Equivariant Pyragas control of discrete waves

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Abstract

Equivariant Pyragas control is a delayed feedback method that aims to stabilize spatio-temporal patterns in systems with symmetries. In this article, we apply equivariant Pyragas control to discrete waves, which are periodic solutions that have a finite number of spatio-temporal symmetries. We prove sufficient conditions under which a discrete wave can be stabilized via equivariant Pyragas control. The result is applicable to a broad class of discrete waves, including discrete waves that are far away from a bifurcation point. Key ingredients of the proof are an adaptation of Floquet theory to systems with symmetries, and the use of characteristic matrix functions to reduce the infinite dimensional eigenvalue problem to a one dimensional zero finding problem.

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1 Introduction

In [Pyr92], Kestitutis Pyragas introduced a delayed feedback control scheme (now known as ‘Pyragas control’) that aims to stabilize periodic motion. Pyragas considers a system without feedback that is described by an ordinary differential equation (ODE)

\[ \dot{x}(t) = f(x(t)), \quad t \geq 0 \]  

with \( f : \mathbb{R}^N \rightarrow \mathbb{R}^N \). Pyragas then introduces a feedback term that measures the difference between the current state and the state time \( p \) ago, and then feeds this difference (multiplied by a matrix) back into the system. Concretely, the system with feedback control becomes

\[ \dot{x}(t) = f(x(t)) + B [x(t) - x(t - p)] \]  

with gain matrix \( B \in \mathbb{R}^{N \times N} \). For a periodic solution with period \( p \), the difference between the current state and the state time \( p \) ago is zero. Hence a \( p \)-periodic solution of the original system (1.1) is also a solution of the feedback system (1.2). However, the global dynamics of the systems (1.1) and (1.2) are radically different, and we can try to choose the matrix \( B \in \mathbb{R}^{N \times N} \) in such a way that an unstable \( p \)-periodic solution of (1.1) is a stable solution of (1.2).

If the original system (1.1) has built-in symmetries, its periodic solutions can satisfy additional spatio-temporal relations. In this case, one can adapt the Pyragas control scheme so that it vanishes on solutions with a prescribed spatio-temporal relation. As in [FFS10], we write the feedback system as

\[ \dot{x}(t) = f(x(t)) + B [x(t) - h x(t - \theta_h)] \]  

with time delay \( \theta_h > 0 \) and \( h \in \mathbb{R}^{N \times N} \) a linear, spatial transformation. The control term now feeds back the difference between the current state and a spatio-temporal transformation of the state, and vanishes on spatio-temporal patterns of the form \( h x(t) = x(t + \theta_h) \). The feedback scheme (1.3) has the advantage that it is able to select a prescribed spatio-temporal pattern amongst a family of periodic solutions with the same

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period, and in such situations can indeed be more successful in stabilizing a specific pattern than Pyragas control [SB16, PBS13]. Since symmetries of the uncontrolled system (1.1) are often described in terms of equivariance relations, we refer to the control scheme (1.3) as **equivariant Pyragas control**.

Implementation of Pyragas control requires knowledge of the period of the targeted solution, but uses no additional information on the original system (1.1). This ‘model-independence’ makes Pyragas control widely applicable; for example in semiconductor lasers [SHW+06, SWH11], CO$_2$-lasers [BDG94] and enzymatic reactions [LFS95]; the paper [Pyr92] has currently (October 2022) more than 3100 citations. The equivariant control scheme (1.3) has recently been experimentally implemented in networks of chemical oscillators [HGTS].

While Pyragas control has many experimental realizations, proving mathematically rigorous results on Pyragas control is challenging. This is mainly because, from a mathematical perspective, the controlled systems (1.2) and (1.3) are delay differential equations (DDE) that generate infinite dimensional dynamical systems. In order to associate to (1.2) (resp. (1.3)) a well-posed initial value problem, we have to provide a function on the interval $[-p,0]$ (resp. $-\theta_h,0]$ as initial condition. So the state space is a function space (that has to be specified more precisely) and the associated dynamical system is infinite dimensional. Although the abstract theory of DDE is well developed [HV93, DvGVW95], the infinite dimensional nature of DDE is still demanding when we want to perform an explicit stability analysis in concrete examples.

This article is concerned with equivariant Pyragas control of **discrete waves**, which are periodic solutions that have a finite number of spatio-temporal symmetries. The main result of this article, Theorem 2.1, provides sufficient conditions under which a discrete wave can be stabilized via equivariant Pyragas control. The sufficient conditions are formulated in terms of eigenvalue properties of the uncontrolled system, and the result is applicable to a broad class of discrete waves.

The results in this article in particular apply to periodic orbits that are not close to a bifurcation point and that are ‘genuine’ periodic orbits, i.e. they cannot be transformed to a ring of equilibria of an autonomous system. This is significant, since in the literature so far, most analytical results on successful stabilization by (equivariant) Pyragas control either concern periodic orbits that bifurcate from an equilibrium [HKRH19, FLR20, VdW17, HBKR17] or concern **rotating waves**, i.e. periodic orbits that can be transformed to equilibria of autonomous systems [PPK14, FFG+08, SB16, PFS10, SdWD22]. In both these cases, the stability analysis simplifies, because we can determine the stability of the periodic orbit by determining the stability of an equilibrium in an autonomous system. These simplifications cannot be made in the setting considered in Theorem 2.1 and consequently the stability problem becomes more involved.

The stability analysis we perform here is based on a combination of equivariant Floquet theory with the theory of characteristic matrix functions. In systems without symmetry, we determine the stability of a $p$-periodic solution using the monodromy operator, which involves solving the linearized equation over a time step $p$. We show that in equivariant settings, we can work with the **twisted monodromy operator**, which involves solving the linearized equation over only a fraction of the period. We then prove that the twisted monodromy operator has a **characteristic matrix function**, a concept that was recently introduced in [KV22]. A characteristic matrix function captures the spectrum of a bounded linear operator in a matrix valued function, and using this concept we rigorously prove that the eigenvalues of the twisted monodromy operator are zeroes of a scalar valued function. This translates the infinite dimensional eigenvalue problem of the twisted monodromy operator to a one dimensional zero finding problem, which means a significant dimension reduction. As the final step, we analyze the scalar valued function and prove sufficient conditions under which the control scheme (1.3) successfully stabilizes a discrete wave.

This article is structured as follows. In Section 2, we state the main result (Theorem 2.1) in mathematically precise form, after having introduced the necessary terminology. In Section 3, we describe the symmetry relations of the controlled system (1.3); we then prove that the stability of discrete wave solutions of (1.3) is determined by the spectrum of the twisted monodromy operator. In Section 4, we introduce the concept of a characteristic matrix function; and show that the eigenvalues of the twisted monodromy operator can be computed as zeroes of a scalar-valued function. We analyze this scalar-valued function in Section 5 and subsequently prove Theorem 2.1.
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2 Setting and statement of the main result
Throughout the rest of this article, we assume that the ODE (1.1) is equivariant with respect to a compact Lie group Γ. This means that there exists a group homomorphism
\[ \rho : \Gamma \to GL(N, \mathbb{R}) \]
(called a representation of Γ) such that
\[ f(\rho(\gamma)x) = \rho(\gamma)f(x) \] (2.1)
for all \( x \in \mathbb{R}^N \) and all \( \gamma \in \Gamma \). If now \( x(t) \) is a solution of (1.1) and \( \gamma \) is an element of \( \Gamma \), then \( \rho(\gamma)x(t) \) is a solution of (1.1) as well. So the group \( \Gamma \) (or rather the group \( \{ \rho(\gamma) \mid \gamma \in \Gamma \} \)) is indeed a group of symmetries of the solutions of (1.1). In many examples, symmetries of an ODE can be effectively described using compact Lie groups; see for example the monographs [GSS88, GS02].

A compact Lie group Γ always has an orthogonal representation, i.e. there always exists a group homomorphism \( \rho : \Gamma \to O(N) \), cf. [GSS88, p. 31]. In the rest of this article, we directly view \( \Gamma \) as a subgroup of the orthogonal group \( O(N) \), instead of viewing it as an abstract compact Lie group with an orthogonal representation. Consequently we also suppress the representation in the notation, e.g. we now write the equivariance condition (2.1) as
\[ f(\gamma x) = \gamma f(x) \]
with \( \gamma \in \Gamma \subseteq O(N) \) and \( x \in \mathbb{R}^N \).

