Non-critical string
Liouville theory
and
geometric bootstrap hypothesis

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Abstract

The applications of the existing Liouville theories for the description of the longitudinal dynamics of non-critical Nambu-Goto string are analyzed. We show that the recently developed DOZZ solution to the Liouville theory leads to the cut singularities in tree string amplitudes. We propose a new version of the Polyakov geometric approach to Liouville theory and formulate its basic consistency condition — the geometric bootstrap equation. Also in this approach the tree amplitudes develop cut singularities.

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1 Introduction

It has been well known since early days of string theory that the covariant quantization of the free Nambu-Goto string [1] leads in non-critical dimensions (1 < d < 25) to free quantum models with longitudinal excitations. However, in spite of numerous attempts no consistent theory of the longitudinal dynamics has been found. Recently the problem was reconsidered in [2]. It was shown that if a consistent theory of the longitudinal dynamics exists it does not satisfy all the axioms of the standard CFT. This explains the failure of previous approaches and makes it difficult to construct a viable alternative.

In the present paper we analyze whether the existing models of quantum Liouville theory provide an adequate description of the longitudinal dynamics. We restrict ourselves to non-critical strings with the space of asymptotic free states coinciding with the space of states of the free non-critical Nambu-Goto string (1 < d < 25) [3]. Although the model is inconsistent due to the presence of a tachyon it is still a good starting point for the analysis of the longitudinal dynamics. The experience with the critical string shows that the tachyon problem is to large extend independent of the consistency of the string perturbation expansion, the covariance and the unitarity issues. Moreover, the standard GSO mechanism [4] can be used to remove the tachyon in the free non-critical RNS string (1 < d < 9) [5]. We may hope therefore that the basic properties of the longitudinal dynamics derived in the context of the noncritical Nambu-Goto string should hold in more realistic models as well.

The relevance of the quantum Liouville theory for the proper description of the longitudinal string excitations has been known since the celebrated papers by A. Polyakov on conformal anomaly [6, 7]. It was conjectured some time ago by A. and Al. Zamolodchikov [8] that the continuum spectrum along with the 3-point functions proposed by Otto and Dorn [9] satisfy the bootstrap consistency conditions of standard (in the sense of BPZ [10]) CFT. In the weak coupling regime c > 25 this conjecture was recently proved by Ponsot and Teschner [11, 12]. It is believed to hold in the strong coupling regime 1 < c < 25 by the analytic continuation argument [13]. Among many recent applications of this elegant solution let us only mention the 1+1 dimensional string models [14–17] (where its predictions are confirmed by the matrix model results [18, 19]), quantization of the Teichmüller space of Riemann surfaces [20–22], relations with WZNW models, [23, 24] and string theory on the AdS$_3$ space [25, 26].

The continuous spectrum plays an essential role in all the applications mentioned above. On the other hand it is the main obstacle in applying the theory to the non-critical Nambu-Goto string where the longitudinal sector consists of a single conformal family. The straightforward application of the DOZZ theory yields consistent perturbation expansion and unitarity. The price one has to pay is the extension of the spectrum of external string states far beyond the spectrum of the free non-critical Nambu-Goto. This leads to the continuous family of intercepts and is not acceptable on physical grounds. A possible way out is to restrict ourselves to the external states from the single conformal family of the longitudinal sector. In this case however the tree amplitudes exhibit cut singularities.
There exist another approach to the quantum Liouville theory originally proposed by A. Polyakov [27] and developed by Takhtajan [28–32]. In this geometric approach the correlation functions are defined in terms of path integral over conformal class of Riemannian metrics with prescribed singularities at the punctures. The semi-classical results obtained in the case of parabolic singularities [28–31] are in perfect agreement with the general properties of the longitudinal dynamics derived in [2]. An additional support for the geometrical approach comes from the fact that some of its geometric predictions can be rigorously proved ([30, 31], and references therein). In spite of considerable achievements the geometric approach is not yet capable to produce the puncture correlators. For this reason both the relation to the DOZZ theory [32] and the application to longitudinal dynamics remain open problems.

As a step toward calculation of puncture correlators we propose to supplement the geometric approach by a new dynamical principle - the geometric bootstrap equation. We are not able to prove that structure constants satisfying this equation exist. However assuming their analytic dependence on the conformal weights one can show that the 4-point correlator leads to the cut singularities of string amplitudes which coincides with the result obtained from the DOZZ solution.

With the standard structure of the string perturbation expansion [33] assumed, the cut singularities of tree amplitudes indicate the continuous spectrum of intercepts. This is a strong evidence that either the consistent non-critical Nambu-Goto string does not exist at all or the existing Liouville models does not provide an adequate description of the longitudinal dynamics. The third possibility that the perturbation expansion of non-critical string does not have the structure known from the critical string theory is a difficult open problem going beyond the scope of the present paper.

