Black String Solutions with Arbitrary Tension

Chul H. Lee *

Department of Physics, and BK21 Division of Advanced Research and Education in Physics,
Hanyang University, Seoul 133-791, Korea

Abstract

We consider 1 + 4 dimensional black string solutions which are invariant under translation along the fifth direction. The solutions are characterized by the two parameters, mass and tension, of the source. The Gregory-Laflamme solution is shown to be characterized by the tension whose magnitude is one half of the mass per unit length of the source. The general black string solution with arbitrary tension is presented and its properties are discussed.

*e-mail: chulhoon@hanyang.ac.kr
I. INTRODUCTION

Spacetimes of dimensions higher than 1+3 have become objects for serious consideration in physics as some physical theories such as the string theory and brane cosmology are necessarily formulated in those higher dimensional spacetimes. In exploring the existence of extra dimensions, the higher dimensional black hole solution can be a useful tool. Along with the higher dimensional black hole, another interesting object is the black string which is obtained by extending the 1 + 3 dimensional black hole to the extra dimensions. The simplest case, first studied by Gregory and Laflamme [1], is the black string obtained by extending uniformly the Schwarzschild black hole to the fifth dimension. Its metric is given by

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]
\[ = -(1 - \frac{\rho_0}{r}) dt^2 + \frac{dr^2}{1 - \frac{\rho_0}{r}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 + dz^2. \]  

(1)

Also Chamblin, Hawking and Real [3] discussed black strings in the models of Randall and Sundrum where our universe is viewed as a domain wall in the 5-dimensional anti-de Sitter space.

The metric in Eq. (1) was shown in Ref. [2] to be unstable under small perturbations. The numerical analysis there showed the existence of unstable modes for a range of time frequency and wavelength. The possibility of the unstable black string finally fragmenting into black holes was mentioned. However, Horowitz and Maeda [4] argued that event horizons could not pinch off. So they conjectured that the unstable black string solution would evolve to settle down to a new type of black string solution which is not invariant under translation along the string. This prompted many works on nonuniform black string solutions (see [5] [6] for recent reviews). However the discussions so far have been limited to perturbative or numerical analyses; no analytic solutions have been found.

In this work we search for alternative uniform black string solutions other than that of Eq. (1). We first note that, as was discussed in [7], [8], two independent asymptotic
quantities are needed to characterize the leading correction to the static metric far away from the source on space $R^{d-1} \times S^1$. One of the quantities is the mass and the other is the tension of the source. The metric in Eq. (1) turns out to be, as is shown in the following, the one produced by a source characterized by the tension of the particular value of one half of the mass per unit length of the source.

Let us restrict our discussion to cases of 1 + 4 dimensional spaces with the coordinates $x^0 = t$, $x^i (i = 1, 2, 3)$, and $x^4 = z$. For the weak gravitational field produced by stationary $z$-independent weak source, the linearized Einstein equation in harmonic coordinates is

$$\partial_i \partial^i h_{\mu\nu} = -16\pi G_5 T_{\mu\nu}$$

where

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \ (|h_{\mu\nu}| << 1),$$

$$T_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} T$$

and $G_5$ is the 5-dimensional gravitational constant. Using the Green’s function for the three dimensional Laplacian, the solution of Eq. (2) can be written by

$$h_{\mu\nu}(x) = 4G_5 \int d^3 y \frac{T_{\mu\nu}(y)}{|x - y|}$$

With the expansion $\frac{1}{|x - y|} = \frac{1}{r} + \frac{x^i y^i}{r^3} + \cdots$ where $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, the leading terms of $h_{\mu\nu}(x)$ are then calculated to be

$$h_{00}(x) \simeq \frac{4G_5}{r} \int d^3 y \left( \frac{2}{3} T_{00} + \frac{1}{3} T_{44} \right)$$

$$h_{0i}(x) \simeq 2G_5 \frac{x^j}{r^3} \int d^3 y (y^j T^{0j} - y^j T^{0i})$$

$$h_{04}(x) \simeq \frac{4G_5}{r} \int d^3 y T_{04}$$

$$h_{ij}(x) \simeq \frac{4G_5}{3} \frac{\delta_{ij}}{r} \int d^3 y (T_{00} - T_{44})$$

$$h_{i4}(x) \simeq 2G_5 \frac{x^j}{r^3} \int d^3 y (y^j T^{4i} - y^j T^{4j})$$

$$h_{44}(x) \simeq \frac{4G_5}{r} \int d^3 y \left( \frac{1}{3} T_{00} + \frac{2}{3} T_{44} \right)$$

