Scalar-tensor propagation of light in the inner solar system at the millimetric level

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In a recent paper [1], motivated by forthcoming space experiments involving propagation of light in the Solar System, we have proposed an extension of the IAU metric equations at the $c^{-4}$ level in General Relativity. However, scalar-tensor theories may induce corrections numerically comparable to the $c^{-4}$ general relativistic terms. Accordingly, one first proposes in this paper an extension of [1] to the scalar-tensor case. The case of a hierarchized system (such as the Solar system) is emphasized. In this case, the relevant metric solution is proposed. Then, the geodesic solution relevant for propagation of light in the inner solar system at the millimetric level is given in explicit form.

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I. INTRODUCTION

Forthcoming space missions and missions in project – such as LATOR [18], TIPO [17], ASTROD [18], PLR [19], ODYSSEY [20] or SAGAS [21] – will require distance measurements at millimetric level in the Solar System. This corresponds to time transfer at the precision $10^{-11}$ s. As argued in [1], this requires a complete Solar System metric at the $c^{-4}$ level, in order to describe the laser links involved in such experiments. This has been proposed in the framework of the General Relativity (GR) theory in [1], leading to an appropriate extension of the metric equations recommended by the IAU2000 resolution [2].

The relative amplitude of the relativistic effects is of the order of $\epsilon = GM/rc^2$, where $M$ and $r$ are some characteristic mass and distance. The so-called first order terms are of order $\epsilon$, the second order terms of order $\epsilon^2$. In the inner Solar System, $\epsilon$ is typically of the order $10^{-8}$, and can be sensitively greater ($10^{-7}$) for photons entering well inside Mercury’s orbit, and can even be as large as $10^{-6}$ for photons grazing the Sun.

On the other hand, there is a surge of interest in scalar-tensor (ST) theories since about two decades. Indeed, the gravitational sector of a lot of tentative fundamental theories, like string or (modern) Kaluza-Klein, turns to be described by a metric tensor plus a scalar field (Brans-Dicke [3, 4] or not [5]). Besides, alternative theories to GR (including ST) are also sometimes required by some authors to deal with the so-called dark energy (cosmological level), dark matter (galactic level), Pioneer or fly-by anomalies (solar system level) problems. Phenomenologically, the divergence between ST theories and GR is quantified by the Post-Newtonian (PN) factor ($= 1$ in GR) entering the $c^{-2}$ term in the space-space components of the metric tensor. From the present observations, $|\gamma - 1|$ can at best reach values of the order of $10^{-5}$ [8]. But, some theoretical considerations strongly suggest $\gamma$ could have been driven from any “initial” value to a value close to unity by the cosmological expansion (more precisely, ST theories are driven to GR, as soon as these theories fulfill some (not very constraining) conditions), and even claim the $1 - \gamma$ current value should be of order $10^{-7}$ or $10^{-8}$ [8, 9]. On the other hand, $c^{-4}$ terms are typically $10^6$ to $10^8$ smaller than $c^{-2}$ terms. All these reasons make relevant the extension of the gravitational framework proposed in [1] to encompass the ST case, as much as $c^{-4}$ space-space metric terms have to be taken into account in light propagation problems.

In principle, this would require new definitions of multipolar moments at the $c^{-2}$ level. But other publications proposed $c^{-2}$ multipolar moments in ST theories [10] or in a parametrized post-newtonian framework [11]. Thus we do not discuss this point in this paper and focus only on the $c^{-4}$ metric side of the problem. Another reason to discard this point is that non-multipolar terms turn out to have numerically negligible contributions for links we are interested in (inner Solar System case).

In section II one defines a terminology relevant to the considered problem. Since the Einstein conformal representation plays a central role in the approach followed in this paper, the section III is devoted to the conformal link between the representations of ST theories, and to the related notations we will use. Section IV is dedicated to the derivation of the ST field equations up to $O(c^{-5})$ terms. In section V these field equations are rewritten up to $O(c^{-5},\omega_0^{-1}c^{-4})$ terms, for applications taking explicitly our present experimental knowledge on gravity into account. Considering applications to Solar System-like systems, one defines hierarchized systems in section VI. This is made by defining a quantity $\mu$ that quantifies how much the system is gravitationally dominated by its most massive body. Accordingly, the field equations are rewritten up to $O(c^{-5},\omega_0^{-1}c^{-4},\mu c^{-4})$ terms. In this case, the explicit metric solution that can be used in relevant applications is written in well-suited coordinates. Finally, in section VII, we give the explicit form

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of isotropic geodesics relevant for time transfer and ranging problems in the inner solar system at the millimetric level of accuracy (required by forthcoming space missions or missions in project).

II. TERMINOLOGY : DEFINITION OF THE PN/BM AND PN/RM METRICS

The PN approximation is based on the assumption of a weak gravitational field and low velocities for both the sources and the (test) body (i.e. velocities of the order $\sqrt{GM/r}$ or less, $M$ being some characteristic mass of the system). It formally consists in looking for solutions under the form of an expansion in powers of $1/c$. The usually so-called $n$PN order terms in the metric, leading to $c^{-2n}$ terms in the equation of motion of a body describing a bounded orbit, are terms of orders $c^{-2n-2}$ in $g_{00}$, $c^{-2n-1}$ in $g_{0i}$ and $c^{-2n}$ in $g_{ij}$. In this case, the Ricci tensor components have to be developed the same way as (18) in [1]. In the present paper, a metric developed this way will be referred as the $n$PN/BM metric (BM meaning “Bounded Motion” for test particles). It is particularly well-adapted for studying bounded motions in systems made by non-relativistic massive bodies, as the Solar System is.

However, since we are interested in the propagation of light, we are led to relax the hypothesis on the velocity of the test particle whose motion is considered. Of course, this doesn’t change the metric, but the terms to be considered in the metric components are not the same as in the PN/BM problem. Indeed, the terms leading to $c^{-2n}$ terms in the equation of motion of a test particle moving at relativistic velocity are terms of order $c^{-2n}$ in both $g_{00}$, $g_{0i}$ and $g_{ij}$. In this case, the Ricci tensor components have to be developed the same way as (19) in [1]. In the present paper, a metric developed this way will be referred as the $n$PN/RM metric (RM meaning ”Relativistic Motion” for test particles). A PN/RM metric is particularly well-suited for studying relativistic motions of test bodies (for instance, the propagation of light) in systems made by non-relativistic bodies, as the Solar System is.

The present paper deals with the PN/RM problem since we are concerned in propagation of light.

III. THE ST THEORIES IN EINSTEIN VS JORDAN REPRESENTATIONS

The Jordan representation of the ST theories is described by the action

$$S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} \left[ \Phi R - \frac{\omega (\Phi)}{\Phi} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi \right] + \int d^4x \sqrt{-g} L_{NG}(\Psi, g_{\mu\nu}).$$

(1)

In this representation, the gravitational sector of the theory is described by the Jordan metric $g_{\alpha\beta}$ and the scalar field $\Phi$, while the non-gravitational fields are symbolically represented by $\Psi$. The scalar field couples directly with the metric, leading to rather complicated field equations. Besides, the kinetic term associated to the scalar field doesn’t have the standard form, involving a scalar field function $\omega$, the function characterizing the ST theory we are dealing with. On the other hand, the non-gravitational lagrangian $L_{NG}$ doesn’t depend on the scalar field, leading to simple equations of motion ($\nabla_\alpha T^{\alpha\beta} = 0$), with the nice consequence that the weak equivalence principle applies in this representation of the theory.

To overcome the just mentioned drawbacks, one could be tempted to resort to a dependent variables change

$$(g_{\alpha\beta}, \Phi) \rightarrow (\tilde{g}_{\alpha\beta}, \varphi)$$

chosen in such a way that the action (1) transforms into

$$S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} - 2\tilde{g}^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \right] + \int d^4x \sqrt{-\tilde{g}} \tilde{L}_{NG}$$

(2)

$\tilde{L}_{NG}$ depending on $\Psi$, $\tilde{g}_{\mu\nu}$ and $\varphi$ in a way to be precised later ($\tilde{R}$ and $\tilde{g}$ correspond to $R$ and $g$, but with $g_{\alpha\beta}$ replaced by $\tilde{g}_{\alpha\beta}$). Since the scalar field doesn’t couple with the metric, (2) is referred to as the Einstein representation of the theory.

