Abstract

We give a new realization of $\mathcal{Y}(\mathfrak{sl}_3)$ via Cartan–Weyl elements. An algebraic description of Yangian Double $\mathcal{DY}(\mathfrak{sl}_3)$, explicit comultiplication formulas and universal $R$-matrix are obtained in these terms.

1 Introduction

Yangian $\mathcal{Y}(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ was introduced by Drinfeld in [D1]. He showed that rational $R$-matrices are obtained via finite dimensional irreducible representations of Yangians. The second realization of Yangians, introduced in [D2], serves best for representations. However, such realization (in terms of Chevalley generators, Serre type relations) doesn’t provide a natural vector space basis. This is an obstacle for the construction of the quantum double $\mathcal{DY}(\mathfrak{g})$ and universal $R$-matrix, which both play an important role in physical applications.

In this paper, we come up with a new collection of generators for $\mathcal{Y}(\mathfrak{sl}_3)$ — an analogue of Cartan–Weyl elements. In their language, we give an explicit description of $\mathcal{DY}(\mathfrak{sl}_3)$, including comultiplication formulas (Theorem 5.2), and find a formula for universal $R$-matrix. The paper has the following structure:

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• Section 1: Definition of Yangian $Y(g)$.

• Section 2: $Y(sl_2)$ in terms of generating functions.

• Section 3: Isomorphism between two realizations of $Y(sl_n)$ (Theorem 4.1), its use for construction of Cartan–Weyl elements, description of $Y(sl_3)$ in their terms (Theorem 4.3 and Corollary 2 from it), and co-multiplication formulas (4.4).

• Section 4: Definition of $DY(g)$ as quantum double of $Y(g)$. The full explicit description of $DY(sl_3)$ (Theorem 5.2).

• Section 5: Universal $R$-matrix.

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# 2 Yangian $Y(g)$

Let $g$ be a simple Lie algebra over complex numbers. Fix a basis of simple roots $\{\alpha_i\}, \ i \in \Gamma$ with Cartan matrix $A_{i,j}$. Denote the positive roots by $\Delta_+(g)$. Yangian is a deformation of the universal enveloping algebra $U(g[t])[CP]$. $Y(g)$ can be defined by generators $[D2] e_{i,j}, f_{i,j}, h_{i,j}, \ i \in \Gamma, \ j \geq 0$ and relations:

\[
[h_{i,k}, h_{j,l}] = 0, \quad [h_{i,0}, e_{j,l}] = (\alpha_i, \alpha_j)e_{j,l},
\]

\[
[h_{i,0}, f_{j,l}] = -(\alpha_i, \alpha_j)f_{j,l}, \quad [e_{i,k}, f_{j,l}] = \delta_{i,j}h_{i,k+l},
\]

\[
[h_{i,k+1}, e_{j,l}] - [h_{i,k}, e_{j,l+1}] = \frac{1}{2}(\alpha_i, \alpha_j)\{h_{i,k}, e_{j,l}\},
\]

\[
[h_{i,k+1}, f_{j,l}] - [h_{i,k}, f_{j,l+1}] = -\frac{1}{2}(\alpha_i, \alpha_j)\{h_{i,k}, f_{j,l}\},
\]

\[
[e_{i,k+1}, e_{j,l}] - [e_{i,k}, e_{j,l+1}] = \frac{1}{2}(\alpha_i, \alpha_j)\{e_{i,k}, e_{j,l}\},
\]

\[
[f_{i,k+1}, f_{j,l}] - [f_{i,k}, f_{j,l+1}] = -\frac{1}{2}(\alpha_i, \alpha_j)\{f_{i,k}, f_{j,l}\},
\]
where \( \{a, b\} = ab + ba \). Two remaining groups of relations will be called Serre type relations in Yangian:

\[
\text{Sym}_{\{k\}}[e_{i,k_1}e_{i,k_2} \ldots e_{i,k_{n_{i,j}}}, e_{j,\ldots}] = 0,
\]

\[
\text{Sym}_{\{k\}}[f_{i,k_1}f_{i,k_2} \ldots f_{i,k_{n_{i,j}}}, f_{j,\ldots}] = 0,
\]

(2.1)

where \( i \neq j \), \( n_{i,j} = 1 - A_{i,j} \), and \( \text{Sym}_{\{k\}} \) stands for symmetrization on \( k_1, \ldots, k_{n_{i,j}} \).

A bialgebra (and then a Hopf algebra) structure is defined thru comultiplication \( \Delta \):

\[
\Delta(x) = x \otimes 1 + 1 \otimes x, \quad x \in g
\]

\[
\Delta(e_{i,1}) = e_{i,1} \otimes 1 + 1 \otimes e_{i,1} + h_{i,0} \otimes e_{i,0} - \sum_{\gamma \in \Delta_+(g)} f_{\gamma} \otimes [e_{i,0}, e_{\gamma}],
\]

\[
\Delta(f_{i,1}) = f_{i,1} \otimes 1 + 1 \otimes f_{i,1} + f_{i,0} \otimes h_{i,0} + \sum_{\gamma \in \Delta_+(g)} [f_{i,0}, f_{\gamma}] \otimes e_{\gamma},
\]

\[
\Delta(h_{i,1}) = h_{i,1} \otimes 1 + 1 \otimes h_{i,1} - \sum_{\gamma \in \Delta_+(g)} (\alpha_i, \gamma) f_{\gamma} \otimes e_{\gamma},
\]

Put

\[
e^+_i(u) = \sum_{k \geq 0} e_{i,k} u^{-k-1}, \quad f^+_i(u) = \sum_{k \geq 0} f_{i,k} u^{-k-1},
\]

\[
h^+_i(u) = 1 + \sum_{k \geq 0} h_{i,k} u^{-k-1}.
\]

3 Yangian via generating functions

This section contains the description of \( Y(sl_2) \) in terms of generating functions [KT][K]. The advantages of such description are discussed along with the difficulties to generalize it to \( Y(g) \).

