Deformations of quadratic algebras and the corresponding quantum semigroups

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Abstract

Let $V$ be a finite dimensional vector space. Given a decomposition $V \otimes V = \bigoplus_i^n I_i$, define $n$ quadratic algebras $(V, J_m)$ where $J_m = \bigoplus_{i \neq m} I_i$. This decomposition defines also the quantum semigroup $M(V; I_1, ..., I_n)$ which acts on all these quadratic algebras. With the decomposition we associate a family of associative algebras $A_k = A_k(I_1, ..., I_n), k \geq 2$. In the classical case, when $V \otimes V$ decomposes into the symmetric and skewsymmetric tensors, $A_k$ coincides with the group algebra of the symmetric group $S_k$. Let $I_{ih}$ be deformations of the subspaces $I_i$. In the paper we give a criteria for flatness of the corresponding deformations of the quadratic algebras $(V[[h]], J_{ih}$ and the quantum semigroup $M(V[[h]]; I_{1h}, ..., I_{nh})$. It says that the deformations will be flat if the algebras $A_k(I_1, ..., I_n)$ are semisimple and under the deformation their dimension does not change.

Usually, the decomposition into $I_i$ is defined by a given Yang-Baxter operator $S$ on $V \otimes V$, for which $I_i$ are its eigensubspaces, and the deformations $I_{ih}$ are defined by a deformation $S_h$ of $S$. We consider the cases when $S_h$ is a deformation of Hecke or Birman-Wenzl symmetry, and also the case when $S_h$ is the Yang-Baxter operator which appears by a representation of the Drinfeld-Jimbo quantum group. Applying the flatness criteria we prove that in all these cases we obtain flat deformations of the quadratic algebras and the corresponding quantum semigroups.

1 Quadratic algebras, quantum semigroup, and notations

Let $V$ be a module over a ring $A$, $I$ a submodule of $V^{\otimes 2} = V \otimes V$. Denote by $I^{i,k}$ the submodule in $V^{\otimes n}$ of the type $V \otimes \cdots \otimes I \otimes \cdots \otimes V$ where $I$ occupies the positions $i, i + 1$. Set $I^k = \sum_i I^{i,k}$ and $I^{(k)} = \cap_i I^{i,k}$. So, $I = I^2 = I^{(2)}$.

Let $V^*$ be the dual module to $V$. Denote by $I^\perp$ the submodule of $V^*$ which consists of all linear mappings $\varphi : V \to A$ such that $\varphi(v) = 0$ for $v \in I$. 


We say that the ordered pair of submodules \((I, J)\), \(I, J \subset V \otimes k\), is well situated if \(V \otimes k = I^{(k)} \oplus J^k\) for all \(k \geq 2\). In particular, \(V \otimes 2 = I \oplus J\). It is easy to see that if the pair \((I, J)\) is well situated then the pair \((J^⊥, I^⊥)\) of submodules in \(V^* \otimes 2\) is also well situated. This follows from the relations \((L + M)^⊥ = L^⊥ \cap M^⊥\) and \((L \cap M)^⊥ = L^⊥ + M^⊥\) which are true for any submodules of an \(A\)-module.

For a submodule \(J \in V \otimes 2\) we denote by \(Q_J = (V, J)\) the quadratic algebra \(T(V)/I_J\), where \(T(V)\) is the tensor algebra over \(V\) and \(I_J\) denotes the ideal generated by \(J\). The algebra \(Q_J\) is a graded one, its \(k\)th homogeneous component \(Q_J^k\) is equal to \(T^k(V)/J^k\) as \(A\)-module. If the pair \((I, J)\) is well situated the restriction of the natural mapping \(T^k(V) \to Q_J^k\) gives an isomorphism \(I^{(k)} \to Q_J^k\) of \(A\)-modules.

In the sequel we will deal with the cases when \(A\) is either a field \(k\) of characteristic zero or the algebra \(k[[h]]\) of formal power series in a variable \(h\). In the latter case we will consider only modules of finite rank and complete in \(h\)-adic topology. In particular, all tensor products will be completed in that topology. Any free \(k[[h]]\)-module of rank \(n\) is isomorphic as \(k[[h]]\)-module to \(E \otimes_k k[[h]] = E[[h]]\), the module of formal power series in \(h\) with coefficients from \(E\).

We say that a submodule \(J_h\) of a \(k[[h]]\)-module \(E_h\) is a splitting submodule if it has a complementary submodule \(I_h\), i.e. \(E_h = J_h \oplus I_h\). It is clear that in case \(E_h\) is a free module any submodule \(J_h\) is free, but \(J_h\) is a splitting one if and only if the module \(E_h/J_h\) is free. We call a morphism of free modules, \(\varphi : E_h \to V_h\), flat if \(Im\varphi\) (or equivalently, \(Ker\varphi\)) is a splitting submodule.

Let \(J\) is a linear subspace in a vector space \(E\) over \(k\). We say that \(J_h\) is a family of subspaces in \(E\), or a (formal) deformation of the subspace \(J\), if \(J_h\) is a splitting submodule in \(E_h = E[[h]]\) such that \(J_h = J\). Here \(J_h\) is the set of elements which obtained from elements of \(J_h\) replacing \(h\) by 0. Note that for a submodule \(J_h\) in \(E[[h]]\) we have \(\dim J_h \leq \dim J_{h \neq 0}\), and \(J_h\) defines a deformation of the subspace \(J_0\) if \(\dim J_0 = \dim J_{h \neq 0}\). Here \(J_{h \neq 0}\) denotes the vector subspace in the “general” point, i.e. the vector subspace \(J_h \otimes_{k[[h]]} k\{\{h\}\}\) in the vector space \(V_h \otimes_{k[[h]]} k\{\{h\}\}\) over the field of formal Laurent series \(k\{\{h\}\}\).

