ON THE ASYMPTOTIC EXPANSION OF THE $q$-Dilogarithm

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Abstract. In this work, we study some asymptotic expansion of the $q$-dilogarithm at $q = 1$ and $q = 0$ by using Mellin transform and adequate decomposition allowed by Lerch functional equation.

1. Introduction

Euler’s dilogarithm is defined by [11]

\begin{equation}
Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| < 1.
\end{equation}

In [7], A. N. Kirillov defines the following $q$-analogue of the dilogarithm $Li_2(z)$

\begin{equation}
Li_2(z; q) = \sum_{n=1}^{\infty} \frac{z^n}{n(1 - q^n)}, \quad |z| < 1, \quad 0 < q < 1,
\end{equation}

and observes the following remarkable formula [7, §2.5, Lemma 8]

\begin{equation}
\sum_{n=0}^{\infty} \frac{z^n}{(q, q)_n} = \exp(Li_2(z, q)), \quad |z| < 1, \quad |q| < 1.
\end{equation}

where

\begin{equation}
(q, q)_0 = 1, \quad (q, q)_n = \prod_{n=0}^{n-1} (1 - q^n), \quad n = 1, 2, \ldots.
\end{equation}

In [14], Hardy and Littlewood proved that for $|q| = 1$, this identity holds inside the radius of convergence of either series. It seems, a precise formulation of (1.3) going back to Ramamujan (see [8, Ch. 27, Entry 6]) gives an asymptotic series for $Li_2(z; q)$. Kirillov [7, §2.5, Corollary 10] and Ueno Nishizawa [6] derive this asymptotic series by using Euler–Maclaurin’s summation formula, see also [11].

The principal aim of this paper is to give a complete asymptotic expansions of the $q$-dilogarithm at $q = 0$ and $q = 1$. We first give by applying Mellin transform technique an integral representation of Barnes type for the $q$-dilogarithm. Secondly, we use the Lerch functional equation to decompose the integrand and to apply Cauchy theorem.

2. $q$-Dilogarithm

The polylogarithm, is defined in the unit disk by the absolutely convergent series [11]

\begin{equation}
Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad |z| < 1
\end{equation}

Several functional identities satisfied by polylogarithm are available in the literature (see [11]). For $n \in \mathbb{N}$, the function $Li_n(z)$ can also be represented as

\begin{equation}
Li_n(z) = \int_0^z \frac{Li_{n-1}(t)}{t} \, dt, \quad n \in \mathbb{N}, \quad Li_1(z) = - \log(1 - z) = \int_0^z \frac{dt}{1 - t}.
\end{equation}
which is valid for all $z$ in the cut plane $\mathbb{C} \setminus [1, \infty)$.

The notation $F(\theta, s)$ is used for the polylogarithm $Li_s(e^{2i\pi \theta})$ with $\theta$ real, called the periodic zeta function (see [11, §25.13]) and is given by the Dirichlet series

$$F(\theta, s) = \sum_{n=1}^{\infty} \frac{e^{2i\pi \theta \, n}}{n^s}, \quad \theta \in \mathbb{R},$$

it converges for $\Re s > 1$ if $\theta \in \mathbb{Z}$, and for $\Re s > 0$ if $\theta \in \mathbb{R}/\mathbb{Z}$. This function may be expressed in terms of the Clausen functions $Ci_s(\theta)$ and $Si_s(\theta)$, and vice versa (see [11, §27.8]):

$$Li_s(e^{\pm i \theta}) = Ci_s(\theta) \pm i Si_s(\theta).$$

In [5], T.H. Koornwinder defines the $q$-analogue of the logarithm function

$$-\log(1 - z) = \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1,$$

as follows

$$\log_q(z) = \sum_{n=1}^{\infty} \frac{z^n}{1 - q^n}, \quad |z| < 1, \quad 0 < q < 1.$$

Recall that, the $q$-analogue of the ordinary integral (called Jackson’s integral) is defined by

$$\int_0^z f(t) \, dq_t = (1 - q)z \sum_{n=0}^{\infty} f(zq^n)q^n.$$

One can recover the ordinary Riemann integral as the limit of the Jackson integral for $q \uparrow 1$.

