Variational Synthesis of Controlled Dynamic Mappings

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Abstract

The article deals with the subject of solving the problem of canonical-map synthesis for Hamiltonian systems. For this purpose, the controlling-function method has been developed that allows appropriate changes of the variables in terms of calculus of canonical variation, starting from their target conditions. To use the canonical formalism, the initial dynamic system that employs changing Lagrange multipliers is reduced to a Hamiltonian system in an expanded phase space, followed by the construction of controlling function. The algorithm suggested for the canonization of controlled mappings has an advantage over the known procedures, and first of all, redundancy in the procedure that chooses regulated coordinate transformations as a base for a goal-seeking synthesis scheme.
Introduction

Below we suggest a new procedure of variational changes in dynamical and mechanical systems, which is named the method of controlled (or synthesized) mappings. In distinction to the known methods of generative functions, those of Lie generators, parametrization in the Hilbert-Courant form, etc. \[1–5\], the method allows a large extent of generality and versatility (for example, with respect to the transformations of coordinate axes, etc.) because it makes possible to synthesize mappings in dependence on the chosen target conditions without resort to some laborious-intensive procedures (as it takes place in a classical variant of the generative-function method).

The formalism of generative functions unfortunately does not permit beforehand a new system of variables suitable to finding solution of the selected problems (normalization, integration, and so on). On the contrary, the method of controlled mapping makes possible to solve these problems constructively basing on the beforehand formulated conditions. It might be caused by some shortcomings in the method itself, which presented, as so often was the case, little more than derivation of the related equations in the required quantities, but did not present their analytical solution.

In Section 1, a controlled change of variables in dynamic systems has been considered. For this purpose, canonical equations are derived variationally, with giving proof of a necessary Weierstrass condition for a minimal action functional in terms of invariant Hilbert integral. The controlled mapping is synthesized in phase space by means of a controlling function, and its discovery becomes the most important problem of the method.

In Section 2 the most distinctive singularities of the suggested controlled transformations are pointed. Here we, first of all, reveal the conditions for appearing the properties of canonicity and invariance. Various forms of canonicity criteria for the controlled mappings are found, and some possible ways of their support are discussed. Great attention is paid to the derivation of Hamilton-Jacobi equation for the controlling potential. For the introduced controlled mappings, the invariance properties are established.

Section 3 presents description of energy transformations in an initial dynamic system, which were induced by the canonical controlled mappings. The relationship between the acting and controlling functions is established, which provides the controlling transformation field in phase space. By a basic target condition imposed on the new Hamiltonian, they present an algorithmic scheme for finding the controlling function.

Section 4 is small in volume but important in theoretical generalizations. The direct analogy between small controlled canonical mappings and infinitesimal canonical transformations is firstly revealed. It is secondly shown that the canonization of controlled transformations provides target conditions without using a special choice of initial data.

1 Controlled change of variables

The controlled transformation of variables in any moving dynamic system is presumed to be a change of system variables in the process of system movement as time passes, i. e. supplying a controlled signal in the form of controlled mapping to an input converter of initial variables \(x(t)\).
Let movement of the dynamic system be described by an $n$-dimensional vector equation

$$\dot{x} = f(x, t), \quad x \in \mathbb{R}^n,$$  \hspace{1cm} (1.1)

where $x = (x_1, ..., x_n)$ is the phase vector, $x_i = x_i(t), \ i = \overline{1,n}$ are phase variables (coordinates, velocities), $t \in [t_0, t_1] \subset \mathbb{R}$ is time. It is thought that $f, \partial f/\partial x, \partial f/\partial t \in C^1[t_0, t_1]$ on the right side of equation (1.1) satisfy the existence and uniqueness condition applied to the Cauchy problem in some limited region $D \subset \mathbb{R}^n \times \mathbb{R}$, i.e. equation (1.1) is satisfied by a unique continuous integral path in the expanded phase space $\mathbb{R}^n \times \mathbb{R}$ that originates at the point $x_{0i} = x_i(t_0)$ and terminates at the point $x_{1i} = x_i(t_1), \ i = \overline{1,n}$.

Given differential constraints in the form of equation of motion (1.1), let us introduce the Lagrange multipliers $\lambda_i(t), \ i = \overline{1,n}$, where $\lambda_i(t)$ is the unknown temporal functions determined as part of the solution of optimization problem for the action functional $S$, with the provision of constrained equation (1.1). The Lagrange multipliers can be seen as weighting action-functional integral multipliers

$$S = \int_{t_0}^{t_1} L(x, \dot{x}, \lambda, t) \, dt,$$  \hspace{1cm} (1.2)

where $L = \lambda (\dot{x} - f)$ is the Lagrange function (Lagrangian).

The variation of variable quantity is, as we know, the difference of values of this quantity on a permissible reference trajectory and on an appropriate permissible trajectory called the comparison trajectory. The variation of the quantity calculated for the same moment of time $t$ is called the isochronous variation and denoted by $\delta$. If in calculating the variation the value of the quantity is considered on the comparison trajectory at time $t + \Delta t$, this variation is called a total variation. Let the total variation be denoted by $\Delta$.

Supply also the quantities taken on the comparison trajectory by a line over. Then the relationship between the total and isochronous variations is specified by the equalities

$$\Delta x_i = \bar{x}_i(t + \Delta t) - x_i(t) = \bar{x}_i(t + \Delta t) - \bar{x}_i(t) + \delta x_i,$$

$$\Delta x_i = \delta x_i + \dot{x}_i \Delta t, \quad i = \overline{1,n},$$  \hspace{1cm} (1.3)

and the quantity $\bar{x}_i$ in formula (1.3) substitutes for $\dot{x}_i$, if they were considered to be diverged infinitesimally. It is here thought that we have the permutation relations $\delta \dot{x}_i = \dot{x}_i - \dot{x}_i = d (\bar{x}_i - x_i)/dt = d (\delta x_i)/dt$.

To seek $\Delta S$, let us go from the Lagrange function $L$ to the corresponding Hamilton function $H$ using the Legendre transformation

$$H(x, \lambda, t) = \frac{\partial L}{\partial \dot{x}} \dot{x} - L, \quad L = L(x, \dot{x}, \lambda, t).$$  \hspace{1cm} (1.4)

According to the definition of function $L$ and by (1.4), we obtain

$$\frac{\partial L}{\partial \dot{x}} = \lambda, \quad H = \lambda f.$$  \hspace{1cm} (1.5)

The vector multiplication is here and further on considered as a scalar product.
With relations (1.4), (1.5) we can write [6–8]

\[ \Delta S = \int_{t_0+\Delta t_0}^{t_1+\Delta t_1} \bar{\lambda} \dot{x} \, dt - \int_{t_0}^{t_1} \lambda \dot{x} \, dt - \Delta \int_{t_0}^{t_1} H \, dt \]

\[ = \int_{t_0}^{t_1} (\bar{\lambda} \dot{x} - \lambda \dot{x}) \, dt + \bar{\lambda} \dot{x} |_{t_1} \Delta t_1 - \bar{\lambda} \dot{x} |_{t_0} \Delta t_0 \]

\[ - \int_{t_0}^{t_1} \delta H \, dt - H_1 \Delta t_1 + H_0 \Delta t_0, \]  

(1.6)

where \( H_0 = H |_{t=t_0}, \ H_1 = H |_{t=t_1}, \ \bar{\lambda} - \lambda = \delta \lambda, \) and all equalities in (1.6) are valid accurate within infinitesimals of the order of smallness higher than first. The first summand in the right side of formula (1.6) may be written with the mentioned accuracy in the form

\[ \int_{t_0}^{t_1} (\bar{\lambda} \dot{x} - \lambda \dot{x}) \, dt = \int_{t_0}^{t_1} \lambda (\dot{x} - \dot{x}) \, dt + \int_{t_0}^{t_1} \dot{x} \delta \lambda \, dt \]

\[ = \int_{t_0}^{t_1} \lambda \frac{d}{dt} (\delta x) \, dt + \int_{t_0}^{t_1} \dot{x} \delta \lambda \, dt = \int_{t_0}^{t_1} (-\dot{\lambda} \delta x + \dot{x} \delta \lambda) \, dt + \lambda \delta x |_{t_0}^{t_1}. \]  

(1.7)

Using relations (1.3), (1.7), and

\[ \delta H(x, \lambda, t) = \frac{\partial H}{\partial x} \delta x + \frac{\partial H}{\partial \lambda} \delta \lambda \]

for the calculation of \( \Delta S \) being accurate with the second and higher orders of smallness, we obtain

\[ \Delta S = - \int_{t_0}^{t_1} \left[ \left( \frac{\partial H}{\partial x} + \dot{\lambda} \right) \delta x + \left( \frac{\partial H}{\partial \lambda} - \dot{x} \right) \delta \lambda \right] \, dt \]

\[ + \left( \lambda \Delta x - H \Delta t \right) |_{t_0}^{t_1}. \]  

(1.8)

If there is an extremal of the functional \( S \), it is necessary to fulfill the stationary condition \( \Delta S = 0 \). When the variations in the integral (1.8) are independent and when we take account to the equalities (1.5), we obtain necessary conditions in the form of:

(1) Euler equations (Euler-Lagrange) with respect to the multipliers \( \lambda \)

\[ \hat{\lambda}_i = - \frac{\partial H}{\partial x_i} = - \sum_{k=1}^{n} \lambda_k \frac{\partial f_k}{\partial x_i}, \quad i = 1, n; \]  

(1.9)

(2) equations of motion (1.1) with respect to the variables \( x \) in the form of Euler equations

\[ \dot{x}_i = \frac{\partial H}{\partial \lambda_i} = f_i, \quad i = 1, n; \]  

(1.10)

(3) the universality condition

\[ \left( \lambda \Delta x - H \Delta t \right) |_{t_0}^{t_1} = 0. \]  

(1.11)
It is more believed that the right sides of equations (1.9), (1.10) satisfy the existence and unique conditions for the Cauchy problem in a limited region $D$, namely, they satisfy the continuous vector-function $f(x,t)$ and limited elements of the Jacobi matrix $f_x(x,t)$. These equations therefore fit a unique continuous integral trajectory in the expanded space $R^n \times R$ that issues out of the point with coordinates $x_0, \lambda_0$ at the initial moment of time $t_0$.

Note that Euler-Lagrange equations (1.9), $\dot{\lambda} = -f_x \lambda, \ f_x = \partial f/\partial x$, are the Euler equations of variational Lagrange problem (1.2) with the fixed ends that serve to finding an extremal of the functional $S$,

$$L_x = \frac{d}{dt} L_x,$$

where

$$L_x = \frac{\partial L}{\partial x} = -f_x \lambda, \quad L_\dot{x} = \frac{\partial L}{\partial \dot{x}} = \lambda.$$

It is clear that the system of equations (1.9), (1.10) is well-evident to form a canonical system with the Hamiltonian $H$, coordinates $x_i$, and factors $\lambda_i$. This system is different from a familiar canonical system by no more than the linear dependence of $H$ on $\lambda_i$. What is more, for a total derivative of the function $H = H(x, \lambda, t)$ with respect to the time, we, according to (1.9) and (1.10), have

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial \lambda} \dot{\lambda} + \frac{\partial H}{\partial t} = \lambda \frac{\partial H}{\partial \lambda} - \frac{\partial H}{\partial \lambda} \frac{\partial H}{\partial x} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}.$$

Because by relations (1.5) the expression

$$\frac{dH}{dt} = \dot{\lambda} f + \lambda \dot{f} = \dot{\lambda} f + \lambda f_x \dot{x} + \lambda f_t = \lambda f_t$$

is valid, we shall from here find that on the extremal obeying the necessary conditions for the extremal the equality $f_x = \partial f/\partial x, \ f_t = \partial f/\partial t$ takes place, we from here find that, the equality

$$H = h + \int_{t_0}^{t_1} \lambda f_t \, dt, \quad h = \text{const} \quad (1.12)$$

must be fulfilled. If the vector function $f$ does not explicitly depend on $t$, it follows evidently from (1.12) that $H = h$.