The symmetries of the ODE (1.1) naturally induce two symmetry groups on a given periodic solution. Suppose that \( x_\ast \) is a periodic solution of (1.1) with minimal period \( p > 0 \); denote by \( O = \{ x_\ast(t) \mid t \in \mathbb{R} \} \) its orbit. Then the group
\[ K := \{ \gamma \in \Gamma \mid \gamma x_\ast(0) = x_\ast(0) \} \]
leaves the initial condition \( x_\ast(0) \) invariant, and the group
\[ H := \{ \gamma \in \Gamma \mid \gamma O = O \} \]
leaves the orbit \( O \) invariant; cf. [PSSS]. If \( k \in K \), then \( kx_\ast(t) \) and \( x_\ast(t) \) are two solutions of (1.1) with the same initial condition, and hence
\[ kx_\ast(t) = x_\ast(t) \]
for all \( t \in \mathbb{R} \). So elements of the group \( K \) leave the orbit of \( x_\ast \) fixed pointwise and hence we refer to the group \( K \) as the group of spatial symmetries of \( x_\ast \). For any element \( h \in H \), there exists a time-shift \( \theta_h \in [0, p) \) such that \( hx_\ast(0) = x_\ast(\theta_h) \). But then \( hx_\ast(t) \) and \( x_\ast(t + \theta_h) \) are both solutions of (1.1) with the same initial condition, and hence
\[ hx_\ast(t) = x_\ast(t + \theta_h) \] (2.2)
for all \( t \in \mathbb{R} \). So every element of \( H \) induces a spatio-temporal relation of the form (2.2) on \( x_\ast \), and hence we refer to the group \( H \) as the group of spatio-temporal symmetries of \( x_\ast \).
If \( h, g \in H \) are two spatio-temporal symmetries of \( x_* \), then \( ghx_*(t) = x_*(t + \theta_h + \theta_g) \) and hence \( \theta_{hg} = \theta_h + \theta_g \mod p \). Thus the map

\[
H \to S^1 \cong \mathbb{R}/p\mathbb{Z} \\
h \mapsto \theta_h
\]

is a group homomorphism. Since the group \( K \) is exactly the kernel of the map \( H \ni h \mapsto \theta_h \), it is in particular a normal subgroup of \( H \), and the quotient group \( H/K \) is a subgroup of \( S^1 \). This implies that

\[
\left\{
\begin{aligned}
H/K & \cong \mathbb{Z}_n & \text{for some } n \in \mathbb{N}, \\
H/K & \cong S^1,
\end{aligned}
\right.
\]

where \( \mathbb{Z}_n \) denotes the cyclic group of order \( n \). If \( H/K \cong S^1 \), the periodic solution \( x_* \) is called a **rotating wave**; if \( H/K \cong \mathbb{Z}_n \) the periodic solution \( x_* \) is often called a **discrete wave**, cf. [Fie88]. For discrete waves, the time-shift \( \theta_h \) associated to spatio-temporal symmetry \( h \in H \) is always rationally related to the minimal period \( p \) of the orbit. Indeed, if \( h \in H \) and \( H/K \cong \mathbb{Z}_n \), then necessarily \( h^n \in K \). So \( n\theta_h = 0 \mod p \) and therefore there exists an integer \( m \in \{1, \ldots, n\} \) such that

\[
\theta_h = \frac{m}{n}p, \tag{2.3}
\]

i.e. \( \theta_h \) and \( p \) are rationally related.

Throughout the rest of this article, we focus on the stabilization of discrete waves. For future reference, we collect the relevant assumptions on the ODE (1.1) in a separate hypothesis.

**Hypothesis 1.**

1. \( f : \mathbb{R}^N \to \mathbb{R}^N \) is a \( C^2 \) function;
2. system (1.1) has a periodic solution \( x_* \) with minimal period \( p > 0 \);
3. system (1.1) is equivariant with respect to a compact symmetry group \( \Gamma \subseteq O(N) \), i.e.

\[
f(\gamma x) = \gamma f(x) \quad \text{for all } x \in \mathbb{R}^N \text{ and } \gamma \in \Gamma.
\]

4. The periodic solution \( x_* \) is a discrete wave, i.e. \( H/K \cong \mathbb{Z}_n \) for some \( n \in \mathbb{N} \).

To determine whether the \( p \)-periodic discrete wave \( x_* \) is a stable solution of (1.1), we consider the linearized equation

\[
\dot{y}(t) = f'(x_*(t))y(t), \tag{2.5}
\]

which is non-autonomous and \( p \)-periodic in its time argument. We denote by \( Y(t) \in \mathbb{R}^{N \times N} \) the **fundamental solution** of (2.5) with \( Y(0) = I \), i.e. \( Y(t) \) is the matrix-valued solution of the initial value problem

\[
\frac{d}{dt}Y(t) = f'(x_*(t))Y(t), \quad Y(0) = I.
\]

Floquet theory implies that the eigenvalues of the **monodromy operator** \( Y(p) \in \mathbb{R}^{N \times N} \) determine whether \( x_* \) is a stable solution of (1.1). The equivariance assumption in Hypothesis 1 allows us to refine Floquet theory for discrete waves. We do this in detail in Section 3; for now we just mention that in the stability analysis of discrete waves, the operator

\[
Y_h : \mathbb{R}^N \to \mathbb{R}^N, \quad Y_h = h^{-1}Y(\theta_h) \quad \text{with } h \in H, \tag{2.6}
\]

plays an important role; we call the operator (2.6) the **twisted monodromy operator** (associated to \( h \)). The twisted monodromy operator \( Y_h \) always has an eigenvalue 1 \( \in \mathbb{C} \), which we call the **trivial eigenvalue**.
This is because differentiating the relation \( \dot{x}_a(t) = f(x_a(t)) \) with respect to time implies that \( \dot{x}_a(t) \) is a solution of the linearized equation (2.5). So \( Y(t)\dot{x}_a(0) = \dot{x}_a(t) \) and together with (2.2) this implies that
\[
Y^{-1}(\theta_h)\dot{x}_a(0) = h^{-1}x_a(\theta_h) = \dot{x}_a(0).
\]
In Section 3 we additionally prove that if \( Y_h \) has an eigenvalue \( |\mu| > 1 \), then \( x_a \) is an unstable solution of (1.1).

With these preparations, we are now ready to state this article’s main result.

**Theorem 2.1.** Consider the ODE (1.1) satisfying Hypothesis 1. Assume that the discrete wave \( x_a \) has a spatio-temporal symmetry \( h \in H \) such that the twisted monodromy operator \( Y_h \) defined in (2.6) has the following properties:

1. The eigenvalue \( 1 \in \sigma(Y_h) \) is algebraically simple and \( Y_h \) has no other eigenvalues on the unit circle;
2. If \( \mu \in \sigma(Y_h) \) and \( |\mu| > 1 \), then
   \[
   -\epsilon^2 < \mu < -1.
   \]

Then there exists an open interval \( I \subseteq (-\infty, 0) \) such that for \( b \in I \), \( x_a \) is a stable solution of
\[
\dot{x}(t) = f(x(t)) + b [x(t) - hx(t - \theta_h)].
\]

Theorem 2.1 addresses equivariant Pyragas control with **scalar control gain**, i.e. in the control term
\[
b [x(t) - hx(t - \theta_h)]
\]
the factor \( b \) is a real number rather than a matrix. The fact that Theorem 2.1 achieves stabilization with a scalar control gain is remarkable since non-equivariant Pyragas control with a scalar control gain fails to stabilize a rather large class of periodic solutions. Indeed, if \( x_a \) is a \( p \)-periodic solution of the ODE (1.1), and the monodromy operator \( Y(p) \) has at least one real eigenvalue \( \mu > 1 \), then \( x_a \) is an unstable solution of the controlled system
\[
\dot{x}(t) = f(x(t)) + b [x(t) - (t - t_p)]
\]
for every choice of \( b \in \mathbb{R} \). Although Theorem 2.1 makes assumptions on the eigenvalues of the twisted monodromy operator \( Y_h \), it does not make assumptions on the eigenvalues of the monodromy operator \( Y(p) \). Given a periodic orbit \( x_a \), it is possible that its monodromy operator \( Y(p) \) has an eigenvalue \( \mu > 1 \), while its twisted monodromy operator \( Y_h \) satisfies the assumptions of Theorem 2.1. In this situation, the equivariant control scheme (2.8) overcomes a limitation to Pyragas control in the sense that the periodic solution can be stabilized using the control (2.8) but is always an unstable solution of (2.9).

A concrete example of this situation occurs in the Lorenz equation
\[
\begin{align*}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2, \\
\dot{x}_2 &= -x_1 x_3 + \lambda x_1 - x_2, \\
\dot{x}_3 &= x_1 x_2 - \epsilon x_3,
\end{align*}
\]
with \( x_1, x_2, x_3 \in \mathbb{R} \) and with parameters \( \sigma, \epsilon, \lambda \in \mathbb{R} \). System (2.10) is symmetric with respect to the group \( \mathbb{Z}_2 = \{e, \gamma\} \), where \( e \) is the identity element of the group and we represent \( \gamma \) on \( \mathbb{R}^3 \) as
\[
(x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3).
\]
In [WS06], Wulff and Schebs numerically show that for parameter values \( \sigma = 10 \), \( \epsilon = 8/3 \) and \( \lambda = 312 \) system (2.10) has a periodic solution with \( H = \mathbb{Z}_2 \) and \( K = \{e\} \). The authors continue this periodic orbit with respect to the parameter \( \lambda \), while keeping the parameters \( \sigma \) and \( \epsilon \) fixed. They find that for \( \lambda \approx 312.97 \), the periodic orbit undergoes a **flip-pitchfork bifurcation** (a term introduced in [Pie88]), which means that there exist parameter values close to \( \lambda \approx 312.97 \) where the twisted monodromy operator \( Y_h \) has an eigenvalue \( \mu < -1 \) and satisfies the assumptions of Theorem 2.1 whereas the monodromy operator \( Y(p) \) has an eigenvalue larger than 1. So in this parameter regime the periodic orbit can be stabilized using the equivariant control scheme (2.8), although it cannot be stabilized using Pyragas control of the form (2.9).
3 Equivariant Floquet theory

Throughout this section, we fix a spatio-temporal symmetry $h \in H$ together with a scalar control gain $b \in \mathbb{R}$ and consider the controlled system (2.8). We first identify the symmetries of system (2.8), and then develop an equivariant Floquet theory for its linearization.

