Abstract

We prove an estimate on the modulus of continuity at a boundary point of a cylindrical domain for local weak solutions to degenerate parabolic equations of $p$-laplacian type. The estimate is given in terms of a Wiener-type integral, defined by a proper elliptic $p$-capacity.

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1 Introduction

Let $E$ be an open set in $\mathbb{R}^N$ and for $T > 0$ let $E_T$ denote the cylindrical domain $E \times (0, T]$. Moreover let

$$S_T = \partial E \times (0, T], \quad \partial_p E_T = S_T \cup (\bar{E} \times \{0\})$$

denote the lateral, and the parabolic boundary respectively.

We shall consider quasi-linear, parabolic partial differential equations of the form

$$u_t - \text{div} A(x, t, u, Du) = 0 \quad \text{weakly in} \ E_T, \quad (1.1)$$

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where the function $A : E_T \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$ is only assumed to be measurable and subject to the structure conditions

$$
\begin{align*}
A(x,t,u,\xi) \cdot \xi \geq C_0 |\xi|^p \\
|A(x,t,u,\xi)| \leq C_1 |\xi|^{p-1}
\end{align*}
$$

a.e. $(x,t) \in E_T, \forall u \in \mathbb{R}, \forall \xi \in \mathbb{R}^N,$ \hspace{1cm} (1.2)

where $C_0$ and $C_1$ are given positive constants, and $p > 2$.

We refer to the parameters $\{p,N,C_0,C_1\}$ as our structural data, and we write $\gamma = \gamma(p,N,C_0,C_1)$ if $\gamma$ can be quantitatively determined a priori only in terms of the above quantities. A function

$$
u \in C(0,T; L^2_{\text{loc}}(E)) \cap L^p_{\text{loc}}(0,T; W^{1,p}_{\text{loc}}(E))$$

is a local, weak sub(super)-solution to (1.1)–(1.2) if for every compact set $K \subset E$ and every sub-interval $[t_1,t_2] \subset (0,T]$

$$
\int_K u \phi dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K [-u \phi_t + A(x,t,u,Du) \cdot D\phi] dx dt \leq (\geq) 0
$$

for all non-negative test functions

$$
\phi \in W^{1,2}_{\text{loc}}(0,T; L^2(K)) \cap L^p_{\text{loc}}(0,T; W^{1,p}_{\text{loc}}(K)).
$$

This guarantees that all the integrals in (1.4) are convergent.

For any $k \in \mathbb{R}$, let

$$(v-k)_- = \max \{- (v-k),0\}, \hspace{1cm} (v-k)_+ = \max \{v-k,0\}.$$

We require (1.1)–(1.2) to be parabolic, namely that whenever $u$ is a weak solution, for all $k \in \mathbb{R}$, the functions $(u-k)_\pm$ are weak sub-solutions, with $A(x,t,u,Du)$ replaced by $\pm A(x,t,k \pm (u-k)_\pm, \pm D(u-k)_\pm)$. As discussed in condition $(A_6)$ of [3, Chapter II] or Lemma 1.1 of [4, Chapter 3], such a condition is satisfied, if for all $(x,t,u) \in E_T \times \mathbb{R}$ we have

$$
A(x,t,u,\eta) \cdot \eta \geq 0 \hspace{1cm} \forall \eta \in \mathbb{R}^N,
$$

which is guaranteed by (1.2).

For $y \in \mathbb{R}^N$ and $\rho > 0$, $K_\rho(y)$ denotes the cube of edge $2\rho$, centered at $y$ with faces parallel to the coordinate planes. When $y$ is the origin of $\mathbb{R}^N$, we simply write $K_\rho$.

We are interested in the boundary behaviour of solutions to the Cauchy-Dirichlet problem

$$
\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t} - \text{div} \ A(x,t,u,Du) = 0 \hspace{0.5cm} \text{weakly in} \hspace{0.5cm} E_T \\
u(\cdot,t) \bigg|_{\partial E} = g(\cdot,t) \hspace{0.5cm} \text{a.e.} \hspace{0.5cm} t \in (0,T] \\
u(\cdot,0) = g(x,0),
\end{cases}
\end{align*}
$$

(1.5)

where
• (H1): \( A \) satisfies (1.2) for \( p > 2 \), as already mentioned before;

• (H2): \( g \in L^p(0, T; W^{1,p}(E)) \), and \( g \) is continuous on \( \overline{E}_T \) with modulus of continuity \( \omega_g(\cdot) \).

We do not impose any \textit{a priori} requirements on the boundary of the domain \( E \subset \mathbb{R}^N \).

A weak sub(super)-solution to the Cauchy-Dirichlet problem (1.5) is a measurable function \( u \in C(0, T; L^2(E)) \cap L^p(0, T; W^{1,p}(E)) \) satisfying

\[
\int_E u \phi(x,t)dx + \iint_{E_T} \left[ -\psi u_t + A(x,t,u,Du) \cdot D\psi \right] dxdt \\
\leq (\geq) \int_E g \phi(x,0)dx
\]  

(1.6)

for all non–negative test functions \( \varphi \in W^{1,2}(0, T; L^2(E)) \cap L^p(0, T; W^{1,p}_0(E)) \).

In addition, we take the boundary condition \( u \leq g \ (u \geq g) \) to mean that \((u - g)^+ (\cdot, t) \in W^{1,p}_0(E) \ ((u - g)^- (\cdot, t) \in W^{1,p}_0(E))\) for a.e. \( t \in (0, T) \). A function \( u \) which is both a weak sub-solution and a weak super-solution, is a solution. Notice that the range we are assuming for \( p \) and the continuity of \( g \) on the closure of \( E_T \) ensure that a weak solution \( u \) to (1.5) is bounded (see, for example, [3, Chapter V, Theorem 3.3]).

Let \((x_o, t_o) \in S_T\); the relative capacity of \( E^c \) at \( x_o \) is defined as

\[
\delta(p) \overset{\text{def}}{=} \frac{\text{cap}_p(K_{\rho}(x_o) \setminus E, K_{\rho}(x_o))}{\text{cap}_p(K_{\rho}(x_o), K_{\rho}(x_o))}.
\]  

(1.7)

We refer to Section 2 for more details on the notion of capacity. In the sequel, we always assume \( x_o \) is a \textit{Wiener point of the domain} \( E \), i.e.,

\[
\int_0^1 \delta(s) \frac{ds}{s} = \infty.
\]  

(1.8)

Let \( \gamma_\ast > 1 \) be the constant claimed in Lemma 3.3 fix \( R_o > 0 \) and \( 0 < \epsilon < 1 \), such that

\[
(t_o - 3\gamma_\ast [\delta(R_o)]^{\frac{2-p}{p-}\epsilon} R_o^{p-\epsilon}, t_o) \subset (0, T],
\]  

(1.9)

and set

\[
Q_{R_o} = K_{2R_o}(x_o) \times (t_o - 3\gamma_\ast [\delta(R_o)]^{\frac{2-p}{p-}\epsilon} R_o^{p-\epsilon}, t_o).
\]

Condition (1.9) can always be realized, since otherwise we would have for all \( s \in (0, 1) \) that

\[
3\gamma_\ast [\delta(s)]^{\frac{2-p}{p-}\epsilon} s^{p-\epsilon} \geq t_o,
\]

and consequently

\[
\int_0^1 \delta(s) \frac{ds}{s} \leq \left( \frac{3\gamma_\ast}{t_o} \right)^{\frac{1}{p-\epsilon}} \int_0^1 s^{\frac{2-p}{p-}\epsilon} ds = \left( \frac{3\gamma_\ast}{t_o} \right)^{\frac{1}{p-\epsilon}} \frac{p-2}{p-\epsilon} < \infty.
\]

