DYNAMIC SELECTION OF P-NORM IN LINEAR ADAPTIVE FILTERING VIA ONLINE KERNEL-BASED REINFORCEMENT LEARNING

Minh Vu  
Yuki Akiyama  
Konstantinos Slavakis

Tokyo Institute of Technology, Japan
Department of Information and Communications Engineering
Emails: {vu.d.aa, akiyama.y.am, slavakis.k.aa}@m.titech.ac.jp

ABSTRACT

This study addresses the problem of selecting dynamically, at each time instance, the “optimal” p-norm to combat outliers in linear adaptive filtering without any knowledge on the potentially time-varying probability density function of the outliers. To this end, an online and data-driven framework is designed via kernel-based reinforcement learning (KBRL). Novel Bellman mappings on reproducing kernel Hilbert spaces (RKHSs) are introduced that need no knowledge on transition probabilities of Markov decision processes, and are nonexpansive with respect to the underlying Hilbertian norm. An approximate policy-iteration framework is finally offered via the introduction of a finite-dimensional affine superset of the fixed-point set of the proposed Bellman mappings. The well-known “curse of dimensionality” in RKHSs is addressed by building a basis of vectors via an approximate linear dependency criterion. Numerical tests on synthetic data demonstrate that the proposed framework selects the “optimal” p-norm for the outlier scenario at hand at every time instance, outperforming several non-RL and KBRL schemes.

1. INTRODUCTION

The least-squares (LS) error/loss between an observed value and its predicted one plays a pivotal role in signal processing, e.g., adaptive filtering [1], and machine learning [2]. For example, the least-mean squares (LMS) and recursive (RLS) [1] are two celebrated algorithms in adaptive filtering and stochastic approximation which are based on the LS-error criterion. Notwithstanding, LS methods are notoriously sensitive to the presence of outliers within data [3], where outliers are defined as (sparsely) contaminating data that do not adhere to a nominal data generation model, and are often modeled as random variables (RVs) with non-Gaussian heavy tailed distributions, e.g., α-stable ones [4, 5]. To combat the negative effects of outliers, several non-LS criteria, such as least mean p-power (LMP) [6–12] and maximum correntropy (MC) [13], have been studied. This work focuses on the LMP criterion, owing to the well-documented merits (e.g., [16–18]) over the alternative RL frameworks of temporal-difference (TD) and Q-learning [15], especially for continuous and high-dimensional state spaces such as the one considered here. PI comprises two stages at every iteration n: policy evaluation and policy improvement. At policy evaluation, the current policy is evaluated by a Q-function [15], which represents, loosely speaking, the long-term cost that the agent would suffer had the current policy been used to determine the next state, whereas at the policy-improvement stage, the agent uses the Q-function values to update the policy. The underlying state space is considered to be continuous and high dimensional, due to the nature of the available data (x_n, y_n), while the action space is considered to be discrete: an action is a value of p taken from a finite grid of the interval [1, 2].

Deep neural networks offer approximating spaces for Q-functions, e.g., [19], but they may require processing of batch data (even re-training) during online-mode operation, since they may face test data generated by probability density functions (PDFs) different from those of the training ones (dynamic environments). Such batch processing incurs large computational times and complexity, discouraging the application of deep neural networks to online learning where a small complexity footprint is desired.

