The Scalar Curvature of a Riemannian Almost Paracomplex Manifold and Its Conformal Transformations

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Abstract: A Riemannian almost paracomplex manifold is a 2n-dimensional Riemannian manifold \((M, g)\), whose structural group \(O(2n, \mathbb{R})\) is reduced to the form \(O(n, \mathbb{R}) \times O(n, \mathbb{R})\). We define the scalar curvature \(\pi\) of this manifold and consider relationships between \(\pi\) and the scalar curvature \(s\) of the metric \(g\) and its conformal transformations.

Keywords: almost paracomplex manifold; conformal transformation; scalar curvature

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1. Introduction

An almost paracomplex structure on a 2n-dimensional smooth manifold \(M\) is a smooth field \(J\) of automorphisms of the tangent spaces, whose square is the identity operator \((J^2 = \text{id}_{TM})\) and two eigenspaces (corresponding to eigenvalues \(\pm 1\)) have dimension \(n\). In this case, the pair \((M, J)\) is called an almost paracomplex manifold, e.g., [1]. An almost paracomplex structure can alternatively be defined as a \(G\)-structure on \(M\) that reduces the structural group \(GL(2n, \mathbb{R})\) to the form \(GL(n, \mathbb{R}) \times GL(n, \mathbb{R})\), see [1]. A paracomplex manifold is an almost paracomplex manifold \((M, J)\) such that the \(G\)-structure defined by \(J\) is integrable. A paracomplex manifold \((M, J)\) is a locally product manifold, i.e., \(M\) is locally diffeomorphic to the product \(M_1 \times M_2\) of two \(n\)-dimensional manifolds. An almost paracomplex structure \(J\) on a Riemannian manifold \((M, g)\) is said to be orthogonal if two eigenspaces of \(J\) are orthogonal. Moreover, every almost paracomplex structure on a Riemannian manifold is always orthogonal with respect to some Riemannian metric, see Section 2.

We can offer an alternative definition of a Riemannian paracomplex manifold. Namely, a 2n-dimensional Riemannian manifold \((M, g)\) admits an orthogonal almost paracomplex structure if its structure group \(O(2n, \mathbb{R})\) can be reduced to the form \(O(n, \mathbb{R}) \times O(n, \mathbb{R})\). A Riemannian manifold \((M, g)\) with an orthogonal paracomplex structure \((g, J)\) will be called a Riemannian almost paracomplex manifold and denoted by \((M, g, J)\).

The theory of paracomplex structures (e.g., [1–3]) has applications (see [4]) to the theory statistical manifolds, see [5]. The long history of the theory of almost paracomplex manifolds and a survey of the results of this theory, as well as examples of almost paracomplex manifolds, can be found in [1,2].

In this article, we define the scalar curvature \(\pi\) of a Riemannian almost paracomplex manifold \((M, g, J)\) and consider the relationship between \(\pi\) and the scalar curvature \(s\) of the metric \(g\) and its image under conformal transformations.
2. A Riemannian Orthogonal Paracomplex Manifold and Its Scalar Curvature

Here, we briefly describe the notation and conventions used in this article, see also [1,2]. We will also prove our first results and give illustrative examples.

An almost paracomplex structure on a smooth manifold $M$ is a tensor field $J \in C^\infty(T^*M \otimes TM)$ such that $J^2 = 1d$ and trace $J = 0$, see [2]. As a result, the direct decomposition holds $T_xM = H_x \oplus V_x$, where $H_x$ and $V_x$ are horizontal and vertical subspaces of the tangent space $T_xM$ at every point $x \in M$. The corresponding distributions $H = \{ H_x \}$ and $V = \{ V_x \}$ on $M$ (i.e., subbundles of $TM$) have equal dimensions and correspond to the eigenvalues $-1$ and $+1$ of the tensor $J$, respectively. Thus, the dimension of a manifold with almost paracomplex structure is necessarily even. It is known that, for example, a four-dimensional sphere has no globally defined almost paracomplex structures, but there exist a non-integrable almost paracomplex structure on a six-dimensional unit sphere with its standard metric, see [3]. An almost paracomplex structure $J$ on a Riemannian manifold $(M, g)$ is called orthogonal, see [1,2], if

$$g(JX, JY) = g(X, Y), \quad X, Y \in TM,$$

and it is denoted by $(g, J)$. In this case, the distributions $H$ and $V$ of $(g, J)$ are orthogonal. Note that even if an almost paracomplex structure $J$ is not orthogonal with respect to $g$, then $J$ is orthogonal with respect to the Riemannian metric $\bar{g}$ defined by

$$\bar{g}(X, Y) := g(X, Y) + g(JX, JY), \quad X, Y \in TM,$$

because $g(JX, JY) = g(X, Y)$, see (1). The triplet $(M, \bar{g}, J)$, where $(g, J)$ is an orthogonal almost paracomplex structure on $M$, is called a Riemannian almost paracomplex manifold.

