Slow and Ordinary Provability for Peano Arithmetic

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Abstract

The notion of slow provability for Peano Arithmetic (PA) was introduced by S.-D. Friedman, M. Rathjen, and A. Weiermann. They studied the slow consistency statement Con₄ asserting that a contradiction is not slow provable in PA. They showed that the logical strength of the theory PA + Con₄ lies strictly between that of PA, and PA together with its ordinary consistency: PA ⊊ PA + Con₄ ⊊ PA + Con₄.

This paper is a further investigation into slow provability and its interplay with ordinary provability in PA. We study three variants of slow provability. The associated consistency statement of each of these yields a theory that lies strictly between PA and PA + Con₄ in terms of logical strength. We investigate Turing-Feferman progressions based on these variants of slow provability. For our three notions, the Turing-Feferman progression reaches PA + Con₄ in a different numbers of steps, namely ε₀, ω, and 2. For each of the three slow provability predicates, we also determine its joint provability logic with ordinary PA-provability.

Keywords: Peano Arithmetic, Provability Logic, Fast-Growing Hierarchy, Turing-Feferman Progressions, Slow Consistency

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1 Introduction

Slow provability, introduced by S.-D. Friedman, M. Rathjen, and A. Weiermann in [9], is a notion of nonstandard provability for Peano Arithmetic (PA) – while we know that it coincides with ordinary provability for PA, this fact is not verifiable in PA itself. This paper is a further investigation into the relation between slow and ordinary provability, as seen from the perspective of PA.

The definition of slow provability relies on a fast-growing hierarchy, also known as the extended Grzegorczyk hierarchy. What we mean by this is, following [9], an ordinal-indexed family of recursive functions \( \{ F_\alpha \}_{\alpha \leq \epsilon_0} \). The functions \( \{ F_\alpha \}_{\alpha < \omega} \) are closely related to a family of classes of functions known as the Grzegorczyk hierarchy (\[12\]). They are primitive recursive, and furthermore every primitive recursive function is dominated by some function in \( \{ F_\alpha \}_{\alpha < \omega} \). Löb and Wainer ([16], [17]) extended the hierarchy into the transfinite. The exact version of the fast-growing hierarchy used in [9] was introduced by Solovay and Ketenen ([14]). The function \( F_{\epsilon_0} \) results from diagonalizing over the functions \( \{ F_{\omega n} \}_{n \in \omega} \), each of which is provably total in PA, and is not provably total in PA itself. This makes it interesting to consider the following r.e. theory:

\[
\text{PA} \upharpoonright F_{\epsilon_0} := \{ I \Sigma_n \mid F_{\epsilon_0}(n) \downarrow \},
\]

(1)

where \( I \Sigma_n \) is as usual PA with the induction schema restricted to \( \Sigma_n \)-formulas. Since \( F_{\epsilon_0} \) is total, we know that \( \text{PA} \upharpoonright F_{\epsilon_0} \) and PA have exactly the same theorems. Arguing in PA, on the other hand, the totality of \( F_{\epsilon_0} \) cannot be assumed, and thus \( \text{PA} \upharpoonright F_{\epsilon_0} \) might seem to be a weaker theory than PA. As shown in [9], there exist indeed models of PA where a contradiction is provable in PA but not in \( \text{PA} \upharpoonright F_{\epsilon_0} \).

A notion of slow provability can be associated to any recursive function \( f \) not provably total in PA, by considering the theory

\[
\text{PA} \upharpoonright f := \{ I \Sigma_n \mid f(n) \downarrow \}.
\]

(2)

Since the equivalence of \( \text{PA} \) and \( \text{PA} \upharpoonright f \) might not be verifiable in \( \text{PA} \), it is interesting to ask how exactly do the two theories relate to each other, as seen from the perspective of PA. This paper offers some ways of answering the above question.
1.1 Results of this paper

Denote by $\text{Con}_{\text{PA}}$, the usual consistency statement for $\text{PA}$, and by $\text{Con}_f$ the statement expressing that a contradiction is not provable in $\text{PA}|_f$. As mentioned above, $\text{Con}_f$ need not be provably equivalent to $\text{Con}_{\text{PA}}$. However it is conceivable that by iterating $\text{Con}_f$ sufficiently many times, a statement equivalent to $\text{Con}_{\text{PA}}$ is reached. We explore this possibility by considering transfinite iterations of slow consistency statements. Given a non-zero ordinal $\alpha \leq \varepsilon_0$, the $\alpha$-iteration $\text{Con}_f^\alpha$ of $\text{Con}_f$ is informally defined as the consistency statement for the theory $\text{PA}|_f + \{\text{Con}^\beta_f \mid 0 < \beta < \alpha\}$. (3)

We adopt the provability logic approach to transfinite iterations, developing the notion of a transfinite iteration $\text{Pr}_f^\alpha(x)$ of a provability predicate $\text{Pr}(x)$ along a Kalmar elementary well-ordering. We show that these iterations satisfy the Hilbert-Bernays-Löb derivability conditions, and can thus be considered as provability predicates themselves (Section 4).

We show that the $\varepsilon_0$-iteration of $\text{Con}^\varepsilon_0$ is equivalent to $\text{Con}_{\text{PA}}$ (Theorem 10), thus answering a question raised in [9, Remark 4.4]. We also show that a small index shift in the definition of $\text{PA}|_{\varepsilon_0}$ yields a slow consistency statement whose $\omega$-iteration is already equivalent to $\text{Con}_{\text{PA}}$ (Theorem 5). While finishing writing this paper, the authors learned that the above results are also contained in Anton Freunds recent paper [8]. The results of our paper were obtained independently from the latter.

We also introduce a variant of slow provability that can be seen as the square root of ordinary $\text{PA}$-provability, in the sense that already the two-fold iteration of the associated slow consistency statement is equivalent to $\text{Con}_{\text{PA}}$ (Theorem 12). Our slow provability variant is the first example of such a provability predicate in the context of $\text{PA}$.

For each of our three notions of slow provability, we determine its joint provability logic with ordinary $\text{PA}$-provability. While the slow provability predicate studied in [9] and its shifted version mentioned above behave very differently when it comes to transfinite iterations, their joint provability logic with ordinary provability is the same, namely Lindström logic (Theorem 16). It was shown in [15] that the latter is also the joint provability logic of ordinary and Parikh provability, which can be seen as a speeded up version of ordinary $\text{PA}$-provability. Our proof or arithmetical completeness is rather
general and works for a large class of pairs of provability predicates, including ordinary and Parikh provability.

1.2 Overview of this paper

Sections 2 and 3 contain the basic results and notions used in the paper. Section 4 introduces transfinite iterations of provability predicates. The notion of slow provability, along with some results from [9], forms the content of Section 5. In Section 6 we show that in some cases provability in PA implies a certain transfinite iteration of slow provability. Section 7 deals with the converse. The joint provability logic of slow and ordinary provability is determined in Section 9. The material in this section relies only on sections 2 and 5.

1.3 History and context

We point out some developments related to the subject matter of this paper.

1.3.1 Nonstandard notions of provability for PA

The method of arithmetization developed by Gödel allows PA to talk about basic syntactical notions. In particular, there is an arithmetical formula $\text{Pr}_{PA}(x)$, the so-called provability predicate, that expresses basic facts about provability in PA. Writing $\text{Con}_{PA}$ for the sentence $\neg \text{Pr}_{PA}(\bot)$, Gödel’s Second Incompleteness Theorem states that $\text{Con}_{PA}$ is not provable in PA.

Since $\text{Pr}_{PA}(x)$ is, prima facie, an arithmetical formula, one may justifiably ask what exactly is meant by calling it a provability predicate. Could there be another provability predicate whose associated consistency statement is provable in PA? Likewise, which properties of $\text{Pr}_{PA}(x)$ does the proof of the Second Incompleteness Theorem rely on?

Such questions were for the first time investigated by Feferman in his influential paper [5]. He constructs a predicate $\text{Pr}^*_{PA}(x)$ that has the same extension as $\text{Pr}_{PA}(x)$ on the natural numbers, whose associated consistency statement is however provable in PA. The existence of such a nonstandard provability predicate illustrates the need for a more careful formulation of the Second Incompleteness Theorem.

In order to demonstrate the difficulty of singling out one “standard” provability predicate for PA, Feferman ([5, Theorem 7.4, 7.5]) provides a rather
general method for modifying a given provability predicate $\text{Pr}(x)$, so as to obtain new provability predicates $\text{Pr}'(x)$ and $\text{Pr}''(x)$ for the same theory, whose associated consistency statements lie strictly below and above the original one respectively:

$$\text{PA} \subset \text{PA} + \text{Con}' \subset \text{PA} + \text{Con} \subset \text{PA} + \text{Con}''.$$  \hspace{1cm} (4)

In particular, we obtain a theory lying between $\text{PA}$ and $\text{PA} + \text{Con}_\text{PA}$ in terms of logical strength. Since the above method relies on self-reference, in the form of a Rosser-style construction, it is reasonable to ask whether a natural theory with this property can also be found. The theory $\text{PA} + \text{Con}_{\epsilon_0}$, obtained by adding to $\text{PA}$ the statement of its slow consistency, may be seen as the first example of such a theory.

Another example of nonstandard provability is the so-called Parikh provability. An arithmetical sentence $\varphi$ is said to be Parikh provable if it is provable in $\text{PA}$ together with Parkih’s rule, where the latter allows one to infer $\varphi$ from the sentence $\text{Pr}_{\text{PA}}(\varphi)$. Since Parikh’s rule is admissible in $\text{PA}$, adding it to $\text{PA}$ does not yield new theorems. As shown in [18], it does yield speed-up, meaning that some theorems have much shorter proofs when Parikh’s rule is allowed. The equivalence of Parikh provability and ordinary provability is however not verifiable in $\text{PA}$.

### 1.3.2 Slow provability for weak theories

A notion of slow provability in the context of Kalmar Elementary Arithmetic ($\text{EA}$) was introduced by Visser in [27]. He uses a superexponential (not provably total in $\text{EA}$) function in order to modify the standard provability predicate of $\text{EA}$. As in the case of slow provability for $\text{PA}$, the associated slow consistency statement lies strictly between $\text{EA}$ and $\text{EA} + \text{Con}_{\text{EA}}$ in terms of logical strength.

A version of square root provability for $I\Delta_0 + \text{Exp}$ was also found by Visser: it is shown in [26] that cut-free or tableaux provability serve as the square root of ordinary provability in the context of $I\Delta_0 + \text{Exp}$.

### 1.3.3 Provability logic

The idea of viewing $\text{Pr}_{\text{PA}}(x)$ as a modal operator $\Box$ goes back to Gödel ([III]). Hilbert and Bernays formulated certain conditions for $\text{Pr}_{\text{PA}}(x)$ that would suffice for the proof of the Second Incompleteness Theorem. These were
later simplified by Löb, and are now referred to as the *Hilbert-Bernays-Löb derivability conditions*:

1. $\text{PA} \vdash \varphi \Rightarrow \text{PA} \vdash \Box \varphi$
2. $\text{PA} \vdash \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$
3. $\text{PA} \vdash \Box \varphi \rightarrow \Box \Box \varphi$

The system GL of propositional modal logic is axiomatized by adding to the basic modal logic $K$ the following, known as Löb’s axiom: $\Box (\Box A \rightarrow A) \rightarrow \Box A$. It was proven by Solovay (23) that GL is the provability logic of PA: its theorems are exactly the propositional schemata involving $\text{Pr}_{\text{PA}}(x)$ that are provable in PA.

Solovay’s method has been used to apply modal logic to study other metamathematical predicates besides $\text{Pr}_{\text{PA}}(x)$. Shavrukov (22) determined the joint provability logic of ordinary and Feferman provability $\text{Pr}^*_{\text{PA}}(x)$. The joint provability logic of ordinary and Parikh provability was established by Lindström (15).

### 1.3.4 Turing-Feferman progressions

The idea of transfinite iterations of consistency statements goes back to Turing (25). Given a sufficiently strong $\Sigma_1$-sound theory $T$, consider the sequence of theories given by: $T^0 := T$, and $T^{n+1} := T^n + \text{Con}_{T^n}$ for all $n$. It follows from the Second Incompleteness Theorem that each $T^{n+1}$ is a strictly stronger theory than $T^n$. In his doctoral thesis [25], Turing introduced the method of transfinite iterations, allowing one to extend the above sequence into the transfinite. Given a theory $T$ and an ordinal $\alpha$, the theory $T^\alpha$ is informally defined as:

$$T^\alpha = T + \text{Con}(\bigcup_{\beta<\alpha} T^\beta).$$

Returning to transfinite iterations of consistency statements introduced informally in (3) above, we note that using the notation of (5), we have that for $\alpha > 0$, $\text{PA}|_2 + \text{Con}_2^\alpha$ is the theory $(\text{PA}|_2)^\alpha$.

A proper construction of the above sequence of theories requires a *recursive ordinal notation system*. As shown in [25, 6], the properties of $T^\alpha$ for infinite $\alpha$ depend significantly on the choice of the ordinal notation system. These difficulties will not influence our paper, however, as we shall only
2 Arithmetical theories

We consider first-order theories formulated in the language $L$ of arithmetic containing 0, $S$ (successor), $+$, $\times$, and $\leq$. As usual, an $L$-formula is said to be $\Delta_0$ (equivalently: $\Sigma_0$ or $\Pi_0$) if all its quantifiers are bounded, and $\Sigma_{n+1}(\Pi_{n+1})$ if it is of the form $\exists x_0 \ldots x_n \phi$, with $\phi$ a $\Pi_n(\Sigma_n)$-formula. We write $\pi$ for the $L$-term corresponding to $n$, i.e. 0 followed by $n$ applications of $S$. Given this, we shall mostly write $n$ instead of $\pi$.

The basic facts concerning 0, $S$, $+$, $\times$, and $\leq$ are given by the axioms of the theory $\mathbb{Q}$ of Robinson Arithmetic ([13, Definition I.1.1]). The theory $\mathbb{Q}$ is $\Sigma_1$-complete, meaning that it proves every true $\Sigma_1$-sentence.

Our main interest in this article is the theory $\mathbb{PA}$ of Peano Arithmetic that results from adding to $\mathbb{Q}$ the induction schema for all arithmetical formulas. As usual, $I\Sigma_n$ denotes the fragment of $\mathbb{PA}$ obtained by restricting the induction schema to $\Sigma_n$-formulas. Clearly, $I\Sigma_n \subseteq I\Sigma_{n+1}$ for all $n$, and $\mathbb{PA} = \bigcup_{n \in \omega} I\Sigma_n$. We shall therefore sometimes also write $I\Sigma_\omega$ for $\mathbb{PA}$. Using that satisfaction for $\Sigma_n$-formulas is definable in $I\Delta_0$ by a $\Sigma_n$-formula, one can show that for all $n > 0$, $I\Sigma_n$ is finitely axiomatisable ([13, Theorem I.2.52], see also Section 2.2 below).