The form (2) is achieved by considering a conformal transformation of the metric

$$g_{\alpha\beta} = A(\varphi)^2 \tilde{g}_{\alpha\beta}.$$ 

(3)

From the induced transformation of the Ricci scalar [12], and up to a divergence term, (2) turns into

$$S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} \left[ \frac{1}{A^2} R + \left\{ \frac{6}{A^2} - 2 \frac{d\varphi}{dA} \right\}^2 g^{\alpha\beta} \partial_\alpha A \partial_\beta A \right] + \int d^4x \sqrt{-g} A^{-2} \tilde{L}_{NG}.$$ 

(4)
Comparing (4) with (1) suggests:

- the link between the Jordan and Einstein representations of the scalar field

\[ \Phi = \frac{1}{A(\varphi)^2}; \]  

(5)

- the identification

\[ \frac{\omega(\Phi) g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi}{\Phi} = \left\{ \frac{2}{A^2} \left( \frac{d\varphi}{dA} \right)^2 - \frac{6}{A^4} \right\} g^{\alpha\beta} \partial_\alpha A \partial_\beta A; \]  

(6)

- the Einstein representation of the non gravitational lagrangian

\[ \bar{L}_{NG} = A(\varphi)^4 L_{NG}(\Psi, g_{\mu\nu}) \]

\[ = A(\varphi)^4 L_{NG}(\Psi, A(\varphi)^2 g_{\mu\nu}). \]  

(5) and (6) lead to the link between the functions \( \omega(\Phi) \) and \( A(\varphi) \) (equivalently characterizing the considered ST theory)

\[ (2\omega + 3) \left( \frac{d\ln |A|}{d\varphi} \right)^2 = 1. \]  

(7)

(7) requires \( \omega > -3/2 \). This results from the fact one has imposed the sign of the scalar kinetic energy term in Einstein representation in order to ensure the dynamical stability of the theory [13].

The link between the stress tensor components in the two representations follows from the general stress tensor definition

\[ \delta g^{\alpha\beta} \rightarrow \delta \int d^4x \sqrt{-\bar{g}} L_{NG} \equiv -\frac{1}{2} \int d^4x \sqrt{-\bar{g}} T^{\alpha\beta} \delta g^{\alpha\beta} \]

\[ \delta \bar{g}^{\alpha\beta} \rightarrow \delta \int d^4x \sqrt{-\bar{g}} \bar{L}_{NG} \equiv -\frac{1}{2} \int d^4x \sqrt{-\bar{g}} \bar{T}^{\alpha\beta} \delta \bar{g}^{\alpha\beta}. \]

Since \( \sqrt{-\bar{g}} L_{NG} = \sqrt{-\bar{g}} \bar{L}_{NG} \), and since the scalar field is not varied in this metric variation process (no ambiguity since the two versions \( \varphi \) and \( \Phi \) of the scalar field are related in the non metric dependent way (5)), it directly turns that

\[ \bar{T}_{\alpha\beta} = A^2 T_{\alpha\beta}. \]

For the mixed and contravariant components, it follows

\[ \bar{T}_\alpha^\beta = A^4 T_\alpha^\beta \]  

\[ \bar{T}^\alpha_\beta = A^6 T^\alpha_\beta \]

the indexes being raised/ lowered by the metric involved in the corresponding representation.

**Eliminating** \( A \)

It is clear \( A \) can be eliminated between the two representations of the scalar field using (5). This way, the considered ST theory is represented by the function \( \Phi(\varphi) \) in Einstein representation. From (7), this function is linked to \( \omega(\Phi) \) by

\[ (2\omega + 3) \left( \frac{d\ln |A|}{d\varphi} \right)^2 = 4. \]  

(8)

The conformal transformation (3), the link between the non gravitational lagrangians and stress tensors now write

\[ \bar{g}_{\alpha\beta} = \Phi g_{\alpha\beta} \]

\[ \bar{L}_{NG} = \Phi^{-2} L_{NG}(\Psi, \Phi^{-1} \bar{g}_{\mu\nu}) \]  

(9)
\[ T^\alpha_{\beta} = \Phi^{-1} T^\alpha_{\beta}, \]
\[ \tilde{T}^\alpha_{\beta} = \Phi^{-2} T^\alpha_{\beta} \quad \Rightarrow \quad \tilde{T} = \Phi^{-2} T \]
\[ \tilde{T}^\alpha_{\beta} = \Phi^{-3} T^\alpha_{\beta}. \quad (10) \]

As it turns from equation (8), \( \varphi \) is defined up to a sign and an additive constant. When needed in the following, the sign will be fixed by the choice
\[ \sqrt{2\omega + 3} \frac{d\ln \Phi}{d\varphi} = 2. \quad (11) \]

**IV. THE FIELD EQUATIONS**

**A. Using the Einstein representation gravitational field variables**

From (2), the field equations can be written
\[ R^\alpha_{\beta} = \frac{8\pi G}{c^4} \left( T^\alpha_{\beta} - \frac{1}{2} T g^\alpha_{\beta} \right) + 2 \gamma^\alpha_{\beta} \phi \delta^\alpha_{\beta} \]
\[ \partial_\alpha \left( \sqrt{-g} \gamma^\alpha_{\beta} \phi \right) = \frac{2\pi G}{c^4} \tilde{T} \sqrt{-g} \frac{d\ln \Phi}{d\varphi} \quad (12) \]

the function \( \Phi(\varphi) \) characterizing the ST theory explicitly entering the scalar field equation. As usual in PN approximation, let us write the scalar field as
\[ \varphi = \varphi_0 + \frac{\varphi^2}{c^2} + \frac{\varphi^4}{c^4} + O \left( c^{-5} \right) \quad (13) \]

where \( \varphi_0 \) is constant and \( \frac{\varphi}{c^2} \) and \( \frac{\varphi}{c^4} \) are zeroth order terms. (Remark that, since \( \varphi \) is defined up to an additive constant, it is not restrictive to set \( \varphi_0 = 0 \).) As a consequence, it turns out that, under the standard PN assumptions, \( \tilde{R}^{ij} = O \left( c^{-4} \right) \), so that the Strong Spatial Isotropy Condition (SSIC) applies in this representation. It is then possible to choose a coordinate system in which the Einstein metric takes the following form at the 2PN/RM level
\[ g^0_0 = -1 + \frac{2w}{c^2} - \frac{2w^2}{c^4} + O \left( c^{-5} \right) \quad (14) \]
\[ g^0_i = -\frac{4w_i}{c^3} + O \left( c^{-5} \right) \]
\[ g^{ij} = \delta^{ij} \left( 1 + \frac{2w}{c^2} + \frac{2w^2}{c^4} \right) + \frac{4\tau}{c^4} + O \left( c^{-5} \right). \]

Putting \( \Phi(\varphi_0) = 1 \) is not restrictive since \( \Phi \) enters through its logarithm derivative. Hence, the scalar field function develops as
\[ \Phi(\varphi) = 1 + \frac{\varphi}{c^2} \Phi_0' \varphi + \frac{\varphi}{c^4} \left( \Phi_0''(\varphi) + \frac{1}{2} \Phi_0''(\varphi)^2 \right) + O \left( c^{-5} \right) \quad (15) \]

where \( \Phi_0' \) and \( \Phi_0'' \) stand for the values of the derivatives of \( \Phi \) at \( \varphi_0 \). Setting
\[ \sigma = \frac{1}{c^2} (T^{00} + T^{kk}) \quad (16) \]
\[ \sigma^i = \frac{1}{c} T^{0i} \]
\[ \sigma^{ij} = T^{ij} - T^{kk} \delta^{ij} \quad \Rightarrow \quad \sigma^{kk} = -2T^{kk} \quad (10) \]

(which, from standard PN assumptions, are \( c^0 \) order quantities) and using (10), the (00), (0i) and (ij) field equations lead respectively to
\[ \triangle w + \frac{1}{c^2} (3\partial_\mu w + 4\partial_{ik} w_k) + \frac{3}{c^2} \Phi_0' \varphi \triangle w = -4\pi G \sigma + O \left( c^{-3} \right) \quad (17) \]
\[ \Delta w_i - \partial_{ik} w_k - \partial_{ti} w = -4\pi G \sigma^i + O (c^{-2}) \]  \tag{18}

\[ \Theta_{ij} (\tau_{kl}) = \partial_i w \partial_j w - \partial_t (\partial_i w_j + \partial_j w_i) - 2\delta_{ij} \partial_t (\partial_k w + \partial_k w_k) + 4\pi G \sigma^{ij} + \partial_i \varphi \partial_j \varphi + O (c^{-1}) \]  \tag{19}

where \( \Theta_{ij} \) is defined, as in \( \text{[1]} \), by

\[ \Theta_{ij} (\tau_{kl}) \equiv \partial_{ik} \tau_{jk} + \partial_{jk} \tau_{ik} - \Delta \tau_{ij} - \partial_{ij} \tau_{kk}. \]

The scalar field equation gives

\[ \Delta (\varphi^2) + \frac{1}{c^2} (-\partial_t \varphi + \Delta \varphi) + \frac{1}{c^2} \left( 4\Phi' - \frac{\Phi''}{\Phi} \right) (\varphi^2) \Delta \varphi = -2\pi G \Phi'(\sigma + \frac{\sigma^{kk}}{c^2}) + O (c^{-3}). \]  \tag{20}

Remark that, in contrast to the GR case, the (00) equation is not linear, because of the \( c^{-2} \varphi \Delta w \) term. The scalar field equation also contains a non-linear \( c^{-2} \) term.