**Remark 1.** An almost paracomplex structure is the antipode of a well-known almost complex structure on a $2n$-dimensional manifold, see [1]. Below, we consider the geometry of Riemannian paracomplex manifolds by analogy with the theory of almost Hermitian manifolds.

The torsion tensor of an almost paracomplex structure $J$ on a smooth manifold $M$ is the $(2, 1)$-tensor field $N_J$ such that (e.g., [3])

$$N_J(X, Y) = [X, Y] + [JX, JY] - J[JX, Y] - J[X, JY], \quad X, Y \in TM,$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields. The tensor $N_J$ is an analog of the Nijenhuis tensor for an almost complex structure on a smooth manifold of even dimension.

The equality $N_J = 0$ holds on $M$ if and only if the distributions $H$ and $V$ are involutive (or integrable, that is the same), see [3] (Theorem 2.4). Then, $M$ is locally the product of two $n$-dimensional smooth manifolds (e.g., [3]). In this case, the almost paracomplex structure $J$ is called integrable and $(M, J)$ is called a paracomplex manifold. Therefore, an integrable paracomplex structure exists on the product of manifolds of the same dimension, e.g., on the product of $n$-dimensional unit spheres (see [1]).

Let $(M, g, J)$ be a Riemannian almost paracomplex manifold with the Levi-Civita connection $\nabla$ of the metric $g$ and the Riemannian curvature tensor $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - [X, Y]$. Let $\sigma_x$ be a plane in $T_xM$, i.e., a two-dimensional subspace of $T_xM$ at an arbitrary point $x \in M$. Choosing an orthonormal basis $X_x, Y_x$ of $\sigma_x$, we define the sectional curvature $sec(\sigma_x)$ in direction of $\sigma_x$ by

$$sec(\sigma_x) = R(X_x, Y_x, X_x, Y_x),$$

where $R(X_x, Y_x, W_x, Z_x) = g(R(X_x, Y_x)Z_x, W_x)$. We shall write also $sec(X_x, Y_x)$ for $sec(\sigma_x)$.

It is known that $R(X_x, Y_x, X_x, Y_x)$ (the right-hand side) depends only on $\sigma_x$, and not on the choice of the orthonormal basis $X_x, Y_x$. The scalar curvature $s$ of the metric $g$ is defined by

$$s = \sum_{i,j=1}^{2n} sec(\epsilon_i, \epsilon_j),$$
where \( \{e_1, \ldots, e_{2n}\} \) is any orthonormal basis of \( T_xM \). On the other hand, if \( X_x, Y_x \) is an orthonormal basis for \( \sigma_x \), then \( JX_x, JY_x \) is an orthonormal basis of another plane \( \sigma'_x \) such that \( \sigma'_x = J\sigma_x \). In this case, \( \sigma_x = J\sigma_x = J^2\sigma_x \). Therefore, given two \( J \)-invariant planes \( \sigma_x \) and \( \sigma'_x \) in \( TM \), we can define the \textit{bisectional curvature} \( \text{bisec}(\sigma_x, J\sigma_x) \) by the equality

\[
\text{bisec}(\sigma_x, J\sigma_x) = R(X_x, Y_x, JX_x, JY_x).
\]

One can verify that \( R(X_x, Y_x, JX_x, JY_x) \) depends only on \( \sigma_x \) and \( \sigma'_x \). The \textit{bisectional curvature} is an analog of the holomorphic bisectional curvature of a Kähler manifold, see [6,7] (pp. 303–313). Using the above, we can consider the scalar curvature \( \pi \) of an orthogonal paracomplex structure \( (g, J) \), or, in other words, of a Riemannian almost paracomplex manifold \( (M, g, J) \), defined by the equality

\[
\pi = \sum_{i,j=1}^{2n} R(e_i, e_j, J\sigma_x, J\sigma_x)
\]

(2)

for a local orthonormal basis \( \{e_1, \ldots, e_{2n}\} \) of \( TM \). Let \( \{e_1, \ldots, e_n\} \) and \( \{e_{n+1}, \ldots, e_{2n}\} \) be local orthonormal bases of the horizontal distribution \( H \) and the vertical distribution \( V \), respectively. Vectors of these bases satisfy the following conditions:

\[
Je_a = -e_a, \quad J\varepsilon_a = \varepsilon_a
\]

for \( a = 1, \ldots, n \) and \( \alpha = n + 1, \ldots, 2n \). Using the above, we can show that

\[
\pi = \sum_{i,j=1}^{2n} R(e_i, e_j, J\sigma_x, J\sigma_x) = \sum_{\alpha=1}^{n} R(e_\alpha, e_\alpha, J\sigma_x, J\sigma_x)
\]

