Research Article

On a Subclass of Analytic Functions That Are Starlike with Respect to a Boundary Point Involving Exponential Function

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In the present exploration, the authors define and inspect a new class of functions that are regular in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$, by using an adapted version of the interesting analytic formula offered by Robertson (unexploited) for starlike functions with respect to a boundary point by subordinating to an exponential function. Examples of some new subclasses are presented. Initial coefficient estimates are specified, and the familiar Fekete-Szegö inequality is obtained. Differential subordinations concerning these newly demarcated subclasses are also established.

1. Introduction and Preliminary Results

Let $\mathcal{H}$ be the class comprising of all holomorphic functions in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $\mathcal{A}$ signify the subclass of $\mathcal{H}$ entailing of functions $h \in \mathcal{A}$ be of the form

$$h(z) = cz + \sum_{n=2}^{\infty} a_n z^n, \quad z \in D,$$

with the normalization $h(0) = h'(0) - 1 = 0$. Denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ comprising univalent functions. Two convergent subclasses of $\mathcal{A}$ are familiarized by Robertson [1], are defined with their analytical description as

$$\mathcal{S}^*(\alpha) = \left\{ h \in \mathcal{A} : \frac{ch'(z)}{h(z)} > \alpha, \quad z \in D \right\},$$

and are correspondingly known as starlike and convex functions of order $\alpha(0 \leq \alpha < 1)$. It is well known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}$ and $\mathcal{C}(\alpha) \subset \mathcal{S}$. In interpretation of Alexander’s relation, $h \in \mathcal{C}(\alpha) \Leftrightarrow ch'(z) \in \mathcal{S}^*(\alpha)$ for $z \in D$. For $\alpha = 0$, the class $\mathcal{S}^* := \mathcal{S}^*(0)$ condenses to the well-known class of normalized starlike univalent functions, and $\mathcal{C} := \mathcal{C}(0)$ reduces to the normalized convex univalent functions.

A function $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ written as $f < g$ if there exists $\omega \in \mathcal{H}$ with $\omega(0) = 0$ and $\omega(D) \subset D$ such that $f(z) = g(\omega(z))$ for every $z \in D$. In precise, if $g$ is univalent, then $f < g$ if and only if $f(0) = g(0)$ and $f(D) \subset g(D)$.

Let $\mathcal{P}$ symbolize the class of functions $p \in \mathcal{H}$ with the normalization $p(0) = 1$, i.e., of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in D,$$

and such that $\Re p(z) > 0$ for $z \in D$. Functions in $\mathcal{P}$ are called familiarly as the Carathéodory class of functions. Ma and Minda [2] proposed a appropriate subclass of $\mathcal{P}$ denoted
by \( \mathcal{P}^{*}(1) \) comprising of all \( \Phi \) that is univalent in \( \Omega \) with

\[
\Phi(0) = 1 ; \Phi'(0) > 0, \quad (4)
\]

\( \Phi(\Omega) \) is symmetric with respect to the real axis

(2) Starlike with respect to 1

He also represented the class \( \Phi \in \mathcal{P}^{*}(1) \) by

\[
\Phi(\zeta) = 1 + \sum_{n=1}^{\infty} B_n \zeta^n, B_1 > 0 ; \zeta \in \Omega. \quad (5)
\]

The class \( \mathcal{P}^{*}(1) \) plays a vital part in defining generalized form of holomorphic functions. Ma and Minda [2] considered the function \( \Phi \in \mathcal{P}^{*}(1) \) and defined \( \mathcal{P}^{*}(\Phi) \) as the class of all \( h \in \mathcal{A} \) such that \( ch'(\zeta)/h(\zeta) < \Phi(\zeta) \) for \( \zeta \in \Omega \). The above functions defined are called as functions of Ma and Minda kind. Observe that \( \mathcal{P}^{*}(\alpha) = \mathcal{P}^{*}(\Phi) \) with \( \Phi(\zeta) = (1 + (1 - 2\alpha)\zeta)/(1 - \zeta), \zeta \in \Omega \).

