Various Exact Solutions for the Conformable Time-Fractional Generalized Fitzhugh–Nagumo Equation with Time-Dependent Coefficients

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1. Introduction

Fractional calculus is considered as a generalization of classical concepts of integration and differentiation. The first appearance of this idea was in 1695. Then, its importance began to increase due to its many applications in several fields such as biology, plasma physics, solid state physics, engineering, economy, finance, liquid crystals, electrical network, numerical analysis, dynamical systems, and control systems [1–9].

Many mathematicians have proposed different kinds for the definition of the fractional derivative. The most popular of these are Riemann–Liouville, Caputo, Grunwald–Letnikov Erdelyi, Riesz, Hadamard, Marchaud, and Kober [10–13].

There are many properties achieved in the classical derivative, but they are not satisfied with the definitions that are mentioned above. The most important of these properties are product rule, quotient rule, chain rule, Rolle theorem, and mean value theorem.

In 2014, Khalil et al. [14] introduced a new definition of fractional derivative, which is called the conformable fractional derivative. Unlike other definitions, this new definition satisfies the properties mentioned above [15].

A number of researchers presented analytical solutions to a large number of NFPDEs with constant coefficients by using the conformable fractional derivative, such as Al-Shawba et al. [16] solved the KdV-ZK equation with time-fractional derivative using the \((G'/G, 1/G)\)-expansion method. The space-time-fractional modified equal-width equation was solved by Zafar et al. [17] using a hyperbolic function method. A time-fractional biological population model was solved by Zhang and Zhang [18] using a fractional subequation method. Also, Çenesiz and Kurt [19] presented the solution of the space-fractional telegraph equation by introducing the conformable fractional complex transform.

Nowadays, the equations with time-dependent coefficients are more important than equations with constant coefficients as they describe cases that are more general. In 2020, Injrou [20] modified the subequation method to obtain a set of exact solutions for the space-time-fractional Zeldovich equation with time-dependent coefficients.

The Fitzhugh–Nagumo equation system has been derived by both Fitzhugh [21] and Nagumo et al. [22]. It is a simplified form of the Hodgkin–Huxley Model because it is too difficult to be solved analytically.
Recently, many researchers have been interested in the time-fractional Fitzhugh–Nagumo equation with different applications in the areas of neurophysiology, logistical population growth, flame spread, catalytic chemical reaction, and nuclear reactor theory where it combines diffusion and nonlinearity which are controlled by the term \( u (1 - u)(\mu - u) \):

\[
D^\alpha_t u - u_{xx} + u(1 - u)(\mu - u) = 0; \quad t > 0, \ 0 < \alpha \leq 1, \ x \in \mathbb{R}.
\]  

(1)

In 2012, Merdan [23] obtained analytical solutions to the time-fractional Fitzhugh–Nagumo equation (1) by a new application of fractional variational iteration method. At the

\[
D^\alpha_t u + \beta(t) u_x - \gamma(t) u_{xx} + \delta(t) u(1 - u)(\mu - u) = 0; \quad t > 0, \ 0 < \alpha \leq 1, \ x \in \mathbb{R},
\]  

(2)

where \( \beta(t), \gamma(t), \) and \( \delta(t) \) are arbitrary real-valued function of \( t, \mu \) is a constant, and \( u(x, t) \) is the unknown function depending on the temporal variable \( t \) and the spatial variable \( x \). For \( \beta(t) = 0, \gamma(t) = \delta(t) = 1 \), equation (2) will be reduced to the standard fractional Fitzhugh–Nagumo equation. Equation (2) has never solved in the case where the coefficients are time-dependent in literature before.

The advantages of these methods are that they are more general because they are used in the predicted solution as time-dependent coefficients instead of using constant coefficients and they are effective and easy to apply to NFPDEs with time-dependent coefficients. However, we have noticed that some fractional differential equations are solved with time-dependent coefficients in a variety of methods but with constant coefficients in the predicted solution, for example [28, 29]. On the other hand, these methods require fewer calculations than other methods like the exp-function method or the \((G'/G, (1/G))-expansion method.

