INSTABILITY FOR RANK ONE FACTORS OF PRODUCT ACTIONS

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Abstract. We provide a counterexample to a standard interpretation of the Katok-Spatzier conjecture, and pose questions which may serve as reasonable replacements.

1. Introduction

The Katok-Spatzier conjecture for higher-rank abelian group actions without rank one factors can be traced back to the work of Burns and Spatzier on compact higher-rank Riemannian manifolds [2], as well as works of Hurder [5], which were extended by Katok and Lewis [6, 7] for actions of higher-rank lattices. The key ideas in the proofs of these rigidity results were associated higher-rank abelian group actions, and their hyperbolicity properties.

These ideas lead to a series of papers in the 90’s, where Katok and Spatzier gave several striking features for irreducible actions on tori and some Weyl chamber flows without rank one factors, including measure and cocycle rigidity [8, 10].

The third prototypical result in the rigidity program provided the basis for the Katok-Spatzier conjecture: smooth local rigidity of some natural higher-rank actions without rank one factors [9], which showed that any $C^\infty$, sufficiently $C^1$-close perturbation was $C^\infty$-conjugate to the original action after a linear change of coordinates. For a more complete history of the rigidity program of higher rank abelian and semisimple Lie group actions, see, [12], which discusses rigidity phenomena very broadly, [4], which is focused on the context of lattices in semisimple Lie groups, or the introduction to [13], which focuses on the history of rigidity program for abelian actions.

Analysis of several of the proofs reveals a similar theme: obtaining isometric behavior on certain dynamically defined foliations coming from hyperbolic behavior allows one to spread invariant structures around. The mixing of these conventional opposites, hyperbolicity and isometric behavior, leads to rigidity.

It is therefore natural to establish two critical assumptions. First, that the action is Anosov, which provides the hyperbolicity assumptions to obtain dynamically-defined foliations, which in algebraic examples have algebraic structure.

Definition 1.1. Let $\alpha : \mathbb{R}^k \curvearrowright X$ be a locally free $C^r$ group action on a $C^\infty$ manifold $X$ and $\mathcal{O}$ denote its orbit foliation. We say that $a \in \mathbb{R}^k$ is an Anosov element if there is an $\mathbb{R}^k$-invariant splitting of the tangent space $TX = E^u_a \oplus E^s_a \oplus T\mathcal{O}$, some $\lambda, C > 0$ such that for all $t > 0$,

$$\|d\alpha(ta)|_{E^u_a}\| \leq Ce^{-\lambda t} \text{ and } \|d\alpha(-ta)|_{E^s_a}\| \leq Ce^{-\lambda t}.$$ 

We say that the action is Anosov if it has at least one Anosov element. We say that an action is totally Anosov if the set of Anosov elements are dense.
An Anosov action is *transitive* if there exists a point with a dense $\mathbb{R}^k$-orbit. We say that an Anosov action is *strongly transitive* if there exists an open cone $C \subset \mathbb{R}^k$ and $x \in X$ such that $C \cdot x$ is dense, and the only non-Anosov element of $\overline{C}$ is 0.

Second, we need a way to rule out well-known perturbative families in the setting of Anosov flows and diffeomorphisms. The following definition does this by saying that no factor of the action is a flow or diffeomorphism.

**Definition 1.2.** If $\alpha : \mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X$ is a locally free action, a $C^r$ rank one factor of $\alpha$ is a $C^r$, fixed-point free flow $\psi_t : Y \to Y$ or diffeomorphism $f : Y \to Y$ with a $C^r$ submersion $\pi : X \to Y$ and surjective homomorphism $\sigma : \mathbb{R}^k \times \mathbb{Z}^\ell \to \mathbb{R}$ (or $\sigma : \mathbb{R}^k \times \mathbb{Z}^\ell \to \mathbb{Z}$) such that

$$\pi(\alpha(a)x) = \psi_{\sigma(a)}(x) \quad \text{or} \quad \pi(\alpha(a)x) = f^{\sigma(a)}(x).$$

We allow for passing to a finite index subgroup of $\mathbb{R}^k \times \mathbb{Z}^\ell$ or a finite cover of $X$.

When the action is homogeneous, it is more clear what is meant by a rank one factor. The definition of a rank one factor in a more general setting has been unclear and nebulous throughout the development of the theory. It is usually used to guarantee some transitivity or ergodicity of actions of subactions (see, [13, Theorem 2.1], Section 4 and Lemma 6.1).

Finally, we need to identify the models for such actions. The following definition includes the two common “building blocks” for Anosov $\mathbb{R}^k$ actions, Weyl chamber flows and actions by toral automorphisms. It is closed under taking products, suspensions and skew products, so is the natural class to consider.

