A Connection of Apparent Horizon and Naked Singularities

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Abstract

We show that the behaviour of the outgoing radial null geodesic congruence on the apparent horizon is related to the property of nakedness in spherical dust collapse justifying the difference in the Penrose diagrams in the naked and covered dust collapse scenarios. We provide arguments suggesting that the relationship could be generally valid.
1 Introduction

Consider a cloud of matter (regular initial Cauchy data) collapsing indefinitely under its own gravity. A singularity eventually develops in the spacetime and it is indicated by the divergence of the Kretschmann scalar. In the advanced stages of collapse trapped regions are formed [1], [2] and there exists a null ray which marginally escapes to infinity (event horizon). It is not clear whether the singular boundary is entirely surrounded by the trapped region. In other words, it is not known if a portion of the boundary is exposed in the untrapped region and non-spacelike geodesics can emanate from it (naked singularities).

Exact solutions to Einstein’s field equations with certain kind of source terms are known to exhibit both naked and covered singularities depending upon the sort of regular initial data chosen. Little progress has been made regarding a general classification of initial data according to the covered or naked consequence of the evolution. It is not known if the data leading to naked singularities has zero measure in the set of all possible (or all possible physically relevant) Cauchy data.

The complexity of the problem lies in the fact that the Cauchy initial value problem for the Einstein field equations with sources is less tractable. The systematics available about the general problem is far too less for any implication for questions like formation of naked singularities. For instance, even the well posedness of the problem is not self evident and has to be proved independently for different types of sources [3], [4], [5]. It would be indeed difficult to find or even expect a conserved or monotonically behaving function of the Cauchy surface with respect to its evolution, which could be expected to provide insight into the process of creation of naked singularities.

As a result of this difficulty, a large number of investigations that have been carried out have been concerning certain exact solutions or numerical simulations. Issues like strength of the singularities, genericity, behaviour with respect to change of source etc. have been studied in examples like dust, null dust, perfect and imperfect fluids and scalar fields. However, they do not suggest any typical geometrical feature which could be expected to arise before a naked singularity forms (In case of a singularity the singularity theorems make use of a typical geometrical feature viz. trapped regions to prove its existence). Such a feature indicating the presence of a naked singularity


would be interesting in the light of the Hoop conjecture or the issue of isoperimetric inequalities and could lead to some geometrical insight into the process. The lack of indication of existence of such a feature is evident in the fact that one is forced to check for the existence of naked singularities in a direct manner even in particular examples. To be precise, one checks if non space-like geodesics emerge from the singular boundary using differential geometry and the form of the metric in the example.

This paper is a first step towards an indirect criterion. Preferably, the criterion should be applicable away from the singular boundary. That is a non-local problem and given the difficulties of Cauchy evolution, there is no indication available for what the criterion could be. It would be perhaps appropriate in such a situation to examine regions near the singularity. There are also parts of the singularity from which geodesics cannot escape. Some criterion applicable to such a portion would also be significant. It would indicate that the information about the exposure of a part of the boundary is contained elsewhere on the boundary.

We illustrate the main features of spherical dust collapse in radial co-ordinates (figures 1 and 2) and in causally correct Penrose diagrams (figures 3 and 4).

In figure 3, there is no portion of the boundary beyond O that is exposed. In figure 4, however, there is a null portion. In this paper, we examine if O would yield the information about the existence of an exposed portion. O is a covered point of the singular boundary (with or without a naked portion). This point is where the apparent horizon meets the singularity. Mathematically, one works with points on the apparent horizon in the approach to the concerned point. This could be looked upon as a property of the marginally trapped regions which constitute the apparent horizon. One is therefore working with outgoing null congruences with zero expansion. If a singularity meets such a region, the point in the strict sense will not have any non-spacelike geodesics emerging from itself and will therefore be covered.

We prove the criterion for the dust case. We offer a simple justification for the difference in the Penrose diagrams in the naked and covered dust cases. However, the criterion is not dust specific and should be further examined in general cases.