To meet the desired computational complexity requirements, this study builds an approximate (API) framework for online RL along the lines of kernel-based (KB)RL [16–18, 20–27]. Central to the API design is the construction of novel Bellman mappings [15, 28]. The proposed Bellman mappings are defined on a reproducing kernel Hilbert space (RKHS) H [29, 30], which serves as the approximating space for the Q-functions. In contrast to the prevailing route in KBRL [16–18, 20–27], which views Bellman mappings as contractions in L_p-norm Banach spaces (by definition, no inner product available), this study introduces nonexpansive [31]
Bellman operators on $\mathcal{H}$ to capitalize on the reproducing property of the inner product of $\mathcal{H}$ [29,30], and to open the door to powerful Hilbertian tools [31]. This path offers also flexibility to the user to choose any point from the potentially non-singleton fixed-point set of the nonexpansive Bellman mapping, as opposed to the case of a contraction mapping which is known to have a unique fixed point. A superset of the fixed-point set of one of the proposed Bellman mappings is designed onto which the proposed API framework is based. To robustify the policy-improvement stage, the well-known methodology of rollout [15] is employed. Moreover, to address the issue of the “curse of dimensionality” that arises naturally in online learning in RKHSs, the proposed framework is equipped with the approximate linear dependency (ALD) criterion [32]. Note that [20,27], being along the lines of TD and Q-learning, do not include any discussion on Bellman mappings.

Unlike the classical Bellman operators, where information on transition probabilities of a Markov decision process is needed [15], the proposed Bellman mappings need neither such information nor any training/online data and past policies, but sample and average the state space on-the-fly, at each $n$, to explore the surrounding environment. This suits the current adaptive-filtering setting, where the presence of outliers, with a possibly time-varying PDF, may render the information obtained offline or from past policies outdated.

As such, the proposed Bellman mappings fall closer to [17] than to studies which use training data collected beforehand (offline), e.g., [18,33,34]. It is worth stressing here that the proposed framework, together with its complementary study [35], appear to be the first attempts to apply RL arguments to robust adaptive filtering. In contrast to [35], where the state space is the low-dimensional $\mathbb{R}^4$, this study considers the high-dimensional $\mathbb{R}^{2L+1}$ ($\mathbb{R}$ is the set of all real numbers). This work constructs a finite-dimensional affine set as a superset of the fixed-point set of one of the proposed Bellman mappings, as opposed to [35] where a potentially infinite-dimensional hyperplane is designed. To reduce the size of the computational footprint of the proposed framework, ALD is used instead of the random Fourier features (RFF) [36] in [35]. Finally, rollout [15] is employed for robustification, whereas experience replay [37] is applied in [35].

Numerical tests on synthetic data demonstrate that the advocated framework identifies the value of $p$ that leads to “optimal” performance, without any knowledge on the PDF of the outliers, which is intentionally made to be time varying. Due to space limitations, any proofs, results on convergence analysis, and further numerical tests will be reported in the journal version of this paper.

2. NONEXPANSIVE BELLMAN MAPPINGS ON RKHSs

2.1. State-Action Space

Following (1), the state space is defined as the following continuous and high-dimensional $\mathcal{S} := \{s := (x,y,\theta) \mid x \in \mathbb{R}^L, y \in \mathbb{R}, \theta \in \mathbb{R}\} := \mathbb{R}^{2L+1}$, where $\mathbb{R}$ stands for the set of all real numbers. The action space $\mathcal{A}$ is defined as any finite grid of the interval $[1,2]$, so that an action $a \in \mathcal{A}$ becomes any value of $p$ taken from that finite grid. The state-action space is defined as $\mathcal{Z} := \mathcal{S} \times \mathcal{A}$, and its element is denoted as $z = (s,a)$.

Along the lines of the general notation in [15], consider now the set of all mappings $\mathcal{M} := \{\mu(\cdot) \mid \mu(\cdot) : \mathcal{Z} \to \mathcal{A} : s \to \mu(s)\}$. In other words, $\mu(s)$ denotes the action that the “system” will take being at state $s$. The set $\Pi$ of policies is defined as $\Pi := \mathcal{M}^\Pi := \{\mu_0,\mu_1,\ldots,\mu_n,\ldots\}$. A policy will be denoted by $\pi \in \Pi$. Given $\mu \in \mathcal{M}$, the stationary policy $\pi_\mu \in \Pi$ is defined as $\pi_\mu := \{\mu,\mu,\ldots,\mu\}$. It is customary for $\mu$ to denote also $\pi_\mu$.