**Lemma 2.1.** The function $\log_q(z)$ has the following $q$-integral representation

$$\log_q(z) = \int_0^z \frac{1}{1 - t} \, dq_t, \quad |z| < 1.$$

Moreover, it can be extended to any analytic function on $\mathbb{C} - \mathbb{C} - \{q^{-n}, n \in \mathbb{N}_0\}$.

**Proof.** Assume that $|z| < 1$, then from (2.4) we have

$$(1 - q) \log_q(z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} z^n q^{nm},$$

$$= (1 - q)z \sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty} z^n q^{nm},$$

$$= (1 - q)z \sum_{m=0}^{\infty} \frac{q^m}{1 - zq^m}.$$

The inversion of the order of summation is permitted, since the double series converges absolutely when $|z| < 1$.

Let $K$ be compact subset of $\mathbb{C} - \{q^{-n}, n \in \mathbb{N}_0\}$. There exist $N \in \mathbb{N}$ such that for all $z \in K$, $|q^N z| < q$. Then for $n \geq N$ we have

$$|\frac{q^m}{1 - zq^m}| \leq \frac{q^m}{1 - q}.$$

Hence, the series $\sum_{m=N}^{\infty} \frac{q^m}{1 - zq^m}$ converges uniformly in $K$. \qed

The $q$-dilogarithm (1.2) is related to Koornwinder’s $q$-Logarithm (2.5) by

$$Li_2(z, q) = \int_0^z \frac{\log_q(t)}{t} \, dt.$$
It follows that for $n \geq 2$, we can also define
\begin{equation}
(2.10) \quad Li_n(z, q) = \int_0^z \frac{Li_{n-1}(t, q)}{t} dt.
\end{equation}
This integral formula proves by induction that $Li_n(z, q)$ has an analytic continuation on $\mathbb{C} - [1, \infty)$. Moreover, for $|z| < 1$, we have
\begin{equation}
Li_n(z, q) = \sum_{k=1}^{\infty} z^k \frac{1}{k^n(1-q^n)}.
\end{equation}
This converges absolutely for $|z| < 1$ and defines a germ of a holomorphic function in the neighbourhood of the origin. Note that
\begin{align*}
\lim_{q \uparrow 1} (1-q) Li_2((1-q)z, q) &= Li_2(z),
\lim_{q \downarrow 0} (1-q) Li_2(z, q) = - \log(1-z), \quad |z| < 1.
\end{align*}
Let $\omega = e^{-zx+2i\pi\theta}$, $\theta \in \mathbb{R}$ and $\text{Re} \ z > 0$, we define
\begin{align*}
(2.11) \quad & Ci_2(\omega, q) = \sum_{n=1}^{\infty} e^{-nz} \cos(2\pi n\theta) \frac{1}{n(1-q^n)}, \\
(2.12) \quad & Si_2(\omega, q) = \sum_{n=1}^{\infty} e^{-nz} \sin(2\pi \theta) \frac{1}{n(1-q^n)}.
\end{align*}
Note that these function can be considered as $q$-analogues of the Clausen functions (2.4) and are related to the $q$-diogarithm by
\begin{equation}
(2.13) \quad Li_2(\omega, q) = Ci_2(\omega, q) + i Si_2(\omega, q).
\end{equation}
Now, we will use the Mellin transform method to obtain the integral representation
\begin{equation}
(2.14) \quad Li_2(\omega, e^{-x}) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \zeta(s, z) F(\theta, s) \Gamma(s) x^{-s} ds, \quad c > 1,
\end{equation}
where
\begin{equation}
\omega = e^{-zx+2i\pi\theta}, \quad x > 0, \quad \text{Re} \ z > 1, \quad 0 < \theta < 1.
\end{equation}
Recall that, the Mellin transform for a locally integrable function $f(x)$ on $(0, \infty)$, is defined by
\begin{equation}
(2.15) \quad M(f, s) = \int_0^{\infty} f(x) x^{s-1} dx.
\end{equation}
which converges absolutely and defines an analytic function in the strip $a < \text{Re} \ s < b$, where $a$ and $b$ are real constants (with $a < b$) such that for $\varepsilon > 0$
\begin{equation}
(2.16) \quad f(x) = \begin{cases} O(x^{-a-\varepsilon}) & \text{as } x \to 0_+ \\ O(x^{-b-\varepsilon}) & \text{as } x \to +\infty. \end{cases}
\end{equation}
The inversion formulas reads
\begin{equation}
(2.17) \quad f(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} M(f, s) x^{-s} ds,
\end{equation}
where $c$ satisfies $a < c < b$. Equation (2.17) is valid at all points $x \geq 0$ where $f(x)$ is continuous.
We first compute the Mellin transform $M(\psi_n(x), s)$, where
\begin{equation}
(2.18) \quad \psi_n(x) = \frac{e^{-nx}}{n(1-e^{-nx})}, \quad x > 0, \quad \text{Re} \ z > 1, \quad n \in \mathbb{N}.
\end{equation}
Since
\[(2.19)\]
\[\psi_n(x) \sim \frac{1}{nx}, \quad x \to 0^+\]
\[(2.20)\]
\[\psi_n(x) \sim \frac{1}{n} e^{-nx(z-1)}, \quad x \to +\infty.\]

