Gravitational Radiation from Rotational Instabilities in Compact Stellar Cores with Stiff Equations of State

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Abstract

We carry out 3-D numerical simulations of the dynamical instability in rapidly rotating stars initially modeled as polytropes with $n = 1.5, 1.0,$ and $0.5$. The calculations are done with a SPH code using Newtonian gravity, and the gravitational radiation is calculated in the quadrupole limit. All models develop the global $m = 2$ bar mode, with mass and angular momentum being shed from the ends of the bar in two trailing spiral arms. The models then undergo successive episodes of core recontraction and spiral arm ejection, with the number of these episodes increasing as $n$ decreases: this results in longer-lived gravitational wave signals for stiffer models. This instability may operate in a stellar core that has expended its nuclear fuel and is prevented from further collapse due to centrifugal forces. The actual values of the gravitational radiation amplitudes and frequencies depend sensitively on the radius of the star $R_{eq}$ at which the instability develops.

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I. INTRODUCTION

The direct detection of gravitational radiation from astrophysical sources is one of the greatest challenges of our day. With interferometers such as LIGO [1], VIRGO [2], and GEO [3] under construction, and a new generation of spherical resonant mass detectors under study [4,5], the detailed modeling of these sources takes a high priority.

One promising source of gravitational waves is the development of rotational instabilities in dense stellar cores or compact objects [6]. Consider, for example, a rapidly rotating stellar core that has expended its nuclear fuel and is unable to collapse to neutron star size due to centrifugal forces. If such an object underwent a rotational instability, it could possibly shed enough angular momentum to allow collapse to a supernova [7,8]. Alternatively, a neutron star that is spun up by accretion of mass from a binary companion may reach fast enough rotation rates to go unstable [9,10].

Global rotational instabilities in fluids arise from nonradial “toroidal” modes $e^{\pm im\phi}$, where $m = 2$ is known as the “bar mode”. It is convenient to parametrize them by

$$\beta = \frac{T_{\text{rot}}}{|W|},$$

where $T_{\text{rot}}$ is the rotational kinetic energy and $W$ is the gravitational potential energy [11,12,13]. We focus on the bar instability since it is expected to be the fastest growing mode. This instability can occur under two different physical mechanisms. The dynamical bar instability is driven by Newtonian hydrodynamics and gravity. It operates for fairly large values of the stability parameter $\beta > \beta_{d}$ and develops on a timescale of approximately one bar rotation period. In contrast, the secular instability arises from dissipative processes such as gravitational radiation reaction and viscosity. It occurs for $\beta_{s} < \beta < \beta_{d}$ and develops on a timescale of several rotation periods or longer [10]. For the constant density, incompressible, uniformly rotating Maclaurin spheroids we have $\beta_{s} \approx 0.14$ and $\beta_{d} \approx 0.27$. In the case of differentially rotating fluids with a polytropic equation of state

$$P = K \rho^{\Gamma} = K \rho^{1+1/n},$$

(2)
where $n$ is the polytropic index and $K$ is a constant that depends on the entropy, early studies indicated that the secular and dynamical bar instabilities should occur at about these same values of $\beta$ [12,13,14,15]. More recent work [16] shows that both the angular momentum distribution and, to a lesser degree, the polytropic index affect the value of $\beta_s$ at which the $m = 2$ secular instability sets in. For the dynamical bar instability Pickett, et al. [17] demonstrate that, for $n = 1.5$ polytropes, the $m = 2$ dynamical stability limit $\beta_d \approx 0.27$ is valid for centrally condensed initial angular momentum distributions that are similar to those of Maclaurin spheroids. However, for angular momentum distributions with somewhat extended disk-like regions, both one- and two-armed spiral instabilities appear at considerably lower values of $\beta$.

The work presented in this paper is part of a research program aimed at calculating the gravitational radiation produced when a rapidly rotating stellar core undergoes the dynamical bar instability. These studies are carried out using 3-D numerical simulations. The gravitational field is purely Newtonian, and the gravitational radiation produced is calculated using the quadrupole approximation; the back reaction of the radiation on the system is not included. The rapidly rotating cores are initially modeled as polytropes with $\beta \approx 0.3$, which is just above the dynamical stability limit. In order to sustain such high rotational kinetic energy they must be rotating differentially [11,18]. Such objects could form, for example, when the cores of massive stars collapse on a dynamical time scale.

Much of the previous work in this area has concentrated on polytropes with $n = 1.5$. This case has been investigated by Tohline and collaborators [19,20,21] and more recently by Pickett, et al. [17] in the context of star formation. These studies primarily use an Eulerian code that imposes the polytropic equation of state [2] throughout the evolution instead of solving an energy equation; thus, the entropy generation by shocks during the later stages of evolution is not taken into account. The work of Houser, Centrella, and Smith [22] was the first to model the fluid using an energy equation and to calculate the resulting gravitational radiation; these calculations were carried out using the Lagrangian smooth particle hydrodynamics (SPH) method. (New [23] has recently performed similar
calculations using an improved version of Tohline’s Eulerian code.) Smith, Houser, and Centrella \cite{24} also updated the earlier work of Ref. \cite{20} by carrying out a detailed comparison study of this model with two different 3-D codes, one using Eulerian techniques and the other based on SPH.

In this paper we extend our calculations to objects having stiff polytropic equations of state using SPH \cite{25}. We use $\beta \approx 0.3$ and consider the cases $n = 1.5, 1.0, \text{and} \ 0.5$, which correspond to $\Gamma = 5/3, 2, \text{and} \ 3$. Previously, Williams and Tohline \cite{21} studied the initial development of the bar instability in similar models with $n = 0.8, 1.0, 1.3, 1.5, \text{and} \ 1.8$; longer evolutions were carried out in Ref. \cite{26} for the cases $n = 0.8 \text{and} \ n = 1.8$. Their work was done with an Eulerian code in cylindrical coordinates $(\varpi, \phi, z)$ that used the diffusive donor cell advection method and imposed the polytropic equation of state (2) throughout the runs. In addition, they modeled only the region $0 \leq \phi < \pi$ in the angular coordinate, so that only even toroidal modes were represented. Our simulations do not suffer from these restrictions, and include the calculation of the gravitational radiation.

This paper is organized as follows. Section \textbf{II} contains a brief description of the techniques used in our simulations. The construction of initial axisymmetric models with $\beta \approx 0.3$ is discussed in Sec. \textbf{III} and the dynamical evolution of the models is presented in Sec. \textbf{IV}. Analysis of the instabilities using Fourier components is given in Sec. \textbf{V} and the gravitational radiation produced by the models is presented in Sec. \textbf{VI}. The paper concludes with a summary and discussion of our results in Sec. \textbf{VII}.

\textbf{II. SIMULATION TECHNIQUES}

Detailed descriptions of the basic techniques used to carry out these simulations have been presented previously in Refs. \cite{24} and \cite{27}. We therefore give a only brief description of these methods in this section, and refer the reader to the literature for further information.

SPH is a Lagrangian method in which the fluid is modeled as a collection of fluid elements of finite extent described by a smoothing kernel \cite{28}. We have used the implementation of
SPH by Hernquist & Katz [29] known as TREESPH, which allows variable smoothing lengths and individual particle timesteps. For the runs discussed in this paper, we smooth over \( N_S = 64 \) neighbors for kernel interpolation. Shocks are handled using an artificial viscosity modified by the curl of the velocity field, with the user-specified coefficients of the linear and quadratic terms taking the values \( \alpha = 0.25 \) and \( \beta_{AV} = 1.0 \); see Refs. [29, 24, 27] for more details. The gravitational forces in this code are purely Newtonian, and are calculated using a hierarchical tree method optimized for vector computers [30]. This leads to a significant gain in efficiency and allows the use of larger numbers of particles than would be possible with methods that simply sum over all possible pairs of particles.

