Improved Field Theoretical Approach to Noninteracting Brownian Particles in a Quenched Random Potential

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(Received 16 July 2020; revised 8 August 2020; accepted 10 August 2020)

We construct a dynamical field theory for noninteracting Brownian particles in the presence of a quenched Gaussian random potential. The main variable for the field theory is the density fluctuation, which measures the difference between the local density and its average value. The average density is spatially inhomogeneous for a given realization of the random potential. It becomes uniform only after being averaged over the disorder configurations. We develop a diagrammatic perturbation theory for the density correlation function and calculate the zero-frequency component of the response function exactly by summing all the diagrams contributing to it. From this exact result and the fluctuation dissipation relation, which holds in equilibrium dynamics, we find that the connected density correlation function always decays to zero in the long-time limit for all values of disorder strength implying that the system always remains ergodic. This nonperturbative calculation relies on the simple diagrammatic structure of the present field theoretical scheme. We compare in detail our diagrammatic perturbation theory with the one used in a recent paper [B. Kim, M. Fuchs and V. Krakoviack, J. Stat. Mech. 2020, 023301 (2020)], which uses the density fluctuation around the uniform average, and discuss the difference in the diagrammatic structures of the two formulations.

Keywords: Diffusion in random media, Brownian motion, Ergodicity breaking
DOI: 10.3938/jkps.77.719

I. INTRODUCTION

The dynamics of fluids in a quenched random environment has been studied in connection with many different research areas ranging from the structural glass transition [1–5] to biological [6,7] and engineering [8–11] applications. Theoretically, the main focus has been on the possible existence of an anomalous diffusion [12–22], which has been studied in connection with the spatial dimension and the range of the random potential and the thermal noise [23–25]. One of the main physical quantities is the late-time diffusion constant of a tagged particle. The calculation of effective transport properties in the presence of disorder has also been extensively studied [26–32]. All these studies are, however, based on the single-particle picture. An alternative way is to use the field theoretical approaches [33–44], which are based on the Martin-Siggia-Rose-Janssen-de Dominicis (MSRJD)-type dynamical field theory [45–47] for the stochastic equations governing collective variables such as the density field. Using the field theoretical formalism has a number of advantages. One can, for example, extract physical information from the symmetry property of the action functional. For instance, the fluctuation dissipation relation (FDR) of equilibrium dynamics is obtained from the invariance of the action functional under time-reversal transformations of fields [38,48]. Another point is the availability of systematic techniques such as the loop expansion and the diagrammatic resummation methods in the field theoretical setting. In fact, a series of attempts [38–43] have been made to obtain the mode coupling theory (MCT) [49] of the structural glass transition as a first-order self-consistent renormalized theory from the field theoretical formulation of dense liquids or colloids.

Nontrivial technical issues arise in the field theoretical formalism. For colloidal fluids described by Brownian particles obeying overdamped Langevin equations for the particle positions, an alternative stochastic equation for the microscopic density known as the Dean-Kawasaki (DK) equation [50,51] can be obtained. The field theoret-
ical formulation of the DK equation even in the simplest case of noninteracting Brownian particles turns out to be nontrivial [44]. Even in the absence of interactions among the particles, the field theory contains an interaction originating from the multiplicative noise in the DK equation. The multiplicative noise is also responsible for the nontrivial form of the response function [37,38] that appears in the FDR. Recently, the effect of quenched disorder on the field theoretical formulation of noninteracting Brownian particles was studied [52]. The quenched disorder produces another form of interaction in addition to the one due to the multiplicative noise. Various schemes of perturbation and self-consistent renormalization theories consistent with the FDR were developed. For some renormalized perturbation schemes, the system is shown to become nonergodic for strong disorder in the sense that the connected correlation function does not decay to zero in the long-time limit while still exhibiting normal diffusion [52].