The paper is organized as follows. In Sect.2 we summarize those properties of the longitudinal dynamics which are necessary for the Lorentz covariance of the tree light-cone amplitudes of non-critical Nambu-Goto string. In Sect.3 we demonstrate that the solution provided by the DOZZ theory leads to the tree non-critical strings with cut singularities. In Sect.4, using general properties of the geometric approach to the Liouville theory we formulate the geometric bootstrap equation. Using the results of Wolf and Wolpert on the asymptotic behaviour of the geodesic length [34] we show that the singularities of string amplitude are of the same type as in the case of the DOZZ solution. Finally Sect.5 contains conclusions and brief discussion of open problems.

2 General properties of the longitudinal dynamics

In the light-cone formulation string amplitudes are constructed in terms of 2-dim field theory on light-cone diagrams describing time ordered sequences of elementary splitting and joining processes [35–37]. The problem of introducing interactions can be seen as a problem of extending the theory from the cylinder, where it is completely determined by the free string
theory at hand, to an arbitrary light-cone diagram. In the case of the Nambu–Goto non-critical string the extension in the transverse sector is given by the tensor product of \(d-2\) copies of the scalar CFT.

According to our assumptions concerning the free theory the space of states in the longitudinal sector is a tensor product of single left and single right Verma module \(\mathcal{V}_h \otimes \mathcal{V}_{\bar{h}}\). The central charges \(c = \bar{c}\) and the highest weights \(h = \bar{h}\) are related to the dimension of the target space by \([3]\):

\[
c = 1 + 6Q^2 = 26 - d, \quad h = \frac{Q^2}{4} = \frac{25 - d}{24}.
\]

The only ground state \(|0\rangle = \omega \otimes \bar{\omega}\) is not \(PSL(2, \mathbb{C})\)-invariant and the energy momentum-tensor\(^3\)

\[
T(z) = \sum_n L_n z^{-n-2}
\]

is singular in the limit \(z \to 0\),

\[
T(z)|0\rangle = \frac{Q^2}{4z^2}|0\rangle + \frac{1}{z}L_{-1}|0\rangle + \text{regular terms}.
\]

The ground state applied to the free end of a semi-infinite cylinder thus corresponds to a puncture on the complex plane with prescribed singularity structure of the energy–momentum tensor at it.

In the case of tree light-cone diagrams with \(N\) arbitrary external states the longitudinal sector is described by correlation functions of appropriate number of the energy–momentum tensor insertions on the Riemann sphere \(S^2(z_1, \ldots, z_N)\) with \(N\) punctures:

\[
\langle \prod_j T(w_j) \prod_k \tilde{T}(\bar{w}_k) \rangle_{S^2(z_1, \ldots, z_N)}.
\]

Such formulation may seem strange from the point of view of the standard CFT. Indeed if we had assumed the operator–state correspondence we could have replaced the correlation functions on \(N\)-punctured sphere by correlation functions on the sphere with no punctures but with additional local operator insertions. As we shall see the space of states in the longitudinal sector is “too small” to accommodate such construction. Even though the operator–state correspondence is not assumed it is convenient to replace the adequate but clumsy notation (2.2) by

\[
\langle \prod_j T(w_j) \prod_k \tilde{T}(\bar{w}_k) \prod_{r=1}^N P(z_r, \bar{z}_r) \rangle.
\]

The properties of the longitudinal sector which are necessary for the Lorentz covariance of tree string amplitudes can be summarized as follows [2]

I \textit{The conformal anomaly has its universal form given by the (regularized) Liouville action on the punctured sphere and depends on the central charge }c = 1 + 6Q^2 \textit{in the standard way.}

\(^3\)The antiholomorphic counterparts of the formulae are assumed.
II For all correlation functions on $N$-punctured spheres the operator–operator product expansion (OOPE) and the operator–puncture product expansion (OPPE) hold:

\[
T(w)T(z) = \frac{1 + 6Q^2}{2(w - z)^4} + \frac{2}{(w - z)^2}T(z) + \frac{1}{w - z} \partial T(z) + \ldots ,
\]

\[
T(w)P(z, \bar{z}) = \frac{Q^2}{4(w - z)^2}P(z, \bar{z}) + \frac{1}{w - z} \partial P(z, \bar{z}) + \ldots .
\]

III The conformal transformations of the correlation functions (2.3) are generated by the energy–momentum tensor:

\[
\delta_l T^l(z) = -\frac{1}{2\pi i} \oint dw \epsilon(w)T^l(w)T^l(z),
\]

\[
\delta_{l\bar{c}}P(z, \bar{z}) = -\frac{1}{2\pi i} \oint dw \epsilon(w)T^l(w)P(z, \bar{z})
\]

