Parisi–Symmetry of the Many–Body Quantum Theory of randomly interacting fermionic systems

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We show that fermion systems with random interactions lead to strong coupling between glassy order and fermionic correlations, which culminates in the implementation of Parisi replica permutation symmetry breaking (RPSB) in their zero temperature quantum field theories. Precursor effects, setting in below fermionic Almeida–Thouless lines, become stronger as the temperature decreases and play a crucial role for many physical properties within the entire low temperature regime. The Parisi ultrametric structure is shown to determine the dynamic behaviour of fermionic correlations (Green’s functions) for large times and for the corresponding low energy excitation spectra, which is predicted to affect transport properties in metallic (and superconducting) spin glasses. Thus we reveal the existence and the detailed form of a number of quantum–dynamical fingerprints of the Parisi scheme. These effects, being strongest as \( T \to 0 \), are contrasted with the replica–symmetric nature of the critical field theory of quantum spin glass transitions at \( T = 0 \), which display only small corrections at low \( T \) from replica permutation symmetry breaking (RPSB). RPSB–effects moreover appear to influence the loci of the ground state transitions at \( O(T^0) \) and hence the phase diagrams. From explicit solutions for arbitrary temperatures we also find a new representation of the zero temperature Green’s function. This leads to a map of the fermionic (insulating) spin glass solution to the local limit solution of a Hubbard model with a random repulsive interaction. This map exists at any number of replica symmetry breaking steps \( K \). We obtain the distribution of the Hubbard interaction fluctuation and its dependence on the order of RPSB. A generalized mapping between metallic spin glass and random U Hubbard model is conjectured. We also suggest that the new representation of the Green’s function at \( T = 0 \) can be used for generalizations to superconductors with spin glass phases. Further generalizations of the fermionic Ising spin glass to models with additional spin and charge quantum–dynamics occurring in metallic spin glasses, or due to Coulomb effects including crossover from 4–state per site to effectively 3–state per site models in the limit of infinite repulsive Hubbard coupling are briefly considered. We compare our spin glass results with recent \( d = \infty \) (clean) Hubbard model analyses, paying particular attention to the common role of the corresponding Onsager reaction fields.
I. INTRODUCTION

In this paper we emphasize and explore the implementation of Parisi replica permutation symmetry breaking in the quantum theory of fermionic systems with random interactions. We present details of the derivation of a many body quantum theory with ultrametric structure and we discuss the results in context of the tricritical phase diagram, which is given in the subsequent paper II.

Calculations based on the 4–state per site extension of the standard SK–model provide the framework for detailed information that helps to analyze a variety of more complicated fermionic spin– and charge–glass quantum models. Spin glasses have been described for a long time by models that isolated glassy magnetic phenomena from other physical phenomena. The source of the frustrated magnetic interaction often hidden in the RKKY–interaction of metallic systems, this modelling was meant to apply to cases that allow to study glassy magnetic order and electronic transport in such a semi–independent way. A first microscopic diagrammatic approach to describe at the same footing the formation of spins and the calculation of their frustrated interaction was designed by Hertz \cite{Hertz} for a system of randomly placed strong Hubbard U centers. A way of phenomenological treatment of spin glass order and transport theory can be found in the book of Fischer and Hertz \cite{FischerHertz}.

Quantum spin glass models possess a genuine richness in their low temperature phase diagrams that is due to the entanglement of a many body interaction and the randomness contained in the couplings. The mixing of spin and charge fluctuations leads to new experimentally observable phenomena and to fundamentally new structures of the many body theory.

These systems link phase transitions of remarkably different character: zero temperature so–called quantum phase transitions displaying quantum dynamical effects, but also interesting types of thermal tricritical transitions \cite{Leuzzi}, which mix spin– and charge–fluctuations and also an interference of spin glass features in charge response and transport properties. The symmetry classification of these QPT’s is totally different from that of thermal phase transitions and also does not appear to resemble, for example, the $T = 0$ QPT universality classes of Anderson localization.

We elaborate details of the manifestation of Parisi symmetry in a $T = 0$ quantum theory of the fermionic Ising spin glass. First extensions to models that possess, beyond the ubiquitous fermion quantum dynamics, also spin– and charge–quantum dynamics are also considered. We derive the long–time behaviour of the fermion propagator, which proves to be one of the ideal quantities that represent quantum–dynamically the Parisi replica permutation symmetry breaking in theories at $T = 0$ and at low temperatures. The order parameter function $q(x)$ is static and displays its nontrivial features only on an interval of order $T$. It needs the linear susceptibility $\chi = \beta (\bar{q} - \int_0^1 q(x))$, where the spin autocorrelation function $\bar{q}$ remains static for any multi–state Ising spin glass, to incorporate an $O(1)$–effect within a correlation function. The fermion correlations describe the fermionic spin glasses in a more detailed way than spin– and charge–correlations. The latter can however all be obtained from the former ones. Even for spin–static models the fermionic correlations, beginning with the fermion propagator, yield a detailed quantum–dynamical description of RPSB. In addition, this is reminiscent of the classical–dynamical theories of spin glasses \cite{Leuzzi} with imposed Glauber–dynamics and despite the fact that the latter was invented to avoid the replica trick thus producing a dynamical image of the classical spin glass solution, which indeed represented the Parisi solution. It is this analogy which we emphasize by elaborating in this paper the quantum–dynamical image of the Parisi solution for the fermionic Ising spin glass, which is spin– and charge–static but has fermion dynamics exclusively determined from the Hamiltonian. Excitation spectra encountered in one– and two–fermion correlators are seen to be determined by RPSB in a qualitative and quantitative way.

While most of our results are given for the 4–state fermionic Ising spin glass. The influence of RPSB on the $T = 0$–quantum theory emerges in a general and apparently exists in a model–independent way. This clearly calls for renewed efforts to find RPSB in the other domains of disordered electronic systems such as the theory of Anderson–, Anderson–Mott–, and similar localization transitions. This is beyond the scope of this paper, but the role of Parisi symmetry in the (insulating) fermionic Ising spin glass together with the Ward identity for charge conservation, which we work out here, indicates the possible relevance of this symmetry.

II. MODELS OF RANDOMLY INTERACTING SYSTEMS

The most fundamental and intensively studied model of random interaction is the Ising model with fully frustrated spin interaction. As any spin model it can be described in a grand canonical description with appropriate imaginary chemical potential. In this way the standard SK Ising spin glass model is viewed as a fermionic spin glass, given on special points of the imaginary axis of the complex $\mu$–plane. On continuing to real chemical potentials one converts the Sherrington Kerkpatrick–model into a physical fermionic Ising spin glass. As an interacting disordered fermion
model the Hamiltonian appears to be simple. On one hand it only takes into account a random magnetic interaction, which, after partial particle–hole transformation is equivalent to a random charge–charge interaction and a magnetic field. It allows however for a spin–charge coupling due to redistribution among magnetic and nonmagnetic states, which depends strongly on the particle pressure and hence on the effective spin density. From the point of view of an interacting disordered fermion system we demonstrate the way in which the known instability against replica symmetry breaking leads to a quantum field theory with Parisi symmetry. A large part of this paper is devoted to explain this by detailed results for the infinite–range fermionic Ising spin glass. Thus the theory presented here starts out from the Hamiltonian, which superficially coincides with the classical SK–model, but its spins are defined as the fermionic objects

\[ \sigma \equiv n_\uparrow - n_\downarrow, \quad n_\sigma \equiv a_\sigma^\dagger a_\sigma \quad \text{with} \quad \{a_\alpha, a_\beta\} = 0, \{a_\alpha^\dagger, a_\beta\} = \delta_{ij} \delta_{\alpha\beta}.\]  

This introduces a new class of observables and correlations which involve odd numbers of fermion operators at particular instants of time. Some new correlations of groups of even numbers of operators at a given time may become indirectly affected, but this type of correlations feels only quantum statistics, which controls the occupation of magnetic states \(|\uparrow,0>, |0,\downarrow>\) in correspondence with the nonmagnetic ones \(|\uparrow,\uparrow>, |0,0>\) and remains static until a noncommuting term such as a transverse field or a fermion hopping term etcetera is added to the Ising Hamiltonian. As for the fermionic Ising spin glass, the grand canonical description with a chemical potential \(\mu\) and given by

\[ K = H - \mu N = -\sum_{ij} J_{ij} \sigma_i \sigma_j - \sum_i h_i \sigma_i - \mu \sum_{i,\sigma} n_{i,\sigma} \]  

appears standard, when finally the condition of independent and gaussian distributed random bonds \(J_{ij}\) is imposed. The physics of this model is however far more complicated firstly than the 2–state parent model on spin space and secondly than related classical spin 1 models, like the BEG–model or the diluted 2–state per site model. Complications that arise in understanding the phase diagram of the are deferred to the subsequent paper II. According to the statement above, the SK–model and this model prove to be very different even at half–filling. Expressing the partition function with the help of Grassman integrals the disorder averaging of the free energy is performed by means of the replica trick \(\beta[F]_{av} = \log Z_{av} = \lim_{\alpha\rightarrow0} \frac{1}{\alpha} (1 - \Pi_{\alpha=1} Z_{av})\). The decoupling of eight–fermion correlations generated by the gaussian disorder average is rather standard and briefly described in the Appendix. The fermionic standard model can be easily generalized or transformed to match more complicated and realistic systems. While this paper is mainly devoted to the quantum–dynamical representation of Parisi symmetry in the field theory of the fermionic Ising spin glass, earlier studies of spin correlations in more complicated models like metallic spin glasses also exist, and the analysis of fermionic correlations subject to RPSB promises many new results in the light of the present results for the insulating cases. Writing a generalized grand canonical hamiltonian

\[ K = \sum_{ij} t_{ij} a_i^\dagger a_{j\sigma} - \sum_{ij} J_{ij}^\sigma \sigma_i^\sigma \sigma_j^\sigma + [[U]_{av} + \delta U] \sum_i n_{i\uparrow} n_{i\downarrow} + \sum_{i,j,\sigma,\sigma'} V(i-j) n_{i\sigma} n_{j\sigma'} - \sum_{i,\alpha} h_i^\alpha \sigma_i^\alpha - \mu \sum_{i,\sigma} n_{i\sigma} \]  

one comprises several subclasses such as itinerant magnetism (with emphasis on itinerant spin glasses), the Hubbard model including effects of disorder in \(U\) for example. Thus the general model takes care of Coulombic effects, both due the Hubbard interaction with and without random fluctuations of the coupling \(U\), and due to the long–range bare Coulomb repulsion \(V_{i-j}\), and secondly allows for transport through either random or nonrandom fermion hopping. While the Hubbard interaction for example controls the crossover from 4–state per site to an 3–state per site space at \(U = \infty\) and adds a second magnetic interaction (without frustration if not due to a lattice), the long–range part of the Coulomb interaction is also important due to its power to affect or even create gaps for example. Generalization due to longitudinal or transverse fields and due to Heisenberg type of interactions are rather standard and not of main interest. Both the metallic Ising spin glass (part 1 and 2) and the Hubbard model are known to be extremely hard problems even in the mean field limits and due to O(1) role of their corresponding Onsager reaction fields. On the other hand it is just this common point which, beyond the physical relevance of Coulomb effects in fermionic spin glasses, calls for a joint treatment. It is the \(d = \infty\)–method of the Hubbard model (called dynamic mean field theory DMFT or local impurity selfconsistent approximation LISA by their inventors), which we want to include to some extent and compare with the spin glass results. One of the interesting questions concerns the type of competition that arises in the limit of vanishing \(U\), which implies an equal weight for local pairing states and for magnetic states. The locality of the responsible interaction renders this problem different from frustration in spin glass problems. A third group of fermionic spin glass models is given by the Kondo–coupling of itinerant carriers to localized and
randomly frozen spins. Since these spins can be represented by pseudofermions coupled to a heat bath by imaginary chemical potential, this model falls into the class of interacting systems with two fermion species. These s–d models which apply as well to CuMn as to the doped II–VI semiconductor CdMnTe, and to heavy fermion systems are written in tight binding form by

$$K = - \sum_{ij} t_{ij} a^\dagger_{i\sigma} a_{j\sigma} - \sum_{i} a^\dagger_{i\sigma} \sigma_{i\sigma}^{\alpha} a_{i\sigma} s_i - \sum_{ij} J_{ij} s_i s_j - i\frac{\pi}{2} \sum_{i,\lambda} d^\dagger_{i\lambda} d_{i\lambda} - \mu \sum_{i,\sigma} a^\dagger_{i\sigma} a_{i\sigma}$$  \hspace{1cm} (4)$$

with the localized spins s given in terms of fermion operators d by \(s_i \equiv d^\dagger_{i\lambda} \sigma_{i\lambda}^{\alpha'} d_{i\lambda'}\) where the simplest imaginary chemical potential \(-i\frac{\pi}{2} T\) given here models the Hilbert space constraint for spin quantum number \(\frac{1}{2}\). Higher spin quantum numbers can be treated with a little more effort.\cite{24} In the present paper we assume that the localized spins are integrated out and end up in a single species model of a type described above. We do not claim that all physical phenomena of Kondo systems can be described in this way, but the extraordinary difficulties encountered in theory of fermionic spin glasses requires at present the limitation to the simplest models. On the other hand the results given here for the fermionic Ising spin glass show that a quantum Parisi phase is to be expected in general.

