PRODUCT CORDIAL LABELING OF DOUBLE PATH UNION OF C₃ RELATED GRAPHS
Mukund V. Bapat *1
*1Hindale, Tal: Devgad, Sindhudurg Maharashtra, 416630, India

Abstract:
In this paper we discuss The double path unions obtained on C₃ related graphs for product cordial labeling given by Pₘ(2G') where G' = C₃, FL(C₃), C₃+, bull, tail(C₃,2P₂) etc.

Keywords: Labeling; Cordial; Product; Bull Graph; Crown; Flag Graph; Tail Graph; Double Path Union.

Subject Classification: 05C78

Cite This Article: Mukund V. Bapat. (2018). “PRODUCT CORDIAL LABELING OF DOUBLE PATH UNION OF C₃ RELATED GRAPHS.” International Journal of Engineering Technologies and Management Research, 5(4), 146-152. DOI: 10.5281/zenodo.1251967.

1. Introduction

The graphs we consider are simple, finite, undirected and connected. For terminology and definitions we depend on Graph Theory by Harary [9], A dynamic survey of graph labeling by J.Gallian [8] and Douglas West.[11]. I.Cahit introduced the concept of cordial labeling [5]. There are variety of cordial labeling available in labeling of graphs. Sundaram, Ponraj, and Somasundaram [10] introduced the notion of product cordial labeling. A product cordial labeling of a graph G with vertex set V is a function f from V to {0,1} such that if each edge uv is assigned the label f(u)f(v), the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a product cordial graph. We use vᵢ(0,1) = (a, b) to denote the number of vertices with label 1 are a in number and the number of vertices with label 0 are b in number. Similar notion on edges follows for eᵢ(0,1) = (x, y).

A lot of work is done in this type of labeling so far. One interested in survey may refer Dynamic survey in Graph labeling by J. Gillian. We mention some part of it. Sundaram, Ponraj, and Somasundaram have shown that trees; unicyclic graphs of odd order; triangular snakes; dragons; helms; PmUPn; CmUPn; PmUK1,n; WmUFn (Fn is the fan Pn+K₁); K₁,mUK₁,n; WmU K₁,n; WmUPn; WmUCn; the total graph of Pn (the total graph of Pn has vertex set V (Pn)UEi(Pn) with two vertices adjacent whenever they are neighbors in Pn); Cn if and only if n is odd; Cₙ⁽ᵣ⁾, the one-point union of t copies of Cₙ, provided t is even or both t and n are even; K₂+mK₁ if and only if m is odd; Cₙ∪Pₙ if and only if m+n is odd; Kₘₙ∪Pₘ if s >mn; Cₙ+2UK₁,n; KnUKₙ,(n−1)/2
when \( n \) is odd; \( K_n \cup K_{n-1,n/2} \) when \( n \) is even; and \( P_2 \) if and only if \( n \) is odd. They also prove that \( K_{m,n} \) (\( m,n>2 \)), \( P_m \times P_n \) (\( m,n>2 \)) and wheels are not product cordial and if a \((p,q)\)-graph is product cordial graph, then \( q \leq (p-1)(p+1)/4 + 1 \). In this paper we show that \( P_m(2G') \) where \( G' = C_3, \ FL(C_3), \ C_3^+ \), bull, tail(\( C_3,2P_2 \)) are product cordial graphs.

2. Preliminaries

2.1. Fusion of Vertex

Let \( G \) be a \((p,q)\) graph. Let \( u \neq v \) be two vertices of \( G \). We replace them with single vertex \( w \) and all edges incident with \( u \) and that with \( v \) are made incident with \( w \). If a loop is formed is deleted. The new graph has \( p-1 \) vertices and at least \( q-1 \) edges. If \( u \in G_1 \) and \( v \in G_2 \), where \( G_1 \) is \((p_1,q_1)\) and \( G_2 \) is \((p_2,q_2)\) graph. Take a new vertex \( w \) and all the edges incident to \( u \) and \( v \) are joined to \( w \) and vertices \( u \) and \( v \) are deleted. The new graph has \( p_1+p_2-1 \) vertices and \( q_1+q_2 \) edges. Sometimes this is referred as \( u \) is identified with the concept is well elaborated in John Clark, Holton[6]

2.2. Crown Graph

It is \( C_n \cup K_2 \). At each vertex of cycle a \( n \) edge was attached. We develop the concept further to obtain crown for any graph. Thus crown \((G)\) is a graph \( G \cup K_2 \). It has a pendent edge attached to each of it’s vertex. If \( G \) is a \((p,q)\) graph then crown(G) has \( q+p \) edges and \( 2p \) vertices.

