Failures of Contingent Thinking

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ABSTRACT

In this paper, we provide a theoretical framework to analyze an agent who misinterprets or misperceives the true decision problem she faces. We show that a wide range of behavior observed in experimental settings manifest as failures to perceive implications, in other words, to properly account for the logical relationships between various payoff relevant contingencies. We present a behavioral definition of perceived implication, thereby providing an elicitation technique, and show that an agent’s account of implication identifies a subjective state-space that underlies her behavior. By analyzing this state-space, we characterize distinct benchmarks of logical sophistication that drive empirical phenomena. We disentangle static and dynamic rationality. Thus, our framework delivers both a methodology for assessing an agent’s level of contingent thinking and a strategy for identifying her beliefs in the absence full rationality.

KEYWORDS: Contingent Thinking; Misspecified Models; Bounded Rationality; Interpretation Dependent Expected Utility.

JEL CLASSIFICATION: C72, D81, D84, D91.

1 Introduction

When facing complicated and uncertain environments, economic agents often fail to act optimally, choosing actions which do not maximize their expected payoff. An intuitive explanation, and one often appealed to in the experimental and behavioral literature, is that agents misinterpret or misperceive the true decision problem they face. In this paper, we provide a theoretical framework to analyze an agent whose subjective representation of

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For example: assessing likelihoods in a manner incompatible with classical probability (Kahneman and Tversky, 1979; Schmeidler, 1989; Tversky and Koehler, 1994; Garfagnini and Walker-Jones, 2023), disregarding the process by which information is generated (Jin et al., 2015; Enke and Zimmermann, 2019; Enke, 2020), failing to properly condition beliefs (Tversky and Kahneman, 1983; Thaler, 1988; Esponda and Vespa, 2014), ignoring the strategic considerations of other agents (Eyster and Rabin, 2005; Esponda, 2008), or simply misunderstanding experimental directions or elicitation techniques (Plott and Zeiler, 2005).
a decision problem may diverge from that of the analyst or experimenter, and which may in general be logically unsound.

In order to organically capture the ability to reason contingently, we posit a simple and experiment-friendly elicitation of perceived implication—the agent’s subjective belief regarding ‘if ... then ...’ relationships between contingencies. This behavioral characterization regards the agent’s betting behavior, a straightforward type of economic data and provide a general (i.e., context agnostic) definition of contingent reasoning that can be applied to a broad set of empirical phenomena. The main contribution of this model is fourfold:

- We show that the agent’s perceived implication identifies the state-space—the possible resolutions of uncertainty—that represents the agent’s subjective view of the decision problem, and which may in general bear no resemblance to the modeler’s view. Our methodology therefore enables the familiar economic analysis using state-spaces without requiring the any ex-ante assumptions on the understanding of agents.

- We exhibit how various restrictions on behavior correspond to different modes of contingent reasoning as captured by distortions in the agent’s subjective state-space. We catalogue how distinct flaws in reasoning give rise to empirically observed patterns, and furnish a set of testable criteria for these empirically relevant benchmarks of rational behavior.

- We disentangle static failures of contingent reasoning (i.e., a logically flawed understanding implication) from errors in the dynamic process of information acquisition. In particular, we formalize the collective wisdom that reducing uncertainty improves contingent thinking.2

- Finally, we examine the relation between logical and probabilistic reasoning by considering expected utility type preferences of an agent who makes probabilistic trade-offs across states of her subjective, and possibly logically inconsistent, state-space. In doing so, we present a tractable way modeling logically flawed agents in economic applications.

Almost universally, economic models specify a state space, \( W \), and model an agent’s probabilistic judgements via a probability distribution, \( \ell \), over \( W \).3 Each state \( w \in W \)

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2See for example, Esponda and Vespa (2014); Martínez-Marquina et al. (2019); Enke (2020); Enke et al. (2021); Araujo et al. (2021).

3Of course, there is a large literature wherein ‘probability distributions’ are replaced by more general objects such as capacities, belief functions, or sets of probabilities, see Machina and Siniscalchi (2014). Eschewing the state-space is less common, although notable exceptions include the literature on subjective state-spaces (Kreps, 1979; Ahn and Sarver, 2013; Dekel et al., 2001) and unawareness (Halpern and Rêgo, 2006; Heifetz et al., 2007; Piermont, 2022).
is interpreted as the resolution of all uncertainties germane to the decision problem and \( \ell \) captures the agent’s beliefs about the likelihood of these various resolutions. Essentially, the prescription of a specific state space is the exogenous determination of the set of possible resolutions of uncertainty, and thus an implicit determination on how agents understand and reason about contingencies. Thus in choosing a state-space to represent the decision environment, the modeler also conditions which inferences can be drawn about an agent’s contingent thinking. Eschewing the provision of an objective state-space is critical to our goal of identifying, rather than imposing, failures of contingent thinking.

Our theory, then, concerns a set, \( \mathcal{L} \), of syntactic (or linguistic) statements such as “the value of the stock exceeds $10” or “the ball drawn from the urn is red.” The observable data of our model is the agent’s preferences regarding bets on these syntactic statements. For \( \Phi \subseteq \mathcal{L} \), let \( x_\Phi \) denote the bet with a payoff of \( x \) if and only if some \( \varphi \in \Phi \) is true (and nothing otherwise).

From this preference, we advance our central conceptual novelty: a definition of perceived implication. An agent perceives that \( \psi \) implies \( \varphi \) if whenever \( \varphi \in \Phi \), then she values \( x_{\Phi \cup \psi} \) equally to \( x_\Phi \) (for \( x > 0 \)). Intuitively, if the agent perceives \( \psi \) to imply \( \varphi \) then she believes that whenever \( \psi \) is true so too is \( \varphi \). Then if she already receives a payoff of \( x \) contingent on \( \varphi \), as for \( x_\Phi \), she will see no further value to the additional winning criterion \( \psi \): whenever this additional contingency obtains, she believes \( x_\Phi \) will as well.

Remarkably, through this definition, we procure a comprehensive view into the agent’s understanding of how uncertainty might resolve, far beyond what is seemingly contained in her betting behavior. Indeed, we show that under exceedingly weak conditions—reflexivity and transitivity of the preference over bets—the agent’s perceived implication identifies her unique interpretation of uncertainty (IOU), \((W,t)\). \( W \) is a state-space and \( t \) a truth assignment that assigns to each statement, \( \varphi \in \mathcal{L} \), the set of states, \( t(\varphi) \subseteq W \), where it is interpreted to be true. An agent’s IOU must faithfully recount her understanding of implication: \( t(\varphi) \subseteq t(\psi) \) exactly when the agent perceives \( \varphi \) to imply \( \psi \).

Thus, from her preferences over bets, we acquire a state-space, \( W \), collecting those resolutions of uncertainty that the agent considers possible, and in doing so, completely expose her understanding of the logical relation between the contingencies: For example, if \( t(\varphi) \cap t(\psi) = \emptyset \), then the agent believes that the statements \( \varphi \) and \( \psi \) are incompatible; if \( t(\varphi) \cup t(\varphi') = t(\psi) \), then the agent believes that the statements \( \varphi \) and \( \varphi' \) are mutually equivalent to \( \psi \), at least one of them is always true whenever \( \psi \) is, etc.

As detailed next, it is by studying the connection between perceived implication and the identified IOU that we can relate different failures of contingent reasoning (manifested

\footnote{Owing to the abstract nature of IOUs, uniqueness is only up to the behaviorally and conceptually irrelevant transformations of relabeling and adding redundant states. When allowing for infinite contingencies, the identification is weaker; see Theorem 3.}
by structural flaws in the IOU) back to observable patterns in choice. This includes both a static understanding of the problem, as well as where the IOU changes in response to information acquisition. We also examine the relation between logical and probabilistic judgements by providing utility functional representations of preference that cohere with the the agent’s IOU.

1.1 Overview

The Structure of Contingent Thinking. For a decision maker to be rational she must interpret statements in a logically consistent way: two logically equivalent statements must map to the same event, a statement and its negation must partition the state-space, the conjunction of two statements must map to their intersection, etc. So, while there is only one way for to be logically rational, as evinced by this laundry-list of requirements, there are many distinct ways of failing to be.

In Section 4 we explore how distinct logical errors, captured by different structural deficiencies in $t$, can account for the different experimentally observed patterns in decision making such as framing effects, the conjunction fallacy, violations of the sure thing principle, failures of negative introspection, and more. This showcases the value of interpretations of uncertainty as a universal model of contingent reasoning: our methodology is flexible enough to accommodate (and interconnect) many disparate empirical observations, without requiring any ex-ante assumption of which type of failures will be relevant in driving behavior.

We construct a taxonomy of failures of contingent thinking, and examine how they result in distinct behavior. To do this, we require structure in our language: if $\varphi, \psi$ are in $\mathcal{L}$ then so too are $\neg \varphi$, the negation of $\varphi$, is interpreted as the statement that $\varphi$ is not true; $\varphi \land \psi$, the conjunction of $\varphi$ and $\psi$, is interpreted as the statement that both $\varphi$ and $\psi$ are true; and $\varphi \lor \psi$, the disjunction of $\varphi$ and $\psi$, is interpreted as the statement that at least one of $\varphi$ and $\psi$ is true.

In section 2.5, we provide behavioral characterizations for the various structural components of logical reasoning, outlining the conditions on betting behavior such that, in the agent’s IOU, $t$ properly reflects the structural facets of the language: i.e., if $\varphi$ and $\psi$ are logically equivalent then $t(\varphi) = t(\psi)$, $t(\neg \varphi) = W - t(\psi)$, $t(\varphi \land \psi) = t(\varphi) \cap t(\psi)$, etc.

Dynamics Preferences and Information Acquisition. In Section 2.6 we explore the dynamics of IDEU preferences under information acquisition. How the agent’s IOU changes in response to learning that some statement $\varphi \in \mathcal{L}$ is true will depend not only on her interpretation of $\varphi$, but also how she interprets this statement relative to the other statements under consideration. We propose a definition of dynamic rationality that is independent of the agent’s initial ability to reason about the contingencies.
Apposite to recent experimental results,\(^5\) we show that this updating process preserves all benchmarks of rationality. In other words, while the process of learning can bestow rationality, it can never make her a worse reasoner. In other words, a reduction of uncertainty should correspond to a (weak) increase in contingent reasoning.

In particular, for any IOU \((W,t)\), we provide dynamic consistency restrictions which allows us to interpret updating as discarding states of the initial state-space. That is, if \((\succeq,\succeq^\varphi)\) represent the agents ex-ante and ex-post preferences, with respect to learning \(\varphi\), and \((W,t)\) is an initial IOU that faithfully captures \(\succeq\), then \(\succeq^\varphi\) is faithfully interpreted by a subset of states \(W^\varphi \subseteq W\), and \(t^\varphi : \psi \mapsto t(\psi) \cap W^\varphi\). It is as if learning \(\varphi\) allowed the agent to discard any states \(w \notin W^\varphi\), but does not otherwise change her interpretation. If in addition the agent properly understood the conditioning event, then \(W^\varphi = t(\varphi)\).

**Probabilistic Trade-offs in IOUs.** Minimal additional structure allows us to further identify the agent’s probabilistic assessment, \(\ell \in \Delta(W)\). In particular, in Section 3.1, we axiomatize interpretation dependent expected utility, the model wherein in an IOU then explains the agent’s preferences:

\[
x_\varphi \succeq y_\psi \iff x\ell(t(\varphi)) \geq y\ell(t(\psi)).
\]  

(IDEU)

In words, the agent prefers to bet on \(\varphi\) rather than \(\psi\), precisely when she believe the set of states where \(\varphi\) is true is more likely than the set of states where \(\psi\) is true.

While IDEU captures the valuations of bets, it is less clear how an agent, with an IOU \((W,t,\ell)\), should value a general syntactic act \(f : \Phi \to \mathbb{R}_+\) (where \(\Phi \subseteq \mathcal{L}\)) which might specify different payoffs for different \(\varphi \in \Phi\). In Section 3.2 we introduce extensions of the IDEU representation to this more general decision making environment.

The main conceptual hurdle is that an act \(f\) might not corresponds to a unique function over \(W\).\(^6\) Flawed hypothetical reasoning creates ambiguity (in the colloquial sense of the word) yielding multiple functions from \(W\) to payoffs which are all consistent with some translation of the syntactic act \(f\). Vagueness in the valuation of an act arises from its inability to be uniquely manifested within the agent’s IOU. We let \(\{\{f\}\}\) collect the set of consistent translations of the syntactic act \(f\) into the agent’s IOU and axiomatize two extensions of IDEU.

First, sparse IDEU, reminiscent of twofold multiprior preferences as introduced by Echenique et al. (2022), assumes that the agent prefers act \(f\) to \(g\) exactly when every translation of \(f\) yields a higher expected payoff (according to \((W,t,\ell)\)) than every translation of

\(^5\)For example, see Esponda and Vespa (2014); Martínez-Marquina et al. (2019); Park (2023).

\(^6\)For example, if the agent mistakenly believes that \(\varphi\) and \(\neg\varphi\) (the negation of \(\varphi\)) are compatible, then in her interpretation of uncertainty, there is some \(w \in t(\varphi) \cap t(\neg\varphi)\). Now for \(f\) which maps \(\varphi \mapsto x\) and \(\neg\varphi \mapsto y\), the value of \(f\) in state \(w\) is not determined.
That is, \( f \succ g \) if and only if

\[
f \succ g \iff \min_{f \in \mathcal{F}} \int f \, d\ell > \max_{g \in \mathcal{G}} \int g \, d\ell
\]  

(S-IDEU)

The interval representation corresponds directly to the bounds on the value of a syntactic act under the possibly many different translations it might have (i.e., the set \( \{ f \} \)). As such, sparse IDEU rigorously demonstrates the intuitive claim that twofold multiprior preferences model agent’s ”who may not be able to fully reason in terms of the underlying state space.”

The sparse IDEU model provides a notion of dominance between acts, but might leave many acts incomparable. As such, we introduce a further extension, MaxMin IDEU, that chooses between incomparable acts by selecting the most pessimistic translation:

\[
f \succeq g \iff \min_{f \in \mathcal{F}} \int f \, d\ell \geq \min_{g \in \mathcal{G}} \int g \, d\ell
\]  

(MM-IDEU)

It is rather immediate that if (S-IDEU) ranks two bets then (MM-IDEU) will preserve this ranking. However, unlike sparse IDEU, the later is a true decision criteria as it recounts a complete preference.

1.2 Relation to the Literature

There is mounting experimental evidence of marked and systematic deviations from rational behavior that extends far beyond what is attributable to subjects’ attitudes towards probabilistic uncertainty. For example, behavior that is dependent on the description of the problem (Tversky and Kahneman, 1981) or that directly contractions logically implication (Tversky and Kahneman, 1983; Garfagnini and Walker-Jones, 2023), the employment of dominated strategies (Kagel and Levin, 1993; Agranov et al., 2020), choices that can only by explained by extreme ambiguity of risk seeking behavior (Jabarian and Lazarus, 2022; Kuzmics et al., 2022; Martínez-Marquina et al., 2019), failures of monotonicity (Schneider and Schonger, 2019), failures to extract or use available information Enke (2020); Araujo et al. (2021), etc. See Niederle and Vespa (2023) for an overview. After introducing our theory, we survey, in Section 4, how it can accommodate many of the above findings.

Many of these authors directly point to failures of contingent thinking as the driving factor in their findings: Esponda and Vespa (2019) write

"subjects are not good at thinking through the state space in the way analysts often assume and [...] incomplete preferences or anomalies may precisely stem from the fact that states are not naturally given, may be hard to construct, or some states may not be salient."

Charness and Levin (2009) state that their results show
that the origin of the winner’s curse does not lie in [...] current theoretical explanations involving beliefs about other players. Instead, it stems from a form of bounded rationality, as to a large extent decision makers fail to recognize that a future contingency is relevant for their current decisions.

Martínez-Marquina et al. (2019) write “aggregating over multiple possible values of the state is especially difficult when there is uncertainty.” and Agranov et al. (2020) conclude “at least part of the [...] behavior observed [...] comes from subjects’ difficulty in thinking contingently.” Calford and Cason (2022) find that although subjects “should take actions that allow their future selves to make use of valuable contingent information, it appears that many have difficulty in effectively solving contingent thinking problems optimally.”

These assertions suggest strong demand for tractable and universal models of contingent reasoning. And yet—in contrast to the advances in modeling errors in probabilistic judgments—general models of decision making in the absence of logical sophistication are few and far between.

Nonetheless, there is a small literature detailing decision making under flawed logical reasoning; in this vein, the closest paper to ours is Lipman (1999) which presents a model in which agents may consider impossible states of affairs—for example where \( p \) and \( \neg p \) are both true—what are called impossible possible worlds in the philosophy literature (Hintikka, 1979). Like us, Lipman (1999) also constructs an endogenous state-space, the inconsistent states of which account for the agent’s flawed or limited reasoning, but, he focuses directly on the problem of recognizing logical equivalence, without analyzing more general forms of flawed reasoning such as the recognition of implication. His methodology is substantially different from ours: the primitive objects of the decision theory is a set of ex-post information indexed preference relations, \( \succ_I \), over acts defined on an objective/consistent state-space. This raises two issues: first, it is unclear how the modeler could observe ex-post preferences contingent on all possible information. Second, the set of acts under consideration are only those that depend on the objective and exogenous state-space—thus, by examining preferences over exactly this set, the modeler is implicitly informing the agent as to the set of ‘real’ states. Our model, by contrast, takes an ex-ante approach and considers syntactic acts unburdened by objective consistency requirements, thereby circumventing these observability issues.