In this paper, we present an improved field theoretical formulation compared to that in Ref. 52 for the noninteracting Brownian particles in a quenched random potential with a Gaussian correlation. The main variable in the field theory is the density fluctuation, which measures the difference of the local density from its average value, which is spatially inhomogeneous depending on a given realization of the random potential. Only after averaging over the disorder does a uniform average density emerge. We present the field theoretical formulation in terms of the density fluctuation around its inhomogeneous average value. In Ref. 52, the field theory for the same problem was formulated in terms of the density fluctuation with respect to the uniform average value. We find that although the two versions of the field theory differ only by the definition of the main variable and eventually give the same physical quantities, the detailed diagrammatic perturbation theories are quite different. We show that within the formalism of Ref. 52, the perturbation expansion in terms of the disorder strength produces a multitude of unnecessary diagrams that either cancel among themselves or vanish. In our version of the field theory, however, the diagrammatic expansion has a much simpler structure. As an example of the simplicity, we show that we can evaluate the zero-frequency limit or the time integral of the response function exactly by summing all the diagrams contributing to it. This quantity gives, via the FDR, the value of the density-density correlation function in the long-time limit. From the exact calculation, we find that the system remains ergodic for all values of the disorder strength in the sense that the connected density-density correlation function always decays to zero in the long time limit.

This paper is organized as follows. In Sec. II, we present the MSRJD field theory for the noninteracting Brownian particles in the presence of a quenched random potential. In the next section, we compare the diagrammatic perturbation theory of our formulation with that used in Ref. 52. In Sec. IV, we calculate the zero-frequency limit of the response function and discuss its implication on the ergodicity of the system. We conclude in the following section with a discussion.

II. MSRJD FIELD THEORY FOR NONINTERACTING BROWNIAN PARTICLES IN A QUENCHED RANDOM POTENTIAL

We consider $N$ noninteracting Brownian particles moving in $d$ dimensions under the influence of an external random potential $\Phi$. The position $X_i(t)$ of the $i$-th particle at time $t$ is described by the overdamped Langevin equations

$$\frac{dX_i(t)}{dt} = -\Gamma \nabla \Phi(X_i(t)) + \eta_i(t),$$

where $\eta_i(t)$ is the Gaussian white noise having zero mean and variance

$$\langle \eta_i^\mu(t) \eta_j^\nu(t') \rangle = 2D \delta_{ij} \delta^\mu\nu \delta(t - t').$$

for $i, j = 1, \ldots, N$ and $\mu, \nu = 1, \ldots, d$ and $D = \Gamma T$ with the temperature $T$ ($k_B = 1$). The quenched random potential $\Phi(x)$ is selected from the Gaussian distribution with zero mean and the variance

$$\Phi(x_1)\Phi(x_2) = \Delta(x_1 - x_2),$$

where $\Delta$ is a short-ranged function of the distance between the two points. Here, the overline indicates the average over the quenched random potential.

In this paper, we study this system by using the microscopic density field

$$\rho(x, t) = \sum_{i=1}^{N} \delta^{(d)}(x - X_i(t))$$

as a main dynamical variable. The above equation can then be rewritten as a stochastic equation with multiplicative noise [50,51] as

$$\frac{\partial \rho(x, t)}{\partial t} = \Gamma \nabla \cdot \{ \rho(x, t) \nabla \Phi(x) \} + D \nabla^2 \rho(x, t) - \nabla \cdot \{ \xi(x, t) \sqrt{\rho(x, t)} \},$$

where

$$\langle \xi^\mu(x, t) \xi^\nu(x', t') \rangle = 2D \delta^\mu\nu \delta(t - t') \delta^{(d)}(x - x').$$

We note that in the above derivation, Ito’s discretization convention was used.

