ON THE SUM FORMULA FOR MULTIPLE $q$-ZETA VALUES

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Abstract. Multiple $q$-zeta values are a 1-parameter generalization (in fact, a $q$-analog) of the multiple harmonic sums commonly referred to as multiple zeta values. These latter are obtained from the multiple $q$-zeta values in the limit as $q \to 1$. Here, we discuss the sum formula for multiple $q$-zeta values, and provide a self-contained proof. As a consequence, we also derive a $q$-analog of Euler’s evaluation of the double zeta function $\zeta(m,1)$.

1. INTRODUCTION

Sums of the form

$$\zeta(n_1,n_2,\ldots,n_r) := \sum_{k_1 > k_2 > \cdots > k_r > 0} \prod_{j=1}^{r} \frac{1}{k_j^{n_j}}$$

have attracted increasing attention in recent years; see eg. [1, 2, 3, 4, 6, 7, 8, 10, 11, 20]. The survey articles [5, 12, 26, 27, 29] provide an extensive list of references. Here and throughout, $n_1, \ldots, n_r$ are positive integers with $n_1 > 1$, and we sum over all positive integers $k_1, \ldots, k_r$ satisfying the indicated inequalities. Note that with positive integer arguments, $n_1 > 1$ is necessary and sufficient for convergence. The sums (1) are sometimes referred to as Euler sums, because they were first studied by Euler [13] in the case $r = 2$. In general, they may be profitably viewed as instances of the multiple polylogarithm [2, 5, 14, 15], and are now more commonly referred to as multiple zeta values, reducing to the Riemann zeta function in the case $r = 1$. A $q$-analog of (1) was independently introduced in [9, 25, 28] as

$$\zeta[n_1,n_2,\ldots,n_r] := \sum_{k_1 > k_2 > \cdots > k_r > 0} \prod_{j=1}^{r} \frac{q^{(n_j-1)k_j}}{|k_j|^{n_j}},$$

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where

\[ [k]_q := \sum_{j=0}^{k-1} q^j = \frac{1 - q^k}{1 - q}, \quad 0 < q < 1. \]

Observe that we now have

\[ \zeta(n_1, \ldots, n_r) = \lim_{q \to 1} \zeta[n_1, \ldots, n_r], \]

so that (2) represents a generalization of (1). In this note, we prove an identity for (2), the \( q = 1 \) case of which was originally conjectured by Moen [17] and Markett [21].

It is convenient to state results in terms of the shifted multiple zeta functions defined by

\[ \zeta^*[n_1, \ldots, n_r] := \zeta[1 + n_1, n_2, \ldots, n_r] = \sum_{k_1 > \cdots > k_r > 0} \frac{q^{k_1}}{[k_1]_q} \prod_{j=1}^{r} q^{(n_j - 1)k_j} \]

and correspondingly,

\[ \zeta^*(n_1, \ldots, n_r) := \zeta(1 + n_1, n_2, \ldots, n_r) = \lim_{q \to 1} \zeta^*[n_1, \ldots, n_r]. \]

The main focus of our discussion is the following result.

**Theorem 1** (\( q \)-sum formula). If \( N \) and \( r \) are positive integers with \( N \geq r \), then

\[ \sum_{\substack{n_1 + \cdots + n_r = N \\forall j, n_j \geq 1}} \zeta^*[n_1, n_2, \ldots, n_r] = \zeta^*[N], \]

where the sum is over all positive integers \( n_1, n_2, \ldots, n_r \) such that \( \sum_{j=1}^{r} n_j = N \).

The limiting case \( q = 1 \) is of course the now familiar

**Corollary 1** (sum formula). If \( N \) and \( r \) are positive integers with \( N \geq r \), then

\[ \sum_{\substack{n_1 + \cdots + n_r = N \\forall j, n_j \geq 1}} \zeta^*(n_1, n_2, \ldots, n_r) = \zeta^*(N), \]

where the sum is over all positive integers \( n_1, n_2, \ldots, n_r \) such that \( \sum_{j=1}^{r} n_j = N \).