Working with \( Y(sl_2) \), we omit subscripts of generating functions so that \( e^+(u) := e^+_1(u), \quad f^+(u) := f^+_1(u), \quad h^+(u) := h^+_1(u) \). One easily checks that relations (2.1) can be rewritten as follows:

\[
[h^+(u), h^+(v)] = 0, \quad [e^+(u), f^+(v)] = -\frac{h^+(u) - h^+(v)}{u - v},
\]

\[
[e^+(u), e^+(v)] = -\frac{(e^+(u) - e^+(v))^2}{u - v},
\]

3
\[
[f^+(u), f^+(v)] = \frac{(f^+(u) - f^+(v))^2}{u-v},
\]
\[
[h^+(u), e^+(v)] = -\frac{\{h^+(u), e^+(u) - e^+(v)\}}{u-v},
\]
\[
[h^+(u), f^+(v)] = \frac{\{h^+(u), f^+(u) - f^+(v)\}}{u-v}.
\] (3.1)

The comultiplication is not so easy to deal with, the corresponding formulas are hard to guess:
\[
\Delta(e^+(u)) = e^+(u) \otimes 1 + \sum_{k=0}^{\infty} (-1)^k (f^+(u+1))^k h^+(u) \otimes (e^+(u))^{k+1},
\]
\[
\Delta(f^+(u)) = 1 \otimes f^+(u) + \sum_{k=0}^{\infty} (-1)^k (f^+(u))^k h^+(u) (e^+(u+1))^k,
\]
\[
\Delta(h^+(u)) = \sum_{k=0}^{\infty} (-1)^k (k+1)(f^+(u+1))^k h^+(u) \otimes h^+(u) (e^+(u+1))^k.
\] (3.2)

We see, that in the case of \(Y(sl_2)\), all the formulas can be rewritten in terms of generating functions. The advantages of this description are difficult to exaggerate. First, the commutation relations (3.1) immediately imply P.B.W. theorem or, putting it in other words, monomials
\[
e_{m_0}^{n_0} e_{m_0}^{n_k} h_{m_0}^{m_0} \ldots h_{m_0}^{m_p} f_{l_0}^{l_0} \ldots f_{l_q}^{l_q}
\]
constitute a vector-space basis for \(Y(sl_2)\). Besides that, let us take a look at formulas for the comultiplication of \(Y(sl_2)\) (3.2). They have simple and compact structure. If we wished to write them down in terms of \(e_i, f_i, h_i\), we would end up in ugly cumbersome relations already for \(i = 2\). In addition, as we will see below, relations among generating functions ”do not change” for dual Hopf algebra case. What is changed - the generating functions themselves: they are now expanded in series at nonnegative powers of \(u\). Note, in turn, that dual algebra is not finitely generated what makes the effort to define comultiplication on generators almost hopeless. So, the problem to write the relations and comultiplication in Yangian thru generating functions is, in fact, equivalent to the problem of finding dual Hopf algebra and, as a matter of fact, to the problem of constructing the quantum double.

Almost all the relations in \(Y(sl_n)\) are analogous to those of \(Y(sl_2)\), so the formulas similar to (3.1) hold. However, there are two additional groups of relations, namely:
\[ [e_{i,k+1}, e_{j,l}] - [e_{i,k}, e_{j,l+1}] = (\alpha_i, \alpha_j) \{e_{i,k}, e_{j,l}\}, \quad (3.3) \]

\[ [f_{i,k+1}, f_{j,l}] - [f_{i,k}, f_{j,l+1}] = -(\alpha_i, \alpha_j) \{f_{i,k}, f_{j,l}\}, \quad (3.4) \]

where \( \alpha_i, \alpha_j \) are neighboring roots ((\( \alpha_i, \alpha_j \) = -1). These relations cannot be resolved with respect to commutator, i.e. \([e_{i,k}, e_{j,l}]\) cannot be expressed so that to give us an appropriate ordering of \(e_{i,k+1}\) and \(e_{j,l}\) for P.B.W. theorem. This is, of course, by no means incidental. Yangian, realized via Chevalley generators, has no natural vector space basis – one needs additional elements.

4 Cartan–Weyl basis for Yangian \( Y(sl_3) \)

In this section, we build Cartan–Weyl elements for \( Y(sl_3) \) and find explicit formulas for comultiplication on them.

If one aims to construct Cartan–Weyl elements in \( Y(sl_3) \) then definition of additional elements \( e_{3,k}, f_{3,k} \) as the components of generating functions \( e^+_3(u) = -[e^+_1(u), e_{2,0}], f^+_3(u) = [f^+_1(u), f_{2,0}] \) looks quite natural. Using definition of Yangian (2.1), one can obtain the relations among so defined generating functions. But it is not clear how to find the comultiplication on them. The key point here is the isomorphism between \( Y(sl_n) \) and Hopf algebra defined by RTT relation[2][FRT]. The isomorphism is constructed from triangle decomposition[DF]. As we will see shortly, the elements neighboring main diagonal in triangle decomposition of \( T(u) \) provide us (up to some shifts) with generating functions for Chevalley generators from the definition of Yangian (2.1). In Theorem 4.3 we show that two remaining (in upper and lower triangular parts respectively) functions give us \( e^+_3(u) \) and \( f^+_3(u) \) introduced above. So the isomorphism naturally extends to Cartan–Weyl elements what allows us to obtain the formulas for the comultiplication. Now, let us describe the isomorphism between two realizations of \( Y(sl_n) \). Consider bialgebra \( Y(R) \) given by generators \( t^{(k)}_{ij}, 1 \leq i, j \leq n, k = 1, 2, \ldots \) and defining relations:

\[ R(u - v)(T(u) \otimes E_n)(E_n \otimes T(v)) = (E_n \otimes T(v))(T(u) \otimes E_n)R(u - v), \quad (4.1) \]

\[ \det_q T(u) = 1, \]
where \( R = R(u - v) = 1 + \frac{P}{u - v} \), \( P \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \) being the flip of two factors, \( T(u) = (t_{ij}(u)) \), \( t_{ij}(u) = \delta_{ij} + \sum_{k \geq 0} t_{ij}^k u^{-k-1} \), \( \text{det}_q T(u) \) — the quantum determinant of \( T(u) \), i.e. \( \text{det}_q T(u) = \sum \text{sgn}(i_1, \ldots, i_n) t_{111}(u + \frac{n-1}{2}) t_{221}(u + \frac{n-3}{2}) \cdots t_{nn1}(u + \frac{1-n}{2}) \) — the sum over all permutations \((i_1, \ldots, i_n)\) of \( 1, 2, \ldots, n \). And the comultiplication has the following form:

\[
\Delta(t_{ij}(u)) = \sum_{k=1}^n t_{kj}(u) \otimes t_{ik}(u).
\]

**Theorem 4.1** ([D2]) Yangian \( Y(sl_n) \) is isomorphic to bialgebra \( Y(R) \), described above.

Comments:
First, one rewrites relation (4.1) as the following quadratic relation:

\[
[t_{ij}(u), t_{kl}(v)] = -\frac{1}{u - v} (t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u)).
\]  

(4.2)

Besides, bialgebra \( Y(R) \) is a Hopf algebra [M][FRT]. So, we induce a Hopf structure to \( Y(sl_n) \).

