Rigorous approaches for spin glass and Gaussian spin glass with P-wise interactions

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Abstract: Purpose of this paper is to face up to P-spin glass and Gaussian P-spin model, i.e. spin glasses with polynomial interactions of degree $P > 2$. We consider the replica symmetry and first step of replica symmetry breaking assumptions and we solve the models via transport equation and Guerra’s interpolating technique, showing that we reach the same results.

Thus, using rigorous approaches, we recover the same expression for quenched statistical pressure and self-consistency equation in both assumption found with other techniques, including the well-known replica trick technique.

At the end, we show that for $P = 2$ the Gaussian P-spin glass model is intrinsically RS.
Introduction

In last few decades statistical mechanics has played an important role in different fields, e.g. spin glasses[13, 28] and neural networks[4, 7, 27]. For this reason there has been a lot of research to tackle increasingly complex models through rigorous mathematical technique[14], alternative to the well-known replica-trick[28]. Furthermore the growing interest in Replica Symmetry Breaking and its connection with ultrametricity, as just proved in [21, 31], has taken the Scientific Community to take into account the results linked to this assumption[8, 29, 30, 34].

In this paper we focus on mathematical methods on spin glasses, considered a real challenge for mathematicians[33]. In particular, in this paper we want to generalize some computations for Sherrington Kirkpatrick and Gaussian models with two-spin interactions in a polynomial interactions of degree $P > 2$, with two different rigorous mathematical approaches, namely Guerra’s interpolating technique, introduced in [23, 24], and transport equation, introduced in Replica Symmetric assumption in [5] and in Replica Symmetry Breaking [1] for the description of Hopfield model and, afterwards,
for deep Hopfield networks [2]. Thus we prove that both methods are consistent and mathematically valid for these models.

Furthermore we face to RS and 1RSB assumptions, showing that, as far as Gaussian spin glass model concerns, we reach the results in [9, 10] for \( P = 2 \) and in [17, 20, 35] for \( P > 2 \). For the 1RSB assumption, we use the broken replica interpolation introduced by Francesco Guerra for the Sherrington Kirkpatrick model [23] and, moreover, we adapt it into a PDE framework (standard transport equation).

We stress that the present paper is focused on mathematical methods to deepen these models. So, we do not face up to physical interpretation of our results, for which we remind to [17, 22, 35].

The paper is structured as follows.

In the first section we analyze P-spin glass model, namely Sherrington-Kirkpatrick with P-wise interactions, starting from the main characters until the expression in RS and 1RSB assumption of quenched statistical pressure and self-consistency equations. For both assumption we compute via Guerra’s interpolating technique and transport equation, showing that we reach the same results. We proceed in similar way for Gaussian P-spin glass in the second section, coming to the emergent property for this model too. The paper is closed by a section concerning the case \( P = 2 \) for Gaussian P-spin glass, in which we show that it is intrinsically RS, and some appendices with computations from results of the paper.

1 P-spin glass model

In this section we deal with P-spin glass model, namely Sherrington-Kirkpatrick (SK) model with \( P \)-wise interactions, introduced by Derrida in [18] and meticulously described afterwards, e.g. [20, 22, 28, 36, 37]. In particular we find in RS and 1RSB assumption the expressions of statistical quenched pressure and self-consistency equations.

1.1 Generalities

Definition 1. Let \( P \in \mathbb{N} \) and \( \sigma \in \{-1; +1\}^N \) be a configuration of \( N \) spins, the Hamiltonian of the P-spin glass model is defined as

\[
H_N^{(P)}(\sigma|J) := -\frac{1}{p^1} \sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p},
\]

(1.1)

where the P-wise quenched couplings \( J = \{J_{i_1,\ldots,i_p}\}_{i_1,\ldots,i_p=1,\ldots,N} \) are given by

\[
J_{i_1,\ldots,i_p} := \frac{J_0}{N^{p-1}} + J \sqrt{\frac{2}{N^{p-1}}} z_{i_1,\ldots,i_p},
\]

(1.2)

with \( J_0 \in \mathbb{R}^+ \) and \( z_{i_1,\ldots,i_p} \) i.i.d. standard random variables \( z_{i_1,\ldots,i_p} \sim N(0; 1) \).

Definition 2. The partition function related to the Hamiltonian (1.1) is given by

\[
Z_N^{(P)}(\beta, J) := \sum_{\sigma} \exp \left[ -\beta H_N^{(P)}(\sigma|J) \right],
\]

(1.3)

where \( \beta \in \mathbb{R}^+ \) is the inverse temperature in proper units such that for \( \beta \to 0 \) the probability distribution for the neural configuration is uniformly spread while for \( \beta \to \infty \) it is sharply peaked at the minima of the energy function (1.1).
We introduce the **Boltzmann average** induced by the partition function (1.3), denoted with $\omega_J$ and, for an arbitrary observable $O(\sigma)$, defined as

$$\omega_J(O(\sigma)) := \frac{\sum_{\sigma} O(\sigma) e^{-\beta H_N(\sigma|J)}}{Z_N(\beta, J)}. \quad (1.4)$$

This can be further averaged over the realization of the $J_{i_1}, \ldots, J_{i_p}$’s (also referred to as **quenched average**) to get

$$\langle O(\sigma) \rangle := \mathbb{E}_{\omega_J(O(\sigma))}. \quad (1.5)$$

**Definition 3.** The intensive quenched pressure of the $P$-spin glass model (1.1) is defined as

$$A_N^{(P)}(\beta, J) := \frac{1}{N} \mathbb{E} \ln Z_N^{(P)}(\beta, J), \quad (1.6)$$

and its thermodynamic limit, assuming its existence, is referred to as

$$A^{(P)}(\beta, J) := \lim_{N \to \infty} A_N^{(P)}(\beta, J). \quad (1.7)$$

**Remark 1.** The existence of the thermodynamical limit of free energy density is proved for $P$-spin glass model in Guerra and Toninelli’s work[25].

**Remark 2.** We stress that the quenched statistical pressure $A^{(P)}$ is equivalent to free energy, namely

$$A^{(P)}(\beta, J) = -\beta f^{(P)}(\beta, J) = -\beta \left( E(\beta, J) - \frac{S(\beta, J)}{\beta} \right) \quad (1.8)$$

where $E$ is the internal energy and $S$ the entropy of the system.

In order to solve the model we want to find out an explicit expression for the quenched pressure (1.7) in terms of the natural order parameters of the theory, namely the magnetization $m$ and the two-replica overlap $q_{12}$, defined in the following

**Definition 4.** The order parameters used to describe the macroscopic behavior of the model are the standard ones [7, 16], namely, the magnetization

$$m := \frac{1}{N} \sum_{i=1}^{N} \sigma_i, \quad (1.9)$$

and the two-replica overlap, introduced as

$$q_{12} := \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(1)} \sigma_i^{(2)}. \quad (1.10)$$

### 1.2 Resolution via Guerra’s interpolation

The purpose of this section is to solve the P-spin glass model through Guerra’s interpolating technique. To do so, we compute the derivative w.r.t. the interpolating parameter $t$ and we apply the Fundamental theorem of Calculus. In the end, we find the expression of statistical pressure in the approximation of replica symmetry (RS) and first step of replica symmetry breaking (RSB).
1.2.1 RS solution

Definition 5. Under the replica-symmetry assumption, the order parameters, in the thermodynamic limit, self-average around their mean values and their distributions get delta-peaked at their equilibrium value (denoted with a bar), independently of the replica considered, namely

\[
\lim_{N \to \infty} \langle (m - \bar{m})^2 \rangle = 0 \Rightarrow \lim_{N \to \infty} \langle m \rangle = \bar{m}, \quad (1.11)
\]

\[
\lim_{N \to \infty} \langle (q_{12} - \bar{q})^2 \rangle = 0 \Rightarrow \lim_{N \to \infty} \langle q_{12} \rangle = \bar{q}. \quad (1.12)
\]

For the generic order parameter \(X\) this can be rewritten as \(\langle (\Delta X)^2 \rangle_{N \to \infty} \to 0\), where

\[
\Delta X := X - \bar{X},
\]

and, clearly, the RS approximation also implies that, in the thermodynamic limit, \(\langle \Delta X \Delta Y \rangle = 0\) for any generic pair of order parameters \(X, Y\). Moreover in the thermodynamic limit, we have \(\langle (\Delta X)^k \rangle \to 0\) for \(k \geq 2\).

Definition 6. Given the interpolating parameter \(t \in [0, 1]\), \(A, \psi \in \mathbb{R}\) and \(z_i \sim \mathcal{N}(0, 1)\) for \(i = 1, \ldots, N\) standard Gaussian variables i.i.d., the Guerra’s interpolating partition function is given as

\[
Z_N^{(P)}(t) := \sum_{\{\sigma\}} \exp \left[ t \beta' J_0 N \frac{m^{(P)}(\sigma)}{2} + (1 - t) N \psi m(\sigma) + \sqrt{\beta' J} \sum_{i_1, \ldots, i_p} z_{i_1} \cdots i_p \sigma_{i_1} \cdots \sigma_p + A \sqrt{1 - t} \sum_{i=1}^N z_i \sigma_i \right],
\]

where \(\beta' = 2 \beta / P!\).

Definition 7. The interpolating pressure for the \(P\)-spin glass model \((1.1)\), at finite \(N\), is introduced as

\[
A_N^{(P)}(t) := \frac{1}{N} \mathbb{E} \left[ \ln Z_N^{(P)}(t) \right],
\]

where the expectation \(\mathbb{E}\) is now meant over \(z_{i_1}, \ldots, i_p\) and \(z_i\), in the thermodynamic limit,

\[
A^{(P)}(t) := \lim_{N \to \infty} A_N^{(P)}(t). \quad (1.15)
\]

By setting \(t = 1\) the interpolating pressure recovers the original one \((1.6)\), that is \(A_N^{(P)}(\beta, J_0, J) = A_N^{(P)}(t = 1)\).

Remark 3. The interpolating structure implies an interpolating measure, whose related Boltzmann factor reads as

\[
B(\sigma; t) := \exp [\beta \mathcal{H}(\sigma; t)]; \quad (1.16)
\]
In this way the partition function is written as $Z_N(t) = \sum_\sigma B(\sigma; t)$. A generalized average follows from this generalized measure as
\[ \omega_t(O(\sigma)) := \sum_\sigma O(\sigma)B(\sigma; t) \]
(1.17)
and
\[ \langle O(\sigma) \rangle_t := E[\omega_t(O(\sigma))] \]
(1.18)
where $E$ denotes the average over $z_1, \ldots, z_p$ and $\{z_i\}_{i=1, \ldots, N}$.

Of course, when $t = 1$ the standard Boltzmann measure and related average is recovered. Hereafter, in order to lighten the notation, we will drop the subindices $t$.

**Lemma 1.** The $t$ derivative of interpolating pressure is given by
\[
\frac{dA(p)(t)}{dt} := \frac{\beta' J_0}{2} \left( \langle m^p \rangle - \frac{2\psi}{\beta} J_0 \langle m \rangle \right) - \frac{\beta'^2 J^2}{4} \left( \langle q_{12}^p \rangle - \frac{2A^2}{\beta^2 J^2} \langle q_{12} \rangle \right)
\]
(1.19)
Proof. Deriving equation (1.14) with respect to $t$, we get
\[
\frac{dA(p)(t)}{dt} = \frac{\beta' J_0}{2} \left( \langle m^p \rangle - \psi \langle m \rangle \right) + \frac{\beta'^2 J^2}{2N\sqrt{1-t}} \left( \sum_i z_i \omega(\sigma_i) \right)
\]
(1.20)
Now, using the Stein’s lemma (also known as Wick’s theorem) on standard Gaussian variable $z_i$ and $z_i$, which is for a standard Gaussian variable $J$, i.e. $J \sim N(0,1)$, and for a generic function $f(J)$ for which $Ezf(z)$ and $E\partial_J f(J)$ both exist, then
\[ E \left( Jf(J) \right) = E \left( \frac{\partial f(J)}{\partial J} \right) \],
we may rewrite the second and the third member of (1.20) as
\[
\frac{\beta' J}{2N\sqrt{1-t}} \left( \sum_i z_i \omega(\sigma_i) \right) - \frac{1}{2N\sqrt{1-t}} E \left[ A \sum_i z_i \omega(\sigma_i) \right] = D_1 + D_2.
\]
Let’s investigate those three terms:
\[
D_1 = \frac{\beta' J}{2N\sqrt{1-t}} \left( \sum_i z_i \omega(\sigma_i) \right) \]
\[
= \frac{\beta'^2 J^2}{4N} \left( \sum_i \omega(\sigma_i) \right)^2 - \sum_i E \left[ \omega(\sigma_i) \right]^2 \]
(1.22)
\[
= \frac{\beta'^2 J^2}{4} \left( 1 - \langle q_{12}^p \rangle \right);
\]
\[
D_2 = -\frac{1}{2N\sqrt{1-t}} A \sum_i E \left[ \partial_i \omega(\sigma_i) \right]
\]
\[
= -\frac{1}{2N} A^2 \left( \sum_i E \left[ \omega(\sigma_i) \right] \right) - \sum_i E \left[ \omega(\sigma_i) \right]^2 \]
(1.23)
\[
= -\frac{1}{2} A^2 \left[ 1 - \langle q_{12} \rangle \right].
\]
Rearranging together (1.22) and (1.23) we obtain the thesis. 

\[ \square \]
Remark 4. We stress that, for the RS assumption presented in Definition (5), we can use the relations
\[
\langle m^P \rangle = \sum_{k=2}^{P} \binom{P}{k} \langle (m - \bar{m})^k \rangle \bar{m}^{P-k} + \bar{m}^P (1 - P) + P \bar{m}^{P-1} \langle m \rangle,
\]
(1.24)
\[
\langle q_{12}^P \rangle = \sum_{k=2}^{P} \binom{P}{k} \langle (q_{12} - \bar{q})^k \rangle \bar{q}^{P-k} + \bar{q}^P (1 - P) + P \bar{q}^{P-1} \langle q_{12} \rangle,
\]
(1.25)
which are proven in Appendix (A). Using these relations, if we fix the constants as
\[
\psi = \frac{P}{2} \beta' J_0 \bar{m}^{P-1},
\]
(1.26)
\[
A^2 = \frac{P}{2} \beta'^2 J^2 \bar{q}^{P-1},
\]
the (1.19) in the thermodynamical limit reads as
\[
\frac{dA^P(t)}{dt} := -\frac{P-1}{2} \beta' J_0 \bar{m}^P + \frac{\beta'^2 J^2}{4} (1 - P \bar{q}^{P-1} + (P - 1) \bar{q}^P)
\]
(1.27)
which is now independent of \( t \).

