BIPARTITE EXTENSION GRAPHS AND THE DUFLO–SERGANNOVA FUNCTOR

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Abstract. We consider several examples when the extension graph admits a bipartition compatible with the action of Duflo–Serganova functors.

0. Introduction

In this paper $g = g_0 \oplus g_1$ is a complex Lie superalgebra and $x \in g_1$ is such that $[x, x] = 0$. This text is based on some results and ideas of [27], [18] and [33]. We give a more detailed presentation of some parts of [27], [18] which are slightly different for $osp$ and $gl$-cases.

This text is not intended for publication; its different parts have been developed in other papers [14] and [12].

0.1. Let $\mathcal{C}$ be a category of representations of a Lie superalgebra $g$ and $\text{Irr}(\mathcal{C})$ be the set of isomorphism classes of simple modules in $\mathcal{C}$. Assume that the modules in $\mathcal{C}$ are of finite length. In many examples the extension graph of $\mathcal{C}$ is bipartite, i.e. there exists a map $\text{dex} : \text{Irr}(\mathcal{C}) \to \mathbb{Z}_2$ such that

$$(\text{Dex1}) \quad \text{Ext}^1_{\mathcal{C}}(L_1, L_2) = 0 \quad \text{if} \quad \text{dex}(L_1) = \text{dex}(L_2).$$

In this paper we are interested in examples when the map $\text{dex}$ is “compatible” with the Duflo–Serganova functors $\text{DS}_x$, i.e.

$$(\text{Dex2}) \quad \text{Hom}(\text{DS}_x(L), L') = 0 \quad \text{if} \quad \text{dex}(L) \neq \text{dex}(L').$$

Note that in (Dex2) we have to choose $\text{dex}$ on $\text{Irr}(\mathcal{C})$ and on $\text{Irr}(\text{DS}_x(\mathcal{C}))$. If $\text{dex}$ satisfies (Dex1) and (Dex2), then $\text{DS}_x(L)$ is completely reducible for each $L \in \mathcal{C}$.

We say that a module $M$ is pure if for any subquotient $L$ of $M$, $\Pi(L)$ is not a subquotient of $M$. Note that (Dex2) implies the purity of $\text{DS}_x(L)$ for each $L \in \text{Irr}(\mathcal{C})$.

In what follows $\mathcal{F}\text{lin}(g)$ stands for the full subcategory of finite-dimensional $g$-modules which are completely reducible over $g_0$. In this paper we consider the case when $g$ is a finite-dimensional Kac-Moody superalgebra and $\mathcal{C} = \mathcal{F}\text{lin}(g)$; for this case we require that $\text{(Dex2)}$ holds for each $x \in X(g)$, where

$$X(g) := \{ x \in g_1 \mid [x, x] = 0 \}.$$
For other cases it make sense to restrict (Dex2) to certain values of $x$: for instance, for affine superalgebras $\mathfrak{g}$ it makes sense to consider $x$ such that $D_x(\mathfrak{g})$ is affine and for $\mathcal{C} = \mathcal{O}(\mathfrak{g})$ it makes sense to consider $x$ ”preserving” the category $\mathcal{O}$ (see [11], Sections 7 and 8 respectively).

The examples with dex satisfying (Dex1) and (Dex2) include $\mathcal{F}\text{in}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}(m|n)$ (this follows from [21]) and the full subcategory of integrable modules in the category $\mathcal{O}(\mathfrak{gl}(1|n)^{(1)})$ (this follows from [16]). In Section 2 we will check the existence of dex for exceptional Lie superalgebras. We will introduce dex satisfying (Dex1) for $\mathfrak{osp}(m|n)$-case; in [13] we will show that (Dex2) holds in this case too. It would be interesting to find other examples with dex satisfying (Dex1) and (Dex2). The strange superalgebras $\mathfrak{p}_n, \mathfrak{q}_n$ do not admit dex satisfying (Dex1), (Dex2). By [17], for $n \leq 3$, $\mathcal{F}\text{in}(\mathfrak{q}_n)$ admits dex satisfying (Dex1); this does not hold for larger $n$, see [26]. Moreover, the module $D_x(L)$ is not pure for each atypical $L \in \text{Irr}(\mathcal{F}\text{in}(\mathfrak{q}_2))$ (and $x \neq 0$), see [11], 5.5.2. By contrast, the module $D_x(L)$ is pure for $L \in \text{Irr}(\mathcal{F}\text{in}(\mathfrak{p}_n))$ if $x$ is of rank 1, see [8].

In [18], C. Gruson and V. Serganova express the character of a simple finite-dimensional $\mathfrak{osp}(M|N)$-module in terms of a basis consisting of “Euler characters”. Using dex we will show that all coefficients in such formula have the same sign (equal to $\text{dex}(L)$) (we call this property “positiveness”). The similar formulæ hold for the simple modules in $\mathcal{F}\text{in}(\mathfrak{g})$, where $\mathfrak{g}$ is an exceptional Lie superalgebra or $\mathfrak{gl}(1|n)^{(1)}$ (see [24],[34]). For $\mathfrak{q}_n$ the character formula of the above form was obtained in [30],[31] (see also [1]). In [14] we will prove a Gruson-Serganova type character formula for $\mathfrak{gl}(m|n)$-case.

The reduced Grothendieck ring is the quotient of the Grothendieck ring modulo the relation $[N] + [\Pi N] = 0$. If $\mathcal{C}$ is rigid, then $*$ induces an involution of the reduced Grothendieck ring. By Hinich’s Lemma, DS-functor induces a homomorphism $d_s$ of the reduced Grothendieck rings; this homomorphism, introduced in [21], is compatible with the above involutions (in many cases the reduced Grothendieck ring is isomorphic to the ring of supercharacters so $d_s$ can be represented as the restriction of supercharacters to a subalgebra of $\mathfrak{h}$; for the algebras from the list (1) the homomorphism $d_s$ was studied in [22]).

0.2. The map $\text{dex}$ for finite-dimensional Kac-Moody superalgebras. Let $\mathfrak{g}$ be one of the following superalgebras:

(1) $\mathfrak{gl}(m|n), \mathfrak{osp}(m|2n)$ for $m, n \geq 0$, $D(2,1|a)$, $F(4)$, $G(3)$, $\mathfrak{sl}(m|n)$ for $m \neq n$.

Note that the list (1) includes $\mathfrak{gl}(0|0) = \mathfrak{osp}(1|0) = \mathfrak{osp}(0|0) = 0$, $\mathfrak{osp}(2|0) = \mathbb{C}$ and the reductive Lie algebras $\mathfrak{gl}_m, \mathfrak{o}_m, \mathfrak{sp}_m$. For each value of $x$, the algebra $\mathfrak{g}_x$ is again one of the algebras from the list (1).

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by $\Lambda_{m|n}$ the integral weight lattice in $\mathfrak{h}^*$; this lattice is equipped by the standard parity function $p : \Lambda_{m|n} \rightarrow \mathbb{Z}_2$ (for
\[ \mathfrak{gl}(m|n), \mathfrak{osp}(M|N) \text{ and } G(3) \text{ the lattice } \Lambda_{m|n} \text{ is spanned by } \varepsilon_i s \text{ and } \delta_j s \text{ with } p(\varepsilon_i) := 0 \text{ and } p(\delta_j) := 1. \] 

For our purposes the study of the category \( \mathcal{F}\text{in}(\mathfrak{g}) \) of finite dimensional representations of \( \mathfrak{g} \) reduces to study the category \( \hat{\mathcal{F}}(\mathfrak{g}) \) with the modules whose weights lie in \( \Lambda_{m|n} \). In its turn, the category \( \hat{\mathcal{F}}(\mathfrak{g}) \) decomposes into a direct sum two equivalent categories

\[ \hat{\mathcal{F}}(\mathfrak{g}) = \mathcal{F}(\mathfrak{g}) \oplus \Pi(\mathcal{F}(\mathfrak{g})), \]

where the grading on the modules in \( \mathcal{F}(\mathfrak{g}) \) is induced by the parity function \( p \). Our goal is to find a map

\[ \text{dex} : \text{Irr}(\hat{\mathcal{F}}(\mathfrak{g})) \to \mathbb{Z}_2 \]

satisfying (Dex1) and (Dex2). By [33], for each \( L \in \text{Irr}(\hat{\mathcal{F}}(\mathfrak{g})) \) there exists \( x \) such that \( DS_x(L) \) is a non-zero typical module. Therefore it is enough to define \( \text{dex} \) on the typical simple modules. If this is done in such a way that \( \text{dex}(\Pi(L)) \neq \text{dex}(L) \), then \( \text{dex} \) satisfying (Dex2) is unique and satisfies

\[ (2) \quad \text{dex}(\Pi(L)) \neq \text{dex}(L) \quad \text{for each } L \in \text{Irr}(\hat{\mathcal{F}}(\mathfrak{g})). \]

0.3. Reduction to \( DS_1 \). In many cases it is enough to check (Dex2) for one particular value of \( x \). We continue to consider the case when \( \mathfrak{g} \) is as in [11] (the same reasoning work for symmetrizable affine Lie superalgebras for \( x \) as in Section 9 of [11]). We set

\[ \text{rank } x := \text{defect } \mathfrak{g} - \text{defect } \mathfrak{g}_x \quad \text{for } x \in X(\mathfrak{g}). \]

By [5], \( \mathfrak{g}_x \cong \mathfrak{g}_y \) if \( x, y \in X(\mathfrak{g}) \) are such that \( \text{rank } x = \text{rank } y \); we set

\[ X(\mathfrak{g})_r := \{ x \in X(\mathfrak{g}) | \text{rank } x = r \}. \]

Using Lemma 2.4.1 in [11], one can reduce (Dex2) to the case \( x \in X(\mathfrak{g})_1 \) (see [13], Section 9 for details). Fix any \( x \in X(\mathfrak{g})_1 \) and denote \( DS_x \) by \( DS_1 \). Using the results of [5], it is easy to see that for each \( y \in X(\mathfrak{g})_1 \) there exists an automorphism \( \phi \in \text{Aut}(\mathfrak{g}) \) satisfying \( \phi(x) = y \); this automorphism induces an isomorphism \( \tilde{\phi} : \mathfrak{g}_x \rightarrow \mathfrak{g}_y \) and \( DS_x(N^x) = (DS_y(N^y))^{\tilde{\phi}} \). Thus \( DS_1 \) is “independent” from the choice of \( x \); in particular, if the formula (Dex2) holds for \( DS_x \), then it holds for each \( y \in X(\mathfrak{g})_1 \). The argument of [13], Section 9 give

\[ \text{if } \text{dex satisfies (Dex1), (2) and (Dex2) holds for some } x \text{ of rank 1, then} \]

\[ (3) \quad DS_{x'}(L) \cong DS_1(DS_1(\ldots DS_1(L)\ldots)) \quad \text{for any } L \in \text{Irr}(\mathcal{F}\text{in}(\mathfrak{g})) \text{ and each } x' \in X(\mathfrak{g}). \]

In this way, the computation of \( DS_x(L) \) reduces to the computation of the multiplicities \([DS_1(L') : L'']\) for each quadruple \( (\mathfrak{g}', \mathfrak{g}'', L', L'') \), where

\[ \mathfrak{g}' := DS_1(DS_1(\ldots DS_1(\mathfrak{g})\ldots)), \quad \mathfrak{g}'' := DS_1(\mathfrak{g}) \]

and \( L' \in \text{Irr}(\mathcal{F}\text{in}(\mathfrak{g}')), L'' \in \text{Irr}(\mathcal{F}\text{in}(\mathfrak{g}'')). \)
0.3.1. The multiplicities $[DS_1(L') : L'']$ were computed for $\mathfrak{gl}(m|n)$ in [21]. For the exceptional cases we compute the multiplicities in Section 2. For the remaining case $\mathfrak{osp}(m|n)$ the multiplicities are computed in [13]. In all these cases the following properties hold:

- the module $DS_1(L')$ is pure;
- $[DS_1(L') : L''] \leq 2$;
- $[DS_1(L') : L''] \leq 1$ for $\mathfrak{g} = \mathfrak{gl}(m|n)$;
- there exists $\text{dex}$ satisfying (Dex1), (2) and the formula (Dex2) for the case when rank $x = 1$.

By above, this implies (Dex2) (for any $x \in X(\mathfrak{g})$) and shows that $DS_x(L)$ is pure, semisimple and can be computed via the formula (3).

0.3.2. In 2.3.3 we compute $DS_1(L)$ for $F(4)$. The results imply that the image of the Grothendieck ring of $\mathcal{F}(F(4))$ under the homomorphism $\text{ds}$ coincides with $\sigma$-invariants in the Grothendieck ring of $\mathcal{F}(\mathfrak{sl}_3)$ for $\sigma$ induced by a Dynkin diagram involution of $\mathfrak{sl}_3$. For all other algebras from the list [11] a similar result is obtained in [22].

0.3.3. Remark. By [8], $DS_1(L)$ is pure and multiplicity free for each $L \in \text{Irr}(\mathcal{F}(\mathfrak{p}_n))$; however, $DS_1(L)$ is not always semisimple, [3] does not hold and $DS_1(DS_1(L))$ is not always pure (see [11], Example 3.4.3).

0.3.4. Question. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra containing $\mathfrak{b}$ with the property that the defect of the Levi subalgebra of $\mathfrak{p}$ is less by one than the defect of $\mathfrak{g}$. We denote by $L(\lambda)$ (resp., $L_p(\lambda)$) a simple $\mathfrak{g}$ (resp., $\mathfrak{p}$) module of the highest weight $\lambda$. For $L(\lambda) \in \mathcal{F}(\mathfrak{g})$ we define $\Gamma_{\mathfrak{p},\mathfrak{p}}^i(L_p(\lambda))$ as in [18] (see 3.1 below) and consider the multiplicities

$$m_{\lambda,\mu}^{(i)} := [\Gamma_{\mathfrak{p},\mathfrak{p}}^i(L_p(\lambda)) : L_\mu(\mu)].$$

One has $m_{\lambda,\mu}^{(0)} = 1$. In all our examples $m_{\lambda,\mu}^{(i)} \in \{0, 1\}$ and, except for the case $m_{\lambda,\mu}^{(0)}$, one has

$$m_{\lambda,\mu}^{(i)} \neq 0 \implies \text{dex}(L(\lambda)) - \text{dex}(L(\mu)) \equiv i + 1 \mod 2.$$

It is interesting to know whether these properties hold in other cases.

0.4. Content of the paper. In Section 1 we recall the construction of $\text{DS}$-functor.

In Section 2 we consider the cases $\mathfrak{g} = D(2;1; a)$, $F(4)$, $G(3)$ and $\mathfrak{osp}(3|2)$. We compute $DS_x(L)$ for $L \in \text{Irr}(\mathcal{F}(\mathfrak{g}))$ and check that $\text{dex}$ satisfies (Dex1),(Dex2) and (2). The modules $DS_x(L)$ can be described as follows. To each atypical block $B$ in $\mathcal{F}(\mathfrak{g})$ we assign a $\mathfrak{g}_x$-module $L'$ (this assignment is injective). By [10], for each atypical block of $\mathfrak{osp}(3|2), G(3)$ the extension graph $\text{Ext}(B)$ if $D_\infty$; for the cases $F(4)$, $D(2;1; a)$ the graphs of atypical blocks are $A_\infty$ and $D_\infty$ (see [10], [25]). If $\text{Ext}(B) = D_\infty$, then $L'$ is simple and $DS_x(L) \cong L'$ if $L \in \text{Irr}(B)$ is an “end vertex” of $D_\infty$ and $DS_x(L) \cong \Pi^i(L')^{\oplus 2}$ if $L$ is the $i$th vertex.
counting from the ends. For \( \text{Ext}(B) = A^\infty \) one has \( DS_x(L) \cong \Pi^f(L') \) for each \( L \in \text{Irr}(B) \).

For \( g = F(4), D(2|1;a) \) the module \( L' \) corresponding to \( D_\infty \)-graph is a simple \( g_x \)-module satisfying \( (L')^* \cong L' \), whereas the module \( L' \) corresponding to \( A^\infty \)-graph is of the form \( L' = V \oplus V^* \), where \( V \) is a simple \( g_x \)-module with \( V^* \not\cong V \).

In Section 3 we construct for each block \( B \) in \( \tilde{F}(g) \) a graph \( \hat{\Gamma}^x \) and its subgraph \( \Gamma^x \) defined in terms of the functors \( \Gamma^x \) introduced in [29] and [18] (we follow the definition in [18]). The extension graph \( \text{Ext}(B) \) is a a subgraph of \( \hat{\Gamma}^x \); the graph \( \Gamma^x \) is useful for Gruson-Serganova type character formulae. In 3.5.7 we introduce a notion of “parametric bipartition” on the graphs \( \hat{\Gamma}^x, \Gamma^x \); a parametric bipartition on \( \hat{\Gamma}^x \) induces a bipartition on \( \text{Ext}(B) \); a parametric bipartition on \( \Gamma^x \) gives a “positive” Gruson-Serganova type character formulae. In Corollary 3.6.3 we show that under a certain conditions (which hold in the \( \text{osp} \)-case) \( \text{Ext}(B) \) is a a subgraph of \( \Gamma^x \) and a parametric bipartition on \( \Gamma^x \) induces a bipartition on \( \text{Ext}(B) \).

The graphs \( \hat{\Gamma}^x, \Gamma^x \) depend on the choice of triangular decomposition; for the case \( \text{osp}(2|2) \cong \text{sl}(2|1) \) the graph \( \text{Ext}(B) \) is a subgraph of \( \Gamma^x \) for the “mixed” triangular decomposition and is not a subgraph of \( \Gamma^x \) for the distinguished one, see Remark 3.9.3. By [27], for \( \text{gl}(m|n) \)-case the map \( \text{dex} \) gives a parametric bipartition on \( \hat{\Gamma}^x \); we check that this also holds for \( \text{osp}(2|2), \text{osp}(3|2), D(2|1;a), F(4) \) and \( G(3) \).

