ABSTRACT. In the first section of this paper, we introduce the notions of fractional and invertible ideals of semirings. We also define Prüfer semirings and prove that a semiring $S$ is Prüfer iff the semiring $S$ is a multiplicatively cancellative semiring and every 2-generated ideal of $S$ is invertible. In the second section, we give a semiring version for the Gilmer-Tasng Theorem, which states that for a suitable family of semirings, the concepts of Prüfer and Gaussian semirings are equivalent. At last we end this paper by giving a plenty of examples of Gaussian and Prüfer semirings, which are not rings.

0. INTRODUCTION

Semirings not only have significant applications in different fields such as automata theory in theoretical computer science, (combinatorial) optimization theory, and generalized fuzzy computation, but are fairly interesting generalizations of two broadly studied algebraic structures, i.e. rings and bounded distributive lattices$^4$. Except the most familiar example of a semiring, i.e. the semiring of nonnegative integers, which mathematicians have worked on it since ancient times, apparently the first examples of semirings appeared in the works of Dedekind [D] when he worked on the algebra of the ideals of commutative rings. Later the other mathematician who worked on semirings in 1930s, was Vandiver [V], but it seems he was not successful to draw the attention of mathematicians to consider semirings as an independent algebraic structure that is worth to be developed. Actually it was in the late 1960s that semirings were considered a more serious topic of research when real applications were found for semirings. Eilenberg and a couple of other mathematicians started developing formal languages and automata theory systematically [E], which have strong connections to semirings. Since then many mathematicians and computer scientists have broadened the theory of semirings and related structures ([Gl]). Multiplicative ideal theoretic methods in ring theory can be a good source of motivations to develop semiring theory (for more see [Go] and [HW]). The main scope of this paper is to develop some ring theoretic methods of multiplicative ideal theory, namely invertible ideals and Gaussian property for the content of polynomials over semirings.

In this paper, by a semiring, we understand an algebraic structure, consisting of a nonempty set $S$ with two operations of addition and multiplication such that the following conditions are satisfied:

(1) $(S, +)$ is a commutative monoid with identity element 0;

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A nonempty subset \( I \) of a semiring \( S \) is said to be an ideal of \( S \), if \( a + b \in I \) for all \( a, b \in I \) and \( sa \in I \) for all \( s \in S \) and \( a \in I \). Similar to the ideal theory of commutative rings, it is easy to see that if \( a_1, a_2, \ldots, a_n \in S \), then the finitely generated ideal \( (a_1, a_2, \ldots, a_n) \) of \( S \) is the set of all linear combinations of the elements \( a_1, a_2, \ldots, a_n \), i.e.
\[
(a_1, a_2, \ldots, a_n) = \{ s_1a_1 + s_2a_2 + \cdots + s_na_n : s_1, s_2, \ldots, s_n \in S \}.
\]

In the first section, we define fractional and invertible ideals and we prove a couple of interesting assertions for them. Particularly we prove that if \( S \) is a semiring, then every nonzero finitely generated ideal of \( S \) is an invertible ideal of \( S \) iff every nonzero principal and every nonzero 2-generated ideal of \( S \) is an invertible ideal of \( S \). This result and a nice example of a semiring, which has this property but it is not a ring, motivate us to define Prüfer semiring, the semiring that each of its finitely generated ideal is invertible.

A nonzero ideal \( I \) of the semiring \( S \) is called a cancellation ideal, if \( IJ = IK \) implies \( J = K \) for all ideals \( J \) and \( K \) of \( S \). It is defined that \( [I: J] = \{ s \in S : sJ \subseteq I \} \). Now let \( f \in S[X] \) be a polynomial over the semiring \( S \). The content of \( f \), denoted by \( c(f) \), is defined to be the ideal generated by the coefficients of \( f \). It is, then, easy to see that \( c(fg) \subseteq c(f)c(g) \) for all \( f, g \in S[X] \). Finally a semiring \( S \) is defined to be a Gaussian semiring if \( c(fg) = c(f)c(g) \) for all \( f, g \in S[X] \). In the second section, we discuss Gaussian semirings and prove the Gilmer-Tsang Theorem for semirings: Let \( S \) be a subtractive semiring such that every nonzero principal ideal of \( S \) is invertible and \( ab \in (a^2, b^2) \) for all \( a, b \in S \). Then the following statements are equivalent:

1. \( S \) is a Prüfer semiring,
2. Each nonzero finitely generated ideal of \( S \) is cancellation,
3. \([IJ : I] = J \) for all ideals \( I \) and \( J \) of \( S \),
4. \( S \) is a Gaussian semiring.

