Pair Production of Open Strings - Relativistic versus Dissipative Dynamics

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Abstract

We study the pair production of open strings in constant external electric fields, using a general framework which encodes both relativistic string theory and generic linearly extended systems as well. In the relativistically invariant case we recover previous results, both for pair production and for the effective Born-Infeld action. We then derive a non-relativistic limit - where the propagation velocity along the string is much smaller than the velocity of light - obtaining quantum dissipation. We calculate the pair nucleation rate for this case, which could be relevant for applications.

1. Introduction and Summary

The dynamics of quantum systems in the presence of external fields has been of interest since the early days of quantum mechanics. Even later, when a quantum description of gauge fields became available, such systems remained appealing, basically because they allowed insights into some nonperturbative physics. One famous example is the Schwinger calculation\textsuperscript{II} of the rate of production of electron-positron pairs.
pairs out of the vacuum, in the presence of a constant electric field. This has been achieved by computing a one-loop effective action in the presence of a background.

In recent years, the dynamics of (real or virtual) strings in electromagnetic backgrounds - coupled to the ends of an open string - has also been studied. In particular, the Born-Infeld effective action for the external field has been obtained in [3, 4], whereas the possibility of string pair creation through the Schwinger mechanism has been discussed in refs. [6, 7]. The string vacuum also develops a 'classical' instability [6, 5], due to the finite extension of the strings. It has been shown that there is an interesting relation between string theory (ST) and quantum dissipation [8]. This opens the possibility of studying the generalization of the Schwinger mechanism in the context of dissipative dynamics, by means of ST methods. Pair creation within dissipative dynamics has been investigated in ref. [12], related to possible applications to vortex pair nucleation in a superconductor. There, quantum dissipation has been introduced by means of the Caldeira-Leggett (CL) formalism [10]. Subsequently in [9], the relation between the CL formalism and the dynamics of linearly extended system has been explicitly worked out, in the context of non-relativistic pair nucleation. This was obtained by integrating out the (free) degrees of freedom along the string, the resulting effective action for the end-point (where the external electric field couples) turning out to be of dissipative type, in some infinite string length limit. For the standard relativistic string this limit, which corresponds to the case where the string worldsheet is a disk, gives the Born-Infeld action for the external field. In order to get pair creation in that case, one has to consider a non-relativistic version of ST. Thus, pair creation within quantum dissipation appears to require some non-relativistic limit, although the Schwinger mechanism describes the antiparticle member of the pair as a particle moving backward in time.

We wish to present here a systematic treatment of the above problems, based on a generalization of the (bosonic) string action, which will allow us to derive in a unified fashion:

a) the above known results for the relativistic string, namely the rate of pair production in external fields for the case of an annulus world-sheet, as well as the Born-Infeld effective action obtained when the world-sheet shrinks to a disk;

b) the dissipative dynamics of the string end-point (which is meant to represent a physical particle, say a pointlike vortex, coupled to the external field) and the pair nucleation rate in a nonrelativistic context.

We provide a continuous interpolation between the results for relativistic string pair production studied in [6, 7] and the dissipation-dominated nucleation of refs [12, 9]. This is done by introducing a physical parameter, having the meaning of a velocity, which is allowed to be much smaller than the speed of light.

Our method is reliable for such different situations because it is field theoretical in essence: we study by path integral methods the dynamics of an oscillating linearly extended object. Actually, for standard relativistic string theory the results are valid
only in the critical dimension, whereas for the non-relativistic case they can apply to any space-time dimensionality. In this paper we focus on a (3+1) dimensional string theory, as demanded by possible applications to situations at an accessible energy. At non-critical dimensionality, of course, we have to face noncanonical values for the zero-point energy (coming from the normal ordering of the string Hamiltonian), which we will anyhow reabsorb into a renormalization of some rest energy. In any case, we observe that the basic result for the Schwinger-like pair creation does not crucially depend on the string dimensionality. As every WKB tunnel effect, the nucleation rate is given by the product of an exponential factor and a mildly varying prefactor. The exponent is not affected by the space-time dimensionality and by the possible inclusion of the fermionic degrees of freedom of the supersymmetric version of the theory, but it crucially depends on the type of dynamics, i.e. whether it is dissipative or not. Namely, for small electric field $E$, it is proportional to $-1/E$ in the usual nondissipative dynamics, including the case of the Schwinger mechanism for relativistic strings, whereas it goes like $-1/E^2$ for the dissipative case.

Having in mind the nonrelativistic application mentioned above (say pair nucleation in a planar system), our computations will include the contribution to the prefactor of (2+1) coordinates. The third one can be regarded as parametrizing the string extension (see fig.1, where the electric field is also indicated), and does not contain physical degrees of freedom.

The plan of this paper is as follows. In section 2 we briefly review the Schwinger computation of the vacuum decay rate, adapted to our case. In section 3 we introduce a general form of the string action, which allows the description of both the usual relativistic string and the nonrelativistic case as well. The later one will arise by taking a suitable limit for our parameters. We clarify that limit, explaining its significance. In section 4 we work out our calculations, which will be made with the general action. Here we evaluate - by path integral methods - the partition function in the presence of the external electric field coupled to the end point of the open string. The appearance of singularities resulting from the path integral, namely poles in the worldsheet modulus, signals the possibility of pair production.
Relativistic and dissipative dynamics will be just particular cases of this unifying picture. In section 5 we use - for the same calculation - the 'shortcut' provided by the boundary state formalism (BSF) [11], where the boundary conditions containing the electric field are explicitly implemented. We get in fact the same result, with a suitable normalization of the boundary state. We can say that the BSF provides a quick way to see what happens, but to get the proper normalization we have to compare with the path integration. Section 6 is dedicated to the discussion of the pole singularity in various limits. We also show how to get the pole from a zero-mode of the action, corresponding to a solution of the classical equations of motion with proper boundary conditions. We compare our results in the relativistic limit with previous work [3, 7] in section 7. Indeed, we check that going to 26 dimensions and including the ghosts, one gets the same rate of pair production as in [7]. The Born-Infeld action arises as in ref [3], in the limit in which the world-sheet has only one boundary. In section 8 we proceed by calculating the rate of pair creation in the dissipative limit, ending with a few comments on the results.

2. General setting

Our final aim is to evaluate the probability of vacuum decay through pair production, in an external constant electric field, in a bosonic string theory or a field theory. So, let us review briefly the way to compute it. The zero temperature vacuum free energy in presence of an external constant electric field $E$ is the logarithm of:

$$e^{-iW(E)_{vac}} = <0|e^{-i\hat{H}\times\text{(time)}}|0> = 0|0 >_{E}= Z.$$ 

For a static field $W_{vac}(E) = \mathcal{E}_{vac}(E) \times (time)$, where (time) is the total time interval.