We can now state the main result of this work.
**Theorem 1.1.** Let $u$ be a weak solution to (1.5), assume that (H1)–(H2) and (1.8) are satisfied, choose $R_o$ and $\epsilon$ such that (1.9) holds true. Then there exist positive constants $\gamma \in (0, 1)$, and $\bar{\gamma} > 0$ that depend only on the data $\{p, N, C_o, C_1\}$, such that for any $\rho \in (0, R_o)$

$$\text{osc}_{Q_{\rho}(\omega_o) \cap E_T} u \leq \omega_o \exp \left\{ -\gamma \int_{R_o}^{R_o \rho} \frac{ds}{s} \right\} + \text{osc}_{Q_{R_o} \cap S_T} g + \bar{\gamma} R_o^{p-2},$$

(1.10)

where $\delta(s)$ is defined in (1.7), and

$$\omega_o \overset{\text{def}}{=} \text{osc}_{Q_{R_o}} u, \quad Q_{\rho}(\omega_o) = K_{2\rho}(x_o) \times [t_o - \omega^2 \rho^p, t_o].$$

By the same argument of proving (1.9), one easily obtains that there is a sequence of positive numbers $\{R_n\}$ converging to zero, such that

$$3\gamma \ast \left[ \delta(R_n) \right]^{2-p} R_n^{p-\epsilon} \to 0 \quad \text{as } n \to \infty.$$

Therefore, from Theorem 1.1 we can conclude the following corollary in a standard way.

**Corollary 1.1.** Let $u$ be a weak solution to (1.5), assume that (H1)–(H2) hold true, that $(x_o, t_o) \in S_T$, and that $x_o$ is a Wiener point of the domain $E$. Then

$$\lim_{(x,t) \to (x_o,t_o)} u(x,t) = g(x_o, t_o).$$

As already remarked in [7], Theorem 1.1 also implies Hölder regularity up to the boundary under a fairly weak assumption on the domain. More specifically, a set $A \subset \mathbb{R}^N$ is uniformly $p$-fat, if for some $\gamma_o, \rho_o > 0$ one has

$$\frac{\text{cap}_p(K_{\rho}(x_o) \cap A, K_{\frac{1}{2}\rho}(x_o))}{\text{cap}_p(K_{\rho}(x_o), K_{\frac{1}{2}\rho}(x_o))} \geq \gamma_o$$

for all $0 < \rho < \rho_o$ and all $x_o \in A$. See [12] for more on this notion. We have the following corollary.

**Corollary 1.2.** Let $u$ be a weak solution to (1.5), assume that (H1)–(H2) hold true, the complement of the domain $E$ is uniformly $p$-fat, and let $g$ be Hölder continuous. Then the solution $u$ is Hölder continuous up to the boundary.

**Remark 1.1.** When $p > N$, then for any $s \in (0, 1)$ we always have $\delta(s) \geq \gamma_o$ for some $\gamma_o \in (0, 1)$ depending only on $N$ and $p$, as explained in Section 2. In such a case, if $g$ is assumed to be Hölder continuous, then Corollary 1.2 is automatically satisfied, and

$$\text{osc}_{Q_{\rho}(\omega_o) \cap E_T} u \leq \omega_o \left( \frac{\rho}{R_o} \right)^{\alpha},$$

(1.11)

where $\alpha \in (0, 1)$ depends only on the data $\{p, N, C_o, C_1\}.$
1.1 Novelty and Significance

The continuity at the boundary of rough sets for solutions to elliptic partial differential equations of $p$-laplacian type is by now basically a settled matter (see, for example, [13]). In the parabolic setting the theory is more fragmented, and still to be fully developed.

Continuity at the boundary for quite general operators with a growth of order $p = 2$ has been considered in [16, 17]. When dealing with a general $p > 1$, the fact that a Wiener point is a continuity point has already been observed in [2] (see also [10]). However, only the prototype parabolic $p$-laplacian is dealt with, and no explicit decay estimate as in (1.10) is provided.

The so-called super-critical singular range, that is when \( \frac{2N}{N+1} < p < 2 \), has been considered in [14] based on the comparison principle, and then, more recently in [7], with different techniques, which are closely related to the method we use here. Coming to the degenerate range $p > 2$, a result similar to ours is stated in [14]. In such a paper, the comparison principle once more plays a fundamental role; this is not the case here, where no use whatsoever of the comparison principle is made, and purely structural estimates are proved. Moreover, we give an explicit modulus of continuity, and therefore, Theorem 1.1 represents a step forward.

Here we also point out a difference between the singular case and the degenerate case, when proving the reduction of oscillation along a family of nested, intrinsically scaled cylinders. In the singular case, we do not require a priori that the Wiener integral (1.8) diverges. However, in the degenerate case, we need to use the divergence of the Wiener integral in order to fit the cylinders in one another, due to the role played by $\delta(\rho)$ in the time scaling (see Lemma 4.1).

As already remarked in [7] for an analogous result, Corollary 1.2 can be seen as an extension of Theorem 1.2 of [3, Chapter III], where the Hölder continuity up to the boundary of weak solutions to the Cauchy-Dirichlet problem (1.5) with Hölder continuous boundary data is proved, assuming that the domain $E$ satisfies a positive geometric density condition. It is a matter of straightforward computations to see that if a domain $E$ has positive geometric density, then the complement of $E$ is uniformly $p$-fat, but the opposite implication obviously does not hold.

As pointed out in Remark 1.1 when $p > N$, and the boundary datum is Hölder continuous, the solution is also Hölder continuous, regardless of the geometry of the domain $E$. This is obvious for the elliptic $p$-laplacian due to the Sobolev embedding, but the parabolic case seems new.

Finally, all the estimates are stable as $p \to 2^+$, and therefore, the continuity result of Corollary 1.1 recovers the analogous one given in [10].

As for the structure of the paper, the proof of Theorem 1.1 is given in Section 4 whereas the previous sections are devoted to introductory material, namely some preliminary results (Section 2), and a couple of auxiliary lemmas (Section 3).

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2 Preliminaries

The first basic fact is taken from [11, Lemma 2.2] (see also [4, Lemma 10.1 on page 116]).

Lemma 2.1. Let \( u \) be a non-negative, local, weak super-solution to the degenerate equation \( u_{t} - \Delta^{p} u = 0 \) in the cylinder \( K \times (t_1, t_2) \) where \( K \) is a cube in \( \mathbb{R}^N \). Then for all \( \varepsilon \in (-1, 0) \),

\[
\sup_{t_1 < t < t_2} \int_K u^{1+\varepsilon} \varphi^p(x, t) \, dx + \int_{t_1}^{t_2} \int_K |Du|^p u^{\varepsilon-1} \varphi^p \, dx \, dt
\]

\[
\leq \left( \frac{C_1 p}{C_0 |\varepsilon|} \right)^p \int_{t_1}^{t_2} \int_K u^{\varepsilon+p-1} |D\varphi|^p \, dx \, dt
\]

\[
+ \frac{p}{C_0(1+|\varepsilon|)} \int_{t_1}^{t_2} \int_K u^{1+\varepsilon} \left( \frac{\partial \varphi^p}{\partial t} \right)_{+} \, dx \, dt
\]

\[
+ \frac{p}{C_0(1+|\varepsilon|)} \int_{t_1}^{t_2} \int_K u^{1+\varepsilon} \varphi^p(x, t_2) \, dx
\]

for every non-negative test function \( \varphi \in W^{1,2}(t_1, t_2; L^2(K)) \cap L^p(t_1, t_2; W^{1,p}_o(K)) \).

Proof. Take the test function \( (u + \nu)^\varepsilon \varphi^p \) in the weak formulation (1.3) where \( \nu \) is a positive constant. Then a routine calculation followed by letting \( \nu \to 0 \) yields the conclusion.

