Simple proofs of Jensen’s, Chu’s, Mohanty-Handa’s, and Graham-Knuth-Patashnik’s identities

Victor J. W. Guo

Department of Mathematics, East China Normal University, Shanghai 200062, People’s Republic of China
jwguo@math.ecnu.edu.cn, http://math.ecnu.edu.cn/~jwguo

AMS Subject Classifications: 05A10; 05A19

Abstract. Motivated by the recent work of Chu [Electron. J. Combin. 17 (2010), #N24], we give simple proofs of Jensen’s identity
\[ \sum_{k=0}^{n} \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{k=0}^{n} \binom{x+y-k}{n-k} z^k, \]
and Chu’s and Mohanty-Handa’s generalizations of Jensen’s identity. We also give a quite simple proof of an equivalent form of Graham-Knuth-Patashnik’s identity
\[ \sum_{k \geq 0} \binom{m+r}{m-n-k} \binom{n+k}{n} x^{m-n-k} y^k = \sum_{k \geq 0} \binom{-r}{m-n-k} \binom{n+k}{n} (-x)^{m-n-k} (x+y)^k, \]
which was rediscovered, respectively, by Sun in 2003 and Munarini in 2005. Finally we give a multinomial coefficient generalization of this identity and raise two open problems.

Keywords: Jensen’s identity, Chu’s identity, Mohanty-Handa’s identity, Graham-Knuth-Patashnik’s, Chu-Vandermonde, multinomial coefficient

1 Introduction

Abel’s identity (see, for example, [8, §3.1])
\[ \sum_{k=0}^{n} \binom{n}{k} x(x+kz)^{k-1}(y-kz)^{n-k} = (x+y)^n \] (1.1)
and Rothe’s identity (or called Hagen-Rothe’s identity, see, for example, [9, §5.4])
\[ \sum_{k=0}^{n} \frac{x}{x-kz} \binom{x-kz}{k} \binom{y+kz}{n-k} = \binom{x+y}{n}, \] (1.2)
are famous in the literature and play an important role in enumerative combinatorics. Recently, Chu [6] gave elementary proofs of Abel’s identity and Rothe’s identity by using the binomial theorem and the Chu-Vandermonde convolution formula respectively.

Motivated by Chu’s work, we shall study Jensen’s identity [17], which is closely related to Rothe’s identity, and can be stated as follows:
\[ \sum_{k=0}^{n} \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{k=0}^{n} \binom{x+y-k}{n-k} z^k. \] (1.3)
Jensen’s identity (1.3) has ever attracted much attention by different authors. Gould [11] obtained the following Abel-type analogue:

\[
\sum_{k=0}^{n} \frac{(x+kz)^k (y-kz)^{n-k}}{k! (n-k)!} = \sum_{k=0}^{n} \frac{(x+y)^k}{k!} z^{n-k}.
\] (1.4)

Carlitz [1] gave two interesting theorem related to (1.3) and (1.4) by mathematical induction. With the help of generating functions, Gould [12] derived the following variation of Jensen’s identity (1.3):

\[
\sum_{k=0}^{n} \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{k=0}^{n} \binom{x+y}{n-k} \frac{x+y-(n-k)z-k}{x+y-k} z^k.
\] (1.5)

E. G.-Rodeja F. [10] deduced Gould’s identity (1.4) from (1.3) by establishing an identity which includes both. Cohen and Sun [7] also gave an expression which unifies (1.3) and (1.4). Chu [4] generalized Jensen’s identity (1.3) to a multi-sum form:

\[
\sum_{k_1+\cdots+k_s=n} \prod_{i=1}^{s} \binom{x_i+k_i z}{k_i} = \sum_{k=0}^{n} \binom{k+s-2}{k} \binom{x_1+\cdots+x_2+nz-k}{n-k} z^k.
\] (1.6)

Moreover, the identities (1.3) and (1.6) were respectively generalized by Mohanty and Handa [20] and Chu [5] to the case of multinomial coefficients (to be stated in Section 4).

The first purpose of this paper is to give simple proofs of Jensen’s identity, Chu’s identity (1.6), Mohanty-Handa’s identity, and Chu’s generalization of Mohanty-Handa’s identity. We shall use the Chu-Vandermonde convolution formula

\[
\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}
\]

and a well-known identity

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^r = \begin{cases} 0, & \text{if } 0 \leq r \leq n-1, \\ n!, & \text{if } r = n. \end{cases} \tag{1.7}
\]

Eq. (1.7) may be easily deduced from the Stirling numbers of the second kind [28, p. 34, (24a)]. The first case of (1.7) was already utilized by the author [13] to give a simple proof of Dixon’s identity and by Chu [6] in his proofs of Abel’s and Rothe’s identities.

