ASYMPTOTIC FORMULA FOR BALANCED WORDS

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Abstract. We give asymptotic formulas for the number of balanced words whose slope \( \alpha \) and intercept \( \rho \) lie in a prescribed rectangle. They are related to uniform distribution of Farey fractions and Riemann Hypothesis. In the general case, the error term is deduced using an inequality of large sieve type.

1. Introduction

Let \( A = \{0,1\} \) and we denote by \( A^* \) the monoid generated by \( A \) by concatenation, where the empty word \( \lambda \) is its identity. The length of \( x \in A^* \) is \( |x| \). We denote \( |x|_1 \) the cardinality of 1 in \( x \in A^* \). \( A^N \) (resp. \( A^Z \)) is the set of right infinite (resp. bi-infinite) words. If \( y \in A^* \) is a factor (a subword) of \( x \in A^* \cup A^N \cup A^Z \), we write \( y \prec x \). A word in \( x \in A^* \cup A^N \cup A^Z \) is balanced if \( |u|_1 - |v|_1 | \leq 1 \) holds for any \( u, v \prec x \) with \( |u| = |v| \). An infinite word \( w \in A^N \) is sturmian if \( \text{Card} \{u \prec w \mid |u| = n\} = n + 1 \) for all \( n \in \mathbb{N} \). Morse and Hedlund characterized a sturmian word as an aperiodic balanced word in \( A^N \). They also characterized sturmian words as a coding of irrational rotation. More precisely a lower mechanical word \((s_n) \in A^N\) is defined by \( s_n(\alpha, \rho) = [\alpha(n+1) + \rho] - [\alpha n + \rho] \) with a given slope \( \alpha \in [0,1] \) and an intercept \( \rho \in [0,1) \). An upper mechanical word is similarly defined by replacing \([\cdot]\) with \(\lceil\cdot\rceil\), which is denoted by \( \hat{s}_n(\alpha, \rho) \). Then a sturmian word is a (lower or upper) mechanical word of an irrational slope \( \alpha \) and vice versa. It is known that every balanced word is a factor of a lower and an upper mechanical word \([17]\). Indeed, for every balanced word \( x = x_1 \ldots x_n \) we can find a slope \( \alpha \) and an intercept \( \rho \) such that \( x_i = s_i(\alpha, \rho) = \hat{s}_i(\alpha, \rho) \). Note that the choice of \( \alpha \) and \( \rho \) is not unique. In fact, a balanced \( x \) word corresponds to \((\rho, \alpha)\) in a convex polygon in \([0,1) \times [0,1] \). This geometric idea for enumeration is found in \([1, 25, 14, 3]\). Let \( \phi(n) \) be the Euler totient function. The formula for the number of balanced words of length \( n \) is given by

\[
1 + \sum_{k=1}^n (n+1-k)\phi(k).
\]

Several different proofs are found in \([17, 16, 1, 5, 18]\).

In this paper we refine the formula (1) and give its asymptotic behavior. Denote by \( B(n, t, u) \) the cardinality of the set of balanced words of length \( n \) whose slope...
\[ \alpha \in [1-t, 1] \] and its intercept \( \rho \in [0, u) \) and let \( B(n) \) be the set of balanced words of length \( n \). Then we show

**Theorem 1.**

\[ B(n, t, u) = 1 + \sum_{m \leq n} A(m, t, u) \]

with

\[ A(m, t, u) = \sum_{\substack{i < j \leq m, (i, j) = 1 \, \, \, \, \, i/j \leq t, \, \, \langle mi/j \rangle < u}} 1, \]

where \( i \) and \( j \) are non negative integers.

Theorem 1 slightly generalizes Yasutomi [25, Proposition 4], shown in a different context. By using Theorem 1, we will derive asymptotic formulas. Hereafter, we use conventional terminology in analytic number theory, i.e., Landau \( O \), \( o \) symbol and Vinogradov symbol \( \ll \). The symbol \( \varepsilon \) is reserved as an arbitrary positive constant, and the symbol \( c \) in Landau \( O \) is a suitably chosen positive constant, which may differ among formulas.

