The volume of the black holes - the constant curvature slicing of the spherically symmetric spacetime

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Abstract

We consider the problem of determination of a volume of some bounded space-like hypersurfaces in the case of spherically symmetric spacetimes. In the case when the hypersurfaces is cut or bounded by a light-like hypersurface the problem may not be well defined. In order to define properly the volume we introduce the volume forms related to the given foliation (observer) of the considered spacetime. In the case of the constant curvature slice the volume of the hypersurface cut by the horizon (light-like surface) becomes composed of the two parts, outer and inner, treated differently. We compute the corresponding volumes outside and inside of the horizon of the eternal Schwarzschild black hole.

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1 Introduction

In order to find a volume of a space-like hypersurface $S$ of a spacetime $M$ one need to determine $S$ and to know a canonical volume form related to the spacetime $M$. This canonical volume form is next pulled back to $S$ by an embedding which gives the parametrization of $S$ and finally one performs an appropriate integral. This procedure is well-known and well-defined. The matter changes if one wants to find the volume of the space-like hypersurface bounded or cut by an event horizon of a black hole, a light-like hypersurface with vanishing volume but the fixed area. How to define the foliation inside the horizon? The problem of foliations in the case of the Schwarzschild spacetime appeared in [1-3]. In the papers [4-14] the problem of a volume was considered for different types of
black holes. Different values of volumes had been found changing from: zero [8] to infinity [13] being dependent on a chosen foliation \( \mathcal{F} \) of the spacetime which gave the space-like slices of \( M \). These slices are leaves of \( \mathcal{F} \).

Motivated by the above mentioned question: what is a volume of a spacelike hypersurface \( S \) cut by an event horizon we will consider in this paper a volume as related to the constant curvature slice in the case of the spherically symmetric spacetimes. We will demand that \( S \) lays on the slice; it is simultaneous with the slice. The condition of the constant curvature of the slice is expressed by a constant value of the covariant divergency of a vector \( n \) normal to \( S \). Vanishing divergency of \( n \) determines maximal value of the volume of \( S \). It results in this case in an equation that is solved in order to determine the embedding of \( S \) into \( M \). The solution contains some parameter \( c \) of the dimension “length\(^2\)”.

Thus to find the volume of \( S \) one needs to pull-back the volume form by this embedding and to perform the integration in given limits. The embedding and also the volume depend on \( c \). This constant will be determined by the claim that the induced volume functional reaches a local maximum with respect the variations that keep fixed the boundary \( \partial S \) of \( S \).

In this paper we will consider the case of the spacetime that is asymptotically Minkowskian, containing an ethernal black hole. In the simplest case it is a given by Schwarzschild solution.

The paper is organized as follows:

In the section 2 we recall spherically symmetric metrics and we will obtain the volume three-form for the foliation given by the unit time-like vector (observer) in a case of a spherically symmetric spacetime. In the section 3 we will determine the embedding of the space-like hypersurface from the condition of the constant mean extrinsic curvature. In the section 4 we will apply this general approach to the case of the Schwarzschild black holes in order to get a volume of hypersurface cut by the horizon and the volume of the black holes in the given foliation (in three different asymptotics) The section 5 is devoted to discussion. In the Appendix we prove that volume is independent on the coordinate systems and we consider hypersurface bounded by the trapped surfaces.

2 Foliations and volume forms

A spacetime \( M \) is spherically symmetric if its isometry group contains a subgroup isomorphic to the group \( SO(3) \), and orbits of this subgroup are 2—dimensional spheres \( S^2 \). In other words: the spacetime \( M \) is spherically symmetric about one point \( p \), if, in some coordinate system, a metric \( g_{\mu\nu} \) is invariant for three-dimensional spatial rotations about \( p \) that is, three-dimensional spatial rotations are isometries for \( g_{\mu\nu} \). However in the case of the Schwarzschild spacetime \( M \) the point \( p \) is singular and not belongs to \( M \). As is well-known the metric \( g_{\mu\nu} \) on \( M \) can be expressed in the coordinates \( (x^0, x^1, \theta, \phi) \) as follows:

\[
ds^2 = g_0 (x^0, x^1) d (x^0)^2 - g_1 (x^0, x^1) d (x^1)^2 - g_2 (x^0, x^1) d\Omega^2_{S^2} \quad (2.1)
\]
and \( d\Omega^2_{S^2} = d\theta^2 + \sin^2\theta d\phi^2 \) is the metric on the unit sphere \( S^2 \). In this space-time \( M \) the vector field \( e_0 \) determines the orthonormal basis which is spanned by vectors \( e_a = X^\mu_a \partial_\mu \) such that \( e_a \cdot e_b = X^\mu_a X^\nu_b \eta_{\mu\nu} = \delta_{ab} = \{diag (+1, -1, -1, -1)\}_{ab} \) and \( a, b = 0, 1, 2, 3 \). We shall consider foliations which have one basis vector fixed in the equatorial plane, \( \theta = \pi/2 \). This vector is: 
\( e_2 = (g_2)^{-1/2} \partial_\theta \). Then, the coefficients \( X^\mu_a \) are given by the matrix (see [15]):

\[
(X^\mu_a) = \begin{pmatrix}
\cosh q & \sinh q \cos \chi & 0 & \sinh q \sin \chi \\
\sinh q & \cosh q \cos \chi & 0 & \cosh q \sin \chi \\
\sqrt{g_0} & 0 & 1 & 0 \\
0 & \sqrt{g_1} & 0 & 0 \\
0 & -\sin \chi / \sqrt{g_2} & 0 & \cos \chi / \sqrt{g_2}
\end{pmatrix},
\]  
(2.2)

