THE REES ALGEBRA FOR CERTAIN MONOMIAL CURVES

Debasish Mukhopadhyay and Indranath Sengupta

Abstract. Let $K$ be a perfect field and let $m_0 < m_1 < m_2 < m_3$ be a sequence of coprime positive integers such that they form a minimal arithmetic progression. Let $\mathcal{P}$ denote the defining ideal of the monomial curve $C$ in $K^4$, defined by the parametrization $X_0 = T^{m_0}, X_1 = T^{m_1}, X_2 = T^{m_2}, X_3 = T^{m_3}$. Let $R = K[X_0, X_1, X_2, X_3]$. In this article, we find the equations defining the Rees algebra $R[\mathcal{P}t]$ explicitly and use them to prove that the blowup scheme $\text{Proj } R[\mathcal{P}t]$ is not smooth. This proves Francia's conjecture in affirmative, which says that a dimension one prime in a regular local ring is a complete intersection if it has a smooth blowup.

Keywords: Monomial Curves, Gröbner Basis, Rees Algebra.
Mathematics Subject Classification 2000: 13P10, 13A30.

1. Introduction

Blowup Algebras, in particular the Rees algebra $R(I) = R[It]$ ($t$ a variable) and the associated graded ring $\mathcal{G}(I) = R(I)/IR(I)$ of an ideal $I$ in a Noetherian ring $R$ play a crucial role in the birational study of algebraic varieties. The scheme $\text{Proj}(R(I))$ is the blowup of $\text{Spec}(R)$ along $V(I)$, with $\text{Proj}(\mathcal{G}(I))$ being the exceptional fiber. Although blowing up is a fundamental operation, an explicit understanding of this process remains an open problem. For example, Francia’s conjecture stated in O’Carroll-Valla (1997) says: If $R$ is a regular local ring and $I$ is a dimension one prime ideal in $R$ then $I$ is a complete intersection if $\text{Proj}(R(I))$ is a smooth projective scheme. A negative answer to this conjecture was given by Johnson-Morey (2001; 1.1.5), for $R = \mathbb{Q}[x, y, z]$. It is still unknown whether the conjecture is true or not for polynomial rings $R$ over an algebraically closed field. It is evident that a good understanding of the defining equations of the Rees algebra is necessary to answer such queries and an explicit computation of these equations is often extremely difficult. In this context, the Cohen-Macaulay and the normal properties of blowup algebras have attracted the attention of several authors because they help in describing these algebras qualitatively.

Our aim in this article is to use the Elimination theorem to explicitly compute the defining equations of the Rees algebra for certain one dimensional prime ideals $\mathcal{P}$, namely those which arise as the defining ideal of the affine monomial curve given by the parametrization $X_0 = T^{m_0}, X_1 = T^{m_1}, X_2 = T^{m_2}, X_3 = T^{m_3}$ such that $m_0 < m_1 < m_2 < m_3$ is a sequence of positive integers with gcd 1, which form an arithmetic progression (see Section 4 for complete technical details). The explicit form of these equations will be used

---

Research supported by the DST Project No. SR/S4/MS: 614/09.
in Section 6, in conjunction with the Jacobian Criterion for smoothness over a perfect field to prove that \( \text{Proj} R[\varphi t] \) is not smooth. It is known from the work of Maloo-Sengupta (2003), that \( \varphi \) is not a complete intersection. Hence Francia’s conjecture is true for \( \varphi \) over any perfect field \( K \).

2. Equations defining the Rees Algebra

In order to compute the equations defining the Rees algebra \( \mathcal{R}(I) \), we view \( \mathcal{R}(I) \) as quotients of polynomial algebras. Thus for a Rees Algebra \( \mathcal{R}(I) \), it amounts to the study of the natural homomorphism associated to the generators \((a_1, \ldots, a_m)\) of \( I \)

\[
\hat{R} = R[T_1, \ldots, T_m] \xrightarrow{\varphi} R[It], \quad \varphi(T_i) = a_i t;
\]

and particularly of how to find \( E = \ker(\varphi) \), and analyze its properties. \( E \) will be referred to as the equations of \( \mathcal{R}(I) \) or the defining ideal of \( \mathcal{R}(I) \). One approach to get at these equations goes as follows. Let

\[
R^r \xrightarrow{\varphi} R^m \to I \to 0,
\]

be a presentation of the ideal \( I \). \( E_1 \) is generated by the 1-forms

\[
[f_1, \ldots, f_r] = [T_1, \ldots, T_m], \varphi = T \cdot \varphi.
\]

The ring \( R[T_1, \ldots, T_m]/(E_1) \) is the symmetric algebra of the ideal \( I \), and we write \( \mathcal{A} = E/(E_1) \) for the kernel of the canonical surjection

\[
0 \to \mathcal{A} \to S(I) \to \mathcal{R}(I) \to 0.
\]

If \( R \) is an integral domain, \( \mathcal{A} \) is the \( R \)-torsion submodule of \( S(I) \). The ideal \( I \) is said to be an ideal of linear type if \( \mathcal{A} = 0 \), i.e., \( E = E_1 \), or equivalently the symmetric algebra and the Rees algebra are isomorphic. We will come across a natural class of such ideals in Section 6. An ideal reference is Vasconcelos (1994) for more on Blowup algebras and related things.

3. Computational Methods

In this section, we assume the basic knowledge of Gröbner bases and recall the Elimination Theorem below, mostly from the book Cox-Little-O’Shea (1996).

3.1. The Elimination Theorem. Let \( K[t_1, \ldots, t_r, Y_1, \ldots, Y_s] \) be a polynomial ring over a field \( K \). Let \( \mathfrak{a} \) be an ideal in \( K[t_1, \ldots, t_r, Y_1, \ldots, Y_s] \). The \( r \)-th elimination ideal is \( \mathfrak{a}(r) = \mathfrak{a} \cap K[Y_1, \ldots, Y_s] \). We can actually compute a Gröbner basis for \( \mathfrak{a}(r) \), if we know that of \( \mathfrak{a} \) and if we choose a monomial order suitably on \( K[t_1, \ldots, t_r, Y_1, \ldots, Y_s] \). Let \( >_\varepsilon \) be a monomial order on \( K[t_1, \ldots, t_r, Y_1, \ldots, Y_s] \), such that

\[
t_1 >_\varepsilon \cdots >_\varepsilon t_r >_\varepsilon Y_1 >_\varepsilon \cdots >_\varepsilon Y_s
\]

and monomials involving at least one of the \( t_1, \ldots, t_r \) are greater than all monomials involving only the remaining variables \( Y_1, \ldots, Y_s \). We then call \( >_\varepsilon \) an elimination order with respect to the variables \( t_1, \ldots, t_r \).

One of the main tools for computing the equations of the Rees algebras is the Elimination Theorem, which is the following:
Theorem 3.1. Let $G$ be a Gröbner basis for the ideal $a$ in $K[t_1, \ldots, t_r, Y_1, \ldots, Y_s]$, where the order is an elimination order $>_{\mathcal{E}}$ with respect to the variables $t_1, \ldots, t_r$. Then $G_r = G \cap K[Y_1, \ldots, Y_s]$ is a Gröbner basis of the $r$-th elimination ideal $a(r)$, with respect to $>_{\mathcal{E}}$.

Proof. See Cox-Little-O'Shea (1996; Chapter 3). □

Let $I = (a_1, \ldots, a_m)$ be an ideal in the polynomial ring $R := K[Z_1, \ldots, Z_n]$, over a field $K$. The presentation of the Rees algebra $R[It]$ is obtained as:

Proposition 3.2. In the ring $R[z_1, \ldots, z_m, t]$, consider the ideal $a$ generated by the polynomials $z_j - ta_j$, $j = 1, \ldots, m$. Then $R[It] = R[z_1, \ldots, z_m]/E$, where $E = a \cap R[z_1, \ldots, z_m]$.

