Quantum Fluctuations in the Inflationary Universe

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Abstract

The recent three-year WMAP data selects large-field models with certain power-law potentials and small-field models for all power-law potentials as consistent inflation models. We study the large-field and small-field inflation model with a quadratic and a quartic potential within the framework of quantum field theory in an expanding Friedmann-Robertson-Walker universe. We find that quantum fluctuations in the small-field model lead to a significant contribution to the effective potential and may have non-negligible effects on the slow-roll parameters predicted by classical theory.

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I. INTRODUCTION

The inflation paradigm assumes a period of accelerated expansion of the universe driven by the inflaton, a homogeneous scalar field or condensate of the scalar field. Inflation models predict that the density of the universe is close to the critical value, the geometry of the universe is flat, the power spectrum of the primordial density perturbations is nearly Harrison-Zeldovich scale-invariant and the CMB radiations are homogeneous and isotropic. Now the three-year WMAP data is able to select certain types of inflation models and at the same time exclude many other models [1].

Inflation models are classified into the single-field models and multi-field models depending on the number of fields involved. The single-field models have no internal space and are described by a single scalar field and are further classified into the large-field and small-field models [2]. The chaotic inflation belongs to the former while the new inflation model belongs to the latter. In large-field models the inflaton obtains the potential energy from Planck scale quantum fluctuations, whereas in small-field models it starts from a local maximum resulted from a symmetry breaking. The WMAP data together with other astronomical observations seems to favor the large-field and small-field models [1, 3, 4, 5].

In large-field or small-field models the inflaton can have the energy scale requiring quantum theory for a proper framework. The necessity of quantum theory for inflation dynamics may be understood even through a quantum-to-classical transition from quantum gravity [6] or a coherent state being close to a classical field [7]. Then the inflaton shows a characteristic behavior that a classical background field drives inflation and its quantum fluctuations lead to density perturbations. Quantum effects on inflation models have recently been studied in small-field models [8] and in large-field models [9].

The main purpose of this paper is to study the inflation models favored by the three-year WMAP data within semiclassical gravity. Either semiclassical gravity from quantum gravity based on the Wheeler-DeWitt equation [6] or quantum field theory in a classical background spacetime may be used. As we concern more about quantum fluctuations of fields than gravity itself, we shall employ quantum field theory in a classical spacetime. Since the inflaton evolves out of equilibrium as the universe expands rapidly during the inflation period, nonequilibrium quantum field theory should be used. One of nonequilibrium formalisms is the closed-time path integral introduced long time ago by Schwinger and
Keldysh [10]. In this paper we shall use a canonical formalism that unifies the functional Schrödinger equation [11] with the Liouville-von Neumann equation for density operator [12]. In this canonical formalism the semiclassical Friedmann equation can keep an analogous form of the classical equation. We find that, in contrast with relatively small quantum fluctuations in large-field models, quantum fluctuations grow exponentially due to the spinodal instability during slow-rolling in small-field models and may contribute non-negligible amounts to the slow-roll parameters predicted by the classical theory. We calculate the effective small-roll parameters including quantum fluctuations, compare with classical values and finally discuss the physical implications in cosmology.

II. SEMICLASSICAL GRAVITY

Inflation models assume the existence of an inflaton (a homogeneous scalar field or a condensate of the scalar field) whose energy density leads a period of accelerated expansion of the universe. The spacetime of the universe is assumed to have the homogeneous and isotropic Friedmann-Robertson-Walker metric

$$ ds^2 = -dt^2 + a^2(t)d\Omega_3^2, $$

where $d\Omega_3^2$ is the metric for three-dimensional space at fixed $a(t)$. In classical gravity the inflation dynamics is described by the Friedmann equation (time-time component of the Einstein equations) (in units of $G = c = 1$)

$$ \left( H = \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi}{3a^3} \left( \dot{\phi}^2 + V(\phi) \right), $$

and the inflaton field equation

$$ \ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0. $$

Here and hereafter overdots denote derivatives with respect to the comoving time $t$.