It is interesting that our proof of Chu’s identity (1.6) will also leads to a very short proof of Graham-Knuth-Patashnik’s identity, which was rediscovered several times in the past few years. The second purpose of this paper is to give a multinomial coefficient generalization of Graham-Knuth-Patashnik’s identity and raise two open problems.

## 2 Proof of Jensen’s identity

By the Chu-Vandermonde convolution formula, we have

\[
\sum_{k=0}^{n} \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{k=0}^{n} \binom{x+kz}{k} \sum_{i=k}^{n} \binom{x+y+1}{n-i} \binom{-x-kz-1}{i-k} \tag{2.1}
\]
Interchanging the summation order in (2.1) and noticing that
\[
\binom{x + kz}{k} \binom{-x - kz - 1}{i - k} = (-1)^{i-k} \binom{i}{k} \binom{x + kz + i - k}{i},
\]
we have
\[
\sum_{k=0}^{n} \binom{x + kz}{k} \binom{y - kz}{n - k} = \sum_{i=0}^{n} \binom{x + y + 1}{n - i} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \binom{x + kz + i - k}{i}
\]
\[
= \sum_{i=0}^{n} \binom{x + y + 1}{n - i} (z - 1)^{i},
\]
where the second equality holds because \(\binom{x + kz + i - k}{i}\) is a polynomial in \(k\) of degree \(i\) with leading coefficient \((z - 1)^{i}/i!\) and we can apply (1.7) to simplify. We now substitute \(x \to -x - 1,\) \(y \to -y + n - 1\) and \(z \to -z + 1\) in (2.2) and observe that
\[
\binom{-x}{k} = (-1)^{k} \binom{x + k - 1}{k}. \tag{2.3}
\]
Then we obtain
\[
\sum_{k=0}^{n} \binom{x + kz}{k} \binom{y - kz}{n - k} = \sum_{i=0}^{n} \binom{x + y - i}{n - i} z^{i}, \tag{2.4}
\]
as desired.

Combining (1.3) and (2.2), we get the following identity:
\[
\sum_{k=0}^{n} \binom{x - k}{n - k} z^{k} = \sum_{k=0}^{n} \binom{x + 1}{n - k} (z - 1)^{k},
\]
which is equivalent to the following identity in Graham et al. [9, p. 218]:
\[
\sum_{k \leq m} \binom{m + r}{k} x^{k} y^{m-k} = \sum_{k \leq m} \binom{-r}{k} (-x)^{k} (x + y)^{m-k}.
\]

3 Proofs of Chu’s and Graham-Knuth-Patashnik’s identities

By repeatedly using the Chu-Vandermonde convolution formula, we have
\[
\binom{x + k_{s}z}{k_{s}} = \binom{x_{s} + (n - k_{1} - \cdots - k_{s-1})z}{n - k_{1} - \cdots - k_{s-1}}
\]
\[
= \sum_{j=k_{1}+\cdots+k_{s-1}}^{n} \sum_{j_{1}+\cdots+j_{s-1}=j}^{n} \binom{x_{1} + \cdots + x_{s} + nz + s - 1}{n - j}
\]
\[
\times \prod_{i=1}^{s-1} \binom{-x_{i} - k_{i}z - 1}{j_{i} - k_{i}}. \tag{3.1}
\]
It follows that
\[
\sum_{k_1+\cdots+k_s=n} \prod_{i=1}^{s} \binom{x_i + k_i z}{k_i} = \sum_{j_1+\cdots+j_{s-1}=0} \sum_{j_1+\cdots+j_{s-1}=j} \left( \sum_{n-j}^{n} (x_1 + \cdots + x_s + nz + s - 1) \right) \times \prod_{i=1}^{s-1} \binom{x_i + k_i z}{k_i} \left( -x_i - k_i z - 1 \right) - \frac{1}{j_i - k_i}.
\]