2. Preliminaries

2.1. Conformable Fractional Calculus

2.1.1. Definition of Conformable Fractional Derivative [14]. Given a function \( f: [0,\infty) \to \mathbb{R} \), the conformable fractional derivative of \( f \) of order \( \alpha \) is defined by

\[
D^\alpha_t f(t) = \lim_{\xi \to 0} \frac{f(t + \xi t^{1-\alpha}) - f(t)}{\xi},
\]  

(3)

for \( t \in [0,\infty), \ \alpha \in (0,1] \). If \( f \) is \( \alpha \)-differentiable in some interval \( (0, a) \), \( a > 0 \), and \( \lim_{\alpha \to 0} D^\alpha_t f(t) \) exists, then \( D^\alpha_t f(0) = \lim_{\alpha \to 0} D^\alpha_t f(t) \).

Furthermore, if \( \alpha = 1 \), the definition is equivalent to the classical definition of the first-order derivative of the function.

2.1.2. Definition of Conformable Fractional Integral [14]. The conformable fractional integral of \( f \) (function continuous) of order \( \alpha \), on interval \([0,t]\), is defined by \( f \) function continuous:  

\[
\mathcal{I}^\alpha_t f(t) = \int_0^t x^{\alpha-1} f(x) \, dx.
\]  

(4)

2.1.3. Properties of Conformable Fractional Derivative [30]. Some important properties of the conformable fractional derivative and the conformable fractional integral are as follows.

Let \( \alpha \in (0,1] \) and \( f, g \) be \( \alpha \)-differentiable at a point \( t > 0 \). Then, there are the following properties

Property 1. (differential of a constant).

\[ D^\alpha_t c = 0, \quad \text{where} \ c \ \text{is a constant}. \]  

(5)

Property 2. (the linearity property).

\[ D^\alpha_t (af(t) + bg(t)) = aD^\alpha_t f(t) + bD^\alpha_t g(t); \quad a, b \ \text{are constants}. \]  

(6)

Property 3. (product rule).

\[ D^\alpha_t (fg)(t) = fD^\alpha_t g(t) + gD^\alpha_t f(t). \]  

(7)

Property 4. (quotient rule).

\[ D^\alpha_t \left( \frac{f}{g} \right)(t) = \frac{gD^\alpha_t f(t) - fD^\alpha_t g(t)}{g^2(t)}. \]  

(8)

Property 5. (chain rule).

\[ D^\alpha_t (f \circ g)(t) = f'(g(t)) D^\alpha_t g(t). \]  

(9)
Property 6. Let $f$ be a function defined in the range of $f$ that is differentiable and also $\alpha$-differentiable, then
\[ D_t^\alpha f(t) = t^{1-\alpha} \frac{df}{dt} \tag{11} \]

Property 7. Let $\alpha \in (0,1]$ and $f$ be $\alpha$-differentiable at a point $t > 0$. If $f$ is differentiable, then
\[ D_t^\alpha f(t) = t^{1-\alpha} \frac{df}{dt} \tag{11} \]

Property 8. Let $\alpha \in (0,1]$ and $f$ is any continuous function in a domain of $I^n_\alpha$, for $t > a$, we have
\[ D_t^\alpha f(t) = f(t), \tag{12} \]
\[ \int_1^t[D_t^\alpha f(t)] = f(t) - f(0); \quad \text{on the interval } [0,t]. \tag{13} \]

2.1.4. Theorem [31]. Let $f: [0,\infty) \rightarrow \mathbb{R}$, be a function such that $f$ is differentiable and also $\alpha$-differentiable. Let $g$ be a function defined in the range of $f$ and also differentiable, then
\[ D_t^\alpha (f \circ g)(t) = t^{1-\alpha} g'(t) f'(g(t)). \tag{14} \]

2.2. Description of Two Methods

2.2.1. Description of the Improved Subequation Method. In this section, we summarize the main steps of the improved subequation method for finding exact solutions of (NFPDEs).

Suppose an NFPDE
\[ F(u, D_t^\alpha u, u_x, D_t^\alpha u_{xx}, \ldots) = 0; \quad 0 < \alpha \leq 1, \tag{15} \]
where $D_t^\alpha u$ is the conformable fractional derivative of $u$, $u = u(x,t)$ is an unknown function, and $F$ is a polynomial in $u$ and its conformable time-fractional partial derivatives and space partial derivatives.

The main steps of the improved subequation method are presented in [18, 20, 32] as follows:

Step 1: suppose that
\[ u(x,t) = U(\xi); \quad \xi = kx + I^n_\alpha t(t). \tag{16} \]

By substituting (15), NFPDEs turn to the ordinary differential equation (ODE):
\[ P(U, U', U'', \ldots) = 0, \tag{17} \]
where $P$ is a polynomial in $U$ and its derivatives, $U' = (dU/d\xi)$.

