Use of primary decomposition of polynomial ideals arising from indicator functions to enumerate orthogonal fractions

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Received: 29 September 2021 / Revised: 6 February 2022 / Accepted: 15 February 2022 / Published online: 3 March 2022
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Abstract
A polynomial indicator function of designs is first introduced by Fontana et al. (J Stat Plan Inference 87:149–172, 2000) for two-level cases. They give the structure of the indicator functions, especially the relation to the orthogonality of designs. These results are generalized by Aoki (J Stat Plan Inference 203:91–105, 2019) for general multi-level cases. As an application of these results, we can enumerate all orthogonal fractional factorial designs with given size and orthogonality using computational algebraic software. For example, Aoki (2019) gives classifications of orthogonal fractions of $2^4 \times 3$ designs with strength 3, which is derived by simple eliminations of variables. However, the computational feasibility of this naive approach depends on the size of the problems. In fact, it is reported that the computation of orthogonal fractions of $2^4 \times 3$ designs with strength 2 fails to carry out in Aoki (2019). In this paper, using the theory of primary decomposition, we enumerate and classify orthogonal fractions of $2^4 \times 3$ designs with strength 2. We show there are 35,200 orthogonal half fractions of $2^4 \times 3$ designs with strength 2, classified into 63 equivalent classes.

Keywords Computational algebraic statistics · Fractional factorial designs · Gröbner bases · Indicator functions · Orthogonal designs · Primary decomposition

1 Introduction

An application of Gröbner basis theory to problems in the field of statistics first arises in the design of experiments. In the first paper of this research field named computational algebraic statistics, Pistone and Wynn reveal the relation between the identifiability problem of polynomial models on designs and the ideal membership problem (Pis-
tone & Wynn, 1996). After this breakthrough paper, various applications of algebraic methods to the problems in the design of experiments are given by researchers both in the field of statistics and algebra. The theory of the indicator function is one of the early, fundamental results in this field.

The indicator function of a fractional factorial design is first introduced for the designs of two-level factors (Fontana et al., 2000). Based on the arguments of Pistone and Wynn (1996) and Fontana et al. (2000) show the one-to-one correspondence between the design and its indicator function. By this correspondence, various statistical concept such as orthogonality, resolution and aberration can be translated to the structure of the indicator functions. As an application of this result, they enumerate and classify all orthogonal fractions of $2^4$ and $2^5$ designs by solving the system of algebraic equations for the coefficients of the indicator functions.

This argument is generalized to the designs of arbitrary multi-level factors in Aoki (2019). In Aoki (2019), algebraic equations for the coefficients of the indicator functions of designs with given orthogonality are given in general settings. As an application, Aoki gives in Aoki (2019) the classification of all orthogonal fractions of $2^3 \times 3$ designs with strength 2 and orthogonal fractions of $2^4 \times 3$ designs with strength 3. These results are derived by simple eliminations of variables, which is one of the fundamental applications of Gröbner basis theory. However, the computational feasibility of this naive approach depends on the size of the problems and apparently fails to carry out for problems of large sizes. In fact, it is reported in Aoki (2019) that the Gröbner basis calculation of the first elimination ideal for the orthogonal fractions of $2^4 \times 3$ designs with strength 2 is difficult to carry out for standard PC. (It is reported that the calculation does not finish in 1 week.) This is the reason why the results for $2^4 \times 3$ designs are given for strength 3, not for strength 2 in Aoki (2019).

In this paper, we report that we have broken through the computational limit of Aoki (2019) described above by a technique of computational algebraic geometry. Using the theory of primary decomposition, we obtain the enumeration and classification of half orthogonal fractions of $2^4 \times 3$ designs with strength 2. The result is summarized as follows: there are 35,200 orthogonal half fractions of $2^4 \times 3$ designs with strength 2, classified into 63 equivalent classes. This result is shown in detail in Sect. 4. The contents of the remaining sections of this paper are as follows. In Sect. 2, we summarize the theory of the indicator functions and describe the problems we consider. In Sect. 3, we give some definitions and facts on the primary decomposition, and explain our approach. In Sect. 5, we give some discussion.

## 2 The indicator functions of fractional factorial designs

In this section, we give necessary tools and results on the indicator functions of fractional factorial designs. Mostly, we follow the notations and definitions of Aoki (2019). The existence, uniqueness and structure of the indicator functions are derived from the results of Pistone and Wynn (1996).

Let $x_1, \ldots, x_n$ be $n$ factors. Let $A_j \subset \mathbb{Q}$ be a level set of a factor $x_j$ for $j = 1, \ldots, n$, where $\mathbb{Q}$ be the field of rational numbers. We denote by $r_j = \# A_j$ the cardinality of $A_j$ and assume $r_j \geq 2$ for $j = 1, \ldots, n$. A full factorial design of the factors $x_1, \ldots, x_n$
is a direct product $D = A_1 \times \cdots \times A_n$. For later use, we introduce an index set $\mathcal{I} = \{i_1, \ldots, i_n\} \in [r_1] \times \cdots \times [r_n]$, where we define $[k] = \{1, 2, \ldots, k\}$ for a positive integer $k$, and represent $D$ by its design points explicitly as $D = \{d_i \in \mathbb{Q}^n : i \in \mathcal{I}\}$. A subset of $D$ is called a fractional factorial design. A fractional factorial design $F \subset D$ is written as $F = \{d_i \in D : i \in \mathcal{I}'\}$ for a subset $\mathcal{I}'$ of $\mathcal{I}$.

Now, we give a definition of an indicator function.

**Definition 2.1** (Fontana et al., 2000) Let $D$ be a full factorial design and $F \subset D$ be a fractional factorial design. The indicator function of $F$ is a $\mathbb{Q}$-valued function $f$ on $D$ satisfying

$$f(d) = \begin{cases} 1, & \text{if } d \in F, \\ 0, & \text{if } d \in D \setminus F. \end{cases}$$

To give the structure of the indicator functions, we prepare some algebraic materials of polynomial rings. Let $\mathbb{Q}[x_1, \ldots, x_n]$ be the polynomial ring with coefficients in $\mathbb{Q}$. For a design $F \subset \mathbb{Q}^n$, we denote by $I(F)$ the set of polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$ which are 0 at every point of $F$, i.e., $I(F) = \{f \in \mathbb{Q}[x_1, \ldots, x_n] : f(d) = 0, \forall d \in F\}$. Clearly, $I(F)$ satisfies the conditions that

- if $f_1 \in I(F)$ and $f_2 \in I(F)$, then $f_1 + f_2 \in I(F)$, and
- if $f \in I(F)$ and $g \in \mathbb{Q}[x_1, \ldots, x_n]$, then $fg \in I(F)$.