If \(J_h\) is not a splitting submodule in \(E[[h]]\), then the module \(P_h = E[[h]]/J_h\) has a decomposition \(P_h = P'_h \oplus P''_h\) where \(P'_h\) is a free module and \(P''_h\) is the torsion submodule of \(P_h\), that is \(b \in P''_h\) if and only if there exists \(m > 0\) such that \(h^mb = 0\). Denote by \(J'_h\) the
kernel of the natural projection $E[[\h]] \to P_\h'$. It is clear that $J_\h'$ is a splitting submodule in $E[[\h]]$ and $\dim J_\h \neq 0 = \dim J_\h' \neq 0 = \dim J_\h''$. So we denote $J_\h'' = J_{\h \to 0}$.

We will only consider quadratic $k[[\h]]$-algebras $(V_\h, J_\h)$ such that $V_\h$ is a free module of finite rank and $J_\h$ is a splitting submodule of $V_\h^{\otimes 2}$. We associate to the quadratic algebra $(V_\h, J_\h)$ the quadratic algebra $(V, J)$ over $k$ taking $V = V/hV_\h$ and $J = J_\h/hJ_\h$ with the natural imbedding $J \to V^{\otimes 2}$. In this case we call the quadratic algebra $Q_\h = (V_\h, J_\h)$ a deformation of the algebra $Q = (V, J)$. We call another deformation $(V', J'_\h)$ of the algebra $(V, J)$ equivalent to $(V_\h, J_\h)$ if there exists an isomorphism $\phi : V'_\h \to V_\h$ which induces the identity isomorphism on $V$ and the isomorphism $\phi \otimes \phi : V'_\h \otimes V'_\h \to V_\h \otimes V_\h$ gives an isomorphism $J'_\h \to J_\h$. Since there exists an isomorphism $V[[\h]] \to V_\h$, any deformation $(V_\h, J_\h)$ of the algebra $(V, J)$ can be given by a formal deformation of the subspace $J$ in $V^{\otimes 2}$, i.e. is equivalent to a deformation of the form $(V[[\h]], J'_\h)$ where $J'_\h = J$.

Let $(V_\h, J_\h)$ be a quadratic algebra. Note that in general $J^k_\h$ need not be a splitting submodule in $V^{\otimes k}_\h$ for $k > 2$, so the homogeneous component $Q^k_\h = V^{\otimes k}_\h/J_\h$ will not be a free module. We call the deformation $Q_\h$ a flat deformation (or quantization) of $Q$ if all modules $Q^k_\h$ are free. Note that this terminology is not completely standard. For many authors the flatness condition is included in the definition of a deformation.

We mention here a theorem due to Drinfeld [DR], which states that in case of the Koszul quadratic algebra $(V, J)$, [Ma], in order for all modules $Q^k_\h$, $k > 2$, to be free it is sufficient that the module $Q^3_\h$ be free, i.e. the submodule $J^3_\h$ be splitting in $V^{\otimes 3}_\h$.

Let $V$ be a vector space over $k$. Suppose $I_i, i = 1, ..., n$, are vector subspaces in $V^{\otimes 2}$ such that $V^{\otimes 2} = \bigoplus_i^n I_i$. Denote $J_k = \bigoplus_{i \neq k} I_i$, so $V^{\otimes 2} = I_1 \oplus J_1$. We associate to the tuple $(I_1, ..., I_n)$ a quantum semigroup $M(V) = M(V; I_1, ..., I_n)$ in the following way. We identify $\text{End}(V) = V \otimes V^*$ and put $M(V) = (\text{End}(V), I)$, the quadratic algebra where the subspace $I$ of $\text{End}(V)^{\otimes 2}$ is defined as $I = \sigma_{2,3}(I_1 \otimes I_1^\perp + \cdots + I_n \otimes I_n^\perp)$ where $\sigma_{2,3}$ is the permutation of the second and third tensor components. The algebra $M(V)$ has the natural bialgebra structure and the algebras $(V, J_i)$ make into comodules over $M(V)$ [Ma].

The quantum semigroup $M(V)$ also admits the description in the spirit of Faddeev-Reshetikhin-Takhtajan [FR1]. Let $\lambda_i, i = 1, ..., n$, be different elements from $k$. Let $S$ be the linear operator acting on $\text{End}(V^{\otimes 2})$, which has $I_i$ as the eigensubspace corresponding to the eigenvalue $\lambda_i$ for all $i$. Identifying $\text{End}(V^{\otimes 2}) \cong \text{End}(V)^{\otimes 2}$ via the Kronecker product
we may view $S$ as an element of $\text{End}(V)^{\otimes 2}$, $S = S_{(1)} \otimes S_{(2)}$ in the Sweedler notation. Then $I$ consists of the elements $X = X_{(1)} \otimes X_{(2)}$ of $\text{End}(V)^{\otimes 2}$ having the form $SX - XS = S_{(1)}X_{(1)} \otimes S_{(2)}X_{(2)} - X_{(1)}S_{(1)} \otimes X_{(2)}S_{(2)}$. This means that coaction of $M(V)$ on $V$ preserves all the subspaces $I_i$.

Denote by $A_2(S)$ the associative subalgebra in $\text{End}(V)$ generated by $S$. It is a semisimple algebra isomorphic to a direct sum of $n$ copies of the base field. Let $A_k(S)$ be the associative subalgebra in $\text{End}(V^{\otimes k})$ generated by the operators $S_i$, $i = 1, ..., k - 1$, where $S_i$ denotes the operator in $\text{End}(V^{\otimes k})$ which coincides with $S$ in the position $i, i + 1$ and is the identity in the other positions. It is clear that all the algebras $A_k(S)$ depend only on the subspaces $I_i$ but not on choosing of $\lambda_i$. So the algebras $A_k(S)$ we also denote by $A_k(I_1, ..., I_n)$.

2 Quadratic algebras and semisimplicity

In what following we suppose that the field $k$ is equal to $\mathbb{R}$ or $\mathbb{C}$.