2.3. Flag of a Graph \( G \) denoted by \( FL(G) \) is obtained by taking a graph \( G=G(p,q) \). At suitable vertex of \( G \) attach a pendent edge. It has \( p+1 \) vertices and \( q+1 \) edges.

2.4. A bull Graph Bull \((G)\) was initially defined for a \( C_3 \)-bull. It has a copy of \( G \) with an pendent edge each fused with any two adjacent vertices of \( G \). For \( G \) is a \((p,q)\) graph, bull\((G)\) has \( p+2 \) vertices and \( q+2 \) edges.

2.5. A Tail Graph (Also called as antenna graph) is obtained by fusing a path \( p_k \) to some vertex of \( G \). This is denoted by tail\((G, P_k)\). If there are \( t \) number of tails of equal length say \((k-1)\) then it is denoted by tail\((G, tp_k)\). If \( G \) is a \((p,q)\) graph and a tail \( P_k \) is attached to it then tail\((G, P_k)\) has \( p+k-1 \) vertices and \( q+k-1 \) edges.

2.6. Path union of \( G \), i.e. \( P_m(G) \) is obtained by taking a path \( p_m \) and take \( m \) copies of graph \( G \). Then fuse a copy each of \( G \) at every vertex of path at given fixed point on \( G \). It has \( mp \) vertices and \( mq +m-1 \) edges. Where \( G \) is a \((p,q)\) graph.

2.7. Double path union of \( G \), i.e. \( P_m(2G) \) is obtained by taking a path \( p_m \) and take \( 2m \) copies of graph \( G \). Then fuse two copies of \( G \) at every vertex of path at given fixed vertex on \( G \). It has \( 2mp \) vertices and \( 2mq +m-1 \) edges. Where \( G \) is a \((p,q)\) graph.

3. Main Results

Theorem 3.1 Double path union of triangles given by \( G = P_m(2C_3) \) is product cordial for all \( m \)
Proof: We define $G$ as a graph with path $P_m$ given by $(v_1,e_1,v_2,e_2,v_3,e_3,...,v_m)$. The one of the copies of $C_3$ fused at $V_i$ is $(u_{i,1},e_{i,1},u_{i,2},e_{i,2},u_{i,3},e_{i,3},u_{i,1})$. The other copy of $C_3$ is given by $(w_{i,1},e_{i,1},w_{i,2},e_{i,2},w_{i,3},e_{i,3},w_{i,1})$. Note that $u_{i,1}$, $v_i$, $w_{i,1}$ are the same vertices as $v_i$; $i = 1, 2, ..m$. (being common vertex to both copies of $C_3$ and path vertex $v_i$).

Define a function $f$: $V(G) \rightarrow \{0,1\}$ as follows:

Case $m = 2x$,

$f(u_{i,j}) = 0, j = 1, 2, 3; i = 1, 2, ..x$.

$f(w_{i,j}) = 0, j = 1, 2, 3; i = 1, 2, ..x$.

$f(u^{'i}) = 0$ for $i = 1, 2, ..x$.

$f(w^{'i}) = 0$ for $i = 1, 2, ..x$.

Case $m = 2x+1$

The $P_{2x}(2C_3)$ part of $P_{2x+1}(2C_3)$ is labeled first as above.

Theorem 3.2 Let $G'$ be FL $(C_3)$ then $P_m(2G')$ is product cordial. (All structures).

Proof: We define $G$ as a graph with path $P_m$ given by $(v_1,e_1,v_2,e_2,v_3,e_3,...,v_m)$. The one of the copies of $C_3$ fused at $V_i$ is $(u_{i,1},e_{i,1},u_{i,2},e_{i,2},u_{i,3},e_{i,3},u_{i,1})$. The other copy of $C_3$ is given by $(w_{i,1},e_{i,1},w_{i,2},e_{i,2},w_{i,3},e_{i,3},w_{i,1})$. Note that $u_{i,1}$, $v_i$, $w_{i,1}$ are the same vertices as $v_i$; $i = 1, 2, ..m$. (being common vertex to both copies of $C_3$ and path vertex $v_i$). The pendant vertices $u^{'i}$ and $w^{'i}$ are attached by pendant edges $e^{'i}$ and $e_i''$ respectively to vertex $u_{i,2}$ and $w_{i,2}$. Thus $G$ has $7m$ vertices and $9m-1$ edges.