With the above lemma at disposal, we are able to show the following reverse Hölder’s inequality. This is done by carefully tracing the dependence in the proof of [11 Lemma 5.3] or [4 Lemma 11.1].

Lemma 2.2. Let \( v \) be a non-negative, local, weak super-solution to the degenerate equation \( v_{t} - \Delta^{p} v = 0 \) in the cylinder

\[
K_{2p}(x_o) \times (t_o - \theta p, t_o),
\]

with \( \theta > 0 \) to be determined later. For any \( \sigma \in (0, 1) \), and for any \( \eta \in (0, 1) \), there exists a constant \( C_0 > 1 \) depending only on the data \( \{p, N, C_0, C_1\} \), \( \sigma \), and
Let \( \eta \), such that

\[
\int_{t_0}^{t_0 - \theta} \int_{K_1(x_0)} v^{p-2+\sigma(1+\frac{2}{p})} \, dx \, dt \\
\leq C_\eta \left[ \sup_{t_0 - \theta < t_0 < t_0 - \theta} \int_{K_{2r}(x_0)} v(x, t) \, dx \right]^{p-2+\sigma(1+\frac{2}{p})} + \eta \left( \frac{1}{\theta} \right)^{\frac{p}{p-2} \left( 1 + \frac{2}{p} \right)}.
\]

Proof. By a change of variables, we may consider this problem in the cylinder

\( Q = K_1 \times (-\theta, 0] \).

Furthermore, for \( i = 1, 2 \) let us set

\( Q_r_i = K_r_i \times (-\theta, 0] \) with \( \frac{1}{2} < r_1 < r_2 < 1 \);

pick a non-negative, piecewise smooth, cutoff function on \( K_{r_2} \), such that

\[ 0 \leq \varphi \leq 1, \quad \varphi = 1 \text{ in } K_{r_1}, \quad |D\varphi| \leq \frac{1}{r_2 - r_1}. \]

An application of the parabolic Sobolev embedding (see, for example, [3, Chapter I, Proposition 3.1]) gives us that

\[
\int \int_{Q_{r_1}} v^{p-2+\sigma(1+\frac{2}{p})} \, dx \, dt \\
\leq \gamma \int \int_{Q_{r_2}} |D(v^{\frac{p-2+\sigma}{p}} \varphi)|^p \, dx \, dt \left( \sup_{-\theta < t < 0} \int_{K_1} v^\sigma(x, t) \, dx \right)^{\frac{1}{p}}.
\]

For simplicity, let

\[ M_\sigma \overset{\text{def}}{=} \sup_{-\theta < t < 0} \int_{K_1} v^\sigma(x, t) \, dx. \]

By Lemma [2.1] with \( \varepsilon = -1 + \sigma \) we have

\[
\int \int_{Q_{r_2}} |D(v^{\frac{p-2+\sigma}{p}} \varphi)|^p \, dx \, dt \\
\leq \gamma \left( M_\sigma + \int \int_{Q_{r_2}} v^{p-2+\sigma} |D\varphi|^p \, dx \, dt \right).
\]

By Young’s inequality

\[
\int \int_{Q_{r_2}} v^{p-2+\sigma} |D\varphi|^p M_\sigma^{\frac{p}{p-2+\sigma}} \, dx \, dt \\
\leq \frac{1}{\gamma} \int \int_{Q_{r_2}} v^{p-2+\sigma(1+\frac{2}{p})} \, dx \, dt \\
+ \gamma M_\sigma^{\frac{p-2+\sigma(1+\frac{2}{p})}{p-2+\sigma}} \int \int_{Q_{r_2}} |D\varphi|^{p+\frac{N(p-2+\sigma)}{2}} \, dx \, dt.
\]
Combine the above estimates to obtain that

$$\int \int_{Q_{r_1}} v^{p-2+\sigma(1+\frac{\sigma}{p})} \, dx \, dt \leq \frac{1}{9} \int \int_{Q_{r_2}} v^{p-2+\sigma(1+\frac{\sigma}{p})} \, dx \, dt$$

$$+ \gamma M^{1+\frac{\sigma}{p}} + \gamma \theta M^{\frac{p-2}{2}+1+\frac{\sigma}{p}} \left( \frac{1}{r_2 - r_1} \right) \left( \frac{N(p-2+\alpha)}{r_2} \right).$$

By an interpolation argument (see, for example, [3, Chapter I, Lemma 4.3]) one arrives at

$$\int_{-\theta}^{0} \int_{K_{\frac{1}{2}}} v^{p-2+\sigma(1+\frac{\sigma}{p})} \, dx \, dt \leq \gamma M^{1+\frac{\sigma}{p}} + \gamma \theta M^{\frac{p-2}{2}+1+\frac{\sigma}{p}};$$

$$\int_{-\theta}^{0} \int_{K_{\frac{1}{2}}} v^{p-2+\sigma(1+\frac{\sigma}{p})} \, dx \, dt \leq \frac{M^{1+\frac{\sigma}{p}}}{\theta} + \gamma M^{\frac{p-2}{2}+1+\frac{\sigma}{p}}.$$

Note that

$$\gamma M^{\frac{p-2}{2}+1+\frac{\sigma}{p}} = \gamma \left( \sup_{-\theta<t<0} \int_{K_1} v^\sigma(x,t) \, dx \right) \left( \frac{p-2+\sigma(1+\frac{\sigma}{p})}{p-2+\sigma(1+\frac{\sigma}{p})} \right),$$

and

$$\frac{\gamma}{\theta} M^{1+\frac{\sigma}{p}} = \frac{\gamma}{\theta} \left( \sup_{-\theta<t<0} \int_{K_1} v^\sigma(x,t) \, dx \right) \left( \frac{p-2+\sigma(1+\frac{\sigma}{p})}{p-2+\sigma(1+\frac{\sigma}{p})} \right),$$

therefore, we conclude that

$$\int_{-\theta}^{0} \int_{K_{\frac{1}{2}}} v^{p-2+\sigma(1+\frac{\sigma}{p})} \, dx \, dt \leq C_\eta \left( \sup_{-\theta<t<0} \int_{K_1} v(x,t) \, dx \right)^{p-2+\sigma(1+\frac{\sigma}{p})} + \eta \left( \frac{1}{\theta} \right)^{1+\frac{p-2}{2}(1+\frac{\sigma}{p})}.$$

Returning to the original variables yields the desired result.

\textbf{Remark 2.1.} If one chooses

$$\theta = \left[ \int_{K_{2\rho}(x_o)} v(x,t_o) \, dx \right]^{-p},$$

8
We will need the following weak Harnack inequality, proved in [11].

**Theorem 2.1.** Let \( u \) be a non-negative, local, weak super-solution to (1.1)–(1.2). There exist positive constants \( c \) and \( \gamma_0 \), depending only on the data \( \{ p, N, C_0, C_1 \} \), such that for a.e. \( s \in (0, T) \)

\[
\int_{K_{p}(y)} u(x, s)dx \leq c \left( \frac{\rho^p}{T - s} \right)^{\frac{1}{p - 2}} + \gamma_0 \inf_{K_{4p}(y)} u(\cdot, t) \quad (2.2)
\]

for all times

\[
s + \frac{1}{2} \theta \rho^p \leq t \leq s + \theta \rho^p
\]

where

\[
\theta = \min \left\{ e^{2-p} \frac{T - s}{\rho^p}, \left( \int_{K_{p}(y)} u(x, s)dx \right)^{2-p} \right\}.
\]

**Remark 2.2.** If \( s \) and \( \rho \) are chosen such that

\[
s + \frac{2e^{\sigma - 2}}{\left( \int_{K_{p}(y)} u(x, s)dx \right)^{p - 2}} \rho^p < T,
\]

then

\[
c \left( \frac{\rho^p}{T - s} \right)^{\frac{1}{p - 2}} < \frac{1}{2^{p - 2}} \int_{K_{p}(y)} u(x, s)dx
\]

\[
\theta = \left( \int_{K_{p}(y)} u(x, s)dx \right)^{2-p},
\]

and therefore,

\[
\int_{K_{p}(y)} u(x, s)dx \leq \tilde{\gamma} \inf_{K_{4p}(y)} u(\cdot, t) \quad (2.4)
\]

for all times

\[
s + \frac{1}{2} \theta \rho^p \leq t \leq s + \theta \rho^p.
\]

Moreover, \( \tilde{\gamma} = \frac{\gamma_0}{1 - \left( \frac{2}{3} \right)^{p - 2}} \), and therefore the constant is stable as \( p \to 2 \).

