On the Information of the Second Moments Between Random Variables Using Mutually Unbiased Bases

Hongyi Yao

Abstract

The notation of mutually unbiased bases (MUB) was first introduced by Ivanovic to reconstruct density matrices [10]. The subject about how to use MUB to analyze, process, and utilize the information of the second moments between random variables is studied in this paper. In the first part, the mathematical foundation will be built. It will be shown that the spectra of MUB have complete information for the correlation matrices of finite discrete signals, and the nice properties of them. Roughly speaking, it will be shown that each spectrum from MUB plays an equal role for finite discrete signals, and the effect between any two spectra can be treated as a global constant shift. These properties will be used to find some important and natural characterizations of random vectors and random discrete operators/filters. For a technical reason, it will be shown that any MUB spectra can be found as fast as Fourier spectrum when the length of the signal is a prime number.

In the second part, some applications will be presented. First of all, a protocol about how to increase the number of users in a basic digital communication model will be studied, which has bring some deep insights about how to encode the information into the second moments between random variables. Secondly, the application of signal analysis will be studied. It is suggested that complete "MUB" spectra analysis works well in any case, and people can just choose the spectra they are interested in to do analysis. For instance, single Fourier spectra analysis can be also applied in nonstationary case. Finally, the application of MUB in dimensionality reduction will be considered, when the prior knowledge of the data isn’t reliable.

INDEX TERMS: Mutually Unbiased bases, Second Moment, Correlation Matrix, Digital Communication, Signal Processing, Dimensionality Reduction

I. INTRODUCTION

Ivanovic first introduced mutually unbiased bases (MUB) to reconstruct density matrices [10].

1Hongyi Yao is a PHD student of Institute of Theoretical Computer Science of Tsinghua University, Beijing, 100084, P. R. China(email: thy03@mails.tsinghua.edu.cn)

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**Definition 1**  Let $M_v = \{v_1, v_2, \ldots, v_d\}$, $M_u = \{u_1, u_2, \ldots, u_d\}$ be two normalized orthogonal bases in the $d$ dimension complex space. They are said to be mutually unbiased bases if and only if $| \langle v_i, u_j \rangle | = \frac{1}{\sqrt{d}}$ for any $i, j = 1, 2, \ldots, d$. A set of normalized orthogonal bases $\{M_1, M_2, \ldots, M_n\}$ are said to be mutually unbiased bases if and only if each pair of bases $M_i$ and $M_j$ are mutually unbiased bases.

MUB is widely used in the areas of quantum physics and quantum information theory, such as the reconstruction of pre-state [12], tomography, Wigner distribution [7], teleportation [6], and quantum cryptograph [2, 3, 4]. But it has only a few classical application such as [21]. This is quite reasonable, because do full MUB spectra analysis need $d+1$ times time and space resources where $d$ equals the length of signals. But it should be noticed that bases from MUB has natural connections with the Fourier base which has plenty of applications, [17] has done some study about it. Intuitively, the relation between any two bases from MUB is the same as that between the standard bases and the Fourier bases if we only concern the inner products of the vectors.

One of the major subjects in this area is to construct MUB for a given dimension $d$. It’s known that, there are no more than $d+1$ MUB for dimension $d$, and when $d$ is the power of prime, all $d+1$ MUB can be explicitly constructed [12]. This paper only focuses on the case when $d+1$ MUB can be found for dimension $d$, and will not study the construction of the. It will be introduced, in Sections II – IV, some mathematical foundations. Then the paper will present some interesting applications of these results in Sections V – VII.

In Section II, the equivalence between autocorrelation matrix and the spectra of mutually unbiased based will be formally presented. Some interesting properties concerning what kinds of spectra can form autocorrelation matrix are studied, such as the generalization of Uncertainty Principle. It will be shown that the equivalent relation is robust, because the effects of small errors are also trivial.

In Section III, some nice properties of the spectra of MUB will be studied. First, the original definition of ”stationary” will be extremely extended, and it’s interesting to see that any discrete random signals can have all kinds of ”stationary” versions of them. Then, the relationship between related random sources and independent random sources will be presented, it will be shown that treating normal random sources as a bunch of independent random sources will bring a lot of convenience. Of course, MUB is the key tool. The third part of this section is going to use the nice properties of MUB to do complete analysis for random operators/filters. This part will introduce a general way to do all kinds of stabilization for random vectors with some compensations on ”white noise”. At last, a filter which only deal with some designated spectra and left others untouched will be presented.

In Section IV, the MUB spectra for a deterministic vector will be studied. In the first part, an algorithm will be shown which tells that any MUB transform can be done as fast as DFT when in prime dimensions. Then some properties
of the MUB spectra for deterministic vectors will be listed.

The main application of above results is a simple digital communication protocol which can significantly increase the number of users without any advanced techniques such as [14]. This will be introduced in Section V. Maybe the theoretical protocol is far from practice, but it provides some deep insights about how to encode information into the second moments between random variables based on above results. Roughly speaking, communication using the first moments of the signals is well studied [16], while our protocol is based on the moment of higher order. When some users are idle, the protocol retrogresses simple ones such as "TDMA"/"FDMA". Based on results of Section III, we will introduce some interesting alternations of this model which suggest we can do many things based on such model.

In Section VI, we study the application of signal analysis. Spectra analysis for stationary signal is useful and well known [18, 1], while nonstationary case are much harder [15, 8]. Using MUB, we suggest complete spectra analysis for discrete signals works well in any case. Actually, it suggests that people can choose the spectra they are interested in to do the analysis. For instance, Fourier spectra analysis also make sense in nonstationary case. We will give an example about how to apply it to signal detection. However, we should do more about the physical meanings of the nonfourier spectra of MUB, because they are important for practical and mathematical reasons.

Finally, we will consider the applications of MUB in dimensionality reduction. In the case when no prior knowledge of the data is known, we will present some local results and a global conjecture. When the prior knowledge is not reliable, we suggest that MUB work well.

We will give some basic notations for the paper. We only work in $d$ dimension complex linear space, where the whole $d+1$ mutually unbiased bases (MUB) can be found. Assume $M_1, M_2, ... M_{d+1}$ are the MUB of $d$ dimension complex linear space where the columns of $M_i$ form the $i^{th}$ base. Without loss of generality, $M_1$ is the standard base for dimension $d$ complex linear space. For all random variables mentioned in this paper, the estimation values of them are zero because constant shift is easy to handle. So in the paper, autocorrelation matrixes has the same meaning of correlation matrixes. Each vector is a vertical vector as default. $R_x$ is assumed the autocorrelation matrix of complex random vector $X = \{x_1, x_2, ..., x_d\}^T$, and $tr(R_x) = 1$ as default. We say $x$ is "white noise" if and only if $E(x) = 0$ and $x$ is independent to all other random variables mentioned in this paper.