The single-field inflation models compatible with WMAP data are classified into large-field models in which the change of the inflaton field during $N$-fold is $\Delta \phi \gg M_P$ ($M_P = 1/\sqrt{8\pi G}$ being the Planck mass) and small-field models in which the change is $\Delta \leq M_P$ [2]. The chaotic inflation with monomial potentials $V = M^4(\phi/M_P)^p$ belongs to the former class and the new inflation with potentials $V = M^4[1 - (\phi/\mu)^p]$ belongs to the latter class.
Here models are characterized by the energy scale $M$ and power-law exponent $p$ and/or $\mu$. It is remarkable that the three-year WMAP data now excludes $p > 3.1$ at 95 % CL in the large-field models but allows all the value of $p$ in the small-field models [3]. In the large-field models the best fit with three-year data is $m^2\phi^2$ but $\lambda\phi^4$ is not completely ruled out and still on the edge of 68 % CL.

Though most inflation models assume classical gravity as the underlying theory, the proper theory should be semiclassical gravity, in which the background spacetime is still classical but matter fields are quantum. It is widely accepted that quantum fluctuations are the seeds for structure formation, primordial black holes and defect formation. Primordial density perturbations are remnants of quantum fluctuations and quantum fluctuations may have some imprints on CMB data such as the non-Gaussianity or on cosmic gravitational waves. Furthermore, the energy scale of the inflaton at the onset of the inflation belongs to a quantum regime. In the new inflation models the inflaton slowly rolls down the potential from an initial false vacuum toward the true vacuum of global minimum whereas in the chaotic inflation models the inflaton gains a Planck-scale potential energy from quantum fluctuations. It is thus legitimate to use semiclassical gravity or quantum field theory in curved spacetimes to study the inflation dynamics. Further it would be interesting to see how the inflation models work in semiclassical gravity.

The very early stage of evolution of the universe would have been described by quantum gravity, symbolically denoted by $\hat{G}_{\mu\nu} = 8\pi\hat{T}_{\mu\nu}$. It is expected to have a quantum-to-classical transition of gravity: $\hat{G}_{\mu\nu} = 8\pi\hat{T}_{\mu\nu} \Rightarrow G_{\mu\nu} = 8\pi\langle\hat{T}_{\mu\nu}\rangle \Rightarrow G_{\mu\nu} = 8\pi\mathcal{T}_{\mu\nu}$ [6]. (From now on we drop the overhats for operators unless they cause confusion.) The universe enters the semiclassical gravity regime when the Planck scale gravity first decoheres and becomes classical but matter fields still keep quantum nature. This is, in the semiclassical gravity regime, the spacetime evolution is governed by the Friedmann equation (in units of $G = c = \hbar = 1$)

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3a^3} \langle H_\phi \rangle.$$  \hfill (4)

Here the inflaton with the Hamiltonian,

$$H_\phi = \int d^3x \left[ \frac{\pi^2}{2a^2} + \frac{a}{2}(\nabla\phi)^2 + a^3V(\phi) \right],$$  \hfill (5)
obeys the functional Schrödinger equation [11]
\[
i\frac{\delta \Psi(\phi)}{\delta t} = H_\phi \Psi(\phi). \tag{6}
\]

It should be noted that the inflaton as well as other matter fields evolves out of equilibrium because the Hamiltonian (5) depends on the scale factor \(a(t)\) of the universe, in particular, when the universe undergoes an accelerated expansion phase. One may use a criterion that a system evolves out of equilibrium when the operator \(\rho_H(t) = e^{-\beta H_\phi(t)}/Z_H\) deviates by a large amount from the true density operator \(\rho_I(t)\) in the sense of \(|||\rho_H(t) - \rho_I(t)||/||\rho_I(t)|| \gg 1\) with respect to an appropriate measure \(|| \cdot ||\) [12]. On the other hand, when \(|||\rho_H(t) - \rho_I(t)||/||\rho_I(t)|| \leq \mathcal{O}(1)\), the system evolves quasi-equilibrium. The true density operator satisfies the Liouville-von Neumann equation
\[
i\frac{\delta \rho_I(t)}{\delta t} + [\rho_I(t), H_\phi(t)] = 0. \tag{7}
\]

The closed-time path integral by Schwinger and Keldysh may be one method to handle this nonequilibrium evolution [10]. The time-dependent functional Schrödinger equation may be another method that makes use of all useful properties of quantum mechanics [11]. It is observed that the Liouville-von Neumann equation may be used to solve not only the density operator but also the time-dependent functional Schrödinger equation [7, 12]. In this paper we shall use the latter approach.

