Free particle on $SU(2)$ group manifold

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Abstract. We consider classical and quantum dynamics of a free particle on $SU(2)$ group manifold. The eigenfunctions of the Hamiltonian are constructed in terms of coordinate free objects

1. Lagrangian description

The dynamics of a free particle on $SU(2)$ group manifold is described by the Lagrangian

$$\mathcal{L} = \langle g^{-1} \dot{g} g^{-1} \dot{g} \rangle$$

where $g \in SU(2)$ and $\langle \rangle$ denotes the normalized trace $\langle \rangle = -\frac{1}{2} Tr(\ )$, which defines a scalar product in $su(2)$. The Lagrangian (1.1) defines the following dynamical equations

$$\frac{d}{dt} \langle g^{-1} \dot{g} \rangle = 0$$

otherwise, one can notice that our Lagrangian has $SU(2)$ "right" and $SU(2)$ "left" symmetry. It means that it is invariant under the following transformations

$$g \rightarrow h_1 g \quad "left" \ symmetry$$

$$g \rightarrow gh_2 \quad "right" \ symmetry$$

According to the Noether’s theorem the corresponding conserving quantities are

$$R = g^{-1} \dot{g} \quad \frac{d}{dt} R = 0 \quad "right" \ symmetry$$

$$L = \dot{g} g^{-1} \quad \frac{d}{dt} L = 0 \quad "left" \ symmetry$$

Now let’s introduce the basis of $su(2)$ algebra

$$T_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (7)$$

The elements of $su(2)$ are traceless anti-hermitian matrices, and any $A \in su(2)$ can be parameterized in the following way

$$A = A^n T_n \quad n = 1, 2, 3 \quad R = R^n T_n \quad (8)$$
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Scalar product $AB = \langle AB \rangle = -\frac{1}{2} Tr(AB)$ provides that
\[ A^n = \langle AT_n \rangle \quad (\langle T_n T_m \rangle = \delta_{nm}) \]  

(9)

Now we can introduce 6 functions
\[ R_n = \langle T_n R \rangle \quad n = 1, 2, 3 \quad R = R^n T_n \] 

(10)

\[ L_n = \langle T_n L \rangle \quad n = 1, 2, 3 \quad L = L^n T_n \] 

(11)

which are integrals of motion. It is easy to find general solution of Euler-Lagrange equation
\[ \frac{d}{dt} g^{-1} \dot{g} = 0 \quad \Rightarrow \quad g^{-1} \dot{g} = \text{const} \] 

(12)

\[ g = e^{Rt} g(0) \] 

(13)

These are well known geodesics on Lie group.

2. Hamiltonian description

Working in a first order Hamiltonian formalism we construct new Lagrangian which is equivalent to the initial one
\[ \tilde{L} = \langle R(g^{-1} \dot{g} - \tilde{v}) \rangle + \frac{1}{2} \langle \tilde{v}^2 \rangle \] 

(14)

in sense that variation of R provides
\[ g^{-1} \dot{g} = \tilde{v} \] 

(15)

and $\tilde{L}$ reduces to $L$. Variation of $\tilde{v}$ gives $R = \tilde{v}$ and therefore we can rewrite equivalent Lagrangian $\tilde{L}$ in terms of R and g
\[ \tilde{L} = \langle R g^{-1} \dot{g} \rangle - \frac{1}{2} \langle R^2 \rangle \] 

(16)

Where function $\frac{1}{2} \langle R^2 \rangle$ plays role of Hamiltonian and one-form $\langle R g^{-1} d g \rangle$ is a symplectic potential $\theta$. External differential of $\theta$ is the symplectic form $\omega$, that determines Poisson brackets and the form of Hamilton’s equation.
\[ \omega = d \theta = -\langle g^{-1} d g \wedge d R \rangle - \langle R g^{-1} d g \wedge g^{-1} d g \rangle \] 

(17)

$\omega$ provides isomorphism between vector fields and one-forms
\[ X \rightarrow X|\omega \] 

(18)