II. THE EQUIVALENCE BETWEEN CORRELATION MATRIXES AND THE SPECTRA OF MUB

Ivanovic first introduced the idea about using the spectra of mutually unbiased
bases to reconstruct density matrices of quantum states [10]. It’s easy to see that when apply a unitary matrix $U$ to random vectors, the change of correlation matrixes is the same as that for density matrixes when apply $U$ to the quantum states. So follow the notations of introduction, we give some basic definitions.

**Definition 2** Let $k$-Spectrum $S_k$ of $Rx$ be the diagonal part of matrix $M_i^H \cdot Rx \cdot M_i$. And the set $\{S_1, S_2, ..., S_{d+1}\}$ form the complete spectra of $Rx$.

Then we present the following theorem which is the base of this paper. Let $I_d$ denotes the identity matrix of dimension $d$, and $\text{Diag}(V)$ is a diagonal matrix with diagonal part equals $V$.

**Theorem 1** Each autocorrelation matrix $Rx$ corresponds to a unique set of $d + 1$ nonnegative real vectors $\{S_1, S_2, ..., S_{d+1}\}$, where $S_k$ is the $k$-Spectrum of $Rx$ and for each $k$, $\sum_{i=1}^{d} (S_k)i = 1$. $\{S_1, S_2, ..., S_{d+1}\}$ can reconstruct $Rx$ by

$$Rx = \sum_{i=1}^{d+1} M_i \cdot \text{Diag}(S_i) \cdot M_i^H - Id \quad (1)$$

But the inverse is not right, i.e there are some set of $d + 1$ nonnegative real vectors $\{V_1, V_2, ..., V_{d+1}\}$ satisfies for each $k$, $\sum_{i=1}^{d} (V_k)i = 1$, but they can’t form the complete spectra of any autocorrelation matrix.

**Proof.** The first part of the theorem is finished by [10], where we only need to switch "density matrixes" to "autocorrelation matrixes". And it’s easy to find a counterexample for the second part. Let $V_i$ is a zero vector except the $i^{th}$ term which is 1, for $i = 1, 2, ..., d$. Then no matter how we choose $V_{d+1}$, $\{V_i, V_2, ..., V_{d+1}\}$ can’t form the spectra of some autocorrelation matrix. \[\Box\]

A trivial observation is that many different real nonnegative vectors $\{S_1, S_2, ..., S_{d+1}\}$ can construct the same $Rx$ use (1). The next theorem says that it’s not interesting except for some constant global shifts to the spectrum. So as default, in the next, we will use definition 1 to define the spectra of MUB. Let $\text{One}$ denotes a $d$ length vector with all term 1

**Theorem 2** Nonnegative real vectors $\{S_1, S_2, ..., S_{d+1}\}$ and $\{S'_1, S'_2, ..., S'_{d+1}\}$ can construct the same $Rx$ use (1) only if for each $i = 1, 2, ..., d + 1$, there exists a real number $u_i$, s.t $S_i = S'_i + u_i \cdot \text{One}$.

**Proof.** Assume $S_k$ is the $k$-spectrum of $Rx$ by definition 1, and:

$$Rx = \sum_{i=1}^{d+1} M_i \cdot \text{Diag}(S'_i) \cdot M_i^H - Id \quad (2)$$
For each $i \neq j$, we can check that the diagonal part of $M^H_j \cdot M_i \cdot \text{Diag}(S'_i) \cdot M^H_i \cdot M_j$ is $u_{j,i} \cdot \text{Id}$, where $u_{j,i}$ is a real number. This finishes the proof. \hfill \square

In theorem 1, we have shown that not all kinds of sets of positive vectors can form a autocorrelation matrix. So what kinds of vectors can form the complete spectra is an interesting question. Two theorems will be presented about this subject and will be used in next sections.

**Theorem 3** Let $\text{tr}(Rx) = 1$, and $\{S_1, S_2, ..., S_{d+1}\}$ form the complete spectra of a autocorrelation matrix $Rx$, then $\{S_1, S_2, ..., S_d, F\}$ also form the complete spectra of another autocorrelation matrix $Rx'$, where $F$ equals $\frac{1}{n} \cdot \text{One}$.

**Proof.** In [10], the author shows that if $\{S_1, S_2, ..., S_{d+1}\}$ form the complete spectra of a autocorrelation matrix $Rx$, then $Rx = \sum_{i=1}^{d+1} M_i \cdot \text{Diag}(S_i) \cdot M_i^H - I$. He also shows that $\sum_{i=1}^{d} M_i \cdot \text{diag}(S_i) \cdot M_i^H - \frac{n-1}{n} \cdot I$ is also a autocorrelation matrix $Rx'$. This finishes the the proof. \hfill \square

The next theorem is the "uncertainty principle" of the complete spectra.

**Theorem 4** Let $\text{tr}(Rx) = 1$, $m_i$ denotes the max value of $S_i$, then:

$$m_j < \sqrt{2} \cdot \sqrt{1 - m_i} + \frac{1}{d}, i \neq j \quad (3)$$

**Proof.** Without loss of generality, we assume $j = 2$ and $i = 1$. Let $Dm(Rx)$ denotes the matrix with diagonal part equaling the diagonal part of $Rx$ and other terms equaling 0. Let $Dv(Rx)$ denotes the vector which equals the diagonal part of $Rx$.

$$S_2 = Dv(M_2^H \cdot Rx \cdot M_2) \quad (4)$$
$$= Dv(M_2^H \cdot Dm(Rx) \cdot M_2) + Dv(M_2^H \cdot (Rx - Dm(Rx))) \cdot M_2) \quad (5)$$
$$= \frac{1}{d} \cdot \text{One} + Dv(M_2^H \cdot (Rx - Dm(Rx))) \cdot M_2) \quad (6)$$

$\text{One}$ denotes a $d$ length vector with each term equals 1. Assume $Dv(Rx) = [d_1, d_2, ... d_d]^T$, and $d_1 = m_1$. Because of cauchy-schwarz inequality, we have
(Rx)_{i,j} \leq \sqrt{d_i \cdot d_j}. For matrix M, let (\text{abs}(M))_{i,j} = |(M)_{i,j}|, then:

\[
\max \ (Dv(M_2^H \cdot (Rx - Dm(R_x)) \cdot M_2)) \\
\leq \ max(Dv(\text{abs}(M_2^H) \cdot \text{abs}((Rx - Dm(R_x))) \cdot \text{abs}(M_2))) \\
\leq \frac{2}{d} \sum_{i \neq j} \sqrt{d_i \cdot d_j} \\
\leq \frac{2}{d} (\sqrt{(d-1) \cdot d_1 \cdot (1-d_1)} + \sqrt{(d-2) \cdot d_2 \cdot (1-d_1 \cdot d_2)} + \cdots + \sqrt{d_{d-1} \cdot d_1}) \\
\leq \frac{2}{d} \cdot 1 \cdot \sqrt{d-1 \cdot d - 2 + \cdots + 1} \\
< \sqrt{2} \cdot \sqrt{1-d_1}
\]

We get (10), (12) from cauchy-schwarz inequality. \quad \Box

The following theorem is about the sensitivity of the equivalence between the two representations of the autocorrelation matrixes. We consider the cases of random error and deterministic error. The proof is trivial, and omitted here.

