Functional integral approach to semi-relativistic Pauli-Fierz models

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Dedicated to Professor Asao Arai on the occasion of his 60th birthday

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Abstract

By means of functional integrations spectral properties of semi-relativistic Pauli-Fierz Hamiltonians

\[ H = \sqrt{(p - \alpha A)^2 + m^2} - m + V + H_{\text{rad}} \]

in quantum electrodynamics is considered. Here \( p \) is the momentum operator, \( A \) a quantized radiation field on which an ultraviolet cutoff is imposed, \( V \) an external potential, \( H_{\text{rad}} \) the free field Hamiltonian and \( m \geq 0 \) describes the mass of electron. Two self-adjoint extensions of a semi-relativistic Pauli-Fierz Hamiltonian are defined. The Feynman-Kac type formula of \( e^{-tH} \) is given. An essential self-adjointness, a spatial decay of bound states, a Gaussian domination of the ground state and the existence of a measure associated with the ground state are shown. All the results are independent of values of coupling constant \( \alpha \), and it is emphasized that \( m = 0 \) is included.
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1 Introduction

1.1 Preliminary

In the past decade a great deal of work has been devoted to studying spectral properties of non-relativistic quantum electrodynamics in the purely mathematical point of view. In this paper we are concerned with the semi-relativistic Pauli-Fierz model (it is abbreviated as SRPF model) in quantum electrodynamics and its spectral properties by using functional integrations. The SRPF model describes a minimal interaction between semi-relativistic electrons and a massless quantized radiation field $\mathcal{A}$ on which an ultraviolet cutoff function is imposed. We assume throughout this paper that the electron is spinless and moves in $d \geq 3$ dimensional Euclidean space for simplicity. In the case where the electron has spin $1/2$, the procedure is similar and we shall publish details somewhere. A Hamiltonian of semi-relativistic as well as non-relativistic quantum electrodynamics is usually described as a self-adjoint operator in the tensor product of a Hilbert space and a boson Fock space. In this paper instead of the boson Fock space we can formulate the Hamiltonian as a self-adjoint operator in the known Schrödinger representation in a functional realization of the boson Fock space as a space of square integrable functions with respect to the corresponding Gaussian measure. Through the Schrödinger representation a Feynman-Kac type formula of the strongly continuous one parameter semigroup generated by the SRPF Hamiltonian is given. A functional integral or a path measure approach is proven to be useful to study properties of bound states associated with embedded eigenvalues in the continuous spectrum. See e.g., [LHB11, Sections 6 and 7]. We are interested in investigating properties of bound states and ground states of the SRPF Hamiltonian by functional integrations.

1.2 Self-adjoint extensions and functional integrations

The SRPF Hamiltonian can be realized as a self-adjoint operator bounded from below in the tensor product of $L^2(\mathbb{R}^d)$ and a boson Fock space $\mathcal{F}$, where $L^2(\mathbb{R}^d)$ denotes the state space of a semi-relativistic electron and $\mathcal{F}$ that of photons. Then the decoupled Hamiltonian is given by

$$\left(\sqrt{p^2 + m^2} - m + V\right) \otimes 1 + 1 \otimes H_{\text{rad}}, \quad (1.1)$$
where \( p = (p_1, \ldots, p_d) = (-i\partial_{x_1}, \ldots, -i\partial_{x_d}) \) denotes the momentum operator, \( m \) electron mass, \( V : \mathbb{R}^d \to \mathbb{R} \) an external potential, and \( H_{\text{rad}} \) the free field Hamiltonian on \( \mathcal{F} \). The SRPF Hamiltonian is defined by introducing the minimal coupling by the quantized radiation field \( A \) with cutoff function \( \hat{\varphi} \), i.e., replacing \( p \otimes 1 \) with \( p \otimes 1 - \alpha A \) and, then

\[
H = \sqrt{(p \otimes 1 - \alpha A)^2 + m^2} - m + V \otimes 1 + 1 \otimes H_{\text{rad}},
\]

where \( \alpha \) is a real coupling constant. In order to investigate the semigroup \( e^{-tH}, \ t \geq 0 \), we redefine \( H \) on \( L^2(\mathbb{R}^d) \otimes L^2(\mathcal{D}) \) instead of \( L^2(\mathbb{R}^d) \otimes \mathcal{F} \), where \( L^2(\mathcal{D}) \) denotes the set of square integrable functions on a Gaussian probability space \( (\mathcal{D}, \mu) \), and is called a Schrödinger representation of \( \mathcal{F} \).

We introduce three classes, \( \mathcal{V}_qf, \mathcal{V}_{\text{Kato}} \) and \( \mathcal{V}_{\text{rel}} \), of external potentials. The definitions of \( \mathcal{V}_qf, \mathcal{V}_{\text{Kato}} \) and \( \mathcal{V}_{\text{rel}} \) are given in Definitions 3.13, 5.3 and 3.11 respectively. Note that \( \mathcal{V}_{\text{Kato}} \) contains relativistic Kato-class potentials (see (1.7)), \( \mathcal{V}_{\text{rel}} \) potentials being relatively bounded with respect to \( \sqrt{p^2 + m^2} - m \), and \( \mathcal{V}_{\text{Kato}} \) contains \( \mathcal{V}_qf \), \( \mathcal{V}_{\text{rel}} \) in \( \mathcal{V}_qf \) hold.

We show in Theorems 4.5 and 4.7 that \( H \) is self-adjoint on \( \mathcal{D} \) for \( V \in \mathcal{V}_qf \). For more singular potentials we shall construct two appropriate self-adjoint extensions of \( H \), which are denoted by \( H_{qf} \) and \( H_K \). The former is defined for \( V \in \mathcal{V}_qf \) by the quadratic form sum and the later for \( V \in \mathcal{V}_{\text{Kato}} \) through Feynman-Kac type formula. See Definition 3.13 for \( H_{qf} \) and Definition 5.3 for \( H_K \). Although \( \mathcal{V}_qf \) is wider than \( \mathcal{V}_{\text{Kato}}, \mathcal{V}_{\text{rel}} \) is defined under weaker condition on cutoff function \( \hat{\varphi} \) than that for \( \mathcal{V}_{qf} \).

In Introduction \( H \) stands for \( H_{qf} \) or \( H_K \) in what follows. We construct the Feynman-Kac type formula of \( e^{-tH} \) in terms of a composition of Euclidean quantum field \( A_E(f) \) with test function \( f \in \mathcal{E} = \bigoplus L_2^d(\mathbb{R}) \), \( d \)-dimensional Brownian motion \( (B_t)_{t \in \mathbb{R}} \) on the whole real line \( \mathbb{R} \) defined on a probability space \( (\Omega_F, \mathcal{B}_F, P^x) \), and a subordinator \( (\mathcal{T}_t)_{t \geq 0} \) on \( (\Omega_\nu, \mathcal{B}_\nu, \nu) \). The Euclidean quantum field \( A_E(f) \) is Gaussian, and the covariance is given by \( \mathbb{E}_{pE}[A_E(f)A_E(g)] = q_E(f, g) \) with some bilinear form \( q_E(\cdot, \cdot) \) on \( \mathcal{E} \times \mathcal{E} \). Hence it is driven in Theorem 3.15 and Corollary 3.16 that

\[
(F, e^{-2tH}G) = \int_{\mathbb{R}^d} dF \mathbb{E}_{pE}^x \left[ \left( J_{-t} F(\mathcal{T}_t), e^{-i\alpha A_E(\mathcal{I}_{-t} f)} e^{-\int_{-t}^t V(B_s)ds} J_t G(\mathcal{T}_t) \right) \right]
\]

for \( F, G \in L^2(\mathbb{R}^d; L^2(\mathcal{D})) \cong L^2(\mathbb{R}^d) \otimes L^2(\mathcal{D}) \). Here \( \mathcal{I}_{-t}, t \) is a limit of \( \mathcal{E} \)-valued stochastic integrals, which is formally written as

\[
\mathcal{I}_{-t} = \bigoplus_{\mu=1}^d \int_{-T_t}^{T_t} \lambda(B_s) dB_s^\mu
\]
with \( \lambda = (\hat{\phi}/\sqrt{\omega}) \). Here \( T^*_s = \inf\{t|T_t = s\} \) is the first hitting time of \((T_t)_{t \geq 0}\) at \( s \). Notations \( J_t \) and \( j_t \) are defined in Section 2.2 below, and the rigorous definition of (1.4) is given in Lemma 3.7, Remarks 3.8 and 3.17.

1.3 Main results

By using the Feynman-Kac type formula (1.3), we study the spectrum of the SRPF Hamiltonian \( H \). The main results of this paper are (a)-(d) below:

(a) Self-adjointness and essential self-adjointness of \( H \) (Theorems 4.5 and 4.7).

(b) Spatial decay of bound states \( \Phi_b \) of \( H \) (Theorem 5.12).

(c) Gaussian domination of the ground state \( \varphi_g \) of \( H \) (Theorem 6.8).

(d) Existence of a probability measure \( \mu_\infty \) associated with \( \varphi_g \) (Theorem 7.3).

The spectrum of non-relativistic versions of \( H \), which is the so-called Pauli-Fierz model, have been studied, and among other things the existence of a ground state is proven in [BFS99, GLL01]. See also [Spo04] and references therein. The spectrum of semi-relativistic versions, \( H \), is also studied in e.g., [FGS01, HH13a, HH13b, KMS09, KMS11, KMS12, MS10, MS09] from an operator-theoretic point of view. In particular the existence of ground states of \( H \) are considered under some conditions in [KMS09, KMS12] for \( m > 0 \) and [HH13b] for \( m \geq 0 \).

Here are outlines of assertions (a)-(d) mentioned above.

(a) Following our previous work [Hir00b], we investigate (a). This can be proven by estimating the scalar product \( |(K,F,e^{-\alpha H}G)| \) for self-adjoint operators \( K = \mathbb{1} \otimes H_{\text{rad}} \) and \( p_\mu \otimes \mathbb{1} \). Let \( V = 0 \). Then a bound \( |(K,F,e^{-\alpha H}G)| \leq C_{K,G}||F|| \), \( F,G \in \text{D}(H) \), is shown with some constant \( C_{K,G} \). Hence \( e^{-\alpha H} \) leaves \( \text{D}([p] \otimes \mathbb{1}) \cap \text{D}(\mathbb{1} \otimes H_{\text{rad}}) \) invariant for \( V = 0 \) and we can conclude that \( H \) is essentially self-adjoint on \( \text{D}([p] \otimes \mathbb{1}) \cap \text{D}(\mathbb{1} \otimes H_{\text{rad}}) \) by Proposition 3.3 for \( V \in \mathcal{V}_{\text{rel}} \) for arbitrary values of \( \alpha \). This is an extension of that of a non-relativistic case established in [Hir00b] and [LHB11, Section 7.4.1]. Furthermore the self-adjointness of \( H \) is shown in Theorem 4.7. Examples include a spinless hydrogen like atom (Example 4.8). It is noted that our method is also available to the SRPF Hamiltonian with spin. We give a comment on known results. Although in [KMS11, MS10] the self-adjointness of the SRPF Hamiltonian with spin 1/2 is considered, it is not sure that the method can be available to spinless cases.
(b) Let
\[ H_p = \sqrt{p^2 + m^2} - m + V \] (1.5)
be the semi-relativistic Schrödinger operator. Let \((z_t)_{t \geq 0}\) be the \(d\)-dimensional Lévy process on a probability space \((Ω_Z, ℬ_Z, P_Z)\) such that \(E^Z_x[e^{-iu \cdot z_t}] = e^{-t(\sqrt{|u|^2 + m^2} - m)}e^{-iu \cdot x}\). Hence the self-adjoint generator of \((z_t)_{t \geq 0}\) is given by \(\sqrt{p^2 + m^2} - m\). The Feynman-Kac type formula for \(H_p\) is thus given by
\[
(f, e^{-tH_p}g) = \int_{\mathbb{R}^d} dx E^Z_x \left[ \bar{f}(z_0)g(z_t)e^{-\int_0^t V(z_s)ds} \right].
\] (1.6)
Conversely taking a potential \(-V\) such that
\[
\sup_{x \in \mathbb{R}^d} E^Z_x[e^{-\int_0^t V(z_s)ds}] < \infty,
\] (1.7)
we can define the strongly continuous one-parameter symmetric semigroup \(s_t\), \(t \geq 0\), on \(L^2(\mathbb{R}^d)\) by
\[
(s_tf)(x) = E^Z_x \left[ f(z_t)e^{-\int_0^t V(z_s)ds} \right].
\] (1.8)
Thus we can define the unique self-adjoint operator \(H^K_p\) by \(s_t = e^{-tH^K_p}, t \geq 0\). A potential \(V\) satisfying \(\sup_{x \in \mathbb{R}^d} E^Z_x[e^{+\int_0^t V(B_s)ds}] < \infty\) is known as a Kato-class potential. Replacing the Brownian motion \(B_t\) with Lévy process \(z_t\), we call a potential \(-V\) satisfying (1.7) a relativistic Kato-class potential. The property (1.7) is also used in the proofs of Lemmas 5.8 and 5.11 and Corollary 5.9. Let \(V = V_+ - V_-\) be such that \(V_+ \geq 0\), \(V_+ \in L^1_{loc}(\mathbb{R}^d)\) and \(V_-\) is a relativistic Kato-class potential. \(\mathcal{Y}_{Kato}\) denotes the set of such potentials. Furthermore let \(\phi_b\) be a bound state of \(H^K_p\) with \(V \in \mathcal{Y}_{Kato}\), i.e., \(H^K_p\phi_b = E\phi_b\) with some \(E \in \mathbb{R}\). Then the stochastic process
\[
\left(e^{tE}e^{-\int_0^t V(z_s+x)ds}\phi_b(z_t+x)\right)_{t \geq 0}
\] (1.9)
is martingale with respect to the natural filtration \(M_t = \sigma(z_s, 0 \leq s \leq t)\). From martingale property we can derive a spatial decay of \(\phi_b(x)\) (CMS90). Furthermore in [HIL13] we can extend these procedures to a semi-relativistic Schrödinger operators of the form: \(\sqrt{(\sigma \cdot (p - a))^2 + m^2} - m + V\) on \(\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)\), where \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\) denotes \(2 \times 2\) Pauli matrices and \(a = (a_1, a_2, a_3)\) a vector potential satisfying suitable conditions.
In a similar manner to [1.8], we define a strongly continuous one-parameter symmetric semigroup and define the SRPF Hamiltonian with \( V \in \mathcal{V}_{\text{Kato}} \). We can show in Theorem 5.2 that the map

\[
(S_t F)(x) = \mathbb{E}_F^x \left[ \int_0^t e^{-i\alpha_x \mathcal{A}_E(t_0, t)} e^{-\int_0^t V(B_{T_s^- + x})ds} J_t F(B_{T_t}) \right]
\]

is the strongly continuous one-parameter symmetric semigroup under the identification \( L^2(\mathbb{R}) \otimes L^2(\mathcal{D}) \cong L^2(\mathbb{R}; L^2(\mathcal{D})) \). Thus we can define the self-adjoint operator \( H_k \) by \( S_t = e^{-tH_k}, t \geq 0 \). To study (b) we also show a martingale property of some stochastic process derived from the Feynman-Kac type formula (1.3). Let \( \Phi_b \) be any bound state of \( H_k \), i.e., \( H_k \Phi_b = E\Phi_b \) with some \( E \in \mathbb{R} \). We can show in Theorem 5.10 that the \( L^2(\mathcal{D}) \)-valued stochastic process

\[
(M_t(x))_{t \geq 0} = \left( e^{tE} e^{-i\alpha_x \mathcal{A}_E(I^0[0,t])} e^{-\int_0^t V(B_{T_s^- + x})ds} J_t \Phi_b(B_{T_t} + x) \right)_{t \geq 0}, \quad t \geq 0, \quad (1.10)
\]

is martingale with respect to a filtration \( (\mathcal{M}_t)_{t \geq 0} \). Suppose that \( |V(x)| \to 0 \) as \( |x| \to \infty \). Then we can show in Theorem 5.12 that \( \|\Phi_b(x)\|_{L^2(\mathcal{D})} \) spatially decays exponentially in the case of \( m > 0 \) and polynomially in the case of \( m = 0 \). As far as we know a polynomial decay of bound states of the SRPF Hamiltonian with \( m = 0 \) is new.

(c) By the phase factor \( e^{-i\alpha_x \mathcal{A}_E(I^0[0,t], t)} \) appeared in the Feynman-Kac type formula (1.3), \( (F, e^{-tH}G) \in \mathbb{C} \) for \( F, G \geq 0 \) in general. However it is established in a similar manner to [Hir00a] that \( (F, e^{-i\frac{E}{2}N}e^{-tH}e^{i\frac{E}{2}N}G) \geq 0 \) for \( F, G \geq 0 \) \((F \neq 0, G \neq 0)\), where \( N \) denotes the number operator. \( e^{-i\frac{E}{2}N}e^{-tH}e^{i\frac{E}{2}N} \) is positivity improving. Then the ground state \( \varphi_g \) satisfies that \( e^{-i\frac{E}{2}N} \varphi_g > 0 \). This is a key point to study the ground state of \( H \) by path measures. By \( e^{-i\frac{E}{2}N} \varphi_g > 0 \), normalizing sequence

\[
\varphi_g^t = e^{-tH}(\phi \otimes \mathbb{1})/\|e^{-tH}(\phi \otimes \mathbb{1})\| \quad (1.11)
\]

strongly converges to a normalized ground state \( \varphi_g \) as \( t \to \infty \) for any \( 0 \leq \phi \in L^2(\mathbb{R}) \) but \( \phi \neq 0 \).

Physically it is interested in observing expectation values of some observable \( \mathcal{O} \) with respect to \( \varphi_g \), i.e., \( (\varphi_g, \mathcal{O}_\varphi \varphi_g) \). Since \( \varphi_g^t \to \varphi_g \) as \( t \to \infty \) strongly, we can see that \( (\varphi_g, \mathcal{O}_\varphi \varphi_g) = \lim_{t \to \infty} (\varphi_g^t, \mathcal{O}_\varphi \varphi_g^t) \). Let \( \mathcal{A}_\xi \) be the quantized radiation field smeared by \( \xi \in \bigoplus^d L^2_\mathbb{R}(\mathbb{R}) \). To show (c) we prove in Lemma 6.7 the bound

\[
(\varphi_g^t, e^{\beta \mathcal{A}_\xi} \varphi_g^t) \leq \frac{1}{\sqrt{1 - 2\beta qE(j_0\xi, j_0\xi)^2}} \quad (1.12)
\]
uniformly in $t$ for some $\beta > 0$. Taking the limit $t \to \infty$ on both sides of (1.12), we show that $\varphi_g \in D(e^{\beta A^2})$ for some $0 < \beta$.

(d) For some important observables $O$, by (1.3) we can see that $(\varphi^t, O \varphi^t) = \mathbb{E}_\mu [F^t_O]$ with an integrant $F^t_O$ and probability measures (we call this as finite volume Gibbs measure) given by

$$
\mu^t_{\text{SRPF}}(A) = \mu_\infty(A) = \frac{1}{Z_t} \int_{\mathbb{R}^d} \mathbb{E}_\mathbb{P}_{\text{SRPF}}^t \left[ \Pi_A e^{-\frac{q^2}{2} q(t,t)} e^{-\int_t^0 V(B_s) ds} \right], \quad t \geq 0, \quad (1.13)
$$

where $Z_t$ denotes the normalization constant. See Definition 6.4. Furthermore it is interesting to show the convergence of measures $\mu_t$, $t \geq 0$, for its own sake in mathematics. Formally we have $(\varphi^t_g, O \varphi^t_g) = \mathbb{E}_\mu_\infty [F^t_O]$. Exponent $q(t,t)$ in (1.13) is called a pair interaction associated with $H$, which is formally given by

$$
W^\text{SRPF} = q(t,t) = \sum_{\mu, \nu = 1}^d \int_{-T_t}^{T_t} dB^\mu_s \int_{-T_t}^{T_t} dB^\nu_r W_{\mu \nu}(T^\nu_s - T^\mu_r, B_s - B_r), \quad (1.14)
$$

where the pair potential $W_{\mu \nu}$ is given by

$$
W_{\mu \nu}(t, X) = \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} \left( \delta_{\mu \nu} - \frac{k_\mu k_\nu}{|k|^2} \right) e^{-ik \cdot x} e^{-|t| \omega(k)} dk. \quad (1.15)
$$

See (6.31) and (6.35) for details. Several limits of some finite volume Gibbs measures associated with models in quantum field theory are considered, e.g., examples include the Nelson model [BHLMS02, OS99], spin-boson model [HHL12] and the Pauli-Fierz model [BH09]. In this paper we consider a limit of finite volume Gibbs measures associated with the SRPF model. The pair interaction associated with a spin-boson model [HHL12], the Nelson model [BHLMS02] and the Pauli-Fierz model [BH09, Hir00a, Spo87] are given by

$$
W^\text{SB} = \int_{-t}^t ds \int_{-t}^t dr \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} (-1)^{N_s - N_r} e^{-|s-r| \omega(k)} dk, \quad (1.16)
$$

$$
W^\text{N} = \int_{-t}^t ds \int_{-t}^t dr \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} e^{-ik \cdot (B_s - B_r)} e^{-|s-r| \omega(k)} dk, \quad (1.17)
$$

$$
W^\text{PF} = \sum_{\mu, \nu = 1}^d \int_{-t}^t dB^\mu_s \int_{-t}^t dB^\nu_r \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} \left( \delta_{\mu \nu} - \frac{k_\mu k_\nu}{|k|^2} \right) e^{-ik \cdot (B_s - B_r)} e^{-|s-r| \omega(k)} dk, \quad (1.18)
$$

8
|                      | Path without jumps | Path with jumps |
|----------------------|--------------------|-----------------|
| Uniformly bounded $W^#$ | $\mu_t^N$       | $\mu_t^{SB}$   |
| Non-uniformly bounded $W^#$ | $\mu_t^{PF}$   | $\mu_t^{SRPF}$ |

Figure 1: Finite volume Gibbs measures

respectively. Let $\mu_t^#$ be the finite volume Gibbs measure with the pair interaction $W^#$, where # stands for SRPF, SB, N, PF. Note that $W^N$ and $W^{SB}$ are uniformly bounded with respect to paths, i.e.,

$$W^# \leq \int_{-t}^{t} ds \int_{-t}^{t} dr \int_{\mathbb{R}} \frac{|\hat{\phi}(k)|^2}{2\omega(k)} e^{-|t-s|\omega(k)} dk,$$

while $W^{SRPF}$ and $W^{PF}$ are not uniformly bounded. In addition, $\mu_t^N$ and $\mu_t^{PF}$ are measures defined on the set of continuous paths, $\mu_t^{SB}$ and $\mu_t^{SRPF}$, however, on the set of paths with jumps. See Figure 1.

Existence of limits of $\mu_t^N$ and $\mu_t^{PF}$ is proven in [LHB11, Theorem 6.12] and [BH09], respectively, by showing the tightness of the family of measures $(\mu_t^N)_{t\geq0}$ and $(\mu_t^{PF})_{t\geq0}$. It is, however, not straightforward to show the convergence of $\mu_t^{SB}$, since $(\mu_t^{SB})_{t\geq0}$ is a measure defined on the set of paths with jumps $\pm1$. Then the local weak convergence of $\mu_t^{SB}$ is shown in [HHL12] instead of a weak convergence. Since both $\mu_t^N$ and $\mu_t^{SB}$ include the uniformly bounded pair interactions, we can fortunately easily use the limit measures to express the ground state expectation with some observable, e.g., $e^{+\beta N}$, etc. See [HHL12] and [LHB11, Section 6]. On the other hand since $\mu_t^{PF}$ includes the non-uniformly bounded pair interaction, it is unfortunately hard to apply the limit measure to express the ground state expectation with some concrete observable. See [Spo04, p.196-197]. It is however worthwhile showing the existence of limit measure itself, since our pair interaction is far singular than that of e.g. [OS99]. The family of probability measures $\mu_t^{SRPF}$, which is our main object in this paper, is defined on the set of càdlàg paths, and its pair interaction is not uniformly bounded. We prove that $\mu_t^{SRPF}$ converges to a probability measure $\mu_\infty^{SRPF}$ in the local weak sense as $t \to \infty$ by using the existence of the ground state of $H$, which is studied in [HH13a, KMS09, KMS11].

This paper is organized as follows: Section 2 is devoted to defining the SRPF Hamiltonian $H_{qf}$ in both a Fock space and a function space to study the semigroup
by a path measure. In Section 3 we construct a Feynman-Kac type formula for $H_{qf}$. In Section 4 we show the essential self-adjointness and the self-adjointness of $H_{qf}$. In Section 5 we define the self-adjoint operator $H_K$ of the SRPF Hamiltonian with a potential in the relativistic Kato-class, and show that some stochastic process is martingale by which a spatial decay of bound states is proven. Section 6 is devoted to showing a Gaussian domination of the ground state. In Section 7 the existence of an infinite volume limit of finite Gibbs measures is shown. In Section 8 we give comments on a model with spin $1/2$ and model with a fixed total momentum. Finally in Appendix we give fundamental tools of probability theory and proofs of some equalities used in this paper.

