A criterion of quasi-infinite divisibility for discrete multivariate probability laws

I. A. Alexeev∗†, A. A. Khartov‡§

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Abstract

Multivariate discrete probability laws are considered. We show that such laws are quasi-infinitely divisible if and only if their characteristic functions are separated from zero. We generalize the existing results for the univariate discrete laws and for the multivariate laws on \( \mathbb{Z}^d \).

Keywords and phrases: multivariate probability laws, characteristic functions, infinitely divisible laws, the Lévy representation, quasi-infinitely divisible laws.

1 Introduction.

Let \( F \) be a distribution function of a multivariate probability law on \( \mathbb{R}^d \), where \( \mathbb{R} \) is the real line, \( d \) is a positive integer. Recall that \( F \) and the corresponding law are called infinitely divisible if for every positive integer \( n \) there exists a distribution function \( F_n \) such that \( F = F_n^* \), where "*" denotes the convolution, i.e. \( F \) is the \( n \)-fold convolution power of \( F_n \). It is known that \( F \) is infinitely divisible if and only if its characteristic function

\[
f(t) := \int_{\mathbb{R}^d} e^{i\langle t, x \rangle} dF(x), \quad t \in \mathbb{R}^d,
\]

admits the following Lévy representation (see [20, Theorem 8.1])

\[
f(t) = \exp \left\{ i\langle t, \gamma \rangle - \frac{1}{2} \langle t, Qt \rangle + \int_{\mathbb{R}^d} \left( e^{i\langle t, x \rangle} - 1 - \frac{i\langle t, x \rangle}{1 + \|x\|^2} \right) \nu(dx) \right\}, \quad t \in \mathbb{R}^d,
\]

(1)

where \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product in \( \mathbb{R}^d \), \( \|x\| := \sqrt{\langle x, x \rangle} \) for any \( x \in \mathbb{R}^d \), \( \gamma \in \mathbb{R}^d \) is a fixed vector, \( Q \) is a symmetric nonnegative-definite \( d \times d \) matrix, and \( \nu \) is a measure on \( \mathbb{R}^d \) that satisfies the following conditions

\[
\nu(\{0\}) = 0, \quad \int_{\mathbb{R}^d} \min\{\|x\|^2, 1\} \nu(dx) < \infty.
\]

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∗St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, 27 Fontanka, Saint-Petersburg
†Institute for Information Transmission Problems of Russian Academy of Sciences, Bolshoy Karetny per. 19, build.1, Moscow, email: vanyalexeev@list.ru
‡Laboratory for Approximation Problems of Probability, Smolensk State University, 4 Przhevalsky st., 214000 Smolensk, Russia, web-site: approlab.org
§Scientific and Educational Center of Mathematics, ITMO University, Kronverksky Pr. 49, 197101, Saint-Petersburg, Russia, e-mail: alexeykhartov@gmail.com
Here and below, we denote by $\bar{0}$ the zero vector of $\mathbb{R}^d$. The vector $(\gamma, Q, \nu)$ is called a characteristic triplet and it is uniquely determined by $f$ and hence by $F$.

In the recent paper by Berger, Kutlu, and Lindner [5], the notion of quasi-infinitely divisible distributions on $\mathbb{R}^d$ was introduced. Following them, a distribution function $F$ and the corresponding law are called quasi-infinitely divisible, if there exist infinitely divisible distribution functions $F_1$ and $F_2$ such that $F_1 = F \ast F_2$. It was proved in [5] that the representation (1) holds for quasi-infinitely divisible distributions, but the measure $\nu$ is a a signed finite measure on $\mathbb{R}^d \setminus (-r, r)^d$ for any $r > 0$ that satisfies $\nu(\{\bar{0}\}) = 0$, and

$$\int_{\mathbb{R}^d} \min\{\|x\|^2, 1\}|\nu|(dx) < \infty,$$

where $|\nu|$ denotes the total variation of the measure $\nu$. It is seen that quasi-infinitely divisible distributions are natural generalizations of infinitely divisible distributions.

The examples of univariate quasi-infinitely divisible laws can be found in the classical monographs [7], [15], and [16]. The first detailed analysis of these laws on $\mathbb{R}$ was performed in [14], and a lot of results for the univariate case are contained in the works [1], [3], [4], [9], [10], and [11]. The multivariate case was considered in the recent papers [5], [6], and [19]. In these works the authors studied questions concerning supports, moments, continuity, and the weak convergence. The most complete results were obtained for probability laws on the set $\mathbb{Z}^d$, where $\mathbb{Z}$ is the set of integers. In particular, the following important fact was stated in [6].

**Theorem 1** Let $F$ be the distribution function of the probability law on $\mathbb{Z}^d$. Let $f$ be its characteristic function. Then $F$ is quasi-infinitely divisible if and only if $f(t) \neq 0$ for all $t \in \mathbb{R}^d$. In that case, $f$ admits the following representation

$$f(t) = \exp \left\{ i\langle t, \gamma \rangle + \sum_{k \in \mathbb{Z}^d \setminus \{\bar{0}\}} \lambda_k (e^{i\langle t, k \rangle} - 1) \right\}, \quad t \in \mathbb{R}^d,$$

(2)

where $\gamma \in \mathbb{Z}^d$, $\lambda_k \in \mathbb{R}$, $k \in \mathbb{Z}^d \setminus \{\bar{0}\}$, and $\sum_{k \in \mathbb{Z}^d \setminus \{\bar{0}\}} |\lambda_k| < \infty$.

It is clear that (2) can be rewritten in the form (1).

The purpose of this article is to generalize Theorem 1 to the arbitrary multivariate discrete probability laws. The corresponding criterion will be formulated in section 2. Sections 3 and 4 are devoted to the proof of the main result. More precisely, the tools for proving the main theorem, which are also of independent interest, will be formulated in section 3. All of the mentioned results will be proved in section 4.

