GROUPS AND RINGS DEFINABLE IN D-MINIMAL STRUCTURES

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Abstract. We study groups and rings definable in d-minimal expansions of ordered fields. We generalize to such objects some known results from o-minimality. In particular, we prove that we can endow a definable group with a unique definable topology making it a definable manifold and a topological group, and that a definable ring of dimension at least 1 and without zero divisors is a skew field.

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1. Introduction

Let $K$ be a first-order expansion of an ordered field. Recall that $K$ is o-minimal if every definable (by “definable” we will always mean “definable in $K$ with parameters”) subset of $K$ is a finite union of points and intervals with endpoints in $K \cup \{ \pm \infty \}$. O-minimal structures have been widely studied (see [vdD98] for an introduction). There is a rich literature on groups definable in o-minimal structures; see for instance [Pil88, OPP96, PPS00] for problems treated in this article (we will discuss those references more in details in the following sections), and [Ote08, Pet10] for an overview. One of the starting points was A. Pillay’s theorem that any such group can be endowed in a unique way with a topology that makes it both a topological group and a definable manifold. An important result in this context is the solution of

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the “Pillay’s conjecture” and its extension to groups definable in NIP structures (see [HPP08]).

In this article, we extend some of the previous results about groups and rings definable in o-minimal structures to groups and rings definable in d-minimal structures.

Recall that $K$ is said to be **definably complete** if every definable subset of $K$ has a supremum in $K \cup \{ \pm \infty \}$ (see e.g. [PS10] and its bibliography). In this article we study the following generalization of o-minimality:

**Definition 1.1.** $K$ is **d-minimal** if it is definably complete, and every definable set $X \subset K$ is the union of an open set and finitely many discrete sets, where the number of discrete sets does not depend on the parameters of definition of $X$.

Notice that $K$ is o-minimal iff it is definably complete and every definable unary set is the union of an open set and finitely many points.

[vdD85] gives the first known example of a d-minimal non o-minimal expansion of $\mathbb{R}$, [FM05] [MT06] give more examples of d-minimal expansions of $\mathbb{R}$ (and introduce the notion of d-minimality), [Mil05] studies general properties of d-minimal expansions of $\mathbb{R}$ (and other such “tameness” notions), and in [For10a] we studied d-minimal structures in general (see §2 for an overview).

Let $K$ be a d-minimal structure. The main results are:

**Theorem 1.2** ([Pil88]). Let $G := \langle G, \cdot, e, -1 \rangle$ be a definable group. Then, we can endow $G$ with a unique differential structure (see Definition 3.1), such that $\cdot$ and $-1$ are differentiable functions.

Similarly, if $F$ is a definable ring, then we can endow $F$ with a unique differential structure, such that the ring operations are differentiable functions.

**Theorem 1.3.** Let $G$ be a definable Abelian group. If $G$ is definably connected, then it is divisible.

**Theorem 1.4** ([PPS00]). Let $G$ be a definable group. Assume that $G$ is definably connected, centerless, and semisimple (i.e., every definable normal Abelian subgroup is discrete). Then, $G$ is definably isomorphic to a semi-algebraic group.

**Theorem 1.5** ([OPP96]). Let $F$ be a definable ring without 0 divisors, such that $\dim(F) \geq 1$. Then, $F$ is a skew field, and it is definably isomorphic to either $K$, $K(\sqrt{-1})$, or the rings of quaternions over $K$.

As far as we know, the following theorem is new even for o-minimal structures.

**Theorem 1.6.** Let $F$ be a definable ring with 1. Then, there exist a definable $K$-algebra $F^0 \subseteq F$ and a definable discrete subring $D \subseteq F$, such that, as rings, $F = D \oplus F^0$. 
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2. Preliminaries on d-minimal structures

2.1. Conventions and notation. \( \mathbb{K} \) will always be a d-minimal expansion of an ordered field \( \mathbb{K} := \langle K, +, \cdot, <, 0, 1 \rangle \), and “definable” will always mean “definable with parameters from \( K \)”. Moreover, \( \overline{X} \) or \( \text{cl}(X) \) denote the topological closure of \( X \). \( \mathbb{R} := \langle \mathbb{R}, +, \cdot, <, 0, 1 \rangle \) is the field of real numbers.

2.2. Some examples.

(1) A definably complete expansion of an ordered field \( K \) is locally o-minimal if every definable subset of \( K \) with empty interior is discrete (see [For10b, Sch11]). Clearly, a locally o-minimal structure is d-minimal, and an ultraproduct of o-minimal structures is locally o-minimal (but not necessarily o-minimal).

(2) Given \( c \in \mathbb{R} \), denote \( c^\mathbb{Z} := \{ c^n : n \in \mathbb{Z} \} \). The structure \( \langle \mathbb{R}, +, \cdot, c^\mathbb{Z} \rangle \) is d-minimal for every \( c \in \mathbb{R} \) (see [vdD85]); on the other, the structure \( \langle \mathbb{R}, +, \cdot, 2^\mathbb{Z}, 3^\mathbb{Z} \rangle \) not only is not d-minimal, but it defines the set of natural numbers (see [Hie10]).

(3) Let \( R \) be an o-minimal expansion of \( \overline{\mathbb{R}} \), \( (a_i)_{i \in \mathbb{N}} \) be a “fast sequence”, let \( A := \{ a_i : i \in \mathbb{N} \} \). Then, the expansion of \( \mathbb{R} \) by a predicate for each subset \( A^n \) for every \( n \in \mathbb{N} \) is d-minimal (see [FM05] for the relevant definitions and proofs).

(4) For other examples of d-minimal expansions of \( \mathbb{R} \), see [Mil05].

Notice that NIP can fail in d-minimal structures: there are d-minimal expansion of \( \mathbb{R} \) that define an isomorphic copy of \( \langle \mathbb{N}, +, \cdot \rangle \) (see Example 3) and there exist ultraproducts of o-minimal structures that do not satisfy NIP (see [For10a]).

2.3. Previous results. See [Mil05] for d-minimal expansions of \( \mathbb{R} \), and [For10a] for general d-minimal structures.

Definition 2.1. Let \( X \subseteq \mathbb{K}^n \) be a definable set. We say that \( X \) is an embedded \( C^k \)-manifold of dimension \( d \) if, for every \( x \in X \) there exists an open box \( U \) containing \( x \), such that, after a permutation of coordinates, \( X \cap U \) is the graph of a \( C^k \)-function \( f : V \to W \), where \( V \subseteq \mathbb{K}^d \) and \( W \subseteq \mathbb{K}^{n-d} \) are the unique boxes satisfying \( U = V \times W \).

Here we will use the following facts (from [For10a]).

Fact 2.2. Let \( X \) be a definable set. Then, \( X \) is the disjoint union of finitely many embedded manifolds.

Definition 2.3. Let \( X \subseteq \mathbb{K}^n \) be a definable set. The dimension of \( X \), denoted by \( \dim(X) \), is \(-\infty\) if \( X \) is empty, and otherwise it is the smallest integer \( d \) such that there exists a \( d \)-dimensional coordinate
space \( L \), such that \( \pi_L(X) \) has nonempty interior in \( L \), where \( \pi(L) \) is the orthogonal projection onto \( L \).

**Fact 2.4.** \( \dim \) satisfies the axioms for a dimension function in [vdD89]. In particular, \( \dim(X \cup Y) = \max(\dim(X), \dim(Y)) \), and if \( f : X \to Y \) is a definable function, such that \( \dim(f^{-1}(y)) = d \) for every \( y \in Y \), then \( \dim(Y) = d + \dim(X) \). Moreover, if \( X \) is an embedded manifold, \( \dim(X) \) coincides with the dimension of \( X \) as a manifold. Finally, \( \dim(X) \leq 0 \) iff \( X \) is a finite union of (definable) discrete sets.

However, unlike the o-minimal case, it is not true in general that, for \( X \) nonempty, \( \dim(\overline{X} \setminus X) < \dim(X) \).

Notice that every embedded manifold is locally closed in \( \mathbb{K}^n \). Thus, we have the following fact.

**Fact 2.5.** Let \( X \subseteq \mathbb{K}^n \) be definable. Then, \( X \) is constructible: that is, \( X \) is the union of finitely many definable locally closed sets.

**Fact 2.6.** Let \( f : \mathbb{K}^n \to \mathbb{K}^m \) be a definable function, and \( k \in \mathbb{N} \). Then, there exists a closed definable set \( C \subseteq \mathbb{K}^n \) with empty interior, such that, outside \( C \), \( f \) is \( C^k \).

For every \( C^1 \) function \( f : \mathbb{K}^m \to \mathbb{K}^n \), define \( \Lambda_f(k) := \{ x \in \mathbb{K}^m : \text{rk}(d_x(f)) \leq k \} \), (where \( \text{rk}(M) \) is the rank of the matrix \( M \) and \( d_x(f) \) is the differential of \( f \) at \( x \)), and \( \Sigma_f(k) := f(\Lambda_f(k)) \). The set of singular values of \( f \) is \( \Sigma_f := \bigcup_{k=0}^{n-1} \Sigma_f(k) \).

**Fact 2.7** (Sard’s Lemma). Let \( f : \mathbb{K}^m \to \mathbb{K}^n \) be definable and \( C^1 \). Then, for every \( d \leq n \), \( \dim(\Sigma_f(d)) \leq d \). In particular, there exists \( c \in \mathbb{K}^m \) that is a regular value for \( f \).