2 Semi-relativistic Pauli-Fierz model

2.1 SRPF model in Fock space

Let us begin by defining fundamental tools of quantum field theory in Fock representation. Let $\mathcal{W} = L^2(\mathbb{R}^d \times \{1, \ldots, d-1\})$ be the Hilbert space of a single photon in the $d$-dimension Euclidean space, where $\mathbb{R}^d \times \{1, \ldots, d-1\} \ni (k, j)$ denotes the pair of momentum $k$ and polarization $j$ of a single photon. We denote the $n$-fold symmetric tensor product of $\mathcal{W}$ by $\otimes^n_{\text{sym}} \mathcal{W}$ for $n \geq 1$ and set $\otimes^0_{\text{sym}} \mathcal{W} = \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. The boson Fock space describing the full photon field is defined then as the Hilbert space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} (\otimes^n_{\text{sym}} \mathcal{W})$$

endowed with the scalar product $(\Psi, \Phi)_\mathcal{F} = \sum_{n=0}^{\infty} (\Psi^{(n)}, \Phi^{(n)})_{\otimes^n \mathcal{W}}$ for $\Psi = \bigoplus_{n=0}^{\infty} \Psi^{(n)}$ and $\Phi = \bigoplus_{n=0}^{\infty} \Phi^{(n)}$. Alternatively, $\mathcal{F}$ can be identified as the set of $\ell^2$-sequences $\{\Psi^{(n)}\}_{n=0}^{\infty}$ with $\sum_{n=0}^{\infty} \|\Psi^{(n)}\|_{\otimes^n \mathcal{W}}^2 < \infty$. The vector $\Omega_b = \{1, 0, 0, \ldots\} \in \mathcal{F}$ is called the Fock vacuum. The finite particle subspace $\mathcal{F}_{\text{fin}}$ is defined by

$$\mathcal{F}_{\text{fin}} = \left\{ \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F} \mid \Psi^{(m)} = 0 \text{ for } \forall m \geq M \text{ with some } M \right\}. \quad (2.2)$$

With each $f \in \mathcal{W}$ a creation operator and an annihilation operator are associated. The creation operator $a^\dagger(f) : \mathcal{F} \to \mathcal{F}$ is defined by

$$(a^\dagger(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)})$$

(2.3)
for \( n \geq 1 \), where \( S_n(f_1 \otimes \cdots \otimes f_n) = (1/n!) \sum_{\pi \in \mathfrak{S}_n} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)} \) is the symmetrizer with respect to the permutation group \( \mathfrak{S}_n \) of degree \( n \). The domain of \( a^\dagger(f) \) is maximally defined by

\[
D(a^\dagger(f)) = \{ \{ \Psi(n) \}^\infty_{n=0} \in \mathcal{F} \mid \sum_{n=1}^\infty n \| S_n(f \otimes \Psi(n-1)) \|^2 < \infty \}. \tag{2.7}
\]

The annihilation operator \( a(f) \) is introduced as the adjoint of \( a^\dagger(f) \), i.e., \( a(f) = (a^\dagger(f))^* \). Both \( a^\dagger(f) \) and \( a(f) \) are closable operators, their closed extensions are denoted by the same symbols. Also, they leave \( \mathcal{F}_{\text{fin}} \) invariant and obey the canonical commutation relations on \( \mathcal{F}_{\text{fin}} \):

\[
[a(f), a^\dagger(g)] = (\tilde{f}, g) \mathbb{1}, \quad [a(f), a(g)] = 0, \quad [a^\dagger(f), a^\dagger(g)] = 0. \tag{2.4}
\]

The dispersion relation considered in this paper is chosen to be \( \omega(k) = |k| \) for \( k \in \mathbb{R}^d \). We denote \( \hat{f} \) the Fourier transformation of \( f \in L^2(\mathbb{R}^d) \). We use the informal expression

\[
\sum_{j=1}^{d-1} \int a^\dagger(k, j) f(k, j) dk \text{ for } a^\dagger(f) \text{ for convenience.}
\]

Then the quantized radiation field smeared by \( f \in L^2(\mathbb{R}^d) \) is defined by

\[
A_\mu(f, x) = \frac{1}{\sqrt{2}} \sum_{j=1}^{d-1} \int \frac{e_\mu(k, j)}{\sqrt{\omega(k)}} \left( a^\dagger(k, j) e^{-ik \cdot x} \hat{f}(k) + a(k, j) e^{ik \cdot x} \hat{f}(-k) \right) dk \tag{2.5}
\]

for each \( x \in \mathbb{R}^d \) and its momentum conjugate by

\[
\Pi_\mu(f, x) = \frac{i}{\sqrt{2}} \sum_{j=1}^{d-1} \int e_\mu(k, j) \sqrt{\omega(k)} \left( a^\dagger(k, j) e^{-ik \cdot x} \hat{f}(k) - a(k, j) e^{ik \cdot x} \hat{f}(-k) \right) dk, \tag{2.6}
\]

where \( e(k, j), k \in \mathbb{R}^d \setminus \{0\}, j = 1, \ldots, d-1 \), are \( d \) dimensional polarization vector such that \( e(k, j) \cdot e(k, j') = \delta_{jj'} \) and \( k \cdot e(k, j) = 0 \). From canonical commutation relations it follows that \( [A_\mu(f, x), \Pi_\nu(g, y)] = i \int \delta^\perp_{\mu\nu}(k) \hat{f}(-k) \hat{g}(k) e^{ik \cdot (x-y)} dk \), where

\[
\delta^\perp_{\mu\nu}(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}, \quad k \neq 0,
\]

denotes the transversal delta function. The quantized radiation field with a fixed ultraviolet cutoff function \( \hat{\varphi} \) is then defined by

\[
A_\mu(x) = A_\mu(\varphi, x). \tag{2.7}
\]

By \( k \cdot e(k, j) = 0 \), the Coulomb gauge condition

\[
\nabla_x \cdot A(x) = 0 \tag{2.8}
\]

holds as an operator. A standing assumption in this paper is as follows.
Assumption 2.1 We suppose that $\hat{\varphi}(k) = \hat{\varphi}(-k)$ and $\hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$.

We also introduce an assumption.

Assumption 2.2 We suppose that $\omega \sqrt{\omega} \hat{\varphi}, \hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$.

Under Assumption 2.1, $A_\mu(x)$ is a well-defined symmetric operator in $\mathcal{F}$. By the fact that $\sum_{n=0}^\infty \| A_\mu(x)^n \Phi \| t^n \leq \infty$ for $\Phi \in \mathcal{F}_{\text{fin}}$ and $t > 0$, and Nelson’s analytic vector theorem [Nel59], the symmetric operator $A_\mu(x)|_{\mathcal{F}_{\text{fin}}}$ is essentially self-adjoint. We denote its closure $A_\mu(x)|_{\mathcal{F}_{\text{fin}}}$ by the same symbol $A_\mu(x)$.

Next we define the free quantum field Hamiltonian on $\mathcal{F}$. The free quantum field Hamiltonian is defined as the infinitesimal generator of a one-parameter unitary group. This unitary group is constructed through a functor $\Gamma$. Let $C(W)\to C(F)$ denote the set of contraction operators from $X$ to $Y$. We set $C(X)$ for $C(X\to X)$ for simplicity.

Functor $\Gamma : C(W) \to C(F)$ is defined as $\Gamma(T) = \bigoplus_{n=0}^\infty [\otimes^n T]$, where $\otimes^0 T = \mathbb{1}$. For a self-adjoint operator $h$ on $W$, $\Gamma(e^{it\hat{h}}), t \in \mathbb{R}$, is a strongly continuous one-parameter unitary group on $\mathcal{F}$. Then by Stone’s theorem there exists a unique self-adjoint operator $d\Gamma(h)$ on $\mathcal{F}$ such that $\Gamma(e^{it\hat{h}}) = e^{itd\Gamma(h)}, t \in \mathbb{R}$. $d\Gamma(h)$ is called the second quantization of $h$. Let $\omega$ be regarded as the multiplication operator $f \mapsto \omega(k)f(k) = |k|f(k,j)$. The operator $d\Gamma(\omega)$ is then the free quantum field Hamiltonian.

The Hilbert space describing a state space of a single electron is $L^2(\mathbb{R}^d)$. The semi-relativistic electron Hamiltonian on $L^2(\mathbb{R}^d)$ with a real-valued external potential $V$ is given by

$$H_p = \sqrt{p^2 + m^2} - m + V.$$  \hspace{1cm} (2.9)

Here $p^2 = \sum_{\mu=1}^d p_\mu^2$, $V$ acts as the multiplication operator in $L^2(\mathbb{R}^d)$, and $m \geq 0$ describes the mass of an electron. We regard $m \geq 0$ as a non-negative parameter and it is allowed to be $m = 0$. The state space of the joint electron-field system is

$$H_{\text{Fock}} = L^2(\mathbb{R}^d) \otimes \mathcal{F}.$$  \hspace{1cm} (2.10)

To define the quantized radiation field $A$ we identify $H_{\text{Fock}}$ with the set of $\mathcal{F}$-valued $L^2$ functions on $\mathbb{R}^d$, i.e., $H_{\text{Fock}} \cong \int_{\mathbb{R}^d} \mathcal{F} dx$ and $A_\mu$ is defined by $A_\mu = \int_{\mathbb{R}^d} A_\mu(x) dx$ with the domain

$$D(A_\mu) = \left\{ F \in \int_{\mathbb{R}^d} \mathcal{F} dx \mid F(x) \in D(A_\mu(x)) \text{ a.e. } x \in \mathbb{R}^d \text{ and } \int_{\mathbb{R}^d} \| A_\mu(x) F(x) \|_{\mathcal{F}}^2 dx < \infty \right\}.$$
Hence \((A_\mu F)(x) = A_\mu(x)F(x)\) for \(F(x) \in D(A_\mu(x))\) and \(A_\mu\) is self-adjoint. The Friedrichs extension of \(\frac{1}{2}(p \otimes I - \alpha A)^2|_{C_c^\infty(\mathbb{R}^d) \otimes \mathcal{F}_{\text{fin}}}\) is denoted by \(h_A\).

**Definition 2.3 (Definition of SRPF Hamiltonian)** Suppose Assumption 2.1. The SRPF Hamiltonian is defined by

\[
(2h_A + m^2)^{1/2} - m + V \otimes I + I \otimes d\Gamma(\omega)
\]

with the domain \(D((2h_A + m^2)^{1/2}) \cap D(V \otimes I) \cap D(I \otimes H_{\text{rad}})\).

**2.2 SRPF model in function space**

In order to construct the Feynman-Kac type formula of the semigroup generated by the SRPF Hamiltonian we prepare some probabilistic tools for the field and the particle. Let us use a \(\mathcal{Q}\)-space representation instead of the Fock representation. Define the field operator \(A_\mu(f)\) by

\[
A_\mu(f) = \frac{1}{\sqrt{2}} \sum_{j=1}^{d-1} \int e_\mu(k, j) \left( \hat{f}(k)a(k, j) + \hat{f}(-k)a(k, j) \right) dk
\]

and the \(d \times d\) matrix \(D(k)\) by \(D(k) = (\delta_{\mu\nu}(k))_{1 \leq \mu, \nu \leq d}\) for \(k \neq 0\). Consider the bilinear form \(q_M : \oplus^d L^2(\mathbb{R}^d) \times \oplus^d L^2(\mathbb{R}^d) \rightarrow \mathbb{C}\) defined by

\[
q_M(f, g) = \frac{1}{2} \int_{\mathbb{R}^d} \langle \hat{f}(k), D(k)\hat{g}(k) \rangle dk,
\]

where \(\langle x, y \rangle = \bar{x} \cdot y\) denotes the standard scalar product on \(\mathbb{C}^d\). Then we have

\[
\sum_{\mu, \nu=1}^{d-1} (A_\mu(f_\mu)\Omega_b, A_\nu(g_\nu)\Omega_b)_{\mathcal{F}} = q_M(f, g).
\]

We introduce another bilinear form \(q_E : \oplus^d L^2(\mathbb{R}^{d+1}) \times \oplus^d L^2(\mathbb{R}^{d+1}) \rightarrow \mathbb{C}\) by

\[
q_E(F, G) = \frac{1}{2} \int_{\mathbb{R}^{d+1}} \langle \hat{F}(k, k_0), D(k)\hat{G}(k, k_0) \rangle dk dk_0.
\]

Note that \(D(k)\) is independent of \(k_0 \in \mathbb{R}\) in the definition of \(q_E\). We denote \(q_{\#}(K, K)\) by \(q_{\#}(K)\) for simplicity, where \(q_{\#}\) stands for \(q_M\) and \(q_E\).

Let \(\mathcal{X}_{\mathbb{R}}(\mathbb{R}^d)\) be the set of real-valued Schwarz test functions on \(\mathbb{R}^d\). Let \(\mathcal{D} = (\oplus^d \mathcal{X}_{\mathbb{R}}(\mathbb{R}^d))^/'\) and \(\mathcal{D}_E = (\oplus^d \mathcal{X}_{\mathbb{R}}(\mathbb{R}^{d+1}))'\). Here \(X'\) denotes the dual space of a locally convex space \(X\). We denote the pairing between elements of \(\mathcal{D}\) and \(\oplus^d \mathcal{X}_{\mathbb{R}}(\mathbb{R}^d)\) by
\( \langle \phi, f \rangle_M \in \mathbb{R} \) for \( \phi \in \mathcal{D} \) and \( f \in \bigoplus^d \mathcal{S}(\mathbb{R}^d) \). We denote the expectation with respect to a probability path measure \( P^x \) starting from \( x \) at \( t = 0 \) by \( \mathbb{E}^x_0[\cdots] = \int \cdots dP^x \). By the Bochner-Minlos Theorem there exists a probability space \( (\mathcal{D}, \Sigma_M, \mu_M) \) such that \( \Sigma_M \) is the smallest \( \sigma \)-field generated by \( \{ \langle \phi, f \rangle_M : f \in \bigoplus^d \mathcal{S}(\mathbb{R}^d) \} \) and \( \langle \phi, f \rangle_M \) is a Gaussian random variable with mean zero and the covariance given by \( \sigma_{\phi,f} = q_M(f,g) \). Then we have

\[
\mathbb{E}_{\mu_M}[e^{i\langle \phi, f \rangle_M}] = e^{-\frac{1}{4}q_M(f,f)}. \tag{2.14}
\]

Since \( \langle \phi, \bigoplus^d \delta_{\nu, f} \rangle \) is a \( \mathcal{D} \)-representation of the quantized radiation field with test function \( f \in \mathcal{S}(\mathbb{R}^d) \), we have to extend \( f \in \mathcal{S}(\mathbb{R}^d) \) to a more general class since our cutoff is \( \langle \hat{\phi}/\sqrt{\omega} \rangle \in L^2(\mathbb{R}^d) \). For any \( f = \Re f + i\Im f \in \bigoplus^d \mathcal{S}(\mathbb{R}^d) \) we set \( \langle \phi, f \rangle_M = \langle \phi, \Re f \rangle_M + i \langle \phi, \Im f \rangle_M \). Let

\[
\mathcal{M} = \bigoplus^d L^2(\mathbb{R}^d). \tag{2.15}
\]

Since \( \mathcal{S}(\mathbb{R}^d) \) is dense in \( L^2(\mathbb{R}^d) \) and the equality \( \int_{\mathcal{D}} |\langle \phi, f \rangle_M|^2 d\mu_M = \frac{1}{2} \|f\|^2_\mathcal{D} \) holds by (2.14), we can define \( \langle \phi, f \rangle_M \) for \( f \in \mathcal{M} \) by \( \langle \phi, f \rangle_M = s\lim_{n \to \infty} \langle \phi, f_n \rangle_M \) in \( L^2(\mathcal{D}) \), where \( \{f_n\}_{n=1}^\infty \subset \bigoplus^d \mathcal{S}(\mathbb{R}^d) \) is any sequence such that \( s\lim_{n \to \infty} f_n = f \) in \( \mathcal{D} \). Thus we define the multiplication operator \( A(f) \) by

\[
(A(f)F)(\phi) = \langle \phi, f \rangle_M F(\phi), \quad f \in \mathcal{M}
\]

in \( L^2(\mathcal{D}) \) with the domain \( \text{D}(A(f)) = \{ F \in L^2(\mathcal{D}) | \int_{\mathcal{D}} |\langle \phi, f \rangle_M F(\phi)|^2 d\mu_M < \infty \} \). Denote the identity function in \( L^2(\mathcal{D}) \) by \( \mathbb{1}_\mathcal{D} \) and the function \( A(f) \mathbb{1}_\mathcal{D} \) by \( A(f) \) unless confusion may arise. It is known as the Wiener-Itô decomposition that

\[
L^2(\mathcal{D}) = \bigoplus_{n=0}^\infty L_n^2(\mathcal{D})
\]

with \( L_n^2(\mathcal{D}) = L^2(\mathcal{D}) \) and \( X: \bigoplus^d \mathcal{S}(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \) denotes Wick product recursively defined by \( :A(f): = A(f) \) and \( :A(f) \prod_{j=1}^n A(f_j): = A(f) \prod_{j=1}^n A(f_j) - \sum_{j=1}^n q_M(f,f_j) \prod_{i \neq j}^n A(f_i) \). We set \( \alpha(f) = A(\bigoplus^d \delta_{\nu, f}) \) for \( f \in L^2(\mathbb{R}^d) \). Let

\[
\mathcal{E} = \bigoplus^d L^2(\mathbb{R}^{d+1}). \tag{2.16}
\]
Similarly we can define the Gaussian random variable $A_E(f)$ labelled by $f \in \mathcal{E}$ on a probability space $(\mathcal{Q}_E, \Sigma_E, \mu_E)$ with $q_M$ replaced by $q_E$ in (2.14). In particular

$$E_{\mu_E} \left[ e^{i\phi, f} \right] = e^{-\frac{1}{2}q_E(f, f)} \tag{2.17}$$

and $(A_E(f)F)(\phi) = \langle \phi, f \rangle_E F(\phi)$ hold for $f \in \mathcal{E}$.

We define the second quantization on $L^2(\mathcal{Q})$. Let $T \in \mathcal{C}(L^2(\mathbb{R}^d))$. Then $\Gamma(T) \in \mathcal{C}(L^2(\mathcal{Q}_E))$ is defined by

$$\Gamma(T)1_{\mathcal{Q}} = 1_{\mathcal{Q}}, \quad \Gamma(T) = \prod_{j=1}^n A(f_j): = \prod_{j=1}^n A(Tf_j):. \tag{2.18}$$

For $T \in \mathcal{C}(L^2(\mathbb{R}^{d+1}))$ (resp. $\mathcal{C}(L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{d+1}))$, $\Gamma(T) \in \mathcal{C}(L^2(\mathcal{Q}_E))$ (resp. $\Gamma(T) \in \mathcal{C}(L^2(\mathcal{Q}) \to L^2(\mathcal{Q}_E))$) is similarly defined. For each self-adjoint operator $h$ in $L^2(\mathbb{R}^d)$ (resp. $L^2(\mathbb{R}^{d+1})$), $\Gamma(e^{i\hbar t})$, $t \in \mathbb{R}$, is a one-parameter unitary group on $L^2(\mathcal{Q})$ (resp. $L^2(\mathcal{Q}_E)$). Then there exists a unique self-adjoint operator $d\Gamma(h)$ in $L^2(\mathcal{Q})$ (resp. $L^2(\mathcal{Q}_E)$) such that $\Gamma(e^{i\hbar t}) = e^{itd\Gamma(h)}$ for all $t \in \mathbb{R}$. We set

$$H_{\text{rad}} = d\Gamma(\omega(p)), \quad P_{\mu} = d\Gamma(p_\mu), \quad N = d\Gamma(1_{L^2(\mathbb{R}^d)}) \tag{2.19}$$

in $L^2(\mathcal{Q})$, where $\omega(p) = |p| = \sqrt{p^2}$. We also set

$$\overline{H}_{\text{rad}} = d\Gamma(1 \otimes \omega(p)), \quad \overline{P}_{\mu} = d\Gamma(1 \otimes p_\mu), \quad \overline{N} = d\Gamma(1 \otimes 1_{L^2(\mathbb{R}^d)}), \tag{2.20}$$

where we identify $L^2(\mathbb{R}^{d+1}) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^d)$. $H_{\text{rad}}$ denotes the free field Hamiltonian of $L^2(\mathcal{Q})$, $P_t$ the momentum operator and $N$ the number operator, and $\overline{H}_{\text{rad}}, \overline{P}_t$ and $\overline{N}$ the Euclidean version of $H_{\text{rad}}, P_t$ and $N$, respectively. The spaces $L^2(\mathcal{Q})$ and $L^2(\mathcal{Q}_E)$ are connected by the family of isometries. Let $j_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{d+1})$, $t \in \mathbb{R}$, be the family of isometries such that $\langle j_t f, j_t g \rangle_{L^2(\mathbb{R}^{d+1})} = \langle \hat{f}, e^{-|t-s|\omega} \hat{g} \rangle_{L^2(\mathbb{R}^d)}$, and then $J_t = \Gamma(j_t)$, $t \in \mathbb{R}$, turns to be the family of isometry transforming $L^2(\mathcal{Q})$ to $L^2(\mathcal{Q}_E)$ such that $(J_t \Phi, J_t \Psi)_{L^2(\mathcal{Q}_E)} = (\Phi, e^{-|t-s|H_{\text{rad}}} \Phi)_{L^2(\mathcal{Q})}$. We have the relations:

$$J_t H_{\text{rad}} = \overline{H}_{\text{rad}} J_t, \quad J_t N = \overline{N} J_t, \quad J_t P_t = \overline{P}_t J_t. \tag{2.21}$$

It is known that $\mathcal{F}$, $A_\mu(f)$ and $d\Gamma(h)$ are isomorphic to $L^2(\mathcal{Q})$, $A_\mu(f)$ and $d\Gamma(h(p))$, respectively, where $h$ is the multiplication operator by $h$. That is, there exists a unitary
operator \( U : \mathcal{F} \to L^2(\mathcal{D}) \) such that (1) \( \bar{U} \Omega = 1_{L^2(\mathcal{D})} \), (2) \( \bar{U} \otimes_{\text{sym}} W = L_n^2(\mathcal{D}) \), (3) \( \bar{U} A_\mu (f) \bar{U}^{-1} = A_\mu (f) \), and (4) \( \bar{U} \alpha f \bar{U}^{-1} = \tilde{\alpha} f \). We set
\[
\mathcal{H} = L^2(\mathbb{R}^d) \otimes L^2(\mathcal{D}).
\]
(2.22)

Through the unitary operator \( U = 1 \otimes \bar{U} : L^2(\mathbb{R}^d) \otimes \mathcal{F} \to \mathcal{H} \) the SRPF Hamiltonian is defined as an operator on \( \mathcal{H} \). Let
\[
\lambda = (\hat{\phi}/\sqrt{\omega})^\vee,
\]
(2.23)

where \( \hat{f} \) denotes the inverse Fourier transform of \( f \) in \( L^2(\mathbb{R}^d) \). Set \( A_\mu (\lambda(-x)) = A(\bigoplus_{\nu=1}^d \delta_{\nu\mu} \lambda(-x)) \). Then the quantized radiation field with cutoff function \( \varphi \) is defined by \( A_\mu = \int_{\mathbb{R}^d} A_\mu (\lambda(x)) \text{d}x \). Then \( A \) is a self-adjoint operator in \( \mathcal{H} \) under the identification: \( \mathcal{H} \cong \int_{\mathbb{R}^d} L^2(\mathcal{D}) \text{d}x \). Let \( L^2(\mathcal{D}) \) be the finite particle subspace of \( L^2(\mathcal{D}) \), i.e.,
\[
L^2_{\text{fin}}(\mathcal{D}) = \text{Linear hull of } \left\{ \prod_{j=1}^n A(f_j) : \| f_j \|_{\mathcal{H}}, j = 1, ..., n, n \geq 1 \right\}.
\]
(2.24)

Then the Friedrichs extension of \( \frac{1}{2}(p - \alpha A)^2 |_{C_0^\infty(\mathbb{R}^d) \otimes L^2_{\text{fin}}(\mathcal{D})} \) is denoted by \( h_A \).

**Definition 2.4 (Definition of \( H_F \))** Suppose Assumption 2.1. The SRPF Hamiltonian in the function space \( \mathcal{H} \) is defined by
\[
H_F = T_{\text{kin}} + V + H_{\text{rad}},
\]
(2.25)
\[
T_{\text{kin}} = (2h_A + m^2)^{1/2} - m
\]
(2.26)

with the domain \( \text{D}(H_F) = \text{D}(T_{\text{kin}}) \cap \text{D}(V) \cap \text{D}(H_{\text{rad}}) \).

We investigate \( H_F \) instead of (2.11) in what follows.

### 3 Feynman-Kac type formula

#### 3.1 Markov properties

Let \( \mathcal{O} \subset \mathbb{R} \) and we set
\[
U_\mathcal{O} = \text{L.H.}\{ f \in L^2_{\mathbb{R}}(\mathbb{R}^{d+1}) | f \in \text{Ran} \hat{j}_t \text{ with some } t \in \mathcal{O} \}
\]
and define the sub-$\sigma$-field $\Sigma_O$ by the minimal $\sigma$-field generated by $A_E(f), f \in U_O$, i.e., $\Sigma_O = \sigma(A_E(f)|f \in U_O)$. We also set $\Sigma_{\{s\}} = \Sigma_s$. Let $\varepsilon_O : L^2_\mathbb{R}(\mathbb{R}^{d+1}) \to U_O$ be the projection and the second quantization $\Gamma(\varepsilon_O) : L^2(\mathcal{O}) \to L^2(\mathcal{O}_E)$ is denoted by $E_O$.

Hence $E_O L^2(\mathcal{O})$ is the set of $\Sigma_O$-measurable functions in $L^2(\mathcal{O}_E)$. Moreover we set $E_s = J_s \Gamma_s^*$. Then $E_s = E_{\{s\}}$ follows. Let $E_{\mu_E} [\Phi | \Sigma_O]$ be the conditional expectation of $\Phi \in L^2(\mathcal{O}_E)$ with respect to $\Sigma_O$, i.e., By the Jensen inequality $\rho = E_{\mu_E} [\Phi | \Sigma_O]$ is the unique $L^2$-function such that it is $\Sigma_O$-measurable and $E_{\mu_E} [\Psi \Phi] = E_{\mu_E} [\Psi \rho]$ for all $\Sigma_O$-measurable function $\Psi$.

