Theta vacuum effects on the pseudoscalar condensates and the \( \eta' \) meson in 2-dimensional lattice QED

Hidenori Fukaya \(^a\), Tetsuya Onogi \(^a\†\)

\(^a\)Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

We study the chiral condensates and the \( \eta' \) meson correlators of the massive Schwinger model in \( \theta \neq 0 \) vacuum. Our data suggest that the pseudoscalar operator does condense in a fixed topological sector and gives long range correlations of the \( \eta' \) meson. We find that this is well understood from the clustering decomposition and statistical picture. Our result also indicates that even in \( \theta = 0 \) case, the long range correlation of \( \eta' \) meson receives non-zero contributions from all the topological sectors and that their cancellation is non-trivial and requires accurate measurement of the reweighting factors as well as the expectation values. It is then clear that the fluctuation of the “disconnected” diagram originates from the pseudoscalar condensates.

1. Introduction

In QCD or the massive Schwinger model in \( \theta = 0 \) vacuum \([1–4]\), it is well known that the scalar operator condense while the pseudoscalar does not;

\[
\langle \bar{\psi} \psi \rangle \neq 0, \quad \langle \bar{\psi} \gamma_5 \psi \rangle = 0 \quad (\theta = 0),
\]

where the second equation follows from parity symmetry. However, if we have a non-zero \( \theta \) term, which violates parity symmetry, both of them have non-zero expectation values;

\[
\langle \bar{\psi} \psi \rangle \neq 0, \quad \langle \bar{\psi} \gamma_5 \psi \rangle \neq 0 \quad (\theta \neq 0),
\]

which indicates that \( \eta' \) meson should have a long-range correlation as

\[
\lim_{|x| \to \text{large}} \langle \eta'(x) \eta(0) \rangle \propto \langle \bar{\psi} \gamma_5 \psi(x) \rangle \langle \bar{\psi} \gamma_5 \psi(0) \rangle \neq 0.
\]

We would like to present our numerical results of the 2-flavor massive Schwinger model with a \( \theta \neq 0 \) term. We investigate \( \bar{\psi} \gamma_5 \psi \) condensates and the \( \eta' \) meson correlators in each topological sector. It is also found that they are non-trivially related each other to reproduce the \( \theta \) dependence. We find that their behavior is well understood by the intuitive picture based on the clustering decomposition and the statistical mechanics.

In particular, our study shows that the accurate contributions from higher topological sectors are essential in order to assure parity symmetry, which never allows the long-range correlation of the \( \eta' \) meson correlators. Moreover, it is also shown that the origin of the fluctuations of disconnected diagram is from pseudoscalar condensates in each topological sector.

2. Strategy for simulations

Our strategy to calculate the \( \theta \) vacuum effects is to separate the integral of the gauge fields into topological sectors;

\[
\langle O \rangle_{\beta, m}^\theta = \frac{\sum_{Q=-\infty}^{+\infty} e^{iQ\theta} \langle O \rangle_{Q, \beta, m}^Q Z^Q(\beta, m)}{\sum_{Q=-\infty}^{+\infty} e^{iQ\theta} Z^Q(\beta, m)},
\]

where \( \beta = 1/g^2 \) denotes a coupling constant and \( \langle O \rangle^Q \) and \( Z^Q \) denote the expectation value and the partition function in a fixed topological sector respectively.

The expectation values with a fixed topological charge; \( \langle O \rangle^Q \) are evaluated by generating link variables with the following gauge action \([5]\);

\[
S_G = \begin{cases} 
\sum_F \frac{(1 - \text{Re} P_{\mu \nu}(x))}{1 - (1 - \text{Re} P_{\mu \nu}(x))/\epsilon} & \text{if admissible} \\
\infty & \text{otherwise},
\end{cases}
\]
where $P_{\mu\nu}$ denotes a plaquette and $\epsilon$ is a fixed constant. This action impose the Lüscher’s bound [5] on the gauge fields, which realize an exact topological charge on the lattice, that is never changed in each step of the the hybrid Monte Carlo updatation.

$Z^Q$ normalized by that of zero topological sector can be evaluated by decomposing it into three parts:

$$
R^Q(\beta, m) = \frac{Z^Q(\beta, m)}{Z^Q(\beta, m)} = e^{-\beta S^Q_{\text{Gmin}}^\text{classical solution}} \times \left( \frac{\Delta S^Q}{\text{moduli integral}} \right) \right.
$$

where $S^Q_{\text{Gmin}}$ denotes the classical minimum of the gauge action with topological charge $Q$, $\int dA_{\beta}^Q$ denotes the moduli integral, and $\Delta S^Q \equiv \langle S^Q_{\beta} - S^Q_{\text{Gmin}} \rangle - \langle S^Q_{\beta} \rangle_{\beta}$, all of which are numerically calculable [6,7).