Define a function $f$: $V(G) \rightarrow \{0,1\}$ as follows:

Case $m = 2x$,

$f(u_{i,j}) = 0, j = 1, 2, 3; i = 1, 2, ..x$.

$f(w_{i,j}) = 0, j = 1, 2, 3; i = 1, 2, ..x$.

$f(u^{'i}) = 0$ for $i = 1, 2, ..x$.

$f(w^{'i}) = 0$ for $i = 1, 2, ..x$.

The label distribution is $v_f(0,1) = (8x,8x)$ and $e_f(0,1) = (9x, 9x-1)$.

Case $m = 2x+1$

The $P_{2x}(2G')$ part of $P_{2x+1}(2G')$ is labeled first as above.
for \( i = 2x+1 \) we have,
\[
\begin{align*}
  f(u_{ij}) &= 1, & \text{for all } j = 1, 2, 3. \\
  f(w_{ij}) &= 1, & \text{for all } j = 2, 3. \\
  f(u_i') &= 1; \\
  f(w_i') &= 0.
\end{align*}
\]

The label distribution is \( v_f(0,1) = (8x+3, 8x+4) \) and \( e_f(0,1) = (9x+4, 9x+4) \). In above labeling the label of pendent vertex is done separately in the sense that it is independent of which vertex on \( C_3 \) the pendent edge is fused with. Hence the same function works for all structures to produce product cordial labeling.

Thus the graph is product cordial for all \( m \). #.

**Theorem 3.3** Double path union on bull graph given by \( P_m(2\text{bull}) \) is product cordial.

Proof: We define \( G \) as a graph with path \( P_m \) given by \( (v_1, e_1, v_2, e_2, v_3, e_3, \ldots, v_m) \). The one of the copies of \( C_3 \) fused at \( v_i \) is \( (u_{i1}, e_{i1}, u_{i2}, e_{i2}, u_{i3}, e_{i3}, u_{i1}) \). The other copy of \( C_3 \) is given by \( (w_{i1}, c_{i1}, w_{i2}, c_{i2}, w_{i3}, c_{i3}, w_{i1}) \). Note that \( u_{i1}, v_i, w_{i1} \) are the same vertices as \( v_i; i = 1, 2, \ldots, m \). (being common vertex to both copies of \( C_3 \) and path vertex \( v_i \)). The pendent vertices are \( u_i, u_i' \) and \( w_i, w_i'' \) with corresponding pendent edges \( (u_{i1}u_i'), (u_{i1}u_i''), (w_{i1}w_i'), (w_{i2}w_i'') \) respectively. Thus \( G \) has has \( 9m \) vertices and \( 11m-1 \) edges.

Define a function \( f: V(G) \rightarrow \{0, 1\} \) as follows:

**Case** \( m = 2x \),
\( f(u_{i,j}) = 0, j = 1, 2, 3; \) \( i = 1, 2, \ldots, x \);
\( f(w_{i,j}) = 0, j = 1, 2, 3; \) \( i = 1, 2, \ldots, x \);
\( f(u'_i) = 0 \) for \( i = 1, 2, \ldots, x \);
\( f(w'_i) = 0 \) for \( i = 1, 2, \ldots, x \);
\( f(u''_i) = 0 \) for \( i = 1, 2, \ldots, x \);
\( f(w''_i) = 0 \) for \( i = 1, 2, \ldots, x \);
\( f(u_{i,j}) = 1, j = 1, 2, 3; \) \( i = x+1, x+2, \ldots, 2x \).
\( f(w_{i,j}) = 1, j = 1, 2, 3; \) \( i = x+1, x+2, \ldots, 2x \).
\( f(u'_i) = 1 \) for \( i = x+1, x+2, \ldots, 2x \);
\( f(w'_i) = 1 \) for \( i = x+1, x+2, \ldots, 2x \).
\( f(u''_i) = 1 \) for \( i = x+1, x+2, \ldots, 2x \);
\( f(w''_i) = 1 \) for \( i = x+1, x+2, \ldots, 2x \).