At last we end this paper by giving a plenty of examples of Gaussian and Prüfer semirings, which are not rings. Actually we prove that if \( S \) is a Prüfer semiring (say for example \( S \) is a Prüfer domain), then \( \text{FId}(S) \) is a Prüfer semiring, where by \( \text{FId}(S) \) we mean the semiring of finitely generated ideals of \( S \).

A connoisseur of multiplicative ideal theory may have noticed that some of these definitions and results emulate with their ring version ones.

In this paper, all semirings are assumed to be commutative with identity and the phrase \( \text{iff} \) always stands for “if and only if”.

### 1. Invertible Ideals of Semirings

Let us recall that an element \( s \) of a semiring \( S \) is said to be multiplicatively-cancellable (abbreviated as MC), if \( sb = sc \) implies \( b = c \) for all \( b, c \in S \). Note that \( r \) is an MC element of the ring \( R \) iff \( r \) is a regular element of \( R \), which means that \( r \) is not a zero-divisor of the ring \( R \). This is not the case in semiring theory, since the idempotent semiring \( S = \{0, 1, u\} \),
where \(1 + u = u + 1 = u\), has not a nonzero zero-divisor while \(u\) is not an MC element of \(S\), because \(u.u = u.1\), but \(u \neq 1\).

We denote \(MC(S)\) all MC elements of \(S\). Using techniques adapted from ring theory, total quotient semiring of the semiring \(S\) can be constructed in the following way:

1. Define the equivalent relation \(\sim\) on \(S \times MC(S)\) by \((a, b) \sim (c, d)\), if \(ad = bc\).
2. Put \(Q(S)\) the set of all equivalence classes of \(S \times MC(S)\) and define addition and multiplication on \(Q(S)\) respectively by \([a, b] + [c, d] = [ad + bc, bd]\) and \([a, b] \cdot [c, d] = [ac, bd]\), where by \([a, b]\), also denoted by \(a/b\), we mean the equivalence class of \((a, b)\).

It is easy to see that \(Q(S)\) with the mentioned operations of addition and multiplication in above is a semiring. Now we proceed to define fractional and invertible ideals of semirings. These concepts emulate with their ring version concepts, defined for example in the books \([LM]\) and \([Gi1]\).

Let us recall that a semiring \(S\) is defined to be a multiplicatively-cancellative (abbreviated as MC semiring), if any nonzero element of the semiring \(S\) is an MC element of \(S\).

Definition 1. Fractional ideal. We define a fractional ideal of a semiring \(S\) to be a subset \(I\) of the total quotient semiring \(Q(S)\) of \(S\) such that:

1. \(I\) is an \(S\)-semimodule of \(Q(S)\), that is, if \(a, b \in I\) and \(s \in S\), then \(a + b, sa \in I\) and
2. there exists an MC element \(d \in S\) such that \(dI \subseteq S\).

Invertible ideal. We define a fractional ideal \(I\) of a semiring \(S\) to be invertible if there exists a fractional ideal \(J\) of \(S\) such that \(IJ = S\).

Let us denote the set of all nonzero fractional ideals of \(S\) by \(\text{Frac}(S)\). Then we have the following:

Theorem 2. Let \(S\) be a semiring with its total quotient semiring \(Q(S)\).

