Sedimentation of non-Brownian suspensions

G C Abade\textsuperscript{1} and B Cichocki\textsuperscript{2}

\textsuperscript{1} Departamento de Engenharia Mecânica, Universidade de Brasília, Campus Darcy Ribeiro, 70910-900, Asa Norte, Brasília-DF, Brazil
\textsuperscript{2} Institute of Theoretical Physics, University of Warsaw, Hoża 69, 00-681 Warsaw, Poland
E-mail: abade@unb.br

Abstract. We describe the sedimentation of a non-Brownian suspension of hard spheres by Stokesian dynamics simulations. Attention is focused on the mean settling velocity and the structure of particles distribution as the suspension attains a non-equilibrium steady sedimentation state. Many-body hydrodynamic interactions between particles are evaluated using the precise multipole method, encoded in the new FAST-HYDROMULTIPOLE program. With the complete neglect of Brownian motion, the sedimentation velocity and the particles configurational distribution in the steady state differ measurably from their equilibrium counterparts. Simulations reproduce short-range particle correlations in good agreement with the statistical description of sedimentation formulated by Cichocki and Sadlej [Europhys. Lett. 72, 936 (2005)].

1. Introduction
We address the sedimentation of a dilute suspension of non-Brownian spheres, with special attention being paid to the mean sedimentation velocity and the underlying spatial distribution of the spheres throughout the fluid.

When the randomizing Brownian motion is absent, the dynamics of sedimenting spheres is governed by the interplay of permanent particle transport (driven by external forces) and the complex long-range hydrodynamic interactions mediated by the background fluid. This purely Stokesian dynamics poses severe difficulties to the determination of the suspension microstructure and the corresponding observable suspension properties, such as the average sedimentation velocity.

The average settling velocity is characterized by the sedimentation coefficient \( \frac{U}{U_0} \), where \( U \) is the mean sedimentation speed and \( U_0 \) is the settling velocity of an isolated sphere. For dilute suspensions, the sedimentation coefficient is represented in terms of an expansion in powers of the particle volume fraction \( \phi \),

\[
\frac{U}{U_0} = 1 + \lambda \phi + O(\phi^2).
\] (1)

Calculation of the linear coefficient \( \lambda \) involves the evaluation of hydrodynamic interactions and presupposes a pair correlation function \( g(r) \) describing the suspension microstructure. This pair correlation function may be assumed or derived.

Batchelor [1] obtained \( \lambda = -6.55 \) by assuming a pair correlation function that corresponds to the equilibrium particle distribution, where all configurations of non-overlapping spheres are
equally probable. Batchelor’s assumption would be plausible if the Brownian motion played a dominant role in determining the particles distribution.

The non-Brownian case was tackled by Cichocki and Sadlej [2, 3, 4]. Instead of assuming a priori an equilibrium structure, they predicted the non-equilibrium pair correlation function using statistical-mechanical considerations, providing \( \lambda = -3.87 \) [2]. This result is in excellent agreement with the value \(-3.9\), obtained by fitting Eq. (1) to the experimental data [5, 6] for the settling velocity of non-Brownian suspensions in the dilute regime. For example, at a volume fraction \( \phi = 0.05 \), Batchelor’s result predicts \( U/U_0 \approx 0.67 \), while Cichocki and Sadlej provide \( U/U_0 \approx 0.81 \). The correction to the average settling velocity arising from relaxation of the particle distribution away from equilibrium is quite considerable.

In this paper we describe, by computer simulations of the Stokesian dynamics, the relaxation of the mean sedimentation velocity as the system develops a non-equilibrium microstructure. To describe the dynamics in the bulk of a suspension which is spatially uniform on the macroscopic scale, we follow an established practice and impose periodic boundary conditions. Hydrodynamic interactions between many spheres are properly evaluated using the FAST-HYDROMULTIPOLE code [7], an accelerated version of the well-established HYDROMULTIPOLE program [8, 9, 10, 11]. The method underlying both numerical codes is based on a multipole moment expansion of the formal integral solution of the Stokes equations. In the FAST-HYDROMULTIPOLE, the speed of calculation is improved by use of the Fast Multipole Method (FMM), developed by Greengard and Rokhlin [12] and widely used for efficient computation of gravitational and electrostatic interactions. For hydrodynamic interactions, the FMM allows for evaluation of the velocity fields generated by a collection \( N \) particles in \( O(N) \) operations.