As our metatheory, we mostly use $I\Delta_0 + \text{Exp}$. In order to introduce the latter, we recall that there is a $\Delta_0$-formula $\psi_e(x, y, z)$ that defines, provably in $I\Delta_0$, the graph of a recursively defined exponentiation function $e(x, y) = x^y$ ([13, Theorem V.3.15]), i.e. we have:

\[
\begin{align*}
I\Delta_0 + \text{Exp} & \vdash \psi_e(x, 0, z) \iff z = 1 \\
I\Delta_0 + \text{Exp} & \vdash \psi_e(x, y + 1, z) \iff \exists w (E(x, y, w) \land z = w \cdot x)
\end{align*}
\]

The sentence stating the totality of this function:

\[
\forall x \forall y \exists ! z \psi_e(x, y, z)
\]  

is not provable in $I\Delta_0$. The theory $I\Delta_0 + \text{Exp}$ is the result from adding (6) as an additional axiom to $I\Delta_0$. We recall that $I\Delta_0 + \text{Exp}$ is finitely axiomatizable ([13, Theorem V.5.6 ]).

Since the formula defining exponentiation in $I\Delta_0$ is $\Delta_0$, $I\Delta_0 + \text{Exp}$ is a conservative extension of Elementary Arithmetic ($\mathbb{EA}$). By $\mathbb{EA}$, we mean the
theory formulated in the language of arithmetic, together with a function symbol \( \exp \) for exponentiation. It contains the basic facts concerning 0, \( S \), +, \( \times \), \( \leq \) and \( \exp \), plus induction for all \( \Delta_0 \)-formulas of the extended language. EA is strong enough to formalize almost all of finitary mathematics outside logic.

2.1 Representing recursive functions in \( \Delta_0^1 + \text{Exp} \)

It is well-known that a coding of sequences can be carried out inside \( \Delta_0^1 + \text{Exp} \). Using that, it is straightforward to show that every primitive recursive relation \( R \) can be represented inside \( \Delta_0^1 + \text{Exp} \) by a \( \Sigma_1 \)-formula \( \varphi_R \), in the sense that for all \( n_0, \ldots, n_k \),

\[
(n_0, \ldots, n_k) \in R \text{ iff } \Delta_0^1 + \text{Exp} \vdash \varphi_R(m_0, \ldots, m_k).
\]  

(7)

We recall that there are primitive recursive functions \( T \) and \( U \) with the property that for all recursive \( f \), there exists some \( e \), such that for all \( n \),

\[
f(n) = U(\mu y T(e, n, y)).
\]  

(8)

Thus we can associate to any recursive function \( f \) a \( \Sigma_1 \)-formula \( \varphi_f \) that defines \( f \) in a natural way, say by mimicking its definition in (8). If \( f \) is \( k \)-ary, then for all \( n_1, \ldots, n_k \), we have that

\[
\Delta_0^1 + \text{Exp} \vdash \varphi_f(n_1, \ldots, n_k, f(n_1, \ldots, n_k)) \quad (9)
\]

\[
\Delta_0^1 + \text{Exp} \vdash \exists z \varphi_f(m_1, \ldots, m_k, z) \quad (10)
\]

Since \( \Delta_0^1 + \text{Exp} \) is \( \Sigma_1 \)-sound, it follows that for any recursively enumerable (r.e.) set \( A \), there is a \( \Sigma_1 \)-formula \( \varphi_A \) such that for all \( n, n \in A \) if and only if \( \Delta_0^1 + \text{Exp} \vdash \varphi_A(n) \). In fact, as was first shown in [4], given any extension \( S \) of \( \Delta_0^1 + \text{Exp} \) and any r.e. set \( A \), there is a \( \Sigma_1 \)-formula \( \varphi_A \) such that for all \( n, n \in A \) if and only if \( S \vdash \varphi_A(n) \).

Given a \( k \)-ary recursive function \( f \), we denote by \( f(x_1, \ldots, x_k) \downarrow \) the formula \( \exists y \varphi_f(x_1, \ldots, x_k, y) \), and say that \( f \) converges on input \( x_1, \ldots, x_k \). Similarly, we denote by \( f(x_1, \ldots, x_k) \uparrow \) the formula \( \neg f(x_1, \ldots, x_k) \downarrow \), and say that \( f \) diverges on input \( x_1, \ldots, x_n \). We use \( f \downarrow \) as shorthand for

\[
\forall x_1 \ldots x_k f(x_1, \ldots, x_k) \downarrow,
\]

and \( f \uparrow \) as shorthand for \( \neg f \downarrow \).
A recursive function $f$ is said to be provably recursive in a theory $S \supset I\Delta_0+\text{Exp}$ if $S \vdash f \downarrow$. The provably recursive functions of $I\Delta_0+\text{Exp}$ are exactly the Kalmar elementary functions. The class of Kalmar elementary functions is the smallest class containing successor, zero, projection, addition, multiplication, substraction, and closed under composition as well as bounded sums and bounded products (\cite{20}). For a characterization of the provably recursive functions of $I\Sigma_n$ for $n \geq 1$, see Theorem \ref{thm:char} in Section \ref{sec:char} below.

2.2 Metamathematics in $I\Delta_0+\text{Exp}$

It is well-known that arithmetization of syntax can be carried out in $I\Delta_0+\text{Exp}$. We assume as given some standard gödelnumbering of $\mathcal{L}$-formulas, and write $\lceil \varphi \rceil$ for the gödelnumber of $\varphi$. We shall often identify a formula with its gödelnumber, writing $\psi(\varphi)$ instead of $\psi(\lceil \varphi \rceil)$.

Let $S$ be a r.e. extension of $I\Delta_0+\text{Exp}$. As explained in Section \ref{sec:arithmetization}, the set of axioms of $S$ can be represented in $I\Delta_0+\text{Exp}$ by a $\Sigma_1$-formula $\varphi_S$. Using the latter, one can define in a natural way a $\Sigma_1$-formula $\text{Pr}_S(\varphi)$ representing provability in $S$ inside $I\Delta_0+\text{Exp}$ (\cite{5, Definition 4.1}). In this paper, we shall write $\text{Pr}_S$ instead of $\text{Pr}_\varphi_S$, having in mind some formula $\varphi_S$ representing the axioms of $S$ in $I\Delta_0+\text{Exp}$ in a natural way, by mimicking their informal definition. We refer to $\text{Pr}_S$ as the standard provability predicate of $S$.

We employ modal notation, writing $\square_S$ instead of $\text{Pr}_S$. We write $\square_0$ as shorthand for $\square_{I\Delta_0+\text{Exp}}$. By $\square_x$ we denote the formula containing $x$ as a free variable, and such that for $n > 0$, $\square_n$ (the result of substituting $\pi$ for $x$ in $\square_x$) is $\square_{I\Sigma_n}$. We write $\square$, or sometimes also $\square_\omega$, for $\square_{\text{PA}}$. We write $\Diamond_S \varphi$ as shorthand for $\neg \square_S \neg \varphi$.

We recall that $\text{PA}$ is essentially reflexive, meaning that it proves the consistency of each of its finite subtheories, and the same holds for every consistent extension of $\text{PA}$ in the language of arithmetic (\cite[Theorem III.2.35]{13}).

We use the dot notation as usual, thus $\square_S \varphi(\bar{x})$ means that the numeral for the value of $x$ has been substituted for the free variable of the formula $\varphi$ inside $\square_S$. If the intended meaning is clear from the context, we will often simply write $\square_S \varphi(x)$ instead of $\square_S \varphi(\bar{x})$. We recall that any theory $S$ extending $I\Delta_0+\text{Exp}$ is provably $\Sigma_1$-complete, meaning that for any $\Sigma_1$-formula $\sigma$,

$$I\Delta_0+\text{Exp} \vdash \sigma(x) \to \square_S \sigma(\bar{x}).$$

It is well-known that if $S$ is as above, then the Hilbert-Bernays-Löb derivability conditions hold for $\square_S$ verifiably in $I\Delta_0+\text{Exp}$.
1. if $S \vdash \varphi$, then $\Delta_0 + \text{Exp} \vdash \Box S \varphi$

2. $\Delta_0 + \text{Exp} \vdash \Box S (\varphi \rightarrow \psi) \rightarrow (\Box S \varphi \rightarrow \Box S \psi)$

3. $\Delta_0 + \text{Exp} \vdash \Box S \varphi \rightarrow \Box S \Box S \varphi$

We note that 2 and 3 also hold with internal variables ranging over $\varphi$ and $\psi$.

**Theorem 1.** Let $\varphi$ be an $L$-formula whose free variables are exactly $x_0, \ldots, x_n$. Then there is an $L$-formula $\psi$ with exactly the same free variables, and such that

$$\Delta_0 + \text{Exp} \vdash \psi (x_1, \ldots, x_n) \leftrightarrow \varphi (\Gamma \psi \neg, x_1, \ldots, x_n).$$  \hspace{1cm} (11)

From the proof of Theorem 1 it is clear that if $\varphi$ is $\Sigma_n (\Pi_n)$, then so is $\psi$.

Verifiability of Löb’s principle for $\Box S$ in $\Delta_0 + \text{Exp}$ follows from the Hilbert-Bernays-Löb derivability conditions for $\Box S$, together with Theorem 1 ([2, Theorem 3.2]). This means that

$$\Delta_0 + \text{Exp} \vdash \Box S (\Box S \varphi \rightarrow \varphi) \rightarrow \Box S \varphi,$$  \hspace{1cm} (12)

and thus modal principles valid in the Gödel-Löb provability logic $GL$ can be used when reasoning about $S$ in $\Delta_0 + \text{Exp}$.

For $n \geq 1$, there is a partial truth definition $\text{True}_{\Pi_n}(x)$ ($\text{True}_{\Sigma_n}(x)$) in $\Delta_0 + \text{Exp}$ for the class $\Pi_n$ ($\Sigma_n$) [13, V.5(b)]. Thus for every $\varphi \in \Pi_n$ ($\varphi \in \Sigma_n$) we have

$$\Delta_0 + \text{Exp} \vdash \varphi \leftrightarrow \text{True}_{\Pi_n}(\varphi) \quad (\Delta_0 + \text{Exp} \vdash \varphi \leftrightarrow \text{True}_{\Sigma_n}(\varphi)).$$

Moreover, $\text{True}_{\Pi_n}(x)$ and $\text{True}_{\Sigma_n}(x)$ satisfy Tarski’s conditions (see [13, Definition I.1.74]). For all $n \geq 1$, $\text{True}_{\Pi_n}(x)$ is a $\Pi_n$-formula, and $\text{True}_{\Sigma_n}(x)$ is a $\Sigma_n$-formula.

Suppose $\alpha \in [1, \omega]$ and $n \geq 1$. By $\Box \alpha$, we denote the provability predicate for the theory $\Sigma_{\alpha}$ extended by all true $\Pi_n$ sentences. We formalize $\Box \alpha \varphi$ in a natural way using a partial truth definition:

$$\exists \psi \in \Pi_n \text{-Sen}(\text{True}_{\Pi_n}(\psi) \land \Box \alpha (\psi \rightarrow \varphi)),$$

where $\Pi_n \text{-Sen}$ denotes the set of all Gödel numbers of $\Pi_n$-sentences. Here we can have quantifiers over $\alpha$ but not over $n$ in the language of arithmetic. We use $\Diamond \alpha$ to denote the dual of $\Box \alpha$, i.e. $\Diamond \alpha \varphi := \neg \Box \alpha \neg \varphi$. The sentence $\Diamond \alpha \varphi$ is equivalent to uniform $\Pi_{n+1}$-reflection for $\Sigma_{\alpha} + \varphi$, i.e. the principle saying that if for some $\Pi_{n+1}$-formula $\psi(x)$ and every $m$ the theory $\Sigma_{\alpha} + \varphi$ proves $\psi(m)$, then $\forall x \psi(x)$ is true.
3 Ordinals and the fast-growing hierarchy

We introduce a certain fast-growing hierarchy of recursive functions indexed by ordinals below $\varepsilon_0$. We recall the basic facts concerning this hierarchy, including a characterization of the provably recursive functions of $\S_\Sigma_n$, for $n > 0$.

In order to define the fast-growing functions, and to talk about them in our arithmetical theories, we need to represent ordinals below $\varepsilon_0$ as natural numbers. For that, it is useful to recall the Cantor normal form theorem:

**Theorem 2.** For every ordinal $\alpha > 0$, there exist unique $\alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_k$ with

$$\alpha = \omega^{\alpha_0} + \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}.$$ 

The above representation of $\alpha$ is called its Cantor normal form. Since $\varepsilon_0$ is the least ordinal $\varepsilon$ for which it holds that $\varepsilon = \omega^\varepsilon$, we see that if $\alpha < \varepsilon_0$, then $\alpha$ has a Cantor normal form with exponents $\alpha_i < \alpha$, and these exponents in turn have Cantor normal form with yet smaller exponents. We represent an ordinal $\alpha$ below $\varepsilon_0$ by either the symbol 0 if $\alpha = 0$, or otherwise its Cantor normal form

$$\omega^{\alpha_0} + \omega^{\alpha_1} + \ldots + \omega^{\alpha_k},$$

where each $\alpha_i$ is represented in the same way. More formally, this means that for any ordinal below $\varepsilon_0$, we fix a term built $\omega^x$, $x + y$, and 0. This method, known as Cantor ordinal notation system, is the most common way of representing ordinals below $\varepsilon_0$.

In order to work with the above terms in arithmetic, we represent them as their Gödel numbers. We note that the predicate $<$ and the standard functions of ordinal arithmetic ($x + y$, $x \cdot y$ and $\omega^x$) on Cantor ordinal notations can be expressed in the language of arithmetic. Basic facts about ordinal arithmetic can be easily proven in $\op{I}_\Delta_0 + \op{Exp}$ ([23 Section 3]); we will omit the details of this formalization in our proofs.

### 3.1 The fast-growing hierarchy

For an ordinal number $\alpha$ and $n < \omega$, we define $\omega_n^\alpha$ by $\omega_0^\alpha := \alpha$, and $\omega_{n+1}^\alpha = \omega^{\omega_n^\alpha}$. We write $\omega_n$ for $\omega_n^1$. Thus $\omega_0 = 1$, $\omega_1 = \omega$, $\omega_2 = \omega^\omega$, etc. It is well-known that the ordinal $\varepsilon_0$ can also be characterized as $\sup\{\omega_n \mid n \in \omega\}$; we therefore define $\omega := \varepsilon_0$. 

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A fundamental sequence for a countable limit ordinal $\lambda$ is a strictly monotone sequence $\{\lambda[n]\}_{n \in \omega}$ converging to $\lambda$ from below, i.e. $\lambda[n] < \lambda[n+1] < \lambda$ for all $n < \omega$, and $\sup\{\lambda[n] \mid n \in \omega\} = \lambda$. We consider the standard assignment of fundamental sequences to limit ordinals below $\varepsilon_0$.

**Definition 1.** Let $\varepsilon_0[n] := \omega_{n+1}$. For a limit ordinal $\lambda < \varepsilon_0$ with Cantor normal form $\lambda = \omega^{\alpha_0} + \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$, we define $\lambda[n]$ as follows:

- If $\alpha_k$ is a successor ordinal, let $\lambda[n] := \omega^{\alpha_0} + \omega^{\alpha_1} + \ldots + \omega^{(\alpha_k-1)} \cdot (n+1)$
- If $\alpha_k$ is a limit ordinal, let $\lambda[n] := \omega^{\alpha_0} + \omega^{\alpha_1} + \ldots + \omega^{\alpha_k[n]}$

Given a function $F : \mathbb{N} \to \mathbb{N}$, we use exponential notation to denote repeated compositions of $F$, thus $F^0(x) = x$, and $F^{n+1}(x) = F(F^n(x))$.