**Lemma 3.1** Let $\Phi \in L^2(\mathcal{O}_E)$. Then $E_O \Phi = E_{\mu_E} [\Phi | \Sigma_O]$.

**Proof:** We see that $\rho = E_O \Phi$ is measurable with respect to $\Sigma_O$ and $E_{\mu_E} [\Psi \Phi] = (\Psi, E_O \Phi) = (\Psi, \Phi) = E_{\mu_E} [\Psi \rho]$ for all $\Sigma_O$-measurable function $\Psi$. Thus the lemma follows. $\square$

The property below is known as Markov property [Sim74]: let $a \leq b \leq t \leq c \leq d$, then $E_{[a,b]} E_t E_{[c,d]} = E_{[a,b]} E_{[c,d]}$ follows. From this property we can see the corollary below:

**Corollary 3.2** It follows that $E_{\mu_E} [\Phi | \Sigma_{(-\infty,s]}] = E_{\mu_E} [\Phi | \Sigma_s]$ for all $\Sigma_{[s,\infty)}$-measurable function $\Phi$.

**Proof:** We note that $E_{(-\infty,s]} E_{[s,\infty)} \Phi = E_{(-\infty,s]} E_s E_{[s,\infty)} \Phi = E_s E_{[s,\infty)} \Phi$ by the Markov property. Then the lemma follows from Lemma 3.1 and $E_s = E_{\{s\}}$. $\square$

### 3.2 Euclidean groups

We introduce the second quantization of Euclidean group $\{u_t, r\}$ on $L^2(\mathbb{R}^{d+1})$, where the time shift operator $u_t$ is defined by $u_t f(x_0, x) = f(x_0 - t, x)$ and the time reflection $r$ by $rf(x_0, x) = f(-x_0, x)$. The second quantization of $u_t$ and $r$ are denoted by $U_t = \Gamma(u_t) : L^2(\mathcal{O}_E) \to L^2(\mathcal{O}_E)$ and $R = \Gamma(r) : L^2(\mathcal{O}_E) \to L^2(\mathcal{O}_E)$, respectively. Note that $r^* = r$, $rr = r^* r = 1$, $u_t^* = u_{-t}$ and $u_t u_t = 1$ and that $U_t$ and $R$ are unitary. The time shift $u_t$, the time reflection $r$ and isometry $j_t$ satisfy the algebraic relations: $u_t j_s = j_{s+t}$ and $r j_s = j_{-s} r$. From these relations it follows that $U_t J_s = J_{s+t}$ and $RU_s = U_{-s} R$ as operators.
3.3 Feynman-Kac type formula and time-shift

Let $(\Omega, \mathcal{F}, P^x)$ be a probability space, and $(B_t)_{t \in \mathbb{R}}$ the $d$-dimensional Brownian motion on whole real line $\mathbb{R}$ on $(\Omega, \mathcal{F}, P^x)$ starting from $x$ at $t = 0$. See Appendix A for the detail of the Brownian motion on whole real line $\mathbb{R}$. We also introduce a subordinator $(T_t)_{t \geq 0}$ on a probability space $(\Omega_\nu, \mathcal{F}_\nu, \nu)$ such that

$$E_\nu^0 \left[ e^{-uT_t} \right] = e^{-t(\sqrt{2u+m^2}-m)}, \quad t \geq 0, \quad u \geq 0. \quad (3.1)$$

The subordinator $(T_t)_{t \geq 0}$ is one-dimensional Lévy process and indeed given by $T_t = \inf \{ s > 0 | B_s^1 + ms = t \}$, where $(B_t^1)_{t \geq 0}$ denotes the one-dimensional Brownian motion. Path $[0, \infty) \ni t \mapsto T_t \in [0, \infty)$ is nondecreasing and right continuous, and the left limit exists almost surely in $\nu$. The distribution $\mu_t$ of $T_t$, $t \geq 0$, on $\mathbb{R}$ is given by

$$\mu_t(s) = \frac{t}{\sqrt{2\pi}} e^{tm}s^{-3/2} \exp \left( -\frac{1}{2} \frac{t^2}{s} + m^2s \right) 1_{[0,\infty)}(s) \quad (3.2)$$

and thus $E_\nu^x[f(T_t)] = \int_{\mathbb{R}} f(s+x) \mu_t(s) ds$. Notice that $E_\nu^0[T_t] < \infty$ if and only if $m > 0$. We need to define a self-adjoint extension of $H_F$, which is constructed through a functional integration. The idea is a combination of Proposition [3.4] below and a subordinator $(T_t)_{t \geq 0}$. In quantum mechanics, the path integral representation of the heat semigroup generated by the semi-relativistic Schrödinger operator $\sqrt{(p-a)^2 + m^2} - m + V$ is given by

$$\left( f, e^{-t(\sqrt{(p-a)^2 + m^2} - m + V)} g \right) = \int_{\mathbb{R}^d} dx E_{p,\nu}^{x,0} \left[ f(B_{T_t}) g(B_{T_t}) e^{-\int_0^{T_t} V(B_s) ds} e^{-i\int_0^{T_t} a(B_s) \circ dB_s} \right]. \quad (3.3)$$

Here $\int_0^{T_t} \circ dB_s$ is defined by $\int_0^T \circ dB_s$ evaluated at $T = T_t$. Although the SRPF Hamiltonian is of a similar form of $\sqrt{(p-a)^2 + m^2} - m + V$, it is not straightforward to construct the Feynman-Kac type formula of $e^{-tH_F}$. The Feynman-Kac type formula for the case of $\alpha = 0$ is however immediately given by

$$\left( F, e^{-t(H_F+H_{\text{rad}})} G \right)_\mathcal{H} = \int_{\mathbb{R}^d} dx E_{p,\nu}^{x,0} \left[ \left( J_0 F(B_{T_t}), J_t G(B_{T_t}) \right)_{L^2(\mathcal{D})} e^{-\int_0^{T_t} V(B_s) ds} \right]. \quad (3.4)$$

We shall extend this formula for an arbitrary value of $\alpha$. The self-adjoint operator $h_A$ is defined by the Friedrichs extension. In general self-adjoint extensions are not
unique, and it is also not trivial to signify an operator core of \( h_A \). As is shown in the proposition below we can however show the essential self-adjointness of \( h_A \) by means of functional integral approach under some conditions. Let \( C^\infty(N) = \cap_{n=1}^\infty D(N^n) \), where we recall that \( N \) denotes the number operator. We define the \( L^2(\mathbb{R}^d) \)-valued stochastic integral
\[
\int_0^t \lambda(\cdot - B_s)dB_s^\mu
\]
in \( L^2(\mathbb{R}^d \times \Omega_D,dx \otimes dP^x) \) with \( t_j = t_j/2^n \).

**Proposition 3.3** Let \( h \) be closed and the generator of a contraction semigroup on a Banach space. Let \( D \) be dense and \( D \subset D(h) \), so that \( e^{-th}D \subset D \). Then \( D \) is a core of \( h \), i.e., \( h|_D = h \).

**Proof:** See [RS75, Theorem X.49]. \( \square \)

To prove an essential self-adjoint of \( h_A \) we apply Proposition 3.3.

**Proposition 3.4** Suppose Assumptions 2.1 and 2.2. Then \( h_A \) is essentially self-adjoint on \( \mathcal{D}(p^2) \cap C^\infty(N) \), and it follows that
\[
(F,e^{-t\lambda_A}G) = \int_{\mathbb{R}^d} dx E^x[F(B_0),e^{-i\alpha A(K^{[0,t]}\big|0,\Omega_D)G(B_t)}], \quad (3.5)
\]
where \( K^{[0,t]} = \bigoplus_{\mu=1}^d \int_0^t \lambda(\cdot - B_s)dB_s^\mu \).

**Proof:** See Appendix B. \( \square \)

The path integral representation of the semigroup generated by the semi-relativistic Schrödinger operator can be constructed by a combination of the \( d \)-dimensional Brownian motion \((B_t)_{t \geq 0}\) and a subordinator \((T_t)_{t \geq 0}\). In a similar manner we can see the lemma below:

**Lemma 3.5** Suppose Assumptions 2.1 and 2.2. Then
\[
(F,e^{-t\lambda_{kin}}G) = \int_{\mathbb{R}^d} dx E^{x,0}_{P^x}[F(B_{T_0}),e^{-i\alpha A(K^{[0,T]}\big|0,T)\big|0,\Omega_D)G(B_{T_t})], \quad (3.6)
\]
where \( K^{[0,T]} = \bigoplus_{\mu=1}^d \int_0^T \lambda(\cdot - B_s)dB_s^\mu \) is defined by \( \bigoplus_{\mu=1}^d \int_0^T \lambda(\cdot - B_s)dB_s^\mu \) evaluated at \( T = T_t \).
Proof: Since \((F, e^{-tT_{\text{kin}}}G) = E_0^\nu \left[ (\Psi, e^{-th\lambda \Phi}) \right] \), by Proposition 3.4 and (3.1), we see that \((F, e^{-tT_{\text{kin}}}G) = E_0^\nu \left[ \int_{\mathbb{R}^d} dx \mathbb{E}_p \left[ (\Psi(B_{T_0}), e^{-i\alpha A(K[0,t])\Phi(B_{T_1}))} \right] \right] \). We can exchange \(\int dv\) and \(\int dx\) by Fubini’s lemma. Then the lemma follows.

By Lemma 3.5 we see that \(D(T_{\text{kin}}) \cap D(H_{\text{rad}})\) is dense. Then we can define the quadratic form sum \(T_{\text{kin}} + H_{\text{rad}}\). Let \(V\) be bounded. Then by the Trotter-Kato product formula [KM78] we have
\[
e^{-t(T_{\text{kin}} + H_{\text{rad}} + V)} = s-\lim_{n \to \infty} \left( e^{-\frac{t}{2^n}T_{\text{kin}}} e^{-\frac{t}{2^n}H_{\text{rad}}} e^{-\frac{t}{2^n}V} \right)^{2^n}, \quad t \geq 0. \tag{3.7}
\]
Using this formula we construct a Feynman-Kac type formula of \(e^{-t(T_{\text{kin}} + H_{\text{rad}} + V)}\) for a bounded \(V\). We define an \(L^2(\mathbb{R}^{d+1})\)-valued stochastic integral \(\int_S^T j_s \lambda(\cdot - B_s) dB^\mu_s\) by the strong limit:
\[
\int_S^T j_s \lambda(\cdot - B_s) dB^\mu_s = s-\lim_{n \to \infty} \sum_{j=1}^{2^n} \int_{S + \Delta_j}^{S + \Delta_{j-1}} j_s \lambda(\cdot - B_s) dB^\mu_s \tag{3.8}
\]
in \(L^2(\mathbb{R}^{d+1} \times \Omega_p, dx \otimes dP^x)\), where \(\Delta_j = (T - S) \frac{j}{2^n}\). We give a remark on notation. Notation \(\lambda(\cdot - B_r)\) denotes the function \(\lambda = \lambda(\cdot)\) shifted by \(B_r\). We denotes the image of \(\lambda(\cdot - B_r)\) by the isometry \(j_t\) by \(j_t \lambda(\cdot - B_r)\). More precisely
\[
j_t \lambda(\cdot - B_s)(k_0, k) = e^{-itk_0} \frac{\sqrt{\omega(k)}}{\sqrt{\omega(k)^2 + |k_0|^2}} \hat{\lambda}(k) e^{-iB_s}, \quad (k_0, k) \in \mathbb{R} \times \mathbb{R}^d.
\]
Let us recall the family of projections: \(E_t = J_t J_t^*, t \in \mathbb{R}\).

**Lemma 3.6** Suppose Assumptions 2.1, 2.2 and that \(V \in C_0^\infty(\mathbb{R}^d)\). Then
\[
(F, e^{-\frac{t}{2^n}T_{\text{kin}}} e^{-\frac{t}{2^n}H_{\text{rad}}} e^{-\frac{t}{2^n}V}) \left( G \right) = \int_{\mathbb{R}^d} dx \mathbb{E}_{p^{x,0}} \left[ (J_0 F(B_{T_0}), e^{-i\alpha A(I_n[0,t]) J_t G(B_{T_1}))} e^{-\sum_{j=0}^{2^n} \frac{t}{2^n} V(B_{T_j})} \right], \tag{3.9}
\]
where
\[
I_n[0,t] = \bigoplus_{\mu=1}^d \sum_{j=1}^{2^n} \int_{T_{j-1}}^{T_j} j_{t_j} \lambda(\cdot - B_s) dB^\mu_s \tag{3.10}
\]
with \(t_j = t/2^n\), and \(\int_{T_{j-1}}^{T_j} j_{t_j} \lambda(\cdot - B_s) dB^\mu_s\) denotes \(L^2(\mathbb{R}^{d+1})\)-valued stochastic integral \(\int_{T_{j-1}}^{T_j} \lambda(\cdot - B_s) dB^\mu_s\) evaluated at \(T = T_{j-1}\) and \(S = T_j\).
Proof: By the formula $J_t^*J_s = e^{-|t-s|H_{rad}}$, we have

$$
(F, \left( e^{-\frac{t}{\kappa}H_{kin}}e^{-\frac{t}{\kappa}H_{rad}}e^{-\frac{t}{\kappa}V} \right)^2 G) = \int_{\mathbb{R}^d} dx \mathbb{E}_P^{x,0} \left[ U_n e^{-\sum_{j=0}^{2n} \frac{t}{\kappa}V(B_{t_j})} \right],
$$

where

$$
U_n = \left( J_0 F(B_{T_0}), \prod_{j=1}^{2n} \left( J_{t_{j-1}} \frac{e^{-i\alpha A}}{\sqrt{n}} \left( \sum_{\mu=1}^{d} J_{T_{t_j-1}} J_{t_{j-1}} \right) J_{t_{j-1}} \right) J_{t_{j-1}} G(B_{T}) \right),
$$

and we see that

$$
J_{t_{j-1}} e^{-i\alpha A} \left( \prod_{\mu=1}^{d} J_{T_{t_j-1}} J_{t_{j-1}} \right) J_{t_{j-1}} = E_{t_{j-1}} e^{-i\alpha \Lambda} \left( \prod_{\mu=1}^{d} J_{T_{t_j-1}} J_{t_{j-1}} \right) E_{t_{j-1}}
$$

by the definition of $J_t$ and $E_t$. Then by the Markov property of $E_{t'} s$ can be removed in (3.11) and thus the lemma follows.

$(I_n[0, t])_{t \geq 0}$ can be regarded as an $\mathcal{F}$-valued stochastic process on the product probability space $(\Omega_\mathcal{P} \times \Omega_\nu, \mathcal{B}_\mathcal{P} \times \mathcal{B}_\nu, P^x \otimes \nu)$. By the Itô isometry we have

$$
\mathbb{E}_P^x \left[ \|I_n[0, t]\|_{\mathcal{F}}^2 \right] = d \sum_{j=1}^{2n} \mathbb{E}_P^x \left[ \int_{T_{t_{j-1}}}^{T_{t_j}} \|\tilde{b}_{t_{j-1}} \lambda(\cdot - B_s)\|^2_{L^2(\mathbb{R}^{d+1})} ds \right] = dT_t \|\tilde{\varphi}/\sqrt{\omega}\|^2.
$$

We will show that $I_n[0, t]$ has a limit as $n \to \infty$ in some sense. Let $\mathcal{N}_0 \in \mathcal{B}_\nu$ be a null set, i.e., $\nu(\mathcal{N}_0) = 0$, such that for arbitrary $w \in \Omega_\nu \setminus \mathcal{N}_0$, the path $t \mapsto T_t(w)$ is nondecreasing and right-continuous, and has the left-limit.

**Lemma 3.7** For each $w \in \Omega_\nu \setminus \mathcal{N}_0$ the sequence $(I_n[0, t])_n$ strongly converges in $L^2(\Omega_\mathcal{P}, P^x) \otimes \mathcal{F}$ as $n \to \infty$, i.e., there exists an $I[0, t] \in L^2(\Omega_\mathcal{P}, P^x) \otimes \mathcal{F}$ such that

$$
\lim_{n \to \infty} \mathbb{E}_P^x \left[ \|I_n[0, t] - I[0, t]\|_{\mathcal{F}}^2 \right] = 0.
$$

**Proof:** Set $I_n = I_n[0, t]$. It is enough to show that $(I_n)_n$ is a Cauchy sequence in $L^2(\Omega_\mathcal{P}, P^x) \otimes \mathcal{F}$. We have $I_{n+1} - I_n = \bigoplus_{\mu=1}^{d} \sum_{m=1}^{2n} \int_{T_{t_{2m-1}}}^{T_{t_{2m}}} (j_{t_{2m-1}} - j_{t_{2m-2}}) \lambda(\cdot - B_s) dB^\mu_s$, where $t_j = t_j/2^{n+1}$. Thus

$$
\mathbb{E}_P^x \left[ \|I_{n+1} - I_n\|_{\mathcal{F}}^2 \right] = d \sum_{m=1}^{2n} \mathbb{E}_P^x \left[ \int_{T_{t_{2m-1}}}^{T_{t_{2m}}} \|j_{t_{2m-1}} - j_{t_{2m-2}}\|_{L^2(\mathbb{R}^{d+1})}^2 ds \right].
$$

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by the Itô isometry (3.12). Notice that 
\[ \| (j_t - j_s) f \|_2^2 = 2(\hat f, (1 - e^{-i|t-s|}\hat f)). \] Thus
\[ \mathbb{E}_\mathbb{P} \left[ \| I_{n+1} - I_n \|_{\mathcal{F}}^2 \right] \leq d \sum_{m=1}^{2^n} 2(\hat \varphi/\sqrt{\omega}, (1 - e^{-\frac{1}{2^n} - |t-s|\omega}) \hat \varphi/\sqrt{\omega}) (T_{t_2m} - T_{t_2m-1}). \]
Since \( T_t = T_t(w) \) is not decreasing in \( t \) for \( w \in \Omega_\nu \setminus \mathcal{N}_\nu, \sum_{m=1}^{2^n} (T_{t_2m} - T_{t_2m-1}) \leq T_t \) follows. Thus \( \mathbb{E}_\mathbb{P} \left[ \| I_{n+1} - I_n \|_{\mathcal{F}}^2 \right] \leq d \mathbb{E}_\mathbb{P} \left[ \frac{1}{\sqrt{2^n}} \right] \mathbb{E} \left[ \| \hat \varphi/\sqrt{\omega} \| \right]^2. \] Hence we have
\[ \mathbb{E}_\mathbb{P} \left[ \| I_m - I_n \|_{\mathcal{F}}^2 \right] \leq \left( \sqrt{d \mathbb{E}_\mathbb{P} \left[ \| \hat \varphi/\sqrt{\omega} \| \right]} \sum_{j=n+1}^{m} \left( \frac{1}{\sqrt{2^n}} \right)^j \right)^2 \]
for \( m > n \). The right-hand side above converges to zero as \( n, m \to \infty \). Then the sequence \( I_n \) is a Cauchy sequence for almost surely \( \nu \). Then the lemma follows. \( \square \)

**Remark 3.8** Integral \( I[0, t] \) is informally written as
\[ I[0, t] = \bigoplus_{\mu = 1}^{d} \int_{0}^{T_t} j_{T_s}^* \lambda(\cdot - B_s) d\mathbb{P}^\mu. \] (3.13)
Here \( T_s^* = \inf\{ t | T_t = s \} \) is the first hitting time of \( (T_t)_{t \geq 0} \) at \( s \).

In a similar way to \( I[0, t] \) we define \( I[s, t] \) by the limit of
\[ I_n[s, t] = \bigoplus_{\mu = 1}^{d} \sum_{j=1}^{2^n} \sum_{j=1}^{2^n} j_{s+(t-s)j-1} \lambda(\cdot - B_{t_j}) d\mathbb{P}^\mu \] (3.14)
with \( (t-s)_j = (t-s)j/2^n \) in \( L^2(\Omega_\mathbb{P}, \mathbb{P}^\otimes \mathcal{F}) \). Moreover it can be straightforwardly seen that \( I[s, t] \) coincides with the limit of subdivisions
\[ I_n[s, t] = \bigoplus_{\mu = 1}^{d} \sum_{j=1}^{a2^n} \sum_{j=1}^{a2^n} j_{s+(t-s)j-1} \lambda(\cdot - B_{t_j}) d\mathbb{P}^\mu \] (3.15)
for arbitrary \( a \in \mathbb{N} \). We show some properties of \( I[a, b] \) in Appendix B.

**Lemma 3.9** Suppose Assumptions \( \mathcal{L} \) and \( \mathcal{K} \). Then
\[ (F, e^{-t(T_{kin} + H_{rad})} G)_{\mathcal{F}} = \int_{\mathbb{R}^d} \mathbb{E}_{\mathbb{P}^0} \left[ J_{[0, t]} F(B_{t_0}), e^{-i\alpha E(I[0, t])} J_{t} G(B_{t_1}) \right]. \] (3.16)
Suppose Assumptions 2.1 and 2.2. Let Corollary 3.10

\[ F = \text{sgn}((T_{\text{kin}} + H_{\text{rad}})) \]

Proof: Since \( |J_t G| \leq |J_t| G \), it is straightforward to see that

\[ |(F, e^{-t(T_{\text{kin}} + H_{\text{rad}})} G)| \leq \int_{\mathbb{R}^d} dx E_{\mathcal{P}_\nu}^x \left[ \|F(BT_0)|, e^{-tH_{\text{rad}}} |G(BT_0)| \right] \]

\[ = \left( |F|, e^{-t(\sqrt{p^2 + m^2} - m + H_{\text{rad}})} |G| \right). \]

Then (1) follows. (2) is similarly proven.

We introduce a class of potentials.

Definition 3.11 V is in \( \mathcal{V}_{\text{rel}} \) if and only if V is relatively bounded with respect to \( \sqrt{p^2 + m^2} \) with a relative bound strictly smaller than one.

Lemma 3.12 Suppose Assumptions 2.1 and 2.2. Let \( V \in \mathcal{V}_{\text{rel}} \). Then V is also relatively form bounded (resp. bounded) with respect to \( T_{\text{kin}} + H_{\text{rad}} \) with a relative bound smaller than a.

Proof: Let \( \text{sgn}F(x) = \frac{F(x)}{\|F(x)\|_{L^2(\mathcal{D})}} \) for \( \|F(x)\|_{L^2(\mathcal{D})} \neq 0 \) and = 0 for \( \|F(x)\|_{L^2(\mathcal{D})} = 0 \). Let \( z > 0 \) be sufficiently large. Let \( \psi \in C_0^\infty(\mathbb{R}^d) \) and \( \psi(x) \geq 0 \). Substituting the vector \( F = \text{sgn}((T_{\text{kin}} + H_{\text{rad}} + z)^{-1/2} G) \cdot \psi \in \mathcal{H} \) in the inequality

\[ |(F, (T_{\text{kin}} + H_{\text{rad}} + z)^{-1/2} G)_0| \leq \left( \|F\|, (\sqrt{p^2 + m^2} - m + z)^{-1/2} \|G\| \right)_{L^2(\mathcal{D})} \]

derived from Corollary 3.10 (2), we see that

\[ (\psi, ((T_{\text{kin}} + H_{\text{rad}} + z)^{-1/2} G)(\cdot))_{L^2(\mathcal{D})} \leq (\psi, (\sqrt{p^2 + m^2} - m + z)^{-1/2} \|G(\cdot)\|_{L^2(\mathcal{D})}). \]

Thus \( \|((T_{\text{kin}} + H_{\text{rad}} + z)^{-1/2} G)(x)\|_{L^2(\mathcal{D})} \leq (\sqrt{p^2 + m^2} - m - z)^{-1/2} \|G(x)\|_{L^2(\mathcal{D})} \) follows for almost every \( x \in \mathbb{R}^d \), and

\[ |||V|||^{1/2}(T_{\text{kin}} + H_{\text{rad}} - z)^{-1/2} G|_{\mathcal{H}} \leq |||V|||^{1/2}(\sqrt{p^2 + m^2} - m - z)^{-1/2} G|_{\mathcal{H}} \]

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are derived. Then $V$ is also form bounded with respect to $T_{\text{kin}} + H_{\text{rad}}$.

$$\| V \| (T_{\text{kin}} + H_{\text{rad}} - z)^{-1} G \| \leq \| V \| \left( \sqrt{p^2 + m^2} - m - z \right)^{-1} G \|$$

is similarly derived. 

If $V \in L^1_{\text{loc}}(\mathbb{R}^d)$, then $D(T_{\text{kin}}) \cap D(H_{\text{rad}}) \cap D(V)$ is dense. Let $V = V_+ - V_-$, where $V_+ = \max\{V, 0\}$ is the positive part of $V$ and $V_- = \max\{-V, 0\}$ the negative part. We introduce a class of potentials:

**Definition 3.13** $V = V_+ - V_-$ is in $\mathcal{V}_{\text{qf}}$ if and only if $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $V_-$ relatively form bounded with respect to $(p^2 + m^2)^{1/2}$ with relative bound strictly smaller than one.

Let $V = V_+ - V_- \in \mathcal{V}_{\text{qf}}$. Define the quadratic form $t$ on $\mathcal{H}$ by

$$t(F, G) = (T_{\text{kin}}^{1/2} F, T_{\text{kin}}^{1/2} G) + (H_{\text{rad}}^{1/2} F, H_{\text{rad}}^{1/2} G) + (V_+^{1/2} F, V_+^{1/2} G) - (V_-^{1/2} F, V_-^{1/2} G) \quad (3.17)$$

with the form domain $Q(t) = D(T_{\text{kin}}^{1/2}) \cap D(H_{\text{rad}}^{1/2}) \cap D(V_+^{1/2})$. By Lemma 3.12 $t$ is semibounded and closed.