Therefore, $I(F)$ is an ideal of $\mathbb{Q}[x_1, \ldots, x_n]$, and is called the design ideal of $F$. A generator of a design ideal of a full factorial design $D$ can be written as $G = \{x_j^{r_j} - g_j, \ j = 1, \ldots, n\}$, where $g_j$ is a polynomial in $\mathbb{Q}[x_j]$ with the degree less than $r_j$, $j = 1, \ldots, n$, and $I(D)$ is written as $I(D) = \langle G \rangle$. This $G$ is a reduced Gröbner basis of $I(D)$ for any monomial order. We write the set of the monomials of $x_1, \ldots, x_n$ that are not divisible by $x_j^{r_j}$, $j = 1, \ldots, n$, the leading monomials of $G$, as

$$\text{Est}(D) = \left\{x^a = \prod_{j=1}^n x_j^{a_j} : a \in L\right\},$$

where

$$L = \{a = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n : 0 \leq a_j \leq r_j - 1, \ j = 1, \ldots, n\}$$

and $\mathbb{Z}_{\geq 0}^n$ is the set of nonnegative integers.

From $D = \{d_i \in \mathbb{Q}^n : i \in \mathcal{I}\}$ and $L$, we define a model matrix by $X = [a_i^a]_{i \in \mathcal{I}, a \in L}$, where $a_i^a = \prod_{j=1}^n d_{ij}^{a_j}$ and $d_{ij}$ is the level of the factor $x_j$ in the experimental run indexed by $i \in \mathcal{I}$. By ordering the element of $\mathcal{I}$ and $L$, $X$ is an $m \times m$ matrix where we denote by $m = \prod_{j=1}^n r_j$, the size of $D$. Note that both the cardinality of $L$ and $\mathcal{I}$ are $m$, and $X$ is non-singular (Theorem 26 of Pistone et al., 2001).
From these materials, the indicator function can be constructed as follows. Let \( F = \{ d_i \in D : i \in I' \} \) be a fractional factorial design for a subset \( I' \subset I \). Then, the indicator function of \( F \) is written uniquely as

\[
f(x_1, \ldots, x_n) = \sum_{a \in L} \theta_a x^a,
\]

where an \( m \times 1 \) column vector \( \theta = (\theta_a)_{a \in L} \) is give by \( \theta = X^{-1} y \) for an \( m \times 1 \) column vector \( y = (y_i)_{i \in I} \) given by

\[
y_i = \begin{cases} 
1, & \text{if } i \in I', \\
0, & \text{if } i \in I \setminus I'.
\end{cases}
\]

In other words, the indicator function of \( F \) is the interpolatory polynomial function on \( D \) for the observation \( y \) given in (2).

The coefficients of the indicator function satisfy the following system of the algebraic equations. For a polynomial \( f \) of the form (1), calculate

\[
\sum_{a_1 \in L} \sum_{a_2 \in L} \theta_{a_1} \theta_{a_2} x^{a_1+a_2}
\]

and divide it by \( G \), the reduced Gröbner basis of \( I(D) \), a unique remainder is obtained in the form \( r = \sum_{a \in L} \mu_a x^a \). This \( r \) is called the standard form of (3) with respect to \( G \). Then, \( f \) is an indicator function of some fractional factorial design if and only if \( \{ \theta_a : a \in L \} \) satisfies the system of algebraic equations

\[
\theta_a = \mu_a, \; a \in L.
\]

See Proposition 3.1 of Aoki (2019) for detail.

In addition to the system of the algebraic equations (4), the orthogonality of the designs can be characterized as the constraints for the coefficients of the corresponding indicator functions as follows. Following to Chapter 7 of Wu and Hamada (2009), we call a design \( F \subset D \) is orthogonal of strength \( t \) \( (t \leq n) \), if for any \( t \) factors, all possible combinations of levels appear equally often in \( F \). To characterize the constraints of a given strength for \( \{ \theta_a : a \in L \} \), we introduce a contrast matrix as follows (Definition 3.4 of Aoki, 2019). We define the \( m \times m \) contrast matrix \( C \) by

\[
C^T = [1_m | C_1^T | C_2^T | \cdots | C_n^T],
\]

where \( 1_m = (1, \ldots, 1)^T \) is an \( m \times 1 \) column vector of the elements 1’s, and \( C_k \) is a \( v_k \times m \) matrix where

\[
v_k = \sum_{J \subset [n], \#J = k} \left( \prod_{j \in J} (r_j - 1) \right).
\]
The set of $m \times 1$ column vector of $C_k^T$ is

$$
\left\{ c_{J(\tilde{\i})} = \{c_{J(\tilde{\i})}(i)\}_{i \in I} : J \subset [n], \# J = k, \tilde{\i} \in \prod_{j \in J}[r_j - 1] \right\},
$$

where

$$
c_{J(\tilde{\i})}(i) = \begin{cases} 
1, & i_J = 1, \\
-1, & i_J = \tilde{i} + 1, \\
0, & \text{otherwise}
\end{cases}
$$

for $\# J = 1$ and

$$
c_{J(\tilde{\i})}(i) = \begin{cases} 
1, & i_J = (\tilde{i}_1, \ldots, \tilde{i}_{k-1}, 1), \\
-1, & i_J = (\tilde{i}_1, \ldots, \tilde{i}_{k-1}, \tilde{i}_k + 1), \\
0, & \text{otherwise}
\end{cases}
$$

for $\# J \geq 2$, $\prod [r_j] = \{1, 2, \ldots, r_j\}, j \in J$. Note that the contrast matrix $C$ is a non-singular $m \times m$ matrix (Corollary 3.8 of Aoki, 2019). Then a polynomial $f$ of the form (1) is an indicator function of size $s$, orthogonal fractional factorial design of strength $t$, if and only if $\{\theta_a : a \in L\}$ satisfies

$$
1_m^T X \theta = s, \quad C_\ell X \theta = 0_{v_\ell}, \quad \ell = 1, \ldots, t,
$$

where $0_{v_\ell} = (0, \ldots, 0)^T$ is a $v_\ell \times 1$ column vector of the element 0’s, in addition to (4). See Theorem 3.6 of Aoki (2019) for detail.

**Example 2.2** Consider designs of 3 factors $x_1, x_2, x_3$, where $x_1, x_2$ are two-level factors and $x_3$ is a three-level factor. As the level sets, we define $A_1 = A_2 = \{-1, 1\}$ and $A_3 = \{-1, 0, 1\}$. The full factorial design of the factors $x_1, x_2, x_3$ is a direct product $D = A_1 \times A_2 \times A_3$. We also represent $D$ by its design points as $D = \{d_i : i \in I\}$, where $I = \{1, 2\} \times \{1, 2\} \times \{1, 2, 3\}$ is the index set, and

$$
d_{(111)} = (-1, -1, -1), \quad d_{(112)} = (-1, -1, 0), \quad d_{(113)} = (-1, -1, 1), \\
d_{(121)} = (-1, 1, -1), \quad d_{(122)} = (-1, 1, 0), \quad d_{(123)} = (-1, 1, 1), \\
d_{(211)} = (1, -1, -1), \quad d_{(212)} = (1, -1, 0), \quad d_{(213)} = (1, -1, 1), \\
d_{(221)} = (1, 1, -1), \quad d_{(222)} = (1, 1, 0), \quad d_{(223)} = (1, 1, 1).
$$