Isomorphism of Theorem 4.1 is constructed via Gauss (triangle) decomposition. Let us do the case of \( Y(sl_3) \) in details. Consider the triangle decomposition of \( T(u) \):

\[
\begin{pmatrix}
  t_{11} & t_{12} & t_{13} \\
t_{21} & t_{22} & t_{23} \\
t_{31} & t_{32} & t_{33}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  1 & 0 & 0 \\
  f_1(u) & 1 & 0 \\
  f_3(u) & f_2(u) & 1
\end{pmatrix}
\begin{pmatrix}
  k_1(u) & 0 & 0 \\
  0 & k_2(u) & 0 \\
  0 & 0 & k_3(u)
\end{pmatrix}
\times
\begin{pmatrix}
  1 & \tilde{e}_1(u) & \tilde{e}_3(u) \\
  0 & 1 & \tilde{e}_2(u) \\
  0 & 0 & 1
\end{pmatrix}
\]

(\*)

**Lemma 4.1** The equality (\*) implies the following expressions:

\[
k_1(u) = t_{11}(u), \quad k_2(u) = t_{22}(u) - t_{21}(u) t_{11}(u)^{-1} t_{12}(u)
\]

\[
\tilde{e}_1(u) = t_{11}(u)^{-1} t_{12}(u), \quad \tilde{f}_1(u) = t_{21}(u) t_{11}(u)^{-1}
\]

\[
\tilde{e}_2(u) = k_2(u)^{-1} (t_{23}(u) - t_{21}(u) t_{11}(u)^{-1} t_{13}(u))
\]

\[
\tilde{f}_2(u) = (t_{32}(u) - t_{31}(u) t_{11}(u)^{-1} t_{12}(u)) k_2(u)^{-1}
\]

\[
\tilde{e}_3(u) = t_{11}(u)^{-1} t_{13}(u), \quad \tilde{f}_3(u) = t_{31}(u) t_{11}(u)^{-1}
\]

\[
k_3(u) = t_{33}(u) - \tilde{f}_3(u) k_1(u) \tilde{e}_3(u) - \tilde{f}_2(u) k_2(u) \tilde{e}_2(u)
\]
The proof of lemma is trivial and is narrowed down to multiplying the matrices and solving a simple system of equations. Lemma conveniently gives us the expressions of $e_i(u)$, $f_i(u)$, $k_l(u)$ via $t_{ij}(u)$. So it can be viewed as definition of components of $\tilde{e}_i(u)$, $\tilde{f}_i(u)$, $k_l(u)$, that belong to $Y(R)$. Now take $Y(sl_3)$, defined in terms of generators $e_i, f_i, h_l, i = 1, 2, k \geq 0$ and relations (2.1). Recall that:

\[ e_i^+(u) = \sum_{k=0}^{\infty} e_{i,k} u^{-k-1}, \quad f_i^+(u) = \sum_{k=0}^{\infty} f_{i,k} u^{-k-1}, \]

\[ h_i^+(u) = 1 + \sum_{k=0}^{\infty} h_{i,k} u^{-k-1}. \]

Let us consider the map $j : Y(sl_3) \to Y(R)$ defined as follows

\[ j(e_1^+(u)) = \tilde{e}_1(u), \quad j(e_2^+(u)) = \tilde{e}_2(u + \frac{1}{2}), \quad (4.3) \]

\[ j(f_1^+(u)) = \tilde{f}_1(u), \quad j(f_2^+(u)) = \tilde{f}_2(u + \frac{1}{2}), \quad (4.4) \]

\[ j(h_1^+(u)) = \tilde{h}_1(u), \quad j(h_2^+(u)) = \tilde{h}_2(u + \frac{1}{2}), \quad (4.5) \]

where $\tilde{h}_1(u) = k_1^{-1}(u)k_2(u)$, $\tilde{h}_2(u) = k_2^{-1}(u)k_3(u)$.

**Theorem 4.2** [D2] The map $j : Y(sl_3) \to Y(R)$, defined above, is an isomorphism satisfying the conditions of Theorem 4.1 for $Y(sl_3)$.

All is quite the same for $Y(sl_n)$. If we denote by $\tilde{e}_i(u)$ the i-th element above the main diagonal in triangle decomposition of $T(u)$, then we have to take the shifts $e_i^+(u) = \tilde{e}_i(u + \frac{i-1}{2})$ in order to obtain $e_i^+(u)$ from the definition of $Y(sl_n)$. In [D2] one finds general formulas for such an isomorphism while we restrict ourselves to the case of $Y(sl_3)$. Let

\[ e_3^+(u) = -[e_1^+(u), e_{2,0}], \quad f_3^+(u) = [f_1^+(u), f_{2,0}], \]

\[ h_3^+(u) = h_1^+(u)h_2^+(u) + \frac{1}{4}(\{h_1^+(u), e_2^+(u)\}, f_2^+(u)). \]

So, the components of $f_i^+(u)$, $e_i^+(u)$, $h_i^+(u)$, $i = 1, 2, 3$ sit inside of $Y(sl_3)$, and the components of $\tilde{e}_i(u)$, $\tilde{f}_i(u)$ are in $Y(R)$. We have the following theorem:
Theorem 4.3  

a) Under the isomorphism of Theorem 4.2

\[ j(e_3^+(u)) = \tilde{e}_3(u), \quad j(f_3^+(u)) = \tilde{f}_3(u), \]
\[ j(h_3^+(u)) = t_{11}^{-1}(u)(t_{33}(u) - t_{31}(u)t_{11}^{-1}(u)t_{13}(u)) \]

b) The following relations hold:\

\[ [e_i^+(u), e_i^+(v)] = -\frac{(e_i^+(u) - e_i^+(v))^2}{u - v}, \quad (4.6) \]
\[ [f_i^+(u), f_i^+(v)] = \frac{(f_i^+(u) - f_i^+(v))^2}{u - v}, \quad (4.7) \]

where \( i = 1, 2, 3 \)

\[ [e_i^+(u), f_j^+(v)] = -\delta_{ij} \frac{h_i^+(u) - h_j^+(v)}{u - v}, \quad (4.8) \]

\[ [h_i^+(u), h_j^+(v)] = 0, \quad (4.9) \]

where \( i, j = 1, 2 \)

\[ [h_i^+(u), e_i^+(v)] = -\left\{ \frac{h_i^+(u) - h_i^+(v)}{u - v} \right\}, \quad (4.10) \]

where \( i = 1, 2 \)

\[ [h_i^+(u), f_i^+(v)] = \frac{h_i^+(u) - f_i^+(v)}{u - v}, \quad (4.11) \]

where \( i = 1, 2 \)

\[ [e_1^+(u), e_2^+(v)] = -\frac{1}{2} \left\{ \frac{e_1^+(u) - e_1^+(v)}{u - v}, e_2^+(v) \right\} + \frac{e_3^+(u) - e_3^+(v)}{u - v}, \quad (4.12) \]

\[ [f_1^+(u), f_2^+(v)] = \frac{1}{2} \left\{ \frac{f_1^+(u) - f_1^+(v)}{u - v}, f_2^+(v) \right\} - \frac{e_3^+(u) - e_3^+(v)}{u - v}, \quad (4.13) \]

\[ [e_1^+(u), e_3^+(v)] = -\frac{1}{u - v}(e_1^+(u) - e_1^+(v))(e_3^+(u) - e_3^+(v)), \quad (4.14) \]
\[
[f_1^+(u), f_3^+(v)] = \frac{1}{u-v}(f_1^+(u) - f_1^+(v))(f_3^+(u) - f_3^+(v)), \quad (4.15)
\]
\[
[h_1^+(u), e_2^+(v)] = \frac{1}{2} \left\{ h_1^+(u), e_2^+(u) - e_2^+(v) \right\}, \quad (4.16)
\]
\[
[h_2^+(u), e_1^+(v)] = \frac{1}{2} \left\{ h_2^+(u), e_1^+(u) - e_1^+(v) \right\}, \quad (4.17)
\]
\[
[h_1^+(u), f_2^+(v)] = -\frac{1}{2} \left\{ h_1^+(u), f_2^+(u) - f_2^+(v) \right\}, \quad (4.18)
\]
\[
[h_2^+(u), f_1^+(v)] = -\frac{1}{2} \left\{ h_2^+(u), f_1^+(u) - f_1^+(v) \right\}, \quad (4.19)
\]
\[
[e_3^+(u), f_3^+(v)] = -\frac{h_3^+(u) - h_3^+(v)}{u-v}. \quad (4.20)
\]

