Coherent energy transport and control of coupled currents in classical nonlinear oscillators

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Energy transport is common to all physical systems out of thermal equilibrium. The standard approach to non-equilibrium thermodynamics describes transport in terms of generalised forces and coupled currents [1][4], a typical example being the Fourier law that relates temperature gradient to the heat flux [3]. In this Letter, we demonstrate the presence of persistent energy currents in a lattice of classical nonlinear oscillators with uniform temperature and chemical potential. In analogy with the well known Josephson effect, the currents are proportional to the sine of the phase differences between the oscillators. Those phases play the role of thermodynamical forces that drive the system out of equilibrium. This general feature indicates novel ways to practically control energy transfer in many different devices, and elucidates a fundamental property nonequilibrium systems. Moreover, it suggests a simple, macroscopic setup for studying a phenomenon which hitherto have only been observed in microscopic, quantum-mechanical systems.

By means of a simple theoretical model and numerical simulations, we demonstrate the presence of persistent energy currents in a lattice of classical nonlinear oscillators with uniform temperature and chemical potential. In analogy with the well known Josephson effect, the currents are proportional to the sine of the phase differences between the oscillators. Those phases play the role of thermodynamical forces that drive the system out of equilibrium. This general feature indicates novel ways to practically control energy transfer in many different devices, and elucidates a fundamental property nonequilibrium systems. Moreover, it suggests a simple, macroscopic setup for studying a phenomenon which hitherto have only been observed in microscopic, quantum-mechanical systems.

The general feature, related to the fluctuations-dissipation theorem, provides a novel way to practically control energy transfer in a large class of systems, such as spin caloritronics [7][10] and nano-phononics [5][11] devices, nanoscale mechanical oscillators, thermal/spin rectifiers [12][17], heat pumps etc. It also elucidates the connection between phase-coherence and energy transport, and shows an intriguing analogy between a purely quantum phenomenon and a simple, macroscopic setup. Possible experimental tests are also discussed.

As illustrating example, we consider here the dynamics of the magnetisation in a ferromagnet, described by the Landau-Lifshitz-Gilbert (LLG) equation of motion [18][20]:

\[ \dot{m} = -m \times H_{\text{eff}} + \alpha (m \times \dot{m}). \] (1)

This equation describes the precession of the unit magnetisation vector field \( m(x, t) \) around the effective field \( H_{\text{eff}} = J \Delta m + (H_{\text{ext}} + m_z) e_z \), see Fig. 1a. The first term of \( H_{\text{eff}} \) is the exchange interaction with coupling strength \( J \), while the other two terms account respectively for applied field and easy axis anisotropy. They are both oriented along \( e_z \), which defines the precession axis of the magnetisation. The second term on the right hand side of Eq. (1) accounts for energy dissipation at a rate proportional to the Gilbert damping parameter \( \alpha \).

Computing the time evolution of \( M_z = \int m_z(x, t) d^3x \) leads to the definition of the magnetisation current \( j_M = J(m \times \nabla m) \) [21]. Eq. (1) is approximated by the nonlinear Schrödinger equation

\[ i \dot{\psi} = (1 + i \alpha) [J \Delta \psi + (|H_{\text{ext}}| + |\psi|^2) \psi] \] (2)

for the complex wave function (SW) amplitude \( \psi = (m_x + im_y)/(m_x + 1) \) [22][24]. In terms of this variable, the magnetisation current reads \( j_M = 2 J m |\psi|^2 \nabla |\psi|^2 \). The latter is associated to the time evolution of \( P = \int p(x, t) d^3x \), where \( p = |\psi|^2 \) is the local oscillator power, that accounts for the precession amplitude of the magnetisation [25].

In the absence of damping, \( P \) and \( M_z \) are conserved charges. In the case considered here, where dissipation is present, one can still write their time evolution in terms of propagating currents, but has to keep into account the effect of the damping. This will be discussed in details below.