We calculate the gravitational radiation produced by the instabilities using the quadrupole approximation, which is valid for nearly Newtonian sources [31]. The reduced (i.e., traceless) quadrupole moment of the source is given by

\[
I_{ij} = \int \rho \left( x_i x_j - \frac{1}{3} \delta_{ij} r^2 \right) d^3 r,
\]

where \( i, j = 1, 2, 3 \) are spatial indices and \( r = (x^2 + y^2 + z^2)^{1/2} \) is the distance to the source. For an observer situated on the axis at \( \theta = 0, \phi = 0 \) of a spherical coordinate system with its origin located at the center of mass of the source, the amplitude of the gravitational waves for the two polarization states takes the simple form

\[
h_+ = \frac{G}{c^4 r} \left( \ddot{I}_{xx} - \ddot{I}_{yy} \right),
\]
\[
h_\times = \frac{G}{c^4 r} \dot{I}_{xy},
\]

where an overdot indicates a time derivative \( d/dt \). The gravitational wave luminosity is given by

\[
L = \frac{dE}{dt} = \frac{1}{5} \frac{G}{c^5} \langle I_{ij}^{(3)} I_{ij}^{(3)} \rangle,
\]

and the rate at which angular momentum is lost through gravitational radiation is

\[
\frac{dJ_i}{dt} = \frac{2 G}{5 c^5} \varepsilon_{ijk} \langle I_{jm}^{(2)} I_{km}^{(3)} \rangle.
\]
Here there is an implied sum on repeated indices, the superscript \((3)\) indicates the third time derivative, and the angle brackets indicate an average over several wave periods. Since such averaging is not well-defined for these burst sources, we display instead the unaveraged quantities \((G/5c^5)I_{ij}^{(3)}I_{ij}^{(3)}\) and \((2G/5c^5)\epsilon_{ijk}I_{jm}^{(2)}I_{km}^{(3)}\) below. The energy emitted as gravitational radiation is

\[
\Delta E = \int L \, dt
\]

and the angular momentum carried away by the waves is

\[
\Delta J_i = \int (dJ_i/dt) \, dt.
\]

Finally, the gravitational wave energy spectrum \(dE/df\), which gives the energy emitted as gravitational radiation per unit frequency interval, takes the form

\[
\frac{dE}{df} = \frac{c^3 \pi}{G^2} (4\pi r^2 f^2 (|\tilde{h}_+ (f)|^2 + |\tilde{h}_\times (f)|^2)),
\]

where \(\tilde{h}(f)\) is the Fourier transform of \(h(t)\). The double angle brackets in Eq. (10) denote an average over all source angles.

We calculate the reduced quadrupole moment \(I_{ij}\) and its derivatives using the methods developed in Ref. [27]. In particular, particle positions, velocities, and accelerations already present in the code are used to obtain \(\dot{I}_{ij}\) and \(\ddot{I}_{ij}\), yielding expressions similar to those in Ref. [33]. This results in waveforms that are very smooth functions of time and require no filtering or smoothing to remove numerical noise. However, the luminosity \(L\) and angular momentum lost by gravitational radiation \(dJ_i/dt\) do contain the time derivative of the particle acceleration; this is taken numerically and therefore introduces some noise. To remove this noise, we smooth the luminosity data using simple averaging over a fixed time interval of \(0.1t_D\) centered on each point. Here, \(t_D\) is the dynamical time for a spherical star of mass \(M\) and (equatorial) radius \(R_{eq}\) and is defined by

\[
t_D = \left( \frac{R_{eq}^3}{GM} \right)^{1/2}.
\]
In general this procedure produces very smooth luminosity profiles \[27\] and makes a negligible change in the integrated luminosity \(\Delta E\), which gives the energy emitted as gravitational radiation. The profiles of \(dJ_i/dt\) are not smoothed.

III. INITIAL MODELS

The initial conditions for our simulations consist of rotating axisymmetric equilibrium fluid models having \(\beta \approx 0.3\) and polytropic index \(n\). The dynamical bar instability then grows from nonaxisymmetric perturbations due to particle discreteness in the models. We use a two-step procedure to generate the initial models. First, a self-consistent field (SCF) method is used to produce an equilibrium model on a grid. Then, a particle fit to the SCF model is performed to generate initial data for TREESPH. In this section, we describe the construction of these equilibrium models.

A. Self-Consistent Field Method

The first step is to use the SCF method (\[34\]; see also \[35,36,37\]) to generate axisymmetric equilibrium models. The SCF procedure derives from an integral formulation of the equations of hydrodynamic equilibrium which automatically incorporates the boundary conditions. We use cylindrical coordinates \((\varpi, z)\) and a uniformly-zoned grid of \(N_\varpi\) radial and \(N_z\) axial zones. An initial “guess” density distribution \(\rho(\varpi, z)\) is given, and the gravitational potential is calculated using a Legendre polynomial expansion to solve Poisson’s equation \[37\]. A rotation law in which angular momentum is constant on cylinders is specified. This takes the general form \(j(m) = j(m(\varpi))\), where \(j(m)\) is the specific angular momentum and \(m(\varpi)\) here denotes the dimensionless mass fraction interior to the cylinder of radius \(\varpi\) \[35\]. Following the convention of earlier work (e.g. Refs. \[19,20,21,26,36\]) we use the rotation law for the uniformly rotating, constant density Maclaurin spheroids,

\[
x(\varpi) = \frac{5}{2} \frac{J}{M} \left[1 - (1 - m)^{2/3}\right],
\]

\[12\]
where $J$ is the total angular momentum and $M$ is the total mass. Applying this rotation law to polytropes, which do not have constant density, produces differentially rotating models. The rotation law (12) is used to calculate a rotational potential, which is then used with the gravitational potential to compute an improved density distribution. This process is then repeated, iterating until convergence is achieved.

Rotation causes the resulting models to be flattened, so that $R_p < R_{eq}$, where $R_p$ is the polar radius and $R_{eq}$ is the equatorial radius. The freely specifiable quantities in this method are the dimensionless form of the rotation law $h(m) = (M/J)j(m)$, $n$, $R_{eq}$, and the axis ratio $R_p/R_{eq}$. Upon convergence to a solution of the equations of hydrodynamic equilibrium, this procedure gives the density $\rho(\varpi, z)$, the angular velocity $\Omega(\varpi)$, $M$, $J$, and $\beta$. To get a dimensional model, we specify the entropy, which is given by the constant $K$ in the polytropic equation of state (2), and the maximum density.

One measure of the accuracy of the initial equilibrium models comes from the virial theorem. For a fluid system this gives

$$2T + W + 3\Pi = 0,$$  \hspace{1cm} (13)

where $T$ is the total kinetic energy and $\Pi = \int PdV$ is the volume integral of the pressure. Using this we define the virial relation $VR$ by

$$VR = \left| \frac{2T + W + 3\Pi}{W} \right|.$$  \hspace{1cm} (14)

Using the SCF method, we generated initial models for the cases $n = 1.5, 1.0, \text{ and } 0.5$; we refer to these as SCF1, SCF2, and SCF3, respectively. The parameters of these initial models are given in Table I. Notice that a finer grid was used for the $n = 1.5$ model SCF1; this was done because it is more centrally condensed than the other two cases.

B. Generation of Particle Models

Once the SCF equilibrium model has been produced on a cylindrical grid, it must be transformed into a form readable by TREESPH. This is done by performing a particle
fit to the density profile $\rho(\varpi, z)$ given by the SCF method. The simplest technique for doing this randomly distributes particles within the probability distribution $\rho(\varpi, z)$ using a “rejection” method [38, 27]; this technique was used to generate the initial conditions for the runs performed in Ref. [22]. Due to the underlying Poisson distribution of particles, these initial models contain relatively large positive and negative density fluctuations. The resulting internal noise causes spurious entropy generation and partially masks the signal of the bar mode instability. Because of these problems, we developed methods to produce “colder” initial particle models with less internal noise [24, 25].

To generate these cold models, we use equipotential surfaces to determine the function $M(\Phi)$, which is the mass interior to an equipotential. This information then allows us to obtain the desired physical properties of the SCF model by reordering the particles from a physically simpler model. This simpler model is created by placing particles within the known stellar boundary to obtain a uniform density particle distribution. Then, using a chosen set of equipotential surfaces, the mass interior to these surfaces $M(\Phi)$ is computed in both the SCF and uniform density models. The actual number of surfaces used is taken to equal the number of zones in the $\varpi$ direction used to generate the SCF model, $N_{pot} = N_{\varpi}$; c.f. Table I. By a direct comparison between the resulting SCF and uniform density mass functions, a systematic contraction, or repositioning, of particle positions from their original locations in the uniform density model can be performed. This results in a particle model which realistically reproduces the SCF density profile, and does not suffer from the density fluctuations found in the random particle method [24].