Proposition 1. In the thermodynamic limit \( (N \to \infty) \) and under RS assumption, applying the Fundamental Theorem of Calculus and using the suitable values of \( A \) and \( \psi \), we find the quenched pressure for the P-spin glass model as
\[
A^P(\beta, J_0, J) = \ln 2 + \langle \ln \cosh \left[ \frac{P}{2} \beta' J_0 \bar{m}^{P-1} + z \beta' J \sqrt{\frac{P}{2} \bar{q}^{P-1}} \right] \rangle_z
\]
(1.28)
\[
= \ln 2 + \langle \ln \cosh \left[ P \beta' J_0 \bar{m}^{P-1} + z \beta' J \sqrt{\frac{P}{2} \bar{q}^{P-1}} \right] \rangle_z.
\]
(1.29)
Proof. Using the Fundamental Theorem of Calculus:
\[
A^P(t = 1) = A^P(t = 0) + \int_0^1 \partial_s A^P(s) \bigg|_{s=t} dt
\]
(1.30)
and computing the one-body terms
\[
A^P(t = 0) = \frac{1}{N} \ln \left\{ \sum_{\sigma} \exp \left[ N \psi m(\sigma) + A \sum_{i=1}^N z_i \sigma_i \right] \right\},
\]
(1.31)
Finally, putting (1.27) and (1.31) in (1.30), we find (1.28).

Extremizing the statistical pressure in (1.28) w.r.t. the order parameters we find the self-consistency equations.  

Remark 5. If we do not take into account the disordered part of the system, namely $J = 0$, in (1.28), we recover for this P-spin ferromagnetic model the same result in [19].

Remark 6. If we don’t consider in (1.28) the ferromagnetic part of the system, namely $J_0 = 0$, we recover the same expressions found in [3].

1.2.2 1RSB solution

In this subsection we turn to the solution of the P-spin glass model via the Guerra’s interpolating technique, restricting the description at the first step of RSB.

Definition 8. In the first step of replica-symmetry breaking, the distribution of the two-replica overlap $q$, in the thermodynamic limit, displays two delta-peaks at the equilibrium values, referred to as $\bar{q}_1$, $\bar{q}_2$, and the concentration on the two values is ruled by $\theta \in [0, 1]$, namely

$$ \lim_{N \to +\infty} P^\theta_N(q) = \theta \delta(q - \bar{q}_1) + (1 - \theta) \delta(q - \bar{q}_2). $$

(1.32)

The magnetization still self-averages at $\bar{m}$ as in (1.11).

Following the same route pursued in the previous sections, we need an interpolating partition function $Z$ and an interpolating quenched pressure $A$, that are defined hereafter.

Definition 9. Given the interpolating parameter $t$ and the i.i.d. auxiliary fields $\{z^{(1)}_i, z^{(2)}_i\}_{i=1,...,N}$, with $z^{(1,2)}_i \sim N(0,1)$ for $i = 1,...,N$ we can write the 1-RSB interpolating partition function $Z_N(t)$ for the P-spin glass model (1.1) recursively, starting by

$$ Z^{(P)}_2(t) := \sum_{\{\sigma\}} \int D\tau \exp \left[ t \beta J_{0N} \frac{1}{2} m^p(\sigma) + \frac{1}{N} \psi m(\sigma) + \sqrt{t} \sum_{i_1,...,i_p=1}^{N} z_{i_1...i_p} \sigma_{i_1} \cdots \sigma_{i_p} + \right. $$

$$ + \sqrt{1-t} \sum_{a=1}^{2} \left( A_a \sum_{i=1}^{N} z^{(a)}_i \sigma_i \right) $$

(1.33)

where the $z_{i_1,...,i_p}$’s i.i.d. standard Gaussian. Averaging out the fields recursively, we define

$$ Z^{(P)}_1(t) := \mathbb{E}_2 \left[ Z^{(P)}_2(t)^{\theta} \right]^{1/\theta} $$

(1.34)

$$ Z^{(P)}_0(t) := \exp \mathbb{E}_1 \left[ \ln Z^{(P)}_1(t) \right] $$

(1.35)

$$ Z^{(P)}_N(t) := Z^{(P)}_0(t) $$

(1.36)

where with $\mathbb{E}_a$ we mean the average over the variables $z^{(a)}_i$’s, for $a = 1, 2$, and with $\mathbb{E}_0$ we shall denote the average over the variables $z_{i_1,...,i_p}$’s.
Definition 10. The 1RSB interpolating pressure, at finite volume $N$, is introduced as
\[ A^{(P)}_N(t) := \frac{1}{N} \mathbb{E}_0 \left[ \ln Z_0^{(P)}(t) \right], \]  
(1.37)
and, in the thermodynamic limit, assuming its existence
\[ A^{(P)}(t) := \lim_{N \to \infty} A^{(P)}_N(t). \]  
(1.38)
By setting $t = 1$, the interpolating pressure recovers the standard pressure (1.6), that is, $A^{(P)}_N(1) = \mathcal{A}^{(P)}(t = 1)$.

Remark 7. In order to lighten the notation, hereafter we use the following
\[ \langle m \rangle = \mathbb{E}_0 \mathbb{E}_1 \mathbb{E}_2 \left[ \frac{1}{N} \sum_{i=1}^{N} \omega(\sigma_i) \right] \]  
(1.39)
\[ \langle q_{12} \rangle_1 = \mathbb{E}_0 \mathbb{E}_1 \left[ \frac{1}{N} \sum_{i=1}^{N} (\mathbb{E}_2 [W_2 \omega(\sigma_i)])^2 \right] \]  
(1.40)
\[ \langle q_{12} \rangle_2 = \mathbb{E}_0 \mathbb{E}_1 \mathbb{E}_2 \left[ W_2 \frac{1}{N} \sum_{i=1}^{N} \omega(\sigma_i)^2 \right] \]  
(1.41)
where the weight $W_2$ is defined as
\[ W_2 := \frac{\mathbb{Z}^g_2}{\mathbb{Z}_2^2}. \]  
(1.42)

Now the next step is computing the $t$-derivative of the statistical pressure. In this way we can apply the fundamental theorem of calculus and find the solution of the original model.

Lemma 2. The derivative w.r.t. $t$ of interpolating statistical pressure can be written as
\[ d_t A^{(P)}_N = \frac{\beta J_0}{2} \langle m^P \rangle - \psi \langle m \rangle + \frac{\beta^2 J^2}{4} \left[ 1 + (\theta - 1)\langle q_{12}^P \rangle_2 - \theta \langle q_{12}^P \rangle_1 \right] \]  
(1.43)
\[ - (A_1)^2 \left[ 1 + (\theta - 1)\langle q_{12}^P \rangle_2 - \theta \langle q_{12}^P \rangle_1 \right] - (A_2)^2 \left[ 1 + (\theta - 1)\langle q_{12}^P \rangle_2 \right] \]

Since the proof is a bit lengthy, we leave it in Appendix (B).

Remark 8. If we want to use the 1RSB assumption in such a way that for $a = 1, 2$
\[ \langle m^P \rangle = \sum_{k=2}^{P} \binom{P}{k} \langle (m - \bar{m})^k \rangle m^{P-k} + \bar{m}^P (1 - P) + P \bar{m}^{P-1} \langle m \rangle \]  
(1.44)
\[ \langle q_{12}^P \rangle_2 = \sum_{k=2}^{P} \binom{P}{k} \langle (q_{12} - \bar{q}_a)^k \rangle q_{12}^{P-k} + \bar{q}_a^P (1 - P) + P \bar{q}_a^{P-1} \langle q_{12} \rangle_2 \]  
(1.45)
we need to fix the constants as
\[ \psi = \frac{\beta J_0}{2} \bar{m}^{P-1}, \]  
(1.46)
\[ A_1 = \frac{\beta^2 J^2 P}{4} \bar{q}_1^{P-1}, \]  
(1.47)
\[ A_2 = \frac{\beta^2 J^2 P}{4} (\bar{q}_2^{P-1} - \bar{q}_1^{P-1}). \]  
(1.48)
Thus, the derivative w.r.t. $t$ in the thermodynamic limit is computed as

$$d_t A_N^{(P)} = \frac{\beta' J_0}{2} \bar{m}^P (1 - P) + \frac{\beta'^2 J^2}{4} \left[ 1 - P \bar{q}_2^{P-1} + (P - 1) \bar{q}_1^P \right]$$

$$- \frac{\beta'^2 J^2}{4} (P - 1) \theta (\bar{q}_2^P - \bar{q}_1^P).$$

(1.49)

**Proposition 2.** At finite size and under 1RSB assumption applying the Fundamental Theorem of Calculus and using the suitable values of $A^{(1)}, A^{(2)}, \psi$, we find the quenched pressure for the $P$-spin glass model as

$$A^{(P)} = \log 2 + \frac{\beta' J_0}{2} \bar{m}^P (1 - P) + \frac{\beta'^2 J^2}{4} \left[ 1 - P \bar{q}_2^{P-1} + (P - 1) \bar{q}_1^P \right]$$

$$+ \frac{1}{\theta} E_1 \log E_2 \cosh^\theta \left[ \beta' \left( \frac{J_0}{2} P \bar{m}^{P-1} + z^{(1)} J \sqrt{\frac{P}{2} \bar{q}_1^{P-1} + z^{(2)} J \sqrt{\frac{P}{2} (\bar{q}_2^{P-1} - \bar{q}_1^{P-1})}} \right) \right]$$

$$- \frac{\beta'^2 J^2}{4} (P - 1) \theta (\bar{q}_2^P - \bar{q}_1^P) + \frac{\beta' J_0 P}{2} \sum_{k=2}^P \binom{P}{k} \langle (\Delta m)_k \rangle \bar{m}^{P-k}$$

$$+ \frac{\beta'^2 J^2}{4} (\theta - 1) \sum_{k=2}^P \binom{P}{k} \langle (\Delta q_2)_k \rangle \bar{q}_2^{P-k} - \frac{\beta'^2 J^2}{4} \theta \sum_{k=2}^P \binom{P}{k} \langle (\Delta q_1)_k \rangle \bar{q}_1^{P-k}$$

(1.50)

where we use $\Delta X_a = X - \bar{X}_a$.

**Proof.** We use the fundamental theorem of Calculus

$$A^{(P)} = A^{(P)}(t = 1) = A^{(P)}(t = 0) + \int_0^1 \partial_s A^{(P)}(s)|_{s=1} dt.$$  

(1.51)

The last step we need is the computation of one-body term. We omit it since it is similar to RS case.

**Theorem 1.** In the thermodynamic limit, under 1RSB assumption, the expression of quenched statistical pressure is

$$A^{(P)} = \log 2 - \frac{\beta' J_0}{2} \bar{m}^P (P - 1) + \frac{\beta'^2 J^2}{4} \left[ 1 - P \bar{q}_2^{P-1} + (P - 1) \bar{q}_1^P \right]$$

$$- \frac{\beta'^2 J^2}{4} (P - 1) \theta (\bar{q}_2^P - \bar{q}_1^P) + \frac{1}{\theta} E_1 \log E_2 \cosh^\theta g(z, \bar{m}),$$

(1.52)

where

$$g(z, \bar{m}) = \frac{\beta' J_0}{2} \bar{m}^{P-1} + z^{(1)} \beta' J \sqrt{\frac{P}{2} \bar{q}_1^{P-1} + z^{(2)} \beta' J \sqrt{\frac{P}{2} (\bar{q}_2^{P-1} - \bar{q}_1^{P-1})}}$$

(1.53)
where the order parameters are ruled by

\[ \bar{m} = E_1 \left[ \mathbb{E}_2 \cosh^\theta g(z, \bar{m}) \tanh g(z, \bar{m}) \right] \tag{1.54} \]

\[ \bar{q}_1 = E_1 \left[ \mathbb{E}_2 \cosh^\theta g(z, \bar{m}) \tanh g(z, \bar{m}) \right]^2 \tag{1.55} \]

\[ \bar{q}_2 = E_1 \left[ \mathbb{E}_2 \cosh^\theta g(z, \bar{m}) \tanh^2 g(z, \bar{m}) \right] \tag{1.56} \]

where \( g(z, \bar{m}) \) is the same defined in (1.53).

Proof. If we use definition 5, the last three terms of expression (1.50) will vanish in the thermodynamic limit, so we recover (1.52).

Extremizing the statistical pressure in (1.52) w.r.t. the order parameters we find the self-consistency equations.

Remark 9. The expression (1.52) with \( J_0 = 0 \) is the same computed by [20, 22] through replica trick and by [3] via Hamilton-Jacobi technique in 1RSB assumption.

1.3 Resolution via transport equation

Now the aim is to show that, using a different strategy, such as the transport equation technique, already introduced in [1] for both assumption in classic SK model with ferromagnetic contribution, we are able to retrieve the same expression of quenched statistical pressure and self-consistency equations found via Guerra’s interpolating technique.

