A note on the limiting entry and return times distributions for induced maps

Nicolai Haydn *

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Abstract

For ergodic measures we consider the return and entry times for a measure preserving transformation and its induced map on a positive measure subset. We then show that the limiting entry and return times distributions are the same for the induced maps as for the map on the entire system. The only assumptions needed are ergodicity and that the measures of the sets along which the limit is taken go to zero.

1 Introduction

In [4] it was shown that on manifolds the limiting return times distributions for an ergodic map is the same for the induced map on a subset if the limit is along a sequence of metric balls. The theorem was proven for measures that satisfy the Lebesgue density theorem, which include absolutely continuous measures and also Radon measures. Here we will show a general version of that statement that only requires ergodicity and does not impose any other restrictions.

The statistics of entry and return times has been studied to quite some degree in the last two decades in particular. A number of results have been achieved for a variety of systems that have good mixing properties as for instance for Axiom A systems or subshifts of finite type and their equilibrium states by using the Laplace transform (Hirata [9], Coelho [5] and Collet) or using the moment method (Pitskel [14] and others as for instance in [7] for rational maps). Galves and Schmitt [6] showed that entry times are exponentially distributed for \( \phi \)-mixing systems and also gave error terms by a method that later was considerable expanded and sharpened by Abadi [1] for \( \phi \)-mixing and later even some \( \alpha \)-mixing systems. A more elementary counting approach by Abadi and Vergne [3] proved that \( \phi \)-mixing measures have exponentially decaying entry and return times. They also have results on multiple return times which are Poissonian for \( \psi \)-mixing measures. In [4] a reduction to induced maps was used to obtain the limiting distribution of return times. The

*Mathematics Department, USC, Los Angeles, 90089-1113. E-mail: <nhaydn@math.usc.edu>. 
important connection there was a result that establishes that the limiting return
times distribution for the given system is the same as the limiting return times
distribution of an induced system. This is the result we expand and sharpen in
this note. It allows to obtain distribution results for many more systems because
instead of checking mixing properties for the given system one can consider the
jump transform which is an induced map on a suitable subset so that the induced
map has good expanding properties that result in exponential decay of correlations.
Such an approach has been particularly successful in the study of parabolic interval
maps like the Manneville-Pommeau map which is non-uniformly expanding because
it has a parabolic fixed point where the derivative is equal to 1. The induced map on
any interval not including the parabolic fixed point is uniformly expanding and its
statistical properties can more easily be analysed exploiting the quasi compactness
of the transfer operator and the consequential exponential decay of correlations.

2 Return times and the induced map

Let $\Omega$ be a measure space with $\sigma$-algebra $\mathcal{B}$ and $\mu$ a probability measure. Moreover $T$ is a measure preserving map on $\Omega$. We assume that $\mu$ is ergodic. For $U \subset \Omega$ we define the function

$$\tau_U(x) = \min\{j \geq 1 : T^j x \in U\},$$

which is the entry time for $x$ if $x \in \Omega$ and is the return time if $x \in U$. We put $\tau_U(x) = \infty$ if the forward orbit of $x$ never enters $U$. Poincaré’s recurrence theorem states that $\tau_U(x) < \infty$ for almost every $x \in U$ for any finite $T$-invariant measure $\mu$ on $\Omega$ and Kac’s theorem \cite{Kac} tells us that $\tau_U$ is integrable on $U$. In fact

$$\int_U \tau_U(x) \, d\mu(x) = 1$$

if $\mu(U) > 0$. Let us note that $\tau_U$ is not necessarily integrable over $\Omega$, in fact

$$\int_\Omega \tau_U \, d\mu < \infty$$

if and only if $\tau_U$ is square integrable over $U$.

For a subset $U \subset \Omega$, $\mu(U) > 0$, let us denote by $\hat{T} = T^{\tau_U} : U \to U$ the induced map. $\hat{T}$ exists by Poincaré’s (or Kac’s) theorem almost everywhere. We also have the induced measure $\hat{\mu}$ which is defined on $U$ by $\hat{\mu}(A) = \frac{\mu(A)}{\mu(U)}$ for all measurable $A \subset U$. The induced measure $\hat{\mu}$ is $\hat{T}$-invariant and ergodic (see e.g. \cite{Liverani}).