**Theorem 2.**

\[ B(n, t, u) = \frac{tu}{\pi^2} n^3 + O \left( n^2 (\log n)^{15/2} \right). \]

Moreover, we have

\[ B(n, 1, 1) = \frac{n^3 + 3n^2}{\pi^2} + O \left( n^2 \exp \left( -c \left( (\log n)^{3/5} (\log \log n)^{-1/5} \right) \right) \right), \]

and

\[ B(n, t, 1) = \frac{tn^3}{\pi^2} + O(n^2). \] (2)

For almost all \( t \), the estimate (2) can be sharpened with the help of Fujii [9], see the discussion in the end of §3. Denote by \( \chi_A(x) \) the indicator function of the set \( A \). For a rectangle \( (a, b) \times [c, d] \) in the unit square \([0, 1] \times [0, 1] \), we see

\[ \text{Card} \{ x \in B(n) \mid (\rho, \alpha) \in (a, b) \times [c, d] \} = \frac{(b - a)(d - c)}{\pi^2} n^3 + O \left( n^2 (\log n)^{15/2} \right), \]

from

\[ \chi_{(a, b) \times [c, d]}(x) = \chi_{(a, b) \times [c, 1]}(x) - \chi_{(0, a) \times [c, 1]}(x) - \chi_{(0, b) \times [d, 1]}(x) + \chi_{(0, a) \times [d, 1]}(x). \]

Moreover for a Jordan measurable region \( W \) in the unit square, we have

**Corollary 3.**

\[ \text{Card} \{ x \in B(n) \mid (\rho, \alpha) \in W \} = \frac{\text{Area}(W)}{\pi^2} n^3 + O \left( n^2 (\log n)^{15/2} \right) \]

where Area is the 2-dimensional Lebesgue measure, since every Jordan measurable set is well approximated by finite union of rectangles.

Farey series \( \{f_m(i)\} \) of order \( m \) is the finite increasing sequence composed of irreducible fractions in \([0, 1) \) whose denominators are not larger than \( m \):

\[ 0 = f_m(1) < f_m(2) < \cdots < f_m(\Phi(m)) < 1 \]

The statement of Theorem 1 is simpler by this choice than taking \([0, t] \) because the lines in the proof do not intersect \((0, 1) \times \{0\} \).
with $\Phi(m) = \sum_{k=1}^{m} \phi(k)$. Clearly $A(m, t, 1) = \max \{j \mid f_m(j) \leq t\}$. It is well-known that Riemann Hypothesis is equivalent to

$$\sum_{i=1}^{\Phi(m)} \exp(2\pi i f_m(i)) \ll m^{1/2+\varepsilon},$$

a strong uniform distribution property of Farey fractions. As pointed out in [8], Franel [7] already noticed that

$$\int_{0}^{1} (A(m, t, 1) - t \Phi(m))^2 dt \ll m^{1+\varepsilon}$$

is also equivalent to Riemann Hypothesis. One can see

$$A(m, t, 1) - t \Phi(m) = O(m),$$

similarly to (8) and (10) in §3, but we expect it is much smaller in average. See also [21, 8, 15]. We state another equivalent statement directly related to the number of balanced words:

**Corollary 4.** The estimate

$$B(n, 1, 1) = \frac{n^3 + 3n^2}{\pi^2} + O\left(n^{3/2+\varepsilon}\right)$$

(3)

is equivalent to Riemann Hypothesis.

2. **Proof of Theorem**

We elucidate a geometric counting discussion of Yasutomi [25, Proposition 4] in our convenient terminology, which is more straightforward than the one in [1]. Let $m$ be a fixed positive integer and put $X := [0, 1) \times [0, 1)$. The map $\psi : (\rho, \alpha) \rightarrow (s_n(1-\alpha, \rho))$ gives a natural partition

$$X := \bigcup_{w \in B(m)} \psi^{-1}(w).$$

(4)

Then $X$ is cut into convex cells by segments:

$$Y := X \setminus \{(x, y) \mid x = ny - \ell, n \in \{1, \ldots, m\}, \ell \in \{0, 1, \ldots, n-1\}\}.$$ 