**Theorem 5** Let \( \{S_1, S_2, \ldots, S_{d+1}\} \) is the complete spectra of \( Rx \), and \( E_R \) is an error matrix, \( E_{S_i} \) is an error vector. Assume that \( Rx + E_R \) is also positive and \( \{S_1 + E_{S_1}, S_2 + E_{S_2}, \ldots, S_{d+1} + E_{S(d+1)}\} \) is also the complete spectra of a autocorrelation matrix.

(i) If \( E_R \) is deterministic error matrix of \( Rx \) satisfies \( |E_R|_\infty < \epsilon \), then the complete spectra of \( E_R + Rx \) is \( \{S_1 + E_{S_1}, S_2 + E_{S_2}, \ldots, S_{d+1} + E_{S(d+1)}\} \), satisfies \( |E_{S_i}|_\infty < d \cdot \epsilon \), for \( i = 1, 2, \ldots, d+1 \).

(ii) If \( E_R \) is random error matrix of \( Rx \) satisfies each term of \( E_R \) are independent, \( E((E_R)_{i,j}) = 0 \), and \( E((E_R)_{i,j})^2 < \epsilon \) for all \( i, j = 1, 2, \ldots, d \). Then the complete spectra of \( E_R + Rx \) is \( \{S_1 + E_{S_1}, S_2 + E_{S_2}, \ldots, S_{d+1} + E_{S(d+1)}\} \), satisfies \( E((E_{S_i})) = 0, E((E(R))_{i,j})^2 < \epsilon \), for \( i, j = 1, 2, \ldots, d+1 \).

(iii) If for each \( i \), \( E_{S_i} \) is deterministic error vector of \( S_i \) satisfies \( |E_{S_i}|_\infty < \epsilon \), then \( \{S_1 + E_{S_1}, S_2 + E_{S_2}, \ldots, S_{d+1} + E_{S(d+1)}\} \) form the complete spectra of \( E_R + Rx \), where \( |E_R|_\infty < n \cdot \epsilon \).

(iv) If for each \( i \), \( E_{S_i} \) is random error vector of \( S_i \) satisfies \( (E_{S_i})_{i,j} \) are independent for each \( i = 1, 2, \ldots, d+1 \) and \( j = 1, 2, \ldots, d \), and \( E((E_{S_i})) = 0, E((E_{S_i})_{i,j})^2 < \epsilon \). Then \( \{S_1 + E_{S_1}, S_2 + E_{S_2}, \ldots, S_{d+1} + E_{S(d+1)}\} \) form the complete spectra of \( E_R + Rx \), where \( E((E(R))_{i,j}) = 0, E((E(R))_{i,j})^2 < \epsilon \).

**III. THINGS BECOMES CLEAR WHEN MUB COMES**
A. the Generalization of the Definition of Stationary

Stationary random signal is easy in the sense that we can apply Fourier spectra analysis. But things become much harder when the signal is nonstationary. In this subsection, the definition of stationary random vector is extremely extended by MUB. This extension is serious, because it concerns which domains we should concern to do complete signal analysis. In this subsection, $X, X'$ are two random complex vectors, $R_x$ and $R_x'$ are autocorrelation matrixes of $X, X'$, and $\{S_1, S_2, ..., S_{d+1}\}$ and $\{S'_1, S'_2, ..., S'_{d+1}\}$ are the complete spectra of $R_x, R_x'$. $F$ also equals $1/n \cdot \text{One}$.

Definition 3 $X$ is $[i_1, i_2, ..., i_k]$-stationary if and only if $S_{i_1} = S_{i_2} = ... = S_{i_k} = \text{tr}(R_x) \cdot F$

Definition 4 $X'$ is $[i_1, i_2, ..., i_k]$-stabilizer of $X$ if and only if $S'_{i_1} = S'_{i_2} = ... = S'_{i_k} = \text{tr}(R_x) \cdot F$, and $S'_j = S_j$ for each $j \notin \{i_1, i_2, ..., i_k\}$

Proposition I Every $X$ can have all kinds of stabilizer because of theorem 3.

One should notice that "stabilization" is an information lossing process. And $[i_1, i_2, ..., i_k]$ stabilizer of $X$ will left the information of $j$ spectrum of $X$ unchanged, when $j \notin \{i_1, i_2, ..., i_k\}$. However, it will be shown that this process can protect the information of some spectra. And a general way to stabilize signals will be presented.

There are two interesting propositions which concerns some traditional important properties of random vector.

Proposition II If $M_2$ is the Fourier base, $X$ is "stationary" (in original sense) if and only if $X$ is $[1, 3, 4, ..., d+1]$ stationary.

Proposition III $X$ is "white noise" (in original sense) if and only if $X$ is $[1, 2, ..., d+1]$ stationary.

B. Correlation and Independent

In this part, some relationships between normal related random sources and independent random sources will be presented. Let $m(R_x)$ denotes the minimum eigenvalue of $R_x$, $m_i(i)$ denote the minimum term of $S_i$. We first give the main theorem of this subsection.

Theorem 6 If $R_x$ is a autocorrelation matrix with complete spectra $\{S_1, S_2, ..., S_{d+1}\}$, and $\text{tr}(R_x) = 1$. If:

$$m_i(i) \geq \frac{1}{n+1}, i = 1, 2, ..., d+1$$

(15)

then we can construct a complex random vector with autocorrelation matrix $R_x$ by $d \cdot (d+1)$ independent random variables.

Proof. From [10], Let we have:

$$R_x = \sum_{i=1}^{d+1} M_i \cdot \text{diag}(S_i) \cdot M_i^H - I$$

(16)

$$R_x = \sum_{i=1}^{d+1} M_i \cdot (\text{diag}(S_i) - \frac{1}{n+1} \cdot I) \cdot M_i^H$$

(17)
If (15) holds, we can construct $d+1$ random vectors $\{Y_1, Y_2, \ldots, Y_{d+1}\}$, satisfies $\{(Y_i)_j, i = 1, 2, \ldots, d+1, j = 1, 2, \ldots, d\}$ are independent random variables. For each $i, j$ of available values, $(Y_i)_j$ satisfies:

$$
E((Y_i)_j) = 0 \quad (18)
$$

$$
E((Y_i)_j^2) = (S_i)_j - \frac{1}{n + 1} \quad (19)
$$

Let:

$$
X = \sum_{i=1}^{d+1} M_i \cdot Y_i \quad (21)
$$

Then autocorrelation matrix of $X$ is $R_x$. □

**Remark I** If $m(R) \geq \frac{1}{n+1}$ then (15) holds.

**Remark II** With theorem 2, one can shows that (15) can be replaced by a weaker one: sum of $m_s(i)$ is no less than 1. But still a lot of autocorrelation matrixes fail to satisfy it.