**Definition 2.8.** Let \( X \subseteq \mathbb{K}^n \) be a definable set, and \( p \in \mathbb{N} \). Let \( \text{reg}^p(X) \) denote the set of all \( x \in X \) such that, for some open box \( U \) containing \( a \), \( X \cap U \) is a \( C^p \) embedded manifold of the same dimension as \( X \).

**Fact 2.9.** Let \( X \subseteq \mathbb{K}^n \) be a definable set, and \( p \in \mathbb{N} \). Then, \( X \setminus \text{reg}^p(X) \) is nowhere dense in \( X \).

**Fact 2.10** (Dimension is local: see [FH12]). Let \( X \subseteq \mathbb{K}^n \) be a definable set. Assume that, for every \( x \in X \), there exists a definable neighbourhood \( U \) of \( x \), such that \( \dim(U \cap X) \leq d \). Then, \( \dim(X) \leq d \).

**Fact 2.11.** \( \mathbb{K} \) has definable Skolem functions and definable choice.

Thus, (almost) all the results about definable sets can be extended to sets that are interpretable in \( \mathbb{K} \).

**Definition 2.12.** (1) Let \( \bar{c} \in \mathbb{K}^n \) and \( \bar{A} \subseteq \mathbb{K} \). We say that \( \text{rk}^Z(\bar{c}/\bar{A}) \leq d \) if there exists a \( d \)-dimensional set \( X \) definable with parameters from \( A \), such that \( \bar{c} \in X \). We say that \( \text{rk}^Z(\bar{c}/\bar{A}) = d \) if \( \text{rk}^Z(\bar{c}/\bar{A}) \leq d \) and \( \text{rk}^Z(\bar{c}/\bar{A}) \neq d - 1 \).
(2) Let \( X \subseteq \mathbb{K}^n \) be a definable set, \( A \subseteq \mathbb{K} \) be a set containing the parameters of definition of \( X \), and \( \bar{c} \in X \). We say that \( \bar{c} \) is generic in \( X \) over \( A \), if for every set \( Y \) definable over \( A \), if \( \bar{c} \in Y \), then \( \dim(Y) \geq \dim(X) \).

**Fact 2.13.** (1) \( \bar{c} \) is generic in \( X \) over \( A \) iff \( \text{rk}^Z(\bar{c}/A) = \dim(X) \).

(2) If \( \mathbb{K} \) is sufficiently saturated and \( X \) is an embedded manifold, then the set of points in \( X \) that are generic over \( A \) is (topologically) dense in \( X \) (notice that, even for o-minimal structures, this may not be true when \( X \) is not an embedded manifold).

(3) \( \text{rk}^Z \) is the rank corresponding to a (unique) matroid on \( \mathbb{K} \) (see [For11b]).

Thus, most of the proofs in o-minimal situations that rely on generic elements can be transferred without much difficulty to d-minimal structures.

2.4. Functions. The results in this subsection (with the same proofs) hold not only for d-minimal structures, but also when \( \mathbb{K} \) is any definably complete expansion of some ordered field.

**Proposition 2.14.** Let \( U := I_1 \times I_2 \subseteq \mathbb{K}^n \times \mathbb{K}^m \) be an open rectangular box, and \( F : U \to M_n(\mathbb{K}) \) be a \( \mathcal{C}^1 \) definable function (where \( M_n(\mathbb{K}) \) is the set of \( n \times n \) matrices over \( \mathbb{K} \)). For \( (a, b) \in U \), consider the system of differential equations

\[
\begin{align*}
\phi(a) &= b, \\
d_x(\phi) &= F(x, \phi(x)).
\end{align*}
\]

Then, there exists at most one function \( \phi : I_1 \to I_2 \), which is definable, \( \mathcal{C}^1 \), and satisfies (1).

**Proof.** The same as [OPP96, Theorem 2.3].

**Proposition 2.15** (Local Submersion Theorem). Let \( f : \mathbb{K}^n \to \mathbb{K}^m \) be a definable function. Assume that \( d_0(f) \) has rank \( m \). Then, \( f \) is an open map in a neighbourhood of 0.

**Proof.** The Implicit Function Theorem for definable functions was proved e.g. in [Ser06]. The Local Submersion Theorem follows in the usual way.

3. Definable groups

**Notation.** In all the article, unless explicitly said otherwise, when we say that \( G \) is a group, we will denote by \( G \) the underlying set, by \( \cdot \) the multiplication, by \( e \) the identity, and by \( ^{-1} \) the inverse operation of \( G \).

\( ^{(1)} \) For definable groups, there is a different notion of “generic” (see [HPP08]), but we will not use it.
3.1. Examples. 1) Let $G$ be a semi-algebraic group (i.e., a real Lie group definable in $\mathbb{R}$); let $\hat{G} \to G$ be the universal cover of $G$, and let $D$ be the kernel of the covering map $\hat{G} \to G$. Notice that $D := \langle D, + \rangle$ is a discrete central subgroup of $\hat{G}$. Remember that the structure $R := \langle \mathbb{R}, 2^Z \rangle$ is d-minimal. Since $D$ is a finitely generated Abelian group, it is definable in $R$. By [HPP11, §8.1], an isomorphic copy of the group $\hat{G}$, together with the extension maps $D \to \hat{G} \to G$, can be interpreted in the structure 2-sorted structure $\langle \mathbb{R}, D \rangle$, and therefore the extension $D \to \hat{G} \to G$ can be interpreted in $R$. For instance, an isomorphic copy of the group $\hat{SL}_2(\mathbb{R})$, together with the extension $Z \to \hat{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R})$, can be defined inside $R$.

2) What can we say about groups definable in a d-minimal structure? For 0-dimensional groups, almost nothing. Let $G$ be a countable abstract group; then, there exists a d-minimal expansion of $\mathbb{R}$ that defines a 0-dimensional group that is isomorphic to $G$ (see §2.2(3)).

The situation changes drastically for higher dimensional groups: more precisely, we will show that a group definable in a d-minimal expansion of $\mathbb{R}$ is, in a canonical way, a Lie group.

3.2. Definable topology. A. Pillay ([Pill]) proved that every group definable in an o-minimal structure can be endowed with a definable topology that makes it both a topological group and a definable manifold. His result was later extended by A. Mosley and R. Wencel to other kinds of structures. While those generalizations do not apply directly to d-minimal structures, a small modification will suffice for our purpose.

Definition 3.1. A finitary $C^k$-manifold (or, simply, finitary manifold when $k = 0$) of dimension $d$ is given by

1. a definable set $X \subseteq \mathbb{K}^n$,
2. a topology $\tau$ on $X$,
3. finitely many definable embedded $C^k$-manifolds $C_1, \ldots, C_m$ of dimension $d$,
4. for each $i = 1, \ldots, m$, a definable map $f_i : C_i \to X$, such that:
   i) for each $i$, $f_i$ is a homeomorphism with its image (with topology $\tau$) and the image of $\tau$-open in $X$;
   ii) $X = \bigcup_i f(C_i)$
   iii) for each $i$ and $j$, the partial function $f_i^{-1} \circ f_j : C_j \to C_i$ is $C^k$ (notice that this condition is superfluous when $k = 0$).

We also say that $(X, \tau)$ is a finitary $C^k$-manifold if we can find $C_1, \ldots, C_m$ and $f_1, \ldots, f_k$ as above; the set $\{ (f(C_1), f_1^{-1}), \ldots, (f(C_m), f_m^{-1}) \}$ is an atlas for the finitary manifold, and a $C^k$ differential structure. Two $C^k$ differential structures on the same set $X$ are to be considered equal
if the identity map on $X$ is a $C^k$ diffeomorphism from the first differential structure to the second. We will often denote in the same way (say, $\tau$) the $C^k$ differential structure and the underlying topology.

Notice that we are not taking position on what is the “correct” definition of an abstract definable manifold (i.e., one that may require infinitely many charts). Notice also that, unlike the o-minimal case, we are not claiming that we can take the set $C_i$ to be open subsets of $K^d_i$; and, in fact, in general we cannot do that: for instance, if $X$ is an infinite definable discrete subset of $K$, then (obviously) $X$ is not a finite union of copies of $K^0$.

**Examples 3.2.** (1) Every definable embedded manifold is a finitary manifold.

(2) The disjoint union of finitely many finitary manifold of the same dimension (with the differential structure of disjoint union) is also a finitary manifold.

(3) The main result of [For11a] can be formulated by saying that a definably complete structure with the discrete topology is not a finitary manifold. It is the “definable counterpart” of the fact that $\mathbb{R}$ with the discrete topology is not a manifold (since it is not second countable).

**Definition 3.3.** Fix $k \geq 0$. Let $X \subseteq K^n$ be a definable set, endowed with a $C^k$ differential structure. A subset $Y \subseteq X$ is large in $X$ if $\dim(X \setminus Y) < \dim X$. We say that $X$ is $k$-tame (or simply “tame” if the $k$ is either clear form the context or unimportant) if, for every $n \geq 1$, for every definable subset $Y \subseteq X^n$ such that $\dim(Y) = \dim(X^n)$, and every definable function $f : Y \to Z$, there exists $V \subseteq Y$, such that $V$ is open in $X^n$, large in $Y$, and $f \upharpoonright V$ is $C^k$.

**Remark 3.4.** If $K$ is locally o-minimal, then every definable set is $k$-tame for every $k$ (see [For10b]). If $K$ is not locally o-minimal, then there exists a definable subset of $K$ that is not 0-tame.

**Remark 3.5.** A finitary $C^k$-manifold is tame. The disjoint union of finitely many tame sets (with the differential structure of the disjoint union) is tame.