We pointed out that from equations (1.9), (1.10) their contingency follows, where the vector $\lambda(t)$ was that of conjugate variables. But other than these, the functions $L$ and $H$ are conjugate characteristic functions derived from the Legendre transformations because, by (1.4), (1.5), and (1.10) on the extremal trajectories of functional (1.2) while differential relation (1.1) is provided, the following equalities are resulting:

$$L = \frac{\partial L}{\partial x} \dot{x} - H = \lambda \frac{\partial H}{\partial \lambda} - H = \lambda f - \lambda f = 0.$$

We also further on need certain results of variational analysis concerning the notion of integral invariance. The calculus of variation [9] for problem (1.2) with the Lagrangian $L$
the class of curves $C$ with the ends given on the interval $[t_0, t_1]$ is considered. It is necessary to find a minimum of the integral $S(C)$ (1.2) along $C$, where $x(t)$ is the representation of curve $C$. The integral is here taken over the interval of time $[t_0, t_1]$, within which the curve $x(t)$ is defined.

It is stated that if $M$ is the class of curves with the given ends $A = x(t_0)$, $B = x(t_1)$, and $C_0$ is a curve from this class, the quantity $S(C_0)$ is their minimum $S(C)$ with respect to $C \in M$ on the assumption that there exists a total derivative $\Phi$ of the function $\Psi = \Psi(x, \lambda, t)$ such that $L = \Phi$ along the curve $C_0$ and $L \geq \Phi$ along all other curves from the class $M$.

Explain that we here have $\Phi = \dot{\Psi} = \Psi_x \dot{x} + \Psi_\lambda \dot{\lambda} + \Psi_t$. The statement in question may be proved straightforwardly: $\forall C \in M$ the value $S(C)$ is as follows:

$$S(C) \geq \int_{(C)} \Phi \, dt = \int_{(C_0)} L \, dt = S(C_0).$$

**Definition 1** The vector function $g = g(x, t)$ is called a geodesic inclination if there exists a total derivative $\Phi = \dot{\Psi} = \Psi_x \dot{x} + \Psi_\lambda \dot{\lambda} + \Psi_t$ of the function $\Psi = \Psi(x, \lambda, t)$ such that $\forall (x, \dot{x}, \lambda, t)$, where $(x, t)$ are lying in the region of definition of the function $g(x, t)$, the inequality

$$L \geq \Phi$$

is valid, and $L = \Phi$ at $\dot{x} = g$. Here we have $g(x, t) : R^n \times R \rightarrow R^n$.

Thus, $\min (L - \Phi) = 0$ is achieved at $\dot{x} = g$. Providing the coupling equation (1.1), we therefore have

$$L = \lambda (\dot{x} - f) = \Psi_x \dot{x} + \Psi_\lambda \dot{\lambda} + \Psi_t,$$

and we from where obtain a chain of equalities

$$\Psi_x = L_{\dot{x}}, \quad \Psi_\lambda = L_{\dot{\lambda}} = 0, \quad \Psi_t = L - L_{\dot{x}} \dot{x} = -H.$$  \hspace{1cm} (1.13)

Let us add to definition 1 [9] that if $g$ is the geodesic inclination, the geodesic inclination curves are called solutions of the differential equations $\dot{x} = g$, where $g = g(x, t)$. The family of such curves is called a geodesic family. The geodesic family is named the geodesic flow if it covers the region, where the function $g$ is only once defined.

Note that in the agreed notation, the integral

$$S = \int_{(C)} [L(x, g, \lambda, t) - (\dot{x} - g) L_g(x, g, \lambda, t)] \, dt$$  \hspace{1cm} (1.14)

defined the invariant Hilbert integral, where $L_g = \partial L / \partial g$, is independent of the integration path $C$, but only dependent on the end of the curve $C$. Due to relations (1.13) at $\dot{x} = g$, integral (1.14) is indeed equal to the value $\int_{(C)} \Phi \, dt = \Psi(B) - \Psi(A)$, where $A$ and $B$ are the beginning and end of the integration path; in our case $\int_{(C)} \Phi \, dt = \Psi(B) - \Psi(A)$.

With the use of inequality of the form $L - \Phi \geq 0$ and the substitution of $\Phi$ as a integrand (1.14), we obtain the Weierstrass condition $E \geq 0$, where the Weierstrass function $E = E(x, g, \dot{x}, \lambda, t)$ is of the form

$$E = L(x, \dot{x}, \lambda, t) - L(x, g, \lambda, t) - (\dot{x} - g) L_g(x, g, \lambda, t).$$  \hspace{1cm} (1.15)
The min \((L - \Phi) = 0\) is thus achieved at \(\dot{x} = g\). When relation equation (1.1) is satisfied, we therefore have

\[
L = \lambda (\dot{x} - f) = \Psi_x \dot{x} + \Psi_\lambda \lambda + \Psi_t,
\]

we from where obtain the chain of equalities

\[
\Psi_x = L_{\dot{x}}, \quad \Psi_\lambda = L_{\lambda} = 0, \quad \Psi_t = L - L_{\dot{x}} \dot{x} = -H. \quad (1.13)
\]

Add to the definition 1 [9] that if \(\dot{x} = g\) is the geodesic inclination, the solutions of the geodesic equation \(\dot{x} = 0\), where \(g = g(x,t)\), are referred to as the geodesic inclination curves. The family of such curves is called a geodesic family. The geodesic family is referred to as a geodesic flow if it only once covers the region of definition for the function \(g = g(x,t)\).

Point out that in the agreed notation the integral

\[
S = \int_{(C)} \left[ L(x,g,\lambda,t) - (\dot{x} - g) L_g(x,g,\lambda,t) \right] dt \quad (1.14)
\]

named the Hilbert integral, where \(L_g = \partial L/\partial g\), is independent of the integration path \(C\), but dependent on the ends of the curve \(C\). By (1.13), the integral (1.13) at \(\dot{x} = g\) is indeed equal to \(\int_{(C)} \Phi dt = \Psi(B) - \Psi(A)\) at \(\dot{x} = g\), where \(A\) and \(B\) are the initial and terminal ends of integral pathes, being \(L(x,g,\lambda,t) = \lambda (g - f)\) in our case.

Using the inequality \(L - \Phi \geq 0\) from definition 1 and substituting the integrand (2.14) for \(\Phi\), we obtain the Weierstrass condition \(E \geq 0\), where the Weierstrass function \(E = E(x,g,\dot{x},\lambda,t)\) has the form

\[
E = L(x,\dot{x},\lambda,t) - L(x,g,\lambda,t) - (\dot{x} - g) L_g(x,g,\lambda,t). \quad (1.15)
\]

Point to the fact that if \(g\) is the geodesic inclination, we have \(L = \Phi + E\), hence, if \(\forall C \in M\), it joins two levels \(\Psi = \Psi_1\) and \(\Psi = \Psi_2\), we from here obtain the Weierstrass formula of calculus of variation

\[
S(C) = \Psi_2 - \Psi_1 + \int_{(C)} E \, dt,
\]

or

\[
S(C) = S(C_0) + \int_{(C)} E \, dt,
\]

where \(C_0\) is the curve of flow, and \(C\) is any curve joining the levels of the curve \(C_0\).

A necessary Weierstrass condition \((E \geq 0)\) for the minimum of the action functional \(S\) (see (1.2)) can be probed as follows. When \(g = g(x,t)\) substitutes for the function \(\dot{x}\), the Lagrange function \(L(x,\dot{x},\lambda,t)\) changes into the function \(L(x,g,\lambda,t)\). Expanding into series, we find

\[
L(x,\dot{x},\lambda,t) = L(x,g,\lambda,t) + \frac{\partial L}{\partial \dot{x}} \bigg|_{\dot{x}=g} (\dot{x} - g) + \alpha (|\dot{x} - g|),
\]

where \(\alpha(\cdot)\) denotes a set of infinitesimals of the order higher than first order with respect to \(|\dot{x} - g|\). The written expression gives the Weierstrass function \(E \ 1.15\) if we neglect \(\alpha\).
We have for the system (1.1) in the corresponding variational problem, according to formulas (1.4) and (1.5),

\[ L = \lambda \left[ \dot{x} - f(x, t) \right], \quad \frac{\partial L}{\partial \dot{x}} = \lambda. \]

Hence,

\[ E = \lambda (g - f) - \lambda (\dot{x} - f) - \lambda (g - \dot{x}) \equiv 0, \]

i.e. the Weierstrass condition is satisfied.

Insert into consideration a scalar function

\[ U = U(x, \lambda, t) \]

doubly continuously differentiable with respect to its arguments for the Hamilton system (1.9), (1.10), which we name the controlling function. That the function \( U(x, \lambda, t) \) is supposedly not given beforehand, but found from quite definite target conditions.

The controlled mapping from the old variables \( x(t) \in \mathbb{R}^n, \lambda(t) \in \mathbb{R}^n \) to the new ones \( y(t) \in \mathbb{R}^n, \mu(t) \in \mathbb{R}^n \) is realized using the relation

\[ x + U_\lambda = y, \quad \lambda - U_x = \mu, \tag{1.16} \]

where \( U_\lambda = \frac{\partial U}{\partial \lambda}, U_x = \frac{\partial U}{\partial x} \). Note that the record \( x = q, \lambda = p, y = Q, \mu = P \) is usual, further we however follow the established notation. In addition, it from expressions (1.16) follows that in case of identical transformations \( x = y, \lambda = \mu \), we have \( U_\lambda \equiv 0, U_x \equiv 0 \), and \( U \equiv 0 \) may be taken as the controlling function \( U = U(x, \lambda, t) \).

Give attention to the fact that the mapping \( (x, \lambda) \rightarrow (y, \mu) \) given by the equalities (1.16) is the generalization of the known Hilbert-Courant mapping [5] for the parametric generative function \( \Gamma(a, b, t) \) in a canonical system (\( a \) and \( b \) are here the vector parameters) at the change from the variables to \( q = q(a, b, t), p = p(a, b, t) \) to those of \( Q = Q(a, b, t), P = P(a, b, t) \), by the rule

\[ q = a - \frac{1}{2} \Gamma_b, \quad p = b + \frac{1}{2} \Gamma_a, \quad Q = a + \frac{1}{2} \Gamma_b, \quad P = b - \frac{1}{2} \Gamma_a, \tag{1.17} \]

where \( \Gamma_a = \frac{\partial \Gamma}{\partial a}, \Gamma_b = \frac{\partial \Gamma}{\partial b} \). The removing parameters \( a \) and \( b \) give in these formulas clearly give rise to the relations of form (1.16).