To implement this procedure, we use the rotational ($\Phi_{rot}$) and gravitational ($\Phi_{grav}$) potentials, which are natural by-products of the SCF method, to define the total surface equipotential $\Phi_{surf}$. A uniform 3-D Cartesian grid centered on the star is then created inside a cube having length $2R_{eq}$. Particles are placed at each of the grid nodes, and a particle is accepted into the model if it lies inside the boundary of the star, producing a 3-D uniform density particle representation with the exact physical shape of the SCF stellar model. This uniform distribution must now be transformed into the more centrally-condensed polytropic
model given by the SCF method. In practice we do this by systematically “contracting” the particle positions in the original uniform density model along radial vectors (i.e., moving them toward the center). Comparison of the mass functions $M(\Phi)$ for the two models tells how to move the particles along radial rays to achieve the SCF density distribution. Since the SCF model is more centrally condensed than the uniform density distribution, the particles are moved toward the stellar center to their new positions. This repositioning of particles then reproduces the SCF density distribution. When the contraction process is completed, each particle is assigned an angular velocity $\Omega$ to reproduce the rotation law given in Eq. (12).

Using this contraction method, we generated initial models with $\beta \approx 0.3$ and $n = 1.5$ using particle numbers in the range $N \sim 2000 - 32,000$. These SPH models form the basis of the comparison study with Eulerian methods reported in Ref. [24]. Overall, these models do not suffer from the large density fluctuations present in the randomly generated models and, when evolved using TREESPH, better reproduce the basic features of the bar mode instability.

To generate the initial models for the runs presented here, we incorporated several improvements to this method. First of all, we added an iterative procedure to the contraction process. The initial repositioning of particles is identical to that presented above. However, once the uniform density model has been contracted, the mass interior to the equipotential surfaces is recalculated for the particle model. The radial contraction is again applied to all particles, and the process is iterated until the difference between the initial and final positions of all particles is less than a given tolerance, here chosen to be typically $\sim 0.5N^{-1}$. We also modified the initial placement of the particles within the uniform density model. A box with extent $\pm R_{eq}$ in the $x$ and $y$ directions and $\pm R_p$ in the $z$ direction was used instead of a cubical box. Squeezing the particles along the $z$ direction to fit within the range of $R_p$ should initially position them closer to their final equilibrium positions, thus making the contraction method more efficient. Also, to eliminate systematic errors due to contraction, the particle planes of constant $z$ were displaced in the $\varpi$ direction by $\pm 1/4$ of the inter-
particle spacing for even and odd \(z\)-planes, respectively. Overall, this iterated contraction method produces particle models that better reproduce the SCF density distribution.

Using this iterated contraction procedure, we constructed three models with \(\beta \approx 0.3\) using \(N \sim 16000\) for different values of the polytropic index \(n\). We refer to these as Run 1 \((n = 1.5)\), Run 2 \((n = 1.0)\), and Run 3 \((n = 0.5)\); see Table II. Fig. 1 shows the normalized equatorial plane density for these initial models. Notice that the density values calculated at the particle positions (shown as dots in the figure) match the SCF profiles (shown as solid lines) with very little scatter. The angular velocity profiles for all particles are shown in Fig. 2 and also reproduce the SCF values with very little scatter.

Rigorously, this repositioning of particles should be carried out by following normal, rather than radial, vectors. We have used radial contraction here for simplicity and computational speed. For spherical systems, the normal and radial vectors coincide. However, as the equilibrium model becomes increasingly oblate due to rotation, contracting the particles along their radial vectors becomes less accurate. In practice, we find the radial contraction method models the equatorial plane density well, as shown in Fig. 1. However, underdensities are observed in the regions around the rotation axis for increasing values of \(|z|\). When the models are evolved forward in time, this causes a slight redistribution of mass and angular momentum in the inner regions. Based on comparisons of the \(n = 1.5\) singly contracted models with Eulerian runs in Ref. 24, we do not believe that this small adjustment of the initial models significantly affects the evolution of the bar mode instability. The improved iterated contraction method reduces this under-density somewhat for the \(n = 1.5\) case. As \(n\) decreases, the polytrope becomes less centrally condensed and hence is closer to the uniform density model prior to contraction. As a result the under-densities decrease for Run 2, and almost disappear for Run 3.
IV. DYNAMICAL EVOLUTION

The initial particle models for Runs 1, 2, and 3 generated using the iterated contraction method were evolved in time using TREESPH. The case \( n = 1.5 \) has already been the subject of our detailed comparison study using Eulerian and SPH methods reported in Ref. [24]. Run 1 comprises the evolution of an improved initial model with \( n = 1.5 \) and shows the same general behavior seen in that previous study; it is included here as a benchmark for comparison with the stiffer equations of state.

The dynamical evolution of these models is displayed visually in Figs. 3 - 8. Plots showing the particle positions projected onto the equatorial plane are shown for Run 1 with \( n = 1.5 \) in Fig. 3, Run 2 with \( n = 1.0 \) in Fig. 5 and Run 3 with \( n = 0.5 \) in Fig. 7. Corresponding contour plots covering the same spatial area in the equatorial plane are shown for Run 1 in Fig. 4, Run 2 in Fig. 6, and Run 3 in Fig. 8. Time is measured in units of the dynamical time \( t_D \) for a spherical star with radius \( R_{eq} \) as defined in Eq. (11); recall that these rapidly rotating models have significant rotational flattening. All models are rotating in the counterclockwise direction. Runs 1 and 2 were stopped at \( t_f = 35t_D \) and \( t_f = 50t_D \), respectively, by which times these models had essentially stopped evolving. Run 3 was still evolving at \( t_f = 60t_D \) when it was stopped because of the need to save computer time. Table II displays several important parameters of these models.

The three models exhibit certain basic features in common; c.f. [13,19,20,22,24,26]. Non-axisymmetric structure grows spontaneously out of deviations from axisymmetry, caused in these models by particle discreteness. A global bar-shaped structure develops in which the amplitude of the \( m = 2 \) mode grows exponentially in time; see Sec. V. During the bar mode’s growth, trailing spiral arms develop as mass is shed from both ends of the bar. The bar and spiral arms exert gravitational torques that cause angular momentum to be transported outward from the core and lost from the ends of the bar. The spiral arms expand supersonically, causing shock heating and dissipation. Careful examination of the contour plots shows that the cores recontract toward an axisymmetric state after the initial
growth of the bar and ejection of spiral arms. These systems undergo two or more such episodes, depending on $n$; we will describe this in more detail below. Table [II] shows that the amounts of mass and angular momentum in the cores at the ends of the Runs 1 and 2 are very similar, as are the final values of the stability parameter $\beta$. Here we define the core to contain material within cylindrical radius $\varpi = R_{eq}$, where $R_{eq}$ is the initial equatorial radius. Since Run 3 was still evolving when the simulation was ended, we do not know what its final values will be. Overall, a significant amount of angular momentum is removed from the core by a relatively small amount of mass.

Figure 9 displays the mass and angular momentum distributions for the three runs in the initial [frames (a) and (b)] and final [frames (c) and (d)] states. Here, $m(\varpi)/M$ and $J(\varpi)/J_0$ are the normalized mass and angular momentum within cylindrical radius $\varpi$, respectively. $M$ is the total mass and $J_0$ is the initial total angular momentum. Notice that at the final times the curves for both the mass and angular momentum distributions intersect near $\varpi \sim R_{eq}$. This is consistent with the fact that the models all shed roughly the same total amount of mass and angular momentum, and have final cores with radii $\varpi \sim R_{eq}$.

Figs. 3(l) and 5(u) show that the final states of both Runs 1 and 2 exhibit a flattened “double halo” structure, consisting of a denser, inner region surrounded by a more diffuse, extended outer distribution of matter. (We do not know if Run 3 will have the same double halo structure at the conclusion of its evolution since the model was not run long enough to reach its final state.) This double halo may result from differences in the angular momentum carried by the mass when it is shed. During the first spiral arm ejection phase, the system has a higher value of $\beta$ and the mass is shed from the ends of the bar with a greater angular momentum. This mass moves into the vacuum carrying a fraction of the initial angular momentum of the system, and eventually distributes itself about the remaining core to produce the outer halo. The inner halo is formed when the system undergoes the second spiral arm ejection episode at a lower value of $\beta$, and the mass is shed with lower angular momentum into a region that already has mass from the first episode.