For a chain of \( m = 1, ..., N \) spins with local SW amplitude \( \psi_m \), it is straightforward to derive the expression \( j_m^E = 2 J \Re[\psi_m^{*} \psi_{m+1}] \), which describes the local particle current in the discrete nonlinear Schrödinger (DNLS) equation. The other current of this model is the energy flow \( j_m^E = 2 J \Re[\psi_m^{*} \psi_{m+1}] \), given by the time evolution of the Hamiltonian [26][27]. A similar model was recently used to describe energy transfer in spin-Seebeck...
and control the flows
phases and chemical potentials (represented by the links). Thermal baths at different temperatures and chemical potentials (T, µ) together with the phases βmn can be used to drive the system out of equilibrium and control the flows jmn\textsuperscript{M/E} between the nodes m and n.

Figure 1: a), macrospin m precessing around the effective field H_eff, described in terms of the stereographic variable ψ. Its phase φ specifies the position in the x-y plane, while p is related to the precession angle θ. b), oscillator network, where the oscillators ψ_m and ψ_n (represented by the dots) are coupled through the complex matrix A_mn = C_mn e^{iβmn} (represented by the links). Thermal baths at different temperatures and chemical potentials (T, µ_m) together with the phases β_mn can be used to drive the system out of equilibrium and control the flows jmn\textsuperscript{M/E} between the nodes m and n.

The two coupled currents j_m\textsuperscript{M/E} can be controlled by a non-uniform distribution of spin temperature and chemical potential (T_m, µ_m), which represent the associated thermodynamical forces. Currents and forces are related by the Onsager matrix, according to the standard formalism of non-equilibrium thermodynamics [14, 20].

The time and ensemble averaged currents have the natural interpretation of correlations between the oscillators (m, n). Transport can occur if the oscillators are phase locked [16, 17]. To describe this general feature, to which we refer as coherent energy transport, we consider the following model

\[i\dot{\psi}_m = \omega_m(p_m)\psi_m - i\Gamma_m(p_m)\psi_m + i\mu_m\psi_m + \sum_n A_{mn}\psi_n + i\xi_m(t)\]

that generalises the DNLS to an ensemble of coupled nonlinear oscillators with arbitrary geometry and local complex amplitudes \(\psi_m = \sqrt{p_m(t)} e^{i\phi_m(t)}\). Eq. (3), which is widely used in magnonics [16, 17, 23], finds application in many other branches of physics, such as BEC, photonics waveguides, lasers and mechanical oscillators [22, 24].

In fact, irrespectively of the underlying physical process, the small amplitude dynamics of a large class of nonlinear oscillators can be written in the form of Eq. (3) [28, 29], which is usually referred to as universal nonlinear oscillator model [25].

The first term on the right hand side of Eq. (3) is the nonlinear frequency, which in spin systems read \(\omega_m(p_m) = \gamma |H_{\text{eff}} \cdot e_z|\) and can be modified by tuning the local applied field. The second term \(\Gamma_m(p_m) = \alpha_m\omega_m(p_m)\) is the nonlinear damping, proportional to the parameter \(\alpha_m\). In the small amplitude regime considered here, those terms are written as expansion in powers of \(p_m [17, 25]\).

Next comes the chemical potential \(\mu_m\), which acts as a torque that controls the relaxation time towards the reservoirs by compensating the damping [17, 25, 27]. In magnonics and spintronics devices, this can be controlled through spin transfer torque [17, 25].

The matrix \(A_{mn} = C_{mn} e^{i\beta_{mn}}\) accounts for the coupling and specifies the geometry of the system, may depend on different mechanisms and includes both long range and short range (nearest neighbour) interactions. The phase \(\beta_{mn}\) also depends on the geometry of the system and on the coupling mechanism. Note that \(A_{mn}\) is hermitian in the absence of dissipation, while in a dissipative system one normally has \(\beta_{mn} = \beta_{nm}\), as it will be clarified below. We will consider for simplicity a linear coupling, but our analysis can be extended without difficulty to more general cases.

