Irreducible Highest Weight Representations Of The Simple n-Lie Algebra

Dana Balibanu, Johan van de Leur

Department of Mathematics,
Utrecht University,
P.O. Box 80.010,
3508 TA Utrecht,
The Netherlands

email: D.M.Balibanu@uu.nl, J.W.vandeLeur@uu.nl

Abstract

A. Dzhumadil’daev classified all irreducible finite dimensional representations of the simple n-Lie algebra. Using a slightly different approach, we obtain in this paper a complete classification of all irreducible, highest weight modules, including the infinite-dimensional ones. As a corollary we find all primitive ideals of the universal enveloping algebra of this simple n-Lie algebra.

1 Introduction

In 1985 Filippov [6] introduced a generalization of a Lie algebra, which he called an n-Lie algebra. The Lie product is taken between n elements of the algebra instead of two. This new bracket is n-linear, anti-symmetric and satisfies a generalization of the Jacobi identity. For n = 3 this product is a special case of the Nambu bracket, well known in physics, which was introduced by Nambu [13] in 1973, as a generalization of the Poisson bracket in Hamiltonian mechanics. In recent years this type of Lie algebra has appeared in the physics literature. For example a metric 3-Lie algebra is used in the Lagrangian description of a certain 2+1 dimensional field theory, the so called Bagger-Lambert-Gustavsson theory [1]-[10]. This has lead to a dramatic increase of interest in n-Lie algebras in recent years.

Let n be a natural number greater or equal to 3. An n-Lie algebra is a natural generalization of a Lie algebra. Namely:

Definition 1.1. A vector space V together with a multi-linear, antisymmetric n-ary operation [,] : V^n → V is called an n-Lie algebra, n ≥ 3, if the n-ary bracket is a derivation with respect to itself, i.e,

\[ [x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n-1}] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, x_{n+1}, \ldots, x_{2n-1}], \ldots, x_n], \quad (1) \]
where \(x_1, \ldots, x_{2n-1} \in V\).

Equation (1) is called the \textit{generalized Jacobi Identity}. We will only consider \(n\)-Lie algebras over the field of complex numbers.

In [12] W. Ling has shown that for every \(n \geq 3\) there is, up to isomorphism, only one simple \(n\)-Lie algebra, namely \(C^{n+1}\) and the operation being given by the generalized vector product. Denote by \(e_1, \ldots, e_{n+1}\) the standard basis of \(C^{n+1}\), then the \(n\)-ary bracket is given by:

\[
[e_1, \ldots, \hat{e_i}, \ldots, e_{n+1}] = (-1)^{n+i+1}e_i,
\]

where \(i\) ranges from 1 to \(n + 1\) and the hat means that \(e_i\) does not appear in the bracket.

A. Dzhumadil’daev studied in [5] the finite-dimensional, irreducible representation of the simple \(n\)-Lie algebra. It is our aim to classify in the present paper both finite- and infinite-dimensional, irreducible, highest weight representations of this algebra. The annihilators of these modules are the primitive ideals of the universal enveloping algebra of the simple \(n\)-Lie algebra.

To every \(n\)-Lie algebra, one can associate a \textit{basic Lie algebra}. For the simple \(n\)-Lie algebra, denoted throughout this paper by \(A\), the associated basic Lie algebra is \(so(n+1)\). Finding irreducible, highest weight representations of \(A\) is equivalent to finding irreducible highest weight representations of \(so(n+1)\) on which some two-sided ideal, of the universal enveloping algebra of \(so(n+1)\), acts trivially. The method used in this paper has the advantage of recovering both finite- and infinite-dimensional, irreducible, highest weight representations. The annihilators of these modules will supply the primitive ideals we are searching for. Below, we give the statement of the main theorem we obtained. The two statements are similar in nature, the only difference being given by the parity of \(n + 1\).

Denote by \(Z(\lambda)\) the unique irreducible quotient of the Verma module of \(so(n+1)\) which has highest weight \(\lambda\).

\textbf{Theorem 1.2.} Let \(n \geq 3\), \(n+1 = 2N\) and \(t \in \{1, \ldots, N\}\). Denote by \(\pi_1, \ldots, \pi_N\) the fundamental weights of \(so(2N)\). Then, \(Z(\lambda)\) is an irreducible representation of the simple \(n\)-Lie algebra \(A\) if and only if

\[
\lambda = \begin{cases} 
  x\pi_1 & t = 1, \\
  (-1 - x)\pi_{t-1} + x\pi_t & 1 < t < N - 1, \\
  (-1 - x)\pi_{N-2} + x\pi_{N-1} + x\pi_N & t = N - 1, \\
  (-1 - x)\pi_{N-1} + (-1 + x)\pi_N & t = N,
\end{cases}
\]

where \(x \in \mathbb{C}\).

\textbf{Theorem 1.3.} Let \(n \geq 3\), \(n+1 = 2N + 1\) and \(t \in \{1, \ldots, N\}\). Denote the fundamental weights of \(so(2N+1)\) by \(\pi_1, \ldots, \pi_N\). Then, \(Z(\lambda)\) is an irreducible highest weight representation of the simple \(n\)-Lie algebra \(A\) if and only if

\[
\lambda = \begin{cases} 
  x\pi_1 & t = 1, \\
  (-1 - x)\pi_{t-1} + x\pi_t & 1 < t \leq N,
\end{cases}
\]

where \(x \in \mathbb{C}\).
Chapter 2 gives an introduction to the theory of n-Lie algebras. It presents all the definitions used in this paper. Chapter 3 describes the strategy adopted for solving the problem. It also introduces a graphical language which will be very useful further on. The details of the strategy are presented in Chapter 4. Here the main results are stated and two ways of arriving to the same conclusion are presented. The final chapter is concerned with primitive ideals. The authors would like to express their gratitude to V. Kac, E. van den Ban and I. Mǎrcuţ for their interest in this work and for all the good suggestions.

2 Main definitions

In this section we will give a short introduction to n-Lie algebras. We will concentrate mainly on the most important definitions and some useful theorems.

2.1 n-Lie algebras and their basic Lie algebras

As mentioned in the introduction, an n-Lie algebra is a vector space V, over the field of complex numbers, together with an n-ary operation that satisfies anti-symmetry, multi-linearity and the generalized Jacobi Identity.

Let V be an n-Lie algebra, n ≥ 3. We will associate to V a Lie algebra called the basic Lie algebra. This construction goes as presented below.

Consider \( \text{ad} : \wedge^{n-1}V \to \text{End}(V) \) given by \( \text{ad}(a_1 \wedge \ldots \wedge a_{n-1})(b) = [a_1, \ldots, a_{n-1}, b] \).

One can easily see that we could have chosen the codomain of \( \text{ad} \) to be \( \text{Der}(V) \) (the set of derivations of \( V \)) instead of \( \text{End}(V) \). \( \text{ad} \) induces a map \( \tilde{\text{ad}} : \wedge^{n-1}V \to \text{End}(\wedge^nV) \) defined as \( \tilde{\text{ad}}(a_1 \wedge \ldots \wedge a_{n-1})(b_1 \wedge \ldots \wedge b_m) = \sum_{i=1}^{n-1} b_1 \wedge \ldots \wedge [a_1, \ldots, a_{i-1}, b_i] \wedge \ldots \wedge b_m. \)

Denote by \( \text{Inder}(V) \) the set of inner derivations of \( V \), i.e. endomorphisms of the form \( \text{ad}(a_1, \ldots, a_{n-1}) \).

Let \( a := a_1 \wedge \ldots \wedge a_{n-1} \) and \( b := b_1 \wedge \ldots \wedge b_{n-1} \) be elements of \( \wedge^{n-1}V \). Define \( [a, b] = \frac{1}{n!} (\text{ad}(a)(b) - \text{ad}(b)(a)). \)

**Proposition 2.1.** \([\cdot, \cdot]\) defines a Lie algebra structure on \( \wedge^{n-1}V \) and \( \tilde{\text{ad}} : \wedge^{n-1}V \to \text{Inder}(V) \) is a surjective Lie algebra homomorphism.

**Proof.** The skew-symmetry of the bracket is obvious, and so is the surjectivity of \( \text{ad} \), thus we only need to prove that the Jacobi identity holds. In order to do this, we first show that \( \tilde{\text{ad}}(\text{ad}(a)(b)) = \text{ad}(a) \circ \text{ad}(b) - \text{ad}(b) \circ \text{ad}(a) = [\text{ad}(a), \text{ad}(b)] \), and from here it will follow that \( \tilde{\text{ad}} \) is an Lie algebra homomorphism, i.e. \( \tilde{\text{ad}}([a, b]) = \text{ad}(a) \circ \text{ad}(b) - \text{ad}(b) \circ \text{ad}(a) = [\text{ad}(a), \text{ad}(b)]. \) Since both the left-hand-side and the right-hand-side of the equation are derivations of the exterior algebra \( \wedge^nV, \wedge \) (as previously mentioned) it suffices to show the equality for some arbitrary \( c \in V \).

\[
\tilde{\text{ad}}(\text{ad}(a)(b))(c) = \tilde{\text{ad}}(\text{ad}(a_1 \wedge \ldots \wedge a_{n-1})(b_1 \wedge \ldots \wedge b_{n-1}))(c) = \\
= \text{ad}(\sum_{i=1}^{n-1} b_1 \wedge \ldots \wedge [a_1, \ldots, a_{i-1}, b_i] \wedge \ldots \wedge b_{n-1})(c) = \\
= \end{equation}

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Hence, \[ n \sum_{i=1}^{n-1} [b_i, \ldots, [a_i, \ldots, a_{n-1}, b_i], \ldots, b_{n-1}, c] = \]
\[ = [a_1, \ldots, a_{n-1}, [b_1, \ldots, b_{n-1}, c]] - [b_1, \ldots, b_{n-1}, [a_1, \ldots, a_{n-1}, c]] = \]
\[ = (\tilde{\text{ad}}(a) \circ \tilde{\text{ad}}(b) - \tilde{\text{ad}}(b) \circ \tilde{\text{ad}}(a))(c). \]

Hence, \[
\tilde{\text{ad}}([a, b]) = \tilde{\text{ad}}(\frac{1}{2}(\tilde{\text{ad}}(a)(b) - \tilde{\text{ad}}(b)(a))) = \\
= \frac{1}{2}(\tilde{\text{ad}}(\tilde{\text{ad}}(a)(b)) - \tilde{\text{ad}}(\tilde{\text{ad}}(b)(a))) = \\
= (\tilde{\text{ad}}(a) \circ \tilde{\text{ad}}(b) - \tilde{\text{ad}}(b) \circ \tilde{\text{ad}}(a)) = \\
= [\text{ad}(a), \text{ad}(b)].
\]

Now the Jacobi identity follows easily and we are done. \(\square\)

**Definition 2.2.** A subspace \( V' \subseteq V \) is called a subalgebra of the \( n \)-Lie algebra \( V \) if \([V', \ldots, V'] \subseteq V'\). A subalgebra \( I \subseteq V \) of an \( n \)-Lie algebra \( V \) is called an ideal if \([I, V, \ldots, V] \subseteq I\). An \( n \)-Lie Algebra \( V \) is called simple if it has no proper ideals besides 0.