Another result we will rely on is the following (see [8, Corollary 3.1]).
Lemma 2.3. Let $u$ be a non-negative, local, weak super-solution to (1.1) - (1.2) in the cylinder $K_{2p}(y) \times [\bar{t}, \bar{t} + T]$. Suppose that

$$\inf_{K_{2p}(y)} u(x, \bar{t}) \geq k$$

for some $k > 0$.

Then for all $t \in (\bar{t}, \bar{t} + T)$ we have

$$\inf_{K_{p}(y)} u(x, t) \geq \frac{k}{2} \left( 1 + \frac{t - \bar{t}}{\nu k^{2-p(2p)}p} \right)^{\frac{1}{p}}$$

where $\nu \in (0, 1)$ is a constant that depends only on the data $\{p, N, C_0, C_1\}$.

Finally, we recall the notion of capacity introduced in [7, §4]. Let $\Omega \subset \mathbb{R}^N$ be an open set, and $Q \overset{\text{def}}{=} \Omega \times (t_1, t_2)$: $Q$ is an open cylinder in $\mathbb{R}^{N+1}$. In the following we will refer to such sets as open parabolic cylinders. For any compact set $K \subset Q$, we define the parabolic capacity of $K$ with respect to $Q$ as

$$\gamma_p(K, Q) = \inf \left\{ \int_{Q} |D\varphi|^p \, dx \, dt : \varphi \in C_{\infty}^\infty(Q), \quad \varphi \geq 1 \quad \text{on a neighborhood of } K \right\},$$

where $D\varphi$ denotes the gradient of $\varphi$ with respect to the space variables only.

The notion of the elliptic capacity is quite standard. Indeed, for every compact set $F \subset \Omega$ we define

$$\text{cap}_p(F, \Omega) = \inf \left\{ \int_{\Omega} |D\psi|^p \, dx : \psi \in C_{\infty}^\infty(\Omega), \quad \psi \geq 1 \quad \text{on } F \right\}.$$ 

For $p \geq N$ one always assume that $\Omega \subset \subset \mathbb{R}^N$, since $\text{cap}_p(F, \mathbb{R}^N) = 0$ for $p \geq N$. It should be remarked that an explicit calculation (see, for example, [9, page 35]) gives us that

$$\text{cap}_p(K_{\rho}(x_0), K_{2\rho}(x_0)) = c_1(N, p)\rho^{N-p}, \quad \forall p > 1,$$

$$\text{cap}_p(\{x_0\}, K_{\rho}(x_0)) = c_2(N, p)\rho^{N-p}, \quad \forall p > N,$$

where $c_1$ and $c_2$ are positive constants with indicated dependence. Hence, when $p > N$, the relative capacity defined in (1.7) is always bounded below, i.e.,

$$\delta(\rho) \geq \gamma_{\alpha}(N, p) \quad \text{for some } \gamma_{\alpha} \in (0, 1).$$

As a result, condition (1.8) always holds when $p > N$. For further details and properties about the elliptic capacity, see for example [2, Chapter 4], [9, Chapter 2], [13, Chapter 2], or [6].

Now we point out the connection between the two notions of capacity. Let $Q = \Omega \times (t_1, t_2)$, and for any set $E \subset \mathbb{R}^{N+1}$ define $E_\tau = E \cap \{t = \tau\}$. Then, we have the following result (see [11, Proposition A.2]).
Proposition 2.1. Let $K \subset Q$ be compact. Then,
\[
\gamma_p(K, Q) = \int_{t_1}^{t_2} \text{cap}_p(K_\tau, \Omega) \, d\tau.
\] (2.7)

3 Auxiliary Lemmas

Fix $(x_0, t_o) \in S_T$, and consider the cylinder
\[
Q = K_{16\rho}(x_0) \times [s, t],
\] (3.1)
where $s, t$ are such that $0 < s < t_o < t < T$, and let $\Sigma \overset{\text{def}}{=} S_T \cap Q$. Our estimates are based on the following lemma, stated and proved in [7, Lemma 2.1].

Lemma 3.1. Take any number $k$ such that
\[
k \geq \sup_{\Sigma} g. \quad (3.2)
\]
Let $u$ be a weak solution to (1.5) in the cylinder $Q$, and define
\[
u_k = \begin{cases}
(u - k)_+, & \text{in } Q \cap E_T, \\
0, & \text{in } Q \setminus E_T.
\end{cases}
\] (3.3)
Then $u_k$ is a (local) weak sub-solution to (1.5) in the cylinder $Q$. The same conclusion holds for the zero extension of $u_h = (h - u)_+$ for truncation levels $h \leq \inf_{\Sigma} g$.

Let $k$ be any number which satisfies (3.2), and
\[
\begin{cases}
def \text{define } u_k = (u - k)_+, \\
\text{choose } \mu > 0 \text{ such that } \mu \geq \sup_Q u_k, \\
\text{define } v : Q \to \mathbb{R}_+, \quad v = \mu - u_k.
\end{cases}
\] (3.3)
It is not hard to verify that $v$ is a weak super-solution to (1.5) in the whole $Q$.

Finally, let
\[
\delta(\rho) \overset{\text{def}}{=} \frac{\text{cap}_p(K_\rho(x_o) \setminus E, K_{2\rho}(x_o))}{\text{cap}_p(K_\rho(x_o), K_{2\rho}(x_o))},
\]
and
\[
\bar{\theta} \overset{\text{def}}{=} \left(\mu[\delta(\rho)]^{p-1}\right)^{2-p}.
\] (3.4)
We have the following.

Lemma 3.2. Let $(x_0, t_o), Q, u_k, \mu, v$ as in (3.1)–(3.3), take $\bar{\theta}$ as in (3.4), and assume that $t_o - \bar{\theta}p^\rho \geq s$. Then there exists a constant $\gamma_1 > 1$, that depends only on the data $\{p, N, C_o, C_1\}$, such that
\[
\mu[\delta(\rho)]^{\frac{1}{p-1}} \leq \gamma_1 \sup_{t_o - \bar{\theta}p^\rho < t < t_o} \int_{K_{2\rho}(x_o)} v(x, t) \, dx.
\] (3.5)
Proof. Without loss of generality, we may assume that \((x_0, t_0) = (0, 0)\). Construct three cylinders:

\[ Q_1 = K_\rho \times \left( -\frac{3}{4} \bar{\theta} \rho^p, -\frac{1}{4} \bar{\theta} \rho^p \right); \]
\[ Q_2 = K_{\frac{3}{2}\rho} \times \left( -\frac{7}{8} \bar{\theta} \rho^p, -\frac{1}{8} \bar{\theta} \rho^p \right); \]
\[ Q_3 = K_{2\rho} \times (-\bar{\theta} \rho^p, 0). \]