The one-step loss $g : \mathcal{Z} \to \mathbb{R} : (s,a) \to g(s,a)$, which quantifies the cost/loss of taking action $a$ while being at state $s$, is defined for the current setting as:

$$g(s,a) := |y - \theta^T x|^2 + |y - \theta^T x|,$$

where $\theta^T \triangleq \theta + \rho a |y - \theta^T x|^2 - (y - \theta^T x) x$, according to (1). Recall that by definition the values of action $a$ are values of $p$ taken from a user-defined grid in $[1,2]$. The loss in (2) is motivated by classical adaptive filtering [1]; namely, the first term in (2) resembles the prior loss, while the second one mimics the posterior loss. Only the prior loss is chosen to be affected by action $a$ in (2), because $a$ applies to the current state $s$ and not to the next one. The long-term loss $Q : \mathcal{Z} \to \mathbb{R} : (s,a) \to Q(s,a)$ represents/estimates the cost that the agent suffers in the long run ($n \to \infty$) had the action $a$ been taken at state $s$.

2.2. Novel Bellman Mappings

Central to dynamic programming and RL [15] is the concept of Bellman mappings which operate on $Q$-functions. Typical definitions are, e.g., [38], $\forall (s,a) \in \mathcal{Z}$:

$$T_\alpha^\mu Q(s,a) := g(s,a) + \alpha E_{W((s,a))}[Q(s',\mu(s'))],$$

$$T^\alpha Q(s,a) := g(s,a) + \alpha E_{W((s,a))}[\min_{s'} Q(s',a')],$$

where $E_{W((s,a))}[\cdot]$ stands for the conditional expectation operator with respect to the potentially next state $s'$ conditioned on $(s,a)$, and $\alpha \in (0,1)$ is the discount factor. In the case where $Q$ is considered an element of the Banach space $L_\infty$ of all (essentially) bounded functions [39], equipped with the norm $\|\cdot\|_\infty$, then it can be shown that the mappings in (3) are contractions [15], which, according to the Banach-Picard theorem [31], possess unique fixed points $Q_\mu^n, Q^n$, i.e., points that solve the Bellman equations $T_\alpha^\mu Q_\mu^n = Q_\mu^n$ and $T^\alpha Q^n = Q^n$, and characterize “optimal” long-term losses [15]. Nevertheless, in most cases of practical interest, there is not sufficient information on the conditional PDF to compute $E_{W((s,a))}[\cdot]$. Motivated by this fact, this study proposes approximations of the Bellman mappings in (3) by assuming that losses $g$ and $Q$ belong to an RKHS $\mathcal{H}$, i.e., a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, norm $\|\cdot\|_{\mathcal{H}} := \langle \cdot, \cdot \rangle_{\mathcal{H}}^{1/2}$, and a reproducing kernel $k(\cdot,\cdot) : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$, such that $k(z(\cdot)) \in \mathcal{H}, \forall z \in \mathcal{Z}$, and the reproducing property holds true: $Q(z) = \langle Q, k(z(\cdot)) \rangle_{\mathcal{H}}, \forall Q \in \mathcal{H}, \forall z \in \mathcal{Z}$. Space $\mathcal{H}$ may be infinite dimensional; e.g., $k(\cdot,\cdot)$ is a Gaussian kernel [29,30]. For compact notations, let $\varphi(z) := k(z(\cdot)), \forall Q \in \mathcal{Q}, Q^\alpha$.

Hereafter, losses $g, Q$ are assumed to belong to $\mathcal{H}$. The proposed Bellman mappings $T_\alpha^\mu : \mathcal{H} \to \mathcal{H} : Q \to T_\alpha^\mu Q$ and $T^\alpha : \mathcal{H} \to \mathcal{H} : Q \to T^\alpha Q$ are defined as:

$$T_\alpha^\mu Q := g + \alpha \sum_{j=1}^{N_\alpha} Q(s^{av}_j, \mu(s^{av}_j)) \cdot \psi_j,$$

$$T^\alpha Q := g + \alpha \sum_{j=1}^{N_\alpha} \inf_{a_j \in A} Q(s^{av}_j, a_j) \cdot \psi_j,$$

