RIESZ TRANSFORMS AND COMMUTATORS
IN THE DUNKL SETTING

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Abstract. In this paper we characterise the optimal pointwise size and regularity estimates for the Dunkl Riesz transform kernel involving both the Euclidean metric and the Dunkl metric, where these two metrics are not equivalent. We further establish a suitable version of the pointwise kernel lower bound of the Dunkl Riesz transform via the Euclidean metric only. Then we show that the lower bound of commutator of the Dunkl Riesz transform is with respect to the BMO space associated with the Euclidean metric, and that the upper bound is respect to the BMO space associated with the Dunkl metric. Moreover, the compactness and the two types of VMO are also addressed.

1. Introduction

The classical Fourier transform, initially defined on $L^1(\mathbb{R}^N)$, extends to an isometry of $L^2(\mathbb{R}^N)$ and commutes with translation, dilation and rotation groups. To study the differential operators associated to reflection groups, Dunkl in [D1, D2] introduced a similar transform, the Dunkl transform, which enjoys properties similar to the classical Fourier transform. The Dunkl transform is given by

$$\mathcal{F}_\kappa f(\xi) := c_\kappa^{-1} \int_{\mathbb{R}^N} E(-i\xi, x) f(x) d\omega(x),$$

where the usual character $e^{-i(x,y)}$ is replaced by $E(x, y) := \int_{\mathbb{R}^N} e^{\langle x, y \rangle} d\mu_x(\eta)$. Here $\mu_x$ is a probability measure supported in the convex hull $O(x)$ of the $G$-orbit of $x$ and the measure $\omega$ are invariant under a finite reflection group $G$ on $\mathbb{R}^N$ and $c_\kappa = \int_{\mathbb{R}^N} e^{-\|x\|^2/2} d\omega(x)$. Corresponding to the Dunkl transform, the Dunkl translation operator $\tau_x$ is defined on $L^2(\mathbb{R}^N, d\omega)$ by

$$\mathcal{F}_\kappa(\tau_x(f))(y) = E(i x, y) \mathcal{F}_\kappa f(y), \quad y \in \mathbb{R}^N. \tag{1.1}$$

See also [BCV, deJ, R1, R2, R3, TX1] for more topics related to the Dunkl setting.

Parallel to classical singular integrals, there is a natural Riesz transform in this Dunkl setting. The case $N = 1$, goes back to the work of S. Thangavelu and Y. Xu [TX2], where they established the $L^p$-boundedness of the associated Riesz transform in the Dunkl setting. This was extended to the case of general dimension $N$ by Amri and Sifi [AS]. See also [DH1, DH2] for singular integrals and multipliers.

Here we recall the setting of $\mathbb{R}^N$. Consider the Euclidean space $\mathbb{R}^N$ equipped with the standard inner product $\langle x, y \rangle = \sum_{j=1}^N x_j y_j$ and the corresponding Euclidean norm $\|x\| = \sqrt{\sum_{j=1}^N |x_j|^2}$.

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where  be the associated measure in (1.2)

In\( \mathbb{R}^N \), the reflection \( \sigma_\alpha \) with respect to the hyperplane \( \alpha_\perp \) orthogonal to a nonzero vector \( \alpha \) is given by

\[
\sigma_\alpha(x) = x - 2 \frac{\langle x, \alpha \rangle}{\|\alpha\|^2} \alpha.
\]

A finite set \( R \subset \mathbb{R}^N \setminus \{0\} \) is called a root system if \( \sigma_\alpha(R) = R \) for every \( \alpha \in R \). Let \( R \) be a root system in \( \mathbb{R}^N \) normalized so that \( \langle \alpha, \alpha \rangle = 2 \) for \( \alpha \in R \) and \( G \) the finite reflection group generated by the reflections \( \sigma_\alpha (\alpha \in R) \), where \( \sigma_\alpha(x) = x - \langle \alpha, x \rangle \alpha \) for \( x \in \mathbb{R}^N \).

Corresponding to this reflection group, we denote by \( \mathcal{O}(x) \) the \( G \)-orbit of a point \( x \in \mathbb{R}^N \). There is a natural metric between two \( G \)-orbits \( \mathcal{O}(x) \) and \( \mathcal{O}(y) \), given by

\[
d(x, y) := \min_{\sigma \in G} \|x - \sigma(y)\|.
\]

It is clear that \( d(x, y) \leq \|x - y\| \) and it is possible that for certain \( x, y \in \mathbb{R}^N \), \( d(x, y) = 0 \) while \( \|x - y\| > 0 \).

For a multiplicity function \( \kappa \) defined on \( R \) (invariant under \( G \)), let

\[
d\omega(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{\kappa(\alpha)} dx
\]

be the associated measure in \( \mathbb{R}^N \), where, here and subsequently, \( dx \) stands for the Lebesgue measure in \( \mathbb{R}^N \).

The Dunkl Riesz transforms \( R_j, j = 1, 2, \ldots, N \), are defined on \( L^2(\mathbb{R}^N, d\omega) \) by

\[
R_j(f)(x) = \lim_{\epsilon \to 0} \int_{|y| > \epsilon} \tau_x(f)(-y) \frac{y_j}{\|y\|^{\kappa}} d\omega(y), \quad x \in \mathbb{R}^N,
\]

where \( d_\kappa = 2^{\frac{n+1}{2}} \sqrt{\pi} p_{\kappa}, \) \( p_\kappa = \gamma_\kappa + N + 1 \) and \( \gamma_\kappa = \sum_{\alpha \in R} \kappa(\alpha) \). In [AS] the authors obtained an explicit expression for the kernel \( R_j(x, y) \) through which (1.3) can be represented as

\[
R_j(f)(x) = \int_{\mathbb{R}^N} R_j(x, y) f(y) d\omega(y).
\]

Indeed, For \( x, y \in \mathbb{R}^N \) and \( \eta \) in the convex hull \( \mathcal{O}(x) \), set \( A(x, y, \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, y \rangle} \).

Denote by

\[
K_j^{(1)}(x, y) = \int_{\mathbb{R}^N} \frac{\eta_j - y_j}{A^{\kappa}(x, y, \eta)} d\mu_\eta(\eta)
\]

and

\[
K_j^{(\alpha)}(x, y) = \frac{1}{\langle y, \alpha \rangle} \int_{\mathbb{R}^N} \left[ \frac{1}{A^{\kappa-2}(x, y, \eta)} - \frac{1}{A^{\kappa-2}(x, \sigma_\alpha \cdot y, \eta)} \right] d\mu_\eta(\eta), \quad \alpha \in R_+.
\]

The kernel \( R_j(x, y) \) is given by

\[
R_j(x, y) := d_\kappa \left\{ K_j^{(1)}(x, y) + \sum_{\alpha \in R_+} \frac{\kappa(\alpha) \alpha_j}{p_\kappa - 2} K_j^{(\alpha)}(x, y) \right\}.
\]

Moreover, \( R_j(x, y) \) satisfies the Hörmander condition: there exists \( C > 0 \) such that

\[
\int_{d(x, y) \geq 2\|y - y_0\|} |R_j(x, y) - R_j(x, y_0)| d\omega(x) \leq C, \quad y, y_0 \in \mathbb{R}^N.
\]
However, the Hörmander condition alone is insufficient to bring in recent progress and techniques in harmonic analysis to this Dunkl Riesz transform, such as the sparse domination and sharp quantitative weighted estimate \([H1, HRT, La, Le]\), and the boundedness and compactness of commutators (and its two weight setting) \([HLW, LL]\). And since we are after more refined estimates other than just the \(L^p\) boundedness, we rectify this lack of information, through the first main result in this paper, the optimal pointwise size and smoothness estimate of the Riesz transform kernel.

**Theorem 1.1.** There exists a constant \(C\) such that for \(j = 1, 2, \ldots, N\) and for every \(x, y\) with \(d(x, y) \neq 0\),

\[
|R_j(x, y)| \leq C \frac{d(x, y)}{\|x - y\|} \frac{1}{\omega(B(x, d(x, y)))},
\]

\[
|R_j(x, y) - R_j(x, y')| \leq C \frac{\|y - y'\|}{\|x - y\|} \frac{1}{\omega(B(x, d(x, y)))} \quad \text{for} \quad \|y - y'\| \leq d(x, y)/2,
\]

\[
|R_j(x', y) - R_j(x, y)| \leq C \frac{\|x - x'\|}{\|x - y\|} \frac{1}{\omega(B(x, d(x, y)))} \quad \text{for} \quad \|x - x'\| \leq d(x, y)/2.
\]

With this result, the door opens to many other questions about the Dunkl Riesz transforms. It would seem that all properties of Dunkl Riesz transform would become clear since the above pointwise size and smoothness estimates are in the standard form of Calderón–Zygmund operators. However, a problem still exists in that there are two different, though related, metrics appearing in the estimates (the same comment holds true for the Hörmander condition) and these metrics are not equivalent. Even with these more standard Calderón–Zygmund estimates, the Dunkl Riesz transforms does not fall into the classical frame of Calderón–Zygmund theory.

A natural question arises: “What is the right version of the corresponding BMO space in the Dunkl setting?” In \([Dz]\), Dziubański characterised the Dunkl Hardy space (in terms of the Euclidean metric and Dunkl measure \(d\omega\)) via the Dunkl Riesz transforms (see also \([ADH]\)). We now investigate the BMO space in this Dunkl setting. A typical question is to consider the BMO space and the commutator of the Dunkl Riesz transform \([b, R_j]\). As is well-known, in the classical setting, Coifman, Rochberg and Weiss \([CRW]\) characterised the boundedness of commutators of Riesz transform via the space of BMO functions. However, due to the conflict of metrics of the size and regularity of the kernel as in Theorem 1.1, the approach in \([CRW]\) and the modern methods as in \([H2, LOR]\) do not directly apply.