Remark II seems a strong constraint, but in the next subsection, we will see that in some place it can be overcome easily, while in others, it will lead some natural results.

For convenience, we define:

**Definition 5** $X$ is a k-domain random vector if and only if $X = M_k \cdot Y$, where $Y$ is a $d$ dimension random vector satisfies $E(Y) = 0$, and the terms of $Y$ are independent.

It will be shown that the alternation between $X$ and $\{Y_i, i = 1, 2, \ldots, d+1\}$ is very useful in various areas. Generally speaking, $X + N$ can be viewed as composition of independent random vectors from different domains, where $N$ is "white noise” with $E(N \cdot N^H) = tr(R_x) \cdot I$. It should be noticed that $N$ and $k$-domain random vector has nothing to do with the $r$-spectrum except for a global incensement/decrement if $r \neq k$. Or we can think of $X$ is a composition of independent random variables from different domains with a denoise procedure in the end. This suggests that we can just treat signals as a set different independent signals from different domains, and energy distribution on each domain won’t change after the composition except for a global constant shift. In other words, $i$-spectrum has nothing to say about the energy distribution of the $j$-spectrum when $i \neq j$.

**C. Linear Random Operator and Some Special Kinds of Filters**

In this subsection, we will do something in the taste of signal processing. The reader will see that linear operators filters for random vectors will be demonstrated clearly with MUB. And we can judge whether a filter is good in the sense that it only do what it should and left other parts untouched.

A general formulation of linear random operators is a good start point to study complete MUB analysis for operators. Reminding that a random variable is "white noise” only if it’s independent with any other random variables in this paper.
Definition 6. \( P \) is a random linear operator for \( d \) dimension complex random vector, if:
\[
P(X) = T \cdot X
\]
(22)
Where \( T \) is a random \( d \cdot d \) matrix. And for each subset \( S_{\alpha} \) of \( \{(X)_1,(X)_2,...,(X)_d\} \), each subset \( S_{\beta} \) of \( \{(T)_{i,j},i,j=1,2,...,d\} \) satisfies:
\[
Pr(S_{\alpha},S_{\beta}) = Pr_{S_{\alpha}} \cdot Pr_{S_{\beta}}
\]
(23)

There are some propositions for \( P \), which are trivial but important.

Proposition I. For random vectors \( X \) and \( X' \), if \( RX = RX' \), then \( RP(X) = RP(X') \).

Proposition II. For random vectors \( X \) and \( X' \), if \( E(X'X^H) = 0 \), then \( RP(X+X') = RP(X) + RP(X') \).

Then we the main theorem of this subsection:

Theorem 7. For random vector \( X \) with \( tr(R) = 1 \), \( \{S_1,S_2,...,S_{d+1}\} \) are the complete spectra of \( RX \), \( \{S_{p1},S_{p2},...,S_{p(d+1)}\} \) are the complete spectra of \( RP(X) \). There exist \( d+1 \) dimension \( d \ast (d^2 + d) \) deterministic real matrices \( \{D_1,D_2,...D_{d+1}\} \), such that for \( i = 1,2,...d+1 \),
\[
S_{pi} = D_i \cdot [(S_1 - \frac{1}{d+1} \cdot One)^T,(S_2 - \frac{1}{d+1} \cdot One)^T,...,(S_{d+1} - \frac{1}{d+1} \cdot One)^T]^T
\]
(24)

One is a \( d \) length vector with each term equals 1

Proof. First assumes that \( m(RX) \geq \frac{1}{n+1} \), then from theorem 5. There exist \( d+1 \) random vectors \( \{Y_1,Y_2,...Y_{d+1}\} \), satisfies \( \{(Y)_i,i = 1,2,...d+1,j = 1,2,...d\} \) are independent random variables, \( E(Y_i) = 0 \), and \( E((Y_i)^2) = (S_i)_j = \frac{1}{d+1} \). Let
\[
X' = \sum_{i=1}^{d+1} M_i \cdot Y_i
\]
(25)
Then \( RX = RX' \). From proposition I, II,
\[
RP(X) = RP(X') = \sum_{i=1}^{d+1} \sum_{j=1}^{d} RP(M_i,z_j) \cdot ((S_i)_j - \frac{1}{d+1})
\]
(26)

\( Z_j \) is the random vector satisfies \( E(Z_j) = 0 \), and \( E((Z_j)^2) = \delta_{ij} \). \( 26 \) has already shown the properties of \( P \) can be determined by some deterministic matrices, but we need to go further.

For each \( RP(M_i,z_j) \), the \( k \) spectrum is \( S_{i,k}^{l,j} \), so the \( k \) spectrum of \( RP(X) \) is:
\[
S_{pk} = \sum_{i=1}^{d+1} \sum_{j=1}^{d} S_{i,k}^{l,j} \cdot ((S_i)_j - \frac{1}{d+1})
\]
(27)
So there exists \( \{D_1,D_2,...D_{d+1}\} \) satisfies \( 24 \).

The second part of the proof will deal with the constraint ”\( m(RX) \geq \frac{1}{n+1} \)”. Let \( X_n = \frac{1}{\sqrt{n+1}} \cdot (X + N) \), \( N \) is \( d \) length ”white noise” with \( E(N \cdot N^H) = I \). Now
\[ m(Rx_n) \geq \frac{1}{m+1} \] holds. Let \( L_{ONE} \) denotes a \( d^2 + d \) length vector with each term is 1. Let \( S_{pk}^{(X_n)} \) denotes the \( k \)-spectrum of \( R_{X_n} \), and \( S_{pk}^{(N)} \) corresponds to the \( k \)-spectrum of \( R_{P(X)} \). Then

\[
S_{pk}^{(X_n)} = \frac{1}{d+1} \cdot D_k \cdot [S_1^T, S_2^T, ..., S_{d+1}^T]^T \quad (28)
\]

\[
S_{pk}^{(X_n)} = \frac{1}{d+1} \cdot (S_{pk} + S_{pk}^{(N)}) \quad (29)
\]

\[
S_{pk}^{(N)} = \frac{1}{d+1} \cdot D_k \cdot L_{ONE}^T \quad (30)
\]

From above equations, \( (24) \) holds.

\[
\{D_1, D_2, ..., D_{d+1}\} \] shows some basic property of \( P \). For example, Let \( I_d \) denote the identity \( d \times d \) matrix. \( I_d \) is a \( d \times d \) matrix with each term equals 1, if there are some real numbers \( \mu_i, i = 1, 2, ..., d + 1 \) such that:

\[
D_k = [\mu_1 \cdot 1_d, \mu_2 \cdot 1_d, ..., \mu_d \cdot 1_d] \quad (32)
\]

Then the output of \( P \) will be \([k]\) stationary. If

\[
D_k = [\mu_1 \cdot 1_d, \mu_2 \cdot 1_d, ..., \mu_{k-1} \cdot 1_d, \mu_k \cdot I_d, \mu_{k+1} \cdot 1_d, ..., \mu_{d+1} \cdot 1_d] \quad (33)
\]

Then \( P \) actually switch the \( i \)-spectrum of \( X \) to the \( k \)-spectrum of \( P(X) \) with a global constant increment/decrement .