Let $\mathcal{F}(SU(2))$ denote the real-valued smooth function on $SU(2)$. For an $f \in SU(2)$ there exists a Hamiltonian vector field satisfying
\[ X_f : \quad X|\omega = -df \] 

(19)

Where $X|\omega$ denotes the contraction of $X$ with $\omega$. $X_f$ is called Hamiltonian vector field associated with $f$. According to the definition Poisson bracket of two function is as follows
\[ \{ f, g \} = \mathcal{L}_{X_f} g = X_f |dg = \omega(X_f, X_g) \] 

(20)
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Where $\mathcal{L}_X g$ denotes Lie derivative of $g$ with respect to $X$. The skew symmetry of $\omega$ provides skew symmetry of Poisson bracket. Hamiltonian vector fields that correspond to $R_n, L_m$ and $g$ functions are the following

$$X_n = X_{R_n} = ( [R, T_n] , g T_n ) = ( X^{(R)}_{R_n} , X^{(g)}_{R_n} )$$

$$Y_n = X_{L_m} = ( [R, g T_m g^{-1}] , T_m g ) = ( X^{(R)}_{L_m} , X^{(g)}_{L_m} )$$

Therefore Poisson brackets are

$$\{ L_n, L_m \} = -2\epsilon_{nm}^k L_k \quad \{ R_n, R_m \} = 2\epsilon_{nm}^k R_k$$

$$\{ R_n, L_m \} = 0$$

$$\{ R_n, g \} = g T_n \quad \{ L_m, g \} = T_m g$$

the results are natural. $R$ and $L$ that correspond respectively to the "right" and "left" symmetry commute with each other and independently form $su(2)$ algebras. It is easy to write down Hamilton’s equations

$$\dot{g} = \{ H, g \} = g R$$

$$\dot{R} = \{ H, R \} = 0$$

We consider case of $SU(2)$, but the same constructions can be applied to the other Lie groups.

3. Quantization

Let’s introduce operators

$$\hat{R}_n = \frac{i}{2} \mathcal{L}_{X_n}$$

$$\hat{L}_m = -\frac{i}{2} \mathcal{L}_{Y_m}$$

They act on the square integrable functions (see Appendix A) on $SU(2)$ and satisfy quantum commutation relations

$$[ \hat{L}_n, \hat{L}_m ] = i\epsilon_{nm}^k \hat{L}_k$$

$$[ \hat{R}_n, \hat{R}_m ] = i\epsilon_{nm}^k \hat{R}_k$$

$$[ \hat{R}_n, \hat{L}_m ] = 0$$

The Hamiltonian is defined as

$$\hat{H} = \hat{R}^2 = \hat{L}^2$$

Therefore the complete set of observables that commute with each other is as follows

$$\hat{H}, \quad \hat{R}_a, \quad \hat{L}_b$$
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Where a and b unlike n and m are fixed. Using a simple generalization of a well known algebraic construction (see Appendix B) one can check that the eigenvalues of the quantum observables $\hat{H}, \hat{R}_a$ and $\hat{L}_b$ are as follows

$$\hat{H} \psi_{lr}^j = j(j+1) \psi_{lr}^j$$

(35)

where $j$ takes positive integer and half integer values

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 ...$$

(36)

$$\hat{R}_a \psi_{lr}^j = r \psi_{lr}^j$$

(37)

$$\hat{L}_b \psi_{lr}^j = l \psi_{lr}^j$$

(38)

with $r$ and $l$ taking values in the following range

$$-j, -j+1, ..., j-1, j$$

(39)

The main aim of the article is construction of the corresponding eigenfunctions $\psi_{lr}^j$. The first step of this construction is proposition 1

**proposition 1.** The function $\langle \tilde{T} g \rangle$ where $\tilde{T} = (I + iT_a)(I + iT_b)$ is an eigenfunction of $\hat{H}, \hat{R}_a$ and $\hat{L}_b$ with eigenvalues, respectively $\frac{3}{4}, \frac{1}{2}$ and $\frac{1}{2}$.

