Infinite-tension strings at $d > 1$

D.V. Boulalov

The Niels Bohr Institute
University of Copenhagen
Blegdamsvej 17, 2100 Copenhagen Ø
Denmark
and
Centre de Physique Theorique, Ecole Polytechnique
91128 Palaiseau, France

Abstract

A matrix model describing surfaces embedded in a Bethe lattice is considered. From the mean field point of view, it is equivalent to the Kazakov-Migdal induced gauge theory and therefore, at $N = \infty$ and $d > 1$, the latter can be interpreted as a matrix model for infinite-tension strings. We show that, in the naive continuum limit, it is governed by the one-matrix-model saddle point with an upside-down potential. To derive mean field equations, we consider the one-matrix model in external field. As a simple application, its explicit solution in the case of the inverted W potential is given.
1 Introduction

Recently the “induced QCD” model proposed by V. Kazakov and A. Migdal [1] has drawn much attention [2, 3, 4, 5]. This model possesses the local $U(1)^{\otimes N}$ symmetry reminiscent to matrix models [6] which makes it inequivalent to QCD. Actually, to see that, it is sufficient to consider only the discrete $Z_N$ subgroup which leads to the local-confinement selection rule [3] incompatible with the usual QCD area law.

However, as will be shown in the present paper, the Kazakov-Migdal (KM) model is interesting by itself. In the planar limit it is soluble by the mean field technique. But, within this approximation, it is equivalent to a matrix model with a Bethe-lattice (BL) embedding space. It means that all possible configurations are tree-like, i.e., have no embedding area. Therefore, it is natural to consider the KM model as an infinite-tension limit of a $d$-dimensional matrix model. Indeed, in this limit, surfaces degenerate to trees (or branched polymers in the other terms). Different aspects of branched polymer physics were discussed from the viewpoint of random surfaces in ref. [7].

In some sense, this model is as interesting as any other non-critical string theory solved so far, because all of them do not have transverse modes on the world sheet. Moreover, except one-dimensional string theory, they do not have a real embedding space and can be interpreted only as the interaction of 2-d quantum gravity with different matter fields. But the $d=1$ matrix model describes trees, because the world sheet has no area by construction. This model was solved in the planar limit in ref. [8, 9] and in the double scaling in [10]. The compactified embedding space was considered in ref. [11], where the exact solution for the singlet sector was obtained and interpreted as vortex-free string theory on a ring. Non-singlet sectors were considered in refs. [12, 13].

Our mean field approach is a direct generalization of the one of refs. [8, 9]. In the Hamiltonian formalism, free fermions interacting through an effective potential appear naturally and the Thomas-Fermi approximation is applicable. We consider only the planar limit, where the connection between KM and BL models holds and the mean field approximation is exact. In the continuum limit the Hamiltonian approach should be equivalent to the Lagrangian one. The latter has been intensively explored in the induced QCD framework. Saddle point equations for the KM model were derived in ref. [2].
and investigated in refs. [2, 4]. The exact solution for the quadratic potential was found in ref. [5].

Mean field gives an exact answer usually for $d$ bigger than some critical dimension and matrix models should not be an exception. It means that, in principle, we can use the BL matrix model as a convenient representation of multi-dimensional non-critical string theory.

The outline of the present paper is the following. In Section 2, we establish the connection between KM and BL models. In Section 3, we describe the mean field approximation within the Hamiltonian formalism. To derive mean field equations, we consider in Section 4 the external field problem for a one-matrix integral and generalize the Brezin-Gross approach to it [4]. So far only a solution for the cubic potential has been known [13]. Our method allows us, in principle, to consider a general potential. In the planar limit, our approach gives a set of equations coinciding with the Migdal’s one. However, a meaning of quantities involved is slightly different and, for finite $N$, we still have rather simple linear equations convenient for the $1/N$ expansion. In Section 5, we investigate the continuum limit of the KM model. Some problems are discussed in Section 6. An explicit solution to the external field problem for the inverted W potential is given in Appendix.

2 The KM model and Bethe lattices

Let us start from a general matrix model with an embedding space being an arbitrary graph $\mathcal{G}$ defined by its incidence matrix $G_{xy}$ ($G_{xy} = 1$, if there is a link connecting vertices $x$ and $y$, and 0, otherwise). The matrix model action reads

$$S = -N \sum_{xy} G_{xy} tr(\Phi(x)\Phi(y)) + N \sum_x tr V_0(\Phi(x))$$

(1)

where $\Phi(x)$ is an $N \times N$ hermitian matrix attached to an $x$'th vertex; the potential, $V_0(\lambda)$, is an arbitrary polynomial. The partition function is defined as the integral over all field configurations

$$Z = \int \prod_{x \in \mathcal{G}} d\Phi(x) \ e^{-S}$$

(2)
In the planar limit \((N \to \infty)\), we adjust coefficients of \(V_0(\lambda)\) so that to find a leading singularity of \(Z\); universal continuum behavior takes place in its vicinity. At each vertex, \(\Phi(x)\) can be decomposed into diagonal \(\varphi(x)\) and angular \(S(x) \in U(N)\) parts:

\[
\Phi(x) = S^+(x)\varphi(x)S(x)
\]  

(3)

It is convenient to introduce the on-link gauge variables

\[
\Omega_{xy} = S^+(x)S(y)
\]

(4)

Obviously they obey the constraint that, for every closed loop \(L\),

\[
\prod_{(xy) \in L} \Omega_{xy} = \Omega_L = I
\]

(5)

where the product runs along \(L\).