where \( g_3 = g_2 (x^0, x^1) \sin^2 \theta \). Thus the foliation is parametrized by two functions \( q \) and \( \chi \) which depend on \( x^0 \) and \( x^1 \). The tetrad of this foliation is given by the four one-forms \( E^a = E^\mu(a) dx^\mu \) dual to the vector fields \( e_a \). It means that: 
\( e_a \cdot E^b = \delta^b_a \). The coefficients \( E^\mu_a \) are given by the relation:

\[
(E^\mu_a) = \left( \left(X^\mu_a\right)^{-1}\right)^T.
\]  
(2.3)

Thus:

\[
(E^\mu_a) = \begin{pmatrix}
\sqrt{g_0} \cosh q & -\sqrt{g_1}u_1 & 0 & -\sqrt{g_1u_2} \\
-\sqrt{g_0} \sinh q & \sqrt{g_1} s_1 & 0 & 0 \\
0 & \sqrt{g_2} & 0 & 0 \\
0 & -\sqrt{g_1} \sin \chi & 0 & \sqrt{g_2} \cos \chi
\end{pmatrix},
\]  
(2.4)

where: \( u_1 = \sinh q \cos \chi, u_2 = \sinh q \sin \chi, s_1 = \cosh q \cos \chi \) and \( s_2 = \cosh q \sin \chi \). In this tetrad \( (E^a) \) the metric has the form:

\[
ds^2 = (E^{(0)})^2 - (E^{(1)})^2 - (E^{(2)})^2 - (E^{(3)})^2
\]  
(2.5)

and the canonical volume four-form \( dV \) is:

\[
dV = E^{(0)} \wedge E^{(1)} \wedge E^{(2)} \wedge E^{(3)} = \det (E^\mu_a) d\theta \wedge d\phi.
\]  
(2.6)

Hence one finds that a volume three-form \( \eta \) related to the foliation determined by \( e_{(0)} \) is given by the inner product of \( e_{(0)} \) and \( dV \):

\[
\eta \equiv i_{e_{(0)}} dV = E^{(1)} \wedge E^{(2)} \wedge E^{(3)}.
\]  
(2.7)

This form \( \eta \) expressed in the coordinates \( (x^0, x^1, \theta, \phi) \) is:

\[
\eta = -g_2 \sqrt{g_0 g_1} \left( \frac{\cosh q \sqrt{g_0}}{\sqrt{g_1}} dx^1 - \frac{\sinh q \cos \chi \sqrt{g_0}}{\sqrt{g_1}} dx^0 \right) \wedge d(\cos \theta) \wedge d\phi \\
-\sqrt{g_0 g_1} g_2 \sinh q \sin \chi dx^0 \wedge dx^1 \wedge d\theta,
\]  
(2.8)
where we used relation: $\sqrt{-\text{deg}(g_{\mu\nu})} = g_2\sqrt{g_{01}}\sin \theta$. One can see that $\eta$ consists of two parts that depend on $q$ and $\chi$. In the considered case of the spherical symmetry $q$ and $\chi$ depend only on coordinates which appear in the metric coefficients.

2.1 Spacetimes with black hole

As is well-known a spacetime $\mathcal{M}$ with a non rotating and uncharged black hole is the spherically symmetric solution of Einstein equations in the vacuum with the cosmological constant $\Lambda$. In the Schwarzschild-like coordinates ($x^0 = T$, $x^1 = R$) the solution is given by metric:

$$ds^2 = h(R) \, dT^2 - \frac{dR^2}{h(R)} - R^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right), \quad (2.12)$$

where

$$h(R) = 1 - \frac{r_S}{R} - \frac{1}{3} \Lambda R^2 \quad (2.12a)$$

and $R \in (r_S, R_{\text{max}})$ the radius $R_{\text{max}}$ depends on $r_S$ and $\Lambda$. The volume form $\eta$ in these coordinates takes the form:

$$\eta = -\frac{R^2}{\sqrt{h(R)}} \left( \cosh q dR - h(R) \sinh q \cos \chi dT \right) \wedge \left( \cos \theta \wedge d\phi \right) - R \sinh q \sin \chi dT \wedge dR \wedge d\theta. \quad (2.13)$$

Hereafter we will discuss the case $\Lambda = 0$.

3 Volume of the space-like slices

A hypersurface $S$ in the spacetime $\mathcal{M}$ is called space-like if a normal vector field $n$ to $S$ is time-like. In our signature it means that $n^2 > 0$. Such a hypersurface $S$ can be represented by means of its embedding $\Phi : S \to \mathcal{M}$ into the spacetime $\mathcal{M}$ via some parametric equations:

$$\Phi (\xi^i) = (x^\mu (\xi^i)) \in \mathcal{M}, \quad (3.1)$$

where $\xi^i$ are local coordinates for $S$ ($i = 1, 2, 3$), while $x^\mu$ are local coordinates in $\mathcal{M}$ ($\mu = 0, \ldots, 3$).