2.1 Entry times distributions

Let $B \subset \Omega$ ($\mu(B) > 0$) and put for (parameter values) $t > 0$

$$F_B(t) = \mathbb{P}\left(\tau_B > \frac{t}{\mu(B)}\right) = \mu\left\{x \in \Omega : \tau_B(x) > \frac{t}{\mu(B)}\right\}$$

for the entry time distribution to $B$. The entry times distribution $F_B(t)$ is locally constant on intervals of length $\mu(B)$ and has jump discontinuities at values $t$ which are integer multiples of $\mu(B)$. For any $s \in \mathbb{N}_0$ one has

$$\{\tau_B > s + 1\} = T^{-1}\{\tau_B > s\} \setminus T^{-1}B$$  \hfill (1)
sets\[\mathbb{B}\](results are equivalent by\[8\].

wild looking and in particular won’t be topological balls or cylinder sets for a given

\(F\) that exists a sequence of positive measure sets \(\mathbb{B}\)\(\mathbb{F}\) satisfies

\(F\)\(\mathbb{t}\)\(\mathbb{B}\)(0) = 1, is continuous, convex, monotonically decreasing on \((0, \infty)\) and consequently \(F\) is continuous.

Lacroix\[12\] has shown that if \(F(t)\) is an eligible limiting distribution, that is it satisfies \(F(0) = 1\), is continuous, convex, monotonically decreasing on \((0, \infty)\) and \(F(t) \to 0^+\) as \(t \to \infty\), then for any ergodic \(T\)-invariant probability measure \(\mu\) there exists a sequence of positive measure sets \(B_j \subset \Omega\) so that \(\mu(B_j) \to 0\) and such that \(F(t) = \lim_{j \to \infty} F_{B_j}(t)\) for every \(t \in (0, \infty)\). The sets \(B_j\) are typically pretty wild looking and in particular won’t be topological balls or cylinder sets for a given partition. A similar result was shown for return times in\[11\] although the two results are equivalent by\[8\].

For a positive measure subset \(U \subset \Omega\) let us now consider the induced system \((U, \hat{T}, \hat{\mu})\) which carries the entry time function \(\hat{\tau}_B(x) = \min\{j \geq 1: \hat{T}^j x \in B\}\) for sets \(B \subset U, \hat{\mu}(B) > 0\). As above we can then define the entry times distribution

\[\hat{F}_B(t) = \hat{\mathbb{P}}\left(\hat{\tau}_B > \frac{t}{\hat{\mu}(B)}\right) = \hat{\mu}\left(\left\{x \in U : \hat{\tau}_B(x) > \frac{t}{\hat{\mu}(B)}\right\}\right)\]

The following theorem shows that a restricted system \((U, \hat{T}, \hat{\mu})\) has the same limiting entry times distribution as the original system \((\Omega, T, \mu)\).

**Theorem 1.** Let \(\mu\) be ergodic, \(U \subset \Omega, \mu(U) > 0\). Assume there exists a sequence of sets \(B_j \subset U, \mu(B_j) \to 0^+\), so that either the limiting entry times distribution for \((\Omega, T, \mu)\)

\[F(t) = \lim_{j \to \infty} F_{B_j}(t)\]

exists, or the limiting entry times limiting distribution

\[\hat{F}(t) = \lim_{j \to \infty} \hat{F}_{B_j}(t)\]

for the induced system exists \((U, \hat{T}, \hat{\mu})\).

Then both limiting entry times distributions exist and moreover \(F(t) = \hat{F}(t)\) for all \(t > 0\).

**Proof.** Let \(B = B_j\). We first relate \(\tau_B\) to \(\hat{\tau}_B\) \((B \subset U, \mu(B) > 0)\). If we put \(m = \hat{\tau}_B(x), x \in U, \text{ then}\)

\[\tau_B(x) = \tau_U(x) + \tau_U(\hat{T}x) + \tau_U(\hat{T}^2 x) + \cdots + \tau_U(\hat{T}^{m-1}x) = n^m(x),\]
where we wrote the ergodic sum of the function \( n = \tau_U \) for the return time on \((U, \hat{T})\). By the Birkhoff ergodic theorem on \((U, \hat{T}, \hat{\mu})\) we get as \( \hat{\mu} \) is ergodic:

\[
\frac{1}{m} \tau_B(x) = \frac{1}{m} n^m(x) \to \int_U n(x) \, d\mu(x) = \int_U \tau_U(x) \frac{d\mu(x)}{\mu(U)} = \frac{1}{\mu(U)}
\]
as \( m \to \infty \) by Kac’s theorem for almost every \( x \in U \).

