On dualities for SSEP and ASEP with open boundary conditions

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Received 24 February 2016, revised 2 December 2016
Accepted for publication 5 January 2017
Published 30 January 2017

Abstract

Duality relations for simple exclusion processes with general open boundaries are discussed. It is shown that a combination of spin operators and bosonic operators enables us to have a unified discussion about duality relations with open boundaries. As for the symmetric simple exclusion process (SSEP), more general results than those from previous studies are obtained. It is clarified that not only the absorbing sites, but also additional sites—called copying sites—are needed for the boundaries in the dual process for the SSEP. The role of the copying sites is to conserve information about the particle states on the boundary sites. Similar discussions are applied to the asymmetric simple exclusion process (ASEP), in which the $q$-analogues are employed, and it is clarified that the ASEP with open boundaries has a complicated dual process on the boundaries.

Keywords: symmetric simple exclusion process, asymmetric simple exclusion process, duality, Doi–Peliti formalism

1. Introduction

The concept of duality relations has been widely used in various areas of research. In particular, in recent years, duality relations have been used to investigate various stochastic processes, ranging from stochastic differential equations [1–4] to interacting particle systems, which include a symmetric simple exclusion process (SSEP) and an asymmetric simple exclusion process (ASEP) [5–9]. For example, the SSEP and ASEP with reflective boundaries have self-duality properties, and it has been shown that the correlations in the original SSEP and ASEP can easily be investigated by using the corresponding dual processes (for example, see [5] and [6]).

Although the dual processes and duality functions have sometimes been derived heuristically, there are also a few systematic ways to investigate duality relations. It has already been
shown that symmetries of the generators are useful for deriving the duality functions and dual processes \cite{10}. In \cite{10}, the usefulness of the symmetries of the generators has been demonstrated; the ‘classical duality’ (in the sense of \cite{5}) has been adequately derived. However, in general, duality studies with boundary-driven cases are difficult. The SSEP with a specific open boundary condition has been discussed in \cite{10}, but the derivation includes some heuristic parts. Furthermore, as far as I know, the duality relations for the ASEP with open boundaries have not yet been discussed. Recently, the ASEP with periodic boundary conditions on a low current has been discussed \cite{11}, but discussing the open boundary cases is an important remaining task.

In the present paper, some discussions on the duality relations are given for the SSEP and ASEP with open boundary conditions. The discussions are based on the symmetries of the quantum Hamiltonian and the recent developments on the usage of bosonic operators, i.e. the so-called Doi–Peliti formalism \cite{12, 13}. Additionally, as for the ASEP, the $q$-analogues of the exponential functions are employed.

Firstly, the SSEP case is discussed, and a more general result than that of the previous work is obtained; only the absorbing states are needed for the dual stochastic process in previous studies \cite{10}, but additional sites should be used for more general open boundary conditions. The additional sites are called ‘copying sites’ in the present paper, and the role is to conserve the particle states on the boundary sites. Secondly, the ASEP with open boundary conditions is discussed: from a derivation of the dual time-evolution operator, it is clarified that the open boundary conditions in the ASEP result in very complicated dual processes. This means that the standard duality relations cannot be used for the ASEP case—at least at this stage. Using some techniques proposed in the present paper, the current work represents the first time that this fact has been revealed.

The construction of the present paper is as follows. In section 2, the formalism, some definitions, and a basic idea for deriving the duality relations are explained. Section 3 gives a re-derivation of the duality relations for the SSEP without open boundaries. The first main contribution of the present paper is given in section 4; the SSEP with open boundaries is discussed, employing a technique with a bosonic operator formalism, and the general duality relations are derived. In section 5, the previously known duality relations for the ASEP without open boundaries are re-derived; this discussion gives us a basis for the cases with open boundaries. Section 6 is the second main contribution of the present paper, where the duality relations in the ASEP with open boundaries are discussed. Finally, some concluding remarks are made in section 7.

2. Definitions, notations, and a basic idea

Firstly, some notations based on the quantum Hamiltonian formalism, which are useful for the following discussions, are introduced. After that, a basic idea for deriving the duality relations will be shown.

2.1. Quantum spin language

Here, we employ the following formulation based on the quantum spin language; for details, see, e.g. \cite{14}. We set

$$
\begin{align*}
{s^+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \quad {s^-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & \quad {s^z} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
$$

(1)
and the number operator is defined as
\[ n = \frac{1}{2} I - s^z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]  
(2)

Note that the above spin operators obey the following commutation relations:
\[ [s^+, s^z] = \pm s^z, \quad [s^+, s^-] = 2s^z. \]  
(3)

### 2.2. Time-evolutions for the SSEP and ASEP

Denote the empty system as \( |0\rangle \), and one can construct an \( n \) particle state with particle positions at \( x_1, \ldots, x_n \) by
\[ |x_1, \ldots, x_n\rangle = s^+_{x_1} \cdots s^+_{x_n} |0\rangle, \]  
(4)

where the operator \( s^+_i \) puts a particle on site \( i \). In contrast, the operator \( s^-_i \) destroys the particle on site \( i \). Hence, the operator corresponding to the particle hopping from site \( i \) to site \( j \) is written as \( s^-_i s^+_j \). The state of the system can also be specified by \( \eta: \eta_i = 1 \) (respectively \( \eta_i = 0 \)) means that site \( i \) is occupied (respectively empty). Sometimes the system state is abbreviated as \( |\eta\rangle \) with \( \eta = \{\eta_i | i \in S\} \), where \( S \) denotes the set of all sites. Note that the state vector for site \( i \) is written explicitly in terms of vectors;
\[ |\eta_i = 1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |\eta_i = 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]  
(5)

In the present paper, we assume that the underlying lattice is one dimensional, that is, a finite lattice \( L = \{1, 2, \ldots, L\} \). If only the SSEP cases are discussed, it is possible to deal with arbitrary lattice structures. However, as we will see in section 6, the discussion for the ASEP relies on a one dimensional structure. Although it might be possible to extend the following discussions to the ASEP in general lattice cases, it would become very complicated, and hence the current study focuses only on the one dimensional lattice.

Let \( P(\eta, t) \) be the probability that the configuration of the system is \( \eta \) at time \( t \). Define
\[ |P(t)\rangle = \sum_\eta P(\eta, t) |\eta\rangle, \]  
(6)

where \( \sum_\eta \) means the summation over all particle configurations. Using a quantum Hamiltonian \( H \), the time-evolution for \( |P(t)\rangle \) in the ASEP is given by
\[ \frac{d}{dt} |P(t)\rangle = -H |P(t)\rangle, \]  
(7)

where \( H \) is defined as follows:
\[ H = H^{\text{bulk}} + H^1 + H^L. \]  
(8)

Here, \( H^{\text{bulk}} \) denotes the transition matrix for the bulk part, and
\[ H^{\text{bulk}} = -\sum_{k=1}^{L-1} \left[ \alpha_k (s^-_k s^+_k - n_k n_{k+1}) + \beta_k (s^+_k s^-_{k+1} - n_k (1 - n_{k+1})) \right]. \]  
(9)

The open boundary conditions are given by \( H^1 \) and \( H^L \) as follows:
Note that when we set $H = H_{\text{bulk}}$, the quantum Hamiltonian gives the ASEP with reflective boundaries.

In the following discussions, as in the previous work [6], we assume that

\[ q_k = \frac{\alpha_k}{\beta_k} = q, \quad \mu_k = \sqrt{\alpha_k / \beta_k}. \tag{12} \]

That is, the asymmetry of the system is uniform, but the mobility can depend on the lattice sites.

If we set $q = 1$, the Hamiltonian (8) gives the time-evolution for the SSEP with open boundaries. Note that the above open boundary conditions are extensions of the previous work in [10]. In [10], the parameters for the boundaries take a restricted form: the boundary condition is interpreted as a particle reservoir whose particle density is assumed to be less than one. Hence, for example, the in-rate of particles at the boundary site 1 only takes $0, 1$ and the out-rate is determined as $\gamma_{1}^{\text{out}} = 1 - \gamma_{1}^{\text{in}}$. In the present paper, this restriction is not needed.