The design ideal of $D$ is written as $I(D) = \langle x_1^2 - 1, x_2^2 - 1, x_3^3 - x_3 \rangle \subset \mathbb{Q}[x_1, x_2, x_3]$, and $G = \{x_1^2 - 1, x_2^2 - 1, x_3^3 - x_3\}$ is a reduced Gröbner basis of $I(D)$ for any monomial order. Therefore, we have

$$
\operatorname{Est}(D) = \{1, x_1, x_2, x_3, x_3^2, x_1 x_2, x_1 x_3, x_2 x_3, x_1 x_2 x_3, x_1 x_3^2, x_2 x_3^2, x_1 x_2 x_3^2\},
$$

and the corresponding $L$ is given by $L = \{(000), (100), (010), (001), (002), (110), (101), (011), (111), (102), (012), (112)\}$. Then the model matrix $X$ is given in Fig. 1.
The fractional factorial design $F \subset D$ is characterized as $F = \{d_i \in D : i \in \mathcal{I}'\}$ for a subset $\mathcal{I}' \subset \mathcal{I}$, and its indicator function is written in the form (1), where the set of the coefficients $\{\theta_a : a \in L\}$ satisfies the relation (4). For this example, we see that the polynomial function

$$f(x_1, x_2, x_3) = \theta_{000} + \theta_{010}x_1 + \theta_{011}x_2 + \theta_{001}x_3 + \theta_{002}x_2^2 + \theta_{110}x_1x_2 + \theta_{101}x_1x_3 + \theta_{011}x_2x_3 + \theta_{111}x_1x_2x_3 + \theta_{102}x_1x_3^2 + \theta_{012}x_2x_3^2 + \theta_{112}x_1x_2x_3$$

is an indicator function of some fractional factorial design of $D$ if and only if the following relations hold.

\[
\begin{align*}
\theta_{000} &= \theta_{100}^2 + \theta_{010}^2 + \theta_{001}^2 + \theta_{110}^2, \\
\theta_{100} &= 2\theta_{100}\theta_{000} + 2\theta_{010}\theta_{110}, \\
\theta_{010} &= 2\theta_{100}\theta_{110} + 2\theta_{010}\theta_{000}, \\
\theta_{001} &= 2\theta_{100}\theta_{010} + 2\theta_{010}\theta_{011} + 2\theta_{001}\theta_{002} + 2\theta_{001}\theta_{000} + 2\theta_{110}\theta_{111} + 2\theta_{110}\theta_{102} + 2\theta_{011}\theta_{012} + 2\theta_{111}\theta_{112}, \\
\theta_{002} &= 2\theta_{100}\theta_{102} + 2\theta_{010}\theta_{012} + \theta_{001}^2 + \theta_{002}^2 + 2\theta_{000}\theta_{000} + 2\theta_{110}\theta_{112} + \theta_{101}^2 + \theta_{011}^2 + \theta_{111}^2 + \theta_{102}^2 + \theta_{012}^2 + \theta_{112}^2, \\
\theta_{110} &= 2\theta_{100}\theta_{010} + 2\theta_{000}\theta_{110}, \\
\theta_{101} &= 2\theta_{100}\theta_{001} + 2\theta_{010}\theta_{011} + 2\theta_{001}\theta_{010} + 2\theta_{002}\theta_{010} + 2\theta_{000}\theta_{101} + 2\theta_{000}\theta_{101} + 2\theta_{010}\theta_{011} + 2\theta_{011}\theta_{112} + 2\theta_{111}\theta_{012}, \\
\theta_{011} &= 2\theta_{100}\theta_{011} + 2\theta_{010}\theta_{001} + 2\theta_{001}\theta_{012} + 2\theta_{002}\theta_{012} + 2\theta_{000}\theta_{111} + 2\theta_{000}\theta_{111} + 2\theta_{110}\theta_{101} + 2\theta_{110}\theta_{101} + 2\theta_{110}\theta_{111} + 2\theta_{111}\theta_{112} + 2\theta_{111}\theta_{102} + 2\theta_{002}\theta_{111} + 2\theta_{000}\theta_{111} + 2\theta_{010}\theta_{101} + 2\theta_{011}\theta_{101} + 2\theta_{011}\theta_{101} + 2\theta_{011}\theta_{102},
\end{align*}
\]
\[ J(\tilde{i}) \backslash \mathcal{I} \]

| Const. | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
|--------|---|---|---|---|---|---|---|---|---|---|---|
| 1(1)   | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 |
| 2(1)   | 1 | 1 | -1 | -1 | -1 | 1 | 1 | 1 | -1 | -1 | -1 |
| 3(1)   | 1 | -1 | 0 | 1 | -1 | 0 | 1 | -1 | 0 | 1 | -1 |
| 3(2)   | 1 | 0 | -1 | 1 | 0 | -1 | 1 | 0 | -1 | 1 | 0 |
| 12(11) | 1 | 1 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13(11) | 1 | -1 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13(12) | 1 | 0 | -1 | 1 | 0 | -1 | 0 | 0 | 0 | 0 | 0 |
| 23(11) | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 |
| 23(12) | 1 | 0 | -1 | 0 | 0 | 0 | 1 | 0 | -1 | 0 | 0 |
| 123(111)| 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 123(112)| 1 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Fig. 2** The contrast matrix of \(2 \times 2 \times 3\) designs

\[
\begin{align*}
\theta_{102} &= 2\theta_{100}\theta_{002} + 2\theta_{010}\theta_{112} + 2\theta_{001}\theta_{101} + 2\theta_{002}\theta_{102} + 2\theta_{000}\theta_{102} + 2\theta_{110}\theta_{012} + 2\theta_{011}\theta_{011} + 2\theta_{012}\theta_{112}, \\
\theta_{012} &= 2\theta_{100}\theta_{112} + 2\theta_{010}\theta_{002} + 2\theta_{001}\theta_{011} + 2\theta_{002}\theta_{012} + 2\theta_{000}\theta_{012} + 2\theta_{110}\theta_{002} + 2\theta_{011}\theta_{001} + 2\theta_{012}\theta_{002}, \\
\theta_{112} &= 2\theta_{100}\theta_{012} + 2\theta_{010}\theta_{102} + 2\theta_{001}\theta_{111} + 2\theta_{002}\theta_{110} + 2\theta_{002}\theta_{112} + 2\theta_{000}\theta_{112} + 2\theta_{101}\theta_{011} + 2\theta_{102}\theta_{012}.
\end{align*}
\]