An important advantage of mapping (1.16) over the parametric form of mapping (1.17) consists in the lack of any intermediate parameters and in the subsequent necessity for their determination.

The Jacobian of mappings (1.16) is assumed to be non-zero:

\[
\begin{align*}
\det \left( \frac{\partial y}{\partial x} \right) &= \frac{D(y_1, \ldots, y_n)}{D(x_1, \ldots, x_n)} \neq 0, \\
\det \left( \frac{\partial \mu}{\partial \lambda} \right) &= \frac{D(\mu_1, \ldots, \mu_n)}{D(\lambda_1, \ldots, \lambda_n)} \neq 0.
\end{align*}
\tag{1.18}
\]

In relations (1.18) there are determinants of the matrices \( E \pm \partial^2 U/\partial x \partial \lambda \) respectively, where \( E \) is a single-valued matrix of \( n \) dimensions.
Instead of relation (1.16), a controlled sympletic mapping may be taken that resembles by its form the Hilbert-Courant mapping

\[ x + \frac{1}{2} IU_x = y, \quad \lambda - \frac{1}{2} IU_x = \mu, \quad I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \]

where \( E \) is the sympletic matrix. It is easy to see that in the presence of (1.19), the Jacobians are the same and equal to

\[ \det \left( \frac{\partial y}{\partial x} \right) = \det \left( \frac{\partial \mu}{\partial \lambda} \right) = 1 + \frac{1}{4} \det \left( \frac{\partial^2 U}{\partial x \partial \lambda} \right). \]

We however confine ourselves in our further arguments to mappings (1.16).

2 Essential traits of controlled mapping

The leading question that interests us at the given stage is what qualitative changes are introduced by the controlled mapping (1.16) into the description of canonical system (1.9), (1.10)

\[ \dot{x} = H_\lambda, \quad \dot{\lambda} = -H_x \]

with the Hamiltonian dependent on the canonical arguments \( x, \lambda, t \).

We have known that the invariant Hilbert integral (1.14) calculated over the extremal transforms into the action functional \( S \) that takes a minimum value, see principle of the least action in the Hamilton form

\[ S = \int_{t_0}^{t_1} L(x, \dot{x}, \lambda, t) \, dt = \int_{(C_0)} L \, dt, \]

independent of the integral path, but dependent only on the ends of this path. We have over the extremal

\[ S = \int_{(C_0)} L \, dt = \int_{(C_0)} \left( \frac{\partial L}{\partial \dot{x}} \dot{x} - H \right) \, dt = \int_{(C_0)} \lambda dx - H \, dt. \quad (2.1) \]

The integral (2.1) is the above treated Hilbert integral. The differential expression \( \lambda dx - H \, dt \) being its member has the form of an integral Poincare-Kartan invariant.

We assume that in the expanded phase space \( R^{2n+1} \) with coordinates \( x \in R^n, \lambda \in R^n, t \in R \) and the Hamilton function \( H = H(x, \lambda, t) \) constructed for the system (1.9), (1.10), the controlled transformation (1.16) is diffeomorphic, where the diffeomorphism is considered as a mutually single-valued and mutually differentiable mapping.

Let us construct 1-form \( \omega^1 = \lambda dx - H \, dt \) and refer to the following definition [10, p. 859].

**Definition 2** Diffeomorphism retaining the external differential form \( \omega^2 = \sum_{i=1}^n d\lambda_i \wedge dx_i \) is referred to as a canonical mapping.
As the phase trajectories (phase flow) \((x, \lambda)\) of system (1.9), (1.10), representing \([1]\) the rotor lines for the form \(\lambda \, dx - H \, dt\) such that by the Stokes lemma \((\oint_{\gamma_1} \omega^1 = \oint_{\gamma_2} \omega^1, )\) the statement (the theorem on the integral Poincare-Kartan invariant) follows that

\[
\oint_{\gamma_1} \lambda \, dx - H \, dt = \oint_{\gamma_2} \lambda \, dx - H \, dt,
\]

where \(\gamma_1, \gamma_2\) are the closed curves enveloping one and the same tube of phase (integral) trajectories of Hamilton system (1.9), (1.10).

In these relations the form \(\lambda \, dx\) is referred to as a relative integral Poincare invariant. For the two-dimensional part of the tube of rotor \(\sigma\), the Stokes formula

\[
\oint_{\gamma} \lambda \, dx = \iint_{\sigma} d\lambda \wedge dx
\]

is valid, the integral invariance then follows of the 2-form \(\omega^2 = d\lambda \wedge dx\) for the phase flow \((x, \lambda)\) \([1]\).

The canonical transformations convert the Hamilton system into a Hamiltonian system as well. We in this connection turn our attention to a known theorem.

**Theorem 1** Let \(F : R^{2n} \rightarrow R^{2n}\) be the canonical transformation of phase space that converts the point with coordinates \((x, \lambda)\) into that with coordinates \((y, \mu)\). In the new phase coordinates \((y, \mu)\), canonical equations (1.9), (1.10) then have the canonical form

\[
\frac{dy}{dt} = \frac{\partial G}{\partial \mu}, \quad \frac{d\mu}{dt} = - \frac{\partial G}{\partial y}, \tag{2.2}
\]

with the former Hamilton function

\[
G(y, \mu, t) = H(x, \lambda, t). \tag{2.3}
\]

This theorem is proved with the use of a canonical property of mapping, namely:

\[
\oint_{\gamma} \lambda \, dx = \oint_{\gamma} \mu \, dy
\]

over any closed curve \(\gamma\), from where

\[
\oint_{\gamma} \lambda \, dx - \mu \, dy = 0
\]

and, therefore, the integral \(\int_{(x_0, \lambda_0)}^{(x_1, \lambda_1)} \lambda \, dx - \mu \, dy = Q\), where \(\int_{(x_0, \lambda_0)}^{(x_1, \lambda_1)} \lambda \, dx - \mu \, dy = Q\) depends on no path of integration, but an initial \((x_0, \lambda_0)\) and final \((x_1, \lambda_1)\) point of path. Here we have

\[
\lambda \, dx - \mu \, dy = dQ.
\]

\(Q = Q(x, \lambda, t)\) is here an arbitrary continuously differentiable function of its arguments. This equality in an expanded phase space \(R^{2n+1}\) transforms into the equality

\[
\lambda \, dx - H \, dt = \mu \, dy - H \, dt + dQ. \tag{2.4}
\]
The trajectories of canonical system (1.9), (1.10) are represented by rotor lines of the form \( \mu \, dy - G \, dt + dQ \) (see details in [1]). Comparing this form with form (2.4) in the right side, we therefore arrive at the conclusion that equality (2.3), \( G(y, \mu, t) = H(x, \lambda, t) \), is valid. If the Hamilton function does not change in the canonical transformation, i.e. \( G = H \), such a transformation calls quite a canonical transformation.

If we have the considered case of controlled mapping (1.16), \( y = y(x, \lambda, t), \mu = \mu(x, \lambda, t) \), using formula (2.4), where on the right there is \( G \) instead of \( H \), we arrive at an important conclusion that in the canonical transformation of phase space dependent on time \( t \), the canonical equations (1.9), (1.10) in the variables \( y, \mu, t \) have a canonical form (2.2) with a new Hamilton function

\[
G(y, \mu, t) = H(x, \lambda, t) + \frac{\partial Q}{\partial t},
\]  

(2.5)

where, as before, we have \((x_0, \lambda_0)\) at the fixed initial point

\[
Q(x_1, \lambda_1, t) = \int_{(x_0, \lambda_0)}^{(x_1, \lambda_1)} \lambda \, dx - \mu \, dy.
\]

So, for the Poincare-Kartan invariant to retain its form, we must require that the changed part of this integral invariant is a total differential:

\[
\lambda \, dx - H \, dt = \mu \, dy - G \, dt + \frac{\partial Q}{\partial x} \, dx + \frac{\partial Q}{\partial \lambda} \, d\lambda + \frac{\partial Q}{\partial t} \, dt,
\]

from where we obtain the system of relations whose fulfillment must be required for providing the canonicity of the controlled transformation:

\[
G - H = Q_t, \quad (\lambda - Q_x) \, dx = \mu \, dy + Q_{x} \, d\lambda.
\]

To use the results of Theorem 1, it needs to demonstrate what conditions are required for controlled transformation (1.16) to be canonical.

**Theorem 2** Let conditions (1.18) be fulfilled and, in addition, the controlling function \( U = U(x, \lambda, t) \) and the variable \( \lambda(t) \in R^n \) in transformation (1.16) satisfy the differential equality

\[
(U_x - \lambda) \, dU_{\lambda} = U_{\lambda} \, d\lambda,
\]

(2.6)

where \( dU_{\lambda} = U_{\lambda x} \, dx + U_{\lambda \lambda} \, d\lambda + U_{\lambda t} \, dt \). The transformation (1.16) of variables \( x, \lambda \rightarrow y, \mu \) will then be canonical. This transformation changes Hamilton system (1.9), (1.10) with the Hamiltonian \( H(x, \lambda, t) \) into the Hamilton system (2.2) with the Hamiltonian \( G(y, \mu, t) \) by rule (2.5), where \( Q(x, \lambda, t) = U(x, \lambda, t) \).

**Proof.** The canonicity criterion similar to canonicity criterion given in [2, 3] actually follows from (2.4), (2.5): if in the transformation of variables \( x, \lambda \rightarrow y, \mu \) the differential form

\[
W = \mu \, dy - \lambda \, dx - (G - H) \, dt
\]

(2.7)
is a total differential for a function \( V = V(x, \lambda, t) \), i.e. \( W = dV \), where \( dV = V_x \, dx + V_\lambda \, d\lambda + V_t \, dt \), we have a canonical transformation.

In the considered case, the arguments \( x, \lambda, y, \mu \) in the Hamiltonians \( H \) and \( G \) are related through expressions (1.16), and the difference \( G - H \) satisfies equality (2.5). We must in this way take the function \(-Q\) as a function \( V \), following equality (2.4). Setting further \( V = -Q = -U \), we obtain the following expression for the form \( W \) (2.6):

\[
W = d(-U). \tag{2.8}
\]

Let us verify the fulfillment of this condition. Substitute relations (1.16) and (2.5) into the form \( W \) (2.7). We then get

\[
W = (\lambda - U_x) \, d(x + U_\lambda) - \lambda \, dx - U_t \, dt = (\lambda - U_x) \, dU_\lambda - U_x \, dx - U_t \, dt.
\]

With regard for the theorem condition (2.6), we obtain for expression (2.8)

\[
W = -U_\lambda \, d\lambda - U_x \, dx - U_t \, dt = -dU.
\]

It has thereby been argued that the controlled mapping (1.16) is a canonical mapping that establishes the theorem itself.

Some arguments of a general nature, which are immediate from Theorem 2 that deals with the canonicity of the controlled mapping, can be proved. Note that the scalar condition (2.6) may be written in a form all the more compact

\[
U_x \, dU_\lambda = d(\lambda U_\lambda).
\]

The condition (2.6) is in any case the equal of the canonicity criterion on the existence of some function \( V(x, \lambda, t) \) such as \( W = dV \), where the form \( W \) is described by equality (2.7).

