COHOMOLOGY ALGEBRA OF ORBIT SPACES OF FREE INVOLUTIONS ON LENS SPACES

MAHENDER SINGH

Abstract. Let $X$ be a finitistic space having the mod-2 cohomology algebra of the lens space $L_{p}^{2m-1}(q_{1},...,q_{m})$. If $4 
mid m$, then we determine the possible mod-2 cohomology algebra of orbit space of any free involution on $X$ using the Leray spectral sequence associated to the Borel fibration $X 	woheadrightarrow X \rightarrow B\mathbb{Z}_{2}$. As an application we show that if $4 
mid m$ then for any free involution on $X$, there does not exist any $\mathbb{Z}_{2}$-equivariant map $S^{n} \rightarrow X$ for $n \geq 2$, where $S^{n}$ is equipped with the antipodal involution.

1. Introduction

An involution on a topological space $X$ is a self homeomorphism $X \rightarrow X$ which is its own inverse, that is, its an action of the group $G = \mathbb{Z}_{2}$ on $X$. This paper is concerned with the study of free involutions on lens spaces, more generally, mod-2 cohomology lens spaces. Lens spaces are odd dimensional spherical space forms described as follows. Let $p \geq 2$ be a positive integer and $q_{1}, q_{2},...,q_{m}$ be integers coprime to $p$, where $m > 1$. Let $S^{2m-1} \subset \mathbb{C}^{m}$ be the unit sphere and let $i^{2} = -1$. Then

$$(z_{1},...,z_{m}) \mapsto (e^{\frac{2\pi i q_{1}}{p}z_{1}},...,e^{\frac{2\pi i q_{m}}{p}z_{m}})$$

defines a free action of the cyclic group $\mathbb{Z}_{p}$ on $S^{2m-1}$. The orbit space is called the Lens space and is denoted by $L_{p}^{2m-1}(q_{1},...,q_{m})$. It is a compact Hausdorff orientable manifold of dimension $(2m - 1)$.

Let $X \simeq_{2} L_{p}^{2m-1}(q_{1},...,q_{m})$ mean that there is an abstract isomorphism of graded algebras

$$H^{*}(X; \mathbb{Z}_{2}) \cong H^{*}(L_{p}^{2m-1}(q_{1},...,q_{m}); \mathbb{Z}_{2}).$$

We call such a space a mod-2 cohomology lens space and refer to dimension of $L_{p}^{2m-1}(q_{1},...,q_{m})$ as its dimension. Involutions on lens spaces have been studied in detail, particularly on three dimensional lens spaces ([9], [11], [12], [13], [14], [18]). Hodgson and Rubinstein [9] obtained a classification of smooth involutions on three dimensional lens spaces having one dimensional...
fixed point sets. Kim [13] obtained a classification of orientation preserving and sense preserving PL involutions on three dimensional lens spaces. Also Kim [14] obtained a classification of free involutions on three dimensional lens spaces whose orbit spaces contains Klein bottles. The work in this paper is motivated by the work of Myers [18], where he showed that every free involution on a three dimensional lens space is conjugate to an orthogonal free involution, in which case the orbit space is again a lens space (see Remark 4.8). We consider free involutions on finitistic mod-2 cohomology lens spaces and determine the possible mod-2 cohomology algebra of the orbit space. Before we state our theorem, recall that, a finitistic space is a paracompact Hausdorff space whose every open covering has a finite dimensional open refinement, where the dimension of a covering is one less than the maximum number of members of the covering which intersect non-trivially (the notion was introduced by Swan in [20]). It is a large class of spaces including all compact Hausdorff spaces and all paracompact spaces of finite covering dimension. The lens space $L^{2m-1}(q_1, \ldots, q_m)$ is a compact Hausdorff space and hence is finitistic. If $X/G$ denote the orbit space, then we prove the following theorem.

**Main Theorem.** Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq L^{2m-1}(q_1, \ldots, q_m)$. If $4 \nmid m$, then $H^*(X/G; \mathbb{Z}_2)$ is one of the following graded algebras:

1. $\mathbb{Z}_2[x]/\langle x^{2m} \rangle$, where $\deg(x) = 1$.
2. $\mathbb{Z}_2[x, y]/\langle x^2, y^m \rangle$, where $\deg(x) = 1$ and $\deg(y) = 2$.
3. $\mathbb{Z}_2[x, y, z]/\langle x^4, y^2, z^m, x^2y - \lambda x^2 \rangle$, where $\deg(x) = 1$, $\deg(y) = 1$, $\deg(z) = 4$, $\lambda \in \mathbb{Z}_2$, $m > 2$ is even.

Our theorem generalises the results known for the orbit spaces of free involutions on the three dimensional lens space $L^3_p(q)$, to that of the large class of finitistic spaces $X \simeq L^{2m-1}_p(q_1, \ldots, q_m)$ (see remarks 5.4, 5.7 and 5.8). We also give an application to equivariant maps $\mathbb{S}^n \to X$, where $\mathbb{S}^n$ is equipped with the antipodal involution.

### 2. Cohomology of lens spaces

For convenience we write $L^{2m-1}_p(q)$ for $L^{2m-1}_p(q_1, \ldots, q_m)$. The homology groups of a lens space can be easily computed using its cell decomposition (see for example [8], p.144) and are given by

$$H_i(L^{2m-1}_p(q); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2m - 1 \\ \mathbb{Z}_p & \text{if } i \text{ is odd and } 0 < i < 2m - 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $p$ is odd, then the mod-2 cohomology groups are
Consider the diagonal action of $G$ cohomology. Let the group lens space $X$.