**Definition 1.3.** An *algebraic action* of $\mathbb{R}^k \times \mathbb{Z}^\ell$ is constructed from the following data:

- a Lie group $G$,
- a compact subgroup $M \subset G$,
- a (cocompact) lattice $\Gamma \subset G$, and
- a homomorphism $i : \mathbb{R}^k \times \mathbb{Z}^\ell \to \text{Aff}_{M,\Gamma}(G)$.

Here $\text{Aff}_{M,\Gamma}(G)$ is the group of affine maps $g \mapsto h\varphi(g)$, where $h \in Z_G(M)$, the centralizer of $M$ in $G$ and $\varphi \in \text{Aut}(G)$ is such that $\varphi$ preserves $M$, $Z_G(M)$ and $\Gamma$. We denote the image of $a$ under $i$ by $i_a$. The action is defined by $\alpha : \mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X = M \backslash G / \Gamma$, where

$$\alpha(a) \cdot Mg\Gamma = M{i_a}(g)\Gamma.$$

One may sometimes expect a topological orbit equivalence to such models in rank one (as in the Smale conjecture for Anosov diffeomorphisms and associated Franks-Manning theorem on tori and nilmanifolds), but usually not a conjugacy. Such topological rigidity fails in the case of Anosov flows, which have several constructions which change the topological orbit structure significantly.

With these definitions in hand, and the proof of local rigidity using them as the “essential” tools to obtain rigidity, the following conjecture was formulated:

**Conjecture 1.4** (Katok-Spatzier). *If $\mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X$ is a transitive, $C^\infty$, Anosov action on a compact manifold without $C^\infty$ rank one factors, then (up to finite cover) it is $C^\infty$ conjugate to an algebraic action.*
Progress toward the conjecture in special cases has been made incrementally over the last 20 years. The optimal results for $\mathbb{Z}^k$-actions were obtained by Rodriguez-Hertz and Wang, who showed the conjecture for actions on nilmanifolds and tori in [11], and for $\mathbb{R}^k$-actions, the author and Spatzier proved the conjecture for strongly transitive totally Cartan actions (see Definition 3.1) [13].

Remark 1.5. The Katok-Spatzier rigidity program is meant to promise of rigidity smooth structures and parameterizations of orbits. In particular, the cocycle rigidity results for genuinely higher-rank actions means that one may not take a nontrivial time change of a homogeneous action without rank one factors, so an important feature of the rigidity program for higher-rank actions has been that the actions are considered, and not just their orbit foliations. We expand on this remark in Section 6.

The main theorem of this paper provides a family of counterexamples to the conjecture:

**Theorem 1.6.** Let $f_s : Y_1 \to Y_1$ and $g_t : Y_2 \to Y_2$ be topologically mixing, $C^\infty$ Anosov flows on 3-manifolds. Then there exists a $C^\infty$ action of $\mathbb{R}^2$ on $\mathbb{X} = Y_1 \times Y_2$ which is Anosov, has no $C^1$ rank one factors and is not homogeneous.

In Section 6 we will comment on features of this family of examples and how a revision to Conjecture 1.4 could be formulated to accommodate these new examples.

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### 2. Cocycles and time changes of abelian group actions

The central idea of the paper is to use a reparameterization of the $\mathbb{R}^2$-orbits of a product action to destroy the product structure of the action. This is analogous to taking a time change of a flow, so we call such perturbations time changes of an $\mathbb{R}^2$-action.

**Definition 2.1.** If $\alpha_0 : \mathbb{R}^k \acts X$ is an action of $\mathbb{R}^k$ on a space $X$, a $C^r$-time change of $\alpha_0$ is an action $\alpha : \mathbb{R}^k \acts X$ such that there exists a $C^r$ map $\varphi : \mathbb{R}^k \times X \to \mathbb{R}^k$ such that

\begin{equation}
\alpha(a)x = \alpha_0(\varphi(a,x))x,
\end{equation}

and $\varphi(\cdot, x)$ is a $C^r$ diffeomorphism from $\mathbb{R}^k \to \mathbb{R}^k$. We say that $\varphi : \mathbb{R}^k \times X \to \mathbb{R}^k$ determines $\alpha$.

Not every function $\varphi$ will determine a time change, so we must be careful in constructing it. The main tool for doing so is the following.

**Definition 2.2.** If $\alpha : \mathbb{R}^k \acts X$ is an action of $\mathbb{R}^k$ on a space $X$, an (abelian) cocycle over $\alpha$ is a map $\beta : \mathbb{R}^k \times X \to \mathbb{R}^\ell$ such that

\begin{equation}
\beta(a + b, x) = \beta(a, x) + \beta(b, \alpha(a)x).
\end{equation}
A cocycle is a **coboundary** if there exists some \( H : X \rightarrow \mathbb{R}^k \) such that \( \beta(a, x) = H(a \cdot x) - H(x) \). We consider cocycles and coboundaries in the \( C^\infty, C^r, C^0 \) and measurable categories when appropriate.