The second main result of this paper is to establish the link between boundedness of the commutator of the Dunkl Riesz transform \([b, R_j]\) and a corresponding BMO space, and show that the BMO defined via the Euclidean metric ball and the associated measure \(d\omega(x)\) is the lower bound of \([b, R_j]\) and the one with \(d(x, y)\) is the upper bound of \([b, R_j]\) in the Dunkl setting. Before addressing this, we first investigate the pointwise kernel lower bound for the Dunkl Riesz transform as follows:

**Theorem 1.2.** For \(j = 1, 2, \ldots, N\) and for every ball \(B = B(x_0, r) \subseteq \mathbb{R}^N\), there is another ball \(\tilde{B} = B(y_0, r)\) such that \(\|x_0 - y_0\| = 5r\), and that for every \((x, y) \in B \times \tilde{B}\),

\[
|R_j(x, y)| \geq \frac{C}{\omega(B(x_0, r))}.
\]
To state our result on commutator, we recall the BMO space in the Dunkl setting as

$$\text{BMO}_{\text{Dunkl}}(\mathbb{R}^N) = \{ b \in L_{\text{loc}}^1(\mathbb{R}^N, d\omega) : \| b \|_* < \infty \},$$

where

$$\| b \|_* = \sup_{B \subset \mathbb{R}^N} \frac{1}{\omega(B)} \int_B |b(x) - b_B|d\omega(x) < \infty$$

with the supremum is taken over all Euclidean balls $$B = B(y, r) = \{ z \in \mathbb{R}^N : \| z - y \| < r \}$$ and

(1.7) $$b_B = \frac{1}{\omega(B)} \int_B b(x)d\omega(x).$$

We also recall the $$\text{BMO}_d(\mathbb{R}^N)$$ space associated with $$d(x, y)$$ as

$$\text{BMO}_d(\mathbb{R}^N) = \{ b \in L_{\text{loc}}^1(\mathbb{R}^N, d\omega) : \| b \|_d < \infty \},$$

where

$$\| b \|_d = \sup_{B \subset \mathbb{R}^N} \frac{1}{\omega(O(B))} \int_{O(B)} |b(x) - b_O(B)|d\omega(x) < \infty.$$ 

Note that $$\text{BMO}_d(\mathbb{R}^N) \subset \text{BMO}_{\text{Dunkl}}(\mathbb{R}^N)$$ (see for example [JL]). We have the first main result.

**Theorem 1.3.** Suppose $$b \in L_{\text{loc}}^1(\mathbb{R}^N, d\omega)$$. Consider the commutator of the Dunkl Riesz transform $$[b, R_j]$$, defined by $$[b, R_j](f)(x) = b(x)R_j(f)(x) - R_j(bf)(x)$$. Suppose $$b \in \text{BMO}_d$$. Then for $$1 < p < \infty$$, $$[b, R_j]$$ is bounded on $$L^p(\mathbb{R}^N, d\omega)$$ with

$$\| [b, R_j] \|_{L^p(\mathbb{R}^N, d\omega) \to L^p(\mathbb{R}^N, d\omega)} \lesssim \| b \|_d.$$ 

Conversely, if $$[b, R_j]$$ is bounded on $$L^p(\mathbb{R}^N, d\omega)$$ for some $$1 < p < \infty$$, then $$b \in \text{BMO}_{\text{Dunkl}}(\mathbb{R}^N)$$ with

$$\| b \|_* \lesssim \| [b, R_j] \|_{L^p(\mathbb{R}^N, d\omega) \to L^p(\mathbb{R}^N, d\omega)}.$$ 

With the boundedness of the commutator now completely understood we can additionally consider additional operator theoretic conditions of the commutator. In particular, we obtain information about the compactness of these commutators. To do so, we define the VMO space in the Dunkl setting as follows:

$$\text{VMO}_{\text{Dunkl}}(\mathbb{R}^N) = \{ b \in \text{BMO}_{\text{Dunkl}}(\mathbb{R}^N) : (1) - (3) \text{ holds} \}$$

where

(1) $$\lim_{r \to 0} \sup_{B \subset \mathbb{R}^N, r_B = r} \frac{1}{\omega(B)} \int_B |b(x) - b_B|d\omega(x) = 0,$$

(2) $$\lim_{r \to \infty} \sup_{B \subset \mathbb{R}^N, r_B = r} \frac{1}{\omega(B)} \int_B |b(x) - b_B|d\omega(x) = 0,$$

(3) $$\lim_{r \to \infty} \sup_{B \subset \mathbb{R}^N, B \cap B(0, r) = \emptyset} \frac{1}{\omega(B)} \int_B |b(x) - b_B|d\omega(x) = 0.$$

We define the VMO space associated the Dunkl metric as follows:

$$\text{VMO}_d(\mathbb{R}^N) = \{ b \in \text{BMO}_d(\mathbb{R}^N) : (4) - (6) \text{ holds} \}$$

where

(4) $$\lim_{r_B \to 0} \sup_{O(B) \subset \mathbb{R}^N} \frac{1}{\omega(O(B))} \int_{O(B)} |b(x) - b_O(B)|d\omega(x) = 0,$$

(5) $$\lim_{r \to \infty} \sup_{B \subset \mathbb{R}^N} \frac{1}{\omega(B)} \int_B |b(x) - b_B|d\omega(x) = 0.$$
Suppose Theorem 1.4. Calderón–Zygmund operators in the Euclidean setting, see [Uch78], with VMO, which can be seen as an extension of the work by Uchiyama or the standard and \( \hat{\omega} \) upper bound for \( \sup_{O(B) \subset \mathbb{R}^N} \frac{1}{\omega(O(B))} \int_{O(B)} |b(x) - b_{O(B)}|d\omega(x) = 0 \).

\[
\lim_{r \to \infty} \sup_{B \subset \mathbb{R}^N, O(B) \cap B(0,r) = \emptyset} \frac{1}{\omega(O(B))} \int_{O(B)} |b(x) - b_{O(B)}|d\omega(x) = 0.
\]

The result we obtain is the following characterisation of compactness and equivalence with VMO, which can be seen as an extension of the work by Uchiyama for the standard Calderón–Zygmund operators in the Euclidean setting, see [Uch78].

**Theorem 1.4.** Suppose \( b \in \text{BMO}_{\text{Dunkl}}(\mathbb{R}^N) \). If \( b \in \text{VMO}_d(\mathbb{R}^N) \), then for \( 1 < p < \infty \), \([b, R_j]\) is compact on \( L^p(\mathbb{R}^N, d\omega) \). Conversely, if \([b, R_j]\) is compact on \( L^p(\mathbb{R}^N, d\omega) \), then \( b \in \text{VMO}_{\text{Dunkl}}(\mathbb{R}^N) \).

Based on the properties for the Dunkl transform, the Dunkl Poisson semigroup and the Dunkl Riesz transform, we obtain the upper bound via the Dunkl Poisson extension and Carleson measure estimates. The proof strategy we utilize was used when studying the standard Laplacian and classical Riesz transforms via the Poisson extension in [LS]. The lower bound follows from proving that the Dunkl Riesz transform kernel satisfies the non-degenerate condition (see for example the standard setting [H2, LOR]) via the Euclidean metric.

The paper is organised as follows. We will prove Theorems 1.1 and 1.2 in Section 2. The upper bound for \([b, R_j]\) will be given in Section 3, and then the lower bound for \([b, R_j]\) in Section 4. Compactness is dealt with in Section 5.

## 2. Proof of Theorems 1.1 and 1.2

Consider the Euclidean space \( \mathbb{R}^N \) equipped with the standard inner product and the corresponding norm. Let \( B(x, r) := \{ y \in \mathbb{R}^N : ||x - y|| < r \} \) stand for the ball with center \( x \in \mathbb{R}^N \) and radius \( r > 0 \). In \( \mathbb{R}^N \), the reflection \( \sigma_\alpha \) with respect to the hyperplane \( \alpha_\perp \) orthogonal to a nonzero vector \( \alpha \) is given by

\[
\sigma_\alpha(x) = x - 2 \frac{\langle x, \alpha \rangle}{||\alpha||^2} \alpha.
\]

A finite set \( R \subset \mathbb{R}^N \setminus \{0\} \) is called a root system if \( \sigma_\alpha(R) = R \).

Let \( R \) be a root system in \( \mathbb{R}^N \) normalized so that \( \langle \alpha, \alpha \rangle = 2 \) for \( \alpha \in R \) and with \( R_+ \) a fixed positive subsystem, and \( G \) the finite reflection group generated by the reflections \( \sigma_\alpha \) (\( \alpha \in R \)). We shall denote by \( O(x) \), resp. \( O(B) \) the \( G \)-orbit of a point \( x \in \mathbb{R}^N \), resp. a subset \( B \subset \mathbb{R}^N \).

We denote by \( N = N + \sum_{\alpha \in R} \kappa(\alpha) \) the homogeneous dimension of the system. The measure \( d\omega \) as in [1.2] satisfies that

\[
\omega(B(tx, tr)) = t^N \omega(B(x, r))
\]

and that there is a constant \( C > 0 \) such that

\[
\omega(B(x, 2r)) \leq C \omega(B(x, r)) < \infty
\]

for all \( x \in \mathbb{R}^N \), \( t, r > 0 \). Moreover,

\[
C^{-1} \left( \frac{r_2}{r_1} \right)^N \leq \frac{\omega(B(x, r_2))}{\omega(B(x, r_1))} \leq C \left( \frac{r_2}{r_1} \right)^N \quad \text{for } 0 < r_1 < r_2,
\]

and

\[
\int_{\mathbb{R}^N} f(x)d\omega(x) = \int_{\mathbb{R}^N} \frac{1}{t^N} f \left( \frac{x}{t} \right) d\omega(x)
\]
for $f \in L^1(\mathbb{R}^N, d\omega(x))$, $t > 0$. By (2.1), it is easy to see $\omega(B(x, \|x - y\|)) \approx \omega(B(y, \|x - y\|))$.

Recall that

$$d(x, y) := \min_{\sigma \in G} \|x - \sigma(y)\|$$

denotes the distance between two $G$-orbits $\mathcal{O}(x)$ and $\mathcal{O}(y)$. Obviously,

$$\mathcal{O}(B(x, r)) = \bigcup_{\sigma \in G} B(\sigma(x), r) = \{y \in \mathbb{R}^N : d(x, y) < r\}$$

and

$$\omega(B(x, r)) \leq \omega(\mathcal{O}(B(x, r))) \leq |G|\omega(B(x, r)).$$

See in [ADH, DH1].