The plan of the paper is as follows. We first motivate the view that the tangent vector to a geodesic in a congruence represents a 'flux den-
sity’ analogous to the situation in ordinary electrostatics where one has lines of force. We wish to investigate the tangent vector on the apparent horizon. The next section describes the self similar dust model where we discuss the above and demonstrate the connection with nakedness. In the next section we show that the results can be extended to the general dust case. Finally we turn to the conformal transformation leading to the Penrose diagram for the naked case from spherical co-ordinates pointing out that it should diverge at the singular boundary. We argue that in a general case, this divergence would prevent ingoing geodesics from reaching the point of interest which would then suggest that an exposed ingoing null boundary exists.

2 Flux of Congruence

The scenario of spherical dust collapse is shown in the radial coordinates in figures 1 and 2. Observe that in the naked case (figure 2) several geodesics emerge from the centre between the event and apparent horizons and fall into the trapped region crossing the apparent horizon (Typically this is a case of a locally naked singularity). All along the apparent horizon in the approach to the centre, one would find null geodesics which would have originated at the naked centre. As drawn in the figure, they appear to intersect the apparent horizon in a sequence of points which agglomerate in the approach to the centre (in the Euclidean sense of the figure). If this idea of agglomeration could be made precise then it would be of interest to study if they (or the geodesics) tend to cluster near a point on the apparent horizon. Such a property would perhaps contain information about the source, if existent, for the lines of the congruence in a manner analogous to electrostatics. One could expect an enhancement of clustering of the null curves in the naked case as there is a source of lines in the vicinity (naked portion of the singular boundary). No such source at the centre exists in the covered case and the congruence could perhaps exhibit a different characteristic behaviour. Motivated by this, we define the quantity of flux density. Given a manifold M, one defines a congruence to be a set of curves such that their union is M and each point of M has a single curve containing it. This is analogous to the concept of

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2 In such naked singularities, causal curves emanating from the singularity need not reach an asymptotic observer.
lines of force in electrostatics. So we define a quantity analogous to electrostatic flux density for a geodesics congruence. We first affinely parametrize the congruence and treat the tangent vector $\xi$ like the electric field of the analogy. For any 3 dimensional hypersurface we define $\xi^\mu ds_\mu$ to be the infinitesimal flux across the infinitesimal three area $ds_\mu$ of the hypersurface. The flux density is obviously the tangent vector itself. It is this tangent vector that we examine in the limit of approach to the singularity (along the apparent horizon).

It might appear from the motivation in the previous paragraph that one would attempt to check for sources of the flux (like taking $\nabla.E$ in electrostatics). Instead of this, we examine it on the apparent horizon in an attempt to move away from the source (naked part of the singular boundary or set of ‘points’ where causal curves originate on the singularity or more precisely set of ideal points to T.I.F.s for searching for a non-local criterion. We find that though we are unable to formulate any criterion away from the singularity, we are able to examine the point where the apparent horizon meets the boundary which is always (marginally) trapped and can never be a source.

The general expression for tangent vector is not available in a closed analytic form. However, one can calculate it in a special case which we describe below and subsequently show how to generalize the results.

3 Self Similar Dust Model

The collapse of a spherical cloud of pressureless fluid is given by the following metric

$$ds^2 = dt^2 - \frac{R'^2}{1 + f(r)}dr^2 - R^2 d\Omega^2$$

where

$t$ and $r$ are the co-moving time and radial co-ordinates respectively. $R(t, r)$ is called the ‘area radius’ and a closed expression for this quantity which results in the dust case has enabled substantial progress in understanding the model.

Two free functions arise viz. $F(r)$, called the mass function since it is the total mass to the interior of a shell of radius $r$ and the total energy function $f(r)$ which is called so because of the constraint below
which resembles a relation between kinetic and gravitational potential energies of a shell.

\[ \dot{R}^2 = \frac{F}{R} + f \]  

(2)

The source energy momentum tensor is \( \text{diag}[\rho(t, r), 0, 0, 0] \).

