FLOCKS OF CONES: STAR FLOCKS

WILLIAM CHEROWITZO

Abstract. The concept of a flock of a quadratic cone is generalized to arbitrary cones. Flocks whose planes contain a common point are called star flocks. Star flocks can be described in terms of their coordinate functions. If the cone is “big enough”, the star flocks it admits can be classified by means of a connection with minimal blocking sets of Rédei type. This connection can also be used to obtain examples of bilinear flocks of non-quadratic cones.

1. Introduction

This is the second (see [8]) in a series of articles devoted to providing a foundation for a theory of flocks of arbitrary cones in $PG(3, q)$. The desire to have such a theory stems from a need to better understand the very significant and applicable special case of flocks of quadratic cones in $PG(3, q)$. Flocks of quadratic cones have connections with several other geometrical objects, including certain types of generalized quadrangles, spreads, translation planes, hyperovals (in even characteristic), ovoids, inversive planes and quasi-fibrations of hyperbolic quadrics. This rich collection of interconnections is the basis for the strong interest in such flocks.

2. Cones and Flocks

Let $\pi_0$ be a plane and $V$ a point not on $\pi_0$ in $PG(3, q)$. Let $S$ be any set of points in $\pi_0$ (including the empty set). A cone, $\Sigma = \Sigma(V, S)$ is the union of all points of $PG(3, q)$ on the lines $VP$ where $P$ is a point of $S$. $V$ is called the vertex and $S$ is called the carrier of $\Sigma$. $\pi_0$ is the carrier plane and the lines $VP$ are the generators of $\Sigma$. In the event that $S = \emptyset$ we call $\Sigma$ the empty cone and by convention consider it to consist of only the point $V$.

A flock of planes in $PG(3, q)$ is any set of $q$ distinct planes of $PG(3, q)$. As $q$ planes can not cover all the points of $PG(3, q)$, there always are

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points of the space which do not lie in any of the planes in a flock of planes. If $\Sigma$ is a cone of $PG(3, q)$, then a flock of planes, $F$, is said to be a flock of $\Sigma$ when the vertex of $\Sigma$ lies in no plane of $F$ and no two planes of $F$ intersect at a point of $\Sigma$. Any flock of planes is a flock of a cone, possibly only the empty cone. In general, however, a given flock of planes will be a flock of several cones. In the literature on flocks of quadratic cones, the approach is always to consider a fixed quadratic cone and study the flocks of that cone. We will change the viewpoint and consider, for a fixed flock of planes, the various cones of which it is a flock. In the sequel we shall refer to a flock of planes simply as a flock and it shall be understood that it is always a flock of a cone, even if the cone is not explicitly indicated. Furthermore, we shall always assume, unless explicitly stated otherwise, that a flock is a flock of a non-empty cone.

Let $F$ be a flock. We can introduce coordinates in $PG(3, q)$ so that the plane $x_3 = 0$ is one of the planes of the flock and the point $V = (0, 0, 0, 1)$ is not in any plane of the flock. Since $V$ is not in any plane of $F$, each of the planes of this flock has an equation of the form $Ax_0 + Bx_1 + Cx_2 - x_3 = 0$. We parameterize the planes of $F$ with the elements of $GF(q)$ in an arbitrary way except that we will require that $0$ is the parameter assigned to the plane $x_3 = 0$. We can now describe the flock as $F = \{ \pi_t: f(t)x_0 + g(t)x_1 + h(t)x_2 - x_3 = 0 \mid t \in GF(q) \}$ with $\pi_0: x_3 = 0$. The functions $f, g$ and $h$ are called the coordinate functions of the flock. Note that the requirement on the parameter $0$ means that $f(0) = g(0) = h(0) = 0$. If $f, g$ and $h$ are the coordinate functions of the flock $F$ we shall write $F = F(f, g, h)$. We remark that the coordinate functions of a flock depend on the parameterization of the flock.

As all cones under consideration have vertex $V = (0, 0, 0, 1)$ and the plane $\pi_0$ as the carrier plane, a cone is determined when its carrier $S$, a point set in $\pi_0$, is specified. Given a flock $F$, there is a largest set $S_0$ of $\pi_0$ such that $F$ is a flock of the cone with carrier $S_0$. This cone is called the critical cone of $F$. If $C$ is any subset of the carrier of the critical cone of a flock $F$, then clearly $F$ is also a flock of the cone with carrier $C$. Thus, determining the critical cone of a flock implicitly determines all cones for which this flock of planes is a flock.

