QCD Equations of State and the QGP Liquid Model

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Recent advances in the study of equations of state of thermal lattice Quantum Chromodynamics obtained at non-zero baryon density allow validation of the quark-gluon plasma (QGP) liquid model equations of state (EoS). We study here the properties of the QGP-EoS near to the phase transformation boundary at finite baryon density and show a close agreement with the lattice results.

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The ab-initio exploration of Quantum Chromodynamics (QCD) on the lattice at finite temperature and baryon density has seen rapid recent progress \cite{1,2,3}. It is already known that for vanishing baryon density lattice results can be described in terms of ideal quantum gases allowing only for lowest order perturbative interactions, provided that a non-perturbative temperature dependence of the coupling constant is introduced, along with vacuum latent heat \cite{4}. We will refine this quark-gluon plasma model, and demonstrate that it agrees extremely well with the finite baryon density lattice results, except in a region of temperature in vicinity of the phase transition/formation domain.

A strong motivation to have an understanding of the thermal QCD lattice results in terms of a set of more intuitive degrees of freedom originates in the need to treat a fast evolving system created in relativistic heavy ion collisions. At the onset of the QCD thermal matter formation, e.g., at $\tau_0 \approx 0.5$ fm, one cannot expect a full chemical equilibrium to have been reached, i.e., the phase space occupancy of quark and/or gluon gases has not approached unity. In a model as we will present these situations can be easily incorporated, allowing study of initial temperature, phase space occupancy evolution, and other physical conditions in the realistic environment of high energy relativistic heavy ion collisions.

More generally, a handy model of QCD matter equations of state employing established physical concepts allows to study many physical properties which are only difficult to infer from the numerical lattice results. But perhaps the most important question we pursue is if one can already in the temperature domain explored by present day experiments describe thermal QCD matter in terms of nearly free quark and gluon degrees of freedom. If this is confirmed, we would be able to verify if the physical system formed in these reactions is the deconfined quark-gluon plasma.

A systematic perturbative expansion within the framework of thermal field theory of the interacting quark-gluon gas converges poorly \cite{5}. The situation is illustrated for the case of a pure gauge SU(3) case in figure \ref{fig:1}, where we see that the ratio of the computed perturbative pressure $P$ to the Stefan-Boltzmann pressure $P_0$ is oscillating around unity depending on the order $n$ of the expansion in the coupling constant $g$ considered. Convergence seems to occur only in the asymptotic freedom limit of relatively high temperature $T$, beyond the experimental reach. Better agreement with the lattice results (solid dots) can be achieved (solid line) at a relatively low $T$, here expressing the temperature in units of the renormalization scale $\Lambda_{\text{MS}}$, provided that the unknown relative coefficient of the $g^6$ term is optimized to the value 0.7.

We advocate a very different approach, combining the perturbative result with key nonperturbative features. Our QGP liquid model arises from the empirical observation that the lowest order thermal perturbation contribution evaluation of the QCD matter properties, combined with a non-perturbative temperature dependent strong
coupling constant agrees with the QCD thermal lattice results, once the other often used non-perturbative effect, e.g. bag constant, has been incorporated.

The QGP-liquid partition function is assumed to have the form,

\[ \frac{T}{V} \ln Z_{\text{QGP}} \equiv P_{\text{QGP}} = -B + \frac{8}{45\pi^2} c_1 (\pi T)^4 \]

\[ + \sum_{i=q,s} \frac{n_i}{15\pi^2} \left[ \frac{7}{4} c_2 (\pi T)^4 + \frac{15}{2} c_3 \left( \mu_i^2 (\pi T)^2 + \frac{1}{2} \mu_i^4 \right) \right] \]

where \( n_q = 2, n_s = 1, \mu_s = 0 \) and:

\[ c_1 = 1 - \frac{15\alpha_s}{4\pi}, \quad c_2 = 1 - \frac{50\alpha_s}{21\pi}, \quad c_3 = 1 - \frac{2\alpha_s}{\pi}. \]

We recall that \( \mu_b = 3\mu_q \) and \( \lambda_b = e^{\mu_b/T} \). A value \( B = (0.211 \text{ GeV})^4 \) fits best the lattice results we consider here.