There are recent articles ([3–6]) where subclasses of \( \mathcal{A} \) were defined by using subordination satisfying the relation \( ch'(\zeta)/h(\zeta) < \Phi(\zeta) \) for \( \zeta \in \Omega \) (see also [7, 8]). In particular, the exponential function \( \Phi(\zeta) = e^\zeta \), an entire function in \( \mathbb{C} \) has positive real part in \( \Omega \), \( \Phi(0) = 1 \), \( \Phi'(0) = 1 \), and \( \Phi_3(\Omega) = \{w \in \mathbb{C} : \log|w| < 1\} \), is symmetric with respect to the real axis and starlike with respect to 1. Further, \( \Phi_3 \in \mathcal{P}^{*}(1) \) and therefore, it is now to make a remark that the class

\[
\mathcal{P}_3 = \left\{ f \in \mathcal{A} : \frac{\zeta f'(\zeta)}{f(\zeta)} < \Phi(\zeta) = e^\zeta, \zeta \in \Omega \right\} \quad (6)
\]

is well defined. For an attractive study on starlike functions connected with the exponential function, an individual can refer to Mendiratta et al. [9, 10] (see also the works of [11–13]).

We recall the class of close-to-convex functions denoted by \( \mathcal{H} \) introduced and studied by Kaplan [14]. A function \( h \in \mathcal{H} \) is called to be close-to-convex if and only if there exist a function \( \psi \in \mathcal{C} \) and \( \beta \in (-\pi/2, \pi/2) \) such that

\[
\Re \left( \frac{e^{i\beta}h'(\zeta)}{\psi(\zeta)} \right) > 0, \quad \zeta \in \Omega. \quad (7)
\]

Remark at this time that even though starlikeness of a fixed order has been discussed and well thought-out in detail in countless articles in excess of a elongated stage of period, class of univalent functions \( g \in \mathcal{H} \) that maps \( \Omega \) onto \( \Omega \), starlike domain with reverence to a boundary point is still a conception that is not exclusively explored. Robertson [15] recognized this examination and introduced a new subclass

\[
\mathcal{G}^* = \left\{ g \in \mathcal{H} : \Re \left( e^{\delta} g(\zeta) \right) > 0 ; \delta \in \mathbb{R}, \zeta \in \Omega \right\}, \quad (8)
\]

with

\[
g(0) = 1, \quad g(1) = \lim_{r \to 1} g(r) = 0, \quad (9)
\]

and maps (univalently) \( \Omega \) onto a domain starlike with respect to the origin. Presume in addition that the constant function \( g \equiv 1 \in \mathcal{G}^* \), in addition, Robertson through a conjecture that \( \mathcal{G}^* \) coincides with the class \( \mathcal{G} \) of all \( g \in \mathcal{H} \) of the structure

\[
g(\zeta) = 1 + \sum_{n=1}^{\infty} B_n \zeta^n, \quad \zeta \in \Omega, \quad (10)
\]

such that

\[
\Re \left( \frac{2c^g g'(\zeta)}{g(\zeta)} + \frac{1 + \zeta}{1 - \zeta} \right) > 0, \quad \zeta \in \Omega, \quad (11)
\]

proving that \( \mathcal{G} \subset \mathcal{G}^* \). Definitely, in the same article Robertson showed that if \( g \in \mathcal{G} \) and \( g \neq 1 \), then \( g \in \mathcal{G} \) and so univalent in \( \Omega \). It is importance of citing that (11) was identified by much erstwhile by Styer [16]. This surmise of Robertson that \( \mathcal{G}^* \) coincide with the class \( \mathcal{G} \) was soon after proved by Lyzzaik [17], where he established that \( \mathcal{G}^* \subset \mathcal{G} \).

A different analytical categorization of starlike functions with respect to a boundary point was proposed by Lecko and Lyzzaik [19] (see [[20], Chapter VII] as well). Encouraged by the article of Robertson [15], Aharanov et al. [21] (see also [22]) investigated about the class of functions that are spirallike with respect to a boundary point. Let

\[
P(\zeta ; M) := \frac{4\zeta}{\sqrt{(1 - \zeta)^2 + 4\zeta/M}} \sqrt{1 + 1} = 1, \quad \zeta \in \Omega, \quad (12)
\]

be the Pick function. By using the Pick function \( P(\zeta ; M) \), the author in [23] considered another closely related class to \( \mathcal{G} \), the family \( \mathcal{G}(M), M > 1 \), comprising of all \( g \in \mathcal{H} \) of the form (10) such that

\[
\Re \left( \frac{2c^g g'(\zeta)}{g(\zeta)} + \frac{c^g P(\zeta ; M)}{P(\zeta ; M)} \right) > 0, \quad \zeta \in \Omega. \quad (13)
\]