We consider the vacuum fluctuations due to creation and annihilation of open strings. The zero temperature vacuum free energy can be expressed by the following formula [1]

$$W_{vac} = \int_{0}^{\infty} dt t \text{Tr} e^{-t\hat{H}_{string}}$$

and the trace is evaluated by means of a (suitably normalised) path integral

$$\text{Tr} e^{-t\hat{H}_{string}} = \int DX_0 DX e^{-S(X_0, X, E)}.$$ 

The action $S(X_0, X, E)$ (discussed in section 3) includes an electric field, which couples to one of the string’s end-points. Due to that coupling, the vacuum energy has an imaginary part

$$\mathcal{E}_{vac}(E) = \text{Re}\mathcal{E}_{vac}(E) - i \frac{\Gamma}{2}$$

which provides us with an expression for the vacuum decay rate per unit volume
\[
\frac{\Gamma}{V} = -\frac{2}{(\text{time})V} \text{Im} \int_0^\infty \frac{dt}{t} \int DX_0 D\vec{X} e^{-S(X_0, \vec{X}, E)}. \quad (2)
\]

Here \( V \) is the space volume spanned by the coordinates on which we path-integrate in eq.(3). Now, the zero modes in the path integral will give precisely the space-time volume, so we obtain a formula for the vacuum decay rate per unit area \( \gamma = \frac{\Gamma}{V} \):

\[
\gamma = -2 \text{Im} \int_0^\infty \frac{dt}{t} \int DX_0 D\vec{X} e^{-S(X_0, \vec{X}, E)}. \quad (3)
\]

The prime means that we have factored out the zero mode part of the action.

The path integral factorizes into a free one (along \( X_2 \)) and a second one including the free terms along the axes 0 and 1, plus the interaction term involving \( E \), which couples \( X_0 \) and \( X_1 \).

### 3. Relativistic versus nonrelativistic limit and quantum dissipation

We wish now to explain in more detail which situations we are going to address, and to see how both relativistic and dissipative nonrelativistic dynamics arise from our formalism. Since we take all the space directions to be on equal footing and the electric field along \( X_1 \), the action is:

\[
S = -\frac{\alpha}{2} \int_0^t d\tau \int_0^l d\sigma [(\frac{\partial X_0}{\partial \tau})^2 + (\frac{\partial X_0}{\partial \sigma})^2] + \frac{\beta}{2} \int_0^t d\tau \int_0^l d\sigma [(\frac{\partial \vec{X}}{\partial \tau})^2 + v^2 (\frac{\partial \vec{X}}{\partial \sigma})^2]
\]

\[-iE \int_0^t d\tau \left[ X_0 \frac{\partial X_1}{\partial \tau} \right]_{\sigma = l}. \quad (4)
\]

For \( v = 1 \) and \( \alpha = \beta \), eq (4) is the standard way of writing the worldsheet action in string theory. But we have in mind possible applications to dissipative quantum dynamics in nonrelativistic situations, thus we consider also \( v < 1 \) and, for generality, \( \alpha \neq \beta \) (although the precise relation between \( \alpha \) and \( \beta \) will not be needed in the following).

Now we rescale by \( \tau \rightarrow \alpha l \tau, \sigma \rightarrow \sigma l \), so that the limits of integration become:

\[
T = \frac{t}{\alpha l}, \quad \Delta \sigma = 1. \quad (5)
\]

\( T \) gets the dimension of a length to the square, and allows the explicit connection with Schwinger’s ”proper time” formalism. The case of quantum dissipation for nonrelativistic nonrelativistic situations corresponds to the limit:

\[
\alpha T >> 1 >> \alpha v T. \quad (6)
\]
We see what that means: the oscillations of $X_0$ along $\sigma_0$ are much suppressed compared to those along $T$. For $\vec{X}$ the reverse happens. So time ($X_0$) is "rigid", but space ($\vec{X}$) not at all. We will compute the decay rate in these limits, while keeping $\beta v$ fixed. $\alpha$ will set the scale.

We now specialize to the (2+1)-dimensional case, that is $\vec{X} = (X_1, X_2, X_3)$, where $X_{1,2}$ describe the physical transverse oscillations of the string, whereas we take $X_3 = \sigma_b$, with $b$ the intermembrane distance (see fig.1). Thus, $X_3$ contributes to the action with the term:

$$\int_0^T d\tau \int_0^1 d\sigma_1 \frac{1}{2} \alpha \beta v^2 \left( \frac{\partial \sigma}{\partial \sigma} \right)^2 = \frac{1}{2} T \alpha \beta v^2 b^2. \quad (7)$$

The partition function will contain as a factor the exponential of minus this term, and (7) will play a role similar to the square rest mass of a string. Thus, we identify

$$E_0 = \sqrt{\frac{1}{2} \alpha \beta v^2 b^2} \quad (8)$$

as being the rest energy of a string.

The physics of the situation is quite transparent, and shows clearly why this limit has to be called 'nonrelativistic'. We have already seen that in the free action for $X_0$, the oscillations along $\tau$ (the 'kinetic term') dominate over those along $\sigma$. Along the spatial directions the opposite happens. This is due just to $\alpha >> \frac{1}{T}$ and is similar to what happens in string theory when $\alpha' = \frac{1}{2\pi} >> E^2$, where $E$ is the available energy. Then, the stringy massive modes are frozen, in the same way in which the oscillations of $X_0$ along $\sigma$ are frozen in our problem. Now, since $\tau$ has the dimensionality of time to the square, and the energy scale of our problem is $E_0$, the natural time scale will be $time = E_0 \tau$. In consequence, we have

$$d(time) = E_0 d\tau \quad d(X_3) = b d\sigma. \quad (9)$$

Using the previous equation in the free action, we obtain the velocities with which $X_0$ and $X_{1,2}$ signals propagate, respectively:

$$v^2_{\text{prop}}(X_0) = \frac{\alpha}{\beta v^2} \quad v^2_{\text{prop}}(X_1) = \frac{\alpha}{\beta}. \quad (10)$$

Furthermore, for our system the length scale is given by $b$, whereas a typical time scale is given by $E_0^{-1}$. Thus a typical speed is of the order of

$$v_{\text{typical}} = b E_0. \quad (11)$$

It is now easy to see that the relation between these three velocities, thanks to the limit (6), is the following:

$$v_{\text{prop}}(X_0) >> v_{\text{typical}} >> v_{\text{prop}}(X_{1,2}). \quad (12)$$
The first inequality means that the propagation of time excitations is practically instantaneous (the time is Galilean) - a nonrelativistic situation. The second one says that the propagation of excitations of space-like coordinates is very slow compared to the typical velocity - or that strings are very long. This corresponds to pure dissipative dynamics, since in this limit one obtains the Caldeira-Leggett action (8, 9).