If the graph \(G\) is a tree, all gauge variables are independent and can be integrated out by the Itzykson-Zuber (IZ) formula [16]:

\[
I(\phi, \psi) = \int d\Omega \ e^{N \text{tr} \phi \Omega \psi^+} = N^{-N(N-1)/2} \prod_{n=1}^{N-1} n! \frac{\det_{ab} e^{N\phi_a \psi_b}}{\Delta(\phi) \Delta(\psi)}
\]

(6)

In eq. (6), \(\phi\) and \(\psi\) without loss of generality are real and diagonal; \(\Delta(\phi) = \prod_{i<j}(\phi_i - \phi_j)\) is the Van-der-Monde determinant.

As a result, one gets a model where a role of dynamical variables is played by \(N\) eigenvalues of \(\Phi(x)\). In the \(N \to \infty\) limit, the mean field approximation is applicable and, in this sense, the model is soluble.

If the graph \(G\) has loops, constraints (5) can be imposed by averaging a number of \(\delta\)-functions with the corresponding KM action:

\[
Z = Z_{KM} \left\langle \prod_{\{L\}} \delta\left( \prod_{(xy) \in L} \Omega_{xy}, I \right) \right\rangle_{KM}
\]

(7)

where \(Z_{KM}\) is the KM partition function for \(G\). More precisely, we treat variables \(\Omega\) as ordinary gauge field, defined on a given lattice \(G\). To recover the original matrix model, this field has to be a pure gauge. However, eq. (7) does not coincide with the weak coupling limit of a lattice gauge partition function, because the latter is singular and we have to impose the constraints
along non-contractible loops as well (if $\mathcal{G}$ is not simply connected as a $d$-dimensional lattice).

Expanding $\delta$-functions in eq. (7) in representations, we obtain the following integral at each link

$$I_{ab}^R(\phi, \psi) = \int dU \, D_{ab}^R(U) \, e^{N \text{tr} \phi U \psi U^+}$$

(8)

where $D_{ab}^R(U)$ is a matrix element of a $U(N)$ irrep $R$; lower indices run over a representation space $V_R$: $a, b = 1, \ldots, d_R$ ($d_R$ is the dimension of $V_R$); $\phi$ and $\psi$ are real and diagonal.

The integral in eq. (8) goes actually over the right/left coset $U(1)^\otimes N \backslash U(N)/U(1)^\otimes N$ rather than over the $U(N)$ Haar measure, since the action in eq. (7) is invariant under left and right shifts by diagonal unitary matrices

$$U_{ab} \rightarrow U_{ab} e^{i(\alpha_a + \beta_b)}$$

(9)

As was shown in ref. [13], the symmetry (9) gives rise to the selection rule for the integral (8):

$$I_{ab}^R(\phi, \psi) = 0, \quad \text{unless} \quad \sum_{k=1}^N m_k = 0$$

(10)

where $m_k, k = 1, \ldots, N$, are highest-weight components of an irrep $R$ (for $U(N)$ they are unrestricted). Actually, as was noticed for the KM model in ref. [3], in order to obtain eq. (10), it is sufficient to consider only transformations from the center of $SU(N)$.

Inside representation spaces, $V_R$, we also have the selection rule

$$I_{ab}^R(\phi, \psi) = 0, \quad \text{unless} \quad a, b \in V_R^{(0)}$$

(11)

where $V_R^{(0)}$ is the subspace of $V_R$ spanned by all zero-weight vectors [13]. Its dimension is equal to

$$d_R^{(0)} = \int_0^{2\pi} \prod_{k=1}^N \frac{d\alpha_k}{2\pi} \chi_R(e^{i\alpha}),$$

(12)
where $\chi_R(e^{i\alpha})$ is the character of an irrep $R$. $d^{(0)}_R \neq 0$ only for irreps obeying eq. (10). When $N \to \infty$,
\[
d^{(0)}_R = 0(\sqrt{d_R}) \tag{13}
\]

It should be stressed that the selection rule (11) cannot be derived from the $Z_N$ symmetry of the action in (8), because the center acts onto representation spaces globally. The general approach to the calculation of integrals (8) was recently given in ref. [17].

We can look at the $d$-dimensional KM model also from another point of view. The absence of conditions on $\Omega$-variables is reminiscent to matrix models with tree-like embedding spaces. We can consider a Bethe lattice, i.e., an infinite tree having the same coordination number for all vertices. This lattice is a covering of a regular lattice. We can obtain the KM model identifying eigenvalues $\varphi(x)$ which cover the same vertices. The $U(N)$ integration includes the sum over all permutations and it makes eigenvalues indistinguishable. But it is sufficient to demand the equality of corresponding momenta of their distributions. These constraints can be interpreted as conditions on correlators in the BL model. Actually, we restrict only a small part of local degrees of freedom. It suggests that, in the $N \to \infty$ limit, the KM model describes trees. Indeed, the mean field approximation is exact in this limit in both models and they are indistinguishable within it as far as their partition functions are concerned.