How could we take advantage of canonicity condition (2.6)? In principle, criterion (2.6) can be seen as the criterion that is satisfied by the controlling function \( U(x, \lambda, t) \) (or rather its vector differential components \( U_x, U_\lambda \)) and vector-function of variables \( \lambda(t) \).

However, to limit the choice of the function \( U \), on the assumption of condition (2.6), where the vector \( \lambda \) is given by equation (1.9), is a rather strict requirement. Because it first of all groups together \( 2n \) unknown vector-functions \( U_x \) and \( U_\lambda \). The condition (2.6) would therefore be appropriate for the choice of a not uniquely defined vector \( \lambda \) rather than in the choice of \( U \).

Now demonstrate how this can be done. Let the Jacobi matrix as a preliminary be denoted as \( f_x = A \), \( A = A(x, t) \). Then it is known [11] that the solution of vector differential equation (1.9) \( \lambda(t_0) = \lambda_0 \) may be written as

\[
\lambda(t) = B(x, t)\lambda_0,
\]

where \( B(x, t) \) is the matrix obeying the matrix differential equation

\[
\dot{B}(x, t) = -A(x, t) \, B(x, t), \quad B(x_0, t_0) = E,
\]

where \( x_0 = x(t_0) \) \( E \) is the unit \( n \)-matrix. The formula (2.9) for the mapping \( \lambda_0 \rightarrow \lambda \) determines the differential homomorphism of class \( C^1 \).
Substitute expressions (1.9) and (2.9) into canonicity criterion (2.6):

\[(\lambda - U_x) dU_\lambda = U_\lambda A_\lambda dt,\]

from where we arrive at the scalar equation

\[(\dot{U}_\lambda - A^* U_\lambda) \lambda = U_x \dot{U}_\lambda,\]

or

\[(\dot{U}_\lambda - A^* U_\lambda) B_\lambda = U_x \dot{U}_\lambda,\]

where * over indicates the transposition operation and all vectors are multiplied as scalars.

Let us also denote the vector \( C = B^* (\dot{U}_\lambda - A^* U_\lambda). \) Then the latter equation is

\[C_\lambda = U_x \dot{U}_\lambda,\]

where \( C_\lambda_0 = (C, \lambda_0) = \sum_{i=1}^{n} C_i \lambda_{0i} \) may be considered as an equation with respect to one, for example, \( k \)-th initial condition \( \lambda_{0k} :\)

\[\lambda_{0k} = \frac{U_x \dot{U}_\lambda - \sum_{i=1}^{n} C_i \lambda_{0i}}{C_k} \bigg|_{t=t_0},\]

where at the top of the formula there is \( i = 1, n, i \neq k, \) and \( t_0 \) is the fixed initial moment of time.

One more point needed to be made that the controlling function \( U \), by the before made premises and especially the assumption that \( U = Q \), \( U(x_1, \lambda_1, t) = f(x_1, \lambda_1) \lambda dx - \mu dy \), is of the nature of potential (energetic) function. In this connection the function \( U \) may be named the controlling potential function, or controlling potential.

It is significant that the canonical condition (2.6) is realized with a feedback. If the initial moment of time \( t_0 \) to be really considered as a fixed one, the expression for \( \lambda_{0k} \) is determined, having regard to the solutions of the corresponding equations of dynamics (1.9), (1.10), i. e. to the action-functional extremals, through the current values of phase variables. We arrive by that at the problem of canonical synthesis as far as here we construct the law for forming the initial value \( \lambda_{0k} \), which provides the fulfillment of the canonicity condition for a controlled mapping and covers all manifold of other initial data.

Let us show that the above obtained condition of canonicity and the Hamilton–Jacobi equation for the controlling function are closely related.

**Theorem 3** Let canonicity criterion (2.6) be fulfilled for mapping (1.16). Then, in order that mapping (1.16) to be a solution of the Cauchy problem for Hamilton equations (2.2)

\[
\begin{align*}
\dot{y} &= \frac{\partial G}{\partial \mu}, & \dot{\mu} &= -\frac{\partial G}{\partial y}, & y_0 = y(t_0), & \mu_0 = \mu(t_0),
\end{align*}
\]

the controlling function \( U \) must satisfy the Hamilton–Jacobi equation with the Hamiltonian \( G \):

\[\frac{\partial U}{\partial t} = G\left(x + \frac{\partial U}{\partial \lambda}, \lambda - \frac{\partial U}{\partial x}, t\right), \tag{2.10}\]

where \( U = U(x, \lambda, t), U(x, \lambda, t_0) = 0. \)
Proof. To prove the theorem, we need equality (2.5) given at \( Q = U \). If canonicity criterion (2.6) is fulfilled, we have

\[
\frac{\partial U}{\partial t} + H(x, \lambda, t) = G(y, \mu, t). \tag{2.11}
\]

It remains to be noted that in the Cauchy problem for the system of canonical equations (2.2), the variables \( x \) and \( \lambda \) became initial points of the canonical system

\[
x = y(t_0), \quad \lambda = \mu(t_0)
\]

for the trajectory \( y = y(t), \mu = \mu(t) \) with the Hamiltonian \( G(y, \mu, t) \). As this takes place, the motion equations are shaped into the simplest form that appropriates to a zero Hamiltonian \( H(x, \lambda, t) = 0 \). Therefore, to determine the controlling function \( U \) from equation (2.11), we shall obtain equation (2.10). This completes the proof.

It is ready to discover that for the Cauchy problem of system (1.9), (1.10) at \( x_0 = x(t_0), \lambda_0 = \lambda(t_0) \). Theorem 3 can be reformulated to terms of an old Hamiltonian \( H(x, \lambda, t) \) for Cauchy’s problem of system (1.9), (1.10). We in this case have \( y = x(t_0), \mu = \lambda(t_0), \) and \( G(y, \mu, t) = 0 \) in equation (2.11).

**Theorem 4** For the Hamilton system (1.9), (1.10), the Cauchy problem solutions may be presented by canonical mappings (1.16) when the function \( U \) satisfies the Hamilton–Jacobi equation with the Hamiltonian \( H \) having the form

\[
\frac{\partial U}{\partial t} + H(x, \lambda, t) = 0,
\]

where, as before, \( U = U(x, \lambda, t), U(x, \lambda, t_0) = 0, \) and the variables \( x \) and \( \lambda \) are solutions of equations (1.9), (1.10) respectively.

**Remarks**

(1) We can see from the structure of the proofs that Theorems 3 and 4 are indeed invertible, i. e. represent the necessary and sufficient conditions for the solutions of corresponding Hamilton–Jacobi controlling function equations (2.10), (2.11) to exist.

(2) Draw our attention to the fact that the functions \( U = U(x, \lambda, t) \) are common in their notation in equations (2.10) and (2.11), but different in values.

(3) The controlling function \( U \) should not be confused with the generating function that is actively used in the classical formalism of the theory of Hamilton equations, canonical transformations, and integration of differential equations by the Hamilton–Jacobi method.

These functions are different in meaning of their formation and further application. Their main distinction consists in the dependence of the controlling function on the old variables \( x, \lambda \) only, including time \( t \). Recall that the generating function \( \tilde{U} \) is an arbitrary function of mixed (old and new) variables.
Of course, the function $U$ can be given the nature of generating function $\tilde{U}$. With this aim in view, let us write mappings (1.16) in a general form,

$$ y = \varphi(x, \lambda, t), \quad \mu = \psi(x, \lambda, t), $$

(2.12)

where $\varphi \equiv x + U\lambda$, $\psi \equiv \lambda - Ux$, assuming that at $\partial^2 \varphi/\partial x \partial \lambda \neq 0$, $\partial^2 \psi/\partial x \partial \lambda \neq 0$ they are solvable in the old variables

$$ x = x(y, \mu, t), \quad \lambda = \lambda(y, \mu, t). $$

(2.13)

The substitution of one of formulas (2.13) into the function $U$ for $x$ or $\lambda$ (there can be nothing but four variants) leads to the appearance of a generating function $\tilde{U}$. The advantage in the use of controlled mapping (1.16) with the controlling function $U$ over the mapping with the generating function $\tilde{U}$ is obvious: the use of $U$ does not provide for the resolution (reversibility) of equations (2.12) in $x$ and $\lambda$ and for the conversion to the explicit dependences (2.13).

It was specified in [1] that the generating function formalism seems to be 'depressive in its non-invariance and essentially uses phase-state coordinate structure'. Taking into account this note, let us study the question on invariant properties at controlled mapping (1.16): $(x, \lambda, t) \rightarrow (y, \mu, t)$.

Consider, for example, the converted form of the integral Poincare-Kartan invariant in the canonical conversion $(x, \lambda) \rightarrow (y, \mu)$ under the generating function $\tilde{U}(x, \mu, t)$

$$ \lambda \, dx - H \, dt = \mu \, dy - G \, dt + d\tilde{U}, $$

(2.14)

where $d\tilde{U} = \tilde{U}_x \, dx + \tilde{U}_\mu \, d\mu + \tilde{U}_t \, dt$. It follows in particular that

$$ G = \tilde{U}_t + H, \quad \lambda = \tilde{U}_x, \quad y = \tilde{U}_\mu. $$

The canonical transformation $(x, \lambda) \rightarrow (y, \mu)$ obtained with the generating function $\tilde{U}(x, \mu, t)$ is admissible if the condition

$$ \det \frac{\partial^2 \tilde{U}}{\partial x \partial \mu} \neq 0 $$

has been fulfilled in relation (2.14). It is obvious that this condition depends on the choice of new canonical variables.

On the contrary, for the controlling function $U(x, \lambda, t)$ with the converted form

$$ \lambda \, dx - H \, dt = \mu \, dy - G \, dt + dU, $$

the existence condition for the canonical transformation directly depends on the condition $\det (E \pm \partial^2 U/\partial x \partial \lambda) \neq 0$ (1.18) applied to the controlled mappings (1.16). This condition is invariant with respect to the new canonical variables and may be affected by the canonical change of variables.

Consider further the Lagrangian $K$ derived from the Lagrangian $L(x, \dot{x}, \lambda, t)$ by means of controlled change of variables (1.16) if we set in addition that relations (1.16) are resolvable over the old variables $x$ and $\lambda$ by formulas (2.13):

$$ L(x, \dot{x}, \lambda, t) = L[x(y, \mu, t), \dot{x}(y, \mu, t), \lambda(y, \mu, t), t] $$
\[ K(y, \dot{y}, \mu, \dot{\mu}, t), \]  
(2.15)

where \( \dot{x}(y, \mu, t) = x_y \dot{y} + x_\mu \dot{\mu} + x_t \). Here we have

\[ \Omega = x_y = \frac{\partial x}{\partial y}, \quad x_\mu = \frac{\partial x}{\partial \mu}, \quad x_t = \frac{\partial x}{\partial t} \]

in standard notation.

**Definition 3** The Lagrangian \( K \) (2.15) obtained after the change of variables in the Lagrangian \( L \), will be called an induced Lagrangian. The corresponding Euler equation for \( K \) will be called the induced Euler equation and its solution the induced extremals.

The following theorem on the invariance of mappings (1.16), (2.13) is valid, which generalizes a known statement of calculus of variation [9].

