Lie groups of bundle automorphisms and their extensions

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Summary. We describe natural abelian extensions of the Lie algebra \( \mathfrak{aut}(P) \) of infinitesimal automorphisms of a principal bundle over a compact manifold \( M \) and discuss their integrability to corresponding Lie group extensions. Already the case of a trivial bundle \( P = M \times K \) is quite interesting. In this case, we show that essentially all central extensions of the gauge algebra \( \mathcal{C}^{\infty}(M, \mathfrak{k}) \) can be obtained from three fundamental types of cocycles with values in one of the spaces \( z := \mathcal{C}^{\infty}(M, V), \Omega^{1}(M, V) \) and \( \Omega^{1}(M, V)/d\mathcal{C}^{\infty}(M, V) \). These cocycles extend to \( \mathfrak{aut}(P) \), and, under the assumption that \( TM \) is trivial, we also describe the space \( H^{2}(\mathcal{V}(M), z) \) classifying the twists of these extensions. We then show that all fundamental types have natural generalizations to non-trivial bundles and explain under which conditions they extend to \( \mathfrak{aut}(P) \) and integrate to global Lie group extensions.

Keywords: gauge group; automorphism group; infinite dimensional Lie group; central extension; abelian extension; affine connection

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Introduction

Two of the most important classes of infinite-dimensional Lie groups are groups of smooth maps, such as the Lie group \( \mathcal{C}^{\infty}(M, K) \) of smooth maps of a compact smooth manifold \( M \) with values in a Lie group \( K \), and groups of diffeomorphisms, such as the group \( \text{Diff}(M) \) of diffeomorphisms of a compact smooth manifold \( M \).

A case of particular importance arises for the circle \( M = \mathbb{S}^{1} \), where \( LK := \mathcal{C}^{\infty}(\mathbb{S}^{1}, K) \) is called the loop group of \( K \). If \( K \) is a compact simple Lie group, then \( LK \) has a universal central extension \( \hat{L}K \) by the circle group \( \mathbb{T} \). Furthermore, the group \( \mathbb{T} \subseteq \text{Diff}(\mathbb{S}^{1}) \) of rigid rotations of \( \mathbb{S}^{1} \) acts smoothly by automorphisms on \( LK \), this action lifts to the central extension \( \hat{L}K \), and we thus obtain the affine Kac–Moody Lie groups \( \hat{L}K \rtimes \mathbb{T} \). The twisted

\[1\] Strictly speaking, these are the “unitary forms” of the affine Kac–Moody groups. Starting with a complex simple Lie group \( K \) instead, we obtain complex versions.
affine Kac–Moody Lie groups can be realized in this picture as the fixed point groups for an automorphism $\sigma$ acting trivially on $T_r$, and inducing on $LK$ an automorphism of the form $\sigma(f)(t) = \varphi(f(\zeta t))$, where $\varphi \in \text{Aut}(K)$ is an automorphism of finite order $m$ and $\zeta \in T_r$ satisfies $\zeta^m = 1$. On each affine Kac–Moody group, the group $\text{Diff}^+(S^1)$ of orientation preserving diffeomorphisms of $S^1$ acts by automorphism, so that $\hat{L}K \rtimes \hat{T}$ embeds into $\hat{L}K \rtimes \text{Diff}^+(S^1)$.

Actually, the latter group permits an interesting twist, where the subgroup $T \times \text{Diff}^+(S^1)$ is replaced by the Virasoro group, a non-trivial central extension of $\text{Diff}^+(S^1)$ by $T$.

The purpose of this note is to discuss several infinite-dimensional Lie groups that are constructed in a similar fashion from higher dimensional compact manifolds $M$. Since the one-dimensional manifold $M = S^1$ is a rather simple object, the general theory leaves much more room for different Lie group constructions, extensions and twistings thereof. Here we shall focus on recent progress in several branches of this area, in particular relating the Lie algebra picture to global objects. We shall also explain how this relates to other structures, such as multiloop algebras, which are currently under active investigation from the algebraic point of view ([ABP06], [ABFP07]). We also present open problems and describe several areas where the present knowledge is far from satisfactory.

The analog of the loop algebra $C^\infty(S^1, \mathfrak{k})$ of a Lie algebra $\mathfrak{k}$ is the Lie algebra of smooth maps $g = C^\infty(M, \mathfrak{k})$ on the compact manifold $M$. Here the compactness assumption is convenient if we want to deal with Lie groups, because it implies that $g$ is the Lie algebra of any group $G = C^\infty(M, K)$, where $K$ is a Lie group with Lie algebra $\mathfrak{k}$. This group can also be identified with the gauge group of the trivial $K$-principal bundle $P = M \times K$, so that gauge groups of principal bundles are natural generalizations of mapping groups. Twisted loop groups are gauge groups of certain non-trivial principal bundles over $S^1$ (cf. Section 4).

For a $K$-principal bundle $q: P \to M$ over the compact manifold $M$, we write $\text{Aut}(P) := \text{Diff}(P)^K$ for the group of all diffeomorphisms of $P$ commuting with the $K$-action, i.e., the group of bundle automorphisms. The gauge group $\text{Gau}(P)$ is the normal subgroup of all bundle isomorphisms inducing the identity on $M$. Writing $\text{Diff}(M)_P$ for the set of all diffeomorphisms of $M$ that can be lifted to bundle automorphisms, which is an open subgroup of the Lie group $\text{Diff}(M)$, we obtain a short exact sequence of Lie groups

$$1 \to \text{Gau}(P) \to \text{Aut}(P) \to \text{Diff}(M)_P \to 1$$

(cf. [ACM85], [Wo06]). On the Lie algebra level, we have a corresponding short exact sequence of Lie algebras of vector fields

$$0 \to \mathfrak{gau}(P) \to \mathfrak{aut}(P) := \mathfrak{V}(P)^K \to \mathfrak{V}(M) \to 1.$$  

It is an important problem to understand the central extensions of gauge groups by an abelian Lie group $Z$ on which $\text{Diff}(M)_P$ acts naturally and the
extent to which they can be enlarged to abelian extensions of the full group \(\text{Aut}(P)\), or at least its identity component. Whenever such an enlargement exists, one has to understand the set of all enlargements, the twistings, which leads to the problem to classify all abelian extensions of \(\text{Diff}(M)_P\) by \(Z\).

Below we shall discuss various partial results concerning these questions. We also describe several tools to address them and to explain what still remains to be done.

The contents of the paper is as follows: After discussing old and new results on Lie group structures on gauge groups, mapping groups and diffeomorphisms groups in Section 1, we turn in Section 2 to central extensions of mapping groups \(G := C^\infty(M,K)\). Here we first exhibit three fundamental classes of cocycles over which all other non-trivial cocycles can be factored, up to cocycles vanishing on the commutator algebra. The fundamental cocycles are defined by an invariant symmetric bilinear form \(\kappa: \mathfrak{k} \times \mathfrak{k} \to V\) and an alternating map \(\eta: \mathfrak{k} \times \mathfrak{k} \to V\) as follows:

\[
\omega_{\kappa}(\xi_1, \xi_2) := \left[\kappa(\xi_1, d\xi_2)\right] \in \Omega^1(M, V) := \Omega^1(M, V)/dC^\infty(M, V),
\]

\[
\omega_{\eta}(\xi_1, \xi_2) := \eta(\xi_1, \xi_2) \in C^\infty(M, V)
\text{ for } d\eta = 0,
\]

and if \(d\kappa(x, y, z) = \kappa([x, y], z)\), we also have

\[
\omega_{\kappa, \eta}(\xi_1, \xi_2) := \kappa(\xi_1, d\xi_2) - \kappa(\xi_2, d\xi_1) - d(\eta(\xi_1, \xi_2)) \in \Omega^1(M, V).
\]

We then discuss the action of \(G_0 \rtimes \text{Diff}(M)\) on the corresponding central Lie algebra extension and explain under which conditions it integrates to an extension of Lie groups. Since we are also interested in the corresponding abelian extensions of the full automorphism group \(\text{Aut}(M \times K) \cong G \rtimes \text{Diff}(M)\) and its Lie algebra, we turn in Section 3 to abelian extensions of \(V(M)\), resp., \(\text{Diff}(M)\) by the three types of target spaces \(\mathfrak{z}\) from above. If \(TM\) is trivial and \(V = \mathbb{R}\), then a full description of \(H^2(V(M), \mathfrak{z})\) in all cases has been obtained recently in [BiNe07] (Theorem 3.8).

In Section 4 we then turn to gauge and automorphism groups of general principal bundles, where the first step consists in the construction of analogs of the three fundamental types of cocycles. A crucial difficulty arises from the fact that one has to consider principal bundles with non-connected structure groups to realize natural classes of Lie algebras such as twisted affine algebras as gauge algebras. We therefore have to consider target spaces \(V\) on which the quotient group \(\pi_0(K) = K/K_0\) acts non-trivially and consider \(K\)-invariant forms \(\kappa: \mathfrak{k} \times \mathfrak{k} \to V\). A typical example is the target space \(V(\mathfrak{t})\) of the universal invariant symmetric bilinear form of a semisimple Lie algebra \(\mathfrak{t}\) on which \(K = \text{Aut}(\mathfrak{t})\) acts non-trivially. Replacing the exterior differential by a covariant derivative, we thus obtain cocycles

\[
\omega_{\kappa}(\xi_1, \xi_2) := \left[\kappa(\xi_1, d\xi_2)\right] \in \Omega^1(M, V) := \Omega^1(M, V)/d\Gamma V,
\]

where \(V \to M\) is the flat vector bundle associated to \(P\) via the \(K\)-module structure on \(V\). For the cocycles \(\omega_{\kappa}\) there are natural criteria for integrability.
whenever $\pi_0(K)$ is finite (NeWo07), but it is not so clear how to extend $\omega_\kappa$ to $\mathfrak{aut}(P)$. However, the situation simplifies considerably for the cocycle $d \circ \omega_\kappa$ with values in $\Omega^2(M, V)$ which extends naturally to a cocycle on $\mathfrak{aut}(P)$ that integrates to a smooth group cocycle on $\text{Aut}(P)$.

The analogs of $C^\infty(M, V)$-valued cocycles are determined by a central extension $\hat{k}$ of $k$ by $V$ to which the adjoint action of $K$ lifts. Here the central Lie algebra extension $\hat{\mathfrak{gau}}(P)$ is a space of sections of an associated Lie algebra bundle with fiber $\hat{k}$. We then show that for any central extension $\hat{K}$ of $K$ by $V$ there exists a $\hat{K}$-bundle $\hat{P}$ with $\hat{P}/V \cong P$, so that $\text{Aut}(\hat{P})$ is an abelian extension of $\text{Aut}(P)$ by $C^\infty(M, V)$ containing the central extension $\mathfrak{gau}(\hat{P})$ of $\mathfrak{gau}(P)$ with Lie algebra $\hat{\mathfrak{gau}}(P)$. There are also analogs of the $\Omega^1(M, V)$-valued cocycles which exist whenever the 3-cocycle $\kappa([x, y], z)$ is a coboundary. For non-trivial bundles the fact that the extension $\mathfrak{aut}(P)$ of $\mathfrak{aut}(V) \ltimes \mathfrak{V}(M)$ does in general not split makes the analysis of the abelian extensions of $\mathfrak{aut}(P)$ considerably more difficult than in the trivial case, where we have a semidirect product Lie algebra. As a consequence, the results on group actions on the Lie algebra extensions and on integrability are much less complete than for trivial bundles.

To connect our geometric setting with the algebraic setup, we describe in Section 5 the connection between multiloop algebras and gauge algebras of flat bundles over tori which can be trivialized by finite coverings. The paper concludes with some remarks on bundles with infinite-dimensional structure groups and appendices on integrability of abelian Lie algebra extensions, extensions of semidirect products (such as $C^\infty(M, \mathfrak{t}) \ltimes \mathfrak{V}(M)$), and the triviality of the action of a connected Lie group on the corresponding continuous Lie algebra cohomology.

Throughout we only consider Lie algebras and groups of smooth maps endowed with the compact open $C^\infty$-topology. To simplify matters, we focus on compact manifolds to work in a convenient Lie theoretic setup for the corresponding groups. For non-compact manifolds, the natural setting is provided by spaces of compactly supported smooth maps with the direct limit $LF$-topology (cf. KM97, ACM99, Mi80, Gl02). However, there are some classes of non-compact manifolds on which we still have Lie group structure on the full group of smooth maps (Theorem 1.4). For groups of sections of group bundles one can also develop a theory for $C^k$- and Sobolev sections, but this has the disadvantage that smooth vector fields do not act as derivations (Sch04).

Notation and basic concepts

A Lie group $G$ is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. We write $1 \in G$ for the identity element and $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on $G$. Then each
$x \in T_1(G)$ corresponds to a unique left invariant vector field $x_l$ with $x_l(g) := d\lambda_g(1)\cdot x, g \in G$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $T_1(G)$ a continuous Lie bracket which is uniquely determined by $[x, y] := [x_l, y_l]$ for $x, y \in T_1(G)$. We write $L(G) = \mathfrak{g}$ for the so obtained locally convex Lie algebra and note that for morphisms $\varphi: G \to H$ of Lie groups we obtain with $L(\varphi) := T_1(\varphi)$ a functor from the category of Lie groups to the category of locally convex Lie algebras. We write $q_G: \tilde{G}_0 \to G_0$ for the universal covering map of the identity component $G_0$ of $G$ and identify the discrete central subgroup $\ker q_G$ of $\tilde{G}_0$ with $\pi_1(G) \cong \pi_1(G_0)$.

In the following we always write $I = [0, 1]$ for the unit interval in $\mathbb{R}$. A Lie group $G$ is called regular if for each $\xi \in C^\infty(I, \mathfrak{g})$, the initial value problem

$$\gamma(0) = 1, \quad \gamma'(t) = \gamma(t) \xi(t) = T(\lambda_{\gamma(t)}) \xi(t)$$

has a solution $\gamma \in C^\infty(I, G)$, and the evolution map

$$\text{evol}_G: C^\infty(I, \mathfrak{g}) \to G, \quad \xi \mapsto \gamma(1)$$

is smooth (cf. [Mil84]). For a locally convex space $E$, the regularity of the Lie group $(E, +)$ is equivalent to the Mackey completeness of $E$, i.e., to the existence of integrals of smooth curves $\gamma: I \to E$. We also recall that for each regular Lie group $G$, its Lie algebra $\mathfrak{g}$ is Mackey complete and that all Banach–Lie groups are regular ([GN07]). For a smooth map $f: M \to G$ we define the left, resp., right logarithmic derivative in $\Omega^1(M, \mathfrak{g})$ by $\delta^l(f)v := f(m)^{-1} \cdot T_m(f)v$ and $\delta^r(f)v := T_m(f)v \cdot f(m)^{-1}$, where $\cdot$ refers to the two-sided action of $G$ on its tangent bundle $TG$.

A smooth map $\exp_G: L(G) \to G$ is said to be an exponential function if for each $x \in L(G)$, the curve $\gamma_x(t) := \exp_G(tx)$ is a homomorphism $\mathbb{R} \to G$ with $\gamma_x'(0) = x$. Presently, all known Lie groups modelled on complete locally convex spaces are regular, hence possess an exponential function. For Banach–Lie groups, its existence follows from the theory of ordinary differential equations in Banach spaces. A Lie group $G$ is called locally exponential, if it has an exponential function mapping an open $0$-neighborhood in $L(G)$ diffeomorphically onto an open neighborhood of $1$ in $G$. For more details, we refer to Milnor’s lecture notes [Mil84], the survey [Neu06a] or the monograph [GN07].

If $q: E \to B$ is a smooth fiber bundle, then we write $GE := \{ s \in C^\infty(B, E): q \circ s = \text{id}_B \}$ for its space of smooth sections.

If $\mathfrak{g}$ is a topological Lie algebra and $V$ a topological $\mathfrak{g}$-module, we write $(C^\bullet(\mathfrak{g}, V), d_\mathfrak{g})$ for the corresponding Lie algebra complex ([Che48]).
1 Lie group structures on mapping groups and automorphism groups of bundles

Before we turn to extensions, we collect in this short section some results on Lie group structures on gauge groups and automorphism group of bundles and introduce notation and conventions used below. Some of the material is classical, but there are also some new interesting results.

1.1 Automorphism groups of bundles

Theorem 1.1. Let $M$ be a compact manifold and $K$ a Lie group with Lie algebra $\mathfrak{k}$. Then the following assertions hold:

1. $\text{Diff}(M)$ carries the structure of a Fréchet–Lie group whose Lie algebra is the space $\mathcal{V}(M)$ of smooth vector fields on $M$.

2. $G := \text{C}^{\infty}(M,K)$ carries a natural Lie group structure with Lie algebra $\mathfrak{g} := \text{C}^{\infty}(M,\mathfrak{k})$, endowed with the pointwise bracket. If $(\varphi,U)$ is a $\mathfrak{k}$-chart of $K$, then $(\varphi G, U G)$ with $U G := \{ f \in G : f(M) \subseteq U \}$ and $\varphi G(f) = \varphi_K \circ f$ is a $\mathfrak{g}$-chart of $G$.

For (1) we refer to [Les67], [Omo70], and [Ha82], and for (2) we refer to [Mil84] and [Mi80] (see also [GN07] for all that).

Theorem 1.2. If $q : P \to M$ is a smooth principal bundle over a compact manifold $M$ with locally exponential structure group $K$, then its gauge group $\text{Gau}(P)$ and its automorphism group $\text{Aut}(P)$ carry Lie group structures turning (1) into a Lie group extension, where the group $\text{Diff}(M)_P$ is open in $\text{Diff}(M)$. In particular, the conjugation action of $\text{Aut}(P)$ on $\text{Gau}(P)$ is smooth.

For the case where $K$ is finite-dimensional this can be found in [KYMO85], [ACM89] and [KM97]. For general locally exponential Lie groups $K$, this has recently been proved by Ch. Wockel [Wo06].

Remark 1.3. (a) If $P = M \times K$ is a trivial bundle, the group extension $\text{Aut}(P)$ splits and we have

$$\text{Aut}(P) \cong \text{Gau}(P) \times \text{Diff}(M) \quad \text{where} \quad \text{Gau}(P) \cong C^\infty(M,K).$$

Here $\text{Diff}(M)$ acts on $P$ by $\varphi.(m,k) := (\varphi(m),k)$ and $C^\infty(M,K)$ acts by $f.(m,k) = (m,f(m)k)$.

(b) For any principal $K$-bundle $P$, each gauge transformation $\varphi \in \text{Gau}(P)$ determines a unique smooth function $f : P \to K$ by $\varphi(p) = p.f(p)$ for $p \in P$. Then $f$ satisfies $f(p.k) = k^{-1}f(p.k)$, and any such function corresponds to a gauge transformation $\varphi_f$, so that

$$\text{Gau}(P) \cong \{ f \in C^\infty(P,K) : (\forall p \in P)(\forall k \in K) f(p.k) = k^{-1}f(p.k) \}.$$
Accordingly, we have on the level of vector fields
\[
gau(P) \cong \{ \xi \in C^\infty(P, \mathfrak{g}): (\forall p \in P)(\forall k \in K) \ xi(p,k) = \text{Ad}(k)^{-1} \xi(p) \}.
\]

(c) For any connected manifold \(M\), the universal covering map \(q_M: \tilde{M} \to M\) defines on \(\tilde{M}\) the structure of a \(\pi_1(M)\)-principal bundle, where \(\pi_1(M)\) denotes the group of deck transformations of this bundle, considered as a discrete group. Then
\[
\text{Gau}(\tilde{M}) = Z(\pi_1(M)) \quad \text{and} \quad \text{Aut}(\tilde{M}) = C_{\text{Diff}(\tilde{M})}(\pi_1(M)).
\]

If \(M\) is compact, then \(\text{Aut}(\tilde{M})\) carries a natural Lie group structure turning it into a covering of the open subgroup \(\text{Diff}(\tilde{M})/\tilde{M}\) of \(\text{Diff}(M)\) with kernel \(Z(\pi_1(M))\). The normalizer \(\text{Diff}(M)\) of \(\pi_1(M)\) in \(\text{Diff}(\tilde{M})\) carries a Lie group structure that leads to a short exact sequence
\[
1 \to \pi_1(M) \to \tilde{\text{Diff}}(M) \to \text{Diff}(M) \to 1.
\]

This is due to the fact that each diffeomorphism \(\varphi\) of \(M\) lifts to some diffeomorphism of \(\tilde{M}\) normalizing \(\pi_1(M)\), but in general it does not centralize \(\pi_1(M)\).

(d) Let \(\rho: \pi_1(M) \to K\) be a homomorphism and
\[
P_\rho := \tilde{M} \times_\rho K := (\tilde{M} \times K)/\pi_1(M)
\]
be the corresponding flat \(K\)-bundle associated to the \(\pi_1(M)\)-bundle \(\tilde{M}\) via \(\rho\). Then \(P_\rho\) is the set of \(\pi_1(M)\)-orbits in \(\tilde{M} \times K\) for the right action \((\tilde{m}, \gamma, k) : (\tilde{m}, \gamma, k) := (\tilde{m}, \gamma, \rho(\gamma)^{-1} k)\). We write \([(\tilde{m}, k)]\) for the orbit of the pair \((\tilde{m}, k)\). Then \(s: \tilde{M} \to P, s(\tilde{m}) := [(\tilde{m}, 1)]\) is a smooth function with \(q \circ s = q_\tilde{M}\), satisfying
\[
s(\tilde{m}, \gamma) = s(\tilde{m}), \rho(\gamma) \quad \text{for} \quad \tilde{m} \in \tilde{M}, \gamma \in \pi_1(M).
\]

In view of \(P = s(\tilde{M}), K\), any bundle automorphism \(\tilde{\varphi}\) is determined by its values on the image of \(s\). If \(\varphi \in \text{Diff}(M)\) is the corresponding diffeomorphism of \(M\) and \(\tilde{\varphi}\) is a lift of \(\varphi\) to a diffeomorphism of \(\tilde{M}\), then \(\tilde{\varphi}\) can be written
\[
\tilde{\varphi}(s(\tilde{m})) = s(\tilde{\varphi}(\tilde{m})). f(\tilde{m})
\]
for some smooth function \(f: \tilde{M} \to K\), satisfying
\[
f(\tilde{m}) \rho(\gamma) = \rho(\tilde{\varphi}(\gamma)^{-1}) f(\tilde{m}) \quad \text{for} \quad \tilde{m} \in \tilde{M}, \gamma \in \pi_1(M).
\]

Conversely, if (4) it satisfied, then (3) defines an automorphism of \(P\). We thus obtain a gauge transformation if and only if \(\varphi = \text{id}_M\), which leads to
\[
\text{Gau}(P) \cong \{ f \in C^\infty(\tilde{M}, K): (\forall \gamma \in \pi_1(M)) f \circ \gamma = e^{-1}_f \circ f \},
\]
where \( c_k(h) := khk^{-1} \). The group \( \text{Aut}(\tilde{M}) \) defined under (c) acts naturally on \( P \), preserving \( s(\tilde{M}) \), via \( \tilde{\varphi}(s(\tilde{m})) := s(\tilde{\varphi}(\tilde{m})) \). An element of this group induces the identity on \( M \) if and only if it comes from some \( \gamma \in Z(\pi_1(M)) \), and this corresponds to the constant function \( \tilde{M} \to K \) with the value \( \rho(\gamma) \).

We thus obtain an open subgroup of \( \text{Aut}(P) \) as the quotient

\[
(Gau(P) \rtimes \text{Aut}(\tilde{M}))/Z(\pi_1(M)).
\]

Accordingly, the horizontal lift \( X \mapsto \tilde{X} \) of vector fields on \( M \) to vector fields on \( P \) yields an isomorphism

\[
\text{aut}(P) \cong \text{gau}(P) \rtimes \mathcal{V}(M),
\]

where \( \text{gau}(P) \cong \{ f \in C^\infty(\tilde{M}, \mathfrak{k}): (\forall \gamma \in \pi_1(M)) f \circ \gamma = \text{Ad}(\rho(\gamma))^{-1} \circ f \} \).

### 1.2 Mapping groups on non-compact manifolds

For a non-compact smooth manifold \( M \), the above constructions of the atlas on \( C^\infty(M, K) \) does no longer work because the sets \( U_G \) are not open. However, it turns out that in some interesting cases there are other ways to obtain charts. To formulate the results, we say that a Lie group structure on \( G = C^\infty(M, K) \) is compatible with evaluations if one of the following conditions hold:

1. The universal covering group \( \tilde{K} \) of \( K \) is diffeomorphic to a locally convex space; which is the case if \( K \) is finite-dimensional solvable. If, in addition, \( \pi_1(M) \) is finitely generated, the Lie group structure is compatible with the smooth compact open topology.
2. \( \dim M = 1 \).
3. \( M \cong \mathbb{R}^k \times C \), where \( C \) is compact.