In $\mathcal{M}$ there is a canonical volume form $\varepsilon = \sqrt{|\det (g_{\mu\nu})|} \, dx^0 \wedge dx^1 \wedge d\theta \wedge d\phi$. In order to get the volume of the hypersurface $S$ one need to pull back the inner product $\varepsilon$ and $n$ by the embedding $\Phi$ and to perform an integral:

$$\text{vol} (S; i_n \varepsilon) = \int_S \Phi^* (i_n \varepsilon). \quad (3.2)$$

An induced metric $h_{ij}$ on $S$ is given by the embedding $\Phi$ and the metric $g_{\mu\nu}$:

$$h_{ij} = \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^j} g_{\mu\nu}. \quad (3.3)$$
Hence the volume form $\Phi^* (i_n \varepsilon)$ reads:

$$\Phi^* (i_n \varepsilon) = \sqrt{| \det (h_{ij}) |} d\xi^1 \wedge d\xi^2 \wedge d\xi^3. \quad (3.4)$$

We shall consider the spherically symmetric space-like hypersurface $S$ embedded in the spherically symmetric spacetime $M$ with the metric (2.1) so $\Phi$ is:

$$\Phi (\xi, \theta, \phi) = (x^0 (\xi), x^1 (\xi), \theta, \phi) \quad (3.5)$$

where $\xi \in [\xi_i, \xi_f]$ and $\theta, \phi \in S^2$. The unit tangent vectors (in the metric (2.1)) to this space-like hypersurface are:

$$t_{(1)} = \frac{1}{\sqrt{g_1 R'^2 - g_0 T'^2}} \left( T' \frac{\partial}{\partial T} + R' \frac{\partial}{\partial R} \right),$$

$$t_{(2)} = \frac{1}{\sqrt{g_2 \partial^\theta}} \frac{\partial}{\partial \theta}, \quad t_{(3)} = \frac{1}{\sqrt{g_2 \sin \theta}} \frac{\partial}{\partial \phi} \quad (3.6)$$

and $t_{(a)} \cdot t_{(b)} = -\delta_{ab}$, where $a, b = 1, 2, 3$ and the prime means differentiation with respect to $\xi : T' = \frac{dT}{d\xi}, R' = \frac{dR}{d\xi}$ (and $x^0 \equiv T, x^1 \equiv R$, see Sec.2.1) Then the embedding $\Phi$ is the space-like if

$$g_1 R'^2 > g_0 T'^2. \quad (3.7)$$

The unit time-like vector $n$ normal to $S$ is:

$$n = \pm \frac{1}{\sqrt{g_0 g_1 (g_1 R'^2 - g_0 T'^2)}} \left( g_1 R'^2 \frac{\partial}{\partial T} - g_0 T'^2 \frac{\partial}{\partial R} \right) \quad (3.8)$$

where $n^2 = +1$. The induced metric $h_{ij}$ on $S$ has the components:

$$h_{11} = g_0 T'^2 - g_1 R'^2 \equiv -h_1 < 0,$$

$$h_{22} = -g_2, \quad h_{33} = -g_2 \sin^2 \theta. \quad (3.9)$$

Thus we obtain that the volume of $S$ related to $\varepsilon$ is given by the formula:

$$\text{vol} (S; i_n \varepsilon) = 4\pi \int_{\xi_i}^{\xi_f} g_2 (T (\xi), R (\xi)) \sqrt{g_1 R'^2 - g_0 T'^2} d\xi. \quad (3.10)$$

On the other hand there is the volume form $\eta$ (2.8) related to an arbitrary foliation which under the embedding $\Phi$ (3.5) takes the form:

$$\Phi^* \eta = -g_2 \sqrt{g_0} \cosh q \left( \sqrt{\frac{g_1}{g_0}} R' - T' \tanh q \cos \chi \right) d\xi \wedge d (\cos \theta) \wedge d\phi. \quad (3.11)$$

Hence the volume of $S$ related to such foliation is:

$$\text{vol} (S; \eta) = 4\pi \int_{\xi_i}^{\xi_f} g_2 \sqrt{g_0} \cosh q \left( \sqrt{\frac{g_1}{g_0}} R' - T' \tanh q \cos \chi \right) d\xi. \quad (3.12)$$
The scalar product of the normal vector $n$ to $S$ and the unit time-like vector $e_{(0)}$ is:

$$n \cdot e_{(0)} = \frac{\cosh q}{\sqrt{h_1}} (\sqrt{g_1} R' - \sqrt{g_0} T' \tanh q \cos \chi).$$  \hspace{1cm} (3.13)

One can say that $S$ lies on the simultaneity surface of the foliation given by $e_{(0)}$ if:

$$n \cdot e_{(0)} = 1. \hspace{1cm} (3.14a)$$

Hence we get the condition:

$$\left( T' - R' \sqrt{\frac{g_1 \cosh q \sinh q \cos \chi}{g_0}} \right)^2 + \frac{g_1 R'^2 \sinh^2 q \sin^2 \chi}{g_0 (1 + \sinh^2 q \cos^2 \chi)} = 0. \hspace{1cm} (3.14b)$$

which has the solution only for $\chi = 0$ (since $g_1/g_0 > 0$) and:

$$T' = R' \sqrt{\frac{g_1}{g_0}} \tanh q (R(\xi)). \hspace{1cm} (3.15)$$

The embedding $\Phi$ takes the form:

$$\Phi (R, \theta, \phi) = \left( \int R' \sqrt{\frac{g_1 (R)}{g_0 (R)}} \tanh q (R(\xi)) \, d\xi, R, \theta, \phi \right) = (T(R), R, \theta, \phi),$$

and $T(R)$ is the solution of (3.15) expressed by the coordinate $R$. The normal vector (3.8) for such an embedding is:

$$n = \frac{\cosh q}{\sqrt{g_0}} \frac{\partial}{\partial T} - \frac{\sinh q}{\sqrt{g_1}} \frac{\partial}{\partial R}.$$  \hspace{1cm} (3.17)

and the volume of $S$ is equal to:

$$\text{vol} (S) = \text{vol} (S; i_n \varepsilon) = \text{vol} (S; \eta) = 4\pi \int_{R_1}^{R} \frac{g_2 \sqrt{g_1}}{\cosh q} dR.$$  \hspace{1cm} (3.18)

