Action of overallgebra in Plancherel decomposition and shift operators in imaginary direction

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In last 50 years, there were solved many problems of explicit spectral decompositions for restrictions of unitary representations to subgroups (see the bibliography in [22]).

This class of problems also contains the following problems, which are formulated in other terms.

1. Decomposition of the tensor product of representations \( \rho_1, \rho_2 \) of a group \( G \). Indeed, this is exactly the problem of restriction of representations of \( G \times G \) to the diagonal subgroup \( G \subset G \times G \).

2. Decomposition of \( L^2 \) on a pseudo-Riemannian symmetric space \( G/H \). As was observed\(^2\) in [23], for each classical pseudo-Riemannian symmetric space \( G/H \), there exists a canonical classical group \( \tilde{G} \supset G \) and a representation \( \rho \) of the group \( \tilde{G} \) of degenerated principal series satisfying one of two following properties (usually the first variant is realized):
   - the restriction of \( \rho \) to \( G \) is \( L^2(G/H) \)
   - the restriction of \( \rho \) to \( G \) is the direct sum of the spaces \( L^2(G/H_j) \), where \( G/H_j \) is a finite collection of symmetric spaces, and \( G/H \) is one of the spaces \( G/H_j \).

Hence the decomposition of \( L^2 \) on a classical Riemannian symmetric space can be considered as a restriction problem.

Description of the spectral type (without explicit Plancherel formula) for all the pseudo-Riemannian symmetric spaces was recently obtained in the works of van den Ban, Schlichtkrull, Delorme, and Oshima (the proof is contained in the union of a large collection of papers, for references, see [2], [7]). It seems that for classical symmetric spaces the problem of evaluation of the Plancherel measure is near the final solution.

For some cases, the explicit Plancherel measure is known; in particular, for \( L^2 \) on semisimple groups ([9], [14]), on Riemannian symmetric spaces ([11], [12], see also [15]), on rank 1 spaces ([20], [21]) and on the spaces \( G_\mathbb{C}/G_\mathbb{R} \), where \( G_\mathbb{C} \) is a complex group, and \( G_\mathbb{R} \) is its real form ([12], [13]).

3. Berezin kernel representations (deformations of \( L^2 \) on Riemannian noncompact symmetric space \( G/K \)) also can be obtained by the restriction from some overgroup \( G^o \supset G \), see [24], [26] and references in [26].

Usually the problem of the noncommutative harmonic analysis is formulated as the problem about the spectrum of a representation or as the more complicated problem of an explicit decomposition of a representation into a direct integral of irreducible representations; the last question includes the explicit

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\(^2\)This phenomenon could be easily observed from the Makarevich paper [18] published in 1973, in a strange way it remained nonformulated for a long time
evaluation of the spectral measure (the so-called Plancherel measure). Recent works of the author \[25\], \[26\] contain an attempt of an investigation of “the analysis after the Plancherel formula”.

This work is a continuation of \[25\]. Here we are trying to understand the answer to following question.

**Question.** Assume that we know the explicit Plancherel formula for the restriction of a unitary representation \(\rho\) of a group \(G\) to a subgroup \(H\). Is it possible to write the action of Lie algebra of \(G\) in the direct integral of representations of \(H\)?

We obtain the positive solution of this problem for one of the simplest possible examples, precisely, for the tensor product of a representation of \(\text{SL}(2, \mathbb{R})\) with a highest weight and the conjugate representation with a lowest weight. This tensor product and its decomposition were widely discussed in the literature on representation theory and special functions in last 40 years (some references: \[27\], \[4\], \[29\], \[19\], \[8\]). Nevertheless, the formula for the action of \(sl_2 \oplus sl_2\) in tensor product were not appear. The reason is the unusual for representation theory form of these Lie algebra operators.