**Proof:**

Let us prove (4.12). We shall use the isomorphism from Theorem 4.2. We have:

\[
[\tilde{e}_1(u), \tilde{e}_2(v)] = [t_{11}^{-1}(u)t_{12}(u), k_2(v)^{-1}(t_{23}(v) - t_{21}(v)t_{11}(v)^{-1}t_{13}(v))]
= (det_2(v))^{-1}[t_{11}^{-1}(u)t_{12}(u), t_{11}(v-1)(t_{23}(v) - t_{21}(v)t_{11}(v)^{-1}t_{13}(v))]
\]

(Here \(det_2(v) = k_2(v)t_{11}(v-1)\) commutes with \(t_{ij}(u), i, j = 1, 2\))

\[
= (det_2(v))^{-1}t_{11}^{-1}(u)[t_{12}(u), t_{11}(v-1)(t_{23}(v) - t_{21}(v)t_{11}(v)^{-1}t_{13}(v))],
\]

then

\[
[t_{12}(u), t_{23}(v) - t_{21}(v)t_{11}(v)^{-1}t_{13}(v)] = A + B + C + D,
\]

where (we exploit (4.2))

\[
A = [t_{12}(u), t_{23}(v)] = -\frac{1}{u-v}(t_{22}(u)t_{13}(v) - t_{22}(v)t_{13}(u)),
\]
\[
B = -[t_{12}(u), t_{21}(v)]t_{11}^{-1}(v)t_{13}(v)
\]
\[
= \frac{1}{u-v}(t_{22}(u)t_{11}(v) - t_{22}(v)t_{11}(u))t_{11}^{-1}(v)t_{13}(v),
\]
\[ C = -t_{21}(v)t_{11}^{-1}(v)[t_{11}(v), t_{12}(u)]t_{11}^{-1}(v)t_{13}(v) \]
\[ = -\frac{1}{u-v} t_{21}(v)t_{11}^{-1}(v)(t_{12}(u)t_{11}(v) - t_{12}(v)t_{11}(u))t_{11}^{-1}(v)t_{13}(v), \]
\[ D = -t_{21}(v)t_{11}^{-1}(v)[t_{12}(u), t_{13}(v)] \]
\[ = \frac{1}{u-v} t_{21}(v)t_{11}^{-1}(v)(t_{12}(u)t_{13}(v) - t_{12}(v)t_{13}(u)). \]

Summing up, we get:
\[(det_{2})(v)^{-1}t_{11}^{-1}(u)t_{11}(v - 1)(A + B + C + D) = \frac{\bar{e}_3(u) - \bar{e}_3(v)}{u-v}.\]

What is left is the remark:
\[ [t_{11}^{-1}(u)t_{12}(u), t_{11}(v - 1)] \]
\[ = [t_{12}(u - 1)t_{11}^{-1}(u - 1), t_{11}(v - 1)] \]
\[ = [t_{12}(u - 1), t_{11}(v - 1)]t_{11}^{-1}(u - 1) \]
\[ = -\frac{1}{u-v} (t_{12}(u - 1)t_{11}(v - 1) - t_{12}(v - 1)t_{11}(u - 1))t_{11}^{-1}(u). \]

This implies:
\[(det_{2})(v)^{-1}t_{11}^{-1}(u)[t_{12}(u), t_{11}(v - 1)](t_{23}(v) - t_{21}(v)t_{11}(v)^{-1}t_{13}(v)) \]
\[ = \frac{\bar{e}_1(v) - \bar{e}_1(u)}{u-v} \bar{e}_2(v), \]
so
\[ [\bar{e}_1(u), \bar{e}_2(v)] = -\frac{\bar{e}_1(u) - \bar{e}_1(v)}{u-v} \bar{e}_2(v) + \frac{\bar{e}_3(u) - \bar{e}_3(v)}{u-v}. \]

Making the shift (recall that \( \bar{e}_2(v) = e_2^+(v - \frac{1}{2}) \), \( \bar{e}_1(v) = e_1^+(v) \)), we have:
\[ [e_1^+(u), e_2^+(v)] = -\frac{1}{2} \frac{\{e_1^+(u), e_1^+(v)}{u-v} + \frac{\bar{e}_3(u) - \bar{e}_3(v)}{u-v}. \]

Multiply this equality by \((u-v)\) and let \(v \to \infty\), then we get \(\bar{e}_3(u) = -[e_1^+(u), e_2, e_3] = e_3^+(u)\). The proof of the other relations is left to the reader. In principal, all the commutation relations are simply a translation of the definition of \(Y(sl_3)\) into the language of generating functions.

The key thing here (not obvious from this messy section) is the connection with the isomorphism of Theorem 4.2, i.e. in our case the formulas \(\bar{e}_3(u) = e_3^+(u), \ f_3(u) = f_3^+ (u)\), which we essentially proved. \(\triangleright\)

**Corollary 1** Put
\[ e_3^+(u) = \frac{1}{2} \{e_1^+(u), e_2^+(u)\} - e_3^+(u), \]
then
\[ f_3^+(u) = \frac{1}{2} \{ f_1^+(u), f_2^+(u) \} - f_3^+(u), \]

then
\[ e_3^+(u) = [e_{1,0}, e_2^+(u)], \quad f_3^+(u) = -[f_{1,0}, f_2^+(u)], \]
hence the following decomposition gives us other than in Theorem 4.2 realization for isomorphism from Theorem 4.1.

\[
\begin{pmatrix}
  t_{11} & t_{12} & t_{13} \\
  t_{21} & t_{22} & t_{23} \\
  t_{31} & t_{32} & t_{33}
\end{pmatrix}(u) = \begin{pmatrix}
  1 & 0 & 0 \\
  f_2^+(u) & 1 & 0 \\
  f_3^+(u) & f_1^+(u - \frac{1}{2}) & 1
\end{pmatrix} \begin{pmatrix}
  \tilde{k}_1(u) & 0 & 0 \\
  0 & \tilde{k}_2(u) & 0 \\
  0 & 0 & \tilde{k}_3(u)
\end{pmatrix}
\times \begin{pmatrix}
  1 & e_2^+(u) & e_3^+(u) \\
  0 & 1 & e_1^+(u - \frac{1}{2}) \\
  0 & 0 & 1
\end{pmatrix},
\]

where
\[
\tilde{k}_1(u)^{-1} \tilde{k}_2(u) = h_2^+(u), \quad \tilde{k}_2(u)^{-1} \tilde{k}_3(u) = h_1^+(u - \frac{1}{2})
\]