Using (2.4) and (2.5) they can be cast in the form of the conformal Ward identities (CWI):

\[
\langle T(w) \prod_r P(z_r, \bar{z}_r) \rangle^L = \sum_r \left( \frac{Q^2}{4(w - z_r)^2} + \frac{1}{w - z_r} \frac{\partial}{\partial z_r} \right) \langle \prod_r P(z_r, \bar{z}_r) \rangle^L ,
\]

\[
\langle T^l(w) \prod_r P(z_r, \bar{z}_r) \rangle^L = \frac{1 + 6Q^2}{2(w - u)^4} \langle \prod_r P(z_r, \bar{z}_r) \rangle^L
\]

\[
+ \left( \frac{2}{(u - w)^2} + \frac{1}{u - w} \frac{\partial}{\partial w} \right) \langle T(w) \prod_r P(z_r, \bar{z}_r) \rangle^L
\]

\[
+ \sum_r \left( \frac{Q^2}{4(u - z_r)^2} + \frac{1}{u - z_r} \frac{\partial}{\partial z_r} \right) \langle T(w) \prod_r P(z_r, \bar{z}_r) \rangle^L.
\]

Apart from the complications related with the non-invariant vacuum and the lack of operator–state correspondence the conditions listed above are exact counterparts of almost all fundamental properties of standard CFT. So are their consequences. Using CWI one can for instance reduce an arbitrary correlation function to the correlation function of punctures alone. As in the standard CFT the form of three puncture correlator is determined up to a constant

\[
\langle P(z_1, \bar{z}_1)P(z_2, \bar{z}_2)P(z_3, \bar{z}_3) \rangle = \frac{C}{|z_1 - z_2|^{q_1^2/4} |z_1 - z_3|^{q_2^2/4} |z_2 - z_3|^{q_2^2/4}}.
\]

The familiar state–operator correspondence can be replaced by the state–puncture correspondence defined by

\[
L_{-n_1} \ldots L_{-n_N} |0 \rangle^L \rightarrow L_{-n_1} \ldots L_{-n_N} \cdot P(z, \bar{z})
\]

\[
= \frac{1}{(2\pi i)^N} \oint_{C_1} dz_1 \frac{T(z_1)}{(z_1 - z)^{n_1 - 1}} \ldots \oint_{C_N} dz_N \frac{T(z_N)}{(z_N - z)^{n_N - 1}} P(z, \bar{z})
\]
where the contours of integration are chosen such that \( C_i \) surrounds \( C_{i+1} \) for \( i = 1, \ldots, N - 1 \), and \( C_N \) surrounds the point \( z \). Using this prescription one can associate to each state \( \xi \otimes \bar{\xi} \in \mathcal{V}_h \otimes \bar{\mathcal{V}}_h \) a uniquely determined object \( V_0(\xi, \bar{\xi}; z, \bar{z}) \) which we shall call the vertex puncture corresponding to \( \xi \otimes \bar{\xi} \).

The only fundamental property of standard CFT we have not yet required is the puncture–puncture product expansion (PPPE). With our choice of the space of states there is only one conformal family of vertex punctures and the PPPE would be necessarily of the form

\[
P(x, \bar{x})P(0, 0) = Cx^{-h}x^{-\bar{h}} [P(0, 0) + \beta_1 x \mathcal{L}_{-1} \cdot P(0, 0) + \beta_1 \bar{x} \bar{\mathcal{L}}_{-1} \cdot P(0, 0) + \ldots],
\]

(2.12)

where \( C \) is the constant appearing in the three puncture partition function (2.10) and the descendant vertex punctures \( \mathcal{L}_{-1} \cdot P(0, 0), \bar{\mathcal{L}}_{-1} \cdot P(0, 0), \) etc. are defined by (2.11). Using PPPE one could in principle reduce all \( N \)-puncture correlators to the 3-puncture functions. The consistency conditions of this procedure yields in the case of 4-puncture functions the bootstrap equation [10]. One of its consequences in standard CFT is the restriction on possible central charge, conformal dimensions, and fusion rules known as Vafa’s condition [38]. Using Lewellen’s derivation of this condition [39] one can show that the PPPE (2.12) cannot be satisfied.

According to the properties of the longitudinal sector derived so far the theory is completely determined once all \( N \)-puncture correlators are known

\[
\left\langle \prod_{r=1}^{N} P(z_r, \bar{z}_r) \right\rangle^L.
\]

(2.13)

It should be stressed that the correlation functions (2.13) can not be expressed as the vacuum expectation values of some chronological product of local operators acting in the Hilbert space \( \mathcal{V}_h \otimes \bar{\mathcal{V}}_h \) of the free theory. Indeed if it were possible it would imply the PPPE which is excluded by Vafa’s condition. This may rise the question on the quantum mechanical meaning of the functions (2.13). In the present paper we adopt the interpretation advocated in the geometric approach to the Liouville theory [28] where the \( N \)-puncture correlators are understood as partition functions on \( N \)-punctured spheres (or in general on higher genus punctured surfaces). The theory we are looking for can then be seen as a theory of only one local field (the energy–momentum tensor) in different noncompact 2-dim geometries.