This article is organized as follows. Section 2 describes our model system and the equations governing the Stokesian dynamics of suspensions. The numerical scheme for calculation of hydrodynamic interactions is outlined. Section 3 presents simulation results for the mean sedimentation velocity and positional particle correlations in a dilute suspension. Section 4 contains discussions.

2. Model system and simulation method

We consider a collection of \( N \) hard spheres, each of radius \( a \), immersed in an unbounded viscous incompressible fluid of shear viscosity \( \eta \). The spheres are made to move through the fluid by an externally applied constant force \( \mathbf{F} \) acting on each of the particles. The motion of the immersed spheres and the suspending fluid are coupled by no-slip velocity boundary condition at the fluid-particle interfaces. The set up fluid flow is assumed to be described by the stationary Stokes equations,

\[
\eta \nabla^2 \mathbf{v} - \nabla p = 0, \quad \nabla \cdot \mathbf{v} = 0, \tag{2}
\]

where \( \mathbf{v} \) is the fluid velocity and \( p \) is the pressure field.

The suspended spheres, centered at \( \mathbf{X} = (\mathbf{R}_1, \ldots, \mathbf{R}_N) \), undergo a rigid body motion with translational velocities \( \mathbf{U} = (\mathbf{U}_1, \ldots, \mathbf{U}_N) \). It follows from linearity of Eq. (2) and boundary conditions that particle velocities \( \mathbf{U} \) are linearly related to the driving forces \( \mathbf{F} = (\mathbf{F}, \ldots, \mathbf{F}) \),

\[
\frac{d\mathbf{X}}{dt} = \mathbf{U} = \mathbf{\mu(X)}F, \tag{3}
\]

where \( \mathbf{\mu} \) is the \( 3N \times 3N \) translational mobility matrix depending on the instantaneous configuration \( \mathbf{X} \). The mobility matrix formally embodies all information on the fluid mediated hydrodynamic interactions. Next we outline the scheme for proper account of these interactions.
2.1. The mobility matrix

The $N$-particle mobility matrix $\mu$ may be expressed formally as follows [13]:

$$\mu = \mu_0 + \mu_0 Z_0 \sum_{k=0}^{\infty} (-G\hat{Z}_0)^k GZ_0 \mu_0,$$

(4)

where $\mu_0 = (6\pi\eta a)^{-1}1$ is the mobility of an isolated sphere and $1$ is the unit matrix. The second term on the right-hand side of Eq. (4) accounts for the hydrodynamic interactions. The hydrodynamic operators $Z_0$, $\hat{Z}_0$, and $G$ are $N$-particle matrices with

$$[Z_0]_{ij} = Z_0(i)\delta_{ij}, \quad [G]_{ij} = G(ij)(1 - \delta_{ij}), \quad i, j = 1, \ldots, N,$$

(5)

where $Z_0(i)$ and $G(ij)$ are, respectively, single- and two-particle integral operators. $\hat{Z}_0$ has the same single-particle structure as $Z_0$. The hydrodynamic operators will be defined below (details may be found e.g. in Ref. [11]).

When acted on by an external force $F$, a source particle $j$ exerts on the fluid a force density, distributed along the fluid-particle interface. This force density gives rise to a disturbing flow field, which is propagated and eventually collides with the other suspended particles. The two-particle Green integral propagator $G(ij)$ appearing in Eq. (5) describes the incident velocity field on the target particle $i$ due to a force density applied on the fluid by the source particle $j$.

When immersed in the incident fluid flow generated by the suspended particles and acted on by $F$, a test particle $i$ resists the flow; it exerts on the fluid a force density, which produces a reflecting flow field. The object $Z_0(i)$ (Eq. (5)) gives the force density on the surface of particle $i$ in response to a given incident field. Finally the operator $\hat{Z}_0$ is the convective counterpart of $Z_0$, i.e. corresponds to the situation when the immersed particle is convected by the incident flow without exerting net force or torque on the fluid.