For complex groups we have (cf. [NeWa07], Thms. III.12, IV.3):

**Theorem 1.4.** Let \( K \) be a connected regular real Lie group and \( M \) a real finite-dimensional connected manifold (with boundary). Then the group \( C^\infty(M, K) \) carries a Lie group structure compatible with evaluations if one of the following conditions hold:

1. The universal covering group \( \tilde{K} \) of \( K \) is diffeomorphic to a locally convex space; which is the case if \( K \) is finite-dimensional solvable. If, in addition, \( \pi_1(M) \) is finitely generated, the Lie group structure is compatible with the smooth compact open topology.
2. \( \dim M = 1 \).
3. \( M \cong \mathbb{R}^k \times C \), where \( C \) is compact.

**Theorem 1.5.** Let \( K \) be a connected regular complex Lie group and \( M \) a finite-dimensional connected complex manifold without boundary. Then the group \( \mathcal{O}(M, K) \), endowed with the compact open topology, carries a Lie group structure with Lie algebra \( \mathcal{O}(M, \mathfrak{k}) \) compatible with evaluations if

1. \( \tilde{K} \) is diffeomorphic to a locally convex space. If, in addition, \( \pi_1(M) \) is finitely generated, the Lie group structure is compatible with the compact open topology.
Problem 1.6. (1) Do the results of Section 3 below on central extensions of mapping group $C^\infty(M,K)$ extend to corresponding groups of smooth and holomorphic maps on non-compact manifolds whenever Theorem 1.4 or Theorem 1.5 provide a suitable Lie group structure? For corresponding results on groups of compactly supported maps we refer to [Ne04b].

(2) Find an example of a connected non-compact smooth manifold $M$ and a finite-dimensional Lie group $K$ for which $C^\infty(M,K)$ does not carry a Lie group structure compatible with evaluations. The first candidate causing problems is the compact group $K = SU_2(C)$ and $M$ should be a manifold that is not a product of some $R^n$ with a compact manifold (cf. Theorem 1.4(3)); which excludes all Lie groups and all Riemannian symmetric spaces.

(3) Do the preceding theorems generalize in a natural way to gauge groups? Which restrictions does one have to impose on the bundles?

2 Central extensions of mapping groups

In this section we discuss several issues concerning central extensions of mapping groups. We start with a description of the fundamental types of 2-cocycles on the Lie algebra $\mathfrak{g} := C^\infty(M,\mathfrak{k})$, where $M$ is a finite-dimensional smooth manifold and $\mathfrak{k}$ a finite-dimensional Lie algebra over $K \in \{R, C\}$. These cocycles have values in spaces like $C^\infty(M,V)$, $\Omega^1(M,V)$ and $\Omega^1(M,V) = \Omega^1(M,V)/dC^\infty(M,V)$. In particular we describe what is known about their integrability to corresponding central Lie group extensions. This is best understood for the identity component $C^\infty(M,K)_0$, but the extendibility to all of $C^\infty(M,K)$ is not clear (cf. Problem 7.5 below). In Section 4 below we outline some generalizations of the results in the present section to gauge groups of non-trivial bundles.

2.1 Central extensions of $C^\infty(M,\mathfrak{k})$

We write $[\alpha] \in \Omega^2(M,V)$ for the image of a 1-form $\alpha \in \Omega^1(M,V)$ in this space. The subspace $dC^\infty(M,V)$ of exact $V$-valued 1-forms is characterized by the vanishing of all integrals over loops in $M$, hence a closed subspace, and therefore $\Omega^1(M,V)$ carries a natural Hausdorff locally convex topology.

For a trivial $\mathfrak{k}$-module $V$, we write $\text{Sym}^2(\mathfrak{k}, V)^\mathfrak{k}$ for the space of $V$-valued symmetric invariant bilinear forms, and recall the Cartan map

$$\Gamma: \text{Sym}^2(\mathfrak{k}, V)^\mathfrak{k} \to \Omega^2(\mathfrak{k}, V), \quad \Gamma(\kappa)(x, y, z) := \kappa([x, y], z).$$

We call $\kappa$ exact if $\Gamma(\kappa)$ is a coboundary.

**Theorem 2.1.** Each continuous 2-cocycle with values in the trivial $\mathfrak{g}$-module $K$ is a sum of cocycles that factor through a cocycle of one of the following types:
(I) Cocycles \( \omega_\kappa \in Z^2(\mathfrak{g}, \mathfrak{t}^1(M, V)) \) of the form

\[
\omega_\kappa(\xi_1, \xi_2) = [\kappa(\xi_1, d\xi_2)], \quad \kappa \in \text{Sym}^2(\mathfrak{t}, V)^t,
\]

where \( \kappa(\xi_1, d\xi_2) \) is considered as a \( V \)-valued 1-form on \( M \).

(II) Cocycles \( \omega_\eta \in Z^2(\mathfrak{g}, C^\infty(M, V)) \) of the form

\[
\omega_\eta(\xi_1, \xi_2) = \eta(\xi_1, \xi_2) := \eta(\xi_1, \xi_2), \quad \eta \in Z^2(\mathfrak{t}, V).
\]

(III) Cocycles \( \omega_{\kappa, \eta} \in Z^2(\mathfrak{g}, \Omega^1(M, V)) \) of the form

\[
\omega_{\kappa, \eta}(\xi_1, \xi_2) = \kappa(\xi_1, d\xi_2) - \kappa(\xi_2, d\xi_1) - d(\eta(\xi_1, \xi_2)),
\]

where \( \eta \in C^2(\mathfrak{t}, V) \) and \( \kappa \in \text{Sym}^2(\mathfrak{t}, V)^t \) are related by \( \Gamma(\kappa) = d_\mathfrak{t} \eta \).

(IV) Cocycles vanishing on \( \mathfrak{g}' = C^\infty(M, \mathfrak{t}') \) is the commutator algebra, i.e., pull-backs of cocycles of an abelian quotient of \( \mathfrak{g} \).

Proof. First we apply \( \text{NeWa00} \) (Lemma 1.1, Thm. 3.1, Cor. 3.5 and Section 6) to \( \mathfrak{g} = C^\infty(M, \mathfrak{k}) \otimes \mathfrak{t} \) to see that after subtracting a cocycle of type (IV), each continuous cocycle \( \omega \in Z^2(\mathfrak{g}, \mathfrak{k}) \) can be written as

\[
\omega(f \otimes x, g \otimes y) = \beta_\alpha(f dg - g df)(x, y) - \beta_s(f g)(x, y), \quad (5)
\]

where

\[
\beta_\alpha: \Omega^1(M, \mathfrak{k}) \rightarrow \text{Sym}^2(\mathfrak{t}, \mathfrak{k})^t \quad \text{and} \quad \beta_s: C^\infty(M, \mathfrak{k}) \rightarrow C^2(\mathfrak{t}, \mathfrak{k})
\]

are continuous linear maps coupled by the condition

\[
\Gamma(\beta_\alpha(df)) = d_\mathfrak{t}(\beta_s(f)) \quad \text{for} \quad f \in C^\infty(M, \mathfrak{k}). \quad (6)
\]

Conversely, any pair \( (\beta_\alpha, \beta_s) \), satisfying (6) defines a cocycle via (5):

\[
(d_\mathfrak{g} \omega)(f \otimes x, f' \otimes x', f'' \otimes x'') = -\sum_{\text{cyc}} \omega(f f' \otimes [x, x'], f'' \otimes x'')
\]

\[
= -\sum_{\text{cyc}} \beta_\alpha(f f' df'' - f'' df(f'))([x, x'], x'') + \sum_{\text{cyc}} \beta_s(f f' f'')(x, x', x'')
\]

\[
= \Gamma(\beta_\alpha(df f f''))(x, x', x'') - d_\mathfrak{t}(\beta_s(f f' f''))(x, x', x'').
\]

Let \( V(\mathfrak{t}) := S^2(\mathfrak{t})/\mathfrak{t} S^2(\mathfrak{t}) \) be the target space of the universal invariant symmetric bilinear form \( \kappa_\alpha(x, y) := [x \vee y] \) and observe that \( \text{Sym}^2(\mathfrak{t}, \mathfrak{k})^t \cong V(\mathfrak{t})^* \). We write \( \text{Sym}^2(\mathfrak{t}, \mathfrak{k})^t_{\text{ex}} \), for the subspace of \( \text{Sym}^2(\mathfrak{t}, \mathfrak{k})^t \) consisting of exact forms, and \( V(\mathfrak{t})_{\text{ex}} \subseteq V(\mathfrak{t})^* \) for its annihilator. Then there exists a linear map

\[
\chi: \text{Sym}^2(\mathfrak{t}, \mathfrak{k})^t_{\text{ex}} \rightarrow C^2(\mathfrak{t}, \mathfrak{k}) \quad \text{with} \quad d_\mathfrak{t}(\chi(\beta)) = \Gamma(\beta).
\]

As \( \beta_\alpha(dC^\infty(M, \mathfrak{k})) \subseteq \text{Sym}^2(\mathfrak{t}, \mathfrak{k})^t_{\text{ex}} \), we may use the Hahn–Banach Extension Theorem to extend \( \beta_\alpha \) from \( dC^\infty(M, \mathfrak{k}) \) to a continuous linear map

\[
\beta_\alpha: \Omega^1(M, \mathfrak{k}) \rightarrow \text{Sym}^2(\mathfrak{t}, \mathfrak{k})^t_{\text{ex}}
\]

and put \( \beta_s(f) := \chi(\beta_\alpha(df)) \), so that
\[ \tilde{\omega}(f \otimes x, f' \otimes x') := \tilde{\beta}_a(f df' - f' df)(x, x') - \tilde{\beta}_s(ff')(x, x') \]

is a 2-cocycle. Further, \( \tilde{\beta}_a := \beta_a - \tilde{\beta}_a \) vanishes on \( \mathcal{d}C^\infty(M, \mathbb{K}) \), so that the values of \( \tilde{\beta}_s := \beta_s - \tilde{\beta}_s \) are cocycles. Hence both pairs \((\tilde{\beta}_a, 0), \) resp., \((0, -\tilde{\beta}_s)\) define cocycles, \( \omega_2, \) resp., \( \omega_3, \) satisfying
\[
\omega = \tilde{\omega} + \omega_2 - \omega_3.
\]

We now show that each of these summands factors through a cocycle of the form (I)-(III). For \( \omega_2 \) we put \( V := V(t) \), so that we may write
\[
\omega_2(f \otimes x, f' \otimes x') = \tilde{\beta}_a(f df' - f' df)(x, x') = \tilde{\beta}_a(df - f' df \otimes \kappa_a(x, x')),
\]
where \( \tilde{\beta}_a : \Omega^1(M, V) \cong \Omega^1(M, \mathbb{K}) \otimes V \to \mathbb{K} \) is a continuous linear map vanishing on \( \mathcal{d}C^\infty(M, V) \). This shows that \( \omega_2 \) factors through a cocycle of type (I) with values in \( \tilde{\Omega}^1(M, V) \).

For \( \omega_3 \) we put \( V := \mathcal{Z}^2(t, \mathbb{K})^* \), and write \( \eta_u \in \mathcal{Z}^2(t, V) \) for the 2-cocycle defined by \( \eta_u(x, x)(f) = f(x, x') \) for \( f \in \mathcal{Z}^2(t, \mathbb{K}) \). Then
\[
\omega_3(f \otimes x, f' \otimes x') = \tilde{\beta}_s(f f')(x, x') = \eta_u(x, x')(\tilde{\beta}_s(ff')) = \tilde{\beta}_s(ff' \otimes \eta_u(x, x')),
\]
where \( \tilde{\beta}_s : \mathcal{C}^\infty(M, V) \cong \mathcal{C}^\infty(M, \mathbb{K}) \otimes V \to \mathbb{K} \) is a continuous linear map. Hence \( \omega_3 \) factors through a cocycle of type (II) with \( \eta = \eta_u \).

Finally, we turn to \( \tilde{\omega} \). Now we put \( V := V(t)/V(t)_0, \) so that \( V^* \cong \text{Sym}^2(t, \mathbb{K}) \) and write \( \kappa : t \times t \to V \) for the symmetric bilinear map obtained from \( \kappa_u \). We note that \( \chi : V^* \to \mathcal{C}^2(t, V) \) defines a map \( \chi^2 \in \mathcal{C}^2(t, V) \) by
\[
\chi(x)(x, y) = \chi^2(x, y) \quad \text{for} \quad x, y \in t, \lambda \in V^*.
\]
For each \( \lambda \in V^* \) we then identify \( \lambda \) with the corresponding bilinear form \( \lambda \circ \kappa \) and obtain
\[
\lambda \circ d_t(\chi^2) = d_t(\lambda \circ \chi^2) = d_t(\chi(\lambda)) = \Gamma(\lambda) = \Gamma(\lambda \circ \kappa) = \lambda \circ \Gamma(\kappa),
\]
showing that \( d_t(\chi^2) = \Gamma(\kappa) \).

Let \( \overline{\beta}_a : \Omega^1(M, V) \cong \Omega^1(M, \mathbb{K}) \otimes V \to \mathbb{K} \) be the continuous linear functional defined by \( \overline{\beta}_a(\alpha \otimes v) = \tilde{\beta}_a(\alpha)(v) \). Then
\[
\overline{\beta}_a(\kappa(f \otimes x, df \otimes x')) - \kappa(f' \otimes x', df \otimes x) - d(\chi^2(f \otimes x, f' \otimes x'))
\]
\[
= \overline{\beta}_a((df - f' df)(x, x'))\kappa(x, x') - d(\chi^2(x, x'))
\]
\[
= \overline{\beta}_a(df - f' df)(x, x') - \overline{\beta}_a(\kappa(df'))(x, x')
\]
\[
= \overline{\beta}_a(df - f' df)(x, x') - \chi(\overline{\beta}_a(df'))(x, x')
\]
\[
= \overline{\beta}_a(df - f' df)(x, x') - \overline{\beta}_a(ff')(x, x') = \tilde{\omega}(f \otimes x, f' \otimes x').
\]
This completes the proof. \( \square \)
Remark 2.2. (a) The central extension \( \hat{\mathfrak{g}} := C^\infty(M,V) \oplus \omega_\eta \mathfrak{g} \) defined by a cocycle \( \omega_\eta \) of type (II) can also be described more directly: If \( \mathfrak{k} = V \oplus \eta \mathfrak{g} \) is the central extension defined by \( \eta \), then \( \hat{\mathfrak{g}} \cong C^\infty(M,\mathfrak{k}) \oplus \mathfrak{k} \), i.e., the \( C^\infty(M,\mathbb{K}) \)-Lie algebra \( \hat{\mathfrak{g}} \) is obtained from \( \mathfrak{k} \) by extension of scalars from \( \mathbb{K} \) to the ring \( C^\infty(M,\mathbb{K}) \).

(b) For any \( \hat{\eta} \in C^\infty(M,Z^2(\mathfrak{t},V)) \),

\[
\omega_{\hat{\eta}}(\xi_1,\xi_2)(m) := \hat{\eta}(m)(\xi_1(m),\xi_2(m))
\]

also defines a continuous \( C^\infty(M,V) \)-valued 2-cocycle on \( \mathfrak{g} \). To see how these cocycles fit into the scheme of the preceding theorem, we observe that \( Z^2(\mathfrak{t},V) \) is finite-dimensional (if \( \mathfrak{k} \) and \( V \) are finite-dimensional), so that \( \hat{\eta} = \sum_{i=1}^n \alpha_i \eta_i \) for some \( \alpha_i \in C^\infty(M,\mathbb{R}) \) and \( \eta_i \in Z^2(\mathfrak{k},V) \). This leads to \( \omega_{\hat{\eta}} = \sum_i \alpha_i \omega_{\eta_i} \), showing that \( \omega_{\hat{\eta}} \) factors through the cocycle \( \omega(\eta_1,...,\eta_n) \) with values in \( C^\infty(M,V)^n \cong C^\infty(M,V^n) \).

All the cocycles \( \omega_{\hat{\eta}} \) are \( C^\infty(M,\mathbb{R}) \)-bilinear and if such a cocycle is a coboundary, there exists a continuous linear map \( \beta: \mathfrak{g} \to C^\infty(M,V) \) with \( \omega_{\hat{\eta}}(\xi_1,\xi_2) = \beta([\xi_1,\xi_2]) \) for \( \xi_1,\xi_2 \in \mathfrak{g} \). This relation easily implies that the restriction of \( \beta \) to the commutator algebra \( \mathfrak{g}' = C^\infty(M,\mathfrak{t}') \) is \( C^\infty(M,\mathbb{R}) \)-bilinear so that we may w.l.o.g. assume that \( \beta \) itself has this property. Then there exists a smooth map \( \beta: M \to \text{Hom}(\mathfrak{t},V) \) with \( \beta(\xi)(m) = \beta(m)(\xi(m)) \) and we obtain

\[
d_\mathfrak{t}(\beta(m)) = \hat{\eta}(m) \quad \text{for} \quad m \in M.
\]

Thus \( \omega_{\hat{\eta}} \) is a coboundary if and only if \( \hat{\eta}(M) \) consists of coboundaries. In particular, \( [\omega_{\eta}] = 0 \) is equivalent to \( [\eta] = 0 \) if \( \hat{\eta} = \eta \) is constant, which corresponds to cocycles of type (II).

(c) Each cocycle \( \omega_{\kappa,\eta} \) of type (III) may be composed with the quotient map \( q: \Omega^1(M,V) \to \Omega^1(M,V) \), which leads to the cocycle \( q \circ \omega_{\kappa,\eta} = 2\omega_{\kappa} \) of type (I). Here the exactness of \( \kappa \) ensures the existence of a lift of \( \omega_{\kappa} \) to an \( \Omega^1(M,V) \)-valued cocycle. If \( \hat{\eta} \in C^2(\mathfrak{t},V) \) also satisfies \( d_\mathfrak{t}\hat{\eta} = \Gamma(\kappa) \), then \( \hat{\eta} - \eta \in Z^2(\mathfrak{t},V) \) and

\[
\omega_{\kappa,\hat{\eta}} = \omega_{\kappa,\eta} + d \circ \omega_{\hat{\eta} - \eta}
\]

is another lift of \( 2\omega_{\kappa} \).

(d) If \( \mathfrak{t} \) is a semisimple Lie algebra and \( \kappa: \mathfrak{t} \times \mathfrak{t} \to V(\mathfrak{t}) \) is the universal invariant symmetric bilinear form, then the corresponding cocycle \( \omega_\kappa \) with values in \( \Omega^1(M,V(\mathfrak{t})) \) is universal, i.e., up to coboundaries each 2-cocycle with values in a trivial module can be written as \( f \circ \omega_\kappa \) for a continuous linear map \( f: \Omega^1(M,V(\mathfrak{t})) \to V \) (cf. [Ma02], [PS86]).

(e) Cocycles of type (III) exist if and only if \( \mathfrak{t} \) carries an exact invariant symmetric bilinear form \( \kappa \) with \( \Gamma(\kappa) \neq 0 \). Typical examples of Lie algebras with this property are cotangent bundles \( \mathfrak{t} = T^*\mathfrak{h} := \mathfrak{h}^* \times \mathfrak{h} \) with \( \kappa((\alpha,\mathfrak{h}),(\alpha',\mathfrak{h}')) := \alpha(\mathfrak{h}') + \alpha'(\mathfrak{h}) \) and \( \eta((\alpha,\mathfrak{h}),(\alpha',\mathfrak{h}')) := \alpha'(\mathfrak{h}) - \alpha(\mathfrak{h}') \) (cf. [NeWa06], Example 5.3).
(f) Cocycles of the types (I)-(III) can also be defined if \( k \) is an infinite-dimensional locally convex Lie algebra. In this case we require \( \kappa \) and \( \eta \) to be continuous. The only point where we have used the finite dimension of \( k \) in the proof of Theorem 2.1 is to show that every \( K \)-valued cocycle is a sum of cocycles factoring through a cocycle of type (I)-(IV); the corresponding arguments do not carry over to the infinite dimensional case.

**Remark 2.3.** Many results concerning cocycles of type (I)-(III) can be reduced from general manifolds to simpler ones as follows.

For cocycles of type (I), the integration maps \( \int_\gamma: \mathcal{T}^t(M,V) \to V, \gamma \in C^\infty(S^1,M) \), separate the points. Accordingly, we have pull-back homomorphisms of Lie groups \( \gamma^*: C^\infty(M,K) \to C^\infty(S^1,K) \), \( f \mapsto f \circ \gamma \) satisfying
\[
\int_\gamma \omega^M_\kappa = \int_{S^1} \omega^{S^1}_\kappa \circ (L(\gamma^*) \times L(\gamma^*)), \tag{8}
\]
which can be used to reduce many things to \( M = S^1 \) (cf. [MN03]).

For cocycles of type (II), the evaluation maps \( \text{ev}^V_m: C^\infty(M,V) \to V, f \mapsto f(m) \) separate the points and the corresponding evaluation homomorphism of Lie groups \( \text{ev}^K_m: C^\infty(M,K) \to K \), \( f \mapsto f(m) \), satisfies
\[
\text{ev}^V_m \circ \eta = \eta \circ (L(\text{ev}^K_m) \times L(\text{ev}^K_m)) = \eta \circ (\text{ev}^V_m \times \text{ev}^V_m). \tag{9}
\]

Finally, we observe that for cocycles of type (III), the integration maps \( \int_\gamma: \Omega^1(M,V) \to V, \gamma \in C^\infty(I,M) \), separate the points and that the pull-back homomorphism \( \gamma^*: C^\infty(M,K) \to C^\infty(I,K) \) satisfies
\[
\int_\gamma \omega^M_{\kappa,\eta} = \int_I \omega^I_{\kappa,\eta} \circ (L(\gamma^*) \times L(\gamma^*)). \tag{10}
\]

### 2.2 Covariance of the Lie algebra cocycles

In this subsection we discuss the covariance of the cocycles of type (I)-(III) under \( \text{Diff}(M) \) and \( C^\infty(M,K) \), i.e., the existence of an actions of these groups on the centrally extended Lie algebras \( \mathfrak{g} = \mathfrak{z} \oplus \omega \mathfrak{g} \), compatible with the actions on \( \mathfrak{g} \) and \( \mathfrak{z} \). For \( \text{Diff}(M) \), the situation is simple because the cocycles are invariant, but for \( C^\infty(M,K) \) several problems arise. For the identity component \( C^\infty(M,K)_0 \), the existence of a corresponding action on \( \mathfrak{g} \) is equivalent to the vanishing of the flux homomorphism (Proposition 7.1), but for the existence of an extension to the full group there is no such simple criterion.

**Remark 2.4. (Covariance under \( \text{Diff}(M) \))** (a) The Lie algebra \( \mathcal{V}(M) \) acts in a natural way on all spaces \( \Omega^p(M,V) \) and \( \mathcal{T}^p(M,V) \) and it acts on the Lie
algebra \( \mathfrak{g} = C^\infty(M,\mathfrak{k}) \) by derivations. With respect to this action, all cocycles \( \omega_\kappa, \omega_\eta \) and \( \omega_{\kappa,\eta} \) are \( \mathcal{V}(M) \)-invariant, hence extend by

\[ \overline{\omega}(f, X), (f', X') := \omega(f, f') \]
to cocycles of the semidirect sum \( \mathfrak{g} \rtimes \mathcal{V}(M) \).

(b) These cocycles are invariant under the full diffeomorphism group \( \text{Diff}(M) \), which implies that the diagonal action of this group on the corresponding central extension defined by \( \varphi(z, f) := ((\varphi^{-1})^*z, \varphi(f) \) is a smooth action by Lie algebra automorphisms \([MN03]\), Thm. VI.3).

Remark 2.5. (Covariance under \( C^\infty(M, \mathfrak{k}) \)) (a) For \( \omega_\kappa \in Z^2(\mathfrak{g}, \Omega^1(M, \mathfrak{v})) \) of type (I) we also have a natural action of the full group \( C^\infty(M, \mathfrak{k}) \) on the extended Lie algebra \( \hat{\mathfrak{g}} = \mathfrak{v} \oplus \omega_\kappa \mathfrak{g} \) \([MN03]\), Thm. VI.3). If \( \delta(f) := \delta^i(f) \) denotes the left logarithmic derivative of \( f \), then \( \theta(f)(x) := [\kappa(\delta(f), x)] \) defines a linear map \( \mathfrak{g} \rightarrow \mathfrak{v} \), satisfying

\[ \text{Ad}(f)^*\omega_\kappa - \omega_\kappa = d_\mathfrak{g}(\theta(f)) \]
so that

\[ \text{Ad}_\mathfrak{g}(f).(z, \xi) := (z - \theta(f)(\xi), \text{Ad}(f)\xi) \]
defines an automorphism of \( \hat{\mathfrak{g}} \) (Lemma 7.6), and we thus obtain a smooth action of the Lie group \( C^\infty(M, \mathfrak{k}) \) on \( \hat{\mathfrak{g}} \).

