Simple harmonic oscillation in non-Hermitian SSH chain at exceptional point

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The balance of gain and loss in an open system may maintain certain Hermitian dynamical behaviors, which can be hardly observed in a popular Hermitian system. In this paper, we systematically study a 1D $\mathcal{PT}$-symmetry non-Hermitian SSH model with open boundary condition based on exact approximate solution. We show that the long-wave length standing-wave modes can be achieved within the linear dispersion region when the system is tuned at the exceptional point (EP). The whole Hilbert space can be decomposed into two quasi-Hermitian subspaces, which are consisted of positive and negative energy levels, respectively. Within each subspace, the system maintains all the features of a Hermitian one. We construct a coherent-like state in a subspace and find that it exhibits perfect simple harmonic motion (SHM). In contrast to a canonical coherent state, the shape of the wavepacket deforms periodically rather than entirely translation. The amplitude of the SHM is not determined by the initial condition but the shape of the wavepacket. Our result indicates that novel Hermitian dynamics can be realized by a non-Hermitian system.

I. INTRODUCTION

The quantum harmonic oscillator as the quantum-mechanical analog of the classical harmonic oscillator is one of the most important model systems in quantum mechanics, not only due to its broad application, but also one of the few exactly solvable quantum system [1–4]. Coherent states (also called Glauber states) take an important role not only to the connection between quantum mechanics and classical mechanics [4], but also to the applications in many aspects.

Recently, non-Hermitian quantum mechanics [5–15] attracted increasing interest in several branches of physics [16–25]. Due to the reality of the spectrum in a non-Hermitian system, the connection to a Hermitian system is an interesting topic. A metric-operator method has been proposed to compose a Hermitian Hamiltonian, which has exactly the same real spectrum with the pseudo-Hermitian Hamiltonian [26]. From the Hermitian counterpart, one can extract the physical meaning of a pseudo-Hermitian Hamiltonian in the viewpoint of spectrum [27–30]. Alternatively, a connection between a non-Hermitian Hamiltonian and an infinite Hermitian system can be established in the viewpoint of eigen state [31–34].

It is found that a non-Hermitian system can exhibit conditional Hermitian dynamical behaviors, which preserves the Dirac probability due to the balance of gain and loss. Such dynamics is hard to realized in a non-trivial Hermitian system. Therefore, it provides a new way to prepare and control the quantum states in the field of quantum information and technology.

In this paper we present a pseudo-Hermitian system to demonstrate that a coherent state can be formed in a discrete system by introducing $\mathcal{PT}$-symmetry staggered imaginary potentials rather than a real parabolic potential as studied in Refs. [35]. The system is described by a 1D $\mathcal{PT}$-symmetry non-Hermitian SSH model with open boundary condition. The exact solution is obtained in the strong dimerization limit, which shows that the equal-level-spacing high-frequency standing-wave modes (EHSW) can be achieved when the system is tuned at the exceptional point (EP). The gapless system supports two quasi-Hermitian subspaces, which are consisted of positive and negative energy levels, respectively. A coherent state constructed in a subspace exhibits perfect simple harmonic motion (SHM). In contrast to a canonical coherent state, the shape of the wavepacket deforms periodically rather than entirely translation. The amplitude of the SHM is not determined by the initial condition but the shape of the wavepacket. Our result indicates that novel Hermitian dynamics can be realized by a non-Hermitian system.

The remainder of this paper is organized as follows. In Sec. II we present a non-Hermitian SSH chain model and the formulation of approximate diagonalization. Sec. III reveals the Hermitian dynamics in sub-spaces with positive and negative eigen energies. Sec. IV demonstrates a peculiar quantum state living in such a non-Hermitian gapless system. Finally, we present a summary and discussion in Sec. V.