Proof. It is clear that $E \supset a \cap R[z_1, \ldots, z_m]$. Conversely, if $f(z_1, \ldots, z_m)$ is an element of $E$, we write

$$f(z_1, \ldots, z_m) = f(ta_1 + (z_1 - ta_1), \ldots, ta_m + (z_m - ta_m))$$

and we can use Taylor expansion to show that $f \in a$. □

Proposition 3.3. Let $R$, $a$ and $E$ be as defined in the Proposition 1.2.3. Let $>_{\mathcal{E}}$ be an elimination order with respect to the variable $t$ on $R[z_1, \ldots, z_m, t]$, with $t >_{\mathcal{E}} Z_i, z_j$. If $G$ is a Gröbner basis for $a$ with respect to $>_{\mathcal{E}}$, then $G \cap R[z_1, \ldots, z_m]$ is a Gröbner basis for $E$.

Proof. Follows from Theorem 3.1 and Proposition 3.2. □

We end this section with the statement of the Jacobian Criterion for smoothness, which will be used for verifying smoothness of the blowup; see Kunz (1985; page 171) for a proof.

Theorem 3.4. Let $R = K[Z_1, \ldots, Z_n]$ be a polynomial ring over a perfect field $K$. Let $J = (f_1, \ldots, f_m)$ be an ideal in $R$ and set $S = R/J$. Let $p$ be a prime ideal of $R$ containing $J$ and write $\kappa(p) = K(R/p)$ for the residue field at $p$. Let $c$ be the codimension of $J_p$ in $R_p$.

1. The Jacobian matrix

$$J := (\partial f_i/\partial Z_j),$$

taken modulo $p$ has rank atmost $c$.

2. $S_p$ is a regular local ring iff the matrix $J$, taken modulo $p$, has rank $c$.

4. Monomial Curves

Let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of nonnegative integers and the set of integers respectively. Assume that $0 < m_0 < m_1 < \ldots < m_p$ form an arithmetic sequence of integers, with $p \geq 2$ and $\gcd(m_0, \ldots, m_p) = 1$. We further assume that $m_i = m_0 + id$ where $d$ is the common difference of the arithmetic sequence and $m_0, m_1, \ldots, m_p$ generate the numerical semigroup $\Gamma := \sum_{i=0}^{p} \mathbb{N} m_i$ minimally. Write $m_0 = ap + b$, where $a$ and $b$ are unique integers such that $a \geq 1$ (otherwise $m_0, m_1, \ldots, m_p$ cannot generate the numerical semigroup $\Gamma$ minimally) and $1 \leq b \leq p$. Let $\wp$ denote the kernel of the map $\eta : R := K[X_0, X_1, \ldots, X_p] \to K[T]$, given by $\eta(X_i) = T^{m_i}$. The prime ideal $\wp$ is an one-dimensional perfect ideal and it is the defining ideal of the affine monomial curve given by the parametrization $X_0 = T^{m_0}, \ldots, X_p = T^{m_p}$. A minimal
binomial generating set $G$ for $\varphi$ was constructed by Patil (1993). It was proved by Sengupta (2003) that it is a Gröbner basis with respect to the graded reverse lexicographic monomial order. It was noted in Maloo-Sengupta (2003) that the set $G$ depends intrinsically on the integer $b$. We therefore write $G_b$ instead of $G$, which is $G_b := \{ \phi(i,j) \mid i, j \in [1, p - 1] \} \cup \{ \psi(b, j) \mid j \in [0, p - b] \}$, such that

(i) $\phi(i, j) := \begin{cases} X_iX_j - X_{\varepsilon(i,j)}X_{i+j-\varepsilon(i,j)}, & \text{if } i, j \in [1, p - 1]; \\ 0, & \text{otherwise}; \end{cases}$

(ii) $\psi(b, j) := \begin{cases} X_{b+j}X_p^a - X_jX_0^{a+d}, & \text{if } j \in [0, p - b]; \\ 0, & \text{otherwise}; \end{cases}$

with

(iii) $\varepsilon(i, j) := \begin{cases} i + j & \text{if } i + j < p; \\ p & \text{if } i + j \geq p \end{cases}$

(iv) $[a, b] = \{ i \in \mathbb{Z} \mid a \leq i \leq b \}$.

We now restrict our attention to $p = 3$, since we will be dealing only with monomial curves in affine 4-space, parametrized by four integers $m_0, \ldots, m_3$ in arithmetic progression. Let us write, $R_b = K[X, \Psi_b, \Phi]$, such that $\Psi_b = \{ \Psi(b, 0), \Psi(b, 1), \ldots, \Psi(b, 3 - b) \}$, $\Phi = \{ \Phi(2, 2), \Phi(1, 2), \Phi(1, 1) \}$ and $X = \{ X_1, X_2, X_3, X_0 \}$ are indeterminates. The indeterminate $X_0$ in the set $X$ has been listed at the end deliberately, keeping the monomial order in mind, to be defined in the next section.

Let $t$ be an indeterminate. We define the homomorphism $\varphi_b : R_b \rightarrow R[\varphi_t]$ as $\varphi_b(X_i) = X_i$, $\varphi_b(\Phi(i,j)) = \phi(i,j)t$, $\varphi_b(\psi(b, j)) = \psi(b, j)t$. Let $E_b$ denote the kernel of $\varphi_b$. Our aim is to construct a minimal Gröbner basis for the ideal $E_b$. Write $S = R_b[t]$ and define the ring homomorphism $\overline{\varphi} : S \rightarrow R[\varphi_t]$ as $\overline{\varphi}(t) = t$ and $\overline{\varphi} = \varphi_b$ on $R_b$. We follow the method of elimination described in Propositions 3.2 and 3.3 and consider the ideal $a_b \subseteq S$ such that $a_b \cap R_b = E_b$. We shall compute a Gröbner basis $\hat{a}_b$ for $a_b$, with respect to an elimination order $>_E$ (with respect to $t$) on $S$. Then, $\hat{a}_b \cap R_b$ is a Gröbner basis for $E_b$, that is, those elements of $\hat{a}_b$ that do not involve the variable $t$. These generators of $E_b$ will be used to decide the non-smoothness of the blowup in section 6. We now define the desired elimination order on $S$.

5. Elimination order on $S = R_b[t]$

A monomial in $S = R_b[t] = R[\Psi(0, 0), \ldots, \Psi(3 - b, 2), \Phi(2, 2), \Phi(1, 2), \Phi(1, 1), t]$ is given by

$$t^dX_0^\alpha \Psi_2^\beta \Phi_1^\gamma = t^d(\prod_{i=0}^{3-b} \Psi(b,i)^{\beta_i}) (\Phi(2, 2)^{\gamma_1} \Phi(1, 2)^{\gamma_2} \Phi(1, 1)^{\gamma_3})$$

which is being identified with the ordered tuple $(d, \alpha, \beta, \gamma) \in \mathbb{N}^{12-b}$, such that

$$\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_0), \quad \beta := (\beta_0, \ldots, \beta_{3-b}), \quad \gamma := (\gamma_1, \gamma_2, \gamma_3).$$