(b) Let \( \mathfrak{k} \) be a connected Lie group with Lie algebra \( \mathfrak{k} \), \( V \) a trivial \( \mathfrak{k} \)-module and \( \eta \in Z^2(\mathfrak{k}, V) \) be a 2-cocycle. For the cocycle \( \omega_\eta \in Z^2(\mathfrak{g}, C^\infty(M, V)) \) of type (II), the situation is slightly more complicated. We recall from Proposition 7.1 below that the action of \( \mathfrak{k} \) on \( \mathfrak{v} \) lifts to an action on \( \hat{\mathfrak{v}} = \mathfrak{v} \oplus_\eta \mathfrak{k} \) if and only if the flux

\[ F_\eta: \pi_1(K) \rightarrow \text{Hom}_{\text{Lie}}(\mathfrak{t}, V) = H^2(\mathfrak{t}, V) \]
vanesishes.

Let \( m_0 \in M \) be a base point and \( C^\infty(M, \mathfrak{k}) \leq C^\infty(M, K) \) be the normal subgroup of all maps vanishing in \( m_0 \). Then this also is a Lie group, and

\[ C^\infty(M, K) \cong C^\infty_*(M, K) \rtimes K \]
as Lie groups. If \( q_K: \widetilde{K} \rightarrow K \) denotes the universal covering group, then a smooth map \( f: M \rightarrow K \) vanishing in \( m_0 \) lifts to a map \( M \rightarrow \widetilde{K} \) if and only if the induced homomorphism \( \pi_1(f): \pi_1(M) \rightarrow \pi_1(K) \) vanishes. We therefore have an exact sequence of groups

\[ 1 \rightarrow C^\infty_*(M, \widetilde{K}) \rightarrow C^\infty_*(M, K) \rightarrow \text{Hom}(\pi_1(M), \pi_1(K)), \tag{11} \]
and since \( \pi_1(M) \) is finitely generated because \( M \) is compact, we may consider \( C^\infty_*(M, \widetilde{K}) \) as an open subgroup of \( C^\infty_*(M, K) \). From the canonical action \( \text{Ad}_\mathfrak{v} \)
of $\tilde{K}$ on $\hat{\mathfrak{f}}$ we immediately get a smooth action of $C^\infty(M,\tilde{K})$ on $\hat{\mathfrak{g}} \cong C^\infty(M,\hat{\mathfrak{f}})$ by

$$(\text{Ad}_\hat{\theta}(f)\xi)(m) := \text{Ad}_\hat{\theta}(f(m))\xi(m).$$

This shows that the flux

$$F_{\omega_\eta} : \pi_1(C^\infty(M,K)) \cong \pi_1(C^\infty_*(M,K)) \times \pi_1(K) \to H^1(\mathfrak{g},C^\infty(M,V))$$

vanishes on the subgroup $\pi_1(C^\infty_*(M,K))$, hence factors through the flux $F_{\eta}$ of $\eta$.

For any $f \in C^\infty(M,K)$ and $m \in M$, the connectedness of $K$ implies that $\text{Ad}(f(m))^*\eta - \eta$ is a 2-coboundary (Theorem 9.1), and from that we derive the existence of a smooth map $\hat{\theta} : M \to C^1(\mathfrak{t},V)$ with $\text{Ad}(f(m)^{-1})^*\eta - \eta = d_\mathfrak{f}(\hat{\theta}(m))$ for $m \in M$. Now the automorphism $\text{Ad}(f)$ lifts to an automorphism of $\tilde{\mathfrak{k}}$ (Lemma 7.6).

(c) To determine the covariance of the cocycles $\omega_{\varsigma,\eta}$ of type (III) under the group $C^\infty(M,K)$, we first observe that for $f \in C^\infty(M,K)$ and $\xi_1,\xi_2 \in \mathfrak{g}$ we obtain similarly as in [MN03], Prop. III.3:

$$(\text{Ad}(f)^*\omega_{\varsigma,\eta})(\xi_1,\xi_2) = \kappa(\text{Ad}(f)\xi_1,d(\text{Ad}(f)\xi_2)) - \kappa(\text{Ad}(f)\xi_2,d(\text{Ad}(f)\xi_1)) - d((\text{Ad}(f)^*\eta)(\xi_1,\xi_2))$$

$$= \kappa(\xi_1,d\xi_2) - \kappa(\xi_2,d\xi_1) + 2\kappa(d^f,[[\xi_2,\xi_1]]) - d((\text{Ad}(f)^*\eta)(\xi_1,\xi_2))$$

$$= \omega_{\varsigma,\eta}(\xi_1,\xi_2) + 2\kappa(d^f,[[\xi_2,\xi_1]]) - d((\text{Ad}(f)^*\eta - \eta)(\xi_1,\xi_2)).$$

It is clear that the term $\kappa(d^f,[[\xi_2,\xi_1]])$ is a Lie algebra coboundary. To see that the second term also has this property, we note that $\Gamma(\kappa) = d_\mathfrak{t}\eta$ implies that for each $x \in \mathfrak{f}$ the cochain $i_x(d\eta)(y,z) = \kappa([x,y],z) = \kappa(x,[y,z])$ is a coboundary. Hence $L_x\eta = i_xd\eta + d(i_x\eta)$ also is a coboundary and since $K$ is connected, for each $m \in M$ the cocycle $\text{Ad}(f(m))\eta - \eta$ is a coboundary. With Lemma 7.6 we now see that for each smooth function $f : M \to K$, the adjoint action on $\mathfrak{g}$ lifts to an automorphism of the central extension $\hat{\mathfrak{g}}$, defined by $\omega_{\varsigma,\eta}$. If $f$ is not in $C^\infty(M,K)_0$, i.e., homotopic to a constant map, this does not follow from Theorem 9.1.

**Problem 2.6.** For which groups $K$ and which connected smooth manifolds $M$ does the short exact sequence

$$1 \to C^\infty(M,K)_0 \to C^\infty(M,K) \to \pi_0(C^\infty(M,K)) \cong [M,K] \to 1$$

split?

If $K \cong \mathfrak{f}/\Gamma_K$ is abelian, then $C^\infty(M,K)_0$ is a quotient of $C^\infty(M,\mathfrak{f})$, hence divisible, and therefore the sequence splits.

If $K$ is non-abelian, the situation is more involved and even the case $M = S^1$ is non-trivial. Then $[M,K] \cong \pi_1(K)$ and if $K$ is abelian, then we obtain a splitting from the isomorphism $\pi_1(K) \cong \text{Hom}(\mathbb{T},K)$. If $K$ is solvable and $T \subset K$ is a maximal torus, then $K \cong \mathbb{R}^m \times T$ as a manifold ([Hof3]), so
that \([M, K] \cong [M, T]\) and we obtain a splitting from the splitting \([M, T] \to C^\infty(M, T) \subseteq C^\infty(M, K)\). The case of compact (semisimple) groups is the crucial one.

If \(M\) is a product of \(d\) spheres (f.i. a \(d\)-dimensional torus), then \([M, K]\) is nilpotent of length \(\leq d\) which is not always abelian ([Whi78], Th. X.3.6).

**Problem 2.7.** Can the automorphisms of the central extensions \(\hat{\mathfrak{g}} = \mathfrak{z} \oplus \mathfrak{g}\) of type (I)—(III) corresponding to elements of \(C^\infty(M, K)\) be chosen in such a way as to define a group action? If \(\mathfrak{t}\) and hence \(\mathfrak{g}\) is perfect, this follows trivially from the uniqueness of lifts to \(\hat{\mathfrak{g}}\). In general, lifts are only unique up to an element of of the group \(H^1(\mathfrak{g}, \mathfrak{z})\), so that only an abelian extension acts. Are the corresponding cohomology classes in \(H^2(C^\infty(M, K), \mathfrak{z})\) arising this way always trivial? Since the action of \(\mathfrak{g}\) on \(\hat{\mathfrak{g}}\) is determined by the cocycle, we obtain an action of the identity component \(C^\infty(M, K)_0\) whenever the flux vanishes, and then the restriction of the cocycle to the identity component is trivial.

Here even the case \(M = S^1\) of loop groups is of particular interest. Then \(\pi_0(C^\infty(S^1, K)) \cong \pi_1(K)\), so that we have a natural inflation map
\[
H^2(\pi_1(K), \mathfrak{z}) \to H^2(C^\infty(S^1, K), \mathfrak{z}),
\]
and it would be nice if all classes under consideration can be determined this way.

If we have a semidirect decomposition \(C^\infty(M, K) \cong C^\infty(M, K)_0 \rtimes [M, K]\), then the present problem simplifies significantly because then semidirect product techniques similar to those discussed in Appendix B for Lie algebras apply.

**Problem 2.8.** Generalize the description of \(H^2(A \otimes \mathfrak{k}, \mathbb{K})\) obtained in [NeWa06] under the assumption that \(A\) is unital to non-unital commutative associative algebras. This would be of particular interest for algebras of the form \(C^\infty(M, \mathbb{R})\) (functions vanishing in one point) or the algebra \(C^\infty_c(M, \mathbb{R})\) of compactly supported functions.

If \(A_+ := A \oplus \mathbb{K}\) is the algebra with unit \(1 = (0, 1)\), then we have
\[
A_+ \otimes \mathfrak{t} \cong (A \otimes \mathfrak{t}) \rtimes \mathfrak{t},
\]
so that we may use the semidirect product techniques in Appendix B below.

### 2.3 Corresponding Lie group extensions

It is a natural question to which extent the cocycles of the form \(\omega_\kappa, \omega_\eta\) and \(\omega_{\kappa, \eta}\) on \(\mathfrak{g} = C^\infty(M, \mathfrak{t})\) actually define central extensions of the corresponding group \(C^\infty(M, K)\).

Cocycles of the form \(\omega_\kappa\) for a vector-valued \(\kappa: \mathfrak{t} \times \mathfrak{t} \to V\) are treated in [MN03], where it is shown that the corresponding period homomorphism
\[
\text{per}_{\omega_\kappa}: \pi_2(C^\infty(M, K)) \to \overline{\text{pi}^1}(M, V)
\]
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(cf. Appendix A) has values in the subspace $H^1_{dR}(M,V)$, and the image consists of cohomology classes whose integrals over circles take values in the image of the homomorphism

$$\text{per}_\kappa: \pi_3(K) \to V, \quad [\sigma] \mapsto \int_\sigma \Gamma(\kappa)^{eq},$$

where $\Gamma(\kappa)^{eq} \in \Omega^3(K,V)$ is the left invariant closed 3-form whose value in 1 is $\Gamma(\kappa)$. The flux homomorphism

$$F_\omega: \pi_1(C^\infty(M,K)) \to H^1(g,\Omega^1(M,V))$$

vanishes because the action of $g$ on the central extension $\hat{g}$ defined by $\omega_\kappa$ integrates to a smooth action of the group $C^\infty(M,K)$ on $\hat{g}$ (Remark 2.5(a), [MN03], Prop. III.3). This leads to the following theorem ([MN03], Thms. I.6, II.9, Cor. III.7):

**Theorem 2.9.** For the cocycles $\omega_\kappa$ and the connected group $G := C^\infty(M,K)_0$, the following are equivalent:

1. $\omega_\kappa$ integrates for each compact manifold $M$ to a Lie group extension of $G$.
2. $\omega_\kappa$ integrates for $M = S^1$ to a Lie group extension of $G$.
3. The image of $\text{per}_{\omega_\kappa}$ in $H^1_{dR}(M,V) \subseteq \Omega^1(M,V)$ is discrete.
4. The image of $\text{per}_{\omega_\kappa}$ in $V$ is discrete.

These conditions are satisfied if $\kappa$ is the universal invariant bilinear form with values in $V(\mathfrak{k})$.

**Remark 2.10.** Composing a cocycle $\omega_\kappa$ with the exterior derivative, we obtain the $\Omega^2(M,V)$-valued cocycle

$$(d \circ \omega_\kappa)(\xi_1, \xi_2) = \kappa(d\xi_1, d\xi_2).$$

In view of [MN03], Thm. III.9, we have a smooth 2-cocycle

$$c(f_1, f_2) := \delta^\mathfrak{k}(f_1) \wedge_\kappa \delta^\mathfrak{k}(f_2) = \delta^V(f_1) \wedge_\kappa \text{Ad}(f_1). \delta^V(f_2)$$

on the full group $C^\infty(M,K)$ which defines a central extension by $\Omega^2(M,V)$ whose corresponding Lie algebra cocycle is $2d \circ \omega_\kappa$. Here $\wedge_\kappa$ denotes the natural product $\Omega^1(M,\mathfrak{k}) \times \Omega^1(M,\mathfrak{k}) \to \Omega^2(M,V)$ defined by $\kappa: \mathfrak{k} \times \mathfrak{k} \to V$.

In Subsection 4.2 below, this construction is generalized to gauge groups of non-trivial bundles.

For the cocycles of type (II), the situation is much simpler (Theorem 7.2):

**Theorem 2.11.** Let $K$ be a connected finite-dimensional Lie group with Lie algebra $\mathfrak{k}$, $\eta \in Z^2(\mathfrak{k},V)$, and $\mathfrak{t} = V \oplus \eta$. Then the following are equivalent:

a) The central Lie algebra extension $\widehat{g} = C^\infty(M,V) \oplus_{\omega_\eta} \mathfrak{g}$ of $g = C^\infty(M,\mathfrak{k})$ integrates to a central Lie group extension of $C^\infty(M,K)_0$. 
(b) $\hat{t} \to t$ integrates to a Lie group extension $V \hookrightarrow \hat{K} \to K$.

(c) The flux homomorphism $F_\eta: \pi_1(K) \to H^1(t,V)$ vanishes.

If this is the case, then

$$1 \to C^\infty(M,Z) \to C^\infty(M,\hat{K}) \to C^\infty(M,K) \to 1$$

defines a central Lie group extension of the full group $C^\infty(M,K)$ integrating $\hat{g}$.

**Proof.** For $m \in M$, the relation $ev^V_m \circ \omega_\eta = L(ev^K_m)^* \eta = (ev^K_m)^* \eta$ (10) in Remark 2.3 leads to $ev^V_m \circ \per_\omega_\eta = \per_\eta \circ \pi_2(ev^K_m)$ (see 17 in Remark 7.4). Since $\pi_2(K)$ vanishes ([CaE36]), the period map $\per_\eta$ is trivial and therefore all maps $ev^V_m \circ \per_\omega_\eta$ vanish, which implies that $\per_\omega_\eta = 0$.

A direct calculation further shows that the flux homomorphisms $F_\eta$ and $F_{\omega_\eta}: \pi_1(G) \to H^1(g, C^\infty(M,V))$ satisfy

$$F_{\omega_\eta}([\gamma])(\xi)(m) = F_\eta([ev^K_m \circ \gamma])(\xi(m)), \quad \xi \in g, [\gamma] \in \pi_1(G)$$

(see 18 in Remark 7.4). Therefore $F_\eta$ vanishes if and only if $F_{\omega_\eta}$ does. Now Theorem 7.2 below shows that (a) is equivalent to (b) which in turn is equivalent to (c) because all period maps vanish.

If these conditions are satisfied, we apply loc. cit. with $V = A$ to obtain a central extension $V \hookrightarrow \hat{K} \to K$. This extension is a principal $V$-bundle, hence trivial as such, so that there exists a smooth section $\sigma: K \to \hat{K}$. Therefore $C^\infty(M,\hat{K})$ is a central Lie group extension of $C^\infty(M,K)$ with a smooth global section.

**Remark 2.12.** If, in the context of Theorem 2.11, $q_K: \hat{K} \to K$ is a connected Lie group extension of $K$ by $Z = V/\Gamma_Z$, $\Gamma_Z \subseteq V$ a discrete subgroup, and $L(\hat{K}) = t = V \oplus_t t$, then $\hat{K}$ does in general not have a smooth global section and the image of the canonical map $C^\infty(M,\hat{K}) \to C^\infty(M,K)$ only is an open subgroup.

In our context, where $K$ is finite-dimensional, the obstruction for a map $f: M \to K$ to lift to $\hat{K}$ can be made quite explicit. The existence of a smooth lift $\hat{f}: M \to \hat{K}$ of $f$ is equivalent to the triviality of the smooth $Z$-bundle $f^*\hat{K} \to M$, and the equivalence classes of these bundles are parametrized by $H^2(M,\Gamma_Z)$ (cf. [Bry93]). Describing bundles in terms of Čech cocycles, it is easy to see that we thus obtain a group homomorphism

$$C^\infty(M,K) \to H^2(M,\Gamma_Z), \quad f \mapsto [f^*\hat{K}]$$

which factors through an injective homomorphism

$$\pi_0(C^\infty(M,K)) \cong [M,K] \to H^2(M,\Gamma_Z), \quad [f] \mapsto [f^*\hat{K}]$$

of discrete groups.
Since $\pi_2(K)$ and $F_\eta$ vanish, Remark 6.12 in [NeWa07] implies that the homology class $[\tilde{K}] \in H^2(K, \Gamma_Z)$ vanishes on $H_2(K)$, so that the Universal Coefficient Theorem shows that it is defined by an element of $\text{Ext}(H_1(K), \Gamma_Z) = \text{Ext}(\pi_1(K), \Gamma_Z)$, we may interpret $[f^*\tilde{K}]$ as an element of the group $\text{Ext}(H_1(M), \Gamma_Z)$. The long exact homotopy sequence of the $Z$-bundle $\tilde{K}$ contains the short exact sequence

$$1 \to \Gamma_Z = \pi_1(Z) \to \pi_1(\tilde{K}) \to \pi_1(K) \to 1,$$

which exhibits $\pi_1(\tilde{K})$ as a central extension of $\pi_1(K)$ corresponding to the class in $\text{Ext}(\pi_1(K), \Gamma_Z)$.

We now see that the homomorphism from above yields an exact sequence of groups

$$C^\infty(M, \tilde{K}) \to C^\infty(M, K) \to \text{Ext}(H_1(M), \Gamma_Z).$$

Here the rightmost map can be calculated by first assigning to $f \in C^\infty(M, K)$ the homomorphism $H_1(f) : H_1(M) \to H_1(K) \cong \pi_1(K)$ and then $H_1(f)^*[\tilde{K}] \in \text{Ext}(H_1(M), \Gamma_Z)$, which can be evaluated easily in concrete cases.

Now we turn to cocycles of type (III). We start with a key example:

**Example 2.13.** Let $\mathfrak{k}$ be a locally convex real Lie algebra and consider the locally convex Lie algebra $\mathfrak{g} := C^\infty(I, \mathfrak{k})$, where $I := [0, 1]$ is the unit interval.

Let $\kappa : \mathfrak{k} \times \mathfrak{k} \to V$ be a continuous invariant symmetric bilinear form with values in the Mackey complete space $V$ and consider $V$ as a trivial $\mathfrak{g}$-module. If $\Gamma(\kappa) = d_\kappa \eta$ for some $\eta \in C^2(\mathfrak{k}, V)$, then the cocycle $\omega_{\kappa, \eta} \in Z^2(\mathfrak{g}, \Omega^1(I, V))$ vanishes on the subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ of constant maps.

Let $K$ be a connected Lie group with Lie algebra $\mathfrak{k}$ and $G := C^\infty(I, K)$. The map $H : I \times G \to G, H(t, f)(s) := f(ts)$ is smooth because the corresponding map $\tilde{H} : I \times G \times I \to K, \tilde{H}(t, f, s) := f(ts)$ is smooth (cf. [NeWa07], Lemma A.2). Since $H_1 = \text{id}_G$ and $H_0(f) = f(0)$, $H$ is a smooth retraction of $G$ to the subgroup $K$ of constant maps. Therefore the inclusion $j : K \to G$ induces an isomorphism $\pi_2(j) : \pi_2(K) \to \pi_2(G)$. The period map $\text{per}_{\omega_{\kappa, \eta}} : \pi_2(G) \to V$ satisfies $\text{per}_{\omega_{\kappa, \eta}} \circ \pi_2(j) = \text{per}_{\kappa, \eta} = 0$ because the cocycle $j^*\omega_{\kappa, \eta} = \omega_{\kappa, \eta}|_{t \in \mathfrak{k}}$ vanishes, so that the period group of $\omega_{\kappa, \eta}$ is trivial.

**Theorem 2.14.** Let $M$ be a smooth compact manifold, $\mathfrak{k}$ a locally convex Lie algebra, $\kappa : \mathfrak{k} \times \mathfrak{k} \to V$ a continuous invariant symmetric bilinear form and $\eta \in C^2(\mathfrak{k}, V)$ with $\Gamma(\kappa) = d_\kappa \eta$. For a connected Lie group $K$ with Lie algebra $\mathfrak{k}$ and $G := C^\infty(M, K)$, the following assertions hold:

(a) The period map $\text{per}_{\omega_{\kappa, \eta}} : \pi_2(G) \to \Omega^1(M, V)$ vanishes.
(b) If $K$ is simply connected, then the flux $F_{\omega_{\kappa, \eta}}$ vanishes.
(c) If $K$ is finite-dimensional, then there exists $\tilde{\eta} \in C^2(\mathfrak{k}, V)$ with $d_\kappa \tilde{\eta} = \Gamma(\kappa)$ for which the flux $F_{\omega_{\kappa, \eta}}$ vanishes.
(d) If $F_{\omega_{\kappa, \eta}} = 0$, then $\omega_{\kappa, \eta}$ integrates to a central extension of $G_0$ by $\Omega^1(M, V)$. 

Proof. (a) Combing (10) in Remark 2.3 with (17) in Remark 7.4, we obtain for each \( \gamma \in C^\infty(I, M) \):
\[
\int_\gamma \circ \text{per}_{\omega_{\kappa, \eta}} = \text{per}_{f_* \omega_{\kappa, \eta}} = \text{per}_{f_* \omega_{\kappa, \eta} \circ \pi_2(\gamma^*)} = \int_f \circ \text{per}_{\omega_{\kappa, \eta} \circ \pi_2(\gamma^*)}
\]
and Example 2.13 implies that \( \text{per}_{\omega_{\kappa, \eta}} \) vanishes. Since \( \gamma \) was arbitrary, all periods of \( \omega_{\kappa, \eta} \) vanish.

(b) Next we assume that \( K \) is 1-connected. In Example 2.13 we have seen that the group \( C^\infty(I, K) \) is also 1-connected, so that the flux of \( \omega_{\kappa, \eta} \) vanishes.

Now we apply (18) in Remark 7.4 with (10) in Remark 2.3 to see that the flux \( F_{\omega_{\kappa, \eta}} \) also vanishes.

(c) Assume that \( K \) is finite-dimensional and pick a maximal compact subgroup \( C \subseteq K \). Since \( \kappa \) is \( K \)-invariant, the affine action of the compact group \( C \) on the affine space \( \{ \eta \in C^2(k, V) : d\eta = \Gamma(\kappa) \} \) has a fixed point \( \tilde{\eta} \). Then \( \omega_{\kappa, \tilde{\eta}} \) is \( C \)-invariant, and therefore \( C \) acts diagonally on \( \hat{g} = \Omega^1(I, V) \oplus \omega_{\kappa, \tilde{\eta}} \mathfrak{g} \).

It follows in particular that the flux vanishes on the image of \( \pi_1(C) \cong \pi_1(K) \) in \( \pi_1(G) \) (Proposition 7.1). For \( M = I = [0, 1] \) we immediately derive that the flux vanishes and in the general case we argue as in (b) by reduction via pull-back maps \( \gamma^* : C^\infty(M, K) \rightarrow C^\infty(I, K) \).

(d) follows from Theorem 7.2.

Remark 2.15. We have seen in the preceding theorem that if \( K \) is simply connected, the period homomorphism and the flux of \( \omega := \omega_{\kappa, \eta} \) vanish. Since \( G := C^\infty(I, K) \) is also 1-connected, one may therefore expect an explicit formula for a corresponding group cocycle integrating \( \omega_{\kappa, \eta} \) because the cohomology class \( [\omega_{\kappa, \eta}] \in H^2_{dR}(C^\infty(I, K), C^\infty(I, \mathbb{R})) \) vanishes (cf. [Ne02], Thm. 8.8).

Such a formula can be obtained by a method due to Cartan, combined with fiber integration and the homotopy \( H : I \times G \rightarrow G \) to \( K \). Since \( \omega \) vanishes on \( \kappa \), the fiber integral \( \theta := \frac{1}{2} H^* \omega_{\kappa} \in \Omega^1(G, V) \) satisfies \( d\theta = \omega_{\kappa} \), so that we may use Prop. 8.2 in [Na04a] to construct an explicit cocycle. For a more general method, based on path groups, we refer to [Vi07].

