More on the Ehrenfencht-Fraïssé game of length $\omega_1$.

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This paper is a continuation of [8]. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two first order structures of the same vocabulary $L$. We denote the domains of $\mathfrak{A}$ and $\mathfrak{B}$ by $A$ and $B$ respectively. All vocabularies are assumed to be relational. The Ehrenfeucht-Fraïssé-game of length $\gamma$ of $\mathfrak{A}$ and $\mathfrak{B}$ denoted by $\text{EFG}_\gamma(\mathfrak{A}, \mathfrak{B})$ is defined as follows: There are two players called $\forall$ and $\exists$. First $\forall$ plays $x_0$ and then $\exists$ plays $y_0$. After this $\forall$ plays $x_1$, and $\exists$ plays $y_1$, and so on. If $\langle (x_\beta, y_\beta) : \beta < \alpha \rangle$ has been played and $\alpha < \gamma$, then $\forall$ plays $x_\alpha$ after which $\exists$ plays $y_\alpha$. Eventually a sequence $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$ has been played. The rules of the game say that both players have to play elements of $A \cup B$. Moreover, if $\forall$ plays his $x_\beta$ in $A$ ($B$), then $\exists$ has to play his $y_\beta$ in $B$ ($A$). Thus the sequence $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$ determines a relation $\pi \subseteq A \times B$. Player $\exists$ wins this round of the game if $\pi$ is a partial isomorphism. Otherwise $\forall$

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wins. The notion of winning strategy is defined in the usual manner. The game \( EFG_\gamma^\delta(A, B) \) is defined like \( EFG_\gamma(A, B) \) except that the players play sequences of length \(< \delta\) at a time. Thus \( EFG_\gamma(A, B) \) is the same game as \( EFG_\delta^\gamma(A, B) \).

It was proved in [8] that, assuming \( \Box_{\omega_1} \), there are models \( \mathfrak{A} \) and \( \mathfrak{B} \) of cardinality \( \aleph_2 \) such that the game \( \mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B}) \) is non-determined. In this paper we weaken the assumption \( \Box_{\omega_1} \), to “\( \omega_2 \) is not weakly compact in \( L \)” (Corollary 13), but we can do this only if we assume CH. We do not know if this is possible without CH. In the other direction, it was proved in [8] that if the \( \omega_1 \)-nonstationary ideal on \( \omega_2 \) has a \( \sigma \)-closed dense subset, then the game \( EFG_{\omega_1}(\mathfrak{A}, \mathfrak{B}) \) is determined for all \( \mathfrak{A} \) and \( \mathfrak{B} \) of cardinality \( \leq \aleph_2 \). The assumption is equiconsistent with the existence of a measurable cardinal. In this paper we weaken the assumption to a condition which is consistent relative to the existence of a weakly compact cardinal (Corollary 13). Thus we establish:

**Theorem 1**  The following statements are equiconsistent relative to ZFC:

1. There is a weakly compact cardinal.

2. CH and \( EF_{\omega_1}(\mathfrak{A}, \mathfrak{B}) \) is determined for all models \( \mathfrak{A} \) and \( \mathfrak{B} \) of cardinality \( \aleph_2 \).

In [8] we proved in ZFC that there are structures \( \mathfrak{A} \) and \( \mathfrak{B} \) of cardinality \( \aleph_3 \) with one binary predicate such that the game \( EFG_{\omega_1}(\mathfrak{A}, \mathfrak{B}) \) is non-determined. We now improve this result under some cardinal arithmetic assumptions. We prove:

**Theorem 2**  Assume that \( 2^{\omega} < 2^{\omega_3} \) and \( T \) is a countable complete first order theory. Suppose that one of (i)-(iii) below holds. Then there are \( \mathfrak{A}, \mathfrak{B} \models T \) of power \( \omega_3 \) such that for all cardinals \( 1 < \theta \leq \omega_3 \), \( EF_{\omega_1}^\theta(\mathfrak{A}, \mathfrak{B}) \) is non-determined.

(i) \( T \) is unstable.

(ii) \( T \) is superstable with DOP or OTOP.

(iii) \( T \) is stable and unsuperstable and \( 2^{\omega} \leq \omega_3 \).
This result complements the result in [8] that if $T$ is an $\omega$-stable first order theory with NDOP, then $EFG_{\omega_1}(\mathcal{A}, \mathcal{B})$ is determined for all models $\mathcal{A}$ of $T$ and all models $\mathcal{B}$. This is actually true under the weaker assumption that $T$ is superstable with NDOP and NOTOP.

**Notation:** We follow Jech [3] in set theoretic notation. We use $S_n^m$ to denote the set \( \{ \alpha < \omega_m : \text{cof}(\alpha) = \omega_n \} \). Closed and unbounded sets are called cub sets. A set of ordinals is $\lambda$-closed if it is closed under supremums of ascending $\lambda$-sequences $\langle \alpha_i : i < \lambda \rangle$ of its elements. A subset of a cardinal is $\lambda$-stationary if it meets every $\lambda$-closed unbounded subset of the cardinal.

## 1 Getting a weakly compact cardinal

In this section we show that if CH holds and $EFG_{\omega_1}(\mathcal{A}, \mathcal{B})$ is determined for all models $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_2$, then $\omega_2$ is weakly compact in $L$ (Corollary [3]). We use the results from [7] that if $\omega_2$ is not weakly compact in $L$, then there is a bistationary $S \subseteq S_2^0$ such that for all $\alpha < \omega_2$ either $\alpha \cap S$ or $\alpha \setminus S$ is non-stationary.

If $I$ is a linear order, we use $(I)^*$ to denote the reverse order of $I$. We call a sequence $s = (s_\xi)_{\xi < \zeta}$ coinitial sequence of length $\zeta$ in $I$, if it is decreasing in $I$ and has no lower bound in $I$. The coinitiality $\text{coinit}(I)$ of a linear order $I$ is the smallest length of a coinitial sequence in $I$.

Let $\theta = \omega + ((\omega_1)^* + \omega) \cdot \omega_1$.

**Lemma 3** There is a dense linear order $I$ such that

(i) $|I| = \aleph_1$.

(ii) $\text{coinit}(I) = \aleph_0$

(iii) $I \cdot (\alpha + 1) \cong I$ for all $\alpha \leq \omega_1$.

(iv) $I \cong I \cdot \omega + I \cdot (\omega_1)^*$.

(v) $I \cdot \theta + I \cong I$.

**Proof.** This is like Lemma 4.7.16 in [9]. If $J_1$ and $J_2$ are linear orders, let $H(J_1, J_2)$ be the set of $f : n_f \to J_1 \cup J_2$, where $n_f < \omega$ is even, $f(2i) \in J_1$.
and $f(2i + 1) \in J_2$ for all $i < n_f$. We can make $H(J_1, J_2)$ a linear order by ordering the functions lexicographically, i.e.

$$f \leq g \iff \exists m \leq n_f(\forall i < m(f(i) = g(i)) \& (m < n_f \rightarrow f(m) < g(m))).$$

Let $I_0 = H(Q, \omega + (\omega_1)^*)$ and $I_1 = H(I_0, \omega_1)$. Thus $I_0 \cong (1 + I_0) \cdot (\omega + (\omega_1)^*) \cdot Q$ and $I_1 \cong (1 + I_1) \cdot \omega_1 \cdot I_0$. By using $Q \cong Q + 1 + Q$, $\omega = 1 + \omega$ and $\omega_1 = 1 + \omega_1$, one gets easily the following, first for $I_0$, and then for $I_1$:

$$I_0 \cong I_0 + 1 + I_0, \quad I_1 \cong I_1 + 1 + I_0. \quad (1)$$

Let $I$ be the set of $f : \omega \rightarrow I_1 \cup \theta$, where $f(2i) \in I_1$ and $f(2i + 1) \in \theta$ for all $i < \omega$ ordered lexicographically. Thus $I \cong I \cdot \theta \cdot I_1$. In fact, $I$ is of the form $J \cdot Q$, so (ii) is true. By (i) and $\theta \cong 1 + \theta$ one gets immediately (v). If we combine this with $I \cong I \cdot \theta \cdot I_1$ and $(\omega_1)^* \cong (\omega_1)^* + 1$, we get (iv).