\[
+ 2 \sum_{\alpha=1}^{n} \sum_{\beta=n+1}^{2n} R(e_\alpha, e_\alpha, J\sigma_x, J\sigma_x) = \sum_{\alpha=1}^{n} \sum_{\beta=n+1}^{2n} R(e_\alpha, e_\alpha, e_\alpha, e_\alpha)
\]

\[
+ 2 \sum_{\alpha=1}^{n} \sum_{\beta=n+1}^{2n} R(e_\alpha, e_\beta, e_\alpha, e_\beta) = \sum_{i,j=1}^{2n} \sec(e_i, e_j) - 4 \sum_{\alpha=1}^{n} \sum_{\beta=n+1}^{2n} \sec(e_\alpha, e_\alpha)
\]

where we denoted by

\[
s_{\text{mix}} = \sum_{\alpha=1}^{n} \sum_{\beta=n+1}^{2n} \sec(e_\alpha, e_\alpha)
\]

the \textit{mixed scalar curvature} of an orthogonal paracomplex structure \( (g, J) \). The concept of the mixed scalar curvature of a distribution on a Riemannian manifold has a long history and many applications [8–11]. By the above calculations, we obtain the following.

**Theorem 1.** Let \( (M, g, J) \) be a Riemannian almost paracomplex manifold. Then,

\[
s = \pi + 4 s_{\text{mix}},
\]

(3)

where \( s \) is the scalar curvature of the metric \( g \), and \( \pi \) and \( s_{\text{mix}} \) are the scalar and mixed scalar curvatures, respectively, of its orthogonal paracomplex structure \( (g, J) \).

By (3), if the metric of \( (M, g, J) \) has constant sectional curvature 1, then \( \pi = -2n \). In contrast, the scalar curvature of such metric \( g \) on \( M \) is \( s = 2n(2n - 1) \).

We consider three examples with the scalar curvature \( \pi \) of a Riemannian almost paracomplex manifold, which is equal to the scalar curvature of its orthogonal paracomplex structure.

**Example 1.** Recall that a distribution on a Riemannian manifold is totally geodesic if any geodesic that is tangent to the distribution at one point is tangent to the distribution at all its points. If
both structure distributions $H$ and $V$ of a Riemannian paracomplex manifold $(M, g, J)$ are totally geodesic, then $s_{\text{mix}} = (1/8)\|\nabla J\|^2$, see [8], and by (3) we obtain

$$\pi = s - (1/2)\|\nabla J\|^2 \leq s.$$  

**Example 2.** Recall that a distribution on a Riemannian manifold is minimal (or, harmonic) if its mean curvature vector field (the trace of the second fundamental form) vanishes, see [12] (p. 149). If a minimal distribution is integrable, then its leaves (maximal integral manifolds) are minimal submanifolds, see [12] (p. 151). Let $(M, g, J)$ be a Riemannian paracomplex manifold, then $M$ is locally the product of two $n$-dimensional manifolds, $M = M_1 \times M_2$. If in addition, maximal integral manifolds of $H$ and $V$ are minimal submanifolds of $(M, g, J)$, then $s_{\text{mix}} = -(1/8)\|\nabla J\|^2$, see [8], and by (3) we obtain

$$\pi = s + (1/2)\|\nabla J\|^2 \geq s.$$  

**Example 3.** Let $(M, J, g)$ be a 2$n$-dimensional Riemannian almost paracomplex manifold. Assume that $\nabla J = 0$ for the Levi-Civita connection $\nabla$ of the metric $g$, then both structure distributions $H$ and $V$ are involutive with totally geodesic integral manifolds. In this case, the Riemannian paracomplex manifold $(M, g, J)$ is locally the product of two $n$-dimensional Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$. The converse is also true. In this case, $s_{\text{mix}} = 0$; therefore, $\pi = s$, see also [13]. In particular, the scalar curvature of an orthogonal paracomplex structure of $(S^n \times S^n, g_0 \oplus g_0)$ can be expressed in terms of the scalar curvature of $g_0$ via the formula

$$s(g_0 \oplus g_0) = s(g_0) + s(g_0) = 2n(n - 1).$$  

Therefore, $\pi = s = 2n(n - 1)$.

**Remark 2.** Theorem 1 can be extended for an almost product structure on an $m$-dimensional Riemannian manifold $(M, g)$. Namely, let $P_i$ ($i = 1, 2$) be orthoprojectors on two complementary orthogonal distributions $D_i$ ($i = 1, 2$), see [12] (p. 146). Set $G = P_2 - P_1$ and define

$$\Pi = \sum_{i,j=1}^m R(e_i, e_j, Ge_i, Ge_j)$$

for a local orthonormal basis $\{e_1, \ldots, e_m\}$ of TM, compare with (2). Then, $g(GX, GY) = g(X, Y)$ for $X, Y \in TM$, compare with (1). Now, let $\{e_1, \ldots, e_m\}$ be a local orthonormal basis of the distribution $D_1$ and $\{e_{n_1+1}, \ldots, e_m\}$ be a local orthonormal basis of the distribution $D_2$. Vectors of these bases satisfy the following conditions:

$$Ge_a = -e_a \quad (a = 1, \ldots, n_1), \quad Ge_a = e_a \quad (a = n_1 + 1, \ldots, m).$$

Using the above, we can prove that (see [14])

$$s = \Pi + 4s_{\text{mix}}.$$  

In particular, if $(M, g)$ has constant sectional curvature 1, then $s = m(m - 1)$ and $s_{\text{mix}} = n_1n_2$; hence, $\Pi = m(m - 1) - 4n_1n_2$.