The solution for \( R \) mentioned above is

\[ t - t_0(r) = -\frac{R^{3/2}}{\sqrt{F}} G \left( \frac{-Rf}{F} \right) \]  

(3)

where a singularity boundary is formed at \( t = t_0(r) \). The central shell focussing singularity, which is the limit as \( r \to 0 \) along this locus is of interest and turns out to be naked for some initial data.

The self-similar model is the one in which \( F(r) = \lambda r \) where \( \lambda \) is a constant (which decides if the central singularity will be naked or not) and \( f(r) = 0 \).

We choose the scaling \( t_0(r) = r \). A self similar co-ordinate \( z = t/r \) is introduced. We note the expressions for \( R \) and \( R' \) which will be useful in the subsequent analysis

\[ R = r \lambda^{-2/3} (3/2 (z - 1))^{2/3} \]  

(4)

\[ R' = \left( \frac{2\lambda/3}{z - 1} \right)^{1/3} \left( \frac{z - 3}{2} \right) \]  

(5)

We cast the metric into double null co-ordinates. It is not difficult to show that

\[ ds^2 = r^2 \left( z^2 - R'^2 \right) \ du \ dv \]  

(6)

where

\[ du = \frac{dr}{r} + \frac{dz}{z - R'(z)} \]  

(7)

\[ dv = \frac{dr}{r} + \frac{dz}{z + R'(z)} \]  

(8)

The double null form \( (ds^2 = C^2(u, v)du dv) \) turns out to be useful when affine parameters along null geodesics are to be calculated. For instance, along an outgoing radial null geodesic \( (du = 0) \), the affine parameter is \( \int_{u=\text{constant}} C^2 dv \) upto a multiplicative and an additive constant.

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\( ^3 \) The model is referred to as self similar since there exists a homothetic Killing field \( t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} \).
Now let us turn to calculating the tangent vector to the outgoing null radial geodesic congruence, which is our primary interest.

Assume the vector to be of the form

$$\xi = (Q(t, r), Q(t, r)\sqrt{1 + f/R'}, 0, 0)$$ \hspace{1cm} (9)

where $Q$ is obtained from the geodesic equation which $\xi$ has to satisfy. That constraint turns out to be

$$Q\dot{Q} + QQ'\sqrt{1 + f/R'} + Q^2\dot{R}'/2R' = 0$$ \hspace{1cm} (10)

We have provided the expressions for the most general dust case here. One may read off the expressions for the self similar case by setting $f$ to zero and using equation 5 for $R'$.

The equation above takes the form

$$1/Q = \int_{u=\text{constant}} \frac{\dot{R}'}{2R'} dk + A(u)$$ \hspace{1cm} (11)

where $k$ is an affine parameter along outgoing radial null geodesics and $u$ is the retarded null co-ordinate. $A$ is an arbitrary function of $u$ resulting because of the partial integration.

As indicated earlier,

$$dk = r^2 \left( z^2 - R'^2 \right) dv$$ \hspace{1cm} (12)

keeping $u$ fixed.

Using this in equation (1/Q) we obtain

$$1/Q = \int_{u=\text{constant}} \frac{r}{3} \left( z - 3 \right)^2 \left( \frac{2\lambda/3}{z - 1} \right)^{1/3} \frac{dz}{z^2 - 1}$$ \hspace{1cm} (13)

or using the fact that $du = 0$ from equation (du)

$$1/Q = \int_{u=\text{constant}} \frac{1}{3} \left( z - 3 \right)^2 \left( \frac{2\lambda/3}{z - 1} \right)^{1/3} e^{-\int \frac{db}{b-R'(b)}} dz$$ \hspace{1cm} (14)

The integral over $z$ is to be evaluated from $r = 0$ to the apparent horizon, where we shall be interested in evaluating the tangent vector. The latter can be shown to be the curve $R = F$ and turns out to be the locus $z = 1 - 2\lambda/3$. 

