Measurable time-restricted sensitivity

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Abstract

We develop two notions of sensitivity to initial conditions for measurable dynamical systems, where the time before divergence of a pair of paths is at most an asymptotically logarithmic function of a measure of their initial distance. In the context of probability measure-preserving transformations on a compact space, we relate these notions to the metric entropy of the system. We examine one of these notions for classes of non-measure-preserving, nonsingular transformations.

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1. Introduction

Sensitivity has been widely studied as a characterization of chaos for topological dynamical systems, see, e.g., [BBC+92, GW93, AAB96]. Recently, sensitivity has been explored in the context of other measurable dynamical properties, such as weak mixing and entropy, for a finite measure-preserving transformation equipped with a metric of full support [ABC02, HYW04, CJ05]. More recently, in [JKL*08], the authors introduced a measure-theoretic version of sensitivity, invariant under measurable isomorphism, for nonsingular transformations. This has been further studied in [GIL*12].

In broad terms, sensitivity asserts that for any point \( x \) in the space, there exists another arbitrarily close point \( y \) such that at some future positive time \( n = n(x, y) \) the points \( T^n(x) \) and \( T^n(y) \) are separated by some predetermined distance. (Let us call this \( n \) a sensitive time.) This simplest notion of sensitivity does not assert anything about the sensitive time other
than its existence. A refinement of this definition, called strong sensitivity, was introduced in [ABC02], where for each point \( x \) the set of sensitive times \( n(x, y) \), for some \( y \), is co-finite. Strong sensitivity was also studied in the measurable context in [JKL+08]. An alternative refinement of (topological) sensitivity was studied in [Moo07], where the set of sensitive times is required to be syndetic. In [CJ05], the authors studied a notion of sensitivity that requires separation for almost every pair of points \( x \) and \( y \), and this was further investigated for topological dynamical systems in [HLY11].

In this paper, we are interested in placing a quantitative, asymptotic bound on the sensitive time, restricting the first sensitive time of a point \( x \) and a point in an \( \varepsilon \)-ball around \( x \) to be at most asymptotically logarithmic in the measure of that \( \varepsilon \)-ball. We develop two notions of measurable sensitivity that restrict the sensitive time in this way, show that these notions are related to positive metric entropy for finite measure-preserving systems, and explore one of the notions in the context of (non-measure-preserving) nonsingular transformations. An outstanding problem in nonsingular ergodic theory is the lack of a theory of entropy, see, e.g., [DS09]. Our definitions of restricted sensitivity are related to entropy in the case of finite measure-preserving systems but can be formulated in the context of nonsingular transformations. Thus they can be thought of as an approach to entropy for nonsingular systems.

In section 2, we define the notions of restricted sensitivity and restricted pairwise sensitivity, drawing from [JKL+08, CJ05], and we explore these notions for Bernoulli shifts. In section 3, we consider the setting of measure-preserving transformations on a compact space with probability measure. We prove that restricted pairwise sensitivity implies positive metric entropy and that positive metric entropy, for a continuous transformation, implies restricted sensitivity, and we place bounds on the metric entropy using the sensitive time. In section 4, we explore restricted sensitivity in the context of (non-measure-preserving) nonsingular transformations. We show that there exist nonsingular transformations, including type III transformations (i.e. transformations not admitting an equivalent \( \sigma \)-finite invariant measure), that are restricted sensitive, and we construct a class of nonsingular rank-one transformations that are not restricted sensitive. It is well-known that finite measure-preserving rank-one transformations have zero entropy; it would be interesting to know if all nonsingular rank-one transformations are not restricted sensitive.

## 2. Time-restricted notions of sensitivity

Throughout, we let \((X, \mathcal{S}(X), \mu)\) be a standard nonatomic Lebesgue space (i.e., \((X, \mathcal{S}(X))\) is a standard Borel space, and \(\mu\) is a \(\sigma\)-finite, nonatomic measure on \(\mathcal{S}(X)\)). We consider metrics \(d : X \times X \to \mathbb{R}\) on \(X\) that are (Borel) measurable. We denote by \(B(x, r)\) the open ball of radius \(r\) centred at \(x \in X\), and by \(B[x, r]\) the closed ball of radius \(r\) centred at \(x \in X\), with the convention \(B[x, 0] = \{x\}\). It follows that all balls are measurable (see [GIL+12]).

We will assume that \(d\) is \(\mu\)-compatible as defined in [JKL+08], i.e. all nonempty open \(d\)-balls have positive \(\mu\)-measure. The topology of a \(\mu\)-compatible metric is separable [JKL+08]. Therefore, all open sets are measurable as they are countable unions of balls, so \(\mathcal{S}(X)\) contains the Borel sigma algebra generated by \(d\) (see [GIL+12]).

We will consider measurable, nonsingular transformations \(T : X \to X\) (i.e., for all \(A \in \mathcal{S}(X)\), \(T^{-1}(A) \in \mathcal{S}(X)\) and \(\mu(A) = 0\) if and only if \(\mu(T^{-1}(A)) = 0\)). In certain explicitly stated situations, we may assume that \(T\) is measure-preserving or ergodic. Recall that \(T\) is conservative and ergodic if and only if for all measurable sets \(A\), if \(T^{-1}(A) \subset A\), then \(\mu(A) = 0\) or \(\mu(A^c) = 0\) (see e.g. [Sil08]).
We will occasionally make the following additional assumption on \( d \) and \( \mu \). Let us say that \( d \) is \( \mu \)-regular if for \( \mu \)-a.e. \( x \in X \), there exists \( c > 0 \) such that

\[
c\mu B(x, r) \subseteq \mu B(x, r)
\]

for all \( r \geq 0 \), i.e. a specified fraction of the measure of each closed ball around \( x \) must come from the interior of that ball. In particular, if all \( d \)-balls are \( \mu \)-continuity sets, then \( d \) is \( \mu \)-regular, and if \( d \) is \( \mu \)-regular, then applying the definition for \( r = 0 \) yields \( \mu(x) = 0 \).