**Theorem 1.** A point $(a,b,c,0)$ is in the carrier of the critical cone of the flock $F = F(f, g, h)$ in $PG(3, q)$ if and only if the function $t \mapsto af(t) + bg(t) + ch(t)$ is a permutation of $GF(q)$.

**Proof.** A point on the line $\langle (0, 0, 0, 1), (a, b, c, 0) \rangle$ other than the vertex has coordinates $(a, b, c, \lambda)$ with $\lambda \in GF(q)$. Such a point is on the
plane \( \pi_t \) of \( \mathcal{F} \) if and only if \( \lambda = af(t) + bg(t) + ch(t) \). No two planes of \( \mathcal{F} \) will meet at the same point of this line if and only if \( t \mapsto af(t) + bg(t) + ch(t) \) is a permutation of \( GF(q) \). Thus, under this condition, the point \((a, b, c, 0)\) will be in the carrier of the critical cone of \( \mathcal{F} \). □

The critical cone of a flock may be fairly “small”. Besides the empty cone, we will consider cones whose carriers consist of collinear points as being “small”. Cones of this type are called flat cones. For the most part, we shall regard flocks whose critical cones are flat as being uninteresting. For any nonempty set \( S \) and any point \( P \) of a projective plane, define \( w_S(P) \) to be the number of lines through \( P \) which contain an element of \( S \). In the projective plane \( \Pi \) we can define the width of a set \( S \) to be \( W_S = \min \{ w_S(P) \mid P \in \Pi \} \). Clearly, \( W_S = 1 \) if and only if \( S \) consists of a set of collinear points. If \( S \) is an oval in a projective plane of order \( q \) (\( q + 1 \) points, no three of which are collinear), then \( W_S = \frac{q+1}{2} \) if \( q \) is odd or \( W_S = \frac{q+2}{2} \) if \( q \) is even. This can be written as \( W_S = \lfloor \frac{q+2}{2} \rfloor \), independent of the parity of \( q \). Using this idea we can provide an admittedly crude classification of critical cones. In \( PG(3,q) \), if \( S \) is the set of points of a carrier of a cone, then if \( W_S < \lfloor \frac{q+2}{2} \rfloor \), we call the cone a thin cone. Cones which are not thin are called wide and a wide cone with at least \( q+1 \) points in a carrier are called thick cones. The class of thick cones contains the quadratic cones as well as all cones whose carrier contains any oval.

3. Definitions and Basic Properties

A linear flock is a flock whose planes share a common line. Any cone whose carrier is not a proper blocking set (a set of points which contains no line and which every line intersects) in its carrier plane admits linear flocks. Linear flocks can easily be characterized by their coordinate functions.

**Theorem 2.** A flock is a linear flock if and only if the three coordinate functions are scalar multiples of each other. □

A star flock is a flock whose planes share a common point and a proper star flock is one for which this common point is unique. The common point of a proper star flock is called the star point. Linear flocks are clearly star flocks, but not proper star flocks.

We may also characterize star flocks in terms of their coordinate functions.

**Theorem 3.** A flock is a star flock if and only if the coordinate functions are linearly dependent over \( GF(q) \). □
Since a flock is a set of planes of $PG(3, q)$ it is natural to define the equivalence of flocks without reference to the cones that they are flocks of. While there are several ways to do this, the following has proved to be most useful. Two flocks, $F_1$ and $F_2$ are equivalent if they are in the same orbit of the group $PGL(4, q)_{V, x_3=0}$. That is, there is a collineation of $PG(3, q)$ fixing the point $V = (0, 0, 0, 1)$ and stabilizing the plane $x_3 = 0$ which maps the planes of $F_1$ to those of $F_2$.

**Theorem 4.** A star flock is equivalent to a flock $F(t, g(t), 0)$, where 0 denotes the function which is identically zero.