We obtain essentially the same partition by using $s_n$, the difference is seen only on the boundary of $Y$. In this paper, we use $s_n$ for the partition. Fixing $\alpha \in [0, 1]$, the intersections of the line $y = 1 - \alpha$ and $x = ny - \ell$ are written as

$$\{R^{-n}(0) \mid n \in \{1, \ldots, m\} \times \{1-\alpha\} = \{-n\alpha (mod 1) \mid n = 1, 2, \ldots, m\} \times \{1-\alpha\}$$

where $R : (x, 1-\alpha) \mapsto (x+\alpha, 1-\alpha)$ is the rotation map acting on the unit interval $[0, 1) \times \{1-\alpha\}$ which is identified with the torus $T := \mathbb{R}/\mathbb{Z}$. The partition of $[0, 1) \times \{1-\alpha\}$ by $R^{-n}(0)$ ($n = 1, \ldots, m$) gives cylinder sets of rotation $R$, i.e., the points in the same cylinder share the same coding

$$\{\chi_{[1-\alpha,1]} \times \{1-\alpha\} \left(\right) R^{-i-1}(x)\}_{i=1}^{m}.$$ 

They are the $m + 1$ different words which correspond to the words of length $m$ appear in the sturmian word of slope $\alpha$ when $\alpha$ is irrational. Slicing $X$ by the line $y = 1 - \alpha$, we observe the $m$-th level cylinder sets of the rotation $R$ on $T$ of slope $\alpha$ acting on the unit interval $[0, 1) \times \{1-\alpha\}$. Considering $\alpha$ as a variable, we reconstruct the partition (4) of $X$ that every convex cell corresponds to an element of $B(m)$, which is consistent with the cylinder partition of $[0, 1) \times \{1-\alpha\}$ for each
\( \alpha \in [0, 1] \). In this manner, the partition (4) is seen as a pile of cylinder sets of level \( m \) for all (rational/irrational) rotations. The case \( m = 4 \) is depicted in Figure 1.

![Figure 1. Partition for \( m = 4 \)](image)

To enumerate \( B(n, t, u) \), we compute \( B(m, t, u) - B(m - 1, t, u) \) for \( m \geq 2 \), that is, the increase of number of cells as we add a new slope \( 1/m \). Equivalently, we count the number of intersections in \( [0, u] \times [0, t] \) which appear by adding new segments of slope \( 1/m \), see Figure 2. The intersection points \((x, y) \in [0, u] \times [0, t]\) of \( x = my - b \) and \( x = \ell y - c \) (\( \ell < m \)) are in one to one correspondence with the set of their \( y \)-coordinates:

\[
P(m) := \left\{ \frac{b-c}{m-\ell} \left| \begin{array}{c}
\frac{b}{m} \leq \frac{b-c}{m-\ell} < \frac{b+1}{m}, \quad \frac{b-c}{m-\ell} \leq t, \\
\frac{\ell b-mc}{m-\ell} < u, \quad 0 \leq b, \ell < m, \quad 0 \leq c < \ell
\end{array} \right. \right\}.
\]

We claim that \( P(m) \) coincides with the set

\[
Q(m) := \{ x \in \mathbb{Q} \cap [0, 1) \mid x \leq t, \quad \langle mx \rangle < u, \quad \text{den}(x) \leq m \}
\]

where \( \langle x \rangle = x - \lfloor x \rfloor \) and \( \text{den}(y) \) is the denominator of \( y \in \mathbb{Q} \). In fact \( P(m) \subseteq Q(m) \) is clear. Let \( x = p/q \in Q(m) \) with \( 1 \leq q \leq m \), \( (p, q) = 1 \), \( p/q \leq t \) and \( \langle mx \rangle < u \). Put \( \ell = m - q \) and choose \( b \) with \( b/m \leq p/q < (b+1)/m \) and let \( c = b - p \). Since \( p/q < (b+1)/m \), we have \( p \leq b \) and \( c \geq 0 \). From the property of Farey fraction, we have \( (b - p)/(m - q) < b/m < p/q \), which implies \( c = b - p < m - q = \ell \). The inequality \( 0 < \frac{\ell b-mc}{m-\ell} < 1 \) follows from \( \frac{b}{m} \leq \frac{b-c}{m-\ell} < \frac{b+1}{m} \). Thus \( \langle mx \rangle = \frac{\ell b-mc}{m-\ell} \) and therefore \( x \in P(m) \). The claim is proved. Set \( A(m, t, u) = \text{Card}(Q(m)) \) for \( m \geq 1 \). From \( B(1, t, u) = 2 \) and \( A(1, t, u) = 1 \), we obtain Theorem 1. \( \square \)
We illustrate this proof by an example.