We wish to see qualitatively why this nonrelativistic limit is related to the Caldeira-Leggett type quantum dissipative dynamics. In [10], dissipative dynamics was obtained by integrating out a thermal bath made of oscillators, on which a spectral condition has been imposed. The way we can reobtain a "thermal bath" is very simple: we just rewrite $X(\sigma, \tau)$ as $X_\sigma(\tau)$. Then, we see that only $X_{1,2}$ depend on $\sigma$, so the integration over a thermal bath is replaced here by path integrating over $\sigma$. The finite spatial extension of the strings amounts to an infinity of harmonic oscillators which can form a suitable bath. On the other hand, $X_0$ is independent (in the nonrelativistic limit!) of $\sigma$, so the time is singled out as a good coordinate for a pointlike object from the beginning. This is what we need, since we want to have the same time coordinate along the string to obtain nonrelativistic quantum mechanics, whereas integrating out the spatial coordinates will provide us with a dissipative dynamics for the point particle which remains after [8, 9]. In CL language, time is already a macroscopical coordinate, whereas the space-like coordinates of the string’s charged end are not; they can be made so only if accompanied by a dissipative term. What is remarkable is the fact that the CL "spectral condition” is not needed. The string seems to automatically provide such a constraint. Of course, some constraint is to be expected because all the dynamics is encoded in a continuum Lagrangian with fewer parameters than the 'many-body’ CL one (for various approaches, see [8, 9, 10]).

4. Path integral evaluation

4.1 Free case ($E = 0$)

For completeness, but also to establish its full physical interpretation, we first evaluate the free string - properly normalized - partition function. We start from the action for a generic uncoupled coordinate $X$:

$$S = \int_0^t d\tau \int_0^l d\sigma [\frac{\alpha}{2}(\frac{\partial X}{\partial \tau})^2 + \frac{\alpha v^2}{2}(\frac{\partial X}{\partial \sigma})^2].$$

We take the boundary conditions to be periodic along $\tau$: $X(t + \tau, \sigma) = X(\tau, \sigma)$ and Neumann along $\sigma : \frac{\partial X}{\partial \sigma}|_{\sigma=0,l} = 0$. Of course, if we want to consider the time-like direction $X_0$, then $v = 1$. The boundary conditions allow us to write
\[ X(\tau, \sigma) = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{N}} X_{nk} \cos(k \pi \sigma) \exp(2\pi i n \frac{\tau}{l}) \] (14)

such that the action becomes

\[ S = \alpha \frac{tl}{2} \left( \sum_{n>0,k>0} |X_{nk}|^2 \left[ \frac{4\pi^2}{l^2} n^2 + v^2 \frac{\pi^2}{l^2} k^2 \right] + \sum_{n>0,k=0} |X_{n0}|^2 \left[ \frac{4\pi^2}{l^2} n^2 \right] + \sum_{n=0,k>0} X_{0k}^2 \left[ \frac{v^2 \pi^2}{l^2} k^2 \right] \right) \]

We drop the zero-mode \((n = 0, k = 0)\) in the path integral, since it gives just the volume of space-time, and we wish to compute the amplitude per unit time and volume. Then we have to evaluate

\[
\int DX e^{-S} = \prod_{k>0} \frac{1}{\sqrt{\frac{4\pi}{l^2} k^2 \alpha v^2}} \prod_{n>0} \frac{1}{\sqrt{\frac{4\pi}{l^2} n^2 \alpha}} \prod_{k>0} \prod_{n>0} \frac{1}{\alpha (\frac{4\pi^2}{l^2} n^2 + \frac{\pi^2}{l^2} k^2)}
\]

\[
= \sqrt{\frac{\alpha l}{2\pi t}} e^{-\pi \frac{1}{12} \sum_{k>0} k} \prod_{k>0} \frac{1}{1 - e^{-2\pi k \frac{vt}{l}}}. \]

We have used - only for the index \(n\), corresponding to the Fourier transform along \(\tau\) - the free particle normalization

\[
\prod_{n>0} \frac{1}{4\pi} \frac{1}{m} \frac{1}{n^2} = \sqrt{\frac{m}{2\pi t}}
\]

as well as the Euler factorization of \(\frac{\sinh x}{x}\):

\[
\prod_{n=1}^{\infty} \frac{1 + x^2}{n^2} = \frac{\sinh \pi x}{\pi x}.
\]

Now we make use of the transformation properties of the Dedekind eta function

\[
\eta(x) = e^{\pi x} \prod_{k=1}^{\infty} \left( 1 - e^{-2\pi i n x} \right) = \frac{1}{\sqrt{-ix}} \eta(-\frac{1}{x})
\] (15)

and get at the end:

\[
\int DX e^{-S} = \sqrt{\frac{\alpha l}{4\pi}} e^{-\frac{\pi}{2\pi} \sum_{k>0} k} e^{-\frac{\pi}{2\pi} \sum_{n>0} n} \prod_{n>1} \frac{1}{1 - e^{-4\pi n \frac{vt}{l}}}. \]

The factor \(e^{-\frac{\pi}{2\pi} \sum_{k>0} k} e^{-\frac{\pi}{2\pi} \sum_{n>0} n}\) becomes one if we use the Riemann \(\zeta\)-function regularization \(\sum_{k=1}^{\infty} = \lim_{s \to 1} \zeta(-s) = -\frac{1}{12}\), as in string theory. The remaining exponential term is important. We will see that it amounts to normal order the
Hamiltonian in the boundary state formalism, and it gives a contribution which we can interpret in a thermodynamical context.

This is so because, if we make the identification $1/t = T = \beta^{-1}$, were $T$ means temperature now ($\tau$ has been already Euclidean), we can interpret the path integral as a partition function of a system in thermal equilibrium:

$$\int DX e^{-S} = Z_0 = \sum_n e^{-\beta E_n} = e^{-\beta F} = e^{-W}.$$  

The specific heat of this system is then given by ($U = \frac{\partial W}{\partial \beta}$):

$$C = \left( \frac{\partial U}{\partial T} \right)_{V=\text{const}} = \frac{\pi}{3} T l v,$$

which is positive (in our case $W = -\frac{\pi}{12} l v$) and raises linearly with the temperature. This indicates that

$$c = \frac{\partial C}{\partial T} = \frac{\pi}{3} l v $$

is a temperature independent physical parameter, which could in principle be measured. It will appear in our result for string pair production in the nonrelativistic case, when the string could be identified with a vortex line.

We may say that a system with an infinity of degrees of freedom, like a string, can be characterized by a peculiar specific heat in a thermodynamical context. Thus, our final result is

$$\int DX e^{-S} = \sqrt{\alpha v} e^{\frac{\pi c}{4\pi}} \prod_{n \geq 1} \frac{1}{1 - e^{-4\pi n^2} l v}. \quad (17)$$

4.2 Interacting case ($E \neq 0$)

The normalization we have obtained up to now was for the free case. If we switch on the electric field, however, we could get a normalization factor depending on the magnitude of $E$, so that we can not rely on what we have just learnt in the free case. In order to see what happens in general, with correct normalizations included, we evaluate the path integral for the interacting case.

The interaction term is quadratic, so that we could hope to eliminate it by a suitable linear transformation of $X_0$ and $X_1$. This does not work directly, since our interaction is a boundary term, not a bulk term.