Many physical quantities such as the intermediate scattering functions are obtained from the correlation functions of the density fluctuation. For a given realization of the external random potential, we consider
In the presence of an external potential, \( \rho_b(x) \) will be inhomogeneous and can be determined from the stationarity condition of Eq. (5) as

\[
\nabla \cdot \{ \rho_b(x) \nabla \Phi(x) \} + D \nabla^2 \rho_b(x) = 0
\]

(8)

with the solution

\[
\rho_b(x) = N \frac{e^{-\Phi(x)/T}}{\int d^d \rho \ e^{-\Phi(x)/T}}.
\]

(9)

We can show that the mean and the variance of the denominator on the right-hand side of Eq. (9) over the Gaussian distribution of the random potential are both proportional to the volume (see Appendix A). Therefore, in the thermodynamic limit, we can replace the denominator average value \( \int d^d \rho e^{-\Phi(x)/T} = V e^{\Delta(0)/(2T^2)} \) with the volume \( V \). This is consistent with the fact that when averaged over the disorder realizations, we have \( \rho_b(x) = \rho_0 = N/V \). For future use, we rewrite Eq. (9) as

\[
\rho_b(x) = \rho_0^* e^{-\Phi(x)/T},
\]

(10)

where \( \rho_0^* = \rho_0 e^{-\Delta(0)/(2T^2)} \). We shall use \( \delta \rho \) defined in Eq. (7) as the main variable for the MSRJD field theoretical development below. An alternative way is to use the density fluctuation around its uniform average as

\[
\Delta \rho(x, t) \equiv \rho(x, t) - \rho_0.
\]

(11)

The field theoretical formulation using \( \Delta \rho \) will be discussed in the next section in detail.

We write \( \rho = \rho_b + \delta \rho \) in Eq. (5) and transform it into a field theoretical setting by using the MSRJD formalism [45–47], for which the generating functional is written as path integrals over the density fluctuation \( \delta \rho(x, t) \) and the auxiliary response field \( \dot{\rho}(x, t) \). For a given external potential \( \Phi \), the average of an observable \( O(\delta \rho, \dot{\rho}) \) is then given by the functional integral

\[
\langle O(\delta \rho, \dot{\rho}) \rangle = \int D \delta \rho \int D \dot{\rho} O(\delta \rho, \dot{\rho}) e^{S_b[\delta \rho, \dot{\rho}]},
\]

(12)

where

\[
S_b[\delta \rho, \dot{\rho}] = \int d^d x \int dt \left[ -i \dot{\rho}(x, t) \left( \frac{\partial \delta \rho(x, t)}{\partial t} - D \nabla^2 \delta \rho(x, t) - \Gamma \nabla \cdot \{ \delta \rho(x, t) \nabla \Phi(x) \} \right) 
\]

\[
+ D \rho_b(x) \{ \nabla i \dot{\rho}(x, t) \}^2 + D \delta \rho(x, t) \{ \nabla i \dot{\rho}(x, t) \}^2 \right].
\]

(13)

The average over the disorder realization can be obtained by integrating over the distribution

\[
\mathcal{P}[\Phi] = \frac{1}{N} \exp \left[ -\frac{1}{2} \int d^d x \int d^d x' \Phi(x) \Delta^{-1}(x - x') \Phi(x') \right]
\]

(14)

for the external potential, where \( N \) is the normalization constant and \( \Delta^{-1} \) is the matrix inverse of \( \Delta \). Therefore, we have

\[
\langle O(\delta \rho, \dot{\rho}) \rangle = \int D \delta \rho \int D \dot{\rho} \int [D\Phi] O(\delta \rho, \dot{\rho}) e^{S[\delta \rho, \dot{\rho}, \Phi]},
\]

(15)

where the functional integral \([D\Phi]\) contains the normalization factor \(1/N\) and the effective action \( S \) is given by

\[
S[\delta \rho, \dot{\rho}, \Phi] = S_b[\delta \rho, \dot{\rho}] - \frac{1}{2} \int d^d x \int d^d x' \Phi(x) \Delta^{-1}(x - x') \Phi(x').
\]

(16)