A finite-dimensional representation $E$ of an algebra $A$ (or $A$-module) is called simple if there are no nontrivial invariant subspaces, and it is called semisimple if $E$ is isomorphic to a direct sum of simple representations. A finite-dimensional algebra is called semisimple if all its finite-dimensional representations are semisimple. An linear operator $B \in \text{End}(E)$ is semisimple if the subalgebra of operators generated by it is semisimple. In general, we call a set of operators $F \subset \text{End}(E)$ semisimple if the subalgebra $A(F)$ generated by this family is semisimple.

As is known [Pie] an algebra $A$ will be semisimple if and only if its semisimple representations separate points, i.e. for any two elements $a, b \in A$ there exists a semisimple representation $\varphi : A \to \text{End}(V)$ such that $\varphi(a) \neq \varphi(b)$. In particular, if $A$ is a subalgebra of $\text{End}(E)$ and the space $E$ is a semisimple $A$-module then $A$ is semisimple. It follows from this that the following algebras are semisimple:

a) $A(\varphi(G))$ for a representation $\varphi : G \to \text{End}(E)$ of a semisimple or compact Lie algebra $G$;

b) $A(\varphi(G))$ for a representation $\varphi : G \to \text{End}(E)$ of semisimple or compact Lie group $G$;

c) $A(\varphi(F))$ for any subset $F$ of a compact Lie algebra or group and $\varphi$ is its representation.

Note that if $\varphi$ is a representation of a connected Lie group $G$ and $\psi$ is the corresponding
Proposition 2.1 Let $E$ be a finite-dimensional vector space over $k$. Suppose $B_1,\ldots,B_m$ is a semisimple set of linear operators on $E$ and $\lambda_1,\ldots,\lambda_m$ are elements from $k$. Denote $L = \sum_i \text{Im}(B_i - \lambda_i)$, $K = \cap_i \text{Ker}(B_i - \lambda_i)$. Then

a) the subspaces $L$ and $K$ are invariant under all the $B_i$;

b) $E = L \oplus K$.

Proof  a) The invariance of $K$ is obvious. Let $v = \sum_i(B_i - \lambda_i)u_i$. Then $B_jv = \sum_i B_i(B_j - \lambda_j)u_i = \sum_i(B_i - \lambda_i)B_ju_i + \sum_i(B_j - \lambda_j)u_i$, and a) follows from the equality of commutators: $[B_j, (B_i - \lambda_i)] = [(B_j - \lambda_j), (B_i - \lambda_i)]$.

b) Because of semisimplicity there exists an invariant subspace $P$ in $E$ complementary to $L$. If $v \in P$ then $(B_i - \lambda_i)v$ has to belong to both $L$ and $P$. Hence $(B_i - \lambda_i)v = 0$ for all $i$. It means that $v \in K$. So $P \subset K$ and $E = L + K$. Let now $T$ be the invariant subspace in $E$ complementary to $K$. It is clear that $L = \sum_i(B_i - \lambda_i)T$, so $L \subset T$ and, therefore, $L \cap K = 0$. It implies that $E = L \oplus K$. 

Now we consider deformations of semisimple algebras and their morphisms. In general, let $A_h$ be an algebra over $k[[h]]$ which is a free $k[[h]]$-module. Then $A_0 = A_h/hA_h$ is an algebra over $k$, and we call $A_h$ a family of algebras, or a deformation of the algebra $A_0$. If $A_h'$ is another deformation of $A_0$ then a morphism of the deformations is a $k[[h]]$-algebra morphism $A_h \to A_h'$ which is the identity for $h = 0$. The deformation is trivial if there exists an $k[[h]]$-algebra isomorphism $A_h \to A_0[[h]] = A_0 \otimes_k k[[h]]$. We say that a subalgebra $B_h \subset A_h$ is splitting if it is a splitting $k[[h]]$-submodule in $A_h$.

Proposition 2.2 a) Let $A_h$ be a family of algebras. Suppose the algebra $A_0$ over $k$ is semisimple. Then $A_h$ is isomorphic to $A_0[[h]]$ as $k[[h]]$-algebra, i.e. the deformation is trivial.

b) Let $\phi_h : A[[h]] \to B[[h]]$ be a morphism of $k[[h]]$-algebras. It induces the morphism $\phi_0 : A \to B$ of $k$-algebras. Suppose $A$ is semisimple and $B$ is an arbitrary unital algebra. Then there exists an element $f_h \in B[[h]]$ such that $f_0 = 1$ and $\phi_h = \text{Ad}(f_h)(\phi_0 \otimes 1)$. Here $\text{Ad}(b)c = bcb^{-1}$ and $\phi_0 \otimes 1 : A \otimes_k k[[h]] \to B \otimes_k k[[h]]$ is the morphism of tensor products induced by $\phi_0$ and the identity morphism.
Proof The proposition follows from the fact that the Hochschild cohomology of any semisimple algebra are equal to zero, using the standard arguments. More precisely, a) follows from $H^2(A, A) = 0$ and b) from $H^1(A, B) = 0$ where $B$ is considered as $A$-bimodule via the morphism $\phi_0$.

Let families of algebras $A_h$ and vector spaces $V_h$ be given. Suppose the algebra $A_h$ acts on $V_h$, i.e. we are given a morphism of $k[[h]]$-algebras $\varphi_h : A_h \to \text{End}(V_h)$. Then $\varphi_h$ induces a morphism $\varphi_0 : A_0 \to \text{End}(V_0)$. On the other hand, any morphism $\psi : A_0 \to \text{End}(V_0)$ generates in the trivial way the morphism $\psi : A_0[[h]] \to \text{End}_{k[[h]]}(V_0[[h]]) = \text{End}_k(V_0)[[h]]$, so as a consequence of the preceding proposition we get that if the algebra $A_0$ is semisimple then the morphisms $\varphi_h$ and $\varphi_0$ are isomorphic.