In [24], Todorov established a structural formula and coefficient estimates by associating \( \mathcal{G} \) with a functional \( f(\zeta)/1 - \zeta \) for \( \zeta \in \Omega \). For \( g \in \mathcal{H} \) in (10), Obradović and Owa [25] and Silverman and Silvia [26] separately introduced the classes

\[
\mathcal{G}_x = \left\{ g \left( \frac{2cg'(\zeta)}{g(\zeta)} + (1 - \alpha) \frac{1 + \zeta}{1 - \zeta} \right) > 0, \quad \zeta \in \Omega \right\}, \quad (14)
\]
where \( \alpha \in [0, 1) \). The authors in [26] confirmed a remarkable fact that for each \( \alpha \in [0, 1) \), the class \( \mathcal{G}_\alpha \) is a subclass of \( \mathcal{G}^* \). Clearly, \( \mathcal{G}_{1/2} = \mathcal{G} \) and appealing coefficient inequalities of \( \mathcal{G} \) were established in [27].

For \( g \in \mathcal{H} \) assumed as in (10) and \(-1 < E \leq 1 \); \(-E < F \leq 1 \), Jakubowski and Włodarczyk [28] defined the class \( \mathcal{G}(E, F) \) as

\[
\mathcal{R}(j(z)) > 0, \quad z \in \mathcal{D},
\]

where

\[
j(z) = \frac{2 \alpha g'(z)}{g(z)} + \frac{1 + Ez}{1 - Fz}.
\]

By desirable quality of the initiative proposed in [2], Mohd and Darus [29] presented a new class \( \mathcal{G}_\alpha^*(\Phi) \), where \( \Phi \in \mathcal{P}^*(1) \), of all \( g \in \mathcal{H} \) of the form (10) such that

\[
\frac{2 \alpha g'(z)}{g(z)} + \frac{1 + \alpha z}{1 - \alpha} \leq \Phi(z), \quad z \in \mathcal{D}.
\]

An additional appealing class on the above direction was in recent times analyzed by Lecko et al. [30].

The most important intend of the present article is to illustrate and do a organized inquiry of the function class defined as below.

**Definition 1.** For \( g \in \mathcal{H} \) and as assumed in (10), we let a new class \( \mathcal{G}_\rho \) as

\[
\mathcal{G}_\rho = \left\{ g \in \mathcal{H} : \frac{2 \rho g'(z)}{g(z)} + \frac{1 + \rho z}{1 - \rho} \leq \Phi(z), \quad z \in \mathcal{D} \right\}.
\]

**Remark 2.** Note that the condition (18) is well defined, for

\[
p(z) = \frac{2 \rho g'(z)}{g(z)} + \frac{1 + \rho z}{1 - \rho}, \quad z \in \mathcal{D}
\]

is holomorphic in \( \mathcal{D} \).

Based on the description of the class \( \mathcal{G}_\rho \), and on the analytical characterization of the class \( \mathcal{G}^* \) of starlike functions with respect to a boundary point, we can prepare the next result.

### 2. Representation Theorem and Coefficient Results

Let us start the section with the following representation theorem which in fact offers a handy procedure to build functions in our new class \( \mathcal{G}_\rho \).

**Theorem 3.** A function \( g \in \mathcal{G}_\rho \) if and only if there exists \( p \in \mathcal{H} \) such that \( p < \mathcal{P}_\rho \) and

\[
g(z) = (1 - \rho) \exp \left( \frac{1}{2} \int_0^z \frac{p(\zeta) - 1}{\zeta} d\zeta \right), \quad \zeta \in \mathcal{D}.
\]

**Proof.** Let us suppose that \( g \in \mathcal{G}_\rho \), then, a function \( p \) defined by (19) is holomorphic and satisfies \( p < \mathcal{P}_\rho \). Also, (19) can be rewritten in the type

\[
\frac{2 \rho g'(z)}{g(z)} + \frac{2}{1 - \rho} = \frac{p(z) - 1}{\rho}, \quad z \in \mathcal{D}.
\]