For example, Fig. 10 shows the mass $\Delta m/M$ and angular momentum $\Delta J/J_0$ distributions
for Run 1 at $t = 13.4t_D$ (solid line) and $t = 18.5t_D$ (dashed line); these times correspond to the first and second episodes of mass shedding through spiral arms and are shown in the contour frames (e) and (i) in Fig. 4. Here $\Delta m$ is the amount of mass within a cylindrical shell of thickness $d\varpi$ at radius $\varpi$, and similarly for $\Delta J$. Fig. 10(a) and (b) show the first set of spiral arms represented as a localized concentration of mass directly outside the stellar core in the region $1 \lesssim \varpi/R_{eq} \lesssim 2$ at $t = 13.4t_D$. As the model evolves, this mass expands into the surrounding vacuum. The amount of mass shed during the second episode is much less than during the first; c.f. Table V. This can be easily seen when we examine Fig. 10(b), which zooms in on the ejected mass. Fig. 10(c) and (d) show that the material ejected from the primary instability carries a higher amount of angular momentum than the mass ejected after the second spiral arm ejection phase.

Comparison of Figs. 3 - 8 shows that several important model properties depend on the stiffness of the equation of state. For example, the spatial deformation of the initially axisymmetric model into an elongated bar-shaped figure increases as $n$ decreases and the fluid description approaches that of an incompressible fluid [21]. The widths of the spiral arms and bar also depend on the polytropic index, both decreasing as $n$ decreases. And, as already mentioned, the system undergoes more spiral arm ejection phases as the equation of state stiffens.

Recall that for stiffer polytropes, the density profiles become less centrally condensed, as shown in Fig. 1. This greater amount of mass near the stellar boundary causes the material at the edge to be more tightly bound. Also, since the same rotation law Eq. (12) is applied to all runs, the less compressible models exhibit a smaller degree of differential rotation. This can be seen by examining Figs. 2 and 11, which show, respectively, the angular velocities of the models at the initial and final times. The results of these effects can be clearly seen in frame (c) of Figs. 3, 5, and 7, which are the particle position plots for Runs 1, 2 and 3, respectively. Comparison of these frames shows the mass at the stellar boundary becoming less diffuse as the polytropic index decreases. Overall, these effects contribute to the development of the less tightly wound spiral arm pattern found in models with lower
polytropic indices \cite{21}.

The behavior of the stability parameter for these runs is shown in Figure 12. The solid lines give $\beta = T_{\text{rot}}/|W|$ and the dashed lines show $T/|W|$, where $T_{\text{rot}}$ is the rotational kinetic energy and $T$ is the total kinetic energy. Comparison with Figs. 3-8 shows that $\beta$ decreases sharply from its initial value as the bar instability develops, dropping to a local minimum as the bar reaches its maximum elongation. For stiffer polytropes, the temporal location of the initial growth of the bar occurs later in the evolution, with the minimum value of $\beta$ occurring at $\sim 13t_D$, $17t_D$, and $19t_D$ for Runs 1, 2, and 3, respectively. When the results of Williams and Tohline \cite{26} are converted to time measured in units of the dynamical time $t_D$, they also show that the instability reaches a nonlinear amplitude later as $n$ is decreased. Notice also that the value of the first local minimum of $\beta$, which corresponds to the end of the first spiral arm ejection phase, decreases with $n$; c.f. \cite{26}. This behavior reflects the fact that as $n$ decreases, the maximum elongation of the central bar-like region, and hence its moment of inertia, increases. Assuming angular momentum conservation in the core, this causes the minimum kinetic energy to decrease with $n$.

At the end of the first spiral arm ejection phase, the core recontracts and $\beta$ increases again. As the models evolve forward in time, they undergo successive periods of spiral arm ejection and core recontraction. The number of these episodes increases as the equation of state stiffens, with Run 1 showing 2 spiral arm ejection phases and Run 2 showing 4. Run 3 undergoes 5 such episodes before the run was stopped; we expect it would exhibit several more if allowed to run to later times. Table IV shows the cumulative amount of mass and angular momentum shed from the core ($\varpi \leq R_{\text{eq}}$) after each spiral arm ejection phase. Notice that the cumulative amount of angular momentum lost by the core after each spiral arm ejection episode decreases as the equation of state stiffens. Therefore, assuming that the cores conserve angular momentum between periods of spiral arm ejection, the stiffer cores recontract to a higher angular velocity (and a larger $\beta$), and deform to a greater elongation (and a smaller $\beta$) than more compressible ones, as shown in Fig. 12. Also, since the cores lose angular momentum with each spiral arm ejection, each successive episode produces a
smaller maximum and a larger minimum value of $\beta$.

Now consider Run 3, which undergoes 5 periods of core recontraction corresponding to the local maxima of $\beta$ seen in Fig. 12. Its contracted core, displayed in the contour plots in frames (g), (k), (o), (r), and (w) of Fig. 8, shows a “parallelogram-like” structure. This feature becomes stronger as $n$ is decreased, with an intercomparison of frame (g) of Figs. 4, 6, and 8 showing the emergence of this feature as we progress through the polytropic sequence toward stiffer equations of state. An explanation of this feature is as follows. When core recontraction begins, the ends of the bar move toward the central regions. A more compressible model is better able to increase its central density in response to this material forced toward the center. However, as $n$ decreases, the material in the center cannot be easily compressed, thus forcing the fluid to move in a direction perpendicular to the contracting bar. This produces the observed parallelogram-like structure, or “anti-bar”. We shall see in Sec. V below that the $m = 4$ mode is also present in these simulations; this may provide a degree of freedom that allows the formation of this feature.

Another important difference observed as $n$ is changed concerns the long term behavior of the models. The spiral arms in Runs 1 and 2 eventually merge as the systems evolve, resulting in a late time state consisting of a nearly axisymmetric central remnant of extent $\sim R_{eq}$ surrounded by a flattened double halo. After a comparable period of time, the core of Run 3 was still quite elongated and the spiral arms were just beginning to merge. One explanation for this longer-lived elongation in the $n = 0.5$ case is as follows. Consider an equilibrium sequence of uniformly rotating axisymmetric polytropes parametrized by $\beta$. As $\beta$ increases along such a sequence, a point is eventually reached at which mass is lost at the equator. Uniformly rotating polytropes with $n \geq 0.808$ ($\Gamma \leq 2.24$) reach this mass-shedding limit before the point at which ellipsoidal configurations can exist [11,18]. Although Fig. 11 shows that the central remnants in these runs are differentially rotating, we believe that a similar mechanism may be operating here (see also [39,40]), causing the cores of Runs 1 and 2 to be nearly axisymmetric at the end of the run.

One major difference observed between our work and that of Tohline, Durisen and collab-
orators is the final outcome of the simulations. In all of our runs with \( n = 1.5 \) and \( n = 1.0 \), we find that the systems evolve to a final state consisting of a nearly axisymmetric central remnant surrounded by an extended disk-like halo \([24,22]\). This behavior was observed in both the \( n = 1.5 \) Eulerian and SPH runs investigated in the comparison study reported in Ref. \([24]\) as well as the work reported here. In contrast, all the long Eulerian evolutions reported by these other researchers (refs. \([17,19,20,26]\)) resulted in a bar-like central core surrounded by a ring of material. (Interestingly, in the very low resolution SPH run reported in Ref. \([20]\), the low-density material did form an extended disk.) Currently, we do not understand the reason for these differences in the final state. One possible explanation is that these other researchers do not evolve an energy equation, and hence cannot model the shocks which occur in the outer, low density regions \([17]\). Clearly, this is an important issue and efforts are underway to resolve these questions.

V. ANALYSIS OF FOURIER COMPONENTS

We quantify the development of the dynamical instability by studying the behavior of various Fourier components in the density using cylindrical coordinates \((\varpi, \phi, z)\). The density in a ring of fixed \( \varpi \) and \( z \) is analyzed using the complex Fourier series

\[
\rho(\varpi, \phi, z) = \sum_{m=-\infty}^{+\infty} C_m(\varpi, z)e^{im\phi}.
\]

(15)

The azimuthal Fourier decomposition of the density distribution for various components \( m \) is expressed in terms of the amplitudes \( C_m \), defined by

\[
C_m(\varpi, z) = \frac{1}{2\pi} \int_0^{2\pi} \rho(\varpi, \phi, z)e^{-im\phi} d\phi.
\]

(16)

The relative normalized amplitude is then defined by

\[
|A_m| = |C_m|/|C_0|,
\]

(17)

where \( C_0(\varpi, z) = \bar{\rho}(\varpi, z) \) is the mean density in the ring under examination. The integration is performed over the azimuthal coordinate \((0 \leq \phi < 2\pi)\) while \( \varpi \) and \( z \) remain fixed. In this way, the analysis can be carried out in “density rings” for different values of \( \varpi \) and \( z \).
To apply this procedure to the SPH simulations, the particle model is interpolated onto a cylindrical grid at pre-chosen time intervals (typically every 0.01\(t_D\)) using kernel estimation \[29\]. The grid used here consists of 66, 34, and 16 zones in the \(\varpi, \phi,\) and \(z\) directions, respectively. Analysis of the density in rings at different values of \(\varpi\) in the equatorial plane \((z = 0)\) gives quantitative information about the development of the global \(m = 2\) mode visually seen in Figures 3 - 8, as well as other Fourier components that may be present.