Recall the following definition of measurable sensitivity:

**Definition 2.1 ([JKL'08]).** A nonsingular dynamical system \((X, S(X), \mu, T)\) is said to be measurably sensitive if whenever a dynamical system \((X_1, S(X_1), \mu_1, T_1)\) is measurably isomorphic to \((X, S(X), \mu, T)\) and \(d\) is a \( \mu_1 \)-supported metric on \( X_1 \), then there exists \( \delta > 0 \) such that for all \( x \in X_1 \) and all \( \varepsilon > 0 \), there exists \( n \in \mathbb{N}_0 \) such that

\[
\mu_1 \{ y \in B(x, \varepsilon) : d(T_1^n x, T_1^n y) > \delta \} > 0.
\]

In this paper, we study the following time-restricted modification of this sensitivity notion:

**Definition 2.2.** A measurable, nonsingular transformation \( T \) on \((X, S_p(X), \mu)\) with metric \( d \) is restricted sensitive at \( x \in X \) if there exist \( \delta > 0 \) and \( a > 0 \) (possibly depending on \( x \)) such that for every \( \varepsilon > 0 \), there exists \( n \in \mathbb{N}_0 \), \( n < -a \log \mu B(x, \varepsilon) \) or \( n = 0 \), with

\[
\mu \{ y \in B(x, \varepsilon) : d(T^n x, T^n y) > \delta \} > 0.
\]

\( T \) is restricted sensitive if it is restricted sensitive at a.e. \( x \in X \).

This definition requires, for each \( d \)-ball around \( x \), a positive-measure subset of the ball to be separated from \( x \) by a specified distance and before a specified time. We will want to also consider a stronger notion of sensitivity in which this separation is required not only for a positive-measure subset of the ball, but for almost every point, similar to the notion of pairwise sensitivity studied in [CJ05]. Hence we define the following:

**Definition 2.3.** A measurable, nonsingular transformation \( T \) on \((X, S_p(X), \mu)\) with metric \( d \) is restricted pairwise sensitive at \( x \in X \) if there exist \( \delta > 0 \) and \( a > 0 \) (possibly depending on \( x \)) such that for a.e. \( y \in X \), there exists \( n \in \mathbb{N}_0 \), \( n < -a \log \mu B(x, d(x, y)) \) or \( n = 0 \), with \( d(T^n x, T^n y) > \delta \). \( T \) is restricted pairwise sensitive if it is restricted pairwise sensitive at a.e. \( x \in X \).

Let us call \( \delta \) and \( a \) from definitions 2.2 and 2.3 a sensitivity constant and an asymptotic rate, respectively, at the point \( x \in X \). For both definitions, if \( \delta \) is a sensitivity constant at \( x \), then any \( \delta' < \delta \) is also a sensitivity constant at \( x \), and if \( a \) is an asymptotic rate at \( x \), then any \( a' > a \) is also an asymptotic rate at \( x \). To verify restricted sensitivity, it suffices to check definition 2.2 for \( \varepsilon \leq \delta \), and to verify restricted pairwise sensitivity, it suffices to check definition 2.3 for \( \varepsilon \leq \delta \).

Note that in contrast to definition 2.1 and some previously established notions of sensitivity to initial conditions, the sensitivity constant \( \delta \) in definitions 2.2 and 2.3 is defined locally at each point \( x \in X \), rather than globally over the entire space. Also in contrast with definition 2.1, we define restricted sensitivity and restricted pairwise sensitivity to be with respect to a single fixed metric \( d \), rather than requiring that the definitions hold for all \( \mu \)-supported metrics. We will assume, for some results in section 3, that \( X \) is compact or that \( T \) is continuous with respect to the topology of \( d \), but definitions 2.2 and 2.3 may be applied in a more general setting, and we do not make these assumptions unless explicitly stated. It is possible to frame notions analogous to definitions 2.2 and 2.3 in the context of topological dynamics without an...
underlying measure, by restricting the sensitive time to be less than \(-a \log \varepsilon\) and \(-a \log d(x, y)\) in these definitions respectively. We will not explore this analogy further in the current paper.

Note that the condition \(n < -a \log \mu B(x, d(x, y))\) in definition 2.3 implies the existence of some \(\varepsilon > 0\) with \(y \in B(x, \varepsilon)\) and \(n < -a \log \mu B(x, \varepsilon)\), so that restricted pairwise sensitivity is a natural strengthening of restricted sensitivity. Indeed, we have the following implication:

**Proposition 2.4.** Suppose \(d\) is \(\mu\)-regular. If a transformation \(T\) is restricted pairwise sensitive, then \(T\) is restricted sensitive.

**Proof.** Take \(x \in X\) such that \(T\) is restricted pairwise sensitive with sensitivity constant \(\delta > 0\) and asymptotic rate \(a > 0\) at \(x\) and such that \(c \mu B(x, r) \leq \mu B(x, r)\) for \(c > 0\).

Consider any \(\varepsilon < \delta\), and let \(\varepsilon_M = \sup\{\varepsilon' < \varepsilon : \mu B(x, \varepsilon') < \mu B(x, \varepsilon)\}\). If \(\mu B(x, \varepsilon_M) = \mu B(x, \varepsilon)\), then choose \(\varepsilon' < \varepsilon_M\) sufficiently close to \(\varepsilon_M\) such that \(\mu(B(x, \varepsilon) \setminus B(x, \varepsilon')) > 0\) and \(\mu B(x, \varepsilon) \geq c \mu B(x, \varepsilon)\). Otherwise if \(\mu B(x, \varepsilon_M) < \mu B(x, \varepsilon)\), then we must have \(\mu B(x, \varepsilon_M) = \mu B(x, \varepsilon)\), and we may choose \(\varepsilon' = s_M\) so that \(\mu(B(x, \varepsilon) \setminus B(x, \varepsilon')) > 0\) and \(\mu B(x, \varepsilon') \geq c \mu B(x, \varepsilon)\) by \(\mu\)-regularity.

For all \(y \in B(x, \varepsilon) \setminus B(x, \varepsilon')\) (which is a set of positive measure), there exists \(n < -a \log \mu B(x, d(x, y))\) such that \(d(T^n x, T^n y) > \delta\). We note that \(\mu B(x, d(x, y)) \geq \mu B(x, \varepsilon') \geq c \mu B(x, \varepsilon)\), so \(n < -a \log \mu B(x, \varepsilon) = -a \log \mu B(x, \varepsilon) - a \log c\). As we may choose the pairwise sensitivity constant \(\delta\) so that \(\mu B(x, \varepsilon) \leq \mu B(x, \varepsilon) < 1\), we may take \(\bar{a}\) so that \(-a \log \mu B(x, \varepsilon) - a \log c < -\bar{a} \log \mu B(x, \varepsilon)\) for all \(\varepsilon \leq \delta\). Hence \(T\) is restricted sensitive with sensitivity constant \(\delta\) and asymptotic rate \(\bar{a}\) at \(x\).