Step 2: suppose that the solution of equation (15) can be expressed in the following form:
\[ U(\xi) = \sum_{i=-m}^m a_i(t) \phi_i(\xi), \tag{18} \]
where $a_i(t)$ $(i = -m, \ldots, m)$ is the function of $t$ to be determined later, and $\phi = \phi(\xi)$ satisfies the following Riccati equation:
\[ \varphi' = \sigma + \varphi^2(\xi), \tag{19} \]
where $\sigma$ is a constant. The following solutions of Riccati equation (19) are given by [33]
\[ \varphi(\xi) = \begin{cases} -\sqrt{-\sigma} \tanh(\sqrt{-\sigma} \xi), \quad \sigma < 0, \\ \sqrt{\sigma} \tan(\sqrt{\sigma} \xi), \quad \sigma > 0, \\ \frac{1}{\xi + \omega}, \quad \sigma = 0, \end{cases} \tag{20} \]
where $\omega$ is a constant.

Step 3: determine the positive integer $m$ by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in equation (17), then substituting (18) with equation (19) into equation (17), then using the properties of the conformable fractional derivative (5)–(14), then collecting each coefficient of $\varphi_i(\xi)$ to zero, one can get an overdetermined system of nonlinear differential equations for $a_i(t)$ and $\xi$.

Step 4: assume that $a_i(t)$ and $\xi$ can be obtained by solving the overdetermined system of Step 3, then substituting these results and the solutions, one can obtain the exact solutions of equation (15) immediately.

2.2.2. Description of the Improved Sine-Cosine Method. In this section, the main steps of the sine-cosine method [34,35] are given as follows:

Step 1: suppose that
\[ u(x,t) = \phi(\xi); \quad \xi = kx + I^n_\alpha t(t), \]
\[ \phi(\xi) = \sum_{i=1}^n \sin^{i-1} \frac{\pi}{2} \phi_i b_i(t) \sin^i + a_i(t) \cos^i + a_0(t), \tag{21} \]
where $a_0(t), \ldots, a_n(t)$ and $b_1(t), \ldots, b_n(t)$ are all functions of $t$ to be determined later; equation (15) reduces to ODE
\[ P(\phi, \phi', \phi'', \ldots) = 0, \tag{22} \]
where $\phi' = d\phi/d\xi$.

Now, taking the target equation as
\[ \frac{d\varphi}{d\xi} = \sin\varphi, \quad \text{or} \quad \frac{d\varphi}{d\xi} = \cos\varphi, \]  
\tag{23}

\[ \frac{d\varphi}{d\xi} = \tanh\xi \]  
\tag{24}

\[ \frac{d\varphi}{d\xi} = -\sech\xi \]  
\tag{25}

Step 2: equating the highest order nonlinear term and highest order linear partial derivative in equation (22), it yields the value of \( n \).

Step 3: substituting (21) and (23) into equation (15), and collecting all the terms with the same power of \( \sin\varphi \), \( \cos\varphi \), \( \sin\varphi \cos\varphi \), \( \sin^2\varphi \), . . . , then setting the coefficients to be zero, one can obtain an overdetermined system of nonlinear differential equations for \( b_i(t) \), \( a_i(t) \) and \( \tau(t) \) for \( i = 1, \ldots, n \).

Step 4: assuming that \( b_i(t) \), \( a_i(t) \), and \( \xi \) can be obtained by solving the overdetermined system of Step 3, then substituting these results and the solutions, the exact solutions of equation (15) can be obtained immediately.

\section{Applications of Two Methods}

\subsection{Applications of the Improved Subequation Method}

Using the generalized traveling wave transformation,
\[ u(x, t) = U(\xi), \]
\[ \xi = kx + t^\alpha \tau(t), \]
and Property 2.1.3 and Theorem 2.1.4, equation (2) can be reduced to the following nonlinear ordinary differential equation:
\[ (\tau(t) + k\beta(t))U' - \gamma(t)k^2U'' + \delta(t)U(1 - U)(\mu - U) = 0, \]
\tag{27}

where \( U' = (dU/d\xi) \) and \( U'' = (d^2U/d\xi^2) \). We suppose that equation (27) has a solution in the form of (18). Balancing the highest order derivative term \( U'' \) and with nonlinear term \( U^3 \) in equation (27), one can get \( m + 2 = 3m \Rightarrow m = 1 \). So, we have
\[ U(\xi) = \sum_{i=1}^{3} a_i(t) \varphi^i(\xi) = a_0(t) + a_1(t) \varphi(\xi) + a_{-1}(t) \varphi^{-1}(\xi). \]
\tag{28}