**Proposition 3.6.** Fix $k \geq 0$. Let $G$ be a group definable with parameters $A$. Assume that $G \subseteq K^n$ is tame and of dimension $d$. Let $\sigma$ be the induce topology on $G$ from $K^n$. Then, there exists an $A$-definable set $V \subseteq G$ and a topology $\tau$ on $G$, such that:

1. $V$ is a $C^k$ embedded manifold;
2. $G$ with the topology $\tau$ is a topological group;
3. $V$ is large and open in $G$ with respect both the topologies $\tau$ and $\sigma$;
4. the topologies $\tau$ and $\sigma$ restricted to $V$ coincide;
5. $V$ is a $d$-dimensional $C^k$ embedded manifold;
6. some $d+1$ left (right) translates of $V$ cover $G$: $G = g_1 \cdot V \cup \cdots \cup g_{d+1} \cdot V$;
(7) $g_1 \cdot V, \ldots, g_{d+1} \cdot V$ (with the obvious maps into $V$) form an atlas inducing the topology $\tau$, and making $G$ a finitary $C^k$-manifold and a $C^k$ group (i.e., this atlas multiplication and inversion are $C^k$-functions).

Proof. The proposition for $k = 0$ is proved in [Wen11, Theorem 3.5]. He states the theorem under the assumption that all definable sets are tame, but by inspecting the proof, one easily sees that it suffices that $G$ is tame. The general proposition can be proved in the same way. □

Assume that $\mathbb{K}$ is locally o-minimal and $G$ is a group definable in $\mathbb{K}$; then, by Remark 3.4, $G$ is tame, and therefore we can apply Proposition 3.6 to $G$ (see [For10b]). This is no longer true in the case when $\mathbb{K}$ is not locally o-minimal.

Example 3.7. By [FM05], there exist a d-minimal expansion $\mathcal{R}$ of $\overline{\mathbb{R}}$, and a group $G$ definable in $\mathcal{R}$, such that

1. as an abstract group, $G$ is isomorphic to $\langle \mathbb{Z} \times \mathbb{Z}, + \rangle$;
2. $\dim(G) = 0$;
3. $G$ is not 0-tame;
4. the conclusion of Proposition 3.6 fails for $G$: let $V \subseteq G$ be the set of isolated points of $G$ (in the topology induced by the ambient space); then, $V$ is not large in $G$, and $G$ is not covered by finitely many translates of $V$.

We can now prove a version of Pillay’s result. Remember that every definable set $X$ is the union of finitely many embedded manifolds (possibly, of different dimensions); hence, after changing the topology of $X$ slightly, we can assume that $X$ is tame; thus, we can prove the following result.

Theorem 3.8. Let $\langle H, \cdot, e \rangle$ be a definable group, with $H \subseteq \mathbb{K}^n$. Then, there exist a definable group $G$ and a definable continuous group isomorphism $f : G \to H$, such that $G$ satisfies the conclusion of Proposition 3.6. In particular, on $H$ we can put a $C^k$ differential structure $\tau$ that makes $\langle H, \cdot \rangle$ a finitary $C^k$-manifold and a $C^k$ differential group; moreover, there exists $V \subseteq H$ definable, open in both the topology induced from $\mathbb{K}^n$ and the topology $\tau$, such that $V$ is a embedded $C^k$-manifold, and such that $\tau$ and the $C^k$ differential structure induced by $\mathbb{K}^n$ coincide on $V$ (however, in general $V$ will not be large in $H$, nor finitely many translate of $V$ will suffice to cover $H$).

Lemma 3.9. Let $G$ and $G' := \langle G', \cdot' \rangle$ be definable groups and $k \in \mathbb{N}$.

1. Let $\sigma$ and $\sigma'$ be topologies on $G$ and $G'$ respectively that make them finitary manifolds, and such that all left multiplications are continuous maps. Let $\phi : G \to G'$ be a definable group homomorphism. Then, $\phi$ is continuous. If moreover we have differential structures on $G$ and $G'$ that make them finitary $C^k$-manifolds, such that all left multiplications are $C^k$-functions, then $\phi$ is also a $C^k$-function.
(2) The $C^k$ differential structure $\tau$ in Theorem 3.8 is the unique $C^k$ differential structure on $G$ that makes it both a a finitary manifold and a $C^k$ differential group.

(3) Let $\phi : G \to G'$ be a definable surjective homomorphism. Then, $\phi$ is an open map (in the group manifold topologies), and therefore $G'$ has the quotient topology. Moreover, if $X \subseteq G$ is clopen and definable, then $\phi(X)$ is clopen.

Proof. (1) Let us show the case $k = 0$. By tameness, there exists $V \subseteq G$ open, definable, and nonempty, such that $\phi \restriction V$ is continuous. Let $a \in G$; we have to show that $\phi$ is continuous in a neighbourhood of $a$. Choose $b \in V$, and let $c := a \cdot b^{-1}$. Then, $c \cdot V$ is an open neighbourhood of $b$, and, since left multiplications by $c$ and $\phi(c^{-1})$ are continuous maps, $\phi$ is continuous on $c \cdot V$. The same proof works for $k > 0$.

(2) is immediate from (1).

(3) By (1), we can assume that $\phi$ is $C^1$. By Sard’s Lemma, there exists $g \in G$ such that $d_g(\phi)$ has rank equal to $\dim(G')$; thus (by the local submersion theorem) $\phi$ is open in a neighbourhood of $g$. Thus, $\phi$ is an open map.

Let $X \subseteq G$ be clopen and definable; we want to prove that $\phi(X)$ is clopen. We know already that $\phi(X)$ is open; thus, we only need to show that it is closed. Since $G'$ has the quotient topology, it suffices to show that $Y := \phi^{-1}(\phi(X))$ is closed in $G$. Let $b \in \text{cl}(Y)$; we want to show that $b \in Y$. Let $V \subseteq U$ be an open, definable, and definably connected neighbourhood of $e$ (the identity of $G$); we know that $b \cdot V \cap X \neq \emptyset$; but $b \cdot V$ is definably connected and $X$ is clopen and definable; therefore, $b \in b \cdot V \subseteq X$.

Thus, by the above lemma, we can talk about the $C^k$ differential structure $\tau$ that makes a definable group $G$ both a finitary $C^k$-manifold and a $C^k$ differential group; we will call $\tau$ the group $C^k$ structure of $G$ (or the group manifold topology on $G$ when $k = 0$). When we say e.g. that a definable group is definably connected, we mean in its group manifold topology.

A similar result holds for a definable group action.

Proposition 3.10. Fix $k \in \mathbb{N}$. Let $\ast : G \times X \to X$ be a transitive definable group action, from a definable group $G$ on a definable set $X$. Then, there exists a $C^k$ differential structure on $X$ such that $\ast$ is a $C^k$-function (with the group $C^k$ structure on $G$).

Proof. See the proofs of either [Wen11, Theorem 4.6] or [PPS00, Theorem 2.11].

Lemma 3.11. Let $G$ be a definable group, and let $\tau$ be its group manifold topology. Let $H < G$ be a definable group. Then, $H$ is a closed subgroup (w.r.t. $\tau$). Moreover, t.f.a.e.:
(1) $H$ is clopen;
(2) $H$ has nonempty interior;
(3) $\dim(G/H) = 0$;
(4) $\dim(G) = \dim(H)$.

Proof. The fact that $H$ is closed and $(2 \Rightarrow 1)$ are as in [Pil87, Fact 2.6 and Proposition 2.7].

$(4 \iff 3)$ is clear, since $\dim$ is additive.

$(2 \Rightarrow 4)$ is clear, since $\langle G, \tau \rangle$ is a finitary manifold.

$(4 \Rightarrow 2)$ is clear, because $\dim$ is local and $\langle G, \tau \rangle$ is a finitary manifold.

$(1 \Rightarrow 2)$ is clear.

$(2 \Rightarrow 1)$ is as in [Pil87, Proposition 2.7]. \qed

Lemma 3.12. Fix $k \in \mathbb{N}$. Let $G$ be a definable group, with its group $C^k$ differential structure. Let $H < G$ be a definable subgroup, with its group $C^k$ differential structure $\tau$. Then, $H$ is an embedded $C^k$-submanifold of $G$, and $\tau$ coincides with the differential structure induced by $G$.

Thus, when we deal with subgroups, we don’t have to distinguish between the intrinsic topology/differential structure and the induced one.

Proof. By Lemma 3.9, it suffices to show that $H$ is an embedded $C^k$-submanifold of $G$. Since left multiplication is a $C^k$-function on $G$, it suffices to show that there exists a definable nonempty open set $V \subseteq G$, such that $V \cap H$ is an embedded $C^k$-submanifold of $G$. Let $W \subseteq G$ be an open definable subset of $G$ that is definably $C^k$-diffeomorphic to $\mathbb{K}^n$ (where $n := \dim(G)$), and $H' := H \cap W$; w.l.o.g., we can assume that $W = \mathbb{K}^n$. By Fact 2.19 $\text{reg}^k(H')$ is nonempty, proving what we wanted. \qed

Definition 3.13. A definable set $X$ is definably connected if it contains no nontrivial definable clopen subsets.

Proposition 3.14. Let $G := \langle G, +, 0 \rangle$ be a definable Abelian group, and $1 \leq n \in \mathbb{N}$. Define $G[n] := \{ g \in G : ng = 0 \}$. Then, $G[n]$ is a discrete subgroup of $G$. If moreover $G$ is definably connected, then $G$ is divisible.

Proof. For every $1 \leq n \in \mathbb{N}$, let $F_n : G \to G$ be the map $x \mapsto nx$, and $P : G \times G \to G$, $(x, y) \mapsto x + y$. Let $d := \dim(G)$.