We will illustrate these two notions by means of the following example.

**Example 2.3.** Let \( V = \{ f : \mathbb{R}^n \to \mathbb{R} | f \text{ of class } C^\infty \}, \ n \geq 3 \), and define the \( n \)-ary bracket on \( V \) as the Jacobian.

\[
[f_1, \ldots, f_n] = \begin{vmatrix} \\
\frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \ldots & \frac{\partial f_n}{\partial x_n}
\end{vmatrix}.
\]

\( V \) together with the operation given by the Jacobian forms an \( n \)-Lie algebra. Denote by \( V' \) the subset of \( V \) containing the polynomial functions in \( V \). Then \( V' \) together with the inherited operation forms a subalgebra of the \( n \)-Lie algebra \( V \). An ideal of this \( n \)-Lie algebra can be obtained as follows. Denote by \( I \) the set of functions in \( V \) which are flat at the origin. Then \( I \) is an ideal of \( V \).

In his PhD-thesis in 1993 [12], Wuxue Ling classified all simple \( n \)-Lie algebras, \( n \geq 3 \). He arrived at the following conclusion: for every \( n \geq 3 \) the only simple \( n \)-Lie algebra is isomorphic to \( \mathbb{C}^{n+1} \), where the bracket is defined as follows. Denote by \( e_1, \ldots, e_{n+1} \) the canonical basis of \( \mathbb{C}^{n+1} \), then:

\[
[e_1, \ldots, e_i, \ldots, e_{n+1}] = (-1)^{n+i+1} e_i,
\]

where \( e_i \) means that \( e_i \) is omitted.

**Remark 1.** From now on we will denote a generic \( n \)-Lie algebra by \( V \) and the simple one by \( A \). The vector space \( A \) comes together with an inner product, \( \langle e_i, e_j \rangle = \delta_{i,j} \), and the standard orientation form, \( e_1 \wedge \ldots \wedge e_{n+1} \). Hence, we can observe that the simple \( n \)-Lie Algebra \( A \) is just the vector space \( \mathbb{C}^{n+1} \) together with the Hodge star operation \(* : \wedge^n \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \). Moreover \( * : \wedge^2 A \to \wedge^{n-1} A \) is an isomorphism of Lie algebras; \( \wedge^{n-1} A \cong \wedge^2 A \cong so(n+1) \).
2.2 n-Lie modules

Definition 2.4. A vector space $M$ is called an $n$-Lie module for the $n$-Lie algebra $V$, if on the direct sum $V \oplus M$ there is the structure of an $n$-Lie algebra, such that the following conditions are satisfied:

- $V$ is a subalgebra;
- $M$ is an abelian ideal, i.e. when at least two slots of the $n$-bracket are occupied by elements in $M$ the result is 0.

Let $M$ be an $n$-Lie module of $V$. Then, $M$ is a module for the basic Lie algebra $\wedge^{n-1}V$, where the action is given by $x_1 \wedge \ldots \wedge x_{n-1} \cdot m = \left[ x_1, \ldots, x_{n-1}, m \right]$. On the direct sum of the two spaces $M$ and $V$ we must have the structure of an $n$-Lie algebra. This means that the generalized Jacobi identity has to hold. If we write out this condition, we obtain a two sided ideal of the universal enveloping algebra of $\wedge^{n-1}V$, generated by the elements

\[
x_{a_1,\ldots,a_{2n-2}} = [a_1,\ldots,a_n] \wedge a_{n+1} \wedge \ldots \wedge a_{2n-2} - \sum_{i=1}^{n} (-1)^{i+n}(a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge a_n)(a_i \wedge a_{n+1} \wedge \ldots \wedge a_{2n-2}).
\]

This two sided ideal must act trivially on $M$. We will denote this ideal by $Q(V)$ and the universal enveloping algebra of the basic Lie algebra by $U := U(\wedge^{n-1}V)$. On the other hand, any module of the basic Lie algebra, on which $Q(V)$ acts trivially, satisfies the conditions defining an $n$-Lie module of $V$. Thus, because of this property, it makes sense to define:

Definition 2.5. The universal enveloping algebra of the $n$-Lie algebra $V$, denoted by $U(V)$, is defined as $U/Q(V)$.

Because of this definition, we obtain the same relation, between the representations of an $n$-Lie algebra and the representations of its universal enveloping algebra, as for Lie algebras. Namely, representations of the $n$-Lie algebra $V$ are in 1-1 correspondence with the representations of the universal enveloping algebra $U(V)$. It was proven in [5] that an $n$-Lie module $M$ of $V$ is irreducible / completely reducible if and only if it is irreducible / completely reducible as a module of the Lie algebra $\wedge^{n-1}V$.

For Lie algebras, primitive ideals are defined to be two sided ideals of the universal enveloping algebra, which are annihilators of irreducible representations. It makes sense to define them the same in the case of $n$-Lie algebras.

Definition 2.6. Let $V$ be an $n$-Lie algebra and $U(V)$ its universal enveloping algebra. A two-sided ideal of $U(V)$ is called primitive if it is the annihilator of some irreducible module of $V$.

Let $M$ be an irreducible module of the $n$-Lie algebra $V$. The annihilator of this module, $\text{Ann}(M)$, is a primitive ideal of $U(V)$, by definition. Since $U(V) = U/Q(V)$, we may conclude that $\text{Ann}(M)$ is a primitive ideal of $U$ which includes $Q(V)$. On the other hand, let $I$ be a primitive ideal of $U$ which includes $Q(V)$. Then, there exist an irreducible module of $U$, call it $M$, which is
annihilated by $I$, and thus by $Q(V)$. This transforms $M$ into a module of $U(V)$ and $I$ into a primitive ideal of $U(V)$. Hence, we may conclude that primitive ideals of $U(V)$ are in $1-1$ correspondence with the primitive ideals of $U$ which contain $Q(V)$.

Primitive ideals of $U$ are, by definition, annihilators of the irreducible modules. It was proven in [3], by M. Duflo, that one does not need to look at all irreducible modules. Namely, it suffices to look only at the highest weight, irreducible ones and these will supply all the primitive ideals. Irreducible modules of an $n$-Lie algebra are in particular irreducible modules of the associated basic Lie algebra. We define highest weight modules of the $n$-Lie algebra similarly. A module of the $n$-Lie algebra $V$ is called a highest weight module, if it is a highest weight module of the basic Lie algebra $\wedge^{n-1}V$. This means that our problem of determining the primitive ideals of $U(A)$, can be reformulated as determining the highest weight, irreducible representations of $A$.

2.3 The simple $n$-Lie algebra $A$

Recall that we denoted by $A$ the simple, $n+1$-dimensional $n$-Lie algebra.

The ideal $Q(A)$ is generated by the elements

$$x_{a_1,\ldots,a_{2n-2}} = [a_1,\ldots,a_n] \wedge a_{n+1} \wedge \ldots \wedge a_{2n-2} -$$

$$- \sum_{i=1}^{n} (-1)^{i+1} (a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge a_n)(a_1 \wedge a_{n+1} \wedge \ldots \wedge a_{2n-2}).$$

For the simple $n$-Lie algebra $A$ we can compute these generators further. Instead of the elements $a_i \in A$ we plug in elements of the standard basis of $A$ and use the isomorphism of Lie algebras which sends the element $e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge e_{n+1} \in \wedge^{n-1}A$ to

$$e_{ij} := e_i \wedge e_j = \frac{e_i \otimes e_j - e_j \otimes e_i}{2} = E_{ij} - E_{ji} \in so(n+1), \quad 1 \leq i < j \leq n+1.$$  

We obtain the following relations:

$$x_{i,k,l,m} = \begin{cases} e^i e^j e^k e^l - e^i e^k e^l e^j + e^i e^m e^l & \text{if } i, k, l, m \text{ are all distinct}, \\ 0 & \text{otherwise}, \end{cases}$$

where $i < k < l < m \in \{1,\ldots,n+1\}$. Here, $x_{i,k,l,m}$ is just a short-hand notation of the generator denoted before by $x_{a_1,\ldots,a_{2n-2}}$. For the detailed computation, we refer the reader to [5].

Remark 2. We will show later on, that the condition on the indices can be dropped. Although it is not necessary, we will still assume them to be ordered.

One can easily see that these relations live in $S^2(so(n+1))$.\footnote{This will play an important role in the near future.}

Denote by $R := \text{Span}\{x_{j,k,l,m}\}_{1 \leq j,k,l,m \leq n+1}$ and recall that $\wedge^{n-1}A \simeq \wedge^2 A$.