**2 Main result**

Let us consider a multivariate discrete probability law with the following distribution function

$$F(x) = \sum_{\substack{k \in \mathbb{N} : \\ x_k \in (-\infty, x]}} p_{x_k}, \quad x \in \mathbb{R}^d.$$

(3)

Here $x_k \in \mathbb{R}^d$, $k \in \mathbb{N}$, are distinct numbers with probability weights $p_{x_k} \geq 0$, $k \in \mathbb{N}$ (the set of positive integers), $\sum_{k=1}^{\infty} p_{x_k} = 1$. We denote by $(-\infty, x]$ with $x = (x^{(1)}, x^{(2)}, \ldots, x^{(d)}) \in \mathbb{R}^d$ the set $(-\infty, x^{(1)}) \times \ldots \times (-\infty, x^{(d)}) \subset \mathbb{R}^d$. Let $f$ be the characteristic function of $F$, i.e.

$$f(t) := \int_{\mathbb{R}^d} e^{i\langle t, x \rangle} dF(x) = \sum_{k \in \mathbb{N}} p_{x_k} e^{i\langle t, x_k \rangle}, \quad t \in \mathbb{R}^d.$$

(4)
We will formulate a criterion for the distribution function $F$ to be quasi-infinitely divisible through condition for characteristic function $f$. For the sharp formulation of the result below we need to introduce the set of all finite $\mathbb{Z}^d$-linear combinations of elements from a set $Y \subset \mathbb{C}^d$ ($\mathbb{C}$ is the set of complex numbers):

$$\langle Y \rangle = \left\{ \sum_{k=1}^{n} z_k y_k^{(1)} : n \in \mathbb{N}, z_k \in \mathbb{Z}, y_k \in Y \right\} \times \ldots \times \left\{ \sum_{k=1}^{n} z_k y_k^{(d)} : n \in \mathbb{N}, z_k \in \mathbb{Z}, y_k \in Y \right\},$$

where $y_k = (y_k^{(1)}, \ldots, y_k^{(d)})$. So $\langle Y \rangle$ is a module over the ring $\mathbb{Z}^d$ with the generating set $Y$. It is easily seen that $Y \subset \langle Y \rangle$, $0 \in \langle Y \rangle$. If a countable set $Y \neq \emptyset$, then $\langle Y \rangle$ is an infinite countable set.

**Theorem 2** Let $F$ be a discrete distribution function of the form (3) with characteristic function $f$ of the form (4). The following statements are equivalent:

(a) $F$ is quasi-infinitely divisible;

(b) $\inf_{t \in \mathbb{R}^d} |f(t)| > 0$.

If one of the conditions is satisfied, and hence all, then $f$ admits the following representation

$$f(t) = \exp \left\{ i (t, \gamma) + \sum_{u \in \langle X \rangle \setminus \{0\} } \lambda_u \left( e^{i(u,t)} - 1 \right) \right\}, \quad t \in \mathbb{R}^d,$$

where $X := \{ x_k : px_k > 0, k \in \mathbb{N} \} \neq \emptyset$, $\gamma \in \langle X \rangle$, $\lambda_u \in \mathbb{R}$ for all $u \in \langle X \rangle \setminus \{0\}$, and $\sum_{u \in \langle X \rangle \setminus \{0\}} |\lambda_u| < \infty$.

Note that Theorem 2 generalizes Theorem 1. Indeed, for characteristic function $f$ of probability law on $\mathbb{Z}^d$ the condition that $f(t) \neq 0, t \in \mathbb{R}^d$, is equivalent to the condition that $\inf_{t \in \mathbb{R}^d} |f(t)| > 0$. It follows due to the continuity and $2\pi$-periodicity of the function $|f(t)|$, $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$, over each $t_j$. Theorem 2 also generalizes the corresponding results from [1] and [10] for the discrete distributions in the univariate case.

3 Tools

We will get the main result from more general positions. Namely, we will consequently study admission of the Lévy type representations for a general almost periodic function $h$ on $\mathbb{R}^d$ that is very similar to $f$.

**Theorem 3** Let $h: \mathbb{R}^d \to \mathbb{C}$ be a function of the following form:

$$h(t) = \sum_{y \in Y} q_y e^{i(t,y)}, \quad t \in \mathbb{R}^d,$$

where $Y \subset \mathbb{R}^d$ is a nonempty at most countable set, $q_y \in \mathbb{C}$ for all $y \in Y$, and $0 < \sum_{y \in Y} |q_y| < \infty$. Assume that $h(0) = \sum_{y \in Y} q_y = 1$. If $\inf_{t \in \mathbb{R}^d} |h(t)| = \mu > 0$, then $h$ admits the following representation

$$h(t) = \exp \left\{ i (t, \gamma) + \sum_{u \in \langle Y \rangle \setminus \{0\} } \lambda_u \left( e^{i(u,t)} - 1 \right) \right\}, \quad t \in \mathbb{R}^d,$$

where $\gamma \in \langle Y \rangle$, $\lambda_u \in \mathbb{C}$ for all $u \in \langle Y \rangle \setminus \{0\}$, and $\sum_{u \in \langle Y \rangle \setminus \{0\}} |\lambda_u| < \infty$.

It should be noted that the function $h$ in Theorem 3 is an almost periodic function on $\mathbb{R}^n$ with absolutely convergence Fourier series. Recall that (see [12, p. 255] or [17, Definition 1]) a function $h: \mathbb{R}^d \to \mathbb{C}$ is called
almost periodic if for any sequence \( \{t_n\}_{n \in \mathbb{N}} \) from \( \mathbb{R}^d \) there exists a subsequence \( (t_{n_k})_{k \in \mathbb{N}} \) and a continuous function \( \varphi : \mathbb{R}^d \to \mathbb{C} \) such that

\[
\sup_{t \in \mathbb{R}^d} |h(t + t_{n_k}) - \varphi(t)| \xrightarrow{k \to \infty} 0.
\]

Detailed information about almost periodic functions on \( \mathbb{R}^n \) can be found in [2], [12], [13], [17], and [18]. Note that in the above literature, the results are formulated in a greater generality, that is, for local compact Abelian (LCA) groups.

