ADDENDUM TO:
“CONSTRUCTING QUANTIZED ENVELOPING ALGEBRAS VIA INVERSE LIMITS OF FINITE DIMENSIONAL ALGEBRAS”

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Abstract. It is shown that the question raised in Section 5.7 of [1] has an affirmative answer.

We use the notation and numbering from [1]. In Section 5.7 we raised the question: does \( R \) \( U \) embed in \( \hat{R} \) \( U \)? The purpose of this addendum is to show that the answer is affirmative. To be precise, we have the following result.

Theorem. The map \( R^\theta : R\ U \rightarrow \hat{R} \) \( U \) defined in 5.7(a) is injective. Hence, \( R\ U \) is isomorphic to the \( R \)-subalgebra of \( \hat{R} \) \( U \) generated by all \( \hat{E}^{(m)}_{\pm i} \) \((i \in I, m \geq 0)\) and \( \hat{K}_h \) \((h \in Y)\).

Proof. Consider the commutative diagram of \( R \)-algebra maps

for any finite saturated \( \pi \subset \pi' \). The universal property of inverse limits guarantees the existence of a unique \( R \)-algebra map \( R^\theta : R\ U \rightarrow \hat{R} \) \( U \) making the diagram commute, and one easily checks that this map coincides with the map defined in 5.7(a). We need to show that \( R^\theta \) is injective.

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We note that from the definitions it follows that for any \( \pi \) the maps \( p_\pi \) and \( \dot{p}_\pi \) are related by the identity \( 1_\pi (1 \otimes p_\pi)(u)_1 = (1 \otimes \dot{p}_\pi)(1_\pi u_1) \), for any \( u \in U, \lambda, \mu \in X \). It follows immediately that
\[
1_\lambda (1 \otimes p_\pi)(u)_1 = (1 \otimes \dot{p}_\pi)(1_\lambda u_1),
\]
for any \( u \in RU, \lambda, \mu \in X \). This is needed below.

Let \( u \in \ker R\theta \) and \( \lambda, \mu \in X \). Then \( \hat{1}_\lambda R\theta(u) \hat{1}_\mu = 0 \) in \( R\hat{U} \). This implies that \( 1_\lambda (1 \otimes p_\pi)(u)_1 = 0 \) in \( RS(\pi) \) for any \( \pi \), and hence that \( (1 \otimes \dot{p}_\pi)(1_\lambda u_1)_1 = 0 \) in \( R\hat{U} \) for any \( \pi \). Thus by Lemma 5.2 we have \( 1_\lambda u_1 \in \cap_\pi R\hat{U}[\pi^c] \). Since the intersection is zero, the equality \( 1_\lambda u_1 \in R\hat{U} \) for any \( \lambda, \mu \in X \). We claim this implies that \( u = 0 \).

To see the claim we observe that the construction of \( \dot{U} \) given in Section 3.1 and [3, Chapter 23] commutes with change of scalars. This is easily verified and left to the reader. It means that \( R\pi_{\lambda,\mu}(u) = 0 \) where
\[
R\pi_{\lambda,\mu} : RU \to RU/\left( \sum_{h \in Y} (K_h - \xi^{(h,\lambda)})RU + \sum_{h \in Y} RU(K_h - \xi^{(h,\mu)}) \right)
\]
is the canonical projection map. Thus it follows that
\[
u \in \sum_{h \in Y} (K_h - \xi^{(h,\lambda)})RU + \sum_{h \in Y} RU(K_h - \xi^{(h,\mu)}).
\]
Since this is true for all \( \lambda, \mu \in X \) it follows that \( u = 0 \) as claimed. \( \square \)

From [3 31.1.5] we recall the category \( RC \) of unital \( R\hat{U} \)-modules. As in [3 23.1.4] one easily checks that this is the same as the category of \( RU \)-modules admitting a weight space decomposition. Following [3 31.2.4], we say that an object \( M \) of \( RC \) is integrable if for any \( m \in M \) there exists some \( n_0 \) such that
\[
E_i^{(n)} m = 0 = E_{-i}^{(n)} m
\]
for all \( n \geq n_0 \).

We have the following consequence of the theorem, which generalizes [2 Proposition 5.11] and [3 Proposition 3.5.4].

**Corollary.** Suppose that \( u \in RU \) acts as zero on all integrable objects of \( RC \). Then \( u = 0 \).

**Proof.** The natural quotient map \( 1 \otimes p_\pi : RU \to RS(\pi) \) makes \( RS(\pi) \) into a left \( RU \)-module, by defining \( u \cdot s = \bar{u} s \) (for \( u \in RU, s \in RS(\pi) \)) where \( \bar{u} \) is the image of \( u \). It is easily checked that, as a left \( RU \)-module, \( RS(\pi) \) is an integrable object of \( RC \). Hence by hypothesis \( u \) acts as zero on \( RS(\pi) \), for any finite saturated subset \( \pi \) of \( X^+ \). It follows
that $u$ lies in the intersection of the kernels of the various $1 \otimes p_r$. By the commutative diagram above this implies that $R\theta(u) = 0$. By the theorem, $u = 0$. □

References

[1] S. Doty, Constructing quantized enveloping algebras via inverse limits of finite dimensional algebras, *J. Algebra* 321 (2009), 1225–1238.
[2] J.C. Jantzen, *Lectures on Quantum Groups*, Amer. Math. Soc. 1996.
[3] G. Lusztig, *Introduction to Quantum Groups*, Birkhäuser Boston 1993.

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