The Dunkl operators $T_\xi$, introduced in [D1], is the following $\kappa$-deformation of the directional derivative $\partial_\xi$ by a difference operator:

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R} \frac{\kappa(\alpha)}{2} \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}.$$  

For fixed $y \in \mathbb{R}^N$ the Dunkl kernel $E(x, y)$ is a unique solution of the system

$$T_\xi f = \langle \xi, y \rangle f, \quad f(0) = 1.$$  

Let $e_j, j = 1, \ldots, N$ denote the canonical orthonormal basis in $\mathbb{R}^N$ and let $T_j = T_{e_j}$. In particular

$$T_{j,x}E(x, y) = y_j E(x, y),$$

where $T_{j,x}$ denotes the action of $T_j$ with respect to the variable $x$.

For $f \in L^1(\mathbb{R}^N, d\omega)$ (the Lebesgue space with respect to the measure $\omega$) the Dunkl transform is defined by

$$\mathcal{F}_k(f)(\xi) = \frac{1}{c_k} \int_{\mathbb{R}^N} E_k(-i \xi, x)f(x)d\omega(x), \quad c_k = \int_{\mathbb{R}^N} e^{-\|x\|^2/2}d\omega(x).$$

The Dunkl translation $\tau_x f$ of a function $f \in \mathcal{S}(\mathbb{R}^N)$ by $x \in \mathbb{R}^N$ is defined by

$$\tau_x f(y) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, x)E(i\xi, y)\mathcal{F}_k f(\xi)d\omega(\xi).$$

If $f$ is a continuous radial function in $L^2(\mathbb{R}^N, \omega)$ with $f(y) = \hat{f}(\|y\|)$, then

$$\tau_x(f)(y) = \int_{\mathbb{R}^N} \hat{f} \left( \sqrt{\|x\|^2 + \|y\|^2 + 2\langle y, \eta \rangle} \right) d\mu_x(\eta).$$

This formula is first proved by M. Rösler [R2] for $f \in \mathcal{S}(\mathbb{R}^N)$ and recently is extended to radial continuous functions by F. Dai and H. Wang [DW].

We collect below some useful facts:

(i) For all $x, y \in \mathbb{R}^N$,

$$\tau_x(f)(y) = \tau_y(f)(x).$$

(ii) For all $x, \xi \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N)$,

$$T_\xi \tau_x(f) = \tau_x T_\xi(f).$$

(iii) For all $x \in \mathbb{R}^N$ and $f, g \in L^2(\omega)$,

$$\int_{\mathbb{R}^N} \tau_x(f)(-y)g(y)d\omega(y) = \int_{\mathbb{R}^N} f(y)\tau_x g(-y)d\omega(y).$$
where the heat kernel \( L \)

The operator \( \Delta \) is essentially self-adjoint on \( L^p(\mathbb{R}^N, \omega) \) and the following holds

\[
\|\tau_x(f)\|_{L^p(\omega)} \leq \|f\|_{L^p(\omega)}.
\]

The Dunkl Laplacian associated with \( G \) and \( \kappa \) is the differential-difference operator \( \Delta = \sum_{j=1}^{N} f_j^2 \), which acts on \( C^{2}(\mathbb{R}^N) \)-functions by

\[
\Delta f(x) = \Delta_{\text{eucl}} f(x) + \sum_{\alpha \in R} \kappa(\alpha) \delta_\alpha f(x),
\]

\[
\delta_\alpha f(x) = \frac{\partial_\alpha f(x)}{\langle \alpha, x \rangle} - \frac{\|\alpha\|^2 f(x) - f(\sigma x)}{2 \langle \alpha, x \rangle^2}.
\]

The operator \( \Delta \) is essentially self-adjoint on \( L^2(\mathbb{R}^N, \omega) \). The semigroup has the form

\[
H_t(f)(x) = e^{t\Delta} f(x) = \int_{\mathbb{R}^N} h_t(x, y) f(y) d\omega(y),
\]

where the heat kernel

\[
h_t(x, y) = \tau_x h_t(-y), \quad \text{with} \quad h_t(x) = c_{\kappa}^{-1}(2t)^{-N/2} e^{-\|x\|^2/(4t)},
\]

is a \( C^\infty \)-function of all variables \( x, y \in \mathbb{R}^N, t > 0 \), and satisfies

\[
0 < h_t(x, y) = h_t(y, x), \quad \int_{\mathbb{R}^N} h_t(x, y) d\omega(y) = 1.
\]

Set

\[
V(x, y, r) := \max\{\omega(B(x, r)), \omega(B(y, r))\}.
\]

The following theorem was proved in [ADH, Theorem 4.1].

**Theorem 2.1 (ADH).**

(a) There are constants \( C, c > 0 \) such that

\[
\frac{1}{C \min\{\omega(B(x, \sqrt{t})), \omega(B(y, \sqrt{t}))\}} e^{-\|x-y\|^2/4t} \leq |h_t(x, y)| \leq CV(x, y, \sqrt{t})^{-1} e^{-cd(x,y)^2/4t},
\]

for every \( t > 0 \) and for every \( x, y \in \mathbb{R}^N \).

(b) There are constants \( C, c > 0 \) such that

\[
|h_t(x, y) - h(x, y')| \leq C \left( \frac{\|y - y'\|}{\sqrt{t}} \right) V(x, y, \sqrt{t})^{-1} e^{-cd(x,y)^2/4t},
\]

for every \( t > 0 \) and for every \( x, y, y' \in \mathbb{R}^N \) such that \( \|y - y'\| < \sqrt{t} \).

We now recall the Riesz transforms in the Dunkl setting defined by

\[
\mathcal{F}_\kappa(R_j f)(\xi) = -i \frac{\xi_j}{\|\xi\|} \mathcal{F}_\kappa(f)(\xi) \quad \text{for} \ j = 1, 2, \ldots, N.
\]

Note that

\[
R_j f = -T_j (-\Delta)^{-1/2} f = -C_1 \int_0^\infty T_j e^{t\Delta} f \frac{dt}{\sqrt{t}},
\]

where the integral converges in \( L^2(\mathbb{R}^N, \omega) \) (See [ADH, page 2391]). In [DH1, Lemma 3.3], for all \( x, y \in \mathbb{R}^N \) and \( t > 0 \),

\[
T_j h_t(x, y) = \frac{y_j - x_j}{2t} h_t(x, y).
\]
We write the Riesz transforms as follows:

$$R_j f(x) = \int_{\mathbb{R}^N} R_j(x, y) f(y) d\omega(y),$$

then the kernel $R_j(x, y)$ satisfies the following smoothness condition (1.4)-(1.6).

**Proof of Theorem 1.1.** To estimate the kernel $R_j(x, y)$, we recall the following estimates for the Dunkl-heat kernel given in [DH1, Theorem 3.1]

(a) There are constants $C, c > 0$ such that

$$|h_t(x, y)| \leq C \frac{1}{V(x, y, \sqrt{t})} \left(1 + \frac{||x - y||}{\sqrt{t}}\right)^{-2} e^{-cd(x, y)^2/t},$$

for every $t > 0$ and for every $x, y \in \mathbb{R}^N$.

(b) There are constants $C, c > 0$ such that

$$|h_t(x, y) - h(x, y')| \leq C \left(\frac{||y - y'||}{\sqrt{t}}\right) \frac{1}{V(x, y, \sqrt{t})} \left(1 + \frac{||x - y||}{\sqrt{t}}\right)^{-2} e^{-cd(x, y)^2/t},$$

for every $t > 0$ and for every $x, y, y' \in \mathbb{R}^N$ such that $||y - y'|| < \sqrt{t}$.

We now estimate the kernel $R_j(x, y)$ as follows.

$$|R_j(x, y)| \lesssim |y_j - x_j| \int_0^\infty \frac{1}{V(x, y, \sqrt{t})} \frac{t}{||x - y||^2} e^{-cd(x, y)^2/t} \frac{dt}{t \sqrt{t}}$$

$$\leq \frac{1}{||x - y||} \left(\int_0^{d(x, y)^2} + \int_0^\infty \frac{1}{V(x, y, \sqrt{t})} e^{-cd(x, y)^2/t} \frac{dt}{\sqrt{t}}\right)$$

$$=: I_1 + I_2.$$

For $t \leq d(x, y)^2$, by using the doubling condition we have that

$$\omega(B(x, d(x, y))) \lesssim \left(\frac{d(x, y)}{\sqrt{t}}\right)^N \omega(B(x, \sqrt{t}))$$

and hence

$$V(x, y, \sqrt{t})^{-1} \lesssim \frac{1}{\omega(B(x, \sqrt{t}))} \lesssim \left(\frac{d(x, y)}{\sqrt{t}}\right)^N \frac{1}{\omega(B(x, d(x, y)))}.$$

We obtain

$$I_1 \lesssim \frac{1}{||x - y||} \frac{1}{\omega(B(x, d(x, y)))} \int_0^{d(x, y)^2} \left(\frac{d(x, y)}{\sqrt{t}}\right)^N e^{-cd(x, y)^2/t} \frac{dt}{\sqrt{t}}$$

$$\lesssim \frac{1}{||x - y||} \frac{1}{\omega(B(x, d(x, y)))} \int_0^{d(x, y)^2} \frac{d(x, y)^N}{t^{\frac{N}{2}}} \left(\frac{t}{d(x, y)^2}\right)^{\frac{N}{2}} dt$$

$$\lesssim \frac{1}{||x - y||} \frac{1}{\omega(B(x, d(x, y)))}.$$

It is clear that for $t \geq d(x, y)^2$, by using the reversed doubling condition,

$$\left(\frac{\sqrt{t}}{d(x, y)}\right)^N \omega(B(x, d(x, y))) \lesssim C \omega(B(x, \sqrt{t})).$$
we get
\[ I_2 \lesssim \frac{1}{\|x-y\|} \int_0^\infty \frac{1}{V(x, y, d(x, y))} \frac{d(x, y)^N}{t^{1+N}} \frac{dt}{t^\frac{1}{2}} \]
\[ \lesssim \frac{d(x, y)}{\|x-y\| \omega(B(x, d(x, y)))}. \]