where $\{\psi_j\}_{j=1}^{N_\alpha} \in \mathcal{H}$, for a user-defined positive integer $N_\alpha$, $\{s^{av}_j\}_{j=1}^{N_\alpha}$ are state vectors chosen by the user for the summations in (4) to approximate the conditional expectations in (3), and $\alpha \geq 0$. For example, $\{s^{av}_j\}_{j=1}^{N_\alpha}$ may be samples drawn from a Gaussian PDF centered at a state of interest (the current state $s_n$, in Section 4). For notational convenience, let $\Psi := [\psi_1, \ldots, \psi_{N_\alpha}]$, and its $N_\alpha \times N_\alpha$ kernel matrix $K_\Psi := \Psi^T \Psi$ whose $(j,j')$ entry is equal to $\langle \psi_j | \psi_{j'} \rangle_{\mathcal{H}}$. 
Consider a subset $\mathcal{B} = \{\mathbf{z}^j\}_{j=1}^{N_b} \subset \mathfrak{X}$, for some $N_b \in \mathbb{N}$. Define also $\Phi^\mathbf{b} := [\phi(\mathbf{z}^1), \ldots, \phi(\mathbf{z}^N_b)]$, $\Phi^\mathbf{a} = [\phi^\mathbf{a}(\mathbf{z}^1), \ldots, \phi^\mathbf{a}(\mathbf{z}^N_b)]$, where $\phi^\mathbf{a}(\mathbf{z}^j) := \phi(\mathbf{z}^j) b$, and let $K^\mathbf{a} := \Phi^\mathbf{a} \Phi^\mathbf{a}_T$. Let also the kernel matrices $K_0 := \Phi^\mathbf{b} \Phi^\mathbf{b}_T$ and $K^\mathbf{b} := \Phi^\mathbf{b} \Phi^\mathbf{b}_T$. Whenever $\mathcal{B}$ is linearly independent, $K_0$ is positive definite. Moreover, consider an $N_b \times N_b$ matrix $\Upsilon$. Define then mappings $T^\mathbf{b}_0, T^\mathbf{b} : \mathbb{R}^{N_b} \rightarrow \mathbb{R}^{N_b}$ as follows: $\forall \mathbf{z} \in \mathbb{R}^{N_b}$,

$$T^\mathbf{b}_0 \mathbf{z} := \eta b + \alpha \Upsilon K^\mathbf{b} \mathbf{z},$$

$$(5a)$$

$$T^\mathbf{b} \mathbf{z} := \eta b + \alpha \Upsilon \text{im}_{\mu \in \mathcal{M}} K^\mathbf{b} \mathbf{z},$$

$$(5b)$$

It can be verified that the fixed-point set of $T^\mathbf{b}_0$ satisfies:

$$\text{Fix } T^\mathbf{b}_0 := \{\mathbf{z} \in \mathbb{R}^{N_b} \mid T^\mathbf{b}_0 \mathbf{z} = \mathbf{z}\} = \{\mathbf{z} \in \mathbb{R}^{N_b} \mid (I_{N_b} - \alpha \Upsilon K^\mathbf{b}) \mathbf{z} = \eta b\},$$

$$(6)$$

where $I_{N_b}$ stands for the $N_b \times N_b$ identity matrix. Since $\text{Fix } T^\mathbf{b}_0$ may be empty, Line 9 of Algorithm 1 defines a non-empty affine superset of (6). More details on (6) are omitted due to lack of space. Note here that [35] focuses on (4a) and follows a different route by defining a potentially infinite-dimensional hyperplane as the superset of the fixed-point of (4a).