This analogy holds also for correlators. Let us consider, for example, the following one
\[
\left\langle \text{tr}(\Phi(x_1)\Phi(x_2))\text{tr}(\Phi(y_1)\Phi(y_2)) \right\rangle_{KM} = \left\langle \text{tr}(\varphi(x_1)\Omega_{L_{x_1x_2}}\varphi(x_2)\Omega_{L_{x_2x_1}})\text{tr}(\varphi(y_1)\Omega_{L_{y_1y_2}}\varphi(y_2)\Omega_{L_{y_2y_1}}) \right\rangle_{KM} \tag{14}
\]

In the KM model we have to connect eigenvalues at different vertices by paths $L_{x_1x_2}$ and multiply $\Omega$-matrices along them:
\[
\Omega_{L_{x_1x_2}} = \prod_{(uv) \in L_{x_1x_2}} \Omega_{uv} \tag{15}
\]

Owing to the selection rule (11), all appearing loops encircle zero area, because the first non-trivial irrep obeying eq. (10) is the adjoint. In the induced QCD framework, this phenomenon is known as the local confinement. If there
are links through which paths from different traces go, we can obtain a non-
trivial answer also for more general loop configurations (an example is shown
in Fig. 1). However, because of the second selection rule (11) and eq. (13),
it has subleading order in $N$. In general, averaging $\Omega$-matrices, we always
lose half powers of $N$, and a correlator has a correct order in $N$ only if the
corresponding index loops are pure backtracks. Therefore, in the planar
limit, the KM model is identical to the BL one.

The ambiguity of the choice of paths can be interpreted as follows. As
far as planar graphs are concerned, the KM model is defined actually on
a covering of a regular lattice, where it is unambiguous and all possible
configurations are tree-like.

In next orders in $N$, we lose this interpretation, as those two models are
not equivalent anymore. The geometrical reason is clear. Already a torus has
non-contractible loops which can wrap around plaquettes without creating
an embedding space area.

3 Mean field in the Hamiltonian approach

Let us consider the Hamiltonian formulation of the KM model. It means
that we treat one of dimensions separately as a time. In this respect, it is
a direct generalization of the standard approach to the $d = 1$ matrix model
\cite{8,9}.

We consider the partition function (2) with the action (1). After the
change of variables (3), (4) let us drop constraints (5) and then integrate
over $\Omega$-matrices in the $t$-direction only. If we introduce the continuous time
keeping a lattice structure in “space directions”, $N$ eigenvalues at each site
of the $(d - 1)$-dimensional lattice become fermions in complete analogy with
refs. \cite{8,9}. We have a fermionic ground state and, in the $N \to \infty$ limit,
the Thomas-Fermi approximation is exact. Each fermion, $\varphi_k$, moves in an
effective potential induced by its neighbours

$$U(\varphi_k) = V_0(\varphi_k) - 2(d - 1) \log I[\varphi, \psi](\varphi_k)|_{\rho(\psi) = \rho(\varphi)}$$  \hspace{1cm} (16)

where $I[\varphi, \psi](\varphi_k)|_{\rho(\psi) = \rho(\varphi)}$ means the IZ integral as a function of a $k$’th eigen-
value $\varphi_k$ when both densities coincide: $\rho(\psi) = \rho(\varphi)$. As we are looking for
the ground state in the planar limit, densities of eigenvalues can be taken
homogeneous in the space. Fermions at different sites interact only through
a collective field, since the IZ integral depends only on traces of powers of $\varphi$ and $\psi$ separately. Hence, we have no Fermi sphere and can consider fermions at each site independently. The one-particle Hamiltonian reads

$$H(p, x) = \frac{1}{2} p^2 + U(x)$$

(17)

$p$ and $x$ are a momentum and a coordinate, respectively. Now, we can repeat standard steps [8]. The Fermi level, $E_F$, is defined by the equation

$$N = \int \frac{dp \, dx}{2\pi} \theta(E_F - H(p, x))$$

(18)

where $\theta(x)$ is the step function.

The ground state energy is given by the formula

$$F = \int \frac{dp \, dx}{2\pi} H(p, x) \theta(E_F - H(p, x))$$

(19)

Differentiating eq. (18) with respect to $E_F$, we find the inverse frequency of a particle moving in the effective potential

$$\frac{\partial N}{\partial E_F} = \frac{1}{\omega(E_F)} = \int \frac{dp \, dx}{2\pi} \delta(E_F - H(p, x))$$

(20)

A simple calculation gives the local density of eigenvalues

$$\rho(x) = \int_{U(x)}^{E_F} \frac{dE}{2\pi} \sqrt{2E - 2U(x)} = \frac{1}{2\pi N} \sqrt{2E_F - 2U(x)}$$

(21)

Critical behavior occurs when the Fermi level reaches a local maximum of the effective potential $U(x)$, which is, in general, quadratic:

$$U(x) = U(x_o) - \frac{a}{2}(x - x_o)^2 + \ldots$$

(22)

However, in our case, there is a possibility to get an answer different from the $d = 1$ model, as the coefficient $a$ can be a function of the Fermi level $E_F$.

Let us introduce the analytic function

$$f(x) = \int dy \frac{\rho(y)}{x - y}$$

(23)
where the integral goes over a support of \( \rho(y) \). From the one-matrix-model solution we know that, in order to get a given density of eigenvalues, \( \frac{1}{\pi} \text{Im} f(x) \), we have to take \( 2 \int \text{Re} f(x) \) as the one-matrix potential. The IZ interactions between lattice sites perturb this simple solution.