Lemma 2.7. $R$ is a finite dimensional $\wedge^2 A$-module.
Proof. An easy computation. \hfill ∎

**Lemma 2.8.** The left-sided ideal generated by \( R \) equals to the right-sided ideal generated by \( R \) and equals \( Q(A) \).

Consider the \( \wedge^2 A \)-module \( \wedge^4 A \) and the map \( \psi : \wedge^4 A \to S^2(\wedge^2 A) \) defined on monomials as:

\[
(v_i \wedge v_j \wedge v_k \wedge v_l) \mapsto (v_i \wedge v_j) \circ (v_k \wedge v_l) - (v_i \wedge v_k) \circ (v_j \wedge v_l) + (v_i \wedge v_l) \circ (v_j \wedge v_k)
\]

**Lemma 2.9.** Let \( \tilde{\psi} : \wedge^4 A \to R \) be the restriction of the map \( \psi \) to \( R \subset S^2(\wedge^2 A) \). Then, \( \tilde{\psi} \) is an isomorphism of Lie modules.

Of course, we can also define the map \( \phi : S^2(\wedge^2 A) \to \wedge^4 A \) defined on monomials as:

\[
(v_i \wedge v_j) \circ (v_k \wedge v_l) \mapsto v_i \wedge v_j \wedge v_k \wedge v_l.
\]

Observe that \( \phi \circ \psi = 3Id \). Then, \( \ker \phi \) is a subrepresentation of \( \wedge^2 A \) in \( S^2(\wedge^2 A) \), complementary to \( R \).

Hence, we have obtained the following decomposition into submodules:

\[
S^2(\wedge^2 A) = \psi(\wedge^4 A) \oplus \ker \phi.
\]

On the other hand, we already know that

\[
U(\text{so}(n + 1)) = \bigoplus_k S^k(\wedge^2 A).
\]

Thus, the universal enveloping algebra of the simple \( n \)-Lie algebra \( A \) is \( U = \bigoplus_k S^k(\wedge^2 A)/Q(A) \).

### 3 Finding irreducible highest weight representation of \( A \)

Our main concern in this section will be to find irreducible, highest weight representations of the simple \( n \)-Lie algebra. Once this is done, it will be easy, by looking at the highest weight, to figure out which of these representations are finite dimensional and which are not.

#### 3.1 A graphical interpretation of the generators of \( Q(A) \)

In order to understand the relations which generate \( Q(A) \) better, we want to view them as graphical diagrams. On the basis elements of \( \text{so}(n+1) \) we define the lexicographical order, namely \( e^{i_1j_1} \leq e^{i_2j_2} \iff i_1 < i_2 \), or \( i_1 = i_2 \) and \( j_1 \leq j_2 \).

In this case, we say that \( (i_1, j_1) \leq (i_2, j_2) \). We always assume that \( i < j \). If this is not the case, we can interchange them by the following rule: \( e^{ij} = -e^{ji} \).

Taking into account that two basis elements of \( \text{so}(n+1) \), which are not in lexicographical order, can be reordered, using the Lie bracket, at the expense of some term of degree one less, it becomes easy to give a PBW-basis of \( U \), the

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2The reader should observe that these are exactly the generators of \( Q(A) \), the only difference being the notation.

3A rigorous proof of this can be found later on.

4This fact will supply the proof to a future statement about the PBW-basis of \( U \).
universal enveloping algebra of the basic Lie algebra $\wedge^{n-1} A \simeq so(n+1)$. Let $U_k = \{ e \in U | e = e^{i_1 j_1} e^{i_2 j_2} \ldots e^{i_k j_k}, \text{ where } (i_1, j_1) \leq (i_2, j_2) \leq \ldots \leq (i_k, j_k) \}$, i.e. all simple elements of degree $k$. Then a Poincare-Birkhoff-Witt basis is given by $\bigcup_k U_k$.

For any simple element of degree 1, i.e. some $e^{ij}$, we can represent it graphically as $n+1$ ordered points with an oriented arc going from the $i$’th point to the $j$’th. (Recall that both $i$ and $j$ range from 1 to $n+1$.)

Changing the orientation of this arc is the same as changing the order of $i$ and $j$, thus it results in a minus sign in front of the diagram. Some arbitrary product in $U$ can be represented similarly as $n+1$ ordered points with arcs connecting them. Multiple arcs between the same two points are allowed, each of this arcs having its own number above it. This number stands for the place the basis element occupies in the product. Multiplication of such elements can be translated, in this graphical language, as the overlapping of such diagrams, where the numbers above the arcs in the second diagram have to be shifted by a number equal to the number of arcs in the first diagram.

In $U$ the following commutation relations hold. These are the relations mentioned in the beginning of this section, represented now graphically.

In view of these relations, the numbers above the arcs can be dropped and the order of the arrows can be assumed to be the lexicographical order.

Now we can give a graphical interpretation to the generators $x_{i,j,k,l}$ of $Q(A)$. They tell us that we can resolve intersections in any diagram:

From here we can conclude that a PBW-basis for the universal enveloping algebra of $A$ can be obtained from the PBW-basis of $U(\wedge^2 A)$ by dropping those elements which in their graphical representation have intersecting arcs.

Although this graphical interpretation does not supply much insight just yet, it will become very useful further on.
3.2 First steps towards a solution

The main algebraic object we will work with is the simple Lie algebra $so(n + 1)$ (respectively the semi-simple one, in the case of $so(4)$). Its universal enveloping algebra will be denoted as before by $U$. Let $H$ be a Cartan subalgebra of $so(n + 1)$ and $\Phi$ the corresponding root system. We denote by $\Phi^+$ the set of positive roots of $\Phi$ and by $\Phi^-$ the set of negative ones. Fix $\lambda$ in $H^*$.

Consider the left ideal of $U$, $I(\lambda)$, generated by all $x_\alpha$, with $\alpha \in \Phi^+$ and all $h - \lambda(h)1$, where $h \in H$. Then $V(\lambda) := U/I(\lambda)$ is a highest weight module of $so(n + 1)$ with highest weight $\lambda$, called the Verma module of weight $\lambda$. $V(\lambda)$ might not be irreducible but it has a unique irreducible quotient. Define $Z(\lambda)$ to be $U/J(\lambda)$, where $J(\lambda)$ is the unique maximal left ideal of $U$ containing the left ideal $I(\lambda)$. Then $Z(\lambda)$ is an irreducible highest weight module with highest weight $\lambda$. Our goal is to determine for which $\lambda \in H^*$, $Z(\lambda)$ is an irreducible module of the $\lambda$-Lie algebra $A$, i.e. for which $\lambda \in H^*$, the two-sided ideal $Q(A)$ acts trivially on $Z(\lambda)$.

Before we do this, we need to show that our results will be independent of the choice of the Cartan subalgebra, i.e. for any two Cartan subalgebras $H$ and $H'$ of $so(n + 1)$, the sets of weights corresponding to these subalgebras, such that $Z(\lambda)$ is an irreducible representation of the $\lambda$-Lie algebra $A$, are related by an isomorphism.

Let $H$ and $H'$ be two Cartan subalgebras of $so(n + 1)$, $\lambda \in H^*$, $\lambda' \in (H')^*$ and $I(H, \lambda), I(H', \lambda')$ the two left ideals corresponding to each of these Cartan subalgebras (the notation now keeps track of the Cartan subalgebra as well). Denote by $J(H, \lambda), J(H', \lambda')$ the maximal left ideals of $U$ which include the two ideals above. As will be clear later on, the problem of $Z(\lambda)$, respectively $Z(\lambda')$, being an irreducible $A$-module translates as $Q(A) \subseteq J(H, \lambda)$, respectively $Q(A) \subseteq J(H', \lambda')$.

Let

$$
\Lambda := \{ \alpha \in H^* | Q(A) \subseteq J(H, \alpha) \},
$$

$$
\Lambda' := \{ \alpha' \in (H')^* | Q(A) \subseteq J(H', \alpha') \}
$$

and $\varphi : so(n + 1) \to so(n + 1)$ be a Lie algebra isomorphism for which $\varphi(H) = H'$. Then $\varphi$ induces an automorphism $\tilde{\varphi} : U \to U$.

We want the following equality to hold:

$$(\varphi|_H)^*(\Lambda') = \Lambda,$$

which is the same thing as

$$Q(A) \subseteq J(H, \lambda) \text{ if and only if } Q(A) \subseteq J(H', \lambda'),$$

for $\lambda' = (\varphi|_H)^*(\lambda)$. Since $\varphi$ is an isomorphism, it is enough to show just one implication of the above equivalence. Assume that $Q(A) \subseteq J(H, \lambda)$. Then,

$$Q(A) \subseteq J(H, \lambda) \Rightarrow \tilde{\varphi}(Q(A)) \subseteq \tilde{\varphi}(J(H, \lambda)) = J(H', \lambda').$$

Hence, if we can show that $\tilde{\varphi}(Q(A)) = Q(A)$ we are done.
We will prove this by finding a canonical description for the two sided ideal \( Q(A) \). It will then follow that any automorphism of \( U \), coming from a Lie algebra isomorphism, will map \( Q(A) \) into itself.

### 3.2.1 The canonical description of \( Q(A) \)

As explained above, finding a canonical description for \( R \), the Span of the relations, is a crucial step in proving that our strategy is independent of the Cartan subalgebra we are working with. For this purpose, we will explain how \( R \) appears as an eigenspace of the Casimir acting on \( so(n + 1) \otimes so(n + 1) \).

Denote by \( c \) the universal Casimir element, given by the formula

\[
c = -\frac{1}{2n} \sum_{i<j} (e^{ij})^2.
\]

Define \( \bar{c} = \frac{2}{n}c - n \cdot Id \). Our goal is to show that \( R = \text{Eigenspace}_{\bar{c}}(-2) \), i.e. the eigenspace of \( \bar{c} \) with eigenvalue \(-2\).