To see the smoothness estimates, we write
\[ |R_j(x, y) - R_j(x, y')| \leq C|y_j - y_j'| \int_0^\infty |h_t(x, y)| \frac{dt}{t^{\sqrt{t}}} + |y_j' - x_j| \int_0^\infty |h_t(x, y) - h_t(x, y')| \frac{dt}{t^{\sqrt{t}}}. \]

By the above method,
\[ |y_j - y_j'| \int_0^\infty |h_t(x, y) - h_t(x, y')| \frac{dt}{t^{\sqrt{t}}} \lesssim \frac{|y - y'|}{\|x - y\| \omega(B(x, d(x, y)))} \]
\[ \text{for} \quad \|y - y'\| < \frac{1}{2}d(x, y), \]
\[ \text{it suffices to show that} \]
\[ |y_j' - x_j| \int_0^\infty |h_t(x, y) - h_t(x, y')| \frac{dt}{t^{\sqrt{t}}} \lesssim \frac{|y - y'|}{\|x - y\| \omega(B(x, d(x, y)))} \quad \text{for} \quad \|y - y'\| < \frac{1}{2}d(x, y). \]

If \( \|y - y'\| < \frac{1}{2}d(x, y) \), then \( |y_j' - x_j| \leq \frac{3}{2} |x - y| \). Hence
\[ |y_j' - x_j| \int_0^\infty |h_t(x, y) - h_t(x, y')| \frac{dt}{t^{\sqrt{t}}} \]
\[ \leq C\|x - y\| \int_0^\infty |h_t(x, y) - h_t(x, y')| \frac{dt}{t^{\sqrt{t}}} \]
\[ \leq C\|x - y\| \left( \int_0^{d(x, y)^2} + \int_0^\infty \right) |h_t(x, y) - h_t(x, y')| \frac{dt}{t^{\sqrt{t}}} \]
\[ =: II_1 + II_2. \]

Note that if \( \|y - y'\| < \sqrt{t} \), then the above condition (b) gives
\[ |h_t(x, y) - h(x, y')| \leq C \left( \frac{\|y - y'\|}{\sqrt{t}} \right) \frac{1}{V(x, y, \sqrt{t})} \left( 1 + \frac{\|x - y\|}{\sqrt{t}} \right)^{-2} e^{-cd(x, y)^2/t}. \]

If \( \|y - y'\| \geq \sqrt{t} \), then
\[ |h_t(x, y) - h(x, y')| \leq \left( \frac{\|y - y'\|}{\sqrt{t}} \right) (|h_t(x, y)| + |h(x, y)|). \]

Since \( \|y - y'\| < \frac{1}{2}d(x, y) \), we have \( d(x, y) \approx d(x, y') \) and \( \|x - y\| \approx \|x - y'\| \) and thus
\[ II_1 \lesssim \|y - y'\| \|x - y\| \int_0^{d(x, y)^2} \frac{1}{V(x, y, \sqrt{t})} \frac{t}{\|x - y\|^2} e^{-cd(x, y)^2/t} \frac{dt}{t^2} \]
\[ \lesssim \|y - y'\| \frac{1}{\|x - y\| \omega(B(x, d(x, y)))} \int_0^{d(x, y)^2} \left( \frac{d(x, y)}{\sqrt{t}} \right)^N e^{-cd(x, y)^2/t} \frac{dt}{t}. \]
To estimate $II$, we have $\|y - y'\| < \frac{1}{2}d(x, y) < \sqrt{t}$ and the above condition (b) gives

$$II \lesssim \frac{\|y - y'\|}{\|x - y\|} \int_0^\infty \frac{1}{V(x, y, \sqrt{t})} e^{-cd(x, y)^2/t} \frac{dt}{t}.$$  

The estimate of the smoothness for $x$ variable is similar. The proof of Theorem 1.1 is complete.

We now prove the pointwise lower bounded of $R_j(x, y)$.

**Proof of Theorem 1.2** Let $B = B(x_0, r)$. We choose $\tilde{B} = B(y_0, r)$ with $\|x_0 - y_0\| = 5r$ and satisfy that $y_j - x_j \geq r$ and $\|x - y\| \approx r$ for $x \in B$ and $y \in \tilde{B}$. Note that

$$R_j(x, y) = -C \int_0^\infty \frac{y_j - x_j}{t} h_t(x, y) \frac{dt}{\sqrt{t}}.$$  

It is clear that

$$\int_0^\infty \frac{1}{t} h_t(x, y) \frac{dt}{\sqrt{t}} \geq \int_0^\infty \frac{1}{t} \min\{\omega(B(x, \sqrt{t})), \omega(B(y, \sqrt{t}))\} e^{-c\|x - y\|^2/t} \frac{dt}{\sqrt{t}}$$

$$= \left( \int_0^\infty e^{s} + \int_0^\infty e^{s} \min\{\omega(B(x, \sqrt{t})), \omega(B(y, \sqrt{t}))\} e^{-c\|x - y\|^2/t} \frac{dt}{\sqrt{t}} \right)$$

$$=: A_1 + A_2.$$  

To estimate $A_1$, we use $s = \|x - y\|^2/t$ to get

$$A_1 \geq \frac{1}{\omega(B(x, \|x - y\|))} \int_0^\infty e^{-c\|x - y\|^2/t} \frac{dt}{t}$$

$$= \frac{1}{\omega(B(x, \|x - y\|))} \|x - y\|^{-1} \int_1^\infty e^{-s} \frac{ds}{\sqrt{s}}.$$  

To estimate $A_2$, we use doubling condition to give

$$\omega(x, \sqrt{t}) \leq \frac{t^{N/2}}{\|x - y\|^N} \omega(x, \|x - y\|)$$  

and hence

$$A_2 \geq \frac{1}{\omega(B(x, \|x - y\|))} \int_0^\infty \frac{\|x - y\|^N}{t^{N/2}} e^{-c\|x - y\|^2/t} \frac{dt}{t\sqrt{t}}$$

$$= \frac{1}{\omega(B(x, \|x - y\|))} \|x - y\|^{-1} \int_0^1 e^{-s} s^{N+1/2} ds.$$
For \((x, y) \in B \times \tilde{B}\), we obtain that
\[
|R_j(x, y)| = C \left| \int_0^\infty \frac{y_j - x_j}{t} b_t(x, y) \frac{dt}{\sqrt{t}} \right| \gtrsim \frac{1}{\omega(B(x, \|x - y\|))} \gtrsim \frac{1}{\omega(B(x, r))} \gtrsim \frac{1}{\omega(B(x_0, r))}.
\]
The proof is completed. \(\square\)

3. Proof of Theorem 1.3: Upper bound of commutator

The maximal function \(Mf\) is defined as
\[
Mf(x) = \sup_{x \in B} \frac{1}{\omega(B)} \int_B |f(y)|d\omega(y).
\]
The sharp function \(f^{\sharp}\) is defined as
\[
f^{\sharp}(x) = \sup_{x \in B} \frac{1}{\omega(B)} \int_B |f(y) - f_B|d\omega(y),
\]
where \(f_B\) is defined in (1.7).

Proof of Theorem 1.3: upper bound of commutator. Suppose \(b \in BMO_d\), \(1 < p < \infty\) and \(f\) in \(L^p(\mathbb{R}^N, d\omega)\).

For any \(x \in \mathbb{R}^N\) and for any ball \(B = B(x_0, r) \subset \mathbb{R}^N\) containing \(x\), we set \(f = f_1 + f_2\) with \(f_1 = f \cdot \chi_{O(5B)}\).

Then for any \(y \in B\), we have that
\[
[b, R_j](f)(y) = b(y)R_j(f)(y) - R_j(bf)(y) = (b(y) - b\chi_{O(B)})R_j(f)(y) - R_j((b - b\chi_{O(B)})f)(y) = (b(y) - b\chi_{O(B)})R_j(f)(y) - R_j((b - b\chi_{O(B)})f_1)(y) - R_j((b - b\chi_{O(B)})f_2)(y) =: I(y) + \Pi(y) + \Pi(y).
\]

For \(I(y)\) we have that
\[
\frac{1}{\omega(B)} \int_B |I(y) - I_B|d\omega(y) \leq \frac{2}{\omega(B)} \int_B |I(y)|d\omega(y) = \frac{2}{\omega(B)} \int_B \left| (b(y) - b\chi_{O(B)})R_j(f)(y) \right|d\omega(y) \leq 2 \left( \frac{1}{\omega(B)} \int_B \left| (b(y) - b\chi_{O(B)}) \right|^{s'}d\omega(y) \right)^{\frac{1}{s'}} \left( \frac{1}{\omega(B)} \int_B |R_j(f)(y)|^{s}d\omega(y) \right)^{\frac{1}{s}} \leq C \left( \frac{1}{\omega(O(B))} \int_{\chi_{O(B)}} \left| (b(y) - b\chi_{O(B)}) \right|^{s'}d\omega(y) \right)^{\frac{1}{s'}} \left( \frac{1}{\omega(B)} \int_B |R_j(f)(y)|^{s}d\omega(y) \right)^{\frac{1}{s}} \leq C \|b\|_d \left( M(|R_j f|^{s})(x) \right)^{\frac{1}{s}},
\]
where \(s\) is chosen to satisfy \(1 < s < p < \infty\) and \(s'\) is the conjugate of \(s\).