In Section 4 we consider the principal block \( B \) for \( g = \text{osp}(2n + t|2n) \) with \( t = 0, 1, 2 \). In this case the extension graph of \( B \) is a subgraph of \( \Gamma^x \). We describe \( \text{dex} \) which gives a parametric bipartition on \( \Gamma^x \); this implies (Dex1) and “positiveness” of the Gruson-Serganova character formula.

In Section 5 we give a description of \( \Gamma^x \) in the \( \text{osp} \)-case using the language of “arch diagrams” introduced in [13]. The results of this section are not used in the rest of the paper.

In Appendix we explain why \( \text{Ext}(B) \) is a subgraph of \( \hat{\Gamma}^x \) (this part essentially follows Sect. 6 of [27]).

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0.6. **Index of definitions and notation.** Throughout the paper the ground field is \( \mathbb{C} \); \( \mathbb{N} \) stands for the set of non-negative integers. We will use the standard Kac’s notation [23] for the root systems.
The DS-functor was introduced in [5]; see also [15] for an expanded exposition. We recall definitions and some results below. In this section \( \mathfrak{g} \) is any superalgebra; we set \( X(\mathfrak{g}) := \{ x \in \mathfrak{g}_1 | [x,x] = 0 \} \).

1. **Construction**. For a \( \mathfrak{g} \)-module \( M \) and \( g \in \mathfrak{g} \) we set \( M^g := \ker_M g. \) For \( x \in X(\mathfrak{g}) \) we set

\[
\text{DS}_x(M) := M^x/xM.
\]

Notice that \( \mathfrak{g}^x \) and \( \mathfrak{g}_x := \text{DS}_x(\mathfrak{g}) = \mathfrak{g}^x/\langle x, \mathfrak{g} \rangle \) are Lie superalgebras. Since \( M^x, xM \) are \( \mathfrak{g}^x \)-invariant and \( \langle x, \mathfrak{g} \rangle M^x \subset xM, \) \( \text{DS}_x(M) \) is a \( \mathfrak{g}^x \)-module and \( \mathfrak{g}_x \)-module. Thus

\[
\text{DS}_x : M \mapsto \text{DS}_x(M)
\]

is a functor from the category of \( \mathfrak{g} \)-modules to the category of \( \text{DS}_x(\mathfrak{g}) \)-modules.

There are canonical isomorphisms \( \text{DS}_x(\Pi(N)) = \Pi(\text{DS}_x(N)) \) and

\[
\text{DS}_x(M) \otimes \text{DS}_x(N) = \text{DS}_x(M \otimes N).
\]

For a finite-dimensional module \( L \) one has \( \text{DS}_x(L^*) \cong \text{DS}_x(L)^* \).

1.2. **Hinich’s Lemma.** The following result is called Hinich’s Lemma (see [5]); a similar result is Lemma 2.1 in [21].

A short exact sequence of \( \mathfrak{g} \)-modules

\[
0 \to M_1 \to N \to M_2 \to 0
\]

induces a long exact sequence of \( \mathfrak{g}_x \)-modules

\[
0 \to Y \to \text{DS}_x(M_1) \to \text{DS}_x(N) \to \text{DS}_x(M_2) \to \Pi(Y) \to 0
\]
where $Y$ is some $g_x$-module. We will use Hinich’s Lemma in the following situation.

Let $N$ be a $g$-module with a three-step filtration

$$0 = F_0(N) \subset F_1(N) \subset F_2(N) \subset F_3(N) = N$$

with the quotients $M_1, M_2, M_3$ (where $M_i := F_i(N)/F_{i-1}(N)$).

1.2.1. **Corollary.** If $DS_x(N) = 0$ and

$$\text{Hom}_{g_x}(DS_x(M_1), \Pi(DS_x(M_3))) = 0,$$

then there exists an exact sequence

$$0 \to \Pi(DS_x(M_3)) \to DS_x(M_2) \to \Pi(DS_x(M_1)) \to 0.$$

*Proof.* Since $DS_x(N) = 0$, the Hinich’s Lemma gives an exact sequence

$$0 \to Y_1 \to DS_x(M_1) \to 0 \to DS_x(N/M_1) \to \Pi(Y_1) \to 0$$

which implies $DS_x(N/M_1) \cong \Pi(DS_x(M_1))$. Using the Hinich’s Lemma for $N/M_1$ we obtain an exact sequence

$$0 \to Y_2 \to DS_x(M_2) \to \Pi(DS_x(M_1)) \to DS_x(M_3) \to \Pi(Y_2) \to 0.$$  

By the assumption, $\psi = 0$, so $DS_x(M_3) \cong \Pi(Y_2)$ which gives the required exact sequence. □

2. **The map $\text{dex}$ for $g$ of defect 1**

The simplest non-trivial extension graphs are $A_\infty, A^\infty_\infty$ and $D_\infty$:

$$A_\infty : \quad L^0 - L^1 - L^2 - L^3 - L^4 - \ldots$$

$$A^\infty_\infty : \quad \ldots - L^{-2} - L^{-1} - L^0 - L^1 - L^2 - \ldots$$

$$D_\infty : \quad L^1 - L^2 - L^3 - L^4 - \ldots$$

we depict $\longleftrightarrow$ by $-$. J. Germoni conjectured that the extension graph of each blocks of atypicality 1 for a basic classical Lie superalgebra is either $A^\infty_\infty$ or $D_\infty$; this conjecture was checked in [9], [10] for all cases except $F(4)$, which was completed in [25].
2.1. Proposition. Take a non-zero \( x \in \mathfrak{g}_1 \) satisfying \( x^2 = 0 \).

Let \( \mathcal{B} \) be a block and \( \text{Irr}(\mathcal{B}) = \{ L^i \}_{i \in I} \). We assume that each \( L^i \in \text{Irr}(\mathcal{B}) \) has a projective cover with a three step radical filtration with the following subquotients

\[
(5) \quad L^1; \bigoplus_{j \in \text{Adj}(i)} L^j; L^i, \quad \text{where } \text{Adj}(i) := \{ j \in I \mid \text{Ext}^1(L^j, L^i) \neq 0 \}.
\]

We set \( M_i := \text{DS}_x(L^i) \). Assume that for some \( s \) the module \( M_s \) is pure and

\[
\text{Ext}^1_{\mathcal{F}(\mathfrak{g}_s)}(M_s, M_s) = 0.
\]

(i) For each \( i \in I \) the module \( M_i \) is pure. Moreover, if \( M_s \neq 0 \), then \( M_i \neq 0 \) for each \( i \in I \).

(ii) If \( M_s \neq 0 \), then the extension graph \( \text{Ext}(\mathcal{B}) \) is bipartite.

(iii) Using the notation of (3) we have

\[
\begin{align*}
\text{if } \text{Ext}(\mathcal{B}) = A_\infty & \text{ then } M_j \cong \Pi^j(M_0)^{\oplus 2} \text{ for } j \geq 1; \\
\text{if } \text{Ext}(\mathcal{B}) = D_\infty & \text{ then } M_j \cong \Pi^{-1}(M_0)^{\oplus 2} \text{ for } j \geq 2; \ M_1 \cong M_0; \\
\text{if } \text{Ext}(\mathcal{B}) = A^\infty & \text{ then } M_j \cong \Pi^j(M_0).
\end{align*}
\]

Proof. By [5], the functor \( \text{DS}_x \) (for \( x \neq 0 \)) kills the projective modules in \( \mathcal{F}(\mathfrak{g}) \). Using Corollary 1.2.1 and (5) we conclude that for any \( i \) the purity of \( M_i \) implies the existence of an exact sequence

\[
(6) \quad 0 \to \Pi(M_i) \to \bigoplus_{j \in \text{Adj}(i)} M_j \to \Pi(M_i) \to 0
\]

In particular, the purity of \( M_i \) implies the purity of \( M_j \) for each \( j \in \text{Adj}(i) \) and \( M_i = 0 \) implies \( M_j = 0 \) for \( j \in \text{Adj}(i) \). Since \( \text{Ext}(\mathcal{B}) \) is connected, this proves (i).

Let \( L' \) be a simple module \( L' \) such that \( [M_s : L'] \neq 0 \). For each \( i \in I \) set \( p_i := [M_i : L'] \), \( q_i := [M_i : \Pi(L')] \). By above, \( p_s \neq 0 \). Using (6) and (i) we obtain

\[
(7) \quad p_i, q_i \geq 0, \quad 2q_i = \sum_{j \in \text{Adj}(i)} p_j, \quad 2p_i = \sum_{j \in \text{Adj}(i)} q_j, \quad p_iq_i = 0
\]

for each \( i \in I \). In particular, if \( p_j = q_j = 0 \) for some \( j \), then \( p_i = q_i = 0 \) for each \( i \), a contradiction. Hence for each \( i \) either \( p_i \neq 0 \) or \( q_i \neq 0 \). It is easy to see from (7) that \( p_i = 0 \) if \( L', L^s \) are connected by a path of odd length. Hence \( \text{Ext}(\mathcal{B}) \) does not have cycles of odd length; this gives (ii). For \( A_\infty \) and \( D_\infty \), (iii) follows from (7) by induction. For the case \( A^\infty \) observe that \( m_i := p_i + q_i \) satisfies \( 2m_i = m_{i-1} + m_{i+1} \) for \( i \in I = \mathbb{Z} \). Since \( m_i \geq 0 \), we get \( m_i = m_s \) for each \( i \). This completes the proof. \( \square \)
2.2. DS-functor for small rank \( \mathfrak{g} \). Let \( \mathfrak{g} \) be one of the Lie superalgebras

\[
\mathfrak{osp}(2|2), \mathfrak{osp}(3|2), D(2|1; a), G(3), F(4).
\]

Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g}_0 \). We denote by \( W \) the Weyl group of \( \mathfrak{g}_0 \) and by \(|-|\) the symmetric non-degenerate form on \( \mathfrak{h}^* \) which is induced by a non-degenerate invariant form on \( \mathfrak{g} \).

2.2.1. Let \( \Sigma \) be a base of \( \mathfrak{g} \) which contains an isotropic root \( \beta \). Fix a non-zero \( x \in \mathfrak{g}_{\beta} \).

Set \( \Delta_x := (\beta^\perp \cap \Delta) \setminus \{\beta, -\beta\} \). By [5], \( \mathfrak{g}_x \) can be identified with a subalgebra of \( \mathfrak{g} \) generated by the root spaces \( \mathfrak{g}_\alpha \) with \( \alpha \in \Delta_x \) and a Cartan subalgebra \( \mathfrak{h}_x \subset \mathfrak{h} \). If \( \Delta_x \) is not empty, then \( \Delta_x \) is the root system of the Lie superalgebra \( \mathfrak{g}_x \) and one can choose \( \Sigma_x \) in \( \Delta_x \) such that \( \Delta^+(\Sigma_x) = \Delta^+ \cap \Delta_x \). If \( \mathfrak{g} = \mathfrak{osp}(m|2) \), then \( \mathfrak{g}_x = \mathfrak{o}_m; -2; \) for \( \mathfrak{g} = D(2|1; a) \), \( G_3, F_4 \) one has \( \mathfrak{g}_x = \mathbb{C}, \mathfrak{sl}_2, \mathfrak{sl}_3 \) respectively.

2.2.2. Lemma. Let \( L := L(\lambda) \) be a finite-dimensional module and \( (\lambda|\beta) = 0 \). Set \( L' := L_{\mathfrak{g}_x}(\lambda|_{\mathfrak{h}_x}) \). One has

\[
\text{DS}_x(L) \cong \begin{cases} L' & \text{for } \mathfrak{osp}(2|2), \mathfrak{osp}(3|2), G(3) \\ L' & \text{for } D(2|1; a), F(4) \text{ if } L' \cong (L')^* \\ L' \oplus (L')^* & \text{for } D(2|1; a), F(4) \text{ if } L' \not\cong (L')^*. \end{cases}
\]

Proof. It is easy to see that \( [\text{DS}_x(L) : L'] = 1 \). Set \( \lambda' := \lambda|_{\mathfrak{h}_x} \). From [5], Sect. 7, \( \text{DS}_x(L) \) is a typical module and each simple subquotient of \( \text{DS}_x(L) \) is of the form \( L_{\mathfrak{h}_x}(\nu) \) with \( \nu \in \{\lambda', \sigma(\lambda')\} \), where \( \sigma = \text{Id} \) for \( \mathfrak{g} = \mathfrak{osp}(2|2), \mathfrak{osp}(3|2) \) and \( G(3) \), \( \sigma = -\text{Id} \) for \( D(2|1; a) \) and \( \sigma \) is the Dynkin diagram automorphism of \( \mathfrak{g}_x = \mathfrak{sl}_3 \) in \( F(4) \)-case. This gives the first formula. For \( D(2|1; a), F(4) \) one has \( L_{\mathfrak{h}_x}(\nu)^* \cong L_{\mathfrak{h}_x}(\sigma(\nu)) \); this gives the second formula. For \( \mathfrak{g} = D(2|1; a), F(4) \) the Weyl group contains \(-\text{Id}, \text{so } L \cong L^* \) and thus \( \text{DS}_x(L) \cong \text{DS}_x(L^*) \) by [1], [11]. This implies the third formula. \( \square \)

2.2.3. We fix a triangular decomposition of \( \mathfrak{g}_0 \) and denote by \( \Delta_0^+ \) the corresponding set of positive roots. We consider all bases \( \Sigma \) for \( \Delta \) which satisfy \( \Delta_0^+ \subset \Delta^+(\Sigma) \). We say that an isotropic root \( \beta \) is of the first type if \( \beta \) lies in a base \( \Sigma \) with \( \Delta_0^+ \subset \Delta^+(\Sigma) \).

Take any base \( \Sigma \) as above and denote by \( \rho \) the corresponding Weyl vector. It is easy to see that a simple atypical module \( L = L(\nu) \) satisfies the assumptions of Lemma 2.2.2 for some \( \Sigma' \) and \( \beta \in \Sigma' \) if and only if \( \nu + \rho \) is orthogonal to an isotropic root of the first type.

2.3. Blocks of atypicality 1. The blocks of atypicality 1 for basic classical Lie superalgebras were studied by J. Germoni in [9], [10] and by L. Martirosyan in [25]. These blocks satisfy the assumption of Proposition 2.2.1 and have the following extension graphs:

- \( A_\infty^\infty \) for \( \mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(2m|2n), F(4), D(2|1; a) \) for \( a \in \mathbb{Q} \);
- \( D_\infty \) for \( \mathfrak{g} = \mathfrak{osp}(2m + 1|2n), \mathfrak{osp}(2m|2n), F(4), G(3), D(2|1; a) \).

Let \( \mathfrak{g} \) be as in [2.2] and \( \mathcal{B} \) be an atypical block.

We call a block containing the trivial module \( L(0) \) a \textit{principal block}. Clearly, \( DS_x(L(0)) \) is the trivial \( \mathfrak{g}_x \)-module, so [2.1] gives \( DS_x(L) \) for each \( L \in \text{Irr}(\mathcal{B}_0) \). For \( \mathfrak{osp}(2|2), \mathfrak{osp}(3|2) \) the principal block is the only atypical block.

Combining 2.1 and 2.2.2, 2.2.3 we see that in order to compute \( DS_x(L) \) for each \( L \) in \( \text{Irr}(\mathcal{B}) \), it is enough to find \( L(\nu) \in \text{Irr}(\mathcal{B}) \) such that \( \nu + \rho \) is orthogonal to an isotropic root of the first type. Below we will list such \( \nu \) for each non-principal atypical block in the remaining cases \( D(2|1; a), F(4) \) and \( G(3) \).

2.3.1. Case \( D(2|1; a) \). For \( \mathfrak{g} := D(2|1; a) \) one has \( \mathfrak{g}_x = \mathbb{C} \). The atypical blocks were described in [10], Thm. 3.1.1.

The extension graph of the principal block \( \mathcal{B}_0 \) is \( D_\infty \), so for \( L^i \in \text{Irr}(\mathcal{B}_0) \) we have \( DS_x(L^i) = \mathbb{C} \) for \( i = 0, 1 \) and \( DS_x(L^i) = \Pi^{-1}(\mathbb{C})^\oplus 2 \) for \( i > 1 \) (where \( \mathbb{C} \) stands for the trivial even \( \mathfrak{g}_x \)-module).

If \( a \) is irrational, the principal block is the only atypical block in \( \mathcal{F}(\mathfrak{g}) \). Consider the case when \( a \) is rational. Recall that \( \mathfrak{h}^* \) has an orthogonal basis \( \{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \} \) with

\[
||\varepsilon_1||^2 = \frac{a}{2}, \quad ||\varepsilon_2||^2 = \frac{1}{2}, \quad ||\varepsilon_3||^2 = \frac{-1 + a}{2},
\]

let \( \varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^* \) be the dual basis in \( \mathfrak{h} \). The lattice \( \Lambda_{2|1} \) is spanned by \( \varepsilon_i \); the parity map is given by \( p(\varepsilon_1) = p(\varepsilon_2) = 0, p(\varepsilon_3) = 1 \). One has

\[
D(2|1; 1) = \mathfrak{osp}(4|2), \quad D(2|1; a) \cong D(2|1; -1 - a) \cong D(2|1; a^{-1})
\]

so we can assume that \( 0 < a < 1 \) and write \( a = \frac{2p}{q}, \) where \( p, q \) are relatively prime positive integers.