Concluding this section let us briefly comment on the unitarity problem. The light-cone approach used in [2] to derive the properties discussed above incorporates the idea of joining–splitting interactions by the assumption that there exists a well defined 2-dimensional theory on light-cone diagrams. The standard proof of unitarity in this formulation is based on the identification of the light-cone amplitudes as terms of a Dyson perturbative expansion in the space of multi-string states [40, 41]. In order to interpret the integration over moduli of punctured sphere as the integration over interaction times one uses the Mandelstam map to transform the theory back onto an appropriate family of light-cone diagrams. If the string degrees of freedom are described by standard CFT (as they are in the transverse sector) one
can use the OPE to factorize the light-cone amplitude on the free spectrum at any moment between interactions. This in order allows to interpret the amplitude in terms of interaction vertices and free propagation yielding the required Dyson expansion structure \[2\]. Since in the longitudinal sector the PPPE does not hold such interpretation is not possible and the standard proof of unitarity breaks down.

3 DOZZ solution

The correlators of operators from the \((\frac{Q^2}{4}, \frac{Q^2}{4})\) conformal family calculated in the DOZZ theory provide a set of functions satisfying all the requirements listed in the previous section. We shall demonstrate that this solution leads to the cut singularities in the non-critical string amplitudes. The DOZZ structure constants \(C(\alpha_3, \alpha_2, \alpha_1)\), which determine the three point correlation function

\[
C\left[ a_{\alpha_1, \alpha_2, \alpha_3} \left( z_1, z_2, z_3 \right) \right] = |z_{12}|^{2\gamma_1}|z_{32}|^{2\gamma_2}|z_{31}|^{2\gamma_3}C(\alpha_3, \alpha_2, \alpha_1)
\]

where \(\gamma_1 = \Delta_\alpha - \Delta_\alpha - \Delta_\alpha, \gamma_2 = \Delta_\alpha - \Delta_\alpha - \Delta_\alpha, \gamma_3 = \Delta_\alpha - \Delta_\alpha - \Delta_\alpha\), are given by

\[
C(\alpha_3, \alpha_2, \alpha_1) = \left( \frac{\pi \mu \tilde{\mu} \gamma \left( \frac{1}{b^2} \right)}{2} \right)^{Q/2} \left( \frac{b}{b-1} \right)^{Q+\alpha_1-\alpha_2-\alpha_3}
\]

The special function \(\Upsilon_b(x)\) has an integral representation convergent in the strip \(0 < \Re x < Q\)

\[
\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[ \left( \frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left( \frac{Q}{2} - x \right)}{\sinh^2 \frac{t}{2}} \right],
\]

\(\Upsilon_0 = \text{res}_{x=0} \frac{d\Upsilon_b(x)}{dx}\), \(\gamma(x) = \frac{\gamma(x)}{\Gamma(1-x)}\) and the “dual” cosmological constant \(\tilde{\mu}\) is related to \(\mu\) by

\[
\left[ \pi \mu \tilde{\mu} \gamma \left( \frac{1}{b^2} \right) \right]^{1/b} = \left[ \pi \mu \gamma \left( \frac{1}{b^2} \right) \right]^{1/b}.
\]

Eq. (3.1) is self-dual: it remains unchanged under \(b \rightarrow 1/b\). Its form agrees with the three-point coupling constant proposed in [8] in the weak coupling region of \(b \in \mathbb{R}\). If one assumes that the product \(\mu \tilde{\mu}\) remains real and positive in the strong coupling region, \(b = e^{i\theta}, \theta \in \mathbb{R}\), (it is straightforward to check that (3.2) admits such solution), then (3.1) defines analytic continuation of the coupling constant to this region respecting the self-duality condition, which is sufficient for \(C\) to be real there.

The four-point function for primary fields can be written as the s-channel integral

\[
C_{a_{\alpha_4, \alpha_3, \alpha_2, \alpha_1}}(z, \bar{z}) = \int D \alpha C(\alpha_4, \alpha_3, \alpha)C(\bar{\alpha}, \alpha_2, \alpha_1) |F_\alpha \left[ a_{\alpha_4, \alpha_1} \right] (z)|^2
\]

where the conformal block \(F_\alpha \left[ a_{\alpha_4, \alpha_1} \right] (z)\) is represented by power series of the form