If we expand Eq. (10) in powers of \( \Phi \), we can separate the effective action into Gaussian and non-Gaussian parts as \( S = S_G + S_v \), where

\[
S_G = \int d^d x \int dt \left[ -i \dot{\rho}(x, t) \left( \frac{\partial \delta \rho(x, t)}{\partial t} - D \nabla^2 \delta \rho(x, t) \right) \right] \int d^d x \int d^d x' \Phi(x) \Delta^{-1}(x - x') \Phi(x'),
\]

and

\[
S_v = \int d^d x \int dt \left[ D \delta \rho(x, t) (\nabla i \dot{\rho}(x, t))^2 + \Gamma (i \dot{\rho}(x, t)) \nabla \cdot \{ \delta \rho(x, t) \nabla \Phi(x) \} + D \rho_b(x) (\nabla i \dot{\rho}(x, t))^2 \sum_{n=1}^{\infty} \frac{\Phi^n(x)}{n!(-T)^n} \right].
\]

(17)

(18)
In the absence of the random potential ($\Phi = 0$ and $\Delta = 0$), we recover the action studied in Ref. 44 for a Brownian gas. The first term in Eq. (18), which we refer to as the noise vertex, comes from multiplicative noise and its effect has been studied in a field theoretical setting in the absence of disorder in Ref. 44. The second and third terms in Eq. (18) are the vertices arising from the quenched random potential. The second term is linear in $\Phi$, but the third term is an infinite series of terms that originate from the expansion of $\rho \Phi$. We denote these as type-$A$ and type-$B_n$ vertices ($n = 1, 2, \cdots$), respectively. Below we study the effects of these vertices on the perturbation theory in detail.

The central quantities in this paper are the two-point propagators $G_{\alpha\beta}(x, x', t - t') = \langle \psi_\alpha(x, t) \psi_\beta(x', t') \rangle_S$, (19) where $\psi_\alpha$ and $\psi_\beta$ represent $\hat{\rho}$ or $\delta \rho$ and the average $\langle \cdots \rangle_S$ is with respect to the effective action $S$ in Eqs. (17) and (18). (In the subscript, we use $\rho$ instead of $\delta \rho$ for brevity.) Their Fourier transforms are given by

$$G_{\alpha\beta}(q, t) = \int d^4x e^{-iqx} G_{\alpha\beta}(x, t).$$

We first note that due to causality, $G_{\hat{\rho} \hat{\rho}} = 0$.

The bare propagators are obtained from the Gaussian action in Eq. (17). These will be denoted by $G_{\psi_\alpha \psi_\beta}^0$ and by the lines shown in Fig. 1. The double lines indicate $\delta \rho$ and the single lines are $i \hat{\rho}$. The bare propagators are given by

$$i \tilde{G}_{\hat{\rho} \hat{\rho}}^0(q, t) = \Theta(t)e^{-Dq^2|t|},$$
$$i G_{\hat{\rho} \hat{\rho}}^0(q, t) = \Theta(-t)e^{Dq^2|t|},$$
$$G_{\hat{\rho} \hat{\rho}}^0(q, t) = \rho_0^2e^{-Dq^2|t|},$$

where $\Theta(t)$ is the step function. The bare propagator involving $\Phi$ is $G_{\Phi \Phi}^0(x, x') = \Delta(x - x')$, for which Fourier transform is given by

$$\tilde{\Delta}(q) = \int d^4x e^{-iqx} \Delta(x),$$

which will be denoted by dashed lines in the diagrams, and carry momentum but not frequency.

The nonlinear vertices in the action, Eq. (18) are represented graphically as in Figs. 2 and 3. At each black dot, momentum conservation holds. The cubic noise vertex involves two $i \hat{\rho}$ fields and one $\delta \rho$ field and is shown on the left panel of Fig. 2. It contributes the factor of $-Dq_1 \cdot q_2$ and does not involve the random potential. The type-$A$ vertex is shown in the right panel of Fig. 2. It gives the factor $\Gamma q_1 \cdot q_2$. Finally, an infinite number of vertices, labelled as type-$B_n$, exist as shown in Fig. 3, each of which carries the factor

$$-\rho_0^2 \frac{Dq_1 \cdot q_2}{n!(−T)^n}.$$  

(25)