**Definition 2.** The fast-growing hierarchy $\{F_\alpha\}_{\alpha \leq \varepsilon_0}$ of recursive functions is given by:

- $F_0(n) = n + 1$
- $F_{\alpha+1}(n) = F_\alpha^{n+1}(n)$
- $F_\lambda(n) = F_{\lambda[n]}(n)$

This exact version of the fast-growing hierarchy was first introduced by Solovay and Ketonen in [14]. Their results, together with results of Paris in [19], imply the following classification of the provably recursive functions of PA:

**Theorem 3.** For $n > 0$, $\Sigma_n \vdash F_\alpha \downarrow \iff \alpha < \omega_n$.

The computation of $F_\alpha(n)$ is closely connected to the following stepdown relation on ordinals.

**Definition 3.** For any ordinals $\alpha, \beta \leq \varepsilon_0$ and numbers $n, r$ we write $\alpha \xrightarrow{n} \beta$ if there exists a sequence $\gamma_0, \ldots, \gamma_r$ such that $\gamma_0 = \alpha$, $\gamma_r = \beta$, and for all $0 \leq i < r$, $\gamma_{i+1} = \gamma_i[n]$ if $\gamma_i$ is a limit ordinal and $\gamma_{i+1} + 1 = \gamma_i$, otherwise. We write $\alpha \xrightarrow{r} \beta$ in case $\alpha \xrightarrow{n} \beta$ for some $r$.

**Lemma 1** ($\Delta_0 + \text{Exp}$). ([9, Lemma 2.3, Lemma 2.4])

1. If $\alpha \xrightarrow{n} \beta$ and $F_\alpha(n) \downarrow$ then $F_\beta(n) \downarrow$ and $F_\alpha(n) \geq F_\beta(n)$.
2. If \( F_\alpha(n) \downarrow \) and \( n > m \), then \( F_\alpha(m) \downarrow \) and \( F_\alpha(n) \geq F_\alpha(m) \).

3. If \( \alpha > \beta \) and \( F_\alpha \downarrow \) then \( F_\beta \downarrow \).

4. If \( i > 0 \) and \( \mathcal{F}_i(n) \downarrow \) then \( n < \mathcal{F}_i(n) \).

5. If \( F_\alpha(n) \downarrow \) then \( \alpha \rightarrow n \rightarrow 0 \).

Lemma 2 (PA). \cite[Lemma 2.10]{Lemma} Suppose \( \alpha, \beta < \varepsilon_0 \), \( n \) is a number, \( \omega^\alpha \rightarrow n \rightarrow 0 \), and \( \alpha \rightarrow n \rightarrow \beta \). Then \( \omega^\alpha \rightarrow n \rightarrow \omega^n \).

Lemma 3 (PA). For all numbers \( k, n, \) and \( s \) if \( \omega^{k+1} \rightarrow n \rightarrow 0 \) then \( \omega^{k+1} \rightarrow n \rightarrow \omega^n \).

Lemma 4 (PA). For all numbers \( k, n, \) and \( s \geq 1 \) if \( \omega^{k+1} \rightarrow n \rightarrow 0 \) then \( \omega^{k+1} \rightarrow n \rightarrow \omega^n + 1 \).

Proof. By the Lemma 3 we have \( \omega^{k+1} \rightarrow n \rightarrow \omega^n \). We show that on the step before \( \omega^n \) on the \( \rightarrow n \rightarrow \omega^n \)-path from \( \omega^{k+1} \rightarrow n \rightarrow \omega^n \), we have \( \omega^n + 1 \) and hence the lemma holds. Case consideration shows that there are at most two possible \( \alpha \)-s such that \( \alpha \rightarrow n \rightarrow \omega^n \): the ordinal \( \alpha = \omega_n + 1 \) and the ordinal \( \alpha = \omega_n + 1 \) if \( s + 1 = k \). But because \( \omega_{n+1} > \omega^{k+1} \), any \( \rightarrow n \rightarrow \omega^n \)-chain from \( \omega^{k+1} \rightarrow n \rightarrow \omega^n \) should go through \( \omega+n+1 \). Hence \( \omega^{k+1} \rightarrow n \rightarrow \omega^n + 1 \).

Lemma 5 (PA). For all numbers \( k, n, \) and \( m \leq n \) if \( \omega^{k+1} \rightarrow k \rightarrow 0 \) then \( \omega^{k+1} \rightarrow n \rightarrow \omega^{k+1} \).

Proof. From the definition of \( \rightarrow k \) it follows that \( k + 1 \rightarrow k \). Thus from Lemma 2 it follows that \( \omega^{k+1} \rightarrow k \rightarrow \omega \). We have \( \omega \rightarrow k + 1 \). Hence \( \omega^{k+1} \rightarrow k \rightarrow k + 1 \). Now we use the latter and Lemma 2 to prove the lemma by induction on \( n \).

Lemma 6 (PA). If \( m \leq n, \alpha \rightarrow m \rightarrow \beta \), and \( \mathcal{F}_\alpha(n) \downarrow \) then \( \mathcal{F}_\beta(n) \downarrow \) and \( \mathcal{F}_\alpha(n) \geq \mathcal{F}_\beta(n) \).
Proof. Using Lemma 1 it is sufficient to show that \( \alpha \rightarrow^n \beta \). Consider the only sequence \( \gamma_0, \ldots, \gamma_r \) such that \( \gamma_0 = \alpha \), \( \gamma_r = \beta \), and \( \gamma_i \rightarrow^m \gamma_{i+1} \), for all \( i < r \). We show by induction on \( i \) that for any \( i < r \) we have \( \gamma_i \rightarrow^n \gamma_{i+1} \) and \( F_{\gamma_{i+1}}(n) \downarrow \). We first show that \( \gamma_i \rightarrow^n \gamma_{i+1} \), assuming that \( F_{\gamma_i}(n) \downarrow \) (we have it either from induction assumption or if \( i = 0 \) we have it because \( F_\alpha(n) \downarrow \)). We consider two cases: \( \gamma_i \) is a limit ordinal and \( \gamma_i \) is a successor ordinal. The case of successor ordinal is trivial. If \( \gamma_i \) is a limit ordinal then from \( F_{\gamma_i}(n) \downarrow \) it follows that \( F_{\gamma_i}[n](n) \downarrow \) and thus by Lemma 1 we have \( \gamma_i[n] \rightarrow^n \gamma_{i+1} \). Hence \( \gamma_i \rightarrow^n \gamma_{i+1} \). Now from Lemma 1 it follows that \( F_{\gamma_{i+1}}(n) \downarrow \). This finishes our inductive proof. Since we have \( \gamma_i \rightarrow^n \gamma_{i+1} \), for any \( i < r \), we clearly have \( \gamma_0 \rightarrow^n \gamma_r \), i.e. \( \alpha \rightarrow^n \beta \). \( \square \)

3.2 Transfinite induction

Using the representation of ordinals in \( \text{PA} \), we can formulate the schema of transfinite induction. For an ordinal \( \alpha \leq \varepsilon_0 \) and a number \( n \geq 0 \), we write \( \text{TI}_{\Pi_n-\alpha} \) for the following schema:

\[
\forall \beta < \alpha \ (\forall \gamma < \beta \varphi(\gamma) \rightarrow \varphi(\beta)) \rightarrow \forall \gamma < \alpha \varphi(\gamma),
\]

(13)

where \( \varphi \) is a \( \Pi_n \)-formula. Since there is a \( \Pi_n \)-partial truth definition \( \text{True}_{\Pi_n} \) in \( I\Delta_0 + \text{Exp} \) (see Section 2.2), there is an instance of the schema that implies all other instances of it in \( I\Delta_0 + \text{Exp} \). We can thus identify \( \text{TI}_{\Pi_n-\alpha} \) with this instance, and consider \( \text{TI}_{\Pi_n-\alpha} \) to be a single formula.

It follows from Gentzen’s work in [10] that \( \text{PA} \) proves \( \text{TI}_{\Pi_n-\alpha} \) for all \( n \) and \( \alpha < \varepsilon_0 \), and that it does not prove \( \text{TI}_{\Pi_0-\varepsilon_0} \). For a treatment of the amount of transfinite induction available in the fragments \( I\Sigma_n \) of \( \text{PA} \), see for example [24].

Suppose we argue in \( \text{PA} \), and want to show that a certain property \( \varphi \) holds for all ordinals less than some \( \alpha < \varepsilon_0 \). By the above, it suffices to show that

\[
\forall \beta < \alpha \ (\forall \gamma < \beta \varphi(\gamma) \rightarrow \varphi(\beta)).
\]

(14)

A formula \( \varphi \) for which (11) holds will be called \textit{progressive}. We note that for any \( \alpha \leq \varepsilon_0 \), \( I\Sigma_1 \) verifies that the formula \( F_\alpha \downarrow \) is progressive, i.e. that

\[
\forall \gamma < \beta F_\gamma \downarrow \rightarrow F_\beta \downarrow.
\]

(15)
To see that (15) holds, note that by Definition 2, the following are verifiable in $I\Sigma_1$:

1. $F_0 \downarrow$
2. $\forall \alpha (F_\alpha \downarrow \rightarrow F_{\alpha+1} \downarrow)$
3. $\forall \lambda \leq \varepsilon_0 (\lambda \in Lim \rightarrow (\forall \alpha < \lambda F_\alpha \downarrow \rightarrow F_\lambda \downarrow))$

Thus whether a function $F_\alpha$ (for some $\alpha \leq \varepsilon_0$) is provably total in some extension $T$ of $I\Sigma_n$ depends on the amount of transfinite induction available in $T$.

4 Transfinite iterations of provability predicates

In the present section we will give precise definitions of transfinite iterations of provability predicates and their duals. These notions are closely related to Turing-Feferman progressions [25, 6]. Our presentation of this subject is based on the approach from [11] which itself is based on [21].

**Definition 4.** We say that $(D, \prec)$ is an elementary linear ordering if $D$ is a subset of the natural numbers, for both $D$ and $\prec$ there are fixed bounded formulas of the language of $EA$ that define them, and $EA$ proves that $(D, \prec)$ is a linear ordering.

Note, that for any elementary well-ordering $(D, \prec)$ there are $\Sigma_1$ formulas of the language of first-order arithmetic $L$ that are equivalent in $EA$ to the standard defining formulas for $D$ and $\prec$. Because $EA$ is a conservative extension of $I\Delta_0 + \text{Exp}$, the choice of the formulas above is unique up to $I\Delta_0 + \text{Exp}$-provable equivalence. Thus we can freely talk about provability of facts about an elementary well ordering within theories containing $I\Delta_0 + \text{Exp}$.

We will define transfinite iterations of provability predicates. Reflexive induction is an important method of reasoning about such iterations.

**Lemma 7.** [1, Lemma 2.4/21] For any elementary linear ordering $(D, \prec)$, any theory $T$ extending $I\Delta_0 + \text{Exp}$ is closed under the following reflexive induction rule:

$$
\forall \alpha \in D \ (\square_T \forall \beta \prec \alpha F(\beta) \rightarrow F(\alpha)) \quad \Rightarrow \quad \forall \alpha \in DF(\alpha)
$$
Proof. Suppose $T \vdash \forall \alpha \in D ((\Box_T \forall \beta \prec \alpha F(\beta)) \rightarrow F(\alpha))$. Then the sentence with stronger assumption is also derivable:

$$T \vdash \forall \alpha \in D ((\Box_T \forall \beta \in D F(\beta)) \rightarrow F(\alpha)).$$

We can also weaken the conclusion:

$$T \vdash \Box_T \forall \alpha \in D F(\alpha) \rightarrow \forall \alpha \in D F(\alpha).$$

Therefore by Löb’s theorem we have

$$T \vdash \forall \alpha \in D F(\alpha).$$

We fix for the rest of the section a $\Sigma_1$-provability predicate $\triangle$ for an arithmetical theory $T$ containing $\text{l}\Delta_0+\text{Exp}$ that satisfies Hilbert-Bernays-Löb derivability conditions verifiably in $\text{l}\Delta_0+\text{Exp}$. We denote by $\triangledown$ the dual consistency predicate for $\triangle$. Also, we fix an elementary linear ordering $(D, \prec)$ such that the least element of $(D, \prec)$ is $0^D$ and the fact that $0^D$ is the least element of $(D, \prec)$ is verifiable $\text{l}\Delta_0+\text{Exp}$.

We define iterations of $\triangle$ along $(D, \prec)$: $\triangle^\alpha \varphi$, where $\alpha \in D$ and $\varphi$ is an arithmetical sentence. An iteration $\triangle^x y$ is an arithmetical formula with two free variables such that

$$\text{l}\Delta_0+\text{Exp} \vdash \forall \varphi \forall \alpha \in D \setminus \{0^D\} (\triangle^\alpha \varphi \leftrightarrow \exists \beta \prec \alpha \triangle \beta \varphi).$$

Here and below, if we refer to Gödel numbers of iterations, we could also use zero times iterations. We define $\triangle^0 \varphi$ to be equal to $\varphi$, i.e. more formally, $\triangle \beta \varphi$ should be written as

$$(\beta = 0^D \rightarrow \varphi) \land (\beta \neq 0^D \rightarrow \triangle \beta \varphi).$$

Existence of iterations follows from the Diagonal Lemma (Theorem 1). Simple inspection of the last argument shows that the resulting formula is $\Sigma_1$. Actually any two iterations are $\text{l}\Delta_0+\text{Exp}$-provably equivalent (this fact resembles uniqueness of smooth progressions [1]).