If a filter only want to do something about the \( j \)-spectrum and keep the information of other spectra unchanged, then it should try to satisfy :

\[
D_k = [\mu_1 \cdot 1_d, \mu_2 \cdot 1_d, ..., \mu_{k-1} \cdot 1_d, \mu_k \cdot I_d, \mu_{k+1} \cdot 1_d, ..., \mu_{d+1} \cdot 1_d], k \neq j \quad (34)
\]

For example, let the matrix \( T \) of operator \( P \) be a deterministic matrix with the form \([V, S_1(V), S_2(V), ..., S_{d-1}(V)]^H\), while \( V \) is a length \( d \) vector and \( S_i(V) \) means the vector which left ring shift \( V \) \( i \) times. This kind of \( P \) is well studied(such as Winner Filter). We could also say that kind of \( P \) is a good 2-spectrum filter if the input signals \( X \) are \([1, 3, 4, ..., d + 1]\) stationary, i.e \( E((X)i \cdot (X)_j) = F(j - i) \), where \( F \) satisfies \( F(k) = F(-k)^H \).

So it’s a very interesting question that what kinds of \( \{D_1, D_2, ..., D_{d+1}\} \) corresponds to a physical realizable random operator, but this paper can’t answer it .

The second part of this subsection will focus on some special kinds of operators.

**Theorem 8** For \( k = 1, 2, ..., d \), there exist a operator \( P_{i_1, i_2, ..., i_k} \), such that for any input \( X \), it will output \( X_{i_1, i_2, ..., i_k} + N \). \( X_{i_1, i_2, ..., i_k} \) is the \([i_1, i_2, ..., i_k]\) stabilizer of \( X \) and \( N \) is "white noise" with \( E(N \cdot N^H) = \text{tr}(R_X) \cdot \frac{2(k-1)}{d} \cdot I_d \).

**Proof.** Let \( Y_{d(j)} = \text{diag}(y_1^{(j)}, y_2^{(j)}, ..., y_d^{(j)}) \), for each \( j \notin \{i_1, i_2, ..., i_k\} \), satisfies that \( Y = \{y_i^{(j)}, j \notin \{i_1, i_2, ..., i_k\}, i = 1, 2, ..., d\} \) are independent random variables,
$E(y_i^{(j)}) = 0$ and $E((y_i^{(j)})^2) = 1$. $Y$ are independent to $X$. Let $X'$ is the output of $P_{i_1,i_2,...,i_k}$ with input $X$. $X'$ is constructed from the following equations.

$$X' = \sum_{j \notin \{i_1,i_2,...,i_k\}} M_j \cdot Y_s^{(j)} \cdot M_j^H \cdot X$$

Then the $t$ spectrum $S'_t$ of $X'$ is:

$$S'_t = S_t + \text{tr}(Rx) \cdot \frac{d-k}{d} \cdot \text{One, } t \notin \{i_1,i_2,...,i_k\}$$

$$S'_t = \text{tr}(Rx) \cdot \frac{d-k+1}{d} \cdot \text{One, else}$$

This finishes the proof.

It’s natural to see that more precise stabilization needs more compensations on "white noise". It is also very interesting to study how to lowerbound the "noise compensation" of "stabilization".

Based on above techniques, we can also construct a special kind of filter mentioned above, the one that only works on designated spectra. For example, think of the case that we only want to do something in the Fourier domain. Based on above technique, we can first choose a suitable value for $E((y_i^{(2)})^2)$, for $i = 1, 2, ..., d$. Then we output $M_2 \cdot Ys^{(2)} \cdot M_2^H \cdot X + X$. This filter only change the Fourier spectrum with compensation on "white noise".

IV. MUB TRANSFORMATION FOR DETERMINISTIC VECTORS

From now on, $X$ becomes a deterministic vector of $d$ dimension complex linear space, and $X'$'s $k$-spectrum is denoted by $S_k = M_k^H \cdot X$. When $k$ is a odd prime number, and $M_1$ denotes $I_d$, $M_k$ with $k > 1$ can be constructed by the formulae [24]:

$$(M_k)_{j,r} = W^{r \cdot (k-2) \cdot \frac{j^2 - j}{2}}$$

$i$ is the square root of $-1$, and $W = e^{2\pi i / d}$. A trivial observation is that $M_2$ is the discrete fourier matrix. The following theorem says that for each $k$, $k$-spectrum $S_k$ of $X$ can be found from $X$ nearly as fast as the 2-spectrum which could use FFT.

**Theorem 9** If for any $k = 1, 2, ... d + 1$, $M_k$ is constructed from (38), $T_k$ denotes the time needed to compute $S_k$ from $X$, $T'_k$ denotes the time needed to compute $X$ from $S_k$, then $T_k \leq T_2 + d \cdot T_m$ and $T'_k \leq T'_2 + d \cdot T_m$, where $T_m$ is the time need complex multiplication.

**Proof.** Let $H_k = \text{diag}(h_1^{(k)}, h_2^{(k)}, ..., h_d^{(k)})$, where $h_j^{(k)} = W^{k \cdot \frac{j^2 - j}{2}}.$ Then :

$$M_k = H_k \cdot M_2$$

$$X = M_k \cdot S_k = H_k \cdot M_2 \cdot S_k$$

$$M_k^H = M_2^H \cdot H_k^H$$

$$S_k = M_k^H \cdot X = M_2^H \cdot H_k^H \cdot X$$
This finishes the proof. □

Similar to DFT, there are also some interesting properties for the MUB spectra of $X$.

