Lagrangian Formulation of an Infinite Derivative Real Scalar Field Theory in the Framework of the Covariant Kempf-Mangano Algebra in a \((D+1)\)-dimensional Minkowski Space-time

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Abstract

In 2017, G. P. de Brito and co-workers suggested a covariant generalization of the Kempf-Mangano algebra in a \((D+1)\)-dimensional Minkowski space-time [A. Kempf and G. Mangano, Phys. Rev. D 55, 7909 (1997); G. P. de Brito, P. I. C. Caneda, Y. M. P. Gomes, J. T. Guaitolini Junior, and V. Nikoofard, Adv. High Energy Phys. 2017, 4768341 (2017)]. It is shown that reformulation of a real scalar field theory from the viewpoint of the covariant Kempf-Mangano algebra leads to an infinite derivative Klein-Gordon wave equation which describes two bosonic particles in the free space (a usual particle and a ghostlike particle). We show that in the low-energy (large-distance) limit our infinite derivative scalar field theory behaves like a Pais-Uhlenbeck oscillator for a spatially homogeneous field configuration \(\phi(t,\vec{x}) = \phi(t)\). Our calculations show that there is a characteristic length scale \(\delta\) in our model whose upper limit in a four-dimensional Minkowski space-time is close to the nuclear scale, i.e., \(\delta_{\text{max}} \sim \delta_{\text{nuclear scale}} \sim 10^{-15} \text{m}\). Finally, we show that there is an equivalence between a non-local real scalar field theory with a non-local form factor \(K(x-y) = -\frac{\Box_x}{(1-\frac{D}{2} \Box_x)^2} \delta^{(D+1)}(x-y)\) and an infinite derivative real scalar field theory from the viewpoint of the covariant Kempf-Mangano algebra.

Keywords: Classical field theories; Relativistic wave equations; Nonlinear or nonlocal theories and models; Higher derivatives; Canonical formalism, Lagrangians, and variational principles; Pais-Uhlenbeck oscillator; Characteristic length scale

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1 Introduction

In classical mechanics, the motion of a point particle is described by the following action functional

$$S[x] = \int_{t_i}^{t_f} \, dt \, L(x(t), \dot{x}(t)).$$

In 1850, the Russian mathematician Mikhail Ostrogradsky proposed a higher-order generalization of Lagrangian mechanics [1]. The action functional for a higher-order derivative mechanical system is

$$S[x] = \int_{t_i}^{t_f} \, dt \, L(x^{(0)}(t), x^{(1)}(t), \ldots, x^{(N)}(t)),$$

where $x^{(i)}(t) := \frac{d^i x(t)}{dt^i}$ and $i \in \{0, 1, \ldots, N\}$.

The variation of the above action functional with respect to $x$ leads to the following generalized Euler-Lagrange equation

$$\sum_{i=0}^{N} \left( -\frac{d}{dt} \right)^i \partial L \partial x^{(i)} = 0. \quad (1)$$

In 1942, B. Podolsky suggested a higher-order derivative generalization of Maxwell electrodynamics, in which the electrostatic self-energy of a point charge was a finite value [2].

Six years later, A. E. S. Green presented a higher-order derivative meson-field theory, in which the potential energy for a point nucleon at the origin was a finite value [3]. In 1950, A. Pais and G. E. Uhlenbeck showed that the appearance of higher-order derivative terms in the Lagrangian density of a quantum field theory can eliminate the ultraviolet divergences that appear in the $S$-matrix elements [4].

Today we know that the addition of higher-order derivative terms to the action functional of a quantum field theory is a possible way of regularizing quantum field theories [5-9]. Recently, the infinite derivative scalar field theory of the form

$$S[\phi] = \frac{1}{2c} \int d^{D+1}x \, d^{D+1}y \, \phi(x) \, K(x-y) \, \phi(y) - \frac{1}{c} \int d^{D+1}x \, V(\phi(x)) + \frac{1}{c} \int d^{D+1}x \, J(x) \, \phi(x),$$

has attracted a considerable attention because of its importance in non-local quantum field theory and string field theory [10].