1.3.1 RS solution

In this subsection we find the RS solution of P-spin glass model via the transport equation technique. The strategy is to introduce an interpolating pressure \( A_N \) living in a fictitious space-time framework and recovering the intensive quenched pressure \( A_N \) of the original model in a specific point of this space, and to show that it fulfills a transport equation in such a way that the solution of the statistical mechanical problem is recast in the solution of a partial differential equation.

The definition of RS assumption for the order parameters is the same as (5).

Definition 11. Given the interpolating parameter \( t, x, w \), and \( z_i \sim \mathcal{N}(0, 1) \) standard Gaussian variables iid, the partition function is given as

\[ Z_N^{(P)}(t, x) := \sum_{\{\sigma\}} \exp \left[ \frac{\beta' J_0 N}{2} m^P(\sigma) + w N m(\sigma) + \right. \]

\[ + \sqrt{\beta' J} \int \sum_{i_1, \ldots, i_P = 1}^{N} z_{i_1} \cdots z_{i_P} \sigma_{i_1} \cdots \sigma_{i_P} + \]

\[ + \sqrt{x} \sum_{i=1}^{N} z_i \sigma_i, \tag{1.57} \]

where we set \( \beta' = 2\beta / P! \).

Similar to RS Guerra’s interpolation, we can define the interpolating pressure, the Boltzmann factor and the generalized measure.
Lemma 3. The partial derivatives of the interpolating pressure (1.14) w.r.t. \( t, x, w \) give the following expectation values:

\[
\frac{\partial A_N^{(P)}}{\partial t} = \frac{\beta J_0}{2} (m_P^+) + \frac{\beta^* J^2}{4} \left( 1 - \langle q_{12}^P \rangle \right),
\]
(1.58)

\[
\frac{\partial A_N^{(P)}}{\partial x} = \frac{1}{2} \left( 1 - \langle q_{12} \rangle \right),
\]
(1.59)

\[
\frac{\partial A_N^{(P)}}{\partial w} = \langle m \rangle.
\]
(1.60)

Proof. Since the procedures for the derivatives w.r.t. each parameter are analogous, we prove only the derivatives w.r.t. \( t \). The partial derivative of the interpolating quenched pressure with respect to \( t \) reads as

\[
\frac{\partial A_N^{(P)}}{\partial t} = \frac{1}{N} \left[ \frac{\beta^* N}{2} \omega (m_P^+) \right] + \\
+ \frac{\beta J}{2N\sqrt{T}} \sqrt{\sum_{k=1}^N} \sum_i E \left[ z_i \omega (\sigma_{k1} \cdots \sigma_{kr}) \right].
\]
(1.61)

Now, using the Stein’s lemma (1.21) on standard Gaussian variable \( z_i \)

\[
\frac{\partial A_N^{(P)}}{\partial t} = \frac{\beta J_0}{2} \langle m_P^+ \rangle + \frac{\beta^* J^2}{4N^2} \sum_i \left( E \left[ \omega (\sigma_{k1} \cdots \sigma_{kr}) \right] - E \left[ \omega (\sigma_{k1} \cdots \sigma_{kr})^2 \right] \right)
\]
(1.62)

\[
= \frac{\beta J_0}{2} \langle m_P^+ \rangle + \frac{\beta^* J^2}{4} \left( 1 - \langle q_{12}^P \rangle \right)
\]

Proposition 3. The interpolating pressure (1.57) at finite size obeys the following differential equation:

\[
\frac{dA_N^{(P)}}{dt} = \frac{\partial A_N^{(P)}}{\partial t} + \dot{x} \frac{\partial A_N^{(P)}}{\partial x} + \dot{w} \frac{\partial A_N^{(P)}}{\partial w} = S(t, r) + V_N(t, r),
\]
(1.63)

where we set

\[
\dot{x} = -\frac{P}{2} \beta^* J^2 q^{P-1}, \quad \dot{w} = -\frac{P}{2} \beta \beta J_0 \bar{m}^{-P-1}
\]
(1.64)

and

\[
S(t, r) := -\frac{P - 1}{2} \beta J_0 \bar{m}^{P} + \frac{\beta^* J^2}{4} (1 - P \bar{q}^{P-1} + (P - 1) \bar{q}^P),
\]
(1.65)

\[
V_N(t, r) := \frac{\beta J_0}{2} \sum_{k=2}^P \left( \begin{array}{c} P \\ k \end{array} \right) m^{P-k} (\Delta m)^k + \frac{\beta^* J^2}{4} \sum_{k=2}^P \left( \begin{array}{c} P \\ k \end{array} \right) q^{P-k} (\Delta q_{12})^k.
\]
(1.66)

Proof. Starting to evaluate explicitly \( \frac{\partial}{\partial t} A_N \) by using (1.59)-(1.60) and Definition (1.24)-(1.25) we write

\[
\frac{\partial}{\partial t} A_N^{(P)} = \frac{\beta J_0}{2} \left( \sum_{k=2}^P \left( \begin{array}{c} P \\ k \end{array} \right) (m - \bar{m})^k \bar{m}^{P-k} + \bar{m}^P (1 - P) + P \bar{m}^{-P-1} \langle m \rangle \right)
\]

\[
+ \frac{\beta^* J^2}{4} \left( 1 - \sum_{k=2}^P \left( \begin{array}{c} P \\ k \end{array} \right) (q_{12} - \bar{q})^k \bar{q}^{P-k} + q^{P}(1 - P) + P q^{-P-1} \langle q_{12} \rangle \right)
\]

(1.67)

\[
= V_N(t, r) + S(t, r) + \frac{P}{2} \beta J_0 \bar{m}^{-P-1} \langle m \rangle + \frac{\beta^* J^2}{2} \bar{q}^{P-1} \frac{1}{2} (1 - \langle q_{12} \rangle)
\]

\[
= V_N(t, r) + S(t, r) + \frac{\beta J_0}{2} \bar{m}^{-P-1} \frac{\partial A_N^{(P)}}{\partial w} + \frac{\beta^* J^2}{2} \bar{q}^{P-1} \frac{\partial A_N^{(P)}}{\partial x}
\]

(1.68)
Thus, by placing $\dot{r} = (\dot{x}, \dot{w})$ as in (1.64) we reach the thesis.

**Remark 10.** Since the RS assumption, which is linked to the fact that in the thermodynamic limit
\[
\langle (m - \bar{m})^k \rangle = 0 \quad k \geq 2
\]
\[
\langle (q_{12} - \bar{q})^k \rangle = 0 \quad k \geq 2
\]
we can say that $V_N$ defined in (1.66) vanishes.

**Proposition 4.** The transport equation associated to the interpolating pressure $A_N(t, r)$ in the thermodynamic limit and under the RS assumption is
\[
\frac{\partial A_{RS}(P)}{\partial t} = \frac{\beta' P q_{12}^P - 1}{2} \left( \frac{\partial A_{RS}(P)}{\partial x} \right) - \frac{P}{2} \beta' J_0 \bar{m} P - 1 \left( \frac{\partial A_{RS}(P)}{\partial w} \right) = \frac{P - 1}{2} \beta' J_0 \bar{m} P + \frac{\beta' P q_{12}^P}{4} (1 - P q_{12}^P - 1 + (P - 1) q_{12}^P),
\]
whose solution is given by
\[
A_{RS}(P)(t, r) = \ln 2 + \left\langle \ln \cosh \left[ w + \frac{P}{2} \beta' J_0 \bar{m} P - 1 t + z \sqrt{x - \frac{P}{2} \beta' P q_{12}^P - 1 t} \right] \right\rangle_z.
\]

Proof. We can find the transport equation applying Remark (10) in Prop. (3).

We compute the solution using the characteristic method on the transport equation:
\[
A_{RS}(P)(t, r) = A_{RS}(P)(0, r - \dot{r} t) + S(t, r)t.
\]
where $\dot{r} = (\dot{x}, \dot{w})$. Along the characteristics, the fictitious motion in the $(t, r)$ time-space is linear and returns
\[
x = x_0 - \frac{P}{2} \beta' P q_{12}^P - 1 t \quad w = w_0 - \frac{P}{2} \beta' J_0 \bar{m} P - 1 t
\]
where $r_0 = (x_0, w_0) = (x(t = 0), w(t = 0))$. The Cauchy condition at $t = 0$ is given by a direct computation at finite $N$ as
\[
A(P)(0, r - \dot{r} t) = A(P)(0, r_0)
\]
\[
= \frac{1}{N} \mathbb{E} \left\{ \sum_{\{\sigma\}} \exp \left[ w_0 N \psi m(\sigma) + \sqrt{2} \sum_{i=1}^{N} z_i \sigma_i \right] \right\}
\]
\[
= \ln 2 + \langle \ln \cosh [w_0 + z_0 \sqrt{x_0}] \rangle_z.
\]

Giving the suitable values of parameters we have the following
Corollary 1. The RS approximation of the quenched pressure for the P-spin glass model is obtained by posing $t = 1$ and $r = 0$ in (1.69), which returns

$$A_{\text{RS}}^{(P)}(t, r) = \ln 2 + \left\langle \ln \cosh \left[ \frac{P}{2} \beta' J_0 \bar{m}^{P-1} + z \sqrt{\frac{P}{2} \beta' J^2 \bar{q}^{P-1}} \right] \right\rangle_z \quad (1.73)$$

where the order parameters are ruled by the following self-consistency equations

$$\bar{m} = \left\langle \tanh \left[ \frac{P}{2} \beta J_0 \bar{m}^{P-1} + z \beta J \sqrt{\frac{P}{2} \bar{q}^{P-1}} \right] \right\rangle_z,$$

$$\bar{q} = \left\langle \tanh^2 \left[ \frac{P}{2} \beta J_0 \bar{m}^{P-1} + z \beta J \sqrt{\frac{P}{2} \bar{q}^{P-1}} \right] \right\rangle_z. \quad (1.74)$$

We stress that the expression found with transport equation method in Corollary (1) is the same found with Guerra’s interpolation in Proposition (1).

1.3.2 1-RSB solution

In this subsection we turn to the solution of the P-spin glass model via the generalized broken-replica interpolating technique, restricting the description at the first step of RSB.

The definition of two-replica overlap $q$ distribution is the same as Guerra’s interpolating technique (See eq. (8)).

Following the same route pursued in the previous sections, we need an interpolating partition function $Z$ and an interpolating quenched pressure $A$, that are defined hereafter.

Definition 12. Given the interpolating parameters $r = (x^{(1)}, x^{(2)}, w), t$ and the i.i.d. auxiliary fields $\{z_i^{(1)}, z_i^{(2)}\}_{i=1,...,N}$, with $z_i^{(1,2)} \sim N(0, 1)$ for $i = 1,...,N$, we can write the 1-RSB interpolating partition function $Z_N(t, r)$ for the P-spin glass model (1.1) recursively, starting by

$$Z_2^{(P)}(t, r) := \sum_{\{\sigma\}} \exp \left[ \frac{\beta' J_0 N}{2} m^P(\sigma) + w N m(\sigma) + \sqrt{\beta' J} \sqrt{\frac{1}{2N^P-1}} \sum_{i_1,\ldots,i_P=1}^{N-1,\ldots,N} z_{i_1\ldots i_P} \sigma_{i_1} \cdots \sigma_{i_P} + \sum_{a=1}^{2} \left( \sqrt{x^{(a)}} \sum_{i=1}^{N} z_i^{(a)} \sigma_i \right) \right]. \quad (1.75)$$

Averaging out the fields recursively, we define

$$Z_1(t, r) := \mathbb{E}_0 \left[ Z_2(t, r)^{\theta} \right]^{1/\theta} \quad (1.76)$$

$$Z_0(t, r) := \exp \mathbb{E}_0 [ \ln Z_1(t, r)] \quad (1.77)$$

$$Z_N(t, r) := Z_0(t, r), \quad (1.78)$$

where with $\mathbb{E}_a$ we mean the average over the variables $z_i^{(a)}$’s for $a = 1, 2$, and with $\mathbb{E}_0$ we shall denote the average over the variables $z_i,...,i_P$’s.
The definition of 1RSB interpolating pressure at finite volume \( N \) and in the thermodynamic limit is the same as Guerra’s interpolating technique (10). The notation for the generalized average are the analogous as Guerra’s interpolation.