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Here the relations $\int d^3 y T^{i\mu} = 0$ and $\int d^3 y y^i T^{\mu j} = - \int d^3 y y^j T^{\mu i}$, which are derived from the energy-momentum conservation $\partial_\nu T^{\mu \nu} = \partial_i T^{\mu i} = 0$ (and $\partial_k (x^i T^{\mu k}) = T^{\mu i}$, $\partial_k (x^i x^j T^{\mu k}) = x^j T^{\mu i} + x^i T^{\mu j}$) are used. For the static case with $T^{0i} = T^{04} = 0$, the leading corrections to the metric far away from the source, up to the order of $1/r$, can be seen to be characterized by the two quantities

$$\lambda \equiv \int d^3 x T_{00}, \quad \tau \equiv - \int d^3 x T_{44}.$$  \hspace{1cm} (7)

$\lambda$ is the mass per unit length and $\tau$ is the tension along the $z$-direction of the source. Then, up to the order of $1/r$,

$$h_{00}(x) \simeq \frac{4G_5}{3} \frac{2\lambda - \tau}{r},$$  
$$h_{ij}(x) \simeq \frac{4G_5}{3} \delta_{ij} \frac{\lambda + \tau}{r},$$  
$$h_{44}(x) \simeq \frac{4G_5}{3} \frac{\lambda - 2\tau}{r}.$$  \hspace{1cm} (8)

and all other components are zero. If the direction $z$ is periodic with $0 \leq z < L$, the four dimensional gravitational constant $G_4$ is given by $G_4 = \frac{G_5}{L}$ and the total mass of the source is $M = \lambda L$.

In section II, we show that the source of the Gregory-Laflamme metric is characterized by the tension whose magnitude is equal to one half of the mass per unit length. A general class of black string solutions with arbitrary tension is presented in section III. The spacetime properties of the new solutions are discussed in section IV.

II. THE CHARACTERISTICS OF THE SOURCE OF THE GREGORY-LAFLAMME METRIC

In order to examine the characteristics of the source that produces the metric given in Eq. (1), one needs to reexpress it in harmonic coordinates. We first replace $r$ with $\rho$ defined by

$$r \equiv \rho (1 + \frac{r g}{4\rho})^2$$  \hspace{1cm} (9)
Then the metric is rewritten by
\[ ds^2 = \frac{(1 - \frac{r_g}{4\rho})^2}{(1 + \frac{r_g}{4\rho})^2} dt^2 + (1 + \frac{r_g}{4\rho})^4 (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2) + dz^2 \] (10)

And finally with the definition
\[ x^1 = \rho \sin \theta \cos \phi, \quad x^2 = \rho \sin \theta \sin \phi, \quad x^3 = \rho \cos \theta \] (11)

the metric becomes
\[ ds^2 = \frac{(1 - \frac{r_g}{4\rho})^2}{(1 + \frac{r_g}{4\rho})^2} dt^2 + (1 + \frac{r_g}{4\rho})^4 \delta_{ij} dx^i dx^j + dz^2 \] (12)

A straightforward calculation shows that this form of the metric satisfies the linearized harmonic coordinate condition, \( \partial_\mu h^\mu - \frac{1}{2} \partial_\mu h = 0 \), up to the second order in \( \frac{1}{\rho} \). The leading corrections of the metric in the asymptotic region can now be read to be

\[ h_{00} = \frac{r_g}{\rho}, \quad h_{ij} = \frac{r_g}{\rho} \delta_{ij}, \quad h_{44} = 0 \] (13)

Comparing Eq. (13) with Eq. (8) gives
\[ \lambda = \frac{r_g}{2G_5}, \quad \tau = \frac{r_g}{4G_5} \] (14)

Therefore one can conclude that the source of the Gregory-Laflamme metric is characterized by the tension whose magnitude is one half of the mass per unit length (\( \tau = \frac{1}{2} \lambda \)).