**Proof.** Let $P$ be a point in the carrier of the critical cone and $Q$ the star point of a star flock. By using a collineation of $PG(3, q)$ which fixes $(0, 0, 0, 1)$ and stabilizes $x_3 = 0$, we may assume that $P = (1, 0, 0, 0)$ and $Q = (0, 0, 1, 0)$. If the star flock is given by $F = F(f, g, h)$, then by Theorem 1, $t \mapsto f(t)$ is a permutation. We can re-parameterize the planes of the flock so that $f(t) = t$. Since all the planes of the flock pass through $Q$, we must have $h \equiv 0$. □

4. Star Flocks of Wide Cones

Viewing flocks in a dual setting has been an effective technique in the classical quadratic cone situation. It is especially useful in studying star flocks.

Let $F$ be a flock with critical cone $\Sigma = \Sigma(V, S)$ in $PG(3, q)$. By passing to the dual, $F$ becomes a set of $q$ points in $PG(3, q)$ and the cone is the set of all planes passing through a set of lines in the plane corresponding to $V$, with the property that no line determined by a pair of the $q$ points lies in any of these planes.

If $F$ is a proper star flock, then the corresponding $q$ points in the dual are coplanar. We fix some notation for this situation. Let $F = F(t, g(t), 0)$ be a star flock of the cone $\Sigma = \Sigma(V, S)$ with $V = (0, 0, 0, 1)$ and the planes of $F$ given by $tx_0 + g(t)x_1 - x_3 = 0$. For the sake of clarity, we employ the notational device of using $(X, Y, Z, W)$ to refer to the generic coordinates of a point when viewed in the dual setting. Using the duality $(x, y, z, w) \leftrightarrow [x, y, z, -w]$, the flock becomes a set of points $D_F = \{(t, g(t), 0, 1): t \in GF(q)\}$. $V$ becomes the plane with equation $W = 0$. The points on a generator of $\Sigma(V, S)$ become the set of all planes passing through a line in $W = 0$. The set of lines in $W = 0$, corresponding to the generators of $\Sigma$ will be denoted by $D_G$, and the set of all planes passing through the lines of $D_G$ will be denoted by $D_\Sigma$. The condition that lines formed by pairs of points of $D_F$ do not lie in the planes of $D_\Sigma$ is equivalent to the condition that these lines do not intersect the lines of $D_G$. The $q$ points of $D_F$ lie in
the plane $Z = 0$, and since $F$ is proper, they are not all collinear. Let $m$ be the line of intersection of the planes $W = 0$ and $Z = 0$.

The $q$ points of $D_F$ lie in the affine plane obtained by removing the line $m$ from $Z = 0$. Since $F$ is a flock, the line joining any two of these points can not intersect $W = 0$ at a point on any line of $D_G$. Thus, the set of points on $m$ at which these lines meet is disjoint from the set of points on $m$ at which the lines of $D_G$ meet $m$. Let $N = |M|$. $N$ can be thought of as the number of “slopes” determined by the set of $q$ points of $D_F$.

A blocking set in a projective plane is a set of points in the plane which every line intersects. Let $n$ be the largest number of collinear points in a blocking set. In a plane of order $q$, a blocking set of size $q + n$ is called a Rédei blocking set.

**Proposition 5.** In $Z = 0$ the points of $D_F \cup M$ form a Rédei blocking set.

**Proof.** Consider a point of $m$ which does not lie in $M$. The $q$ lines through this point other than $m$ must each contain exactly one point of $D_F$. Therefore, $D_F \cup M$ is a blocking set. Since the points of $D_F$ are not collinear, no collinear set of points of $D_F$ can contain more than $N - 1$ points. Thus $D_F \cup M$ is of Rédei type of size $q + N$. □

The problem of determining the number of slopes determined by $q$ points in an affine plane has been well studied ([24],[21],[4],[3],[1]) due to the connection with blocking sets of Rédei type. This problem is completely settled in the cases we are interested in and the relevant theorem (paraphrased in our terminology) is due to Ball [1] who settled two open cases of the main theorem first given by Blokhuis, Ball, Brouwer, Storme and Szönyi [3].