**Theorem 5** Let all suppositions about the existence of mappings (1.16), (2.13) be fulfilled. Then the extremals for the variational problem with the Lagrangian \( L \) are the induced extremals for the variational problem with the induced Lagrangian \( K \).

**Proof.** Let us use the notation \( v = K_y \), i.e.

\[ v = \frac{\partial K}{\partial y} = \frac{\partial L}{\partial x} \frac{\partial \dot{x}}{\partial y} = \lambda \Omega, \]  
(2.16)

as far as \( \dot{x}_y = x_y \). We find now \( K_y \):

\[ \frac{\partial K}{\partial y} = \frac{\partial L}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial L}{\partial \mu} \frac{\partial \dot{\mu}}{\partial y} + \frac{\partial L}{\partial \lambda} \frac{\partial \lambda}{\partial y} \]  
(2.17)

\[ = L_\xi \Omega + \lambda \Xi + (\dot{x} - f) \Lambda, \]

where the notation

\[ \Xi = \frac{\partial \dot{x}}{\partial y}, \quad \Lambda = \frac{\partial \lambda}{\partial y} \]

has been used.

Let us find the value of matrix \( \Xi \) from expression (2.17):

\[ \Xi = \frac{\partial \dot{x}}{\partial y} = \frac{\partial}{\partial y} (x_y \dot{y} + x_\mu \dot{\mu} + x_t) = \frac{\partial^2 x}{\partial y \partial y} \dot{y} + \frac{\partial^2 x}{\partial y \partial \mu} \dot{\mu} + \frac{\partial^2 x}{\partial y \partial t}. \]

On the other hand, the value of matrix \( \dot{\Omega} \), where \( \Omega = x_y = \Omega(y, \mu, t) \), may be written as

\[ \dot{\Omega} = \frac{\partial \Omega}{\partial y} \dot{y} + \frac{\partial \Omega}{\partial \mu} \dot{\mu} + \frac{\partial \Omega}{\partial t} \dot{t} = \frac{\partial^2 x}{\partial y \partial y} \dot{y} + \frac{\partial^2 x}{\partial y \partial \mu} \dot{\mu} + \frac{\partial^2 x}{\partial t \partial y}. \]

It from here follows that we have the equality \( \Xi = \dot{\Omega} \).
Then we construct an induced Euler equation with the induced Lagrangian $K$ using equations (2.16), (2.17):

$$
\frac{d}{dt} \frac{\partial K}{\partial \dot{y}} - \frac{\partial K}{\partial y} = \dot{v} - K_y = \frac{d}{dt} (\lambda \Omega) - L_x \Omega - \lambda \dot{\Omega} + (\dot{x} - f) \Lambda
$$

It thus follows that the equations $\dot{\lambda} = L_x$, $\dot{x} = f$, and consequently $\dot{v} = K_y$, take place on the extremals of the Euler equations generated by the Lagrangian $L$. That establishes the theorem.

Note in passing that we would arrive at the conclusion which has been formulated in Theorem 5 that mappings (1.16), (2.13) are invariant if we use the invariance in the Poincare-Kartan form when the choice of variables are canonical.

The representation of the controlled mapping is, of course, not be limited by formulas (1.16). They were entered for definiteness sake and for the demonstration of the relation with Hilbert-Courant mapping (1.17). The same treatment may be applied very well to the controlled mapping of the form

$$
x \pm U_x = y, \quad \lambda \pm U_\lambda = \mu,
$$

or

$$
x \pm U_x = y, \quad \lambda \pm U_\lambda = \mu
$$

may be as readily entered, the signs on the left parts of equations (2.18), (2.19) can be either identical or opposite. Still more exotic combinations seem to be possible when constructing new variables $y, \mu$.

The main requirement for the new variables, nevertheless, as well as for the old ones, is their canonicity. Take, for example, a controlled mapping of the form

$$
x + U_x = y, \quad \lambda - U_\lambda = \mu.
$$

Satisfy that the transformations (2.20) lead in specific situations to the canonical variables $y, \mu$. In this case, a theorem analogous to Theorem 2 may be proved.

**Theorem 6** Let conditions (1.18) be fulfilled with the controlling function $U = U(x, \lambda, t)$ such that its partial derivatives $U_x, U_\lambda$ satisfy equations (2.20). If the equality

$$
(\lambda - U_\lambda) dU_x = (U_\lambda - U_x) dx - U_\lambda d\lambda
$$

is then satisfied, the transformation of the variables $x, \lambda \to y, \mu$ will be canonical. System (2.2) has here the Hamiltonian $G(y, \mu, t)$, (2.5), where $Q(x, \lambda, t) \equiv U(x, \lambda, t)$.

**Proof.** Theorem 6 schematically appears in the following form. We formulate differential form (2.7), where $W = dV$, $V = V(x, \lambda, t)$ is any function. Further it needs to check that if the canonicity criterion (2.21) is fulfilled, the function $V = -U$, where $U = U(x, \lambda, t)$ is the controlling function, it is a desired one. We have by mapping (2.20) for form (2.7)

$$
W = (\lambda - U_\lambda) d(x + U_x) - \lambda dx - U_t dt
$$
\[\begin{align*}
\lambda dU_x - U_\lambda dx - U_\lambda dU_x - U_t dt &= 0 \\
&= (\lambda - U_\lambda) dU_x - U_\lambda dx - U_t dt \\
&= -U_x dx - U_\lambda d\lambda - U_t dt = d(-U).
\end{align*}\]

The latter record is, obviously, completes the proof of the theorem.

In addition, the canonicity criterion (2.21) can be provided by the choice of a certain \(k\)-th initial condition \(\lambda_{0k}\). In this connection to solve equation (1.9), we take equality (2.9). Considering initial condition (1.1), criterion (2.21) can be written in the following form:

\[\begin{align*}
(\lambda - U_\lambda) dU_x = [(U_\lambda - U_x) f + U_\lambda A\lambda] dt,
\end{align*}\]

hence, we get the equation

\[\begin{align*}
(\lambda - U_\lambda) \dot{U}_x = (U_\lambda - U_x) f + U_\lambda A\lambda,
\end{align*}\]

or

\[\begin{align*}
(\dot{U}_x - A^* U_\lambda) B\lambda_0 = U_\lambda \dot{U}_x + (U_\lambda - U_x) f.
\end{align*}\]

An expression for \(\lambda_{0k}\) remains to be written. Resolving the latter scalar equation in \(\lambda_{0k}\), we obtain the formula

\[\begin{align*}
\lambda_{0k} = \frac{U_\lambda \dot{U}_x + (U_\lambda - U_x) f - \sum_{i=1}^n D_i \lambda_{0i}}{D_k} \bigg|_{t=t_0},
\end{align*}\]

where \(i \neq k, k = \overline{1,n}, D = B^* (\dot{U}_x - A^* U_\lambda)\).

**Example 1** The controlled mapping (2.2) was chosen not accidentally. It is convenient with its help to make rotations of the phase coordinate system \((x, \lambda)\).

For example, the rotation of axes \(x\) and \(\lambda\) through the right angle in an anticlockwise direction that is executed by transformations (2.20) must be consistent with the equalities \((y = \lambda, \mu = -x)\) : \(x + U_x = \lambda, \lambda - U_\lambda = -x\), from where the expressions for \(U_x\) and \(U_\lambda\) follow:

\[\begin{align*}
U_x = \lambda - x, \quad U_\lambda = \lambda + x.
\end{align*}\]

The controlling function

\[\begin{align*}
U = U(x, \lambda, t) = \frac{\lambda^2}{2} - \frac{x^2}{2} + \lambda x + u(t), \quad \lambda^2 = (\lambda, \lambda), \quad x^2 = (x, x)
\end{align*}\]

obviously provides an appropriate rotation of axes \(x\) and \(\lambda\) for every continuously differentiable function \(u(t)\).
3 Energy transformations and controlling fields

By the controlling field is meant a scalar functional field in $R$ that is given by the values of function $U(x, \lambda, t)$. We previously mentioned that the controlling function $U = U(x, \lambda, t)$ was of energy nature from the assumptions made on the canonicity of a controlled mapping. Turn to this point.

Take once more expression (1.2) for the action functional $S$. The action functional, where $t = t_1$ in integral (1.2), $t$ being the running time, and the initial point $(x_0, \lambda_0, t_0)$ fixed, is said to be an action function

$$S(x, \lambda, t) = \int_{t_0}^{t} L(x, \dot{x}, \lambda, t) \, dt. \quad (3.1)$$

Try to discover a relation between the action function $S$ (3.1) and the controlling function $U$. If such a relation will be estimated, we shall be able to form an algorithm for determining the controlling function.

Denote through $\gamma$ the extremal joining its initial point $(x_0, t_0)$ with its terminal at the running point $(x, t)$. For the action function $S(x, \lambda, t)$ (3.1), we thus have as well

$$S(x, \lambda, t) = \int_{\gamma} L(x, \dot{x}, \lambda, t) \, dt.$$ 

According to [1], we may demonstrate that if the initial point $(x_0, t_0)$ is fixed, the differential of the action function $S(x, \lambda, t)$ has the form of integral Poincare-Kartan invariant (cf. with expression (2.1)):

$$dS = \lambda dx - H dt, \quad (3.2)$$

where

$$dS = \frac{\partial S}{\partial x} dx + \frac{\partial S}{\partial \lambda} d\lambda + \frac{\partial S}{\partial t} dt, \quad \lambda = \frac{\partial L}{\partial \dot{x}},$$

and the quantity $H = \lambda \dot{x} - L$ is determined in view of a finale velocity $\dot{x}$ of the trajectory $\gamma$.

The proof of equality (3.2) is based on lifting the extremal $\gamma$ from the space $(x, t)$ to the expanded phase space $(x, \lambda, t)$, where $\lambda = \partial L/\partial \dot{x}$. In this case, the extremal is substituted by a phase trajectory that is among the variety of rotor lines having the form $\lambda dx - H dt$.

By the fact that the relations

$$\frac{\partial S}{\partial t} = -H(x, \lambda, t), \quad \left(\frac{\partial S}{\partial x} - \lambda\right) dx + \frac{\partial S}{\partial \lambda} d\lambda = 0 \quad (3.3)$$

follow from equality (3.2), we conclude that the action function $S = S(x, \lambda, t)$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H \left( x, \frac{\partial S}{\partial x}, t \right) = 0. \quad (3.4)$$

The form (1.2) can indeed be written as

$$dS = L \, dt,$$
or $\dot{S} = L$, where $L = L(x, \dot{x}, \lambda, t) = \lambda (\dot{x} - f)$. We have then

$$S_x \dot{x} + S_\lambda \dot{\lambda} + S_t = L,$$

from where the equalities

$$S_x = L_{\dot{x}} = \lambda, \quad S_\lambda = L_{\dot{\lambda}} = 0, \quad S_t = L - L_{\dot{x}} \dot{x} = -H = -\lambda f$$

follow (compare with equalities (1.13)). With an additional condition $S_\lambda = 0$, the first equation in system (3.3) is therefore converted to form (3.4).