**Definition 3.14 (Definition of $H_{\text{qf}}$)** Suppose Assumptions 2.1 and 2.2. Let $V \in \mathcal{V}_{\text{qf}}$. Then the self-adjoint operator associated with the quadratic form $t$ is denoted by $H_{\text{qf}}$ and written as

$$H_{\text{qf}} = T_{\text{kin}} + H_{\text{rad}} + V_+ - V_- \quad (3.18)$$

Note that the form domain of $H_{\text{qf}}$ coincides with $Q(t)$.

We now construct a Feynman-Kac type formula of $e^{-tH_{\text{qf}}}$. 

**Theorem 3.15** Suppose Assumptions 2.1 and 2.2. Let $V \in \mathcal{V}_{\text{qf}}$. Then

$$(F, e^{-tH_{\text{qf}}} G)_{\mathcal{H}} = \int_{\mathbb{R}^d} \mathbb{E}_{\mathbb{P}_{\nu}} \left[ (J_0 F(B_{T_0}), e^{-i\alpha A_E(I_{[0,t]}]} J_t G(B_{T_t})) e^{-\int_0^t V(B_{T_s}) ds} \right]. \quad (3.19)$$

**Proof:** By the Trotter product formula (3.9) we have

$$ (F, e^{-tH_{\text{qf}}} G) = \lim_{n \to \infty} \left( F, \left( e^{-\frac{t}{n} T_{\text{kin}}} e^{-\frac{t}{n} H_{\text{rad}}} e^{-\frac{t}{n} V} \right)^2 G \right) $$

$$ = \lim_{n \to \infty} \int_{\mathbb{R}^d} \mathbb{E}_{\mathbb{P}_{\nu}} \left[ (J_0 F(B_{T_0}), e^{-i\alpha A_E(I_{[0,t]}]} J_t G(B_{T_t})) e^{-\sum_{j=0}^{2^n} \frac{t}{n} V(B_{T_{ij}})} \right].$$
Suppose that $V$ is in $C^0_0(\mathbb{R}^d)$. By Lemma 3.6 and the dominated convergence theorem we can show that the right-hand side above converges to that of (3.19). For general $V$, by monotone convergence theorems for both integrals and quadratic forms, we can establish (3.19). See [Sim05, Theorem 6.2] and [LHB11, Theorem 3.31].

We can shift the time in the Feynman-Kac type formula. We see it in the corollary below.

**Corollary 3.16** Suppose Assumptions 2.1 and 2.2. Let $V \in \mathcal{V}_0$. Then

\[
(F, e^{-2t\mathbf{H}_{Vt}} G)_{\mathcal{M}} = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{\mathcal{P}_{\mathcal{M}}} \left[ (J_{-t}F(B_{-T}), e^{-i\alpha A_{E}(I[-t,0]+I[0,t])} J_t G(B_{T_t})) e^{-j_0^t V(B_{-T_s}) ds} - j_0^t V(B_{T_s}) ds \right],
\]

where $I[-t,0]$ is defined by

\[
I[-t,0] = \bigoplus_{\mu=1}^d \lim_{n \to \infty} \sum_{j=1}^{2^n} \int_{-T_s-(t+1-t)} \int_{(t_j-t)} \lambda(\cdot - B_s) dB^\mu_s. \tag{3.21}
\]

**Proof:** This is proven by means of the shift $U_t$ in the field and the facts that $T_s - T_t = T_{s-t}$ in law. By Theorem 3.15 we have

\[
(F, e^{-2t\mathbf{H}_{Vt}} G)_{\mathcal{M}} = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{\mathcal{P}_{\mathcal{M}}} \left[ (J_{0}F(B_{T_0}), e^{-i\alpha A_{E}(I[0,2t])} J_t G(B_{T_t})) e^{-j_0^t V(B_{T_s}) ds} \right]
\]

and

\[
= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{\mathcal{P}_{\mathcal{M}}} \left[ (J_{-t}F(B_{-T}), e^{-i\alpha A_{E}(I[0,2t])} U_t J_t G(B_{T_2})) e^{-j_0^t V(B_{T_s}) ds} \right].
\]

By the shift of the Brownian motion, $B_t \to B_{t-T_t}$, we have

\[
= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{\mathcal{P}_{\mathcal{M}}} \left[ (J_{-t}F(B_{-T}), e^{-i\alpha A_{E}(S)} J_t G(B_{T_2-T_1})) e^{-j_0^t V(B_{T_s-T_1}) ds} \right],
\]

where $S = \lim_{n \to \infty} \bigoplus_{\mu=1}^d \sum_{j=1}^{2^n} \int_{T_{j-1}-T_1} \int_{T_{j-1}-T_1} \lambda(\cdot - B_s) dB^\mu_s$ and, since $T_s - T_t = T_{s-t}$ for $s \geq t$ in law, we can check that

\[
\int_0^{2t} V(B_{T_s-T_t}) ds = \int_0^t V(B_{-T_{t-s}}) ds + \int_t^{2t} V(B_{T_{s-t}}) ds = \int_0^0 V(B_{-T_s}) ds + \int_0^t V(B_{T_s}) ds.
\]
Furthermore we have
\[
\sum_{j=1}^{2-2^n} \int_{T_j - T_{t}} j_{t} - j_{t-1} \lambda (-B_s) dB_s^\mu \\
= \sum_{j=1}^{2^n} \int_{T_j - (t_j - t)} j_{t} - j_{t-1} \lambda (-B_s) dB_s^\mu + \sum_{j=2^{n+1}}^{2^n} \int_{(T_{j-1} - t)} j_{t} - j_{t-1} \lambda (-B_s) dB_s^\mu.
\]

Then the theorem follows. □

**Remark 3.17** For the notational convenience we denote \(I[-t,0] + I[0,t] = \bigoplus_{\mu=1}^{d} \int_{-t}^{T_t} i_{t-1} \lambda (-B_s) dB_s^\mu, \) and \(\int_{-t}^{T_t} V(B_{-t,s}) \text{d}s + \int_{t}^{0} V(B_{t,s}) \text{d}s \) by \(\int_{-t}^{T_t} V(B_{t,s}) \text{d}s \)

For later use we construct a functional integral representation of the Green function of the form:
\[
(F_0, e^{-(t_1-t_0)H_{q,t}} F_1 e^{-(t_2-t_1)H_{q,t}} \cdots F_{n-1} e^{-(t_n-t_{n-1})H_{q,t}} F_n)_{\mathcal{H}}. \tag{3.22}
\]

**Corollary 3.18** Suppose Assumptions 2.1 and 2.2. Let \(V \in \mathcal{Y}_{q,t}. \) Let \(-\infty < t_0 < t_1 < \cdots < t_n < \infty. \) For \(F_0, F_n \in \mathcal{H} \) and \(F_j = F_j(x,A(\rho)) \in L^\infty(\mathbb{R}^d) \otimes L^\infty(\mathcal{D}), \) it follows that
\[
(F_0, e^{-(t_1-t_0)H_{q,t}} F_1 e^{-(t_2-t_1)H_{q,t}} \cdots F_{n-1} e^{-(t_n-t_{n-1})H_{q,t}} F_n)_{\mathcal{H}} = \int_{\mathbb{R}^d} \text{d}x E_{\mathcal{D}} \left[ \left( J_0 F_0(B_{T_0}), \prod_{j=1}^{n-1} \tilde{F}_j \right) e^{-i\alpha E[I(t_0,t_n)]} J_{t_0} F_n(B_{T_{T_n}}) \right] e^{-\int_{t_0}^{T_{T_n}} V(B_{T_s}) \text{d}s}. \tag{3.23}
\]

Here \(\tilde{F}_j = F_j(B_{T_{T_j}}, A_{E}(\rho_j(\rho_j))), \) \(j = 1, \ldots, n-1, \) and \(T_s = -T_{-s} \) for \(s < 0. \) In particular
\[
(f \otimes 1, e^{-(t_1-t_0)H_{q,t}} \prod_{j=1}^{n-1} 1_{A_j}(B_{T_{T_j}}) e^{-(t_n-t_{n-1})H_{q,t}} g)_{\mathcal{H}} = \int_{\mathbb{R}^d} \text{d}x E_{\mathcal{D}} \left[ f(B_{T_0}) \prod_{j=1}^{n-1} 1_{A_j}(B_{T_{T_j}}) g(B_{T_{T_n}}) \prod_{j=1}^{n-1} e^{-i\alpha E[I(t_0,t_n)]} \right] e^{-\int_{t_0}^{T_{T_n}} V(B_{T_s}) \text{d}s}. \tag{3.24}
\]

**Proof:** Note that \(F_j, j = 1, \ldots, n-1, \) can be regarded as bounded operators. Thus the corollary can be proven in a similar manner to Theorem 3.13 and Corollary 3.16 □
4 Self-adjointness

4.1 Burkholder type inequalities

In this section by using the functional integral representation derived in Theorem 3.15 we show the essential self-adjointness of \( H_{qf} \) for arbitrary values of coupling constants. To prove this we find an invariant domain \( D \) so that \( D \subset D(H_{qf}) \) and \( e^{-tH_{qf}}D \subset D \). Then \( H_{qf} \) is essentially self-adjoint on \( D \) by Proposition 3.3. Let \( T \) be a self-adjoint operator. The strategy is to estimate the scalar product \((TF, e^{-tH_{qf}}G)\) as

\[
|\langle TF, e^{-tH_{qf}}G \rangle| \leq c(G,T) \|F\| \text{ for all } F, G \in D(T),
\]

which implies that \( e^{-tH_{qf}}G \in D(T) \) for \( G \in D(T) \).

By the Itô isometry we have

\[
\mathbb{E}^x_\nu \| 1 \otimes \omega(p)^{\alpha/2}I[0,t] \|_2^2 = d \mathbb{E}^x_\nu \left[ \int_0^T \| \omega(p)^{\alpha/2} \lambda(\cdot - B_r) \|_{L^2(\mathbb{R}^d)}^2 dr \right]. \tag{4.1}
\]

In particular

\[
\mathbb{E}^x_\nu \| 1 \otimes \omega(p)^{\alpha/2}I[0,t] \|_2^2 \leq d \mathbb{E}^0_\nu[T_t] \| \omega^{(\alpha-1)/2} \hat{\varphi} \|_{L^2(\mathbb{R}^d)}^2 \tag{4.2}
\]

and the right-hand side above is finite in the case of \( m > 0 \), since \( \mathbb{E}^0_\nu[T_t] < \infty \). We can also estimate \( \mathbb{E}^x_\nu \| 1 \otimes \omega(p)^{\alpha/2}I[0,t] \|_4^4 \).

**Lemma 4.1** Suppose \( m > 0 \). Then the Burkholder type inequalities hold:

\[
\mathbb{E}^x_\nu \| 1 \otimes \omega(p)^{\alpha/2}I[0,t] \|_4^4 \leq C \| \omega^{(\alpha-1)/2} \hat{\varphi} \|_{L^2(\mathbb{R}^d)}^4, \tag{4.3}
\]

where \( C \) is a constant.

**Proof:** It is known that by [Hir00b, Theorem 4.6]

\[
\mathbb{E}^x \left[ \left\| 1 \otimes \omega(p)^{\alpha/2} \int_0^t j_s \lambda(\cdot - B_s) dB^\mu_s \right\|_{L^2(\mathbb{R}^{d+1})}^{2m} \right] \leq \frac{(2m)!}{2^{m}} t^m \| \omega^{(\alpha-1)/2} \hat{\varphi} \|_{L^2(\mathbb{R}^d)}^{2m}, \tag{4.4}
\]

Notice that \( I[0,t] = s \lim_{n \to \infty} \sum_{\mu=1}^{d} \sum_{j=1}^{2^n} a_j^\mu \) with \( a_j^\mu = \int_{T_{ij-1}}^{T_{ij}} j \lambda(\cdot - B_s) dB^\mu_s \in L^2(\mathbb{R}^{d+1}) \), and \( \lambda = \omega(p)^{\alpha/2} \lambda \) and \( \hat{\lambda} = \omega^{(\alpha-1)/2} \hat{\varphi} \). We fix a \( \mu \) and set \( a_j^\mu = a_j \) for simplicity. \( a_j \)
and $a_i$ are independent for $i \neq j$ and then we have

$$
\mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \left\| \sum_{j=1}^{2^n} a_j \right\|_{L^2(\mathbb{R}^{d+1})}^4 \right] 
= \sum_{j,j'} \sum_{i,i'} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \left( \int_{\mathbb{R}^{d+1}} a_j(x) a_{j'}(x) dx \right) \left( \int_{\mathbb{R}^{d+1}} a_i(y) a_{i'}(y) dy \right) \right]
= \sum_{j=1}^{2^n} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \left( \int_{\mathbb{R}^{d+1}} a_j(x)^2 dx \right)^2 \right] + \sum_{j=1}^{2^n} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \int_{\mathbb{R}^{d+1}} a_j(x)^2 dx \right] \sum_{i \neq j} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \int_{\mathbb{R}^{d+1}} a_i(x)^2 dx \right] 
+ \sum_{j=1}^{2^n} \sum_{i \neq j} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \int_{\mathbb{R}^{d+1}} a_j(x) a_i(x) dx \int_{\mathbb{R}^{d+1}} a_j(y) a_i(y) dy \right].
$$

We estimate the first term of the right-hand side above. We have by (4.4)

$$
\sum_{j=1}^{2^n} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \left( \int_{\mathbb{R}^{d+1}} a_j(x)^2 dx \right)^2 \right] = \sum_{j=1}^{2^n} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left\| a_j \right\|^4 \leq 6 \| \varphi \|^4 \sum_{j=1}^{2^n} \mathbb{E}_{P_{\lambda^{0}}}^{0} \left[ T_{\frac{t}{2^n+i}} \right]^2.
$$

By using the distribution (3.2) of $T_t$ and the assumption $m > 0$ we have

$$
\sum_{j=1}^{2^n} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \left( \int_{\mathbb{R}^{d+1}} a_j(x)^2 dx \right)^2 \right] \leq 6 \| \varphi \|^4 \frac{t}{2 \sqrt{2 \pi}} \int_0^{\frac{m}{2^n+i}} \int_0^\infty \sqrt{s} \exp \left( -\frac{1}{2} \left( \frac{t}{s} + m^2 s \right) \right) ds.
$$

The right-hand side converges to

$$
\frac{3t}{\sqrt{2 \pi}} \| \varphi \|^4 \int_0^\infty \sqrt{s} \exp \left( -\frac{1}{2} m^2 s \right) ds
$$
as $n \to \infty$. The second term is estimated as

$$
\sum_{j=1}^{2^n} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \int_{\mathbb{R}^{d+1}} a_j(x)^2 dx \right] \sum_{i \neq j} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \int_{\mathbb{R}^{d+1}} a_j(x)^2 dx \right] \leq \left( \sum_{j=1}^{2^n} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \int_{\mathbb{R}^{d+1}} a_j(x)^2 dx \right] \right)^2.
$$

By the Itô isometry we have

$$
\sum_{j=1}^{2^n} \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \int_{\mathbb{R}^{d+1}} a_j(x)^2 dx \right] = \mathbb{E}_{P_{\lambda^{0}}}^{x,0} \left[ \int_0^{T_t} \| \lambda \alpha \cdot - B_s \|^2 ds \right] \leq \mathbb{E}_0^{x} [T_t] \| \varphi \|^2.
$$
Hence

\[ \sum_{j=1}^{2^n} E_{P_{O}}^{x,0} \left[ \int a_j(x)^2 \, dx \right] \sum_{i \neq j} E_{P_{O}}^{x,0} \left[ \int a_j(x)^2 \, dx \right] \leq (\mathbb{E}_{\nu}[T_{i}])^2 \| \omega^{(\alpha-1)/2} \hat{\varphi} \|^4. \]

Finally we estimate the third term. We see that

\[ \sum_{j=1}^{2^n} \sum_{i \neq j} E_{P_{O}}^{x,0} \left[ \int a_j(x) a_i(x) \, dx \int a_j(y) a_i(y) \, dy \right] \leq \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} \left| \sum_{j=1}^{2^n} E_{P_{O}}^{x,0} [a_j(x) a_j(y)] \right|^2. \]

Note that \( E_{P_{O}}^{x,0} [a_j(x) a_j(y)] = E_{P_{O}}^{x,0} \left[ \int_{T_j} A_s(x, j) A_s(y, j) \, ds \right] \), where we set \( A_s(x, j) = (S_{j-1} \lambda_{\alpha}(\cdot - B_s))(x) \). By the Schwarz inequality we have

\[
\begin{align*}
&\leq \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} dy \left[ \sum_{j=1}^{2^n} \int_{T_j} A_s(x, j) A_s(y, j) \, ds \right]^2 \\
&\leq \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} dy \left[ \sum_{j=1}^{2^n} \int_{T_j} A_s(x, j)^2 \, ds \right] \left( \sum_{j=1}^{2^n} \int_{T_j} A_s(y, j)^2 \, ds \right) \\
&= E_{P_{O}}^{x,0} \left[ \int_{\mathbb{R}^{d+1}} \left( \sum_{j=1}^{2^n} \int_{T_j} A_s(x, j)^2 \, ds \right) \int_{\mathbb{R}^{d+1}} \left( \sum_{j=1}^{2^n} \int_{T_j} A_s(y, j)^2 \, ds \right) \right] \\
&= E_{P_{O}}^{x,0} \left[ T_i^2 \| \omega^{(\alpha-1)/2} \hat{\varphi} \|^4 \right] = \mathbb{E}_{\nu}[T_{i}^2] \| \omega^{(\alpha-1)/2} \hat{\varphi} \|^4.
\]

Note that \( \mathbb{E}_{\nu}[T_{i}^n] = \frac{\nu_{x,0}}{\sqrt{2\pi}} \int_0^\infty \frac{s^n}{\sqrt{s}} \exp \left( -\frac{1}{2} (\frac{i^2}{s} + m^2 s) \right) \, ds < \infty \) for \( n \geq 0 \). Then the lemma follows.

\[ \square \]

### 4.2 Invariant domain and self-adjointness

Let \( P_{\mu} = p_{\mu} \otimes 1 + 1 \otimes P_{t_{\mu}} \) be the total momentum operator in \( H \).

**Lemma 4.2** Let \( V = 0 \). Then \( e^{-itP_{\mu}} e^{-sH_{\mu}} e^{itP_{\mu}} = e^{-sH_{\mu}} \).

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Lemma 4.3 Suppose Assumptions 2.1 and 2.2. Let

\[ (F, e^{-itp_\mu} e^{-sH_{\text{rad}}} e^{itp_\mu} G) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{p_\mu} \left[ (J_0 F(B_{T_0}), e^{-itp_\mu} e^{-itp_\mu^\dagger} e^{-iA_E(I[0,t])} e^{itp_\mu} J_t G(B_{T_t})) \right]. \]

Since \( e^{-itp_\mu} e^{-itp_\mu^\dagger} e^{-iA_E(I[0,t])} e^{itp_\mu} = e^{-iA_E(I[0,t])} \), the lemma follows.

Proof: By the Feynman-Kac type formula we have

\[ (F, e^{-itp_\mu} e^{-sH_{\text{rad}}} e^{itp_\mu} G) \]

\[ \leq C \left( \left\| \sqrt{\omega} \hat{\varphi} \right\| + \left\| \hat{\varphi} \right\| \right) \left( \left\| H_{\text{rad}} + \mathbb{I} \right\|^{1/2} G \right) \left\| F \right\|. \] (4.5)

Proof: Notice that \((e^{isp_\mu} F, e^{-iH_{\text{rad}}} G) = (e^{-is\tilde{p}_\mu} F, e^{-iH_{\text{rad}}} e^{-is\tilde{p}_\mu} G)\). Then

\[ (e^{isp_\mu} F, e^{-iH_{\text{rad}}} G) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{p_\mu} \left[ (J_0 F(B_{T_0}), e^{+i\tilde{p}_\mu^\dagger} e^{-iA_E(I)} e^{-is\tilde{p}_\mu} J_t e^{-isp_\mu} G(B_{T_t})) \right]. \] (4.6)

Here and in what follows in this proof we set \( I = \bigoplus_{\mu} I^\mu = I[0,t] \). We see that \( e^{+i\tilde{p}_\mu^\dagger} e^{-iA_E(I)} e^{-is\tilde{p}_\mu} = e^{-iA_E(I)} e^{is(\mathbb{I} \otimes p_\mu)} \). Take the derivative at \( s = 0 \) on both sides of (4.6). We have

\[ (i p_\mu F, e^{-iH_{\text{rad}}} G) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{p_\mu} \left[ (J_0 F(B_{T_0}), -iA_E \mu (ip_\mu) e^{-iA_E(I)} J_t G(B_{T_t})) \right] \]

\[ + \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{p_\mu} \left[ (J_0 F(B_{T_0}), e^{-iA_E(I)} J_t(-ip_\mu G)(B_{T_t})) \right]. \] (4.7)

It is trivial to see that

\[ \left| \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{p_\mu} \left[ (J_0 F(B_{T_0}), e^{-iA_E(I)} J_t(-ip_\mu G)(B_{T_t})) \right] \right| \leq \left\| F \right\||p_\mu G||. \]

We can estimate the first term on the right-hand side of (4.7) as

\[ \left| \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{p_\mu} \left[ (J_0 F(B_{T_0}), A_E \mu (ip_\mu) e^{-iA_E(I)} J_t G(B_{T_t})) \right] \right| \]

\[ \leq \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{p_\mu} \|A_E \mu (ip_\mu) J_0 F(B_{T_0})\| \|J_t G(B_{T_t})\|  \]
By the bound $\|A_{E\mu}(f)\Phi\| \leq C(\|\hat{f}\| + \|\hat{f}/\sqrt{\omega}\|)(H_{\text{rad}} + \mathbb{1})^{1/2}\Phi\|$ with some constant $C > 0$, we have

$$\leq C \int_{\mathbb{R}^d} dx \mathbb{E}_{P_{x\nu}}^{x,0} [\|p\mu I\| + \|\omega(p)^{-1/2}p\mu I\|](H_{\text{rad}} + \mathbb{1})^{1/2} F(B_{T_0})\|\|G(B_{T_1})\|]$$

and by the Schwarz inequality,

$$\leq C \left( \int_{\mathbb{R}^d} dx \mathbb{E}_{P_{x\nu}}^{x,0} [\|\omega(p)I\mu\| + \|\omega(p)^{1/2}I\mu\|^2]\|(H_{\text{rad}} + \mathbb{1})^{1/2} F(x)\|^2 \right)^{1/2} \|G\|$$

$$\leq C(\|\omega^{1/2}\hat{\varphi}\| + \|\hat{\varphi}\|)(H_{\text{rad}} + \mathbb{1})^{1/2} F\|\|G\|.$$

Then the lemma follows. \hfill \Box

We define the momentum conjugate of $A_E(f)$ by $\Pi_E(f) = i[H_{\text{rad}}, A_E(f)]$ in the function space.

**Lemma 4.4** Suppose Assumptions \[\text{2.1}\] and \[\text{2.2}\]. Let $V = 0$. Then for $F, G \in D(H_{\text{rad}})$ it follows that

$$(H_{\text{rad}} F, e^{-iH_{\text{rad}} G})$$

$$\leq \left( \|H_{\text{rad}} G\| + |\alpha| (\|\sqrt{\omega}\hat{\phi}\| + \|\hat{\phi}\|)(H_{\text{rad}} + \mathbb{1})^{1/2} G\| + |\alpha|^2 \|\hat{\phi}/\sqrt{\omega}\|^2 \|G\| \right) \|F\|.$$

**PROOF:** By the Feynman-Kac type formula we have

$$(H_{\text{rad}} F, e^{-iH_{\text{rad}} G}) = \int_{\mathbb{R}^d} dx \mathbb{E}_{P_{x\nu}}^{x,0} \left[ (J_0 F(B_{T_0}), e^{-i\alpha A_{E}(I[0,t])} S J_t G(B_{T_t})) \right],$$

where $S = e^{i\alpha A_{E}(I[0,t])} H_{\text{rad}} e^{-i\alpha A_{E}(I[0,t])} = H_{\text{rad}} - \alpha \Pi_E(I[0,t]) + \alpha^2 g$ with the constant $g = q_E(I[0,t])$. It is trivial to see that

$$\int_{\mathbb{R}^d} dx \mathbb{E}_{P_{x\nu}}^{x,0} \left[ (J_0 F(B_{T_0}), e^{-i\alpha A_{E}(I[0,t])} H_{\text{rad}} J_t G(B_{T_t})) \right] \leq \|F\| \|H_{\text{rad}} G\|. \quad (4.8)$$

In the same way as the estimate of the first term of the right-hand side of (4.7) we can see that

$$\left| \int_{\mathbb{R}^d} dx \mathbb{E}_{P_{x\nu}}^{x,0} \left[ (J_0 F(B_{T_0}), e^{-i\alpha A_{E}(I[0,t])} \Pi_E(I[0,t]) J_t G(B_{T_t})) \right] \right|$$

$$\leq C(\|\sqrt{\omega}\hat{\phi}\| + \|\hat{\phi}\|) \|F\| \|(H_{\text{rad}} + \mathbb{1})^{1/2} G\|$$

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with some constant $C > 0$. Here we used the fundamental bound $\| \Pi_{E_\mu}(f)\Phi \| \leq C \left( \| \sqrt{\omega} \hat{f} \| + \| \hat{f} \| \right) \| (H_{\text{rad}} + I)^{1/2}\Phi \|$ and Lemma 4.1. Finally we see that $g \leq C \| I_{[0,t]} \|_E$ and by Lemma 4.1 again,

$$\left| \int \mathcal{E}^{x,0}_{P_{\phi'}} \left[ \int \rho F(B_{T_0}), e^{-i\alpha \mathcal{H}_{[0,t]} g} \mathcal{J}_G(B_{T_0}) \right] \right| \leq C \left( \int \mathcal{E}^{x,0}_{P_{\phi'}} \| I_{[0,t]} \|^2 \| F(x) \|^2 \right)^{1/2} \| G \| \leq C \| \hat{\psi} / \sqrt{\omega} \|^2 \| F \| \| G \|. \quad (4.9)$$

Then the lemma follows.