Mukerji (1997), Esponda and Vespa (2019), and Piermont (2021) all explore the relation between contingent and probabilistic reasoning. However, in all of these works, the choice objects relate to a state-space that is exogenously imposed. As discussed above, this choice

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7Such models include ambiguity aversion (Gilboa and Schmeidler, 1989; Schmeidler, 1989; Machina and Siniscalchi, 2014), over weighting small probabilities (Kahneman and Tversky, 1979; Quiggin, 1982), base-rate neglect (Kahneman and Tversky, 1973; Benjamin et al., 2019), correlation neglect (Golub and Jackson, 2010; Enke and Zimmermann, 2019), etc.
limits the type of inferences that can be drawn from such a model. Piermont (2021), like this paper, focuses on a preference-based definition of subjective implication, analogous to the present notion of ‘perceiving an implication.’

Closely related to the methodology of our paper are models of syntactic decision theory, where the primitive relates to statements about the world rather than semantic acts or lotteries, principally Blume et al. (2021) and Björndahl and Halpern (2021) and to a lesser extent Piermont (2017); Kochov (2018) and Minardi and Savochkin (2019). Methodologically, the closest paper to ours is Blume et al. (2021) (BEH). They also take as primitive a preference relation over syntactic objects and construct a state-space as a representation object, specifically, their choice objects are recursively defined conditional assignments of the form “if \( \varphi \) then \( a_1 \) else \( a_2 \),” where \( a_1 \) and \( a_2 \) can be either other such choice objects or primitive (i.e., unconditional) outcomes.

The BEH characterization centers around cancellation axioms which rely critically on the additive structure of their representation—thus their setup does not speak to the interplay between the logical and the probabilistic modes of reasoning. Moreover, the bulk of their analysis centers around a logically rational decision maker; consequentially, they do not apply their methodology to the study of contingent reasoning, and have no concept of perceived implication, a central object of the present paper and one critical to our identification results, namely Theorems 3 and 6.

Our paper, in particular the identification of the agent’s subjective model, relates closely to the emerging literature on misspecified models. Many of these papers, for example Eliaz et al. (2020); Eliaz and Spiegler (2020); Ellis and Thysen (2021), regard an agent’s misperception of causality. While this is clearly related to the perception of logical implication, it comes with a a very different set of dynamic and interventional concerns. Others regard information acquisition and learning under misspecified models, for example Acemoglu et al. (2016); Mailath and Samuelson (2020); Frick et al. (2020). In contrast to this paper, the extant literature on misspecified models assumes that agents, despite having an incorrect view of the world, are logically consistent. Our methodology, by contrast, allows both incorrect knowledge and incorrect logical reasoning, and indeed, allows the analyst to disentangle these two forms of ignorance.

Possibility correspondences (and knowledge operators) form an alternative way of modeling an agent’s subjective understanding of contingencies, common in the foundations of game theory (Aumann, 1999), and have been used to model flawed logical reasoning (Geanakoplos, 1989; Dekel et al., 1998). More broadly, our paper shares much, both mathematically and philosophically, with logical representations of knowledge and awareness, see Halpern (2017) for an overview.
1.3 Contents

The reminder of the paper is structured as follows: The next section contains introduces the domain and primitive and contains our analysis of perceived implication, relating the agent’s observable behavior to her ability to preform contingent thinking. This includes the behavior characterization of implication (Section 2.2), interpretations of uncertainty (Section 2.3), identification results (Section 2.4), our taxonomy of the structural components to static rationality (Section 2.5), and properties of dynamic rationality (Section 2.6). Next, Section 3 contains our representation theorems, connecting perceived implication to global choice behavior, both over bets (Section 3.1) and over general actions (Section 3.2). Finally, after our theoretical exploration, Section 4 highlights how this model can account for many empirical phenomena within consolidated framework. The Appendix contains all proofs and auxiliary results.

2 Interpretation and Implication

2.1 Decision Environment

Our analysis begins with a language \( \mathcal{L} \), which is an arbitrary set of pay-off relevant statements. We think of these statements as real linguistic statements, expressed in human language, for example “It is raining” or “The S&P500 went up today.” For now, we impose absolutely no restrictions on \( \mathcal{L} \), but will later (in Section 2.5) consider additional structural assumption on the relation between statements. Let \( \mathcal{K} \) collect the set of all non-empty subsets of \( \mathcal{L} \), which we will refer to as contingency. We use lower cases \( \phi, \psi, \ldots \) for generic statements, elements of \( \mathcal{L} \), and uppercase \( \Phi, \Psi, \ldots \) for generic contingencies, elements of \( \mathcal{K} \). When there is no risk of confusion we will associate each statement \( \mathcal{L} \) with the set of singleton contingencies.

To understands how an agent interprets the uncertainty embodied by the linguistic statements, the modeler must observe her choices over syntactic objects, actions that relate payoffs to the statements in \( \mathcal{L} \). We take as the primitive the agent’s preferences over bets, acts which pay a fixed payoff contingent on a set of winning conditions, and nothing otherwise.\(^8\) Formally, a syntactic bet is a pair, \( x_{\Phi} = (x, \Phi) \), where \( x \in \mathbb{R}_+ \) is a payoff and \( \Phi \in \mathcal{K} \) is a contingency; we write \( x_\phi \) instead of \( x_{\{\phi\}} \) when the domain is a singleton. The idea is that \( x_{\Phi} \) represents a bet that pays a payoff of \( x \geq 0 \), denominated in utils,\(^9\) should

\(^8\)Later, in Section 3.2, we investigate how a agent’s interpretation of uncertainty affects her decisions over more complex choice objects which have possibly many distinct outcomes.

\(^9\)The assumption that payoffs are in utils is harmless given the body of decision theoretic results allowing for the conversion of lotteries or other alternatives into utils in a cardinally unique manner.
any of the statements \( \varphi \in \Phi \) turn out to be true, and pays 0 otherwise. The observable of the model is a preference relation \( \succcurlyeq \) the set of syntactic bets.

### 2.2 Subjective Implication

When observing an agent’s choices in a decision problem, an analyst will want to make inference about the agent’s understanding of specific contingencies and her ability to reason about the relation between them. We here posit a behavioral definition of implication that directly regards the primitive observable data. As such, this definition is universally applicable to any larger model of decision making as it is agnostic to any further assumptions about the agent’s behavior.

To gather intuition for this central concept, consider an agent who is indifferent between betting on \( \varphi \) alone or on betting on both \( \varphi \) and \( \psi \): 

\[
x_{\varphi} \sim x_{\{\varphi, \psi\}}
\]

The agent’s indifference—her belief that the additional criteria for receiving \( x \) contingent on \( \psi \) has no added benefit—exposes her understanding that whenever \( \psi \) is true, \( \varphi \) is also true. In other words, \( \psi \implies \varphi \). Our general definition of implication requires that this relation hold also in the presence of any other winning conditions, \( \Gamma \).

**Definition 1.** For two contingencies, \( \Psi, \Phi \in \mathcal{K} \), say that the agent (given by \( \succcurlyeq \)) perceives that:

- \( \Phi \implies \Psi \), written \( \Phi \implies \Psi \), if \( x_{\Psi \cup \Gamma} \sim x_{\Phi \cup \Psi \cup \Gamma} \) for all \( x \in \mathbb{R}_+ \) and \( \Gamma \in \mathcal{K} \);

- \( \Phi \in \mathcal{K} \) is **null** if \( \Phi \implies \Psi \) for all \( \Psi \in \mathcal{K} \);

- \( \Phi \) and \( \Psi \) **disjoint**, written \( \Phi \perp \Psi \), if for all \( \Gamma \in \mathcal{L} \), \( \Gamma \implies \Phi \) and \( \Gamma \implies \Psi \) implies \( \Gamma \) is null.

When discussing singleton sets, we will omit the brackets, i.e., we write \( \varphi \implies \Psi \) rather than \( \{\varphi\} \implies \Psi \), etc.

In addition to implication, Definition 1 also defines **null** contingencies as those that are perceived to trivialize the interpretation, in that if they were true, everything else would follow. In classical models of logical inference, \( \Phi \) is null iff it is a contradiction, that is, if its is necessarily false. Then, two statements are perceived to be **disjoint** if the only there are no non-null contingencies that jointly imply them; in other words, \( \Phi \) and \( \Psi \) are disjoint if they can only simultaneously hold in the trivial state-of-affairs where everything is true.

As a concise and essential example of flawed contingent thinking, take the following experimental findings adapted from Esponda and Vespa (2014). As we introduce the elements of our model, we will frequently return to this example.

**Example (Esponda and Vespa (2014)).** The state of the world is either \( R \) or \( B \) with equal probability. Each subject must cast a vote for either \( R \) or \( B \) without observing the state but
after observing a signal—the possible signal realizations are $R$ or $B$, with accuracy $\frac{2}{3}$. In addition, two computers observe the state and are programmed to follow the specific rule:

- If the state is $R$: vote $R$
- If the state is $B$: vote $B$ with probability $\frac{2}{3}$ and $R$ with $\frac{1}{3}$.

If a simple majority votes for the correct state, the subject’s payoff is $20$; otherwise, the payoff is $0$. Subjects are made fully aware of the rule used by the computers.

It is a dominate strategy to vote $B$ independently of the observed signal; for the subject’s vote to affect the outcome, the computers must disagree, and hence the state must be $B$. Nonetheless, 80% of subjects do not play strategically even after 40 rounds of play. On the other hand, in a further treatment, when subjects observed the votes of the computers before casting their own, they were much more likely to play the dominate strategy.

The description of the decision problem presented to subjects requires the following relevant statements:

- $s_c = “The state is c.”$
- $o_c = “The observed signal is c.”$
- $v_{i,c} = “Computer i voted c.”$

for $c \in \{R, B\}$ and $i \in \{0, 1\}$. To directly implement our theory, would require eliciting a subject’s valuations for bets over this language; nonetheless, from the experimental results we can make reasonable assumptions about how the subject’s behavior fits within our model.

It seems likely that subject understands negation and that the state/signal/vote is either red or blue, so that we observe $s_R \perp s_B$, $v_R \perp v_B$ etc. Moreover assume that the subject understands that she is pivotal if and only if the computers disagree, so that

$$P =: def \left( v_{0,R} \land v_{1,B} \right) \lor \left( v_{1,R} \land v_{0,B} \right)$$

denotes the statement that the subject’s vote determines the outcome.

In fact, we can assume that in an immediate sense, the subject understands the computers’ rules so that we observe the following implications: $s_R \Rightarrow v_{1,R}$ for $i \in \{0, 1\}$. Using the calculus of standard logical deduction, from these implications one could deduce the, payoff relevant, implication $P \Rightarrow s_B$. Critically, however, we assume this is not observed; despite being a consequence of her primitive understanding of the experiment, the subject’s flawed reasoning does not lead her to this essential relationship between contingencies.

### 2.3 Interpretations of Uncertainty

It turns out that under exceedingly weak conditions on $\Rightarrow$, the agent’s purely subjective notion of implication can be viewed as arising from a state-space interpretation of uncertainty.
That is, there exists a state space, $W$ and an assignment $t : \mathcal{L} \to 2^W$ such that the agent’s exposed understanding of implication is represented by the state-space. Towards this:

**Definition 2 (Interpretation of Uncertainty).** An **interpretation of uncertainty (IOU)** is a pair $(W, t)$ where: where $W$ is a set of states and $t : \mathcal{K} \to 2^W$ is a truth assignment such that for all $w \in W$, $w \in t(\Phi)$ for some $\Phi \in \mathcal{K}$.

An IOU, $(W, t)$, is a particular way of interpreting contingencies and how they relate to one another. Each state $w \in W$ represents a possible way that uncertainty can resolve, where exactly the statements $\{ \varphi \in \mathcal{L} \mid w \in t(\varphi) \}$ are true. Thus, identifying how the agent understands the relation between statements, and thus more generally her ability to reason about the consequences of statements, is tantamount to identifying an IOU that faithful captures her preferences.

**Definition 3 (Faithful).** Say that an IOU is faithful to $\succ$ if

(i) $t(\Phi) \subseteq t(\Psi)$ if and only if $\Phi \succ \Psi$,

(ii) $t(\Phi) = \emptyset$ if and only if $\Phi$ is null, and

(iii) $w \in t(\Phi) \cap t(\Psi)$ implies $w \in t(\Gamma)$ for some $\Gamma \in \mathcal{K}$ such that $\Gamma \succ \Phi$ and $\Gamma \succ \Psi$.

A faithful interpretation is one that accurately reflects the agent’s betting behavior, if particular, her revealed subjective understanding of implication. The first dictate of faithfulness states that the agent believes that $\varphi$ implies $\psi$ exactly when the every state where $\varphi$ is true is also a state where $\psi$ is true, reflecting our colloquial understanding of implication. The other two dictates require that properties of nullness and disjointness, as defined through the primitive, are also characterized by their set theoretic analogs.\(^{10}\)

**Example (E&V, continued).** Returning to the example, consider the following IOU, $(W, t)$, where

$$W = \bigcup \left\{ \{S_R\} \times \{O_R, O_B\} \times \{V_{0,R}, V_{1,R}\} \times \{P, \text{ NOT } P\} \right\}$$

and $t$ maps each primitive statement to the states that contain it.\(^{11}\) This particular state-space reflects the subjects perception of the relation between the statements as indicated earlier in the example: $t(v_{1,R}) \cap t(v_{1,B}) = \emptyset$ and $t(S_R) \subseteq t(v_{1,R})$, but $t(p) \not\subseteq t(S_R)$. \(\Box\)

---

\(^{10}\)Note that property (iii) implies $t(\Phi) \cap t(\Psi) = \emptyset$ if and only if $\Psi \perp \Psi$. It is in fact slightly stronger this, also requiring that every state in $\Psi$ and $\Psi$’s intersection witnesses their disjointness.

\(^{11}\)Moreover, if we wish to include additional statements like negations, conjunctions and disjunctions, then $t$ maps these via complements, intersections and unions, respectively, except for statements containing $P$, as required by $t(\varphi)$. 

12
Given the complete lack of structure on \( L \), finding a faithful interpretation seems an onerous request. At first glance, there is little reason to believe that an agent’s abstract understanding of statements, as revealed by her betting behavior, can be so succinctly embodied via a state-space model. Contrary to our intuition, however, it turns out that the requirements on \( \succsim \) to ensure a faithful interpretation are extremely weak.

**Theorem 1.** If \( \succsim \) is reflexive and transitive then there exists an interpretation of uncertainty, \((W,t)\), faithful to \( \succsim \).

Implicit in Theorem 1 is the impossibility of disciplining behavior without also requiring some level of logical rationality; even the minimal structure on preference inherent in an arbitrary IOU requires a rather developed sense of logical implication on behalf of the decision maker. Notice an immediate consequence of the Theorem is the transitivity of \( \succ \).

Classically, the transitivity of logical implication—understood since antiquity under the moniker *hypothetical syllogism*—is central to the idea of deductive reasoning; it is via this transitivity that a reasoner can chain together intermediate deductions so as to synthesize understanding from inchoate assumption.

### 2.4 Identifying Interpretations

A faithful interpretation is an understanding of the world that generates the agent’s behavior, each state of which captures a possible resolution of uncertainty (i.e., determines exactly which contingencies are true). Thus, pinning down an agent’s interpretation is tantamount to identifying exactly how she believes uncertainty might resolve. Given our very weak requirements of preference, it is not in general possible to uniquely determine the interpretation. Nonetheless, we can meaningfully identify the state-space up to behaviorally and conceptually irrelevant transformations, such as the addition of redundant states.

**Definition 4.** Let \((W,t)\) and \((W',t')\) be two IOUs. Say that \((W',t')\) extends \((W,t)\) if for all \( w \in W \) there is some \( w' \in W' \) such that

\[
\{ \Phi \in \mathcal{K} \mid w \in t(\Phi) \} = \{ \Phi \in \mathcal{K} \mid w' \in t(\Phi) \}.
\]

One interpretation extends another if every state in the later can be thought of as some state in the former. That is, up to relabeling, the extension simply adds additional states. Up to this relabeling, the IOUs faithful to some \( \succsim \) are highly structured, allowing a modeler to meaningfully identify the uncertainty perceived by the agent.

**Theorem 2.** Let \( \succ \) be reflexive and transitive. Then

- There exists a maximal faithful interpretation of \( \succ \), i.e., that extends any other faithful interpretation.
• If \((W, t)\) extends, and is extended by, resp., two faithful interpretations of \(\succ\), it is itself a faithful interpretation of \(\succ\).

This bound identifies the set of possible resolutions of uncertainty could be considered by the agent: A maximal IOU contains all states that are consistent with the agent’s betting behavior (i.e., any interpretation with any state not in the maximal IOU is inconsistent). The second claim also shows that the bounds of the set of faithful interpretations is the best we can hope for, since if we find two IOUs, one extending the other, than any set of states contained in the larger but not the smaller can be removed without behavioral consequence.

States of maximal interpretation are often redundant, not only in the sense that they not necessary to explain behavior, but also in the further conceptual sense that they add nothing to the modeler’s understanding of how the agent envisions uncertainty might resolve. Specifically, \(w\) is redundant if there is some event \(E\) such that knowing that \(w\) obtains would provide the same information as knowing that \(E\) obtained. As such, removing \(w\) from the state-space does not alter the set of eventualities considered by the agent.

A key insight of our paper is that by ignoring redundant states, we can sharpen our identification claims: When the domain is finite, the addition of redundant states is the only impediment to identification, so there exists a unique redundancy-free interpretation. Moreover, this IOU is minimal, it captures those states that must be present to explain the agent’s behavior.