Substituting (28) into equation (2), collecting all the terms with the same power of \( \varphi^i \) \((i = -3, -2, \ldots, 2, 3) \), then setting the coefficients of \( \varphi^i \) \((i = -3, -2, \ldots, 2, 3) \) to be zero, one can obtain the overdetermined system of nonlinear differential equations for \( a_{-1}(t) \), \( a_0(t) \), \( a_1(t) \), and \( \tau(t) \) as follows:
\begin{align*}
\varphi^{-3}: & -2k^2\sigma^2\gamma(t)a_{-1}(t) + \delta(t)a_{-1}^3(t) = 0, \tag{29} \\
\varphi^{-2}: & -k\sigma\beta(t)a_{-1}(t) + 3\delta(t)a_0(t)a_{-1}^2(t) - \sigma\tau(t)a_{-1}(t) - (1 + \mu)\delta(t)a_{-1}^2(t) = 0, \tag{30} \\
\varphi^{-1}: & t^\alpha \frac{da_{-1}(t)}{dt} - 2(1 + \mu)\delta(t)a_0(t)a_{-1}(t) + 3\delta(t)a_{-1}(t)a_0^2(t) + 3\delta(t)a_1(t)a_{-1}^2(t) + \mu\delta(t)a_{-1}(t) - 2k^2\sigma\gamma(t)a_{-1}(t) = 0, \tag{31} \\
\varphi^0: & t^\alpha \frac{da_0(t)}{dt} + k\sigma\beta(t)a_1(t) + 6\delta(t)a_0(t)a_{-1}(t)a_1(t) - 2(1 + \mu)\delta(t)a_0(t)a_{-1}(t) - (1 + \mu)\delta(t)a_0^2(t) + \tau(t)\sigma a_1(t) - \tau(t)a_{-1}(t) - (1 + \mu)\delta(t)a_0^2(t) + \delta(t)a_0^3(t) + \mu\delta(t)a_0(t) = 0, \tag{32} \\
\varphi^1: & t^\alpha \frac{da_1(t)}{dt} - 2(1 + \mu)\delta(t)a_0(t)a_1(t) + 3\delta(t)a_0^2(t)a_1(t) + 3\delta(t)a_0(t)a_{-1}(t)a_1(t) + 3\delta(t)a_1(t)a_{-1}(t) + \mu\delta(t)a_1(t) - 2k^2\sigma\gamma(t)a_1(t) = 0, \tag{33} \\
\varphi^2: & \tau(t)a_1(t) + k\beta(t)a_1(t) + 3\delta(t)a_0(t)a_1^2(t) + (1 + \mu)\delta(t)a_1^3(t) = 0, \tag{34} \\
\varphi^3: & -2k^2\gamma(t)a_1(t) + \delta(t)a_1^3(t) = 0. \tag{35}
\end{align*}
\[
\Psi(t) = \exp(-\int t^{\sigma-1} f(t) dt), \quad \text{where } f(t) \text{ is a continuous function. Solving this system by Maple software, and for simplicity, we introduce the notation.}
\]

Finally, one can obtain these four cases:

**Case 1.**

If we assume that \( a_{-1}(t) = 0, \ a_0(t) = \Psi(t) \), it can be obtained that

\[
a_1(t) = c_1 \exp \left[ \int t^{\sigma-1} \left( 2(1 + \mu)\delta(t)\Psi(t) - 3\delta(t)\Psi^2(t) - \mu\delta(t) + 2k^2\sigma\Psi(t) dt \right) \right],
\]

\[
\tau(t) = \frac{-1}{\sigma a_1(t)} \left[ \beta(t)\sigma a_1(t) - f(t)\Psi(t) + \delta(t) \left( -(1 + 2\mu)\Psi^2(t) + \Psi^3(t) + \mu\Psi(t) \right) \right].
\]

When \( \sigma < 0 \), the following hyperbolic function solution of equation (2) is obtained as follows:

\[
u(x,t) = \Psi(t) - \sqrt{-\sigma} a_1(t) \ \tanh \left( \sqrt{-\sigma} \ (kx + I_1^e \tau(t)) \right),
\]

(38)

\[
u(x,t) = \Psi(t) - \sqrt{-\sigma} a_1(t) \ \coth \left( \sqrt{-\sigma} \ (kx + I_1^e \tau(t)) \right).
\]