Not every function \( \varphi : \mathbb{R}^k \times X \rightarrow \mathbb{R}^k \) determines an \( \mathbb{R}^k \)-action via the formula (2.1). Indeed, while such a function \( \varphi \) always reparameterizes orbits, it must satisfy the cocycle property over the new candidate action \( \alpha \) to determine a time change.

**Lemma 2.3.** If \( \alpha \) is a time change of \( \alpha_0 \) with determining function \( \varphi \), then \( \varphi \) is a cocycle over \( \alpha \).

**Proof.** We verify (2.2) directly from the condition that \( \alpha \) is an action of \( \mathbb{R}^k \):

\[
\alpha_0(\varphi(a + b, x)) x = \alpha(a + b) x = \alpha(b) \alpha(a) x = \alpha_0(\varphi(b, \alpha(a) x)) \alpha(a) x = \\
\alpha_0(\varphi(b, \alpha(a) x)) \alpha_0(\varphi(a, x)) x = \alpha_0(\varphi(b, \alpha(a) x) + \varphi(a, x)) x.
\]

Since the action is locally free, we conclude the cocycle equation for small values of \( \mathbb{R}^k \), and hence for large values of \( \mathbb{R}^k \) by writing them as integer multiples of small values and applying the cocycle equation the correct number of times. \( \square \)

If we wish to construct a time change of an action \( \alpha_0 \) from a cocycle \( \beta \), Lemma 2.3 suggests that we interchange the roles of which one is a time change of the other. In particular, we may think of \( \alpha \) as the original action and \( \alpha_0 \) as the time change, so that the determining function is a cocycle over \( \alpha_0 \). The cost is that we must be able to invert the function \( \beta \) as a map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) with fixed \( x \). This is formalized in the following lemma.

**Lemma 2.4.** There exists \( \varepsilon_0 > 0 \) with the following property: Let \( \alpha : \mathbb{R}^k \rightrightarrows X \) be a \( C^\infty \) action of \( \mathbb{R}^k \) and \( \beta : \mathbb{R}^k \times X \rightarrow \mathbb{R}^k \) be a \( C^\infty \) cocycle such that \( ||d_a \beta(0, x) - \text{Id}|| \leq \varepsilon_0 \) for every \( x \in X \), where \( d_a \) represents the derivative of \( \beta \) is the \( \mathbb{R}^k \) coordinate. Then there exists a \( C^\infty \) function \( \varphi : \mathbb{R}^k \times X \rightarrow X \) such that:

1. \( \varphi(\beta(a, x), x) = a = \beta(\varphi(a, x), x) \) for every \( a \in \mathbb{R}^k, x \in X \),
2. \( \varphi(\cdot, x) \) is a \( C^\infty \) diffeomorphism from \( \mathbb{R}^k \) to \( \mathbb{R}^k \), and
3. \( \varphi \) determines a \( C^\infty \) time change of \( \alpha_0 \).

**Proof.** We first construct the function \( \varphi \) at a fixed \( x \) by showing the map \( \beta(\cdot, x) : \mathbb{R}^k \rightarrow \mathbb{R}^k \) has a global inverse. Indeed, by picking \( \varepsilon_0 \) sufficiently small, we may assume that \( d\beta(a, 0) \) is invertible and hence that there is a local inverse to the function \( \beta(\cdot, x) \) defined on a neighborhood \( B(0, \eta_x) \subset \mathbb{R}^2 \) for some \( \eta_x > 0 \), and the inverse function is \( C^\infty \). To see that it has a global inverse, notice that by the cocycle equation gives that \( \beta(a + b, x) = \beta(a, \alpha_0(b) x) + \beta(b, x) \). By fixing \( b \) and letting \( a \) vary, we get that \( d_a \beta(b, x) = d_a \beta(0, \alpha_0(b) x) \), so \( d_a \beta(b, x) \) is close to the identity for all \( b \in \mathbb{R}^k \) as well. In particular, \( \eta_x \) can be chosen uniformly in \( x \), and the function \( \beta \) is surjective since its image can always be extended by a ball of uniform size.

To see that it is globally injective, note that integrating the closeness of the derivative yields that \( ||\beta(a, x) - a|| \leq \varepsilon_0 ||a|| \) for all \( a \in \mathbb{R}^k, x \in X \). If \( \beta(a, x) = \beta(b, x) \), then \( \beta(a, x) - \beta(b, x) = \beta(a - b, \alpha_0(b) x) = 0 \). But \( ||\beta(a - b, \alpha_0(b) x) \geq (1 - \varepsilon_0)||a - b|| \), so this is not possible unless
\(a = b\). Therefore, for a fixed \(x\), the map \(\beta\) has a global \(C^\infty\) inverse in the coordinate \(a\), which we denote by \(\varphi\).