Corollary 1 was proved for \( r = 2 \) by Euler, for \( r = 3 \) by Hoffman and Moen [18], and in full generality by Granville [16]. Then Ohno derived Corollary 1 as a consequence of his generalized duality relation [23], and later as a consequence of an auxiliary result used in his proof of the cyclic sum formula [19]. Corollary 1 is also derived in [24] by specializing the height relation given there. Subsequently and independently [9, 25], \( q \)-analogos of all these results were discovered and proved. For example, Theorem 1 is derived in [9] as a consequence of generalized \( q \)-duality [9, 25] (a \( q \)-analogue of the main result in [23],...
but proved using an entirely different technique). Likewise, a $q$-analog of the cyclic sum formula \[25\] also leads to a quick proof \[9\] of Theorem \[1\]. Finally, in \[25\], a $q$-analog of the height relation is also given; we show below how this too can be used to derive Theorem \[1\]. However, as all these proofs of Theorem \[1\] depend on comparatively more sophisticated results for \[2\], we feel it may be of interest to give a self-contained proof, more in the spirit of \[16\].

2. Self-Contained Proof of Theorem \[1\]

By expanding both sides in powers of $z$ and comparing coefficients, one readily sees that Theorem \[1\] is equivalent to the following result.

**Theorem 2.** If $r$ is a positive integer and $z \in \mathbb{C} \setminus \{q^{-m}[m]_q : m \in \mathbb{Z}^+\}$, then

$$
\sum_{k_1 > \cdots > k_r > 0} q^{k_1} \prod_{j=1}^{r} \frac{1}{[k_j]_q - z q^{k_j}} = \sum_{m=1}^{\infty} \frac{q^{m}}{[m]_q ([m]_q - z q^m)}.
$$

(3)

**Proof of Theorem 2.** Let $L_r = L_r(z)$ denote the left hand side of (3). By partial fractions,

$$
L_r = \sum_{j=1}^{r} S_j
$$

(4)

where

$$
S_j = S_{j,r}(z) := \sum_{k_1 > \cdots > k_r > 0} q^{k_1} \prod_{i=1}^{r} \frac{1}{[k_i - k_j]_q} \frac{q^{k_i}}{[k_j]_q ([k_j]_q - z q^{k_j})}.
$$

Now rename $k_j = m$ and sum first on $m$, so that

$$
S_j = \sum_{m=1}^{\infty} A(m, j - 1) B(m, r - j)
$$

(5)

where $A(m, 0) := q^m/[m]_q$,

$$
A(m, j - 1) := \sum_{k_1 > \cdots > k_{j-1} > m} q^{k_1} \prod_{i=1}^{j-1} \frac{1}{[k_i - m]_q}
$$

for $2 \leq j \leq r$,

$B(m, 0) := 1$ and for $1 \leq j \leq r - 1$,

$$
B(m, r - j) := \sum_{m > k_j + 1 > \cdots > k_r > 0} \prod_{i=j+1}^{r} \frac{1}{[k_i - m]_q} = (-1)^{r-j} \sum_{m > k_j + 1 > \cdots > k_r > 0} \prod_{i=j+1}^{r} \frac{q^{m-k_i}}{[m - k_i]_q}.
$$
From (4) and (5) we now get that
\[
L_r = \sum_{j=0}^{r-1} S_{j+1} = \sum_{m=1}^{\infty} \frac{1}{[m]_q - zq^m} \sum_{j=0}^{r-1} A(m, j) B(m, r - 1 - j),
\]
and hence
\[
\sum_{r=1}^{\infty} x^{r-1} L_r = \sum_{m=1}^{\infty} \frac{A_m(x) B_m(x)}{[m]_q - zq^m},
\]
where the generating functions \(A_m\) and \(B_m\) are defined by
\[
A_m(x) := \sum_{n=0}^{\infty} x^n A(m, n), \quad B_m(x) := \sum_{n=0}^{\infty} x^n B(m, n).
\]
The proof of Theorem 2 now follows more or less immediately from the representations
\[
A_m(x) = q^m \prod_{c=1}^{m} \left(1 - \frac{xq^c}{[c]_q}\right)^{-1} \quad \text{and} \quad B_m(x) = \prod_{b=1}^{m-1} \left(1 - \frac{xq^b}{[b]_q}\right).
\]
To see this, observe that (7) gives
\[
A_m(x) B_m(x) = \sum_{n=0}^{\infty} x^n A(m, n) = \sum_{n=0}^{m-1} x^{r-1} q^{r_m} [m]_q,
\]
and hence from (6),
\[
\sum_{r=1}^{\infty} x^{r-1} L_r = \sum_{r=1}^{\infty} x^{r-1} \sum_{m=1}^{\infty} \frac{q^{r_m} [m]_q}{([m]_q - zq^m)}.
\]
It now remains only to prove the representations (7). First, note that
\[
B_m(x) = \sum_{n=0}^{\infty} (-1)^n m^{k_1 \ldots k_n} \prod_{j=1}^{n} \frac{q^{m-k_j}}{[m-k_j]_q} = \sum_{n=0}^{\infty} (-1)^n \sum_{m>b_n \ldots >b_1>0} \prod_{j=1}^{n} \frac{q^{b_j}}{[b_j]_q}.
\]
Next, we define
\[
A(m, n, k) := \sum_{b_1 > \ldots > b_n > k} \frac{q^{m+b_1}}{[m+b_1]_q} \prod_{j=1}^{n} \frac{1}{[b_j]_q},
\]
and note that \(A(m, n) = A(m, n, 0)\). We have
\[
A(m, 1, k) = \sum_{b<k} \frac{q^{m+b}}{[m+b]_q [b]_q} = \frac{q^m}{[m]_q} \sum_{b<k} \left(\frac{q^b}{[b]_q} - \frac{q^{m+b}}{[m+b]_q}\right) = \frac{q^m}{[m]_q} \sum_{c \geq 1} \frac{q^{c+k}}{[c+k]_q},
\]
where
and if for some positive integer $n$,