Proof of this proposition is straightforward. Using $\langle \tilde{T} g \rangle$ we construct the complete set of eigenfunctions of $\hat{H}, \hat{R}_a$ and $\hat{L}_b$ operators

$$\psi_{lr}^j = \hat{L}_b^{j-l} \hat{R}_a^{j-r} \langle \tilde{T} g \rangle^{2j}$$

(40)

(for the definition of the $\hat{R}_a$ and $\hat{L}_b$ operators see Appendix) that are defined up to a constant multiple. Indeed, acting on (40) with $\hat{H}, \hat{R}_a$ and $\hat{L}_b$ operators and using commutation relations (see Appendix B) one can prove that equations (35-38) hold for $\psi_{lr}^j$, defined by (40)

4. Free particle on $S^2$ as a $SU(2)/U(1)$ coset model

Free particle on 2D sphere can be obtained from our model by gauging $U(1)$ symmetry. In other words let’s consider the following local gauge transformations

$$g \rightarrow h(t) g$$

(41)

Where $h(t) \in U(1) \subset SU(2)$ is an element of $U(1)$. Without loss of generality we can take

$$h = e^{\beta(t) T_3}$$

(42)

Since $T_3$ is antihermitian $h(t) \in U(1)$ and since $h(t)$ depends on $t$ Lagrangian

$$\mathcal{L} = \langle g^{-1} \dot{g} g^{-1} \dot{g} \rangle$$

(43)
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is not invariant under (41) local gauge transformations. To make (43) gauge invariant we should replace \( \frac{d}{dt} \) with a covariant derivative \( \nabla g = (\frac{d}{dt} + B) g \) Where \( B \) can be represented as follows

\[
B = b T_3 \in su(2)
\]

with transformation rule

\[
B \rightarrow h B h^{-1} - \dot{h} h^{-1}
\]

in the other words

\[
b \rightarrow b - \dot{\beta}
\]

The new Lagrangian

\[
\mathcal{L}_G = \langle g^{-1} \nabla g g^{-1} \nabla g \rangle
\]

is invariant under (41) local gauge transformations. But this Lagrangian as well as every gauge invariant Lagrangian is singular. It contains additional non-physical degrees of freedom. To eliminate them we should eliminate \( B \) using Lagrange equations

\[
\frac{\partial \mathcal{L}_G}{\partial B} = 0 \implies b = -\langle \dot{g} g^{-1} T_3 \rangle
\]

put it back in (47) and rewrite last obtained Lagrangian in terms of gauge invariant (physical) variables.

\[
\mathcal{L}_G = \langle (g^{-1} \dot{g} - L_3 T_3)^2 \rangle
\]

It’s obvious that the following

\[
X = g^{-1} T_3 g \in su(2)
\]

element of \( su(2) \) algebra is gauge invariant. Since \( X \in su(2) \) it can be parameterized as follows

\[
X = x^a T_a
\]

where \( x^a \) are real functions on \( SU(2) \)

\[
x_a = \langle X T_a \rangle
\]

So we have three gauge invariant variables \( x^a \ a = 1, 2, 3 \) but it’s easy to check that only two of them are independent. Indeed

\[
\langle X^2 \rangle = \langle g^{-1} T_3 g g^{-1} T_3 g \rangle = \langle T_3^2 \rangle = 1
\]

otherwise

\[
\langle X^2 \rangle = \langle x^a T_a x^b T_b \rangle = x^a x_a
\]

So physical variables take values on \( 2D \) sphere. In other words configuration space of \( SU(2)/U(1) \) model is sphere. By direct calculations one can check that having been rewritten in terms of gauge invariant variables \( \mathcal{L}_G \) takes the form

\[
\mathcal{L}_G = \frac{1}{4} \langle X^{-1} \dot{X} X^{-1} \dot{X} \rangle
\]
This Lagrangian describes free particle on the sphere. Indeed, since \( X = x^a T_a \) it’s easy to show that

\[
\mathcal{L}_G = \frac{1}{4} \langle X^{-1} \dot{X} X^{-1} \dot{X} \rangle = \frac{1}{4} \langle X \dot{X} X \dot{X} \rangle = \frac{1}{2} \dot{x}^a \dot{x}_a
\]

(56)

So \( SU(2)/U(1) \) coset model describes free particle on \( S^2 \) manifold.