The operator $K(x-y)$ in Eq. (2) has the following general form [10]:

$$K(x-y) = F(\Box_x) \, \delta^{(D+1)}(x-y),$$

where $F(\Box_x)$ is an entire analytic function of the d’Alembertian operator $\Box_x$. On the other hand, different theories of quantum gravity such as string theory and loop quantum gravity predict the
existence of a minimal length scale of the order of the Planck length [11-15]. Today we know that the existence of a minimal length scale in different theories of quantum gravity leads to a generalization of Heisenberg uncertainty principle as follows [12]:

$$\Delta X \Delta P \geq \frac{\hbar}{2} \left[ 1 + \beta (\Delta P)^2 + \cdots \right], \quad \beta > 0.$$

The above generalized uncertainty principle leads to a minimal length scale \( \Delta X_0 = \hbar \sqrt{\beta} (\Delta X_0 > 0) \) in the measurement of space intervals [12]. It should be noted that the reformulation of quantum field theory in the presence of a minimal length scale is another possible way for regularizing a quantum field theory [12]. The aim of this paper is reformulation of a real scalar field theory from the viewpoint of a covariant generalization of the Kempf-Mangano algebra which was proposed by G. P. de Brito and co-workers in Ref. [14].

This paper is organized as follows. In Section 2, a covariant generalization of the Kempf-Mangano algebra in a \((D + 1)\)-dimensional Minkowski space-time is presented briefly. In Section 3, we show that reformulation of a real scalar field theory in the framework of the covariant Kempf-Mangano algebra leads to a generalized Klein-Gordon wave equation with infinitely many derivatives. The free space solutions of this generalized Klein-Gordon wave equation describe two bosonic particles. In Section 4, we show that in the low-energy (large-distance) limit the infinite derivative scalar field theory which was formulated in Section 3 behaves like a Pais-Uhlenbeck oscillator for a spatially homogeneous field configuration \( \phi(t, \vec{x}) = \phi(t) \) [4].

Our calculations in Sections 3 and 4 together with numerical evaluations in Section 5 show that there is a characteristic length scale in our generalized real scalar field theory whose upper limit is very near to the nuclear scale, i.e., \(10^{-15} \text{ m} \). Finally, it should be emphasized that the results of this paper in the low-energy regime are compatible with the results of the standard Klein-Gordon theory. We use SI units in this paper. The flat space-time metric has the signature \( \eta_{\mu\nu} = \eta^{\mu\nu} = diag(+, -, \ldots, -) \).

2 The Covariant Kempf-Mangano Algebra

In 1997, A. Kempf together with G. Mangano proposed a one-parameter extension of the Heisenberg algebra [12]. The Kempf-Mangano algebra in a \(D\)-dimensional Euclidean space is described
by the following generalized commutation relations:

\[
\left[\hat{X}_i,\hat{P}_j\right] = \frac{i\hbar}{(1 + 2\ell^2\tilde{P}^2)^{\frac{1}{2}}} \delta^{ij} + \ell^2 \hat{P}_i \hat{P}_j,
\]

(3)

\[
\left[\hat{X}_i,\hat{X}_j\right] = 0,
\]

(4)

\[
\left[\hat{P}_i,\hat{P}_j\right] = 0,
\]

(5)

where \(i, j = 1, 2, \ldots, D\) and \(\ell\) is a non-negative constant parameter with dimension of \([\text{momentum}]^{-1}\) [12]. The reformulation of non-relativistic quantum mechanics and a charged scalar field in the framework of Kempf-Mangano algebra have been studied in details in Ref. [13].

In 2017, G. P. de Brito and co-workers proposed a covariant generalization of the Kempf-Mangano algebra [14]. The covariant Kempf-Mangano algebra in a \((D+1)\)-dimensional Minkowski space-time is described by the following generalized commutation relations:

\[
\left[\hat{X}^\mu,\hat{P}^\nu\right] = -i\hbar \left( \frac{1 + (1 - 2\ell^2\tilde{P}^2)^{\frac{1}{2}}}{2} \eta^{\mu\nu} - \ell^2 \hat{P}^\mu \hat{P}^\nu \right),
\]

(6)

\[
\left[\hat{X}^\mu,\hat{X}^\nu\right] = 0,
\]

(7)

\[
\left[\hat{P}^\mu,\hat{P}^\nu\right] = 0,
\]

(8)