The atypical blocks are \( \mathcal{B}_k \) for \( k \in \mathbb{N} \) (the principal block is \( \mathcal{B}_0 \)). Consider the block \( \mathcal{B}_k \) with \( k > 0 \). The extension graph of \( \mathcal{B}_k \) is \( A_\infty^k \). By [10], Thm. 3.1.1, the block \( \mathcal{B}_k \) contains a simple module \( L \) with the highest weight \( \lambda_{k;0} \) satisfying \( (\lambda_{k;0} + \rho|\beta) = 0 \) for

\[
\beta := -\varepsilon_1 + \varepsilon_2 + \varepsilon_3.
\]

Taking \( x \in \mathfrak{g}_x \) we can identify \( \mathfrak{g}_x \) with \( \mathbb{C} h \) for \( h := q\varepsilon_1^* + p\varepsilon_2^* \). Combining 2.2.2 and 2.2.3 we get

\[
DS_x(L) = L_{q_x}(k) \oplus L_{p_x}(-k),
\]

where \( L_{q_x}(u) \) stands for the even one-dimensional \( \mathfrak{g}_x \)-module with \( h \) acting by \( u(p^2 + q^2) \). By Proposition 2.1, \( DS_x(L') \cong \Pi'(DS_x(L)) \) for each \( L' \in \text{Irr}(\mathcal{B}_k) \) (for \( k > 0 \)).

2.3.2. Case \( G(3) \). For \( \mathfrak{g} := G(3) \) the atypical blocks were described in [10], Thm. 4.1.1. The atypical blocks in \( \mathcal{F}(\mathfrak{g}) \) are \( \mathcal{B}_k \) for \( k \in \mathbb{N} \); the extension graphs are \( D_\infty \). The block \( \mathcal{B}_k \) contains a simple module with the highest weight \( \lambda_{k;0} \) satisfying \( (\lambda_{k;0} + \rho|\beta) = 0 \) for

\[
\beta := -\varepsilon_1 + \delta.
\]
Taking \( \Sigma := \{ \delta - \varepsilon_1, \varepsilon_2 - \delta, \delta \} \) and \( x \in \mathfrak{g}_{\beta} \) we can identify \( \mathfrak{g}_x \) with \( \mathfrak{sl}_3 \)-triple corresponding to the root \( \alpha = \varepsilon_1 + 2 \varepsilon_2 \). One has \( \lambda_{k;0} = k \alpha \). Combining 2.2.2 and 2.1 we get

\[
\text{DS}_x(L^0) \cong \text{DS}_x(L^1) \cong L_{\mathfrak{sl}_3}(2k), \quad \text{DS}_x(L^i) = \Pi^{i-1}(L_{\mathfrak{sl}_3}(2k)) \oplus^2 \quad \text{for } i > 1.
\]

2.3.3. Case \( F(4) \). For \( \mathfrak{g} := F(4) \) we have \( \mathfrak{g}_x \cong \mathfrak{sl}_3 \). The integral weight lattice is spanned by \( \varepsilon_1, \varepsilon_2, \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \) and \( \frac{1}{2} \delta \); the parity is given by \( p(\frac{\delta}{2}) = 0 \) and \( p(\frac{\delta}{2}) = 1 \).

The atypical blocks are described in [25], Thm. 2.1. These blocks are parametrized by the pairs \((m_1, m_2)\), where \( m_1, m_2 \in \mathbb{N}, m_1 - m_2 \in 3\mathbb{N} \). We denote the corresponding block by \( \mathcal{B}_{(m_1; m_2)} \).

The extension graph of \( \mathcal{B}_{(i;i)} \) is \( D_{\infty} \); the block \( \mathcal{B}_{(0,0)} \) is principal. For \( i > 0 \) the block \( \mathcal{B}_{(i;i)} \) contains a simple module \( L(\lambda) \) with

\[
\lambda + \rho = (i + 1)(\varepsilon_1 + \varepsilon_2) - \beta_1, \quad \text{where} \quad \beta_1 := \frac{1}{2}(2 \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \delta).
\]

One has \( (\lambda + \rho)|\beta_1 \) = 0. Take \( x \in \mathfrak{g}_{\beta_1} \) and consider the base

\[
\Sigma_1 := \{ \beta_1; \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \delta) \} = \{ \beta_1; \varepsilon_2 \}.
\]

Then \( \mathfrak{g}_x \) can be identified with \( \mathfrak{sl}_3 \) corresponding to the set of simple roots \( \{ \varepsilon_2; \varepsilon_3; \varepsilon_1 - \varepsilon_3 \} \) and Lemma 2.2.2 gives

\[
\text{DS}_x(L(\lambda)) = L_{\mathfrak{sl}_3}(i \omega_1 + i \omega_2),
\]

where \( \omega_1, \omega_2 \) are the fundamental weights of \( \mathfrak{sl}_3 \). By 2.1 we get for \( L^j \in \text{Irr}(\mathcal{B}_{(i;i)}):\)

\[
\text{DS}_x(L^0) \cong \text{DS}_x(L^1) \cong L_{\mathfrak{sl}_3}(i \omega_1 + i \omega_2), \quad \text{DS}_x(L^j) \cong \Pi^{j-1}(L_{\mathfrak{sl}_3}(i \omega_1 + i \omega_2)) \oplus^2 \quad \text{for } j > 1.
\]

Consider a block \( \mathcal{B}_{(i_1,i_2)} \) for \( i_1 \neq i_2 \). The extension graph of this block is \( A_{\infty} \) and this block contains a simple module \( L := L(\lambda') \) with

\[
\lambda' + \rho = i_1 \varepsilon_1 + i_2 \varepsilon_2 + (i_1 - i_2) \varepsilon_3.
\]

In particular, \( (\lambda' + \rho)|\beta_2 \) = 0 for \( \beta_2 := \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta) \). Taking \( x \in \mathfrak{g}_{\beta_1} \) and

\[
\Sigma_2 := \{ \beta_2; \varepsilon_2 \} = \{ \beta_2; \varepsilon_2 - \varepsilon_3 - \delta \}, \quad \text{where} \quad \beta_1 := \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \delta).
\]

we identify \( \mathfrak{g}_x \) with \( \mathfrak{sl}_3 \) corresponding to the set of simple roots \( \{ \varepsilon_2 - \varepsilon_3; \varepsilon_1 + \varepsilon_3 \} \). Combining Lemma 2.2.2 and 2.1 we get

\[
\text{DS}_x(L) = L_{\mathfrak{sl}_3}(i_1 \omega_1 + i_2 \omega_2) \oplus L_{\mathfrak{sl}_3}(i_2 \omega_1 + i_1 \omega_2), \quad \text{DS}_x(L^j) \cong \Pi^{j}(\text{DS}_x(L))
\]

for each \( L^j \) in the block \( \mathcal{B}_{(i_1,i_2)} \).

**Corollary.** The image of the Grothendieck ring of \( \mathcal{F}(F(4)) \) under the homomorphism \( ds \) coincides with \( \sigma \)-invariants in the Grothendieck ring of \( \mathcal{F}(\mathfrak{sl}_3) \).

**Proof.** The condition \( m_1 - m_2 \) divisible by 3 is equivalent to \( m_1 \omega_1 + m_2 \omega_2 \) lies in the root lattice of \( \mathfrak{sl}_3 \). \( \square \)
2.4. Conclusion. Let \( t \) be one of the superalgebras in \([2.2]\) or one of Lie algebras \( \mathbb{C},\mathfrak{sl}_2 \) or \( \mathfrak{sl}_3 \). We introduce the map \( \text{dex} \) for \( t \) by

- for a typical \( L \in \text{Irr}(\tilde{\mathcal{F}}(t)) \) we take \( \text{dex}(L) := 0 \) for \( L \in \text{Irr}(\mathcal{F}(t)) \);
- for an atypical \( L \in \text{Irr}(\tilde{\mathcal{F}}(t)) \) we set \( \text{dex}(L) := 0 \) if \( DS_x(L) \) is an even vector space.
- \( \text{dex}(\Pi(L)) \equiv \text{dex}(L) + 1 \mod 2 \).

One readily sees that \( \text{dex} \) satisfies (Dex1) and (Dex2).

3. Functors \( \Gamma^i_{\mathfrak{g},\mathfrak{q}} \)

In this section \( \mathfrak{g} \) is one of the superalgebras \( F(4), G(3), \mathfrak{gl}(m|n), \mathfrak{osp}(m|2n) \) for \( m, n \geq 0 \). We fix any triangular decomposition \( \Delta = \Delta^+ \coprod (-\Delta^+) \) and denote by \( \mathfrak{b} \) the corresponding Borel subalgebra. We consider the standard partial order \( \nu_1 \leq \nu_2 \) for \( \nu_2 - \nu_1 \in \mathbb{N}\Delta^+ \).

3.1. Let \( \mathfrak{q} \subseteq \mathfrak{p} \) be parabolic subalgebras containing \( \mathfrak{b} \) and \( \mathfrak{l} \) be the Levi factor of \( \mathfrak{p} \).

For a finite-dimensional \( \mathfrak{q} \)-module \( V \) denote by \( \Gamma^i_{\mathfrak{g},\mathfrak{p}}(V) \) the maximal finite-dimensional quotient of the induced module \( U(\mathfrak{p}) \otimes_{U(\mathfrak{q})} V \). We denote by \( \tilde{\mathcal{F}}(\mathfrak{p}) \) the category of finite-dimensional \( \mathfrak{p} \)-modules with the restriction lying in \( \tilde{\mathcal{F}}(\mathfrak{g}) \) and by \( \text{Ext}^1_{\mathfrak{p}} \) the functor \( \text{Ext}^1 \) in this category. For \( \lambda \in \Lambda_{\mathfrak{m}|\mathfrak{n}} \) we denote by \( L_{\mathfrak{p}}(\lambda) \) a simple \( \mathfrak{p} \)-module of the highest weight \( \lambda \) with the grading induced by the parity function on \( \Lambda_{\mathfrak{m}|\mathfrak{n}} \).

In \([18]\), Sect. 3 the authors introduce for \( i = 0, 1, \ldots \) an additive functor

\[
\Gamma^i_{\mathfrak{g},\mathfrak{p}} : \tilde{\mathcal{F}}(\mathfrak{p}) \to \tilde{\mathcal{F}}(\mathfrak{g})
\]

(in \([18]\) these functors are denoted by \( \Gamma_i(G/P; -) \)) in the following way. For each \( V \in \tilde{\mathcal{F}}(\mathfrak{q}) \) we take the vector bundle \( G \times_{\mathfrak{p}} V \) over the generalized Grassmanian \( G/P \) and consider the cohomology groups \( H^i(G/P, G \times_{\mathfrak{p}} V) \) as \( \mathfrak{g} \)-module. We set

\[
\Gamma^i_{\mathfrak{g},\mathfrak{p}}(V) := (H^i(G/P, G \times_{\mathfrak{p}} V^*))^*.
\]

Below we recall several properties of the functors \( \Gamma^i_{\mathfrak{g},\mathfrak{p}} \); for the proofs and other properties see \([18]\), Sections 3, 4.

3.1.1. One has \( \Gamma^0_{\mathfrak{g},\mathfrak{p}}(V) = \Gamma_{\mathfrak{g},\mathfrak{p}}(V) \). For each \( i \) the module \( \Gamma^i_{\mathfrak{g},\mathfrak{p}}(L_{\mathfrak{p}}(\lambda)) \) has the same central character as \( L(\lambda) \).

3.1.2. Each short exact sequence of \( \mathfrak{q} \)-modules

\[
0 \to U \to V \to U' \to 0
\]

induces a long exact sequence

\[
\ldots \to \Gamma^i_{\mathfrak{g},\mathfrak{q}}(V) \to \Gamma^i_{\mathfrak{g},\mathfrak{q}}(U') \to \Gamma^0_{\mathfrak{g},\mathfrak{q}}(U) \to \Gamma^0_{\mathfrak{g},\mathfrak{q}}(V) \to \Gamma^0_{\mathfrak{g},\mathfrak{q}}(U') \to 0.
\]
3.1.3. If \([\Gamma^i_{\mathfrak{g},\mathfrak{h}}(L_p(\lambda)) : L(\nu)] \neq 0\), then there exist \(I \subset \Delta^+_i\) and \(w \in W\) of length \(i\) such that 
\[
\nu + \rho = w(\lambda + \rho) - \sum_{\alpha \in I} \alpha.
\]

3.2. Poincaré polynomials. Let \(\mathfrak{b} \subset \mathfrak{q} \subset \mathfrak{p}\) be as in 3.1. We set
\[
\mathfrak{q}' := \mathfrak{q} \cap I, \quad \mathfrak{h}' := \mathfrak{h} \cap I, \quad \mathfrak{h}'' = \{ h \in \mathfrak{h} \mid [h, I] = 0 \}
\]
and notice that \(\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''\). Let \(\lambda, \mu \in \Lambda_{m/\nu}\) be such that \(L_q(\lambda) \in \mathcal{F}(\mathfrak{q})\) and \(L_p(\mu) \in \mathcal{F}(\mathfrak{p})\). For \(i = 0, 1, \ldots\) we define
\[
K^\lambda_{\mathfrak{p}, \mathfrak{q}}(z) := \sum_{i=0}^{\infty} i K^\lambda_{\mathfrak{p}, \mathfrak{q}} z^i.
\]
(It is easy to see that this Poincaré polynomial is equal to the Poincaré polynomial defined in [18], Section 4.) When the term \(K^\lambda_{\mathfrak{p}, \mathfrak{q}}\) appears in a formula it is always assumed that \(L_q(\lambda) \in \mathcal{F}(\mathfrak{q})\) and \(L_p(\mu) \in \mathcal{F}(\mathfrak{p})\).

3.2.1. By 3.1.1 we have
\[
[\Gamma^i_{\mathfrak{g},\mathfrak{h}}(L_q(\lambda)) : L_p(\mu)] = K^\lambda_{\mathfrak{p}, \mathfrak{q}} = K^\lambda_{\mathfrak{p}, \mathfrak{q}}(0).
\]
In particular, \(K^\lambda_{\mathfrak{p}, \mathfrak{q}}(0) = 1\) and \(K^\lambda_{\mathfrak{p}, \mathfrak{q}}(0) \neq 0\) implies \(\mu \leq \lambda\). By [18], Thm. 1 one has
\[
K^\lambda_{\mathfrak{p}, \mathfrak{q}}(-1) = \sum_{\nu} K^\nu_{\mathfrak{p}, \mathfrak{q}}(-1) K^\nu_{\mathfrak{p}, \mathfrak{q}}(-1)
\]
where the summation is taken on \(\nu \in \mathfrak{h}^*\) with \(\dim L_p(\nu) < \infty\).

3.3. Euler characteristic formula. Let \(\rho\) be the Weyl vector and \(R\) be the Weyl denominator, i.e.
\[
2\rho = \sum_{\alpha \in \Delta^+} (-1)^{p(\alpha)} \alpha, \quad R = \prod_{\alpha \in \Delta^+} (1 - (-1)^{p(\alpha)} e^{-\alpha})(-1)^{p(\alpha)}.
\]
We denote by \(\text{sgn} : W \to \mathbb{Z}_2\) the standard sign homomorphism and set
\[
E_{\lambda, \rho} := R^{-1} e^{-\rho} \sum_{w \in W} \text{sgn}(w) w(\frac{e^\rho \text{ch} L_p(\lambda)}{\prod_{\alpha \in \Delta^+_i(\lambda)} (1 + e^{-\alpha})}).
\]
By [18], Prop.1, if \(L_p(\lambda) \in \mathcal{F}(\mathfrak{p})\), then
\[
\sum_{\mu} K^\lambda_{\mathfrak{p}, \mathfrak{q}}(-1) \text{ch} L(\mu) = E_{\lambda, \rho}.
\]
3.3.1. Notice that $E_{\lambda,\rho}$ can be zero. For instance, take $g = \mathfrak{osp}(m|2n)$ with $m \geq 4$ and $b$ corresponding to the “mixed base”. Then $E_{0,b} = R^{-1} e^{-\rho} \sum \text{sgn}(w) e^{\omega\rho} = 0$. Since $\text{ch} L(\mu)$ are linearly independent, we have $K_{g,b}^0(-1) = 0$ for all $\mu$.

3.4. **Marked graphs.** Consider a directed graph $(V, E)$ where $V$ is at most countable and the number of edges between any two vertices is finite.

We say that $\iota : V \to \mathbb{N}$ (resp., $\iota : V \to \mathbb{Z}$) defines a $\mathbb{N}$-grading (resp., $\mathbb{Z}$-grading) on this graph if for each edge $\nu \xrightarrow{e} \lambda$ one has $\iota(\nu) < \iota(\lambda)$. Notice that for a $\mathbb{Z}$-graded graph the number of paths between any two vertices is finite.

Assume that the set of edges $E$ is equipped by two functions $b$ and $\kappa$, where $b : E \to \mathbb{Z}$ and $\kappa$ is a function from $E$ to a commutative ring.

3.4.1. For a path $P := \nu_1 \xrightarrow{e_1} \nu_2 \xrightarrow{e_2} \nu_3 \ldots \xrightarrow{e_s} \nu_{s+1}$ we define

$$\text{length}(P) := s, \quad \kappa(P) := \prod_{i=1}^{s} \kappa(e_i).$$

We call the path $P$ $b$-decreasing (resp., $b$-increasing) if $b(e_1) > b(e_2) > \ldots > b(e_s)$ (resp., $b(e_1) < \ldots < b(e_s)$). We consider $P = \nu$ as a $b$-decreasing/increasing path of zero length with $\kappa(P) = 1$. We denote the set of decreasing (resp., increasing) paths from $\nu$ to $\lambda$ by $P_b^>(\nu, \lambda)$ (resp., $P_b^<\nu, \lambda\)).$

3.4.2. **Definition.** We call two functions $b, b' : E \to \mathbb{Z}$ decreasingly-equivalent if for each path $\nu_1 \xrightarrow{e_1} \nu_2 \xrightarrow{e_2} \nu_3$ one has

$$b(e_1) > b(e_2) \iff b'(e_1) > b'(e_2).$$

Notice that $b, b'$ are decreasingly equivalent if and only if $P_b^>(\nu, \lambda) = P_{b'}^>(\nu, \lambda)$.