5. Quantization of the coset model.

Working in a first order Hamiltonian formalism (see (14)-(16)) we get

\[
\tilde{\mathcal{L}}_G = \langle R(g^{-1} \dot{g} - \tilde{u}) \rangle + \frac{1}{2} \langle (\tilde{u} + g^{-1} B g)^2 \rangle
\]

(57)

\[
\tilde{\mathcal{L}} = \langle R g^{-1} \dot{g} \rangle - \frac{1}{2} \langle R^2 \rangle
\]

(58)

variation of \( \tilde{u} \) provides:

\[
R = \tilde{u} + g^{-1} B g
\]

(59)

\[
\tilde{u} = R - g^{-1} B g
\]

(60)

Rewriting \( \tilde{\mathcal{L}}_G \) in terms of \( R \) and \( g \) leads to

\[
\tilde{\mathcal{L}}_G = \langle R g^{-1} \dot{g} \rangle - \frac{1}{2} \langle R^2 \rangle - \langle B g R g^{-1} \rangle = \langle R g^{-1} \dot{g} \rangle - \frac{1}{2} \langle R^2 \rangle - b \langle g R g^{-1} T_3 \rangle
\]

\[= \langle R g^{-1} \dot{g} \rangle - \frac{1}{2} \langle R^2 \rangle - b L_3 \]

(61)

Due to the gauge invariance of (47) we obtain constrained Hamiltonian system, where \( \langle R g^{-1} dg \rangle \) is symplectic potential, \( \frac{1}{2} \langle R^2 \rangle \) plays role of Hamiltonian and \( b \) is a Lagrange multiple, variation of which leads to the first class constrain:

\[
\phi = \langle g R g^{-1} T_3 \rangle = \langle L T_3 \rangle = L_3 = 0
\]

(62)

Therefore coset model is equivalent to the initial one with (62) constrain. Using technique of the constrained quantization, instead of quantization of the coset model we can submit quantum model , that corresponds to the free particle on \( SU(2) \), to the following operator constrain

\[
\hat{L}_3 |\psi\rangle = 0
\]

(63)

\[
\text{Free particle on } SU(2) \quad \rightarrow \quad \text{Quantum particle on } SU(2)
\]

\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \text{reduction} \quad \text{reduction}
\]

\[
\text{Free particle on } S^2 \quad \rightarrow \quad \text{Quantum particle on } S^2
\]

Hilbert space of the initial sistem (that is linear span of \( \psi^j_{\tau l} \), \( j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ... \) wave functions) reduces to the linear span of \( \psi^j_{\tau l_0} \), \( j = 0, 1, 2, 3, ... \) wave functions. Indeed, \( \hat{L}_3 \psi^j_{\tau l} \) implies \( l = 0 \), and since \( l = 0 \) \( j \) is integer. Therefore \( r \) takes \(-j, -j+1, ..., j-1, j \) integer values only. Wave functions \( \psi^j_{\tau l} \) rewritten in terms
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of gauge invariant variables up to a constant multiple should coincide with well known spherical harmonics

$$\psi_{r0}^j \sim J_{jr}$$ (64)

One can check the following

$$\psi_{r0}^j \sim \hat{L}^j \hat{R}^{-r} (\hat{T} g)^{2j} \sim \hat{R}^{-r} (T_+ g^{-1} T_3 g)^j \sim \hat{R}^{-r} \sin^j \theta e^{ij \theta} \sim \hat{R}^{-r} J_{jj} \sim J_{jr}$$ (65)

This is an example of using large initial model in quantization of coset model.