The correlation functions are obtained by connecting the lines in the vertices using the bare propagators. Before going into this discussion, however, we first note that the $\Delta$-dependent part in the average density $\rho_0^2$ appearing in Eqs. (17), (18), (23) and (25) can be eliminated by considering the renormalized type-$B_n$ vertices in the following way: We first consider the type-$B_{n+2m}$ vertex for given $n, m = 1, 2, \cdots$. If we connect the $m$ pairs of dashed lines in the vertex as in Fig. 4, we end up with another type-$B_n$ vertex. Because $(n + 2m)!/(2^m n! m!)$ distinct ways of doing this are possible, this new type-$B_n$ vertex carries a factor of

$$-\rho_0^2 \frac{Dq_1 \cdot q_2 \Delta^n(0)}{2^m n! m! (−T)^{n+2m}}.$$  

(26)

Now, if we sum all these contributions for $m = 1, 2, \cdots$ and combine with the bare vertex in Eq. (25), which corresponds to $m = 0$ case, we have

$$-\rho_0^2 \frac{Dq_1 \cdot q_2 \Delta^n(0)}{n!(−T)^n} \sum_{m=0}^{\infty} \frac{\Delta^m(0)}{m!2^m (−T)^{2m}} = -\rho_0^2 \frac{Dq_1 \cdot q_2}{n!(−T)^n}.$$  

(27)

We note that the effect of this renormalization is just the use of $\rho_0$ instead of $\rho_0^2$ in the type-$B_n$ vertex. This
In terms of the number of $\hat{\delta}\rho$ and the type-$B_n$ vertices, we regard the pairings of these propagators and vertices is that the number of single lines ($\hat{\rho}$'s and $\hat{\delta}\rho$'s) and the type-$B_n$ vertices are that the number of single lines ($\hat{\rho}$) in every vertex is greater than or equal to that of the double lines ($\hat{\delta}\rho$). Because no loops with respect to the single and the double solid lines are possible, all the diagrams are tree diagrams with the connected dashed lines originating from a single vertex as done in Fig. 4 to have already been taken care of by using the renormalized type-$B_n$ vertex.

We now investigate the diagrams contributing to the two-point functions, $G_{\rho\rho}(q,t)$, in the perturbation theory. An interesting feature of the field theory constructed from these propagators and vertices is that the number of single lines ($\hat{\rho}$) in every vertex is greater than or equal to that of the double lines ($\hat{\delta}\rho$). Because no loops with respect to the single and the double solid lines are possible, all the diagrams are tree diagrams with the connected dashed lines originating from a single vertex as done in Fig. 4 to have already been taken care of by using the renormalized type-$B_n$ vertex.

For $G_{\rho\rho}(q,t)$, in addition to the type-$A$ vertex, the type-$B_n$ vertices may contribute because the two external $\delta\rho$'s can make up for the two extra $\hat{\rho}$'s in that vertex. In terms of the number of $\hat{\rho}$'s and $\hat{\delta}\rho$'s, the noise vertex may appear to contribute to $G_{\rho\rho}(q,t)$. This is, however, not the case. Because only two external lines are used for $G_{\rho\rho}(q,t)$ and because the noise vertex has three legs, the only way for the noise vertex to contribute to $G_{\rho\rho}(q,t)$ is to form a closed loop. However, the closed loops that appear in this field theory are all in the form of response loops described in Fig. 5, because every vertex has only one double line ($\delta\rho$) at most. These loop diagrams, however, vanish due to the causal structures of $G^0_{\rho\rho}(q,t)$ and $G^0_{\delta\rho\delta\rho}(q,t)$. The absence of closed loops is an important feature of the present field theory, which has already been noted in Ref. 44 for a Brownian gas without disorder. This feature is also shared by the diagrammatic field theory developed in Ref. 52 which we discuss in detail in Sec. III.