2.3. Basic idea for the duality relations

In [10], the duality relations have been discussed based on the symmetries of the generators. In the present paper, a different derivation using bra–ket notation is employed, which might be more familiar to many physicists. Note that the explanation here is only the basic and formal one; concrete examples for the derivations are given in the following sections. That is, as for the purpose of the illustration of the derivations, a simple dual process, i.e. a self-dual process, is used here. As explained later, the open-boundary cases need different kinds of dual processes.

Firstly, we introduce the following dual (bra) vectors:

\[ \langle \eta' | = (0 1), \quad \langle \eta' = 0 | = (1 0), \tag{13} \]

where $\langle \eta' = 1 |$ means that a particle is on site $i$ in the dual process, and $\langle \eta' = 0 |$ corresponds to the absence of a particle on site $i$. Hence, for the dual process, the spin operator $s_i^-$ makes a particle on site $i$, and $s_i^+$ destroys a particle on site $i$; the role of the spin operators is changed compared with the original (ket) vectors.

As in the original process, we abbreviate the particle state of the dual process as $\langle \eta' |$ with $\eta' = \{ \eta'_i | i \in \mathcal{S} \}$. In addition, the state vector $\langle P(t) |$ for the dual process is defined as

\[ \langle P(t) | = \sum_{\eta'} P(\eta', t) \langle \eta' |, \tag{14} \]

where $P(\eta', t)$ corresponds to the probability for the dual process with which the state is $\eta'$ at time $t$. Note that, at this stage, we have not mentioned the time-evolution of the dual process; the explicit time-evolution will be given by using examples in the following discussions. In addition, note again that the dual processes should have additional sites if we have the open boundary conditions; these extensions are discussed later.
Secondly, choose a time-independent operator $A$, which acts on the state vectors. In general, the operator $A$ does not commute with the time-evolution operator $H$ in (8). Then, the duality relations can be considered as an explicit expression for the following quantity:

$$\langle P'(t = 0)|A|P(t) \rangle = \langle P'(t = 0)|e^{-itH}|P(t = 0) \rangle$$

$$\langle P'(t = 0)|e^{-itH}|P(t = 0) \rangle = \langle P'(t)|A|P(t = 0) \rangle.$$ (15)

Here, the new time-evolution operator $\widetilde{H}$, which stems from the interchange of $e^{-itH}$ with $A$, is introduced. If $-\widetilde{H}$ is the generator of a Markov process, (15) gives the conventional duality relations between two stochastic processes [5]. Instead of the time-evolution of the original process, that of the dual process is available to evaluate the quantity $A$ in the original process at time $t$. Although there is no guarantee that $-\widetilde{H}$ will give a Markov process in general, sometimes we can make generators of Markov processes using some of the techniques introduced in section 4.3.

3. Re-derivation of the duality in the SSEP without open boundaries

Here, we briefly explain the derivation of the duality relations in the SSEP without open boundaries, based on the basic idea in (15). In this section, we neglect $H'$ and $H_L$ in (8), and set $H = H_{\text{bulk}}$; in addition, we set $q = 1$.

To begin with, we must choose the operator $A$. If the operator $A$ is chosen adequately, we can calculate important quantities for the original SSEP by solving the dual process. In addition, for SSEP problems, the time-evolution operator has the following special property:

$$H = H_{\text{bulk}} = (H_{\text{bulk}})^T.$$ (16)

Hence, if we choose the operator $A$ as satisfying the commutative property with $H = H_{\text{bulk}}$, we have $\widetilde{H} = H_{\text{bulk}}$ in (15), and then the dual process obeys the same time-evolutions with the original SSEP. Because of this self-dual property, the operator $A$ satisfying $[H, A] = 0$ is expected to be easy to discuss.

The simplest example is $A = I$, i.e. the identity operator. In this case, the duality relation reduces to a simple transition probability; when $\langle \eta' \rangle = (x'_1, \ldots, x'_n)$ and $|\eta \rangle = |x_1 \ldots x_n \rangle$, this corresponds to the probability that $n$ particles starting from $x_1, \ldots, x_n$ at time 0 are on sites $x'_1, \ldots, x'_n$ at time $t$. In this case, defining the following duality function

$$D(\eta, \eta') = \prod_{i \in S} \delta_{\eta_i, \eta'_i},$$ (17)

we have the duality relation

$$\mathbb{E}_\eta [D(\eta, \eta')] = \mathbb{E}^\text{dual}_{\eta'} [D(\eta, \eta')],$$ (18)

where $\mathbb{E}_\eta$ means the expectation for the time-evolution in the original SSEP process $\eta_i$ starting from the initial state $\eta_i$; $\mathbb{E}^\text{dual}_{\eta'}$ corresponds to the expectation in the dual stochastic process $\eta'_i$ starting from $\eta'_i$ at $t = 0$; see (15) and its interpretation. Note that $H_{\text{bulk}} = (H_{\text{bulk}})^T$, and hence the time-evolution of the dual process is the same as the original one. Of course, it would not be common to call this simplest case a duality relation.

As for a nontrivial example of the duality relations in the SSEP, here we choose $A = \exp(\sum_{i \in S} S_i^+)$ (see, for example, [10]). Note that the operator $A$ commutes with the time-evolution operator $H$ [10];
In this case, we have the following nontrivial duality function:

\[ D(\eta, \eta') = \prod_{i \in S, \{i', \in 1\}} \eta_i. \]  

(20)

The product of the above duality function means that if only site \(i\) in the dual SSEP has a particle, the product is taken according to the variable \(\eta_i\) for the original SSEP. In order to understand the derivation, the following discussions would be helpful. That is, an empty site in the dual process does not contribute to the duality function because \(0|0\rangle = 0, (0|s^+|0\rangle = 0, \ldots, (0|1\rangle = 0, (0|s^+|1\rangle = 1, (0|(s^+)^2|1\rangle = 0, and so on; after the expansion of \(\exp(s^+)\), the empty site in the dual process equally gives a factor 1, regardless of the particle state in the original SSEP. In contrast, we have \(1|0\rangle = 0, (1|s^+|0\rangle = 0, \ldots, (1|1\rangle = 1, (1|s^+|1\rangle = 0, and so on. Hence, the occupied site \(\eta' = 1\) makes different contributions depending on the particle state in the original process \(\eta\). Using the facts, the duality relation (20) is derived (see [10] for other explicit derivations).

Based on these nontrivial duality relations, it is possible to calculate the \(m\)th correlation function for the original SSEP with \(n\) particles by using the dual SSEP with only \(m\) particles. For example, if we want to calculate the 2-body correlation function, the dual SSEP always has only two particles; compared with the original SSEP, the dual SSEP is easy to deal with.

Note that the above derivation for the duality relations based on the bra and ket notations gives the same result as that based on the symmetries of the generators in [10]. However, we will show that the bra and ket notations are suitable for extending the discussion to open boundary cases.

4. Duality in the SSEP with open boundaries

4.1. Non-commutative property of the quantity

As in the previous section, the quantity \(A = \exp(\sum_{i \in S} s_i^+)\) is considered here because this quantity enables us to calculate the correlation functions. However, when there are open boundary conditions, the operator \(A\) does not commute with the time-evolution operator \(H\);

\[
\left[ \exp\left( \sum_{i \in S} s_i^+ \right), H \right] = \left[ \exp\left( \sum_{i \in S} s_i^+ \right), H^{\text{bulk}} + H^l + H^l \right] = 0.
\]

(21)

Hence, the time-evolution operator \(\tilde{H}\) for the dual process is different from the original one, provided it exists.