### III. BASIC ELEMENTS AND SYMMETRIES OF THE MANY BODY THEORY

Conventional many body theory usually discards effects of statistical fluctuations of interactions such as exchange– or Hubbard–interactions. This type of randomness is however not a rare event; a typical example in the case of substitutional disorder is given by the so–called randomly placed U–centers, a picture used by Hertz\cite{24} in an early and pioneering diagrammatic approach to itinerant spin glasses.

Random interactions can lead to unique and decisive physical properties in all sorts of fermionic quantum spin glasses. The entire class of these systems cannot be exhaustively described on the spin level as the example of effects of complex magnetic order on low energy excitations and transport behaviour showed.\cite{24} In this chapter we provide details on technicalities and on physical results obtained by the use of the elements of this new type of many body theory.

One of the major novelties stems from the necessity to include broken replica permutation symmetry of Parisi–type \cite{24} in the fermionic many body theory.\cite{24} We shall demonstrate that the long–time behaviour of averaged fermion propagators is determined by the highly nontrivial Parisi scheme. This bears an interesting relation with the time–dependent formalism intended to avoid the replica trick and applied to classical spin glasses with additional Glauber dynamics. Our present theory however maintains the replica trick together with quantum dynamics, which is enforced by the Hamiltonian rather than being added due to the effect of external degrees of freedom which provide a heat bath.

#### A. Green’s functions and generating functionals for fermionic spin glasses

The fermionic Green’s function, denoted by \(G_{i,\sigma}(\epsilon_i)\), and whose standard definition in terms of fermion operators on a lattice reads

$$G_{ij}(\tau, \tau') = \left< T_{\tau}(a_{\tau\sigma}(\tau)a^\dagger_{\tau'\sigma}(\tau')) \right>_{av}$$  \hspace{1cm} (5)$$

is studied in detail in this paper. It involves apart from the usual quantum statistical average \(< \ldots >\) an additional disorder average which uses probability distributions \(P(J_{ij})\) for spin glasses or \(P(U)\) for a random U Hubbard interaction for example. In order t be able to study a Parisi type quantum solution, we choose the replica formalism to cope with these disorder averages. This formalism builds into fermion field theory the technicalities necessary to deal with the typical problems that arise from randomness present in many body interactions. Those are first the presence of at least 8–fermion correlations in the replicated action and secondly the broken replica symmetry in connection with glassy order.

The method of time slicing allows to start out from a representation of the fermion propagator in terms of derivatives of a generating functional with respect to anticommuting, generating (replicated) fields \(\eta, \bar{\eta}\) as

$$G_{i}(\epsilon_i) = \frac{\delta}{\delta \eta_{i,\sigma}} \frac{\delta}{\delta \bar{\eta}_{i,\sigma}} \ln \int d\psi d\bar{\psi} \exp[A(\psi, \bar{\psi}) + \eta \bar{\psi} - \bar{\eta} \psi] |_{\eta=0, \bar{\eta}=0},$$  \hspace{1cm} (6)$$
where $\mathcal{A}$ denotes the effective Grassmannian action. This action is exactly of eighth order in the Grassmann fields, since we have chosen a complete gaussian distribution of statistical fluctuations in two–body interactions.

a) The fermionic Ising spin glass
We start with the fermionic Ising spin glass \textit{ISG}_f as our first example, thus avoiding complications from quantum–spin dynamics present for example in the fermionic Heisenberg spin glasses or in metallic spin glasses of any kind. Quantum spin dynamics, originating from (fermion) spin operators that do not commute with the Hamiltonian, are interesting in itself but avoided here to clarify the role of fermion quantum dynamics. The latter is shown to be determined on the long time scale by the ultrametric structure of the Parisi solution so far only known to be present in (static) spin overlap order parameters. We believe that this manifestation of ultrametricity in the $T=0$ quantum field theory of frustrated interacting systems could also indicate a route to the (to our best knowledge) open question of Parisi RPSB in the theory of fermion localization. This refers to the standard part of the theory, whereas the question of Anderson localization caused by disorder of frustrated interactions is new and arises at a different level; a treatment independent of the precise knowledge of the phase diagram and from the magnetic transition would be hazardous. It is clear that this expected phenomenon may be different from Anderson localization of noninteracting systems as well as from Anderson–Mott transitions.

The model of gaussian distributed interaction constants allows to decouple the effective eighth order fermion correlations by Q–fields, whose average is static for models without quantum spin dynamics. Fluctuation fields

$$\delta Q_{ij}^{ab}(t,t') \equiv Q_{ij}^{ab}(t,t') - \langle Q_{ij}^{ab} \rangle$$

are totally irrelevant in this case.

Taking care of an additional Coulomb interaction (including the local Hubbard part) this situation remains unchanged. This will be included below following the effects of a nonrandom Hubbard interaction that plays an interesting role even in the absence of a fermion hopping hamiltonian. The cooperation with the nonlocal spin interaction and the complications from quantum–spin dynamics present for example in the fermionic Heisenberg spin glasses or in metallic spin glasses of any kind. Quantum spin dynamics, originating from (fermion) spin operators that do not commute with the Hamiltonian, are interesting in itself but avoided here to clarify the role of fermion quantum dynamics.

$$\mathcal{A}_{\text{eff}} \left( \frac{\beta}{2} \right) = \frac{\beta}{h} \sum_{a,b} M_{a,b} \left( \frac{r_i}{r_j} - \frac{r_j}{r_i} \right)$$

is the effective Grassmannian action. This action is exactly of eighth order in the Grassmann fields, since we have chosen a complete gaussian distribution of statistical fluctuations in two–body interactions.

B. Replicated Grassmannian action of the models with gaussian distributed interaction constants

All models listed in section 2 can be represented by the use of anticommuting fields. The partition function reads

$$Z = \int \mathcal{D}\Psi(\tau)e^{-\mathcal{A}}$$

where $\Psi(\tau)$ denotes a master Grassmann fields with imaginary time $\tau$ generated by time–slicing; this field $\Psi$ contains all Grassmann variables that may carry different components in order to take care of different fermion species. Its conjugate $\bar{\Psi}$ is related to $\Psi$ by charge conjugation.

The action can thus be written as

$$\mathcal{A} = \int_0^{\beta h} d\tau \left[ \psi_{\gamma \tau}^* \dot{\psi}_{\gamma \tau} - \mu_{\gamma \tau} + \overline{\chi}_{\gamma \tau} \dot{\chi}_{\gamma \tau} - i\pi \frac{T}{2} \right]$$

$$+ H(\overline{\psi}_{\gamma \tau}, \psi_{\gamma \tau}, \overline{\chi}_{\gamma \tau}, \chi_{\gamma \tau})$$

where all $\psi$ and $\chi$ variables anticommute with each other. The nonhermitean part, which serves to reduce the 4–state per site Fock space to a spin $\frac{1}{2}$ space, can be absorbed by U(1) transformation into the fields

$$\tilde{\chi}_{\gamma}(\tau) \equiv e^{i\pi \frac{T}{2}} \chi_{\gamma}(\tau).$$

In this way the periodicity of the new anticommuting field $\tilde{\chi}$ becomes

$$\tilde{\chi}(\tau = \beta) = -i\tilde{\chi}(\tau = 0)$$

a periodicity that induces semionic imaginary frequencies of the Fourier transformed fields $\chi(\epsilon_l)$ with $\epsilon_l \equiv (2l + \frac{1}{2})\pi k_B T/h$.

The frustrated magnetic interaction of the fermionic Ising spin glass contributes

$$\mathcal{A}_{\text{eff}}^{\text{int}} = - \sum_{a,b} \sum_{i,j} \frac{M_{a,b}}{h^2} \int_0^{\beta h} d\tau \int_0^{\beta h} d\tau' \sigma_i^a(\tau) \sigma_i^b(\tau')$$

$$\times \sigma_j^a(\tau') \sigma_j^b(\tau)$$

where $\sigma_i^a$ denotes the effective Grassmannian action. This action is exactly of eighth order in the Grassmann fields, since we have chosen a complete gaussian distribution of statistical fluctuations in two–body interactions.
to the effective interaction, provided the distribution of the $J_{ij}'s$ is gaussian with 2nd moment $M(r_i - r_j)$. The spin fields are given in terms of the anticommuting fields by

$$\sigma_i^a(\tau) = \sum_{\lambda = \pm 1} \bar{\psi}_i^a(\tau) \lambda \psi_i^a(\tau)$$

IV. RESULTS OBTAINED FROM REPLICA SYMMETRIC APPROXIMATIONS

A. Generating functional for the fermionic Ising spin glass

The derivation of the Green’s function is first given in replica symmetric formalism in order to provide the simplest possible and transparent introduction. The replica–symmetric results are stable solutions only above the Almeida–Thouless line and unstable below. The sole presence of random fluctuations in a many body interaction already complicates the form of fermion propagators and hence deserves attention before the ordered phase and Parisi RPSB in addition will be described.

We will hence ignore until the next section all sorts of replica symmetry breaking, ie the most important Parisi RPSB, also vector replica symmetry breaking (RVSB), recently brought up by Dotsenko and M´ezard in the different context of classical random field systems, which would mean that different copies of the averaged one fermion propagator have to be studied. Another type of RPSB could as well arise in form of a nonzero fermion propagator between between different replicas.

The replica symmetric saddle point solution of the $Q$–field

$$Q_{ab} = q \quad \text{for} \quad a \neq b, \quad \text{and} \quad Q_{aa} = \tilde{q}$$

allows to find the Green’s function in the form

$$G_a^\sigma(\epsilon_i) = \frac{\delta}{\delta \eta_i^a} \ln \int_z \prod_a \int_{y_a} \prod_a e^{\Phi_a(\eta, \eta')} |_{\eta = \eta'}$$

where $y$ and $z$ denote replica–local and replica–global spin–decoupling fields and the gaussian integrations over these fields are written in the abbreviated notation $\int_z \equiv \int_{-\infty}^{\infty} \sqrt{2\pi} e^{-\frac{1}{2}x^2}$. The global field $z$ decouples $(\sum_a \bar{\psi}_a^a \sigma_i^a)^2$ and $y$–integration takes care of VRSB quite naturally within the grassmannian formalism.

The functional can now be expressed in terms of the Green’s function operator $g$ by

$$\exp \left[ \sum_a \Phi_a(\eta, \eta') \right] = \prod_a \int d\psi d\bar{\psi} e^{\sum_a \bar{\psi}_a^a g_{i,a}(z, y^a) \psi_a^a + \eta_i^a \bar{\psi}_a^a \eta_i^a}$$

$$= \exp \left[ \sum_a \Phi_a(0, 0) \right] \exp \left[ - \sum_{a,i} g_{i,a}(z, y^a) \eta_i^a \eta_i^a \right]$$

with

$$g_{i,a}(z, y^a) \equiv \sigma_i(\epsilon_i|z, y^a)$$

$$= \frac{1}{i\epsilon_i + \mu + \sigma(h + J\sqrt{q}z + J\sqrt{q - \tilde{q}}y^a)}$$

For brevity we shall use below the effective field $\tilde{H}(z, y^a) = h + J\sqrt{q}z + J\sqrt{q - \tilde{q}}y^a$. Using the above equations the Green’s function assumes the form

$$G_a^\sigma(\epsilon_i) = \frac{\int_z \prod_{a'} \int_{y'} g_{i,a'}(\epsilon_i|z, y^a') \exp(\sum_{a'} \Phi_a'(z, y^a'|0, 0))}{\int_z \prod_{a'} \int_{y'} \exp(\sum_{a'} \Phi_a'(z, y^a'|0, 0))}$$

which turns into
\[
G_\sigma(\epsilon_l) = \int_G \int_y \frac{g_\sigma(\epsilon_l|z,y) \exp(\Phi(z,y|0,0))}{\int_y \exp(\Phi(z,y|0,0))}
\]  

in the replica limit.