\[
F_\alpha \left[ a_{\alpha_4, \alpha_1} \right] (z) = z^{\Delta_\alpha-\Delta_\alpha-\Delta_\alpha} \sum_{n=0}^\infty z^n F_\alpha \left[ a_{\alpha_4, \alpha_1} \right] (n).
\]
For \( \alpha_i = \frac{Q}{2} \), the set \( \mathcal{D} \) coincides with the spectrum \( \mathcal{S} = \frac{Q}{2} + i\mathbb{R}^+ \) which we shall parameterize with \( \frac{Q}{2} + iP \). The coefficients \( F_{\alpha} \frac{[\alpha_1 \alpha_2]}{[\alpha \alpha]}(n) \) are rational functions of \( \Delta_\alpha = \frac{Q^2}{4} + P^2 \) with poles located (for \( 1 < c < 25 \)) outside \( \mathcal{S} \). For \( b = e^{i\theta} \) the formula (3.1) implies

\[
C \left( \frac{Q}{2}, \frac{Q}{2}, \frac{Q}{2} + iP \right) C \left( \frac{Q}{2} - iP, \frac{Q}{2}, \frac{Q}{2} \right) = q e^{2H_\theta(P)},
\]

where \( q \) does not depend on \( P \) and

\[
H_\theta(P) = \int_0^\infty \frac{dt}{t} \left[ \cos^2 \theta e^{-t} + \frac{1 - 8 \sin^2 \frac{Pt}{2} - \cos(2Pt) \cosh(t \cos \theta) - \cos(t \sin \theta)}{\cosh(t \cos \theta) - \cos(t \sin \theta)} \right] \sim -P^2
\]

for \( P^2 > 0 \). The 4-string amplitude for the tachionic external states can be written as [2]

\[
A = (2\pi)^\frac{d}{2} |\alpha|^{-\frac{d}{2}} \prod_{\mu=0}^{d-1} \delta \left( \sum_{r=1}^{4} p_r^\mu \right) \int_{\mathbb{C}} d^2z \left| z \right|^{\frac{p_1}{2\alpha}} \left| 1 - z \right|^{\frac{p_2}{2\alpha}} G^L(z, \bar{z}) \equiv G^L(z, \bar{z})
\]

where \( G^L(z, \bar{z}) = G^L_{Q, \frac{Q}{2}+iP, \frac{Q}{2}}(z, \bar{z}) \) and \( p_r \) denote external momenta satisfying the on-mass-shell condition \( \frac{d^2}{4\alpha} = \frac{\bar{Q}^2}{Q^2} \). Using (3.5) we write

\[
\mathcal{I} = \int_{\mathbb{C}} d^2z \left| z \right|^{\frac{p_1}{2\alpha}} \left| 1 - z \right|^{\frac{p_2}{2\alpha}} G^L(z, \bar{z})
\]

\[
= q \int_{\mathbb{C}} d^2z \left| z \right|^{\frac{p_1}{2\alpha}} \left| 1 - z \right|^{\frac{p_2}{2\alpha}} \int_0^\infty dP e^{2H_\theta(P)} \left\{ F_{\frac{Q}{2}+iP} \left[ \frac{\bar{Q}}{2}, \frac{Q}{2} \right] (m) \right\} \left\{ F_{\frac{Q}{2}-iP} \left[ \frac{\bar{Q}}{2}, \frac{Q}{2} \right] (n) \right\}.
\]

If one can change the order of integration, then — expanding the integrand in the power series of \( z, \bar{z} \) and integrating in the vicinity of \( z = 0 \) one gets

\[
\mathcal{I} \sim \sum_{n} \int_0^\infty dP \frac{d_{n,n}(P)}{\frac{Q}{4\alpha} + \frac{Q^2}{2} + 2n + 2P^2 - 2} e^{2H_\theta(P)}
\]

up to terms analytic in the Mandelstam variable \( s = (p_1 + p_2)^2 \) at the finite part of the complex plane. \( d_{n,n}(P) \) are rational functions of \( P \) without poles on the real axis. The presence of poles on the integration contour for \( s < -4\alpha \left( \frac{Q^2}{2} + 2P^2 - 2 \right) \) leads to cuts in \( \mathcal{I} \) in this region.

The cut structure of the scattering amplitude can be confirmed by choosing a different route in calculating (3.7). Taking into account the convergence of the series (3.4), the properties of its coefficients \( F_{\frac{Q}{2}+iP} \left[ \frac{\bar{Q}}{2}, \frac{Q}{2} \right] (n) \) and of \( H_\theta(P) \) one can change the order of integration and summation in calculating the four-point function \( G^L(z, \bar{z}) \) in the vicinity of \( z = 0 \). Integrals of the form

\[
\int_0^\infty dP \left| z \right|^{2P^2} e^{2H_\theta(P)} F_{\frac{Q}{2}+iP} \left[ \frac{\bar{Q}}{2}, \frac{Q}{2} \right] (m) F_{\frac{Q}{2}-iP} \left[ \frac{\bar{Q}}{2}, \frac{Q}{2} \right] (n)
\]

that arise in this procedure contain terms which for \( z \to 0 \) behave like powers of \( 1/\log |z| \). Inserted into (3.7) they produce cuts in the complex \( s \) plane.
4 Geometric bootstrap hypothesis