The temperature dependence \( \alpha_s(T) \) is obtained from \( \alpha_s(x) \) setting the energy scale \( x \) to be:

\[ x = 2\pi \beta^{-1} \sqrt{1 + \frac{1}{2} \ln^2 \lambda_q} = 2\sqrt{(\pi T)^2 + \mu_q^2} \]

\[ \alpha_s(x) \text{ is obtained integrating the renormalization group equation, incorporating physical thresholds for heavy flavor. Use of semi-analytical formulas with fixed active number of quarks introduces an unacceptable error in the value of } \alpha_s \text{ and has lead to false conclusions in some earlier work. We evaluate:} \]

\[ \frac{dx}{\partial x} = -b_0 \alpha_s^2 - b_1 \alpha_s^3 + \ldots \equiv \beta^\text{pert}_2. \]

\( \beta^\text{pert}_2 \) is the beta-function of the renormalization group in two loop approximation, and

\[ b_0 = \frac{11 - 2n_3/3}{2\pi}, \quad b_1 = \frac{51 - 19n_3/3}{4\pi^2}. \]

\( \beta^\text{pert}_2 \) does not depend on the renormalization scheme, and solutions of Eq. (3) differ from higher order renormalization scheme dependent results by less than the error introduced by the experimental uncertainty in the measured value of \( \alpha_s(\mu = M_Z) = 0.118 \pm 0.001 - 0.0016 \), used as the initial value in numerical integration of Eq. (1). We note for \( \mu_q \to 0 \) the empirical form:

\[ \alpha_s(T/T_c) \simeq \frac{0.47}{1 + 0.72 \ln(T/T_c)}, \quad T_c = 172 \text{ MeV}. \]

When we explore the effect of the finite quark masses, we introduce the correction \( \delta_m \ln Z_{\text{QGP}} ^q \) to the partition function,

\[ \frac{1}{T^3 V} \delta_m \ln Z_{\text{QGP}} ^q = \frac{n_i}{T^3} \frac{m_i^3}{3^3 \pi^2} \]

\[ \times \int_0^\infty dx x^2 \left[ \ln \left( \frac{1 + \lambda_i e^{-x/\lambda}}{1 + \lambda_i e^{-x/\mu}} \right) + \frac{1}{\lambda_i} \right], \]

where \( i = q, s \). We use \( m_s = 170 \text{ MeV} \). Note that we did not allow for the QCD-thermal \( \alpha_s \) correction in \( \delta_m \ln Z_{\text{QGP}} ^q \). When we compare with lattice results, we use the values \( m_q = 65 \text{ MeV} \) and \( m_s = 135 \text{ MeV} \), as reported to have been used in lattice simulations we compare with. Strange quarks enter below only when we consider the absolute pressure. Similarly, when we explore how an effective gluon mass alters the pressure, we consider a correction

\[ \frac{\delta_m \ln Z_{\text{QGP}} ^G}{T^3 V} = -\frac{m_i^3}{T^3 \pi^2} \int_0^\infty dx x^2 \ln \left( \frac{1 - e^{-x/\lambda}}{1 - e^{-x/\mu}} \right). \]

We need to introduce a finite thermal glue mass \( m_G \simeq 0.2 \text{ GeV} \) in order to reach for \( T > T_c \) a line width agreement with lattice results. This is not the thermal gluon mass which, as we have discussed elsewhere, can be used as an alternative way to express the perturbative QCD effect.

In order to be able to consider our results in currently ongoing experiments, it is important that the QGP-liquid equations of state are verified at chemical potential which is relatively small, up to \( \mu_b \simeq 100 \text{ MeV} \) at initial conditions. However, we extend this study to all values considered in the lattice simulation.\(^{1}\) \( \mu_b = 100, 210, 330, 410 \) and \( 530 \text{ MeV} \), which aside of RHIC, also encompasses the physical reach of CERN-SPS experiments. Our objective is to test if for \( T > 1.5T_c \) the model and lattice-‘experiment’ agree, while near to the critical temperature domain significant modifications due to the non-perturbative features of QCD must be expected.

Physical properties we consider will be rendered dimensionless by considering a suitable ratio with either \( T^4 \) (pressure \( P \)) or \( T^3 \) (baryon density \( n_b \)). To assess if the model liquid has the right density dependence we consider in turn the pressure \( P \) modification at finite baryon density, \( \Delta P = P(T, \mu_b) - P(T, \mu_b = 0) \), the baryon density \( n_b \), the pressure \( P(T, \mu_b = 0) \) and finally \( \epsilon(T, \mu_b) - 3P(T, \mu_b) \), which vanishes for ideal gases. In the following figures all ‘experimental’ points are taken directly from the postscript format of the figures of Ref.\(^{1}\), into which we add our results.

The reader should note that for \( T < T_c \) equations of state with hadrons must be employed. Hence we do not here include results for this domain of lattice results. The study of this confined phase domain involves also modeling of excluded volume effect, and/or within the bootstrap model, the understanding of the singularity of the hadron mass spectrum. Given the availability of lattice results we hope to return to discuss these intricate matters in near future.