Our strategy will be to decompose in Fourier modes, and evaluate the determinant in momentum space. In order to evaluate:

$$Z = \int DX_0 DX_1 e^{-\left[ \int_0^t dr \int_0^l d\sigma \left[ -\frac{1}{2} \left( \frac{\partial X_0}{\partial r} \right)^2 - \frac{1}{2} \left( \frac{\partial X_0}{\partial \sigma} \right)^2 + \frac{1}{2} \left( \frac{\partial X_1}{\partial r} \right)^2 + \frac{1}{2} \left( \frac{\partial X_1}{\partial \sigma} \right)^2 \right] - i \int_0^t dr \left[ EX^0 \frac{\partial X_1}{\partial r} \right]_{\sigma = i} \right].}$$
we develop in an arbitrary basis, independent of the interaction, as in the free case:

\[ X(\tau, \sigma) = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{N}} X_{nk} \cos(k\pi \sigma / \ell) \exp(2\pi i n \tau / \ell). \]

Then \( S \) becomes:

\[ S = S(0) + \sum_{n>0} S(n), \]

with

\[ S(0) = -\frac{tl}{2} \sum_{k>0} [(X_{0k}^0)^2 \frac{\alpha^2 \pi^2}{2 l^2} k^2 - (X_{0k}^1)^2 \frac{\beta v^2 \pi^2}{2 l^2} k^2] \]

and

\[ S(n > 0) = -\frac{tl}{2} [ |X_{n0}^0|^2 4 \frac{\alpha^2 \pi^2}{2 l^2 n^2} - |X_{n0}^1|^2 4 \frac{\beta v^2 \pi^2}{2 l^2 n^2}] + X^\dagger A X. \]

The term \( X^\dagger A X \) encodes the modes which couple through the electric field; we use the notation:

\[ X^\dagger = (X_{-n,1}^0 \ldots X_{-n,k}^0 \ldots X_{-n,1}^1 \ldots X_{-n,k}^1 \ldots) \]

and

\[ A = \begin{pmatrix}
  a_1 & 0 & 0 & \ldots & D & D & D & \ldots \\
  0 & a_2 & 0 & \ldots & D & D & D & \ldots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
  -D & -D & -D & \ldots & b_1 & 0 & 0 & \ldots \\
  -D & -D & -D & \ldots & 0 & b_2 & 0 & \ldots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}. \]

Here the ‘coupling term’ is \( D = -2\pi n E \), whereas \( a_{k>0} = \alpha \frac{tl}{2} (4\pi^2 n^2 + \frac{\pi^2}{2} k^2) \) and similarly for \( b_k \), with \( \alpha, \alpha \rightarrow \beta, \beta v^2 \). It turns out that (after some algebra and a proof by induction for any \( n \)):

\[ \det(A) = \prod_{i=1}^{n} a_i \prod_{j=1}^{n} b_j \left[ 1 + D^2 (\sum_{i=1}^{n} \frac{1}{a_i}) (\sum_{j=1}^{n} \frac{1}{b_j}) \right]. \]

Using that, together with the identity

\[ \sum_{k=1}^{\infty} \frac{1}{a^2 + k^2} = \frac{\pi}{2a} \text{cth}(\pi a) - \frac{1}{a^2}, \]

our path integral becomes
\[ Z = (Z_{\text{free}}) \prod_{n \geq 1} \left( 1 - \frac{E^2}{\alpha \beta v} \cosh(2\pi n L) \cosh(2\pi n \frac{L}{vt}) \right)^{-1} \]
\[ = \left[ 1 - \frac{E^2}{\alpha \beta v} \sqrt{\frac{\alpha \beta v}{(4\pi)^2}} e^{\frac{\pi}{\alpha} + (1 + \frac{\beta}{\alpha})} \right] \times \prod_{n \geq 1} \frac{1}{[1 + e^{-4\pi n L}(1 + \frac{\beta}{\alpha}) - \frac{E^2}{\alpha \beta v} (e^{-4\pi n L} + e^{-4\pi n \frac{L}{vt}})]} \]

We have \( \zeta \)-function regularised the divergent sum \( \sum_{k > 0} 1 = \lim_{s \to 0} \zeta(s) = -\frac{1}{2} \).

**5. Boundary State Formalism (BSF) evaluation**

We are going to repeat the former computation, this time through the boundary state formalism, that is, implementing in an operatorial way the boundary conditions in presence of the electric field. A review is presented in Appendix A. The result eq. (41) can be written as:

\[ \int DX^0(\sigma, t) DX^1(\sigma, t) e^{-S} = N(E) < B_E(l)|B_0> = N(E) < B_E|e^{-lH}|B_0>, \]  

where \( N(E) \) is a normalization factor to be obtained later by comparing with the path integral result (18). The boundary state \( |B_E> \) is obtained in eq.(40) of Appendix A. Using the notation \( \frac{E}{\alpha} = A, \frac{p}{l} = l_0 \)
\( \frac{E}{\beta v} = B, \frac{p}{vt} = l_1 \)

which will simplify the writing, we obtain:

\[ < B_E(l)|B_0(0)> = 0 | e^{\sum_{n \geq 1} \left( \frac{AB + 1}{AB - 1} (X_n^1 \tilde{X}_n^1 e^{-2nl_1} - X_n^0 \tilde{X}_n^0 e^{-2nl_0}) \right) + \frac{2}{AB - 1} (AX_n^0 \tilde{X}_n^1 - BX_n^1 \tilde{X}_n^0 e^{-nl_1}) | e^{\sum_{n \geq 1} (X_n^0 \tilde{X}_n^1 - X_n^1 \tilde{X}_n^0)} | 0 > \]

where \( l \) plays now the role of ‘time’. The former expression will be given by an infinite product of terms of the form

\[ < 0| e^{\kappa b \bar{a}} e^{\lambda a \bar{a}} e^{\mu b \bar{a}} e^{\nu \bar{a} \bar{d}} e^{\rho a \bar{d} l} e^{\sigma b \bar{d} l} | 0 >, \]
the operators $a, a^\dagger$ and $b, b^\dagger$ satisfying the usual harmonic oscillator commutation relations. Using the formula (proved in Appendix B)

$$<0|e^{k\hat{b}\dagger}e^{\lambda\hat{a}\dagger}e^{\mu\hat{b}\dagger}e^{\nu\hat{a}\dagger}|0> = \frac{1}{(1-\kappa\sigma)(1-\lambda\rho)-\mu\nu\rho\sigma}$$  \hspace{1cm} (21)

and substituting back, we get

$$Z = N(E) <B_E(l)|B_0> = N(E)e^{\frac{\alpha v}{4\pi}}e^{\frac{\beta v}{4\pi}}$$

$$\times \prod_{n>0} \frac{1}{1+e^{-4\pi n\frac{\alpha}{\alpha\beta v}}(1+\frac{\beta}{\alpha}) - \frac{1+E^2}{1-E^2}(e^{-4\pi n\frac{\alpha}{\alpha\beta v}} + e^{-4\pi n\frac{\beta}{\alpha\beta v}})}.$$  \hspace{1cm} (22)

In order to reproduce the path integral result we have to take $N(E) = N_0 \sqrt{1 - \frac{E^2}{\alpha\beta v}}$, $N_0 = \frac{\sqrt{\alpha\beta v}}{4\pi}$. We will see that the term $\sqrt{1 - \frac{E^2}{\alpha\beta v}}$ is going to be just the Born-Infeld action, specialized for our case.