As an application of these notions of restricted sensitivity and restricted pairwise sensitivity, let us consider the standard one-sided and two-sided Bernoulli shift transformations.

**Example 2.5.** Consider the space \(\Sigma_N^0 = \prod_{n=0}^{\infty} \{1, \ldots, N\}\) with its product \(\sigma\)-algebra and the probability measure \(\mu = \bigotimes_{n=0}^{\infty} \mu\), where \(\mu\) is a probability measure on \([1, \ldots, N]\) of full support. Consider the metric \(d(\sigma, \tau) = 2^{-I(\sigma, \tau)}\) where \(I(\sigma, \tau) = \min\{|i : \sigma_i \neq \tau_i\}\). Let \(T\) be the one-sided Bernoulli shift transformation \(T((\sigma_0, \sigma_1, \sigma_2, \ldots)) = (\sigma_1, \sigma_2, \sigma_3, \ldots)\). Let \(p = \max_{k} \mu' (k), \delta = \frac{1}{2}\), and \(a > -\frac{1}{\log p}\). Consider any two points \(\sigma, \tau \in \Sigma_N^0\) and suppose that \(I(\sigma, \tau) = n\). Then \(\mu B[\sigma, d(\sigma, \tau)] \leq p^n\), so \(-a \log \mu B[\sigma, d(\sigma, \tau)] > n\). Since \(d(T^n \sigma, T^n \tau) = 1 > \delta\), \(T\) is restricted pairwise sensitive with sensitivity constant \(\delta\) and asymptotic rate \(a\) at each \(\sigma \in \Sigma_N^0\). As \(d\) is \(\mu\)-regular with \(c = \min_k \mu' (k)\) for all \(\sigma \in \Sigma_N^0\), \(T\) is also restricted sensitive.

**Example 2.6.** Consider the space \(\Sigma_N = \prod_{n=-\infty}^{\infty} \{1, \ldots, N\}\) with its product \(\sigma\)-algebra and the probability measure \(\mu = \bigotimes_{n=-\infty}^{\infty} \mu\), where \(\mu\) is a probability measure on \([1, \ldots, N]\) of full support. Consider the metric \(d(\sigma, \tau) = 2^{-I(\sigma, \tau)}\) where \(I(\sigma, \tau) = \min\{|i : \sigma_i \neq \tau_i\}\). Let \(T\) be the two-sided Bernoulli shift transformation such that \(T(\sigma)_0 = \sigma_{t+1}\). Let \(p_k = \mu' (k)\) for each \(k = 1, \ldots, N\).

For any \(\sigma \in \Sigma_N\), \(\delta < 1\), and \(a > 0\), choose an integer \(k_1 > 0\) such that \(2^{-k_1} < \delta\), and let \(P = \prod_{k=k_1}^{\infty} \rho_{k+1}\). Choose an integer \(k_2 > k_1\) such that \(2^{-k_2 a \log p} < \delta\). Consider any \(r\) in the cylinder set \([\bar{\sigma}_{-k_1} \sigma_{-k_1+1} \ldots \sigma_0 \ldots \sigma_{k_2}]\), where \(\bar{\sigma}_{-k_1}\) is some symbol not equal to \(\sigma_{-k_1}\). Then by the construction of \(k_1\) and \(k_2\), for all \(n < -a \log P = -a \log \mu B(\sigma, d(\sigma, \tau)), d(T^n \sigma, T^n \tau) \leq \max(2^{-k_2 a \log p}, 2^{-k_1 a \log p}) < \delta\). Hence \(T\) is not restricted pairwise sensitive at any \(\sigma \in \Sigma_N\).

On the other hand, \(p = \max k \rho_k\), \(\delta < \frac{1}{2}\), and \(a = -\frac{1}{2 \log p}\). Consider any \(\sigma \in \Sigma_N\) and any ball \(B(\sigma) = [\sigma_{-n} \ldots \sigma_0 \ldots \sigma_n]\). We have that \(\mu B(\sigma) \leq 2^n\), so \(n < -a \log \mu B(\sigma)\). Since \(d(T^n \sigma, T^n \tau) = \frac{1}{2} > \delta\) for all \(\tau \in [\sigma_{-n} \ldots \sigma_0 \bar{\sigma}_{n+1}] \subset [\sigma_{-n} \ldots \sigma_n]\) where \(\bar{\sigma}_{n+1}\) is some symbol not equal to \(\sigma_{n+1}\), \(T\) is restricted sensitive.
In particular, whereas certain non-time-restricted notions of sensitivity and pairwise sensitivity can be shown to be equivalent for topological dynamical systems as in [HLY11], example 2.6 shows that the notions of restricted sensitivity and restricted pairwise sensitivity are not equivalent, even when considering a continuous transformation on a compact metric space with a $\mu$-regular metric. One can also easily construct transformations that are not restricted sensitive but that are measurably sensitive according to definition 2.1; examples of such transformations include the rank-one cutting and stacking transformations that we will study in section 4.

3. Sensitivity and entropy for measure-preserving transformations

In the setting of probability measure-preserving systems, the notion of metric entropy measures the rate at which a transformation disorganizes the space. In this section, we show that for a compact metric space, metric entropy is closely related to sensitivity as characterized in definitions 2.2 and 2.3. To quantify the relationship between metric entropy and restricted sensitivity, it will be convenient for us to make the following two definitions:

Definition 3.1. The minimal restricted asymptotic rate $a^*_T(x)$, for each $x \in X$, is the infimum over all $a > 0$ such that $T$ is restricted sensitive at $x$ with asymptotic rate $a$ and some sensitivity constant $\delta > 0$. If $T$ is not restricted sensitive at $x$, then $a^*_T(x) = \infty$.

Definition 3.2. The minimal restricted pairwise asymptotic rate $a^+_T(x)$, for each $x \in X$, is the infimum over all $a > 0$ such that $T$ is restricted pairwise sensitive at $x$ with asymptotic rate $a$ for some sensitivity constant $\delta > 0$. If $T$ is not restricted pairwise sensitive at $x$, then $a^+_T(x) = \infty$.