Now, combining \( \text{[17]} \) and \( \text{[20]} \) leads to

\[ \Delta \left( \frac{(2) \varphi}{2} - \frac{1}{2} \Phi'_0 w \right) = O (c^{-2}). \]

Accordingly, let us choose

\[ (2) \varphi = \frac{1}{2} \Phi'_0 w. \]  \tag{21}

Hence, defining \( \chi \equiv \frac{(4) \varphi}{\Phi'_0} \)

\[ \varphi = \varphi_0 + \frac{\Phi'_0}{2c^2} w + \frac{\Phi''_0}{c^2} \chi + O (c^{-5}). \]  \tag{22}

The metric field variables \( w, w_i \) and \( \tau_{ij} \) are now decoupled from the scalar field \( \chi \). The system constraining \( w, w_i \) and \( \tau_{ij} \) now writes

\[ \Delta w + \frac{1}{c^2} (3\partial_t w + 4\partial_{ik} w_k) + \frac{3}{2c^2} \Phi'^2 \omega_0 \Delta w = -4\pi G \sigma + O (c^{-3}) \]  \tag{23}

\[ \Delta w_i - \partial_{ik} w_k - \partial_{ti} w = -4\pi G \sigma^i + O (c^{-2}) \]  \tag{24}

\[ \Theta_{ij} (\tau_{kl}) = -\partial_{iw} \partial_j w_i + \left( 1 + \frac{1}{4} \Phi'^2 \right) \partial_i w \partial_j w - 2\delta_{ij} \partial_t (\partial_k w + \partial_k w_k) + 4\pi G \sigma^{ij} + O (c^{-1}) \]  \tag{25}

\( \chi \) being obtained in a second step, by solving

\[ \Delta \chi - 2 (\partial_t w + \partial_{ik} w_k) + \frac{1}{4} (\Phi'^2 - \Phi''_0) \omega_0 \Delta w = -2\pi G \sigma^{kk} + O (c^{-1}). \]  \tag{26}

Remark that \( \Delta w \) may be replaced by \(-4\pi G \sigma \) in the non linear terms of equations \( \text{[23]} \) and \( \text{[26]} \).

**B. Back to Jordan representation**

Let us use the function \( \omega (\Phi) \) and its derivative \( \omega' (\Phi) \) instead of \( \Phi' (\varphi) \) and \( \Phi'' (\varphi) \). One finds, using \( \text{[11]} \)

\[ \Phi'_0 = \frac{2}{\sqrt{2\omega_0 + 3}} \]

\[ \Phi''_0 = \frac{4}{2\omega_0 + 3} \left( 1 - \frac{\omega'_0}{2\omega_0 + 3} \right) = \Phi'^2 - \frac{4\omega'_0}{(2\omega_0 + 3)^2}. \]
One now goes back to Jordan representation using (9), with, from (15) and (22),
\[ \Phi^{-1} = 1 - \frac{2w}{c^2(2\omega_0 + 3)} + \frac{1}{c^4(2\omega_0 + 3)} \left[ \frac{2}{2\omega_0 + 3} \left( 1 + \frac{\omega'_0}{2\omega_0 + 3} \right) w^2 - 4\chi \right] + O(c^{-5}) . \] (27)

Now let us put
\[ \gamma = \frac{\omega_0 + 1}{\omega_0 + 2} \]
\[ \beta = 1 + \frac{\omega'_0}{(2\omega_0 + 3)(2\omega_0 + 4)^2} \]
\[ G_{eff} = \frac{2\omega_0 + 4}{2\omega_0 + 3} \]
and let us define
\[ (U, U_i, U_{ij}, P) = \frac{2\omega_0 + 4}{2\omega_0 + 3} (w, w_i, \tau_{ij}, \chi) \]
and the related quantities \((W, W_i, W_{ij})\) by
\[ W = U + (1 - \gamma) \frac{P}{c^2} \]
\[ W_i = U_i \]
\[ W_{ij} = U_{ij} - (1 - \gamma) P \delta_{ij} . \]

Using (14) and (27), one gets the Jordan metric
\[ g_{00} = -1 + \frac{2W}{c^2} - \frac{\beta W^2}{c^4} + O(c^{-5}) \]
\[ g_{0i} = - (\gamma + 1) \frac{2W_i}{c^3} + O(c^{-5}) \]
\[ g_{ij} = \delta_{ij} \left\{ 1 + \gamma \frac{2W}{c^2} + (\gamma^2 + \beta - 1) \frac{2W^2}{c^4} \right\} + (\gamma + 1) \frac{2W_{ij}}{c^4} + O(c^{-5}) . \] (28)

where the functions \((W, W_i, W_{ij}, P)\) satisfy the following field equations – after some algebra from (23-26) and (16)
\[ \Box W + \frac{1 + 2\beta - 3\gamma}{c^2} W \Box W + \frac{2}{c^4} (1 + \gamma) \partial_t J = -4\pi G_{eff} \Sigma + O(c^{-3}) \]
\[ \Delta W_i - \partial_i J = -4\pi G_{eff} \Sigma^i + O(c^{-2}) \]
\[ \Delta W_{ij} + \partial_i W \partial_j W + 2(1 - \beta) \delta_{ij} W \Box W - \partial_i J - \partial_j J - 2\gamma \delta_{ij} \partial_t J = -4\pi G_{eff} \Sigma^{ij} + O(c^{-1}) \]
\[ \Delta P + \frac{2\beta - 1}{1 - \gamma} W \Box W - 2\partial_t J = -4\pi G_{eff} \Sigma^{kk} \frac{\Sigma^{kk}}{3\gamma - 1} + O(c^{-1}) . \] (29)

In (29), one has set
\[ J = \partial_t U + \partial_k U_k \]
\[ J_i = \partial_i W + \partial_k W_{ik} + O(c^{-2}) \]
\[ J_{ij} = \partial_{ik} U_k - \frac{1}{2} \partial_i U_{kk} + \partial_t U_i \]
\[ = \partial_i W_{kk} - \frac{1}{2} \partial_i W_k + \partial_t W_i - \frac{1 - \gamma}{2} \partial_t P \] (30)

and, for the matter part of the equations
\[ \Sigma = \frac{1}{c^2} (T^{00} + \gamma T^{kk}) \]
\[ \Sigma^i = \frac{1}{c} T^{0i} \]
\[ \Sigma^{ij} = T^{ij} - \gamma T^{kk} \delta_{ij} \quad (\implies \Sigma^{kk} = - (3\gamma - 1) T^{kk}) \]
Let us remark that the quantity $2(\beta - 1)/(1 - \gamma)$ ($= \omega_0^2(2\omega_0 + 3)^{-1}(2\omega_0 + 4)^{-1}$) is not diverging, even if $\gamma$ is (arbitrarily) close to unity. Besides, no new PN parameter appears neither in the $c^{-4}$ space-space part of the metric nor in the corresponding field equations, as stressed in [3].

This form is relevant in all sufficiently weak gravitational field, even in systems where the ST theory is not very close to GR, i.e. where the PN parameters $\gamma$ and $\beta$ are not close to unity. A priori, this may occur even if $\gamma$ and $\beta$ are close to unity in some (other) regions of the universe, as in the Solar System, as soon as the ST theory is not (in some sense) close to the Brans-Dicke one (in Brans-Dicke gravity, $\omega$ doesn’t depend on the scalar field, so that it has the same value in all the space-time regions of the universe).

Let us point out that the numerical values of the coefficients $\gamma^2 + \beta - 1$ and $\gamma + 1$ entering $g_{ij}$ in (29) can be chosen independently one to another, since both $\beta$ and $\gamma$ enter these coefficients. Hence, there is no a priori relation between the coefficients of the $c^{-4}$-terms $W^2$ and $W_{ij}$, contrary to what may be suggested by the form of the metric chosen in [34] (in the one-mass case). More precisely, this doesn’t mean the form chosen by [34] is uncorrect, but rather that this form doesn’t encompass the (general) ST case (but it encompasses the GR case, as it must be).