Remark 2.16. In view of Remarks 2.4 and 2.5 for cocycles \( \omega \) of type (I)-(III), the diffeomorphism group \( \text{Diff}(M) \) acts diagonally on \( \hat{g} \). If, in addition, \( K \) is 1-connected, then the flux \( F_{\omega} \) vanishes, so that the identity component \( G_0 := C^\infty(M, K)_0 \) also acts naturally on \( \hat{g} \) (Proposition 7.1), and this action is uniquely determined by the adjoint action of \( \mathfrak{g} \) on \( \hat{g} \) (cf. [GN07]). From that is easily follows that these two actions combine to a smooth action of \( C^\infty(M, K)_0 \times \text{Diff}(M) \) on \( \hat{g} \) and the Lifting Theorem 7.7 provides a smooth action of this group by automorphisms on any corresponding central extension \( \hat{G} \) of the simply connected covering group \( \tilde{G} \).

Problem 2.17. Do the central extensions of the connected group \( C^\infty(M, K)_0 \) constructed above extend to the full group?
The answer is positive for \( M = S^1 \), \( K \) compact simple and \( \omega = \omega_\kappa \) if \( \kappa \) is universal ([PS86], Prop. 4.6.9). See Problem 7.3 for the general framework in which this problem can be investigated.

### 3 Twists and the cohomology of vector fields

We have seen above how to obtain central extensions of the Lie algebra \( C^\infty(M, \mathfrak{k}) \) and that the three types of cocycles \( \omega_\kappa, \omega_\eta \) and \( \omega_{\kappa, \eta} \) are \( \mathcal{V}(M) \)-invariant, which leads to central extensions of the semidirect sum \( C^\infty(M, \mathfrak{k}) \rtimes \mathcal{V}(M) \) by spaces of the type \( \mathfrak{z} := C^\infty(M, \mathcal{V}) \), \( \Omega^1(M, \mathcal{V}) \), and \( \Omega^1(M, \mathcal{V}) \). The bracket on such a central extension has the form

\[
[(z_1, (\xi_1, X_1)), (z_2, (\xi_2, X_2))] = (X_1 \cdot z_2 - X_2 \cdot z_1 + \omega(\xi_1, \xi_2), (X_1, \xi_2 - X_2, \xi_1 + [\xi_1, \xi_2], [X_1, X_2])),
\]

and there are natural twists of these central extensions that correspond to replacing the cocycle \( \hat{\omega}(\xi_1, X_1), (\xi_2, X_2)) := \omega(\xi_1, \xi_2) \) by \( \hat{\omega} + \hat{\eta} \), where

\[
\hat{\eta}(\xi_1, X_1), (\xi_2, X_2)) := \eta(X_1, X_2) \quad \text{for some} \quad \eta \in \mathbb{Z}^2(\mathcal{V}(M), \mathfrak{z})
\]

(cf. Appendix B on extensions of semidirect products). To understand the different types of twists, one has to determine the cohomology groups

\[
H^2_c(\mathcal{V}(M), \mathbb{R}), \quad H^2_c(\mathcal{V}(M), \Omega^1(M, \mathbb{R})) \quad \text{and} \quad H^2_c(\mathcal{V}(M), \overline{\Omega}^1(M, \mathbb{R})).
\]

For a parallelizable manifold \( M \), these spaces have been determined in [BiNe07], and below we describe the different types of cocycles showing up.

We also discuss the integrability of these twists to abelian extensions of \( \text{Diff}(M) \), or at least of the simply connected covering \( \tilde{\text{Diff}}(M)_0 \) of its identity component \( \text{Diff}(M)_0 \).

#### 3.1 Some cohomology of the Lie algebra of vector fields

The most obvious source of 2-cocycles of \( \mathcal{V}(M) \) with values in differential forms is described in the following lemma, applied to \( p = 2 \) ([Ne06c], Prop. 6):

**Lemma 3.1.** For each closed \((p + q)\)-form \( \omega \in \Omega^{p+q}(M, V) \), the prescription

\[
\omega^{[p]}(X_1, \ldots, X_p) := [i_{X_p} \ldots i_{X_1} \omega] \in \overline{\Omega}^q(M, V)
\]

defines a continuous \( p \)-cocycle in \( Z^p(\mathcal{V}(M), \overline{\Omega}^q(M, V)) \).

The preceding lemma is of particular interest for \( p = 0 \). In this case it associates Lie algebra cohomology classes to closed differential forms. According to Cor. 3.2 in [Lec85], we have the following theorem:
Theorem 3.2. Each smooth $p$-form $\omega \in \Omega^p(M, \mathbb{R})$ defines a $p$-linear alternating continuous map $\mathcal{V}(M)^p \to C^\infty(M, \mathbb{R})$, which leads to an inclusion of chain complexes

$$(\Omega^\bullet(M, \mathbb{R}), \partial) \hookrightarrow (C^\bullet(\mathcal{V}(M), C^\infty(M, \mathbb{R})), d_{\mathcal{V}(M)})$$

inducing an algebra homomorphism

$$\Psi: H^\bullet_{dR}(M, \mathbb{R}) \to H^\bullet(\mathcal{V}(M), C^\infty(M, \mathbb{R}))$$

whose kernel is the ideal generated by the Pontrjagin classes of $M$, i.e., the characteristic classes of the tangent bundle.

Remark 3.3. There are several classes of compact manifolds for which the Pontrjagin classes all vanish but the tangent bundle is non-trivial. According to [GHV72], II, Proposition 9.8.VI, this is in particular the case for all Riemannian manifolds with constant curvature, hence in particular for all spheres.

A second source of cocycles is the following ([Kosz74]):

Lemma 3.4. Any affine connection $\nabla$ on $M$ defines a 1-cocycle

$$\zeta: \mathcal{V}(M) \to \Omega^1(M, \text{End}(TM)), \quad X \mapsto \mathcal{L}_X \nabla,$$

where $(\mathcal{L}_X \nabla)(Y)(Z) = [X, \nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y [X, Z]$. For any other affine connection $\nabla'$, the corresponding cocycle $\zeta'$ has the same cohomology class.

Definition 3.5. We use the cocycle $\zeta$, associated to an affine connection $\nabla$ to define $k$-cocycles $\Psi_k \in Z^k_{\mathcal{V}}(\mathcal{V}(M), \Omega^k(M, \mathbb{R}))$. Note that $A \mapsto \text{Tr}(A^k)$ defines a homogeneous polynomial of degree $k$ on $\text{gl}_d(\mathbb{R})$, invariant under conjugation. The corresponding invariant symmetric $k$-linear map is given by

$$\beta(A_1, \ldots, A_k) = \sum_{\sigma \in S_k} \text{Tr}(A_{\sigma(1)} \cdots A_{\sigma(k)}),$$

and we consider it as a linear $\text{GL}_d(\mathbb{R})$-equivariant map $\text{gl}_d(\mathbb{R})^\otimes k \to \mathbb{R}$, where $\text{GL}_d(\mathbb{R})$ acts trivially on $\mathbb{R}$. This map leads to a vector bundle map

$$\beta_M : \text{End}(TM)^\otimes k \to M \times \mathbb{R},$$

where $M \times \mathbb{R}$ denotes the trivial vector bundle with fiber $\mathbb{R}$. On the level of bundle-valued differential forms, this in turn yields an alternating $k$-linear $\text{Diff}(M)$-equivariant map

$$\beta^1_M : \Omega^1(M, \text{End}(TM))^k \to \Omega^k(M, \mathbb{R}).$$

The $\text{Diff}(M)$-equivariance implies the invariance of this map under the natural action of $\mathcal{V}(M)$, so that we can use $\beta^1_M$ to multiply Lie algebra cocycles (cf.}
If $\tilde{\theta}$ satisfies connection on $M$ functions, and for the corresponding cocycle $\zeta$, $\nabla(\text{[BiNe07], Example II.3}).$ Moreover, $X \in V$ that each map $\tau$ its cohomology class in $\text{Lie algebra cocycle}$ $\text{Remark 3.6.}$ $\text{Remark 3.7.}$ (cf. [Ne04a], Lemma F.3).

$k$ smooth $\text{Lie groups of bundle automorphisms and their extensions}$ $23$ $\text{Remark 3.6.}$ The Lie algebra 1-cocycle $\zeta_D: \text{Diff}(M) \to \Omega^1(M, \text{End}(TM))$, $\varphi \mapsto \varphi, \nabla - \nabla$, where

$$(\varphi, \nabla)_X Y := \varphi(\nabla_{\varphi^{-1}} X \varphi^{-1}, Y) \text{ for } \varphi X = T(\varphi) \circ X \circ \varphi^{-1}.$$ 

Now the $k$-fold cup product of $\zeta_D$ with itself defines an $\Omega^k(M, \mathbb{R})$-valued smooth $k$-cocycle on $\text{Diff}(M)$ whose corresponding Lie algebra cocycle is $\Psi_k$ (cf. [Ne04a], Lemma F.3).

$\text{Remark 3.7.}$ Assume that $M$ is a parallelizable $d$-dimensional manifold and that the 1-form $\tau \in \Omega^1(M, \mathbb{R}^d)$ implements this trivialization in the sense that each map $\tau_m$ is a linear isomorphism $T_m(M) \to \mathbb{R}^d$. Then, for each $X \in \mathcal{V}(M)$, $L_X \tau$ can be written as $L_X \tau = -\theta(X) \cdot \tau$ for some smooth function $\theta(X) \in C^\infty(M, \mathfrak{gl}_d(\mathbb{R}))$ and $\theta$ is a crossed homomorphism, i.e.,

$$\theta([X, Y]) = X \theta(Y) - Y \theta(X) + [\theta(X), \theta(Y)]$$

([BiNe07], Example II.3). Moreover, $\nabla_X Y := \tau^{-1}(X, \tau(Y))$ defines an affine connection on $M$ for which the parallel vector fields correspond to constant functions, and for the corresponding cocycle $\zeta$, the map

$$\tilde{\tau}: \Gamma(\text{End}(TM)) \to C^\infty(M, \mathfrak{gl}_d(\mathbb{R})), \quad \tilde{\tau}(\varphi)(m) = \tau_m \circ \varphi_m \circ \tau_m^{-1}$$

satisfies $\tilde{\tau} \circ \zeta(X) = -d(\theta(X)) \in \Omega^1(M, \mathfrak{gl}_d(\mathbb{R}))$. This leads to

$$\Psi_k(X_1, \ldots, X_k) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \text{ Tr } (d\theta(X_{\sigma(1)}) \wedge \ldots \wedge d\theta(X_{\sigma(k)})). \quad (12)$$

If $TM$ is trivial, we further obtain the cocycles $\Phi_k \in Z^{2k-1}_c(\mathcal{V}(M), C^\infty(M, \mathbb{R}))$,

$$\Phi_k(X_1, \ldots, X_{2k-1}) = \sum_{\sigma \in S_{2k-1}} \text{sgn}(\sigma) \text{ Tr } (\theta(X_{\sigma(1)}) \cdots \theta(X_{\sigma(2k-1)}))$$
and $\overline{\Psi}_k \in Z^k(V(M), \Omega^{k-1}(M, \mathbb{R}))$, defined by

$$
\overline{\Psi}_k(X_1, \ldots, X_k) = \sum_{\sigma \in S_k} \text{sgn}(\sigma)[\text{Tr}(\theta(X_{\sigma(1)})d\theta(X_{\sigma(2)}) \wedge \ldots \wedge d\theta(X_{\sigma(k)}))]
$$

([BiNe07], Def. II.7). For each $k \geq 1$ we have $d \circ \overline{\Psi}_k = \overline{\Psi}_k$.

For $k = 1$ we have in particular

$$
\overline{\Psi}_1(X) = \text{Tr}(\theta(X)) \in C^\infty(M, \mathbb{R}) = \Omega^0(M, \mathbb{R}), \quad \overline{\Psi}_1(X) = \text{Tr}(d\theta(X)),
$$

and

$$
\overline{\Psi}_2(X_1, X_2) = [\text{Tr}(\theta(X_1)d\theta(X_2) - \theta(X_2)d\theta(X_1))] \in \Omega^1(M, \mathbb{R}).
$$

Collecting the results from [BiNe07], Section IV, we now obtain the following classification result which, for a parallelizable manifold $M$, provides in particular all the information on the twists of the fundamental central extensions of the mapping Lie algebras defined by cocycles of type (I)-(III).

**Theorem 3.8.** Let $M$ be a compact $d$-dimensional manifold with trivial tangent bundle. Then the following assertions hold:

1. For $p \geq 2$ and $d \geq 2$ the map

$$
H^{p+2}_{dR}(M, \mathbb{R}) \to H^2(V(M), \Omega^p(M, \mathbb{R})), \quad [\omega] \mapsto [\omega^{[2]}]
$$

is a linear isomorphism.

2. For $d \geq 2$ we have

$$
H^2(V(M), \Omega^1(M, \mathbb{R})) \cong H^3_{dR}(M, \mathbb{R}) \oplus \mathbb{R}[\overline{\Psi}_1], \quad [\alpha] \mapsto [\alpha] \oplus [\overline{\Psi}_1],
$$

where $H^3_{dR}(M, \mathbb{R})$ embeds via $[\omega] \mapsto [\omega^{[2]}]$.

3. For $d \geq 2$ the map

$$
H^2_{dR}(M, \mathbb{R}) \oplus H^1_{dR}(M, \mathbb{R}) \to H^2(V(M), C^\infty(M, \mathbb{R})), \quad ([\alpha], [\beta]) \mapsto [\alpha + \beta \wedge \overline{\Psi}_1]
$$

is a linear isomorphism.

4. For $M = S^1$, $\dim H^2(V(M), C^\infty(M, \mathbb{R})) = 2$ and $\dim H^2(V(M), \Omega^1(M, \mathbb{R})) = \dim H^2(V(M), \mathbb{R}) = 1$.

In addition, for any compact connected manifold $M$,

$$
H^2(V(M), \Omega^2(M, \mathbb{R})) = \mathbb{R}[\overline{\Psi}_2] \oplus \mathbb{R}[\overline{\Psi}_1 \wedge \Psi_1]
$$

is 2-dimensional and

$$
H^2(V(M), \Omega^1(M, \mathbb{R})) = \mathbb{R}[\overline{\Psi}_1 \wedge \Psi_1] \oplus \{[\alpha \wedge \Psi_1] : [\alpha] \in H^1_{dR}(M, \mathbb{R})\},
$$

is of dimension $1 + b_1(M)$. 
Remark 3.9. The main point of the preceding theorem is the precise description of the cohomology spaces, which is obtained under the assumption that the tangent bundle is trivial. The cocycles $\Psi_k$ exist for all manifolds, whereas the construction of the cocycles $\Phi_k$ and $\overline{\Psi}_k$ requires at least the existence of a flat affine connection on $M$: If $\nabla$ is a flat affine connection on the $d$-dimensional manifold $M$, then we have a holonomy homomorphism $\rho: \pi_1(M) \to \text{GL}_d(\mathbb{R})$, for which the tangent bundle $TM$ is equivalent to the associated bundle $\tilde{M} \times_{\rho} \mathbb{R}^d$. We then obtain a trivializing 1-form $\tilde{\tau} \in \Omega^1(\tilde{M})$ satisfying $\gamma^*\tilde{\tau} = \rho(\gamma)^{-1} \circ \tilde{\tau}$ for $\gamma \in \pi_1(M)$ and a corresponding crossed homomorphism $\tilde{\theta}: \mathcal{V}(\tilde{M}) \to C^\infty(\tilde{M}, \mathfrak{gl}_d(\mathbb{R}))$.

For $X \in \mathcal{V}(M)$, let $\tilde{X} \in \mathcal{V}(\tilde{M})^{\pi_1(M)}$ denote the canonical lift of $X$ to $\tilde{M}$. Then

$$\gamma^*\tilde{\theta}(\tilde{X}) = \rho(\gamma)^{-1} \circ \tilde{\theta}(\tilde{X}) \circ \rho(\gamma) \quad \text{for} \quad \gamma \in \pi_1(M),$$

which means that the cocycles $\Psi_k$, $\Phi_k$ and $\overline{\Psi}_k$ lead on vector fields of the form $\tilde{X}$ to $\pi_1(M)$-invariant differential forms, hence that they actually define cocycles on $\mathcal{V}(M)$ with values in $\Omega^k(M, \mathbb{R})$, $C^\infty(M, \mathbb{R})$ and $\Omega^{k-1}(M, \mathbb{R})$.

Problem 3.10. Compute the second cohomology of $\mathcal{V}(M)$ in $\Omega^1(M, \mathbb{R})$ for all (compact) connected smooth manifolds. The cohomology with values in $\Omega^p(M, \mathbb{R})$ can be reduced with results of Tsujishita ([Tsu81], Thm. 5.1.6; [BiNe07], Thm. III.1) to the description of $H^*(\mathcal{V}(M), C^\infty(M, \mathbb{R}))$ which depends very much on the topology of $M$ ([Hae76]).

3.2 Abelian extensions of diffeomorphism groups

We now address the integrability of the cocycles described in the preceding subsection. Here a key method consists in a systematic use of crossed homomorphisms which are used to pull back central extensions of mapping groups (cf. [Bi03]).

Definition 3.11. Let $G$ and $N$ be Lie groups and $S: G \to \text{Aut}(N)$ a smooth action of $G$ on $N$. A smooth map $\theta: G \to N$ is called a crossed homomorphism if

$$\theta(g_1 g_2) = \theta(g_1) \cdot g_1 \theta(g_2),$$

i.e., if $\tilde{\theta} := (\theta, \text{id}_G): G \to N \rtimes_S G$ is a morphism of Lie groups.

Crossed homomorphisms are non-abelian generalizations of 1-cocycles. They have the interesting application that for any smooth $G$-module $V$ (considered as a module of $N \rtimes_S G$ on which $N$ acts trivially), we have a natural pull-back map

$$\tilde{\theta}^* = (\theta, \text{id}_G)^*: H^2(N \rtimes_S G, V) \to H^2(G, V).$$
Two crossed homomorphisms \( \theta_1, \theta_2 : G \to N \) are said to be equivalent if there exists an \( n \in N \) with \( \theta_2 = c_n \circ \theta_1 \), \( c_n(x) = n x n^{-1} \). Since \( N \) acts trivially by conjugation on the cohomology group \( H^2(N \rtimes_S G, V) \) (cf. [Ne04a, Prop. D.6]), equivalent crossed homomorphisms define the same pull-back map on cohomology.

**Remark 3.12.** (a) If \( \theta : G \to N \) is a crossed homomorphism, then \( \tilde{\theta} = (\theta, \text{id}_G) \) yields a new splitting on the semidirect product group \( N \rtimes_S G \), which leads to an isomorphism \( N \rtimes_S G \cong N \rtimes_{S'} G \), where \( S'(g) = c_{\theta(g)} \circ S(g) \).

As we shall see below, it may very well happen that a Lie group extension of \( N \rtimes_S G \) is trivial on \( \{1\} \times G \) but not on \( \tilde{\theta}(G) \), which leads to a non-trivial extension of \( G \) (Example 3.16).

(b) If \( \theta : G \to N \) is a crossed homomorphism of Lie groups, \( L(\theta) := T_{\theta}(\theta) : L(G) \to L(N) \) is a crossed homomorphism of Lie algebras because \( (L(\theta), \text{id}_{L(G)}) = L(\tilde{\theta}) : L(G) \to L(N) \rtimes L(G) \) is a morphism of Lie algebras.

(c) If the central extension \( \hat{n} = V \oplus n \) is defined by a \( g \)-invariant cocycle \( \omega \) and \( \tilde{\omega} \) is the corresponding cocycle of the abelian extension \( \hat{n} \rtimes g \), corresponding to the diagonal action of \( g \) on \( \hat{n} \), then the pull-back of this cocycle with respect to a crossed homomorphism \( \theta : g \to n \) has the simple form

\[
(\tilde{\theta}^* \tilde{\omega})(\theta(x), \theta(y)) = \omega(\theta(x), \theta(y)) = (\theta^* \omega)(x, y).
\]

As we shall see below, crossed homomorphisms can be used to prove the integrability of many interesting cocycles of the Lie algebra of vector fields to the corresponding group of diffeomorphisms.

**Example 3.13.** (a) Let \( q : V \to M \) be a natural vector bundle over \( M \) with typical fiber \( V \). Then the action of \( \text{Diff}(M) \) on \( M \) lifts to an action on \( V \) by bundle automorphisms. Any trivialization of \( V \) yields an isomorphism

\[
\text{Aut}(V) \cong \text{Gau}(V) \rtimes \text{Diff}(M) \to \text{Aut}(M \times V) \cong C^\infty(M, \text{GL}(V)) \rtimes \text{Diff}(M),
\]

and the lift \( \rho : \text{Diff}(M) \to \text{Aut}(V) \) can now be written as \( \rho = (\theta, \text{id}_{\text{Diff}(M)}) \) for some crossed homomorphism \( \theta : \text{Diff}(M) \to C^\infty(M, \text{GL}(V)) \).

(b) Let \( \tau \in \Omega^1(M, \mathbb{R}^d) \) be such that each \( \tau_m \) is invertible in each \( m \in M \), so that \( \tau \) defines a trivialization of the tangent bundle of \( M \). In [BiNe07], Ex. II.3, it is shown that \( L_X \tau = -\theta(X) \cdot \tau \) defines a crossed homomorphism

\[
\theta : V(M) \to C^\infty(M, \mathfrak{gl}_d(\mathbb{R}))
\]

(cf. Remark 3.7). A corresponding crossed homomorphism on the group level is given by

\[
\Theta : \text{Diff}(M) \to C^\infty(M, \text{GL}_d(\mathbb{R})), \quad \varphi \cdot \tau = (\varphi^{-1})^* \tau = \Theta(\varphi)^{-1} \cdot \tau.
\]

It satisfies \( L(\Theta) = \theta \), taking into account that \( V(M) \) is the Lie algebra of the opposite group \( \text{Diff}(M)^{\text{op}} \), which causes a minus sign.
We shall use this crossed homomorphism to write the cocycles \( \Psi_2 \) and \( \Psi_1 \wedge \Psi_1 \) as pull-backs (cf. [Bi03] for the case where \( M \) is a torus). For the Lie algebra \( \mathfrak{k} := \mathfrak{gl}_d(\mathbb{R}) \), the space \( \text{Sym}^2(\mathfrak{k}, \mathbb{R}) \) is 2-dimensional, spanned by the two forms

\[
\kappa_1(x, y) := \text{tr}(x) \text{tr}(y) \quad \text{and} \quad \kappa_2(x, y) := \text{tr}(xy).
\]

Then the relations

\[
\begin{align*}
[\text{tr}(\theta(X_1)) \text{tr}(d\theta(X_2)) - \text{tr}(\theta(X_2)) \text{tr}(d\theta(X_1))] &= (\Psi_1 \wedge \Psi_1)(X_1, X_2), \\
[\text{tr}(\theta(X_1)d\theta(X_2) - \theta(X_2)d\theta(X_1))] &= \Psi_2(X_1, X_2)
\end{align*}
\]

imply that

\[
2\theta^*\omega_{\kappa_1} = \Psi_1 \wedge \Psi_1 \quad \text{and} \quad 2\theta^*\omega_{\kappa_2} = \Psi_2.
\]

With these preparations, we are now ready to prove the following:

**Proposition 3.14.** If the tangent bundle \( TM \) is trivial, then the cocycles

\[
\Psi_1 \wedge \Psi_1 \quad \text{and} \quad \Psi_2
\]

integrate to abelian Lie group extensions of some covering of \( \text{Diff}(M)_0 \).

**Proof.** Fix \( i \in \{1, 2\} \). Since the invariant form \( \kappa_i \) is one of the two components of the universal invariant symmetric bilinear form of \( \mathfrak{gl}_d(\mathbb{R}) \), Theorem 2.9 implies that the cocycles \( 2\omega_{\kappa_i} \) of \( C^\infty(M, \mathfrak{gl}_d(\mathbb{R})) \) are integrable to a central extension \( \hat{G} \) of the simply connected covering group \( \tilde{G} \) of \( G := C^\infty(M, \text{GL}_d(\mathbb{R}))_0 \) by \( Z := \text{Im}(\omega_{\kappa_i})/\text{Im}(\text{per}_{\kappa_i}) \) on which \( \text{Diff}(M) \) acts smoothly by automorphisms (Remark 2.3 and the Lifting Theorem 7.7). We thus obtain a semidirect product group \( \hat{H} := \hat{G} \times \text{Diff}(M)_0 \) which is an abelian extension of \( H := \tilde{G} \times \text{Diff}(M)_0 \) by \( Z \).

The crossed homomorphism \( \Theta : \text{Diff}(M)_0 \to G \) has a unique lift to a crossed homomorphism \( \Xi : \text{Diff}(M)_0 \to \hat{G} \) and the pull back extension \( \Xi^* \hat{H} \) integrates \( 2\theta^*\omega_{\kappa_i} \). In view of the preceding example, the assertion follows. \( \Box \)

**Problem 3.15.** It would be nice to know if we really need to pass to a covering group in the preceding proposition. As the argument in the proof shows, this is not necessary if the crossed homomorphism \( \Theta \) lifts to a crossed homomorphism into \( \tilde{G} \). The obstruction for that is the homomorphism

\[
\pi_1(\Theta) : \pi_1(\text{Diff}(M)) \to \pi_1(G),
\]

but this homomorphism does not seem to be very accessible.