We now turn to the following general version of the Theorem 2.

**Theorem 4** Let \( h : \mathbb{R}^d \to \mathbb{C} \) be a function of the following form

\[
h(t) = \sum_{y \in Y} q_y e^{i(t,y)}, \quad t \in \mathbb{R}^d,
\]

where \( Y \subset \mathbb{R}^d \) is a nonempty at most countable set, \( q_y \in \mathbb{C} \) for all \( y \in Y \), and \( 0 < \sum_{y \in Y} |q_y| < \infty \). Suppose that \( h(0) = \sum_{y \in Y} q_y = 1 \). Then the following statements are equivalent:

(i) \( \inf_{t \in \mathbb{R}^d} |h(t)| > 0 \);

(ii) There exist a countable set \( Z \subset \mathbb{R}^d \) and coefficients \( r_z \in \mathbb{C} \), \( z \in Z \), \( \sum_{z \in Z} |r_z| < \infty \), such that

\[
\frac{1}{h(t)} = \sum_{z \in Z} r_z e^{i(t,z)}, \quad t \in \mathbb{R}^d;
\]

(iii) \( h \) admits the representation

\[
h(t) = \exp\left\{ i\langle t, \gamma \rangle + \sum_{u \in (Y) \setminus \{0\}} \lambda_u \left(e^{i\langle t,u \rangle} - 1\right) \right\}, \quad t \in \mathbb{R}^d,
\]

where \( \gamma \in (Y) \), \( \lambda_u \in \mathbb{C} \) for all \( u \in (Y) \setminus \{0\} \), and \( \sum_{u \in (Y) \setminus \{0\}} |\lambda_u| < \infty \);

(iv) \( h \) admits the representation

\[
h(t) = \exp\left\{ i\langle t, \gamma \rangle - \frac{1}{2} \langle t, Qt \rangle + \int_{\mathbb{R}^d} \left( e^{i\langle t,u \rangle} - 1 - \frac{i\langle t,u \rangle}{1 + |u|^2} \right) \nu(du) \right\}, \quad t \in \mathbb{R}^d,
\]

where \( \gamma \in \mathbb{C}^d \), \( Q \in \mathbb{C}^{d \times d} \) is a matrix, \( \nu \) is a complex measure on \( \mathbb{R}^d \) such that

\[
\nu(\{0\}) = 0, \quad \text{and} \quad \int_{\mathbb{R}^d} \min\{\|x\|^2, 1\} |\nu|(dx) < \infty.
\]

**4 Proofs**

**Proof of Theorem 3.** We will sequentially consider the following cases: 1) \( Y = \mathbb{Z}^d \), 2) \( Y \) is a finite subset of \( \mathbb{R}^d \), 3) \( Y \) is at most countable subset of \( \mathbb{R}^d \) (the general case). We always assume that \( Y \neq \emptyset \). Each subsequent case will be based on the previous one.

1) Suppose that \( Y = \mathbb{Z}^d \). It is easy to see that the function \( h \) is \( 2\pi \)-periodic in all coordinates, i.e. for any \( k = 1, \ldots, d \) and \( t \in \mathbb{R}^d \) we have \( h(t + 2\pi e_k) = h(t) \), where \( \{e_1, e_2, \ldots, e_d\} \) denotes the canonical basis in \( \mathbb{R}^d \). Let us consider the distinguished logarithm \( t \mapsto \ln h(t), \quad t \in \mathbb{R}^d \), which satisfies \( \exp\{\ln h(t)\} = h(t), \)
where \( \lambda \in \mathbb{C} \) for all \( u \in \mathbb{Z}^d \setminus \{ \emptyset \} \), and \( \sum_{u \in \mathbb{Z}^d \setminus \{ \emptyset \}} |\lambda_u| < \infty \). Note that in this case \( \langle Y \rangle = \mathbb{Z}^d \).

2) Assume that \( Y = \{ y_1, \ldots, y_n \} \), where \( y_1, \ldots, y_n \) are distinct elements from \( \mathbb{R}^d \). So we have \( h(t) = \sum_{k=1}^{n} q_{yk} e^{i(t \cdot y_k)} \), \( t \in \mathbb{R}^d \). If \( n = 1 \) then \( Y = \{ y_1 \} \) and \( q_{y_1} = 1 \). For this case representation (6) holds with \( \gamma = y_1 \) and \( \lambda_u = 0 \) for all \( u \in \{ Y \} \setminus \{ \emptyset \} \). We next suppose that \( n \geq 2 \). We set \( y_k = (y_{k(1)}, \ldots, y_{k(d)}) \), \( k = 1, \ldots, n \). Without loss of generality, we can assume that for every \( k = 1, \ldots, n \) there exist \( j \) such that \( y_{k(j)} \neq 0 \), since otherwise we can turn to the space \( \mathbb{R}^{d'} \) with some \( d' < d \). Next, for every \( j = 1, \ldots, d \) we can choose non-zero \( \beta_1^{(j)}, \ldots, \beta_{m_j}^{(j)} \in Y^{(j)} = \{ y_{1(j)}, \ldots, y_{n(j)} \} \subset \mathbb{R} \) that consist a basis in \( Y^{(j)} \) over \( \mathbb{Z} \), i.e. for every \( j = 1, \ldots, d \) and for every \( k = 1, \ldots, n \) there exist unique real values \( c_{k,1}^{(j)}, \ldots, c_{k,m_j}^{(j)} \in \mathbb{Z} \) such that

\[
y_{k(j)} = \sum_{l=1}^{m_j} c_{k,l}^{(j)} \beta_l^{(j)}.
\]

So it is easy to check that

\[
\langle Y \rangle = \left\{ \left. \left( \sum_{l=1}^{m_1} z_1^{(1)} \beta_1^{(1)}, z_1^{(1)} \in \mathbb{Z} \right) \times \ldots \times \left( \sum_{l=1}^{m_d} z_d^{(d)} \beta_l^{(d)}, z_d^{(d)} \in \mathbb{Z} \right) \right| z_i^{(j)} \in \mathbb{Z} \right\}.
\]

Note that for every \( j = 1, \ldots, d \) the values \( \beta_1^{(j)}, \ldots, \beta_{m_j}^{(j)} \) are linearly independent over \( \mathbb{Z} \), i.e. the equation

\[
l_1 \beta_1^{(j)} + \ldots + l_{m_j} \beta_{m_j}^{(j)} = 0
\]

holds with \( l_1, \ldots, l_{m_j} \in \mathbb{Z} \) if and only if \( l_1 = \ldots = l_{m_j} = 0 \).

