Construction of cyclic calorons

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Abstract. Analytic Nahm data of calorons, \textit{i.e.}, Yang-Mills instantons on $\mathbb{R}^3 \times S^1$, with spatial $C_N$-symmetries are considered. The Nahm equations for the bulk data are reduced to the periodic Toda equation by applying the $C_N$-symmetric ansatz for monopoles by Sutcliffe. It is found that the bulk data are solved by elliptic theta functions which enjoy the hermiticity and the reality conditions. The defining relation to the boundary data are given and the $C_3$-symmetric Nahm data are found as an example.

1. Introduction

The Atiyah-Drinfeld-Hitchin-Manin (ADHM) \cite{1} and the Nahm \cite{2} constructions are powerful tool for getting the information of the moduli space to the anti-selfdual (ASD) Yang-Mills solitons, such as the instantons and the Bogomol’nyi-Prasad-Sommerfield (BPS) monopoles \cite{3}. The principal ingredients of the ADHM/Nahm construction are the ADHM/Nahm data, respectively. One can obtain the gauge fields through the so called Nahm transform once the ADHM/Nahm data are determined. Although the ADHM/Nahm data are significant objects in the analysis on the ASD Yang-Mills solitons, the construction of the exact forms to the ADHM/Nahm data is unexplored field.

Among the ASD Yang-Mills solitons, calorons, or periodic instantons, are quite interesting object \cite{4}. They are ASD instantons on $\mathbb{R}^3 \times S^1$, so that if we take the circumference of $S^1$ be infinitely large, the calorons are reduced to instantons. On the other hand, it is naturally expected to be the BPS monopoles, if we take the ratio of the size of the calorons to the circumference be infinity. Hence, calorons are expected to give the connection between instantons and BPS monopoles. There are articles demonstrating this interpolation in analytic \cite{5, 6, 7, 8, 9}.

In this paper, we give an outline for the construction of the Nahm data of calorons with instanton charge $N$ associated to spatial $C_N$-symmetries around an axis. It is known that the Nahm data of calorons are composed of the bulk data and the boundary data, respectively. We will apply the $C_N$-symmetric ansatz for monopole Nahm data given by Sutcliffe \cite{10, 11} as the bulk Nahm data of calorons, and find that the defining equations for the bulk data are the well-known periodic Toda equations. As we will see, it is not appropriate to fix the boundary data in the basis on which the Toda equations are solved. Hence we will make a unitary transformation into another basis, in which the reality conditions are manifest. We will give a conjecture on the unitary transformations between these basis. We restrict ourselves to consider the calorons of the $SU(2)$ gauge theory, and of trivial holonomy cases.
This paper is organized as follows. In section 2, we find the bulk Nahm data by solving the reduced Nahm equations, and confirm its hermiticity. In section 3, we consider the reality conditions for the bulk data. We give a conjecture on the manifest form of the unitary transformation from the $C_N$-symmetric basis, on which the Toda equations are solved, into the reality basis. In section 4, we consider the fixing of the boundary data. As an illustration, we give the $C_3$-symmetric boundary data explicitly. Section 5 is devoted to the conclusion.

2. $C_N$-symmetric ansatz for the bulk data

The Nahm data of calorons are compounded by two parts, $T_j(s)$ and $W$, where $j = 1, 2, 3$. The former is three $N \times N$ matrix valued regular functions periodic in $s$, referred to as the bulk data; we take the fundamental period $[-\mu, \mu]$ here. Throughout this work, we take the gauge in which $T_0(s)$ is gauged away. The latter is an $N$-dimensional row vector of quaternion entries, referred to as the boundary data. In the Nahm construction of calorons [2], the Nahm data $\{T_j(s), W\}$ are defined by the following four set of conditions,

\[
T_j^i(s) = \frac{i}{2} \epsilon_{ijk} [T_j(s), T_k(s)], \quad (1)
\]

\[
T_j^j(s) = T_j(s), \quad (2)
\]

\[
T_j^j(s) = T_j(-s), \quad (3)
\]

\[
T_j(-\mu) - T_j(\mu) = \frac{1}{2} \text{tr}_2 \sigma_j W^\dagger W, \quad (4)
\]