**Theorem 6 ([1]).** Let $D_F$ consist of $q$ points of $PG(2, q)$ whose coordinates are given by $(t, g(t), 1)$, $t \in GF(q)$, where $g$ is a permutation polynomial with $g(0) = 0$. If $q = p^n$ for some prime $p$, let $e$ be the largest integer so that any line of the plane containing at least two points of $D_F$ contains a multiple of $p^e$ points of $D_F$. If $N$ is the number of “slopes” determined by the points of $D_F$ then we have one of the following:

(i) $e = 0$ and $q^3 + q - q^2 \leq N \leq q + 1$,
(ii) $e \mid n$ and $p^{n - e} + 1 \leq N \leq \frac{q - 1}{p^e - 1}$,
(iii) $e = n$ and $N = 1$.

Moreover, if $p^e > 2$, then $g$ is a $p^e$-linearized polynomial. □
A $p^e$-linearized polynomial is one of the form:
\[ g(t) = \sum_{i=0}^{k-1} \alpha_i t^{ie}, \] where $\alpha_i \in GF(p^n)$, $n = ke$.

In this theorem, all the bounds for $N$ are sharp. The $p^e$-linearized permutation polynomials form a group under composition mod $(x^q - x)$. This is the Betti-Mathieu group (see [20]) and it is known to be isomorphic to $GL(k, p^e)$ where $q = p^{ek}$. Thus,

**Proposition 7.** Let $q = p^n$ and $s = p^e$ with $n = ek$. Then the number of monic $s$-linearized permutation polynomials over $GF(q)$ is:

\begin{equation}
(1) \quad s^{\frac{k(k-1)}{2}} \prod_{i=1}^{k-1} (s^i - 1).
\end{equation}

Theorem 6 has the following implication for star flocks of wide cones:

**Theorem 8.** If $q = p^n$ with $p$ a prime, then $\mathcal{F}$ is a proper star flock with a wide critical cone if and only if there exists a coordinatization such that $\mathcal{F} = F(t, g(t), 0)$ where $g$ is a non-linear $p^e$-linearized permutation polynomial for some $e | n$ with $e < n$. Furthermore, the critical cone of this star flock contains the points $(x, f(x), 1)$ for $x \neq 0$ if and only if

\begin{equation}
(2) \quad f(x)g(t) + xt = 0 \text{ has no solutions with } xt \neq 0.
\end{equation}

**Proof.** By Theorem 4 we may assume that $\mathcal{F} = F(t, g(t), 0)$ is a proper star flock of a wide cone with star point $P = (0, 0, 1, 0)$ and $g$ a non-linear permutation polynomial (non-linearity follows from Theorem 2).

Since the cone is wide, there are at least $\left\lfloor \frac{q+2}{2} \right\rfloor$ lines in $x_3 = 0$ through $P$ which contain the points of the critical cone of this flock. Thus, there are at least $\left\lfloor \frac{q+2}{2} \right\rfloor$ planes determined by these lines and the vertex of the cone. We now consider the dual setting and use the notation introduced at the beginning of this section. These planes correspond to points in $W = 0$ which are also in $Z = 0$, i.e., points on $m$. Since these planes contain points of the carrier, the points on $m$ that they correspond with are points where lines of $D_G$ meet $m$. Thus, for a wide cone, we must have $N \leq q+1 - \left\lfloor \frac{q+2}{2} \right\rfloor = \left\lfloor \frac{q+1}{2} \right\rfloor$. Since the star flock is proper, we must also have $N > 1$. We can identify $Z = 0$ with the projective plane $\Pi$ obtained by suppressing the third coordinates. Thus, the points of $D_F$ have coordinates $(t, g(t), 1)$ in $\Pi$. By Theorem 6 if $p^e = 1$ or 2, the number of slopes determined by $D_F$ does not satisfy $1 < N \leq \left\lfloor \frac{q+1}{2} \right\rfloor$. Thus, $p^e > 2$, and we have that $g$ is a $p^e$-linearized polynomial.
Now, suppose that \( g \) is a non-linear \( p^e \)-linearized polynomial for some \( e \mid n \) with \( e < n \). Such functions are additive (i.e., \( g(t+s) = g(t)+g(s) \)), so to calculate \( N \) we need only calculate the number of distinct values of \( g(t)/t \) for \( t \neq 0 \). Let \( K \) be the proper subfield of \( GF(q) \) of order \( p^e \). For each \( c \in K \) we have \( g(ct) = cg(t) \). Thus, for \( c \in K \setminus \{0\} \), \( g(ct)/ct = g(t)/t \), and the number of distinct non-zero values of \( g(t)/t \) is at most \( (q-1)/(p^e-1) \). Therefore, \( N \leq (q-1)/(p^e-1) + 1 < \left\lfloor \frac{q+1}{2} \right\rfloor \) if \( p^e > 2 \). We obtain a star flock of a wide cone from such a \( g \), and it is proper since \( g \) is not linear.