C. Harmonic gauges

Since the use of the harmonic gauge (HG) is recommended by the IAU, let us consider the field equations in this gauge. Of course one has to specify the representation in which the HG is prescribed. The Jordan HG condition reads

$$g^{\alpha\beta} \Gamma^\sigma_{\alpha\beta} = 0$$

and it leads to, for the space ($\sigma = k$) component

$$(\gamma - 1) \partial_k U = O\left(c^{-2}\right). \tag{32}$$

As expected from known results in GR [14], this condition reduces to a triviality in the case $\gamma = 1$. On the other hand, if $\gamma \neq 1$, [32] shows that the coordinate system in which the metric takes the (Jordan) form [28], corresponding to SSIC in Einstein representation, doesn’t encompass (Jordan) harmonic coordinates in the ST case. In other terms, (Jordan) harmonic coordinates are incompatible with the SSIC in Einstein representation.

One could rather choose to impose the HG condition on the metric in Einstein representation

$$\tilde{g}^{\alpha\beta} \tilde{\Gamma}^\sigma_{\alpha\beta} = 0$$

since the Einstein metric [14] satisfies the SSIC. From [14], this means one imposes $w$, $w_i$ and $\tau_{ij}$ to satisfy

$$\partial_i w + \partial_k w_k = O\left(c^{-2}\right)$$

$$\partial_k \tau_{ik} - \frac{1}{2} \partial_i \tau_{kk} + \partial_i w_i = O\left(c^{-1}\right).$$

Translated in terms of $(U, U_i, U_{ij})$, this takes exactly the same form, i.e., using [30,31]

$$J = O\left(c^{-2}\right), \quad J_i = O\left(c^{-1}\right). \tag{33}$$

It turns out this corresponds to the Nutku gauge constraints [10,15], meaning that imposing the HG in the Einstein representation is equivalent to impose the Nutku gauge in the Jordan representation. Using [33], the three first equations of (29) take the reduced form

$$\Box W + \frac{1 + 2\beta - 3\gamma}{c^2} W \Delta W = -4\pi G_{eff} \Sigma + O\left(c^{-3}\right) \tag{34}$$

$$\Delta W_i = -4\pi G_{eff} \Sigma^i + O\left(c^{-2}\right) \tag{35}$$

$$\Delta W_{ij} + \partial_i W \partial_j W + 2(1 - \beta) \delta_{ij} W \Delta W = -4\pi G_{eff} \Sigma^{ij} + O\left(c^{-1}\right) \tag{36}$$

while the fourth equation of (29) and the harmonic constraints read, using [30] and [31]

$$\Delta P + \frac{2\beta - 1}{1 - \gamma} W \Delta W = -4\pi G_{eff} \frac{\Sigma^{kk}}{3\gamma - 1} + O\left(c^{-1}\right) \tag{37}$$

$$\partial_i W + \partial_k W_k = O\left(c^{-2}\right) \tag{38}$$

$$\partial_k W_{ik} - \frac{1}{2} \partial_i W_{kk} + \partial_i W_i - \frac{1 - \gamma}{2} \partial_i P = O\left(c^{-1}\right) \tag{39}$$
or equivalently (after elimination of the scalar field $P$)

$$\partial_t W + \partial_k W_k = O \left( c^{-2} \right)$$  \hspace{1cm} (40)

$$\partial_t W_{ik} - \frac{1}{2} \triangle W_{kk} + \partial_t W_i + (\beta - 1) W \triangle W = -2 \pi G_{eff} \frac{1 - \gamma}{3 \gamma - 1} \Sigma^{kk} + O \left( c^{-1} \right)$$  \hspace{1cm} (41)

$$\partial_t W_{jk} + \partial_t W_j = \partial_{jk} W_{ik} + \partial_{ij} W_i + O \left( c^{-1} \right).$$  \hspace{1cm} (42)

The last equation refers to the fact that $\partial_t P$ (given by [39]) is a gradient.

Note these equations are coherent with 1.5PN/BM equations assumed in [11].

V. RELEVANT FIELD EQUATIONS CONSIDERING PRESENT CONSTRAINTS ON GRAVITY

A. Without making the HG choice

In the inner solar system, the gravitational field is such that

$$\frac{2U}{c^2} \sim 10^{-6} \text{ to } 10^{-8}.$$  

On the other hand, from experimental/observational constraints [7]

$$|\gamma - 1| \lesssim 10^{-5} \text{ i.e. } \omega_0 \geq 10^5.$$  

This means $\gamma - 1$ (or $\omega_0^{-1}$) could be considered numerically as a $c^{-1}$ (at best) order quantity. Hence, it is convenient to present the metric under the form of a generalized development in both powers of $c^{-1}$ and $\omega_0^{-1}$. The useful metric resulting from [28] reads (if $\omega_0^{-1}$ is not "unreasonably large")

$$g_{00} = -1 + \frac{2W}{c^2} - \frac{2W^2}{c^4} + O \left( c^{-5}, \omega_0^{-1} c^{-4} \right)$$

$$g_{0i} = - (\gamma + 1) \frac{2W_i}{c^2} + O \left( c^{-5} \right)$$

$$g_{ij} = \delta_{ij} \left[ 1 + \frac{2W}{c^2} + \frac{2W^2}{c^4} \right] + \frac{4W_{ij}}{c^4} + O \left( c^{-5}, \omega_0^{-1} c^{-4} \right)$$

where $W$, $W_i$ and $W_{ij}$ satisfy, from [29]

$$\square W + \frac{4}{c^2} \partial_t J = -4 \pi G_{eff} \sigma + O \left( c^{-3}, \omega_0^{-1} c^{-2} \right)$$

$$\triangle W_i - \partial_i J = -4 \pi G_{eff} \sigma_i + O \left( c^{-2} \right)$$

$$\triangle W_{ij} + \partial_i W \partial_j W - \partial_i J_j - \partial_j J_i - 2 \delta_{ij} \partial_t J = -4 \pi G_{eff} \sigma^{ij} + O \left( c^{-1}, \omega_0^{-1} \right)$$

and where $J_i$ reduces to

$$J_i = \partial_t W_{ik} - \frac{1}{2} \partial_k W_{kk} + \partial_t W_i + O \left( \omega_0^{-1} \right).$$  \hspace{1cm} (43)

One remarks the field equations take exactly the same form as the GR case [1] (with $G$ replaced by $G_{eff}$). The only remaining reference to the scalar field is reduced to the presence of the $\gamma$ PN coefficient in the metric tensor. Related to this, the field equation on $P$ is dropped out.

B. Making the HG choice

The corresponding harmonic equations to be used when considering known constraints on $\omega_0$ reads

$$\square W = -4 \pi G_{eff} \sigma + O \left( c^{-3}, \omega_0^{-1} c^{-2} \right)$$

$$\triangle W_i = -4 \pi G_{eff} \sigma_i + O \left( c^{-2} \right)$$

$$\triangle W_{ij} + \partial_i W \partial_j W = -4 \pi G_{eff} \sigma^{ij} + O \left( c^{-1}, \omega_0^{-1} \right)$$  \hspace{1cm} (44)
with gauge conditions

$$\partial_t W + \partial_k W_k = O\left(c^{-2}\right)$$

$$\partial_k W_{ik} - \frac{1}{2} \partial_i W_{kk} + \partial_i W_i = O\left(c^{-1}, \omega_0^{-1}\right).$$

### VI. EXPLICIT HARMONIC METRIC RELEVANT FOR HIERARCHIZED SYSTEMS

#### A. Hierarchized systems

Let us consider the case where the system is composed by bodies of masses $M_A$. Let us consider one of these bodies, named $S$, of mass $M_S$. Let us define the parameter

$$\mu = \frac{1}{M_S} \sum_{A \neq S} M_A.$$ 

One defines a hierarchized system as a system in which the body $S$ can be chosen in such a way that $\mu \ll 1$.

In such a system, the body $S$ will be hereafter referred as the "star", while the other bodies will be referred as the "planets".

In the general relativistic $N$-body problem, multipolar moments of a body $A$ are defined in the coordinate system in which this body is, in some sense, at rest. These moments are affected by coordinate transforms through a "Lorentz-like length contraction effect". These effects being of order $(u/c)^2$, where $u$ is the relative velocity between the two frames, the induced effects in the metric components are of order $c^{-4}$, since potentials are at least $c^{-2}$ terms.

In hierarchized systems, the velocity of the body $S$ is of the order of

$$v_S \sim \mu v_B \sim \mu \sqrt{\frac{GM_S}{r_{S-B}}}$$

where $B$ is the most massive planet (and $r_{S-B}$ the distance between $B$ and the star). All the Lorentz-like contraction terms have a form like

$$\frac{GM_A v_A^2}{r c^2} \mu c^{-2}.$$ 

If $A$ is a planet ($A \neq S$), this term is at best of order $O\left(\mu c^{-4}\right)$, since $M_A \lesssim \mu M_S$. If $A$ is the star ($A = S$), this term is of order $O\left(v_S^2 c^{-4}\right)$, i.e. $O\left(\mu^2 c^{-4}\right)$. Hence all these terms are, at least, of order $O\left(\mu c^{-4}\right)$.