Now we consider the function

\[
\varphi(t^{(1)}_1, \ldots, t^{(1)}_{m_1}, \ldots, t^{(d)}_1, \ldots, t^{(d)}_{m_d}) := \sum_{k=1}^{n} q_{yk} \exp \left\{ i \sum_{j=1}^{d} \sum_{l=1}^{m_j} c_{k,l}^{(j)} \beta_l^{(j)} t_l^{(j)} \right\},
\]

for any \( t^{(j)}_l \in \mathbb{R} \), \( l = 1, \ldots, m_j \), and \( j = 1, \ldots, d \). If for any such \( j \) and \( l \) we set \( t_l^{(j)} := t^{(j)}_l \in \mathbb{R} \), then

\[
\varphi(t^{(1)}_1, \ldots, t^{(1)}_{m_1}, \ldots, t^{(d)}_1, \ldots, t^{(d)}_{m_d}) = h(t),
\]

where \( t = (t^{(1)}, \ldots, t^{(d)}) \). We set \( M := m_1 + \ldots + m_d \). Let us fix an arbitrary \( \varepsilon > 0 \). Since the function \( \varphi \) is uniformly continuous, there exists \( \delta_\varepsilon > 0 \) such that for any \( t_1 \) and \( t_2 \) from \( \mathbb{R}^M \) satisfying \( \| t_1 - t_2 \| < \delta_\varepsilon \) we have \( |\varphi(t_1) - \varphi(t_2)| < \varepsilon \). Let us arbitrarily fix the vector \( t := (t^{(1)}_1, \ldots, t^{(1)}_{m_1}, \ldots, t^{(d)}_1, \ldots, t^{(d)}_{m_d}) \in \mathbb{R}^M \). We set \( b_j := \min \{ |\beta_1^{(j)}|, \ldots, |\beta_{m_j}^{(j)}| \} > 0 \) for every \( j = 1, \ldots, d \). Since for every \( j \) the values \( \beta_1^{(j)}, \ldots, \beta_{m_j}^{(j)} \) are linearly independent over \( \mathbb{Z} \), then, by the Kronecker theorem (see [13, p.37]), we conclude that the inequalities

\[
|\beta_l^{(j)} t_l^{(j)} - 2 \pi n_l^{(j)}| < \frac{\delta_\varepsilon b_j}{\sqrt{m_j}} , \quad l = 1, \ldots, m_j,
\]
have a common solution \( s^{(j)} \in \mathbb{R} \) for some \( n_l^{(j)} \in \mathbb{Z} \). We fix these numbers and we conclude that
\[
\left| s^{(j)} - \frac{t_l^{(j)} + 2\pi n_l^{(j)}}{\beta_l^{(j)}} \right| < \frac{\delta_\varepsilon}{\sqrt{m_j d}} \quad l = 1, \ldots, m,
\]
and
\[
\sum_{j=1}^{d} \sum_{l=1}^{m_j} \left| s^{(j)} - \frac{t_l^{(j)} + 2\pi n_l^{(j)}}{\beta_l^{(j)}} \right|^2 < \delta_\varepsilon^2.
\]
The latter inequality means that \( \| s - \tilde{t} \| < \delta_\varepsilon \), where
\[
s := (s^{(1)}, \ldots, s^{(1)}, \ldots, s^{(d)}, \ldots, s^{(d)}) \in \mathbb{R}^M,
\]
\[
\tilde{t} := \left( \frac{t_1^{(1)} + 2\pi n_1^{(1)}}{\beta_1^{(1)}}, \ldots, \frac{t_{m_1}^{(1)} + 2\pi n_{m_1}^{(1)}}{\beta_{m_1}^{(1)}}, \ldots, \frac{t_1^{(d)} + 2\pi n_1^{(d)}}{\beta_1^{(d)}}, \ldots, \frac{t_{m_d}^{(d)} + 2\pi n_{m_d}^{(d)}}{\beta_{m_d}^{(d)}} \right) \in \mathbb{R}^M,
\]
in the vector \( s^{(1)} \) repeats \( m_1 \) times, \( s^{(2)} \) repeats \( m_2 \) times, \ldots, \( s^{(d)} \) repeats \( m_d \) times. Therefore \( |\varphi(s) - \varphi(t)| < \varepsilon \). It is easily seen from (8) that
\[
\varphi(\tilde{t}) = \varphi\left( \frac{t_1^{(1)} + 2\pi n_1^{(1)}}{\beta_1^{(1)}}, \ldots, \frac{t_{m_1}^{(1)} + 2\pi n_{m_1}^{(1)}}{\beta_{m_1}^{(1)}}, \ldots, \frac{t_1^{(d)} + 2\pi n_1^{(d)}}{\beta_1^{(d)}}, \ldots, \frac{t_{m_d}^{(d)} + 2\pi n_{m_d}^{(d)}}{\beta_{m_d}^{(d)}} \right)
\]
i.e.
\[
\tilde{\varphi}(t) = \sum_{k=1}^{n} q_{yk} \exp \left\{ i \sum_{j=1}^{d} \sum_{l=1}^{m_j} c_{k,l} t_l^{(j)} \right\};
\]
since \( t \) was fixed arbitrarily, we consider \( \tilde{\varphi} \) as a function on \( \mathbb{R}^M \). So we have that \( |\varphi(s) - \tilde{\varphi}(t)| < \varepsilon \). Thus, due to (9), we get that for any \( \varepsilon > 0 \) and \( t \in \mathbb{R}^M \) there exists \( s' = (s^{(1)}, \ldots, s^{(d)}) \in \mathbb{R}^d \) such that \( |h(s') - \tilde{\varphi}(t)| < \varepsilon \). According to the assumption \( \inf_{s \in \mathbb{R}^d} |h(s)| > 0 \), we conclude that \( \inf_{t \in \mathbb{R}^M} |\tilde{\varphi}(t)| > 0 \).