The contrast matrix \(C\) for this case is given in Fig. 2. Note that \(C\) has the structure \(C^T = [1_{12} \mid C_1^T \mid C_2^T \mid C_3^T]\), which is explicitly shown in Fig. 2. The set of the coefficients of the indicator function of the orthogonal fractional factorial design with the given strength is characterized in the form (5). For example, along with the relation (4), the coefficients of the fractional factorial design with the size 6, (i.e., a half fraction) satisfy

\[
1_{12}^T \mathbf{X} \theta = 6 \iff 12\theta_{000} + 8\theta_{002} = 6,
\]

and the coefficients of the orthogonal fractional factorial design of strength 1, (i.e., equi-replicated design), satisfy

\[
\begin{align*}
1_{12}^T \mathbf{X} \theta = 6 & \iff \begin{cases} 12\theta_{000} + 8\theta_{002} = 6 \\
-12\theta_{100} - 8\theta_{102} = 0 \\
-12\theta_{010} - 8\theta_{012} = 0 \\
-4\theta_{001} + 4\theta_{002} = 0 \\
-8\theta_{001} = 0. \end{cases}
\end{align*}
\]

Now, we can describe the problem that we consider in this paper. Let \(\mathbb{Q}[\theta]\) be the polynomial ring for the variables \(\{\theta_a : a \in L\}\). For given size \(s\) and strength of
orthogonality $t$, we can define a polynomial ideal of $\mathbb{Q}[\theta]$ from (4) and (5) by

$$I = \left\{ \theta_a - \mu_a, \ a \in L, \ 1_m^T X \theta - s, \ C_\ell X \theta, \ \ell = 1, \ldots, t \right\}. \tag{6}$$

Then, the variety defined by the polynomial ideal $I$,

$$V(I) = \{(a_1, \ldots, a_m) \in \mathbb{Q}^m : f(a_1, \ldots, a_m) = 0, \ \forall f \in I\}, \tag{7}$$

corresponds to the set of the coefficients of the indicator functions of all orthogonal fractional factorial designs with size $s$ and strength $t$. By definition, the cardinality of $V(I)$ is finite, i.e., $I$ is a 0-dimensional ideal. In addition, all the zeros of $f \in I$ is rational, i.e., for all $f \in I$,

$$f(a_1, \ldots, a_m) = 0, \ (a_1, \ldots, a_m) \in \mathbb{R}^m \Rightarrow (a_1, \ldots, a_m) \in \mathbb{Q}^m$$

holds, where $\mathbb{R}$ be the field of real numbers.

### 3 Prime decomposition of the radicals

To enumerate points of the variety (7), we use the theory of the primary decomposition. We summarize fundamental theory of the prime decomposition of the radicals in this section.

Let $I \subset \mathbb{Q}[\theta]$ be a 0-dimensional polynomial ideal $I \subset \mathbb{Q}[\theta]$ given in (6). Let $\sqrt{I} = \{f \in \mathbb{Q}[\theta] : \exists k \in \mathbb{Z}_{>0}, \ f^k \in I\}$ be the radical ideal of $I$. Then, $V(I) = V(\sqrt{I})$ holds. An ideal $P \subset \mathbb{Q}[\theta]$ is called a prime ideal if

$$f, g \in \mathbb{Q}[\theta], \ fg \in P \Rightarrow f \in P \text{ or } g \in P$$

holds. For prime ideals $P_1, \ldots, P_u$ satisfying

$$\sqrt{I} = P_1 \cap \cdots \cap P_u \tag{8}$$

and $P_i$ is not included in any $P_j$ ($j \neq i$) for $i = 1, \ldots, u$, the decomposition (8) is called a prime decomposition of $\sqrt{I}$. See Chapter 4 of Cox et al. (2007) for detail on the primary decomposition of ideals. If we have the prime decomposition (8) of $\sqrt{I}$, we can obtain the expression for the variety $V(I)$ as

$$V(I) = V(\sqrt{I}) = V(P_1) \cup \cdots \cup V(P_u).$$

Here, we use a relation $V(P_i \cap P_j) = V(P_i) \cup V(P_j)$.

Our approach is to derive the prime decomposition of $\sqrt{I}$. However, a direct computation seems to be infeasible for our problem shown in Sect. 4. Therefore, we divide the problem as follows. For our 0-dimensional ideal $I \subset \mathbb{Q}[\theta]$, suppose

$$I = I_1 + I_2 \tag{9}$$
holds, where we define \( I_1 + I_2 = \{ f_1 + f_2 : f_1 \in I_1, \ f_2 \in I_2 \} \). Then we have a relation \( V(I) = V(I_1) \cap V(I_2) \). Suppose we obtain the prime decomposition of \( \sqrt{I_1} \) as \( \sqrt{I_1} = \bigcap_k P_k \). In this case, from the relation \( V(I_1) = \bigcup_k V(P_k) \), we have

\[
V(I) = \left( \bigcup_k V(P_k) \right) \cap V(I_2) = \bigcup_k (V(P_k) \cap V(I_2)) = \bigcup_k V(P_k + I_2).
\]

Therefore, if we obtain the prime decomposition \( \sqrt{P_k + I_2} = \bigcap_j Q_{kj} \) for each \( k \), we have \( V(P_k) \cap V(I_2) = \bigcup_j V(Q_{kj}) \) for each \( k \), and we derive the decomposition

\[
V(I) = \bigcup_k \bigcup_j V(Q_{kj}).
\]

We see that our problem is solved in this way in Sect. 4.

### 4 Classifications of half orthogonal fractions of \( 2^4 \times 3 \) designs with strength 2

Now, we show the computational results for our problem. We consider the design of 5 factors with level sets \( A_1 = \cdots = A_4 = \{ -1, 1 \} \) and \( A_5 = \{ -1, 0, 1 \} \). Our aim is to enumerate and classify all the half orthogonal fractional factorial designs with strength 2. Suppose \( I \subset \mathbb{Q}[\theta] \) be the corresponding polynomial ideal defined in (6), where \( m = 48, \ s = 24, \ t = 2 \). All the computations are done by Risa/Asir (Noro et al., 1992) installed in MacBook Pro, 2.3 GHz, Quad-Core, Intel Core i7. Our computation is described in the following 3 steps.

**Step 1: Preprocessing** Using the constraints (5), some of the variables in \( \theta = \{ \theta_a : a \in L \} \) can be eliminated as follows. Following to the expression (7), we write \( \{ \theta_a : a \in L \} = (a_1, \ldots, a_m) \) for simplicity, and suppose for some variable, say \( a_1, \ a_1 - g(a_2, \ldots, a_m) \in I \) holds. Then, we can eliminate \( a_1 \) and have \( V(I) = \{ (g(a_2, \ldots, a_m), a_2, \ldots, a_m) : (a_2, \ldots, a_m) \in V' \} \), where \( V' = \{ f(g(a_2, \ldots, a_m), a_2, \ldots, a_m) : f(a_1, \ldots, a_m) \in I \} \). In this way, from \( m = 48 \) variables, we eliminate 21 variables and have the variety \( V(I) \subset \mathbb{Q}^{27} \).