(39)

When \( \sigma > 0 \), we obtain the following trigonometric function solution of equation (2), as follows:

\[
u(x,t) = \Psi(t) + \sqrt{\sigma} a_1(t) \ \tan \left( \sqrt{\sigma} \ (kx + I_1^e \tau(t)) \right),
\]

(40)

\[
u(x,t) = \Psi(t) - \sqrt{\sigma} a_1(t) \ \cot \left( \sqrt{\sigma} \ (kx + I_1^e \tau(t)) \right).
\]

(41)

**Case 2.**

If we assume that \( a_1(t) = 0, \ a_0(t) = \Psi(t) \), it can be obtained that

\[
a_{-1}(t) = c_2 \exp \left[ \int t^{\sigma-1} \left( 2\delta(t)\Psi(t) - 3\delta(t)\Psi^2(t) - \mu\delta(t) + 2k^2\sigma\Psi(t) dt \right) \right],
\]

\[
\tau(t) = \frac{-1}{a_{-1}(t)} \left[ \beta(t)ka_1(t) + f(t)\Psi(t) + \delta(t) \left( -\Psi^3(t) + (1 + \mu)\Psi^3(t) - \mu\Psi(t) \right) \right].
\]

When \( \sigma = 0 \), substituting \( \sigma = 0 \) into the above-mentioned system and then solving the obtained system, one can find \( a_1(t) \) will not change but \( \tau(t) \) is changed and becomes equal to

\[
t_1(t) = -\beta(t)k + (-3\Psi(t) + 1 + \mu) a_1(t)\delta(t).
\]

When \( \sigma = 0 \), we have the following rational solution of equation (2), as follows:

\[
u(x,t) = \Psi(t) - \frac{1}{kx + I_1^e \tau_1(t) + \omega} a_1(t).
\]

(42)

**Case 3.**

If we assume that \( a_1(t) = 0, \ a_0(t) = \Psi(t) \), it can be obtained that

\[
\nu(x,t) = \Psi(t) - \frac{a_{-1}(t)}{\sqrt{\sigma} \ \tan \left( \sqrt{\sigma} \ (kx + I_1^e \tau(t)) \right)}.
\]

(46)

\[
u(x,t) = \Psi(t) - \frac{a_{-1}(t)}{\sqrt{\sigma} \ \cot \left( \sqrt{\sigma} \ (kx + I_1^e \tau(t)) \right)}.
\]

(47)

When \( \sigma = 0 \), we have the following solution of equation (2), as follows:

\[
u(x,t) = \Psi(t) - \left( kx + I_1^e \tau(t) + \omega \right) a_{-1}(t).
\]

(48)

**Case 4.**

If we assume that \( a_0(t) = c_0\Psi(t), \ a_1(t) = c_1\Psi(t) \), it can be obtained that
\[ a_{-1}(t) = \frac{\exp\left[\int t^{\alpha-1} \epsilon(t) \, dt\right]}{c_5 + 3 \int \exp\left[\int \epsilon(t)^{\alpha-1} \, dt\right] \delta(t) c_4 \Psi(t) \, t^{\alpha-1} \, dt} \]  

where

\[ \epsilon(t) = 2c_3 \delta(t) \Psi(t) - 3c_3^2 \delta(t) \Psi^2(t) - \mu \delta(t) + 2k^2 \sigma^2(\Psi(t) + 2c_3 \mu \delta(t)) \Psi(t), \]

\[ \tau(t) = \frac{1}{\sigma c_4 \Psi(t) - a_{-1}(t)} \]

\[ \left[ -k \sigma \beta(t) c_4 \Psi(t) + 6 \sigma \delta(t) c_4 \Psi^2(t) c_4 a_{-1}(t) + \delta(t) (1 + \mu) \left( 2c_4 \Psi(t) a_{-1}(t) + c_3^2 \Psi^2(t) \right) \right. \]

\[ + \beta(t) k a_{-1}(t) + c_4 \Psi(t) f(t) \right] - \delta(t) \left( c_4^3 \Psi^3(t) + \mu c_4 \Psi(t) \right). \]

When \( \sigma < 0 \), we obtain the following hyperbolic function solution of equation (2), as follows:

\[ u(x, t) = c_3 \Psi(t) - c_4 \sqrt{-\sigma} \Psi(t) \tanh(\sqrt{-\sigma}(kx + I^\alpha_t \tau(t))) - \frac{a_{-1}(t)}{\sqrt{-\sigma} \tanh(\sqrt{-\sigma}(kx + I^\alpha_t \tau(t)))}. \]  