To see that \(\varphi\) determines a time change, we check that \(\alpha(a) \cdot x := \alpha_0(\varphi(a, x))x\) is an abelian action:

\[
\alpha(a) \cdot (\alpha(b)x) = \alpha(a) \cdot (\alpha_0(\varphi(b, x))x) = \alpha_0(\varphi(a, \alpha_0(\varphi(b, x)))\alpha_0(\varphi(b, x))x \\
= \alpha_0(\varphi(a, \alpha_0(\varphi(b, x))) + \varphi(b, x)x).
\]

We therefore need to check that \(\varphi(a + b, x) = \varphi(a, \alpha_0(\varphi(b, x))) + \varphi(b, x)\). Since we have shown that \(\beta\) is invertible in the \(a\) coordinate, it suffices to check equality after applying \(\beta(\cdot, x)\) to each side. Then the desired equality follows exactly from the cocycle equation for \(\beta\) over \(\alpha_0\):

\[
\beta(\varphi(b, x) + \varphi(a, \alpha_0(\varphi(b, x)))x) = \beta(\varphi(b, x), x) + \beta(\varphi(a, \alpha_0(\varphi(b, x))), \alpha_0(\varphi(b, x)))x \\
= b + a = \beta(\varphi(a + b, x), x).
\]

Therefore, \(\varphi\) determines an \(\mathbb{R}^k\) group action \(\alpha\) which is a time change of \(\alpha_0\). It is clear from the definition that \(\alpha\) is a \(C^\infty\) group action, since \(\beta\) is assumed to be \(C^\infty\) in all coordinates, and the derivatives of \(\varphi\) can be computed explicitly from the definition. \(\Box\)

3. Anosov actions and coarse Lyapunov foliations

We now summarize the theory of coarse Lyapunov foliations, for details see [13, Section 4.1]. Given an Anosov action, through standard constructions from the theory of normal hyperbolicity theory, each Anosov element has a pair of Hölder foliations \(W^s_a\) and \(W^u_a\) with \(C^r\) leaves. \(W^s_a\) are unique integral foliations of the distributions \(E^s_a\), \(s = s, u\). By considering the action of other elements on such foliations, one may refine them to find common stable manifolds \(W^s_{a_1, \ldots, a_n}\) for any collection of Anosov elements \(a_1, \ldots, a_n\), which are characterized as

\[
W^s_{a_1, \ldots, a_n}(x) = \left\{ y \in X : d(\alpha(ka_i)x, \alpha(ka_i)y) \xrightarrow{k \to \infty} 0 \text{ for } i = 1, \ldots, n \right\}.
\]

Each common stable manifold is a Hölder foliation with \(C^r\) leaves, and has tangent bundle \(TW^s_{a_1, \ldots, a_n} = \cap_{i=1}^n E^s_{a_i}\).

**Definition 3.1.** A common stable manifold \(W^\beta = W^s_{a_1, \ldots, a_n}\) is a coarse Lyapunov foliation of an action \(\alpha : \mathbb{R}^k \curvearrowright X\) if for any Anosov element \(a \in \mathbb{R}^k\), \(W^\beta \subset W^s_a\) or \(W^\beta \subset W^u_a\). We call \(E^\beta = TW^\beta\) the corresponding coarse Lyapunov distribution. Let \(\Delta\) denote an indexing set for the collection of coarse Lyapunov foliations.

We say that an Anosov action is Cartan if for every \(\beta \in \Delta\), \(\dim(W^\beta) = 1\). We say that an action is totally Cartan if it is Cartan and totally Anosov.

The following is analogous to the following fact from linear algebra: if \(A \in GL(N, \mathbb{R})\), \(\mathbb{R}^N\) splits as a direct sum of generalized eigenspaces (allowing for Jordan blocks within the eigenspaces), and any \(A\)-invariant subspace is refined by the subspaces.
Lemma 3.2. If $\alpha : \mathbb{R}^k \acts X$ is an Anosov $C^r$ group action on a $C^\infty$ manifold $X$, then $TX = T\mathcal{O} \oplus \bigoplus_{\beta \in \Delta} W^\beta$. If $\alpha$ is Cartan, and $E \subset TX$ is an $\mathbb{R}^k$-invariant distribution, then there exists a subset $\Phi \subset \Delta$ and a subbundle $E_\Phi \subset T\mathcal{O}$ such that $E = E_\Phi \bigoplus_{\beta \in \Phi} E^\beta$.