We concluded that \(M(\psi_n(x), s)\) is defined in the half–plane \(\Re s > 0\). That is, the constants \(a\) and \(b\) satisfy \(a = 1\) and \(b = +\infty\), which values can be used for all \(n \geq 1\) and \(\Re z > 1\). The Mellin transform of \(\psi_n(x)\) can be obtained from the following integral representation of the Hurwitz zeta function \(\zeta(s, z)\)
\[(2.21)\]
\[\zeta(s, z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-zx}}{1-e^{-x}} x^{s-1} dx,\]
\[\text{(Re } s > 0, \quad |\text{arg}(1-z)| < \pi; \quad \text{Re } s > 1, \quad z = 1).\]

Note that \(\zeta(s, z)\) is expressed also by the series
\[(2.22)\]
\[\zeta(s, z) = \sum_{k=1}^{\infty} \frac{1}{(z+k)^s}, \quad \text{Re } s > 1, \quad z \neq 0, -1, -2, \ldots.\]

For the other values of \(z\), \(\zeta(s, z)\) is defined by analytic continuation. It has a meromorphic continuation in the \(s\)–plane, its only singularity in \(\mathbb{C}\) being a simple pole at \(s = 1\)
\[(2.23)\]
\[\zeta(s, z) = \frac{1}{s-1} - \psi(z) + \mathcal{O}(s-1).\]

Applying the Mellin inversion theorem to the integral \[(2.21),\] we then find
\[(2.24)\]
\[\psi_n(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s, z) \Gamma(s)(nx)^{-s} ds.\]

From the Stirling’s formula, which shows that for finite \(\sigma\)
\[(2.25)\]
\[\Gamma(\sigma + it) = \mathcal{O}(|t|^{\sigma-1} e^{-\frac{1}{2}\pi |t|}) \quad (|t| \to +\infty)\]
and from the well–known behaviour of \(\zeta(s, z)\) (see [9])
\[(2.26)\]
\[\zeta(s, z) = \mathcal{O}(|t|^{\tau(\sigma)} \log |t|),\]
where
\[
\tau(\sigma) = \begin{cases} 
\frac{1}{2} - \sigma, & \sigma \leq 0, \\
\frac{1}{2}, & 0 \leq \sigma \leq \frac{1}{2}, \\
1 - \sigma, & \frac{1}{2} \leq \sigma \leq 1, \\
0, & \sigma \geq 1.
\end{cases}
\]

