $V_1$.

Finally, we obtain that $l = l' + c - \sigma = c - l_0$, which is in $S_2(X)$ by Proposition 5.10. \qed
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