4.2. Boundary terms and BCH formula

In order to derive the time-evolution operator \(\tilde{H}\) for the dual stochastic process, the following Baker–Campbell–Hausdorff (BCH) formula is employed:
For example, for the boundaries on site 1,
\[
[x^+_1, H^b] = -2\gamma_1^{in} s^+_1 + (\gamma_1^{out} - \gamma_1^{in}) s^+_1, \tag{23}
\]
\[
[x^+_1, [s^+_1, H^b]] = 2\gamma_1^{in} s^+_1, \tag{24}
\]
\[
[x^+_1, [s^+_1, [s^+_1, H^b]]] = 0, \tag{25}
\]
and hence after some calculations, we have
\[
\tilde{H} = H^{bulk} + \tilde{H}^I + \tilde{H}^L, \tag{26}
\]
where
\[
\tilde{H}^I = -\gamma_1^{in} s^-_1 + (\gamma_1^{in} - \gamma_1^{out}) n_1, \tag{27}
\]
\[
\tilde{H}^L = -\gamma_1^{in} s^-_L + (\gamma_1^{in} + \gamma_1^{out}) n_L. \tag{28}
\]
As discussed above, the Hamiltonian for the bulk parts has a symmetric property, i.e. \( H^{bulk} = (H^{bulk})^T \), and then the transposed quantum Hamiltonian \( (H^{bulk})^T \) can be directly interpreted as the transition matrix for the usual SSEP. On the other hand, the boundary parts, \( \tilde{H}^I \) and \( \tilde{H}^L \), are inadequate as stochastic processes; there is no probability conservation law. Hence, it is impossible to consider the operator \( \tilde{H} \) as the time-evolution operator for the dual stochastic process. In order to recover the characteristics as the stochastic processes, we need an additional theoretical framework, as described below.

### 4.3. Bosonic operators and birth-death processes

The technique based on the bosonic operators, the so-called Doi–Peliti formalism \([15–17]\), is mainly used to investigate reaction–diffusion processes (see, for example, \([18]\)). The formulation is based on algebraic probability theory \([19, 20]\), and recently the technique has been used to derive dual birth-death processes from stochastic differential equations \([12, 13]\). It has been shown that the bosonic formulation can connect differential operators in the Fokker–Planck equations (corresponding to the stochastic differential equation) and the creation and annihilation operators in the birth-death processes.

Here, we employ the bosonic operators in order to deal with the open boundary conditions. It will be shown that a combination of the spin operators and the bosonic operators can derive an adequate dual stochastic process, even in the case of open boundaries.

Different from the previous works in \([12, 13]\), the bra vectors in the Fock space play important roles in deriving the dual process for the SSEP problem. Firstly, the following bosonic operators and bra vectors in the Fock space are introduced:

\[
\langle \xi' | a^\dagger = \xi' (\xi' - 1), \quad \langle \xi' | a = (\xi' + 1), \tag{29}
\]
where \( a^\dagger \) and \( a \) are creation and annihilation operators, respectively, and \( \xi' \in \mathbb{N} \). Note that the roles of the ‘creation’ and ‘annihilation’ operators are changed because here we operate them with the bra vectors; if we apply them to ket vectors, the names and roles are directly connected. The bra vector \( \langle \xi' \rangle \) corresponds to the number of particles in the birth-death process.
The vacuum state $|0\rangle$ is characterized by $0|a^\dagger = 0$, and the creation and annihilation operators satisfy the following commutation relations:

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0.$$  \hfill (30)

The ket vector $|\zeta\rangle$ is naturally introduced from the inner product defined as

$$\langle \xi'|\zeta \rangle = \delta_{\xi|\zeta}.$$  \hfill (31)

As shown later, in order to deal with the cases with boundary conditions, the following property of the coherent states is important:

$$a|z\rangle = z|z\rangle,$$  \hfill (32)

where $|z\rangle$ is the coherent state with parameter $z \in \mathbb{R}$, which is defined as

$$|z\rangle \equiv e^{iz}|0\rangle.$$  \hfill (33)

### 4.4. Additional sink sites

In this section, only the cases with $\gamma^{\text{in}} - \gamma^{\text{out}}_\ell \neq 0 (\ell \in \{1, L\})$ are discussed. The case with $\gamma^{\text{in}} - \gamma^{\text{out}}_\ell = 0$, is discussed in appendix A.

In order to obtain an adequate dual ‘stochastic’ process for the SSEP problem, here we consider the following state vector $|\tilde{P}(t)\rangle$:

$$|\tilde{P}(t)\rangle = \sum_\eta P(\eta, t) |\eta, z^{(1)}_1, x^{(1)}_1, z^{(2)}_1, x^{(2)}_1\rangle,$$  \hfill (34)

where $|\eta, z^{(1)}_1, x^{(1)}_1, z^{(2)}_1, x^{(2)}_1\rangle$ is the abbreviation for $|\eta|z^{(1)}_1|z^{(2)}_1\rangle$, and $|z^{(1)}_1\rangle$ and $|z^{(2)}_1\rangle$ correspond to the coherent states with parameter $z^{(1)}_1$ and $z^{(2)}_1$, respectively ($\ell \in \{1, L\}$). The coherent states, $|z^{(1)}_1\rangle$ and $|z^{(2)}_1\rangle$, are created by $a^{(1)}_1$ and $a^{(2)}_1$, respectively. In addition, the corresponding annihilation operators are $a^{(1)}_1$ and $a^{(2)}_1$. Corresponding to the ‘extension’ of the state vector $|\tilde{P}(t)\rangle$, the following time-evolution operator is introduced:

$$\mathbf{\Pi} = \mathbf{H}^{\text{bulk}} + \mathbf{H}^1 + \mathbf{H}^L,$$  \hfill (35)

where

$$\mathbf{H}^1 = \frac{1}{2} d^{(1)}_1 s^+_1 - \frac{1}{2} d^{(1)}_1 (1 - n_1) + (2 - d^{(1)}_1 - d^{(2)}_1) s^+_1 - (2 - d^{(1)}_1 - d^{(2)}_1) n_1,$$  \hfill (36)

$$\mathbf{H}^L = \frac{1}{2} d^{(1)}_L s^+_L - \frac{1}{2} d^{(1)}_L (1 - n_L) + (2 - d^{(1)}_L - d^{(2)}_L) s^+_L - (2 - d^{(1)}_L - d^{(2)}_L) n_L,$$  \hfill (37)

and the annihilation operator $a^{(1)}_1$ (respectively $a^{(2)}_1$) acts only on the coherent state $|z^{(1)}_1\rangle$ (respectively $|z^{(2)}_1\rangle$). The reason why we take these forms will become clear below.

If we set $z^{(1)}_1 = \gamma^{\text{in}}_\ell$ and $z^{(2)}_1 = 2 - \gamma^{\text{in}}_\ell - \gamma^{\text{out}}_\ell$ for $\ell \in \{1, L\}$, the time-evolution operator (35) becomes the same as the original one in (8) by employing the property of the coherent state. Here, note that $a^{(1)}_1|z^{(1)}_1\rangle = z^{(1)}_1|z^{(1)}_1\rangle = \gamma^{\text{in}}_\ell|z^{(1)}_1\rangle = \gamma^{\text{in}}_\ell$ and

$$\begin{align*}
(2 - d^{(1)}_1 - d^{(2)}_1) |z^{(1)}_1\rangle &= \gamma^{\text{in}}_\ell, \\
(2 - \gamma^{\text{in}}_\ell) |z^{(1)}_1\rangle &= \gamma^{\text{in}}_\ell, \\
(2 - \gamma^{\text{in}}_\ell - \gamma^{\text{out}}_\ell) |z^{(1)}_1\rangle &= \gamma^{\text{in}}_\ell, \\
(2 - \gamma^{\text{in}}_\ell - \gamma^{\text{out}}_\ell) |z^{(2)}_1\rangle &= 2 - \gamma^{\text{in}}_\ell - \gamma^{\text{out}}_\ell.
\end{align*}$$  \hfill (38)
Furthermore, by using the modified quantum Hamiltonian (35), the basic idea in (15) gives the following dual Hamiltonian $\tilde{H}'$:

$$\tilde{H}' = H^{\text{bulk}} + \tilde{H}'^{1} + \tilde{H}'^{L},$$

(39)

where

$$\tilde{H}'^{1} = -(a_{1}^{\dagger}(s_{1} - n_{1} + a_{1}^{(2)}n_{1} - n_{1})),$$

(40)

$$\tilde{H}'^{L} = -(a_{L}^{\dagger}(s_{L} - n_{L} + a_{L}^{(2)}n_{L} - n_{L})).$$

(41)

(We can easily verify (39) by using the correspondences with $\gamma_{\ell}^{\text{in}} \leftrightarrow a_{1}^{(1)}(\ell)$ and $2 - \gamma_{\ell}^{\text{in}} \rightarrow \gamma_{\ell}^{\text{out}} \rightarrow a_{L}^{(2)}(\ell)$ in (27) and (28).) Because the role of the annihilation operators in the dual process is the creation of particles, the above boundary terms are interpreted as follows:

(I) If a boundary site $\ell$ has a particle, the particle can hop into an additional sink site (1) attached to site $\ell$ at rate 1 (this additional sink site stems from the action of $a_{1}^{(1)}(\ell)$). This is caused by $a_{1}^{\dagger}(s_{\ell} - n_{\ell})$. Hence, we can interpret the sink sites as absorbing ones.

