A study of the transient dynamics of perturbations in Keplerian discs using a variational approach

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ABSTRACT
We study linear transient dynamics in a thin Keplerian disc, employing a method based on a variational formulation of the optimization problem. It is shown that in a shearing sheet approximation due to a prominent excitation of density waves by vortices, the most rapidly growing shearing harmonic has an azimuthal wavelength $\lambda_\gamma$, of the order of the disc thickness $H$, and its initial shape is always nearly identical to a vortex having the same potential vorticity. Also, in the limit $\lambda_\gamma \gg H$, the optimal growth $G \propto (\Omega/\kappa)^4$, where $\Omega$ and $\kappa$ denote local rotational and epicyclic frequencies, respectively. This suggests that the transient growth of large-scale vortices can be much stronger in areas with non-Keplerian rotation (e.g. in the inner parts of relativistic discs around black holes). We estimate that if the disc is already in a turbulent state with effective viscosity given by the Shakura parameter $\alpha < 1$, the considered large-scale vortices with wavelengths $H/\alpha > \lambda_\gamma > H$ have the most favourable conditions to be transiently amplified before they are damped. At the same time, turbulence is a natural source of the potential vorticity for this transient activity. We extend our study to a global spatial scale, showing that global perturbations with azimuthal wavelengths more than an order of magnitude greater than the disc thickness are still able to attain the growth of dozens of times in a few Keplerian periods at the inner boundary of the disc.

Key words: accretion, accretion discs – hydrodynamics – instabilities – turbulence.

1 INTRODUCTION
A conventional way to study non-stationary phenomena in accretion discs is spectral (i.e. modal) analysis, when disc eigenfrequencies are determined by looking for the set of solutions that vary exponentially with time (see Kato 2001 for a basic review). The next problem is whether the modes are really excited because of spectral instabilities, turbulent motions or external forcing. However, as we show in this paper, a prominent transient growth of amplitudes of global vortices with azimuthal wavelengths larger than the disc thickness can be obtained employing the non-modal approach to the dynamics of small perturbations in geometrically thin discs.

Contrary to the modal framework, the non-modal approach sorts out perturbations according to the amount of energy they gain from the background during a specified time interval (see Schmid & Henningsson 2001; Schmid 2007). Mathematically, the set of eigenvectors of the underlying dynamical operator is replaced by the set of its singular vectors. The key point is that the latter is time-dependent and strongly differs from the former in the case of shear (i.e. non-normal) flow. The projection of an arbitrary initial perturbation on to eigenvectors gives information about its long-term behaviour, while the projection on to singular vectors additionally represents its potential to transient dynamics (i.e. disc response shortly after some abrupt event triggered perturbation). Clearly, this response can be quite different from what is expected from spectral analysis. The component corresponding to the highest singular value is usually called an optimal perturbation. In previous works, the study of transient dynamics has concentrated mainly on local analysis made in the shearing sheet approximation (e.g. Chagelishvili et al. 2003). A few investigations have been devoted to global transient dynamics, including a real accretion disc geometry in order to reveal an additional mechanism for enhanced angular momentum transfer (Umurhan et al. 2006; Rebusco et al. 2009; Shtemler et al. 2010). Global perturbations have been also examined for transient dynamics by Ioannou & Kakouris (2001), which remains a unique study of the global optimal growth in Keplerian discs made so far. However, Ioannou & Kakouris (2001) considered the problem of angular momentum transport, restricting their analysis to incompressible perturbations. This is quite a strong restriction if one addresses the issue of the non-stationary appearance of geometrically thin disc as a result of transient effects. Thus, here we would like to tackle the optimal configurations of compressible perturbations and the corresponding optimal growth.
factors. We would also like to illustrate the potential of a relatively new technique for the optimization, which is based on the variational formalism and has been successfully applied to a number of complex hydrodynamical flows (Luchini & Bottaro 1998; Corbett & Bottaro 2001; Guegan, Schmid & Huerre 2006). The advantage of the variational approach is that, in contrast to the usual optimization method, it does not rely on the representation in the basis of modes or on any other discretization procedure. This implies that it can be easily generalized to non-stationary background flows and even to a non-linear problem.

This paper is organized as follows. First, we describe a general formalism of the optimization method. This shows that optimal perturbation can be determined by using an iterative loop in which the set of basic dynamical equations is integrated forward in time and the set of corresponding adjoint equations is integrated backward in time. We discuss the fact that this method is applicable to perturbations in stationary as well as non-stationary flows. Next, we consider the dynamics of the perturbed rotating flow, specifying the sets of equations for linear perturbations on a global spatial scale as well as in the shearing sheet approximation. We also discuss a choice of norm to measure the transient growth in the case of compressible fluid. We suggest the use of the norm that equals the canonical energy of perturbations in the axisymmetric case. For all options, we derive the adjoint equations. Then, we present the optimal growth calculations in a geometrically thin Keplerian disc, comparing the results in different approaches. Additionally, in Appendix A, we analytically study in detail a problem of the non-modal growth of optimal axisymmetric perturbations measured by their acoustic energy. Finally, in Appendix B, we describe the optimization procedure when applied to a model case of incompressible perturbations in the global and local contexts.

2 VARIATIONAL FORMULATION OF OPTIMIZATION PROBLEM

2.1 Method of Lagrange multipliers

In order to solve an optimization problem, it is necessary to find the initial conditions that excite the most powerful perturbation at a given time interval $t$. While dealing with a linear problem for stationary background flow, a common strategy is to consider a linear subspace of solutions of dynamical equations, which is represented by linear span of a finite number of modes. By modes, we mean spectral solutions with exponential dependence on time $e^{\text{int}}$. Then, it is necessary to determine the coefficients in the combination of modes that maximize the growth of perturbation. This can be done, for example, by means of the singular value decomposition of the relevant matrix exponential projected on to the orthonormal basis (see Schmid & Henningson 2001). Butler & Farrell (1992) use this method to study the optimal transient growth in classical Couette, Poiseuille and Blasius boundary layer flows. Mukhopadhyay, Afshordi & Narayan (2005) use it to investigate Keplerian flow in the local approximation and Zhuravlev & Shkaura (2009) and, subsequently, Razdoburdin & Zhuravlev (2012) apply it to the study of the global optimal growth in a quasi-Keplerian torus with free boundaries. However, this strategy entails all the technical difficulties related to the calculation of modes and eigenfrequencies, among which are complications that are encountered when solving the boundary problem in the vicinity of singular points, such as corotational or Lindblad resonances. Furthermore, because the modes are non-orthogonal, a high dimension of their linear span might be required to obtain reliable results. Also, the contribution of the continuous spectrum to transient dynamics remains unclear, which is particularly important when the unbounded flows are considered. Finally, this method cannot be generalized to investigate the behaviour of perturbations in non-stationary flows as well as the behaviour of perturbations with finite amplitude.

Instead, the optimization problem can be alternatively formulated in terms of a variational principle. For a brief introduction to this subject, see Schmid (2007); a more detailed exposition can be found in the book by Gunzburger (2003). Here, we would like to give a generic view on the variational framework, which will be applied to concrete astrophysical flow later.

So, all that we need to do is to maximize the functional

$$ G(t) = \frac{\|q(t)\|^2}{\|q(0)\|^2}, \qquad (1) $$

which is defined in the space of perturbation state vectors, $q$. Each element $q = \{q^i(t)\}$ consists of the set of perturbation quantities evolving with time (e.g. perturbations of pressure and velocity components as functions of time and spatial coordinates). Here, $G$ is usually called an objective, or cost, functional of the problem. By equation (1), we imply that the inner product is defined in this functional space,

$$ (q_1, q_2) = \int M_{ij} \text{Re}[q_i^*\tilde{q}_j] \, dV, \qquad (2) $$

which is real for an arbitrary pair of vectors. Here, an overbar indicates complex conjugation and $M_{ij}$ is a certain real, symmetric and positive definite matrix, so that the norm $\|q\| = \sqrt{\langle q, q \rangle}$ characterizes the amplitude of perturbations. This can be either the kinetic energy, in the case of incompressible dynamics, or the acoustic energy if we include a finite sound speed, or it can be any other positive definite physically motivated quantity.

The key ingredient of the method is that the maximum of $G$ that we are looking for is the conditional one, because $q$ is constrained by the requirement that it obeys the basic dynamical equations,

$$ \frac{\partial q}{\partial t} = Aq, \qquad (3) $$

The system (3) contains a dynamical (different) operator $A$ that controls the evolution of perturbation quantities. This suggests that actually we have to implement a constrained optimization, which can be done using the method of Lagrangian multipliers. This method is the generalized version of finding the conditional extrema of functions when the Lagrangian multiplier emerges as the proportionality factor between the gradients of an objective function and a constraint function. In the calculus of variations, functionals replace functions whereas the Lagrangian multipliers become functions themselves. In other words, we formally change to the extended space of vectors without the restriction (3) and introduce an additional adjoint vector, $\tilde{q}$, which serves as the Lagrangian multiplier. After this, we can define what is usually called an augmented Lagrangian, involving both $q$ and $\tilde{q}$ in the following way,

$$ L(q, \tilde{q}) = G(q) - \int_0^t (\tilde{q}, \dot{q} - Aq) \, dt, \qquad (4) $$

where the partial time derivative is denoted by a dot. The second term in equation (4) is called the penalty term; that is, it penalizes the objective Lagrangian, $G$, each time when $q(t)$ does not conform equation (3).

All that remains is to find an unconditional extremum of $L$, which is given by the zero variations of $L$ with respect to an arbitrary
variation of both $q$ and $\dot{q}$. For example, in the case of the adjoint vector, it is necessary for
\[
\delta \mathcal{L} = \lim_{\epsilon \to 0} \frac{\mathcal{L}(q, \dot{q} + \epsilon \delta q) - \mathcal{L}(q, \dot{q})}{\epsilon} = 0
\]
to disappear for an arbitrary function $\delta \dot{q}$.

Obviously, the zero variation of $\mathcal{L}$ with respect to $\dot{q}$ recovers the system (3), whereas in order to vary $\mathcal{L}$ over $q$ we have to integrate the second term in equation (4) by parts. This is
\[
\int_0^\tau (\dot{q}, \dot{q} - Aq) \, dt = (\dot{q}, q)_{\tau} = \int_0^\tau (\dot{q} + A^\dagger \dot{q}, q) \, dt,
\]
where we use the ordinary definition of the adjoint operator, $(\dot{q}, Aq) = (A^\dagger \dot{q}, q)$, through the inner product (2). It is now straightforward to see that the variation of $\mathcal{L}$ with respect to an arbitrary deviation $\delta q$ gives the set of so-called adjoint equations,
\[
\frac{\partial \delta q}{\partial t} = -A^\dagger \delta q,
\]
and the following additional relations:
\[
\dot{q}(\tau) = \frac{2}{||q(0)||^2} q(\tau),
\]
\[
q(0) = \frac{2}{||q(\tau)||^2} \dot{q}(0).
\]
Indeed, first, we take arbitrary $\delta q$, which vanish in the neighborhood of $t = 0$ and $t = \tau$. Then, only the second ('volume') term in equation (6) contributes to $\delta \mathcal{L}$, which implies equation (7). Once we find that equation (7) holds inside the interval $(0, \tau)$, we see that the remaining 'edge' terms in $\delta \mathcal{L} = 0$ imply that
\[
2q(\tau) \frac{1}{||q(\tau)||^2} - 2q(0) \frac{||q(\tau)||^2}{||q(0)||^4} - 1_{\tau}^{\tau} = 0.
\]
This gives equations (8) and (9) because $\delta q$ can vanish at $t = 0$ and $t = \tau$ independently.

Additionally, we assume here that both $\dot{q}$ and $q$ satisfy appropriate boundary conditions. Note that for the physical problem considered below, where we derive an explicit form of $A^\dagger$, the boundary conditions for the adjoint variables are obtained using both the zero variation of $\mathcal{L}$ and the boundary conditions for the state variables. This is shown using the integration by parts in the spatial domain (i.e. similar to what was done above with a time dependence).

Thus, equations (3) and (7) are coupled through the conditions (8) and (9) and must be solved together. The unique solution gives both the state and adjoint vectors that correspond to a maximum of the cost functional (1), $G = ||g||_2$, provided that the dynamical equations (3) are satisfied. Physically, this means that we find an optimal initial perturbation attaining the highest possible energy growth at a given time interval, $\tau$. Usually, $G(\tau)$ itself is called the optimal growth. Besides, while considering the evolution of the particular perturbation, no matter whether optimized or not, we characterize it by the growth factor, $g(\tau) = ||q(\tau)||^2/||q(0)||^2$.

### 2.2 Operator solutions and iterative scheme of optimization

At least in the linear case, the natural method of solution of these coupled sets of equations stems from the fact that the optimal state vector is the first singular vector of propagator, $U$, which advances perturbations up to $t = \tau$, that is, $q(\tau) = Uq(0)$; for a short but clear account, see also Luchini (2000).

An explicit form of $U$ can be obtained solving equation (3). First, let us suppose that $A$ is independent of time (an autonomous operator; see Farrell & Ioannou 1996a for reference). Then, $U = e^{\tau A}$, which can be seen from equation (3). From the operator theory, it is known that the first singular value of any operator is the square root of the largest eigenvalue of a positive definite composite operator, which is the original times its adjoint. Thus, in order to solve an optimization problem, we have to determine the largest eigenvalue of $U \cdot U^\dagger = e^{\tau A} \cdot e^{\tau A^\dagger}$, which is equivalent to the advance of the perturbation first forward in time using equation (3) and then backward in time using equation (7), because of the minus appearing in equation (7). The direct way to converge to the largest eigenvalue of $U \cdot U^\dagger$ is to iterate an arbitrary initial perturbation, advancing it recurrently by the operator itself, $(U \cdot U^\dagger)^{p-\infty}q(0)$, where $p$ is a natural number. This procedure is usually called the power iteration (for details, see Golub & Van Loan 1996). It can be shown that the power iteration is equivalent to a steepest descent algorithm for finding an extremum of $\mathcal{L}$ (e.g. Gunzburger & Hyung 1994).