**Lemma 8.** For any two iterations $(\triangle^x y)_1$ and $(\triangle^x y)_2$ of $\triangle$ along $(D, \prec)$ we have

$$\text{l}\Delta_0+\text{Exp} \vdash \forall \alpha \in D \setminus \{0^D\} \forall \varphi((\triangle^\alpha \varphi)_1 \leftrightarrow (\triangle^\alpha \varphi)_2)).$$

(16)
Proof. We use reflexive induction to prove it. We need to show that
\[ I \Delta_0 + \text{Exp} \vdash \forall \alpha \in D \setminus \{0^D\} (\Box_0 \forall \beta < \alpha \forall \varphi (\Delta^\beta \varphi)_1 \leftrightarrow (\Delta^\beta \varphi)_2) \rightarrow \forall \varphi ((\Delta^\alpha \varphi)_1 \leftrightarrow (\Delta^\alpha \varphi)_2). \]  \hspace{1cm} (17)

By definition of an iteration the latter will follow from
\[ I \Delta_0 + \text{Exp} \vdash \forall \alpha \in D \setminus \{0^D\} (\Box_0 \forall \beta < \alpha \forall \varphi (\Delta^\beta \varphi)_1 \leftrightarrow (\Delta^\beta \varphi)_2) \rightarrow \forall \varphi (\exists \beta < \alpha \Delta(\Delta^\beta \varphi)_1 \leftrightarrow \exists \beta < \alpha \Delta(\Delta^\beta \varphi)_2)). \]  \hspace{1cm} (18)

Because there is a symmetry between \((\Delta x y)_1\) and \((\Delta x y)_2\), it is enough to show that
\[ I \Delta_0 + \text{Exp} \vdash \forall \alpha \in D \setminus \{0^D\} (\Box_0 \forall \beta < \alpha \forall \varphi (\Delta^\beta \varphi)_1 \rightarrow (\Delta^\beta \varphi)_2) \rightarrow \forall \varphi (\exists \beta < \alpha \Delta(\Delta^\beta \varphi)_1 \rightarrow \exists \beta < \alpha \Delta(\Delta^\beta \varphi)_2)). \]  \hspace{1cm} (19)

Clearly, we have
\[ I \Delta_0 + \text{Exp} \vdash \forall \alpha \in D \setminus \{0^D\} (\Box_0 \forall \beta < \alpha \forall \varphi (\Delta^\beta \varphi)_1 \rightarrow (\Delta^\beta \varphi)_2) \rightarrow \forall \beta < \alpha \forall \varphi \Box_0 ((\Delta^\beta \varphi)_1 \rightarrow (\Delta^\beta \varphi)_2)). \]  \hspace{1cm} (20)

Because \(T\) contains \(I \Delta_0 + \text{Exp}\) we have
\[ I \Delta_0 + \text{Exp} \vdash \forall \varphi \forall \beta \in D \setminus \{0^D\} (\Box_0 ((\Delta^\beta \varphi)_1 \rightarrow (\Delta^\beta \varphi)_2) \rightarrow \Delta((\Delta^\beta \varphi)_1 \rightarrow (\Delta^\beta \varphi)_2)). \]

Thus
\[ I \Delta_0 + \text{Exp} \vdash \forall \varphi \forall \beta \in D \setminus \{0^D\} (\Box_0 ((\Delta^\beta \varphi)_1 \rightarrow (\Delta^\beta \varphi)_2) \rightarrow (\Delta(\Delta^\beta \varphi)_1 \rightarrow \Delta(\Delta^\beta \varphi)_2)). \]

Hence (19) holds and we have (16). \(\Box\)

In the same fashion as iterations of \(\triangle\) we define the dual notion of iterations \(\nabla^x y\) of \(\nabla\). \(\nabla^x y\) is an arithmetical formula with two free variables such that
\[ I \Delta_0 + \text{Exp} \vdash \forall \varphi \forall \alpha \in D \setminus \{0^D\} (\nabla^\alpha \varphi \leftrightarrow \forall \beta < \alpha (\nabla \nabla^\beta \varphi)). \]

Existence of iterations \(\nabla^x y\) again follows from Diagonal Lemma.
Lemma 9. For any two iterations \((\nabla^x y)_1\) and \((\nabla^x y)_2\) of \(\nabla\) along \((D, \prec)\) we have
\[
!\Delta_0 + \text{Exp} \vdash \forall \alpha \in D \setminus \{0_D\} \forall \varphi ((\nabla^\alpha \varphi)_1 \leftrightarrow (\nabla^\alpha \varphi)_2)).
\] (21)

Proof. Can be proved in the same fashion as Lemma 8.

Because we have existence and uniqueness (up to provable equivalence), we use iterations \(\Delta^\alpha \varphi\) and \(\nabla^\alpha \varphi\) freely, without specifying explicit formulas.

Let us denote by \(\text{Succ}_D(\alpha, \beta)\) the formula
\[
\alpha \in D \land \beta \in D \land \alpha \prec \beta \land \forall \gamma \in D \neg (\alpha \prec \gamma \land \gamma \prec \beta).
\]

Let us denote by \(D^\text{lim}\) the set of all \(\alpha \in D\) such that
\[
\alpha \neq 0_D \land \forall \beta \in D \neg \text{Succ}_D(\beta, \alpha).
\]

Lemma 10 \((!\Delta_0 + \text{Exp})\). The following properties of iterations of \(\Delta\) and \(\nabla\) hold:

1. \(\forall \varphi \forall \alpha \in D \setminus \{0_D\} (\neg \Delta^\alpha \neg \varphi \leftrightarrow \nabla^\alpha \varphi)\);
2. \(\forall \varphi \forall \alpha \in D \setminus \{0_D\} (\Delta \varphi \rightarrow \Delta^\alpha \varphi)\);
3. \(\forall \varphi \forall \alpha, \beta \in D \setminus \{0_D\} (\alpha \prec \beta \rightarrow (\Delta^\alpha \varphi \rightarrow \Delta^\beta \varphi))\);
4. \(\forall \alpha \in D \setminus \{0_D\} \forall \varphi, \psi (\Delta^\alpha (\varphi \rightarrow \psi) \rightarrow (\Delta^\alpha \varphi \rightarrow \Delta^\alpha \psi))\);
5. \(\forall \alpha \in D \setminus \{0_D\} \forall \varphi \in \Sigma_1\text{-Sen}(\text{True}_{\Sigma_1}(\varphi) \rightarrow \Delta^\alpha \varphi)\);
6. \(\forall \alpha \in D \setminus \{0_D\} \forall \varphi (\Delta^\alpha \varphi \rightarrow \Delta^\alpha \Delta^\alpha \varphi)\);
7. \(\forall \alpha \in D \setminus \{0_D\} \forall \beta (\text{Succ}_D(\beta, \alpha) \rightarrow \forall \varphi (\Delta^\alpha \varphi \leftrightarrow \Delta^\beta \varphi))\);
8. \(\forall \alpha \in D^\text{lim} \forall \varphi (\Delta^\alpha \varphi \leftrightarrow \exists \beta \prec \alpha (\beta \in D \setminus \{0_D\} \land \Delta^\beta \varphi))\).

Proof. The proof of items 1, 2, 3, 4, and 5 is straightforward by using reflexive induction. Item 6 follows from item 5. We prove item 7 using item 3 and we prove item 8 using items 7 and 3.

Lemma 10 gives us a number of facts about iterations of provability predicates and their duals. We will frequently use them below without explicitly referring to them.

Note that items 2, 4, 6 of Lemma 10 yield \(!\Delta_0 + \text{Exp}\)-verifiable Hilbert-Bernays-Löb derivability conditions for each \(\Delta^\alpha\). The latter with Lemma 10 item 8 means that each \(\nabla^\alpha\) is dual for the provability predicate \(\Delta^\alpha\).
Lemma 11. Suppose $T$ is an arithmetical theory such that $\Delta_0+\text{Exp} \subseteq T$, $\Delta_1$ and $\Delta_2$ are $\Sigma_1$ provability predicates that satisfy $\Delta_0+\text{Exp}$-verifiable Hilbert-Bernays-Löb derivability conditions, and $T \vdash \forall \varphi (\Delta_1 \varphi \to \Delta_2 \varphi)$. Then

$$T \vdash \forall \alpha \in D \setminus \{0^0\} \forall \varphi (\Delta_1^\alpha \varphi \to \Delta_2^\alpha \varphi).$$

Proof. By reflexive induction. \qed

For the rest of the section we assume that $(D, \prec)$ is Cantor ordinal notations for the ordinals $\leq \varepsilon_0$ as defined in Section 3.

Lemma 12 ($\Delta_0+\text{Exp}$). $\forall \alpha \geq 1 \forall \beta \leq \varepsilon_0 \forall \varphi (\Delta^\alpha \Delta^\beta \varphi \leftrightarrow \Delta^{\beta+\alpha} \varphi)$

Proof. By reflexive induction on $\alpha$. \qed

5 Slow provability

Suppose that the theory $S$ is given by a uniform r.e. enumeration $\{S_n\}_{n \in \omega}$. We can use any (partial) recursive function $f$ to “slow down” provability in $S$, by considering the theory

$$S|_f := \Delta_0+\text{Exp} \cup \bigcup \{S_n \mid f(n) \downarrow\}. \tag{22}$$

The reason for adding $\Delta_0+\text{Exp}$ is to ensure that all our theories exhibit a minimal amount of nice behaviour (see Section 2). From the definition, it is clear that $S|_f$ is a r.e. subtheory of $S$. If $f$ is total, then $S|_f$ has exactly the same theorems as $S$. However this fact may not be verifiable in a theory where $f$ is not provably total.

Since $\{S_n\}_{n \in \omega}$ is uniformly r.e., there is an arithmetical formula $\Box_{S_x}$, containing $x$ as a free variable, such that $\Box_{S_n}$ is the provability predicate of $S_n$. We assume that, verifiably in $\Delta_0+\text{Exp}$, $\Box_{S} \varphi$ is provably equivalent to $\exists x \Box_{S_x} \varphi$. The provability predicate $\Box_{S|_f}$ of $S|_f$ can be defined in a natural way:

Definition 5. The provability predicate $\Box_{S|_f}$ of $S|_f$ is defined as

$$\Box_{S|_f} \varphi := \Box_0 \varphi \lor \exists y \leq x (\Box_{S_y} \varphi \land f(x) \downarrow).$$

If $\{S_n\}_{n \in \omega}$ is $\{\Sigma_n\}_{n \in \omega}$, we write $\Box_f$ instead of $\Box_{S|_f}$. 

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If \( f \) is total but not provably total in \( T \), then from the point of view of \( T \) the formula \( \Box_{S \cdot f} \) is a nonstandard provability predicate for \( S \). On the other hand, \( \Box_{S \cdot f} \) is a standard \( \Sigma_1 \)-provability predicate for the r.e. theory \( S \rceil_f \). It therefore satisfies the Hilbert-Bernays-Löb derivability conditions verifiably in \( \Delta_0 + \text{Exp} \) (Section 2.2).

With Definition 5, the usual provability predicate for \( \text{PA} \) can be written as \( \Box_f \), where \( f \) is any Kalmar elementary function (assuming \( \Delta_0 + \text{Exp} \) as our metatheory). It is easy to see that for any \( f \), \( \Box_f \phi \) is provably equivalent in \( \Delta_0 + \text{Exp} \) to the formula:

\[
\exists x (\Box x \phi \land f(n) \downarrow).
\]

(23)

We recall the slow provability predicate studied in [9], defined as:

\[
\exists x (\Box x \phi \land F_{\varepsilon_0}(x) \downarrow).
\]

(24)

We define, for \( z \in \mathbb{Z} \),

\[
F^{(z)}_{\varepsilon_0}(x) := F_{\varepsilon_0}(x - z),
\]

(25)

and consider the provability predicates \( \Box_{F^{(z)}_{\varepsilon_0}} \). For the sake of readability, we let

\[
\Box_z \phi := \Box_{F^{(z)}_{\varepsilon_0}} \phi.
\]

(26)

Thus the provability predicate in (24) becomes \( \Box_0 \).

**Remark 1.** For any \( z \), we can define a “shifted” enumeration \( \{T^z_n\}_{n \in \omega} \) of \( \text{PA} \), such that \( \Box_z \) is provably equivalent to \( \Box_{T^z_{\varepsilon_0}} \), by simply defining \( T^z_z \) as \( \Sigma_x + z \).

In [9], Theorem 4 below is proven for \( \Box_0 \). In order to consider the more general case, we need one more definition.

**Definition 6.** We say that \( \{S_n\}_{n \in \omega} \) is a recursive sequence of finitely axiomatizable theories if there is a recursive sequence \( \{S_n\rceil_{\text{Ax}}\}_{n \in \omega} \) such that for all \( n \), \( S_n \) is axiomatized by \( S_n\rceil_{\text{Ax}} \), and \( S_n\rceil_{\text{Ax}} \) is finite.

**Theorem 4.** Suppose \( S_n \) is a recursive sequence of finitely axiomatizable theories such that \( \text{PA} \) proves that \( \Delta \equiv \bigcup_{n \in \omega} S_n \). Let \( \Delta \) denote the provability predicate \( \Box_{S \cdot f_{\varepsilon_0}} \). Then \( \text{PA} \vdash \forall \phi (\Box \Delta \phi \rightarrow \Box \phi) \).

**Proof.** Essentially the same as [9, Theorem 4.1]. See also Theorem 11 in Section 7 below.
It follows that, from the point of view of $\mathsf{PA}$, any provability predicate $\triangle$ as in the statement of Theorem 4 defines in fact a weaker theory than $\Box$:

**Corollary 13.** Suppose $S_n$ is a recursive sequence of finitely axiomatizable theories such that $\mathsf{PA}$ proves that $\mathsf{PA} = \bigcup_{n \in \omega} S_n$. Let $\triangle$ denote the provability predicate $\Box_{S_n,F_{\varphi}}$. Then $\mathsf{PA} \not\vdash \Box \bot \rightarrow \triangle \bot$.

**Proof.** Suppose that $\mathsf{PA} \vdash \Box \bot \rightarrow \triangle \bot$. Since $\triangle \varphi$ implies $\Box \varphi$ for all $\varphi$, we have

$$\mathsf{PA} \vdash \triangle \bot \rightarrow \Box \bot, \quad (27)$$

whence by Theorem 4

$$\mathsf{PA} \vdash \triangle \bot \rightarrow \Box \bot. \quad (28)$$

Combining this with our assumption yields $\mathsf{PA} \vdash \triangle \bot \rightarrow \triangle \bot$. By Löb’s Theorem for $\triangle$ (this follows from the Hilbert-Bernays-Löb derivability conditions for $\triangle$), we now have that $\mathsf{PA} \vdash \triangle \bot$ whence also $\mathsf{PA} \vdash \Box \bot$, contradiction. $\Box$

Theorem 4 holds for a rather wide class of provability predicates $\triangle$. In Section 9 below, we determine the joint provability logic of any such $\triangle$ and ordinary $\mathsf{PA}$-provability. In contrast, the following sections provide examples of properties where the exact axiomatization of slow provability leads to a radical difference in the behaviour of the corresponding provability predicates. In particular, we show that

$$\mathsf{PA} \vdash \Box_{1}^{\varepsilon_0} \varphi \leftrightarrow \Box \varphi, \quad (29)$$

whereas

$$\mathsf{PA} \vdash \Box_{2}^{\omega} \varphi \leftrightarrow \Box \varphi, \quad (30)$$

where $\Box_{\alpha}^{\omega}$ denotes the $\alpha$-iteration of $\Box_{\omega}$ (see Section 4). In Section 8, we shall furthermore show that there is a function $r$ such that

$$\mathsf{PA} \vdash \Box_{r} \Box_{r} \varphi \leftrightarrow \Box \varphi. \quad (31)$$

Thus the slow provability predicates $\Box_{1}$, $\Box_{2}$, and $\Box_{r}$ may be seen as the $\varepsilon_0$-root, the $\omega$-root, and the square root of ordinary provability, respectively.
6 Provability implies iterated slow provability

In the section we will show that in some cases provability of an arithmetical sentence in PA implies provability of the same sentence with respect to certain transfinite iterations of some slow provability predicates.

Lemma 14 (IΔ₀ + Exp). For all n and k we have □ₙFₙ(₁)(k) ↓.

Proof. In the case of n = 0 the statement of lemma holds because F₀(₁)(x) = F₁(x) = 2x + 1. In the rest of the proof we consider the case of n ≥ 1.

Note that we have □ₙ(Fₙ(₁)(k) ↓ ↔ Fₙ₊₁(k) ↓). Thus it is enough to show that for every α < ωₙ we have □₁∀α ≤ ε₀ (TIΠ₂−₁ → Fₐ ↓). Hence it is enough to show that for all α < ωₙ we have □nTIΠ₂−₁.

If n = 1 then we only need to show that for any number m we have □₁TIΠ₂−₁. Clearly, for every m we have IΔ₀ + Exp ⊢ TIΠ₂−₁ → TIΠ₂−₁ + 1. Therefore for any m we have IΔ₀ + Exp ⊢ TIΠ₂−₁ and hence □₁TIΠ₂−₁. Thus we will consider only the case of n ≥ 2.