This upon integration give

\[
\log \left( \frac{g(z)}{1 - \rho} \right)^2 = \int_0^z \frac{p(\zeta) - 1}{\zeta} d\zeta, \quad z \in \mathcal{D}, \quad \log 1 = 0.
\]

This in essence gives

\[
(g(z))^2 = (1 - \rho)^2 \exp \left( \int_0^z \frac{p(\zeta) - 1}{\zeta} d\zeta \right), \quad z \in \mathcal{D},
\]

which imply (20). \( \square \)

Let us presume \( p < \mathcal{P}_\rho \). By defining a function \( g \) as in (20), and by observing that \( p(0) = 1 \), it is noticeable that \( g \) is holomorphic in \( \mathcal{D} \). A working out shows that \( g \) satisfies (21); so, (19). Thus, \( g \in \mathcal{G}_\rho \), which ends the confirmation of the theorem.

Let \( \Psi_\rho \) be a holomorphic function which is the solution of the differential equation (see also [[10], p. 367])

\[
\frac{\Psi_\rho''(z)}{\Psi_\rho(z)} = e^z, \quad z \in \mathcal{D}, \quad \Psi_\rho(0) = 0, \quad \Psi_\rho'(0) = 1,
\]

i.e.,

\[
\Psi_\rho(z) = e^z + \int_0^z e^\zeta - 1 \frac{d\zeta}{\zeta} = z + e^z + \int_0^z \left( \frac{e^\zeta - 1}{\zeta} \right) d\zeta + \int_0^z \frac{e^\zeta - 1}{\zeta^2} d\zeta + \cdots, \quad z \in \mathcal{D}.
\]

Next, we present few examples for the class \( \mathcal{G}_\rho \).

**Example 4.**

(1) For a specified \( A \in \mathbb{R} \) and \( c \in \mathcal{D} \), let us name

\[
p_A(z) = 1 + Ac, \quad g_A(z) = (1 - c) \exp \left( \frac{Ac}{2} \right), \quad c \in \mathcal{D}.
\]
Note down that \( g_A \in \mathcal{H} \) with \( g_A(0) = 1 \). Observe that

\[
\frac{2 \zeta g_A'(\zeta)}{g_A(\zeta)} + \frac{1 + \zeta}{1 - \zeta} = p_A(\zeta), \quad \zeta \in \mathcal{D}.
\] (27)

We finish that \( g_A \in \mathcal{D}_e \) for \( |A| \leq 1/e \).

(2) Given \(-1 < A \leq 1\) and \(-A < B < 1\), define

\[
w = p_{A,B}(\zeta) = \frac{1 + A\zeta}{1 - B\zeta}, \quad \zeta \in \mathcal{D}.
\] (28)

Then, we identify that \( p_{A,B}(\mathcal{D}) \) is an open disk symmetrical with respect to the real axis centered at \((1 + AB)/(1 - B^2)\) of radius \((A + B)/(1 - B^2)\). In particular, for \( B = A \), this disk is given by

\[
|w - \frac{1 + A^2}{1 - A^2}| < \frac{2A}{1 - A^2},
\] (29)

with diametrical end points \( x_L = (1 - |A|)/(1 + |A|) \) and \( x_R := (1 + |A|)/(1 - |A|) \). Since \( x_L \geq 1/e \) and \( x_R \leq e \) if \( |A| \leq (e - 1)/e + 1 \), we perceive that then \( p_{A,A} \prec \Phi_e \). As a result, a function \( g \in \mathcal{H} \) with \( g(0) = 1 \) defined by

\[
\frac{2 \zeta g'(\zeta)}{g(\zeta)} + \frac{1 + \zeta}{1 - \zeta} = p_{A,A}(\zeta), \quad \zeta \in \mathcal{D},
\] (30)

i.e., the function

\[
g(\zeta) = \frac{1 - \zeta}{1 - A\zeta}, \quad \zeta \in \mathcal{D},
\] (31)

belongs to the class \( \mathcal{D}_e \) for \( |A| \leq (e - 1)/(e + 1) \).