In a more general case when $A$ depends on time (non-autonomous operator; see Farrell & Ioannou 1996b for reference), $U$ can be represented as an ordered product of infinitesimal propagators,
\[
U(t) = \lim_{n \to \infty} \prod_{j=1}^{n} e^{\lambda_j \delta t},
\]
where $\delta t = t/n$ and $(j - 1)\delta t < t_j < j\delta t$. Then, according to the rule of taking the adjoint of composite operator, the adjoint of propagator, $U^\dagger$, is given by
\[
U^\dagger(t) = \lim_{n \to \infty} \prod_{j=1}^{n} e^{\lambda_j \delta t},
\]
where the adjoint infinitesimal propagators are in the reverse order; that is, to advance some initial vector, we take $A^\dagger(t)$', starting from the final point of the time interval and moving back to 0. Again, the action of $e^{\lambda_j \delta t}$ is equivalent to the integration of the system (7) backward in time from $j\delta t$ to $(j - 1)\delta t$. Thus, according to equation (12) the action of $U^\dagger$ is identical to the integration of the system (7) backward in time. We see that the action of $U \cdot U^\dagger$ is equivalent to the forward and backward advance of perturbation solving equations (3) and (7), respectively, just as in the case of $A$ independent of time.

It should be noted that the existence of the largest eigenvalue of $U \cdot U^\dagger$ is guaranteed by the Krein–Rutman theorem of functional analysis (see Krein & Rutman 1950). For details on the related subject of compact operators, the reader is referred to Kolmogorov & Fomin (1961).

As a result, no matter whether the system (3) contains coefficients dependent on time or not, the underlying optimization problem is naturally solved by integrating the basic and the adjoint dynamical equations (3) and (7), forward and backward in time, respectively, with the conditions (8) and (9) linking the state and adjoint vectors at the turning points of the loop. Note that we have not made any assumptions about the background flow. So, the iteration scheme described above can be employed in a wide class of complex flows when the solution of the spectral problem commonly used to evaluate the transient growth can be quite an involved task. Moreover, as shown above, there is no stationarity restriction of the background. Hence, there is no technical obstacle to investigating the transient dynamics that might be triggered in non-stationary acceleration discs or other types of astrophysical shearing flows (i.e. jets and winds). The situation is more complicated if we try to apply an iterative loop to a non-linear problem, but nevertheless a number of
technical improvements have been devised for this case (see section 6 of the review by Schmid, 2007, and references therein).

3 PERTURBATIONS IN Rotating SHEAR FLOW

To apply the general formalism described in Section 2 in an astrophysical context, we would like to consider small perturbations in a disc (i.e. in axisymmetric rotating flow). If we neglect the effects of viscosity and consider only the model case of barotropic equation of state, then the dynamics of small perturbations is described by the set of linear equations

\[ \frac{\partial \delta v}{\partial t} + (v \cdot \nabla) \delta v + (\delta v \cdot \nabla)v = -\nabla \delta h; \]  

(13)

\[ \frac{\partial \delta \rho}{\partial t} + \nabla \cdot (\rho \delta v) + \nabla \cdot (\delta \rho v) = 0. \]  

(14)

Here, \( \delta v \) and \( \delta h \) are the Eulerian perturbations of velocity and enthalpy, respectively, and \( \delta \rho \) is the Eulerian perturbation of density. In our case, \( \delta h = \delta p/\rho \), where \( \delta p \) is the Eulerian perturbation of pressure and \( \rho(r,z) \) is the background density. We use the cylindrical coordinates \((r, \psi, z)\) in which the background flow is described by azimuthal motion \( v = (0, v_\psi, 0) \) with angular velocity \( \Omega = v_\psi/r \), which depends on the radial coordinate only.

In this study, we also assume that perturbations preserve vertical hydrostatic equilibrium. For barotropic flow, the vertical hydrostatic equilibrium results in perturbations with no dependence on \( z \), which makes it possible to integrate the dynamical equations along the vertical direction (e.g. Goldreich, Goodman & Narayan 1986). Let us note that, in general, the assumption of the absence of vertical motions in the perturbed flow can be strictly justified only if \( t_{\text{pert}} \gg \Omega^{-1} \) and \( \lambda_{\text{pert}} \gg H \), where \( t_{\text{pert}} \) and \( \lambda_{\text{pert}} \) are the characteristic time and length of perturbations, respectively, and \( H \) is the thickness of the disc. However, even without this restriction, there are still particular cases when perturbations with no node in the vertical direction can exist in the flow. See, for example, Okazaki, Kato & Fukue (1987) who have shown that vertical and planar perturbed motions can be separated from each other in a thin disc with an isothermal vertical structure. Thus, in this study, we would like not to restrict ourselves with the above rigorous assumption about \( t_{\text{pert}} \) and \( \lambda_{\text{pert}} \). An additional argument in favour of the model case of vertically independent perturbations in the context of non-modal analysis comes from the study by Yecko (2004), who investigated the transient growth of three-dimensional local incompressible perturbations in viscous Keplerian shears. The largest growth factors were found for perturbations uniform along the \( z \)-axis. Thus, in what follows, we consider the planar perturbed velocity field, \( \delta v = (\delta v_\eta, \delta v_\psi, 0) \), and we work with the set of equations (13) and (14) integrated over \( z \). Because the background flow is rotationally symmetrical, we are dealing with the azimuthal Fourier harmonic of perturbations \( e^{im\psi} \) hereafter, implying that perturbation quantities are functions of time and radial coordinates.\(^1\)

In the rest of this section, we discuss the specific equations that have to be solved in order to determine the optimized perturbations. We measure perturbations using two different norms and derive the specific adjoint equations for both of them. Along with the full optimization problem, which is formulated to investigate global perturbations and accurately accounts for the cylindrical geometry as well as for the shear rate distribution across the flow, we study its local spatial limit using the well-known shearing sheet approximation.

Although our objective is to consider transient dynamics in hypersonic flow, a complementary description of the optimization problem in the model case of incompressible fluid can be found in Appendix B. There, we give necessary equations constructed for perturbations of vorticity and stream function. Numerical tests carried out using these equations have allowed us to make an additional check of our primary numerical scheme for compressible dynamics. In the shearing sheet limit, the variational procedure for the divergence-free velocity perturbations becomes especially simple and can be performed fully analytically. Hence, we obtain an exact analytical expression for optimal growth as a function of time in this case.

3.1 Optimization on a global spatial scale

Let us assume that the state vector \( \mathbf{q} \) is constructed from the Eulerian perturbations of the velocity components and the enthalpy, \( \mathbf{q} = \{\delta v_\eta, \delta v_\psi, \delta h\} \). The explicit form of \( A \) in equation (3) is

\[ \begin{pmatrix} -i m \Omega & \frac{2\Omega}{r} & -\Omega \kappa \\ -\kappa^2 & -i \Omega & -\text{m}^2 \frac{1}{r} \\ -a_n^2 \left[ (\Sigma \Omega)^{-1} \Omega \check{\omega} + \frac{1}{r} \right] & -a_n^2 \left[ \frac{m}{r} \Omega \right] & -\text{m}^2 \frac{1}{r} \end{pmatrix}. \]  

(15)

where \( \Sigma = \int \rho \, dz \) is the surface density and \( a_n^2 = \frac{n a_{\text{eq}}}{n + 1/2} \), where \( a_{\text{eq}} \) is the sound speed in the equatorial plane of the flow and \( n \) is the polytropic index. Both \( \Sigma \) and \( a_{\text{eq}} \) have a specified dependence on \( r \) and \( \kappa^2 = (2\Omega/r)\text{d}/\text{d}(r \Omega^2) \) is the epicyclic frequency squared.

3.1.1 Choice of norm and adjoint equations

It is not a matter of course which norm of compressible perturbations to choose in order to measure their growth appropriately. The very first idea that comes to mind is to choose \( M_r \) in equation (2) in such a way that the norm of each state vector equals to the total acoustic energy of perturbation, which is

\[ ||\mathbf{q}||^2_{M_r} = \pi \int \Sigma \left[ (\delta v_\eta)^2 + (\delta v_\psi)^2 + \frac{|\delta h|^2}{a^2_n} \right] r \, dr. \]  

(16)

In equation (16), it is implied that integration is performed over the azimuthal and vertical coordinates. The variant to measure perturbations by their acoustic energy seems to be natural because this quantity is physically meaningful and it is conserved in the absence of shear.

However, it turns out that the norm (16) indicates the non-modal growth of axisymmetric compressible perturbations, which have an oscillatory rather than transient behaviour. This is not difficult to show in the shearing sheet approximation, as done in present work and relegated to Appendix A.

Strictly speaking, the general oscillatory solution can also be referred to as a non-modal growth; see, for example, section 4.2 of Afshordi, Mukhopadhyay & Narayan (2005), who considered axisymmetric perturbations in the incompressible limit but including vertical motions. Despite this, we would like to exclude this particular case, reserving the pure transient dynamics. Fortunately, this is possible because the basic equations (3) with \( A \) given by equation (15) allow for an energy-like integral for \( m = 0 \).

\(^1\) In Section 3.2, which is devoted to local dynamics, we start to consider perturbations with general dependence on \( \psi \).
Indeed, equation (15) taken with $m = 0$ yields
\[
\frac{\partial (\bar{v}_r)}{\partial t} + \frac{4\Omega^2}{k^2} \frac{\delta v_r}{\bar{v}_r} + \frac{\delta h}{\bar{h}} \frac{\partial (\bar{h}h)}{\partial t} = \frac{1}{r \Sigma} \frac{\partial (r \Sigma \delta v_r \delta h)}{\partial r},
\]
which leads to conservation of the following quantity
\[
E_c = \pi \int \Sigma \left( \left| \delta v_r \right|^2 + \frac{4\Omega^2}{k^2} \left| \delta v_\phi \right|^2 + \left| \delta h \right|^2 \right) \frac{k^2}{\Omega^2} \frac{r \delta \bar{r}}{\partial \bar{r}} dr,
\]
provided that $\Sigma$ disappears at the boundaries of the flow.

A further inspection reveals that equation (17) is nothing but the canonical energy in the particular case of axisymmetric perturbations in the $(r\theta)$ domain. It is not difficult to verify this fact by looking at the general expression for $E_c$ derived by Friedman & Schutz (1978a, hereafter FS) in the Lagrangian framework concerning linear perturbations settled in the rotating axially symmetric flow (see their equation 45).

Let us first note that the sixth term in square brackets in equation (45) by FS yields
\[
\left| \xi_r \right|^2 (\partial_r p + \rho \partial_r \Phi) + \left| \xi_\phi \right|^2 (\partial_r p + \rho \partial_r \Phi) = \rho \left[ \frac{\Omega^2}{k^2} \left| \xi_\phi \right|^2 + 2\Theta \partial_r \xi_r \Omega |\xi_r|^2 + \frac{1}{\rho^2} (\partial_r \rho)^2 |\xi_r|^2 \right],
\]
(18)
where we keep the notations of FS. The first term in square brackets in equation (18) cancels the geometric terms coming from $-\rho v_r \cdot \nabla |\xi|^2$, which enters equation (45) of FS. At the same time, the last term therein, along with the rest of the thermal terms entering equation (45) of FS, gives the thermal contribution to $E_c$, $\Sigma |\delta h|^2 / \alpha^2$, including the usual change to the two-dimensional polytropic index, $n \rightarrow n + 1 / 2$, after the integration over the disc thickness. Finally, with the help of the kinematic relation (11) of FS and the explicit form of $\xi_\phi$ derived by Friedman & Schutz (1978b) directly below their equation (21), we find that
\[
\left| \xi_r \right|^2 + \left| \xi_\phi \right|^2 + 2\Theta \partial_r \xi_r \Omega |\xi_r|^2 = |\delta v_r|^2 + \frac{4\Omega^2}{k^2} |\delta v_\phi|^2.
\]
This confirms that equation (17) is the canonical energy of axisymmetric compressible (barotropic) perturbations expressed in Eulerian variables. Note that $E_c$ is positive definite, which allows us to use it as a norm for our optimization method.

Thus, the definition
\[
\left[|q|\right]_E = E_c
\]
(19)
for the norm of an arbitrary perturbation (including those with non-zero $m$) itself excludes the non-modal growth of axisymmetric perturbations.

In this work, it is instructive to use both norms, equations (16) and (19), in equal rights in order to obtain a broader picture of transient perturbations. Also, the variant (19) is used for the first time in the context of the non-modal approach in discs in contrast to the more canonical energy of axisymmetric compressible perturbations expressed in Eulerian variables. Note that $E_c$ is positive definite, which allows us to use it as a norm for our optimization method. Thus, the definition
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Once the inner product is defined, we can derive the explicit form of $A^\dagger$. This is done by writing the penalty term in equation (4) with the help of the explicit form of $k^2$ and $A$ given by equation (15). Using the equalities $\delta h \nabla \cdot (\Sigma \delta \rho v) = \nabla \cdot (\delta h \Sigma \delta \rho v) = \Delta \delta \rho v \cdot \nabla \delta h$ and $\nabla \delta h = \nabla \delta p - \delta h \nabla (\Sigma \delta \rho v)$ in the $r$-domain, implementing the variation of the final expression over arbitrary deviations of $\delta \rho v$ and $\delta h$ and setting the result to zero, we obtain from the edge term that $\delta \bar{p} = 0$ at the boundary provided that $\delta p = 0$ ibidem. The volume integral gives the adjoint set of equations so that $A^\dagger$ takes the form
\[
\begin{align*}
\left. \begin{array}{l}
\text{im}\Omega & -\kappa^2 \frac{r}{2\Omega} \partial_r \varepsilon & \partial_r \\
2\Omega & \text{im}\Omega & \text{im} \frac{\kappa}{r} \\
\alpha^2 & \frac{4\Omega^2}{r} & \frac{k^2}{4\Omega^2} \text{im}\Omega
\end{array} \right\} & \quad \text{for the norm given by equation (16)}
\end{align*}
\]
(20)
and
\[
\begin{align*}
\left. \begin{array}{l}
\text{im}\Omega & -2\Omega & \partial_r \\
\frac{\kappa^2}{2\Omega} & \text{im}\Omega & \text{im} \frac{\kappa}{r} \\
\alpha^2 & \frac{4\Omega^2}{r} & \frac{k^2}{4\Omega^2} \text{im}\Omega
\end{array} \right\} & \quad \text{for the norm given by equation (19)}.
\end{align*}
\]
(21)
Looking at equation (20), we see that it slightly differs from equation (15). Apart the opposite sign, which actually annihilates with minus in equation (7) when we write the set of equations, it has two off-diagonal elements flipped over. Namely, $\kappa^2 / (2\Omega)$ in the second row of equation (15) takes the place of $2\Omega$ in the first row of equation (15), and vice versa. This means that for a rigidly rotating flow we must have $A = -A^\dagger$, and consequently $U \cdot U^T = I$ (because for rigid rotation $\kappa = 2\Omega$), and no transient growth effects because there is no shear in that case. A different thing happens while changing to the inner product according to equation (19): in equation (21), we find another two off-diagonal terms multiplied (divided) by the factor $4\Omega^2 / \kappa^2$, which differs from unity in the presence of shear.