In general, we may not know that $a^*_T$ and $a^+_T$ are measurable. To construct integrals of these functions and avoid issues of measurability, we will consider where appropriate the outer integral defined as

$$\int^* f \, d\mu = \inf \left\{ \int g \, d\mu : g \geq f, g \text{ measurable} \right\},$$

and $\int^*_A f \, d\mu = \int^*_A f \chi_A \, d\mu$ for arbitrary $A$ and $f$.

The following theorem places a lower bound on metric entropy using $a^+_T$, and it implies that a restricted pairwise sensitive system has positive metric entropy:

Theorem 3.3. Let $T$ be a measure-preserving transformation on $(X, S_\mu(X), \mu)$ with metric $d$, where $\mu$ is a probability measure, $d$ is $\mu$-regular, and $X$ is compact under $d$. If $h_\mu(T)$ is the metric entropy of $T$, then

$$h_\mu(T) \geq \int^*_A \frac{1}{a^+_T(x)} \, d\mu(dx).$$

Proof. Fix $\delta > 0$ and let $A_\delta \subset X$ be the set of points $x$ were $T$ is restricted pairwise sensitive with some asymptotic rate $a$ and sensitivity constant $\delta$. Let $A_\delta = \{X_1, \ldots, X_k\}$ be a measurable partition of $X$ such that $\text{diam}X_i < \delta$, whose existence is guaranteed by compactness of $X$. Suppose $x \in A_\delta$ and $d$ is $\mu$-regular at $x$. Let $C_n(x)$ denote the element of the partition $\{\cap_{i=0}^n T^{-i}A\}$ containing $x$. If $y \in C_n(x)$, then $T^n x$ and $T^n y$ are in the same element of $A$ for all $i \leq n$, so $d(T^n x, T^n y) < \delta$ for all $i \leq n$.

Let $c > 0$ be such that $e^{-\mu B(x, r)} \leq \mu B(x, r)$ for all $r \geq 0$, whose existence is given by $\mu$-regularity. Then $\mu(x) = 0$ and so $\mu B(x, r) \to 0$ as $r \to 0$. For each integer $n \geq 1$, we may take $\delta_n$ such that $e^{-(n-1)c} > \mu B(x, \delta_n) \geq e^{-nc}$, for otherwise there must exist $\varepsilon'$ with...
\( \mu B[x, e'] \geq e^{-(n-1)c} \) and \( \mu B(x, e') < e^{-nc} \), contradicting the definition of \( c \). For any \( n \), if \( y \notin B(x, \varepsilon'_{\lfloor nac \rfloor}) \), then

\[
\mu B[x, d(x, y)] \geq \mu B(x, \varepsilon'_{\lfloor nac \rfloor}) \geq e^{-(n-1)c},
\]

so for a.e. \( y \notin B(x, \varepsilon'_{\lfloor nac \rfloor}) \), there exists \( i < -a \log \mu B[x, d(x, y)] \leq n \) such that \( d(T'x, T'y) > \delta \), and so \( y \notin C_n(x) \). Hence \( C_n(x) \subset B(x, \varepsilon'_{\lfloor nac \rfloor}) \) mod \( \mu \), so

\[
\mu C_n(x) \leq \mu B(x, \varepsilon'_{\lfloor nac \rfloor}) < e^{-(\lfloor nac \rfloor - 1)c}.
\]

Letting \( h(x) = \lim \inf_{n \to \infty} -\frac{1}{n+1} \log \mu C_n(x) \), we have

\[
h(x) \geq \lim \inf_{n \to \infty} \frac{c}{n+1} \left( \left\lfloor \frac{n}{ac} \right\rfloor - 1 \right) = \frac{1}{a}.
\]

As \( \delta \to 0 \), we may let \( a \) decrease to \( a^+_f(x) \), so that \( h(x) \geq \frac{1}{a^+_f(x)} \). This holds for a.e. \( x \in A_\delta \), so by Fatou’s lemma,

\[
h_\mu(T, A_\delta) = \lim_{n \to \infty} \int -\frac{1}{n+1} \log \mu C_n(x) \, d\mu \geq \int_{A_\delta^+} h(x) \mu(dx) \geq \int_{A_\delta^+} \frac{1}{a^+_f(x)} \mu(dx).
\]

The sets \( A_\delta \) increase to the set \( A \) of points where \( T \) is restricted pairwise sensitive, and \( \frac{1}{a^+_f(x)} = 0 \) outside of \( A \), so the result follows.

**Corollary 3.4.** Under the conditions of theorem 3.3, if \( T \) is restricted pairwise sensitive on a set of positive measure, then \( h_\mu(T) > 0 \).

Examining the above proof, we note that we may obtain a stronger bound using the Shannon–McMillan–Breiman theorem if \( T \) is ergodic:

**Theorem 3.5.** Suppose, in addition to the conditions of theorem 3.3, that \( T \) is ergodic. Then \( h_\mu(T) \geq \operatorname{ess sup} x \in A \frac{1}{a^+_f(x)} \).

**Proof.** The Shannon–McMillan–Breiman theorem gives

\[
h_\mu(T, A_\delta) = \lim_{n \to \infty} -\frac{1}{n+1} \log \mu C_n(x)
\]

for a.e. \( x \in X \), where \( A_\delta \) and \( C_n(x) \) are defined as in the proof of theorem 3.3. Taking any \( x \in A_\delta \) for which this relation holds, we obtain as in the preceding proof that

\[
h_\mu(T) \geq h_\mu(T, A_\delta) \geq \frac{1}{a}.
\]

This holds for arbitrarily small \( \delta \) and hence for \( a \) arbitrarily close to \( a^+_f(x) \).