**Theorem 4.5 (Essential self-adjointness)** Let $V \in \mathcal{V}_{\text{rel}}$. Suppose that $m > 0$, and Assumptions 2.1 and 2.2 hold. Then $H_{\text{qf}}$ is essentially self-adjoint on $D(|p|) \cap D(H_{\text{rad}})$.

**Proof:** Suppose $V = 0$. Let $F \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}_{\text{fin}}$. Then we see that

$$\| (T_{\text{kin}} + H_{\text{rad}}) F \|^2 \leq C_1 \| |p| F \|^2 + C_2 \| H_{\text{rad}} F \|^2 + C_3 \| F \|^2$$

with some constants $C_1, C_2$ and $C_3$. Since $C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}_{\text{fin}}$ is a core of $|p| + H_{\text{rad}}$,

$$D(T_{\text{kin}} + H_{\text{rad}}) \supset D(|p|) \cap D(H_{\text{rad}}) \quad (4.10)$$

follows from a limiting argument. By Lemmas 4.3 and 4.4 we also see that

$$e^{-t(T_{\text{kin}} + H_{\text{rad}})} (D(|p|) \cap D(H_{\text{rad}})) \subset (D(|p|) \cap D(H_{\text{rad}})). \quad (4.11)$$

(4.10) and (4.11) imply that $T_{\text{kin}} + H_{\text{rad}}$ is essentially self-adjoint on $D(|p|) \cap D(H_{\text{rad}})$ by Proposition 3.3. Next we suppose that $V$ satisfies assumptions in the theorem. By Lemma 3.12 $V$ is also relatively bounded with respect to $T_{\text{kin}} + H_{\text{rad}}$ with a relative bound strictly smaller than one. Then the theorem follows by the Kato-Rellich theorem.

Furthermore in Hidaka and Hiroshima [HH13b] the self-adjointness of $H_{\text{qf}}$ for arbitrary $m \geq 0$ is established. The key inequality is as follows.

**Lemma 4.6** Suppose that $m > 0$, and Assumptions 2.1 and 2.2 hold. Let $V = 0$. Then there exists a constant $C$ such that

$$\| |p| F \|^2 + \| H_{\text{rad}} F \|^2 \leq C \| (T_{\text{kin}} + H_{\text{rad}} + I) F \|^2 \quad (4.12)$$

for all $F \in D(|p|) \cap D(H_{\text{rad}})$.
Theorem 4.7 (Self-adjointness [HH13b]) Suppose that \( m \geq 0 \), and Assumptions 2.1 and 2.2 hold. Let \( V \in \mathcal{V}_{\text{rel}} \). Then \( \mathcal{H}_{\text{qf}} \) is self-adjoint on \( D(|p|) \cap D(H_{\text{rad}}) \).

**Proof:** We show an outline of the proof. See [HH13b] for detail. Suppose that \( V = 0 \) and \( m > 0 \). We write \( H_m \) for \( \mathcal{H}_{\text{qf}} \) to emphasize \( m \)-dependence. By (4.12), \( H_m \lceil_{D(|p|) \cap D(H_{\text{rad}})} \) is closed on \( D(|p|) \cap D(H_{\text{rad}}) \). Then \( H_m \) is self-adjoint on \( D(|p|) \cap D(H_{\text{rad}}) \). Note that \( H_0 = H_m + (H_0 - H_m) \) and \( H_0 - H_m \) is bounded. Then \( H_0 \) is also self-adjoint on \( D(|p|) \cap D(H_{\text{rad}}) \) for \( V = 0 \). Finally let \( V \in \mathcal{V}_{\text{rel}} \). Then \( V \) is also relatively bounded with respect to \( H_m \) with a relative bound strictly smaller than one. Then the theorem follows from Kato-Rellich theorem.

Example 4.8 (Hydrogen like atom) Let \( d = 3 \). A spinless hydrogen like atom is defined by introducing the Coulomb potential \( V_{\text{Coulomb}}(x) = -g/|x|, g > 0 \), which is relatively form bounded with respect to \( \sqrt{p^2 + m^2} \) with a relative bound strictly smaller than one if \( g \leq 2/\pi \) by [Her77] (see also [BE11, Theorem 2.2.6]). Furthermore if \( g < 1/2 \), \( V_{\text{Coulomb}} \) is relatively bounded with respect to \( \sqrt{p^2 + m^2} \) with a relative bound strictly smaller than one. Let \( A_{\Lambda} \) be the quantized radiation field with the cutoff function \( \hat{\varphi}(k) = 1_{|k| < \Lambda}(k)/\sqrt{(2\pi)^3} \), where \( \Lambda > 0 \) describes a UV cutoff parameter. By Lemma 3.12, when \( g < 2/\pi \), \( V \) is relatively form bounded with respect to \( T_{\text{kin}} + H_{\text{rad}} \) and \( \mathcal{H}_{\text{qf}} \) is well defined as a self-adjoint operator. Furthermore by Theorem 4.7 when \( g < 1/2 \), \( \mathcal{H}_{\text{qf}} \) is self-adjoint on \( D(|p|) \cap D(H_{\text{rad}}) \). All the statements mentioned above are true for arbitrary values of \( \alpha \in \mathbb{R} \) and \( \Lambda > 0 \).

5 Martingale properties and fall-off of bound states

5.1 Semigroup and relativistic Kato-class potential

In this subsection we define the self-adjoint operator \( \mathcal{H}_K \) with a potential \( V \) in the so-called relativistic Kato-class through the Feynman-Kac type formula. Let us define the relativistic Kato-class.
Definition 5.1 (Relativistic Kato-class) (1) Potential $V$ is in the relativistic Kato-class if and only if
\[
\sup_x \mathbb{E}^{x,0}_P \left[ e^{\int_0^T V(B_T) \, ds} \right] < \infty.
\]
(5.1)

(2) $V = V_+ - V_-$ is in $\mathcal{V}_{\text{Kato}}$ if and only if $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $V_-$ is in relativistic Kato-class.

The property (5.1) is used in the proofs of Lemmas 5.8 and 5.11 and Corollary 5.9. When $V \in \mathcal{V}_{\text{Kato}}$, we can see that
\[
|r_t(F,G)| \leq c_t \|F\|\|G\|
\]
follows with some constant $c_t$. Then the Riesz representation theorem yields that there exists a bounded operator $S_t$ such that $r_t(F,G) = (F, S_t G)$ for $F, G \in \mathcal{H}$ and $\|S_t\| \leq c_t$. By the Feynman-Kac type formula (3.15) we indeed see that $(S_t G)(x) = \mathbb{E}^{x,0}_P \left[ J[0,t] G(B_T) \right]$, where
\[
J[0,t] = J_0^* e^{-\int_0^t V(B_T) \, dr} e^{-i \alpha A_E(\mathbb{I}[0,t])} J_t.
\]
(5.2)

Theorem 5.2 Let $V \in \mathcal{V}_{\text{Kato}}$. Suppose Assumption 2.1. Then $S_t, t \geq 0$, is a strongly continuous one-parameter symmetric semigroup.

Definition 5.3 (Definition of $H_K$) Let $V \in \mathcal{V}_{\text{Kato}}$. Suppose Assumption 2.1. The unique self-adjoint generator of $S_t, t \geq 0$, is denoted by $H_K$, i.e., $S_t = e^{-tH_K}$, $t \geq 0$.

Remark 5.4 Note that
\[
\mathcal{V}_{\text{rel}} \subset \mathcal{V}_{\text{qf}}, \quad \mathcal{V}_{\text{Kato}} \subset \mathcal{V}_{\text{qf}}.
\]
(5.3)

It is easy to see that $\mathcal{V}_{\text{rel}} \subset \mathcal{V}_{\text{qf}}$. See Appendix C for the inclusion $\mathcal{V}_{\text{Kato}} \subset \mathcal{V}_{\text{qf}}$. We give a remark on the difference between $H_{\text{qf}}$ and $H_K$. In order to define $H_{\text{qf}}$ we need Assumptions 2.1 and 2.2, an extra Assumption 2.2 is, however, not needed to define $H_K$.

In order to prove Theorem 5.2 we need several lemmas:
Lemma 5.5 Let $V \in \mathcal{V}_\text{Kato}$. Suppose Assumption \[2.7\]. Then $S_t, t \geq 0$, satisfies the semigroup property, i.e., $S_s S_t = S_{s+t}$ for all $s, t \geq 0$.

Proof: We have $(F, S_s S_t G) = \left( F, \mathbb{E}_{P_{x,\nu}}^{s,0} \left[ J_{[0,s]} \mathbb{E}_{P_{x,\nu}}^{B_{T_t},0} \left[ J_{[0,t]} G(B_{T_t}) \right] \right] \right)$. By Lemma \[E.1\] we show that

$$
\mathbb{E}_{P_{x,\nu}}^{s,0} \left[ J_{[0,s]} \mathbb{E}_{P_{x,\nu}}^{B_{T_t},0} \left[ J_{[0,t]} G(B_{T_t}) \right] \right] = \mathbb{E}_{P_{x,\nu}}^{0,0} \left[ J_{[0,s]} J_0^* e^{-f_{s+t} V(B_{T_t})} e^{-i \alpha A_{E}(I_{[0,s+t]})} J_t G(B_{T_{s+t}}) \right],
$$

where

$$
I_{[s,s+t]} = \lim_{n \to \infty} \sum_{j=1}^{2^n} \int_{T_{\frac{j}{2^n}}(j-1)+s}^{T_{\frac{j}{2^n}}(j-1)+s} j_{\frac{j}{2^n}} \lambda(x - R_t) dB^e_R.
$$

Since it is obtained that

$$
J_{[0,s]} J_0^* e^{-f_{s+t} V(B_{T_t})} e^{-i \alpha A_{E}(I_{[0,s+t]})} J_t G(B_{T_{s+t}})
$$

and $J_s J_0^* = U_s J_0 J_s = U_s E_s$, we have

$$(F, S_s S_t G) = \left( F, \mathbb{E}_{P_{x,\nu}}^{s,0} \left[ J_0^* e^{-i \alpha A_{E}(I_{[0,s]})} U_s E_s e^{-i \alpha A_{E}(I_{[0,s+t]})} e^{-f_{s+t} V(B_{T_t})} \right] d \mathbb{E}_{P_{x,\nu}}^{s+t} \right).
$$

By the Markov property of projection $E_s$, $E_s$ can be deleted, and $U_s$ satisfies that $U_s e^{-i \alpha A_{E}(I_{[0,s+t]})} J_t G(B_{T_{s+t}}) = e^{-i \alpha A_{E}(I_{[0,s+t]})} J_{s+t} G(B_{T_{s+t}})$. Then by Proposition \[D.2\] we have

$$(F, S_s S_t G) = \left( F, \mathbb{E}_{P_{x,\nu}}^{s,0} \left[ J_0^* e^{-f_{s+t} V(B_{T_t})} \right] d \mathbb{E}_{P_{x,\nu}}^{s+t} \right) = (F, S_{s+t} G).
$$

Then the semigroup property, $S_s S_t = S_{s+t}$, follows.

Lemma 5.6 Let $V \in \mathcal{V}_\text{Kato}$. Suppose Assumption \[2.7\]. Then $S_t, t \geq 0$, is strongly continuous in $t$ and $s - \lim_{t \to 0} S_t = \mathbb{1}$.

Proof: It is enough to show that $(F, S_t G) \to (F, G)$ as $t \to 0$ for $F, G \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}_\text{fin}$. Let $F = f \otimes \Psi$ and $G = g \otimes \Phi$. Since $V \in \mathcal{V}_\text{Kato}$, we have

$$
|\mathbb{E}_{P_{x,\nu}}^{s,0} \left[ (e^{-i \alpha A_{E}(I_{[0,t]})} - 1) g(B_{T_t}) \Phi \right]|^2
$$

$$
\leq C \| F \|_{\mathcal{F}} \left( \int d \mathbb{E}_{P_{x,\nu}}^{s,0} \left[ (e^{-i \alpha A_{E}(I_{[0,t]})} - 1) g(B_{T_t}) \Phi \right] \right)^{1/2}.
$$

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Since $g \in C_0^\infty(\mathbb{R}^d)$, $|g(x)| \leq a \mathbb{1}_K(x)$ with some $a$ and a compact domain $K \subset \mathbb{R}^d$, we have

$$|(F, (S_t - \mathbb{1}G)| \leq aC\|F\|_{\mathcal{M}} \left\{ \int_K dx^2 \mathbb{E}_{P^0}[|e^{-iuA_0(l[0,t])} - 1|\Phi|^2] \right\}^{1/2}. $$

By the bound $\mathbb{E} \left[ |(e^{-iuA_0(l[0,t])} - 1)|\Phi|^2 \right] \leq |\alpha||I[0,t]|\|((N + \mathbb{1})^{1/2}\Phi\right\|$, we have

$$(F, (S_t - \mathbb{1}G)| \leq |\alpha|aC\|F\|_{\mathcal{M}} \left\{ \int_K dx^2 \mathbb{E}_{P^0}[|I[0,t]|\Phi|^2] \right\}^{1/2} \leq \sqrt{Ta}|\alpha|C\|\hat{\phi}/\sqrt{\omega}\|\|F\|_{\mathcal{M}} \left\{ \int_K dx \right\}^{1/2} \|((N + \mathbb{1})^{1/2}\Phi\right\|. $$

Then $|(F, (S_t - \mathbb{1}G)| \to 0$ as $t \to 0$.

\[\square\]

**Lemma 5.7** Let $V \in \mathcal{V}_{\text{Kato}}$. Suppose Assumption 2.1. Then $S_t$, $t \geq 0$, is symmetric, i.e., $S_t^* = S_t$ for all $t \geq 0$.

**Proof:** Recall that $R = \Gamma(r)$ is the second quantization of the reflection $r$. We have

$$(F, S_tG) = \int dx^2 \mathbb{E}_{P^0}[e^{-f_0^t V(B_{r_T})dr} (J_0 F(B_{r_T}), e^{-iuA_0(r[0,t])}J_t G(B_{r_T}))],$$

and by the time-shift $U_t = \Gamma(u_t)$,

$$= \int dx^2 \mathbb{E}_{P^0}[e^{-f_0^t V(B_{r_T})dr} (J_0 F(B_{r_T}), e^{-iuA_0(u_t r[0,t])}J_0 G(B_{r_T}))].$$

Notice that $u_t r[0,t] = \lim_{n \to \infty} \sum_{j=1}^{2^n} \int_{T_{j-1}}^{T_j} \int_{T_{j-1}}^{T_j} I_{t-j-1} \lambda(\cdot - B_s)dB^\mu_s$. Exchanging integrals $\int dP^0$ and $\int dx$ and changing the variable $x$ to $y - B_{r_T}$, we can have

$$= \mathbb{E}_{P^0}[\int dy e^{-f_0^t V(B_{r_T}, -B_{r_T} + y)dr} (J_t F(y - B_{r_T}), e^{-iuA_0(r[0,t])}J_0 G(y))], \quad (5.6)$$

where $u_t r[0,t] = \lim_{n \to \infty} \sum_{j=1}^{2^n} \int_{T_{j-1}}^{T_j} I_{t-j-1} \lambda(\cdot - (B_s - B_{r_T} + y))dB^\mu_s$. By Lemma E.2, we can see that

$$(5.6) = \int dy \mathbb{E}_{P^0}[\int dy e^{-f_0^t V(B_{r_T})dr} (J_0 e^{-iuA_0(r[0,t])}J_t F(y + B_{r_T}), G(y)) = (S_t F, G).$$

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Then the lemma follows. \hfill \Box

**Proof of Theorem 5.2**

Lemmas 5.5-5.7 yield that \( S_t \) is symmetric and strongly continuous one-parameter semigroup. Then there exists the unique self-adjoint operator such that \( S_t = e^{-tH_K} \) by a semigroup version of the Stone theorem [LHB11, Proposition 3.26]. \hfill \Box

### 5.2 Martingale properties

Let \( \Phi_b \) be a bound state of \( H_K \) and \( E \in \mathbb{R} \) the eigenvalue associated with \( \Phi_b \):

\[
H_K \Phi_b = E \Phi_b.
\]

In this section we study the spatial decay of \( \|\Phi_b(x)\|_{L^2(\mathcal{X})} \) as \( |x| \to \infty \). In order to do that we show the martingale property of the stochastic process \( (M_t(x))_{t \geq 0} \):

\[
M_t(x) = e^{tE} e^{-\int_0^t V(B_{T_s} + x) ds} e^{-i\alpha \Lambda_E (\Gamma^x[0,t])} J_t \Phi_b(B_{T_t} + x), \quad t \geq 0,
\]

on \( \Omega_P \times \Omega_x \times \mathcal{D}_E \). Here \( \Gamma^x[0,t] \) is defined by \( I[0,t] \) with \( B_s \) replaced by \( B_s - x \), i.e., \( \Gamma^x[0,t] = \bigoplus_{|\mu|=1}^d \int_0^{T_s} j_{T_s} \lambda(-B_s - x) dB_s^\mu \). Using the stochastic process \( (M_t(x))_{t \geq 0} \), bound state \( \Phi_b \) can be represented as

\[
\Phi_b(x) = \mathbb{E}^{0,0}_{P_{\mathcal{X}_E}, \mu_E} \| J_0^* M_t(x) \|
\]

for arbitrary \( t \geq 0 \). We can also obtain that \( (u \otimes \Phi, \Phi_b) = (u \otimes \Phi, e^{-t(H_K - E)} \Phi_b) = \int_{\mathbb{R}^d} du \mathbb{E}^{0,0}_{P_{\mathcal{X}_E}, \mu_E}[J_0 \Phi \cdot M_t(x)] \). Then we have \( (\Phi, \Phi_b(x))_{L^2(\mathcal{X})} = \mathbb{E}^{0,0}_{P_{\mathcal{X}_E}, \mu_E}[J_0 \Phi \cdot M_t(x)] \).

**Lemma 5.8** Let \( V \in \mathcal{V}_{\text{Kato}} \). Suppose Assumption 2.1. Then \( \|\Phi_b(\cdot)\|_{L^2(\mathcal{X})} \in L^\infty(\mathbb{R}^d) \).

**Proof:** By \( \Phi_b(x) = \mathbb{E}^{0,0}_{P_{\mathcal{X}_E}, \mu_E} \| J_0^* M_t(x) \| \) for arbitrary \( t > 0 \), we have

\[
\|\Phi_b(x)\|_{L^2(\mathcal{X})} \leq e^{tE} \left( \mathbb{E}^{0,0}_{P_{\mathcal{X}_E}} \left[ e^{-2 \int_0^t V(B_{T_s} + x) ds} \right] \right)^{1/2} \left( \mathbb{E}^{0,0}_{\mathcal{X}_E} \| \Phi_b(B_{T_t} + x) \| \right)^{1/2}.
\]

We have \( \sup_{x \in \mathbb{R}^d} \mathbb{E}^{0,0}_{P_{\mathcal{X}_E}} \left[ e^{-2 \int_0^t V(B_{T_s} + x) ds} \right] < \infty \), since \( V \) is relativistic Kato-class, and

\[
\mathbb{E}^{0,0}_{\mathcal{X}_E} \| \Phi_b(B_{T_t} + x) \|^2 = \int_{\mathbb{R}^d} dy \int_0^\infty ds \frac{\rho_t(s)e^{-|y|^2/(2s)}}{(2\pi s)^{d/2}} \| \Phi_b(x + y) \|^2 \leq C \| \Phi_b \|_{\mathcal{H}}^2.
\]

Then \( \sup_{x \in \mathbb{R}^d} \|\Phi_b(x)\|^2 \leq C \|\Phi_b\|^2_{\mathcal{H}} \) follows. \hfill \Box
Corollary 5.9 It follows that \( \mathbb{E}_{P_\nu}^{0,0,0} \mathbb{E}_{\mu_\nu} \left[ \|M_t(x)\|_{L^2(\mathcal{F})} \right] < \infty \) for all \( x \in \mathbb{R}^d \).

PROOF: This follows from Lemma 5.8.

We define a filtration under which \((M_t(x))_{t \geq 0}\) is martingale. Let

\[
\mathcal{F}^{(1)}_{[0,t]} = \left\{ \bigcup_{w_1 \in \Omega_\nu} (A(w_1), w_1) \big| A(w_1) \in \sigma(B_r, 0 \leq r \leq T_t(w_1)) \right\} \subset \mathcal{B}_\nu \times \mathcal{B}_\nu
\]

and

\[
\mathcal{F}^{(2)}_{[0,t]} = \left\{ \bigcup_{w_2 \in \Omega_\mu} (w_2, B(w_2)) \big| B(w_2) \in \sigma(T_r, 0 \leq r \leq t) \right\} \subset \mathcal{B}_\nu \times \mathcal{B}_\nu.
\]

Then we set \( \mathcal{F}_{[0,t]} = \mathcal{F}^{(1)}_{[0,t]} \cap \mathcal{F}^{(2)}_{[0,t]}, t \geq 0 \), and define a filtration in \( \mathcal{B}_\nu \times \mathcal{B}_\nu \times \Sigma_\nu \) by

\[
(M_t)_{t \geq 0} = (\mathcal{F}_{[0,t]} \times \Sigma_{(-\infty,t]}_{t \geq 0}.
\]

Theorem 5.10 (Martingale property of \((M_t(x))_{t \geq 0}\)) Let \( V \in \mathcal{V}_{\text{Kato}} \). Suppose Assumption 2.1. Then the stochastic process \((M_t(x))_{t \geq 0}\) is martingale with respect to the filtration \((\mathcal{M}_t)_{t \geq 0} \). I.e., \( \mathbb{E}_{P_\nu}^{0,0,0} \mathbb{E}_{\mu_\nu} \left[ M_t(x)|\mathcal{M}_s \right] = M_s(x) \) for \( t \geq s \).