3.1.2 Implementation of iterative loop

To test the variational technique and the underlying iterative procedure for the determination of global optimal perturbations, we implement a numerical integrator of the Cauchy problem for the basic and adjoint equations. Because $A$ and $A^\dagger$ are given by equations (15) and (20) (or equation 21 if changing to a different norm), respectively, we follow Frank & Robertson (1988) and choose a leapfrog scheme, because this is a simple explicit second-order method that is stable for wave-like dynamics. Four different meshes with constant coordinate and time steps, $\Delta x$ and $\Delta t$, are introduced on the $(r,t)$-plane. The second and third meshes are shifted for $h$- and $r$-planes. The second mesh is shifted for $\Delta t / 2$ along the time axis and for $\Delta r / 2$ along the coordinate axis relative to the first. The fourth mesh is shifted in both time and space for $\Delta t / 2$ and $\Delta r / 2$. Then, we split the basic and adjoint sets of equations into real and imaginary parts and assign real and imaginary parts of $\delta v_r$, $\delta v_\phi$, $\delta h$, $\delta h_\phi$ to corresponding meshes in such a way that for a particular equation, approximations of time and spatial derivatives are centred at the same nodes. In order to advance the large-scale perturbations with $\lambda_{pert}$ comparable to the radial scale of variations of background quantities, we have to impose the boundary conditions. We require that the Lagrangian perturbation of enthalpy vanishes at the inner boundary of the flow. Note that if $\Sigma \rightarrow 0$ at the boundary, it is sufficient to impose the regularity condition on the perturbed quantities therein. At the same time, the outer boundary condition is of no concern, because the outer boundary is assumed to be located far beyond the radial domain occupied by perturbations evolved until $t = \tau$.

Regarding the background flow, we use mainly a Keplerian disc with uniform distribution of $\Sigma$ and $a_{out}$, which should be suitable to make comparisons with local dynamics in order to see how the
distribution of the shear rate and the cylindrical geometry affects transient growth. Thus, we set
\[ \Sigma = \text{const}, \quad a_\text{eq} = \frac{\delta}{\sqrt{2n}} \quad \text{and} \quad \Omega = r^{-3/2}, \]
which we refer to as the homogeneous disc model hereafter. In the global approach, it is assumed that \( r \) is given in units of the inner radius of the disc and all time intervals are measured in units of the inverse Keplerian frequency at \( r = 1 \). The constant \( \delta \sim H/r \ll 1 \) specifies the aspect ratio of a geometrically thin disc. Also, we would like to check another variant of the background flow, which is specified by the following profiles
\[ \Sigma \propto r^{-3/2}(1 - r^{-1/2})^{3/2}, \]
\[ a_\text{eq} = \left( \frac{\delta}{\sqrt{2n}} \right) \Omega \left[ 1 - r^{-1/2} \right]^{1/5}, \]
where \( \Omega \) takes its Keplerian value as well. Equations (23) and (24) are adopted from Shakura & Sunyaev (1973) as a representative of the thin accretion disc model.

Now, in order to find an optimal perturbation, we specify an arbitrary initial state vector; that is, we take an arbitrary initial condition, \( q_i(t = 0) \), and integrate the basic equations forward in time up to some \( t = \tau \). Then, we substitute the result, \( q_i(t = \tau) \), as the initial condition to the adjoint equations (depending on the norm chosen for perturbations) and integrate them backward in time up to \( t = 0 \), obtaining the next variant of the initial state vector, \( q_i(t = 0) \). Finally, \( q_i(t = 0) \) must be renormalized, that is, divided by its own norm calculated according to equation (16) (or, alternatively, to equation 19). The first iteration is accomplished now. The iterative loop consists of a number of such iterations necessary to achieve the desirable accuracy of determination of the optimal vector \( q_{\text{opt}}(t) = \lim_{p \to \infty} q_{p}(t) \) and the corresponding optimal growth
\[ G(\tau) \equiv \frac{||q_{\text{opt}}(\tau)||^2}{||q_{\text{opt}}(0)||^2} = \lim_{p \to \infty} G_p(\tau), \]
where \( G_p(\tau) \equiv \frac{||q_{p}(\tau)||^2}{||q_{p}(0)||^2} \).

First, let us check a convergence of this iterative loop. This is illustrated in Fig. 1 for the norm given by equation (16) and the background profiles given by equations (23) and (24). To launch iterations, we take two distinct starting shapes of \( \text{Re}[\delta h] \) as the initial condition for the Cauchy problem, that is, the single and double Gaussian functions, which have arbitrary positions and radial dispersions,

\[ \text{Re}[\delta h]_{r=0} \propto \exp \left[ -\frac{(r^2 - \mu_1^2)}{2\sigma_1^2} \right] \]

and

\[ \text{Re}[\delta h]_{r=0} \propto \exp \left[ -\frac{(r^2 - \mu_2^2)}{2\sigma_2^2} \right] \]
\[ + \exp \left[ -\frac{(r^2 - \mu_3^2)}{2\sigma_3^2} \right], \]
where \( \mu_{1,2,3} \) and \( \sigma_{1,2,3} \) are arbitrary numbers. The rest of the quantities constructing the state vector are set to zero at \( t = 0 \). Using our numerical scheme after \( 10^2 \) iterations, we obtain a unique optimal initial shape of perturbation with the optimization time-span, \( \tau \approx 3 \). At the same time, the iterative value of optimal growth, \( G_p(\tau) \), converges to the highest possible value, \( G(\tau) \), as seen in the bottom-right panel of Fig. 1. Let us stress that we obtain the same optimal initial shape of \( \delta h \) independently of the particular values of \( \mu_{1,2,3} \) and \( \sigma_{1,2,3} \). A similar situation takes place for the norm given by equation (19).

Figure 1. Convergence of arbitrary initial profiles of \( \delta h \) to an optimal initial profile in a Keplerian disc with structure specified by equations (23) and (24). It is implied that perturbations are measured according to equation (16). In the top-left and top-right panels, middle-left and middle-right panels and bottom-left panel, the initial profiles of \( \delta h \) are presented corresponding to \( p = 0, 25, 80, 120 \) and 350 iterations, respectively. In the bottom-right panel, there is an iterative value of optimal growth, \( G_p(\tau) \), versus the number of iterations. The dashed and solid curves are used for two different iterative loops that launch with single and double Gaussian functions given by equations (26) and (27). Note that \( \tau = 3, m = 5, n = 1.5 \) and \( \delta = 0.05 \).

3.2 Optimization in a shearing sheet model

In the local spatial limit, the evolution of compressible perturbations can be considered in the shearing sheet approximation. The corresponding equations have been derived by Goldreich & Lynden-Bell (1965); see also Umurhan & Regev (2004) for a detailed account of this derivation. Bodo et al. (2005, B05 hereafter) have studied linear perturbations in the context of the non-modal approach. Equations that we need can be quoted from B05. These are the following

\[ \left( \frac{\partial}{\partial t} - q \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right) u_x - 2 \frac{\partial}{\partial y} u_y = - \frac{\partial W}{\partial x}, \]

\[ \left( \frac{\partial}{\partial t} - q \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right) u_y + (2 - q) \frac{\partial}{\partial y} u_x = - \frac{\partial W}{\partial y}, \]

\[ \left( \frac{\partial}{\partial t} - q \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right) W + a_z^2 \left( \frac{\partial}{\partial x} u_x + \frac{\partial}{\partial y} u_y \right) = 0, \]

where \( u_x, u_y \) are the Eulerian perturbations of the velocity components, \( W \) is the Eulerian perturbation of enthalpy excited in a small patch of the disc, \( x \equiv r - r_0, y \equiv r_0(\rho - \Omega_0 t) \) are the local Cartesian coordinates, \( x, y \ll r_0 \), which correspond to the reference frame rotating with the angular velocity \( \Omega_0 = \Omega(r_0) \), \( q \equiv -(r/\Omega(\partial \Omega/\partial r))_{r=r_0} \) is the constant shear rate that defines the background velocity as \( v^\text{loc} = -q \Omega_0 x \) and all other background quantities, such as \( \Sigma \) and \( a_z \), are assumed to be constant.
Let us employ the iterative procedure elucidated in Section 3.1. In order to do this, we have to find equations that are adjoint to the set (28)–(30). Evidently, the shearing sheet limit of equation (20) is reproduced by the following set of equations:

\[
\frac{\partial}{\partial t} - q\Omega_0 \frac{\partial}{\partial y} \tilde{u}_s - (2 - q)\Omega_0 \tilde{u}_s = -\frac{\partial \tilde{W}}{\partial x};
\]

\[
\frac{\partial}{\partial t} - q\Omega_0 \frac{\partial}{\partial y} \tilde{u}_s + 2\Omega_0 \tilde{u}_s = -\frac{\partial \tilde{W}}{\partial x};
\]

\[
\frac{\partial}{\partial t} - q\Omega_0 \frac{\partial}{\partial y} \tilde{W} + a_s^2 \left( \frac{\partial \tilde{u}_s}{\partial x} + \frac{\partial \tilde{u}_s}{\partial y} \right) = 0.
\]

Alternatively, the shearing sheet limit of equation (21) is reproduced by the following set of equations:

\[
\frac{\partial}{\partial t} - q\Omega_0 \frac{\partial}{\partial y} \tilde{u}_s - 2\Omega_0 \tilde{u}_s = -\frac{\partial \tilde{W}}{\partial x};
\]

\[
\frac{\partial}{\partial t} - q\Omega_0 \frac{\partial}{\partial y} \tilde{u}_s + (2 - q)\Omega_0 \tilde{u}_s = -\frac{2 - q}{2} \frac{\partial \tilde{W}}{\partial y};
\]

\[
\frac{\partial}{\partial t} - q\Omega_0 \frac{\partial}{\partial y} \tilde{W} + a_s^2 \left( \frac{\partial \tilde{u}_s}{\partial x} + \frac{2}{2 - q} \frac{\partial \tilde{u}_s}{\partial y} \right) = 0.
\]

In equations (31)–(36), it is assumed that \(\tilde{u}_s, \tilde{u}_s, \tilde{W}\) denote the adjoint velocity and enthalpy perturbations.

Finally, we would like to change to the dimensionless comoving Cartesian coordinates, \(x' = \Omega_0 x/a_s, y' = \Omega_0 (y + q\Omega_0 x)/a_s\), and \(r = \Omega_0 r^2\). This leads us to a spatially homogeneous set of equations that is, however, inhomogeneous in time. Nevertheless, considering partial solutions in the form of the shear harmonic, or, more accurately, the spatial Fourier harmonic (SFH), \(f = \tilde{f} (k_x, k_y, r) \exp(ik_x x + ik_y y)\), where \(f\) is any of the unknown quantities, \(\tilde{f}\) is its Fourier amplitude and \((k_x, k_y)\) are the dimensionless wavenumbers along the \(x\) and \(y\)-axis expressed in units of \(\Omega_0/a_s\), we obtain the corresponding set of ordinary differential equations (ODEs):

\[
\frac{d\tilde{u}_s}{dr^2} = 2\tilde{u}_s - i(k_x + k_y q t')\tilde{W};
\]

\[
\frac{d\tilde{u}_s}{dr} = -(2 - q)\tilde{u}_s - ik_y \tilde{W};
\]

\[
\frac{d\tilde{W}}{dr} = -i(k_x + k_y q t')\tilde{u}_s + k_x \tilde{u}_s;
\]

\[
\frac{d\tilde{u}_s}{dr} = (2 - q)\tilde{u}_s - i(k_x + k_y q t')\tilde{W};
\]

\[
\frac{d\tilde{w}}{dr} = -2\tilde{u}_s - ik_y \tilde{W};
\]

\[
\frac{d\tilde{W}}{dr} = -i(k_x + k_y q t')\tilde{u}_s + k_x \tilde{u}_s;
\]

Here, equations (37)–(39) have to be solved in order to determine the state vector, \(q(t') = (\tilde{u}_s(t'), \tilde{u}_s(t'), \tilde{W}(t'))\), and equations (40)–(42) have to be solved in order to determine the adjoint vector, \(\tilde{q}(t') = (\tilde{u}_s(t'), \tilde{u}_s(t'), \tilde{W}(t'))\). Equations (37)–(42) contain the adjoint part that results from equations (31)–(33), whereas in equations (34)–(36) this has to be replaced by the following set of equations:

\[
\frac{d\tilde{u}_s}{dr^2} = 2\tilde{u}_s - i(k_x + k_y q t')\tilde{W};
\]

\[
\frac{d\tilde{u}_s}{dr} = -(2 - q)\tilde{u}_s - \frac{2 - q}{2} ik_y \tilde{W};
\]

\[
\frac{d\tilde{W}}{dr} = -i \left( k_x + k_y q t' \tilde{u}_s + \frac{2}{2 - q} k_y \tilde{u}_s \right).
\]

Throughout equations (37)–(45), it is implied that the SFH of velocity perturbations and the SFH of enthalpy perturbations are expressed in units of \(a_s\) and \(a_s^2\), respectively. We omit the prime after \(t\) hereafter. Seeking a solution to equations (37)–(42), we use the surface density of the acoustic energy of a single SFH,

\[
E = \frac{1}{2S} \int_\Omega \left( \text{Re}[u_s]^2 + \text{Re}[\dot{u}_s]^2 + \frac{\text{Re}[\dot{W}]}{a_s^4} \right) dx dy,
\]

as a norm for local perturbations. Equation (46) leads to the following expression for the SFH,

\[
||q||_1^2 = \frac{1}{2} \left( ||\tilde{u}_s||^2 + ||\dot{\tilde{u}}_s||^2 + ||\tilde{W}||^2 \right),
\]

which is the local analogue of equation (16).

In order to find a local counterpart of optimal perturbations measured by equation (19), we have to solve equations (37)–(39) and (43)–(45) by employing the norm, which is

\[
||q||_2^2 = \frac{1}{2} \left( ||\tilde{u}_s||^2 + \frac{2}{2 - q} ||\dot{\tilde{u}}_s||^2 + ||\tilde{W}||^2 \right).
\]

3.2.1 On the relevant parametrization of the problem

Two types of shearing harmonics are allowed to exist in a compressible medium: vortices and density waves (see Chagelishvili, Rogava & Segal 1994; Chagelishvili et al. 1997a; B05). Both are decoupled from each other when the perturbed motion is subsonic (i.e., when the difference of shear velocities on the length-scale of the problem is less than the speed of sound). The relevant length-scale in the shearing sheet is defined by the wavelength of the SFH, \(\lambda_s = \pi/k_x + k_y q t'\), across the shear. Thus, the condition that vortices and density waves live separately in the shearing box is

\[
\lambda_s q \Omega_0/a_s = \frac{q}{|k_x(k_x + k_y q t')|} = \frac{R}{|k_x + k_y q t'|} < 1,
\]

where we introduce the new parameters, \(R\) and \(\beta\), expressed through the usual wavenumbers as \(R = q/k_x\) and \(\beta = k_y/k_x\). At least for Keplerian shear, \(R\) is of the order of \(\lambda_s/H\) characterizing the azimuthal scale of the SFH relative to the disc thickness. At the same time, for the shearing harmonic with \(\beta < 0\), the latter parameter defines the time of swing, \(t_s = -\beta/q\) (i.e., the instant when the SFH changes its form from a leading to a trailing spiral). Previous studies have shown that during this event, vortices stop gaining energy from the background and switch to the decay phase, whereas density waves exhibit exactly the opposite behaviour. The initially leading
spiral vortices are of particular interest in this study because their vertical configurations are subject to transient growth.