We now apply the previous part 1 to the function (10) (it is valid, because there are \( c_{k,l}^{(j)} \in \mathbb{Z} \) in (10)). So we have the following representation:
\[
\ln \tilde{\varphi}(t) = \ln \tilde{\varphi}\left( t_1^{(1)}, \ldots, t_{m_1}^{(1)}, \ldots, t_1^{(d)}, \ldots, t_{m_d}^{(d)} \right) = i \sum_{j=1}^{d} \sum_{l=1}^{m_j} \gamma_l^{(j)} t_l^{(j)} + \sum_{z \in \mathbb{Z}^M \setminus \{0\}} \lambda_z \left( \exp \left\{ i \sum_{j=1}^{d} \sum_{l=1}^{m_j} z_l^{(j)} t_l^{(j)} \right\} - 1 \right),
\]
where \( z = (z_1^{(1)}, \ldots, z_{m_1}^{(1)}, \ldots, z_1^{(d)}, \ldots, z_{m_d}^{(d)}) \in \mathbb{Z}^M \setminus \{0\}, \gamma_l^{(j)} \in \mathbb{Z}, \lambda_z \in \mathbb{C} \) for all \( z \in \mathbb{Z}^M \setminus \{0\} \), and \( \sum_{z \in \mathbb{Z}^M \setminus \{0\}} |\lambda_z| < \infty \). From the above, we get
\[
\ln \varphi( t_1^{(1)}, \ldots, t_{m_1}^{(1)}, \ldots, t_1^{(d)}, \ldots, t_{m_d}^{(d)} ) = i \sum_{j=1}^{d} \sum_{l=1}^{m_j} \gamma_l^{(j)} \beta_l^{(j)} t_l^{(j)}
\]

\[
+ \sum_{z \in \mathbb{Z}^M \setminus \{0\}} \lambda_z \left( \exp \left\{ i \sum_{j=1}^{d} \sum_{l=1}^{m_j} z_l^{(j)} \beta_l^{(j)} t_l^{(j)} \right\} - 1 \right).
\]
Due to (9), for every \( t = (t^{(1)}, \ldots, t^{(d)}) \) we have
\[
\ln h(t) = i \sum_{j=1}^{d} \left( \sum_{l=1}^{m_j} \gamma_l^{(j)} \beta_l^{(j)} \right) t^{(j)} + \sum_{z \in \mathbb{Z}^m \setminus \{0\}} \lambda_z \left( \exp \left\{ i \sum_{j=1}^{d} \left( \sum_{l=1}^{m_j} z_l^{(j)} \beta_l^{(j)} \right) t^{(j)} \right\} - 1 \right).
\]

For every \( j = 1, \ldots, d \) we set \( \gamma^{(j)} := \sum_{l=1}^{m_j} \gamma_l^{(j)} \beta_l^{(j)} \in \langle Y^{(j)} \rangle \), \( \gamma := \langle \gamma^{(1)}, \ldots, \gamma^{(d)} \rangle \in \langle Y \rangle \), \( \lambda_u := \lambda_z \) for \( u = (u^{(1)}, \ldots, u^{(d)}) \in \langle Y \rangle \setminus \{0\} \) with \( u^{(j)} := \sum_{l=1}^{m_j} z_l^{(j)} \beta_l^{(j)} \in \langle Y^{(j)} \rangle \setminus \{0\} \) (\( u \) determines \( z \) uniquely, because \( \beta_l^{(j)} \) consist a basis, see above). Thus we come to the representation (6) for \( h \).

**3** We now turn to the general case: \( Y \) is at most countable subset of \( \mathbb{R}^d \). Without loss of generality we can set \( Y := \{y_1, y_2, \ldots\} \) with distinct \( y_k \in \mathbb{R}^d \). So \( A := \sum_{k=1}^{\infty} |q_{y_k}| < \infty \) and \( h(t) := \sum_{k=1}^{\infty} q_{y_k} e^{i(t,y_k)} \), \( t \in \mathbb{R}^d \). We will approximate \( h \) by the following functions:
\[
h_n(t) := \sum_{k=1}^{n} q_{n,y_k} e^{i(t,y_k)}, \quad t \in \mathbb{R}^d, \quad n \in \mathbb{N},
\]
where
\[
q_{n,y_k} := \frac{q_{y_k}}{\sum_{m=1}^{n} q_{y_m}}, \quad k = 1, \ldots, n, \quad n \in \mathbb{N}.
\]
Since \( \sum_{k=1}^{\infty} q_{y_k} = 1 \), here \( |\sum_{m=1}^{n} q_{y_m}| \geq \frac{1}{2} \) for all \( n \geq n_0 \) with a positive integer \( n_0 \). Let us estimate the approximation error for every \( n \geq n_0 \):
\[
\sup_{t \in \mathbb{R}^d} |h(t) - h_n(t)| = \sup_{t \in \mathbb{R}^d} \left| \sum_{k=1}^{n} (q_{y_k} - q_{n,y_k}) e^{i(t,y_k)} + \sum_{k=n+1}^{\infty} q_{y_k} e^{i(t,y_k)} \right| \\
\leq \sum_{k=1}^{n} |q_{y_k} - q_{n,y_k}| + \sum_{k=n+1}^{\infty} |q_{y_k}|.
\]