**Theorem 5.** Let \( 0 < r < 1 \). If \( g \in \mathcal{D}_e \), then

\[
(i) \quad \sqrt{\frac{-\Psi_e(-r)}{r}}(1 - r) \leq |g(\zeta)| \leq \sqrt{\frac{\Psi_e(-r)}{r}}(1 + r), \quad |\zeta| = r.
\] (32)

\[
(ii) \quad \left| \arg \frac{g(\zeta_0)}{1 - \zeta_0} \right| \leq \frac{1}{2} \max_{|k| \leq r} \frac{\Psi_e(\zeta)}{\zeta}, \quad |\zeta_0| = r, \quad \arg 1 = 0.
\] (33)

**Proof.** Let \( g \in \mathcal{D}_e \).

(i) Describe the function

\[
h(\zeta) = \frac{\zeta(g(\zeta))^2}{(1 - \zeta)^2}, \quad \zeta \in \mathcal{D}.
\] (34)

Obviously, \( h \) is a holomorphic function in \( \mathcal{D} \), and an uncomplicated working out yields

\[
\frac{ch'(\zeta)}{h(\zeta)} = \frac{2czg'(\zeta)}{g(\zeta)} + \frac{1 + \zeta}{1 - \zeta}, \quad \zeta \in \mathcal{D}.
\] (35)

It is straightforward to witness from the above that \( g \in \mathcal{D}_e \) if and only if

\[
\frac{ch'(\zeta)}{h(\zeta)} < e^\zeta, \quad \zeta \in \mathcal{D}.
\] (36)

By the result of Corollary 1' of [2], we obtain

\[
-\Psi_e(-r) \leq |h(\zeta)| \leq \Psi_e(r), \quad |\zeta| = r,
\] (37)

i.e., by using (34),

\[
-\Psi_e(-r) \leq \left| \frac{\zeta(g(\zeta))^2}{(1 - \zeta)^2} \right| \leq \Psi_e(r), \quad |\zeta| = r,
\] (38)

which gives (32).

(ii) By (36), a function \( h \) defined by (34) belongs to \( \mathcal{D}^* \) \( (\Phi_e) \). Due to Corollary 3' of [2], the inequality

\[
\left| \arg \frac{h(\zeta_0)}{\zeta_0} \right| \leq \max_{|k| \leq r} \frac{\Psi_e(\zeta)}{\zeta}, \quad |\zeta_0| = r
\] (39)

is valid. Using now (34) in turn yields (33). \( \square \)

Next, we ascertain some coefficient results for the class \( g \in \mathcal{D}_e \). Let \( \mathcal{B} = \{ \omega \in \mathcal{H} : |\omega(0)| \leq 1, \omega \in \mathcal{D} \} \) and \( \mathcal{B}_0 \) be the subclass of \( \mathcal{B} \) consisting of functions \( \omega \) such that \( \omega(0) = 0 \). We comment at this time that the elements of \( \mathcal{B}_0 \) are termed as Schwarz functions.

We will pertain two lemmas below to prove our main results.

**Lemma 6.** (see [2]). If \( p \in \mathcal{P} \) is of the form (3), then for \( \mu \in \mathbb{C} \),

\[
|p_2 - \mu p_1^2| \leq 2 \max \{ 1, |2\mu - 1| \}.
\] (40)

In particular, if \( \mu \) is a real number, then

\[
|p_2 - \mu p_1^2| \leq \begin{cases} 
-4\mu + 2, & \mu \leq 0, \\
2, & 0 \leq \mu \leq 1
\end{cases}, \quad 4\mu - 2, \quad \mu \geq 1.
\] (41)

When \( \mu < 0 \) or \( \mu > 1 \), the equality holds true if and only if \( p(\zeta) = (1 + \zeta)/(1 - \zeta) = \mathcal{L}(\zeta), \zeta \in \mathcal{D} \), or one of its rotations. If \( 0 < \mu < 1 \), then the equality holds true if and only if \( p(\zeta) = \mathcal{L}(\zeta^2), \zeta \in \mathcal{D} \), or one of its rotations. If \( \mu = 0 \), the equality
Theorem 8. If $g \in \mathcal{E}_c$ is of the form (10), then
\[
|\theta_1 + 1| \leq \frac{1}{2},
\]
(44)
\[
|\theta_1| \leq \frac{3}{2},
\]
(45)
\[
|2\theta_2 - \theta_1^2 + 1| \leq \frac{1}{2},
\]
(46)
\[
|\theta_2| \leq \frac{3}{4},
\]
(47)
\[
|3\theta_3 - 3\theta_1 \theta_2 + \theta_1^2 + 1| \leq \frac{1}{2},
\]
(48)
and for $\delta \in \mathbb{R}$,
\[
|\theta_2 - \delta \theta_1^2| \leq \frac{1}{4} \left( \max \{|\delta| - 1| + 2|2\delta - 1| + 4|\delta| \right).
\]
(49)

Inequalities (44), (45), (46), (47), and (48) are sharp.