Finally, For each \( x \neq 0 \), \( f(x)g(t) + xt \) is a \( p^e \)-linearized polynomial (in \( t \)). Such polynomials represent permutations if and only if they vanish only at \( t = 0 \) (Dickson [13]). Thus, (2) insures that each of the points \( (x, f(x), 1, 0) \) is in the critical cone of \( \mathcal{F} \) by Theorem [11].

**Example 1.** Let \( q = p^h \), \( p \) an odd prime, and \( h > 1 \). Let \( \sigma = p^i, 1 \leq i \leq h - 1 \). Then \( t^\sigma \) is a \( p^{\gcd(i,h)} \)-linearized polynomial. In \( x_3 = 0 \) the points \( (x, -m/x, 1, 0) \), \((1,0,0,0)\) and \((0,1,0,0)\) form a conic \( (xy = -m) \). \(-\frac{m}{x}t^\sigma + xt = xt(-\frac{m^{p^i-1}}{x^2} + 1)\) will have no solutions with \( xt \neq 0 \) if and only if \( m \) is a non-square in \( GF(q) \). Thus, \( \mathcal{F}(t^\sigma, 0) \) will be a (proper) star flock of the quadratic cone \( xy = -m \) when \( m \) is a non-square. These star flocks are known as the Kantor-Knuth (or K1) flocks of a quadratic cone. These examples come from a special case \((k = 0)\) of the fact that the points \((x, f(x), 1, 0), x \neq 0 \) where \( f(x) = -\frac{m}{x^{p^e+1}}, 0 \leq k \leq \frac{q-1}{2} \) are in the critical cone of this star flock when \( m \) is a non-square.

**Corollary 9.** If \( q \) is a prime, then all star flocks of wide cones in \( PG(3, q) \) are linear.

**Proof.** Since a prime field has no proper subfields, there are no \( p^e \)-linearized polynomials other than the linear ones, so by Theorem [8] there are no proper star flocks. □

**Corollary 10.** If \( q = 2^p \), with \( p \) a prime, then all star flocks of wide cones in \( PG(3, q) \) are linear.

**Proof.** If a star flock were not linear in this situation, then we would have \( p^e = 2 \) which is ruled out since it gives too large an \( N \) for a wide cone. □

We examine two classes of examples of proper star flocks of wide cones which correspond to extreme values of \( N \). Let \( E = GF(q) = GF(p^\alpha) \) with a subfield \( K = GF(p^e) \), then the permutation function \( g(t) = k(t^{p^e} - ct) \), where \( c \neq \beta^{p^e-1} \) with \( k, \beta \in GF(q)^*, \) gives \( N = (q-1)/(p^e-1) \) and for \( p^e > 2 \) we have \( N < \left\lfloor \frac{q+1}{2} \right\rfloor \), and so,
\( \mathcal{F}(t, g(t), -(at + bg(t))) \) for \( a, b \in GF(q) \) is a star flock of a wide cone. These examples include those of Example 1 in odd characteristic, so we shall call them (or any flocks equivalent to them, see [8]) Kantor-Knuth star flocks. Under the same assumptions about \( q \) and \( e \), the function \( g(t) = k_1(Tr_{E/K}(\frac{t}{k_2})) - ct \), where \( c \neq \frac{1}{\beta}Tr_{E/K}(\frac{t}{k_2}) \), \( k_1, k_2, \beta \in GF(q)^* \) and \( Tr_{E/K} \) is the relative trace function from \( E \) onto \( K \), gives \( N = p^{n-e} + 1 \), and again, if \( p^e > 2, N < [\frac{q+1}{2}] \). We call these proper star flocks of wide cones, Holder-Megyesi star flocks (L. Megyesi gave the example in the Rédei blocking set context and L. Holder, independently, gave the example in the conic blocking set context (see [17]).