For \(\Pi(y)\), since \(R_j\) is bounded on \(L^q(\mathbb{R}^N, d\omega)\), \(1 < q < \infty\), we have
\[
\frac{1}{\omega(B)} \int_B |\Pi(y) - \Pi_B|d\omega(y) \leq C \|b\|_d \left( M(|R_j f|^{s})(x) \right)^{\frac{1}{s}}.
\]
\[
\leq \frac{2}{\omega(B)} \int_B |\Pi(y)| d\omega(y) \\
= \frac{2}{\omega(B)} \int_B |R_j((b - b_{\mathcal{O}(B)})f_1)(y)| d\omega(y) \\
\lesssim \left( \frac{1}{\omega(B)} \int_{\mathcal{O}(5B)} |b(y) - b_{\mathcal{O}(B)}|^q |f(y)|^q d\omega(y) \right)^{1/q} \\
\lesssim \left( \frac{1}{\omega(B)} \int_{\mathcal{O}(5B)} |b(y) - b_{\mathcal{O}(B)}|^q |f(y)|^q d\omega(y) \right)^{1/q} \\
\lesssim \left( \frac{1}{\omega(B)} \int_{\mathcal{O}(5B)} |b(y) - b_{\mathcal{O}(B)}|^q |f(y)|^q d\omega(y) \right)^{1/q} \\
\lesssim \|b\|_d \left( M(|f|^p)(x) \right)^{1/p},
\]
where we have chosen \( q, v \in (1, \infty) \) such that \( 1 < qv < p < \infty \) and have set \( \beta := qv \).

Finally, we turn our attention to term \( \mathbf{III}(y) \). For \( \omega \in \mathbb{R}^N \setminus \mathcal{O}(5B) \), it is clear that for \( y \in B, \|x_0 - y\| \leq \frac{1}{2} d(x_0, w) \). Since \( \omega(B(w, d(w, x_0))) \approx \omega(B(x_0, d(w, x_0))) \), we have

\[
|\mathbf{III}(y) - \mathbf{III}(x_0)| = |R_j((b - b_{\mathcal{O}(B)}f_2)(y) - R_j((b - b_{\mathcal{O}(B)}f_2)(x_0)|
\lesssim \int_{\mathbb{R}^N \setminus \mathcal{O}(5B)} |R_j(w, y) - R_j(w, x_0)||b(w) - b_{\mathcal{O}(B)}||f(w)| d\omega(w)
\lesssim \int_{\mathbb{R}^N \setminus \mathcal{O}(5B)} \frac{\|y - x_0\|}{\omega(B(w, d(w, x_0)))} |b(w) - b_{\mathcal{O}(B)}||f(w)| d\omega(w)
\lesssim r \left( \int_{\mathbb{R}^N \setminus \mathcal{O}(5B)} \frac{1}{d(w, x_0)} \frac{1}{\omega(B(x_0, d(w, x_0)))} |b(w) - b_{\mathcal{O}(B)}|^s d\omega(w) \right)^{1/s}
\times \left( \int_{\mathbb{R}^N \setminus \mathcal{O}(5B)} \frac{1}{d(w, x_0)} \frac{1}{\omega(B(x_0, d(w, x_0)))} |f(w)|^s d\omega(w) \right)^{1/s},
\]
where \( 1 < s < p < \infty \). Hence,

\[
\int_{\mathbb{R}^N \setminus \mathcal{O}(5B)} \frac{1}{d(x_0, w)} \frac{1}{\omega(B(x_0, d(w, x_0)))} |f(w)|^s d\omega(w)
\lesssim \sum_{j=0}^{\infty} \int_{2^{j+1}r \leq d(w, x_0) \leq 2^{j+2}r} \frac{1}{d(w, x_0)} \frac{1}{\omega(B(x_0, d(w, x_0)))} |f(w)|^s d\omega(w)
\lesssim \sum_{j=0}^{\infty} 2^{-j} r^{-1} \frac{\omega(B(x_0, 2^j r))}{\omega(B(x_0, 2^{j+1} r))} \int_{d(w, x_0) \leq 2^{j+1} r} |f(w)|^s d\omega(w)
\lesssim r^{-1} M_d(|f|^s)(x).
\]
Similarly, by the John-Nirenberg inequality, we have

\[
\int_{\mathbb{R}^N \setminus \mathcal{O}(5B)} \frac{1}{d(x_0, w)} \frac{1}{\omega(B(x_0, d(w, x_0)))} |b(w) - b_{\mathcal{O}(5B)}|^s d\omega(w).
\]
The upper bound of commutator is complete.

Thus,

\[ \int_{d(w, x_0) \leq 2^{j+1+5r}} |b(w) - b_{\mathcal{D}(5B)}|^\gamma d\omega(w) \]

\[ \lesssim r^{-1} \|b\|_d^\gamma. \]

Therefore,

\[ \int_B \|\mathbb{I}(y) - \mathbb{I}(x_0)\| d\omega(y) \lesssim \|b\|_d \left( M_d(|f|^\gamma)(x) \right)^{\frac{1}{\gamma}}. \]

By the above estimates we obtain that

\[ |(b, R_j f)^2(x)| \lesssim \|b\|_d \left( M(|R_j f|^\gamma)(x) \right)^{\frac{1}{\gamma}} + \left( M(|f|^\gamma)(x) \right)^{\frac{1}{\gamma}} + \left( M_d(|f|^\gamma)(x) \right)^{\frac{1}{\gamma}}. \]

Since \(M, M_d\) and \(R_j\) are bounded on \(L^p(\mathbb{R}^N, d\omega)\), we obtain

\[ \|b, R_j f\|_{L^p(\mathbb{R}^N, d\omega)} \lesssim \|b\|_d \|f\|_{L^p(\mathbb{R}^N, d\omega)}. \]

The upper bound of commutator is complete.

\[ \square \]

4. Proof of Theorem 1.3 Lower bound of commutator

In this section, we want to prove the lower bound of commutator \([b, R_j]\).

**Definition 4.1.** Let \(f\) be finite almost everywhere on \(\mathbb{R}^N\). For \(B \subseteq \mathbb{R}^N\) with \(\omega(B) < \infty\), we define a median value \(m_f(B)\) of \(f\) over \(B\) to be a real number satisfying

\[ \omega\{x \in B : f(x) > m_f(B)\} \leq \frac{1}{2}\omega(B) \quad \text{and} \quad \omega\{x \in B : f(x) < m_f(B)\} \leq \frac{1}{2}\omega(B). \]

**Proof of Theorem 1.3 lower bound of commutator.** For given \(b \in L^1_{\text{loc}}(\mathbb{R}^N, d\omega)\) and for any ball \(B\), let \(\Omega_N(b, B)\) be the oscillation defined by

\[ \Omega_N(b, B) := \frac{1}{\omega(B)} \int_B |b(x) - b_B| d\omega(x), \]

where \(b_B\) is the average value of \(b\) in \(B\). Under the assumption of Theorem 1.3, we will show that for any ball \(B\),

\[ |\Omega_N(b, B)| \lesssim 1. \]

Let \(B = B(x_0, r)\) with \(x_0 \in \mathbb{R}^N\) and \(r > 0\). Note that

\[ [b, R_j]f(x) = b(x)R_j f(x) - R_j(bf)(x) \]

\[ = \int_{\mathbb{R}^N} (b(x) - b(y))R_j(x, y) f(y) d\omega(y), \]

where

\[ R_j(x, y) = -c \int_0^\infty \frac{y_j - x_j}{t} h_t(x, y) \frac{dt}{\sqrt{t}}. \]
We choose \( \tilde{B} = B(\bar{x}_0, r) \) such that \( y_j - x_j \geq r \) and \( \| x - y \| \approx r \) for \( x \in B \) and \( y \in \tilde{B} \). Then based on Definition 1.1, we now choose two measurable sets

\[
E_1 \subset \{ y \in \tilde{B} : b(y) < m_b(\tilde{B}) \} \quad \text{and} \quad E_2 \subset \{ y \in \tilde{B} : b(y) \geq m_b(\tilde{B}) \}
\]

such that \( \omega(E_i) = \frac{1}{2} \omega(\tilde{B}), i = 1, 2 \), and that \( E_1 \cup E_2 = \tilde{B}, E_1 \cap E_2 = \emptyset \).

Moreover, we define

\[
B_i := \{ x \in B : b(x) \geq m_b(\tilde{B}) \} \quad \text{and} \quad B_2 := \{ x \in B : b(x) \leq m_b(\tilde{B}) \}.
\]

Now based on the definition of \( E_i \) and \( B_i \), for \( (x, y) \in B_i \times E_i, i = 1, 2 \), we have

\[
|b(x) - b(y)| = |b(x) - m_b(\tilde{B}) + m_b(\tilde{B}) - b(y)|
= |b(x) - m_b(\tilde{B})| + |m_b(\tilde{B}) - b(y)| \geq |b(x) - m_b(\tilde{B})|.
\]

Hence, we have the following facts.

\[
\begin{align*}
(i) & \quad B = B_1 \cup B_2, \tilde{B} = E_1 \cup E_2 \quad \text{and} \quad \omega(E_i) \geq \frac{1}{2} \omega(\tilde{B}), i = 1, 2; \\
(ii) & \quad b(x) - b(y) \text{ does not change sign for all } (x, y) \in B_i \times E_i, i = 1, 2; \\
(iii) & \quad |b(x) - m_b(\tilde{B})| \leq |b(x) - b(y)| \text{ for all } (x, y) \in B_i \times E_i, i = 1, 2.
\end{align*}
\]

By Theorem 1.2 we obtain that, for \( (x, y) \in B_i \times E_i, i = 1, 2 \),

\[
|R_j(x, y)| \geq \frac{1}{\omega(B(x_0, r))}.
\]

Let \( f_i = \chi_{E_i}, i = 1, 2 \). Then the facts \((4.2)\) give

\[
\frac{1}{\omega(B)} \sum_{i=1}^{2} \int_B |[b, R_j]f_i(x)|d\omega(x) \geq \frac{1}{\omega(B)} \sum_{i=1}^{2} \int_{B_i} |[b, R_j]f_i(x)|d\omega(x)
= \frac{1}{\omega(B)} \sum_{i=1}^{2} \int_{B_i} \int_{E_i} |b(x) - b(y)|R_j(x, y)|d\omega(y)d\omega(x)
\geq \frac{1}{\omega(B)} \sum_{i=1}^{2} \int_{B_i} |b(x) - m_b(\tilde{B})| \frac{1}{\omega(B(x_0, r))} \int_{E_i} d\omega(y)d\omega(x)
\geq \frac{1}{\omega(B)} \sum_{i=1}^{2} \int_{B_i} |b(x) - m_b(\tilde{B})|d\omega(x)
\geq |\Omega_N(b, B)|.
\]