III. NONPERTURBATIVE METHOD FOR NONEQUILIBRIUM QUANTUM FIELDS

To treat the nonequilibrium quantum field, we first divide the field into a classical background field and quantum fluctuations, \(\phi(t, x) = \phi_c(t) + \phi_f(t, x)\), or use the (squeezed or thermal) coherent state representation, \(\langle \phi \rangle_C = \phi_c\) and \(\langle (\phi - \phi_c)^2 \rangle_C = \langle \phi_f^2 \rangle_{V/T} [7, 12]\). We then decompose the field and momentum into Fourier modes as
\[
\phi_f(t, x) = \int \frac{d^3k}{(2\pi)^3} \phi_k(t)e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \phi_k(t, x) = \int d^3x \phi_f(t, x)e^{-i\mathbf{k} \cdot \mathbf{x}}. \tag{8}
\]

In the oscillator representation
\[
\phi_k(t) = \varphi_k(t)a_k(t) + \varphi_k^*(t)a_k^\dagger(t), \quad \pi_k(t) = a^3\left(\dot{\varphi}_k(t)a_k(t) + \dot{\varphi}_k^*(t)a_k^\dagger(t)\right). \tag{9}
\]
the Hamiltonian separates into a quadratic part $H_0$ and the part $H_P$ higher than quadratic in $a_k$ and $a_k^\dagger$ as

$$H_\phi(t, a_k, a_k^\dagger) = H_0(t, a_k, a_k^\dagger) + \lambda H_P(t, a_k, a_k^\dagger). \quad (10)$$

It may be possible to find exactly the Green function for $H_0$,

$$\left[ i \frac{\partial}{\partial t} - H_0 \right] G_0(t, x; t', x') = \delta(t - t') \delta(x - x'), \quad (11)$$

in terms of an auxiliary field variable that obeys a mean-field type equation, as will be shown in the next section. Then the full wave functional is expanded perturbatively as

$$\Psi(t, x) = \Psi_0(t, x) + \lambda \int dt' d^3 x' G_0(t, x; t', x') H_P(t', x') \Psi(t', x'). \quad (12)$$

We compare our formalism with the Hartree method, another canonical formalism, which has been used to study large-field and small-field inflation models in semiclassical gravity [8, 9]. The Hartree method approximates the nonlinear Heisenberg equation by a linear equation by factorizing nonlinear terms by powers of the two-point correlation function, which includes some part of higher-order loop corrections from the factorization. However, it has a disadvantage that the other higher-order loop corrections cannot be found beyond the linear evolution equation.

**IV. LOOP CORRECTIONS OF QUANTUM FLUCTUATIONS**

In semiclassical gravity, the energy density and the effective potential are obtained by taking the expectation value of the corresponding operators with respect to the vacuum or thermal state that satisfies the time-dependent functional Schrödinger equation:

$$\rho_c(\phi) = \langle H_\phi(\phi_c(t) + \phi_f(t, x)) \rangle_{V/T} = \rho_c(\phi_c) + \rho_q(\phi_c, \phi_f), \quad (13)$$

$$V_c(\phi) = \langle V(\phi_c(t) + \phi_f(t, x)) \rangle_{V/T} = V_c(\phi_c) + V_q(\phi_c, \phi_f). \quad (14)$$

Here $\rho_c$, $V_c$ and $\rho_q$, $V_q$ denote the classical energy density, potential and the loop corrections to the energy density and potential, respectively. The expectation value with respect to, for instance, a Gaussian vacuum state or a coherent-thermal state, takes the simple form [7, 12]

$$\langle (\phi_c(t) + \phi_f(t, x))^{2n} \rangle_{V/T} = \sum_{k=0}^{n} \frac{(2n)!}{2^k k!(2n - 2k)!} \left( \langle \phi_f^2 \rangle_{V/T} \right)^k \phi_c^{2n-2k}. \quad (15)$$
where the two-point correlation function for quantum fluctuations is given by

$$\langle \phi_j^2 \rangle_{V/T} = \int \frac{d^3k}{(2\pi)^3} (2n_k(t_0) + 1) \phi^*_k(t) \phi_k(t).$$  \hspace{1cm} (16)$$

Here $n_k(t_0)$ is the Bose-Einstein distribution of fluctuations at the initial time $t_0$ and $\phi_k(t)$ is an auxiliary field that will be determined below by the functional Schrödinger equation and/or Liouville-von Neumann equation. In fact, the two-point correlation function is the evolution of that of the initial time $t_0$ in the thermal or vacuum state.