**3. Conformal Transformations of Metrics of Riemannian Almost Paracomplex Manifolds**

An identity map $\text{id} : M \to M$ from a differentiable manifold $M$ into itself, also known as an identity transformation, is defined as the map with domain and range $M$, which satisfies $\text{id}(x) = x$ for any $x \in M$, and it is the simplest map, which is both continuous and bijective (see [15]). Here, we will consider the conformal geometry of the identity map on a manifold $M$, and we assume that the domain $M$ and the range $M$ of $\text{id}$ are equipped with metrics $g$ and $\bar{g}$, respectively. The identity map $\text{id} : M \to M$ is called a conformal transformation of the metric $g$ if

$$\bar{g} = e^{2\sigma} g$$

(4)
for some smooth function \( \sigma \) on \( M \), e.g., \([6]\) (p. 115) and \([7]\) (p. 269).

In this case, the metric \( \hat{g} \) is called a conformal transformation of \( g \); and if \( \sigma = \text{const} \), then this transformation is called a homothety. The converse statement (i.e., \( g \) is a conformal transformation of \( \hat{g} \)) is also true, because the equality \( g = e^{-2\sigma} \hat{g} \) holds. In addition, the equality \( \hat{g}^{-1} = e^{-2\sigma} g^{-1} \) holds. For such a rescaled metric \( \hat{g} \), there is a unique symmetric connection, \( \nabla \), compatible with \( \hat{g} \), i.e., \( \nabla \hat{g} = 0 \). Under a conformal transformation (4), the following relation (between two connections) holds, see \([6]\) (p. 115) and \([7]\) (p. 270):

\[
\nabla_X Y = \nabla_X Y + X(\sigma) Y + Y(\sigma) X - g(X, Y) \nabla \sigma, \quad X, Y \in C^\infty(TM).
\]

We will consider conformal deformations of metrics of a Riemannian almost paracomplex manifold \((M, g, J)\). Obviously,

\[
\hat{g}(JX, JY) = e^{2\sigma} g(JX, JY) = e^{2\sigma} g(X, Y) = \hat{g}(X, Y), \quad X, Y \in C^\infty(TM).
\]

Hence, a conformal deformation of \( g \) preserves the orthogonal decomposition \( TM = H \oplus V \) of the tangent bundle of \((M, g, J)\), i.e., it preserves the orthogonal almost paracomplex structure. On the other hand, a diffeomorphism \( \pi : M \to M \) is called a paraholomorphic transformation of \((M, g, J)\), if it preserves the almost paracomplex structure \( J \), see \([1]\). Therefore, we have the following.

**Proposition 1.** Let \((M, g, J)\) be a Riemannian almost paracomplex manifold. Then, the conformal transformation of metric \( \text{id} : M \to M \), see (4), represents a paraholomorphic transformation of \((M, g, J)\).

On the contrary, a conformal transformation of the metric of a Riemannian almost paracomplex manifold \((M, g, J)\) does not preserve its scalar curvature \( \pi \). Thus, below, we study the relationship between the scalar curvatures \( \pi \) and \( \pi \) of orthogonal paracomplex structures \((g, J)\) and \((\hat{g}, J)\), respectively. By the theory of conformal mappings, e.g., \([7]\) (p. 271), the relationship between the curvature tensors (of the Levi-Civita connections \( \nabla \) and \( \hat{\nabla} \)) of the metrics \( g \) and \( \hat{g} \), respectively, has the following form, e.g., \([6]\) (p. 115) and \([7]\) (p. 271):

\[
e^{-2\sigma} R_{ijkl} = R_{ijkl} + \bar{g}_{ik}\sigma_{lj} - \bar{g}_{lj}\sigma_{ik} + \bar{g}_{ik}\bar{g}_{lj} - \sigma_{ij}\bar{g}_{ik} + (\bar{g}_{ik}\bar{g}_{lj} - \bar{g}_{ij}\bar{g}_{ik})\|d\sigma\|^2
\]

with respect to local coordinates \((x^1, \ldots, x^{2n})\), where \( \sigma_{ij} \) and \( \bar{g}_{ij} \) are components of metrics \( \hat{g} \) and \( g \). In (6), we denote by \( \hat{R}_{ijkl} \) and \( R_{ijkl} \), the components of Riemannian curvature tensors \( \hat{R} \) and \( R \) of metrics \( \hat{g} \) and \( g \), respectively. The components \( \sigma_{ij} \) in (6) are given by