where $i, j$ and $k$ run through 1 to 3, $\epsilon_{ijk}$ is totally anti-symmetric tensor, and the derivative is with respect to the variable $s$. The first conditions (1) are differential equaitons for the bulk data, known as Nahm equations. The second (2) and the third (3) are the hermiticity conditions and the reality conditions for the bulk data, respectively. The reality conditions are necessary due to the group isomorphism $SU(2) \simeq Sp(1)$. Those conditions (1)–(3) are commonly required to the definition of the Nahm data of the $SU(2)$ BPS monopoles [12]. The fourth conditions (4) are the relations between the bulk data and the boundary data, called the matching conditions, where the trace is taken for the quaternions. In the Nahm construction for the BPS monopoles, the matching conditions (4) are taken place by the boundary conditions on the bulk data. The $SU(2)$ caloron gauge fields can be obtained by the Nahm data $\{T_j(s), W\}$ through the Nahm transform.

Let us now consider the symmetry of caloron Nahm data under the action of $SO(3)$, a rotation in the configuration space. Let us denote $R$ an element of the subgroup of $SO(3)$, i.e., for a spatial rotation of a position vector we have $x_j \mapsto x_j' = R_{jk} x_k$, where $R_{jk}$ is an image of $R$ in the 3-dimentional orthogonal representation of $SO(3)$. We also denote $R_2$ the image of $R$ in the 2-dimensional irreducible representation of $SU(2)$, which gives rise to a spatial rotation of quaternions, $x \mapsto x' = R_2 x R_2^{-1}$, where $x = x_\mu e_\mu$ and the quaternion basis are defined by $e_\mu = (1, -i\sigma_1, -i\sigma_2, -i\sigma_3)$.

A caloron Nahm data is said to be symmetric under the action of $R$, if the Nahm data $\{T_j, W\}$ enjoys

\[
R_N T_j R_N^{-1} = R_{jk} T_k, \quad (5)
\]

and

\[
R_N \otimes R_2 W^\dagger = W^\dagger \hat{q}, \quad (6)
\]

where $R_N$ denotes the image of $R$ in $SL(n, \mathbb{C})$, and $\hat{q}$ is a unit quaternion, such that $\hat{q}^\dagger \hat{q} = 1$. For the case of $C_N$-symmetris, it is given by

\[
R_N = \omega^j \text{diag.} [\omega^{N-1}, \omega^{N-2}, \ldots, \omega, 1], \quad (7)
\]
where $\omega$ is an $N$-th root of unity and $l = 0, 1, \ldots, N - 1$.

So far, we have given a very short review on the caloron Nahm data with a spatial symmetry. Now we determine the bulk Nahm data satisfying the $C_N$-symmetric condition (5). For this purpose, we will apply the ansatz for the monopole Nahm data with $C_N$-symmetries given by Sutcliffe over a decade ago [10, 11], whose uniqueness is proved in [13]. The form of the $N \times N$ bulk data is given in terms of the functions $f_j(s)$ and $p_j(s)$ ($j = 0, 1, 2, \ldots, N - 1$) as

$$T_1 = \frac{1}{2} \begin{bmatrix} f_1 & f_2 & \cdots & f_0 \\ f_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{N-1} \\ f_0 & \cdots & f_{N-1} & f_1 \end{bmatrix},$$

(8)

$$T_2 = i\frac{1}{2} \begin{bmatrix} -f_1 & f_0 & \cdots & f_{N-1} \\ f_0 & -f_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_1 \\ -f_0 & \cdots & f_1 & f_0 \end{bmatrix},$$