It turns out that the operators (12)–(14) of Lie algebra \(sl_2 \oplus sl_2\) are the second order differential operators with respect to one variable and the second order difference operators with respect to the another variable; moreover, it turns out that the difference operators are defined in the terms of a shift in the imaginary direction, i.e., in our formula, there appear the operators of the form

\[
T f(x) = f(x + i)
\]

for \(f \in L^2(\mathbb{R})\) (\(i^2 = -1\)). There is no self-contradiction in this expression, the shift operators are well defined on functions admitting the holomorphic continuation to an appropriate strip. Operators \(f \to xf\) and \(f \to \frac{d}{dx}\) also are not defined on the whole \(L^2(\mathbb{R})\).

First, these Lie algebra operators were obtained by author using theorems on the operational calculus for the index hypergeometric transform (17) from \[25\]. But the final formulas are elementary and admit a direct verification by elementary tools.

Since the approximate structure of formulas for action of overalgebra now became more clear, it is natural to formulate the general problem given above.

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1. **Group \(\text{SL}(2, \mathbb{R})\).** We realize the group \(\text{SL}(2, \mathbb{R})\) as the group of complex \(2 \times 2\) matrices having the form

\[
\begin{pmatrix}
a & b \\
b & \pi
\end{pmatrix},
\]

where \(|a|^2 - |b|^2 = 1\).

By \(D\) we denote the disk \(|z| < 1\) on the complex plane \(\mathbb{C}\), by \(S^1\) we denote the circle \(|z| = 1\); we represent the points of the circle in the form \(z = e^{i\varphi}\).
The group $\text{SL}(2, \mathbb{R})$ acts on the disk by the Möbius transformations

$$
\begin{pmatrix}
  a & b \\
  \bar{b} & \bar{a}
\end{pmatrix} : \quad z \mapsto \frac{az + b}{\bar{b}z + \bar{a}}.
$$

If $z \in S^1$, then its image under (1) also is contained in $S^1$.

2. **Highest weight representations of** $\text{SL}(2, \mathbb{R})$. Fix $\alpha > 1$. Consider the space $H_\alpha$ of holomorphic functions $f$ on the circle $D$ satisfying the condition

$$
\int_D |f(z)|^2 (1 - |z|^2)^{\alpha - 2} \{dz\} < \infty,
$$

where $\{dz\}$ denotes the Lebesgue measure on $D$. Define the inner product in the space $H_\alpha$ by

$$
\langle f, g \rangle_\alpha = \frac{\alpha - 1}{\pi} \int_D f(z) \overline{g(z)} (1 - |z|^2)^{\alpha - 2} \{dz\};
$$

the pre-integral factor provides the identity $\langle 1, 1 \rangle_\alpha = 1$. In terms of the Taylor coefficients, the inner product is given by

$$
\langle \sum c_k z^k, \sum c'_k z^k \rangle_\alpha = \sum_{k \geq 0} c_k c'_k \frac{k!}{\alpha(\alpha + 1) \ldots (\alpha + k - 1)},
$$

and condition (2) for $f(z) = \sum c_k z^k$ has the form

$$
\sum |c_k|^2 k^{-(\alpha - 1)} < \infty.
$$

It can easily be checked that the space $H_\alpha$ is complete with respect to the norm defined by this inner product, i.e., $H_\alpha$ is a Hilbert space.

The group $\text{SL}(2, \mathbb{R})$ acts in the space $H_\alpha$ by the unitary operators

$$
T_\alpha \begin{pmatrix}
  a & b \\
  \bar{b} & \bar{a}
\end{pmatrix} (z) = f \left( \frac{az + b}{\bar{b}z + \bar{a}} \right) \left( \frac{\bar{b}z + \bar{a}}{az + b} \right)^{-\alpha}.
$$