We denote slightly from those introduced by Patil (1993) in the following manner: The embedding dimension in our case is $p + 1$ and not $p$; the indeterminates $X_0, \ldots, X_p, Y$ have been replaced by $X_0, \ldots, X_p$; the binomials $\xi_{ij}$ occur in our list of binomials $\phi(i,j)$; the binomial $\theta$ is $\psi(b, p - b)$ in our list.
Let us define the weight function \( \hat{\omega} \) on the non-zero monomials of \( S \) to be the function with the property
\[
\hat{\omega}(fg) = \hat{\omega}(f) + \hat{\omega}(g),
\]
for any two non-zero monomials \( f \) and \( g \) in \( S \), and that
\[
\hat{\omega}(t) = 1, \quad \hat{\omega}(X_i) = m_i, \quad \hat{\omega}(\Phi(i,j)) = \hat{\omega}(X_iX_j), \quad \hat{\omega}(\Psi(b,j)) = \hat{\omega}(X_{3}^{a}X_{b+j}).
\]

We say that \( t^{d}X^{a}\Psi^{\beta}\Phi^{\gamma} \geq \varepsilon \ t^{d'}X^{a'}\Psi^{\beta'}\Phi^{\gamma'} \), if one of the following holds:

(i) \( d > d' \); 
(ii) \( d = d' \) and \( \hat{\omega}(X^{a}\Psi^{\beta}\Phi^{\gamma}) > \hat{\omega}(X^{a'}\Psi^{\beta'}\Phi^{\gamma}) \); 
(iii) \( d = d' \), \( \hat{\omega}(X^{a}\Psi^{\beta}\Phi^{\gamma}) = \hat{\omega}(X^{a'}\Psi^{\beta'}\Phi^{\gamma}) \) and \( \sum \beta_i > \sum \beta'_i \); 
(iv) \( d = d' \), \( \hat{\omega}(X^{a}\Psi^{\beta}\Phi^{\gamma}) = \hat{\omega}(X^{a'}\Psi^{\beta'}\Phi^{\gamma}) \), \( \sum \beta_i = \sum \beta'_i \) and in the difference \( (\beta - \beta') \), the rightmost non-zero entry is negative;
(v) \( d = d' \), \( \hat{\omega}(X^{a}\Psi^{\beta}\Phi^{\gamma}) = \hat{\omega}(X^{a'}\Psi^{\beta'}\Phi^{\gamma}) \), \( \beta = \beta' \) and \( \sum \gamma_i > \sum \gamma'_i \); 
(vi) \( d = d' \), \( \hat{\omega}(X^{a}\Psi^{\beta}\Phi^{\gamma}) = \hat{\omega}(X^{a'}\Psi^{\beta'}\Phi^{\gamma}) \), \( \beta = \beta' \), \( \sum \gamma_i = \sum \gamma'_i \) and in the difference \( (\gamma - \gamma') \), the rightmost non-zero entry is negative;
(vii) \( d = d' \), \( \hat{\omega}(X^{a}\Psi^{\beta}\Phi^{\gamma}) = \hat{\omega}(X^{a'}\Psi^{\beta'}\Phi^{\gamma}) \), \( \beta = \beta' \), \( \gamma = \gamma' \) and in the difference \( (\alpha - \alpha') \), the rightmost non-zero entry is negative.

Then \( \geq \varepsilon \) is the desired elimination order on \( S \), with respect to the variable \( t \).

**Theorem 5.1.** Given \( b \in \{1, 2, 3\} \), let \( a_b \) be the ideal in \( S \), generated by

\[
\begin{align*}
\bullet \ P(i,j) &= \begin{cases} tX_iX_j - tX_{\epsilon(i,j)}X_{i+j-\epsilon(i,j)} - \Phi(i,j) & , \quad i, j \in [1, 2], \\
0 & , \quad \text{otherwise};
\end{cases} \\
\bullet \ P(\Psi(b,l)) &= \begin{cases} tX_{b+l}X_{3}^{a} - tX_{l}X_{0}^{a+d} - \Psi(b,l) & , \quad l \in [0, 3 - b], \\
0 & , \quad \text{otherwise};
\end{cases}
\end{align*}
\]

A Gröbner Basis for the ideal \( a_b \) is the set
\[
\tilde{a}_b = \{ P(i,j), P(\Psi(b,j)), M(b,j), L(i), B(i,j), A(i;b,j), D, Q(b,i) \}
\]
such that,

\[
\begin{align*}
\bullet \ D &= (X_{2}^{2} - X_{2}X_{0})\Phi(1, 2) - (X_{1}X_{2} - X_{3}X_{0})\Phi(1, 1); \\
\bullet \ B(i,j) &= \begin{cases} (X_{i}X_{j} - X_{\epsilon(i,j)}X_{i+j-\epsilon(i,j)})\Psi_{b,3-b} - (X_{3}^{a}X_{b} - X_{0}^{a+d+1})\Phi(i,j) & , \quad \text{if} \quad i, j \in [1, 2], \\
0 & , \quad \text{otherwise};
\end{cases}
\end{align*}
\]
The following Lemma will be used for proving Theorem 5.1.

For our convenience let us set the following:

- \( A(i; b, j) = \begin{cases} 
    X_i \Psi(b, j) - X_{b+i-j-(i,b+j)} \Psi(b, \epsilon(i, b + j) - b) - X_3^a \Phi(i, b + j) \\
    + X_0^{a+d} [\Phi(i, j) - \Phi(b + i + j - 3, 3 - b)] \\
    0 & \text{if } i \in [1, 3], j \in [0, 2 - b] \text{ and } b \neq 3, 
  \end{cases} \)

- \( L(i) = \begin{cases} 
    X_i \Phi(2, 2) - X_{i+1} \Phi(1, 2) + X_{i+2} \Phi(1, 1) & \text{if } i \in [0, 1], \\
    0 & \text{otherwise}; 
  \end{cases} \)

- \( Q(b, i) = \begin{cases} 
    \Psi(1, 0) \Phi(2, 2) + \Psi(1, 2) \Phi(1, 1) - \Psi(1, 1) \Phi(1, 2) & \text{if } b = 1 \text{ and } i = 1, \\
    \Psi(1, 1)^2 - \Psi(1, 2) \Phi(1, 0) - X_3^{a-1} \Psi(1, 2) \Phi(2, 2) + X_0^{a+d-1} \Psi(1, 0) \Phi(1, 1) \\
    - X_3^{a-1} X_0^{a+d-1} (\Phi(1, 2)^2 - \Phi(2, 2) \Phi(1, 1)) & \text{if } b = 1 \text{ and } i = 2, \\
    \Psi(2, 0)^2 \Phi(2, 2) - X_3^{a-1} \Psi(2, 1) \Phi(2, 2)^2 - \Psi(2, 1) \Psi(2, 0) \Phi(1, 2) - X_3^{a-1} X_0^{a+d-1} \Phi(1, 2)^3 \\
    + \Psi(2, 1)^2 \Phi(1, 1) + X_3^{a-1} X_0^{a+h-1} \Phi(2, 2) \Phi(1, 2) \Phi(1, 1) + X_0^{a+d-1} \Phi(2, 0) \Phi(1, 1)^2 \\
    \text{if } b = 2 \text{ and } i = 1, \\
    0 & \text{otherwise}; 
  \end{cases} \)

- \( M(b, i) = \begin{cases} 
    t X_0^{a+d+1} \Psi(b, i) + \Psi(b, 0) \Psi(b, i) - t X_3^{a-1} X_{1+b+i} X_{b-1} \Psi(b, 3 - b) \\
    - t X_3^{a} \Phi(b, b + i) + (-1)^{i+1} t X_3^{a} X_0^{a+d} X_{3b+3i-3} \Phi(3 - b - i, 3 - b - i) \\
    \text{if } i \in [0, 2 - b] \text{ and } b \neq 3, \\
    0 & \text{otherwise}. 
  \end{cases} \)