II. MODEL AND SOLUTION

We consider a non-Hermitian SSH chain with staggered balanced gain and loss. The simplest tight-binding model with these features is

\[
H = (1 + \delta) \sum_{j=1}^{N} a_j^\dagger b_j + (1 - \delta) \sum_{j=1}^{N-1} b_{j+1}^\dagger a_j + \text{H.c.}
\]

\[+ i\gamma \sum_{j=1}^{N} (a_j^\dagger a_j - b_j^\dagger b_j), \tag{1}\]

where $\delta$ and $i\gamma$, are the distortion factor with unit tunneling constant and the alternating imaginary potential
magnitude, respectively. Here $a_i^\dagger$ and $b_i^\dagger$ are the creation operators of the particle at the $i$th site in $A$ and $B$ sub-lattices. The particle can be fermion or boson, depending on their own commutation relations. A sketch of the lattice is shown in Fig. 1(a). In the absence of the staggered potentials, the SSH model [39] has served as a paradigmatic example of the 1-D system supporting topological character [37]. It has an extremely simple form but well manifests the typical feature of topological insulating phase, and the transition between non-trivial and trivial topological phases, associated with the number of zero energy edge states as the topological invariant [38]. For nonzero $\gamma$, it is still a $\mathcal{PT}$-symmetry. Here, the time reversal operation $\mathcal{T}$ is such that $\mathcal{T} i \mathcal{T} = -i$, while the effect of the parity is such that $\mathcal{P} a_i^\dagger = b_{N+1-i}$ and $\mathcal{P} b_i = a_{N+1-i}$. Applying operators $\mathcal{P}$ and $\mathcal{T}$ on the Hamiltonian [1], one has $[\mathcal{T}, H] \neq 0$ and $[\mathcal{P}, H] \neq 0$, but $[\mathcal{PT}, H] = 0$. According to the non-Hermitian quantum theory, such a Hamiltonian may have fully real spectrum within a certain parameter region. The boundary of the region is the critical point of quantum phase transition associated with $\mathcal{PT}$-symmetry breaking. The system with periodic boundary condition has been studied in Ref. [39]. We will show that open boundary condition leads to different dynamical behavior.

According to the Appendix, in the strong dimerization limit $1 + \delta \gg 1 - \delta$, the Hamiltonian can be diagonalized as the form

$$H = \sum_k \varepsilon_k (\alpha_k \alpha_k^\dagger - \beta_k \beta_k^\dagger),$$

where the operators are

$$\alpha_k = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin (kj) \frac{a_j + \exp(-i\varphi_k) b_j^\dagger}{1 + i \exp(-i\varphi_k)},$$

$$\beta_k = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin (kj) \frac{a_j - \exp(i\varphi_k) b_j}{1 + i \exp(i\varphi_k)}.$$

and their counterparts are

$$\alpha_k = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin (kj) \frac{a_j + \exp(-i\varphi_k) b_j^\dagger}{1 + i \exp(-i\varphi_k)},$$

$$\beta_k = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin (kj) \frac{a_j - \exp(i\varphi_k) b_j}{1 + i \exp(i\varphi_k)}.$$

Here the dispersion relation and phase are

$$\varepsilon_k = \sqrt{[(1+\delta) - (1-\delta) \cos k]^2 - \gamma^2},$$

$$\tan \varphi_k = \frac{\varepsilon_k}{\gamma},$$

with $k = \frac{n+1}{N+1}$, $n = 0, 1, 2, 3, ..., N-1$.