**Theorem 3.** Let \(\succ\) be reflexive and transitive and let there be a finite set of \(\equiv_{\succ}\)-equivalence classes. Then there exists a unique (up to relabeling) faithful interpretation, \((W, t)\), of \(\succ\) that is redundancy-free, that is, such that there is no \(w \in W\) with

\[
\{\Phi \in \mathcal{K} \mid w \in t(\Phi)\} = \bigcap_{w' \in E} \{\Phi \in \mathcal{K} \mid w' \in t(\Phi)\}
\]

for some \(w \notin E \subset W\). Moreover, any other faithful interpretation extends \((W, t)\).

### 2.5 The Structure of Contingent Thinking

So far we have examined how an agent’s preferences can be reflected via a state-space based interpretation of syntactic statements. To further investigate how such interpretations (or, misinterpretations) can give rise to the empirically documented patterns in choice cataloged above, we must consider specific deviations from rational behavior. This requires two ingredients: first, adding structure to the language, so as to be able to classify behavior

\[12\] While a redundancy-free IOU always exists (this is a consequence of our constructive proofs), this many not be unique in infinite domains. This embarrassment of riches is related to the existence of non-principal ultrafilters, and thus hinges on the axiom of choice, waters in which the authors prefer not to swim at present.
along structural lines, and second, delineating what rational behavior would entail, so as to understand what it means to deviate from it.

To the first point, we now assume the language is closed under several connectives. In particular, if $\varphi, \psi$ are in $L$ then we assume also the existence of the statements: $\neg \varphi$, $(\varphi \land \psi)$, and $(\varphi \lor \psi)$. The interpretation is the usual one: $\neg \varphi$, the negation of $\varphi$, is interpreted as the statement that $\varphi$ is not true, $\varphi \land \psi$, the conjunction of $\varphi$ and $\psi$, is interpreted as the statement that both $\varphi$ and $\psi$ are true, and $\varphi \lor \psi$, the disjunction of $\varphi$ and $\psi$, is interpreted as the statement that at least one of $\varphi$ and $\psi$ is true.

To specify rational behavior, we take as given an objective implication operator, $\Rightarrow$, meant to capture the modeler’s understanding of implication. We write $\varphi \Rightarrow \psi$ to mean that, under the modeler’s interpretation of statements, $\varphi$ implies $\psi$.\footnote{While our results require no particular assumption regarding $\Rightarrow$, it is often convenient to think of $\Rightarrow$ as resulting from some traditional (classical) logical deduction: i.e., $\varphi \Rightarrow \psi$ whenever $\psi$ can be deduced from $\varphi$ under the deduction rules (and some set of axioms including logical tautologies) of propositional logic. For example, $p \Rightarrow p \lor q$ would be valid implication.} Take as short-hand $\varphi \Leftrightarrow \psi$ to mean that $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$, i.e., that $\psi$ and $\varphi$ are logically equivalent.

When the agent’s interpretation coincides with the modeler’s, as is tacit in routine economic analysis, it is without loss of generality suppress the syntactic component of the model, working only with $W$. We carry around this extra baggage, therefore, specifically because failures of contingent thinking are hereby given a precise definition as the logical invalidities manifest in $t$. By retaining some structure on the truth assignment, we provide benchmarks of rationality. We call a truth assignment:

- **exact** if $\varphi \Leftrightarrow \psi$ implies $t(\varphi) = t(\psi)$,
- **monotone** if $\varphi \Rightarrow \psi$ implies $t(\varphi) \subseteq t(\psi)$,
- **symmetric** if $t(\neg \varphi) = W - t(\varphi)$,
- **$\land$-distributive** if $t$ is exact and $t(\varphi \land \psi) = t(\varphi) \cap t(\psi)$
- **$\lor$-distributive** if $t$ is exact and $t(\varphi \lor \psi) = t(\varphi) \cup t(\psi)$
- **sound** if it is all of the above.

In addition, we consider the following property about how $t$ aggregates over sets of statements.

- **set-consistent** if $t(\Phi \cup \Psi) = t(\Phi) \cup t(\Psi)$.

In what follows, we characterize these benchmarks of contingent reasoning, as captured by each of the properties of $t$, via conditions on agents’ preference. Before doing so, we illuminate the value of these structural considerations within the example:
Example (E&V, continued). Notice that the interpretation given by the \((W, t)\) presented above, find \(t\) exact and symmetric. It fails, however, to be satisfy implication (with respect to the logically sound interpretation), as is self-evident by our focus on the failure to perceive \(p\) implies \(s_b\). This failure, however, can be more intrinsically examined, and is, at least given our unsubstantiated assumptions, traceable back to a failure of distributivity. Recall that

\[ p = \text{def} \bigvee_{i \in \{0,1\}} v_{i,R} \land v_{1-i,B} \]

So, the subject can correctly infer implications between simple statements, like \(s_r \Rightarrow v_{i,R}\), but fails to properly asses the entailments of compound statement \(p\), understanding the compound statement as something other than its constituent parts. \(\square\)

The first property characterizes those agents that understand objective logical equivalence.

\((E)\) — EQUVALENCE. If \(\psi \Leftrightarrow \varphi\) then \(\psi = \varphi\).

Applying the axiom twice, we see that \(E\) requires that the agent’s perceived implication reflects objective logically equivalence. The next property strengthens \(E\), characterizing agents who recognize not only equivalence but also single directional implication:

\((I)\) — IMPLICATION. If \(\psi \Rightarrow \varphi\) then \(\psi = \varphi\).

Our next property states captures the behavior of an agent who understands conjunction: if she understands that something implies both \(\varphi\) and \(\psi\), she can conclude that they must jointly hold.

\((C)\) — CONJUNCTION. \(\Gamma = \varphi\) and \(\Gamma = \psi\) if and only if \(\Gamma = \varphi \land \psi\).

Each of these properties in reflected by an analogous restriction on the set of faithful interpretations:

**Theorem 4.** Let \(\succ\) be reflexive and transitive. Then \(\succ\) satisfies \(E\) (resp., \(I\), \(C\)) iff and only if every faithful interpretation, \((W, t)\), of \(\succ\) finds \(t\) exact (resp., monotone, \(\land\)-distributive).

While Theorem 4 makes inroads in characterizing the sundered components of rationality, it does not yield a full characterization. To advance further, we require an additional richness condition on the set of statements, which will yield, the hitherto unsecured, set-consistency.

**Definition 5 (Articulation).** We call \(\succ\) articulate if, for every \(\Phi, \Psi \in \mathcal{K}\) the following two hold:

\((i)\) If \(\Phi \not\succ \Psi\) then there exists a non-null \(\Gamma \in \mathcal{K}\) such that \(\Gamma = \varphi\) and \(\Gamma \perp \Psi\).
(ii) For any collection \( \{ \Psi_i \}_{i \in I} \subseteq \mathcal{K} \), if \( \Phi \perp \Psi_i \) for all \( i \in I \) then, \( \Phi \perp \bigcup_{i \in I} \Psi_i \).

The first point states that when the agent believes \( \Phi \) does not imply \( \Psi \), she can articulate why: she can point out a specific state of affairs (\( \Gamma \)) where \( \Phi \) would hold but is incompatible with \( \Psi \). Similarly, the second point states that if the agent believes \( \Phi \) is compatible with some aggregate principle \( \Psi \), then the agent can articulate a specific state of affairs where both are true: whenever \( \Psi \) is broken down into subsets \( \{ \Psi_i \}_{i \in I} \), the agent considers some subset compatible with \( \Phi \).

Articulation, alone, ensures the existence of a set-consistent interpretation of \( \succeq \). In addition, it allows us to capture the other structural properties of \( t \) in a straightforward manner. In a dual axiom to \( C \), the following captures an understanding of disjunction:

\[ (D) \quad \varphi \Rightarrow \Gamma \text{ and } \psi \Rightarrow \Gamma \text{ if and only if } \varphi \lor \psi \Rightarrow \Gamma \]

The last axiom requires the agent understand negation—that in every possible resolution of uncertainty either a statement or its negation (but not both) is true.

\[ (N) \quad \varphi \perp \neg \varphi \text{ and } \Gamma \Rightarrow \{ \varphi, \neg \varphi \} \]

**Theorem 5.** Let \( \succeq \) be reflexive and transitive and articulate. Then there exists a set-consistent faithful interpretation of \( \succeq \). Moreover, \( \succeq \) satisfies \( C \) (resp., \( N \)) iff and only if every set-consistent faithful interpretation, \( (W, t) \), of \( \succeq \) finds \( t \lor \)-distributive (resp., symmetric).

These results exhibit the virtue of the model as reifying the intuition that many disparate empirical observations all fall within the purview of “failures of contingent thinking.” Moreover, by linking distinct failures of contingent thinking to specific behavioral patterns, Theorems 4 and 5 serves the modeler with a diagnostic tool for uncovering the particular driving force behind observed deviations from rationality.

### 2.6 Dynamics: Conditional Interpretations

In this section, we examine how flawed interpretations can alter the structure of belief dynamics, how an agent’s understanding changes in the face of new information. We now assume that the modeler observes not only \( \succeq \) but also \( \succeq^\varphi \) for some \( \varphi \in \mathcal{L} \), under the interpretation that \( \succeq^\varphi \) is the agents preferences after learning that \( \varphi \) is true. Let \( \Rightarrow^\varphi \) be the agents updated understanding of implication, as given by \( \succeq^\varphi \). We assume throughout that both \( \Rightarrow \) and \( \Rightarrow^\varphi \) are reflexive and transitive.

We will consider a very general class of dynamically linked preferences, where \( \succeq \) and \( \succeq^\varphi \) are linked by a updating rule: \( \mathcal{R}^\varphi : \mathcal{K} \to \mathcal{K} \), the idea being that \( \mathcal{R}^\varphi (\Psi) \) is an element of \( \mathcal{K} \) in the agents initial interpretation (i.e., before learning \( \varphi \)) that represents the agent’s
interpretation of $\Psi$ after learning $\varphi$. For a completely rational decision maker, one has $R^\varphi : \psi \to \psi \land \varphi$.

**Definition 6.** Call $R^\varphi : L \to L$ an **updating rule** for $(\succsim, \succsim^\varphi)$ if

$$\Psi \succsim^\varphi \Gamma \text{ if and only if } R^\varphi(\Psi) \succsim R^\varphi(\Gamma)$$

for all $\Psi, \Gamma \in K$. Moreover, when an updating rule exists for $(\succsim, \succsim^\varphi)$, we say that $\succsim^\varphi$ is **foreseen** (via $R^\varphi$) from $\succsim$.

It is rather straightforward to see that $\succsim^\varphi$ is foreseen from $\succsim$ if and only if it can be represented by a subset of the initial state-space: that is if $(W, t)$ is faithful to $\succsim$, then there exists a faithful interpretation of $\succsim^\varphi$, $(W^\varphi, t^\varphi)$ with $W^\varphi \subseteq W$. This makes sense of the nomenclature **foreseen**, as our definition precludes learning unforeseen information that would alter the structure of what is considered possible.

An agent’s dynamic rationality, her ability to incorporate new information into her interpretation, can be understood through the structure of the updating rule that connects her ex-ante and ex-post preferences. We posit the following as a minimum requirement in the consistent incorporation of acquired information.

**Definition 7.** Call an updating rule: $R^\varphi : K \to K$, **discriminate** if the following three properties hold:

**D1** **Veracity:** $R^\varphi(\Psi) \succsim \Psi$

**D2** **Specificity:** If $R^\varphi(\Psi) \succsim \Gamma$ then $R^\varphi(\Psi) \succsim R^\varphi(\Gamma)$

**D3** **Minimality:** If $\Psi \succsim R^\varphi(\Gamma)$ for any $\Gamma \in K$, then $R^\varphi(\Psi) = \Psi$

Moreover, call $(\succsim, \succsim^\varphi)$ **discriminate** if $\succsim^\varphi$ is foreseen from $\succsim$ via a discriminate updating rule.

To better intuit the three properties of a discriminate updating rule, consider the following. A number $x$ can take values in $\{2, 3, 4, 5\}$. And consider learning the statement $\varphi = \text{“}x \text{ is prime.}$$$. **Veracity** requires that the interpretation of any $\psi$ after learning $\varphi$ implies the initial interpretation of $\psi$ is true. For example, if $\psi = \text{“}x \text{ is even,}$$"$, then after learning $\varphi$, we have the ex-post interpretation $R^\varphi(\psi) = \text{“}x = 2\text{”}$ which of course implies that $x$ is even. **Specificity** requires that if the ex-post interpretation of $\psi$ implies $\gamma$ in the general context (without knowing the truth of $\psi$), then it will continue to in the more specific context where $\varphi$ is known. For example, if $\gamma = \text{“}x \neq 5\text{”}$, then $R^\varphi(\psi)$ implies $\gamma$, so too it implies the ex-post interpretation of $\gamma$ (that is, rationally, $R^\varphi(\gamma) = \text{“}x \in \{2, 3\}\text{”}$). Finally, **minimality** requires that if, ex-ante, $\psi$ implies something consistent with having already learned $\varphi$ is true, then
ψ does not get re-interpreted. For example, if ψ′ = “x ≤ 3”, then ψ′ implies Rψ(γ). So, learning ϕ does not change the interpretation of ψ′.

One might worry that the abstract nature of an updating rule leaves them observational meaningless: there may be, in general, many transformations of the language that satisfy (1). While this is true, our focus on discriminate rules palliates this concern, as within this scope the updating rule is identified:

**Theorem 6.** Let Rψ and R̄ψ be two discriminate updating rules for (≽, ≽ϕ). Then Rψ(Ψ) ≽ R̄ψ(Ψ) and R̄ψ(Ψ) ≽ Rψ(Ψ) for all Ψ ∈ K.

(D1)–(D3), while normative appealing individually, collectively correspond dynamically behavior, whereby updating is the result of discarding the states of the initial state-space inconsistent with the learned information. That is, if (≽, ≽ϕ) is discriminate and (W, t) is a faithful representation of≽, then ≽ϕ is faithfully interpreted by a subset of states Wϕ ⊆ W, and t : Ψ → t(Ψ) ∩ Wϕ. It is as if learning ϕ allowed the agent to discard any states w /∈ Wϕ, but does not otherwise change her interpretation.

**Theorem 7.** Let (W, t) be a faithful representation of≽. Then

(i) ≽ϕ is foreseen from≽ via a discriminate updating rule, Rψ
implies

(ii) there exists some Wϕ ∈ t(K) such that (Wϕ, Ψ → t(Ψ) ∩ Wϕ) is a faithful interpretation of≽ϕ,

(iii) there exists some Φϕ ∈ K such that Ψ ≽ϕ Ψ′ if and only if for all Γ : Γ ≽ Φϕ and Γ ≽ Ψ then Γ ≽ Ψ′.

Moreover, if≽ is articulate, then either (ii) or (iii) imply (i).

As we make no demands of the ex-ante interpretation, this result shows that we can cleanly separate dynamic rationality concerns—updating via conditioning—from static concerns revealed by the logical (un)soundness of the initial interpretation. Indeed, we here show that even a flawed reasoner can incorporate new information in consistent way, where consistency is relative to her own initial interpretation. Further, notice that Wϕ may, in general, bare no relation to ϕ; this captures an agent who misinterprets the updating event, but is then consistent given her misinterpretation. Below, we provide an additional restriction, (D4), to ensure that Wϕ = t(ϕ).

Theorem 7(iii) also shows that this conditioning process can be equivalently captured via the existence of a Φϕ that ‘localizes’ ex-ante implication: that is if ex-post implication can be recovered as ex-ante implication local to those contingencies that imply Φϕ. Of course, the connection between these viewpoints is that t(Φϕ) = Wϕ.
With this notion of information acquisition in place, we can return one last time to our running example, finally speaking to the experimental findings of Esponda and Vespa (2019).

Example (E&V, completed). The subject prefers to vote red whenever she prefers a bet on \( S_R \land P \) to a bet on \( S_B \land P \). To analyze this decision problem, we must consider the subject’s conditional preferences: Upon learning \( O_R \) her new state-space is \( W^{O_R} = t(O_R) \) and her new interpretation \( t^{O_R}(\psi) : \psi \rightarrow t(\psi) \cap t(O_R) \) (and, her probabilities update via conditional probability, see Section 3.1 for an analysis including probabilities).

So, conditional on seeing the signal \( O_R \), the subject considers the state-space

\[
W^{O_R} = \bigcup \left\{ \left\{ S_R \right\} \times \left\{ O_R \right\} \times \left\{ v_{0,R} \right\} \times \left\{ v_{1,R} \right\} \times \left\{ P, \neg P \right\}, \left\{ S_B \right\} \times \left\{ O_R \right\} \times \left\{ v_{0,B}, v_{0,B} \right\} \times \left\{ v_{1,R}, v_{1,B} \right\} \times \left\{ P, \neg P \right\} \right\}
\]

where, assuming the probabilistic structure induced by the description on the problem, \( S_R \land P \) is indeed more likely than \( S_B \land P \), rationalizing a vote for red. This is precisely because the subject does not intuit the true relation between her ability to influence the election and the state.

Finally, consider what happens when the computers’ votes are themselves observed by the subject. Conditional on seeing, for example, 0 vote blue and 1 vote red, and seeing a red signal, the subject considers the state-space:

\[
W^{(V_{0,B}, V_{1,R}, O_R)} = \left\{ S_B \right\} \times \left\{ O_R \right\} \times \left\{ v_{1,R} \right\} \times \left\{ v_{2,R} \right\} \times \left\{ P, \neg P \right\}
\]

Although the subject still does not correctly assess the event that she is pivotal, she does understand that the state is certainty blue. This makes sense of the observed increase is rational behavior in the sequential treatment, where the computers votes are observed. ■

As is suggested by the example, an agent’s ability to engage in contingent thinking will increase subsequent to a reduction of uncertainty (at least when updating is discriminate). In particular, an important, if immediate, consequence of Theorem 7 is that updating via a discriminate updating rule will preserve all of the rationality benchmarks presented in Section 2.5. Formally:

**Corollary 8.** If \((\succeq, \succeq^\varphi)\) are discriminate, then whenever \( \succeq \) satisfies any of subset of \( \{ E, I, C, D, N \} \), so to does \( \succeq^\varphi \).