We can write the controlled system (2.8) as

$$\dot{x}(t) = g(x(t), x(t - \theta h))$$

with $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ defined as

$$g(x, y) = f(x) + b[x - hy].$$

The function $g$ is equivariant with respect to the group generated by $h$, in the sense that

$$g(h^j x, h^j y) = h^j g(x, y)$$

holds for all $x, y \in \mathbb{R}^N$ and for all $j \in \mathbb{N} \cup \{0\}$. If now $x(t)$ is a solution of the DDE defined by (3.1a)–(3.1b), then

$$\frac{d}{dt}(h^j x(t)) = h^j g(x(t), x(t - \theta h))$$

and hence $h^j x(t)$ is a solution of (3.1a)–(3.1b) as well. So the DDE defined by (3.1a)–(3.1b) (or, equivalently, the DDE (2.8)) is equivariant with respect to the group generated by $h$ in the sense that elements of this group send solutions to solutions.

In general, not every symmetry of the uncontrolled system (1.1) is also a symmetry of the controlled system (3.1a)–(3.1b): if the symmetry group $\Gamma$ of the ODE (1.1) is not abelian, and $\gamma h \neq h \gamma$, then $\gamma$ is not a symmetry of (3.1a)–(3.1b). So in general the symmetry group of (3.1a)–(3.1b) contains a subgroup of $\Gamma$ (namely the group generated by $h$) but it need not be the entire group $\Gamma$.

We next consider the linearized equation

$$\dot{y}(t) = f'(x_\ast(t)) y(t) + b [y(t) - hy(t - \theta h)];$$

if we fix $s \in \mathbb{R}$ and supplement (3.2a) with the initial condition

$$y(s + t) = \varphi(t) \quad \text{for } t \in [-\theta h, 0] \text{ and } \varphi \in C([-\theta h, 0], \mathbb{R}^N),$$

then the system (3.2a)–(3.2b) has a unique solution $y(t)$ for $t \geq s$. Given a time $t \geq s$, we define the history segment $y_t \in C([-\theta h, 0], \mathbb{R}^N)$ at time $t$ as $y_t(\vartheta) = y(t + \vartheta)$, $\vartheta \in [-\theta h, 0]$. We then associate to (3.2a)–(3.2b) a two-parameter system of operators

$$U(t, s) : C([-\theta h, 0], \mathbb{R}^N) \to C([-\theta h, 0], \mathbb{R}^N), \quad t \geq s$$

with the property that $y_t = U(t, s) \varphi$ is the solution of (3.2a) with initial condition (3.2b) at time $s$. We refer to (3.3) as the family of solution operators of (3.2a). Since the non-autonomous system (3.2a) is $p$-periodic in its time argument, standard Floquet theory for DDE implies that

$$U(t + p, s + p) = U(t, s)$$

for all $t \geq s$; see [DvGVW95, Chapter 13]. The next lemma shows that the symmetry relations on (2.8) induce additional spatio-temporal relations on the family of solution operators $U(t, s)$.
Lemma 3.1. Consider the ODE \([1,1]\) satisfying Hypothesis \([1]\). For a fixed spatio-temporal symmetry \(h \in H\) and scalar control gain \(b \in \mathbb{R}\), let \(U(t,s), t \geq s\), be the family of solution operators associated to the linearized system \([3.2a]\); let \(n \in \mathbb{N}\) be such that \(H/K \simeq \mathbb{Z}_n\). Then
\[
hU(t,s) = U(t + \theta_h, s + \theta_h)h
\]
and
\[
h^nU(t,s) = U(t,s)h^n
\]
for all \(t \geq s\).

Proof. The equivariance relation \([2.4]\) implies that
\[
hf(x) = f(hx) \quad \text{for all} \quad x \in \mathbb{R}^N
\]
and differentiating \([3.6]\) with respect to \(x\) yields
\[
hf'(x) = f'(hx)h \quad \text{for all} \quad x \in \mathbb{R}^N.
\]
Since \(hx_0(t) = x_0(t + \theta_h)\), this implies that
\[
hf'(x_0(t)) = f'(hx_0(t))h = f'(x_0(t + \theta_h))h
\]
for all \(t \in \mathbb{R}\). Now fix \(s \in \mathbb{R}\) and \(\varphi \in C\([-\theta_h, 0], \mathbb{R}^N\)\), and let \(y(t)\) be the unique solution of the initial value problem
\[
\begin{cases}
y(t) = f'(x_0(t))y(t) + b[y(t) - y(t - \theta_h)], & t \geq s, \\
y(t) = \varphi(t), & t \in [s - \theta_h, s]
\end{cases}
\]
so that \(y_t = U(t, s)\varphi\). Then \(hy(t)\) satisfies
\[
\frac{d}{dt}(hy(t)) = hy(t)
\]
\[
= hf'(x_0(t))y(t) + hy(t - \theta_h)
\]
\[
= f'(x_0(t + \theta_h))hy(t) + b[hy(t) - h[hy(t - \theta_h)]].
\]
So \(hy(t)\) is a solution of the initial value problem
\[
\begin{cases}
\dot{z}(t) = f'(x_0(t + \theta_h))z(t) + b[z(t) - hz(t - \theta_h)], & t \geq s \\
z(t) = h\varphi(t), & t \in [s - \theta_h, s]
\end{cases}
\]
But then uniqueness of solutions implies that \(h\varphi = U(t + \theta_h, s + \theta_h)\varphi\). So \(hU(t,s)\varphi = U(t + \theta_h, s + \theta_h)h\varphi\) for all \(\varphi \in C\([-\theta_h, 0], \mathbb{R}^N\)\), which proves \([3.5a]\).

Iteratively applying \([3.5a]\) gives
\[
h^nU(t,s) = U(t + n\theta_h, s + n\theta_h)h^n.
\]
Since we have assumed that \(x_0\) is a discrete wave with \(H/K \simeq \mathbb{Z}_n\), there exists an \(m \in \{1, \ldots, n\}\) such that \(n\theta_h = mp\), cf. \([2.3]\). Substituting this into \([3.7]\) gives that
\[
h^nU(t,s) = U(t + mp, s + mp)h^n.
\]
But iteratively applying \([3.4]\) implies that
\[
U(t + mp, s + mp) = U(t, s)
\]
and hence
\[
h^nU(t,s) = U(t,s)h^n,
\]
as claimed.
For $U(t,s), t \geq s$, the family of solution operators of (3.2a), and $h \in H$ the spatio-temporal symmetry used for control in (2.8), we define the **twisted monodromy operator (associated to $h$)** as

$$U_h := h^{-1}U(\theta_h,0).$$

(3.8)

Here we slightly abuse notation and view the matrix $h^{-1} \in \mathbb{R}^{N \times N}$ as an operator on the Banach space $C([\theta_h, 0], \mathbb{R}^N)$ acting as $(h^{-1}\phi)(\theta) = h^{-1}\phi(\theta), \ \theta \in [\theta_h, 0]$. The notion of the twisted monodromy operator is inspired by an equivariant version of the Poincaré map introduced for ODE in [Fie88, p. 55]. The difference with [Fie88] is that we work with the flow of the linearized system, rather than with the Poincaré map, and work in the context of DDE, rather than ODE.

The next lemma provides a relation between the **monodromy operator** $U(p,0)$, which plays an important role in non-equivariant Floquet theory, and the twisted monodromy operator $U_h$.

**Lemma 3.2.** Consider the ODE (1.1) satisfying Hypothesis 2. For a fixed spatio-temporal symmetry $h \in H$ and scalar control gain $b \in \mathbb{R}$, let $U(t,s), t \geq s$, be the family of solution operators associated to the linearized system (3.2a). Let $n \in \mathbb{N}$ be such that $H/K \simeq \mathbb{Z}_n$ and let $m \in \{1, \ldots, n\}$ be such that $\theta_h = \frac{m}{n}p$. Moreover, let

$$U_h := h^{-1}U\left(\frac{m}{n}p,0\right)$$

be the twisted monodromy operator. Then

$$U(p,0)^m = h^nU_h^n.$$

**Proof.** Iteratively applying (3.4) gives that

$$U(mp,0) = U((mp, (m-1)p)U((m-1)p, (m-2)p) \ldots U(2p, p)U(p,0)$$

for $j \geq 1$. This implies that

$$U(mp,0) = U(p,0)^m.$$  

(3.9)

Next, iteratively applying (3.5a) with $\theta_h = \frac{m}{n}$ gives that

$$U\left(\frac{jm}{n}p, \frac{(j-1)m}{n}p\right) = h^{j-1}U\left(\frac{m}{n}p,0\right)h^{1-j}$$

for $j \geq 1$. This implies that

$$U(mp,0) = U\left(mp, mp - \frac{m}{n}p\right) \ldots U\left(\frac{2m}{n}p, \frac{m}{n}p\right)U\left(\frac{m}{n}p,0\right)$$

$$= h^{n-1}U\left(\frac{m}{n}p,0\right)h^{1-n} \left[h^{n-2}U\left(\frac{m}{n}p,0\right)h^{2-n} \ldots \right. U\left(\frac{m}{n}p,0\right)h^{1-n}] U\left(\frac{m}{n}p,0\right)$$

$$= h^{n-1}U\left(\frac{m}{n}p,0\right)h^{1-n}U\left(\frac{m}{n}p,0\right) \ldots h^{1-n}U\left(\frac{m}{n}p,0\right)$$

$$= h^n\left(h^{-1}U\left(\frac{m}{n}p,0\right)\right)^n$$

so

$$U(mp,0) = h^nU_h^n.$$  

(3.10)

Combining this with (3.9) yields that

$$U(p,0)^m = h^nU_h^n,$$

as claimed.