Note that \( so(n + 1) \otimes so(n + 1) = \wedge^2 so(n + 1) \oplus S^2(so(n + 1)) \), where

\[
x \otimes y = \frac{x \otimes y - y \otimes x}{2} + \frac{x \otimes y + y \otimes x}{2}.
\]

We denote the first term of the sum on the right-hand-side by \( x \wedge y \) and the second one by \( x \circ y \). It is easy to see that \( \forall r \in R : \bar{c} \cdot r = -2r \). This shows that \( R \subseteq \text{Eigenspace}_{\bar{c}}(-2) \). So, it remains to show that the converse inclusion also holds. We will prove this by explicitly exhibiting a basis of \( S^2(so(n + 1)) \) composed of eigenvectors of \( \bar{c} \).

To do this we will use the following formulas:

\[
\bar{c}(e^{ab} \circ e^{cd}) = \begin{cases} 
-\frac{1}{2} \sum_{i=1}^{n+1} (e^{ai} \otimes e^{bi} + e^{bi} \otimes e^{ai}) + e^{ab} \circ e^{ab} & a = c, b = d, \\
-\frac{1}{2} \sum_{i=1}^{n+1} e^{ab} \circ e^{cd} + e^{bc} \circ e^{bd} & a = c, b \neq d, \\
-\frac{1}{2} \sum_{i=1}^{n+1} e^{ab} \circ e^{cd} & a \neq c, b = d, \\
-\frac{1}{2} \sum_{i=1}^{n+1} e^{ab} \circ e^{cd} + e^{bc} \circ e^{bd} & b = c, \\
-\frac{1}{2} \sum_{i=1}^{n+1} e^{ab} \circ e^{cd} + e^{ca} \circ e^{ac} & a = d, \\
-\frac{1}{2} \sum_{i=1}^{n+1} e^{ab} \circ e^{cd} + e^{bd} \circ e^{ad} & \text{otherwise},
\end{cases}
\]

where \( a < b \) and \( c < d \).

Denote by

\[
B := \langle e^{ab} \circ e^{cd} | a < b, c < d \in \{1, \ldots, n + 1\}; \|\{a, b, c, d\}\| = 4 >
\]

\[
C := \langle e^{ab} \circ e^{cd} | a < b, c < d \in \{1, \ldots, n + 1\}; \|\{a, b, c, d\}\| = 3 >
\]

and by

\[
D := \langle e^{ab} \circ e^{ab} | 1 \leq a < b \leq n + 1 >.
\]
Then, \( S^2(so(n+1)) = B \oplus C \oplus D \) and \( \dim(S^2(so(n+1))) = \dim(B) + \dim(C) + \dim(D) = 3\left(\frac{n+1}{4}\right) + 3\left(\frac{n+1}{3}\right) + \frac{(n+1)}{2} \). We will find a basis of eigenvectors for each of the three subspaces \( B, C \) and \( D \).

For fixed \( a, b, c, d \in \{1, \ldots, n+1\} \), all distinct, denote by
\[
V_{abcd} := \langle e^{ab} \circ e^{cd}, e^{ad} \circ e^{bc}, e^{bd} \circ e^{ac} \rangle,
\]
the 3-dimensional subspace of \( B \) generated by the three vectors. Then, a basis of eigenvectors for \( V_{abcd} \) is given by the 3 vectors
\[
e^{ab} \circ e^{cd} + e^{ad} \circ e^{bc} - e^{ac} \circ e^{bd} \in \text{Eigenspace}_{c}(-2),
\]
\[
e^{ab} \circ e^{cd} + e^{ac} \circ e^{bd} \in \text{Eigenspace}_{c}(1),
\]
\[
e^{ad} \circ e^{bc} + e^{ac} \circ e^{bd} \in \text{Eigenspace}_{c}(1).
\]

Since
\[
B = \bigoplus_{1 \leq a < b < c < d \leq n+1} V_{abcd},
\]
we obtain a basis of eigenvectors of \( \vec{c} \) acting on \( B \). The reader is urged to note that the eigenvectors corresponding to the eigenvalue -2 are the relations \( x_{a,b,c,d} \).

For the subspace \( C \) the procedure is similar. For \( a < c \in \{1, \ldots, n+1\} \) denote by
\[
V_{ac} = \langle e^{ai} \circ e^{ci} | i \in \{1, \ldots, n+1\} - \{a,c\} \rangle.
\]
Then
\[
C = \bigoplus_{1 \leq a < c \leq n+1} V_{ac}.
\]

We regard \( V_{ac} \) as the subspace of \( C \) spanned by \( n-1 \) vectors: \( v_k := e^{ak} \circ e^{ck} \), for \( k \in \{1, \ldots, n+1\} - \{a, c\} \); and consider \( n-2 \) eigenvectors of \( \vec{c} \) given by \( v_1 - v_2, \ldots, v_{n-1} - v_{n+1} \). These will be linearly independent eigenvectors of \( \vec{c} \) with eigenvalue 1. To obtain a basis of \( V_{ac} \) we still need one more eigenvector, namely
\[
\sum_{i=1}^{n+1} e^{ai} \circ e^{ci} \in \text{Eigenspace}_{c}\left(-\frac{n-1}{2}\right).
\]

By taking the union of these bases, we obtain a basis of \( C \) given by eigenvectors, none of which belongs to the eigenspace of \( \vec{c} \) with eigenvalue -2, for generic \( n \). By this we mean that for \( n \neq 5 \) none of the eigenvalues obtained so far will be equal to -2. On the other hand if \( n = 5 \), then \( -\frac{n-1}{2} = -2 \). Hence, this case has to be treated separately.

The last subspace we need to take into account is the subspace \( D \):
\[
D = \langle e^{ij} \circ e^{ij} | 1 \leq i < j \leq n+1 \rangle.
\]

We use the following notation
\[
X_{a,b} := e^{ab} \circ e^{ab},
\]
\[ S_n = \sum_{i=1}^{n+1} e^{ai} \odot e^{ai}. \]

With this notation, a basis of eigenvectors is given by:

\[ X_{a,b} - X_{a+1,b} - X_{a,b+1} + X_{a+1,b+1} \in \text{Eigenspace}_c(1), \text{ for } 1 \leq a < b - 1 \leq n, \]

\[ S_a - S_{a+1} \in \text{Eigenspace}_c\left(-\frac{n-1}{2}\right), \text{ for } 1 \leq a \leq n, \]

\[ \sum_{a=1}^{n+1} S_a \in \text{Eigenspace}_c(-n), \]

\[ X_{p-1,p} - X_{p-1,p+1} - X_{p,p+2} + X_{p+1,p+2} \in \text{Eigenspace}_c(1), \text{ for } p \in \{2, \ldots, n-1\}. \]

Hence, for \( n \neq 5 \), we have obtained a basis of eigenvectors for \( S^2(\text{so}(n+1)) \) and it becomes clear that \( R = \text{Eigenspace}_c(-2) \).

Next we treat the case \( n = 5 \), when the eigenvalues \(-2\) and \(-\frac{n-1}{2}\) coincide.

Thus, we obtain that \( R \subsetneq \text{Eigenspace}_c(-2) \).

If \( n = 5 \) then \( \text{so}(n+1) = \text{so}(6) \) and \( \wedge^4 \mathbb{C}^6 \) is an irreducible representation of \( \text{so}(6) \) isomorphic to the adjoint representation and isomorphic to \( R \). Let \( \pi_i, i \in \{1,2,3\} \), be the fundamental weights of \( \text{so}(6) \) (as presented in the Dynkin diagram below) and \( V^d_{(a,b,c)} \) the irreducible highest weight \( \text{so}(6) \)-module of dimension \( d \) and with highest weight \( a\pi_1 + b\pi_2 + c\pi_3 \). Then, the decomposition of \( S^2 \text{so}(6) \) into irreducible modules is given by:

\[ S^2 \text{so}(6) = V^1_{(0,0,0)} + V^{15}_{(0,1,1)} + V^{84}_{(0,2,2)} + V^{20}_{(2,0,0)}. \]

It is easy to see that \( R \) appears only once in this decomposition \( (R \cong V^{15}_{(0,1,1)}) \), hence we can conclude that our results will be independent of the Cartan subalgebra.

Figure 1: The Dynkin diagram of \( \text{so}(6) \).

3.2.2 Some useful lemmas

Our main goal is to find those highest weights \( \lambda \in H^* \) for which \( \text{Z}(\lambda) \) is an irreducible highest weight module of the simple \( n \)-Lie algebra \( A \).

We start by showing that if \( Q(A) \subseteq J(\lambda) \), then the universal enveloping algebra \( U \) of the \( n \)-Lie algebra \( A \), acts on \( U/J(\lambda) = Z(\lambda) \), which means that \( Z(\lambda) \)
is an irreducible $n$-Lie module. In the following Lemma we will prove this claim and its converses.

**Lemma 3.1.** $Q(A) \subseteq J(\lambda)$ if and only if $U(V)$ acts on $U/J(\lambda)$.

**Proof.** Let $\rho : U \to \text{End}(U/J(\lambda))$ be the representation, given by left multiplication

$$\rho(u)(x) = u.x.$$ 

Then, obviously, $\ker \rho \subseteq J(\lambda)$.

\[ \Rightarrow \]

Assume that $U/Q(A)$ acts on $Z(\lambda)$. Then, $Q(A) \subseteq \ker \rho \subseteq J(\lambda)$.

\[ \Leftarrow \]

Assume that $Q(A) \subseteq J(\lambda)$. We want to show that $Q(A)$ is also a subset of $\ker \rho$. Let $q \in Q(A)$ and $[u] \in U/J(\lambda)$, where $u \in U$. Then

$$\rho(q)([u]) = \rho(q \cdot u)(1) \in \rho(Q(V))(1) \subseteq \rho(J(\lambda))(1) = J(\lambda).$$

Thus $\rho(q)([u]) = 0$ and we are done.

We can rephrase this Lemma as follows: $Z(\lambda)$ is a representation of $A$ if and only if $Q(A) \subseteq J(\lambda)$. Thus, we want to see for which $\lambda \in H^*$ the inclusion above holds. The next Lemma gives us a useful method for checking this inclusion.