(II) If a boundary site $\ell$ has a particle, a copy of the particle is created, at rate 1, for another additional sink site (2) attached to site $\ell$ (which stems from $a_{2}^{(2)}(\ell)$). Note that, in this case, the particle on site $\ell$ is not destroyed. This is caused by $a_{2}^{\dagger}(s_{\ell}).$ In this sense, these sink sites can be interpreted as ‘copying’ sites.

According to the above interpretation of the operators on the boundaries, the following ‘extended’ state space for the bra vectors should be used in order to obtain an adequate dual ‘stochastic’ process:

- Each boundary site $\ell \in \{1, L\}$ has two sink sites, i.e. the absorbing and copying ones.
- The number of particles in the sink sites attached to the boundary site $\ell$ is denoted as $\xi_{\ell}^{(1)}(t) \in \mathbb{N}$ (the absorbing sites) and $\xi_{\ell}^{(2)}(t) \in \mathbb{N}$ (the copying sites).
- Each bra vector $|\xi_{\ell}^{(1)}(t)\rangle$ for $\ell \in \{1, L\}$ and $t \in \{1, 2\}$ is connected to the creation and annihilation operators $(a_{1}^{(1)}(\ell))$ and $(a_{1}^{(2)}(\ell)).$
- Initially, the numbers of particles in these sink sites are set to zero; $\xi_{\ell}^{(1)}(0) = 0$ and $\xi_{\ell}^{(2)}(0) = 0$ for $\ell \in \{1, L\}$ at $t = 0.$
- The extended dual process has the following state vector:

$$|\tilde{P}(t)| = \sum_{\xi_{1}^{(1)}, \xi_{1}^{(2)}, \xi_{2}^{(1)}, \xi_{2}^{(2)}} P(\xi_{t}', \xi_{1}^{(1)}, \xi_{2}^{(1)}, \xi_{1}^{(2)}, \xi_{2}^{(2)}, t)|\xi_{t}', \xi_{1}^{(1)}, \xi_{2}^{(1)}, \xi_{1}^{(2)}, \xi_{2}^{(2)}, t|,$$

(42)

where $P(\xi_{t}', \xi_{1}^{(1)}, \xi_{2}^{(1)}, \xi_{1}^{(2)}, \xi_{2}^{(2)}, t)$ is the probability distribution for the extended dual process.

- The dual stochastic process obeys the same time-evolution with the original SSEP for the bulk part, and the time-evolution on the boundaries corresponds to the above explanation (I) and (II).

From the identity

$$\langle \xi_{\ell}^{(1)} | a_{\ell}^{(1)} \rangle = (\xi_{\ell}^{(1)})^{\dagger} a_{\ell}^{(1)},$$

(43)

we finally obtain the following duality function
and the duality relation
\[
\mathbb{E}_0 \left[ D(\eta^r, \xi^{(1)}, \xi^{(2)}, \xi_L^{(1)}, \xi_L^{(2)}) \right] = \mathbb{E}^{\text{dual}} \left[ D(\eta, \xi^{(1)}, \xi^{(2)}, \xi_L^{(1)}, \xi_L^{(2)}) \right].
\] 
(45)

This duality relation is an extension of the previous result in [10]. Note that the above discussions can adequately recover the previous result in [10] for cases with \( \gamma_\ell^{\text{in}} = \rho_\ell \) and \( \gamma_\ell^{\text{out}} = 1 - \rho_\ell \) (\( \rho_\ell \in [0, 1] \)); in this case, only the sink site 1 for each boundary site \( \ell \) plays a special role. That is, the particle copy process does not affect the duality function because \( 2 - \gamma_\ell^{\text{in}} - \gamma_\ell^{\text{out}} = 1 \).

At least at this stage, it is difficult to show an explicit relation between the above derivation and the standard form of the duality relation in [10]. In the current derivation, we can find the forms of the duality functions after some concrete calculations. The main point here is that, once one chooses the time-independent operator \( A \) (which is called 'symmetry' in [10]), the bra and ket notations derive the duality relations by employing the bosonic operators. Note that the natural choice of the operator \( A \) is still unclear, which is the same problem as in [10]. In principle, there could be several choices for the operators, as we will see in the following ASEP cases.

There is a final comment for the SSEP case: in [4], a different type of dual process has been derived ((4.2) and (4.3) in [4]). In brief, the difference from the current result is as follows: the dual process in [4] contains the rate parameters \( \gamma_\ell^{\text{in}} \) and \( \gamma_\ell^{\text{out}} \). It is possible to derive the same result in the current theoretical framework with the bosonic operators. From (27), we have
\[
H_s^{\text{in}} = -\left( \gamma_\ell^{\text{in}} + \gamma_\ell^{\text{out}} \right) \frac{\gamma_\ell^{\text{in}}}{\gamma_\ell^{\text{in}} + \gamma_\ell^{\text{out}}} \frac{1}{x_1} - (\gamma_\ell^{\text{in}} + \gamma_\ell^{\text{out}}) n_1
\] 
(46)

Hence, when we replace \( x_1 \rightarrow x_1 / (\gamma_\ell^{\text{in}} + \gamma_\ell^{\text{out}}) \) with \( a_1 \) and use the coherent state parameter \( z_1 = \gamma_\ell^{\text{in}}(\gamma_\ell^{\text{in}} + \gamma_\ell^{\text{out}}) \), we can recover the results in [4] and there is no need to introduce the copying sites. Of course, in this case, the rate parameters, \( \gamma_\ell^{\text{in}} + \gamma_\ell^{\text{out}} \), still remain in the dual process. As seen above, use of the copying sites enables us to remove the rate parameters from the dual process. The current theoretical framework with bosonic operators can deal with these various types of dual processes in a unified manner.

5. Re-derivation of the duality in the ASEP without open boundaries

As for the ASEP without open boundaries, i.e. with reflective boundaries, the self-dual property has already been derived [6]. In this section, a slightly different derivation is given, which becomes the basis for the discussions for the open boundary cases.

5.1. Similarity transformation and some notations

Different from the SSEP case, even in the absence of the open boundary conditions, the time-evolution operator for the dual stochastic process cannot be obtained easily. If we choose a certain quantity \( A \), which commutes with \( H^{\text{bulk}} \), we have

\[
D(\eta^r, \xi^{(1)}, \xi^{(2)}, \xi_L^{(1)}, \xi_L^{(2)}) = \prod_{i \in \Delta \cup \{0 \}} \frac{\gamma_1^{(i)}}{\gamma_1^{(i)} - \gamma_1^{(i)}} \left( 2 - \gamma_1^{(i)} - \gamma_1^{(i)} \right) \xi_L^{(i)}
\] 
(44)

and the duality relation
\[
\mathbb{E}_0 \left[ D(\eta^r, \xi^{(1)}, \xi^{(2)}, \xi_L^{(1)}, \xi_L^{(2)}) \right] = \mathbb{E}^{\text{dual}} \left[ D(\eta, \xi^{(1)}, \xi^{(2)}, \xi_L^{(1)}, \xi_L^{(2)}) \right].
\] 
(45)
\[ \langle P(t = 0) | A | P(t) \rangle = \langle P(t = 0) | A e^{-H_{\text{bulk}} t} | P(t = 0) \rangle = \langle P(t = 0) | e^{-t H_{\text{bulk}}} A | P(t = 0) \rangle, \]

because \( H_{\text{bulk}} A = A H_{\text{bulk}} \). However, \( H_{\text{bulk}} \) does not correspond to the time-evolution operator for the dual stochastic process; the left action of \(-H_{\text{bulk}}\) does not satisfy the probability conservation law.