(9)

$$T_3 = \frac{1}{2} \text{diag.} [p_1, p_2, \cdots, p_{N-1}, p_0],$$

(10)

where we have omitted the argument of the functions. One can find that the hermiticity conditions (2) are enjoyed if $f_j, p_j \in \mathbb{R}$. Substituting the ansatz (8)–(10) into the Nahm equations (1), we obtain the differential equations for $f_j(s)$’s and $p_j(s)$’s

$$f'_j = \frac{1}{2} f_j (p_{j+1} - p_j)$$

(11)

$$p'_j = f_{j-1}^2 - f_j^2,$$

(12)

where the periodicity, $f_{j+N} = f_j$ and $p_{j+N} = p_j$, is taking into account. The system of differential equations (11) and (12) is well-known as the periodic Toda lattice. Note that it is necessary $\text{tr}T_3^2 = \frac{1}{2} \sum_{j=0}^{N-1} p'_j = 0$ due to the Nahm equations, which will be confirmed later. Eliminating the $p_j(s)$’s in (11) and (12), the equations for $f_j$’s are found to be

$$\frac{d^2}{ds^2} \log f_j^2 = -f_{j+1}^2 + 2f_j^2 - f_{j-1}^2,$$

(13)

which are well-known form of Toda lattice.

Let us now find special solutions to (13) appropriate for the caloron Nahm data. By introducing $\tau$-functions $\tau_j := \tau(s, j)$, ($j = 0, 1, \ldots, N - 1$) as

$$f_j^2 = -C^2 \frac{\tau_j - \tau_{j+1}}{\tau_j^2},$$

(14)

the differential equations for $\tau_j$’s from (13) turn out to be

$$\frac{d^2}{ds^2} \log \tau_j = C^2 \frac{\tau_j - \tau_{j+1}}{\tau_j^2},$$

(15)

where $C$ is a constant defined below. Simultaneously, we find the expression for $p_j$’s by the $\tau$-functions

$$p_j = \frac{d}{ds} \left( \log \frac{\tau_j}{\tau_{j-1}} \right),$$

(16)
from (11). Now let us suppose the following form to the \( \tau \)-functions in terms of elliptic, or Jacobi, theta functions \( \vartheta_{\nu}(u, q) \), where \( u \in \mathbb{C} \) and \( \nu = 0, 1, 2, 3 \) and \( q \) is so-called the modulus parameter \(^1\). The definition of the elliptic theta functions are given in the literatures, see e.g., \([17]\). The assumption we suppose is

\[
\tau_j(s) = \exp \left( \frac{1}{2} \tilde{A}s^2 + bs + \tilde{b}j \right) \vartheta_{\nu}(\pm s + \kappa j + a, q),
\]

(17)

where \( \tilde{A}, b, \tilde{b}, \kappa \) and \( a \) are constants. This form of the \( \tau \)-functions for the periodic Toda lattice was originally introduced by M.Toda in 1967 \([16]\). Substituting (17) into (15), we find a differential equation for the theta functions,

\[
A + C^{-2} (\log \vartheta_{\nu}(s_j))'' = \frac{\vartheta_{\nu}(s_j - \kappa) \vartheta_{\nu}(s_j + \kappa)}{\vartheta_{\nu}(s_j)^2},
\]

(18)

where \( s_j := \pm s + \kappa j + a \), \( A := \tilde{A}C^{-2} \), and we have omitted the modulus dependence. This differential equation (18) is solved if the constants are given by the special values of theta functions \([15, 16]\),

\[
C^{-2} = \left( \frac{\vartheta_1(\kappa)}{\vartheta_1(0)} \right)^2,
\]

(19)

\[
A = \tilde{A}C^{-2} = \left( \frac{\vartheta_0(\kappa)}{\vartheta_0(0)} \right)^2 - \frac{\vartheta_0''(0)}{\vartheta_0(0)} \left( \frac{\vartheta_1(\kappa)}{\vartheta_1(0)} \right)^2.
\]

(20)

We therefore have found the elliptic theta function solution to the Nahm equations (11) and (12),

\[
f_j^2 = -C^2 \frac{\vartheta_{\nu}(s_{j-1}) \vartheta_{\nu}(s_{j+1})}{\vartheta_{\nu}(s_j)^2},
\]

(21)

\[
p_j = \frac{d}{ds} \left( \log \frac{\vartheta_{\nu}(s_j)}{\vartheta_{\nu}(s_{j-1})} \right).
\]

(22)

Note that the exponential factors are not appeared in the expression. By taking the square root of (21), \( f_j \)’s are rewritten as

\[
f_j = \pm iC \frac{\sqrt{\vartheta_{\nu}(s_{j-1}) \vartheta_{\nu}(s_{j+1})}}{\vartheta_{\nu}(s_j)}.
\]

(23)

Without loss of generality, we can now take the plus sign in (23).