The next step is building a transport equation for the interpolating quenched pressure, for which we preliminary need to evaluate the related partial derivatives, as discussed in the next

**Lemma 4.** The partial derivative of the interpolating quenched pressure with respect to a generic variable \( \rho \) reads as

\[
\frac{\partial}{\partial \rho} A_{N}^{(p)}(t, r) = \frac{1}{N} E_0 E_1 E_2 \left[ \mathcal{W}_2 \omega(\partial_{\rho} B(\sigma; t, r)) \right].
\]  

(1.79)

In particular,

\[
\frac{\partial}{\partial t} A_{N}^{(p)} = \frac{1}{2} \left( 1 - (1 - \theta) \langle q_{12} \rangle_2 - \theta \langle q_{12} \rangle_1 \right)
\]  

(1.80)

\[
\frac{\partial}{\partial x^{(1)}} A_{N}^{(p)} = \frac{1}{2} \left( 1 - (1 - \theta) \langle q_{12} \rangle_2 - \theta \langle q_{12} \rangle_1 \right)
\]  

(1.81)

\[
\frac{\partial}{\partial x^{(2)}} A_{N}^{(p)} = \frac{1}{2} \left( 1 - (1 - \theta) \langle p_{12} \rangle_2 \right)
\]  

(1.82)

\[
\frac{\partial}{\partial \omega} A_{N}^{(p)} = \langle m \rangle
\]  

(1.83)

**Proof.** The proof is pretty lengthy and basically requires just standard calculations similar to RS case, since we omit it. Here we just prove that, in complete generality

\[
\frac{\partial}{\partial \rho} A_{N}^{(p)}(t, r) = \frac{1}{N} E_0 E_1 [\partial_{\rho} \ln Z_1]
\]  

\[
= \frac{1}{N} E_0 E_1 \left[ \frac{1}{\theta} Z_1 \left[ Z_0^\theta \right]^{1/\theta - 1} E_2 [\partial_{\rho} Z_2^\theta] \right]
\]  

\[
= \frac{1}{N} E_0 E_1 E_2 \left[ \frac{Z_0^\theta}{E_2} \frac{\partial_{\rho} Z_2}{Z_2} \right]
\]  

\[
= \frac{1}{N} E_0 E_1 E_2 \left[ \mathcal{W}_2 \partial_{\rho} Z_2 \right].
\]  

(1.84)

**Proposition 5.** The streaming of the 1-RSB interpolating quenched pressure obeys, at finite volume \( N \), a standard transport equation, that reads as

\[
\frac{dA^{(p)}}{dt} = \partial_t A^{(p)} + \dot{z}^{(1)} \partial_{x^1} A^{(p)} + \dot{z}^{(2)} \partial_{x^2} A^{(p)} + \dot{w} \partial_{\omega} A^{(p)} = S(t, r) + V_N(t, r)
\]  

(1.85)

where

\[
S(t, r) := \frac{\beta^2 J^2}{4} \left[ 1 - P q_2 P - 1 \right] \]  

\[
= \left( \theta - 1 \right) \sum_{k=2}^{P} \left( \begin{array}{c} P \\ k \end{array} \right) q_2^{P-k}(\langle \Delta q_2 \rangle_2^k - \theta \langle \Delta q_2 \rangle_1^k) \right) \right] - \left( P - 1 \right) \frac{\beta J_0}{2} \bar{m} P,
\]  

(1.86)

\[
V_N(t, r) := \frac{\beta^2 J^2}{4} \left\{ \left( \theta - 1 \right) \sum_{k=2}^{P} \left( \begin{array}{c} P \\ k \end{array} \right) q_2^{P-k}(\langle \Delta q_2 \rangle_2^k - \theta \langle \Delta q_2 \rangle_1^k) \right) \right] \right] + \frac{\beta J_0}{2} \sum_{k=2}^{P} \left( \begin{array}{c} P \\ k \end{array} \right) \langle \Delta m \rangle^k,
\]  

(1.87)
where, as usual, we use $\Delta X_a = X - \bar{X}_a$.

**Proof.** Similar to the case RS (1.24-1.25), we have, for $a = 1, 2$

\[
\langle \eta_{12}^P \rangle_a = \sum_{k=1}^P \left( \begin{array}{c} P \\ k \end{array} \right) \eta_k a^{-k}(\Delta \eta_k)^k + \eta_k a (1 - P) + P \eta_k a^{-1}(\eta_{12})_a. \quad (1.88)
\]

Now, starting to evaluate explicitly $\frac{\partial}{\partial t} A_N^{(P)}$ by using (1.81 - 1.83) we write

\[
\frac{\partial}{\partial t} A_N^{(P)} = \frac{\beta' j_0}{2} \left[ \sum_{k=2}^P \left( \begin{array}{c} P \\ k \end{array} \right) (\Delta m)^k \bar{m}^{P-k} + \bar{m}^P (1 - P) + P \bar{m}^{-1}(m) \right] + \frac{\beta^2 j_2}{4} \left\{ 1 + (\theta - 1) \left[ \sum_{k=1}^P \left( \begin{array}{c} P \\ k \end{array} \right) \eta_k a^{-k}(\Delta \eta_k)^k + \eta_k a (1 - P) + P \eta_k a^{-1}(\eta_{12}) \right] \right\} = \frac{\beta' j_0 P}{2} \bar{m} m^{-1} \partial \omega A_N^{(P)} + \frac{\beta^2 j_2 P}{2} \eta_k a^{-1}(\eta_{12}) \partial x_2 A_N^{(P)} + \frac{\beta^2 j_2 P}{2} (\bar{q_2} - \eta_k a^{-1}) \partial x_2 A_N^{(P)} . \quad (1.89)
\]

Thus, by placing

\[
\dot{x}_{(1)} = -\frac{\beta^2 j_2 P}{2} \bar{q_2} a^{-1}, \quad (1.90)
\]

\[
\dot{x}_{(2)} = -\frac{\beta^2 j_2 P}{2} (\bar{q_2} a^{-1} - \eta_k a^{-1}), \quad (1.91)
\]

\[
\dot{w} = -\frac{\beta' j_0 P}{2} \bar{m} m^{-1}. \quad (1.92)
\]

thus, we reach the thesis. \[\square\]

**Remark 11.** In the thermodynamic limit, in the 1RSB scenario, we have

\[
\lim_{N \to \infty} \langle (m - \bar{m})^2 \rangle = 0, \quad (1.93)
\]

\[
\lim_{N \to \infty} \langle (q_{12} - \bar{q})^2 \rangle_i = 0; \quad i = 1, 2. \quad (1.94)
\]

Similar to the RS approximation, in the thermodynamic limit we have that the central moments greater than 2 tend to zero; in this way

\[
\lim_{N \to \infty} V_N(t, r) = 0. \quad (1.95)
\]

Exploiting Remark 11 we can prove the following

**Proposition 6.** The transport equation associated to the interpolating pressure function $A^{(P)}(t, r)$, in the thermodynamic limit and under the 1RSB assumption, can be written as

\[
\frac{\partial}{\partial t} A^{(P)} + \dot{x}_{(1)} \partial x_1 A^{(P)} + \dot{x}_{(2)} \partial x_2 A^{(P)} + \dot{z} \partial z A^{(P)} + \dot{w} \partial w A^{(P)} = \frac{\beta' j_0 \bar{m} m^{-1}(P - 1)}{2} \quad (1.96)
\]
whose explicit solution is given by
\[
\mathcal{A}^{(P)} = \log 2 + \frac{1}{\theta} \mathbb{E}_1 \log \mathbb{E}_2 \cosh^\theta \left( w_0 + z^{(1)} \sqrt{x_0^{(1)}} + z^{(2)} \sqrt{x_0^{(2)}} \right) \\
+ t \left\{ \frac{\beta^2 J^2}{4} \left[ 1 - P\bar{q}_2^{P-1} + (P - 1)\bar{q}_2^P - \theta(p - 1)(\bar{q}_2^P - \bar{q}_1^P) \right] - \frac{\beta^2 J_0 \bar{m}^P (P - 1)}{2} \right\} 
\] (1.97)

Proof. If we consider (1.95) we have (1.96) We compute the transport equation through characteristic method:
\[
\mathcal{A}_{1RSB}^{(P)}(t, r) = \mathcal{A}_{1RSB}^{(P)}(0, r - \dot{r}t) + S_{1RSB}(t, r)t. 
\] (1.98)

By putting (1.90)-(1.92) into (1.85) we find
\[
\mathcal{A}^{(P)}(t, r) = \mathcal{A}^{(P)}(0, r_0) + t \left\{ -\frac{\beta^2 J_0 \bar{m}^P (P - 1)}{2} \\
+ \frac{\beta^2 J^2}{4} \left[ 1 - P\bar{q}_2^{P-1} + (P - 1)\bar{q}_2^P - \theta(p - 1)(\bar{q}_2^P - \bar{q}_1^P) \right] \right\} 
\] (1.99)

where \( r_0 \) can be obtained by using the equation of motion
\[
r = r_0 + \dot{r}t 
\] (1.100)

where the velocities are defined in (1.90)-(1.92). Then, all we have to compute is \( \mathcal{A}_0^{(P)}(0, r_0) \), that can be easily done because at \( t = 0 \) the two body interaction vanishes and it can be written as
\[
\mathcal{A}_0^{(P)}(0, r_0) = \log 2 + \frac{1}{\theta} \mathbb{E}_1 \log \mathbb{E}_2 \cosh^\theta \left( w_0 + z^{(1)} \sqrt{x_0^{(1)}} + z^{(2)} \sqrt{x_0^{(2)}} \right). 
\] (1.101)

We omit the computation since it is similar to one-body term in Guerra’s interpolating scheme, 1RSB assumption.

Then, putting together (1.99)-(1.101) and (1.90)-(1.92), we finally achieve an explicit expression for the interpolating pressure of the P-spin glass model in the 1RSB approximation.

To sum up, we have the following main theorem for the 1RSB scenario

**Theorem 2.** The 1-RSB quenched pressure for P-spin glass model, in the thermodynamic limit, reads as
\[
\mathcal{A}^{(P)}(\beta) = \log 2 + \frac{1}{\theta} \mathbb{E}_1 \log \mathbb{E}_2 \cosh^\theta g(z, \bar{m}) - (P - 1)\frac{\beta^2 J_0}{2} \bar{m}^P \\
+ \frac{\beta^2 J^2}{4} \left[ 1 + \theta - 1)\bar{q}_2^P (1 - P) - \theta\bar{q}_1^P (1 - P) - P\bar{q}_2^{P-1} \right] 
\] (1.102)

where
\[
g(z, \bar{m}) = \beta^2 J_0 \frac{P}{2} \bar{m}^{P-1} + \beta^2 J_2^{(1)} \sqrt{\frac{P}{2} \bar{q}_2^{P-1} + \beta^2 J_2^{(2)} \sqrt{\frac{P}{2} (\bar{q}_2^{P-1} - \bar{q}_1^{P-1})}. 
\] (1.103)
The order parameters are ruled by

\[
\begin{align*}
\bar{m} &= E_1 \left[ \frac{E_2 \cosh \theta g(z, \bar{m}) \tanh g(z, \bar{m})}{E_2 \cosh \theta g(z, \bar{m})} \right] \\
\bar{q}_1 &= E_1 \left[ \frac{E_2 \cosh \theta g(z, \bar{m}) \tanh g(z, \bar{m})}{E_2 \cosh \theta g(z, \bar{m})} \right]^2 \\
\bar{q}_2 &= E_1 \left[ \frac{E_2 \cosh \theta g(z, \bar{m}) \tanh^2 g(z, \bar{m})}{E_2 \cosh \theta g(z, \bar{m})} \right]
\end{align*}
\] (1.104, 1.105, 1.106)

with \( g(z, \bar{m}) \) is the same defined in (1.103).

Proof. By taking \( r = 0 \) and \( t = 1 \) we find the P-spin glass pressure in the 1RSB approximation. \( \square \)

We stress that the expression in (1.102) is the same found with Guerra’s interpolating scheme in 1RSB assumption.

2 Gaussian P-Spin Glass

In this section we face with Gaussian P-spin glass model, namely spin glass model with Gaussian spins with \( P \)-wise interactions. Several works describe the principal aspects of this model, either in our assumption [11, 15] or in the so-called spherical version, where the Gaussian spin are on a spherical surface[17, 26, 32]. We stress that two assumptions on Gaussian spins are equivalent, as shown in [9]. We deepen the model through Guerra’s interpolating technique and transport equation for RS and 1RSB assumptions.

2.1 Generalities

Definition 13. Let \( P \in \mathbb{N} \) and \( z_i \in \mathbb{R}, \ i = 1, 2, \ldots, N \) be a configuration of \( N \) gaussian spins, the Hamiltonian of the \( P \)-spin Gaussian model is defined as

\[
H_N^{(P)}(z|J) := -\frac{1}{P!} \prod_{i_1, \ldots, i_P = 1}^{N, \ldots, N} J_{i_1, \ldots, i_P} z_{i_1} \cdots z_{i_P}
\] (2.1)

where the \( P \)-wise quenched couplings \( J = \{ J_{i_1, \ldots, i_P} \}_{i_1, \ldots, i_P = 1, \ldots, N} \) are given by

\[
J_{i_1, \ldots, i_P} := \sqrt{\frac{2}{N! - 1}} J_{i_1, \ldots, i_P}
\] (2.2)

with \( J_{i_1, \ldots, i_P} \) i.i.d. standard random variables drawn from \( P(j_{i_1, \ldots, i_P}) = N(0; 1) \).

Definition 14. The partition function related to the Hamiltonian (2.1) is given by

\[
Z_N(\beta, J) := \int d\mu(z) \exp \left[ -\beta H_N^{(P)}(z|J) \right] = \mathbb{E}_z e^{-\beta H_N^{(P)}(z|J)},
\] (2.3)

where, \( d\mu(z) = \prod_i (2\pi)^{-1/2} e^{-\sum_i z_i^2/2} \) and \( \beta \in \mathbb{R}^+ \) is the inverse temperature in proper units.
As early pointed out for instance in [12], unfortunately these kind of models need to be regularized; in fact, the right side of (2.3) is not always well defined as the P-wise interactions bridges soft spins which are both Gaussian distributed. It will be clear soon that a good definition is

\[
Z_N(\beta, J, \lambda) := \mathbb{E}_z \exp \left[ -\beta H_N^{(P)}(z|J) - \left( \frac{\beta}{P!} \right)^2 \frac{1}{N^{P-1}} \left( \sum_{i=1}^{N} z_i^2 \right)^P + \frac{\lambda}{2} \sum_{i=1}^{N} z_i^2 \right],
\]

(2.4)

where the first additional term is needed for convergence of the integral over the Gaussian measure \(d\mu(z)\). Instead, the new parameter \(\lambda\), within the last term of (2.4), is just to modify the variance of the soft spins, as in several applications this can sensibly vary.

