A REMARK ON WU’S REMARK ON
HOLOMORPHIC SECTIONAL CURVATURE

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Wu [3] proved that if \( g \) and \( h \) are Hermitian metrics on a complex manifold and their holomorphic sectional curvatures satisfy \( H_g \leq -K_g < 0 \) and \( H_h \leq -K_h < 0 \) then the holomorphic sectional curvature of their sum satisfies

\[
H_{g+h} \leq -\frac{K_gK_h}{K_g+K_h} < 0.
\]

The proof shows that given any tangent vector \( \xi \in T_{X,x} \) there exists a holomorphic embedding \( f : D \to X \) of the unit disk in \( X \) such that \( f(0) = x \) and \( f'(0) = \xi \) and such that the holomorphic sectional curvature of a given metric at \( \xi \) is the curvature of the pullback metric to the disk at its center. Thus we reduce to the case of one complex variable, which had already been proven by Grauert and Reckziegel [1].

In this note we point out that there is a direct route to Wu’s theorem by using an expression for the curvature tensor of the sum of Hermitian metrics one obtains from Griffiths’ theorem on the curvatures of sub- and quotient bundles [2]. Our starting point is the following result, which is given as an exercise in Zheng’s lovely textbook [4].

**Proposition 1.** Let \( E \to X \) be a holomorphic vector bundle over a complex manifold. Let \( g \) and \( h \) be Hermitian metrics on \( E \). Then the curvature tensor of \( g+h \) is

\[
R_{g+h} = R_g + R_h - \sigma^*q,
\]

where \( q \) is a Hermitian metric on \( E \) and \( \sigma(\xi)s = D_{g,\xi}s - D_{h,\xi}s. \)

**Proof.** Consider short exact sequence

\[
0 \to E \to E \oplus E \to E \to 0,
\]

where the first nonzero arrow is \( j(s) = s \oplus s \) and the second is \( \pi(s \oplus t) = s-t \). We equip \( E \oplus E \) with the metric \( g \oplus h \), the subbundle with its pullback \( g+h \), and the quotient with the induced metric \( q \). The second fundamental form of \( E \) in \( E \oplus E \) is then

\[
\sigma(\xi)s = \pi(D_{g \oplus h,\xi}j(s)) = D_{g,\xi}s - D_{h,\xi}s.
\]

The curvature tensor of \( g+h \) is then \( j^*R_{g \oplus h} - \sigma^*q \), which unravels to what we wanted to show. \( \square \)

It is possible to write down exactly what this “quotient” metric \( q \) is; in fact

\[
q = ((g+h)^{-1}h)^*g + ((g+h)^{-1}g)^*h,
\]

where we view Hermitian metrics on \( T_X \) as smooth isomorphisms \( T_X \to \overline{T}_X \). We won’t need to know this, but it is useful for analyzing what happens when one of the metrics varies and leads to the basic estimate \( q \leq g+h \).

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Theorem 2 (Wu [3]). Let $g$ and $h$ be Hermitian metrics on a complex manifold $X$ and assume both have nonpositive holomorphic sectional curvature. Then the holomorphic sectional curvature of $g + h$ is nonpositive, and satisfies

$$H_{g+h} \leq \frac{H_g H_h}{H_g + H_h} < 0$$

if the holomorphic sectional curvatures of $g$ and $h$ are negative.

Proof. We know that

$$R_{g+h} = R_g + R_h - \sigma^* q$$

and that $q$ is positive-definite. Then

$$H_{g+h}(\xi) \leq \frac{H_g(\xi) |\xi|^4_g + H_h(\xi) |\xi|^4_h}{|\xi|_{g+h}^4} = \frac{H_g(\xi) |\xi|^4_g + H_h(\xi) |\xi|^4_h}{(|\xi|^2_g + |\xi|^2_h)^2}.$$ 

Thus $H_{g+h} \leq 0$ if $H_g \leq 0$ and $H_h \leq 0$.

As Wu notes, for any positive real numbers $a, b, x, y$ we have

$$\frac{xy}{x+y} \leq \frac{xa^2 + yb^2}{(a+b)^2}$$

by elementary algebra. The inequality is reversed if $x$ and $y$ are negative. If the holomorphic sectional curvatures of $g$ and $h$ are negative we thus get

$$H_{g+h} \leq \frac{H_g H_h}{H_g + H_h} < 0.$$

One can show that if a metric on a compact manifold has negative (or positive) holomorphic sectional curvature, then so does any sufficiently small deformation of the metric. On a Kähler manifold this implies that the set of Kähler classes with a representative with negative holomorphic sectional curvature is an open cone inside the Kähler cone. Wu’s theorem further shows that convex combinations of metrics in this cone are again in it, so the cone is connected. It would be unexpected if these two cones were the same but I don’t know of an example where they differ.

References

[1] Hans Grauert and Helmut Reckziegel. Hermiteische metriken und normale familien holomorpher abbildungen. Mathematische Zeitschrift, 89(2):108–125, 1965.

[2] Phillip A Griffiths. Hermitian differential geometry and the theory of positive and ample holomorphic vector bundles. J. Math. Mechanics, 14(1):117–140, 1965.

[3] H Wu. A remark on holomorphic sectional curvature. Indiana University Mathematics Journal, 22(11):1103–1108, 1973.

[4] Fangyang Zheng. Complex differential geometry. Number 18 in AMS/IP Studies in Advanced Mathematics. Amer. Math. Soc., 2000.

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