We now recall the periodicity of the theta functions \( \vartheta_{\nu}(u + 1) = \vartheta_{\nu}(u) \), where the sign depends on \( \nu \) \(^2\). Hence, we easily find that the periodicity \( f_{j+N} = f_j \) and \( p_{j+N} = p_j \) follow if we take \( \kappa = 1/N \).

Let us now fix the hermiticity (2) of the theta function solutions given above. For this, it is necessary that \( f_j, p_j \in \mathbb{R} \) as mentioned earlier. From (22) and (23), it is sufficient to take \( \vartheta_{\nu}(s_j) \) is real and positive valued on the region \( s \in [-\mu, \mu] \), and \( C \in i\mathbb{R} \), i.e., pure imaginary valued. For these to be enjoyed, we choose the modulus parameter \( q \) takes real values \( 0 < q < 1 \), then we find \( \vartheta_{\nu}(s_j) \) is positive real valued on \( s \in [-\mu, \mu] \) for \( \nu = 0 \) and \( 3 \). On the other hand, for \( \nu = 1 \) or \( 2 \), \( \vartheta_{\nu}(s_j) \) has a zero and then changes sign on the real axis, so that the numerical value

\(^1\) \( \vartheta_0 \) is also expressed as \( \vartheta_4 \) in the literatures.

\(^2\) Note that the period in \([17]\) is \( \pi \) instead of 1, which is only due to the scaling of the variable \( u \).
in the square root of (23) will be able to negative. One can find there does not exist the case in which all of the $f_j$’s take real valued simultaneously for $\nu = 1$ or 2. Hence we eliminate $\nu = 1$ and 2 and concentrate hereafter on the solutions of $\nu = 0$ or 3, both of which are even functions. Next we fix the constant $C$ to be pure imaginary. From (19), we find

$$C = \pm \frac{\vartheta'_1(0)}{\vartheta_1(\kappa)},$$

(24)

where both the numerator and the denominator have a factor $q^{1/4}$,

$$\vartheta'_1(0) = 2\pi q^{1/4} \prod_{m=1}^{\infty} (1 - q^{2m})^3,$$

(25)

$$\vartheta_1(\kappa) = 2q^{1/4} \sin \kappa \pi \prod_{m=1}^{\infty} (1 - 2q^{2m} \cos 2\kappa \pi + q^{4m})^3.$$

(26)

Hence, if we take the relative branch of $q^{1/4}$ in (24) such that $C \in i\mathbb{R}$, then the hermiticity (2) holds.

3. The reality conditions

Let us now consider the reality conditions (3), i.e., $T_j^\dagger(s) = T_j(-s)$. It can easily be found that the bulk Nahm data obtained above do not satisfy these conditions. For example, the elements of the diagonal matrix $T_3(s)$ have to be even functions for the reality condition, however, the elements $p_j(s)$, (22), are not so in general. Therefore, we have to show that there is a basis of the bulk Nahm data in which the reality conditions are apparent. The situation is similar to the monopoles with Platonic symmetries [14]. The new basis are given by a unitary transformation $U_N$,

$$T_j \mapsto \tilde{T}_j = U_N T_j U_N^{-1},$$

(27)

where

$$\tilde{T}_j^\dagger(s) = \tilde{T}_j(-s).$$

(28)

In the new basis of the bulk data $\tilde{T}_j$, however, the $C_N$-symmetries are not apparent. We hereafter refer the primary $T_j$ as the bulk data in the “$C_N$-symmetric basis”, and $\tilde{T}_j$ as that in the “reality basis”. In addition, the transformation of the boundary data from the $C_N$-symmetric basis to the reality basis is read from (4) as

$$\tilde{W}^\dagger = U_N W^\dagger, \quad \tilde{W} = W U_N^{-1},$$

(29)

where we define $\tilde{W}$ as the boundary data in the reality basis.