As to (i), we only have $|I| = 2^\omega$. We use this lemma in a context where CH is assumed, so we could simply assume it here. But actually the lemma is true without CH, as we can construct $I$ in $L$. Then $|I| = \aleph_1$. Note that our $I_0$ and $I_1$ are in $L$, and the only property of $\omega_1$ that we used was that it is a limit ordinal. \(\Box\)

**Definition 4** Suppose $S \subseteq S_0^2$. We define

$$\Phi(S) = \sum_{i < \omega_2} \eta_i,$$

where

$$\eta_i = \begin{cases} I \cdot (\omega_1)^*, & \text{if } i \in S \\ I, & \text{if } i \notin S. \end{cases}$$

Let $\Phi_{\alpha, \beta}(S)$ be the suborder $\sum_{\alpha \leq i < \beta} \eta_i$ of $\Phi(S)$. The rank of $x \in \Phi(S)$ is the least $\alpha$ such that $x \in \Phi_{\alpha, \alpha+1}(S)$. We denote this $\alpha$ by $\text{rnk}(\Phi(S), x)$.

**Lemma 5** Assume $S \subseteq S_0^2$ is such that there is no $\alpha \in S_1^2$ with both $S \cap \alpha$ and $(S \cap S_0^2) \setminus S$ stationary. Then

$$\Phi_{\alpha, \alpha+1}(S) \cong I$$

whenever $\alpha < \beta < \omega_2$ and $\alpha \notin S$.  

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Lemma 6. Assume $S \cap I$ is lost for $\exists \alpha$ such that $S$ stationary. Then there are models sequence of length $\omega$ strategy. By CH we have an EF and $\xi$. So $n < \omega$ round number $\omega$ during the first $\omega$ rounds of the game, $\forall$ plays elements of the models of rank $< \alpha$, then so does $\exists$ following $\tau$. Let $\delta \in C \cap S$. Let $(\delta_n)_{n<\omega}$ be an increasing cofinal sequence in $\delta$. Now we let $\forall$ play against $\tau$ as follows: On round number $n < \omega$ we let $\forall$ play some element of $A$, if $n$ is even, and of $B$, if $n$ is odd, of rank $\delta_n$. During rounds $\omega + n$, $n < \omega$, we let $\forall$ play a coinitial sequence of length $\omega$ in $\Phi_{\delta,\delta+1}(\emptyset) \subseteq A$. As $\coinf(\Phi_{\delta,\delta+1}(S)) = \omega_1$, the game is lost for $\exists$. So $\tau$ could not be a winning strategy.

Suppose then $\rho$ is a strategy of $\forall$. We show that this cannot be a winning strategy. By CH we have an $\omega_1$-cub set $D$ of ordinals $\delta < \omega_2$ such that if

**Proof.** This is like Lemma 4.7.19 in [1]. We use Lemma $\exists$ and induction on $\beta$.

Let us first assume $\beta \notin S$. If $\beta$ is a successor ordinal, then $\Phi_{\alpha,\beta+1}(S) \cong I + I = I$ by (iii). If $\beta$ has cofinality $\omega$, then $\Phi_{\alpha,\beta+1}(S) \cong I \cdot \omega + I \cong I$. If $\beta$ has cofinality $\omega_1$ and $\beta \cap S$ is non-stationary, then $I \cong I \cdot \omega + I \cong I$. Finally, if $\beta$ has cofinality $\omega_1$ and $\beta \setminus S$ is non-stationary, then $I \cong I \cdot \theta + I \cong I$, by $\langle v \rangle$.

Let us then assume $\beta \in S$. Thus $\beta$ has cofinality $\omega$. Therefore $\Phi_{\alpha,\beta+1}(S) \cong I \cdot \omega + I \cdot (\omega_1)^+ \cong I$, by (iv). $\square$

**Lemma 6** Assume $S \subseteq S^3_0$ is such that there is no $\alpha \in S^3_1$ with both $S \cap \alpha$ and $(S \cap S^3_0) \setminus S$ stationary. Then $\Phi_{0,\alpha}(S) \cong \Phi_{0,\alpha}(\emptyset)$ whenever $\alpha \in S^3_1$ and $S \cap \alpha$ is not stationary.

**Proof.** Let $(\alpha_\xi)_{\xi < \omega_1}$ by a continuously increasing cofinal sequence in $\alpha$ such that $\alpha_\xi \notin S$ for all $\xi < \omega_1$. By Lemma $\exists$ there is an isomorphism

$$f_\xi : \Phi_{\alpha_\xi,\alpha_\xi+1}(S) \rightarrow \Phi_{\alpha_\xi,\alpha_\xi+1}(\emptyset).$$

Let $f = \cup_{\xi < \omega_1}f_\xi$. This is the required isomorphism. $\square$

**Proposition 7** Assume CH and that there is $S \subseteq S^3_0$ such that both $S$ and $S^3_0 \setminus S$ are stationary but there is no $\alpha \in S^3_1$ with both $S \cap \alpha$ and $(S \cap S^3_0) \setminus S$ stationary. Then there are models $A$ and $B$ of cardinality $\aleph_2$ such that $EF_{\omega_1}(A, B)$ is non-determined.

**Proof.** We may assume, that $\{\alpha \in S^3_1 : \alpha \cap S$ is non-stationary $\}$ is stationary, for otherwise we work with $S' = S^3_0 \setminus S$. Let $A = \Phi(S)$ and $B = \Phi(\emptyset)$. We first show that $\exists$ cannot have a winning strategy in $EF_{\omega+\omega_1}(A, B)$. Suppose $\tau$ is a strategy of $\exists$. Let $C$ be the cub of ordinals $\alpha < \omega_2$ such that if during the first $\omega$ rounds of the game, $\forall$ plays elements of the models of rank $< \alpha$, then so does $\exists$ following $\tau$. Let $\delta \in C \cap S$. Let $(\delta_n)_{n<\omega}$ be an increasing cofinal sequence in $\delta$. Now we let $\forall$ play against $\tau$ as follows: On round number $n < \omega$ we let $\forall$ play some element of $A$, if $n$ is even, and of $B$, if $n$ is odd, of rank $\delta_n$. During rounds $\omega + n$, $n < \omega$, we let $\forall$ play a coinitial sequence of length $\omega$ in $\Phi_{\delta,\delta+1}(\emptyset) \subseteq A$. As $\coinf(\Phi_{\delta,\delta+1}(S)) = \omega_1$, the game is lost for $\exists$. So $\tau$ could not be a winning strategy.

Suppose then $\rho$ is a strategy of $\forall$. We show that this cannot be a winning strategy. By CH we have an $\omega_1$-cub set $D$ of ordinals $\delta < \omega_2$ such that if
exists plays only elements of rank $< \delta$, then $\rho$ directs $\forall$ to play also elements of rank $< \delta$ only. Let $\delta \in D \cap S_2^\omega$ such that $\delta \cap S$ is non-stationary. By Lemma 8 there is an isomorphism $f : \Phi_0,\alpha(S) \to \Phi_0,\alpha(\emptyset)$. Now $\exists$ can beat $\rho$ by using $f$. \lq\lq

Corollary 8 If CH holds and $EFG_{\omega_1}(A,B)$ is determined for all models $A$ and $B$ of cardinality $\aleph_2$, then $\omega_2$ is weakly compact in $L$.

2 Getting determinacy from a weakly compact cardinal

In this section we show that if $\kappa$ is weakly compact, then there is a forcing extension in which the game $EFG_{\omega_1}(A,B)$ is determined for all $A$ and $B$ of cardinality $\leq \aleph_2$.

We shall consider models $A, B$ of cardinality $\aleph_2$, so we may as well assume they have $\omega_2$ as universe. For such a model $A$ and any ordinal $\alpha < \omega_2$ we let $A_\alpha$ denote the structure $A \cap \alpha$. Similarly $B_\alpha$. Let us first recall the following basic fact from [8]:

Lemma 9 [8] Suppose $A$ and $B$ are structures of cardinality $\aleph_2$. If $\forall$ does not have a winning strategy in $EFG_{\omega_1}(A,B)$, then

$$S = \{ \alpha : A_\alpha \cong B_\alpha \}$$

is $\omega_1$-stationary.