In Equation (4), the term

$$\sum_{k=0}^{\infty} (-G\hat{Z}_0)^k = 1 - G\hat{Z}_0 + G\hat{Z}_0 G\hat{Z}_0 - \cdots$$

(6)

describes the infinite sequence of reflected (or scattered) flow fields. The scattering series (6) is responsible for the full account of the many-body hydrodynamic interactions.

2.2. Fast evaluation of particle velocities

Fast schemes for Stokesian dynamics simulations avoid explicit calculation of the $3N \times 3N$ mobility matrix $\mu$. Evaluation of the $3N$-dimensional vector of particle velocities $U$ in response to known driving forces $F$ suffices to generate particle trajectories according to (3).

Recalling Eq. (3), the determination of particle velocities $U$ is done by applying the mobility matrix to $F$. This operation entails several matrix-vector multiplications involving $Z_0$, $\hat{Z}_0$ and $G$ (see (4) and (6)). The single-particle matrices, $Z_0$ and $\hat{Z}_0$, are diagonal in particle indices, and their application to an $3N$-dimensional vector is a fast $O(N)$ operation. The $N$-particle matrix $G$ of propagators (defined in Eq. (5)) has however $O(N^2)$ nonzero elements.

An accelerated scheme for evaluating particle velocities is based on a sparse representation of the matrix $G$ suitable for fast computation of matrix-vector multiplication. This representation is constructed by employing the ideas underlying the Fast Multipole Method (FMM) [12].

Application of the FMM to hydrodynamic interactions [7] yields a formal representation of matrix $G$ in the form

$$G = [G]_N + S^\dagger G S,$$

(7)
where $\mathbf{G}_N$ is a restriction of $\mathbf{G}$ which includes only two-particle propagators $\mathbf{G}(ij)$ between nearest-neighbours, and thereby contains $O(N)$ nonzero elements. The term $\mathbf{S}^\dagger \tilde{\mathbf{G}} \mathbf{S}$ is formally an $O(N)$ sparse matrix accounting for hydrodynamic interactions between the remaining well-separated particles. Matrix $\tilde{\mathbf{G}}$ is formed by Green propagators connecting well-separated groups of particles, arranged at different length scales in a self-similar ($\mathbf{X}$-independent) structure. Hence $\tilde{\mathbf{G}}$ is completely decoupled from the details of the configuration of particles centres and may be pre-calculated. The operator $\mathbf{S}$ and its adjoint $\mathbf{S}^\dagger$ (defined in Ref. [7]) are responsible for this decoupling. Details of the numerical implementation of this scheme will be published separately.

2.3. Multipole method
Numerical calculation of the mobility matrix requires transformation of the integral operators $\mathbf{Z}_0$, $\hat{\mathbf{Z}}_0$ and $\mathbf{G}$ (Eq. (4)) into algebraic ones. Transformation is done by projecting and expanding them in spherical multipole functions around particle centers (see e.g. Refs. [10, 11, 14] for details). Multipole functions are indexed by $l = 1, 2, \ldots$, $m = -l, \ldots, l$, and $\sigma = 0, 1, 2$. Explicit expressions for the multipole elements of the hydrodynamic operators may be found in Ref. [14].

It follows from the multipole expansion of the Green propagator [15] that

\begin{equation}
\langle lm\sigma|\mathbf{G}(\mathbf{R})|l'm\sigma'\rangle \sim \frac{1}{R^{l+l'+\sigma+\sigma'-1}},
\end{equation}

where $\mathbf{R}$ is the relative distance vector of two centers of expansion. These may be the centers of two interacting spheres or two interacting groups of particles in the fast scheme.

This method has the distinguishing characteristic that the multipole functions are expressed in terms of an irreducible representation. Therefore Eq. (8) furnishes a complete insight into the structure of the spatial decay of interactions between point multipoles of different orders (labeled by $l m \sigma$ and $l'm\sigma'$). To account for essential collective effects due to hydrodynamic interactions, a reasonable prescription is to keep all long-range multipole matrix elements of the propagator $\mathbf{G}(\mathbf{R})$, i.e. decaying as slowly as $1/R^3$ [10, 16]. This is done by including all multipoles with $l \leq 3$.