For our convenience let us set the following:

1. \( V(i, j; q) = \begin{cases} 
    B(i, j) & \text{if } q = 1, \\
    P(i, j) & \text{if } q = 2; 
  \end{cases} \)

2. \( U(q) = \begin{cases} 
    \Psi(b, 3 - b) & \text{if } q = 1, \\
    t & \text{if } q = 2; 
  \end{cases} \)

3. \( u(q) = \begin{cases} 
    \psi(b, 3 - b) & \text{if } q = 1, \\
    1 & \text{if } q = 2. 
  \end{cases} \)

4. \( X_i = 0 \text{ if } i \notin [0, 3]; \)

5. \( \Phi(i, j) = \Phi(j, i); \)

6. \( \Phi(i, j) = 0 \text{ if } i, j \notin [1, 2]; \)

7. \( \Psi(b, j) = 0 \text{ if } j \notin [0, 3 - b]; \)

8. \( \phi(i, j) = \phi(j, i) \text{ and } V(i, j; q) = V(j, i; q). \)

The following Lemma will be used for proving Theorem 5.1.
Lemma 5.2. Given $b \in \{1, 2, 3\}$, let $\mathfrak{Q}_b$ be the ideal in $S$, generated by $\{P(i, j), P(\Psi(b, 3-b))\}$. A Gröbner Basis for $\mathfrak{Q}_b$ is the set $\widehat{\mathfrak{Q}}_b = \{P(i, j), P(\Psi(b, 3-b)), L(i), B(i, j), D\}$.

Proof. We apply the Buchberger's criterion and show that all the $S$-polynomials reduce to zero modulo $\widehat{\mathfrak{Q}}_b$. If $\gcd(L\text{m}(f), L\text{m}(g)) = 1$, then the $S$-polynomials reduce to 0 modulo $\widehat{\mathfrak{Q}}_b$. Let us consider the other cases, that is when the gcd is not one.

1. $S(V(1, 1; q), L(1)) = \Phi(2, 2)V(1, 1; q) - X_i\Phi(b, 0)L(1)$, where $i \in [1, 2]
   = -u(q)[X_1X_0\Phi(2, 2) - X_iX_2\Phi(1, 2)]
   = -X_{1+i}U(q)q + X_2\Phi(1, 1)

2. $S(V(1, 1; q), D) = X_i^{-1}\Phi(1, 2)V(1, 1; q) - X_i^{-1}U(q)D$, where $i \in [1, 2]
   = -X_0U(q)\Phi(1, 2)[X_i^{-1}X_{1+i} - X_i^{-1}X_2] - u(q)X_i^{-1}\Phi(1, 2)\Phi(1, 1)
   = X_0^{-1}\Phi(1, 1)V(i, 2; q) + X_2\Phi(1, i - 1)V(1, 2; q) + u(q)\Phi(1, 2)L(i - 2)

Note that the LT is $X_i^{-1}U(q)X_1X_2\Phi(1, 1)$ if $i = 1$, and the LT is $X_i^{-1}U(q)X_2X_0\Phi(1, 2)$ if $i = 2$.

3. $S(V(1, 1; q), V(2, 2; q)) = X_2^2V(1, 1; q) - X_1^2V(2, 2; q)
   = -u(q)[X_2X_0 - X_i^2X_3] + u(q)[X_2^2\Phi(2, 2) - X_2^2\Phi(1, 1)]
   = X_1X_3V(1, 1; q) - X_2X_0V(2, 2; q) + u(q)[X_1L(1) - X_2L(0)]

4. $S(V(1, 1; q), V(2, q)) = X_1V(1, 1; q) - X_jV(i, i; q)$, where $i \in [1, 2]$ and $j \in \{1, 2\} \setminus \{i\}
   = -u(q)[X_iX_3X_0 - X_jX_{\epsilon(1,i)}X_{\epsilon(1,2)}] - u(q)[X_1\Phi(1, 2) - X_j\Phi(1, 1)]
   = X_{3i-3}V(j; j; q) + u(q)L(2 - j)

5. $S(L(0), L(1)) = X_1L(0) - X_0L(1) = -[X_1^2 - X_2X_0]\Phi(1, 2) + \Phi(1, 2)\Phi(1, 1) = -D$

6. $S(L(1), D) = X_1\Phi(1, 2)L(1) - \Phi(2, 2)D
   = -\Phi(1, 2)[X_1X_2\Phi(1, 2) - X_1X_3\Phi(1, 1) - X_2X_0\Phi(2, 2)] + \Phi(1, 2)\Phi(2, 2)\Phi(1, 1)
   = X_2\Phi(1, 2)L(0) - \Phi(1, 1)[X_3L(0) - X_2L(1)]$

7. $S(P(i, j), B(i, j)) = \Psi(b, 3 - b)P(i, j) - tB(i, j)$, where $i, j \in [1, 2]
   = -\Psi(b, 3 - b)\Phi(i, j) + tz\Psi(b, 3 - b)\Phi(i, j) = \Phi(i, j)P(\Psi(b, 3 - b))$

8. $S(P(i, j), B(l, j)) = X_i\Psi(b, 3 - b)P(i, j) - tX_iB(l, j)$, where $i, j, l \in [1, 2]$ with $l \neq i$
Note that, by Lemma 5.2, every non-zero polynomial \( H \in \hat{(\mathcal{Q}_{\hat{b}})} \), such that \( \mu(3) = 4 = 1 + \text{ht}(3) \), and therefore an ideal of linear type by Huneke (1981) and Valla (1980, 1980/81). It is interesting to note that \( \mu(\mathfrak{q}_b) = 4 = 1 + \text{ht}(\mathfrak{q}_b) \), and what we have proved above shows that \( \mathfrak{q}_b \) is an ideal of linear type for \( b \in [1, 3] \), but \( \mathfrak{q}_b \) is not a prime ideal if \( b \neq 3 \). This produces a class of non-prime ideals of linear type which have the property that \( \mu(\cdot) = 1 + \text{ht}(\cdot) \).

**Proof of Theorem 5.1.**

Proof. We apply the Buchberger’s criterion and show that all the S-polynomials reduce to zero modulo \( \hat{\mathfrak{a}}_{\hat{b}} \). If \( \gcd(\text{Lm}(f), \text{Lm}(g)) = 1 \), then the S-polynomials reduce to 0 modulo \( \hat{\mathfrak{a}}_{\hat{b}} \). Let us consider the other cases, that is when the gcd is not one.

Note that, by Lemma 5.2, every non-zero polynomial \( H \in K[t, X_i, \Psi(b, 3 - b), \Phi(i, j)] \subseteq R_b[t] \), with \( \overline{\text{Lm}}(H) = 0 \), can be expressed as \( H = \sum c_i H_i \), with \( c_i \in R \), \( H_i \in \hat{\mathfrak{a}}_{\hat{b}} \) and \( \text{Lm}(H) \geq \text{Lm}(c_i H_i) \), whenever \( c_i \neq 0 \). Henceforth, the symbols \( G \) and \( H \) will only denote polynomials in \( K[t, X_i, \Psi(b, 3 - b), \Phi(i, j)] \subseteq R_b[t] \), such that \( \overline{\text{Lm}}(H) = 0 \). We use this observation below to prove that the S-polynomials converge to zero. We only indicate the proof for the S-polynomial \( S(A(i; b, j), A(l; b, j)) \), for all other cases the proof is similar.
\( (1) \) \( S(A(i; b, j), A(l; b, j)) = X_i A(i; b, j) - X_i A(l; b, j), \) with \( i < l \) and \( i, l \in [1, 3] \)

\[
= -X_i [X_{b+j-i} \Phi(b, \epsilon(i, b + j) - 1) + X_3 \Phi(b, \epsilon(i, b + j) - 1)] - X_0 \Phi(b, \epsilon(i, b + j) - 1)
\]

where \( \Phi \) is an element of \( R[t, X_1, X_2, X_3, X_0, \Psi(b, 3 - b), \Phi(i, j)] \). Note that the only monomial of \( X_i A(i; b, j) - X_i A(l; b, j), \) which does not belong to \( R[t, X_1, X_2, X_3, X_0, \Psi(b, 3 - b), \Phi(i, j)] \)

\[
= -X_i X_{b+j-i} \Psi(b, \epsilon(i, b + j) - 1) + X_i X_{b+j-i} \Phi(b, \epsilon(i, b + j) - 1) + G
\]

Moreover, \( X_i A(i; b, j), X_i A(l; b, j) \) and \( X_{b+j-i} A(l; b, \epsilon(i, b + j) - 1) \) belong to \( \ker(\Phi) \).