**Corollary 11.** If \( q = p^2 \) for any prime \( p \), or \( p = 4 \), then all star flocks of wide cones in \( PG(3, q) \) are either linear or Kantor-Knuth star flocks.

**Proof.** All flocks of wide cones in \( PG(3, 4) \) are linear (see [8]) so we may assume that \( p > 2 \). Up to scalar multiples the only \( p \)-linearized permutation polynomials are of the form \( t^p - ct \), where \( c \neq \beta^{p-1} \) for any \( \beta \in GF(q)^* \) or \( t \). In the case of \( PG(3, 16) \), the 2-linearized polynomials do not give star flocks of wide cones, so the only \( s \)-linearized polynomials giving star flocks of wide cones have \( s = 4 \). \( \square \)

**Example 2.** In \( PG(2, 16) \) there are two projectively inequivalent hyperovals, the hyperconic (a.k.a. regular hyperoval) and the Lunelli-Sce hyperoval. The cone over a hyperconic (quadratic cone) admits only linear star flocks (Thas [26]), but a cone over the Lunelli-Sce hyperoval admits both linear and Kantor-Knuth star flocks. Let \( \lambda \) be a primitive element of \( GF(16) \) satisfying \( \lambda^4 = \lambda + 1 \). The point sets of \( x_3 = 0 \) given by \( \mathcal{H}_i = \{(x, f_i(x), 1, 0) : x \in GF(16)\} \cup \{(0, 1, 0, 0), (1, 0, 0, 0)\} \) with \( i = 1, 2 \), where

\[
\begin{align*}
f_1(x) &= \lambda^{13}x^{14} + \lambda^3x^{12} + \lambda^6x^{10} + x^8 + \lambda^6x^6 + \lambda^3x^4 + \lambda^{13}x^2 \\
f_2(x) &= \lambda^4x^{14} + \lambda^{10}x^{12} + \lambda^{11}x^{10} + \lambda^{11}x^8 + x^6 + \lambda^2x^4,
\end{align*}
\]

represent Lunelli-Sce hyperovals which intersect in nine points, the maximum number of points in the intersection of two hyperovals in \( PG(2, 16) \). Among the points of intersection are the points \((0, 0, 1, 0), (1, 1, 1, 0), (\lambda^{10}, \lambda^{13}, 1, 0), (\lambda^8, \lambda^2, 1, 0), (\lambda^{14}, \lambda^5, 1, 0) \) and \((\lambda^{13}, \lambda^{10}, 1, 0) \). The slopes of the lines through \((0, 0, 0, 1)\) and each of the other five points form a set of values \( \{\lambda^3i : 0 \leq i \leq 4\} \), i.e., the non-zero cubes in \( GF(16) \). The symmetric difference of these two hyperovals, \( \mathcal{H}_1 \Delta \mathcal{H}_2 \) is another Lunelli-Sce hyperoval (see [5]) with these five lines as exterior lines through the point \((0, 0, 1, 0) \). The proper Kantor-Knuth star flock,
Theorem 13. In $\mathbb{P}G(3,q)$ there are no proper bilinear star flocks of wide cones. \hfill \Box

This result is sharp as the following examples show.
Example 3. Let $q$ be odd. A well known Rédei blocking set in $PG(2, q)$ is the projective triangle of side $(q + 3)/2$ (see [19]). This consists of $3(q + 1)/2$ points with $(q + 3)/2$ points on each side of a triangle (including the vertices) such that the line joining any two points of the set on different sides of the triangle meets the third side at a point of the set. A representation of a projective triangle which includes the point $(0, 0, 1)$ has affine points of the form $(x, x^{q+2}/2, 1)$ and infinite points $(1, 1, -z, 0) \cup (1, -1, 0) \cup (1, 1, 0)$ for each non-square $z \in GF(q)$. Note that the affine points of the set lie on the two lines $y = x$ (for square $x$) and $y = -x$ (for non-square $x$). This representation gives rise to the proper star flock $F(t, t^{q+1}/2, 0)$ which can only be a flock of a thin cone. In particular, since the star point is $(0, 0, 1, 0)$ the critical cone consists only of points other than $(0, 0, 1, 0)$ on the lines $y = cx$ where $1/c$ is a non-zero square. There are $q−1/2$ such values of $c$ if $−1$ is not a square in $GF(q)$ and $q−3/2$ otherwise. The point $(0, 1, 0, 0)$ is in the critical cone, and hence the points of the line $x = 0$ other than $(0, 0, 1, 0)$ if and only if $−1$ is a square in $GF(q)$, that is, $q \equiv 1 \pmod{4}$. Therefore, for any odd $q$, the width of the base of this cone is $q−1/2 < [q−3/2]$ and so the cone is thin, but any larger cone would be wide.