Where \(\mu, \nu = 0, 1, 2, \ldots, D\), \(\hat{X}^\mu\) and \(\hat{P}^\mu\) are the generalized position and momentum operators, and \(\tilde{P}^2 = \hat{P}_\mu \hat{P}^\mu = (\hat{P}_0)^2 - \sum_{i=1}^{D}(\hat{P}_i)^2\). In the coordinate representation, the generalized position and momentum operators \(\hat{X}^\mu\) and \(\hat{P}^\mu\) in Eqs. (6)-(8) have the following exact representations [14]:

\[
\hat{X}^\mu = \hat{x}^\mu,
\]

(9)

\[
\hat{P}^\mu = \frac{1}{1 + \frac{\ell^2}{2} \tilde{P}^2} \hat{p}^\mu,
\]

(10)

where \(\hat{x}^\mu\) and \(\hat{p}^\mu\) are the conventional position and momentum operators which satisfy the conventional covariant Heisenberg algebra \((i.e., [\hat{x}^\mu, \hat{p}^\nu] = -i\hbar \eta^{\mu\nu}\) and \([\hat{x}^\mu, \hat{x}^\nu] = [\hat{p}^\mu, \hat{p}^\nu] = 0\). In Eq. 

\[\text{In 2006, C. Quesne and V. M. Tkachuk introduced a two-parameter extension of the covariant Heisenberg algebra in a \((D+1)\)-dimensional space-time [15]. There are many papers about reformulation of quantum field theory from the viewpoint of the Quesne-Tkachuk algebra. For a review, we refer the reader to Refs. [16,17].} \]
Equations (9) and (10) show that in order to reformulate quantum field theory from the viewpoint of covariant Kempf-Mangano algebra, the conventional position and derivative operators \((\hat{x}^\mu, \partial_\mu)\) must be replaced as follows:

\[
\begin{align*}
\hat{x}^\mu \rightarrow & \quad \hat{X}^\mu = \hat{x}^\mu, \\
\partial_\mu \rightarrow & \quad \nabla_\mu := \frac{1}{1 - \frac{(\hbar \ell)^2}{2}} \partial_\mu.
\end{align*}
\]

It is important to note that in the limit of \(\hbar \ell \rightarrow 0\), the generalized derivative operator \(\nabla_\mu\) in Eq. (12) becomes the conventional derivative operator, i.e., \(\lim_{\hbar \ell \rightarrow 0} \nabla_\mu = \partial_\mu\). In the next section, we will introduce a Lorentz-invariant infinite derivative scalar field theory in the framework of the covariant Kempf-Mangano algebra.

### 3 Lagrangian Formulation of an Infinite Derivative Scalar Field Theory Based on the Covariant Kempf-Mangano Algebra

The Lagrangian density for a real scalar field in a \((D + 1)\)-dimensional flat space-time can be written as follows [18]:

\[
\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} (\partial_\mu \phi(x)) (\partial^\mu \phi(x)) - \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 \phi^2(x).
\]

Using Eq. (11) together with the transformation rule for a scalar field, we obtain

\[
\phi(x) \rightarrow \Phi(X) = \phi(x).
\]

If we use Eqs. (12)-(14), we will get the generalized Lagrangian density for a real scalar field as follows:

\[
\mathcal{L} = \frac{1}{2} (\nabla_\mu \Phi(X)) (\nabla^\mu \Phi(X)) - \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 \Phi^2(X)
= \frac{1}{2} \left( \frac{1}{1 - \frac{(\hbar \ell)^2}{2}} \partial_\mu \phi(x) \right) \left( \frac{1}{1 - \frac{(\hbar \ell)^2}{2}} \partial^\mu \phi(x) \right) - \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 \phi^2(x).
\]

Now, let us consider a classical scalar field theory whose action functional is given by [19]

\[
S[\phi] = \int_{\mathbb{R}^{1,D}} d^D x \ dt \ \mathcal{L}(\phi, \partial_{\nu_1} \phi, \partial_{\nu_1} \partial_{\nu_2} \phi, \partial_{\nu_1} \partial_{\nu_2} \partial_{\nu_3} \phi, \cdots).
\]
The variation of (16) with respect to $\phi$ leads to the following generalized Euler-Lagrange equation [19]

$$\frac{\partial L}{\partial \phi} - \left( \frac{\partial L}{\partial \phi_{\mu_1}} \right)_{\mu_1} + \left( \frac{\partial L}{\partial \phi_{\mu_1\mu_2}} \right)_{\mu_1\mu_2} - \cdots + (-1)^k \left( \frac{\partial L}{\partial \phi_{\mu_1\mu_2\cdots \mu_k}} \right)_{\mu_1\mu_2\cdots \mu_k} + \cdots = 0,$$  

