TOPOLOGICAL GRAVITY IN GENUS 2
WITH TWO PRIMARY FIELDS

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Abstract. We calculate the genus 2 correlation functions of two-dimensional topological
gravity, in a background with two primary fields, using the genus 2 topological recursion
relations.

In this paper, we calculate the genus 2 correlation functions of two-dimensional topological
gravity in a background with two primary fields \( \mathcal{O}_0 \) and \( \mathcal{O}_1 \); this extends the work of Eguchi,
Yamada and Yang [8], who considered the case of the \( A_2 \)-model.

The most interesting example of such a theory is the Gromov-Witten theory of \( \mathbb{CP}^1 \); in this
case, there is a rigorous construction of the correlation functions (see Manin [13]). For \( \mathbb{CP}^1 \),
our calculation may be made into a rigorous proof. One of our motivations was to confirm
that the resulting potential is consistent with the Toda conjecture of Eguchi and Yang [5].

In the general case, in order to complete the proof, we must use the equation \( L_1 Z = 0 \),
which is part of the Virasoro conjecture of Eguchi, Hori and Xiong [6]. We verify that the
Virasoro conjecture then holds in genus 2 for these models.

Our results agree with those of Dubrovin and Zhang [4], who use the method of Eguchi
and Xiong [9]; in particular, they use the Virasoro constraints \( L_n Z = 0, n \leq 10 \).

1. Topological recursion relations

1.1. Notation. The correlators of the theory are denoted \( \langle \tau_{k_1, a_1} \ldots \tau_{k_n, a_n} \rangle_g \). We denote \( \tau_{0, a} \)
by \( \mathcal{O}_a \). The labels on the primaries are fixed in such a way that the puncture operator is
\( \mathcal{O}_0 \). Let \( \eta_{ab} \) be the intersection form, \( \eta^{ab} \) its inverse, and let \( \mathcal{O}^a = \eta^{ab} \mathcal{O}_b \). In the case of two
primaries, the intersection form equals \( \eta_{ab} = \delta_{a+b, 1} \).

Let \( F_g \) be the genus \( g \) potential on the large phase space:

(1.1) \[ F_g = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1 \ldots k_n, a_1 \ldots a_n} t_{k_1}^{a_1} \ldots t_{k_n}^{a_n} \langle \tau_{k_1, a_1} \ldots \tau_{k_n, a_n} \rangle_g. \]

we use the summation convention with respect to the indices \( a_i \) labelling the primaries.

Denote \( \partial / \partial t_k \) by \( \partial_{k,a} \). The vector field \( \partial = \partial_{0,0} \), corresponding to the puncture operator
\( \mathcal{O}_0 \), plays a special role in the theory. The partial derivatives of the potential \( F_g \) are denoted

\[ \langle \langle \tau_{k_1, a_1} \ldots \tau_{k_n, a_n} \rangle \rangle_g = \partial_{k_1, a_1} \ldots \partial_{k_n, a_n} F_g. \]
1.2. The topological recursion relation in genus 0. The simplest example of a topological recursion relation is obtained by taking the relation \( \psi_1 = 0 \) on the zero-dimensional moduli space \( \mathcal{M}_{0,3} \). The resulting topological recursion relation is the equation

\[
\langle\langle \tau_{k,a} \tau_{\ell,b} \tau_{m,c} \rangle\rangle_0 = \langle\langle \tau_{k-1,a} \mathcal{O}^d \rangle\rangle_0 \langle\langle \mathcal{O}_{d} \tau_{\ell,b} \tau_{m,c} \rangle\rangle_0.
\]

Let \( \Theta \) be the power series

\[
\Theta(z)^b_a = \delta^b_a + \sum_{k=0}^{\infty} z^{k+1} \langle\langle \tau_{k,a} \mathcal{O}^b \rangle\rangle_0;
\]

it is an orthogonal matrix, in the sense that \( \Theta^{-1}(z) = \Theta^*(z) \). Let \( \mathcal{U} \) be the matrix with components \( \mathcal{U}^b_a = \langle\langle \mathcal{O}_a \mathcal{O}^b \rangle\rangle_0 \). The topological recursion relation (1.2) with \( m = 0 \) may be rewritten as

\[
\partial_{k,a} \Theta(z) = z \Theta(z) \partial_{k,a} \mathcal{U}.
\]

Let \( \partial_a(z) = \sum_{k=0}^{\infty} z^k \partial_{k,a} \), and define vector fields \( \{ D_{k,a} \mid k \geq 0 \} \) on the large phase space by

\[
D_a(z) = \sum_{k=0}^{\infty} z^k D_{k,a} = \Theta^{-1}(z)^b_a \partial_b(z).
\]

For example, \( D_{0,a} = \partial_{0,a} \) and \( D_{1,a} = \partial_{1,a} - \mathcal{U}^b_a \partial_{0,b} \).

**Lemma 1.1.** We have \( D_a(z) \mathcal{U} = \partial_a \mathcal{U} \) and \( D_a(z) \Theta(w) = w \Theta(w) \partial_a \mathcal{U} \). In particular, \( D_{k,a} \mathcal{U} = D_{k,a} \Theta = 0 \) if \( k > 0 \).

**Proof.** It follows easily from (1.2) that \( D_a(z) \mathcal{U} = \partial_a \mathcal{U} \); since

\[
D_a(z) \Theta(w) = w \Theta(w) D_a(z) \mathcal{U},
\]

the result follows. \( \square \)

**Corollary 1.2.** The vector fields \( D_{k,a} \) and \( D_{\ell,b} \) commute if both \( k \) and \( \ell \) are positive, while

\[
[D_{k,a}, \partial_{0,b}] = \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}^c \rangle\rangle_0 D_{k-1,c}.
\]

**Proof.** By Lemma 1.1,

\[
D_a(w) D_b(z) = D_a(w) \Theta^{-1}(z)^c_b \partial_c(z)
\]

\[
= \Theta^{-1}(z)^c_b D_a(w) \partial_c(z) - z \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}^c \rangle\rangle_0 D_c(z)
\]

\[
= \Theta^{-1}(z)^c_b \Theta^{-1}(w)^d_c \partial_d(w) \partial_c(z) - z \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}^c \rangle\rangle_0 D_c(z).
\]

It follows that \( [D_a(w), D_b(z)] = \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}^c \rangle\rangle_0 (w D_c(w) - z D_c(z)) \). \( \square \)