The last term of Eq. (3) describes thermal fluctuations in terms of a complex Gaussian random variable with the usual statistical properties \(\langle \xi_m(t) \rangle = 0\) and \(\langle \xi_m(t)\xi_n(t') \rangle = D_m(p_m)T_m\delta_{mn}\delta(t - t') [17, 23, 27]\). The nonlinear diffusion constant \(D_m(p_m) = \lambda m(p_m)/\omega_m(p_m)\) specifies the coupling strength with the underlying phonon bath, as prescribed by the fluctuation dissipation theorem [25, 27]. \(T_m\) is the bath temperature, while \(\lambda\) is a parameter that depends on the geometry of the oscillators [25].

Eq. (3) is conveniently written in the phase-amplitude representation as [16, 17, 23, 27].

\[\dot{p}_m = 2[\mu_m - \Gamma_m(p_m)]p_m + 2\sqrt{p_m} \text{Re}[\tilde{\xi}_m(t)]\]

\[\dot{\phi}_m = -\omega_m(p_m) + \frac{1}{\sqrt{p_m}} \text{Im}[\tilde{\xi}_m(t)] - \sum_n K_{mn} \cos(\phi_m - \phi_n + \beta_{mn}).\]

Eq. (4) is the continuity equation which relates the time evolution of \(p_m = |\psi_m|^2\) to the currents \(j_m\textsuperscript{M} = 2\text{Im}[A_{mn}\psi^*_m\psi_n]\) between oscillators (m, n). The local chemical potentials \(\mu_m\) and dampings \(\Gamma_m(p_m)\) act respectively as sources and drains for \(p_m\). The constant term \(2D_m(p_m)T_m\) comes from the coupling with the baths and ensures that the average values of \(p_m\) is never zero at finite temperature [17].

For convenience, we will keep the term magnetisation current, even if this can describe other types of transport processes, such as the particle flow in a Bose-Einstein condensate (BEC) [30]. The stochastic variable \(\tilde{\xi}_m(t) = \xi_m(t)e^{-i\phi_m(t)}\) models a thermal bath with the same properties as \(\xi_m(t) [27]\).

Eq. (5) describes the dynamics of the local phase \(\phi_m [31, 32]\), which depends on the coupling \(K_{mn} = \sqrt{\frac{E_m}{p_m}}C_{mn}\).
between oscillators \((m, n)\). When all the oscillators have the same coupling, Eq. [5] reduces to the well known Kramers model [33].

The conservative part of Eq. [4] is given by the functional derivative \(\dot{\psi}_m = -i\partial H/\partial \psi^*_m\) of the total Hamiltonian \(H = \sum_{m,n}[\omega_m(p_m)p_m + A_{mn}\psi^*_m\psi_n + c.c.]\) [25] [27]. Computing the time evolution of \(H\) gives the energy current \(j^E_{mn} = 2\text{Re}[A_{mn}\psi^*_m\psi_n]\) [5] [10] [17] [20] [27].

Using the phase-amplitude representation, the two currents are conveniently written as

\[
\begin{align*}
J^M_{mn} &= 2C_{mn}\sqrt{p_m}p_n \sin (\phi_m - \phi_n + \beta_{mn}) \\
J^E_{mn} &= -C_{mn}p_m\sqrt{p_n}/p_m \cos (\phi_m - \phi_n + \beta_{mn}) \\
&+ 2C_{mn}\sqrt{p_m}p_n\omega_m(p_m) \sin (\phi_m - \phi_n + \beta_{mn}).
\end{align*}
\]

In many situations the system reaches a steady state in which the condition \(p_m = 0\) is satisfied [17] [25], such that the only second term of \(j^E_{mn}\) survives. Note that the two currents are independent quantities. In particular, \(J^M_{mn}\) is proportional the oscillator frequency, while \(J^E_{mn}\) is not. This suggests a mechanism to control them separately, as it will be discussed in the numerical simulations below.