**Algorithm 1 Approximate policy-iteration framework.**

1. Arbitrarily initialize $\mathcal{B}_0 \subset \mathfrak{X}$, with $N_0[0] = |\mathcal{B}_0|$, $\xi_0 \in \mathbb{R}^{N_b[0]}$, $\Phi^\mathbf{b}_0$, $Q_0$, $\mu_0 \in \mathcal{M}$, and $\theta b \in \mathbb{R}^\cdot$.
2. while $n \in \mathbb{N}$ do
3. Data $(x_n, y_n)$ become available. Let $s_n = (x_n, y_n, \theta b)$.
4. Define $\{\mathbf{s}_n[n]\}_{n=1}^{N_b[n]}$.
5. **Policy improvement:** Update $\mu_n(s_n)$ by (9).
6. where $p := \mu_n(s_n)$.
7. Update $\mathbf{b}_n$, $\Phi^\mathbf{b}_n$, and identify $\Upsilon_n$ and $K^\mathbf{b}$ as in Section 3.
8. Update $\eta_n$ by (8).
9. Compute

$$\xi_{n+1} := \arg \min_{\xi} \| (I_{N_b[n]} - \alpha \Upsilon_n K^\mathbf{b}_n) \mathbf{z} - \eta_n \|^2 .$$

10. **Policy evaluation:** Update $Q_{n+1} := \Phi^\mathbf{b}_n \xi_{n+1}$.
11. Increase $n$ by one, and go to Line 2.
12. end while

**Theorem 1** Let $\psi_\eta(\mathbf{x}) \geq 0 \forall \mathbf{x} \in \mathfrak{X}, \forall j \in \{1, \ldots, N_b\}$.

(i) If $\alpha \leq \|K^\mathbf{a}\|^{-1/2} (\sup_{\mu \in \mathcal{M}} \|K^\mathbf{b}\|)^{-1/2}$, then $\forall \mu \in \mathcal{M}$, the mapping $T^\mathbf{b}_0$ in (4a) is affine nonexpansive and $T$ in (4b) is nonexpansive within the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$. The norms $\|K^\mathbf{a}\|$, $\|K^\mathbf{b}\|$ are the spectral norms of $K^\mathbf{a}$, $K^\mathbf{b}$.

(ii) Let $\alpha \leq \|K^\mathbf{a}\|^{-1/2} (\sup_{\mu \in \mathcal{M}} \|K^\mathbf{b}\|)^{-1/2}$. Consider also the case where vectors $\{\phi(\mathbf{z}^j)\}_{j=1}^{N_b}$ are linearly independent, and $\{\psi(\mathbf{z}^j)\}_{j=1}^{N_b} \subset \mathcal{H}$, $\text{span} (\phi(\mathbf{z}^j))_{j=1}^{N_b}$, that is, there exist $\eta \in \mathbb{R}^{N_b}$ and $\Upsilon \in \mathbb{R}^{N_b \times N_b}$ such that $\eta = \Phi^\mathbf{a}_T \eta$ and $\Upsilon = \Phi^\mathbf{a}_T \Upsilon$. Then, for any $Q \in \mathcal{H}$, $T^\mathbf{b}_0 Q = \Phi^\mathbf{b} T^\mathbf{b}_0 \xi$, $\forall \mu \in \mathcal{M}$, and $T^\mathbf{b} Q = \Phi^\mathbf{b} T^\mathbf{b} \xi$. Moreover, $\forall \mu \in \mathcal{M}$, the mapping $T^\mathbf{b}_0$ in (5a) is affine nonexpansive and $T^\mathbf{b} Q$ in (5b) is nonexpansive within the Euclidean space $(\mathbb{R}^{N_b}, \langle \cdot, \cdot \rangle_{\mathcal{K}_b})$. Furthermore, the fixed-point set of $T^\mathbf{b}_0$ satisfies:

$$\text{Fix } T^\mathbf{b}_0 := \{\mathbf{z} \in \mathbb{R}^{N_b} \mid T^\mathbf{b}_0 \mathbf{z} = \mathbf{z}\} = \{\mathbf{z} \in \mathbb{R}^{N_b} \mid (I_{N_b} -\alpha \Upsilon K^\mathbf{b}) \mathbf{z} = \eta b\},$$