This corollary leads to an algorithm for the calculation of \( D_a(z) \langle\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle\rangle_g \) by induction on \( n \) in terms of \( D_a(z) \mathcal{F}_g \), using the formula

\[
D_a(z) \langle\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle\rangle_g = \sum_{i=1}^{n} \partial_{0,a_i} \ldots [D_a(z), \partial_{0,a_i}] \ldots \partial_{0,a_n} \mathcal{F}_g + \partial_{0,a_1} \ldots \partial_{0,a_n} D_a(z) \mathcal{F}_g.
\]

\[
D_a(z) \langle\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle\rangle_g = \sum_{i=1}^{n} \partial_{0,a_i} \ldots [D_a(z), \partial_{0,a_i}] \ldots \partial_{0,a_n} \mathcal{F}_g + \partial_{0,a_1} \ldots \partial_{0,a_n} D_a(z) \mathcal{F}_g.
\]
1.3. **The string equation in genus 0 and coordinates on the large phase space.** The genus 0 string equation says that $\mathcal{L}_0 \Phi_0 + \frac{1}{2} \eta_{ab} t^a_0 t^b_0 = 0$, where $\mathcal{L}_0$ is the vector field

$$\mathcal{L}_0 = \sum_{k=0}^{\infty} t_{k+1} \partial_{k,a} - \partial_{0,0}.$$ 

The string equation implies the following lemma.

**Lemma 1.3.** The restriction of $\partial \mathcal{U}$ to the small phase space $\{ t^a_k = 0 \mid k > 0 \}$ equals the identity, while for $n > 1$, the restriction of $\partial^n \mathcal{U}$ to the small phase space vanishes.

**Proof.** The vector fields $\partial_{0,a}$ commute with $\mathcal{L}_0$; it follows that $\mathcal{L}_0 \Phi_0 = \partial_{0,a} \partial_{0,b} \mathcal{L}_0 \Phi_0 = -\eta_{ab}$.

Written out explicitly, this equation says that

$$\partial \mathcal{U}_a = \delta^b_a + \sum_{k=0}^{\infty} t_{k+1} \langle \langle O_a O^b \tau_k c \rangle \rangle_0.$$ 

Applying the operator $\partial^{n-1}$, $n > 0$, we obtain

$$\partial^n \mathcal{U}_a = \sum_{k=0}^{\infty} t_{k+1} \partial^{n-1} \langle \langle O_a O^b \tau_k c \rangle \rangle_0.$$ 

The lemma is an immediate consequence of these formulas.

In conjunction with the genus 0 topological recursion relation, this implies the following theorem.

**Theorem 1.4.** Let $u^a = \partial \langle \langle O^a \rangle \rangle_0$. The functions $u^a_n = \partial^n u^a$, $n \geq 0$, form a coordinate system in a neighbourhood of the small phase space, and

$$(1.5) \quad \mathcal{D}_a(z) = \sum_{n=0}^{\infty} ((\partial + z \partial \mathcal{U})^n \partial \mathcal{U})_a^b \frac{\partial}{\partial u^b_n}.$$ 

**Proof.** Since $u^b = \mathcal{U}_b^0$, Lemma [1.1] implies that

$$\mathcal{D}_a(z) u^b_n = (\Theta^{-1}(z) \partial^n \Theta(z) \mathcal{U})_a^b = ((\Theta^{-1}(z) \partial \Theta(z))^n \partial \mathcal{U})_a^b.$$ 

Since $\Theta^{-1}(z) \partial \Theta(z) = \partial + z \partial \mathcal{U}$ by (1.3), we conclude that $\mathcal{D}_a(z) u^b_n = ((\partial + z \partial \mathcal{U})^n \partial \mathcal{U})_a^b$.

By Lemma [1.3], the restriction of $(\partial + z \partial \mathcal{U})^n \partial \mathcal{U}$ to the small phase space equals $z^n$. It follows that the restriction of $\mathcal{D}_a(z) u^b_n$ to the small phase space equals $\delta_{k,n} \delta^b_a$; hence the functions $u^a_n$ form a coordinate system in a neighbourhood of the small phase space.

Note that $(\partial + z \partial \mathcal{U})^n \partial \mathcal{U} = z^{-1} p_{n+1}(z \partial \mathcal{U})$, where $p_{n+1}(f) = (\partial + f)^n f$ is the $(n+1)$st Faà di Bruno polynomial.

**Corollary 1.5.** If $D_{k,a} f = 0$ for $k > n$, then $\partial f / \partial u^a_k = 0$ for $k > n$. 
Corollary 1.6. In terms of the coordinates $u^a_n$, the small phase space $\{t_k^a = 0 \mid k > 0\}$ is the submanifold

$$u^a_n = \begin{cases} \delta_0^a & n = 1, \\ 0 & n > 1. \end{cases}$$

Theorem 1.4 shows that the large phase space may be defined for any Frobenius manifold $M$, as the infinite jet space $\mathcal{J}^\infty M$ (i.e. Dubrovin’s “loop space”). This is seen by rewriting the matrix $\partial U^a_{bc}$ as $\partial u^c A^a_{b}$(1.6), where $A^a_{b} = \partial U^a_{bc} \partial u^c$ is the tensor describing the product on the tangent bundle of $M$.

An attractive feature of the vector fields $D_k^a$ is that they commute with $L_{-1}$:

$$[L_{-1}, D_k^a(z)] = [L_{-1}, \Theta^{-1}(z)^b_a \partial_b(z)] = [L_{-1}, \Theta^{-1}(z)^b_a \partial_b(z)] + \Theta^{-1}(z)^b_a [L_{-1}, \partial_b(z)]$$

$$= (z \Theta^{-1}(z)^b_a \partial_b(z) - \Theta^{-1}(z)^b_a (z \partial_b(z))) = 0.$$ 

By the genus 0 string equation, $L_{-1}u^a_n$ vanishes for $n > 0$, while $L_{-1}u^a = -\delta_0^a$: it follows that in the coordinate system $\{u^a_n\}$, the vector field $L_{-1}$ is given by the formula

$$L_{-1} = -\frac{\partial}{\partial u^0}.$$ 

In the coordinate system $\{u^a_n\}$, the string equation $L_{-1} \mathcal{F}_g = 0$ says that $\mathcal{F}_g$ is independent of $u^0$.

Lemma 1.3 shows that $\partial U$ is invertible in a neighbourhood of the small phase space: denote its inverse by $\mathcal{C}$. We also see that its determinant $\Delta = \det(\partial U)$ equals 1 on the small phase space.