It is interesting to point out the origin of the various factors. If we have a closer look at the free string computation (15-16), we can see that $(\frac{\alpha\beta v}{4\pi})^{\frac{1}{2}}$ comes from the zero modes along $\sigma$, i.e. it is related to the pointlike component of our object, not to its stringy excitations. Also, it is the part of the free string partition function not cancelled by the ghosts. The exponential term has already been interpreted as a specific heat contribution, characterizing the way energy is distributed among the string modes (degrees of freedom). Finally, $\sqrt{1 - \frac{E^2}{\alpha\beta v}}$ is a boundary term (the electric field acts on the world sheet boundary, not on the bulk), and it has a long story. Its form will give rise - for the relativistic string - to the Born-Infeld action. That factor has already been obtained in a variety of ways, for instance doing the path integral (after integrating out the bulk) in configuration space [3], or through operatorial methods, developing in a Fourier basis adapted to the form of the interaction [8].

6. Poles

6.1 General pole equation and particular limits

From now on, we switch to the notation of the rescaling (5). That simply amounts to set $l = 1$ and replace $t = \alpha T$. First, let us consider $\alpha, \beta, v$ arbitrary. In the general case (18,23) we have poles in $T$ whenever

$$\frac{E^2}{\alpha \beta v} = th2\pi n \frac{1}{\alpha T} \cdot th2\pi n \frac{1}{v\alpha T}.$$  \hspace{1cm} (23)
This is the general case from which, taking various limits, we obtain different physical situations.

1) We can look at the case of the relativistic string $\alpha = \beta$, $v = 1$. Poles are located, for a given $n$, at:

$$\frac{E}{\alpha} = \text{th} 2\pi n \frac{1}{\alpha T}$$

For either small $E$ or small $\frac{1}{\alpha T}$ we get

$$T = \frac{2\pi n}{E}. \quad (24)$$

As we will see, this is the situation studied in [7].

2) A second possibility is $\frac{1}{\alpha T} >> 1$, $\frac{1}{\alpha Tv} >> 1$, either for the relativistic invariant case or in general. In that case the pole is independent of $T$, and is located at the 'critical' value of $E$:

$$\frac{E^2}{\alpha \beta v} = 1.$$ 

It corresponds to the case studied in [3], were the Born-Infeld action was obtained.

3) If we take the limit $\frac{1}{\alpha T} >> 1$ and either $\frac{1}{\alpha T} << 1$ or $E$ small, we get

$$T = \frac{2\pi n \beta v}{E^2}.$$ 

We will see that this case corresponds to a nonrelativistic string, from which dissipative point particle quantum mechanics can be obtained by integrating out the string degrees of freedom [3]. We will have more to say about this later.

6.2 Another method

We note that it is possible to obtain the expression for the poles also by solving the Euclidean equations of motion, subject to the boundary conditions established in Appendix A. Our expression for the poles (23) is just the consistency condition needed in order for the boundary conditions at $\sigma = 0, 1$ to be satisfied by both $X_0$ and $X_1$.

Taking

$$\frac{\partial^2 X_0^0}{\partial \sigma^2} = \omega_n^2 X_0^0 \quad \frac{\partial^2 X_1^0}{\partial \sigma^2} = \frac{1}{v^2} \omega_n^2 X_1^0$$

we obtain ($\omega_n = \frac{2\pi n}{\alpha T}$):

$$X_0^0(\sigma) = A_0 \text{ch} \omega_n \sigma + B_0 \text{sh} \omega_n \sigma \quad X_1^1(\sigma) = A_1 \text{ch} \frac{\omega_n}{v} \sigma + B_1 \text{sh} \frac{\omega_n}{v} \sigma.$$
Using \( \frac{\partial X^0}{\partial \sigma} |_{\sigma=0} = \frac{\partial X^1}{\partial \sigma} |_{\sigma=0} = 0 \) we get \( B_0 = B_1 = 0 \). Finally, from the boundary conditions at \( \sigma = l \equiv 1 \) (see Appendix A) we get

\[
A_0 \omega_n sh(\omega_n) = \frac{\omega_n E}{\alpha} A_1 ch\left(\frac{\omega_n}{v}\right)
\]

\[
A_1 \frac{\omega_1}{v} sh\left(\frac{\omega_1}{v}\right) = \frac{\omega_n E}{\beta v} A_0 ch(\omega_n).
\]

For consistency then:

\[
\frac{E^2}{\alpha \beta v} = th(2\pi n \frac{1}{\alpha T}) th(2\pi n \frac{1}{\nu \alpha T})
\]

which is our former pole condition. In fact, the possibility of having a classical solution for a particular \( T \) for which the boundary conditions at both ends can be satisfied implies that the (Euclidean) action of the corresponding modes is zero. Thus the gaussian integration over those modes produces a singularity.

7. Comparison with previous results

We wish now to compare the result we have obtained with previous ones. First we remark that we can rewrite our partition function \( Z \) in the form:

\[
Z \sim \prod_{n=1}^{\infty} \frac{1}{E^2_{\alpha \beta v} - th2\pi n \frac{1}{\alpha T} th2\pi n \frac{1}{\nu \alpha T}}
\]

\[
= \prod_{n=1}^{\infty} \det \left| \begin{array}{cc} th(2\pi n \frac{1}{\alpha T}) & -\frac{E}{\alpha} \frac{1}{\nu \alpha T} \\ \frac{E}{\beta v} & -th(2\pi n \frac{1}{\nu \alpha T}) \end{array} \right|^{-1}
\]

\[
= \prod_{n=1}^{\infty} \det \left( g^{\mu \nu} th(2\pi n \frac{1}{v_{\mu} \alpha_{\mu} T}) - \frac{F_{\mu \nu}}{\alpha_{\mu} v_{\mu}} \right)^{-1},
\]

where \( \alpha_{\mu}, v_{\mu} \) are the string tension and propagation velocity for the coordinate \( X_{\mu} \). This makes clear the way the Born-Infeld action emerges: when the hyperbolic tangent above goes to one. Of course, we have proved the equality in the last line only for one electric field \( F_{01} \) along the \( X_1 \) direction. Nevertheless, the formula can be extended to a general constant \( F_{\mu \nu} \), the generalization requiring just to put an antisymmetric matrix into its block diagonal form. This way of presentation is appropriate for studying various particular cases.
We made our calculations on a cylinder, of circumference $t$ and length $l$. Through a conformal mapping, it can be transformed into an annulus, on which some of the previous calculations have been done. We remark that the limit $l \to \infty$ (the present $\alpha T \to 0$) corresponds to the shrinking of the interior circle of the annulus to zero radius, thus obtaining a disk.