The next task is to find the unitary transformation from the $C_N$-symmetric basis to the reality basis. We can guess the transformation matrices heuristically as follows. First, we consider the odd $N = 2k + 1$ ($k = 1, 2, \ldots$) cases. By defining $(2k + 1) \times (2k + 1)$ matrices

$$J_{2k+1} := \begin{bmatrix} 1 & \ldots & 1 \\ & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix},$$

(30)
we find unitary matrices

\[ U_{2k+1} = \frac{1}{\sqrt{2}} (1 + iJ_{2k+1}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 + i & i \\ 1 & 1 \end{bmatrix} \]  

(31)

For even \( N = 2k + 2 \), defining \((2k + 2) \times (2k + 2)\) matrices \(J_{2k+2} := 1 \oplus J_{2k-1}\), then we find unitary matrices

\[ U_{2k+2} = \frac{1}{\sqrt{2}} (1 + iJ_{2k+2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 + i & i \\ 1 & 1 \end{bmatrix} \]  

(32)

By using these unitary matrices, we give a conjecture that the transformed bulk Nahm data \( \tilde{T}_j \), (27), enjoys the reality conditions (28) manifestly. We have confirmed the conjecture is correct at least \( N \leq 6 \). The detail will be published elsewhere.

To illustrate, we consider the cases \( N = 3 \) and 4. The \( N = 3 \) bulk Nahm data in the \( C_3 \)-symmetric basis from the previous section are

\[ T_1 = \frac{1}{2} \begin{bmatrix} 0 & f_1 & f_0 \\ f_1 & 0 & f_2 \\ f_0 & f_2 & 0 \end{bmatrix}, \]

(33)

\[ T_2 = \frac{i}{2} \begin{bmatrix} 0 & -f_1 & f_0 \\ f_1 & 0 & -f_2 \\ -f_0 & f_2 & 0 \end{bmatrix}, \]

(34)

\[ T_3 = \frac{1}{2} \text{ diag. } [p_1, p_2, p_3]. \]

(35)

where \( f_j \) and \( p_j \) are given by (23) and (22). Here we choose \( \kappa = 1/3 \) and \( a = 0 \) for \( s_j = \pm s + \kappa j + a \) so that \( \vartheta_\nu(s_{j+3}) = \vartheta_\nu(s_j) \). The ambiguity of the sign in \( s_j \) is fixed to be consistent with the matching conditions (4). Performing the unitary transformation into the reality basis

\[ \tilde{T}_j = U_3 T_j U_3^{-1}, \]  

(36)

by using a unitary matrix

\[ U_3 = \frac{1}{\sqrt{2}} (1 + iJ_3) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & i \\ 0 & 1 + i & 0 \\ i & 0 & 1 \end{bmatrix}, \]

(37)
we find the $N = 3$ bulk data in the reality basis are

$$
\tilde{T}_1 = \frac{1}{2} \begin{bmatrix}
0 & f_+ - if_- & f_0 \\
f_+ + if_- & 0 & f_+ - if_-
\end{bmatrix},
$$

(38)

$$
\tilde{T}_2 = \frac{1}{2} \begin{bmatrix}
f_0 & -f_+ - if_- & 0 \\
-f_+ + if_- & 0 & f_+ + if_-
\end{bmatrix},
$$

(39)

$$
\tilde{T}_3 = \frac{1}{4} \begin{bmatrix}
p_2 & 0 & i(p_0 - p_1) \\
0 & 2p_2 & 0 \\
-i(p_0 - p_1) & 0 & -p_2
\end{bmatrix},
$$

(40)

where $f_{\pm}(s) = (f_1(s) \pm f_2(s))/2$. To confirm the reality conditions (3), we notice the behavior of the theta functions under the inversion $s \rightarrow -s$,

$$
\vartheta_\nu(s_j) = \vartheta_\nu(\mp s + j/3) \longrightarrow \vartheta_\nu(\mp s + j/3) = \vartheta_\nu(\pm s - j/3) = \vartheta_\nu(s_{-j}),
$$