Equation (49) leads to an apparent but important conclusion worth discussing here. We see that the leading spirals always pass a period when vortex motion is inseparable from acoustic motion. This ‘swing interval’ is confined by the instants

\[ t_{\text{swing}} = (-\beta/q)(1 \pm R/\beta), \]

(50)

with \( t_s < t_i < t_{\text{swing}} \). This is small compared to a characteristic evolution time of the leading spiral given by \( t_s \) if

\[ R \ll |\beta|/2, \]

(51)

and thus, not necessarily in the case of small azimuthal wavelengths, \( R \ll 1 \). Generally, the sufficiently tightly wound (either leading or trailing) spirals with \( \lambda_y \gg H (R \gg 1) \) can still be a well-defined vortex or density wave. However, it must be noted that the swing interval becomes short with respect to the dynamical time-scale, \( (t_{\text{swing}} - t_s) \ll 1 \), only if the SFH is truly small-scaled compared to the disc thickness, that is, if \( \lambda_y \gg H (R \ll 1) \). Also note that equation (49) is modified when \( \alpha_1 \) becomes comparable or less than \( k(\lambda_y^2 + \lambda_y^2)^{1/2} \), where \( \lambda^2 = 2(\epsilon - q)^2 \Omega_0^2 \) in the local case, because epicyclic oscillations become significant in the last case. However, it can be checked that the correction to estimate (50) is always of the order of unity.

Furthermore, Chagelishvili et al. (1997b) and B05 have described a phenomenon of the generation of density waves by vortices as they swing from leading to trailing spirals. This process is asymmetric in the sense that vortices are able to excite density waves, but not vice versa. Later, Heinemann & Papaloizou (2009a, HP hereafter) developed an analytical theory of density wave excitation within the Wenzel–Kramers–Brillouin–Jeffreys (WKBJ) framework. They obtained analytical expressions for amplitude and phase of the density wave that emerges at the swing time of the vortex, \( t_s \). The amplitude of the density wave is proportional to \( \epsilon^{-1/2} \exp(-4\pi^2/\epsilon) \) (see equation 53 of HP), where \( \epsilon \) is assumed to be a small parameter of the theory,

\[ \epsilon = \frac{q \kappa}{\kappa^2 + q^2 / \Omega_0^2} = \frac{R}{1 + 2(\epsilon - q)^2 R^2 / q}. \]

(52)

From equation (52), we see that density wave excitation is suppressed in the limit of small azimuthal wavelengths, \( R \ll 1 \) (because \( \epsilon \sim R \)), as well as in the limit of large azimuthal wavelengths, \( R \gg 1 \) (because \( \epsilon \sim R^{-1} \) if \( q \) is not too close to 2).

Summarizing this section, we expect that outside the swing interval defined by equation (50), an arbitrary initial SFH can always be represented as the combination of a vortex and a density wave. If the vortex constituting a part of the SFH is a leading spiral, it generates an additional density wave at the swing time, \( t_s \). However, the latter event is substantial only if \( R \sim 1 \). Bearing in mind the general picture briefly exposed in this section, we regard the parameters \( \beta \) and \( R \) as suitable to make a subsequent analysis of the optimal SFH, and we use them below to present our results.

### 3.2.2 Transient growth of vortices in compressible medium

As has been discussed by HP and by others, the vortical perturbations in compressible shear flow can be recognized as the slowly evolving solutions with non-zero potential vorticity. Indeed, equations (37)–(39) can be reformulated as the second-order inhomogeneous equations for \( \tilde{u} \) (see equation 32 of B05 or, alternatively, equation 23 of HP), \( \tilde{u} \), and \( \tilde{W} \) (see equation 22 of HP). The right-hand sides of these equations are proportional to the SFH of the potential vorticity being time-invariant. In our notations and in a form defined by B05, this is

\[ \tilde{u}_s = -\frac{K + q}{K^2 + 4q^2 k_s^2} \]

(54)

\[ \tilde{u}_s = \frac{k_s + q k t}{K} \]

(55)

and

\[ \tilde{W} = 2i \frac{q k_s^2 - K}{K^2 + 4q^2 k_s^2} I, \]

(56)

where \( K = k_s^2 + (k_s + q k t)^2 + \kappa^2 / \Omega_0^2 \). It is implied that equations (53)–(56) are given for dimensionless quantities.

As follows from the reasoning in Section 3.2.1, equations (54)–(56) are a good approximation to an accurate solution of equations (37)–(39) obtained for some initial vortex perturbation if the following two conditions are satisfied: it is considered to be outside the swing interval and its azimuthal wavelength differs significantly from the disc thickness (i.e. \( \epsilon \ll 1 \)). Despite such strict limitations to solution (54)–(56), we would like to check to what estimations of growth factors it leads. Using equation (47), we find for the acoustic energy of the vortex that

\[ \frac{2E}{a^2 I^2} = \frac{(k_s + q k t)^2}{K^2} + \frac{4}{K^2 + 4q^2 k_s^2} \]

(57)

with \( K_s = 4 + k_s^2 \).

Equation (57) provides us with an expression for \( g \) as a function of \( k_s, k_t \), and \( t \). In order to obtain the local analogue of \( G \) introduced by equation (25), we have to find the maximum of \( g(k_s) \) for fixed \( k_s \) and \( t \). To avoid straightforward but cumbersome calculations redundant in our current estimations, we assume that it is close to the growth factor of the SFH swinging at \( t \). Thus, we approximate the optimal growth by \( G \approx g(k_s = -k, q t) \), which yields

\[ G \approx \frac{K_s [K_s + (q k t)^2]}{[K_s + (k_s + q k t)^2]^2 + 4(k_s^2 q^2 t^2)} \times \frac{[K_s + (k_s + q k t)^2]^2 + 4q^2 k_s^2}{K_s^2 + 4q^2 k_s^2}, \]

(58)

with \( K_0 = K(t = -k_s/(q k_t)) \).

Furthermore, equation (58) should be considered in the limits of small \( (k_s, q k_t) \), and large \( (k_s, q k_t) \) azimuthal wavenumbers when the excitation of density waves is suppressed and non-modal growth is represented solely by vortices.

Particularly, in the limit \( k_s, q k_t \ll 1 \) (strictly, as long as \( R \gg q \Omega_0 / \kappa \)), we obtain

\[ G \approx \frac{4q^2 k_s^2 (q k_t^2 / R^2 + \kappa^2 / \Omega_0^2)^2}{k_s^2 (q k_t^2 / R^2 + 4)}, \]

(59)

where \( k_s \), is replaced by \( R \).

Note that for a sufficiently long \( t \), equation (59) becomes especially simple. Strictly, in the case \( t \gg 2R / q \),

\[ G \approx \frac{4q^2 k_s^2 q k_t^2}{k_s^2 R^2}, \]

(60)
which demonstrates that transient growth drops inversely to the second power of $R$, whilst it remains constant for constant ratio $t/R$. Another important point is that $G$ rapidly increases as the background flow tends to constant angular momentum shear. Indeed, given that $k^2 = (2 - q)\Omega_0^2$, we see that $G \propto q^2/(2 - q^2) \to \infty$ as $q \to 2$. This result indicates that the transient growth of large-scale vortices might be of primary importance close to the last stable orbit in relativistic discs around black holes. However, this is not the case in the opposite limit. For $k_0 \gg 1$ (strictly, as long as $R \ll q/2$) equation (58) yields

$$G \approx 1 + (qk)^2,$$  \hspace{1cm} (61)

which is consistent with the basic conclusions of Afshordi et al. (2005), who also treated the more realistic cases analytically by adding the viscosity and vertical component of small-scale vortices. See also the results of Yecko (2004) in this context. Note that this case is equivalent to $a_0 \to \infty$ (i.e. to the limit of incompressible dynamics when the velocity perturbation takes a divergence-free form). There exists a simple analytical solution of the corresponding initial value problem and it becomes possible to obtain an exact analytical expression for $G(t)$ (see equation B10 and Appendix B for details).

Finally, let us assess the influence of non-zero viscosity, which can effectively emerge through turbulent motions in the disc. If it were not for unwinding because of the shear, the initially tightly wound leading spiral would disperse in the time-scale $\Delta t \sim \lambda_y^2/\nu$, where $\nu$ is a kinematic viscosity coefficient. We employ its usual parametrization through the Shakura-$\alpha$-parameter, $\nu = \alpha a_0 H$, finding that the dimensionless $\Delta t \sim (\alpha k_0^2)^{-1}$ rapidly decreases as $k_0$ becomes larger. At the same time, it takes a longer time for the transient growth to occur because $\Delta t_{\text{tg}} \sim [k_0/(q k_0)]$. However, while the spiral unwinds, its radial scalelength increases, allowing the viscous dispersal to be suspended. Thus, the condition $\Delta t_{\text{tg}} = \Delta t$ puts a lower limit on the longest duration of non-modal growth of vortex in viscous flow. Using the latter equality, we obtain

$$\max(\Delta t_{\text{tg}}) \gtrsim \alpha^{-1/3} \left( \frac{R}{q^2} \right)^{2/3}.$$  \hspace{1cm} (62)

We can check that equation (62) recovers an estimate given by Afshordi et al. (2005); see their equation (81). The upper bound on optimal growth corresponding to $\max(\Delta t_{\text{tg}})$ is given by its inviscid value, $G_{\text{max}}$, which is as follows:

$$G_{\text{max}} \approx \frac{4\Omega_0^4}{k^2} \left( \frac{q^2}{\alpha R} \right)^{2/3}.$$  \hspace{1cm} (63)

Estimate (63) is obtained for large-scale vortices ($R \gg 1$) by substituting equation (62) into equation (60). We see that according to this approximate expression, the transient growth ceases only for vortices with $H/\lambda_y \sim \alpha$, which, in turn, becomes a marginal condition for moderately viscous discs with $\delta \sim \alpha$.

3.2.3 Implementation of iterative loop

In the shearing sheet model, we construct a local counterpart of the scheme described in Section 3.1.2. This time, an arbitrary initial condition, $\phi_0(t = 0)$, consists of a single SFH of velocity and enthalpy perturbations and is used to integrate equations (37)–(39) forward in time. Again, we use the result, $\phi_0(t = \tau)$, as the initial condition to equations (40)–(42) – alternatively, to equations (43)–(45) – and we integrate them backward in time. After $\phi_0(t = 0)$ is renormalized, that is, divided by its own norm given by equation (47) (alternatively, by equation 48), it is used in the next iteration.

Note that in this way $G_p$ and $G$ are determined for particular values of $k_0$ and $k_\tau$, or, alternatively, for particular values of $\beta$ and $R$. Thus, we imply hereafter that the local optimal growth is the quantity obtained in the iterative loop for a single SFH. However, we denote it further explicitly as $G(\beta)$ because, by default, $G$ is defined as the optimal growth for the specified azimuthal wavenumber ($k_0$ in the local context or $m$ in the global context).

The numerical parametrical study of optimal growth carried out with the help of the standard GNU Scientific Library routine for the integration of sets of ODEs shows that there always exists some value of $\beta = \beta_{\text{max}}(p)$ where $G_p(\beta)$ attains an absolute maximum, provided that the other parameters (including $R$) are fixed. Consequently, renormalizing $\phi_0(t, \beta)$ by the norm of $\phi_0(t, \beta = \beta_{\text{max}}(p))$ for each iteration we eventually obtain the optimal state vector $\phi_0(t, \beta = \beta_{\text{max}})$ corresponding to a single SFH. This result is independent of the initial state vector, $\phi_0(t, \beta)_{\text{const}}$, which can be any packet of shearing harmonics. At the same time, the optimal growth corresponding to optimal vector obtained in this way, $G$, is equal to absolute maximum of $G(\beta)$ mentioned previously.

4 OPTIMAL SOLUTIONS IN A KEPLERIAN DISC

4.1 Inspection of optimal perturbations in shearing sheet model

We start our research of optimal shearing harmonics in the particular case of Keplerian shear by calculating $G(\beta)$ for different values of $R$. First, in order to reveal the basic features of non-modal growth, only the acoustic energy is used to measure perturbations. Setting the optimization time to the particular dynamical value $t = 10$, we find that the non-modal growth is mostly an attribute of the SFH with $\beta < 0$, as would be expected from the theory of transiently growing vortices (see Fig. 2). Indeed, the solid curve obtained for small $R \ll 1$ virtually represents the incompressible growth factor given by equation (B9) in the range of $\beta \lesssim -1$. This result hints that by performing the optimization we just reproduce vortices in this case. Noting that in this range of parameters, $R \ll 1$ and $\beta \lesssim -1$, vortices and density waves exist separately from each other and the wave excitation is suppressed. Presumably, this is why the optimization
scheme approaches the pure vortex solution because any density wave can only return its energy to the flow while being a leading spiral. The solid curve attains maximum at \( \beta = -15 \approx - (3/2) \pi \) (i.e. for the SFH that swings at \( t = \tau \)). At the same time, we notice that the optimal growth also exceeds unity for \( \beta \gtrsim 1 \). In the domain \( \beta \gtrsim 1 \), the two types of SFH are well distinguished again and we should suppose that the optimization scheme approaches the density wave because the vortex can only return its energy to the flow while being a trailing spiral. However, the non-modal growth for \( \beta \gtrsim 1 \) is highly reduced in comparison with the case \( \beta \lesssim -1 \). We attribute this to the fact that the growth rate of density waves is proportional to \( q_t \) rather than to \( (q_t)^2 \), as in the case of a vortical SFH (see equation 61). Indeed, we can make use of the analytical results of Chagelishvili et al. (1997a), who derived how the acoustic energy of a density wave grows with time in the (non-rotating) shear (see their equation 3.7a). In our notations, this leads to the following dependence,

\[
g = \left[ \frac{1 + (\beta + q t)^2}{1 + \beta^2} \right]^{1/2},
\]

and it is assumed here that \( R \ll 1 \). Interestingly, it can be checked that equation (64) perfectly recovers the solid curve in Fig. 2 for \( \beta > 0 \). For a sufficiently long time, \( t \gg \beta/q \), equation (64) yields \( g \approx qt/(1 + \beta^2) \) affirming that \( G \propto qt - \eta \propto (q t)^2 \) as for the maximum of \( G(\beta) \) in the domain of negative \( \beta - \) and decreases monotonically when proceeding to large \( \beta \).