Let \( \varepsilon > 0 \), then there exists \( G_\varepsilon \subset U \), and \( M_\varepsilon \in \mathbb{N} \) so that

\[
\left| \frac{1}{m} n^m(x) - \frac{1}{\mu(U)} \right| < \varepsilon \quad \forall \ x \in G_\varepsilon, \ m \geq M_\varepsilon
\]

and \( \mu(G_\varepsilon^c) < \varepsilon \). Thus

\[
\tau_B(x) = \sum_{j=0}^{\hat{\tau}_B(x)-1} \tau_U \circ \hat{T}^j = \frac{\hat{\tau}_B(x)}{\mu(U)} + O(\hat{\tau}_B(x)\varepsilon)
\]

for all \( x \in G_\varepsilon \) such that \( \hat{\tau}_B(x) \geq M_\varepsilon \). Since \( \tau_U \) is integrable on \( U \) there exists a \( \delta > 0 \) (depending on \( \varepsilon \)) so that \( \int_S \tau_U \, d\mu < \varepsilon \) for any set \( S \subset U \) for which \( \mu(S) < \delta \).

We can assume that \( \mu(G_\varepsilon^c) < \min(\delta, \varepsilon) \).

For \( j = 0, 1, 2, \ldots \) put \( A_j = \Omega \setminus T^{-j}U = T^{-j}U^c \) and

\[
D_j^k = \bigcap_{\ell=j}^k A_\ell = \{ x \in \Omega : T^\ell x \notin U \ \forall \ell = j, \ldots, k \}
\]

for \( 0 \leq j \leq k \). Then for any \( j \in \mathbb{N} \)

\[
\{ x \in \Omega : \tau_U(x) = j \} = T^{-j} U \cap D_j^{j-1} = D_j^{j-1} \setminus D_j^j.
\]

On the other hand we also have

\[
\{ x \in U : \tau_U(x) \geq j \} = U \cap D_1^{j-1}.
\]

We now do the following decomposition (as \( D_1^{j-1} = \{ x \in \Omega : \tau_U(x) \geq j \} \)):

\[
F_B(t) = \int_\Omega \chi_{\tau_B > s} \, d\mu
\]

\[
= \sum_j \int_{\{ \tau_U = j \}} \chi_{\tau_B > s} \, d\mu
\]

\[
= \sum_j \left( \int_{D_1^{j-1}} \chi_{\tau_B > s} \, d\mu - \int_{D_j^j} \chi_{\tau_B > s} \, d\mu \right),
\]
as \( D_1^j \subset D_1^{j-1} \), where we wrote \( s = \frac{t}{m(B)} \). For the second term in the last line consider