This shows that to get determinacy of $EFG_{\omega_1}(A,B)$ it suffices to give a winning strategy of $\exists$ under the assumption that the above set $S$ is $\omega_1$-stationary. In [3] an assumption $I^*(\omega)$ was used. This assumption says that the non-$\omega_1$-stationary ideal on $\omega_2$ has a $\sigma$-closed dense set. The rough idea was that $\exists$ uses the Pressing Down Lemma on $S$ to ”normalize” his moves so that he always has an $\omega_1$-stationary sets of possible continuations of the game. We use now the same idea. The hypothesis $I^*(\omega)$ is equiconsistent with a measurable cardinal. Since we assume only the consistency of a weakly compact cardinal, we have to work more.

Suppose $\kappa$ is a weakly compact cardinal. Let $\mathcal{I}$ denote the $\Pi^1_1$-ideal on $\kappa$, i.e. the ideal of subsets of $\kappa$ generated by the sets $\{ \alpha : (H(\alpha), \epsilon, A \cap H(\alpha)) \models
\[ \exists \neg \phi \}, \text{ where } A \subseteq H(\kappa) \text{ and } \phi \text{ is a } \Pi^1_1 \text{-sentence such that } (H(\kappa), \epsilon, A) \models \phi. \text{ We collapse } \kappa \text{ to } \omega_2 \text{ and then force a cub to the complement of every set } S \subseteq S_1^2 \text{ in } \mathcal{I}. \text{ In the resulting model the above "normalization" strategy of } \exists \text{ works even though the non-}\omega_1 \text{-stationary ideal on } \omega_2 \text{ may not have a } \sigma \text{-closed dense set.}

**Definition 10** Let } \mathcal{F} \text{ be a set of cardinality } \kappa \text{ of regressive functions } \kappa \to \kappa \text{ and } S \subseteq \kappa. \text{ The game } PDG_{\omega_1}(S, \mathcal{F}) \text{ has two players called } \forall \text{ and } \exists. \text{ They alternately play } \omega_1 \text{ rounds. During each round } \forall \text{ first chooses } f_i \in \mathcal{F}. \text{ Then } \exists \text{ chooses a subset } S_i \text{ of } \bigcap_{j<i} S_j \text{ (of } S, \text{ if } i=0) \text{ such that it is unbounded in } \kappa \text{ and } f_i \text{ is constant on } S_i. \text{ Player } \exists \text{ wins if he can play all } \omega_1 \text{ moves following the rules.}

**Lemma 11** Suppose } S = \{ \alpha < \omega_2 : \alpha \neq 0, \aleph_\alpha \cong \aleph_\alpha \} \text{ and } h_\alpha : \aleph_\alpha \cong \aleph_\alpha \text{ for } \alpha \in S. \text{ Let } \mathcal{F} = \{ f_\alpha : \alpha \in S \} \cup \{ g_\alpha : \alpha \in S \}, \text{ where } f_\alpha : \omega_2 \to \omega_2 \text{ is the regressive function mapping } \xi (\neq 0) \text{ to } h_\xi(\alpha) \text{ if } \xi > \alpha, \text{ and to } 0 \text{ otherwise, and } g_\alpha \text{ is the regressive function mapping } \xi (\neq 0) \text{ to } (h_\xi)^{-1}(\alpha) \text{ if } \xi > \alpha, \text{ and to } 0 \text{ otherwise. Suppose } \exists \text{ has a winning strategy in } PDG_{\omega_1}(S, \mathcal{F}). \text{ Then } \exists \text{ has a winning strategy in the game } EFG^2_{\omega_1}(\mathfrak{A}, \mathfrak{B}).

**Proof.** We present the proof for } EFG^2_{\omega_1}(\mathfrak{A}, \mathfrak{B}). \text{ The case of } EFG^2_{\omega_1}(\mathfrak{A}, \mathfrak{B}) \text{ is similar. } \mathcal{H} = \{ h_\alpha : \alpha \in S \}, \text{ where } h_\alpha : \aleph_\alpha \cong \aleph_\alpha \text{ for } \alpha \in S. \text{ Let } \tau \text{ be a winning strategy of } \exists \text{ in the game } PDG_{\omega_1}(S, \mathcal{F}). \text{ Suppose the sequence } ((x_i, y_i) : i < \alpha) \text{ has been played, where } \alpha < \omega_1, \text{ } x_i \text{ denotes a move of } \forall \text{ and } y_i \text{ a move of } \exists. \text{ Suppose } \forall \text{ plays next } x_\alpha. \text{ During the game } \exists \text{ also plays } PDG_{\omega_1}(S, \mathcal{F}). \text{ Let us denote his moves in } PDG_{\omega_1}(S, \mathcal{F}) \text{ by } S_\alpha. \text{ Thus } S_\alpha \subseteq S_i \text{ for } i < j < \alpha. \text{ The point of the sets } S_i \text{ is that } \exists \text{ has taken care that for all } i < \alpha \text{ and } j \in S_i \text{ we have } y_i = h_j(x_i) \text{ or } x_i = h_j(y_i) \text{ depending on whether } x_i \in \mathfrak{A} \text{ or } x_i \in \mathfrak{B}. \text{ Let } S_\alpha' = \bigcap_{i<\alpha} S_i \setminus \alpha_i. \text{ The winning strategy } \tau \text{ gives an } S_\alpha \subseteq S_\alpha' \text{ and a } y_\alpha \text{ such that } f_i(x_\alpha) = y_\alpha \text{ for all } i \in S_\alpha, \text{ if } x_\alpha \in \mathfrak{A}, \text{ and } g_i(x_\alpha) = y_\alpha \text{ for all } i \in S_\alpha, \text{ if } x_\alpha \in \mathfrak{B}. \text{ This element } y_\alpha \text{ is the next move of } \exists. \text{ Using this strategy } \exists \text{ cannot lose and hence wins. } \square

**Theorem 12** It is consistent relative to the consistency of a weakly compact cardinal, that for every } \omega_1 \text{-stationary } S \subseteq \omega_2 \text{ and every set } \mathcal{F} \text{ of cardinality } \aleph_2 \text{ of regressive functions } \omega_2 \to \omega_2, \exists \text{ has a winning strategy in the game } PDG_{\omega_1}(S, \mathcal{F}).
Proof. We may assume GCH. Suppose $\kappa$ is weakly compact. Let $Q$ be the Levy-collapse of $\kappa$ to $\aleph_2$. In $V^Q$ we define by induction a sequence $P_\alpha$, $\alpha < \kappa^+$, of forcing notions. Let $(A_\alpha)_{\alpha < \kappa^+}$ be a complete list of all sets in the $\Pi^1_1$-ideal $I$ on $\kappa$ such that every element of $A_\alpha$ has uncountable cofinality. If $\alpha$ is limit of cofinality $\leq \omega_1$, then $P_\alpha$ is the inverse limit of all $P_\beta$, $\beta < \alpha$. For other limit $\alpha$, $P_\alpha$ is the direct limit of $P_\beta$, $\beta < \alpha$. At successor stages we let $P_{\alpha+1} = P_\alpha \ast R_\alpha$, where $R_\alpha$ is defined as follows: $q \in R_\alpha$ iff $q$ is a bounded closed sequence of elements of $\kappa$ such that $q \cap A_\alpha = \emptyset$. $R_\alpha$ is ordered by the end extension relation. Thus each $P_\alpha$ is countably closed. Let $P = P_{\kappa^+}$. Now $Q \ast P$ satisfies the $\kappa^+$-chain condition. Note also that for all $\alpha < \kappa^+$, $Q \ast P_\alpha$ has power $\kappa$. We prove that it is true in $V^Q$ that $P_\alpha$ does not add new subsets of $\kappa$ of cardinality $\leq \aleph_1$, hence $\kappa$ remains $\aleph_2$ also after forcing with $P$. It follows also that $Q \ast P$ and each $Q \ast P_\alpha$ are countably closed.