Summarizing, Eqs. (3), (4), and (7), with an irreducible multipole representation of the hydrodynamic operators, form the basis of the FAST-HYDROMULTIPOLE simulation tool. Our treatment combines proper high-order multipole truncation and a full account of many-body hydrodynamic interactions through the multiple scattering series (6). This account includes all essential effects to assure the nonvanishing of the divergence of particle velocities in the Stokes-Liouville equation (see e.g. Ref. [2]), the statistical-mechanical equivalent of (3). Important lubrication effects are also included as explained in Ref. [17].

2.4. Simulations
The Stokesian dynamics equations (3) were numerically integrated using the fourth-order Adams-Bashforth algorithm [18]. Since inherent numerical errors in the solution of Eq. (3) by finite-difference integration algorithms may yield unrealistic overlaps between sedimenting spheres, we have devised an alternative scheme for stepping in time particle configurations containing close spheres in relative motion. In our approach, the numerical integration algorithm is supplemented with the physical information that hydrodynamic lubrication forces (which diverge at contact) prevent close spheres from touching each other. The proposed method is fast and effectively prevents particle overlaps without the requirement of prohibitively small integration time-steps. We will provide a detailed description of this algorithm in a separate publication.

Simulations were carried out for the volume fraction $\phi = 0.05$ and for two system sizes, $N = 512$ and 1024. Integration of Eq. (3) considered a time-step $\Delta t = 0.05a/U_0$ (where $U_0$ is the settling velocity of an isolated sphere). The time evolution started from two different initial
distributions of non-overlapping spheres: (i) equilibrium with periodic boundary conditions and (ii) random distribution of hard spheres inside a box with impenetrable walls. The condition of impenetrable walls is imposed only for the purpose of generating an initial configurational distribution of the particles which is different from the equilibrium one. Stokesian dynamics simulations in this work were carried out under periodic boundary conditions only. In this way one may verify if the system evolves to the same stationary state, independent of initial conditions. Time evolutions were carried out until measurements of averaged variables were deemed to have reached steady values.

3. Results
3.1. Evolution of the mean settling velocity
This section presents results for the time evolution of the mean sedimentation velocity as the sedimenting suspension develops a non-equilibrium structure.

![Figure 1](image)

**Figure 1.** Evolution of the $N$-particle average sedimentation velocity for the volume fraction $\phi = 0.05$ and two system sizes: (a) $N = 512$; (b) $N = 1024$. The lower curves correspond to simulations starting from equilibrium distributions.

Figure 1 shows the evolution of the $N$-particle mean sedimentation velocity for $N = 512$ [Fig. 1(a)] and $N = 1024$ [Fig. 1(b)], starting from two different initial particle distributions. The lower curves correspond to simulations starting from equilibrium configurations. For each system size studied a stationary sedimentation state has been established, being this state independent (within statistical errors) of the initial conditions considered. For $N = 512$, the settling velocity attains a steady value after an evolution time of 300 $a/U_0$ approximately. For the system comprised of 1024 particles, the sedimentation velocity has been relaxed for times larger than 200$a/U_0$.

We emphasize that the sedimentation velocity $U_N/U_0$ plotted in Fig. 1 is for a finite $N$-particle system. It exhibits considerable system-size dependence due to periodicity (values of $U_N/U_0$ for $N = 1024$ are slightly larger than those for $N = 512$). To compare with experiments and theory, $U_N/U_0$ must be extrapolated to its infinite-system counterpart, $U_\infty/U_0$. Extrapolation is performed according to [19]

$$
\frac{U}{U_0} = \frac{U_N}{U_0} + 1.76S(0) \frac{\eta}{\eta_\infty} \left( \frac{\phi}{N} \right)^{1/3}.
$$