(41)

then we observe $f_0(s), f_+(s)$ and $p_2(s)$ are even functions, $f_(s)$ and $p_0(s) - p_1(s)$ are odd functions in $s$, respectively. We therefore find that the reality conditions hold for the bulk data (38)--(40). We show the profile to the components of the bulk data in the reality basis in Figure 1, where the case of the $\vartheta_0$ solution with $q = 0.5$ is displayed.
For the case $N = 4$, the bulk Nahm data in the $C_4$-symmetric basis are

$$T_1 = \frac{1}{2} \begin{bmatrix} 0 & f_1 & 0 & f_0 \\ f_1 & 0 & f_2 & 0 \\ 0 & f_2 & 0 & f_3 \\ f_0 & 0 & f_3 & 0 \end{bmatrix}, \quad (42)$$

$$T_2 = \frac{i}{2} \begin{bmatrix} 0 & -f_1 & 0 & f_0 \\ f_1 & 0 & -f_2 & 0 \\ 0 & f_2 & 0 & -f_3 \\ -f_0 & 0 & f_3 & 0 \end{bmatrix}, \quad (43)$$

$$T_3 = \frac{1}{2} \text{diag. } [p_1, p_2, p_3, p_0], \quad (44)$$

where $f_j$ and $p_j$ are given by (23) and (22), as in the $N = 3$ case. Here we choose $\kappa = 1/4$ and $a = -1/8$ in $s_j = \pm s + \kappa j + a$. By using a unitary matrix

$$U_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 + i & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 1 + i & 0 \\ 0 & i & 0 & 1 \end{bmatrix}, \quad (45)$$

we perform the unitary transformation $\tilde{T}_j = U_4 T_j U_4^{-1}$, and find the transformed bulk data

$$\tilde{T}_1 = \begin{bmatrix} 0 & f_{+1} - if_{-1} & 0 & f_{+1} + if_{-1} \\ f_{+1} + if_{-1} & 0 & f_{+3} - if_{-3} & 0 \\ 0 & f_{+3} + if_{-3} & 0 & f_{+3} - if_{-3} \\ f_{+1} - if_{-1} & 0 & f_{+3} + if_{-3} & 0 \end{bmatrix}, \quad (46)$$

$$\tilde{T}_2 = \begin{bmatrix} 0 & f_{+1} + if_{-1} & 0 & -f_{+1} + if_{-1} \\ f_{+1} - if_{-1} & 0 & -f_{+3} - if_{-3} & 0 \\ 0 & -f_{+3} + if_{-3} & 0 & f_{+3} + if_{-3} \\ -f_{+1} - if_{-1} & 0 & f_{+3} - if_{-3} & 0 \end{bmatrix}, \quad (47)$$

$$\tilde{T}_3 = \frac{1}{4} \begin{bmatrix} 2p_1 & 0 & 0 & 0 \\ 0 & p_0 + p_2 & 0 & i(p_0 - p_2) \\ 0 & 0 & 2p_3 & 0 \\ 0 & -i(p_0 - p_2) & 0 & p_0 + p_2 \end{bmatrix}, \quad (48)$$

where we have defined

$$f_{\pm 1} := \frac{1}{4}(f_0 \pm f_1), \quad (49)$$

$$f_{\pm 3} := \frac{1}{4}(f_2 \pm f_3). \quad (50)$$

As in the case of $N = 3$, one can find $f_{+1}, f_{+3}, p_1, p_3$ and $p_0 + p_2$ are even functions, and $f_{-1}, f_{-3}$ and $p_0 - p_2$ are odd functions, respectively, so that the bulk data (46)–(48) enjoy the reality conditions (3).

For larger $N$, we claim a conjecture that $\tilde{T}_j = U_N T_j U_N^{-1}$ satisfies the reality conditions (3) with an appropriate choice of $\kappa$ and $a$ in $s_j = \pm s + \kappa j + a$. Here the unitary matrix $U_N$ is defined by (31) and (32) for odd and even $N$, respectively.
4. The boundary data