6. Appendix A

Scalar product in Hilbert space is defined as follows

$$\langle \psi_1 | \psi_2 \rangle = \int_{SU(2)} \prod_{a=1}^{3} \langle g^{-1} \, dg \, T_a \rangle \psi_1^* \psi_2$$ (A.66)

It’s easy to prove that if scalar product is (A.66) operators $$\hat{R}_n$$ and $$\hat{L}_m$$ are hermitian. Indeed

$$\langle \psi_1 | \hat{R}_n \psi_2 \rangle = \int_{SU(2)} \prod_{a=1}^{3} \langle g^{-1} \, dg \, T_a \rangle \psi_1^* \left( \frac{i}{2} \mathcal{L}_{X_n} \psi_2 \right) = \int_{SU(2)} \prod_{a=1}^{3} \langle g^{-1} \, dg \, T_a \rangle \left( \frac{i}{2} \mathcal{L}_{X_n} \psi_1 \right)^* \psi_2$$ (A.67)

Where integration by part have been used. It’s easy to check that the additional term coming from measure

$$\prod_{a=1}^{3} \langle g^{-1} \, dg \, T_a \rangle$$ (A.68)

vanishes since

$$\mathcal{L}_{X_n} \langle g^{-1} \, dg \, T_a \rangle$$ (A.69)

For more transparency one can introduce the following parameterization of SU(2). For any $$g \in SU(2)$$,

$$g = e^{aq^a T_a}$$ (A.70)

Then the symplectic potential takes the form

$$\langle R g^{-1} \, dg \rangle = R_a \, dq^a$$ (A.71)

and scalar product

$$\langle \psi_1 | \psi_2 \rangle = \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} d^3 q \, \psi_1^* \psi_2$$ (A.72)

that coincides with (A.66) because of

$$dq_a = \langle g^{-1} \, dg \, T_a \rangle$$ (A.73)
7. Appendix B

Without loss of generality we can take $\hat{H}$, $\hat{L}_3$ and $\hat{R}_3$ as a complete set of observables. Assuming that there exist at least one eigenfunctions of $\hat{H}, \hat{L}_3$ and $\hat{R}_3$ operators:

$$\hat{H}\psi = E\psi$$  \hspace{1cm} (B.74)
$$\hat{R}_3\psi = r\psi$$  \hspace{1cm} (B.75)
$$\hat{L}_3\psi = l\psi$$  \hspace{1cm} (B.76)

It is easy to show that eigenvalues of $\hat{H}$ are non-negative

$$E \geq 0$$  \hspace{1cm} (B.77)

and

$$E - r^2 \geq 0$$  \hspace{1cm} (B.78)
$$E - l^2 \geq 0$$  \hspace{1cm} (B.79)

Indeed operators $\hat{R}$ and $\hat{L}$ are selfadjoint so

$$\langle \psi | \hat{R}^2 | \psi \rangle = \| \hat{R}_1 \psi \| + \| \hat{R}_2 \psi \| \geq 0$$  \hspace{1cm} (B.80)

To prove (B.78)-(B.79) we shall consider $\hat{R}_1^2 + \hat{R}_2^2$ and $\hat{L}_1^2 + \hat{L}_2^2$ operators

$$\langle \psi | \hat{R}_1^2 + \hat{R}_2^2 | \psi \rangle = \| \hat{R}_1 \psi \| + \| \hat{R}_2 \psi \| \geq 0$$  \hspace{1cm} (B.81)

and

$$\langle \psi | \hat{R}_1^2 + \hat{R}_2^2 | \psi \rangle = \langle \psi | \hat{H} - \hat{R}_3^2 | \psi \rangle = (E - r^2)\langle \psi | \psi \rangle$$  \hspace{1cm} (B.82)

Therefore $E - r^2 \geq 0$ Now let’s introduce new operators

$$\hat{R}_+ = i\hat{R}_1 + \hat{R}_2 \hspace{1cm} \hat{R}_- = i\hat{R}_1 - \hat{R}_2$$  \hspace{1cm} (B.83)
$$\hat{L}_+ = i\hat{L}_1 + \hat{L}_2 \hspace{1cm} \hat{L}_- = i\hat{L}_1 - \hat{L}_2$$  \hspace{1cm} (B.84)