□

8
The next proposition shows that the stability of discrete wave solutions is determined by the eigenvalues of the twisted monodromy operator. For the second statement of the proposition, the proof strategy is to first relate the spectrum of the twisted monodromy operator \( U_h \) to the spectrum of the monodromy operator \( U(p, 0) \). We then in turn relate the spectrum of the monodromy operator \( U(p, 0) \) to the stability of the periodic orbit \( x_\ast \) in (2.8) by invoking stability theory for DDE (see [HV93 Chapter 10] and [DvGVW95 Chapter XIV]).

In order to discuss the spectrum of the twisted monodromy operator \( U_h \), we first have to complexify the operator \( U_h \) via a canonical procedure as detailed in, for example, [DvGVW95 Chapter 3]. However, we do not make the complexification explicit in notation, i.e. we write \( U_h \) for both the real operator on the real Banach space \( C([-\theta_h, 0], \mathbb{R}^N) \) and the complexified operator on the complex Banach space \( C([-\theta_h, 0], \mathbb{C}^N) \).

**Proposition 3.3.** Consider the ODE (1.1) satisfying Hypothesis (1). For a fixed spatio-temporal symmetry \( h \in H \) and scalar control gain \( b \in \mathbb{R} \), consider the controlled system (2.8). Let \( U(t, s), t \geq s, \) be the family of solution operators associated to the linearized system (3.2a) and let

\[
U_h = h^{-1}U(\theta_h, 0)
\]

be the twisted monodromy operator. Then the following statements hold:

1. The non-zero spectrum of \( U_h \) consists of isolated eigenvalues of finite algebraic multiplicity.

2. If \( U_h \) has an eigenvalue strictly outside the unit circle, then \( x_\ast \) is an unstable solution of (2.8). If the trivial eigenvalue \( 1 \in \sigma_{pt}(U_h) \) is algebraically simple and all other eigenvalues of \( U_h \) lie strictly inside the unit circle, then \( x_\ast \) is a stable solution of (2.8).

**Proof.** We divide the proof into three steps:

**Step 1:** We start by proving the first statement of the proposition.

For \( s = 0 \) and \( t \in [0, \theta_h] \), the initial value problem (3.2a)–(3.2b) becomes

\[
\dot{y}(t) = f'(x_\ast(t))y(t) + b[y(t) - h\varphi(t - \theta_h)] , \quad y(0) = \varphi(0). \tag{3.11}
\]

Let \( X(t) \) be the fundamental solution of the ODE \( \dot{y}(t) = f'(x_\ast(t))y(t) + by(t) \), i.e. \( X(t) \) is the matrix-valued solution to the initial value problem

\[
\frac{d}{dt}X(t) = f'(x_\ast(t))X(t) + bX(t), \quad X(0) = I.
\]

Then we can solve (3.11) by variation of constants as

\[
y(t) = X(t)\varphi(0) - \int_0^t X(t)X(\zeta)^{-1}bh\varphi(\zeta - \theta_h)d\zeta.
\]

Therefore \( (U_h\varphi)(\vartheta) := h^{-1}y(\theta_h + \vartheta) \) with \( \vartheta \in [-\theta_h, 0] \) is given by

\[
(U_h\varphi)(\vartheta) = X(\theta_h + \vartheta)\varphi(0) - \int_0^{t(\theta_h + \vartheta)} X(t + \theta_h + \vartheta)X(\zeta)^{-1}bh\varphi(\zeta - \theta_h)d\zeta.
\]

The operator

\[
R : C([-\theta_h, 0], \mathbb{R}^N) \rightarrow C([-\theta_h, 0], \mathbb{R}^N)
\]

defined by

\[
(R\varphi)(\vartheta) = X(\theta_h + \vartheta)\varphi(0), \quad \vartheta \in [-\theta_h, 0]
\]

has finite dimensional range and is hence compact. Moreover, the operator

\[
V : C([-\theta_h, 0], \mathbb{R}^N) \rightarrow C([-\theta_h, 0], \mathbb{R}^N)
\]
decomposition to relate the spectrum of twisted monodromy operator $U$ under both the twisted monodromy operator $U_h$ and the monodromy operator $U(p, 0)$. We then use this decomposition to relate the spectrum of twisted monodromy operator $U_h$ to the spectrum of the monodromy operator $U(p, 0)$.

By assumption, $h^n \in \mathbb{R}^{N \times N}$ is an orthogonal matrix and hence it is diagonalizable. So if $\lambda_1, \ldots, \lambda_d$ are the distinct eigenvalues of $h^n$, then we can decompose $\mathbb{C}^N$ as

$$\mathbb{C}^N = Y_1 \oplus \cdots \oplus Y_d$$

with $Y_i = \ker (\lambda_i I - h^n)$, $1 \leq i \leq d$.

Next we decompose the Banach space $X = C (\mathbb{R}, \mathbb{C}^N)$ as

$$X = X_1 \oplus \cdots \oplus X_d$$

with $X_i = C (\mathbb{R}, Y_i)$, $1 \leq i \leq d$.

Now fix $1 \leq j \leq d$ and let $\varphi \in X_j$, i.e. $\lambda_j \varphi - h^n \varphi = 0$. Then equality (3.5b) implies that

$$\lambda_j U_h \varphi - h^n U_h \varphi = U_h (\lambda_j \varphi - h^n \varphi) = 0.$$

So $U_h \varphi \in X_j$ and hence the space $X_j$ is invariant under the twisted monodromy operator $U_h$. Similarly, if $\varphi \in X_j$, then

$$\lambda_j U(p, 0) \varphi - h^n U(p, 0) \varphi = U(p, 0) (\lambda_j \varphi - h^n \varphi) = 0.$$

So $U(p, 0) \varphi \in X_j$ and hence the space $X_j$ is invariant under the monodromy operator $U(p, 0)$ as well.

By Lemma 3.2 there exists a $n \in \mathbb{N}$ and a $m \in \{1, \ldots, n\}$ such that $U(p, 0)^m = h^n U_h^n$. But since both the operator $U(p, 0)$ and the operator $U_h$ leave the spaces $X_i$, $1 \leq i \leq d$ invariant, it holds that

$$U(p, 0)^m|_{X_j} = \left(h^n U_h^n\right)|_{X_j}.$$

But $h^n \varphi = \lambda_j \varphi$ for $\varphi \in X_j$, and hence

$$U(p, 0)^m|_{X_j} = \lambda_j \left(U_h|_{X_j}\right)^n$$

for all $1 \leq j \leq d$. By the first statement of the lemma, all non-zero spectrum of $U_h|_{X_j}$ consists of isolated eigenvalues of finite algebraic multiplicity. Therefore equality (3.12) implies that also all the non-zero spectrum of $U(p, 0)^m|_{X_j}$ consists of isolated eigenvalues of finite algebraic multiplicity, and moreover it holds that

$$\sigma_{pt} \left(U(p, 0)^m|_{X_j}\right) = \lambda_j \sigma_{pt} \left(U_h|_{X_j}\right)^n = \lambda_j \sigma_{pt} \left(U_h|_{X_j}\right)^n$$

for all $1 \leq j \leq d$, (3.13)

where all equalities count algebraic multiplicities.

**Step 3:** We now prove the second statement of the proposition. First suppose that twisted monodromy operator $U_h$ has an eigenvalue $\mu$ with $|\mu| > 1$. Since

$$U_h = U_h|_{X_1} + \cdots + U_h|_{X_d},$$

there exists a $1 \leq j \leq d$ such that $U_h|_{X_j}$ has an eigenvalue $\mu$. Equality (3.13) then implies that $U(p, 0)^m|_{X_j}$ has an eigenvalue $\lambda_j \mu^n$. Since the matrix $h^n$ is orthogonal, its eigenvalue $\lambda_j$ has norm one; hence $|\lambda_j \mu^n| = |\mu|^n > 1$ and $U(p, 0)^m$ has an eigenvalue strictly outside the unit circle. This implies that $U(p, 0)$ has an
eigenvalue strictly outside the unit circle as well. Stability theory for DDE (see [HV93, Chapter 10] and [DvGVW95, Chapter XIV]) then implies that \( x^* \) is an unstable solution of (2.8), as claimed.

Vice versa, assume that the eigenvalue 1 \( \in \sigma_{pt}(U_h) \) is algebraically simple and all other eigenvalues of \( U_h \) lie inside the unit circle. Since

\[
U(p, 0)^m = U(p, 0)^m|_{X_1} + \ldots + U(p, 0)^m|_{X_d},
\]
equality (3.13) implies that \( U(p, 0)^m \) has an algebraically simple eigenvalue; all other spectrum of \( U(p, 0)^m \) lies strictly inside the unit circle. But then also \( U(p, 0) \) has an an algebraically simple eigenvalue and all other spectrum of \( U(p, 0) \) lies strictly inside the unit circle. Stability theory for DDE then implies that \( x^* \) is a stable solution of (2.8).