In all cases where we have a compact subgroup \( C \subseteq \text{Diff}(M) \) for which \( \pi_1(C) \to \pi_1(\text{Diff}(M)) \) is surjective (this is in particular the case for \( \text{dim } M \leq 2; \) [EE69]), we may choose the trivialization \( \tau \) in a \( C \)-invariant fashion, so that \( \Theta(C) \) is trivial. This clearly implies that \( \pi_1(\Theta) \) vanishes.
We now take a closer look on some examples of extensions of $\text{Diff}(M)_0$ defined by these the two cocycles discussed above.

**Example 3.16.** (cf. [Bi03]) (The Virasoro group) For $M = S^1$, the preceding constructions can be used in particular to obtain the Virasoro group, resp., a corresponding global smooth 2-cocycle.

The simply connected cover of $C^\infty(S^1, GL_1(\mathbb{R}))_0$ is the group $N := C^\infty(S^1, \mathbb{R})$ on which we have a smooth action of $G := \text{Diff}(S^1)_0^{\text{op}}$ by $\varphi.f := f \circ \varphi$. In the notation of the preceding example, we have $\kappa := \kappa_1 = \kappa_2$, and, identifying $V(S^1)$ with $C^\infty(S^1, \mathbb{R})$, we obtain the 1-cocycle $\theta(X) = X'$. The corresponding 1-cocycle on $G$ with values in $N$ is given by $\Theta(\varphi) = \log(\varphi')$.

The cocycle $\omega := 2\omega_\kappa$ on $V(S^1)$ with values in $\Omega^1(S^1, \mathbb{R}) \cong \mathbb{R}$ is

$$\omega(\xi_1, \xi_2) = \int_0^1 \xi_1^\prime \xi_2^\prime - \xi_2^\prime \xi_1^\prime \, dt = 2 \int_{S^1} \xi_1 d\xi_2,$$

and $\frac{1}{2} \omega = \omega_\kappa$ is a corresponding group cocycle, defining the central extension $\hat{N} := \mathbb{R} \times_\omega N$ on which $G$ acts by $\varphi.(z, f) := (z, \varphi.f)$, so that the semi-direct product $\hat{N} \rtimes G$ is a central extension of $N \rtimes G$ (cf. [Ne02]).

We now pull this extension back with $\tilde{\Theta} : G \to \hat{N} \rtimes G$. Since we have in $\hat{N} \rtimes G$ the relation

$$(0, \Theta(\varphi), \varphi)(0, \Theta(\psi), \psi) = \left(\omega_\kappa(\Theta(\varphi), \varphi, \Theta(\psi)), \Theta(\varphi) \varphi, \Theta(\psi), \varphi \psi\right),$$

this leads to the 2-cocycle

$$\Omega(\varphi, \psi) = \omega_\kappa(\Theta(\varphi), \varphi, \Theta(\psi)) = \omega_\kappa(\Theta(\varphi), \Theta(\varphi \psi) - \Theta(\varphi))$$

$$= \omega_\kappa(\Theta(\varphi), \Theta(\varphi \psi)) = -\omega_\kappa(\log(\psi \circ \varphi)', \log \varphi') = - \int_{S^1} \log(\psi \circ \varphi)' \, d(\log \varphi').$$

This is the famous Bott–Thurston cocycle for $\text{Diff}(S^1)_0^{\text{op}}$, and the corresponding central extension is the Virasoro group. On the Lie algebra level, the pull-back cocycle is

$$(\theta^* \omega)(\xi_1, \xi_2) = \omega(\xi_1', \xi_2') = \int_0^1 \xi_1^\prime \xi_2^\prime - \xi_2^\prime \xi_1^\prime \, dt$$

(cf. [FF01]).

**Example 3.17.** (Bi03) The preceding example easily generalizes to tori $M = T^d$. Then we can trivialize $TM$ with the Maurer–Cartan form $\tau$ of the group $T^d$, which leads after identification of $V(M)$ with $C^\infty(M, \mathbb{R}^d)$ to $\theta(X) = -X'$ (the Jacobian matrix of $X$). A corresponding crossed homomorphism is

$$\Theta : \text{Diff}(T^d) \to C^\infty(T^d, GL_d(\mathbb{R})), \quad \Theta(\varphi) := (\varphi^{-1})'. $$
where the derivative $\varphi'$ of an element of $\text{Diff}(T^d) \subseteq C^\infty(T^d, T^d)$ is considered as a $\text{GL}_d(\mathbb{R})$-valued smooth function on $T^d$.

The main difference to the one-dimensional case is that here we do not have an explicit formula for the action of $\text{Diff}(T^d)$ on the central extension of the mapping group $C^\infty(M, \text{GL}_d(\mathbb{R})).$ As a consequence, we do not get explicit cocycles integrating $\overline{\psi}_2$ and $\overline{\psi}_1 \wedge \overline{\psi}_1$.

On the integration of cocycles of the form $\omega^{[2]}$

Now we turn to the other types of cocycles listed in Theorem 3.8(2), namely those of the form $\omega^{[2]}$, where $\omega$ is a closed $V$-valued 3-form on $M$.

Let $G$ be a Lie group and $\sigma : M \times G \to M$ be a smooth right action on the (possibly infinite dimensional) smooth manifold $M$. Let $\omega \in \Omega^{p+q}(M, V)$ be a closed $(p + q)$-form with values in a Mackey complete space $V$. If $L(\sigma) : \mathfrak{g} \to V(M)$ is the corresponding homomorphism of Lie algebras, then

$$\omega_g := L(\sigma)^* \omega^{[p]} \in Z^p(\mathfrak{g}, \mathcal{T}^G(M, V))$$

is a Lie algebra cocycle and we have a well-defined period homomorphism

$$\text{per}_{\omega_g} : \pi_p(G) \to \mathcal{T}^G(M, V)^G.$$ 

If $M$ is smoothly paracompact, which is in particular the case if $M$ is finite-dimensional, then any class $[\alpha] \in \mathcal{T}^G(M, V)$ is determined by its integrals over smooth singular $q$-cycles $S$ in $M$ which permits us to determine $\text{per}_{\omega_g}$ geometrically.

In the following we write for an oriented compact manifold $F$:

$$\int_F : \Omega^{p+q}(M \times F, V) \to \Omega^q(M, V)$$

for the fiber integral (cf. [GHV72], Ch. VII).

**Theorem 3.18.** (Period Formula) For any smooth singular $q$-cycle $S$ on $M$ and any smooth map $\gamma : S^p \to G$, we have the following integral formula

$$\int_S \text{per}_{\omega_g}([\gamma]) = (-1)^{pq} \int_{S \bullet \gamma} \omega,$$

where $S \bullet \gamma \in H_{p+q}(M)$ denotes the singular cycle obtained from the natural map $H_q(M) \otimes H_p(G) \to H_{p+q}(M), [\alpha] \otimes [\beta] \mapsto [\sigma \circ (\alpha \times \beta)]$ induced by the action map $M \times G \to M$.

**Proof.** First, let $\gamma : S^p \to G$ be a smooth map and $\alpha : \Delta_q \to M$ be a smooth singular simplex. We then obtain a smooth map

$$\alpha \bullet \gamma := \sigma \circ (\alpha \times \gamma) : \Delta_q \times S^p \to M, \quad (x, y) \mapsto \alpha(x).\gamma(y).$$
Integration of $\omega$ yields with 7.12/14 in [GHV72] (on pull-backs and Fubini’s Theorem):

$$
\int_{\Delta_q} \omega = \int_{\Delta_q \times SP} (\alpha \cdot \gamma)^* \omega = \int_{\Delta_q \times SP} (\alpha \times \gamma)^* \sigma^* \omega
$$

$$
= \int_{\Delta_q \times SP} (\alpha \times \text{id}_{SP})^*(\text{id}_M \times \gamma)^* \sigma^* \omega = \int_{\Delta_q \times SP} (\alpha \times \text{id}_{SP})^*(\text{id}_M \times \gamma)^* \sigma^* \omega
$$

$$
= \int_{\Delta_q} \alpha^* \int_{SP} (\text{id}_M \times \gamma)^* \sigma^* \omega = \int_{\Delta_q \times SP} (\text{id}_M \times \gamma)^* \sigma^* \omega \quad \text{for} \quad \gamma \in \Omega^n(M, V)
$$

It therefore remains to show that

$$
\text{per}_{\omega_q}[\gamma] = (-1)^{pq} \left( \int_{SP} (\text{id}_M \times \gamma)^* \sigma^* \omega \right).
$$

(15)

So let $Y_1, \ldots, Y_q \in \mathcal{V}(M)$ be smooth vector fields on $M$ and write $j^q_m : F \rightarrow M \times F, x \mapsto (m, x)$ for the inclusion map. We then have

$$
\int_{SP} (\text{id}_M \times \gamma)^* \sigma^* \omega(Y_1, \ldots, Y_q)_m = \int_{SP} (j^q_m)^*(i_{Y_q} \cdots i_{Y_1}((\text{id}_M \times \gamma)^* \sigma^* \omega))
$$

$$
= \int_{SP} (j^q_m)^*((\text{id}_M \times \gamma)^* (i_{Y_q} \cdots i_{Y_1}(\sigma^* \omega))) = \int_{\gamma} (j^q_m)^*((i_{Y_q} \cdots i_{Y_1}(\sigma^* \omega)).
$$

We further observe that for $x_1, \ldots, x_p \in \mathfrak{g} = T_1(G)$ we have

$$
(i_{Y_q} \cdots i_{Y_1}(\sigma^* \omega))_{(m, g)}(g.x_1, \ldots, g.x_p)
$$

$$
= (\sigma^* \omega)_{(m, g)}(Y_1(m), \ldots, Y_q(m), g.x_1, \ldots, g.x_p)
$$

$$
= \omega_{(m, g)}(Y_1(m).g, \ldots, Y_q(m).g, m.(g.x_1), \ldots, m.(g.x_p))
$$

$$
= \omega_{(m, g)}(Y_1(m).g, \ldots, Y_q(m).g, L(\sigma)(x_1)(m.g), \ldots, L(\sigma)(x_p)(m.g))
$$

$$
= (-1)^{pq}(\sigma_\gamma^*(i_{L(\sigma)(x_1)}(\cdot) \cdots i_{L(\sigma)(x_p)}(\cdot)))\omega_{(m, g)}(Y_1(m), \ldots, Y_q(m))
$$

$$
= (-1)^{pq}(g.\omega_{\gamma}(x_1, \ldots, x_p))(Y_1, \ldots, Y_q)_m
$$

$$
= (-1)^{pq}g.\omega_{\gamma}(g.x_1, \ldots, g.x_p)(Y_1, \ldots, Y_q)_m.
$$

We thus obtain in $\Omega^q(M, V)$ the identity:

$$
\int_{SP} (\text{id}_M \times \gamma)^* \sigma^* \omega = (-1)^{pq} \int_{\gamma} \omega_{\gamma} = (-1)^{pq} \text{per}_{\omega_q}([\gamma]),
$$

which in turn implies (15) and hence completes the proof. □

Example 3.19. For $q = 0$ and a closed $p$-form $\omega \in \Omega^p(M, V)$, the preceding theorem implies in particular that per$_{\omega_q} : \pi_p(G) \rightarrow C^\infty(M, V)$ satisfies

$$
\text{per}_{\omega_q}([\gamma])(m) := \int_{m \cdot \gamma} \omega \quad \text{with} \quad (m \cdot \gamma)(x) = m.\gamma(x).
$$
As an important consequence of the period formula in the preceding theorem, we derive that the 2-cocycles $\omega^{[2]}$ of integral $(p + 2)$-forms integrate to abelian extensions of a covering group of $\text{Diff}(M)_0$. This applies in particular to $p = 1$, which leads to the integrability of the cocycles occurring in Theorem 3.8(2).

**Corollary 3.20.** Let $\omega \in \Omega^{p+2}(M, V)$ be closed with discrete period group $\Gamma_\omega$. Then there exists an abelian extension of the simply connected covering group $\tilde{\text{Diff}}(M)_0$ by the group $Z := \Omega^p(V(M), \Omega^1(M, V)) / \Gamma_Z$, where $\Gamma_Z := \{ [\alpha] \in H^p_\text{dr}(M, V) : \int_{S_1} \alpha \leq \Gamma_\omega \} \cong \text{Hom}(H^p(M), \Gamma_\omega)$.

whose Lie algebra cocycle is $\omega^{[2]} \in Z^2(V(M), \mathcal{T}^p(M, V))$.

**Proof.** Applying the period formula in Theorem 3.18 to $\omega_\gamma = \omega^{[2]}$ and $G = \text{Diff}(M)^{\text{op}}$, which acts from the right on $M$, we see that the image of the period map is contained in the group $\Gamma_Z$, which is discrete. Now Theorem 7.2 below applies.

It is a clear disadvantage of the preceding theorem that it says only something about the simply connected covering group, and since $\pi_1(\text{Diff}(M))$ is not very accessible, it would be nicer if we could say more about the flux homomorphism of $\omega^{[2]}$. Unfortunately it does not vanish in general, so that we cannot expect an extension of the group $\text{Diff}(M)_0$ itself. However, the following result is quite useful to evaluate the flux:

**Theorem 3.21.** (Flux Formula) Let $\omega \in \Omega^{p+2}(M, V)$ be a closed $(p + 2)$-form and $\gamma : S^1 \to \text{Diff}(M)$ a smooth loop. Then $\eta_\gamma := \int_0^1 \gamma^*(i_{\delta^l(\gamma)} \omega) \, dt$ is a closed $(p + 1)$-form and the flux of $\omega^{[2]}$ can be calculated as

$$F_{\omega^{[2]}}([\gamma]) = [\eta^{[2]}_\gamma] \in H^1(V(M), \mathcal{T}^p(M, V)).$$

**Proof.** We recall from Appendix A that $F_{\omega^{[2]}}([\gamma]) = [I_\gamma] = -[I_{\gamma^{-1}}]$ for

$$I_\gamma(X) = \int_0^1 \gamma(t)^* \omega^{[2]}(\text{Ad}(\gamma(t))^{-1}.X, \delta^l(\gamma(t))) \, dt.$$

Hence $I_{\gamma^{-1}}(X) \in \mathcal{T}^p(M, V)$ is represented by the $p$-form $\bar{I}_{\gamma^{-1}}(X)$, defined on vector fields $Y_1, \ldots, Y_p$ by...
\[ I_{\gamma^{-1}}(X)(Y_1, \ldots, Y_p) = \int_0^1 \gamma(t)^* \left( \omega(\text{Ad}(\gamma(t)).X, \delta^t(\gamma^{-1}).t, \text{Ad}(\gamma(t)).Y_1, \ldots, \text{Ad}(\gamma(t)).Y_p) \right) dt \]
\[ = \int_0^1 \gamma(t)^* \left( (i_{\delta^t(\gamma)} \omega)(\text{Ad}(\gamma(t)).X, \text{Ad}(\gamma(t)).Y_1, \ldots, \text{Ad}(\gamma(t)).Y_p) \right) dt \]
\[ = -\int_0^1 \left( \gamma(t)^* (i_{\delta^t(\gamma)} \omega) \right)(X, Y_1, \ldots, Y_p) dt \]
\[ = -\eta_\gamma(X, Y_1, \ldots, Y_p) = -\eta^{[1]}_\gamma(X, Y_1, \ldots, Y_p). \]

That \( \eta_\gamma \) is closed follows directly from the fact that for any smooth path \( \gamma: I \to \text{Diff}(M) \), we have
\[ \gamma(1)^* \omega - \omega = \int_0^1 \gamma(t)^* (\mathcal{L}_{\delta^t(\gamma)} \omega) dt = \int_0^1 \gamma(t)^* (d(i_{\delta^t(\gamma)} \omega)) dt \]
\[ = d \int_0^1 \gamma(t)^* (i_{\delta^t(\gamma)} \omega) dt = d\eta_\gamma. \]

\[ \square \]

In general it is not so easy to get good hold of the flux homomorphism, but there are cases where it factors through an evaluation homomorphism \( \pi_1(\text{ev}_m): \pi_1(\text{Diff}(M)) \to \pi_1(M, m) \). According to Lemma 11.1 in [Ne04a], this is the case if \( \omega \) is a volume form on \( M \).

At this point we have discussed the integrability of all the 2-cochains of \( \mathcal{V}(M) \) that are relevant to understand the twists of the fundamental central extensions of the mapping algebras with values in \( \Omega^1(M, V) \).

The twists relevant for cocycles with values in \( \Omega^2(M, V) \) and \( C^\infty(M, V) \) are easier to handle. First we recall that a product of two integrable Lie algebra 1-cocycles is integrable to the cup product of the corresponding group cocycles (Ne04a, Lemma F.3), so that, combined with the integrability of the \( \Psi_k \) (Remark 3.6), the relevant information is contained in the following remark.

**Remark 3.22.** (a) If \( \alpha \in \Omega^2(M, \mathbb{R}) \) is a closed 1-form, then Example 3.19 applied to \( p = 1 \) provides a condition for integrability to a 1-cocycle on \( \text{Diff}(M) \). Passing to a covering \( q: \hat{M} \to M \) of \( M \) for which \( q^* \alpha \) is exact shows that there exists a covering of the whole diffeomorphism group to which \( \alpha \) integrates as a 1-cocycle with values in \( C^\infty(M, V) \) (cf. Remark 1.3(c)).

(b) If \( \omega \in \Omega^2(M, \mathbb{R}) \) is a closed 2-form, then Example 3.19 applied with \( p = 2 \), shows that if \( \omega \) has a discrete period group \( \Gamma^\omega \), then it integrates to a group cocycle on some covering group of \( \text{Diff}(M)_0 \). This can be made more explicit by the observation that in this case \( Z := \mathbb{R}/\Gamma^\omega \) is a Lie group and there exists a \( Z \)-bundle \( q: P \to M \) with a connection 1-form \( \theta \) satisfying \( q^* \omega = d\theta \). Then the group \( \text{Aut}(M) \) is an abelian extension of the open subgroup.
Diff(M)$_P$ of all diffeomorphisms lifting to bundle automorphisms of $P$ by $C^\infty(M, Z)$ and $\omega$ is a corresponding Lie algebra cocycle (cf. [Ko70], [NV03]).

For these cocycles the flux can also be made quite explicit (cf. [Ne04a], Prop. 9.11): For $\alpha \in C^\infty(S^1, M)$ and $\gamma \in C^\infty(S^1, \text{Diff}(M)_{0})$, we have

$$\int_{\alpha} F_\omega([\gamma]) = \int_{\gamma^{-1} \bullet \alpha} \omega, \quad \text{where} \quad (\gamma^{-1} \bullet \alpha)(t, s) := \gamma(t)^{-1}(\alpha(s)).$$

(c) If $\mu$ is a volume form on $M$, then the cocycle $\bar{\nu}_1$ satisfies $\mathcal{L}_X \mu = -\bar{\nu}_1(X)\mu$ ([BiNe07], Lemma III.3), so that $\bar{\nu}_1(X) = -\text{div} X$. Therefore $\zeta(\varphi)$, defined by $\varphi.\mu = \zeta(\varphi)^{-1}\mu$, defines a $C^\infty(M, \mathbb{R})$-valued group cocycle integrating $\bar{\nu}_1$.

If $M$ is not orientable, then there is a 2-sheeted covering $q: \hat{M} \to M$ such that $\hat{M}$ is orientable. Accordingly we have a 2-fold central extension $\hat{\text{Diff}}(M)$ of $\text{Diff}(M)$ by $\mathbb{Z}/2$ acting on $\hat{M}$ and the preceding construction provides a cocycle $\zeta: \hat{\text{Diff}}(M) \to C^\infty(\hat{M}, \mathbb{R})$. If $\sigma: \hat{M} \to M$ is the non-trivial deck transformation, then $\sigma$ reverses the orientation of $\hat{M}$ and there exists a volume form $\mu$ on $\hat{M}$ with $\sigma^* \mu = -\mu$. Since $\sigma$ commutes with $\hat{\text{Diff}}(M)$, all functions $\zeta(\varphi)$ are $\sigma$-invariant, hence in $C^\infty(\hat{M}, \mathbb{R})^\sigma \cong C^\infty(M, \mathbb{R})$. We also note that $\sigma^* \mu = -\mu$ implies that $\sigma$ is not contained in the identity component of $\hat{\text{Diff}}(M)$, so that $\text{Diff}(M)_{0} \cong \text{Diff}(M)_{0}$ and $\bar{\nu}_1$ also integrates to a cocycle on $\text{Diff}(M)_{0}$.

**Toroidal groups and their generalizations**

For the spaces $\mathfrak{z} = \overline{\mathfrak{g}^{1}}(M, V)$, $C^\infty(M, V)$ and $\Omega^{1}(M, V)$ the cocycles $\omega$ of type (I)-(III) on $\mathfrak{g} = C^\infty(M, \mathfrak{k})$ are $\mathcal{V}(M)$-invariant, hence extend trivially to cocycles on $\mathfrak{h} := \mathfrak{g} \rtimes \mathcal{V}(M)$ and we have also seen how to classify the $\mathfrak{z}$-cocycles $\tau$ on $\mathcal{V}(M)$ that occur as twists (cf. Theorems 3.8, 5.5). We write $\omega_\tau$ for the corresponding twisted cocycle on $\mathfrak{g} \rtimes \mathcal{V}(M)$.

Let $G := C^\infty(M, K)_{0}$. If $\omega$ is integrable to a Lie group extension, then we write $\hat{G}$ for a corresponding extension of $G$ by $Z \cong \mathfrak{z}/\mathfrak{g}_{Z}$. This extension is uniquely determined by $[\omega]$, which has the consequence that the natural action of $\text{Diff}(M)$ on $\hat{\mathfrak{g}}$ integrates to a smooth action on $\hat{G}$, which leads to a semidirect product $\hat{G} \rtimes \text{Diff}(M)$.

If $\rho: \text{Diff}(M) \to \text{Diff}(M)_{0}$ is an abelian extension of $\text{Diff}(M)_{0}$ by $Z$ corresponding to $\tau$, then $\hat{G} \rtimes \text{Diff}(M)$ is an extension of $\hat{H} := G \rtimes \text{Diff}(M)_{0}$ by $Z \times Z$, and the antidiagonal $\Delta_Z$ in $Z$ is a normal split Lie subgroup, so that

$$\hat{H} := (\hat{G} \rtimes \text{Diff}(M))/\Delta_Z$$

carries a natural Lie group structure, and it is now easy to verify that $L(\hat{H}) \cong \mathfrak{z} \oplus \omega_\tau \oplus 0$. 

For the special case where \( M = \mathbb{T}^d \) is a torus, \( \mathfrak{k} \) is a simple complex algebra, \( \kappa \) is the Cartan–Killing form, \( V = \mathbb{C} \), \( \omega = \omega_\kappa \) and \( \tau \) is a linear combination of the two cocycles \( \Psi_2 \) and \( \Psi_1 \wedge \Psi_1 \), we thus obtain Lie groups \( \hat{H} \) whose Lie algebras are Fréchet completions of the so-called toroidal Lie algebras. These algebras (with the twists) have been introduced by Rao and Moody in [ERMo94] and since then their representation theory has been an active research area (cf. [Bi98], [Lar99], [Lar00], [ERC04], [FJ07]).

4 Central extensions of gauge groups

In this section we shall see how, and under which circumstances, the fundamental types of cocycles on \( C^\infty(M, \mathfrak{k}) \) generalize to gauge algebras \( \text{gau}(P) \). Therefore it becomes a natural issue to understand the corresponding central extensions of the gauge group \( \text{Gau}(P) \), resp., its identity component. Depending on the complexity of the bundle \( P \), this leads to much deeper questions, most of which are still open.

4.1 Central extensions of \( \text{gau}(P) \)

In this section we describe natural analogs of the cocycles of type (I)-(III) for gauge Lie algebras of non-trivial bundles.

Let \( q: P \to M \) be a \( K \)-principal bundle and \( \text{gau}(P) \) its gauge algebra, which we realize as

\[
\text{gau}(P) \cong \{ f \in C^\infty(P, \mathfrak{k}) : (\forall p \in P)(\forall k \in K) f(p,k) = \text{Ad}(k)^{-1} f(p) \}
\]

which is the space of smooth sections of the associated Lie algebra bundle \( \text{Ad}(P) = P \times_{\text{Ad}} \mathfrak{k} \) (cf. Remark 1.3(b)). Further, let \( \tilde{M} := P/K_0 \) denote the corresponding squeezed bundle, which is a, not necessarily connected, covering of \( M \) and a \( \pi_0(K) \)-principal bundle. The squeezed bundle is associated to the \( \pi_1(M) \)-bundle \( q_M: \tilde{M} \to M \) by the homomorphism \( \tilde{\pi}_1: \pi_1(M) \to \pi_0(K) \) from the long exact homotopy sequence of \( P \). Sometimes it is convenient to reduce matters to connected structure groups, which amounts to considering \( P \) as a \( K_0 \)-principal bundle over the covering space \( \tilde{M} \) of \( M \).