PROOF: By Proposition 5.2, we have \( A_E(\mathcal{I}[0,t]) = A_E(\mathcal{I}[0,s]) + A_E(\mathcal{I}[s,t]) \) for \( s \leq t \). Since \( e^{-i\alpha A_E(\mathcal{I}[s,t])}e^{-\int_0^s V(B_t)\,dr} \) is \( \mathcal{M}_s \)-measurable, we have

\[
\mathbb{E}_{P_\nu}^{0,0,0} \mathbb{E}_{\mu_\nu} \left[ M_t(x)|\mathcal{M}_s \right] = e^{\int_0^t e^{-i\alpha A_E(\mathcal{I}[0,s])}e^{-\int_0^s V(B_t)\,dr} \, J_t \Phi_b(B_t \, x)|\mathcal{M}_s \right].
\]

By the definition of \( \mathcal{I}[s,t] \) it is seen that

\[
\frac{\mathbb{E}_{P_\nu}^{0,0,0} \mathbb{E}_{\mu_\nu} \left[ e^{-i\alpha A_E(\mathcal{I}[s,t])}e^{-\int_0^s V(B_t)\,dr} \, J_t \Phi_b(B_t \, x)|\mathcal{M}_s \right]}{\lim_{n \to \infty} \mathbb{E}_{P_\nu}^{0,0,0} \mathbb{E}_{\mu_\nu} \left[ e^{-i\alpha A_E(\mathcal{I}[s,t])}e^{-\int_0^s V(B_t)\,dr} \, J_t \Phi_b(B_t \, x)|\mathcal{M}_s \right]},
\]

and then

\[
\mathbb{E}_{P_\nu}^{0,0,0} \mathbb{E}_{\mu_\nu} \left[ e^{-i\alpha A_E(\mathcal{I}[s,t])}e^{-\int_0^s V(B_t)\,dr} \, J_t \Phi_b(B_t \, x)|\mathcal{M}_s \right] = \mathbb{E}_{\mu_\nu} \left[ \mathbb{E}_{P_\nu}^{0,0,0} \mathbb{E}_{\mu_\nu} \left[ e^{-i\alpha A_E(\mathcal{I}[s,t])}e^{-\int_0^s V(B_t)\,dr} \, J_t \Phi_b(B_t \, x)|\mathcal{F}^{(1)}_{[0,s]} \big| \mathcal{F}^{(2)}_{[0,s]} \big| \Sigma_{(-\infty,s]} \right].
\]
By the Markov property of the Brownian motion we see that
\[
\mathbb{E}^y_p \left[ e^{-i\alpha A\epsilon(I_n^+[s,t])} e^{-\int_s^t V(B_{T_s}+x)\,dr} J_t \Phi_b(B_{T_t} + x) \big| \mathcal{F}^{(1)}_{[0,s]} \right] 
\]
\[
= \lim_{n \to \infty} \mathbb{E}^{B_{T_s}}_p \left[ e^{-i\alpha A\epsilon(I_n^+[s,t])} e^{-\int_s^t V(B_{T_s}-T_s+x)\,dr} J_t \Phi_b(B_{T_t}-T_s + x) \right],
\]
where \(\mathbb{E}^{B_{T_s}}_p\) means \(\mathbb{E}^y_p\) evaluated at \(y = B_{T_s}\) and
\[
I_n^{(1)}[s,t] = \sum_{j=s}^{t} \int_{(j-1)+s}^{j+s} \frac{\xi_{(j-1)+s}}{2n} \lambda(-B_r-x) dB^\mu_r.
\]
Since the subordinator \((T_t)_{t \geq 0}\) is also a Markov process, we have
\[
\mathbb{E}^0_p \left[ \mathbb{E}^{B_{T_0}}_p \left[ e^{-i\alpha A\epsilon(I_n^{(2)}[s,t])} e^{-\int_s^t V(B_{T_s}-T_0+x)\,dr} J_t \Phi_b(B_{T_t}-T_0 + x) \right] \big| \mathcal{F}^{(2)}_{[0,s]} \right] 
\]
\[
= \mathbb{E}^{T_0}_p \mathbb{E}^{B_{T_0}}_p \left[ e^{-i\alpha A\epsilon(I_n^{(2)}[s,t])} e^{-\int_s^t V(B_{T_s}-T_0+x)\,dr} J_t \Phi_b(B_{T_t}-T_0 + x) \right],
\]
where \(\mathbb{E}^{T_0}_p\) also means \(\mathbb{E}^y_p\) evaluated at \(y = T_s\) and
\[
I_n^{(2)}[s,t] = \sum_{j=s}^{t} \int_{(j-1)+s}^{j+s} \frac{\xi_{(j-1)+s}}{2n} \lambda(-B_r-x) dB^\mu_r.
\]
Again the Markov property of the Euclidean field yields that
\[
\mathbb{E}^{Y}(T_0)_{t \geq 0} \mathbb{E}^{B_{T_0}}_p \left[ e^{-i\alpha A\epsilon(I_n^{(3)}[s,t])} e^{-\int_s^t V(B_{T_s}-T_0+x)\,dr} J_t \Phi_b(B_{T_t}-T_0 + x) \right] | \mathcal{F}^{(3)}_{[0,s]} 
\]
\[
= \mathbb{E}^{Y}(T_0)_{t \geq 0} \mathbb{E}^{B_{T_0}}_p \left[ e^{-i\alpha A\epsilon(I_n^{(3)}[s,t])} e^{-\int_s^t V(B_{T_s}-T_0+x)\,dr} J_t \Phi_b(B_{T_t}-T_0 + x) \right] | \mathcal{F}_s.
\]
The right-hand side above equals to
\[
= E_s \mathbb{E}^{T_0}_p \mathbb{E}^{B_{T_0}}_p \left[ e^{-i\alpha A\epsilon(I_n^{(3)}[s,t])} e^{-\int_s^t V(B_{T_s}-T_0+x)\,dr} J_t \Phi_b(B_{T_t}-T_0 + x) \right] 
\]
\[
= J_s J_0 U_{-s} \mathbb{E}^{T_0}_p \mathbb{E}^{B_{T_0}}_p \left[ e^{-i\alpha A\epsilon(I_n^{(3)}[s,t])} e^{-\int_s^t V(B_{T_s}-T_0+x)\,dr} J_t \Phi_b(B_{T_t}-T_0 + x) \right].
\]
Since \(U_{-s}\) is the shift by \(-s\), we have
\[
= J_s J_0 U_{-s} \mathbb{E}^{T_0}_p \mathbb{E}^{B_{T_0}}_p \left[ e^{-i\alpha A\epsilon(I_n^{(3)}[s,t])} e^{-\int_s^t V(B_{T_s}-T_0+x)\,dr} J_{t-s} \Phi_b(B_{T_{t-s}}-T_0 + x) \right],
\]

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where

\[ I_n^{(3),x}[s,t] = \bigoplus_{\mu=1}^d \sum_{j=1}^{2^n} \int \frac{T_{(t-s)}^{(j-2n)+1}}{2^n} \frac{T_{(t-s)}^{(j-1)-T_0}}{2^n} \frac{\lambda(\cdot - B_{t-s} - x)dB_{t-s}^\mu}{2^n}. \]

We notice that the random variable \( T_t + y \) under \( \nu \) has the same law as \( T_t \) under \( \nu^y \), i.e., \( \mathbb{E}_\nu[f(T_t)] = \mathbb{E}_\nu[f(T_t + y)] \), we can see that

\[
= J_s J_0^* \mathbb{E}_\nu^{B_{u+t_0}} \left[ e^{-i\alpha A_E(t_n^{(3),x}[s,t])} e^{-f_t^s V(B_{T_{T_{t-s}} - T_0} + x) dr} J_{t-s} \Phi_b(B_{T_{T_{t-s}} - T_0} + x) \right]_{u=T_s} \\
= J_s J_0^* \mathbb{E}_\nu^{B_{T_{t-s}} \nu} \left[ e^{-i\alpha A_E(t_n^{(4),x}[s,t])} e^{-f_t^s V(B_{T_{T_{t-s}}} + x) dr} J_{t-s} \Phi_b(B_{T_{T_{t-s}}} + x) \right],
\]

where

\[ I_n^{(4),x}[s,t] = \bigoplus_{\mu=1}^d \sum_{j=1}^{2^n} \int \frac{T_{(t-s)}^{(j-2n)+1}}{2^n} \frac{T_{(t-s)}^{(j-1)-T_0}}{2^n} \frac{\lambda(\cdot - B_{t-s} - x)dB_{t-s}^\mu}{2^n}. \]

Taking the limit \( n \to \infty \), we finally obtain that

\[
\mathbb{E}_{\nu^0}^{0,0} \mathbb{E}_{\mu_E}^{M_t(x)|M_s} = e^{sE} e^{-i\alpha A_E(t^\nu[s,t])} e^{-\int_0^s V(B_{t-s} + x) dr} J_s \\
\times e^{(t-s)E} \mathbb{E}_{\nu^0}^{B_{T_{t-s}}^\nu,0} \left[ J_0^* e^{-i\alpha A_E(t^\nu[s,t-s])} e^{-\int_0^s V(B_{t-s} + x) dr} J_{t-s} \Phi_b(B_{T_{T_{t-s}}} + x) \right].
\]

Notice that

\[
e^{(t-s)E} \mathbb{E}_{\nu^0}^{B_{T_{t-s}}^\nu,0} \left[ J_0^* e^{-i\alpha A_E(t^\nu[s,t-s])} e^{-\int_0^s V(B_{t-s} + x) dr} J_{t-s} \Phi_b(B_{T_{T_{t-s}}} + x) \right] = \Phi_b(B_{T_{t-s}} + x)
\]

and hence

\[
\mathbb{E}_{\nu^0}^{0,0} \mathbb{E}_{\mu_E}^{M_t(x)|M_s} = e^{sE} e^{-i\alpha A_E(t^\nu[s,t])} e^{-\int_0^s V(B_{t-s} + x) dr} J_s \Phi_b(B_{T_{t-s}} + x) = M_s(x).
\]

Then the proof is complete. \[\square\]

Since we show that \( (M_t(x))_{t \geq 0} \) is a martingale, for an arbitrary stopping time \( \tau \) with respect to \( (M_t)_{t \geq 0} \), \( (M_{t \wedge \tau}(x))_{t \geq 0} \) is also a martingale. By using this fact we can show a spatial decay of bound state \( \Phi_b \) of \( H_K \).

### 5.3 Fall-off of bound states

Let us recall that \( (z_t)_{t \geq 0} \) is the \( d \)-dimensional Lévy process on a probability space \( (\Omega_z, \mathcal{F}_z, \mathbb{P}^z) \) such that \( \mathbb{E}^z [e^{-i u \cdot z}] = e^{-t(\sqrt{|u|^2 + m^2} - m)} e^{-i u \cdot x} \). Hence the generator of \( (z_t)_{t \geq 0} \)
is given by $\sqrt{p^2 + m^2} - m$, and the distribution $k_{t,m}(x)$ of $z_t$ by

$$k_{t,m}(x) = 2 \left( \frac{m}{2\pi} \right)^{d+1} e^{tm} K_{d+1}(m \sqrt{t^2 + |x|^2}), \quad m > 0,$$

$$k_{t,0}(x) = \frac{\Gamma(d+1)}{(2\pi)^{d+1}} \frac{t}{(t^2 + |x|^2)^{d+1/2}}, \quad m = 0.$$

Here $\Gamma(m)$ denotes the Gamma function, $K_\nu(z)$ is the modified Bessel function of the third kind of order $\nu$, and it is known that $K_\nu(z) \sim \frac{1}{2} \Gamma(\frac{1}{2}) z^{-\nu}$ as $z \sim 0$.

**Lemma 5.11** Let $V \in \mathcal{H}_{Kato}$. Suppose Assumption [2.1]. Let $\tau$ be a stopping time with respect to the filtration $(\mathcal{M}_t)_{t \geq 0}$. Then

$$\|\Phi_b(x)\| \leq \|\Phi_b\|_{\mathcal{H}} \mathbb{E}_Z \left[ e^{-\int_0^\tau (V(z_r) - E) dr} \right]. \quad (5.13)$$

**Proof:** Since $(J_0 \Phi \cdot M_t(x))_{t \geq 0}$ is a martingale with respect to the filtration $(\mathcal{M}_t)_{t \geq 0}$, also is $(J_0 \Phi \cdot M_{t \wedge \tau}(x))_{t \geq 0}$. Then $\mathbb{E}_{P_\alpha} [J_0 \Phi \cdot M_t(x)] = \mathbb{E}_{P_\alpha} [J_0 \Phi \cdot M_{t \wedge \tau}(x)]$ follows. It is immediate to see by Lemma [5.8] that

$$\|\mathbb{E}_{P_\alpha} [J_0 \Phi \cdot M_{t \wedge \tau}(x)]\| \leq C \|\Phi\|_{\mathcal{H}} \mathbb{E}_Z \left[ e^{-\int_0^\tau (V(z_r) - E) dr} \right],$$

where $C = \sup_{x \in \mathbb{R}^d} \|\Phi_b(x)\|$. Since $B_{\tau_t} = z_t$ in law, we then have

$$\|\mathbb{E}_{P_\alpha} [J_0 \Phi \cdot M_t(x)]\| \leq \|\Phi\|_{\mathcal{H}} \mathbb{E}_Z \left[ e^{-\int_0^\tau (V(z_r) - E) dr} \right]. \quad (5.14)$$

From $\|\Phi_b(x)\|_{L^2(\mathcal{G})} = \sup_{\Phi \in L^2(\mathcal{G}), \Phi \neq 0} \mathbb{E}_{P_\alpha} [J_0 \Phi \cdot M_t(x)] / \|\Phi\|$, the lemma follows. \quad $\square$

**Theorem 5.12 (Fall-off of bound states)** Let $V = V_+ - V_- \in \mathcal{H}_{Kato}$. Suppose Assumption [2.1].

1. Suppose that $\lim_{|x| \to \infty} V_-(x) + E = a < 0$. Then

   **Case** $m = 0$ : there exists $C > 0$ such that $\frac{\|\Phi_b(x)\|_{L^2(\mathcal{G})}}{\|\Phi_b\|_{\mathcal{H}}} \leq \frac{C}{1 + |x|^{d+1}}$;

   **Case** $m > 0$ : there exist $C > 0$ and $c > 0$ such that $\frac{\|\Phi_b(x)\|_{L^2(\mathcal{G})}}{\|\Phi_b\|_{\mathcal{H}}} \leq C e^{-c|x|}$. 41
(2) Suppose that $\lim_{|x| \to \infty} V(x) = \infty$. Then there exist $C > 0$ and $c > 0$ such that $\|\Phi_b(x)\|_{L^2(\mathcal{F})} \leq C e^{-c|x|} \|\Phi_b\|_\mathcal{F}$.

**Proof:** (1) Suppose that $V_-(x) + E < a + \epsilon < 0$ for all $x$ such that $|x| > R$, and $\tau_R = \inf\{s| |z_s| < R\}$ is a stopping time with respect to the filtration $(\mathcal{M}_t)_{t \geq 0}$. By (5.13) we have $\|\Phi_b(x)\| \leq \|\Phi_b\|_\mathcal{F} \mathbb{E}_Z^x [e^{-2(\epsilon + a)(t \wedge \tau_R)}]$ for $|x| > R$. In a similar way to [CMS90 Proposition IV.1] we have

$$\mathbb{E}_Z^x [e^{-2(\epsilon + a)(t \wedge \tau_R)}] \leq \frac{C}{1 + |x|^{d+1}}, \quad m = 0, \quad m > 0. \tag{5.15}$$

Thus (1) follows.

(2) Let $\tau_R = \inf\{s| |z_s| > R\}$, which is the stopping time with respect to the filtration $(\mathcal{M}_t)_{t \geq 0}$. Let $W(x) = \inf\{V(y)| |x - y| < R\}$. Then it can be shown in [HIL13 Theorem 4.7] and [CMS90 Proposition IV.4] that

$$\mathbb{E}_Z^x \left[ e^{(t \wedge \tau_R)E - \int_0^{t \wedge \tau_R} V(z_r)dr} \right] \leq e^{-t(W(x) - E)} + C e^{-\alpha R e^{ct}} \tag{5.16}$$

with some constants $\alpha, c$ and $C$. Inserting $R = p|x|$ with any $0 < p < 1$, we see that $W(x) \to \infty$ as $|x| \to \infty$. Substituting $t = \delta|x|$ for sufficiently small $\delta > 0$ and $R = p|x|$ with some $0 < p < 1$, (2) follows. $\square$

### 6 Gaussian domination of ground states

Let $H = H_{qf}$ or $H_K$ in this section. Throughout this section, when we consider $H_{qf}$ we suppose Assumptions 2.1 and 2.2 and when we consider $H_K$ we suppose Assumption 2.1. A fundamental assumption in this section is that $H$ has a ground state $\varphi_g$.

**Assumption 6.1** Suppose that $m \geq 0$ and $H$ has a ground state $\varphi_g$, i.e.,

$$H \varphi_g = E \varphi_g, \quad E = \inf \sigma(H). \tag{6.1}$$

The existence of ground state is studied in [HH13a] [KMS09] [KMS11].
Corollary 6.2 The operator $e^{i\frac{\pi}{2} N t} e^{-ih} e^{-i\frac{\pi}{2} N t}$ is positivity improving for $t > 0$, i.e., $(F, e^{i\frac{\pi}{2} N t} e^{-ih} e^{-i\frac{\pi}{2} N t} G) > 0$ for any $F \geq 0$ and $G \geq 0 \ (F \not= 0, G \not= 0)$. In particular $e^{i\frac{\pi}{2} N} \varphi_g$ is strictly positive and then the ground state of $H$ is unique up to multiplication constants.

Proof: It is established in [Hir00a] that $J_0 e^{i\frac{\pi}{2} N} e^{-i\alpha E} e^{-i\frac{\pi}{2} N} J_t$ is positivity improving for arbitrary $t \in \mathcal{T} \ L^2_{\mathbb{R}}(\mathbb{R}^d)$. Thus the first statement follows. Since $e^{i\frac{\pi}{2} N}$ is unitary, the statement on the uniqueness also follows from the Perron-Frobenius theorem. $\square$

For an arbitrary fixed $0 \leq \phi \in L^2(\mathbb{R}^d)$ but $\phi \not= 0$, we define

$$\phi_t = e^{-t(H-E)(\phi \otimes 1)} , \quad \varphi^t_g = \phi_t / \|\phi_t\|.$$  \hspace{1cm} (6.2)

Then it follows that $\varphi^t_g \to \varphi_g$ strongly as $t \to \infty$, since $(\phi \otimes 1, \varphi_g) \not= 0$. Let

$$\mathcal{L}_t = \phi(B_{-T} \phi(B_{T}) e^{-\frac{\pi^2}{2q_e[I[-t,t]} e^{-\int_{-t}^t V(B_s) ds}}, \quad t \geq 0.$$  \hspace{1cm} (6.3)

Remark 6.3 We formally write the pair interaction $W_{SRPF} = q_e(I[-t,t])$ by

$$q_e(I[-t,t]) = -\frac{\alpha^2}{2} \sum_{\mu,\nu=1}^d \int_{-T}^{T_t} dB^\mu_s \int_{-T}^{T_t} dB^\nu_r W_{\mu\nu}(T^*_s - T^*_r, B_s - B_r),$$  \hspace{1cm} (6.4)

where the pair potential, $W_{\mu\nu}(t,X)$, is given by

$$W_{\mu\nu}(t,X) = \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} \left( \delta_{\mu\nu} - \frac{k_k k_\nu}{|k|^2} \right) e^{-ik \cdot X} e^{-\omega(k)|t|} dk.$$  \hspace{1cm} (6.5)

Definition 6.4 Define the probability measure $\mu_{t_{SRPF}} = \mu_t$ on the measurable space $(\Omega_P \times \Omega_P, \mathcal{B}_P \times \mathcal{B}_P)$ by

$$\mathcal{B}_P \times \mathcal{B}_P \ni A \mapsto \mu_t(A) = \frac{1}{Z_t} \int_{\mathbb{R}^d} dx E^{x,0}_{P\otimes P}[\mathbb{1}_A L_d], \quad t \geq 0.$$  \hspace{1cm} (6.6)

Here $Z_t$ is the normalizing constant such that $\mu_t(\Omega_P \times \Omega_P) = 1$.

We define the self-adjoint operator $A_\xi$ in $\mathcal{H}$ by $A_\xi = \int_{\mathbb{R}^d} A(\xi, -x) dx$, where $\xi \in \mathcal{T} \ L^2_{\mathbb{R}}(\mathbb{R}^d)$. Then we have

$$\langle \varphi_g, e^{-i\beta A_\xi} \varphi_g \rangle = \lim_{t \to \infty} \frac{(e^{-iH} \phi \otimes 1, e^{-i\beta A_\xi} e^{-iH} \phi \otimes 1)}{(e^{-iH} \phi \otimes 1, e^{-iH} \phi \otimes 1)}, \quad \beta \in \mathbb{R}.$$  \hspace{1cm} (6.7)
Lemma 6.5 Let \( \beta \in \mathbb{R} \). Then it follows that

\[
\frac{(e^{-tH} \phi \otimes \mathbb{1}, e^{-i\beta A\xi} e^{-tH} \phi \otimes \mathbb{1})}{(e^{-tH} \phi \otimes \mathbb{1}, e^{-tH} \phi \otimes \mathbb{1})} = \mathbb{E}_{\mu_t} [e^{-(2\alpha \beta q_0)(I[-t, t], j_0\xi) + \beta^2 q_0(j_0\xi)}].
\]

(6.8)

Proof: This follows from Corollary 3.18

Note that both \( q_0(I[-t, t], j_0\xi) \) and \( q_0(j_0\xi) \) do not depend on \( x \).

Corollary 6.6 Let \( \xi = \bigoplus_{\nu=1}^d \delta_{\mu, \xi} \) and \( A_{\mu} = \int_{\mathbb{R}^d} A(\xi(\cdot - x)) dx \). We suppose that \( \text{supp}\hat{\xi}_\mu \cap \text{supp}\hat{\phi} = \emptyset \). Then

\[
(\varphi_g, A_{\mu}^n \varphi_g)_{\mathcal{H}} = (\mathbb{1}, A_{\mu}(0)^n \mathbb{1})_{L^2(\mathbb{R})}
\]

\[
= \begin{cases} (-1)^m (2m - 1)! (\frac{1}{2} \int_{\mathbb{R}^d} |\hat{\xi}_\mu(k)|^2 (1 - \frac{k^2}{|k|^2}) dk)^m & n = 2m \\ 0 & n = 2m - 1 \end{cases}
\]

(6.9)

where \( A_{\mu}(0) = A(\xi) \).

Proof: Formally we see that

\[
q_0(I[-t, t], j_0\xi) = \frac{1}{2} \sum_{\nu=1}^d \int_{-T_t}^{T_t} dB_\nu \left( \int_{\mathbb{R}^d} \hat{\xi}_\mu(k) \hat{\phi}(k) \frac{1}{\sqrt{\omega(k)}} e^{-T_t^* \omega(k)} e^{-ikB_{\nu}} \left( \delta_{\mu, \nu} - \frac{k_{\mu}k_{\nu}}{|k|^2} \right) dk \right) = 0.
\]

This is proven rigorously from the definition of \( I[-t, t] \). By (6.8) and taking the limit \( t \to \infty \), we have \( (\varphi_g, e^{-i\beta A_{\mu} \varphi_g}) = e^{-\beta^2 q_0(j_0\xi)} \). Since \( \varphi_g \in D(A_{\mu}^n) \) by Theorem 6.8 below, we derive (6.9) by taking \( n \)-times derivative at \( \beta = 0 \).

Lemma 6.7 Suppose that \( \beta < (2q_0(j_0\xi))^{-1} \). Then \( \varphi_g^t \in D(e^{\beta A_{\mu}^2 / 2}) \) and

\[
\| e^{\beta A_{\mu}^2 / 2} \varphi_g^t \|^2 = (1 - 2\beta q_0(j_0\xi))^{-1/2} \mathbb{E}_{\mu_t} \left[ \frac{e^{-\beta \alpha q_0(I[-t, t], j_0\xi)}}{(1 - 2\beta q_0(j_0\xi))} \right].
\]

(6.10)

Proof: We have \( (\varphi_g^t, e^{-i\beta A_{\mu} \varphi_g^t}) = \mathbb{E}_{\mu_t} [e^{-\alpha q_0(I[-t, t], j_0\xi)}] e^{-\frac{1}{2}k^2 q_0(j_0\xi)} \). By the Gaussian transformation with respect to \( k \), we see that

\[
(\varphi_g^t, e^{-A_{\mu}^2 / 2} \varphi_g^t) = \langle 2\pi \rangle^{-1/2} \int_{\mathbb{R}} e^{-\frac{k^2}{2}} \mathbb{E}_{\mu_t} [e^{-\alpha q_0(I[-t, t], j_0\xi)}] e^{-\frac{1}{2}k^2 q_0(j_0\xi)} dk,
\]

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and by Fubini’s lemma, we can exchange $\int dk$ and $\int d\mu_t$. Then

$$
(\varphi^t_g, e^{-A^2_t/2} \varphi^t_g) = \frac{1}{\sqrt{1 + qE(j_0\xi)}} \mathbb{E}_{\mu_t} \left[ e^{\alpha^2 qE(I_{[t-\epsilon,t]}, j_0\xi)^2} \right].
$$

(6.11)

Replacing $\xi$ with $\sqrt{-2\beta} \xi$ for $\beta < 0$, we have (6.10) with $\beta < 0$. We can extend this to $\beta < (2qE(j_0\xi))^{-1}$ by an analytic continuation. For notational simplicity we set $b = qE(j_0\xi)$. Let

$$
\chi(z) = (\varphi^t_g, e^{-zA^2_t} \varphi^t_g), \quad \rho(z) = \mathbb{E}_{\mu_t} \left[ \exp \left( z\alpha^2 qE(I_{[t-\epsilon,t]}, j_0\xi)^2 \right) \right], \quad \theta(z) = \frac{2zb}{1 + 2zb}.
$$

Then (6.10) is realized as

$$
\chi(z) = \frac{1}{\sqrt{1 + 2zb}} \rho \circ \theta(z)
$$

(6.12)

for $z \geq 0$. Notice that $\mathbb{E}_{\mu_t} \left[ \exp \left( z\alpha^2 qE(I_{[t-\epsilon,t]}, j_0\xi)^2 \right) \right] < \infty$ for all $z > 0$. Then we know that

$$
\rho(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_{\mu_t} \left[ \left( \alpha^2 qE(I_{[t-\epsilon,t]}, j_0\xi)^2 \right)^n \right] z^n
$$

(6.13)

for $z \geq 0$, and hence $\rho(z)$ can be analytically continued to the whole complex plane $\mathbb{C}$, which is denoted by $\bar{\rho}(z)$ and it follows that $\bar{\rho}(z) = \mathbb{E}_{\mu_t} \left[ \exp \left( z\alpha^2 qE(I_{[t-\epsilon,t]}, j_0\xi)^2 \right) \right]$ for $z \in \mathbb{C}$. Then $\frac{1}{\sqrt{1 + 2zb}} \rho \circ \theta(z)$ can be analytically continued to the domain: (Fig.2)

$$
D = \{ z \in \mathbb{C} | |z| < (2b)^{-1} \} \cup \{ z \in \mathbb{C} | \Re z > 0 \}.
$$

In particular the radius of convergence $r$ of $\frac{1}{\sqrt{1 + 2zb}} \rho \circ \theta(z)$ at $z = 0$ satisfies that $1 - \epsilon < r < 1$ for an arbitrary $\epsilon > 0$. By the equality (6.12), $\chi$ can be also analytically continued to the domain $D$, which is denoted by $\bar{\chi}$. Let $\epsilon > 0$. Then

$$
\chi(z) = \sum_{n=0}^{\infty} \left( \frac{-1}{n!} \int_0^\infty \lambda^n e^{-\epsilon\lambda} dE(\lambda) \right) (z - \epsilon)^n
$$

(6.14)

for $0 < \epsilon - z$, where $dE(\lambda)$ denotes the spectral resolution of the self-adjoint operator $A^2_{\xi}$ with respect to $\varphi^t_g$. Since we have

$$
\frac{1}{\sqrt{1 + 2zb}} \rho \circ \theta(z) = \sum_{n=0}^{\infty} a_n (z - \epsilon)^n
$$

(6.15)

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for \( z \in \mathbb{C} \) such that \(|z - \epsilon| < \sqrt{\frac{1}{(2b)^2} + \epsilon^2}\). Comparing both expansions (6.14) and (6.15) we see that
\[
a_n = \frac{(-1)^n}{n!} \int_0^\infty \lambda^n e^{-\epsilon \lambda} dE(\lambda)
\]
and by (6.15) we have
\[
\sqrt{1 + 2zb} \bar{\rho} \circ \theta(z) = \sum_{n=0}^\infty \left( \frac{(-1)^n}{n!} \int_0^\infty \lambda^n e^{-\epsilon \lambda} dE(\lambda) \right) (z - \epsilon)^n.
\]
(6.16)

In particular it follows that for \(-\delta < 0\) with \(\epsilon + \delta < \sqrt{\frac{1}{(2b)^2} + \epsilon^2}\),
\[
\bar{\chi}(\delta) = \sum_{n=0}^N \left( \frac{1}{n!} \int_0^\infty \lambda^n e^{-\epsilon \lambda} dE(\lambda) \right) (\delta + \epsilon)^n < \infty.
\]

Thus
\[
\sum_{n=0}^\infty \left( \frac{1}{n!} \int_0^N \lambda^n e^{-\epsilon \lambda} dE(\lambda) \right) (\delta + \epsilon)^n = \int_0^N e^{\delta \lambda} dE(\lambda)
\]
and take \(N \to \infty\) on both sides we have \(\bar{\chi}(\delta) = \int_0^\infty e^{\delta \lambda} dE(\lambda) < \infty\). Since \(\epsilon > 0\) is arbitrary, then it follows that \((\varphi_t', e^{zA_{\xi}^2} \varphi_t') < \infty\) for \(\beta > (2b)^{-1}\).

\[\square\]

**Theorem 6.8 (Gaussian domination of the ground state)** Let \(\beta < (2q_E(j_0 \xi))^{-1}\). Then \(\varphi \in D(e^{\beta A_{\xi}^2/4})\) follows.