(17)

where

$$\phi_{\mu_1\mu_2\cdots \mu_k} := \partial_{\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_k} \phi,$$  

(18)

$$\frac{\partial \phi_{\mu_1\mu_2\cdots \mu_k}}{\partial \phi_{\nu_1\nu_2\cdots \nu_k}} = \delta_{\mu_1}^{\nu_1} \delta_{\mu_2}^{\nu_2} \cdots \delta_{\mu_k}^{\nu_k}.$$  

(19)

If we substitute (15) into (17), we will obtain the following generalized Klein-Gordon wave equation

$$\frac{1}{\left(1 - \frac{(\hbar \ell)^2}{2} \Box \right)^2} \Box \phi(x) + \left( \frac{mc}{\hbar} \right)^2 \phi(x) = 0.$$  

(20)

Note that in the low-energy limit ($\hbar \ell \to 0$), the generalized Klein-Gordon equation (20) becomes the conventional Klein-Gordon equation, i.e.,

$$\Box \phi(x) + \left( \frac{mc}{\hbar} \right)^2 \phi(x) + \frac{3}{4} (\hbar \ell)^4 \Box \Box \phi(x) + \mathcal{O} \left( (\hbar \ell)^6 \right) = 0.$$  

(21)

The generalized Klein-Gordon equation (20) has a plane-wave solution as follows:

$$\phi(x) = A e^{-\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{x}},$$  

(22)

where $A$ is the amplitude of the scalar field. After inserting Eq. (22) into Eq. (20), we obtain the following generalized dispersion relation:

$$\frac{\mathbf{p}^2}{\left(1 + \frac{\ell^2}{2} \mathbf{p}^2\right)^2} = m^2 c^2.$$  

(23)

Equation (23) leads to the following generalized energy-momentum relations:

$$E_{+}^2 (\ell) = m_{+}^2 (\ell) c^4 + c^2 \mathbf{p}^2,$$  

(24)

$$E_{-}^2 (\ell) = m_{-}^2 (\ell) c^4 + c^2 \mathbf{p}^2,$$  

(25)
where the effective masses $m_+(\ell)$ and $m_- (\ell)$ have the following definitions

$$m_+(\ell) := 1 + \frac{\sqrt{1 - 2m^2c^2\ell^2}}{mc^2\ell^2}, \quad (26)$$

$$m_- (\ell) := 1 - \frac{\sqrt{1 - 2m^2c^2\ell^2}}{mc^2\ell^2}. \quad (27)$$

In order to obtain the real values for the effective masses $m_{\pm}(\ell)$ in Eqs. (26) and (27) the parameter $\ell$ must satisfy the following inequality

$$\ell \leq \frac{1}{\sqrt{2} \, mc}. \quad (28)$$

It must be emphasized that the parameter $\hbar \ell$ which has a dimension of $[\text{length}]$ defines a characteristic length scale $\delta := \hbar \ell$ in our model. Equation (28) shows that the upper limit of $\delta$ is

$$\delta_{\text{max}} = \frac{1}{\sqrt{2} \, mc}. \quad (29)$$

If we expand the effective masses (26) and (27) around $\delta = 0$, we will obtain the following low-energy expressions for $m_{\pm}(\ell)$

$$m_+(\ell) = \frac{2}{mc^2\ell^2} - m + O(\ell^2), \quad (30)$$

$$m_- (\ell) = m + \frac{1}{2} m^3 c^2 \ell^2 + O(\ell^4). \quad (31)$$

Therefore, the low-energy limit of our model describes two particles, one with the usual mass $m$ and the other a ghostlike particle of mass $\frac{2}{mc^2\ell^2}$.

4 Relationship between the Low-Energy Behavior of the Model for a Spatially Homogeneous Field Configuration and the Pais-Uhlenbeck Oscillator

In this section, we want to study the low-energy behavior of the infinite derivative scalar field theory which was introduced in the previous section for a spatially homogeneous field configuration.