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The integral being over $z$, it is important to know which value of $z$ along the outgoing null curve yields $r = 0$, the lower limit of the integral. This issue as it is shown further leads to the difference in the behaviour of $Q$ in the naked and covered cases.

Consider then the equation

$$r = e^{\left[-\int_{b}^{\infty} \frac{db}{b-R'(b)}\right]z}$$  \hspace{1cm} (15)$$

which we examine for $r = 0$. That will happen when the integral in the square bracket diverges positively. Two cases can be immediately seen to arise.

Case(i) $b - R'(b) = 0$ has no real root.

The integrand therefore does not diverge anywhere and also remains positive (or entirely negative) all over the real line. It can be checked that $b - R'(b) > 0$ for any one real $b$ which would be sufficient to claim that the integrand is positive. Also, $b - R'(b)$ is bounded since $b$ is to be limited to the nonsingular region $z < 1$ ($z = 1$ is the singularity curve itself). So, in order that the integral diverge, the range of integration should be infinite. We have chosen to limit the final point to the apparent horizon $z = 1 - \frac{2\lambda}{3}$ and hence the initial point must be

$$z = -\infty.$$

In fact a shorter intuitive argument is possible. It is known from the Tolman Bondi dust model that central singularity forms at $(t_0(0), 0)$. If one assumes the Penrose diagram for the covered case (which is indeed what this case turns out to be), the null rays crossing the apparent horizon begin at the centre at $t < t_0$ which makes $z = -\infty$ there.

Case(ii) $b - R'(b) = 0$ has atleast one real root.

In this case, the equation implies that $r = 0$ at the value of $z$ for which the integral in the exponent diverges. The range of integration for equation (1/Q second one) would then be limited at the lower end by that value of $z$. This will be the root (in fact the one closest to $1 - 2\lambda/3$, the apparent horizon).

That this is so is seen as follows. Equation (r) can then be re-cast using an expansion for the integrand in the exponent as outlined below.

Expanding $R'(b)$ using the Taylor series about the root (called $z_-$) it can be shown that the leading order behaviour of $r$ is as follows.
\[ r = (z - z_-) \frac{1}{\alpha} + O(z - z_-) \quad (16) \]

where
\[ \alpha = \left( \frac{dR'(b)}{db} \right)_{b=z_-} \quad (17) \]

(It can be easily checked that \( \alpha < 0 \))

At \( z = z_- \), therefore, \( r \) vanishes.

Thus in conclusion of this analysis, we note that the lower limit of integral for \( 1/Q \) differs. It is \( -\infty \) when \( R'(b) - b = 0 \) can never have a real solution and is the root (closest to apparent horizon) when a solution exists.

This observation plays the key role in further analysis. Making note of this consider equation \( (1/Q \) second one). Analyzing the various factors in the integrand one finds that the integrand would diverge if \( z = -1 \) ( \( z \neq 1 \) since we are not on the singular boundary).

\( z \) will take the value \( -1 \) in case i. In case ii, the following takes place. Consider \( b - R' \) using equation \( (R') \). It is easy to see that \( b - R' < 0 \) for all \( b < 0 \). So, the root \( b_- \) cannot be negative. Hence it is certainly greater than \( -1 \). Thus \( z \) cannot take the value \( -1 \) in case ii in the integral for \( 1/Q \).

Thus the integrand diverges as \( 1/(z + 1) \) in case i and is finite in case ii.

Expanding the rest of the integrand factor in a Taylor series about \( z = -1 \), one can easily check that integral diverges logarithmically in case i while staying finite in case ii.

Thus, \( Q \) vanishes in case ii and stays non zero (and finite) in case ii.

Returning now to equation 9, we can now see that \( \xi \) behaves in different ways in case i and case ii on the apparent horizon, in particular as one approaches the point \( O \) on the Penrose diagrams shown (figures 3 and 4). It can be checked that the factor \( \sqrt{1 + f/R'} \) tends to is a non-zero finite quantity on the apparent horizon.