Interchanging the summation order in (3.2) and observing that
\[
\binom{x_i + k_i z}{k_i} \left( -x_i - k_i z - 1 \right) = (-1)^{j_i-k_i} \binom{j_i}{k_i} \binom{x_i + k_i z + j_i - k_i}{j_i}
\]
and \((x_i+k_i z+j_i-k_i)\) is a polynomial in \(k_i\) of degree \(j_i\) with leading coefficient \((z-1)^{j_i}/j_i!\), by (1.7) we get
\[
\sum_{k_1+\cdots+k_s=n} \prod_{i=1}^{s} \binom{x_i + k_i z}{k_i} = \sum_{j=0}^{n} \left( \sum_{n-j}^{n} (x_1 + \cdots + x_s + nz + s - 1) \right) \sum_{j_1+\cdots+j_{s-1}=j} (z-1)^{j} \binom{j+s-2}{j} \binom{x_1 + \cdots + x_s + nz + s - 1}{n-j} (z-1)^{j}. \tag{3.3}
\]
Substituting \(x_i \to -x_i - 1\) (\(i = 1, \ldots, s\)) and \(z \to -z + 1\) in (3.3) and using (2.3), we immediately get Chu's identity (1.6).
Comparing (1.6) with (3.3) and replacing \(s\) by \(s+2\), we immediately get
\[
\sum_{k=0}^{n} \binom{k+s}{k} \binom{x-k}{n-k} z^k = \sum_{j=0}^{n} \left( \sum_{n-j}^{n} (x_1 + \cdots + x_s + nz + s - 1) \right) \sum_{j_1+\cdots+j_{s-1}=j} (z-1)^{j} \binom{j+s-2}{j} \binom{x_1 + \cdots + x_s + nz + s - 1}{n-j} (z-1)^{j}. \tag{3.4}
\]
It is easy to see that the identity (3.4) is equivalent to each of the following known identities:

- Graham-Knuth-Patashnik's identity [9, p. 218]
  \[
  \sum_{k \geq 0} \binom{m+r}{m-n-k} \binom{n+k}{n} z^{m-n-k} y^k = \sum_{k \geq 0} \binom{-r}{m-n-k} \binom{n+k}{n} (-x)^{m-n-k} (x+y)^k.
  \]

- Sun's identity [30]
  \[
  \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{n+k}{a} (1+x)^{n+k-a} = \sum_{k=0}^{n} \binom{n}{k} \binom{m+k}{a} x^{m+k-a}. \tag{3.5}
  \]

- Munarini's identity [21]
  \[
  \sum_{k=0}^{n} (-1)^{n-k} \binom{\beta - \alpha + n}{n-k} \binom{\beta + k}{k} (1+x)^k = \sum_{k=0}^{n} \binom{\alpha}{n-k} \binom{\beta + k}{k} x^k. \tag{3.6}
  \]

Moreover, the following special case
\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{k} (1+x)^k = \sum_{k=0}^{n} \binom{n+k}{k} x^k \tag{3.7}
\]
was reproved by Simons [27], Hirschhorn [16], Chapman [2], Prodinger [22], Wang and Sun [31].
4 Mohanty-Handa’s identity and Chu’s generalization

Let \( m \) be a fixed positive integer. For \( \mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m \) and \( \mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{C}^m \), set \( |\mathbf{a}| = a_1 + \cdots + a_m \), \( \mathbf{a}! = a_1! \cdots a_m! \), \( \mathbf{a} + \mathbf{b} = (a_1 + b_1, \ldots, a_m + b_m) \), \( \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_m b_m \), and \( b^\mathbf{a} = b_1^{a_1} \cdots b_m^{a_m} \). For any variable \( x \) and \( \mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}^m \), the multinomial coefficient \( \binom{x}{\mathbf{n}} \) is defined by
\[
\binom{x}{\mathbf{n}} = \begin{cases} 
    x(x - 1) \cdots (x - |\mathbf{n}| + 1)/|\mathbf{n}|!, & \text{if } \mathbf{n} \in \mathbb{N}^m, \\
    0, & \text{otherwise}.
\end{cases}
\]

Moreover, we let \( \mathbf{0} = (0, \ldots, 0) \) and \( \mathbf{1} = (1, \ldots, 1) \).

In 1969, Mohanty and Handa [20] established the following multinomial coefficient generalization of Jensen’s identity
\[
\sum_{k=0}^{n} \binom{x + k \cdot \mathbf{z}}{k} \binom{y - k \cdot \mathbf{z}}{\mathbf{n} - k} = \sum_{k=0}^{n} \binom{x + y - |k|}{\mathbf{n} - k} \binom{|k|}{k} \mathbf{z}^k. \tag{4.1}
\]

Twenty years later, Mohanty-Handa’s identity was generalized by Chu [5] as follows:
\[
\sum_{k_1 + \cdots + k_s = n} \prod_{i=1}^{s} \binom{x_i + k_i \cdot \mathbf{z}}{k_i} = \sum_{k=0}^{n} \binom{|k| + s - 2}{k} \binom{x_1 + \cdots + x_s + n \cdot \mathbf{z} - |k|}{\mathbf{n} - k} \mathbf{z}^k, \tag{4.2}
\]
which is also a generalization of (1.6).