A special case of this example occurs in $PG(2, q^2)$ for $q$ an odd prime power. The map $x \mapsto x^q$ is an involutory automorphism of $GF(q^2)$. The flock $F(t, At^{q+1}/2, 0)$, where $A^q = -A$ has a critical cone which contains the points of $x^qy = z^{q+1}$ in the carrier plane $w = 0$. In the dual setting, the points of $Z = 0$ on the lines of $D_G$ are (with $Z$ coordinate suppressed) $(0, 1, 0)$ and $\{(1, \alpha, 0) \mid \alpha \in GF(q)\}$. The points of $W = 0$ on the lines determined by pairs of points of $D_F$ are (again with $Z$ coordinate suppressed): $(1, A, 0), (1, -A, 0)$ and $\{(1, A(1+z), 0) \mid z$ is a nonsquare in $GF(q^2)\}$. All of these points lie on the line $m = \{W = 0\} \cap \{Z = 0\}$ and we wish to determine $A$ so that the sets are disjoint. Clearly, $A$ can not lie in $GF(q)$. The condition that $A(1+z)/z$ is a critical cone is (with $\lambda$ a primitive element of $GF(q^2)$), $(A^{q−1}−1)(\lambda^{q+1}/(2k+1)−1) = (A^{q−1}+1)(\lambda^{q−1}/(2k+1)−1)$ for all integer $k$. Assuming $A^q = -A$, this reduces to $\lambda^{(q−1)/(2k+1)} = 1$, which implies that $q − 1 \mid 2k + 1$, a contradiction since $q$ is odd.

Example 4. The analogous example for even $q$ is the projective triad of side $(q + 2)/2$ (see [19]). This set consists of $(3q + 2)/2$ points with $(q + 2)/2$ points on each of three concurrent lines (point of intersection included) such that the line joining any two points of the set on different lines of the triad meets the third line at a point of the set. A representation of a projective triad which includes the point $(0, 0, 1)$
has affine points \((x, \text{tr}(x), 1)\), where “\(\text{tr}\)” is the absolute trace function from \(GF(q^2)\) to \(GF(2)\), and infinite points \((1, \frac{1}{a}, 0) \cup (1, 0, 0)\) where \(\text{tr}(a) = 1\). The critical cone of the proper start flock which arises from this example consists only of points other than \((0, 0, 1, 0)\) on the lines \(y = cx\) where \(\text{tr}(c) = 0\). Note that \((0, 1, 0, 0)\) is never in the critical cone since the absolute trace function is not a permutation. The width of the base of the critical cone is therefore \(\frac{q^2}{2}\) and the critical cone is thin, but again any larger cone would be wide.

In the special case of \(PG(2, 2^{2e})\), the flock \(F(t, \text{tr}(t), 0)\) has a critical cone which contains the points of \(x^3y = z^{q+1}\) for \(q = 2^e\) except for \((0, 1, 0, 0)\) in the carrier plane \(u = 0\). This follows easily since an affine point of the line \(y = cx\) other than \((0, 0, 1, 0)\) which lies on this curve must satisfy \(c = (\frac{z}{x})^{q+1} \in GF(q)\). This implies that in \(GF(q^2)\) we have \(\text{tr}(c) = 0\). Note that if the relative trace function (from \(GF(q^2)\) to \(GF(q)\)) instead of the absolute trace function is used in the definition of the flock, we would have obtained a \(q\)-linear (instead of bilinear) flock which is of the Megyesi-Holder type.

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Department of Mathematical and Statistical Sciences, University of Colorado Denver, Campus Box 170, P.O. Box 173364, Denver, CO 80217-3364, U.S.A.
E-mail address: william.cherowitzo@ucdenver.edu