(9)
Application of Eq. (9) requires as inputs the static structure factor at zero wavenumber, \( S(0) \), and the effective suspension viscosity, \( \eta_{\text{eff}} \). Both \( S(0) \) and \( \eta_{\text{eff}} \) depend on the suspension microstructure. As an approximation, we use Einstein’s expression for the suspension viscosity [20], \( \eta_{\text{eff}}/\eta = 1 + 2.5\phi + O(\phi^2) \). For the steady-state structure factor, we employ the result \( S(0) = 1 - 1.64\phi + O(\phi^2) \) derived by Cichocki and Sadlej [2] from their short-range pair correlation function in the steady state. For equilibrium microstructures, \( S(0) \) may be evaluated by the Carnahan-Starling expression [19].

Finally we remark that the mean settling rate increases as the suspension evolves away from the equilibrium, i.e. as the suspension structure relaxes towards the non-equilibrium stationary particle distribution. After extrapolation according to Eq. (9), simulations provide the value \( U/U_0 = 0.78 \pm 0.01 \) for the mean sedimentation velocity in the stationary state, which is larger than sedimentation coefficient \( U/U_0 = 0.72 \pm 0.01 \) for an equilibrium distribution at \( \phi = 0.05 \). The uncertainty (±0.01) in the stationary value of the mean settling velocity was estimated by use of the “blocking” method [21].

### 3.2. Particle correlations in the steady-state
In this section we present simulation results for the steady-state pair correlation function, \( g(r) \).

![Figure 2](image-url)

**Figure 2.** Particle correlations for a steadily sedimenting suspension. (a) The simulated pair correlation function in the steady sedimentation state (solid symbols) compared with its equilibrium counterpart (solid line) at the same volume fraction \( \phi = 0.05 \). The dashed line represents the stationary-state theoretical result by Cichocki and Sadlej [2]. Both steady-state and equilibrium pair correlation functions vanish for \( r/a < 2 \). (b) Structure factor for a steadily sedimenting suspension (solid symbols) being compared with the equilibrium distribution (solid line) of hard spheres.

Figure 2(a) shows the spherically averaged numerical results for \( g(r) \) in the stationary state in comparison with its equilibrium counterpart, \( g_{\text{eq}}(r) \). It may be observed that sedimentation produces significant changes in pair correlations relative to equilibrium. It induces the formation of close pairs, which tend to stay together for long times. Prevalence of close pairs explains the increase in the mean settling rate (shown in Fig. 1) as the suspension evolves away from the equilibrium distribution. For our hard-sphere model system, both equilibrium and steady-state pair correlation functions vanish for \( r/a < 2 \).
Remarkably, Figure 2(a) displays an excellent agreement between our numerical results for $g(r)$ and the isotropic pair correlation function derived theoretically by Cichocki and Sadlej [2] for a dilute suspension (denoted henceforth in the text by $g_{cs}(r)$ for convenience). Agreement with $g_{cs}(r)$ is important since it allows for further exploration of our computer simulation results in the light of the transparent structure of homogeneous sedimentation provided by the Cichocki-Sadlej description. We also point out that our numerical results for $g(r)$ qualitatively agree with the pair distribution obtained by Ladd [22] via large-scale Lattice-Boltzmann simulations. However, Ladd’s results were obtained for a larger volume fraction, $\phi = 0.10$.

Further insight into the suspension microstructure at larger length scales is obtained by examining the static structure factor, $S(q)$, related to $g(r)$ by Fourier transform. Figure 2(b) shows our simulation results for the non-equilibrium $S(q)$ in comparison with its equilibrium counterpart, $S_{eq}(q)$. The shape of $S_{eq}(q)$ presented in Fig. 2(b) indicates particle correlations arising from hard-core interactions (recall that $g_{eq}(r) = 0$ for $r < 2a$).

At wavenumbers $qa \geq 1$, the steady-state data for $S(q)$ obtained from different system sizes and initial particle distributions collapsed into the same curve, with a smooth and well-established wavenumber dependence. At such large wavenumbers, where the short-range structure in a length scale up to $r \sim 6a$ is probed, we were able to obtain an stabilized structure in the stationary state which measurably differs from the equilibrium one. Relative to $S_{eq}(q)$, the position of first peak of $S(q)$ has been slightly displaced to the right, what evidences that sedimentation increases pair correlations in a length scale of the order of the particles diameter.