Having regard to the written relations, the Hamilton–Jacobi equation (3.4) of the scalar action function $S(x, \lambda, t)$ can then be presented in the form

$$\frac{\partial S}{\partial t} = -\frac{\partial S}{\partial x} f(x, t).$$

The system of $2n$ equations

$$\frac{\partial S}{\partial x} = \lambda, \quad \frac{\partial S}{\partial \lambda} = 0$$

has in this case to be thought of as a system of equations in $2n$ unknowns $x(t)$ and $\lambda(t)$.

If the Cauchy problem with the initial condition $S(x, \lambda, t_0) = S_0(x, \lambda)$ is formulated for equation (3.4), its solution is then reduced to the solution of canonical Hamilton equations (1.9), (1.10):

$$\dot{x} = \frac{\partial H}{\partial \lambda}, \quad \dot{\lambda} = -\frac{\partial H}{\partial x}$$

with the initial conditions

$$x(t_0) = x_0, \quad \lambda(t_0) = \lambda_0 = \frac{\partial S_0}{\partial x} \bigg|_{x=x_0}.$$

(3.5)

The solution of this problem on the space $(x, t)$ comprises, as we know, the extremal $x = x(t)$ for the given variational principle $\Delta \int L \, dt = 0$. This extremal issues out of the initial point $x_0$ and is called the characteristic of problem (3.4). We integrate equality (3.2) along the characteristic that joins the points $A_0 = (x_0, t_0)$ and $A = (x, t)$. And for the action function $S$ with the initial condition $S_0$, we find the expression

$$S(A, \lambda) = S_0(A_0, \lambda_0) + \int_{A_0}^A L(x, \dot{x}, \lambda, t) \, dt,$$

(3.6)

which gives the solution of the problem (3.4). Note in addition that the initial condition $\lambda_0$ must be correlated to the initial condition $\lambda_{0k}$ (3.5).

Let us briefly consider the most important autonomous case. Let time $t$ now not come explicitly in the expression of Hamilton function $H$, i.e. $\partial H/\partial t = 0$. The equality $dH/dt = \partial H/\partial t$ was above obtained in view of the Hamilton equations (1.9), (1.10). We therefore have $dH/dt = 0$ in the given variant, from where it follows that $H = h = \text{const}$ is the first integral (Jacobi integral) of equations (1.9), (1.10).
By the Legendre transformation, \( H = (\partial L/\partial \dot{x}) \dot{x} - L = \lambda f \) and \( f = f(x) \). The Legendre function \( L \) is then independent of time too; \( L = L(x, \dot{x}, \lambda) = \lambda (\dot{x} - f) \) and \( \partial H/\partial t = -\partial L/\partial t = 0 \).

Let the surface \( H(x, \lambda) = h \) be projected from the expanded phase space \((x, \lambda, t)\) to the space \((x, \lambda)\). In this case, the time \( t \) actually does not vary \( ((dt = 0)) \), the total variation \( \Delta \) changes into the isochronous variation \( \delta \), and the expression (3.2) takes a shortened form as compared to the relative integral Poincare invariant:

\[
    d S = \lambda \, dx. \tag{3.7}
\]

Hence, the trajectories of Hamilton system (1.9),(1.10) are extremals for the variational principle corresponding to form (3.7).

The phase trajectories of canonical equations (1.9),(1.10) lying on the surface \( H(x, \lambda) = h \) are in this way extremals of the integral

\[
    S = \int_{\gamma} \lambda \, dx, \tag{3.8}
\]

and these extremals joint the points \( x_0 \) and \( x_1 \). The formulated principle forms the contents of the Maupertuis–Lagrange least (stationary) action principle validated by Lagrange [12] with regard to the Lagrange action \( J \), namely,

\[
    J = \int_{t_1}^{t_2} 2T \, dt,
\]

where \( T = mv^2/2 \) is the kinetic energy of a material point of mass \( m \) and velocity \( v = ds/dt \), namely

\[
    J = \int_{t_1}^{t_2} mv^2 \, dt = \int_{t_1}^{t_2} mv \frac{ds}{dt} \, dt = \int_{s_1}^{s_2} mv \, ds.
\]

It is clear that in the agreed notation the Lagrange action \( J \) will be equal to the shortened action integral \( S \) (4.8) if the impulse \( mv = \lambda \) and distance \( s = x \) are taken.

Let the controlled mapping (1.16) be in the autonomous case given by the controlling function \( U = U(x, \lambda) \). If this mapping is quite a canonical one, then by Theorem 1, the conversion from Hamilton equations (1.9),(1.10) to those of form (2.2) takes place according to equality (2.3) for the Hamiltonians \( G = H \), where \( G = G(y, \mu) \), \( H = H(x, \lambda) \).

In so doing, the relation \( \lambda \, dx - \mu \, dy = dQ \) is valid (see expression (2.4), where \( Q = Q(x, \lambda) \)). We have by that

\[
    \lambda \, dx = \mu \, dy + \frac{\partial Q}{\partial x} \, dx + \frac{\partial Q}{\partial \lambda} \, d\lambda,
\]

we get from where

\[
    (\lambda - Q_x) \, dx = \mu \, dy + Q_\lambda \, d\lambda.
\]

Select as a function \( Q : Q = U, U = U(x, \lambda) \). Then for the canonicity of transformations (1.16), the equality

\[
    (\lambda - U_x) \, dx = (\lambda - U_x) \, d(x + U_\lambda) + U_\lambda \, d\lambda
\]
needs to be required, or
\[ d\left(\lambda U_\lambda\right) = U_x \, dU_\lambda, \]
i. e. we again arrive at the canonicity criterion (2.6).

We shall return to basic relation (2.4), where \( Q(x, \lambda, t) \equiv U(x, \lambda, t) \) and the function \( U(x, \lambda, t) \) specifies controlled mappings (1.16). These mappings are canonical when criterion (2.6) is fulfilled, i. e. at a certain choice of the initial vector \( \lambda_0 \).

Further, let us use the concepts of Lagrangian, Hamiltonian, action function, canonical variables, and write energy the equality (2.4) being main in the controlled mapping method in the form
\[ L \, dt = K \, dt + dU, \quad (3.9) \]
where \( L \, dt = \lambda \, dx - H \, dt \), \( L = L(x, \dot{x}\lambda, t) \) is an initial Lagrange function, \( x, \lambda \) are initial canonical variables, \( H = H(x, \lambda, t) \) is the initial Hamilton function, \( K \, dt = \mu \, dy - G \, dt \), \( K = K(y, \dot{y}\mu, t) \) is a new Lagrange function:
\[ K = \mu (\dot{y} - g), \quad \mu = K_{\dot{y}}, \quad G = \mu g. \quad (3.10) \]

\( G = G(y, \mu, t) = \mu g \) is here a new Hamilton function, where \( g = g(y, t) \), which is connected with the function \( K \) by a Legendre transformation and \( y, \mu \) are new canonical variables in the new \((2n + 1)\)-dimensional expanded phase space \((y, \mu, t)\).

The function \( K(y, \dot{y}, \mu, t) \) is corresponded by a new action function \( R = R(y, \mu, t) \):
\[ R(y, \mu, t) = \int_{t_0}^t K(y, \dot{y}, \mu, t) \, dt = \int_{\delta} K(y, \dot{y}, \mu, t) \, dt, \quad (3.11) \]
where \( \delta \) is the extremal of new variational principle \( \Delta \int K \, dt = 0 \), \( \delta \) being an induced extremal. The new canonical equations with a new Hamiltonian \( G \) have the form of equations (2.2)
\[ \dot{y} = \frac{\partial G}{\partial \mu} = g, \quad \dot{\mu} = -\frac{\partial G}{\partial y} = -g_y \mu, \quad (3.12) \]
where \( g = g(y, t) \) is a certain function continuously differentiable in its arguments.

It is important to note that the choice of the new Hamiltonian \( G = G(y, \mu, t) \) and of the right parts in canonical equations (3.12), respectively (for example for a more convenient integration or reduction of the systems), in a more simple form (its normalization) is a main purpose condition of the formation of the controlling function \( U = U(x, \lambda, t) \).

Using the formalism of action functions, we write relation (3.9) that determines the controlling potential \( U(x, \lambda, t) \) in the form
\[ dS = dR + dU, \quad S - R = U, \quad (3.13) \]
from where we derive the expression for \( U(x, \lambda, t) \):
In formula (3.14) by the canonicity of controlled transformations, the written line integrals do not lie on the path of integration but depend only on the finale values \((x, \lambda), (y, \mu)\) at the fixed initial values \((x_0, \lambda_0), (y_0, \mu_0)\).

In addition, in (3.14) the new canonical variables \(y, \mu\) are connected with the old canonical variables \(x, \lambda\) by controlled mappings (1.16). Note also that for the new action function \(R\) (3.11) (from equation (3.13)) there takes place a Hamilton-Jacobi equation of the form

\[
R_t = -G, \quad R_y = \mu, \quad R_\mu = 0,
\]

where \(G = G(y, R_y, t)\).

**Example 2**

Consider an autonomous dynamic system that is square integrable and described by the equation

\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n
\]

with the given initial condition \(x_0 = x(t_0)\) and suitable requirements to the vector-function \(f(x)\). It is now necessary, using controlled transformations of the form (1.16), to reduce it to the given system presented by the equation

\[
\dot{y} = g(y), \quad y \in \mathbb{R}^n,
\]

where the vector-function \(g(y)\) is determined from the given Hamilton function \(G(y, \mu)\).

Use all previous notation and constructions to solve this problem. We have

\[
L(x, \dot{x}, \lambda) = \lambda (\dot{x} - f), \quad H(x, \lambda) = \lambda f = h,
\]

where \(h > 0\) is a constant equal to the value of total system energy. Canonical equations in the variables \(x, \lambda\) have the form

\[
\dot{x} = \frac{\partial H}{\partial \lambda} = f(x), \quad \dot{\lambda} = -\frac{\partial H}{\partial x} = -f_x(x) \lambda.
\]

It is considered that the initial data \(x_0, \lambda_0\) are given and \(\lambda_0\) is coordinated with the canonicity criterion (2.6).

Let the new Hamiltonian \(G\) meet the requirement

\[
G(y, \mu) = a\mu,
\]

where \(a \in \mathbb{R}^n\) is a given constant vector. Due to the autonomy, \(H = G = h\). The new canonical equations (3.12) may be written in the form

\[
\dot{y} = \frac{\partial G}{\partial \mu} = a, \quad \dot{\mu} = -\frac{\partial G}{\partial y} = 0,
\]

from where it follows that \(y(t) = at + b, \mu(t) = c\). Here \(a, b, c\) are the vectors of known components. Thus, we have \(g(y) = a, G(y, \mu) = ac = h\).

It is required, starting from these purpose conditions, to determine the controlling function \(U = U(x, \lambda)\) such that it is reduced to these new Hamilton equations. Let there be controlling mappings (1.16) and by the choice of \(\lambda_0\) canonicity criterion (2.6) be provided.
Here with regard to (3.9)–(3.14), we have for the situation of finding the controlling function $U(x, \lambda)$:

$$\lambda \, dx = \mu \, dy + dU, \quad U = S - R,$$

from where, since $H = G = h$, we get a chain of equivalent relations

$$U(x, \lambda) = \int_{\gamma} L \, dt - \int_{\delta} K \, dt$$

$$= \int_{\gamma} \lambda [\dot{x} - f(x)] \, dt - \int_{\delta} c (\dot{y} - a) \, dt$$

$$= \int_{(x_0, \lambda_0)}^{(x, \lambda)} \lambda \, dx - \int_{(y_0, c)}^{(y, c)} c \, dy = \int_{(x_0, \lambda_0)}^{(x, \lambda)} \lambda \, dx - c (y - y_0),$$

where $y = y(t)$, $y_0 = y(t_0)$, and by the canonicity of transformations, all written line integrals do not depend on the path of integration, but depend only on the finale values at the fixed initial data.