For the functions $G_{\rho\delta}$ and $G_{\delta\rho}$, as discussed already, only the type-$A$ vertex is relevant. Because no loops with respect to the single and the double solid lines are possible, all the diagrams are tree diagrams with the connected dashed lines representing the disorder strength $\Delta(q)$. We can present a general recipe for constructing a generic diagram contributing to, say $G_{\rho\rho}$, at an arbitrary order as in Fig. 6. We first put $2n$ dots between the external lines and make a single line by connecting them all with the bare $G^0_{\rho\rho}$'s. We then use dashed lines to connect $n$ pairs of the dots in all possible ways to generate a general diagram contributing to $G_{\rho\rho}$.

For $G_{\rho\rho}$, by counting the number of $\rho$'s and $\hat{\rho}$'s, we note that the type-$B_n$ vertices can only appear once at most in a diagram contributing to this function. We can classify the diagrams contributing to $G_{\rho\rho}$ into two distinct categories: (a) those in which type-$B_n$ vertices are not used, and (b) those where it is used only once. The structures of these two kinds of diagrams are described in Fig. 7. They are constructed in a similar way to the previous case by connecting all possible pairs of dashed lines.

We develop the perturbation theory for the two-point functions in terms of the strength of the disorder poten-
Fig. 9. Diagrams contributing to the first-order correction $\tilde{G}_{\rho \rho}^{1}(\rho, t)$ to the density-density correlation function.

The large-$t$ behavior of $\tilde{G}_{\rho \rho}^{1}(\rho, t)$ is much slower than an exponential decay. Therefore, $\tilde{\Delta} (\rho)$ can then be evaluated once the explicit form of $\tilde{\Delta} (\rho)$ is known. Without explicitly evaluating the integral, we can extract the long-time behavior of $\tilde{G}_{\rho \rho}^{1}(\rho, t)$. We find that the right-hand side of Eq. (29) in the large-$t$ limit is dominated by

$$-\Gamma^{2} \Theta(t) \int \frac{d^{d}q'}{(2\pi)^{d}} \frac{e^{-Dq'^{2}t}}{D^{2}(q^2 - q'^2)^{2}} \cdot \tilde{\Delta}(q - q') \cdot (q - q') \cdot \tilde{\Delta}(q - q').$$

The integrals over $t'$ and $t''$ can be done explicitly, yielding the overall factor $\Theta(t)$, which is consistent with the causal structure of $G_{\rho \rho}(\rho, t)$. The remaining integral over $\rho$ can then be evaluated once the explicit form of $\tilde{\Delta}(\rho)$ is known. Without explicitly evaluating the integral, we can extract the long-time behavior of $\tilde{G}_{\rho \rho}^{1}(\rho, t)$. We find that the right-hand side of Eq. (29) in the large-$t$ limit is dominated by

$$i\tilde{G}_{\rho \rho}^{1}(\rho, t) = -\Gamma^{2} \int \frac{d^{d}q'}{(2\pi)^{d}} \frac{e^{-Dq'^{2}t}}{D^{2}(q^2 - q'^2)^{2}} \cdot \tilde{\Delta}(q - q') \cdot (q - q') \cdot \tilde{\Delta}(q - q').$$

All the other terms fall exponentially fast as $e^{-Dq'^{2}t}$. In the large-$t$ limit, only the small-$q'$ behavior of the integrand is important. If we expand the integrand for small-$q'$, we find that the leading order term is of $O(q'^{2})$. Therefore, $\tilde{G}_{\rho \rho}^{1}(\rho, t)$ in the long-time limit behaves as

$$\int_{0}^{\infty} dq' q'^{d-1} q'^{2} e^{-Dq'^{2}t} \sim t^{-\frac{d}{2}-1},$$

which is much slower than an exponential decay.