**Example 5.** Let $t = 0.7$ and $u = 0.59$. Then $(A(m, t, u))_{m=1}^8 = (1, 2, 4, 4, 7, 8, 10, 13)$ and $B(8, t, u) = 50$. In Figure 2, we are counting the number of cells in the shaded region $[0, u] \times [0, t]$. Dashed segments are of slope $1/8$ intersecting at 13 points indicated by dots, which contribute to the increase of the number of cells.

**Figure 2.** $B(8, 0.7, 0.59) = 50$

Hereafter we show an asymptotic formula of $B(n, t, u)$.

3. **Theorem 2 for $u = 1$ and Corollary 4**

We show a

**Lemma 6.**

$$
\sum_{m \leq x} (x - m)\phi(m) = \frac{x^3}{32} + O \left( x^2 \exp \left( -c (\log x)^{3/5} (\log \log x)^{-1/5} \right) \right).
$$

**Proof.** This is a variation of the argument to deduce prime number theorem [20, Theorem 6.9]. We use the Mellin inversion formula

$$
\sum_{m \leq x} (x - m)\phi(m) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\zeta(s-1)x^s}{\zeta(s)s(s+1)} ds
$$

for $a > 2$. Here $\zeta(s)$ is Riemann zeta function with a complex variable $s = \sigma + i\tau$ and put $\tau = |\omega| + 4$. Since $|\zeta(s-1)| \ll \tau^{1/2} \log \tau$ and $1/\zeta(s) \ll \log \tau$ in the required
range, shifting the path to \( \sigma = 1 + (\log x)^{-1} =: a_0 \), we can pick the residue at \( s = 2 \):

\[
\sum_{m \leq x} (x - m) \phi(m) = \frac{x^3}{\pi^2} + \frac{1}{2\pi i} \int_{a_0 - i\infty}^{a_0 + i\infty} \frac{\zeta(s - 1)x^{s+1}}{\zeta(s)s(s+1)} ds. 
\]

(5)

Truncating this formula,

\[
\sum_{m \leq x} (x - m) \phi(m) = \frac{x^3}{\pi^2} + \frac{1}{2\pi i} \int_{a_0 - iT}^{a_0 + iT} \frac{\zeta(s - 1)x^{s+1}}{\zeta(s)s(s+1)} ds + O \left( \frac{x^2(\log T)^2}{T^{1/2}} \right)
\]

and shifting the path to the zero-free region

\[
\sigma > 1 - \frac{c}{(\log \tau)^{2/3}(\log \log \tau)^{1/3}} =: b
\]

of \( \zeta(s) \) due to Vinogradov-Korobov \[24, 11\], we get a rectangular contour to be studied. The contribution from horizontal segments is \( O(x^2/T^{3/2-\epsilon}) \) and the vertical one at \( \sigma = b \) is \( O \left( x^{b+1} \right) \). Lemma 6 is obtained by the choice \( T = \exp \left( c(\log x)^{3/5}(\log \log x)^{-1/5} \right) \). \( \square \)

Let us start with the easiest case \( t = u = 1 \). From

\[
A(m, 1, 1) = \sum_{i<j \leq m, (i,j)=1} 1 = \Phi(m)
\]

and Theorem\[1\] we have

\[
B(n, 1, 1) = 1 + \sum_{m=1}^{n} \Phi(m) = 1 + \sum_{j=1}^{n} (n+1-j)\phi(j) = \frac{n^3 + 3n^2}{\pi^2} + O \left( n^2 \exp \left( -c(\log n)^{3/5}(\log \log n)^{-1/5} \right) \right).
\]

(7)

Here we used Lemma\[6\] and the Mertens formula

\[
\Phi(n) = \frac{3n^2}{\pi^2} + O(n \log n).
\]

The error term \( E(n) = \Phi(n)-(3/\pi^2)n^2 \) is well studied in literature \[22, 6, 24, 19, 13\]. However in \( 6 \), the effect of this error term cancels out and we find the second main term in \( 7 \). We retrieved the formula \[1\] as well.