In most of inflation models it is assumed that the classical field $\phi_c$, which plays an order parameter here, dominates the energy density and the potential and therefore determines the slow-roll parameters for inflation. So the loop corrections of quantum fluctuations to the slow-roll parameters may be written as

$$\epsilon_e = \frac{M_p^2}{2} \left( \frac{dV_c}{d\phi_c} \right)^2 = \epsilon_c \left( 1 + \frac{V''}{V'c} \right)^2 = \epsilon_c (1 + \epsilon_q),$$  \hspace{1cm} (17)$$

$$\eta_e = \frac{M_p^2}{2} \left( \frac{d^2V_c}{d\phi_c^2} \right)^2 = \eta_c \left( 1 + \frac{V''''}{V''c} \right)^2 = \eta_c (1 + \eta_q).$$  \hspace{1cm} (18)$$

Note that $\epsilon_q$ and $\eta_q$ measure the relative amount of quantum corrections to the slow-roll parameters. If $\epsilon_q$ and $\eta_q$ are order of unity or larger than one, then quantum fluctuations need to be included in the data analysis or dominate over the classical parameters so the classical theory may not be valid. In the latter case one needs a full nonperturbative quantum theory such as the canonical method in Sec. 3 for inflation models.

In this paper we shall confine our attention to the potential of the form $V = V_0 \pm m^2 \phi^2/2 + \lambda \phi^4/4!$, in which $V_0 = M^4$, $-m^2 = -M^4/\mu^2$ and $\lambda = M_\lambda^4/\mu^4$ for a small-field model while $V_0 = 0$, $+m^2 = M^4/M_p^2$ and $\lambda = M_\lambda^4/M_p^4$ for a large-field model, both of which are favored by three-year WMAP data. The quartic term $\lambda \phi^4/4!$ is necessary to stop the inflaton from rolling down eternally in the small-field model but may be a minor modification of the massive scalar field model in the large-field model. The large-field and small-field model based on these potentials have a sufficient $e$-folding necessary for successful inflation [13].

The classical background field obeys the field equation

$$\dddot{\phi}_c + 3H \ddot{\phi}_c \pm m^2 \phi_c + \frac{\lambda}{6} \left( \phi_c^2 + 3 \langle \phi_j^2 \rangle_{V/T} \right) \phi_c = 0.$$  \hspace{1cm} (19)$$

In the Schrödinger picture of Sec. 3, the auxiliary field obeys the $c$-number equation [12]

$$\dddot{\varphi}_k + 3H \ddot{\varphi}_k + \left( \pm m^2 + \frac{k^2}{a^2} + \frac{\lambda}{2} \phi_c^2 + \frac{\lambda}{2} \langle \phi_j^2 \rangle_{V/T} \right) \varphi_k = 0.$$  \hspace{1cm} (20)$$
From the equal-time commutation relation $[\phi_k(t), \pi_k(t)] = i$ in Eq. (9), the complex solution should satisfy the Wronskian condition
\[ a^3(t)(\dot{\varphi}_k^*(t)\varphi_k(t) - \dot{\varphi}_k(t)\varphi_k^*(t)) = i. \tag{21} \]

The two-point correlation function is then obtained by putting the complex solution $\varphi_k$ in Eq. (20) satisfying Eq. (21) into Eq. (16).

Finally, the loop corrections to the slow-roll parameters are given by
\[
\begin{align*}
\frac{V_q}{V_c} &= \frac{\lambda}{4} \frac{(\phi_f^2)_{V/T} \phi_c^2 \pm m^2(\phi_f^2)_{V/T} + \frac{\lambda}{8}((\phi_f^2)_{V/T})^2}{V_0 \pm \frac{m^2}{2} \phi_c^2 + \frac{\lambda}{24} \phi_c^4}, \\
\frac{V_q'}{V_c'} &= \frac{\lambda}{2} \frac{(\phi_f^2)_{V/T} \phi_c}{\pm m^2 \phi_c + \frac{\lambda}{6} \phi_c^3}, \\
\frac{V_q''}{V_c''} &= \frac{\lambda}{4} \frac{(\phi_f^2)_{V/T}}{\pm m^2 + \frac{\lambda}{2} \phi_c^2}.
\end{align*}
\tag{22} \tag{23} \tag{24}
\]