Proof. The part of the lemma for Anosov actions follows from [13, Corollary 4.6]. Let $V$ be an $\mathbb{R}^k$ invariant distribution. Then fix a periodic point $p \in X$ and some $v \in T_p X$, and an Anosov element $a \in \mathbb{R}^k$ such that $a \cdot p = p$. Since the set of Anosov elements are open, convex and a union of lines, given any Anosov element $a'$, we may choose $a$ which fixes $p$ and shares the same splitting into stable and unstable distributions. Then since $TX = T\mathcal{O} \oplus E_a^s \oplus E_a^u$, and $d\alpha(a)V(p) = V(p)$, $V(p)$ has a common refinement with the stable and unstable splitting at $p$ since they are sums of the generalized eigenspaces for $d\alpha(a)$. So there exists corresponding subspaces of $V(p)$ such that $V(p) = V_a^0(p) \oplus V_a^s(p) \oplus V_a^u(p)$, and $V_a^s(p) = E_a^s(p) \cap V(p)$.

Now, since all distributions are continuously varying and this splitting holds at periodic orbits, since the periodic orbits are dense, $V$ splits everywhere as $V^0 \oplus V^s \oplus V^u$. This procedure can be repeated for another Anosov element $b$ to refine each new invariant distribution into the stable and unstable distributions for $b$. In particular, since each common stable manifold is either a coarse Lyapunov distribution or can be refined, the final refinement gives a subspace of each coarse Lyapunov distribution. In particular, since the dimension of the coarse Lyapunov distributions are assumed to be 1 for Cartan actions, either the distribution appears fully as part of the final splitting of $V$, or does not appear at all. 

Definition 3.3. Let $\alpha : \mathbb{R}^k \acts X$ be a Cartan action. A coarse Lyapunov path based at $x \in X$ is a finite sequence $\rho = (x = x_0, x_1, \ldots, x_n)$ such that $x_{i+1} \in W^{\beta_i}(x_i)$ for some coarse Lyapunov foliation $W^{\beta_i}$. $c(\rho) = n$ is called the combinatorial length of the path $\rho$, and $L(\rho) = \sum_{i=0}^{n-1} d_{W^{\beta_i}}(x_i, x_{i+1})$ is called the geometric length of $\rho$. $e(\rho) = x_n$ is called the endpoint of $\rho$.

$\alpha$ is said to have locally transitive coarse Lyapunov foliations if for every $x \in X$, there is some $N \in \mathbb{N}$ and $L_0 > 0$ such that

$$\{e(\rho) : L(\rho) \leq L_0, c(\rho) \leq N\}$$

contains a neighborhood of $x$.

Lemma 3.4. If $\alpha_0 : \mathbb{R}^k \acts X$ is a Cartan defined as a $k$-fold product of topologically mixing Anosov flows on 3-manifolds, there exists a $C^1$-neighborhood $U$ of $\alpha_0$ in the space of $C^1 \mathbb{R}^k$-actions such that every $\alpha \in U$ is an Anosov action with locally transitive coarse Lyapunov foliations.

Proof. Following [11], it follows that each of the Anosov flows for $\alpha_0$ satisfies an engulfing property, which establishes a continuously varying family of paths in the stable and unstable foliations around a point which stably fill out a neighborhood (see Section 3, and in particular, Proposition 3.4 of [11] and the subsequent remark). One may generalize this by insisting that the paths are tangent to coarse Lyapunov foliations, and as a consequence, each Anosov element $a \in \mathbb{R}^k$ will satisfy the engulfing property for $\alpha_0$, and hence the coarse Lyapunov foliations are locally transitive. Since engulfing is stable under perturbations, the claim follows. □
4. Rank one factors and hyperplanes

Lemma 4.1. Let \( \mathbb{R}^k \curvearrowright X \) be a transitive, \( C^1 \) Cartan action and \( \psi_t : \mathbb{R} \curvearrowright Y \) be a \( C^1 \) rank one factor, with corresponding homomorphism \( \sigma : \mathbb{R}^k \to \mathbb{R} \) and projection map \( \pi : X \to Y \). Assume that \( E^x \) is a coarse Lyapunov distribution. Then if \( E^x \cap \ker d\pi = \{0\} \) at some \( x \in X \), there exists a continuous metric on \( E^x \) such that for every \( a \in \ker \sigma \), \( da|_{E^x} \) is an isometry.

Proof. Notice that \( \ker d\pi \) is a continuous \( \mathbb{R}^k \)-invariant subbundle since \( \pi \) is a submersion. It therefore has a common refinement with the coarse Lyapunov splitting \( TX = T\mathcal{O} \oplus \bigoplus_{\gamma \in \Delta} E^\gamma \) by Lemma 3.2. In particular, either \( E^x \subset \ker d\pi \) for every \( x \in X \), or it is transverse to \( \ker d\pi \) for every \( x \in X \).