3.4.3. Let $(V, E)$ be a $\mathbb{Z}$-graded graph. We introduce the square matrices $A^<\kappa) = (a_{\lambda,\mu}\kappa_{\lambda,\mu})_{\lambda,\mu \in V}$ and $A^>\kappa) = (a_{\lambda,\mu}\kappa_{\lambda,\mu})_{\lambda,\mu \in V}$ by

$$a_{\lambda,\mu}^<\kappa := \sum_{P \in P_b^>(\nu, \lambda)} \kappa(P), \quad a_{\lambda,\mu}^<\kappa := \sum_{P \in P_b^>(\nu, \lambda)} (-1)^{\text{length}(P)} \kappa(P).$$

Since the graph is a $\mathbb{Z}$-graded, these matrices are lower-triangular with $a_{\lambda,\lambda}^<\kappa = a_{\lambda,\lambda}^>\kappa = 1$.

3.4.4. **Lemma.** Let $(V, E)$ be a $\mathbb{Z}$-graded graph with a finite number of edges between any two vertices. Assume that $b : E \to \mathbb{Z}$ satisfies the property

$$(BB)\text{ for each path } \nu_1 \xrightarrow{e_1} \nu_2 \xrightarrow{e_2} \nu_3 \text{ one has } b(e_1) \neq b(e_2).$$

Then $A^>\kappa) \cdot A^<\kappa) = A^<\kappa) \cdot A^>\kappa) = \text{Id}$. 
Proof. The proof is similar to [18], Thm. 4. The entries of $A^<(\kappa) \cdot A^>(\kappa)$ are of the form

$$\sum_{\mu} \sum_{P \in \mathcal{P}_\omega^< (\nu, \mu)} \sum_{Q \in \mathcal{P}_\omega^> (\mu, \lambda)} (-1)^{\text{length}(Q)} \kappa(PQ),$$

where $PQ$ stands for the concatenation of $P$ and $Q$. The property (BB) implies that each path of non-zero length which can be presented as the concatenation $PQ$, where $P$ is $b$-increasing and $Q$ is $b$-decreasing, has exactly two presentations of this form: $PQ = P'Q'$ with $\text{length} Q' = \text{length} Q \pm 1$ (for instance, for a path of length 5 with $b(e_1) = 1, b(e_2) = 2, b(e_3) = 4, b(e_4) = 2, b(e_5) = 1$ the increasing part can be either $e_1, e_2$ or $e_1, e_2, e_3$). This implies the statement. □

3.5. Useful graphs. We fix a sequence of parabolic subalgebras in $\mathfrak{g}$:

(9)

$$\mathfrak{b} = \mathfrak{p}^{(0)} \subset \mathfrak{p}^{(1)} \subset \ldots \subset \mathfrak{p}^{(k)} = \mathfrak{g}$$

and denote by $\mathfrak{l}^{(p)}$ the Levy subalgebra of $\mathfrak{p}^{(p)}$. We also fix a central character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ and denote by $\Lambda^\chi$ the set of dominant weights corresponding to $\chi$:

$$\Lambda^\chi := \{ \lambda \in \Lambda_{m|n} \mid \text{dim} L(\lambda) < \infty \text{ and } \text{Ann}_{Z(\mathfrak{g})} L(\lambda) = \text{Ker} \chi \}.$$ 

For each $\lambda \in \Lambda^\chi$ we denote by $\text{tail}(\lambda)$ the maximal $s$ such that $\text{tail}(\lambda) < b(e)$ for each edge $\nu \xrightarrow{e} \lambda$ in $\Gamma^\chi$.

Note that each two vertices in $\hat{\Gamma}^\chi$ are connected by at most $k$ edges.

3.5.1. Graph $\hat{\Gamma}^\chi$. Let $\hat{\Gamma}^\chi(z)$ be a graph with the set of vertices $V := \Lambda^\chi$ and the following edges: if $K^\chi_{\mathfrak{p}^{(s)}, \mathfrak{p}^{(s-1)}} \neq \delta_{\nu, \lambda}$ (where $\delta_{\nu, \lambda}$ is the Kronecker symbol) we join $\nu, \lambda$ by the edge of the form $\nu \xrightarrow{e} \lambda$ with $b(e) = s$, $\kappa(e) := K^\chi_{\mathfrak{p}^{(s)}, \mathfrak{p}^{(s-1)}}(z) - \delta_{\nu, \lambda} \in \mathbb{Z}[z]$.

Note that each two vertices in $\hat{\Gamma}^\chi$ are connected by at most $k$ edges.

For each $z_0 \in \mathbb{C}$ we denote by $\hat{\Gamma}^\chi(z_0)$ the subgraph where the edges with $\kappa(z_0) = 0$ are deleted and the function $\kappa_{z_0} : E \to \mathbb{C}$ is given by $\kappa_{z_0}(e) := \kappa(e)(z_0)$.

Note that $\hat{\Gamma}^\chi(0)$ does not have loops, see [3,2,1].

3.5.2. Graph $\Gamma^\chi$. Let $\Gamma^\chi$ (resp., $\Gamma^\chi(z_0)$) be the graph obtained from $\hat{\Gamma}^\chi$ (resp., from $\hat{\Gamma}^\chi(z_0)$) by removing the edges of the form $\nu \xrightarrow{e} \lambda$ with $b(e) \leq \text{tail}(\lambda)$. Thus we have $\text{tail}(\lambda) < b(e)$ for each edge $\nu \xrightarrow{e} \lambda$ in $\Gamma^\chi$.

We will always assume that $\Gamma^\chi$ satisfies the following condition:

(Tail) $\text{tail}(\nu) \leq b(e)$ for each edge $\nu \xrightarrow{e} \lambda$ in $\Gamma^\chi$.

which is is tautological for $k = 1$. 

3.5.3. **Notation.** We denote by $P^>_b(\nu, \lambda)$ (resp., by $\hat{P}^>_b(\nu, \lambda)$) the set of $b$-decreasing paths from $\nu$ to $\lambda$ in the graph $\Gamma^x$ (resp., $\hat{\Gamma}^x$).

3.5.4. **Corollary.** Take $\lambda, \nu \in \Lambda^x$.

(i) Assume that for each $\mu \in \Lambda^x$ one has

$$(\exists i \text{ s.t. } K_{p(i+1), p(i)}^\nu(\mu, \eta)(z) \neq 0) \implies \eta \in \Lambda^x.$$

Then

$$K^\lambda,\nu_{b^{-1}}(-1) = \sum_{P \in \hat{P}^>_b(\nu, \lambda)} \kappa_{-1}(P).$$

(ii) Assume that $\Gamma^x$ satisfies (Tail) and that for each $\mu \in \Lambda^x$ one has

$$(\exists i \geq \text{tail}(\mu) \text{ s.t. } K_{p(i+1), p(i)}^\nu(\mu, \eta)(z) \neq 0) \implies \eta \in \Lambda^x.$$

Then

$$K^\lambda,\nu_{b^{-1}}(-1) = \sum_{P \in P^>_b(\nu, \lambda)} \kappa_{-1}(P).$$

**Proof.** The assertions follow from the formula (8). □

3.5.5. **Remark.** The graph $\Gamma^x$ is useful for character formulae. Retain notation of 3.3 and notice that $E_{\lambda, p}^\mu$ has a particularly nice formula if $\lambda |_{\mathfrak{h}_b} = 0$ (in this case $\text{ch} L^\mu_\lambda = e^\lambda$). Thus it makes sense to express $\text{ch} L(\mu)$ in terms of $E_{\lambda, p}^\mu(j)$ for $j \leq \text{tail}(\lambda)$. By 3.3.1, $E_{\lambda, p}^\mu$ can be zero if “$p$ is too small”; thus it makes sense to consider the maximal “nice” $p$ for each $\lambda$, which is $p_\lambda$.

3.5.6. **Lemma.** If the zero weight is a minimal dominant weight for $l^p$ for each $p$, then $\hat{\Gamma}^x(0) = \Gamma^x(0)$.

**Proof.** By 3.1.1, as a $l^p$-module $\Gamma^0_{p(p), p(p-1)}(L_{p(p-1)}(\lambda))$ is a finite-dimensional quotient of the Verma $l^p$-module $M_{l^p(\lambda)}$. If the zero weight is a minimal dominant weight for $l^p$, then for each $p \leq \text{tail}(\lambda)$ the module $L_{p(p)}(\lambda)$ is a unique finite-dimensional quotient of $M_{l^p}(\lambda)$, so $\Gamma^0_{p(p), p(p-1)}(L_{p(p-1)}(\lambda)) = L_{p(p)}(\lambda)$. Therefore $\hat{\Gamma}^x(0) = \Gamma^x(0)$ as required. □

3.5.7. **Definitions.** Let $\text{dex} : V \to \mathbb{Z}_2 = \{0, 1\}$ be any map.

We say that $\Gamma^x(0)$ (resp., of $\hat{\Gamma}^x(0)$) is bipartite with respect to $\text{dex}$ if for each edge $\nu \xrightarrow{e} \lambda$ in this graph $\text{dex}(\lambda) \neq \text{dex}(\nu)$.

Recall that $\kappa(e)$ is a polynomial with non-negative integral coefficients. We say that $\text{dex}$ gives a parametric bipartition of $(\Gamma^x, \kappa)$ if for each edge $\nu \xrightarrow{e} \lambda$ one has

$$z^{\text{dex}(\lambda) - \text{dex}(\nu) + 1} \kappa(e) \in \mathbb{Z}[z^2].$$
We say that \( \text{dex} \) gives a \textit{signed bipartition} of \( (\Gamma^x(-1), \kappa_{-1}) \) if for each edge \( \nu \overset{e}{\rightarrow} \lambda \)

\[
(11) \quad (-1)^{\text{deg}(\lambda) - \text{deg}(\nu) + 1} \kappa_{-1}(e) \in \mathbb{Z}_{\geq 0}
\]
or, equivalently,

\[
(12) \quad (-1)^{\text{deg}(\lambda) - \text{deg}(\nu)} (-1)^{\text{length}(P)} \kappa_{-1}(P) \in \mathbb{Z}_{\geq 0} \quad \text{for each path } P \text{ from } \nu \text{ to } \lambda.
\]

We say that \( \text{dex} \) gives a \textit{parametric bipartition} of \( (\hat{\Gamma}^x, \kappa) \) if each edge \( \nu \overset{e}{\rightarrow} \lambda \) with \( \nu \neq \lambda \) satisfies \( (10) \). Similarly, we say that \( \text{dex} \) gives a \textit{signed bipartition} of \( (\hat{\Gamma}^x(-1), \kappa_{-1}) \) if \( (11) \) holds for each \( \nu \overset{e}{\rightarrow} \lambda \) with \( \nu \neq \lambda \) or, equivalently if \( (12) \) holds for any paths without loops.

3.5.8. Recall that \( \hat{\Gamma}^x(0) \) does not have loops. If \( \text{dex} \) is a parametric bipartition of \( \Gamma^x \) (resp., \( \hat{\Gamma}^x \)), then \( \text{dex} \) is a signed bipartition of \( (\Gamma^x(-1), \kappa_{-1}) \) (resp., of \( (\hat{\Gamma}^x(-1), \kappa_{-1}) \)) and \( \Gamma^x(0) \) (resp., \( \hat{\Gamma}^x \)) is bipartite with respect to \( \text{dex} \).

3.5.9. \textit{Remark.} In the examples \([3,4,3,9,5]\) below \( \Gamma^x \) is a \( \mathbb{Z} \)-graded graph admitting a parametric bipartition; the same is true for the graph \( \hat{\Gamma}^x \) for the dense flag for a distinguished Borel in \( \mathfrak{gl}(n|n) \)-case, see \([27]\). The graph \( \hat{\Gamma}^x \) has a loop \( 0 \overset{e}{\rightarrow} 0 \) (and thus is not \( \mathbb{Z} \)-graded) for the dense flag for a mixed Borel for \( g = \mathfrak{osp}(2|2) \) (= \( \mathfrak{sl}(1|2) \)) and for \( g = \mathfrak{osp}(4|2) \) see \([3,9,2]\) and \([3,9,5]\). By \([18]\), Lemma 26, \( \kappa(e) = z \) for \( \mathfrak{osp}(2|2) \) and \( \kappa(e) = z^2 \) for \( \mathfrak{osp}(4|2) \) so the formula \( (10) \) holds for \( e \) if \( g = \mathfrak{osp}(2|2) \) and does not hold if \( g = \mathfrak{osp}(4|2) \). The graph \( \hat{\Gamma}^x \) admits a parametric bipartition in both cases.

3.6. \textbf{Graph Ext(\( \chi \))}. By \([6,3]\) \( \dim \text{Ext}^1(L(\lambda), L(\nu)) = \dim \text{Ext}^1(L(\nu), L(\lambda)) \). We denote by \( \text{Ext}(\chi) \) the graph without loops, with the set of vertices \( \Lambda^x \) and \( \dim \text{Ext}^1(L(\lambda), L(\nu)) \) edges between \( \nu \) and \( \lambda \) for \( \nu \neq \lambda \) (we will usually consider the undirected edges).

3.6.1. We say that \( \text{Ext}(\chi) \) is a subgraph of a directed graph if \( \text{Ext}(\chi) \) is a subgraph of the “undirected version” of this graph (we forget the directions of edges).

3.6.2. Recall that \( \Gamma^0_{p(p),p(p-1)}(L_{p(p-1)}(\lambda)) \) is the maximal finite-dimensional quotient of \( \text{Ind}_{p(p-1)}^{p(p)}(L_{p(p-1)}(\lambda)) \); this is indecomposable module with the cosocle is isomorphic to \( L_{p(p)}(\lambda) \). Using Corollary \([6,5]\) we obtain the

\textbf{Corollary.} The graph \( \text{Ext}(\chi) \) is a subgraph of \( \hat{\Gamma}^x(0) \).

3.6.3. \textbf{Corollary.} Assume that \( \hat{\Gamma}^x \) admits a parametric bipartition \( \text{dex} \).

(i) \( \text{Ext}(\chi) \) is bipartite with respect to \( \text{dex} \).

(ii) \( \Gamma^i_{p(p),p(p-1)}(L_{p(p-1)}(\lambda)) \) is a semisimple \( p(p) \)-module for \( i > 0 \) and has a semisimple radical for \( i = 0 \).
(iii) Assume that $\Gamma^\chi$ admits a parametric bipartition $\text{dex}$ and that the zero weight is a minimal dominant weight for $l^p$ for each $p = 0, 1, \ldots, k - 1$. Then $\text{Ext}(\chi)$ is a subgraph of $\Gamma^\chi(0)$ and $\text{dex}$ defines a bipartition of $\text{Ext}(\chi)$. Moreover, the claims of (ii) hold for $p > \text{tail}(\lambda)$.

Proof. Corollary 3.6.2 implies (i). For (ii) let $\text{Ext}(p)$ be the “Ext”-graph for $p^\chi$: the set of vertices for this graph is $\Lambda^\chi$ and the multiplicity of the edge $\lambda \rightarrow \nu$ is $\dim \text{Ext}^1(L_p(\nu), L_p(\lambda))$. By Corollary 6.5, $\text{Ext}(p)$ is a subgraph of $\hat{\Gamma}^\chi(0)$, so $\text{dex}$ gives a bipartition on $\text{Ext}(p)$. For $i > 0$ one has

$$[\Gamma^i_p, p(p-1)](L_p(p-1)(\lambda) : L_p(p)(\nu)] \neq 0 \implies \text{dex}(\nu) + \text{dex}(\lambda) \equiv i + 1 \mod 2.$$ 

Therefore there are no non-splitting extensions between the subquotients of the $p^\chi$-module $\Gamma^i_p, p(p-1)$, thus this module is completely reducible. For $i = 0$ the same holds for $\nu \neq \lambda$ and $L_p(p)(\lambda)$ is a unique simple quotient of $\Gamma^0_p, p(p-1)(L_p(p-1)(\lambda))$. Hence the radical of $\Gamma^0_p, p(p-1)(L_p(p-1)(\lambda))$ is semisimple. This gives (ii). If the zero weight is a minimal dominant weight for $l^p$ for each $p$, then $\hat{\Gamma}^\chi(0) = \Gamma^\chi(0)$ (see Lemma 3.5.6) and so (iii) has the same proof as (ii).

3.6.4. Remark. We see that in order to have a parametric partition on $\hat{\Gamma}^\chi$ one has to take a “dense enough” chain of the parabolic subalgebras, since if $\hat{\Gamma}^\chi$ admits such grading, then the maximal finite-dimensional quotient of $\text{Ind}^p_p, p(p-1)(L_p(p-1)(\lambda))$ has a Loewy filtration of length $\leq 2$. In the examples below we take $l^p$ of the defect $p$.

3.7. The Gruson-Serganova algorithm. We assume that $\Gamma^\chi = (A^\chi, E)$ is a $\mathbb{Z}$-graded graph which satisfies the assumptions of Corollary 3.5.4 (ii). The following construction is a slight reformulation of the construction described in [18], Sect. 12.