$$A(m, n, k) = \frac{q^m}{[m]_q} \sum_{m \geq c_1 \geq \cdots \geq c_n \geq 1} \frac{q^{c_n + k}}{[c_n + k]_q} \prod_{j=1}^{n-1} \frac{q^{c_j}}{[c_j]_q},$$

then

$$A(m, n + 1, k) = \sum_{b_1 > \cdots > b_{n+1} > k} \frac{q^{m+b_1}}{[m+b_1]_q} \prod_{j=1}^{n+1} \frac{1}{[b_j]_q}$$

$$= \sum_{b_2 > \cdots > b_{n+1} > k} \left( \prod_{j=2}^{n+1} \frac{1}{[b_j]_q} \right) \sum_{b_1 > b_2} \frac{q^{m+b_1}}{[m+b_1]_q[b_1]_q}$$

$$= \sum_{b_2 > \cdots > b_{n+1} > k} \left( \prod_{j=2}^{n+1} \frac{1}{[b_j]_q} \right) A(m, 1, b_2)$$

$$= \sum_{b_2 > \cdots > b_{n+1} > k} \left( \prod_{j=2}^{n+1} \frac{1}{[b_j]_q} \right) \frac{q^m}{[m]_q} \sum_{c_0 = 1}^{m} \frac{q^{c_0 + b_2}}{[c_0 + b_2]_q}$$

$$= \frac{q^m}{[m]_q} \sum_{c_0 = 1}^{m} \frac{q^{c_0 + b_2}}{[c_0 + b_2]_q} \prod_{j=2}^{n+1} \frac{1}{[b_j]_q}$$

$$= \frac{q^m}{[m]_q} \sum_{c_0 = 1}^{m} A(c_0, n, k)$$

$$= \frac{q^m}{[m]_q} \sum_{m \geq c_0 \geq \cdots \geq c_n \geq 1} \frac{q^{c_n + k}}{[c_n + k]_q} \prod_{j=1}^{n-1} \frac{q^{c_j}}{[c_j]_q},$$

by the induction hypothesis. It follows that

$$A(m, n) = A(m, n, 0) = \frac{q^m}{[m]_q} \sum_{m \geq c_1 \geq \cdots \geq c_n \geq 1} \prod_{j=1}^{n} \frac{q^{c_j}}{[c_j]_q},$$

and hence

$$A_m(x) = \frac{q^m}{[m]_q} \prod_{c=1}^{m} \left( 1 + xq^c + \left( xq^c \right)^2 + \left( xq^c \right)^3 + \cdots \right) = \frac{q^m}{[m]_q} \prod_{c=1}^{m} \left( 1 - xq^c \right)^{-1}. \quad \square$$
3. Evaluation of $\zeta[m, 1]$

Euler [13, 22] (see also [11, eq. (31)]) proved that for all integers $m \geq 2$,

$$2\zeta(m, 1) = m\zeta(m + 1) - \sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1),$$

thereby expressing $\zeta(m, 1)$ in terms of values of the Riemann zeta function. The following $q$-analog of Euler’s formula is an easy consequence of the $r = 2$ case of Theorem 1 and the $q$-stuffle multiplication rule [9].