**Proposition 2.3** Let $E$ be a finite-dimensional vector space over $k$. Suppose $B_1, ..., B_m$ is a semisimple set of semisimple linear operators on $E$, $\lambda_1, ..., \lambda_m$ are elements from $k$. Let $B_{ih} \in \text{End}(E[[h]])$ and $\lambda_{ih} \in k[[h]]$ be deformations of $B_i$ and $\lambda_i$ ($i = 1, ..., m$) such that

i) all the subalgebras $A_{ih} = A(B_{ih})$ are splitting submodules;

ii) the subalgebra $A_h = A(B_{1h}, ..., B_{mh})$ is a splitting submodule;

iii) all the submodules $K_{ih} = \text{Ker}(B_{ih} - \lambda_{ih})$ are splitting ones.

Denote $L_h = \sum_i \text{Im}(B_{ih} - \lambda_{ih})$, $K_h = \cap_i \text{Ker}(B_{ih} - \lambda_{ih})$. Then

a) the submodules $L_h$ and $K_h$ are invariant under all $B_{ih}$;

b) $E[[h]] = L_h \oplus K_h$. In particular, $L_h$ and $K_h$ are splitting submodules.

Proof The invariance of $L_h$ and $K_h$ can be proven as in Proposition 2.1. At first, suppose that the algebra $A_h$ has the form $A_0[[h]]$ there $A_0 = A(B_1, ..., B_m)$. Denote $L_0 = \sum_i \text{Im}(B_i - \lambda_i)$, $K_0 = \cap_i \text{Ker}(B_i - \lambda_i)$. If $K_0 = 0$ then $L_0 = E$ by Proposition 2.2, and b) is obvious from the fact that $\dim L_0 \leq \dim L_{h \neq 0}$. Suppose $K_0 \neq 0$. Then, since $K_0$ is an eigensubspace for all elements from $A_0$, the elements $\lambda_i$ define an algebra homomorphism $\chi_0 : A_0 \to k$ by $\chi_0(B_i) = \lambda_i$. In the same way the element $\lambda_{ih}$ and submodule $K_{ih}$ define a morphism $\rho_{ih} : A_{ih} \to k[[h]]$ for all $i = 1, ..., m$. Consider the morphism $\chi_h = \chi_0 \otimes 1 : A_0[[h]] \to k[[h]]$. Since the algebras $A(B_i)$ are semisimple and the restriction of $\chi_0$ onto $A(B_i)$ coincides with $\rho_{0i}$ for all $i$, it follows from i) and Proposition 2.2 b) that $\chi_h = \rho_{ih}$ on $A_{ih}$ for all $i$. This implies that

$$L_h = \sum_{B \in A_h} \text{Im}(B - \chi_h(B)), K_h = \bigcap_{B \in A_h} \text{Ker}(B - \chi_h(B)).$$
Taking into account that $\chi_h = \chi_0 \otimes 1$ we get that $L_h = L_0[[h]]$ and $K_h = K_0[[h]]$, which proves the proposition in the case $A_h = A_0[[h]]$.

Suppose now that $A_h$ is arbitrary. Then, by ii) and Proposition 2.2 there exists an element $f_h \in \text{End}(V)[[h]]$ such that $f_h A_h f_h^{-1} = A_0[[h]]$. Constructing the spaces $L'_h = \sum_i Im(B'_{ih} - \lambda_{ih})$, $K'_h = \cap_i Ker(B'_{ih} - \lambda_{ih})$ for $B'_{ih} = f_h B_{ih} f_h^{-1}$ we obtain that the modules $L_h = f_h^{-1} L'_h f_h$ and $K_h = f_h^{-1} K'_h f_h$ satisfy the proposition. \quad \blacksquare

Let $B_1, ..., B_m$ is a semisimple set of semisimple operators in a vector space $E$. We say that deformations of these operators, $B_{1h}, ..., B_{mh}$, form a flat deformation of the set if the conditions i) and ii) from Proposition 2.3 hold.

**Remark 2.1.** It is clear that if a semisimple operator $B$ on $E$ and its flat deformation $B_h$ are given then for any eigenvalue $\lambda$ of $B$ its deformation $\lambda_h$ is uniquely defined. Furthermore, $K_h = Ker(B_h - \lambda_h)$ and $L_h = Im(B_h - \lambda_h)$ form deformations of the subspaces $K = Ker(B - \lambda)$ and $L = Im(B - \lambda)$ in $E$. Indeed, $\lambda$ defines a character $\chi : A(B) \to k$, $\chi(B) = \lambda$, which, by Proposition 2.2, has the unique extension $\chi_h : A(B_h) \to k[[h]]$. Then, $\lambda_h = \chi_h(B_h)$. So, it follows from this that if $B_i$, $\lambda_i$, $i = 1, ..., m$, is a set of semisimple operators on $E$ with fixed eigenvalues and $B_{ih}$ is a flat deformation of the set, then the deformations of the eigenvalues, $\lambda_{ih}$, exist and are uniquely defined such that the condition (iii) of Proposition 2.3 is satisfied and, therefore, for these $\lambda_{ih}$ the proposition holds.

Let $\hat{E}$ be a tensor space over $E$, i.e. a tensor product of a number of copies of $E$ and $E^*$. The Lie algebra $\text{End}(E)$ acts on $\hat{E}$ in the usual way. For example, if $B$ is a linear operator in $\text{End}(E)$ and $v \otimes u \in E \otimes E$ then by definition $\hat{B}(v \otimes u) = Bv \otimes u + v \otimes Bu$, if $w \in E^*$ then $\hat{B}(w(v)) = -w(B(v))$ for all $v \in E$. In particular, if we identify $\text{End}(E) \cong E \otimes E^*$ and $M \in \text{End}(E)$ then $\hat{B}(M) = BM - MB$.

**Proposition 2.4** Let $B_1, ..., B_m$ be a semisimple set of semisimple linear operators on $E$. Suppose that their deformations $B_{1h}, ..., B_{mh}$ form a flat deformation of the set. Then for any tensor space $\hat{E}$ over $E$ the deformations $\hat{B}_{1h}, ..., \hat{B}_{mh}$ form a flat deformation of the set $\hat{B}_1, ..., \hat{B}_m$.