In [24, Theorem 4.1] it has been shown that if 0 < m ≤ n and ω ≤ α < ε₀ then IΔ₀ proves that TIΠ₂−₁ implies TIΠ₂−₁ for all β < ωₙ−₁. A simple inspection of the proof shows that the argument could be formalized in IΔ₀ + Exp. Thus for all α < ωₙ we have □₁(TIΠ₂−₁ → TIΠ₂−₁). But TIΠ₂−₁ is just Π₂-Induction for natural numbers and is well-known to be equivalent in IΔ₀ to Σₙ-Induction [13, Theorem I.2.4]. Therefore for all α < ωₙ we have □ₙTIΠ₂−₁. ⊠

Corollary 15. Suppose Sₙ is a recursive sequence of finitely axiomatizable theories such that PA proves that PA = ∪ Sₙ. Let △ denote the provability predicate □ₙₚ,ₚ₀. Then PA ⊢ ∀φ (□φ → □△φ).

Lemma 16 (IΣ₁). Suppose φ is an arithmetical sentence and we have □φ. Then there exists a number n such that □ₙφ.

Proof. Because we have □φ we also have □ₙφ for some n. If n < 2 then because Fₚ₀(₁) ↓, we have □₁φ. Thus it remains to consider the case of n ≥ 2.
Now we prove by induction on $m > 0$ that $\Box_2^m F_{\varepsilon_0} (m - 1) \downarrow$. We have $F_{\varepsilon_0} (0) \downarrow$ and by $\Sigma_1$-completeness of $\Box_2$ we have $\Box_2 F_{\varepsilon_0} (0) \downarrow$. Thus induction base holds. We recall that $\Box_2^m$ satisfies Hilbert-Bernays-Löb conditions. From $\Box_2^m F_{\varepsilon_0} (m - 1) \downarrow$ it follows that $\Box_2^m \forall \psi (\Box_{m+1} \psi \rightarrow \Box_2 \psi)$. From Lemma 14 we have Theorem 5 theorem holds: $\Box_2^m \forall \psi (\Box_{m+1} \psi \rightarrow \Box_2 \psi)$ and by $\Sigma_1$-completeness of $\Box_2^m$ we have $\Box_2^m \Box_{m+1} F_{\varepsilon_0} (m) \downarrow$. Therefore we have $\Box_2^m (\forall \psi (\Box_{m+1} \psi \rightarrow \Box_2 \psi) \land \Box_m F_{\varepsilon_0} (m))$ and hence $\Box_2^m \Box_2 F_{\varepsilon_0} (m) \downarrow$. Therefore the induction step holds.

Hence we have $\Box_2^{m-1} F_{\varepsilon_0} (n - 2) \downarrow$. Thus we have $\Box_2^{m-1} \forall \psi (\Box_n \psi \rightarrow \Box_2 \psi)$. From $\Box_n \varphi$ it follows that $\Box_2^{m-1} \Box_n \varphi$. Therefore we have $\Box_2^m \varphi$.

Combining Theorem 4 and Lemma 16 we conclude that the following theorem holds:

**Theorem 5 (PA).** Suppose $\varphi$ is an arithmetical sentence. Then $\Box \varphi$ iff $\Box_2^0 \varphi$.

We clearly have the following generalization:

**Theorem 6.** Suppose $S_n$ is a recursive sequence of finitely axiomatizable theories such that PA proves that $\text{PA} = \bigcup S_n$ and $I \Sigma_{n+2} \subseteq S_n$. Let $\triangle$ denote the provability predicate $\Box_{S_n} \varphi$. Then $\text{PA}$ proves that for every arithmetical sentence $\varphi$ we have $\Box \varphi$ iff $\Box_2^0 \varphi$.

**Lemma 17 (\text{I} \Sigma_1).** Suppose $\alpha < \varepsilon_0$ and $n$ is a number. Then $\Box_1 \omega^n F_\alpha (n) \downarrow$.

**Proof.** We use reflexive induction and hence it is sufficient to show (while reasoning in $\text{I} \Sigma_1$) that for every $\alpha \leq \varepsilon_0$, if $\Box_1 \forall \beta < \alpha \forall x \Box_1 \omega^n F_\beta (x) \downarrow$ then for every $m$ we have $\Box_1 \omega^n F_\alpha (m) \downarrow$.

We reason in $\text{I} \Sigma_1$. Let us consider three cases: $\alpha = 0$, $\alpha$ is a limit ordinal, and $\alpha$ is a successor ordinal. In the first case we need to show that $\Box_1 F_0 (n) \downarrow$ which is clearly true. In the second case $\Box_1 \omega^n F_\alpha (m) \downarrow$ is equivalent to $\Box_1 \omega^n F_\alpha [m] (m) \downarrow$. The latter follows from $\Box_1 \Box_1 \omega^n F_\alpha [m] (m) \downarrow$ which itself follows from $\Box_1 \forall \beta < \alpha \forall x \Box_1 \omega^n F_\beta (x) \downarrow$.

Now let us consider the case of successor $\alpha = \gamma + 1$. In order to show that $\Box_1 \omega^{\gamma+1} F_{\gamma+1} (m) \downarrow$ it is enough to show that $\Box_1 \Box_1 \omega^{\gamma+1} F_{m+1} (m) \downarrow$. In order to prove latter we prove by induction on $k \geq 1$ that $\Box_1 \forall x \Box_1 \omega^\gamma k F_\gamma (x) \downarrow$. The base of induction follows directly from reflexive induction assumption. Suppose we have $\Box_1 \forall x \Box_1 \omega^\gamma k F_\gamma (x) \downarrow$. Then we have $\Box_1 \Box_1 \omega \forall x \Box_1 \omega^\gamma F_\gamma (x) \downarrow$. Using reflexive induction assumption we obtain

$$\Box_1 \forall x \Box_1 \omega^\gamma F_\gamma (x) \downarrow \land \Box_1 \forall x \Box_1 \omega^{\gamma+1} k F_{\gamma+1} (x) \downarrow.$$
Thus we have
$$\square_1 \forall x \square_1^{\omega^\gamma} (F_\gamma(x) \downarrow \land \forall y \square_1^{\omega^{\gamma-k}} F_\gamma^k(y) \downarrow).$$
Hence we have
$$\square_1 \forall x \square_1^{\omega^\gamma} (\exists y (y = F_\gamma(x) \land \square_1^{\omega^{\gamma-k}} F_\gamma^k(y) \downarrow)).$$
Therefore
$$\square_1 \forall x \square_1^{\omega^\gamma} F_\gamma^{k+1}(x) \downarrow.$$ This finishes the proof in the successor case.

**Lemma 18.** Suppose $S_n$ is a recursive sequence of finitely axiomatizable theories such that $\text{PA}$ proves that $\text{PA} = \bigcup_{n \in \omega} S_n$. Let $\triangle$ denote the provability predicate $\square_{S_n} F_{\varepsilon_0}$. Then $\text{PA}$ proves that for every arithmetical sentence $\varphi$ such that $\square \varphi$ we have $\triangle^{\varepsilon_0} \varphi$.

**Proof.** Let us reason in $\text{PA}$. For some $n$ we have $\square S_n \varphi$. From Lemma 17 it follows that we have $\square_1^{\omega^{n+2}} F_{\varepsilon_0} (n) \downarrow$. Since for some $n$ the theory $\Sigma_1 \subset S_n$, in $\text{PA}$ the predicate $\triangle$ is at least as strong as $\square$. Therefore by Lemma 10 and Lemma 11 we have $\triangle^{\varepsilon_0} F_{\varepsilon_0} (n) \downarrow$. Hence we have $\triangle^{\varepsilon_0} (F_{\varepsilon_0} (n) \downarrow \land \square_{S_n} \varphi)$. Thus $\triangle^{\varepsilon_0} \triangle \varphi$. Finally, by Lemma 12 we have $\triangle^{\varepsilon_0} \varphi$. $\square$

7 Models for slow consistency

In this section we will show that for the slow provability predicate $\square_1$, the theory $\text{PA}$ proves that $\square_1^{\varepsilon_0} \varphi$ is equivalent to $\square \varphi$. We also show that in addition to $\square_1$ a large family of provability predicates has the same property.

We will use model-theoretic techniques while reasoning in $\text{PA}$. We briefly present basic definitions of formalization of model theory within arithmetic. A *model* of finite signature is a tuple of formulas that give the domain, as well as interpretations of the constant, functional, and predicate symbols. A *full model* is a model with a satisfaction relation for all first-order sentences of the signature of the model given by their Gödel numbers. Note that for a model without satisfaction relation there is no straightforward way to formalize in $\text{PA}$ whether the model satisfies some infinite set of axioms, hence when we will talk about models of $\text{PA}$ within $\text{PA}$, we will assume that the models are full models. We will formalize our model-theoretic arguments within $\text{PA}$ and since full induction schema is present, our arguments will not depend
on the complexity of formulas giving models. We also recall that Gödel Completeness theorem is formalizable in PA, i.e., in PA the consistency of a theory $T$ implies the existence of a full model of $T$. Also, in PA the existence of some full model for a theory $T$ implies the consistency of $T$.

For models of arithmetical theories we also consider partial satisfaction relation that are defined only for $\Pi_n$ (or equivalently, $\Sigma_n$) formulas. Note that if we fix a number $k$ externally, then in PA we can construct a $\Pi_{n+k}$ partial satisfaction relation from a $\Pi_n$ partial satisfaction relation. Also note that since PA proves the Cut Elimination Theorem, it also proves that if $T$ is an arithmetical theory axiomatizable by a recursive set of $\Pi_n$-formulas and there is a model of $T$ with a $\Pi_n$ partial satisfaction relation, then $T$ is consistent.

For a proper presentation of the basic definitions and the proofs of basic theorems of model theory in formal arithmetic the reader is referred to [13, I.4(b)].

Below we work with non-standard models of arithmetical signature, all of them will be models of $I\Delta_0 + \text{Exp}$. Because standard natural numbers are embeddable as initial segments in every model of $I\Delta_0 + \text{Exp}$ we freely assume that every such model contains the standard numbers.

Suppose that $\mathcal{M}$ is a model of $I\Delta_0 + \text{Exp}$. A cut $\mathcal{I}$ of $\mathcal{M}$ is a submodel of $\mathcal{M}$ such that for any $a \in \mathcal{M}$ and $b \in \mathcal{I}$, if $\mathcal{M} \models a < b$, then we have $a \in \mathcal{I}$. For every cut $\mathcal{I}$ and an element $a \in \mathcal{M}$ we write $\mathcal{I} < a$ if $\forall b \in \mathcal{I} (\mathcal{M} \models b < a)$ and we write $a < \mathcal{I}$ if $a \in \mathcal{I}$.

**Theorem 7.** [24, Theorem 5.25] Suppose $n \geq 1$ is a number, $\alpha < \varepsilon_0$ is an ordinal, $\mathcal{M}$ is a model of $I\Delta_0 + \text{Exp}$, and $a, b, c \in \mathcal{M}$ are non-standard numbers such that $\mathcal{M} \models F_{\omega^{\omega^{\omega^{\alpha}}}}(a) = b$. Then there exists a cut $\mathcal{I}$ of $\mathcal{M}$ such that $a < \mathcal{I} < b$ and $\mathcal{I}$ is a model of $I\Delta_0 + \text{Exp} + T\Pi_n, \omega^{\omega^{\alpha}}$.

**Remark 2.** It is possible to formalize Theorem 7 in PA as a theorem schema. We assume that the model $\mathcal{M}$ is given by some fixed tuple of arithmetical formulas, possibly with additional parameters. Then there are arithmetical formulas $\varphi(x, n, \alpha, a, b, c)$ and $\psi(y, n, \alpha, a, b, c)$ such that PA proves that if $n, \alpha, a, b, c$ satisfy the conditions of Theorem 7 then the set of all $x$ from $\mathcal{M}$ for which $\varphi(x, n, \alpha, a, b, c)$ holds is a cut $\mathcal{I}$, the formula $\psi(y, n, \alpha, a, b, c)$ gives a $\Pi_n$ partial satisfaction relation for $\mathcal{I}$, and the cut $\mathcal{I}$ satisfies the conclusion of Theorem 7.
Theorem 8. ([24, Theorem 5.2]]) For all $n \geq 1$ and all $\alpha, \beta < \varepsilon_0$,

$$\text{I}\Delta_0 + \text{Exp} + \text{TI}_{\Pi_n} \omega^{\omega^\alpha} \vdash F_{\beta} \downarrow \iff \beta < \omega^{\alpha+1}.$$ 

Remark 3. We will employ only $\Leftarrow$ direction of Theorem 8. Examination of the proof of [24, Lemma 5.5] shows that $\Leftarrow$ direction of Theorem 8 can be formalized in $\text{I}\Delta_0 + \text{Exp}$.

A result close to the following theorem goes back to Paris [19]. In the form given below the theorem can be derived from results of Beklemishev [1, Theorem 1, Proposition 7.3, Remark 7.4], Freund has proved this theorem explicitly [7] for the case of $\alpha < \omega$.

Theorem 9. [1, 7]

$$\text{I}\Sigma_1 \vdash \forall \alpha \in [1, \omega] (\Diamond_\alpha \top \leftrightarrow F_{\omega_\alpha} \downarrow).$$

Proof. In [7] it has been proved that

$$\text{I}\Sigma_1 \vdash \forall \alpha \in [1, \omega] (\Diamond_\alpha \top \leftrightarrow F_{\omega_\alpha} \downarrow).$$

We need to prove that $\text{I}\Sigma_1 \vdash \Diamond_\omega \top \leftrightarrow F_{\varepsilon_0} \downarrow$. Clearly,

$$\text{I}\Sigma_1 \vdash \Diamond_\omega \top \leftrightarrow \forall x \Diamond_x \top \leftrightarrow \forall x F_{\omega_x} \downarrow.$$ 

From Lemma 1 it follows that $\text{I}\Delta_0 + \text{Exp} \vdash \forall x F_{\omega_x} \downarrow \iff F_{\varepsilon_0} \downarrow$. Thus indeed

$$\text{I}\Sigma_1 \vdash \Diamond_\omega \top \leftrightarrow F_{\varepsilon_0} \downarrow. \quad \Box$$

Lemma 19. (PA) For every arithmetical sentence $\varphi$ if $\Diamond \varphi$ then $\Diamond_{1}^{\varepsilon_0} \varphi$.

Proof. We reason in PA. In order to prove our claim we consider two cases $\Box \bot$ and $\neg \Box \bot$, i.e. $\Diamond \top$.

We start with the case of $\Box \bot$. From Theorem 9 for the case of $\alpha = \omega$ it follows that there exists $n_0$ such that $F_{\varepsilon_0}(n_0) \uparrow$ and hence $F_{\varepsilon_0}(n) \uparrow$, for every $n \geq n_0$. Thus for all $n \geq n_0$ and arithmetical sentences $\psi$ we have $\Diamond_1 \psi$ if $\Diamond_n \psi$. By straightforward calculations we exclude cases $n_0 = 0, 1$. Hence $n_0 \geq 2$.

Suppose we have been given an arithmetical sentence $\varphi$ such that $\Diamond \varphi$. We need to show that $\Diamond_{1}^{\varepsilon_0} \varphi$ holds. By definition $\Diamond_{1}^{\varepsilon_0} \varphi$ iff for every $\alpha < \varepsilon_0$
we have $\Diamond \varphi$. Thus it is sufficient to show that for every $\alpha < \varepsilon_0$ we have $\Diamond \varphi$.