Corollary 8 follows immediately from (ii) and fact that all relevant properties of truth assignments are preserved under intersection with a subset of the state-space. Of course, the converse does not hold: it is possible that an agent who does not satisfy some benchmark
of rationality before learning \( \varphi \) will satisfy it after learning. In particular, her preferences might be ‘locally’ rational over the subset of states where \( \varphi \) is true; see Example ??.

By requiring slightly more of the updating rule, we recover rational behavior with respect to the incorporation of new information. Consider:

**Corollary 9.** Let \((W,t)\) be a faithful representation of \(\succsim\); then \(\succsim^{\varphi}\) is foreseen from \(\succsim\) via a discriminate updating rule such that

\[(D4)\] **Relevance:** If \(\varphi \Rightarrow \Psi\) then \(R^{\varphi}(\Psi) = \varphi\).

if and only if \((t(\varphi), \Psi \mapsto t(\Psi) \cap t(\varphi))\) is a faithful interpretation of \(\succsim^{\varphi}\). Notice that if \(\succsim\) also satisfies A3, then \(R^{\varphi}(\psi) = \varphi \land \psi\).

Finally, it is also instructive to consider weaker restrictions on updating rules. Consider the following, all of which are implied by \((D1)\) and \((D2)\).

\[(D5)\] **Preservation:** If \(\Psi \Rightarrow \Gamma\) then \(R^{\varphi}(\Psi) = R^{\varphi}(\Gamma)\)

\[(D6)\] **Weak Preservation:** If \(\Psi \Rightarrow \Gamma\) and \(\Gamma \Rightarrow \Psi\) then \(R^{\varphi}(\Psi) = R^{\varphi}(\Gamma)\) and \(R^{\varphi}(\Gamma) = R^{\varphi}(\Psi)\).

Notice immediately that \((D5)\) implies \((D6)\). Let be \(\succsim^{\varphi}\) is foreseen from \(\succsim\) via a weakly preserving updating rule, \(R^{\varphi}\) and let \((W,t)\) be a faithful representation of \(\succsim\). Then it is rather straightforward to see that there exists a function \(\xi : \mathcal{2}^W \rightarrow \mathcal{2}^W\) such that \((W, \xi \circ t)\) is a faithful representation of \(\succsim^{\varphi}\). As such, we can think of learning \(\varphi\) as inducing a operation on the state-space. Moreover, if \(R^{\varphi}\) is preserving, then \(\xi\) is monotone with respect to set inclusion; if it is veracious then \(\xi(A) \subseteq A\), etc. *Piermont* (2021) studies these operations over a state-space in a static context.

### 3 Representation Theorems

#### 3.1 Interpretation Dependent Expected Utility

So far we have provide any representation of betting behavior, analyzing only the derived implication operator, \(\Rightarrow\). In what follows we consider additional axiomatic restrictions on \(\succsim\) so as to ensure that an agent’s preferences over bets are consistent with some probabilistic assessment over her interpretation of uncertainty. That is, we delineate behavior that results from expected utility maximization over the entirely subjective state-space that arises out of the agent’s revealed perception of implication.

We consider a preference relation, \(\succsim\), over syntactic bets, that satisfies the following axioms:
**A1**—Order. \(\succeq\) is a continuous weak order.

Here, continuity means that the upper and lower contour sets of \(\succeq\) are closed, endowing \(\mathcal{K}\) with the discrete topology, and subsequently, \(\mathcal{K}^\mathbb{R}\) with the product topology.\(^{14}\)

Next, we ensure that the agents prefers to bet on larger events, as understood by her perceived implication.\(^{15}\)

**A2**—Event Monotonicity. If \(\Phi \supseteq \Psi\) then \(x_\Psi \succeq x_\Phi\).

The next axiom ensures that getting a payoff of 0 does not depend on the winning condition; this is tantamount to assuming that preferences over different winning conditions can be jointly normalized.

**A3**—Groundedness. For all \(\Phi, \Psi \in \mathcal{K}\): \(0_\Phi \sim 0_\Psi\).

Payoff Monotonicity requires the agent prefers higher payoffs in the obvious sense: as long as the winning condition is not null, then a higher payoff is strictly better:

**A4**—Payoff Monotonicity. \(x > y\) implies \(x_\Phi \succ y_\Phi\), with \(x_\Phi \succ y_\Phi\) if and only if \(\Phi \in \mathcal{K}\) is not null.

Next, we impose that preferences are linear in payoffs holding the winning conditions fixed. Importantly, our linearity axioms says nothing about how the agent makes trade-offs across distinct events, only that she can scale up the payoffs without reversing preference.

**A5**—Linearity. For \(\alpha \in \mathbb{R}_+: \) if \(x_\Phi \succeq y_\Psi\) then \(\alpha x_\Phi \succeq \alpha y_\Psi\).

The axioms **A1**–**A5**, while relatively simple, imply the existence of and IOU and a likelihood assessment, that rationalize the agent’s preference over bets. To make this formal:

**Definition 8** (Probabilistic Interpretation of Uncertainty). A **probabilistic interpretation of uncertainty** (P-IOU) is a list \((W, t, \ell)\) where \((W, t)\) is an IOU and \(\ell : t(\mathcal{K}) \to [0, 1]\) is a likelihood assessment on \(t(\mathcal{K})\), the image of \(t\). That is \(\ell : t(\mathcal{K}) \to [0, 1]\), such that

- \(A \subseteq B\) implies \(\ell(A) \leq \ell(B)\),
- \(t(\emptyset) = 0\) whenever \(\emptyset \in t(\mathcal{K})\), and
- \(t(W) = 1\) whenever \(W \in t(\mathcal{K})\).

\(^{14}\)The upper contour set of \(x_\Phi\) is the set of all preferred bets: \(\{y_\Psi : y_\Psi \succeq x_\Phi\}\). The lower contour set collect the dis-preferred bets: \(\{y_\Psi : x_\Phi \succeq y_\Psi\}\).

\(^{15}\)This ensures that an agent’s likelihood assessment is monotone. By dropping **A2**, we could consider a model with non-monotone set functions as likelihood functions, although we would need to replace it with a boundedness axiom that ensures that there does not exist a sequence of events that becomes infinitely valuable to bet on.
An probabilistic interpretation of uncertainty can make sense of the DM’s preferences over bets:

**Definition 9 (IDEU preference).** Call $\succeq$ an interpretation dependent expected utility preference (IDEU) if there exits some $P$-IOU $(W,t,\ell)$ such that $(W,t)$ is a faithful interpretation of $\succeq$ and such that for all syntactic bets $x_\Phi$ and $y_\Psi$,

\[
x_\Phi \succeq y_\Psi \Leftrightarrow x\ell(t(\Phi)) \geq y\ell(t(\Psi)).
\]

Thus, $\succeq$ is an IDEU preference if the value of $x_\Phi$ is the DM’s subjective expectation of the bet: the value, $x$, times the probability of winning, $\ell(t(\Phi))$. Unlike the classical model, however, the probability of winning depends not only on the DM’s beliefs about the likelihood of states, but also her subjective understanding of when the various winning conditions are true or false.

**Theorem 10.** A relation $\succeq$ satisfies A1–A5 if and only if it is a IDEU preference.

### 3.1.1 Additive IDEU

In order to better reveal the distinction between logical reasoning and probabilistic reasoning, materialized by the quality of $t$ and of $\ell$, respectively, we here consider impositions so as to require rationality properties on $\ell$. Probabilistic reasoning (or sophistication) has long been understood through the lens of additivity.\(^\text{16}\)

**Definition 10 (Additive IOUs).** Call a likelihood assessment, $\ell$ weakly additive if for all $A, B \in t(K)$ such that $A \cup B \in t(K)$ and $A \cap B = \emptyset$,

\[
\ell(A \cup B) = \ell(A) + \ell(B).
\]

(W)

Weak additivity follows from the following axiom:

**A6—Subjective Independence.** For all $\Phi, \Psi \in K$ with $\Phi \perp \Psi$

\[
1_\Phi \sim x_{\Phi \cup \Psi} \text{ implies } 1_\Psi \sim (1 - x)_{\Phi \cup \Psi}.
\]

A weakly additive IOU satisfies the additivity criterion on all events on which it can be directly defined, that is, on the image of $t$. Although weakly additive IDEU preferences never display direct violations of additivity, there may not exist any probability measure that extends $\ell$ to the algebra generated by $t(K)$. This is owing to the paucity of events on which any restriction is made, as the following example shows:

\[^{16}\text{See for example Savage (1954); Anscombe and Aumann (1963); Fishburn (2013); Machina and Schmeidler (1992); Marinacci (2002); Strzalecki (2011).}\]
Example 1. Consider \((W, t, \ell)\) where \(W = \{w_1, w_2, w_3\}\), the image of \(t\) equal the family of sets:

\[ \{ W, \{w_1, w_2\}, \{w_2, w_3\}, \{w_2\}, \emptyset \} \]

and, let \(\ell(W) = \ell(\{w_1, w_2\}) = \ell(\{w_2, w_3\}) = 1, \ell(\{w_2\}) = \frac{1}{2}\) and \(\ell(\emptyset) = 0\). Then this IOU is weakly additive since the image of \(t\) contains no non-empty disjoint sets! But clearly, it does not extend to any measure since 1 + 1 = \(\ell(\{w_1, w_2\}) + \ell(\{w_2, w_3\}) > \ell(\{w_1, w_2\} \cup \{w_2, w_3\}) + \ell(\{w_1, w_2\} \cap \{w_2, w_3\}) = 1 + \frac{1}{2}\). \(\square\)

Of course, if the image of \(t\) is well behaved—for example, if has a lattice structure—then a slight strengthening of weak additivity ensures \(\ell\) can be extended from this image to the algebra there generated.

This stronger axiom, equivalent to \(A_6\) when the image of \(t\) is at least a semi-ring, eliminates pathologies of the type expressed in Example 1:

\[ A_7—\text{Strong Subjective Independence.} \] Let \(\Phi, \Phi', \Psi \in K\) be such that \(\Gamma = \succ \Psi\) if and only if \(\Gamma = \succ \Phi, \Gamma = \succ \Psi\). Then:

\[ 1_\Phi \sim x_{\Phi \cup \Phi'} \text{ and } 1_{\Phi'} \sim y_{\Phi \cup \Phi'} \text{ implies } x + y \geq 1 \text{ and } 1_\Psi \sim (x + y - 1)_{\Phi \cup \Phi'}. \]

We obtain:

**Theorem 11.** Let \((W, t, \ell)\) be a faithful interpretation of \(\succ\). Then

- \(\succ\) satisfies \(A_6\) if and only \(\ell\) is weakly additive.
- If \(\succ\) is articulate and \(t\) set-consistent, then \(\succ\) satisfies \(A_7\) if and only if there exists a unique finitely additive measure extending \(\ell\) to the algebra of sets generated by \(t(K)\).

### 3.2 Choice over General Acts

The preferences we consider in the paper so far are rather limiting, in the sense that they related to a limited class of choice objects. Indeed, the theory presented hitherto remains silent on how the DM contemplates more complex actions that might yield many distinct outcomes. In this section, we consider a more standard (Savage like) choice environment, and provide several choice criteria that expose how a DM’s interpretation of uncertainty will dictate her preferences.

The primitive of this decision theory is a preference over *syntactic acts*, which are partial functions from the language \(\mathcal{L}\) into utils; in other words, an act is a function whose domains is a *subset* of the language. For simplicity, we assume that the image of each syntactic act is finite. The syntactic bets which formed the basis of the IDEU preferences correspond to
constant acts in this domain. Let $\text{DOM}(f) \in \mathcal{K}$ denote the domain of $f$, the set of statements on which it is defined.

An interpretation dependent expected utility maximizer compares the linguistic description of more complex acts by first translating them into her subjective model (i.e., into her interpretation) and then by computing the expected payoff. The agent with the interpretation $(W, t, \ell)$, and who faces the syntactic act $f$, first translates it into her model as a function $f : W \rightarrow \mathbb{R}$, then computes the expectation of $f$ with respect to $\ell$ à la Choquet integration.

The second step, aggregating across subjective states, is straightforward: despite their generality, likelihood assessments carry well-defined theory of integration: for each measurable partial function $f : W \rightarrow \mathbb{R}_+$ and likelihood appraisal $\ell$ let,

$$
\int f d\ell = \int_0^\infty \ell(\{w \in W \mid f(w) \geq r\})dr.
$$

where the integral on the right hand side is the standard Lebesgue integral over $\mathbb{R}$.

The translation step, however, is not trivial. When $t$ is not sound—when the agent is not a perfect logical reasoner—a syntactic act, even a well specified one, may not have a unique translation as a semantic act (i.e., a function $W \rightarrow \mathbb{R}$); the DM might perceive ambiguity when her interpretation is not sound. For example, if $\varphi$ and $\psi$ are pairwise contradictory, but the agent fails to recognize this, she may envision a state of affairs, $w$, where both are simultaneously true: an act which yields different outcomes on the basis of these statements does not uniquely determine the outcome at $w$. Because of this, we deal with the notion of consistent translations.

Say that $f : W \rightarrow \mathbb{R}_+$ is consistent with $f$ if for all $w \in W$

$$
f(w) \in f \circ t^{-1}(w) = \{f(\varphi) \mid \varphi \in \mathcal{L} \text{ such that } w \in t(\varphi)\}
$$

For example see Figure 1. Fixing the interpretation of uncertainty let \{f\} collect the set of all (semantic) acts consistent with $f$. 

Figure 1: $W = \{w_1, w_2, w_3\}$ and $t : p \mapsto \{w_1, w_2\}$ and $t : \neg p \mapsto \{w_2, w_3\}$. The for the syntactic act $f$ given by $f : p \mapsto 2$ and $f : \neg p \mapsto 3$, there are two consistent acts: delineated by the solid black and dotted yellow lines.
Under the interpretation \((W, t, \ell)\), the verbal description of an act, \(f\), does not distinguish between the elements of \(\{f\}\). Thus, the agent’s inability to properly interpret the contingencies on which \(f\) is predicated creates ambiguity. When \(f\) is partitional and \(t\) is sound, then \(\{f\}\) is a singleton. It is in this sense that the use of semantic acts is without loss of generality under the assumption that the agent is rational.

### 3.3 Sparse IDEU

The first choice criteria we consider is *sparse IDEU*, reminiscent of *twofold multiprior preferences* as introduced by Echenique et al. (2022) in the domain of purely probabilistic uncertainty. Twofold multiprior preferences model are represented by an set of probabilistic beliefs over an objective state-space, and where the agent prefers one act to another when the former’s lower bound valuation over the set of beliefs beats the latter’s upper bound.

In this model, \(f\) is (strictly) preferred to \(g\) if and only if every consistent translation of \(f\) has a higher expected payoff than that of \(g\). Formally:

**Definition 11 (Sparse IDEU preferences).** Call \(\succ\) an sparse IDEU preference if there exits some interpretation of uncertainty \((W, t, \mu)\), with additive \(\mu\), such that for all syntactic acts \(f\) and \(g\),

\[
\min_{f \in \{f\}} \int f \, d\ell > \max_{g \in \{g\}} \int g \, d\ell \quad \text{(s-ideu)}
\]

The sparse IDEU representation is an interval representation a la Fishburn (1970): each act \(f\) is associated with an interval of utilities \([f, \bar{f}] \subset \mathbb{R}_+\) and \(f \succ g\) when the \(f\)’s interval lies entirely above \(g\)’s. The interval of utilities, is, of course, generated by the set of consistent translations—it represents the range of possible expected payoffs taking into account the ambiguity arising from the agent’s inability to precisely interpret the act. Sparse IDEU reduces to subjective expected utility when (i) \(t\) is sound (ii) attention is restricted to partitional acts and (iii) \(\ell\) is a measure, for generalized IDEU. We provide an axiomatization for sparse IDEU preferences in Section 3.5.

### 3.4 MaxMin IDEU

Thought of as a criteria on strict preference (i.e., taking \(\succ\) as the observable), sparse IDEU preferences are often far from complete. In other words, the representation \((\text{s-ideu})\) is agnostic about how to compare acts when different translations of the acts can be ranked in different ways (i.e., when the interval of constant equivalent bets overlap).

The next criteria we consider is *MaxMin IDEU*, which can be thought of as the cautious completion of this order. Whenever two acts are not strictly ranked by sparse IDEU, MaxMin selects the act with the higher worst-case consistent translation. Formally:
Definition 12 (Sparse IDEU preferences). Call $\succ$ an sparse IDEU preference if there exits some interpretation of uncertainty $(W, t, \mu)$, with additive $\mu$, such that for all syntactic acts $f$ and $g$,

$$f \succ g \iff \min_{f \in \{f\}} \int f \, d\ell \geq \min_{g \in \{g\}} \int g \, d\ell$$

(Max-M IDEU)

MaxMin IDEU preferences are also axiomatized in Section 3.5, below. It is rather immediate that if $(s\text{-ideu})$ ranks two bets then $(\text{MM-ideu})$ will preserve this ranking. However, unlike sparse IDEU, the latter is a true decision criteria as it recouts a complete preference.

3.5 Axiomatic Foundation

Here we provide axioms on a preference relation (also denoted $\succ$) over syntactic acts. The two choice criteria discussed above share a lot of the same structure, and so we introduce the axioms for both all at once.

We will consider notation for some special acts: For $\Lambda \subseteq \Phi \subset L$ and $x \in \mathbb{R}^+\cdot$, let $x_\Phi^\Lambda$ denote the acts that maps $\Lambda$ to $x$ and $\Phi \setminus \Lambda$ to 0. That is, the act whose domain is $\Phi$, and which pays $x$ if and only if $\Lambda$. Further, identify $x_\Phi$ with the syntactic bet of $x$ on $\Phi$.