The non-Hermitian operator $H$ in Eq. (2) is in diagonal form, since $\alpha_k$, $\alpha^\dagger_k$, $\beta_k$, and $\beta^\dagger_k$ are canonical conjugate operators, obeying the canonical commutation relations

$$[\alpha_k, \alpha_{k'}^\dagger] = \delta_{k, k'},$$

$$[\alpha_k, \alpha_k^\dagger] = \delta_{k, k'},$$

$$[\alpha^\dagger_k, \alpha^\dagger_{k'}] = [\beta_k, \beta_{k'}] = 0,$$

$$[\alpha_k, \beta_{k'}^\dagger] = [\alpha^\dagger_k, \beta_k] = [\beta^\dagger_k, \beta^\dagger_{k'}] = 0.$$

The system is pseudo-Hermitian since it can either have fully real spectrum or complex spectrum with complex conjugation pair imaginary levels. We emphasize that the canonical conjugate pairs appearing in Eqs. (9) are not simply defined by the Hermitian conjugate operation, i.e. $\alpha_k^\dagger = \alpha_k^\dagger$ and $\beta_k^\dagger = \beta_k^\dagger$, which is differ from that in a Hermitian regime. Although the Eqs. (9) is obtained by the approximation in the strong dimerization limit, it should be hold within all the range of parameter when the exact expression of operators ($\alpha_k$, $\alpha_k^\dagger$, $\beta_k$, $\beta_k^\dagger$) is applied.

We note that the spectrum $\varepsilon_k$ consists of two branches separated by an energy gap $\Delta = \sqrt{4\delta^2 - \gamma^2}$. Obviously, it displays a full real spectrum within the region of $4\delta^2 \geq \gamma^2$, which is refer to as unbroken $\mathcal{PT}$-symmetry breaking.

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**FIG. 1.** (Color online) (a) Schematic for the non-Hermitian SSH chain with $A$ (golden) and $B$ (blue) sub-lattices. Black thick and gray thin lines indicate the hopping between two nearest neighbor sites with amplitudes $(1+\delta)$ and $(1-\delta)$, respectively. Two sub-lattices have opposite site imaginary $\mathcal{PT}$-symmetry breaking. The system is pseudo-Hermitian since it can either have fully real spectrum or complex spectrum with complex conjugation pair imaginary levels. We emphasize that the canonical conjugate pairs appearing in Eqs. (9) are not simply defined by the Hermitian conjugate operation, i.e. $\alpha_k^\dagger = \alpha_k^\dagger$ and $\beta_k^\dagger = \beta_k^\dagger$, which is differ from that in a Hermitian regime. Although the Eqs. (9) is obtained by the approximation in the strong dimerization limit, it should be hold within all the range of parameter when the exact expression of operators ($\alpha_k$, $\alpha_k^\dagger$, $\beta_k$, $\beta_k^\dagger$) is applied.
FIG. 2. (Color online) Plots of profiles of Dirac norm of the coherent state from Eq. (19) with $|\alpha| = 0.1$ (black), $|\alpha| = 0.3$ (blue), and $|\alpha| = 0.5$ (red) with different phases (a) 0, (b) $\pi/2$ and (c) $\pi$. The same plot for the cases with $|\alpha| = 1$ (black), and $|\alpha| = 3$ (red). We can see that shape of the state for small $|\alpha|$ is similar to the canonical coherent state, while deforms strongly for large $|\alpha|$. The position of center of mass arises from the deformation of the wavepacket. (g) 3D plot of the center of mass of the wavepackets of the states as the function of $\alpha$ obtained from Eq. (20). It is periodic function of the phase of $\alpha$. (h) The center of mass of the wavepackets obtained from Eq. (20) with $|\alpha| = 0.5$ (black) and $|\alpha| = 3$ (red), which correspond to the black and white lines in (g). It indicates that the periodic function is sinusoidal wave for small $|\alpha|$ but tends to triangle wave as $|\alpha|$ increases. (i) Amplitudes of the center of mass as the function of $|\alpha|$. We see that it is linear for small $|\alpha|$, which is similar to the canonical coherent state. The parameters for the SSH chain are $N = 500$, $\delta = 0.9$ and $\gamma = 1.8$. Beyond this region, the imaginary eigenvalues appears and the $\mathcal{PT}$ symmetry of the corresponding eigenfunction is broken simultaneously according to the non-Hermitian quantum theory. Notice that, when the onset of the $\mathcal{PT}$ symmetry breaking begins, the band gap vanishes. We are interested in the system at the criti-
cal point, at which the gap close, separating two quantum phases, with full real and complex spectra, respectively. The system reaches the critical point when $\gamma$ takes $\gamma_c = 2\delta$. The zero modes corresponds to $k_c = \frac{\pi}{N+1}$, which results in $\tan \varphi_{k_c} \to \infty$ and