We obtain the following majorization of the modulus of the integrand in \[(2.24),\]
\[(2.27)\]
\[\mathcal{O}(|t|^{\tau(\sigma)+\sigma-1} \log |t|).\]

Consequently, the integral \[(2.24),\] converges absolutely in all vertical strip of the half–plane \(\Re s > 0\). Then we replace \(x\) by \(nx\), where \(n\) is a positive integer, and sum over \(n\), we then obtain
\[(2.28)\]
\[Li_2(\omega, e^{-x}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s, z) F(\theta, s + 1) \Gamma(s) x^{-s} ds, \quad c > 1,\]
where
\[\omega = e^{-zx+2i\pi\theta}, \quad x > 0, \quad \Re z > 1, \quad 0 < \theta < 1.\]
3. Asymptotic at \( q = 1 \)

The integral \((2.28)\) will be used to derive asymptotic expansions of the \( q \)-dilogarithm. The contour of integration is moved at first to the left to obtain asymptotic expansion at \( q = 1 \) and then to the right to get asymptotic expansion at \( q = 0 \).

Let us consider the function
\[
g(s) = \zeta(s, z) F(\theta, s + 1) \Gamma(s).
\]

The periodic function zeta function \( F(\theta, s) \) has extension to an entire function in the \( s \)-plane (see\[13\]). Hence, the function \( g(s) \) has a meromorphic continuation in the \( s \)-plane, its only singularity in \( \mathbb{C} \) coincide with the pole of \( \Gamma(s) \) and \( \zeta(s, z) \) being a simple pole at \( s = 1, 0, -1, -2, \ldots \).

Now we compute the residues of the poles. The special values at \( s = -1, -2 \ldots \) of the periodic zeta function are reduced to the Apostol–Bernoulli polynomials (see\[13\]). Hence, the function \( g \) coincides with the pole of \( \Gamma(s) \).

\[
F(\theta, -n) = -\frac{B_{n+1}(1, e^{2\pi i \theta})}{n + 1}.
\]

We need also the following asymptotic expansion of \( \Gamma(s) \) and \( \zeta(s) \) at \( s = 0 \)
\[
\Gamma(s) = \frac{1}{s} - \gamma + \mathcal{O}(s^2),
\]
\[
\zeta(s) = \frac{1}{2} - z + s \log \frac{\Gamma(z)}{2\pi} + \mathcal{O}(s^2).
\]

Hence,
\[
\lim_{s \to 0} (s - 1)g(s) = \text{Li}_2(e^{2\pi i \theta}), \quad \lim_{s \to -n} (s + n)g(s) = \frac{(-1)^n}{(n + 1)(n + 1)!} B_{n+1}(z) B_{n+1}(1, e^{2\pi i \theta}).
\]

Here \( B_n(z) \) is the Bernoulli polynomial (see\[11\]).

Let \( N \) be an integer and \( d \) real number such that \(-N - 1 < d < -N\). We consider the integral taken round the rectangular contour with vertices at \( d \pm iA \) and \( c \pm iA \), so that the side in \( Re(s) < 0 \) parallel to the imaginary axis passes midway between the poles \( s = 1, 0, -1, -2, \ldots, -N \).

The contribution from the upper and lower sides \( s = \sigma \pm iA \) vanishes as \( |A| \to +\infty \), since the modulus of the integrand is controlled by
\[
\mathcal{O}(|A|^{\sigma+\sigma-1/2}) \log |A| e^{-\frac{1}{2}\pi|A|}.
\]

This follows from Stirling’s formula (2.25), the behaviour \( \zeta(s, z) \) given by (2.26), and the following estimation
\[
|F(\theta, s + 1)| \leq \zeta(\sigma + 1) = \mathcal{O}(1) \quad |A| \to +\infty.
\]