III. A GENERAL CLASS OF BLACK STRING SOLUTIONS

We now search for the metric whose source has some arbitrary tension. Let us say the tension and the mass per unit length of the source are related by
\[ \tau = a \lambda \] (15)

where \( a \) is some arbitrary constant. Then we know, from Eq. (8), that the leading corrections of the metric far away from this source are
\[
    h_{00} = \frac{B(2 - a)}{\rho}, \quad h_{ij} = \delta_{ij} \frac{B(1 + a)}{\rho}, \quad h_{44} = \frac{B(1 - 2a)}{\rho}
\]

where \( B = \frac{4G_5 \lambda}{3} \). That is, the asymptotic form of the metric is

\[
    ds^2 \approx -(1 - \frac{B(2 - a)}{\rho}) dt^2 + (1 + \frac{B(1 + a)}{\rho})(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2) + (1 + \frac{B(1 - 2a)}{\rho}) dz^2
\]

(17)

We finally have to find the solutions of the vacuum Einstein field equations which reduce to the asymptotic form of Eq. (17) at large \( \rho \). We start with the ansatz

\[
    ds^2 = -F(\rho) dt^2 + G(\rho)(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2) + H(\rho) dz^2,
\]

(18)

and substitute this form of metric into the vacuum Einstein field equations to derive differential equations for the three functions \( F(\rho) \), \( G(\rho) \) and \( H(\rho) \). The solutions turn out to be

\[
    F = (1 - \frac{K_a}{\rho})^s (1 + \frac{K_a}{\rho})^{-s}
\]

\[
    G = (1 - \frac{K_a}{\rho})^{2 - \frac{1 + a}{2 - a} s} (1 + \frac{K_a}{\rho})^{2 + \frac{1 + a}{2 - a} s}
\]

\[
    H = (1 - \frac{K_a}{\rho})^{-\frac{1 - 2a}{2 - a} s} (1 + \frac{K_a}{\rho})^{\frac{1 - 2a}{2 - a} s}
\]

(19)

where

\[
    s = \frac{2(2 - a)}{\sqrt{3}(1 - a + a^2)}, \quad K_a = \sqrt{1 - a + a^2} G_5 \lambda
\]

(20)

For \( a = \frac{1}{2} \), these solutions in Eq’s (19) and (20) can be seen to give Eq. (10), the Gregory-Laflamme metric, with \( r_g = 4K_{1/2} = 2G_5 \lambda = 2G_4 M \). For \( a = 0 \) (no tension), the metric becomes

\[
    ds^2 = -(1 - \frac{K_a}{\rho})^{4/\sqrt{3}} dt^2 + (1 - \frac{K_0}{\rho})^{2 - \frac{2}{\sqrt{3}}} (1 + \frac{K_0}{\rho})^{2 + \frac{2}{\sqrt{3}}} (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2) + (1 + \frac{K_0}{\rho})^{2/\sqrt{3}} dz^2
\]

(21)

with \( K_0 = \frac{1}{\sqrt{3}} G_5 \lambda \). For \( a = 1 \), the metric becomes
\[ ds^2 = -\frac{(1 - \frac{K_1}{\rho})^{2/\sqrt{3}}}{(1 + \frac{K_1}{\rho})^{2/\sqrt{3}}} dt^2 + \frac{(1 + \frac{K_1}{\rho})^{\frac{4\sqrt{3}}{-2}}}{(1 - \frac{K_1}{\rho})^{\frac{4\sqrt{3}}{-2}}} (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2) + \frac{(1 - \frac{K_1}{\rho})^{2/\sqrt{3}}}{(1 + \frac{K_1}{\rho})^{2/\sqrt{3}}} dz^2 \]

with \( K_1 = \frac{1}{\sqrt{3}}G_5\lambda \).

**IV. SPACETIME PROPERTIES OF THE NEW SOLUTIONS**

Using, instead of \( \rho \), a new coordinate \( r \) defined by

\[ r = \rho G^{1/2} \]

the metric in Eq. (18) is transformed to

\[ ds^2 = -F dt^2 + \frac{dr^2}{(1 + \frac{r}{2G_d} \frac{dG}{d\rho})^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + Hz^2 \]  

One can see that the event horizon is formed at the surface satisfying the condition

\[ 1 + \frac{\rho}{2G_d} \frac{dG}{d\rho} = 0, \]

or explicitly

\[ \left( \frac{\rho}{K_a} - \frac{1 + a}{\sqrt{3}(1 - a + a^2)} \right)^2 + \frac{(1 - 2a)(2 - a)}{3(1 - a + a^2)} = 0 \]  