\[
\sigma_{ij} = \nabla_i \nabla_j \sigma - (\nabla_i \sigma)(\nabla_j \sigma),
\]

where \( \nabla_i = \nabla \frac{\partial}{\partial x_i} \). From (6), we obtain

\[
e^{2\sigma} \hat{\pi} = e^{-2\sigma} \hat{R}_{ijkl}(\hat{e}^{2\sigma} \hat{g}_{ij}^{kl})(\hat{e}^{2\sigma} \hat{g}_{jk}^{lp})
\]

\[
= R_{ijkl}(\hat{e}^{2\sigma} \hat{g}_{ij}^{kl})(\hat{e}^{2\sigma} \hat{g}_{jk}^{lp}) + 2\bar{g}^{lk}\sigma_{ik} + 2n\|d\sigma\|^2
\]

\[
= \pi + 2\Delta \sigma + 2(n - 1)\|d\sigma\|^2,
\]

where \((\hat{g}^{ij}) = (g^{ij})^{-1}, \Delta \sigma = \hat{g}^{ij}\nabla_i \nabla_j \sigma \) and \( \Delta = \text{div} \circ \nabla \) is the Laplace–Beltrami operator. We can rewrite (7) as

\[
\Delta \sigma = \frac{1}{2} (e^{2\sigma} \hat{\pi} - \pi) - (n - 1)\|d\sigma\|^2.
\]
The total scalar curvature $\pi(M)$ of a compact Riemannian almost paracomplex manifold $(M, g, J)$ is defined by the integral equality

$$\pi(M) = \int_M \pi \, d\text{vol}_g,$$

where $d\text{vol}_g$ is the volume form of the metric $g$. Note that $\pi(M)$ is an analog of the total scalar curvature of a compact Riemannian manifold $(M, g)$, see [16] (p. 119) and [9,14],

$$s(M) = \int_M s \, d\text{vol}_g.$$ Integrating (8) over $M$ and using the Green’s formula $\int_M \Delta \sigma \, d\text{vol}_g = 0$ yields

$$\pi(M) = \int_M (e^{2\sigma} \pi - 2(n - 1) \|d\sigma|^2) \, d\text{vol}_g.$$

The above integral equality yields the inequality

$$\pi(M) \leq \int_M (e^{2\sigma} \tilde{\pi}) \, d\text{vol}_g.$$

By the above inequality, if $\pi \leq 0$ on $M$ and $\pi(M) \geq 0$, then $\pi(M) = 0$ and $\pi \equiv 0$. In this case, $\sigma = \text{const}$.

**Theorem 2.** Let $(M, g, J)$ be a compact Riemannian almost paracomplex manifold with nonnegative total scalar curvature, $\pi(M) \geq 0$, and let $\bar{g} = e^{2\sigma} g$ be another metric conformally related to $g$ for some $\sigma \in C^2(M)$. If $\bar{\pi} \leq 0$ on $M$, then $\sigma$ is constant. Thus, the conformal transformation of $g$ to the metric $\bar{g}$ is a homothety; furthermore, $\bar{\pi} = \pi = 0$ on $M$.

Setting $\sigma = \frac{1}{n-1} \ln u$ for a positive scalar function $u \in C^2(M)$, from (4) we obtain $\bar{g} = u^{2/(n-1)} g$ with $u > 0$. In this case, (6) can be rewritten as

$$\Delta u = \frac{n - 2}{2} (u^{\frac{n+1}{n-1}} \pi - u \pi).$$

Integrating (9) over compact manifold $M$ and using the Green’s formula, gives

$$\int_M u^{\frac{n+1}{n-1}} \bar{\pi} \, d\text{vol}_g = \int_M u \pi \, d\text{vol}_g.$$ We can formulate the following theorem supplementing Theorem 2.

**Theorem 3.** Let $(M, J, g)$ be a compact Riemannian almost paracomplex manifold with scalar curvature $\pi \leq 0$ on $M$, and let a metric $\bar{g}$ be conformally related to $g$. If $\pi \geq 0$ on $M$, then the conformal deformation of the metric $g$ to $\bar{g}$ is a homothety; furthermore, $\bar{\pi} = \pi = 0$ on $M$.

**Corollary 1.** Let $(M, J, g)$ be a compact Riemannian almost paracomplex manifold, and let a metric $\bar{g}$ be conformally related to $g$. If both orthogonal paracomplex structures $(g, J)$ and $(\bar{g}, J)$ have nonvanishing scalar curvatures, i.e., $\pi \neq 0$ and $\bar{\pi} \neq 0$ on $M$, then these scalar curvatures have the same sign.

