Using the New Mathematical Method to Find the Third Fundamental Form of Surface

Abdel Radi Abdel Rahman Abdel Gadir Abdel Rahman¹, Tasabeh Awadallah Abdel Majid Ali², Sulima Ahmed Mohammed Zubir³, Hassan Abdelrhaman Mohammed Elnaeem⁴

¹² Department of Mathematics, Faculty of Education, Omdurman Islamic University, Omdurman, Sudan
³ Department of Mathematics, College of Science and Arts, Qassim University, Ar Rass, Saudi Arabia
⁴ Department of Information Security, College of Computer Science and Information Technology, Karary University Khartoum, Sudan

ARTICLE INFO

Published Online: 19 April 2022
Corresponding Author: Abdel Radi Abdel Rahman Abdel Gadir Abdel Rahman

ABSTRACT

The Third Fundamental Form of Surface which consider simple form is a differential geometry through which it is possible to know the measurement of surface symbolize it III. The third fundamental form is the intrinsic Property of surface. The program MATLAB is on of the most famous Mathematical program helps us to verify the validity of any calculation performed manually. The main MATLAB stands for matrix laboratory. MATLAB has advantages compared to conventional computer language for solving technical problems. This study aims to find the Third Fundamental Form of Surface using the new mathematical method. We followed the applied mathematical method using the new mathematical method and we found the following some results: To find the Third Fundamental Form of Surface we must be calculated the First and Second Natural Curvature. Solving the Third Fundamental Form of Surface using the new mathematical method is more accurate and fast but we cant find the solution graph.

KEYWORDS: New mathematical method, Surface, Third Fundamental Form, MATLAB.

1. INTRODUCTION

In daily life, we see many surfaces such as balloons, tubes, tea cups and thin sheets such as soap bubble, which represent physical models. To study these surfaces, we need coordinates for the work of the necessary calculations. These surfaces exist in the triple space, but we cannot think of them as three dimensional. For example, if we cut a cylinder longitudinal section, it can be individual or spread to become a flat on a desktop. This shows that these surfaces are two dimensional inheritance and this is described by the coordinates. This gives us the first impression of how the geometric description of the surface. The regular surface can be obtained by distorting the pieces of flat paper and arranging them in such away that the resulting shape is free of sharp dots or pointed letters or intersections (the surface cuts it self) and thus can be talked about the tangent level at the shape points. The surface is said to be in the triple vacuum R³ is subgroup from R³ (Le, a special pool of points) of course, not all particle groups are surfaces and certainly mean smooth and two dimensional surfaces[12,pp216]

MATLAB is a software package for computation in engineering, science, and applied mathematics. It offers a powerful programming language excellent graphics and a wide range of expert knowledge. MATLAB is published by a trademark of the math works. The focus in MATLAB is on computation, not mathematics: Symbolic expressions and manipulations are not possible (except through the optional symbolic tool box, a clever interface to Maple). All results are not only numerical but inexact. The limitation to numerical computation can be seen as a drawback, but it is a source of strength too: MATLAB is much preferred to Maple, Mathematic, and the like when it comes to numeric. On the other hand compared to other numerically oriented languages like C++ and Fortran, MATLAB is much easier to use and comes with a huge standard library. The unfavorable comparison here is a gap in execution speed. This gap is not always as dramatic as
2. THE CONCEPT OF SURFACE

Definition (2.1):
Let the components of the position vector \( r \) be single valued and continuously differentiable function of two variable parameters \( u \) and \( v \), where
\[
(2.1)
\]
if in some region the variable \( u \) and \( v \) have \( r_{1r2}\ne0 \) where \( ri=\partial ri/\partial ui \) (i=1,2)
The end point of \( r \) defines a point set which is called a continuously differentiable surface.

Note: In what follows the notation \( ri(i=1,2) \), or \( ru=\partial ru/\partial u1 \), \( rv=\partial rv/\partial v1 \), where \( r=r(u,v) \), will be used

The parametric for the surface (2.1) is
\[
(2.2)
\]
Eliminating the parameters \( u \) and \( v \) form (2.2) one obtains the implicit form
\[
F(x1,x2,x3) = 0 \quad (2.3)
\]
Which is satisfied for all points \( \{x1,x2,x3\} \) of the surfaces (2.2).