So, \( \xi \) vanishes in case i and tends to a non-zero quantity in case ii.

Using its interpretation of flux density of the congruence, we find that the congruence tends to cluster in case ii as against case i.

\[ \text{Figure 4 is a diagram of a locally naked singularity. The self similar cloud which we examine here turns out to be globally naked. However, the structure near O is the same as any locally naked case and figure 4 can be used.} \]
From previous analysis of naked singularities (self similar cases) using analysis for emergence of geodesics (roots analysis), it can be checked that case i corresponds to the covered case and case ii corresponds to the naked singular metric.

4 Extension to the general dust case

In the general dust case, the equation 10 yields no closed analytic solution which would have clearly been useful. However, we note that we are interested only in the behaviour of $Q$ in the limit of approach to point $O$ on the apparent horizon.

To this end the following observation plays an important role. It is shown that given a dust solution, one can construct a modified dust solution (modified distribution) which in a suitable limit approaches the given dust solution [7]. The key result that makes this construction useful is that it is proved that naked modified distributions reproduce naked dust solutions given and covered modified distributions reproduce covered ones. One can then work with the modified distribution for the given dust solution and take the limit which preserves naked or covered nature. We outline the construction in [7] below

a) Marginally bound case ($f = 0$)

Imagine a shell of radius $r_c$ in the given Tolman Bondi dust model. Replace the interior of the shell by a self similar dust metric, matching the first and second fundamental forms at the interface $r = r_c$. It can be shown that this restricts the self similarity parameter $\lambda$ which appears in the mass function. This specifies the self similar solution completely. Now taking the limit as $r_c$ tends to zero, one can show [4] that the matching constraint does imply that the interior self similar solution stays naked in the limit if the original dust solution was naked and wise in the covered case.

b) Non Marginally bound case ($f \neq 0$)

The construction is similar in this case except for an additional interface. Two shells, $r_c1$ and $r_c2$ (say $r_c1 < r_c2$) are now considered. To the interior of $r_c1$, we replace by a self similar metric. Between the two shells, we replace with a dust portion having $f$ so behaved that it increases smoothly from zero at $r_c1$ to $f(r_c2)$ of the original dust metric. The $F$ function for this extra portion of dust however is the same as that of the original dust metric. We match the first and
second fundamental forms at each of the interfaces. As before, this can be shown to constrain the interior self similar solution uniquely given \( r_c \) and the original dust solution. Again, the property of being naked or covered is preserved in the limit (\( r_c \to 0 \)) like the previous case.

We now consider \( Q \) in the modified distribution for any given dust solution. In the self similar part of the latter, results of the previous section apply. Since the congruence of outgoing geodesics is smooth, so is \( Q \). This makes \( Q \) continuous across the interface/s in the modified distribution. Now imagine the given dust solution as the limiting case of the modified distribution. In the limit of approach to point \( O \) on the apparent horizon, one has to evaluate \( Q \) in the self similar part. Because of continuity of \( Q \), the same behaviour will continue to hold in the limit of the interface/s tending to zero when the original dust solution is reproduced. Making use of the fact that the property of being naked or covered is preserved in this limit, one concludes that the behaviour of \( Q \) in the self similar naked and covered cases continues to hold in the general dust scenario as well.

5 Conformal transformation and Penrose diagram

The tendency of the null geodesics of the congruence to cluster in the approach to the singular boundary is basically due to the inappropriate nature of the co-ordinate system at the boundary. If one wishes to depict the boundary as a curve in a particular co-ordinate system, the null congruence has to be well defined (in the sense that the property that one and only one curve passes through every point should hold even when the congruence is extended to the boundary). For instance, in the naked dust case, when one uses spherical co-ordinates it can be seen that several radial null geodesics appear to emerge from the central singularity with the same tangent vector.

The issue about the co-ordinate system being appropriate for such an extension could thus related to the behaviour of \( \xi \).