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2The appearance of these ghostlike particles in the theory of fourth-order derivative wave equations such as $\square \phi(x) + (\frac{d^2}{dt^2})\phi(x) + \frac{1}{\Lambda^2} \square^2 \phi(x) = 0$ ($\Lambda$ is a regulator (cutoff)) is a well-known problem in higher-derivative quantum field theories [16,20].
The action functional (16) for the generalized Lagrangian density (15) is

$$S_\delta[\phi] = \int_{\mathbb{R}^{1,D}} d^D x \, dt \left[ \frac{1}{2} \left( \frac{1}{1 - \delta_2 \Box} \partial_\mu \phi(x) \right) \left( \frac{1}{1 - \delta_2 \Box} \partial^\mu \phi(x) \right) - \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 \phi^2(x) \right].$$  \hspace{1cm} (32)

For a spatially homogeneous real scalar field $\phi(t)$ Eq. (32) becomes

$$S_\delta[\phi] = \int_{\mathbb{R}^D} d^D x \int_{\mathbb{R}} dt \left[ \frac{1}{2c^2} \left( \frac{1}{1 - \frac{1}{2} \left( \frac{\delta}{c} \right)^2 \frac{d^2}{dt^2}} \dot{\phi}(t) \right) \left( \frac{1}{1 - \frac{1}{2} \left( \frac{\delta}{c} \right)^2 \frac{d^2}{dt^2}} \dot{\phi}(t) \right) - \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \phi^2(t) \right],$$  \hspace{1cm} (33)

where $V_D = \int_{\mathbb{R}^D} d^D x$ is the volume of the spatial part of the space-time and dot denotes derivative with respect to $t$.

Using the field redefinition $\psi(t) := \sqrt{V_D} \phi(t)$ Eq. (33) can be rewritten as follows:

$$S_\delta[\psi] = \int_{\mathbb{R}} dt \left[ \frac{1}{2} \left( \frac{1}{1 - \frac{1}{2} \left( \frac{\delta}{c} \right)^2 \frac{d^2}{dt^2}} \dot{\psi}(t) \right) \left( \frac{1}{1 - \frac{1}{2} \left( \frac{\delta}{c} \right)^2 \frac{d^2}{dt^2}} \dot{\psi}(t) \right) - \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \psi^2(t) \right].$$  \hspace{1cm} (34)

The action functional $S_\delta[\psi]$ in Eq. (34) has the following low-energy expansion

$$S_\delta[\psi] = \sum_{n=0}^{\infty} \delta^{2n} S_n[\psi]$$

$$= \int_{\mathbb{R}} dt \left[ \frac{1}{2} \left( \psi^2(t) + \left( \frac{\delta}{c} \right)^2 \ddot{\psi}(t) \right) - \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \psi^2(t) \right] + \mathcal{O}(\delta^4)$$

$$= \int_{\mathbb{R}} dt \left[ \frac{1}{2} \psi^2(t) - \left( \frac{\delta}{c} \right)^2 \ddot{\psi}(t) \right] - \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \psi^2(t) \right] + \int_{\mathbb{R}} dt \frac{d\Omega(t)}{dt} + \mathcal{O}(\delta^4),$$  \hspace{1cm} (35)

where

$$\Omega(t) := \frac{1}{2} \left( \frac{\delta}{c} \right)^2 \dot{\psi}(t) \ddot{\psi}(t).$$  \hspace{1cm} (36)

If we neglect terms of order $\delta^4$ and higher in Eq. (35) and dropping out the boundary term $\int_{\mathbb{R}} dt \frac{d\Omega(t)}{dt}$, we will find

$$S_\delta[\psi] = \int_{\mathbb{R}} dt \left[ \frac{1}{2} \psi^2(t) - \frac{1}{2} \left( \frac{\delta}{c} \right)^2 \ddot{\psi}(t) - \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \psi^2(t) \right].$$  \hspace{1cm} (37)
Straightforward but tedious calculations show that $S_\delta[\psi]$ in Eq. (37) can be written as follows:

$$S_\delta[\psi] = -\frac{1}{2} \left( \frac{\delta}{c} \right)^2 \int_{\mathbb{R}} dt \left[ \ddot{\psi}^2(t) - \left( \omega_+^2 + \omega_-^2 \right) \dot{\psi}^2(t) + \omega_+^2 \omega_-^2 \psi^2(t) \right], \quad (38)$$

where the effective frequencies $\omega_\pm$ are defined as follows:

$$\omega_+ := \frac{c}{2\delta} \left[ \sqrt{1 + \frac{2mc\delta}{\hbar}} + \sqrt{1 - \frac{2mc\delta}{\hbar}} \right], \quad (39)$$

$$\omega_- := \frac{c}{2\delta} \left[ \sqrt{1 + \frac{2mc\delta}{\hbar}} - \sqrt{1 - \frac{2mc\delta}{\hbar}} \right]. \quad (40)$$