We show now that in $V^{Q \ast P}$ the claim is true. Suppose $S$ and a set $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$, of regressive functions $\kappa \to \kappa$ are given in $V^{Q \ast P}$ such that (in $V^{Q \ast P}$) $S \subseteq S_1^2$ is $\omega_1$-stationary. Suppose $\alpha < \kappa^+$ is such that $\bar{S}$, $\bar{\mathcal{F}}$ and $\bar{f}_\alpha$ are $Q \ast P_\alpha$-names for $S, \mathcal{F}$ and $f_\alpha$, correspondingly. Since $S$ is $\omega_1$-stationary in $V^{Q \ast P}$, $S$ is not in the ideal generated by $I$ in $V^{Q \ast P_\alpha}$. Suppose $(p, q) \Vdash \bar{S} \notin I$. For a contradiction, suppose also that $(p, q)$ forces that $\exists \alpha$ does not have a winning strategy in the game $BM_{\omega_1}(S, \mathcal{F})$.

Let $(B_0', \in)$ be a sufficiently elementary substructure of $(V, \in)$ such that $|B_0'| = \kappa$, $B_0'^{<\kappa} \subseteq B_0'$, $Q, P_\alpha, \alpha, \kappa, \bar{\mathcal{F}}, \bar{f}_\alpha, \bar{S}$, are in $B_0'$, and $\alpha \cup \kappa \subseteq B_0'$. Let $B_0$ be the transitive collapse of $B_0'$. Thus $Q, P_\alpha, \alpha, \kappa \in B_0$, $A_j \in B_0$ for $i \leq \alpha$ and $\bar{f}_i \in B_0$ for $i < \kappa$. Let

$$T = \{\alpha < \kappa \mid \exists (p', q') \leq (p, q)((p', q') \Vdash_{Q \ast P_\alpha} \alpha \in \bar{S})\}. $$

Clearly $T \in B_0$ and $T \notin I$. By weak compactness, there are a transitive $B_1$ and an elementary embedding $j : B_0 \to B_1$ such that $\kappa$ is the critical point of $j$, $\kappa \in j(T)$ and $\kappa \notin j(A_i)$ for $i \leq \alpha$. So there is some $(p', q') \in j(Q \ast P_\alpha)$ such that $(p', q') \leq j((p, q)) = (p, q)$ and $(p', q') \Vdash_{j(Q \ast P_\alpha)} \kappa \in j(\bar{S})$. Note that $Q, P_\alpha \in B_1$ and $\bar{f}_i \in B_1$ for $i < \kappa$.

By (the proof of) Lemma 3 in [7], there are a $Q \ast P_\alpha$-generic $G$ over $B_1$ and a forcing notion $R \in B_1[G]$ such that $(p, q) \in G$, in $B_1[G]$, $R$ is countably closed, for all $R$-generic $K$ over $B_1[G]$, there is a canonical $j(Q \ast P_\alpha)$-generic $G_K$ over $B_1$ such that $B_1[G_K] = B_1[G][K]$ and for some $K$, $G_K$ is such that $(p', q') \in G'$.

Then for every $Q \ast P_\alpha$-name $\bar{X} \in B_0$, there is a canonical $R$-name $\bar{Y} \in B_1[G]$ such that for all $R$-generic $K$ over $B_1[G]$, $j(\bar{X})$ and $\bar{Y}$
have the same interpretation in $B_1[G][K]$. We do not distinguish $j(\bar{X})$ and $\bar{Y}$. With this notation, there is $r \in \mathbb{R}$ which forces in $B_1[G]$, that $\kappa \in j(\bar{S})$. Then there is some $(p^*, q^*) \leq (p, q)$ in $G$ that in $B_1$ forces the existence of such $\mathbb{R}$ and $r$. So we may assume that $G$ is generic over $V$ and our $V^{\mathbb{Q} \ast \mathbb{P}_\alpha}$ is the same as $V[G]$.

We describe in $B_1[G]$ a winning strategy of $\exists$ in the game $BM_{\omega_1}(S, \mathcal{F})$. This is a contradiction since all possible winning plays of $\forall$ are in $B_1[G]$ and being unbounded is absolute in transitive models. The strategy of $\exists$ is to play on the side conditions $q^i$ in $B_1[G]$ and sets $S_i \in B_0[G]$ with $\mathbb{Q} \ast \mathbb{P}_\alpha$-names $\bar{S}_i$ in $B_\emptyset$ such that

1. $q^i \in \mathbb{R}$.
2. $q^0 \leq r$.
3. $i < k < \omega_1$ implies $q^k \leq q^i$.
4. $i < k < \omega_1$ implies $S_k \subseteq S_i \subseteq S$.
5. $q^i \Vdash_\mathbb{R} \kappa \in j(\bar{S}_i)$ in $B_1[G]$.

Suppose $\exists$ has followed this strategy, forming conditions $q^i$ and sets $S_i$ for $i < k$. Let $p = \inf\{\{q^i : i < k\}\}$. If we let $S$ to be $\bigcap_{i < k} S_i$ and $\bar{S}$ a name for this, then in $B_1[G]$,

$$p \Vdash_\mathbb{R} \kappa \in j(\bar{S}).$$

Suppose then $\forall$ moves $f_k \in \mathcal{F}$. Let $q^k \leq p$ such that for some $\delta < \kappa$ we have $q^k \Vdash_\mathbb{R} j(f_k)(\kappa) = \delta$ in $B_1[G]$ and let $S_k$ be $\{\beta \in S : f_k(\beta) = \delta\}$ and $\bar{S}_k$ a name for this. Then $q^k \Vdash_\mathbb{R} \kappa \in j(\bar{S}_k)$ in $B_1[G]$.

Finally we have to prove that $\mathbb{Q} \ast \mathbb{P}_\alpha$ does not add new subsets of $\kappa$ of cardinality $\leq \aleph_1$ over and above those added by $\mathbb{Q}$. The proof of this is, mutatis mutandis, like the proof of the Main fact (page 761) in [7]. Here we use the assumption $\kappa \notin j(A_i)$ for $i \leq \alpha$. Thus, if $C$ is a generic sequence in the complement of $j(A_\beta)$ in $V^{j(\mathbb{Q} \ast \mathbb{P}_\alpha)}$, then we can continue it to a closed condition $C \cup \{\kappa\} \in \mathbb{R}_{j(\beta)}$. □

Results similar to Theorem 12 have been treated also in [13] and [14].

**Corollary 13** It is consistent relative to the consistency of a weakly compact cardinal, that the game $EFG_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is determined for all $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\leq \aleph_2$. 9
3 Non-determinacy and structure theory

In this section we prove Theorem 2, which essentially establishes, under cardinality assumptions concerning the continuum, the existence of non-determined Ehrenfeucht-Fraïssé games of length $\omega_1$ for models of non-classifiable theories. This complements the observation, made in [8], that the Ehrenfeucht-Fraïssé game of length $\omega_1$ is determined for models of classifiable theories.

We start by proving Theorem 2 under assumption (iii), which we consider the most interesting case. That is, we start with a countable complete stable and unsuperstable first order theory and show that, assuming $2^\omega \leq \omega_3$, it has two models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_3$ for which $EFG_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is non-determined. Actually, we construct $\mathfrak{A}$ and $\mathfrak{B}$ so that $\exists$ does not have a winning strategy even in $EFG_{2^\omega + \omega}(\mathfrak{A}, \mathfrak{B})$ and $\forall$ does not have a winning strategy even in $EFG_{\omega_3}(\mathfrak{A}, \mathfrak{B})$.

We then prove Theorem 2 under assumption (i), that is, we now start with a countable complete unstable first order theory and show that, assuming $2^\omega < 2^{\omega_3}$, it has two models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_3$ for which $EFG_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is non-determined.

Theorem 2 under assumption (ii) can be dealt with in the same way as under assumption (i). The section ends with some remarks on possible improvements.

3.1 The stable unsuperstable case

We will prove Theorem 2, case (iii), in a series of lemmas. We assume $\omega_3^\omega = \omega_3$ all the time. Let $T$ be a countable complete stable and unsuperstable first order theory. As usual, we work inside a large saturated model $M$ of $T$. We start by fixing some notation. By a tree $I$ we mean a lexicographically ordered downwards closed subtree of $\theta^{<\omega+1}$ for some linear order $\theta$, that is, $I = (I, \ll, P_a, <, H)_{a \leq \omega} \in K^\omega_T(\theta)$, see [4] Definition 8.2 or [11]. For a while, we fix a tree $I \in K^\omega_T(\lambda)$, where $\lambda$ is some large enough cardinal, so that $(I, \ll)$ is isomorphic to $\lambda^{<\omega+1}$. As in [3], for $u, v \in P_\omega(I)$ ($=\text{finite subsets of } I$), we define $r(u, v)$ to be the unique set $R$ which satisfies

(I) $R \subseteq X_{u,v} = \{H(\eta, \xi) \mid \eta \in u, \xi \in v\}$,

(II) For all $\nu \in X_{u,v} - R$, there is $\nu' \in R$ such that $\nu \ll \nu'$,

(III) If $\eta$ and $\xi$ are distinct elements of $R$, then $\eta \not\ll \xi$. 