Examples (2.2):
Consider the local coordinate chart:
\[
(2.4)
\]
Clearly, the surface represented by this chart is part of the sphere
\[
x2+y2+z2=1.
\]
The chart cannot possibly represent the whole sphere because although a sphere is locally Euclidean. There is certainly a topological difference between a sphere and a plane. Indeed, if one analyzes the coordinate chart carefully, one will note that at the North pole \( u=0,z=1 \) the coordinates become singular. This happen because \( u=0 \) implies that \( x=y=0 \) regardless of the value of \( v \), so that the North pole has an infinite number of labels. The fact that it is required to have two parameters to describe a patch on a surface in \( R3 \) is a manifestation of the 2-dimensional nature of the surfaces. If one holds one of the parameters constant while varying the other, then the resulting 1-parameter equations describe a curve on the surface. Thus, for example, letting \( u=0 \) constant in equation (2.4), we get the equation of a meridian great circle.

Notation (2.3):
Given a parameterization of a surface in local chart
\[
x(u,v)=x(u1,u2)=x(u), \text{ we denote the partial derivatives by any of the following notations :}
\]
\[
xu=x1=\partial x/\partial x1u=x11=2xu2
\]
\[
xv=x2=\partial x/\partial x2u=x22=2xv2
\]
\[
x=\partial xuxu1=2xu
\]

3. THE FIRST FUNDAMENTAL FORM

In the last unit you have studied about the metric. The metric of a surface determines the first fundamental form of the surface. Thus the quadratic differential from
\[
Edu2+2Fdudv+Gdv2
\]
is called the first fundamental form and the quantities \( E,F,G \) are called the first order fundamental magnitudes or first fundamental coefficients. Here it should be noted that since the quantities \( E,F,G \) depend on \( U \) and therefore, in general, they vary from point to point on the surface.

Definition (3.1):
The differential of arc length \( ds \) of curve of a curve \( u=u1=1,2 \)
\[
\]
On the surface \( r=r(u1,u2) \) is defined by
\[
ds2=g11(du1)2+g12du1du2+g22(du2)2
\]
Where \( gik=ri.rk \) This expression for \( ds2 \) is called the first fundamental form of the surface. Using the summation convention, this expression can be written in the form
\[
ds2=gikdukduk \quad (i,k=1,2)
\]
Writing \( u,v \) instead of \( u1,u2 \) and \( E,F,G \), instead of \( g11,g12,g22 \), one obtains the classical expression for \( ds2 \)
\[
ds2=Edu2+2Fdudv+Gdv2 \quad [5,pp65]
\]

Proposition (3.2):
The parametric lines are orthogonal if \( F=0 \) [1, pp45]

4. THE SECOND FUNDAMENTAL FORM

The next step in the development of the theory of surfaces is taken in studying the deviation of the surface from its tangent plane in the neighborhood of the point of tangency, and this in turn leads to various developments touching on the curvature of the surface. It is convenient to introduce the function \( \rho,\mu,\nu=|x,\mu,\nu,-x(0,0)|,x=3(0,0) \), where \( x=3 \), is the unit normal vector Thus \( \rho(u,v) \), is the perpendicular distance (with an appropriate sign) from the tangent plane to the point of the surface \( S \) fixed by \( u,v \). Assuming \( (u,v) \) to have continuous third derivatives, it can be developed as follows:
\[
xu,v=x0,0+x1u+x2v+12(x11u2+2x12uv+x22v2)
\]
Here the vector functions \( xi=xiu \) and \( xj=2xuuj \) are evaluated at \( (0,0) \) and the dots refer to terms that are cubic at least in \( u \) and \( v \). Hence \( \rho(u,v) \) is given by
\[
u=12(x11x3u2+2x12x3uv+x22x3v2)
\]
Since \( x1x3=0 \), The quantities
\[
L=x11x3=x11,M=x12x3=L12,N=x22x3=L22
\]
are introduced, and the second fundamental form \( II \) is defined as the quadratic form
\[
II=Lu2+2Muv+Nv2
\]
For small values of \( u \) and \( v \) the function \( 2p(u,v) \) is approximated by the quadratic form \( II \) with errors of third or higher order in \( u \) and \( v \); a study of the form \( II \) will therefore give information about the shape of the surface \( S \) near the point of tangency, as will be seen in the next and later sections.