In order to recover the probability conservation law, the following similarity transformation is employed \([6, 11]\). Defining

\[ V = q^{\sum_{k=1}^{L} n_k}, \]

it has been shown that the following relation is satisfied:

\[ (H_{\text{bulk}})_{\chi} = V^2 H_{\text{bulk}} V^{-2}. \]

Note that \( \langle P(t = 0) | e^{-t H_{\text{bulk}}} \rangle = \langle V^2 | V^{-2} \rangle \) gives an adequate time-evolution of the ASEP for the bra state; the dual stochastic process obeys the same time-evolutions with the original ASEP.

Setting the initial states for the bra and ket states as

\[ \langle x_1, \ldots, x_m | P(t = 0) \rangle \equiv | y_1, \ldots, y_N \rangle, \]

we have

\[ \langle x_1', \ldots, x_m' | A | P(t) \rangle = \langle x_1', \ldots, x_m' | V^2 e^{-t H_{\text{bulk}}} V^2 A | P(t = 0) \rangle = q^{-2 \sum_{k=1}^{m} x_k'} \langle x_1', \ldots, x_m' | e^{-t H_{\text{bulk}}} V^2 A | y_1', \ldots, y_N \rangle = q^{-2 \sum_{k=1}^{m} x_k'} \langle P(t) | V^2 A | y_1', \ldots, y_N \rangle. \]

As already denoted, the derived duality function can be varied depending on the choice of the operator \( A \).

Here, for later use, let us define the following quantities \([6]\):

\[ S^+ = \sum_{k=1}^{L} s^+_k(q), \quad S^- = \sum_{k=1}^{L} s^-_k(q), \quad S^z = \sum_{k=1}^{L} s^z_k = \sum_{k=1}^{L} \left( \frac{1}{2} I - n_k \right), \]

where

\[ s^+_k(q) = q^{-\sum_{j=1}^{k-1} n_j - \sum_{j=k+1}^{L} n_j} s^+_k q^{\sum_{j=1}^{k-1} n_j}, \quad s^-_k(q) = q^{\sum_{j=1}^{k-1} n_j} s^-_k q^{-\sum_{j=k+1}^{L} n_j - 1}. \]

In addition, note the following useful identities:

\[ q^m s^+_k = s^+_k, \quad s^+_k q^m = q^m s^+_k, \quad q^m s^-_k = q s^-_k, \quad s^-_k q^m = s^-_k. \]

5.2. An example of the duality function

As for the operator \( A \), here we choose \( e^{S^z} \); it has already been shown that this quantity commutes with the quantum Hamiltonian \( H_{\text{bulk}} \) \([6]\).

Define \( P'_x(t) \) as the probability for the ASEP in configuration \( x = \{ x_1, \ldots, x_m \} \) at time \( t \), and \( P'_y(t) \) as that in \( y = \{ y_1, \ldots, y_N \} \). In addition, here we assume that \( m < N \). From (51), we have
\begin{equation}
\langle x'_1, \ldots, x'_n | e^{\mathcal{P}t} | P(t) \rangle = q^{-2\sum_{i=1}^{n} x'_i} \langle P'(t) | V^2 e^{\mathcal{V}t} | y'_1, \ldots, y'_N \rangle.
\end{equation}

Note that the quantum Hamiltonian $H_{\text{bulk}}$ in (9) does not change the total number of particles. Hence, the total number of particles for the ket state $P(t)$ is still $N$ even at time $t$ (and of course that for the bra state $P'(t)$ is $m$). Therefore, (55) can be rewritten as

\begin{equation}
\frac{1}{(N-m)!} \sum_{1 \leq q_1 < \cdots < q_{m} \leq L} P_q(t) \langle x'_1, \ldots, x'_n | (S^+)^{N-m} | y'_1, \ldots, y'_N \rangle
\end{equation}

\begin{equation}
= \frac{1}{(N-m)!} q^{-2\sum_{i=1}^{n} x'_i} \sum_{1 \leq q_1 < \cdots < q_{m} \leq L} P'_q(t) q^{2\sum_{i=1}^{m} q_{n_i}} \langle x'_1, \ldots, x'_n | (S^+)^{N-m} | y'_1, \ldots, y'_N \rangle.
\end{equation}

Here, we used the following facts: if the bra and ket states have different particle numbers, the inner product immediately gives zero, and the operator $S^+$ generates only one particle for the bra state. Next, we introduce the projection state, which is of an equal weight to any $N$-particle configuration [6]:

\begin{equation}
\sum_{\eta \sum_{j=1}^{n} \eta_j = N} \langle \eta \rangle \equiv \langle N \rangle = \frac{1}{[N]_q!} \langle 0 | (S^+)^N = \langle 0 | \frac{q - q^{-1}}{q^2 - 1} \frac{q - q^{-1}}{q^2 - 1} \cdots \frac{q - q^{-1}}{q^N - q^{-N}} (S^+)^N,
\end{equation}

where

\begin{equation}
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}
\end{equation}

and

\begin{equation}
[m]_q! = [m]_q \cdot [m-1]_q \cdots [1]_q.
\end{equation}

Then, using the following notations, which were introduced in [6],

\begin{equation}
N_e = \sum_{j=1}^{k} n_j,
\end{equation}

\begin{equation}
Q_x = q^{-2N_e},
\end{equation}

\begin{equation}
Q'_x = \frac{Q_x - Q_{x-1}}{q^2 - 1} = q^{-2N_e-n_x},
\end{equation}

we have

\begin{equation}
\langle N | \tilde{Q}'_{n_1} \cdots \tilde{Q}'_{n_k} = q^{-m(N-1)} \langle x'_1, \ldots, x'_n | (S^+)^{N-m} | [N-m]_q!.
\end{equation}

(This identity in (63) has been verified in [6].) Using these facts, by multiplying an adequate constant, (56) becomes

\begin{equation}
\sum_{1 \leq q_1 < \cdots < q_{m} \leq L} P_q(t) \langle N | \tilde{Q}'_{n_1} \cdots \tilde{Q}'_{n_k} | y'_1, \ldots, y'_N \rangle
\end{equation}

\begin{equation}
= q^{-2\sum_{i=1}^{n} x'_i} \sum_{1 \leq q_1 < \cdots < q_{m} \leq L} P'_q(t) q^{2\sum_{i=1}^{m} q_{n_i}} \langle N | \tilde{Q}'_{n_1} \cdots \tilde{Q}'_{n_k} | y'_1, \ldots, y'_N \rangle.
\end{equation}
immediately gives the duality relation for the ASEP with reflective boundaries, which has already been derived in [6, 11]; the duality function is given as

\[ D_{\eta, \eta'} = \prod_{i=1}^{m} q^{2N_{i-1} - 2n_{i}}. \]

where \( n_{i} \) means the number operator for site \( x_{i}' \) in the ket state \( \eta \), and \( N_{i-1} \) the number of particles up to site \( x_{i}' - 1 \) in the ket state \( \eta \) (note that \( x_{i}' \) is the \( i \)th particle position in \( \eta' \)).

5.3. Comments on the duality in the reflective boundaries

In the above discussion, we selected \( e^{S^{+}} \) as the commutative quantity with \( H_{\text{bulk}} \). Of course, it is possible to consider different quantities; similar discussions have already been given in [21].

Here, there is a comment on the connection with a previous work [7]. In [7], instead of \( e^{S^{+}} \), a slightly different operator, \( \exp(q^{S^{+}}) \), has been used to derive the duality relation for the ASEP with reflective boundaries; \( \exp(q^{S^{+}}) \) also commutes with \( H_{\text{bulk}} \) [7]. However, it is possible to show that this different quantity gives the same duality relation in (64); the derivation is written in appendix B. The important point for the derivations is as follows: for the reflective boundary cases, the numbers of particles are conserved in the time-evolution both for the bra and ket states. In addition, the difference between \( e^{S^{+}} \) and \( \exp(q^{S^{+}}) \) is the factor \( q^{S^{+}} \), which only depends on the total number of particles, and the difference can be removed by multiplying certain constants on the l.h.s. and r.h.s. in (51); as a result, \( e^{S^{+}} \) and \( \exp(q^{S^{+}}) \) give the same quantity to be calculated. As we will see later, this special characteristic, i.e. the conservation of the total number of particles, is not available to the open boundary cases.