We introduce the Boltzmann average induced by the partition function (2.3), denoted with \(\omega_J\) and, for an arbitrary observable \(O(\sigma)\), defined as

\[
\omega_J(O(\sigma)) := \mathbb{E}_z O(\sigma)e^{-\beta H_N(\sigma|J)} / Z_N(\beta, J, \lambda).
\]

(2.5)

This can be further averaged over the realization of the \(J_{i_1, \ldots, i_P}\)'s (also referred to as quenched average) to get

\[
\langle O(\sigma) \rangle := \mathbb{E}\omega_J(O(\sigma)).
\]

(2.6)

**Definition 15.** The intensive quenched pressure of the Gaussian P-spin glass model (2.1) is defined as

\[
A_N(\beta, \lambda) := \frac{1}{N} \mathbb{E} \ln Z_N(\beta, J, \lambda),
\]

(2.7)

and its thermodynamic limit, assuming its existence, is referred to as

\[
A(\beta, \lambda) := \lim_{N \to \infty} A_N(\beta, \lambda).
\]

(2.8)

In order to solve the model we want to find out an explicit expression for the quenched pressure (2.8) in terms of the natural order parameter of the theory, namely the two-replica overlap \(q_{12}\), defined in the following

**Definition 16.** The order parameter used to describe the macroscopic behavior of the model is the two-replica overlap, introduced as

\[
q_{12} := \frac{1}{N} \sum_{i=1}^{N} z_i^{(1)} z_i^{(2)}.
\]

(2.9)

### 2.2 Resolution via Guerra’s interpolation

Mirroring P-spin glass model, the purpose of this section is to solve the Gaussian P-spin glass model via Guerra’s interpolating technique. To do so, we recover the expressions of statistical pressure (which is equivalent to free energy) and self consistency equations in the approximation of replica symmetry (RS) and first step of replica symmetry breaking (RSB).

#### 2.2.1 RS solution

Using the same definition for \(q_{12}\) presented for the P-spin glass model (def. 5), we can introduce the Guerra’s interpolating partition function of the Gaussian P-spin glass model.
Definition 17. Given the interpolating parameter $t \in [0,1]$, $A$, $B \in \mathbb{R}$ and $J \sim \mathcal{N}(0,1)$ for $i = 1, \ldots, N$ standard Gaussian variables i.i.d., the Guerra's interpolating partition function is given as

$$Z^{(p)}_N(t) := \mathbb{E}_z \exp \left[ \beta \sqrt{1 - \frac{2}{2N^p - 1}} \sum_{i_1, \ldots, i_p = 1}^{N} J_{i_1, \ldots, i_p} z_{i_1} \cdots z_{i_p} - \frac{\beta^2}{4N^p - 1} \left( \sum_{i=1}^{N} z_i^2 \right)^p \right]$$

(2.10)

where $\beta' = 2 \beta / P!$.

Definition 18. The interpolating pressure for the Gaussian P-spin glass model (2.1), at finite $N$, is introduced as

$$A^{(p)}_N(t) := \frac{1}{N} \mathbb{E} \left[ \ln Z^{(p)}_N(t) \right]$$

(2.11)

where the expectation $\mathbb{E}$ is now meant over $J_{i_1, \ldots, i_p}$ and $\tilde{J}_i$, in the thermodynamic limit,

$$A^{(p)}(t) := \lim_{N \to \infty} A^{(p)}_N(t).$$

(2.12)

By setting $t = 1$ the interpolating pressure recovers the original one (2.7), that is $A^{(p)}_N(\beta, \lambda) = A^{(p)}_N(t = 1)$.

Remark 12. The interpolating structure implies an interpolating measure, whose related Boltzmann factor reads as

$$B(z; t) := \exp \left[ \beta H(z; t) \right];$$

(2.13)

In this way the partition function is written as $Z_N(t) = \mathbb{E}_z B(z; t)$.

A generalized average follows from this generalized measure as

$$\omega_i(O(z)) := \mathbb{E}_z O(z) B(z; t)$$

(2.14)

and

$$\langle O(z) \rangle_t := \mathbb{E}[\omega_i(O(z))]$$

(2.15)

where $\mathbb{E}$ denotes the average over $\tilde{J}_i, J_{i_1, \ldots, i_p}$ and $\{J_i\}_{i = 1, \ldots, N}$.

Of course, when $t = 1$ the standard Boltzmann measure and related average is recovered. Hereafter, in order to lighten the notation, we will drop the subindices $t$.

Lemma 5. The $t$ derivative of interpolating pressure is given by

$$\frac{dA^{(p)}(t)}{dt} := \beta' \frac{2}{4} \left( \langle q_{12} \rangle^2 - \frac{2A^2}{\beta^2} \langle q_{12} \rangle \right) - \frac{1}{2N} (A^2 + B) \sum_{i=1}^N \langle z_i^2 \rangle$$

(2.16)

Proof. Deriving equation (2.11) with respect to $t$, we get

$$\frac{dA^{(p)}(t)}{dt} = \frac{\beta'}{2N \sqrt{1 - t}} \mathbb{E} \left[ \sum_i J_i \omega_i(z_i) \cdots z_{i_p} \right] - \frac{\beta^2}{4N^p} \mathbb{E} \left[ \sum_i \omega_i(z_i^2) \right] +$$

(2.17)

$$- \frac{1}{2N \sqrt{1 - t}} \mathbb{E} \left[ A \sum_i \tilde{J}_i \omega_i(z_i) \right] - \frac{1}{2N} \mathbb{E} \left[ B \sum \omega_i(z_i^2) \right].$$


Now, using the Stein’s lemma (also known as Wick’s theorem) on standard Gaussian variable \( \tilde{J}_i \) and \( J_i \), we may rewrite the second and the third member of (2.17) as
\[
\frac{\beta'}{2N\sqrt{T}} \sqrt{\frac{1}{2N^{P-1}}} \mathbb{E} \left[ \sum_i J_i \omega(z_{i1} \cdots z_{iP}) \right] - \frac{1}{2N\sqrt{1-t}} \mathbb{E} \left[ \sum_i \tilde{J}_i \omega(z_i) \right] = D_1 + D_2.
\]

Let’s investigate those three terms:
\[
D_1 = \frac{\beta'}{2N\sqrt{T}} \sqrt{\frac{1}{2N^{P-1}}} \mathbb{E} \left[ \sum_i \partial_j \omega(z_{i1} \cdots z_{iP}) \right]
= \frac{\beta'^2}{4NP} \left( \sum_i \mathbb{E} [\omega((z_{i1} \cdots z_{iP})^2)] - \sum_i \mathbb{E} [\omega(z_{i1} \cdots z_{iP})^2] \right) \tag{2.18}
\]
\[
D_2 = -\frac{1}{2N\sqrt{1-t}} \mathbb{E} \left[ A \sum_i \partial_j \omega(z_i) \right]
= -\frac{1}{2N} A^2 \left( \sum_i \mathbb{E} [\omega(z_i^2)] - \sum_i \mathbb{E} [\omega(z_i)^2] \right) \tag{2.19}
\]
Rearranging together (2.18) and (2.19) we obtain the thesis.

**Remark 13.** We stress that, for the RS assumption presented in Definition (5), we can use the relations (1.24) and (1.25) to fix the two constants as
\[
A^2 = \frac{P}{2} \beta'^2 \bar{q}^{P-1}, \tag{2.20}
\]
\[
B = -A^2
\]
the (2.16) in the thermodynamical limit reads as
\[
\frac{dA^{(P)}(t)}{dt} := (P-1) \frac{\beta'^2}{4} \bar{q}^P \tag{2.21}
\]
which is now independent of \( t \).

**Theorem 3.** In the thermodynamic limit \((N \to \infty)\) and under RS assumption, applying the Fundamental Theorem of Calculus and using the suitable values of \( A \) and \( B \), we find the quenched pressure for the Gaussian \( P \)-spin glass model as
\[
A^{(P)}(\beta', \lambda) = \frac{P}{2} \bar{q}^{P-1} \left( 1 - \frac{1}{2} \ln \left[ 1 - \lambda + \frac{P}{2} \beta'^2 \bar{q}^{P-1} \right] \right) + \frac{1}{2} \ln \left[ 1 - \lambda + \frac{P}{2} \beta'^2 \bar{q}^{P-1} \right] + (P-1) \frac{\beta'^2}{4} \bar{q}^P. \tag{2.22}
\]
where the self-consistency equation which rule the order parameter are given by the resolution or the implicit equation
\[
\bar{q} = \frac{\frac{P}{2} \beta'^2 \bar{q}^{P-1}}{\left( 1 - \lambda + \frac{P}{2} \beta'^2 \bar{q}^{P-1} \right)^2}. \tag{2.23}
\]
Proof. Using the Fundamental Theorem of Calculus (1.30) and computing the one-body terms

\[
A(P)(t = 0) = \frac{1}{N} \mathbb{E}_z \ln \left\{ \mathbb{E}_z \exp \left[ A \sum_{i=1}^{N} \tilde{j}_i z_i + \frac{1}{2} (B + \lambda) \sum_{i=1}^{N} z_i^2 \right] \right\},
\]

\[
= \frac{A^2}{2(1 - \lambda - B)} - \frac{1}{2} \ln |1 - \lambda - B| \tag{2.24}
\]

\[
= \frac{2 \beta' \gamma^2 q^{P-1}}{2(1 - \lambda + \frac{P}{2} \beta' \gamma^2 q^{P-1})} - \frac{1}{2} \ln \left[ 1 - \lambda + \frac{P}{2} \beta' \gamma^2 q^{P-1} \right].
\]

Finally, putting (2.21) and (2.24) in (1.30), we find (2.22).

Extremizing the statistical pressure in (2.22) w.r.t. the order parameter we find equation (2.23). 

A, briefly, deeper study of the interesting case of \( P = 2 \) is done in Section 2.4, for a complete discussion we remand to [9].

Remark 14. We highlight that we recover the same results in [17] in RS assumption computed via replica trick.

Without going into details of implicit equation (2.23), if we focus on \( P > 2 \) case, we stress that the solution \( \bar{q} = 0 \) is always a saddle point; for \( \bar{q} \neq 0 \) we can see numerically that there is only an acceptable solutions, which maximize the quenched statistical pressure. For every details we remind to [17].

2.2.2 1RSB solution

In this subsection we turn to the solution of the Gaussian P-spin glass model via the Guerra’s interpolating technique, restricting the description at the first step of RSB.

Following the same route pursued in the previous sections, we need an interpolating partition function \( Z \) and an interpolating quenched pressure \( A \), that are defined hereafter.

Definition 19. Given the interpolating parameter \( t \) and the i.i.d. auxiliary fields \( \{ \tilde{j}_i^{(1)}, \tilde{j}_i^{(2)} \}_{i=1,...,N} \), with \( \tilde{j}_i^{(1,2)} \sim \mathcal{N}(0, 1) \) for \( i = 1, ..., N \) we can write the 1-RSB interpolating partition function \( Z_N(t) \) for the P-spin Gaussian model (2.4) recursively, starting by

\[
Z_2^{(P)}(t) := \mathbb{E}_z \exp \left[ \sqrt{\beta'} \sqrt{\frac{1}{2N^{P-1}}} \sum_{i_1, \ldots, i_P=1}^{N} J_{i_1 \cdots i_P} z_{i_1} \cdots z_{i_P} - t \frac{\beta' \gamma^2}{4N^{P-1}} \left( \sum_{i=1}^{N} z_i^2 \right)^P \right]
\]

\[
+ \sqrt{1 - t} \sum_{a=1}^{2} \left( A^{(a)} + (1 - t) B \right) \frac{1}{2} \sum_{i=1}^{N} z_i^2 + \frac{\lambda}{2} \sum_{i=1}^{N} z_i^2 \right), \tag{2.25}
\]

where the \( J_{i_1 \cdots i_P} \)'s i.i.d. standard Gaussian. Averaging out the fields recursively, we define

\[
Z_2^{(P)}(t) := \mathbb{E}_z \left[ Z_2^{(P)}(t)^\theta \right]^{1/\theta} \tag{2.26}
\]

\[
Z_0^{(P)}(t) := \exp \mathbb{E}_z \left[ \ln Z_1^{(P)}(t) \right] \tag{2.27}
\]

\[
Z_N^{(P)}(t) := Z_0^{(P)}(t), \tag{2.28}
\]
where with $E_a$ we mean the average over the variables $\tilde{J}_i^{(a)}$’s, for $a = 1, 2$, and with $E_0$ we shall denote the average over the variables $J_1, \ldots, J_p$’s.

**Definition 20.** The 1RSB interpolating pressure, at finite volume $N$, is introduced as

$$A_N^{(P)}(t) := \frac{1}{N}E_0[\ln Z_0^{(P)}(t)],$$

and, in the thermodynamic limit, assuming its existence

$$A^{(P)}(t) := \lim_{N \to \infty} A_N^{(P)}(t).$$

By setting $t = 1$, the interpolating pressure recovers the standard pressure (2.7), that is, $A_N^{(P)}(t = 1)$.

Now the next step is computing the $t$-derivative of the statistical pressure. In this way we can apply the fundamental theorem of calculus and find the solution of the original model.

**Lemma 6.** The derivative w.r.t. $t$ of interpolating statistical pressure can be written as

$$d_t A_N^{(P)} = \frac{\beta P}{4} \left[ (\theta - 1)\langle q_2^{P-1} \rangle_2 - \theta \langle q_1^{P-1} \rangle_1 \right] - \frac{(A^{(1)})^2}{2} \left[ \frac{1}{N} \sum_i \langle z_i^2 \rangle + (\theta - 1)\langle q_2_2 \rangle - \theta \langle q_1_2 \rangle \right]$$

$$- \frac{(A^{(2)})^2}{2} \left[ \frac{1}{N} \sum_i \langle z_i^2 \rangle + (\theta - 1)\langle q_2 \rangle \right] - \frac{B}{N} \sum_i \langle z_i^2 \rangle$$

Since the proof is similar to SK spin glass case, we omit it.