Now, turning to \( R \lesssim 1 \), we find that the maximum of \( G(\beta) \) takes a greater value and slightly shifts towards the SFH that swings a little before the optimization instant. Moreover, a significant hump in \( G(\beta) \) appears for \(-15 < \beta < 0 \). We suspect that this occurs as a result of the phenomenon of wave excitation by the vortex, which arises for \( R \sim 1 \). This is what can enhance the growth factor of a vortical SFH right after it swings from a leading to a trailing spiral, because the emerged density wave extends the non-modal growth to a trailing spiral phase. However, before we check our suspicion, we would like to reveal the physical nature of optimal solutions using a rigorous algorithm. For this, we decompose the initial optimal SFH on to the vortex and the density wave. First, the potential vorticity perturbation of the optimal SFH, \( I \), is determined using equation (53). Using the iterative method proposed in the appendix of B05, we obtain values of \( \dot{u}_x, \dot{u}_y \) and \( \dot{W} \) at \( t = 0 \) proportional to \( I \) that extract the vortex from the optimal solution. This method converges, provided that at \( t = 0 \) the SFH is outside the swing interval (see equation 50). If so, the remainder is nothing but a density wave, with \( I = 0 \) being a solution of the homogeneous parts of equations (22) and (23) of HP. The result of decomposition is presented in Fig. 3, where we take several values of \( \beta \) and the corresponding optimal solutions lying on the dashed curve in Fig. 2. It is shown that the optimal solution is in fact a mixture of a vortex and a density wave. Note that we have numerically checked that the accuracy of vortex determination and the accuracy of optimization loop lie well under the smallest of the amplitudes on to which the optimal solution is decomposed. As can be seen in Fig. 3, both the vortex and wave constituting each pair swing simultaneously at the time defined by the value of \( \beta \). As expected, the leading spiral density wave changes from decay to growth as opposed to the vortex. While \( \beta \ll -1 \), the optimal SFH is almost a vortex rather than a wave. At the same time, when we set \( \beta > 0 \), there is no swing and the trailing spiral vortex decays from the very beginning as opposed to the density wave, which becomes a cardinal component of the optimal solution. Furthermore, if an absolute value of \( \beta \) becomes smaller, the contribution of the secondary component increases. Thus, the overall conclusion is that in the domain where the non-modal growth attains its highest rates (i.e. for \( \beta \ll -1 \)), the initial optimal SFH is virtually indistinguishable from the vortex carrying the same potential vorticity.

With this knowledge, we pay attention to the solid curves of pairs (1) and (2) in Fig. 3. As pointed out, these growth factors, \( g(t) \), correspond to the pure vortex solution at \( t = 0 \). It becomes clear that because these vortices inevitably excite density waves at the instant of the swing, these density waves are the reason for an additional hump in the dashed curve in Fig. 2 in comparison with the solid curve in Fig. 2. As can be seen in Fig. 3, for \( R = 0.5 \), the amplitude of the excited density wave is so large that \( g \) grows almost monotonically all the time except for a small peak at the time of the swing, whereas the small-scale vortex with \( \lambda_c \ll H \) decays after it swings. This is why the difference in \( G(\beta) \) between the cases \( R = 0.1 \) and \( R = 0.5 \) is so large in the range \(-15 < \beta < 0 \). However, the rise of optimal growth is also noticeable for \( \beta \approx -15 \) corresponding to the swing right at the optimization instant. The magnitude of this rise directly characterizes the additional amplitude of the density wave right after it was excited. As we have already mentioned, this amplitude is defined by the parameter \( \epsilon \) (see equation 52), and attains maximum for \( R \sim 1 \). This result has been obtained analytically by HP. For \( R = 1 \) (dash-dotted curve in Fig. 2), the amplitude and the growth rate of the excited density wave become so high that the optimal SFH, which swings approximately 1.5 times earlier than the optimization instant, attains the highest optimal growth throughout the range of \( \beta \). The corresponding value of \( G(\beta) \) by 1.5 times exceeds its value in the small wavelength limit, \( R \ll 1 \). Finally, we plot \( G(\beta) \) for the case \( R = 4 \), representing a longer wavelength behaviour. We find that the maximum of \( G(\beta) \) returns to the position corresponding to the SFH swinging at the optimization instant, which apparently confirms the fact that the amplitude of the emerged density wave strongly decreases as \( \epsilon \) becomes less than unity again. At the same time, the growth factor of the vortex itself also decreases, as expected according to equation (59), as a function of \( R \).

In order to illustrate how the evolution of the growth factors of the optimal SFH changes with \( R \), we plot \( g(t) \) for the same values of \( R \) as in Fig. 2, taking \( \beta = \beta_{\text{max}} \) corresponding to the maxima of curves of \( G(\beta) \) in Fig. 2. Because \( \beta_{\text{max}} \ll -1 \) for all curves,
$g(t)$ virtually represents the evolution of corresponding vortices with the same potential vorticity perturbation (see the top panel of Fig. 4). As soon as we take $R$ close to unity, the vortices excite density waves with the acoustic energy growing linearly with $t$ (see equation 64) at long times. The dot-dashed curve in Fig. 4 demonstrates the case $R = 1$, which, in fact, gives the highest possible growth over all $R$ for a fixed optimization time. Moreover, the optimal solution for $R = 1$ is perfectly recovered by the analytical solution of HP, in spite of the fact that $R = 1$ and, consequently, $\epsilon \sim 1$ for the Keplerian flow corresponds to the worst case for the asymptotical analytics employed by HP. Indeed, this is shown in the middle and bottom panels of Fig. 4 where we plot the sum of equation (54) and equation (57b) of HP standing for $\tilde{u}_r(t)$ and the sum of equation (55) and equation (52) of HP standing for $\tilde{u}_r(t)$ (dotted curves). Note that these expressions are evaluated for the potential vorticity perturbation of the optimal SFH. They are plotted for $t$ longer than the optimization time because before the optimal SFH swings from a leading to a trailing spiral, it is described by a nearly vortical solution.

From the top panel of Fig. 4, we find that $g(t)$ for $R = 1$ is tangent to the dot-dot-dashed curve in the vicinity of the optimization instant. The dot-dot-dashed curve is defined as an optimal growth, $G(t)$, maximized over all $R$ (i.e. over all SFH with the specific $k_y$ and $k_y$); see also Fig. 5. None of the initial local perturbations can grow in acoustic energy higher than is limited by this curve. In Fig. 5, this general bound of transient growth in compressible Keplerian flow is plotted for both norms of perturbations that we have defined in this paper. There are also two additional curves that represent the optimal growth in the limit of small-scale perturbations, $\lambda_y \ll H$, measured either by equation (47) or by equation (48). Let us remember that the case $\lambda_y \ll H$ is equivalent to the transient growth of local perturbations with a divergence-free velocity field (see Appendix B for details). It turns out that independent of the choice of norm, the compressible perturbations are able to grow significantly faster (by a factor increasing with time-span) than incompressible perturbations. Again, independent of the norm choice, perturbations with a characteristic length-scale of the order of the disc thickness ($\lambda_y \sim H$) grow most rapidly at the fixed time interval, which takes place as a result of the ability of vortices with $\lambda_y \sim H$ to excite the strongest density waves. Thus, the compressibility of the medium is a factor that cannot be ignored while studying the transient growth phenomenon in astrophysical discs. Here, we should mention that the simulations performed by Heinemann & Papaloizou (2009b) in the shearing sheet model confirmed the emergence of density waves generated by vortices having a turbulent origin. They also verified that this phenomenon becomes most prominent for the wavelengths $\sim H$ (see fig. 7 of their paper). In this study, we come to similar conclusions when looking for an extreme solution to the initial value problem for linear perturbations. However, the non-linear evolution of the excited density wave extracted by Heinemann & Papaloizou (2009b) shows a significant damping, contrary to the linear solution. This indicates that the solutions represented by the dot-dashed curve in Fig. 4 must be more saturated than all the others by the non-linear effects in a well-developed turbulence.

Despite the argument that the scale of the order of the disc thickness has been revealed to be optimal in the context of non-modal growth in a Keplerian flow, we believe it is important to consider in more detail the case of large-scale perturbations, $R \gg 1$, when the density wave excitation is suppressed, just as in the incompressible case, $R \ll 1$. First, this can be justified by the fact that the large-scale perturbations are not subject to fast dissipation in realistic viscous (turbulent) shear flow. Our estimate shows that dissipation
time \( \propto \alpha^{-1/3}R^{2/3} \) (see equation 62), and consequently the highest possible growth for all time-spans, \( G_{\text{max}} \), does not depend so sharply on \( R \) as it is for \( G \) for the fixed time-span in the inviscid fluid (see equation 60). Moreover, as we have already assessed at the very end of Section 3.2.2, for realistic thin accretion discs with an aspect ratio 0.01 \( \lesssim \delta \lesssim 0.1 \) with insufficient turbulent viscosity, \( \alpha \lesssim \delta, G_{\text{max}} > 1 \) for all possible values of \( \lambda_y \), up to \( \lambda_y \sim r \), where \( r \) is the radial scale of the disc. Second, we have shown above that the most rapidly growing (i.e. optimal) perturbations are nearly the vortices, and thus any preliminary chaotic motions in the disc acquiring large-scale potential vorticity perturbation become natural seeds for large-scale (compressible) vortices. Through linear transient growth, these vortices can provide extra angular momentum transfer in weakly turbulent discs.

Additionally, we would like to mention that recent magnetohydrodynamics (MHD) simulations of turbulence in accretion discs, performed on intermediate and global spatial scales, show that a significant fraction of the accretion stress is contained in azimuthal modes with \( k \sim H \lesssim 1 \). For example, see the plots with autocorrelation functions of Maxwell and Reynolds stresses in Simon, Beckwith & Armitage (2012) and the plot with the toroidal power spectrum of the Maxwell stress, as well as the plot with the total accretion stress on small azimuthal scales relative to the same quantity on large azimuthal scales in Beckwith, Armitage & Simon (2011). Although it is unclear to what degree the large-scale magnetic field is important in this situation, the transiently growing large-scale vortices considered in this work (see the next section) can give an independent contribution to the non-local transfer of disc angular momentum. However, in order to confirm or discard this guess, it could be worth performing a study similar to that of Heinemann & Papaloizou (2009b) who considered particular spatial Fourier amplitudes of perturbations extracted from their (local) simulations of magnetorotational instability (MRI) turbulence. A similar procedure applied to the result of simulations on intermediate and global spatial scales, \( \lambda_y \gg H \), could reveal whether the large-scale vortices are responsible for additional accretion stress because of their transient growth as they swing from leading to trailing spirals. Another simplified approach to this issue is to regard the non-linear contribution of turbulence as an external noise imposed in the disc.

The latter has been successfully employed by Ioannou & Kakouris (2001) applied to incompressible global perturbations in a Keplerian disc. An extension of their study to perturbations with \( \lambda_y \gg H \) in thin discs would be worthwhile.

For these reasons, in the next section we investigate the large-scale vortex transient dynamics by employing the shearing sheet approximation as well as the global treatment of the optimization problem formulated previously (see Section 3.1). Let us emphasize that the optimization scheme in the global approach allows us to determine a unique radial profile of the optimal solution with a specified azimuthal wavenumber only (see Fig. 1). At the same time, in the shearing sheet model, the optimization scheme converges to a single azimuthal wavenumber (i.e. to the solution with specified \( k_y \)).

### 4.2 Extension to global spatial scale

In this section, whenever we mention \( R \) in the context of global dynamics, we mean its following analogue

\[
R = \frac{3/2}{\max_{\beta=1}\lambda_{m}}.
\]

which equals the local version of \( R \) in the vicinity of \( r = 1 \) in the case of Keplerian shear. In order to make comparisons with the shearing sheet model, we employ the homogeneous disc with uniform \( \Sigma \) and aspect ratio, \( \delta \) (see Section 3.1.2). This allows us to study solely the influence of the non-zero background vorticity gradient discarded in the shearing sheet as well as the corrections due to the cylindrical geometry. It is implied that local dynamics is considered in the vicinity of the inner boundary of the disc, \( r = 1 \), and the time in the local problem is measured in units of the inverse Keplerian frequency at \( r = 1 \).

First, in the top panel of Fig. 6, we show the slice of \( G(R) \) obtained in the shearing sheet model of the Keplerian flow for fixed optimization time, \( \tau = 10 \). Clearly, the maxima of curves of \( G(R) \) representing two different norms of perturbations lie on the dot-dash-dotted and dot-dashed curves in Fig. 5. For the norm (16), the maximum of \( G \) is attained close to \( R = 1 \), whereas for the norm (19) it is shifted to a smaller value of \( R \). As discussed in the previous section, the humps that we find in \( G(R) \) for both norms of perturbations emerge as a result of the excitation of density waves by vortices, which are the major components of the initial optimal SFH. Also, we find the breaks on each curve of \( G(R) \). For the norm (16), the break is located approximately at \( R \approx 0.7 \), whereas for the norm (19) it is shifted to \( R \approx 0.2 \). These breaks are caused by a jump from the local maximum of \( G(\beta) \), corresponding to the SFH swinging closely to the optimization instant, to another local maximum of \( G(\beta) \), corresponding to the SFH swinging earlier in time. This happens when the density wave excited by the latter SFH is so high that the latter maximum becomes larger than the former maximum (see Fig. 2). In the top panel of Fig. 6, we also plot an additional curve that reproduces our estimate of \( G \) given by equation (58), which represents solely the vortices. This curve distinctly shows the effect of the emergence of density waves for perturbations with \( \lambda_y \sim H \). Note that for \( R \gg 1 \), the analytical results for vortices yield a moderate underestimation of \( G \), despite the fact that the density wave excitation is already exponentially suppressed. This happens because the swing interval becomes too wide and the analytical solution for vortices (see equations 54–56) diverges from a precise numerical solution.