\[
\int_{D_0^{j-1}} \chi_{\tau_B > s} \, d\mu = \int_\Omega \left( \chi_{D_0^{j-1}} \chi_{\tau_B > s} \right) \circ T \, d\mu = \int_{D_j^j} \chi_{\tau_B > s} \circ T \, d\mu
\]
as $T^{-1}D_{0}^{j-1} = D_{1}^{j}$. The inclusions
\[
\{\tau_B > s + 1\} \subset T^{-1}\{\tau_B > s\} \subset \{\tau_B > s + 1\} \cup T^{-1}B
\]
imply the inequalities
\[
\int_{D_{1}^{j}} \chi_{\tau_B > s + 1} d\mu \leq \int_{D_{1}^{j}} \chi_{\tau_B > s} \circ T d\mu \leq \int_{D_{1}^{j}} \chi_{\tau_B > s + 1} d\mu \leq \int_{D_{1}^{j}} \chi_{T^{-1}B} d\mu
\]
where the last integral is equal to zero as $T^{-1}B \cap D_{1}^{j} = T^{-1}(B \cap D_{0}^{j-1}) = \emptyset$ because $D_{0}^{j-1} \subset U \subset B^c$ for $j \geq 1$. Thus
\[
\int_{D_{1}^{j}} \chi_{\tau_B > s} d\mu = \int_{D_{1}^{j-1}} \chi_{\tau_B > s - 1} \circ T d\mu = \int_{D_{0}^{j-1}} \chi_{\tau_B > s - 1} d\mu
\]
and
\[
F_{B}(t) = \sum_{j} \left( \int_{D_{1}^{j-1}} \chi_{\tau_B > s} d\mu - \int_{D_{0}^{j-1}} \chi_{\tau_B > s - 1} d\mu \right)
\]
\[
= \sum_{j} \int_{C_{j}} \chi_{\tau_B > s} d\mu + E_{0}
\]
\[
= \int_{U} \tau_{U} \chi_{\tau_B > s} d\mu + E_{0}
\]
where we put $C_{j} = D_{1}^{j-1} \setminus D_{0}^{j-1} = \{x \in U : \tau_{U}(x) \geq j\}$ and used that $\sum_{j} \chi_{C_{j}} = \tau_{U}$ on $U$ (as $|\{j : x \in C_{j}\}| = \tau_{U}(x) \forall x \in U$). To estimate the error term $E_{0}$ note that
\[
P(\tau_B = s) = P(\tau_B > s - 1) - P(\tau_B > s) \leq \mu(B)
\]
(see the remark preceding the theorem). Since $D_{0}^{j-1} = \{x \in U^c : \tau_{U}(x) \geq j\}$ one has
\[
-E_{0} = \sum_{j} \int_{D_{0}^{j-1}} \chi_{\tau_B = s} d\mu \leq \int_{\Omega} \tau_{U} \chi_{\tau_B = s} d\mu.
\]
In order to show that $E_{0}$ goes to zero as $\mu(B)$ decreases to zero let us put $B_{k,j} = U_{k} \cap T^{-j}\{\tau_B = s\}$ for $j = 0, 1, \ldots, k - 1$ and $k = 1, 2, \ldots$, where $U_{k} = \{x \in U : \tau_{U}(x) = k\}$. Then $B_{k,j} \cap B_{k,i} = \emptyset$ if $i \neq j$ because if there were an $x \in B_{k,j} \cap B_{k,i}$ it would imply that $T^{j}x, T^{i}x \in \{\tau_B = s\}$ and therefore, assuming $j > i$, one would get the contradiction $s = \tau_B(T^{j}x) > \tau_B(T^{i}x) = s$. Hence the sets $B_{k,j}$ are pairwise disjoint for $k \in \mathbb{N}$ and $j = 0, \ldots, k - 1$. In other words, for every $x \in B_{k}$ there is a unique $j \in [0, k)$ so that $T^{j}x \in \{\tau_B = s\}$. Consequently $\{\tau_B = s\} = \bigcup_{k=1}^{\infty} \bigcup_{j=0}^{k-1} T^{j}B_{k,j}$, and since $\mu(T^{j}B_{k,j}) \geq \mu(B_{k,j})$ we obtain
\[
\mu(\tilde{B}) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \mu(B_{k,j}) \leq \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \mu(T^{j}B_{k,j}) = P(\tau_B = s) \leq \mu(T^{-s}B) = \mu(B),
\]
where $\tilde{B} = \bigcup_{k=1}^{\infty} \bigcup_{j=0}^{k-1} B_{k,j} (\tilde{B} \subset U)$. For $x \in \tilde{B}$ for which $T^{j}x \in \{\tau_B = s\}$ one has $\tau_{U}(x) \geq \tau_{U}(T^{j}x)$ and therefore
\[
|E_{0}| \leq \int_{\tilde{B}} \tau_{U} d\mu < \varepsilon
\]
as we can assume that \( \mu(B) < \delta \).

Replacing \( \tau_B \) by \( \hat{\tau}_B \) using the relation \( \frac{\tau_B}{\mu(U)} = \frac{1}{\mu(U)} + O(\varepsilon) \) implies \( \tau_B = \frac{\varepsilon \mu(U)}{\mu(B)} + \eta \) where \( \eta : U \to \mathbb{R} \) has the bound \( |\eta| \leq \varepsilon \mu(U) \tau_B \leq \varepsilon \hat{\tau}_B \). Hence

\[
F_B(t) = \int_U \tau_U \chi_{\hat{\tau}_B > \frac{t}{\mu(B)} + \eta} d\mu + E_0.
\]

Now we want to introduce a power \( k \) of the induced map \( \hat{T} \) so that we can average over \( k \) and use the ergodic theorem on \((U, \hat{T}, \hat{\mu})\). By \( \hat{T} \)-invariance of \( \hat{\mu} \)

\[
F_B(t) = \int_U \left( \tau_U \chi_{\hat{\tau}_B > \frac{t}{\mu(B)} + \eta} \right) \circ \hat{T}^k d\mu + E_0
\]

\[
= \int_{G_{\varepsilon}} \left( \tau_U \chi_{\hat{\tau}_B > \frac{t}{\mu(B)} + \eta} \right) \circ \hat{T}^k d\mu + E_0 + H_k,
\]

where we get for the error

\[
H_k = \int_{G_{\varepsilon}} \left( \tau_U \chi_{\hat{\tau}_B > \frac{t}{\mu(B)} + \eta} \right) \circ \hat{T}^k d\mu \leq \int_{G_{\varepsilon}} \tau_U \circ \hat{T}^k d\mu = \int_{\hat{T}^{-k} G_{\varepsilon}} \tau_U d\mu < \varepsilon
\]

since by assumption \( \mu(\hat{T}^{-k} G_{\varepsilon}^c) = \mu(G_{\varepsilon}^c) < \delta \) as \( \mu \) restricted to \( U \) is \( \hat{T} \)-invariant.