**Step 2: Primary decomposition** After Step 1, we find that the generating set of the ideal \( I \) is composed of 48 polynomials. Let this generating set be \( G \) and write \( I = \langle G \rangle \). We divide \( G = G_1 \cup G_2 \), where \( G_1 \) is the set of polynomials with less than or equal to 4 terms, and \( G_2 \) is the set of polynomials with greater than 4 terms. From these sets, we define \( I_1 = \langle G_1 \rangle \) and \( I_2 = \langle G_2 \rangle \) in (9). We find that there are 15 polynomials in \( G_1 \), and 33 polynomials in \( G_2 \), respectively. Following to the strategy given in Sect. 3, first we compute the prime decomposition of \( \sqrt{I_1} \), which we obtain the union of the varieties generated by 111 prime ideals. For each prime ideal, we combine \( I_2 \) and calculate prime decomposition again. After these computations, we obtain the result that the variety \( V(I) \) is composed of 35,200 points by computation of about 2 h.

**Step 3: Classification to equivalence classes** We classify the obtained 35,200 points to equivalence classes. The group which we consider is a group of permutations of
Table 1 The classification of equivalence classes of the half orthogonal fractions of $2^4 \times 3$ designs with strength 2 ($i, j, k \in \{1, 2, 3, 4\}$)

| Num of non-orthogonal triplet of $(x_i, x_j, x_k)$ | $J$ | Num of non-orthogonal triplet of $(x_i, x_j, x_3)$ |
|-----------------------------------------------|-----|-------------------------------------------------|
|                                               | 0   | 1 2 3 4 5                                       |
| 0                                             | [0, 0, 0] | 3$^a$ 4 2 3 1 1   |
| 1                                             | [24, 0, 0] | 1$^b$ 1                                             |
|                                               | [16, 0, 0] | 1 1                                             |
|                                               | [8, 0, 0] | 4 8 8 5 2                                   |
| 2                                             | [16, 8, 0] | 1 1                                             |
|                                               | [8, 8, 0] | 1 4 3 1                                   |
| 3                                             | [8, 8, 8, 0] | 1 1 3 2                                   |

$^a$Orthogonal fractional factorial designs with strength 3 shown in Aoki (2019)

$^b$Regular fractional factorial design with the defining relation $x_ix_jx_k = \pm 1$

levels for each factor and permutations of factors among 2-level factors. Let $S \subset S_I$ be the group we consider, where $S_I$ is a group of permutations of $I$. Then, the equivalence classes for $\theta$ with respect to $S$ is $[\theta] = \{ X^{-1} P_g X : g \in S \}$, where $P_g$ is an $m \times m$ permutation matrix for $g \in S$. See Proposition 3.11 of Aoki (2019) for detail. After classification, we find that the 35,200 points are classified into 63 equivalence classes.

We summarize the obtained 63 equivalence classes by their orthogonality of strength 3. As for the orthogonality of 3 two-level factors $x_1, \ldots, x_4$, we calculate $J$-statistics (Tang, 2001) given by

$$J_s(F) = \sum_{i \in I'} \prod_{j \in s} d_{i,j}$$

for $s \subset \{1, 2, 3, 4\}$, where we define $F = \{ d_i : i \in I' \}$ to be the fractional factorial design. Because $d_{i,j} \in \{-1, 1\}$ for $j \in \{1, 2, 3, 4\}$ and the orthogonality of strength 2, $J_{j_1,j_2,j_3}(F) = 0$ hold if and only if 8 possible combinations of levels of $x_{j_1}, x_{j_2}, x_{j_3}$ appear equally often (i.e., 3 times, respectively) in the design. Let $J$ be the set $J = \{|J_{123}(F)|, |J_{124}(F)|, |J_{134}(F)|, |J_{234}(F)|\}$, which is an invariant for the equivalence classes. We calculate the set $J$, in addition to the number of three pair of $\{x_1, x_2, x_3, x_4\}$ which is not orthogonal. As for the orthogonality among 2 two-level factors and three-level factor $x_5$, we simply count the number of three pair of $\{x_i, x_j, x_5\}, i, j \in \{1, 2, 3, 4\}$, which is not orthogonal. We summarize the number of equivalence classes in Table 1.

We give a list of the indicator functions of the representative elements for the equivalence classes. In the list below, we use an index “Type $T_1(J)$-$T_2$”, meaning that $T_1$ is a number of non-orthogonal triplet of $(x_i, x_j, x_k)$ and $T_2$ is a number of non-orthogonal triplet of $(x_i, x_j, x_5)$.

- Type 0-0
  - 2 relations: $\frac{1}{2} + \frac{1}{2} x_1x_2x_3x_4$;
○ 6 relations: $\frac{1}{2} - \frac{1}{2} x_1 x_2 x_3 x_4 + x_1 x_2 x_4 x_5^2$;
○ 48 relations: $\frac{1}{2} - \frac{1}{2} x_1 x_2 x_4 + \frac{1}{4} x_1 x_2 x_3 x_4 x_5 + \frac{1}{4} x_1 x_2 x_3 x_4 x_5 - \frac{3}{4} x_1 x_2 x_4 x_5^2 - \frac{1}{4} x_1 x_2 x_3 x_4 x_5^2$.

- Type 0-1
  ○ 72 relations: $\frac{1}{2} + \frac{1}{2} (x_1 x_2 x_3 x_4 + x_3 x_4 x_5 - x_1 x_2 x_4 x_5^2)$;
  ○ 144 relations: $\frac{1}{2} - \frac{1}{2} x_1 x_2 + \frac{3}{4} x_1 x_2 x_5^2 + \frac{1}{4} (x_1 x_2 x_3 x_5 - x_1 x_2 x_4 x_5 + x_1 x_2 x_3 x_4 x_5^2)$;
  ○ 288 relations: $\frac{1}{2} - \frac{1}{2} x_1 x_2 x_4 + \frac{1}{4} (x_2 x_4 x_5 + x_2 x_3 x_4 x_5 - x_1 x_2 x_3 x_4 x_5^2) + \frac{3}{4} x_1 x_2 x_4 x_5^2$;
  ○ 288 relations: $\frac{1}{2} - \frac{1}{2} (x_1 x_2 x_3 x_4 - x_1 x_2 x_3 x_4 x_5^2) - \frac{1}{4} (x_3 x_4 x_5 - x_1 x_3 x_4 x_5 + x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5)$.

- Type 0-2
  ○ 576 relations: $\frac{1}{2} + \frac{1}{4} (x_1 x_2 x_3 x_4 - x_1 x_2 x_3 x_4 x_5^2) - \frac{1}{4} (x_2 x_3 x_5 - x_3 x_4 x_5 + x_1 x_2 x_3 x_5 + x_1 x_3 x_4 x_5)$;
  ○ 1152 relations: $\frac{1}{2} + \frac{1}{4} (x_1 x_2 - x_1 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 - x_1 x_2 x_3 x_5 - x_1 x_2 x_3 x_4 x_5^2) + \frac{1}{8} (x_1 x_2 x_5 + x_1 x_3 x_5 + x_1 x_2 x_4 x_5 - x_1 x_3 x_4 x_5) - \frac{3}{8} (x_1 x_2 x_5 - x_1 x_3 x_5^2 + x_1 x_2 x_4 x_5^2 + x_1 x_3 x_4 x_5^2)$.