**Theorem 10** For a normalized complex vector $X$, let $m_i = |S_j|_\infty$, then the following holds:

$$m_j < \frac{1}{\sqrt{d}} \cdot m_i + \sqrt{1 - m_i^2}, j \neq i$$  \hspace{1cm} (43)

**Proof.** With out loss of generality, assumes $|(S_i)_1| = m_i$. If $j \neq i$, we have:

$$m_j \leq \frac{1}{\sqrt{d}} \cdot m_i + \frac{1}{\sqrt{d}} \cdot \sum_{k=1}^{d} |(S_i)_k|$$  \hspace{1cm} (44)

$$\leq \frac{1}{\sqrt{d}} \cdot m_i + \frac{\sqrt{d - 1}}{\sqrt{d}} \cdot \sqrt{1 - m_i^2}$$  \hspace{1cm} (45)

$$\leq \frac{1}{\sqrt{d}} \cdot m_i + \sqrt{1 - m_i^2}$$  \hspace{1cm} (46)

$$< \frac{1}{\sqrt{d}} \cdot m_i + \sqrt{1 - m_i^2}$$  \hspace{1cm} (47)

It’s easy to see that (46) comes from cauchy-schwarz inequality. □

The above theorem can be thought of the generalization of original "uncertainty principle" for deterministic vectors. While the next theorem is a positive result about the MUB spectra.

**Theorem 11** For any normalized complex vector $X$, there exists $k \in [1, d + 1]$, satisfies $|S_k|_\infty > \frac{1}{\sqrt{d}}$

**Proof.** Let $V_x$ is the vector which contradict the theorem, construct a $d \times d$ matrix $A = [V_x, V_x, ... V_x]$, and $B = A \cdot A^H$. All the $d \times d$ matrix forms a $d^2$ linear space, and $B$ is not in the subspace of all diagonal matrixes. It’s easy to check $B$ is orthogonal to all the non-diagonal matrixes constructed in theorem 3.4 of [24], which implies there are at least $d^2 + 1$ orthogonal bases for $d \times d$ matrixes. □

If $d$ is prime and MUB are constructed from (38), numerical analysis suggests that complete MUB spectra of $X$ also have many interesting properties similar to the spectrum of DFT, such as symmetry of $X$ will leads interesting symmetries for all MUB spectra, and ring shifts of $X$ also cause some shifts of all MUB spectra in the sense of absolute values.

V. ENCODE INFORMATION INTO SECOND MOMENTS

12
The main application of above results is a simple digital communication protocol which can significantly increase the number of users who can use the channel simultaneously and worst case bounded. Although the theoretical protocol is far from practice, it has provided some deep insights about how to encode information into the second moments between random variables. Based on the results of Section III, we will introduce some interesting alternations of the model which suggest we can do many things based on such model.

First we assume that \{A_1, A_2, ..., A_n\} are all nodes who want to communicate with others. There is only a public discrete complex channel C for them to communicate. In the first half of each time interval, C collects a complex message \(\text{Mes}_i\) from \(A_i\), sums \(\text{Mes}_i\) all to \(\text{Mes}\), and sends \(\text{Mes}\) to each \(A_i\) in the second half of the interval.

We assume for every \(d\) intervals, C will give an synchronous impulse to each \(A_i\) which can be distinguished from messages. The abilities of \(A_i\) are constraint, they can’t count the impulses from the start. Actually, the impulses can be thought as the frame synchronous signal of the channel, and this model is the base for multiple access digital communication\[16\], such as TDMA/FDMA\[22, 23\]. Since the number of digital communication users grows fast, scientists invent many advance techniques to handle large size systems, such as the one which combine TDMA and FDMA together \[14\]. In this part, we present a easier way to increase the number of users. We will also study some interesting alternations of \(C\) later.

We define the protocol is \((n, d, m)\) worst case good on \(g\) if there exists a function \(g\), such that from the start time when \(A_i\) wants to send a \(k\) bit message, there only needs \(g(n, d, k)\) time intervals to make sure that the probability that \(A_j\) can get right information from \(A_i\) is at least \(2/3\), for each \(j \neq i\).

It’s easy to see that when \(n = O(d)\), the protocol is good because of TDMA or FDMA. If we have more users, we can use the idea of arithmetic coding\[13\], but it’s hard to be applied to large system because \(t\) times users needs \(2^t\) times power cost for some users. Now we introduce a easily applied protocol which can square the number of users. We will also study some interesting alternations of \(C\) later.

We define the protocol is \((n, d, m)\) worst case good on \(g\) if there exists a function \(g\), such that from the start time when \(A_i\) wants to send a \(k\) bit message, there only needs \(g(n, d, k)\) time intervals to make sure that the probability that \(A_j\) can get right information from \(A_i\) is at least \(2/3\), for each \(j \neq i\).

The idea of the protocol is very simple. We first assume the messages are all positive real numbers. Then we assign each \(A_i\) a special range, such as time range or frequency range. When \(A_i\) want to send some messages, he first flips coins and gives some random signs to the messages. After that, \(A_i\) sends the message which are coded in his designated range. The key is that if \(X\) is a composition of random vectors \(\{V_k, k \in [1, d + 1]\}\) from different domains (see definition 6), then the energy distribution of \(X\) on domain \(k\) is the same as \(V_k\) except for a global constant shift. For example, we will give the protocol when \(d = 4, n = 10\).

Assume for \(i = 1, 2, ..., 10\), the the messages of \(A_i\) are two real numbers \(Mo_1^{(i)}\) and \(Mo_2^{(i)}\) of \([0, 1]\), and he wants to tell others which one is larger. Let \(M_1, M_2, ..., M_5\) are the MUB of dimension 4. For \(A_i\), we assign \(M_{\lfloor x \rfloor + 1}\) to him, where \([x]\) means the integer part of \(x\). To communicate, \(A_i\) first compute \(Mes_{(i)} = \sqrt{Mo^{(i)}}\). Then for each round, \(A_i\) flips two coins, and change the sign of \(Mes_{(i)}\) if he got ”heads” at the \(j^{th}\)
flipping, \( j = 1, 2 \). Then he computes \( V_i \) by

\[
V_i = M_1 \cdot [M_{e_1}^{(i)}, M_{e_2}^{(i)}, 0, 0]^T, \quad i = \text{odd}
\]

\[
V_i = M_1 \cdot [0, 0, M_{e_1}^{(i)}, M_{e_2}^{(i)}]^T, \quad i = \text{even}
\]

When each synchronous impulse comes, \( A_i \) send \( V_i \) one by one to the channel \( C \). For \( A_j \), he receives signals one by one from \( C \). Assume the signals in this round form a \( d \) length complex vector \( X \). For \( A_j \), he needs to keep \( |(M_1^T \cdot X)_1|^2 \) and \( |(M_1^T \cdot X)_2|^2 \) for the information of \( A_1 \), and keep the data for other \( A_i \) in a similar way. \( i \neq j \). Then after a 1000 rounds, \( A_j \) can tell whether \( Mo_1 \) is larger than \( Mo_2 \) correctly with high probability.

In general case, we count the rounds needed for \( A_j \) theoretically. We assume \( A_j \) wants to recover \( E(|(M_1^T \cdot X)_1|^2) \). In the communication, some users of domain \( k \), \( k \neq 1 \), may start/stop to send signals to \( C \). It doesn’t matter, because for \( A_j \), they are global looked same noise and won’t effect the relation between \( E(|(M_1^T \cdot X)_1|^2) \) and \( E(|(M_1^T \cdot X)_2|^2) \). So we only consider the case when the total energy of all domains except 1 is upper bounded by \( K \).