Introduce the standard cut-off functions \(\zeta\) and \(\varphi\) such that

\[
\zeta(x,t) = \begin{cases} 
1 & \text{if} \quad (x,t) \in Q_1 \\
0 & \text{if} \quad (x,t) \notin Q_2 
\end{cases}
\]

and

\[
\varphi(x,t) = \begin{cases} 
1 & \text{if} \quad (x,t) \in Q_2 \\
0 & \text{if} \quad (x,t) \notin Q_3. 
\end{cases}
\]

We use the test function \(u_k\zeta^p\) in the weak formulation, modulus a standard Steklov average; a straightforward calculation similar to the one in [7, Lemma 5.1] gives us that

\[
\iint_{Q_2} |D(v\zeta)|^p \, dxdt \leq \gamma \mu \int \iint_{Q_2} |Dv|^{p-1}|D\zeta| \, dxdt + \gamma \int \iint_{Q_2} v^p |D\zeta|^p \, dxdt
\]

\[+ \gamma \mu \int \iint_{Q_2} v|\zeta_t| \, dxdt.\]

The last term on the right-hand side is estimated by

\[
\mu \int \iint_{Q_2} v|\zeta_t| \, dxdt \leq \frac{|Q_2|}{\rho^p} \mu \int \iint_{Q_2} v \, dxdt
\]

\[\leq \gamma \mu \bar{\theta} \rho^{N} \left[ \sup_{-\bar{\theta} \rho^{p-1} t < 0} \int_{K_{2\rho}} v(x,t) \, dx \right] \frac{1}{\bar{\theta}}
\]

\[= \gamma \mu \bar{\theta}^{\frac{1}{p-1}} \bar{\theta}^{\frac{1}{p-1}} \rho^{N} \left[ \sup_{-\bar{\theta} \rho^{p-1} t < 0} \int_{K_{2\rho}} v(x,t) \, dx \right] \mu^{\frac{p-2}{p-1}} \bar{\theta}^{\frac{p-2}{p-1}} \rho^{\frac{N}{p-2} + \frac{p-1}{p-2}} \frac{1}{\bar{\theta}}
\]

\[\leq C \eta_1 \mu \bar{\theta} \rho^{N} \left[ \sup_{-\bar{\theta} \rho^{p-1} t < 0} \int_{K_{2\rho}} v(x,t) \, dx \right]^{p-1} + \eta_1 \mu \bar{\theta} \rho^{N} \frac{1}{\bar{\theta}^{\frac{p-1}{p-2}}},\]

where \(\eta_1 \in (0, 1)\) will be determined later. The second term is estimated by Lemma 2.2 choosing \(\sigma\) such that

\[p - 2 + \sigma \left( 1 + \frac{p}{N} \right) = p - 1.\]

We have

\[
\iint_{Q_2} v^p |D\zeta|^p \, dxdt \leq \mu \int \iint_{Q_2} v^{p-1} \, dxdt
\]

\[\leq \gamma \frac{|Q_2|}{\rho^p} \mu \int \iint_{Q_2} v^{p-1} \, dxdt.
\]
\[
\begin{align*}
&\leq \gamma \mu \bar{\rho}^{N} \left[ C_{\eta_{2}} \left( \sup_{-\bar{\theta} \rho \rho < t < 0} \int_{K_{2\rho}} v(x, t) \, dx \right)^{p-1} + \frac{\eta_{2} - 1}{\gamma} \frac{1}{\bar{\theta}^{p-2}} \right] \\
&= \gamma C_{\eta_{2}} \mu \bar{\rho}^{N} \left( \sup_{-\bar{\theta} \rho \rho < t < 0} \int_{K_{2\rho}} v(x, t) \, dx \right)^{p-1} + \eta_{2} \mu \bar{\rho}^{N} \frac{1}{\bar{\theta}^{p-2}},
\end{align*}
\]
where once more \( \eta_{2} \in (0, 1) \) will be determined later. Next, the first term is estimated by Hölder’s inequality
\[
\begin{align*}
\int_{Q_{2}} |Dv|^{p-1} \, dx \, dt &\leq \left( \int_{Q_{2}} |Dv|^{p-1-\epsilon} \, dx \, dt \right)^{1-\frac{1}{p}} \left( \int_{Q_{2}} v^{(1+\epsilon)(p-1)} \, dx \, dt \right)^{\frac{1}{p}}, \\
\int_{Q_{2}} |Dv|^{p-1-\epsilon} \, dx \, dt &\leq \gamma (\epsilon) \left[ \int_{Q_{3}} v^{p-\epsilon-1} |D\varphi|^{p} \, dx \, dt + \int_{Q_{3}} v^{1-\epsilon} |\varphi_{t}| \, dx \, dt \right],
\end{align*}
\]
where \( \epsilon \) is a positive number such that
\[
0 < \epsilon < p - 2.
\]
Combining the above two inequalities yields
\[
\begin{align*}
\frac{\mu}{\rho} \int_{Q_{2}} |Dv|^{p-1} \, dx \, dt &\leq \frac{\gamma \mu}{\rho} \left( \int_{Q_{3}} v^{p-\epsilon-1} |D\varphi|^{p} \, dx \, dt \right)^{1-\frac{1}{p}} \left( \int_{Q_{2}} v^{(1+\epsilon)(p-1)} \, dx \, dt \right)^{\frac{1}{p}} \\
&+ \frac{\gamma \mu}{\rho} \left( \int_{Q_{3}} v^{1-\epsilon} |\varphi_{t}| \, dx \, dt \right)^{1-\frac{1}{p}} \left( \int_{Q_{2}} v^{(1+\epsilon)(p-1)} \, dx \, dt \right)^{\frac{1}{p}}.
\end{align*}
\]
Let us focus on the first term on the right-hand side. Choosing \( \epsilon \) smaller if necessary, and \( \sigma \) such that first \( p - \epsilon - 1 = p - 2 + \sigma (1 + \frac{\rho}{\rho}) \) and then \( (1 + \epsilon)(p - 1) = p - 2 + \sigma (1 + \frac{\rho}{\rho}) \), by Lemma 2.2 and repeated applications of Young’s inequality we have
\[
\begin{align*}
\frac{\gamma \mu}{\rho} \left[ \int_{Q_{3}} v^{p-\epsilon-1} \, dx \, dt \right]^{1-\frac{1}{p}} \left[ \int_{Q_{2}} v^{(1+\epsilon)(p-1)} \, dx \, dt \right]^{\frac{1}{p}} &\leq \frac{\gamma \mu}{\rho} \left[ \int_{Q_{3}} v^{p-\epsilon-1} \, dx \, dt \right]^{\frac{p-1}{p}} \left[ C_{\eta_{3}} \left( \sup_{-\bar{\theta} \rho \rho < t < 0} \int_{K_{2\rho}} v(x, t) \, dx \right)^{p-\epsilon-1} + \eta_{3} \frac{1}{\bar{\theta}} \left( \frac{1}{\theta} \right)^{\frac{p-1}{p-2}} \right]^{\frac{p}{p-1}}.
\end{align*}
\]
\[
\times |Q_2|^{\frac{1}{\theta}} \left[ C_{q_4} \left( \sup_{\tilde{\rho}^p \rho < 0} \int_{K_2} v(x,t) \, dx \right)^{(1+\epsilon)(p-1)} + \eta_4^p \left( \frac{1}{\theta} \right)^{\frac{(1+\epsilon)(p-1)}{p-2}} \right]^{\frac{1}{p}}
\]
\[
\leq \gamma \tilde{\rho}^N \left[ C_{q_3} \left( \sup_{\tilde{\rho}^p \rho < 0} \int_{K_2} v(x,t) \, dx \right)^{p-\epsilon-1} + \eta_3^p \left( \frac{1}{\theta} \right)^{\frac{(1+\epsilon)(p-1)}{p-2}} \right]^{\frac{1}{p}}
\]
\[
\times \left[ C_{q_4} \left( \sup_{\tilde{\rho}^p \rho < 0} \int_{K_2} v(x,t) \, dx \right)^{(1+\epsilon)(p-1)} + \eta_4^p \left( \frac{1}{\theta} \right)^{\frac{(1+\epsilon)(p-1)}{p-2}} \right]^{\frac{1}{p}}
\]
\[
\leq \gamma \tilde{\rho}^N \left[ C_{q_3} \left( \sup_{\tilde{\rho}^p \rho < 0} \int_{K_2} v(x,t) \, dx \right)^{(p-\epsilon-1)(p-1)} + \eta_3^p \left( \frac{1}{\theta} \right)^{\frac{(p-\epsilon-1)(p-1)}{p-2}} \right]^{\frac{1}{p}}
\]
\[
\times \left[ C_{q_4} \left( \sup_{\tilde{\rho}^p \rho < 0} \int_{K_2} v(x,t) \, dx \right)^{(1+\epsilon)(p-1)} + \eta_4^p \left( \frac{1}{\theta} \right)^{\frac{(1+\epsilon)(p-1)}{p-2}} \right]^{\frac{1}{p}}
\]
\[
= \gamma \tilde{\rho}^N \left[ \sup_{\tilde{\rho}^p \rho < 0} \int_{K_2} v(x,t) \, dx \right]^{p-1} + \eta_5 \tilde{\rho}^N \left( \frac{1}{\theta} \right)^{\frac{(p-\epsilon-1)(p-1)}{p-2}}
\]