7.1 Relativistic string

If we take a relativistic string, $\alpha = \beta$ and $v = 1$, and keep $\alpha T$ finite, our results should be the same as the ones of [7]. To check this, we rewrite our partition function $Z$, for $X_0$ and $X_1$ - see eq.(18), in terms of $\eta$ and $\Theta$ functions:

$$Z = -\frac{i}{2\pi} E \frac{\eta(\tau = 2i\frac{1}{\alpha T})}{\Theta_1(u = -\frac{i}{2\pi}ln(\frac{1+B}{\alpha^{1/2}})|\tau)}.$$  \hfill (25)

To prove that, put $e^{2\pi i u} = e^w = \frac{1+B}{\alpha^{1/2}}$ and $q = e^{-4\pi \frac{1}{\alpha T}} = e^{2\pi i v}$. Remembering that

$$\Theta_1(u|\tau) = 2q^{\frac{1}{2}} \sin \pi u \prod_{n=1}^{\infty} (1-q^n)(1-e^{2\pi i u}q^n)(1-e^{-2\pi i u}q^n)$$

we can rewrite our two-dimensional partition function as:

$$Z = \sqrt{1 - \frac{E^2}{\alpha^2}} \frac{2\sin \pi u}{4\pi} \frac{\eta(\tau = 2i\frac{1}{\alpha T})}{\Theta_1(u = -\frac{i}{2\pi}ln(\frac{1+B}{\alpha^{1/2}})|\tau)}.$$  \hfill (25)

Using the relationship between $u$ and $\frac{E^2}{\alpha^2}$ and the fact that $\sin \pi u = -i \sinh \frac{w}{2}$, we obtain (25).

The free one-dimensional partition function (along a space-like direction, $X_2$ say) becomes

$$Z_2 = \sqrt{\frac{\beta v}{4\pi}} \eta(\frac{2i}{\alpha T}).$$

Now, if we take into account the other 24 free dimension and the ghosts, and also use the way $\eta$ and $\Theta$ functions behave under modular transformations, we obtain the full 26-dimensional bosonic partition function

$$Z_{26} = -\frac{i}{2\pi} E \frac{\alpha^{13}}{(4\pi)^{13}} \frac{\eta^{-21}(2i\frac{1}{\alpha T})}{\Theta_1^{-1}(u\frac{2i}{\alpha T})}.$$  \hfill (26)

This is exactly the Bachas and Porrati amplitude for the case of an open bosonic string with a non-zero charge only at one end. Although a calculation involving superstrings would give rise to additional factors in the amplitude, it does not change the pole structure [7], which is in fact given by eq.(24). For this reason we restrict ourselves to the bosonic case. As discussed in ref [7], this amplitude has poles in $T$,
due to the zeroes of the $\theta$-function $\Theta_1$. These poles induce the imaginary part in the r.h.s. of eq.(3) and thus the vacuum decay rate.

### 7.2 Born-Infeld action

Let us take the relativistic string case again, but now supplement it with the limit $\alpha T \to 0$ ($l \to \infty$) - as we said, the annulus is shrinking to a disk. Then, since $th(\infty) = 1$, we see that our expression for $Z$ reduces to the Born-Infeld action. In this way we reobtain the result of [3]. In this case there is no pair production, at least for $E < \alpha$. We remark that in our case the string has a charge only at one end, whereas in [3] it had equal and opposite charges at the two ends, both coupled to the electric field.

We notice that it has been also observed in ref [9] that the Born-Infeld action can be obtained by tracing out the string degrees of freedom in the relativistic case and the limit $l \to \infty$.

In this limit, we have already seen that there are no poles (except for $E \to E_{\text{critical}}$), hence no pair production. Indeed, in the limit $\alpha T \to 0$ the 26-dimensional partition function (26) reduces to

$$\frac{\alpha^{13}}{(4\pi)^{13}}(e^{4\pi \frac{\alpha}{T}})^2 \sqrt{1 - \frac{E^2}{\alpha^2}}$$

which is (modulo a constant) a Born-Infeld action.

### 8. Vacuum Decay Amplitude in a Dissipative Context

We proceed now with the nonrelativistic case, where instead $v << 1$ will be a crucial condition. We put everything together to evaluate the decay rate, which is given by

$$\gamma = -2Im\int_0^\infty \frac{dT}{T} \int DX_0 DX_1 DX_2 e^{-S},$$

multiplied by the rest mass factor (cf. section 3) coming from the fact that our strings are stretched along $X_3$. We will work with the rescaled variable $T = \frac{t}{\alpha}$. We obtain the vacuum transition amplitude (over unit space and time, since we have already subtracted the zero mode - cf. eq. (4))

$$\gamma = -2Im\int_0^\infty \frac{dT}{T} \sqrt{\frac{\alpha(\beta v)^2}{(4\pi)^3}} e^{\frac{1}{2}\alpha(1 + \frac{1}{2})} e^{-\frac{1}{2}\alpha \beta v^2 T} \sqrt{1 - \frac{e^{2E^2}}{\alpha \beta v}}$$

$$\times \prod_{n > 0} \frac{1}{[1 - e^{-4\pi n \frac{1}{\alpha T}}][1 + e^{-4\pi n \frac{1}{\alpha T}}(1 + \frac{1}{4}) - \frac{1 + e^{2E^2}}{1 - e^{2E^2}}(e^{-4\pi n \frac{1}{\alpha T}} + e^{-4\pi n \frac{1}{\alpha T}})]}. \quad (27)$$
In order to get the rate of pair production we focus on the imaginary part - due to the presence of poles - of the former expression. Taking the limits \( \alpha T \gg 1 \) and \( \alpha v T \ll 1 \) while keeping \( \alpha \) and \( \beta v \) fixed we remain with:

\[
\gamma = -2Im \int_0^\infty \frac{dT}{T} \sqrt{\frac{\alpha (\beta v)^2}{(4\pi)^3}} e^{\frac{\pi}{4\sqrt{T}}(1+\frac{2}{\alpha T})} e^{-\frac{1}{2}\alpha \beta v^2 b^2 T} \sqrt{1 - \frac{e^2 E^2}{\alpha \beta v} \prod_{n>0} \frac{1}{1 - \frac{E^2}{\alpha \beta v} (e^{-4\pi n \frac{1}{\alpha T}})} [1 - \frac{1 + \frac{E^2}{\alpha \beta v}}{1 - \frac{E^2}{\alpha \beta v} (e^{-4\pi \frac{1}{\alpha T}})}]}
\]

We take into account only the first, dominant pole \( (n = 1) \) and evaluate the residue there using the expansion:

\[
\frac{1}{1 - e^{-\frac{E^2}{\alpha \beta v} - 4\pi \frac{1}{\alpha T}}} \approx -\frac{1}{\frac{2E^2}{\alpha \beta v} + 4\pi \frac{1}{\alpha T}} = \frac{-\alpha \beta v T}{2\pi \beta v E^2}.
\]

Thus we have a pole for \( T = T_P = \frac{2\pi \beta v}{E^2} \). Using now the identity \( \frac{1}{x-i\epsilon} = P\left(\frac{1}{x}\right) + i\pi \delta(x) \) in order to obtain the imaginary part of the \( T \)-integral in eq (27), we get the vacuum decay rate:

\[
\gamma = \frac{1}{8\pi} \frac{\alpha (\beta v)^2}{E} \sqrt{1 - \frac{E^2}{\alpha \beta v} e^{\frac{E^2}{6\alpha \beta v}} e^{\frac{E^2}{2\pi \beta v}} e^{-\pi \alpha \beta v^2 \frac{b^2}{2 T^2}}}. \tag{28}
\]

We remark that we have used the transformation properties of the Dedekind \( \eta \)-function, eq (15).