We know that \( X \) is constructed from independent random variables \( \{n_j^{(i)}\} \) from different domains, where \( n_j^{(i)} \) denotes the \( j^{th} \) random variable from domain \( i \). Because \( n_j^{(i)} \) won’t effect \( |(M_1^T \cdot X)_1|^2 \) for \( j = 2, 3, ..., d \), we compute the standard deviation \( \sigma(|(M_1^T \cdot X)_1|^2) \) by:

\[
\sigma^2(|(M_1^T \cdot X)_1|^2) \leq O(E(|(M_1^T \cdot X)_1|^4)) \leq O\left(\sum_{j_1,j_2:j_1 \neq 1, j_2 \neq 1} \frac{1}{d^2} \cdot E(n_{j_1}^{(1)})^2 \cdot E(n_{j_2}^{(2)})^2\right) \leq O\left(\frac{K^2}{d^2}\right)
\]

\( K \) denotes the total energy from all the domains except 1. Let \( M(|(M_1^T \cdot X)_1|^2) \) denotes the mean value of \( |(M_1^T \cdot X)_1|^2 \) in \( r \) rounds. Using chernoff bound, we conclude when \( r = O(K^2 \cdot \log(k)) \), we have:

\[
\Pr(|M(|(M_1^T \cdot X)_1|^2) - E(|(M_1^T \cdot X)_1|^2)| > O(\frac{1}{d})) \leq O(\frac{1}{k})
\]

So the probability that all \( k \) bits are correct is more than a constant positive value. In the worst case, when \( K = d^2 \), we need \( O(d^2 \cdot \log(k) \cdot k) \) time intervals to make sure \( A_j \) can receive the right information from \( A_i \) with probability larger than \( \frac{1}{k} \).

Next we consider the error from quantification. It’s easy to check that when the error of \( (X)_i \) is less than \( \varepsilon \), for \( i = 1, 2, ..., d \), then error of \( |(M_1^T \cdot X)_1|^2 \) is less than \( O(d^2 \cdot \varepsilon) \). So if \( \varepsilon < \frac{1}{d^2} \), the mean error of \( |(M_1^T \cdot X)_1|^2 \) from quantification will be less than \( O(\frac{1}{d^2}) \). For each \( A_i \), he need to quantify the the signal(sent or received) to \( O(d^3) \) discrete magnitude values and \( O(d^3) \) discrete phase values to satisfy \( \varepsilon < \frac{1}{d^2} \).

If only time/frequency resources are allowed, the protocol is just "TDMA/FDMA".

When the case that more than one domains from MUB are used, we must bounds the
total energy of each domain because it’s the ”noise” of other domains. There is a trade off in this model, when more users work simultaneously, more noise comes, so more rounds are needed. But the rounds needed for $A_i$ will be upper bounded by a function which only concerns $n, d, k$. Although each user can choose any time to start or end a communication process, a better choice is to choose a time when the energy of his designated range is low, which may bring an average optimization to the whole system. So when the frequency resource is in shortage, and it’s not suitable to apply some advanced techniques to the system, it seems a reasonable way to allocate resources to great numerous of users, for the reason that it’s adaptable, analyzable, and worst case bounded.

Actually, traditional protocols such as ”TDMA” are based on the first moments of the signal, while the highlight of our protocol is that it can fully utilize the information of the second moments of the signals.

Next we’ll focus on some special kinds of channels/filters $C$ based on subsection $C$ of Section III. We study how can $C$ process the information of each $A_i$.

First, when $C$ has ”white noise” $N$, then $N$ effects all the users equivalently as ”white noise”.

Second, if $C$ can be described by some deterministic matrixes $\{D_1, D_2, ..., D_{d+1}\}$ (See theorem 7), $C$ will do what we claimed in the part following theorem 7. So we can choose the domains that have nice properties to realize the protocol.

Third, follow the idea of theorem 8, $C$ can do something special to $A_i$. Such as $C$ can change the information of $A_i$ without effect others except for some global looked same ”noise”. Actually, $C$ can stabilizes the range designated to $A_i$ so nobody can know the information from $A_i$.

Compared to traditional protocol, such as ”TDMA”, $C$ can almost do all the job the channel $C_T$ of ”TDMA” can do. Even more, $C$ also can do things $C_T$ can’t do, such as $C$ can switch the information from different domains. However, almost every special thing $C$ can do will bring ”noise”. So the question raised before that ”what kinds of $\{D_1, D_2, ..., D_{d+1}\}$ correspond to a physical realizable filter” becomes important.

VI. DISCRETE SIGNAL ANALYSIS WITH MUB

In subsection $B$ of Section III, the traditional definition of ”Stationary” is extremely extended by MUB. And subsection $C$ of Section III suggests the spectra which are far from stationary must implies some nontrivial information in their domains. Actually, if we treat discrete signals as a composition of independent signals from different domains, then spectra analysis in any domain has its own meaning: the $k$-spectrum uniquely describe the energy distribution of the $k$-domain random vector except for a global constant shift. So Fourier spectrum analysis also makes sense when the signal is nonstationary.

Subsection $C$ of Section III gives some ideas about how to construct filters to process statistic signals. These filters are different from traditional ones in the sense that they must concerns all the spectra which we are interested in.
Next, for signal detection, we give a definition regarded to how to judge whether a signal is meaningful.

**Definition 7** The \( k \)-spectrum entropy of \( X \) is defined as
\[
E_k(X) = \sum_{i=1}^{d} (-\log(\frac{|S_k(x_i)|}{\text{tr}(R_x)})),
\]
the complete entropy of \( X \) is defined as
\[
E_c(X) = \sum_{j=1}^{d+1} E_j(X).
\]

So meaningful signals should have \( E_c \) less than \( d \cdot (d+1) \cdot \log(d) \). And a signal with \( E_2 \) much less than \( d \cdot \log(d) \) must imply some important information in the Fourier domain, no matter whether the signal is stationary or not.

However, the most important thing left in this part is how to justify the physical meanings of each base. This paper failed to achieve it. Unlike the Fourier base, for other bases from MUB, it’s looks impossible to correspond them to continuous functional transformations when we only use the the construction when \( d \) is prime. Roughly speaking, the MUB spectra based on the constructions when \( d \) is prime is very sensitive to \( d \). For instance, when a vector has only a single point in the \( k \)-spectrum for dimension \( d \), then it will change a lot when consider the \( d' > d \) dimension’s \( k \)-spectrum, and the larger \( k \), the more change. Whatever, the paper suggests that if the physical meaning of a base (such as the Fourier base) has been found, then do spectra analysis of such base will always make sense.