Substituting the expression for $y$ (1.16) into the latter equality, we obtain an equation for the required function $U(x, \lambda)$:

$$U(x, \lambda) = \int_{(x_0, \lambda_0)}^{(x, \lambda)} \lambda \, dx - c (x + U_\lambda) + cy_0.$$

This equation is a first-order partial linear differential equation with respect to the unknown function $U(x, \lambda)$. This has the form

$$U(x, \lambda) + c \frac{\partial U(x, \lambda)}{\partial \lambda} = F(x, \lambda), \quad (3.15)$$

where the notation is

$$c \frac{\partial U(x, \lambda)}{\partial \lambda} = \sum_{i=1}^{n} c_i \frac{\partial U(x, \lambda)}{\partial \lambda_i}, \quad F(x, \lambda) = \int_{(x_0, \lambda_0)}^{(x, \lambda)} \lambda \, dx + c (y_0 - x),$$

$F(x, \lambda)$ being a known function of variables $x$ and $\lambda$.

An attempt to get an analytical solution of equation (3.15) meets certain difficulties [13, 14] caused by $2n$ independent variables $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n$ and $n$ partial derivatives $U_{\lambda_1}, \ldots, U_{\lambda_n}$. To obtain the relation between $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n$ and $U$, we must find a solution for the corresponding system of ordinary differential equations (characteristic equations of a specific form) that satisfy the given initial conditions. If we disregard the theoretical questions concerning the solution of equation (3.15), we can point to the fact that in the given case this equation can be considered as a final algorithmic equation for determining the controlling potential that gives a mapping in the initial phase space $(x, \lambda)$ with subsequent numerical resolution.

Let us apply the obtained results of the method of dynamical controlled-mapping method to the most important class of mechanical systems, Hamiltonian systems, which in standard notation of generalized coordinates $q(t) \in R^n$ and generalized momenta $p(t) \in R^n$, have the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, n, \quad (3.16)$$
where \( H = H(q,p,t) = p\dot{q} - L(q,\dot{q},t) \) is the Hamilton function, \( L = L(q,\dot{q},t) \) is the Lagrange function, \( T = T - \Pi \), \( T \) is the kinetic energy of the system, \( \Pi = \Pi(q,t) \) is its potential energy, \( T = T_2 + T_1 + T_0 \), \( T_2 = (1/2) \sum a_{ij} \dot{q}_i \dot{q}_j \) is the quadratic form of generalized velocities, \( A = (a_{ij}) \) is the kinetic energy matrix, \( T_1 = \sum b_i \dot{q}_i \) is a linear form of generalized velocities, and \( T_0 = T_0(q,t) \) is its null-form. Thus, we have

\[
L = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \dot{q}_i \dot{q}_j + \sum_{i=1}^{n} b_i \dot{q}_i + T_0 - \Pi,
\]

where the coefficients \( a_{ij}, b_i \) depend on the generalized coordinates and time. Hence, the generalized momenta are expressed by

\[
p_i = \frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^{n} a_{ij} \dot{q}_j + b_i.
\]

Since the kinetic energy matrix \( A \) is non-degenerated, solving then the latter linear equation with respect to \( \dot{q}_i \), we get

\[
\dot{q}_i = \sum_{j=1}^{n} \alpha_{ij} (p_j - b_j),
\]

where \( A^{-1} = (\alpha_{ij}) \). If to substitute this expression into the expression for \( H \), we shall find

\[
H = \sum_{i=1}^{n} p_i \dot{q}_i - L = \sum_{i,j=1}^{n} \alpha_{ij} (p_i - b_i) p_j - \frac{1}{2} \sum_{i,j,k,l=1}^{n} a_{ij} \alpha_{ik} \alpha_{jl} (p_k - b_k)(p_l - b_l)
\]

\[
- \sum_{i,j=1}^{n} b_i \alpha_{ij} (p_j - b_j) - T_0 + \Pi
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{n} \alpha_{ij} (p_i p_j + b_i b_j) - \sum_{i,j=1}^{n} \alpha_{ij} b_i p_j - T_0 + \Pi.
\]

In case when the Lagrange function does not depend on time (\( \partial L/\partial t = 0 \)), we have the first integral (generalized energy integral, Painleve–Jacobi integral) as follows:

\[
\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{const},
\]

or \( T_2 - T_0 + \Pi = \text{const} \). If the system is conservative, \( T_1 = T_0 = 0 \) and the generalized energy potential is the same as the total energy \( T_2 + \Pi = \text{const} \) (the total energy conservation law in the conservative systems).

The Hamilton equations (3.16) are equivalent to the Lagrange equations

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, n,
\]

(3.17)
where $\partial L/\partial \dot{q}_i = p_i$ are the generalized momenta, $\partial L/\partial q_i$ are the generalized forces, the trajectories of motion for mechanical systems (3.16), (3.17) agreeing with the extremals of the action functional $\int_{t_0}^{t_1} L(q, \dot{q}, t) \, dt$ (principle of least action in the Hamilton form).

Further, we must rewrite the dynamic relations of mechanical systems in terms of customary standard notation $x(t)$ and $\lambda(t)$. Let us take for this aim $x(t) = (q(t), \dot{q}(t))$ or $x(t) = (\dot{q}(t), q(t))$ and attach the meaning of vector Hamilton multiplier to $\lambda(t)$. We then resolve equation (3.17) with respect to a higher derivative and introduce the vector of new derivatives $x(t)$. Now we arrive at a normal formulation of the mechanical system in the form of equation of motion (1.1). The neat process of controlled canonical transformation of system is completely corresponded to the above described scheme.

It should be particularly emphasized that the controlled mapping method is based on the representation the system of equations of motion and Euler equations as a normal system of differential Cauchy equations (1.9), (1.10) respectively, giving the Hamilton function $H$ in a special form (1.5), $H = \lambda f$. Such a formulation sometimes leads to problems.

**Example 3** In the problem dealing with the determination of Lagrange multipliers for a ballistic flight (it is a section of path, where an engine is excluded) in a central gravitation field [7], it is necessary to present the equations of motion and Euler equations (Euler–Hamilton ones) in a normal Cauchy form and rewrite the Hamiltonian in the form of (1.5).

The trajectory of motion, as we know from theoretical mechanics, is here represented by a conic section that in polar coordinates $r, \varphi$ satisfies the equation

$$r = \frac{p}{1 + e \cos (\varphi - \omega)},$$

where we have the notation: $p$ is a focal parameter ($(p = r$ at $\varphi = \pi/2 + \omega)$, $e$ is the eccentricity, $\omega$ is a polar angle periphery center, i.e. the trajectory point closest to the focus of conic section, the angle $a = \varphi - \omega$ being said to be a true anomaly. The change of polar angle is determined by the area integral

$$r^2 \dot{\varphi} = \sigma \sqrt{p},$$

where $\sigma = \sqrt{\gamma M}$, $\gamma$ is the gravitation constant, $M$ is the mass of a central body.

In order to compile the Euler equations for the multipliers $\lambda$, we must first write the equations of motion in the normal form. Let us write them in polar coordinates. Use some known kinematic expressions for the projections of point acceleration on the generalized axes of curvilinear polar coordinate system

$$w_r = \ddot{r} - r \dot{\varphi}^2, \quad w_\varphi = \frac{1}{r} \frac{d}{dt} \left( \frac{r^2 \dot{\varphi}}{\dot{r}} \right).$$

Since the point of mass $m$ is under the action of only Newton force of attraction to the fixed center $\vec{F} = -m\sigma^2 \ddot{r}/r^3$, we obtain the equations of motion

$$\ddot{r} - r \dot{\varphi}^2 = -\left( \frac{\sigma}{r} \right)^2, \quad \frac{d}{dt} \left( \frac{r^2 \dot{\varphi}}{\dot{r}} \right) = 0.$$
Note simultaneously that the second equation leads to an area integral, and the first one in passing to a new variable $1/r$ and new argument $\varphi$ instead of $t$ leads to the already known equation of conic sections.

Of course, the equation that describes the motion of a point in a gravitation field in polar coordinate can also be found with the second-order Lagrange equations. For this aim, we must write the kinetic equation $T$ and the generalized forces $Q_r$ and $Q_\varphi$:

$$T = \frac{1}{2} m (r^2 + r^2 \varphi^2), \quad Q_r = -m \left(\frac{\sigma}{r}\right)^2, \quad Q_\varphi = 0$$

and then construct the equations

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_{q_i}, \quad i = 1, 2, \quad q_1 = r, \quad q_2 = \varphi$$

being coincident with the above cited equations of motion.

Construct now the equations of motion in a form solvable with respect to derivatives, i.e. found in their normal form. To this end, take the projections of the velocity of a point on the polar axes, namely $v_r = \dot{r}$, $v_\varphi = r \dot{\varphi}$, as velocity phase coordinates. As a result, we obtain the system of four equations of motion

$$\dot{v}_r = \frac{v_\varphi^2}{r} - \left(\frac{\sigma}{r}\right)^2, \quad \dot{v}_\varphi = -\frac{v_r v_\varphi}{r}, \quad \ddot{r} = v_r, \quad \ddot{\varphi} = \frac{v_\varphi}{r}.$$  

The Hamilton function $H$ (1.5) then takes the form

$$H = \lambda_1 \left[ \frac{v_\varphi^2}{r} - \left(\frac{\sigma}{r}\right)^2 \right] + \lambda_2 \left(\frac{-v_r v_\varphi}{r}\right) + \lambda_3 v_r + \lambda_4 \frac{v_\varphi}{r},$$

and the system of Euler equations for the determination the multipliers $\lambda(t)$ will be written in the form

$$\dot{\lambda}_1 = \lambda_2 \frac{v_\varphi}{r} - \lambda_3, \quad \dot{\lambda}_2 = -2 \lambda_1 \frac{v_r}{r} - \lambda_2 \frac{v_r}{r} - \lambda_4 \frac{1}{r},$$

$$\dot{\lambda}_3 = \lambda_1 \frac{v_\varphi^2}{r^2} - 2 \lambda_1 \frac{\sigma^2}{r^3} - \lambda_2 \frac{v_r v_\varphi}{r^2} + \lambda_4 \frac{v_\varphi}{r^2}, \quad \dot{\lambda}_4 = 0.$$  

The latter formula can be represented in the form $\lambda_4 = \text{const}$. The latter three form a system of linear equations in $\lambda_1, \lambda_2, \lambda_3$ with alterable coefficients.

### 4 Canonical controlled mappings

The controlled mappings (1.16) are constructed on the base of additive occurrence of controlling functions. The efficiency of their use may change from task to task. Point out in this connection that the transformation by formulas (1.16) can be of a more general nature being independent of the choice of initial data.