where once more $\eta_5 \in (0,1)$ will be chosen later. The second term on the right-hand side is estimated by

\[
\frac{\gamma \mu}{\rho} \left( \int \int Q_3 |v|^{1-\epsilon} v \, dx \, dt \right)^{\frac{1}{p-1}} \left( \left[ \int \int Q_2 v^{(1+\epsilon)(p-1)} \, dx \, dt \right]^{\frac{1}{p}} \right)
\]
\[
\leq \frac{\gamma \mu}{\rho} \left( \left[ \int \int Q_3 v \, dx \, dt \right]^{1-\epsilon} \left( \int \int Q_2 v^{(1+\epsilon)(p-1)} \, dx \, dt \right) \right)^{-\frac{1}{p}}
\]
\[
\times |Q_3|^{\frac{1}{p}} \left[ C_{q_3} \left( \sup_{\tilde{\rho}^p \rho < 0} \int_{K_2} v(x,t) \, dx \right)^{(1+\epsilon)(p-1)} + \eta_4^p \left( \frac{1}{\theta} \right)^{\frac{(1+\epsilon)(p-1)}{p-2}} \right]^{\frac{1}{p}}
\]
\[
\leq \gamma \tilde{\rho}^N \left[ \left( \sup_{\tilde{\rho}^p \rho < 0} \int_{K_2} v(x,t) \, dx \right)^{1-\epsilon} \left( \frac{1}{\theta} \right)^{\frac{1}{p-2}} \right]^{\frac{p}{p-1}}
\]
\[
\times \left[ C_{q_3} \left( \sup_{\tilde{\rho}^p \rho < 0} \int_{K_2} v(x,t) \, dx \right)^{(1+\epsilon)(p-1)} + \eta_4^p \left( \frac{1}{\theta} \right)^{\frac{(1+\epsilon)(p-1)}{p-2}} \right]^{\frac{1}{p}}
\]
\[
\leq \gamma \tilde{\rho}^N \left[ C_{q_3} \left( \sup_{\tilde{\rho}^p \rho < 0} \int_{K_2} v(x,t) \, dx \right)^{p-\epsilon-1} + \eta_3^p \left( \frac{1}{\theta} \right)^{\frac{(p-\epsilon-1)(p-1)}{p-2}} \right]^{\frac{1}{p}}
\]
\[
\times \left[ C_{q_4} \left( \sup_{\tilde{\rho}^p \rho < 0} \int_{K_2} v(x,t) \, dx \right)^{(1+\epsilon)(p-1)} + \eta_4^p \left( \frac{1}{\theta} \right)^{\frac{(1+\epsilon)(p-1)}{p-2}} \right]^{\frac{1}{p}}
\]

14
\[
\leq \gamma \mu \bar{\theta} N \left[ C_{\rho_0} \left( \sup_{-\bar{\theta}_p < t < 0} \int v(x, t) \, dx \right)^{p-1} + \eta_8 \frac{1}{\gamma} \frac{1}{\bar{\theta}} \right]
= \gamma C_{\rho_0} \mu \bar{\theta} N \left( \sup_{-\bar{\theta}_p < t < 0} \int v(x, t) \, dx \right)^{p-1} + \eta_8 \mu \bar{\theta} N \frac{1}{\bar{\theta}^{\frac{p}{p-1}}},
\]

where as before \( \eta_8 \in (0, 1) \) is still to be chosen. Therefore, combining all the above estimates we arrive at

\[
\iint_{Q_2} |D(v\zeta)|^p \, dxdt \leq \bar{C} \mu \bar{\theta} N \left[ \sup_{-\bar{\theta}_p < t < 0} \int v(x, t) \, dx \right]^{p-1} + \eta_1 \eta_2 \eta_5 \eta_8 \mu \bar{\theta} N \frac{1}{\bar{\theta}^{\frac{p}{p-1}}},
\]

where \( \bar{C} \) takes into account all the \( \gamma C_{\eta} \)-terms. On the other hand, the left-hand side is bounded from below as

\[
\iint_{Q_2} |D(v\zeta)|^p \, dxdt \geq \mu^p \gamma N (Q_1 \setminus E, Q_2)
= \mu^p \int_{-\bar{\theta}_p}^0 \text{cap}_p(K_{\rho} \setminus E, K_{\frac{1}{2}\rho})\chi_{\rho}((\frac{3}{4}\bar{\theta}_p, \frac{1}{2}\bar{\theta}_p))(t) \, dt
= \frac{1}{2} \mu^p \bar{\theta}^p \text{cap}_p(K_{\rho} \setminus E, K_{\frac{1}{2}\rho}).
\]

Thus, recalling the definition of \( \delta(\rho) \), we obtain

\[
\mu^{p-1} \frac{\text{cap}_p(K_{\rho} \setminus E, K_{\frac{1}{2}\rho})}{\text{cap}_p(K_{\rho}, K_{\frac{1}{2}\rho})} \leq \gamma \left[ \sup_{-\bar{\theta}_p < t < 0} \int v(x, t) \, dx \right]^{p-1} + \gamma (\eta_1 \eta_2 \eta_5 \eta_8) \mu^{p-1} \frac{\text{cap}_p(K_{\rho} \setminus E, K_{\frac{1}{2}\rho})}{\text{cap}_p(K_{\rho}, K_{\frac{1}{2}\rho})}.
\]

Choosing \( \eta_1, \eta_2, \eta_5, \eta_8 \) such that \( \gamma (\eta_1 \eta_2 \eta_5 \eta_8) \leq \frac{1}{2} \), the above estimate yields

\[
\mu |\delta(\rho)|^{\frac{1}{p-1}} \leq \gamma \sup_{-\bar{\theta}_p < t < 0} \int v(x, t) \, dx.
\]

We conclude this section, with a second lemma, which will be crucial in the proof of our main result.