Hence, \( \Phi_i(H) = 0 \). Therefore, we can write

\[
S(A(i; b, j), A(l; b, j)) = X_i A(i; b, j) - X_i A(l; b, j), \text{ with } i < l \text{ and } i, l \in [1, 3];
\]

\[
= -X_i X_{b+j-i} \Psi(b, \epsilon(i, b + j) - 1) + X_i X_{b+j-i} \Phi(b, \epsilon(i, b + j) - 1) + G
\]

Now one can apply Lemma 5.3 to conclude that there exist \( c_i \in R \) and \( H_i \in \Phi_i \), such that

\[ \Phi_i = \sum_i c_i H_i. \]

\( (2) \) \( S(D, A(1; b, j))) = \Phi(b, j)D - X_1 \Phi(1, 2)A(1; b, j) \)

\[
= -X_2 X_0 \Phi(b, j) \Phi(1, 2) - \Phi(1, 2) \Psi(b, j) \Phi(1, 1) + X_1 X_0 \Phi(b, 1 + j) \Phi(1, 2) + G
\]

\[
= -X_2 X_0 \Phi(1, 2) + X_1 \Phi(1, 1)]A(2; b, j) + X_0 \Phi(1, 1)A(3; b, j) + X_0 \Phi(1, 2)A(1; b, 1 + j) + H
\]

\( (3) \) \( S(L, A(1; b, j)) = \Phi(b, j)L(1) - \Phi(2, 2)A(1; b, j) \)

\[
= -X_2 \Phi(b, j) \Phi(1, 2) + X_3 \Psi(b, j) \Phi(1, 1) + X_0 \Phi(b, 1 + j) \Phi(2, 2) + G
\]

\[
= -\Phi(1, 2)[A(2; b, j) - A(1; b, 1 + j)] + \Phi(1, 1)[A(3; b, j) - A(2; b, 1 + j)] + \Psi(b, 1 + j)L(0) + H
\]

\( (4) \) \( S(P, P(b, j)) = X_{i+j} X_{i+l} \Phi(b, j) \), where \( b + j \notin \{i, l\} \) and \( i \leq l \)

\[
= -X_{i+j} X_{i+l} \Phi(b, j) - X_{i+l} X_{i+l} \Phi(b, j) + X_{i+j} X_{i+l} \Phi(b, j) + X_{i+l} X_{i+l} \Phi(b, j) + H
\]

\[
= -X_{i+j} X_{i+l} \Phi(b, j) - X_{i+l} X_{i+l} \Phi(b, j) + X_{i+j} X_{i+l} \Phi(b, j) + X_{i+l} X_{i+l} \Phi(b, j) + H
\]
\( S(P(\Psi(b,j)), P(b + j, l)) = X_1 P(\Psi(b,j)) - X_3^d P(b + j, l) = tX_{e(b+j,l)}X_{b+j-l-\epsilon(b,j,l)}X_3^a - X_1 P(\Psi(b,j)) + G \)
\( = -A(l; b, j) - X_0 P(\Psi(b, 3b + 5j + 3l - 5)) + H \)

\( S(P(\Psi(b,j)), B(b + j, l)) = X_1 P(\Psi(b, 3 - b)) - tX_3^d B(b + j, l) \)
\( = X_{e(b+j,l)}X_{b+j-l-\epsilon(b,j,l)}X_3^a \Psi(b, 3 - b) - X_1 P(\Psi(b,j)) + G \)
\( = -\Psi(b, 3 - b) A(l; b, j) - X_0 P(\Psi(b, 2)) + tX_{e(b+j,l)}X_{b+j-l-\epsilon(b,j,l)}X_3^a - X_1 P(\Psi(b,j)) + G \)
\( = -A(l; b, j) - X_0 P(\Psi(b, 3b + 5j + 3l - 5)) + H \)

\( S(\Psi(b,i), P(\Psi(b,j))) = X_{b+i} P(\Psi(b,i)) - X_{b+i} P(\Psi(b,j)) \) assume \( i < j \)
\( = -tX_{e(b+i,l)}X_3^a + tX_{e(b+i,l)}X_3^a - X_{b+j} \Psi(b,i) + X_{b+i} \Psi(b,j) \)
\( = -X_0 + d[P(i, b + j) - P(j, b + i)] - A(b + j; b, i) + A(b + i; b, j) + H \)

\( S(A(i; b, j), P(i, l)) = tX_i A(i; b, j) - \Psi(b,j) P(i, l) \)
\( = -tX_{e(i,j,l)}X_3^a X_{b+i} \Psi(b,i) - tX_{e(i,j,l)}X_3^a X_{b+i} \Psi(b,i) + \Psi(b,j) P(i, l) + G \)
\( = -tX_{e(i,j,l)}X_{b+i} \Psi(b,j) P(i, l) + \Psi(b,j) P(i, l) + G \)
\( = -X_0 \Psi(b, 2) A(l; b, 3b + 5j + 3i - 5) + X_{i+l-\epsilon(i,l)} \Psi(b, 3 - b) A(e(i,l); b, j) \)
\( + \Psi(i,l) \left[ X_3^a A(3; b, j) - X_0 + d A(3 - b; b, j) \right] + H \)

\( S(M(b,j), P(\Psi(b,i))) = X_3^a X_{b+i} M(b,j) - X_0 + d^+ \Psi(b,j) P(\Psi(b,i)) \)
\( = X_3^a X_{b+i} M(b,j) - X_0 + d^+ \Psi(b,j) P(\Psi(b,i)) \)
\( = X_0 + d^+ [tX_i X_3^a + \Psi(b,i)] + X_3^a X_{b+i} \Psi(b,0) + X_3^a X_{b+i} \Psi(b,0) + G \)
\( = X_0 + d^+ M(b,i) + X_3^a X_{b+i} \Psi(b,0) - X_0 + d^+ \Psi(b,j) P(\Psi(b,j)) \left[ X_0 + d^+ + X_0 + d^+ \right] \}
\( + X_0 + d^+ \left[ \left( Q(b, 2j) \Psi(b, 2j - 2) \right) - X_0 + d^+ \Phi(2b - 1, 2b - 1) P(\Psi(b, 2j - 2)) \right] \}
\( + X_0 + d^+ X_{b+i} \Psi(b,j) A(3; b, 0) - X_0 + d^+ X_{b+i} \Psi(b,j) A(3; b, 0) - X_0 + d^+ \Phi(b, 3 - b) P(\Psi(b,j)) + H \)