The $\sigma$-functional $\Phi(z,y^n) \equiv \Phi(z,y^n|0,0)$ is regularized at $\tilde{H} = 0$ and $\mu = 0$ in order to retain any physically relevant dependence. This results in

\[
\Phi(z,y) = \sum_{\epsilon_i} \left[ \ln((\epsilon - i\mu)^2 + \tilde{H}^2(z,y)) - \ln \epsilon_i^2 \right]
\]

The frequency sum evaluated in the replica limit yielded $e^{\Phi(z,y)} = 2 \left[ \cosh(\beta\mu) + \cosh(\beta\tilde{H}(z,y)) \right]$ which, together with the result $\int_y e^{\Phi(z,y)} = 2 \left[ \cosh(\beta\mu) + \cosh(\beta\tilde{H}(z,0))e^{\frac{1}{2}\beta^2J^2(q-\bar{q})} \right]$, shows that the replica–symmetric fermion Green’s function follows from

\[
G_\sigma(\epsilon_l) = \int_G \int_y \frac{\cosh(\beta\mu) + \cosh(\beta\tilde{H}(z,y))}{\cosh(\beta\mu) + \cosh(\beta\tilde{H}(z,0))e^{\frac{1}{2}\beta^2J^2(q-\bar{q})}} g_\sigma(\epsilon_l|z,y)
\]  

\[\text{1. Fermion Green’s Function in the disordered phase (} T > T_f)\]

The fermionic Ising spin glass does not allow for spin dynamics unless Glauber dynamics introduces it by coupling to a heat bath. This would be analogous to the SK–model. The ISG$_f$’s Green’s function however is always dynamic and the way in which dynamical behaviour shows up in the spin glass is however nontrivial. The simplest task is to determine first the Greens function above the freezing temperature and to compare it for example with that of a random Hubbard model in the local limit. The fermionic Ising spin glass allows for an exact evaluation of the bare fermion propagator $G_{ij,\sigma} = [\left\langle \frac{1}{\epsilon_{ij} + \mu + \tilde{H}} \right\rangle]_{av}$ in the disordered phase: here $\tilde{H}$ denotes the usual effective field and the double average refers to the replica–local and the Parisi block decoupling fields. The result can be written in the form

\[
G_{ij,\sigma}(\epsilon_n) = -i \sqrt{\frac{\pi}{2J^2\tilde{q}}} \sum_{\lambda=0,\pm 1} \frac{(2 - \lambda^2)e^{\frac{1}{2}(\beta\lambda)^2}ch((1 - \lambda^2)\beta\mu_\sigma)}{\exp(\frac{1}{2}\beta^2J^2\tilde{q})} ch((1 - \lambda^2)\beta\mu_\sigma) (1 - Erff(\frac{\epsilon_n - i\mu_\sigma + i\lambda\beta\tilde{J}^2\tilde{q}}{\sqrt{2}\tilde{q}J})e^{\frac{(\epsilon_n - i\mu_\sigma + i\lambda\beta\tilde{J}^2\tilde{q})^2}{\sqrt{2}\tilde{q}J}}) \delta_{ij},
\]

where $\mu_\sigma = \mu + \sigma H$ includes a magnetic field $H$ and $\epsilon_n = (2n + 1)\pi k_B T/h$. By analytical continuation to real energies (the error function with imaginary argument may then be expressed by the confluent hypergeometric function with real argument) one easily extracts the disorder averaged electronic density of states

\[
\rho_\sigma(\epsilon) = -\frac{1}{\pi} Im G_\sigma^R(E) = \frac{1}{\sqrt{2\pi qJ}} e^{\frac{(\epsilon + \mu)^2}{2J^2\tilde{q}}} \frac{ch(\beta\mu_\sigma) + ch(\beta(\epsilon + \mu))}{ch(\beta\mu_\sigma) + ch(\beta\tilde{H})exp(\frac{1}{2}\beta^2J^2\tilde{q})}
\]

and the real part of the Green’s function as

\[
Re G_\sigma(E) = \frac{ch(\beta\mu_\sigma) E^2}{ch(\beta\mu_\sigma) + e^{\frac{1}{2}\beta^2J^2\tilde{q}} J^2\tilde{q}^2} |F_1(\frac{1}{2}, \frac{3}{2}, \frac{E^2}{2J^2\tilde{q}})|
\]

\[
\quad + \frac{exp(\frac{1}{2}\beta^2J^2\tilde{q} - \frac{(E + \beta\tilde{J}^2\tilde{q})^2}{2J^2\tilde{q}})}{ch(\beta\mu_\sigma) + exp(\frac{1}{2}\beta^2J^2\tilde{q})} |F_1(\frac{1}{2}, \frac{3}{2}, \frac{(E + \beta\tilde{J}^2\tilde{q})^2}{2J^2\tilde{q}})|
\]

\[
\quad + \frac{exp(\frac{1}{2}\beta^2J^2\tilde{q} - \frac{(E - \beta\tilde{J}^2\tilde{q})^2}{2J^2\tilde{q}})}{ch(\beta\mu_\sigma) + exp(\frac{1}{2}\beta^2J^2\tilde{q})} |F_1(\frac{1}{2}, \frac{3}{2}, \frac{(E - \beta\tilde{J}^2\tilde{q})^2}{2J^2\tilde{q}})|
\]

where $1F1$ denotes the confluent hypergeometric function.
2. Spin– and charge–response function derived from the spin glass fermion propagator

As a basis for calculations in the spin glass phase, and in particular under respect of replica symmetry broken fermion propagators, we demonstrate the easiest example of spin– and charge–susceptibilities above the freezing temperature. Both correlations given by

\[ \chi_\sigma \equiv \frac{2}{T^2} \left( \sum_{\sigma, \epsilon} g_\sigma^2 (\epsilon_i) \right) \Phi(y) - \frac{1}{T} \left( \sum_{\sigma} \sigma^\epsilon g_\sigma (\epsilon_i) \right) \Phi(y) \]

\[ = \frac{2}{T} \sum_{\sigma, \epsilon} G_\sigma^2 (\epsilon_i) - \left( \frac{1}{T} \sum_{\sigma} \sigma^\epsilon \tilde{G} (\epsilon) \right)^2 \]

are static and satisfy the relation \( \chi_\sigma + \chi_\rho \equiv 2 \nu \equiv 2 [n] \).

3. Fermion Green’s function in the spin glass ordered phase \((T < T_f)\) (replica–symmetric approximation)

Despite the fact that the replica symmetric result is unstable below the fermionic Almeida Thouless line, hence in any case below the freezing temperature, the fermion propagator is complicated to an extent that requires insight from its simplest realization. This is the extension of Eq. (22) by including the effect of one replica symmetric spin glass order parameter \( q \). By evaluating Eq. (21) we obtain on the imaginary frequencies

\[ G_\sigma^{(0)} (\epsilon_i) = \frac{1}{4J^{\sqrt{q - q}}} \int_{-\infty}^\infty dz e^{-\frac{1}{2} z^2} \frac{1}{ch(\beta \mu) + \text{Exp}(\frac{1}{2} \beta^2 J^2 (\tilde{q} - q) ch(\beta \sqrt{q} z))} \]

\[ \times [2 ch(\beta \mu) e^{\frac{(\epsilon_i - i(\mu + J \sqrt{q} z + \beta J (\tilde{q} - q))}{2 J (\sqrt{q} z - q)}} (1 - \text{Erf}(\frac{\epsilon_i - i(\mu + J \sqrt{q} z + \beta J (\tilde{q} - q))}{J \sqrt{2} (\tilde{q} - q)})]. \]

The asymptotic behaviour either for large energies or for \( T \to 0 \) can be checked by the use of \( 1 - \text{Erf}(x) \cong \frac{1}{\sqrt{\pi}} \frac{\exp(-x^2)}{x} \) and the real part of \( G^{(0)} (E) \) given by

\[ \text{Re} \left[ G^{(0)} (E) \right] = \frac{1}{\sqrt{2 \pi} J (\sqrt{q} - q)} \int_{-\infty}^\infty dz \text{Exp} \left( -\frac{1}{2} z^2 \right) \frac{1}{ch(\beta \mu) + \text{Exp}(\frac{1}{2} \beta^2 J^2 (\tilde{q} - q) ch(\beta \sqrt{q} z))} \]

\[ \times \left[ \text{Exp} \left( -\frac{(E + J \sqrt{q} z)^2}{2 J^2 (\tilde{q} - q)} \right) (E + J \sqrt{q} z) \right] \text{F}_1 \left( \frac{1}{2}, 1; \frac{3}{2}, \left( E + J \sqrt{q} z + \beta J (\tilde{q} - q) \right)^2 \right) \]

\[ + \sum_{\lambda=\pm 1} e^{\lambda \beta J \sqrt{q} z + \frac{1}{2} \beta^2 J^2 (\tilde{q} - q)} e^{\epsilon_i - i(\mu + J \sqrt{q} z + \beta J (\tilde{q} - q))} \left( 1 - \text{Erf}(\frac{\epsilon_i - i(\mu + J \sqrt{q} z + \beta J (\tilde{q} - q))}{J \sqrt{2} (\tilde{q} - q)}) \right) \]

\[ \left. \frac{\left( E + J \sqrt{q} z + \beta J (\tilde{q} - q) \right)^2}{2 J^2 (\tilde{q} - q)} \right) \text{F}_1 \left( \frac{1}{2}, 1; \frac{3}{2}, \left( E + J \sqrt{q} z + \beta J (\tilde{q} - q) \right)^2 \right) \]

The result Eq. (28) is displayed in Fig. (1). Comparison with the extrapolation of the (paramagnetic) \( q = 0 \)–result below \( T_c \), the role of the (replica–symmetric) susceptibility as a measure of the spread of just two wells forming below the freezing temperature is shown. This spread is seen to be related to the size of the magnetic hardgap in this approximation. We shall come back to the essential changes of the fermion propagator in the small energy– and in the long–time regime below in the section on replica permutation symmetry breaking and Parisi symmetry. The fermion density of states in replica symmetric approximation already reveals the strong coupling between spin glass order and fermionic properties. The strongest effect emerges in the zero temperature limit. After analytic continuation \( i\epsilon_i \to \epsilon + i0 \) we find

\[ \rho_\sigma (\epsilon) = \frac{1}{2 \pi J^{\sqrt{q - q}}} \int_{-\infty}^\infty dz e^{-\frac{1}{2} z^2 - \frac{1}{2} \left( \epsilon + J \sqrt{q} z + \beta E \right)^2} \frac{\text{Exp}(\beta J (\tilde{q} - 0)) e^{\frac{1}{2} \beta^2 J^2 (\tilde{q} - q)}}{\text{Exp}(\beta (\sqrt{q} z + \beta J (\tilde{q} - q)))} \]

\[ \text{F}_1 \left( \frac{1}{2}, 1; \frac{3}{2}, \left( \epsilon + J \sqrt{q} z + \beta J (\tilde{q} - q) \right)^2 \right) \]

\[ \text{F}_1 \left( \frac{1}{2}, 1; \frac{3}{2}, \left( \epsilon + J \sqrt{q} z + \beta J (\tilde{q} - q) \right)^2 \right) \]

The \( T \to 0 \) limit is not simple. We first consider the regime
\[ |\mu| < \frac{1}{2} \beta J^2 (q - q). \]  

(30)

One should note that \( q - q = O(T) \), whence this condition requires the Fermi energy to satisfy
\[ E_F < \frac{J}{\sqrt{2\pi}} \]  

(31)

This value was obtained from the rs–selfconsistent order parameter equations above. One may prefer to identify \( \beta J (q - q) \) with the linear susceptibility \( \chi \) of the fermionic Ising spin glass \( ISG_f \).

\[ \rho_\sigma(\epsilon) = \frac{e^{-\frac{1}{2} \beta^2 J^2 (q - q)} (\cosh(\beta \mu) + \cosh(\beta E))}{2\pi J \sqrt{q - q}} \int_0^\infty dz \left[ e^{-\frac{1}{2} z^2 - \frac{1}{2} (\frac{J}{\sqrt{2}} q - q)^2}{\cosh(\beta H(z, 0))} + ... \right] \]

(32)

The \( T \to 0 \) limit is most easily obtained by integrating over the rescaled variable \( u = \beta H(z, 0) \). This isolates the leading contribution as
\[ \rho_\sigma(\epsilon) = \frac{\cosh(\beta \mu) + \cosh(\beta E)}{2\pi \beta J^2 \sqrt{q - q}} e^{-\frac{1}{2} (\beta J^2 (q - q) + \frac{\mu^2}{2 q} + \frac{E^2}{2 J^2 (q - q)})} I(T, E), \]

(33)

where the integral
\[ I(T, E) = \int_{-\infty}^\infty du \frac{1 + O(T u^2)}{\cosh(u) e^{\frac{\mu^2}{2 q}}} \]

(34)

converges for energies satisfying the condition
\[ E < \beta J^2 (q - q). \]

(35)

Since the maximal \( |E| \) within this regime coincides with the limit of a vanishing prefactor and hence a vanishing density of states in the \( T \to 0 \) limit, this energy is to be identified with the gap edge. We call the gap energy \( E_g(h) \).

The replica–symmetric solution for energies inside the \( T = 0 \) gap (and Fermi energies within half of this gap) is hence
\[ E_g(H) := \sqrt{\frac{2}{\pi} J e^{-\frac{1}{2} \frac{\mu^2}{2}}} \]

(36)

and the final result for the leading low temperature DoS–contribution, which vanishes exponentially within the gap \( |E| < E_g(H) \), reads
\[ \rho_\sigma(\epsilon) = \frac{\cosh(\beta \mu) + \cosh(\beta E)}{2 J \sqrt{E_g(H) \cos(\frac{E}{2 E_g(H)})}} e^{-\frac{1}{2} \frac{\mu^2}{2} + \frac{1}{2} (1 - \frac{\mu^2}{2}) (\frac{E}{E_g(H)})^2} \]

(37)

Two striking aspects emerge:

i) there is neither a spin dependence of the gapwidth nor of the leading and exponentially decaying low temperature density of states despite the magnetic field dependences;

ii) the given result is valid only for \( |\mu| < \frac{1}{2} E_g(H) \), while there is no limitation on \( |E| \equiv |\epsilon + \mu| \). Thus the conclusion on the gapwidth is limited to a Fermi energy varying over half the gapwidth.

iii) one would like to see how the Fermi energy can be moved through a gap edge in order to obtain a non–half–filled ground state. From the analysis of the free energy we know however that phase separation occurs beyond \( |\mu| = \frac{1}{2} E_g(H) \), at least in the replica–symmetric approximation. Hence we believe that instanton solutions may have to be used, replacing the homogeneous saddle point solutions above. We conjecture that an instability, identified as a second negative eigenvalue of the Hessian matrix – the first being just the known Parisi–RSB instability –, indicates just this breakdown of the spatially constant solution in the regime of first order transition from a half–filled to a completely filled (or empty respectively) system.