The bottom panel of Fig. 6 shows the dependences \( G(R) \) obtained in the global problem. Setting a particular value for the (global) azimuthal wavenumber, \( m = 5 \), and a particular polytropic...
index, \( n = 3/2 \), we alter \( R \) by varying the disc aspect ratio, \( \delta \). For example, in order to obtain the optimal solution for \( R = 10 \), we must set \( \delta = 0.06 \) in dynamical equations, etc. Conversely, \( R < 1 \) formally corresponds to a thick disc with \( \delta > 1 \). Basically, in the global configuration, the transient growth of perturbations with \( R < 1 \) is considerably suppressed in comparison with the local case. Qualitatively, the curves of \( G(R) \) in the global and local approaches resemble each other, except that for the second norm (equation 19), the hump produced by the excitation of density waves almost vanishes. Also, we see that the second norm always yields a smaller optimal growth in comparison with perturbations measured by the acoustic energy (see also Figs 5 and 7). Despite this, it is found that \( G \gg 1 \) in all possible variants. However, we find that in the opposite case, \( R \gg 1 \), the difference between global and local optimal growth is noticeably reduced. In particular, if perturbations are measured by the acoustic energy, the optimal growth evaluated at \( R = 0.1 \) is larger than that evaluated at \( R = 10 \) by roughly a factor of 10 in the shearing sheet model (see the solid curve in Fig. 6). The same difference for global perturbations is given by roughly a factor of 3.5. It is better to check this property of global transient growth by looking at how \( G \) depends on time for particular values of \( R \). We plot \( G(t) \) in Fig. 7, using both local and global optimization methods, again employed for two choices of norm of perturbations (top and bottom panels). Indeed, the curves obtained for large \( R = 12 \) settle much closer to each other than those obtained for small \( R = 0.12 \) in both panels of the figure. The optimal growth in the marginal case \( m = 1 \) is not plotted in Fig. 7, but, as we have checked, for small \( R = 0.12 \) \( G(t = 20) \approx 30 \), whereas for large \( R = 12 \), \( G(t = 20) \approx 20 \), which is only 1.5 times smaller. Actually, this implies that the global large-scale vortices, \( R \gg 1 \), that we consider in this work exhibit transient growth almost comparable to those we have known since the paper by Ioannou & Kakouris (2001), who presented \( G(t) \) for incompressible global perturbations in a Keplerian flow (see their fig. 1). However, as we have found previously, this is not the case in the local approach: the small-scale optimal SFH, \( R \ll 1 \), grows much more rapidly than the large-scale optimal SFH, \( R \gg 1 \) (see the results of Section 4.1 and the top panel of Fig. 6). Because we have mentioned the calculations by Ioannou & Kakouris (2001), we should bear in mind that we present the dynamics with quite short optimization time-spans \( \tau < 20 \), which correspond to less than \( \sim \)three Keplerian orbits at the inner boundary of the disc. For comparison, Ioannou & Kakouris (2001) measured time in Keplerian periods at \( r = 10r_\text{in} \), which corresponds to a rescaling of dimensionless time by a factor of \( \sim 180 \) if changing from their units to our units. Thus, we would actually see the comparable magnitudes of transient growth of our large-scale global vortices with \( R \gg 1 \) if the corresponding \( G(t) \) were plotted in their fig. 1.

Additionally, we test our global numerical scheme by checking that in the limit of large \( m \gg 1 \) (fixing the value of \( R \)) and the small size of computational domain (near \( r = 1 \)), it reproduces the solid curves in both panels of Fig. 7; that is, we check that the dashed curves approach the solid curves in the limit \( m \gg 1 \) (and fixed \( R \)). Furthermore, the optimization of global incompressible perturbations described in Appendix B is employed to carry out one more independent check of our basic numerical scheme. We make sure that for \( m = 5 \) the iterative loop based on \( A \) and \( A^\dagger \) given by...
equations (B5) and (B6), respectively, yields \( G(t) \), which virtually recovers the dashed curve in the top panel of Fig. 7. Also, note that the solid curves in Fig. 5 and in the top panel of Fig. 7 virtually recover the analytical \( G(t) \) given by equation (61).

Returning to the global large-scale perturbations, \( R \gg 1 \), it can be seen that for short time intervals of less than one Keplerian period at the inner boundary of the disc (\( t \ll 5 \)), the optimal growth of perturbations measured by the acoustic energy acquires a flat segment corresponding to \( G \approx 4 \) in both local and global problems (see the dotted and dash-dotted curves in the top panel of Fig. 7). In contrast, for \( R \ll 1, G \rightarrow 1 \) self-similarly while \( t \rightarrow 0 \) (see the dashed curve in the same panel). This discordance would become particularly distinctive if we plotted the marginal case of \( m = 1 \), with \( a_0 \rightarrow \infty \) standing for the Fourier global mode with the least azimuthal wavenumber in incompressible fluid, contrary to the marginal case of \( m = 0 \), with finite \( a_0 \ll \infty \) standing for compressible axisymmetric perturbation \( R \rightarrow \infty \). In the former situation, we would obtain \( G < 4 \) up to \( t \approx 5 \), whereas, in the latter situation, we would obtain \( G = 4 \) of all time intervals \( t \geq 1 \). The point is that in contrast to solenoidal planar perturbations, compressibility allows for the existence of one-dimensional radial motions in the perturbed flow. Moreover, these one-dimensional radial motions have optimal configurations, which are able to grow by a factor of \( \sim 4 \) in a Keplerian disc, as measured by their total acoustic energy. This factor is the squared ratio of epicyclic frequency for rigid rotation to epicyclic frequency of shear flow under consideration. In Appendix A, we interpret this fact and give a detailed description of the optimal axisymmetric perturbations. We argue that at short time intervals, the epicyclic motions in the rotating shear flow (see Appendix A) are responsible for the flat segment that emerges on the curves of \( G(t) \) as \( R \) goes to infinity. At the same time, when the compressible perturbations are optimized according to equation (19) (see the bottom panel in Fig. 7), the curves of optimal growth have a similar shape as those for perturbations in incompressible fluid. Thus, as \( R \) increases, \( G \rightarrow 1 \) for all time intervals. This is an expected result because the norm (19) has been introduced to exclude the non-modal behaviour of axisymmetric perturbations (see Section 3.1.1).

To conclude this section, we focus once again on the large-scale vortices, \( R \gg 1 \). Our incentive is to verify the validity of their analytics outlined in Section 3.2.2. Fig. 8 plots the growth factors, \( g(t) \), obtained with various approaches. We choose a particular value \( R = 12 \) for the shearing sheet model. We set \( m = 5 \) for global calculations, which means that we take \( \delta = 0.05 \) for the background because the same value of the global analogue of \( R \) is implied. The optimization time-span is \( \tau = 20 \). First, the optimal SFH is represented by the solid curve. This is obtained using the optimization procedure in the shearing sheet approximation for \( \beta \approx -30 \), which corresponds to \( \max \{ G(\beta) \} \) for fixed \( R \). Because \( |\beta| > R \), the iterative method described in the appendix of B05 has a good convergence at the moment \( t = 0 \), and we employ it to determine the initial conditions for the pure vortex with the same potential vorticity as for our optimal SFH (see also the description in Section 4.1 and Fig. 3). Using these initial conditions, we advance the perturbations numerically and obtain the dashed curve in Fig. 8. Clearly, we obtain excellent agreement with the solid curve because for \( \beta \ll -1 \) the optimal solution is almost a vortex. Further, in accordance with the theory of HP, we find that a weak signature of density wave excitation appears well after the vortex decays, giving its energy back to the flow (see the intensity of the excited density wave with \( R = 4 \), plotted in the top panel of Fig. 4). Then, we plot the analytical solution of HP employing the norm (47) and constructing the result from two different curves. The first corresponds to the analytical solution for the vortex (equation 31 of HP or equations 54–56 in this paper), obtained for known \( k_i \) and \( k_y \). The second corresponds to the sum of the analytical solution for the vortex and the analytical solution for the excited density wave (equations 52–57 of HP). As can be seen in the plot, this analytical (composite) curve recovers the numerical solutions well everywhere except for the zone around the instant of the swing. This is expected because, as discussed in Section 3.2.1, the existence of vortices becomes ill-defined inside the swing interval given by equation (50). In the particular case displayed in Fig. 8, \( t_1 = 12 \) and \( t_2 = 28 \). However, the actual size of the interval where analytics diverge with the precise solution luckily appears to be at least two times less than expected from equation (50). This is why the underestimation of \( G \) from analytics is not dramatic, in spite of the fact that the condition (51) is not fulfilled and \( R \sim \beta/2 \). Another encouraging thing is that the actual growth of the vortex has been found to be larger than expected from analytics (see also Fig. 6). Moreover, the approximate expression for \( G \) given by equation (60), which is valid in the limit of large \( R \gg 1 \) and long \( t \gg 1 \) (see Section 3.2.2), yields the value of \( G \approx 57 \) in our particular case, which is intermediate between the analytical and numerical values. Finally, we plot the result of global optimization, which exhibits transient growth close to the analytical estimate.\(^3\)

As discussed at the end of Section 4.1, in a disc with small viscosity, \( \alpha < \delta \), the vortices \( \lambda_\alpha \gg H \) have the ability to exhibit transient growth on all scales up to the highest, \( \lambda_\alpha \sim H/\delta \). Additionally to the analytical estimations of \( G_{\text{max}} \) given in Section 3.2.2, we check its magnitude using the numerical optimization in the shearing sheet model. We determine \( G \) for the time-span (62). In particular, we find that the maximum value of \( G_{\text{max}} \) in the whole range of \( R > 1 \) attains \( \sim 250 \) and \( \sim 1000 \) for \( \alpha = 10^{-3} \) and \( \alpha = 10^{-4} \), respectively.

\(^3\) Also note that inertial-acoustic modes would have almost constant \( g \approx 1 \) if plotted here, because their increments (decrements) are too small in a thin Keplerian disc (e.g. Goldreich & Narayan 1985; Kato 1987).
4.2.1 Particular case of formally inviscid Shakura–Sunyaev accretion disc

Finally, we take the Keplerian disc model with the structure specified by equations (23) and (24) and determine the optimal large-scale vortex employing both norms (equations 16 and 19). We set all parameters to the same values as in Fig. 8. We find that the optimal growth becomes approximately two times smaller than for the homogeneous disc model. The instant profiles of $\Re[\delta h]$ for optimal solutions are plotted in Fig. 9. Also, we show the slices of $\Re[\delta h]$ for perturbation triggered at $t = 0$ by the profile (26). Interested readers are invited to watch the corresponding movie at the web site given in the caption to Fig. 9. Comparing the panels in Fig. 9, we can see that the incidental perturbation demonstrates a typical wave-like evolution splitting into two waves running in the opposite directions (top panels). Although it can be checked that this perturbation has non-zero potential vorticity (i.e. it is a mixture of vortex and density wave), there is no signature of transient growth measured by any of two norms used in this work. At the same time, the optimized shape of $\Re[\delta h]$ first exhibits the correlated transient enhancement in the vicinity of the disc’s inner boundary. According to the results of our study in the local framework, the initial optimal perturbation is nearly identical to a pure vortex. Moreover, at $t = 0$, it is a leading spiral, which is shrinking because of the shear during the phase of transient growth. Additionally, because we are concerned with large-scale perturbations with an azimuthal wavelength much larger than the disc thickness, the excitation of the density wave at the instant of swing of the spiral is quite insignificant. Note that in the case of scaling by the total acoustic energy (middle panels), the profiles of $\Re[\delta h]$ show small-scale variations at the instant of swing (see the solid curve in the middle-right panel of Fig. 9). This is a distinctive feature of modal solutions obtained in the WKBJ approximation in a thin disc (e.g. Kato 1987). Clearly, the radial size of these variations is dictated by the disc thickness. We presume that they emerge because of the contribution of oscillatory motions with frequencies of the order of the Keplerian frequency, which is a hint of epicyclic deviations included in the optimization with the norm given by equation (16). Indeed, the change to norm (19) eliminates these small-scale variations producing the smooth instant profile of the optimal perturbation at the instant of swing (see the solid curve in the bottom-right panel of Fig. 9). Both optimal instant shapes strongly differ from modal solutions as well as from the evolution of randomly taken perturbations (top panels).

5 CONCLUSIONS

In this work, we have studied the transient dynamics of linear perturbations in thin Keplerian discs. We show that substantial non-modal growth can exist at all spatial scales including those when the azimuthal wavelength of perturbations is much larger than the disc thickness. Moreover, this remains true if the azimuthal wavelength becomes comparable to the radial scale in the disc. In the latter situation, we have been dealing with global perturbations. The most reasonable way to illustrate the transient activity is to solve the optimization problem, in this way finding the configuration of the initial perturbations that determines the largest possible non-stationary response of the disc. Such optimals have been calculated for compressible perturbations in a Keplerian thin disc without viscosity.

First, this is done in the shearing sheet model, assuming that the perturbation azimuthal wavelength is small compared to the disc radial scale but that it can be in any ratio with the disc thickness, which is fixed by a parameter $R$ (see Section 3.2.1 and equation 49). Before presenting the numerical results for optimal shearing harmonics, we consider analytically two opposite cases of small-scale local perturbations, $R \ll 1$, and large-scale local perturbations, $R \gg 1$. We derive analytical expressions for the magnitude of transient growth of vortices in both situations; see equation (B10) (or simplified equation 61) and equation (59), respectively. It is important to note that in the latter case the magnitude of transient growth is $\propto 4\Omega^2/\kappa^2$, which suggests that vortices with $R \gg 1$ can exhibit much stronger amplification in a shear flow that approaches a uniform specific angular momentum distribution (e.g. in the inner parts of relativistic accretion discs around black holes). Besides, we formulate a condition of separability of vortices and density waves in compressible accretion flow, which, in particular, leads to a condition of the validity of equation (59). This requires that the leading spiral corresponding to a vortex must be initially tightly wound (see equation 51). This restriction is particularly strong.
for large-scale vortices, $R \gg 1$, but luckily the comparison of analytical and numerical growth factors of vortices indicates that actually a reasonable agreement holds up to $R \sim |\beta|/2$ (see the comments about Fig. 8 in Section 4.2). We also assess the influence of non-zero viscosity on the transient growth of these large-scale vortices and find that its absolute maximum is given approximately by equation (63). This leads us to estimate that in a weakly viscous disc with $\alpha < \delta$, the amplification of vortices exists for all possible azimuthal wavelengths up to the largest, $\sim H/\beta$ (see the end of Section 3.2.2).