To achieve to goal, we need the efforts from various areas. Such as we need scientists from the areas of signal processing, physics, and bioinformatics to find some physical meanings of spectra which are definitely different from frequency. And we also need mathematicians to tell us how to construct MUB which have as many good properties as possible (such as the Fourier bases).

VII. DIMENSIONALITY REDUCTION WITH MUB

For information lossy data compression such as dimensionality reduction, sometimes it’s hard to have a good compression ratio when few prior knowledge is known, and things become even worse when the data looks like "white noise" \[19\]. In this section, we claim that Mutually Unbiased Based can do the looks impossible job in some sense.

In the following, compress \( X \) with MUB means choosing a subset \( \text{Sub}_m \) of all MUB bases, and find an optimal MUB spectrum of \( \text{Sub}_m \) to express \( X \), which need only \( \log(d) \) bits to denote which base has been chosen.

Theorem 8 is a technical reason that engineers can choose any unbiased base to do data transformation, theorem 9 suggest that not all spectra can look good, and theorem 10 makes sure that the worst case won’t happen when whole MUB spectra are considered. Next, we will do something different.

\( Sp \) denotes the unit sphere of \( d \) dimension complex linear space, i.e \( Sp = \{V | <V,V> = 1, V \in C^d\} \). For any subset \( \text{Sub}_Sp \) of \( Sp \), \( V(\text{Sub}_Sp) \) denotes its standard volume metric of \( d \) dimension complex sphere \[19\].

A normalized uniform random vector is a good start point to analysis the case when no prior knowledge is known.
Definition 8 \( X \) is a normalized uniform random vector if and only if:

\[
Pr(X \in \text{Sub}_S) = \frac{V(\text{Sub}_S)}{V(Sp)}
\]  

(55)

In the following, compressing \( X \) with \( k \) normalized unitary matrices \( \{B_1, B_2, ..., B_k\} \) means choosing a optimal spectrum of these bases to express \( X \), which needs only \( \log(k) \) bits to denote which base has been chosen. First we assume \( k \leq d + 1 \) bases from MUB are chosen, and the max absolute value of \( X \)'s \( i \)-spectrum is \( m_i \). Then we arbitrarily choose \( k \) unitary normalized matrixes \( U_1, U_2, ..., U_k \), and let \( u_i = |U_i \cdot X| \omega \). We often wants to find some spectrum with large entry to express \( X \). The following theorem justifies that the bases from MUB will do better than any \( \{U_i\} \) locally.

**Theorem 12** When \( X \) is a normalized uniform random vector, then:

\[
Pr(\max(m_1, m_2, ..., m_k) \geq C) \geq Pr(\max(u_1, u_2, ..., u_k) \geq C)
\]  

(56)

**Proof.** First, a lemma will be shown:

**Lemma 13** If \( V_1, V_2 \) are two normalized \( d \) length complex vectors satisfies \( | < V_1, V_2 > | \leq \frac{1}{\sqrt{d}} \). Then for any normalized vector \( V \), if \( | < V, V_1 > | = | < V, V_2 > | = C \), we have:

\[
C \leq \frac{d}{2d + 1 - 2\sqrt{d}}
\]  

(57)

**Proof.** There exist some vector normalized \( W, | < W, V_1 > | = 0 \), and \( V = e^{i\theta_1} \cdot C \cdot V_1 + \sqrt{1 - C^2}W, i \) is the square root of -1. Then we have:

\[
C = | < V, V_2 > | = \frac{1}{\sqrt{d}} \cdot C \cdot e^{i\theta_1} + \sqrt{1 - C^2} \cdot | < W, V_2 > | \leq C \cdot \frac{1}{\sqrt{d}} + \sqrt{1 - C^2}
\]  

(58)

(59)

From above inequality, we can prove the lemma. \( \square \)

For any vector \( V_0 \) and constant \( C \), let:

\[
De(V_0, C) = \{ V : |V|_2 = 1, | < V, V_0 > | > C, V \in C^d \}
\]  

(60)

If \( C = \sqrt{\frac{d}{2d + 1 - 2\sqrt{d}}} \), and \( V_i, V_j \) are any two unequal vectors from MUB, then:

\[
De(V_i, C) \cap De(V_j, C) = \emptyset
\]  

(61)

So

\[
Pr(\max(m_1, m_2, ..., m_k) \geq C) \geq k \cdot d \cdot \frac{V(De(V_0, C))}{V(Sp)} \geq Pr(\max(u_1, u_2, ..., u_k)
\]

(62)

(63)
Remark I When \( d \) goes to infinity, \( \sqrt{\frac{d}{2d+3-2\sqrt{d}}} \) limits to \( \sqrt{2} \).

Remark II When \( d \) goes to infinity, \( d \cdot (d + 1) \cdot \frac{V(D(\nu^0, C))}{V(S_p)} \) goes to zero when \( C > 0 \).

Since Remark II is a negative news for large size data. In this case, we can cut the total vector into shorter ones, with the compensation on more bits to denote which bases have been used. The next conjecture try to support MUB globally, where \( m_i, u_i \) has the same meaning.

**Conjecture 1** When \( X \) is a normalized uniform random vector, then:

\[
E(\max(m_1, m_2, \ldots, m_k)) \geq E(\max(u_1, u_2, \ldots, u_k))
\]

(64)

Numerical analysis by the author strongly support the conjecture.

When the autocorrelation matrixes \( R_X \) of \( X \) is known, Principal Component Analysis (PCA) [11, 9] really works well. However, it’s hard to change the PCA base when \( R_X \) is changed. It’s interesting to consider MUB when \( R_X \) is known, and choose the unbiased bases following the information of the complete spectra of \( R_X \). As the discussion above, we could treat \( X \) a bunch of independent random vectors from different domains. So engineers only need to choose the bases which have nice spectra to get an average optimization. Theorem 5 implies that some inaccuracy about the autocorrelation matrixes won’t effect much. But theorem 3 says that there must be some MUB spectra of \( R_X \) looks bad.

**VIII. CONCLUSIONS**

In this paper, we studied the subject about how to analyze, process, and utilize the information in the second order moments between random variables. We presented a number of applications of this subject. However, many problems remain open, and we list some important ones here:

(i) What about the information in moments of order higher than 2?

(ii) How do we find MUB when \( d \) is not power of prime? In particular, for prime dimension \( d \), there are simple formulas to compute MUB and has fast algorithm to do transformation, what can we say about the case when \( d \) is not prime?

(iii) What about the physical meaning of the nonfourier bases?

(iv) What kind of second moments filter (see subsection C of Section III) are physical realizable?

We should noticed that Symmetric Informationally Complete Sets (SICs) [20, 5] can do a similar job. But we don’t know whether SICs exists for dimension larger than 45 complex linear space. It should be interesting to ask which one (SICs or MUB) is more fundamental to express discrete statistic signals.
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