$$\begin{align*}
\alpha_{k_c} = \beta_{k_c} &= \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin (k_c j) \frac{a_j - ib_j}{2}, \\
{\bar{\alpha}}_{k_c} = {\bar{\beta}}_{k_c} &= \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin (k_c j) \frac{a_j^\dagger - ib_j^\dagger}{1 - e^{-i\varphi_{k_c}}},
\end{align*}$$

i.e., two zero modes become identical, where $\ket{0}$ is vacuum state of particle operators $(a_j, b_j)$. In contrast to a Hermitian system, two zero-energy eigen states coalesce at the EP point, rather than degeneracy point. At this situation, the spectrum around zero energy can be expressed as

$$\varepsilon_k \approx \sqrt{2\delta(1-\delta)k},$$

which is linear in $k$. The spectra of the lattice is shown in Fig. 3(a).

### III. HERMITIAN DYNAMICS IN SUB-SPACE

In this section, we investigate the dynamics in the system within unbroken region. For simplicity, we only focus on single-particle invariant space. The obtained result can be extended to the many-particle invariant subspace due to the canonical commutation relations in Eqs. (9). According to the non-Hermitian quantum theory, the eigenstates of a pseudo-Hermitian Hamiltonian can construct a set of biorthogonal bases in association with the eigenstates of its Hermitian conjugate. Similar to the works [34][39], eigenstates $\{\bar{\alpha}_k | 0\}, \{\bar{\beta}_k | 0\}$ of $H$ and eigenstates $\{\alpha_k^\dagger | 0\}, \{\beta_k^\dagger | 0\}$ of $H^\dagger$ are the biorthogonal bases of the single-particle invariant subspace. In general, the eigenstates of a non-Hermitian Hamiltonian are not orthogonal under the Dirac inner product due to the non-Hermiticity of the Hamiltonian. However, we note that orthogonality between the eigenstates with different $k$ in the Dirac inner product still maintain due to following quasi-canonical commutation relations

$$\begin{align*}
[-\beta_k, \beta_{k'}^\dagger] &= \frac{1 - e^{i\varphi_{k'}}}{2(1 + \sin \varphi_k)} \delta_{kk'}, \\
[-\beta_k, \beta_{k'}] &= \frac{1 - e^{i\varphi_k}}{2(1 - \sin \varphi_k)} \delta_{kk'}, \\
[-\beta_k, \beta_{k'}^\dagger] &= \frac{1 - e^{i\varphi_{k'}}}{2(1 - \sin \varphi_k)} \delta_{kk'}, \\
[-\beta_k, \beta_{k'}] &= \frac{1 - e^{i\varphi_{k'}}}{2(1 + \sin \varphi_k)} \delta_{kk'}, \\
[-\beta_k, \beta_{k'}^\dagger] &= [\beta_k, \beta_{k'}] = 0,
\end{align*}$$

We use the term quasi due to the absence of orthogonality between the eigenmodes of $\bar{\alpha}_k$ and $\bar{\beta}_k$. We note that if one only consider the particle with positive energy (particle) or particle with negative energy (hole), the non-Hermitian system appears as a Hermitian one due to the commutation relations for two particles (or holes). This is crucial for this work.