1.4. The topological recursion relation in genus 1. We now illustrate the way in which use of the vector fields $D_k^a$ simplifies the discussion of topological recursion relations, using as an example the topological recursion relation in genus 1:

$$(\langle \tau_{k,a} \rangle)_1 = \langle \langle \tau_{k-1,a} \mathcal{O}^b \rangle \rangle_0 \langle \langle \mathcal{O}_b \rangle \rangle_1 + \frac{1}{24} \langle \langle \tau_{k-1,a} \mathcal{O}_b \mathcal{O}^b \rangle \rangle_0.$$ (1.7)

Multiplying by $z^k$ and summing over $k$, we obtain

$$\partial_a(z) \mathcal{F}_1 = \Theta(z)^b_a \langle \langle \mathcal{O}_b \rangle \rangle_1 + \frac{1}{24} z \partial_a(z) \text{Tr}(U),$$

hence, by Lemma 1.1,

$$D_a(z) \mathcal{F}_1 = \langle \langle \mathcal{O}_b \rangle \rangle_1 + \frac{1}{24} z D_a(z) \text{Tr}(U) = \langle \langle \mathcal{O}_b \rangle \rangle_1 + \frac{1}{24} z \partial_{0,a} \text{Tr}(U).$$

This may be written as the sequence of differential equations

$$D_{k,a} \mathcal{F}_1 = \begin{cases} \frac{1}{24} \partial_{0,a} \text{Tr}(U) & k = 1, \\ 0 & k > 1. \end{cases}$$ (1.8)
The equations (1.8) have the particular solution $\frac{1}{24} \log(\Delta)$. Let $\psi = F_1 - \frac{1}{24} \log(\Delta)$; we see that $D_{k,a} \psi = 0$ for all $k > 0$. Hence, by Corollary 1.3, $\psi$ depends only on the coordinates $u_0^a$; by the string equation, it is independent of $u_0^0$. In this way, we recover a result of Dijkgraaf and Witten [2]: there is a function $\psi$ of the coordinates $\{u_0^a\}$ such that $F_1 = \frac{1}{24} \log(\Delta) + \psi$.

1.5. The dilaton equation. The dilaton equation is another important constraint on the potentials of topological gravity. Let $D$ be the vector field

$$D = \partial_{1,0} - \sum_{k=0}^{\infty} t^a_k \partial_{k,a}.$$ 

The dilaton equation says that

$$DF_g = \begin{cases} 
(2g - 2)F_g, & g \neq 1, \\
\chi/24, & g = 1, 
\end{cases}$$

where $\chi$ is the Euler characteristic of the background.

**Proposition 1.7.** In the coordinate system $\{u_0^a\}$, the dilaton vector field $D$ equals

$$D = \sum_{n=1}^{\infty} n u_n^a \frac{\partial}{\partial u_n^a}.$$ 

**Proof.** By the genus 0 dilaton equation $DF_0 = -2F_0$, we have $Du_n^a = n u_n^a$, and the formula for $D$ follows.

2. The $A_2$ and $\mathbb{CP}^1$ models in genus 2

In genus 2, there are two topological recursion relations [11]. The first is

$$\langle \tau_{k,a} \rangle_2 = \langle \tau_{k-1,a} O_b^a \rangle_0 \langle O_b \rangle_2 + \langle \tau_{k-2,a} O_b^a \rangle_0 \left( \langle \tau_{1,b} \rangle_2 - \langle O_b O_c \rangle_0 \langle O_c \rangle_2 \right)$$

$$+ \langle \tau_{k-2,a} O_b^a O_c^a \rangle_0 \left( \frac{13}{20} \langle O_b \rangle_1 \langle O_c \rangle_1 + \frac{1}{10} \langle O_b O_c \rangle_1 \right)$$

$$+ \frac{13}{200} \langle \tau_{k-2,a} O_b^a O_c^a O_c^a \rangle_0 \langle O_b \rangle_1 - \frac{1}{240} \langle \tau_{k-2,a} O_b^a \rangle_1 \langle O_b O_c O_c \rangle_0$$

$$+ \frac{1}{800} \langle \tau_{k-2,a} O_b^a O_b O_c O_c \rangle_1.$$ 

Using the topological recursion relations in genus 0 and 1, (2.1) may be rewritten as the sequence of differential equations

$$D_{k,a} F_2 = R_{k,a},$$

(2.2)
where

\[
R_{k,a} = \begin{cases}
\langle\langle O_a O_b O_c \rangle\rangle_0 \left( \frac{7}{10} \langle\langle O^b \rangle\rangle_1 + \frac{1}{10} \langle\langle O^b O^c \rangle\rangle_1 \right) \\
+ \frac{13}{240} \langle\langle O_a O_b O_c O^c \rangle\rangle_0 \langle\langle O^b \rangle\rangle_1 - \frac{1}{240} \langle\langle O_a O_b O_c O^c \rangle\rangle_0 \\
+ \frac{1}{960} \langle\langle O_a O_b O_b O_c O^c \rangle\rangle_0 & k = 2, \\
\langle\langle O_a O_b O_c \rangle\rangle_0 \left( \frac{1}{20} \langle\langle O^b \rangle\rangle_1 + \frac{1}{280} \langle\langle O_b O^c O^d O_d \rangle\rangle_0 \right) + \frac{1}{720} \langle\langle O_a O_b O_c O^c \rangle\rangle_0 \langle\langle O_b O_c O^d O_d \rangle\rangle_0 & k = 3, \\
\frac{1}{1152} \langle\langle O_a O_b O_c \rangle\rangle_0 \langle\langle O_b O_c O^c \rangle\rangle_0 \langle\langle O_d O^c \rangle\rangle_0 & k = 4, \\
0 & k > 4.
\end{cases}
\]