\( S(M(b,j), P(i,l)) = X_i M(b,j) - X_0 + d^+ \Psi(b,j) P(i,l) \) assume that \( i \leq l \)
\( = X_i M(b,j) - X_0 + d^+ [tX_i X_3^a + \Psi(b,i)] + \Psi(b,j) + G \)
\( = X_i M(b,j) - X_i \left[ X_0 + d^+ A(l; b, j) + X_3^a \Psi(b, 0) \right] + \Psi(b,j) + G \)
\( + X_i X_0 + d^+ [\Phi(i,j) - \Phi(b + l + j - 3, 3 - b)] P(\Psi(b,0)) \)
\[ + X_0^{a+d+1} [tX_{i+l-\epsilon(i,l)}A(\epsilon(i,l); b, j) - \Phi(i, l)P(\Psi(b, j))] \]
\[ + tX_0 \psi(b, 0)A(3j + 1; b, 7 - 2b - 2j - 2l) + H \]

(12) \[ S(L(0), M(b, j)) = tX_0^{a+d}\Psi(b, j)L(0) - \Phi(2, 2)M(b, j) \]
\[ = -tX_0^{a+d}[X_1 \Phi(1, 2) - X_2 \Phi(1, 1)]\Psi(b, j) - \Psi(b, 0)\Psi(b, j)\Phi(2, 2) + \mathbf{G} \]
\[ = -tX_0^{a+d}[\Phi(1, 2)A(1; b, j) - \Phi(1, 1)A(2; b, j)] - \Psi(b, j)Q(b, b) - Q(b, b - 1) \]
\[ - \Phi(1, 2)Q(b, j) - M(b, j + 1)\Phi(1, 2) + (-1)^b\Phi(b, 3 - b)\Phi(1, b)P(\Psi(b, j)) \]
\[ + \Psi(1, 2)\Phi(1, 2)P(\Psi(b, j - 1)) - X_0^{a+d-1}\Phi(1, j + 1)\Phi(1, 1)P(\Psi(b, b + j - 2)) + \mathbf{H} \]

(13) \[ S(P(\Psi(b, j)), A(3; b, l)) = \Psi(b, l)P(\Psi(b, j)) - tX_0^3A(b + j; b, l) \text{ when } b + j \neq 3 \]
\[ = -\Psi(b, l)[tX_jX_0^{a+d} + \Psi(b, j)] + tX_0X_2^{b+j+l-\epsilon(b+j,b+l)}\Psi(b, \epsilon(b + j, b + l) - b) + \mathbf{G} \]
\[ = -tX_0^{a+d}A(j; b, l) - M(b, j + l) - Q(b, 4l + 2j - 4) + tX_0^{a+d-1}A(3; b, b + l) \]
\[ + \Psi(b, 2)P(\Psi(b, l + j - 2)) + \Psi(b, 1)P(\Psi(b, 4b + j - 9)) - X_0^{a+d-1}\Phi(1, 1)P(\Psi(b, 4l + 3j - 7)) + \mathbf{H} \]

(14) \[ S(M(b, j), A(3; b, l)) = X_1M(b, j) - tX_1^{a+d+1}A(i; b, j) \]
\[ = X_1[\Psi(b, 0)\Psi(b, j) - tX_{b-1}X_{b+j+1}X_3^{a-1}\Psi(b, 3 - b)] + tX_{b+i+j-\epsilon(i,b,j)}X_0^{a+d+1}\Psi(b, \epsilon(i, b+j) - b) + \mathbf{G} \]
\[ = \Psi(b, 0)A(i; b, j) + X_0M(b, b + i + j - 1) \]
\[ - X_0\Psi(b, 3 - b)P(\Psi(b, b + i + j - 3) + \Psi(b, 3 - b)A(b + i + j - 3, b, 0) \]
\[ - X_0^{a+d}[\Phi(i, j) - \Phi(b + i + j - 3, 3 - b)]P(\Psi(b, 0)) + \mathbf{H} \]

Note that the LT is \[ -tX_1X_{b-1}X_{b+j+1}X_3^{a-1}\Psi(b, 3 - b) \text{ if } (i; b, j) \neq (1; 1, 0), \]
and the LT is \[ tX_{b+i+j-\epsilon(i,b,j)}X_0^{a+d+1}\Psi(b, \epsilon(i, b+j) - b) \text{ if } (i; b, j) = (1; 1, 0). \]

Rest of the \( S \)-polynomial computations and their reductions modulo \( \hat{\mathfrak{s}}_0 \) is divided into two cases, depending on \( b = 1 \) and \( b = 2 \).

Case (i): \( b = 1 \)

(1) \[ S(A(i; 1, 0), A(i; 1, 1)) = \Psi(1, 1)A(i; 1, 0) - \Psi(1, 0)A(i; 1, 1) \]
\[-X_{1+i-\epsilon(i,1)}\Psi(1,\epsilon(i,1)-1)\Psi(1,1) + X_{i-1}\Psi(1,0)\Psi(1,2)\]
\[-X_3^0[\Psi(1,1)\Phi(i,1) - \Psi(1,0)\Phi(i,2)] - X_0^{a+d}[\Phi(1,1)\Phi(i-2,2) + \Psi(1,0)\{\Phi(i,1) - \Phi(i-1,2)\}]\]
\[= -X_3^0[-\Phi(i,1)A(3;1,1) - \Phi(i,2)A(3;1,0)] - \Psi(1,2)[A(1+i-\epsilon(i,1);1,1) - A(i-1;1,0)]\]
\[+ X_0^{a+d}Q(1,i-2) - X_0Q(1,i+1) + H\]
Note that the LT is 
\[-X_{1+i-\epsilon(i,1)}\Psi(1,1)\Psi(1,\epsilon(i,1)-1)\] if \(i = 1\),
and the LT is 
\[X_{i-1}\Psi(1,0)\Psi(1,2)\] if \(i \neq 1\).

(2) \[S(P(\Psi(1,0)),L(1)) = \Phi(2,2)P(\Psi(1,0)) - tX_3^0L(1)\]
\[-tX_0^{a+d+1}\Phi(2,2) - \Phi(2,2)\Psi(1,0) + G = -Q(1,1) + \Phi(1,2)P(\Psi(1,1)) + H\]

(3) \[S(P(\Psi(1,0)),D) = X_1\Phi(1,2)P(\Psi(1,0)) - tX_3^0D\]
\[-tX_2X_3^0X_0\Phi(1,2) - X_1\Psi(1,0)\Phi(1,2) + G = -\Phi(1,2)A(1;1,0) + X_0\Phi(1,2)P(\Psi(1,1)) + H\]

(4) \[S(Q(1,1),L(i)) = X_iQ(1,1) - \Psi(1,0)L(i)\]
\[= -\Phi(1,2)[X_i\Psi(1,1) - X_{i+1}\Psi(1,0)] - X_{2+i}\Psi(1,0)\Phi(1,1) + G\]
\[= -\Phi(1,2)[A(i;1,1) - A(i+1;1,0)] - \Phi(1,1)A(i+2;1,0) + H\]

(5) \[S(Q(1,1),A(i;1,0)) = X_iQ(1,1) - \Phi(2,2)A(i;1,0)\]
\[= -X_i\Psi(1,1)\Phi(1,2) + X_{1+i-\epsilon(i,1)}\Psi(1,\epsilon(i,1)-1)\Phi(2,2) + G\]
\[= -\Phi(1,2)[A(i;1,1) - A(1;1,i)] - \Phi(1,1)A(2;1,i) + \Psi(1,i)L(0) + H\]
Note that the LT is 
\[-X_i\Psi(1,1)\Phi(1,2)\] if \(i \neq 1\),
and the LT is 
\[X_{1+i-\epsilon(i,1)}\Psi(1,\epsilon(i,1)-1)\Phi(2,2)\] if \(i = 1\).