To obtain a closed set of equations, we have to express \( U(x) \) through \( \rho(x) \). One of possible ways to do it is to consider the one-matrix model in external field.

4 External field problem revised

Let us consider the matrix integral with the linear source term:

\[
\mathcal{I}[X] = \int d^{N^2}Y \exp N\text{tr}\{XY - V(Y)\}
\]  

(24)

where \( X \) and \( Y \) are \( N \times N \) hermitian matrices, and \( V(Y) \) is an arbitrary function defined by its Taylor expansion \( V(y) = \sum v_k y^k \).

It is obvious that \( \mathcal{I}[X] \) depends only on eigenvalues of \( X \). The standard way to deal with the integral (24) is to write down the Schwinger-Dyson equation

\[
\int d^{N^2}Y \text{tr}\left[ X - V'(Y) \right] \exp N\text{tr}\{XY - V(Y)\} = 0
\]

(25)

and, then, rewrite it in terms of eigenvalues of \( X \).

It is convenient to introduce the resolvent

\[
G_{ij}(z) = \int d^{N^2}Y \left( \frac{1}{z - Y} \right)_{ij} \exp N\text{tr}\{XY - V(Y)\}
\]

(26)

Then, eq. (24) can be rewritten as follows

\[
\oint \frac{dz}{2\pi i} \text{tr} \left[ (X - V'(z))G(z) \right] = 0
\]

(27)

The integral goes along a small circle around \( z = 0 \).

Eq. (27) has to be accompanied by an equation for \( G(z) \) which can be obtained from the following obvious identity

\[
\mathcal{I}[X] = \int d^{N^2}Y \frac{1}{N} \text{tr}\left\{ \left( z - \frac{1}{N} \frac{\partial}{\partial X^i} \right) \left( \frac{1}{z - Y} \right) \right\} e^{N\text{tr}(XY - V(Y))}
\]

(28)
Let us take $X$ almost diagonal

$$X = x + i[A, x] + \ldots$$  \hspace{1cm} (29)

where $A$ is an infinitesimal hermitian matrix: $\|A\| \ll 1$. $[\cdot, \cdot]$ is the commutator. Then, eq. (28) can be rewritten in the form

$$I[X] = \frac{1}{N} \sum_{i,j=1}^{N} \left\{ \left( z - \frac{1}{N} \frac{\partial}{\partial x_i} \right) \delta_{i,j} + \frac{1}{N} \frac{1}{x_i - x_j} \frac{\partial}{\partial A_{ij}} \right\}$$

$$\int d^{N^2} Y \left( \left( \frac{1}{z - Y} \right)_{ij} + [iA, \frac{1}{z - Y}]_{ij} \right) e^{N\text{tr}(xY - V(Y))} \bigg|_{A=0}$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left\{ \left( z - \frac{1}{N} \frac{\partial}{\partial x_i} \right) g_i(z) - \frac{1}{N} \sum_{j \neq i} \frac{1}{x_i - x_j} (g_i(z) - g_j(z)) \right\}$$  \hspace{1cm} (30)

where

$$g_i(z) = \int d^{N^2} Y \left( \frac{1}{z - Y} \right)_{ii} e^{N\text{tr}(xY - V(Y))}$$

are diagonal elements of the resolvent matrix $G_{ij}(z)$. If we normalize them as

$$W_k(z) = \frac{g_k(z)}{I[X]}$$  \hspace{1cm} (32)

then we find the system of equations

$$x_k = \oint \frac{dz}{2\pi i} V'(z) W_k(z)$$  \hspace{1cm} (33)

$$1 = \left( z - \frac{1}{N} \frac{\partial}{\partial x_k} \right) W_k(z) - W_k(z) \frac{1}{N} \frac{\partial}{\partial x_k} \log I[X] - \sum_{j \neq k} \frac{W_k(z) - W_j(z)}{x_k - x_j}$$  \hspace{1cm} (34)

Eq. (34) gives the following Laurent expansion

$$W_k(z) = \frac{1}{z} + \frac{1}{z^2} \frac{1}{N \partial x_k} \log I[X] + O\left( \frac{1}{z^3} \right)$$  \hspace{1cm} (35)

and, hence, we can recover $I[X]$ from a residue of $z W_k(z)$.  

9
Eqs. (33), (34) are convenient for the large $N$ limit to be taken and also provide a suitable framework for the $1/N$ expansion.

Let us suppose that, in the $N \to \infty$ limit, there exists the density of eigenvalues having a finite support

$$\rho(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \delta(x - x_k)$$  \hspace{1cm} (36)

Following ref. [14], we can introduce a continuous variable $x$ instead of the lower index of the function $W_k(z)$: $w(z, x)$. Then, in the $N \to \infty$ limit, eqs. (33), (34) take the form

$$x = \oint \frac{dz}{2\pi i} V'(z) w(z, x)$$  \hspace{1cm} (37)

$$1 = (z - w_1(x))w(z, x) - \int dy \rho(y) \frac{w(z, x) - w(z, y)}{x - y}$$  \hspace{1cm} (38)

where

$$w_1(x) = \frac{1}{N} \frac{d}{dx} \frac{\delta}{\delta \rho(x)} \log \mathcal{I}[X]$$ \hspace{1cm} (39)

is the second coefficient in the Laurent expansion

$$w(z, x) = \sum_{k=0}^{\infty} \frac{w_k(x)}{z^{k+1}}$$ \hspace{1cm} (40)

It should be noted that eq. (38) coincides with the one obtained in ref. [2] in a slightly different context.