First, consider the case, when $\alpha$ is an integer. The factor $\left( \frac{\bar{b}z + \bar{a}}{az + b} \right)^{-\alpha}$ is a degree of the derivative of the function (1), this implies that $T_\alpha$ is a representation:

$$
T_\alpha(g_1)T_\alpha(g_2) = T_\alpha(g_1g_2).
$$

The operators $T_\alpha(g)$ are unitary, i.e.,

$$
\langle T_\alpha(g)f_1, T_\alpha(g)f_2 \rangle_\alpha = \langle f_1, f_2 \rangle_\alpha,
$$

this can be easily checked by a change of the variable.
For a noninteger $\alpha$, $T_\alpha(g)$ also is an unitary representation; we only must explain the meaning of the expression
\[
(bz + \overline{\mu})^{-\alpha} = (1 + b\overline{\mu}^{-1}z)^{-\alpha} = (1 + b\overline{\mu}^{-1}z)^{-\alpha}e^{-\alpha(\ln z + 2\pi ik)}.
\] (7)

Obviously, $|b\overline{\mu}^{-1}| < 1$. Hence the function
\[
(1 + b\overline{\mu}^{-1}z)^{-\alpha} = 1 + \frac{\alpha}{1!}b\overline{\mu}^{-1}z + \frac{\alpha(\alpha - 1)}{2!}(b\overline{\mu}^{-1})^2z^2 + \ldots
\]
is well defined.

Therefore the operator (5) is defined up to the factor $e^{-2\pi i\alpha}$, the absolute value of this factor is 1. Hence equality (6) is replaced by
\[
T_\alpha(g_1)T_\alpha(g_2) = \theta \cdot T_\alpha(g_1g_2),
\]
where $|\theta| = 1$. Thus, $T_\alpha(g)$ is an unitary projective representation of the group $\text{SL}(2, \mathbb{R})$.

**Remark.** Clearly, $T_\alpha(g)$ can be also considered as a linear representation of the universal covering group of $\text{SL}(2, \mathbb{R})$.

3. **Tensor product.** By $\overline{T}_\alpha(g)$ we denote the representation complex conjugate to $T_\alpha(g)$, it acts in the space $\overline{H}_\alpha$ of antiholomorphic functions in the circle $|u| < 1$ by the formula
\[
T_\alpha \left( \begin{array}{cc} a & b \\ \overline{b} & \overline{a} \end{array} \right) f(z) = f \left( \frac{\overline{a}z + \overline{b}}{b\overline{a} + a} \right)(b\overline{a} + a)^{-\alpha}.
\]
The inner product in $\overline{H}_\alpha$ is given by (3).

Consider the space $H_\alpha \otimes \overline{H}_\alpha$. It consists of functions $f(z, \overline{\mu})$ on the bidisk $D \times D$, holomorphic with respect to the variable $z$ and antiholomorphic in $\overline{\mu}$; the inner product in $H_\alpha \otimes \overline{H}_\alpha$ is given by
\[
\langle f_1, f_2 \rangle = \frac{(\alpha - 1)^2}{\pi^2} \int_{D \times D} f_1(z, \overline{\mu})\overline{f_2(z, \overline{\mu})}(1 - z\overline{\mu})^{\alpha - 2}(1 - \overline{\mu})^{\alpha - 2} \{dz\} \{d\overline{\mu}\}.
\]
The group $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ acts in $H_\alpha \otimes \overline{H}_\alpha$ by the operators $T_\alpha(g_1) \otimes \overline{T}_\alpha(g_2)$ given by
\[
T_\alpha \left( \begin{array}{cc} a_1 & b_1 \\ \overline{b}_1 & \overline{a}_1 \end{array} \right) \otimes \overline{T}_\alpha \left( \begin{array}{cc} a_2 & b_2 \\ \overline{b}_2 & \overline{a}_2 \end{array} \right) f(z, u) =
= f \left( \frac{a_1z + b_1}{b_1z + \overline{a}_1}, \frac{a_2\overline{\mu} + \overline{b}_2}{b_2\overline{\mu} + a_2} \right) \left( \frac{b_1z + \overline{\mu}}{b_2\overline{\mu} + a_2} \right)^{-\alpha}(b_2\overline{\mu} + a_2)^{-\alpha}.
\]
We restrict this representation to the diagonal subgroup $\text{SL}_2(\mathbb{R}) \subset \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$, this corresponds to the substitution $a_1 = a_2 = a$, $b_1 = b_2 = b$ to the
last formula. This representation of \( SL_2(\mathbb{R}) \) is a linear (nonprojective) representation. Indeed,
\[
(b\bar{z} + \bar{a})^\alpha (b\bar{u} + a)^\alpha = (1 + \bar{a}^{-1}b\bar{z})^\alpha (1 + a^{-1}b\bar{u})^\alpha (a\bar{a})^{-\alpha},
\]
and this expression is a single-valued function for \(|z| < 1, |u| < 1\).