Claim 1. $d_0(F_n) = nI$, where $I$ is the $d \times d$ identity matrix in $M_d(\mathbb{K})$, and in particular $d_0(F_n)$ is surjective.

As in [OPP96, Lemma 4.3], one sees that $d_0(P)$ is the matrix $\langle I, I \rangle \in M(2d \times d, \mathbb{K})$. Since

$$d_0(F_n) = d_{(0,0)}P \cdot \left( \begin{array}{c} I \\ d_0(F_{n-1}) \end{array} \right),$$
the claim follows by induction on $n$.

Thus, by the local submersion theorem, $F_n$ is an open map around 0; since moreover $F_n$ is a homomorphism, $F_n$ is an open map, and in particular $F_n(G)$ is open. Thus, by additivity of dimension, $\dim(G[n]) = 0$, and $G[n]$ is discrete.

If moreover $G$ is definably connected, then, since $F_n(G)$ is an open definable subgroup of $G$, $F_n(G) = G$. Since the above is true for every $n > 1$, $G$ is divisible. \[\square\]

**Definition 3.15.** $G$ is **definably compact** if, for every definable decreasing family $(X_t : t \in \mathbb{K})$ of closed nonempty subsets of $G$, we have $\bigcap_t X_t \neq \emptyset$.

Notice that the above definition generalizes the usual one for definable subsets of $\mathbb{K}^n$ with the subspace topology.

**Conjecture 3.16 (EO04).** Let $G := \langle G, +, 0 \rangle$ be Abelian, definably compact, definably connected, and of dimension $d$. Then, for every $1 \leq n \in \mathbb{N}$, $G[n] \cong (\mathbb{Z}/n\mathbb{Z})^d$.

**Example 3.17.** We now show that some Lie group is not definable in any $d$-minimal expansion of the reals. Fix an integer $d \geq 1$. Let $\mathbb{Q}^*$ be the group of nonzero rational numbers, with the usual multiplication and the discrete topology. Consider the action of $\mathbb{Q}^*$ on $\mathbb{R}^d$ by scalar multiplication, and define $H := \mathbb{R}^d \rtimes \mathbb{Q}^*$ to be the corresponding semidirect product. We claim that $H$ is not definable in any $d$-minimal expansion of the real field. In fact, assume, for a contradiction, that $H$ were definable in some $d$-minimal structure. Since $H$ is uncountable, $\dim(H) \geq 1$. Let $F$ be a definable Abelian subgroup of $H$ of dimension at least 1 ($F$ exists by Proposition 3.32). It is easy to see that $F$ is a subgroup of $\mathbb{R}^d \times \{1\}$; thus, its connected component $F^0$ is a torsion-free Abelian Lie subgroup of $\mathbb{R}^d$ of dimension at least 1 (unfortunately, we don’t know whether $F^0$ is definable); therefore, as a Lie group, $F^0$ is isomorphic (not necessarily in a definable way) to $\mathbb{R}^e$ for some $e \geq 1$. Let $d := \langle v, 1 \rangle$ be any nonzero element in $F^0$ and $C$ be the centralizer of $d$ inside $H$: notice that $C = \{\langle 0, q \rangle : q \in \mathbb{Q}^* \}$. Define, $D := \{\langle 0, 1 \rangle\} \cup C \cdot d = \{\langle qv, 1 \rangle : q \in \mathbb{Q} \}$. Thus, $D$ is a definable subgroup of $F^0$ isomorphic to $(\mathbb{Q}, +)$; let $R$ be the topological closure of $D$ inside $H$. By working inside the Lie group $F^0$, we easily see that $D$ is uncountable, and therefore $\dim(D) \geq 1$, while $\dim(C) = 0$, which is impossible inside a $d$-minimal structure.

**Question 3.18.** Let $H$ be a connected Lie group. Is there some $d$-minimal expansion $\mathcal{R}$ of the real field, such that $H$ is isomorphic (as a Lie group) to a group definable in $\mathcal{R}$?

Notice that if $H$ is compact, then the answer to be above question is yes, since we can take $\mathcal{R}$ to be the o-minimal structure $\mathbb{R}_{an}$; see also §3.1.
3.3. Connected components.

**Definition 3.19.** Let $X$ be a definable set and $a \in X$. The **definable quasi-component** of $a$ in $X$ is the intersection of all definable clopen subsets of $X$ containing $a$. \(^{(2)}\)

Let $G$ be a definable group. Define $G^0$ to be the quasi-component of $e$ in the group manifold topology.

**Warning 3.20.** (1) Let $G$ be a definable group. Then, $G^0$ is type-definable, but, unlike in the o-minimal case, we don’t know whether $G^0$ is definable.

(2) There exists a d-minimal expansion of $\bar{\mathbb{R}}$ that defines a set $X$ that is a 1-dimensional submanifold of $\mathbb{R}^3$, and such that $X$ has 2 (arc-)connected components, but it is definably connected (see [For12]). Thus, unlike the o-minimal case, even for “nice” subsets of $\mathbb{R}^n$, definably connected does not imply connected.

Notice that if $G$ is a 0-dimensional definable group, then $G^0 = \{e\}$.

**Lemma 3.21.** Let $G$ be a definable group, and $H = G^0$. Then, $H$ is a normal closed subgroup of $G$. If moreover $H$ is definable, then $H$ is the smallest definable subgroup of $G$ such that $\dim(G/H) = 0$.

**Proof.** Let us show first that $H$ is a subgroup; since $x \mapsto x^{-1}$ is a homeomorphism, it is clear that $H^{-1} = H$. Thus, we only need to show that, given $a \in H$, $a \cdot H \subseteq H$. Let $X \subseteq G$ be clopen and definable, such that $1 \in X$. Notice that $a \in H \subseteq X$, and therefore $e \in a^{-1} \cdot X$; thus, since $a^{-1} \cdot X$ is also clopen and definable, we have $H \subseteq a^{-1} \cdot X$, that is, $a \cdot H \subseteq X$. Taking the intersection of all the possible $X$, we get $a \cdot H \subseteq H$.

The fact that $H$ is normal and closed is clear.

Moreover, since $(G, \tau)$ is a finitary manifold, $H$ is clopen in $G$; thus, if $H$ is moreover definable, it must be smallest definable subgroup of $G$ such that $\dim(G/H) = 0$. \(\square\)

**Lemma 3.22.** Let $A \subseteq \mathbb{K}$ and $G$ be a group definable over $A$. Then, there exists a family $\{H_i : i \in I\}$, such that each $H_i$ is a clopen normal subgroup of $G$ definable over $A$, and $G^0 = \bigcap_{i \in I} H_i$.

**Proof.** Let $X \subseteq G$ be a clopen definable set, containing $e$. It suffices to show that there exists a clopen normal subgroup $H < G$ that is definable over $A$, and such that $X \subseteq H$. W.l.o.g., we can assume $A = \emptyset$. Moreover, after replacing $X$ with $X \cap X^{-1}$, we can assume that $X = X^{-1}$.

Let $H(X) := \{g \in X : g \cdot X = X\}$. Clearly, $H(X)$ is a definable subgroup of $G$.

\(^{(2)}\) In classical topology, given a topological space $X$ and $a \in X$, the quasi-component of $a$ in $X$ is the intersection of all clopen subsets of $X$ containing $a$; it can be larger than the connected component of $a$, unless $X$ is locally connected.
Claim 2. $H(X) \subseteq X$.

In fact, since $e \in X$, $H(X) = H(X) \cdot e \subseteq H(X) \cdot X = X$.

Claim 3. $H(X)$ is open (and therefore clopen).

Let $U \subseteq G$ be an open, definable, and definably connected neighbourhood of $e$. Let $V \subseteq U$ be an open, definable, and definably connected neighbourhood of $e$, such that $V^{-1} \subseteq U$. It suffices to show that $U \subseteq H(X)$. Let $g \in V$; we want to show that $g \cdot X = X$. Let $x \in X$; since $X$ is clopen, we have $U \cdot x \subseteq X$; thus, $g \cdot X \subseteq X$. Similarly, $g^{-1}X \subseteq X$, and therefore $g \cdot X = X$.

Let $\tilde{c}$ be a finite tuple of parameters, and $\phi(\tilde{x}, \tilde{y})$ be a formula, such that $\phi(G, \tilde{c}) = X$, and, for every $\tilde{c}'$, $\phi(G, \tilde{c}')$ is a clopen subset of $G$ containing $e$.

Let $H_0 := \bigcap_{\tilde{c}} H(\phi(G, \tilde{c}'))$, and $H := \bigcap_{g \in G} g \cdot H_0 \cdot g^{-1}$. Clearly, $H$ is a normal subgroup of $G$, definable without parameters, and contained in $X$. Thus, it suffices to prove that $H$ is open. Fix a parameter $\tilde{c}'$ and $g \in G$, and let $H' := g \cdot H(\phi(G, \tilde{c}) \cdot g^{-1}$. By Claim 2, $g \cdot H(\phi(G, \tilde{c})) \cdot g^{-1}$ is clopen, and therefore contains $G^0$. Thus, $G^0$ is contained in $H$; since $G^0$ is open in $G$, $H$ contains an open neighbourhood of $e$, and therefore it is open.

Lemma 3.23. Let $\mathbb{G}$ and $\mathbb{G}' := \langle G', \cdot', e' \rangle$ be definable groups, and let $\phi$ and $\phi' : G \to G'$ be definable homomorphisms. If $d_v(\phi) = d_v(\phi')$, then $\phi \upharpoonright G^0 = \phi' \upharpoonright G^0$.

Proof. Same as [OPP96 Lemma 3.2]. The uniqueness of definable solutions to differential equations is Proposition 2.14.