For the first-order correction $G_{\rho \rho}^{1}(\rho, t)$, five diagrams shown in Fig. 9 contribute. The first two diagrams belong to the category (b) we have discussed while the rest belong to the category (a). Using the identity $G_{\rho \rho}^{0} = \rho_{0} [iG_{\rho \rho}^{0} + iG_{\rho \rho}^{0}]$ in the third and fifth diagrams in Fig. 9, we can see that some of them are exactly $iG_{\rho \rho}^{1} + iG_{\rho \rho}^{1}$. Combining with the remaining part, we have

$$\tilde{G}_{\rho \rho}^{1}(\rho, t) = \rho_{0} \int \frac{d^{d}q'}{(2\pi)^{d}} \frac{e^{-Dq'^{2}t}}{D^{2}(q^2 - q'^2)^{2}} \cdot \tilde{\Delta}(q - q') \cdot (q - q') \cdot \tilde{\Delta}(q - q').$$

In a similar way to the case of $\tilde{G}_{\rho \rho}^{1}(\rho, t)$, we can show that $G_{\rho \rho}^{1}(\rho, t)$ again exhibits a power law behavior in the large-$t$ limit. After evaluating the time integrals in Eq. (32), we expand the integrand around $q' = 0$ to extract the long-time behavior. Unlike the previous case, the leading order term in this case is of $O(1)$, which amounts to

$$\int_{0}^{\infty} dq' q'^{d-1} e^{-Dq'^{2}t} \sim t^{-\frac{d}{2}}.$$

This decay is slower than that of $\tilde{G}_{\rho \rho}^{1}(\rho, t)$ and determines the long-time behavior of $G_{\rho \rho}(\rho, t)$. This long-time behavior was reported in Ref. 52. This is confirmed in the numerical evaluation of $G_{\rho \rho}(\rho, t)$ from Eq. (32) as shown in Fig. 10, where we considered the case of a simple Gaussian form

$$\tilde{\Delta}(\rho) = \Delta_{0} \exp(-\xi^{2}q^{2})$$

for the disorder correlation with some constants $\xi$ and $\Delta_{0}$.

Fig. 10. First-order correction $\tilde{G}_{\rho \rho}^{1}(\rho, t)$ to the density-density correlation function for various dimensions $d = 2, 3$ and 4 for the case where the disorder correlation $\tilde{C}(\rho)$ is given by Eq. (34). The time $t$ and $G_{\rho \rho}^{1}(\rho, t)$ are in units of $\xi^{2}/D$ and $\rho_{0}\Delta_{0}/(\xi^{2}q^{2})$, respectively. The plot shows the case where the momentum is $q = 1$. The slope in the long-time limit clearly shows the predicted $t^{-d/2}$ behavior.
III. FIELD THEORY FOR THE DENSITY FLUCTUATION AROUND THE UNIFORM AVERAGE VALUE

Before going to the nonperturbative calculation, we study in this section the field theory using the density fluctuation $\Delta \rho(x,t)$ around the uniform average density $\rho_0$, which is defined in Eq. (11). This is the formalism used in Ref. 52. We present a detailed comparison of the field theory with respect to this variable with the one developed in the previous section. We will show that both formalisms give the same correlation functions and developed in the previous section. We will show that the perturbative field theory using $\Delta \rho$ is much more complicated than that for $\delta \rho$.

For a given realization of the disorder potential, the thermal average $\langle \Delta \rho(x,t) \rangle = \rho(x) - \rho_0 \equiv \delta \rho_0(x)$ does not vanish. As we have seen in Eq. (10), only after being averaged over the disorder potential does this quantity vanish, $\langle \Delta \rho(x,t) \rangle = 0$. Higher order moments of $\Delta \rho$’s are related to those of $\delta \rho$’s, because we can write $\Delta \rho(x,t) = \delta \rho(x,t) + \delta \rho_0(x)$. For example, we have [5, 52]

$$
\langle \Delta \rho(x,t) \Delta \rho(x',t') \rangle = \langle \delta \rho(x,t) \delta \rho(x',t') \rangle + \delta \rho_0(x) \delta \rho_0(x'),
$$

where we can evaluate explicitly the last term as

$$
\delta \rho_0(x) \delta \rho_0(x') = \rho_0^2 \left( e^{\Delta(x-x')/T^2} - 1 \right).
$$

The first term on the right-hand side of Eq. (35) is just $G_{\rho\rho}(x-x',t-t')$ defined in the previous section. Therefore, the correlation function on the left-hand side of Eq. (35) differs from $G_{\rho\rho}(x-x',t-t')$ by the time-independent quantity given by Eq. (36).