**Lemma 3.3.** Let \( (x_0, t_0), Q, u_k, v \) as in (3.1), take \( \bar{\theta} \) as in (3.3), and assume that \( s \leq t_o - 3\gamma \bar{\theta} \rho^p < t_o \leq t \) for some \( \gamma > 1 \) to be determined only in terms of the data \( \{p, N, C_0, C_1\} \). Then there exists a constant \( \gamma_2 > 1 \), that depends only on the data \( \{p, N, C_0, C_1\} \), such that

\[
\mu |\delta(\rho)|^{\frac{1}{p-1}} \leq \gamma_2 \inf_{K_{2\rho}(x_0)} v(\cdot, t),
\]

for all \( t \in [t_0 - \gamma_2 \bar{\theta} \rho^p, t_0] \).
Proof. We may assume that \((x_o, t_o) = (0, 0)\). By our notion \((1.3)\) of solutions, it is not hard to verify that 

\([-\gamma \tilde{\theta} \rho^p, 0] \ni t \rightarrow \int_{K_{2\rho}} v(x, t) \, dx \) is a continuous function.

Let \(t_1 \in [-\tilde{\theta} \rho^p, 0]\) be the point where the supremum in \((3.5)\) is achieved, namely

\(\mu[\delta(\rho)] \frac{1}{p-1} \leq \gamma_1 \int_{K_{2\rho}} v(x, t_1) \, dx. \)  \tag{3.7}

On the other hand, by the weak Harnack inequality \((2.4)\), we have

\[\int_{K_{2\rho}} v(x, t_1) \, dx \leq \bar{\gamma} \inf_{K_{\rho}} v(\cdot, t) \]  \tag{3.8}

for any \(t \in [t_1 + \frac{1}{2} \tilde{\theta} \rho^p, t_1 + \tilde{\theta} \rho^p]\), where

\(\tilde{\theta} = \left[\int_{K_{2\rho}} v(x, t_1) \, dx\right]^{2-p} \rho^p.\)

Combining \((3.7)\) and \((3.8)\) yields

\[\frac{1}{\gamma_1} \mu[\delta(\rho)] \frac{1}{p-1} \leq \inf_{K_{\rho}} v(\cdot, t) \]  \tag{3.9}

for any \(t \in [t_1 + \gamma_* \tilde{\theta} \rho^p, t_1 + \gamma_* \tilde{\theta} \rho^p]\) with \(\gamma_* = \gamma_1^{p-2}.\) At this stage, the time interval where the infimum is taken is somewhat undefined, since a precise value of \(t_1\) is not known. The next argument is meant to provide a precise localization in time of a lower bound for \(v\).

By its definition, \(t_1 + \gamma_* \tilde{\theta} \rho^p \geq 0.\) On the other hand,

\[t_1 + \gamma_* \tilde{\theta} \rho^p = t_1 + \gamma_* ([\delta(\rho)] \frac{1}{p-1} \mu)^{2-p} \rho^p \leq \gamma_* ([\delta(\rho)] \frac{1}{p-1} \mu)^{2-p} \rho^p.\]

Therefore, if we apply Lemma \((2.3)\) with \(\bar{t} = t_1 + \gamma_* \tilde{\theta} \rho^p,\) and take

\[t \in [\gamma_* ([\delta(\rho)] \frac{1}{p-1} \mu)^{2-p} \rho^p, 2 \gamma_* ([\delta(\rho)] \frac{1}{p-1} \mu)^{2-p} \rho^p],\]

we have \(t - \bar{t} \leq 2 \gamma_* ([\delta(\rho)] \frac{1}{p-1} \mu)^{2-p} \rho^p,\) and substituting in \((2.5)\), we conclude, where \(\gamma_2\) depends on \(\nu, \gamma, \gamma_1,\) and \(p.\)

\[\square\]

4 Proof of Theorem \((1.1)\)

Let \((x_o, t_o) \in S_T,\) and for \(R_o > 0\) set

\[Q_{R_o} = K_{2R_o}(x_o) \times (t_o - 3 \gamma_* [\delta(R_o)] \frac{2}{p-1} R_o^{p-\epsilon}, t_o],\]
where $0 < \epsilon < 1$ and $\delta(R_o)$ has been defined in (1.7). As discussed in §1, we may take $R_o$ so small that

$$(t_o - 3\gamma_* \delta(R_o) \frac{2-p}{p-1} R_o^{-p}, t_o) \subset (0, T].$$

Next, if we choose the level

$$k = \sup_{Q_{R_o} \cap \Sigma_T} g,$$

then Lemma [3.1] can be applied. From now on, we deal with such a level, and with the corresponding truncated function $u_k \overset{\text{def}}{=} (u-k)_+$. Moreover, we assume that $u_k$ has been extended to zero in $Q_{R_o} \setminus E_T$.

### 4.1 The First Step

Consider $u_k$, and choose $\mu_o > 0$ such that

$$\mu_o = \sup_{Q_{R_o}} u_k. \quad (4.1)$$

Without loss of generality, we may assume that

$$\mu_o^{2-p} R_o^p \leq R_o^{-\epsilon}. \quad (4.2)$$

Indeed, if (4.2) is not satisfied, then $\mu_o$ has a power-like decay with respect to $R_o$, and there is nothing to prove. If we let

$$v \overset{\text{def}}{=} \mu_o - u_k,$n_and

$$\bar{\theta}_o \overset{\text{def}}{=} \left( \mu_o \delta(R_o) \right)^{\frac{1}{p-1}} \overset{\text{def}}{=} \left( \mu_o \delta(R_o) \right)^{\frac{1}{p-1}},$$

by (4.2), the assumptions of Lemma [3.3] are satisfied, and we conclude that

$$\mu_o \delta(R_o)^{\frac{2-p}{p-1}} \leq \gamma_2 \inf_{K_{2R_o}(x_o)} v(\cdot, t)$$

for all $t \in \left[ t_o - \gamma_* \left( \mu_o \delta(R_o)^{\frac{1}{p-1}} \right)^{2-p} R_o^p, t_o \right]$, that is

$$\sup_{Q_1} u_k \leq \mu_o \left( 1 - \frac{1}{\gamma_2} \delta(R_o)^{\frac{2-p}{p-1}} \right), \quad (4.3)$$

where

$$Q_1 = K_{2R_o}(x_o) \times \left[ t_o - \gamma_* \left( \mu_o \delta(R_o)^{\frac{1}{p-1}} \right)^{2-p} R_o^p, t_o \right],$$

$$= K_{2R_o}(x_o) \times \left[ t_o - \gamma_o \bar{\theta}_o R_o^p, t_o \right]. \quad (4.4)$$
4.2 The Induction

We now proceed by induction. In order to do that, we first need the following result which is based on the fact that we assume \textit{a priori} the Wiener integral \[1.8\] is divergent. The idea of selecting a specific subsequence is taken from [14]. For ease of notation, we set \( A(s) = \left[ \delta(s) \right]^{\frac{1}{p-1}} \).

**Lemma 4.1.** Assume that
\[
\int_0^1 A(s) \frac{ds}{s} = \infty.
\]
Then there exist \( \bar{c} \in (0, 1) \) depending only the data, and a subsequence \( \{ \rho_i \} \) of the sequence \( \{ \rho_i = c^ER_o \} \), such that
\[
3 \left[ \mu_{i+1}(A(\rho_{i+1})) \right]^{2-p} \rho_{i+1}^p \leq \left[ \mu_i(A(\rho_i)) \right]^{2-p} \rho_i^p, \tag{4.5}
\]
where
\[
\mu_{i+1} = \mu_i \left[ 1 - \frac{1}{\gamma_2} A(\rho_i) \right].
\]
Moreover,
\[
\sum_{i=0}^{i_{k+1}-1} A(\rho_i) \leq 2 \sum_{j=0}^{k} A(\rho_j) \text{ for any } k = 0, 1, 2, \ldots. \tag{4.6}
\]

**Proof.** First, we observe that the divergence of the Wiener integral implies the divergence of the series
\[
\sum_{i=0}^{\infty} A(\rho_i),
\]
which does not require any quantitative information about \( \bar{c} \). Next, for any non-negative integer \( i \), there exists \( j \in \mathbb{N} \) such that
\[
\frac{A(\rho_{i+j})}{A(\rho_i)} > \left( \frac{1}{2} \right)^j;
\]
otherwise, it would lead to the convergence of the series \( \sum_{i=0}^{\infty} A(\rho_i) \).