Throughout we shall use Čech rest of the paper without mentioning explicitly. For details of most of the content in this section we refer to [4] and [16].

We now construct a free involution on the lens space $L^2_{2m-1}(q)$. Let $q_1, ..., q_m$ be odd integers coprime to $p$. Consider the map $\mathbb{C}^m \to \mathbb{C}^m$ given by

$$(z_1, ..., z_m) \mapsto (e^{2\pi i q_1 / 2p} z_1, ..., e^{2\pi i q_m / 2p} z_m).$$

This map commutes with the $Z_p$ action on $S^{2m-1}$ defining the lens space and hence descends to a map $\alpha : L^2_{2m-1}(q) \to L^2_{2m-1}(q)$ such that $\alpha^2 = \text{identity}$. Thus $\alpha$ is an involution. Denote elements of $L^2_{2m-1}(q)$ by $[z]$ for $z = (z_1, ..., z_m) \in S^{2m-1}$. If $\alpha([z]) = [z]$, then

$$(e^{2\pi i q_1 / 2p} z_1, ..., e^{2\pi i q_m / 2p} z_m) = (e^{2\pi i k q_1 / p} z_1, ..., e^{2\pi i k q_m / p} z_m)$$

for some integer $k$. Let $1 \leq i \leq m$ be an integer such that $z_i \neq 0$, then $e^{2\pi i q_i / 2p} z_i = e^{2\pi i q_i / 2p} z_i$ and hence $e^{2\pi i q_i / 2p} = e^{2\pi i q_i / 2p}$. This implies

$$q_i - \frac{k q_i}{p} = q_i (1 - 2k)$$

is an integer, a contradiction. Hence the involution $\alpha$ is free. Observe that the orbit space of the above involution is $L^2_{2m-1}(q)/\langle \alpha \rangle = L^2_{2m-1}(q)$.

3. Free involutions on lens spaces

4. Preliminaries

In this section, we recall some basic facts that we shall be using in the rest of the paper without mentioning explicitly. For details of most of the content in this section we refer to [4] and [16]. Throughout we shall use Čech cohomology. Let the group $G = \mathbb{Z}_2$ act on a finitistic mod-2 cohomology lens space $X$. Let $G \hookrightarrow E_G \twoheadrightarrow B_G$ be the universal principal $G$-bundle. Consider the diagonal action of $G$ on $X \times E_G$, then the projection $X \times E_G \to E_G$ is $G$-equivariant and gives a fibration $X \hookrightarrow X_G \twoheadrightarrow B_G$ called the Borel fibration (see [6], Chapter IV), where $X_G = (X \times E_G)/G$ is the orbit space of the diagonal action on $X \times E_G$. We shall exploit the Leray spectral sequence associated to the Borel fibration $X \hookrightarrow X_G \twoheadrightarrow B_G$.

**Proposition 4.1.** Let $X \hookrightarrow X_G \twoheadrightarrow B_G$ be the Borel fibration, then there is a first quadrant spectral sequence of algebras $\{E_r^{*,*}, \delta_r\}$, converging to $H^*(X_G; \mathbb{Z}_2)$ as an algebra, with

$$E_2^{k,l} = H^k(B_G; \mathcal{H}^l(X; \mathbb{Z}_2)),$$
the cohomology of the base $B_G$ with locally constant coefficients $\mathcal{H}^l(X;\mathbb{Z}_2)$ twisted by a canonical action of $\pi_1(B_G)$.

See [16], Theorem 5.2.

The graded commutative algebra $H^*(X_G;\mathbb{Z}_2)$ is isomorphic to $\text{Tot}E^*_\infty$, the total complex of $E^*_\infty$. Since the fundamental group $\pi_1(B_G) = \mathbb{Z}_2$ acts trivially on the cohomology $H^*(X;\mathbb{Z}_2)$, the system of local coefficients is constant and hence by [16] Proposition 5.5 we have

$$E^2_{k,l} \cong H^k(B_G;\mathbb{Z}_2) \otimes H^l(X;\mathbb{Z}_2).$$

Note also that $H^*(X_G)$ is a $H^*(B_G)$-module with the multiplication given by $(b, x) \mapsto \rho^*(b) \cup x$.

**Proposition 4.2.** The edge homomorphisms

$$H^k(B_G;\mathbb{Z}_2) = E^k_2 \longrightarrow E^k_3 \longrightarrow \cdots \longrightarrow E^k_k = E^k_\infty \subset H^k(X_G;\mathbb{Z}_2)$$

and

$$H^l(X_G;\mathbb{Z}_2) \longrightarrow E^{0,l}_\infty = E^{0,l}_{i+1} \subset E^{0,l}_{i} \subset \cdots \subset E^{0,l}_2 = H^l(X;\mathbb{Z}_2)$$

are the homomorphisms

$$\rho^*: H^k(B_G;\mathbb{Z}_2) \rightarrow H^k(X_G;\mathbb{Z}_2) \text{ and } i^*: H^l(X_G;\mathbb{Z}_2) \rightarrow H^l(X;\mathbb{Z}_2).$$

See [16], Theorem 5.9.

We now recall some results regarding $\mathbb{Z}_2$ actions on finitistic spaces.

**Proposition 4.3.** Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X$. Suppose that $H^j(X;\mathbb{Z}_2) = 0$ for all $j > n$, then $H^j(X_G;\mathbb{Z}_2) = 0$ for $j > n$.

See [4], Chapter VII, Theorem 1.5.

Let $h : X_G \rightarrow X/G$ be the map induced by the $G$-equivariant projection $X \times E_G \rightarrow X$. Then the following is true.