The coefficients \( L_k = x_3 k_3 \) are given by:

\[
L_k = g_{ik} N_i C
\]

Where \( g_{ik} \) is the space covariant metric tensor and the indexed \( b \) are the coefficients of the surface covariant curvature tensor.\cite{15, pp81}

Definition (5.1):

It is possible to derive another interesting formula with the aid of the formulas (5.4) again assuming the coordinate vectors \( X_1 \) and \( X_2 \) to be in orthogonal principal directions. By combining these formulas the following relations result with respect to differentiation along an arbitrary curve:

\[
X_3 + k_1 X_1 = a X_2 \quad X_3 + k_2 X_2 = b X_1
\]

With \( a \) and \( b \) certain scalars, the values of which play no role in what follows. Scalar multiplication of these equations with one another yields the equation

\[
X_3 X_3 + k_1 X_1 + k_2 X_2 = 0
\]

Where \( g_{ij} \) is the space covariant metric tensor and the indexed \( b \) are the coefficients of the surface covariant curvature tensor.\cite{15, pp81}

The form \( III \) evidently furnishes the line element of the spherical image of the surface. In a moment it will be shown that \( X_3 X_3 = II - I \), and since \( I = X_3 X_3 \) (5.5) yields the following identity that holds for the three fundamental forms:

\[
III = X_3 X_3 = (X_3)^2
\]

The form \( III \) evidently furnishes the line element of the spherical image of the surface. In a moment it will be shown that \( X_3 X_3 = II - I \), and since \( I = X_3 X_3 \) (5.5) yields the following identity that holds for the three fundamental forms:

\[
III = X_3 X_3 = (X_3)^2
\]

It is of interest to observe that (5.7) is a relation that is invariant with respect to parameter transformations since the forms \( II \), \( I \), and the Gaussian curvature \( K \) have that property, and although the form \( II \) changes its sign if the Jacobian of the transformation is negative, so also does \( H \): thus while the derivation of the formula made use of special parameters the end result is seen to be invariant with respect to all parameter transformations.\cite{7, pp98}

Theorem (5.2):

The first, second and third fundamental forms are related through the Gaussian curvature \( K \) and the mean curvature \( H \), by the following relation:

\[
K I - 2 H I I + III = 0
\]

proof:

Let \( (L_1 = g_{12} = 0) \)

\[
d N = N u d u + N v d v = - k_1 u d u - k_2 v d v
\]

\[
= k_1 u d u + v d v + k_1 k_2 v d v = - k_1 d \sigma + k_1 - k_2 v d v \Rightarrow d N + k_1 d = (k_1 - k_2) u d u
\]

Similarly

\[
d N + k_2 d \sigma = k_2 - k_1 u d u
\]

Since \( g_{12} = u v = 0 \) then

\[
d N + k_1 d \sigma = 0 \Rightarrow d N + k_2 d N . d \sigma = 0
\]

\[
d N . d N + k_1 d N + k_1 k_2 d \sigma . d \sigma = 0
\]

\[
III = 2 H I I + K I = 0
\]

The coefficients of first, second and third fundamental forms are correlated, through the mean curvature \( H \) and the Gaussian curvature \( K \), by the following relation

\[
K g_{ij} = II + K I
\]

By multiplying both sides with \( d N . d \sigma \) and contracting we obtain

\[
K g_{ij} = II + K I
\]

That is:

Trace (\( c j i \)) \( = 4 H^3 - 2 K \) \cite{12, pp349}

Example (5.3):

If we have the following sphere

\[
r = \sin \theta \cos \phi + \cos \theta \sin \phi \sin \theta + \cos \phi
\]

Find the third fundamental form of the surface

Solution:

\[
r = \cos \theta \sin \phi \sin \theta + \cos \phi \cos \theta \sin \phi \cos \theta + \cos \phi
\]

\[
N = r r \theta \phi = a_2 \cos \theta \sin \phi \sin \theta + a_2 \sin \theta \cos \phi
\]

\[
N = \sin \theta \cos \phi + \sin \theta \sin \phi \cos \phi
\]

\[
N = 1 = r \theta \phi
\]

\[
N = \sin \theta \cos \phi \sin \theta \cos \phi \sin \theta \cos \phi \cos \phi \sin \phi
\]

\[
N = \sin \theta \cos \phi \sin \theta \cos \phi \sin \theta \cos \phi \cos \phi \sin \phi
\]

\[
N = \sin \theta \cos \phi \sin \theta \cos \phi \sin \theta \cos \phi \cos \phi \sin \phi
\]

\[
N = \sin \theta \cos \phi \sin \theta \cos \phi \sin \theta \cos \phi \cos \phi \sin \phi
\]

\[
N = \sin \theta \cos \phi \sin \theta \cos \phi \sin \theta \cos \phi \cos \phi \sin \phi
\]

\[
N = \sin \theta \cos \phi \sin \theta \cos \phi \sin \theta \cos \phi \cos \phi \sin \phi
\]

\[
N = \sin \theta \cos \phi \sin \theta \cos \phi \sin \theta \cos \phi \cos \phi \sin \phi
\]

\[
N = \sin \theta \cos \phi \sin \theta \cos \phi \sin \theta \cos \phi \cos \phi \sin \phi
\]

\[
N = \sin \theta \cos \phi \sin \theta \cos \phi \sin \theta \cos \phi \cos \phi \sin \phi
\]

\[
N = \sin \theta \cos \phi \sin \theta \cos \phi \sin \theta \cos \phi \cos \phi \sin \phi
\]

\[
N = \sin \theta \cos \phi \sin \theta \cos \phi \sin \theta \cos \phi \cos \phi \sin \phi
\]
Using the New Mathematical Method to Find the Third Fundamental Form of Surface