6. Duality in the ASEP with open boundaries

In this section, we discuss the duality relation in the ASEP with open boundaries. Firstly, some discussions for the commutative quantities with \( H_{\text{bulk}} \) are given. Secondly, boundary effects on the dual process are investigated using the \( q \)-analogues of the exponential functions. The final conclusion is a slightly disappointing one; the obtained dual process could become very complicated, and hence, at this stage, the duality relations could not be useful. In order to find this out, the tools developed in the previous sections were employed; it may be impossible to find this out in a heuristic way.

6.1. What physical quantities should we use?

In section 5.3, we discussed that two different types of quantities, \( e^{S^{+}} \) and \( \exp(q^{S^{+}}) \), give the same duality relation. Here, we will show that a similar discussion cannot be used for the open boundary cases.

For the open boundary cases, there is no guarantee that the number of particles is conserved in the processes; because of the in- and out-effects on the boundaries (site 1 and \( L \)), the number of particles can vary with time. Hence, the l.h.s. in (55) becomes

\[ \langle x'_{1}, \ldots, x'_{m} | e^{S^{+}} | P(t) \rangle = \sum_{N=0}^{\infty} \sum_{n=0}^{\infty} \sum_{0 \leq y_{1} < \cdots < y_{N} \leq L} \langle x'_{1}, \ldots, x'_{m} | \frac{1}{n!} (S^{+})^{n} P(t) | y_{1}, \ldots, y_{N} \rangle. \]

Note that the summation for \( N \) is necessary, which is different from section 5.

After some calculations, we have
\[ \langle x_1', \ldots, x_m' | e^{S^+} | P(t) \rangle \]
\[ = \sum_{N=m}^{\infty} \sum_{1 \leq y_1 < \cdots < y_N \leq L} q^{m(N-1)(N-m)}! P_N(t) \langle \ddot{Q}_{y_1} \cdots \ddot{Q}_{y_N} \mid y_1, \ldots, y_N \rangle, \quad (67) \]

but this does not give the expectation for \( \ddot{Q}_{y_1} \cdots \ddot{Q}_{y_N} \); the coefficients \( \frac{q^{m(N-1)(N-m)}!}{(N-m)!} \) remain, and a kind of weighted expectation is then obtained.

In order to obtain a usual expectation for \( \ddot{Q}_{y_1} \cdots \ddot{Q}_{y_N} \), the following quantity, which is based on the \( q \)-analogue of exponential functions [22], is available:

\[ A = e_q \left( 1 - q^{-2} q^\sum_{i=1}^m S^+ \right). \quad (68) \]

where

\[ e_q(x) \equiv \sum_{n \geq 0} \frac{x^n}{(q; q)_n}, \quad 0 < |q| < 1, \quad |x| < 1, \quad (69) \]

and

\[ (q; q)_n \equiv \prod_{i=1}^n (1 - q^i). \quad (70) \]

Note that \( q^{S^+}S^+ \) commutes with \( H^{\text{bulk}} \), and then we have

\[ \left[ e_q \left( 1 - q^{-2} q^\sum_{i=1}^m S^+ \right), H^{\text{bulk}} \right] = 0. \quad (71) \]

As shown in appendix C, this quantity gives the usual expectation for \( \ddot{Q}_{y_1} \cdots \ddot{Q}_{y_N} \):

\[ \langle x_1', \ldots, x_m' | e_q \left( 1 - q^{-2} q^\sum_{i=1}^m S^+ \right) | P(t) \rangle \]
\[ = \sum_{N=m}^{\infty} \sum_{1 \leq y_1 < \cdots < y_N \leq L} \langle \ddot{Q}_{y_1} \cdots \ddot{Q}_{y_N} P_N(t) \mid y_1, \ldots, y_N \rangle. \quad (72) \]

Notice the following facts: in order to use the \( q \)-analogues, we must restrict the following discussions for the cases with \( q > 1 \), because \( |q^{-2}| < 1 \) is needed for the definition of the \( q \)-analogues. (The \( q = 1 \) case immediately reduces to the SSEP case, and the following discussions can easily be obtained by using the usual exponential functions. The following formulations are formally available— even in the \( q = 1 \) case— so here we consider \( q \geq 1 \).) That is, \( \alpha_k > \beta_k \) for all \( k \in \mathcal{S} \). The discussions for the \( q < 1 \) cases need the order of the lattice structures to be changed.

6.2. The effects of open boundaries on the dual process

In order to discuss the duality relations, the following different type of \( q \)-analogue of the exponential functions is useful [21]:
\[ \exp_q(x) \equiv \sum_{n \geq 0} \frac{x^n}{(n)_q!}. \]  

(73)

where

\[ (n)_q \equiv \frac{1 - q^n}{1 - q}. \]  

(74)

and

\[ (n)_q! \equiv (n)_q \cdot (n-1)_q \cdot \ldots \cdot 1_q. \]  

(75)

That is, two types of the \( q \)-exponentials in (69) and (73) are related to each other as follows:

\[ e_q^{-z((1 - q^{-2})\zeta)} = \exp_q^{-z(\zeta)}. \]  

(76)

For the open boundary cases, we must consider the quantum Hamiltonian including \( H_1 \) and \( H_L \). As for the quantity \( A \) in (68), we have

\[ \left[ \exp_q^{-z((1 - q^{-2})\zeta)} \sum_{i=1}^{L} \exp_q^{-z(i\zeta)} \right] H \equiv 0. \]

Different from the SSEP cases in section 4, it is impossible to employ the BCH formula directly, because here we use the \( q \)-analogues of the exponential functions. Hence, if we want to perform similar discussions in (47), it is necessary to seek the following alternative quantum Hamiltonian \( \widetilde{H} \):

\[ \exp_q^{-z\sum_{i=1}^{L} i n_i} \widetilde{H} \exp_q^{-z\sum_{i=1}^{L} i n_i} \equiv H. \]  

(77)

Then, we have

\[ \exp_q^{-z\sum_{i=1}^{L} i n_i} e^{-H t} = e^{-\widetilde{H} t} \exp_q^{-z\sum_{i=1}^{L} i n_i}. \]  

(78)

In the discussions for the effects of the open boundaries, the following proposition (proposition 5.1 in [21]) is useful:

**Proposition (Pseudo-factorization [21]).** Let \( \{g_i, \ldots, g_L\} \) and \( \{k_i, \ldots, k_L\} \) be operators such that for \( L \in \mathbb{N} \) and \( r \in \mathbb{R} \)

\[ k g_i = r g k_i \text{ for } i = 1, \ldots, L. \]  

(79)

Define

\[ \hat{g}^{(L)} \equiv \sum_{i=1}^{L} g_i h^{(i+1)} \]  

(80)

with

\[ h^{(i)} \equiv k_i^{-1} \cdots k_{L}^{-1} \text{ for } i \leq L \text{ and } h^{(L+1)} = 1. \]  

(81)
then
\[ \exp(g^{(L)}) = \exp(g_1 h^{(2)}) \cdots \exp(g_{L-1} h^{(L-1)}) \cdot \exp(g_L). \] (82)

Now, setting
\[ g_i = s_i^+, \quad k_i = q^{2n_i}, \quad r = q^{-2}, \] (83)
then
\[ q^{2n_i} s_i^+ = q^{-2} s_i^+ q^{2n_i}, \] (84)
and
\[ g^h(i+1) = g_{k+1}^{-1} \cdots k_L = s_i^+ q^{-2n_{i+1}} \cdots q^{-2n_L} = s_i^+ q^{-2 \sum_{j=1}^{i+1} n_j}. \] (85)

Therefore, we have the following factorization:
\[
\exp_q \left( \sum_{k=1}^{L} s_k^+ q^{-2 \sum_{j=k+1}^{L} n_j} \right) \\
= \exp_q \left( s_1^+ q^{-2 \sum_{j=1}^{L} n_j} \right) \exp_q \left( s_1^+ q^{-2 \sum_{j=1}^{L} n_j} \right) \cdots \exp_q \left( s_L^+ \right). \] (86)

Because
\[
\left[ \exp_q \left( (1 - q^{-2}) q^{-\sum_{j=1}^{L} n_j} S^+ \right) \right] H^{\text{bulk}} = 0, \] (87)

it is enough to consider the interchange with \( \exp_q \left( (1 - q^{-2}) q^{-\sum_{j=1}^{L} n_j} S^+ \right) \) and \( H^1 \) (and \( H^2 \)).