**Lemma 3.2.** $Q(A) \not\subseteq J(\lambda)$ if and only if $Q(A) + I(\lambda) = U$.

**Proof.** \[ \Leftarrow \]

Assume that $Q(A) + I(\lambda) = U$. Let us suppose that $Q(A) \subseteq J(\lambda)$. It follows that $Q(A) + I(\lambda) \subseteq J(\lambda)$. But $J(\lambda)$ is strictly included in $U$. Contradiction.

\[ \Rightarrow \]

Next we prove the converse implication. Assume that $Q(A) \not\subseteq J(\lambda)$ and suppose that $Q(A) + I(\lambda) \neq U$. Let $M$ be the maximal left ideal of $U$ such that $Q(A) + I(\lambda) \subseteq M$. By definition $J(\lambda)$ is the unique maximal left ideal of $U$ which contains $I(\lambda)$, hence $M = J(\lambda)$. It follows that $Q(A) + I(\lambda) \subseteq J(\lambda) \Rightarrow Q(A) \subseteq J(\lambda)$. Contradiction.

$Q(A) + I(\lambda) = U$ is equivalent to the fact that $1 \in Q(A) + I(\lambda)$, where 1 is the unit in $U$. Denote by $\hat{Q}(A) := \frac{Q(A) + I(\lambda)}{I(\lambda)} \subseteq V(\lambda) = U/I(\lambda)$ and by $\hat{1} := 1 + I(\lambda)$.

We can rewrite the above equality as $\hat{1} \in \hat{Q}(A)$.

We define the following map:

$$pr_\lambda : V(\lambda) \to \text{Span}\hat{1} = V(\lambda)_\lambda,$$

where $V(\lambda)_\lambda$ denotes the 1-dimensional subspace of $V(\lambda)$ with weight $\lambda$. With this notation we obtain the following equivalence:

$$\hat{1} \in \hat{Q}(A) \text{ if and only if there exists } x \in \hat{Q}(A) \text{ s.t. } pr_\lambda(x) \neq 0.$$ 

The implication "$\Rightarrow$" is trivial, hence in order to convince ourselves that the above equivalence holds we only need to check the converse implication.

Let $x \in \hat{Q}(A)$ with $pr_\lambda(x) \neq 0$, then $\hat{Q}(A) \not\subseteq \ker(pr_\lambda)$.

\[ ^5 \text{We have already mentioned this equivalence above.} \]
On the other hand \( \tilde{J}(\lambda) = J(\lambda)/I(\lambda) \subset \text{Ker}(pr_\lambda) \)
\[ \Rightarrow \tilde{Q}(A) \not\subset \tilde{J}(\lambda) \Rightarrow \exists \in \tilde{Q}(A). \]

The next lemma is the most important one in this section.

**Lemma 3.3.** A has a representation on \( \tilde{Z}(\lambda) \) if and only if \( pr_\lambda(\tilde{R}) = 0 \), where
\[ \tilde{R} = \frac{R + I(\lambda)}{J(\lambda)}. \]

**Proof.** By the last equivalence above and the lemmas before, it follows that

A has a representation on \( \tilde{Z}(\lambda) \) if and only if \( \tilde{Q}(A) \subset \text{Ker}(pr_\lambda) \).

Observe that the last equivalence holds in general. Hence, we just have to show that

\[ \tilde{Q}(A) \subset \text{Ker}(pr_\lambda) \text{ if and only if } \tilde{R} \subset \text{Ker}(pr_\lambda). \]

Since \( \tilde{R} \subset \tilde{Q}(A) \), the implication "\( \Rightarrow \)" is obvious.

"\( \Leftarrow \)"
Assume that \( \tilde{R} \subset \text{Ker}(pr_\lambda) \) and let \([q] \in \tilde{Q}(A)\), where \( q \in Q(A) \). Since \( Q(A) = URU = UR = RU \), the ideal \( Q(A) \) is spanned by elements of the form \( q = r \cdot u \), where \( r \in R \) and \( u \in U \), hence \([q] = [r \cdot u] = r \cdot [u] \), where \([u] \in U/I(\lambda) = V(\lambda) \). This implies that \([u] = \sum [x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_k}] \), where \( \alpha_i \in \Phi^- \), for all \( i \) in the set \{1, \ldots, k\}. This shows that \( Q(A) \) is spanned by elements of the form \( r \cdot x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_k} \cdot 1 \), where \( r \in R \) and \( \alpha_i \in \Phi^- \). Since \( R \) is a finite dimensional \( \Lambda^{n-1}A \cong so(n+1) \)-module and \( so(n+1) \) is \( (\text{semi}) \)-simple, it follows that \( R \) is decomposable into weight spaces. Let \( \{r_1, \ldots, r_k\} \) be a basis of \( R \), where \( \text{weight}(r_i) = \mu_i \). Then, \( \tilde{Q}(A) \) is spanned by elements of the form \( r_1 \cdot x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_k} \cdot 1 \). To see if \( \tilde{Q}(A) \) is indeed contained in \( \text{Ker}(pr_\lambda) \), we have to compute \( pr_\lambda(r_1 \cdot x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_k} \cdot 1) \). Since \( \text{weight}(1) = \lambda \), it follows that we need to compute this projection for those elements for which \( \text{weight}(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_k}) + \text{weight}(r_1) = 0 \), i.e. \( \alpha_1 + \alpha_2 + \cdots + \alpha_k + \mu_1 = 0 \):

\[ pr_\lambda(r_1 \cdot x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_k} \cdot 1) = pr_\lambda([r_1, x_{\alpha_1}] x_{\alpha_2}, \ldots, x_{\alpha_k} \cdot 1) + pr_\lambda(x_{\alpha_1}, r_1 \cdot x_{\alpha_2}, \ldots, x_{\alpha_k} \cdot 1). \]

We already know that in the first term of the sum, \([r_1, x_{\alpha_1}] \) is a linear combination of generators of \( Q(A) \) (\( R \) being a module for \( so(n+1) \)), hence \([r_1, x_{\alpha_1}] = \sum a_j r_j \). The second term of the sum is zero, since \( \text{weight}(r_1 \cdot x_{\alpha_2}, \ldots, x_{\alpha_k} \cdot 1) \succ \lambda \). Thus, by an inductive argument, we obtain

\[ pr_\lambda(r_i \cdot x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_k} \cdot 1) = \sum_j a_j pr_\lambda(r_j \cdot 1) = 0. \]

This proves the claim of the Lemma. \( \square \)

Let \( r_i \in R \). If \( \text{weight}(r_i) = \mu_i \neq 0 \) then \( \text{weight}(r_i \cdot 1) \neq \lambda \) and it follows that \( pr_\lambda(r_i \cdot 1) = 0 \). This shows that in order for \( \tilde{Z}(\lambda) \) to be an \( n \)-Lie module, we just have to look at those elements \( r_0 \in R \) for which \( \text{weight}(r_0) = 0 \) and insure that \( pr_\lambda(r_0 \cdot 1) = 0 \).
4 Main theorems and their proofs

In this section we will concentrate on the important results of this paper. The basic Lie algebra of the simple \( n \)-Lie algebra \( A \) is \( \text{so}(n+1) \) and we denote by \( H \) a Cartan subalgebra of this Lie algebra. Recall that our goal is to determine for which \( \lambda \in H^* \), the two-sided ideal \( Q(A) \) acts trivially on the irreducible, highest weight module \( Z(\lambda) \). This will insure that \( Z(\lambda) \) is an \( n \)-Lie module of \( A \). We will first explain the notation used in the statements of the theorems.

4.1 \( \text{so}(n+1) \)

Until now, the \( (n+1) \)-dimensional, complex vector space \( A \) came attached with the standard basis \( e_1, \ldots, e_{n+1} \), the inner product \( \langle e_i, e_j \rangle = \delta_{i,j} \) and the orientation form \( e_1 \wedge \ldots \wedge e_{n+1} \). This was useful for us because of several reasons: the easy expression of the \( n \)-ary bracket, the simple form of the generators of \( Q(A) \) being just two examples we have encountered so far. It will be useful for us, further on, to have a complex bilinear form instead of an inner product on \( A \). Denote by \( (\cdot, \cdot) \) the bilinear form on \( A \) defined by

\[
(v_i, v_j) = \frac{i e_2 j}{\sqrt{2}}, \quad j = 1, 2, \ldots, N,
\]

then a basis of \( A \) will be denoted by \( v_N, \ldots, v_1, v_0, v_N, \ldots, v_1 \), and again \( (v_i, v_j) = \delta_{i+j,0} \).