### IV. SIMPLE HARMONIC OSCILLATION

In this section, we investigate the dynamics in the gapless system. The single-particle eigen vectors can be expressed as

$$\ket{\psi_n^+} = \frac{1}{\sqrt{N+1}} \sum_{j=1}^{N} (-1)^j \sin (k_j) \left(a_j^\dagger \pm e^{i\varphi_{k}} b_j^\dagger\right) \ket{0},$$

where $k$ is redefined as

$$k = \frac{(n+1)\pi}{N+1}, \quad n \in [0, N-1].$$

The corresponding eigen energy is

$$E_n^\pm = \pm (n+1)\omega, \quad \omega = \frac{\sqrt{2\delta(1-\delta)}\pi}{N+1},$$

approximately for not large $n$. Here $|\psi_n^\pm\rangle$ is normalized in the framework of Dirac inner product, i.e., $\langle \psi_m^+ | \psi_n^+ \rangle = \delta_{mn}$. We note that $|\psi_n^0\rangle = |\psi_n^-\rangle$ is the coalescing eigenstates and $\langle \psi_n^+ | \psi_m^- \rangle \neq 0$. When we focus on the positive energy region, it is similar to the light field mode in a rectangular cavity, standing wave with uniform energy-level spacing. In the rest of paper, we concentrate on the positive energy Hilbert space and denote $|\psi_n\rangle = |\psi_n^+\rangle$.

Following the idea for a quantum simple harmonic oscillator system, we construct a coherent-like state

$$|\alpha\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\psi_n\rangle.$$
FIG. 3. (Color online) Plots of numerical simulations for the profiles of Dirac norm of the time evolution for initial coherent states with several typical $\alpha$ for the SSH chain with parameters $N = 150$, $\delta = 0.9$ and $\gamma = 1.8$. (a) $\alpha = 0.5$. (b) $\alpha = 1$, (c) $\alpha = 3$, (d) $\alpha = 4$, (e) $\alpha = 6$, and (f) $\alpha = 12$, respectively. The white solid lines indicate the center of mass of the wavepackets. It shows that the trajectory is sinusoidal wave for small $\alpha$, while close to triangle wave for large $\alpha$. Here the time is in units of $20J^{-1}$, where $J$ is the scale of the Hamiltonian and we take $J = 1$.

We define the number operator $\hat{n}$ by

$$\hat{n} |\psi_n\rangle = n |\psi_n\rangle,$$  \hspace{1cm} (17)

and find that

$$\bar{n} = \langle \alpha | \hat{n} | \alpha \rangle = |\alpha|^2,$$ \hspace{1cm} (18)

which indicates the significance of $\alpha$. The position state $|l\rangle$ with $l \in [1,2N]$ is

$$|2j-1\rangle = a_j^\dagger |0\rangle, \quad |2j\rangle = b_j^\dagger |0\rangle,$$

with $j \in [1,N]$. The profile of distribution of Dirac probability of the state can be obtained as

$$P_D (l) = |\langle l | \alpha \rangle|^2,$$ \hspace{1cm} (19)

and the center of mass of the wavepacket is

$$r_c = \sum_{l=1}^{2N} l |\langle l | \alpha \rangle|^2.$$ \hspace{1cm} (20)

To demonstrate the property of coherent-like state, we plot Eqs. \hspace{1cm} (19) and \hspace{1cm} (20) for some typical $\alpha$ in Fig. 2. It shows that $r_c$ is determined by the phase $\text{arg}(\alpha)$ and the amplitude $|\alpha|$ of $\alpha$. The detailed features of the state can be understood by the following analysis since the phase of $\alpha$ is equivalent to the phase factor arising from the time evolution.