**Corollary 2 (Cartan–Weyl basis for \( Y(sl_3) \))**

\( Y(sl_3) \) can be defined by generators – the components of \( e_i^+(u), f_i^+(u), h_i^+(u) \) (for \( i = 1, 2 \)), \( e_3^+(u), f_3^+(u) \) — and relations (4.6)–(4.19), (4.21)–(4.22), where \( e_3^+(v), f_3^+(v) \) are defined in Corollary 1.

\[
[e_2^+(u), e_3^+(v)] = -\frac{1}{u - v} (e_2^+(u) - e_2^+(v))(e_3^+(u) - e_3^+(v)) \quad (4.21)
\]

\[
[f_2^+(u), f_3^+(v)] = \frac{1}{u - v} (f_2^+(u) - f_2^+(v))(f_3^+(u) - f_3^+(v)) \quad (4.22)
\]

In this way, we added relations with \([e_2^+(u), e_3^+(v)]\) and \([f_2^+(u), f_3^+(v)]\).

They are obtained immediately from Corollary 1. To prove Corollary 2 completely one also has to note (well, noticing requires some more technical calculations that are not difficult) that Serre type relations are followed from relations (4.6)–(4.19), (4.21)–(4.22).

**Theorem 4.4** The following formulas for comultiplication in \( Y(sl_3) \) hold:

\[
\Delta(e_1^+(u)) = e_1^+(u) \otimes 1 + \sum_{i,j=0}^{\infty} (-1)^{i+j} (i+j) f_1^+(u + 1)^i f_3^+(u + 1)^j \\
\otimes e_1^+(u)^i e_3^+(u)^j (h_1^+(u) \otimes e_1^+(u) + \frac{1}{2} (h_1^+(u), f_2^+(u)) \otimes e_3^+(u)) \quad (4.23)
\]

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\[ \Delta(e^+_2(u)) = e^+_2(u) \otimes 1 + \left( \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+j}{i} f^+_2(u + 1)^i f^+_3(u + 1)^j \right. \\
\left. \otimes e^+_2(u)^i e^+_3(u)^j (h^+_2(u) \otimes e^+_2(u) + \frac{1}{2} \{ h^+_1(u), f^+_1(u) \} \otimes e^+_3(u)) \right) \tag{4.24} \]

\[ \Delta(e^+_3(u)) = e^+_3(u) \otimes 1 + \left( \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+j}{i} f^+_1(u + 1)^i f^+_3(u + 1)^j \right. \\
\left. \otimes e^+_1(u)^i e^+_3(u)^j (h^+_3(u) \otimes e^+_3(u) + \frac{1}{2} \{ h^+_1(u), e^+_2(u) \} \otimes e^+_1(u)) \right) \tag{4.25} \]

\[ \Delta(f^+_1(u)) = 1 \otimes f^+_1(u) + (f^+_2(u) \otimes h^+_1(u) + \frac{1}{2} f^+_3(u) \otimes \{ h^+_1(u), e^+_2(u) \}) \\
\times \left( \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+j}{i} f^+_1(u)^i f^+_3(u)^j \right. \\
\left. \otimes e^+_1(u + 1)^i e^+_3(u + 1)^j \right) \tag{4.26} \]

\[ \Delta(f^+_2(u)) = 1 \otimes f^+_2(u) + (f^+_2(u) \otimes h^+_2(u) + \frac{1}{2} f^+_3(u) \otimes \{ h^+_2(u), e^+_1(u) \}) \\
\times \left( \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+j}{i} f^+_2(u)^i f^+_3(u)^j \right. \\
\left. \otimes e^+_2(u + 1)^i e^+_3(u + 1)^j \right) \tag{4.27} \]

\[ \Delta(f^+_3(u)) = 1 \otimes f^+_3(u) + (f^+_2(u) \otimes h^+_3(u) + \frac{1}{2} f^+_1(u) \otimes \{ h^+_3(u), f^+_2(u) \}) \\
\times \left( \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+j}{i} f^+_3(u)^i f^+_1(u)^j \right. \\
\left. \otimes e^+_1(u + 1)^i e^+_3(u + 1)^j \right) \tag{4.28} \]

**Proof:**

Notice that

\[ \Delta(e^+_1(u)) = \Delta(t^{-1}_{11}(u)t_{12}(u)) = \Delta(t_{11}(u))^{-1} \Delta(t_{12}(u)) \]

\[ = (1 \otimes 1 + t^{-1}_{11}(u)t_{21}(u) \otimes t^{-1}_{11}(u)t_{12}(u) \]
\[ + t^{-1}_{11}(u)t_{31}(u) \otimes t^{-1}_{11}(u)t_{13}(u))^{-1} \]
\[ \times (t^{-1}_{11}(u)t_{12}(u) \otimes 1 + r^{-1}_{11}(u)t_{22}(u) \otimes t^{-1}_{11}(u)t_{12}(u) \]
\[ + t^{-1}_{11}(u)t_{32}(u) \otimes t^{-1}_{11}(u)t_{13}(u)). \]
Playing with the last expression we find:

\[
\Delta(e_1^+(u)) = (1 \otimes 1 + f_1^+(u + 1) \otimes e_1^+(u) + f_3^+(u + 1) \otimes e_3^+(u))^{-1} \\
\times (e_1^+(u) \otimes 1 + (h_1^+(u) + f_1^+(u + 1)e_1^+(u)) \otimes e_1^+(u) + \\
(f_3^+(u + 1)e_1^+(u) + h_1^+(u)f_2^+(u - \frac{1}{2})) \otimes e_3^+(u)) \\
= e_1^+(u) \otimes 1 + (1 \otimes 1 + f_1^+(u + 1) \otimes e_1^+(u) \\
+ f_3^+(u + 1) \otimes e_3^+(u))^{-1} \\
\times (h_1^+(u) \otimes e_1^+(u) + \frac{1}{2}\{h_1^+(u), f_2^+(u)\} \otimes e_3^+(u)).
\]

It is easy to see that

\[
(1 \otimes 1 + f_1^+(u + 1) \otimes e_1^+(u) + f_3^+(u + 1) \otimes e_3^+(u))^{-1} \\
= \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+j}{i} (f_1^+(u + 1))^i (f_3^+(u + 1))^j \otimes (e_1^+(u))^i (e_3^+(u))^j;
\]

it implies

\[
\Delta(e_1^+(u)) = e_1^+(u) \otimes 1 + (\sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+j}{i} f_1^+(u + 1)^i f_3^+(u + 1)^j) \\
\otimes e_1^+(u)^i e_3^+(u)^j) (h_1^+(u) \otimes e_1^+(u) + \frac{1}{2}\{h_1^+(u), f_2^+(u)\} \otimes e_3^+(u)).
\]

So we get formula (4.23). The proof for the other formulas is similar.