One should nevertheless realize that only \( \mu = 0 \) corresponds to half–filling at any temperature; moreover it is interesting to compare the domain of continuous thermal spin glass transitions which extends to the tricritical points at either \( |\mu(T_{c1})| \approx 0.96125 \) at \( T_{c1} = \frac{J}{2} \) with the \( |\mu(T = 0)| = \frac{J}{2\pi} \approx 0.398942 J \).

Real and imaginary parts of the Green’s function are plotted for various cases in Figs.
FIG. 1. The real part of the replica-symmetric fermion Green's function above and below the freezing temperature $0.6767J$ ($-\text{Re}[G]$ is shown) as a function of $\beta J = \frac{J}{T}$.

FIG. 2. Cross sections through preceding figure (rotated) of $\text{Re}[G]$ at temperatures $T = \infty$ (curve a), $T = T_f$ (b), $T = 0.5J$ (c), and $T = 0.2J$ (d).
FIG. 3. The imaginary part of the replica–symmetric $G^A \sim \text{DoS}$ as a function of energy $E = \epsilon (\mu = 0)$ and $\beta J = \frac{1}{T}$.

$\rho(\mathbf{E}, \mathbf{H} = 0)$

FIG. 4. Creation of the spin glass related gap as the temperature falls below $T_f$ (cross sections of preceding figure): curve a: $T = \infty$, b: $T \approx T_f$, c: $T = J/2$, d: $T = J/3$, f: $T = J/6$, T = .1J, g: $T = .05J$, and h: $T = .025J$.

4. The magnetic hardgap of the DoS at $T = 0$

The density of states $\rho(T = 0, |E| > E_g(h))$ is nonzero; it can be evaluated by the saddle point method. We start out from Eq. (29). The very same condition $|\mu| < \frac{1}{2}E_g(h)$ applies again. We shall come back to this below.
While \( q, \chi, \) and \( \nu(\mu) \) can be obtained by extremizing the free energy, ie without explicit knowledge of the underlying density of states (DoS), this fermionic DoS is of course related to the magnetic properties of the system and its analysis requires the solution of several selfconsistency equations for spin– and for charge correlations. We obtained these solutions for low temperatures and for the entire range of magnetic fields and of single fermion excitation energies. The existence of a hardgap of size \( 2E_g(H) \) is best appreciated in our \( T = 0 \)–result for the density of states, which is given by

\[
\rho_\sigma(\epsilon; H) = \frac{1}{\sqrt{2\pi}J} \Theta(|E| - E_g(H)) \exp \left[ -\frac{1}{2} \left( \frac{E}{J} \right)^2 \right],
\]

with \( E := \epsilon + \mu \). For finite range models we expect exponentially small corrections within the gap and due to Griffiths singularities. Eq.(38) is valid within the regime \( |\mu| < \frac{1}{2} E_g \), which corresponds to a half filled system at \( T = 0 \) with filling factor

\[
\nu = 1 + (1 - q - T\chi )tanh(\beta \mu).
\]

The zero field ratio \( E_g(0)/T_c \) increases with the chemical potential from 1.179 at \( \mu = 0 \) to 1.238 at \( \mu = J/\sqrt{2\pi} \). In terms of \( E_g(H) \) the following low temperature expansions (for linear susceptibility and order parameter), required for the exact evaluation of the DoS formula (29) at low \( T \), were obtained as

\[
q = 1 - \frac{E_g(H)}{J} + O(T^2/J^2),
\]

\[
\chi = \frac{E_g(H)}{J^2} + \frac{1}{2} \frac{E_g^2(H)}{J^3} \left( 1 - \frac{H^2}{J^2} \right) T + O(T^2/J^2).
\]

These results show that i) the density of states is zero at \( T = 0 \) in the finite interval given by \( |\epsilon + \mu| < E_g(H) \) and ii) the gapwidth shrinks as the magnetic field is increased. This will be the source of the negative magnetoresistance in the extended Ising spin glass model with charge transport as discussed below. The physical consequences of these spin glass related properties are hence in agreement with the experimental observations of crossover behavior in the low \( T \) resistivities mentioned above. For single fermion energies \( E \equiv \epsilon + \mu \) smaller than the gap energy \( E_g(H) \) and \( |\mu| < \frac{1}{2} E_g(H) \) the density of states decays to zero exponentially as given by

\[
\rho_\sigma(\epsilon) = \frac{1}{2 J \cos(\pi E_g(H)/2E_g(H))} e^{-\frac{E_g^2(H) + E^2}{2J^2} - \frac{\epsilon^2}{2J^2}} \left[ \tanh(\beta \mu) + ch(\beta E) \right] e^{-\frac{\beta}{2} \left( \frac{E_g^2(H) + E^2}{E_g^2(H)} \right)}.
\]

We find that the hardgap persists also in itinerant models with a fermion hopping term added to the nonconducting Ising Hamiltonian, until the bandwidth exceeds a critical value. The gap and its related properties of this extended itinerant spin glass model, whose magnetic phase diagram and \( \nu \)–transitions have previously been analysed\cite{[16][10]} are discussed below.
FIG. 5. Single particle density of states (DoS) for the fermionic Ising spin glass (ISG) versus energy and magnetic field in units of J. The hardgap centered around zero energy is due to random magnetic correlations and manifests itself in an activated behavior of the low temperature hopping conductivity of disordered magnetic materials. The reduction of the gapwidth by an external magnetic field explains the observed negative magnetoresistance.

Coulomb interaction effects are of diverse nature: the long range part will still tend to depress the (remaining) DoS near the Fermi level and thus stay particularly relevant when either spin glass order is weak or absent, or (in the case of fully developed spin glass order) when the Fermi energy lies close to the gap edges. The Hubbard coupling U leads to a shift of the chemical potential \((\mu \rightarrow \mu - \frac{U}{2}\nu)\), and also to a shift \(H \rightarrow H' \equiv H + \frac{U}{4m}(\theta)\) of the applied magnetic field. This implies for finite temperatures that the Almeida Thouless line depends on the Hubbard coupling too.

5. The replica–symmetric approximation of the \(T = 0\) fermion propagator

The density of states could be used as the spectral weight in the Kramers–Kronig relation for the fermion Green’s function. One may as well extract the zero temperature limit of Eq. (27). This results in the (bare \(K = 0\)) fermion propagator \(G_{\sigma}^{(K=0)}(\epsilon)\)

\[
G_{\sigma}^{(0)}(\epsilon) = -i \int_{0}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{4}z^2} \left\{ \frac{1}{\epsilon - i(\mu + z + \chi_0)} + \frac{1}{\epsilon - i(\mu - z - \chi_0)} \right\}
\]

with energies \(\epsilon_l = (2l + 1)\pi T \rightarrow \epsilon\) lying dense on the imaginary axis and \(\chi_0 = \lim_{T \to 0} \beta(\langle q - q \rangle)\) representing the RS–approximation of the susceptibility (here and in the following we set \(J = 1\) for brevity). The most remarkable fact being that the gaussian distribution of say self–energies \(z + \chi_0\) are now cutoff, which is responsible for the hardgap described above. Recall that in Eq. (27) one starts off from a complete gaussian average over all \(z\).

Insertion the zero temperature results for the density of states, Eqs. (29) and (38), in the spectral representation

\[
G_{\sigma}(\epsilon_l) = \int_{-\infty}^{\infty} d\epsilon' \frac{\rho_{\sigma}(\epsilon')}{i\epsilon_l - \epsilon'}
\]

confirms the representation of Eq. (43). The final integral in Eq. (43) can be evaluated to give the following explicit solution

\[
G_{\sigma}^{(0)}(\epsilon) = -\frac{1}{\sqrt{8\pi}} \sum_{\lambda=\pm 1} \left[ e^{-\frac{A^2}{2}} Ei(-\frac{A^2}{2}) + i\pi\lambda \text{sign}(\epsilon) e^{\frac{A^2}{2}} \left[ 1 - \text{Erf}(\frac{A\lambda}{\sqrt{2}}) \right] \right]
\]
with
\[ A_\lambda(\epsilon) \equiv \epsilon - i [\mu + \lambda \chi_0], \] (46)
and \( \text{Ei}(x) \) denoting the exponential integral function. The results are given for zero magnetic field; the fact that the chemical potential appears in the form \( \mu \pm \chi_0 \) shows the effect of the Onsager reaction field. The oversimplification of the replica–symmetric approximation is revealed by the \( \mu \)–shift, resembling only contributions from up and down local fields.

The time–dependence is obtained from
\[
G^{(0)}_{\sigma}(\tau) = \int_0^\infty \frac{dz}{\sqrt{2\pi}} e^{-\frac{z^2}{2\tau}} \left\{ \left[ e^{-\mu_+(z)\tau} \theta(-\mu_+(z)) + e^{-\mu_-(z)\tau} \theta(-\mu_-(z)) \right] \theta(\tau) \\
+ \left[ e^{-\mu_+\tau} \theta(\mu_+(z)) + e^{-\mu_-\tau} \theta(\mu_-(z)) \right] \theta(-\tau) \right\},
\] (47)
where we introduced effective chemical potentials
\[ \mu_{\pm}(z) \equiv \mu \pm (z + \chi_0) \] (48)
that depend on the spin decoupling field and on the lowest order susceptibility. The latter represents the Onsager field contribution.

We recall that the given solution is valid for \( \mu < \frac{1}{2} \chi_0 \) (apart from its instability against RPSB that is taken into account below), which means half–filling at \( T = 0 \). Evaluation of the imaginary–time dependent propagator yields
\[
G^{(0)}_{\sigma}(\tau) = \frac{1}{2} e^{-\mu_0\tau} \tau^{-\frac{1}{2}} \left[ 1 + \text{Erf}\left( \frac{\tau}{\sqrt{2}} \right) \right] \theta(\tau) + \frac{1}{2} e^{-\mu_0\tau} \tau^{-\frac{1}{2}} \left[ 1 - \text{Erf}\left( \frac{\tau}{\sqrt{2}} \right) \right] \theta(-\tau)
\] (49)
Alteration of this time–dependent behaviour under RPSB will be an important conclusion below. Before this problem is attacked, it is interesting to compare the replica symmetric fermionic spin glass propagator with a propagator of random U Hubbard models in the local limit. Fermion hopping, which introduces the full complexity of the Hubbard model, plays also an important role in promoting the insulating fermionic spin glass to a metallic spin glass model. Since all of the complicated physics of both model classes is sandwiched between the (identical and solvable) fermion hopping model and the (different and solvable, provided an infinite–range magnetic interaction is considered) pure interaction limits, the latter limit offers the most interesting case for comparing the two solutions.

6. Comparison and mapping with the Green’s function of a random U local Hubbard limit

The wellknown exact solution of the local limit played an important role as starting points of recent many body theories of the Hubbard model. The field–theoretic decoupling given by Vollhardt\cite{28} can be extended to account for disorder fluctuations in the coupling \( U \). We shall compare the spin glass and the random U Hubbard model field theoretic techniques in Appendix B. For the purpose of comparing the replica–symmetric fermion propagator of the fermionic spin glass with a potential random U Hubbard analog in the local limit, we do not need this apparatus. It is sufficient to consider a gaussian average over \( U \) with a properly chosen cutoff to see the similarity with Eq. (43). As well one may compare the Green’s function obtained above the freezing temperature with the local Hubbard limit. At half–filling the Green’s function reads
\[
G_{\sigma}(\epsilon_l) = \frac{1}{2 \iota \epsilon_l + U/2 + \frac{1}{\iota \epsilon_l - U/2}}
\] (50)
Since this is the exact solution for any half–filled \( U \)–realization in the local random U model, it is easy to evaluate not only gaussian distributions but also Lorentz–, box–, and semicircular–distributions of random \( U \)–couplings in order to obtain the disorder–averaged one particle Green’s function. The average \( < \log Z > \) can also be derived. Thus these results serve also as a interesting test field for the replicated field theory, which usually runs into formal difficulties whenever nongaussian distributions are involved.

For unrestricted gaussian–distributed \( U \) (equal weight of attractive and repulsive interaction) one obtains
\[
[G_{\sigma}(\epsilon_l)]_{P_g(U)} = \pi e^{\frac{\epsilon_l^2}{4MU}} (1 - \text{Erf}\left( \frac{\epsilon_l}{\sqrt{4MU}} \right)) \text{sgn}(\epsilon_l)
\] (51)
and the density of states is a simple gaussian
\[ [\rho]_{P_c(U)}(\epsilon) = \sqrt{\frac{MU}{8\pi}} e^{-2\epsilon^2/\delta U} \] 

(52)

In order to generate a solution that matches the fermionic spin glass propagator, Eq. (43), one needs to cut off the gaussian distribution.