**Proof** From semisimplicity it follows that there exists an element $f \in \text{End}(E)[[h]]$ such that the algebra $\text{Ad}(f) A(B_{1h}, ..., B_{mh})$ is equal to the algebra $A(B_1, ..., B_m)[[h]]$. The group $\text{Aut}(E)$ acts on $\hat{E}$ in the usual way. Let $\hat{f}$ be the image of $f$ by the corresponding mapping.
\( \text{Aut}(E)[[h]] \to \text{Aut}(\hat{E})[[h]] \). It is clear that the algebra \( \text{Ad}(f)A(\hat{B}_1, ..., \hat{B}_{mh}) \) is equal to the algebra \( A(\hat{B}_1, ..., \hat{B}_m)[[h]] \). The algebra \( A(\hat{B}_1, ..., \hat{B}_m) \) is generated by the image of the Lie subalgebra \( L = L(B_1, ..., B_m) \) of \( \text{End}(E) \) spanned on \( B_1, ..., B_m \). The action of \( L \) on \( E \) and, therefore, on \( \hat{E} \) is semisimple. Hence, the algebra \( A(\hat{B}_1, ..., \hat{B}_m) \) is semisimple, which proves the proposition.

**Remark 2.2.** Proposition 2.4 remains true if the operators \( B_1, ..., B_m \) are invertible and the operators \( \hat{B}_{1h}, ..., \hat{B}_{mh} \) are defined as the images of \( B_{1h}, ..., B_{mh} \) by the mapping \( \text{Aut}(E)[[h]] \to \text{Aut}(\hat{E})[[h]] \). To prove this we replace in the proof above the Lie subalgebra \( L = L(B_1, ..., B_m) \) by the Lie subgroup generated by \( B_1, ..., B_m \).

Now we return to the setting of the end of Section 1.

**Proposition 2.5** Let \( V \) be a finite-dimensional vector space over \( k \), \( S \) a linear operator on \( V^\otimes 2 \), and \( S_h \) a deformation of \( S \). Suppose \( \lambda_1, ..., \lambda_n \) are the eigenvalues of \( R \) and \( I_1, ..., I_n \) are the corresponding eigensubspaces. Suppose that the subalgebras \( A_k(S) \) in \( \text{End}(V^\otimes k) \) are semisimple and the subalgebras \( A_k(S_h) \) in \( \text{End}(V^\otimes k)[[h]] \) are splitting submodules for all \( k \geq 2 \). Then deformations of the eigenvalues, \( \lambda_{ih} \), and eigensubspaces, \( I_{ih}, i = 1, ..., n \), are uniquely defined and

a) the pairs of submodules \( (I_{mh}, J_{mh}), m = 1, ..., n, \) are well situated, where \( J_{mh} = \bigoplus_{i \neq m} I_{ih} \);

b) the quadratic algebras \( (V[[h]], J_{mh}) \) form flat deformations of the quadratic algebras \( (V, J_m) \) for all \( m \);

c) the quantum semigroup \( M(V[[h]]; I_{1h}, ..., I_{nh}) \) is a flat deformation of the quantum semigroup \( M(V; I_1, ..., I_n) \).

**Proof** It is clear that \( S_{1h}, ..., S_{(k-1)h} \) form a flat set of semisimple operators in \( \text{End}(V^\otimes k) \) for all \( k \). The deformations of the eigenvalues, \( \lambda_{ih} \), and eigensubspaces, \( I_{ih}, i = 1, ..., n, \) are uniquely defined by Remark 2.1. Noting that \( I_m^{(k)} = \cap_{i=1}^{k-1} \text{Ker}(S_i - \lambda_m) \) and \( J_m^{(k)} = \sum_{i=1}^{k-1} \text{Im}(S_i - \lambda_m) \) and applying Proposition 2.3 we obtain a) and b). Condition c) follows from Proposition 2.4.

**Remarks 2.3** 1. Proposition 2.5 gives the following criteria of flatness for deformations of quadratic algebras and the corresponding quantum semigroups. Let \( V \) be a vector space over \( k \). Suppose that \( I_i, i = 1, ..., n, \) are vector subspaces in \( V^\otimes 2 \) such that \( V^\otimes 2 = \bigoplus I_i. \)
Denote $J_k = \oplus_{i \neq k} I_i$, so $V^{\otimes 2} = I_i \oplus J_i$. Suppose deformations $I_{ih}, i = 1, ..., n$, of the subspaces are given and the subalgebras $A_k(I_{1h}, ..., I_{nh})$ of $\text{End}(V^{\otimes k})[[h]]$ are splitting submodules for all $k$. Then all the deformations $(V[[h]], J_{mh})$ of the quadratic algebras $(V, J_m)$ are flat. Moreover, the deformation $M(V[[h]]; I_{1h}, ..., I_{nh})$ of the quantum semigroup $M(V; I_1, ..., I_n)$ is flat as well.

2. One can consider the case when the variable $h$ runs through a complex or real analytic manifold $X$, the subspaces $I_{ih}$ depend on $h$ analytically, and one has the decomposition $V^{\otimes 2} = \oplus^n_i I_{ih}$ at any point $h \in X$. Suppose $\dim A_k(I_{1h}, ..., I_{nh})$ does not depend on $h$ (this condition replaces the condition of splitting of the subalgebra in the formal case). Suppose $A_k(I_{1h}, ..., I_{nh})$ is semisimple at one point of $X$. Then there exists an analytic subset $Y_k \subset X$ such that for $h \in X \setminus Y_k$ all the algebras $A_k(I_{1h}, ..., I_{nh})$ are semisimple and isomorphic to each other (cf. Proposition 2.2 a). Following the arguments of this section one can prove that for $h \in X \setminus \cup Y_k$ all the quadratic algebras $(V, J_{mh})$ have the same dimension of their homogeneous components. The same is true for the corresponding quantum semigroups.