**Proposition 4.4.** Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X$, then

$$h^*: H^*(X/G;\mathbb{Z}_2) \xrightarrow{\cong} H^*(X_G;\mathbb{Z}_2).$$

See [4], Chapter VII, Proposition 1.1. In fact $X/G$ and $X_G$ have the same homotopy type.

**Proposition 4.5.** Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X$. Suppose that $\sum_{i \geq 0} r_i H^i(X) < \infty$ and the induced action on $H^*(X;\mathbb{Z}_2)$ is trivial, then the Leray spectral sequence associated to $X \xrightarrow{\pi} X_G \rightarrow B_G$ do not degenerate.
See [4], Chapter VII, Theorem 1.6.

Recall that for $G = \mathbb{Z}_2$, $H^*(B_G; \mathbb{Z}_2) \cong \mathbb{Z}_2[t]$, where $\deg(t) = 1$.

From now onwards our cohomology groups will be with $\mathbb{Z}_2$ coefficients and we will suppress it from the cohomology notation.

5. Proof of the Main Theorem

This section is divided into three subsections according to the various possibilities for $p$. The Main Theorem follows from a sequence of propositions proved in this section.

4.1. When $p$ is odd.

Recall that for $p$ odd, $L^{2m-1}_p(q) \simeq_2 S^{2m-1}$. It is well known that the orbit space of any free involution on a mod-2 cohomology sphere is a mod-2 cohomology real projective space of same dimension (see for example Bredon [5], p.144). For the sake of completeness we give a quick proof using the Leray spectral sequence.

**Proposition 5.1.** If $G = \mathbb{Z}_2$ acts freely on a finitistic space $X \simeq_2 S^n$, where $n \geq 1$ is any positive integer, then

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/\langle x^{n+1} \rangle,$$

where $\deg(x) = 1$.

**Proof.** Note that $E^{k,l}_2$ is non-zero only for $l = 0$, $n$. Therefore the differentials $d_r = 0$ for $2 \leq r \leq n$ and for $r \geq n + 2$. As there are no fixed points, the spectral sequence do not degenerate and hence

$$d_{n+1} : E^{k,n}_{n+1} \to E^{k+n+1,0}_{n+1}$$

is non-zero and it is the only non-zero differential. Thus $E^* = E_{n+2}^*$ and

$$H^k(X_G) = E^{k,0}_\infty = \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = \rho^*(t) \in E^{1,0}_\infty \subset H^1(X_G)$ be determined by $t \otimes 1 \in E^{1,0}_2$. Since the cup product

$$x \cup (-) : H^k(X_G) \to H^{k+1}(X_G)$$

is an isomorphism for $0 \leq k \leq n - 1$, we have $x^k \neq 0$ for $1 \leq k \leq n$ and therefore $H^*(X_G) \cong \mathbb{Z}_2[x]/\langle x^{n+1} \rangle$. As the action of $G$ is free, $H^*(X/G) \cong H^*(X_G)$. This gives the case (1) of the main theorem. □

4.2. When $p$ is even and $4 \nmid p$.

Let $p$ be even, say $p = 2p'$ for some integer $p' \geq 1$. Since $q_1, \ldots, q_m$ are coprime to $p$, all of them are odd. Also all of them are coprime to $p'$. Note that $L^{2m-1}_p(q) = L^{2m-1}_{2p'}(q) = L^{2m-1}_{2p'}(q)/(\alpha)$, where $\alpha$ is the involution on $L^{2m-1}_{2p'}(q)$ as defined in Section 3. When $4 \nmid p$, that is $p'$ is odd, we have
Proposition 5.2. If $G = \mathbb{Z}_2$ acts freely on a finitistic space $X \simeq \mathbb{R}P^{2m-1}$, then

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y]/\langle x^2, y^m \rangle,$$

where $\deg(x) = 1$ and $\deg(y) = 2$.

Proof. Note that if $m = 1$, then the proposition is obvious. Assume $m > 1$. Let $a \in H^1(X)$ be the generator of the cohomology ring $H^*(X)$. As there are no fixed points, the spectral sequence do not degenerate at the $E_2$ term. Therefore $d_2(1 \otimes a) = t^2 \otimes 1$. One can see that $d_2 : E_2^{k,l} \rightarrow E_2^{k+2,l-1}$ is the trivial homomorphism for $l$ even and an isomorphism for $l$ odd. Note that $d_r = 0$ for all $r \geq 3$ and for all $k, l$. Hence $E_\infty^{*,*} = E_3^{*,*}$. This gives

$$E_\infty^{k,l} = \begin{cases} \mathbb{Z}_2 & \text{for } k = 0, 1 \text{ and } l = 0, 2, \ldots, 2m - 2 \\ 0 & \text{otherwise.} \end{cases}$$

But

$$H^j(X_G) = \begin{cases} E_\infty^{0,j} & \text{if } j \text{ even} \\ E_\infty^{1,j-1} & \text{if } j \text{ odd.} \end{cases}$$

Therefore

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq j \leq 2m - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = \rho^*(t) \in E_\infty^{1,0}$ be determined by $t \otimes 1 \in E_2^{1,0}$ and $x^2 \in E_\infty^{2,0} = 0$. The element $1 \otimes a^2 \in E_2^{0,2}$ is a permanent cocycle and determines an element $y \in E_\infty^{0,2} = H^2(X_G)$. Also $i^*(y) = a^2$ and $y^m = 0$. Since the multiplication

$$x \cup (-) : H^k(X_G) \rightarrow H^{k+1}(X_G)$$

is an isomorphism for $0 \leq k \leq 2m - 2$, we have $xy^r \neq 0$ for $0 \leq r \leq m - 1$. Therefore we get

$$H^*(X_G) \cong \mathbb{Z}_2[x, y]/\langle x^2, y^m \rangle,$$

where $\deg(x) = 1$ and $\deg(y) = 2$. As the action of $G$ is free, $H^*(X/G) \cong H^*(X_G)$, which gives the case (2) of the main theorem. \qed