The other topological recursion relation in genus 2 is,

\[
(2.3) \quad \langle\langle \tau_{k,a} \tau_{\ell,b} \rangle\rangle_2 = \langle\langle \tau_{k,a} O_c \rangle\rangle_2 \langle\langle O^c \tau_{\ell-1,b} \rangle\rangle_0 + \langle\langle \tau_{k-1,a} O_c \rangle\rangle_0 \langle\langle O^c \tau_{\ell,b} \rangle\rangle_2 \\
- \langle\langle \tau_{k-1,a} O_c \rangle\rangle_0 \langle\langle \tau_{\ell-1,b} O_d \rangle\rangle_0 \langle\langle O^c O^d \rangle\rangle_2 \\
+ 3 \langle\langle \tau_{k-1,a} \tau_{\ell-1,b} O^c \rangle\rangle_0 \langle\langle \tau_{1,c} \rangle\rangle_2 - \langle\langle O_c O^d \rangle\rangle_0 \langle\langle O_d \rangle\rangle_2 \\
+ \frac{13}{10} \langle\langle \tau_{k-1,a} \tau_{\ell-1,b} O_c O_d \rangle\rangle_0 \langle\langle O^c \rangle\rangle_1 \langle\langle O^d \rangle\rangle_1 \\
+ \frac{1}{24} \langle\langle \tau_{k-1,a} \tau_{\ell-1,b} O_c O_d \rangle\rangle_1 + \frac{1}{24} \langle\langle \tau_{k-1,a} O_c O_d \rangle\rangle_1 \langle\langle O^c \rangle\rangle_0 \langle\langle O_d \rangle\rangle_0 \\
+ \frac{1}{24} \langle\langle \tau_{k-1,a} \tau_{\ell-1,b} O_c O_d \rangle\rangle_0 \langle\langle O^c \rangle\rangle_1 \langle\langle O_d \rangle\rangle_1 \\
- \frac{1}{24} \langle\langle \tau_{k-1,a} \tau_{\ell-1,b} O_c O_d \rangle\rangle_0 \langle\langle O^c \rangle\rangle_1 \langle\langle O_d \rangle\rangle_1 \\
+ \frac{1}{720} \langle\langle \tau_{k-1,a} \tau_{\ell-1,b} O_c O_d \rangle\rangle_0 \langle\langle O^c O^d \rangle\rangle_1 + \frac{1}{80} \langle\langle \tau_{k-1,a} \tau_{\ell-1,b} O_c O_d \rangle\rangle_0 \langle\langle O^c O^d \rangle\rangle_0 \\
+ \frac{29}{280} \langle\langle \tau_{k-1,a} \tau_{\ell-1,b} O_c O_d \rangle\rangle_0 \langle\langle O^c \rangle\rangle_1 - \frac{1}{80} \langle\langle \tau_{k-1,a} \tau_{\ell-1,b} O_c O_d \rangle\rangle_0 \langle\langle O^c O^d \rangle\rangle_0 \\
+ \frac{7}{360} \langle\langle \tau_{k-1,a} \tau_{\ell-1,b} O_c O_d \rangle\rangle_0 \langle\langle O^c O^d \rangle\rangle_1 + \frac{7}{360} \langle\langle \tau_{k-1,a} \tau_{\ell-1,b} O_c O_d \rangle\rangle_0 \langle\langle O^c O^d \rangle\rangle_0.
\]

Taking \( k \) and \( \ell \) equal to 1 and using the topological recursion relations in genus 0 and 1, we obtain the system of differential equations

\[
(2.4) \quad D_{1,a,b} F_2 = R_{1,a,b},
\]

where \( D_{1,a,b} = D_{1,a} D_{1,b} - 3 \langle\langle O_a O_b O^c \rangle\rangle_0 D_{1,c} \), and

\[
R_{1,a,b} = \frac{13}{10} \langle\langle O_a O_b O_c O_d \rangle\rangle_0 \langle\langle O^c \rangle\rangle_1 \langle\langle O^d \rangle\rangle_1 \\
+ \frac{1}{5} \langle\langle O_a O_c \rangle\rangle_1 \langle\langle O_d \rangle\rangle_1 + \frac{1}{24} \langle\langle O_a O_c O_d \rangle\rangle_1 \langle\langle O_b O^c O^d \rangle\rangle_0 \\
+ \frac{1}{5} \langle\langle O_a O^c O^d \rangle\rangle_0 \langle\langle O_b O_c \rangle\rangle_1 \langle\langle O_d \rangle\rangle_1 + \frac{1}{24} \langle\langle O_b O_c O_d \rangle\rangle_1 \langle\langle O^c \rangle\rangle_1 \langle\langle O^d \rangle\rangle_1 \\
- \frac{1}{24} \langle\langle O_a O_b O_c \rangle\rangle_0 \langle\langle O_d O^c \rangle\rangle_1 + \frac{1}{24} \langle\langle O_b O_c O_d \rangle\rangle_1 \langle\langle O^c \rangle\rangle_0 \langle\langle O_d \rangle\rangle_0 \\
+ \frac{1}{120} \langle\langle O_a O_c O_d O^d \rangle\rangle_0 \langle\langle O^c O_b \rangle\rangle_1 + \frac{1}{280} \langle\langle O_a O_c O_d O^d \rangle\rangle_0 \langle\langle O^c \rangle\rangle_1 - \frac{1}{80} \langle\langle O_a O_b O_c \rangle\rangle_1 \langle\langle O^c O^d O_d \rangle\rangle_0 \\
+ \frac{29}{280} \langle\langle O_a O_b O_c O_d O^d \rangle\rangle_0 \langle\langle O^c \rangle\rangle_1 + \frac{7}{360} \langle\langle O_a O_b O_c O_d O^d \rangle\rangle_0 \langle\langle O^c O^d \rangle\rangle_1 + \frac{1}{180} \langle\langle O_a O_b O_c O_d O^d \rangle\rangle_0 \langle\langle O^c O^d \rangle\rangle_0.
\]
We now specialize to the case of the $A_2$ model. In this model, there are two primary fields $\mathcal{O}_0$ and $\mathcal{O}_1$, with intersection form $\eta_{ab} = \delta_{a+b,1}$. Denote the associated coordinates by $u = \langle\langle \mathcal{O}_0 \mathcal{O}_0 \rangle\rangle_0 = \partial^2 F_0$ and $v = \langle\langle \mathcal{O}_0 \mathcal{O}_1 \rangle\rangle_0$. The matrix $\mathcal{U}$ is given by the formula

$$
\mathcal{U} = \begin{bmatrix} U_0^0 & U_0^1 \\ U_1^0 & U_1^1 \end{bmatrix} = \begin{bmatrix} v & u \\ u^2 & v \end{bmatrix},
$$

and $\mathcal{F}_1 = \frac{1}{24} \log(\Delta)$. As was shown by Eguchi, Yamada, and Yang [8], the genus 2 potential of the $A_2$-model is given by the formula

$$
\mathcal{F}_2 = \frac{1}{1152} \partial^2 \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \mathcal{O}_d \rangle\rangle_0 C^{ab} C^{cd} - \frac{1}{1152} \partial^2 \langle\langle \mathcal{O}_e \mathcal{O}_f \rangle\rangle_0 \partial \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \mathcal{O}_d \mathcal{O}_e \mathcal{O}_f \rangle\rangle_0 C^{ab} C^{ef} - \frac{1}{5760} \partial^2 \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \mathcal{O}_d \rangle\rangle_0 \partial \langle\langle \mathcal{O}_e \mathcal{O}_f \mathcal{O}_g \mathcal{O}_h \rangle\rangle_0 C^{ab} C^{ef} + \frac{1}{5760} \partial^2 \langle\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \mathcal{O}_d \rangle\rangle_0 \partial \langle\langle \mathcal{O}_e \mathcal{O}_f \mathcal{O}_g \mathcal{O}_h \rangle\rangle_0 C^{ab} C^{ef}.
$$

(2.5)

It may be checked that this function solves the equations (2.2) and (2.4).