We reinterpret the exponents in terms of physical quantities. The quantum dissipation coefficient is \( \eta = \beta v \) (see [10, 9]). The rest energy of the nucleated object is \( \mathcal{E}_0 = \sqrt{\frac{1}{2} \alpha \beta v^2 b^2} \) (see eq.(8)), thus \( \pi \alpha \beta v^3 \frac{b^2}{2 T} = \mathcal{E}_0^2 \cdot T_P \). Further, we write

\[
\frac{\pi^2 \alpha \beta v}{12 E^2} \equiv -\Delta \mathcal{E}^2 \cdot T_P
\]

(with \( \Delta \mathcal{E}^2 = -\frac{\pi^2}{72} \)), reabsorbing it into a redefinition of the rest energy: \( \mathcal{E}^2 = \mathcal{E}_0^2 + \Delta \mathcal{E}^2 \). Concerning the first term, we rewrite it in terms of the temperature derivative of the specific heat, \( c = \frac{\pi}{3\alpha v} \) (see eq.(16)):

\[
\frac{E^2}{6\alpha \beta v^2} = \frac{E^2}{4\pi \eta}.
\]

In physical applications, both \( \mathcal{E}^2 \) and \( c \) will be taken as physical parameters to be determined experimentally.

We note one further point : in order to fix the normalization of the electric field \( E \) we have to remember that in the Schwinger method the space-time trajectory of a particle is described by a path integral action \( S \) which includes a term \( \frac{1}{4} \int_0^T d\tau (\frac{\partial X_\mu}{\partial \tau})^2 \).
Thus in eq. (28) we have to rescale $X_0$, which ultimately implies rescaling $E \rightarrow E \sqrt{2}$. Finally we get the decay rate:

$$\gamma = \frac{1}{8\sqrt{2}\pi} \frac{\alpha(\eta)\frac{4}{E}}{e^{\frac{\pi \eta}{E_2}} e^{-\pi \eta \frac{E_2}{2}}}. \quad (29)$$

We note that our result has the same form as equation (13) of the second reference in [12]. It is remarkable that while there a cut-off on the frequency for which dissipation occurs has been introduced by hand, here the way we calculate the path-integral starting with an underlying string theory takes care of everything.

We now discuss in which conditions a production rate should be observable. For that, we would need $T\mathcal{E}^2 \sim 1$. Due to our assumption $\alpha T >> 1$ (with $T = T_P \sim \frac{\eta}{E^2}$) we get the condition

$$\mathcal{E}^2 \sim \frac{E^2}{\eta} << \alpha. \quad (30)$$

Now, using the fact that $\frac{E^2}{\alpha \eta} << 1 << \frac{E^2}{\alpha \eta v}$ - which follows from the pole equation and the nonrelativistic limit - we end up with:

$$v << \frac{\mathcal{E}^2}{\alpha} << 1. \quad (31)$$

Other relationships are possible among our parameters. For instance the relation $\alpha v T << 1 << \alpha T$ could make us infer that $\alpha T \sqrt{v} \sim 1$ which is a reasonable assumption. These constraints can be further refined if we assume a definite relationship between $\alpha$ and $\beta$ ($\alpha \sim \beta$, or $\alpha \sqrt{v} \sim \beta$, for instance). However, we prefer not to add any further assumption for the time being. We just stress again that in order for the pair creation of vortices in a thin superconductor to be observable we need (30) to be satisfied. We postpone further elaborations for the time when experimental results will be in sight.

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Appendix A

We review here the boundary state formalism [11], applying it to our case. The strategy is the following: We first establish the boundary conditions along $\sigma$, in the open string channel. We then switch to the closed string channel and obtain the operatorial boundary conditions there. Our amplitude will be the scalar product between the state satisfying them at $\sigma = 0$ and at $\sigma = l$. Since the uncoupled case
\( (X_2) \) is easily obtainable from the coupled one, we restrict to the action along \( X_0 \) and \( X_1 \):

\[
S = \int_0^t dt \int_0^l d\sigma \left[ -\frac{\alpha}{2} \left( \frac{\partial X_0}{\partial \tau} \right)^2 - \frac{\alpha}{2} \left( \frac{\partial X_0}{\partial \sigma} \right)^2 + \frac{\beta}{2} \left( \frac{\partial X_1}{\partial \tau} \right)^2 + \frac{\beta v^2}{2} \left( \frac{\partial X_1}{\partial \sigma} \right)^2 \right] - i \int_0^t dt \sigma X_0 \frac{\partial X_1}{\partial \tau} \bigg|_{\sigma=l}
\]

The boundary conditions at \( \sigma = l \) are:

\[
\left( \alpha \frac{\partial X_0}{\partial \sigma} + i E \frac{\partial X_1}{\partial \tau} \right)_{\sigma=l} = 0 \quad (32)
\]

\[
\left( \beta v^2 \frac{\partial X_1}{\partial \sigma} + i E \frac{\partial X_0}{\partial \tau} \right)_{\sigma=l} = 0. \quad (33)
\]

At \( \sigma = 0 \) we just put \( E = 0 \). Assuming now periodic boundary conditions along \( \tau \), \( X(\tau, \sigma) = X(t+\tau, \sigma) \), we Fourier expand \( X \) with respect to \( \tau \):

\[
X(\tau, \sigma) = \sum_{n \in \mathbb{Z}} e^{i\omega_n \tau} X_n(\sigma) \quad \text{where} \quad \omega_n = \frac{2\pi}{l} n.
\]

The boundary conditions for the \( X_n \)'s are given by the Fourier transform of eqs.(32-33):

\[
\left. \frac{\partial X_0}{\partial \sigma} \right|_{\sigma=0} = 0 \quad \alpha \left. \frac{\partial X_0}{\partial \sigma} \right|_{\sigma=l} - \omega_n E X_1 \left. \right|_{\sigma=l} = 0 \quad \text{(34)}
\]

\[
\left. \frac{\partial X_1}{\partial \sigma} \right|_{\sigma=0} = 0 \quad \beta v^2 \left. \frac{\partial X_1}{\partial \sigma} \right|_{\sigma=l} - \omega_n E X_0 \left. \right|_{\sigma=l} = 0. \quad \text{(35)}
\]

The free equations of motion (there is no electric field in the bulk) are:

\[
\frac{\delta S(n)}{\delta X_{0,n}(\sigma)} = -\alpha \omega_n^2 X_0^n + \alpha \frac{\partial^2 X_0^n}{\partial \sigma^2} = 0
\]

\[
\frac{\delta S(n)}{\delta X_{1,n}(\sigma)} = \beta \omega_n^2 X_1^n - \beta v^2 \frac{\partial^2 X_1^n}{\partial \sigma^2} = 0.
\]