Lemma 3.24. Let $\mathbb{G}$ and $\mathbb{G}' := \langle G', \cdot', e' \rangle$ be definable groups, and let $\phi : G \to G'$ be a definable homomorphism. Then, $\phi(\mathbb{G}^0) \subseteq \mathbb{G}'^0$. If moreover $\phi$ is an open map and $\mathbb{K}$ is $\omega$-saturated, then $\phi(\mathbb{G}^0) = \mathbb{G}'^0$.

Proof. Let $a \in \mathbb{G}^0$ and $b := \phi(a)$. Assume, for a contradiction, that $b \notin \mathbb{G}'^0$; let $X \subseteq G'$ be clopen and definable, such that $e' \in X$ and $b \notin X$. Then, $Y := \phi^{-1}(X)$ is clopen and definable, $e \in Y$, and $a \notin Y$, absurd.

Assume now that $\phi$ is open and $\mathbb{K}$ is $\omega$-saturated. Let $\bar{a} \in \mathbb{K}^m$ be the parameters of definition of $\mathbb{G}$, $\mathbb{G}'$, and $\phi$. Let $b \in \mathbb{G}'^0$; we want to prove that $\phi^{-1}(b) \cap \mathbb{G}^0 \neq \emptyset$. Assume not. By Lemma 3.22 there exists a family $\{ H_i : i \in I \}$ of clopen subgroups of $G$, definable over $\bar{a}$, such that $\mathbb{G}^0 = \bigcap_i H_i$. By saturation, there exists $H < G$ clopen subgroup of $G$, such that $\phi^{-1}(b) \cap H = \emptyset$, and hence $b \notin \phi(H)$. However, since $\phi$ is an open map, $\phi(H)$ is a definable clopen subgroup of $G'$, and therefore $\phi(H) \subseteq \mathbb{G}'^0$, absurd.

We can now refine Proposition 3.14.
Lemma 3.25. Let $\mathbb{G} := \langle G, +, 0 \rangle$ be a definable Abelian group. Assume that $\mathbb{K}$ is $\omega$-saturated. Then, $\mathbb{G}^0$ is divisible.

Proof. Fix $1 \leq n \in \mathbb{N}$, and consider the map $\phi : G \to G$, $x \mapsto nx$. Since $\mathbb{G}$ is Abelian, $\phi$ is a group homomorphism. By Proposition 3.14 $\phi$ is an open map. Thus, by Lemma 3.24 $\phi(\mathbb{G}^0) = \mathbb{G}^0$. □

We don’t know if in the above lemma the assumption that $\mathbb{K}$ is $\omega$-saturated is necessary. We don’t know if $\mathbb{G}^0$ is definable or not; however, we have the following conjecture.

Conjecture 3.26. Let $\mathcal{R}$ be a $d$-minimal expansion of $\bar{\mathbb{R}}$. Let $X \subseteq \mathbb{R}^n$ be a manifold definable in $\mathcal{R}$. Let $Y \subseteq X$ be a clopen subset of $X$. Then, $\langle \mathcal{R}, Y \rangle$ (the expansion of $\mathcal{R}$ with a predicate for $Y$) is also $d$-minimal.

3.4. The Lie algebra of a group. Let $\mathbb{G}$ be a definable group of dimension $n$. Following [PPS00], we can endow its tangent space $\mathfrak{g} := T_e(\mathbb{G})$ with the “usual” Lie algebra structure, in the following way. For every $g \in \mathbb{G}$, let $\chi_g : \mathbb{G} \to \mathbb{G}$ be the map $x \mapsto gxg^{-1}$. Let $\text{Ad} : \mathbb{G} \to \text{GL}_n(\mathbb{K})$, $g \mapsto d_e\chi_g$ be the adjoint representation of $\mathbb{G}$, and $\text{ad} := d_e(\text{Ad}) : \mathfrak{g} \to M_n(\mathbb{K})$. Let $[\ , \ ]$ be the Lie bracket on $T_e(\mathbb{G})$: that is, $[v, w] := \text{ad}(v)(w)$. Almost everything in [PPS00][§2.1–2.4] goes through for $d$-minimal structures, with very similar proofs (the only inconvenience is that $\mathbb{G}^0$ might not be definable, and hence some small changes are needed, as shown in the proofs of the following results).

Fact 3.27. Let $\mathbb{G}$ be a definable group with Lie algebra $\mathfrak{g}$.

(1) Let $\mathfrak{h}$ be a linear subspace of $\mathfrak{g}$. Then, the subalgebra $\{ v \in \mathfrak{g} : [v, \mathfrak{h}] = 0 \}$ is the Lie algebra of the subgroup $\{ g \in \mathbb{G} : \text{Ad}(g) \upharpoonright \mathfrak{h} = \text{id} \}$.
(2) $\mathbb{G}^0$ is Abelian iff $\mathfrak{g}$ is Abelian (that is, $[v, w] = 0 \ \forall v, w \in \mathfrak{g}$).
(3) If $H$ is a subgroup of $\mathbb{G}$, then $H^0$ is normal in $\mathbb{G}$ iff its Lie algebra is an ideal of $\mathfrak{g}$.

Proof. See the proofs of [PPS00] Claims 1.31 and 1.32. □

Corollary 3.28. Let $\mathbb{G}$ be a definable group. If $\mathbb{G}$ is definably connected and of dimension 1, then $\mathbb{G}$ is Abelian.

Proof. The Lie algebra of $\mathbb{G}$ has dimension 1, and therefore it is Abelian. Thus, by Fact 3.27 $\mathbb{G}$ is Abelian. □

Lemma 3.29. Let $\mathbb{G}$ be a definable group. Let $H$ and $L$ be definable subgroups of $\mathbb{G}$. Then, $T_e(H \cap L) = T_eH \cap T_eL$.

Proof. $T_e(H \cap L) \subseteq T_eH \cap T_eL$ is true for any differential manifolds $H$ and $L$.

For the opposite inclusion, we only need to show that $\dim(L \cap H) \geq \dim(T_eL \cap T_eH)$. Consider the map $f : L \to G/H$, $l \mapsto l\cdot H$, where on $G/H$ we put the quotient $C^1$ structure given by Proposition 3.10. Let $d := \dim(L/(L \cap H))$: notice that, for every $l \in L$, $d = \dim L -$
dim(L \cap l \cdot H). By Fact 3.27, there exists \( l \in L \) such that \( \text{rk}(d_l f) \geq d \). Thus, \( \dim(T_l(L \cap l \cdot H)) \geq \dim(L \cap l \cdot H) \). However, \( \dim(T_l(L \cap l \cdot H)) = \dim(L \cap H) \), and \( \dim(L \cap l \cdot H) = \dim(L \cap H) \), and we are done. \( \square \)

Since we don’t know if \( \mathbb{G}^0 \) is definable, we will use the next two lemmas.

Lemma 3.30. Let \( \mathbb{G} \) be a definable group. Let \( H := C_\mathbb{G}(\mathbb{G}^0) \) be the centralizer of \( \mathbb{G}^0 \), and \( \mathfrak{Z} \) be the center of \( T_e(\mathbb{G}) \), that is \( \mathfrak{Z} := \{ v \in T_e(\mathbb{G}) : \forall \mathfrak{g} \in T_e(\mathbb{G}) \ [v, \mathfrak{g}] = 0 \} \). Then, \( H \) is definable, and \( T_e(H) = \mathfrak{Z} \).

Proof. Notice that \( H = \{ g \in \mathbb{G} : \operatorname{Ad}(g) = 0 \} \), and hence \( H \) is definable. By applying Fact 3.27 to the subalgebra \( \mathfrak{h} := T_e(\mathbb{G}) \), we get that \( \mathfrak{Z} = T_e(H) \). \( \square \)

Lemma 3.31. Let \( \mathbb{G} \) be a definable group, and let \( H < \mathbb{G} \) be a definable subgroup.

1. If \( H^0 \) is normal in \( \mathbb{G} \), then there exists a definable subgroup \( H' < H \), such that \( H^0 < H' < H \) and \( H' \) is normal in \( \mathbb{G} \).
2. If \( H^0 \) is Abelian and normal in \( \mathbb{G} \), then there exists a definable subgroup \( H' < H \), such that \( H^0 < H' < H \) and \( H' \) is Abelian and normal in \( \mathbb{G} \).

Proof. (1) Let \( H' \) be the intersection of all \( \mathbb{G} \)-conjugates of \( H \).

(2) Let \( L := C_\mathbb{G}(H^0) \); by Lemma 3.30, \( L \) is definable. Let \( L' \) be the center of \( L \); by assumption, \( H^0 < L' \), and clearly \( L' \) is definable and Abelian. Let \( H' \) the intersection of all \( \mathbb{G} \)-conjugates of \( L' \). \( \square \)

Proposition 3.32. Let \( \mathbb{G} \) be a definable group, and \( v \in T_e(\mathbb{G}) \). Then, there exists a definable subgroup \( H < \mathbb{G} \), such that \( H \) is Abelian and \( v \in T_e(H) \).

In particular, if \( \dim(G) \geq 1 \), then there exists a definable Abelian subgroup \( H < \mathbb{G} \), such that \( \dim(H) \geq 1 \).

Proof. Let \( n := \dim(G) \). Let \( L := \{ g \in \mathbb{G} : \operatorname{Ad}(g)(v) = v \} \).

By Fact 3.27, \( T_e(L) = \{ w \in T_e(\mathbb{G}) : [v, w] = 0 \} \).