By Completeness Theorem we have a model $\mathcal{M}$ of $\text{PA} + \varphi$. Let us consider arbitrary $n > n_0$. We will construct a cut $\mathcal{I}_n$ of $\mathcal{M}$ such that $\mathcal{I}_n$ is a model of $\Delta_0 + \text{Exp} + \text{T}_{\Pi_1} - \omega_{n+1} + F_{\varepsilon_0}(n) \uparrow$.

First, assume that we have already constructed the cut $\mathcal{I}_n$. Given that $\text{PA}$ is essentially reflexive, $\mathcal{M} \models \Diamond \varphi$. Hence because $\Diamond \varphi$ is $\Pi_1$ we have $\mathcal{I}_n \models \Diamond \varphi$. Combining Theorem 8, Remark 3, and Theorem 9 we see that $\Delta_0 + \text{Exp} + \text{T}_{\Pi_1} - \omega_{n+1} + F_{\varepsilon_0}(n) \uparrow$.

Let us reason in $\Delta_0 + \text{Exp} + \text{T}_{\Pi_1} - \omega_{n+1} + F_{\varepsilon_0}(n) \uparrow$. We claim that $\Diamond \varphi$, for all non-zero $\alpha < \omega_{n+1}$. We prove by transfinite induction that $\Diamond \varphi$, for all non-zero $\alpha < \omega_{n+1}$. The base holds because $\Diamond \varphi$ follows from $\Diamond \varphi$. The limit case follows directly from definition. For the successor case we use the fact that $\Diamond \varphi$ is $\Pi_1$ and that $\Sigma_n$ is consistent with all true $\Pi_1$-sentences. Thus we obtain $\Diamond \varphi$ and next $\Diamond \varphi$. This finishes the inductive proof. In order to prove our claim we note that $\Diamond \varphi$ follows from $\Diamond \varphi$ since $\Diamond \varphi$ is $\Pi_1$. Thus if $\mathcal{I}_n$ will have the above properties, we will have $\mathcal{I}_n \models \Diamond \varphi$ and next $\Diamond \varphi$.

Now we will construct the cut $\mathcal{I}_n$. From Theorem 6 it follows that $\text{PA} \vdash F_{\omega_{n+1} \uparrow}$ and thus $\mathcal{M} \models F_{\varepsilon_0}(n) \downarrow$. Let us denote by $u \in \mathcal{M}$ the non-standard number such that $\mathcal{M} \models F_{\varepsilon_0}(n) \downarrow = u$.

Since $F_{\varepsilon_0}(n) \downarrow$ is a $\Sigma_1$-sentence, for every cut $\mathcal{J}$ such that $\mathcal{J} < u$, we have $\mathcal{J} \models F_{\varepsilon_0}(n) \uparrow$.

Thus it is sufficient to take as $\mathcal{J}_n$ any cut $\mathcal{J}$ such that $\mathcal{J} < u$ and $\mathcal{J} \models \Delta_0 + \text{Exp} + \text{T}_{\Pi_1} - \omega_{n+1}$.

Let us denote by $a \in \mathcal{M}$ the nonstandard number such that $\mathcal{M} \models a = F_{\varepsilon_0}(n_0) = F_{\omega_{(n_0+1)}}(n_0)$.

We denote by $b$ the nonstandard number such that $\mathcal{M} \models b = F_{\omega_n . a}(a)$.

Now we can apply Theorem 6 with $c = a$ and obtain a cut $\mathcal{J}$ such that $a < \mathcal{J} < b$ and $\mathcal{J} \models \Delta_0 + \text{Exp} + \text{T}_{\Pi_1} - \omega_{n+1}$.
We claim that $\mathfrak{M} \models u > b = \mathcal{F}_{\omega_n a}(a)$ and thus that $\mathcal{J}_n < u$. Clearly, for any $k < n + 1$ we will get a limit ordinal by applying operation $\alpha \mapsto \alpha[n]$ exactly $k$ times to $\omega_{n+1}$. Thus $n + 1$ is the least $r$ such that $\omega_{n+1} \frac{n+1}{n} \beta + 1$ for some $\beta$. We denote by $\beta$ the ordinal such that $\omega_{n+1} \frac{n+1}{n} \beta + 1$. Since $n > n_0 \geq 2$, the ordinal $\beta$ is greater than $\omega^2$. Also, $\beta$ is greater than $\omega_{n_0 + 1}$.

Using induction we derive that $\omega_{n+1} \omega_n^2$ and $\omega_{n+1} \omega_n^{n+1}$ from Lemma 2. Since $\beta > \omega^2$ and $\beta > \omega_{n_0 + 1}$, we have $\beta \rightarrow \omega^2_n$ and $\beta \rightarrow \omega_{n_0 + 1}$. Since $\mathfrak{M}$ satisfies $\Sigma_0 + \exp$, we can use the definition of fast-growing hierarchy to obtain

$$\mathfrak{M} \models u = \mathcal{F}_{\omega_0}(n) = \mathcal{F}_{\beta}^{n+1}(n).$$

From Lemma 1 it follows that $\mathfrak{M} \models \omega_{n-1} \rightarrow 0$. By Lemma 4 we have $\mathfrak{M} \models \omega_{n-1}^2 \rightarrow \omega_{n-1} + 1$. Therefore by Lemma 2 we have $\mathfrak{M} \models \omega_n^2 \rightarrow \omega_{n-1} + 1$. Thus $\mathfrak{M} \models \omega_n^2 \rightarrow \omega_n \cdot a$. From Lemma 1 and Lemma 6 it follows that

$$\mathfrak{M} \models \mathcal{F}_{\beta}^{n+1}(n) > \mathcal{F}_{\beta}^2(n) \geq \mathcal{F}_{\omega_0}^2(\mathcal{F}_{\omega_{n_0 + 1}}(n)) \geq \mathcal{F}_{\omega_0}^2(\mathcal{F}_{\omega_{n_0 + 1}}(n_0)) = \mathcal{F}_{\omega_0}(a) \geq \mathcal{F}_{\omega_n}(a).$$

Hence we have proved our claim.

Thus for every $n$ and $\alpha < \omega_{n+1}$ we have a model of $\Sigma_0 + \exp + \Diamond_1 \varphi$. Thus for every $n$ and $\alpha < \varepsilon_0$ we have a model of $\Sigma_0 + \exp + \Diamond_1 \varphi$, since for every $\psi$ and $n_1 < n_2$ we have $\Sigma_0 + \exp \vdash \Diamond_{n_2} \psi \rightarrow \Diamond_{n_1} \psi$. Note that every $\Pi_1$-sentence that holds in some model of $\Sigma_0 + \exp$ is true. Thus for every $\alpha < \varepsilon_0$ and number $n$ we have $\Diamond_1 \varphi$. Therefore, for every $\alpha < \varepsilon_0$ we have $\Diamond_1 \varphi$. And therefore $\Diamond_1 \varphi$.

Now assume that $\Diamond \top$. Suppose we have been given an arithmetical sentence $\varphi$ such that $\Diamond \varphi$. By L"ob’s Theorem for $\Diamond$ we have $\Diamond \neg \Diamond \top$. Thus because $\Diamond \varphi$ is $\Pi_1$, we have $\Diamond (\neg \Diamond \top \land \Diamond \varphi)$. Therefore we have $\Diamond (\neg \Diamond \top \land \Diamond \varphi)$. Note that formalization of the first part of the proof gives us

$$\text{PA} \vdash \neg \Diamond \top \rightarrow \forall \psi (\Diamond \psi \rightarrow \forall \alpha < \varepsilon_0 \Diamond_1 \psi).$$

Using that we conclude that $\Diamond (\forall \alpha < \varepsilon_0 \Diamond_1 \varphi)$. Next we see that $\forall \alpha < \varepsilon_0 \Diamond_1 \varphi$ is a $\Pi_1$-sentence. Therefore we have $\forall \alpha < \varepsilon_0 \Diamond_1 \varphi$, i.e. $\Diamond_0 \varphi$. }

Using Lemma 19 and Lemma 18 we obtain the following theorem:
Theorem 10 (PA). Suppose $\varphi$ is a sentence and $\alpha < \varepsilon_0$ is a non-zero ordinal. Then the following sentences are equivalent:

1. $\Box \varphi$,
2. $\Box^\varepsilon_0 \varphi$,
3. $\Box \varepsilon_0 \varphi$.

Using Lemma 11 and Lemma 18 we can generalize the previous theorem to a wider spectrum of provability predicates:

Theorem 11. Suppose $S_n$ is a recursive sequence of finitely axiomatizable theories such that \( \text{PA} \) proves that $\text{PA} = \bigcup_{n \in \omega} S_n$ and $S_n \subset \Sigma_{n+1}$. Let $\Delta$ denote the provability predicate $\Box^{\varepsilon_0} S_x F_{\varepsilon_0}$. Then $\text{PA}$ proves that for every arithmetical sentence $\varphi$, and non-zero ordinal $\alpha < \varepsilon_0$ the following sentences are equivalent:

1. $\Box \varphi$,
2. $\Delta^{\varepsilon_0} \varphi$,
3. $\Box \Delta^{\alpha} \varphi$.

8 Square root of $\text{PA}$-provability

Recall that for a recursive function $f$ we denote by $\Box_f$ the slowdown of standard $\text{PA}$-provability predicate by the function $f$:

$$\Box_f \varphi : = \Box_0 \varphi \lor \exists y \leq x (\Box_y \varphi \land f(x) \downarrow).$$

We are going to prove the following theorem.

Theorem 12. There exists a fast-growing recursive function $r$ such that $\text{PA} \vdash \forall \varphi (\Box_r \Box_r \varphi \leftrightarrow \Box \varphi)$.

First we define the required function $r$ and auxiliary function $l$. The function $r$ will be a function that grows faster than any $F_{\omega_n}$, but considerably slower than $F_{\varepsilon_0}$.
Definition 7. We define function $1$ by recursion. We give $1(n)$ under the assumptions that for all $m < n$ the values $1(m)$ are already defined:

$$1(n) = \max(\{0\} \cup \{m \mid 0 < m < n \text{ and } F_{\omega_1(n)}(m) \leq n\}).$$

We define the function $r$ as following:

$$r(n) = F_{\omega_1(n)}(n).$$

Lemma 20 ($I\Delta_0 + \text{Exp}$). The following holds:

1. $1$ is total;
2. there exists $n$ such that $1(n) = 1$;
3. function $1$ is monotone-nondecreasing;
4. for every $n$, we have $r(1(n)) \downarrow$;
5. for every $n$, we have $r(1(n)) \leq n$ if $1(n) \geq 1$;
6. for every $n$ such that $r(n) \downarrow$, we have $1(r(n)) = n$.

Proof. Straightforward using definition of $1$, $r$ and Lemma 1.

Lemma 21 ($I\Sigma_1$). For every $n$, if $r(n) \downarrow$ then for all $m < n$ we have $r(m) \downarrow$ and $r(m) < r(n)$.

Proof. Follows from Lemma 1 Item 1 and easily provable fact that if $F_\alpha(n) \downarrow$ and $n > m$ then $F_\alpha(n) > F_\alpha(m)$.

Lemma 22 ($I\Sigma_1$). For every arithmetical sentence $\varphi$, if we have $\Box \varphi$ then we have $\Box r \varphi$.

Proof. Suppose we have $\Box \varphi$. Then there is a number $n$ such that $\Box_n \varphi$. Let $m = 1(n)$. By Lemma 14 we have $\Box_m F_{\omega_1(n)} \downarrow$. Hence directly from the definition of $r$ we have $\Box_m r(n) \downarrow$. Thus $I\Sigma_m$ proves that $I\Sigma_n \subseteq PA|_r$, i.e. $\Box_m \forall \psi(\Box_n \psi \rightarrow \Box_r \psi)$. By $\Sigma_1$-completeness of $I\Sigma_m$ we have $\Box_m \Box_n \varphi$. Combining conclusions of the last two sentences, we obtain $\Box_m \Box_r \varphi$. By Lemma 20 Item 4 we have $r(m) \downarrow$ and therefore $I\Sigma_m \subseteq PA|_r$. Thus $\Box_r \Box \varphi$. 

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Theorem 13. [9, Theorem 3.7] Suppose $\mathcal{M}$ is a non-standard model of the theory $\Delta_0 + \text{Exp}$, $n \geq 1$ is a standard integer, and $a, b, e \in \mathcal{M}$ are non-standard integers such that $\mathcal{M} \models F_{\omega^{n-1}}(a) = b$. Then there is a cut $\mathfrak{J}$ of $\mathcal{M}$ such that $a < \mathfrak{J} < b$ and $\mathfrak{J}$ is a model of $I \Sigma_n$.

Remark 4. Theorem 13 can be formalized in $\text{PA}$ in the same fashion as Theorem 7 in Remark 2. We assume that the model $\mathcal{M}$ is given by some fixed tuple of arithmetical formulas, possibly with additional parameters. Then there are arithmetical formulas $\varphi(x, n, a, b, e)$ and $\psi(y, n, a, b, e)$ such that $\text{PA}$ proves that if $n, a, b, e$ satisfy the conditions of Theorem 13 then the set of all $x$ from $\mathcal{M}$ for which $\varphi(x, n, a, b, e)$ is a cut $\mathfrak{I}$, the formula $\psi(y, n, a, b, e)$ gives a $\Pi_n$ partial satisfaction relation for $\mathfrak{I}$, and the cut $\mathfrak{J}$ satisfies the conclusion of Theorem 13.

Lemma 23 (PA). For every arithmetical sentence $\varphi$, if $\Diamond \varphi$ then $\Diamond_r \Diamond_r \varphi$.

Proof. Let us consider a model $\mathcal{M}$ of $\text{PA} + \varphi$. If $\mathcal{M} \models \Diamond_r \varphi$ then we are already done. So we assume that $\mathcal{M} \not\models \Diamond_r \varphi$. Thus there exist $u \in \mathcal{M}$ such that $\mathcal{M} \models \Diamond_u \varphi \land \neg \Diamond_{u+1} \varphi \land r(u+1)$. From Lemma 21 it follows that $\mathcal{M} \models r(u) \downarrow$ and $r(u) < r(u+1)$. Note that $u$ is a non-standard number because $\text{PA}$ is an essentially reflexive theory and thus for every standard $m$ we have $\mathcal{M} \models \Diamond_m \varphi$.

We are going to prove that for every $n$ such that $r(n) \downarrow$ we have a cut $\mathfrak{J}_n$ of $\mathcal{M}$ such that $r(u) < \mathfrak{J}_n < r(u+1)$, the cut $\mathfrak{J}_n$ have $\Pi_n$ satisfaction relation, and $\mathfrak{J}_n \models I \Sigma_n$. If we prove our claim then for every cut $\mathfrak{J}_n$, we will have $\text{PA} \models_\mathfrak{J} = I \Sigma_n$ within it and hence $\mathfrak{J}_n \models \Diamond_r \varphi$. Since every cut $\mathfrak{J}_n$ that we will construct will have $\Pi_n$ partial satisfaction relation and will be a model of $I \Sigma_n$, we can conclude that $\Diamond_n \Diamond_r \varphi$, for every $n$ such that $r(n) \downarrow$. Hence we will have $\Diamond_r \Diamond_r \varphi$, contradiction.