From a technical point of view, the calculations using the radial co-ordinates could be performed in a conformally related metric which avoids the problem of clustering if it occurs. The conformal transformation would be the one leading to the structure of the singularity as
depicted in the Penrose diagram.

We now argue that the under a conformal transformation which diverges in the limit of $O$, $\xi$ which tend to a non-zero limit transform to vector fields which vanish in the limit.

Recall that we defined $\xi$ for any geodesic congruence using an affine parametrization. Under conformal transformations, affine parameters along null geodesics change (unlike timelike geodesics which do not remain geodesic curves, null geodesics do stay so provided the affine parameter changes appropriately). Infinitesimal parameter $ds$ transforms to $\Omega^2(x^\mu)ds$ \[3\], where $\Omega^2$ is a conformal transformation. Thus it is obvious that $\xi^\mu = dx^\mu/ds$ if finite and non-vanishing in the limit will vanish under $\Omega^2$ transformation provided the latter diverges there.

Thus we find that at least in the dust case, one requires a conformal transformation which diverges on the apparent horizon in the limit of approach to the singularity in the naked case as against the covered case where the radial co-ordinates are appropriate to describe the singularity structure\[5\]. This justifies the difference in the structure of the singular boundary near $O$ in figures 3 and 4.

## 6 A possible general scenario

Consider the cases of collapse in which the singularity formed meets the boundary of the trapped region (or even crossing it as in naked cases) Now it would be of interest to examine $\xi$ in general on the apparent horizon and check if it vanishes or not in the approach to the singular boundary. If it does not, then one invokes the diverging conformal transformation to obtain the correct causal depiction. The immediate question would be the naked or covered nature of such a singularity. We certainly know that it is naked in the dust case when the conformal transformation diverges. We present an argument suggesting its validity in a general scenario relaxing the assumption of dust and spherical symmetry.

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\[5\] It is evident from the definitions that the difference in the behaviour of flux is due to difference in the metrics, one being conformally related to the other. This raises the question of the flux being an appropriate characterization of the causal structure. It certainly is not appropriate at any event within the spacetime but in the limit of approach of the boundary its behaviour indicates if a metric with the correct limiting causal structure has been employed in its calculation.
Let point $O$ be the intersection of the singularity and the apparent horizon as before.

**Theorem:** No ingoing null geodesic can reach point $O$ after a conformal transformation if the conformal transformation diverges at $O$.

**Proof:**
Consider a space-like hypersurface from which an ingoing null geodesic reaches $O$ if possible. From the Raychaudhari equations, it can be shown that once a null geodesic has negative expansion, it will reach a conjugate point after a finite amount of affine parameter has elapsed. If the conformal transformation diverges in the limit, then a null geodesic reaching $O$ would imply an elapse of infinite amount of affine parameter. This is a contradiction. Hence the conjugate point must occur before $O$ on the null curve, beyond which the geodesic cannot be extended. So the geodesic cannot reach $O$. □

**Lemma:** There exists an ingoing null boundary (including $O$) to the past of $O$ if no ingoing geodesic reaches $O$.

**Proof:**
Consider a sequence of ingoing null geodesic segments $\{\Lambda_n\}$ with future endpoints on the apparent horizon, the endpoints approaching $O$ as $n \to \infty$. Let there be no ingoing null boundary to the past of $O$, if possible. Then there will be a limiting null geodesic of $\{\Lambda_n\}$ which reaches $O$. This contradicts the previous theorem. □

In the Penrose diagram (figure 4) one can imagine an ingoing null geodesic which reaches the null singular boundary at point $P$. This is the conjugate point for that geodesic. We have simply justified that there will be a boundary to the spacetime in place of the geodesic curve between $P$ and $O$.

If the apparent horizon is spacelike in the approach to $O$, the above portion of boundary is certainly exposed into the untrapped region and is therefore naked. One may ask if the apparent horizon is always

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6 See previous section
spacelike in the naked case. (It is certainly true for dust \[9\].) If a singular portion to the past of O exists, then it cannot be trapped. If such a portion existed then it would simply appear from a Penrose diagram with such a portion that causal curves would emanate from them. These arguments are made precise below (For definitions of IFs, Proper IFs, and TIFs, see \[2\],\[4\]).