(6) \[S(Q(1,1),M(1,0)) = tX_0^{a+d+1}Q(1,1) - \Phi(2,2)M(1,0)\]
\[-tX_0^{a+d+1}\Psi(1,1)\Phi(1,2) - \Psi^2(1,0)\Phi(2,2) + G\]
\[= -\Phi(1,2)M(1,1) - \Psi(1,0)Q(1,1) - \Psi(1,2)\Phi(1,1)P(\Psi(1,0)) + H\]

(7) \[S(Q(1,2),A(i;1,1)) = X_iQ(1,2) - \Psi(1,1)A(i;1,1)\]
\[= -\Psi(1,2)[X_i\Psi(1,0) - X_{i-1}\Psi(1,1)] + X_3\Psi(1,1)\Phi(i,2)\]
\[+ X_0^{a+d-1}[X_i\Psi(1,0)\Phi(1,1) - X_0\{\Phi(i,1) - \Phi(i-2,2)\}\Psi(1,1)] + G\]
\[= -\Psi(1,2)[A(i;1,0) - A(i-1;1,1)] + X_3^{a-1}\Phi(i,2)A(3;1,1)\]
\( S(Q(1,2), M(1,1)) = tX_0^{a+d+1}Q(1,2) - \Psi(1,1)M(1,1) \)
\[
= -tX_0^{a+d+1}[\Psi(1,0)\Psi(1,2) - X_0^{a+d-1}\Psi(1,0)\Phi(1,1)] - \Psi(1,0)\Psi^2(1,1) \\
+ tX_0^a\Psi(1,1)[X_0\Psi(1,2) - X_0^{a+d}\Phi(1,1) + X_0^a\Phi(1,2)] + G \\
= -[\Psi(1,2) - X_0^{a+d-1}\Phi(1,1)]M(1,0) + X_0^{a+d-1}X_3^{-1}[\Psi^2(1,2) - \Phi(2,2)\Phi(1,1)]P(\Psi(1,0)) \\
+ X_3^{a-1}\Psi(1,2)\Phi(2,2)P(\Psi(1,0)) - \Psi(1,0)Q(1,2) + tX_3^{a-1}[X_0\Psi(1,2) + X_0^{a+d}\Phi(1,1) + X_0^a\Phi(1,2)]A(3; 1, 1) \\
- tX_3^{2a-1}[\Psi(1,2)L(1) - X_0^{a+d-1}\{(\Phi(1,2)L(0) + \Phi(1,1)L(1)) \}} + H \\
\]

(9) \( S(M(1,0), M(1,1)) = \Psi(1,1)M(1,0) - \Psi(1,0)M(1,0) \)
\[
= -tX_0^{a-1}\Psi(1,1)[X_2X_0\Psi(1,2) + X_0^a\Phi(1,1) + X_0^{a+d}X_0\Phi(2,2)] \\
+ tX_0^{a-1}\Psi(1,0)[X_3X_0\Psi(1,2) + X_0^a\Phi(1,2) - X_0^{a+d}X_3\Phi(1,1)] + G \\
= -tX_0^{a-1}X_0\Psi(1,2)[A(2; 1, 1) - A(3; 1, 0)] - tX_3^{2a-1}[\Phi(1,1)A(3; 1, 1) - \Phi(1,2)A(3; 1, 0)] \\
- tX_3^{a-1}X_0^{a+d}[\Psi(1,1)L(0) + \Phi(1,2)A(1; 1, 1) - \Phi(1,1)A(2; 1, 1) + \Phi(1,1)A(3; 1, 0)] + H \\
\]

**Case(ii):** \( b = 2 \)

(1) \( S(M(2,0), Q(2,1)) = \Psi(2,0)\Phi(2,2)M(2,0) - tX_0^{a+d+1}Q(2,1) \)
\[
= \Psi(2,0)\Phi(2,2)[\Psi(2,0)\Psi(2,0) - tX_3^aX_1\Psi(2,1) - tX_3^{2a}\Phi(2,2) - tX_3^{a-1}X_0^{a+d}X_3\Phi(1,1)] \\
+ tX_0^{a+d+1}[X_3^{a-1}\Psi(2,1)\Phi^2(2,2) + \Psi(2,0)\Psi(2,1)\Phi(1,2)] + G \\
= \Psi(2,0)Q(2,1) + \Psi(2,1)\Phi(1,2)M(2,0) - X_0^{a+d-1}\Phi^2(1,1)M(2,0) - X_3^{a-1}\Psi(2,0)\Phi^2(2,2)P(\Psi(2,1)) \\
+ \Psi^2(2,1)\Phi(1,1)P(\Psi(2,0)) - X_3^{a-1}X_0^{a+d-1}[\Psi^3(2,2) - \Phi(2,2)\Phi(1,2)\Phi(1,1)]P(\Psi(2,0)) \\
- tX_3^{a-1}\Phi(2,2)[X_1\Psi(2,1)A(3; 2, 0) + X_0^{a+d}\{(\Phi(1,1)A(3; 2, 0) + \Phi(2,2)A(1; 2, 0)) \}} + H \\
\]

(2) \( S(Q(2,1), A(i; 2, 0)) = X_iQ(2,1) - \Psi(2,0)\Phi(2,2)A(i; 2, 0) \)
\[
= -X_i\Psi(2,0)[\Psi(2,1)\Phi(1,2) - X_0^{a+d-1}\Phi(1,1)^2] \\
+ \Psi(2,0)\Phi(2,2)[X_{i-1}\Psi(2,1) + X_0^a\Phi(i, 2) + X_0^{a+d}\Phi(i - 1, 1)] + G \\
= -[\Psi(2,1)\Phi(1,2) - X_0^{a+d-1}\Phi^2(1,1)]A(i; 2, 0) + \Psi(2,1)\Phi(2,2)A(i - 1; 2, 0) \\
+ X_3^{a-1}\Phi(i, 2)\Phi(2,2)A(3; 2, 0) + X_0^{a+d-1}\Phi(i - 1, 1)\Psi(2,0)L(0) \\
+ X_0^{a+d-1}\Phi(i - 1, 1)\Phi(1,2)A(1; 2, 0) - \Phi(1,1)A(2; 2, 0)] + H \\
\]

(3) \( S(Q(2,1), L(i)) = X_iQ(2,1) - \Psi(2,0)^2L(i) \)
\[ \begin{align*}
&= -X_i \Psi(2, 0)[\Psi(2, 1)\Phi(1, 2) - X_0^{a+d-1}\Phi(1, 1)^2] + \Psi^2(2, 0)[X_{1+1}\Phi(1, 2) - X_{2+1}\Phi(1, 1)] + G \\
&= \Psi(2, 0)[\Phi(1, 2)A(i + 1; 2, 0) - \Phi(1, 1)A(i + 2; 2, 0)] \\
&- \Psi(2, 1)[\Phi(1, 1)A(i + 1; 2, 0) - \Phi(1, 2)A(i; 2, 0)] \\
&+ X_3^a[\Phi(2, 1 + i)\Phi(1, 2) - \Phi(2, 2 + i)\Phi(1, 1)]A(3; 2, 0) \\
&+ X_0^{a+d-1}\Phi(1, 1)^2A(i; 2, 0) + H
\end{align*} \]

Hence the proof. \(\square\)

**Theorem 5.6.** Given \(b \in \{1, 2\}\), a Gröbner Basis for the ideal \(E_b\) is the set

\[ \tilde{E}_b = \{A(i; b, j), B(i, j), D, L(i), Q(b, i)\}. \]

**Proof.** Note that, \(\tilde{E}_b = \hat{a}_b \cap R_b\). \(\square\)

Furthermore, if \(b \in \{1, 2\}\), \(b + l = 2\) and \(i, j \in [1, 2]\), we have

\[ B(i, j) = X_{i+1}A(j; b, l) - X_jA(i + 1; b, l) - X_3^aL(i + j - 1) - X_0^{a+d}L(2i + 2j - 5b) + X_0^{a+d}L(7 - b - i - j). \]

Therefore, a smaller set \(\tilde{E}_b = \{A(i; b, j), L(i), Q(b, i), D\}\) generates the ideal \(E_b\).