We shall start with some remarks on the spectrum of conformal weights. In the weak coupling regime $c > 25$ we are interested in correlators of local operators with conformal weights satisfying the Seiberg bound $\Delta < \frac{Q^2}{4}$. If the theory is coupled to conformal matter such operators are necessary for the gravitational dressing. They are supposed to correspond to microscopic, non-normalizable states with imaginary Liouville momenta [42]. In the geometric approach they are described by conical (elliptic) singularities with the opening angle $\nu$ related to the imaginary Liouville momentum $P$ by

$$\Delta = \frac{Q^2}{4} + P^2 = \frac{Q^2}{4} \left( 1 - \left( \frac{\nu}{2\pi} \right)^2 \right)$$

These operators do not appear when the Liouville theory is regarded as the model for the longitudinal string excitations in physical dimensions $1 < c < 25$. Indeed the derivation of the light-cone amplitudes shows that only the parabolic singularities (limiting case of the conical singularity with the opening angle $\nu = 0$) are relevant for the description of external states. The states with imaginary Liouville momenta are also not expected in factorization of puncture correlators [18, 42].

In order to describe the factorization one has to extend the geometric approach to surfaces with finite holes. We assume that the Liouville action and the space of metrics in the path integral are chosen in such a way that the classical solution corresponds to the hyperbolic metric with the curvature $-\mu < 0$ and the geodesic boundary. We also assume that the conformal weight of the hole is

$$\Delta_\ell = \frac{Q^2}{4} \left( 1 + \frac{\mu \ell^2}{8\pi^2} \right)$$

where $\ell$ is the length of the hole circumference measured with respect to the metric with constant negative curvature $R = -\mu$. It depends only on the conformal class of metrics over which the path integral is taken. This assumption is motivated by the properties of the energy–momentum tensor of the classical hyperbolic solutions on the cylinder and on sphere with holes (black-hole solutions in 3-dim gravity). The puncture corresponds to the limiting case of the hole with zero circumference. Since in the case of puncture the classical conformal dimension does not receive quantum corrections one may expect that this is so for the holes as well (up to the renormalization of the cosmological constant $\mu$).

Let us now consider the simplest case of 4-puncture correlator. In the geometric approach it is given by the path integral over the conformal class of metrics with parabolic singularities at puncture locations. Since the conformal weight of the puncture is fixed the space of metric fluctuations at each puncture is completely described by $L_n$ operators defined in (2.11) and coincides with the space of longitudinal excitations of the free string. The Liouville action and the space of metrics in the path integral are chosen in such a way that for each configuration of punctures one gets a unique classical solution corresponding to the hyperbolic metric with scalar curvature $-1$. There are three closed geodesics $\Gamma_s, \Gamma_t, \Gamma_u$ in this geometry separating the
punctures into pairs (12,34), (13,24), (14,23), respectively. Let us cut the path integral open along the geodesic $\Gamma_s$ dividing $S$ into the spheres $S_{12}, S_{34}$ with one hole and two punctures ($S = S_{12} \cup S_{34}, \Gamma_s = S_{12} \cap S_{34}$). The classical solution $g$ on $S$ determines classical solutions $g_{12}$ on $S_{12}$ and $g_{34}$ on $S_{34}$. The initial path integral factorizes into a path integral over the conformal class of $g_{12}$, a path integral over the conformal class of $g_{34}$ and the integration over all possible intermediate states. According to our assumptions the holes on $S_{12}$ and $S_{34}$ have the same conformal dimension $\Delta_s = Q^2 \left(1 + \frac{\ell^2_s}{8\pi^2}\right)$ uniquely determined by the length $\ell_s$ of the common boundary. It follows that factorization in each channel involves exactly one conformal family. Its conformal weight depends on the channel and the moduli of the surface. This is in contrast with the DOZZ description of the weak coupling regime where the factorization is independent both of the channel and of the moduli and involves integration over continuous spectrum of conformal families.

Before we formulate the consistency conditions of the geometric factorization introduced above some comments on the hole-state correspondence are in order. First of all since the length of the geodesic boundary can assume any positive value one needs a continuous spectrum of conformal weights $\Delta \geq Q^2$ of intermediate states. This however does not necessary mean that the space of states has to be extended. The crucial observation is that the geometric factorization on exactly one conformal family allows to interpret the conformal weight as a characteristic of the energy–momentum tensor behavior around the hole rather than a characteristic of intermediate states attached to its boundary. Such interpretation is consistent with our choice of the space of free string states. Indeed for the central charge in the range $1 \leq c \leq 25$, the left Verma module $V_{Q^2}$ can be realized as the Fock space $F^L$ generated out of the vacuum state $\omega$ by the oscillators

$$[c_m, c_n] = m\delta_{m,-n}, \quad m, n \in \mathbb{Z} \setminus \{0\}$$

with the Virasoro generators given by [43]

$$L_n = \frac{1}{2} \sum_{k \neq 0, -n} :c_{-k}c_{n+k}: + i \frac{Q}{\sqrt{2}} n c_n + \frac{Q^2}{4} \delta_{n,0}.$$ 