Since \( [v, w] = 0 \), we have \( v \in T_e(L) \). Define \( M := C_L(L^0) \) to be the centralizer of \( L^0 \) inside \( L \).

Claim 4. \( v \in T_e(M) \).

By Lemma 3.30, \( T_e(M) = \{ w \in T_e(L) : [v, w] = 0 \} \).

It is clear that \( M^0 \) is an Abelian subgroup of \( \mathbb{G} \), and that \( v \in T_e(M^0) \).

By Lemma 3.31, there exists \( H < M \) such that \( H \) is definable and Abelian, and \( M^0 < H \), and hence \( v \in T_e(H) \). \( \square \)

Lemma 3.33. Let \( \mathbb{G} \) be a definable group of dimension \( n \). Assume that \( \mathbb{G} \) is definably connected and centerless. Then, the adjoint map \( \operatorname{Ad} \) is a (definable and \( \mathcal{C}^\infty \)) embedding into \( \text{GL}_n(\mathbb{K}) \).
Proof. The proof is in [OPP96]; remember that the uniqueness of definable solutions to differential equations is Proposition 2.14. □

Definition 3.34. Let $G$ be a definable group. We say that $G$ is semisimple if every definable, normal, Abelian subgroup is discrete. We say that $G$ is definably simple if it has no definable, normal, nontrivial subgroups.

A Lie algebra is semisimple if it has no nontrivial Abelian ideal, and it is simple if it has no nontrivial ideal.

Lemma 3.35. Let $G$ be a definable, definably connected, semisimple group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a an ideal of $\mathfrak{g}$. Then, there exists a definable normal subgroup $H \trianglelefteq G$ whose Lie algebra is $\mathfrak{h}$.

Proof. As in [PPS00, Claim 2.35]. □

Lemma 3.36. Let $G$ be a definable and definably connected group.

(1) $G$ is semisimple iff its Lie algebra is semisimple.
(2) $G$ is definably simple iff its Lie algebra is simple.

Proof. (1) See the proof of [PPS00 Theorem 2.34], using Lemma 3.31 to obviate to the fact that $G^0$ might not be definable.

(2) See the proof of [PPS00 Theorem 2.36], using Lemma 3.35, and replacing everywhere “finite” with “has dimension 0”. □

Theorem 3.37. Let $G$ be a definable group of dimension $n$. Assume that $G$ is definably connected, centerless, and semisimple. Then,

(1) The map $\text{Ad} : G \to \text{GL}_n(K)$ is an injective homomorphism, and its image is a semi-algebraic linear group.
(2) Identify $G$ with $\text{Ad}(G) \vartriangleleft \text{GL}_n(K)$. $G$ is the direct product of finitely many subgroups $H_1, \ldots, H_m$, such that each $H_i$ is semi-algebraic, definably simple, and definably connected.
(3) There exists a semi-algebraic group $G'$ defined without parameters, such that $G$ is definably isomorphic to $G'$.

Proof. The fact that the map $\text{Ad}$ is an injective homomorphism is Lemma 3.33. The fact that the image of $\text{Ad}$ is semi-algebraic is as in the proof of [PPS00 Theorem 2.37] (notice that we do have to assume that $G$ is definably connected to conclude that $\text{Ad}(G)$ is semi-algebraic: e.g., if $G$ had infinitely many definably connected components, then no definably homeomorphic copy of $G$ could be semi-algebraic.).

Thus, w.l.o.g. we can assume that $G$ is semi-algebraic; therefore we can work inside the structure $\bar{K} := \langle K, +, \cdot \rangle$, and use [PPS00 Theorem 2.38] to conclude that $G$ is the direct product of finitely many subgroups $H_1, \ldots, H_m$, such that, for each $i \leq m$, $H_i$ is semi-algebraic and definably simple in the structure $\bar{K}$ (notice that we are using [PPS00 Theorem 2.38], not its proof). Fix $i \leq m$. By Lemma 3.36
the Lie algebra $T_e(H)$ is simple, and, again by Lemma 3.36, $H$ is definably simple in the structure $\mathbb{K}$. Since $G$ is definably connected, each $H$ must also be definably connected.

The proof [PPS02, Theorem 5.1] gives the third part. □

3.5. **Type-definable connected component.** Let $G$ be a group definable in some $\kappa$-saturated structure, for some large cardinal $\kappa$. Let $G^{00}$ be the intersection of all type-definable subgroups of $G$ of bounded index in $G$ (i.e., of index less than $\kappa$). People say that “$G^{00}$ exists” if $G^{00}$ itself is of bounded index. It is known that $G^{00}$ exists when $\mathbb{K}$ is o-minimal, and more in general when it has NIP (see e.g. [HPP08] for definitions and properties of $G^{00}$). In this subsection we will give an example of a $d$-minimal structure such that $G^{00}$ does not exist, where $G$ is the group $\mathbb{G} := \langle \{0, 1\}, + \text{ (mod 1)} \rangle$.

Let $\langle \mathbb{R}^*, \mathbb{N}^* \rangle$ be a $\kappa$-saturated elementary extension of $\langle \mathbb{R}, \mathbb{N} \rangle$ (of course, it is not a $d$-minimal structure), where $\kappa$ is a sufficiently large cardinal. Let $n$ be a “non-standard natural number”, i.e. $n \in \mathbb{N}^* \setminus \mathbb{N}$. Let $P := \{m \in \mathbb{N}^* : 1 \leq m \leq n \}$. Finally, let $\mathbb{K} := \langle \mathbb{R}^*, P \rangle$. Notice that $\mathbb{K}$ is locally o-minimal, and a fortiori $d$-minimal. We now prove that $G^{00}$ does not exist. Assume, for a contradiction, that $G^{00}$ exists.

Let $H < G$ be the subgroup of infinitesimal elements; notice that $H$ is type-definable and of bounded index in $G$, and therefore $G^{00} < H$. For every $m \in P$, let $\phi_m : G \to G$ be the multiplication by $m$, i.e. $\phi_m(g) = mg \pmod{1}$. Define $\bar{H}_m := \phi^{-1}_m(H)$. Notice that $\bar{H}_m$ is type-definable and of bounded index in $G$, and therefore $G^{00} < \bar{H}_m$. Thus, to reach a contradiction it suffices to show that $\bigcap_{m \in P} \bar{H}_m$ does not have bounded index in $G$. Let $\lambda$ be an infinite cardinal, such that $[G : G^{00}] < \lambda < \kappa$. Let $(p_i : i < \lambda)$ be a sequence of elements in $\mathbb{N}^*$, such that, for every $i < \lambda$, $p_i$ is non-standard, $p_i < n$, and, for each $j < i$, $p_j > 2p_jp_0$. For each $i < \lambda$, let $H_i := H \cap \bigcap_{0 < j < i} \bar{H}_{p_j}$. It suffices to show that, for every $i < \lambda$, $\bar{H}_{p_i} \cap H_i$ is a proper subgroup of $\bar{H}_{p_i}$. By saturation, it suffices to prove the above for $i$ finite. Thus, we have to show that, for each $i \in \mathbb{N}$, there exists $g \in H \cap \bigcap_{p_j < p_i} \bar{H}_{p_j} \setminus \bar{H}_{p_i}$. Let $I := (0, 1) \pmod{1}$; notice that $I \subseteq \bar{H}_{p_1} \cap \cdots \cap \bar{H}_{p_i}$. Notice that $\phi_{p_{i+1}}(I) = (0, p_{i+1}/p_0p_i) \pmod{1} \subseteq \mathbb{N}$ (mod 1) = $G$, and therefore there exists $g \in I$ such that $\phi_{p_{i+1}}(g) = 1/2$, and thus $g \in I \setminus \bar{H}_{p_{i+1}}$, and we are done.

We conclude this subsection with a conjecture.

**Conjecture 3.38.** Assume that $\mathbb{K}$ has NIP. Let $G$ be a definable group. Assume that $G$ is definably connected and definably compact (see Definition 3.15). Then, $G$ has finitely satisfiable generics and satisfies compact domination (see [HPP08] for the relevant definitions and properties).
4. Definable rings

A ring will always be associative, but not necessarily commutative or with 1; a ring homomorphism will not necessarily send 1 to 1; a \( \mathbb{K} \)-algebra \( F \) will not necessarily contain a copy of \( \mathbb{K} \) (but the 1 if \( \mathbb{K} \) will act as the identity on \( F \)); remember that a division \( \mathbb{K} \) algebra is the same as a \( \mathbb{K} \)-algebra that is also a skew field.

**Notation.** In all this section, unless explicitly said otherwise, when we say that \( F \) is a ring, we will denote by \( F \) the underlying set, by + the sum, by − the minus, by \( \cdot \) the multiplication, by 0 the identity of +, and by 1 the identity of \( \cdot \), if it exists. We define \( F^* := F \setminus \{0\} \). If \( F \) is definable, then \( F^0 \) will be the definably connected component of \( F \) containing 0 (in the group topology of \( \langle F, +, 0 \rangle \)).

**Theorem 4.1.** Fix \( k \in \mathbb{N} \). Let \( F \) be a ring definable over \( A \subseteq \mathbb{K} \), with \( F \subseteq \mathbb{K}^n \) and \( \dim(F) = d \). Let \( \tau \) be the group \( C^k \) differential structure on \( \langle F, + \rangle \).

If \( F \) is tame, then:

1. \( F \) with the differential structure \( \tau \) is a \( C^k \) ring;
2. we can find a definable subset \( V \subseteq F \) that is large in \( F \), open with respect to both \( \tau \) and the topology on \( F \) induced by \( \mathbb{K}^n \), and an embedded \( C^k \)-manifold in \( \mathbb{K}^n \);
3. the restriction to \( V \) of \( \tau \) and the \( C^k \) differentials structures induced by \( \mathbb{K}^n \) coincide;
4. some \( d + 1 \) additive translates of \( V \) cover \( F \).