These equations are valid, strictly speaking, only on a support of $\rho(x)$. Let us suppose, following ref. [14], that they can be regarded as dispersion relations and, in such a way, can be given a meaning on the whole complex plane.

Let us introduce the function

$$F(z, x) = \int dy \frac{\rho(y)w(z, y)}{x - y} = \sum_{k=0}^{\infty} \frac{f_k(x)}{z^{k+1}}$$ \hspace{1cm} (41)
where the integral runs over a support of \( \rho(x) \). \( F(z,x) \) is analytic at the infinity and

\[
\text{Im} \, F(z,x) = \frac{1}{\pi} \rho(x) w(z,x)
\]

(42)

In terms of residues, eq. (38) can be written as follows

\[
w_{k+1}(x) = (w_1(x) + f_0(x)) w_k(x) - f_k(x)
\]

(43)

The functions \( f_k(x) \) are not known (except \( f_0(x) \), which is determined by \( \rho(x) \)). However, if the potential \( V(z) \) is a polynomial, we have a finite number of equations and a bootstrap solution can be, in principle, found, provided an analytic structure of all functions is guessed. No further information is needed. In this respect, our method is a direct generalization of the Brezin-Gross technique [14].

A singularity of \( w_1(x) \) at the infinity is determined by the equation \( V'(w_1(x)) = x \). Therefore, for potentials of a degree bigger than 5, this singularity is not algebraic and the problem is very complicated. However, for the inverted W potential, \( w_1(x) \) can be found in a closed form by using the well-known solution to the cubic equation. For reader’s convenience we give it in Appendix.

## 5 Continuum limit and scaling at \( d > 1 \)

In the case under consideration, the density of eigenvalues of external field coincides with the one of the integrand and we have the following equality

\[
\int dx \, \rho(x) \, w(z,x) = \int dy \frac{\rho(y)}{z-y}
\]

(44)

In ref. [2] Migdal solved eq. (18) for \( w(z,x) \) after expressing \( w_1(x) \) from the Lagrangian saddle-point equation

\[
w_1(x) = \frac{1}{2d} V_0'(x) - \frac{1}{d} \text{Re} f(x)
\]

(45)

Then, the condition (44) gave the following Master Field Equation

\[
\text{Re} f(x) = \int \frac{dy}{\pi} \, \text{Im} \, \log \left[ x - \frac{1}{2} W'(y) + i \, \text{Im} f(y) \right]
\]

(46)
where the integral goes over a support of \( \text{Im} \, f(y) \); \( W'(y) \) is the derivative of the effective potential

\[
U(x) = V_0(x) - 2(d-1) \int dx \, w_1(x) \tag{47}
\]

which, after the substitution (45) for \( w_1(x) \), takes the form

\[
W(x) = \frac{1}{d} V_0(x) + \frac{2(d-1)}{d} \int \text{Re} \, f(x) \, dx \tag{48}
\]

In the Hamiltonian approach we easily obtain the square root form of the density (21), while in the Lagrangian one the simple saddle-point equation (45) holds. Both approaches are compatible in the continuum limit. In the simplest, \( d=1 \), case it was demonstrated in ref. [5]. Let us use the same technique also for \( d > 1 \).

To take the continuum limit, we have to rescale the field and the potential

\[
x = \lambda \varepsilon^{d/2-1}; \quad V_0(x) = \lambda^2 d \varepsilon^{d-2} + \varepsilon^d v(\lambda) \tag{49}
\]

where \( \varepsilon \to 0 \) is a lattice spacing. To preserve the normalization, we define

\[
f(x) = \varepsilon^{1-d/2} \varphi(\lambda) \tag{50}
\]

In terms of new variables, eq. (46) takes the form

\[
\text{Re} \, \varphi(\lambda) = \frac{1}{\varepsilon^{2-d}} \int \frac{d \mu}{\pi} \text{Im} \, \log \left[ \lambda - \mu - \frac{\varepsilon^2}{2d} v'(\mu) - \frac{d-1}{d} \varepsilon^{2-d} \text{Re} \, \varphi(\mu) + \varepsilon^{2-d} \text{Im} \, \varphi(\mu) \right] \tag{51}
\]

In the \( \varepsilon \to 0 \) (or \( d \to \infty \)) limit, the term \( \varepsilon^2 v'(\mu) \) seems to be negligible with respect to \( \varepsilon^{2-d} \text{Re} \, \varphi(\mu) \). However, for the sensible continuum limit to exist, \( \text{Re} \, \varphi(\lambda) \) should become \( 0(\varepsilon^d) \) dynamically.