4 A characteristic matrix function for the twisted monodromy operator

We next show that we can compute the eigenvalues of the twisted monodromy operator \( U_h \) by computing the zeroes of a scalar-valued function. This translates the infinite dimensional eigenvalue problem of \( U_h \) to a one dimensional zero finding problem. The concept of a characteristic matrix function, as introduced in [KV22], provides the theoretical framework for this dimension reduction. We start this section by giving an overview of the relevant definitions and results from [KV22], and then discuss how the concept of a characteristic matrix function can be applied to the twisted monodromy operator (3.8).

In the following, we denote by \( \mathcal{L}(X, X) \) the space of bounded linear operators on a Banach space \( X \). We denote by \( I_X \) the identity operator on \( X \), but suppress the subscript whenever the underlying space is clear.

**Definition 4.1** ([KV22, Definition 5.2.1]). Let \( X \) be a complex Banach space, \( T : X \to X \) a bounded linear operator and \( \Delta : \mathbb{C} \to \mathbb{C}^{N \times N} \) an analytic matrix-valued function. We say that \( \Delta \) is a characteristic matrix function for \( T \) if there exist analytic functions \( E, F : \mathbb{C} \to \mathcal{L}(\mathbb{C}^N \oplus X, \mathbb{C}^N \oplus X) \) such that \( E(z), F(z) \) are invertible operators for all \( z \in \mathbb{C} \) and such that

\[
\begin{pmatrix}
\Delta(z) & 0 \\
0 & I_X
\end{pmatrix}
= F(z)
\begin{pmatrix}
I_{\mathbb{C}^N} & 0 \\
0 & I - zT
\end{pmatrix}
E(z)
\] (4.1)

holds for all \( z \in \mathbb{C} \).

Given a non-zero complex number \( \mu \in \mathbb{C} \), the equality (4.1) implies that the operator \( I - \mu T \) is not invertible if and only if \( \Delta(\mu) \) is not invertible, i.e. if and only if \( \det \Delta(\mu) = 0 \). But the operator \( I - \mu T \) is not invertible if and only if \( \mu^{-1} \) is in the spectrum of \( T \). So if \( \Delta \) is a characteristic matrix function for \( T \), then \( \mu^{-1} \) is in the spectrum of \( T \) if and only if \( \mu \) is a zero of the equation

\[
\det \Delta(z) = 0
\] (4.2)

and computing the roots of the equation (4.2) is equivalent to computing the non-zero spectrum of the operator \( T \).

The characteristic matrix function also captures other properties of the spectrum of \( T \), such as geometric and algebraic multiplicity of the non-zero eigenvalues. We have summarized the relevant results on the connection between the spectrum of \( T \) and the characteristic matrix function in the following lemma, which we cite without proof from [KV22].

**Lemma 4.2** (cf. [KV22, Theorem 5.2.2]). Let \( X \) be a complex Banach space, \( T : X \to X \) a bounded linear operator and \( \Delta : \mathbb{C} \to \mathbb{C}^{N \times N} \) a characteristic matrix function for \( T \). Then the following statements hold:
1. The non-zero spectrum of $T$ consists of isolated eigenvalues of finite algebraic multiplicity, and $\mu^{-1} \in \mathbb{C}\setminus\{0\}$ is a non-zero eigenvalue of $T$ if and only if
   \[ \det \Delta(\mu) = 0. \]

2. If $\mu^{-1} \in \mathbb{C}\setminus\{0\}$ is a non-zero eigenvalue of $T$, then its geometric multiplicity equals the dimension of the space $\ker \Delta(\mu)$.

3. If $\mu^{-1} \in \mathbb{C}\setminus\{0\}$ is a non-zero eigenvalue of $T$, then its algebraic multiplicity equals the order of $\mu$ as a root of $\det \Delta(z) = 0$.

Characteristic matrix functions have applications in the stability analysis of periodic solutions of DDE: under additional conditions on the relation between the period and the delay, characteristic matrix functions can be used to determine the stability of non-symmetric periodic orbits and of discrete waves. Suppose that the equation
   \[ \dot{x}(t) = g(x(t), x(t-\tau)) \] (4.3)
has no symmetries, but does have a periodic solution $x_*$ with period $\tau > 0$, i.e. the period of $x_*$ is equal to the time delay in (4.3). If we denote by $U(t,s)$, $t \geq s$, the family of solution operators associated to the linearized equation
   \[ \dot{y}(t) = \partial_1 g(x_*(t), x_*(t))y(t) + \partial_2 g(x_*(t), x_*(t))y(t-\tau), \]
then the monodromy operator $U(\tau, 0)$ has a characteristic matrix function, as was proven in [Ver92] and [KV22, Section 11.4]. The spectrum of the monodromy operator determines whether $x_*$ is a stable solution of (4.3), and hence we can determine the stability of the periodic solution $x_*$ by computing roots of a scalar-valued function. This result is particularly useful in the context of non-equivariant Pyragas control, since in system (1.2) the period of the targeted periodic solution is indeed equal to the time delay; see [FLR+20, SWLY13] for applications of this fact.

If the function $g$ in (4.3) is equivariant with respect to a group $\Gamma \subseteq O(N)$, i.e. if
   \[ g(\gamma x, \gamma y) = \gamma g(x, y) \] for all $x, y \in \mathbb{R}^N$ and $\gamma \in \Gamma$, (4.4)
then $\Gamma$ is a symmetry group of the solutions of (4.3) and we can describe the symmetries of its periodic solutions in the same way as we have done for the ODE setting in Section 2. If a period solution $x_*$ has a spatio-temporal symmetry of the form
   \[ hx_*(t) = x_*(t+\tau), \] (4.5)
i.e. the time shift in the spatio-temporal pattern (4.5) is equal to the time delay in the DDE (4.3), then there exists a characteristic matrix for the twisted monodromy operator $U_h = h^{-1}U(\tau, 0)$. Since the spectrum of the twisted monodromy operator determines whether $x_*$ is a stable solution of (4.3), we can also in this case determine the stability of $x_*$ by computing roots of a scalar-valued function. We formally state this in the following proposition, where we also give an explicit expression for a characteristic matrix function for $U_h = h^{-1}U(\tau, 0)$. We cite the proposition without proof from [dW22]; the proof builds upon the recent result [KV22, Theorem 6.1.1], which provides a characteristic matrix function for any operator that is the sum of a Volterra operator and a finite rank operator.

**Theorem 4.3** (cf. [dW22, Theorem 2.3]). Consider the DDE
   \[ \dot{x}(t) = g(x(t), x(t-\tau)) \] (4.6)
with $f : \mathbb{R}^N \times \mathbb{R}^N$ a $C^2$-function and with time delay $\tau > 0$. Assume that

1. system (4.6) is equivariant with respect to a compact symmetry group $\Gamma \subseteq O(N)$, i.e. (4.4) holds;
2. System (4.6) has a periodic solution $x_\ast$;
3. The periodic solution has a spatio-temporal symmetry $h \in H$ with
   \[ hx_\ast(t) = x_\ast(t + \tau). \]

Let $U(t,s), \ t \geq s$ be the family of solution operators of the linearized DDE
   \[ \dot{y}(t) = \partial_1 g(x_\ast(t), x_\ast(t - \tau))y(t) + \partial_2 g(x_\ast(t), x_\ast(t - \tau))y(t - \tau). \]

For $z \in \mathbb{C}$, let $F(t,z)$ be the fundamental solution of the ODE
   \[ \dot{y}(t) = \partial_1 g(x_\ast(t), x_\ast(t - \tau))y(t) + z \cdot \partial_2 g(x_\ast(t), x_\ast(t - \tau))y(t) \]
with $F(0,z) = I_{\mathbb{C}^N}$. Then the analytic function
   \[ \Delta(z) = I_{\mathbb{C}^N} - zh^{-1}F(\tau,z) \]
is a characteristic matrix function for the operator
   \[ U_h = h^{-1}U(\tau,0). \]

Returning to the topic of equivariant Pyragas control, we can write the controlled system (2.8) as
   \[ \dot{x}(t) = g(x(t), x(t - \theta h)) \]
with $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ given by
   \[ g(x,y) = f(x) + k[x - hy], \quad x,y \in \mathbb{R}^N. \]

For $h$ as in (2.8) and fixed $j \in \mathbb{N} \cup \{0\}$, the function $g$ defined in (4.9) satisfies
   \[ g(h^jx, h^jy) = h^jg(x,y) \quad \text{for all } x,y \in \mathbb{R}^N \]
and hence the controlled system (2.8) is equivariant with respect to the group generated by $h$ (cf. Section 3). By construction, the system (2.8) has a discrete wave solution $x_\ast$ which satisfies
   \[ hx_\ast(t) = x_\ast(t + \theta h), \]
i.e. the time shift in the spatio-temporal relation (4.10) is equal to the time delay in (4.8). So the system (2.8) satisfies the assumptions of Proposition 4.3, and applying Proposition 4.3 gives a characteristic matrix function $\Delta$ for the twisted monodromy operator.

The expression for $\Delta$ involves the fundamental solution of the parametrized family of ODE
   \[ \dot{y}(t) = f'(x_\ast(t))y(t) + b[y(t) - zy(t)], \quad z \in \mathbb{C}. \]

Due to the fact that the control gain $b$ in (4.11) is a scalar, we can relatively explicitly compute the fundamental solution of (4.11), and hence also obtain a concrete expression of the characteristic matrix $\Delta$. We do this in the following corollary.