Hence the volume of $S$ depends on the function $q$. But the foliation is determined by $q$. As it is well known not every foliation is consistent with the Einstein equations. In the ADM formalism the consistency conditions are given by the constraint equations. However the dynamic equations give the relation between the volume $\text{vol} (V)$ of a some three-dimensional domain $V$ with a boundary $\partial V$ on the leaf $\Sigma_t$ of the given foliation and an extrinsic curvature $K_{ij}$ of $\Sigma_t$ (eg. [16]) as follows:

$$\frac{d\text{vol} (V)}{dn} = \int_V h^{ij} K_{ij} d^3v,$$

where $d^3v$ is the volume form on $\Sigma_t$ and $d\text{vol} (V)/dn$ means the change of the volume with the respect to the normal vector $n$ to $\Sigma_t$. One can say that the volume $\text{vol} (V)$ enclosed in $V$ is extremal with respect to variations of the domain.
delimited by \( \partial V \), if the mean extrinsic curvature \( K \equiv h^{ij}K_{ij} \) is vanishing. It is called maximal slicing condition. We will apply this condition in our case and extend it to the case when \( K \) takes the constant value. The mean extrinsic curvature \( K \) has a dimension of length\(^{-1}\) and is expressed by the vector \( n \) (see eq. (3.8)) as follows:

\[
K = -\nabla_{\mu} n^{\mu} = \frac{-1}{\sqrt{-\det(g_{\alpha\beta})}} \partial_{\mu} \left[ \sqrt{-\det(g_{\alpha\beta})} n^{\mu} \right]. \tag{3.19}
\]

Thus if \( K \) is constant, then the function \( q(T, R) \) can be determined from (3.19) in the special cases. If the vector field \( n \) were the time-like Killing vector field, then one would obtain the thermodynamic volumes \([6,7,10,12]\). But in our case \( n \) does not need to be the Killing vector (see Sec. 4.2).

4 Volume outside and inside horizon of the black hole

In this section we will consider applications of this hypersurface volume maximal value as given by means of the vanishing mean extrinsic curvature, \( K = 0 \) in the case of the exterior and interior of the Schwarzschild black hole. In both cases one demands that the hypersurface \( S \) is space-like, i.e. the condition (3.14a) is satisfied. Hence the volume of \( S \) is given by (3.12).

4.1 The Schwarzschild black hole in Minkowski space-time

1) Outside horizon

Let us start from an exterior of the Schwarzschild black hole in the asymptotically flat Minkowski space-time. Thus in the metric (2.12) we put \( \Lambda = 0 \). In this case, when the metric (2.12) expressed in Schwarzschild coordinates, depends on \( r \) but not on \( t \), one can apply a conventional notation, \( T = t \), and \( R = r \). Hence the space-like hypersurface \( S \) may be characterized as:

\[
S = \{ t = t(r), \ r \in [r_1, r_2] \text{ and } (\theta, \phi) \in S^2 \} \subset M. \tag{4.1}
\]

It is the three dimensional manifold bounded by two spheres \( S^2_1 \) and \( S^2_2 \) with radii \( r_1 \) and \( r_2 \). Thus the volume (3.18) of \( S \) related to the foliation given by the function \( q \) is:

\[
\text{vol}(S) = 4\pi \int_{r_1}^{r_2} \frac{g_2(r)}{\cosh q(r)} \sqrt{g_1(r)} \, dr = 4\pi \int_{r_1}^{r_2} r^2 \frac{dr}{\sqrt{r - r_S} \cosh q(r)}, \tag{4.2}
\]

for \( r_2 > r_1 > r_S \). Then the Eq. (3.19) takes the form:

\[
K = \frac{1}{g_2 \sqrt{g_0 g_1}} \partial_r \left[ g_2 \sqrt{g_0} \sinh q \right].
\]
leading to the solution:

\[
\sinh q(r) = \frac{c}{g_2 \sqrt{g_0}} + \frac{K}{g_2 \sqrt{g_0}} \int g_2 \sqrt{g_0 g_1} dr',
\]

(4.3)

where \( c \) is a constant of dimension: length\(^2\). One can see that the function \( q \) depends on two parameters \( c \) and \( K \) with dimensions: length\(^2\) and length\(^{-1}\), respectively. Hence the volume (3.18) of \( S \) also depends on these two parameters:

\[
\text{vol} (S; c, K) = 4\pi \int_{r_i}^{r_f} \frac{g_2^2 (r) \sqrt{g_0 g_1}}{\sqrt{g_0 g_2^2 + (c + K \int g_2 \sqrt{g_0 g_1} dr')^2}} dr. 
\]

(4.4)