Let us expand the r.h.s. of eq. (51) in powers of \( \varepsilon \),

\[
\text{Re} \, \varphi(\lambda) = \int \frac{d \mu}{\pi} \frac{\text{Im} \, \varphi(\mu)}{\lambda - \mu} \left\{ 1 + \frac{1}{\lambda - \mu} \left( \frac{\varepsilon^2}{2d} v'(\mu) + \frac{d-1}{d} \varepsilon^{2-d} \text{Re} \, \varphi(\mu) \right) \right. \\
\left. - \varepsilon^{2(2-d)} \left[ \frac{\text{Im} \, \varphi(\mu)^2}{3(\lambda - \mu)^2} + \ldots \right] \right\} \tag{52}
\]

as well as \( \varphi(\lambda) \).
Re \( \varphi(\lambda) = \varepsilon^{\kappa_1} r_1(\lambda) + \varepsilon^{\kappa_2} r_2(\lambda) + \ldots \)

Im \( \varphi(\lambda) = \varepsilon^{\kappa_1} j_1(\lambda) + \ldots \) \hspace{1cm} (53)

Substituting these expansions in eq. (52), we find

\[ \kappa_1 = d \quad r_1(\lambda) = -\frac{v'(\lambda)}{2(d - 1)} \]
\[ \kappa_2 = d + 2 \quad j_1(\lambda) = \sqrt{\frac{4(d - 1)}{d}} \int d\lambda r_2(\lambda) \] \hspace{1cm} (54)

in obvious agreement with eq. (21).

The leading term gives the equation for \( r_2(\lambda) \)

\[ -\frac{v'(\lambda)}{2(d - 1)} = \int d\mu \frac{4(d - 1)}{d} \int d\mu r_2(\mu) \] \hspace{1cm} (55)

It has the form of the one-matrix-model saddle-point equation with the upside-down potential. It is tempting to connect such a strange phenomenon with the appearance of a tachyon in multidimensional string theories. Its technical reason is clear: the effective potential tunes itself to cancel a bare one up to higher order terms. Though this branch corresponds formally to the continuum limit, it seems to describe an effective zero-dimensional system having pathological properties. It is unstable for most potentials, when the initial lattice model is obviously well-defined.

If we keep only a quadratic term in the potential, \( v(\mu) = \frac{m^2}{2} \mu^2 \), the solution is known from ref. \[4\] to be

\[ \varphi(\lambda) = \frac{c\lambda}{2} \sqrt{\frac{c^2 \lambda^2}{4} - c} \] \hspace{1cm} (56)

In the \( \varepsilon \to 0 \) limit, there are two possible values of \( c \)

\[ c_1 = -\frac{\varepsilon^d m^2}{d \cdot (d - 1)} \] \hspace{1cm} (57)
and
\[
c_2 = \frac{4d(d - 1)}{2d - 1} \varepsilon^{d-2}
\] (58)

The latter, \(c_2\), does not correspond to the continuum limit. However, this branch is always stable. If eq. (55) has no solution, the model should become gaussian in the \(\varepsilon \to 0\) limit regardless of a form of the bare potential. Of course, one could interpret this saddle-point as a non-trivial background and construct an \(\varepsilon\) expansion around it. But it is doubtful that this would have connection with continuum theory. This solution corresponds to the strong coupling phase of the lattice model. In the case of a quadratic potential it was discussed by Gross [5].

However, there is some hope to obtain non-trivial behavior at least in the Hamiltonian approach for \(d \gg 1\). A possible critical scaling could correspond here to the “upside-down” effective potential (22) in which case

\[
\text{Im} \varphi(\lambda) \sim \sqrt{-e + a\lambda^2}
\] (59)

In the scaling limit, \(e \to 0\), we have

\[
\text{Re} \varphi(\lambda) = 2 \int_\Lambda^{\tilde{\Lambda}} \frac{d\mu}{\sqrt{\pi/\alpha}} \sqrt{-e + a\mu^2} \frac{1}{\lambda - \mu} \approx -\lambda \sqrt{a} \log \left( \frac{\tilde{\Lambda}^2}{e} \right) + 0(1)
\] (60)

where we have introduced the cutoff \(\tilde{\Lambda} = 2\Lambda\sqrt{a}\).

In the \(d \to \infty\) limit, the presence of the continuous time should not influence an effective potential which is proportional (as follows from eq. (48)) to \(2 \int \text{Re} f(x)\). If we assume that the quadratic top (22) is really created by non-linearities at large distances, then, from the form of the effective potential, we find the following selfconsistency equation

\[
a \cong 2\sqrt{a} \varepsilon^d \log \left( \frac{\tilde{\Lambda}^2}{e} \right)
\] (61)

from which it follows that

\[
a \cong \left( 2 \varepsilon^d \log \left( \frac{\tilde{\Lambda}^2}{e} \right) \right)^2
\] (62)

up to log-log terms.
Now, we are in a position to calculate the singularity of $\frac{1}{\omega(E_F)}$. From eq. (22) we have $e \sim U(x_o) - E_F$ and, hence,

$$\frac{\partial}{\partial e} \left[ \frac{1}{\omega(E_F)} \right] \sim \frac{\partial}{\partial e} \int_{\sqrt{e/a}}^\Lambda dx \frac{1}{\sqrt{-e + ax^2}} \sim \frac{1}{e|\log e|}$$

(63)

Finally, we find that

$$\frac{1}{\omega(E_F)} \sim \log |\log e|$$

(64)

Repeating standard steps [9], we can introduce a coupling constant, $g$, by rescaling $x$ and find

$$\frac{\partial F}{\partial E_F} = \frac{E_F}{\omega(E_F)} = E_F \frac{\partial g}{\partial E_F}$$

(65)

A critical value, $g_c$, corresponds to $E_F$ touching the top of the potential. Then, we get for the second derivative of the free energy with respect to the coupling constant the following formula

$$F'' \sim \frac{1}{\log |\log \delta g|}$$

(66)

where $\delta g = g_c - g$. Let us remind that we have assumed that $d \gg 1$.