Remark 5.3. It is well known that there is no free involution on a finitistic space $X \simeq \mathbb{R}P^{2m}$. For, the Floyd’s Euler characteristic formula ([4], p.145)

$$\chi(X) + \chi(X_G) = 2\chi(X/G)$$

gives a contradiction as $\chi(X) = 1$ and $\chi(X_G) = 0$. 
Remark 5.4. The above result follows easily for free involutions on $\mathbb{R}P^3$. Let there be a free involution on $\mathbb{R}P^3$. This lifts to a free action on $S^3$ by a group $H$ of order 4 and $\mathbb{R}P^3/\mathbb{Z}_2 = S^3/H$. There are only two groups of order 4, namely, the cyclic group $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. By Milnor [17], $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ cannot act freely on $S^3$. Hence $H$ must be the cyclic group $\mathbb{Z}_4$. Now by Rice [19], this action is equivalent to an orthogonal free action and hence $\mathbb{R}P^3/\mathbb{Z}_2 = L_4^3(q)$.

4.3. When $p$ is even and $4 \mid p$.
As above $L_{p'}^{2m-1}(q) = L_{p'}^{2m-1}(q) = L_{p'}^{2m-1}(q)/\langle \alpha \rangle$. Since $4 \mid p$, that is $p'$ is even, the cohomology groups $H^i(L_{p'}^{2m-1}(q)) = \mathbb{Z}_2$ for every $0 \leq i \leq 2m - 1$ and 0 otherwise. The Smith-Gysin sequence of the orbit map $\eta : L_{p'}^{2m-1}(q) \to L_{p'}^{2m-1}(q)$, which is a 0-sphere bundle, is given by

$$0 \to H^0(L_{2p'}^{2m-1}(q)) \xrightarrow{\eta^*} H^0(L_{p'}^{2m-1}(q)) \xrightarrow{\tau} H^0(L_{2p'}^{2m-1}(q)) \xrightarrow{\cup \eta} H^1(L_{p'}^{2m-1}(q)) \xrightarrow{\eta^*} \cdots$$

where $\tau$ is the transfer map.
By exactness the cup-square $v^2$ of the characteristic class $v \in H^1(L_{2p'}^{2m-1}(q))$ is zero. This gives the cohomology ring

$$H^*(L_{2p'}^{2m-1}(q)) \cong \wedge [v] \otimes \mathbb{Z}_2[w]/\langle w^m \rangle \cong \mathbb{Z}_2[v, w]/\langle v^2, w^m \rangle,$$

where $v \in H^1(L_{2p'}^{2m-1}(q))$ and $w \in H^2(L_{2p'}^{2m-1}(q))$.

Let $v, w$ be generators of $H^*(X) = H^*(L_{2p'}^{2m-1}(q))$. As the group $G = \mathbb{Z}_2$ acts freely on $X$ with trivial action on $H^*(X)$, the spectral sequence do not degenerate at the $E_2$ term and we have $d_2 \neq 0$ with one of the following possibilities:

(a) $d_2(1 \otimes v) = t^2 \otimes 1$ and $d_2(1 \otimes w) = t^2 \otimes v$,
(b) $d_2(1 \otimes v) = t^2 \otimes 1$ and $d_2(1 \otimes w) = 0$ and
(c) $d_2(1 \otimes v) = 0$ and $d_2(1 \otimes w) = t^2 \otimes v$.

We consider the above possibilities one by one. First we consider (a). By the multiplicative property of $d_2$, we have

$$d_2(1 \otimes w^q) = \begin{cases} t^2 \otimes vw^{q-1} & \text{if } q \text{ odd} \\ 0 & \text{if } q \text{ even} \end{cases}$$

and $d_2(1 \times vw^q) = t^2 \otimes w^q$. This shows that

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}$$
is an isomorphism if \( l \) even and \( 2 \nmid \frac{l}{2} \) or \( l \) odd. And \( d_2 \) is zero if \( l = 4q \). This gives

\[
E_3^{k,l} = \begin{cases} 
E_2^{k,l} & \text{if } k = 0, 1 \text{ and } l = 4q \\
0 & \text{otherwise.}
\end{cases}
\]

Note that \( d_r = 0 \) for all \( r \geq 3 \) and for all \( k, l \) as \( E_r^{k+r,l-r+1} = 0 \). Therefore \( E_\infty^{*,*} = E_3^{*,*} \), which is a contradiction by Proposition 4.3. Hence (a) does not arise.

For the possibility (b), we have the following proposition.

**Proposition 5.5.** Let \( G = \mathbb{Z}_2 \) act freely on a finitistic space \( X \simeq_2 \mathbb{L}^{2m-1}(q) \) where \( 4 \mid p \). Let \( \{ E_r^{*,*}, d_r \} \) be the Leray spectral sequence associated to the fibration \( X \xrightarrow{i} X_G \xrightarrow{p} B_G \). If \( v, w \) are generators of \( H^*(X) \) such that \( d_2(1 \otimes v) \neq 0 \) and \( d_2(1 \otimes w) = 0 \), then

\[
H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y]/\langle x^2, y^m \rangle,
\]

where \( \deg(x) = 1 \) and \( \deg(y) = 2 \).