C=N2N2=-a sin cos i-asin sin cos j+a k . -asin cos i-asin sin cos j+a k
=a2 + + a2
C= a2 + =a2
B=N1.N2
B=(-sin i+acos j)-asin cos i-asin sin cos j+a k
B=a2sin sin cos -a2sin sin cos
B = 0
III = Ad2+2Bdθdϕ+Cd2
III= a2d2+20+a2 d2
III= a2d2+a2 d2

6. METHODOLOGY AND DISCUSSION

Definition (6.1):
MATLAB is a high performance language for technical computing it integrates computation visualization and programming in easy to use environment where problems and solution are expressed in familiar mathematical notation.

Typical useinclude:
Math and computation, Algorithm development, Date analysis exploration and visualization
And Application development [10]

Solve Part Differential Equation with MATLAB (6.2):
The MATLAB partial differential equation solver pdepe initial boundary value problems for systems of Parabolic and elliptic part differential equation in the one space variable x and time t. there must be at lead one parabolic equation in the system. The pdepe solver converts the PDF to ODE using a second order accurate spatial discretization based on a set of nodes specified by the use. [11]

Part Differential Equation Solver Basic Syntax (6.3):
The basic syntax of the solver is:
sol=pdepe(m,pde fun,ic fun,bc fun,xmesh,tspan)
m is specifies the symmetry of problem.
pde fun is a function handle that computes m, f and s

M=m*rt first normal curvarte
N=N^rp secand normal curvarte
rt=diff(r,t)

N=(rt*rp)/(abs((rt*rp)));
N=simplify(N)
N1=[i j k*sin(p)*cos(t)*sin(t)*sin(p)*cos(t)*a*cos(t)*cos(p)*acos(t)*cos(p)]
N1=det(N1)
A=N1*N1;
A=simplify(A)
N2=[i j k*sin(p)*cos(t)*sin(t)*sin(p)*cos(t)*a*sin(t)*sin(p);
a*sin(t)*cos(p)*cos(t)*a*sin(t)*sin(p)]
N2=det(N2)
B=N1*N2;
B=simplify(B)
C=N2*N2;
C=simplify(C)
III = A*d2t^2 + 2*B*dt*dp + C*dp^2;
III=simplify(III)

Result
r =
a^k*cos(t) + a^i*cos(p)*sin(t) + a^j*sin(p)*sin(t)
rt =
a^i*cos(p)*cos(t) - a^k*sin(t) + a^j*cos(t)*sin(p)

N =
(a^2*sin(t)*sin(p)*cos(t) - a^k*sin(t) + a^j*cos(t)*sin(p)*cos(t) - k*sin(t) + j*cos(t)*sin(p))*abs(sin(t))*(j*cos(p) - i*sin(p))

N1 =
a^j*cos(p)*cos(t)*a^2 + a^j*cos(p)*sin(t)^2
A =
(a^i*cos(p) - a^j*cos(p) - a^i*cos(p)*cos(t)*sin(t)*sin(p) + a^i*cos(t)*cos(t)*sin(p)*sin(t) + a^i*cos(t)*cos(t)*sin(p)*sin(t))

N2 =
a^k*cos(p)*cos(t)*sin(t) - a^j*cos(t)*sin(t) - a^j*cos(t)*sin(t) - k*cos(t)*cos(t)*sin(p)*sin(t)

B =
a^k*cos(p)*sin(t)^2 + a^k*cos(p)*sin(t)^2 + a^i*cos(p)*cos(t)*sin(t)*sin(p) + a^i*cos(p)*cos(t)*sin(t)*sin(p)

C =
a^2*(k*cos(t)*a^2 - k + i*cos(p)*cos(t)*sin(t) + j*cos(t)*sin(p)*sin(t))^2

III =
“Using the New Mathematical Method to Find the Third Fundamental Form of Surface”

(a*dt*i*sin(p) - a*dt*j*cos(p) - a*dp*k + a*dp*k*cos(t)^2 - a*dt*i*cos(t)^2*sin(p) + dt*i*acos(t)*cos(t)*sin(p) + a*dp*i*cos(t)*sin(t) + a*dp*j*cos(t)*sin(p)*sin(t) - dt*k*cos(p)*acos(t)*sin(p)*sin(t) + a*dt*k*cos(p)*cos(t)*sin(p)*sin(t))^2