Now we show Corollary\[4\] If Riemann Hypothesis is valid, then \( 5 \) holds with \( a_0 = 1/2 + \varepsilon \). Thus from

\[
\lim_{T \to +\infty} \int_{a_0 - iT}^{a_0 + iT} \frac{\zeta(s - 1)}{|\zeta(s)s(s+1)|} ds < \infty,
\]

we get the estimate \[3\]. Conversely Mellin transformation shows

\[
\int_{1}^{\infty} \left( \sum_{n \leq x} (x-n)\phi(n) - \frac{x^3}{\pi^2} \right) x^{-s-2} dx = \frac{\zeta(s - 1)}{\zeta(s)s(s+1)} - \frac{1}{\pi^2(s-2)}
\]

for \( \sigma > 2 \). If \[3\] is valid, then the parenthesis in the integrand is \( O(x^{3/2+\varepsilon}) \). This gives the holomorphic continuation of the right side to \( \sigma > 1/2 + \varepsilon \), which finishes the proof.
Let us discuss a small counting issue. Since $i = 0$ implies $j = 1$ in the sum $A(m, t, u)$, we have

$$A(m, t, u) = 1 + \sum_{i < j \leq m, \frac{i}{j} \leq t^\prime} \sum_{(mi/j) < u, (i,j)=1} 1$$

where $i$ and $j$ are positive integers hereafter. Therefore

$$B(n, t, u) = 1 + n + \sum_{i < j \leq n, \frac{i}{j} \leq t} \sum_{(mi/j) < u, (i,j)=1} 1.$$ 

If $t < 1$ then, it is the same as

$$B(n, t, u) = 1 + n + \sum_{i,j \leq m \leq n, \frac{i}{j} \leq t} \sum_{(mi/j) < u, (i,j)=1} 1.$$ 

In the case $t = 1$, $i = j$ happens only when $i = j = 1$, and we may write

$$B(n, 1, u) = 1 + \sum_{i,j \leq m \leq n} \sum_{(mi/j) < u, (i,j)=1} 1.$$ 

Let $\mu$ be the M"obius function and assume $u = 1$ and $t < 1$. With positive integers $a, b$, we have

$$B(n, t, 1) - 1 = \sum_{i,j \leq m \leq n} \sum_{k\backslash(i,j)} \mu(k) = \sum_{k \leq m \leq n} \mu(k) \sum_{b \leq m/k} \sum_{a/b \leq t} 1$$

$$= \sum_{k \leq m \leq n} \mu(k) \sum_{b \leq m/k} \lfloor bt \rfloor$$

$$= t \sum_{k \leq m \leq n} \mu(k) \sum_{b \leq m/k} b - \sum_{kb \leq m \leq n} \mu(k)(bt). \quad (8)$$

While $t = 1$, the same computation gives

$$B(n, 1, 1) - 1 = \sum_{k \leq m \leq n} \mu(k) \sum_{b \leq m/k} b,$$

we see

$$B(n, t, 1) = 1 - t + n + tB(n, 1, 1) - \sum_{kb \leq m \leq n} \mu(k)(bt) \quad (9)$$

for $t < 1$. Since

$$\sum_{kb \leq m \leq n} \mu(k)(bt) = O(m) \quad (10)$$

by Niederreiter [21] (6) and Lemma 3], we have

$$\sum_{kb \leq m \leq n} \mu(k)(bt) = O(n^2). \quad (11)$$

Note that the implied constant does not depend on $t$. From (7), (9) and (11), we have shown

$$B(n, t, 1) = \frac{ln^3}{\pi^2} + O(n^2). \quad (12)$$

We do not know whether

$$\sum_{kb \leq m \leq n} \mu(k)(bt) = o(n^2)$$
holds for all \( t \). If this estimate is valid for a fixed \( t \), we observe the second main term:
\[
B(n, t, 1) = \frac{t(n^3 + 3n^2)}{\pi^2} + o(n^2).
\]
Fujii \[9\] elaborated the improvement of (10) by using Hecke’s Dirichlet series \[10\]:
\[
Z_t(s) := \sum_{b=1}^{\infty} \left( \frac{bt}{b^s} \right) - \frac{1}{2}b^s.
\]
The analytic property of \( Z_t(s) \) heavily depends on the Diophantine approximation property of \( t \) by rationals. The refinement of (10) in the proofs of \[9, Theorem 1 and 2\] imply
\[
B(n, t, 1) = \frac{t(n^3 + 3n^2)}{\pi^2} + O\left(n^2 \exp\left(-c \frac{\log n \cdot \log \log n}{n}^{1/3}\right)\right)
\]
for almost all \( t \), including all algebraic numbers.\(^2\)

Note that the proofs are rather different between rational \( t \) and algebraic irrational \( t \), and the implied \( O \) constant could be very sensitive to the choice of \( t \).