4. Principal series of representations. Fix \( s \in \mathbb{R} \). Consider the representation \( \rho_s \) of the group \( SL(2, \mathbb{R}) \) in \( L^2 \) on the circle \( S^1 \) given by
\[
\rho_s \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) f(e^{i\varphi}) = f \left( \frac{a e^{i\varphi} + b}{\bar{b} e^{i\varphi} + \bar{a}} \right) \bar{b} e^{i\varphi} + \bar{a}^{-1-2is}.
\]
These unitary representations are the so-called representations of the principal series. Recall, that the representation \( T_s \) is equivalent to \( T_{-s} \) (see [3]).

5. Spectral decomposition. Let \( \alpha \) be the same as in pp.2–3. Consider the kernel
\[
K_\alpha(\varphi, s; z, u) := \frac{(1 - z e^{i\varphi})^{-1/2-i\alpha}(1 - u e^{-i\varphi})^{-1/2-i\alpha}}{(1 - \bar{z}u)^{\alpha+1/2+is}},
\]
where \(|z| < 1, |u| < 1, \varphi \in [0, 2\pi], s \in \mathbb{R}\). Consider the integral operator \( J_\alpha \), that takes a function \( f \in H_\alpha \otimes \overline{H}_\alpha \) to the function \( F(\varphi, s) \) given by
\[
F(\varphi, s) = \iint_{D \times D} K_\alpha(\varphi, s; z, u) f(z, \bar{z}) (1 - z \bar{z})^\alpha (1 - u \bar{u})^{2s} \{dz\} \{du\}. \quad (8)
\]
Consider the action of the group \( SL(2, \mathbb{R}) \) in space of functions in the variables \((\varphi, s)\) defined by
\[
R \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) F(e^{i\varphi}, s) = F \left( \frac{a e^{i\varphi} + b}{\bar{b} e^{i\varphi} + \bar{a}} \right) \bar{b} e^{i\varphi} + \bar{a}^{-1-2is}.
\]
Note, that for a fixed \( s \in \mathbb{R} \) the function \( F(e^{i\varphi}, s) \) (as a function in \( \varphi \)) transforms by the formula for principal series representations.

Simple calculation show, that the operator \( J_\alpha \) intertwines the representations \( T_\alpha(g) \otimes \overline{T}_\alpha(g) \) and \( R \) of \( SL(2, \mathbb{R}) \):
\[
J_\alpha \cdot (T_\alpha(g) \otimes \overline{T}_\alpha(g)) = R(g) \cdot J_\alpha \quad (9)
\]

**Theorem 1.** Operator \( J_\alpha \) is a unitary operator from \( H_\alpha \otimes \overline{H}_\alpha \) to the space \( L^2 \) on \( \varphi \in [0, 2\pi], s \geq 0 \) with respect to the measure
\[
\frac{\Gamma(\alpha - 1/2 + is)^2}{\Gamma(\alpha)^2} \frac{\sinh(\pi s)}{\cosh(\pi s)} ds \, d\varphi = \iint \frac{\Gamma(\alpha - 1/2 + is)\Gamma(1/2 + is)}{\Gamma(\alpha)\Gamma(is)}^2 \, ds \, d\varphi. \quad (10)
\]
Thus $T_\alpha \otimes T_\alpha$ is a multiplicity free integral over principal series representations (this fact was obtained by Pukanszky [27]). Various ways for obtaining the Plancherel measure (9) are contained in [29], [19], [8], [25].