While proceeding to the optimization in the shearing sheet model, we find that generally the optimal shearing harmonic (SFH) is a mixture of a vortex and a density wave (see Fig. 3). However, the contribution of the density wave sharply decreases with transition to negative $\beta$ (i.e. to initially leading spirals). Because for reasonable optimization time-spans $\tau > 1$, the largest optimal growth, $G(\beta)$, is produced by SFH with $\beta < -1$. We find that performing the optimization for a particular $R$ we always obtain the initial shape of the optimal SFH nearly identical to the vortex having the same potential vorticity. However, for $\beta > 0$, we also find $G(\beta) > 1$, which is explained by the non-modal growth of (zero potential vorticity) density waves (see also the fourth pair of curves in Fig. 3). Further, $G(R)$ by itself attains a maximum value at $R \approx 1$, which is a consequence of the density wave excitation by vortices. The excitation of density waves becomes most prominent for $R \approx 1$ (see the analytical investigation by HP). The maximum of $G(R)$ is provided by a vortex that swings from a leading to trailing spiral approximately 1.5 times earlier than the optimization time. This is explained by the fact that the amplitude and growth rate of the excited density wave become sufficiently high to exceed the magnitude of another vortex that swings at the optimization time. The enhancement of optimal growth as a result of this effect is quite substantial and increases with time (see Fig. 5). Additionally, we demonstrate that optimal SFH is indistinguishable from the analytical solution obtained by HP taken with the same potential vorticity (see Fig. 4).

Determining numerically the optimal large-scale SFH, $R \gg 1$, we find an approximate agreement with our analytical estimates for large-scale vortices; see equations (59) and (60), Fig. 6 (top panel) and Fig. 8, as well as our discussion of these. As already mentioned, this agreement holds despite the fact that the swing interval is comparable to the duration of the growth phase for the particular values of the parameters used in the calculations. Locally, for $R \gg 1$, the optimal growth (approximately, $G \propto R^{-2}$) is highly suppressed in comparison with its incompressible ($R \ll 1$) magnitude (see the top panel of Fig. 6). However, the situation changes when we extend our study to a global spatial scale, taking into account the background vorticity gradient and the disc’s cylindrical geometry. We find that while $m \rightarrow 1$ the optimal growth falls much more sharply for $R \ll 1$ rather than for $R \gg 1$ (see the bottom panel of Fig. 6 and Fig. 7). For example, in the particular case $m = 1$, $\tau = 20$ and $R = 12$, the optimal growth is just 1.5 times smaller than its counterpart at $R \ll 1$ (i.e. formally in our disc with $\delta > 1$). Actually, the last limit corresponds to global incompressible dynamics considered by Ioannou & Kakouris (2001); see the inviscid dependence in their fig. 1. Thus, contrary to what we have in the local problem, the global vortices with the lowest azimuthal wavenumbers, $m = 1, 2, \ldots$, at least in the range $R \lesssim 10$, exhibit transient growth comparable to what has been found previously in the simplified model of incompressible Keplerian flow. Thus, any kind of persistent source of the potential vorticity on the scales above the disc thickness can lead to the formation and growth of global vortices, providing an enhanced angular momentum transfer to the disc’s periphery. At the same time, any kind of weak pre-existing turbulence can become such a natural sower of the potential global vorticity perturbations in the disc. With regards to the incompressible Keplerian flow, this was shown by Ioannou & Kakouris (2001) who introduced the action of pre-existing turbulence as an external stochastic forcing in hydrodynamical equations. Importantly, they found that the coherent structures that emerge in the steady state of the perturbed disc are similar to the structures of global optimal perturbations sliced at the instant of swing from leading to trailing spirals. More precisely, the instant of swing of these optimal perturbations corresponds to the time-span at which the curve of optimal growth attains its maximum, provided that the disc is viscous. Because in this work we show that the transient growth of global vortices preserves its strength in thin compressible discs, it is tempting to suggest that they are responsible for an additional accretion stress on scales well above the disc thickness. As we discuss at the end of Section 4.1, the recent results of global MHD simulations of disc turbulence probably provide some evidence for this by detecting the non-local accretion stress.

Finally, let us highlight the results of a more methodical and technical nature. First, we find that non-modal dynamics in the perturbed flow measured by the total acoustic energy of perturbations leads to a distinctive feature in the curve of optimal growth as a function of time. Particularly, at time intervals shorter than one rotational period at the inner disc boundary, there is a quick rise of optimal growth up to a flat segment at the magnitude which approximately equals the squared ratio of epicyclic frequency for rigid rotation to epicyclic frequency for the given shear. This feature is independent of the value of azimuthal wavenumber and we suggest that it is related to the existence of epicyclic motions in the disc. In order to support this conclusion, in Appendix A, we consider the non-modal growth of axisymmetric compressible perturbations. We carry out an analytical treatment and find that the optimals are the standing density waves with the optimal growth given by equation (A6). The standing wave is the configuration that naturally leads to oscillations of the acoustic energy integrated over its wavelength. Then, because in the shear flow perturbations of centrifugal and centripetal forces acting on a displaced fluid particle are not balanced, their non-zero difference serves as the source of the kinetic energy for the fluid particle. Thus, we obtain an additional, inertial mechanism of non-modal growth in differentially rotating flows, which dominates in the limit of small sound speed and long wavelength and yields optimal growth, given by equation (A7). Note that this is not the case for non-axisymmetric perturbations of either type because their dynamics are mostly dictated by the perturbed pressure gradients and are subject to the classical lift-up effect. However, it is important to note that these findings for the optimal axisymmetric perturbations are true as long as the acoustic energy is employed to measure the compressible dynamics. It turns out that we are able to choose the norm of perturbations in such a way that the optimal growth identically equals unity in the axisymmetric case (see equation 19). This is an energy-like quantity that becomes equivalent to the canonical energy for axisymmetric perturbations. It allows us to exclude the degenerate limit of the non-modal growth associated with a general oscillatory solution for epicyclic motions in the disc and to keep only transient growth. Consequently, the optimal growth $G \rightarrow 1$ as $R \rightarrow \infty$ (or $m = 0$) and the slices of the particular optimal solutions acquire a smooth shape (see Fig. 9 for a comparison of optimal perturbations measured by two alternative norms).

Second, we describe a useful optimization technique that has not been applied previously to the theory of astrophysical discs. It allows us to study the transient effects directly by solving the Cauchy
problem for perturbations in the framework of an iterative scheme when the basic set of equations is advanced forward in time, whereas the adjoint set of equations is advanced backward in time. This is done without referring to modal solutions, which can be quite an involved task, especially in complex flows. For example, Zhuravlev & Shakura (2009) and Razdoburdin & Zhuravlev (2012) treated the optimals in the form of finite linear combinations of neutral acoustic modes in a quasi-Keplerian torus. The optimal perturbation obtained corresponds to a wave packet localized initially in the vicinity of the outer boundary of the torus and moving towards the inner boundary. At the moment of the reflection from the inner boundary, the total acoustic energy attains its maximum. The constituent modes are phased in such a way that the shape of the wave packet corresponds to the highest possible energy profit at this maximum for the given time interval. However, these results are not robust, in the sense that to make a decisive conclusion about the transient dynamics we have to cover all possible combinations of modes, including those with corotational and Lindblad resonances inside the flow. Note that neutral and damping modes with corotation inside the flow must also be considered, which is an involved task because it is necessary to extend the calculations in the complex plane according to the Lin rule. This is a complex problem, especially if one would like to investigate the influence of stratification and even baroclinity of the flow on its capability for transient behaviour. In contrast, in this work we have avoided all such issues because we have not been obliged to obtain modal solutions in order to study the optimal growth in the disc.

Using arguments from operator theory, we show that the variational technique can be applied to non-stationary accretion flows when we have to solve differential equations with time-dependent coefficients. In this case, the spectral problem cannot be formulated at all. However, the transient growth and optimal perturbations can exist in such a flow and could be determined using the corresponding iterative procedure. Also note that the variational technique can also be applied to non-linear problems. Finally, let us point out that it can be used to find the whole set of singular vectors. In order to do this, we can apply the same procedure in the functional subspace normal to the one spanned by previously obtained singular vectors.

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APPENDIX A: PARTICULAR CASE OF AXISYMMETRIC PERTURBATIONS MEASURED BY THEIR ACoustIC ENERGY

In comments to Fig. 7, we discuss the fact that $G = 4$ for axisymmetric perturbations (i.e. with $m = 0$), measured by their total acoustic energy, provided that the shear is Keplerian. The question that should be addressed is the reason for non-modal growth in this situation when the classical lift-up effect does not work. In order to discuss this issue in detail, we would like to carry out an analytical investigation, which is not difficult in the shearing sheet
approximation. The general solution to equations (37)–(42) with \( k = 0 \) is
\[
\tilde{u}_t = C_4 e^{i\omega t} + C_5 e^{-i\omega t}
\]
\[
\tilde{u}_s = \frac{ik^2}{2\sigma^2} (C_4 e^{i\omega t} - C_5 e^{-i\omega t}) + \frac{k^2}{2} C_3
\]
\[
\tilde{W} = -\frac{k^2}{\sigma} (C_4 e^{i\omega t} - C_5 e^{-i\omega t}) - i C_3
\] (A1)
\[
\tilde{u}_s = K_4 e^{i\omega t} + K_5 e^{-i\omega t}
\]
\[
\tilde{u}_s = \frac{2i}{\sigma} (K_4 e^{i\omega t} - K_5 e^{-i\omega t}) + \frac{k^2}{2} K_3
\]
\[
\tilde{W} = -\frac{k^2}{\sigma} (K_4 e^{i\omega t} - K_5 e^{-i\omega t}) - \frac{i k^2}{4 \sigma^2} K_3
\] (A2)
where \((C_1, C_2, C_3)\) and \((K_1, K_2, K_3)\) are complex constants to be determined by the iterative procedure and the dimensionless \( \sigma = k^2/\Omega_0^2 + k_\star^2 \).

The analytical expression for the norm of the perturbations state vector, \( |q| \), is derived using equation (47). It has the form
\[
|q|^2 = (|C_1|^2 + |C_2|^2) \left( 1 + \frac{k^2 s^{-1}/\Omega_0^2 + k_\star^2}{\sigma^2} \right)
\]
\[
+ |C_1|^2 \left( 1 + \frac{k_\star^2}{4} \right) + 2 \Re \left[ C_1^* C_2 e^{2i\sigma t} \right]
\]
\[
\times \left( 1 - \frac{k^2 s^{-1}/\Omega_0^2 + k_\star^2}{\sigma^2} \right) + \frac{k^2}{\sigma} (s^{-1} - 1)
\]
\[
\times \Re \left[ i C_3^* (C_4 e^{i\omega t} - C_5 e^{-i\omega t}) \right],
\] (A3)
where the asterisk denotes complex conjugation and \( s \equiv 4\Omega_0^2 k^2 \) is a parameter that characterizes the shear magnitude (e.g. for rigid rotation \( s = 1 \) and for Keplerian rotation \( s = 4 \). From equation (A3), we conclude that whenever \( s \neq 1 \), \( |q| \) becomes a time-dependent quantity. Now, to obtain optimal perturbations corresponding to some moment \( t = \tau \), we can proceed in two different ways.

(i) Let us construct an iterative scheme for the coefficients \( C_1, C_2, C_3 \). Specifically, matching the state and the adjoint vectors at \( t = \tau \), \( \tilde{q}_\star(\tau) = \tilde{q}_s(\tau) \), with the help of equations (A1) and (A2) we express \((K_1, K_2, K_3)p\) through \((C_1, C_2, C_3)p\), where \( p \) is the number of iterations. Then, matching \( q_{p+1}(0) = \tilde{q}_s(0) \) and dividing \( q_{p+1}(0) \) by its own norm according to equation (A3), we finally obtain the recursive relations for coefficients \((C_1, C_2, C_3)p\) expressed in terms of \((C_1, C_2, C_3)p\) known from the previous step. Starting from an arbitrary set of \((C_1, C_2, C_3)\), we obtain a converging sequence of coefficients, which gives us an optimal perturbation corresponding to \( t = \tau \) and the value of the optimal growth, \( G(\tau) \).

(ii) Let us determine the maximum of \( g(t) \equiv ||q(t)||^2/||q(0)||^2 \), considering it as a function of coefficients \( C_1, C_2, C_3 \). This gives us an optimal growth curve. Below, we describe the details of the analytical derivation of \( G(\tau) \) for small shear \((s \approx 1) \) and discuss the physical reasons for the non-modal growth of axisymmetric perturbations.

### A1 Optimal growth of axisymmetric perturbations in a rotating flow with a small shear

Let us suppose that \( s = 1 + \epsilon > 1 \), where \( \epsilon \ll 1 \). Also, we notice that if we regards \((C_1, C_2, C_3)\) as vectors with components \( C_i = (X_i, Y_i) \) in some Cartesian reference frame, then \( C_1 C_2^* = |C_1|\overline{|C_2|^2}(\cos \psi_1 - i \sin \psi_1) \) and \( C_2 C_3^* = |C_2|\overline{|C_3|^2}(\cos \psi_2 - i \sin \psi_2) \), where \( \psi_{1,2} \) are the angles between vectors \((C_1, C_2), (C_2, C_3) \) and \((C_1, C_3) \), respectively. Note that, by definition, \( \psi_3 = \psi_1 + \psi_2 \).

These equalities allow us to write the growth factor of some perturbation, \( g(t, k) \), expanding equation (A3) over small \( \epsilon \) and retaining the linear term only,
\[
(g - 1)e^{-1} = \frac{2}{2(A_1^2 + 1) + A_2^2(1 + k^2)} \left[ A_1 + \frac{|k|A_2}{(1 + k^2)^{1/2}} \right] e^{i\omega t} \sin(\sigma t - \psi_1 - \psi_2) + \sin(\psi_1 + \psi_2) + \sin(\sigma t + \psi_2) - \sin(\psi_1),
\] (A4)
where \( k \equiv k_\star/2 \) and \( A_1 \equiv |C_1|/|C_2|, A_2 \equiv |C_1|/|C_2| \).

We determine the maximum of \( g \), provided that \( t \) and \( k_\star \) are fixed. First, let us set \( A_1 = 1 \). Then, it is straightforward to obtain the values of \( \psi_{1,2} \) and \( A_2 \) corresponding to the maximum of equation (A4) as a function of \( \psi_1, \psi_2 \) and \( A_2 \). We have
\[
\psi_1 = -2\psi_2,
\]
\[
\sin(\psi_2 + \sigma t/2) = \frac{A_2 |k|(1 + k^2)^{1/2} - [A_2^2 k^2(1 + k^2) + 32 \cos^2(\sigma t/2)]^{1/2}}{8 \cos(\sigma t/2)}.
\] (A5)

Finally, it is not difficult to check that equations (A5) turn into the identities, the conditions of maximum of \( g \) (from equation A4) over the varying \( A_1 \). Thus, we make sure that \( A_1 = 1 \) along with equations (A5) correspond to the optimal growth case.

Substituting equations (A5) along with \( A_1 = 1 \) into equation (A4) yields
\[
(g - 1)e^{-1} = 2 |\sin(\sigma t/2)| \frac{|k^2 + \cos^2(\sigma t/2)|^{1/2}}{1 + k^2}.
\] (A6)

Finally, we note that setting \( k = 0 \) (which corresponds to \( k_\star \rightarrow 0) \), equation (A6) turns into a simple expression for \( G_0 \), explicitly
\[
(G - 1)e^{-1} = |\sin(\sigma t)|.
\] (A7)

Thus, \( G \) oscillates with both time and radial wavenumber.