Proof. In view of (18), there exists $\omega \in \mathcal{B}_0$ such that
\[
\frac{2\omega \phi'(\zeta)}{g(\zeta)} + \frac{1 + \zeta}{1 - \zeta} = \Phi_2(\omega(\zeta)) = \exp(\omega(\zeta)), \quad \zeta \in \mathcal{D}.
\]
(50)

By an application of (10), one can easily obtain with simple computation that
\[
\frac{2\omega \phi'(\zeta)}{g(\zeta)} + \frac{1 + \zeta}{1 - \zeta} = 1 + 2(\theta_1 + 1)\zeta + 2(2\theta_2 - \theta_1^2 + 1)\zeta^2
\]
\[
+ 2(3\theta_3 - 3\theta_1 \theta_2 + \theta_1^2 + 1)\zeta^3 + \cdots, \quad \zeta \in \mathcal{D}.
\]
(51)

Define the function $p$ by
\[
p(\zeta) = \frac{1 + \omega(\zeta)}{1 - \omega(\zeta)} = 1 + p_1 \zeta + p_2 \zeta^2 + \cdots, \quad \zeta \in \mathcal{D}.
\]
(52)

Clearly, $p \in \mathcal{P}$. Moreover,
\[
\omega(\zeta) = \frac{p(\zeta) - 1}{p(\zeta) + 1} = \frac{p_1}{2} \zeta + \left( \frac{p_2}{2} - \frac{p_1^2}{4} \right) \zeta^2
\]
\[
+ \left( \frac{p_3}{2} - \frac{p_1 p_2}{4} + \frac{p_1^3}{48} \right) \zeta^3 + \cdots, \quad \zeta \in \mathcal{D}.
\]
(53)

Hence,
\[
\exp(\omega(\zeta)) = 1 + \omega(\zeta) + \frac{(\omega(\zeta))^2}{2} + \frac{(\omega(\zeta))^3}{6} + \cdots = 1 + \frac{p_1 \zeta}{2}
\]
\[
+ \left( \frac{p_2}{2} - \frac{p_1^2}{8} \right) \zeta^2 + \left( \frac{p_3}{2} - \frac{p_1 p_2}{4} + \frac{p_1^3}{48} \right) \zeta^3 + \cdots, \quad \zeta \in \mathcal{D}.
\]
(54)

Substituting (51) and (54) into (50), by comparing the corresponding coefficients, we obtain
\[
2(\theta_1 + 1) = \frac{p_1}{2},
\]
(55)
\[
2(2\theta_2 - \theta_1^2 + 1) = \frac{p_2}{2} - \frac{p_1^2}{8},
\]
(56)
\[
2(3\theta_3 - 3\theta_1 \theta_2 + \theta_1^2 + 1) = \frac{p_3}{2} - \frac{p_1 p_2}{4} + \frac{p_1^3}{48}.
\]
(57)

Since (e.g., ([32], Vol. I, p. 80)),
\[
|p_n| \leq 2, \quad n \in \mathbb{N}.
\]
(58)

From (55), we obtain (44). Rewriting (55) as $\theta_1 = p_1/4 - 1$, (45) easily follows. Further, (56) together with (40) yields
\[
|2(2\theta_2 - \theta_1^2 + 1)| = \left| \frac{p_2}{2} - \frac{p_1^2}{8} \right| \leq 1,
\]
(59)
which proves (46).

Upon applying (55) for $\theta_1$ in (56), we get
\[
4\theta_2 = \frac{p_2}{2} - p_1.
\]
(60)

Hence, by applying (41), we obtain (47).