If $\pi \leq 0$ and $\bar{\pi} \geq 0$, then by (9) we obtain $\Delta u \leq 0$ on $M$. Thus, from (8), we conclude that $\sigma$ is a superharmonic function. On the other hand, a complete Riemannian manifold $(M, g)$ is called a parabolic manifold if it does not admit a non-constant positive superharmonic function, e.g., [17] (p. 313). For example, a complete Riemannian manifold $(M, g)$ of finite volume is a parabolic manifold because it does not carry non-constant positive superharmonic functions, see [18]. Using the above, we can formulate the following.
Theorem 4. Let \((M, g, J)\) be a parabolic Riemannian almost paracomplex manifold (in particular, \((M, g, J)\) be a complete manifold of finite volume) with scalar curvature \(\pi \geq 0\) on \(M\), and let a metric \(\bar{g}\) be conformally related to \(g\). If \(\pi \leq 0\) on \(M\), then the conformal deformation of the metric \(g\) to the metric \(\bar{g}\) is a homothety; furthermore, \(\bar{\pi} = \pi = 0\) on \(M\).

If \(\pi \geq 0\) and \(\pi \leq 0\) then \(\Delta u \geq 0\) on \(M\), then from (9), we conclude that \(u\) is subharmonic function. We recall the following famous theorem by C. Yau: let \(u\) be a nonnegative smooth subharmonic function on a complete Riemannian manifold \((M, g)\), then \(\int_M u^p \, d\text{vol}_g = \infty\) for any \(p > 1\), unless \(u\) is a constant function, see \([19]\) (Theorem 3).

Therefore, we can formulate the following statement on complete Riemannian almost paracomplex manifolds.

Theorem 5. Let \((M, g, J)\) be a complete Riemannian almost paracomplex manifold with scalar curvature \(\pi \leq 0\) on \(M\), and let \(\bar{g}\) be another metric conformally related to \(g\) by the formula \(\bar{g} = u^{2/(n-1)} g\) for some positive function \(u \in C^2(M)\). If \(\pi \geq 0\) on \(M\) and \(u \in L^p(M, g)\) for some \(p > 1\), then the conformal deformation of the metric \(g\) to the metric \(\bar{g}\) is a homothety; furthermore, \(\bar{\pi} = \pi = 0\) on \(M\).

A Riemannian manifold \((M, g)\) is locally conformally flat if for each point \(x \in M\), there exists a neighborhood \(U\) of \(x\) and a smooth function \(\sigma : U \to \mathbb{R}\) such that \((U, \sigma^2 g)\) is flat, i.e., the curvature of the metric \(\sigma^2 g\) vanishes on \(U\). In the case of a Riemannian almost paracomplex manifold \((M, g, J)\), we can formulate the following.

Theorem 6. Let \((M, g, J)\) be a Riemannian almost paracomplex manifold such that \(g\) is a locally conformally flat metric with vanishing scalar curvature \(s\), then its scalar curvature \(\pi\) vanishes on \(M\).

Proof. Following \([20]\), denote by \(\text{sec}(D_x)\), the sectional curvature of a Riemannian manifold \((M, g)\) associated with an \(r\)-plane section \(D_x \subset T_x M\) for an arbitrary point \(x \in M\). Then, for any orthonormal basis \(\{e_1, \ldots, e_r\}\) of \(D_x\), the scalar curvature \(s(D_x)\) of the \(r\)-plane section \(D_x\) is defined by, see also \([20]\),

\[
s(D_x) = \sum_{p,q=1}^r \text{sec}(e_p, e_q).
\]

Now, let \((M, g)\) be a \(2r\)-dimensional locally conformally flat manifold with vanishing scalar curvature \(s\) of the metric \(g\), then \(s(D_x) = -s(D_x^\perp)\), where \(D_x^\perp\) is the orthogonal complement of \(D_x\), see \([21]\). In the case of a \(2n\)-dimensional Riemannian almost paracomplex manifold \((M, g, J)\), the scalar curvature \(s\) of the metric \(g\) can be presented as

\[
s = s(H) + 2s_{\text{mix}} + s(V),
\]

where \(s(H) = \sum_{a,b=1}^n \text{sec}(e_a, e_b)\) and \(s(V) = \sum_{a,b=1}^n \text{sec}(e_a, e_b)\) are scalar curvatures of the horizontal and vertical distributions. Moreover, if \((M, g, J)\) is a locally conformally flat manifold with vanishing scalar curvature \(s\) of the metric \(g\), then \(s(H) = -s(V)\). In this case, from (10) we obtain \(s_{\text{mix}} = 0\). Thus, by our Theorem 1, \(\pi = 0\). \(\square\)

For example (see \([16]\) [p. 61]), the product of two Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\), one with sectional curvature 1, and the other with sectional curvature \(-1\), is locally conformally flat. In particular, if \(\dim M_1 = \dim M_2 = n\), then we have \(s = s_1 + s_2 = n(n - 1) - n(n - 1) = 0\) and \(s_{\text{mix}} = 0\). Therefore, \(\pi = 0\).