**Corollary 4.4.** Consider the ODE (1.1) satisfying Hypothesis 1. For a fixed spatio-temporal symmetry $h \in H$ and scalar control gain $b \in \mathbb{R}$, let $U(t,s), \ t \geq s$, be the family of solution operators associated to the linear DDE (3.2a). Moreover, let $Y(t), \ t \in \mathbb{R}$, be the fundamental solution of the linear ODE
   \[ \dot{g}(t) = f'(x_\ast(t))y(t) \]
(4.12)
with \( Y(0) = I_{CN} \). Then the analytic function
\[
\Delta(z) = I_{CN} - z \left[ h^{-1}Y(\theta_h) \right] e^{b(1-z)\theta_h}
\]
is a characteristic matrix function for the operator
\[
U_h = h^{-1}U(\theta_h, 0).
\] (4.13)

In particular, this means the following: Denote by \( \mu_1, \ldots, \mu_N \) the (possibly non-distinct) eigenvalues of the matrix \( h^{-1}Y(\theta_h) \in \mathbb{R}^{N \times N} \), i.e.
\[
\{ \mu_1, \ldots, \mu_N \} = \sigma(h^{-1}Y(\theta_h))
\]
and define the function
\[
d(z) = \prod_{j=1}^{N} \left( 1 - z\mu_j e^{b(1-z)\theta_h} \right).
\] (4.14)

Then \( \mu^{-1} \in \mathbb{C}\setminus\{0\} \) is an eigenvalue of the twisted monodromy operator (4.13) if and only if \( \mu \) is a root of \( d(z) = 0 \).

Proof. If \( Y(t) \) is the fundamental solution of the ODE (4.12) with \( Y(0) = I \), then
\[
F(t, z) = Y(t)e^{b(1-z)t}, \quad z \in \mathbb{C}
\]
is the fundamental solution of the ODE (4.11) with \( F(0, z) = I \). So by Proposition 4.3, the analytic function
\[
\Delta(z) = I_{CN} - h^{-1}F(\theta_h, z)
\]
\[
= I_{CN} - z \left[ h^{-1}Y(\theta_h) \right] e^{b(1-z)\theta_h}
\]
is a characteristic matrix function for the operator (4.13), as claimed.

Since the multiplicative factor \( e^{b(1-z)\theta_h} \) is a complex scalar, we have that
\[
\det \Delta(z) = \prod_{j=1}^{N} \left( 1 - z\mu_j e^{b(1-z)\theta_h} \right),
\]
where \( \mu_1, \ldots, \mu_N \) are the eigenvalues of \( h^{-1}Y(\theta_h) \in \mathbb{R}^{N \times N} \). Therefore, Lemma 4.2 implies that \( \mu^{-1} \in \mathbb{C}\setminus\{0\} \) is an eigenvalue of (4.13) if and only if \( \mu \) is a root of \( d(z) = 0 \), with \( d \) as in (4.14).

5 Eigenvalues analysis and proof of the main theorem

The function \( d \) defined in (4.14) is the product of \( N \) factors. So to find the zeroes of \( d \), it suffices to find the zeroes of each of the individual factors, i.e. to find roots of equations of the form
\[
1 - z\mu e^{b(1-z)\theta_h} = 0 \quad \text{with } \mu \in \mathbb{C} \text{ and } b \in \mathbb{R}
\]
or, by setting \( b_* = b\theta_h \), equations of the form
\[
1 - z\mu_* e^{b_*(1-z)} = 0 \quad \text{with } \mu_* \in \mathbb{C} \text{ and } b_* \in \mathbb{R}.
\] (5.1)

The equation (5.1) was previously studied in [MNS11]. In fact, Lemma 5.3 and Lemma 5.4 in this section are similar to [MNS11, Lemma 7.4], but we provide new and simpler proofs. Our strategy here is to prove a connection between the roots of (5.1) and roots of the equation
\[
0 = -z + \alpha + \beta e^{-z}
\] (5.2)
with \(\alpha, \beta \in \mathbb{R}\). Equation (5.2) also appears in the stability analysis of equilibria of autonomous DDE; in that context, study of (5.2) dates back at least at least to [Hay50, BC63], see also [HV93, DvGVW95] for more recent accounts. So we can analyze the equation (5.1) by first exploiting the connection with (5.2) and then applying the results from [Hay50, BC63, HV93, DvGVW95]. This significantly simplifies the stability analysis compared to [MNS11] and makes the argument more systematic. In addition, the connection between (5.1) and (5.2) provides a new link between Floquet multipliers of a non-autonomous equation and eigenvalues of an autonomous equation. This connection unexpected in the context of DDE and we therefore believe it to be interesting in its own right; see also the discussion in Section 6.

We first relate solutions of the equation

\[
0 = 1 - ze^\alpha e^{\beta z}
\]  

(5.3)

with \(\alpha, \beta \in \mathbb{R}\) to solutions of the equation

\[
0 = -z + \alpha + \beta e^{-z}.
\]

**Lemma 5.1.** Let \(\alpha, \beta \in \mathbb{R}\) and consider the analytic functions

\[
F : \mathbb{C} \to \mathbb{C}, \quad F(z) = 1 - ze^\alpha e^{\beta z}
\]

\[
G : \mathbb{C} \to \mathbb{C}, \quad G(z) = -z + \alpha + \beta e^{-z}.
\]

Then \(\mu \neq 0\) is a solution of \(F(z) = 0\) if and only if \(\mu = e^{-\lambda}\), where \(\lambda\) is a solution of \(G(z) = 0\). So

\[
\{\mu \in \mathbb{C}\backslash \{0\} \mid F(\mu) = 0\} = \{e^{-\lambda} \mid G(\lambda) = 0\}.
\]  

(5.4)

**Proof.** For \(z \in \mathbb{C}\), it holds that

\[
F(e^{-z}) = 1 - e^{G(z)}.
\]  

(5.5)

So if \(G(z) = 0\), then \(F(e^{-z}) = 0\), and hence the map

\[
\lambda \mapsto e^{-\lambda}
\]

(5.6)

is well-defined. We show that the map (5.6) is bijective.

**Injective:** suppose that \(\lambda, \nu\) satisfy \(G(\lambda) = 0, G(\nu) = 0\) and \(e^{-\lambda} = e^{-\nu}\). Then

\[
\lambda = \alpha + \beta e^{-\lambda}
\]

\[
\nu = \alpha + \beta e^{-\nu}
\]

but since \(e^{-\nu} = e^{-\mu}\), this implies that \(\lambda = \nu\). Hence the map (5.6) is injective.

**Surjective:** let \(\mu \in \mathbb{C}\backslash \{0\}\) be such that \(F(\mu) = 0\) and let \(\tilde{\lambda} \in \mathbb{C}\) be such that \(e^{-\tilde{\lambda}} = \mu\). Then

\[
1 = e^{-\lambda} e^\alpha e^{\beta e^{-\lambda}}
\]

and hence

\[
-\tilde{\lambda} + \alpha + \beta e^{-\tilde{\lambda}} = 2\pi ik
\]

for some \(k \in \mathbb{Z}\). But then \(\lambda := \tilde{\lambda} + 2\pi ik\) satisfies

\[
-\lambda + \alpha + \beta e^{-\lambda} = 0
\]

and \(e^{-\lambda} = \mu\). So the map (5.6) is surjective. We conclude that the map (5.6) is bijective and the statement of the lemma follows. \(\square\)
Substituting \( z = \pm i \omega, \omega \geq 0 \) in (5.1) and solving for \( \alpha, \beta \) gives that \( \pm i \omega \) is a pair of roots of (5.1) if

\[
\alpha = \frac{\omega \cos(\omega)}{\sin(\omega)}, \quad \beta = -\frac{\omega}{\sin(\omega)}, \quad \text{with } \omega \in (\pi k, \pi (k + 1)) \text{ and } k \in \mathbb{N} \cup \{0\}.
\] (5.7)

Moreover, \( z = 0 \) is a solution of (5.1) if

\[
\alpha + \beta = 0.
\] (5.8)

For \( \omega = 0 \), the curve (5.7) intersects the line (5.8) at \((\alpha, \beta) = (1, -1)\), corresponding to \( z = 0 \) being a double root of (5.1). Otherwise the curves defined by (5.7)–(5.8) do not intersect each other, and they divide the \((\alpha, \beta)\)-parameter plane into regions, as is graphically depicted in Figure 1.

A more detailed analysis, as was for example done in [DvGVW95, Chapter XI], shows that the the curve

\[
\alpha = \frac{\omega \cos(\omega)}{\sin(\omega)}, \quad \beta = -\frac{\omega}{\sin(\omega)}, \quad 0 \leq \omega < \pi
\] and the half line

\[
\alpha + \beta = 0, \quad \alpha < 1
\] bound the **stability region** of (5.2), i.e the region of parameter values for which (5.2) has no eigenvalue in the strict right half of the complex plane. We make this precise in the following proposition, which we take without proof from [DvGVW95, Chapter XI].

**Proposition 5.2** (cf. [DvGVW95, Chapter XI]). Consider the equation (5.2) with \( \alpha, \beta \in \mathbb{R} \) and \( z \in \mathbb{C} \). In the \((\alpha, \beta)\)-plane, consider the half-line

\[
R = \{(\alpha, \beta) \mid \alpha + \beta = 0, \alpha < 1\}
\] (5.9)

and the curve

\[
C = \\{ (\alpha, \beta) \mid \alpha = \frac{\omega \cos(\omega)}{\sin(\omega)}, \beta = -\frac{\omega}{\sin(\omega)}, 0 \leq \omega < \pi \}\.
\] (5.10)

Denote by \( S \) the region in the \((\alpha, \beta)\)-plane bounded by \( R \) and \( C \) (see Figure 2). It holds that

1. **For** \((\alpha, \beta)\) **on** the half-line \( R \), \( z = 0 \) is a solution of (5.2) and all other solutions \( z \) satisfy \( \text{Re}z < 0 \).