If moreover \( F \) is a skew field, then the restriction of \( \tau \) to \( F^* \) is the group \( C^k \) differential structure of \( \langle F^*, \cdot \rangle \), and some \( d + 1 \) multiplicative translates of \( V \setminus \{0\} \) cover \( F^* \).

If \( F \) is not tame, then there exist a definable ring \( F' := \langle F', +, \cdot, 0 \rangle \) and a definable continuous isomorphism \( h : F' \to F \), such that \( F' \) is tame.

In particular, there exists a definable \( C^k \) differential structure on \( F \) that makes \( F \) a definable \( d \)-dimensional \( C^k \)-manifold and \( F \) a \( C^k \) ring.

**Proof.** See the proof of [OPP96, Lemma 4.1]. Alternatively, one can adapt the proof of [Wen11, Theorem 5.1].

We would like to adapt some of the known results about rings definable in o-minimal structures to rings definable in d-minimal structures; we will follow the blueprints of [Pil88,OPP96].

About 0-dimensional fields, we can say almost nothing.

**Example 4.2.** Let \( F \) be a countable abstract ring. Then, there exists a d-minimal expansion of \( \overline{\mathbb{R}} \) that defines a 0-dimensional ring isomorphic to \( F \) (see §2.2[3]).

However, notice the following fact (which we will use later).
Remark 4.3. Let $G \leq \langle \mathbb{K}, + \rangle$ be a definable additive subgroup. Then, $G$ is a trivial subgroup.

For higher dimensional rings instead we can say much more. Notice that, unlike the o-minimal case, we don’t have the Descending Chain Condition for definable groups; however, we are still able to prove Theorem 4.9.

Definition 4.4. Let $F$ be a definable ring, and $a \in F$. We say that $a$ is (left-)trivial if $a \cdot F = 0$; $a$ is almost trivial if $a \cdot F^0 = 0$.

Lemma 4.5. Let $F$ be a definable ring of dimension $n \geq 1$. Fix $1 \leq k \in \mathbb{N}$ and put on $F$ the corresponding structure of group $\mathcal{C}^k$-manifold. Define $\mu : F \to M_n(\mathbb{K}), g \mapsto d_0(\lambda_g)$, where $\lambda_g : F \to F$ is the left multiplication by $g$, and $M_n(\mathbb{K})$ is the ring of $n \times n$ matrices over $\mathbb{K}$. Then, $\mu$ is a definable $\mathcal{C}^{k-1}$-ring homomorphism. $\text{Ker} \mu$ is the set of almost trivial elements of $F$ (hence, the set of almost trivial elements of $F$ is definable). In particular, if either $F$ has no zero-divisors, or $F$ is definably connected and has no trivial elements, then $\mu$ is an isomorphism with the image.

Proof. See [OPP96, Lemma 4.3]. □

Lemma 4.6. (1) Let $G \leq \langle \mathbb{K}^n, + \rangle$ be a definable additive subgroup. Then, $G$ is a $\mathbb{K}$-linear subspace of $\mathbb{K}^n$, and in particular it is definably connected.

(2) Let $F \subseteq M_n(\mathbb{K})$ be a definable subring (not necessarily containing 1). Then, $F$ is $\mathbb{K}$-subalgebra of $M_n(\mathbb{K})$, and it is definably connected.

Proof. Clearly, it suffices to prove (1). Let $c \in G \setminus \{0\}$, and define $S_c := \{ t \in \mathbb{K} : t \cdot c \in G \}$. Notice that $S_c$ is a definable nontrivial additive subgroup of $\mathbb{K}$, and therefore $S_c = \mathbb{K}$, proving that $G$ is a $\mathbb{K}$-linear subspace of $\mathbb{K}^n$. Thus, $G$ is a finite dimensional $\mathbb{K}$-linear space, and hence it is definably connected. □

Lemma 4.7. Let $F$ be a definable $\mathbb{K}$-algebra. Then, $F$ is definably connected. Let $a \in F$ be a nonzero-divisor. Then, $F$ has a 1, and $a$ has a multiplicative inverse.

Proof. By assumption, $\langle F, +, 0 \rangle$ is a finite-dimensional $\mathbb{K}$-vector space, and hence it is definably connected. Consider the map $\lambda_a : F \to F$ (the left multiplication by $a$). Then, $\lambda_a$ is a $\mathbb{K}$-linear endomorphism of $\langle F, + \rangle$; by assumption, $\lambda_a$ is injective, and therefore it is surjective. Hence, there exists $u \in F$ such that $a \cdot u = a$. Thus, for every $b \in F$, we have $a \cdot u \cdot b = a \cdot b$; since $\lambda_a$ is injective, we have that $u \cdot b = b$ for every $B \in F$. Similarly, using right multiplication by $a$, we find $v \in F$ such that, for every $b \in F$, $b \cdot v = b$. Thus, $u = v$ is the unit of $F$. Finally, since $\lambda_a$ is surjective, $a$ has a multiplicative inverse. □
Lemma 4.8. Let $F$ be a definable ring, with no zero-divisors, and of dimension $n \geq 1$. Then, $F$ is a skew field, and, in a canonical way, a $K$-subalgebra of $M_n(K)$, containing the 1 of $M_n(K)$, and it is definably connected.

Proof. Let $\mu : F \to M_n(K)$ be the function defined in Lemma 4.5. By Lemma 4.5, $\mu$ is a ring isomorphism; therefore, w.l.o.g. we can assume that $F$ is a subring of $M_n(K)$. Thus, by Lemma 4.6, $F$ is a $K$-subalgebra of $M_n(K)$; hence, by Lemma 4.7, $F$ is a definably connected skew field. Moreover, by definition, $\mu(1) = 1$. 

Denote by $\sqrt{-1}$ one of the imaginary units; remember that $\overline{K}$ denotes the underlying field of $K$. We now state the analogue of [OPP96, Theorem 1.1].

Theorem 4.9. Let $F$ be a definable ring. Assume that $F$ has no zero-divisors, and dim$(F) \geq 1$. Then $F$ is a skew field and

1. either dim$F = 1$ and $F$ is definably isomorphic to $K$,
2. or dim$F = 2$ and $F$ is definably isomorphic to $K(\sqrt{-1})$,
3. or dim$F = 4$ and $F$ is definably isomorphic to the ring of quaternions over $K$.

Proof. By Lemma 4.8, $F$ is a finite-dimensional division $K$-algebra (containing $K$ in its center). Conclude, as in [OPP96], by using Frobenius’ Theorem.

We will now study more general definable rings. First, we will consider the definably connected ones.

First of all, notice that if $F$ is a $K$-subalgebra of $M_n(K)$, then $F$ is definable in the language of fields (since it is enough to specify a $K$-linear basis of $F$).

Corollary 4.10. Let $F$ be a definably connected definable ring of dimension $n \geq 1$. Assume that $F$ has no trivial elements. Then, via the map $\mu$, $F$ is definably isomorphic to a $K$-subalgebra of $M_n(K)$. If moreover there exists $a \in F$ that is a nonzero-divisor, then $F$ contains the unit of $M_n(K)$.

Proof. By Lemma 4.5, $\mu$ is an isomorphism with the image, and $\mu(F)$ is a definably subring of $M_n(K)$. By Lemma 4.6, $\mu(F)$ is a $K$-subalgebra of $M_n(K)$. If $F$ contains a nonzero-divisor, then $F$ contains 1 by Lemma 4.7 and definition of $\mu$.

Thus, we have “full” understanding of definably connected definable rings with no trivial elements (i.e., each such a ring is definably isomorphic to a “classical” one).

(3) In [OPP96, Theorem 1.1] they forgot the assumption that the ring is infinite, which here is replaced by the assumption that is has dimension $> 0$. 

Lemma 4.11. Assume that $\mathbb{K}$ is $\omega$-saturated. Let $F$ be a definable ring, and $D := \text{Ker } \mu$. Then, $F = F^0 + D$.

Proof. Let $a \in F$. By Lemma 3.9, $\mu : F \rightarrow \mu(F)$ is an open map, and thus, by Lemma 3.24 there exists $b \in F^0$ such that $\mu(b) = \mu(a)$. Since $a - b \in \text{Ker } \mu$, we are done.

We now give a structure theorem for definable rings with 1 (but not necessarily connected). This result is, as far as I know, is new even for $\omega$-minimal structures.

Theorem 4.12. Let $F$ be a definable ring with 1. Then, $F^0$ is a definable subring (also with 1, but the unit of $F^0$ may be different from the one of $F$). Define $D := \text{Ker } \mu$ and $D' := (D^0, +, 0)$. Then, $D$ is definable discrete subring of $F$ (also with 1), and as definable rings, $F = D \oplus F^0$. Then, $F^0 \cap D = \{0\}$ and $F^0 + D = F$.

Moreover, $\mu(F^0) = \mu(F)$; and $\mu \upharpoonright F_0 : F^0 \rightarrow \mu(F)$ is a ring isomorphism; thus, $F^0$ is a $\mathbb{K}$-algebra.

Proof. Since every $\mathbb{K}$-algebra is definably connected, w.l.o.g. we can assume that $\mathbb{K}$ is $\omega$-saturated.

Claim 5. $\mu(F_0) = \mu(F)$.

By Lemma 3.24.

In particular, there exists $u_0 \in F^0$ such that $\mu(u_0) = \mu(1) = 1$.