3.7.1. Recall that $\text{ch} L_{p, \lambda}(\lambda) = e^\lambda$. Set

$$\mathcal{E}_\nu := \mathcal{E}_{\nu, p, \nu} = R^{-1}e^{-\rho} \sum_{w \in W} \text{sgn}(w)w\left(\prod_{\alpha \in \Delta(l_\nu)^*_u} (1 + e^{-\alpha})\right).$$

Combining 3.3 and Corollary 3.5.3 (ii) we get

$$\sum_{\mu} a_{\lambda, \nu}^\nu \text{ch} L(\mu) = \mathcal{E}_{\nu},$$

for $A^\nu(\kappa-1) = (a_{\lambda, \nu}^\nu)$ defined as in 3.4.3. The matrix $A := A^\nu(\kappa-1)$ is lower-triangular with $a_{\lambda, \lambda} = 1$. Thus $A$ is invertible that is

$$\text{ch} L(\lambda) = \sum_{\mu} a_{\lambda, \mu}^\nu \mathcal{E}_{\nu},$$
for $(a'_{\lambda,\mu}) := A^{-1}$. In the light of Lemma 3.4.4 the entries of $A^{-1}$ can be expressed in terms of $b$-increasing paths from $\nu$ to $\lambda$ if $b : E \to \mathbb{Z}$ would satisfy the property (BB). Unfortunately, $b$ almost never satisfy (BB); however, it is often possible to find a decreasingly-equivalent function $b'$ satisfying (BB) (we do not require that $b'$ satisfies (Tail)). For $\mathfrak{gl}(M|N)$ and $\mathfrak{osp}(M|N)$ the function $b'$ is given in [27] and [18] respectively; in 4.4.3 below we describe $b'$ for $\mathfrak{osp}(M|N)$-case. Denoting by $\mathcal{P}_b^< (\nu, \lambda)$ the set of $b'$-increasing paths in $\Gamma^x$ we obtain

$$a'_{\lambda,\mu} = \sum_{P \in \mathcal{P}_b^< (\mu, \lambda)} \kappa_1(P) = \sum_{P \in \mathcal{P}_b^< (\mu, \lambda)} \kappa_{-1}(P), \quad a'_{\lambda,\mu} = \sum_{P \in \mathcal{P}_b^< (\mu, \lambda)} (-1)^{|\text{length}(P)|} \kappa_{-1}(P).$$

3.7.2. Assume that $\text{dex} : V \to \{0,1\}$ is a signed bipartition of $(\Gamma^x(1), \kappa_{-1})$ (see 3.5.7 for definition). By (12) the number $(-1)^{\text{dex}(\lambda) - \text{dex}(\mu)} a'_{\lambda,\mu}$ is a non-negative integer. (i.e. the Gruson-Serganova character formula is “positive”). These number can be interpreted as follows. Consider the following modification of the graph $\Gamma^x = (V,E)$: the graph $D^x$ with the same set of vertices $V = \Lambda^x$ and the set of egdes $E'$ obtained from $E$ by taking each edge $\nu \xrightarrow{-\lambda} \lambda$ with the multiplicity $(-1)^{\text{dex}(\lambda) - \text{dex}(\nu)} \kappa_{-1}(e)$ (this number is non-negative since $\text{dex}$ is a signed bipartition). By above,

$$(-1)^{\text{dex}(\lambda) - \text{dex}(\mu)} a'_{\lambda,\mu}$$

is the number of $b'$-increasing paths from $\mu$ to $\lambda$ in $D^x$.

For $\mathfrak{osp}(M|N)$-case the graph $D^x$ is described in [18]; we give some details in 4.5 below; the case $\mathfrak{gl}(m|n)$ will be treated in [14].

3.7.3. The assumption that $\Gamma^x$ is $\mathbb{Z}$-graded can be weaken using the following trick. Fix a set of “bad vertices” $\Lambda' \subset \Lambda(\chi)$ and consider a graph $\Gamma'(\chi)$ obtained from $\Gamma^x$ by erasing all edges ending at $\lambda \in \Lambda'$. Assume that $\Gamma'(\chi)$ is $\mathbb{Z}$-graded. The above reasoning allows to express $\text{ch } L(\mu)$ in terms of $E_\lambda$ for $\lambda \in \Lambda(\chi) \setminus \Lambda'$ and $\text{ch } L(\nu)$ for $\nu \in \Lambda'$, see 3.9.5 for examples.

3.8. Examples. The Poincaré polynomials for certain chains of parabolic subalgebras were computed for the finite-dimensional Kac-Moody superalgebras in [32], [27], [10], [25] and for $\mathfrak{q}_n$ in [30]. In all these cases the chain satisfies the following condition: $[p]n$ has defect $p$. We list some properties of the corresponding graphs (for the $\mathfrak{gl}$-case we consider only the principal block in $\mathfrak{gl}(n|n)$).

In all these examples the Poincaré polynomials have the following property: the polynomial $K_{p(s), p(s-1)}^{\lambda,\mu} - \delta_{\lambda,\mu}$ is non-zero for at most one value of $s > \text{tail}(\lambda)$. We denote by $\kappa_{\lambda\mu}$ the corresponding non-zero polynomial (if it exists) and set $\kappa_{\lambda\mu} = 0$ otherwise.

The above property implies that $\Gamma^x$ does not have multi-edges (and that $\kappa(\mu \to \lambda) = \kappa_{\lambda\mu}$). The graph $\Gamma^x$ admits a $\mathbb{N}$-grading and satisfies (Tail).

For $\mathfrak{g} \neq \mathfrak{q}_n$ the graph $\Gamma^x$ admits a parametric bipartition $\text{dex}$. 

\text{For } \mathfrak{g} \neq \mathfrak{q}_n \text{ the graph } \Gamma^x \text{ admits a parametric bipartition } \text{dex}.
For \( g \neq q_n, \mathfrak{osp}(2m|2n) \), the polynomials \( \kappa^{\lambda,\mu} \) are monomials, so the condition on the parametric bipartition simply means that \( \kappa^{\lambda,\mu} \) is zero or \( z^i \) for \( i \equiv \text{dex}(\lambda) - \text{dex}(\mu) + 1 \) modulo 2. In these cases \( D^x = \Gamma^x \).

For \( g = \mathfrak{osp}(2m|2n), q_n \) with \( \chi \) of atypicality greater than one, \( \kappa^{\lambda,\mu} \in \{0, z^i, z^i + z^j\} \) and the condition on the parametric bipartition takes the form \( i \equiv \text{dex}(\lambda) - \text{dex}(\mu) + 1 \) and \( i \equiv j \mod 2 \). This holds for \( \mathfrak{osp}(2m|2n) \) in this case \( D^x \) is obtained from \( \Gamma^x \) by doubling the edges with \( \kappa^{\lambda,\mu} = z^i + z^j \).

An interesting example is the \( q_n \)-case. For \( \chi \) of atypicality greater than one, \( i - j \) can be odd, so \( \Gamma^x \) does not admit a parametric bipartition and \( \Gamma^x(0) \) is not bipartited. By [26] the Ext-graph is not bipartite. However, the \( (\Gamma^x(-1), \kappa_{-1}) \) admits a signed bipartition \( \text{dex} \). In this case \( D^x \) is obtained from \( \Gamma^x \) by doubling the edges with even \( j \) and deleting the edges with odd \( j \) \((\kappa^{\lambda,\mu}(-1) = 0 \text{ if } j \text{ is odd})\). The Gruson-Serganova algorithm gives the Su-Zhang character formula [35].

3.9. Examples of defect one. We start form the examples when \( g \) has defect 1 and the chain is \( b = p^{(0)} \subset p^{(1)} = g \).

In this case \( b(e) = 1 \) for each \( e \in \hat{\Gamma}^x \), so the condition (BB) does not hold if \( \Gamma^x \) contains paths of length two. Moreover, the decreasing paths in \( \Gamma^x \) are the paths containing at most one edge. Thus any function \( b' : E \to \mathbb{Z} \) satisfying \( b'(e_1) < b'(e_2) \) for each path \( \cdot \xrightarrow{e_1} \cdot \xrightarrow{e_1} \cdot \) in \( \Gamma^x \) is descreasingly equivalent to \( b \) and satisfies (BB).

We will depict an edge \( e \) in \( \Gamma^x \) (resp., in \( \hat{\Gamma}^x \)) as \( \nu \xrightarrow{j,\kappa(e)} \lambda \) (resp., as \( \nu \xrightarrow{\kappa(e)} \lambda \)) with \( j = b'(e) \).

Except for \( \mathfrak{gl}(1|1) \) one has tail \( \lambda = \delta_{0,\lambda} \); for \( \mathfrak{gl}(1|1) \) one has tail \( \lambda = 1 \) for each atypical weight \( \lambda \). Recall that the condition (Tail) in this case is tautological.

Except for \( F(4), G(3) \) we take \( \chi \) corresponding to the principal block (i.e., \( \chi \) is the central character of the trivial module \( L(0) \)).

In all these examples the Poincaré polynomials \( \kappa(e) \in \{1, z\} \); the graph \( \Gamma^x \) admits a parametric partition \( \text{dex} \) (which means that \( \kappa(e) = 1 \) if \( e \) connects the vertices with different value of \( \text{dex} \) and \( \kappa(e) = z \) otherwise). In particular, \( D^x = \Gamma^x \).

3.9.1. Example: \( \mathfrak{gl}(1|1) \). We take \( g := \mathfrak{gl}(1|1) \) with \( \Delta^+ = \{\alpha\} \).

The simple modules in the principal block are \( \{L(s\alpha)|s \in \mathbb{Z}\} \). As a vector space \( L(s\alpha) \cong \Pi^s(\mathbb{C}) \), so \( \text{sdim} L(s\alpha) = (-1)^s \); we define \( \text{dex}(L(s\alpha)) := s \mod 2 \). For a non-zero \( x \in \mathfrak{g}_1 \) one has \( \text{DS}_x(L(s\alpha)) = \Pi^s(\mathbb{C}) \). We have

\[
\begin{align*}
\hat{\Gamma}^x & \quad \ldots \quad (1) \quad -\alpha \quad (1) \quad 0 \quad (1) \quad \alpha \quad (1) \quad \ldots \\
\Gamma^x & \quad \ldots \quad -\alpha \quad 0 \quad \alpha \quad \ldots \\
\text{Ext}(\chi) & \quad \ldots \quad \leftrightarrow \quad -\alpha \quad \leftrightarrow \quad 0 \quad \leftrightarrow \quad \alpha \quad \leftrightarrow \quad \ldots
\end{align*}
\]
Clearly, \( \text{dex}(L(s\alpha)) := s \) modulo 2 defines a parametric partition on \( \hat{\Gamma}^x \) and a bipartition on \( \text{Ext}(\chi) \); the graph \( \hat{\Gamma}^x \) does not admit a \( \mathbb{N} \)-grading.

3.9.2. **Example: \( \mathfrak{osp}(2|2) \).** Take \( \mathfrak{g} = \mathfrak{osp}(2|2) \cong \mathfrak{sl}(2|1) \) with the base \( \Sigma = \{ \delta_1 \pm \varepsilon_1 \} \).

The simple modules in the principal block are \( \{ L(\lambda_s) \mid s \in \mathbb{Z} \} \), where \( \lambda_s := |s|\delta_1 + s\varepsilon_1 \). Note that 0 is a minimal dominant weight, so the assumption of Corollary 3.6.3 (iii) holds. The extension graph \( \text{Ext}(\chi) \) is \( A_\infty \). The Poincaré polynomials \( K_{\lambda,\nu}^{g,b} \) were computed in [18], Sect. 12. One has

\[
\begin{array}{c}
\Gamma^x \\
\text{Ext}(\chi) = A_\infty \\
\end{array}
\]

By [18], Lemmata 25, 26 the graph \( \hat{\Gamma}^x \) can be obtained from the graph \( \Gamma^x \) by adding a loop around \( \lambda_0 \) which is marked by \((1; z)\). Observe that \( \text{dex}(\lambda_i) := p(\lambda) \equiv i \) modulo 2 is a parametric partition on \( \hat{\Gamma}^x \) and is a bipartition on \( \text{Ext}(\chi) \). The function \( ||\lambda_i||_{\text{gr}} := |i| \) gives a \( \mathbb{N} \)-grading on \( \Gamma^x \); the graph \( \hat{\Gamma}^x \) does not admit a \( \mathbb{N} \)-grading (since it has a loop).

The Gruson-Serganova formula is

\[
\text{ch} L(\lambda_j) = \sum_{s=0}^{j} (-1)^{j-s} E_{\lambda_{s}} \quad \text{for } j \geq 0.
\]

3.9.3. **Example: \( \mathfrak{osp}(3|2) \).** Take \( \mathfrak{g} = \mathfrak{osp}(3|2) \) with the base \( \Sigma = \{ \varepsilon_1 - \delta_1, \delta_1 \} \). The simple modules in the principal block are \( \{ L(\lambda_s) \}_{s=0}^{\infty} \), where \( \lambda_0 := 0 \) and \( \lambda_s := (s - 1)\delta_1 + s\varepsilon_1 \) for \( s > 1 \); 0 is a minimal dominant weight. The Poincaré polynomials \( K_{\lambda,\nu}^{g,b} \) were computed in [10]. One has

\[
\hat{\Gamma}^x : \quad \lambda_1 \xleftarrow{(1)} \lambda_2 \xrightarrow{(1)} \lambda_3 \xleftarrow{(1)} \ldots
\]

The map \( \text{dex}(\lambda_0) := 0, \text{dex}(\lambda_i) := i - 1 \) modulo 2 for \( i \neq 0 \) is a parametric partition on \( \hat{\Gamma}^x \) (one has \( \text{dex}(\lambda_i) = p(\lambda_i) \)). The graph \( \Gamma^x \) is

\[
\lambda_1 \xleftarrow{(2:1)} \lambda_2 \xrightarrow{(3:1)} \lambda_3 \xleftarrow{(4:1)} \ldots
\]

and the “undirected version” of \( \text{Ext}(\chi) \) coincides with \( \Gamma^x(0) \). The function \( ||\lambda_i||_{\text{gr}} := i \) gives an \( \mathbb{N} \)-grading \( \Gamma^x \) (note that the graph \( \hat{\Gamma}^x \) is not \( \mathbb{Z} \)-graded).
The Gruson-Serganova formula is \( L(\lambda_0) = \mathcal{E}_0 = 1 \) and
\[
(15) \quad \text{ch } L(\lambda_1) = \mathcal{E}_{\lambda_0} + \mathcal{E}_{\lambda_1}, \quad \text{ch } L(\lambda_j) = 2(-1)^{j-1}\mathcal{E}_{\lambda_0} + \sum_{s=1}^{j} (-1)^{j-s}\mathcal{E}_{\lambda_s} \quad \text{for } j > 1.
\]

3.9.4. **Remark.** For \( \mathfrak{osp}(3|2) \) we have two bases: the “mixed” base \( \{\varepsilon_1 - \delta_1; \delta_1\} \) and the base \( \{\delta_1 - \varepsilon_1; \varepsilon_1\} \). The computations in [10] are performed for the second base; it is not hard to see that for the first base the results are the same.

For \( \mathfrak{os p}(2|2) \cong \mathfrak{sl}(1|2) \) we have two bases: the “mixed” base \( \{\delta_1 \pm \varepsilon_1\} \) and the distinguished base \( \{\varepsilon_1 - \delta_1; 2\delta_1\} \). The graphs for the mixed base are given in 3.9.2; the graphs for the distinguished base are the same as in 3.9.1 (notice that 0 is a minimal dominant weight for the mixed base, whereas for the distinguished base the set of dominant weights does not have minimal elements).

3.9.5. **Cases** \( \mathfrak{osp}(4|2), G(3) \) and \( F(4) \). Recall that \( \text{Ext}(\chi) \) is either \( D_{\infty} \) or \( A_{\infty} \), see 2.3. The graphs \( \hat{\Gamma}(\chi) \) for a certain distinguished Borel subalgebras were computed in [10], [25]: this graph is is the same as for \( \mathfrak{osp}(3|2) \) (resp., as for \( \mathfrak{osp}(2|2) \)) if \( \text{Ext}(\chi) = D_{\infty} \) and (resp., if \( \text{Ext}(\chi) = A_{\infty} \)).

For the principal blocks the graph \( \text{Ext}^1(\chi) \) is \( D_{\infty} \) and so the graph \( \hat{\Gamma}(\chi) \) is the same as for \( \mathfrak{osp}(3|2) \). Since tail \( 0 = 1 \) and tail \( \lambda = 0 \) for each \( \lambda \neq 0 \), the graph \( \Gamma^x \) is the same as for \( \mathfrak{osp}(3|2) \).

For the non-principal blocks one has \( \hat{\Gamma}^x = \Gamma^x \) (since tail \( \lambda = 0 \) for each \( \lambda \in \Lambda^x \)).

The character formulae for these cases were obtained in [10], [25]. The above approach give other type of character formulae.

By above, for the principal block \( \Gamma^x \) is the same as for \( \mathfrak{osp}(3|2) \), so we obtain the same character formulae [15]. Consider a non-principal block. By above, \( \Gamma^x = \hat{\Gamma}^x \) is the same as the graph \( \hat{\Gamma}^x \) for \( \mathfrak{osp}(2|2) \) or for \( \mathfrak{osp}(3|2) \). In both cases \( \Gamma^x \) have cycles and all these cycles contain \( \lambda_0 \); the graph \( \Gamma' \) which is obtained from \( \Gamma^x \) by erasing all edges ending at \( \lambda_0 \) is \( \mathbb{N} \)-graded. Using 3.7.3 we get Gruson-Serganova type character formulae which can be obtained from (14) and (15) respectively by changing \( \mathcal{E}_0 \) by \( \text{ch } L(\lambda_0) \) (notice that \( \text{ch } L(\lambda_0) \) is given by the Kac-Wakimoto formula).

3.10. **Remark on Gruson-Serganova type character formulae.** Let \( B \) one of the blocks of atypicality 1 considered in Section 2. Then \( \Gamma^x \) is one of the graphs appeared in 3.9.2 3.9.3 and we call \( \lambda_0 \) a Kostant weight; notice that \( \lambda_0 \) is uniquely defined in terms of \( \Gamma^x \) which has fewer automorphisms than \( \text{Ext}(\chi) \).

Take \( L(\lambda_i) \in \text{Irr}(B) \) and write \( \text{ch } L(\lambda_i) = \sum a_i \mathcal{E}_{\lambda_i} \) using (14), (15). By 2.8 one has \( \text{DS}_x(L(\lambda_i)) = \Pi^s(\text{DS}_x(L(\lambda_0)))a_0 \) for \( s = 0 \) if \( a_0 > 0 \) and \( s = 1 \) if \( a_0 < 0 \). This can be translated to the language of supercharacters in the following manner.
Retain notation of 0.12. Denote by $\text{Sch}(\mathfrak{g})$ the image of the map $\text{sch} : \hat{\mathcal{F}}(\mathfrak{g}) \to \mathbb{Z}[[\Lambda_{m|n}]]$. Since $\text{sch}(\Pi(V)) = -\text{sch}(V)$ one has $\text{sch}(\hat{\mathcal{F}}(\mathfrak{g})) = \text{sch}(\mathcal{F}(\mathfrak{g}))$. For Lie superalgebras the ring $\text{Sch}(\mathfrak{g})$ is isomorphic to the reduced Grothendieck ring of $\hat{\mathcal{F}}(\mathfrak{g})$ and $\text{DS}_x$ induces an algebra homomorphism $ds'_x : \text{Sch}(\mathfrak{g}) \to \text{Sch}(\mathfrak{g}_x)$ given by $f \mapsto f|_{h_x}$, see [22].