For any real positive number $P$ (the Liouville momentum) one can construct on $F^L$ the local operator

$$T_p(z) = \sum_n L_n^p z^{-n-2}, \quad L_n^p = \begin{cases} L_n^L + 2Pc_n & \text{for } n \neq 0 \\ L_0^L + P^2 & \text{for } n = 0 \end{cases}$$

satisfying

$$T_p(w)T_p(z) = \frac{1}{4} (1 + 6Q^2) \frac{1}{(w-z)^4} + \frac{2}{(w-z)^2} T_p(z) + \frac{1}{w-z} \partial T_p(z) + \ldots ,$$

$$T_p(z)\Omega = \frac{\Delta}{z^2} \Omega + \frac{1}{z} L_{-1}^{P} \Omega + \text{regular terms} . \quad (4.1)$$

where $\Delta = \frac{Q^2}{4} + P^2$. With an appropriate choice of the coordinates around the hole, the state-hole correspondence takes the form

$$L_{-n_1}^P \ldots L_{-n_N}^P |0\rangle \Leftrightarrow \mathcal{L}_{-n_1} \ldots \mathcal{L}_{-n_N} \cdot H_\ell(z, \bar{z}) \quad (4.2)$$
with the central charge

\[ \omega \]

in the formulae above up to the structure constant \( C \) same as in the standard CFT. In particular an arbitrary three vertex correlator is determined by vacuum expectation values of local operators all their conformal properties are exactly the same as in the standard CFT. In particular an arbitrary three vertex correlator is determined up to the structure constant \( C(\ell, \ell, \ell) \). It takes the form

\[
\langle V_{\ell_1}^I(\xi_1, \xi_2|z_1, \bar{z}_1) V_{\ell_2}^J(\xi_2, \xi_3|z_2, \bar{z}_2) V_{\ell_3}^K(\xi_3, \xi_3|z_3, \bar{z}_3) \rangle = C(\ell_1, \ell_2, \ell_3)
\]

\[
\times \frac{\rho_{\ell_1 \ell_2 \ell_3}(\xi_1, \xi_2, \xi_3)}{(z_1 - z_2)^{\Delta_1 + \Delta_2 - \Delta_3}(z_1 - z_3)^{\Delta_1 + \Delta_3 - \Delta_2}(z_2 - z_3)^{\Delta_2 + \Delta_3 - \Delta_1}}
\]

\[
\times \frac{\rho_{\ell_1 \ell_2 \ell_3}(\xi_1, \xi_2, \xi_3)}{(z_1 - z_2)^{\Delta_1 + \Delta_2 - \Delta_3}(z_1 - z_3)^{\Delta_1 + \Delta_3 - \Delta_2}(z_2 - z_3)^{\Delta_2 + \Delta_3 - \Delta_1}}
\]

where \( \Delta_i = \Delta_{\ell_i} + |\xi_i|, \bar{\Delta}_i = \Delta_{\ell_i} + |\bar{\xi}_i| \). The trilinear forms \( \rho_{\ell_1 \ell_2 \ell_3}(\xi_1, \xi_2, \xi_3) \), universal for all CFT, are uniquely determined by CWI [13]. Also the notion of the conformal block remains unchanged

\[
F_{\ell_1, \ell_2, \ell_3}(z) = z^{\Delta_1 + \Delta_2 - \Delta_3} \sum_{n=0}^{\infty} z^n F_{\ell_1, \ell_2, \ell_3}(n)
\]

\[
F_{\ell_1, \ell_2, \ell_3}(n) = \sum_{I, J} \rho_{\ell_1 \ell_2 \ell_3}(\omega, \bar{\omega}, \xi_I, \bar{\xi}_J) B_{\ell_1, \ell_2, \ell_3}(n) \rho_{\ell_1 \ell_2 \ell_3}(\xi_J, \omega, \bar{\omega})
\]

(In the formulae above \( \omega \) denotes the vacuum state in \( F^L \), \( \{\xi_I\}_{I \in I_n} \) is a basis on the level \( n \) in \( F^L \), and \( B_{\ell_1, \ell_2, \ell_3}(n) \) is the inverse to the Gram matrix on the level \( n \) in the Verma module with the central charge \( c = 1 + 6Q^2 \) and the highest weight \( \Delta_{\ell} = \frac{Q^2}{4} \left( 1 + \frac{mc^2}{8\pi} \right) \)).