For any arbitrary theory of topological gravity, let $\mathcal{F}_{2,0}$ be the function on the large phase space given by formula (2.3). For all theories of topological gravity for which we know the genus 2 potential, the function $\mathcal{F}_{2,0}$ appears to be a major contribution to this potential.

We now turn to the case of $\mathbb{C}P^1$. As in the $A_2$-model, there are two primary fields $\mathcal{O}_0$ and $\mathcal{O}_1$, with intersection form $\eta_{ab} = \delta_{a+b,1}$. Again, denote the associated coordinates by $u = \langle\langle \mathcal{O}_0 \mathcal{O}_0 \rangle\rangle_0 = \partial^2 F_0$ and $v = \langle\langle \mathcal{O}_0 \mathcal{O}_1 \rangle\rangle_0$. The matrix $\mathcal{U}$ is now given by the formula

$$
\mathcal{U} = \begin{bmatrix} v & u \\ u^2 & v \end{bmatrix},
$$

and $\mathcal{F}_1 = \frac{1}{24} \log(\Delta) - \frac{1}{24} u$.

The correlators $\langle\tau_{1,a_1} \mathcal{O}_{a_2} \ldots \mathcal{O}_{a_n} \rangle_2$ and $\langle\mathcal{O}_{a_1} \mathcal{O}_{a_2} \ldots \mathcal{O}_{a_n} \rangle_2$ vanish in the $\mathbb{C}P^1$-model for dimensional reasons. It follows that the following solution to the equations (2.2) and (2.4) is the genus 2 potential:

$$
\mathcal{F}_2 = \mathcal{F}_{2,0} - \frac{1}{480} \partial^3 \langle\langle \mathcal{O}_a \mathcal{O}_b \rangle\rangle_0 C^{ab} + \frac{7}{5760} \partial^3 \langle\langle \mathcal{O}_a \rangle\rangle_0 \partial^2 \langle\langle \mathcal{O}_b \rangle\rangle_0 C^{ab} + \frac{11}{5760} \partial^2 \langle\langle \mathcal{O}_a \mathcal{O}_b \rangle\rangle_0 \partial^2 \langle\langle \mathcal{O}_c \mathcal{O}_d \rangle\rangle_0 C^{ac} C^{bd}.
$$

(2.6)

The three additional terms reflect the fact that, unlike in the $A_2$-model, the function $\psi(u) = -\frac{1}{24} u$ is nonzero in the $\mathbb{C}P^1$-model.

The Toda conjecture of Eguchi and Yang ([3], [5], [16]) provides conjectural formulas for the functions $\langle\langle \mathcal{O}_1 \mathcal{O}_1 \rangle\rangle_0, \ldots, \langle\langle \mathcal{O}_1 \mathcal{O}_1 \rangle\rangle_0$ of the $\mathbb{C}P^1$-model:

$$
\sum_{g=0}^{\infty} \lambda^{2g} \langle\langle \mathcal{O}_1 \mathcal{O}_1 \rangle\rangle_0 = \exp \left( \frac{2}{\lambda^2} (\cosh(\lambda \partial) - 1) \right) \sum_{g=0}^{\infty} \lambda^{2g} F_g.
$$

In genus 2, this yields the equation

$$
\langle\langle \mathcal{O}_1 \mathcal{O}_1 \rangle\rangle_2 = e^u \left( \partial^2 F_2 + \frac{1}{12} \partial^4 F_1 + \frac{1}{360} \partial^6 F_0 + \frac{1}{2} (\partial^2 F_1 + \frac{1}{12} \partial^4 F_0)^2 \right).
$$

(2.7)
It is easily checked, using the explicit formula for \( F_2 \), that this equation holds.

3. Models with Two Primaries

In this section, we consider topological gravity in a general background with two primary fields \( \mathcal{O}_0 \) and \( \mathcal{O}_1 \), and intersection form \( \eta_{ab} = \delta_{a+b,1} \). It is not clear to what extent such a model, even if it possesses a consistent loop expansion, corresponds to a physical theory: it may be that only the \( A_2 \) and \( \mathbb{CP}^1 \)-models are physical theories. The fact that our equations remain consistent in this setting is nevertheless very suggestive.

Denote the associated coordinates \( u = \langle \langle \mathcal{O}_0 \mathcal{O}_0 \rangle \rangle_0 \) and \( v = \langle \langle \mathcal{O}_0 \mathcal{O}_1 \rangle \rangle_0 \). The genus 0 sector is characterized by the function \( \langle \langle \mathcal{O}_1 \mathcal{O}_1 \rangle \rangle_0 \); by the string equation, this is a function of \( u \) alone, and we denote it by \( \phi(u) \). The matrix \( \mathcal{U} \) is given by the formula

\[
\mathcal{U} = \begin{bmatrix} v & u \\ \phi(u) & v \end{bmatrix}.
\]

In this section, the correlation functions \( \langle \langle \tau_{k_1,a_1} \ldots \tau_{k_n,a_n} \rangle \rangle_g \) are assumed to have the following form: they are holomorphic functions of \( \{(v, u) \in \mathbb{C}^2 \mid u \notin (-\infty, 0]\}, \) Laurent polynomials in \( \Delta \), and polynomial in the remaining coordinates \( \{\partial^n v, \partial^n u \mid n > 0\} \).