Let us now consider the boundary data. The boundary data are defined by the matching conditions (4), in which the contribution to the left hand side is only from the odd function parts of the bulk data. Thus, it is convenient to fix the boundary data in the reality basis considered in the previous section. The boundary data of the calorons with charge \(N\) in the reality basis are of the following form

\[
\tilde{W} = (\lambda_1, \lambda_2, \ldots, \lambda_N),
\]

where \(\lambda_m\) \((m = 1, \ldots, N)\) are quaternions with components \(\lambda_m = (\lambda_{m0}, \lambda_{m1}, \lambda_{m2}, \lambda_{m3})\). Thus the matching conditions are

\[
\tilde{T}_j(-\mu) - \tilde{T}_j(\mu) = \frac{1}{2} \mathrm{tr}_2 \sigma_j \tilde{W}^\dagger \tilde{W} = \frac{1}{2} \mathrm{tr}_2 \sigma_j \begin{bmatrix}
\lambda_1^j \lambda_1 & \lambda_1^j \lambda_2 & \cdots & \lambda_1^j \lambda_N \\
\lambda_2^j \lambda_1 & \lambda_2^j \lambda_2 & \cdots & \lambda_2^j \lambda_N \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_N^j \lambda_1 & \lambda_N^j \lambda_2 & \cdots & \lambda_N^j \lambda_N
\end{bmatrix},
\]

(52)

where we have used the fact \(\lambda_m \lambda_m = \sum_{k=0}^3 \lambda_{mk}^2 \in \mathbb{R}\) so that \(\mathrm{tr}_2 \sigma_j \lambda_m^j \lambda_m = 0\). If we solve \(\tilde{W}\) to (52) for a given bulk data \(\tilde{T}_j(s)\), then the Nahm data \(\{\tilde{T}_j(s), \tilde{W}\}\) gives a caloron of instanton charge \(N\), however it has no specific symmetry in general. For the calorons to be \(C_N\)-symmetric, it is necessary that the boundary data enjoys (6) in addition to (52). To confirm the symmetry of the boundary data, we should make an investigation in the \(C_N\)-symmetric basis. From (29), the boundary data in the \(C_N\)-symmetric basis are given by

\[
W^\dagger = U_N^{-1} \tilde{W}, \quad W = \tilde{W} U_N.
\]

(53)

For example, we consider the case \(N = 3\) with \(C_3\)-symmetry. Let us first rename the quaternion entries of the boundary data for simplicity

\[
\tilde{W} = (\lambda, \rho, \chi),
\]

(54)

where \(\lambda, \rho\) and \(\chi\) are quaternions with components \(\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3), \ldots\). The matching conditions (52) turn out to be

\[
\frac{1}{2} \mathrm{tr}_2 \sigma_j \tilde{W}^\dagger \tilde{W} = \frac{1}{2} \mathrm{tr}_2 \sigma_j \begin{bmatrix}
0 & \lambda^j \rho & \lambda^j \chi \\
\rho^j \lambda & 0 & \rho^j \chi \\
\chi^j \lambda & \chi^j \rho & 0
\end{bmatrix}.
\]

(55)

From the odd function parts of the bulk data, we define

\[
g(\mu) := \frac{1}{2} (f_-(\mu) - f_-(\mu)) = -f_-(\mu),
\]

(56)

\[
h(\mu) := \frac{1}{4} \{(p_0(\mu) - p_1(\mu)) - (p_0(\mu) - p_1(\mu))\}
\]

\[
= -\frac{1}{2} (p_0(\mu) - p_1(\mu)),
\]