In the principal term we want to exploit the identity \( \sum_j \chi_{C_j} = \tau_U \) on \( U \). For that purpose note that

\[
\left\{ x \in U : \hat{\tau}_B(\hat{T}^k x) \geq s \right\} \setminus \bigcup_{\ell=1}^{k-1} \hat{T}^{-\ell} B = \{ x \in U : \hat{\tau}_B(x) \geq s + k \}
\]

which yields (here we use \( s = \frac{t}{\mu(B)} + \eta \))

\[
F_B(t) = \int_{G_{\varepsilon}} \left( \tau_U \chi_{\hat{\tau}_B > \frac{t}{\mu(B)} + \eta + k} \right) \circ \hat{T}^k d\mu + E_0 + H_k + K_k,
\]

where the individual errors are bounded by:

\[
K_k \leq \int_{G_{\varepsilon}} \tau_U \circ \hat{T}^k \sum_{\ell=1}^{k-1} \chi_B \circ \hat{T}^\ell d\mu.
\]

We now estimate the average error over \( k \in \{0, 1, \ldots, n - 1\} \):

\[
\hat{K}_n = \frac{1}{n} \sum_{k=0}^{n-1} K_k
\]

\[
= \int_{G_{\varepsilon}} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\ell=1}^{k-1} (\tau_U \circ \hat{T}^k)(\chi_B \circ \hat{T}^\ell) d\mu
\]

\[
= \int_{G_{\varepsilon}} \sum_{\ell=1}^{n-2} (\chi_B \circ \hat{T}^\ell) \left( \frac{1}{n} \sum_{k=\ell+1}^{n-1} \tau_U \circ \hat{T}^k \right) d\mu
\]

\[
\leq c_1 \frac{1}{\mu(U)} \int_{G_{\varepsilon}} \sum_{\ell=1}^{n-2} (\chi_B \circ \hat{T}^\ell) d\mu
\]

\[
\leq c_1 n \hat{\mu}(B)
\]
where we used the estimate

$$\frac{1}{n} \sum_{k=\ell+1}^{n-1} \tau_U \circ \hat{T}^k \leq \frac{1}{n} \sum_{k=0}^{n-1} \tau_U \circ \hat{T}^k \leq \frac{1}{\mu(U)} + \varepsilon \leq c_1 \frac{1}{\mu(U)}$$

for some constant $c_1$ and for all $x \in G_\varepsilon$ provided $n \geq M_\varepsilon$. Thus

$$F_B(t) = \frac{1}{n} \sum_{k=1}^{n} \int_{G_\varepsilon} \left( \tau_U \circ \hat{T}^k \right) \chi_{\hat{T}_B > \frac{1}{\varepsilon} + \frac{1}{\mu(U)}} d\mu + E_0 + \hat{H}_n + \hat{K}_n,$$

where $\hat{H}_n = \frac{1}{n} \sum_{k=0}^{n-1} H_k < \varepsilon$ and $\eta'' : U \rightarrow \mathbb{R}$ satisfies the bound $|\eta''| < \varepsilon$. Consequently (as $|E_0 + \hat{K}_n| < \varepsilon + c_1 n \hat{\mu}(B)$)

$$F_B(t) = \int_{G_\varepsilon} \frac{1}{n} \sum_{k=1}^{n} \left( \tau_U \circ \hat{T}^k \right) \chi_{\hat{T}_B > (1+\eta'') \frac{1}{\mu(U)}} d\mu + O(\varepsilon + n \hat{\mu}(B))$$

as $\frac{1}{n} \sum_{k=1}^{n} \tau_U \circ \hat{T}^k \rightarrow \hat{T}_B$ on $G_\varepsilon$, where $\eta'' : U \rightarrow \mathbb{R}$ satisfies $|\eta''| < c_2 |\eta'| + \frac{\varepsilon}{\mu(U)} (c_2 > 0)$. To adjust for the ‘time shift’ in the lower bound of the entry function, we use the fact that $|\hat{F}_B(t) - \hat{F}_B(s)| \leq |t - s| + \hat{\mu}(B)$ and thus obtain (for a $c_3$)