In the large-field model, the classical background field slowly rolls down from a large initial value $\phi_c(t_0) \gg M_p$ as $\phi_c(t) \approx \phi_c(t_0) e^{-\int m^2/3H}$ but quantum fluctuations oscillate due to the positive frequency squared $\omega_k^2(t) = m^2 + k^2/a^2 + \lambda(\phi_c^2/2 + (\phi_f^2)_{V/T})/2$ as
\[
\varphi_k(t) \approx \frac{1}{\sqrt{2a^3(t)\Omega_k(t)}} e^{-i\int \Omega_k(t)},
\]
\[
\Omega_k(t) = \left[ \omega_k^2 - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H}\right) + \left(\frac{1}{4} \left(\frac{\dot{\Omega}_k}{\Omega_k}\right)^2 - \frac{1}{2} \left(\frac{\ddot{\Omega}_k}{\Omega_k}\right)^2\right)\right]^{1/2}. \tag{25}
\]

In the lowest-order WKB approximation $\Omega_k \approx \omega_k$. Hence $\varphi_k^* \varphi_k \approx 1/(2a^3\Omega_k)$ and quantum corrections during the inflation period are suppressed as $V_q/V_c \approx V_q'/V_c' \approx V_q''/V_c'' \approx \lambda (\phi_f^2)_{V/T}/(2m^2) \ll 1$. It is shown that the loop corrections indeed change one percent of classical slow-roll parameters [9].

On the other hand, in the small-field model quantum fluctuations grow exponentially due to the spinodal instability from the negative curvature dominated by $-m^2$ at the onset of the inflation. This growth of quantum fluctuations due to the spinodal instability overcomes the damping due to the expansion of the universe. At the onset of inflation the inflaton slowly rolls down the potential, so as long as the self-interaction is small compared with the negative mass squared, the inflaton approximately becomes a scalar field with the negative mass squared in an expanding spacetime. The scalar field in the de Sitter spacetime $a(t) = e^{Ht}$ may shed light on the dynamics of the inflaton. The mode equation
\[
\ddot{\varphi}_k + 3H\dot{\varphi}_k + \left(k^2e^{-2Ht} - m^2\right)\varphi_k = 0, \tag{26}
\]
has the solution

\[ \varphi_k = \sqrt{\frac{\pi}{4H}} e^{-3Ht/2} H^{(1)}_{\nu} (k e^{-Ht}/H), \quad \nu = \sqrt{\frac{9}{4} + \frac{m^2}{H^2}}. \]  

(27)

At \( t = -\infty \), all the modes are inside the Hubble horizon and behave as the Minkowskian modes \( \varphi_k \approx e^{-ikt}/\sqrt{2k} \). However, at later times, while ultraviolet modes, which matters renormalization, are still inside the Hubble horizon, infrared modes cross the Hubble horizon and grow as \( |\varphi_k|^2 \approx \Gamma(\nu)(2H/k)^{2\nu} e^{2(\nu-3/2)Ht/(4\pi H)} \). Therefore the loop corrections may not be negligible in small-field models.

V. CONCLUSION

Cosmology now has become a science of precision in the sense that measurements of CMB select the inflation models. In particular, the three-year WMAP data favors large-field models with power-law potentials for \( p \leq 3.1 \) and small-field models for all values of \( p \) [3, 4]. This analysis is based on the classical dynamics of the inflaton. However, at the onset of inflation, the inflaton would be in a quantum state and afterward evolve out of equilibrium due to a rapid expansion of the universe. It is likely that quantum fluctuations modify the slow-roll parameters.

In this paper we have shown that the nonequilibrium quantum evolution of the inflaton changes the slow-roll parameters as \( \epsilon_e = \epsilon_e (1 + \epsilon_q), \eta_e = \eta_e (1 + \eta_q) \), etc. As the contribution of quantum fluctuations is comparable to the classical one particularly for small-field models and thus invalidates the classical inflation models, it would be worthy to study inflation models in semiclassical gravity. It is likely that the nonequilibrium quantum evolution may distinguish the small-field models owing to the spinodal instability of quantum fluctuations from the large-field models with small fluctuations. In small-field models quantum fluctuations require a full nonperturbative treatment. The detailed calculation of quantum fluctuations on slow-roll parameters in the small-field and/or large-field inflation models and the comparison with WMAP data and other astronomical observations will be addressed in a future publication.

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