5 Quantum Double for \(DY(sl_3)\)

This section is concerned with the construction of the quantum double of Yangian. We construct Cartan–Weyl basis for \(DY(sl_3)\) and find the comultiplication on it. In fact, our description implies (for the case of \(DY(sl_3)\)) the conjecture about algebraic structure of \(DY(sl_n)\) made in [KT].

The definition of quantum double is contained in the following theorem:

Theorem 5.1 [R] Let \(A\) be a Hopf algebra, \(A^\circ\) be the dual Hopf algebra \(A^*\) with the flipped comultiplication. There exists a unique quasi-triangular Hopf algebra \((D(A), R)\) such that the following holds: 1) \(A, A^\circ\) are Hopf subalgebras of \(D(A)\), 2) The linear map \(A \otimes A^\circ \rightarrow D(A)\), \(a \otimes b \rightarrow ab\) is bijective, 3) \(R\) is the image of the canonical element under inclusion \(A \otimes A^\circ \rightarrow D(a) \otimes D(A)\).

The permutation relations in double are given by the formula

\[
\]
\[ a \cdot b = \langle a^{(1)}, b^{(1)} \rangle > \langle S^{-1}(a^{(3)}), b^{(3)} \rangle > b^{(2)} \cdot a^{(2)} \], \quad (5.1)\]

where \( a \in A, \ b \in A^\circ, \ \Delta^2(x) = (\Delta \otimes id)\Delta(x) = (id \otimes \Delta)\Delta(x) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \), \( S \) is the antipode \([FRT][M]\) in \( A \).

Now we get back to \( Y(sl_n) \). Consider an algebra \( C \), generated by elements \( e_{i,k}, f_{i,k}, h_{i,k}, 1 \leq i \leq n - 1, k \in \mathbb{Z} \) and relations (2.1). Let \( \bar{C} \) be the formal completion of \( C \) corresponding to the filtration

\[ \cdots \subset C_{-n} \subset \cdots \subset C_0 \subset \cdots \subset C_n \cdots \subset C, \quad (5.2)\]

defined by \( \deg e_{i,k} = \deg f_{i,k} = \deg h_{i,k} = k \); \( \deg x \in C_m \leq m \). In the paper \([KT]\), there was made a conjecture that \( DY(sl_n) \), i.e. double of \( Y(sl_n) \), is isomorphic to \( \bar{C} \) as an algebra. For \( DY(sl_3) \) this conjecture is followed from Theorem 5.2 proved below.

Let

\[ e^+_i(u) := \sum_{k \geq 0} e_{i,k} u^{-k-1}, \quad e^-_i(u) := - \sum_{k < 0} e_{i,k} u^{-k-1}, \quad (5.3)\]

\[ f^+_i(u) := \sum_{k \geq 0} f_{i,k} u^{-k-1}, \quad f^-_i(u) := - \sum_{k < 0} f_{i,k} u^{-k-1}, \quad (5.4)\]

\[ h^+_i(u) := 1 + \sum_{k \geq 0} h_{i,k} u^{-k-1}, \quad h^-_i(u) := 1 - \sum_{k < 0} h_{i,k} u^{-k-1}. \quad (5.5)\]

We have formulas 4.4. The crucial step to describe the algebra dual to \( Y(sl_3) \) is to extend the comultiplication to all generating functions we just introduced. Let us do it as follows

\[ \Delta(e^+_1(u)) = e^+_1(u) \otimes 1 + (\sum_{i,j=0}^\infty (-1)^{i+j} \binom{i+j}{i} f^+_1(u+1)^i f^+_3(u+1)^j e^+_1(u) e^+_3(u)^2) (h^+_1(u) \otimes e^+_1(u) + \frac{1}{2} \{ h^+_1(u), f^+_2(u) \} \otimes e^+_3(u)), \quad (5.6)\]

\[ \Delta(e^+_2(u)) = e^+_2(u) \otimes 1 + (\sum_{i,j=0}^\infty (-1)^{i+j} \binom{i+j}{i} f^+_2(u+1)^i f^+_3(u+1)^j e^+_2(u) e^+_3(u)^2) (h^+_2(u) \otimes e^+_2(u) + \frac{1}{2} \{ h^+_2(u), f^+_1(u) \} \otimes e^+_3(u)), \quad (5.7)\]
\[ \Delta(e_3^\pm(u)) = e_3^\pm(u) \otimes 1 + \left( \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+j}{i} f_1^\pm(u+1)^i f_3^\pm(u+1)^j \right) \otimes e_1^\pm(u)^i e_3^\pm(u)^j (h_3^\pm(u) \otimes e_3^\pm(u) + \frac{1}{2} \{ h_1^\pm(u), e_2^\pm(u) \} \otimes e_1^\pm(u)), \]

\[ \Delta(f_1^\pm(u)) = 1 \otimes f_1^\pm(u) + (f_1^\pm(u) \otimes h_1^\pm(u) + \frac{1}{2} f_3^\pm(u) \otimes \{ h_1^\pm(u), e_2^\pm(u) \}) \]
\[ \times \left( \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+j}{i} f_1^\pm(u)^i f_3^\pm(u)^j \right) \otimes e_1^\pm(u+1)^i e_3^\pm(u+1)^j, \]

\[ \Delta(f_2^\pm(u)) = 1 \otimes f_2^\pm(u) + (f_2^\pm(u) \otimes h_2^\pm(u) + \frac{1}{2} f_3^\pm(u) \otimes \{ h_2^\pm(u), e_1^\pm(u) \}) \]
\[ \times \left( \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+j}{i} f_2^\pm(u)^i f_3^\pm(u)^j \right) \otimes e_2^\pm(u+1)^i e_3^\pm(u+1)^j, \]

\[ \Delta(f_3^\pm(u)) = 1 \otimes f_3^\pm(u) + (f_3^\pm(u) \otimes h_3^\pm(u) + \frac{1}{2} f_1^\pm(u) \otimes \{ h_1^\pm(u), f_2^\pm(u) \}) \]
\[ \times \left( \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+j}{i} f_1^\pm(u)^i f_3^\pm(u)^j \right) \otimes e_1^\pm(u+1)^i e_3^\pm(u+1)^j. \]

Now, we have got all to formulate and then prove the main theorem:

**Theorem 5.2** (Cartan–Weyl basis for DY(sl_3)) The quantum double of Yangian DY(sl_3) is completed by the filtration (5.2) algebra, given by generators \( e_{i,k}, f_{i,k}, h_{i,k}, i = 1, 2, e_{3,k}, f_{3,k}, k \in \mathbb{Z} \) and relations (5.12)-(5.28), where \( \varepsilon, \delta \in \{ +, - \}; e_1^\varepsilon(u), f_1^\varepsilon(u), e_2^\varepsilon(u), f_2^\varepsilon(u), e_3^\varepsilon(u), f_3^\varepsilon(u), h_1^\varepsilon(u), h_2^\varepsilon(u), h_3^\varepsilon(u) \) are generating functions defined by (5.3)-(5.5), and
\[ e_3^\varepsilon(u) = \frac{1}{2} \{ e_1^\varepsilon(u), e_2^\varepsilon(u) \} - e_3^\varepsilon(u), \]
\[ f_3^\varepsilon(u) = \frac{1}{2} \{ f_1^\varepsilon(u), f_2^\varepsilon(u) \} - f_3^\varepsilon(u), \]
\[ h_3^\varepsilon(u) = h_1^\varepsilon(u) h_2^\varepsilon(u) + \frac{1}{2} \{ h_1^\varepsilon(u), e_2^\varepsilon(u) \}, f_2^\varepsilon(u) \} \). The comultiplication is computed by formulas (5.6)-(5.11).

\[ [e_1^\varepsilon(u), e_1^\delta(v)] = - \frac{(e_1^\varepsilon(u) - e_1^\delta(v))^2}{u-v}, \]

\[ [f_1^\varepsilon(u), f_1^\delta(v)] = \frac{(f_1^\varepsilon(u) - f_1^\delta(v))^2}{u-v}, \]
where $i = 1, 2, 3$

\[
[e^\epsilon_i(u), f^\delta_j(v)] = -\delta_{ij} \frac{h^\epsilon_i(u) - h^\delta_j(v)}{u - v},
\]

(5.14)

\[
[h^\epsilon_i(u), h^\delta_j(v)] = 0,
\]

(5.15)

where $i, j = 1, 2$

\[
[h^\epsilon_i(u), e^\delta_i(v)] = -\frac{h^\epsilon_i(u), e^\epsilon_i(u) - e^\delta_i(v)}{u - v},
\]

(5.16)

\[
[h^\epsilon_i(u), f^\delta_i(v)] = \frac{h^\epsilon_i(u), f^\epsilon_i(u) - f^\delta_i(v)}{u - v},
\]

(5.17)

where $i = 1, 2$

\[
[e^\epsilon_1(u), e^\delta_2(v)] = -\frac{1}{2} \left\{ e^\epsilon_1(u) - e^\delta_2(v) \right\} + \frac{e^\epsilon_3(u) - e^\delta_3(v)}{u - v},
\]

(5.18)

\[
[f^\epsilon_1(u), f^\delta_2(v)] = \frac{1}{2} \left\{ f^\epsilon_1(u) - f^\delta_2(v) \right\} - \frac{f^\epsilon_3(u) - f^\delta_3(v)}{u - v},
\]

(5.19)

\[
[e^\epsilon_1(u), e^\delta_2(v)] = -\frac{1}{u - v} \left( e^\epsilon_1(u) - e^\delta_2(v) \right) \left( e^\epsilon_3(u) - e^\delta_3(v) \right),
\]

(5.20)

\[
[f^\epsilon_1(u), f^\delta_3(v)] = \frac{1}{u - v} \left( f^\epsilon_1(u) - f^\delta_3(v) \right) \left( f^\epsilon_3(u) - f^\delta_3(v) \right),
\]

(5.21)

\[
[e^\epsilon_2(u), e^\delta_3(v)] = -\frac{1}{u - v} \left( e^\epsilon_2(u) - e^\delta_3(v) \right) \left( e^\epsilon_3(u) - e^\delta_3(v) \right),
\]

(5.22)

\[
[f^\epsilon_2(u), f^\delta_3(v)] = \frac{1}{u - v} \left( f^\epsilon_2(u) - f^\delta_3(v) \right) \left( f^\epsilon_3(u) - f^\delta_3(v) \right),
\]

(5.23)

\[
[h^\epsilon_1(u), e^\delta_2(v)] = \frac{1}{2} \left\{ h^\epsilon_1(u), e^\epsilon_2(u) - e^\delta_2(v) \right\},
\]

(5.24)
\[ [h_2^\varepsilon(u), e_1^\delta(v)] = \frac{1}{2} \left\{ h_2^\varepsilon(u), e_1^\delta(v) - e_1^\delta(v) \right\}, \quad (5.25) \]

\[ [h_1^\varepsilon(u), f_2^\delta(v)] = -\frac{1}{2} \left\{ h_1^\varepsilon(u), f_2^\delta(u) - f_2^\delta(v) \right\}, \quad (5.26) \]

\[ [h_2^\varepsilon(u), f_1^\delta(v)] = -\frac{1}{2} \left\{ h_2^\varepsilon(u), f_1^\delta(u) - f_1^\delta(v) \right\}, \quad (5.27) \]

\[ [e_3^\varepsilon(u), f_3^\delta(v)] = -\frac{h_3^\varepsilon(u) - h_3^\delta(v)}{u - v}, \quad (5.28) \]

**Proof:**

Let the subalgebra \( Y^+ = Y(sl_3) \subset D Y(sl_3) \) be generated by components of \( e_i^+(u), f_i^+(u), h_i^+(u) \), and \( Y^- \) be the formal completion by (5.2) of subalgebra generated by components of \( e_i^-(u), f_i^-(u), h_i^-(u) \). Let us describe the pairing between \( Y^+ \) and \( Y^- \). Denote by \( E^\pm, F^\pm, H^\pm \) the subalgebras (or their completions in case of \( Y^- \)), generated by components of \( e_i^\pm(u), f_i^\pm(u), h_i^\pm(u), i = 1, 2 \). We agree that \( E^+ \) and \( F^+ \) do not contain the unit. We are going to use the following proposition [KT]

**Lemma 5.1** There exist a pairing \( <, > : Y^+ \otimes Y^- \rightarrow \mathbb{C} \) such that

- The pairing \( <, > \) preserves decompositions

\[ Y^+ = E^+H^+F^+, \quad Y^- = F^-H^-E^-, \]

i.e.

\[ < e^+h^+f^+, f^-h^-e^- >= < e^+, f^- > < h^+, h^- > < f^+, e^- > \]

for all \( e^\pm \in E^\pm, h^\pm \in H^\pm, f^\pm \in F^\pm \).

- \( <, > \) is as follows on generators:

\[ < e_1^+(u), f_1^-(v) > = \frac{\delta_{ij}}{(u - v)}, \quad < f_1^+(u), e_1^-(v) > = \frac{\delta_{ij}}{(u - v)} \]

\( i, j = 1, 2, 3 \)

\[ < h_1^+(u), h_1^-(v) > = \frac{u - v + b_{ij}}{u - v - b_{ij}} \]

where \( i, j = 1, 2, b_{ii} = 1, b_{i(i+1)} = -\frac{1}{2} \)
\( (Y^+)^\circ = Y^- \) in the sense that the following holds:

\[
<x, y_1 y_2> = <\Delta(x), y_1 \otimes y_2>, \quad <x_1 x_2, y> = <x_2 \otimes x_1, \Delta(y)>
\]

There is a proof of the lemma in [KT]. The explicit formula for \( R \)-matrix given in the end of this paper implies that the pairing is non-degenerate. In this way, we only left with the burden of proving the permutation relations that follow from formula (5.1). For this purpose we need a few terms of formulas (5.6)–(5.11).