4. A Riemannian Almost Paracomplex Manifold Conformally Related to the Product of Riemannian Manifolds

Let a \(2n\)-dimensional Riemannian almost paracomplex manifold \((M, J, g)\) satisfy the following conditions: \(M = M_1 \times M_2\) and \(g = \sigma^2 (g_1 \oplus g_2)\) for some \(n\)-dimensional Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\), respectively, and \(\sigma \in C^2(M)\). In this case,
the metric of \((M, J, g)\) arises as a result of the conformal transformation of the metric \(g_1 \oplus g_2\) of the product of Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). At the same time, there exists a natural integrable orthogonal paracomplex structure \(J\) of \((M_1 \times M_2, g_1 \oplus g_2)\) and the Levi-Civita connection \(\nabla\) of its metric \(\tilde{g} = g_1 \oplus g_2\) such that \(\nabla \tilde{g} = 0\) and \(\nabla J = 0\) (see Example 3). Applying (5), we obtain the following relationship between the covariant derivatives \(\nabla J\) and \(\nabla J\):

\[
g((\nabla \chi)Y, Z) = g((\nabla \chi)Y, Z) - \chi(Y)g(JX, Z) - \chi(Z)g(JX, Y) + g(J(\nabla \chi), Y)g(X, Z) + g(J(\nabla \chi), X)g(Y, Z), \quad X, Y, Z \in TM.
\]

In the case of \(\nabla J = 0\), this formula has the following form:

\[
g((\nabla \chi)Y, Z) = \chi(Y)g(JX, Z) + \chi(Z)g(JX, Y) - g(J(\nabla \chi), Y)g(X, Z) - g(J(\nabla \chi), X)g(Y, Z), \quad X, Y, Z \in TM. \tag{11}
\]

The converse is true only in a local sense. By the above, we can formulate the following:

**Theorem 7.** Let a 2n-dimensional Riemannian almost paracomplex manifold \((M, g, J)\) be conformal to the product of n-dimensional Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\), then the structural tensor \(J\) satisfies (11). The converse is true only in a local sense.

Let a 2n-dimensional Riemannian almost paracomplex manifold \((M, g, J)\) be the product of n-dimensional Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, \(\nabla J = 0\) and \(\pi = s\) on \(M = M_1 \times M_2\). After the conformal deformation \(\tilde{g} = e^{2\sigma}(g_1 \oplus g_2)\) for some \(\sigma \in C^2(M)\) of the metric \(\tilde{g} = g_1 \oplus g_2\), we obtain the equation, see [7] (p. 271):

\[
e^{2\sigma} s = s - 2(2n - 1) \Delta \sigma - 2(n - 1)(2n - 1)\|d\sigma\|^2. \tag{12}
\]

for the scalar curvature \(s\) of the metric \(\tilde{g} = e^{2\sigma}(g_1 \oplus g_2)\). We rewrite (6) as

\[
e^{2\sigma} \pi = \pi + 2\Delta \sigma + 2(n - 1)\|d\sigma\|^2. \tag{13}
\]

From (12) and (13), it follows that

\[
\Delta \sigma = \frac{1}{4n} e^{2\sigma} (\pi - s) - \|d\sigma\|^2. \tag{14}
\]

Setting \(\sigma = \ln u\) for a positive scalar function \(u \in C^2(M)\), the equality \(\tilde{g} = e^{2\sigma} g\) can be rewritten as \(\tilde{g} = u^2 g, u > 0\). In this case, (14) can be rewritten as

\[
\Delta u = \frac{1}{2n} u^2 (\pi - s).
\]

If \(M = M_1 \times M_2\) is a compact manifold (in particular, if \(M_1\) and \(M_2\) are compact manifolds), then from the above formula we obtain the following integral equation:

\[
\frac{1}{2n} \int_M u^2 (\pi - s) \, d\text{vol}_\tilde{g} = 0. \tag{15}
\]

Note that conditions \(\pi \leq s\) and \(\pi \leq s\) (or, \(\pi \geq s\) and \(\pi > s\)) for at least one point \(x \in M_1 \times M_2\) contradict (15). Thus, the following theorem holds.