The mean extrinsic curvature \( K \) is the property of \( S \) but the constant \( c \) is arbitrary. However the volume (4.4) makes sense if its value is unique and depends only on the geometry of the spacetime which is given by the metric (2.1) and the geometry of \( S \). If \( K = 0 \), then

\[
\sinh q(r) = \frac{c}{g_2 \sqrt{g_0}} 
\]

(4.5a)

the volume (4.4) is:

\[
\text{vol} (S; c, 0) = 4\pi \int_{r_i}^{r_f} \frac{g_2^2 (r) \sqrt{g_0 g_1}}{\sqrt{g_0 g_2^2 + c^2}} dr. 
\]

(4.5b)

and in this case it takes the form:

\[
\text{vol} (S; c, 0) = 4\pi \int_{r_1}^{r_2} \frac{r^4 dr}{\sqrt{r^4 - r^4 r_S + c^2}}. 
\]

(4.6)

Hence the volume of \( S \), becomes the function of \( c \). Introducing a dimensionless variable \( x = r/r_S \) the eq. (4.6) becomes:

\[
\text{vol} (S; C, 0) = 4\pi r_S^3 \int_{r_1/r_S}^{r_2/r_S} \frac{x^4 dx}{\sqrt{x^4 - x^4 + C^2}}, 
\]

(4.7)

where:

\[
C = c/r_S^2 \geq 0. 
\]

(4.8)

The expression (4.7) has been obtained in [13] for the interior of the event horizon and led to the conclusion that the volume of the black hole is infinite. In order to determine \( C \) here we use the condition that in the limit \( r_S \to 0 \) one reproduces the well-known result, volume in the flat space. It means that (4.7) becomes

\[
\text{vol}_0 (S; C, 0) = 4\pi \int_{r_1}^{r_2} \frac{r^4 dr}{\sqrt{r^4 + c^2}} 
\]

(4.9)

(the index "0" labels the flat spacetime case). As the volume of \( S \) in the flat space is:

\[
\text{vol}_0 (S) = \frac{4\pi}{3} (r_2^3 - r_1^3). 
\]

(4.10)
Hence we obtain:
\[ c = c_m = 0. \]  
(4.11)

Let us underline that this value of \( c \) also extremises the volume (4.7). Therefore, Eq.(4.7) takes the form:
\[ \text{vol} (S; C = 0) = 4 \pi r_S^3 \int_{r_1/r_S}^{r_2/r_S} x^2 \sqrt{\frac{x}{x-1}} dx. \]  
(4.12)

The above integral is elementary thus the volume of \( S \) is:
\[ \text{vol} (S; C = 0) = 4 \pi r_S^3 \left[ I (x_2) - I (x_1) \right], \]  
(4.13)

where \( x_{1,2} = r_{1,2}/r_S \) and:
\[ I (x) = \frac{5}{8} \ln \left( \sqrt{x} + \sqrt{x-1} \right) + \sqrt{x(x-1)} \left( \frac{1}{3} x^2 + \frac{5}{12} x + \frac{5}{8} \right), \quad x \geq 1. \]  
(4.14)

Expression (4.13) generalizes the meaning of the volume between the two spheres. Indeed, in the limit of the flat spacetime, \( r_S \to 0 \), (4.13) reduces to the flat space-time result (4.10).

One can notice that the function under the integral in the eq. (4.7) has the asymptotic expansion:
\[ \frac{x^4}{\sqrt{x^4 - x^3 + C^2}} \xrightarrow{x \to \infty} x^2 + \frac{x}{2} + \frac{3}{8}, \]
which does not depend on the constant \( C \). Hence the volume between the two spherical shells with the radii \( r_2 > r_1 >> r_S \) is:
\[ \text{vol} (S; c, 0) = \frac{4 \pi}{3} (r_2^3 - r_1^3) + \pi r_S (r_2^2 - r_1^2) + \frac{3 \pi r_S^2}{2} (r_2 - r_1). \]

2) Inside horizon
In the case of the interior of the Schwarzschild black hole, \( r < r_S \), radial \( r \) and temporal, \( t \) coordinates interchange their roles. Therefore one can apply the following notation, \( T = r \) and \( R = t \). Thus the metric is:
\[ ds^2 = \frac{T}{r_S - T} dT^2 - \left( \frac{r_S}{T} - 1 \right) dR^2 - T^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]
and has the components:
\[ g_0 (T) = g_1^{-1} (T) = \frac{1}{\frac{T}{r_S} - 1}, \quad g_2 (T) = T^2. \]  
(4.15)

Then the parameterizations (and the foliations) undertaken in sections 2 and 3 are left unchanged. We will consider the space-like hypersurface \( S \) parametrized as follows:
\[ S = \{ T = T (\xi) , \quad R = R (\xi) \mid \xi \in [\xi_1, \xi_2] \quad \text{and} \quad (\theta, \phi) \in S^2 \} \subset \mathcal{M}. \]  
(4.16)
The normal vector \( n \) is given by (3.17) Thus the volume (3.18) of \( S \) is extremized by the condition \( K = 0 \):

\[
\partial_T [g_2 \sqrt{g_1} \cosh q (T)] = 0. \tag{4.17}
\]

And one finds that the function \( q \) is given by the relation:

\[
\cosh q (T) = \frac{c}{g_2 \sqrt{g_1}}. \tag{4.18}
\]

In this case the condition (3.15) reads as follows:

\[
R' = T' \left[ g_0 \sqrt{g_1} \coth q (T) \right], \tag{4.19}
\]

and the equation (3.18) becomes:

\[
\text{vol} (S; \eta) = 4\pi \int_{T_1}^{T_2} g_2 (T) \sqrt{g_0 (T)} \sinh q (T) dT. \tag{4.20}
\]