An application of (43) in (57) gives
\[
|6\theta_3 - 6\theta_1 \theta_2 + 2\theta_1^2 + 2| = \left| \frac{p_3}{2} - \frac{p_1 p_2}{4} + \frac{p_1^3}{48} \right| \leq 1,
\]
(61)
i.e., the inequality (48).
Using (60) and making use of the expression for \( \delta_1 \) and \( \delta_2 \), we get
\[
|\delta_2 - \delta\delta_1| \leq \frac{1}{8} \left( f^{2} - \frac{\delta}{2} f^{1} + 2|2\delta - 1|p_{1} + 8|\delta| \right), \quad \delta \in \mathbb{R},
\]
(62)
which leads to the inequality (49).

Equalities in (44) and (45) hold for the function \( \rho = \mathcal{L} \); in (46) for the function \( \rho(\zeta) = \mathcal{L}(\zeta^2), \zeta \in \mathcal{D} \), in (47) for the function \( \rho(\zeta) = \mathcal{L}(-\zeta), \zeta \in \mathcal{D} \) and in (48) for the function \( \rho(\zeta) = \mathcal{L}(\zeta^3), \zeta \in \mathcal{D} \).

### 3. Differential Subordination Results Involving \( \mathcal{G}_\varepsilon \)

In this segment, we derive certain differential subordination results concerning the class \( \mathcal{G}_\varepsilon \).

To demonstrate differential subordination results, we recollect the next lemma (see [[33]], Theorem 8.4.h, p. 132).

**Lemma 9.** Suppose \( q \) is univalent in \( \mathcal{D} \), \( \theta \) and \( \varphi \) be holomorphic in a domain \( T \) containing \( q(\mathcal{D}) \) with \( q(w) \neq 0 \) when \( w \in q(\mathcal{D}) \). Let \( Q(\zeta) = \varphi(\zeta)q(q(\zeta)) \) and \( h(\zeta) = \theta(q(\zeta)) + Q(\zeta) \) for \( \zeta \in \mathcal{D} \). Suppose that either

\[ Q(\zeta) = \varphi^\prime(\zeta)\theta(q(\zeta)) + Q(\zeta), \quad \zeta \in \mathcal{D}, \]

Assume also that
\[ \Re \left( \frac{\tilde{Q}(\zeta)}{Q(\zeta)} \right) > 0, \quad \zeta \in \mathcal{D}. \tag{63} \]

Let \( p \in \mathcal{H} \) with \( p(0) = q(0), p(\mathcal{D}) \subset \mathcal{D} \) and
\[ \theta(p(\zeta)) + \varphi^\prime(\zeta)p(p(\zeta)) < \theta(q(\zeta)) + \varphi^\prime(\zeta)\varphi(q(\zeta)), \quad \zeta \in \mathcal{D}, \tag{64} \]
then \( \rho < q \) and \( q \) are the best dominant.

**Theorem 10.** Let \( g \in \mathcal{H} \) with \( g(0) = 1 \). If \( g \) satisfies the subordination condition,
\[ 2g^\prime(\zeta) g(\zeta) + \frac{1 + \zeta}{1 - \zeta} < 1 + \zeta, \quad \zeta \in \mathcal{D}. \tag{65} \]

Then,
\[ p(\zeta) = \frac{(g(\zeta))^2}{(1 - \zeta)^2} < e^\zeta, \quad \zeta \in \mathcal{D}. \tag{66} \]

**Proof.** Let \( D = \mathbb{C} \setminus \{0\} \). Let \( \theta(w) = 1, w \in \mathbb{C} \) and \( \varphi(w) = 1/ w, w \in \mathcal{D} \). Note that \( \Phi_c(\mathcal{D}) \subset \mathcal{D} \) and \( \theta \) and \( \varphi \) are holomorphic in \( \mathcal{D} \). Thus,
\[ Q(\zeta) = \varphi^\prime(\zeta)\theta(q(\zeta)) + Q(\zeta) = \frac{\varphi^\prime(\zeta)}{\Phi_c(\zeta)} \varphi(q(\zeta)) = \zeta, \quad \zeta \in \mathcal{D} \tag{67} \]
is well defined and holomorphic. Clearly, \( Q \) is a univalent starlike function and so for a function \( h(\zeta) = \theta(q(\zeta)) + Q(\zeta) = 1 + Q(\zeta), \zeta \in \mathcal{D} \), we achieve
\[ \Re \left( \frac{\tilde{Q}(\zeta)}{Q(\zeta)} \right) = 1 > 0, \quad \zeta \in \mathcal{D}. \tag{68} \]