The action functional (38) is a well-known model in the theory of higher-order time derivative models and is called the Pais-Uhlenbeck (PU) oscillator [4,21-25]. Therefore, in the low-energy limit ($\delta \to 0$) our model behaves like a Pais-Uhlenbeck oscillator for a spatially homogeneous field configuration $\phi(t, \vec{x}) = \phi(t)$. In order to obtain the real values for the effective frequencies $\omega_\pm$ in Eqs. (39) and (40) the characteristic length scale $\delta$ must satisfy the following condition

$$\delta_{PU} \leq \frac{1}{2} \frac{\hbar}{mc}. \quad (41)$$

According to Eq. (41) the upper limit of $\delta_{PU}$ is

$$\delta_{PU}^{max} = \frac{1}{2} \frac{\hbar}{mc}. \quad (42)$$

Equations (29) and (42) show that the upper limit of the characteristic length scale $\delta$ in this work is proportional to $\frac{\hbar}{mc}$, i.e.,

$$\delta_{max} \sim \frac{\hbar}{mc}. \quad (43)$$

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3The Pais-Uhlenbeck oscillator describes a one-dimensional harmonic oscillator coupled to a higher-order time derivative term whose action functional is

$$S_{PU} = \frac{\gamma}{2} \int_{\mathbb{R}} dt \left[ \ddot{\psi}^2(t) - \left( \omega_+^2 + \omega_-^2 \right) \dot{\psi}^2(t) + \omega_+^2 \omega_-^2 \psi^2(t) \right],$$

where $\gamma$ is an arbitrary parameter [21]. This model has a wide applications in several areas of theoretical physics [21-25].
5 Summary and Conclusions

More than 70 years ago the American physicist H. S. Snyder introduced a one-parameter extension of the covariant Heisenberg algebra in a four-dimensional space-time in order to remove the infinities which appear in quantum field theories [26]. In 2006 a two-parameter extension of the covariant Heisenberg algebra in a \((D + 1)\)-dimensional Minkowski space-time was presented by Quesne and Tkachuk [15]. The Quesne-Tkachuk algebra contains the Snyder algebra as a subalgebra [15]. In addition, the reformulation of Maxwell equations and Dirac equation from the viewpoint of the Quesne-Tkachuk algebra have been studied for the first time in Ref. [27].

In 2017, G. P. de Brito and his co-workers introduced a modification of the Quesne-Tkachuk algebra [14]. This modified algebra is a covariant generalization of the Kempf-Mangano algebra in a \((D + 1)\)-dimensional Minkowski space-time. In this work, by using the methods of Ref. [27], after Lagrangian formulation of an infinite derivative scalar field theory in the framework of the covariant Kempf-Mangano algebra, it was shown that the infinite derivative field equation (20) describes two particles with the effective masses 

\[
m_\pm(\ell) = \frac{1}{mc^2} \left[ \sqrt{1 - \frac{2m^2c^2}{\ell^2}} \pm \sqrt{1 - \frac{2m^2c^2}{\ell^2}} \right].
\]

We showed that in the low-energy (large-distance) limit the infinite derivative scalar field theory in Eq. (32) for a spatially homogeneous field configuration \(\phi(t, \vec{x}) = \phi(t)\) behaves like a Pais-Uhlenbeck oscillator with the effective frequencies 

\[
\omega_\pm = \frac{c}{2\delta} \left[ \sqrt{1 + \frac{2m^2\delta}{\hbar}} \pm \sqrt{1 - \frac{2m^2\delta}{\hbar}} \right],
\]

where \(\delta = \hbar \ell\) is the characteristic length scale in this paper. Our calculations in Sections 3 and 4 show that the upper limit of \(\delta\) must be proportional to \(\frac{\hbar}{mc}\) (see Eq. (43)).