When \( \sigma > 0 \), we obtain the following trigonometric function solution of equation (2), as follows:

\[ u(x, t) = c_3 \Psi(t) + c_4 \sqrt{\sigma} \Psi(t) \tan(\sqrt{\sigma}(kx + I^\alpha_t \tau(t))) + \frac{a_{-1}(t)}{\sqrt{\sigma} \tan(\sqrt{\sigma}(kx + I^\alpha_t \tau(t)))}. \]  

When \( \sigma = 0 \), we have the following rational solution of equation (2), as follows:

\[ u(x, t) = c_3 \Psi(t) - \frac{1}{(kx + I^\alpha_t \tau(t) + \omega) a_{-1}(t)}. \]

Case 4.

If we assume that \( a_0(t) = c_3 \Psi(t), \ a_1(t) = c_4 \Psi(t) \), it is obtained that

\[ \tau(t) = -k \sigma \beta(t) + 3c_4 \delta(t) c_3 \Psi^2(t) (\mu + 1) \delta(t) c_4 \Psi(t), \]

\[ a_{-1}(t) = \frac{\delta(t) \Psi^2(t) \left( 3c_4^2 \sigma c_3 + c_3^2 \right) + \delta(t) (1 + \mu) \Psi(t) \left( c_3^2 \sigma - c_3^2 \right) - c_4 f(t) + c_3 \mu \delta(t)}{3c_4 \delta(t) \left( -3c_4 \Psi(t) + \mu + 1 \right)}. \]
When $\sigma < 0$, we obtain the following hyperbolic function solution of equation (2), as follows:

$$u(x, t) = c_3 \Psi(t) - c_4 \sqrt{-\sigma} \Psi(t) \tanh \left( \sqrt{-\sigma} \ (kx + I_1^a \tau(t)) \right) - \frac{a_{-1}(t)}{\sqrt{-\sigma} \tanh \left( \sqrt{-\sigma} \ (kx + I_1^a \tau(t)) \right)},$$

$$u(x, t) = c_3 \Psi(t) - c_4 \sqrt{-\sigma} \Psi(t) \coth \left( \sqrt{-\sigma} \ (kx + I_1^a \tau(t)) \right) - \frac{a_{-1}(t)}{\sqrt{-\sigma} \coth \left( \sqrt{-\sigma} \ (kx + I_1^a \tau(t)) \right)}.$$ (57)

(58)

When $\sigma > 0$, we obtain the following trigonometric function solution of equation (2), as follows:

$$u(x, t) = c_3 \Psi(t) + c_4 \sqrt{\sigma} \Psi(t) \tan \left( \sqrt{\sigma} \ (kx + I_1^a \tau(t)) \right) + \frac{a_{-1}(t)}{\sqrt{\sigma} \tan \left( \sqrt{\sigma} \ (kx + I_1^a \tau(t)) \right)},$$

$$u(x, t) = c_3 \Psi(t) - c_4 \sqrt{\sigma} \Psi(t) \cot \left( \sqrt{\sigma} \ (kx + I_1^a \tau(t)) \right) - \frac{a_{-1}(t)}{\sqrt{\sigma} \cot \left( \sqrt{\sigma} \ (kx + I_1^a \tau(t)) \right)}.$$ (59)

(60)

When $\sigma = 0$, we have the following rational solution of equation (2), as follows:

$$u(x, t) = c_3 \Psi(t) - \frac{1}{kx + I_1^a \tau(t) + \omega} c_4 \Psi(t)$$

$$- (kx + I_1^a \tau(t) + \omega) a_{-1}(t),$$

$$u(x, t) = c_3 \Psi(t) - c_4 \sqrt{\sigma} \Psi(t) \cot \left( \sqrt{\sigma} \ (kx + I_1^a \tau(t)) \right) - \frac{a_{-1}(t)}{\sqrt{\sigma} \cot \left( \sqrt{\sigma} \ (kx + I_1^a \tau(t)) \right)}.$$ (61)

Using the generalized traveling wave transformation,

$$u(x, t) = \phi(\xi),$$

$$\xi = kx + I_1^a \tau(t),$$

and Property 2.1.3 and Theorem 2.1.4, equation (2) can be reduced to the following nonlinear ordinary differential equation:

$$(\tau(t) + k\beta(t))\phi'' - \gamma(t)k^2 \phi'' + \delta(t)(\mu\phi - \phi^2 - \mu\phi^2 + \phi^3) = 0.$$ (62)