**Proof.** Let \( d_2(1 \otimes v) = t^2 \otimes 1 \) and \( d_2(1 \otimes w) = 0 \). Consider

\[
d_2: E_2^{k,l} \to E_2^{k+2,l-1}.
\]

If \( l = 2q \), then \( d_2(t^k \otimes w^q) = 0 \) and if \( l = 2q+1 \), then \( d_2(t^k \otimes vw^q) = t^{k+2} \otimes w^q \) for \( 0 \leq q \leq m-1 \). This gives

\[
E_3^{k,l} = \begin{cases} 
E_2^{k,l} & \text{if } k = 0, 1 \text{ and } l = 0, 2, \ldots, 2m-2 \\
0 & \text{otherwise.}
\end{cases}
\]

Note that

\[
d_r: E_r^{k,l} \to E_r^{k+r,l-r+1}
\]

is zero for all \( r \geq 3 \) and for all \( k, l \) as \( E_r^{k+r,l-r+1} = 0 \). This gives \( E_\infty^{*,*} = E_3^{*,*} \). But

\[
H^j(X_G) = \begin{cases} 
E_\infty^{0,j} & \text{if } j \text{ even} \\
E_\infty^{1,j-1} & \text{if } j \text{ odd.}
\end{cases}
\]

Therefore

\[
H^j(X_G) = \begin{cases} 
\mathbb{Z}_2 & \text{for } 0 \leq j \leq 2m-1 \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( x = \rho^*(t) \in E_\infty^{1,0} \) be determined by \( t \otimes 1 \in E_2^{1,0} \). As \( E_\infty^{2,0} = 0 \) we have \( x^2 = 0 \). Note that \( 1 \otimes w \) is a permanent cocycle and therefore it determines an element say \( y \in E_\infty^{0,2} = H^2(X_G) \). Also \( i^*(y) = w \) and \( y^m = 0 \). Since the multiplication

\[
x \cup (-): H^k(X_G) \to H^{k+1}(X_G)
\]

is an isomorphism for \( 0 \leq k \leq 2m-2 \), we have \( xy^r \neq 0 \) for \( 0 \leq r \leq m-1 \). Therefore

\[
H^*(X_G) \cong \mathbb{Z}_2[x, y]/\langle x^2, y^m \rangle,
\]

where \( \deg(x) = 1 \) and \( \deg(y) = 2 \). As the action of \( G \) is free, \( H^*(X/G) \cong H^*(X_G) \) and again we get the case (2) of the main theorem. \( \square \)
Finally for the possibility (c), we have the following proposition.

**Proposition 5.6.** Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq L^m_2(q)$ where $4 \mid p$ and $4 \nmid m$. Let $\{E_r, d_r\}$ be the Leray spectral sequence associated to the fibration $X \xrightarrow{i} X_G \xrightarrow{\rho} B_G$. If $v, w$ are generators of $H^*(X)$ such that $d_2(1 \otimes v) = 0$ and $d_2(1 \otimes w) \neq 0$, then

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, z]/(x^4, y^2, z^m, x^2y - \lambda x^3),$$

where $\deg(x) = 1$, $\deg(y) = 1$, $\deg(z) = 4$, $\lambda \in \mathbb{Z}_2$ and $m > 2$ is even.

**Proof.** Let $d_2(1 \otimes v) = 0$ and $d_2(1 \otimes w) = t^2 \otimes v$. The derivation property of the differential gives

$$d_2(1 \otimes w^q) = \begin{cases} t^2 \otimes vw^{q-1} & \text{if } 0 < q < m \text{ odd} \\ 0 & \text{if } 0 < q < m \text{ even.} \end{cases}$$

Also $d_2(1 \otimes vw^q) = 0$ for $0 < q < m$. Note that $w^m = 0$. If $m$ is odd, then

$$0 = d_2(1 \otimes w^m) = d_2((1 \otimes w^{m-1})(1 \otimes w)) = t^2 \otimes vw^{m-1},$$

a contradiction. Hence $m$ must be even, say $m = 2n$ for some $n \geq 1$. From this we get

$$d_2 : E^{k,l}_2 \to E^{k+2,l-1}_2$$

is an isomorphism if $l$ even and $2 \nmid \frac{l}{2}$, and is zero if $l$ odd or $2 \mid \frac{l}{2}$. This gives

$$E^{k,l}_3 = \begin{cases} E^{k,l}_2 & \text{for all } k \text{ if } l = 4q \text{ or } l = 4q + 3, \text{ where } 0 \leq q \leq n - 1 \\ E^{k,l}_2 & \text{for } k = 0, 1 \text{ and } l = 4q + 1, \text{ where } 0 \leq q \leq n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

We now consider the differentials one by one. First we consider

$$d_3 : E^{k,l}_3 \to E^{k+3,l-2}_3.$$ 

Clearly $d_3 = 0$ for all $k$ and for $l = 4q, 4q + 3$ as $E^{k+3,l-2}_3 = 0$ in this case. For $k = 0, 1$ and for $l = 4(q + 1) + 1 = 4q + 5$,

$$d_3 : E^{k,4q+5}_3 \to E^{k+3,4q+3}_3$$

is also zero, because if $a \in E^{k,4q+5}_3$ and $d_3(a) = [t^{k+3} \otimes vw^{2q+1}]$, then for $b = [t^2 \otimes 1] \in E^{2,0}_3$, we have $a \cdot b \in E^{k+2,4q+5}_3 = 0$ and hence

$$0 = d_3(a \cdot b) = d_3(a) \cdot b + a \cdot d_3(b) = d_3(a) \cdot b + 0 = [t^{k+5} \otimes vw^{2q+1}],$$

which is a contradiction. Now suppose that

$$d_4 : E^{0,3}_4 \to E^{4,0}_4$$

is zero, then we claim that

$$d_4 : E^{k,l}_4 \to E^{k+4,l-3}_4$$
is zero for all $k$, $l$. Note that $E^{k+4,l-3}_4 = 0$ for all $k$ and for $l = 4q$. Similarly $E^{k+4,l-3}_4 = 0$ for $k = 0$, 1 and for $l = 4q + 1$. Now if $l = 4q + 3$, then
\[ d_4 : E^{k,4q+3}_4 \to E^{k+4,4q}_4 \]
is given by
\[ d_4([t^k \otimes v^2q+1]) = (d_4[t^k \otimes v])[1 \otimes w^{2q}] + [t^k \otimes v].(d_4[1 \otimes w^q]) = 0. \]
This proves our claim that $d_4 = 0$.