1. If \(I \in \text{Frac}(S)\) is invertible, then \(I\) is finitely generated \(S\)-semimodule of \(Q(S)\).
2. If \(I, J \in \text{Frac}(S)\) and \(I \subseteq J\) and \(J\) is invertible, then there is an ideal \(K\) of \(S\) such that \(I = JK\).
3. If \(I \in \text{Frac}(S)\), then \(I\) is invertible iff there is a fractional ideal \(J\) of \(S\) such that \(IJ = S\) is principal and generated by an MC element of \(Q(S)\).

Proof. The proof of this theorem is nothing but the mimic of the proof of its ring version \([LM]\) Proposition 6.3.

Lemma 3. Let \(S\) be a semiring and \(a \in S\). Then the principle ideal \((a)\) is invertible iff \(a\) is an MC element of \(S\).

Proof. \((\rightarrow)\): Let \((a)\) be invertible. So there exists a fractional ideal \(J\) of \(S\) such that \((a)J = S\). This means that there are \(s, t \in S\) and \(c \in MC(S)\) such that \((s \cdot a) \cdot (t/c) = 1\). This causes \(c = s \cdot a \cdot t\). But \(c\) is an MC element of \(S\) and this implies that \(a\) is also an MC element of \(S\).

\((\leftarrow)\): Let \(a\) be an MC element of \(S\). Obviously \((1/a)\) is a fractional ideal of \(S\) such that \((a)(1/a) = S\) and the proof is complete.

\(^2\)Note this semiring has been mentioned in \([L]\).
Corollary 4. A semiring $S$ is an MC semiring iff each nonzero principle ideal of $S$ is an invertible ideal of $S$.

Remark 5. From commutative ring theory, we know that if $D$ is a domain, then so is $D[X]$. Obviously if $R$ is a ring, then $R$ is an MC semiring iff $R$ is a domain and this implies that if $R$ is a ring, then $R$ is an MC semiring iff $R[X]$ is an MC semiring. The question may arise if this can be generalized for arbitrary semirings. The answer is negative! Observe that the semiring $\text{Id}(\mathbb{Z})$ is an MC semiring, since any element of $\text{Id}(\mathbb{Z})$ is of the form $(n)$ such that $n$ is a nonnegative integer and $(a)(b) = (ab)$, while $\text{Id}(\mathbb{Z}[X])$ is not an MC semiring and the proof is as follows: Consider the polynomials $f = a + bX$, $g = c + dX + eX^2$ and $h = c + d'X + eX^2$, where $a = (2), b = (3), c = (3), d = (7), d' = (5)$ and $e = (1)$. A simple calculation shows that $fg = fh = (6) + (1)X + (1)X^2 + (3)X^3$, while $g \neq h$. Actually the semiring $\mathbb{B}$ is a semifield, but still $\mathbb{B}[X]$ is not an MC semiring, since $(1 + X)(1 + X^2) = (1 + X)(1 + X + X^2)$. By the way, the assertion “$D$ is a domain iff $D[X]$ is a domain”, can be generalized in another meaning and we express that in Proposition 6.

Let us recall that a semiring $S$ is called an entire semiring if $ab = 0$ implies either $a = 0$ or $b = 0$ for all $a, b \in S$.

Proposition 6. Let $S$ be a semiring. Then $S$ is an entire semiring iff $S[X]$ is an entire semiring.

Proof. ($\rightarrow$): Suppose that $S$ is an entire semiring and imagine $f, g \in S[X]$ are nonzero polynomials. So the leading coefficients $a_n$ and $b_n$ of $f$ and $g$ respectively are nonzero and therefore the leading coefficient $a_nb_n$ of $fg$ is nonzero. This means that $fg \neq 0$ and $S[X]$ is an entire semiring. The other direction of the above assertion holds obviously. \qed

Let us recall that a semiring $(S, +, \cdot)$ is said to be additively-idempotent if $a + a = a$ for all $a \in S$.

Proposition 7. Let $S$ be an additively-idempotent semiring. Then $(a + b)(b + c)(c + a) = (a + b + c)(ab + bc + ca)$ for all $a, b$ and $c \in S$.