For two acts $f$ and $g$ such that $\text{DOM}(f) = \text{DOM}(g)$, let $\alpha f + \beta g$, for $\alpha, \beta \in \mathbb{R}^+$, denote the point-wise mixture of $f$ and $g$.

In what follows, it will be helpful to consider properties of the strict component of preference: Given $\succ$, for any pair of acts $f$ and $g$ write $f \sim g$ if $\{h \mid h \succ f\} = \{h \mid h \succ g\}$, and $f \sim g$ if $\{h \mid f \succ h\} = \{h \mid g \succ h\}$. Finally, write $f \approx g$ if $f \sim g$ and $f \sim g$: that is, if $f$ and $g$ satisfy the exactly same $\succ$-relations. Note that $f \approx g$ is not the same as $f \sim g$; this equivalence does however follow if $\succ$ is complete and transitive.

The first axiom is the requirement that the strict component of the preference is a interval order. This holds for both choice criteria (but is redundant in the case of MaxMin IDEU).

A8—Strict Interval Order. $\succ$ is continuous irreflexive and transitive and has the interval property: if $f \succ g$ then for all $x \in \mathbb{R}^+$ and $\Phi \in \mathcal{K}$ either $f \succ x_\Phi$ or $x_\Phi \succ g$.

The next several axioms are joint to both representations. Interpretability states that when we restrict the preference to syntactic bets, this restrictions satisfies the axioms from Section 3.1.

A9—Interpretability. $\succ$, restricted to syntactic bets is an IDEU preference.

Next, we link the preferences over more complex acts to simple bets by assuming the existence of an (upper) certainty equivalent.
A10—Bet Equivalence. For all \( f \), and non-null \( \Phi \), there exists an \( x \in \mathbb{R}_+ \) such that \( f \sim x_\Phi \).

The next axiom extends the IDEU representation to more complex acts. Even when the agent aggregates in a non-linear way across states (i.e., the representation as a non-probability \( \mu \)), aggregation of co-monotone acts will be linear.

A11—Co-Monotone Additivity. For \( \Phi \), let \( f,g : \Phi \rightarrow \mathbb{R} \) be co-monotone. Then

- Let \( g \sim y_\Phi \). Then \( f \succ x_\Phi \) iff \( f + g \succ (x + y)_\Phi \).
- Let \( y_\Phi \sim g \). Then \( x_\Phi \succ f \) iff \( (x + y)_\Phi \succ f + g \).

The next axiom is the only one that is not joint to the two different representations. The axioms dictates how a agent’s valuation (i.e., the her upper and lower bound valuations as given by \( \sim \) and \( \dot{\sim} \)) respond to expanding the set of contingencies on which an act pays a positive payoff. Sparse IDEU will satisfy both the upper and lower versions, as the choice procedure depends jointly on both bounds, whereas MaxMin IDEU will only satisfy the lower version of the axiom.

A12—Expansion Consistency. Let \( \Phi, \Gamma \subset \Lambda \in \mathcal{K} \) be disjoint and \( \Psi \in \mathcal{K} \) be non-null. Then:

(upper) Let \( x_{\Phi \cup \Gamma} \sim y_\Psi \). Then \( z_\Psi \succ x_\Phi \) iff \( (z + y)_\Psi \succ x_{\Lambda \cup \Gamma} \).

(lower) Let \( x_{\Lambda \setminus \Phi} \sim y_\Psi \). Then \( x_\Phi \succ z_\Psi \) iff \( x_{\Lambda \cup \Gamma} \succ (z + y)_\Psi \).

When moving from betting on \( \Phi \) to betting on \( \Phi \cup \Gamma \), the agent’s upper-bound valuation increases directly by the value of a bet on \( \Gamma \). However, her lower-bound valuation can increase by more, since by changing the payoff contingent on \( \Gamma \) from 0 to 1, the act also becomes less ambiguous—if the agent perceived that \( \Phi \) and \( \Gamma \) might overlap, then the bet on \( \Phi \) alone is not well defined in the agent’s head, but additionally betting on \( \Gamma \) can alleviate this concern and thereby increase the value of the act by more than the direct value of a bet on \( \Gamma \).

Theorem 12.

1. A relation \( \succ \) satisfies A8–A12 if and only if it is a sparse IDEU preference.

2. A relation \( \succ \) is complete and transitive and satisfies A9–A11 and A12-(lower) if and only if it is an MaxMin IDEU preference.
As is apparent by the overlap in their axiomatization, these two choice criteria are closely related. Indeed, given the same IOU, these two preferences will agree on any strict preference indicated by sparse-IDEU. However, to extend the choice procedure to all acts, as in MaxMin IDEU, clearly requires the preference be complete and transitive. Since a complete relation satisfies A8, we can drop this dictate from conditions for MaxMin IDEU.

4 Empirical Failures of Contingent Reasoning

In this section, we explore how our model can accommodate a wide range of observed behavior attributed failures of contingent thinking. Each example concerns a decision problem described to experimental subjects in colloquial language, formalized here as a set of (possibly logically dependent) statements that describe the payoff relevant contingencies.

These examples, which provide an interpretation of uncertainty that gives rise to various empirical patterns, serve to showcase the flexibility and generality in our theory. Nonetheless, there may be other interpretations that also explain the observed behavior. In light of this, we point out that, given rather straightforward economic data, an agent’s interpretation of uncertainty—her subjective state-space, truth assignment, and probabilistic assessment—is identified (Theorem 3). Thus, in principle, each of these examples can be falsified, and in the event they do not hold, our identification results would yield an alternative explanation.

Failures of Omniscience and Framing Effects. Framing effects, introduced by Tversky and Kahneman (1981), wherein different—yet logically equivalent—descriptions of a choice problem yield different choices, constitute a simple and evident violation of rational decision making. For example consider an agent valuing bets over the following statements:

\[ p = \text{"the S&P500 goes up tomorrow"}, \text{ or } \]

\[ q = \text{"the S&P500 goes up tomorrow and the 1 millionth prime number is greater than 15000000."} \]

While \( p \) and \( q \) are in fact equivalent, as the additional criterion in \( q \) is tautological, it is far from unreasonable for an agent to strictly prefer betting on the former. As humans are not perfect reasoners, we often fail to observe logical equivalence, and thus do not treat logically equivalent statements as such. This is easily captured in our model by a \( t \) that maps \( p \) and \( q \) to different events in the state space.

While the effect above is likely due to the computational complexity of the statements, framing effects abound for myriad other, often emotional, reasons. For example, experimental subjects tend to view the efficacy of a drug differently depending on its description:
• 100 patients took the medicine, and 68 patients found it beneficial, or
• 100 patients took the medicine, and 32 patients saw no improvement.

Again, such differentiation must stem failures of extensionality, whereby agents understanding different descriptions of the same event as different events.

**The Conjunction Fallacy.** Tversky and Kahneman (1983) provided subjects with the following vignette:

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. Linda is a teacher in elementary school. Linda works in a bookstore and takes Yoga classes.

and asked them rank statements in order of likelihood including the following three:

\[ F = \text{“Linda is active in the feminist movement.”} \]

\[ t^o = \text{“Linda is a bank teller.”} \]

\[ t^o \wedge F = \text{“Linda is a bank teller and is active in the feminist movement.”} \]

85% of subjects ranked \( F \) more likely that \( t^o \wedge F \) more likely than \( t^o \), i.e., that \( \ell(t(F)) > \ell(t(t^o \wedge F)) > \ell(t(t^o)) \).

A common explanation for this behavior is subjects perceives the statement “Linda is a bank teller” as “Linda is a bank teller and not active in the feminist movement” and makes the analogous inference for \( F \). In other words, subjects mistakenly believe there is a tacit qualification in each statement. We can represent this misunderstanding via a subjective interpretation of uncertainty \((W, t)\) with \( W = \{w_1, w_2, w_3, w_4\} \) and \( t : F \mapsto \{w_1\}, t^o \wedge F \mapsto \{w_2\} \) and \( t^o \mapsto \{w_3\} \).

**Support Theory and Unpacking.** Tversky and Koehler (1994) collect many examples of unpacking wherein decision makers estimate the probability an implicit disjunction (e.g., death from a natural cause) to be less than the sum of its mutually exclusive components (e.g., death from heart disease, cancer, or other natural causes). In other words, subjects assessed the collective probability of a set of specific statements as more likely than a statement comprising their disjunction.

This can be captured by a subject who correctly understands that each component statement is contained in the more general case, but fails to properly account for the mutual exclusivity of statements. Consider the following:
H = “The primary cause of death is heart disease.”
C = “The primary cause of death is cancer.”
H \land C = “The primary cause of death is heart disease and the primary cause of death is cancer.”
N = “The primary cause of death was natural.”

and assume that the subject understands that H \land C is impossible, so that t(H \land C) = \emptyset, and that both H and C imply N: t(H) \subseteq t(N) and t(C) \subseteq t(N).

Now, if t distributes over logical conjunction, it follows that t(H) \cap t(C) = \emptyset, and so it must be that U(H) + U(C) \leq U(N). However, consider (W, t) with W = \{w_1, w_2, w_3\} and t: H \mapsto \{w_1, w_2\}, C \mapsto \{w_2, w_3\}, H \land C \mapsto \emptyset and N \mapsto W:

Here, t is not \land-distributive, so that although the subject, in direct contemplation of H \land C recognizes the statements as disjoint, she implicitly considers states where neither statement is ruled out.

**Violations of Independence & The Sure Thing Principle.** Violations of the independence axiom, or the highly related sure thing principle (Savage, 1954), are often associated with failures of probabilistic reasoning. Esponda and Vespa (2019) show that failures of Savage’s sure-thing-principle are inextricably related to failures of contingent reasoning.

Consider the simple Ellsberg single urn thought experiment: There is an urn with 100 balls, each of which can be white or black, with unknown proportion. A single ball is drawn from the urn, and subjects evaluate bets based on the Language constructed from the following atomic statements:

w = “The drawn ball is white.”
¬w = “The drawn ball is not white.”
w \lor ¬w = “The drawn ball is white or it is not white”

A common pattern of choice is as follows: U(w) = U(¬w) < \frac{1}{2}U(w \lor ¬w).

This pattern can be explained by an IOU in which there are states where neither w nor its negation are true: for example W = \{w_1, w_2, w_3\} and t: w \mapsto w_1, t: ¬w \mapsto w_2 and t: \{w, ¬w\} \mapsto W.

Here the subject understands that w and ¬w are mutually exclusive, and further understands that together they are always true. However, the subject is not able to cleanly delineate the boundary between w and ¬w, and so, entertains states where neither are considered true.
Notice also this provides where an agent’s world view becomes completely rational subsequent to obtaining some information. Conditioning on $w$ she considers only state $w_1$, believing $w$ and $w \lor \neg w$ to be true; conditioning on $\neg w$ she considers only state $w_3$, believing $\neg w$ and $w \lor \neg w$ to be true. Nonetheless, her ex-ante preferences violate $N$ and $D$.

**Redundant Evidence.** In a recent working paper, Garfagnini and Walker-Jones (2023) endow subjects with a lottery that has a $x \in \{0, 1, 2, \ldots, 100\}$ percent chance of paying $20 and a $1 - x$ chance of paying nothing. Subjects do not know $x$, but know that each value of $x$ is equally likely. The experimenters elicit from the subjects their probability equivalent, that is, the minimum $y \in [0, 100]$ such that they are willing to trade the unknown (or compound) lottery for a $y$ percent chance of $20. This equivalent is elicited before and after providing the subjects with various pieces of information about the initial, endowed, lottery of the form

$$P_n = \text{“}x \text{ is greater than } n\text{”}$$

Within this rather simple environment, subjects displayed clear departures from rational behavior. In particular, they often changed their valuation upon receiving redundant evidence. Specifically, they are initially told $P_n$, reveal a probability equivalent $y$, and are subsequently told $P_{n'}$, with $n' \leq n$, and reveal a new probability equivalent $y' \neq y$.

Here, clearly, subjects behavior is inconsistent with an understanding that $P_n$ implies $P_{n'}$. After conditioning on $P_n$, the subjects should consider only states where $x \geq n$, and therefore, the additional revelation of $P_{n'}$ provides no new information. There are two (not-necessarily mutually exclusive) explanations: the first is that in the initial state-space, there are inconsistent states, akin to the states in the conjunction fallacy example, where $P_n$ is true, but $P_{n'}$ is false. The second explanation is that subjects do not update by excluding states, but use some other rationally-limited heuristic.

A further result from the authors suggests strongly that the latter is a contributing factor: subjects often revise their probability equivalents in the opposite direction as expected. Over 60% of subjects with an baseline probability equivalent $y$ lowered their response to $y' < y$ after being informed of $P_{20}$ (or erred analogously in the opposite direction). While properly disentangling static v. dynamic rationality would require a more intentional task, this evidence indicates that reducing uncertainty is not necessarily accompanied by higher standard of rational thinking.

**What you see is all there is.** The following is a slight adaption of Enke (2020): Subjects were presented with bets regarding 6 independent, uniform draws from

$$X = \{-3, -2, -1, 1, 2, 3\}.$$
Subjects observed the first draw, and subsequently indicated whether they believe the average was positive or negative. Thereafter, they observe additional signals by interacting with a computerized information source that (transparently) shares all signals that “align” with their first stated belief (e.g., are positive if the first belief is positive) but not all signals that “contradict” the first belief. Subjects then must again state their beliefs about the average. Whenever subjects’ first signal is positive, their final stated beliefs tend to be upward biased, as if they were ignoring the information contained in *not observing* a given signal (and conversely for initially negative assessments).

Let \( X^+ = \{1, 2, 3\} \) and \( X^- = \{-1, -2, -3\} \), and without loss of generality assume this initial draw is \( x \in X^+ \). Further assume the participant guesses that the average is above positive.

Consider the following statements

\[
\text{OBS}_n = \text{“} n \text{ signals are observed”}
\]

\[
\text{NEG}_{5-n} = \text{“} 5 - n \text{ signals are negative”}
\]

Then it seems that participant’s behavior is being driven by a failure to perceive the equivalence “\( \text{OBS}_n \) if and only if \( \text{NEG}_{5-n} \)” for each \( n \leq 5 \).

To see how this failure of contingent thinking can be captured by an IOU, consider the following state-space: \( W = \{ \pi : X \to \mathbb{N} \mid \sum_X \pi(x) = 5 \} \), denote the state-space, where each state is identified with a function \( \pi \) such that \( \pi(x) \) counts the number of draws of \( x \).

Thus, \( t \) maps the statement, for \( x \in X \) and \( m \leq 5 \),

\[
d_{x,m} = \text{“there were } m \text{ draws of } x\text{”}
\]

to the event \( \{ \pi \in W \mid \pi(x) = m \} \). Now consider the statement, for \( x \in X^+ \) and \( m \leq 5 \),

\[
o_{x,m} = \text{“there were } m \text{ observations of } x\text{”}
\]

For a purely rational subject this should map to the event \( \{ \pi \in W \mid \pi(x) = m \} \); however, a participant who does not intuit that the missing signals must have been negative will understand this statement as the event \( t : o_{x,m} \mapsto \{ \pi \in W \mid \pi(x) \geq m \} \).

So long as the participants are otherwise rational, they will understand the statement \( \text{OBS}_n \) as mapping to the union of all different ways they could have observed \( n \) signals, and likewise the statement \( \text{NEG}_{5-n} \) as the union of the different ways of have drawn \( 5 - n \) negative draws. That is:
\[ t(\text{OBS}_n) = \bigcup \left\{ \bigcap_{x \in X^+} t(O_{x,m_x}) \mid \sum_{x \in X^+} m_x = n \right\} \]

and

\[ t(\text{NEG}_{5-n}) = \bigcup \left\{ \bigcap_{x \in X^-} t(D_{x,m_x}) \mid \sum_{x \in X^-} m_x = 5 - n \right\} \]

It is straightforward to see that for the rational subject, these two sets coincide, whereas for the subject with the flawed \( t \) as given above, \( t(\text{NEG}_{5-n}) \subset t(\text{OBS}_n) \).

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## A Proofs

### A.1 Preliminaries: Taxonomy of Filters

Before proving things, we set some preliminary notation, to be used throughout this appendix.

For any partial order, $R$, over a set $X$, generalize the language from the body of the paper: call $x \in X$ null if $xRy$ for all $y \in X$ and call $x,y$ disjoint, written $x \perp y$, if there exists no non-null $z$
such that $zRx$ and $zRy$. A subset $Y \subseteq X$ is an $R$-up-set (or, simply up-set when the relation is obvious) if whenever $x \in Y$ and $xRy$ then $y \in Y$. A non-empty up-set $Y$ is called a $R$-filter, (or filter) whenever $x, y \in Y$ then there exists a $z \in Y$ such that $zRx$ and $zRy$. A filter is called proper if it does not contain any null elements.\footnote{Our definition of proper differs slightly from the standard usage, where a proper filter is one that is not the totality of $X$. Should there exist any null element, these two definitions coincide. However, when there are no null elements (for example $Z$ under $\leq$), then we allow the full set $X$ to be proper.} Let $\mathfrak{F}(R, X)$ collect all proper filters over $R$. It is easy to verify that $\mathfrak{F}(R, X)$ is closed under intersections. A proper filter is called an ultrafilter if it cannot be extended (by set inclusion) to any other proper filter. Let $\mathfrak{U}(R)$ collect all ultrafilters over $R$.