Due to \( \sum_{m=1}^{\infty} q_{y_m} = 1 \), we have
\[
\sum_{k=1}^{n} |q_{y_k} - q_{n,y_k}| = \left| 1 - \frac{1}{\sum_{m=1}^{n} q_{y_m}} \right| \cdot \sum_{k=1}^{n} |q_{y_k}| = \left| \frac{\sum_{m=n+1}^{\infty} q_{y_m}}{\sum_{m=1}^{n} q_{y_m}} \right| \cdot \sum_{k=1}^{n} |q_{y_k}| \leq 2A \sum_{m=n+1}^{\infty} |q_{y_m}|.
\]

We used \( \sum_{k=1}^{\infty} |q_{y_k}| = A \) and \( |\sum_{m=1}^{n} q_{y_m}| \geq \frac{1}{2} \) in the last inequality. Thus we obtain
\[
\sup_{t \in \mathbb{R}^d} |h(t) - h_n(t)| \leq (2A + 1) \sum_{m=n+1}^{\infty} |q_{y_m}|, \quad n \geq n_0.
\]

Since \( \sum_{k=1}^{n} |q_{y_k}| < \infty \), we have that \( \sup_{t \in \mathbb{R}^d} |h(t) - h_n(t)| \to 0, \ n \to \infty \). Hence for any fixed \( \varepsilon \in (0, \frac{1}{2}) \) there exists a positive integer \( n_\varepsilon \geq n_0 \) such that for every \( n \geq n_\varepsilon \) we have
\[
\sup_{t \in \mathbb{R}^d} |h(t) - h_n(t)| \leq \varepsilon \mu, \quad (11)
\]
where we set \( \mu := \inf_{t \in \mathbb{R}^d} |h(t)| > 0 \). So for every \( n \geq n_\varepsilon \)
\[
\inf_{t \in \mathbb{R}^d} |h_n(t)| \geq \inf_{t \in \mathbb{R}^d} |h(t)| - \sup_{t \in \mathbb{R}^d} |h(t) - h_n(t)| \geq (1 - \varepsilon) \mu. \quad (12)
\]
We now fix \( n \geq n_\varepsilon \) and we represent \( h(t) = h_n(t) \cdot R_n(t) \) with \( R_n(t) := h(t)/h_n(t), \ t \in \mathbb{R}^d \). Since \( h, h_n, R_n \) are continuous functions without zeroes on \( \mathbb{R}^d \) and they equal 1 at \( t = 0 \), we can proceed to the distinguished logarithms:

\[
\ln h(t) = \ln h_n(t) + \ln R_n(t), \quad t \in \mathbb{R}^d.
\]

Let us consider the function \( \ln h_n \). By the result of part 2), we have

\[
\ln h_n(t) = i\langle t, \gamma_n \rangle + \sum_{u \in \langle Y_n \rangle \setminus \{0\}} \lambda_{n,u}(e^{i\langle t,u \rangle} - 1), \quad t \in \mathbb{R}^d,
\]

with a set \( Y_n := \{ y_k : q_{yk} \neq 0, k = 1, \ldots, n \} \), and numbers \( \gamma_n \in \langle Y_n \rangle, \lambda_{n,u} \in \mathbb{C} \) for all \( u \in \langle Y_n \rangle \setminus \{0\} \), \( \sum_{u \in \langle Y_n \rangle \setminus \{0\}} |\lambda_{n,u}| < \infty \). Setting \( \lambda_{n,0} := -\sum_{u \in \langle Y_n \rangle \setminus \{0\}} \lambda_{n,u} \in \mathbb{C} \), we represent \( \ln h_n \) in the following form

\[
\ln h_n(t) = i\langle t, \gamma_n \rangle + \sum_{u \in \langle Y_n \rangle} \lambda_{n,u}e^{i\langle t,u \rangle}, \quad t \in \mathbb{R}^d.
\]

Observe that \( Y_n \subset Y \), and hence \( \langle Y_n \rangle \subset \langle Y \rangle \). So we can write

\[
\ln h_n(t) = i\langle t, \gamma_n \rangle + \sum_{u \in \langle Y \rangle} \lambda_{n,u}e^{i\langle t,u \rangle}, \quad t \in \mathbb{R}^d,
\]

where for every \( u \in \langle Y \rangle \setminus \langle Y_n \rangle \) we define \( \lambda_{n,u} := 0 \) for the case \( \langle Y \rangle \setminus \langle Y_n \rangle \neq \emptyset \).

We next consider the function \( \ln R_n \). Observe that

\[
\ln R_n(t) = \ln \left( 1 + \frac{h(t) - h_n(t)}{h_n(t)} \right), \quad t \in \mathbb{R}^d,
\]

where the latter is the principal value of the logarithm. Indeed, due to (11) and (12),

\[
\sup_{t \in \mathbb{R}^d} \left| \frac{h(t) - h_n(t)}{h_n(t)} \right| \leq \sup_{t \in \mathbb{R}^d} \left| \frac{h(t) - h_n(t)}{h_n(t)} \right| \leq \frac{\varepsilon}{1 - \varepsilon} < 1,
\]

and the function in the right-hand side of (15) is continuous and it equals 0 at \( t = 0 \). Therefore we get the decomposition

\[
\ln R_n(t) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left( \frac{h(t) - h_n(t)}{h_n(t)} \right)^m, \quad t \in \mathbb{R}^d,
\]

which yields the estimate

\[
\sup_{t \in \mathbb{R}} |\ln R_n(t)| \leq \sum_{m=1}^{\infty} \frac{1}{m} \sup_{t \in \mathbb{R}^d} \left| \frac{h(t) - f_n(t)}{f_n(t)} \right|^m \leq \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\varepsilon}{1 - \varepsilon} \right)^m.
\]

Since \( \varepsilon \in (0, \frac{1}{2}) \), we have

\[
\sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\varepsilon}{1 - \varepsilon} \right)^m \leq \sum_{m=1}^{\infty} \left( \frac{\varepsilon}{1 - \varepsilon} \right)^m = \frac{\frac{\varepsilon}{1 - \varepsilon}}{1 - \frac{\varepsilon}{1 - \varepsilon}} = \frac{\varepsilon}{1 - 2\varepsilon} < 2\varepsilon.
\]