Unfortunately, we have no real derivation of this result, and it should be regarded only as a reasonable proposal. If this possibility fails, it will presumably mean that, in the continuum limit, the KM model is effectively gaussian (if we do not take eq. (55) seriously).

For usual branched polymers, the second derivative of the free energy has a square root singularity, which corresponds to the string susceptibility exponent $\gamma_{str} = 1/2$ [7]. In our case $\gamma_{str} = 0$. The reason of it could be that random surfaces have internal degrees of freedom which produce some weights at branching points of polymers. The double logarithm should not be surprising, as for random surfaces $\gamma_{str} \leq 0$ [7], and, at $d = 1$, we have already had $F'' \sim 1/\log \delta g$. 

15
6 Discussion

The Bethe lattices are considered in statistical mechanics as simple systems for which mean field gives exact answers. In a sense, they correspond to the limit $d \to \infty$, i.e., their effective dimension is infinite. It is known that, for local $\phi^4$-type interactions (for example), this regime is exact actually for all dimensions bigger than 4. Presumably, it should be the case for matrix models as well. If it is really so, the obtained mean field solution is exact for all dimensions above a critical one and large dimensional string theory indeed describes branched polymers as was argued in ref. [7]. For the KM model itself, the critical dimension is obviously equal to 1. If it is independent of a tension, non-trivial string theory does not exist.

However, the matrix models do not contain a bare string tension as a parameter. If we substitute the heat kernel

$$K_\beta(\Omega) = \sum_R d_R e^{-\beta C_R} \chi_R(\Omega)$$

for $\delta$-functions in eq. (7) ($C_R$ is a second Casimir), we obtain a model interpolating between the matrix and KM models\textsuperscript{1}. Though we have no clear interpretation for intermediate values of $\beta$, it may bear some properties of a tension dependent matrix model. Of course, its dynamics is extremely complicated, as we have a hybrid of a matrix model and lattice QCD. But, if QCD string really exists, it could be used to introduce a bare string tension in a matrix model. And, from this point of view, the interpretation of $\beta$ as a tension may be quite reasonable.

To conclude, we should say that the mean field problem for matrix models is still far from being well understood. Our attempt to construct a non-trivial continuum limit for the KM model at $d > 1$ has obviously failed. At least, eq. (55) does not seem to be good for it. On the other hand, eq. (64), though looks rather natural, was not rigorously derived. It is even not clear whether it really corresponds to a physical solution of the model. However, in order to prove (or disapprove) it, we would have to solve eq. (51) with a non-trivial potential, which does not look to be a simple task. Also an interpretation of the KM model in subleading orders in $N$ remains to be done. Even for the $d = 1$ model compactified on a circle, a clear understanding of non-singlet

\textsuperscript{1}Another proposal can be found in ref. [18].
sectors is still absent. Let us hope that subsequent works will clarify all these problems.

7 Acknowledgments

I would like to thank J.Ambjørn, E.Brezin, V.Kazakov and I.Kostov for the discussions and A.A.Migdal and B.Rusakov for e-mail correspondence. I also thank the Centre de Physique Theorique de l’Ecole Polytechnique, where a part of this work was done, for hospitality. Financial support from the EEC grant CS1-D430-C is gratefully acknowledged.

References

[1] V.A.Kazakov and A.A.Migdal, Induced QCD at large N, preprint LPTENS-92/15, PUPT-1322 (June 1992).

[2] A.A.Migdal, Exact solution of induced lattice gauge theory at large N, Princeton preprint PUPT-1323 (June 1992).

[3] I.I.Kogan, G.W.Semenoff and N.Weiss, Induced QCD and hidden local $Z_N$ symmetry, UBC preprint UBCTP-92-022 (June 1992);
I.I.Kogan, G.W.Semenoff, A.Morozov and N.Weiss, Area law and continuum limit in induced QCD, preprint UBCTP-92/22 (July 1992).

[4] A.A.Migdal, 1/N expansion and particle spectrum in induced QCD, Princeton preprint PUPT-1332 (July 1992); Phase transitions in induced QCD, ENS preprint LPTENS-92/22 (August 1992); Mixed models of induced QCD preprint LPTENS-92/23 (August 1992)
S.Khokhlachov and Yu.Makeenko, The problem of large-N phase transition in Kazakov-Migdal model of induced QCD, ITEP preprint ITEP-YM-5-92 (July 1992);
Yu.Makeenko, Large N reduction, Master Field and Loop Equations in Kazakov-Migdal model, preprint ITEP-YM-6-92 (August 1992).

[5] D.Gross, Some remarks about induced QCD, Princeton preprint PUPT-1335 (August 1992).
[6] D.V.Boulatov, *Local symmetry in the Kazakov-Migdal gauge model*, preprint NBI-HE-92-62 (September 1992).