Note that if $n = 1$, then
\[ d_r : E^{k,l}_r \to E^{k+r,l-r+1}_r \]
is zero for all $r \geq 5$ and for all $k$, $l$. Hence $E^{*,*}_\infty = E^{*,*}_3$. But $E^{k,3}_3 = E^{k,0}_3 = Z_2$, that is, the top and the bottom lines survives to infinity and $H^j(X_G) \neq 0$ for $j > 3$, a contradiction by Proposition 4.3. Hence $n > 1$ and therefore $m > 2$. Now if
\[ d_5 : E^{0,4}_5 \to E^{5,0}_5 \]
is non-zero, say $d_5([1 \otimes w^2]) = [t^5 \otimes 1]$, then $d_5([1 \otimes w^2q]) = q[t^5 \otimes w^{2(q-1)}]$ and $d_5([1 \otimes v^w^{2q+1}]) = q[t^5 \otimes v^{w^{2q-1}}]$ for $1 \leq q \leq n - 1$. Note that $1 \otimes v^w^{m+1}$ is zero, we get
\[ 0 = d_5([1 \otimes v^w^{m+1}]) = n[t^5 \otimes v^w^{m-1}], \]
which is a contradiction as $2 \nmid n$. Hence $d_5 : E^{0,4}_5 \to E^{5,0}_5$ is zero.

Now consider
\[ d_5 : E^{k,l}_5 \to E^{k+5,l-4}_5. \]
Note that $E^{k+5,l-4}_5 = 0$ for $k = 0$, 1 and for $l = 4q + 1$.

For any $k$ and for $l = 4q$ or $l = 4q + 3$, $d_5$ is zero by the derivation property of $d_5$ and the above. Hence $d_5 = 0$ for all $k$, $l$.

Note that $d_r = 0$ for all $r \geq 6$ and for all $k$, $l$ as $E^{k+r,l-r+1}_r = 0$. This gives $E^{*,*}_\infty = E^{*,*}_3$, a contradiction as before. Hence
\[ d_4 : E^{0,3}_4 \to E^{4,0}_4 \]
must be non-zero, say $d_4([1 \otimes v]) = [t^4 \otimes 1]$. This gives
\[ d_4 : E^{k,l}_4 \to E^{k+4,l-3}_4 \]
is an isomorphism for $l = 4q + 3$ where $0 \leq q \leq n - 1$ and zero otherwise. Hence, we have $E^{*,*}_\infty = E^{*,*}_5$ and
\[ H^j(X_G) = \begin{cases} 
Z_2 & \text{if } j = 4q, 4q + 3, \text{ where } 0 \leq q \leq n - 1 \\
Z_2 \oplus Z_2 & \text{if } 4q < j < 4q + 3, \text{ where } 0 \leq q \leq n - 1 \\
0 & \text{if } j > 4n - 1 
\end{cases} \]
We see that $1 \otimes w^2 \in E^{0,4}_2$ and $1 \otimes v \in E^{0,1}_2$ are permanent cocycles. Hence, they determine elements $z \in E^{0,4}_\infty$ and $u \in E^{0,1}_\infty$ respectively. As $H^4(X_G) = E^{0,4}_\infty = E^{0,4}_2$, we have $\iota^*(z) = w^2$. Since $w^{2n} = w^m = 0$, we get $z^n = 0$. Also $v^2 = 0$ implies $u^2 = 0$. Let $x = \rho^*(t) \in E^{1,0}_\infty$ be determined by $t \otimes 1 \in E^{1,0}_2$. 


As $E_{\infty}^{4,0} = 0$, we have $x^4 = 0$. Similarly $E_{\infty}^{2,1} = 0$ gives $x^2u = 0$. This shows that the total complex $\text{Tot}E_{\infty}^{\ast,\ast}$ is the following graded commutative algebra

$$\text{Tot}E_{\infty}^{\ast,\ast} \cong \mathbb{Z}_2[x, y, z]/(x^4, u^2, z^n, x^2u).$$

Note that $x \in E_{\infty}^{1,0} \subset H^1(X_G)$ and $z \in E_{\infty}^{0,1} = H^4(X_G)$. Let $y \in H^1(X_G)$ represent $u$, then $i^*(y) = v$ and $y^2 = 0$. The cup product

$$H^2(X_G) \otimes H^1(X_G) \to H^3(X_G)$$

is given by $x^2y = \lambda x^3$ for some $\lambda \in \mathbb{Z}_2$. Hence,

$$H^\ast(X_G) \cong \mathbb{Z}_2[x, y, z]/(x^4, y^2, z^n, x^2y - \lambda x^3),$$

where $\deg(x) = 1 = \deg(y)$ and $\deg(z) = 4$. As the action of $G$ is free, $H^\ast(X/G) \cong H^\ast(X_G)$. This is the case (3) of the main theorem. ⊓⊔

With this we have completed the proof of the main theorem.