Thus we obtain

\[
\sup_{t \in \mathbb{R}^d} |\ln R_n(t)| < 2\varepsilon.
\]
Let us consider the function \((h - h_n)/h_n\) from (15). It is clear that \(h - h_n\) is an almost periodic function with absolutely convergent Fourier series. Due to [2, Theorem 3.2] the function \(1/h_n\) is also an almost periodic with absolutely convergent Fourier series. Since the function \(z \mapsto \ln(1 + z), z \in \mathbb{C}\), is analytic on the unit disk, due to (16), [2, Theorem 3.2], and [8, Corollary 5.15], we get that \(\text{Ln} R_n\) is an almost periodic function with absolutely convergent Fourier series:

\[
\text{Ln} R_n(t) = \sum_{u \in \Delta_n} \beta_{n,u} e^{i(t,u)}, \quad t \in \mathbb{R}^d,
\]

where \(\Delta_n\) is at most countable set of vectors from \(\mathbb{R}^d\), \(\beta_{n,u} \in \mathbb{C}\) for \(u \in \Delta_n\), and \(\sum_{u \in \Delta_n} |\beta_{n,u}| < \infty\).

We now return to the function \(\text{Ln} h\) and (13). The formulas (14) and (18) yield

\[
\text{Ln} h(t) = i(t, \gamma) + \sum_{u \in \langle Y \rangle} \lambda_{n,u} e^{i(t,u)} + \sum_{u \in \Delta_n} \beta_{n,u} e^{i(t,u)}, \quad t \in \mathbb{R}^d.
\]

This formula is valid for every \(n \geq n_\varepsilon\). Let us fix \(e \in S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}\), that is the unit sphere in \(\mathbb{R}^d\). Since \(\sum_{u \in \langle Y \rangle} |\lambda_{n,u}| < \infty\) and \(\sum_{u \in \Delta_n} |\beta_{n,u}| < \infty\), \(n \geq n_\varepsilon\), it is easy to see that

\[
\lim_{T \to \infty} \frac{\text{Ln} h(Te)}{iT} = (\gamma, e), \quad n \geq n_\varepsilon.
\]

Since the vector \(e\) is chosen arbitrarily from \(S^{d-1}\), \(\gamma_n\) are equal for \(n \geq n_\varepsilon\), and we set \(\gamma := \gamma_n\). Due to \(\gamma_n \in \langle Y_n \rangle \subset \langle Y \rangle\), we have \(\gamma \in \langle Y \rangle\). Thus for every \(n \geq n_\varepsilon\) we obtain

\[
\text{Ln} h(t) = i(t, \gamma) + \sum_{u \in \langle Y \rangle} \lambda_{u} e^{i(t,u)} + \sum_{u \in \Delta_n} \beta_{n,u} e^{i(t,u)}, \quad t \in \mathbb{R}^d.
\]

Due to the uniqueness theorem for Fourier coefficients (see [2, Lemma 3.1]), one can conclude that

\[
\text{Ln} h(t) = i(t, \gamma) + \sum_{u \in \langle Y \rangle} \lambda_{u} e^{i(t,u)} + \sum_{u \in Z} \lambda_{u} e^{i(t,u)}, \quad t \in \mathbb{R}^d,
\]

where \(Z\) is at most countable subset of \(\mathbb{R}^d\) such that \(\langle Y \rangle \cap Z = \emptyset\), \(\lambda_u \in \mathbb{C}\) for all \(u \in \langle Y \rangle \cup Z\), \(\sum_{u \in \langle Y \rangle \cup Z} |\lambda_u| < \infty\). So for every \(n \geq n_\varepsilon\) the following estimate is true:

\[
\sum_{u \in Z} |\lambda_u|^2 \leq \sum_{u \in \Delta_n} |\beta_{n,u}|^2.
\]

Using the Parseval identity (see [12, Ch. VI, §4] or [17, Theorem 28]) and (17), we get

\[
\sum_{u \in Z} |\lambda_u|^2 \leq \lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} |\text{Ln} R_n(t)|^2 dt < (2\varepsilon)^2, \quad n \geq n_\varepsilon.
\]

Since \(\varepsilon > 0\) can be chosen arbitrarily small, we conclude that \(Z = \emptyset\) or \(Z \neq \emptyset\), but \(\lambda_u = 0\) for all \(u \in Z\). Thus

\[
\text{Ln} h(t) = i(t, \gamma) + \sum_{u \in \langle Y \rangle} \lambda_u e^{i(t,u)}, \quad t \in \mathbb{R}^d,
\]

with \(\gamma \in \langle Y \rangle\), \(\lambda_u \in \mathbb{C}\) for all \(u \in \langle Y \rangle\), and \(\sum_{u \in \langle Y \rangle} |\lambda_u| < \infty\). According to \(\left(\text{Ln} h(t) - i\langle t, \gamma \rangle\right)|_{t=0} = 0\), we get \(\lambda_0 = -\sum_{u \in \langle Y \rangle \setminus \{0\}} \lambda_u\) and we come to the required representation (6). □
We now return to the proof of the Theorem 4.

**Proof of Theorem 4.** The proof will be carried out in the following sequence: (ii) $\xrightarrow{I} (i) \xrightarrow{II} (iii) \xrightarrow{III} (iv) \xrightarrow{IV} (i) \xrightarrow{V} (ii)$.

I. Due to (ii), we have

$$\sup_{t \in \mathbb{R}^d} \left| \frac{1}{h(t)} \right| \leq \sum_{z \in \mathbb{Z}} |\tau_z| = \frac{1}{\mu} < \infty.$$ 

It follows that

$$\inf_{t \in \mathbb{R}^d} |h(t)| = \frac{1}{\sup_{t \in \mathbb{R}^d} |1/h(t)|} = \mu > 0.$$ 

II. This implication directly follows from Theorem 3.

III. It is clear that (iii) yields (iv) with zero matrix $Q$ and the signed measure

$$\nu(B) = \sum_{u \in \mathbb{R} \setminus \{0\}} \lambda_u$$

for every Borel set $B$.