For \( \mu = 0 \) it is just the Onsager reaction field\(^{30}\) of the spin glass, which requires to introduce a finite average \(< U >\) by

\[ U = < U > + \delta U, \] 

(53)

where the fluctuations obey a gaussian distribution of nonnegative \( \delta U \)-fluctuations

\[ P^c(\delta U) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{8}(\delta U)^2) \Theta(\delta U) \] 

(54)

which maps the fermion propagator of this random U Hubbard interaction onto the replica–symmetric approximation of the fermionic Ising spin glass propagator at \( T = 0 \) and \( \mu = 0 \), provided (again given in dimensionless quantities)

\[ < U > = 2\chi_0 \] 

(55)

This mapping involves only repulsive but random interactions. Small random deviations from the average positive U are strongly weighted, which renders the model perhaps a realistic one. The relation (55) also provides an illustration in terms of the Hubbard interaction for the gap–width \( 2\chi_0 \) of the fermionic spin glass.

It is interesting to imagine an additional fermion hopping term, thus a Hubbard model with the given U–fluctuations: realiztions of large U can be described by a \( t-J \) model and the entire regime down to perturbative small U is contained with increasing weight.

The random U local Hubbard limit, introduced here for purely formal reasons, is a zero–dimensional problem and one cannot expect to find a spin glass transition like the one described for the Ising spin glass model with nonlocal or even infinite–range interaction. Allowance for fermion transport however reopens this possibility and requires further analysis, also in relation with other metallic spin glasses. Moreover, the problem of RPSB can be raised for the local limit of the random U Hubbard model too, which means that the exact solution in the random case may not simple. We defer this analysis to another publication.

In contrast to the described situation, which contains a density of states gap identical to the fermionic spin glass by construction, a gaussian distribution without cutoff removes the gap and poses hence a problem to the existence of a Mott transition of the Hubbard model. A striking difference with respect to the interaction gap of the frustrated nonlocal spin interaction is to be noticed. In this respect the statistical fluctuations of the Hubbard interaction compete with both the nonrandom part of U and with the spin coupling \( J_{ij} \). We first need to evaluate exactly the partition function and the generating functional of the random local limit, which passes through a time–dependent Grassmann field theory.

In contrast to fermionic spin glass models it is easy to consider distributions other than gaussian in the local Hubbard limit. The latter may thus become helpful for the spin glass problem.

For example, the box distribution \( P_b(U) \) yields the averaged Green’s function

\[ [G_\sigma(\epsilon_l)]_{P_b(U)} = \frac{-2i}{U_2 - U_1} \left[ \arctan \frac{U_2}{2\epsilon_l} - \arctan \frac{U_1}{2\epsilon_l} \right] \] 

(56)

which turns into the retarded function

\[ [G^R(\epsilon)]_{P_b(U)} = \frac{1}{U_2 - U_1} \ln \left[ \frac{2\epsilon - U_2}{2\epsilon - U_1} \right] \] 

(57)

Depending on the parameters of the box distribution the logarithmic cuts may either overlap or remain separated. In the latter case the Mott transition induced by sufficiently strong fermion hopping will occur; the case of a local limit with arbitrarily small gap should help to study the Mott transition also arbitrarily close to the local limit.

7. Some physical quantities related to the gapped fermion density of states for \( T < T_f \)

Below the freezing temperature \( O(q^2) \)–corrections occur in \( < \rho > \). Using the exact low temperature solutions of section 2.2 one can obtain some information about the density of states deep in the ordered phase. From the relation
\[ \nu_{av} = 2 - \bar{q} \] (valid in this form only at \( T = 0 \)) and \( \bar{q}(T = 0, 0 < \mu < \frac{1}{\sqrt{2\pi}}) = 0 \) follows that the system stays half filled independent of the chemical potential. This quantity can also be obtained from the disorder averaged one particle density of states by

\[
[\nu]_{av}(\mu) = \sum_{\sigma = \uparrow, \downarrow} \int_{-\infty}^{\infty} d\epsilon f(\epsilon) [\rho]_{av}(\epsilon, \mu)
\]  

(58)

and hence display consequences of the interaction induced gap around zero energy. The situation may be comparable to the antiferromagnetic gap in the clean Hubbard model\(^\text{16}\).
V. QUANTUM FIELD THEORY WITH BROKEN REPLICA PERMUTATION SYMMETRY

This chapter describes the central results of the present paper. The Parisi symmetry, described in terms of symmetric groups and more commonly known for its ultrametric structure, is now shown to characterize a quantum field theory. The problems which arise in the QFT due to the effect of RPSB are first resolved in a one–step breaking scheme. Then general relations for arbitrary steps of RPSB are found in addition, which allow conclusions on the low temperature behaviour.

The fermion Green’s function can be derived, like any other correlation function and at any order K of Parisi–RPSB, from the generating functional \( \Xi \). The one–fermion Green’s function can be obtained according to

\[
G_{\alpha\alpha}(\epsilon_l) = \frac{\delta}{\delta \eta_{\alpha,\sigma}} \frac{\delta}{\delta \eta_{\gamma,\sigma}} \ln \Xi_n(\{\eta\}, \{\bar{\eta}\})
\]

employing derivatives with respect to the generating fields \( \eta \) and \( \bar{\eta} \). This functional involves gaussian averages over \((K+1)\) fluctuation fields \( z_\gamma \). Since details are given in Appendix 1, we simply introduce its structure by the following symbolic shorthand notation

\[
\Xi_n(\eta, \bar{\eta}) = e^{-\frac{1}{2} \beta^2 J^2 Tr Q_{Parisi}} \left[ \prod \int_{z_\gamma}^G \right] \int D\Psi Exp \left[ \bar{\Psi} g^{-1} \Psi + \eta \bar{\Psi} - \bar{\eta} \Psi \right]
\]

where

\[
g_\sigma(\underline{\eta}, \epsilon_l|\{z_\gamma\}) = (i\epsilon_l + \mu + \sigma \hat{H}(\{z_\gamma^0\}))^{-1}
\]

denotes the unaveraged fermion Greens function of the localized fermion exposed to the statistically fluctuating effective field

\[
\hat{H}(\{z_\gamma^0\}) = H + \sum_\gamma \sqrt{q_\gamma - q_{\gamma+1}} z_\gamma^{(\alpha_\gamma)}
\]

which depends on all fluctuation fields \( z_\gamma \). The fermion–propagator of the insulating models is of course local in real space. Spin (decoupling)–fields \( z_\gamma \), carrying a Parisi block index \( \gamma \) (\( \alpha_\gamma \) runs over all replicas which belong to a particular Parisi block), explore the random magnetic order. In the metallic case, the bare Green’s function \( g_0 = g(\hat{H} = 0) \) and hence \( g \) itself become nonlocal, a complication that need not be studied now in order to understand the Parisi symmetry of the QFT.

The Parisi matrix \( Q_{Parisi} \) has the wellknown form \( \eta \) apart from the nonvanishing diagonal elements \( \tilde{\eta} \) (which we identify with \( g_0 \)); their presence is required by the fact that \((\tilde{\delta}^*)^2 = (\tilde{\eta_1} - \tilde{\eta}_2)^2 \neq 1 \). The structure of the Parisi–matrix is of course responsible for the rather complicated form of the Lagrangian (see Appendix 1); despite this complication the fermion fields can be eliminated in the standard way, which leads to the selfconsistent equations given below.

**A. One–step replica permutation symmetry breaking**

It is instructive to start with a one step replica symmetry breaking (denoted by \( K = 1 \)), which is also considered to be the correct result in the regime of discontinuous thermal transitions. Here we wish to develop the full Parisi solution from this first nontrivial starting point. Denoting by \( q_1 \) and by \( q_2 \) the new offdiagonal block elements, the decoupling field \( z \) splits up into \( z_1 \) and \( z_2 \). The Green’s function assumes the suggestive form

\[
G_\sigma(\epsilon_l) = [[[i\epsilon_l + \mu + \sigma \hat{H}(z_1, z_2, \eta^0)^{-1}]_{y^a}]_{z_1}]_{z_2}
\]

which includes three successive averages to be specified below. The instability of the replica–symmetric solutions becomes strongest as \( T \to 0 \), but only recently this has found its explicit representation in a zero temperature quantum field theory including the low temperature effects. While quantum magnetic transitions are so far unaffected by this instability to a large extent, the fermionic density of states and hence the fermion Green’s function were observed to depend strongly on RPSB. This leads to a quantum field theory with Parisi symmetry, ie with ultrametric structure. The fermion propagator had acquired a nonsimple form already in the replica–symmetric approximation due to the random interaction on one hand and furthermore through the effects of spin glass order. We now wish to show in
more detail how this solution is improved at first–step breaking $K = 1$ and furthermore how a $K$–invariant relation can be derived to prove the one–fermion pseudogap reported in\[.\]

We have recently presented the first solution to the question how Parisi replica permutation symmetry breaking (RPSB) and the related nonconstant part of the Parisi spin glass order parameter function $q(x)\equiv 0$ are displayed in the low temperature many body theory of fermionic systems with frustrated Ising–interactions, emphasizing the $T = 0$–limit in particular. Despite the fact that the interval $0 \leq x \leq x_1$ of nonconstant $q(x)$ representing RPSB vanishes with temperature $T$ as $T \to 0$, we find a large $O(T^0)$–effect to persist in many important physical quantities. This includes replica–diagonal fermion Green’s function and fermion density of states, where at any step $K$ of RPSB the set of different order parameters is seen to determine the quantum–dynamical behaviour of the fermion propagator and of vertex functions. These effects are complementary to and not in contradiction with recent replica–symmetric descriptions of $T = 0$ quantum spin glass transitions\[.\] Parasi–RPSB is seen to decide the qualitative and quantitative features of the low energy excitation spectrum. While results are presented for an insulating model, the effect appears to be rather model–independent and should hence be felt in transport properties of models with additional hopping hamiltonian for example.

We wish to provide results which evidence the fact that fermionic spin glasses also link closely glassy magnetic order and transport behaviour; further similarities between Hubbard model and the fermionic spin glass have been traced back to the particular role of the Onsager–Brout–Thomas reaction field and of vertex functions, thus on the entire ensemble of quantities that provide the basis of many body theories for fermionic systems with frustrated interactions.

It is known that replica–diagonal quantities like the linear equilibrium susceptibility $\chi$ feel Parisi symmetry breaking even at $T = 0$ despite the fact that the nontrivial part of the Parisi function only lives on an interval of width $T$. The susceptibility had been analysed by Parisi for the standard SK–model who found a rapid convergence towards the exact result as the number of order parameters increased, this number being equal to $K + 1$ in the SK– and equal to $K + 2$ in fermionic models. While the low temperature regime of the SK–model had not been of particular interest from the point of view of phase transition theory, it becomes highly important for fermionic spin glasses, since the $T = 0$–theory of excitation spectra plays a crucial role and, for the additional reason that some models exhibit quantum phase transitions along the $T = 0$–axis. Parisi nevertheless analysed the low $T$ regime of the SK–model finding that K–step RPSB on one hand provided increasingly good approximations but failed to completely remove the negative entropy and the instability problem at low enough temperatures unless $K \to \infty$.

In this paper the effect of one step RPSB on the density of states is presented in detail, followed by the derivation of an analytical relation valid for all $K$, which allows to determine the type of excitation spectrum present in the full Parisi solution for the fermionic Ising spin glass. Despite the fact that the regime of deviation from a replica–symmetric spin glass order parameter is only of $O(T)$, we find that it has a large $O(T^0)$–effect on the density of states, the one–particle–, and many–particle Greens functions at $T = 0$. This density of states is derived as usual from the imaginary–time (disorder–averaged) fermion Green’s function $\langle \bar{\psi}(\tau)\psi(\tau')\rangle_{av}$, which is one of the decisive quantum–dynamical elements of any many body theory of fermionic spin glasses. This illustrates that, unlike the usual picture of a Parisi solution being just a static order parameter function, the fermionic picture must include the qualitative extension to dynamical quantities. Those become drastically altered by the nontrivial part of the Parisi solution, which is otherwise invisible at $T = 0$, hence providing a quantum–dynamical image of RPSB.

Let us consider the Parisi solution of the infinite–range fermionic Ising spin glass defined as the SK–model extension in a grand canonical ensemble with spin one half operators represented as $\hat{\sigma}^z = \hat{n}^\uparrow - \hat{n}^\downarrow$.