Proof. Straightforward \qed

Now we prove the following important theorem, which its ring version can be found in [LM Theorem 6.6].

Theorem 8. Let $S$ be a semiring. Then the following statements are equivalent:

1. Every nonzero finitely generated ideal of $S$ is an invertible ideal of $S$,
2. The semiring $S$ is an MC semiring and every nonzero $2$-generated ideal of $S$ is an invertible ideal of $S$.

Proof. Obviously the first assertion implies the second one. We prove that the second assertion implies the first one. The proof is by induction. Let $n > 2$ be a natural number and suppose that all nonzero ideals of $S$ generated by less than $n$ generators are invertible ideals and $L = (a_1, a_2, \ldots, a_{n-1}, a_n)$ be an ideal of $S$. If we put $I = (a_1), J = (a_2, \ldots, a_{n-1})$ and $K = (a_n)$, then by induction’s hypothesis the ideals $I + J, J + K$ and $K + I$ are all invertible ideals. On the other hand, since $(\text{Id}(S), +, \cdot)$ is an additively-idempotent semiring,
by Proposition\(^7\), the identity \((I + J)(J + K)(K + I) = (I + J + K)(I + JK + KI)\) holds.
Also since product of fractional ideals of \(S\) is invertible if and only if every factor of this product is invertible, the ideal \(I + J + K = L\) is invertible and the proof is complete. \(\square\)

A ring \(R\) is said to be a Prüfer domain if every nonzero finitely generated ideal of \(R\) is invertible. It is, now, natural to ask if there is any semiring \(S\) with this property that every nonzero finitely generated ideal of \(S\) is invertible and still is not a ring. In the following remark we give such an example:

**Remark 9.** Example of a semiring with this property that every nonzero finitely generated ideal of \(S\) is invertible, while it is not a ring: Obviously \((\text{Id}(\mathbb{Z}), +, \cdot)\) is an MC semiring, since any element of \(\text{Id}(\mathbb{Z})\) is of the form \((n)\) such that \(n\) is a nonnegative integer and \((a)(b) = (ab)\). Let \(I\) be an arbitrary ideal of \(\text{Id}(\mathbb{Z})\). Define \(A_I\) to be the set of all positive integers \(n\) such that \((n) \in I\) and put \(m = \min A_I\). Our claim is that \(I\) is the principal ideal of \(\text{Id}(\mathbb{Z})\), generated by \((m)\), i.e. \(I = ((m))\). For doing so, let \((d)\) be an element of \(I\). But then \((d) + (m) = (\gcd(d, m)) \in I\). This means that \(m \leq \gcd(d, m)\), since \(m = \min A_I\), while \(\gcd(d, m) \leq m\) and this implies that \(\gcd(d, m) = m\) and so \(m\) divides \(d\) and therefore there exists a natural number \(r\) such that \(d = rm\). Hence, \((d) = (r)(m)\) and the proof of our claim is finished. From all we said we learn that each ideal of the semiring \(\text{Id}(\mathbb{Z})\) is a principal and, therefore, an invertible ideal, while obviously it is not a ring.

By Theorem\(^8\) and the example given in Remark\(^9\) we are inspired to give the following definition:

**Definition 10.** We define a semiring \(S\) to be a Prüfer semiring if every nonzero finitely generated ideal of \(S\) is invertible.

It is good to mention that in the Theorem\(^{19}\) we construct a plenty of Prüfer semirings that are not rings. Actually we prove that if \(S\) is a Prüfer semiring (say for example \(S\) is a Prüfer domain), then \(\text{FId}(S)\) is a Prüfer semiring, where by \(\text{FId}(S)\) we mean the semiring of finitely generated ideals of \(S\). Now we bring a theorem that is somehow the semiring version of [LM] Theorem 6.6.

**Theorem 11.** Let \(S\) be an MC semiring. Then \(S\) is a Prüfer semiring, if either (i) \((A + B)(A \cap B) = AB\) for all ideals \(A, B\) of \(S\), or (ii) \((A + B) : C = A : C + B : C\) for all ideals \(A, B, C\) of \(S\) with \(C\) finitely generated, or (iii) \(C : (A \cap B) = C : A + C : B\) for all ideals \(A, B, C\) of \(S\) with \(A\) and \(B\) finitely generated.