We say that a proper filter $f \in \mathfrak{F}(R, X)$ is irreducible, if, for every $g_1, g_2 \in \mathfrak{F}(R, X)$,

$$g_1 \cap g_2 = f \implies g_1 \subseteq f \text{ or } g_2 \subseteq f.$$  
Similarly, we say that $f$ is prime, if, for every $g_1, g_2 \in \mathfrak{F}(R, X)$,

$$g_1 \cap g_2 \subseteq f \implies g_1 \subseteq f \text{ or } g_2 \subseteq f.$$  
Let $\mathfrak{F}(R, X)$ and $\mathfrak{P}(R, X)$ collect all irreducible and prime filters, respectively, of $R$ on $X$.

Say that $R$ has meets if for all $x, y \in X$ such that there exists a $z$ such that $zRx$ and $zRy$, then there exists an element $x \land y \in X$, referred to as the meet of $x$ and $y$, such that $zRx$ and $zRy$ if and only if $zR(x \land y)$. The following are immediate to derive: $(x \land y) = (y \land x)$, $(x \land x) = x$, $(x \land y)Ry$, and if $yRy'$ then $(x \land y)R(x \land y')$.

Say that $R$ has covers if for all $x, y \in X$ there exists an element $x \lor y \in X$, referred to as a cover of $x$ and $y$, such that $zR(x \lor y)$, $yR(x \lor y)$ and such that $z \land x$ and $z \land y$ if and only if $z \land (x \lor y)$.

**Lemma 13.** Let $R$ be a partial order over a set $X$ that has meets. Let $f \in \mathfrak{F}$ and $z \in X$ be such that $z$ is not disjoint from any $x \in f$. Then

$$f_z = \{y \in X \mid (x \land z)Ry, x \in f\}$$

is a proper filter containing $f$.

**Proof.** Let $y \in f_z$ and $yRy'$. Then there is some $x \in f$ such that $(x \land z)RyRy'$. $f_z$ is an up-set. Now let $y, y' \in f_z$ so that there is some $x, x' \in f$ such that $(x \land z)Ry$ and $(x' \land z)Ry'$. Since $f$ is a filter, there exists some $x'' \in f$ such that $x''Rx$ and $x''Rx'$. By the properties of meets, $(x'' \land z)R(x'' \land z)Ry$ and $(x'' \land z)R(x'' \land z)Ry'$. Since $(x \land z)Rx$, we have that $f \subseteq f_z$. \(\blacksquare\)

**Lemma 14.** Let $R$ be a partial order over a set $X$ and let $f \in \mathfrak{F}$ and $x \in X$ be such that $x \notin f$. Then there exists an irreducible filter, $f^* \in \mathfrak{F}(R, X)$, such that $f \subseteq f^*$ and $x \notin f^*$. Moreover, if $R$ has meets and covers, and if $x \land y_0$ for some $y_0 \in f$, then it can be assumed that $f^*$ is prime: $f^* \in \mathfrak{P}(R, X)$.

**Proof.** Let $F = \{f' \in \mathfrak{F} \mid f \subseteq f', x \notin f'\}$, ordered by set-inclusion. Let $C$ be a chain of $F$ and let $c = \bigcup C$. We claim that $c$ is an upper-bound for $C$, which clearly follows if $c \in F$. That $f \subseteq c$ and $x \notin c$ are immediate. Clearly $c$ is an $R$-up-set, and if $y', y'' \in c$ then there exists some $f', f'' \in C$ such that $y' \in f'$ and $y'' \in f''$. Since $C$ is a chain, either $f' = f'' \cup f'$ or $f'' = f' \cup f''$; in either case $f \cup f'' \subseteq c$ is a filter containing $y'$ and $y''$, and so contains some $z$ such that $zRy'z$ and $zRy''$. Thus $c$ is a filter, and hence in $F$.

We can thus apply Zorn’s lemma to find a maximal element, $f^*$, of $F$. We must show that $f^*$ is irreducible. Since $f^*$ is maximal in $F$, it follows that for any $f' \in \mathfrak{F}(R)$ if $f^* \subseteq f'$ then $x \in f'$. Thus $x \in \bigcap\{f' \in \mathfrak{F}(R) \mid f \subseteq f'\}$ but $x \notin f^*$; $f^*$ is irreducible.

Now assume in addition that $R$ has meets and covers and that $x \land y_0$ for some $y_0 \in f$. First, we claim that for each $z \notin f^*$, $z$ is disjoint to some element of $f^*$. Assume to the contrary this was not the case, then we have by Lemma 13 that $f^*_z = \{y' \in X \mid (x \land z)Ry', y \in f^*\}$ is a filter containing $f^*$,
so by our assumption of maximality, must contain \( x \) (if it did not contain \( x \), it would be proper filter extending \( f' \)). So then: \((y' \odot z)R x\) for some \( y \in f'\). Since \( f' \) is a filter, there exists some \( y' \in f' \) such that \( y'R y_0 \) and \( y'R y_0 \); moreover, \((y' \odot z)\) is not null since \( y' \) is not disjoint from \( z \), by assumption. Finally, we have \((y' \odot z)R (y' \odot z)Rx\) and \((y' \odot z)R y'R y_0\), contradicting the assumption that \( y_0 \) was disjoint from \( x \).

We now show that \( f' \) is prime. To see this: pick two filters \( g_1, g_2 \in \mathfrak{F}(R, X) \) where \( g_1 \cap g_2 \subseteq f' \). Then, proceed by contradiction and suppose that \( g_1 \not\subseteq f' \) and \( g_2 \not\subseteq f' \) we can then pick \( z_1 \in g_1 \setminus f' \) and \( z_2 \in g_2 \setminus f' \) (notice that \( z_1 \neq z_2 \)). Then, as seen in the prior claim, there exists \( y_1, y_2 \in f' \) such that \( y_1 \perp z_1 \) and \( y_2 \perp z_2 \). Since \( f' \) is a filter, there is some \( y' \in f' \) where \( y'R y_1 \) and \( y'R y_2 \) and notice, then, \( y' \perp z_1 \) and \( y' \perp z_2 \). There exists some cover of \( z_1 \) and \( z_2 \), \( (z_1 \cup z_2) \), and we have \( y' \perp (z_1 \cup z_2) \). Moreover, since \( z_1 R (z_1 \cup z_2) \) and \( z_2 R (z_1 \cup z_2) \), we have \((z_1 \cup z_2) \in g_1 \cap g_2 \subseteq f' \), a contradiction, since it is disjoint to \( y' \in f' \). ■

**Lemma 15.** If \( f \in \mathfrak{F}(R, X) \) is an irreducible filter, then for \( g_1 \ldots g_n \in \mathfrak{F}(R, X) \), with \( n \geq 2 \),

\[
\bigcap_{i \leq n} g_i = f \text{ implies } g_i = f \text{ for some } i \leq n
\]

**Proof.** We prove this by induction on \( n \). The base cases, for \( n = 2 \) is irreducibility. Assume this holds for all intersections of \( n - 1 \) and fix some \( g_1 \ldots g_n \in \mathfrak{F}(R, X) \) such that \( \bigcap_{i \leq n} g_i = f \). Clearly, for each \( i \leq n \), \( f \subseteq g_i \). Let \( g = \bigcap_{2 \leq i \leq n} g_i \); since the intersections of filters is a filter, the fact that \( g_1 \cap g = f \) implies via irreducibility that either \( g_1 \subseteq f \) or \( g \subseteq f \). By the inductive hypothesis, we have the result. ■

**A.2 Properties of Implication and Interpretation**

**Lemma 16.** If \( \Rightarrow \) is reflexive and transitive then \( \Rightarrow \) is reflexive and transitive.

**Proof.** We must show \( \Rightarrow \) is transitive (reflexivity is immediate): so let \( \Phi \Rightarrow \Psi \) and \( \Psi \Rightarrow \Lambda \). Thus we have, for \( x > 0 \), and \( \Gamma \in \mathcal{K} \):

\[
x_{\Delta \cup \Gamma} \sim x_{\Delta \cup \Psi \cup \Gamma} = x_{\Delta \cup \Psi} \cup \Gamma \quad \text{(since } \Psi \Rightarrow \Lambda) \\
x_{\Delta \cup \Psi \cup \Phi \cup \Gamma} \sim x_{\Delta \cup \Phi \cup \Gamma} \quad \text{(since } \Phi \Rightarrow \Psi) \\
x_{\Delta \cup \Phi \cup \Gamma} \quad \text{(since } \Psi \Rightarrow \Lambda)
\]

and so \( \Phi \Rightarrow \Lambda \), as needed. ■

**Lemma 17.** Let \( \Rightarrow \) be a reflexive and transitive, then the following hold

(i) For any \( \Rightarrow \), \( \Phi \Rightarrow \Psi \) if and only if \( \Phi \cup \Psi \Rightarrow \Psi \).

(ii) \( \Phi \subseteq \Psi \), then \( \Phi \Rightarrow \Psi \).

(iii) \( \Phi \downarrow \Psi \) and \( \Phi' \Rightarrow \Phi \) implies \( \Phi' \downarrow \Psi \).

(iv) \( \Phi \Rightarrow \Psi \) and \( \Psi \Rightarrow \Phi \) if and only if \( \Phi \cup \Phi' \Rightarrow \Psi \).

**Proof.** (i) follows immediately from the fact that, for all \( x \in R \) and \( \Gamma \in \mathcal{K} \): \( x_{\Phi \cup \Phi \cup \Psi \cup \Gamma} = x_{\Phi \cup (\Phi \cup \Psi \cup \Gamma)} \). (ii) follows from reflexivity of \( \Rightarrow \) and part (i). To see (iii), let \( \Gamma \Rightarrow \Psi \), and \( \Gamma \Rightarrow \Phi' \), then by the transitivity of \( \Rightarrow \), \( \Gamma \Rightarrow \Phi \), so by the disjointness of \( \Phi \) and \( \Psi \), \( \Gamma \) must be null.

The ‘if’ direction of (iv) follows from (ii) and transitivity. Towards the only if, let \( \Phi \Rightarrow \Psi \) and \( \Phi' \Rightarrow \Psi \). Then we have by definition of \( \Rightarrow \), that for all \( x \in R \) and \( \Gamma \in \mathcal{K} \), that \( x_{\Phi \cup \Gamma} \sim x_{\Phi \cup \Phi' \cup \Gamma} \) and \( x_{\Phi \cup (\Phi \cup \Psi) \cup \Gamma} \sim x_{\Phi \cup \Phi \cup (\Phi' \cup \Gamma)} \). Thus, by transitivity, we have \( x_{\Phi \cup \Gamma} \sim x_{\Phi \cup \Phi \cup (\Phi' \cup \Gamma)} \), or that \( \Phi \cup \Phi' \Rightarrow \Psi \). ■

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By Lemma 16, $\Rightarrow$ is a pre-order, and thus, subjective equivalence is an congruence relation. Let \[ [\Phi] = \{ \Psi \in K \mid \Psi \Rightarrow \Phi \text{ and } \Phi \Rightarrow \Psi \} \]
collect the equivalence class containing $\Phi$. In an abuse of notation, we write $[\varphi]$ rather than $[\{\varphi\}]$ when regarding singletons. Let $K^\dagger = \{ [\Phi] \subseteq K \mid \Phi \in K \}$ denote the quotient of $K$ under subjective equivalence. $K^\dagger$ inherits the pre-order, $\Rightarrow$, as $[\Phi] \Rightarrow [\Psi]$ if $\Phi \Rightarrow \Psi$; over $K^\dagger$, $\Rightarrow$ is partial order.

For each $\Phi \in K^\dagger$, let $f_\Phi = \{ [\Psi] \in K^\dagger \mid \Phi \Rightarrow [\Psi] \}$. It is trivial to show that $f_\Phi$ is a filter and it is proper if and only if $\Phi$ is not null.

**Lemma 18.** If $\Rightarrow$ is articulate, then the partial order $\Rightarrow$, over $K^\dagger$, has meets and covers.

**Proof.** Let $[\Phi], [\Phi'] \in K^\dagger$ be such that there exists some $[\Psi] \in K^\dagger$ such $[\Psi] \Rightarrow [\Phi]$ and $[\Psi] \Rightarrow [\Phi']$. Now let $\Psi^* = \{ \psi \in \mathcal{L} \mid \psi \Rightarrow \Phi \text{ and } \psi \Rightarrow \Phi' \}$ which is non-empty by our assumption and Lemma 17.

We claim that $[\Psi^*]$ is a meet for $[\Phi], [\Phi']$. To see that this is indeed a meet, let $[\Psi^*] \subseteq K^\dagger$ such $[\Psi^*] \Rightarrow [\Phi]$ and $[\Psi^*] \Rightarrow [\Phi']$. Then by Lemma 17(ii), we have $[\psi] \Rightarrow [\Phi]$ and $[\psi] \Rightarrow [\Phi']$ for all $\psi \in \Psi^*$, and thus, $\Psi^* \subseteq \Psi^*$. It follows from yet one more application of Lemma 17(ii) that $[\Psi] \Rightarrow [\Psi^*]$.

It remains to show that $[\Psi^*] = [\Phi]$ and $[\Psi^*] = [\Phi']$. Let $\Gamma \subseteq \Phi$, then we have that $\Gamma \subseteq \Psi^*$ for all $\psi \in \Psi^*$ by Lemma 17(iii). Thus, by part (ii) of articulation, we have $\Gamma \subseteq \Psi^*$. Thus, there is no non-null $\Gamma$ such that $\Gamma \Rightarrow \Psi^*$ and $\Gamma \subseteq \Phi$. By (the contra-positive of) part (i) of articulation, $\Psi^* \Rightarrow \Phi$, as needed. An analogous argument shows $\Psi^* \Rightarrow \Phi'$ as well.

It is trivial to check that for any $\Phi, \Psi \in K$, part (ii) of articulation (and Lemma 17) implies $[\Phi \cup \Psi]$ is a cover for $[\Phi]$ and $[\Psi]$.

**Definition 13** (Canonical interpretation). Let $\Rightarrow$, defined over syntactic acts, be reflexive and transitive. Then, the canonical interpretation of $\Rightarrow$ is the pair $(W^*, t^*)$, where:

\[
W^* := \begin{cases} 
\mathcal{J}(\Rightarrow, K^\dagger) & \text{if } \Rightarrow \text{ is not articulate,} \\
\mathcal{Q}(\Rightarrow, K^\dagger) & \text{if } \Rightarrow \text{ is articulate,} 
\end{cases}
\]

and $t^* : K \to 2^{W^*}$, where $t^*(\Phi) := \{ w \in W^* \mid [\Phi] \in w \}$ for every $\Phi \in K$.

**Proof of Theorem 1.** We will now show the three properties of faithfulness hold for the canonical interpretation:

(i) Let $\Phi \Rightarrow \Psi$, and $w \in t^*(\Phi)$. Then by construction $[\Phi] \in w$, and so since $w$ is an $\Rightarrow$-up-set, $[\Psi] \in w$, and therefore, $w \in t^*(\Psi)$, so $t^*(\Phi) \subseteq t^*(\Psi)$.

Now assume $\Phi \Rightarrow \Psi$. Clearly $[\Phi] \in f_\Phi$ and $[\Psi] \notin f_\Phi$, and so, by Lemma 14, we have the existence of some irreducible filter $w \in \mathcal{J}(\Rightarrow, K^\dagger)$ such that $[\Phi] \in w$ and $[\Psi] \notin w$. So if $\Rightarrow$ is not articulate, then $w \in t^*(\Phi)$ and $w \notin t^*(\Psi)$, or, $t^*(\Psi) \not\subseteq t^*(\Phi)$.

If in addition $\Rightarrow$ is articulate, then $\Phi \Rightarrow \Psi$ implies the existence of some $\Gamma$ such that $\Gamma \Rightarrow \Phi$ and $\Gamma \subseteq \Psi$. Consider $f_\Gamma$, which is $\Rightarrow$-up-cover, and $\Gamma \in f_\Gamma$. By Lemma 18, $\Rightarrow$ (over $K^\dagger$) has meets and covers, and $\Gamma \in f_\Phi$ is disjoint from $\Psi$, so by Lemma 14 we have the existence of some prime filter $w' \in \mathcal{Q}(\Rightarrow, K^\dagger)$ such that $[\Phi] \in w'$ and $[\Psi] \notin w'$. So if $\Rightarrow$ is articulate, then $w' \in t^*(\Phi)$ and $w' \notin t^*(\Psi)$, or, $t^*(\Psi) \not\subseteq t^*(\Phi)$.

(ii) If $\Phi$ is null then $f_\Phi = K^\dagger$. As such, no proper (hence, irreducible or prime) filter can contain $[\Phi]$: $t^*(\Phi) = \emptyset$.

Now assume $\Phi$ is not null. Thus there exists some $\Psi$ such that $\Phi \Rightarrow \Psi$. By part (i), $t^*(\Phi) \not\subseteq t^*(\Psi)$ and thus, $t^*(\Phi) \neq \emptyset$.

(iii) Now assume $w \in t^*(\Phi) \cap t^*(\Psi) \neq \emptyset$. By construction $w$ is a proper filter that contains $[\Phi]$ and $[\Psi]$. By virtue of being a filter, $w$ contains some $[\Gamma]$ such that $\Gamma \Rightarrow \Phi$ and $\Gamma \Rightarrow \Psi$, and by construction, $w \in t^*(\Gamma)$.
Lemma 19. If \((W,t)\) is a faithful interpretation of \(\triangleright\), then for all \(w \in W\),

\[
\{ \Phi \} \in \mathcal{K}^{\downarrow} \mid w \in t(\Phi) \}
\]

is a proper \(\triangleright\)-filter.