- Type 0-3
  ○ 192 relations: $\frac{1}{2} - \frac{1}{4} (x_1 x_2 x_3 x_4 - x_1 x_2 x_3 x_4 x_5^2) - \frac{1}{4} (x_1 x_4 x_5 - x_2 x_4 x_5 + x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5)$;
  ○ 288 relations: $\frac{1}{2} + \frac{1}{4} x_1 x_2 - \frac{3}{4} x_1 x_2 x_5^2 + \frac{1}{4} (x_2 x_3 x_5 - x_2 x_4 x_5 - x_1 x_2 x_3 x_4 x_5^2)$;
  ○ 1152 relations: $\frac{1}{2} + \frac{1}{4} (x_1 x_2 - x_1 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_1 x_4 x_5 - x_1 x_2 x_3 x_4 x_5^2) + \frac{1}{8} (x_1 x_2 x_5 - x_1 x_3 x_5 - x_1 x_2 x_4 x_5 - x_1 x_3 x_4 x_5) - \frac{3}{8} (x_1 x_2 x_5^2 - x_1 x_3 x_5^2 + x_1 x_2 x_4 x_5^2 + x_1 x_3 x_4 x_5^2)$.

- Type 0-4
  ○ 144 relations: $\frac{1}{2} - \frac{1}{4} (x_1 x_2 x_3 x_4 - x_1 x_2 x_3 x_4 x_5^2) - \frac{1}{4} (x_1 x_2 x_5 - x_1 x_4 x_5 + x_2 x_3 x_5 + x_3 x_4 x_5)$.

- Type 0-5
  ○ 576 relations: $\frac{1}{2} - \frac{1}{4} (x_1 x_2 - x_1 x_3 + x_1 x_4 + x_1 x_2 x_3 x_4 - x_2 x_3 x_5 + x_3 x_4 x_5) - \frac{1}{8} (x_1 x_2 x_5 + x_1 x_3 x_5 + x_1 x_4 x_5 + x_1 x_2 x_3 x_4 x_5^2 - x_1 x_2 x_3 x_5^2 + x_1 x_4 x_5^2)$.

- Type 1, {24, 0, 0, 0}-0
  ○ 8 relations: $\frac{1}{2} + \frac{1}{2} x_1 x_2 x_4$.

- Type 1, {16, 0, 0, 0}-0
  ○ 48 relations: $\frac{1}{2} - \frac{1}{2} x_1 x_2 x_4 - \frac{1}{4} (x_1 x_2 x_4 x_5 - x_1 x_2 x_3 x_4 x_5 - x_1 x_2 x_4 x_5^2 + x_1 x_2 x_3 x_4 x_5^2)$.

- Type 1, {16, 0, 0, 0}-1
\[ \frac{1}{2} - \frac{1}{2}x_1x_2x_4 - \frac{1}{4}(x_2x_4x_5 - x_2x_3x_4x_5 - x_1x_2x_4x_5^2 + x_1x_2x_3x_4x_5^2). \]

- **Type 1, \{8, 0, 0, 0\}-0**
  - 24 relations: \[ \frac{1}{2} - \frac{1}{2}x_1x_2x_4 + x_1x_2x_4x_5^2; \]
  - 48 relations: \[ \frac{1}{2} - \frac{1}{2}(x_1x_2x_4 - x_1x_2x_4x_5^2 + x_1x_2x_3x_4x_5^2); \]
  - 48 relations: \[ \frac{1}{2} - \frac{1}{2}(x_1x_2x_4 - x_1x_2x_3x_4x_5 - x_1x_2x_4x_5^2); \]
  - 144 relations: \[ \frac{1}{2} - \frac{1}{2}(x_1x_2x_4 - x_2x_3x_4x_5 - x_1x_2x_4x_5^2). \]

- **Type 1, \{8, 0, 0, 0\}-1**
  - 144 relations: \[ \frac{1}{2} - \frac{1}{2}(x_1x_2x_4 - x_3x_4x_5 - x_1x_2x_4x_5^2); \]
  - 144 relations: \[ \frac{1}{2} + \frac{1}{2}(x_1x_2x_4 + x_2x_4x_5 - x_1x_2x_4x_5^2); \]
  - 288 relations: \[ \frac{1}{2} - \frac{1}{2}x_1x_2x_4 + \frac{1}{4}(x_1x_4x_5 + x_1x_2x_3x_4x_5 - x_1x_3x_4x_5^2) + \frac{3}{4}x_1x_2x_4x_5^2; \]
  - 288 relations: \[ \frac{1}{2} + \frac{1}{2}x_1x_2x_3x_4 - \frac{1}{4}(x_3x_4x_5 - x_2x_3x_4x_5 + x_1x_3x_4x_5^2) - \frac{3}{4}x_1x_2x_3x_4x_5^2; \]
  - 288 relations: \[ \frac{1}{2} + \frac{1}{2}x_1x_2x_3x_4 + \frac{1}{4}(x_3x_4x_5 + x_2x_4x_5 - x_1x_3x_4x_5^2) - \frac{3}{4}x_1x_2x_3x_4x_5^2; \]
  - 288 relations: \[ \frac{1}{2} - \frac{1}{2}x_1x_2x_4 + \frac{3}{4}x_1x_2x_4x_5^2 + \frac{1}{4}(x_1x_2x_4x_5 + x_1x_2x_3x_4x_5 - x_1x_2x_3x_4x_5^2); \]
  - 576 relations: \[ \frac{1}{2} - \frac{1}{2}x_1x_2x_4 - x_1x_2x_4x_5^2 - \frac{1}{4}x_3x_4x_5 - x_1x_3x_4x_5^2 + x_1x_2x_3x_4x_5 + x_1x_2x_3x_4x_5^2; \]
  - 576 relations: \[ \frac{1}{2} - \frac{1}{2}x_1x_2x_4 - x_1x_2x_4x_5^2 + \frac{1}{4}(x_1x_4x_5 + x_1x_2x_3x_4x_5 - x_1x_2x_3x_4x_5^2); \]
  - 576 relations: \[ \frac{1}{2} + \frac{1}{2}x_1x_2x_4 - x_1x_2x_4x_5^2 + \frac{1}{4}(x_1x_4x_5 + x_1x_2x_3x_4x_5 - x_1x_2x_3x_4x_5^2); \]