In the low-$q$ region, however, large fluctuations in $S(q)$ may be found. For a finite system, accurate measurements of the low-$q$ behaviour of $S(q)$ from computer simulation data is particularly troublesome. Considering a simulation of $N$ particles at density $n$ in a periodic cube of side $L = (N/n)^{1/3}$, any information on the microstructure at long-wavelengths beyond $L$ will be suppressed by periodicity. Accordingly, $q = 2\pi/L$ is the smallest accessible wavenumber. Therefore, model systems as large as possible are required to probe the structure in the limit of macroscopic range, i.e. in the limit $q \to 0$ or $L \to \infty$.

We further discuss and summarize our results in the next section.

4. Discussion
The present work reported on computer simulation results of non-Brownian sedimentation that accompany the Cichocki-Sadlej (C-S) statistical description. The C-S treatment is based on the Stokes-Liouville equation [2] (the statistical-mechanical equivalent of Eq. (3)) in terms of reduced distribution functions describing the dynamics of two hydrodynamically interacting spheres under the intervening influence of a third one. Simulations otherwise follow in detail the full many-particle Stokesian dynamics (3). Consistently, present simulations and the C-S theory share the same careful account of hydrodynamic interactions, using the irreducible multipole description corrected for lubrication effects.

At the studied volume fraction ($\phi = 0.05$), simulations predicted (within statistical uncertainties) a unique non-equilibrium stationary sedimentation state, i.e. independent of the starting particle distribution. With the complete neglect of Brownian motion, the sedimentation velocity and the positional particle correlations in the steady state differ measurably from their equilibrium counterparts.

The short-range pair correlation function in the steady state was found to be in agreement with that predicted by Cichocki and Sadlej [2] (denoted by $g_{cs}(r)$). We have avoided any further analysis on the long-range structure of a sedimenting suspension, since the low-$q$ behaviour of the structure factor $S(q)$ cannot be properly probed from our simulation data. However, the C-S description provides a framework for speculating on the simulated long-range structure by examining the pair correlation function in the short-range. In fact not much is known about the long-range structure of a steadily sedimenting suspension, apart from the fundamental
requirement that it must be consistent with the locality of the suspension bulk properties (i.e. they have a well-defined value in the thermodynamic limit). Locality is guaranteed if the pair correlation function \( g(r) \) satisfies the Koch-Shaqfeh screening condition \([23]\),

\[
n \int dr[g(r) - 1] = -1, \tag{10}
\]

where \( n = N/V \) is the particle number density.

We remark that the condition (10) is a prescription underlying the Cichocki-Sadlej derivation of \( g_{CS}(r) \). Since \( g_{CS}(r) \) fits well to the simulated short-range particle correlations (as Fig. 2 shows), it is tempting to suggest that the simulated microstructure would be consistent with the criterion (10) if the large-scale structures were not cut off at the length of the periodic simulation box. However, this speculation is not conclusive; it must be confirmed by simulations of much larger model systems. For the present, due to system size limitations, agreement with \( g_{CS}(r) \) is the only available way to assess consistency of the simulated pair correlation with the Koch-Shaqfeh screening condition (10).

Further analysis of \( g_{CS}(r) \) provides an insightful picture on how sedimentation builds up short-range pair correlations in dilute suspensions. As remarked in Ref. \([2]\), the shape of \( g_{CS}(r) \) coincides with the pair correlation function calculated by Batchelor and Green \([24]\) for the problem of two spheres immersed in a pure straining flow. This suggests that a pair of close spheres experiences the influence of a third far particle through an effective pure straining flow \([2]\). Conforming our simulation results to this picture, one may say that simulated long-range structure in the steady state yields an ambient flow pattern reminiscent of a straining fluid motion, which increases pair correlations at small separations and determines the characteristic structure of \( g(r) \) represented in Fig. 2(a).