1. Fermionic density of states

It is known since Parisi’s work\[.\] that an analytical low temperature expansion is hard to obtain even for the standard SK–model and its smaller set of selfconsistent parameters. First insight is gained by the one–step RPSB ($K = 1$). The standard three parameter set of the SK–model for $K = 1$, order parameters $q_1$ and $q_2$, and $m = m_1 \sim T$, is enlarged in the fermionic space by $\tilde{q} - q_1 \sim T$, where $\tilde{q} := |< \sigma(\tau)\sigma(\tau') > |_{av}$ represents a spin correlation, which remains static unless a fermion hopping mechanism or other noncommuting parts are included in the Hamiltonian. For the fermionic Ising spin glass the $(K = 1)$–DoS reads
\[
\rho_s(E, H) = \frac{ch(\beta \mu) + ch(\beta E) e^{-\frac{1}{2} \beta^2 (\bar{q} - q_1)} \int_{-\infty}^{\infty} dv_2 e^{-\frac{v_2^2}{2\beta^2}} \int_{-\infty}^{\infty} dv_1 e^{-\frac{(v_1 - \bar{q})^2}{2\beta^2} - \frac{v_1 + \mu + E^2}{\beta^2}} c^{m-1}}{\sqrt{2\pi}(\bar{q} - q_1)} \sqrt{2\pi q_2} \int_{-\infty}^{\infty} dv_1 e^{-\frac{(v_1 - \bar{q})^2}{2\beta^2} - \frac{v_1 + \mu + E^2}{\beta^2}} c^{m}}
\]

where

\[
C = \cosh(\beta H) + \zeta, \quad \zeta = \cosh(\beta \mu) \exp(-\frac{1}{2} \beta^2 (\bar{q} - q_1)).
\]

This expression reveals the competition between the particle "pressure" exerted by the chemical potential \(\mu\) and the single-valley susceptibility

\[
\bar{\chi} = \beta (\bar{q} - q_1)
\]

leading to a crossover at \(|\mu| = \frac{1}{2} \bar{\chi}\) in the \(T \to 0\)–limit. The \(\zeta\)–term is a fermionic feature which is absent from the standard SK–model. It is closely related to the fermion filling; this filling factor behaves discontinuously on the \(T = 0\)–axis. The zero temperature limit allows to perform the \(v_1\)–integrations, which results in the exact \(T = 0\) formula for the density of states

\[
\rho_s(E) = \frac{e^{\frac{1}{2} \beta^2 (\bar{q} - q_1) + \Delta_E - \frac{\bar{q}^2}{2\beta^2}}}{\pi \sqrt{1 - q_2(H)}} \int_{-\infty}^{\infty} dz e^{-\frac{1}{2} \beta^2 (\bar{q} - q_1) + \Delta_E - \frac{\bar{q}^2}{2\beta^2}} \Theta(\Delta_E)
\]

using the definitions

\[
d(z) \equiv e^{a(H)\sqrt{\pi}z} \left[1 + \text{Erf} \left( \frac{a(H)(1 - q_2) + \sqrt{\pi}z}{\sqrt{2(1 - q_2)}} \right) \right],
\]

\[
\Delta_E \equiv |E| - w(H),
\]

\[
a(H) \equiv \lim_{T \to 0} \frac{d}{dT} m(T, H).
\]

These equations allow to derive the first improved approximation beyond the replica–symmetric solution, given in the preceding section, which displays a magnetic hardgap of width \(2E_g(H)\) in the DoS and the fact that half-filling at \(T = 0\) extends over the finite interval of chemical potentials given by \(|\mu| < \frac{1}{2} E_g(H)\). Beyond this interval phase separation occurs together with a discontinuous transition into a full or an empty system. More details about the phase diagram are presented in the subsequent paper II.

A stable and spatially–homogeneous saddle–point solution could only be found for the half–filled case. Thus the following analysis is restricted to this interval of chemical potentials. Its width is determined selfconsistently and seen to decrease to zero as \(K\) tends to infinity. For \(|\mu| \leq \frac{1}{2} \lim_{T \to 0} (\beta (\bar{q} - q_1))\) we first derive the coupled set of selfconsistent equations

\[
\bar{q} = 1 - \int_2 \frac{C^{m-1} \zeta}{f_1 C^m} \left(1 \right)
\]

\[
q_1 = \int_2 \frac{C^{m-2} \sinh^2(\beta H)}{f_1 C^m} \left(1 \right)
\]

\[
q_2 = \int_2 \frac{C^{m-1} \sinh(\beta H)}{f_1 C^m} \left(1 \right)
\]

\[
0 = \frac{\partial f}{\partial m} = \frac{\beta J^2}{4} (q_1^2 - q_2^2) + T \ln \int_2 C^{m} - \frac{T}{m} \int_2 C^{m} \ln C
\]

The shorthand gaussian integral notation

\[
\int_1^G \equiv \int_{-\infty}^{\infty} \frac{du_k}{\sqrt{2\pi(q_k - q_{k+1})}} \exp \left[-\frac{u_k^2}{2(q_k - q_{k+1})} \right]
\]

adopts the normalization used by Parisi. For \(T = 0\), we solve the selfconsistency equations up to one final integral, finding the \(T = 0\)–set of coupled equations
\[
\tilde{q} = q_1 = 1, \quad \lim_{T \to 0} \frac{\tilde{q} - q_1}{T} = \tilde{\chi}, \quad q_2 = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{(z - H/\sqrt{q_2})^2}{2}} \left[ \frac{d(z) - d(-z)}{d(z) + d(-z)} \right]^2
\]
(76)

\[
0 = 1 - q_2^2 - \frac{4}{a} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{(H/\sqrt{q})^2}{2}} \left( -\frac{1}{a} \ln \left[ \frac{1}{2} e^\frac{H^2}{2q_2} (d(z) + d(-z)) \right] + \left[ (at + \sqrt{q_2} z)d(z) + (at - \sqrt{q_2} z)d(-z) + \sqrt{8t/\pi} \ e^{-\frac{1}{4} a^2 - \frac{1}{2} q_2 z^2 / t} \right] / [d(z) + d(-z)] \right)
\]
(77)

where \( t \equiv q_1 - q_2 \). All parameters represent the temperature– and magnetic field–dependent solutions. The solutions for \( T = 0 \) and \( H = 0 \) are given by

\[
q_2 = 0.476875
\]
(78)

\[
a = \lim_{T \to 0} \frac{d}{dT} m(T) = 1.36104
\]
(79)

\[
w = \lim_{T \to 0} \frac{\tilde{q} - q_1}{T} = 0.239449
\]
(80)

FIG. 6. Field dependence of \( dm/dT \) (top), of the order parameter \( q_2 \), and of gapwidth parameter \( w \) (bottom) for 1RPSB and zero temperature.
The H–dependent solutions shown in fig. 6 are then used in evaluating eq. (67) for the density of states. $T = 0$–results are shown in figs. 7 and 8, while the result at finite low temperature of fig. 9 illustrates the presence of plateaus of constant slope, each corresponding to Parisi order parameter separations (here: $\tilde{q} - q_1$ and $q_1 - q_2$). The number of these plateaus of constant slope increases with the order $K$ of Parisi–RPSB. Hence, the time–dependence of the Green’s function should characteristically depend on the order parameter separations $q_k - q_{k-1}$.

FIG. 7. Zero temperature density of states as a function of energy and magnetic field for 1–step RPSB

FIG. 8. Effect of one step replica symmetry breaking on the fermionic DoS for magnetic fields $H = 0$ (curve c: 1RPSB, a: 0RPSB) and $H/J = 0.6$ (d: 1RPSB, b: 0RPSB)
FIG. 9. The 1–step RPSB solution for the zero field density of states at very low temperatures ($T = 0.01J$) ($\rho$ is symmetric w.r.t. $E$). The $T = 0$ hardgap of reduced size is already digged out, yet two different slopes $\frac{d\rho}{dE}$ corresponding to $q_0 - q_1 = O(T)$ (steep portion) and to $q_1 - q_2 = O(1)$ remain visible.

FIG. 10. DoS at 1–step RPSB for magnetic fields $H/J = .2$ (curve a), .3 (b), .5 (c), and .6 (d).

If we compare with the replica–symmetric result a reduction of the gapwidth is observed. Analytically one finds a gapwidth

$$E_g(H) = \tilde{\chi} = \lim_{T \to 0} \beta(\tilde{q} - q_1)$$

(81)

which turns into $\lim_{T \to 0} \beta(\tilde{q} - q(1))$ in terms of the Parisi function $q(x)$ at $K = \infty$. Only in absence of RPSB this susceptibility coincides with the equilibrium one, denoted by $\chi$. In fact for 1–step RPSB the fermionic Ising spin glass approaches $\chi = \beta(\tilde{q} - q_1) + \beta m(q_1 - q_2) \to .95$, the same numerical value as the one for the SK–model.

2. The fermion propagator

We are interested in the explicit form of the fermion propagator at one–step RPSB for several reasons. To mention a few, this function $G^{-1}(\epsilon_1)$ is an indispensable element of the quantum field theory with Parisi symmetry, it is directly
linked to some observables, it contributes by \( \text{Re}(G) \) a second quantity which illuminates the crossover to a pseudogap, it is also needed to formulate the Ward identity, and it adds another challenge for the mapping on a random \( U \) Hubbard model in the local limit. We start out from Eq.\( \text{(124)} \) (as derived in Appendix 1) which yields in the replica limit for \( K = 1 \)

\[
G^1(\epsilon_i) = \int_2^1 C_{m-1} \frac{e^{F_{\epsilon_i + \mu + \sigma H(z_1 = 0, z_2)}}}{C_m},
\]

where \( \text{exp}(\Phi) = 2(\cosh(\beta \tilde{H}) + \cosh(\beta \mu) e^{-\frac{1}{2}\beta \xi}) \).

In the regime \( \mu < \frac{1}{2} \xi \) one obtains

\[
G^1(\epsilon_i) = -i \left[ \frac{1}{2} \int_{z_2}^{\tilde{H}} \frac{e^{G(z_1)}}{C_m} \int_{z_1}^{G(z_2)} C_{m-1} \right] 2 \cosh(\beta \mu) e^{-\frac{1}{2}(q + q_1)} \left[ 1 - \text{erf} \left( \frac{\epsilon_i - i(\mu + \tilde{H}_0)}{\sqrt{2}(q - q_1)} \right) \right] \left( \epsilon_1 - \frac{i(\mu + \tilde{H}_0)}{\sqrt{2}(q - q_1)} \right)
\]

The zero temperature limit (using \( \lim_{T \to 0} \beta m = a \) finite) simplifies this result and can be cast into the form

\[
G^1(\epsilon_i) = \int_{-\infty}^{\infty} dz_1 \text{exp} \left( -\frac{1}{2} z_1 \right) \int_{-\infty}^{\infty} dz_2 \Theta(\tilde{H}(0, z_1, z_2)) e^{-\frac{1}{2} z_1^2} e^{a\tilde{H}_0} \frac{1}{\text{erf} \left( \frac{\epsilon_i - i(\mu + \tilde{H}_0)}{\sqrt{2}(q - q_1)} \right)} \left( \epsilon_1 - \frac{i(\mu + \tilde{H}_0)}{\sqrt{2}(q - q_1)} \right)
\]

where \( \tilde{H}_0 \equiv \tilde{H}(0, z_1, z_2) \). The spectral representation \( \text{[44]} \) now employs the 1-step RPSB-result for the density of states Eq.\( \text{(64,67)} \).

Before evaluating the \( z_1 \)-integrals we can reconsider the mapping with the local limit of a random \( U \) Hubbard model and the changes that occur due to RPSB in the spin glass. Now

\[
< U > \to \sqrt{\frac{dU}{2}} \tilde{H} \equiv \sqrt{q_1 - q_2} \left( z_1 + \sqrt{q_2} \right) z_2
\]

The second correspondence shows that only positive \( U \)-fluctuations around a positive mean value occur, which decreases with each step of RPSB like the gapwidth of the fermionic spin glass. However, modelling the latter by the random \( U \) interaction requires \( (K + 1) \) fluctuation–fields \( \delta u_\alpha \) at \( K \)-th order RPSB and subject to the constraint \( \delta U \equiv \sum \delta u_\alpha > 0 \). Apart from the overall condition of positive \( U \) and \( \delta U \) the fluctuation–fields are allowed to assume any negative value. Let us express the speculation that the existence of realizations of arbitrarily negative attractive interactions might open the way to a pairing creation due to Parisi symmetry breaking. These pairs might become delocalized due to the introduction of a fermion hopping term. We do not pretend to have an answer at the moment, but we think it is justified to raise the question \( \text{whether a superconducting transition might become possible due to the Parisi replica symmetry breaking at} \) and beyond the instability line towards RPSB.

The random \( U \) local Hubbard limit was constructed to match the fermionic spin glass propagator at \( T = 0 \). It is clear that this mapping cannot be achieved selfconsistently for all temperatures, since the zero dimensional local limit does not allow for a thermal phase transition. We consider it possibly important that one may achieve a mapping at zero temperature between a random \( U \) Hubbard model and a metallic Ising spin glass with identical hopping hamiltonians. This will involve the question of replica symmetry breaking in the random \( U \) Hubbard model.

The role of frustration and the analog of the Almeida Thouless instability in the random \( U \) Hubbard model are to be studied, which requires to study the replicated field theory with incomplete or nongaussian disorder average. This is a problem for later studies.

We also consider interesting the question, whether frustration in clean systems leads to time–dependent behaviour that resembles the one generated by replica symmetry breaking.

3. Quantum–dynamical image of RPSB

The results for low energy excitations indicate that the time–dependence of fermion correlation functions must be affected by replica symmetry breaking. We study now the fermion propagator and in particular the retarded Green’s function. The Fourier transformation
\[ G^R(t) = \int_{-\infty}^{\infty} \frac{de}{2\pi} e^{-i\epsilon t} G^R(\epsilon) \] (87)

yields

\[ G^{(1)R}(t) = -ie^{i\mu t} \left\{ \sqrt{\frac{2}{\pi q_2}} e^{-\frac{1}{2}a^2 (q_1 - q_2)} \int_{-\infty}^{\infty} du e^{-\frac{1}{2} \left( \frac{u}{2} + \frac{1}{\sqrt{2}} \right)^2 + R^2(u,t)} e^{iu [1 + Erf[R(u,t)]]} e^{\chi \frac{t}{\sqrt{2}}} \right\} + c.c. \] (88)

where

\[ R(u,t) \equiv \sqrt{\frac{q_1 - q_2}{2}} (a + \frac{u}{q_1 - q_2} + it). \] (89)

This result is to be compared with the replica–symmetric result evaluated as

\[ G^{(0)R}(t) = e^{-\frac{\chi^2 t}{2}} \left( \cos(\chi_0 t) - Erfi\left(\frac{t}{\sqrt{2}}\right) \sin(\chi_0 t) \right) \frac{e^{i\mu t}}{i} \Theta(t) \] (90)

Both results are compared with each other for short times in Fig. (11) and for long times in Fig. (12).