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We write $u \leq v$ if $r(u, v) = r(u, u)$. For more on these definitions, see [3]. In [3], it is shown that there are models $A$ and $A_u$, $u \in \mathcal{P}_\omega(I)$, and sequences $a_\eta$ from $A_{(\eta)}$, $\eta \in I$, such that

(i) $A = \bigcup_{u \in \mathcal{P}_\omega(I)} A_u \models T$,

(ii) if $u \leq v$, then $A_u \subseteq A_v$,

(iii) for all $u, v \in \mathcal{P}_\omega(I)$, $A_u \downarrow A_{r(u,v)} A_v$,

(iv) for all $u \in \mathcal{P}_\omega(I)$, $|A_u| \leq \omega_3$,

(v) if $P_\omega(\eta)$ holds and $\xi \ll \eta$ is an immediate successor of $\xi'$, then

$$a_\eta \not\in A_{(\xi')} a_\xi.$$ 

These models are exactly what we want except that they are too large, we want the models $A_u$, $u \in \mathcal{P}_\omega(I)$, to be countable. In order to get this, we use the Ehrenfeucht-Mostowski construction.

We extend the signature $L$ of $T$ to $L_*$ by adding $\omega_3$ new function symbols, some of which will be interpreted in $\mathcal{M}$ so that they provide Skolem-functions for the $L$-formulas. In addition we interpret the functions so that if we write $SH_*(u)$ for the $L_*$-Skolem-hull of $\{a_\eta | \{\eta\} \leq u\}$ then

(vi) for all $u \in \mathcal{P}_\omega(I)$, $SH_*(u) = A_u$.

By the usual argument (using [11, Appendix Theorem 2.6] and compactness) we can interpret the new function symbols so that $\mathcal{M}$ remains sufficiently saturated and the following holds

(vii) if $U$ is a downwards closed subtree of $I$ and $f$ is an automorphism of $U$, then there is an $L_*$-automorphism $g$ of $\bigcup_{u \in \mathcal{P}_\omega(U)} A_u$ such that for all $\eta \in U$, $g(a_\eta) = a_{f(\eta)}$.

Finally, it is easy to see that we can choose countable $L_1 \subseteq L_*$ so that $L \subseteq L_1$, $L_1$ contains the Skolem-functions for the $L$-formulas and if we write $SH_1(u)$ for the $L_1$-Skolem-hull of $\{a_\eta | \{\eta\} \leq u\}$ then

(viii) for all $u, v \in \mathcal{P}_\omega(I)$, $SH_1(u) \downarrow SH_1(v) SH_*(v)$.

So we have proved the following lemma (for the notion $\Phi$ proper for $K_\omega^\omega$ and the Ehrenfeucht-Mostowski models $EM^1(J, \Phi)$, see [4] Definition 8.1 or [11]).
Lemma 14 There are countable $L_1 \supseteq L$ and $\Phi$ proper for $K^\omega_{tr}$ such that the following holds:

(a) For all $J \in K^\omega_{tr}$ there are an $L_1$-model $EM^1(J, \Phi) \models T$ and sequences $a_\eta \in EM^1(J, \Phi)$, $\eta \in J$, such that $EM^1(J, \Phi)$ is the $L_1$-Skolem-hull of $\{a_\eta \mid \eta \in J\}$ (i.e. $\{a_\eta \mid \eta \in J\}$ is the skeleton of $EM^1(J, \Phi)$ and as before for $u \subseteq J$, $SH_1(u)$ denotes the $L_1$-Skolem hull of $\{a_\eta \mid \eta \leq u\}$).

(b) If $U$ is a downwards closed subtree of $J$ and $f$ is an automorphism of $U$, then there is an $L_1$-automorphism $g$ of $SH_1(U)$ such that for all $\eta \in U$, $g(a_\eta) = a_{f(\eta)}$.

(c) Assume $(\eta_i)_{i < \omega}$ is a strictly $\ll$-increasing sequence of elements of $J$, $\eta_{i+1}$ is an immediate successor of $\eta_i$ and $\eta_0$ is the root. Then $(\eta_i)_{i < \omega}$ has an upper bound in $J$ iff there is a sequence $a \in EM(J, \Phi)$ such that for all $i < \omega$, $a^{\downarrow SH_1(\{\eta_i\})}_{a_{\eta_{i+1}}}$.

We will write $EM(J, \Phi)$ for $EM^1(J, \Phi) \upharpoonright L$.

Our next goal is to define the skeletons for the models $A$ and $B$ in the theorem. For this we use the weak box from [8]. By $S^n_m$ we denote the set $\{\alpha < \omega_n \mid cf(\alpha) = \omega_m\}$.

Theorem 15 ([8, Lemma 16]) There are sets $S$, $U$ and $C_\alpha$, $\alpha \in S$, such that the following holds:

(a) $S \subseteq S_0^3 \cup S_1^3$ and $S \cap S_1^3$ is stationary,
(b) $U \subseteq S_0^3$ is stationary and $S \cap U = \emptyset$,
(c) for all $\alpha \in S$, $C_\alpha \subseteq \alpha \cap S$ is closed in $\alpha$ and of order-type $\leq \omega_1$,
(d) for all $\alpha \in S$, if $\beta \in C_\alpha$, then $C_\beta = C_\alpha \cap \beta$,
(e) for all $\alpha \in S \cap S_1^3$, $C_\alpha$ is unbounded in $\alpha$.

We will construct trees $I_\alpha$ and $J_\alpha$, $\alpha < \omega_3$, so that the following holds:

(1) if $\alpha < \beta$ then $I_\alpha$ is a submodel of $I_\beta$ and $J_\alpha$ is a submodel of $J_\beta$; now for $\eta \in I_\alpha$, we will write $rk(\eta)$ for the least $\beta$ such that $\eta \in I_\beta$ and similarly for $\eta \in J_\alpha$.

(2) for all $\alpha \in S$, there is an isomorphism $G_\alpha : I_\alpha \to J_\alpha$.

(3) if $\alpha \in C_\beta$, then $G_\alpha \subseteq G_\beta$. 

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for all $\alpha \leq \beta$ and $\eta \in I_\alpha$, if $P_\omega(\eta)$ does not hold, then there is an immediate successor $\xi$ of $\eta$ such that $\xi \in I_{\beta+1} - I_\beta$.

(5) if $(\eta_i)_{i<\omega}$ is an increasing sequence of elements of $I_\alpha$ (for some $\alpha$) and the sequence has an upper bound $\xi$ in $I_\alpha$, then $r_k(\xi) = sup_{i<\omega} r_k(\eta_i)$ and similarly for sequences from $J_\alpha$.

(6) if $(\eta_i)_{i<\omega}$ is an increasing sequence of elements of $I_\alpha$, $(r_k(\eta_i))_{i<\omega}$ is not eventually constant and the sequence has an upper bound $\xi$ in $I_\alpha$, then $r_k(\xi) (= sup_{i<\omega} r_k(\eta_i)) \in U$; in $J_\alpha$ such sequences never have an upper bound.

(7) $|I_\alpha| \leq \omega_3$ and $|J_\alpha| \leq \omega_3$.

(8) $I_\alpha, J_\alpha \subseteq H_\omega(\omega_3)$, where $H_\omega(\omega_3)$ is the least set $H$ such that $\omega_3 \subseteq H$ and if $E \subseteq H$ is of power $\leq \omega$, then $E \in H$.

It is easy to see that such trees can be constructed by induction on $\alpha$. However, in order to get what we want we need to do a bit more work when we define $I_\alpha$ and $J_\alpha$ in the case $\alpha \in U$. In order to decide, which branches like the one in (6) above, we want to have an upper bound, we use a guessing machine from [12] called black box, which we formulate so that it fits exactly to our purposes.