Examining how the normalized amplitude changes in time yields the growth rate \(d\ln|A_m|/dt\) of the various Fourier components in our models. In practice, this is obtained by fitting a straight line through the data points in the time interval during which the function \(\ln|A_m|\) is linearly growing (thus giving an exponential growth rate for \(|A_m|\)). The endpoints of this time interval are chosen “by eye”. A clearly defined linear region typically lasts for a relatively short time interval, and the value of the slope is sensitive to the endpoints defining the interval.

Also, by examining the complex phase \(\phi_m\) of a Fourier component, where

\[
\phi_m(\varpi, z) = \tan^{-1} \left[ \frac{\text{Im}(-C_m)}{\text{Re}(C_m)} \right], \tag{18}
\]

we can describe global non-axisymmetric structure propagating in the azimuthal direction. The development out of the initial noise of such a global mode with a well-defined angular eigenfrequency allows us to write

\[
\phi_m = \sigma_m t, \tag{19}
\]

where \(\phi_m\) is the phase angle of the disturbance, and \(\sigma_m\) is the eigenfrequency. The relation between the pattern speed \(\Omega_{\text{pat},m}\) of the \(m^{th}\) structure and the phase angle \(\phi_m\) is then

\[
\Omega_{\text{pat},m}(\varpi, z) \equiv \frac{1}{m} \frac{d\phi_m}{dt} = \frac{\sigma_m}{m}. \tag{20}
\]

Notice that for the \(m = 2\) bar mode, the eigenfrequency \(\sigma_2\) is twice the rotational speed of the bar, and the bar rotation period is \(T_{\text{bar}} = 2\pi m/\sigma_m = 4\pi/\sigma_2\).

The eigenfrequency is thus obtained by a simple calculation once the period is known. The period \(T_m\) of the \(m^{th}\) disturbance is determined directly from the period of the cosine
of the phase angle $\phi_m$ versus time. We use the function $\cos \phi_m$ rather than $\phi_m$ itself due to the multi-valued nature of the inverse trigonometric function arctan (Eq. (18)), which would require us to artificially insert multiples of $\pi$ in order to keep the function continuous. In practice, the half-period is obtained by locating successive differences in time between pairs of neighboring extrema of $\cos \phi_m$. Once the half-period is known, the eigenfrequency $\sigma_m$ can be obtained from Eq. (20). Overall, we find that the $m = 2$ instability grows on a time scale of approximately one bar rotation period, as expected.

The linearized tensor viral analysis (TVE) can also be used to calculate the bar mode amplitude $|A_m|$ and phase angle $\phi_m$. Although this method is exact only for small oscillations of uniform density, incompressible ellipsoids [42], it has proven a useful point of comparison for numerical simulations when adapted to the study of rotating compressible fluids [11,19,21]. Table V shows the TVE growth rates $d \ln |A_2|/dt$ and eigenfrequencies $\sigma_2$ for the $m = 2$ mode for the cases $n = 1.5$ and $n = 1.0$ with $\beta = 0.31$ reported by Williams and Tohline in Ref. [21], where we have converted from their units. Notice that as $n$ decreases, both the growth rate and eigenfrequency also decrease. We were unable to find TVE results in the literature for $n = 0.5$.

Fig. 13 shows the amplitudes of the Fourier components $m = 1, 2, 3,$ and 4 for Run 1 in the density ring $\varpi = 0.36 R_{eq}$ in the equatorial plane $z = 0$ as functions of time. As expected, the development of the $m = 2$ disturbance dominates the initial evolution, with the other components growing at later times. Both the $m = 2$ and $m = 4$ components show an initial period of exponential growth. Since this takes place at various cylindrical radii $\varpi$ throughout the model, we identify these disturbances as global modes. The initial peak in the $m = 2$ amplitude corresponds to the maximum elongation of the bar and the minimum value of $\beta$. The detailed structure after the initial growth of the bar mode varies somewhat with $\varpi$, as the density in various parts of the star fluctuates due to the complex motions involved in the contraction and re-expansion of the core.

The growth rate calculated for the $m = 2$ and $m = 4$ modes is rather sensitive to the specific time interval over which the linear fit to $\ln |A_m|$ is performed. For the $m = 2$ mode in
Run 1, we typically get \( d \ln |A_2|/dt \sim 0.6t_D^{-1} \), and \( d \ln |A_4|/dt \sim 0.8-1t_D^{-1} \). The calculation of the eigenfrequencies is more robust, yielding \( \sigma_2 \sim 1.9t_D^{-1} \) and \( \sigma_4 \sim 3.8t_D^{-1} \). Both modes reach their peak amplitudes at about the same time, then drop to local minima and grow again. Since the pattern speeds of these modes are nearly the same, \( \Omega_{\text{pat},2} \sim \Omega_{\text{pat},4} \sim 0.95 \), this suggests that the \( m = 4 \) mode is a harmonic of the bar mode and not an independent mode [21].

We also see that the \( m = 1 \) and \( m = 3 \) Fourier components grow somewhat, although not in the global and coherent fashion exhibited by the \( m = 2 \) and \( m = 4 \) modes. Recent work in the area of star formation [17,43] has highlighted the importance of the \( m = 1 \) case. Fig. [14] displays the amplitudes of the Fourier components \( m = 1, 2, 3, \) and 4 for Run 2 at the same value of \( \varpi \) used above. Again, the \( m = 2 \) and \( m = 4 \) components emerge as exponentially growing global modes, with the bar mode dominating the early stages of the evolution. The growth rates in this case are even more sensitive to the time interval chosen for the linear fit than was the case for Run 1. We find \( d \ln |A_2|/dt \sim 0.5-0.8t_D^{-1} \) and \( d \ln |A_4|/dt \sim 0.9-1.3t_D^{-1} \). Again, the eigenfrequencies are less dependent on the time interval chosen and take the values \( \sigma_2 \sim 1.5t_D^{-1} \) and \( \sigma_4 \sim 2.9t_D^{-1} \). The pattern speeds are thus \( \Omega_{\text{pat},2} \sim \Omega_{\text{pat},4} \sim 0.7t_D^{-1} \), implying again that the \( m = 4 \) mode is a harmonic of the bar mode.

Finally, the amplitudes of the Fourier components \( m = 1, 2, 3, \) and 4 for Run 3 are shown in Fig. [15]. The \( m = 1 \) and \( m = 3 \) Fourier components are stronger in this case, although they do not appear to develop into global modes. The \( m = 2 \) and \( m = 4 \) disturbances do develop into global modes and appear to be more strongly coupled than before. For example, the initial exponential growth rate of the bar mode (in the time interval \( 11.5t_D^{-1} \leq t \leq 15t_D^{-1} \)) is \( d \ln |A_2|/dt \sim 0.4t_D^{-1} \). Then, the bar mode growth rate increases sharply to \( d \ln |A_2|/dt \sim 2.2t_D^{-1} \); this may due to coupling with the \( m = 4 \) mode, which initially grows at the rate \( d \ln |A_4|/dt \sim 2.2t_D^{-1} \). The eigenfrequencies are \( \sigma_2 \sim 1.2t_D^{-1} \) and \( \sigma_4 \sim 2.3t_D^{-1} \), so that \( \Omega_{\text{pat},2} \sim \Omega_{\text{pat},4} \sim 0.6t_D^{-1} \).
Overall, the amount of structure seen in the Fourier components increases as \( n \) decreases. This reflects the fact that the stiffer fluids show more internal fluctuations as the cores expand and recontract. The eigenfrequencies \( \sigma_2 \) do show the decrease with \( n \) predicted by the TVE analysis as given in Table \( \text{V} \). This is due to the fact that the stiffer equations of state produce longer bars, which rotate more slowly. However, it is more difficult to assess the trends in the growth rates of the bar mode. If we consider the initial exponential growth period of the bar mode in Run 3, then it does grow at a slower rate than in Run 1 until the \( m = 4 \) mode starts to grow. As the fluids become stiffer and the number of spiral arm ejection episodes increases, the matter in the cores oscillates more. The coupling between the \( m = 2 \) and \( m = 4 \) also grows stronger; this may be linked to the development of the anti-bar discussed in Sec. \( \text{IV} \). In particular, the elongation of the anti-bar in Run 3 seen in Fig. \( 8 \) (g) at \( t = 25.2 t_{D} \) occurs at roughly the same time as the second maximum in \( \ln |A_4| \) shown in Fig. \( 15 \).