6. Smoothness of Blowups

Let \(E\) and \(\mathfrak{P} = \langle Y_1, \ldots, Y_{n-1} \rangle\) be prime ideals of a ring \(N = K[Y_1, \ldots, Y_n]\) and \(E \subseteq \mathfrak{P}\). Let \(\mathcal{J}_\mathfrak{P}\) denote the Jacobian matrix of the ideal \(E\), taken modulo \(\mathfrak{P}\). Given an indeterminate \(\zeta \in \{Y_1, \ldots, Y_n\}\), let \(C_\zeta\) denote the column in the matrix \(\mathcal{J}_\mathfrak{P}\), corresponding to the indeterminate \(\zeta\). Then, it is obvious from the construction of \(\mathfrak{P}\), that the column \(C_\zeta\) is non-zero if and only if there exists a polynomial \(F \in E\) such that \(F\) has at least one term of the form \(k\zeta Y_n^l\), for some \(k \in K\) and \(l \in \mathbb{N}\).

Before we prove our last theorem let us record the following observations:

1. \(F \in \tilde{E}_1\) implies that no term of \(F\) is an element of the set

\[ \{X_2\Phi(2, 2), X_3\Phi(2, 2), \Phi(1, 1)\Phi(2, 2), \Phi(1, 2)\Phi(2, 2), \Phi(1, 1)\Phi(2, 2), \Phi(2, 2)\}. \]

2. \(F \in \tilde{E}_2\) implies that no term of \(F\) is an element of the set

\[ \{X_2\Phi(2, 2), X_3\Phi(2, 2), \Phi(2, 1)\Phi(2, 2), \Phi(2, 0)\Phi(2, 2), \Phi(1, 2)\Phi(2, 2), \Phi(1, 1)\Phi(2, 2), \Phi(2, 2)\}. \]

3. \(F \in \tilde{E}_3\) implies that no term of \(F\) is an element of the set

\[ \{X_1\Psi(3, 0), X_2\Psi(3, 0), X_3\Psi(3, 0), X_0\Psi(3, 0), \Psi(2, 2)\Psi(3, 0), \Phi(1, 2)\Psi(3, 0), \Phi(1, 1)\Psi(3, 0), \Psi(3, 0)\}. \]

**Theorem 6.1.** \(\text{Proj } \mathcal{R}(\mathfrak{P})\) is not smooth.
Proof. Let us write
\[
\mathfrak{P}_b = \begin{cases} 
\langle X_1, X_2, X_3, X_0, \Psi(1, 0), \Psi(1, 1), \Phi(1, 2), \Phi(1, 1) \rangle, & \text{if } b = 1; \\
\langle X_1, X_2, X_3, X_0, \Psi(2, 0), \Psi(2, 1), \Phi(1, 2), \Phi(1, 1) \rangle, & \text{if } b = 2; \\
\langle X_1, X_2, X_3, X_0, \Phi(2, 2), \Phi(1, 2), \Phi(1, 1) \rangle, & \text{if } b = 3.
\end{cases}
\]

It is clear that \( \mathfrak{P}_b \) is a homogeneous prime ideal of \( R_b \), containing \( E_b \). Let \( \mathcal{J}_{\mathfrak{P}_b} \) denote the Jacobian matrix, taken modulo \( \mathfrak{P}_b \). Now we use the preceding observations to conclude that

- \( C_\zeta \) is non-zero if and only if \( \zeta \in \{ X_0, X_1, \Psi(1, 0) \} \), for \( b = 1 \).
- \( C_\zeta \) is non-zero if and only if \( \zeta \in \{ X_0, X_1 \} \), for \( b = 2 \).
- \( C_\zeta \) is zero if \( \zeta \in \{ X_1, X_2, X_3, X_0, \Phi(2, 2), \Phi(1, 2), \Phi(1, 1) \} \), for \( b = 3 \).

Hence, the rank of the matrix \( \mathcal{J}_{\mathfrak{P}_b} \) is
\[
\begin{cases} 
3 & \text{when } b = 1; \\
2 & \text{when } b = 2; \\
0 & \text{when } b = 3;
\end{cases}
\]
and the height of the ideal \( (E_b)_{\mathfrak{P}_b} \) in the localized ring \( (R_b)_{\mathfrak{P}_b} \) is \( 6 - b \). Therefore, \( (\mathcal{R}(\wp))_{\mathfrak{P}_b} = (R_b/E_b)_{\mathfrak{P}_b} \) is not regular by Theorem 3.4. Hence, \( \text{Proj} \mathcal{R}(\wp) \) is not smooth. \( \square \)

References

[1] Bayer, D., Stillman, M. (1982-1990). “Macaulay”. A system for computation in algebraic geometry and commutative algebra. Source and object code available at [http://www.zariski.harvard.edu](http://www.zariski.harvard.edu).

[2] Cox, D., Little, J., O’Shea, D. (1996). Ideals, Varieties and Algorithms. New York: Springer-Verlag.

[3] Huneke, C. (1981). Symbolic Powers of Prime Ideals and Special Graded Algebras. Communications in Algebra 9(4): 339-366.

[4] Johnson, M.R., Morey, S. (2001). Normal Blow-ups and their expected defining equations. Journal of Pure and Applied Algebra 162: 303-313.

[5] Kunz, E. (1985). Introduction to Commutative Algebra and Algebraic Geometry. Boston: Birkhäuser.

[6] Maloo, A.K., Sengupta, I. (2003). Criterion for Complete Intersection for Certain Monomial Curves. Advances in Algebra and Geometry, University of Hyderabad Conference 2001, Edited by C.Musili, Hindustan Book Agency, pp. 179-184.

[7] O’Carroll, L., Valla, G. (1997). On The Smoothness of Blowups. Communications in Algebra 25(6): 1861-1872.

[8] Patil, D. P. (1993). Minimal sets of generators for the relation ideal of certain monomial curves. Manuscripta Math. 80: 239-248.

[9] Sengupta, I. (2003). A Gröbner bases for certain affine monomial curves. Communications in Algebra 31(3): 1113-1129.

[10] Valla, G. (1980). On the Symmetric and Rees Algebras of an Ideal. Manuscripta Math. 30(3): 239-255.

[11] Valla, G. (1980/81). Correction and Complements to “On the Symmetric and Rees Algebras of an Ideal”. Manuscripta Math. 33(1): 59-61.

[12] Vasconcelos, W. (1994). Arithmetic of Blowup Algebras. LMS Lecture Note Series 195, UK: Cambridge University Press.
Acharya Girish Chandra Bose College, 35, Scott Lane, Kolkata, WB 700009, INDIA.

E-mail address: mdebasish01@yahoo.co.in

School of Mathematical Sciences, Ramakrishna Mission Vivekananda University, Belur Math, Howrah, WB 711 202, INDIA

Current address: DEPARTMENT OF MATHEMATICS, JADAVPUR UNIVERSITY, KOLKATA, WB 700 032, INDIA.

E-mail address: sengupta.indranath@gmail.com