There is a universal differential equation \([10]\) in topological gravity relating the potentials \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \). In the case of two primary fields, this equation says that

\[
\frac{1}{24} \phi'''' + \phi'' \psi' - 2 \phi' \psi'' = 0.
\]

It turns out that this equation is also the necessary and sufficient condition for the system of equations \((2.2)\) and \((2.4)\) to have a solution. The necessity follows from the formula

\[
D_{1,1,0,0} \mathcal{R}_{2,0} - D_{2,0} \mathcal{R}_{1,1,0,0} = \frac{2}{15} (\partial u)^3 (4(\partial v)^2 + (\partial u)^2 \phi') \left( \frac{1}{24} \phi'''' + \phi'' \psi' - 2 \phi' \psi'' \right).
\]

**Theorem 3.1.** Suppose that \( \frac{1}{24} \phi'''' + \phi'' \psi' - 2 \phi' \psi'' = 0 \). Then the equations \((2.2)\) and \((2.4)\) have the solution \( \mathcal{F}_{2,0} + \mathcal{F}_{2,1} \), where \( \mathcal{F}_{2,0} \) is given by \((2.5)\), and

\[
\mathcal{F}_{2,1} = \frac{1}{5760} (\partial \partial_{0,a} \partial_{0,b} \psi + \frac{4}{5} \partial \partial_{0,a} \partial_{0,b} \psi) C^{ab} + \partial^2 \langle \langle \mathcal{O}_a \mathcal{O}_b \rangle \rangle_0 \left( \frac{3}{10} \partial \partial_{0,c} \partial_{0,d} \psi - \frac{1}{10} \partial \partial_{0,c} \partial_{0,d} \psi \right) C^{ab} C^{cd} + \partial^2 \langle \langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle \rangle_0 \partial \partial_{0,d} \psi \partial \partial_{0,d} \psi \left( \frac{3}{10} C^{af} C^{bc} C^{de} - \frac{23}{10} C^{ac} C^{bd} C^{ef} \right) + \frac{1}{10} (\partial u)^4 \phi'' \psi'' \Delta^{-1}.
\]

This solution may be characterized by the property that its restriction to the small phase space, together with the restrictions of the functions \( \partial_{1,a} (\mathcal{F}_{2,0} + \mathcal{F}_{2,1}) \), vanish.

All of the terms in the formula for \( \mathcal{F}_{2,0} + \mathcal{F}_{2,1} \) except the last one \( \frac{1}{5760} (\partial u)^4 \phi'' \psi'' \Delta^{-1} \) are associated to Feynman graphs with propagator \( C \) and vertices \( \partial^n \partial_{a_1} \ldots \partial_{a_{k-2}} \mathcal{U}_{a_{k-1} a_k} \) and \( \partial^n \partial_{a_1} \ldots \partial_{a_k} \psi \). From this point of view, the last term is an instanton, which vanishes if \( \psi \) is a linear function of \( u \), that is, for the \( A_2 \) and \( \mathbb{CP}^1 \)-models.
One calculates that $\mathcal{F}_{2,1}$ is given by the explicit formula

$$
\mathcal{F}_{2,1} = \frac{1}{360} \left( \frac{1}{2} (\partial u)^2 \psi'' + \frac{9}{5} (\partial u)^2 \psi'' \psi' + \frac{13}{2} \partial^2 u \psi'' + \frac{7}{10} \partial^2 u (\psi')^2 
- ((\partial v)^2 + \frac{7}{5} (\partial u)^2) (\partial u)^2 \phi' \psi' \Delta^{-1} + \frac{3}{10} (\partial u)^4 \phi'' (\psi')^2 \Delta^{-1}
+ \left( \frac{2}{5} (\partial^2 v \partial u - \partial^2 u \partial v) \partial u - \frac{1}{10} (\partial u)^4 \phi'' \Delta^{-1}
+ \left( \frac{7}{5} \partial^2 v \partial u - \frac{12}{5} \partial^2 u \partial u \phi' - \frac{1}{5} \partial^2 u (\partial u)^2 \phi'' \Delta^{-1}
+ \frac{1}{5} \left( 4 \partial^2 v \partial^2 u \partial v \partial u \phi' - (\partial^2 v + \partial^2 u \phi')((\partial u)^2 + (\partial u)^2 \phi' + 2 (\partial^2 v \partial u - \partial^2 u \partial v) (\partial u)^2 \partial v \phi'' - \frac{1}{2} (\partial u)^6 \phi'' (\psi')^2 \Delta^{-2} \right) \right). \right)
$$

Now let $\mathcal{F}_2$ be a general solution of (2.2) and (2.4). Write $\mathcal{F}_2 = \mathcal{F}_{2,0} + \mathcal{F}_{2,1} + f_2$. By the equations (2.2), $D_{k,a} f_2 = 0$ for $k > 1$; thus, $f_2$ is a function of the coordinates $\{u, \partial v, \partial u\}$. 

**Theorem 3.2.** Define the functions $h_a = h_a(u)$ by the formula $h_a = \frac{\partial f_2}{\partial (\partial u^a)} \big|_{(\partial u, \partial v) = (1,0)}$. Then

$$
f_2 = \frac{1}{2} \partial u^a \partial U^b_a h_b = \frac{1}{2} \partial u^a \partial u^b A^b_{ab} h_c = \frac{1}{2} ((\partial v)^2 + \phi' (\partial u)^2) h_0(u) + \partial v \partial u h_1(u).$$

**Proof.** Let $\tilde{f}_2 = \frac{1}{2} \partial u^a \partial U^b_a h_b$; then $D_{1,1,a,b} \tilde{f}_2 = 0$ and $h_a = \frac{\partial \tilde{f}_2}{\partial (\partial u^a)} \big|_{(\partial u, \partial v) = (1,0)}$. Thus $f_2 - \tilde{f}_2$ satisfies the equations $D_{k,a} (f_2 - \tilde{f}_2) = 0$ for $k > 1$, and $D_{1,1,a,b} (f_2 - \tilde{f}_2) = 0$, as well as the dilaton equation $D (f_2 - \tilde{f}_2) = 2 (f_2 - \tilde{f}_2)$; and is thus determined by the restrictions of the partial derivatives $\partial f_2 - \partial \tilde{f}_2$ to the small phase space. But these vanish; we conclude that $f_2 = \tilde{f}_2$. $\blacksquare$

In the next section, we determine the functions $h_a$.

### 4. Virasoro Constraints

We now show that the Virasoro constraints $L_0 Z = L_1 Z = 0$ of Eguchi, Hori and Xiong [3], as generalized to arbitrary Frobenius manifolds by Dubrovin and Zhang [3], may be used to complete the determination of the genus 2 potential in two-primary models of topological gravity.