**VI. GRAVITATIONAL RADIATION**

The time-changing quadrupole moment caused by the development of the bar instability generates gravitational waves. The initial development of the bar mode produces a burst of radiation, followed by a weaker signal due to the subsequent expansions and recontractions of the core. Overall, the gravitational wave signal lasts for a longer time as the equation of state stiffens and the systems undergo more episodes of spiral arm ejection. Some interesting properties of the gravitational radiation produced by these models are given in Table \( \text{VI} \).

The gravitational waveform \( r h_+ \) for an observer on the axis at \( \theta = 0, \phi = 0 \) of a spherical coordinate system centered on the source is shown in Fig. \( 16 \) for these runs. Comparison of the waveforms with the contour plots in Figs. \( 4 \), \( 6 \) and \( 8 \) shows that indeed the onset of the burst coincides with the development of the primary instability. Notice that the maximum amplitude of the waveform does not vary significantly with the equation of state; see Table \( \text{VI} \). As the core retracts back to a more axisymmetric state, the amplitude of
The waveform decreases. The successive periods of spiral arm ejection and core recontraction produce additional bursts of gravitational waves; these have decreasing amplitudes because the maximum elongation of the core drops with each episode. Runs 1 and 2 show the weak signal of a slightly non-axisymmetric remnant in their final states, whereas Run 3 shows the much stronger signal of its still-evolving, more elongated core. Also, as the equation of state stiffens the frequency of the waves decreases, since the more elongated bars produced for smaller $n$ rotate more slowly.

The gravitational wave luminosity $L$ for these runs is displayed in Fig. 17. Notice that the peak amplitude of the luminosity decreases as $n$ decreases. For a non-axisymmetric object rotating rigidly about the $z$ axis, the luminosity takes the form

$$L = \frac{dE}{dt} = -\frac{32}{5} \frac{G}{c^5} (I_x - I_y)\Omega^6,$$

where $I_x$ and $I_y$ are the moments of inertia about the $x$ and $y$ axes, respectively, and $\Omega$ is the rotational angular velocity \[\text{[12]}\]. Since the term in $\Omega^6$ dominates, the peak luminosity should decrease as $n$ decreases and the central bar-like structures rotate more slowly, as shown in Fig. 17.

It is interesting to examine the structure of the luminosity profiles. The luminosity of Run 1 shows peaks at $t \sim 13t_D$ and $t \sim 19t_D$; these correspond to the primary and secondary spiral arm ejection episodes. Run 2 shows two closely spaced peaks at $t \sim 15t_D$ and $t \sim 19t_D$, followed by other peaks at $t \sim 25t_D$ and $t \sim 33t_D$. The first two peaks are associated with the initial period of spiral arm ejection; c.f. Fig. 12 (b). The remaining two peaks correspond to subsequent episodes. The successively smaller amplitudes reflect the fact that the angular velocity of the core decreases as angular momentum is shed on each subsequent episode. Finally, Fig. 17 (c) shows that Run 3 has two closely spaced luminosity peaks at $t \sim 17t_D$ and $t \sim 21t_D$, which are again associated with the initial burst. The local minimum in the luminosity at $t \sim 25t_D$ occurs at the time of core recontraction, as can be seen by comparing with Figs. 8 (g) and 12 (c). At later times it is more difficult to discern individual bursts in the luminosity function, a trend that is also seen in the waveform shown.
in Fig. 16 (c).

The energy emitted as gravitational waves $\Delta E/Mc^2$ is shown in Fig. 18. In the case of Runs 1 and 2, $\Delta E/Mc^2$ grows due to the initial and secondary bursts, and levels off when the cores reach their nearly axisymmetric final states. For Run 3, this quantity grows almost linearly with time and has not yet leveled off by the end of the simulation, indicating that the core is still quite nonaxisymmetric.

Fig. 19 shows the rate at which angular momentum is carried by the waves, $dJ_z/dt$. As was the case with the luminosity, we see structure in this quantity that corresponds to periods of spiral arm ejection and core recontraction. The angular momentum $\Delta J_z$ carried by the gravitational waves is displayed in Fig. 20 and shows features similar to those found in $\Delta E/M$.

Finally, the gravitational wave energy spectrum $dE/df$ is displayed in Fig. 21 and shows that the peak frequency of the gravitational radiation $f_{\text{grav}}$ decreases as the equation of state stiffens. Table VII shows that, as expected, $2f_{\text{bar}} \sim f_{\text{grav}}$, where $f_{\text{bar}} = (1/2)\sigma_2/2\pi$ is the rotational frequency of the bar. Notice, however, that the rotational frequencies $2f_{\text{bar}}$ are slightly lower than $f_{\text{grav}}$. This is due to the fact that, while the eigenfrequency $\sigma_2$ is calculated only during the initial development of the bar instability, $f_{\text{grav}}$ is computed for the entire evolution of the model and thus includes the higher rotational velocities obtained when the cores recontract.

VII. SUMMARY AND DISCUSSION

We have carried out numerical simulations of the dynamical instability in rapidly rotating stars initially modeled as polytropes with $n = 1.5, 1.0$, and $0.5$. These calculations have been done using a 3-D SPH code with $N \sim 16,000$ particles. The code has a purely Newtonian gravitational field, and the gravitational radiation is calculated in the quadrupole approximation. The back reaction of the gravitational radiation is not included.

All models exhibit the growth of the global $m = 2$ bar mode, with mass and angular
momentum being shed from the ends of the bar to form two trailing spiral arms. In general, as \( n \) decreases the central bar becomes narrower and more elongated. Once the central core has reached its maximum elongation, it begins to recontract toward a more axisymmetric state. This primary instability is followed by successive episodes of spiral arm ejection and core recontraction, with the number of these episodes increasing for stiffer equations of state. At the end of the simulations, the models with \( n = 1.5 \) and \( n = 1.0 \) have settled into a state with a nearly axisymmetric core of radius \( \sim R_{\text{eq}} \), where \( R_{\text{eq}} \) is the initial equatorial radius, surrounded by a flattened disk-like halo that contains \( \sim 10\% \) of the total mass and \( \sim 30\% \) of the total angular momentum. Since these models have \( \beta_a < \beta < \beta_d \), they are expected to continue evolving under the secular instability \[44\]. The model with \( n = 0.5 \) had a fairly elongated core and was still evolving when that run was terminated.

The development of the instability produces a burst of gravitational radiation. The maximum amplitude of the waveform \( r|h| \) does not vary significantly with the polytropic index, whereas the frequency of the waves decreases somewhat as \( n \) decreases. This lowering of the frequency with \( n \) reflects the fact that the stiffer polytropes produce more elongated bars, which rotate more slowly; it also results in a decrease in the peak gravitational wave luminosity with \( n \). Since the stiffer models undergo more episodes of spiral arm ejection and core recontraction, they produce longer-lived gravitational wave signals from the dynamical instability, with the total amount of energy and angular momentum emitted in the form of gravitational radiation increasing as \( n \) decreases. The nearly axisymmetric final cores (for \( n = 1.5 \) and \( n = 1.0 \)) will continue to emit gravitational radiation as they evolve under the secular instability; this has been calculated by Lai and Shapiro \[44\].

The actual values of the gravitational wave quantities depend sensitively on the equatorial radius \( R_{\text{eq}} \) of the stellar core when the dynamical instability takes place. This in turn depends on the astrophysical scenario in which the instability develops. Consider, for example, the collapse of a rotating stellar core of mass \( M = 1.4M_\odot \) that has \( \beta > \beta_d \) and is prevented from collapsing further due to centrifugal forces. The equatorial radius \( R_{\text{eq}} \) of the core at which this centrifugal hangup occurs determines the amplitude and frequency of the resulting
gravitational radiation. The simulations presented here use stiff equations of state, which are appropriate only for stellar cores that have collapsed to near neutron star densities. We therefore calculated the gravitational wave amplitudes and frequencies from our models for two representative values of this parameter, $R_{eq} = 10$ km and $R_{eq} = 20$ km. We remind the reader that since these simulations have been done in the Newtonian limit, which breaks down for $R_{eq} \sim 10 - 20$ km, these results must be viewed with appropriate caution.