Proof. Let \([\Phi] \in \{ w \}\) and \(\Phi \triangleright \Psi\). Then \(w \in t(\Phi) \subseteq t(\Psi)\), by property (i) of faithfulness. So \([\Phi] \in \{ w \}\); \([\Phi]\) is an \(\triangleright\)-upset. Now let \([\Phi], [\Psi] \in \{ w \}\). So by property (iii) of faithfulness, we have that \(w \in t(\Gamma)\) for some \(\Gamma\) such that \(\Gamma \triangleright \Phi\) and \(\Gamma \triangleright \Psi\). Thus, \([\Gamma] \in \{ w \}\), and \([\Gamma] \triangleright [\Phi]\) and \([\Gamma] \triangleright [\Psi]\). That is \(\{ w \}\) is proper follows immediately from property (ii) of faithfulness.

Lemma 20. If \((W,t)\) is a faithful interpretation of \(\triangleright\), and \(\Gamma\) is such that \([\Gamma]\) is the meet of \([\Phi]\) and \([\Psi]\), then \(t(\Gamma) = t(\Phi) \cap t(\Psi)\).

Proof. By definition of a meet, we have \(\Gamma \triangleright \Phi\) and \(\Gamma \triangleright \Psi\). It follows by property (i) of faithfulness, \(t(\Gamma) \subseteq t(\Phi) \cap t(\Psi)\). Further, let \(w \in t(\Phi) \cap t(\Psi)\); by property (iii) of faithfulness, \(w \in t(\Gamma')\) for some \(\Gamma'\) such that \(\Gamma' \triangleright \Phi\) and \(\Gamma' \triangleright \Psi\). Thus, by definition of a meet, \(\Gamma' \triangleright \Gamma\), so by property (i) of faithfulness \(w \in t(\Gamma') \subseteq t(\Gamma)\). It follows that \(t(\Phi) \cap t(\Psi) \subseteq t(\Gamma)\).

Proof of Theorem 2. Fix \(\triangleright\). We will first define a maximal state-space: let \((W^+, t^+)\) be defined as

\[
W^+ := \mathcal{F}(\triangleright, \mathcal{K}^{\downarrow})
\]

and \(t^+ : \mathcal{K} \to 2^{W^+}\) as \(t^+(\Phi) := \{ w \in W^m \mid [\Phi] \in w \} \) for every \(\Phi \in \mathcal{K}\). It is a routine translation of the proof of Theorem 1 to verify that this is indeed a faithful interpretation of \(\triangleright\). Given this, the maximality of \((W^+, t^+)\) follows immediately from Lemma 19.

Now, assume \((W,t)\) extends some \((W^-, t^-)\) and is extended by some \((W^+, t^+)\), both faithful to \(\triangleright\). We must show that \((W,t)\) is faithful \(\triangleright\). This is routine, so we will show only property (i), omitting the others:

Let \(\Phi \triangleright \Psi\). Let \(w \in W\) be such that \(w \in t(\Phi)\). Then there exists a \(w^+ \in W^+\) with \(\{ w^+ \} = \{ w \}\). By the faithfulness of \((W^+, t^+)\), we have \(\Psi \in \{ w^+ \}\), so \(w \in t(\Psi)\), and \(t(\Phi) \subseteq t(\Psi)\).

Let \(\Phi \not\triangleright \Psi\). By faithfulness, there exists a \(w^- \in W^-\) with \(w \in t^-(\Phi) \setminus t^-(\Psi)\). Then there exists a \(w \in W\) with \(\{ w \} = \{ w^- \}\). So \(w \in t(\Phi) \setminus t(\Psi)\), and \(t(\Phi) \not\subseteq t(\Psi)\).

Proof of Theorem 3. Assume further that \(\mathcal{K}^{\downarrow}\) is finite and consider the canonical interpretation \((W^*, t^*)\).

That \((W^*, t^*)\) is redundancy-free follows immediately from the definition of irreducible filters (and the fact that, since \(\mathcal{K}^{\downarrow}\) is finite, \(\mathcal{F}(\triangleright, \mathcal{K}^{\downarrow})\) is as well).

To see that \((W^*, t^*)\) is also minimal, let \((W,t)\) denote any faithful interpretation of \(\triangleright\) and \(\Phi \in W^*\) be an irreducible filter. We will show that \(\{ \Psi \} \in \mathcal{K}^{\downarrow}\) for some \(\Psi \in W\). Since \(\mathcal{K}^{\downarrow}\) is finite, \(\Phi\) is of the form \(\{ \Psi \} \in \mathcal{K}^{\downarrow}\) for some non-null \(\Phi\) (i.e., all filters are principal).

Now, for all \(w \in t(\Phi)\), \(\{ w \}\) is a proper filter containing \(\Phi\) (by Lemma 19) and thus \(\Phi \not\triangleright \Gamma\). Moreover, if \([\Gamma] \in \bigcap_{w \in t(\Phi)} \{ w \}\) then \(t(\Phi) \subseteq t(\Gamma)\). By faithfulness, therefore, \(\Phi \triangleright \Gamma\), and so \([\Gamma] \in \mathcal{F}(\triangleright, \mathcal{K}^{\downarrow})\). Together these observations imply

\[
\{ w \} = \bigcap_{w \in t(\Phi)} \{ w \}.
\]

Since \(\mathcal{F}(\triangleright, \mathcal{K}^{\downarrow})\) Lemma 15 delivers that \(\{ w \}\) for some \(w \in t(\Phi)\).

Finally, we show that \((W^*, t^*)\) is the unique redundancy-free interpretation. Indeed, let \((W,t)\) denote any other faithful interpretation and let \(w \in W\) be such that \(\{ w \} \not\in \mathcal{F}(\triangleright, \mathcal{K}^{\downarrow})\). It is an
intimdate corollary of Lemma 14 that every filter is equal to the intersection of all irreducible filters that contain it. Thus \( \langle w \rangle \) is the intersection of irreducible filters, all of which are correspond to states of \( W \) by the minimality of \( (W^*, t^*) \); \( w \) is redundant.

**Proof. Proof of Theorem 4.** Let \( (W, t) \) denote an arbitrary faithful interpretation of \( \gg \).

**(E)** Assume that \( \gg \) satisfies \( E \), and let \( \varphi \leftrightarrow \psi \). Then \( \varphi = \gg \psi \) and \( \psi = \gg \varphi \) by \( E \), so by property (i) of faithfulness, \( t(\varphi) \subseteq t(\psi) \) and \( t(\psi) \subseteq t(\varphi) \). \( t \) is exact.

Now assume that \( t \) is exact let \( \varphi \leftrightarrow \psi \). Then \( t(\psi) = t(\varphi) \). So by property (i) of faithfulness, \( \varphi = \gg \psi \) and \( \psi = \gg \varphi \), \( \gg \) satisfies \( E \).

**(I)** Assume \( \gg \) satisfies \( I \) and let \( \varphi \Rightarrow \psi \). Then \( \varphi = \gg \psi \) by \( I \), so by property (i) of faithfulness, \( t(\varphi) \subseteq t(\psi) \). \( t \) is monotone.

Now assume that \( t \) is monotone let \( \varphi \Rightarrow \psi \). Then \( t(\psi) \subseteq t(\varphi) \). So by property (i) of faithfulness, \( \varphi = \gg \psi \), \( \gg \) satisfies \( I \).

**(C)** Assume \( \gg \) satisfies \( C \). Then \( [\varphi \wedge \psi] \) is a meet for \([\varphi]\) and \([\psi]\); that is \( \wedge \)-distributive follows from Lemma 20.

Now assume \( t \) is \( \wedge \)-distributive and let \( \Gamma \gg \varphi \) and \( \Gamma = \gg \psi \), so \( t(\Gamma) \subseteq t(\varphi) \cap t(\psi) = t(\varphi \land \psi) \), so \( \Gamma \gg \varphi \land \psi \). Now let \( \Gamma \gg \varphi \land \psi \). or, \( t(\Gamma) \subseteq t(\varphi \land \psi) = t(\varphi) \cap t(\psi) \), implying \( \Gamma \gg \varphi \) and \( \Gamma \gg \psi \). So \( \gg \) satisfies \( C \).

**Proof. Proof of Theorem 5.** We will show that the canonical interpretation is set-consistent. First notice that Lemma 17(iv) implies that \( f_{\Phi \cup \Psi} = f_{\Phi} \cap f_{\Psi} \).

Now, for any \( w \in W^* \), we know that \( w \in t^*(\Phi \cup \Psi) \) if and only if \( \Phi \cup \Psi \models w \), that is, if and only if \( f_{\Phi \cup \Psi} \subseteq w \). As \( w \) is a prime filter, the latter holds if and only if \( f_{\Phi} \subseteq w \), or \( f_{\Psi} \subseteq w \), and hence, if and only if \( w \in t^*(\Phi) \) or \( w \in t^*(\Psi) \). Thus, we conclude that \( t^*(\Phi \cup \Psi) = t^*(\Phi) \cup t^*(\Psi) \). Hence \( t^* \) is set-consistent.

Now, let \( (W, t) \) denote an arbitrary set-consistent faithful interpretation of \( \gg \).

**(D)** Assume \( \gg \) satisfies \( D \). By the reflexivity of \( \gg \), we have that \( \varphi \land \psi \gg \varphi \land \psi \) and thus by \( D \), \( \varphi \gg \varphi \land \psi \) and \( \psi \gg \varphi \land \psi \). By Lemma 17(iv), \( \{\varphi, \psi\} \gg \varphi \lor \psi \). Similarly, \( \varphi = \gg \{\varphi, \psi\} \) and \( \psi = \gg \{\varphi, \psi\} \) by Lemma 17(ii), and thus by \( D \), \( \varphi \lor \psi = \gg \{\varphi, \psi\} \). Together, via property (i) of faithfulness, we have \( t(\varphi \land \psi) = t(\{\varphi, \psi\}) \). By set-consistency we can then conclude \( t(\varphi \lor \psi) = t(\{\varphi, \psi\}) \land t(\psi) : t \) is \( \lor \)-distributive.

Take some \( \Gamma \). Then \( \varphi = \gg \Gamma \) and \( \psi = \gg \Gamma \) if and only if \( t(\varphi) \subseteq t(\Gamma) \) and \( t(\psi) \subseteq t(\Gamma) \), if and only if \( t(\varphi \lor \psi) = t(\varphi) \cup t(\psi) \subseteq t(\Gamma) \) (by \( \lor \)-distributivity), if and only if \( \varphi \lor \psi = \gg \Gamma \). So \( \gg \) satisfies \( D \).

**(N)** Assume \( \gg \) satisfies \( N \): Since \( \varphi \land \neg \varphi \), no proper filter contains both \( \varphi \) and \( \neg \varphi \). Since every \( \Gamma = \gg \{\varphi, \neg \varphi\} \), every proper filter contains \( \{\varphi, \neg \varphi\} \). Thus by Lemma 19: \( t(\varphi) \cap t(\neg \varphi) = \emptyset \) and \( t(\{\varphi, \neg \varphi\}) = W \). The latter implies via set-consistency that \( t(\varphi) \cup t(\neg \varphi) = W \), and so \( t(\varphi), t(\neg \varphi) \) partitions \( W \), or, \( t \) is symmetric.

Now assume that \( t \) is symmetric. Then \( t(\varphi) \cap t(\neg \varphi) = \emptyset \) so by faithfulness, \( \varphi \land \neg \varphi \). Moreover, \( t(\{\varphi, \neg \varphi\}) = W \) so \( \Gamma = \gg \{\varphi, \neg \varphi\} \) for all \( \Gamma \in \mathcal{K} \).

**Proof of Theorem 6.** Let \( R^\varphi \) and \( \bar{R}^\varphi \) both be discriminate updating rules for \( (\gg, \gg^\varphi) \). Take any \( \Psi \in \mathcal{K} \). We have \( R^\varphi(\Psi) = R^\varphi(\bar{R}^\varphi(\Psi)) \) by (D3), and so, in particular \( R^\varphi(\Psi) = \gg \gg^\varphi(R^\varphi(\Psi)) \). Applying (1) twice, for \( R^\varphi \) and then \( \bar{R}^\varphi \), we conclude first \( \Psi = \gg^\varphi R^\varphi(\Psi) \) and then \( \bar{R}^\varphi(\Psi) = \gg \gg^\varphi(R^\varphi(\Psi)) \). So by (D1) and transitivity we obtain \( \bar{R}^\varphi(\Psi) = \gg R^\varphi(\Psi) \). A symmetric argument furnishes the opposing implication.
Proof of Theorem 7. (i) implies (iii). Let \( \mathcal{R}^\varphi \) be discriminate and let \( \vartriangleleft \mathcal{R}^\varphi \) be foreseen from \( \triangleright \) via \( \mathcal{R}^\varphi \). Set \( \Phi = t(\mathcal{R}^\varphi(\mathcal{L})) \). First note that (D1) and (D2) imply (D5).

Now let \( \Psi =_{\triangleright} \mathcal{R}^\varphi \Psi \) and \( \Gamma =_{\triangleright} \Phi \) and \( \Gamma =_{\triangleright} \Psi \). We have

\[
\Gamma = \mathcal{R}^\varphi(\Gamma) \\
=_{\triangleright} \mathcal{R}^\varphi(\Psi) \\
=_{\triangleright} \mathcal{R}^\varphi(\Psi') \\
=_{\triangleright} \Psi'
\]

(by (D3), since \( \Gamma =_{\triangleright} \Phi \))

(by (D5), since \( \Gamma =_{\triangleright} \Psi \))

(since \( \Psi =_{\triangleright} \mathcal{R}^\varphi \Psi' \))

(since (D1))

For the other direction, assume that for all \( \Gamma \in \mathcal{K} \), if \( \Gamma =_{\triangleright} \Phi \) and \( \Gamma =_{\triangleright} \Psi \), then \( \Gamma =_{\triangleright} \Psi' \). Notice that \( \mathcal{R}^\varphi(\Psi) =_{\triangleright} \Psi \) by (D1) and \( \Psi =_{\triangleright} \mathcal{L} \) so \( \mathcal{R}^\varphi(\Psi) =_{\triangleright} \Phi \) by (D5). Thus by assumption we have \( \mathcal{R}^\varphi(\Psi) =_{\triangleright} \Psi' \), yielding \( \mathcal{R}^\varphi(\Psi) =_{\triangleright} \mathcal{R}^\varphi(\Psi') \) by (D2), as desired.

(iii) implies (ii). Set \( \mathcal{W}^\varphi = t(\Phi) \), where \( \Phi \) is the given by (iii). We must show that \( (\mathcal{W}^\varphi, \Psi \rightarrow t(\Psi) \cap \mathcal{W}^\varphi) \) is a faithful interpretation of \( \triangleright \mathcal{R}^\varphi \). Set \( \mathcal{W}^\varphi_t \) as the map \( \Psi \rightarrow t(\Psi) \cap \mathcal{W}^\varphi \).

(i) First let \( \Psi =_{\triangleright} \mathcal{W}^\varphi \Psi' \). Let \( w \in t^\varphi(\Psi) = t(\Psi) \cap t(\Phi) \). Thus by property (iii) of faithfulness, \( w \in t(\Gamma) \) for some \( \Gamma \) such that \( \Gamma =_{\triangleright} \Psi \) and \( \Gamma =_{\triangleright} \Phi \). Thus by our assumption, \( \Gamma =_{\triangleright} \Psi' \). Therefore, by our assumption, \( \Phi =_{\triangleright} \mathcal{W}^\varphi \Psi \).

(ii) Let \( t^\varphi(\Psi) = \emptyset \), then by (i), \( \Psi =_{\triangleright} \mathcal{W}^\varphi \) null.

Let \( t^\varphi(\Psi) \neq \emptyset \), so then there is some \( w \in t(\Phi) \cap t(\Psi) \). By property (iii) of faithfulness, \( w \in t(\Gamma) \) for some \( \Gamma \) such that \( \Gamma =_{\triangleright} \Psi \) and \( \Gamma =_{\triangleright} \Phi \). This in particular implies \( \emptyset \neq t^\varphi(\Gamma) \subseteq t^\varphi(\Psi) \), indicating that \( \Psi \) is not \( =_{\triangleright} \mathcal{W}^\varphi \) null.

(iii) Let \( w \in t^\varphi(\Psi) \cap t^\varphi(\Psi') \). Then \( w \in t(\Psi) \cap t(\Psi') \) so property (iii) of faithfulness, \( w \in t(\Gamma) \) for some \( \Gamma \) such that \( \Gamma =_{\triangleright} \Psi \) and \( \Gamma =_{\triangleright} \Phi \). We have shown, similarly, \( w \in t(\Gamma') \) for some \( \Gamma' =_{\triangleright} \Psi' \) and \( \Gamma' =_{\triangleright} \Phi \). This indicates that \( t(\Gamma') \subseteq t(\Phi) \) and \( t(\Psi) \cap t(\Psi') \subseteq t(\Psi) \cap t(\Psi') \cap t(\Gamma') = t^\varphi(\Psi) \cap t^\varphi(\Psi') \). So by (i) \( \Gamma' =_{\triangleright} \mathcal{W}^\varphi \Psi' \). Therefore, by our assumption, \( \Phi =_{\triangleright} \mathcal{W}^\varphi \Psi' \), as desired.