We choose the domain-wall fermion action with Pauli-Villars regulators for sea quarks. The link variables are updated by the hybrid Monte Carlo algorithm. The parameters are chosen as $\beta = 1.0$, $m = 0.1, 0.15, 0.2, 0.25, 0.3$. We take $L_x = L_y = 16$ and $L_z = 6$ lattice where $L_z$ denotes the size of the extra dimension of domain-wall fermions. 50 molecular dynamics steps with a step size $\Delta t = 0.035$ are performed in one trajectory. Configurations are updated every 10 trajectories. For each topological sector, around 500 trajectories are taken for the thermalization staring from the initial configuration which is the classical instanton solution with topological charge $Q$. We generate 300 configurations in $-|Q_{\text{max}}| \leq Q \leq |Q_{\text{max}}|$ sectors for the measurements and from 1000 to 10000 for the reweighting factors at various $\beta$, where $|Q_{\text{max}}| = 4$ at $m = 0.1, 0.15, 0.2$ and $|Q_{\text{max}}| = 5$ at $m = 0.25, 0.3$.

### 3. Results

The topological charge dependence of pseudoscalar condensate is derived by the anomaly equation;

$$
-\langle \bar{\psi} \gamma_5 \psi \rangle = \frac{Q}{V},
$$

where $V$ denotes the volume of the torus. As seen in Fig. 1, our data show a good agreement with this equation.

Then the $\eta'$ meson correlators should have long-range correlations. From the clustering decomposition, this can be expressed as

$$
\langle \eta'^{(x)}(0) \rangle_{Q, Q'}^{x \to \text{large}} \rightarrow - \sum_{Q'} P_{Q, Q'} \langle \sum_f \bar{\psi}_f \gamma_5 \psi_f \rangle_B^{Q'} (\sum_f \bar{\psi}_f \gamma_5 \psi_f)_A^{Q-Q'},
$$

where $\langle \rangle_{A,B}$ means the expectation value with the topological charge $Q'$ in the region $A, B$ which denote the half of the large box, where the pseudoscalar operators reside, respectively and $P_{Q, Q'}$ denotes the probability of the distribution where $Q'$ instantons appear in the box $B$ and $Q - Q'$ appear in the box $A$.

In $Q = 0$ case, one obtains

$$
\langle \eta'^{(x)}(0) \rangle_{Q, Q'}^{x \to \text{large}} = + \sum_{Q'} P_{0, Q'} \left( \langle \sum_f \bar{\psi}_f \gamma_5 \psi_f \rangle_B^{Q'} \right)^2 > 0,
$$

where we assume $\langle \rangle_A = \langle \rangle_B$ and use the antisymmetry; $\langle \bar{\psi}_f \gamma_5 \psi_f \rangle_{-Q'} = - \langle \bar{\psi}_f \gamma_5 \psi_f \rangle_{Q}$ as seen in Fig. 1. On the other hand, at large $Q$, assuming the distribution $P_{Q, Q'}$ to be Gaussian around $Q' \sim Q/2$, the correlation can be evaluated as follows,

$$
\langle \eta'^{(x)}(0) \rangle_{Q, Q'}^{x \to \text{large}} = - \sqrt{\frac{\alpha}{\pi}} \int dQ' e^{-\alpha (Q' - Q/2)^2} \times \langle \sum_f \bar{\psi}_f \gamma_5 \psi_f \rangle_B^{Q'} (\sum_f \bar{\psi}_f \gamma_5 \psi_f)_A^{Q-Q'} \sim 4Q^2/mV + 4Q^2/mV,
$$

where $1/\sqrt{\alpha} << Q/2$ is a numerical constant. As seen in Fig 2, it is surprising that these very simple arguments describe the data quite well.
dependence of the \( \eta' \) correlators are evaluated by substituting the data into Eq.(4). Fig. 3 shows the result. It is obvious that there are long-range correlations at \( \theta \neq 0 \) while \( \theta = 0 \) case is consistent with zero, which suggests our reweighting method works well at small \( \theta \).

4. Summary

We study \( Q \) and \( \theta \) dependence of the pseudoscalar condensates and the \( \eta' \) meson correlators. We find that pseudoscalar does condense in each topological sector;

\[-\langle \bar{\psi} \gamma_5 \psi \rangle^Q = \frac{Q}{mV} \neq 0, \quad (11)\]

and there exists a long-range correlation of \( \eta' \) meson;

\[\lim_{|x| \to \text{large}} \langle \eta'^\dagger(x) \eta'(0) \rangle^Q \neq 0 \]

\[\lim_{|x| \to \text{large}} \langle \eta'^\dagger(x) \eta'(0) \rangle^{Q>2} \sim -\frac{4Q^2}{m^2V^2} < 0, \quad (12)\]

which are well understood by the clustering properties.

It is also found that each contribution from different topological sectors plays very important role to produce non-trivial \( \theta \) dependence of these observables. In particular, the cancellation the long-range correlation of \( \eta' \) meson requires accurate measurements of higher topological sectors. It is also obvious that the fluctuation of the disconnected diagrams originates from these pseudoscalar condensates.

It would be interesting to extend our studies to 4-dimensional QCD.

REFERENCES

1. S. R. Coleman, R. Jackiw and L. Susskind, Annals Phys. 93, 267 (1975).
2. S. R. Coleman, Annals Phys. 101, 239 (1976).
3. A. V. Smilga, Phys. Rev. D 55, 443 (1997).
4. J. E. Hetrick, Y. Hosotani and S. Iso, Phys. Lett. B 350, 92 (1995).
5. M. Lüscher, Nucl. Phys. B 549, 295 (1999).
6. H. Fukaya and T. Onogi, Phys. Rev. D 68, 074503 (2003); H. Fukaya and T. Onogi, arXiv:hep-lat/0403024.