Now we turn to the dynamics of the state $|\alpha\rangle$. Under the strong dimerization approximation, the evolved state is

$$U(t) |\alpha\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i(n+1)\omega t} |n\rangle$$

$$= e^{-|\alpha|^2 - i\omega t} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle,$$ \hspace{1cm} (21)

which shows that factor $\omega t$ takes the role of the phase of $\alpha$. Then the trajectory of the centroid

$$r_c(t) = \sum_{l=1}^{2N} l |\langle l | U(t) |\alpha\rangle|^2.$$ \hspace{1cm} (22)
can be estimated directly from above analysis. In the Appendix, we get the following results: (i) In the small $|\alpha|$ limit, we have

$$r_c(t) \approx -\frac{64N}{9\pi^2} |\alpha| \cos(\omega t - \arg \alpha) + N, \quad (23)$$

which is a sinusoidal wave. (ii) In the $|\alpha| \gg 1$ limit, we have

$$r_c(t) = 2N \left( \frac{\omega t}{\pi} - 2n \right) \times \left\{ \begin{array}{ll} 1, & t \in [0, \frac{\pi}{2}) + \frac{2n\pi}{\omega}, n \in \mathbb{Z} \\ -1, & t \in [-\frac{\pi}{2}, 0) + \frac{2n\pi}{\omega} \end{array} \right., \quad (24)$$

which is a triangle wave. It accords with the plots in Fig. 2.

All the above analysis are based on the assumption of the strong dimerization limit. We are interested in the case with moderate value of $\delta$. To verify our result and demonstrate the extent to which the harmonic oscillation also exists, we perform by exact diagonalization for finite-size SSH chain with different $\alpha$. Plots in Fig. 3 is the profile of Dirac probability distribution of evolved state for several typical cases. It shows that the trajectory is sinusoidal wave for small $\alpha$, while close to triangle wave for large $\alpha$. Surprisingly, the trajectory is always smooth even the wavepacket spreads out in the chain.

V. SUMMARY

In summary, we studied the dynamics of non-Hermitian SSH chain at EP. We have shown that a pre-engineered state can exhibit perfect SHM. The underlying mechanism of such dynamic behavior are based on two conditions (EHSM): (i) linear dispersion relation and (ii) long-wave length standing-wave modes. Such two features even appear in a simple uniform chain ($H$ with $\delta = \gamma = 0$) but not coexist in a same set of eigenstates. It is hard to design a Hermitian tight-binding chain satisfying such two conditions. This fact highlights the advantage of the non-Hermitian system. In the present system, the balance of distortion and staggered imaginary potentials leads to a set of eigenstates possessing such two features. It is also associated with the concept of quasi-Hermitian sub-space. If one only consider the particle with positive energy (particle) or particle with negative energy (hole), the non-Hermitian system appears as a Hermitian one, preserving the Dirac probability. This feature also appears in a ring system [39], in which a wavepacket moves periodically but with constant speed. The peculiar dynamical behavior is a demonstration of rich potential resource of the non-Hermitian system at EP. Our result indicates that novel Hermitian dynamics can be realized by a non-Hermitian system.

VI. APPENDIX

A. Approximate solutions

We start with the Hermitian Hamiltonian

$$H_0 = (1 + \delta) \sum_{j=1}^{N} a_j^\dagger b_j + (1 - \delta) \sum_{j=1}^{N-1} b_j^\dagger a_{j+1} + H.c. \quad (25)$$

Introducing particle operators

$$a_j^\dagger = \frac{1}{\sqrt{2}}(a_j^\dagger + b_j), \beta_j^\dagger = \frac{1}{\sqrt{2}}(a_j^\dagger - b_j), \quad (26)$$

or inversely

$$a_j^\dagger = \frac{1}{\sqrt{2}}(a_j^\dagger + \beta_j^\dagger), b_j^\dagger = \frac{1}{\sqrt{2}}(a_j^\dagger - \beta_j^\dagger), \quad (27)$$

we have

$$a_j^\dagger b_j + b_j^\dagger a_j = a_j^\dagger \alpha_j - \beta_j^\dagger \beta_j, \quad (28)$$

$$b_j^\dagger a_{j+1} + a_j^\dagger b_{j+1} = \frac{1}{2}(a_j^\dagger a_{j+1} - \beta_j^\dagger \beta_{j+1} + 1) \alpha_j \beta_{j+1} + \frac{1}{2} \alpha_j^\dagger \beta_{j+1} \beta_j + \frac{1}{2} \beta_j^\dagger \alpha_{j+1} + H.c.. \quad (29)$$