- **Type 1, \{8, 0, 0, 0\}-2**
  - 576 relations: \[ \frac{1}{2} - \frac{1}{2}x_1x_2x_4 + \frac{1}{4}(x_2x_4x_5 + x_3x_4x_5 - x_1x_3x_4x_5^2) + \frac{3}{4}x_1x_2x_4x_5^2; \]
  - 576 relations: \[ \frac{1}{2} - \frac{1}{2}(x_1x_2x_4 - x_1x_2x_4x_5^2) - \frac{1}{4}(x_2x_3x_5 - x_3x_4x_5 + x_1x_2x_3x_5 + x_1x_3x_4x_5); \]
  - 576 relations: \[ \frac{1}{2} + \frac{1}{2}(x_1x_2x_4 - x_1x_2x_4x_5^2) + \frac{1}{4}(x_1x_2x_5 + x_2x_4x_5 + x_1x_2x_3x_5 - x_2x_3x_4x_5); \]
  - 576 relations: \[ \frac{1}{2} - \frac{1}{2}x_1x_2 + \frac{3}{4}x_1x_2x_5^2 + \frac{1}{4}(x_2x_4x_5 + x_1x_2x_3x_5 - x_2x_3x_4x_5^2); \]
  - 576 relations: \[ \frac{1}{2} - \frac{1}{2}x_1x_2 + \frac{3}{4}x_1x_2x_5^2 + \frac{1}{4}(x_2x_4x_5 - x_2x_3x_4x_5 + x_1x_2x_3x_5^2); \]
  - 1152 relations: \[ \frac{1}{2} - \frac{1}{2}(x_1x_2x_4 - x_1x_2x_4x_5^2) + \frac{1}{4}(x_2x_4x_5 + x_3x_4x_5 + x_1x_2x_4x_5 - x_1x_3x_4x_5); \]
  - 1152 relations: \[ \frac{1}{2} + \frac{1}{4}(x_1x_2 - x_1x_3 + x_1x_2x_4 + x_1x_3x_4 + x_1x_2x_3x_4x_5 - x_1x_2x_3x_4x_5^2) - \frac{1}{8}(x_1x_2x_5 + x_1x_2x_3x_4x_5 - x_1x_3x_4x_5) - \frac{3}{8}(x_1x_2x_5^2 - x_1x_3x_5^2 + x_1x_2x_4x_5^2 + x_1x_3x_4x_5^2); \]
  - 2304 relations: \[ \frac{1}{2} + \frac{1}{4}(x_1x_2 - x_1x_3 + x_1x_2x_4 + x_1x_3x_4 + x_1x_2x_3x_4x_5 + x_1x_2x_3x_4x_5^2) + \frac{1}{4}(x_1x_2x_5 + x_1x_3x_5 - x_1x_2x_4x_5 - x_1x_3x_4x_5 - x_1x_2x_4x_5^2) - \frac{3}{8}(x_1x_2x_5^2 - x_1x_3x_5^2 + x_1x_3x_4x_5^2). \]

- **Type 1, \{8, 0, 0, 0\}-3**
  - 192 relations: \[ \frac{1}{2} - \frac{1}{2}(x_1x_2x_4 - x_1x_2x_4x_5^2) - \frac{1}{4}(x_1x_3x_5 - x_2x_3x_5 - x_3x_4x_5 - x_1x_2x_3x_4x_5); \]

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576 relations: $\frac{1}{2} + \frac{1}{4}(x_1 x_2 x_4 - x_1 x_2 x_4 x_5^2) - \frac{1}{4}(x_1 x_4 x_5 - x_2 x_4 x_5 + x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5)$;
1152 relations: $\frac{1}{2} - \frac{1}{4}(x_1 x_2 - x_1 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_5 + x_2 x_3 x_4 x_5^2) - \frac{1}{8}(x_1 x_2 x_5 + x_1 x_3 x_5 + x_1 x_2 x_4 x_5 - x_1 x_3 x_4 x_5) + \frac{3}{8}(x_1 x_2 x_5^2 - x_1 x_3 x_5^2 + x_1 x_2 x_4 x_5^2 + x_1 x_3 x_4 x_5^2)$;
1152 relations: $\frac{1}{2} + \frac{1}{4}(x_1 x_2 - x_1 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_1 x_4 x_5 - x_1 x_2 x_3 x_5^2) - \frac{1}{8}(x_1 x_2 x_5 + x_1 x_3 x_5 - x_1 x_2 x_4 x_5 + x_1 x_3 x_4 x_5) - \frac{3}{8}(x_1 x_2 x_5^2 - x_1 x_3 x_5^2 + x_1 x_2 x_4 x_5^2 + x_1 x_3 x_4 x_5^2)$;
2304 relations: $\frac{1}{2} + \frac{1}{4}(x_1 x_2 - x_1 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_1 x_4 x_5 + x_1 x_2 x_3 x_5 + x_1 x_2 x_3 x_4 x_5) + \frac{1}{8}(x_1 x_2 x_5 - x_1 x_3 x_5 + x_1 x_2 x_4 x_5 - x_1 x_3 x_4 x_5 - x_1 x_2 x_4 x_5^2) - \frac{3}{8}(x_1 x_2 x_5^2 - x_1 x_3 x_5^2 + x_1 x_3 x_4 x_5^2)$.

- Type 1, {8, 0, 0, 0}-4
  576 relations: $\frac{1}{2} + \frac{1}{4}(x_1 x_2 x_4 - x_1 x_2 x_4 x_5^2) - \frac{1}{4}(x_1 x_2 x_5 - x_1 x_4 x_5 + x_2 x_3 x_5 + x_3 x_4 x_5)$;
1152 relations: $\frac{1}{2} + \frac{1}{4}(x_1 x_2 - x_1 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_1 x_4 x_5 - x_1 x_2 x_3 x_5^2) - \frac{1}{8}(x_1 x_2 x_5 + x_1 x_3 x_5 + x_1 x_2 x_4 x_5 - x_1 x_3 x_4 x_5) - \frac{3}{8}(x_1 x_2 x_5^2 - x_1 x_3 x_5^2 + x_1 x_2 x_4 x_5^2 + x_1 x_3 x_4 x_5^2)$.

- Type 2, {16, 8, 0, 0}-0
  144 relations: $\frac{1}{2} - \frac{1}{2}(x_1 x_2 x_4 - x_1 x_2 x_4 x_5^2) + x_1 x_3 x_4 x_5^2$.

- Type 2, {16, 8, 0, 0}-1
  288 relations: $\frac{1}{2} - \frac{1}{2}(x_1 x_2 x_4 - x_1 x_2 x_4 x_5^2) - \frac{1}{4}(x_1 x_4 x_5 - x_2 x_3 x_4 x_5 - x_1 x_2 x_4 x_5^2 + x_1 x_3 x_4 x_5^2)$.

- Type 2, {16, 8, 0, 0}-2
  576 relations: $\frac{1}{2} - \frac{1}{2}(x_1 x_2 x_4 - x_1 x_2 x_4 x_5^2) - \frac{1}{4}(x_2 x_4 x_5 - x_3 x_4 x_5 - x_1 x_2 x_4 x_5^2 + x_1 x_3 x_4 x_5^2)$.

- Type 2, {8, 0, 0, 0}-0
  288 relations: $\frac{1}{2} - \frac{1}{2}(x_1 x_2 x_4 - x_1 x_2 x_4 x_5^2) - \frac{1}{4}(x_1 x_3 x_4 x_5 - x_1 x_2 x_3 x_4 x_5 + x_1 x_3 x_4 x_5^2 + x_1 x_2 x_3 x_4 x_5^2)$.