**Theorem:** Existence of an ingoing null boundary to the past of O implies the existence of TIFs. (The portion of the boundary contains the ideal point of a TIF)

**Proof:**
Consider a sequence \(\{\Lambda_n\}\) as before. This choice is fixed throughout this proof. Also note that \(J^+(\Lambda_i) \subset J^+(\Lambda_j)\) for all \(j > i\) (The null curves are successively to the past). Since each \(\Lambda_i\) has a future endpoint on the apparent horizon \(E_i\), all points of \(\Lambda_i\) except \(E_i\) are untrapped. Choose one such point \(Q_i\). \(I^+(Q_i)\) is non empty. By choosing \(Q_{i+1}\) to the causal past of \(Q_i\) for every \(i\) (we can always do that since the null curves are successively to the past), one obtains a sequence of (proper) indecomposable future sets of \(Q_i\) which are nested. (One may begin the sequence at any \(i\)) The limiting IF as \(i \to \infty\) is therefore non empty. This IF will be proper iff there is a limit of \(\{Q_n\}\) which is a part of spacetime. There is a boundary to the past of O. Therefore at least one Q sequence (constructed as described above) exists which fails to have a limiting Q within spacetime. \(\Box\) Consider this Q sequence (there are actually an uncountably infinite of them). The corresponding limiting IF of the \(Q_i\)'s will be a TIF since there is no point in spacetime of which it is the future. \(\square\)

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7 Observe that if \(\{Q_n\}\) are constructed as defined above, then for every neighbourhood of each \(E_i\), one can find a sequence \(\{Q_n\}\) such that \(Q_i\) lies in that neighbourhood (to the past of \(E_i\)) simply by beginning the sequence at that \(i\). This is true for every \(i\). Next, the union of limit points of all possible Q sequences defines the ‘first’ ingoing null geodesic to reach the singularity (ingoing null curve at P in figure 6). If the limit for every Q sequence exists in spacetime, then one obtains limit points within spacetime as close to O as possible. However, there is a null boundary to the past of O and so this is not possible.
7 Summary and Conclusion

The tangent vector field to a null geodesic congruence being thought of as ‘flux density’ of a congruence of geodesics is examined for behaviour on the apparent horizon in the approach to the singularity (point O) in the dust collapse model. There is a correlation with the property of nakedness with this behaviour. Demanding that the vector vanishes at the covered point O forces the divergence of the conformal transformation at O which leads to the Penrose diagram for the naked scenario. Since the flux vanishes in the covered case, there is no such divergence and hence the Penrose diagrams in the two cases differ.

One demands that the flux vanishes at O in a general collapse scenario on the grounds that O is covered when calculated using a metric exhibiting the correct causal structure, and in case it does not, one uses a suitable conformal transformation (i.e one which diverges at O) in order to obtain the correct causal structure near O. We show that there will be no ingoing null geodesic reaching O if the latter is the case and argue that it indicates the existence of a portion of singularity which is untrapped.

In conclusion, we have shown that the information about whether the singularity formed in collapse is naked is contained at the intersection of the apparent horizon and singular boundary in the spherical dust case. We also suggest that it holds in the case of a general collapse. It should also be noted that the procedure of checking if an appropriate conformal transformation is necessary does not directly involve checking for emergence of causal curves from the singularity.

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Figure 1: Collapse of spherical dust leading to a covered singularity
Figure 2: Collapse of spherical dust leading to a naked singularity
Figure 3: Penrose Carter diagram for collapse of spherical dust leading to a covered singularity
Figure 4: Penrose-Carter diagram for collapse of spherical dust leading to a (locally) naked singularity
Figure 5: Construction of a sequence of ingoing null geodesics approaching O.