6.2.1. Discussion for site 1. The aim here is to seek \( \tilde{H}^1 \), which is obtained by
\[
\exp_q \left( q^{-\sum_{j=1}^{L} n_j} S^+ \right) H^1 = \tilde{H}^1 \exp_q \left( q^{-\sum_{j=1}^{L} n_j} S^+ \right). \] (88)

On the r.h.s. in (86), it is only the first factor, \( \exp_q \left( s_1^+ q^{-2 \sum_{j=1}^{L} n_j} \right) \), that does not commute with \( H^1 \). Hence, we focus on the interchange between \( \exp_q \left( s_1^+ q^{-2 \sum_{j=1}^{L} n_j} \right) \) and \( \exp_q \left( q^{-\sum_{j=1}^{L} n_j} S^+ \right) \).

For notational simplicity, let us define
\[ \zeta = q^{-2 \sum_{j=1}^{L} n_j}. \] (89)

Then, because of \( s_1^+ s_1^+ = 0 \),
\[
\exp_q \left( s_1^+ q^{-2 \sum_{j=2}^{L} n_j} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( s_1^+ q^{-2 \sum_{j=1}^{L} n_j} \right)^n = I + s_1^+ \zeta. \] (90)

In addition, introducing
\[ E_q(z) \equiv \sum_{n=0}^{\infty} \frac{q(z)}{(q;q)_n} z^n, \]  

(91)

where

\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta + 1)},
\]

we have [22]

\[ e_q(z)E_q(-z) = 1. \]  

(92)

Using this ‘inverse’ function of \( e_q(z) \), we can factor out \( \exp_{q,-}\left(s^+_{i} \zeta \right) \). Hence, after some tedious calculations, the following factorized form using \( \exp_{q,-}\left(s^+_{i} \zeta \right) \) is obtained:

\[ \exp_{q,-}\left(s^+_{i} \zeta \right)H^L \]

\[ = \left( -\gamma_{1}^{\text{in}} s_{i}^+ + [\gamma_{1}^{\text{in}} \gamma_{1}^{\text{out}} - (\gamma_{1}^{\text{in}} - \gamma_{1}^{\text{out}}) \zeta] s_{i}^+ + (\gamma_{1}^{\text{in}} - \gamma_{1}^{\text{out}}) \zeta] s_{i}^+ \right. \]

\[ \left. \times \exp_{q,-}\left(s^+_{i} \zeta \right), \right) \]  

(93)

and then

\[ \tilde{H}^L = -\gamma_{1}^{\text{in}} s_{i}^+ + [\gamma_{1}^{\text{in}} \gamma_{1}^{\text{out}} - (\gamma_{1}^{\text{in}} - \gamma_{1}^{\text{out}}) \zeta] s_{i}^+ \]

\[ + (\gamma_{1}^{\text{in}} - \gamma_{1}^{\text{out}}) \zeta I + [- (\gamma_{1}^{\text{in}} - \gamma_{1}^{\text{out}}) + 2 \gamma_{1}^{\text{in}}] \zeta I. \]  

(94)

6.2.2. Discussion for site L. Here, we seek \( \tilde{H}^L \), which satisfies

\[ \exp_{q,-}\left(q^- \sum_{i=1}^{L} n s^+ \right)H^L = \tilde{H}^L \exp_{q,-}\left(q^- \sum_{i=1}^{L} n s^+ \right) \]  

(95)

Different from site 1, all factors on the r.h.s. in (86) must be taken into consideration.

Firstly, it is easy to confirm the following relation, by using a similar discussion for the site 1 case;

\[ \exp_{q,-}(s^+_{L})H^L = (-\gamma_{L}^{\text{in}} s_{L}^+ + (\gamma_{L}^{\text{in}} + \gamma_{L}^{\text{out}}) n_L) \exp_{q,-}(s^+_{L}). \]  

(96)

Secondly,

\[ \exp_{q,-}(s^+_{L,q^-2\alpha})(-\gamma_{L}^{\text{in}} s_{L}^+ + (\gamma_{L}^{\text{in}} + \gamma_{L}^{\text{out}}) n_L) \]

\[ = \left[ -\gamma_{L}^{\text{in}} s_{L}^+ + (\gamma_{L}^{\text{in}} + \gamma_{L}^{\text{out}}) n_L - \gamma_{L}^{\text{in}} q^{-2}(1-s_{L}^+ s_{L}^-) \right] \exp_{q,-}(s^+_{L,q^-2\alpha}). \]  

(97)

Although the successive calculations may become very complicated, using the following fact that

\[ [s_{L}^+ q^{-2 \sum_{j=1}^{L} \eta_j}, s_{L}^- s_{L}^+] = s_{L}^+ q^{-2 \sum_{j=1}^{L} \eta_j} s_{L}^- s_{L}^+ - s_{L}^- s_{L}^+ q^{-2 \sum_{j=1}^{L} \eta_j} \]

\[ = s_{L}^+ q^{-2 \sum_{j=1}^{L-1} \eta_j} q^{-2} q^{-2 \sum_{j=1}^{L-1} \eta_j} - s_{L}^- q^{-2 \sum_{j=1}^{L-1} \eta_j} q^{-2} q^{-2 \sum_{j=1}^{L-1} \eta_j} = 0, \]  

(98)
we finally have the following result from the interchange:

$$\exp\left(-g\sum_{j=1}^{L}n_jS_j\right)H_{\text{L}} = \left(-\gamma_{\text{L}}^{\text{in}}s_{\text{L}}^- + (\gamma_{\text{L}}^{\text{in}} + \gamma_{\text{L}}^{\text{out}})n_{\text{L}} - \sum_{i=1}^{L-1}\gamma_{\text{L}}^{\text{in}}(q^{-2} - 1)s_{\text{L}}^- - q^{-2}\sum_{j=\text{L}+1}^{\infty}n_jS_j\right)$$

and hence

$$\hat{H}_{\text{L}} = -\gamma_{\text{L}}^{\text{in}}s_{\text{L}}^- + (\gamma_{\text{L}}^{\text{in}} + \gamma_{\text{L}}^{\text{out}})n_{\text{L}} - \sum_{i=1}^{L-1}\gamma_{\text{L}}^{\text{in}}(q^{-2} - 1)s_{\text{L}}^- - q^{-2}\sum_{j=\text{L}+1}^{\infty}n_jS_j.$$  \hspace{1cm} (99)

6.3. Some comments on the derived results

Note that the obtained quantum Hamiltonian cannot be directly interpreted as the transition matrix for the dual process; the obtained quantum Hamiltonian does not satisfy the probability conservation law. As in section 4, it may be possible to use the Doi–Peliti formalism to derive an adequate time-evolution operator for the dual stochastic process. However, even from the dual quantum Hamiltonian in (94) and (100), the following facts are immediately obtained:

- The transition rates for the in-flow and out-flow of the particles on site 1 will have the factor $\zeta = q^{-2}\sum_{j=1}^{n_j}$, and hence the transition rates depend on the configurations. It would be difficult to solve the ASEP analytically with such complicated boundary conditions.
- Because the third term of $\hat{H}_{\text{L}}$ in (100) includes $s_{\text{L}}^-s_{\text{L}+1}$, the long-range particle hopping from site $\text{L}$ to site $\text{L} + 1$ will occur in the dual process.

Hence, the dual process will become very complicated, and then, at this stage, there might not be any benefit in considering the duality relation for the ASEP with open boundary conditions. Note that this fact can be revealed by using some techniques introduced in the present paper.