Now let us prove the claim. We take $a, b, e \in \mathcal{M}$:

$$\mathcal{M} \models a = r(u), \quad \mathcal{M} \models b = F_{\omega^{n-1}}(a), \quad \text{and} \quad \mathcal{M} \models e = u$$

and apply Theorem 13 to construct the required cut. Now we need to check that we indeed have $r(u) < \mathfrak{J}_n < r(u+1)$. For this, it is enough to show that

$$\mathcal{M} \models F_{\omega^{n-1}}(r(u)) \leq r(u+1).$$

Since $u$ is a non-standard number, we have $\mathcal{M} \models r(n) < u$. From Lemma 20 item 3, it follows that $\mathcal{M} \models 1(u) \geq 1(r(n))$. From Lemma 20 item 6, it
follows that \(1(r(n)) = n\). Thus \(M \models r(u) = F_{\omega_1(u)}(u) = F_{\omega_{1(u)} - 1}^u(u)\) and
\[
M \models r(u + 1) = F_{\omega_1(u + 1)}(u + 1) = F_{\omega_{1(u + 1)} - 1}^{u+1}(u + 1).
\]
From Lemma 1 it follows that \(M \models \omega_{1(u + 1) - 1}^u \rightarrow 0\). Now we use Lemma 5 and Lemma 4 and deduce that
\[
M \models \omega_{1(u + 1) - 1}^u \rightarrow \omega_{1(u) - 1}^{u + 1} \rightarrow \omega_{1(u) - 1}^{u + 1} + 1.
\]
Therefore from Lemma 1 it follows that
\[
M \models F_{\omega_{1(u + 1) - 1}^u}^{u+1}(u + 1) \geq F_{\omega_{1(u) - 1}^{u + 1}}^{u+1}(u + 1) \geq F_{\omega_{1(u) - 1}^{u + 1}}(u).
\]
Now using the fact that \(M \models F_{\omega_{1(u) - 1}^{u + 1}}(u) \downarrow\) and Lemma 1 we conclude that
\[
M \models \omega_{1(u) - 1}^{u + 1} + 1 \rightarrow 0.
\]
Next using Corollary 5 and Lemma 5 we deduce that
\[
M \models \omega_{1(u) - 1}^{u + 1} \rightarrow \omega_{1(u) - 1}^{u} \rightarrow \omega_{n - 1}^{u}.
\]
From Lemma 1 and Lemma 6 it follows that that
\[
M \models F_{\omega_{1(u) - 1}^{u + 1}}(u) = F_{\omega_{1(u) - 1}^{u + 1}}^{u+1}(u) \geq F_{\omega_{1(u) - 1}^{u + 1}}^{u+1}(u) \geq F_{\omega_{1(u) - 1}^{u + 1}}(u).
\]
Therefore
\[
M \models F_{\omega_{1(u) - 1}^{u + 1}}(r(u)) = F_{\omega_{1(u) - 1}^{u + 1}}(u) \leq F_{\omega_{1(u + 1) - 1}^{u + 2}}(u + 1) = r(u + 1).
\]
Thus we have obtained the required cut \(J_n\).

From Lemma 22 and Lemma 23 it follows that Theorem 12 holds.
9 Bimodal provability logics

We determine the joint provability logic of a wide class of pairs of provability predicates, including ordinary provability together with any of the slow provability predicates $\Box_1$. This result was obtained in cooperation with Volodya Shavrukov. The relevant bimodal system is GLT, or Lindström logic, first studied by Lindström ([15]) due to its relation to Parikh’s rule. The latter allows one to infer $\varphi$ from $\text{Pr}_{\text{PA}}(\varphi)$. Since Parikh’s rule is admissible in $\text{PA}$, adding it to $\text{PA}$ does not yield new theorems. As shown in [18], it does yield speed-up, meaning that some theorems have much shorter proofs when Parikh’s rule is allowed. The equivalence of Parikh provability and ordinary provability is however not verifiable in $\text{PA}$.

In Section 9.4 we establish the joint provability logic of ordinary provability together with square root provability.

9.1 The system GLT

We work with the languages $\mathcal{L}_\Box$ and $\mathcal{L}_\Box\,\Box$ of propositional unimodal and bimodal logic respectively. The symbols $\Box$ and $\triangle$ are thus used for the modalities, not as abbreviations for arithmetical formulas as until now. As before, $\Diamond A$ and $\nabla A$ are written as shorthand for $\neg\Box\neg A$ and $\neg\triangle\neg A$ respectively.

**Definition 8.** The axiom schemata of GL include all propositional tautologies in the language $\mathcal{L}_\Box$, and furthermore:

- (K) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- (L) $\Box(\Box A \rightarrow A) \rightarrow \Box A$

The inference rules of GL are modus ponens and necessitation: if $\vdash_{\text{GL}} A$ then $\vdash_{\text{GL}} \Box A$.

**Lemma 24.** $\vdash_{\text{GL}} \Box A \rightarrow \Box\Box A$

**Proof.** See [2, Theorem 1.18].

A relation $\prec$ on a set $W$ is said to be *conversely well-founded* if for every $S \subseteq W$ with $S \neq \emptyset$, there is some $a \in W$ such that $a \not\prec b$ for all $b \in S$. A conversely well-founded relation is, in particular, irreflexive. We write $a \preceq b$ if either $a \prec b$ or $a = b$.  

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Definition 9. A GL-frame $F$ is a tuple $\langle W, \prec \rangle$, where $\prec$ is a transitive conversely well-founded relation on $W$. $F$ is said to be tree-like if there is a root $r \in W$ such that for all $a \in W$, $r \preceq a$, and furthermore each $a \neq r$ has a unique immediate $\prec$-predecessor $b$.

Definition 10. A GL-model is a triple $\langle W, \prec, \models \rangle$, where $\langle W, \prec \rangle$ is a GL-frame, and $\models$ a valuation assigning to every propositional letter a subset of $W$. $\models$ is extended to all formulas of $L_\Box$ by requiring that it commutes with the propositional connectives, and interpreting $\prec$ as the accessibility relation for $\Box$:

$\mathcal{M}, a \models \Box A$ if for all $b$ with $a \prec b$, $b \models A$.

Given $\mathcal{M} = \langle W, \prec, \models \rangle$, we write $\mathcal{M} \models A$ if $\mathcal{M}, a \models A$ for every $a \in W$. We write $F \models A$ if $\mathcal{M} \models A$ for any model $\mathcal{M}$ whose underlying frame is $F$.

Theorem 14. $\vdash_{\text{GL}} A$ iff for every tree-like GL-frame $F$, $F \models A$.

Proof. See for example Chapter 5 of [2].

Definition 11. The axiom schemata of GLT include all propositional tau-tologies in the language $L_\Box\Delta$, the rules and axiom schemata of GL for $\Delta$, as well as:

$(K^\Box) \quad \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
$(T1) \quad \Delta A \rightarrow \Box A$
$(T2) \quad \Box A \rightarrow \Delta \Box A$
$(T3) \quad \Box A \rightarrow \Box \Delta A$
$(T4) \quad \Box \Delta A \rightarrow \Box A$

It is not difficult to show that $\vdash_{\text{GLT}} \Box(\Box A \rightarrow A) \rightarrow \Box A$. Thus GLT contains GL for both $\Delta$ and $\Box$.

Given a tree-like GL-frame $\langle W, \prec \rangle$ and $a, b \in W$, we write $a \prec_{\infty} b$ if the set $\{c \mid a \prec c \prec b\}$ is infinite. It is shown in [15] that GLT is sound and complete with respect to the class all of tree-like GL-frames, with $\prec$ and $\prec_{\infty}$ as the accessibility relations for $\Delta$ and $\Box$ respectively. We consider a slightly different semantics for GLT that — while arguably less neat than the one just described — has the advantage of allowing us to work with finite models. Indeed, it was introduced by Lindström in order to obtain decidability of GLT.
Definition 12. For $A \in L_{\square \triangle}$, an $A$-sound model is a quadruple $\langle W, \prec, \prec_R, \models \rangle$, where

1. $\langle W, \prec \rangle$ is a finite tree-like GL-frame
2. $a \prec_R b \Rightarrow a \prec b$
3. $a \prec b \prec_R c \Rightarrow a \prec_R c$
4. $a \prec_R b \prec c \Rightarrow a \prec_R c$
5. if $a \prec_R b$, there is some $c$ with $a \prec_R c \preceq b$, and such that $c \models \triangle B \rightarrow B$ for every subformula $B$ of $A$. Such a node $c$ shall be referred to as reflexive.

Finally, $\models$ is a valuation satisfying the usual clauses, with $\prec$ and $\prec_R$ as the accessibility relations for $\triangle$ and $\square$ respectively.

Lemma 25. [L5, Lemma 9] Let $n$ be the cardinality of the set of subformulas of $A$. $\vdash_{GLT} A$ iff $M \models A$ for every $A$-sound model of cardinality $\leq n^{n^2+1}$.

9.2 Arithmetical interpretations of modal logic

In order to formulate the connection between GLT and our arithmetical provability predicates, we use the notion of an arithmetical realization.

Definition 13. Let $Pr_0$ and $Pr_1$ be provability predicates in the language $L$ of arithmetic. An arithmetical realization $*$ is an assignment of $L$-sentences to all modal formulas. The values of $*$ at propositional letters of the modal language can be arbitrary. It is required that $*$ commutes with the propositional connectives, and furthermore $(\square A)^* := Pr_0(\Box A^*)$ and $(\Delta A)^* = Pr_1(\Diamond A^*)$.

We note that the values of an arithmetical realization $*$ are determined by its values at the propositional letters of $L_{\Box \Diamond}$.

Definition 14. Let $Pr_0$ and $Pr_1$ be provability predicates for a theory $T$ containing $\Delta_0 + \text{Exp}$. We say that GL (GLT) is sound for $Pr_0$ (and $Pr_1$) if:

$$\vdash_{GL(GLT)} A \Rightarrow T \vdash A^* \text{ for all arithmetical realizations } *.$$ 

We say that GL (GLT) is complete for $Pr_0$ (and $Pr_1$) if:

$$T \vdash A^* \text{ for all arithmetical realizations } * \Rightarrow \vdash_{GL(GLT)} A.$$ 

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If GL (GLT) is both sound and complete with respect to Pr₀ (and Pr₁), we say that GL (GLT) is the provability logic of Pr₀ (and Pr₁). This means that GL (GLT) contains exactly those principles of provability in T — as given by Pr₀ and Pr₁ respectively — that can be verified in IΔ₀ + Exp.

**Theorem 15.** Let T ⊇ IΔ₀ + Exp be recursively axiomatizable and Σ₁-sound. Then GL is the provability logic of Prₜ.

**Proof.** The case where T = PA was proven by Solovay in [23]. Extension to Σ₁-sound recursively axiomatizable theories containing IΔ₀ + Exp is due to de Jongh, Jumelet, and Montagna in [3].

Throughout the rest of this section, we use the symbols □ and △ for both provability predicates and modalities. Since lower-case Greek letters range over arithmetical and upper-case Latin letters over modal sentences, the intended meaning will always be clear from the context. With this notation, we also imply that we are interested in arithmetical realizations interpreting the modalities △ and □ as the corresponding provability predicates. Bearing this in mind we shall, from now on, mostly skip referring to the explicit formalization of Definition 13.

### 9.3 Arithmetical soundness and completeness

We prove the following arithmetical completeness theorem.

**Theorem 16.** Let □ and △ be Σ₁-provability predicates numerating the same sound theory T extending IΔ₀ + Exp, i.e. for all ϕ,

1. T ⊢ ϕ if and only if IΔ₀ + Exp ⊢ □ϕ
2. N ⊨ □ϕ ↔ △ϕ, where N is the standard model

Assume furthermore that GLT is sound for □ and △. Then GLT is also complete for □ and △.

**Remark 5.** With the above interpretation of □, we depart from the convention that □ stands for the ordinary provability predicate of PA.

**Lemma 26.** Let \{S_n\}_{n∈ω} be a recursive sequence of finitely axiomatizable theories such that PA proves PA = \bigcup_{n∈ω} S_n. Write △ for □_{s∈F_{eq}}, and □ for the usual provability predicate of PA. Then PA verifies the following, for all ϕ:

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1. \( \triangle \varphi \rightarrow \Box \varphi \)
2. \( \Box \varphi \rightarrow \triangle \Box \varphi \)
3. \( \Box \varphi \rightarrow \Box \triangle \varphi \)
4. \( \triangle \Box \varphi \rightarrow \Box \varphi \)

\textit{Proof.} Item 1 is clear from the definition, item 2 follows by provable \( \Sigma_1 \)-completeness of \( \triangle \), \( \Box \) is Corollary 15 in Section 6 and 4 is Theorem 4 in Section 5.

By Remark 1 in Section 5, it follows that the requirements of Theorem 16 are satisfied when taking for \( \Box \) the usual provability predicate of \( \text{PA} \), and for \( \triangle \) any of the slow provability predicates \( \Box_2 \). We note that the requirements of Theorem 16 are also satisfied when taking for \( \triangle \) the usual provability predicate of \( \text{PA} \), and for \( \Box \) the provability predicate for \( \text{PA} \) together with Parikh’s rule. \( \Box \)

We assume all \( \Sigma_1 \)-sentences to be of the form \( \exists y \psi \), with \( \psi \) a \( \Delta_0 \)-formula. If \( \varphi \) is a \( \Sigma_1 \)-sentence, we write \( n : \varphi \) to mean that \( n \) is a witness for \( \varphi \), i.e. if \( \psi(n) \) holds. We also assume:

1. if \( n : \Box \varphi \), then for any \( \psi \neq \varphi \) it is not the case that \( n : \Box \psi \)
2. if there is some \( n \) with \( n : \Box \varphi \), then there are arbitrarily large \( n' \) with \( n' : \Box \varphi \)

The above requirements hold for any reasonable arithmetization of syntax in arithmetic. For every \( \Sigma_1 \)-formula \( \varphi \), there exists a formula \( \exists y \psi \) satisfying 1 and 2 that is \( \text{I} \Delta_0 + \text{Exp} \)-provably equivalent to \( \varphi \); thus the above assumption does not restrict us in any way.

Let \( T, \Box \) and \( \triangle \) be as in the statement of Theorem 16. Our proof of arithmetical completeness proceeds, as usual, by constructing a suitable Solovay function moving along the accessibility relations of a \( \text{GLT} \)-model. For the remainder of this section, fix some \( A \)-sound model \( \mathcal{M} = \langle W, \prec, \prec_R, \models \rangle \). We

\footnote{Lindström’s proof of arithmetical completeness of \( \text{GLT} \) with respect to ordinary provability \( \Box \) and Parikh provability \( \Box_p \) made essential use of the fact that \( \text{PA} \vdash \forall \varphi (\Box_p \varphi \leftrightarrow \Box^p \varphi) \). While, as follows from the results of Section 6, the same relation holds between \( \Box \) and \( \Box_1 \), it fails for \( \Box \) and \( \Box_1 \), where we only have \( \text{PA} \vdash \forall \varphi (\Box \varphi \leftrightarrow \Box_1^p \varphi) \). Our proof method does not rely on \( \triangle \) being a certain ordinal iteration of \( \Box \), and is therefore applicable to a wider class of predicates.}
assume that $\mathcal{M}$ has a root, i.e. that there is a node $0 \in W$ such that $0 \prec a$ for every $0 \neq a \in W$. We also assume as given $\mathcal{L}$-formulas representing $\mathcal{M}$ in $\text{I}\Delta_0+\text{Exp}$ in a natural way.