2. **For** \((\alpha, \beta)\) **on** the curve \( C \), (5.2) has two solutions on the imaginary axis and all other solutions \( z \) satisfy \( \text{Re}z < 0 \).

3. **For** \((\alpha, \beta)\) **in** the interior of \( S \), all roots \( z \) of (5.2) satisfy \( \text{Re}z < 0 \).

4. **For** \((\alpha, \beta) \in \mathbb{C}\setminus S\), (5.2) has at least one root \( z \) satisfying \( \text{Re}z > 0 \).
We are now ready to analyze the zeroes of the function (4.14), or, equivalently, to analyze the roots of the equation (5.1). By Lemma 4.2 and Corollary 4.4, a non-zero number $\mu \neq 0$ is an eigenvalue of the twisted monodromy operator if and only if $\mu$ is a zero of the characteristic equation (4.14). Therefore, to prove that all non-trivial eigenvalues of the twisted monodromy operator are inside the unit circle, we have to prove that all non-trivial zeroes of (4.14) are outside the unit circle.

**Corollary 5.3** (Case $\mu_* = 1$). Let $b_* < 0$. Then the equation

$$1 - ze^{b_*(1-z)} = 0$$

has a simple root $z = 1$; all other roots of (5.11) lie strictly outside the unit circle.

**Proof.** Equation (5.11) is of the form

$$1 - ze^{\alpha e^{\beta z}} = 0$$

with

$$\alpha(b_*) = b_*, \quad \beta(b_*) = -b_*.$$

For $b_* < 0$, the point $(\alpha(b_*), \beta(b_*))$ lies on the line $R$ as defined in (5.9). Therefore, the equation

$$-z + \alpha(b_*) + \beta(b_*)e^{-z} = 0$$

has a solution $z = 0$ and all other solutions lie in the strict left half plane. Lemma 5.1 then implies that equation (5.11) has a solution $z = 1$ and that all other roots of (5.11) lie strictly outside the unit circle.

To prove that $z = 1$ is a simple solution of (5.11) for $b_* < 0$, we compute

$$\frac{d}{dz} \bigg|_{z=1} \left(1 - ze^{b_*(1-z)}\right) = -e^{b_*(1-z)} + b_*ze^{b_*(1-z)} \bigg|_{z=1} = -1 + b_* \neq 0$$

for $b_* < 0$. 

\[ \square \]
Corollary 5.4 (Case $\mu_* < -1$). Consider the equation
\[ 1 - \mu_*ze^{b_* (1-z)} = 0 \] (5.12)
and for fixed $\mu_*$ define the set
\[ I(\mu_*) := \{ b_* \in \mathbb{R} \mid \text{all solutions of (5.12) lie strictly outside the unit circle} \}. \] (5.13)
Then the following two statements hold:

1. If $-e^2 < \mu_* < -1$, the set $I(\mu_*)$ is a non-empty open interval with $I(\mu_*) \subseteq (-\infty, 0)$.
2. If $\mu_{*,1}$, $\mu_{*,2}$ are two real numbers satisfying $-e^2 < \mu_{*,1} < \mu_{*,2} < -1$, then it holds that
\[ I(\mu_{*,1}) \subseteq I(\mu_{*,2}). \] (5.14)

Proof. To prove the first statement of the lemma, let $\mu_*$ be a real number with $-e^2 < \mu_* < -1$. A complex number $\mu \in \mathbb{C}$ is a solution of (5.12) if and only if $\nu := -\mu$ is a solution of
\[ 1 - ze^{-\nu (1+z)} = 0. \] (5.15)
Hence all solutions of (5.12) lie strictly outside the unit circle if and only if all solutions of (5.15) lie strictly outside the unit circle. Since by assumption $-e^2 < \mu_* < -1$, we can write $-\mu_* = e^{\lambda_*}$ with $0 < \lambda_* < 2$. So (5.15) is of the form
\[ 1 - ze^{\alpha b_* e^{\beta z}} = 0 \]
with
\[ \alpha(b_*) = \lambda_* + b_*, \quad \beta(b_*) = b_* \] (5.16)
By Lemma 5.1, the solutions of (5.15) lie strictly outside the unit circle if and only if all solutions of
\[ -z + \alpha(b_*) + \beta(b_*)e^{-z} = 0 \] (5.17)
lie in the strict left half of the complex plane. But by Proposition 5.2, all roots of (5.17) lie in the strict left half of the complex plane if and only if the point (5.16) lies in the region $S$ as defined in Proposition 5.2. So to determine the elements of the set $I(\mu_*)$, we have to find the values of $b_*$ for which the point (5.16) lies in the set $S$ as defined in Proposition 5.2.

If we view (5.16) as a path parametrized by $b_*$, then for $b_* = -\lambda_*/2$ this path crosses the line $R = \{ (\alpha, \beta) \mid \alpha + \beta = 0, \alpha < 1 \}$ in the point
\[ (\alpha, \beta) = \left( \frac{\lambda_*}{2}, -\frac{\lambda_*}{2} \right), \]
and there exists a $\bar{b} < -\lambda_*/2$ such that for $b_*$ in the interval
\[ \left( \bar{b}, -\frac{\lambda_*}{2} \right) \]
the path (5.16) lies inside the set $S$ as defined in Proposition 5.2, see Figure 3. So the set $I(\mu_*)$ defined in (5.13) is of the form $I(\mu_*) = (\bar{b}, -\lambda_*/2)$, which proves the first statement of the lemma.

To prove the second statement of the lemma, we let $\mu_{*,1}, \mu_{*,2}$ be two real numbers with
\[ -e^2 < \mu_{*,1} < \mu_{*,2} < -1 \]
(so $\mu_{*,1}$ has larger modulus than $\mu_{*,2}$) and write
\[ -\mu_{*,1} = e^{\lambda_{*,1}}, \quad -\mu_{*,2} = e^{\lambda_{*,2}} \]
with $\lambda_{*,1} > \lambda_{*,2}$. Now let $b_* \in I(\mu_{*,1})$, which means that the point

$$(\alpha, \beta) = (\lambda_{*,1} + b_*, b_*)$$

(5.18)

lies inside the region $S$. Since $\lambda_{*,2} < \lambda_{*,1}$, we can obtain the point

$$(\alpha, \beta) = (\lambda_{*,2} + b_*, b_*)$$

(5.19)

by translating the point (5.18) horizontally to the left. But we then see from Figure 4 that if (5.18) lies inside the region $S$, then (5.19) lies inside the region $S$ as well, and hence $b_* \in I(\mu_{*,2})$. So it holds that $I(\mu_{*,1}) \subseteq I(\mu_{*,2})$, as claimed.

![Figure 3: The stability region $S$ (grey) together with the path defined by (5.16) for $b_* \leq 0$ (black). The path defined by (5.16) enters the region $S$ at the point $(-\lambda_*/2, \lambda_*/2)$.](image)

![Figure 4: The stability region $S$ (grey) together with the paths defined by (5.18) for $b_* \leq 0$ (black solid) and (5.19) with $b_* \leq 0$ (black dashed). The path (5.19) is a left translate of the path (5.18): if for a given value of $b_*$ the point (5.18) lies inside the region $S$, then for that value of $b_*$ the point (5.19) lies inside the region $S$ as well.](image)

In the case where $\mu_* \in \mathbb{C}\setminus\mathbb{R}$, the equation (5.1) can be only brought into the form $1 - z e^{\alpha} e^{\beta z} = 0$ by choosing $\alpha$ to be complex. So in this case, Proposition 5.2 does not give information on the roots of (5.1). However, for $|\mu_*| < 1$, we can analyze the roots by direct estimates.

**Corollary 5.5** (Case $|\mu_*| < 1$, cf. statement (v) in Lemma 7.4 in [MNS11]). Let $\mu_* \in \mathbb{C}$, $|\mu_*| < 1$ and $b_* < 0$. Then all solutions of

$$1 - \mu_* z e^{b_*(1-z)} = 0$$

(5.20)

lie strictly outside the unit circle.

**Proof.** Let $\mu \in \mathbb{C}$ be a solution of (5.20), then

$$1 = |\mu_*| |\mu| e^{b_*} |e^{-b_* \mu}|.$$
Because $b_* < 0$, it holds that $|-b_* \mu| = -b_* |\mu|$ and hence $|e^{-b_* \mu}| \leq e^{-b_* |\mu|}$. So $\mu$ satisfies the estimate

$$1 \leq |\mu| |\mu| e^{b_* e^{-b_* |\mu|}}. \quad (5.21)$$

Now assume by contradiction that $|\mu| \leq 1$. Since $b_* < 0$, we can estimate the right hand side of (5.21) as

$$|\mu| |\mu| e^{b_* e^{-b_* |\mu|}} \leq |\mu| e^{b_* e^{-b_* |\mu|}} \leq |\mu| < 1.$$

But this contradicts the estimate (5.21). We conclude that if $\mu$ is a solution of (5.12), then $|\mu| \geq 1$. \hfill \Box

Having assembled all the necessary ingredients, we now prove Theorem 2.1.