Now, let us evaluate the numerical value of \(\delta_{\text{max}}\) in Eq. (43).

In nuclear and low-energy particle physics a real scalar field theory describes a neutral \(\pi^0\) meson [28]. The mass of the \(\pi^0\) meson is [18]

\[
m_{\pi^0} = 134.977 \text{ MeV}/c^2.
\]

Inserting (44) into (43), we find

\[
\delta_{\text{max}} \sim 10^{-15} m.
\]

It should be noted that the numerical value of \(\delta_{\text{max}}\) in Eq. (45) is near to the nuclear scale (see page 174 in Ref. [29]), i.e.,

\[
\delta_{\text{max}} \sim \delta_{\text{nuclear scale}} \sim 10^{-15} m.
\]

The above estimations show that in the low-energy limit, the conventional real scalar field theory in Eq. (13) is recovered, while in the high-energy limit the real Klein-Gordon theory in Eq. (13) must be replaced by Eq. (15), i.e.,

\[
\mathcal{L} = \begin{cases}
\frac{1}{2} (\partial_{\mu} \phi(x)) (\partial^{\mu} \phi(x)) - \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 \phi^2(x) & \text{(low-energy regime)}, \\
\frac{1}{2} (\partial_{\mu} \phi(x)) (\partial^{\mu} \phi(x)) - \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 \phi^2(x) - \frac{1}{2} \delta^2(\square \phi(x))(\square \phi(x)) + \text{boundary term} + \mathcal{O}(\delta^4) & \text{(high-energy regime)}.
\end{cases}
\]
The action functional (32) in the presence of an external current \( J(x) \) is

\[
S_{\delta}[\phi] = \frac{1}{c} \int_{\mathbb{R}^{1,D}} d^{D+1}x \left[ \frac{1}{2} \left( \frac{1}{1 - \frac{4\omega}{2}} \partial_\mu \phi(x) \right) \left( \frac{1}{1 - \frac{4\omega}{2}} \partial^\mu \phi(x) \right) - \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 \phi^2(x) + J(x)\phi(x) \right].
\]

(32)

The action functional (48) can be rewritten as follows:

\[
S_{\delta}[\phi] = \frac{1}{c} \int_{\mathbb{R}^{1,D}} d^{D+1}x \left[ \frac{1}{2} \phi(x) \left( \nabla_\mu \nabla^\mu + \left( \frac{mc}{\hbar} \right)^2 \right) \phi(x) + \partial_\mu \Upsilon^\mu + J(x)\phi(x) \right],
\]

(49)

where \( \nabla_\mu \) has been defined in Eq. (12) and \( \Upsilon^\mu \) has the following definition

\[
\Upsilon^\mu := \frac{1}{2} \phi \partial^\mu \phi + \frac{\delta^2}{2} \phi \Box \partial^\mu \phi + \frac{\delta^4}{8} (3\phi \Box^2 \phi - \partial^\mu \phi \Box^2 \phi + \Box \phi \Box^\mu \phi) + O(\delta^6).
\]

(50)

After dropping out the boundary term \( \frac{1}{c} \int_{\mathbb{R}^{1,D}} d^{D+1}x \partial_\mu \Upsilon^\mu \) in (49), we will find

\[
S_{\delta}[\phi] = \frac{1}{c} \int_{\mathbb{R}^{1,D}} d^{D+1}x \left[ -\frac{1}{2} \phi(x) \left( \nabla_\mu \nabla^\mu + \left( \frac{mc}{\hbar} \right)^2 \right) \phi(x) + J(x)\phi(x) \right].
\]

(51)

A comparison between the action functionals (2) and (51) shows that for \( V(\phi(x)) = \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 \phi^2(x) \) and \( F(\Box_x) = -\frac{\Box_x}{(1 - \frac{4\omega}{2})^2} \) there is an equivalence between a non-local real scalar field theory and an infinite derivative real scalar field theory in the framework of the covariant Kempf-Mangano algebra.

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\[\text{References}\]

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\[\text{Note that the form factor } F(z) = -\frac{\Box_x}{(1 - \frac{4\omega}{2})^2} \text{ is not an entire function. It must be emphasized that there are many examples in the literatures about non-local quantum field theory in which the form factor } F(z) \text{ is not an entire function (see Refs. [10,30,31] for more details).}\]
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