Theorem 16 ([12]) ($\omega_3^\omega = \omega_3$.) There are $(M_\alpha^\alpha, \eta_\alpha^\alpha)$, $\alpha < \omega_3$, such that

(i) $M_\alpha^\alpha = (M_\alpha^\alpha)_{i<\omega}$ is an increasing elementary chain of elementary submodels of some $(H_\omega(\omega_3), A, B, \sigma)$, such that $A, B \subseteq H_\omega(\omega_3)$ and $\sigma$ is a strategy of $\exists$ in $EF_2^2(A, B)$ ($A$ and $B$ can be viewed as models of the empty signature),

(ii) $M_i^\alpha = (M_i^\alpha, A_i^\alpha, B_i^\alpha, \sigma_i^\alpha) \in H_\omega(\omega_3)$,

(iii) $\eta^\alpha$ is an increasing function from $\omega$ to $\omega_3$, $M_i^\alpha \in H_\omega(\eta^\alpha(i+1))$ and $sup_{i<\omega} \eta^\alpha(i) \in U$,

(iv) $(\eta^\alpha(j))_{j\leq i}, (M_j^\alpha)_{j\leq i} \in M_{i+1}^\alpha$,

(v) if $\alpha \neq \beta$, then $\eta^\alpha \neq \eta^\beta$, \vfill
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player I does not have a winning strategy for the following game: The length of the game is $\omega$. At each move $i < \omega$, first I chooses $M_i$ and then II chooses $\alpha_i < \omega_3$. I must play so that in the end (i), (ii) and (iv) above are satisfied. I wins if he has played according to the rules and there is no $\alpha < \omega_3$ such that $((M_i)_{i<\omega}, (\alpha_i)_{i<\omega}) = (M', \eta^\alpha)$.

First we uniformize (partially) the Ehrenfeucht-Mostowski construction: We assume that for all $I, I' \in K^\omega_I$, if $I$ is a substructure of $I'$ and $I' \subseteq H_\omega(\omega_3)$, then there is a unique model $EM^1(I, \Phi)$, it is a substructure of $EM^1(I', \Phi)$ and $EM^1(I', \Phi) \subseteq H_\omega(\omega_3)$.

So let $\alpha \in U$ and assume that $I_\beta$ and $J_\beta$ are defined for all $\beta < \alpha$. Write $I^*_\alpha = \bigcup_{\beta < \alpha} I_\beta$ and $J^*_\alpha = \bigcup_{\beta < \alpha} J_\beta$. For $\gamma < \omega_3$, we write $M_\gamma$ for $\bigcup_{i<\omega} M_i^\gamma$ and $A_\gamma, B_\gamma$ and $\sigma_\gamma$ are defined similarly. Let $W^\alpha$ be the set of all $\gamma < \omega_3$ such that

(a) $A^\gamma = EM(I^*_\alpha \cap M^\gamma, \Phi)$ and $B^\gamma = EM(J^*_\alpha \cap M^\gamma, \Phi)$,

(b) $\sup_{i<\omega} \eta^\gamma(i) = \alpha$,

(c) there are $\xi_i^\gamma \in I^*_\alpha \cap M^\gamma$, $i < \omega$, such that $\xi_0^\gamma$ is the root of $I^*_\alpha$, $\xi_{i+1}^\gamma$ is an immediate successor of $\xi_i^\gamma$ and $\xi_i^\gamma \in I_{\eta^\gamma(i)+1} - I_{\eta^\gamma(i)}$.

Notice that by Theorem 16 (v), if $\gamma \neq \delta$, then $\xi_i^\gamma \neq \xi_i^\delta$. Let $C_i^\gamma = SH(\{\xi_i^\gamma\})$. Then we can find a partial function $g^\gamma : A^\gamma \to B^\gamma$ such that

(d) $\text{dom}(g^\gamma) = \bigcup_{i<\omega} C_i^\gamma$,

(e) $g^\gamma$ is a result of a play of $EF^2_\omega(A^\gamma, B^\gamma)$ in which $\exists$ has used $\sigma^\gamma$.

We let $W^\gamma_f$ be the set of those $\gamma \in W^\alpha$ such that

(f) $g^\gamma$ is a partial isomorphism from $EM(I^*_\alpha \cap M^\gamma, \Phi)$ to $EM(J^*_\alpha \cap M^\gamma, \Phi)$,

(g) there is $J$ such that if we let $J_\alpha = J$, then (1),(5)-(8) above are satisfied and there is a sequence $a \in EM(J, \Phi)$ such that for all $i < \omega$, $a \ Ad_{g^\gamma(c_i^\gamma)}$.

We let $W^\gamma_j$ be the set of all $\gamma \in W^\alpha - W^\gamma_f$ such that $g^\gamma$ satisfies (f) above.

Now we can define $I_\alpha$ and $J_\alpha$. First we choose $I_\alpha$ so that it consists of all $\eta \in I^*_\alpha$ together with the supremums for the branches $(\xi_i^\gamma)_{i<\omega}$, $\gamma \in W^\alpha$. $J_\alpha$ is chosen so that it satisfies (g) for all $\gamma \in W^\gamma_j$ (and so especially (1),(5)-(8)).

Then we let $I = \bigcup_{\alpha<\omega_3} I_\alpha$, $J = \bigcup_{\alpha<\omega_3} J_\alpha$, $A = EM(I, \Phi)$ and $B = EM(J, \Phi)$. Clearly $A$ and $B$ can be chosen so that $A, B \subseteq H_\omega(\omega_3)$.
Lemma 17 \( \forall \) does not have a winning strategy for \( EF_{\omega_1}^{\omega_3}(A, B) \).

**Proof.** For this it is enough to show that \( A \) does not have a winning strategy for \( EF_{\omega_1}^{\omega_3}(I, J) \), which is clear by (2) and (3) above and Theorem 15. \( \square \)

Lemma 18 \( \exists \) does not have a winning strategy for \( EF_{\omega+\omega}^2(A, B) \).

**Proof.** For a contradiction, assume \( \sigma \) is a winning strategy of \( \exists \) for the game \( EF_{\omega+\omega}^2(A, B) \). We play a round of the game defined in Theorem 16 (vi). We let player I play so that he follows the rules and

(i) for all \( i < \omega \), \( M_i < (H_\omega(\omega_3), A, B, \sigma \restriction \omega) \),

(ii) for all \( \delta, \delta' \in M_i \), if \( \delta \leq \delta' \), \( \eta \in I_\delta \cap M_i \) and \( P_\omega(\eta) \) does not hold, then there is \( \xi \in (I_{\delta'} - I_\delta) \cap M_{i+1} \) such that \( \xi \) is an immediate successor of \( \eta \),

(iii) the Skolem-hulls of \( \{a_\eta \mid \eta \in I \cap M_i\} \) and \( \{a_\eta \mid \eta \in J \cap M_i\} \) are subsets of \( M_{i+1} \),

(iv) \( A \cap M_i \) is a subset of the Skolem hull of \( \{a_\eta \mid \eta \in I \cap M_{i+1}\} \) and \( B \cap M_i \) is a subset of the Skolem hull of \( \{a_\eta \mid \eta \in J \cap M_{i+1}\} \),

(v) \( \bigcup \{rk(\eta) \mid \eta \in I \cap M_i\} \cup \bigcup \{rk(\eta) \mid \eta \in J \cap M_i\} \in M_{i+1} \).

By Theorem 16 (vi), the round can be played so that \( \forall \) loses. Let \( \alpha_i, i < \omega \), be the choices \( \exists \) made and \( \gamma \) such that \( ((M_i)_{i<\omega}, (\alpha_i)_{i<\omega}) = (M_\gamma, \eta_\gamma) \). Finally, let \( \alpha = \bigcup_{i<\omega} \alpha_i, (\in U) \).