Therefore \( E \) projects to a nontrivial, continuous, \( \psi_t \)-invariant subbundle on \( Y \). Choose any metric \( ||| \cdot |||_Y \) on \( Y \) and if \( v \in E \), let \( ||v||_E := ||d\pi(v)||_Y \). Since \( d\pi|_E \) is injective, this is a well-defined continuous norm. By construction, if \( a \in \ker \sigma \), the action of \( a \) on \( Y \) is trivial, so:

\[
||da(v)||_E = ||d\pi da(v)||_Y = ||d\pi(v)||_Y = ||v||_E.
\]

Lemma 4.2. Let \( k \geq 2 \) and \( \alpha : \mathbb{R}^k \curvearrowright X \) be a strongly transitive, \( C^r \) Cartan action with locally transitive coarse Lyapunov foliations such that for every \( \beta \in \Delta \) and \( a \in \mathbb{R}^k \setminus \{0\} \), there exists \( x \in X \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \log ||d(\alpha^n a(x))||_E \neq 0.
\]

Then \( \alpha \) has no nontrivial \( C^r \) rank one factors.

Proof. Note that any coarse Lyapunov distribution \( E^\beta \) is continuous with a complementary \( \mathbb{R}^k \)-invariant subbundle. Our assumption implies that there is no subgroup of \( \mathbb{R}^k \) which acts isometrically on \( E^\beta \) for any continuous metric. Therefore, if \( \pi : X \to Y \) determines a rank one factor, then \( E^\beta \subset \ker d\pi \) for every coarse Lyapunov distritution by Lemma 4.1. Therefore, \( \pi^{-1}(x) \) contains all coarse Lyapunov foliations, and since they are locally transitive, must be all of \( X \). That is \( \pi \) is a projection onto a point and there are no nontrivial rank one factors.

Lemma 4.3. Under the same assumptions as Lemma 4.2, the action \( \alpha \) is not homogeneous.

Proof. This follows almost immediately. Notice that the derivative of a homogeneous action is always determined by the adjoint representation. Since the coarse Lyapunov distribution is 1-dimensional, it must be spanned by a joint eigenvector of the \( \mathbb{R}^k \)-action. Since every functional from \( \mathbb{R}^k \) to \( \mathbb{R} \) has a nontrivial kernel, there must exists some \( a \) such that \( ||da||_{E^x} = 1 \) with respect to any right-invariant metric. This is incompatible with the assumptions.

5. Construction of the example

Let \( f_s : Y_1 \to Y_1 \) and \( g_t : Y_2 \to Y_2 \) be topologically mixing Anosov flows. Let \( M = Y_1 \times Y_2 \) and consider the product action \( \alpha_0 : \mathbb{R}^2 \curvearrowright M \times M \) defined by

\[
\alpha_0(s, t)(x_1, x_2) = (f_s(x_1), g_t(x_2)).
\]
Then \( \alpha_0 \) is (totally) Cartan with four coarse Lyapunov distributions, \( W^{\pm \chi_1}, W^{\pm \chi_2} \) corresponding to the stable and unstable bundles in each factor of the action. That is, \( E^{\chi_1} = E^+_t \times \{0\}, E^{-\chi_1} = E^-_t \times \{0\}, E^{\chi_2} = \{0\} \times E^s_t \) and \( E^{-\chi_2} = \{0\} \times E^u_t \). Furthermore, the elements \((\pm 1, \pm 1) \in \mathbb{R}^2 \) are Anosov elements of the action. Let \( \varepsilon_1 \) be such that if \( F : \mathcal{M} \to \mathcal{M} \) is such that \( d_{C^1}(F, a) < \varepsilon_1 \) for \( a = (\pm 1, \pm 1) \), then \( F \) acts normally hyperbolically with respect to a nearby foliation, and nearby distributions (such a \( \varepsilon_1 \) exists by Hirsch-Pugh-Shub normal hyperbolicity theory). Let \( \varepsilon_0 \) be as in Lemma 2.4 and choose \( \delta < \min \{\varepsilon_0/4, \varepsilon_1/100\} \) and two points \( p_1, p_2 \in Y_1, q_1, q_2 \in Y_2 \) which lie on distinct periodic orbits of \( f_t \) and \( g_s \), respectively. We may assume that a continuous Riemannian metric on \( Y_1 \) and \( Y_2 \) has been chosen so that there exist coefficients \( \lambda_i, \mu_i, i = 1, 2 \) and \( * = s \) or \( u \) such that

\[
\left\lVert df_t|_{E^+_t}\right\rVert (p_i) = e^{\lambda_i \tau}, \quad \left\lVert dg_s|_{E^s_t}\right\rVert (q_i) = e^{\mu_i \tau} \quad \text{for } i = 1, 2, \ * = s \text{ or } u.
\]