[7] J.Ambjørn, B.Durhuus, J.Fröhlich and P.Orland, Nucl. Phys. **B270** (1986) 457;
    J.Ambjørn, B.Durhuus and J.Jonsson, Phys. Lett. **B244** (1990) 403.

[8] E.Brezin, C.Itzykson, G.Parisi and J.-B.Zuber, Comm. Math. Phys. **59** (1978) 35.

[9] V.A.Kazakov and A.A.Migdal, Nucl. Phys. **B311** (1988) 171.

[10] E.Brezin, V.A.Kazakov and Al.B.Zamolodchikov, Nucl. Phys. **B338** (1990) 673;
     D.Gross and N.Milkovic, Phys. Lett. **B238** (1990) 217;
     P.Ginsparg and J.Zinn-Zustin, Phys. Lett. **B240** (1990) 333.

[11] D.Gross and I.Klebanov, Nucl. Phys. **B344** (1990) 475.

[12] D.Gross and I.Klebanov, Nucl. Phys. **B354** (1990) 459.

[13] D.V.Boulatov and V.A.Kazakov, *One dimensional string theory with vortices as the upside-down oscillator*, preprint LPTENS 91/24 (August 1991) to appear in Int. J. Mod. Phys. A.

[14] E.Brezin and D.Gross, Phys.Lett. **B97** (1980) 120.

[15] V.A.Kazakov and I.K.Kostov, unpublished;
     Yu.Makeenko and G.W.Semenoff, Mod. Phys. Lett. **A6** (1991) 3455.

[16] C.Itzykson and J.-B.Zuber, *J.Math.Phys. 21* (1980) 411.

[17] S.Shatashvily, *Correlation functions in the Itzykson-Zuber model*,
     preprint IASSNS-HEP-92/61 (September 1992).

[18] B.Rusakov, *From hermitian matrix model to lattice gauge theory*,
     preprint TAUP 1996-92 (September, 1992).
A Appendix

We consider eq. (24) with $V(y)$ substituted by

$$V(y) = \frac{a}{2} y^2 - \frac{1}{4} y^4$$

(68)

Then eq. (37) takes the form

$$w_1^3(x) + 2f_0(x)w_1^2(x) - (a + f_1(x) - f_0^2(x))w_1(x) - f_2(x) - f_0(x)f_1(x) = x$$

(69)

A solution to this equation can be written in the form

$$w_1(x) = r^{1/3}(x)u(x) + \frac{v(x)}{r^{1/3}(x)} - \frac{2}{3} f_0(x)$$

(70)

where

$$r(x) = \frac{x}{2} + i \sqrt{\left(\frac{a}{3}\right)^{\frac{3}{2}} - \left(\frac{x}{2}\right)^{\frac{3}{2}}}$$

(71)

and $u(x)$ and $v(x)$ have the following asymptotics

$$u(x) = 1 + 0(1/x), \quad v(x) = \frac{a}{3} + 0(1/x)$$

(72)

From the condition that $\text{Im} \ w_1(x) = 0$, when $\text{Im} \ f_0(x) \neq 0$, we find the only possible real solution

$$w_1(x) = r^{1/3}(x) + \frac{a}{3r^{1/3}(x)} + \frac{1}{3} \int dy \frac{\rho(y)}{x - y} \left[ \left( \frac{r(x)}{r(y)} \right)^{\frac{1}{3}} + \left( \frac{r(y)}{r(x)} \right)^{\frac{1}{3}} - 2 \right]$$

(73)

By substituting this ansatz in eq. (69), one can check after a tedious algebra that the former obeys the latter only on a support of $\rho(x)$. For a cubic potential, the corresponding equation was satisfied on the whole complex plane [15]. The other peculiarity of the quartic solution is that we have no additional equation for a position of the cut.

Integrating over $x$ we find the formula for the integral (24) in the planar limit

$$\lim_{N \to \infty} \frac{1}{N^2} \log \mathcal{I}[X] = \int dx \ \rho(x) \left[ \frac{3}{4} r^{4/3}(x) + \left( \frac{a}{3} \right)^4 r^{-4/3}(x) \right]$$
\[
\frac{a}{2} (r^{2/3}(x) + \left(\frac{a}{3}\right)^2 r^{-2/3}(x)) + \frac{1}{2} \int dx \, \rho(x) \int dy \, \rho(y) \\
\left\{ \log \left[ \left(\frac{r(x)}{r(y)}\right)^3 + \frac{1}{3} \left(\frac{r(y)}{r(x)}\right)^3 + 1 \right] + \log \left[ \frac{3}{a} \left(\frac{r(x) r(y)}{r(x) r(y)}\right)^{1/3} + \frac{a}{3} \left(\frac{r(x) r(y)}{r(x) r(y)}\right)^{1/3} + 1 \right] \right. \\
- \left. \frac{1}{2} \left[ \left(\frac{r(x)}{r(y)}\right)^3 + \left(\frac{r(y)}{r(x)}\right)^3 + \frac{3}{a} \left(\frac{r(x) r(y)}{r(x) r(y)}\right)^{1/3} + \frac{a}{3} \left(\frac{r(x) r(y)}{r(x) r(y)}\right)^{1/3} \right] \right\} \quad (74)
\]