For $V \in \mathcal{F}(\mathfrak{g})$ one has $\text{sch} V = \pi(\text{ch} V)$, where $\pi : \mathbb{Z}[[\Lambda_{m|n}]] \to \mathbb{Z}[[\Lambda_{m|n}]]$ is the involution $\pi(e^\mu) := p(\mu)e^\mu$. In particular, $\{\pi(\mathcal{E}_\lambda) \mid \lambda \in \text{Irr}(\mathcal{F}(\mathfrak{g}))\}$ forms a basis of $\text{Sch}(\mathfrak{g})$. If $\mathfrak{g}$ has defect 1, then the kernel of the map $ds'_x : \text{Sch}(\mathfrak{g}) \to \text{Sch}(\mathfrak{g}_x)$ is spanned by the basis elements $\pi(\mathcal{E}_\lambda)$ for $\lambda$ which are not Kostant weights. A similar property hold for the $\mathfrak{gl}(1|n)^{(1)}$-case; in [14] we will show that this holds for $\mathfrak{osp}(m|n)$-case as well (the situation is more complicated for $\mathfrak{gl}(m|n)$).

3.11. **Example:** $\mathfrak{g} = \mathfrak{gl}(k|k)$. Take $\mathfrak{g} = \mathfrak{gl}(k|k)$ with a distinguished Borel subalgebra $\mathfrak{b}$. For $p = 0, \ldots, k$ we denote by $\mathfrak{p}^{(p)}$ a parabolic subalgebra containing $\mathfrak{b}$ with the Levi factor $\mathfrak{l}^{(p)} \cong \mathfrak{gl}(p|p)$ ($\mathfrak{p}^{(p)}$ is unique since $\mathfrak{b}$ is a distinguished Borel). We consider the corresponding chain of parabolic subalgebras [9].

The Poincaré polynomials $K_{\mathfrak{p}^{(p+1)}, \mathfrak{p}^{(p)}}^{\lambda, \nu}$ were computed in [32], [27] Cor. 3.8. The graph $\hat{\Gamma}^x$ is $\mathbb{Z}$-graded and does not have multi-edges. For the principal block the condition (Tail) holds. The map $p(\lambda)$ defines a parametric partition on $\hat{\Gamma}^x$. By [27], Sect. 6, $\text{Ext}(\chi) = \Gamma^x(0)$ (this can be also deduced from [2]). One has $\text{Ext}(\chi) \neq \Gamma^x(0)$ (see the example of $\mathfrak{gl}(1|1)$ above).

4. **Case $\mathfrak{osp}(M|N)$**

In this section $\mathfrak{g} = \mathfrak{osp}(M|N)$. The category $\mathcal{F}\text{in}(\mathfrak{g})$ was studied in [18] and [6]. In this section we deduce the existence of $\text{dex}$ satisfying (Dex1) from the results of [18]. Another approach is developed in [6], [7]. By [18], each block of atypicality $k$ in $\hat{\mathcal{F}}(\mathfrak{g})$ is equivalent either to a principal $\mathfrak{osp}(2k + t|2k)$-block $\mathcal{B}$ or to $\Pi(\mathcal{B})$, where $t = 1$ for odd $M$ and $t = 0, 2$ for even $M$, and this equivalence is “compatible with character formula”, see Remark 4.6.

We fix a “mixed” base consisting of odd roots, see 4.1.1 below. We denote by $\chi$ the central character of the principal block $\mathcal{B}$ and retain notation of 3.5. For each $\lambda \in \Lambda^x$ we set $\text{dex}(L(\lambda)) = p(\lambda)$ if $t = 0, 1$; for $t = 2$ we define $\text{dex}(L(\lambda))$ via a one-to-one correspondence between the simple modules in the principal blocks for $\mathfrak{osp}(2n + 1|2n)$ and $\mathfrak{osp}(2n + 2|2n)$, see 4.3 below.

The multiplicities $iK_{\mathfrak{p}^{(p)}, \mathfrak{p}^{(p-1)}}^{\lambda, \nu}$ were computed by C. Gruson and V. Serganova in [18] (see [20] for small rank examples). We will recall their results and describe the graphs $\Gamma^x, D^x$ in 4.4. We will see that $\Gamma^x$ is $\mathbb{N}$-graded and satisfies (Tail). We will check that $\text{dex}$ is a parametric partition and that $b$ and $b'$ are decreasingly-equivalent. As a result Corollary 3.6.3 holds for the block $\mathcal{B}$ and the character formula (15) in [18] can be rewritten in the form 177.
Everywhere in this section, except for Remark 4.6, we take \( g = \mathfrak{osp}(2k + t|2k) \) for \( t = 0, 1, 2 \).

4.1. **Notation.** We take \( g := \mathfrak{osp}(2k + t|2k) \) for \( t = 0, 1, 2 \). The integral weight lattice \( \Lambda_{k+\ell k} \) is spanned by \( \{ \varepsilon_i \}_{i=1}^{k+\ell} \cup \{ \delta_i \}_{i=1}^{k} \), where \( \ell := 0 \) for \( t = 0, 1 \) and \( \ell := 1 \) for \( t = 2 \); the parity function is given by \( p(\varepsilon_i) = \overline{T} \), \( p(\delta_j) = T \) for all \( i, j \).

4.1.1. We fix a triangular decomposition corresponding to the “mixed” base:

\[
\Sigma := \left\{ \begin{array}{ll}
\varepsilon_1 - \delta_1 - \varepsilon_2, \ldots, \varepsilon_k - \delta_k, \delta_k & \text{for } \mathfrak{osp}(2k+1|2k) \\
\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \ldots, \varepsilon_{k-1} - \delta_k, \delta_k \pm \varepsilon_k & \text{for } \mathfrak{osp}(2k|2k) \\
\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \ldots, \varepsilon_k - \delta_k, \delta_k \pm \varepsilon_{k+1} & \text{for } \mathfrak{osp}(2k+2|2k).
\end{array} \right.
\]

We have \( \rho = 0 \) for \( t = 0, 2 \) and \( \rho = \frac{1}{2} \sum_{i=1}^{k} (\delta_i - \varepsilon_i) \) for \( t = 1 \).

4.1.2. We consider the embeddings

\( \mathfrak{osp}(t|0) \subset \mathfrak{osp}(2 + t|2) \subset \mathfrak{osp}(4 + t|4) \subset \ldots \subset \mathfrak{osp}(2k + t|2k) = g \)

where \( \mathfrak{osp}(2p + t|2p) \) corresponds to the last \( 2p + \ell \) roots in \( \Sigma \); we denote the subalgebra \( \mathfrak{osp}(2p + t|2p) \) by \( t^{(p)} \). Note that \( t^{(k)} = g \) and \( t^{(0)} = 0 \) for \( t = 0, 1, t^{(0)} = \mathbb{C} \) to \( t = 2 \).

For \( p = 0, \ldots, k \) we consider the parabolic subalgebra \( p^{(p)} := t^{(p)} + \mathfrak{b} \). Notice that \( t^{(p)} \) is the Levi subalgebra of \( p^{(p)} \); as in [3,5] we denote by \( \text{tail}(\lambda) \) the maximal index \( q \) such that \( \lambda|_{\mathfrak{b} \cap \mathfrak{u}_q} = 0 \).

4.2. **Highest weights in the principal block.** For \( \lambda \in \Lambda_{k+\ell k} \) we set

\[
a_i := - (\lambda|\delta_i)
\]

and notice that \( p(\lambda) = \sum_{i=1}^{k} a_i \). By [18], \( \lambda \in \Lambda^x \) if and only if \( a_1, \ldots, a_k \) are non-negative integers with \( a_{i+1} > a_i \) or \( a_i = a_{i+1} = 0 \), and

\[
\lambda + \rho = \left\{ \begin{array}{ll}
\sum_{i=1}^{k-1} a_i (\varepsilon_i + \delta_i) + a_k (\delta_k + \xi \varepsilon_k) & \text{for } t = 0 \\
\sum_{i=1}^{k} a_i (\varepsilon_i + \delta_i) & \text{for } t = 2 \\
\sum_{i=1}^{k} (a_i + \frac{1}{2}) (\varepsilon_i + \delta_i) + \frac{1}{2} (\delta_s + \xi \varepsilon_s) + \sum_{i=s+1}^{k} \frac{1}{2} (\delta_i - \varepsilon_i) & \text{for } t = 1
\end{array} \right.
\]

for \( \xi \in \{ \pm 1 \} \). For \( t = 1 \) we have \( 1 \leq s \leq k + 1 \) and \( a_s = a_{s+1} = \ldots = a_k = 0 \) if \( s \leq k \) (for \( s = k + 1 \) we have \( \lambda + \rho = \sum_{i=1}^{k} (a_i + \frac{1}{2}) (\varepsilon_i + \delta_i) \)).
4.2.1. Take $\lambda \in \Lambda^\chi$ and define $a_i$ for $i = 1, \ldots, k$ as above. We assign to $\lambda$ a “weight diagram”, which is a number line with one or several symbols drawn at each position with non-negative integral coordinate:

we put the sign $\times$ at each position with the coordinate $a_i$;
for $t = 2$ we add $>$ at the zero position;
we add the “empty symbol” $\circ$ to all empty positions.

For $t \neq 2$ a weight $\lambda \in \Lambda^\chi$ is not uniquely determined by the weight diagram constructed by the above procedure. Therefore, for $t = 0$ with $a_k \neq 0$ and for $t = 1$ with $s \leq k$, we write the sign of $\xi$ before the diagram (+ if $\xi = 1$ and $-$ if $\xi = -1$).

Notice that each position with a non-zero coordinate contains either $\times$ or $\circ$. For $t = 0, 1$ the zero position is occupied either by $\circ$ or by several symbols $\times$; we write this as $\times^i$ for $i \geq 0$. Similarly, for $t = 2$ the zero position is occupied by $>$ with $i \geq 0$.

4.2.2. Notice that $\text{tail}(\lambda)$ is equal to the number of symbols $\times$ at the zero position of the weight diagram for all cases except when $t = 1$ and the diagram has the sign $+$; in the latter case the number of symbols $\times$ at the zero position is $\text{tail}(\lambda) + 1$.

4.2.3. Examples. The weight diagram of $0$ is $\times^k$ for $t = 0$, $-\times^k$ for $t = 1$ and $\times^k$ for $t = 2$; one has $\text{tail}(0) = k$.

The diagram $+\circ\times\times$ corresponds to the $\mathfrak{osp}(4|4)$-weight $\lambda = \lambda + \rho = (\varepsilon_2 + \delta_2) + 2(\varepsilon_1 + \delta_1)$ with $\text{tail}(\lambda) = 0$.

The diagram $+\times^3$ corresponds to $\mathfrak{osp}(7|6)$-weight $\lambda = \varepsilon_1$ with $\text{tail}(\varepsilon_1) = 2$.

The empty diagram correspond to $\mathfrak{osp}(0|0) = \mathfrak{osp}(1|0) = 0$; the diagram $>$ corresponds to the weight $0$ for $\mathfrak{osp}(2|0) = \mathbb{C}$.

4.2.4. For $t = 0, 1, 2$ we denote by $\text{Diag}_{k,t}$ the set of (signed) weight diagrams. The above procedure gives a one-to-one correspondence between $\Lambda^\chi$ and $\text{Diag}_{k,t}$. For each diagram $f \in \text{Diag}_{k,t}$ we denote by $\lambda(f)$ the corresponding weight in $\Lambda^\chi$.

In all cases the weight diagrams in $\text{Diag}_{k,t}$ contains $k$ symbols $\times$.

For a diagram $f$ and $a \in \mathbb{N}$ we denote by $f(a)$ the symbols at the position $a$. For $t = 0$ (resp., $t = 1$) a diagram in $\text{Diag}_{k,t}$ has a sign if and only if $f(0) = \circ$ (resp., $f(0) \neq \circ$).

4.2.5. Map $\tau$. Following [18], we introduce a bijection $\tau : \text{Diag}_{k;2} \to \text{Diag}_{k;1}$. For $f \in \text{Diag}_{k;2}$ the diagram $\tau(f) \in \text{Diag}_{k;1}$ is constructed by the following procedure:

we remove $>$ and then shift all entries at the non-zero positions of $f$ by one position to the left; then we add a sign in such a way that $\text{tail}(f) = \text{tail}(\tau(f))$: the sign $+$ if $f(1) = \times$
and the sign $-$ if $f(1) = \circ$ and $f(0) \neq >$. For instance,
\[ \tau(> \circ \times) = - \times \times, \quad \tau(\times > \times) = - \times, \quad \tau(> \times \circ) = + \times, \quad \tau(> \circ \times) = \circ \times. \]

One readily sees that $\tau$ is a one-to-one correspondence.

4.3. The maps $|||\lambda|||, |||\lambda|||_{gr}, \text{dex}$. Let $\{a_i\}_1^k$ be the coordinates of the symbols $\times$ in a diagram of $\lambda$. We set
\[
|||\lambda||| := \left\{ \begin{array}{ll}
\sum_{i=1}^{k} a_i & \text{for } t = 0, 1 \\
|||\tau(f)||| & \text{for } t = 2,
\end{array} \right.
\]
and
\[
\text{dex}(\lambda) :\equiv |||\lambda||| \mod 2.
\]

Clearly, $|||\lambda|||, |||\lambda|||_{gr} \in \mathbb{N}$ and $|||\lambda|||_{gr} = 0$ if and only if $\lambda = 0$.

4.4. Graph $\Gamma^\chi$. Retain notation of Section 3. Consider the chain of parabolic subalgebras (40) with $p^{(i)}, p = 0, 1, \ldots, k$ defined in 4.1.2.

The Poincaré polynomials $K_{p^{(i)},p^{(i-1)}}^{\lambda,\nu}(z)$ for $p > \text{tail } \lambda$ were computed in [18], Sect. 11. It is proven that the map $\tau: Diag_{k,2} \sim \rightarrow Diag_{k,1}$ (see 4.2.5) preserves these polynomials (i.e., $K_{p^{(i)},p^{(i-1)}}^\tau(\lambda,\nu) = K_{p^{(i)},p^{(i-1)}}^{\tau(\lambda),\nu}$); the coefficients $K_{p^{(i)},p^{(i-1)}}^{\lambda,\nu}$ are 0 or 1 and for $t = 0, 2$ one has $K_{p^{(i)},p^{(i-1)}}^{\lambda,\nu} = 1$ if and only if the diagram of $\lambda$ can be obtained from the diagram of $\nu$ by a “move” of degree $i$ which ends at the $p$th symbol $\times$ in the diagram of $\lambda$; we will give some details in 4.4.1 below and give a description in terms of “arch diagrams” in Section 5.

4.4.1. Moves for $t = 0, 2$. Consider the cases $t = 0, 2$. Take $f \in Diag_{k,t}$. For each $p, q \in \mathbb{N}$ denote by $l_f(p, q)$ the number of symbols $\times$ minus the number of symbols $\circ$ strictly between the positions $p$ and $q$ in $f$.

A diagram $f \in Diag_{k,t}$ can be transformed to a diagram $g$ by a “move” of degree $d$ if $f$ satisfies certain conditions, and $g$ is obtained from $f$ by moving either one symbol $\times$ from a position $p$ to an empty position $q$ with $q > p$ or moving two symbols $\times$ from the zero position to empty positions $p, q$ with $p < q$. If $f$ has a sign, then $g$ has the same sign. In both cases we say that the move “ends at the position $q$”. We will not specify all conditions on $f$, but notice that these conditions depend only on $l_f(s, q)$ for $s < q$.