**Proof.** Let \(S\) be an MC semiring. By Theorem\(^8\) in order to prove that \(S\) is a Prüfer semiring, it is sufficient to prove that any nonzero 2-generated ideal of \(S\) is invertible. Now we proceed to prove that any of those three statements causes \(S\) to be a Prüfer semiring.

(1): Let \(C = (c_1, c_2)\) be a nonzero ideal of \(S\). If \(c_1 = 0\) or \(c_2 = 0\), then \(C\) is invertible. So we can assume \(c_1 \neq 0\) and \(c_2 \neq 0\). Then \(A = (c_1)\) and \(B = (c_2)\) are invertible and if (i) holds, we have: \(C(A \cap B) = (A + B)(A \cap B) = AB\). But \(AB = (c_1c_2)\) is an invertible ideal of \(S\). Thus \(C\) is an invertible ideal of \(S\).

(2): Let \(a, b \in S\). If (ii) holds, then
\[
S = (a, b) : (a, b) = ((a) + (b)) : (a, b) = (a) : (a, b) + (b) : (a, b)
\]
\[= (a) : (b) + (b) : (a).\]

Let \(1 = x + y\) where \(xb \in (a)\) and \(ya \in (b)\). Then \((xb)b \subseteq (ab)\) and \((ya)a \subseteq (ab)\). Hence \((a,b)(bx,ay) \subseteq (ab)\). But \(ab = abx + aby\), so \((ab) = (a,b)(bx,ay)\). We may assume \(a \neq 0\) and \(b \neq 0\). Then \((ab)\) is invertible and so \((a,b)\) is invertible.

(3) Let \(a, b \in S\). If (iii) holds, then

\[S = ((a) \cap (b)) : ((a) \cap (b)) = ((a) \cap (b)) : (a) + ((a) \cap (b)) : (b) = (b) : (a) + (a) : (b).\]

Now we can proceed as the case the statement (ii) holds, Q.E.D. □

We end this section with the following straightforward, but, in some senses, important result.

**Proposition 12.** Let \(S\) be a Prüfer semiring. Then the following statements hold:

1. If \(I\) and \(K\) are ideals of \(S\), with \(K\) finitely generated, and if \(I \subseteq K\), then there is an ideal \(J\) of \(S\) such that \(I = JK\).
2. If \(IJ = IK\), where \(I, J\) and \(K\) are ideals of \(S\) and \(I\) is finitely generated and nonzero, then \(J = K\).

**Proof.** By considering Theorem 2, the assertion (1) holds. The assertion (2) is straightforward. □

The second property in Proposition 12 is the concept of cancellation ideal for semirings, introduced by S. LaGrassa in her PhD Thesis [L]. Therefore, in this terminology, each invertible ideal of a semiring is cancellation. While the topic of cancellation ideals is itself interesting, in this paper, we do not go through them deeply. Now we pass to the next section that is on Gaussian semirings.

### 2. Gaussian Semirings

In this section, we discuss Gaussian semirings. For doing so, we need to recall some concepts. An ideal \(I\) of a semiring \(S\) is said to be subtractive, if \(a + b \in I\) and \(a \in I\) implies \(b \in I\) for all \(a, b \in S\). A semiring \(S\) is said to be subtractive if every ideal of the semiring \(S\) is subtractive. For a polynomial \(f \in S[X]\), the content of \(f\), denoted by \(c(f)\), is defined to be the finitely generated ideal of \(S\) generated by the coefficients of \(f\). In \([N,\text{Theorem 3}]\), the semiring version of Dedekind-Mertens lemma (cf. \([P,\text{p. 24}]\) and \([AG]\)) has been proved. We state that in the following only for the convenience of the reader:

**Theorem 13.** Dedekind-Mertens Lemma for Semirings. Let \(S\) be a semiring. Then the following statements are equivalent:

1. \(S\) is a subtractive semiring.
2. If \(f, g \in S[X]\) and \(\deg(g) = m\), then \(c(f)^{m+1}c(g) = c(f)^mc(fg)\).