The phase-amplitude formulation elucidates the importance of synchronisation for energy transport. When the oscillators \((m, n)\) are not phase locked, one has \(\phi_m(t) \neq \phi_n(t)\), so that the currents oscillate in time around zero and they vanish in average. In this free running condition, there is no net transport of energy and magnetisation through the system [16] [17]. On the contrary, when the oscillators are phase locked, \(\phi_m(t) - \phi_n(t) \approx 0\), the net currents are proportional to \(\sin \beta_{mn}\).

This result suggests that energy can flow between two sources with the same temperatures and chemical potentials whenever the phase \(\beta_{mn}\) is finite. This situation is similar to the DC Josephson effect [6], where the phase difference between the superconducting order parameters allows a persistent electrical current to propagate through a superconducting tunnel junctions, without any external bias.

The crucial observation in our case is that the system reaches thermal equilibrium (relaxing to the usual Gibbs distribution) only if the coupling has the form \(A_{mn} = A_{mn}[1 - i\Gamma_m(p_m)]\), where \(A_{mn}\) is a real matrix [27]. This condition of dissipative coupling, prescribed by the fluctuation dissipation theorem, amounts to fixing the imaginary part of \(A_{mn}\) and consequently \(\beta_{mn}\). Modifying this phase drives the system out of equilibrium, in a way similar to a non-uniform temperature or chemical potential.

This fact is most intriguing since the Josephson effect is a purely quantum phenomenon. Normally quantum effects are lost in the classical limit, and a decisive role in this transition is played by the coupling with the environment, which brings decoherence through thermal fluctuations and irreversibility (dissipation) [34].

However, in the present case, fluctuations and dissipation are the key ingredients to observe this effect. In fact, the thermal baths ensure non-vanishing values of \(p_m\) [17], while the phase \(\beta_{mn}\) makes the phase-locking more robust. This can be see from Eq. [6], where \(\beta_{mn}\), which appears in the cosine terms controls the evolution of the phase, and consequently the phase mismatch \(\phi_m(t) - \phi_n(t)\) between oscillators \((m, n)\).

Fluctuations typically break the exact synchronisation and lead to phase diffusion [17] [31], so that this Josephson-like effect, and in general the coherent energy transport, should hold at low temperature.

The generality of the discussed model strongly suggests that this Josephson-like effect should be a general feature common to many physical systems, that can be described in terms of oscillations of a complex wave function. The main difference between the quantum and classical case consists in the nature of fluctuations and in the different distribution (respectively Bose and Gibbs) attained at thermal equilibrium [25].

To substantiate the above arguments, we turn now to numerical simulations. We consider a Schrödinger dimer, the simplest possible realisation of Eq. [3], consisting of only two coupled nonlinear oscillators. This model was recently used to study energy and spin currents in magnonics devices [16] [17]. The two nonlinear frequencies and damping rates read respectively \(\omega_n(p_n) = \omega_0^1 \times (1+2p_n)\) and \(\Gamma_n(p_n) = \alpha \omega_n(p_n)\), \(n = 1, 2\). The oscillators are connected to two Langevin baths with temperatures \(T_n\) and coupling strength \(D_1 = D_2 = 2\alpha\). In all the simulations, the chemical potentials \(\mu_n\) are set to zero, and we consider a symmetric coupling \(A_{12} = A_{21} = C(1-i\alpha)e^{i\beta}\). The equations of motion for the dimer were solved numerically using a fourth order Runge-Kutta algorithm with a time step \(dt = 10^{-3}\) model units. The other parameters, used in all our simulations are \(C = 0.1\) and \(\alpha = 0.02\). The relevant observables (local powers and currents) were time averaged in the stationary state over an interval of \(2 \times 10^6\) time steps and then ensemble-averaged over 100 samples with different realisations of the thermal field.