By above, $\text{tail}(f) - \text{tail}(g)$ is 0, 1 or 2; the degree $d$ satisfies the formula
\[
d = \left\{ \begin{array}{ll}
l_f(p, q) & \text{if } \text{tail}(f) - \text{tail}(g) \neq 1 \\
l_f(p, q) \text{ or } 2 \text{tail}(g) + l_f(p, q) & \text{if } t = 0, \text{tail}(f) - \text{tail}(g) = 1 \\
2 \text{tail}(g) + l_f(p, q) + 1 & \text{if } t = 2, \text{tail}(f) - \text{tail}(g) = 1,
\end{array} \right.
\]
The conditions on \( f \) imply that \( d \geq 0 \). Except for the case \( t = 0 \) with \( \text{tail}(f) - \text{tail}(g) = 1 \), \( g \) can be obtained from \( f \) by at most one move; for \( t = 0 \) it is possible sometimes to obtain \( g \) from \( f \) by two moves of different degrees. We give below examples of several moves and their degrees

\[
\begin{align*}
\cdots \times \circ \cdots & \rightarrow \cdots \times \circ \cdots \quad d = 0 \\
\cdots \times \times \circ \circ \cdots & \rightarrow \cdots \times \times \circ \times \cdots \quad d = 1 \\
\times \circ \cdots & \rightarrow (\pm) \circ \cdots \quad d = 0 \\
x^2 \circ \cdots & \rightarrow \times \cdots \quad d = 0, 2.
\end{align*}
\]

4.4.2. Corollary. Take \( t = 0 \) or \( t = 2 \). Let \( \nu, \lambda \in \Lambda^x \) be two weights with the diagrams \( f \) and \( g \) respectively. Assume that \( g \) is obtained from \( f \) by a move of degree \( d \). Then

\[(i) \ ||\lambda||_{\text{gr}} > |\nu||_{\text{gr}} \text{ and } \lambda > \nu;\]

\[(ii) \text{ if the move ends in the } i\text{th symbol } \times \text{ in } g, \text{ then } \text{tail}(\nu) \leq i;\]

\[(iii) \text{ dex}(\lambda) - \text{dex}(\nu) + d \equiv 1 \mod 2.\]

**Proof.** The inequality \(||\lambda||_{\text{gr}} > |\nu||_{\text{gr}} \) follows from the fact that \( \tau \) preserves \(||\ ||_{\text{gr}} \) and that we move symbol(s) \( \times \) to the right. The inequality \( \lambda > \nu \) follows from the fact that we move the symbol(s) \( \times \) to the right; (ii) is obvious. For (iii) retain notation of 4.4.1 and observe that

\[l_f(p, q) \equiv q - p + 1, \quad \text{dex}(\lambda) - \text{dex}(\nu) \equiv ||\lambda|| - |\nu|| \mod 2.\]

For \( t = 0 \) one has \( d \equiv l_f(p, q) \) by (16), and \(||\lambda|| - |\nu|| = q - p \) if \( \text{tail}(g) - \text{tail}(g) \neq 2 \) and \( q + p \) otherwise. For \( t = 2 \) the formula (16) gives \( d \equiv l_f(p, q) + \text{tail}(\lambda) - \text{tail}(\nu) \); in this case \(||\lambda|| - |\nu|| \equiv q - p + \text{tail}(\lambda) - \text{tail}(\nu) \). This gives (iii). \( \square \)

4.4.3. Retain notation of 3.5. For \( t = 0, 2 \) the graph \( \Gamma^x = (\Lambda^x, E) \) has the edges \( \nu \xrightarrow{b} \lambda \) with \( b(e) = j \) if and only if the diagram of \( \lambda \) can be obtained from the diagram of \( \nu \) by a move which ends at the \( j \)th symbol \( \times \) in the diagram of \( \lambda \); in this case we denote by \( b'(e) = q \) the coordinate of the \( j \)th symbol \( \times \) in the diagram of \( \lambda \). For the edge as above the Poincaré polynomial is the sum of \( z^d \) for all \( d \) such that \( \lambda \) can be obtained from the diagram of \( \nu \) by a move of degree \( d \). By (16) \( \kappa(e) = z^d \) except for the case \( t = 0 \) with \( \text{tail}(\nu) - \text{tail}(\lambda) = 1 \); in the latter case \( \kappa(e) = z^d \) or \( z^d(1 + z^{2\text{tail}(\lambda)}) \) (see Section 5 for details).

By above, \( \tau \) gives a bijection between the graphs \( \Gamma^x \) for \( t = 2 \) and \( t = 1 \) and this bijection is compatible with the functions \( b \) and \( \kappa \). For \( t = 1 \) we define \( b' \) on \( \Gamma^x \) using this bijection.

4.4.4. Corollary.

(i) The map \( \text{dex}(\lambda) \) is a parametric bipartition on \( (\Gamma^x, \kappa) \).

(ii) If \( \nu \xrightarrow{\ell} \lambda \) is an edge in \( \Gamma^x \), then \( \nu < \lambda \) and \(||\nu||_{\text{gr}} < ||\lambda||_{\text{gr}} \). In particular, \(||\lambda||_{\text{gr}} \) defines a \( \mathbb{N} \)-grading on \( \Gamma^x \).
(iii) The graph $\Gamma^x$ satisfies the assumption (Tail).
(iv) The functions $b, b'$ are decreasingly equivalent and $b'$ satisfies the property (BB) of Lemma 3.4.4.

Proof. Consider the cases $t = 0,2$. Corollary 4.4.2 implies (i)–(iii). For (iv) take a path $\lambda(f_1) \xrightarrow{e_1} \lambda(f_2) \xrightarrow{e_2} \lambda(f_3)$ in $\Gamma^x$. Since $f_3$ is obtained from $f_2$ by a move which ends at the symbol $\times$ with the coordinate $b'(e_2)$, the position with this coordinate in $f_2$ is empty, so $b'(e_1) \neq b'(e_2)$. Hence $b'$ satisfies (BB). It remains to verify that $b$ and $b'$ are decreasingly equivalent. Set $j := b(e_1), q := b'(e_1)$. Then $q$ is the coordinate of the $j$th symbol $\times$ in $f_2$ and $q > 0$. The condition $b(e_1) > b(e_2)$ means that for $i \geq j$ the $i$th symbols $\times$ in $f_2$ and $f_3$ have the same coordinates, whereas the condition $q > b'(e_2)$ means that for $s \geq q$ one has $f_2(s) = f_3(s)$. Clearly, these conditions are equivalent, so $b$ and $b'$ are decreasingly equivalent.

Consider the remaining case $t = 1$. Since $\tau$ preserves dex, tail and $||| gr |||$, almost all assertions for $t = 1$ follows from the corresponding assertions for $t = 2$. The only exception is the inequality $\nu < \lambda$ in (ii), which follows from the following observation: for an edge $\nu \rightarrow \lambda$ in $\Gamma^x$ for $t = 1$, the diagram of $\lambda$ is obtained from the diagram of $\nu$ either by moving symbol(s) $\times$ to the right or by changing the sign from $-$ to $+$. □

4.4.5. Corollary. Take $\lambda, \nu \in \Lambda^x$ with $\text{Ext}^1(L(\lambda), L(\nu)) \neq 0$. Then

(i) either $\lambda$ can be obtained from $\nu$ by a move of zero degree or $\nu$ can be obtained from $\lambda$ by a move of zero degree;
(ii) $\text{dex}(\lambda) \neq \text{dex}(\nu)$.

Proof. By Corollary 4.4.4, the graph $\Gamma^x$ satisfies the assumptions of Corollary 3.6.3 (iii), which implies the assertions. □

4.5. Gruson-Serganova character formula. We retain notation of 3.3; for $\nu \in \Lambda^x$ we introduce the “Euler character” $\mathcal{E}_\nu$ by (13).

4.5.1. A character formula is given by Theorem 4 in [18]. Using Corollary 4.4.4 we can write this formula for $\lambda \in \Lambda^x$ in the following way:

$$\text{ch } L(\lambda) = \sum_{\nu \in \Lambda^x} (-1)^{\text{dex}(\lambda) - \text{dex}(\nu)} d^{\lambda,\nu} \mathcal{E}_\nu,$$

where $d^{\lambda,\nu}$ is the number of increasing paths from $\nu$ to $\lambda$ in the graph $D^x$, where $D^x$ is obtained from $\Gamma^x$ by doubling the edges $e$ with $\kappa(e) \neq z^d$ (i.e., $D^x = \Gamma^x$ for $t \neq 0$, see 4.4.3).

Notice that $d^{\lambda,\nu}$ are non-negative integers, $d^{\lambda,\lambda} = 1$ and $d^{\lambda,\nu} \neq 0$ implies $\nu \leq \lambda$ and $||\nu|| \leq ||\lambda||$ (in particular, the right-hand side of (17) is finite).
4.6. Remark. Recall that each block of atypicality $k$ for $\mathfrak{osp}(M|N)$ is equivalent to the principal block of $\mathfrak{osp}(2k + t|2k)$ where $t = 1$ for odd $M$ and $t = 0, 2$ for even $M$. For a dominant weight $\lambda$ of atypicality $k$ let $\overline{\lambda}$ be the image of $\lambda$ in $\Lambda^x$ (that is the $\mathfrak{osp}(2k + t|2k)$-module $L(\overline{\lambda})$ is the image of $\mathfrak{osp}(M|N)$-module $L(\lambda)$ under the above equivalence). It turns out that this equivalence “preserve tails”, i.e. $\text{tail}(\lambda) = \text{tail}(\overline{\lambda})$.

Introducing $\text{dex}(\lambda) := \text{dex}(\overline{\lambda})$ we obtain $\text{Ext}^1(\lambda, \nu) = 0$ if $\text{dex}(\lambda) = \text{dex}(\nu)$. By [18], the formula (17) holds for an arbitrary dominant weight $\lambda$ if we introduce $E_\nu$ by the formula (13) and set $d_{\lambda, \nu}^\delta := d_{\lambda, \nu}^\delta$.

5. Arch diagrams

In the cases when $g$ is not exceptional and the flag of parabolic is standard, the description of $\Gamma^x$ in [27], [18] can be conveniently presented in terms of arch diagrams introduced in [19], [13], where the examples are presented. Below we will present this description of $\Gamma^x$ for the principal blocks in $\mathfrak{gl}(k|k), \mathfrak{osp}(2k + t|2k)$.

Our diagrams differs from the arc- or cup diagrams of [6]; we will call these diagrams “arch diagram”.

We take $g = \mathfrak{gl}(k|k), \mathfrak{osp}(2k + t|2k)$ and the central character $\chi$ corresponding to the principal block. For $g = \mathfrak{gl}(k|k)$ we take $\Sigma = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_k - \delta_1, \delta_1 - \delta_2, \ldots, \delta_k - 1 - \delta_k\}$ and the flag (9) with $l^{(i)} \cong \mathfrak{gl}(i|i)$. For $\mathfrak{osp}(2k + t|2k)$ we retain notation of 4.1.

5.1. Arch diagram. Take $\lambda \in \Lambda^x$.

For $\mathfrak{osp}(2k + t|2k)$ we assign to $\lambda$ the weight diagram as in [12]. For $\mathfrak{gl}(k|k)$-case $\lambda + \rho = \sum_{i=1}^k a_i (\varepsilon_i - \delta_{i+1-i})$ and we assign to $\lambda$ a weight diagram with the symbols $\times$ at the positions $a_1, \ldots, a_k$ and the empty symbols $\circ$ in other positions.

A generalized arch diagram is the following data:

- a weight diagram $f$, where the symbols $\times$ at the zero position are drawn vertically and $>$ (if it is present) is drawn in the bottom,
- a collection of non-intersecting arches, where each arch is
  - either $\text{arc}(a; b)$ connecting the symbol $\times$ with the empty symbol at the position $b$;
  - or $\text{arc}(0; b, b')$ connecting the symbol $\times$ at the zero position with two empty symbols at the positions $b < b'$;

An empty position is called free if this position is not an end of an arch.

We call $\text{arc}(a; b)$ a two-legged arch supported at $a$ and $\text{arc}(0; b, b')$ a three-legged arch supported at 0.

A generalized arch diagram is called arch diagram if
• each symbol $\times$ is the left end of exactly one arch;
• there are no free positions under the arches;
• for the $\mathfrak{gl}$-case all arches are two-legged;
• for the $\mathfrak{osp}(2k|2k), \mathfrak{osp}(2k+1|2k)$-cases the lowest $\times$ at the zero position supports a two-legged arch and the other symbols $\times$ at the zero position support three-legged arches;
• for the $\mathfrak{osp}(2k+2|2k)$-case all symbols $\times$ at the zero position support three-legged arches.

Each weight diagram $f$ admits a unique arch diagram which we denote by $\text{Arc}(f)$; this diagram can be constructed in the following way: we pass from right to left through the weight diagram and connect each symbol $\times$ with the next empty symbol(s) to the right by an arch.

5.1.1. Partial order. We consider a partial order on the set of arches by saying that one arch is smaller than another one if the first one is "below" the second one; one has

$$\text{arc}(a; b) > \text{arc}(a'; b') \iff a < a' < b$$
$$\text{arc}(0; b_1, b_2) > \text{arc}(a'; b') \iff a' < b_2,$$

in addition, any two distinct three-legged arches are comparable.

5.1.2. For a weight diagram $f$ we denote by $l_f(p,q)$ the number of $\times$ minus the number of $\circ$ strictly between the positions $p$ and $q$. We denote by $(f)_p^q$ the diagram which obtained from $f$ by moving $\times$ from the position $p$ to a free position $q > p$; such diagram is defined only if $f(p) \in \{\times^i, \times^i\}$ for $s \geq 1$, $f(q) = \circ$.

For instance, for $f = \times^2 \circ \times$ one has $(f)_0^3 = \times \circ \times \times$ and $(f)_0^1, (f)_1^5$ are not defined. If $f(0) = \times^i$ or $\times^i$ for $i > 1$, we denote by $(f)_0^a$ the diagram which obtained from $f$ by moving two symbols $\times$ from the zero position to free positions $p$ and $q$ with $p < q$; for example, $(\times^2 \times)_0^3 = \circ \circ \circ \times$.

5.2. Description of $\Gamma^\chi$ in diagrammatic terms. Below we present the description for the cases $\mathfrak{gl}(k|k), \mathfrak{osp}(2k|2k)$ and $\mathfrak{osp}(2k+2|2k)$ (the map $\tau$ gives a bijection between the graphs for $\mathfrak{osp}(2k+1|2k)$ and $\mathfrak{osp}(2k+2|2k)$).

Take $\nu, \lambda \in \Xi$ and denote by $f$ and $g$ their weight diagrams. We denote by $l_f(a,b)$ the number of $\times$ minus the number of $\circ$ with the coordinates strictly between $a$ and $b$. Note that $l_f(a,b) = l_f(a,b)$ if $f' = (f)_b^a$.

The graph $\Gamma^\chi$ contains an edge $\nu \rightarrow \lambda$ if and only if $g = (f)_a^p$ or $g = (f)_{0,0}^{p,q}$ and the following conditions hold.
5.2.1. Case $g = (f)^a_p$ and $f(a) = \times$. In this case $\text{Arc}(f)$ contains a two-legged arch $\text{arc}(a; a_-)$ with $a < p \leq a_-$; one has $\kappa(e) = z^{l_f(a, p)}$ and $l_f(a, b) = l_g(a, b)$.

5.2.2. Case $g = (f)^p_0$ and $f(0) \neq \times$. In this case $\text{Arc}(f)$ contains three-legged arches (i.e., either $g = \mathfrak{osp}(2k|2k)$ and $\text{tail}(f) \geq 2$ or $g = \mathfrak{osp}(2k + 2|2k)$ and $\text{tail}(f) \geq 1$). For this case we denote by $\text{arc}(0; a', a_+)$ the highest three-legged arch in $\text{Arc}(f)$ (for example, for $\nu = 0$ this is $\text{arc}(0; 2k - 2, 2k - 1)$ if $g = \mathfrak{osp}(2k|2k), k > 1$ and $\text{arc}(0; 2k - 1, 2k)$ if $g = \mathfrak{osp}(2k + 2|2k)$). In this case $p \leq a_+$.

If $f(0) = ^{x^{i+1}}>$ for $i \geq 0$ (then $t = 2$) we have $\kappa(e) = z^{2i+1+l_f(0,p)}$. In the remaining case $f(0) = ^{x^{i+1}}, i > 0$ (then $t = 0$) there is a unique two-legged $\text{arc}(0; a_-)$ supported at the zero position and $\kappa(e) = z^{2i+l_f(0,p)}$ if $p > a_-$ and $\kappa(e) = z^{2i+l_f(0,p)} + z^{l_f(0,p)}$ if $p \leq a_-$. For example, for $f = x^3 \circ \circ \circ \times$ and $g = (f)^2_0 = x^2 \times \circ \circ \times \times$ one has $\kappa(e) = z + z^5$ and for $g' = (f)^4_0 = x^2 \times \circ \circ \times \times \times$ one has $\kappa(e') = z^3$.

5.2.3. Case $g = (f)^p_0$. In this case $\text{Arc}(f)$ contains a three-legged arch $\text{arc}(0; p, a_2)$ supported at the zero position which is not the highest arch and $p < q \leq a_2$; one has $\kappa(e) = z^{l_f(p,q)}$.

Examples.

For $t = 0$, $\text{tail}(f) \leq 2$ (resp., $t = 2$, $\text{tail}(f) \leq 1$) there are no suitable three-legged arches.

For $f = x^3 \circ \circ \times$ there is only one suitable three-legged arch, which is $\text{arc}(0; 2, 7)$. Thus $p = 2, q \in \{5, 6, 7\}$ and $\kappa(e) = z^{7-q}$.

For $f = ^{x^2} \circ \circ \circ \times$ there is only one suitable three-legged arch, which is $\text{arc}(0; 1, 8)$. Thus $p = 1, q \in \{4, 6, 7, 8\}$ with $\kappa(e) = 2$ for $q = 5, 6$ and $\kappa(e) = 8 - q$ for $q = 7, 8$.

5.3. Applications to Ext-graph. Take $\nu, \lambda \in \Lambda^+$ with $\nu < \lambda$ and denote by $f$ and $g$ their weight diagrams.

By [27], in the $\mathfrak{gl}$-case $\text{Ext}^1(L(\lambda), L(\nu)) \neq 0$ if and only if $g = (f)^a_p$, where $\text{arc}(a, p) \in \text{Arc}(f)$ (i.e. $g$ can be obtained from $f$ by moving a symbol $\times$ along the arch supported by this symbol).

By above, for $g = \mathfrak{osp}(2k + t|2k)$ with $t = 0, 2$ if $\text{Ext}^1(L(\lambda), L(\nu)) \neq 0$, then either $g = (f)^p_0$, where $\text{arc}(a, p) \in \text{Arc}(f)$ or $g = (f)^p_{a+}^+$ where $\text{arc}(0; a', a_+)$ is the highest arch supported at 0, or $g = (f)^p_{0,a^2}$, where $\text{arc}(0; a_1, a_2)$ is not the highest arch supported at 0.

6. Appendix: useful facts about Ext

6.1. We start from the following lemma which express $\dim \text{Ext}^1(L', L)$ in terms of indecomposable extensions.
**Lemma.** Let $A$ be an associative algebra and $L, L'$ be simple non-isomorphic modules over $A$ with $\text{End}_A(L) = \text{End}_A(L') = \mathbb{C}$. Let $N$ be a module with

$$(18) \quad \text{Soc } N = L^\oplus m, \quad N/\text{Soc } N = L'. $$

(i) If $N$ is indecomposable, then $m \leq \dim \text{Ext}^1(L', L)$.

(ii) If $m \leq \dim \text{Ext}^1(L', L)$, then there exists an indecomposable $N$ satisfying $(18)$.