Let \( \rho: K \to \text{GL}(V) \) be a homomorphism with \( K_0 \subseteq \ker \rho \), so that \( V \) is a \( \pi_0(K) \)-module. Then the associated bundle

\[
\nabla := P \times_{\rho} V
\]

is a flat vector bundle. It is associated to the squeezed bundle \( P/K_0 \) via the representation \( \overline{\rho}: \pi_0(K) \to \text{GL}(V) \). We have a natural exterior derivative on the space \( \Omega^*(M, \nabla) \) of \( \nabla \)-valued differential forms and we define \( \overline{\Omega}^1(M, \nabla) := \Omega^1(M, \nabla)/d(\Gamma V) \). If \( V \) is finite-dimensional, then \( d(\Gamma V) \) is a closed subspace of the Fréchet space \( \Omega^1(M, \nabla) \), so that the quotient inherits a natural Hausdorff
locally convex topology (cf. [NeWo07]). The Lie algebra $\mathfrak{aut}(P) \subseteq \mathcal{V}(P)$ acts in a natural way on all spaces of sections of vector bundles associated to $P$, via their realization as smooth functions on $P$, hence in particular on the spaces $\Omega^p(M, \mathcal{V})$, which can be realized as $V$-valued differential forms on $P$. Since $\mathcal{V}$ is flat, the action on $\Omega^p(M, \mathcal{V})$ factors through an action of the Lie algebra $\mathcal{V}(M) \cong \mathfrak{aut}(P)/\mathfrak{gau}(P)$.

Remark 4.1. The passage from $M$ to the covering $\tilde{M} := P/K_0$, the squeezed bundle associated to $P$, provides a simplification of our setting to the case where the structure group under consideration is connected if we consider $P$ as a $K_0$-bundle over $\tilde{M}$. One has to pay for this reduction by changing $M$. In particular, if $\pi_0(K)$ is infinite, the manifold $\tilde{M}$ is non-compact.

We now take a closer look at analogs of the cocycles of types (I)-(III) for gauge algebras $\mathfrak{g} := \mathfrak{gau}(P)$ of non-trivial bundles. Here the natural target spaces are $\Gamma\mathcal{V}, \Omega^1(M, \mathcal{V})$ and $\Gamma\Omega^1(M, \mathcal{V})$, replacing $C^\infty(M, V), \Omega^1(M, V)$ and $\Omega^1(M, V)$.

(I): First we choose a connection 1-form $\theta \in \Omega^1(P, \mathfrak{k})$ and write $\nabla$ for the corresponding covariant derivative, which induces covariant derivatives on all associated vector bundles such as $\text{Ad}(P)$. For the flat bundle $\mathcal{V}$ the corresponding covariant derivative coincides with the Lie derivative. Let $\kappa: \mathfrak{k} \times \mathfrak{k} \to \mathfrak{V}$ be a $K$-invariant symmetric bilinear map. For $\xi \in \mathfrak{gau}(P)$, the 1-form $\nabla\xi \in \Omega^1(P, \mathfrak{k})$ is $K$-equivariant and basic, hence defines a bundle-valued 1-form in $\Omega^1(M, \text{Ad}(P))$. Now $\kappa$ defines a 2-cocycle

$$\omega_\kappa \in Z^2(\mathfrak{gau}(P), \Gamma\Omega^1(M, \mathcal{V})), \quad \omega_\kappa(\xi_1, \xi_2) := [\kappa(\xi_1, \nabla\xi_2)]$$

with values in the trivial module $\Gamma\Omega^1(M, \mathcal{V})$ (cf. [NeWo07]; and [LMNS98] for bundles with connected $K$).

(II): Let $K$ be a connected Lie group with Lie algebra $\mathfrak{k}$ and $\eta \in Z^2(\mathfrak{k}, \mathfrak{V})$ be a 2-cocycle, defining a central Lie algebra extension $\eta: \hat{\mathfrak{k}} \to \mathfrak{k}$ of $\mathfrak{k}$ by $\mathfrak{V}$ which integrates to a central Lie group extension $\hat{K}$ of $K$ by some quotient $Z := V/I_2$ of $V$ by a discrete subgroup. Then $K$ acts trivially on $V$, so that the bundle $\mathcal{V}$ is trivial.

The $K$ action on $\hat{\mathfrak{k}}$ defines an associated Lie algebra bundle $\hat{\text{Ad}}(P)$ with fiber $\hat{\mathfrak{k}}$ which is a central extension of the Lie algebra bundle $\text{Ad}(P)$ by the trivial bundle $M \times V$ (cf. [Ma05]). Now the Lie algebra $\hat{\mathfrak{gau}}(P) := \Gamma\hat{\text{Ad}}(P)$ of smooth sections of this bundle is a central extension of $\mathfrak{gau}(P)$ by the space $C^\infty(M, V) \cong \Gamma\mathcal{V}$.

Any splitting of the short exact sequence of vector bundles

$$0 \to M \times V \to \hat{\text{Ad}}(P) \to \text{Ad}(P) \to 0$$

leads to a description of the central extension $\hat{\mathfrak{gau}}(P)$ by a 2-cocycle $\omega \in Z^2(\mathfrak{gau}(P), C^\infty(M, V))$ which is $C^\infty(M, \mathbb{R})$-bilinear, hence can be identified with a section of the associated bundle with fiber $Z^2(\mathfrak{k}, \mathfrak{V})$. 
(III): To see the natural analogs of cocycles of type (III), we first note that the natural analogs of the cocycles \( \omega_{d \phi} = d \circ \omega_{\phi} \) are cocycles of the form

\[ d \omega \in Z^2(gau(P), \Omega^1(M, V)) \]

where \( \omega \in Z^2(gau(P), C^\infty(M, V)) \) corresponds to a central extension \( \text{Ad}(P) \) of the Lie algebra bundle \( \text{Ad}(P) \) by the trivial bundle \( M \times V \).

Let \( \kappa \) be an invariant \( V \)-valued symmetric bilinear form \( \kappa \) and \( \theta \) a principal connection 1-form on \( P \) with associated covariant derivative \( \nabla \). We are looking for a \( C^\infty(M, R) \)-bilinear alternating map \( \tilde{\eta} \in Z^2(gau(P), C^\infty(M, V)) \) for which

\[ \omega_{\kappa, \nabla, \tilde{\eta}}(\xi_1, \xi_2) := \kappa(\xi_1, \nabla \xi_2) - \kappa(\xi_2, \nabla \xi_1) - d(\tilde{\eta})(\xi_1, \xi_2) \]

defines a Lie algebra cocycle with values in \( \Omega^1(M, V) \).

For \( \tilde{\omega}_{\kappa}(\xi_1, \xi_2) := \kappa(\xi_1, \nabla \xi_2) - \kappa(\xi_2, \nabla \xi_1) \) we obtain as in Section 3:

\[ d_\eta(\tilde{\omega}_{\kappa})(\xi_1, \xi_2, \xi_3) = d(\Gamma(\kappa))(\xi_1, \xi_2, \xi_3). \]

Here we use that in the realization of \( g = ga(u(P)) \) in \( C^\infty(P, \mathfrak{t}) \), we have

\[ \nabla \xi = d\xi + [\theta, \xi], \]

so that

\[ \tilde{\omega}_{\kappa}(\xi_1, \xi_2) = \kappa(\xi_1, d\xi_2) - \kappa(\xi_2, d\xi_1) + 2\kappa(\theta, [\xi_2, \xi_1]). \]

Therefore \( d_\eta(\tilde{\omega}_{\kappa}, \nabla, \tilde{\eta}) = 0 \) is equivalent to \( \Gamma(\kappa) - d_\eta \tilde{\eta} \) having values in constant functions, and since this map is \( C^\infty(M, R) \)-trilinear, we find the condition

\[ \Gamma(\kappa) = d_\eta \tilde{\eta} \quad \text{in} \quad Z^3(gau(P), C^\infty(M, V)). \]

This means that for each \( m \in M \), we have fiberwise in local trivializations \( \Gamma(\kappa) = d_\eta \tilde{\eta}(m) \). In particular, \( \kappa \) has to be exact.

**Lemma 4.2.** If \( \Gamma(\kappa) \) is a coboundary, then there exists a \( C^\infty(M, R) \)-bilinear alternating map \( \tilde{\eta} \in Z^2(gau(P), C^\infty(M, V)) \) for which \( \omega_{\kappa, \nabla, \tilde{\eta}} \) is a 2-cocycle.

**Proof.** The set \( C^2(\mathfrak{t}, V)_\kappa := \{ \eta \in C^2(\mathfrak{t}, V) : d_\eta = \Gamma(\kappa) \} \) is an affine space on which \( K \) acts by affine map. Now the associated bundle \( A := P \times_K C^3(\mathfrak{t}, V)_\eta \) has a smooth section \( \tilde{\eta} \), and any such section defines a \( C^\infty(M, R) \)-bilinear element of \( C^3(\mathfrak{g}, C^\infty(M, V)) \) with \( d_\eta(d \circ \tilde{\eta}) = d \circ \Gamma(\kappa) = d_\eta(\tilde{\omega}_{\kappa}) \).

In view of the preceding lemma, analogs of the type (III) cocycles on the Lie algebra \( ga(u(P)) \) with values in \( \Omega^1(M, V) \) always exist if \( \kappa \) is exact.

**Problem 4.3.** (Universal central extensions) Suppose that \( \mathfrak{t} \) is a semisimple Lie algebra and \( q: P \to M \) a principal \( K \)-bundle. Find a universal central extension of \( ga(u(P)) \). If \( P \) is trivial, then we have a universal central extension by \( T^1(M, V(\mathfrak{t})) \), given by the universal invariant symmetric bilinear form \( \kappa_\mathfrak{t} \) of \( \mathfrak{t} \) (Remark 222(d)). The construction described above yields a central
extension of $\mathfrak{gau}(P)$ by $\Omega^1(M, \mathcal{V})$ for the associated bundle with fiber $V(\mathfrak{k})$, but it is not clear that this extension is universal.

A class of gauge algebras closest to those of trivial bundles are those of flat bundles which can be trivialized by a finite covering of $M$. It seems quite probable that the above central extension is indeed universal. The analogous result for multiloop algebras (cf. Section 5 below) has recently been obtained by E. Neher ([Neh07], Thm. 2.13; cf. also [PPS07], p.147).

4.2 Covariance of the Lie algebra cocycles

Remark 4.4. (Covariance for type (I)) (a) Realizing $\mathfrak{g} = \mathfrak{gau}(P)$ in $\mathcal{C}^\infty(P, \mathfrak{k})$, we have $\nabla \xi = d\xi + [\theta, \xi]$, so that $\omega(\xi_1, \xi_2) = \omega(\xi_1, \nabla \xi_2) = [\kappa(\xi_1, \nabla \xi_2)] = [\kappa(\xi_1, d\xi_2) + \kappa(\theta, [\xi_2, \xi_1])].$

If $X = X_\xi \in \mathfrak{aut}(P)$ corresponds to $\xi \in \mathfrak{gau}(P)$ in the sense that $\theta(X_\xi) = -\xi$, then $\mathcal{L}_{X_\xi} \xi' = -[\theta(X_\xi), \xi'] = [\xi, \xi']$ and the fact that the curvature $d\theta + \frac{1}{2}[\theta, \theta]$ is a basic 2-form leads to

$$\mathcal{L}_{X_\xi} \theta = d(\xi \cdot \theta) + \xi \cdot d\theta = -d\xi - \frac{1}{2} \xi \cdot [\theta, \theta] = -d\xi - [\theta(X_\xi), \theta]$$

$$= -d\xi + [\xi, \theta] = -\nabla \xi,$$

so that $\omega(\xi)$ can also be written as

$$\omega(\xi_1, \xi_2) = -[\kappa(\xi_1, \mathcal{L}_{X_{\xi_2}} \theta)].$$

The group $\text{Aut}(P)$ acts on the affine space $\mathcal{A}(P) \subseteq \Omega^1(P, \mathfrak{k})$ of principal connection 1-forms by $\phi.\theta := (\phi^{-1})^* \theta$ and on $\mathfrak{g} = \mathfrak{gau}(P)$ by $\phi.\xi = \xi \circ \phi^{-1}$. We then have

$$\phi.\nabla \xi = \phi.(d\xi + [\theta, \xi]) = d(\phi.\xi) + [\phi.\theta, \phi.\xi] = \nabla(\phi.\xi) + [\phi.\theta - \theta, \phi.\xi].$$

This leads to

$$(\phi.\omega(\xi_1, \xi_2)) = \phi.\omega(\phi^{-1}\xi_1, \phi^{-1}\xi_2) = \omega(\xi_1, \xi_2) + [\kappa(\phi.\theta - \theta, [\xi_2, \xi_1])].$$

Note that

$$\zeta: \text{Aut}(P) \to \Omega^1(M, \text{Ad}(P)), \quad \phi \mapsto \phi.\theta - \theta$$

is a smooth 1-cocycle, so that

$$\Psi: \text{Aut}(P) \to \text{Hom}(\mathfrak{g}, \Omega^1(M, \mathcal{V})), \quad \Psi(\phi)(\xi) := [\kappa(\phi.\theta - \theta, \xi)].$$

is a 1-cocycle with $d_\phi(\Psi(\phi)) = \phi.\omega - \omega$. Therefore Lemma 7.6 below implies that

$$\varphi.([\alpha], \xi) := ([\varphi.\alpha] + [\kappa(\varphi.\theta - \theta, \varphi.\xi)]), \varphi.\xi)$$
defines an automorphism of the central extension \( \hat{\mathfrak{g}} \) and it is easy to see that we thus contains a smooth action of \( \text{Aut}(P) \) on \( \hat{\mathfrak{g}} \) (cf. [NeWo07]).

If \( \varphi_f \in \text{Gau}(P) \) is a gauge transformation corresponding to the smooth function \( f: P \to K \), then \( \varphi_f^* \theta = \delta^!(f) + \text{Ad}(f)^{-1} \theta \) implies

\[
\varphi_f \cdot \theta = \delta^!(f^{-1}) + \text{Ad}(f) \cdot \theta = -\delta^!(f^{-1}) + \text{Ad}(f) \cdot \theta.
\]

(b) If \( P = M \times K \) is the trivial bundle and \( p_M: P \to M \) and \( p_K: P \to K \) are the two projection maps, then the identification of gauge and mapping groups can be written as

\[
C^\infty(M, K) \to \text{Gau}(P) \subseteq C^\infty(P, K), \quad f \mapsto \tilde{f} := p_K^{-1} \cdot p_M^!(f) \cdot p_K,
\]

and the canonical connection 1-form is \( \theta = \delta^!(p_K) \in \Omega^1(P, \mathfrak{k}) \). We further write \( \tilde{\alpha} := \text{Ad}(p_K)^{-1} \cdot p_M^* \alpha \) for the realization of \( \alpha \in \Omega^1(M, \mathfrak{k}) \) as an element of \( \Omega^1(M, \text{Ad}(P)) \subseteq \Omega^1(P, \mathfrak{k}) \).

If \( \tilde{\varphi} \in \text{Aut}(P) \) is the lift of \( \varphi \in \text{Diff}(M) \) defined by \( \tilde{\varphi}(m, k) := (\varphi(m), k) \), then \( p_K \circ \tilde{\varphi} = p_K \) implies that \( \tilde{\varphi} \cdot \theta = \tilde{\theta} \), so that \( \zeta(\varphi) = 0 \). To determine \( \zeta \) on the gauge group, we calculate:

\[
\begin{align*}
\varphi_f \cdot \theta &= -\delta^!(f) + \text{Ad}(f) \cdot \theta = -\delta^!(f) - \text{Ad}(f) \cdot \delta^!(p_K^{-1}) \\
&= -\delta^!(f \cdot p_K^{-1}) = -\delta^!(p_K^{-1} \cdot p_M^!(f)) = -\delta^!(p_K^{-1}) \cdot \text{Ad}(p_K)^{-1} \cdot \delta^!(p_M^!(f)) \\
&= \delta^!(p_K) - \text{Ad}(p_K)^{-1} \cdot \delta^!(f) = \theta - \delta^!(f).
\end{align*}
\]

In this sense, the restriction of the cocycle \( \zeta \) to \( \text{Gau}(P) \) is \( \zeta(\varphi_f) = -\delta^!(f) \), so that it corresponds to the 1-cocycles defined by the right logarithmic derivative

\[
\delta^r: C^\infty(M, K) \to \Omega^1(M, \mathfrak{k}).
\]

We therefore recover the formulas for the actions of \( C^\infty(M, K) \times \text{Diff}(M) \) on \( \hat{\mathfrak{g}} \) from Remarks [2.4] and [2.5].

For later reference, we also note that the action of the gauge group on 1-forms is given in these terms by

\[
\begin{align*}
\varphi_f \cdot \tilde{\alpha} &= (\varphi_f^{-1})^* \tilde{\alpha} = (\varphi_f^{-1})^* (\text{Ad}(p_K^{-1}) \cdot p_M^* \alpha) \\
&= \text{Ad}(p_K \cdot \tilde{f}^{-1}) \cdot p_M^* (\tilde{f} \cdot p_M^{-1} \cdot \varphi_f^{-1}) \cdot \alpha = \text{Ad}(p_K \cdot \tilde{f}^{-1} \cdot \varphi_f^{-1}) \cdot p_M^* \alpha \\
&= \text{Ad}(p_K^{-1}) \cdot p_M^* (\text{Ad}(f) \cdot \alpha) = (\text{Ad}(f) \cdot \alpha)^- \cdot \text{Ad}(\tilde{f}) \cdot \tilde{\alpha}.
\end{align*}
\]

Remark 4.5. (Covariance for type (II)) In the type (II) situation we have constructed the Lie algebra extension \( \hat{\mathfrak{g}} = \hat{\mathfrak{gau}}(P) \) as \( \Gamma \text{Ad}(P) \), and since \( \text{Ad}(P) \) is a Lie algebra bundle associated to \( P \) via an action of \( K \) on \( \mathfrak{f} \) by automorphisms, the group \( \text{Aut}(P) \) and its Lie algebra \( \text{aut}(P) \) act naturally on \( \hat{\mathfrak{gau}}(P) \) by automorphisms, resp., derivations, lifting the action on \( \mathfrak{gau}(P) \).
Problem 4.6. (Covariance for type (III)) It would be interesting to see if the type (III) cocycles on $\text{gau}(P)$ with values in $\Omega^1(M, V)$ are $\text{aut}(P)$-covariant in the sense that $\text{aut}(P)$ acts on $\hat{\text{gau}}(P)$.

Problem 4.7. (Corresponding crossed modules) For cocycles of type (I) and (II) we have seen that the action of $\text{aut}(P)$ on $\text{gau}(P)$ lifts to an action by derivations on the central extension $\hat{\text{gau}}(P)$, which defines a crossed module

$$\mu: \hat{\text{gau}}(P) \to \text{aut}(P)$$

of Lie algebras.

For type (I), the characteristic class of this crossed module is an element of $H^3(\mathcal{V}(M), \Omega^1(M, V))$ and for type (II) in $H^3(\mathcal{V}(M), C^\infty(M, V))$. The vanishing of these characteristic classes is equivalent to the embeddability of the central extension $\hat{\text{gau}}(P)$ to an abelian extension of $\text{aut}(P)$ (cf. [Ne06b], Thm. III.5).

For type (II) it has been shown in [Ne06b], Lemma 6.2, that the characteristic class can be represented by a closed 3-form, so that it comes from an element of $H^3(\mathcal{V}(M), \Omega^1(M, \mathbb{R}))$. It seems that a proper understanding of the corresponding situation for type (I) requires a description of the third cohomology space $H^3(\mathcal{V}(M), \Omega^1(M, \mathbb{R}))$, but they can be described with Tsujishita’s work ([Tsu81], Thm. 5.1.6; [BiNe07], Thm. III.1) which asserts that $H^\bullet(\mathcal{V}(M), \Omega^\bullet(M, \mathbb{R}))$ is a free module of $\text{cohomology in degree } 3$.

For type (III) cocycles, one similarly needs the cohomology spaces $H^3(\mathcal{V}(M), \Omega^1(M, \mathbb{R}))$, but they can be described with Tsujishita’s work ([Tsu81], Thm. 5.1.6; [BiNe07], Thm. III.1) which asserts that $H^\bullet(\mathcal{V}(M), \Omega^\bullet(M, \mathbb{R}))$ is a free module of $\text{cohomology in degree } 3$.

If $TM$ is trivial, we obtain with Theorem 4.8

$$H^3(\mathcal{V}(M), \Omega^1(M, \mathbb{R})) = H^3(\mathcal{V}(M), C^\infty(M, \mathbb{R})) \cdot [\Psi_1].$$

4.3 Corresponding Lie group extensions

(I): For cocycles of the form $\omega_\kappa \in Z^2(\text{gau}(P), \Omega^1(M, V))$, it seems quite difficult to decide their integrability. However, if $\pi_0(K)$ is finite, we have the following generalization of Theorem 2.9 ([NeWo07]):

**Theorem 4.8.** If $\pi_0(K)$ is finite and $G := \text{Gau}(P)_0$, then the following are equivalent:

1. $\omega_\kappa$ integrates for each principal $K$-bundle $P$ over a compact manifold $M$ to a Lie group extension of $G$.
2. $\omega_\kappa$ integrates for the trivial $K$-bundle $P = S^1 \times K$ over $M = S^1$ to a Lie group extension of $G$.
(3) The image of \( \pi_3(K) \rightarrow V \) is discrete. These conditions are satisfied if \( \kappa \) is the universal invariant bilinear form with values in \( V(\mathfrak{t}) \).

As we have already seen for trivial bundles, the cocycle \( \mathfrak{d} \circ \omega \), with values in \( \Omega^2(M, V) \) can be integrated quite explicitly. We now explain the geometric background for that in the general case. We start with a general group theoretic remark.

**Remark 4.9.** Let \( E \) and \( F \) be vector space and \( \omega : E \times E \rightarrow F \) an alternating bilinear form.

(a) We write \( H := H(E, F, \omega) \) for the corresponding Heisenberg group, which is the product set \( F \times E \), endowed with the product

\[
(f_1, e_1)(f_2, e_2) = (f_1 + f_2 + \omega(e_1, e_2), e_1 + e_2).
\]

The invariance group of \( \omega \) is

\[
\text{Sp}(E, F, \omega) := \{(\varphi, \psi) \in \text{GL}(E) \times \text{GL}(F) : \varphi^* \omega = \psi \circ \omega\},
\]

and the natural diagonal action of this group on \( F \times E \) preserves \( \omega \), so that we may form the semidirect product group

\[
\text{HSp}(E, F, \omega) := H(E, F, \omega) \rtimes \text{Sp}(E, F, \omega)
\]

with the product

\[
(f_1, e_1, \varphi_1, \psi_1)(f_2, e_2, \varphi_2, \psi_2) = (f_1 + \psi_1 f_2 + \omega(e_1, \varphi_1 e_2), e_1 + \varphi_1 e_2, \varphi_1 \varphi_2, \psi_1 \psi_2).
\]

This group is an abelian extension of the group

\[
E \rtimes \text{Sp}(E, F, \omega) \subseteq \text{Aff}(E) \times \text{GL}(F)
\]

by \( F \) and the corresponding cocycle is

\[
\Omega((e_1, \varphi_1, \psi_1), (e_2, \varphi_2, \psi_2)) = \omega(e_1, \varphi_1 e_2).
\]

(b) Now let \( G \) be a group, \( \rho_E : G \rightarrow \text{Aff}(E) \) and \( \rho_F : G \rightarrow \text{GL}(F) \) group homomorphisms such that \( \rho := (\rho_E, \rho_F) \) maps \( G \) into \( E \rtimes \text{Sp}(E, F, \omega) \). Then the pull-back \( G := \rho^* \text{HSp}(E, F, \omega) \) is an abelian extension of \( G \) by \( V \). To make the cocycle \( \omega_G \) of this extension explicit, we write \( \rho_E(g) = (\mu(g), \varphi(g)) \) and obtain

\[
\omega_G(g_1, g_2) = \omega(\mu(g_1), \rho(g_1) \mu(g_2)).
\]

This 2-cocycle \( \omega_G \in Z^2(G, F) \) is the cup-product \( \omega_G = \mu \cup \omega \mu \) of the 1-cocycle \( \mu \in Z^1(G, E) \) with itself with respect to the equivariant bilinear map \( \omega \) (cf. [Ne04a], App. F).