Hence, for any function \( p \) belonging to \( \mathcal{H} \) with \( p(0) = \Phi_c(0) = 1 \) such that \( p(\mathcal{D}) \subset \mathcal{D} \), i.e., for \( p \) nonvanishing in \( \mathcal{D} \), by applying Lemma 9, we infer that from the subordination
\[ 1 + \varphi^\prime(\zeta)\varphi(p(\zeta)) < 1 + \varphi^\prime(\zeta)\varphi(q(\zeta)), \quad \zeta \in \mathcal{D}, \tag{69} \]
then \( \rho < q \) and \( q \) are the best dominant.

**Theorem 11.** Let \( g \in \mathcal{H} \) with \( g(0) = 1 \). If \( g \) satisfies
\[ \frac{2g^\prime(\zeta) g(\zeta)}{g(\zeta)} + \frac{1 + \zeta}{1 - \zeta} < e^\zeta, \quad \zeta \in \mathcal{D}, \tag{71} \]
then
\[ p(\zeta) = \frac{(g(\zeta))^2}{(1 - \zeta)^2} < e^\zeta, \quad \zeta \in \mathcal{D}. \tag{72} \]

**Proof.** Let \( D = \mathbb{C} \setminus \{0\} \). Let \( \psi(w) = w, w \in \mathbb{C} \), and \( \psi(w) = 1/ w, w \in \mathcal{D} \). Note that \( \Phi_c(\mathcal{D}) \subset \mathcal{D} \) and \( \psi \) and \( \psi \) are holomorphic in \( \mathcal{D} \). Thus, the function \( Q \) defined by (67), i.e., the identity function, is univalent starlike. Hence, for a function \( h(\zeta) = \theta(q(\zeta)) + Q(\zeta) = \Phi_c(\zeta) + Q(\zeta), \zeta \in \mathcal{D} \), we obtain
\[ \Re \left( \frac{\tilde{Q}(\zeta)}{Q(\zeta)} \right) = \Re \left( \frac{\varphi^\prime(\zeta)}{\Phi_c(\zeta)} \varphi(q(\zeta)) \right) = \Re \left( \frac{\varphi^\prime(\zeta)}{\Phi_c(\zeta)} \right) \varphi(q(\zeta)) \equiv Q(\zeta) > 0, \quad \zeta \in \mathcal{D}. \tag{73} \]

Thus, for any function \( p \in \mathcal{H} \) with \( p(0) = \Phi_c(0) = 1 \) such
that \( p(\mathbb{D}) \subset D \), i.e., \( p(\zeta) \neq 0 \) for \( \zeta \in \mathbb{D} \), by applying Lemma 9, we deduce that from the subordination

\[
p(\zeta) + \frac{cp'(\zeta)}{p(\zeta)} < \Phi_\zeta(\zeta) + \frac{c\Phi'_\zeta(\zeta)}{\Phi_\zeta(\zeta)} = e^\zeta + \zeta, \quad \zeta \in \mathbb{D},
\]

it follows the subordination \( p \not\subset \Phi_\zeta \). □

Let now take \( g \in \mathcal{H} \) with \( g(0) = 1 \) and \( g(\zeta) \neq 0 \) for \( \zeta \in \mathbb{D} \) satisfying (65). Define a function \( p \) as in (72). We see that

\[
p(0) = \lim_{\zeta \to 0} \zeta \left( \frac{g(\zeta)}{1-\zeta} \right)^2 \left( \int_0^\zeta \frac{g(\xi)}{1-\xi} \, d\xi \right)^{-1}
= (g(0))^2 \lim_{\zeta \to 0} \zeta \left( \int_0^\zeta \frac{g(\xi)}{1-\xi} \, d\xi \right)^{-1} = 1 = \Phi_\zeta(0),
\]

\( p(\zeta) = 0 \) for \( \zeta \in \mathbb{D} \) and \( p \) is holomorphic. Since

\[
p(\zeta) + \frac{cp'(\zeta)}{p(\zeta)} = \frac{2cg'(\zeta)}{g(\zeta)} + \frac{1+\zeta}{1-\zeta}, \quad \zeta \in \mathbb{D},
\]

from (74), (71) follows which completes the proof.

Data Availability

No data sets were used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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