Let us also point out that, since at this level the metric depends on time through the positions of the star and the planets only, all the terms containing the operator $\partial_t$ are at least of order $O(\mu)$. Hence, equations (44)-(46) lead to

$$\Delta W = -4\pi G_{eff} \sigma + O\left(c^{-3}, \omega_0^{-1} c^{-2}, \mu c^{-2}\right)$$

$$\Delta W_i = -4\pi G_{eff} \sigma^i + O\left(c^{-2}, \mu c^{-1}\right)$$

$$\Delta W_{ij} + \partial_i W \partial_j W = -4\pi G_{eff} \sigma^{ij} + O\left(c^{-1}, \omega_0^{-1}, \mu\right)$$

with gauge conditions

$$\partial_t W + \partial_k W_k = O\left(c^{-2}, \mu c^{-1}\right)$$

$$\partial_k W_{ik} - \frac{1}{2} \partial_i W_{kk} = O\left(c^{-1}, \omega_0^{-1}, \mu\right).$$

Related to this, $\Delta_A$ defined in [2] leads to numerically negligible terms (see (11.4.8) in [10] for the ST version).
B. Application to the solar system

In the Solar system, the most massive body $S$ is the Sun and one has

$$\mu \sim 10^{-3}.$$ 

Thus, it is legitimate to consider the Solar system as a hierarchized system. Note that, at best, only $J_2$, $J_4$ and $J_6$ planetary multipolar coefficients (for giant planets) could have a significant impact on laser ranging experiments at the required accuracy (see [22–25] for giant planets’ multipole moments values). Hence, taking advantage that the Solar multipolar terms are very weak, the solution of the field equations (47), suitable for millimetric accuracy in the required accuracy (see [22–25] for giant planets’ multipole moments values). Thus, it is legitimate to consider the Solar system as a hierarchized system. Note that, at best, only the planetary multipolar coefficients (for giant planets) could have a significant impact on laser ranging experiments at the millimetric level. While the rotational term in the time-space component of the metric is given as the usual Lense-Thirring term, \( \dot{r}_A(t, \bar{x}) = \ddot{r}_A(t, \bar{x}) \) and \( r_A(t, \bar{x}) = |\vec{r}_A(t, \bar{x})| = \sqrt{(x^i - x_A^i)(x^i - x_A^i)} \). $M_A$, $r_a$, $v_A$ and $S_A$ are respectively the mass, the position and the velocity in barycentric coordinates, and the total angular momentum of the body $A$. $R_A$ and $J_n^A$ are the radius and the mass multipole coefficients of the body $A$. $P_n$ are the Legendre polynomials and \( \vec{k}_A \) denotes the unit vector along the local $Z_A$ axis of each body $A$. The differences with the IAU2000 metric \[2\] lie in the presence of both the PN parameter $\gamma$ and the $c^{-4}$ space-space metric term. The multipolar term $W_L$ in $g_{ij}$ that has been neglected in the IAU2000 metric – thanks to numerical considerations in the 1PN/BM case – has to be considered here as well. Accordingly, multipolar terms enter also the time-space component of the metric ($g_{ij}$), and could lead to measurable effects, depending on $J_{2n}$ orders of magnitude.

While the rotational term in the time-space component of the metric is given as the usual Lense-Thirring term, slight modifications (spin multipoles) can in principle appear due to the differential rotation of the bodies. However, Solar seismology suggests \[26\] that the Sun’s tachoclyne is at about 0.7 Sun radius. Then, the mass concerned by the differential rotation is of order of a few percent of the total mass and thus, it might not lead to measurable effects. But, incidentally, time transfer and laser ranging experiments could suggest a new way (independent of Solar seismology) to test our knowledges on the solar interior dynamics. Another point coming from the solar seismology is that the solar core ($r < 0.2$ Sun radius) may rotate faster than the external layers \[28\]. Since, the core represents a great amount of the total mass, this could affect the propagation of light at a level depending on the total angular momentum value ($\vec{S}_S$). Depending on the solar internal structure model, this may happen at the millimetric level. Using a simplified model, we show in appendix \[A\] how the non-rigidity of the rotation affects the metric.
Let us note that the $W_L$ terms can be neglected for inner solar system millimetric laser ranging experiments, such as Mars laser ranging for instance.

We emphasize again that neither the $\beta$ parameter nor any $\epsilon$ parameter are required ($\epsilon$ – corresponding sometimes to $\Lambda$ [31] or $\delta$ [16, 33] – being some PN parameter often considered in the $c^{-4}$ space-space metric term [32, 34]). This is because both the former and the latter give too small deviations from GR to be considered in $c^{-4}$ Solar system photon's trajectories calculations. This fact is known for the former from [7] and is then obvious for the latter since it is a function of $\gamma$ and $\beta$ in the (non-massive-)ST theories considered here (as expressed in equation (28)).

VII. ISOTROPIC GEODESIC SOLUTIONS RELEVANT IN THE INNER SOLAR SYSTEM

A. The metric to be used for time transfer at the millimetric level

Space missions like LATOR [16], TIPO [17], ASTROD [18], Phobos Laser Ranging [19] require laser links at the millimetric level in the inner solar system. As discussed in [1], a full $c^{-4}$ metric like (50-52) is needed, and the transfer requires writing the isotropic geodesic equations attached to this metric. However, it turns that some terms in the list (53-56) can be neglected. Indeed, an order of magnitude analysis (relevant for propagation of light restricted to the inner solar system) shows the following terms are not of numerical relevance:

- non-monopolar terms ($W_L$ and $J_{2n}^A$ terms in $W^i$);
- planetary spin terms ($\bar{\epsilon}$ terms in $W^i_{Planets}$).

Hence, the metric components we consider are restricted to (with $S^k_S = (constant)$ solar moments)

\[
\begin{align*}
  g_{00} &= -1 + \frac{(2)}{2} \frac{g_{00}}{g_{00}} + \frac{(4)}{4} \frac{g_{00}}{g_{00}} \\
  g_{0i} &= \frac{(3)}{3} \frac{g_{0i}}{g_{0i}} \\
  g_{ij} &= \frac{(2)}{2} \frac{g_{ij}}{g_{ij}} + \frac{(4)}{4} \frac{g_{ij}}{g_{ij}}
\end{align*}
\]  

with

\[
\begin{align*}
  \frac{(2)}{2} g_{00} &= \sum_A 2GM_A \frac{1}{c^2 r_A} = \sum_A \frac{(2)}{2} g_{(A)00} \\
  \frac{(4)}{4} g_{00} &= \frac{1}{2} \left[ \frac{(2)}{2} g_{00} \right]^2 \\
  \frac{(3)}{3} g_{0i} &= -2 (1 + \gamma) \frac{G}{c^3} \epsilon_{ijk} S^j_S \frac{D^k_S}{r^3_S} - (1 + \gamma) \sum_A \frac{2GM_A}{c^2} \frac{1}{r_A} \\
  \frac{(2)}{2} g_{ij} &= \gamma \delta_{ij} \frac{(2)}{2} g_{00} \\
  \frac{(4)}{4} g_{ij} &= -\delta_{ij} \frac{(4)}{4} g_{00} + \xi_{ij} \quad \text{with} \quad \xi_{ij} = \frac{G^2 M_S^2}{c^4} \left( \frac{D^i_A D^j_A}{r^3_S} - \delta_{ij} \right)
\end{align*}
\]

where $D^i_A (t, x^i) \equiv x^i - x^i_A (t)$ (components of $\vec{r}_A (t, \vec{x})$), $r^2_A = D^i_A D_A^i$ and $\beta^i_A = \frac{v^i_A}{c}$.