For large times we obtain

\[ G^{(1),R}(t \to \infty) \sim \frac{c(q_1, q_2, a)}{t} \] (91)

with an amplitude smaller than the one of the replica–symmetric solution (90), which gives

\[ G^{(0),R}(t \to \infty) \sim -i \sqrt{\frac{2}{\pi}} \frac{\sin(\chi_0 t)}{t} \] (92)

The behaviour at large times of this \( T = 0 \)–quantum theory is obviously marked by RPSB. As will become clear below the susceptibility \( \chi \), on which the \( O(\frac{1}{t}) \)–term of the fermion propagator depends, becomes smaller and smaller with increasing steps \( K \) of RPSB and finally vanishes in the presumed exact solution at \( K = \infty \). The oscillations similarly become slower and disappear at \( K = \infty \).

One may in this context recall the correspondence between long–time behaviour in the case of classical Glauber dynamics, which was seen to correspond to the nontrivial part of the Parisi function.

\[ \text{FIG. 11. Comparison of short–time dependence of the replica–symmetric retarded Green’s function } i e^{-1\mu t} G^{(0),R}(t) \text{ and of the 1–step broken } i e^{-1\mu t} G^{(1),R}(t) \]
FIG. 12. Long–time decay of the fermion Green’s function showing gap–induced faster (0RPSB) and slower (1RPSB) oscillations. The amplitude $c$ of the $\tilde{t}$ long time decay is reduced by 1–step RPSB. Oscillations and amplitude $c$ vanish as $(K \to \infty)$–step RPSB (not shown).

B. Higher order RPSB and $K = \infty$

The result in 1–step RPSB, although still unstable towards higher RPSB, can be viewed as a much better approximation than 0–RPSB, since it already contains features of the full Parisi solution at $K = \infty$. In this section we consider the RPSB equations at arbitrary $K$.

1. $K$–invariant ratios

We now derive two $K$–invariant relations. The first one, between zero temperature gap and nonequilibrium susceptibility both calculated in $K$–th order RPSB, is contained in

$$E_g^{(K)} = \chi^{(K)} = \lim_{T \to 0} \beta (\tilde{q}(K) - q_1^{(K)}).$$  \hfill (93)

In order to obtain the second invariant we compare the $T \to 0$–limit of the selfconsistent equations for $\tilde{q} - q_1$ and for the DoS at arbitrary order $K$ of RPSB. The Green’s function at $K$–th order RPSB can be written as

$$G(\epsilon) = \int_{K+1}^\Lambda \zeta_{K+1} \int_K \zeta_K \cdots \int_2 \zeta_2 \int_1 e^\Phi g(\epsilon)$$

where

$$\zeta_{K+1} = \left[ \int_K \left[ \int_{K-1} \left[ \int_{K-2} \cdots \left( \int_1 C^{m_1} \right)^{m_{K-2}} \right] \right] \right]^{-1}$$

while $\int_{K}$ and $C$ have been defined before by Eq.(65) and Eq.(75) respectively. In the zero temperature limit Eq.(94) turns into

$$G(\epsilon) = \int_{K+1}^\Lambda \zeta_{K+1} |_{T=0} \int_K \zeta_K |_{T=0} \cdots \int_2 \int_1 \Theta(\bar{H}_0) e^{a_1 \bar{H}_0} \frac{1}{e^{\epsilon + \mu + H_0 + \chi} + e^{\epsilon + \mu - H_0 - \chi}}$$

\hfill (96)
The density of states at the gap edge $E_g^{(K)}$ is hence given by

\[ \rho^{(K)}(E_g^{(K)} + 0) = \int_{K+1}^{K} \zeta \kappa \mid_{T=0} \int_{K}^{K} \zeta \kappa \mid_{T=0} \ldots \int_{-\infty}^{T} \frac{du_1 e^{-\frac{(u_2-u_3)^2}{2T^2}}}{\sqrt{2\pi(u_2-u_3)}} \int_{-\infty}^{T} \frac{e^{-\frac{u_1^2}{2T^2}}}{\sqrt{2\pi(u_1^2+T^2)}} du_1. \]  

(97)

A similar expression is obtained for the $T=0$ gap energy $E_g^{(K)}$, since

\[ E_g^{(K)} = \lim_{T\to 0} \beta(q^{(K)} - q^{(1)}) \]

\[ = \int_{K+1}^{K} \zeta \kappa \mid_{T=0} \int_{K}^{K} \zeta \kappa \mid_{T=0} \ldots \int_{-\infty}^{T} \frac{du_1 e^{-\frac{(u_2-u_3)^2}{2T^2}}}{\sqrt{2\pi(u_2-u_3)}} \lim_{T\to 0} \beta \int_{-\infty}^{T} du_1 e^{-\frac{(u_1-u_3)^2}{2T^2}} \cosh^{m_1-2}(\beta u_1) \]

(98)

Thus we obtain the second K–invariant ratio

\[ \frac{\lim_{E \uparrow E_g(H)} \rho^{(K)}(E)}{E_g^{(K)}} \]

\[ = \frac{1}{2} \]

(99)

Together with the fact that, for each given K, the gapwidth equals twice the fermionic nonequilibrium susceptibility $\chi$, which turns into $\tilde{\chi} = \beta(q - q(1))$ differing only by exponentially small terms from the spin–space–analogue $\tilde{\chi} = \beta(1 - q(1)) \sim T^4$, where $q(1)$ denotes the Parisi function $q(x)$ at $x = 1$, the DoS–hardgaps at finite K terminate in a softgap for $K \to \infty$. Note that we did not have to evaluate the $T = 0$–Parisi function $q(x)$ in order to reach this conclusion. Assuming that the relation between gapwidth and $\tilde{\chi}$ remains valid (at least in good approximation) for finite–range models, fluctuation effects should harden the gap. This requires further analysis.

We have proved the existence of a spin–glass hardgap at any finite $K > 0$ with

\[ \delta \rho^{(K)}(T = 0, E) \sim |E - E_g^{(K)}| \]

(100)

for $|E| \geq E_g^{(K)}$ and K finite, the pseudogap of the infinite–range model leads to

\[ \rho(T = 0, E) \sim |E|^x \quad (K = \infty) \]

(101)

with a scaling exponent x, which could eventually become different from one and remains to be determined. This pseudogap together with x=1 would be slightly reminiscent of the exponent found for a superconducting glass unitary nonlinear sigma model.\(^{20}\) However exponent x differing from one are known even in mean fields solutions for gapless superconductors and also at our spin glass thermal tricritical point.\(^{44}\) We remark that the pseudogap–solution given for the fermionic spin glass model refers precisely to $\mu = 0$ whereas the hardgap–solutions at finite K happened to be stable within finite intervals $|\mu| \leq \tilde{\chi}/2$ corresponding to half–filling only at $T = \infty$. The regime beyond half–filling, identified as the domain of phase separation in the replica–symmetric solution,\(^{44}\) requires further analysis at $T = 0$ as well as several metallic–, Kondo–type–extensions. Given the experimental facts about phase diagrams in HighTc superconductors\(^{44}\) the allowance for superconductivity in fermionic spin glass models is interesting too.

In more difficult models than those treated by us here, the new method of Fourier–Transformations in replica space\(^{12}\) is hoped to facilitate further insight into the highly nontrivial $K = \infty$–solutions.

We finally note that an overlap distribution function for data clustering shown in\(^{12}\) and interpreted as a pseudo $T = 0$–problem in a classical spin analogy, revealed, apart from the ratio discussed in Eq. (100), a remarkable similarity with the $(H = 0)$–density of states. It appears interesting to explore pseudo–$(T = 0)$ neural network problems\(^{12}\) as potential classical partners of fermionic spin glasses.

VI. WARD IDENTITY

Conservation of charge and the corresponding continuity equation provide an exact relation which has a bearing on low energy two–particle excitations, connecting them also with the one–particle excitations discussed before. Its feature, which is to connect (directly or indirectly) one–fermion propagators with vertices and higher order correlations, is particularly important in the present context, since RPSB–breaking is thus transferred into many–particle Green’s functions. One may not be surprised to find RPSB in spin– or charge–overlap correlations like $[\sigma^a(\tau)\sigma^b(\tau')]_{av}$ for
shows that the Parisi form of the DoS, depending on all \( q \) function. As usual the three–point function involving the four –current vertex
\[ j^{aa}(r_i) = -i\epsilon r_i \sum_{j,\sigma} \left[ \overline{\psi}_{i,\sigma}^a t_{i,j} \psi_{j,\sigma}^a - \overline{\psi}_{j,\sigma}^a t_{j,i} \psi_{i,\sigma}^a \right] \]
links current– and charge–correlations.

It does not seem necessary to invent replica–overlap currents and densities, if not for the purpose of studying the existence of replica nondiagonal Green’s functions \( G^{ab} \) involving nonhermitian realizations. It is clear that beyond replica permutation symmetry on the level of spins there is the possibility to write, formally, another RPSB on the level of fermions. This is however not the subject of the present paper.

Let us recall the two–fermion propagators \( K^{ab}(q,\omega) \) and the density response function, which played a crucial role in the replica–symmetric theory of Anderson localization. Hopping disorder related diffusion of electrons was closely linked to the Ward identity. Even if we would not question the stability of those theories against RPSB or perhaps other kinds of RSB, the theory of localization due to a random interaction will as well rely on the Ward identity.

The Ward identity for metallic or superconducting systems with allowance for spin glass order and RPSB has the usual form
\[ i\omega \Lambda_\rho(k = 0, \epsilon + \omega, \epsilon) = \mathcal{G}(\epsilon + \omega) - \mathcal{G}(\epsilon) \]

\[ \lim_{\omega \to 0} \omega \lim_{k \to 0} \Lambda^{RA}_\rho(k, \epsilon + \omega, \epsilon) = 2\pi i \rho(\epsilon, \{q, q - r\}) \]

shows that the Parisi form of the DoS, depending on all \( q - q - r \) or on \( q(x) \) for \( K = \infty \), also enters the vertex function. As usual the three–point function involving the four–current vertex \( < T_r [a_i(\tau) a_i(\tau') j^a_{ri}(0)] > \) defines the two–legged quantity \( \Lambda_\rho \).

The Ward identity for metallic or superconducting systems with allowance for spin glass order and RPSB has the usual form
\[ i\omega \Lambda_\rho(k + p, \epsilon + \omega, \epsilon) + i\omega \Lambda_\rho(p + k, \epsilon + \omega, \epsilon) = \mathcal{G}(p + k, \epsilon + \omega) - \mathcal{G}(p, \epsilon), \]

where \( \mathcal{G} \) denotes again the disorder averaged and replica–diagonal Green’s function, which incorporates even at \( T = 0 \) strong RPSB–effects.

It is very tempting to mention the consequences of a representation (84), although we do not yet know, whether it holds beyond the insulating model. It appears likely that it may hold within the Q–static approximation (since the Parisi type average given here involves quantities that are static in the insulating case but become time–dependent in the metallic case). Thus, within the limits of its validity, this representation could be generalized in a rather simple way to the metallic or superconducting cases at zero temperature, and one would obtain the \( (T = 0) \) Ward identity in the form similar to the one of clean systems but including an disorder average with constrained (cutoff) Parisi type distributions (depending on the number of RSB–steps). Of course all order parameters would have to be determined selfconsistently for each particular case. Inferring the form of the \( K = 1 \) propagator, Eq. (84), the vertex (with external legs) \( \Lambda \) depends on the Parisi order parameters as (in shorthand notation)
\[ i\omega \Lambda_\mu(p + k, \epsilon + \omega, \epsilon) = \int_2^G \Theta(\hat{H}_0) e^{a \hat{H}_0} \left[ \sum_{s=\pm 1} \frac{1}{i\omega - s(\mu + \chi)} - \frac{1}{i\omega - s(H_0 + \chi)} \right] \int_1^G e^{a \hat{H}_0} \]

This will also occur in metallic spin glasses, whence diffusive modes and conductivity are expected to depend on Parisi symmetry breaking.
VII. MESOSCOPIC FLUCTUATIONS IN QUANTUM SPIN GLASS PHASES

Spin glasses have been discussed in the context of mesoscopic systems. Nanostructuring of materials with frustrated magnetic interactions promises a large field of applications. We contributed a recent example of a semimagnetic layered semiconductor system. Mesoscopic fluctuations have been studied intensively for electronic systems with random potentials. The possibility of RPSB was ignored or assumed to be irrelevant or nonexisting. We provide here the technique that helps to reconsider this question. This section however briefly evokes only a few examples of mesoscopic fluctuations in spin glass systems. A typical example is given by statistical fluctuations of the density of states. Multi–valley correlations described by

\[ C^{(k)}(E) = \rho_a(E) \rho_B E \]

(108)

where the \( \rho_a \) denote distinct replicas, do not factorize within the spin glass phase. For the choice \(|E_k| = E < E_g(H)|\) we obtain

\[ C^{(k)}(E) = (\rho(E))^k \frac{B(k, d E, \frac{k}{d E})}{(B(\frac{k}{d E}, \frac{k}{d E}))^k} \]

(109)

with \( d E \equiv \frac{E(\mu N)}{E_g(H)} \) and Beta–function B. These multi–valley correlations of arbitrary order \( k \) also vanish exponentially everywhere in the gap as \( T \to 0 \) and, much weaker than the averaged DoS though, they increase as either gap edge is approached at fixed low temperatures.