Remark. Note that the corresponding multinomial coefficient generalization of Rothe’s identity was already obtained by Raney [23] (for a special case) and Mohanty [18]. The reader is referred to Strehl [29] for a historical note on Raney-Mohanty’s identity.

In what follows, we will give an elementary proof of Chu’s identity (4.2) similar to that of (1.6). First note that the Chu-Vandermonde convolution formula has the following trivial generalization
\[
\sum_{k=0}^{n} \binom{x}{k} \binom{y}{\mathbf{n} - k} = \binom{x + y}{\mathbf{n}}, \tag{4.3}
\]
as mentioned by Zeng [32], while (1.7) can be easily generalized as
\[
\sum_{k=0}^{n} (-1)^{|\mathbf{n}| - |k|} \binom{n}{k} \mathbf{k}^r = \begin{cases} 
    0, & \text{if } r_i < n_i \text{ for some } 1 \leq i \leq m. \\
    n!, & \text{if } r = \mathbf{n},
\end{cases} \tag{4.4}
\]
where
\[
\binom{n}{k} := \prod_{i=1}^{m} \binom{n_i}{k_i}.
\]

Lemma 4.1 For \( \mathbf{n} \in \mathbb{N}^m \) and \( s \geq 1 \), there holds
\[
\sum_{k_1 + \cdots + k_s = n} \prod_{i=1}^{s} \binom{|k_i|}{k_i} = \binom{|\mathbf{n}| + s - 1}{\mathbf{n}}. \tag{4.5}
\]
Proof. For nonnegative integers \(a_1, \ldots, a_s\) such that \(a_1 + \cdots + a_s = |n|\), by the Chu-Vandermonde convolution formula (4.4), the following identity holds

\[
\sum_{k_1 + \cdots + k_s = n} \prod_{i=1}^{s} \binom{a_i}{k_i} = \binom{|n|}{n}.
\] (4.6)

Moreover, in this case, for \(k_1 + \cdots + k_s = n\), we have

\[
\prod_{i=1}^{s} \binom{a_i}{k_i} \neq 0 \quad \text{if and only if} \quad |k_i| = a_i \quad (i = 1, \ldots, s).
\]

Thus, the identity (4.6) may be rewritten as

\[
\sum_{k_1 + \cdots + k_s = n} \prod_{i=1}^{s} \binom{a_i}{k_i} = \binom{|n|}{n}.
\]

It follows that

\[
\sum_{k_1 + \cdots + k_s = n} \prod_{i=1}^{s} \binom{|k_i|}{k_i} = \sum_{a_1 + \cdots + a_s = |n|} \sum_{k_1 + \cdots + k_s = n} \prod_{i=1}^{s} \binom{a_i}{k_i}
\]

\[
= \sum_{a_1 + \cdots + a_s = |n|} \binom{|n|}{n}
\]

\[
= \binom{|n| + s - 1}{n},
\]

as desired.

By repeatedly using the convolution formula (4.3), we may rewrite the left-hand side of (4.2) as

\[
\sum_{k_1 + \cdots + k_{s-1} = 0}^{n} \sum_{j = k_1 + \cdots + k_{s-1}}^{n} \binom{x_1 + \cdots + x_s + n \cdot z + m - 1}{n - j}
\]

\[
\times \prod_{i=1}^{s-1} \binom{x_i + k_i \cdot z}{k_i} \binom{-x_i - k_i \cdot z - 1}{j_i - k_i}.
\] (4.7)

Interchanging the summation order in (4.7), observing that

\[
\binom{x_i + k_i \cdot z}{k_i} \binom{-x_i - k_i \cdot z - 1}{j_i - k_i} = (-1)^{|j_i| - |k_i|} \binom{j_i}{k_i} \binom{x_i + k_i \cdot z + |j_i| - |k_i|}{j_i}
\]

and

\[
\binom{x_i + k_i \cdot z + |j_i| - |k_i|}{j_i}
\]

as desired.