**Remark 15.** If we use (1.88), in 1RSB assumption we fix the constants as

$$A_1 = \beta^2 P \frac{\bar{q}_2^{P-1}}{2},$$

$$A_2 = \beta^2 P \left( \bar{q}_2^{P-1} - \bar{q}_1^{P-1} \right),$$

$$B = -\beta^2 \frac{P}{2} \bar{q}_2^{P-1}.$$  

Thus, the derivative w.r.t. $t$ in the thermodynamic limit is computed as

$$d_t A_N^{(P)} = \frac{\beta^2}{4} (P - 1)\bar{q}_2^P - \frac{\beta^2}{4} \theta(P - 1)(\bar{q}_2^P - \bar{q}_1^P).$$

**Proposition 7.** At finite size and under 1RSB assumption applying the Fundamental Theorem of Calculus and using the suitable values of $A^{(1)}, A^{(2)}, B$, we find the quenched pressure for the Gaussian
Proof. We use the fundamental theorem of Calculus (1.30). The last step we need is the computation of one-body term. Computing the Gaussian integral, we have

\[
A^{(P)} = \frac{\beta'2}{4}(\theta - 1) \left[ \sum_{k=2}^{P} \langle q_1^k \rangle \bar{q}_2^{P-k} + (1 - P) \bar{q}_2^P \right]
\]

\[-\frac{\beta'^2}{4} \theta \left[ \sum_{k=2}^{P} \langle q_1^k \rangle \bar{q}_1^{P-k} + (1 - P) \bar{q}_1^P \right] - \frac{1}{2} \log \left( 1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_2^{P-1} \right) - \frac{1}{2} \log \left( 1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_1^{P-1} \right) - \frac{1}{2} \log \left( 1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_2^{P-1} \right) - \frac{1}{2} \log \left( 1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_1^{P-1} \right) \]

\[
+ \frac{1}{2\theta} \log \left( \frac{1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_2^{P-1}}{1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_1^{P-1}} \right) \]

\[
+ \frac{\beta'^2 P}{4} \left( \bar{q}_2^{P-1} - \bar{q}_1^{P-1} \right)
\]

In this way, we obtain the thesis.

Now we have the following

Theorem 4. In the thermodynamic limit, under 1RSB assumption, the expression of quenched statistical pressure for Gaussian P-spin glass model (2.4) is

\[
A^{(P)}(\beta, \lambda) = \frac{\beta'^2}{4} (P - 1) \bar{q}_2^P - \frac{\beta'^2}{4} \theta (P - 1) (\bar{q}_2^P - \bar{q}_1^P)
\]

\[-\frac{1}{2} \log \left( 1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_2^{P-1} \right) - \frac{1}{2} \log \left( 1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_1^{P-1} \right) - \frac{1}{2} \log \left( 1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_2^{P-1} \right) - \frac{1}{2} \log \left( 1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_1^{P-1} \right) \]

\[
+ \frac{1}{2\theta} \log \left( \frac{1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_2^{P-1}}{1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_1^{P-1}} \right) - \frac{1}{2} \log \left( 1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_2^{P-1} \right) - \frac{1}{2} \log \left( 1 - \lambda + \beta'^2 \frac{P}{2} \bar{q}_1^{P-1} \right) \]

\[
+ \frac{\beta'^2 P}{4} \left( \bar{q}_2^{P-1} - \bar{q}_1^{P-1} \right)
\]
where the order parameters are ruled by the implicit equations

\[ \begin{align*}
\bar{q}_1 &= \frac{\beta' \sqrt{2} q_1^{P-1}}{\left(1 - \lambda + \beta' \sqrt{2} q_2^{P-1} - \beta' \sqrt{2} \left(q_2^{P-1} - q_1^{P-1}\right)\right)^2} \\
\bar{q}_2 &= \bar{q}_1 + \frac{\beta' \sqrt{2} \bar{q}_2^{P-1} - \beta' \sqrt{2} \left(q_2^{P-1} - q_1^{P-1}\right)}{\left(1 - \lambda + \beta' \sqrt{2} q_2^{P-1}\right) \left(1 - \lambda + \beta' \sqrt{2} q_2^{P-1}\right)}
\end{align*} \tag{2.39} \]

Proof. If we use (2.35) and (2.37) in (1.30) we recover (2.38).

Extremizing the statistical pressure in (2.38) w.r.t. the order parameters we find the self-consistency equations.

Remark 16. In the case of pairwise interaction \((P = 2)\) the 1RSB solution coincide with the RS one, as shown in Section 2.4.

Moreover, we recover the same expression in [17]; we remind to it for the study of solutions behaviour of order parameters (2.39).

2.3 Resolution via transport equation

Now the aim is to show that, using the transport equation technique, already introduced in Section 1.3 for the RS assumption and for 1RSB in P-spin glass network, we are able to retrieve the same expression of statistical pressure above found by Guerra’s Interpolation.

2.3.1 RS solution

As previous presented in Section 1.3, now we introduce the interpolating partitions function for the Gaussian P-spin model.

Definition 21. Given the interpolating parameter \(t, x, w\), and \(\tilde{J}_i \sim \mathcal{N}(0, 1)\) standard Gaussian variables iid, the partition function is given as

\[ \begin{align*}
Z_N^{(P)}(t, r) &= \mathbb{E}_z \exp \left[ \beta' \sqrt{T} \frac{1}{2NP-1} \sum_{i_1, \ldots, i_P=1}^N J_{i_1, \ldots, i_P} z_{i_1} \cdots z_{i_P} - t \frac{\beta' \sqrt{T}}{4NP-1} \left( \sum_{i=1}^N z_i^2 \right)^P \\
&\quad + \sqrt{T} \sum_{i=1}^N \tilde{J}_i z_i + \frac{w}{2} \sum_{i=1}^N z_i^2 + \frac{\lambda}{2} \sum_{i=1}^N z_i^2 \right],
\end{align*} \tag{2.40} \]

where we set \(\beta' = 2\beta/P!\).

Similar to RS Guerra’s interpolation, we can define the interpolating pressure, the Boltzmann factor and the generalized measure.

Lemma 7. The partial derivatives of the interpolating pressure (2.11) w.r.t. \(t, x, w\) give the following expectation values:

\[ \begin{align*}
\frac{\partial A_N^{(P)}}{\partial t} &= -\frac{\beta' \sqrt{T}}{4} \langle q_{12}^P \rangle, \\
\frac{\partial A_N^{(P)}}{\partial x} &= \frac{1}{2N} \sum_{i=1}^N \langle z_i^2 \rangle - \frac{1}{2} \langle q_{12} \rangle, \\
\frac{\partial A_N^{(P)}}{\partial w} &= \frac{1}{2N} \sum_{i=1}^N \langle z_i^2 \rangle.
\end{align*} \tag{2.41-2.43} \]
Proof. We prove only the derivatives w.r.t. \( t \), as long as the procedure is analogous for each parameter. The partial derivative of the interpolating quenched pressure with respect to \( t \) reads as

\[
\frac{\partial A_N^{(P)}}{\partial t} = \frac{\beta'}{2N\sqrt{t}} \sqrt{\frac{1}{2N^{P-1}}} \sum_i \mathbb{E} \left[ \omega(z_{i_1}, \ldots, z_{i_P}) \right] - \frac{\beta'^2}{4N^P} \mathbb{E} \left[ \omega \left( \sum_{i=1}^{N} z_i^2 \right)^P \right].
\]  

(2.44)

Now, using the Stein’s lemma (1.21) on standard Gaussian variable \( z_{i_1}, \ldots, z_{i_P} \)

\[
\frac{\partial A_N^{(P)}}{\partial t} = \frac{\beta'^2}{4N^P} \left\{ \mathbb{E} [\omega((z_{i_1}, \ldots, z_{i_P})^2)] - \mathbb{E} [\omega(z_{i_1}, \ldots, z_{i_P})^2] - \mathbb{E} \left[ \omega \left( \sum_{i=1}^{N} z_i^2 \right)^P \right] \right\} - \frac{\beta'^2}{4} (\langle q_{12}^2 \rangle).
\]  

(2.45)

\[\]

Proposition 8. The interpolating pressure (2.40) at finite size obeys the following differential equation:

\[
\frac{dA_N^{(P)}}{dt} = \frac{\partial A_N^{(P)}}{\partial t} + \dot{x} \frac{\partial A_N^{(P)}}{\partial x} + \dot{w} \frac{\partial A_N^{(P)}}{\partial w} = S(t, r) + V_N(t, r),
\]  

(2.46)

where we set

\[\]

\[
\dot{x} = -\frac{P}{2} \beta'^2 q^{P-1}, \quad \dot{w} = -\dot{x}
\]  

(2.47)

and

\[
S(t, r) := \frac{\beta'^2}{4} (P - 1) q^P,
\]  

(2.48)

\[
V_N(t, r) := \frac{\beta'^2}{4} \sum_{k=2}^{P} \binom{P}{k} q^{P-k} (\Delta q_{12})^k.
\]  

(2.49)

Proof. Starting to evaluate explicitly \( \frac{\partial}{\partial t} A_N \) by using (2.42) - (2.43) and Definition (1.24)-(1.25) we write

\[
\frac{\partial}{\partial t} A_N^{(P)} = -\frac{\beta'^2}{4} \left( \sum_{k=2}^{P} \binom{P}{k} (q_{12} - \bar{q})^k q^{P-k} - q^P (1 - Pq^{P-1} - \langle q_{12} \rangle) \right)
\]

(2.50)

\[
= V_N(t, r) + S(t, r) + \frac{\beta'^2}{2} Pq^{P-1} \frac{1}{2} (q_{12})
\]

Thus, by placing \( \dot{\hat{r}} = (\dot{x}, \dot{w}) \) as in (2.47) we reach the thesis.

\[\]

Remark 17. As shown for the P-spin glass model in the RS assumption (Remark 10), even for the Gaussian P-spin glass model, we can say that \( V_N \), defined in (2.49), vanishes.

Proposition 9. The transport equation associated to the interpolating pressure \( A_N(t, r) \) in the thermodynamic limit and under the RS assumption is

\[
\frac{\partial A_{\text{RS}}^{(P)}}{\partial t} + \frac{\beta'^2}{2} Pq^{P-1} \left( \frac{\partial A_{\text{RS}}^{(P)}}{\partial x} \right) - \frac{\beta'^2}{2} Pq^{P-1} \left( \frac{\partial A_{\text{RS}}^{(P)}}{\partial w} \right) = \frac{\beta'^2}{4} (P - 1) q^P,
\]  

(2.51)

whose solution is given by

\[
A_{\text{RS}}^{(P)}(t, r) = \frac{1}{2} - \ln \left[ 1 - \lambda \left( w - \frac{P}{2} \beta' q^{P-1} t \right) \right] + \frac{x + \frac{P}{2} \beta' q^{P-1} t}{2 (1 - \lambda \left( w - \frac{P}{2} \beta' q^{P-1} t \right))} + \frac{\beta'^2}{4} t (P - 1) q^P.
\]  

(2.52)
Proof. We can find the transport equation applying Remark (17) in Prop. (8).

We compute the solution using the characteristic method on the transport equation:

$$A_{RS}^{(P)}(t, r) = A_{RS}^{(P)}(0, r - \dot{r} t) + S(t, r) t. \quad (2.53)$$

where $\dot{r} = (\dot{x}, \dot{w})$. Along the characteristics, the fictitious motion in the $(t, r)$ time-space is linear and returns

$$x = x_0 - \frac{P}{2} \beta' 2q^{P-1} t \quad w = w_0 + \frac{P}{2} \beta' 2q^{P-1} t \quad (2.54)$$

where $r_0 = (x_0, w_0) = (x(t = 0), w(t = 0))$. The Cauchy condition at $t = 0$ is given by a direct computation at finite $N$ as

$$A^{(P)}(0, r - \dot{r} t) = A^{(P)}(0, r_0)$$

$$= \frac{1}{N} \mathbb{E} \left\{ \mathbb{E}_z \exp \left[ \frac{1}{2} (w_0 + \lambda) \sum_{i=1}^{N} z_i^2 + \sqrt{x_0} \sum_{i=1}^{N} \tilde{J}_i z_i \right] \right\}$$

$$= -\frac{1}{2} \ln [1 - \lambda - w_0] + \frac{x_0}{2 (1 - \lambda - w_0)} \quad (2.55)$$

Giving the suitable values of parameters we have the following

**Corollary 2.** The RS approximation of the quenched pressure, in the thermodynamic limit, for the Gaussian $P$-spin glass model is obtained by posing $t = 1$ and $r = 0$ in (2.52), which returns

$$A_{RS}^{(P)}(\beta', \lambda) = -\frac{1}{2} \ln [1 - \lambda + P \beta' 2q^{P-1}] + \frac{P \beta' 2q^{P-1}}{2 (1 - \lambda + P \beta' 2q^{P-1})}$$

$$+ \frac{\beta' 2}{4} (P - 1) q^P. \quad (2.56)$$

where the self-consistency equation which rule the order parameter are

$$\bar{q} = \frac{P \beta' 2q^{P-1}}{(1 - \lambda + P \beta' 2q^{P-1})^2}. \quad (2.57)$$

We stress that the expression found with transport equation method in Cor. (2) is the same found with Guerra’s interpolation in Prop. (3).

### 2.3.2 1-RSB solution

In this subsection we turn to the solution of the Gaussian spin glass model through transport equation via the generalized broken-replica interpolating technique, restricting the description at the first step of RSB.

The definition of two-replica overlap $q$ distribution is the same as Guerra’s interpolation (See Def. (8)).