The change on \( \Delta P \) at finite baryochemical potential is shown in figure 2. We note that the agreement with the lattice results at large \( T \) is very satisfactory. This result depends on the empirical choice made for the dependence on chemical potential of the scale of \( \alpha_s \), see Eq. (3). Had we omitted \( \mu_q \), or doubled the coefficient of \( \mu_q \), there would be a well visible deviation in figure...
FIG. 2: \( \Delta P \equiv P(T, \mu_b) - P(T, \mu_b = 0) \) normalized by \( T^4 \) as function of \( T/T_c \) for \( \mu_b = 100, 210, 330, 410 \) and 530 MeV from bottom to top. Data points from ref. [1], solid lines massless liquid of quarks. Dashed (and mostly invisible) results with finite mass correction applied for \( m_q = 65 \) MeV as used in lattice data.

Thus the empirical choice of the Matsubara frequency as the combination of temperature and chemical potential is qualitatively verified by lattice results. For small chemical potential \( \mu_b = 100 \) MeV, we see that there is agreement for \( T \geq 1.1T_c \), at a baryochemical potential \( \mu_b = 530 \) MeV, agreement with lattice data is assured for \( T \geq 1.5T_c \). The practically invisible dashed lines (hidden mostly under solid lines) show the here negligible effect of finite quark masses. By definition, \( \Delta P \) depends only on light quark degrees of freedom. The agreement we see is thus not testing strange quark, or glue behavior.

A similarly remarkable agreement is obtained for the baryon density, shown in figure 3. Here a finite quark mass used in lattice simulations becomes visible (dashed lines). Once we allow for it, we see agreement within the lattice data error for \( T \geq 1.2T_c \) for all baryochemical potentials. For very small baryon density we can expect agreement down to near the critical temperature.

Since both the pressure difference \( \Delta P \) and baryon density \( n_B \) show satisfactory behavior as function of chemical potential, it is expected that the change of energy density, and entropy, with chemical potential is well described, both being derived of \( \Delta P \) with respect to \( \beta = 1/T \) and \( T \) and suitable linear combination with \( n_B \). However, in order to fully certify the liquid model, we need now to fine-tune the behavior of pressure \( P \) at zero chemical potential.

In figure 4 we show the pressure \( P(T, \mu_b = 0)/T^4 \) as function of temperature \( T/T_c \). We find that there is some difference in the lattice result shown in [1] and earlier work [2], which practically agreed with our solid line result in figure 4. To obtain the QGP model agreement with the more recent lattice data [1] we need to:

a) change the value of the Bag constant, from our earlier value \((0.195 \text{ GeV})^4 \) to \( B = (0.211 \text{ GeV})^4 \), in response to

FIG. 3: Baryon density \( n_B \) normalized by \( T^3 \) as function of \( T/T_c \) for \( \mu_b = 100, 210, 330, 410 \) and 530 MeV from bottom to top. Data points from ref. [1], solid lines massless liquid of quarks. Dashed lines: allowance is made for \( m_q = 65 \) MeV as is used to obtain the lattice data.

FIG. 4: The pressure \( P(T, \mu_b = 0) \) normalized by \( T^4 \) as function of \( T/T_c \). Data points from ref. [1]. Solid lines: massless gluons with \( B = (0.211 \text{ GeV})^4 \). Dashed line allows for a finite mass \( m_G = 200 \) MeV.
somewhat higher value of $T_c$, see solid line in figure 4. b) introduce an effective gluon mass $m_G = 0.2$ GeV in order to reduce the number of effective degrees of freedom as is shown by the dashed line in figure 4. This procedure does indeed produce the expected agreement within the line-width.

We do not have a lattice entropy figure to compare with, but we are assured of a valid result by our ability to reproduce the shape of the pressure functions, see figures 2 and 4. A sensitive test of this assertion is obtained considering $(\epsilon - 3P)/T^4$ as function of $T/T_c$ in figure 4. All finite chemical curves we plot coincide and hence only one is visible in the diagram. This agreement with lattice results for $T > 1.15T_c$ confirms that we have obtained a remarkably precise representation of the behavior of equations of state of deconfined QCD matter, except in direct vicinity of the critical temperature.

We conclude that thermal lattice QCD matter behaves just like quark gluon plasma, and thus a naive use of the QGP model is appropriate, provided that suitable coupling strength $\alpha_s(T, \mu_B)$ and vacuum bag constant $B$ is introduced. Moreover, considering that at RHIC the following initial conditions have been reached $\lambda_q > 0$, $T > 1.5T_c \approx 260$ MeV for $\lambda_q = 1.09$, corresponding to the initial baryochemical potential $\mu_b = 3T \ln \lambda_q \approx 45$ MeV, the results presented confirm that the QGP state is established in these reactions. Moreover, we are reassured that we can explore in detail the RHIC initial conditions, allowing for chemical composition dynamics. This should lead us to understanding of QGP properties and conditions in RHIC reactions.

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