For \( a = 1 \), Eq. (26) becomes

\[ \left( \frac{\rho}{K_1} - \frac{2}{\sqrt{3}} \right)^2 - \frac{1}{3} = 0, \]

and the solution is

\[ \rho = \sqrt{3}K_1 = G_5\lambda \]

or

\[ r = \frac{(1 + \frac{1}{\sqrt{3}})^{\frac{4\sqrt{3}}{-1} + 1}}{(1 - \frac{1}{\sqrt{3}})^{\frac{4\sqrt{3}}{-1}}} G_5\lambda \]
For $a = \frac{1}{2}$, Eq. (26) becomes

\[
\left( \frac{\rho}{K_{1/2}} - 1 \right)^2 = 0
\]

(30)

with $K_{1/2} = \frac{1}{2} G_5 \lambda$, and the solution is

\[
\rho = K_{1/2} = \frac{1}{2} G_5 \lambda
\]

(31)

or

\[
r = 4 K_{1/2} = 2 G_5 \lambda
\]

(32)

For $a = 0$, Eq. (26) becomes

\[
\left( \frac{\rho}{K_0} \right) - \frac{1}{\sqrt{3}})^2 + \frac{2}{3} = 0,
\]

(33)

and we see there is no solution to it. Therefore no event horizon is formed in this case. In fact, for $a < \frac{1}{2}$

\[
\frac{(1 - 2a)(2 - a)}{3(1 - a + a^2)} > 0,
\]

(34)

and there is no solution to Eq. (26). We conclude that the event horizon is formed only when the tension is larger than or equal to one half of the mass per unit length of the source.

We now concentrate on the case of no tension ($a = 0$) in the source string. In this case, as was shown above, no event horizon is formed. However there appears a spacetime singularity at $\rho = K_0 (r = 0)$. This conclusion is confirmed by showing the existence of singular components of the Riemann tensor with respect an orthonormal frame. With the basis vectors of the orthonormal frame taken to be

\[
e_t^\mu = \frac{(1 + K_0 \rho)}{(1 - K_0 \rho)^{1/\sqrt{3}}} \delta_t^\mu, \quad e_\rho^\mu = \frac{\delta_\rho^\mu}{(1 - K_0 \rho)^{1-\sqrt{3}}(1 + K_0 \rho)^{1+\sqrt{3}}}, \quad e_\theta^\mu = \frac{\delta_\theta^\mu}{\rho(1 - K_0 \rho)^{1-\sqrt{3}}(1 + K_0 \rho)^{1+\sqrt{3}}},
\]

\[
e_\phi^\mu = \frac{\delta_\phi^\mu}{\rho \sin \theta (1 - K_0 \rho)^{1-\sqrt{3}}(1 + K_0 \rho)^{1+\sqrt{3}}}, \quad e_z^\mu = \frac{(1 - K_0 \rho)^{1/\sqrt{3}}}{(1 + K_0 \rho)^{1/\sqrt{3}}} \delta_z^\mu,
\]

(35)

a straightforward calculation gives
They are indeed singular at $\rho = K_0 (r = 0)$. Here we seen an explicit example in which the cosmic censorship hypothesis is not valid in the higher dimensional spacetime.

The metric of Eq. (21) is the one generated by a string source with no tension located at $\rho = K_0 (r = 0)$. The relevant range of the coordinate $\rho (r)$ is $K_0 \leq \rho < \infty \ (0 \leq r < \infty)$. When the source string has a finite thickness, the location $\rho = K_0 (r = 0)$ is within the source and the spacetime singularity does not materialize.

We leave the similar discussions for the general case of no event horizon $(a < \frac{1}{2})$ for further studies. Also the stability of the solution given in Eq’s (18), (19) and (20) is an important issue of further investigation.

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REFERENCES

[1] R. Gregory and R. Laflamme, Phys. Rev. D 37, 305 (1988)

[2] R. Gregory and R. Laflamme, Phys. Rev. Lett. 70, 2837 (1993)

[3] A. Chamblin, S. Hawking and H. Reall, Phys. Rev. D 61, 065007 (2000)

[4] G. Horowitz and K. Maeda, Phys. Rev. D 87, 131301 (2001)

[5] B. Kol, Phys. Rep. 422, 119 (2006)

[6] T. Hanmark and N. Obers, hep-th/0503020

[7] T. Hanmark and N. Obers, Class. Quan. Grav. 21, 1709 (2004)

[8] B. Kol, E. Sorkin and T. Piran, Phys. Rev. D 69, 064031 (2004)