Finally we remark that a fully developed stationary sedimentation state presupposes relaxation of both short and long-range microstructures. Here, under the limitations of a finite-system, attention was restricted to the short-range structure and its correspondent mean sedimentation velocity. Computer simulations to explore the long-range structure and the behaviour of particle velocity fluctuations in the limit of large systems are in progress. We should be able then to investigate if sedimentation produces an anisotropic pair correlation in the long-range. On the theoretical side, a tentative non-equilibrium statistical-mechanical description of particle correlations in the long-range is available \([25]\). A description of the long-range structure based on the Stokes-Liouville equation still remains to be formulated.

Acknowledgments

This work is part of G.C. Abade’s doctoral thesis, elaborated at the University of Warsaw, Institute of Theoretical Physics, under the supervision of Prof. Bogdan Cichocki, and with financial support from CAPES Foundation/Ministry of Education of Brazil. G.C. Abade acknowledges Dr. Maria Ekiel-Je˙zewska for her kind invitation and support for participating in the Symposium “Microparticles in Stokes flow” in Honor of François Feuillebois’ 65th Birthday. The authors are much indebted to Dr. Eligiusz Wajnryb for indispensable assistance throughout the implementation of the FAST-HYDROMULTIPOLE simulation tool. Valuable discussions with Prof. B. U. Felderhof are gratefully acknowledged. Numerical calculations were carried out at the Centre of Excellence BioExploratorium in Warsaw, Poland.

References

[1] Batchelor G K 1972 *J. Fluid Mech.* **52** 245–268
[2] Cichocki B and Sadlej K 2005 *Europhys. Lett.* **72** 936–942
[3] Cichocki B and Sadlej K 2006 *Arch. Mech.* **58** 327–337
[4] Cichocki B and Sadlej K 2008 *J. Stat. Phys* **132** 129–151
[5] Ham J M and Homsy G M 1988 *Int. J. Multiphase Flow* **14** 533–546
[6] Hanratty T J and Bandukwala A 1957 *AIChE J.* **3** 293–296
[7] Abade G C 2009 *Analysis of transport phenomena in concentrated suspensions by numerical simulations* Ph.D. thesis Warsaw University
[8] Felderhof B U 1988 *Physica A* **151** 1
[9] Cichocki B, Felderhof B U and Schmitz R 1988 *PhysicoChem. Hyd.* **10** 383–403
[10] Cichocki B, Felderhof B, Hinsen K, Wajnryb E and Blawzdziewicz J 1994 *J. Chem. Phys.* **100** 3780
[11] Cichocki B, Jones R B, Kutteh R and E W 2000 *J. Chem. Phys.* **112** 2548–2561
[12] Greengard L and Rokhlin V 1987 *J. Comput. Phys.* **73** 325–348
[13] Szmyczak P and Cichocki B 2004 *J. Chem. Phys.* **121** 3329
[14] Wajnryb E, Szmyczak P and Cichocki B 2004 *Physica A* **335** 339
[15] Felderhof B U and Jones R B 1989 *J. Math. Phys.* **30** 339
[16] Cichocki B 1996 *Proceedings of the Eighth International Symposium on Continuum Models and Discrete Systems* ed Markov K Z pp 15–20
[17] Cichocki B, Ekiel-Jeżewska M L and Wajnryb E 1999 *J. Chem. Phys.* **111** 3265
[18] Press W H, Teukolsky S A, Vetterling W T and Flannery B P 1992 *Numerical Recipes in Fortran 77, Vol. 1* (Cambridge University Press)
[19] Mo G and Sangani A S 1994 *Phys. Fluids* **6** 1637–1652
[20] Einstein A 1956 *Investigations on the theory of the Brownian motion* (Dover)
[21] Flyvbjerg H and Petersen H G 1989 *J. Chem. Phys.* **91** 461–466
[22] Ladd A J C 1993 *Phys. Fluids A* **5** 299–310
[23] Koch D L and Shaqfeh E S G 1991 *J. Fluid Mech.* **224** 275
[24] Batchelor G K and Green J T 1972 *J. Fluid Mech.* **56** 375–400
[25] Felderhof B U 2005 *Physica A* **348** 16–36