$$\text{Fix } T^\mathbf{b} := \{\mathbf{z} \in \mathbb{R}^{N_b} \mid T^\mathbf{b} \mathbf{z} = \mathbf{z}\} = \{\mathbf{z} \in \mathbb{R}^{N_b} \mid (I_{N_b} -\alpha \Upsilon K^\mathbf{b}) \mathbf{z} = \eta b\}.$$
where (8a) offers the minimum-norm solution of (7), while (8b) computes the (metric) projection [31] of $\mathbf{r}_{n-1}$ onto the hyperplane in (7), after zero-padding the $N_0[n-1] \times 1$ vector $\mathbf{r}_{n-1}$ to meet the length $N_0[n]$, in case $N_0[n] \geq N_0[n-1] + 1$. Following Theorem 1(ii), the inner product $\langle \mathbf{e}_{n}, \mathbf{e}_{n-1} \rangle_{K_{n}}$ was utilized in (8).

To identify $\mathbf{y}_{n}$, needed for the computations in Line 9 of Algorithm 1, Theorem 1(ii) is followed where $\mathbf{y}_{n}$ is computed via $\mathbf{Y}_{n}$. Although $\mathbf{Y}_{n}$ can be chosen in many ways such that $\{\psi_{j}[n]\}_{j=1}^{N_0[n]} \subset \mathcal{H}_{n}[n], \{\psi_{j}[n]\}_{j=1}^{N_0[n]}$ are selected in Section 4 simply by taking the first $N_0[n]$ vectors from $\mathbf{Y}_{n}$, under the assumption that $N_0[n] \leq N_{0}[n]$. Hence, according to Theorem 1(ii), $\mathbf{Y}_{n} = [\mathbf{I}_{N_0[n]}; 0_{N_0[n] \times (N_0[n] - N_{0}[n])}]^\top$ in Section 4. With regards to $\varphi_{\mu,\rho}[n] := \varphi_{\mu}[n], \mu_{n}(\mathbf{s}_n^{(p)}[n])$, needed in the computation of $K_{n}^{(\mu,\rho)}$, since $\mathbf{s}_n^{(p)}[n]$ is drawn from a Gaussian PDF centered at $\mathbf{s}_n$, $\mu_{n}(\mathbf{s}_n^{(p)}[n])$ is set equal to $\mu_{n}(\mathbf{s}_n)$ in Section 4.

Typically, the greedy rule $\mu_{n}(\mathbf{s}_n) = \arg \min_{\mathcal{A}} Q_n(\mathbf{s}_n, \mathcal{A})$ is used for policy improvement [15, 18, 33]. However, it has been observed that this greedy rule may lead into instabilities of the RL agent’s behavior and hinder its cognition about the surrounding environment [15]. The $\varepsilon$-greedy strategy [15], a variant of the greedy one, has not performed well in the numerical tests of Section 4. To address this potential drawback of greedy strategies, the popular rollout methodology [15] is employed in Algorithm 1. With the user-defined $M \in \mathbb{N}$, denoting the number of steps in rollout, the following rule is employed for policy improvement:

$$
\mu_{n}(\mathbf{s}_n) = \arg \min_{\mathcal{A}} \left[ g(\mathbf{s}_n, \mathcal{A}) + \sum_{m=1}^{M-1} \alpha^{m} g(\mathbf{s}_{n+m}, \mu^{(\mathbf{s}_{n+m})}) + \alpha^{M} Q_{n}(\mathbf{s}_{n+m}, \mu^{(\mathbf{s}_{n+m})}) \right],
$$

(9)

where $\mu^{(\mathbf{s})} \in \mathcal{H}$ is called the heuristic (stationary) policy, which could be chosen to be random or $\mu_{n-1}$, and $\{\mathbf{s}_{n+m}\}_{m=1}^{M}$, needed to possible successor states of $\mathbf{s}_n$ under action $\mathcal{A}$ and policy $\mu^{(\mathbf{s})}$. A large $M$ (long trajectory) may inflict long computational times and diminish the effect of $Q_{n}$ on policy improvement through $\alpha^{M}$ in (9) if $\alpha < 1$ [15]. Note here that experience replay [37], instead of rollout, was employed in [35].