Fig. 2 shows the phase portrait of the two oscillators at zero temperature, starting from the initial condition \(\psi_1(0) = \psi_2(0) = 0.1(1+i)\). When \(\beta = 0\) (panels a and b), the system is dissipative and the phase space collapses to a point. However, when \(\beta = 1.2\pi\) (panels c and d), the imaginary part of the coupling changes sign and acts as a gain. In this condition, the two oscillators are self-sustained and reach a limit cycle close to \(|\psi_n|^2 \approx 1\). This corroborates our model, showing that \(\beta\) acts as a real torque that forces the dynamics.

The transport properties of the system at finite temperature are displayed in Fig. 3 (a) and (b), which show respectively energy and magnetisation currents as a function of \(\beta\), for different values of the bath temperature. Green circles and blue triangles show the case where the two baths have the same temperatures \(T_1 = T_2 = 0.8\).
discussed in the model, both currents are proportional to sin β and have similar profiles. Note in particular that the currents vanish when β = 0. In this case the coupling has the correct form that respects the fluctuation-dissipation theorem and allows the system to reach thermal equilibrium.

The orange squares correspond the case where T_1 = 1.2 and T_2 = 0.2. In this situation the system is not at thermal equilibrium when β = 0, and a positive current flows from the hot to the cold reservoir. The interesting feature here is that, changing β allows to reverse the direction of the currents, so that the system operates as a heat pump[17]. Note also that the main effect of the gradient is a phase shift of the current, which is the same effect obtained by changing β. The difference p_1 - p_2 between the oscillator powers is displayed in Fig.3 (c). It represents the variation of oscillator amplitudes (or “particle numbers” in the BEC and DNLS language[26, 30]) as a function of β.

Fig.4(a) reports j^M vs ΔT = T_1 - T_2, for different values of β. The computations were performed starting from thermal equilibrium, with T_1 = T_2 = T_0. Then, we have increased one of the two temperatures at a time, while keeping the other fixed at T_0 = 0.2. The black circles corresponds to the case β = 0, and display the usual rectification effect for heat and spin rectifiers [12–17], due to stochastic phase synchronisation. Here the current increases only for ΔT > 0 and remains close to 0 when ΔT < 0. However, note that the profiles change dramatically with β. The case β = 3π/2 (green diamonds) shows a downwards shift of the current, which now remains constant at negative ΔT. The more striking feature occurs when β = π/2 and the current is reversed (red squares), becoming negative at positive ΔT. This is consistent with the model, since between the two cases there is a phase difference of π, which corresponds to a change of sign in the current.

We discuss now the independent control of the coupled currents. Fig.4(b) displays the currents as a function of the natural frequencies of the system. The frequencies are defined as ω^0_1 = 1 + Δω and ω^0_2 = 2 + Δω and the data are plotted vs Δω. For this computation, the temperatures are kept fix at T_1 = 1.2, T_2 = 0.2 and β = 0. As expected from the theoretical model, j^M (blue up triangles) remains constant while j^E (red down triangles) increases linearly with the frequency. In magnonics systems, this splitting could be realised by keeping fix the thermal gradient and tuning the frequency through an applied field.

Our results could be tested in a variety of physical systems, including nanophononics devices, magnetic systems and mechanical nano oscillators. The transport phenomena described here, which are essentially due to the phase coherence between wave functions, appear as general properties of systems out of thermal equilibrium.

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Figure 4: Rectification effect and current splitting. a) Magnetization current $j^M$ vs temperature difference $\Delta T$, computed for different values of the coupling phase $\beta$. The lines are guides to the eye. b) Magnetisation and energy currents $j^M/E$ vs frequency shift $\Delta \omega$. $j^E$ increases linearly, while $j^M$ remains constant, showing that the two currents can be controlled independently. $j^E$ is shifted downwards of 0.08 with respect to its actual value for better visibility. The dashed lines are linear fits. In both simulations, the parameters $\omega_1 = 1$ and $\omega_2 = 2$ were used.

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