Along with controlled mapping (1.16)

$$y = x + U_\lambda, \mu = \lambda - U_x, U = U(x, \lambda),$$
let us consider a quasi-canonical transformation such that its vector variables \( x \) and \( \lambda \) change by an infinitesimal value. Such a transformation is called the infinitesimal canonical transformation \([1, 15]\).

Suppose

\[
y = x + \eta(x, \lambda) \epsilon, \quad \mu = \lambda + \varsigma(x, \lambda) \epsilon, \tag{4.1}
\]

where \( \eta(x, \lambda) \), \( \varsigma(x, \lambda) \) are the vector \( n \)-dimensional functions of \( x, \lambda \), and \( \epsilon \) is a small parameter. If \( \eta \) and \( \varsigma \) are given arbitrarily, the transformation (4.1) is clearly non-canonical.

We therefore set forth the requirements imposed on the vector-functions \( \eta \) and \( \varsigma \) that provide the canonicity of transformation (4.1), and find a formal analogy between the vector-functions \( U_{\lambda, \eta} \) and \( U_{x, \varsigma} \).

Let us fall back on equality (2.4) of the form

\[
\lambda \, dx = \mu \, dy + dQ,
\]

where \( Q = Q(x, \lambda) \) is an arbitrary function of \( x \) and \( \lambda \). Substitute (4.1) into its right side:

\[
\lambda \, dx = (\lambda + \epsilon \varsigma)(dx + \epsilon \, d\eta) + dQ.
\]

Dropping the terms that contain \( \epsilon^2 \), we obtain

\[
\epsilon (\varsigma \, dx + \lambda \, \eta) = -dQ,
\]

or

\[
(\varsigma \, dx - \eta \, d\lambda) + d(\eta \lambda) = -\frac{1}{\epsilon} \, dQ.
\]

The latter equality may be written in the form

\[
\varsigma \, dx - \eta \, d\lambda = -d\Omega(x, \lambda),
\]

where the notation

\[
\Omega(x, \lambda) = \frac{1}{\epsilon} Q + \lambda \eta
\]

is entered for the scalar function \( \Omega(x, \lambda) \).

With regard to the fact that \( d\Omega = \Omega_x \, dx + \Omega_\lambda \, d\lambda \), we obtain from (4.2)

\[
\varsigma = -\Omega_x, \quad \eta = \Omega_\lambda
\]

due to the independence of the quantities \( dx \) and \( d\lambda \). Hence, canonical transformation (4.1), being infinitesimal, will be seen as

\[
y = x + \Omega_\lambda \varsigma, \quad \mu = \lambda - \Omega_x \varsigma, \tag{4.3}
\]

where \( \Omega = \Omega(x, \lambda) \) is an arbitrary scalar function of variables \( x, \lambda \), which plays a great role (compared with (1.16)) of a controlling function with a small parameter or, by another formalism \([16–18]\), of a small controlled perturbation (test signal).

As for a general non-stationary case, all canonical transformations according to S. Lie, see \([15]\), might be defined with one differential condition. Namely, the reversible transformation \( x, \lambda \rightarrow y, \mu \) (in this case the functional determinant \( \partial(y, \mu)/\partial(x, \lambda) \neq 0 \)) is canonical if we have the differential form

\[
\lambda \, dx = \mu \, dy + H_0 \, dt + dQ,
\]
where $H_0$ and $Q$ are arbitrary functions of $4n + 1$ variables $x, \lambda, y, \mu, t,$ are satisfied identically.

This form should be given as another form that is well known if to subtract the quantity $H dt$ from both parts of equation. Then we shall have

$$\lambda dx - H dt = \mu dy - G dt + dQ, \quad G = H - H_0.$$ 

It should be noted that independent in these forms will be only $4n + 1$ variables $x, \lambda, y, \mu, t$ from $2n + 1$ because there exist $2n$ ratios between the variables of the form $y = y(x, \lambda, t), \mu = \mu(x, \lambda, t)$.

If we apply the canonical controlled transformation (4.3), where $U_\lambda = U_\lambda(x, \lambda, \varsigma) = \Omega_\lambda(x, \lambda) \varsigma, U_\mu = U_\mu(x, \lambda, \varsigma) = \Omega_\mu(x, \lambda) \varsigma,$ to a scalar function $\psi(x, \lambda)$, then it is possible to find its variation correct to the magnitudes of the order of $\varsigma^2$ smallness:

$$\psi(x_i, \lambda_i) = \psi(y_i - U_{\lambda_i}, \mu_i + U_{\mu_i}) = \psi(y_i - \Omega_{\lambda_i} \varsigma, \mu_i + \Omega_{\mu_i} \varsigma)$$

$$= \psi(y_i, \mu_i) + \varsigma \sum_{i=1}^{n} \left( - \frac{\partial \psi}{\partial x_i} \frac{\partial \Omega}{\partial \lambda_i} + \frac{\partial \psi}{\partial \lambda_i} \frac{\partial \Omega}{\partial x_i} \right),$$

or in the notation of Poisson bracket

$$\{ \psi, \Omega \} = \sum_{i=1}^{n} \left( \frac{\partial \psi}{\partial x_i} \frac{\partial \Omega}{\partial \lambda_i} - \frac{\partial \psi}{\partial \lambda_i} \frac{\partial \Omega}{\partial x_i} \right),$$

we obtain the formula

$$\psi(x_i, \lambda_i) = \psi(y_i, \mu_i) - \{ \psi, \Omega \} \varsigma = \psi(y_i, \mu_i) - \{ \psi, U \}.$$

Interest is provoked by the case of small canonical controlled transformation when we take the Hamilton function $H(x, \lambda)$ as a function $\Omega$ and infinitesimal interval of time $dt$ as an arbitrary small parameter. Suppose we have

$$U(x, \lambda, \varsigma) = \Omega(x, \lambda) \varsigma = H(x, \lambda) dt.$$ 

For the infinitesimal canonical transformation (4.3), we may then write

$$y = x + H_\lambda dt, \quad \mu = \lambda - H_\mu dt$$

from where it follows by canonical Hamilton equations (1.9), (1.10) that

$$dx = H_\lambda dt, \quad d\lambda = -H_\mu dt$$

according to Hamilton canonical equations (1.9), (1.10)

$$dx = H_\lambda dt, \quad d\lambda = -H_\mu dt.$$ 

These relations actually mean that in a time $dt$ the system $(x(t), \lambda(t))$ will pass to a new state $(x(t + dt), \lambda(t + dt))$, and this change will be induced by the infinitesimal controlled canonical transformation such that the basic controlling function of the form

$$\Omega(x, \lambda) = \frac{\partial}{\partial \varepsilon} U(x, \lambda, \varepsilon)$$
is the Hamilton function $H(x,\lambda)$. We here assume that the initial function $U$ linearly depends on the parameter $\varsigma$.

In this way, it can be concluded, starting from this analysis, that the movement of this system is a sequence of infinitesimal controlled mappings, where the basic controlling function (at a small parameter $\varsigma$) must be the Hamilton function itself. Such a result only confirms an important conclusion [19] that the system can be led to the given specific state by means of sequential small probe signals (controlled perturbations).

Turn to basic controlled transformations $(1.16)$. We said in the preface to Section 4 that it was possible to construct canonical controlled mappings without recourse to specially selected initial data, as was done before.

**Theorem 7** Suppose the conditions $(1.18)$ are fulfilled for transformation $(1.16)$. Transformations $(1.16)$ of the variables $x, \lambda \to y, \mu$ are canonical if simultaneously:

(i) to choose the vector $U_{\lambda}$ from the equation

$$\dot{U}_{\lambda} = A^{*} U_{\lambda}, \quad (4.4)$$

where $A = A(x, t) = f_{x}(x, t);$  

(ii) to choose the vector $U_{x}$ using the orthogonality of vectors $U_{x}$ and $\dot{U}_{\lambda}$.

**Proof.** We shall, as before, start from a key equality $(2.6)$ to which controlled mapping $(1.16)$ leads. If the equality is fulfilled identically, the transformation $(1.16)$ is canonical.

Using initial Hamilton equation $(1.9)$, we succeed in writing equality $(2.6)$ in the form

$$(\dot{U}_{\lambda} - A^{*} U_{\lambda}) \lambda = U_{x} \dot{U}_{\lambda}. \quad (4.5)$$

The relation $(4.5)$ will be fulfilled identically if we choose the vector $U_{\lambda}$ such that equation $(4.4)$ is valid, from where it automatically follows the requirement that the equality

$$U_{x} \dot{U}_{\lambda} = 0 \quad (4.6)$$

to be fulfilled.

The vector differential equation $(4.5)$ is identical in its structure to $(1.9)$ $\dot{\lambda} = -A\lambda$. The solution of equation $(4.5)$ is given by the formula

$$U_{\lambda} = D(x, t) U_{\lambda 0}, \quad U_{\lambda 0} = U_{\lambda} |_{t=t_0}, \quad \quad (4.7)$$

where $D(x, t)$ is the $n \times n$-matrix that satisfies the matrix differential equation

$$\dot{D}(x, t) = A^{*}(x, t) D(x, t), \quad D(x_0, t_0) = E.$$  

The theorem is completed.

In particular, it follows from $(4.7)$ that the vector-function $U_{\lambda}$ is a function of variables $x$ and $t$. This means that the controlling function $U(x, \lambda, t)$ is linearly dependent on the $\lambda$-vector elements:

$$U(x, \lambda, t) = \int_{\lambda_0}^{\lambda} D(x, t) U_{\lambda 0} d\lambda + u(x, t),$$

where $u(x, t)$ is an arbitrary scalar function of $x$, and $t$ that can be chosen, for example, from the orthogonality condition $(4.6)$.  

Conclusions

The idea to reduce a dynamic system to an expanded Hamilton form with simultaneous attachment of the formalism of classical calculus of variation was realized before as well (see, for example, [6–8] and [20, 21] for adaptive systems).

It turned to be that such structures were able to work in the construction of controlled canonical mappings in Hamilton systems for the purpose of writing the equations of motion in a more comfortable form.

For the purpose of constructing canonical coordinate mappings in Hamilton systems, the controlling-function method was developed in the given article, whose leading property was the dependence on the old canonical variables.

This property is, in the end, allowed additional possibilities in forming an algorithm for a canonical change of variables, which render it redundant and less well defined as compared with the algorithm based on the generating-function method.

Note also a broad range of new variational results, which, being of a general theoretic importance, were arranged as some theorems. These results may have practical use in the solution of various problems of canonicity and study of dynamic Hamilton–Jacobi relations.

Notation

$R^n$ is $n$-dimensional Euclidian space.

$\Delta x_i, \delta x_i$ are total and isochronous variations of the component $x_i$ respectively.

$\Delta S$ is a total variation of action functional $S$.

$A^*$ is the transpose with respect to the matrix $A$.

$S(C) = \int_{(C)} L(x, \dot{x}, \lambda, t) \, dt$ is the integral $S$ along the curve $C$, where $x = x(t)$ is the representation of curve $C, t \in [t_0, t_1]$.

$\{\psi, \Omega\}$ is the Poisson bracket with the elements $\psi$ and $\Omega$.

$B_x = \partial B/\partial x$ is the vector (matrix) of partial derivative of the scalar function (vector) $B$ with respect to the elements of vector $x$. 
All items are published in Russian.

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