If, on the other hand, \( n + 1 = 2N + 1 \) then we need one more element, viz. \( v_0 := e_{2N+1} \). Then the basis of \( A \) is given by \( v_N, \ldots, v_1, v_0, v_1, \ldots, v_N \), and again \( (v_i, v_j) = \delta_{i+j,0} \). Hence, a basis of \( \text{so}(n+1) \simeq \wedge^2 A \) is given by elements of the form (cf. (2))

\[
v_j \wedge v_k = \frac{v_j \otimes v_k - v_k \otimes v_j}{2}, \quad -N \leq j < k \leq N
\]

and \( j, k \neq 0 \) if \( n + 1 \) is even. The relation between the two notations used is given by:

\[
v_{ij} \wedge v_{jk} = \frac{1}{2} (e^{2j-1,2k-1} - \nu \mu e^{2j,2k} - i (\nu e^{2j,2k-1} + \mu e^{2j-1,2k})),
\]

\[
v_0 \wedge v_{ij} = \frac{1}{\sqrt{2}} (e^{2j-1,2N+1} - i \nu e^{2j,2N+1}).
\]

The natural bilinear form on \( \text{so}(n+1) \) is the normalized trace form \( (M_1 | M_2) := \frac{1}{2} \text{trace}(M_1 M_2) \), which is nondegenerate, invariant and symmetric. This form is induced by a natural form on \( \text{End}(A) \simeq A \otimes A \):

\[
((w_1 \otimes w_2) | (w_1 \otimes w_2)) = 2(u_1, w_2)(u_2, w_1).
\]
Define
\[ \epsilon_j := ie^{2j-1,2j} = i(e_{2j-1} \otimes e_{2j} - e_{2j} \otimes e_{2j-1}), \quad \text{where } 1 \leq j \leq N. \] (5)

Then \( \epsilon_j = v_j \wedge v_{-j} \) and \( H := \bigoplus_{i=1}^{N} C \epsilon_i \) is a Cartan subalgebra of the Lie algebra \( so(n+1) \). Let \( h \in H \), then the commutator of \( h \) and \( v_{\nu j} \wedge v_{\mu k} \) is:

\[ [h, v_{\nu j} \wedge v_{\mu k}] = (\nu \epsilon_j + \mu \epsilon_k | h) v_{\nu j} \wedge v_{\mu k}. \]

\[ [h, v_0 \wedge v_{\nu j}] = (\nu \epsilon_j | h) v_0 \wedge v_{\nu j}. \]

Remark 3. \( v_0 \wedge v_{\nu j} \) should be considered to be \( v_{\nu j} \wedge v_0 \) in case \( \nu = -1 \), and not \(-v_{\nu j} \wedge v_0\) as would result from the antisymmetry of the exterior product. For this reason we will write \( \nu(v_0 \wedge v_{\nu j}) \).

Thus, if \( n + 1 = 2N \), we define the set of roots

\[ \Phi = \{ \pm(\epsilon_i \pm \epsilon_j) | 1 \leq i \neq j \leq N \} \]

and a base for this root system is given by

\[ \Delta = \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{N-1} - \epsilon_N, \epsilon_N, \epsilon_{N-1} + \epsilon_N \}. \]

Hence, we obtain the following root space decomposition for \( so(2N) \):

\[ so(2N) = \bigoplus_{1 \leq j<k \leq N, \mu \in \{+1,-1\}} C(v_{-j} \wedge v_{\mu k}) \bigoplus_{1 \leq j \leq N} C\epsilon_j \bigoplus_{1 \leq j<k \leq N, \mu \in \{+1,-1\}} C(v_j \wedge v_{\mu k}). \]

If \( n + 1 = 2N + 1 \), then the set of roots is

\[ \Phi = \{ \pm(\epsilon_i \pm \epsilon_j) | 1 \leq i \neq j \leq N \} \cup \{ \pm \epsilon_i | 1 \leq i \leq N \}, \]

while a base for this root system is given by

\[ \Delta = \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{N-1} - \epsilon_N, \epsilon_N \}. \]

This allows us to give the following root space decomposition of the Lie algebra \( so(2N + 1) \):

\[ so(2N + 1) = \bigoplus_{1 \leq j<k \leq N, \mu \in \{+1,-1\}} C(v_{-j} \wedge v_{\mu k}) \bigoplus_{1 \leq j \leq N} C(v_{-j} \wedge v_0) \bigoplus_{1 \leq j \leq N} C\epsilon_j \bigoplus_{1 \leq j<k \leq N, \mu \in \{+1,-1\}} C(v_j \wedge v_{\mu k}). \]

The first line of the formula above contains the negative side of the root space decomposition and the Cartan subalgebra, while on the second line just the positive side is listed.
4.2 Statements of the main theorems

Recall that $H := \oplus_{i=1}^{N} C\epsilon_i$ is a Cartan subalgebra of the Lie algebra $so(n+1)$. Let $\lambda$ be the highest weight of the highest weight module $Z(\lambda)$ and denote by $\lambda_i$ the value of $\lambda$ on $\epsilon_i$.

**Theorem 4.1.** Let $n \geq 3$. The highest weight, irreducible representation $Z(\lambda)$ of $so(n+1)$ is a highest weight, irreducible representations of the simple $n$-Lie algebra $A$ if and only if $\lambda \in H^*$ is such that $\lambda_1 = \lambda_2 = \ldots = \lambda_{t-1} = -1$, $\lambda_t = x \in \mathbb{C}$ and $\lambda_{t+1} = \ldots = \lambda_{\lfloor \frac{n+1}{2} \rfloor} = 0$, for some $1 \leq t \leq \lfloor \frac{n+1}{2} \rfloor$.

From general theory of irreducible Lie algebra representations it follows that if $x \in \mathbb{Z}^+$ and $t = 1$ we obtain a finite dimensional, irreducible representation of $A$ and otherwise an infinite dimensional, irreducible one. Thus, we have recovered the result in [5]. Our proof however, will be different from the one presented there.

When stating the above theorem in terms of the fundamental weights of $so(n+1)$, one needs to be careful about the distinction between the two cases: $n + 1$ is even or $n + 1$ is odd. This gives Theorems 1.2 and 1.3.

The rest of this section will contain the proof of Theorem 4.1. We will follow the strategy described above, namely we will determine the generators of $Q(A)$ of weight zero and impose the condition that they must act trivially on $Z(\lambda)$. These conditions can be expressed as the zero set of a set of polynomials in $\lambda_1, \ldots, \lambda_{\lfloor \frac{n+1}{2} \rfloor}$. These solutions can be read in the statement of the theorem. There will be two proofs given: a computational proof which will differentiate between the two cases: $n + 1$ is even or odd; and a second proof which will make use of the graphical language introduced before. This second proof has the advantage of working in both cases while no additional distinction needs to be made. This graphical proof also has a disadvantage: as required, it does supply the polynomials we where talking about, but it does not supply any indication why the obtained polynomials suffice.

4.3 $n+1$ is even

First, we treat the case: $n + 1 = 2N$. This means that the Lie algebra $\wedge^{n-1}A$ is $so(2N)$.

We want to compute the generators of $Q(A)$ in terms of the elements $v_{\nu j} \wedge v_{\mu k}$. By inverting the matrix which defines the $v_{\nu j} \wedge v_{\mu k}$’s in terms of the $e^{j,k}$’s we obtain the following equality.

$$
\begin{pmatrix}
e^{2j-1,2k-1} & e^{2j,2k} \\
e^{2j,2k-1} & e^{2j-1,2k}
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & 1 & -1 \\
i & -i & i & -i \\
i & i & -i & -i
\end{pmatrix}
\begin{pmatrix}
v_{\nu j} \wedge v_{\mu k} \\
v_{-\nu j} \wedge v_{\mu k} \\
v_{\nu j} \wedge v_{-\mu k} \\
v_{-\nu j} \wedge v_{-\mu k}
\end{pmatrix}
$$

We will avoid to write long, tedious computations, and jump ahead to the final result. The most important piece of information is the matrix above.
Recall that \( x_{i_1,i_2,i_3,i_4} = e^{i_1,i_2}e^{i_3,i_4} - e^{i_1,i_3}e^{i_2,i_4} + e^{i_1,i_4}e^{i_2,i_3} \). This formula can also be rewritten as:

\[
x_{i_1,i_2,i_3,i_4} = \frac{1}{8} \sum_{\sigma \in S_4} sgn(\sigma)e^{i_\sigma(1,i_\sigma(2))e^{i_\sigma(3),i_\sigma(4)}}.
\]

It follows that for any \( \tau \) in \( S_4 \),

\[
x_{i_\tau(1),i_\tau(2),i_\tau(3),i_\tau(4)} = sgn(\tau)x_{i_1,i_2,i_3,i_4}.
\]

Therefore, if two indexes are equal, the formula above tells us that \( x_{i_1,i_2,i_3,i_4} = 0 \). Hence

\[
\text{Span}\{x_{i_1,i_2,i_3,i_4}\}_{1 \leq i_1,i_2,i_3,i_4 \leq 2N} = \text{Span}\{ \frac{1}{8} \sum_{\sigma \in S_4} sgn(\sigma)e^{i_\sigma(1,i_\sigma(2))e^{i_\sigma(3),i_\sigma(4)}} \}_{1 \leq i_1,i_2,i_3,i_4 \leq 2N} = R.
\]

Now we are ready to give the generators of \( Q(A) \) expressed in the new notation:

\[
\text{Span}\{x_{i_1,i_2,i_3,i_4}\}_{1 \leq i_1,i_2,i_3,i_4 \leq 2N} = \text{Span}\{ \frac{1}{2}((v_i \land v_j)(v_k \land v_l) - (v_i \land v_k)(v_j \land v_l) + (v_i \land v_l)(v_j \land v_k) + (v_k \land v_l)(v_i \land v_j) - (v_j \land v_l)(v_i \land v_k) + (v_j \land v_k)(v_i \land v_l))\},
\]

where \( i, j, k, l \) are considered to be ordered and they range from \(-N\) to \(N\), excluding 0. A generator, expressed in this notation, will be denoted by \( v_{a,b,c,d}(\alpha, \beta, \gamma, \delta) \), where \( \alpha, \beta, \gamma \) and \( \delta \) represent the signs of the indices. These indices range now between 1 and \(N\).

Recall that \( x_{a,b,c,d} \) is zero as soon as any two indices are equal. Without loss of generality we may assume that \( a \leq b \leq c \leq d \). For \( v_{a,b,c,d}(\alpha, \beta, \gamma, \delta) \) this fact does not hold anymore, i.e. if any two indices are equal, then \( v_{a,b,c,d}(\alpha, \beta, \gamma, \delta) \) is zero if the corresponding signs are also equal.