(57)
then we find that the left hand sides of (52) are
\[
\begin{align*}
\tilde{T}_1(-\mu) - \tilde{T}_1(\mu) &= \begin{bmatrix} 0 & -ig(\mu) & 0 \\ ig(\mu) & 0 & -ig(\mu) \\ 0 & ig(\mu) & 0 \end{bmatrix}, \\
\tilde{T}_2(-\mu) - \tilde{T}_2(\mu) &= \begin{bmatrix} 0 & -ig(\mu) & 0 \\ ig(\mu) & 0 & ig(\mu) \\ 0 & -ig(\mu) & 0 \end{bmatrix}, \\
\tilde{T}_3(-\mu) - \tilde{T}_3(\mu) &= \begin{bmatrix} 0 & 0 & ih(\mu) \\ 0 & 0 & 0 \\ -ih(\mu) & 0 & 0 \end{bmatrix},
\end{align*}
\]  
respectively. We notice that the matching condition (52) is invariant under the multiplication of a unit quaternion \( h \), i.e., \( h^\dagger h = 1 \), to \( \tilde{W} \) from the left. By using this degree of freedom, we can fix one of the quaternion component being zero. We choose, say, the real component of \( \lambda \) to be zero. Thus, the remaining components of the boundary data are
\[
\lambda = -i(\lambda_1 \sigma_1 + \lambda_2 \sigma_2 + \lambda_3 \sigma_3), \\
\rho = \rho_0 - i(\rho_1 \sigma_1 + \rho_2 \sigma_2 + \rho_3 \sigma_3), \\
\chi = \chi_0 - i(\chi_1 \sigma_1 + \chi_2 \sigma_2 + \chi_3 \sigma_3).
\]  
Then one can observe that a solution to (55) is given in terms of \( \lambda_1, \lambda_2 \) and \( \lambda_3 \), with \( \lambda_1 \neq \lambda_2 \) subject to a constraint
\[
h(\mu) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 =: \Lambda^2,
\]  
as
\[
\rho_0 = -\frac{\lambda_1 + \lambda_2}{\Lambda^2} g(\mu), \\
\rho_1 = \rho_2 = \frac{\lambda_3}{\Lambda^2} g(\mu), \\
\rho_3 = \frac{\lambda_1 - \lambda_2}{\Lambda^2} g(\mu) \\
\chi_0 = \lambda_3, \\
\chi_1 = -\lambda_2, \\
\chi_2 = \lambda_1, \\
\chi_3 = 0.
\]  
This is the general boundary data without specific symmetry corresponding to the bulk data (38)–(40). There are two independent parameters in this boundary data due to the constraint (62). Note that the constraint (62) gives a restriction \( h(\mu) > 0 \) on the bulk data, which reads
\[
h(\mu) = -\frac{1}{2} (\rho_0(\mu) - \rho_1(\mu)) > 0.
\]  
For this condition to be enjoyed, we have to take an appropriate sign in the argument of theta function, such as \( s_j = s + j/3 \) for the \( \vartheta_3 \) solution, and \( s_j = -s + j/3 \) for the \( \vartheta_0 \) solution, respectively.

Let us now consider the \( C_3 \)-symmetric case, for which the boundary data in the \( C_3 \)-symmetric basis satisfies
\[
R_3 \otimes R_2 W^\dagger = W^\dagger \hat{q},
\]  
where \( W = \tilde{W} U_3 \), and \( R_3 \) and \( R_2 \) are the images of \( C_3 \)-rotation. One can observe the boundary data (63) enjoys (65), if we impose additional constraints \( \lambda_1 = \lambda_2 \) and \( \lambda_3 = 0 \). In this case, the boundary data (63) reduces to
\[
\lambda = i\lambda_1 (\sigma_1 + \sigma_2), \\
\rho = \rho_0 = -\frac{2}{\chi_1} g(\mu), \\
\chi = -i\lambda_1 (\sigma_1 - \sigma_2),
\]  
where
with a constraint $h(\mu) = 2\lambda_1^2$. There is no independent parameter in the boundary data here. The exact form of $R_2$ and $\hat{q}$ are

$$R_2 = \hat{q} = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \sigma_3. \quad (67)$$

We therefore have confirmed the existence of the $C_3$-symmetric caloron Nahm data.

5. Conclusion
To summarise, we have considered the caloron Nahm data with instanton charge $N$ focusing on their spatial $C_N$-symmetries. Having applied the ansatz for the $C_N$-symmetric monopoles by Sutcliffe, we have reduced the bulk Nahm equations to the periodic Toda equations, which are solved by the elliptic theta functions. The hermiticity of the bulk data are also confirmed by a specific choice of the branch of the theta functions. We have also given a conjecture on the unitary transformation into a new basis of the Nahm data in which the reality conditions are manifest. In the new basis, as an illustration, the conditions on the boundary data have been written down explicitly for the $N = 3$ case, and the general solution is found. The boundary data of the $N = 3$ calorons with $C_3$-symmetry is also found, as a special case.

Acknowledgments Atsushi Nakamula is grateful to the conference organizers of ISQS-XX for their kind accommodation and hospitality.

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