A semiring \(S\) is said to be Gaussian if \(c(fg) = c(f)c(g)\) for all polynomials \(f, g \in S[X]\) ([N, Definition 7]). This is the ring version of the concept Gaussian rings defined for example in [AC].

Note that by Theorem 13 a Gaussian semiring needs to be subtractive. With this background, it is, now, easy to see that if every nonzero finitely generated ideal of a subtractive
semiring $S$, is invertible, then $S$ is Gaussian. Also note that a famous theorem in commutative ring theory, known as Gilmer-Tsang Theorem (cf. [Gi2] and [T]), states that $D$ is a Prüfer domain iff $D$ is a Gaussian domain. The question may arise if a semiring version for Gilmer-Tsang Theorem can be proved. This is what we are going to do in the rest of the paper. First we prove the following interesting theorem:

**Theorem 14.** Let $S$ be a semiring. Then the following statements are equivalent:

1. $S$ is a Gaussian and an MC semiring and $ab \in (a^2, b^2)$ for all $a, b \in S$,
2. $S$ is a subtractive and Prüfer semiring.

**Proof.** $(1) \rightarrow (2)$: Since $ab \in (a^2, b^2)$, there exists $r, s \in S$ such that $ab = ra^2 + sb^2$. Now put $f = a + bX$ and $g = sb + raX$. It is easy to check that $fg = sab + abX + rabX^2$. Since $S$ is Gaussian, $S$ is subtractive by Theorem 13 and we have $c(fg) = c(f)c(g)$, i.e. $(ab) = (a, b)(sb, ra)$. But $(ab) = (a)(b)$ is invertible and therefore $(a, b)$ is also invertible and by Theorem 8, $S$ is a Prüfer semiring.

$(2) \rightarrow (1)$: Since $S$ is a subtractive and Prüfer semiring, by Theorem 13, $S$ is a Gaussian semiring. On the other hand, one can verify that $(ab)(a, b) \subseteq (a^2, b^2)(a, b)$ for any $a, b \in S$ and if $a = b = 0$, then there is nothing to be proved. Otherwise since $(a, b)$ is an invertible ideal of $S$, we have $ab \in (a^2, b^2)$ and this completes the proof. \[\square\]

Now let us recall the definition of cancellation ideals.

**Definition 15.** A nonzero ideal $I$ of the semiring $S$ is called a cancellation ideal, if $IJ = IK$ implies $J = K$ for all ideals $J$ and $K$ of $S$ [L].

For any ideals $I, J$ of the semiring $S$, it is defined that $[I : J] = \{s \in S : sJ \subseteq I\}$. The following lemma is the semiring version of an assertion mentioned in [Gi1, Exercise 4, p. 66].

**Lemma 16.** Let $S$ be a semiring and $I$ be a nonzero ideal of $S$. Then the following statements are equivalent:

1. $I$ is a cancellation ideal of $S$,
2. $[IJ : I] = J$ for any ideal $J$ of $S$,
3. $IJ \subseteq IK$ implies $J \subseteq K$ for all ideals $J, K$ of $S$.

**Proof.** By considering this point that the equality $[IJ : I]I = IJ$ holds for all ideals $I, J$ of $S$, it is, then, easy to see that (1) implies (2). The rest of the proof is straightforward. \[\square\]

**Theorem 17.** **Gilmer-Tsang Theorem for Semirings.** Let $S$ be a subtractive and an MC semiring such that $ab \in (a^2, b^2)$ for all $a, b \in S$. Then the following statements are equivalent:

1. $S$ is a Prüfer semiring,
2. Each nonzero finitely generated ideal of $S$ is cancellation,
3. $[I : J] = J$ for all finitely generated ideals $I$ and $J$ of $S$,
4. $S$ is a Gaussian semiring.