By repeatedly using the convolution formula (4.3), we may rewrite the left-hand side of (4.2) as
is a polynomial in $k_1, \ldots, k_m$ with the coefficient of $k_i^j$ being $\binom{\lfloor k_i \rfloor}{j!}(z - 1)^j/j!$. Applying (4.4), we get

$$
\sum_{k_1, \ldots, k_s = -n}^n \prod_{i=1}^s \left( x_i + k_i \cdot z \right) = \sum_{j=0}^n \left( x_1 + \cdots + x_s + n \cdot z + s - 1 \right) (z - 1)^j \sum_{j_1 + \cdots + j_s = j} \prod_{i=1}^m \binom{j_i}{j_i},
$$

where the second equality follows from (4.5). Substituting $x_i \rightarrow -x_i - 1$ ($i = 1, \ldots, s$) and $z \rightarrow -z + 1$ in (4.8) and observing that $(\frac{z^n}{k}) = (-1)^{|k|}(x+|k|-1)$, we immediately get (4.2).

Comparing (4.2) with (4.8) and replacing $s$ by $s + 2$, we obtain the following result.

**Theorem 4.2** For $n \in \mathbb{N}^m$ and $z \in \mathbb{C}^m$, there holds

$$
\sum_{k=0}^n \binom{|k| + s}{k} \binom{x - |k|}{n - k} z^k = \sum_{k=0}^n \binom{|k| + s}{k} \binom{x + s + 1}{n - k} (z - 1)^k.
$$

It is easy to see that (4.9) is a multinomial coefficient generalization of (3.4). Substituting $s \rightarrow \beta$, $x \rightarrow \alpha - \beta - 1$ and $z \rightarrow 1 + x$ in (4.9), we get

$$
\sum_{k=0}^n (-1)^{|n| - |k|} \binom{\beta - \alpha + |n|}{n - k} \binom{\beta + |k|}{k} (1 + x)^k = \sum_{k=0}^n \binom{\alpha}{n - k} \left( \frac{\beta + |k|}{k} \right) x^k,
$$

which is a generalization of Munarini's identity (3.6). If $\alpha = \beta = |n|$, then (4.10) reduces to

$$
\sum_{k=0}^n (-1)^{|n| - |k|} \binom{|n|}{n - k} \binom{|n| + |k|}{k} (1 + x)^k = \sum_{k=0}^n \binom{|n|}{n - k} \left( \frac{|n| + |k|}{k} \right) x^k,
$$

which is generalization of Simons’ identity (3.7). Note that Shattuck [26] and Chen and Pang [3] have given different combinatorial proofs of (3.6). It is natural to ask

**Problem 4.3** Find a combinatorial interpretation of (4.10).

### 5 Concluding remarks

We know that binomial coefficient identities usually have nice $q$-analogues. However, there are only curious (not natural) $q$-analogues of Abel’s and Rothe’s identities (see [25] and references therein) up to now. There seems to have no $q$-analogues of Jensen’s identity in the literature.

It is interesting that Hou and Zeng [15] gave a $q$-analogue of Sun’s identity (3.5):

$$
\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{n+k}{a} (-xq^a; q)_{n+k-a}q^{k+1 - \frac{1}{2}} = \sum_{k=0}^n \binom{n}{k} \binom{m+k}{a} x^{m+k-a} q^{mn+\frac{k}{2}},
$$

(5.1)
where \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\) and
\[
\left[ \frac{\alpha}{k} \right] = \begin{cases} 
\frac{(q^{\alpha-k+1}; q)_k}{(q; q)_k}, & \text{if } k \geq 0, \\
0, & \text{if } k < 0.
\end{cases}
\]
Clearly, (5.1) may be written as a \(q\)-analogue of Munarini’s identity (3.6):
\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{\beta - \alpha + n}{n - k} \binom{\beta + k}{k} q^{\binom{n-k}{2} - \binom{n}{2}} (-x; q)_k
= \sum_{k=0}^{n} \binom{\alpha}{n - k} \binom{\beta + k}{k} q^{\binom{n-k+1}{2} + (\beta-\alpha)(n-k)} x^k,
\]
(5.2)
as mentioned by Guo and Zeng [14]. We end this paper with the following problem.

**Problem 5.1** Is there a \(q\)-analogue of (4.10)? Or equivalently, is there a multi-sum generalization of (5.2)?

**Acknowledgments.** This work was partially supported by the Fundamental Research Funds for the Central Universities, Shanghai Rising-Star Program (#09QA1401700), Shanghai Leading Academic Discipline Project (#B407), and the National Science Foundation of China (#10801054).