Displacement of the contour (2.28) to the left then yields
\[
F(\theta, s) = C_1(\theta) \frac{1}{x} + (1 - z)C_2(\theta) + \sum_{n=1}^{N} \frac{(-1)^{n+1}}{(n + 1)(n + 1)!} B_{n+1}(z) B_{n+1}(1, e^{2\pi i \theta})x^n + R_N(x).
\]

where the remainder integral \( R_N(z) \) is given by
\[
R_N(x) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \zeta(s, z) F(\theta, s + 1) \Gamma(s)x^{-s} ds, \quad x > 0, \quad \text{Re} \ z > 1.
\]

From (3.3), we find
\[
|R_N(x)| = \mathcal{O}(\frac{1}{x^{N+1}}),
\]
and we obtain the asymptotic expansion
\begin{equation}
F(\theta, s) \sim Ci_2(\theta) \frac{1}{x} + \left( \frac{1}{2} - x \right) Ci_1(\theta) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)(n+1)!} B_{n+1}(z) B_n(1, e^{2i\pi \theta} x^n).
\end{equation}

4. ASYMPTOTIC AT \( q = 0 \)

Recall that the periodic zeta function satisfies the functional equation (see [13])
\begin{equation}
F(\theta, s) = \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \left\{ e^{\frac{\pi i(1-s)}{2}} \zeta(1-s, \theta) + e^{\frac{\pi i(s-1)}{2}} \zeta(1-s, 1-\theta) \right\},
\end{equation}
(Re \( s > 1, \quad 0 < \theta < 1 \),

first given by Lerch, whose proof follows the lines of the first Riemann proof of the functional equation for \( \zeta(x) \).

It’s well known that the asymptotic expansion near infinity via Mellin transform is obtained by displacement of the contour of integration in the Mellin inversion formulas (2.16) to the right-hand side (see [31]). However, the integrand (4.1) has no poles in half-plane \( \text{Re } s > 1 \). Since, the periodic zeta function \( F(\theta, s) \) has an analytic continuation to the whole \( s \)-space for \( 0 < \theta < 1 \). Moreover, the poles of \( \Gamma(1 - s) \) in the equation (4.1) at \( s = -1, -2, \ldots \), are canceled by the zeros of the function
\[ e^{\frac{\pi i(1-s)}{2}} \zeta(1-s, \theta) + e^{\frac{\pi i(s-1)}{2}} \zeta(1-s, 1-\theta). \]

On the other hand from (4.1) we easily obtain
\begin{equation}
\Gamma(s)\{ F(\theta, s+1) + F(1-\theta, s+1) \} = -\frac{(2\pi)^{s+1}}{2s \sin \frac{\pi s}{2}} \{ \zeta(-s, \theta) + \zeta(-s, 1-\theta) \}.
\end{equation}
where we are able to simplify (4.2) by the well-known reflection formulas
\[ \frac{\pi}{\sin \pi s} = \Gamma(s) \Gamma(1-s), \quad \frac{\sin \pi s}{\pi} = \frac{2}{\pi} \sin \frac{\pi s}{2} \sin \frac{\pi(1-s)}{2}. \]

Proceeding similar as above we also obtain
\begin{equation}
\Gamma(s)\{ F(\theta, s) - F(1-\theta, s) \} = \frac{(2\pi)^{s+1}}{2s \cos \frac{\pi s}{2}} \{ \zeta(-s, \theta) - \zeta(-s, 1-\theta) \}.
\end{equation}

Moreover, the integral representation (2.28) is valid for all \( 0 < \theta < 1 \). So we can replace \( \theta \) by \( 1 - \theta \) in its integrand. Using the above decomposition (1.2) and (4.2), we then obtain
\begin{equation}
Ci_2(\omega, e^{-x}) = -\frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{(2\pi)^{s+1} \zeta(s, z)}{2s \sin \frac{\pi s}{2}} \{ \zeta(-s, \theta) + \zeta(-s, 1-\theta) \} \frac{ds}{x^s}.
\end{equation}
and
\begin{equation}
Si_2(\omega, e^{-x}) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{(2\pi)^{s+1} \zeta(s, z)}{2s \cos \frac{\pi s}{2}} \{ \zeta(-s, \theta) - \zeta(-s, 1-\theta) \} \frac{ds}{x^s}.
\end{equation}
where \( \omega = e^{-zx+2i\pi \theta}, \quad 0 < x, \quad 0 < \theta < 1, \quad 0 < \text{Re } \omega \quad \text{and} \quad 1 < c < 2. \)