3 Applications

1. Let $V$ be a finite-dimensional vector space over $k$ ($k = \mathbb{R}$ or $\mathbb{C}$). Let $S$ be an invertible linear operator on $V \otimes V$ with two eigenvalues $\lambda$ and $\mu$ satisfying the braid relation (or quantum Yang-Baxter equation)

$$S_1S_2S_1 = S_2S_1S_2$$

on $V^{\otimes 3}$. In this case the subalgebras $A_k(S) \subset V^{\otimes k}$ are images of the Hecke algebras. The Hecke algebra $H_k(\lambda, \mu)$ is defined as the quotient algebra of the free algebra $T(x_1, ..., x_{k-1})$ of $k - 1$ variables by the relations

$$x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1}, \quad x_ix_j = x_jx_i \text{ for } |i - j| \geq 2,$$

$$x_i - \lambda)(x_i - \mu) = 0.$$

It is known, [Co], that for almost all pairs $(\lambda, \mu)$ (excepting an closed algebraic subset) this algebra is semisimple and isomorphic to the group algebra, $H_k$, of the symmetric group (the case $\lambda = 1, \mu = -1$). Moreover, in a neighborhood of each point $(\lambda_0, \mu_0)$ this isomorphism can be chosen analytically dependent on $\lambda$ and $\mu$. We suppose that the eigenvalues, $\lambda$ and $\mu$,
of $S$ correspond to the semisimple Hecke algebra. In this case $S$ is called a Hecke symmetry. Gurevich [Gu] considered the case in details. Now we consider deformations of the Hecke symmetry.

Let $S_h$ be a deformation of the operator $S$ satisfying the relations

$$S_h S_{h2} S_{h1} = S_{h2} S_{h1} S_{h2},$$

$$\left( S_h - \lambda_h \right) \left( S_h - \mu_h \right) = 0,$$

where $\lambda_h$ and $\mu_h$ are deformations of $\lambda$ and $\mu$. Let us prove that in this case the subalgebras $A_k(S_h)$ for all $k \geq 2$, are splitting. Indeed, due to relations (4) and (5) there exists an algebra homomorphism $\phi_h : H_k[[h]] \rightarrow \text{End}(V^\otimes k)[[h]]$ such that $\text{Im}(\phi_h) = A_k(S_h)$. Using Proposition 2.2 we conclude that the algebra $A_k(S_h)$ is isomorphic to $A_k(S)[[h]]$ and, therefore, splitting.

2). We obtain the same result if $S$ satisfies the Birman-Wenzl relations:

a) the braid relation (1);

b) the cubic relation $(S - \lambda)(S - \mu)(S - \nu) = 0$ for $\lambda, \mu, \nu \neq 0$;

c) $P_1 S_2 P_1 = a P_1$, where $P = (S - \lambda)(S - \mu)$ and $a$ is a constant;

d) $P_1 P_2 P_1 = b P_1$, where $b$ is a constant.

It follows from b) that $S$ has three eigenvalues and eigensubspaces.

In this case the subalgebras $A_k(S) \subset V^\otimes k$ are images of the Birman-Wenzl (BW) algebras $BW_k$ [BW]. The algebra $BW_k$ is defined as the quotient algebra of the free algebra $T(x_1, \ldots, x_{k-1})$ of $k - 1$ variables by the relations (2) and

$$(x_i - \lambda)(x_i - \mu)(x_i - \nu) = 0,$$

$$p_i x_{i \pm 1} p_i = a p_i,$$

$$p_i p_{i \pm 1} p_i = b p_i,$$

where $p_i = (x_i - \lambda)(x_i - \mu)$. One can show that the constants $a$ and $b$ are uniquely defined, and $a = \lambda \mu (\lambda + \mu)$, $b = (\lambda + \mu)^2 \nu^2$. Note that in [BW] BW algebras are defined by eleven relations, see [Ke] where it is proven that the algebra $BW_k$ can be defined as above.

It is known, [BW], that for almost all triples $\lambda$, $\mu$, and $\nu$ this algebra is semisimple and analytically depended on $\lambda$, $\mu$, $\nu$. We suppose that $\lambda$, $\mu$, and $\nu$ form such a triple. In this case $S$ satisfying the relations a)-d) is called a Birman-Wenzl symmetry. So, in the case of BW symmetry algebras $A_k(S)$ are semisimple as well.
If by a deformation of $S$ the relations a)-d) hold (with deformed eigenvalues $\lambda$, $\mu$, $\nu$) we say that this is a deformation of Birman-Wenzl symmetry. Using the same arguments as in 1) we obtain that by a deformation $S_h$ of the BW symmetry $S$ the algebras $A_k(S_h)$ are splitting.

Applying Proposition 2.3 we obtain

**Proposition 3.1** Let $S$ be a Hecke (BW) symmetry on the space $V$, $I$ its eigensubspace in $V \otimes V$, and $J$ is the sum of other eigensubspaces. Suppose $S_h$ is a deformation of the Hecke (BW) symmetry. Then the deformation defines a flat deformation $(V[[h]], J_h)$ of the quadratic algebra $(V, J)$ and the pair $(I_h, J_h)$ is well situated. Moreover, the deformation of the quantum semigroup corresponding to the eigensubspaces of $S$ is flat.

This proposition for the case $S = \sigma$ is proven in [GGST]. Note that Gurevich proved in [Gr] that in case of Hecke symmetry the algebra $(V, I)$ is Koszul. He also constructed Hecke symmetries with nonclassical dimensions of homogeneous components of $(V, I)$.

In particular, deformations of the Hecke and BW symmetries appears in [FRT] by construction of the quantum analogs (deformations) of the classical Lie groups. Namely, the Hecke symmetry corresponds to the case of general linear group, while the BW symmetry corresponds to the orthogonal and symplectic cases.