Let us prove the formula

\[
[e_1^+(u), e_2^-(v)] = -\frac{1}{2} \left\{ e_1^+(u) - e_1^-(v), e_2^-(v) \right\} + \frac{e_3^+(u) - e_3^-(v)}{u - v}
\]

We will settle for the following information about comultiplication and antipode extracted from (5.6)–(5.11):

\[
S^{-1} e_1^+(u) = 0 \mod E^+ Y^+
\]

\[
\Delta^2(e_1^+(u)) = e_1^+(u) \otimes 1 \otimes 1 + h_1^+(u) \otimes e_1^+(u) \otimes 1
\]

\[
+ \frac{1}{2} \{ h_1^+(u), f_1^+(u) \} \otimes e_3^-((u) \otimes 1 - f_1^+(u + 1) h_1^+(u) \otimes (e_1^+(u))^2 \otimes 1
\]

\[
= \mod Y^+ \otimes Y^+ \otimes E^+ + (F^+)^2 H^+ \otimes Y^+ \otimes Y^+
\]

\[
\Delta^2(e_2^-(v)) = e_2^-(v) \otimes 1 \otimes 1 + h_2^-(v) \otimes e_2^-(v) \otimes 1
\]

\[
+ \frac{1}{2} \{ h_2^-(v), f_2^-((v) \otimes e_3^-((v) \otimes 1 - f_2^-((v + 1) h_2^-((v) \otimes (e_2^-(v))^2 \otimes 1
\]

\[
= \mod Y^- \otimes Y^- \otimes E^- + (F^-)^2 H^- \otimes Y^- \otimes Y^-
\]

By using (5.1) we get:

\[
e_1^+(u) e_2^-(v) = <e_1^+(u), \frac{1}{2} \{ h_2^-(v), f_1^-((v) > e_3^-(v)
\]

\[
+ <h_1^+(u), h_2^-((v) > e_2^-(v) e_1^+(u) + <\frac{1}{2} \{ h_1^+(v), f_2^+(u) \} e_2^-((v) > e_3^+(u).
\]

Then,

\[
<e_1^+(u), \frac{1}{2} \{ h_2^-(v), f_1^-((v) >= <e_1^+(u), f_1^-((v - \frac{1}{2} h_1^-((v) >
\]

\[
= <e_1^+(u), f_1^-((v - \frac{1}{2}) >= \frac{1}{u - v + \frac{1}{2}}.
\]
\[ <h^+_1(u), h^{-}_2(v)> = \frac{u - v - \frac{1}{2}}{u - v + \frac{1}{2}}, \]
\[ <\frac{1}{2}\{h^+_1(v), f^+_2(u)\}, e^{-}_2(v)> = <f^+_1(u + \frac{1}{2}), e^{-}_2(v)> = \frac{1}{u - v + \frac{1}{2}}. \]

Hence
\[ e^+_1(u)e^{-}_2(v) = \frac{e^{-}_3(v) + e^+_3(u)}{u - v + \frac{1}{2}} + \frac{u - v - \frac{1}{2}}{u - v + \frac{1}{2}}e^{-}_2(v)e^+_1(u). \]

Recalling that
\[ e^{-}_3(v) = \frac{1}{2}\{e^{-}_1(v), e^{-}_2(v)\} - e^{-}_3(v), \]
we get the desired relation. The other relations can be proved in the same manner. \(\triangleright\)

6 Universal \(R\)-matrix

There is an analogue of P.B.W. theorem for \(Y(g)\) in [CP]. Reformulation of this result for \(Y(sl_3)\) gives us Theorem 6.1.

Denote by \(Y^+_+, Y^+_-, Y^+_0\) the subalgebras of \(Y(sl_3)\) with the unit, generated by \(e_{i,k} (i = 1, 2; k \geq 0)\); \(f_{i,k} (i = 1, 2; k \geq 0)\); \(h_{i,k} (i = 1, 2; k \geq 0)\) correspondingly.

**Theorem 6.1** The ordered monomials in \(e_{i,k} (i = 1, 2, 3; k \geq 0)\); \(f_{i,k} (i = 1, 2, 3; k \geq 0)\); \(h_{i,k} (i = 1, 2; k \geq 0)\) form vector space bases of \(Y^+_+; Y^+_--; Y^+_0\) respectively. The multiplication induces an isomorphism of vector spaces
\[ Y^+_+ \otimes Y^+_0 \otimes Y^- \rightarrow Y(sl_3). \]

As it was mentioned, \(R\)-matrix in \(DY(sl_3) \otimes DY(sl_3)\) is the canonical element \(\xi_1 \otimes \xi^1\), where \(\xi_i\) and \(\xi^i\) are dual bases of spaces \(Y^+\) and \(Y^-\). Denote by \(Y^+_+, Y^-_-, Y^+_0\) the subalgebras generated by components of \(e^{-}_1(u), f^{-}_1(u), h^{-}_1(u), i = 1, 2\). From the paper [KT] we know that \(R\)-matrix factors as follows (cf. Theorem 6.1):
\[ R = R_ER_HR_F, \]
where
\[ R_E \in Y^+_+ \otimes Y^-_-, \quad R_F \in Y^+_+ \otimes Y^+_-, \quad R_H \in Y^+_0 \otimes Y^-_0. \]

The matrix \(R_H\) is computed in [KT] for \(Y(g)\). However, \(R_E\) and \(R_F\) are not known in general case by the moment. Using the explicit formulas for comultiplication we obtained
Theorem 6.2 In quantum double $DY(sl_3)$ $R$-matrix is expressed by virtue of formulas

$$R_E = \prod_{k \geq 0} \exp(-e_{1,k} \otimes f_{1,-k-1}) \prod_{k \geq 0} \exp(-f_{3,k} \otimes f_{3,-k-1}) \prod_{k \geq 0} \exp(-e_{2,k} \otimes f_{2,-k-1}),$$

$$R_F = \prod_{k \geq 0} \exp(-f_{2,k} \otimes e_{2,-k-1}) \prod_{k \geq 0} \exp(-f_{3,k} \otimes e_{3,-k-1}) \prod_{k \geq 0} \exp(-f_{1,k} \otimes e_{1,-k-1}),$$

where

$$\prod_{k \geq 0} \exp(-e_{i,k} \otimes f_{i,-k-1}) = \exp(-e_{i,0} \otimes f_{i,-1}) \cdot \exp(-e_{i,1} \otimes f_{i,-2}) \cdot \ldots,$$

$$\prod_{k \geq 0} \exp(-e_{i,k} \otimes f_{i,-i-1}) = \ldots \cdot \exp(-f_{i,1} \otimes e_{i,-2}) \cdot \exp(-f_{i,0} \otimes e_{i,-1}).$$

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