(i) implies (ii) (given \( \triangleright \) is articulate). Let \( \mathcal{W}^\varphi \subseteq t(\mathcal{K}) \) be such that \( (\mathcal{W}^\varphi, t^\varphi) \) is a faithful interpretation of \( \triangleright \mathcal{R}^\varphi \), where \( t^\varphi \) is the map \( \Psi \rightarrow t(\Psi) \cap \mathcal{W}^\varphi \). Take \( \Phi \) such that \( t(\Phi) = \mathcal{W}^\varphi \). Then for each \( \Psi \in \mathcal{K} \), set \( \mathcal{R}^\varphi(\Psi) \in \[ \Phi \] \circ \[ \Psi \] \), which exists by Lemma 18. By Lemma 20, \( t(\mathcal{R}^\varphi(\Psi)) = t^\varphi(\Psi) \). So then:

\[
\Psi =_{\triangleright} \mathcal{R}^\varphi \Gamma \\
\text{iff } t^\varphi(\Psi) \subseteq t^\varphi(\Gamma) \quad \text{(by faithfulness)}
\]

\[
\text{iff } t(\mathcal{R}^\varphi(\Psi)) \subseteq t(\mathcal{R}^\varphi(\Gamma)) \quad \text{(since } t \circ \mathcal{R}^\varphi = t^\varphi)\)
\]

\[
\text{iff } \mathcal{R}^\varphi(\Psi) =_{\triangleright} \mathcal{R}^\varphi(\Gamma) \quad \text{(by faithfulness)}
\]

So \( \mathcal{R}^\varphi \) is an updating rule for \( (\triangleright, \triangleright \mathcal{R}^\varphi) \).

It remains to show that \( \mathcal{R}^\varphi \) is discriminate. This is straightforward—for example, that \( t(\mathcal{R}^\varphi(\Psi)) = t^\varphi(\Psi) \subseteq t(\Psi) \) shows (D1)—and so is omitted. ■

A.3 Representation Theorems

Proof of Theorem 10. If \( \Phi \) is null for all \( \Phi \in \mathcal{K} \), it is straightforward to see that \( \triangleright \) is trivial, and \( (\emptyset, t) \), where, clearly, \( t \) sends all statements to the empty-set will represent \( \triangleright \). So assume the existence of the some non null element of \( \mathcal{K} \); this implies that \( \mathcal{L} \) is non-null. In an abuse of notation, associate each \( x \in \mathbb{R}_+ \) with the act \( xL \).

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First some preliminary lemmas:

**Lemma 21.** For all $\Psi$, there exists a unique $x \in [0,1]$ such that $x \sim 1_\Psi$.

*Proof.* Consider, $A = \{ y \mid y \succ 1_\Psi \}$ and $B = \{ y \mid 1_\Psi \succ y \}$, which are closed by $A_1$ (continuity). $A$ cannot be empty: for if it was then for all $n \in \mathbb{N}$, $n \nmid 1_\Psi$ implying by $A_1$ (completeness) that $1_\Psi \nmid n$ and by $A_5$ that $\frac{1}{n} 1_\Psi \succ 1$. Then $A_1$ (continuity) and $A_3$ together imply $0 \sim 0_\Psi \succ 1$. But since $\mathcal{L}$ is non-null; $A_4$ requires $1 \succ 0$, a contradiction. $B$ is also non-empty since it contains 0.

By $A_1$ (completeness), $A \cup B = \mathbb{R}_+$. Then, since $\mathbb{R}_+$ is connected, these sets must not be disjoint, so there exists an $x \in A \cap B$. That $x$ is unique is immediate from $A_4$ and $x \leq 1$ is immediate from $A_2$.\hfill\qed

Now, to prove sufficiency: Let $(W, t)$ be a faithful interpretation of $\succ$, guaranteed to exist by Theorem 1 and $A_1$. Let $t(\mathcal{K})$ denote the image of $t$.

For each event in $t(\mathcal{K})$, set $\ell(t(\Phi)) = y$ where $y \in \mathbb{R}_+$ is such that $y \sim 1_\Phi$. This $y$ exists by Lemma 21, and is well defined: If $t(\Phi) = t(\Psi)$, then $1_\Phi \sim 1_\Psi$. To see this notice that $t(\Phi) = t(\Psi)$ implies, by property (i) of faithfulness, that $\Phi \Rightarrow \Psi$ and $\Psi \Rightarrow \Phi$. In particular, we have $1_\Phi \sim 1_\Psi \cup \Phi \sim 1_\Psi$.

**Lemma 22.** For all $A, B \in t(\mathcal{K})$, we have $\ell(A) = 0$ if and only if $A = \emptyset$, $A \subseteq B$ implies $\ell(A) \leq \ell(B)$, and $\ell(W) = 1$.

*Proof.* Indeed, consider some such $A, B \in t(\mathcal{K})$. Then $A = t(\Phi)$ and $B = t(\Psi)$ for some $\Phi, \Psi \in \mathcal{K}$.

Let $A = \emptyset$, then by faithfulness, $\Phi$ is null, so $1_\Phi \sim 0_\Phi \sim 0$ by $A_4$ and $A_3$: $\ell(\emptyset) = \ell(t(\Phi)) = 0$.

Now assume $A \neq \emptyset$, then by faithfulness, $\Phi$ is not null, so $1_\Phi > 0_\Phi \sim 0$, again by $A_4$ and $A_3$: therefore $\ell(t(\Phi)) > 0$.

Now assume that $A \subseteq B$. By faithfulness we have $\Phi \Rightarrow \Psi$, so by $A_2$, $\ell(B) \sim 1_\Psi \succ 1_\Phi \sim \ell(A)$, and so by $A_4$, $\ell(A) \leq \ell(B)$.

Finally, by reflexivity $1 \sim 1$, so $\ell(t(\emptyset)) = \ell(W) = 1$.\hfill\qed

This completes the proof. For, let $x_\Phi \succ y_\Psi$. Then, by Lemma 21 and $A_5$, $x_\ell(t(\Phi)) \sim x_\Phi \succ y_\Psi \sim y_\ell(t(\Psi))$. $A_4$ then implies that $x_\ell(t(\Phi)) \geq y_\ell(t(\Psi))$. The reverse implication is analogous.

Necessity is straightforward.\hfill\qed

*Proof of Theorem 11.* We will first show that $\Rightarrow$ satisfies $A_6$ if and only if $\ell$ is weakly additive.

Necessity is obvious, we show sufficiency. Consider some such $A, B \in t(\mathcal{K})$ such that $A \cup B \in t(\mathcal{K})$. If either $A$ or $B$ (or both) is empty, the result is immediate. So assume this is not the case. Then $A = t(\Phi)$ and $B = t(\Psi)$ for some $\Phi, \Psi, \Gamma \in \mathcal{K}$.

Then notice that by faithfulness, $t(\Gamma) \subseteq t(\Phi \cup \Psi)$, and moreover, by Lemma 17(iv), $t(\Phi \cup \Psi) \subseteq t(\Gamma)$. Thus, without loss of generality, we can take $\Gamma = \Phi \cup \Psi$. Set $x = \ell(A)$, $y = \ell(B)$ and $z = \ell(A \cup B)$. We must show that $x + y = z$. If $z = 0$, then by the above observation $\Phi \cup \Psi$ is null, and therefore by Lemma 17(ii), so too are $\Phi$ and $\Psi$, thus $x = y = z = 0$.

Since $\Phi \cup \Psi \neq \emptyset$, we have $z > 0$ by Lemma 22. We have $x = \frac{z}{2}$, and so by the representation $1_\Psi = \frac{z}{2} \Phi \cup \Psi$. By faithfulness, $\Phi \cup \Psi$, allowing us to apply $A_6$ to obtain $1_\Psi = (1 - \frac{z}{2}) \Phi \cup \Psi$. Appealing again to the representation furnishes $y = (1 - \frac{z}{2})z$, or, rearranging, $y = z - x$.

Now assume that in addition $t$ is set-consistent. We will now show that $\Rightarrow$ satisfies $A_7$ if and only if $\ell$ can be extended to a finitely additive measure on the algebra generated by $t(\mathcal{K})$. Necessity is again obvious, we show sufficiency.

Let $\mathcal{C} = t(\mathcal{K}) \cup \emptyset$, and extend $\ell$ to $\mathcal{C}$ by setting $\ell(\emptyset) = 0$, which is well defined in light of Lemma 22. Then $\mathcal{C}$ is a lattice of subsets of $W$ that contains the empty-set and $W$ itself: That $\mathcal{C}$ contains (finite) unions is the obvious consequence of set-consistency; that it contains (finite) intersections is the consequence that $\Rightarrow$ has meets (Lemma 18) and Lemma 20.
Now, given \( A_7 \), \( \ell \) is strongly additive over \( \mathcal{C} \), that is, satisfies, for all \( A, B \in \mathcal{C} \)

\[
\ell(A) + \ell(A') = \ell(A \cup A') + \ell(A \cap A').
\]

Indeed, consider some such \( A, A' \in \mathcal{C} \). If either (or both) set is empty, the result is immediate. So assume this is not the case. Then \( A = t(\Phi) \) and \( A' = t(\Phi') \) for some \( \Phi, \Phi' \in \mathcal{K} \). By set consistency \( t(\Phi \cup \Phi') = A \cup B \) and \( t(\Psi) = A \cap B \) where \( \Psi \) is the meet of \( \Phi \) and \( \Phi' \).

Set \( x = \ell(t(\Phi)), y = \ell(t(\Phi')) \), \( z = \ell(t(\Phi \cup \Phi')) \) and \( w = \ell(t(\Psi)) \). We must show that \( x + y = z + w \).

Since \( \Phi \cup \Phi' \neq 0 \), we have \( z > 0 \) by Lemma 22. We have \( x = \frac{z}{2} z \), and \( y = \frac{z}{2} z \) and so by the representation \( 1_\Phi = \frac{z}{2} \) and \( 1_{\Phi'} = \frac{z}{2} \). Thus, we can apply \( A_7 \) to obtain \( 1_\Phi = (\frac{z}{2} + \frac{z}{2} - 1)_{\Phi \cup \Phi'} \).

Appealing again to the representation furnishes \( w = (\frac{z}{2} + \frac{z}{2} - 1) z \), or, rearranging, \( w + z = x + y \).

Now, by Rao and Rao (1983) Theorem 3.5.1, \( \ell \) can be uniquely extended to a finitely additive measure over the smallest field containing \( \mathcal{C} \).

\( \blacksquare \)

### A.4 Choice over General Acts

As in the above proof, we consider only the non-trivial case wherein \( \mathcal{L} \) is non-null, and, retaining our abuse of notation, associate each \( x \in \mathbb{R}_+ \) with the act \( x_\mathcal{L} \). For the remainder of this section, let \( (W, t, \ell) \) denote the representation of \( \succ \) required by \( A_9 \).

**Lemma 23.** Let \( \succ \) satisfy \( A_8 \) and \( A_9 \). Let \( \Phi \) be non-null and let \( f \sim x_\Phi \) and \( y_\Phi \sim g \). Then \( x \succ y \) if and only if \( f \succ g \).

*Proof.* Set \( z = \frac{1}{2} x + \frac{1}{2} y \). If \( x \succ y \) then since \( x_\Phi \succ z_\Phi \) also \( f \succ z_\Phi \). Likewise, \( z_\Phi \succ g \), so by transitivity, \( f \succ g \). If \( y \succ x \), then \( z_\Phi \succ g \) and \( f \succ z_\Phi \). Therefore, by the contra-positive of the interval property, \( f \not\succ g \).

\( \blacksquare \)

**Lemma 24.** Let \( \succ \) satisfy \( A_8, A_9, A_{10} \) and \( A_{11} \). Then for \( \Phi \in \mathcal{K} \) \( f \succ x_\Phi \) if and only if \( \alpha f \succ \alpha x_\Phi \) for \( \alpha > 0 \) and likewise: \( x_\Phi \succ f \) if and only if \( \alpha x_\Phi \succ \alpha f \) for \( \alpha > 0 \).

*Proof.* Let \( \frac{1}{\alpha} f \sim y \) and \( f \sim x \). These are guaranteed to exist by \( A_{10} \). By \( A_{11} \), we have \( z \succ f \) iff \( z - y \succ \frac{1}{\alpha} f \). Hence \( y = \frac{1}{\alpha} x \). Repeating as necessary, we have \( \frac{1}{\alpha^p} f \sim \frac{1}{\alpha^p} x \). Now the set of finite sums of (possibly repeating) elements of the set \( \{\frac{1}{\alpha^p} \}_{p \in \mathbb{N}} \) is dense in \( \mathbb{R}_+ \), so additivity and continuity seal the deal.

\( \blacksquare \)

**Lemma 25.** Set some \( \Phi \subseteq \Lambda \subseteq \mathcal{K} \). If \( \succ \) satisfies \( A_9 \) and \( A_{12} \) (lower) then

\[
1_\Lambda^\Phi \sim \ell(t(\Lambda)) - \ell(t(\Lambda \setminus \Phi)).
\]

If in addition, \( \succ \) satisfies \( A_{12} \) (upper) then

\[
\ell(t(\Phi)) \sim 1_\Lambda^\Phi
\]

*Proof.* Let \( \succ \) satisfy \( A_{12} \) (lower). Set \( \Gamma \) to denote \( \Lambda \setminus \Phi \). Then by the representation for bets,

\[
1_\Lambda^{\Lambda \setminus \Phi} = 1_\Gamma \sim \ell(t(\Gamma))
\]

and

\[
1_\Lambda^{\Phi \setminus \Gamma} = 1_\Lambda \sim \ell(t(\Lambda))
\]

So by \( A_{12} \) (lower) we have \( 1_\Lambda^\Phi \sim \ell(t(\Lambda)) - \ell(t(\Gamma)) = \ell(t(\Lambda)) - \ell(t(\Lambda \setminus \Phi)) \).

Now assume \( \succ \) also satisfies \( A_{12} \) (upper). Again, by definition

\[
\ell(t(\Lambda)) \sim 1_\Lambda
\]

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Applying the just established result yields $1_\Phi^\Lambda \sim \ell(t(\Lambda)) - \ell(t(\Phi))$. So by A12(upper) we have

$$\ell(t(\Lambda)) - \ell(t(\Phi)) - \ell(t(\Phi)) = \ell(t(\Phi)) \sim 1^\Phi_\Lambda.$$ ■

Define $f = w \mapsto \min\{f(\varphi) \mid w \in t(\varphi)\}$ and $\overline{f} = w \mapsto \max\{f(\varphi) \mid w \in t(\varphi)\}$. Clearly, $\underline{f}, \overline{f} \in \{f\}$ and

$$\int \underline{f} d\ell \leq \int f d\ell \leq \int \overline{f} d\ell$$

for any other $f \in \{f\}$.

**Lemma 26.** Let $f, g : \Lambda \to \mathbb{R}_+$ be co-monotone, set $h = \alpha f + \beta g$. Then $\underline{h} = \alpha \underline{f} + \beta \underline{g}$ and $\overline{h} = \alpha \overline{f} + \beta \overline{g}$.

**Proof.** This is immediate from that fact that $\arg\min_{t^{-1}(w)} f(\varphi) \cap \arg\min_{t^{-1}(w)} g(\varphi) \neq \emptyset$, by co-monotonicity, so for each $w$ there exists a $\Phi \in t^{-1}(w)$ such that $\alpha f(\Phi) + \beta g(\Phi) = \alpha f(w) + \beta g(w)$ which is clearly less than $h(\Psi)$ for any other $\Psi \in t^{-1}(w)$. By analogy we have this also for $\overline{h}$. ■

For $1^\Phi_\Lambda$, we have that

$$\underline{1}^\Phi_\Lambda : w \mapsto \begin{cases} 1 & \text{if } w \in t(\Phi) \setminus t(\Lambda \setminus \Phi) \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{1}^\Phi_\Lambda : w \mapsto \begin{cases} 1 & \text{if } w \in t(\Phi) \\ 0 & \text{otherwise} \end{cases}$$

(6)

**Proof.** Proof of Theorem 12 (Sparse IDEU). Let $\succcurlyeq$ satisfy A8–A12. Applying Lemma 25 to (6) we have:

$$\int \underline{1}^\Phi_\Lambda d\ell \sim 1^\Phi_\Lambda \quad \text{and} \quad \overline{1}^\Phi_\Lambda \sim \int \overline{1}^\Phi_\Lambda d\ell$$

Now for any $f$ with $\Lambda = \text{dom}(f)$, we can find $\{\Gamma_n\}_{n=1}^m$ with $\Gamma_{n+1} \subseteq \Gamma_n \subseteq \Lambda$ such that

$$f = \sum_{n=1}^m \alpha_n 1^\Gamma_n_\Lambda$$

for $\alpha_n \in \mathbb{R}_+$. By their nestedness these bets are pairwise co-monotone, we have

$$f = \sum_{n=1}^m \alpha_n 1^\Gamma_n_\Lambda$$

$$\sim \sum_{n=1}^m \alpha_n \int 1^\Gamma_n_\Lambda d\ell \quad \text{(by A11 and Lemma 24)}$$

$$= \int \sum_{n=1}^m \alpha_n 1^\Gamma_n_\Lambda d\ell \quad \text{(by linearity of Choquet integral)}$$

$$= \int f d\ell \quad \text{(by Lemma 26)}$$

A repetition of the argument shows also that

$$\int \overline{f} d\ell \sim f.$$
Hence, $f \succ g$ if and only if (by Lemma 23) $\int f \, d\ell > \int g \, d\ell$, which given the construction of $f$ and $g$ implies exactly the desired representation.

Proof. Proof of Theorem 12 (MaxMin IDEU). Let $\succsim$ be complete and transitive and satisfy $\text{A9–A11}$ and $\text{A12-(lower)}$.

First, we claim that if $f \sim g$ then $f \sim g$. Since the asymmetric component of a relation is necessarily irreflexive, we have $f, g \notin \{h \mid f \succ h\} = \{h \mid g \succ h\}$ (where the equality is the definition of $\sim$) or that $f \not\succ g$ and $g \not\succ f$ so by completeness $f \sim g$.

Following the proof of Theorem 12, we have that $f \sim \int f \, d\ell$, which by the above observation implies $f \sim \int f \, d\ell$. This completes the proof as: $f \succsim g$ if and only if $\int f \, d\ell \succsim \int g \, d\ell$ if and only if $\int f \, d\ell \geq \int g \, d\ell$, which given the construction of $f$ implies exactly the desired representation.