6. Holomorphic continuation of $J_\alpha f(\varphi, s)$. Denote by $W$ the space of all $f \in H_\alpha \otimes H_\alpha$ admitting analytic continuation to a bidisk $|z| < 1 + \delta$, $|u| < 1 + \delta$.

For $f \in W$, consider its Taylor series

$$f(z, u) = \sum_{k,l} c_{kl} z^k u^l.$$ 

Obviously, its coefficients $c_{kl}$ exponentially decrease for $k + l \to \infty$.

**Lemma.** Fix $\varphi$. For $f \in W$, the function $J_\alpha f(\varphi, s)$ can be extended holomorphically to the whole complex plane $s \in \mathbb{C}$.

**Proof.** Fix $s \in \mathbb{C}$, $\varphi \in [0, 2\pi]$. The function $K_\alpha(\varphi, s; z, u)$ as a function in the variables $z, u$ has a polynomial growth near the boundary of the bidisk $D \times D$:

$$|K_\alpha(\varphi, s; z, u)| \leq \exp\{4\pi (1 + |\text{Im } s|)(1 - |z|)^{-1+\alpha+2|\text{Re } s|}(1 - |u|)^{-1+\alpha+2|\text{Re } s|}\}$$

Hence, for fixed $s, \varphi$, the coefficients $a_{kl} = a_{kl}(\varphi, s)$ of the series

$$K_\alpha(\varphi, s; z, u) = \sum a_{kl}(s, \varphi) z^k u^l$$

have the polynomial growth as $k, l \to +\infty$.

Indeed, let $q(z, u)$ be a function in the bidisk satisfying

$$|q(z, u)| \leq C \cdot \delta^{-h} \quad \text{for} \quad |z| \leq 1 - \delta, \ |u| \leq 1 - \delta.$$

The Taylor coefficients $b_{kl}$ of $q(z, u)$ are given by the formula

$$b_{kl} = \frac{1}{(2\pi i)^2} \iint_{|z|=1-\delta, |u|=1-\delta} \frac{q(z) \, dz \, du}{z^{k+1} u^{l+1}}.$$ 

Hence,

$$|b_{kl}| \leq \text{const} \cdot \delta^{-h} (1 - \delta)^{-k-l-2},$$

for all $\delta$. We choose $\delta = h/(h + k + l + 2)$ and obtain polynomial growth for $b_{kl}$.

For the kernel $K_\alpha(z, u)$, we obtain in this way the uniform estimates of the form $|a_{kl}| \leq A \cdot (1 + k + l)^{\tau}$ in each rectangle

$$|\text{Im } s| \leq N, \quad |\text{Re } s| \leq M. \quad (11)$$

Since the Taylor coefficients $c_{kl}$ for $f \in W$ exponentially decrease, the series

$$J_\alpha f(\varphi, s) = \sum c_{k,l} a_{k,l}(s, \varphi)$$

6
is absolutely convergent (see (4)), the summands are holomorphic with respect to \( s \), and the series is uniformly convergent on rectangles (10).

7. Correspondence of differential operators. The operators of the Lie algebra \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \) in the space \( H_\alpha \oplus \overline{H}_\alpha \) have the form

\[
L_0^{(z)} = z \frac{\partial}{\partial z} + \frac{\alpha_1}{2}; \quad L_1^{(z)} = z^2 \frac{\partial}{\partial z} + \alpha z; \quad L_{-1}^{(z)} = \frac{\partial}{\partial z}
\]

\[
L_0^{(u)} = \frac{u}{\alpha} \frac{\partial}{\partial u} + \frac{\alpha_2}{2}; \quad L_1^{(u)} = \frac{\partial}{\partial u}; \quad L_{-1}^{(u)} = \frac{u^2}{\alpha} \frac{\partial}{\partial u} + \alpha \frac{u}{u}
\]