Following the same route pursued in the previous sections, we need an interpolating partition function $Z$ and an interpolating quenched pressure $A$, that are defined hereafter.
Definition 22. Given the interpolating parameters \( r = (x^{(1)}, x^{(2)}, w), t \) and the i.i.d. auxiliary fields \( \{\tilde{J}_i^{(1)}, \tilde{J}_i^{(2)}\}_{i=1,\ldots,N} \), with \( J_i^{(1,2)} \sim N(0,1) \) for \( i = 1, \ldots, N \), we can write the 1RSB interpolating partition function \( Z_N(t, r) \) for the Gaussian P-spin glass model (2.1) recursively, starting by

\[
Z_2^{(p)}(t, r) := \mathbb{E}_\delta \left[ \sqrt{\beta'} \sum_{i=1}^{N} \tilde{J}_i \tilde{J}_i \exp \left( \sum_{i} \tilde{z}_i^2 \right) \right].
\]

Averaging over the fields recursively, we define

\[
Z_1(t, r) := \mathbb{E}_2 \left[ Z_2(t, r)^\theta \right]^{1/\theta}
\]

\[
Z_0(t, r) := \exp \mathbb{E}_1 \left[ \ln Z_1(t, r) \right]
\]

\[
Z_N(t, r) := Z_0(t, r),
\]

where with \( \mathbb{E}_n \) we mean the average over the variables \( \tilde{J}_i^{(a)} \)'s for \( a = 1, 2 \), and with \( \mathbb{E}_0 \) we shall denote the average over the variables \( J_{i_1 \ldots i_p} \)'s.

The definition of 1RSB interpolating pressure at finite volume \( N \) and in the thermodynamic limit and the notation for the generalized average are the same as Guerra’s interpolating technique (20).

The next step is building a transport equation for the interpolating quenched pressure, for which we preliminary need to evaluate the related partial derivatives, as discussed in the next

Lemma 8. The partial derivative of the interpolating quenched pressure with respect to each variable is

\[
\frac{\partial}{\partial t} A^{(p)}_N = \frac{\beta'^2}{2} (\theta - 1) \langle q_{12}^2 \rangle_2 - \frac{\beta'^2}{2} \theta \langle q_{12}^2 \rangle_1.
\]

\[
\frac{\partial}{\partial x^{(1)}} A^{(p)}_N = \frac{1}{2} \left( \frac{1}{N} \sum_i \langle z_i^2 \rangle_1 - (1 - \theta) \langle q_{12} \rangle_2 - \theta \langle q_{12} \rangle_1 \right)
\]

\[
\frac{\partial}{\partial x^{(2)}} A^{(p)}_N = \frac{1}{2} \left( \frac{1}{N} \sum_i \langle z_i^2 \rangle_1 - (1 - \theta) \langle q_{12} \rangle_2 \right)
\]

\[
\frac{\partial}{\partial w} A^{(p)}_N = \frac{1}{2N} \sum_i \langle z_i^2 \rangle_1
\]

The proof is similar to Guerra’s interpolation technique, since we omit it.

Proposition 10. The streaming of the 1-RSB interpolating quenched pressure obeys, at finite volume \( N \), a standard transport equation, that reads as

\[
\frac{dA^{(p)}}{dt} = \partial_t A^{(p)} + \dot{x}^{(1)} \partial_{x_1} A^{(p)} + \dot{x}^{(2)} \partial_{x_2} A^{(p)} + \dot{w} \partial_w A^{(p)} = S_{1RSB}(t, r) + V_N(t, r)
\]

where

\[
\dot{x}^{(1)} = -\frac{\beta'^2}{2} \frac{q_{1}^{P-1}}{q_{1}^{P-1}}
\]

\[
\dot{x}^{(2)} = -\frac{\beta'^2}{2} \frac{q_{2}^{P-1} - q_{1}^{P-1}}{q_{2}^{P-1}}
\]

\[
\dot{w} = \frac{\beta'^2}{2} \frac{q_{2}^{P-1}}{q_{2}^{P-1}}
\]
and

\[ S_{1\text{RSB}}(t, r) := \frac{\beta'^2}{4}(P - 1)q^P_2 - \frac{\beta'^2}{4} \theta(P - 1)(\bar{q}_2^P - \bar{q}_1^P) \] (2.70)

\[ V_N(t, r) := \frac{\beta'^2}{4}(\theta - 1) \left( \sum_{k=2}^{P} \langle \Delta q_{12}^k \rangle \bar{q}_2^{P-k} - \frac{\beta'^2}{4} \theta \sum_{k=2}^{P} \langle \Delta q_{12}^k \rangle \bar{q}_1^{P-k} \right) \] (2.71)

**Proof.** Keeping in mind the expression (1.88), we start to evaluate explicitly \( \frac{\partial}{\partial \theta} A_N^{(P)} \) by using (2.63 - 2.65), so we get

\[
\frac{\partial}{\partial t} A_N^{(P)} = \frac{\beta'^2}{4} \left\{ \theta - 1 \right\} \left[ \sum_{k=2}^{P} \langle \Delta q_{12}^k \rangle \bar{q}_2^{P-k} + \bar{q}_2^P (1 - P) + P \bar{q}_2^{P-1} \langle q_{12} \rangle_2 \right] \\
- \theta \left[ \sum_{k=2}^{P} \langle \Delta q_{12}^k \rangle \bar{q}_1^{P-k} + \bar{q}_1^P (1 - P) + P \bar{q}_1^{P-1} \langle q_{12} \rangle_1 \right]
\]

\[ = V_N(t, r) + S(t, r) + \frac{\beta'^2}{4} P(\theta - 1)q_2^{P-1} \langle q_{12} \rangle_2 - \theta(1 - P)q_1^P \\
= V_N(t, r) + \frac{\beta'^2}{4} \left( \theta - 1 \right) (1 - P)q_2^P - \frac{\beta'^2}{4} \bar{q}_1^P + \frac{\beta'^2}{2} P(1 - P)q_2^{P-1} \partial_x A_N^{(P)} \\
+ \frac{\beta'^2}{2} P(1 - P)q_2^{P-1} \partial_{x(2)} A_N^{(P)} - \frac{\beta'^2}{2} q_2^{P-1} \partial_w A_N^{(P)}
\] (2.72)

Thus, using (2.67)-(2.69) we reach the thesis.

\[ \Box \]

**Remark 18.** In the thermodynamic limit, in the 1RSB scenario, we have that the central moments greater than 2 tend to zero; in this way

\[ \lim_{N \to \infty} V_N(t, r) = 0. \] (2.73)

Exploiting Remark 18 we can prove the following

**Proposition 11.** The transport equation associated to the interpolating pressure function \( A_N^{(P)}(t, r) \), in the thermodynamic limit and under the 1RSB assumption, can be written as

\[
\partial_t A^{(P)} + x^{(1)} \partial_x A^{(P)} + x^{(2)} \partial_{x_2} A^{(P)} + w \partial_w A^{(P)}
= \frac{\beta'^2}{4} (\theta - 1) (1 - P)q_2^P - \frac{\beta'^2}{4} \theta (1 - P)q_1^P
\] (2.74)

whose explicit solution is given by

\[
A_{1\text{RSB}}^{(P)} = \ell \left[ \frac{\beta'^2}{4} (P - 1)q_2^P - \frac{\beta'^2}{4} \theta (P - 1)(\bar{q}_2^P - \bar{q}_1^P) \right] - \frac{1}{2} \log (1 - \lambda - w_0) \\
+ \frac{1}{2\theta} \log \left( \frac{1 - \lambda - w_0}{1 - \lambda - w_0 - \theta x_0^{(2)}} \right) + \frac{x_0^{(1)}}{2(1 - \lambda - w_0 - \theta x_0^{(2)})}
\] (2.75)

**Proof.** We compute through characteristic method:

\[
A_{1\text{RSB}}^{(P)}(t, r) = A_{1\text{RSB}}^{(P)}(0, r - \dot{r} t) + S_{1\text{RSB}}(t, r)t
\] (2.76)
By putting (2.67)-(2.69) into (2.66) we find
\[
A_0^{(P)}(r, t) = A_0^{(P)}(0, r_0) + t \left[ \frac{\beta'^{2}}{4} (P-1) q^P_2 - \frac{\beta'^{2}}{4} \theta(P-1)(q^P_2 - q^P_1) \right]
\]
(2.77)
where \( r_0 \) can be obtained by using the equation of motion
\[
r = r_0 + \dot{r} t
\]
(2.78)
where the velocities are defined in (2.67)-(2.69). Then, all we have to compute is \( A_0^{(P)}(0, r_0) \), that can be easily done because at \( t = 0 \) the two body interaction vanishes and the (2.58) can be written as
\[
A_0^{(P)}(0, r_0) = -\frac{1}{2} \log (1 - \lambda - w_0) + \frac{1}{2\theta} \log \left( \frac{1 - \lambda - w_0}{1 - \lambda - w_0 - \theta x_0^{(2)}} \right) + \frac{\alpha^{(1)}_0}{2(1 - \lambda - w_0 - \theta x_0^{(2)})}.
\]
(2.79)
We omit the computation since it is similar to one-body term in Guerra’s interpolating scheme, 1RSB assumption.
Then, putting together (2.77)-(2.79) and (2.67)-(2.69), we finally achieve an explicit expression for the interpolating pressure of the Gaussian P-spin glass model in the 1RSB approximation.

To sum up, we have the following main theorem for the 1RSB scenario.

**Theorem 5.** The 1-RSB quenched pressure for Gaussian P-spin glass model, in the thermodynamic limit, reads as
\[
A^{(P)}(\beta', \lambda) = \frac{\beta'^{2}}{4} (P-1) q^P_2 - \frac{\beta'^{2}}{4} \theta(P-1)(q^P_2 - q^P_1) - \frac{1}{2} \log \left[ 1 - \lambda + \beta'^{2} \frac{P}{T} q^P_2 \right]
\]
\[
+ \frac{1}{2\theta} \log \left[ \frac{1 - \lambda + \beta'^{2} \frac{P}{T} q^P_2 - \beta'^{2} \frac{P}{T} q^P_1}{1 - \lambda + \beta'^{2} \frac{P}{T} q^P_2 - \beta'^{2} \frac{P}{T} \theta (q^P_2 - q^P_1)} \right]
\]
\[
+ \frac{\beta'^{2} \frac{P}{T} q^P_1}{1 - \lambda + \beta'^{2} \frac{P}{T} q^P_2 - \beta'^{2} \frac{P}{T} \theta (q^P_2 - q^P_1)}
\]
(2.80)
where the order parameters are ruled by
\[
\tilde{q}_1 = \frac{\beta'^{2} \frac{P}{T} q^P_2 - \beta'^{2} \frac{P}{T} q^P_1}{(1 - \lambda + \beta'^{2} \frac{P}{T} q^P_2 - \beta'^{2} \theta \frac{P}{T} (q^P_2 - q^P_1))^2}
\]
\[
\tilde{q}_2 = \tilde{q}_1 + \frac{\beta'^{2} \frac{P}{T} (q^P_2 - q^P_1)}{(1 - \lambda + \beta'^{2} \frac{P}{T} q^P_2 - \beta'^{2} \theta \frac{P}{T} (q^P_2 - q^P_1))(1 - \lambda + \beta'^{2} \frac{P}{T} q^P_2 - \beta'^{2} \theta \frac{P}{T} (q^P_2 - q^P_1))}
\]
(2.81)
\[\]
Proof. By taking \( r = 0 \) and \( t = 1 \) we find the P-spin Gaussian pressure in the 1RSB approximation. Extremizing (2.80) we obtain the self-consistency equations.

We stress that the expression in (2.80) is the same found with Guerra’s interpolating scheme (2.38).
2.4 A deeper analysis of Gaussian spin glass model

In this section we will focus more specifically on the case of $P = 2$ for the Gaussian P-spin glass model under the assumption RS and 1-RSB, showing that we find the same results in [9].

Starting from the RS assumption, if we set $P = 2$ in the quenched pressure of the Gaussian P-spin glass model (2.22), we get

$$A_{RS} (\beta', \lambda) = \frac{\beta'^2 \bar{q}}{2(1 - \lambda + \beta'^2 \bar{q})} - \frac{1}{2} \ln \left[ 1 - \lambda + \beta'^2 \bar{q} \right] + \frac{\beta'^2}{4} \bar{q}^2. \quad (2.82)$$

Extremizing the statistical pressure in (2.82) w.r.t. the order parameter $\bar{q}$ we find the following

**Corollary 3.** The minimum of the quenched statistical pressure in (2.82) is achieved for

$$\bar{q} = 0 \quad \text{if } \beta' \leq 1 - \lambda,$$

$$\bar{q} = \frac{\beta' - (1 - \lambda)}{\beta'^2} \quad \text{if } \beta' > 1 - \lambda. \quad (2.83)$$

**Proof.** By setting equal to zero the derivative with respect to $\bar{q}$ of the eq. (2.82), we get

$$\bar{q} = \frac{\beta'^2 \bar{q}}{1 - \lambda - \beta'^2 \bar{q}} \quad (2.84)$$

this equation give us two possible solution of the order parameter, namely

$$\bar{q} = 0 \text{ and } \bar{q} = \frac{\beta' - (1 - \lambda)}{\beta'^2} \quad (2.85)$$

Now, computing the second derivative of the quenched pressure and replacing the values of $\bar{q}$ found in (2.85), we get

$$\left. \frac{\partial^2 A(\beta', \lambda)}{\partial \bar{q}^2} \right|_{\bar{q} = 0} = \frac{\beta'^2}{2(1 - \lambda)^2} \left[ (1 - \lambda)^2 - \beta'^2 \right],$$

$$\left. \frac{\partial^2 A(\beta', \lambda)}{\partial \bar{q}^2} \right|_{\bar{q} = (\beta' - 1 + \lambda) / \beta'^2} = \beta'(\beta' - (1 - \lambda)). \quad (2.86)$$