Table VIII shows the maximum amplitudes $|h|$ of the gravitational waveforms and the characteristic frequencies $f_{grav}$ for these representative values of $R_{eq}$. Wave amplitudes are given for sources within the Milky Way ($r = 15$ kpc), the Local Group ($r = 1$ Mpc), and the Virgo Cluster ($r = 20$ Mpc). If the dynamical instability occurs at $R_{eq} \sim 20$ km, which is about twice the typical neutron star radius, $f_{grav}$ lies just outside the frequency range of the broad-band interferometers [7]. However, if such objects exist, they may potentially be observed using specially designed narrow-band interferometers [15,16] or resonant detectors [4,5]. Of course, if hangup occurs at about the typical neutron star radius of $R_{eq} \sim 10$ km, the characteristic frequencies become much larger. Since the star must be rotating differentially to achieve $\beta > \beta_d$, this last scenario could only occur in a newly-formed neutron star before its rotation becomes uniform (cf. [12]).

The maximum luminosity $L/L_0$, the energy emitted as gravitational radiation $\Delta E/Mc^2$, and the angular momentum carried by the waves $\Delta J/J_0$ is given for a core with mass $M = 1.4M_\odot$ and these same representative values of $R_{eq}$ in Table IX. Here, $L_0 = c^5/G$ and $J_0$ is the initial total angular momentum. The largest integrated energy and angular momentum losses are produced by the model with $n = 0.5$. Since this model was still evolving when this run was stopped, the final values will be larger.

There are several ways in which these calculations need to be improved to provide greater understanding of gravitational radiation from rotational instabilities. In this paper, we have concentrated on models with stiff equations of state. As noted above, these models are relevant for cores that have already collapsed to radii near the typical neutron star radius, $R_{eq} \sim 10 - 20$ km. However, if centrifugal hangup occurs at $R_{eq} \sim 100$ km the equation
of state is expected to be much softer, with $n \gtrsim 3$. Simulations of this important case are currently in progress; these are being done using Eulerian techniques since we have found it easier to model the softer equations of state in this manner \[17\]. Also, the very important and interesting question of the final state of the objects following the dynamical instability still remains to be fully resolved. In addition to longer runs with $n = 0.5$, this will involve a more detailed understanding of the differences between our results and those of Tohline, Durisen, and collaborators; we are making plans to pursue answers to these questions. Finally, gravitational radiation reaction and other general relativistic effects need to be included in order to have good physical models for comparison with future observations. We intend to include these effects in our future work.

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TABLES

| Model | $n$ | $N_\infty$ | $N_z$ | $N_{it}$ | $\beta$ | $R_p/R_{eq}$ | $VR$ |
|-------|-----|------------|-------|---------|---------|--------------|------|
| SCF1  | 1.5 | 225        | 225   | 20      | 0.30    | 0.20         | $7.6 \times 10^{-5}$ |
| SCF2  | 1.0 | 129        | 70    | 16      | 0.30    | 0.23         | $4.5 \times 10^{-5}$ |
| SCF3  | 0.5 | 129        | 70    | 22      | 0.30    | 0.25         | $4.0 \times 10^{-5}$ |

TABLE I. Properties of the initial axisymmetric equilibrium models created using the SCF method. $N_\infty$ and $N_z$ are the number of uniform grid zones in the $\infty$ and $z$ directions, respectively. $N_{it}$ is the number of iterations required for convergence to a solution with a tolerance of $10^{-5}$. The axis ratio is $R_p/R_{eq}$. The value of the virial parameter calculated on the cylindrical grid is $VR$.

| Model | $n$ | $N$ | $VR|_{ii}$ | $\beta_i$ | time | $|E_i-E_f|$ | $|J_i-J_f|$ | CPU |
|-------|-----|-----|-------------|----------|------|------------|------------|-----|
|        |     |     | $[t_D]$     |          |      |            |            |     |
| Run 1  | 1.5 | 16096 | $4.5 \times 10^{-2}$ | 0.32 | 35 | 0.020 | $\leq .001$ | 18.8 |
| Run 2  | 1.0 | 16619 | $3.7 \times 10^{-2}$ | 0.32 | 50 | 0.015 | .002 | 25.2 |
| Run 3  | 0.5 | 16526 | $1.5 \times 10^{-4}$ | 0.31 | 60 | 0.018 | .003 | 29.0 |

TABLE II. Properties of the SPH models. $N$ is the total number of particles in each model. The fact that the method used to generate the initial particle models does not allow strict control over the the number of particles accepted into each model results in somewhat unusual values of $N$. The subscripts “i” and “f” denote the initial and final states of the model, respectively. The stability parameter of the particle model at the initial time is $\beta_i$. The duration of the run in units of the dynamical time $t_D$ is given in the column labeled “time”. $E$ is the total energy, and $J$ is the total angular momentum. All models were run on a Cray C90; the amount of CPU time used is given for the duration of the run.
| Model | $M_{\text{core},f}$ [%] | $J_{\text{core},f}$ [%] | $\beta_{\text{core},f}$ | $\beta_f$ |
|-------|----------------|----------------|----------------|---------|
| Run 1 | 90 | 70 | 0.24 | 0.26 |
| Run 2 | 91 | 70 | 0.24 | 0.25 |
| Run 3 | 92 | 71 | 0.23 | 0.24 |

TABLE III. Hydrodynamical results for the models. The core refers to matter within cylindrical radius $\varpi = R_{\text{eq}}$, where $R_{\text{eq}}$ is the initial equatorial radius, and the subscript “f” denotes the final state of the model.
## TABLE IV. Properties of the models after each successive spiral arm ejection phase.

The mass $M_{\text{shed}}$ and angular momentum $J_{\text{shed}}$ shown are the *cumulative* mass and angular momentum lost after each such episode. The core is defined as mass within $\varpi = R_{\text{eq}}$; see Fig. 9. The values of $\beta$ are obtained directly from the successive peaks corresponding to core recontraction in Fig. 12.

The last temporal point in each series corresponds to the end of the run, and is not necessarily a time at which the core has reached maximum recontraction.

| Run   | $t$  | $M_{\text{shed}}$ | $J_{\text{shed}}$ | $\beta$ |
|-------|------|-------------------|-------------------|---------|
|       | $[t_D]$ | [%]               | [%]               |         |
| Run 1 | 0    | 0.0               | 0.0               | 0.32    |
|       | 17   | 7.3               | 24                | 0.27    |
|       | 23   | 8.6               | 28                | 0.26    |
|       | 35   | 9.5               | 30                | 0.26    |
| Run 2 | 0    | 0.0               | 0.0               | 0.32    |
|       | 22   | 5.0               | 18                | 0.28    |
|       | 30   | 7.0               | 25                | 0.26    |
|       | 36   | 7.9               | 27                | 0.26    |
|       | 42   | 8.5               | 29                | 0.25    |
|       | 50   | 9.1               | 30                | 0.25    |
| Run 3 | 0    | 0.0               | 0.0               | 0.31    |
|       | 25   | 4.1               | 12                | 0.29    |
|       | 35   | 6.2               | 19                | 0.28    |
|       | 43   | 6.7               | 22                | 0.26    |
|       | 50   | 7.1               | 24                | 0.26    |
|       | 58   | 7.4               | 26                | 0.25    |
|       | 60   | 8.4               | 29                | 0.24    |
TABLE V. TVE bar mode growth rates $d\ln |A_2|/dt$ and eigenfrequencies $\sigma_2$ for $n = 1.5$ and $n = 1.0$. These values are taken from Ref. [21], where we have converted from their units.

| $n$  | $d\ln |A_2|/dt$ | $\sigma_2$ |
|------|----------------|-----------|
| 1.5  | 0.73           | 1.7       |
| 1.0  | 0.56           | 1.5       |

TABLE VI. Gravitational wave results for Runs 1, 2, and 3. The peak values of $|r_h|$, $L/L_0$, and $dJ_z/dt$ throughout the run, and the final (cumulative) values of $(\Delta E/Mc^2)_t$ and $(\Delta J_z/J_0)_t$ are given. $L_0 = c^5/G$ and $J_0$ is the initial total angular momentum. To obtain dimensional quantities, the scalings given in the axis labels of the corresponding Figs. 16 - 20 must be applied; see Tables VIII and IX.

| Model | $\max |r_h|$ | $\max L/L_0$ | $(\Delta E/Mc^2)_t$ | $\max dJ_z/dt$ | $(\Delta J_z/J_0)_t$ |
|-------|-----------|---------------|------------------|----------------|------------------|
| Run 1 | 0.57      | 0.13          | 0.87             | 0.066          | 1.02             |
| Run 2 | 0.58      | 0.091         | 1.1              | 0.057          | 1.53             |
| Run 3 | 0.58      | 0.059         | 2.2              | 0.044          | 3.41             |
### TABLE VII. Frequencies for the models.