Example 2.6 shows that a converse to corollary 3.4 is not true, for the two-sided Bernoulli shift has positive entropy but is not restricted pairwise sensitive at any point. The following result shows, however, that the implication in this direction is true if we replace restricted pairwise sensitivity with restricted sensitivity, under the additional assumption of continuity of \( T \):

**Theorem 3.6.** Let \( T \) be a measure-preserving transformation on \( (X, S_\mu(X), \mu) \) with metric \( d \), where \( \mu \) is a nonatomic probability measure and \( X \) is compact and \( T \) is continuous under \( d \). If \( h_\mu(T) \) is the metric entropy of \( T \), then \( h_\mu(T) \leq \int \frac{1}{a^+_f(x)} \mu(dx) \).
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**Proof.** Suppose that at \( x \in X, T \) is not restricted sensitive with asymptotic rate \( a \) for any sensitivity constant \( \delta \). Then for any \( \delta > 0 \), there exists \( \delta_0 \leq \delta \) such that \( \mu \{ y \in B(x, \delta_0) : d(T^n x, T^n y) > \delta \} = 0 \) for all \( n \leq -a \log \mu B(x, \delta_0) \). For each \( n \), let

\[
C(x, n, \delta) = \{ y \in X : d(T^n x, T^n y) \leq \delta \} \quad \text{mod} \quad \mu.
\]

Then \( B(x, \delta_0) \subset C(x, \lceil -a \log \mu B(x, \delta) \rceil, \delta) \mod \mu \), so \( \mu B(x, \delta_0) \leq \mu C(x, \lceil -a \log \mu B(x, \delta) \rceil, \delta) \).

As \( \mu \) is nonatomic, \( \lim_{\delta \to \infty} -a \log \mu B(x, \delta) = \infty \). Letting \( h_{\mu}(T, x) \) be the Brin–Katok local entropy at \( x \) as defined in [BK83], we have

\[
h_{\mu}(T, x) = \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{-1}{n} \log \mu C(x, n, \delta) \leq \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{-\log \mu C(x, \lceil -a \log \mu B(x, \delta) \rceil, \frac{1}{m})}{\lceil -a \log \mu B(x, \delta) \rceil}
\]

where in the third line we have used the inequality \( \mu C(x, n, \delta) \leq \mu C(x, n, \delta') \) if \( \delta \leq \delta' \). Letting \( a \) increase to \( a^*_T(x) \), we obtain \( h_{\mu}(T, x) \leq \frac{1}{a^*_T(x)} \), and the desired result follows from the Brin–Katok entropy theorem upon integrating over \( X \).

**Corollary 3.7.** Under the conditions of theorem 3.6, if \( h_{\mu}(T) > 0 \), then \( T \) is restricted sensitive on a set of positive measure.

We note that we may obtain a stronger bound using the Brin–Katok entropy theorem if \( T \) is ergodic:

**Theorem 3.8.** Suppose, in addition to the conditions of theorem 3.6, that \( T \) is ergodic. Then \( h_{\mu}(T) \leq \inf_{x \in A} \frac{1}{a^*_T(x)} \).

**Proof.** The Brin–Katok entropy theorem gives \( h_{\mu}(T) = h_{\mu}(T, x) \) for a.e. \( x \in X \). As we have \( h_{\mu}(T, x) \leq \frac{1}{a^*_T(x)} \) by the proof of theorem 3.6, the result follows.

**Corollary 3.9.** Under the conditions of theorem 3.6, if \( h_{\mu}(T) > 0 \) and \( T \) is ergodic, then \( T \) is restricted sensitive.

Regarding the converse of theorem 3.6, the following example constructs an ergodic transformation which is restricted sensitive but has zero metric entropy:

**Example 3.10.** Let \( X \) be a disjoint union of two copies of \([0, 1/2] \), labelled \( I_1 \) and \( I_2 \), equipped with their Borel sigma algebras \( B_1 \) and \( B_2 \) and Lebesgue measures. Consider the \( \sigma \)-algebra on \( X \) given by \( S = \{ S_1 \cup S_2 : S_1 \in B_1 \text{ and } S_2 \in B_2 \} \) and probability measure \( \mu \) on \( X \) given by \( \mu(A) = \lambda(A \cap I_1) + \lambda(A \cap I_2) \). Define a metric \( d \) on \( X \) by \( d(x, y) = |x - y| \) if \( x \) and \( y \) are in the same copy of \([0, 1/2] \), and \( d(x, y) = 2 \) if not.
Let $A \subset [0, 1/2]$ be a Borel set of Lebesgue measure $\lambda(A) = \frac{1}{2}$ with the property that for any subinterval $K$ of $[0, 1/2]$, $\lambda(K \cap A) > 0$ and $\lambda(K \cap A^c) > 0$. (For a construction of such a set, see appendix B of [Sil08].) By theorem 3.4.23 in [Spe98], there exist measure-preserving Borel isomorphisms $\phi : A \to [0, 1/4]$ and $\psi : A^c \to [0, 1/4)$. Let $A_1$ and $A_2$ be the copies of $A$ inside $I_1$ and $I_2$, respectively, and let $\phi_1, \psi_1$ and $\phi_2, \psi_2$ be copies of the maps $\phi, \psi$ on $I_1, I_2$. Define a transformation $T : X \to X$ by

$$T(x) = \begin{cases} 
\psi_1^{-1} \circ \phi_1(x) & \text{for } x \in A_1 \\
\phi_2^{-1} \circ \phi_1(x) & \text{for } x \in A_1^c \cap I_1 \\
\psi_2^{-1} \circ \phi_2(x) & \text{for } x \in A_2 \\
\phi_1^{-1} \circ R \circ \psi_2(x) & \text{for } x \in A_2^c \cap I_2,
\end{cases}$$

where $R : [0, 1/4) \to [0, 1/4)$ is an irrational rotation. Note that $T$ is finite measure-preserving. Also, note that when $T^4$ is restricted to any one of the four “segments” $A_1, A_1^c \cap I_1, A_2, A_2^c \cap I_2$, it is isomorphic to an irrational rotation. We claim that $T$ is ergodic and restricted sensitive and that $h_\mu(T) = 0$.

To see that $T$ is ergodic, let $C \subset X$ and $D \subset X$ have positive measure. Then we can find positive-measure subsets $C^* \subset C$ and $D^* \subset D$ each of which is completely contained inside one of the four segments $A_1, A_1^c \cap I_1, A_2, A_2^c \cap I_2$. Let $k < 4$ be the integer such that $T^k(C^*)$ and $D^*$ are in the same segment. Then, since $T^4$ is isomorphic to an irrational rotation when restricted to that segment, there exists $n$ a multiple of 4 such that $\mu(T^{k+n}(C^*) \cap D^*) > 0$. Therefore, $\mu(T^{k+n}(C) \cap D) > 0$.