**Proof of Theorem 2.1.** Let $Y(t) \in \mathbb{R}^{N \times N}$ be the fundamental solution of the ODE

$$\dot{y}(t) = f'(x_*(t))y(t)$$

with $Y(0) = I$. We denote by $\mu_1, \ldots, \mu_N$ the (possibly non-distinct) eigenvalues of the matrix $h^{-1} Y(\theta_h) \in \mathbb{R}^{N \times N}$, i.e.

$$\{\mu_1, \ldots, \mu_N\} = \sigma(h^{-1} Y(\theta_h)).$$

Moreover, we let $U(t, s), \ t \geq s$, be the family of solution operators associated to the DDE

$$\dot{y}(t) = f'(x_*(t))y(t) + b [y(t) - hy(t - \theta_h)]$$

and we let

$$U_h = h^{-1} U(\theta_h, 0)$$

be the twisted monodromy operator.

By Proposition 3.3, $x_*$ is stable solution of

$$\dot{x}(t) = f(x(t)) + b [x(t) - hx(t - \theta_h)] \quad (5.22)$$

if the trivial eigenvalue $1 \in \sigma_{pt}(U_h)$ is algebraically simple and all other eigenvalues of $U_h$ lie strictly inside the unit circle. By Corollary 4.4, a non-zero complex number $\mu^{-1}$ is an eigenvalue of $U_h$ if and only if $\mu$ is a zero of the function

$$d(z) := \prod_{j=1}^{N} \left(1 - z \mu_j e^{b(1-z)\theta_h}\right), \quad (5.23)$$

and the trivial eigenvalue $1 \in \sigma(U_h)$ is algebraically simple if $z = 1$ is a simple zero of (5.23). So if there exists an open interval $I \subseteq \mathbb{R}$ such that for $b \in I$, $z = 1$ is a simple zero of (5.23) and all other zeroes are strictly outside the unit circle, then the statement of the theorem follows.

We first consider the case in which $h^{-1} Y(\theta_h)$ has at least one eigenvalue strictly outside the unit circle. Denote by $d$ the number of eigenvalues of $h^{-1} Y(\theta_h)$ strictly outside the circle; then we can relabel the eigenvalues $\mu_1, \ldots, \mu_N$ in such a way that

1. the eigenvalue $\mu_1$ has largest modulus, i.e. $|\mu_j| \leq |\mu_1|$ for $2 \leq j \leq N$;
2. the eigenvalues $\mu_1, \ldots, \mu_d$ lie strictly outside the unit circle;
3. the eigenvalues $\mu_{d+1}, \ldots, \mu_{N-1}$ lie strictly inside the unit circle;
4. $\mu_N = 1$.  

20
Next we consider the equation
\begin{equation}
1 - \mu_1 ze^{b(1-z)\theta_h} = 0
\end{equation}
and define the set
\begin{equation}
I := \{ b \in \mathbb{R} \mid \text{all solutions of (5.24) lie strictly outside the unit circle} \}. \quad (5.25)
\end{equation}
So with the notation of Corollary 5.4 and with \( I(\mu_1) \) as defined in (5.13) for \( \mu_* = \mu_1 \), we have that
\begin{equation}
I = \{ b \in \mathbb{R} \mid b_* := b\theta_h \in I(\mu_1) \}
\end{equation}
and Corollary 5.4 implies that \( I \) is a nonempty open interval with \( I \subseteq (-\infty, 0) \). We now claim that for \( b \in I \), the zero \( z = 1 \) of (5.23) is algebraically simple, and all other zeroes of (5.23) lie strictly outside the unit circle.

If \( 1 \leq j \leq d \), i.e. if \( \mu_j \) lies strictly outside the unit circle, then the second assumption of Theorem 2.1 implies that \( -e^2 < \mu_j < -1 \). Moreover, since \( |\mu_1| \geq |\mu_j| \), it holds that \( -e^2 < \mu_1 \leq \mu_j < -1 \). So in particular, the second statement of Corollary 5.4 implies that if \( b \in I \) and \( 1 \leq j \leq d \), then all solutions of
\begin{equation}
1 - z\mu_je^{b(1-z)\theta_h} = 0
\end{equation}
lie strictly outside the unit circle.

If \( b \in I \), then in particular \( b < 0 \) and also \( b_* := b\theta_h < 0 \). So if \( d + 1 \leq j \leq N - 1 \), i.e. if \( \mu_j \) lies strictly inside the unit circle, then Corollary 5.5 implies that all solutions of
\begin{equation}
1 - z\mu_je^{b(1-z)\theta_h} = 0
\end{equation}
lie strictly outside the unit circle. Moreover, if \( j = N \), i.e. for \( \mu_N = 1 \), Corollary 5.3 implies that
\begin{equation}
1 - ze^{b(1-z)\theta_h} = 0
\end{equation}
has a solution \( z = 1 \), which is simple, and all its other solutions lie strictly outside the unit circle.

We conclude that if \( b \in I \), the zero \( z = 1 \) of (5.23) is simple, and all other zeros of (5.23) lie strictly outside the unit circle. In this case the statement of the theorem follows.

In the case where the matrix \( h^{-1}Y(\theta_h) \) does not have a eigenvalue strictly outside the unit circle, the assumptions of the theorem imply that \( h^{-1}Y(\theta_h) \) has an algebraically simple eigenvalue 1 and all other eigenvalues lie strictly inside the unit circle (this in particular means that \( x_* \) is a stable solution of the ODE (1.1)). In this case, Corollaries 5.3 and 5.5 imply that for any \( b < 0 \), the zero \( z = 1 \) of (5.23) is algebraically simple and all other zeroes lie strictly outside the unit circle. So also in this case the statement of the theorem follows.

6 Discussion and outlook

A key ingredient in the proof of Theorem 2.1 is the introduction of the twisted monodromy operator (cf. equation (3.8)), which can be viewed as an adaptation of the monodromy operator to equivariant settings. The introduction of this twisted monodromy operator is crucial for two reasons.

Firstly, the properties of the twisted monodromy operator, and not the properties of monodromy operator, determine whether stabilization via equivariant Pyragas control is possible. We see this in the statement of Theorem 2.1 which provides clear conditions on the location of the spectrum of the twisted monodromy operator (namely on the negative real axis), while the location of the spectrum of the monodromy operator can either be on the positive or negative real axis. As another point in case, [4WS21] contains an invariance principle for non-equivariant Pyragas control in terms of eigenvalues of the monodromy operator; this result (and the consequences thereof) apply verbatim to equivariant Pyragas control when we replace the monodromy operator by the twisted monodromy operator (see also [4W21] Chapter 10 for more precise statements).
Secondly, Theorem 4.3 provides a $N$-dimensional characteristic matrix function for the twisted monodromy operator. However, we cannot so easily find a $N$-dimensional characteristic matrix function for the monodromy operator. This is because for equivariant Pyragas control, the time delay equals the time step in the spatio-temporal symmetry, but the time delay is a fraction of the full period. The existence of a characteristic matrix function for the twisted monodromy operator implies that we can compute its eigenvalues as roots of a scalar valued equation, which is advantageous from a computational point of view.

Lemma 5.1 shows that eigenvalues of the twisted monodromy operator of (3.2a) are exponentially related to roots of equations of the form

$$z = \alpha + \beta e^{-z}$$

with $\alpha, \beta \in \mathbb{R}$. Equation (6.1) is in turn related to the linear, autonomous DDE

$$\dot{y}(t) = \alpha y(t) + \beta y(t - 1) \quad \text{with } y(t) \in \mathbb{R},$$

because roots of (6.1) are exactly the eigenvalues of the generator of the semiflow to (6.2) [DvGVW95, Chapter IV]. So Lemma 5.1 shows that eigenvalues of the twisted monodromy operator of (3.2a) are exponentially related to eigenvalues associated with a linear, autonomous DDE of the form (6.2).

This is reminiscent of the situation in ODE, where there is a time-periodic transformation that transforms the periodic ODE

$$\dot{y}(t) = A(t)y(t), \quad A(t+1) = A(t) \in \mathbb{R}^{N \times N}$$

(6.3)

to an autonomous ODE of the form

$$\dot{y}(t) = By(t), \quad B \in \mathbb{R}^{N \times N};$$

(6.4)

through this transformation, eigenvalues of the monodromy operator of (6.3) are exponentially related to eigenvalues of the generator of (6.4). The crucial difference with DDE, however, is that there is in general no time-periodic transformation that transforms the DDE (3.2a) to an autonomous DDE of the form (6.2); cf [DvGVW95, Chapter XIII]. So it surprising that, although there is no relation between the solution operators of (3.2a) and (6.2), there is a relation between their spectra.

It is natural to ask whether this is true for a larger class of equations, e.g. whether the spectrum associated with a time-periodic DDE of the form

$$\dot{y}(t) = A(t)y(t) + B(t)y(t - 1), \quad \text{with } A(t+1) = A(t), \ B(t+1) = B(t) \text{ and } A(t), \ B(t) \in \mathbb{R}^{N \times N}$$

(6.5)

is related to the spectrum of an autonomous equation, possibly with a distributed delay. Since the eigenvalue problem of autonomous DDE is relatively well understood (see e.g. [YWPT22] for recent developments in this topic), such a correspondence would contribute to a further understanding of the eigenvalue problem associated to (6.5), and in particular to a further analytical understanding of Pyragas control.

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