3). Let $S$ be a Yang-Baxter (YB) operator on $V \otimes V$, i.e $S$ is invertible and satisfies the braid relation (1). Let $\hat{V} = U \otimes \cdots \otimes W$ be a tensor space over $V$, the spaces $U,...,W$ are equal to $V$ or $V^*$. The group $\text{Aut}(V)$ acts on $\hat{V}$ in the usual way. Denote by $\hat{B}$ the image of $B \in \text{Aut}(V)$ by the corresponding homomorphism $\text{Aut}(V) \rightarrow \text{Aut}(\hat{V})$. The operator $S$ defines the operator $\hat{S} = \hat{S}_{(1)} \otimes \hat{S}_{(2)}$ on $\hat{V} \otimes \hat{V}$. Here we use the Sweedler notation, $S = S_{(1)} \otimes S_{(2)}$. It is easy to see that $\hat{S}$ also satisfies the braid relation.

Suppose now that $S$ is a Hecke symmetry. The operator $\hat{S}$ will not be a Hecke symmetry (it may have more than two eigenvalues), but all the algebras $A_k(\hat{S})$ are semisimple. Let $S_h$ be a deformation of $S$. This deformation defines a deformation, $\hat{S}_h$, of $\hat{S}$. By 1) $\hat{S}_h$ defines flat deformations of the algebras $A_k(S)$. So, using Remark 2.2 and Proposition 2.3, we obtain flat deformations of the quadratic algebras and the quantum semigroup corresponding to the decomposition of $\hat{V} \otimes \hat{V}$ into eigensubspaces of $\hat{S}$. Of course, the similar statement is fulfilled for Birman-Wenzl symmetry $S$. 
4). Let \( U_h(\mathcal{G}) \) be the Drinfeld-Jimbo quantized universal enveloping algebra (DJ quantum group), for a semisimple Lie algebra \( \mathcal{G} \) over \( k = \mathfrak{g}^* \). Let \( R \in U_h(\mathcal{G}) \otimes U_h(\mathcal{G}) \) be the corresponding quantum R-matrix. Suppose, a representation \( V_h \) of \( U_h(\mathcal{G}) \) is given, which is a deformation of the finite dimensional representation \( V \) of \( U(\mathcal{G}) \), so \( V_h \) is isomorphic to \( V[[h]] \) as \( k[[h]] \)-module, and the representation can be presented as a homomorphism \( \rho : U_h(\mathcal{G}) \to \text{End}(V)[[h]] \). Consider the operator \( S_h = \sigma R_h \) where \( R_h = (\rho \otimes \rho)(R) \subset \text{End}(V)^{\otimes 2}[[h]] \) and \( \sigma \) is the standard permutation. It is known that \( S_h \) is a Yang-Baxter operator, i.e. satisfying the braid relation (1). But it is not necessarily a flat deformation of semisimple operator, because at \( h = 0 \) the operator \( S_0 \) is equal to \( \sigma \), so has two eigenvalues, \( \pm 1 \), while at the general point \( h \neq 0 \) it is semisimple but may have more than two eigenvalues, \( \lambda_i, i = 1, \ldots, n \), such that \( \lambda_0 = \pm 1 \). Nevertheless, there is the decomposition \( V[[h]] = \oplus I_{ih} \) where \( I_{ih} \) are eigensubmodules of \( S_h \) corresponding to \( \lambda_{ih} \), and, therefore, all \( I_{ih} \) are splitting. We will prove that also in this setting the decomposition defines flat deformations of quadratic algebras, \( (V[[h]], J_{mh}) \), \( J_{mh} = \oplus_{i \neq m} I_{ih} \), and of the corresponding quantum semigroup, \( M(V[[h]]; I_{1h}, \ldots, I_{nh}) \). For this, according to Remark 2.3.1 we will show that the algebras \( A_k(I_1, \ldots, I_n) \) are semisimple, \( I_i = I_{i0} \), and \( A_k(I_{1h}, \ldots, I_{nh}) \) are splitting for \( k \geq 2 \).

We recall some results of Drinfeld from \([\text{Dr}1]\) and \([\text{Dr}2]\). Additional structures on the category \( \text{Rep}_A \) of representations of an associative algebra \( A \) and morphisms of these structures can be given by the additional structures on the algebra \( A \) itself. Thus, the structure of quasitensor monoidal category on \( \text{Rep}_A \) can be given with the help of an algebra homomorphism \( A \to A \otimes A \) (comultiplication), an element \( \Phi \in A^{\otimes 3} \) (associativity constraint), and R-matrix \( R \in A^{\otimes 2} \) (commutativity constraint), satisfying the certain conditions. A morphism of such two structures can be given by an element \( F \in A^{\otimes 2} \). Drinfeld defined such a structure on \( A = U(\mathcal{G})[[h]] \) for any semisimple Lie algebra \( \mathcal{G} \) with the usual comultiplication \( \Delta \) but nontrivial \( R \) and \( \Phi \). He then proved that the corresponding quasitensor category is isomorphic by some \( F_h \) to the category of representations of the Drinfeld-Jimbo quantum group \( U_h(\mathcal{G}) \) which coincides with \( U(\mathcal{G})[[h]] \) as an algebra but has a noncommutative comultiplication \( \tilde{\Delta} \). We denote the corresponding quasitensor categories by \( \mathcal{C} \) and \( \tilde{\mathcal{C}} \), respectively. We keep the notations \( \mathcal{R}_h \) and \( \Phi_h \) for the R-matrix and the associativity constraint in the category \( \mathcal{C} \), while \( R_h \) denote the R-matrix for \( \tilde{\mathcal{C}} \). Further, Drinfeld proved that \( \mathcal{R}_h \) and \( \Phi_h \) may be chosen as \( \mathcal{R}_h = e^{ht} \) where \( t \in \mathcal{G} \otimes \mathcal{G} \) is the split Casimir, and \( \Phi_h = e^{L(ht_1,ht_2)} \in U(\mathcal{G})^{\otimes 3}[[h]] \) where \( L(ht_1,ht_2) \) is a Lie expression of \( t_1 = t \otimes 1 \) and \( t_2 = 1 \otimes t \). The element \( F_h \in U(\mathcal{G})^{\otimes 2}[[h]] \).
is congruent to $1 \otimes 1$ modulo $h$ and satisfies the equation

$$(F_h \otimes 1) \cdot (\Delta \otimes id)(F_h) = (1 \otimes F_h) \cdot (id \otimes \Delta)(F_h) \cdot \Phi_h.$$ \hspace{1cm} (6)

According to this, the commutativity constraints in the categories $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are given by the elements

$$S_h = \sigma \mathcal{R}_h = \sigma e^{ht} \quad \text{and} \quad S_h = F_h S_h (F_h)^{-1} = F_h \sigma e^{ht} (F_h)^{-1},$$ \hspace{1cm} (7)

respectively.