$$|F_B(t) - \hat{F}_B(t)| \leq c_3 \varepsilon + \left( \frac{n}{t} + 1 \right) \hat{\mu}(B) + c_1 n \hat{\mu}(B)$$

for all $n \geq M_\varepsilon$. If $\mu(B_j)$ is small enough so that $\hat{\mu}(B_j) < \min\left( \frac{\varepsilon t}{M_\varepsilon}, \frac{\varepsilon}{(c_1+1)n} \right)$ (if we choose $n = M_\varepsilon$ this requires $\hat{\mu}(B_j) < \frac{\varepsilon}{M_\varepsilon} \min(t, \frac{1}{c_1+1})$) then

$$|F_{B_j}(t) - \hat{F}_{B_j}(t)| < (c_3 + 1) \varepsilon$$

and as $\mu(B_j) \rightarrow 0 (j \rightarrow \infty)$ we obtain $|F(t) - \hat{F}(t)| < (c_3 + 1) \varepsilon$ for any positive $\varepsilon$. Thus if the limiting distribution $F(t)$ exists then also the limiting distribution $\hat{F}(t)$ exists and vice versa. Moreover we obtain equality: $F(t) = \hat{F}(t)$ for all $t > 0$.

### 2.2 Return times distributions

The restriction of the function $\tau_B$ to the set $B \subset \Omega$ is called the return time function and we correspondingly call

$$\hat{F}_B(t) = \mathbb{P}_B \left( \tau_B > \frac{t}{\mu(B)} \right)$$

the return times distribution. For instance, if $\Omega$ is the shiftspace $\Sigma$ and $B = U(x_0x_1 \cdots x_{n-1})$ is an $n$-cylinder then $\tau_B(\bar{x})$ for $\bar{x} \in B$ measures the ‘time’ it takes to see the word $x_0x_1 \cdots x_{n-1}$ again, that is

$$\tau_B(\bar{x}) = \min\{ j \geq 1 : x_j \bar{x}_{j+1} \cdots x_{j+n-1} = x_0x_1 \cdots x_{n-1} \}.$$
The function \( \tilde{F}_B(t) \) then measures the probability to see the first \( n \)-word again after rescaled time \( t/\mu(B) \).

Similarly for the induced system \( (U, \hat{T}, \hat{\mu}) \) we have the return times distribution

\[
\hat{F}_B(t) = \hat{\mathbb{P}}_B \left( \hat{\tau}_B > \frac{t}{\hat{\mu}(B)} \right) = \hat{\mu} \left( \left\{ x \in B : \hat{\tau}_B(x) > \frac{t}{\hat{\mu}(B)} \right\} \right).
\]

In order to get a similar result on the relation between return times for the original system and the induced system, we will need the following result.

**Proposition 2.** [8] Let \( B_j \subset \Omega \) (\( \mu(B_j) > 0 \)) be a sequence of sets so that \( \mu(B_j) \to 0^+ \). If one of the limits \( F(t) = \lim_{j \to \infty} F_{B_j}(t) \), \( \tilde{F}(t) = \lim_{j \to \infty} \tilde{F}_{B_j}(t) \) exists (point-wise) then so does the other limit and moreover

\[
F(t) = \int_t^\infty \tilde{F}(s) \, ds.
\]

While the limiting entry times distribution \( F(t) \) is always Lipshitz continuous with Lipshitz constant 1, the same does not apply to the limiting return times distribution \( \tilde{F}(t) \) which in fact can have (at most countable many) discontinuities. In particular, if the sets \( B_j \) contract to a periodic point, then \( \tilde{F}(t) \) will have a discontinuity at \( t = 0 \) with \( \lim_{t \to 0^+} \tilde{F}(t) < 1 \). Also note that since the limiting entry distribution \( F \) is Lipshitz continuous the limiting return distribution \( \tilde{F}(t) \) is monotonically decreasing to zero which implies that \( F(t) \) is in fact always convex.

One consequence of this result is that the limiting entry times distribution and return times distribution are the same only if they are exponential, that is \( \tilde{F} = F \) if only if \( F(t) = \tilde{F}(t) = e^{-t} \). We use this proposition to obtain the corresponding result of Theorem 1 for the limiting return times distribution.