**Proof:** By Lemma 6.7 we have the uniform bound \(\|e^{\beta A_{\xi}^2} \varphi_t\|^2 \leq \frac{1}{\sqrt{1 - 2\beta q_E(j_0 \xi)^2}}\) in \(t\). Thus there exists a subsequence \(t'\) such that \(\|e^{\beta A_{\xi}^2/4} \varphi_{t'}\|^2\) converges to some \(c\) as
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\( t' \to \infty \). We reset \( t' \) as \( t \). We claim that \( \{e^{\beta A_t^2/4} \varphi_t^t\}_t \) is a Cauchy sequence. Directly we have \( \|e^{\beta A_t^2/4} \varphi_t^t - e^{\beta A_s^2/4} \varphi_s^s\|^2 = \|e^{\beta A_t^2/4} \varphi_t^t\|^2 + \|e^{\beta A_s^2/4} \varphi_s^s\|^2 - 2(\varphi_t^t, e^{\beta A_t^2/2} \varphi_t^t). \) Note that \( \varphi_t^t \) strongly converges to \( \varphi_g \) as \( t \to \infty \). Since the uniform bound of \( \|e^{\beta A_t^2/4} \varphi_t^t\|^2 \) implies that

\[
(\varphi_g^s, e^{\beta A_t^2/2} \varphi_t^t) = (\varphi_g^s - \varphi_t^t, e^{\beta A_t^2/2} \varphi_t^t) + \|e^{\beta A_t^2/4} \varphi_t^t\|^2 \to c
\]

as \( t, s \to \infty \), we obtain that \( \lim_{t,s \to \infty} \|e^{\beta A_t^2/4} \varphi_t^t - e^{\beta A_s^2/4} \varphi_s^s\| = 0 \) and \( e^{\beta A_t^2/4} \varphi_t^t, t > 0, \) is a convergent sequence. Hence the closedness of \( e^{\beta A_t^2/4} \) yields the desired results. \( \square \)

7 Measures associated with the ground state

Similar to Section 6 in this section let \( H = H_{\text{qf}} \) or \( H_K \), and we suppose that \( H \) has a ground state \( \varphi_g \).

7.1 Outline

We set \( \mathcal{X} = \Omega_p \times \Omega_v \) and \( W^x = P^x \otimes \nu \) in what follows. Let \( X_t = B_{T_t} \) for \( t \geq 0 \) and \( X_{-t} = B_{-T_t} \) for \( -t < 0 \). Thus \( t \mapsto X_t(\omega_1, \omega_2) = B_{T_t(\omega_2)}(\omega_1) \) for \( (\omega_1, \omega_2) \in \mathcal{X} \) is a cádlág path, i.e., paths are right continuous and the left limits exist. Let \( \mathcal{F}_{[-s,s]} = \sigma(X_r; r \in [-s, s]) \). Then

\[
\mathcal{G}_t = \bigcup_{0 \leq s \leq t} \mathcal{F}_{[-s,s]}, \quad \mathcal{G} = \bigcup_{0 \leq s} \mathcal{F}_{[-s,s]} \tag{7.1}
\]

are finitely additive families of sets. We define the correction of probability spaces by

\[
(\mathcal{X}, \sigma(\mathcal{G}), \mu_t), \quad t > 0, \tag{7.2}
\]

where \( \mu_t \) is given by \( \{6.6\} \). We show in this section that there exists a probability measure \( \mu_\infty \) on \( (\mathcal{X}, \sigma(\mathcal{G})) \) such that \( \mu_t \to \mu_\infty \) as \( t \to \infty \) in the local weak sense.

The outline of the idea to show the convergence is as follows. First by using \( \varphi_g^t \) we define the family of finitely additive set functions \( \rho_t \) on \( (\mathcal{X}, \mathcal{G}_t) \), \( t > 0 \), and we denote the extension to the probability measure on \( (\mathcal{X}, \sigma(\mathcal{G}_t)) \) by \( \tilde{\rho}_t \). Thus we define the probability space

\[
(\mathcal{X}, \sigma(\mathcal{G}_t)), \tilde{\rho}_t). \tag{7.3}
\]
We show in Lemma 7.5 by using functional integrations that
\[ \bar{\rho}_t(A) = \rho_t(A) = \mu_t(A) \] (7.4)
for \( A \in \mathcal{G} \) for all \( s \leq t \). Next by using the ground state \( \varphi_g \) we define a finitely additive set function \( \mu \) on \((\mathcal{X}, \mathcal{G})\) and denote the extension to the probability measure on \((\mathcal{X}, \sigma(\mathcal{G}))\) by \( \mu_\infty \). Thus we define the probability space
\[ (\mathcal{X}, \sigma(\mathcal{G})), \mu_\infty). \] (7.5)
By applying the fact that \( \varphi_g^t \) strongly converges to \( \varphi_g \) as \( t \to \infty \), we prove that
\[ \rho_t(A) \to \mu(A), \quad t \to \infty, \] (7.6)
for \( A \in \mathcal{G} \) in Lemma 7.6 which, together with (7.4), implies that
\[ \mu_t(A) \to \mu_\infty(A), \quad A \in \mathcal{G} \] (7.7)
and \( \mu_t \) converges to the measure \( \mu_\infty \) in the sense of local weak. By the construction of \( \mu_\infty \) we can show an explicit form of \( \mu_\infty(A) \) for \( A \in \mathcal{G} \). See Figure 3.

7.2 Local weak convergences

Let us define
\[ J_{[-t,t]} = J^* - t e^{-\int_{-t}^t V(X_s) ds} e^{-i A_E([s])} J_t. \] (7.8)
Note that for a.s. \((\omega_1, \omega_2) \in \mathcal{X}, J_{[-t,t]} : L^2(\mathcal{D}) \to L^2(\mathcal{D})\) is a bounded linear operator. Define an additive set function \( \mu : \mathcal{G} \to \mathbb{R} \) by
\[ \mu(A) = e^{2Et} \int_{\mathbb{R}^d} d\mathbb{P}^\mathbb{P}_x [\|A(\varphi_g(X_{-t})), J_{[-t,t]}(\varphi_g(X_t))\|], \quad A \in \mathcal{F}_{[-t,t]}. \] (7.9)

Lemma 7.1 It follows that \( \mu(A) \geq 0 \) for \( A \in \mathcal{F}_{[-t,t]} \).
Proof: We note that $e^{i\frac{N}{2}\varphi} > 0$ and $e^{i\frac{N}{2}J_{[-t,t]}e^{-i\frac{N}{2}}}$ is positivity improving by Corollary 6.2. Then

$$
\mu(A) = e^{2Et} \int_{\mathbb{R}^d} dx E_W^x \left[ \mathbb{I}_A(e^{i\frac{N}{2}\varphi}(X_{-t}), e^{i\frac{N}{2}J_{[-t,t]}e^{-i\frac{N}{2}}\varphi}(X_t)) \right] \geq 0,
$$

the lemma follows. \ \Box

Lemma 7.2 The set function $\mu$ is well defined, i.e., for $A \in \mathcal{F}_{[-s,s]} \subset \mathcal{F}_{[-s,s]}

$$
\mu(A) = e^{2Es} \int_{\mathbb{R}^d} dx E_W^x \left[ \mathbb{I}_A(\varphi(X_{-s}), J_{[-s,s]}\varphi(X_s)) \right]
$$

Proof: Let $\mu(t) = \mu[\mathcal{F}_{[-t,t]}].$ Then $\mu(t)$ is a probability measure on $(\mathcal{F}, \mathcal{F}_{[-t,t]}).$ Let $-s < -t = t_0 < t_1 < \cdots < t_n = t < s.$ Then by Corollary 3.18 the finite dimensional distribution is given by

$$
\mu^{t_0,\ldots,t_n}(A_0 \times \cdots \times A_n) = \mu(X_{t_0} \in A_0, \cdots, X_{t_n} \in A_n)
$$

$$
= e^{2Et} \int_{\mathbb{R}^d} dx E_W^x \left[ \prod_{j=0}^n \mathbb{I}_{A_j}(X_{t_j}) \right] (\varphi(X_{-t}), J_{[-t,t]}\varphi(X_t))
$$

$$
= (\varphi, \mathbb{I}_{A_0} e^{-(1-t_0)(H-E)} \cdots e^{-(t_0-t_n)(H-E)} \mathbb{I}_{A_n} \varphi).
$$

By $e^{-(t_0+s)(H-E)}\varphi = \varphi$ we have

$$
= (\varphi, e^{-(t_0+s)(H-E)} \mathbb{I}_{A_0} e^{-(t_1-t_0)H} \cdots e^{-(t_n-t_{n-1})H} \mathbb{I}_{A_n} e^{-(s-t_n)(H-E)} \varphi)
$$

$$
= e^{2Es} \int_{\mathbb{R}^d} dx E_W^x \left[ \prod_{j=0}^n \mathbb{I}_{A_j}(X_{t_j}) \right] (\varphi(X_{-s}), J_{[-s,s]}\varphi(X_s))
$$

$$
= \mu_{t_0,\ldots,t_n}(A_0 \times \cdots \times A_n).
$$

It can be also seen that the finite dimensional distributions $\mu_{(t)}^{\Lambda}, \Lambda \subset [-t,t], \#\Lambda < \infty,$ satisfy the consistency condition, i.e.,

$$
\mu_{(t)}^{t_0,\ldots,t_n}(A_0 \times \cdots \times A_n) = \mu_{(t)}^{t_0,\ldots,t_n,t_{n+1},\ldots,t_{n+l}}(A_0 \times \cdots \times A_n \times \prod_{i=1}^l \mathbb{R}^d).
$$

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By the Kolmogorov extension theorem there exists a unique probability space \((\mathcal{Y}, \mathcal{B}_q, q)\) and a stochastic process \((Y_s)_{s \in [-t, t]}\) up to isomorphisms (e.g., \cite[Theorem 2.1]{Sim05}) such that \(\mathcal{B}_q = \sigma(Y_s, s \in [-t, t])\), and \(\mu_{(t)}^{t_0, \ldots, t_n}(A_0 \times \cdots \times A_n) = q(Y_{t_0} \in A_0, \ldots, Y_{t_n} \in A_n)\). By the uniqueness, \((\mathcal{Y}, \mathcal{B}_q, q)\) and \((\mathcal{Y'}, \mathcal{F}_{[-t, t]}, \mu_{(t)})\) are isomorphic, and also is \((\mathcal{Y}, \mathcal{B}_q, q)\) and \((\mathcal{Y'}, \mathcal{F}_{[-t, t]}, \mu_{(s)}|_{\mathcal{F}_{[-t, t]}})\). Hence \(q(A) = \mu_{(s)}(A) = \mu_{(t)}(A)\) for \(A \in \mathcal{F}_{[-t, t]}\) follows.

Clearly \(\mu\) is a completely additive set function on \((\mathcal{Y'}, \mathcal{G})\). There exists a unique probability measure \(\mu_{\infty}\) on \((\mathcal{Y'}, \sigma(\mathcal{G}))\) such that \(\mu_{\infty}(A) = \mu(A)\) for \(A \in \mathcal{G}\) by the Hopf theorem.

**Theorem 7.3 (Local weak convergence and uniqueness)** The probability measures \(\mu_t\) converges to \(\mu_{\infty}\) in the local weak sense, i.e., \(\mu_t(A) \to \mu_{\infty}(A)\) as \(t \to \infty\) for each \(A \in \mathcal{G}\), and \(\mu_{\infty}\) is independent of \(\phi\).

Before giving a proof of Theorem 7.3 we need several lemmas. We define an additive set function \(\rho_t : \mathcal{G} \to \mathbb{R}\) by

\[
\rho_t(A) = e^{2Es} \int_{\mathbb{R}^d} dx \mathbb{E}^x_{W} \left[ \mathbb{I}_A \left( \frac{\phi_{t-s}(X_0)}{\|\phi_t\|}, J_{[-s, s]} \frac{\phi_{t-s}(X_s)}{\|\phi_t\|} \right) \right] \tag{7.10}
\]

for \(A \in \mathcal{F}_{[-s, s]}\) with \(s \leq t\).

**Lemma 7.4** The set function \(\rho_t\) satisfies \(\rho_t(A) \geq 0\) and is well defined, i.e.,

\[
\rho_t(A) = e^{2Er} \int_{\mathbb{R}^d} dx \mathbb{E}^x_{W} \left[ \mathbb{I}_A \left( \frac{\phi_{t-r}(X_0)}{\|\phi_t\|}, J_{[-r, r]} \frac{\phi_{t-r}(X_r)}{\|\phi_t\|} \right) \right] = e^{2Es} \int_{\mathbb{R}^d} dx \mathbb{E}^x_{W} \left[ \mathbb{I}_A \left( \frac{\phi_{t-s}(X_0)}{\|\phi_t\|}, J_{[-s, s]} \frac{\phi_{t-s}(X_s)}{\|\phi_t\|} \right) \right] \tag{7.11}
\]

for all \(r \leq s \leq t\).

**Proof:** \(\rho_t(A) \geq 0\) follows in a similar way to Lemma 7.1. The proof of the second statement is similar to that of Lemma 7.2. The left-hand side of (7.11) is denoted by \(\rho_{(r)}(A)\) and the right-hand side by \(\rho_{(s)}(A)\). The finite dimensional distribution of \(\rho_{(r)}\) is given by

\[
\rho_{(r)}^{t_0, \ldots, t_n}(A_0 \times \cdots \times A_n) = \rho_{(r)}(X_{t_0} \in A_0, \ldots, X_{t_n} \in A_n)
= \frac{e^{2Er}}{\|\phi_t\|^2} \int_{\mathbb{R}^d} dx \mathbb{E}^x_{W} \left[ \prod_{j=0}^{n} \mathbb{I}_{A_j}(X_{t_j}) \right] \left( \phi_{t-r}(X_0), J_{[-r, r]} \phi_{t-r}(X_r) \right).
\]

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By Corollary 3.18, the right-hand side above can be represented as

\[
\frac{1}{\|\phi_t\|^2} \left( \phi_{t-r} e^{-(t_0+r)(H-E)} \mathbb{I}_{A_0} e^{-(t_1-t_0)(H-E)} \cdots e^{-(t_{n-1}-t_{n-1})(H-E)} \mathbb{I}_{A_n} e^{-(r-t_n)(H-E)} \phi_{t-r} \right)
\]

\[
= \frac{1}{\|\phi_t\|^2} \left( \phi \otimes \mathbb{I}, e^{-(t_0+\epsilon)(H-E)} \mathbb{I}_{A_0} e^{-(t_1-t_0)(H-E)} \cdots e^{-(t_{n-1}-t_{n-1})(H-E)} \mathbb{I}_{A_n} e^{-(r-t_n)(H-E)} \phi \otimes \mathbb{I} \right)
\]

\[
= \frac{1}{\|\phi_t\|^2} \left( \phi_{t-s} e^{-(t_0+s)(H-E)} \mathbb{I}_{A_0} e^{-(t_1-t_0)(H-E)} \cdots e^{-(t_{n-1}-t_{n-1})(H-E)} \mathbb{I}_{A_n} e^{-(s-t_n)(H-E)} \phi_{t-s} \right)
\]

\[
= \frac{e^{2Es}}{\|\phi_t\|^2} \int_{\mathbb{R}^d} d\mathbb{E}^x_W \left[ \prod_{j=0}^{n} \mathbb{I}_{A_j}(X_{t_j}) \left( \phi_{t-s}(X_{-s}), J_{[-s,s]} \phi_{t-s}(X_s) \right) \right]
\]

\[
= \rho_{t_0 \ldots t_n}(A_0 \times \cdots \times A_n).
\]

Note that \( \rho^A \) and \( \rho^A \), \( A \subset [-t, t], \#A < \infty \), satisfy the consistency condition. Note that \( \rho(r) \mid \mathcal{F}_{[-r, r]} \) and \( \rho(s) \mid \mathcal{F}_{[-r, r]} \) are probability measures on \((\mathcal{F}, \mathcal{F}_{[-r, r]} \). By the Kolmogorov extension theorem we see that \( \rho(r)(A) = \rho(s)(A) \) for \( A \in \mathcal{F}_{[-r, r]} \subset \mathcal{F}_{[-s, s]} \). Then the lemma follows.

By the Hopf theorem there exists a probability measure \( \tilde{\rho}_t \) on \((\mathcal{F}, \sigma(\mathcal{F}))\) such that \( \rho_t = \tilde{\rho}_t \).

**Lemma 7.5** Let \( s \leq t \) and \( A \in \mathcal{G}_s \). Then \( \tilde{\rho}_t(A) = \mu_t(A) \).

**Proof:** For \( A = \{t_0, t_1, \ldots, t_n\} \subset [-s, s] \) and \( A_0 \times \cdots \times A_n \in \times_{j=0}^{n} \mathcal{B}(\mathbb{R}^d) \), we define

\[
\rho^A_t(A_0 \times \cdots \times A_n) = \rho_t(X_{t_0} \in A_0, \ldots, X_{t_n} \in A_n)
\]

\[
= \frac{e^{2Es}}{\|\phi_t\|^2} \int_{\mathbb{R}^d} d\mathbb{E}^x_W \left[ \prod_{j=0}^{n} \mathbb{I}_{A_j}(X_{t_j}) \left( \phi_{t-s}(X_{-s}), J_{[-s,s]} \phi_{t-s}(X_s) \right) \right]
\]

and

\[
\mu^A_t(A_0 \times \cdots \times A_n) = \mu_t(X_{t_0} \in A_0, \ldots, X_{t_n} \in A_n) = \frac{1}{Z_t} \int_{\mathbb{R}^d} d\mathbb{E}^x_W \left[ \prod_{j=0}^{n} \mathbb{I}_{A_j}(X_{t_j}) \right] \mathcal{L}_t.
\]

Both \( \rho^A_t \) and \( \mu^A_t \) are probability measures on \((\mathcal{F}^d, \mathcal{B}(\mathbb{R}^d)^A)\). We have

\[
\mu^A_t(A_0 \times \cdots \times A_n) = \frac{\langle \phi \otimes \mathbb{I}, e^{-(t_0+t_1)(H-E)} \cdots e^{-(t_{n-1}+t_n)(H-E)} \phi \otimes \mathbb{I} \rangle}{\|\phi_t\|^2}
\]

\[
= \frac{e^{2Es} \langle \phi_{t-s}, e^{-(t_0+s)(H-E)} \cdots e^{-(t_{n-1}+t_n)(H-E)} \phi \otimes \mathbb{I} \rangle}{\|\phi_t\|^2}
\]

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by the definition of $\phi_{t-s}$. The right-hand side above can be expressed as
\[
e^{2Es} \int_{\mathbb{R}^d} dx E_W \left[ \prod_{j=0}^{n} \mathbb{1}_{A_j}(X_{t_j}) \left( \frac{\phi_{t-s}(X_0)}{\|\phi_t\|}, J_{[-s,s]} \frac{\phi_{t-s}(X_s)}{\|\phi_t\|} \right) \right].
\]
Then $\rho_t^A(A_0 \times \cdots \times A_n) = \mu_t^A(A_0 \times \cdots \times A_n)$ follows. The probability measures $\mu^A_t$ and $\rho^A_t$ satisfy the consistency condition. Then by the Kolmogorov extension theorem there exists a unique probability space $(\mathcal{Y}, \mathcal{B}, q)$ and stochastic process $Y_s$ such that $\mathcal{B} = \sigma(Y_s, s \in [-t,t])$ and $q(Y_0 \in A_0, \ldots, Y_n \in A_n) = \mu^A_{t_0,\ldots,t_n}(A_0 \times \cdots \times A_n) = \rho^A_{t_0,\ldots,t_n}(A_0 \times \cdots \times A_n)$. On the other hand it holds that $\mu^A_{t_0,\ldots,t_n}(A_0 \times \cdots \times A_n) = \rho^A_{t_0,\ldots,t_n}(A_0 \times \cdots \times A_n) = \bar{\rho}_t(A_0 \times \cdots \times A_n) = \mu_t[\mathbb{g}_t(A_0 \times \cdots \times A_n)]$. Hence $\bar{\rho}_t = q = \mu_t[\mathbb{g}_t]$ follows by the uniqueness of extensions.

\[\text{Lemma 7.6} \quad \text{Let } A \in \mathcal{G}. \text{ Then } \lim_{t \to \infty} \mu_t(A) = \mu_\infty(A).\]

\textbf{Proof:} Suppose that $A \in \mathcal{G}_s$ with some $s$. By Lemma 7.5 we have
\[
\lim_{t \to \infty} \mu_t(A) = \lim_{t \to \infty} \bar{\rho}_t(A) = \lim_{t \to \infty} e^{2Es} \int_{\mathbb{R}^d} dx E_W \left[ \prod_{j=0}^{n} \mathbb{1}_{A_j}(X_{t_j}) \left( \frac{\phi_{t-s}(X_0)}{\|\phi_t\|}, J_{[-s,s]} \frac{\phi_{t-s}(X_s)}{\|\phi_t\|} \right) \right].
\]
Since $\phi_t \to \varphi_g$ strongly as $t \to \infty$, we have
\[
\lim_{t \to \infty} \mu_t(A) = e^{2Es} \int_{\mathbb{R}^d} dx E_W \left[ \prod_{j=0}^{n} \mathbb{1}_{A_j}(\varphi_g(X_0)), J_{[-s,s]} \varphi_g(X_s) \right] = \mu_\infty(A).
\]
Then the lemma follows. \\[\square\]

Now we state the proof of Theorem 7.3.

\textbf{Proof of Theorem 7.3} By Lemma 7.6 it follows that $\mu_t(A) \to \mu_\infty(A)$ for $A \in \mathcal{G}$. Next we show that $\mu_\infty$ is independence of the choice of $\phi$. Suppose that $\mu'_\infty$ is a local weak limit of $\mu'_t$ defined by $\mu_t$ with $\phi$ replace by $\phi'$ such that $0 \leq \phi' \in L^2(\mathbb{R}^d)$. By the construction of $\mu_\infty$, $\mu_\infty(A) = \mu'_\infty(A)$ for $A \in \mathcal{G}$. The uniqueness of Hopf’s extension implies $\mu_\infty = \mu'_\infty$. Thus $\mu_\infty$ is independent of the choice of $\phi$. Then the theorem follows. \\[\square\]
8 Concluding remarks

8.1 Translation invariant models

Let $H = H_{K}$ or $H_{qf}$. Suppose that $V = 0$. Then we already see that $e^{-itP}e^{-itH}e^{itP} = e^{-itH}$. Then $H$ can be decomposable with respect to the spectrum of $P$. Thus we have

$$H = \int_{\mathbb{R}^d} H(p) dp. \quad (8.1)$$

Here $H(p)$ is defined by

$$H(p) = \sqrt{L(p) + m^2} - m + H_{\text{rad}} \quad (8.2)$$

and

$$L(p) = (p - P_{f} - \alpha A(0))^2|_{D(P_{f}^2) \cap D(H_{\text{rad}})}. \quad (8.3)$$

It is established that $(p - P_{f} - \alpha A(0))^2$ is essentially self-adjoint on $D(P_{f}^2) \cap D(H_{\text{rad}})$ in [Hir07, Theorem 2.3]. We can construct the functional integral representation of $e^{-tH(p)}$ for each $p \in \mathbb{R}^d$ in a similar manner to [Hir07].

**Theorem 8.1** Let $F, G \in L^2(\mathcal{Q})$. Then it follows that

$$(F, e^{-tH(p)}G) = \mathbb{E}_{P_{\alpha'}} [e^{-ip \cdot B_{T_{f}}}(j_0 F(B_{T_{f}}), e^{ip \cdot B_{T_{f}}} e^{-i\alpha A(1[0,t])}j_t G(B_{T_{t}}))]. \quad (8.4)$$

From this functional integral representation we can show the self-adjointness of $H(p)$ in a similar manner to $H$.

**Corollary 8.2** Suppose Assumptions 2.1 and 2.2. Then for all $p \in \mathbb{R}^d$, $H(p)$ is self-adjoint on $D(|P_{f}|) \cap D(H_{\text{rad}})$.