Now it is easy to see that \( \gamma \in W_\alpha \), in fact \( \gamma \in W_I^\alpha \) or \( \gamma \in W_J^\alpha \) (otherwise we have demonstrated that \( \sigma \) is not a winning strategy). In the first case, there is a sequence \( a \in A \) such that for all \( i < \omega \), \( \bigwedge_{C_i^\gamma} a_{\xi_{i+1}} \), but in \( B \) there is no sequence \( b \) such that for all \( i < \omega \), \( \bigwedge_{g^\gamma(C_i^\gamma)} g^\gamma(\xi_{i+1}) \), a contradiction. In the latter case, there is a sequence \( b \in B \) such that for all \( i < \omega \), \( \bigwedge_{g^\gamma(C_i^\gamma)} g^\gamma(\xi_{i+1}) \), but by (the construction,) Lemma 2.3 (c) and Theorem 16 (v), there is no sequence \( a \in A \) such that for all \( i < \omega \), \( \bigwedge_{C_i^\gamma} a_{\xi_{i+1}} \), a contradiction. \( \square \)

Now Lemmas 2.6 and 2.7 imply Theorem 3 (iii).
3.2 The unstable case

We will prove Theorem 19, case (i), again in a series of lemmas. We assume \( \omega_3 < 2^{\omega_3} \). Let \( T \) be a countable complete unstable first order theory. Let \( L \) be the signature of \( T \).

**Theorem 19** ([14]) Assume \( T \) is a countable unstable theory in the signature \( L \). There are a countable signature \( L_1 \supseteq L \), a complete Skolem theory \( T_1 \supseteq T \) in the signature \( L_1 \), a first-order \( L \)-formula \( \phi(x, y) \) and \( \Phi \) proper for \( (\omega, T_1) \) (see [Sh1] Definition VII 2.6) such that for every linear order \( I \) there is an Ehrenfeucht-Mostowski model \( EM^1(I, \Phi) \) of \( T_1 \) with a skeleton \( \{a_\eta \mid \eta \in I\} \) such that

\[
EM^1(I, \Phi) \models \phi(a_\eta, a_\xi) \text{ iff } I \models \eta < \xi.
\]

We write \( EM(I, \Phi) \) for \( EM^1(I, \Phi) \upharpoonright L \). Notice that by using the terminology from [12, Definition III 3.1], \( \{a_\eta \mid \eta \in I\} \) is weakly \((\omega, \phi)\)-skeleton-like in \( EM(I, \Phi) \).

In order to use Theorem 19, linear orders are needed. If \( A \) is a linear ordering, \( x \in A \) and \( B \subseteq A \), then by \( x < B \) we mean that for every \( y \in B \), \( x < y \), \( x > B \) and \( C > B \), \( C \subseteq A \) are defined similarly. By \( A^* \) we mean the inverse of \( A \). Again let \( S, U \) and \( C_\alpha, \alpha \in S \), be as in [8, Lemma 16], i.e. Theorem 13 above, with the exception that \( 0 \in S \) and for all \( \alpha \in S - \{0\} \), \( 0 \in C_\alpha \). By induction on \( i < \omega_3 \), we will define linear orders \( A^i_\alpha \) and \( B^i_\alpha \), \( \alpha < \omega_3 \), and for \( i \in S \), isomorphisms

\[
G_i : \Sigma_{\beta < \gamma \leq \omega_3} A^i_\beta \rightarrow \Sigma_{\beta < \gamma \leq \omega_3} B^i_\beta.
\]

We write \( A^i(\beta, \alpha) \) for \( \Sigma_{\beta \leq \gamma < \alpha} A^i_\gamma \) and similarly \( B^i(\beta, \alpha) \). We will do the construction so that the following holds:

1. \( A^0_\alpha \cong \omega^* \) for all \( \alpha < \omega_3 \) and if \( \alpha \not\in U \), then \( B^0_\alpha \cong \omega^* \) and otherwise \( B^0_\alpha \cong (\omega_1)^* \),

2. If \( i < j \), then \( A^i_\alpha \subseteq A^j_\alpha \) and \( B^i_\alpha \subseteq B^j_\alpha \) and otherwise the sets are distinct and if \( j \in C_i \), then \( G_j \subseteq G_i \),

3. If \( cf(\alpha) = \omega \), then \( A^0_\alpha \) is coinitial in \( A^i_\alpha \) and similarly for \( B \).

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We will do this by induction on \( i \). However, in order to be able to show that (3) holds in each step, we need additional machinery.

Let \( C \in \{ A, B \} \). We say that \((I, J)\) is a \((C, i, \beta)\)-cut if \( I \) is an initial segment of \( C^i_\beta \) and \( J = C^i_\beta - I \). We say that the cut is basic if \( I = \emptyset \). We define a notion of forbidden cut by induction on \( i \) as follows (we should talk about \( i \)-forbidden cuts, but \( i \) is always clear from the context):

(a) for all limit \( \beta \), the basic \((C, 0, \beta)\)-cut is forbidden,

(b) if \((I, J)\) is a \((C, i, \beta)\)-cut, \( j < i \) and \((C^j_\beta \cap I, C^j_\beta \cap J)\) is forbidden, then \((I, J)\) is forbidden,

(c) if \((I, J)\) is a forbidden \((A, i, \beta)\)-cut, \( I^* = I \cup \bigcup_{\gamma < \delta} A^i_\gamma \) and \( G_i(I^*) \) is not bounded by any \( x \in \bigcup_{\gamma < \delta} B^i_\gamma \) but some \( y \in B^i_\delta \) bounds it, then \((G_i(I^*) \cap B^i_\delta, B^i_\delta - G_i(I^*))\) is forbidden and similarly for \( A \) and \( B \) reversed (and \( G_i \) replaced by \((G_i)^{-1})\).

Now we can state the additional properties we want our construction have. Let \( E \in \{ A, B \} \), \( i, \beta < \omega_3 \) and \((I, J)\) be a \((E, i, \beta)\)-cut.

(4) If \((I, J)\) is forbidden, then there is no \( j < \omega_3 \) and \( x \in E^i_\beta \) such that \( I < x < J \).

(5) Assume \((I, J)\) is forbidden and \( j \in S \) is such that \( j < i \) and either \( E^j_\beta \cap I \) is cofinal in \( I \) or \( E^j_\beta \cap J \) is coinitial in \( J \) (we say that \( \emptyset \) is both cofinal and coinitial in \( \emptyset \)). Then \((E^j_\beta \cap I, E^j_\beta \cap J)\) is forbidden.

(6) If \( \beta \) is successor, then \( E^0_\beta \) is coinitial in \( E^i_\beta \).

**Lemma 20** Let \( E \in \{ A, B \} \).

(i) For all \( i, \beta < \omega_3 \), if (5) holds upto the stage \( i \), then \((E^i_\beta, \emptyset)\) is not forbidden and neither is \((\emptyset, E^i_\beta)\), if \( \beta \) is successor.

(ii) For limit \( \beta \), every basic \((E, i, \beta)\)-cut is forbidden.

(iii) The property (4) implies the property (3).

(iv) If \( i + 1 < \beta \) and \((I, J)\) is a forbidden \((E, i, \beta)\)-cut, then it is basic (and \( \beta \) is limit).
Proof. Immediate. \(\Box\)

Now we are ready to do the construction: For \(i = 0\), the linear orders are defined by (1) and we let \(G_0\) be the only possible one. Clearly (1)-(6) hold. If \(i \notin S\) or \(sup C_i = i\), then we let \(A^i_\alpha = \cup_{j<i} A^0_j, B^i_\alpha = \cup_{j<i} B^0_j\) and if \(i \in S\) (and \(sup C_i = i\)), then \(G_i = G \cup \bigcup_{j \in C_i} G_j\), where \(G\) is the obvious isomorphism from \(A^i(i, i+2)\) to \(B^i(i, i+2)\) (both are isomorphic to \(\omega^* + \omega^*\)). Now (1), (2), (4) and (6) hold trivially. By Lemma 2.9 (iii), (3) holds. For \(I, J\) the reason why (4) holds trivially. By Lemma 2.9 (i), this does not violated (4). Then we add \(A, B\) and \(B\) as an end segment. By Lemma 2.9 (i), this does not violated (4). Then we add \(A, B\) and \(B\) and by Lemma 2.9 (iv), the sets \(A^0_k = \cup_{k<i} A^0_k\) and \(B^0_k = \cup_{k<i} B^0_k\) are not forbidden, where \(A(\delta) = \{x \in A^0_k | x > C'\}\) and \(B(\delta) = \{x \in B^0_k | x < D'\}\). So we have not violated the property (4).