Using (4.18) we arrive to the equation:

\[
\text{vol} (S; C, 0) = 4\pi \int_{T_1}^{T_2} g_2 \sqrt{g_0 g_1} dT = 4\pi \int_{T_1/r_S}^{T_2/r_S} \frac{x^4 dx}{\sqrt{x^4 - x^3 + C^2}}, \tag{4.21}
\]

where \( C = c/r_S^2 \geq 0 \). In this range, \( T < r_S \), the volume turns out to be extremized by different values of parameter \( c_m \) being dependent on the range \([T_1, T_2]\). One can notice that the quadric polynomial \( w(x; C) = x^4 - x^3 + C^2 \) has a minimum at \( x_m = 3/4 \) and its value is: \( w(3/4; C) \equiv w_m = C^2 - 3^3/2^8 \). Thus \( w(x; C) \) is greater or equal to zero for all \( x \geq 0 \) if \( C^2 \geq C_0^2 \equiv 3^3/2^8 \). It means that for \( C > C_0 \) the polynomial \( w \) has no real roots. In the case when:

\( 0 < C < C_0 \) there are two distinct positive roots \( x_1 \) and \( x_2 \), ordered as follows: \( x_1 > x_2 > x_1 \). It means that \( w > 0 \) for \( x \in [0, x_1) \cup (x_2, +\infty) \) and \( w < 0 \) for \( x \in (x_1, x_2) \). If \( C = 0 \), then \( w < 0 \) for \( x \in (0, 1) \) and \( w > 0 \) for \( x \in (1, +\infty) \). In the case when \( C = C_0 \) the polynomial \( w \) has the decomposition:

\[
w (x, C_0) = \left( x - \frac{3}{2^2} \right)^2 \left( x^2 + \frac{1}{2} x + \frac{3}{2^2} \right). \tag{4.22}
\]

Thus the volume of the black hole \( S_{BH} \) depends on \( C \):

\[
\text{vol} (S_{BH}; C, 0) = 4\pi r_S^3 \int_0^1 \frac{x^4 dx}{\sqrt{x^4 - x^3 + C^2}}, \tag{4.23}
\]

Hence for \( C^2 = 3^3/2^8 \) the above integral is divergent and the volume of the black hole is infinite (cf. [13]). Thus one can see that if \( C = C_0 \), then for \( T_2 = r_S \) and for \( T_1 > \frac{3}{4} r_S \) the volume is finite but grows to infinity as \( T_1 \) approaches \( \frac{3}{4} r_S \); then it becomes infinite for \( T_1 < \frac{3}{4} r_S \). However the regions between \( T_1 = 0 \) and \( T_2 < \frac{3}{4} r_S \) have the finite volumes.
4.2 The hypersurface cut by the horizon

The obtained result may appear confusing. Above we have applied a unique procedure leading to the determination of the volumes outside and inside horizon. The set of formulae in both cases turned out to be different (cf. Eqs. (4.2), (4.20)) but the final outcomes were the same. Indeed the volumes outside horizon, Eq. (4.7) and the volume inside horizon, Eq.(4.21) are expressed by the same integrals. One can ask then for the volume of a spherically symmetric hypersurface \( S \), \( r_1 < r < r_2 \), which is cut by the horizon. The volume of \( S \) is then given formally by:

\[
\text{vol}(S) = 4\pi r_S^3 \int_{r_1/r_S}^{r_2/r_S} \frac{x^4 dx}{\sqrt{x^4 - x^3 + C^2}}.
\]

Assuming the continuity of the space outside and inside horizon (we consider only the classical black hole) a choice \( c = 0 \) might be made. Such an approach leads however to the complex-value expression for the volume

\[
\text{vol}(S; c = 0) = 4\pi r_S^3 \left[ I \left( \frac{r_2/r_S}{1} \right) + i \left( \frac{5\pi}{16} - J \left( \frac{r_1/r_S}{1} \right) \right) \right],
\]

where the functions \( I \) (see 4.14) and \( J \) are:

\[
I(x) = \frac{5}{8} \ln \left( \sqrt{x} + \sqrt{x-1} \right) + \sqrt{x(x-1)} \left( \frac{1}{3} x^2 + \frac{5}{12} x + \frac{5}{8} \right), \text{ for } x \geq 1,
\]

\[
J(y) = \frac{5}{8} \arctan \sqrt{\frac{1}{1-y} - \sqrt{y(1-y)}} \left( \frac{1}{3} y^2 + \frac{5}{12} y + \frac{5}{8} \right), \text{ for } 0 \leq y \leq 1,
\]

so that \( I(1) = J(0) = 0 \) and \( J(1) = \frac{5\pi}{16} \). One can make a desperate step and to define the real volume of the \( S \) as the modulus of \( \text{vol}(S; c = 0) \):

\[
|\text{vol}(S; c = 0)| \equiv \text{Vol}(S; r_1, r_2) = 4\pi r_S^3 \left[ I^2 \left( \frac{r_2/r_S}{1} \right) + \left( \frac{5\pi}{16} - J \left( \frac{r_1/r_S}{1} \right) \right) \right]^{1/2}.
\]