Finally, pick functions \( u_i : Y_i \to \mathbb{R}, \ i = 1, 2 \) such that

1. \( u_i \) is \( C^\infty \), \( i = 1, 2 \)
2. \( u_1(f_t(p_1)) \equiv u_2(g_s(q_1)) \equiv \delta \) for all \( s, t \in \mathbb{R} \)
3. \( u_1(f_t(p_2)) \equiv u_2(g_s(q_2)) \equiv -\delta \) for all \( s, t \in \mathbb{R} \)
4. \( |u_1|, |u_2| \leq 2\delta \)

Such functions \( u_i \) generate cocycles \( \theta_i \) over the flows \( f_t \) and \( g_s \) via the formula \( \theta_1(t, x) = \int_0^t u_1(f_s(x)) \, d\tau \) and \( \theta_2(s, x) = \int_0^s u_2(g_t(x)) \, d\tau \).

Then define a cocycle \( \beta \) over \( \alpha_0 \) by:

\[
\beta(s, t; x) = (s - \theta_2(t, x_2), t - \theta_1(s, x_1)).
\]

One easily verifies that \( \beta \) satisfies property (2.2) for the action \( \alpha_0 \). Furthermore, by the smallness assumption on \( f \), the cocycle \( \beta \) also satisfies the assumptions of Lemma 2.4. Let \( \alpha \) be the corresponding time change of \( \alpha_0 \). We may further assume that \( \delta \) is chosen small enough so that \( \alpha \in \mathcal{U} \), where \( \mathcal{U} \) is the neighborhood in Lemma 3.4.

**Theorem 5.1.** \( \alpha \) is a \( C^\infty \) Cartan action without rank one factors, and which is not homogeneous.

**Proof.** First, notice that \((\pm 1, \pm 1)\) are still Anosov elements since our cocycle \( \beta \) was sufficiently close to \( \text{Id} \). Therefore, we have the same indexing set for the coarse Lyapunov foliations \( \{\pm \chi_1, \pm \chi_2\} \), even though their distributions and foliations may be perturbed.

By Lemma 3.4, \( \alpha \) has locally transitive coarse Lyapunov foliations. So by Lemmas 4.2 and 4.3, it suffices to show that given any \( \chi \in \Delta \), every \( a \in \mathbb{R}^2 \setminus \{0\} \) has some point \( x \in X \) such that \( \lim_{n \to \infty} n^{-1} \log \left| da(\alpha^n(\chi)(p))\right| \neq 1 \). We work with \( E^{\chi_1} \), since all other coarse Lyapunov distributions will have a symmetric argument. Consider the derivatives of \( a \) at the points \( x = (p_1, q_2) \in M \times M \) and \( y = (p_2, q_1) \in M \times M \). By assumption, for fixed \( (s, t) \) we may explicitly compute \( \beta \) near \( x \), \( \beta(s, t; x) = (s + \delta t, t - \delta s) \). Therefore, with \( (s, t) \) fixed and \( x' \) near \( x \),

\[
\varphi(s, t; x') = \frac{1}{1 + \delta^2}(s - \delta t, t + \delta s).
\]
Fix $a = (s, t)$, so that the function $\varphi$ is constant in a neighborhood of the $p_1$-orbit. Therefore, since the time change is constant in a neighborhood, $E^{x_1}$ is exactly a coarse Lyapunov distribution for $\alpha$ along the orbit $x$, as it is an invariant distribution transverse to $O$ at $x$. Denote $(s', t') = \varphi(s, t; x)$. Now, we get that if $v \in E^{x_1}(x)$,

$$d\alpha(s, t)v = d\alpha_0(s', t')v = e^{\lambda_1, u(s-\delta t)/(1+\delta^2)}.$$  

By a symmetric computation, for fixed $a = (s, t)$, $\varphi(s, t; y') = \frac{1}{1+\delta^2}(s + \delta t, t - \delta s)$ with $y'$ near $y$. Therefore, if $v \in E^{x_1}(y)$

$$d\alpha(a)v = e^{\lambda_2, u(s+\delta t)/(1+\delta^2)}.$$  

It is not possible that $s - \delta t = s + \delta t = 0$ unless $s = t = 0$. Therefore, no non-identity element has zero exponents for $\chi_1$ at every $x \in X$. We may repeat this process for $-\chi_1$ and $\pm \chi_2$. Then by Lemma 4.2 the action $\alpha$ has no rank one factors.  

6. Remarks on the example and conjecture

We begin by briefly noting that this example was discovered in the context of several other unexpected examples, and is indirectly related to them. In [13], such examples are discussed at length. Another important example of a $\mathbb{Z}^2$ action with nontrivial coexistence of rigidity and flexibility properties was recently constructed by Damjanovic, Wilkinson and Xu [3].