It is easy to see that the 6 terms in the expression of the generator \( v_{a,b,c,d}(\alpha, \beta, \gamma, \delta) \) all have the same weight, namely

\[
\text{weight}(v_{a,b,c,d}(\alpha, \beta, \gamma, \delta)) = \alpha\epsilon_a + \beta\epsilon_b + \gamma\epsilon_c + \delta\epsilon_d.
\]

Since we are looking for those elements \( r_0 \in R \) for which \( \text{weight}(r_0) = 0 \), it follows that for \( v_{a,b,c,d}(\alpha, \beta, \gamma, \delta) \) to be such an element we must have

\[
a = b, c = d \text{ and } \alpha = \gamma = 1, \beta = \delta = -1. \quad \text{[8]}
\]

In this case, our generator of \( Q(A) \) becomes

\[
v_{a,a,c,c}(1, -1, 1, -1) = \text{[8]}
\]

\[8\] Recall that \( a \leq b \leq c \leq d \) and that the equality of 3 consecutive indexes is not possible. Observe that for different choices of signs, the resulting generator gets multiplied by \( \pm 1 \).
By using the fact that $h \mathfrak{h} = \lambda(h) \mathfrak{h}$ for all $h \in H$, while $v_j \wedge v_{jk} \mathfrak{h} = 0$ for any $1 \leq j < k \leq 2N$ and $\nu \in \{+1, -1\}$, we can compute $pr_\lambda(v_{a,a,c,c}(1, -1, 1, -1) \cdot \mathfrak{h})$.

$$v_{a,a,c,c}(1, -1, 1, -1) \cdot \mathfrak{h} =$$

$$\frac{1}{2} (2\epsilon_a \epsilon_c + (-\epsilon_a + \epsilon_c) - (-\epsilon_a - \epsilon_c)) \cdot \mathfrak{h} =$$

$$(\epsilon_a \epsilon_c + \epsilon_c) \cdot \mathfrak{h}$$

Thus,

$$pr_\lambda(v_{a,a,c,c}(1, -1, 1, -1) \cdot \mathfrak{h}) = \lambda(\epsilon_c)(\lambda(\epsilon_a) + 1).$$

Above we have established that $Z(\lambda)$ is an irreducible $A$-module if and only if $\hat{R} \subseteq \text{Ker}(pr_\lambda)$. We have already seen that the only elements $r \in \hat{R}$ which might cause a problem are those of weight equal to zero. For the case "$n + 1$ is even" the only such element is $v_{a,a,c,c}(1, -1, 1, -1)$, whose projection is equal to $\lambda(\epsilon_c)(\lambda(\epsilon_a) + 1)$. Hence, if $\lambda \in \hat{H}^*$ satisfies the condition that for any $a < c$, $\lambda(\epsilon_c)(\lambda(\epsilon_a) + 1) = 0$, then $Z(\lambda)$ is indeed an irreducible $A$-module.

### 4.4 $n+1$ is odd

Next, we treat the case: $n + 1 = 2N + 1$. Our basic Lie algebra $\wedge^{n-1}A$ is now $so(2N + 1)$ with basis given by elements of the form $v_i \wedge v_j$, where $i$ and $j$ are included in the set $\{-N, \ldots, 0, \ldots, N\}$. We will apply the same strategy as in the even case, namely: rewrite the generators of the ideal $Q(A)$ in terms of the elements above, and afterwards find those which have weight zero.

Observe first that the elements $v_{ij} \wedge v_{jk}$ have the same definition as in the previous case. This proves that:

$$\text{Span}\{x_{i_1,i_2,i_3,i_4}\}_{1 \leq i_1, i_2, i_3, i_4 \leq 2N} =$$

$$\text{Span}\{(v_{aa} \wedge v_{bc})(v_{bc} \wedge v_{dd}) + (v_{ab} \wedge v_{bd})(v_{bc} \wedge v_{cd}) - (v_{ab} \wedge v_{ac})(v_{bd} \wedge v_{cd}) + (v_{ac} \wedge v_{bd})(v_{bc} \wedge v_{ac}) - (v_{ab} \wedge v_{cd})(v_{bc} \wedge v_{ac})\},$$

where $1 \leq a, b, c, d \leq N$ and $\alpha, \beta, \gamma, \delta \in \{\pm 1\}$. This shows that if all indices involved in $x_{a,b,c,d}$, are strictly less than $2N + 1$, then the "weight-zero" relations we are looking for, are those obtained in the "$n + 1$-even" case. Hence, we only need to look for those "weight-zero" relations, $x_{a,b,c,d}$, which involve the indices $a \leq b \leq c < d = 2N + 1$.

By inverting the matrix which defines the elements $v_0 \wedge v_a$ and $v_{-a} \wedge v_0$, we obtain that:

$$e^{2a-1+\alpha,2N+1} = \frac{1}{\sqrt{2}} \sum_{\nu \in \{-1,1\}} \nu(\nu)^a v_0 \wedge v_{\nu a},$$

where $\alpha \in \{0,1\}$. This gives us the following equality:

$$x^{2a-1+\alpha,2b-1+\beta,2c-1+\gamma,2N+1} =$$
\[
\frac{1}{4\sqrt{2}} \sum_{\nu, \mu, \omega \in \{1, -1\}} (i\nu)^\alpha (i\mu)^\beta (i\omega)^\gamma.
\]

\[
\left( \omega(v_{\nu a} \wedge v_{\mu b})(v_0 \wedge v_{\nu c}) + \nu(v_0 \wedge v_{\nu a})(v_{\mu b} \wedge v_{\nu c}) - \mu(v_{\nu a} \wedge v_{\mu b})(v_0 \wedge v_{\nu c}) 
+ \omega(v_0 \wedge v_{\nu c})(v_{\nu a} \wedge v_{\mu b}) + \nu(v_{\mu b} \wedge v_{\nu c})(v_0 \wedge v_{\nu a}) - \mu(v_0 \wedge v_{\mu b})(v_{\nu a} \wedge v_{\nu c}) \right).
\]

At this point we could prove that the span of the elements on the left-hand-side is equal to the span of the elements on the right-hand-side and follow the same guide line as before. But, there is no need for this, since the span of the elements on the left is obviously included in the span of the elements on the right; moreover the elements on the right have weight:

\[
\text{weight} = \nu \epsilon_a + \mu \epsilon_b + \omega \epsilon_c.
\]

For this weight to be zero, we would need that \(a = b = c\). However, if this is the case, then \(\nu + \mu + \omega\) can not be 0, since \(\nu, \mu, \omega \in \{1, -1\}\). Thus, no new polynomials are added to the set previously obtained.

### 4.5 Making use of the graphical interpretation

In the following we will demonstrate another way of obtaining the same polynomials.

By the definition of the ideal \(I(\lambda)\), it follows that in \(V(\lambda)\) the elements \(v_j \wedge v_k\) and \(v_j \wedge v_{-k}\) are zero, where \(j < k\). We use the graphical interpretation of these elements to obtain the following equalities:

\[
\begin{align*}
v_j \wedge v_k &= \frac{1}{2}(\circ \circ \circ \circ - \circ \circ \circ \circ - \circ \circ \circ \circ - \circ \circ \circ \circ) = 0, \\
v_j \wedge v_{-k} &= \frac{1}{2}(\circ \circ \circ \circ + \circ \circ \circ \circ + \circ \circ \circ \circ + \circ \circ \circ \circ) = 0.
\end{align*}
\]

Remember that we chose \(j < k\) and observe that in these equalities the four points are numbered by \(2j - 1, 2j, 2k - 1, 2k\). The computations presented below, should be seen as taking place in \(U/(Q(A) + \text{Ann}(V(\lambda)))\). By adding and subtracting the two equalities we obtain the following two relations:

\[
\begin{align*}
\circ \circ \circ \circ &= i\circ \circ \circ \circ, \\
\circ \circ \circ \circ &= -i\circ \circ \circ \circ.
\end{align*}
\]

They tell us that if an arc connects two points in the diagram which have different parities, then we can shift the left leg of the arc, such that the result is an arc between two points of the same parity. By doing this we acquire either \(+i\) or \(-i\) in front of the diagram.

The ideal \(I(\lambda)\) is a left ideal, hence:

\[
(\circ \circ \circ \circ + i\circ \circ \circ \circ)(\circ \circ \circ \circ + i\circ \circ \circ \circ) = 0
\]
and
\[(\circ \circ \circ \circ - i \circ \circ \circ \circ)(\circ \circ \circ \circ - i \circ \circ \circ \circ) = 0.\]

Recall that the Cartan subalgebra \(H\) was defined as \(\bigoplus_{i=1}^{N} C\epsilon_{i}\), where \(\epsilon_{j} = i \epsilon^{2j-1, 2j} = i \circ \circ \circ \circ\) (a simple diagram with one arc connecting the points \(2j - 1\) and \(2j\)). The element \(\epsilon_{j}\) acts on the highest weight module by multiplication with \(\lambda(\epsilon_{j})\), which we will denote by \(\lambda_{j}\). Using this fact we compute the two equalities above.

The first equality becomes:

\[
\begin{align*}
(\circ \circ \circ \circ + i \circ \circ \circ \circ)(\circ \circ \circ \circ + i \circ \circ \circ \circ) &= \\
= \circ \circ \circ \circ - \circ \circ \circ \circ + i(\circ \circ \circ \circ + \circ \circ \circ \circ) &= \\
= \circ \circ \circ \circ + i(\circ \circ \circ \circ + \circ \circ \circ \circ) &= \\
= (i \circ \circ \circ \circ)(-i \circ \circ \circ \circ) + i(\circ \circ \circ \circ + \circ \circ \circ \circ) &= \\
= -\lambda_{j}\lambda_{k} + i(\circ \circ \circ \circ + \circ \circ \circ \circ) &= \\
= -\lambda_{j}\lambda_{k} + i(\circ \circ \circ \circ + \circ \circ \circ \circ) &= \\
= -\lambda_{j}\lambda_{k} + i(\circ \circ \circ \circ + \circ \circ \circ \circ) - 2(\circ \circ \circ \circ) = 0,
\end{align*}
\]

while the second can be rewritten as:

\[
\begin{align*}
(\circ \circ \circ \circ - i \circ \circ \circ \circ)(\circ \circ \circ \circ - i \circ \circ \circ \circ) &= \\
= \circ \circ \circ \circ - \circ \circ \circ \circ - i(\circ \circ \circ \circ + \circ \circ \circ \circ) &= \\
= \circ \circ \circ \circ - i(\circ \circ \circ \circ + \circ \circ \circ \circ) &= \\
= (i \circ \circ \circ \circ)(-i \circ \circ \circ \circ) - i(\circ \circ \circ \circ + \circ \circ \circ \circ) &= \\
= -\lambda_{j}\lambda_{k} - i(\circ \circ \circ \circ + \circ \circ \circ \circ) = 0.
\end{align*}
\]

Summing the two up, we obtain the following equation:

\[-2\lambda_{j}\lambda_{k} - 2i(\circ \circ \circ \circ) = -2\lambda_{j}\lambda_{k} - 2\lambda_{k} = 0,
\]

hence the same polynomials in \(\lambda\):

\[\lambda_{k}(\lambda_{j} + 1) = 0.\]
5 Primitive ideals

As previously mentioned, primitive ideals of the universal enveloping algebra $U(A)$, where $A$ is the simple $n$-Lie algebra, correspond to the primitive ideals of the universal enveloping algebra of the basic Lie algebra $\wedge^{n-1}A$ which include the two-sided ideal $Q(A)$. Moreover, these primitive ideals are the annihilators of highest weight, irreducible modules of $\wedge^{n-1}A$.