**References**

[1] L. Carlitz, Some formulas of Jensen and Gould, Duke Math. J. 27 (1960), 319–321.
[2] R. Chapman, A curious identity revisited, Math. Gazette 87 (2003), 139–141.
[3] W. Y. C. Chen and S. X. M. Pang, On the combinatorics of the Pfaff identity, Discrete Math. 309 (2009), 2190–2196.
[4] W. Chu, On an extension of a partition identity and its Abel analog, J. Math. Rese. Exposition 6 (4) (1986), 37–39.
[5] W. Chu, Jensen’s theorem on multinomial coefficients and its Abel-analog, Appl. Math. J. Chinese Univ. 4 (1989), 172–178 (in Chinese).
[6] W. Chu, Elementary proofs for convolution identities of Abel and Hagen-Rothe, Electron. J. Combin. 17 (2010), #N24.
[7] M. E. Cohen and H. S. Sun, A note on the Jensen-Gould convolutions, Canad. Math. Bull. 23 (1980), 359–361.
[8] L. Comtet, Advanced Combinatorics, D. Reidel Publishing Company, Dordrecht-Holland, 1974.
[9] R. L. Graham, D. E. Knuth and O. Patashnik, Concrete Mathematics, Addion-Wesley Publishing Co., 1989.
[10] E. G.-Rodeja F., On identities of Jensen, Gould and Carlitz, in: Proc. Fifth Annual Reunion of Spanish Mathematicians (Valencia, 1964), Publ. Inst. “Jorge Juan” Mat., Madrid, 1967, pp. 11–14.
[11] H. W. Gould, Generalization of a theorem of Jensen concerning convolutions, Duke Math. J. 27 (1960) 71–76.
[12] H. W. Gould, Involving sums of binomial coefficients and a formula of Jensen, Amer. Math. Monthly, 69 (5) (1962), 400–402.
[13] V. J. W. Guo, A simple proof of Dixon’s identity, Discrete Math. 268 (2003), 309–310.
[14] V. J. W. Guo and J. Zeng, Combinatorial proof of a curious $q$-binomial coefficient identity, Electron. J. Combin. 17 (2010), #N13.

[15] S. J. X. Hou and J. Zeng, A $q$-analog of dual sequences with applications, European J. Combin. 28 (2007), 214–227.

[16] M. Hirschhorn, Comment on a curious identity, Math. Gazette 87 (2003), 528–530.

[17] J. L. W. V. Jensen, Sur une identité d’Abel et sur d’autres formules analogues, Acta Math. 26 (1902), 307–318.

[18] S. G. Mohanty, Some convolutions with multinomial coefficients and related probability distributions, SIAM Rev. 8 (1966), 501–509.

[19] S. G. Mohanty, Lattice Path Counting and Applications, Academic Press, New York, 1979.

[20] S.G. Mohanty and B.R. Handa, Extensions of Vandermonde type convolutions with several summations and their applications, I. Canad. Math. Bull. 12 (1969), 45–62.

[21] E. Munarini, Generalization of a binomial identity of Simons, Integers 5 (2005), #A15.

[22] H. Prodinger, A curious identity proved by Cauchy’s integral formula, Math. Gazette 89 (2005), 266–267.

[23] G. N. Raney, Functional composition patterns and power series reversion, Trans. Amer. Math. Soc. 94 (1960), 441–451.

[24] H. A. Rothe, Formulæ de serierum reversione demonstratio universalis signis localibus combinatorio-analyticorum vicariis exhibita, Leipzig, 1793.

[25] M. Schlosser, Abel-Rothe type generalizations of Jacobi’s triple product identity, in: Theory and Applications of Special Functions, Dev. Math., 13, Springer, New York, 2005, pp. 383–400.

[26] M. Shattuck, Combinatorial proofs of some Simons-type binomial coefficient identities, Integers 7 (2007), #A27.

[27] S. Simons, A curious identity, Math. Gazette 85 (2001), 296–298.

[28] R. P. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge Studies in Advanced Mathematics, 49, Cambridge University Press, Cambridge, 1997.

[29] V. Strehl, Identities of Rothe-Abel-Schläfi-Hurwitz-type, Discrete Math. 99 (1992), 321–340.

[30] Z.-W. Sun, Combinatorial identities in dual sequences, European J. Combin. 24 (2003), 709–718.

[31] X. Wang and Y. Sun, A new proof of a curious identity, Math. Gazette 91 (2007), 105–106.

[32] J. Zeng, Multinomial convolution polynomials, Discrete Math. 160 (1996), 219–228.