Claim 6. $u_0$ is a 1 of $F^0$: that is, $u_0 \cdot x = x$ for every $x \in F^0$.

In fact, $d_0(\lambda_{u_0}) = 1 = d_0(\text{id}_{F_0})$; the claim follows from Lemma 3.24.

By applying the same reasoning to the opposite ring of $F$, we can conclude that there exists $u'_0 \in F_0$ that is a right 1 for $F^0$. However, $u_0 = u_0 \cdot u'_0 = u'_0$; and therefore $u_0$ is a 1 of $F^0$.

Claim 7. $F^0 = u_0 \cdot F$, and in particular $F^0$ is definable.

Since $F^0$ is a bilateral ideal of $F$, and $u_0 \in F^0$, we have $u_0 \cdot F \subseteq F^0$. Moreover, $u_0 \cdot F \supseteq u_0 \cdot F^0 = F^0$.

Remember that $D = \text{Ker } (\mu)$.

Claim 8. $D \cap F^0 = \{0\}$, and therefore $\mu \upharpoonright F^0$ is injective.

Let $d \in D \cap F^0$, thus, $\mu(d) = 0$, and hence $\lambda_d = 0$ on $F^0$; in particular, $d \cdot u_0 = 0$; but, since $d \in F^0$, $d \cdot u_0 = d$.

Claim 9. $D$ is discrete subring of $F$ (and therefore dim$(D) = 0$).

In fact, $D^0 \subseteq D \cap F^0 = \{0\}$, and thus $D$ is discrete.

Claim 10. $D \cdot F^0 = F^0 \cdot D = \{0\}$.

In fact, both $D$ and $F^0$ are bilateral ideals of $F$; thus, $D \cdot F^0 \subseteq D \cap F^0 = \{0\}$.

By Lemma 3.23, $F^0 + D = F$, and thus, as definable rings, $F = F^0 \oplus D$. □
Notice that the analogue of the above theorem for Lie rings is false, as the following example show.

**Example 4.13.** Let $F$ be the ring $\langle \mathbb{Z} \times \mathbb{R}, +, \cdot \rangle$, where $+$ is defined component-wise, while $\cdot$ is given by $\langle a, b \rangle \cdot \langle a', b' \rangle := \langle ad, ab' + ba' \rangle$; it is easy to verify that $F$ is a 1-dimensional commutative ring, with $1 = \langle 1, 0 \rangle$, and $F^0 = \{0\} \times \mathbb{R}$. Moreover, as additive groups, $F = \mathbb{Z} \oplus \mathbb{R}$, but as rings $F \neq \mathbb{Z} \oplus \mathbb{R}$. Notice also that $F^0$ is a trivial ring. The point where the proof of Theorem 4.12 does not go through is that $\mu(\langle a, b \rangle) = a$, and therefore $\mu(F) = \mathbb{Z}$, which is not an $\mathbb{R}$-algebra.

Conversely, the proof of Theorem 4.12 shows that if $F$ is a Lie ring with 1 and $\mu(F)$ is connected, then $F = \ker\mu \oplus F^0$ as Lie rings.

In general, the following construction might be useful either in finding counterexamples, or in giving structure theorems.

**Example 4.14.** Let $F$ be a definable (resp. Lie) ring and $\mathbb{A}$ be a definable (resp. Lie) bilateral $F$-algebra. Let $L := F \times A$. Let $+$ be the component-wise addition on $L$. Define a product $\cdot$ in the following way: $\langle f, a \rangle \cdot \langle f', a' \rangle := \langle f \cdot f', fa' + a f' + a \cdot a' \rangle$. Then, $L := \langle L, +, \cdot \rangle$ is a definable (resp. Lie) ring, and, via the identification $F = F \times \{0\}$, a bilateral $F$-algebra. Moreover, if $F$ has a 1, then $\langle 1, 0 \rangle$ is the 1 of $L$.

We know give some partial results and conjectures for definable rings without 1.

**Proposition 4.15.** Let $F$ be a definable ring of dimension $n \geq 1$. Let $D := \ker\mu$. Assume that:

1. either $D$ contains no nilpotent elements;
2. or $D$ has a 1;
3. or $D \cap F^0 = (0)$.

Then, $D$ is a (definable) discrete subring of $F$. Besides, if $\mathbb{K}$ is $\omega$-saturated, then $F = D \oplus F^0$ and the map $\mu_0 := \mu \upharpoonright F^0$ is a ring isomorphism between $F^0$ and $\mu(F)$, and hence $F^0$ is (in a canonical way) a $\mathbb{K}$-algebra. Moreover, in cases (1) and (2) $F^0$ is definable and (3) holds.

*Proof.* Assume (1). Let $u_1$ be the 1 of $D$. Let $a \in D \cap F^0$. Then, $a = u_1 \cdot a = 0$, and (3) holds. Moreover, $F^0 = \{ x \in F : u_1 \cdot x = 0 \}$, and hence it is definable.

Assume (2). Let $a \in D \cap F^0$. Then, $a \cdot a = 0$, and, since $D$ has no nilpotent elements, $a = 0$, and (3) holds. Moreover, $F^0 = \{ x \in F : D \cdot x = (0) \}$, and hence it is definable.

Assume now (3). Since $D^0 \subseteq D \cap F^0 = (0)$, we have that $D$ is a discrete subring of $F$. Moreover, since both $D$ and $F^0$ are bilateral ideals of $F$, $D \cdot F^0 \subseteq D \cap F^0 = (0)$, and similarly $F^0 \cdot D = (0)$; thus, $F = D \oplus F^0$. Since $\ker(\mu_0) \subseteq D \cap F^0 = (0)$, we have that $\mu_0$ is injective. If $\mathbb{K}$ is $\omega$-saturated, Lemma 3.21 implies that $\mu_0$ is also surjective. \(\square\)
Definition 4.16. Let $F$ be a definable ring. $F$ is trivial if all elements are trivial (i.e., if $x \cdot y = 0$ for every $x, y \in F$), and $F$ is almost trivial if every element is almost trivial (i.e., if $F \cdot F^0 = (0)$).

Examples 4.17. (1) If $G$ is a definable Abelian group, then $G$ can be made into a trivial ring by defining $x \cdot y = 0$.
(2) Every definable discrete ring is almost trivial.
(3) If $F$ is a definable discrete ring, and $G$ is a trivial ring, then $F \oplus G$ is an almost trivial ring.

Lemma 4.18. Let $F$ be a definable ring. T.f.a.e.:
(1) $\mu = 0$;
(2) $F$ is almost trivial;
(3) the opposite of $F$ is almost trivial (i.e., $F^0 \cdot F = (0)$);
(4) $F^0 \cdot F^0 = (0)$.

Proof. (2 $\Rightarrow$ 1), (2 $\Rightarrow$ 4), and (3 $\Rightarrow$ 4) are clear.
(1 $\Rightarrow$ 2) is Lemma 4.5.
(4 $\Rightarrow$ 1): by assumption, $F^0 \subseteq \text{Ker} \mu$; thus, $\dim(\text{Ker} \mu) = \dim(F)$, and therefore $\dim(\mu(F)) = 0$; but $\mu(F)$ is definably connected, and hence $\mu(F) = \{0\}$.
(3 $\Rightarrow$ 1): apply (4 $\Rightarrow$ 1) to the opposite ring $F^{\text{op}}$. □

Proposition 4.19. Let $F$ be a definable ring of dimension $n$.
(1) We have a short exact sequence of definable rings
$$0 \to L \to F \overset{\mu}{\to} A \to 0,$$
where $A := \mu(F)$ is a $K$-subalgebra of $M_n(K)$ and $L := \text{Ker} \mu$ is an almost trivial ring.
(2) If $F$ is almost trivial, then we have a short exact sequence of definable rings $0 \to G \to F \to D \to 0$, where $D$ is discrete and $G$ is trivial.

Proof. 1) We only have to check that $L$ is almost trivial. However, $L^0 \subseteq F^0$, and, by Lemma 4.5, $L \cdot F^0 = 0$.
2) Define $G := \{ x \in F : \forall y \in F \ x \cdot y = y \cdot x = 0 \}$. Notice that $F^0 \subseteq G$, and therefore $D := F/G$ is discrete. □

Open problem 4.20. (1) Proposition 4.19 shows that a definable ring is built using a $K$-algebra, a definable discrete ring, and a definable trivial ring. How are these rings “put together”? Is $F = D \oplus G \oplus A$, for some definable rings $D$, $G$, and $A$, with $D$ discrete, $G$ trivial, and $A$ $K$-algebra?
(2) Let $F$ be a definable ring (not necessarily with 1). Is $F^0$ definable? Is $F$ of the form $D \oplus F^0$, where $D$ is a definable discrete subring?

Finally, we show that a definable discrete ring has only trivial definable connected modules.
Proposition 4.21. Let $F$ be a definable discrete ring, and $G$ be a definable definably connected (left) $F$-module. Then, $F$ acts trivially on $G$, i.e. $fg = 0$ for every $f \in F$ and $g \in G$.

Proof. Define a ring $L$ in the following way: let $\langle L, + \rangle := \langle F, + \rangle \times \langle G, + \rangle$, with multiplication $\langle \alpha, a \rangle \cdot \langle \beta, b \rangle := \langle \alpha \beta, \alpha b \rangle$. We identify $G$ with $\{0\} \times G \subseteq L$ and $F$ with $F \times \{0\} \subseteq L$. Notice that $L^0 = G$, and, since $G \cdot G = 0$, Lemma 4.18 implies that $L \cdot G = 0$, and in particular $F \cdot G = 0$, proving that $G$ is a trivial $F$-module. □

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