On the other hand, from Hölder’s inequality and the boundedness of \([b, R_j]\), we deduce that

\[
\frac{1}{\omega(B)} \sum_{i=1}^{2} \int_B |[b, R_j]f_i(x)|d\omega(x) \lesssim \frac{1}{\omega(B)} \sum_{i=1}^{2} \left( \int_B |[b, R_j]f_i(x)|^p d\omega(x) \right)^{1/p} \omega(B)^{1/p'}
\lesssim \frac{1}{\omega(B)} \sum_{i=1}^{2} \|[b, R_j]\|_{L^p(\mathbb{R}^N, \omega) \rightarrow L^p(\mathbb{R}^N, \omega)} \omega(E_i)^{1/p'} \omega(B)^{1/p'}.
\]
Since \( \|x - y\| \approx r \) for \( x \in B \) and \( y \in \bar{B} \), we have \( \omega(\bar{B}) \lesssim \omega(B) \) and then
\[
\frac{1}{\omega(B)} \sum_{i=1}^{2} \int_B \|[b, R_j]|f_i(x)|d\omega(x) \lesssim \|[b, R_j]\|_{L^p(\Omega^N, \omega) \to L^p(\Omega^N, \omega)}.
\]
Therefore,
\[
|\Omega_N(b, B)| \lesssim \|[b, R_j]\|_{L^p(\Omega^N, \omega) \to L^p(\Omega^N, \omega)}.
\]
The proof is complete. \( \square \)

5. Proof of Theorem 1.4: The compactness of \([b, R_j]\)

It follows from [CDLV] that the VMO\(_d(\mathbb{R}^N)\) are equivalent the BMO\(_d\)-closure of the set \( \Lambda_{d,0}(\mathbb{R}^N) \) of \( \Lambda_d(\mathbb{R}^N) \), the Lipschitz function space on space of homogeneous type \((\mathbb{R}^N, d, d\omega)\), with the compact support.

Sufficiency:

A set \( S \) is precompact if its closure is compact. A common way to check precompactness is to use the Riesz–Kolmogorov theorem [GM, Theorem 1], which we recall in below.

Theorem 5.1 (GM). (Riesz–Kolmogorov theorem) Let \( \mu \) be a doubling measure such that
\[
h(r) := \inf\{\mu(B(x, r)) : x \in X\} > 0 \quad \text{for each } r > 0
\]
and assume \( 1 < p < \infty \). Let \( x_0 \in X \), then the subset \( E \) of \( L^p(X, \mu) \) is relatively compact if and only if the following conditions are satisfied:
(a) \( E \) is bounded;
(b) \[
\lim_{R \to \infty} \int_{X \setminus B(x_0, R)} |f(x)|^p d\mu(x) = 0 \quad \text{uniformly for } f \in E;
\]
(c) \[
\lim_{r \to 0} \int_X |f(x) - f_{B(x, r)}|^p d\mu(x) = 0 \quad \text{uniformly for } f \in E.
\]

Let \( X = \mathbb{R}^N \) and \( \mu = \omega \). Then \((\mathbb{R}^N, \| \cdot \|, \omega)\) is metric space with doubling measure. Note that
\[
\omega(B(x, r)) \sim r^N \prod_{\alpha \in R} (|\langle \alpha, x \rangle| + r)^{\kappa(\alpha)} \geq r^N.
\]
Thus, we see that \( \inf\{\omega(B(x, r)) : x \in \mathbb{R}^N\} > 0 \) for each \( r > 0 \).

We first show that when \( b \in \text{VMO}_d(\mathbb{R}^N) \), the commutator \([b, R_j]\) is compact on \( L^p(\mathbb{R}^N) \). By a density argument, it suffices to show that \([b, R_j]\) is a compact operator for \( b \in \Lambda_{d,0}(\mathbb{R}^N) \).

For \( b \in \Lambda_{d,0}(\mathbb{R}^N) \), to show \([b, R_j]\) is compact on \( L^p(\mathbb{R}^N) \), it suffices to show that for every bounded subset \( E \subset L^p(\mathbb{R}^N) \), the set \([b, R_j]E\) is precompact. Thus, we only need to show that \([b, R_j]E\) satisfies the hypotheses (a)–(c) in the Riesz–Kolmogorov Theorem (Theorem 5.1). We first point out that by Theorem 1.3 and the fact that \( b \in \text{BMO}_d(\mathbb{R}^N) \), \([b, R_j]\) is bounded on \( L^p(\mathbb{R}^N) \), which implies that \([b, R_j]E\) satisfies hypothesis (a) in Theorem 5.1.

Next, we will show that \([b, R_j]E\) satisfies hypothesis (b) in Theorem 5.1. We may assume that \( b \in \Lambda_{d,0}(\mathbb{R}^N) \) with \( \text{supp}\ b \subset \mathcal{O}(B(0, R)) \). For \( t > 2 \), set \( K^c := \{x \in \mathbb{R}^N : d(x, 0) > tR\} \).
There exists an increasing function $\phi$ such that $\{x \in \mathbb{R}^N : d(x,0) \leq tR\} \subseteq B(0, \phi(tR))$. Then we have
\[
\|b, R_j\|_{L^p(B(0,\phi(tR))c,d\omega)} \leq \|b, R_j\|_{L^p(K^c,d\omega)} = \|b R_j(f) - R_j(bf)\|_{L^p(K^c,d\omega)} \\
\leq \|b R_j(f)\|_{L^p(K^c,d\omega)} + \|R_j(bf)\|_{L^p(K^c,d\omega)}.
\]
Since $\text{supp } b \cap K^c = \emptyset$, we have
\[
\int_{d(x,0) > tR} |b R_j(f)(x)|^p \, d\omega(x) = 0,
\]
and so
\[
\|b, R_j\|_{L^p(K^c,d\omega)} \leq \|R_j(bf)\|_{L^p(K^c,d\omega)}.
\]
Using the size condition of $R_j(x, y)$ and the fact that $\text{supp } b \subset O(B(0, R))$ we have
\[
|R_j(bf)(x)| \leq \int_{d(y,0) < R} |R_j(x, y)||b(y)||f(y)| \, d\omega(y)
\]
\[
\leq \int_{d(y,0) < R} \frac{1}{\omega(B(x,d(x,y)))} |b(y)||f(y)| \, d\omega(y).
\]
(5.1)
Notice that for $d(x,0) > tR$, $t > 2$ and $d(y,0) < R$ we have $d(x,y) > d(x,0) - d(y,0) > d(x,0)/2$. Using this and Hölder’s inequality, inequality (5.2) yields
\[
|R_j(bf)(x)| \leq C \frac{1}{\omega(B(x,d(x,0)))} \int_{d(y,0) < R} |b(y)||f(y)| \, d\omega(y)
\]
\[
\leq C \frac{1}{\omega(B(x,d(x,0)))} \left( \int_{d(y,0) < R} |b(y)|^p \, d\omega(y) \right)^{1/p'} \left( \int_{d(y,0) < R} |f(y)|^p \, d\omega(y) \right)^{1/p}
\]
\[
\leq C \frac{1}{\omega(B(0,d(x,0)))} \|b\|_{L^\infty(\mathbb{R}^N)} \|f\|_{L^p(\mathbb{R}^N,d\omega)} \omega(O(B(0,R)))^{1/p'},
\]
since $b \in L_{d,0}(\mathbb{R}^N)$ and $\omega(B(x,d(x,0))) \approx \omega(B(0,d(x,0)))$. Using this estimate of $|R_j(bf)(x)|$, (5.1) becomes
\[
\|b, R_j\|_{L^p(K^c,d\omega)}
\]
\[
\leq C \omega(B(0,R))^{1/p'} \|b\|_{L^\infty(\mathbb{R}^N)} \|f\|_{L^p(\mathbb{R}^N,d\omega)} \left( \int_{d(x,0) > tR} \frac{1}{\omega(B(0,d(x,0)))^p} \, d\omega(x) \right)^{1/p}
\]
\[
\leq C \omega(B(0,R))^{1/p'} \|b\|_{L^\infty(\mathbb{R}^N)} \|f\|_{L^p(\mathbb{R}^N,d\omega)} \left( \sum_{j=0}^{\infty} \int_{2^j tR < d(x,0) < 2^{j+1} tR} \frac{1}{\omega(B(0,d(x,0)))^p} \, d\omega(x) \right)^{1/p}
\]
\[
\leq C \omega(B(0,R))^{1/p'} \|b\|_{L^\infty(\mathbb{R}^N)} \|f\|_{L^p(\mathbb{R}^N,d\omega)} \sum_{j=0}^{\infty} \omega(B(0,2^{j+1} tR))^{1/p} \omega(B(0,2^j tR))^{1/p'}
\]
\[
\leq C \|b\|_{L^\infty(\mathbb{R}^N)} \|f\|_{L^p(\mathbb{R}^N,d\omega)} 2^{N/p} t^{-N/p'} \sum_{j=1}^{\infty} 2^{-Nj/p'},
\]
where the last inequality follows from (2.2). Finally, given each $\varepsilon > 0$, we can choose $t$ large enough such that $C 2^{N/p} t^{-N/p'} \sum_{j=1}^{\infty} 2^{-Nj/p'} < \varepsilon$. Here the constant $C$ depends on $b$ and on the bound on $\|f\|_{L^p(\mathbb{R}^N,d\omega)}$ for $f \in E$. Hence hypothesis (b) in Theorem 5.1 holds for $[b, R_j] E$. 
It remains to prove that \([b, R_j]E\) also satisfies hypothesis (c) of Theorem 5.1. Let \(\varepsilon\) be a fixed positive constant in \((0, 1/8)\). Since \(b \in \Lambda_{d,0}(\mathbb{R}^N)\), \(b\) is uniformly continuous. Choose \(r = r(b, \varepsilon)\) sufficiently small that for all \(z \in B(x, r)\), we have both \(\|x - z\| < \varepsilon^2\) and for all \(x \in \mathbb{R}^N\), \(|b(x) - b(z)| < \varepsilon\). Fix \(z \in B(x, r)\). Then for all \(x \in \mathbb{R}^N\),

\[
[b, R_j]f(x) - ([b, R_j]f)_{B(x,r)} = \frac{1}{\omega(B(x, r))} \int_{B(x,r)} ([b, R_j]f(x) - [b, R_j]f(z)) d\omega(z).
\]