(63)

Balancing the highest order derivative term $\phi''$ with nonlinear term $\phi^3$, we can get $n + 2 = 3n \Rightarrow n = 1$. So, we have

$$\phi(\xi) = b_1(t) \sin q + a_1(t) \cos q + a_0(t).$$ (64)

Substituting (64) and (23) into equation (2) and collecting all the terms with the same power of $\sin^i q \cos^j q$ for $i = 0, 1$ and $j = 0, 1, 2, 3$, then setting the coefficients to be zero, we can obtain overdetermined system of nonlinear differential equations for $b_1(t)$, $a_0(t)$, $a_1(t)$, and $\tau(t)$ as follows:

$$\sin q: \ t^{-a} \frac{\partial b_1(t)}{\partial t} + k^2 \gamma(t)b_1(t) + \mu \delta(t)b_1(t) + 3 \delta(t)a_0^2(t)b_1(t)$$

$$- 2(1 + \mu)\delta(t)a_0(t)b_1(t) + \delta(t)b_1^2(t) = 0,$$ (65)

$$\sin q \cos q: \ \tau(t)b_1(t) + k\beta(t)b_1(t) - 2(1 + \mu)\delta(t)a_1(t)b_1(t)$$

$$+ 6\delta(t)a_0(t)b_1(t)a_1(t) = 0,$$ (66)

3.2. Applications of the Improved Sine-Cosine Method.
\[
\cos: \ t^{1-a} \frac{d a_1}{d t} + 2k^2 y(t) a_1(t) + 3\delta(t) a_1(t) a_0^2(t) + \mu \delta(t) a_1(t) \\
+ 3\delta(t) a_1(t) b_1(t) - 2(1+\mu) \delta(t) a_0(t) a_1(t) = 0,
\]

\[
\sin^0 \cos^0: \ t^{1-a} \frac{d a_0}{d t} - k\beta(t) a_1(t) + 3\delta(t) b_1^2(t) a_0(t) - a_1(t) \tau(t) \\
- (1+\mu) \delta(t) b_1^2(t) - (1+\mu) \delta(t) a_0^2(t) + \delta(t) a_0(t) + \mu \delta(t) a_0(t) = 0,
\]

\[
\sin \cos^2: \ -2k^2 y(t) b_1(t) - \delta(t) b_1^2(t) + 3\delta(t) a_1^2(t) b_1(t) = 0,
\]
\[
\cos^2 g: \quad \tau(t) a_1(t) + k \beta(t) a_1(t) - 3\delta(t) a_0(t) b_1^2(t) + 3\delta(t) a_0(t) a_1^2(t) - (1 + \mu)\delta(t) a_1(t) + (1 + \mu)\delta(t) b_1^2(t) = 0, 
\]

(70)

\[
\cos^3 g: \quad -2k^2 \gamma(t)a_1(t) + \delta(t) a_1^3(t) - 3\delta(t) a_1(t) b_1^2(t) = 0. 
\]

(71)

Solving this system by Maple, for simplicity, we introduce the notation \(\Psi(t) = \exp(-\int t^{a-1} f(t)dt)\), where \(f(t)\) is a continuous function.

Finally, one finds these four cases as follows:

Case 1.
If we assume that \(b_1(t) = 0, a_0(t) = \Psi(t)\), it can be obtained that

\[
a_1(t) = c_6 \exp \left[ \int t^{a-1} \left( -2k^2 \gamma(t) + 2\delta(t)\Psi(t) - 3\delta(t)\Psi^2(t) - \mu\delta(t) + 2\mu\delta(t)\Psi(t) \right) dt \right]. 
\]

\[
\tau(t) = (1 + \mu - 3\Psi(t))\delta(t)c_6 \exp \left[ \int t^{a-1} \left( -2k^2 \gamma(t) + 2\delta(t)\Psi(t) - 3\delta(t)\Psi^2(t) - \mu\delta(t) + 2\mu\delta(t)\Psi(t) \right) dt \right] - k\beta(t). 
\]

(72)

We substitute what mentioned before into (64) and using (24) then, we obtain the hyperbolic functions solution of equation (2), as follows:

\[
u_1(x,t) = \pm a_1(t) \tanh(kx + I^t_1(t)) + \Psi(t). 
\]

(73)