Hence the volume of the black hole given by the relation \( r_1 = 0, r_2 = r_S = 2M \) is:

\[
\text{Vol}(S_{BH}) = \frac{5}{4}\pi^2 r_S^3 = 10\pi^2 M^3
\]

as opposed to the conclusion of [13] but this seems to be a very superficial attempt.
4.3 The volume in the anti-de Sitter and de Sitter

These considerations are easily extended to the case when $\Lambda \neq 0$ (anti-de Sitter or de Sitter). Thus (3.23) becomes:

$$\text{vol} (S; 0, K; \Lambda) = 12\pi \int_{r_i}^{r_f} r^2 \sqrt{\frac{r}{(K^2 - 9\Lambda)r^3 + 9r - 9r_S}} dr. \quad (4.30)$$

Here we also can take the constant $c = 0$. As one finds there is a hypersurface $S$ with a special value of $K$ in the de Sitter that the volume of $S$ becomes the same as in the Minkowski spacetime One obtains that if $K^2 = 9\Lambda$, then the hypersurface $S$ with $K^2 = 9\Lambda$ in the de Sitter has the same volume as the hypersurface $\tilde{S}$ with $\tilde{K} = 0$ in Minkowski spacetime. The integral (4.30) can be expressed by the elliptic integrals. However the final result is intricate and does not bring new insights to the considered problem.

5 Discussion

The main motivation of this paper was the following question: what is the volume of a spacelike spherically symmetric hypersurface $S$ cut by an event horizon of the Schwarzschild black hole. This problem is obviously related to the problem of the volume of a black hole recently discussed within variety of approaches. Our proposal is to introduce different foliations given by a two-parameter, $q, \chi$, velocity vector fields of a class of observers. Such foliations are accompanied by the corresponding volume forms, $\eta$ (2.8). On the other hand for a given spacelike hypersurface $S$ one can define an appropriate embedding $\Phi$ and related volume form determining volume of $S$ (3.10). Another derivation of the expression for the volume of $S$ is given by using pull back of $\eta$ by $\Phi$ (3.12). Demanding that $S$ lays on the simultaneity surface (3.14a) of the specific foliation one finds that these two volumes (3.10) and (3.12), are equal.

The first interesting issue arises then as one finds, $\chi = 0$. In fact this corresponds to the claim of the spherical symmetry of the foliation as the condition of vanishing $\chi$ represents the restoration of the spherical symmetry of the world line $\gamma$ and simplifies the volume form $\eta$.

One invokes then a maximum volume requirement: the volume of $S$ is maximal if the mean extrinsic curvature $K$ vanishes. This is manifested differently outside and inside horizon: geometry is spherically symmetric in both cases but an interior being homogeneous along one spatial direction is dynamically changing, whereas exterior is obviously static. In result one obtains conditions different outside (4.3) and inside (4.18) horizon, both expressed in terms of an unspecified constant $c$ (with dimension of length$^2$) parameterizing the volume of $S$. Then the second interesting observation may be made: although the condition $K = 0$ for the exterior and interior of the horizon is manifested differently, via (4.3) and (4.18), the final expressions for the volume are the same in these two apparently distinct regions and expressed in terms of variables $r$ and $T$ having apparently different, spatial- and temporal-like, respectively, meaning, cf. Eqs. (4.7) and (4.21).
There are two different ways to determine the parameter $c$ outside horizon: apart of the demand to extremize the volume, one can chose $c$ in such a way that in the flat spacetime limit, $r_S \to 0$, (Minkowski geometry) the well-known result is reproduced. Both these ways lead to the same result: $c_m = 0$. As underlined above inside horizon the expression for a volume of a spherically symmetric hypersurface $S$ is the same as the one obtained outside horizon, but the extremising procedure provides $c_m$ being dependent on the boundaries of $S$, $(\xi_1, \xi_2)$, corresponding to $(T_1, T_2)$ (see Sec. 4.1.2). So for the following boundaries:

- $T_1 < T_2 = 2M$

  $T_1$ decreases from $2M$ to $3/2M$, then $C_m(T_1)$ increases from 0 up to $C^2 = C_0^2 = 3^3/2^8$; then the volume grows indefinitely, i.e. it tends to infinity

- $T_1 < \frac{3}{2} M < T_2 = 2M$

  $T_1$ further decreasing, from $3/2M$, to $0$, then $C_m(T_1)$ becomes equal $C_0$ and the corresponding volume becomes infinite;

- $T_1 < T_2 = 3/2M$

  then $C_m$ becomes a function of $T_2$ and its value drops below $C_0$, then a finite volume in such a case is restored. (The dimensionless parameter $C$ has been defined as follows: $C = c/r_2^2$)

This leads to the conclusion concerning a volume of an ethernal black hole. The volume of an ethernal Schwarzschild black hole is given by the maximal value of the expression (4.21), attained for $C^2 = C_0^2 = 3^3/2^8 = 27/256$ and turns out to be infinite – hence both the expression (4.21) as well as the following conclusions coincide with the results derived by Christodoulou and Rovelli [13]. However, two important differences between above considerations and those of Ref. [13] should be emphasized. First, the discussion presented in the Ref. [13] in terms of Eddington-Filkenstein coordinates dealt with the case of a black hole being formed due to the gravitational collapse and it was shown that in fact the volume tends to infinity during this process. Second, the maximizing condition was differently defined in Ref. [13] from the one given here $K = 0$, but the final expressions for the volume became identical.