To see that $T$ is restricted sensitive, choose $\delta = 1$ and $a = 2$ for all $x \in X$. For any $x \in I_1$ and $\varepsilon \leq 1$, by construction, $\mu(B(x, \varepsilon) \cap A_1) > 0$ and $\mu(B(x, \varepsilon) \cap (A_1^c \cap I_1)) > 0$, so $\mu(y \in B(x, \varepsilon) : d(Tx, Ty) > \delta) > 0$ and we note that $n = 1 < 0 \log \frac{1}{2} \leq \log \mu(B(x, \varepsilon))$. A similar argument holds for any $x \in I_2$, so $T$ is restricted sensitive.

To see that $T$ has zero entropy, simply note that $h_\mu(T^4) = 0$ because $T^4$ is isomorphic to a disjoint union of 4 irrational rotations. This implies that $h_\mu(T) = 0$.

We note that the transformation in the above example is not continuous. The continuity condition of theorem 3.6 was used to apply the theory of Brin–Katok local entropy, but we do not know whether it is necessary for the conclusion of theorem 3.6 to hold, or whether a continuous transformation that is restricted sensitive but has zero entropy exists.

Theorems 3.3, 3.5, 3.6 and 3.8 bound the metric entropy of a system using the minimal asymptotic rates $a^+_T$ and $a^-_T$. We conclude this section by computing these quantities for the one-sided Bernoulli shift transformation of example 2.5 and showing that these bounds are tight.

**Proposition 3.11.** Let $(\Sigma_N^+, B, \mu)$, $d$, and $T$ be as in example 2.5. Then

$$\frac{1}{a_T^+(\sigma)} = \frac{1}{a_T^-(\sigma)} = h_\mu(T)$$

for a.e. $\sigma \in \Sigma_N^+$.

**Proof.** Suppose $T$ is restricted sensitive at $\sigma \in \Sigma_N^+$ with sensitivity constant $\delta(\sigma) > 0$ and asymptotic rate $a(\sigma) > 0$. Let $c(\sigma) \geq 0$ be the integer such that $2^{-c(\sigma)} > \delta(\sigma) \geq 2^{-c(\sigma)-1}$. For each $i = 1, \ldots, N$, let $k_{i}(\sigma)$ be the number of occurrences of the symbol $i$ in $\sigma_0$ through $\sigma_{n-1}$. For any ball $B(\sigma) = [\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}]$, $\min[k : \mu(\tau \in B(\sigma) : d(T^k\sigma, T^k\tau) > \delta(\sigma)) > 0] = n - c(\sigma)$. Hence we must have that $n - c(\sigma) < a(\sigma) \log(p_{1}^{k_{1}(\sigma)} \ldots p_{N}^{k_{N}(\sigma)})$.
for all \( n > c(\sigma) \), so for fixed \( \delta(\sigma) \),
\[
\inf a(\sigma) = \sup_{n > c(\sigma)} -(n - c(\sigma)) \left( \log(p^{k_n(\sigma)}_1 \cdots p^{k_n(\sigma)}_N) \right)^{-1}.
\]

Thus
\[
a^*_T(\sigma) = \inf_{\delta(\sigma)} \sup_{n > c(\sigma)} -(n - c(\sigma)) \left( \log(p^{k_n(\sigma)}_1 \cdots p^{k_n(\sigma)}_N) \right)^{-1}
\]
\[
= \left( \sup_{n > c(\sigma)} \inf_{\delta(\sigma)} \frac{n}{n - c} \cdot \frac{1}{n} \sum_{i=1}^N -k_i(\sigma) \log p_i \right)^{-1}
\]
\[
= \left( \lim_{c \to \infty} \inf_{n > c} \frac{n}{n - c} \cdot \frac{1}{n} \sum_{i=1}^N -k_i(\sigma) \log p_i \right)^{-1},
\]
where the first supremum is taking over all values \( \delta(\sigma) > 0 \) such that \( T \) is restricted sensitive at \( \sigma \) with sensitivity constant \( \delta(\sigma) \) and some asymptotic rate \( a(\sigma) \). By the Birkhoff ergodic theorem, for a.e. \( \sigma \in \Sigma_N^+ \),
\[
\lim_{n \to \infty} \frac{k_n(\sigma)}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^N \chi_{[i]}(T^n(\sigma)) p_i,
\]
so
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^N -k_i(\sigma) \log p_i = \sum_{i=1}^N -p_i \log p_i = h(\mu(T)).
\]
Then
\[
\frac{1}{a^*_T(\sigma)} \leq \lim_{c \to \infty} \lim_{n \to \infty} \frac{n}{n - c} \cdot \frac{1}{n} \sum_{i=1}^N -k_i(\sigma) \log p_i = h(\mu(T)).
\]

For any \( \varepsilon > 0 \), there exists \( c \) sufficiently large so that \( \frac{1}{n} \sum_{i=1}^N -k_i(\sigma) \log p_i \geq h(\mu(T)) - \varepsilon \) for all \( n > c \); hence
\[
\frac{1}{a^*_T(\sigma)} \geq \lim_{c \to \infty} \inf_{n > c} \frac{n}{n - c} \cdot (h(\mu(T)) - \varepsilon) = h(\mu(T)) - \varepsilon.
\]
As \( \varepsilon \) was arbitrary, \( \frac{1}{a^*_T(\sigma)} = h(\mu(T)) \).

Now suppose \( T \) is restricted pairwise sensitive at \( \sigma \in \Sigma_N^+ \) with sensitivity constant \( \delta(\sigma) > 0 \) and asymptotic rate \( a(\sigma) > 0 \), and let \( c(\sigma) \) and \( k_n(\sigma) \) be as above. For any \( \tau \in \Sigma_N^+ \), if \( d(\sigma, \tau) = 2^{-n} \), then \( \min\{k : d(T^k(\sigma), T^k(\tau)) > \delta(\sigma)\} = n - c(\sigma) \) and \( \mu_B[\sigma, d(\sigma, \tau)] = p^{k_n(\sigma)}_1 \cdots p^{k_n(\sigma)}_N \). For each \( n > c(\sigma) \), there must exist \( \tau \in \Sigma_N^+ \) such that \( d(T^k(\sigma), T^k(\tau)) > \delta(\sigma) \) for some \( k < -a(\sigma) \log(p^{k_n(\sigma)}_1 \cdots p^{k_n(\sigma)}_N) \).