Solution:

Clear all
cLC
SymstabcijkN1N2ABCrtrpNdtdpIII
% t=90
% p=180
r =
a*(sin(t)*(cos(p))*i)+(a*(sin(t)*(sin(p)))*j)+(a*cos(t))*k
% III=Adt^2 + 2Bdtdp + Cdp^2
% N1=N^rt first normal curvarte
% N2=N^rp second normal curvarte
rt=diff(r,t);
rt=simplify(rt)
 rp=diff(r,p);
 rp=simplify(rp)
 N=(rt*rp)/(abs((rt*rp))
% t=90
% p=180
N1=[i j k;sin(t)*cos(p) sin(t)*sin(p) cos(t);acos(t)*cos(p)
N1=[a*j]
A=N1*N1
N2=[i j k;sin(t)*(cos(p)) sin(t)*(sin(p)) cos(t);-a*sin(t)*sin(p)
 a*sin(t)*(cos(p) 0];
N2=det(N2)
B=N1*N2;
 B=simplify(B)
C=N2*N2;
 C=simplify(C)
 III = A*dt^2 + 2*B*dt*dp + C*dp^2;
III=simplify(III)

Result
r =
a*k*cos(t) + a*i*cos(p)*sin(t) + a*j*sin(p)*sin(t)
rt =
a*(i*cos(p)*cos(t) - k*sin(t) + j*cos(t)*sin(p))
 rp =
a*sin(t)*(j*cos(p) - i*sin(p))
 N =
(a^2*sin(t)*(j*cos(p) - i*sin(p))*(i*cos(p)*cos(t) - k*sin(t) + j*cos(t)*sin(p)))/(abs(sin(t))*(j*cos(p)
 i*sin(p))*(i*cos(p)*cos(t) - k*sin(t) + j*cos(t)*sin(p)))*abs(a)^2)
N1 =
-a*j
A =
a^2*j^2
N2 =
a*k*cos(p)^2*sin(t)^2 + a*k*sin(p)^2*sin(t)^2 - a*i*cos(p)*cos(t)*sin(t) - a*j*cos(t)*sin(p)*sin(t)
B =
a^2*j*(k*cos(t)^2 - k + i*cos(p)*cos(t)*sin(t) + j*cos(t)*sin(p)*sin(t))
 C =
a^2*(k*cos(t)^2 - k + i*cos(p)*cos(t)*sin(t) + j*cos(t)*sin(p)*sin(t))
 III =
a^2*(dt*j - dp*k + dp*k*cos(t)^2 + dp*i*cos(p)*cos(t)*sin(t) + dp*j*cos(t)*sin(p)*sin(t))

CONCLUSION

At last we can find the Third Fundamental Form of Surface using the new mathematical method but we cant find the solution graphically so we hope that to do our best in the future to find the solution of the Third Fundamental Form of Surface graphically.

REFERENCES
1. Alan wenstin and Joseph Oesterle, Analysis on CR Manifolds,My10013,united States of America.
2. Barret O’Neill, Elementary Differential Geometry,MA018303 ,Burlington ,USA,2006.
3. Brian D.Hahn and DarieLl.valentine ,ESSENTIAL MATLAB for English and scientists,068417, Burlington ,Amsterdam.
4. Dietmar A. Salamon and Joel w.Robin, introduction to differential geometry ,Newyork,5 September 2013.
5. D.S MitrinOvi and j Ulcar, Differential Geometry ,tutorial text No5, Belegrade,1969, p61
6. Gabriel Lugo ,differential geometry in physics ,NC28403, Wilmington ,2006,pp44.
7. J. J. STOKER,DIFFERENTIA L GEOMETRY, Wiley ,1,inited state of America ,1976 ,p98.
8. JofferyM.lee, Differential analysis and physics , Boston , 2000.
9. Marian Fecko ,Deferential Geometry and Groups for Physcists, 84507 , camberidge university , Melbourne 2001.
10. Math work, Getting stated with MATLAB, second printing, may 1997.
11. math work, using MATLAB, versin, 6,2001.
12. Nasar solame, Differential Geometry, 2,Asuit, 2008, pp349.
13. Rao.v.Dukkipati, MATLAB on Introduction with Applications, new Age international (p),new delhi,2004.
14. Soren Bottcher, Solving ODE and PDE in MATLAB SCIE ,2009.
15. Taha Sochi, Principles of Differential Geometry, London, September 9, 2016, pp81.