**Theorem 3.** Let \( \mu \) be ergodic, \( U \subset \Omega \), \( \mu(U) > 0 \). Assume there exists a sequence of sets \( B_j \subset U \), \( \mu(B_j) \to 0^+ \), so that one of the two limiting return times distribution

\[
either \quad \tilde{F}(t) = \lim_{j \to \infty} \tilde{F}_{B_j}(t), \quad or \quad \hat{F}(t) = \lim_{j \to \infty} \hat{F}_{B_j}(t)
\exists.
\]

Then both limiting return times distributions exist and moreover at every point of continuity \( t \in \mathbb{R}^+ \) one has equality \( \tilde{F}(t) = \hat{F}(t) \).

This is the result that was proven in [4] in 2003 for Radon measures on Riemann manifolds using the Lebesgue Density theorem. The limit there was along metric balls \( B_j \) that shrink to a point \( x \) and with the implication that the existence of the limiting return times distribution in the induced system (plus the non-degeneracy condition \( \tilde{F}(0^+) = 1 \)) implies the limiting return times distribution for the entire system and that the two limiting distributions are equal at points of continuity.

**Proof of Theorem 3.** Assume that, say, the limit \( \tilde{F}(t) = \lim_{j \to \infty} \tilde{F}_{B_j}(t) \) exists. By Proposition 2 this implies the also the limiting distribution \( F(t) = \lim_{j \to \infty} F_{B_j}(t) \) exists. By Theorem 1 we get that the limit \( \hat{F}(t) = \lim_{j \to \infty} \hat{F}_{B_j}(t) \) exists and satisfies
Thus, since \( \hat{F} = F \) again by Proposition 2 this implies the limit \( \hat{F}(t) = \lim_{j \to \infty} \hat{F}_{B_j}(t) \) exists. Thus, since

\[
\int_t^\infty \hat{F}(s) \, d\mu(s) = F(t) = \hat{F}(t) = \int_t^\infty \hat{F}(s) \, d\mu(s)
\]

for all \( t > 0 \) we conclude that \( \hat{F}(t) = \hat{F}(t) \) at all points \( t \) of continuity.

Similarly, we show that the limit \( \hat{F}(t) = \lim_{j \to \infty} \hat{F}_{B_j}(t) \) implies the return times limiting distribution \( \hat{F}(t) = \lim_{j \to \infty} \hat{F}_{B_j}(t) \) for the whole system and also equality of the limiting distributions \( \hat{F}(t) = \hat{F}(t) \) at points of continuity.

Let us note that since only the requirements in Theorem 1 and 2 are the existence of the limiting distribution \( \tilde{F} \) for the induced map. We consider the shift space \( \Omega = \mathbb{N}^\mathbb{Z} \) with the shift transformation \( \sigma \). To define the transition probabilities: Let \( p_i \in (0, 1), i = 1, 2, \ldots, \) be a sequence, then we allow for the transition \( i \to i + 1 \) with probability \( p_i \) and for the transition \( i \to 1 \) with probability \( q_i = 1 - p_i \). In other words, we can define a stochastic matrix \( M \) by

\[
\begin{align*}
M_{j,1} &= q_j \\
m_{j,j+1} &= p_j \\
m_{j,k} &= 0 \text{ otherwise, i.e. if } k \neq 1 \text{ or } k \neq j + 1
\end{align*}
\]

where the transition probability of the transition \( j \to k \) is given by the entry \( M_{j,k} \). Then \( M \mathbf{1} = \mathbf{1} \) as \( \sum_{k=1}^{\infty} M_{j,k} = M_{j,1} + M_{j,j+1} = q_j + p_j = 1 \forall j \) and \( M \) has the left eigenvector \( \tilde{x} = (x_1, x_2, \ldots) \) (for the dominant eigenvalue 1) which satisfies

\[
q_1 x_1 + q_2 x_2 + q_3 x_3 + \cdots = x_1
\]

\[
x_j p_j = x_{j+1} \text{ for } j = 1, 2, \ldots
\]

One sees that the components of the left eigenvector are \( x_j = x_1 P_j, j = 2, 3, \ldots, \)

where \( P_j = \prod_{i=1}^{j-1} p_i \) (\( P_1 = 1 \)) and \( x_1 \) is chosen to make \( \tilde{x} \) a probability vector \( x_1^{-1} = \sum_j P_j \). We assume \( x_1 > 0 \). The first equation above is satisfied as \( \sum_j q_j x_j = x_1 \sum_j (P_j - P_{j+1}) = x_1 P_j \) if \( P_j \to 0 \) as \( j \to \infty \). In this way we obtain a shift invariant probability measure \( \mu \) on \( \Omega \) which is ergodic as one can go from any state \( i \) to any other state \( j \) with positive probability.