The calculations have been made with a tracking coefficient $\epsilon$ in front of $\frac{(4)}{4} g_{00}$ and $\frac{(4)}{4} g_{ij}$ in order to separate terms comming from the fourth order part of the metric from those comming from (2nd order)-(2nd order) geodesic equation coupling terms effects in the geodesic solution. However, for convenience, one has put it to unity in the following.
One gets

\[
\begin{align*}
\Gamma^0_{00} &= (3) \\
\Gamma^k_{00} &= \Gamma^k_{00} + \Gamma^0_{00} \\
\Gamma^0_{0i} &= \Gamma^0_{0i} + \Gamma^0_{0i} \\
\Gamma^k_{0i} &= \Gamma^k_{0i} \\
\Gamma^0_{ij} &= \Gamma^0_{ij} \\
\Gamma^k_{ij} &= \Gamma^k_{ij} + \Gamma^k_{ij}
\end{align*}
\]

with, taking into account only relevant terms

\[
\begin{align*}
\Gamma^0_{0i} &= (2) Z_i \\
\Gamma^k_{00} &= (2) Z_k \\
\Gamma^k_{ij} &= -\gamma \delta_{ik} Z_j - \gamma \delta_{jk} Z_i + \gamma \delta_{ij} Z_k \\
\Gamma^0_{00} &= -\sum_{(A)} \beta^l_{Ac} (2) Z_{(A)l} \\
\Gamma^0_{ij} &= (3) Z_{ij} + (3) Z_{ji} + \gamma \delta_{ij} \sum_{(A)} \beta^l_{Ac} (2) Z_{(A)l} \\
\Gamma^0_{0i} &= -Z_{ik} + (3) Z_{ki} + \gamma \delta_{ik} \sum_{(A)} \beta^l_{Ac} Z_{(A)l} \\
\Gamma^0_{00} &= 0 \\
\Gamma^k_{00} &= -2 g^0_{00} Z_k \\
\Gamma^k_{ij} &= -Z_{ijk} - Z_{jik} + Z_{kij}.
\end{align*}
\]

In these expressions, one has set

\[
\begin{align*}
Z_i &= -\frac{1}{2} \partial_i g^0_{00} = \sum_{(A)} \frac{GM_A D^i_A}{c^2 r^3_A} \\
Z_{ij} &= -\frac{1}{2} \partial_i g^0_{0j} = (1 + \gamma) G c^3 \epsilon_{jmk} S^m_{(S)} \left[ \frac{\delta_{ik}}{r^3_S} - 3 \frac{D^k_S}{r^5_S} \right] - (1 + \gamma) \sum_{(A)} \beta^l_{Ac} \frac{GM_A D^i_A}{c^2 r^3_A} \\
Z_{ijk} &= -\frac{1}{2} \partial_i \xi_{jk} = -\frac{1}{2} G c^2 M^2_S \left[ \delta_{ij} \frac{D^k_S}{r^3_S} + \delta_{ik} \frac{D^j_S}{r^3_S} + 2 \delta_{jk} \frac{D^i_S}{r^3_S} - 4 \frac{D^j_S D^k_S D^i_S}{r^5_S} \right]
\end{align*}
\]

B. Isotropic geodesic solutions

In the following, let us put

\[ T \equiv x^0 = ct. \]
One has now to solve the geodesic equation, written in the following (non affine) form

$$\frac{d^2 x^k}{dT^2} + \left( \Gamma^k_{\alpha\beta} - \Gamma^0_{\alpha\beta}\frac{dx^k}{dT} \right) \frac{dx^\alpha}{dT} \frac{dx^\beta}{dT} = 0$$

with the isotropic condition

$$g_{\alpha\beta} \frac{dx^\alpha}{dT} \frac{dx^\beta}{dT} = 0.$$

Let us first consider the geodesic equation. It develops as

$$\frac{d^2 x^k}{dT^2} = -(2)^k \Gamma^k_{00} + (4)^k \Gamma^k_{00} - 2 \Gamma^k_{00} \frac{dx_i^k}{dT} + 2 \Gamma^k_{00} \frac{dx^k}{dT} \frac{dx^k}{dT}$$

The solutions can be written under the form

$$x^k = x^k_e + N^k T + \frac{(2-4)^k}{x}$$

where $x^k_e$ is the position at $t = 0$ (emission). \{N^k\} is any set of three constants related by

$$N^k N^k = 1.$$  \hspace{1cm} (61)

The upper symbol $(2 - 4)$ in $(2 - 4)^k$ means this part of the solution contains all the (numerically required) $c^{-2}$, $c^{-3}$ and $c^{-4}$ contributions. When useful, the notation $(2)^k$ will represent a part of the solution that is only required to contain all the $c^{-2}$ contributions.

Inserting the connexion (58) and (60) in the geodesic equation (59), one gets

$$\frac{d^2 (2-4)^k}{dT^2} = - (1 + \gamma) \frac{(2)^2}{Z_k} + 2 (1 + \gamma) N^k N^1 \frac{(4)}{Z_i}$$

$$- (1 + \gamma) N^k \sum_{(A)} \beta^l_{Ae} \frac{(2)}{Z_{(A)i}} + 2 N^i \left[ \frac{(3)}{Z_{ik}} - \frac{(3)}{Z_{ki}} + N^j N^k \frac{(3)}{Z_{ij}} \right]$$

$$+ 2 \frac{(2)^2}{g_{00} Z_k} + 2 N^i N^j \frac{(4)}{Z_{ijk}} - 4 N^i N^j \frac{(4)}{Z_{kij}} + 4 N^i N^j \frac{(4)}{Z_{kij}} \frac{dx_i^k}{dT} + 4 N^i N^j \frac{(4)}{Z_{kij}} \frac{dx_j^k}{dT} - 2 \frac{(2)^2}{Z_k N^j \frac{dx_i^k}{dT}}$$

The expressions of $D^l_A$ to consider in this equation read, setting $D^l_{Ae} = x^l_e - x^l_{Ae}$ and $\beta^l_{Ae} = v^l_{Ae}/c$ (values at $t = 0$)

$$D^l_A(t, x^i) = D^l_{Ae} + (N^l - \beta^l_{Ae}) T + \frac{(2)^l}{x} \text{ in the } c^{-2} \text{ terms}$$

$$D^l_A(t, x^i) = D^l_{Ae} + N^l T \text{ in the } c^{-3} \text{ and } c^{-4} \text{ terms}$$

since, for a hierarchized system, all the $\beta$ terms are at least of order $O(\mu)$ (see section [VIA]). Now, let us define

$$\rho^2_A \equiv \left( T + T_{Ae} \right)^2 + K^2_{Ae}$$

where (with $K_{Ae} \geq 0$)

$$T_{Ae} \equiv N^l D^l_{Ae} - \beta^l_{Ae} D^l_{Ae} + 2 N^l D^l_{Ae} N^m \beta^m_{Ae}$$

$$K^2_{Ae} \equiv (1 + 2 N^l \beta^l_{Ae}) D^l_{Ae} D^l_{Ae} - T^2_{Ae}$$

$$= (1 + 2 N^l \beta^l_{Ae}) D^l_{Ae} D^l_{Ae} - \left( 1 + 4 N^m \beta^m_{Ae} \right) \left( N^l D^l_{Ae} \right)^2 + 2 \beta^m \beta^m_{Ae} D^m_{Ae} N^l D^l_{Ae}$$

Considering only numerically relevant terms, it turns that

$$v^2_A = (1 - N^l \beta^l_{Ae})^2 \left[ \rho^2_A + 2 \left( D^l_{Ae} + N^l T \right)^2 \right]$$
In the $c^{-2}$ terms, one develops $D_A^i/r_A^3$ and gets
\[
\frac{D_A^i}{r_A^3} = (1 + 3N^m \beta_A^{m}) \frac{D_A^i}{r_A^3} + (N^i - \beta_A^i) T + \left\{ \begin{array}{l} \frac{(2)^i}{x} \frac{D_A^i}{r_A^3} - 3 \left( D_A^i + N^i T \right) \frac{(2)^i}{x} \frac{D_A^i}{r_A^3} + N^i T \end{array} \right. \]
(63)

This way, the geodesic’s equation now reads, discarding non-relevant terms
\[
\frac{d^2(2x)}{dT^2} = (1 + \gamma) \left( \delta_{ik} - 2N^i N^k \right) \sum_{(A)} \frac{GMA}{c^2} (1 + 3N^m \beta_A^{m}) \frac{D_A^i}{r_A^3} + (N^i - \beta_A^i) T
\]
\[- (1 + \gamma) \sum_{(c)} \frac{GM}{c^2} \left( N^i, \frac{D_A^i}{r_A^3} + 2\beta_A^i \frac{N^i N^i D_A^i}{r_A^3} - 2N^i \beta_A^i \frac{D_A^i}{r_A^3} + 2N^i N^j \beta_A^i \frac{N^i D_A^i}{r_A^3} \right)
\]
\[- (1 + \gamma) \frac{G}{c^2} \left( 4N^i \epsilon_{imk} \frac{1}{\rho_S^i} + 6\epsilon_{kiml} \frac{N^i N^j D^k}{\rho_S^i} - 6N^i \epsilon_{iml} \frac{D^i S^k + 6N^i \epsilon_{jml} N^i D^k}{\rho_S^i} \right)
\]
\[- \frac{GM_S}{c^2} \left[ -4N^k \frac{D_S^k}{\rho_S^i} \frac{d^2}{dT^2} - 4N^i \frac{D_S^i}{\rho_S^i} \frac{d^2}{dT^2} + 2 \frac{D_S^k}{\rho_S^i} N^i \frac{d^2}{dT^2} \right]
\]
\[- 2 \left( \delta_{ik} - 2N^i N^k \right) \frac{GM_S}{c^2} \left[ \frac{(2)^i}{x} \frac{D_S^i}{\rho_S^i} - 3 \left( D_S^i + N^i T \right) \frac{(2)^i}{x} \frac{D_S^i}{\rho_S^i} + \frac{N^i N^i T}{\rho_S^i} \right] \]

Let us point out the development (63) is justified only if one has
\[
\frac{15}{2} \left( \frac{(2)^i}{x} \frac{D_A^i}{r_A^3} + N^i T \right) \frac{(2)^i}{x} \frac{D_A^i}{r_A^3} \leq \frac{2}{9} \rho_S^i \times f
\]
(65)
on the whole photon’s orbit. In fact, since only the solar terms are required for the highest order contributing terms, we consider this condition with respect to the Sun only. Besides, considering applications aimed in this paper, for which the contribution of the $c^{-4}$ terms is expected to be close to the limit of detectability, it is sufficient to require
\[
\left| \left( D_S^i + N^i T \right) \frac{(2)^i}{x} \frac{D_A^i}{r_A^3} \right| \leq \frac{2}{9} \rho_S^i \times f
\]

where $f \sim 1/10$, or even $1/100$, to keep a safety margin.