$f_{\text{bar}}$ is the rotational frequency of the bar and is calculated from the eigenfrequency $\sigma_2$. $f_{\text{grav}}$ is obtained from the gravitational wave energy spectrum $dE/df$ shown in Fig. [21].

| Model | $2f_{\text{bar}}$ | $f_{\text{grav}}$ |
|-------|----------------|------------------|
|       | $[t_D^{-1}]$   | $[t_D^{-1}]$     |
| Run 1 | 0.30           | 0.32             |
| Run 2 | 0.24           | 0.30             |
| Run 3 | 0.19           | 0.23             |

### TABLE VIII. The maximum amplitudes of the gravitational waveform $|h|$ and the characteristic frequencies $f_{\text{grav}}$ are given for two representative values of the equatorial radius $R_{\text{eq}}$. The core is taken to have mass $M = 1.4M_\odot$. The waveform amplitudes $|h|$ are given for sources located within the Milky Way ($r = 15$ kpc), the Local Group ($r = 1$ Mpc), and the Virgo Cluster ($r = 20$ Mpc). Notice that $|h|$ is essentially independent of the polytropic index $n$. These values were obtained by applying the appropriate scalings to the data given in Tables VI and VII.

| $R_{\text{eq}}$ | max $|h|_{\text{MW}}$ | max $|h|_{\text{LG}}$ | max $|h|_{\text{VC}}$ | $f_{\text{grav}} (n = 1.5)$ | $f_{\text{grav}} (n = 1.0)$ | $f_{\text{grav}} (n = 0.5)$ |
|----------------|----------------------|----------------------|----------------------|----------------------------|----------------------------|----------------------------|
| (r = 15 kpc)   | $5 \times 10^{-19}$  | $8 \times 10^{-21}$  | $4 \times 10^{-22}$  | 4900 Hz                   | 4100 Hz                    | 3100 Hz                    |
| (r = 1 Mpc)    | $3 \times 10^{-19}$  | $4 \times 10^{-21}$  | $2 \times 10^{-22}$  | 1700 Hz                   | 1400 Hz                    | 1100 Hz                    |
| (r = 20 Mpc)   |                      |                      |                      |                           |                           |                            |
TABLE IX. The maximum luminosity $L/L_0$, the energy emitted as gravitational radiation $\Delta E/Mc^2$, and the angular momentum carried by the waves $\Delta J/J_0$ are given for two representative values of the equatorial radius $R_{eq}$. $L_0 = c^5/G$ and $J_0$ is the initial total angular momentum. The core is taken to have mass $M = 1.4M_\odot$. The lower and upper limits for $L/L_0$ are produced for $n = 0.5$ and $n = 1.5$, respectively; the values for $n = 1.0$ are between these two limits. However, the lower values of $\Delta E/Mc^2$ and $\Delta J/J_0$ correspond to the case $n = 1.5$. The larger values are produced by the model with $n = 0.5$; since this model was still evolving when it was stopped, these values will be larger once the model reaches its final state. These values were obtained by applying the appropriate scalings to the data given in Table VI.

| $R_{eq}$ | $\max L/L_0$ | $\Delta E/Mc^2$ | $\Delta J/J_0$ |
|---------|-------------|----------------|----------------|
| 10 km   | $2 - 5 \times 10^{-5}$ | $4 - 9 \times 10^{-3}$ | $2 - 7 \times 10^{-2}$ |
| 20 km   | $6 - 20 \times 10^{-7}$ | $4 - 8 \times 10^{-4}$ | $4 - 10 \times 10^{-3}$ |
FIGURES

FIG. 1. The normalized equatorial plane density is shown for the iterated contraction initial models Run 1 ($n = 1.5$), Run 2 ($n = 1.0$), and Run 3 ($n = 0.5$). Here, the equatorial plane is taken to include all particles within $z = \pm0.01R_{eq}$. The solid curve in each frame represents the SCF equatorial plane density, and $\rho_c$ is the central SCF density.

FIG. 2. The normalized angular velocity is shown for the initial models of Runs 1, 2 and 3. All particles are plotted in this figure. In each frame, the solid curve gives the angular velocity for the corresponding SCF initial model and $\Omega_c$ is the SCF central angular velocity.

FIG. 3. Particle positions are shown projected onto the equatorial plane for various times during the evolution of Run 1 with $n = 1.5$. All particles are plotted. The vertical axis is $y/R_{eq}$ and the horizontal axis is $x/R_{eq}$. The system rotates in the counterclockwise direction.

FIG. 4. Density contours in the equatorial plane are shown for Run 1 with $n = 1.5$. The frames are taken at the same times as the corresponding particle plots in Fig. 3. The contour levels are the same in all frames, and are spaced a factor of 10 apart, going down 4 decades below the maximum (central) SCF initial density. The contours were calculated using kernel interpolation on a $100 \times 100$ Cartesian grid covering the same spatial area in the equatorial plane as the frames in Figure 3.

FIG. 5. Same as Fig. 3 for Run 2 with $n = 1.0$.

FIG. 6. Same as Fig. 4 for Run 2 with $n = 1$. These frames correspond to the particle plots shown in Fig. 5.

FIG. 7. Same as Fig. 3 for Run 3 with $n = 0.5$.

FIG. 8. Same as Fig. 4 for Run 3 with $n = 0.5$. These frames correspond to the particle plots shown in Fig. 7.
FIG. 9. The distributions of mass $m(\varpi)/M$ and angular momentum $J(\varpi)/J_0$ are shown for Run 1 (solid line), Run 2 (dashed line), and Run 3 (dot-dashed line). Frames (a) and (b) show the initial models, and frames (c) and (d) show the final states. Here, $M$ is the total mass and $J_0$ is the initial total angular momentum.

FIG. 10. The mass $\Delta m/M$ and angular momentum $\Delta J/J_0$ distributions are shown for Run 1. Here, $\Delta m$ is the mass within a cylindrical shell of thickness $d\varpi$ at radius $\varpi$, and similarly for $\Delta J$; $J_0$ is the total initial angular momentum. The solid lines show the values at time $t = 13.4t_D$ and the dashed lines at $t = 18.5t_D$. Frames (b) and (d) show enlargements of (a) and (c), respectively.

FIG. 11. The normalized angular velocity is shown for the final states of Runs 1, 2, and 3. Here, $\Omega_c$ is the central angular velocity for the initial SCF model. All particles are plotted.

FIG. 12. The behavior of the stability parameter $\beta = T_{\text{rot}}/|W|$ (solid line) and $T_{\text{tot}}/|W|$ (dashed line) is shown as a function of time for Runs 1, 2, and 3. Here, $T_{\text{rot}}$ is the rotational kinetic energy, $T_{\text{tot}}$ is the total kinetic energy, and $W$ is the gravitational potential energy.

FIG. 13. The growth of the Fourier components $m = 1, 2, 3,$ and 4 for Run 1 with $n = 1.5$. These values were obtained in the density ring at $\varpi = 0.36R_{\text{eq}}$ in the equatorial plane $z = 0$.

FIG. 14. Same as Fig. 13 for Run 2 with $n = 1.0$.

FIG. 15. Same as Fig. 13 for Run 3 with $n = 0.5$.

FIG. 16. The gravitational waveform $rh_+$ for an observer located at $\theta = \phi = 0$ at distance $r$ from the source for Runs 1, 2, and 3.

FIG. 17. The gravitational wave luminosity $L/L_0$ for Runs 1, 2, and 3. Here, $L_0 = c^5/G$.

FIG. 18. The energy $\Delta E/Mc^2$ emitted as gravitational radiation for Runs 1, 2, and 3.
FIG. 19. The rate $dJ_z/dt$ at which angular momentum is carried away by gravitational radiation for Runs 1, 2, and 3.

FIG. 20. The angular momentum $\Delta J/J_0$ carried by the gravitational waves for Runs 1, 2, and 3. Here, $J_0$ is the total initial angular momentum.

FIG. 21. The gravitational wave energy spectrum $dE/df$ is shown as a function of frequency $f$ for Runs 1, 2, and 3.
(a) Run 1

(b) Run 2

(c) Run 3