We now make some remarks.

Remark 5.7. For the 3-dimensional lens space $L_3^3(q)$, where $p = 4k$ for some $k$, Kim ([12], Theorem 3.6) has shown the orbit space of any sense-preserving free involution on $L_3^3(q)$ to be the lens space $L_3^3(q')$, where $q \equiv \pm 1$ or $q' \equiv \pm q \pmod{p}$ and an involution is sense-preserving if the induced map on $H_1(L_3^3(q); \mathbb{Z})$ is the identity map. This is the case (2) of the main theorem.

Remark 5.8. If $T$ is a free involution on $L_3^3(q)$ where $p$ is an odd prime, then $\mathbb{Z}_p$ and the lift of $T$ to $S^3$ generate a group $H$ of order 2p acting freely on $S^3$. $T$ is said to be orthogonal if the action of $H$ on $S^3$ is orthogonal. Myers [18] showed that every free involution on $L_3^3(q)$ is conjugate to an orthogonal free involution. It is well known that there are only two groups of order 2p, namely the cyclic group $\mathbb{Z}_{2p}$ and the dihedral group $D_{2p}$ (see for example [10], p.97). But by Milnor [17], the dihedral group cannot act freely and orthogonally on $S^3$. Hence $H$ must be the cyclic group $\mathbb{Z}_{2p}$ acting freely and orthogonally on $S^3$. Therefore the orbit space $L_3^3(q)/\langle T \rangle = S^3/H = L_3^3(q)$. Since $p$ is odd, $L_3^3(q) \cong 2 \mathbb{S}^3$ and $L_3^3(q)/\langle T \rangle \cong \mathbb{R}P^3$, which is the case (1) of the main theorem.

Remark 5.9. The condition $4 \nmid m$ is used only in Proposition 5.6, which gives case (3) of the main theorem. If $4 \mid m$, then the higher differentials may not vanish and our approach may not work.

Remark 5.10. The case (1) of the main theorem can be realized by taking the antipodal involution on a sphere. The Smith-Gysin sequence shows that the example discussed in Section 2 realizes the case (2) of the main theorem. We however do not have an example realizing the case (3) of the main theorem.
6. APPLICATION TO $\mathbb{Z}_2$-EQUIVARIANT MAPS

Let $X$ be a paracompact Hausdorff space with a fixed free involution, that is, with a fixed free action of the group $G = \mathbb{Z}_2$. Conner and Floyd [7] asked the following question.

For which integer $n$ is there a $\mathbb{Z}_2$-equivariant map of $S^n$ into $X$, but no such map of $S^{n+1}$ into $X$?

In view of the Borsuk-Ulam theorem, the answer to the question for $X = S^n$ is $n$. Motivated by the classical results of Lyusternik-Schnirelmann [15], Borsuk-Ulam [2], Yang [21][22][23] and of Bourgin [3], Conner and Floyd defined the index of the involution on $X$ as

$$\text{ind}(X) = \max \{ n \mid \text{there is a } \mathbb{Z}_2\text{-equivariant map } S^n \to X \}.$$ 

It is natural to consider the purely cohomological criteria to study the above question. The best known and most easily managed cohomology class are the characteristic classes with coefficients in $\mathbb{Z}_2$. Generalizing the Yang’s index [22], Conner-Floyd defined

$$\text{co-ind}_{\mathbb{Z}_2}(X) = \text{largest integer } n \text{ such that } w^n \neq 0,$$

where $w \in H^1(X/G; \mathbb{Z}_2)$ is the Whitney class of the principal $G$-bundle $X \to X/G$. Since $\text{co-ind}_{\mathbb{Z}_2}(S^n) = n$, we have (by [7], (4.5))

$$\text{ind}(X) \leq \text{co-ind}_{\mathbb{Z}_2}(X).$$

Since $X$ is paracompact Hausdorff, we can take a classifying map $c : X/G \to B_G$ for the principal $G$-bundle $X \to X/G$. If $k : X/G \to X_G$ is a homotopy equivalence, then $pk : X/G \to B_G$ also classifies the principal $G$-bundle $X \to X/G$ and hence it is homotopic to $c$. Therefore it suffices to consider the map $\rho^* : H^1(B_G; \mathbb{Z}_2) \to H^1(X_G; \mathbb{Z}_2)$. The image of the Whitney class of the universal principal $G$-bundle $G \to E_G \to B_G$ is the Whitney class of the principal $G$-bundle $X \to X/G$.

Let $X \cong_{\mathbb{Z}_2} L_p^{2m-1}(q)$ be a finitistic space with a free involution. The Smith-Gysin sequence associated to the principal $G$-bundle $X \to X/G$ shows that the Whitney class is non-zero. In case (1), $x \in H^1(X/G; \mathbb{Z}_2)$ is the Whitney class with $x^{2m} = 0$. This gives $\text{co-ind}_{\mathbb{Z}_2}(X) \leq 2m - 1$ and hence $\text{ind}(X) \leq 2m - 1$. Therefore in this case there is no $\mathbb{Z}_2$-equivariant map from $S^n \to X$ for $n \geq 2m$. Taking $X = S^k$ with the antipodal involution, by Proposition 4.1, we get the Borsuk-Ulam theorem which states that there is no map from $S^n \to S^k$ equivariant with respect to the antipodal involutions when $n \geq k + 1$.

In case (2), $x \in H^1(X/G; \mathbb{Z}_2)$ is the Whitney class with $x^2 = 0$. This gives $\text{co-ind}_{\mathbb{Z}_2}(X) = 1$ and $\text{ind}(X) \leq 1$. Hence in this case there is no $\mathbb{Z}_2$-equivariant map from $S^n \to X$ for $n \geq 2$.