Under the condition

$$1 + \delta \gg 1 - \delta, \quad (30)$$

we have

$$b_j^\dagger a_{j+1} + a_j^\dagger b_{j+1} \approx \frac{1}{2}(a_j^\dagger a_{j+1} - \beta_j^\dagger \beta_{j+1}) + H.c., \quad (31)$$

neglecting the transition terms between sites with opposite potentials. Then we have

$$H_0 = \frac{1}{2}(1 - \delta) \sum_{j=1}^{N-1} \left( a_j^\dagger a_{j+1} - \beta_j^\dagger \beta_{j+1} \right) + H.c.
+ (1 + \delta) \sum_{j=1}^{N} (a_j^\dagger a_j - \beta_j^\dagger \beta_j). \quad (32)$$

The original system reduces to two independent uniform chains with opposite chemical potentials. It can be diagonalized as

$$H_0 = \varepsilon_k^0 (\alpha_k^\dagger \alpha_k - \beta_k^\dagger \beta_k), \quad (33)$$

by taking the linear transformations

$$\alpha_k = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin \frac{n\pi j}{N+1} a_j, \quad (34)$$

$$\beta_k = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin \frac{n\pi j}{N+1} \beta_j, \quad (35)$$
where the real spectrum
\[ \varepsilon_k^0 = (1 + \delta) - (1 - \delta) \cos k, \]
(36)
with \( k = \frac{(n+1)\pi}{N+1}, \) \( n = 0, 1, 2, 3, ..., N - 1. \)
Now we turn to the non-Hermitian Hamiltonian
\[ H = H_0 + i\gamma \sum_{j=1}^{N} (a_j^\dagger a_j - b_j^\dagger b_j), \]
(37)
According to the result from Ref. [40], we still have the similar form
\[ H = \sum_k \varepsilon_k (\alpha_k \alpha_k^\dagger - \beta_k^\dagger \beta_k), \]
(38)
with
\[ \varepsilon_k = \sqrt{\left(\varepsilon_k^0\right)^2 - \gamma^2}. \]
(39)
Here the operators are defined as
\[ \alpha_k = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin (kj) A_{k,j}, \]
(40)
\[ \beta_k = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin (kj) B_{k,j}, \]
(41)
and
\[ \alpha_k^\dagger = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin (kj) \overline{A}_{k,j}, \]
(42)
\[ \beta_k^\dagger = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} (-1)^j \sin (kj) \overline{B}_{k,j}, \]
(43)
where
\[ A_{k,j} = \frac{a_j + e^{-i\varphi_k} b_j}{1 + ie^{-i\varphi_k}}, \quad B_{k,j} = \frac{a_j - e^{i\varphi_k} b_j}{1 + ie^{i\varphi_k}}, \]
(44)
\[ \overline{A}_{k,j} = \frac{a_j^\dagger + e^{-i\varphi_k} b_j^\dagger}{1 - ie^{-i\varphi_k}}, \quad \overline{B}_{k,j} = \frac{a_j^\dagger - e^{i\varphi_k} b_j^\dagger}{1 - ie^{i\varphi_k}}, \]
(45)
and
\[ \tan \varphi_k = \frac{\gamma}{\varepsilon_k}. \]
(46)
Such relations can be directly obtained by the following relations
\[ [A_{k,j}, A_{k,l}]_\pm = [B_{k,j}, B_{k,l}]_\pm = \delta_{jl}, \]
\[ [A_{k,j}, A_{k,l}]_\pm = [B_{k,j}, B_{k,l}]_\pm = 0, \]
\[ [\overline{A}_{k,j}, \overline{A}_{k,l}]_\pm = [\overline{B}_{k,j}, \overline{B}_{k,l}]_\pm = 0, \]
\[ [A_{k,j}, B_{k,l}]_\pm = [\overline{A}_{k,j}, \overline{B}_{k,l}]_\pm = 0, \]
(48)
which are always held without depending on the form of \( \varphi_k. \)