If we think of \( E \) as an abstract affine space without specifying an origin, then any point \( o \in E \) leads to a 1-cocycle \( \mu(g) := g.o - o \) with values in the translation group \( (E, +) \) and, accordingly, we get

\[
\omega_G(g_1, g_2) = \omega(g_1.o - o, \rho(g_1)(g_2.o - o)).
\]
Remark 4.10. On the translation space $\Omega^1(M, \text{Ad}(P))$ of the affine space $\mathcal{A}(P)$ of principal connection 1-forms, any $V$-valued $K$-invariant symmetric bilinear map $\kappa$ on $\mathfrak{k}$ defines an alternating $C^\infty(M, \mathbb{R})$-bilinear map

$$\Omega^1(M, \text{Ad}(P)) \times \Omega^1(M, \text{Ad}(P)) \to \Omega^2(M, V), \quad (\alpha, \beta) \mapsto \alpha \wedge_\kappa \beta,$$

so that $\mathcal{A}(P)$ is an affine space with a translation invariant $\Omega^2(M, V)$-valued 2-form that is invariant under the affine action of $\text{Aut}(P)$ by pull-backs.

**Proposition 4.11.** The smooth 2-cocycle

$$\zeta_\kappa \in Z^2(\text{Aut}(P), \Omega^2(M, V)), \quad \zeta_\kappa(\varphi_1, \varphi_2) = (\varphi_1, \theta - \varphi_1) \wedge_\kappa (\varphi_1 \varphi_2 - \theta)$$

defines an abelian extension of $\text{Aut}(P)$ by $\Omega^2(M, V)$ whose corresponding Lie algebra cocycle is given by

$$(X_1, X_2) \mapsto (\mathcal{L}_{X_1} \theta) \wedge_\kappa (\mathcal{L}_{X_2} \theta) - (\mathcal{L}_{X_2} \theta) \wedge_\kappa (\mathcal{L}_{X_1} \theta) = 2(\mathcal{L}_{X_1} \theta) \wedge_\kappa (\mathcal{L}_{X_2} \theta).$$

Remark 4.12. If $P = M \times K$ is a trivial bundle, then we obtain on $\text{Gau}(P) \cong C^\infty(M, K)$ with Remark [1,3] b):

$$\zeta_\kappa(\varphi, \varphi_2) = \delta^\kappa(\tilde{f}) \wedge_\kappa \text{Ad}(\tilde{f}). \delta^\kappa(\tilde{g}) = \delta^\kappa(\tilde{f}) \wedge_\kappa \delta^\kappa(\tilde{g}) = (\delta^\kappa(\tilde{f}) \wedge_\kappa \delta^\kappa(\tilde{g}))^{-1}.$$

**Problem 4.13.** From the existence of the cocycle $\zeta_\kappa$, it follows that $\mathfrak{d} \circ c = 0$ holds for the obstruction class $c$ of the corresponding crossed module (cf. Problem [4,7]). Therefore the long exact cohomology sequence associated to the short exact sequence

$$0 \to H^1_{\text{dR}}(M, V) \to \mathcal{T}^1(M, V) \to B^2_{\text{dR}}(M, V) \to 0$$

implies that $c$ lies in the image of the space $H^3(V(M), H^1_{\text{dR}}(M, V))$. As the action of $V(M)$ on $H^1_{\text{dR}}(M, V)$ is trivial, we have to consider $H^3(V(M), \mathbb{R})$. This space vanishes for $\dim M \geq 3$ (cf. [BcNe07], Cor. D.6, resp. [Hae76], Thm. 3/4), so that in this case the central extension $\tilde{\text{gau}}(P)$ corresponding to $\omega_\kappa$ embeds into an abelian extension of $\text{aut}(P)$. It is an interesting problem to describe these extensions more explicitly. Are there any 2-dimensional manifolds for which the obstruction class in $c$ is non-zero?

(II): Now we turn to cocycles of type (II). As before, we fix a central Lie group extension $\tilde{K}$ of $K$ by $Z \cong V/\Gamma_Z$. Then there is an obstruction class $\chi_{\tilde{K}} \in H^2(M, \underline{Z})$ with values in the sheaf of germs of smooth $Z$-valued functions that vanishes if and only if there exists a $\tilde{K}$-principal bundle $\tilde{P}$ with $\tilde{P}/Z \cong P$, as a $K$-bundle ([GP78]). Using the isomorphisms

$$\hat{H}^2(M, Z) \cong H^3(M, \Gamma_Z) \cong H^3_{\text{sing}}(M, \Gamma_Z),$$

we see that $\chi$ maps to some class $\chi \in H^3_{\text{dR}}(M, V)$, and it has been shown in [LW06] that $\chi$ yields (up to sign) the characteristic class $\chi(\mu)$ of the Lie algebra crossed module $\mu : \tilde{\text{gau}}(P) \to \text{aut}(P)$. (cf. Theorem [4.2]).

On the level of de Rham classes, the following theorem can also be found in [LW06]: the present version is a refinement.
Theorem 4.14. Let $K$ be a connected finite-dimensional Lie group and $\hat{q}_K: \hat{K} \to K$ a central Lie group extension by $Z$ with $Z_0 \cong V/\Gamma_Z$ and $q: P \to M$ a principal $K$-bundle. Then the following assertions hold:

(a) If $Z$ is connected and $\pi_2(K)$ vanishes, which is the case if $\dim K < \infty$, then there is a $\hat{K}$-bundle $\hat{P}$ with $\hat{P}/Z \cong P$ and we obtain an abelian Lie group extension

$$1 \to C^\infty(M, Z) \to \text{Aut}(\hat{P}) \to \text{Aut}(P) \to 1$$

containing the central extension

$$1 \to C^\infty(M, Z) \to Gau(\hat{P}) \to Gau(P) \to 1$$

integrating the Lie algebra $\hat{\mathfrak{g}}au(P)$.

(b) If $\dim K < \infty$, then the obstruction class in $H^3_{\text{dR}}(M, V)$ vanishes.

(c) If $K$ is 1-connected and $\hat{K}$ is connected, then $Z$ is connected.

Proof. (a) That $\pi_2(K)$ vanishes if $K$ is finite-dimensional is due to É. Cartan ([CaE36]). If $\pi_2(K)$ vanishes and $Z$ is connected, then Theorem 7.12 of [Ne02] implies that the central extension $\hat{K}$ of $K$ by $Z \cong V/\Gamma_Z$ can be lifted to a central extension $K^Z$ of $K$ by $V$ with $\hat{K} \cong K^Z/\Gamma_Z$. This implies that the obstruction class to lifting the structure group of the bundle $P$ to $\hat{K}$, which is an element of the sheaf cohomology group $\check{H}^2(M, Z)$, is the image of an element of $\check{H}^2(M, V) = \{0\}$, hence trivial. Here we have used that the sheaf cohomology groups $\check{H}^p(M, V)$, $p > 0$, vanish because the sheaf of germs of smooth maps with values in the vector space $V$ is soft (cf. [God73]).

(b) Since $Z_0$ is divisible, we have a direct product decomposition $Z \cong Z_0 \times \pi_0(Z)$ and, accordingly, the central extension $\hat{K}$ is a Baer sum of an extension by $Z_0$ and an extension by $\pi_0(Z)$. According to (a), the extension by $Z_0$ does not contribute to the obstruction class, so that it is an element of $H^2(M, \pi_0(Z))$, and therefore the corresponding de Rham class in $H^3(M, V)$ vanishes.

(c) The group $\hat{K}/Z_0$ is a connected covering group of $K$, hence trivial, and therefore $Z = Z_0$. \hfill \square

Remark 4.15. If $\hat{K}$ is a central $T$-extension of some infinite-dimensional group for which the corresponding de Rham obstruction class in $H^3_{\text{dR}}(M, \mathbb{R})$ is non-zero, Lecomte’s Theorem 3.2 implies that the corresponding class in $H^3(\mathcal{V}(M), C^\infty(M, \mathbb{R}))$ is also non-zero because the degrees on the Pontrjagin classes are multiples of 4, so that the ideal they generate vanishes in degrees $\leq 3$. This means that the class of the corresponding crossed module $\mu: \hat{\mathfrak{g}}au(P) \to \mathfrak{g}au(P)$ is non-trivial, hence the central extension $\hat{\mathfrak{g}}au(P)$ does not embed equivariantly into an abelian extension of $\mathfrak{g}au(P)$.

In a similar fashion as the group $H^2(M, \mathbb{Z})$ classifies $T$-bundles over $M$, the group $H^3(M, \mathbb{Z})$ classifies certain geometric objects called bundle
gerbes ([Bry93], [Mu96]). As the preceding discussion shows, non-trivial bundle gerbes can only be associated to the central $\mathbb{T}$-extensions of infinite-dimensional Lie groups. For more on bundle gerbes and compact groups we refer to [SW07] in this volume.

5 Multiloop algebras

In this section we briefly discuss the connection between gauge algebras of flat bundles and the algebraic concept of a (multi-)loop algebra. These are infinite-dimensional Lie algebras which are presently under active investigation from the algebraic point of view ([ABP06], [ABFP07]). To simplify matters, we discuss only the case where $\mathfrak{k}$ is a complex Lie algebra.

5.1 The algebraic picture

Let $\mathfrak{k}$ be a complex Lie algebra, $m_1, \ldots, m_r \in \mathbb{N}$, $\zeta_i \in \mathbb{T}$ with $\zeta_i^{m_i} = 1$ and $\sigma_1, \ldots, \sigma_r \in \text{Aut}(\mathfrak{k})$ with $\sigma_i^{m_i} = \text{id}_\mathfrak{k}$. Then we obtain an action of the group $\Delta := \mathbb{Z}/m_1 \times \ldots \times \mathbb{Z}/m_r$ on the Lie algebra $\mathfrak{k} \otimes \mathbb{C}[t_1^{-1}, \ldots, t_r^{-1}]$ by letting the $i$-th generator $\hat{\sigma}_i$ of $\Delta$ acts by

$$\hat{\sigma}_i(x \otimes t^n) := \sigma_i(x) \otimes \zeta_i^{n} t^n.$$  

The algebra $M(\mathfrak{k}, \sigma_1, \ldots, \sigma_r) := (\mathfrak{k} \otimes \mathbb{C}[t_1^{-1}, \ldots, t_r^{-1}])^\Delta$ of fixed points of this action is called the corresponding multi-loop algebra. For $r = 1$ we also write $L(\mathfrak{k}, \sigma) := M(\mathfrak{k}, \sigma)$ and call it a loop algebra.

The loop algebra construction can be iterated: If we start with $\sigma_1 \in \text{Aut}(\mathfrak{k})$ with $\sigma_1^{m_1} = \text{id}$ and pick $\sigma_2 \in \text{Aut}(L(\mathfrak{k}, \sigma_1))$ with $\sigma_2^{m_2} = \text{id}$, we put $L(\mathfrak{k}, \sigma_1, \sigma_2) := L(L(\mathfrak{k}, \sigma_1), \sigma_2)$.

Repeating this process, we obtain the iterated loop algebras

$$L(\mathfrak{k}, \sigma_1, \ldots, \sigma_r) := L(L(\mathfrak{k}, \sigma_1, \ldots, \sigma_{r-1}), \sigma_r),$$

where $\sigma_j \in \text{Aut}(L(\mathfrak{k}, \sigma_1, \ldots, \sigma_{j-1}))$ is assumed to satisfy $\sigma_j^{m_j} = \text{id}$. Clearly, every multiloop algebra can also be described as an iterated loop algebra, but the converse is not true (cf. Example 5.3 below).
5.2 Geometric realization of multiloop algebras

There is a natural analytic variant of the loop algebra construction. Here the main idea is that the analytic counterpart of the algebra \( C[t, t^{-1}] \) of Laurent polynomials is the Fréchet algebra \( C^\infty(S^1, \mathbb{C}) \) of smooth complex-valued functions on the circle. Its dense subalgebra of trigonometric polynomials is isomorphic to the algebra of Laurent polynomials, so that it is a completion of this algebra. A drawback of this completion process is that the algebra of smooth function has zero divisors, but this in turn facilitates localization arguments.

The analytic analog of the algebra \( k \otimes C[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}] \) is the Fréchet–Lie algebra \( C^\infty(T^r, k) \) of smooth \( k \)-valued functions on the \( r \)-dimensional torus \( T^r \), endowed with the pointwise Lie bracket. To describe the analytic analogs of the multi-loop algebras, let \( \Delta := \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_r \) and fix a homomorphism \( \rho: \Delta \to \text{Aut}(k) \), which is determined by a choice of automorphisms \( \sigma_i \) of \( k \) with \( \sigma_i^{m_i} = \text{id}_k \). Let \( m := (m_1, \ldots, m_r) \) and consider the corresponding torus

\[
T^r_m := (\mathbb{R}/m_1 \mathbb{Z}) \times \cdots \times (\mathbb{R}/m_r \mathbb{Z}),
\]

on which the group \( \Delta \cong \mathbb{Z}^r/(m_1 \mathbb{Z} \oplus \cdots \oplus m_r \mathbb{Z}) \) acts in the natural fashion from the right by factorization of the translation action of \( \mathbb{Z}^r \) on \( \mathbb{R}^r \). Now we obtain an action of \( \Delta \) on the Lie algebra \( C^\infty(T^r_m, k) \) by

\[
(\gamma, f)(x) := \gamma \cdot f(x, \gamma)
\]

Then we have a dense embedding \( M(\mathfrak{k}, \sigma_1, \ldots, \sigma_r) \hookrightarrow C^\infty(T^r_m, \mathfrak{k})^\Delta \). To realize this Lie algebra geometrically, we note that the quotient map \( q: T^r_m \to T^r \) is a regular covering for which \( \Delta \) acts as the group of deck transformations on \( T^r_m \). Then the homomorphism \( \rho: \Delta \to \text{Aut}(\mathfrak{k}) \) from above defines a flat Lie algebra bundle

\[
\mathcal{R} := (T^r_m \times \mathfrak{k})/\Delta, \quad [t, x] \mapsto q(t),
\]

over \( T^r \), where \( \Delta \) acts by \( \gamma \cdot (t, x) := (t \gamma^{-1}, \gamma \cdot x) \), and the space \( \Gamma \mathcal{R} \) of smooth sections of this Lie algebra bundle can be realized as

\[
\{ f \in C^\infty(T^r_m, \mathfrak{k}) : (\forall \gamma \in \Delta)(\forall x \in T^r_m) f(x, \gamma) = \gamma^{-1} f(x) \} = C^\infty(T^r_m, \mathfrak{k})^\Delta.
\]

In this sense, the Lie algebra \( \Gamma \mathcal{R} \) is a natural geometric analog of a multiloop algebra.

5.3 A generalization of multiloop algebras

Our geometric realization of multiloop algebras suggests the following geometric generalization. Let \( \Delta \) be a discrete group and \( q: \tilde{M} \to M \) a regular covering, where \( \Delta \) acts from the right via \( (t, \gamma) \mapsto t \gamma \) on \( \tilde{M} \) as the group of deck transformations. Let \( \mathfrak{k} \) be a locally convex Lie algebra and \( \rho: \Delta \to \text{Aut}(\mathfrak{k}) \) an action of \( \Delta \) by topological automorphisms of \( \mathfrak{k} \). Let \( K_0 \) be a 1-connected
(regular or locally exponential) Lie group with Lie algebra \( \mathfrak{t} \). Then the action of \( \Delta \) integrates to an action \( \rho_K : \Delta \to \text{Aut}(K_0) \) by Lie group automorphisms \((\text{GN07})\), so that we may form the semidirect product Lie group \( K := K_0 \rtimes \Delta \) with \( \pi_0(K) = \Delta \).

Next we form the flat Lie algebra bundle
\[
\mathfrak{g} := (\tilde{M} \times \mathfrak{t})/\Delta, \quad q_\mathfrak{g} : \mathfrak{g} \to M, \quad [t,x] := \Delta.(t,x) \mapsto q(t),
\]
over \( M \), where \( \Delta \) acts on \( M \times \mathfrak{g} \) by \( \gamma.(t,x) := (t,\gamma^{-1},\gamma.x) \). Each section of \( \mathfrak{g} \) can be written as \( s(q(t)) = [t,f_s(t)] \) with a smooth function \( f : \tilde{M} \to \mathfrak{t} \), satisfying \( f(t,\gamma) = \gamma^{-1},f(t) \) for \( t \in \tilde{M}, \gamma \in \Delta \). We thus obtain a realization of the space \( \Gamma \mathfrak{g} \) of smooth sections of this Lie algebra bundle as
\[
\{ f \in C^\infty(\tilde{M},\mathfrak{t}) : (\forall \gamma \in \Delta) f(t,\gamma) = \gamma^{-1},f(t) \} = C^\infty(\tilde{M},\mathfrak{t})^\Delta, \quad (16)
\]
where the \( \Gamma \)-action on \( C^\infty(\tilde{M},\mathfrak{t}) \) is defined by \( (\gamma,f)(t) := \gamma.f(t,\gamma) \).

**Proposition 5.1.** If \( P := (\tilde{M} \times K)/\Delta \) is the flat \( K \)-bundle associated to the inclusion \( \Delta \hookrightarrow K \), then \( \text{gau}(P) \cong \Gamma \mathfrak{g} \).

Hence the Lie algebra of sections of a flat Lie algebra bundle can always be realized as a gauge algebra of a flat principal bundle; the converse is trivial.

**Proof.** Our assumptions on \( K_0 \) imply that \( \text{Aut}(K_0) \cong \text{Aut}(\mathfrak{t}) \) \((\text{GN07})\), so that the action of \( \Delta \) on \( \mathfrak{t} \) integrates to an action \( \rho_K \) on \( K_0 \) by automorphisms.

Clearly, \( P \cong \tilde{M} \times K_0 \) is a trivial \( K_0 \)-bundle, but it is not trivial as a \( K \)-bundle, because the group \( \Delta \) acts on \( \tilde{M} \times K_0 \) by
\[
(\tilde{m},k) \cdot \gamma = (\tilde{m},\gamma,\rho_K(\gamma)^{-1}(k)).
\]

Let \( \text{Ad}(P) = (P \times \mathfrak{t})/K \) denote the corresponding gauge bundle with \( \Gamma \text{Ad}(P) \cong \text{gau}(P) \). Then
\[
\text{gau}(P) \cong \{ f \in C^\infty(P,\mathfrak{t}) : (\forall p \in P)(\forall k \in K) f(p,k) = \text{Ad}(k)^{-1}.f(p) \}
\begin{align*}
&\cong \{ f \in C^\infty(\tilde{M},\mathfrak{t}) : (\forall \gamma \in \Delta) f(\tilde{m},\gamma) = \text{Ad}(\gamma)^{-1}.f(\tilde{m}) \} \\
&= \{ f \in C^\infty(\tilde{M},\mathfrak{t}) : (\forall \gamma \in \Delta) f(\tilde{m},\gamma) = \rho(\gamma)^{-1}.f(\tilde{m}) \}.
\end{align*}
\]
\( \Box \)

### 5.4 Connections to forms of Lie algebras over rings

In this subsection we assume that \( \Delta \) is finite and describe briefly the connection to forms of Lie algebras over rings. For more details on this topic we refer to \cite{PPS07}. The Lie algebra \( \Gamma \mathfrak{g} \) is a module of the algebra \( \mathcal{R} := C^\infty(M,\mathbb{C}) \) and the Lie bracket is \( \mathcal{R} \)-bilinear. The pull-back bundle
Example 5.3. We consider $\mathfrak{k} = \mathfrak{sl}_n(\mathbb{C})$ for $n \geq 2$ and the iterated loop algebra $L(\mathfrak{k}, \sigma_1, \sigma_2)$ defined as follows. As $\sigma_1 \in \text{Aut}(\mathfrak{k})$ we choose an involution not contained in the identity component of Aut($\mathfrak{k}$), so that $L(\mathfrak{k}, \sigma_1)$ is a non-trivial loop algebra corresponding to the affine Kac–Moody algebra of type $A^{(2)}_{n-1}$ (cf. [Ka90], Ch. 8). As $\sigma_2$, we take the involution on

$$L(\mathfrak{k}, \sigma_1) = \{ f \in C^\infty(\mathbb{T}, \mathfrak{k}) : f(-t) = \sigma_1(f(t)) \},$$

where $\mathbb{T}$ is realized as the unit circle in $\mathbb{C}^\times$, defined by

$$\sigma_2(f)(t_1) := f(t_1^{-1})$$

and put

$$L := L(\mathfrak{k}, \sigma_1, \sigma_2) = \{ f \in C^\infty(\mathbb{T}, L(\mathfrak{k}, \sigma_1)) : f(-t_2) = \sigma_2(f(t_2)) \}.$$

We may thus identify elements of $L$ with smooth functions $f : \mathbb{T}^2 \to \mathfrak{k}$ satisfying

$$f(t_1, -t_2) = f(t_1^{-1}, t_2) \quad \text{and} \quad f(-t_1, t_2) = \sigma_1(f(t_1, t_2)).$$

For $\tau_1(t_1, t_2) := (-t_1, t_2)$ and $\tau_2(t_1, t_2) := (t_1^{-1}, -t_2)$ we thus obtain a fixed point free action of $\Delta := \langle \tau_1, \tau_2 \rangle$ on $\mathbb{T}^2$ and a representation $\rho : \Delta \to \text{Aut}(\mathfrak{k})$, defined by $\rho(\tau_1) := \sigma_1$ and $\rho(\tau_2) := \text{id}$. Then
\[ L \cong \{ f \in C^\infty(T^2, t) : (\forall \gamma \in \Delta)(\forall t \in T^2) \quad f(t, \gamma) = \gamma^{-1} f(t) \}. \]

This means that \( L \cong \Gamma K \) for the Lie algebra bundle \( \mathcal{A} \) over \( M := T^2/\Delta \), where \( M \) is the Klein bottle (\( \tau_2 \) is orientation reversing). This implies that the centroid of \( L \) is \( C^\infty(M, \mathbb{C}) \) ([Lee80]), hence not isomorphic to \( C^\infty(T^2, \mathbb{C}) \), so that \( L \) cannot be a multiloop algebra.

6 Concluding remarks

From the structure theoretic perspective on infinite-dimensional Lie groups, it is natural to ask for a classification of gauge groups of principal bundles and associated structural data, such as their central extensions, invariant forms etc.

One readily observes that two equivalent \( K \)-principal bundles \( P_i \to M \) have isomorphic gauge groups, so that a classification of gauge groups or automorphism groups of \( K \)-principal bundles can be achieved, in principle, by a classification of all \( K \)-principal bundles and then identifying those whose gauge groups are isomorphic. A key observation is that if a bundle \( P \) is twisted by a \( Z(K) \)-bundle \( P_Z \), then the corresponding gauge groups are isomorphic Lie groups, where the isomorphism on the Lie algebra level is \( C^\infty(M, \mathbb{R}) \)-linear. Taking this refined structure into account leads to the concept of (transitive) Lie algebroids and the perspective on \( \text{Gau}(P) \) as the group of sections of a Lie group bundle (cf. [Ma89] for results in this direction and [Ma05] for more on Lie groupoids). Presently the theory of Lie groupoids is only available for finite-dimensional structure groups \( K \). It would be natural to develop it also for well-behaved infinite-dimensional groups over finite-dimensional manifolds.

If \( K \) is finite-dimensional with finitely many connected components, then the classification of \( K \)-principal bundles can be reduced to the corresponding problem for a maximal compact subgroup, so that the well developed theory of bundles with compact structure group applies ([Hu94]). If \( K \) is an infinite-dimensional Lie group, it might not possess maximal compact subgroups \( \mathbb{K} \), so that a similar reduction does not work. However, recent results of Müller and Wockel show that each topological \( K \)-principal bundle over a finite-dimensional paracompact smooth manifold \( M \) is equivalent to a smooth one and that two smooth \( K \)-principal bundles which are topologically equivalent are also smoothly equivalent ([MW06]). Therefore the set \( \text{Bun}(M, K) \) of equivalence classes of smooth \( K \)-principal bundles over \( M \) can be identified with the set \( \text{Bun}(M, K)_{\text{top}} \) of topological \( K \)-bundles over \( M \), and the latter set can be identified with the set \( [M, BK] \) of homotopy classes of continuous maps of \( M \) into a classifying space \( BK \) of the topological group \( K \) ([Hu94], Thms. 9.9, 12.2, 12.4).

\[ \text{The representation theory of compact groups implies that the unitary group of an infinite-dimensional Hilbert space is such an example.} \]
On the topological level, each continuous map \( \sigma : M \to BK \) defines an algebra homomorphism

\[
H^\bullet(\sigma, \mathbb{R}) : H^\bullet(BK, \mathbb{R}) \to H^\bullet(M, \mathbb{R}) \cong H^\bullet_{\text{dR}}(M, \mathbb{R}),
\]

whose range are the characteristic classes of the corresponding bundle \( P \).