The rest of the argument proceeds as above. \( \square \)

4. Restricted sensitivity for nonsingular transformations

In the preceding section, the notions of restricted sensitivity and restricted pairwise sensitivity were used to examine measure-preserving transformations, for which there is a well-developed theory of metric entropy. These notions can be applied as well to nonsingular transformations;
let us consider restricted sensitivity in this section. We show that a general class of nonsingular rank-one transformations (including measure-preserving rank-one transformations) are not restricted sensitive, and we construct a class of nonsingular type III transformations that are restricted sensitive.

Let us recall the definition of rank-one transformations. This class is known to contain finite measure-preserving mixing transformations [Orn72] and type III power weakly mixing nonsingular transformations [AFS01]. A nonsingular transformation \( T : (X, \mu) \to (X, \mu) \) is type III if there are no \( \sigma \)-finite measures invariant under \( T \) that are equivalent to \( \mu \). The first example of a type III transformation was rank-one [Orn60]. By [JKL+08], it follows that the class of rank-one transformations further includes strong measurably sensitive finite measure-preserving transformations and measurably sensitive type III transformations.

We give the cutting and stacking definition of these transformations and follow the notation of [CS04, DS09]. Our presentation includes nonsingular transformations. A column consists of a finite ordered collection of disjoint intervals in \( \mathbb{R} \). Each interval is called a level, and the levels may be of different lengths. The height of the column is the number of levels in the column. Each column defines an associated column map, defined on all levels except the top, by mapping each interval of the column to the next interval in the column by the unique orientation-preserving affine map that takes one interval to the other. Hence the column map is defined on all but the last level.

A rank-one nonsingular transformation is specified by a sequence of integers \( r_n \geq 2 \), a sequence of functions \( s_n : [0, \ldots, r_n - 1] \to \mathbb{N}_0 \), and a sequence \( \{p_n\} \) of probability vectors on \([0, \ldots, r_n - 1]\). In the case of a measure-preserving transformations the probability vectors are all uniform, i.e., \( p_n(i) = 1/r_n \) for all \( i \in [0, \ldots, r_n - 1] \). We now describe the inductive procedure that constructs a sequence of columns \( C_n \). Start by letting \( C_0 \) consist of a single interval. Assume that column \( C_n = \{I_{n,i}\}_{i=0}^{r_n-1} \) of height \( h_n \) has been constructed. Then we have a column map \( T_n \), where \( T_n(I_{n,i}) = I_{n,i+1} \) for \( i \neq h_n - 1 \). To construct column \( C_{n+1} \) subdivide \( C_n \) into \( r_n \) subcolumns by cutting each level \( I_{n,i} \) into \( r_n \) subintervals or sublevels, \( \{I_{n,j}\}_{j=0}^{r_n-1} \), where \( I_{n,0}^{[0]} \) is the leftmost sublevel and \( I_{n,r_n-1}^{[r_n-1]} \) is the rightmost) whose lengths are in the proportions

\[
p_n(0) : p_n(1) : \cdots : p_n(r_n - 1).
\]

(For example, if \( r_n = 2 \), and \( p_n(0) = 1/3 \), \( p_n(1) = 2/3 \), then every level is cut in the proportions \( 1/3 : 2/3 \).) Then the subcolumns of \( C_n \) are \( C_n^{[j]} = \{I_{n,j}\}_{j=0}^{r_n-1} \). By preserving the order on the levels, each subcolumn, \( C_n^{[j]} \), is a column in its own right with the associated map \( T_n^{[j]} \) which is the restriction of \( T_n \) to \( C_n^{[j]} \). The next step is to place new intervals the size of the top sublevel above each subcolumn by adding \( s_n,j \) levels above \( C_n^{[j]} \); these new intervals are called spacer levels. To obtain the next column then we stack the resulting subcolumns with spacers right on top of left yielding the new column \( C_{n+1} \) with height

\[
h_{n+1} = r_n h_n + \sum_{j=0}^{r_n-1} s_n(j).
\]

Let \( S_n \) denote the union of spacer levels added to \( C_n \), the collection of levels in \( C_{n+1} \) that are not sublevels of levels in \( C_n \); hence, \( C_{n+1} \setminus S_n \) equals \( C_n \). We denote by \( S_n^{[j]} \) the collection of spacers added over \( C_n^{[j]} \), so that \( S_n = \bigcup_{j=0}^{r_n-1} S_n^{[j]} \). The associated column map \( T_{n+1} \) restricts to \( T_n \) on the levels in \( C_n \). Let \( X \) be the union of all the levels in all columns. We assume that as \( n \to \infty \) the maximal length of the intervals in \( C_n \) converges to 0, so we may define a transformation \( T \) of \((X, \mu)\) by

\[
T(x) := \lim_{n \to \infty} T_n(x).
\]
One can verify that $T$ is well-defined and invertible a.e. and that it is nonsingular and ergodic. $T$ is measure-preserving if all the probability vectors $p_n$ are uniform, and $\mu(X) < \infty$ if and only if the total measure of the added spacers is finite.

The following proposition shows that nonsingular rank-one transformations constructed in this way are not restricted sensitive if there is a uniform lower bound on the elements of the probability vectors $\{p_n\}$.

**Proposition 4.1.** Let $T$ be a nonsingular rank-one transformation with the Euclidean metric $d$ and Lebesgue measure $\lambda$. Suppose that column $C_n$ is divided into $r_n$ subcolumns with proportions $p_n(0), \ldots, p_n(r_n - 1)$. If there exists $c > 0$ such that $p_n(j) \geq c$ for all $n$ and $j$, then $T$ is not restricted sensitive.

**Proof.** For each $n$, let us further divide the leftmost subcolumn $C_n^{[0]}$ of column $C_n$ into three equal subcolumns, labelled from left to right as $C_n^{[0],1}$, $C_n^{[0],2}$, and $C_n^{[0],3}$. Let $S_n = C_n^{[0],2}$ and let $S = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} S_n$. As $\lambda S_n \geq \frac{1}{3}\lambda C_n$, $\lambda(\bigcup_{n=k}^{\infty} S_n) \geq \frac{1}{3}\lambda C_k$ for each $k$, and thus $\lambda S > 0$.