For us it will be more convenient the following collection of the operators

\[
L_0 := L_0^{(z)} - L_0^{(u)}; \quad L_{-1} := L_1^{(z)} - L_1^{(u)}; \quad L_{-1} := L_1^{(z)} - L_{-1}^{(u)};
\]

\[
M_0 := L_0^{(z)} + L_0^{(u)}; \quad M_{-1} := L_1^{(z)} + L_{-1}^{(u)}; \quad M_{1} := L_1^{(z)} + L_{-1}^{(u)};
\]

The operators \( L_0, L_1, L_{-1} \) span the diagonal subalgebra \( \mathfrak{sl}_2 \) in \( \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \). Their images under the operator \( J_\alpha \) are defined by the formulas

\[
J_\alpha \left[ z \frac{\partial}{\partial z} - \frac{u}{\alpha} \frac{\partial}{\partial u} \right] f(\varphi, s) = \left[ \frac{\partial}{i \partial \varphi} \circ J_\alpha \right] f(\varphi, s)
\]

\[
J_\alpha \left[ \frac{u^2}{\alpha} \frac{\partial}{\partial u} + \alpha \frac{u}{u} - \frac{\partial}{\partial z} \right] f(\varphi, s) = -\left[ e^{i \psi} \frac{\partial}{i \partial \varphi} + \left( \frac{1}{2} + is \right) e^{i \psi} \right] J_\alpha f(\varphi, s)
\]

\[
J_\alpha \left[ z^2 \frac{\partial}{\partial z} + \alpha z - \frac{\partial}{\partial \varphi} \right] = \left[ e^{-i \psi} \frac{\partial}{i \partial \varphi} - \left( \frac{1}{2} + is \right) e^{-i \psi} \right] J_\alpha f(\varphi, s)
\]

These three formulas easily follow from (8).

Theorem 2. The unitary operator \( J_\alpha \) transform the operator \( M_0 \) to the operator

\[
Q_0 f(\varphi, s) = -\left( -\frac{1}{2} + is \right) \left( -\alpha + \frac{1}{2} + is \right) f(\varphi, s + i) + \frac{2is}{2is} \left( -\alpha + \frac{1}{2} + is \right) f(\varphi, s - i) - \frac{\alpha + \frac{1}{2} + is}{2is} \frac{\partial^2}{\partial \varphi^2} f(\varphi, s + i), \quad (12)
\]

i.e., for any \( f \in W \) (see our Section 6),

\[
Q_0 J_\alpha f = J_\alpha M_0 f.
\]

The operator \( M_1 \) transforms to

\[
Q_1 f(\varphi, s) = e^{i \psi} \left[ \frac{1}{2} + is \right] \left( -\alpha + \frac{1}{2} + is \right) f(\varphi, s + i) + \frac{2is}{2is} \left( \frac{1}{2} + is \right) \left( -\alpha + \frac{1}{2} + is \right) f(\varphi, s - i) - \frac{\alpha + \frac{1}{2} + is}{2is} \frac{\partial}{\partial \varphi} f(\varphi, s + i) + \frac{\alpha + \frac{1}{2} + is}{\frac{1}{2} + is} \frac{\partial}{i \partial \varphi} f(\varphi, s + i), \quad (13)
\]
and the operator \( M_{-1} \) transforms to

\[
Q^{-1}f(\varphi, s) = e^{-i\varphi}\left[\frac{(\frac{1}{2} + is)(-\alpha + \frac{1}{2} + is)}{2is}f(\varphi, s + i) + \frac{(\frac{1}{2} + is)(\alpha - \frac{1}{2} + is)}{2is}f(\varphi, s - i) - \frac{-\alpha + \frac{1}{2} + is}{2is(-\frac{1}{2} + is)}d^2f(\varphi, s + i) - \frac{-\alpha + \frac{1}{2} + is}{-\frac{1}{2} + is}i\partial_\varphi f(\varphi, s + i)\right].
\] (14)