**Corollary 2** (Corollary 8 of [9]). Let $2 \leq m \in \mathbb{Z}$. Then

$$2\zeta[m, 1] = m\zeta[m + 1] + (1 - q)(m - 2)\zeta[m] - \sum_{k=1}^{m-2} \zeta[m-k] \zeta[k+1].$$

In particular, when $m = 2$ we get $\zeta[2, 1] = \zeta[3]$, which is probably the simplest non-trivial identity satisfied by the multiple $q$-zeta function.

**Proof.** For $1 \leq k \leq m - 2$ the $q$-stuffle multiplication rule [9] implies that

$$\zeta[m-k]\zeta[k+1] = \zeta[m+1] + (1 - q)\zeta[m] + \zeta[m-k, k+1] + \zeta[k+1, m-k].$$

Summing on $k$, we find that

$$\sum_{k=1}^{m-2} \zeta[m-k] \zeta[k+1] = (m - 2) (\zeta[m+1] + (1 - q)\zeta[m]) + 2 \sum_{s+t=m+1, s,t \geq 2} \zeta[s,t].$$

But Theorem 1 gives

$$\sum_{s+t=m+1, s,t \geq 2} \zeta[s,t] = \sum_{s+t=m+1, s \geq 2, t \geq 1} \zeta[s,t] - \zeta[m, 1] = \zeta[m + 1] - \zeta[m, 1].$$

It follows that

$$\sum_{k=1}^{m-2} \zeta[m-k] \zeta[k+1] = m\zeta[m+1] + (1 - q)(m - 2)\zeta[m] - 2\zeta[m, 1].$$

□
4. Height Relation

Corollary 2 is also derived in [9] as a consequence of the more general double generating function identity

\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^{m+1} y^{n+1} \zeta[m+2, \{1\}^n] = 1 - \exp \left\{ \sum_{k=2}^{\infty} \left\{ x^k + y^k - (x + y + (1 - q)xy)^k \right\} \frac{1}{k} \sum_{j=2}^{k} (q - 1)^{k-j} \zeta[j] \right\}, \]  

which implies, among other things, that \( \zeta[m+2, \{1\}^n] = \zeta[n + 2, \{1\}^m] \) can be expressed in terms of sums of products of single q-zeta values for every pair of non-negative integers \( m \) and \( n \). In fact (8) is just the constant term of an even more general result.

For any multi-index \( \vec{n} = (n_1, \ldots, n_r) \) of positive integers, the weight, depth, and height of \( \vec{n} \) are the integers \( n = n_1 + n_2 + \cdots + n_r, r, \) and \( s = \#\{ j : n_j > 1 \} \), respectively. Denote the set of multi-indices of weight \( n \), depth \( r \) and height \( s \) with the additional requirement \( n_1 > 1 \) by \( I_0(n, r, s) \), and set

\[ G_0[n, r, s] := \sum_{\vec{n} \in I_0(n, r, s)} \zeta[\vec{n}], \quad \Phi_0[x, y, z] := \sum_{n, r, s=0}^{\infty} G_0[n, r, s] x^{n-r-s} y^{r-s} z^{s-1}. \]

Okuda and Takeyama [25] proved that

\[ 1 + (z - xy) \Phi_0[x, y, z] = \prod_{n=1}^{\infty} \left( \frac{[n]_q - \alpha q^n}{[n]_q - xq^n} \right) \left( \frac{[n]_q - \beta q^n}{[n]_q - yq^n} \right) = \exp \left\{ \sum_{k=2}^{\infty} \left( x^k + y^k - \alpha^k - \beta^k \right) \frac{1}{k} \sum_{j=2}^{k} (q - 1)^{k-j} \zeta[j] \right\}, \]

where \( \alpha \) and \( \beta \) are determined by

\[ \alpha + \beta = x + y + (q - 1)(z - xy), \quad \alpha \beta = z. \]

The limiting case \( q \to 1 \) reduces to the height relation of [24]. The case \( z = 0 \) gives (8).

As with the \( q = 1 \) case [24], taking the limit as \( z \to xy \) gives

\[ \Phi_0[x, y, xy] = \sum_{m=1}^{\infty} \frac{q^m}{([m]_q - xq^m)([m]_q - yq^m)} = \sum_{n > r > 0} \zeta[n] x^{n-r-1} y^{r-1}. \]

On the other hand, by definition,

\[ \Phi_0[x, y, xy] = \sum_{n > r > 0} G_0[n, r] x^{n-r-1} y^{r-1}, \]
where $G_0[n, r]$ is the sum of all multiple $q$-zeta values of weight $n$ and depth $r$. Thus, we obtain $G_0[n, r] = \zeta[n]$ i.e. Theorem again. □

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