4. Preliminaries

Let \( e(x) = \exp(2\pi ix) \). Let \( d \) be the natural metric on the torus \( T = \mathbb{R}/\mathbb{Z} \). For a given interval \( [\alpha, \beta] \subset [0, 1) \) and a positive \( J > 2/(1 - \beta + \alpha) \) there exists a smooth function \( V_J \) of period 1 such that
\[
\begin{align*}
V_J(z) &= 1 \quad z \in [\alpha, \beta] \\
V_J(z) &= 0 \quad d(z \mod \mathbb{Z}, [\alpha, \beta]) \geq 1/J \\
0 &\leq V_J(z) \leq 1 \quad \text{otherwise},
\end{align*}
\]
whose Fourier expansion is
\[
V_J(z) = \sum_{h \in \mathbb{Z}} v_J(h)e(hz)
\]
with
\[
|v_J(h)| \leq \min\left(\frac{2}{|h|}, \frac{2J}{(\pi h)^2}\right),
\]
see \[23, Chapter 1, Lemma 12\]. We shall use a large sieve inequality \[12, Theorem 7.2\], \[2, Lemma 2.4\].

**Lemma 7.** For any real numbers \( x_m, y_m \) with \( |x_m| \leq X \) and \( |y_m| \leq Y \) and \( \alpha_m, \beta_m \in \mathbb{C} \), we have
\[
\left|\sum_m \sum_n \alpha_m \beta_n e(x_my_n)\right| \leq 5\sqrt{1+XY} \left(\sum_{|x_i-x_j|<1/Y} |\alpha_i\alpha_j| \sum_{|y_i-y_j|<1/X} |\beta_i\beta_j|\right)^{1/2}.
\]
We prepare two more lemmas.

\(^2\)The error term of (13) can be replaced by the one in (11), because we need to improve (11) but not necessarily (10). This may be discussed elsewhere.
Lemma 8. 
\[
\left| \frac{1}{\pi} \int_{-T}^{T} \exp(i \alpha x) \frac{\sin \beta x}{x} dx - \delta(\beta - \alpha) \right| = O \left( \min \left( 1, \frac{1}{T |\beta - \alpha|} \right) \right)
\]
for \( T, \alpha, \beta > 0 \) with
\[
\delta(y) = \begin{cases} 
1 & y > 0 \\
1/2 & y = 0 \\
0 & y < 0.
\end{cases}
\]

Proof. This follows from the sine integral formula
\[
\frac{2}{\pi} \int_{0}^{T} \sin \frac{x}{x} dx = 1 + O \left( \min \left( 1, \frac{1}{T} \right) \right).
\]
c.f. [4, p.166], [12, Lemma 13.11]. \( \Box \)

Lemma 9. 
\[
\sum_{H < h \leq 2H \ M < a \leq 2M} (h, a) = \frac{6HM}{\pi^2} \log(2 \min(H, M)) + O(HM)
\]

Proof. Putting \( d = (h, a) \), we have
\[
\sum_{H < h \leq 2H \ M < a \leq 2M} (h, a) = \sum_{d \leq 2 \min(H, M)} d \sum_{H/d < i \leq 2H/d} \sum_{M/d < j \leq 2M/d} \mu(k) \sum_{H/(dk) < i \leq 2H/(dk)} 1,
\]
where the last sum is
\[
\frac{HM}{dk} + O \left( \frac{H}{dk} \right) + O \left( \frac{M}{dk} \right) + O(1).
\]
The proof follows from \( \sum_{d \leq X} \log(X/d) = X + O(\log(X)) \). \( \Box \)

5. Proof of Theorem

Our strategy is to treat \( B(n, t, u) \) as a function on \( t \). Then \( B(n, t, u) \) is a non-decreasing step function having a finite number of rational discontinuities. Every gap between the discontinuities of \( B(n, s, t) \) as a function on \( t \) is greater than \( 1/n^2 \). Depending on \( n \), we will choose \( t_1 \leq t \leq t_2 \) that \( t_2 - t_1 \) is small and \( t_1, t_2 \) have a suitable property to estimate error terms. If we get the same asymptotic formulas and their implied constants of the Landau and Vinogradov symbols are independent of this choice of \( t_i \), we obtain the estimate for \( t \) by the non-decreasing property, because it is sandwiched by the same formula. Therefore if we could perturb \( t \) and get the same error term (up to negligible terms), then we are done. Hereafter we shall be cautious on the above implied constants, whether they can be independent of the choices of \( t_i \).