**Proof.** These formulas can be checked by direct calculations. For instance, let us consider \( M_0 \).

Obviously, the operator \( M_0 \) is selfadjoint in \( H_\alpha \otimes \overline{H}_\alpha \). Hence

\[
J_\alpha M_0 f(\varphi, s) = \int\int_{D \times D} K(\varphi, s; \overline{z}, u)\left[\left(\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + \alpha\right)f(z, \overline{u})\right] (1 - z\overline{z})^{\alpha - 2} (1 - u\overline{u})^{\alpha - 2} \{dz\} \{du\} = \int\int_{D \times D} \left[\left(\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + \alpha\right)f(z, \overline{u}))(1 - z\overline{z})^{\alpha - 2} (1 - u\overline{u})^{\alpha - 2} \{dz\} \{du\}
\]

Thus the first statement of the theorem is equivalent to the identity

\[
\left(u\frac{\partial}{\partial u} + \overline{z}\frac{\partial}{\partial z} + \alpha\right)K(\varphi, s; \overline{z}, u) - Q_0[K(\varphi, s; \overline{z}, u)] = 0 \tag{15}
\]

After division by \( K \), this identity transforms to the form

\[
K^{-1}\left\{u\frac{\partial}{\partial u} + z\frac{\partial}{\partial z} + \alpha\right\}K + \alpha + K^{-1}\left(\frac{-\frac{1}{2} + is}{2is}\right)(-\alpha + \frac{1}{2} + is)K(s + i) - K^{-1}\left(\frac{-\frac{1}{2} + is}{2is}\right)(1 - \frac{1}{2} + is)\frac{\partial^2}{\partial \varphi^2}K(s + i) \tag{16}
\]

The function in the left side is a long rational expression in \( z, \overline{u}, e^{i\varphi}, s \); the identity can be easily verified by MAPLE.

Let us explain how to verify the identity (15) "by hands". Each summand of (15) can be represented as a linear combination

\[
a(s) + b(s) \cdot \frac{1}{1 - z\overline{u}} + c(s) \cdot \left[\frac{1}{1 - z e^{i\varphi}} + \frac{1}{1 - \overline{u} e^{-i\varphi}}\right] + d(s) \cdot \frac{(1 - z e^{i\varphi})(1 - \overline{u} e^{-i\varphi})}{1 - z\overline{u}}
\]

After this, it remains to sum the coefficients.

**Remark.** Let us apply the operator \( Q_0 \) to the functions \( f(\varphi, s) = g(s) \).

Then the equations

\[
Q_0 g(s) = (k + \alpha)g(s)
\]
coincide with a partial case of the difference equations for the continuous dual Hahn polynomials (see [1], (6.10.9)).

8. Some remarks. In the work of the author [25], there were obtained some elements of an operational calculus for the index hypergeometric transform (it is called also by Olevsky transform or Jacobi transform, see [30], [17])

\[ g(x) \mapsto \hat{g}(s) = \frac{1}{\Gamma(b + c)} \int_0^\infty g(x) \, \, _2F_1(b + is, b - is; b + c; -x)x^{b+c-1}(1 + x)^{b-c} dx, \]  

(17)

In [25], it was shown that the index hypergeometric transform maps the differential operators

\[ Ag(x) = xg(x); \quad Bg(x) = x \frac{\partial}{\partial x} g(x) \]

(and hence all the operators admitting polynomial expression in \( x, x \frac{\partial}{\partial x} \)) to difference operators in imaginary direction; see also the work of Cherednik [6] containing some similar statements for symmetric functions in multidimensional case. Existence of the formulas (11)–(13) more or less follows from these results, but this way for obtaining the expressions (11)–(13) also is not very simple.

There arises the following question.

Question. Is it possible to write explicitly operators of the overalgebra for the case of \( L^2 \) on a pseudo-Riemannian symmetric space and for the kernel representations?

Is it possible to do this at least for rank 1 symmetric spaces?

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