Note that

\[
[b, R_j]f(x) - [b, R_j]f(z) = \int_{\mathbb{R}^N} R_j(x, y)[b(x) - b(y)] f(y) d\omega(y) - \int_{\mathbb{R}^N} R_j(z, y)[b(z) - b(y)] f(y) d\omega(y)
\]

\[
= \int_{d(x,y) > \varepsilon^{-1} \|x - z\|} R_j(x, y)[b(x) - b(z)] f(y) d\omega(y)
\]

\[
+ \sum_{j=0}^{\infty} \int_{d(x,y) < 2^j \varepsilon^{-1} \|x - z\|} R_j(x, y)[b(x) - b(y)] f(y) d\omega(y)
\]

\[
= \sum_{i=1}^{4} L_i(x, z).
\]

We begin with estimating \(L_2\). Since \(\varepsilon \in (0, 1/2)\), it follows that

\[
d(x,y) > \varepsilon^{-1} \|x - z\| \Rightarrow \|x - z\| < \frac{d(x,y)}{2}.
\]

Thus we may apply the smoothness condition of the kernel \(R_j(x, y)\) (as in Theorem 1.1), concluding that

\[
|R_j(x, y) - R_j(z, y)| \leq \frac{1}{\omega(B(x, d(x,y)))} \leq \frac{1}{\omega(B(x, d(x,y)))}.
\]

Using this inequality, together with the fact that \(b \in \Lambda_{d,0}(\mathbb{R}^N)\), we have

\[
|L_2(x, z)| \leq \|x - z\| \int_{d(x,y) > \varepsilon^{-1} \|x - z\|} \frac{|f(y)|}{d(x,y)\omega(B(x, d(x,y)))} d\omega(y)
\]

\[
\leq \|x - z\| \sum_{j=0}^{\infty} \int_{2^j \varepsilon^{-1} \|x - z\|}^{2^{j+1} \varepsilon^{-1} \|x - z\|} \frac{|f(y)|}{d(x,y)\omega(B(x, d(x,y)))} d\omega(y)
\]

\[
\leq \varepsilon \sum_{j=0}^{\infty} 2^{-j} \frac{1}{\omega(B(x, 2^j \varepsilon^{-1} \|x - z\|))} \int_{d(x,y) < 2^{j+1} \varepsilon^{-1} \|x - z\|} |f(y)| d\omega(y)
\]

\[
\leq C \varepsilon \sum_{j=0}^{\infty} 2^{-j} M_\omega(f)(x)
\]

\[
\leq C \varepsilon M_\omega(f)(x),
\]
where $M_\omega$ is the Hardy–Littlewood maximal operator on $(\mathbb{R}^N, d, \omega)$. Hence,

$$
\frac{1}{\omega(B(x, r))} \int_{B(x, r)} |L_2(x, z)| d\omega(z) \leq C \varepsilon M_\omega(f)(x).
$$

This further gives

$$
\begin{align*}
\int_{\mathbb{R}^N} \left| \frac{1}{\omega(B(x, r))} \int_{B(x, r)} |L_2(x, z)| d\omega(z) \right|^p d\omega(x) \\
&\leq C \varepsilon^p \int_{\mathbb{R}^N} |M_\omega(f)(x)|^p d\omega(x) \\
&\leq C \varepsilon^p \|f\|_{L^p(\mathbb{R}^N, d\omega)}^p.
\end{align*}
$$

(5.3)

Turning to $L_3$, by the size condition of the kernel $R_j(x, y)$ (as in Theorem 1.1) and the fact that $b \in \Lambda_{d, 0}(\mathbb{R}^N)$, we conclude that

$$
|L_3(x, z)| \lesssim \int_{d(x, y) \leq \varepsilon^{-1} \|x - z\|} \frac{|f(y)| d(x, y)}{\omega(B(x, d(x, y)))} d\omega(y)
$$

$$
= \sum_{i=-\infty}^{-1} \int_{2^{i+1} \varepsilon^{-1} \|x - z\| < d(x, y) \leq 2^{i+1} \varepsilon^{-1} \|x - z\|} \frac{|f(y)| d(x, y)}{\omega(B(x, d(x, y)))} d\omega(y)
$$

$$
\lesssim \sum_{i=-\infty}^{-1} 2^{i+1} \varepsilon^{-1} \|x - z\| \frac{1}{\omega(B(x, 2^i \varepsilon^{-1} \|x - z\|))} \int_{d(x, y) \leq 2^{i+1} \varepsilon^{-1} \|x - z\|} |f(y)| d\omega(y)
$$

$$
\lesssim \varepsilon^{-1} \|x - z\| M_\omega(f)(x)
$$

By our choice of $z$,

$$
\begin{align*}
\int_{\mathbb{R}^N} \left| \frac{1}{\omega(B(x, r))} \int_{B(x, r)} |L_3(x, z)| d\omega(z) \right|^p d\omega(x) &\lesssim \varepsilon^p \|f\|_{L^p(\mathbb{R}^N, d\omega)}^p.
\end{align*}
$$

(5.4)

Note that $\|z - \sigma(y)\| \leq \|x - \sigma(y)\| + \|x - z\|$ for $\sigma \in G$ gives $d(z, y) \leq d(x, y) + \|x - z\|$. We have $d(z, y) \leq 2\varepsilon^{-1} \|x - z\|$ if $d(x, y) < \varepsilon^{-1} \|x - z\|$ and $0 < \varepsilon < 1/8$. Hence,

$$
|L_4(x, z)| \lesssim \int_{d(z, y) \leq 2\varepsilon^{-1} \|x - z\|} |R_j(z, y)| |b(z) - b(y)| |f(y)| d\omega(y)
$$

$$
\lesssim \int_{d(x, y) \leq 2\varepsilon^{-1} \|x - z\|} \frac{|f(y)| d(z, y)}{\omega(B(z, d(z, y)))} d\omega(y)
$$

$$
= \sum_{i=-\infty}^{-1} \int_{2^i \varepsilon^{-1} \|x - z\| < d(z, y) \leq 2^{i+1} \varepsilon^{-1} \|x - z\|} \frac{|f(y)| d(z, y)}{\omega(B(z, d(z, y)))} d\omega(y)
$$

$$
\lesssim \sum_{i=-\infty}^{0} 2^{i+1} \varepsilon^{-1} \|x - z\| \frac{1}{\omega(B(z, 2^i \varepsilon^{-1} \|x - z\|))} \int_{d(x, y) \leq 2^{i+1} \varepsilon^{-1} \|x - z\|} |f(y)| d\omega(y)
$$

$$
\lesssim \varepsilon^{-1} \|x - z\| M_\omega(f)(z)
$$

and then

$$
\frac{1}{\omega(B(x, r))} \int_{B(x, r)} |L_4(x, z)| d\omega(z) \leq \frac{\varepsilon}{\omega(B(x, r))} \int_{B(x, r)} M_\omega(f)(z) d\omega(z) \leq C \varepsilon M_{\text{non}}(M_\omega(f))(x),
$$
where $M_{non}$ is the non-central Hardy–Littlewood maximal operator on $(\mathbb{R}^N, \| \cdot \|, \omega)$. This implies that

\begin{equation}
\left( 5.5 \right) \int_{\mathbb{R}^N} \left| \frac{1}{\omega(B(x, r))} \int_{B(x, r)} |L_1(x, z)| d\omega(z) \right|^p d\omega(x) \lesssim \varepsilon^p \| f \|^p_{L^p(\mathbb{R}^N, d\omega)}.
\end{equation}

As the last step, we consider $L_1$:

$$|L_1(x, z)| \leq |b(x) - b(z)| \sup_{t > 0} \left| \int_{d(x,y)>t} R_j(x, y) f(y) d\omega(y) \right|.$$ Thanks to [THL, Theorem 1.3], we choose $0 < r < p$ such that

$$\sup_{t > 0} \left| \int_{d(x,y)>t} R_j(x, y) f(y) d\omega(y) \right| \lesssim M(|R_j(f)|^r)(x) \frac{1}{r} + \sum_{\sigma \in G} M(f(\sigma(x))).$$

Recall that $|b(x) - b(z)| < \varepsilon$ by our choice of $z$. Hence

\begin{equation}
\left( 5.6 \right) \int_{\mathbb{R}^N} \left| \frac{1}{\omega(B(x, r))} \int_{B(x, r)} |L_1(x, z)| d\omega(z) \right|^p d\omega(x) \\
\lesssim C_p \varepsilon^p \int_{\mathbb{R}^N} \left( M(|R_j(f)|^r)(x) \frac{1}{r} + M(f(\sigma(x))) \right) d\omega(x) \\
\lesssim C_p \varepsilon^p \int_{\mathbb{R}^N} \left( M(|R_j(f)|^r)(x) \frac{1}{r} + M(f(\sigma(x))) \right) d\omega(x) \\
\lesssim C_p \varepsilon^p \| f \|^p_{L^p(\mathbb{R}^N, d\omega)}.
\end{equation}

Combining the estimates (5.3)–(5.6) of $L_i$, $i \in \{1, 2, 3, 4\}$, we conclude that

$$\int_{\mathbb{R}^N} |P_i b, R_j | f(x) - |P_i b, R_j | f(z)|^p d\omega(x) \lesssim C_p \varepsilon^p \| f \|^p_{L^p(\mathbb{R}^N, d\omega)}.$$ This shows that $[b, R_j]E$ satisfies hypothesis (c) in Theorem 5.1. Hence, $[b, R_j]$ is a compact operator.