One first obtains the solution for the $c^{-2}$ and $c^{-3}$ contributions. Inserting the $c^{-2}$ solution in the rhs of (63), one gets the $c^{-4}$ contribution. Since one considers isotropic geodesics only, the solution has also to satisfy the isotropic condition, that writes, using (57) and (60) (and writing $\gamma = 1$ in the $c^{-4}$ terms)
\[
(1 + \gamma) g_{00}^{(2)} + 2N^i g_{0i}^{(2)} + 2N^i \frac{d}{dT} \frac{(2)^i}{x} - 2g_{00}^{(2)} N^i \frac{d}{dT} \frac{(2)^i}{x} + \frac{d}{dT} \frac{(2)^i}{x} + \frac{d}{dT} \frac{(2)^i}{x} + N^i \xi_{ij} = 0
\]

Finally, after some tedious calculations, one gets the solution
\[
x^k (T) = x^k_c + \left( 1 - \frac{2G^2M^2_S}{c^4 K_{Sc}} \right) N^k T + (1 + \gamma) \sum_{(A)} \frac{GMA}{c^2} \left[ \frac{(2)^k}{x} X_A (T) - \frac{(2)^k}{x} X_A (T = 0) \right]
\]
\[+ (1 + \gamma) \sum_{(A)} \frac{GMA}{c^2} \left[ \frac{(3)^k}{x} Y_A (T) - \frac{(3)^k}{x} Y_A (T = 0) \right] + (1 + \gamma) \frac{G}{c^2} \frac{S^m}{S} \left[ \frac{(3)^km}{Z} (T) - \frac{(3)^km}{Z} (T = 0) \right]
\]
\[+ \frac{G^2M^2_S}{c^4} \left[ \frac{(4)^k}{x} X (T) - \frac{(4)^k}{x} X (T = 0) \right]
\]
(66)
where one has set
\[
(2)^k_A = - \frac{D^k_{Ae} - N^k N^l D^l_{Ae}}{K^2_{Ae}} \rho_A - N^k \ln (\rho_A + T + T_{Ae})
\]
\[
(3)^k_A = - \frac{(\delta^{km} - N^k N^m)(\beta^{l}_{Ae} D^m_{Ae} + D^l_{Ae} \beta^m_{Ae}) N^l}{K^2_{Ae}} \rho_A + \beta^k_{Ae} \ln (\rho_A + T + T_{Ae})
\]
\[
(3)^{km} \frac{Z}{K^2_{Se}} = 2 \frac{D^k_{Se} - N^k N^l D^l_{Se}}{K^2_{Se}} \epsilon_{imn} \left( \frac{2 D_{Se}^n \rho_S - D_{Se}^n}{\rho_S} \right) + 2 \epsilon_{km} \left( \frac{N^l}{K^2_{Se} \rho_S} - \frac{1}{\rho_S} \right) + \frac{D^l_{Se} - N^l N^l D^l_{Se} T + T_{Se}}{\rho_S}
\]
\[
(4)^k \frac{X}{K^2_{Se}} = \left[ \frac{4 K_{Se}}{\rho_S} \ln \frac{\rho_S + T + T_{Se}}{\rho_S + T_{Se}} - \left( 4 - \frac{\epsilon}{4} \right) \frac{T + T_{Se}}{K_{Se}} \arc\tan \frac{T + T_{Se}}{4 \rho_S} \rho_S + \frac{\epsilon}{4} \frac{K^2_{Se}}{\rho_S^2} \frac{T + T_{Se}}{K_{Se}} \rho_S + \frac{4}{\rho_S} \rho_{Se} \right]
\]
\[
(3)^{km} \frac{Z}{K^2_{Se}} = 2 \frac{D^k_{Se} - N^k N^l D^l_{Se}}{K^2_{Se}} \epsilon_{imn} \left( \frac{2 D_{Se}^n \rho_S - D_{Se}^n}{\rho_S} \right) + 2 \epsilon_{km} \left( \frac{N^l}{K^2_{Se} \rho_S} - \frac{1}{\rho_S} \right) + \frac{D^l_{Se} - N^l N^l D^l_{Se} T + T_{Se}}{\rho_S}
\]

and with \( \rho^2_{Ae} = \rho_A^2 (T = 0) = T^2_{Ae} + K^2_{Ae} = (1 + 2 N^l \beta^l_{Ae}) D^m_{Ae} D^m_{Ae} \). The last expression is given with the tracking coefficient \( \epsilon (= 1) \) in applications in order to isolate terms coming from the \( c^{-4} \) part of the metric in the \( c^{-4} \) part of the geodesic solution.

The solution (66) has been written in such a way that:

- all the velocity and spin source terms have been brought together in the \( c^{-3} \) contribution (apart from the \( \beta \) terms entering the definitions of \( T_{Ae} \) and \( K_{Ae} \) in (63));

- terms parallel to \( N \) and orthogonal to \( N \) (i.e. \( \delta^{km} - N^k N^m \), \( D^k_{Ae} = N^k N^l D^l_{Ae} \) and \( \epsilon_{km} N^i \)) appear explicitly.

Let us also point out that:

- thanks to the hierarchy, it is sufficient to put

\[
T_{Ae} = N^l D^l_{Ae} \quad (67)
\]
\[
K^2_{Ae} = D^m_{Ae} D^m_{Ae} - T^2_{Ae} = D^m_{Ae} D^m_{Ae} - (N^l D^l_{Ae})^2
\]

in \((3)^k \), \((3)^{km} \) and \((4)^k \), instead of the full definitions (62);

- the \( \beta \ln (...) \) term in \((3)^k \), \( \epsilon_{km} S^m_S D^l_{Se} \) term entering \((3)^{km} \) and \( S^m_S \) are the only terms that are neither parallel nor orthogonal to \( N \).

For geometrical interpretations, let us remark that:

- in some sense (among others, since \( K_{Ae} \) is defined from the initial position, and not from the (virtual) position at \( t = -\infty \), see the discussion in (34)), \( K_{Ae} \) can be interpreted (from (67)) as some impact parameter, or minimal approach distance, with respect to the body \( A \);

- to the same extent, the quantity \( (D^k_{Ae} - N^k N^l D^l_{Ae}) / K^2_{Ae} \) can be interpreted as a vector of modulus \( 1 / K_{Ae} \) that is pointing toward the closest approach point (with respect to the body \( A \)) ;

- to the same extent, the quantity \( T_{Ae} \) can be interpreted as the opposite of time needed to reach the point of closest approach to the body \( A \) from the initial position \( x^k_{Ae} = x^k (T = 0) \).