The label distribution is \( v_f(0,1) = (9x, 9x) \) and \( e_f(0,1) = (11x, 11x-1) \).

The \( P_{2x}(2\text{bull}) \) part of \( P_{2x+1}(2\text{bull}) \) is labeled first as above, for \( i = 2x+1 \) we have,
\( f(u_{i,j}) = 1, \) for all \( j = 1, 2, 3 \).
\( f(w_{i,j}) = 0, \) for all \( j = 2, 3 \).
\( f(u'_i) = 1; \)
\( f(w'_i) = 0. \)
\( f(u''_i) = 1; \)
\( f(w''_i) = 0. \)

The label distribution is \( v_i(0,1) = (9x+4, 9x+5) \) and \( e_i(0,1) = (11x+5, 11x+5) \).

Thus the graph is product cordial for all \( m \).

**Theorem 3.4** Double path union on crown \( C_3^+ \) given by \( G = P_m(2C_3^+) \) is product cordial.

Proof: We define \( G \) as a graph with path \( P_m \) given by \( (v_1, e_1, v_2, e_2, v_3, e_3, \ldots, v_m) \). The one of the copies of \( C_3 \) fused at \( v_i \) is \( (u_{i,1}, e_{i,1}, u_{i,2}, e_{i,2}, u_{i,3}, e_{i,3}, u_{i,4}) \). The other copy of \( C_3 \) is given by \( (w_{i,1}, c_{i,1}, w_{i,2}, c_{i,2}, w_{i,3}, c_{i,3}, w_{i,4}) \). Note that \( u_{i,1}, v_i, w_{i,1} \) are the same vertices as \( v_i; i = 1, 2, \ldots, m \) (being common vertex to both copies of \( C_3 \) and path vertex \( v_i \)). At each \( u_{i,j} \) a pendent vertex is attached is
ui,j’, j = 1, 2, 3. At each w_i,j a pendent vertex is attached is w_j,j’, j = 1, 2, 3. Thus G has |V(G)|=11m and |E(G)|= 13m - 1.

Define a function f: V(G) \rightarrow \{0,1\} as follows:

Case m = 2x,

f(u_i,j) = 0, j = 1, 2, 3; i = 1, 2, ..x;
f(w_i,j) = 0, j = 1, 2, 3; i = 1, 2, ..x;
f(u_i,) = 0 for i = 1, 2, ..x; j=1,2, 3.
f(w_i,) = 0 for i = 1, 2, ..x; j = 1, 2, 3.
f(u_i,j) = 1, j = 1, 2, 3; i = x+1,x+ 2, ..2x.
f(w_i,j) = 1, j = 1, 2, 3; i = x+1, x+2, ..2x.
f(u_i,) = 1 for i = x+1,x+2, ..2x; j = 1, 2, 3.
f(w_i,) = 0 for i = x+1, x+2, ..2x; j = 1, 2, 3.

The label distribution is \(vf(0,1) = (11x,11x)\) and \(ef(0,1) = (13x, 13x-1)\).

The \(P_{2x}(2bull)\) part of \(P_{2x+1}(C_3^+)\) is labeled first as above. For i = 2x+1 we have,

f(u_i,j) = 1, j = 1, 2, 3;
f(w_i,j) = 0, j = 2, 3;
f(u_i,) = 1 for j =1, 2, 3,
f(w_i,) = 0fo j= 1, 2, 3,

The label distribution is \(vf(0,1) = (11x+5,11x + 6)\) and \(ef(0,1) = (13x+6, 13x+6)\).

Thus the graph is product cordial for all m. #.

**Theorem 3.4** Double path union on \(G' = \text{tail } (C_3,2P_2)\) given by \(P_m(2G')\) is product cordial.

Proof: We define G as a graph with path \(P_m\) given by (v_1,e_1, v_2, e_2, v_3, e_3...v_m). The one of the copies of C_3 fused at v_i is (u_i,1, e_i,1, u_i,2, e_i,2, u_i,3, e_i,3, u_i,1). The other copy of C_3 is given by (w_i,1, c_i,1, w_i,2, c_i,2, w_i,3, c_i,3, w_i,1). Note that u_i,1, v_i, w_i,1 are the same vertices as v_i; i = 1, 2, ...m. (being common vertex to both copies of C_3 and path vertex v_i). At u_i,2 two pendent vertices are fused u_i,1’,u_i,1”with edges given by (u_i,1u_i,1’) and (u_i,1u_i,1”). At w_i,2 two pendent vertices are fused w_i,1’,w_i,1”with edges given by (w_i,1w_i,1’) and (w_i,1w_i,1”). Thus, G has |V(G)|= 9m and |E(G)|= 11m-1.