Put \( A_j = \{ \tilde{\omega} \in \Omega : \omega_0 = j \}, j = 1, 2, \ldots, \) and let \( U = A_1 \) be the return set with return/entry time function \( \tau_U \). If we put \( A_{j,k} = A_j \cap \{ \tau_U = k \} \) then \( \tilde{\omega} \in A_{j,k} \) is of the form \( \omega_0 \omega_1 \cdots \omega_k = j(j + 1)(j + 2) \cdots (j + k - 2)(j + k - 1) \) (symbol sequence of length \( k + 1 \)). One has

\[
\mu(A_{j,k}) = \mu(A_j) p_j p_{j+1} \cdots p_{j+k-2} q_{j+k-1} = x_1 P_j \frac{P_{j+k-1}}{P_j} q_{j+k-1} = x_1 P_{j+k-1} q_{j+k-1}
\]
as \( \mu(A_j) = x_j = x_1 P_j \). Let \( D \) be the countably infinite partition of \( U \) whose partition elements are \( D_j = \{ \omega \in U : \tau_U(\omega) = j \} \) \( (D_j = A_{1,j}) \). The induced map \( \hat{\sigma} : U \to U \) is a Bernoulli shift on \( \hat{\Omega} = D \mathbb{Z} \) and the induced measure \( \hat{\mu} \) is the Bernoulli measure with weights \( \hat{\mu}(P_j) = \frac{1}{\mu(U)} x_1 q_j P_j \), where \( \mu(U) = \sum_j x_1 q_j P_j \). If we denote by \( B_n \) the \( n \)-cylinder which contains a given point \( \hat{\omega} \in \hat{\Omega} \) then the entry times \( \hat{F}_{B_n}(t) \) converge to the exponential distribution \( e^{-t} \) as \( n \to \infty \) for almost every \( \hat{\omega} \). Hence we conclude that entry times \( F_{B_n} \) for the map \( \sigma \) on \( \Omega \) also converge to the limiting distribution \( e^{-t} \) almost surely.

**Remark.** Kac’s theorem states that the return time function \( \tau_U \) is integrable over \( U \) and also gives the value of the integral. We can use this example to achieve that \( \tau_U \) is not integrable over the entire space \( \Omega \) although the measure is ergodic. The integral of \( \tau_U \) over the entire space is

\[
\int \tau_U d\mu = \sum_{j,k} k \mu(A_{j,k}) = \sum_{j,k} k x_1 P_{j+k-1} q_{j+k-1}.
\]

If we choose \( p_i = \left( \frac{i}{i+1} \right)^\alpha \) for some \( \alpha \in (1, 2) \) then \( P_j = \prod_{i=1}^{j-1} \left( \frac{i}{i+1} \right)^\alpha = \frac{1}{j^\alpha} \) and since the \( P_j \) are summable, \( x_1 = \left( \sum_j P_j \right)^{-1} \) is well defined and positive. Then

\[
\int \tau_U d\mu = \frac{1}{x_1} \sum_k \sum_j \frac{1}{(j+k-1)^\alpha} q_{j+k-1}
\]

\[
\geq \frac{c_1}{x_1} \sum_k \sum_j \frac{1}{(j+k-1)^{\alpha+1}}
\]

\[
\geq c_2 \sum_k \frac{k}{k^\alpha} = \infty,
\]

as \( \alpha < 2 \), where we used that \( q_{j+k-1} = 1 - \left( 1 - \frac{1}{j+k-1} \right)^\alpha \geq c_1 \frac{1}{j+k-1} \) for some \( c_1 > 0 \). We thus see that the integral of \( \tau_U \) over the entire space \( \Omega \) diverges.

This can be converted to an example on a two-state shiftspace \( \Sigma \subset \{0, 1\} \mathbb{Z} \) by the single element mapping \( \pi : \Omega \to \Sigma \) which maps \( \pi(1) = 1 \) and collapses all other symbols to 0, i.e. \( \pi(j) = 0, j = 2, 3, \ldots \). The measure \( \mu \) is sent to the probability measure \( \nu = \pi^* \mu \) which is invariant under the shift map.

In fact \( \int \tau_U d\mu \) is finite if and only if \( \int U \tau_U^2 d\mu \) is finite. So the above example is an example where the return time to \( U \) is not square integrable over \( U \).

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