C. Discussion

In the inner Solar system, the equation (66) gives the time-dependent position for a photon starting from the position \( x^k_{Ae} \) at \( t = 0 \). The millimetric level expected in forthcoming experiments has required the post-post-newtonian level of approximation. The trajectory is characterized by two arbitrary parameters, giving the initial "direction" in some sense. These parameters are, for instance, two any components of the "vector" \( N^k \), the third being given by the
Since \( \rho \) as soon as time-space components of the metric (\( g_{\alpha\beta} \)) level. This includes kinetic source terms effects (velocities of the sources, rotation of the Sun), leading to non-zero

The current work includes in a coherent way the different gravity sources of the solar system at the required numerical level. This includes kinetic source terms effects (velocities of the sources, rotation of the Sun), leading to non-zero

However, it has been stressed that \( (65) \) has to be satisfied on the whole trajectory to ensure the validity of the proposed geodesic solution. One has pointed \( K_{Se} \) can be crudely interpreted as the closest approach distance to the Sun \( \rho_{S,min} \) on the photon’s orbit, and \( T_{Se} \) as the opposite of time needed to reach the point of closest approach to the Sun from the initial position. The condition \( (65) \) leads to

\[
\frac{2GMs}{c^2} \left| \rho_S - \rho_{Se} + (T + T_{Se}) \ln \frac{T + T_{Se} + \rho_S}{T_{Se} + \rho_{Se}} \right| \lesssim \frac{2}{5} \rho_S^2 \times f.
\]

For a given orbit (thence a given \( \rho_{S,min} \)), and if \( \rho_{Se} \gg \rho_{Sm} \), the most severe condition is reached at the closest approach, leading to

\[
\rho_{Se} \lesssim \rho_{S,min} \times \left( \frac{2GMs}{\rho_{S,min} c^2} \right)^{-1} \times \frac{2f}{5}.
\]

Since \( \rho_{S,min} \) cannot be smaller than the solar radius, it turns the validity of the proposed geodesic solution is ensured as soon as

\[
\rho_{Se} \lesssim (7.10^5 \text{ km}) \times \left( \frac{3 \text{ km}}{7.10^9 \text{ km}} \right)^{-1} \times \frac{2f}{5} \approx 10^9 \text{ km}
\]

the last estimation being obtained taking \( 2f/3 \approx 1/100 \). This ensures the validity of the proposed solution for inner solar system links.

While it could seem natural to claim that any position on the photon’s orbit can be regarded as the initial one, a glance at the formula \( (66) \) makes obvious this cannot be the case. Indeed, at great ”distance” (i.e. \( T \rightarrow \pm \infty \)), one has

\[
\left( \frac{dX}{dT} \right)^k_s = s_T \frac{D_k^s}{K_{Se}^2} - N^k N^i D_i^{s,c} \left[ - \left( 4 - \frac{\epsilon}{4} \right) \frac{1}{K_{Se}^2} + \frac{4\rho_{Se}}{K_{Se}^2} \right] + \text{(terms that \( \rightarrow 0 \) when \( |T| \rightarrow \infty \))}
\]

where \( s_T \) is the sign of \( T \). On the other hand, it is clear on physical grounds that \( dx^k/dT \) should go to values that don’t depend on the chosen initial point on the orbit when \( T \rightarrow \pm \infty \), these limits corresponding to ”initial” and ”final” directions of the photon (the difference of these limits giving the so-called ”deviation” of the photon induced by the gravitational field). It is clear from \( (69) \) that \( dx^k/dT \) computed from \( (66) \) includes a \( c^{-4} \) term that goes to infinity for an initial position chosen (on any given orbit) arbitrarily far from the Sun. This is obviously physically unacceptable (and incompatible with the fact that \( c^{-4} \) terms should be corrective terms). This term corresponds to one of the so-called ”enhanced terms” in \( (34) \) (also present in \( (32) \)). The validity condition \( (65) \), rewritten \( (68) \), prevents to fall in this case. The tracking coefficient \( \epsilon \) shows the involved term comes from coupling contributions in the geodesic equation \( (33) \) (i.e from \( (34) \) r.h.s. terms depending explicitly on \( x^i \)), and not from the fourth order terms \( (4) \) in the metric, in accordance with \( (34) \).

Since the validity condition is ensured only for initial conditions at distances not exceeding some astronomical units, this shows one should be careful in any extention of such analytical post-post-newtonian approaches to both the deviation of light and light transfer problems at the external solar system scale, at least if photons grazing the Sun are considered (remark it is not the case for the GAIA mission).
VIII. CONCLUSION

In this paper, one has get the explicit geodesic equation relevant for propagation of light at the millimetric level in the inner Solar system, and for related problems. These expressions should be useful for experiments like [16–19]. One has also explicitly given the condition of validity of this expression, and pointed out how this validity condition is related to the method used to solve the geodesic equation at the post-post-newtonian level.

This work can be extended to missions involving links at the whole Solar system scale. The planetary $e^{-4}$ metric contributing terms are expected to be negligible in this case too, since these terms generate effects of order of $10^{-16}$ s at best, i.e. well under the mm level, even with a large safety margin. Hence, the metric (50-52) is relevant in this case too. However, giant planets multipolar terms have to be taken into account in (53-56), at least at the $c^{-2}$ level, as shown by previous studies (see [36] for instance). Besides, the validity condition of the method has to be carefully checked in this case, especially when one deals with photons grazing the Sun before/after travelling the external Solar system as argued in section VII C.

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Appendix A: Effects on the metric of a non-rigid rotation

Recent results coming from Solar seismology suggest that the Solar core rotates faster than the external layers \[28, 30\]. In the following toy model we consider the mass density as spherical \((\rho(\bar{x}) = \rho(r))\). As usual in Solar models \[20\], we make the distinction between three main regions: the core region \((r \in [0, R_C] \approx 0, 2R_S]\), where \(R_C\) is the core radius), the radiative region \((r \in [R_C, R_R] \approx 0, 7R_S]\), where \(R_R\) is the tachocline radius) and the convection region \((r \in [R_R, R_S]\), where \(R_S\) is the Sun radius). \(\bar{\Omega}\) being the angular velocity, we modelize the differential rotation as follows

\[
\bar{\Omega}(\bar{x}) = \Omega(r)\hat{k}_z,
\]

with

\[
\Omega(r) = \Omega_C(\theta) \Pi \left(\frac{r}{R_C}\right) + \Omega_R(\theta) \Pi \left(\frac{r - R_C}{R_R - R_C}\right) + \Omega_D(\theta) \Pi \left(\frac{r - R_R}{R_D - R_R}\right),
\]

where

\[
\Pi(x) = \begin{cases} 1 & \forall \ x \in [0; 1] \\ 0 & \text{everywhere else,} \end{cases}
\]

and where we modelize the differential rotation by a simple model in accordance with the usual model \[27\]

\[
\Omega_B(\theta) = \Omega_B (1 + \epsilon_B \cos^2 \theta),
\]

where \(\Omega_B\) and \(\epsilon_B\) are constants, \(B\) being any of the three previous regions \((C, R \text{ or } D)\).

Solar seismology suggests that the radiative region rotates as a solid – meaning \(\Omega_R(\theta) = \Omega_R\) (ie. \(\epsilon_R = 0\)). However, since we are interested in testing solar seismology results, let us relax this assumption. Let us write

\[
\bar{W}_S^j(t, \bar{x}) = \bar{W}_C^j(t, \bar{x}) + \bar{W}_R^j(t, \bar{x}) + \bar{W}_D^j(t, \bar{x}),
\]

with

\[
\bar{W}_S^j(t, \bar{x}) = G \int_B d^3 X \left(\Omega_B(R) \times \bar{X}\right)^i \frac{\rho(R)}{|\bar{x} - \bar{X}|}.
\]

\(\bar{W}_S^j\) being the Sun spin part of \(W_i\) in \[51\]. Then the solution writes

\[
\bar{W}_S^j(t, \bar{x}) = 4\pi G \sum_B \left(\bar{\Omega}_B \times \bar{x}\right)^i \left[\left(\frac{1}{3} + \frac{\cos^2 \theta}{15}\right) \frac{M_B^j}{r^3} + \frac{\epsilon_B}{33} \frac{M_B^j}{r^5} \left(5 \cos^2 \theta - 1\right)\right],
\]

where

\[
M_B^j = M_N(0, R_C), \quad M_R^j = M_N(R_C, R_R), \quad M_D^j = M_N(R_R, R_S),
\]

with

\[
M_N(X, Y) = \int_X Y R^{N+2} \rho(R) dR.
\]

First, note that a faster rotation of a rigid core will modify the value of the total angular momentum only. However, this value could be affected by a differential rotation as well.

But, one also may have to consider a term like the last term of the r.h.s. of \[A6\] in \[53\], in order to measure possible weak effects due to differential rotations of the different stages of the Sun – then, giving a characteristic way to put constraints on such differential rotations, which will be independent of Solar seismology and neutrino detection results. In what follows, we will refer to this term as the differential-Lense-Thirring term.

As an illustration, let us consider the following toy model. Assume (1) the density decreases linearly with the distance to the center of the Sun (2) the differential rotation is independent of the distance from the center. Now consider that the photons – used for the time transfer – graze the Sun (ie. \(b = \alpha R_S\), where \(b\) is the impact parameter and \(\alpha(> 1)\) a parameter ideally close to 1). Then the effect of the differential-Lense-Thirring term is about \(\alpha^{-2}\epsilon_B/11\) times the usual Lense-Thirring effect.

However, realistic models of the Sun and its rotation are expected to substantially decrease this effect. A specific study, that considers different realistic models of the Sun, should be done in order to clarify this issue.