**Definition 15.** ($\text{I}\Delta_0+\text{Exp}$) Define the primitive recursive function $h : \omega \to W$:

$$h(0) = 0$$

$$h(n + 1) = \begin{cases} 
  b & \text{if } h(n) \prec_R b, \text{ } b \text{ is reflexive, and } n : \square L \neq b, \text{ else:} \\
  c & \text{if } h(n) \prec c \text{ and } n : \triangle L \neq c \\
  h(n) & \text{otherwise}
\end{cases}$$

For $a \in W$, $L = a$ is the formula expressing that $a$ is the limit of $h$. As usual, the apparent circularity in the definition of $h$ is dealt with by using the Diagonal Lemma (see [23] or [2]).

The function $h$ starts at the root 0, and moves along the relations $\prec$ and $\prec_R$. It makes an $\prec_R$-step to some node $b$ only if there is a $\square$-proof that it will not stay there, and it makes an $\prec$-step to some node $c$ only if there is a $\triangle$-proof that it will not stay there. We refer to the elements of the domain of $h$ as *stages*, saying for example that $h$ moved to $a$ at stage $n$ if $h(n-1) \neq h(n) = a$.

**Lemma 27.** ($\text{I}\Delta_0+\text{Exp}$) $h$ has a unique limit.

**Proof.** Since $a \prec_R b$ implies $a \prec b$, it is clear from the definition that $h$ moves along the $\prec$-relation in $\mathcal{M}$. Since $\prec$ is transitive, we thus have that $h(x) \preceq h(y)$ whenever $x < y$. If $h$ were to keep moving forever, there would be an infinite ascending $\prec$-chain in $\mathcal{M}$, contradicting the assumption that $\langle W, \prec \rangle$ is a GL-frame.

Let $\varphi_h(x, y)$ be an $\mathcal{L}$-formula representing $h$ in $\text{I}\Delta_0+\text{Exp}$ in a natural way. According to Lemma 27

$$\text{I}\Delta_0+\text{Exp} \vdash \exists!y \exists x_0 \forall x \geq x_0 \varphi_h(x, y). \quad (32)$$

In the remainder of this section, we work in a definitional expansion of $\text{I}\Delta_0+\text{Exp}$ that contains a term $L$ denoting the limit of $h$. We note that by (32) such a definitional expansion is a conservative extension of $\text{I}\Delta_0+\text{Exp}$. 38
Lemma 28. (I$_{\Delta_0+\text{Exp}}$) If $h(n) = a$, then $a \preceq L$.

Proof. If $h(n) = a$, then either $h$ stays at $a$, i.e. we have $L = a$, or it moves away from $h$, in which case all its further values are $\prec$-successors of $a$, and so $a \prec L$ (as in the proof of Lemma 27).

Lemma 29. (I$_{\Delta_0+\text{Exp}}$) If $L = a$ and $a \prec b$, then $\Diamond L = b$.

Proof. Assume $L = a$ and $a \prec b$, and let $t$ be such that $\forall x \geq t \ h(x) = a$. Suppose for a contradiction that $\Diamond L \neq b$ holds. Then there also exists some $n \geq t$ with $n : \Delta L \neq b$. But this means that $h$ moves away from $a$ to $b$ at stage $n$, contradicting our assumption that $L = a$.

Lemma 30. (I$_{\Delta_0+\text{Exp}}$) If $L = a$ and $a \prec R b$, then $\Box L = b$.

Proof. Assume $L = a$ and $a \prec R b$. By properties of $M$, there exists some reflexive $c$ with $a \prec_R c$ and $c \preceq b$. Arguing as in the proof of Lemma 29, we have that $\Diamond L = c$. If $c = b$, we are done. If $c \prec b$, we have $L = c \rightarrow \nabla L = b$ by Lemma 29, whence also

$$\Diamond L = c \rightarrow \Diamond \nabla L = b$$

by modal reasoning, using the soundness of GL for $\Box$. An application of modus ponens yields $\Diamond \nabla L = b$, and therefore $\Diamond L = b$ by principle T3 of GLT.

Lemma 31. (I$_{\Delta_0+\text{Exp}}$) If $L = a$, then $\Delta a \preceq L$.

Proof. From $L = a$ it follows that $h(n) = a$ for some $n$. By $\Sigma_1$-completeness of $\Delta$,

$$\Delta h(\hat{n}) = a$$

Since the theory defined by $\Delta$ contains I$_{\Delta_0+\text{Exp}}$, Lemma 28 gives us

$$\Delta (h(\hat{n}) = a \rightarrow a \preceq L) .$$

Combining the above yields $\Delta a \preceq L$ as required.

Lemma 32. (I$_{\Delta_0+\text{Exp}}$) If $a \neq 0$ is not reflexive, then $L = a$ implies $\Delta a \prec L$.

Proof. If the limit of $h$ is some irreflexive $a \neq 0$, then $h$ must have moved to $a$ due to some number witnessing $\Delta L \neq a$. By lemma 31 we also have $\Delta a \preceq L$. Combining these, we get $\Delta a \prec L$. 

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Lemma 33. (PA) If \(a \neq 0\), then \(L = a\) implies \(\square a \prec_R L\).

Proof. Assume \(L = a\). By Lemma 31, we have \(\triangle a \preceq L\), whence also \(\square a \preceq L\). Since \(a \neq 0\) and \(h\) moved to \(a\), it must be that \(\square L \neq a\). It therefore suffices to show that if \(b\) is such that \(a \prec b\) but \(a \not\prec_R b\), then \(\square L \neq b\). We assume that the claim has been established for all \(b'\) with \(a \prec b' \prec b\), i.e. that we have \(\square L \neq b'\) for all such \(b'\).

Since \(L = a \neq 0\), \(h\) must have moved to \(a\) at some stage \(t\). Then either \(\square L \neq a\) or \(\triangle L \neq a\) holds, but in either case \(\square L \neq a\). Argue in \(\square\):

1. \(c = b\): the reason for moving could only have been that \(s - 1 : \triangle L \neq b\).
   If the move had been due to the first clause of Definition 15, we would have \(a \prec c \prec_R b\), and so \(a \prec_R b\) by properties of \(\mathcal{M}\), contradicting our assumption that \(a \not\prec_R b\).

2. \(c \neq b\): by the proof of Lemma 31, \(h(s) = c\) implies \(\triangle c \preceq L\). Our choice of \(c\) excludes \(a \prec b\), and we have assumed \(c \neq b\). Thus \(\triangle L \neq b\) also in this case.

Returning to the outside world, we have shown that \(\square \triangle L \neq b\). With principle \(T4\), we obtain \(\square L \neq b\) as desired. ☐

Lemma 34. Let \(\mathfrak{N}\) be the standard model. Then

1. \(\mathfrak{N} \models L = 0\)

2. For any \(a \in W\), the sentence \(L = a\) is consistent with \(T\)

Proof. By Lemma 27 (and soundness of \(I\Delta_0 + \text{Exp}\)), \(\mathfrak{N} \models L = a\) holds for some \(a\). If \(a \neq 0\), then either \(\mathfrak{N} \models \square L \neq a\) or \(\mathfrak{N} \models \triangle L \neq a\), depending on how \(h\) moved to \(a\). Since \(\square\) and \(\triangle\) have the same extension in \(\mathfrak{N}\), we have \(\mathfrak{N} \models \square L \neq a\) in both cases, and thus \(T \models L \neq a\). By soundness of \(T\), we get \(\mathfrak{N} \models L \neq a\), contradicting our assumption. Thus it must be that \(a = 0\). For \(2\), note that by Lemma 29 (and soundness of \(I\Delta_0 + \text{Exp}\)) \(\mathfrak{N} \models L = 0 \rightarrow \nabla L = a\) holds for all \(a \in W\). Since \(\mathfrak{N} \models L = 0\), it follows that \(\mathfrak{N} \models \nabla L = a\). 40
i.e. $\mathfrak{N} \vDash \neg \triangle L \neq a$ for any $a \in W$. Suppose that $T \vdash L \neq a$. Then $\mathfrak{N} \vDash \square L \neq a$ and thus also $\mathfrak{N} \vDash \triangle L \neq a$, a contradiction. Thus we conclude that $T \not\vDash L \neq a$.  

**Lemma 35.** Define the arithmetical realization $\ast$ by:

$$p^\ast := \bigvee_{M, a \neq p} L = a.$$  

Then for every subformula $B$ of $A$, and every $a \in M$, $a \neq 0$

$$M, a \vDash B \quad \Rightarrow \quad T \vdash L = a \rightarrow B^\ast.$$  

**Proof.** This is proven in the standard manner by induction on the complexity of $B$, using Lemmas 27-33 (see e.g. [23] or [2]). We treat the only slightly deviant case of $\triangle B$. Assume $a \vDash \triangle B$, i.e. $b \vDash B$ for all $b$ with $a \prec b$. By the induction assumption, we have $T \vdash L = b \rightarrow B^\ast$ for all such $b$, and thus $T \vdash a \prec L \rightarrow B^\ast$.  

By modal reasoning, this implies

$$T \vdash \triangle a \prec L \rightarrow \triangle B^\ast.$$  

Now argue in $T$, assuming $L = a$. If $a$ is irreflexive, then $L = a$ implies $\triangle a \prec L$ by Lemma 32, and so $\triangle B^\ast$ by (36). If $a$ is reflexive, then we have $a \not\vDash B$, whence the induction assumption additionally gives $L = a \rightarrow B^\ast$, and therefore $a \preceq L \rightarrow B^\ast$ and $\triangle a \preceq L \rightarrow \triangle B^\ast$. With Lemma 31 $L = a$ implies $\triangle a \preceq L$, thus we obtain $\triangle B^\ast$ also in this case.  

We are now ready to combine the results of this section to prove Theorem 16.

**Proof.** We need to show that if $\not\vDash_{\text{GLT}} A$, there is some arithmetical realization $\ast$ with $T \not\vDash A^\ast$. Suppose that $\text{GLT} \not\vDash A$. By Lemma 25 let $\mathcal{M}$ be an $A$-sound model with $\mathcal{M}, w \not\vDash A$ for some $w \in \mathcal{M}$. We append a root 0 to $\mathcal{M}$, and apply Definition 15 to the resulting model. By Lemma 33 $T \vdash L = w \rightarrow \neg A^\ast$. Therefore, since $T \not\vDash L \neq w$ by Lemma 34 it must be that $T \not\vDash A^\ast$.  

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9.4 Square root provability

We conclude with a characterization of the provability logic of square root and ordinary provability. In Section 8, it was shown that there is a provability predicate \( \Box_r \) such that

\[
\text{PA} \vdash \forall \varphi (\Box_r \Box_r \varphi \leftrightarrow \Box \varphi),
\]

where \( \Box \) denotes the usual provability predicate of \( \text{PA} \). By Theorem 15, \( \Box_r \) satisfies the Hilbert-Bernays-Löb derivability conditions (verifiably in \( \text{I} \Delta_0 + \text{Exp} \)). As we will see, the equivalence in (37) is in fact sufficient for obtaining all propositional schemata concerning \( \Box \) and \( \Box_r \) that are provable in \( \text{PA} \). In the remainder of this section, we shall introduce the bimodal logic \( \text{GL2} \), and sketch the proof of its arithmetical completeness with respect to \( \Box \) and \( \Box_r \).

**Definition 16.** The axiom schemata of \( \text{GL2} \) include all propositional tautologies in the language \( \mathcal{L}_{\Box \Box} \), the rules and axioms of \( \text{GL} \) for \( \Box \), as well as:

\[
(2) \quad \Box A \leftrightarrow \Box \Box A
\]

Given a tree-like \( \text{GL} \)-frame \( \langle W, \prec \rangle \) and \( a, b \in W \), write \( a \prec_2 b \) if there is some \( c \) with \( a \prec c \prec b \).

**Lemma 36.** \( \text{GL2} \) is sound and complete with respect to the class of tree-like \( \text{GL} \)-frames, with \( \prec \) and \( \prec_2 \) as the accessibility relations for \( \Box \) and \( \Box \) respectively.

**Proof.** Easy exercise. \( \square \)

It is proven in [26, Section 7] that \( \text{GL2} \) is the joint provability logic of \( \text{I} \Delta_0 + \text{Exp} \)-provability and cut-free provability in \( \text{I} \Delta_0 + \text{Exp} \).

**Theorem 17.** \( \text{GL2} \) is the provability logic of \( \Box_r \) and \( \Box \).

**Proof.** Suppose that \( \not \vDash_{\text{GL2}} A \). By Lemma 36, there is a \( \text{GL2} \) model \( \mathcal{M} = \langle W, \prec, \Vdash \rangle \) with \( \mathcal{M} \not \Vdash A \). By applying the usual Solovay construction for \( \text{GL} \), we find for all \( a \in \mathcal{M} \) a statement \( L = a \) in the language of arithmetic, such that for all modal sentences \( B \) not containing any occurrences of \( \Box \):

\[
a \Vdash B \quad \Rightarrow \quad \text{PA} \vdash L = a \rightarrow B',
\]

\( 42 \)
where $\ast$ is an arithmetical realization\footnote{$p^\ast$ is defined to be $\bigvee_{M, a \vdash p} L = a$. For more details, see [23], or the proof of Theorem 16 in the previous section.} mapping the modality $\triangle$ to square root provability, i.e. to $\square_t$. Now, using that $\mathcal{M} \models \triangle \triangle B \leftrightarrow \square B$ on the modal side, and $\text{PA} \vdash \square_t \square \ast B^\ast \leftrightarrow \square B^\ast$ on the arithmetical side, it is easy to check that (38) can be extended to all $B \in L_{\triangle \triangle}$ (with $\ast$ mapping $\square$ to ordinary provability). Using this, $\text{PA} \not\vdash A^\ast$ follows by the usual argument.

\section{Open Problems}

1. In the present paper we have shown that there exist two large groups of slow provability predicates of the form $\square S_n F_{\varepsilon_0}$: the predicates that are $\omega$-“roots” and $\varepsilon_0$-“roots” of the ordinary provability in $\text{PA}$. For which ordinals $\alpha \in (\omega, \varepsilon_0)$ there exist sequence $S_n$ such that $\text{PA}$ proves that and $\bigcup_{n \in \omega} S_n = \text{PA}$ and for every $\varphi$ we have $\square S_n F_{\varepsilon_0} \varphi \leftrightarrow \square \varphi$?

2. In theorems 12, 6, and 11 we have established a correspondence between ordinary and slow provability. One could also considered slow variants of $[n]$. Would the analogues of the mentioned theorem holds for $[n]$ and its slow variant?

3. In Section 9 we have proved that Lindström Logic is the provability logic of pairs of provability predicates from a rather large family. One could easily generalize Lindström logic to a polymodal case with linearly ordered family of modal connectives by stating all the copies of axioms of Lindström logic, where $\triangle$ correspond to a modality with a smaller index and $\square$ to a modality with larger index. Would the analogous of Theorem 16 holds for this logics?

4. It is easy to see that any pair of provability predicates of the form $(\triangle, \triangle^\alpha)$, where $\alpha \geq \omega$ satisfy the conditions of Theorem 16. Is it true that for every pair of provability predicates $(\triangle, \square)$ that satisfy conditions of Theorem 16 we can find elementary well-ordering and $\alpha$ from it such that $\square$ is equivalent to $\triangle^\alpha$?

5. For which natural $n > 2$ there exist a recursive $f$ such that $\text{PA}$ proves that for every $\varphi$ we have $\square_t^n \varphi \leftrightarrow \square \varphi$?
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