IV. Let us assume the contrary, i.e., $h$ has the representation (7), however $\inf_{t \in \mathbb{R}^d} |h(t)| = 0$. Since $e^z \neq 0$ for all $z \in \mathbb{C}$, then $h(t) \neq 0$ for all $t \in \mathbb{R}^d$. Hence it is sufficient to focus on the case when $h$ has the representation (7), $h(t) \neq 0$ for all $t \in \mathbb{R}^d$, and $\inf_{t \in \mathbb{R}^d} |h(t)| = 0$.

Due to (7), for every fixed $\tau \in \mathbb{R}^d$ we have the following representation

$$\frac{h(t + \tau)h(t - \tau)}{h^2(t)} = \exp\left\{ -\frac{1}{2}\langle \tau, Q\tau \rangle + 2 \int_{\mathbb{R}^d \setminus \{0\}} e^{i(t, u)}(\cos(\langle (\tau, u) \rangle) - 1)\nu(du) \right\}, \quad t \in \mathbb{R}^d.$$ 

It follows that

$$\left| \frac{h(t + \tau)h(t - \tau)}{h^2(t)} \right| \leq \exp\left\{ \frac{1}{2}||Q|| + \int_{0 < ||u|| < 1} ||u||^2 ||\nu||(du) ||\tau||^2 + \int_{||u|| > 1} ||\nu||(du) \right\}, \quad t \in \mathbb{R}^d.$$ 

Hence, for every $\tau \in \mathbb{R}^d$ there exists $C_\tau$ such that

$$\sup_{t \in \mathbb{R}^d} \left| \frac{h(t + \tau)h(t - \tau)}{h^2(t)} \right| \leq C_\tau.$$ 

Let $(t_n)_{n \in \mathbb{N}}$, $t_n \in \mathbb{R}^d$, be a sequence such that $h(t_n)$ tends to 0 as $n \to \infty$. If there exists $R > 0$ such that $||t_n|| < R$ for every $n \in \mathbb{N}$, then there exists subsequence $(n_k)_{k \in \mathbb{N}}$ satisfying $t_{n_k} \to t_* \in \mathbb{R}^d$ as $k \to \infty$. Since $h$ is a continuous function, $h(t_*) = 0$ that contradicts with the (iv). It follows that $||t_n|| \to \infty$ as $n \to \infty$.

Since $h$ is an almost periodic function, the sequence $(h(\cdot + t_n))_{n \in \mathbb{N}}$ is dense in the set of continuous functions, i.e., there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ and a continuous function $\varphi$ such that

$$\sup_{\tau \in \mathbb{R}^d} \left| h(t_{n_k} + \tau) - \varphi(\tau) \right| \to 0, \quad k \to \infty.$$ 

It is obvious that $|\varphi(\tau)| \leq C := \sup_{t \in \mathbb{R}^d} |h(t)| < \infty$ for all $\tau \in \mathbb{R}^d$. Then

$$\Delta_k := \sup_{\tau \in \mathbb{R}^d} \left| h(t_{n_k} + \tau)h(t_{n_k} - \tau) - \varphi(\tau)\varphi(-\tau) \right|$$

$$\leq \sup_{\tau \in \mathbb{R}^d} \left| h(t_{n_k} - \tau) \right| \left| h(t_{n_k} + \tau) - \varphi(\tau) \right| + \sup_{\tau \in \mathbb{R}^d} \left| h(t_{n_k} - \tau) - \varphi(\tau) \right| \cdot |\varphi(\tau)|$$

$$\leq 2C \sup_{\tau \in \mathbb{R}^d} \left| h(t_{n_k} + \tau) - \varphi(\tau) \right| \to 0, \quad k \to \infty.$$
Let us assume that $\varphi(\tau)\varphi(-\tau) = 0$ for all $\tau \in \mathbb{R}^d$. It follows that
\[
\sup_{\tau \in \mathbb{R}^d} |h(t_{nk} + \tau)h(t_{nk} - \tau)| \longrightarrow 0.
\]
So for any fixed $s \in \mathbb{R}^d$,
\[
h(t_{nk} + \tau)h(t_{nk} - \tau) \bigg|_{\tau = -t_{nk} - s} = h(-s)h(2t_{nk} + s) \longrightarrow 0.
\]
Since $h(s) \neq 0$ for every $s \in \mathbb{R}^d$, we have
\[
h(2t_{nk} + s) \longrightarrow 0. \quad (19)
\]
Next, it is easy to see that the function $h(2t_{nk} + \cdot)$ is an almost periodic one. It means that there exists a subsequence $(n_{km})_{m \in \mathbb{N}}$ such that a sequence $(h(2t_{nk} + \cdot))_{m \in \mathbb{N}}$ has a uniform limit. From (19) one can conclude that
\[
\sup_{s \in \mathbb{R}^d} |h(2t_{nk} + s)| \longrightarrow 0.
\]
Applying this with $s = -2t_{nk}$, we come to a contradiction with $h(0) = 1$. Thus the assumption $\inf_{t \in \mathbb{R}^d} |h(t)| = 0$ is false, i.e. $(i)$ follows from $(iv)$.

V. If $(i)$ holds, then $(ii)$ follows directly from [2, Theorem 3.2]. \square

**Proof of Theorem 2.** The implication $(a) \rightarrow (b)$ directly follows from the implication $(iv) \rightarrow (i)$ of Theorem 4. The converse $(b) \rightarrow (a)$ holds due to $(i) \rightarrow (iv)$ of Theorem 4 with applying [5, Theorem 2.7] (so $\gamma \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d \times d}$, $\nu$ is real-valued measure). The representation (5) holds due to $(iii)$ of Theorem 4 and [5, Theorem 2.7] (so $\gamma \in \mathbb{R}^d$ and $\lambda_u \in \mathbb{R}$). \square

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