Case 2.
If we assume that \(a_0(t) = 0, a_1(t) = \Psi(t)\), it can be obtained that

\[
\tau(t) = -k\beta(t) + 2(\mu + 1)\delta(t)\Psi(t), 
\]

(74)

\[
b_1(t) = \pm \frac{1}{\omega_1 + c_7} \sqrt{\omega_1(t) + c_7} \exp \left( -\frac{2t^a k^2 \gamma(t)}{\alpha} - 2\mu\omega_2(t) \right), 
\]

(75)

where

\[
\omega_1(t) = \int \frac{2t^{a-1}\delta(t)}{\exp \left[ 2 \int t^{a-1} \left( k^2 \gamma(t) + \mu\delta(t) \right) dt \right]} dt, 
\]

(76)

\[
\omega_2(t) = \int t^{a-1}\delta(t)dt. 
\]

We substitute the solutions of the Case 2 into (64) and using (24), and then we give the hyperbolic functions solution of equation (2), as follows:

\[
u_2(x,t) = \pm \Psi(t) \tanh(kx + I^t_1(t)) + b_1(t) \sech(kx + I^t_1(t)). 
\]

(77)

Case 3.
If we assume that \(a_0(t) = c_8\Psi(t), a_1(t) = c_9\Psi(t)\), it is obtained that

\[
b_1(t) = c_8 \Psi(t), 
\]

(78)

\[
\tau(t) = -6c_8c_9\delta(t)\Psi^3(t) - k\beta(t) + 2c_8(\mu + 1)\delta(t)\Psi(t). 
\]

We substitute the solutions of the Case 3 into (64) and using (24), and then we give the hyperbolic functions solution of equation (2), as follows:

\[
u_3(x,t) = \pm c_9\Psi(t) \tanh(kx + I^t_1(t)) + b_1(t) \tanh(kx + I^t_1(t)) + c_9\Psi(t). 
\]

(79)

Case 4.
If we assume that \(a_0(t) = c_8\Psi(t), a_1(t) = c_9\Psi(t)\), it is obtained that

\[
b_1(t) = 0, \tau(t) = -6c_8c_9\delta(t)\Psi^3(t) - k\beta(t) + 2c_8(\mu + 1)\delta(t)\Psi(t). 
\]

(80)
We substitute the solutions of the Case 4 into (64) and using (24), and then we give the hyperbolic functions solution of equation (2), as follows:

\[ u_{4}(x,t) = \left[ \pm c_{6}\tanh(kx + I_{\alpha}^{t} \tau(t)) \right] \Psi(t), \tag{81} \]

where \( c_{6}, c_{7}, c_{8}, c_{9}, \) and \( k \) are arbitrary constants.

In Figure 3, it is clearly shown that (a) the space-time graph of solution (73) when \( \alpha = 0.5 \), \( f(t) = t^{0.5} \), \( c_{6} = k = \mu = 1 \), \( \delta(t) = e^{t} \), \( \beta(t) = e^{-t} \), and \( \gamma(t) = (1/2)(4 - 3e^{t} - e^{t} + t^{0.5}) \) clarify the propagation of the kink-shape wave soliton solution and (b) the space-time graph of solution (79) when \( \alpha = 0.5 \), \( f(t) = t^{0.5} \), \( c_{6} = c_{9} = k = \mu = 1 \), \( \delta(t) = e^{-2t} \), \( \beta(t) = 4e^{-3t} \), and \( \gamma(t) = t^{-0.5} \), represents the propagation of the anti-kink-shape wave soliton solution.

4. Conclusions

In this work, we have obtained three kinds of exact solutions that include the trigonometric function solutions, the hyperbolic function solutions, and the rational solutions which are successfully established by using the subequation method. Also, a kind of exact analytical solutions which are the hyperbolic function solutions are found by using the sine-cosine method and conformable fractional derivative of the time-fractional generalized Fitzhugh–Nagumo equation with time-dependent coefficients. We have found that the subequation method gives more general solutions than the solutions that are given by the sine-cosine method. Moreover, it is observed that the solutions obtained in this research may be important to describe certain nonlinear phenomena in mathematical physics, engineering, and biology. Remarkably, these solutions and the proposed traveling wave transformation have not been reported in other literature. Also, some 3D graphical representations are offered for the obtained results with the different time-dependent coefficients and different values of \( \alpha \). We can conclude that the used method is a very powerful, convenient, and efficient technique and it can be used for many other partial fractional nonlinear differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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