Similarly in case (3) of the main theorem, where $4 \nmid m$, $x \in H^1(X/G; \mathbb{Z}_2)$ is the Whitney class with $x^4 = 0$. This gives $\text{co-ind}_{\mathbb{Z}_2}(X) \leq 3$ and hence $\text{ind}(X) \leq 3$. In this case also there is no $\mathbb{Z}_2$-equivariant map from $S^n \to X$ for $n \geq 4$. 


Thus, if $4 \nmid m$, then there does not exist any $\mathbb{Z}_2$-equivariant map from $S^n \to X$ for $n \geq 2$. However for $n = 1$, there does exist a $\mathbb{Z}_2$-equivariant map from $S^n \to L^m_{p-1}(q)$. The inclusion $S^1 \to S^{m-1}$ given by $z \mapsto (z,0,...,0)$ commutes with the $\mathbb{Z}_p$ action defining the lens space and hence gives a map $f : L^1_p(q_1) \to L^m_{p-1}(q)$. Consider the involution $\alpha$ of Section 3 on $L^m_{p-1}(q)$ and the same involution on $L^1_p(q_1)$ by taking $m = 1$. Then $f$ is $\mathbb{Z}_2$-equivariant with respect to the above involutions, where $L^1_p(q_1) = S^1$.

Thus, we have the following.

**Theorem 6.1.** Let $X \simeq L^m_{p-1}(q)$ be a finitistic space with a free involution. If $4 \nmid m$, then there does not exist any $\mathbb{Z}_2$-equivariant map from $S^n \to X$ for $n \geq 2$.

However for $n = 1$, there does exist a $\mathbb{Z}_2$-equivariant map from $S^n \to L^m_{p-1}(q)$. The inclusion $S^1 \to S^{m-1}$ given by $z \mapsto (z,0,...,0)$ commutes with the $\mathbb{Z}_p$ action defining the lens space and hence gives a map $f : L^1_p(q_1) \to L^m_{p-1}(q)$. Consider the involution $\alpha$ of Section 2 on $L^m_{p-1}(q)$ and the same involution on $L^1_p(q_1)$ by taking $m = 1$. Then $f$ is $\mathbb{Z}_2$-equivariant with respect to the above involutions, where $L^1_p(q_1) = S^1$. Thus, we have the following.

**Corollary 6.2.** Let $L^m_{p-1}(q)$ be a lens space with a free involution. If $4 \nmid m$, then $\text{ind}(L^m_{p-1}(q)) = 1$.

**Remark 6.3.** We note that Jaworowski [11] has computed the index of the free involution $\alpha$ on the lens space $L^m_{p-1}(q)$ and we have computed it for any free involution when $4 \nmid m$.

**References**

1. C. Allday, V. Puppe, *Cohomological Methods in Transformation Groups*, Cambridge Studies in Advanced Mathematics 32, Cambridge University Press, Cambridge, 1993.
2. K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. 20 (1933), 177-190.
3. D. G. Bourgin, *On some separation and mapping theorems*, Comment. Math. Helv. 29 (1955), 199-214.
4. G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.
5. G. E. Bredon, *Sheaf Theory*, Second Edition, Springer-Verlag, New York, 1997.
6. A. Borel et al., *Seminar on Transformation Groups*, Annals of Math. Studies 46, Princeton University Press, 1960.
7. P. E. Conner, E. E. Floyd *Fixed point free involutions and equivariant maps-I*, Bull. AMS. 66 (1960), 416-441.
8. A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
9. C. Hodgson, J. H. Rubinstein, *Involutions and isotopies of lens spaces*, LNM 1144, Springer (1985), 60-96.
10. T. W. Hungerford, *Algebra*, 12th printing, Springer-Verlag, New York, 2003.
11. J. Jaworowski, *Involutions in lens spaces*, Topology Appl. 94 (1999), 155-162.
12. P. K. Kim, *Periodic homeomorphisms of the 3-sphere and related spaces*, Michigan Math. J. 21 (1974), 1-6.
13. P.K.Kim, *PL involutions on lens space and other 3-manifolds*, Proc. Amer. Math. Soc. 44 (1974), 467-473.
14. P.K.Kim, *Some 3-manifolds which admit Klein bottles*, Trans. Amer. Math. Soc. 244 (1978), 299-312.
15. L. Lyusternik, Schnirelmann, *Topological methods in variational problems*, Moscow, 1930.
16. J.McCleary, *A User’s Guide to Spectral Sequences*, Cambridge Studies in Advanced Mathematics 58, Second Edition, Cambridge University Press, Cambridge, 2001.
17. J.Milnor, *Groups which act on $S^n$ without fixed points*, Amer. J. Math. 79 (1957), 623-630.
18. R.Myers, *Free involutions on lens spaces*, Topology 20 (1981), 313-318.
19. P.M.Rice, *Free actions of $Z_4$ on $S^3$*, Duke Math. J. 36 (1969), 749-751.
20. R.G.Swan, *A new method in fixed point theory*, Comment. Math. Helv. 34 (1960), 1-16.
21. C. T. Yang, *Continuous functions from spheres to Euclidean spaces*, Ann. of Math. 62 (1955), 284-292.
22. C. T. Yang, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobo and Dyson-I*, Ann. of Math. 60 (1954), 262-282.
23. C. T. Yang, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobo and Dyson-II*, Ann. of Math. 62 (1955), 271-283.

School of Mathematics, Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad-211019, INDIA.

E-mail address: msingh@mri.ernet.in