### B. Wavepacket trajectory

We estimate the wavepacket trajectory in two limit cases with \(|\alpha| \ll 1\) and \(|\alpha| \gg 1\). In the first case, the initial state is approximately expressed as
\[ |\psi(0)\rangle = \frac{1}{\sqrt{1 + |\alpha|^2}} (|0\rangle + \alpha |1\rangle), \]
(49)
so
\[ |\psi(t)\rangle = U(t) |\psi(0)\rangle = \frac{1}{\sqrt{1 + |\alpha|^2}} e^{-i\omega t} (|0\rangle + \alpha e^{-i\omega t} |1\rangle), \]
(50)
where
\[ |n\rangle = \sqrt{\frac{2}{1+N}} \sum_{j=1}^{N} \sin \left( \frac{n+1}{N+1} \pi j \right) |j\rangle, \]
(51)
then we have
\[ r_c(t) = \sum_{l=1}^{N} |\langle l |\psi(t)\rangle|^2 \]
\[ = -8 \frac{|\alpha| \cos \left( \frac{\pi}{1+N} \right) \cot \left( \frac{\pi}{2+2N} \right) \cos(\omega t - \arg \alpha)}{(1 + |\alpha|^2)(1 + N)[1 + 2 \cos \left( \frac{\pi}{1+N} \right)]^2} \]
\[ + \frac{1}{2} (1 + N), \]
(52)
in the small \(|\alpha|\) and large \(N\) limit, by using
\[ \lim_{N \to \infty} \frac{2 \cos \left( \frac{\pi}{1+N} \right) \cot \left( \frac{\pi}{2+2N} \right)}{(1 + N)^2[1 + 2 \cos \left( \frac{\pi}{1+N} \right)]^2} = \frac{8}{9\pi^2}, \]
(53)
we get the wavepacket trajectory
\[ r_c(t) \approx -\frac{32N}{9\pi^2} |\alpha| \cos(\omega t - \arg \alpha) + \frac{N}{2}, \]
(54)
Secondly, in the large \(\alpha = |\alpha|\) limit, we have
\[ |\alpha(t)\rangle \approx \left( \frac{1}{\alpha \sqrt{2\pi}} \right)^{1/2} N^{-1} \sum_{n=0}^{N-1} e^{-\left(\frac{n-\alpha^2}{4\alpha^2}\right)^2} e^{-i(n+1)\omega t} |n\rangle, \]
(55)
The wavepacket trajectory is

$$r_c(t) = \sum_{j=1}^{N} jP(t,j), \quad (56)$$

with

$$P(t,j) \approx \frac{\sum_{n=0}^{N-1} 2e^{-(n-\alpha^2)/4\alpha^2}e^{-in\omega t}}{\sqrt{\alpha^2N \sqrt{2\pi}}} \sin\left(\frac{n\pi j}{N}\right)^2. \quad (57)$$

The oscillating terms have no contribution to the summation, leading to

$$P(t,j) = \begin{cases} 1, & j = \pm \frac{N\omega t}{\pi} \\ 0, & \text{otherwise} \end{cases}. \quad (58)$$

Then we have

$$r_c(t) = \frac{N(\omega t - 2n\pi)}{\pi} \begin{cases} 1, & t \in (0, \frac{\pi}{\omega}) + \frac{2n\pi}{\omega}, n \in Z \\ -1, & t \in (-\frac{\pi}{\omega}, 0) + \frac{2n\pi}{\omega} \end{cases}, \quad (59)$$

which represents a triangular wave with period $\frac{2\pi}{\omega}$.

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