**The constraint** $L_0 Z = 0$. According to Dubrovin and Zhang [3], an Euler vector on a Frobenius manifold determines matrices $\mu$ and $R[n]$, $n > 0$, which satisfy the commutation relations $[\mu, R[n]] = n R[n]$ and the symmetry conditions $\mu_{ab} + \mu_{ba} = 0$ and

$$
R[n]_{ab} + (-1)^n R[n]_{ba} = 0.
$$

The basis $O_a$ of primary fields may be chosen in such a way that the matrix $\mu$ is diagonal

$$
\mu^b_a = \delta^b_a \mu_a,
$$

and $\mu_0 < \mu_a$ for $a \neq 0$. Setting $d_a = \mu_a - \mu_0$ and $d = -2 \mu_0$, we have $\mu_a = d_a - d/2$.
For the Gromov-Witten invariants of a Kähler manifold $X$, the primaries $O_a$ form a basis of the De Rham cohomology $H^*(X, \mathbb{C})$, and the number $d_a$ is the holomorphic degree of $O_a$, that is $O_a \in H^{d_a,*}(X, \mathbb{C})$. (In particular, $d$ equals the complex dimension of $X$.) In this case, $R[1]$ is the matrix of multiplication by $c_1(X)$, and $R[n] = 0$ for $n > 1$.

Introduce the vector field

$$L_0 = \sum_{k=0}^{\infty} \left\{ (\mu^b + k + \frac{1}{2})\tilde{t}_k^a \frac{\partial}{\partial t_k^a} + \sum_{\ell=1}^{k} R[\ell]_{\ell}^b \tilde{t}_k^a \frac{\partial}{\partial t_{\ell}^b} \right\},$$

where $\tilde{t}_k^a$ are the shifted coordinates $\tilde{t}_k^a = t_k^a - \delta_{k,1}\delta_0^a$. The Virasoro constraint $L_0 Z = 0$ in genus $g = 0$ may be expressed as the following equation:

$$(4.1) \quad L_0 U + [\mu, U] + R[1] = 0.$$ 

In genus $g > 0$, the Virasoro constraint $L_0 Z = 0$ says that

$$(4.2) \quad L_0 F_g + \frac{1}{4} \delta_{g,1} \text{Tr}(\frac{1}{4} - \mu^2) = 0.$$ 

These equations are known to hold for Gromov-Witten invariants [14].

Let $E_a = E_a \partial/\partial u_a$ be the Euler vector field, where

$$(4.3) \quad E_a = (1 - d_a)u_a + R[1]_{0}^a.$$ 

Then (4.1) implies that

$$(4.4) \quad L_0 u^a + E^a = 0.$$ 

In calculating the action of the vector field $L_0$ in the coordinate system $\{u_n^a\}$, we use (4.4) together with the commutation relation $[\partial, L_0] = \frac{1}{2}(1 - d)\partial$.

In the case of two primary fields, we have $\mu = \frac{1}{2} \begin{bmatrix} -d & 0 \\ 0 & d \end{bmatrix}$. Consider first the case in which $d$ equals 1; then $R[1] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. By (4.1), we see that $\phi = c e^{2u/r}$; redefining $u$, we may assume that $r = 2$, and we recover the $\mathbb{C}P^1$-model. Since $\text{Tr}(\mu^2 - \frac{r}{4}) = 0$, we see from (4.2) that $L_0 F_1 = 0$; (3.1), now shows that $\psi = -\frac{1}{24}u$, consistent with the known form of $F_1$ in the $\mathbb{C}P^1$-model.

The equation $L_0 F_2 = 0$ of (4.2) constrains the functions $h_a(u)$ of Theorem 3.2; if $d = 1$, it forces them to have negative degree in $e^u$, and hence to vanish, as we have already observed.

If $d \neq 1$, the matrix $R[1]$ vanishes. By (4.1), we see that $\phi(u) = u^{(1+d)/(1-d)}$, up to a constant which we take to equal 1. (For example, the $A_2$-model, has $d = \frac{1}{3}$ and $\phi(u) = u^2$.) In genus 1, the equation (4.2) shows that $\psi(u)$ is proportional to $\log(u)$; both (3.1) and (4.2) yield the same answer for this constant,

$$\psi(u) = \frac{d(3d - 1)}{24(d - 1)} \log(u).$$

Note that the $A_2$-model, for which $d = \frac{1}{3}$, has $\psi = 0$. The equation $L_0 F_2 = 0$ imposes the homogeneities $h_a(u) = C_a u^{(1+a)d-3/(1-d)}$. 
The constraint $L_1 Z = 0$. Let $\mathcal{L}_1$ be the vector field

\[
\mathcal{L}_1 = -(\mu_a - \frac{1}{2})(\mu_a + \frac{1}{2}) \langle \langle \mathcal{O}_a \rangle \rangle_0 \partial_{\partial \theta_0} + \sum_{k=0}^{\infty} \left\{ (\mu_a + k + \frac{1}{2})(\mu_a + k + \frac{3}{2}) \tilde{F}_k \partial_{k+1,a} \right. \\
+ \sum_{\ell \leq k+1} 2(\mu_a + k + 1)R[\ell]_a \tilde{F}_k \partial_{k+1-\ell,b} + \sum_{\ell_1 + \ell_2 \leq k+1} (R[\ell_1]R[\ell_2])_a \tilde{F}_k \partial_{k+1-\ell_1-\ell_2,b} \right\}.
\]

Let $\mathcal{V} = \mathcal{E} \mathcal{U}$; by (4.1), $\mathcal{V} = \mathcal{U} + [\mu, \mathcal{U}] + R[1]$. The constraint $L_1 Z = 0$ in genus 0 may be written (Dubrovin and Zhang [3]; cf. Theorem 5.7 of [12])

(4.5) \hspace{1cm} \mathcal{L}_1 \mathcal{U} + \mathcal{V}^2 = 0.

In particular, we see that

(4.6) \hspace{1cm} \mathcal{L}_1 u^a + \mathcal{E}^b \mathcal{E}^c A^a_{bc} = 0.

In genus $g > 0$, the constraint $L_1 Z = 0$ is

(4.7) \hspace{1cm} \mathcal{L}_1 \mathcal{F}_g + \frac{1}{2} (\frac{1}{4} - \mu^2) \langle \langle \mathcal{O}_a \mathcal{O}_b \rangle \rangle \langle \langle \mathcal{O}_a \mathcal{O}_b \rangle \rangle_{g-h} = 0.