Let us finally answer our initial the question about the volume of the spherically symmetric hypersurface $S$, $r_1 < r_S < r_2$ cut by the horizon of the Schwarzschild black hole, $r = r_S = 2M$. One finds that such a region should be regarded as consisted of two distinct ranges: one, outside horizon, $(r_S, r_2)$ and another one, inside horizon $(r_1 = T_1, r_S = T_2)$. The volumes of these two ranges parameterized with $c$ are determined via extremizing procedure with $c_m = 0$ and $c_m(T_1)$ (see above) in the former and in the latter range, respectively. In this context the third interesting outcome should be pointed out. As presented in Sec.4.2 in such a case one can naively impose the flat spacetime condition $c_m = 0$ for the whole range $r_1 < r_S < r_2$ obtaining however complex-valued volume.
One should remember however that $c$ arises due to the requirement of vanishing mean curvature, but on the other hand it represents the velocity vector field, cf. Eqs. (4.3) and (2.2). The condition $c_m = 0$ defines outside horizon an observer who is in the rest (cf. Eq (2.2)); such an observer couldn’t exists any longer inside horizon, hence one can not impose such a demand for $r < r_S$. Similarly, appropriate demand $c_m(T_1)$ inside horizon (see above discussion) represents a condition for an allowed observer (foliation) defined by a unit (time-like!) velocity vector field.

6 Appendix

6.1 Transformation of the volume form

The coordinates transformations which preserves the spherical symmetry have the form:

$$y^\alpha = y^\alpha (x^A),$$  \hspace{1cm} (a1)

where $\alpha = 0, 1$ and $A = 0, 1$. Under the above transformations the tetrads $E^{(a)}$ and $e^{(0)}$ are:

$$E^{(a)} = E_A^{(a)} \frac{\partial x^A}{\partial y^\alpha} dy^\alpha + E_i^{(a)} dx^i,$$

$$e^{(0)} = e_A^{(0)} \frac{\partial}{\partial x^A} + e_i^{(0)} \frac{\partial}{\partial x^i},$$

where $x^i = \theta, \phi$. Hence the volume form $\eta$ transforms as follows:

$$\tilde{\eta} = -g_2 \sqrt{g_0 g_1} \left( \frac{\cosh q}{\sqrt{g_0}} \frac{\partial x^1}{\partial y^\alpha} - \frac{\sinh q \cos \chi}{\sqrt{g_1}} \frac{\partial x^0}{\partial y^\alpha} \right) \wedge (\cos \theta) \wedge d\phi$$

$$- \sqrt{g_0 g_1} g_2 \sinh q \sin \chi \det \left( \frac{\partial x^A}{\partial y^\alpha} \right) dy^0 \wedge dy^1 \wedge d\theta, \hspace{1cm} (a2)$$

where the metric coefficients $g_{\mu}$ are functions of the new coordinates $y^\alpha$. As one can see the three-form $\eta$ is form-invariant under the transformations given by (a1).

Here we show that volume is invariant under the coordinate transformation. Let in the coordinates $(t, r, \theta, \phi)$ the hypersurface $S$ be given by the equation:

$$S = \{(t, r, \theta, \phi) \mid (t(r), r, \theta, \phi) \text{ and } r \in [r_1, r_2]\},$$

where $t(r)$ is given function. Then under the transformation:

$$T(t, r) = t + f(r),$$

$$R = r, \hspace{1cm} (a3)$$

$S$ transforms into $S_*$ which is given by :

$$S_* = \{(T, r, \theta, \phi) \mid (t(r) + f(r), r, \theta, \phi) \text{ and } r \in [r_1, r_2]\}.$$
Hence the pull-back of \((a2)\) by \(\tilde{\Phi}\) is invariant:

\[
\tilde{\Phi}^* \eta = \Phi^* \eta, \tag{a4}
\]

where \(\tilde{\Phi}(r, \theta, \phi) = (t(r) + f(r), r, \theta, \phi)\). Thus the volume of the hypersurface is independent on the coordinates used in the computations. It is also obvious that the condition \((3.14a)\) is invariant under \((a3)\). In this way the results obtained in the section 3 remain valid in any coordinate systems related by \((a3)\).

### 6.2 Hypersurfaces bounded by the trapped surfaces

In the section 3 we considered hypersurface \(S\) with the boundary \(\partial S\) which is sum of the two spheres and the volume of \(S\) is given by \((3.22)\) for foliation with the constant mean extrinsic curvature. Here we express a volume of a space-like hypersurface \(S\) by the property of the boundary. It is well know that the volume is given by:

\[
\text{vol}(S) = \int_{\partial S} \sigma,
\]

where \(\sigma\) is a two-form, such that: \(d\sigma = \Phi^* (i_n \varepsilon) = \Phi^* (\eta)\). Thus:

\[
\text{vol}(S) = \int_{M_1} \sigma - \int_{M_2} \sigma,
\]

where \(M_{1,2}\) are the space-like two-surfaces \(\partial S = M_1 \cup M_2\) and:

\[
\sigma = F(r) ds_1 \wedge ds_2,
\]

\(s_1, s_2\) parametrize the boundary \(\partial S\). The function \(F(r)\) is constant on \(\partial S\). In the special case of the spherical symmetry there are relations:

\[
\frac{dF(r)}{dr} = g_2(r) \sqrt{g_1(r)} \cosh q(r)
\]

and \(ds_1 \wedge ds_2 = -d(\cos \theta) \wedge d\phi\)

which leads to \((3.18)\).

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