Let \( i_o = 0 \) and choose \( i_1 > i_o \) to be the smallest positive integer satisfying
\[
\frac{A(\rho_{i_1})}{A(\rho_o)} > \left( \frac{1}{2} \right)^{i_1-i_o};
\]
by induction, we obtain a subsequence \( \{ i_j \} \) such that \( i_{j+1} > i_j \) is the smallest positive integer satisfying
\[
\frac{A(\rho_{i_{j+1}})}{A(\rho_{i_j})} > \left( \frac{1}{2} \right)^{i_{j+1}-i_j}. \tag{4.7}
\]
Next, we observe that

\[ 3 \left[ \mu_{ij+1} A(\rho_{ij+1}) \right]^{2-p} \rho_{ij+1}^p \leq 3 \left[ \left( 1 - \frac{1}{\gamma_2} \right) \mu_{ij} A(\rho_{ij+1}) \right]^{2-p} \rho_{ij+1}^p. \]

Hence, in order to show (4.5), we need only to show

\[ 3 \left[ \left( 1 - \frac{1}{\gamma_2} \right) A(\rho_{ij+1}) \mu_{ij} \right]^{2-p} \rho_{ij+1}^p \leq \left[ \mu_{ij} A(\rho_{ij}) \right]^{2-p} \rho_{ij}^p. \]

This is equivalent to

\[ \frac{A(\rho_{ij+1})}{A(\rho_{ij})} \geq 3^{\frac{1}{p}} \left( 1 - \frac{1}{\gamma_2} \right)^{-1} \frac{\bar{c}}{\bar{c} - (i_{j+1} - i_j)}. \]

Comparing this with (4.7), one easily obtains (4.5) by choosing \( \bar{c} = 2^{-\lambda} \) with some large \( \lambda \) satisfying

\[ 2^{\frac{3\lambda}{\gamma_2} - 1} \geq 3^{\frac{1}{p}} \left( 1 - \frac{1}{\gamma_2} \right)^{-1}. \]

Finally, according to the way of choosing \( i_{j+1} \), we must have

\[ \frac{A(\rho_i)}{A(\rho_j)} \leq \left( \frac{1}{2} \right)^{i_{j+1} - i_j} \text{ for any } i_j \leq i \leq i_{j+1} - 1. \]

This implies

\[ \sum_{i=i_j}^{i_{j+1}-1} A(\rho_i) \leq A(\rho_{i_j}) \sum_{i=i_j}^{\infty} \left( \frac{1}{2} \right)^{i_{j+1} - i_j} \leq 2A(\rho_{i_j}). \]

Summing the above inequality over \( j \) from 0 to \( k \) yields

\[ \sum_{i=0}^{i_{k+1}-1} A(\rho_i) = \sum_{j=0}^{k} \sum_{i=i_j}^{i_{j+1}-1} A(\rho_i) \leq 2 \sum_{j=0}^{k} A(\rho_{i_j}) \text{ for any } k = 0, 1, 2, \ldots. \]

This concludes the proof. \( \square \)

Now, assume that up to step \( l \) we have shown

\[ \sup_{Q_{i_j}} u_k \leq \mu_{ij}, \quad j = 1, \ldots, l, \]

where

\[ Q_{ij} = K_{2\rho_{ij-1}}(x_o) \times (t_o - \gamma_{\bar{c}} \tilde{\theta}_{ij-1}, t_o) \rho_{ij-1}^p \]

and

\[ \tilde{\theta}_{ij} = \left[ \mu_{ij-1} A(\rho_{ij-1}) \right]^{2-p}, \quad \mu_{ij} = \mu_{ij-1} \left[ 1 - \frac{1}{\gamma_2} A(\rho_{ij-1}) \right]^p. \]
Then by (4.5) and Lemma 3.3 we have

$$\sup_{Q_{i+1}} u_k \leq \mu_{i+1},$$

where

$$Q_{i+1} = K_{2\rho_i} (x_o) \times (t_o - \gamma_s \bar{\theta}_i \rho_i^{p}, t_o)$$

and

$$\bar{\theta}_i = [\mu_i, A(\rho_i)]^{2-p}, \quad \mu_{i+1} = \mu_i \left[1 - \frac{1}{\gamma_2} A(\rho_i)\right].$$

Employing (4.6), we can now conclude as in [7, Section 6.4]: there exists a constant $\gamma_3 > 1$ that depends only on the data $\{p, N, C_o, C_1\}$, such that

$$\sup_{Q_{i+1}} u_k \leq \mu_{i} \left[1 - \frac{1}{\gamma_2} A(\rho_i)\right] \leq \mu_{o} \exp\left\{-\frac{1}{\gamma_3} \int_{\rho_{i+1}}^{R_o} A(s) \frac{ds}{s}\right\};$$

taking into consideration the reverse case of (4.2) actually yields that

$$\sup_{Q_{i+1}} (u - k)_+ \leq \mu_{o} \exp\left\{-\frac{1}{\gamma_3} \int_{\rho_{i+1}}^{R_o} A(s) \frac{ds}{s}\right\} + \gamma_3 R_o^{-\frac{\gamma_3}{2}}. \quad (4.8)$$

Now fix $\rho \in (0, R_o)$; there is an integer $l \geq 0$ such that

$$\rho_{i+1} \leq \rho < \rho_i.$$

As a result, it is easy to check that

$$Q_\rho(\mu_o) = K_{2\rho} (x_o) \times [t_o - \mu_o^{2-p} \rho^p, t_o] \subset Q_{i+1}.$$

Hence, we may conclude from (4.8) that

$$\sup_{Q_{\rho}(\mu_o)} (u - k)_+ \leq \mu_{o} \exp\left\{-\frac{1}{\gamma_3} \int_{\rho}^{R_o} A(s) \frac{ds}{s}\right\} + \gamma_3 R_o^{-\frac{\gamma_3}{2}}. \quad (4.9)$$

Similarly, if we set

$$h = \inf_{Q_{\mu_o}} g,$$
and work with \( u_h = (h - u)_+ \), an analogous argument as above gives that

\[
\sup_{Q_\rho(\tilde{\mu}_o)} (h - u)_+ \leq \tilde{\mu}_o \exp\left\{ -\frac{1}{\gamma_3} \int_{\rho}^{R_\omega} A(s) \frac{ds}{s} \right\} + \gamma_3 R_\omega^{\frac{\gamma_3}{\gamma_3 - 1}} , \tag{4.10}
\]

where

\[
\tilde{\mu}_o = \sup_{Q_{\rho_o}} u_h.
\]

Note that

\[
\max\{\mu_o, \tilde{\mu}_o\} \leq \mu_o + \tilde{\mu}_o \leq \omega_o - \operatorname{osc} g \leq \omega_o.
\]

Combining \( (4.9) \) and \( (4.10) \) yields

\[
\operatorname{osc}_{Q_{\rho(\omega_o) \cap E_T}} u \leq \omega_o \exp\left\{ -\frac{1}{\gamma_3} \int_{\rho}^{R_\omega} A(s) \frac{ds}{s} \right\} + \operatorname{osc}_{Q_{\rho_o} \cap S_T} g + 2\gamma_3 R_\omega^{\frac{\gamma_3}{\gamma_3 - 1}}.
\]

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