In the case of two primaries, this becomes

(4.8) \hspace{1cm} \mathcal{L}_1 \mathcal{F}_g + \frac{1}{8} (1 - d^2) \left( \sum_{h=1}^{g-1} \langle \langle \mathcal{O}_a \mathcal{O}_b \rangle \rangle \langle \langle \mathcal{O}_a \mathcal{O}_b \rangle \rangle_{g-h} \right) = 0.

In calculating the action of the vector field $\mathcal{L}_1$ in the coordinate system $\{u^a_n\}$, we use (1.6) and the commutation relation

(4.9) \hspace{1cm} [\partial_{0,a}, \mathcal{L}_1] = ((\mu + \frac{1}{2})(\mu + \frac{3}{2}))_a D_{1,b} + ((\mu + \frac{1}{2}) \mathcal{V} + \mathcal{V}(\mu + \frac{1}{2}))_a D_{0,b}.

In the case of two primaries, this implies that

\[
[\partial, \mathcal{L}_1] = \begin{cases} (1 - d) \left( \frac{1}{4} (3 - d) \right) D_{1,0} + v D_{0,0} + u D_{0,1}, & d \neq 1, \\ 2 D_{0,1}, & d = 1. \end{cases}
\]

Using these formulas, we see that the case $g = 2$ of (4.8) yields the equation

\[
0 = \mathcal{L}_1 \mathcal{F}_2 + \frac{1}{4} (1 - d^2) \left( \langle \langle \mathcal{O}_0 \mathcal{O}_0 \rangle \rangle_1 \langle \langle \mathcal{O}_1 \rangle \rangle_1 + \langle \langle \mathcal{O}_0 \mathcal{O}_1 \rangle \rangle_1 \right)
\]

\[
= - 6((d + 1) C_0 + \frac{1}{5760} d(3d - 1)(3d - 5)(d - 2)) u^{-2} \partial v \partial u + 3 C_1 (d - 1) u^{(d - 2)/(1-d)} ((\partial v)^2 + \phi' (\partial u)^2).
\]

It follows that $h_1 = 0$ and

(4.10) \hspace{1cm} h_0 = - \frac{d(3d - 1)(3d - 5)(d - 2)}{5760(d + 1)} u^{(d - 3)/(1-d)}.

completing the determination of $\mathcal{F}_2$. 
Our formula for $F_2$ agrees with that of Dubrovin and Zhang \cite{4}, who apply the method of Eguchi and Xiong \cite{9}; in other words, they use the constraints $D_{k,a}F_2 = 0$, $k > 4$, and $L_nZ = 0$, $n \leq 10$.

**The higher Virasoro constraints.** The higher Virasoro constraints are given by formulas involving a Lie algebra of vector fields $L_n$, $n \geq -1$, on the large phase space, which satisfy the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n}.$$  

This Lie algebra is generated by $L_{-1}$ and $L_2$, for any $n > 1$.

Just as for $L_0$ and $L_1$, we can avoid using the explicit formula for $L_n$. The Virasoro constraint $L_nZ = 0$ in genus 0 may be written

$$L_nU + \mathcal{V}^{n+1} = 0.  \tag{4.11}$$

In calculating the action of the vector field $L_2$ in the coordinate system $\{u^a_0\}$, we use (4.11) and the commutation relation \cite{13}

$$[\partial_{0,a}, L_n] = \sum_{i=0}^{n} (B_{n,i})^b_a D_{i,b},  \tag{4.12}$$

where the matrices $B_{n,i}$ are determined by the recursion

$$B_{n,i} = (\mu + i + \frac{1}{2})B_{n-1,i-1} + \mathcal{V}B_{n-1,i},$$

with initial condition $B_{-1,i} = \delta_{i+1,0}$.

In the case of two primaries, with $n = 2$, this implies that

$$[\partial, L_2] = \begin{cases} 
(1 - d)\left(\frac{1}{8}(3 - d)(5 - d)D_{2,0} + \frac{3}{4}(3 - d)vD_{1,0} + \frac{1}{4}(d^2 - 2d + 9)uD_{1,1} \right), & d \neq 1, \\
6D_{1,1} + 4e^uD_{0,0} + 6vD_{0,1}, & d = 1.
\end{cases}$$

In genus $g > 0$, the constraint $L_2Z = 0$ is

$$L_2F_g + (\mu_a - \frac{3}{2})(\mu_a - \frac{1}{2})\eta^{ab}\left(\sum_{h=1}^{g-1} \langle \langle \tau_{1,a} \rangle \rangle_h \langle \langle O_b \rangle \rangle_{g-h} + \langle \langle \tau_{1,a}O_b \rangle \rangle_{g-1} \right)$$

$$- \frac{1}{2} (3\mu_a^2 + 3\mu_a - \frac{1}{4})R[1]^{ab}\left(\sum_{h=1}^{g-1} \langle \langle O_a \rangle \rangle_h \langle \langle O_b \rangle \rangle_{g-h} + \langle \langle O_aO_b \rangle \rangle_{g-1} \right) = 0.$$  

It may be verified that $F_2$ satisfies this equation.

Since the differentials operators $L_n$, $n > -1$, lie in the Lie algebra generated by $L_{-1}$ and $L_2$, it follows that the Virasoro conjecture holds to genus 2 for two-primary models.
The Belorousski-Pandharipande equation. The Belorousski-Pandharipande equation is a differential equation satisfied by the genus 2 potential, analogous to the equation (3.1) in genus 1; it may be expressed as saying that a certain cubic polynomial in the coordinates \( \{ u^9 \} \) vanishes. It turns out that in the case of backgrounds with two primaries, the equation gives a second (and thus rigorous) derivation of the above formula for \( C_0 \), but leaves \( C_1 \) undetermined.

Taking Theorem 3.2 and the equations \( \mathcal{L}_1 F_2 = 0 \) and \( \mathcal{L}_0 F_2 = 0 \) into account, the Belorousski-Pandharipande equation reduces to a single equation

\[
\phi' h_0' - \frac{1}{2} \phi'' h_0 - \frac{1}{48} \psi''' - \frac{3}{5} \psi'' \psi' + \frac{9}{10} (\psi')^2 = 0.
\]

With \( \phi(u) = u^{(1+d)/(1-d)} \) and \( \psi(u) = \frac{d(3d-1)}{24(d-1)} \log(u) \), the function \( h_0 \) of (4.10) satisfies this equation.

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