4. NUMERICAL TESTS

The proposed PI framework is compared numerically against (i) (1), where $p$ is fixed throughout iterations with $p \in \mathbb{R} := \{1, 1.25, 1.5, 1.75, 2\}$, (ii) CAC-RLP [10], which uses two $p$-norm adaptive filters, with the same $p$ but with different forgetting factors, (iii) RLP [7], which combines two LMP filters with different $p$s, and (iv) KLSPI [18], which extends the least-squares temporal-difference methodology [33] into kernel learning, with no rollout employed. Performance is measured by the normalized deviation $\|\mathbf{e}_{\text{test}} - \mathbf{e}_{n}\| \|\mathbf{e}_{\text{test}}\|$ vs. $n$. Multiple (100) independent tests were performed, with their uniformly averaged results reported in Figures 1 and 2. The software code was written in Julia [40], with $\alpha$-stable outliers generated by [5].

The length $L$ of the system $\mathbf{Q}$, is set equal to $10$ in the data-generation model of Section 1. The entries of $\mathbf{Q}$, and $\mathbf{x}_n, \forall n$, are generated by independent and identically distributed (IID) normal RVs. In the data-generation model of Section 1, RV $\mathbf{x}_n$ describes both outlier and noise scenarios. In the case where no outliers appear in the data-generation model, $\mathbf{x}_n$ obeys the Gaussian PDF, and whenever outliers appear, $\mathbf{x}_n$ follows the $\alpha$-stable PDF [5]. In all figures, no outliers appear for time instances $n < 2 \times 10^4$ and only Gaussian noise, with SNR $= 20$dB, corrupts the data. Outliers appear for $n \geq 2 \times 10^4$, following the $\alpha$-stable PDF [4].

Note that when the “stability” parameter of the $\alpha$-stable PDF takes the values of 2, then the PDF boils down to the Gaussian one. Two types of $\alpha$-stable outliers are examined: (i) “Gauss-like” ones, with parameters “stability” $= 1.95$, “skewness” $= 0.5$, “location” $= 0.5$, “scale” $= 10^{-2}$, which make the tails of the PDF slightly heavier than those of the Gaussian PDF [5]; and (ii) “Cauchy-like” ones, with parameters “stability” $= 1$, “skewness” $= 0.5$, “location” $= 0.5$ and “scale” $= 1$, which make the tails of the PDF heavy [5].

Number $N_{\text{ox}}[n] = 10, \forall n, M = 2$ in (9), while (8a) is employed. The learning rate $\rho$ of LMP in (1) is set equal to $10^{-3}$. A homogeneous polynomial kernel of degree 2 is used for both Algorithm 1 and KLSPI [18]. Note that this kernel satisfies the condition stated in the first line of Theorem 1. Results on other kernels will be reported elsewhere. The ALD criterion is employed with parameter $\delta_{\text{ALD}} = \sin(40\pi/180)$. The parameters of CAC-RLP are set as $p_{\text{RLP}} = 1.4, \lambda_{\text{RLP}} = 0.9$ and $\lambda_{\text{RLP}} = 0.9$. All tests show that Algorithm 1 succeeds in identifying the “optimal” $p$-norm, regardless of the PDF of the outliers. Even if outliers are absent from the data-generation model and only Gaussian noise corrupts the data, Algorithm 1 chooses the $2$-norm, which is well-known to be optimal for Gaussian noise. In contrast, KLSPI identifies fast the $2$-norm in the case where only Gaussian noise appears, but performs poorly when Cauchy-like outliers appear. CAC-RLP [10] converges fast in cases where only Gaussian noise appears, with sub-optimal performance in terms of the normalized deviation, but diverges in the case where Cauchy-like outliers corrupt the data. Due to space limitations, further numerical tests will be reported in the journal version of this manuscript.
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