**Theorem 8.** Let \((M, g, J)\) be a 2n-dimensional Riemannian paracomplex manifold such that \(M = M_1 \times M_2\) for n-dimensional compact manifolds \(M_1\) and \(M_2\). If the scalar curvatures \(\pi\) and \(s\) satisfy the following condition: \(\pi \leq s\) (resp., \(\pi \geq s\)) on \(M\) and \(\pi < s\) (resp., \(\pi > s\)) for at least one point \(x \in M\), then \(M\) does not admit a metric \(\tilde{g} = g_1 \oplus g_2\) arising as a result of a conformal transformation of \(g\).
Let \((M, J, g)\) be a \(2n\)-dimensional integrable Riemannian almost paracomplex manifold with \(M = M_1 \times M_2\) and
\[
g = e^{2\sigma_1} g_1 \oplus e^{2\sigma_2} g_2
\]  
(16)
for some scalar functions \(\sigma_1, \sigma_2 \in C^2(M)\). In this case, (16) defines a biconformal deformation (see [22]) of the product metric \(\tilde{g} = g_1 \oplus g_2\) on the product of \(n\)-dimensional Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). At the same time, for a Riemannian manifold \((M, g)\) such that \(M = M_1 \times M_2\) and \(g = g_1 \oplus g_2\), there is a unique symmetric connection, \(\nabla\), compatible with \(\tilde{g}\) and \(J\), i.e., \(\nabla \tilde{g} = 0\) and \(\nabla J = 0\). Applying (5), we can obtain a relationship between the covariant derivatives \(\nabla J\) and \(\nabla J\). In the case of condition \(\nabla J = 0\), this result has the following form.

\[
g((\nabla_X Y) Z) = \varphi(Y) g(JX, Z) + \varphi(Z) g(JX, Y) + \psi(Y) g(X, Z) + \psi(X) g(Y, Z)
\]
(17)
for all \(X, Y, Z \in TM\) and for some nonzero differentiable 1-forms \(\varphi\) and \(\psi\). The converse is true only in a local sense. Using the above, we can formulate the following.

**Theorem 9.** Let a \(2n\)-dimensional Riemannian almost paracomplex manifold \((M, g, J)\) be biconformal to the product \((M_1 \times M_2, g_1 \oplus g_2)\) of two \(n\)-dimensional Riemannian manifolds. Then, its structural tensor \(J\) satisfies (17). The converse is true only in a local sense.

**Remark 3.** Formula (17) is similar to (11). In particular, assuming \(\varphi = d\sigma\) and \(\psi = J(\nabla \sigma)\), from (17), we obtain (11).

Recall that a distribution on a Riemannian manifold is totally umbilical if its second fundamental form is proportional to the metric restricted on the distribution, see [12] (p. 151). By the above, an orthogonal almost paracomplex structural \((g, J)\) is integrable and maximal integrable manifolds of its structural distributions \(H\) and \(V\) are totally umbilical submanifolds of \((M, g, J)\). The converse is also true, see [23].

Using (17), we have proved the integral formula, see [8, 24], which for the case \(m = 2n\) can be rewritten as
\[
\pi(M) = s(M) - \frac{n-1}{n} \int_M \|\nabla^* J\|^2 \, d \text{vol}_g,
\]
(18)
where \(\nabla^*\) is the operator formally adjoint to \(\nabla\), and the norm of the tensor field \(\nabla^* J\) is defined using \(g\). From (18), we conclude that \(\pi(M) \leq s(M)\). In addition, for \(\pi(M) = s(M)\), we obtain from (18) that \(\nabla^* J = 0\). In this case, both \(H\) and \(V\) have totally geodesic maximal integrable manifolds, see [8, 24], and the Riemannian almost paracomplex manifold \((M, g, J)\) is locally the product of two \(n\)-dimensional Riemannian manifolds.

Using the above, we can formulate the following.

**Theorem 10.** Let \((M, g, J)\) be a \(2n\)-dimensional Riemannian paracomplex manifold such that \(M\) is the product of two compact \(n\)-dimensional manifolds \(M_1\) and \(M_2\). If its metric \(g\) is obtained from the metric of the product \((M_1 \times M_2, g_1 \oplus g_2)\) of two \(n\)-dimensional Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\) by a biconformal deformation, then \(\pi(M) \leq s(M)\). Moreover, if \(\pi(M) = s(M)\), then \((M, g)\) is locally isometric to \((M_1 \times M_2, g_1 \oplus g_2)\).

In [10], we proved a generalization of theorems [25] on two orthogonal complete totally umbilical foliations on a compact and oriented Riemannian manifold. In our case, this result has the following form.

**Theorem 11.** Let \((M, g, J)\) be a \(2n\)-dimensional Riemannian paracomplex manifold such that \(M\) is the product of two \(n\)-dimensional manifolds \(M_1\) and \(M_2\), and let \(g\) be obtained from the metric of the product \((M_1 \times M_2, g_1 \oplus g_2)\) of two complete Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\) by the biconformal deformation \(g = e^{2\sigma_1} g_1 \oplus e^{2\sigma_2} g_2\). If \(s \leq \pi\) and
\[
\|h_*(\nabla \sigma_1) + v_*(\nabla \sigma_2)\| \in L^1(M, g),
\]
where $h_\ast : TM \to TM_1$ and $v_\ast : TM \to TM_2$ are natural projections and $\pi \geq s$ on $M$, then $(M, g)$ is locally isometric to $(M_1 \times M_2, g_1 \oplus g_2)$.

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