At the cost of an additional error term \( O(n^2) \) which takes care of the case \( i/j = t \), we may assume that \( 0 < t < 1 \), \( t \) is rational and \( B(n, t, u) \) is continuous at \( t \), i.e.,
constant in the neighborhood of $t$. Recall that from the middle of §3, $i$ and $j$ are positive integers and

$$B(n, t, u) = 1 + n + \sum_{i,j \leq m \leq n, i/j \leq t} 1.$$

Taking $V_J(x)$ with respect to the interval $[0, u]$, we have

$$\sum_{i,j \leq m \leq n, i/j \leq t} V_J \left( \frac{mi}{j} \right) = B(n, t, u) - n + O \left( \frac{n^3}{J} \right). \quad (16)$$

As (14) implies $v_J(0) = u + O(1/J)$, the main term is

$$\sum_{i,j \leq m \leq n, i/j \leq t} \left( u + O \left( \frac{1}{J} \right) \right) = u \sum_{i,j \leq m \leq n, i/j \leq t} 1 + O \left( \frac{n^3}{J} \right)$$

$$= \frac{tun^3}{\pi^2} + O(n^2) + O \left( \frac{n^3}{J} \right) \quad (17)$$

from (12). The remainder is

$$\sum_{0 \neq h \in \mathbb{Z}} v_J(h) \sum_{i,j \leq m \leq n, i/j \leq t} e \left( \frac{hmi}{j} \right) \sum_{k | (i,j)} \mu(k)$$

$$= \sum_{k \leq n} \mu(k) \sum_{0 \neq h \in \mathbb{Z}} v_J(h) \sum_{a,b \leq m/k, a/b \leq t} e \left( \frac{hma}{b} \right) \quad (18)$$

with positive integers $a, b$. From (15), our target is to estimate

$$C(n, k, t, u) := \sum_{0 \neq h \in \mathbb{Z}} v_J(h) \sum_{a,b \leq m/k, a/b \leq t} e \left( \frac{hma}{b} \right)$$

$$= \sum_{0 \neq h \leq H} v_J(h) \sum_{a,b \leq m/k, a/b \leq t} e \left( \frac{hma}{b} \right) + O \left( \frac{Jn^3}{Hk^2} \right). \quad (19)$$

By Lemma 8 and (14), we have

$$\int_{-U}^{U} \int_{-T}^{T} \sum_{1 \leq h \leq H} v_J(h) \sum_{a,b \leq m/k, a/b \leq t} e \left( \frac{hma}{b} \right) \left( \frac{a}{b} \right)^{ix} \left( \frac{b}{m} \right)^{iy}$$

$$\times \sin(x \log(t)) \sin(y \log(1/k)) \pi x \pi y \, dxdy$$

$$= \sum_{1 \leq h \leq H} v_J(h) \sum_{a,b \leq m/k, a/b \leq t} e \left( \frac{hma}{b} \right) \quad (20)$$

$$+ O \left( \log U \log H \sum_{a,b \leq m/k, a/b \leq t} \frac{1}{T|\log \frac{H}{a}|} \right) + O \left( \log H \sum_{a,b \leq m/k, a/b \leq t} \min \left( 1, \frac{1}{U|\log \frac{H}{m}|} \right) \right). \quad (21)$$
It is important to note that the variables \( h, a, b, m \) are independent in the sum in the double integral (20). We may further assume that \( t \) has a denominator \( p \), which is the smallest prime exceeding \( n^2 \). Then we have \( |\log(1 + (\frac{tb}{a} - 1))| \geq 1/(2np) \) and \( \frac{1}{T|\log \frac{T}{a}|} \leq \frac{4n^3}{p} \), which implies

\[
O \left( \frac{\log U \log H}{a, b \leq n/k, m \leq n} \sum \frac{1}{T|\log \frac{T}{a}|} \right) = O \left( \frac{n^6}{k^2T \log U \log H} \right). \tag{22}
\]

Since one can show that the case \( bk = m \) is negligible, we have

\[
O \left( \log H \sum_{a \leq n/k, a/b \leq t} \min \left( 1, \frac{1}{U|\log \frac{bk}{m}|} \right) \right) = O \left( \frac{n^4}{k^2U \log H} \right). \tag{23}
\]