Let us come back to the setting from the beginning of 4). Using (7) we obtain that the eigenvalues of the operators $S_h$ and $S_h$ acting on $(V \otimes V)[[h]]$ are $\lambda_i h = \pm e^{h\lambda_i}$, where $\lambda_i$ are the eigenvalues of $t$ on the same space.

Now we transfer the setting to the category $\mathcal{C}$, i.e. we consider $V[[h]]$ as an object of $\mathcal{C}$. Instead of $S_h$ we consider $S_h$, but the tensor products depend on the placement of parentheses and the connection between two bracketing, $\phi_h : (V^\otimes k)' \rightarrow (V^\otimes k)'$, of the $k$-fold tensor product is generated by the operator

$$\Phi_h : ((V \otimes V) \otimes V)[[h]] \rightarrow (V \otimes (V \otimes V))[[h]]$$ \hspace{1cm} (8)

and looks like an expression $\phi_h \in \text{End}(V^\otimes k)[[h]]$ depending on the elements $t_{i,j} = 1 \otimes \cdots \otimes t_{(1)} \otimes \cdots \otimes t_{(2)} \otimes \cdots \otimes 1$ ($t_{(1)}$ and $t_{(2)}$ at the places $i$ and $j$, $t = t_{(1)} \otimes t_{(2)}$ in the Sweedler notations). It is easy to see that the eigensubmodules of the operator $S_h$ have the form $\mathcal{I}_i[[h]]$ where $\mathcal{I}_i$ are the common eigensubmodules of $t$ and $\sigma$. Denote by $A_k(\sigma,t)$ the subalgebra in $\text{End}(V^\otimes k)$ generated by the elements $\sigma_{i,i+1}$ and $t_{i,i+1}$. So we get that the algebra $A_2(\mathcal{I}_1[[h]],\ldots,\mathcal{I}_n[[h]])$ is equal to $A_2(\sigma,t)[[h]]$. The algebras $A_k'(\mathcal{I}_1[[h]],\ldots,\mathcal{I}_n[[h]])$ and $A_k''(\mathcal{I}_1[[h]],\ldots,\mathcal{I}_n[[h]])$ for two bracketings $(V^\otimes k)'$ and $(V^\otimes k)''$ are connected by

$$A_k'(\mathcal{I}_1[[h]],\ldots,\mathcal{I}_n[[h]]) = \phi_h^{-1} A_k''(\mathcal{I}_1[[h]],\ldots,\mathcal{I}_n[[h]]) \phi_h.$$

Using the relations of the type $t_{1,3} = \sigma_{2,3} t_{1,2} \sigma_{2,3}$ we conclude that $\phi_h \in A_k(\sigma,t)[[h]]$, and using the fact that $\phi_h$ is congruent to $1 \otimes 1 \otimes 1$ modulo $h^2$, we conclude by induction on $k$ that $A_k'(\mathcal{I}_1[[h]],\ldots,\mathcal{I}_n[[h]]) = A_k''(\mathcal{I}_1[[h]],\ldots,\mathcal{I}_n[[h]]) = A_k(\sigma,t)[[h]]$.

Passing to the category $\tilde{\mathcal{C}}$ we obtain that $A_k(I_{1h},\ldots,I_{nh}) = f_h' A_k'(\mathcal{I}_1[[h]],\ldots,\mathcal{I}_n[[h]])(f_h')^{-1} = f_h' A_k(\sigma,t)[[h]](f_h')^{-1}$ where $f_h'$ is the composition (depending on the bracketing $'$) of a number of $F_h$ with $\Delta$ applied appropriate factors. We will obtain the same result applying to
$A_k(\sigma, t)$ the element $f_h''$ related to the bracketing $''$. So, we have proved that $A_k(I_{1h}, ..., I_{nh})$ is splitting.

For $h = 0$ this algebra is equal to $A_k(\sigma, t)$. Let us prove that it is a semisimple algebra. Indeed, $t$ may be presented as $t = \sum_i d_i \otimes d_i$ where $d_i$ form an orthogonal (with respect to the Killing form) basis in the maximal compact subalgebra $K$ of $G$. Hence, there exists a Hermitian metric on $V$ invariant under action of $K$. This metric induces naturally the metric on $V^\otimes k$ which will be invariant under the operators $t_{i,j}$ and $\sigma$. So these operators are unitary ones, therefore the algebra $A_k(\sigma, t)$ generated by them is semisimple.

Applying proposition 2.3 we obtain

**Proposition 3.2** Let $S_h$ be the Yang-Baxter operator on a space $V$, which is obtained by the representation of the Drinfeld-Jimbo quantum group. Then $S_h$ defines the decomposition $V^\otimes 2 = \oplus_{i=1}^n I_{1h}$ into eigensubmodules. Let $J_{mh} = \oplus_{i \neq m} I_{ih}$. Then $(V[[h]], J_{mh})$ are flat deformations of the quadratic algebras and the pairs $(I_{mh}, J_{mh})$ are well situated. Moreover, the deformation of the quantum semigroup $M(V[[h]]; I_{1h}, ..., I_{nh})$ is flat.

Another proof of this proposition is contained in [DS]. **Acknowledgments.** The authors are grateful to J. Bernstein and D. Gurevich for their interest to the paper and very helpful discussions.

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