The advantage of these characteristic classes is that they can be represented by closed differential forms on \( M \), hence can be evaluated by integrals over compact submanifolds or other piecewise smooth singular cycles.

These characteristic classes are quite well understood for finite-dimensional connected Lie groups \( K \), because in this case essentially everything can be reduced to compact Lie groups, for which a powerful theory exists. However, for infinite-dimensional structure groups, the corresponding theory of characteristic classes has only been explored in very special cases. We refer to Morita’s excellent textbook for more details and references (cf. [Mo01]). Central extensions of gauge groups of bundles with infinite-dimensional fiber over the circle have recently been studied in the context of integrable systems in [OR06].

7 Appendix A. Abelian extensions of Lie groups

In this appendix we result some facts on the integration of Lie algebra 2-cocycles from [Ne04a]. They provide a general set of tools to integrate abelian extensions of Lie algebras to extensions of connected Lie groups.

Let \( G \) be a connected Lie group and \( V \) a Mackey complete \( G \)-module. Further, let \( \omega \in Z^k(\mathfrak{g}, V) \) be a \( k \)-cocycle and \( \omega_{\text{eq}} \in \Omega^k(G, V) \) be the corresponding left equivariant \( V \)-valued \( k \)-form with \( \omega_{\text{eq}}^1 = \omega \). Then each continuous map \( S^k \to G \) is homotopic to a smooth map, and

\[
\text{per}_{\omega} : \pi_k(G) \to V^G, \quad [\sigma] \mapsto \int_{\mathbb{S}^k} \sigma^* \omega_{\text{eq}} = \int_{S^k} \omega_{\text{eq}} = \int_{S^k} \sigma^* \omega_{\text{eq}}
\]

defines the period homomorphism whose values lie in the \( G \)-fixed part of \( V \) ([Ne02], Lemma 5.7).

For \( k = 2 \) we define the flux homomorphism

\[
F_\omega : \pi_1(G) \to H^1(\mathfrak{g}, V), \quad [\gamma] \mapsto [I_\gamma],
\]

where we define for each piecewise smooth loop \( \gamma : S^1 \to G \) the 1-cocycle

\[
I_\gamma : \mathfrak{g} \to V, \quad I_\gamma(x) := \int_{S^1} i_x \omega_{\text{eq}} = \int_0^1 \gamma(t) \omega(\text{Ad}(\gamma(t))^{-1}.x, \delta(\gamma)_t) dt,
\]

where \( x_r \) is the right invariant vector field on \( x \) with \( x_r(1) = x \). If \( V \) is a trivial module, then \( \mathfrak{d} \mathfrak{g} V = \{0\} \), so that \( H^1(\mathfrak{g}, V) \) consists of linear maps \( \mathfrak{g} \to V \) and we may think of the flux as a map \( F_\omega : \pi_1(G) \times \mathfrak{g} \to V \).
Proposition 7.1. (\cite{Ne02}, Prop. 7.6) If $V$ is a trivial $G$-module, then the adjoint action of $g$ on the central extension $\hat{g} := V \oplus \omega g$ integrates to a smooth action $Ad_{\hat{g}}$ of $G$ if and only if $F_\omega = 0$.

Theorem 7.2. Let $G$ be a connected Lie group, $A$ a smooth $G$-module of the form $A \cong a/\Gamma_A$, where $\Gamma_A \subseteq a$ is a discrete subgroup of the Mackey complete space $a$ and $q_A : a \rightarrow A$ the quotient map. Then the Lie algebra extension $\hat{g} := a \oplus \omega g$ of $g$ defined by the cocycle $\omega$ integrates to an abelian Lie group extension of $G$ by $A$ if and only if $q_A \circ \per_\omega : \pi_2(G) \rightarrow A$ and $F_\omega : \pi_1(G) \rightarrow H^1(\hat{g}, a)$ vanish.

Remark 7.3. If only $q_A \circ \per_\omega$ vanishes, then the preceding theorem applies to the simply connected covering group $\tilde{G}$ of $G$, so that we only obtain an extension of $G$ by a non-connected group $A'$ which itself is a central extension of the discrete group $\pi_1(G)$ by $A$. For a more detailed discussion of these aspects, we refer to Section 7 of \cite{Ne04a}.

Remark 7.4. To calculate period and flux homomorphisms, it is often convenient to use related cocycles on different groups. So, let us consider a morphism $\phi : G_1 \rightarrow G_2$ of Lie groups and $\omega_i \in Z^2(g_i, V)$, $V$ a trivial $G_i$-module, satisfying $L(\phi)^* \omega_2 = \omega_1$. Then a straightforward argument shows that

$$per_{\omega_2} \circ \pi_2(\phi) = per_{\omega_1} : \pi_2(G_1) \rightarrow V.$$  \hspace{1cm} (17)

For the flux we likewise obtain

$$F_{\omega_2} \circ (\pi_1(\phi) \times L(\phi)) = F_{\omega_1} : \pi_1(G_1) \times g_1 \rightarrow V,$$  \hspace{1cm} (18)

if we consider the flux as a bihomomorphism $\pi_1(G) \times g \rightarrow V$.

Problem 7.5. It is a crucial assumption in the preceding theorem that the group $G$ is connected and it would be very desirable to have a suitable generalization to non-connected groups $G$.

The generalization to non-connected Lie groups $G$ means to derive accessible criteria for the extendibility of a 2-cocycle, resp., abelian extensions, from the identity component $G_0$ to the whole group $G$. From the short exact sequence $G_0 \rightarrow G \rightarrow \pi_0(G)$, we obtain maps

$$H^2(\pi_0(G), A) \xrightarrow{L} H^2(G, A) \xrightarrow{R} H^2(G_0, A) = H^2(G_0, A) \cong H^2(\pi_0(G), A),$$

but it seems to be difficult to describe the image of the restriction map $R$.

If the identity component $G_0$ of $G$ splits, i.e., if $G \cong G_0 \times \pi_0(G)$, then it is easy to see that $R$ is surjective, because the invariance of the cohomology class of $G_0$ permits to lift the conjugation action of $\pi_0(G)$ to an action on $G_0$,
and then the semidirect product $\hat{G}_0 \rtimes \pi_0(G)$ is an abelian extension of $G$ by $A$, containing $\hat{G}_0$ as a subgroup.

If $A$ is a trivial module, one possible approach is to introduce additional structures on a central extension $\hat{G}_0$ of $G_0$ by $A$, so that the map $q: \hat{G}_0 \to G$ describes a crossed module of groups (cf. [Ne07]), which requires an extension of the natural $G_0$-action of $G$ on $\hat{G}_0$, resp., its Lie algebra $\hat{g}$, to an action of $G$. If we have such an action lifting the conjugation action of $G$ on $G_0$, then the characteristic class $\chi(\hat{G}_0)$ of this crossed module is an element of the cohomology group $H^3(\pi_0(G),A)$ which vanishes if there exists an $A$-extension $\hat{G}$ of $G$ into which $\hat{G}_0$ embeds in a $\hat{G}$-equivariant fashion (cf. [Ne07], Thm. III.8).

The following lemma is easy to verify ([MN03], Lemma V.1; see also [Ne06b], Lemma II.5) for a generalization to general Lie algebra extensions).

**Lemma 7.6.** Let $\hat{g} = V \oplus f \mathfrak{g}$ be a central extension of the Lie algebra $\mathfrak{g}$ and $\gamma = (\gamma_V, \gamma_{\mathfrak{g}}) \in \text{GL}(V) \times \text{Aut}(\mathfrak{g})$. For $\theta \in C^1(\mathfrak{g}, V)$ the formula

$$\hat{\gamma}(z,x) := (\gamma_V(z) + \theta(\gamma_{\mathfrak{g}}(x)), \gamma_{\mathfrak{g}}(x)), \quad x \in \mathfrak{g}, z \in V,$$

defines a continuous Lie algebra automorphism of $\hat{\mathfrak{g}}$ if and only if

$$\gamma \cdot \omega - \omega = d_{\omega} \theta, \quad \text{resp.} \quad d_{\omega} \tilde{\theta} = \omega - \gamma^{-1} \cdot \omega \quad \text{for} \quad \tilde{\theta} := \theta \circ \gamma_{\mathfrak{g}}.$$

The following theorem can be found in [MN03], Thm. V.9:

**Theorem 7.7.** (Lifting Theorem) Let $q: \hat{G} \to G$ be a central Lie group extension of the 1-connected Lie group $G$ by the Lie group $Z \cong \mathfrak{z}/\Gamma_Z$. Let $\sigma_G: H \times G \to G$, resp., $\sigma_Z: H \times Z \to Z$ be smooth automorphic actions of the Lie group $H$ on $G$, resp., $Z$ and $\sigma_{\mathfrak{g}}$ a smooth action of $H$ on $\mathfrak{g}$ compatible with the actions on $\mathfrak{z}$ and $\mathfrak{g}$. Then there is a unique smooth action $\sigma_{\hat{g}}: H \times \hat{G} \to \hat{G}$ by automorphisms compatible with the actions on $Z$ and $G$, for which the corresponding action on the Lie algebra $\hat{\mathfrak{g}}$ is $\sigma_{\mathfrak{g}}$.

### 8 Appendix B. Abelian extensions of semidirect sums

We consider a semidirect product of topological Lie algebras $\mathfrak{h} = \mathfrak{n} \rtimes_{\mathfrak{s}} \mathfrak{g}$ and a topological $\mathfrak{h}$-module $V$. We are interested in a description of the cohomology space $H^2(\mathfrak{h}, V)$ in terms of $H^2(\mathfrak{n}, V)$ and $H^2(\mathfrak{g}, V)$.

To this end, we have study the inflation, resp., the restriction map

$$I: H^2(\mathfrak{g}, V^n) \to H^2(\mathfrak{h}, V), \quad \text{resp.,} \quad R_n: H^2(\mathfrak{h}, V) \to H^2(\mathfrak{n}, V)^{\mathfrak{g}},$$
satisfying $RI = 0$. We further have a restriction map

$$R_\mathfrak{g}: H^2(\mathfrak{h}, V) \to H^2(\mathfrak{g}, V).$$
The composition $R_0 \circ f : H^2(g, V^n) \to H^2(g, V)$ is the natural map induced by the inclusion $V^n \hookrightarrow V$ of $g$-modules. If we interpret the elements of $H^2(h, V)$ as abelian extensions of $h$ by $V$, then the inflation map leads to twistings of these extensions by extensions of $g$ by $V^n$.

We now construct an exact sequence describing kernel and cokernel of the map $(R_n, R_g)$ which provides a quite accessible description of $H^2(h, V)$.

**Definition 8.1.** (a) We write $C^p(n, V)$ for the space of continuous Lie algebra $p$-cochains with values in $V$ and

$$C^p(g, C^q(n, V))_c \subseteq C^p(g, C^q(n, V))$$

for the subspace consisting of those cochains defining a continuous $(p+q)$-linear map $g^p \times n^q \to V$. Accordingly, we define $Z^p(g, C^q(n, V))_c$, $H^p(g, C^q(n, V))_c$ etc.

(b) We further write $H^2(n, V)_{[\theta]} \subseteq H^2(n, V)_\theta$ for the subspace of those cohomology classes $[f]$ for which there exists a $\theta \in C^1(g, C^1(n, V))_c$ with

$$\delta_n(\theta(x)) = x.f \quad \text{for } x \in g.$$  \hfill (19)

Because of the continuity requirement for $\theta$, this is stronger that the $g$-invariance of the cohomology class $[f] \in H^2(n, V)$.

**Remark 8.2.** Any continuous $g$-module action by derivations on a central extension $\tilde{n} = V \oplus_f n$, compatible with the actions on $V$ and $n$ is of the form

$$x.(v, n) = (x.v + \theta(x)(n), x.n),$$  \hfill (20)

where $\theta \in Z^1(g, C^1(n, V))_c$ satisfies (19). In particular, the existence of such an action implies that $[f] \in H^2(n, V)_{[\theta]}$.

**Lemma 8.3.** For $[f_n] \in H^2(n, V)_{[\theta]}$ we choose $\theta \in C^1(g, C^1(n, V))_c$ as in (19). We thus obtain a well-defined linear map

$$\gamma : H^2(n, V)_{[\theta]} \to H^2(g, Z^1(n, V))_c, \quad [f_n] \mapsto [\delta_g \theta].$$

**Proof.** Using $\theta$, we obtain a linear map

$$\hat{\psi} : g \to \text{der}(\tilde{n}), \quad \hat{\psi}(x)(v, n) = (x.v + \theta(x)(n), x.n)$$

which is a linear lift of the given Lie algebra homomorphism

$$\psi = (S, \rho_V) : g \to (\text{der}(n) \times \text{gl}(V)).$$

In view of [Ne06b], Prop. A.7, the corresponding Lie algebra cocycle on $g$ is given by $\bar{f}_g = \delta_g \theta \in Z^2(g, Z^1(n, V))_c$. The corresponding extension

$$0 \to Z^1(n, V) \to \tilde{g} = Z^1(n, V) \times_{\bar{f}_g} g \to g \to 0$$
acts naturally on \( \hat{n} \) by \((\alpha, x).{(v, n)} = (\alpha(n) + x.v + \theta(x)(n), x.n)\).

Next we observe that \( \gamma \) is well-defined. Indeed, if \( \theta' \in C^1(\mathfrak{g}, C^1(\mathfrak{n}, V))_c \) also satisfies (19), then \( \theta' - \theta \in C^1(\mathfrak{g}, Z^1(\mathfrak{n}, V))_c \), so that \([d_{\mathfrak{g}}(\theta' - \theta)] = 0 \) in \( H^2(\mathfrak{g}, Z^1(\mathfrak{n}, V))_c \).

Moreover, if \( \tilde{f}_n = f_n + d_n \beta \) for some \( \beta \in C^1(\mathfrak{n}, V) \), then

\[
x.\tilde{f}_n = x.f_n + d_n(x.\beta) = d_n(\theta(x) + x.\beta),
\]

so that \( \tilde{\theta}(x) := \theta(x) + x.\beta \) leads to \( d_n(\tilde{\theta}(x)) = x.\tilde{f}_n \). Then \( d_{\mathfrak{g}}\tilde{\theta} = d_{\mathfrak{g}}\theta + d^2_n\beta = d_{\mathfrak{g}}\theta \). This shows that \( \gamma \) is well-defined. The linearity is clear.

**Lemma 8.4.** For each \( \theta \in Z^1(\mathfrak{g}, Z^1(\mathfrak{n}, V))_c \), \( x.(v, n) := (x.v + \theta(x)(n), x.n) \) defines a continuous action of \( \mathfrak{g} \) on the semidirect sum \( \hat{n} := V \ltimes \mathfrak{n} \), so that \( \hat{h}_\theta := \hat{n} \ltimes \mathfrak{g} \) is an extension of \( \mathfrak{h} \) by \( V \). We thus obtain a well-defined map

\[
\varphi: H^1(\mathfrak{g}, Z^1(\mathfrak{n}, V)) \to H^2(\mathfrak{h}, V), \quad [\theta] \mapsto [\hat{h}_\theta].
\]

**Proof.** A 2-cocycle defining the extension \( \hat{h}_\theta \) is given by

\[
\omega_\theta((n_1, x_1), (n_2, x_2)) = [(0, (n_1, x_1)), (0, (n_2, x_2))] - (0, [n_1, n_2] + x_1.n_2 - x_2.n_1, [x_1, x_2]) = (0, \theta(x_1)(n_2) - \theta(x_2)(n_1)).
\]

If \( \theta \) is a coboundary, i.e., there exists a \( \beta \in Z^1(\mathfrak{n}, V) \), with \( \theta(x) = x.\beta, x \in \mathfrak{g} \), then \( \theta(x)(n) = x.\beta(n) - \beta(x.n) \) leads to

\[
\omega_\theta((n_1, x_1), (n_2, x_2)) = \theta(x_1)(n_2) - \theta(x_2)(n_1) = x_1.\beta(n_2) - x_2.\beta(n_1) + \beta(x_1.n_2 - x_2.n_1).
\]

If \( \tilde{\beta}: \mathfrak{h} \to V \) denotes the continuous linear map extending \( \beta \) and vanishing on \( \mathfrak{g} \), then \( \omega_\theta \) is a coboundary:

\[
(d_{\mathfrak{h}}\tilde{\beta})(n_1, x_1), (n_2, x_2)) = x_1.\beta(n_2) - x_2.\beta(n_1) + d_n(\beta)(n_1, n_2) - \beta(x_1.n_2 - x_2.n_1)
\]

\[
= x_1.\beta(n_2) - x_2.\beta(n_1) - \beta(x_1.n_2 - x_2.n_1) = \omega_\theta((n_1, x_1), (n_2, x_2)).
\]

**Theorem 8.5.** With the linear map

\[
\eta = (d_n)_*: H^2(\mathfrak{g}, V) \to H^2(\mathfrak{g}, Z^1(\mathfrak{n}, V))_c, \quad [f_\mathfrak{g}] \mapsto [-d_n \circ f_\mathfrak{g}],
\]

the following sequence is exact

\[
H^1(\mathfrak{g}, Z^1(\mathfrak{n}, V)) \xrightarrow{\varphi} H^2(\mathfrak{h}, V) \xrightarrow{(R_n, R_\mathfrak{g})} H^2(\mathfrak{n}, V)[a] \oplus H^2(\mathfrak{g}, V) \xrightarrow{\gamma - \eta} H^2(\mathfrak{g}, Z^1(\mathfrak{n}, V))_c.
\]
Proof. To see that the sequence is exact in $H^2(\mathfrak{h}, V)$, we simply note that a $V$-extension corresponds to a class $[f] \in \ker(R_n, R_g)$ is and only if its restrictions to both subalgebras $\mathfrak{n}$ and $\mathfrak{g}$ are trivial. This is equivalent to $\widehat{\mathfrak{h}}$ being of the form $\widehat{\mathfrak{h}} = (V \times \mathfrak{n}) \times \mathfrak{g}$, hence equivalent to $\mathfrak{h}_\theta$ for some $\theta \in Z^1(\mathfrak{g}, Z^1(\mathfrak{n}, V))_c$.

It remains to verify that $\text{im}(R_n, R_g) = \ker(\gamma - \eta)$. If $[f_n] \in \text{im}(R_n)$, then there exists a topologically split extension $\widehat{\mathfrak{h}}$ of $\mathfrak{h}$ by the topological $\mathfrak{h}$-module $V$. Then $\widehat{\mathfrak{h}}$ contains $\widehat{\mathfrak{n}}$ as an ideal, so that the adjoint action defines an action of $\widehat{\mathfrak{h}}$ on $\widehat{\mathfrak{n}}$. As $V$ acts non-trivially on $\widehat{\mathfrak{n}}$, this action does in general not factor through an action of $\mathfrak{h}$. Let $p: \widehat{\mathfrak{h}} \to \mathfrak{h}$ denote the quotient map. Then $\widehat{\mathfrak{n}} = p^{-1}(\mathfrak{n})$, and we put $\widehat{\mathfrak{g}} := p^{-1}(\mathfrak{g})$. As we have seen in Remark 8.2, the action of $\widehat{\mathfrak{g}}$ is described by some $\widehat{\theta} \in Z^1(\widehat{\mathfrak{g}}, C^1(\mathfrak{n}, V))_c$ satisfying

$$d_n(\widehat{\theta}(x)) = x.f \quad \text{for } x \in \widehat{\mathfrak{g}}. \quad (21)$$

Writing $\widehat{\mathfrak{g}}$ as $V \oplus_{f_n} \mathfrak{g}$, it follows that $\theta(x) := \widehat{\theta}(0, x)$ satisfies (19), so that $[f_n] \in H^2(\mathfrak{n}, V)[\mathfrak{g}]$. For $x, y \in \mathfrak{g}$ we then obtain the relation

$$x.\theta(y) - y.\theta(x) = x.\widehat{\theta}(0, y) - y.\widehat{\theta}(0, x) = \widehat{\theta}([0, x], [0, y])$$

$$= -d_n(f_n(x, y)) + \theta([x, y]),$$

i.e., $d_\mathfrak{g}\theta = -d_n \circ f_n$. Hence, any element $([f_n], [f_\mathfrak{g}]) \in \text{im}(R_n, R_g)$ is contained in the kernel of

$$\gamma - \eta: H^2(\mathfrak{n}, V)[\mathfrak{g}] \oplus H^2(\mathfrak{g}, V) \to H^2(\mathfrak{g}, Z^1(\mathfrak{n}, V))_c.$$

If, conversely, $[f_n] \in \ker(\gamma - \eta)$, then there exists $\theta \in C^1(\mathfrak{g}, C^1(\mathfrak{n}, V))_c$ with $d_\mathfrak{g}\theta = -d_n \circ f_n$, and then $\widehat{\theta}(v, x) := \theta(x) - d_n v$ defines a representation $\widehat{\mathfrak{g}} := V \oplus_{f_n} \mathfrak{g} \to \text{der} \mathfrak{n}$, compatible with the actions of $\mathfrak{g}$ on $\mathfrak{n}$ and $V$.

Then the semidirect product $\widehat{\mathfrak{n}} \rtimes \widehat{\mathfrak{g}}$ is an extension of $\mathfrak{n} \rtimes \mathfrak{g} = \mathfrak{h}$ by the space $V \times V$. Since both $V$-factors act in the same way on $\widehat{\mathfrak{n}}$, the anti-diagonal $\Delta_V$ acts trivially, hence commutes with $\widehat{\mathfrak{n}}$. Since $\widehat{\mathfrak{g}}$, resp., $\mathfrak{g}$, acts diagonally on $V \times V$, it also preserves the anti-diagonal. We thus obtain the $V$-extension $\widehat{\mathfrak{h}} := (\widehat{\mathfrak{n}} \rtimes \widehat{\mathfrak{g}}) / \Delta_V$, of $\mathfrak{h}$. This extensions contains $\widehat{\mathfrak{n}}$ and $\widehat{\mathfrak{g}}$ as subalgebras. This completes the proof. \qed

Corollary 8.6. If $\mathfrak{n}$ is topologically perfect and $V = V^n$, then

$$(R_n, R_g): H^2(\mathfrak{h}, V) \to H^2(\mathfrak{n}, V)[\mathfrak{g}] \oplus H^2(\mathfrak{g}, V)$$

is a linear bijection.

Proof. In this case $Z^1(\mathfrak{n}, V)$ vanishes, and the assertion follows from Theorem 8.5. \qed
Appendix C. Triviality of the group action on Lie algebra cohomology

Theorem 9.1. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and $V$ a Mackey complete smooth $G$-module. Then the natural action of $G$ on the Lie algebra cohomology $H^\bullet(\mathfrak{g}, V)$ is trivial.

Proof. Since the space $H^\bullet(\mathfrak{g}, V)$ carries no natural locally convex topology for which the action of $G$ on this space is smooth, we cannot simply argue that the triviality of the $G$-action follows from the triviality of the $\mathfrak{g}$-action.

The assertion is clear for $p = 0$, because $H^0(\mathfrak{g}, V) = V^G = V^G$ follows from the triviality of the $\mathfrak{g}$-action on the closed $G$-invariant subspace $V^G$ of $V$ (Ne06a, Rem. II.3.7). We may therefore assume $p > 0$. Let $\omega \in Z^p(\mathfrak{g}, V)$ be a continuous $p$-cocycle and $g \in G$. We have to find an $\eta \in C^{p-1}(\mathfrak{g}, V)$ with $g.\omega - \omega = d_g \eta$. Let $\gamma : [0, 1] \to G$ be a smooth curve with $\gamma(0) = 1$ and $\gamma(1) = g$. We write $\delta^I(\gamma)_t := \gamma(t)^{-1}.\gamma(t)$ for its left logarithmic derivative and recall the Cartan formula $L_x = i_x \circ d_x + d_x \circ i_x$ for the action of $\mathfrak{g}$ on $C^\bullet(\mathfrak{g}, V)$. We thus obtain in the pointwise sense on each $p$-tuple of elements of $\mathfrak{g}$:

$$g.\omega - \omega = \int_0^1 \frac{d}{dt} \gamma(t).\omega \, dt = \int_0^1 \gamma(t).\left( L_{\delta^I(\gamma)} \omega \right) \, dt$$

$$= \int_0^1 \gamma(t).\left( i_{\delta^I(\gamma)} d_g \omega + d_g i_{\delta^I(\gamma)} \omega \right) \, dt = \int_0^1 d_g \left( \gamma(t).i_{\delta^I(\gamma)} \omega \right) \, dt$$

$$= d_g \int_0^1 \left( \gamma(t).i_{\delta^I(\gamma)} \omega \right) \, dt.$$

For $p = 2$ we obtain in particular $g.\omega - \omega = d_g \eta_g$ for

$$\eta_g(x) := \int_0^1 \gamma(t).\omega(\delta^I(\gamma)_t, \text{Ad}(\gamma(t))^{-1}.x) \, dt. \quad (22)$$

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