To deal with (20), let us apply Lemma 7 to

\[
G(K, L, M, N) := \sum_{K < m < 2K} \sum_{M < a < 2M} \sum_{L < h < 2L} v_j(h) a^{x} b^{y-x} m^{-iy} e \left( \frac{hma}{b} \right)
\]

where \( M, N \leq n/(2k) \), \( K \leq n/2 \) and \( K, L, M, N \in \mathbb{N} \). Define \( x_m \) and \( y_n \) by rearranging multi-sets

\[
\{ ha \mid L < h \leq 2L, \ M < a \leq 2M \}
\]

and

\[
\{ \frac{m}{b} \mid K < m \leq 2K, \ N < b \leq 2N \}
\]

in the non-decreasing order, keeping multiplicity. Since \( |x_m| \leq 4LM, \ |y_n| \leq 2KN \), we have a bound:

\[
|G(K, L, M, N)|^2 \leq 25 \left( 1 + \frac{8LMK}{N} \right) \left( \sum_{|h_1a_1 - h_2a_2| < \frac{N}{4K}} |v_j(h_1) v_j(h_2)| \right) \left( \sum_{|m/b_1 - m/b_2| < \frac{1}{4N}} 1 \right)
\]

\[
\ll \left( 1 + \frac{8LMK}{N} \right) \frac{1}{L^2} \left( \begin{array}{c} N \vspace{1mm} \frac{2K}{L} \end{array} \right) + 1 \left( \sum_{L < h_1 < 2L} \sum_{M < a_2 < 2M} (h_1, a_2) \right) KN \left( \frac{N^2}{4LMK} + 1 \right)
\]

\[
\ll \left( 1 + \frac{LMK}{N} \right) \frac{MNK}{L} \left( \begin{array}{c} N \vspace{1mm} \frac{K}{L} \end{array} \right) + 1 \left( \frac{N^2}{2LMK} + 1 \right) \log \min(L, M)
\]

by Lemma 9. If \( L \leq N/(8MK) \), then

\[
|G(K, L, M, N)|^2 \ll \left( \frac{N^4}{KL^2} + \frac{N^3}{L^2} \right) \log \min(L, M) \ll \frac{N^4}{L^2} \log \min(L, M). \tag{24}
\]

If \( N/(8MK) < L \leq N^2/(4MK) \), we have

\[
|G(K, L, M, N)|^2 \ll \left( \frac{MN^3}{L} + \frac{MN^2K}{L} \right) \log \min(L, M). \tag{25}
\]

When \( L > N^2/(4MK) \), we get the estimate

\[
|G(K, L, M, N)|^2 \ll (M^2NK + M^2K^2) \log \min(L, M). \tag{26}
\]
Now (24), (25), (26) shows
\[
G(K, L, M, N) \ll \frac{n^2(\log n)^{1/2}}{k}
\]
for any \( K, L, M, N \). Set \( J = \max(n, 3/(1-t)) \) and \( H = J^2 \). Summing up (19), (21), (22), (23), (27) and a similar computation for negative \( h \), we obtain
\[
C(n, k, t, u) \ll \frac{n^2}{k^2} + \frac{n^6}{k^2 T} \log U \log n + \frac{n^4}{k^2} \log T \log U \log n^{9/2}
\]
where the increase of the exponent of the last \( \log n \) comes from the number of dissections
\[
\sum_{1 \leq b \leq N} = \sum_{j \geq 1} \sum_{2^{-j} N < b \leq 2^{-j + 1} N}
\]
and the similar ones for \( a, m \) and \( h \). Taking \( U = n^2 \) and \( T = n^4 \), we obtain
\[
C(n, k, t, u) = O \left( \frac{n^2}{k} \log n^{13/2} \right).
\]
Therefore from (16), (17) and (18), we have
\[
B(n, t, u) = \frac{tun^3}{\pi^2} + O \left( \frac{n^2}{k} \log n^{15/2} \right).
\]

**Remark 10.** One can apply the same method to \( A(m, t, u) \). A slightly easier computation gives
\[
A(m, t, u) = \frac{3tm^2}{\pi^2} + O \left( m^{3/2} \log m^{9/2} \right)
\]
which falls short in showing the above bound for \( B(n, t, u) \).

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