Note that the special values \( \zeta(n, z) \) \((n \in \mathbb{N}_0)\) are expressed in terms of the polygamma function \( \psi(z) \)
\begin{equation}
\zeta(n+1, z) = \frac{(-1)^{n+1}}{n!} \psi^{(n)}(z), \quad z \neq 0, -1, -2, \ldots ,
\end{equation}
and \( \zeta(-n, z) \) \((n \in \mathbb{N})\) is reduced to Bernoulli polynomial
\begin{equation}
\zeta(-n, z) = -\frac{B_{n+1}(z)}{n+1}.
\end{equation}
Applying the identities for the Bernoulli polynomial
\[ B_n(1-\theta) = (-1)^n B_n(\theta), \]
we obtain
\begin{equation}
\zeta(-n, \theta) + \zeta(-n, 1 - \theta) = ((-1)^{n+1} - 1) \frac{B_{n+1}(\theta)}{n+1},
\end{equation}
\begin{equation}
\zeta(-n, \theta) - \zeta(-n, 1 - \theta) = ((-1)^{n} - 1) \frac{B_{n+1}(\theta)}{n+1},
\end{equation}
The integrand in (4.3) has a meromorphic continuation in the $s$-plane, its only singularity in the half-plane $\text{Re} \ s > 0$ coincide with the pole of $1/\sin^{2} \frac{\pi s}{2}$ being a simple pole at $s = 2, 4, \ldots$. Then by Cauchy integral, we can shift the contour in (4.3) to the right, picking up the residues at $s = 2, \ldots, 2N$, with result
\begin{equation}
Ci_{2}(\omega, e^{-x}) = 4 \sum_{k=1}^{N} (-1)^{n} \frac{\psi^{(2n-1)}(z) B_{2n+1}(\theta)}{(2n+1)!} \frac{2\pi}{x} 2^{n} + Q_{N}(x),
\end{equation}
where
\begin{equation}
Q_{N}(x) = \frac{1}{2 \pi i} \int_{c+2N-i\infty}^{c+2N+i\infty} \frac{(2\pi)^{s+1} \zeta(s, z)}{2s \sin \frac{\pi s}{2}} \left(\zeta(-s, \theta) + \zeta(-s, 1 - \theta)\right) \frac{ds}{x^{s}}.
\end{equation}
Using the following estimations in a vertical strip $s = \sigma + it, \quad \sigma \neq 0, \pm 1, \pm 2, \ldots$,
\begin{equation}
\frac{1}{\sin \frac{\pi s}{2}} = O(|t|^{-1} e^{-\frac{\pi}{2}|t|}),
\end{equation}
We obtain
\begin{equation}|Q_{N}(x)| = O\left(\frac{1}{x^{2N+1}}\right).
\end{equation}
Similarly,
\begin{equation}
Si_{2}(\omega, e^{-x}) = \frac{4\gamma}{\pi} B_{2}(\theta) \frac{1}{x} + 4 \sum_{n=1}^{N} (-1)^{n} \frac{\psi^{(2n)}(z) B_{2n+2}(\theta)}{(2n+2)!} \frac{2\pi}{x} 2^{n+1} + O\left(\frac{1}{x^{2N+2}}\right).
\end{equation}
Hence,
\begin{equation}
Li_{2}(\omega, e^{-x}) \sim \frac{4\gamma}{\pi} B_{2}(\theta) \frac{1}{x} + 4 \sum_{n=1}^{\infty} \frac{\psi^{(n-1)}(z) B_{n+1}(\theta)}{(n+1)!} \frac{2\pi}{x} n.
\end{equation}

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