For any $x \in S$, there is an increasing sequence $(n_k)$ such that $x \in S_{n_k}$. Let $h_n$ and $w_n$ be the height and the width of the smallest level, respectively, of column $C_n$. For any $\delta > 0$ and $a > 0$, there exist $n_k$ sufficiently large such that $w_{n_k} < \delta$ and $a(n_k \log \frac{1}{c} + \log \frac{3}{2w_0}) < 2w_{n_k} - 1$.

By the construction of $S_{n_k-1}$, if $h$ is the smallest number such that $T^h(x)$ is in the highest level of $C_{n_k}$, and $w$ is the distance from $x$ to the closer of the two endpoints of the level containing $x$ in $C_{n_k}$, then $h > \frac{w}{\lambda}$ and $w > \frac{\lambda}{h}$. We also note that $w_{n_k} \geq c^w w_0$ and $h_{n_k} \geq 2^{w_n}$.

Consider the ball $B(x, w)$. We note that $-a \log \lambda B(x, w) = -a \log 2w \leq -a \log \frac{2w_{n_k}}{3} \leq -a \log \frac{2c^w w_0}{3}$

\[= a \left(n_k \log \frac{1}{c} + \log \frac{3}{2w_0}\right) < 2w_{n_k} - 1 \leq \frac{h_{n_k}}{2} \leq h.\]

Hence, for all $y \in B(x, w)$, $T^n y \in B(T^n x, w)$ for all $n \leq -a \log \lambda B(x, w)$, so $d(T^n x, T^n y) < w < \delta$.

Hence $T$ is not restricted sensitive for any $x \in S$. \[\square\]

This proposition addresses a large class of nonsingular rank-one transformations. As measure-preserving transformations are those for which the probability vectors $p_n$ are uniform, measure-preserving transformations for which $r_n$ is bounded above over all $n$ satisfy the conditions of this proposition. In fact, the argument in the above proof can be modified to hold for all measure-preserving rank-one transformations:

**Proposition 4.2.** If $T$ is a measure-preserving rank-one transformation with the Euclidean metric $d$ and Lebesgue measure $\lambda$, then $T$ is not restricted sensitive.

**Proof.** The proof is similar to that for proposition 4.1. Let us divide each subcolumn $C_n^{[j]}$ of column $C_n$ into three equal subcolumns, labelled from left to right as $C_n^{[j],1}$, $C_n^{[j],2}$, and $C_n^{[j],3}$. Let

\[S_n = \bigcup_{j=0}^{\left[\frac{w_{n_k}}{2}\right] - 1} C_n^{[j],2},\]

and let $S = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} S_n$ as before. We have $\lambda S_n \geq \frac{1}{3}\lambda C_n$ for all $n$, so $\lambda S > 0$.

The rest of the proof is the same as for proposition 4.1, except that for any $\delta > 0$ and $a > 0$, we choose $n_k$ sufficiently large such that $w_{n_k} < \delta$ and $a \log \frac{h_{n_k}}{2w_0} < \frac{w_{n_k}}{2}$. As $w_{n_k} h_{n_k} \geq 2w_0 h_0$.
for all $k$, we use the bound $w_{n_k} \geq \frac{\sqrt{n_k}}{h_{n_k}}$ in place of the bounds $w_{n_k} \geq c_{n_k} w_0$ and $h_{n_k} \geq 2 n_k$ from the proof of proposition 4.1.

We end by constructing type III nonsingular transformations that are restricted sensitive. The construction is a consequence of the following proposition:

**Proposition 4.3.** Let $T$ be a transformation on a probability space $(X, \mu)$ with metric $d_X$ and $S$ be a transformation on a probability space $(Y, \nu)$ with metric $d_Y$, and let $d_Y$ be $\nu$-supported. If $T$ is restricted sensitive, then the transformation $T \times S$ on $X \times Y$ (with the product $\sigma$-algebra and product measure) is restricted sensitive under the metric given by

$$d((x_1, y_1), (x_2, y_2)) = \max \{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

**Proof.** Suppose $T$ is restricted sensitive at $x \in X$ with sensitivity constant $\delta > 0$ and asymptotic rate $a > 0$. For any $\epsilon > 0$, suppose $n \geq$ is such that

$$\mu\left\{ x' \in B^{d_X}_\epsilon (x) : d_X(T^n(x), T^n(x')) > \delta \right\} > 0.$$

Then for any $y \in Y$,

$$(\mu \times \nu)\left\{ (x', y') \in B^{d_Y}_\epsilon (x, y) : d((T \times S)^n(x, y), (T \times S)^n(x', y')) > \delta \right\} \geq \nu(B^{d_Y}_\epsilon (y)) \mu\left\{ x' \in B^{d_X}_\epsilon (x) : d_X(T^n(x), T^n(x')) > \delta \right\} > 0.$$

As $\nu$ is a probability measure, $(\mu \times \nu)B^{d_Y}_\epsilon (x, y) \leq \mu B^{d_X}_\epsilon (x)$ for any $\epsilon > 0$, so $-a \log(\mu \times \nu)B^{d_Y}_\epsilon (x, y) \geq -a \log \mu B^{d_X}_\epsilon (x)$. Hence $T \times S$ is restricted sensitive at $(x, y)$ with the same sensitivity constant $\delta$ and asymptotic rate $a$. This holds for a.e. $x \in X$ and all $y \in Y$, so $T \times S$ is restricted sensitive. \hfill $\square$

**Corollary 4.4.** Let $T$ be a measure-preserving mixing (or mildly mixing) transformation on a probability space $(X, \mu)$ with metric $d_X$, and suppose that $T$ is restricted sensitive. Let $S$ be a type III nonsingular conservative ergodic invertible transformation on a probability space $(Y, \nu)$ with metric $d_Y$. Then $T \times S$ on $X \times Y$ with the metric $d$ given in proposition 4.3 is a type III conservative ergodic transformation that is restricted sensitive.

**Proof.** By proposition 5.4 and theorem 5.2 of [HS98] the transformation $T \times S$ is conservative ergodic and type III. Proposition 4.3 implies that $T \times S$ is restricted sensitive. \hfill $\square$

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