These operators are not selfadjoint, but $\hat{R}_\pm = \hat{R}_+$ and $\hat{L}_\pm = \hat{L}_+$ and they fulfill the following commutation relations

$$[\hat{R}_\pm, \hat{R}_3] = \pm \hat{R}_\pm$$  \hspace{1cm} (B.85)
$$[\hat{L}_\pm, \hat{L}_3] = \pm \hat{L}_\pm$$  \hspace{1cm} (B.86)
$$[\hat{R}_+, \hat{R}_-] = 2\hat{R}_3$$  \hspace{1cm} (B.87)
$$[\hat{L}_+, \hat{L}_-] = 2\hat{L}_3$$  \hspace{1cm} (B.88)
$$[\hat{R}_+, \hat{L}_3] = 0$$  \hspace{1cm} (B.89)

where $\ast$ takes values $+, -, 3$ using these commutation relations it is easy to show that if $\psi_{\lambda}^{\lambda}$ is eigenfunction of $\hat{H}, \hat{L}_3$ and $\hat{R}_3$ with corresponding eigenvalues:

$$\hat{H}\psi_{\lambda}^{\lambda} = \lambda\psi_{\lambda}^{\lambda}$$  \hspace{1cm} (B.90)
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\[ \hat{L}_3 \psi^\lambda_{rl} = l \psi^\lambda_{rl} \]  \hfill (B.91)
\[ \hat{R}_3 \psi^\lambda_{rl} = r \psi^\lambda_{rl} \]  \hfill (B.92)

then \( \hat{R}_\pm \psi^\lambda_{rl} \) and \( \hat{L}_\pm \psi^\lambda_{rl} \) are the eigenfunctions with corresponding eigenvalues \( \lambda, l \pm 1, r \) and \( \lambda, l, r \pm 1 \). Consequently using \( \hat{R}_\pm, \hat{L}_\pm \) operators we construct a family of eigenfunctions with eigenvalues

\[ l, \; l \pm 1, \; l \pm 2, \; l \pm 3, \; \ldots \]  \hfill (B.93)
\[ r, \; r \pm 1, \; r \pm 2, \; r \pm 3, \; \ldots \]  \hfill (B.94)

but conditions (B.78) and (B.79) give restrictions on a possible range of eigenvalues. We should have

\[ \lambda - r^2 \geq 0 \]  \hfill (B.95)
\[ \lambda - l^2 \geq 0 \]  \hfill (B.96)

In other words, in order to interrupt (B.93)-(B.94) sequences we should have

\[ \hat{L}_+ \psi^\lambda_{rj} = 0 \quad \hat{L}_- \psi^\lambda_{r,-j} = 0 \]  \hfill (B.97)
\[ \hat{R}_+ \psi^\lambda_{kl} = 0 \quad \hat{R}_- \psi^\lambda_{-kl} = 0 \]  \hfill (B.98)

and for some \( j \) and \( k \) therefore \( l \) and \( r \) take the following values

\[ -j, \; -j + 1, \; \ldots \; j - 1, \; j \]  \hfill (B.99)
\[ -k, \; -k + 1, \; \ldots \; k - 1, \; k \]  \hfill (B.100)

The number of values is \( 2j + 1 \) and \( 2k + 1 \) respectively. Since number of values should be integer , \( j \) and \( k \) should take integer or half integer values

\[ j = 0, \; \frac{1}{2}, \; 1, \; \frac{3}{2}, \; 2, \; \ldots \]  \hfill (B.101)
\[ k = 0, \; \frac{1}{2}, \; 1, \; \frac{3}{2}, \; 2, \; \ldots \]  \hfill (B.102)

Now using commutation relations we can rewrite \( \hat{H} \) in terms of \( \hat{R}_\pm, \hat{R}_3 \) operators:

\[ \hat{H} = \hat{R}_+ \hat{R}_- + \hat{R}_3^2 + \hat{R}_3 \]  \hfill (B.103)

(B.103) provides that \( \lambda = j(j+1) = k(k+1) \) so \( j = k \) and \( \lambda = j(j+1) \)

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