Until now, we spoke about the periodic propagation of an open string and consequently developed \( X \) on open-string modes. Now we reverse the picture (i.e. the roles of Euclidean \( \tau \) and \( \sigma \)) and look at a closed string parametrised by \( \tau \), propagating in the 'time' \( \sigma \), for \( \sigma \) from 0 to \( l \). Hence let us develop \( X \) in closed-string modes:

\[
X(t, \sigma) = X_0 + \frac{i}{\sqrt{4\pi}} \sum_{n \geq 1} \frac{1}{\sqrt{n}} \left( e^{i\omega_n \tau - \omega_n \sigma \frac{1}{v}} X_n - e^{-i\omega_n \tau + \omega_n \sigma \frac{1}{v}} X_{-n} + X_{-n} e^{i\omega_n \tau + \omega_n \sigma \frac{1}{v}} X_{-n} + \hat{X}_n e^{-i\omega_n \tau - \omega_n \sigma \frac{1}{v}} X_{-n} e^{i\omega_n \tau + \omega_n \sigma \frac{1}{v}} \right).
\]

Using that we obtain the boundary conditions for the closed string modes:
\begin{align*}
X_n^0(l) + \tilde{X}_{-n}^0(l) &= -\frac{E}{\alpha}(X_n^1(l) - \tilde{X}_{-n}^1(l)) \quad \forall n \quad (37) \\
X_n^1(l) + \tilde{X}_{-n}^1(l) &= -\frac{E}{\beta v}(X_n^0(l) - \tilde{X}_{-n}^0(l)) \quad \forall n.
\end{align*}

At $\sigma = 0$ the right-hand-side vanishes. For $E = 0$ we get the boundary condition for an uncoupled coordinate $X$:

\[ X_n(\sigma = 0, l) + \tilde{X}_{-n}(\sigma = 0, l) = 0 \quad \forall n. \quad (39) \]

Next we find for the boundary state $|B_E>$ satisfying (37,38) the expression (up to a normalization constant $N_E^2(E)$):

\[ |B_E> = \exp \left\{ \frac{1}{E^2} \sum_{n>0} \left[ (1 + \frac{E^2}{\alpha\beta v})(X_{-n}^1\tilde{X}_{-n}^1 - X_{-n}^0\tilde{X}_{-n}^0) \\
+ 2E\left( \frac{X_{-n}^0\tilde{X}_{-n}^1}{\alpha} - \frac{X_{-n}^1\tilde{X}_{-n}^0}{\beta v} \right) \right] \right\} |0> \quad (40) \]

the state and the operators $X$ being taken at $\sigma = l$. At $s = 0$ this becomes

\[ |B_0> = \exp \sum_{n \geq 1} \left( X_{-n}^0\tilde{X}_{-n}^0 - X_{-n}^1\tilde{X}_{-n}^1 \right) |0> \]

Now, the quantity of interest is, up to a normalization factor $N(E)$:

\[ <B_E|B_0> = <B_E|e^{-lH}|B_0> = N^{-1}(E) \int DX^0(\sigma, t) DX^1(\sigma, t) e^{-S} \quad (41) \]

which we are going to calculate in the main body of the paper, obtaining thus the required path integral.
Appendix B

We prove here the equality - eq.(22) in the text:

\[ <0| e^{\lambda \hat{a} \hat{a} \hat{b} \hat{b}} e^{\nu \hat{b} \hat{a}} e^{\rho \hat{a} \hat{a}^{\dagger} \hat{b}^{\dagger}} e^{\sigma \hat{b} \hat{b}^{\dagger}} |0> = \frac{1}{(1 - \kappa \sigma)(1 - \lambda \rho) - \mu \nu \rho \sigma} \]  \tag{42}

where \( \hat{a},\hat{b},\hat{b}^{\dagger} \) are independent annihilation operators, while \( \hat{a}^{\dagger},\hat{b}^{\dagger},\hat{b}^{\dagger} \) are the corresponding creation operators.

We expand the exponentials in power series in the left-hand-side of (42) and keep only the non-zero terms. Then, by using the harmonic oscillator normalization

\[ <0| a^{n}(a^{\dagger})^{n}|0> = n! \] , as well as a trick - differentiating and then resuming - we obtain what follows:

\[
<0| e^{\lambda \hat{a} \hat{a} \hat{b} \hat{b}} e^{\nu \hat{b} \hat{a}} e^{\rho \hat{a} \hat{a}^{\dagger} \hat{b}^{\dagger}} e^{\sigma \hat{b} \hat{b}^{\dagger}} |0> = \\
= \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\kappa \lambda \rho \sigma}{s! r! m! (m + r)! (m + s)!} [s + m + r + 1]! (s + m)! [r + m]! = \\
= \sum_{m=0}^{\infty} \frac{\mu \nu}{m!} \frac{d^m}{d\kappa^m} \frac{1}{1 - \kappa} \frac{d^m}{d\lambda^m} \frac{1}{1 - \lambda} = \\
= \sum_{m=0}^{\infty} \frac{\mu \nu}{m!} \frac{d^m}{d\rho^m} \frac{1}{1 - \rho} \frac{1}{1 - \rho} = \\
= \frac{1}{(1 - \kappa \sigma)(1 - \lambda \rho) - \mu \nu \rho \sigma}.
\]

For \( \mu = \nu = 0 \) we also obtain as a particular case:

\[
<0| e^{\lambda \hat{a} \hat{a} \hat{b} \hat{b}} e^{\rho \hat{a} \hat{a}^{\dagger} \hat{b}^{\dagger}} e^{\sigma \hat{b} \hat{b}^{\dagger}} |0> = \frac{1}{(1 - \kappa \sigma)(1 - \lambda \rho)}.
\]
References

[1] J.Schwinger, Phys. Rev. 82 (1951) 664.

[2] M.Born and L.Infeld, Proc. R. Soc. London A 144 (1934) 425.

[3] E.S.Fradkin and A.A.Tseytlin, Phys. Lett. B163 (1985) 123.

[4] A.Abouelsaood, C.G.Callan, C.R.Nappi and S.A.Yost, Nucl. Phys B280 [FS18] (1987) 599.

[5] V.V.Nesterenko, Intern. J. Mod. Phys A4 (1989) 2627.

[6] C.P.Burgess, Nucl. Phys. B294 (1987) 427.

[7] C.Bachas and M.Porrati, Phys. Lett. B296 (1992) 77, C.Bachas, Phys.Lett. B374 (1996) 37; hep-th/9511043.

[8] C.G.Callan and L.Thorlacius, Nucl. Phys B329 (1990) 117.

[9] R.Iengo and G.Jug, Phil.Mag.B, to be published; hep-th/9702021.

[10] A.O.Caldeira and A.J.Leggett, Ann. Phys. 149 (1983) 374.

[11] C.G.Callan, C.Lovelace, C.R.Nappi and S.A.Yost, Nucl. Phys. B288 (1987) 525, Nucl. Phys. B293 (1987) 83, J.Polchinski and Y.Cai Nucl. Phys. B296 (1988) 91.

[12] R.Iengo and G.Jug, Phys. Rev. B52 (1996) 7536, Phys. Rev. B54 (1996) 9465.

[13] R.Iengo and C.Scrucca, Phys.Rev. B57 (1998) 6046; cond-mat/9710005.