Let $Z(\lambda)$ be such a module, and denote by $\text{Ann}(Z(\lambda))$ its annihilator. For $\text{Ann}(Z(\lambda))$ to be a primitive ideal of $U(A)$, we would need that $Q(A) \subseteq \text{Ann}(Z(\lambda))$, i.e. $Q(A)$ acts trivially on $Z(\lambda)$. Hence, in order to find the primitive ideals of $U(A)$, we want to find those weights $\lambda$, such that the annihilator of the irreducible, highest weight module of weight $\lambda$ includes $Q(A)$. These are precisely the weights we have determined in the previous sections. Hence, we have already proved the following theorem.

**Theorem 5.1.** Let $I$ be a primitive ideal of $U$, the universal enveloping algebra of the basic Lie algebra $\wedge^{n-1}A$. $I$ is a primitive ideal of $U(A)$, the universal enveloping algebra of $A$, if and only if $I$ is the annihilator of an irreducible, highest weight module of $A$ with highest weight $\lambda$ of the form mentioned in either Theorem 1.2 or Theorem 1.3, depending on the parity of $n+1$.

As an example, we will show that the Joseph ideal is a primitive ideal of $U(A)$, the enveloping algebra of the simple $n$-Lie algebra $A$. To avoid complications we fix $n > 4$.

In [11], Joseph constructed a primitive, completely prime ideal in $U(so(n + 1))$ corresponding to the closure of the minimal nilpotent orbit of the coadjoint action; and computed its infinitesimal character. We will denote this ideal by $J$.

To show that $J$ is a primitive ideal of $U(A)$, we must prove that $Q(A) \subseteq J$. Denote by $\alpha$ the highest root of $so(n + 1)$ and recall that $S^2(so(n + 1)) = V(2\alpha) \oplus V(0) \oplus W$. In [7] it was shown that the Joseph ideal $J$ is equal to the ideal generated by $W$ and $C - c_0$, where $c_0$ is the eigenvalue for the Casimir $C$ for the infinitesimal character that Joseph obtained.

Above, we have proven that $R \subseteq S^2(so(n + 1))$ and that $S^2(so(n + 1))$ decomposes as $R \oplus \text{Ker}\phi$, where $\phi : S^2(so(n + 1)) \rightarrow \wedge^3A$ is defined on monomials as

$$(v_i \wedge v_j) \circ (v_k \wedge v_l) \mapsto v_i \wedge v_j \wedge v_k \wedge v_l.$$

Hence, in order to show $Q(A) \subseteq J$, we must prove that $R \cap (V(2\alpha) \oplus V(0)) = \{0\}$, or equivalently that $V(2\alpha) \oplus V(0) \subseteq \text{Ker}\phi$. Taking into account that the highest root of the Lie algebra $so(n + 1)$ is given by $\epsilon_1 + \epsilon_2$ and that $V(2\alpha)$ is generated by $(v_1 \wedge v_2) \circ (v_1 \wedge v_2)$, this follows directly. Thus, we may conclude that the Joseph ideal $J$ is indeed a primitive ideal of $U(A)$.

References

[1] J. Bagger and N. Lambert, Modeling multiple M2’s, Phys. Rev. D 75, 045020 (2007).
A Another method for the smallest case

For the case \( n + 1 = 4 \) we will give another method for obtaining the possible highest weights which transform \( Z(\lambda) = U/J(\lambda) \) into an irreducible representation of \( A \). If \( n + 1 = 4 \) it follows that \( A \) is the simple 3-Lie algebra and its basic Lie algebra is \( so(4) \). Unlike before, \( so(4) \) is not a simple Lie algebra, but it is isomorphic to \( sl(2) \oplus sl(2) \). Our approach will be to find the two \( sl(2) \)'s sitting inside \( so(4) \) and compute the generator of \( Q(A) \) in terms of the elements of these two simple Lie algebras.\(^7\) We will then impose the condition that \( Q(A) \) acts trivially on the highest weight module \( Z(\lambda) \).

Let \( \bar{x}_1 = v_1 \wedge v_2 \wedge v_3 \), \( \bar{y}_1 = v_1 \wedge v_2 \wedge v_3 \) and \( \bar{x}_2 = v_1 \wedge v_2 \wedge v_3 \). Then \([\bar{x}_1, \bar{y}_1] = -\epsilon_1 + \epsilon_2 =: \bar{h}_1 \) and \([\bar{x}_2, \bar{y}_2] = -\epsilon_1 - \epsilon_2 =: \bar{h}_2 \), while \([\bar{x}_1, \bar{y}_2] = [\bar{x}_2, \bar{y}_1] = 0 \). If we compute the rest of the commutators we obtain:

\[ [\bar{h}_1, \bar{x}_1] = -2\bar{x}_1 \quad [\bar{h}_1, \bar{y}_1] = 2\bar{y}_1 \]

\(^7\)Notice that in this case \( Q(A) \) will have only one generator.

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\[ [\bar{h}_2, \bar{x}_2] = -2\bar{x}_2 \quad [\bar{h}_2, \bar{y}_2] = 2\bar{y}_2 \]
\[ [\bar{h}_1, \bar{x}_2] = [\bar{h}_1, \bar{y}_2] = [\bar{h}_2, \bar{x}_1] = [\bar{h}_2, \bar{y}_1] = 0. \]

Thus, if we define
\[ x_j = i\bar{x}_j, \quad y_j = i\bar{y}_j \quad \text{and} \quad h_j = -\bar{h}_j, \forall j \in \{1, 2\} \]
we obtain two \( sl(2) \)'s, namely \( \{x_1, h_1, y_1\} \) and \( \{x_2, h_2, y_2\} \).

In \( so(4) \) the only generator of the two-sided ideal \( Q(A) \) is the relation:
\[ X = e^{12}e^{34} + e^{14}e^{23} - e^{13}e^{24}. \]

Since \( h_1 = \epsilon_1 - \epsilon_2 \) and \( h_2 = \epsilon_1 + \epsilon_2 \) it follows that
\[ e^{12} = \frac{h_1 + h_2}{2i} \quad \text{and} \quad e^{34} = \frac{h_2 - h_1}{2i}. \]

Next, we compute the rest of the products in the relation \( X \). We obtain
\[ e^{23} = \frac{1}{2}(x_1 + x_2 - y_1 - y_2) \]
\[ e^{14} = \frac{1}{2}(-x_1 + x_2 + y_1 - y_2) \]
\[ e^{13} = \frac{1}{2i}(x_1 + x_2 + y_1 + y_2) \]
\[ e^{24} = -\frac{1}{2i}(-x_1 + x_2 - y_1 + y_2), \]

which in turn gives us:
\[ e^{14}e^{23} - e^{13}e^{24} = \frac{h_1 + 2y_1x_1 - h_2 - 2y_2x_2}{2}. \]

The relation \( X \) becomes then:
\[ X = \frac{h_1^2 - h_2^2}{4} + \frac{h_1 - h_2}{2} + y_1x_1 - y_2x_2. \]

Denote by \( V(\mu_1, \mu_2) = \langle 1 \rangle \) some highest weight module of \( so(4) \). For this module to be a module for the simple 3-Lie algebra \( A \) we want the relation \( X \) to act trivially on \( V(\mu_1, \mu_2) \). Since \( h_j1 \Rightarrow \mu_j1 \) for all \( j \in \{1, 2\} \) and \( x_j1 = 0 \), we can conclude that \( X \) acts on \( V(\mu_1, \mu_2) \) as:
\[ \mu_1^2 - \mu_2^2 + 2\mu_1 - 2\mu_2. \]

We impose the condition that
\[ \mu_1^2 - \mu_2^2 + 2\mu_1 - 2\mu_2 = 0 \Rightarrow (\mu_1 + 1)^2 = (\mu_2 + 1)^2. \]

Hence, we have obtained two solutions, namely
\[ \mu_1 = \mu_2 \text{ and } \mu_1 + \mu_2 = -2. \]

Thus, \( V(\mu_1, \mu_2) = V(\mu_1) \otimes V(\mu_2) \) is a 3-Lie algebra module if either both weights coincide, or their sum is -2.

There still remains the matter of determining \( \lambda_1 \) and \( \lambda_2 \) (in the notation previously used). The formulas for \( h_1 \) and \( h_2 \) tell us that \( \mu_1 = \lambda_1 - \lambda_2 \) while \( \mu_2 = \lambda_1 + \lambda_2 \). If \( \mu_1 = \mu_2 \) we obtain that \( \lambda_2 = 0 \) and if \( \mu_1 + \mu_2 = -2 \) we obtain the solution \( \lambda_1 = -1 \). We observe that these are also the solutions of the polynomial: \( \lambda_2(\lambda_1 + 1) = 0 \).