**Necessity:**

We start by assuming $b \in \text{BMO}_{Dunkl}(\mathbb{R}^N)$ is such that $[b, R_j]$ is compact from $L^p(\mathbb{R}^N, d\omega)$ to $L^p(\mathbb{R}^N, d\omega)$. We will use the method of proof by contradiction and hence let us suppose that $b \notin \text{VMO}_{Dunkl}(\mathbb{R}^N)$. Here we follow the main idea from [LL].

As we assume that $b \notin \text{VMO}_{Dunkl}(\mathbb{R}^N)$, at least one of the three conditions presented in the definition of $\text{VMO}_{Dunkl}(\mathbb{R}^N)$ fails to hold. Since a similar argument will work for all three conditions, let us suppose that the first condition does not hold.

That is, there exists some $\delta_0 > 0$ and a sequence of balls $\{Q_i\}_{i \in I} \subset \mathbb{R}^N$ such that $l(Q_i) \to 0$ as $i \to \infty$ and we have that

\begin{equation}
\left( 5.7 \right) \frac{1}{\omega(Q)} \int_Q |b(x) - b_Q| d\omega(x) \geq \delta_0.
\end{equation}

We will also further assume without loss of generality that

\begin{equation}
\left( 5.8 \right) 4l(Q_{j+1}) \leq l(Q_j).
\end{equation}
Note that
\[ R_h(x, y) = -c \int_0^\infty \frac{y_k - x_k}{t} h_t(x, y) \frac{dt}{\sqrt{t}} \]
and \( h_t(x, y) \) has lower bounded \( \frac{1}{\min(\omega(B(x, \sqrt{t})), \omega(B(y, \sqrt{t})))} e^{-c|y-x|^2/t} \). We choose \( \bar{Q}_j = Q_j(\bar{x}_0, r) \)
such that \( y_k - x_k \geq r \) and \( \|x - y\| \approx r \) for \( x \in Q_j \) and \( y \in \bar{Q}_j \). Let us denote by \( m_b(\bar{Q}_j) \) a median value of \( b \) on the ball \( \bar{Q}_j \). That is \( m_b(\bar{Q}_j) \) is a real number such that the two sets below have a measure at least \( \frac{1}{2} \omega(\bar{Q}_j) \)
\[(5.9) \quad F_{j,1} \subset \{y \in \bar{Q}_j : b(y) \leq m_b(\bar{Q}_j)\}, \quad F_{j,2} \subset \{y \in \bar{Q}_j : b(y) \geq m_b(\bar{Q}_j)\}.
\]
Also define the sets
\[(5.10) \quad E_{j,1} \subset \{x \in Q_j : b(x) \geq m_b(\bar{Q}_j)\}, \quad E_{j,2} \subset \{x \in Q_j : b(x) < m_b(\bar{Q}_j)\}.
\]
So we have that \( Q_j = E_{j,1} \cup E_{j,2} \) and \( E_{j,1} \cap E_{j,2} = \emptyset \) and we also have the following
\[ b(x) - b(y) \geq 0, \quad (x, y) \in E_{j,1} \times F_{j,1} \]
\[ b(x) - b(y) < 0, \quad (x, y) \in E_{j,2} \times F_{j,2}.
\]
For \( (x, y) \in E_{j,1} \times F_{j,1} \cup E_{j,1} \times F_{j,2} \), we have that
\[
|b(x) - b(y)| = |b(x) - m_b(\bar{Q}_j)| + |m_b(\bar{Q}_j) - b(y)| \geq |b(x) - m_b(\bar{Q}_j)|.
\]
Define the following sets
\[(5.11) \quad \tilde{F}_{j,1} := F_{j,1} \setminus \cup_{t=j+1}^\infty \tilde{Q}_t, \quad \tilde{F}_{j,2} := F_{j,2} \setminus \cup_{t=j+1}^\infty \tilde{Q}_t, \quad \forall j = 1, 2, \ldots.
\]
Now using the decay condition for the lengths of \( \{Q_j \} \) as given by (5.8), we have for each \( j \) the following
\[(5.12) \quad \omega(\tilde{F}_{j,1}) \geq \omega(F_{j,1}) - \omega(\cup_{t=j+1}^\infty \tilde{Q}_t) \geq \frac{1}{2} \omega(\bar{Q}_j) - \frac{1}{3} \omega(\bar{Q}_j) = \frac{1}{6} \omega(\bar{Q}_j).
\]
We can obtain a similar estimate for the set \( \tilde{F}_{j,2} \). Observe now for every \( j \), the following holds
\[(5.13) \quad \frac{1}{\omega(Q_j)} \int_{Q_j} |b(x) - b_Q| \, d\omega(x) \leq \frac{2}{\omega(Q_j)} \int_{Q_j} \left| b(x) - m_b(\bar{Q}_j) \right| \, d\omega(x)
\]
\[= \frac{2}{\omega(Q_j)} \int_{E_{j,1}} \left| b(x) - m_b(\bar{Q}_j) \right| \, d\omega(x) + \frac{2}{\omega(Q_j)} \int_{E_{j,2}} \left| b(x) - m_b(\bar{Q}_j) \right| \, d\omega(x).
\]
From (5.11) we have that at least one of these inequalities holding
\[ \frac{2}{\omega(Q_j)} \int_{E_{j,1}} \left| b(x) - m_b(\bar{Q}_j) \right| \, d\omega(x) \geq \frac{\delta_0}{2}, \quad \frac{2}{\omega(Q_j)} \int_{E_{j,2}} \left| b(x) - m_b(\bar{Q}_j) \right| \, d\omega(x) \geq \frac{\delta_0}{2}.
\]
Let us suppose that the first of these inequalities holds, i.e.,
\[ \frac{2}{\omega(Q_j)} \int_{E_{j,1}} \left| b(x) - m_b(\bar{Q}_j) \right| \, d\omega(x) \geq \frac{\delta_0}{2}.
\]
Hence for every \( j \), using (5.12) we have that
\[(5.14) \quad \frac{\delta_0}{4} \leq \frac{1}{\omega(Q_j)} \int_{E_{j,1}} \left| b(x) - m_b(\bar{Q}_j) \right| \, d\omega(x) \leq \frac{1}{\omega(Q_j)} \omega(\bar{Q}_j) \int_{E_{j,1}} \left| b(x) - m_b(\bar{Q}_j) \right| \, d\omega(x)
\]
Hence, 

\[ \frac{1}{\omega(Q_j)} \int_{E_{j,1}} \int_{\tilde{F}_{j,1}} \frac{1}{\omega(Q_j)} |b(x) - m_b(\bar{Q}_j)| \, d\omega(y) \, d\omega(x). \]

Hence, 

\[ (5.15) \quad \delta_0 \lesssim \frac{1}{\omega(Q_j)} \int_{E_{j,1}} \int_{\tilde{F}_{j,1}} \frac{1}{\omega(Q_j)} |b(x) - m_b(\bar{Q}_j)| \, d\omega(y) \, d\omega(x) \]

\[ \lesssim \frac{1}{\omega(Q_j)^{\frac{2}{3}}} \int_{E_{j,1}} \int_{\tilde{F}_{j,1}} \frac{1}{\omega(Q_j)^{\frac{2}{3}}} |b, R_k| \left( \frac{\chi_{\tilde{F}_{j,1}}}{\omega(Q_j)^{\frac{2}{3}}} \right) \, d\omega(x) \]

\[ = \frac{1}{\omega(Q_j)^{\frac{2}{3}}} \int_{E_{j,1}} \int_{\tilde{F}_{j,1}} \frac{1}{\omega(Q_j)^{\frac{2}{3}}} |b, R_k| \left( \frac{\chi_{\tilde{F}_{j,1}}}{\omega(Q_j)^{\frac{2}{3}}} \right) \, d\omega(x). \]

Consider \( f_j := \frac{\chi_{\tilde{F}_{j,1}}}{\omega(Q_j)^{\frac{2}{3}}} \), observe that this is a sequence of disjointly supported functions and using (5.12) also satisfy \( \|f_j\|_{L^p(\mathbb{R}^N, d\omega)} \approx 1 \). Using the H"{o}lder's inequality yields 

\[ (5.16) \quad \delta_0 \lesssim \frac{1}{\omega(Q_j)^{\frac{2}{3}}} \int_{E_{j,1}} |b, R_k| (f_j)(x) \, d\omega(x) \]

\[ \lesssim \frac{1}{\omega(Q_j)^{\frac{2}{3}}} \int_{E_{j,1}} \left( \int_{\mathbb{R}^N} |b, R_k| (f_j)(x) \, d\omega(x) \right)^{\frac{1}{p}} \]

\[ \lesssim \left( \int_{\mathbb{R}^N} |b, R_k| (f_j)(x) \, d\omega(x) \right)^{\frac{1}{p}}. \]

Let us consider \( \psi \) in the closure of \( \{b, R_k|(f_j)\}_j \), then we have \( \|\psi\|_{L^p(\mathbb{R}^N, d\omega)} \gtrsim 1 \). Now choose some \( j_i \) such that 

\[ \|\psi - [b, R_k] (f_{j_i})\|_{L^p(\mathbb{R}^N, d\omega)} \leq 2^{-i}. \]

To complete the proof consider a non-negative numerical sequence \( \{c_i\} \) with \( \|\{c_i\}\|_{L^p} < \infty \) but \( \|\{c_i\}\|_{L^1} = \infty \). Then consider \( \phi = \sum_i c_i f_{j_i} \in L^p(\mathbb{R}^N, d\omega) \) and 

\[ (5.17) \quad \left\| \sum_i c_i \psi - [b, R_k]\phi \right\|_{L^p(\mathbb{R}^N, d\omega)} \leq \left\| \sum_i c_i [\psi - [b, R_k] (f_{j_i})] \right\|_{L^p(\mathbb{R}^N, d\omega)} \]

\[ \leq \|c_i\|_{L^p} \left[ \sum_i \|\psi - [b, R_k] (f_{j_i})\|_{L^p(\mathbb{R}^N, d\omega)} \right]^{\frac{2}{p}} \lesssim 1. \]

Hence we conclude that \( \sum_i c_i \psi \in L^p(\mathbb{R}^N, d\omega) \), but \( \sum_i c_i \psi \) is infinite on set of positive measure which is contradiction that completes our proof.

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