Define a function f: V(G) \rightarrow \{0, 1\} as follows:

Case m = 2x,

f(u_i,j) = 0, j = 1, 2, 3; i = 1, 2, ..x;
f(w_i,j) = 0, j = 1, 2, 3; i = 1, 2, ..x;
f(u_i,) = 0 for i = 1, 2, ..x;
f(w_i,) = 0 for i = 1, 2, ..x;
f(u_i,) = 0 for i = 1, 2, ..x;
f(w_i,) = 0 for i = 1, 2, ..x;
f(u_i,j) = 1, j = 1, 2, 3; i = x+1,x+ 2, ..2x.
f(w_i,j) = 1, j = 1, 2, 3; i = x+1, x+2, ..2x.
f(u_i,) = 1 for i = x+1,x+2, ..2x;
f(w_i,) = 1 for i = x+1, x+2, ..2x.
The label distribution is \( v_f(0,1) = (9x,9x) \) and \( e_f(0,1) = (11x, 11x-1) \).

Case \( m = 2x+1 \)

The \( P_{2x} \(2G'\) part of \( P_{2x+1}(2G') \) is labeled first as above. For \( i = 2x+1 \) we have,

\[
\begin{align*}
f(u_{i,j}) &= 1, \quad j = 1, 2, 3; \\
f(w_{i,j}) &= 0, \quad j = 2, 3; \\
f(u_{i,1'}) &= 1, \\
f(w_{i,1'}) &= 0, \\
f(u_{i,1''}) &= 1, \\
f(w_{i,1''}) &= 0,
\end{align*}
\]

The label distribution is \( v_f(0,1) = (9x+4,9x + 5) \) and \( e_f(0,1) = (11x+5, 11x+5) \).

Thus the graph is product cordial for all \( m \).

### 4. Conclusions

We have obtained double path union \( G = P_m(2G') \) by fusing two copies of same graph \( G' \) at every vertex of a path \( P_m \). We have shown that \( G \) is product cordial when \( G' \) is 1) triangle \( C_3 \), 2) \( Fl(C_3) \), 3) bull\((C_3)\), 4) \( C^*_n\), 5) tail\((C_3,2P_2)\). We conjecture that \( G = P_m(G') \) is product cordial when \( G' \) is 1)cycl \( C_n \), 2) \( Fl(C_n) \), 3) bull\((C_n)\), 4) \( C^*_n \), 5) tail\((C_n,2P_2)\) etc.

### References

1. Bapat M.V. Some new families of product cordial graphs, Proceedings, Annual International conference, CMCGS 2017, Singapore, 110-115
2. Bapat M.V. Some vertex prime graphs and a new type of graph labelling Vol 47 part 1 yr2017 pg 23-29 IJMTT
3. Bapat M. V. Some complete graph related families of product cordial graphs. Arya bhatta journal of mathematics and informatics vol 9 issue 2 july-Dec 2018.
4. Bapat M.V. “Extended Edge Vertex Cordial Labelling Of Graph”, International Journal Of Math Archives IJMA Sept 2017 issue
5. Bapat M.V. Ph.D. Thesis, University of Mumbai 2004.
6. John Clark and D. Holton, A book “A first look at graph Theory”, world scientific.
7. I.Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, ArsCombin., 23 (1987) 201-207.Harary, Theory, Narosa publishing, New Delhi
8. J. Gallian Electronic Journal Of Graph Labeling (Dynamic survey)2016
9. Harary, Graph Theory, Narosa publishing, New Delhi
10. M. Sundaram, R. Ponraj, and S. Somasundaram, “Product cordial labeling of graph,” Bulletin of Pure and Applied Science, vol. 23, pp. 155–163, 2004.
11. D West Introduction to graph Theory, Pearson Education Asia, Bapat Mukund V. At and Post: Hindale, Tal. : Devgad, Dist.: Sindhudurg, Maharashtra. India 416630.

*Corresponding author.

E-mail address: algayanpascua@yahoo.com