Thus, from the study of the sign of the previous equations we get the proof. \hfill \Box

Moving on the 1RSB case, if we set $P = 2$ in (2.38), we find

$$A_{1RSB} (\beta', \lambda) = \frac{\beta'^2}{4} \bar{q}_2^2 - \frac{\beta'^2}{4} \log \left( \frac{\bar{q}_2^2 - \bar{q}_1^2}{1 - \lambda + \beta'^2 \bar{q}_2 - \beta'^2 \theta (\bar{q}_2 - \bar{q}_1)} \right)$$

$$+ \frac{1}{2} \log \left( \frac{1 - \lambda + \beta'^2 \bar{q}_2 - \beta'^2 \theta (\bar{q}_2 - \bar{q}_1)}{1 - \lambda + \beta'^2 \bar{q}_2 - \beta'^2 \theta (\bar{q}_2 - \bar{q}_1)} \right)$$

$$+ \frac{1}{2} \log \left( \frac{1 - \lambda + \beta'^2 \bar{q}_2 - \beta'^2 \theta (\bar{q}_2 - \bar{q}_1)}{1 - \lambda + \beta'^2 \bar{q}_2 - \beta'^2 \theta (\bar{q}_2 - \bar{q}_1)} \right) \quad (2.87)$$

Following the same steps presented for the RS assumption, we set to zero the derivatives of (2.87) respect to the order parameters $\bar{q}_1$ and $\bar{q}_2$

$$\bar{q}_1 = \frac{\beta'^2 \bar{q}_1}{(1 - \lambda + \beta'^2 \bar{q}_2 - \beta'^2 \theta (\bar{q}_2 - \bar{q}_1))^2}, \quad (2.88)$$

$$\bar{q}_2 - \bar{q}_1 = \frac{\beta'^2 (\bar{q}_2 - \bar{q}_1)}{(1 - \lambda + \beta'^2 \bar{q}_2 - \beta'^2 \theta (\bar{q}_2 - \bar{q}_1))(1 - \lambda + \beta'^2 \bar{q}_2)}.$$
It is immediate to verify that the previous system of equations admits only the solution $\bar{q}_1 = \bar{q}_2 = \bar{q}$, with $\bar{q}$ of the form presented in (2.83). Therefore, the solution in 1RSB approximation coincides with the one in RS approximation. As a matter of fact, the other solution of (2.88) for $\bar{q}_1 \neq \bar{q}_2$, reads as

$$(\bar{q}_1; \bar{q}_2) = \left(0; \frac{(\theta - 2)(\lambda - 1) - \sqrt{\theta^2(\lambda - 1)^2 - 4\beta^2(\theta - 1)}}{2\beta^2(\theta - 1)}\right),$$

(2.89)

however, it can be shown that this solution is neither a minimum nor a maximum of (2.87), but a saddle point.

Thus, we can conclude this section with the following theorem.

**Theorem 6.** For the pairwise Gaussian spin glass model defined by the Hamiltonian

$$H_N(z|J) := -\frac{1}{2N} \sum_{i,j=1}^{N, N} J_{ij} z_i z_j \quad \text{where} \quad J_{ij} \sim \mathcal{N}(0, 1)$$

(2.90)

the RS solution is exact.

**Proof.** The proof consists in showing that the 1RSB bound for the free energy gives the same result of the RS approximation. \(\square\)

### 3 Conclusions and outlooks

In this paper, we have generalized two rigorous mathematical methods for two different models, namely P-spin glass model and Gaussian P-spin model with $P > 2$. We face up to RS and 1RSB assumptions, comparing an approach closer to Mathematical Physics, namely transport equations, and another linked to Statistics and Probability, namely Guerra’s interpolating technique. We reach the same results via both techniques, proving that they are both valid for these models and we recovered the same expression in \([20, 22, 36, 37]\) for P-spin glass and \([17, 35]\) for Gaussian P-spin glass models.

Having the possibility to investigate the problem with mathematical tools from different fields could be powerful and could give us non-trivial results.

In addition, we showed that, as far as Gaussian spin glass model concerns, the replica symmetry expression for quenched statistical pressure is exact for $P = 2$ case, in contrast to $P > 2$ case.

Further researches should be addressed to a deeper physical interpretation of this results, in particular on replica symmetry breaking assumptions. Another lines could be the rigorous mathematically confirmation of results in KRSB assumption, with K finite in \([22]\) and the approach of more challenging models, such as dense associative networks\([6]\).

### A Proof of (1.24)-(1.25)

In this Appendix, we show the computation of expression which is used either in Guerra’s interpolating scheme or transport equations. Let’s start with (1.24). Using the notation $\Delta X = X - \overline{X}$ and exploiting
Newton’s binomial, we get
\[
    \langle m^P \rangle = \left\langle (m - \bar{m} + \bar{m})^P \right\rangle
\]
\[
    = \sum_{k=0}^{P} \binom{P}{k} \bar{m}^{P-k} \langle (m - \bar{m})^k \rangle
\]
\[
    = (1 - P)\bar{m}^P + P\bar{m}^{P-1}\langle m \rangle + \sum_{k=2}^{P} \binom{P}{k} \bar{m}^{P-k} \langle (m - \bar{m})^k \rangle.
\]

In the same way (using \( q_{12} \) instead of \( m \) and \( \bar{q} \) instead of \( \bar{m} \)) we prove (1.25).

\section*{B Proof of Lemma 2}

We prove the derivative of statistical pressure in 1RSB assumption.

\[
d_t A = \frac{1}{N} E_0 E_1 E_2 W_2 \omega \left[ \frac{\beta J_0}{2} m^P (\sigma) - N \psi m (\sigma) \right]
\]
\[
+ \frac{\beta J}{2\sqrt{t}} \frac{1}{2N^{P-1}} \sum_{i_1, \ldots, i_p=1}^{N-1} z_{i_1} \cdots z_{i_p} \sigma_{i_1} \cdots \sigma_{i_p} - \frac{1}{2\sqrt{1-t}} \sum_{a=1}^{2} A_a \sum_{i=1}^{N} \sigma_i (a) \sigma_i
\]
\[
= -\psi \langle m \rangle + \frac{\beta J_0}{2} \langle m^P \rangle + B_1 + B_2 + B_3
\]
\[
B_1 = \sqrt{\frac{1}{2N^{P-1}}} \sqrt{2N} \frac{\beta J}{W_2} E_0 E_1 E_2 W_2 \omega \left( \sum_{i_1, \ldots, i_p=1}^{N-1} z_{i_1} \cdots z_{i_p} \sigma_{i_1} \cdots \sigma_{i_p} \right) = \sqrt{\frac{1}{2N^{P-1}}} \sqrt{2N} \frac{\beta J}{W_2} E_0 E_1 E_2 \partial_{z_{12}} (W_2 \omega (\sigma_{12} \cdots \sigma_{12})) = \frac{\beta J^2}{4} \left[ 1 + (\theta - 1)(q_{12})_2 - \theta (q_{12})_1 \right]
\]
\[
B_2 = \frac{A_1}{2\sqrt{1-t}} \sum_i E_0 E_1 E_2 W_2 \omega \left( z_{i}^{(1)} \sigma_i \right) = -\frac{A_1}{2\sqrt{1-t}} E_0 E_1 E_2 \partial_{z_{i}^{(1)}} W_2 \omega (\sigma_i)
\]
\[
= \frac{A_1^2}{2} \left[ 1 + (\theta - 1)(q_{12})_2 - \theta (q_{12})_1 \right]
\]
\[
B_3 = -\frac{A_2}{2\sqrt{1-t}} \sum_i E_0 E_1 E_2 W_2 \omega \left( z_{i}^{(2)} \sigma_i \right) = -\frac{A_2}{2\sqrt{1-t}} E_0 E_1 E_2 \partial_{z_{i}^{(2)}} W_2 \omega (\sigma_i)
\]
\[
= \frac{A_2^2}{2} \left[ 1 + (\theta - 1)(q_{12})_2 \right]
\]

Rearranging together we obtain the thesis.

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References

[1] E. Agliari, et al. *Replica symmetry breaking in neural networks: a few steps toward rigorous results*, J. of Phys. A: Mathematical and Theoretical, 53(41), (2020).

[2] E. Agliari, et al. *A transport equation approach for deep neural networks with quenched random weight*, Journal of Physics A: Mathematical and Theoretical (2021).

[3] E. Agliari, et al. *Notes on the p-spin glass studied via Hamilton-Jacobi and smooth-cavity techniques*, Journal of mathematical physics 53.6(063304) (2012).

[4] E. Agliari, et al. *Neural networks with a redundant representation: detecting the undetectable*. Physical review letters 124.2 (028301) (2020).

[5] E. Agliari, F. Aleamanno, A. Barra, A. Fachechi, *Generalized Guerra’s interpolating techniques for dense associative memories*, Neur. Nets. 128, 254-267, (2020).

[6] L. Albanese, F. Aleamanno, A. Alessandrelli, A. Barra, *Replica Symmetry Breaking in dense associative networks*, submitted to Neural Networks (2021).

[7] D.J. Amit, *Modeling brain functions*, Cambridge Univ. Press (1989).

[8] C. Baldassi, F. Pittorino, R. Zecchina. *Shaping the learning landscape in neural networks around wide flat minima*, Proceedings of the National Academy of Sciences 117.1(161-170) (2020).

[9] A. Barra et al. *About a solvable mean field model of a Gaussian spin glass*, Journal of Physics A: Mathematical and Theoretical 47.15(155002) (2014).

[10] A. Barra et al. *How glassy are neural networks?* Journal of Statistical Mechanics: Theory and Experiment 2012.07(P07009) (2012).

[11] G.B. Arous, A. Dembo, A. Guionnet. *Aging of spherical spin glasses*. Probability theory and related fields 120.1(1-67) (2001).

[12] T. H. Berlin, M. Kac, *The Spherical Model of a Ferromagnet*, Phys. Rev. 86, 821 (1952).

[13] E. Bolthausen, A. Bovier, *Spin glasses* Springer (2007).

[14] A. Bovier, P. Picco, *Mathematical aspects of spin glasses and neural networks*, Springer Science & Business Media (Vol. 41.) (2012).

[15] A. Bovier, A. Klimovsky. *The Aizenman-Sims-Starr and Guerra schemes for the SK model with multidimensional spins*. Electronic Journal of Probability 14(161-241) (2009).

[16] A.C.C. Coolen, R. Kuhn, P. Sollich, *Theory of neural information processing systems*, Oxford Press (2005).

[17] A. Crisanti, H.J. Sommers. *The spherical p-spin interaction spin glass model: the statics*. Zeitschrift für Physik B Condensed Matter 87.3(341-354) (1992).

[18] B. Derrida, *Random-energy model: An exactly solvable model of disordered systems*. Physical Review B 24.5 (2613) (1981).

[19] A. Fachechi, *PDE/Statistical Mechanics Duality: Relation Between Guerra’s Interpolated p-Spin Ferromagnets and the Burgers Hierarchy*, Journal of Statistical Physics 183.1(1-28) (2021).

[20] E. Gardner, *Spin glasses with P-spin interactions*, Nuclear Physics B257(747-756) (1985).

[21] S. Ghirlanda, F. Guerra. *General properties of overlap probability distributions in disordered spin systems. Towards Parisi ultrametricity*. Journal of Physics A: Mathematical and General 31.46(9149) (1998).
[22] D.J. Gross, M. Mézard. *The simplest spin glass*. Nuclear Physics B **240.4**(431-452) (1984).

[23] F. Guerra, *Broken replica symmetry bounds in the mean field spin glass model*, Comm. Math. Phys. **233**(1), 1, (2003).

[24] F. Guerra, *Sum rules for the free energy in the mean field spin glass model*, Fiel. Inst. Comm. **30**, 11, (2001).

[25] F. Guerra, F.L. Toninelli, *The thermodynamic limit in mean field spin glass models*, Comm. Math. Phys. **230**(1), 71-79, (2002).

[26] J.M. Kosterlitz, D. J. Thouless, R. C. Jones. *Spherical model of a spin-glass*. Physical Review Letters **36.20**(1217) (1976).

[27] F.E. Leonelli, et al. *On the effective initialisation for restricted Boltzmann machines via duality with Hopfield model.*, Neural Networks (2021).

[28] M. Mézard, G. Parisi, M.A. Virasoro *Spin Glass Theory and Beyond*, World Scientific, Singapore (1987).

[29] R. Monasson, D. O’Kane. *Domains of solutions and replica symmetry breaking in multilayer neural networks*. EPL (Europhysics Letters) **27.2**(85) (1994).

[30] R. Monasson, and R. Zecchina. *Weight space structure and internal representations: a direct approach to learning and generalization in multilayer neural networks.*, Physical review letters **75.12**(2432) (1995).

[31] D. Panchenko, *A connection between the Ghirlanda–Guerra identities and ultrametricity*. The Annals of Probability **38.1**(327-347) (2010).

[32] D. Panchenko, M. Talagrand. *On the overlap in the multiple spherical SK models.*, The Annals of Probability **35.6**(2321-2355) (2007).

[33] M. Talagrand, *Spin glasses: a challenge for mathematicians: cavity and mean field models*. Vol. 46. Springer Science & Business Media, 2003.

[34] M. Talagrand, *Multiple levels of symmetry breaking*. Probability theory and related fields **117.4**(449-466) (2000).

[35] M. Talagrand, *Free energy of the spherical mean field model*. Probability theory and related fields **134.3**(339-382) (2006).

[36] M. Talagrand, *On the p-spin interaction model at low temperature*. Comptes Rendus de l’Académie des Sciences-Series I-Mathematics **331.11**(939-942) (2000).

[37] M. Talagrand, *Rigorous low-temperature results for the mean field p-spins interaction model*. Probability theory and related fields **117.3**(303-360) (2000).

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