VIII. APPENDIX A: QUANTUM FIELD THEORY AND DECOUPLING PROCEDURE

A. A1: the fermionic Ising spin glass

The gaussian average over random fluctuations in the magnetic coupling \( J_{ij} \) renders the thermodynamic potential \( \Omega = \text{Tr} e^{-\beta(H-\mu N)} \) of the fermionic Ising spin glass as

\[ \Omega = \int \exp \sum_{i,j} \int_{\tau, \tau'} M_{ij}(i-j) \left[ \sum_a X_i^{aa}(\tau, \tau') X_j^{aa}(\tau', \tau) + 2 \sum_{a<b} X_i^{ab}(\tau, \tau') X_j^{ba}(\tau', \tau) \right] \]

(110)

where

\[ X_i^{ab}(\tau, \tau') \equiv \sum_{\lambda=\pm 1} \overline{\psi}_{i\lambda}(\tau) \lambda \psi_{i\lambda}(\tau) \sum_{\lambda' = \pm 1} \overline{\psi}_{i\lambda'}(\tau') \lambda' \psi_{i\lambda'}(\tau') \]

(111)

We decouple this eight–fermion correlation by a matrix field \( Q_i^{a,a'}(\tau, \tau') \). Since the fermionic Ising model does neither have spin– nor charge–quantum dynamics, and since \( Q \) represents a coarse–grained spin overlap field with the same average as the X–field, its static and spatially homogeneous saddle point solution, which has to be determined selfconsistently, is extracted by

\[ Q_i^{a,a'}(\tau, \tau') \equiv \Lambda^{a,a'} + \delta Q_i^{a,a'}(\tau, \tau') \]

(112)

where the saddle point matrix \( Q_{sp} \equiv \Lambda \) turns out to be of the usual Parisi form plus a diagonal matrix taking care of \( (\sigma^a)^2 \)–averages, hence

\[ \Lambda^{a,a'} = q_{par,aa'} + \bar{q}_{aa'} \]

(113)

The infinite–range fermionic Ising spin glass is then described by

\[ \Omega = \lim_{n \to 0} \frac{1}{n} \left< Z_n \right> \]

(114)

with
\[ <Z_n> = \int DQ \int Dme^{-nNA(Q,m)} \] (115)

The saddle point contribution for the infinite system \((N \to \infty)\) can be extracted by

\[ = \int D\Psi \exp \left[ -\frac{N}{3}(\beta J)^2 \sum_a (q^{aa})^2 + \frac{1}{2} \sum_a \int \tau \int \tau' X_{\tau \tau'}^{aa} \right] \exp \left[ -\frac{N}{2}(\beta J)^2 \sum_{a < b} (q^{ab})^2 + J^2 \sum_{a < b} \int \tau \int \tau' X_{\tau \tau'}^{ab} \right] , \] (116)

while the fluctuation part is expanded in terms of the fluctuation fields \(\delta Q\). Time–independence of the spin correlations for the Ising case including charge interactions and grand canonical description, allows to perform time integrations on the fluctuation fields. Hence only the \(\delta Q(\omega = 0, \omega = 0)\) components remain. This is different in the metallic case and the separation of a dynamic saddle point together with the dynamic fluctuation theory in terms of fields \(\delta Q(\omega, \omega')\) was described in [4].

The second class of decouplings, needed to reduce 4th–order Grassmann field products to the integrable 2nd order products, must take into account the RPSB in the \(Q\)-matrix. With each step \(K\) of RPSB–steps the number of new decoupling fields increases. The procedure is clearly seen for example at \(K = 2\), where

\[ <Z_n> = e^{-\frac{N}{4} \beta^2 J^2 \tau \tau RPSB^{2}_{partisi}} \prod \int \bar{d}\psi d\psi \exp \left[ \bar{\psi} i\psi + \frac{1}{2} \beta^2 J^2 \tau \tau RPSB(\bar{Q}_{partisi}, X) \right] \] (117)

\[ \equiv \prod_{\mu} \int_{\lambda_{\mu}}^{G} \prod_{\alpha_{\mu}}^{G} \int_{\lambda_{\alpha_{\mu}}}^{G} \prod_{\alpha_{0}}^{G} \int \bar{d}\psi d\psi \exp \left\{ \sum_{\alpha_{k-1} = (\alpha_{k-2} - 1)}^{\alpha_{k-1} = 1} \sum_{\alpha_{k} = (\alpha_{k-1} - 1)}^{\alpha_{k} = 1} \sum_{\alpha_{k-1} = (\alpha_{k-2} - 1)}^{\alpha_{k-1} = 1} \sum_{\alpha_{0} = (\alpha_{k} - 1)}^{\alpha_{0} = 1} \right\} , \] (118)

where the decoupling fields carry a Parisi block index \(k\) (summations over \(\alpha_{k}\) run over replicas pertaining to this block) and \(q_{\alpha} \equiv \bar{q}, \ z_{\alpha} \equiv y^{\alpha}\). The generalization to arbitrary \(K\) is now obvious. Generating fermion fields \(\eta, \bar{\eta}\) can be added to allow for the calculation of Green’s functions. Let us define

\[ \prod_{\mu} \prod_{\alpha_{\mu}}^{G} \int_{\lambda_{\mu}}^{G} \prod_{\alpha_{0}}^{G} \int_{\lambda_{\alpha_{0}}}^{G} \int \bar{d}\psi d\psi \exp \left\{ \sum_{\alpha_{k-1} = (\alpha_{k-2} - 1)}^{\alpha_{k-1} = 1} \sum_{\alpha_{k} = (\alpha_{k-1} - 1)}^{\alpha_{k} = 1} \sum_{\alpha_{k-1} = (\alpha_{k-2} - 1)}^{\alpha_{k-1} = 1} \sum_{\alpha_{0} = (\alpha_{k} - 1)}^{\alpha_{0} = 1} \right\} . \] (119)

The generating functional for an arbitrary number \(K\) of RPSB–steps is then obtained as

\[ \Xi_{n}(\eta, \bar{\eta}) = e^{-\frac{N}{4} \beta^2 J^2 \tau \tau RPSB^{2}_{partisi}} \prod_{\mu} \prod_{\alpha_{\mu}}^{G} \int_{\lambda_{\mu}}^{G} \int \bar{d}\psi d\psi \exp \left[ \sum_{\gamma} \sum_{\alpha_{\gamma}} \sum_{\bar{\alpha}_{\gamma}} \bar{\psi}^{\alpha_{\gamma}} g^{-1}_{\gamma, \bar{\alpha}_{\gamma}} (\epsilon_{\gamma}, \{ z_{\alpha_{\gamma}} \}) \bar{\psi}^{\alpha_{\gamma}} + \eta^{\alpha_{\gamma}} \bar{\psi}^{\alpha_{\gamma}} + \bar{\eta}^{\alpha_{\gamma}} \psi^{\alpha_{\gamma}} \right] \] (120)

where the statistically fluctuating Green’s function \(g\) is given by

\[ g^{-1}_{\gamma, \bar{\alpha}_{\gamma}} (\epsilon_{\gamma}, \{ z_{\alpha_{\gamma}} \}) = \left[ g^{-1}_{\gamma, \bar{\alpha}_{\gamma}} (\epsilon_{\gamma}) + \sigma H(\{ z_{\alpha_{\gamma}} \}) \right]^{-1}. \] (121)

The bare propagator \(g(\epsilon_{\gamma}) = 1/(i\epsilon_{\gamma} + \mu)\) becomes spatially dependent in metallic models, while all of the Ising spin glass complications including Parisi RPSB are taken care of by the fields \(z_{\gamma}\). A shift of the fields leads to the result

\[ \exp(\Phi(\eta, \bar{\eta})) = \left( \prod \int d\psi \bar{\psi} \right) \exp \left( \sum_{\gamma} \bar{\psi}^{\alpha_{\gamma}} g^{-1}_{\gamma, \bar{\alpha}_{\gamma}} (\epsilon_{\gamma}, \{ z_{\alpha_{\gamma}} \}) \psi^{\alpha_{\gamma}} + \eta^{\alpha_{\gamma}} \bar{\psi}^{\alpha_{\gamma}} - \bar{\psi} \psi \right) = \exp(\Phi(0, 0)) \exp(\exp(\eta^{\alpha_{\gamma}} g_{\sigma, \sigma} (\epsilon_{\gamma}, \{ z_{\alpha_{\gamma}} \}) \eta^{\alpha_{\gamma}})), \] (122)

which shows that the averaged Green’s function is obtained by the given \(K+1\) gaussian integrations over the fluctuation fields, contained in the function \(g\), according to

\[ \text{29} \]
\[ G_{\sigma}(\epsilon_i) = \frac{\int_{z_K} G (\prod \int_{z_K} G) \cdots (\prod \int_{z_K} G) Exp [\Phi(0,0)] g_\sigma(\epsilon_i, \{z_\lambda\})}{\int_{z_K} G (\prod \int_{z_K} G) \cdots (\prod \int_{z_K} G) Exp [\Phi(0,0)]}. \]  

(124)

Apart from the gaussian weight indicated by the upper index G, one needs the regularized (provided the continuous time formalism is used; an example for the application of the discrete time formalism is given in Appendix 2 for the local Hubbard limit) solution of

\[ \text{Exp} [\Phi(0,0)] = \prod_{\{b_n\}} \prod_{\epsilon_i} \left[ e^{\frac{1}{2} + \mu^2 + \tilde{H}^2(z_0, z_1, \ldots, z_K)} \right] \text{reg} \]

(125)

\[ = \prod_{\{b_n\}} \left[ 2cosh(\beta \tilde{H}(z_0, z_1, \ldots, z_K)) + 2cosh(\beta \mu) \right] \]

(126)

B. Appendix 2: The local Hubbard limit with random U

The mapping between the Green’s function of the fermionic Ising spin glass and the one for a Hubbard interaction with random positive U and properly chosen mean value of U, both taken at \( T = 0 \) and at half–filling is surprising in several respects. A search for more general relations is necessary. A general method is of course the comparison of the field theories. A closer look on the field theoretic representation of a random U Hubbard interaction with cutoff gaussian \( \delta U \) distribution is needed.

The decoupling of the Hubbard interaction, using the operator identity \( n_\uparrow n_\downarrow = \frac{1}{4}((\sum \lambda n_\lambda)^2 - (\sigma^z)^2) \), was given by Vollhardt\(^\text{13}\). Let us first reconsider the Grassmann integrals using the discrete times (introduced by time slicing), since we know that the infinite–range fermionic spin glass cannot be represented at all temperatures by one zero–dimensional random U Hubbard model, we are looking for the differences in those models field theoretic descriptions. Nevertheless field theories provide the best way to find general relations, for which we are looking here. In fact, since we know that the infinite–range fermionic spin glass cannot be represented at all temperatures by one zero–dimensional random U Hubbard model, we are looking for the differences in those models field theoretic descriptions. Before discussing the effect of cutoff gaussian distributions in replicated extensions of the given partition function, let
us the generating functional in the discrete time formalism.

The generating functional

\[ \Xi(\eta, \bar{\eta}) = \ln \lim_{M \to \infty} \prod_{\sigma, k} \int \mathcal{D} \Psi_i \int_G e^{-\Psi B \Psi + \eta \bar{\Psi} - \bar{\eta} \psi} \]

only requires to invert matrix B. This yields

\[ G(\tau_f - \tau_i) = -\frac{1}{Z} \prod_k \int_{\alpha_k}^G \int_{\gamma_k}^G \frac{1}{B_{l+1,l}^{(\sigma)}} \det B^{-\sigma}, \]

This is readily evaluated to be (for \( \tau_f > \tau_i \))

\[ G^\dagger(\tau_f - \tau_i) = -\frac{1}{Z} \lim_{M \to \infty} \prod_{k=0}^{M-1} \int_{\alpha_k}^G \int_{\gamma_k}^G \prod_{l=1}^{G} \left[ 1 + \epsilon \mu + \sqrt{\frac{\epsilon \mu}{2}} (i \alpha_k + \gamma_k) \right] \left[ 1 + \prod_{l=0}^{M-1} \left( 1 + \epsilon \mu + \sqrt{\frac{\epsilon \mu}{2}} (i \alpha_k - \gamma_k) \right) \right] \]

\[ = -\left[ e^{\mu(\tau_f - \tau_i)} + e^{\beta \mu (\mu - U)(\tau_f - \tau_i)} \right] / Z \]

Since we have now at hand the field theory which correctly reproduces the known results for \( Z \) and Green’s function, we know that the replica formalism applied to a random \( U \) local Hubbard limit with the incomplete gaussian distribution obtained in (54) must reproduce the average of \( \ln Z \) and \( G \). It will be interesting and necessary to study in more detail the field theory of the random Hubbard model with an incomplete gaussian distribution and its relation with the metallic spin glass.

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