In the simplest case of the four puncture correlator,

\[
G^L(x, \bar{x}) = \lim_{z, \bar{z} \to \infty} z^{Q^2/4} \bar{z}^{Q^2/4} \langle P(z, \bar{z})P(1, 1)P(x, \bar{x})P(0, 0) \rangle,
\]

the geometric factorization in the s-channel reads

\[
G^L(x, \bar{x}) = C(0, 0, \ell_s(x, \bar{x}))C(\ell_s(x, \bar{x}), 0, 0) \left| F_{\ell_s(x, \bar{x})} \right| \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \left( \begin{array}{c} x \end{array} \right) \right|^2,
\]

where \( \ell_s(x, \bar{x}) \) is the length of the closed geodesic \( \Gamma_s \). Let us note that the amplitude does not factorizes into a holomorphic and an anti-holomorphic part.

The formula (4.3) involves the length of closed geodesic in the s-channel as a function of the fourth puncture location \( x \). In the vicinity of \( x = 0 \) it can be expanded in powers of \( 1/|x| \) [34]. If we assume that the structure constants are real analytic at \( \ell_i = 0 \) then
the geometric factorization (4.3) yields the same behavior of the Liouville correlator and the
same analytic structure of the string amplitude as in the case of the DOZZ solution.

The basic consistency condition of the presented approach is the obvious requirement
that the geometric factorization in each channel yields the same result. In the slightly more
general case of the four hole correlator it leads to the geometric bootstrap equations:

$$C(\ell_4, \ell_3, \ell_s) C(\ell_s, \ell_2, \ell_1) \left| \mathcal{F}_{Q, \ell} \left[ \frac{\ell_1 \ell_2}{\ell_4 \ell_1} \right] (x) \right|^2$$

$$= C(\ell_4, \ell_1, \ell_t) C(\ell_t, \ell_2, \ell_3) \left| \mathcal{F}_{Q, \ell} \left[ \frac{\ell_1 \ell_2}{\ell_4 \ell_1} \right] (1 - x) \right|^2$$

$$= |x|^{-4\Delta_2} C(\ell_1, \ell_3, \ell_u) C(\ell_u, \ell_2, \ell_4) \left| \mathcal{F}_{Q, \ell} \left[ \frac{\ell_1 \ell_2}{\ell_4 \ell_1} \right] \left( \frac{1}{x} \right) \right|^2$$

where $\ell_s, \ell_t, \ell_u$ are the lengths of the closed geodesics in corresponding channels. As in the
standard CFT one can promote the geometric bootstrap equations to the basic dynamical
principle of the theory.

5 Discussion and conclusions

Whether the geometric bootstrap equations provides a plausible solution of the Liouville
theory depends on our ability to calculate the structure constants. Due to the complicated
non-linear nature of the equations a direct analysis of this problem is prohibitively difficult. It
seems that the only strategy available is to construct a candidate for the structure constants
and then to verify whether it satisfies the geometric bootstrap equations.

The simplest possibility is to consider the DOZZ structure constant. This proposal might
be motivated by the existence of the analytic continuation of the 3-point function from the
weak to the strong coupling region and from the elliptic to the hyperbolic weights. Another
problem is to verify a slightly stronger hypothesis that not only the structure constants but
also the correlators of the geometric approach and of the DOZZ theory are identical. In the
case of 4-point correlators it takes the form of the following equation

$$C(\ell_4, \ell_3, \ell_s(x, \bar{x})) C(\ell_s(x, \bar{x}), \ell_2, \ell_1) \left| \mathcal{F}_{Q, \ell_s} \left[ x \right] \left( \frac{\ell_3 \ell_2}{\ell_4 \ell_1} \right) (x) \right|^2$$

$$\propto \int_0^\infty d\ell C(\ell_4, \ell_3, \ell) C(\ell, \ell_2, \ell_1) \left| \mathcal{F}_{Q, \ell} \left[ \frac{\ell_3 \ell_2}{\ell_4 \ell_1} \right] (x) \right|^2.$$
Even if the structure constants were known, the verification of the geometric bootstrap equations would still be a challenging task. First of all the equations involve the lengths of closed geodesics in each channel as functions of the locations of punctures. Up to our knowledge even in the simplest case of 4-punctured sphere such functions are not known [34, 44]. Secondly the conformal block function is also not known in this case and can be studied only by the numerical methods developed in [45, 46]. Both problems are of their own interest going beyond the present context.

Let us finally comment on the problem of consistent interactions of non-critical Nambu-Goto string. As one could expect from the continuous spectrum the DOZZ approach leads to the tree string amplitudes with cut singularities. The same result was obtained in a more speculative way in the geometric approach. It follows that the Liouville theory description of the longitudinal dynamics applied in the framework of the standard string perturbation expansion does not lead to a consistent interacting non-critical Nambu-Goto string. This leaves almost no room to maneuver for constructing such a theory.

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