### A2 What causes non-modal growth of axisymmetric perturbations?

We plot the results of calculations in Fig. A1. For strictly axisymmetric perturbations, we show the curves of optimal growth obtained by the iterative scheme for coefficients \((C_1, C_2, C_3) \) and based on the analytical solutions (A1) and (A2). As can be seen, the analytical expression (A6) gives the profile of \( G(k_\star) \), which is in good agreement with the iterative scheme for small shear (see the left-hand panel of Fig. 6). Also, the analytical expression (A7) describes well the behaviour of \( G(\tau) \) for the case of long-wavelength perturbations, \( k_\star \rightarrow 0 \) (see the right-hand panel of Fig. 6). Note that, in the latter case, the shear is not small and is set to its Keplerian value (i.e. \( s = 4 \)). Clearly, the optimal growth always attains the value of \( s \). It is symmetric with respect to change.
$k_i \rightarrow -k_i$ and gradually tends to 1 while $|k| \rightarrow \infty$, which corresponds to ordinary sound waves in the absence of shear. To illustrate the transition to non-axisymmetric optimal perturbations, we add curves corresponding to azimuthal wavenumbers $k_i = 0.125$ and 0.25, that is, to $R = 12$ and $R = 6$, respectively.\textsuperscript{4} Non-modal growth is no longer limited by the value of $s$ and becomes larger as $R$ decreases. The shape of $G(\tau)$ transforms to its familiar form (see Fig. 2, dot-dash-dashed curve). Specifically, $G(k_i)$ becomes asymmetric with respect to change $k_i \rightarrow -k_i$, becoming larger in the domain of negative $k_i$, because the non-modal growth appears in this case as a result of the leading spirals being shrunk by the shear.

The variant of the iterative scheme used for axisymmetric perturbations, as well as the results of the analytical consideration in the previous section, show that optimal perturbations are standing axisymmetric inertial-acoustic waves. Indeed, we find that in the optimized solution, $|C_1| = |C_2|$ (i.e. the optimal perturbation is a combination of two monochromatic waves running in opposite directions).

The limit of $k_i \rightarrow 0$ gives an especially simple version of the optimal perturbations. These are nothing but cophased epicyclic perturbations, as well as the results of the analytical consideration in the previous section, show that optimal perturbations are standing axisymmetric inertial-acoustic waves. Indeed, we find that in the optimized solution, $|C_1| = |C_2|$ (i.e. the optimal perturbation is a combination of two monochromatic waves running in opposite directions).

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\begin{equation}
\frac{\partial u_\xi}{\partial t} = 2\Omega\xi, \quad (A8)
\end{equation}

\begin{equation}
\frac{\partial u_\eta}{\partial t} = -(2 - q)\Omega_1 \xi, \quad (A9)
\end{equation}

Let us change to the Lagrangian approach and rewrite equations (A8) and (A9) in terms of the Lagrangian displacement, $\xi$, associated with the particular fluid particle. In this simple case, the Lagrangian velocity perturbations are $u_\xi = u_\xi$ and $u_\eta = u_\eta - q\Omega_1 \xi$, and $\xi = \mathbf{w}$ (e.g. Lynden-Bell & Ostriker 1967). The Lagrangian time derivative is denoted by the dot. We obtain

\begin{equation}
\dot{\xi} = 2\Omega_1(\dot{\xi} + q\Omega_1 \xi), \quad (A10)
\end{equation}

\begin{equation}
\dot{\xi} = -2\Omega_1 \xi, \quad (A11)
\end{equation}

From equations (A10) and (A11), we see that along with the Coriolis force acting on the particle, there is an additional conservative force, $f$, with a non-zero radial component, $f_\xi = 2q\Omega_1 \xi$, which is proportional to the shear. The presence of $f_\xi$ is explained as follows. In the absence of perturbations of the pressure gradient, the dynamics of the fluid particle is determined by the difference between a perturbation of the centrifugal force, $\Omega_1^2 \xi$, and a perturbation of the centripetal force, $2q\Omega_1^2 \xi + \Omega_2 \xi$, which vanishes only for rigid rotation.

Thus, there is an ‘energy’ integral of motion

\begin{equation}
E = u^2/2 - q\Omega_1^2 \xi^2 = \text{const}. \quad (A12)
\end{equation}

Equation (A12) indicates that the kinetic energy of the fluid particle, $E_\xi = u^2/2$, changes because of the work done by $f$. This work is positive when the particle moves away from its unperturbed position. So, $f$ is a destabilizing force. In a limiting case, $q = 2$, the frequency of radial oscillations (i.e. the epicyclic frequency $\kappa$) vanishes and the motion becomes marginally stable. At the same time, $E_\xi$ can increase infinitely, together with $\xi_\xi$. Thus, $E_\xi$ remains constant only in the absence of shear because the Coriolis force does not do work.

Goldreich & Tremaine (1980), for example, have remarked on the energy integral (A12), see their equation (35), where they represented the epicyclic motion problem from the mechanical point of view. The epicyclic trajectory of the fluid particle is an ellipse elongated along the radial direction so when $\xi_\xi = 0$, the azimuthal component of the velocity, $\xi_\eta$, vanishes. This allows us to find $\xi_\eta = -\xi_\xi/(2\Omega_1)$ integrating equation (A11). Using equation (A12), we find that the ratio of maximum and minimum values of the kinetic energy during one orbit equals to $x = 4\Omega_1^2/k^2$. This fact explains the non-modal growth of axisymmetric perturbations with $k \rightarrow 0$ (see the right-hand panel of Fig. A1). When $k \neq 0$, the standing epicyclic waves are modified by a sonic component (see the left-hand panel of Fig. A1). However, when $k_i \neq 0$, the standing epicyclic waves become shrunk by the shear, which causes
an additional enhancement of the kinetic energy of perturbations as a result of the lift-up mechanism (see both panels in Fig. A1).

**APPENDIX B: INCOMPRESSIBLE PERTURBATIONS**

**B1 Optimization on a global spatial scale**

For planar motions, it is convenient to rewrite the dynamical equations for perturbations of vorticity, $\delta \omega = \nabla \times \delta \psi$, and stream function, $\delta \phi$, defined through $\delta \psi = \nabla \times \delta \phi$. The stream function fully describes the velocity field because the latter is solenoidal for incompressible dynamics. Because both $\delta \omega$ and $\delta \psi$ have only one non-zero component along the $z$-axis, we omit the subscript $z$ below.

Taking the curl of equation (13), we obtain for the azimuthal Fourier harmonic of perturbations,

$$\frac{\partial \delta \omega}{\partial t} = -i m \Omega \delta \omega - \frac{m}{r} \frac{d}{dr} \left( k^2 \nabla^2 \right) \delta \psi$$

(B1)

and

$$\delta \omega = \frac{m^2}{r^2} \psi - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \delta \psi}{\partial r} \right).$$

(B2)

Clearly, equation (B1) is derived by constructing a combination

$$-\frac{im}{r} \times (3) + \frac{1}{r} \frac{\partial}{\partial r} [r \times (3)_z],$$

where (3), and (3) denote the first and second equations in the set (3) taken with $A$ in the form of equation (15). Note that by substituting the modal partial solutions $\propto \exp(-i \rho t)$ into the set (B1) and (B2), we obtain the well-known Rayleigh equation for $\delta \psi$ that poses the two-dimensional spectral problem in the rotating shear flow, provided appropriate boundary conditions are imposed. The Rayleigh equation yields an inflexion point criterion for spectral stability (e.g. Landau & Lifshitz 1987).

The norm of the state vector is given by its total kinetic energy,

$$\|q\|^2 = \pi \int (|\delta \psi|^2 + |\delta \omega|^2) dr,$$

(B3)

and here we assume uniform surface density.

To derive the adjoint equation for adjoint vorticity, $\delta \tilde{\omega}$, we now construct exactly the same combination of the first and second equations in the system (7) taken with $A^*$ in the form of equation (20).

We obtain the following result:

$$\frac{\partial \delta \tilde{\omega}}{\partial t} = -i m \Omega \delta \tilde{\omega} + 2im \frac{d}{dr} \frac{\partial}{\partial r} \left( \frac{\delta \psi}{r} \right).$$

(B4)

Again, equation (B4) must be solved together with the relation (B2) where the adjoint quantities, $\tilde{\omega}$ and $\tilde{\psi}$, must be substituted.

Thus, the state vector $q = [\delta \omega \delta \psi]$ contains solely the Eulerian perturbation of vorticity and the operator $A$ in equation (3) can be expressed in the form (see equation 9 of Ioannou & Kakouris 2001)

$$-i m \Omega - \frac{im}{r} \frac{d}{dr} \left( \frac{k^2}{252} \right) \nabla^2 \psi.$$  

(B5)

The adjoint operator $A^*$ is

$$im \Omega - 2im \frac{d}{dr} \frac{\partial}{\partial r} \left( \frac{\nabla^2 \psi}{r} \right).$$

(B6)

where the differential operator $(\nabla^2)^{-1}$ is the inverse of the $\nabla^2$ operator, which specifies the direct relation between the vorticity and the stream function perturbations in equation (B2). The inverse relation given by $(\nabla^2)^{-1}$ is well defined, provided that the appropriate boundary conditions are imposed.

For incompressible perturbations, when $A$ and $A^*$ are given by equations (B5) and (B6), respectively, we choose another numerical scheme because the type of differential equation changes. We use two meshes shifted for $\Delta t/2$ relative to each other along the time axis. Then, to evaluate the vorticity at each time slice, we invert the set of difference equations connecting vorticity and stream function according to equation (B2). To close this set of difference equations, we require perturbation of radial velocity to vanish at the boundaries.

**B2 Optimization in a shearing sheet model**

Locally, the general initial value problem for incompressible perturbations has an exact analytical solution. This was first shown by Lominadze et al. (1988), who also examined two-dimensional perturbations in the disc plane changing to the comoving Cartesian coordinates and considering a particular SFH.

In the limit $a_\star \rightarrow \infty$ (i.e. $k_\star, k_\parallel \gg 1$), the set (37)–(39) gives that each SFH of the radial velocity perturbation obeys the following ODE (we omit the prime after the dimensionless time)

$$\frac{d \tilde{u}_r}{dr} + 2q \frac{\beta + q \tau}{(\beta + q \tau)^2 + 1} \tilde{u}_s = 0.$$  

(B7)

In this way, we obtain a simple solution of equation (B7)

$$\tilde{u}_s(t) = \tilde{u}_s(0) \frac{\beta^2 + 1}{(\beta + q \tau)^2 + 1},$$  

(B8)

which, of course, can also be reproduced from equation (54) in the limit $k_\parallel \gg 1$.

An incompressible relation between $\tilde{u}_s$ and $\tilde{u}_s$ yields the SFH energy density evolution that has exactly the same form as in equation (B8):

$$g(t) = \tilde{u}_s^2(t) = \frac{\tilde{u}_s(0) + \tilde{u}_s^2(0)}{\beta^2 + 1} = \frac{\beta^2 + 1}{(\beta + q \tau)^2 + 1}.$$  

(B9)

Now, we treat equation (B9) as a function of $\beta$. For a fixed value of $t = \tau$, it attains maximum at $\beta = 1/2 \{ - q \tau - [q \tau^2 + 4]^{1/2} \}$ and the maximum value of equation (B9) is the optimal growth defined as maximized $\tilde{u}$, which, in the present idealized case, is

$$G(\tau) = \frac{(q \tau^2)^2 + q \tau^2 [q \tau]^{2} + 4}{(q \tau^2)^2 - q \tau^2 [q \tau]^{2} + 4}.$$  

(B10)

For large times, $q \tau \gg 1$, equation (B10) gives $G \approx (q \tau^2)^2$, which recovers equation (61).

Let us employ an iterative procedure for local vortex perturbations. We have to find a solution of the adjoint set of equations (40)–(42) taken in the limit $a_\star \rightarrow \infty$ (i.e. $k_\star, k_\parallel \gg 1$).

It is not difficult to show that this leads to a trivial equation for the SFH of $\tilde{u}_s$

$$\frac{d \tilde{u}_s}{dr} = 0.$$  

(B11)

Now, omitting the following steps of the iterative procedure, equations (8) and (9), which are formal for the linear problem, we obtain a factor

$$\left[ \frac{\beta^2 + 1}{(\beta + q \tau)^2 + 1} \right]^p,$$  

(B12)

in front of an arbitrary initial profile of $\tilde{u}_s$, $\tilde{u}_s(\beta, t = 0)$, which is used to launch the procedure. Here, $p$ is a natural number that equals...
the number of iterations. With a proper renormalization applied when $p \to \infty$, the factor (B12) discards all SFHs from $\hat{u}^\omega(\beta, t = 0)$ except the optimal one that corresponds to a maximum of (B12) as a function of $\beta$. With this obtained, we evidently mimic the optimal growth profile (B10).

It is instructive to come to the same conclusion by considering the local limit of the sets (B1), (B2) and (B4), (B1) or, equivalently, taking the curl of equations (28) and (29) and the curl of equations (31) and (32) (implying the incompressible limit again) for state and adjoint quantities, respectively. Then, changing to the shearing coordinates, we obtain for $\hat{\omega}$

$$\frac{d\hat{\omega}}{dr} = 0,$$

(B13)

whereas $\hat{\bar{\omega}}$ obeys the following equation

$$\frac{d\hat{\bar{\omega}}}{dr} - 2q \frac{\beta + qt}{1 + (\beta + qt)^2} \hat{\omega} = 0.$$

(B14)

Equation (B13) represents the law of vorticity conservation, which is expected for a perfect incompressible fluid. However, we conclude that the adjoint vorticity does not conserve, as far as there is non-zero shear in the flow. Comparing equation (B14) with equation (B7), we find that they are identical to each other up to a change $t \to -t$. However, according to the iterative procedure, equations (B14) and (B7) must be integrated in opposite directions in time, which leads us again to a factor (B12) after $p$ iterations.

Let us also note that in the limit $a_c \to \infty$, an alternative norm that we use to study the growth of perturbations, given by equation (48), yields another expression for $g$,

$$g(t) = \frac{(\beta^2 + 1)^2}{((\beta + qt)^2 + 1)^2} \frac{1 + [2/(2 - q)](\beta + qt)^2}{1 + [2/(2 - q)]\beta^2},$$

(B15)

which is less tractable analytically. However, along with equation (B9), it recovers well the transient evolution of optimal perturbations for $k_y, k_x \gg 1$.

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