The Katok-Spatzier conjecture can be reformulated in a variety of settings. One way to adjust the conjecture is to strengthen the assumptions. In the formulation of Conjecture 1.4, one assumes that the $\mathbb{R}^k$-action has no $C^\infty$ rank one factors. Notice every $C^\infty$ rank one factor is a continuous rank one factor, and in the measure-preserving setting, every continuous rank one factor is a measurable rank one factor. Therefore, one may consider asking the action to have no continuous, or no measurable rank one factors (with respect to an invariant volume) in order to guarantee rigidity.

One should expect that these examples remain counterexamples with to such revisions of Conjecture 1.4. Indeed, one may see the destruction of a measurable rank one factor when one only destroys one smooth factor. When one uses the cocycle $\beta(s, t; x) = (s, t - \theta_1(s, x_1))$, we may explicitly compute the corresponding function $\varphi(s, t; x) = (s, t + \theta_1(s, x_1))$. Then the time change $\alpha$ induced by $\varphi$ contains a skew product action: the horizontal direction $(s, 0)$ is exactly a skew product. Skew products determined by cocycles not cohomologous to a constant are ergodic. Combined with the following, this shows that the projection onto the second factor of $M = Y_1 \times Y_2$ is no longer a rank one factor.

**Lemma 6.1.** A measure-preserving action $\mathbb{R}^2 \acts (X, \mu)$ has a nontrivial measurable rank one factor if and only if there exists a line $L \subset \mathbb{R}^2$ such that the restriction of the action to $L$ is not ergodic.

**Proof.** First, assume that there exists a rank one factor $\psi_1 : (Y, \nu) \to (Y, \nu)$ determined by a measurable map $\pi : X \to Y$ such that $\pi_* \mu = \nu$ and homomorphism $\sigma : \mathbb{R}^2 \to \mathbb{R}$. Then if $L = \ker \sigma$, $L$ acts trivially on $Y$. Since $Y$ is nontrivial, any function on $X$ defined by $\psi \circ \pi$, for some measurable function $\psi : Y \to \mathbb{R}$ is invariant under $L$. Since $Y$ is not trivial, there exist nontrivial $L$-invariant functions, and the $L$-action is not ergodic.
Now, assume that the restriction of the action to $L$ is not ergodic. By the ergodic decomposition theorem, there exists a $\mu$-almost-everywhere defined map to the space of $L$-invariant measures $\mathcal{M}(L), \phi : (X, \mu) \to (\mathcal{M}(L), \nu)$ such that for any $f \in L^1(X, \mu)$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f((t\ell) \cdot x) \, dt = \int_X f \, d\phi(x).$$

By construction, the action of $\mathbb{R}^k$ descends to $(\mathcal{M}(L), \nu)$ via $a \cdot m = a_* m$, and the $L$-action is trivial. That is, $\phi$ determines a measurable rank one factor of the $\mathbb{R}^k$ action. \hfill \Box

Another way to account for these new examples would be to ask for no rank one factors, even after the modifications used to produce these new examples.

**Question 1.** If no $C^\infty$ time change of a strongly transitive, $C^\infty$ Anosov action $\alpha : \mathbb{R}^k \curvearrowright X$ has a $C^\infty$ rank one factor, is $\alpha C^\infty$ conjugate to an algebraic system?

Another interpretation would be to ignore the parameterization of orbits induced by the action altogether, and consider only the orbit foliations.

**Question 2.** Assume that $\alpha : \mathbb{R}^k \curvearrowright X$ is a strongly transitive, $C^\infty$, Anosov action, and that there does not exist a nontrivial $C^\infty$ flow $\psi_t : Y \to Y$ and submersion $\pi : X \to Y$ such that $\pi(\mathbb{R}^k \cdot x) = \{\psi_t(\pi(x)) : t \in \mathbb{R}\}$ for all $x \in X$. Is $\alpha C^\infty$ conjugate to an algebraic system?

In view of Remark 1.5, allowing for time changes is a more accurate reflection of the spirit of the rigidity program. A time change is determined by a cocycle over the new action. If an action $\alpha$ is cocycle-rigid, as are the homogeneous actions without rank one factors, any $C^r$ time change should be smoothly conjugate to a linear time change of $\alpha$. Therefore, even if one allows for a time change, if rigidity holds, that time change would be trivial. Thus, one still obtains a smooth conjugacy with the original action.

Finally, when the action is strongly transitive and totally Cartan (see Definitions 1.1 and 3.1), the main theorem of [13] implies that if the action has no rank one factor, it is $C^\infty$ conjugate to an algebraic system. This immediately implies that the examples here are not totally Cartan (which is also observable directly from computations), and motivates the following

**Question 3.** Let $\mathbb{R}^k \curvearrowright X$ be a strongly transitive, Cartan action. Is there a $C^\infty$ time change of the action which is totally Cartan?

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