For all \(\alpha > j + 1\), we let \(A^i_\alpha = \cup_{k<i} A^k_\alpha\) and \(B^i_\alpha\) is defined similarly. However we will still make changes to \(B^i_{j+1}\) and \(A^i_{i+1}\). Let \(A\) be a copy of \(B(i, j+3, i+2)\) and \(B\) be a copy of \(A(i, j+2, i+1)\). Furthermore, extend \(A_{i+1}\) so that there is an isomorphism \(g : A^i_{i+1} \rightarrow B^i_{i+2}\) such that \(g(A^i_{i+1}) = B^i_{j+2}\) (this is not a problem since \(A^0_{i+1} = \cup_{k<i} A^k_{i+1} \cong \omega^* \cong B^0_{j+2}\) and by Lemma 2.9 (iv), the sets \(A^k_{i+1}, k < i\), do not contain forbidden \(A, k, i+1\)-cuts; so we do not violate (4)). Then we add \(A\) to (the extended) \(A^i_{i+1}\) as an end segment and \(B\) to \(B^i_{j+1}\) as an end segment. By Lemma 2.9 (i), this does not violated (4). Now it is easy to extend \(G^i_{i+1}\) to \(G_i\) so that \(G_i((A^i(i, j+2, i+1)) = B, G_i(A^i_{i+1} - A) = B^i_{j+2}\).
Now (1), (2) and (6) hold trivially, (4) is already shown to hold and by Lemma 2.9 (iii), (4) implies (3). So we are left to show that

**Lemma 21** (5) holds.

**Proof.** Assume \((I, J)\) is a forbidden \((E, i, \beta)\)-cut, \(E \in \{A, B\}\), and \(\delta \in S\) is such that \(\delta < i\) and \(E^\delta \cap J\) is coinitial in \(J\), the other case is similar. If \(\beta \geq j + 1\) and both \(J \cap (\bigcup_{k<j}A_{j+k1})\) and \(J \cap (\bigcup_{k<j}B_{j+k1})\) are empty, then the claim follows easily from Lemma 2.9 and the induction assumption. So we assume that this is not the case. If \((I, J)\) is forbidden because of (b) in the definition of forbidden cut, the claim follows from the definition of forbidden cut if \(\delta < j\) is such that \((I, J)\) is forbidden by (c) applied to \((C, D)\) (the case \(A\) and \(B\) reversed is symmetric). Since \((I, J)\) is not forbidden by (b) in the definition of forbidden cut, \((C, D)\) must be forbidden because of it, i.e. for some \(\alpha < i\), \((A^\alpha \cap C, A^\alpha \cap D)\) is a forbidden \((A, \alpha, \gamma)\)-cut.

If 

\((\star)\) For no \(y \in A^\alpha \cap D, y \prec A^j \cap D,\)

then by the induction assumption, \((B^j \cap I, B^j \cap J)\) is a forbidden \((B, j, \beta)\)-cut and the claim follows from the definition of forbidden cut if \(\delta \geq j\) and from (5) in the induction assumption if \(\delta < j\). So we assume that (\(\star\)) fails. Let \(y\) be the bound. Then \(\emptyset \neq D' = \{z \in A^j \cap D \mid z \leq y\} \subseteq A^j - \text{dom}(G_j)\). So by the construction, \(G_i(D') \subseteq J - B^\delta\) and for all \(x \in B^\delta\), either \(x < G_i(D')\) or \(x > G_i(D')\). By the choice of the cut \((C, D)\), there can not be \(x \in J \cap B^\delta\) such that \(x < G_i(D')\). But then \(G_i(D') < J \cap B^\delta\), which contradicts the assumption that \(J \cap B^\delta\) is coinitial in \(J\). \(\square\)

Let \(A = \Sigma_{\alpha < \omega_1} \cup_{i < \omega_3} A^i_{\alpha}\) and \(B = \Sigma_{\alpha < \omega_1} \cup_{i < \omega_3} B^i_{\alpha}\). Notice that by (1) and (3), \(inv^1_{\alpha}(A)\) differs from \(inv^1_{\beta}(B)\) in a stationary set which consists of ordinals of cofinality \(\omega\) (for the definition of \(inv^\alpha_{\omega}\), see [12, Definition III 3.4]. Let \(S_\alpha \subseteq S^3_{\omega_1}, i < 2^{\omega_3}\), be stationary sets such that for \(\alpha < \beta < 2^{\omega_3}\), \(S_\alpha \Delta S_\beta\) is stationary and define \(\Psi_{\alpha} = \Sigma_{\alpha < \omega_3} \tau_{\alpha}\), where \(\tau_{\alpha} = A^*\) if \(\alpha \not\in S_\alpha\) and otherwise \(\tau_{\alpha} = B^*\). Notice that for \(\alpha \neq \beta\), \(inv^2_{\alpha}(\Psi_{\alpha})\) differs from \(inv^2_{\beta}(\Psi_{\beta})\) in a stationary set which consists of ordinals of cofinality \(\omega\).

Finally, let \(A_\alpha = EM((\Psi_{\alpha})^* \cdot \omega_1, \Phi)\).

**Lemma 22** For all \(\alpha, \beta < 2^{\omega_3}\), \(A\) does not have a winning strategy for \(EF_{\omega_1}^{\omega_3}(A_\alpha, A_\beta)\).
Proof. For this, it is enough to show that $A$ does not have a winning strategy for $EF_{\omega_1}^\omega((\Psi_\alpha)^* \cdot \omega_1, (\Psi_\beta)^* \cdot \omega_1)$, which follows easily from (2) in the construction of $A$ and $B$ and Theorem 13 (see e.g. [8, Claim 3 in the proof of Theorem 17]). 

Lemma 23 There are $\alpha < \beta < 2^\omega_3$ such that $E$ does not have a winning strategy for $EF_{\omega_1}^2(A_\alpha, A_\beta)$.

Proof. By using the usual forcing notion, we collapse $\omega_3$ to an ordinal of power $\omega_1$. Since this forcing notion does not kill those stationary subsets of $\omega_3$ which consist of ordinals of cofinality $\omega$ and cofinalities $\leq \omega_1$ are preserved, in the extension, $inv_2^\omega(\Psi_\alpha) \neq inv_2^\omega(\Psi_\beta)$ for all $\alpha \neq \beta$. Clearly, the skeletons of the models $A_\alpha$, remain weakly $(\omega, \phi)$-skeleton-like in $A_\alpha$. So by (the proof of) [12, Lemma III 3.15 (1)], $inv_2^\omega(\Psi_\alpha) \in INV_2^\omega(A_\alpha, \phi)$ in the extension (for the definition of $INV_\omega^n$, see [12, Definition III 3.11] and notice that $A \cong B$ implies $INV_\omega^n(A, \phi) = INV_\omega^n(B, \phi)$). Also by [Sh2] Lemma III 3.13 (1), $|INV_\omega^2(A_\alpha, \phi)| = \omega_1$. Since $\omega_3^\omega < 2^{\omega_3}$ in the ground model, in the generic extension, $(2^{\omega_3})^V$ is a cardinal $> \omega_1$. So there are $\alpha < \beta < (2^{\omega_3})^V$ such that $A_\alpha \not\cong A_\beta$ in the extension. Since countable subsets are not added, $E$ does not have a winning strategy for $EF_{\omega_1}^2(A_\alpha, A_\beta)$ (in the ground model). 

Now Lemmas 2.11 and 2.12 imply Theorem 2 (i). 

Before proving the theorem, we make some remarks which follow from the proof.

Remark 24 In many cases in Theorem 4, the assumption on $2^\omega$ can be removed. For example, this is true of linear orders. An easy proof for this is given in [3], alternatively this follows immediately from the proof of Theorem 2 (i) by checking where the assumption $2^\omega < 2^{\omega_3}$ was needed. Another case where the assumption on $2^\omega$ can be removed is the case that $\theta = \omega_3$ in the stable unsuperstable case. This follows from the proof of Theorem 2 (iii) by noticing that the black box can now be replaced by an argument from [3]. Another remark is that in Theorem 2 (i) and (ii), $\omega_3$ can be replaced by any cardinal $\kappa \geq \omega_3$ such that $\kappa$ is a successor of a regular cardinal and $2^\kappa > \kappa^\omega$. Finally, in Theorem 2 (iii), $\omega_3$ can be replaced by any cardinal $\kappa \geq \omega_3$ such that $\kappa$ is a successor of a regular cardinal and $\kappa^\omega = \kappa$. 

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