**Proof.** Obviously $(1) \rightarrow (2)$ and $(2) \rightarrow (3)$ hold by Proposition 12 and Lemma 16 respectively.
(3) → (4): Let \( f, g \in S[X] \). By Theorem \([13]\) we have \( c(f)c(g)c(f)^m = c(fg)c(f)^m \). So
\[
[c(f)c(g)c(f)^m : c(f)^m] = [c(fg)c(f)^m : c(f)^m].
\]
This means that \( c(f)c(g) = c(fg) \) and \( S \) is Gaussian.

Finally \( (4) \to (1) \) by Theorem \([14]\) and this finishes the proof. \( \square \)

**Remark 18.** In \([N]\) Theorem 9], it has been proved that every bounded distributive lattice is a Gaussian semiring. Also note that if \( L \) is a bounded distributive lattice with more than two elements, it is neither a ring nor an MC semiring, since if it is a ring then the idempotency of addition causes \( L = \{0\} \) and if it is an MC semiring, the idempotency of multiplication causes \( L = \mathbb{B} = \{0, 1\} \). With the help of the following theorem, we give a plenty of examples of Gaussian and Pr"ufer semirings, which are not rings. Let us recall that if \( S \) is a semiring, then by \( \text{FId}(S) \), we mean the semiring of finitely generated ideals of \( S \).

**Theorem 19.** Let \( S \) be a Pr"ufer semiring. Then the following statements holds for the semiring \( \text{FId}(S) \):

1. \( \text{FId}(S) \) is a Gaussian semiring.
2. \( \text{FId}(S) \) is a subtractive semiring.
3. \( \text{FId}(S) \) is an AIMC semiring and for all \( IJ \in (I^2, J^2) \).
4. \( \text{FId}(S) \) is a Pr"ufer semiring.

**Proof.**

(1): Let \( I, J \in \text{FId}(S) \). Since \( S \) is a Pr"ufer semiring and \( I \subseteq I + J \), by Theorem \([2]\) there exists an ideal \( K \) of \( S \) such that \( I = K(I + J) \). On the other hand, since \( I \) is invertible, \( K \) is also invertible. This means that \( K \) is finitely generated and therefore \( K \in \text{FId}(S) \) and \( I \in (I + J) \). Similarly it can be proved that \( J \in (I + J) \). So we have \( (I, J) = (I + J) \) and by \([N]\) Theorem 8], \( \text{FId}(S) \) is a Gaussian semiring.

(2): By Theorem \([13]\) every Gaussian semiring is subtractive.

(3): Obviously \( \text{FId}(S) \) is additively-idempotent and since \( S \) is a Pr"ufer semiring, \( \text{FId}(S) \) is an AIMC semiring. By Theorem \([Go]\) Proposition 4.43], we have \((I + J)^2 = I^2 + J^2 \) and so \((I + J)^2 \in (I^2, J^2) \). But \((I + J)^2 = I^2 + J^2 + IJ \) and \( \text{FId}(S) \) is subtractive. So \( IJ \in (I^2, J^2) \).

(4): Since \( \text{FId}(S) \) is a Gaussian and an MC semiring such that \( IJ \in (I^2, J^2) \) for all \( I, J \in \text{FId}(S) \), by Theorem \([14]\) \( \text{FId}(S) \) is a Pr"ufer semiring and this is what we wanted to prove. \( \square \)

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SHABAN GHALANDARZADEH, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, K. N. TOOSI UNIVERSITY OF TECHNOLOGY, TEHRAN, IRAN

E-mail address: ghalandarzadeh@kntu.ac.ir

PEYMAN NASEHPOUR, DEPARTMENT OF ENGINEERING SCIENCE, FACULTY OF ENGINEERING, UNIVERSITY OF TEHRAN, TEHRAN, IRAN

E-mail address: nasehpour@gmail.com

RAFIEH RAZAVI, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, K. N. TOOSI UNIVERSITY OF TECHNOLOGY, TEHRAN, IRAN

E-mail address: rrazavi@kntu.ac.ir