**Proof:** The proof is similar to that of Theorems 4.5 and 4.7 i.e, it can be show that $e^{-tH(p)}$ leaves $D(|P_{f}|) \cap D(H_{\text{rad}})$ invariant for $m > 0$, and that by using the inequality $||p - P_{f}|\Phi||^2 + ||H_{\text{rad}}\Phi|| \leq C||(H(p) + I)\Phi||$ we can show the self-adjointness of $H(p)$ for $m \geq 0$. See [HH13].

\[\square\]
8.2 Spin 1/2 and generalizations

Let us assume that the space dimension \( d = 3 \). The SRPF Hamiltonian with spin 1/2 is defined by

\[
H_{\mathrm{SR}} = \sqrt{(\sigma \cdot (p - \alpha A))^2 + m^2} - m + V + H_{\mathrm{rad}}
\]  

(8.5)
on the Hilbert space \((\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)) \otimes L^2(\mathcal{D})\). Here \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) are the \( 2 \times 2 \) Pauli matrices given by

\[
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. 
\]

(8.6)

Let \((N_t)_{t \geq 0}\) be the Poisson process with the unit intensity on a probability space \((\Omega, \mathcal{B}, \nu)\). We define the stochastic process \( \sigma_t = \sigma(-1)^{N_t}, \, t \geq 0 \), where \( \sigma \in \{-1, +1\} \). Under some condition we can construct a functional integral representation of \( e^{-tH} \) in terms of stochastic processes \((B_t)_{t \geq 0}, (T_t)_{t \geq 0}\) and \((\sigma_t)_{t \geq 0}\). We can identify \((\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)) \otimes L^2(\mathcal{D})\) with \(L^2(\mathbb{R}^3 \times \{\pm 1\}; L^2(\mathcal{D}))\). Under this identification we can construct the Feynman-Kac type formula for \( e^{-tH} \) with

\[
H_{\mathrm{NR}} = \frac{1}{2}(\sigma \cdot (p - \alpha A))^2 + V + H_{\mathrm{rad}}
\]

in [HL08]. By a minor modification we can also construct the Feynman-Kac type formula for \( H \) in (8.3).

**Theorem 8.3** Let \( F, G \in L^2(\mathbb{R}^3 \times \{\pm 1\}; L^2(\mathcal{D})) \). Then

\[
(F, e^{-tH_{\mathrm{SR}}} G) = e^{T_t} \sum_{\sigma=\pm 1} \int_{\mathbb{R}^3} dx e_{P \times \mu \times \nu}^{0,0,0} \left[ e^{-\int_0^T V(B_s)ds} \left( I_0 F(B_{T_t}, \sigma_{T_t}), e^{S J_t G(B_{T_t}, \sigma_{T_t})} \right) \right],
\]

(8.7)

where

\[
S = -i\alpha A_F(\mathcal{I}[0,t]) - \frac{\alpha}{2} \int_0^{T_t} \sigma_s B_3(\lambda \mathcal{F} - B_s)ds \\
+ \int_0^{T_t} \log \left( \frac{\alpha}{2} (B_1(\lambda \mathcal{F} - B_s) - i\sigma_s B_2(\lambda \mathcal{F} - B_s)) \right) dN_s
\]

and \( B(x) = \nabla x \times A_F(x) \) describes the quantized magnetic field.
We can furthermore consider general Hamiltonians of the form:

\[ \Psi \left( \frac{1}{2}(\sigma \cdot (p - \alpha A))^2 \right) + V + H_{\text{rad}}, \tag{8.8} \]

where \( \Psi \) denotes a Bernstein function. The standard Pauli-Fierz Hamiltonian is realized by \( \Psi(u) = u \), and the SRPF Hamiltonian with spin 1/2 by \( \Psi(u) = \sqrt{2u + m^2} - m \). Equation (8.8) can be also investigated by path measures, and only the difference from (8.7) is to take the subordinator \((T^\Psi_t)_{t \geq 0}\) associated with Bernstein function \( \Psi \) instead of \((T_t)_{t \geq 0}\). See Appendix F for relationship between Bernstein functions and subordinatos. We will publish details somewhere in near future.

**Remark 8.4** We give comments on both of semigroups (8.4) and (8.7).

1. The semigroup (8.4) is not positivity improving for \( p \neq 0 \) and positivity improving for \( p = 0 \), since the semigroup includes \( e^{-ip \cdot B_t} \).
2. Let \( V \) and \( \phi \) be rotation invariant. Then in a similar manner to [LHB11, Corollary 7.70] it can be shown that (8.5) has degenerate ground state if it exists. In particular in this case (8.7) cannot be positivity improving.

### 8.3 Gaussian domination and local weak convergence

We can see that \( q_E(I[-t,t],j_0 \xi) \) in (6.10) converges as \( t \to \infty \).

**Lemma 8.5** Sequence \( \{q_E(I[-t,t],j_0 \xi)\}_t \) is a Cauchy sequence in \( L^2(\mathcal{B}, W^0) \).

**Proof:** Let \( s < t \) and we estimate \( \mathbb{E}_W^0[q_E(I[s,t],j_0 \xi)^2] \). By the definition of \( I[s,t] \) we have

\[
\mathbb{E}_W^0[q_E(I[s,t],j_0 \xi)^2] \leq \lim_{n \to \infty} \mathbb{E}_W^0 \left[ \sum_{j=1}^{2^n} \int_{T_{j-1}}^{T_j} (j \omega - B_s) \lambda \|j \xi\| dB_s \right]^2.
\]

By the independent increments of the Brownian motion we have

\[
\leq \lim_{n \to \infty} \sum_{j=1}^{2^n} \mathbb{E}_W^0 \left[ \int_{T_{j-1}}^{T_j} (j \omega - 2t - 1) \lambda \|j \xi\| dB_s \right] \|\lambda\| \leq \lim_{n \to \infty} \sum_{j=1}^{2^n} \mathbb{E}_W^0 \left[ (T_{j-1} - T_j) (j \xi, e^{-2t - 1} \omega \xi) \right] \|\lambda\|^2.
\]

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Since $T_{t_j-t_{j-1}}$ and $T_{t_j}-T_{t_{j-1}}$ have the same low, we see that

$$= \left( \xi, \lim_{n \to \infty} \sum_{j=1}^{2^n} \mathbb{E}_W^0 \left[ T_{t_j-t_{j-1}} e^{-2t_{j-1}\omega} \right] \xi \right) \| \lambda \|^2.$$

Using the distribution of $T_t$ we have

$$= \left( \xi, \lim_{n \to \infty} \sum_{j=1}^{2^n} \int_0^{\infty} ds \frac{\Delta t_j}{\sqrt{2\pi}} \frac{1}{\sqrt{s}} \exp \left( -\frac{1}{2} \left( \frac{2t_{j-1} - \Delta t_j}{s} \right)^2 \right) e^{-2t_{j-1}\omega} \right) \xi \| \lambda \|^2,$$

where $\Delta t_j = t_j - t_{j-1}$. Since $m > 0$ we obtain that

$$\leq C \left( \xi, \lim_{n \to \infty} \sum_{j=1}^{2^n} \Delta t_j e^{-2t_{j-1}\omega} \right) = C \left( \xi, \xi \int_{s}^{t} e^{-2r\omega} dr \right) = C \left( \xi, \frac{e^{-2s\omega} - e^{-2t\omega}}{2\omega} \xi \right)$$

with some constant $C$. Then $q_{E}(I[-t, t], j_0\xi)$ is a Cauchy sequence. 

By Lemma 8.5 there exists $q_{E}(I(-\infty, \infty), j_0\xi)$ such that $\lim_{t \to \infty} q_{E}(I[-t, t], j_0\xi) = q_{E}(I(-\infty, \infty), j_0\xi)$ in $L^2(\mathcal{X}, W^0)$.

Remark 8.6 By Theorem 7.3 and Lemma 8.5 we conjecture that

$$(\varphi_g, e^{\beta A_t^2} \varphi_g) = \frac{1}{\sqrt{1 - 2\beta q_{E}(j_0\xi)}} \mathbb{E}_{\mu_{\infty}} \left[ e^{\frac{\beta_0 a^2 q_{E}(I[-\infty, \infty], j_0\xi)^2}{1 - 2\beta q_{E}(j_0\xi)}} \right]$$

and $\lim_{t \to \infty} \beta q_{E}(j_0\xi)/2 \| e^{\beta A_t^2/2} \varphi_g \| = \infty$. This type of results are derived for a spin-boson model [HHL12].

A Brownian motion on $\mathbb{R}$

Let $(B_t)_{t \in \mathbb{R}}$ be $d$-dimensional Brownian motion on a probability space $(\Omega_p, \mathcal{B}_p, P^x)$. The properties of Brownian motion on the whole real line can be summarized as follows.

Let $N_t$ be the Gaussian random variable with mean zero and covariance $t$.

1. $P^x(B_0 = x) = 1$;

2. the increments $(B_{t_i} - B_{t_{i-1}})_{1 \leq i \leq n}$ are independent Gaussian random variables for any $0 = t_0 < t_1 < \cdots < t_n$ with $B_t - B_s \overset{d}{=} N_{t-s}$, for $t > s$.
(3) the increments \((B_{-t_{i-1}} - B_{-t_i})_{1 \leq i \leq n}\) are independent Gaussian random variables for any \(0 = -t_0 > -t_1 > \cdots > -t_n\) with \(B_{-t} - B_{-s} \overset{d}{=} N_{s-t}\), for \(-t > -s\);

(4) the function \(\mathbb{R} \ni t \mapsto B_t(\omega) \in \mathbb{R}\) is continuous for almost every \(\omega\);

(5) \(B_t\) and \(B_s\) for \(t > 0\) and \(s < 0\) are independent;

(6) the joint distribution of \(B_{t_0}, \ldots, B_{t_n}, -\infty < t_0 < t_1 < \cdots < t_n < \infty\), with respect to \(dx \otimes dP^x\) is invariant under time shift, i.e.,

\[
\int_{\mathbb{R}^d} dx \mathbb{E}_P^x \left[ \prod_{i=0}^{n} f_i(B_{t_i}) \right] = \int_{\mathbb{R}^d} dx \mathbb{E}_P^x \left[ \prod_{i=0}^{n} f_i(B_{t_i+s}) \right] 
\] (1.1)

for all \(s \in \mathbb{R}\).

## B Proof of Proposition 3.4

**Proof of Proposition 3.4** We show an outline of a proof. This is a modification of [Hir00b, Theorem 2.7] and [LHB11, Lemma 7.53]. By the Riesz theorem the right-hand side of (3.5) can be expressed as \((F, S_t G)\) with some bounded operator \(S_t\). We can check that \(S_t, t \geq 0\), is symmetric and strongly continuous one-parameter semigroup. Thus there exists a self-adjoint operator \(K\) such that \(S_t = e^{-tK}\). It is also shown [Hir97] the proof of Lemma 4.8] that

\[
\frac{1}{t}((e^{-tK} - \mathbb{1})F, G) = \int_0^1 (-h_A F, e^{-tsK}G)ds \tag{2.1}
\]

for \(F, G \in C_0^\infty(\mathbb{R}^d) \otimes L^2_{\text{fin}}(\mathcal{D})\). By the inequality \(\|h_A F\| \leq C(\|p^2 F\| + \|(N + \mathbb{1})F\|)\) with some positive constant \(C\), (2.1) can be extended for \(F, G \in D(p^2) \cap D(N)\). Thus \(K = h_A\) on \(D(p^2) \cap C^\infty(N)\). We also see that \(|(UF, e^{-tK}G)| \leq C(U, K, G)\|F\|\) for \(F, G \in D(U)\), where \(C(U, K, G)\) is a positive constant, \(U = p^2\) and \(U = N^n\) for any \(n \geq 1\). Thus \(e^{-tK}\) leaves \(D(p^2) \cap C^\infty(N)\) invariant. Thus the proposition follows from Proposition 3.3.
C Relativistic Kato-class

Let \( m \geq 0 \). Set \( h = \sqrt{p^2 + m^2} - m \). It is known that \( V \geq 0 \) is in the relativistic Kato-class if and only if \( \lim_{E \to \infty} \sup_{x \in \mathbb{R}^d} |((h - E)^{-1}V)(x)| = 0 \). See e.g. [HIL13, Proposition 4.5].

Lemma C.1 Let \( V > 0 \) be in the relativistic Kato-class. Then \( V \) is infinitesimally small form bounded with respect to \( h \), i.e., for arbitrary \( \varepsilon \) there exists \( b_\varepsilon \geq 0 \) such that \( \|V^{1/2}f\| \leq \varepsilon \|h^{1/2}f\| + b_\varepsilon \|f\| \) for arbitrary \( f \in D(h^{1/2}) \). In particular \( \gamma_k \subset \gamma_{\text{rf}} \).

Proof: Let \( \| \cdot \|_{p,p} \) be bounded operator norm on \( L^p(\mathbb{R}^d) \). By duality it is seen that \( \|(h + E)^{-1}V\|_{1,1} = \|(h + E)^{-1}V\|_{\infty,\infty} \). By the Stein interpolation theorem we have \( \|V^{1/2}(h - E)^{-1}V^{1/2}\|_{2,2} \leq \|(h + E)^{-1}V\|_{1,1} \) and notice that \( \|(h + E)^{-1}V\|_{\infty,\infty} = \sup_{x \in \mathbb{R}^d} |((h - E)^{-1}V)(x)| \) → 0 as \( E \to \infty \). From \( \|V^{1/2}f\| \leq \|V^{1/2}(h - E)^{-1/2}\|_{2,2} \|(h - E)^{1/2}f\| \) it follows that \( V \) is form bounded with an infinitesimally small relative bound. \( \square \)

D Integral \( I[a,b] \)

Proposition D.1 Let \( \Gamma_n[0,t] = \bigoplus_{\mu=1}^{d} \sum_{j=1}^{2^n} \int_{T_{ij-1}}^{T_{ij}} j_{ij} \lambda(\cdot - B_s)dB^\mu_s \). Then \( \lim_{n \to \infty} \Gamma_n[0,t] = I[0,t] \) in \( L^2(\Omega_\nu, P^x) \otimes \mathcal{E} \).

Proof: We have \( \|\Gamma_n'[0,t] - \Gamma_n[0,t]\|^2 = d(T_t - T_0)(\lambda, 2(1 - e^{-t\omega/2^n})\lambda) \to 0 \) as \( n \to \infty \). Then the proof is complete. \( \square \)

Proposition D.2 For each \( w \in \Omega_\nu \setminus \mathcal{N}_\nu \), \( I[0,t] = I[0,s] + I[s,t] \) for \( 0 < s < t \) follows in the sense of \( L^2(\Omega_\nu, P^x) \otimes \mathcal{E} \), i.e.,

\[
\mathbb{E}_P^x \|I[0,t] - I[0,s] - I[s,t]\|_{d_x}^2 = 0. \tag{4.1}
\]

Proof: By a limiting argument we see that

\[
\mathbb{E}_P^x \|I[0,t]\|_{d_x}^2 = dT_s \|\phi/\sqrt{\omega}\|^2 \tag{4.2}
\]
for almost surely in $\nu$. We suppose that $s = at/2^k$ with some $a, k \in \mathbb{N}$. Then by the definition of $I_n[0, t]$ we have $I[0, t] = \lim_{n \to \infty} \bigoplus_{\mu=1}^{d} \sum_{j=1}^{2n+k} \int_{T_{j-1}}^{T_j} j_{t_j} \lambda(-B_s) dB_s^\mu$ with $t_j = \frac{t_j}{2n+k}$, and

$$
\sum_{j=1}^{2^{n+k}} \int_{T_{j-1}}^{T_j} j_{t_j} \lambda(-B_s) dB_s^\mu = \sum_{j=1}^{2^{n+a}} \int_{T_{\frac{t}{2^n}(j-1)}}^{T_{\frac{t}{2^n}(j)}} \frac{j_{\frac{t}{2^n}}}{2^n} \lambda(-B_s) dB_s^\mu + \sum_{j=0}^{2^b} \int_{T_{\frac{t}{2^n}(j-1)}}^{T_{\frac{t}{2^n}(j)}} \frac{j_{\frac{t}{2^n}}}{2^n} \lambda(-B_s) dB_s^\mu,
$$

where $b = 2^k - a$. Hence $I[0, t] = I[0, s] + I[s, t]$ follows. Let $0 < s < t$. Then there exists $s(\epsilon) > s$ such that $s(\epsilon) = a/2^k$ with some $a, k \in \mathbb{N}$ and $s(\epsilon) \downarrow s$ as $\epsilon \to 0$. Hence $I[0, t] = I[0, s(\epsilon)] + I[s(\epsilon), t]$. Note that $I[0, s(\epsilon)] - I[0, s] = I[s, s(\epsilon)]$ and $\mathbb{E}_P[I[s, s(\epsilon)]^2] = (T_s - T_s) \|\hat{\phi}/\sqrt{\omega}\|^2$ by the Itô isometry (4.2). Since $T_s = T_s(w)$ is right continuous in $s$ for $w \in \Omega_\nu \setminus \mathcal{N}_\nu$, (4.1) follows.

\[\square\]

Proposition D.3 Let $a \leq b$ and $c \leq d$, and suppose that $[a, b] \cap [c, d] = [c, b]$. Then for each $w \in \Omega_\nu \setminus \mathcal{N}_\nu$, $\mathbb{E}_P[I[a, b], I[c, d]) = d(T_b - T_c) \|\hat{\phi}/\sqrt{\omega}\|^2_{L^2(\mathbb{R}^d)}$.

**Proof:** Suppose that $[a, b] \cap [c, d] = \emptyset$. Then it follows that

$$
\mathbb{E}_P[I[a, b], I[c, d]] = \lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E}_P[I_n[a, b], I_m[c, d]] = 0.
$$

Thus by Proposition D.2 we see that

$$
\mathbb{E}_P[I[a, b], I[c, d]] = \mathbb{E}_P[I[a, c], I[b, d]] + \mathbb{E}_P[I[a, c], I[b, c]] + \mathbb{E}_P[I[b, d], I[c, b]] + \mathbb{E}_P[I[c, b]]^2.
$$

Then the lemma follows from $\mathbb{E}_P[I[c, b]]^2 = d \mathbb{E}_P[\int_{T_c}^{T_b} \lambda(-B_r) \|\hat{\phi}/\sqrt{\omega}\|^2 dr]$ by the Itô isometry (4.2). \[\square\]

**E**

**Proofs of (5.4) and (5.7)**

Lemma E.1 (5.4) follows.
PROOF: From the proof of Theorem 5.10 and (5.12), it follows that

$$
\mathbb{E}_{\mathcal{P} \times \nu}^{0,0} \left[ e^{-i \alpha \mathbb{E} (1^*[s,t]) e^{- \int_s^t V(B_{\tau}) + x) d\tau} \right] \mathcal{F}[0,s] \\
= \mathbb{E}_{\mathcal{P} \times \nu}^{B_{\tau},0} \left[ e^{-i \alpha \mathbb{E} (1^[2]^*[t,s]) e^{- \int_{t-s}^t V(B_{\tau}) + x) d\tau} \right] \mathcal{F}[0,t] 
$$

(5.1)

for arbitrary $G \in \mathcal{H}$. Then we have

$$
\mathbb{E}_{\mathcal{P} \times \nu}^{x,0} \left[ J_{[0,s]} \mathbb{E}_{\mathcal{P} \times \nu}^{B_{\tau},0} \left[ J_{[0,t]} G(B_{\tau}) \right] \right] = \mathbb{E}_{\mathcal{P} \times \nu}^{x,0} \left[ J_{[0,s]}(x) \mathbb{E}_{\mathcal{P} \times \nu}^{B_{\tau},0} \left[ J_{[0,t]}(x) G(B_{\tau} + x) \right] \right] \\
= \mathbb{E}_{\mathcal{P} \times \nu}^{x,0} \left[ J_{[0,s]}(x) \mathbb{E}_{\mathcal{P} \times \nu}^{0,0} \left[ J_{0} e^{- \int_{t-s}^t V(B_{\tau}) d\tau} e^{-i \alpha \mathbb{E} (1[^*[s,s+t]) J_{t} G(B_{\tau+t} + x) \right] \mathcal{F}[0,s] \right] 
$$

Here $I_0^*[s,s+t]$ (resp. $J_{[0,s]}(x)$) denotes $I_0[s,s+t]$ (resp. $J_{[0,s]}$) with $B_{\tau}$ replaced by $B_{\tau} + x$. Since a conditional expectation leaves expectation invariant, we have

$$
= \mathbb{E}_{\mathcal{P} \times \nu}^{x,0} \left[ J_{[0,s]}(x) J_{0} e^{- \int_{t-s}^t V(B_{\tau}) d\tau} e^{-i \alpha \mathbb{E} (1[^*[s,s+t]) J_{t} G(B_{\tau+t} + x) \right] \\
= \mathbb{E}_{\mathcal{P} \times \nu}^{x,0} \left[ J_{[0,s]}(x) J_{0} e^{- \int_{t-s}^t V(B_{\tau}) d\tau} e^{-i \alpha \mathbb{E} (1[^*[s,s+t]) J_{t} G(B_{\tau+t} + x) \right]
$$

and (5.4) follows. \(\square\)

**Lemma E.2 (5.7) follows.**

**PROOF:** Note that $B_{\tau} - B_{\tau} = B_{\tau} - B_{\tau}$ and $y - B_{\tau} = y + B_{\tau}$ in law. We investigate $u_T \tilde{I}_n[0,t]$. We see that

$$
u_T \tilde{I}_n[0,t] = \bigoplus_{\mu=1}^d \bigoplus_{j=1}^{2^n} \int_{T_{j-1}}^{T_j} \tilde{j}_{t-t+j} \lambda(\cdot - (B_{s} - B_{T_{j}} + y)) dB_{t}^{\mu} \\
= \lim_{m \to \infty} \bigoplus_{\mu=1}^d \bigoplus_{i=1}^{2^m} \bigoplus_{j=1}^{2^n} \tilde{j}_{t-t+j} \lambda(\cdot - (B_{T_{j}-1} + (i-1)\Delta_{j-1} - B_{T_{j}} + y)) \\
\times \left( B_{T_{j}-1} + i\Delta_{j-1} - B_{T_{j}-1} + (i-1)\Delta_{j-1} \right) \\
= \lim_{m \to \infty} \bigoplus_{\mu=1}^d \bigoplus_{i=1}^{2^m} \bigoplus_{j=1}^{2^n} \tilde{j}_{t-t+j} \lambda(\cdot - B_{T_{j}-1} -(i-1)\Delta_{j-1} - y) \\
\times \left( B_{T_{j}-1} - i\Delta_{j-1} - B_{T_{j}-1} - (i-1)\Delta_{j-1} \right),
$$

60
\[ \Delta_j = \frac{1}{2} (T_j - T_{j-1}). \] Since \( T_t - T_s \) has the same law as \( T_t - T_s \), we can replace the right-hand side above with

\[ -\lim_{m \to \infty} \sum_{\mu=1}^{d} \sum_{i=1}^{2^n} \sum_{j=1}^{m} \int_t-t_{j-1} \lambda \left( \int - B_{T_t-t_{j-1}} - \frac{1}{2^n} (T_t-t_{j-1}-T_t-t_{j-1}) - y \right) \times \left( B_{T_t-t_{j-1}} - \frac{1}{2^n} (T_t-t_{j-1}-T_t-t_{j-1}) - B_{T_t-t_{j-1}} - \frac{1}{2^n} (T_t-t_{j-1}-T_t-t_{j-1}) \right). \] (5.2)

By the definition of \( \int_T T_s \lambda(\cdot - B_s) dB^\mu_s \) and the Coulomb gauge condition (2.8) it follows that

\[ = -\sum_{\mu=1}^{d} \sum_{i=1}^{2^n} \int_{T_t-t_{j-1}} \int_t-t_{j-1} \lambda(\cdot - B_s - y) dB^\mu_s = -\sum_{\mu=1}^{d} \sum_{i=1}^{2^n} \int_{T_t-t_{j-1}} \int_t-t_{j-1} \lambda(\cdot - B_s - y) dB^\mu. \] (5.3)

Finally we have by Proposition D.1

\[ -\sum_{\mu=1}^{d} \sum_{i=1}^{2^n} \int_{T_t-t_{j-1}} \int_t-t_{j-1} \lambda(\cdot - B_s - y) dB^\mu = -\sum_{\mu=1}^{d} \sum_{i=1}^{2^n} \int_{T_t-t_{j-1}} \int_t-t_{j-1} \lambda(\cdot - B_s - y) dB^\mu. \] (5.4)

Then the proof is complete.

\[ \square \]

## F Subordinators

A subordinator \((T_t)_{t \geq 0}\) is a 1-dimensional Lévy process which has a almost surely nondecreasing path \( t \mapsto T_t \). Subordinator may be thought as a random time, since \( T_t \geq 0 \) and \( T_t \leq T_s \) for \( t \leq s \). The subordinator \((T_t)_{t \geq 0}\) satisfies that \( \mathbb{E}[e^{-u T_t}] = e^{-\psi(u)} \), where

\[ \psi(u) = bu + \int_0^\infty (1 - e^{-uy}) \lambda(dy) \] (6.1)

for \( u > 0 \), where \( b \geq 0 \) a constant and \( \lambda(dy) \) denotes a Lévy measure such that \( \lambda((-\infty, 0)) = 0 \) and \( \int_0^\infty (y \wedge 1) \lambda(dy) < \infty \). Let \( f \in C^\infty((0, \infty)) \) with \( f \geq 0 \). \( f \) is a Bernstein function if and only if \( (-1)^n d^n f / dx^n \leq 0 \) for all \( n = 1, 2, 3, \ldots \). For each Bernstein function \( \psi \) such that \( \lim_{u \downarrow 0} \psi(u) = 0 \) can be realized as (6.1). The examples of Bernstein functions are \( \psi(u) = u^\alpha \) with \( 0 < \alpha < 1 \) and \( \psi(u) = \sqrt{u^2 + m^2} - m \).
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