**Proof.** Consider any exact sequence of the form

$$(19) \quad 0 \to L^\oplus m \xrightarrow{\iota} N \xrightarrow{\phi} L' \to 0. $$

For each $i = 1, \ldots, m$ let $p_i$ be the projection from $L^\oplus m$ to the $i$th component and $\theta_i$ is the corresponding embedding $p_i \theta_i = \text{Id}_L$. Consider a commutative diagram

$$(20) \quad \begin{array}{ccc}
0 & \xrightarrow{\iota} & L^\oplus m
\downarrow \theta_i
\downarrow p_i
\downarrow
0 & \xrightarrow{\iota_i} & M^i
\downarrow \psi_i
\downarrow \phi_i
\downarrow
0 & \xrightarrow{\iota} & \text{Id}_L
\downarrow
0.
\end{array} $$

where $\psi_i : N \to M^i$ is a surjective map with $\text{Ker } \psi_i = \text{Ker } p_i$.

The bottom line is an element of $\text{Ext}^1(L', L)$, which we denote by $\Phi_i$. If $m > \dim \text{Ext}^1(L', L)$, then $\{\Phi_i\}_{i=1}^m$ are linearly dependent and we can assume that $\Phi_1 = 0$, so $\Phi_1$ splits. Let $\tilde{p} : M^1 \to L$ be the projection, i.e. $\iota_1 \tilde{p} = \text{Id}_L$. Consider the maps

$$ L \xrightarrow{\iota \theta_1} N \xrightarrow{\tilde{p} \psi_1} L. $$

The composed map

$$ \tilde{p} \circ \psi_1 \circ \iota \circ \theta_1 : L \to L $$

is surjective, so it is an isomorphism. Hence $N$ is decomposable. This establishes (i).

For (ii) let $\{\Phi_i\}_{i=1}^m$ be linearly independent elements in $\text{Ext}^1(L', L)$, i.e.

$$ \Phi_i : \quad 0 \to L \xrightarrow{\iota_i} M^i \xrightarrow{\phi_i} L' \to 0. $$

Consider the exact sequence

$$ 0 \to L^\oplus m \xrightarrow{\iota} \oplus M_i \to (L')^\oplus m \to 0. $$

Let $\text{diag}(L')$ be the diagonal copy of $L'$ in $(L')^\oplus m$ and let $N$ be the preimage of $\text{diag}(L')$ in $\oplus M^i$. This gives the exact sequence of the form $(19)$ and the commutative diagram $(20)$. Let us show that $N$ is indecomposable. Assume that $N$ decomposable, so $N = N_1 \oplus N_2$. Since $L' \not\cong L$ one has $\phi(N_1) = 0$ or $\phi(N_2) = 0$, so $N_1$ or $N_2$ lies in the socle of $N$. Therefore $N$ can be written as $N = L^1 \oplus N''$ with $L^1 \cong L$. Since $\dim \text{Hom}(L, N) = m$ one has $\dim \text{Hom}(L, N'') = m - 1$. Changing the basis in the span of $\{\Phi_i\}_{i=1}^m$, we can assume that $\text{Ker } p_1 \subset N''$. Since $\text{Ker } \psi_1 = \text{Ker } p_1$, the exact sequence $\Phi_1$ splits, a contradiction. Hence $N$ is indecomposable. This completes the proof. $\square$
6.2. Notation. Let \( g \) be a Lie superalgebra of at most countable dimension.

Let \( h \subset g_0 \) be a finite-dimensional subalgebra satisfying

(H1) \( h \) acts diagonally on \( g \) and \( g^h_0 = h \).

We choose \( h \in h \) satisfying

H2) \( g^h = g^h \) and each non-zero eigenvalue of \( \text{ad} h \) has a non-zero real part.

(The assumption on \( \dim g \) ensures the existence of \( h \)).

We write \( g = g^h \oplus (\oplus_{\alpha \in \Delta(g)} g_\alpha) \), with \( \Delta(g) \subset h^* \) and

\[ g_\alpha := \{ g \in g \mid [h, g] = \alpha(h)g \quad \text{for all} \quad h \in h \}. \]

We introduce the triangular decomposition \( \Delta(g) = \Delta^+(g) \coprod \Delta^-(g) \), with

\[ \Delta^\pm(g) := \{ \alpha \in \Delta(g) \mid \pm \Re \alpha(h) > 0 \}, \]

and define the partial order on \( h^* \) by \( \lambda > \nu \) if \( \nu - \lambda \in \mathbb{N} \Delta^- \). We set \( n^\pm := \oplus_{\alpha \in \Delta^\pm} g_\alpha \) and consider the Borel subalgebra \( b := g^h \oplus n^+ \).

6.2.1. Take \( z \in h \) satisfying

\[ \alpha(z) \in \mathbb{R}_{\geq 0} \quad \text{for} \quad \alpha \in \Delta^+ \quad \text{and} \quad \alpha(z) \in \mathbb{R}_{\leq 0} \quad \text{for} \quad \alpha \in \Delta^- . \]

Consider the superalgebras

\[ t := t(z) := g^z, \quad p := p(z) := g^z + b. \]

Notice that \( p = t \times m \), where \( m := \oplus_{\alpha \in \Delta^+(z) > 0} g_\alpha \). Both triples \( (p(z), h, h) \), \( (t(z), h, h) \) satisfy (H1), (H2). One has \( t^h = p^h = g^h \) and

\[ \Delta^+(p) = \Delta^+(g), \quad \Delta^+(t) = \{ \alpha \in \Delta^+(g) \mid \alpha(z) = 0 \} \]
\[ \Delta^-(p) = \Delta^-(t) = \{ \alpha \in \Delta^-(g) \mid \alpha(z) = 0 \} \]

6.2.2. Modules \( M(\lambda), L(\lambda) \). For a semisimple \( h \)-module \( N \) we denote by \( N_\nu \) the weight space of the weight \( \nu \) and by \( \Omega(N) \) the set of weights of \( N \).

We denote by \( \mathcal{O} \) the full category of finitely generated modules with a diagonal action of \( h \) and locally nilpotent action of \( n \).

By Dixmier generalization of Schur’s Lemma (see [4]), up to a parity change, the simple \( g^h \)-modules are parametrized by \( \lambda \in h^* \); we denote by \( C_\lambda \) a simple \( g^h \)-module, where \( h \) acts by \( \lambda \). We view \( C_\lambda \) as a \( b \)-module with the zero action of \( n \) and set

\[ M(\lambda) := \text{Ind}_n^b C_\lambda; \]

this module has a unique simple quotient which we denote by \( L(\lambda) \). (The module \( M(\lambda) \) is a Verma module if \( g^h = 0 \)). We introduce similarly the modules \( M_p(\lambda), L_p(\lambda) \) for the algebra \( p \) and \( M_t(\lambda), L_t(\lambda) \) for the algebra \( t \).
6.2.3. Set $N(g; m)$. Let $\lambda \neq \nu \in \mathfrak{h}^*$ be such that $Re(\lambda - \nu)(h) \geq 0$. If
\[ 0 \to L(\nu) \to E \to L(\lambda) \to 0 \]
is a non-split exact sequence, then $E$ is generated by $E_\lambda \cong C_\lambda$, so $E$ is a quotient of $M(\lambda)$ and $\nu < \lambda$.

For $\lambda, \nu \in \mathfrak{h}^*$ we denote by $\mathcal{N}(g; m)$ the set of indecomposable $g$-modules $N$ such that
\begin{equation}
\text{Soc } N = L(\nu)^{\oplus m}, \quad N/\text{Soc } N = L(\lambda)
\end{equation}
By Lemma 6.1 one has
\begin{equation}
\dim \text{Ext}^1(L(\lambda), L(\nu)) = \max \{m | \mathcal{N}(g; m) \neq \emptyset\}.
\end{equation}

Note that each module $N \in \mathcal{N}(g; m)$ is a quotient of $M(\lambda) = \text{Ind}_g^h L_\lambda(\lambda)$. We set
\begin{equation}
m(g; p; \lambda; \nu) := \max \{m | \exists N \in \mathcal{N}(g; m) \text{ which is a quotient of } \text{Ind}_g^h L_\lambda(\lambda)\}.
\end{equation}

6.2.4. Corollary. Take $\lambda \neq \nu \in \mathfrak{h}^*$ with $Re(\lambda - \nu)(h) \geq 0$.

(i) If $\text{Ext}^1(L(\lambda), L(\nu)) \neq 0$, then $\lambda > \nu$;
(ii) $\text{Ext}^1(L(\lambda), L(\nu)) = \text{Ext}^1_\mathcal{O}(L(\lambda), L(\nu))$;
(iii) $\dim \text{Ext}^1(L(\lambda), L(\nu)) = m(g; h; \lambda; \nu) \leq \dim M(\lambda)_\nu$.

6.3. Remark. If $g$ is a Kac-Moody superalgebra, then $g$ admits antiautomorphism which stabilizes the elements of $\mathfrak{h}$ and the category $\mathcal{O}(g)$ admits a duality functor $\#$ with the property $L^\# \cong L$ for each simple module $L \in \mathcal{O}(g)$. In this case
\[ \dim \text{Ext}^1(L(\lambda), L(\nu)) = \dim \text{Ext}^1(L(\nu), L(\lambda)). \]

6.4. The following lemma is a slight reformulation of Lemma 6.3 in [27].

Lemma. Take $\lambda, \nu \in \mathfrak{h}^*$ with $\lambda > \nu$.

(i) $m(g; b; \lambda; \nu) \leq m(p; b; \lambda; \nu)$ if $\nu - \lambda \in \mathcal{N}\Delta^- (p)$;
(ii) $m(g; b; \lambda; \nu) = m(g; p; \lambda; \nu)$ if $\nu - \lambda \not\in \mathcal{N}\Delta^- (p)$.

Proof. Write $p = g^c \times m$ as in 6.2.1. For each $g$-module $M$ set
\[ \text{Res}(M) := \{v \in M | vz = \lambda(z)v\}. \]
Note that $\text{Res}(M)$ is a $g^c$-module; we view $\text{Res}(M)$ as a $p$-module with the zero action of $m$. This defines an exact functor $\text{Res} : g - \text{Mod} \to p - \text{Mod}$. By the PBW Theorem
\[ \text{Res}(M(\mu)) = \begin{cases} M_p(\mu) & \text{if } \mu(z) = \lambda(z), \\
0 & \text{if } (\lambda - \mu)(z) < 0 \end{cases} \]
Let us show that
\begin{equation}
\text{Res}(L(\mu)) = \begin{cases} L_p(\mu) & \text{if } \mu(z) = \lambda(z), \\
0 & \text{if } (\lambda - \mu)(z) < 0. \end{cases}
\end{equation}
Indeed, since Res is exact, \( \text{Res}(L(\mu)) \) is a quotient of \( \text{Res}(M(\mu)) \); this gives the second formula. For the first formula assume that \( \mu(z) = \lambda(z) \) and that \( E \) is a proper submodule of \( \text{Res}(L(\mu)) \). Since \( E \) is a \( p \)-module we have \( \mathcal{U}(g)E = \mathcal{U}(n^-)E \). Since \( \text{Res}(L(\mu)) \) is an exact quotient of \( \text{Res}(M(\mu)) \) and

\[
(\text{Res}(M(\mu)))_\mu = (M_p(\mu))_\mu
\]

is a simple \( g^p \)-module, one has \( \gamma < \mu \) for each \( \gamma \in \Omega(E) \). Therefore \( (\mathcal{U}(n^-)E)_\mu = 0 \), so \( \mathcal{U}(g)E \) is a proper \( g \)-submodule of \( L(\mu) \). Hence \( E = 0 \), so \( \text{Res}(L(\mu)) \) is simple. This establishes \((23)\).

Now we fix a non-negative integer \( m \leq m(g; b; \lambda; \nu) \) and \( N \in \mathcal{N}(g; m) \).

Consider the case when \( \nu - \lambda \in \mathbb{N}\Delta^-(p) \). Then \( \lambda(z) = \nu(z) \), so \( \text{Res}(L(\nu)) = L_p(\nu) \). Since \( \text{Res} \) is exact one has

\[
\text{Soc}(\text{Res} N) = L_p(\nu)^{\oplus m}, \quad \text{Res}(N)/\text{Soc}(\text{Res}(N)) = L_p(\lambda)
\]

and \( \text{Res}(N) \) is a quotient of \( M_p(\lambda) \). Therefore \( m \leq m(p; b; \lambda; \nu) \). This gives (i).

For (ii) \( \nu - \lambda \notin \mathbb{N}\Delta^-(p) \). Let us show that \( N \) is a quotient of \( \text{Ind}_p^g L_p(\lambda) \). Write

\[
\text{Ind}_p^g L_p(\lambda) = M(\lambda)/J, \quad L_p(\lambda) = M_p(\lambda)/J'
\]

where \( J \) (resp., \( J' \)) is the corresponding submodule of \( M(\lambda) \) (resp., of \( M_p(\lambda) \)). Since \( \text{Ind}_p^g \) is exact and \( \text{Ind}_p^g M_p(\lambda) = M(\lambda) \) one has \( J \cong \text{Ind}_p^g J' \); in particular, each maximal element in \( \Omega(J) \) lies in \( \Omega(J') \). Note that

\[
\Omega(J') \subset \lambda - \mathbb{N}\Delta^+(l).
\]

Let \( \phi : M(\lambda) \rightarrow N \) be the canonical surjection. Since \( J_\lambda = 0 \), \( \phi(J) \) is a proper submodule of \( N \), so \( \phi(J) \) is a submodule of \( \text{Soc}(N) = L(\nu)^{\oplus m} \).

If \( \phi(J) \neq 0 \), then \( \nu \) is a maximal element in \( \Omega(J) \) and so \( \lambda - \nu \in \mathbb{N}\Delta^+(l) \), which contradicts to \( \nu - \lambda \in \mathbb{N}\Delta^-(p) \). Therefore \( \phi \) induces a map \( \text{Ind}_p^g L_p(\lambda) = M(\lambda)/J \rightarrow N \). Therefore

\[
m(g; b; \lambda; \nu) \leq m(g; p; \lambda; \nu).
\]

Since \( \text{Ind}_p^g L_p(\lambda) \) is a quotient of \( M(\lambda) \), we have \( m(g; b; \lambda; \nu) \geq m(g; p; \lambda; \nu) \). Thus \( m(g; b; \lambda; \nu) = m(g; p; \lambda; \nu) \) as required.

\[\Box\]

6.5. Corollary. Let \( z_1, \ldots, z_{k-1} \in \mathfrak{h} \) satisfying \((27)\) be such that \( t^{z_i} \subset t^{z_{i+1}} \). Set \( t^{(i)} := t^{z_i} \) and consider the chain

\[
\mathfrak{h} =: t^{(0)} \subset p^{(1)} \subset p^{(2)} \subset \ldots \subset t^{(k)} := g.
\]

For \( \lambda \neq \nu \in \mathfrak{h}^* \) with \( \lambda > \nu \) one has

\[
\text{Ext}^1(L(\lambda), L(\nu)) \leq m(t^{(s)}; p; \lambda; \nu),
\]

where \( s \) is minimal such that \( \nu - \lambda \in \mathbb{N}\Delta^-(t^{(s)}) \) and \( p := (t^{(s-1)} + b) \cap t^{(s)} \).
Proof. Take $p^{(i)} := t^{(i)} + h$. Combining Corollary 6.2.4 and Lemma 6.4 we obtain
\[
\text{Ext}^1(L(\lambda), L(\nu)) = m(g; b; \lambda; \nu) \leq m(p^{(s)}; p^{(s-1)}; \lambda; \nu).
\]
Recall that $p^{(i)} = t^{(i)} \cdot m^{(i)}$, where $m^{(i)}$ is the maximal ideal in $p^{(i)}$ which lie in $n$. One has $m^{(s)} \subset m^{(s-1)}$. In particular, $m^{(s)}$ annihilates $\text{Ind}_{p^{(s-1)}}^{p^{(s)}}(L_{p^{(s-1)}}(\lambda))$, so
\[
m(p^{(s)}; p^{(s-1)}; \lambda; \nu) = m(t^{(s)}; p^{(s-1)}/m^{(s)}; \lambda; \nu).
\]
Since $p^{(s-1)} = t^{(s-1)} + b \subset p^{(s)} = t^{(s)} \cdot m^{(s)}$ the image of $p^{(s-1)}$ in $t^{(s)} = p^{(s)}/m^{(s)}$ coincides with $p \subset t^{(s)}$. \hfill \Box

6.5.1. Remark. Consider the special case when $g^b = h$ and $t^{(e)} = l' \times h''$, where $h'' \subset h$. Take
\[
b' := l \cap h, \quad b' := l \cap b, \quad p' := l' \cap p.
\]
and let $h'$ be the image of $z_s$ in $h'$. Then the triple $(l'; b'; h')$ satisfies (H1), (H2) and $b'$ is the Borel subalgebra of $l'$. For each $\lambda \in h^*$ we set
\[
\lambda' := \lambda|_{b'}.
\]
Since $\nu - \lambda \in \mathbb{N} \Delta^-(t^{(s)})$ one has
\[
m(t^{(s)}; p; \lambda; \nu) = m(l', p', \lambda', \nu').
\]
Assume, in addition, that $L_{l'}(\lambda'), L_{l'}(\nu')$ are finite-dimensional. It is not hard to see that $\text{Ind}^{l'}_{l''}(L_{l''}(\lambda'))$ admits a unique maximal finite-dimensional subquotient which we denote by $\Gamma_{l', l''}(L_{l''}(\lambda'))$ and that for any finite-dimensional quotient $N'$ of $\text{Ind}^{l'}_{l''}(L_{l''}(\lambda'))$ there exists an epimorphism $\Gamma_{l', l''}(L_{l''}(\lambda')) \to N'$. This implies
\[
\text{Ext}^1(L(\lambda), L(\nu)) \leq m(l', p', \lambda', \nu') \leq [\Gamma_{l', l''}(L_{l''}(\lambda')) : L_{l''}(\nu')].
\]

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