7. Concluding remarks

In a systematic way, based on a combination of the quantum spin language and the Doi–Peliti formalism, the open boundary effects on the duality relations in the SSEP and ASEP were discussed. Here, the word ‘systematic’ means that once one chooses a quantity (a symmetry in [10]) to be evaluated, the dual stochastic process is derived by employing the bosonic operators. In order to discuss the boundary condition cases, the previous work needs some heuristic discussions; in contrast, the derivation in the present work gives the dual stochastic processes naturally. As a result, the discussions give us a general result for the SSEP with open boundaries; it was clarified that not only the absorbing sites, but also the copying sites are necessary in general. As for the ASEP, it was clarified that the open boundary conditions result in a complicated dual process, which would be difficult to solve analytically. Hence, at this stage, there might not be any merit in considering the duality relations for the study of the ASEP with open boundary conditions. However, as we saw in the present paper, there would be little hope of finding out the complicated dual process in heuristic ways; the discussion in the present paper reveals the characteristics of the dual process in a systematic way.

Up to now, the usage of the bosonic operators, i.e. the Doi–Peliti formalism, has basically been restricted to the duality studies between the stochastic differential equations and the birth-death processes [12, 13], in the context of the duality relations. In the present paper, it
was clarified that the bosonic operators are also useful in combination with the quantum spin language. This technique could be hopeful for discussing duality relations for other types of stochastic processes. For example, in [10], the SSEP with higher spins and open boundaries has been discussed. The application of the bosonic operators for higher spin cases is straightforward: a slightly different duality function with (44) is obtained (for bulk parts, see (67) in [10]). Note that the forms of the additional factors could depend on the form of the boundary conditions of the original processes, and they are obtained after some explicit calculations.

In the present paper, we focused on the expectation $Q_x$, which has been investigated in other duality works for the ASEP. It might be possible to obtain useful and simple dual processes when we consider other physical quantities, although this is beyond the scope of the current work. In addition, there may be a different type of theoretical framework for finding dual processes, and it could be possible to derive a useful dual process. Although the theoretical framework with the bosonic operators seems to be useful for dealing with the boundary conditions in a unified manner, seeking the different formulations will be an important future work.

**Remark.** An anonymous referee suggested the work by Schwartz [23], in which, essentially, the SSEP with the open boundaries was studied. In [23], the probability measures were also discussed. However, the dual process used in (45) does not include the boundary parameters, and these discussions were not used in [23]. Hence, the obtained results in the present work are completely different from [23]. Although it might be possible to understand the discussions in [23] by using similar techniques from the present work, it is beyond the scope of this paper.

**Acknowledgments**

The author is extremely grateful to Tomohiro Sasamoto for useful discussions. This work was supported in part by MEXT KAKENHI (grant nos. 25870339 and 16K00323) and by the JSPS Core-to-Core program ‘Non-equilibrium dynamics of soft-matter and information.’

**Appendix A. Duality in SSEP with specific parameters**

If $2 - \gamma^\text{in}_\ell - \gamma^\text{out}_\ell = 0$ for some $\ell$, we must change the discussion slightly. For simplicity, assume here that $2 - \gamma^\text{in}_\ell - \gamma^\text{out}_\ell = 0$ is satisfied for both $\ell = 1$ and $\ell = L$; it is straightforward to deal with more general cases.

In this case, we must go back to the boundary terms in the original dual process in (27) and (28):

\[
\tilde{H}^1 = -\gamma^\text{in}_1 \hat{s}_1^- + 2n_1 = -2\left(\frac{1}{2} \gamma^\text{in}_1\right) \hat{s}_1^- + 2n_1, \\
\tilde{H}^L = -\gamma^\text{in}_L \hat{s}_L^- + 2n_L = -2\left(\frac{1}{2} \gamma^\text{in}_L\right) \hat{s}_L^- + 2n_L.
\]

(H.1) (A.2)

Hence, it is only necessary to introduce one sink site at each boundary site; we replace $\frac{1}{2} \gamma^\text{in}_\ell$ with the annihilation operator $a_\ell$ for the boundary site $\ell$. In addition, the corresponding coherent state parameter should be set to $\frac{1}{2} \gamma^\text{in}_\ell$. This boundary operator corresponds to the particle hopping from the boundary site $\ell$ to the sink site attached to site $\ell$ ‘with rate 2’. Hence, the duality function is as follows:
where \( \xi'_\ell \) is the number of particles in the sink site \( \ell \in \{1, L\} \) in the dual stochastic process.

**Appendix B. The duality based on \( \exp(q^S S^*) \) in the ASEP with reflective boundaries**

From (55), we have

\[
\frac{1}{(N - m)!} \sum_{1 \leq y_1 < \cdots < y_N \leq L} P_2(t)(x'_1, \ldots, x'_m \mid (q^S S^*)^N \mid y_1, \ldots, y_N) = \frac{1}{(N - m)!} \sum_{1 \leq y_1 < \cdots < y_N \leq L} P'_2(t)q^{2 \sum_{i=1}^{m} x'_i}(x_1, \ldots, x_m \mid (q^S S^*)^N \mid y'_1, \ldots, y'_N). \quad (B.1)
\]

Here, we use the following projection state introduced in [7]:

\[
\langle \eta \rangle = \langle N \rangle = C_N(0)(q^S S^*)^N, \quad (B.2)
\]

where

\[
C_N = (q^{-2})^{\frac{N}{2}} \frac{(1 - q^{-2})^N}{(1 - q^{-2}) \cdots (1 - (q^{-2})^N)}. \quad (B.3)
\]

From (2.27) in [7],

\[
\langle N \mid \tilde{Q}_{i_1} \cdots \tilde{Q}_{i_n} \rangle = C_{N,m}(x_1', \ldots, x_m' \mid (X^-)^N \mid y_1, \ldots, y_N), \quad (B.4)
\]

where

\[
C_{N,m} = \frac{q^{-\frac{1}{2}N-m} \cdot (1 - q^{-2})^{N-m}}{(1 - q^{-2}) \cdots (1 - (q^{-2})^{N-m})}. \quad (B.5)
\]

Hence, we have

\[
\sum_{1 \leq y_1 < \cdots < y_N \leq L} P_2(t)C_{N,m}(x'_1, \ldots, x'_m \mid (q^S S^*)^N \mid y_1, \ldots, y_N) = q^{-2 \sum_{i=1}^{m} x'_i} \sum_{1 \leq y_1 < \cdots < y_N \leq L} P'_2(t)q^{2 \sum_{i=1}^{m} x_i}C_{N,m}(x_1, \ldots, x_m \mid (q^S S^*)^N \mid y'_1, \ldots, y'_N), \quad (B.6)
\]

and finally

\[
\sum_{1 \leq y_1 < \cdots < y_N \leq L} P_2(t)(N \mid \tilde{Q}_{i_1} \cdots \tilde{Q}_{i_n} \mid y_1, \ldots, y_N) = q^{-2 \sum_{i=1}^{m} x'_i} \sum_{1 \leq y_1 < \cdots < y_N \leq L} P'_2(t)q^{2 \sum_{i=1}^{m} x_i}(N \mid \tilde{Q}_{i_1} \cdots \tilde{Q}_{i_n} \mid y'_1, \ldots, y'_N). \quad (B.7)
\]

Note that the above derivation is based on the fact that the number of particles is conserved.
Appendix C. Verification of (72)

\[
\sum_{N=0}^{\infty} \langle x_1', \ldots, x_m' \mid e_q^{-1} \left( (1 - q^{-2}) q^{x_1 + \cdots + x_m}  \right) \sum_{1 \leq y_1 < \cdots < y_N \leq L} \phi_N(t) \mid y_1, \ldots, y_N \rangle \\
= \sum_{N=0}^{\infty} \langle x_1', \ldots, x_m' \mid \frac{1}{q^{-2}; q^{-2}} \left( (1 - q^{-2}) q^{x_1 + \cdots + x_m}  \right) \sum_{1 \leq y_1 < \cdots < y_N \leq L} \phi_N(t) \mid y_1, \ldots, y_N \rangle \\
= \sum_{N=m}^{\infty} q^{(LN-m)/2} (1 - q^{-2})^{N-m} 1 \sum_{1 \leq y_1 < \cdots < y_N \leq L} \phi_N(t) \mid y_1, \ldots, y_N \rangle.
\]

(C.1)

Using the following equality,

\[
q^{(LN-m)/2} (1 - q^{-2})^{N-m} 1 (q^{-2}; q^{-2})_{N-m} C_{N,m} = 1,
\]

we have the usual expectation in (72).

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