- Type 2, {8, 0, 0, 0}-1
  144 relations: $\frac{1}{2} - \frac{1}{2}(x_1 x_2 x_4 - x_1 x_2 x_4 x_5^2) + \frac{1}{4}(x_1 x_2 x_3 x_5^2 - x_1 x_2 x_4 x_5^2 + x_1 x_2 x_3 x_4 x_5^2)$;
288 relations: $\frac{1}{2} + \frac{1}{4}(x_1 x_2 x_3 x_4 - x_1 x_2 x_3 x_4 x_5^2) - \frac{1}{4}(x_1 x_3 x_5 - x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_5^2 + x_1 x_3 x_4 x_5^2)$;
576 relations: $\frac{1}{2} - \frac{1}{2}(x_1 x_2 x_4 - x_1 x_2 x_4 x_5^2) - \frac{1}{4}(x_1 x_4 x_5 - x_1 x_2 x_4 x_5 + x_1 x_3 x_4 x_5^2 + x_1 x_2 x_3 x_4 x_5^2)$;
1152 relations: $\frac{1}{2} - \frac{1}{2}(x_1 x_2 x_4 - x_1 x_2 x_4 x_5^2) - \frac{1}{4}(x_3 x_4 x_5 - x_2 x_3 x_4 x_5 + x_1 x_3 x_4 x_5^2 + x_1 x_2 x_3 x_4 x_5^2)$.
• Type 2, (8, 8, 0, 0)-2
  - 576 relations: $\frac{1}{2} + \frac{1}{4}(x_1x_2x_3x_4 - x_1x_2x_3x_4x^2) + \frac{1}{4}(-x_2x_3x_5 + x_3x_4x_5 - x_1x_2x_3x_4x^2 - x_1x_2x_3x_4x^2);$
  - 1152 relations: $\frac{1}{2} + \frac{1}{4}(x_1x_2 - x_1x_3 + x_1x_2x_4 + x_1x_3x_4 - x_1x_4x_5 - x_1x_2x_3x_4x^2 - \frac{1}{8}(x_1x_2x_5 + x_1x_3x_5 - x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_1x_2x_4x^2 + x_1x_3x_4x^2) - \frac{3}{8}(x_1x_2x^2_5 - x_1x_3x^2_5);$
  - 2304 relations: $\frac{1}{2} - \frac{1}{4}(x_1x_2 - x_1x_3 + x_1x_2x_4 + x_1x_3x_4 - x_1x_2x_3x_4x^2 - x_1x_2x_3x_4x^2 + \frac{1}{8}(-x_1x_2x_5 - x_1x_3x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_1x_2x_4x^2) + \frac{3}{8}(x_1x_2x^2_5 - x_1x_3x^2_5 + x_1x_3x_4x^2_5).$
• Type 2, (8, 8, 0, 0)-3
  - 1152 relations: $\frac{1}{2} + \frac{1}{4}(x_1x_2 - x_1x_3 + x_1x_2x_4 + x_1x_3x_4 - x_1x_4x_5 - x_1x_2x_3x_4x^2 - \frac{1}{8}(x_1x_2x_5 - x_1x_3x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_1x_2x_4x^2 + x_1x_3x_4x^2) - \frac{3}{8}(x_1x_2x^2_5 - x_1x_3x^2_5);$
• Type 3-0
  - 192 relations: $\frac{1}{2} - \frac{1}{2}(x_1x_2x_4 - x_1x_2x_4x^2) + \frac{1}{4}(x_1x_3x_4x_5 + x_2x_3x_4x_5 - x_1x_3x_4x^2_5 + x_2x_3x_4x^2_5).$
• Type 3-1
  - 576 relations: $\frac{1}{2} - \frac{1}{2}(x_1x_2x_4 + x_1x_2x_4x^2) - \frac{1}{4}(x_1x_3x_5 - x_1x_2x_3x_4x_5 + x_1x_2x_3x^2_5 + x_1x_3x_4x^2_5).$
• Type 3-2
  - 576 relations: $\frac{1}{2} - \frac{1}{2}(x_1x_2x_4 - x_1x_2x_4x^2) - \frac{1}{4}(x_2x_3x_5 - x_3x_4x_5 + x_1x_2x_3x^2_5 + \frac{1}{8}(x_1x_2x_5 + x_2x_4x_5 + x_1x_3x_4x^2 - x_2x_3x_4x^2);$
  - 576 relations: $\frac{1}{2} + \frac{1}{2}(x_1x_2x_4 - x_1x_2x_4x^2) + \frac{1}{4}(x_1x_2x_5 + x_2x_4x_5 + x_1x_3x_4x^2 - x_2x_3x_4x^2);$
  - 1152 relations: $\frac{1}{2} + \frac{1}{4}(x_1x_2 - x_1x_3 + x_1x_2x_4 + x_1x_3x_4 + x_1x_2x_3x_4x_5 - x_1x_2x_3x^2_5) + \frac{1}{8}(x_1x_2x_5 + x_1x_3x_5 - x_1x_2x_4x_5 + x_1x_3x_4x_5 - x_1x_2x_4x^2 - x_1x_3x_4x^2) - \frac{3}{8}(x_1x_2x^2_5 - x_1x_3x^2_5).$
• Type 3-3
  - 384 relations: $\frac{1}{2} - \frac{1}{4}(x_1x_2 - x_1x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4x^2 + \frac{1}{8}(x_1x_2x_5 + x_1x_3x_5 - x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_1x_2x_4x^2 + x_1x_3x_4x^2) + \frac{3}{8}(x_1x_2x^2_5 - x_1x_3x^2_5);$
  - 384 relations: $\frac{1}{2} + \frac{1}{4}(x_1x_2 - x_1x_3 + x_1x_2x_4 + x_1x_3x_4 + x_1x_4x_5 - x_1x_2x_3x^2_5) + \frac{1}{8}(x_1x_2x_5 + x_1x_3x_5 + x_1x_2x_4x_5 - x_1x_3x_4x_5 - x_1x_2x_4x^2 - x_1x_3x_4x^2) - \frac{3}{8}(x_1x_2x^2_5 - x_1x_3x^2_5).$
5 Discussion

In this paper, we consider using the theory of primary decomposition to enumerate fractional factorial designs with given orthogonality. As shown in Sect. 4, we have broken through the limit of the previous work Aoki (2019) in this paper. Our approach is to divide generators of the ideal according to the number of terms. We first compute the primary decomposition of the ideal generated by the polynomials with less than or equal to 4 terms, and then combine the result to the other polynomials. This heuristic approach works well for our problem. Of course, it may be difficult to carry out this heuristic approach for problems of larger sizes.

The enumeration and classification of orthogonal designs is one of the fundamental problems in design of experiments. For practical contribution, we can apply the classification to the theory of the optimal design. For the result given in Sect. 4, however, we see that the simple regular design $x_1x_2x_3x_4 = \pm 1$ is the optimal, for the criterion such as $D$-, $A$-, $E$-optimalities under some statistical models. Therefore, the contribution of this paper is restricted to the theoretical one. For more practical contribution, the problems for the designs where there does not exist regular fractions should be considered, which is our future work.

Funding  Funding was provided by JSPS KAKENHI (JP17K00048).

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