We write $\rho = \rho_0 + \Delta \rho$ in Eq. (5) and again transform it into a field theory as done before. The average of an observable $O(\Delta \rho, \hat{\phi})$ is then given by

$$
O(\Delta \rho, \hat{\phi}) = \int D\Delta \rho \int D\hat{\phi} \int [D\Phi] \ O(\Delta \rho, \hat{\phi}) \ e^{S_\Delta[\Delta \rho, \hat{\phi}]},
$$

(37)

where $S_\Delta = S_\Delta^G + S_\Delta^\Lambda$ with

$$
S_\Delta^G = \int d^d x \int dt \left[ -i \hat{\rho}(x,t) \left( \frac{\partial \Delta \rho(x,t)}{\partial t} - D \nabla^2 \Delta \rho(x,t) \right) + D \rho_0 \left( \nabla i \hat{\rho}(x,t) \right)^2 \right] - \frac{1}{2} \int d^d x \int d^d x' \Phi(x) \Delta^{-1}(x-x') \Phi(x')
$$

(38)

and

$$
S_\Delta^\Lambda = \int d^d x \int dt \left[ D \nabla \Delta \rho(x,t) \left( \nabla i \hat{\rho}(x,t) \right)^2 + \Gamma \left( i \hat{\rho}(x,t) \right) \nabla \cdot \left( \Delta \rho(x,t) \right) \nabla \Phi(x) \right].
$$

(39)

The Gaussian part of the action $S_\Delta^G$ takes a form similar to $S_G$ in Eq. (17). However, unlike $S^\rho$, $S_\Delta^G$ does not contain an infinite number of terms and takes a much simpler form. Here, we regard the last term in Eq. (39), which is quadratic in fields, as a part of vertices. As we will see below, due to the unusual nature of this vertex, the perturbation expansion for this action is much more complicated than the corresponding scheme for $\delta \rho$ despite the apparent simplicity of the action. We note that the functional integral over $\Phi$ in Eq. (37) can actually be carried out to yield an effective action that depends only on $\Delta \rho$ and $\hat{\rho}$. This effective action has been used in Ref. 52. For the present discussion, we find it more convenient to consider the $\Phi$-dependent action as in Eq. (39). We will pair up the dashed lines in the perturbation expansion as before, which will produce the same diagrams as the method used in Ref. 52.

We now develop the perturbation theory for the two-point function

$$
F_{\alpha\beta}(x-x',t-t') = \langle \psi_\alpha(x,t) \psi_\beta(x',t') \rangle_{S_\Delta}
$$

(40)

evaluated with respect to $S_\Delta$ for the variables $\psi_\alpha, \psi_\beta$ representing $\hat{\rho}$ or $\Delta \rho$ (denoted by $\rho$ in the subscript again). Because the Gaussian part $S_\Delta^G$ is identical to $S_G$ in Eq. (17) except for $\Delta \rho$ and $\rho_0$ playing the roles of $\delta \rho$ and $\rho_0^0$, respectively, the bare propagators $F_{\alpha\beta}^0$ take the same form as the renormalized $G_{\rho\rho}^0$. The noise vertex and the type-$A$ vertex in $S_\Delta^\Lambda$ have the same structures as before, with $\Delta \rho$ taking the place of $\delta \rho$. Instead of the the type-$B_a$ vertex, the action contains a new vertex as shown in Fig. 11, that comes from the last term in Eq. (39). We denote it as a type-$A_0$ vertex. This carries the factor of $-\rho_0 \Gamma \hat{q}^2$.

The type-$A_0$ vertex contains just one $\hat{\rho}$ field. Therefore, the only way it can contribute to $F_{\hat{\rho}\hat{\rho}}$ is to appear
at most twice in a diagram. Otherwise, we would have too many $\tilde{\rho}$’s to make a nonzero diagram. We can, therefore, classify the diagrams contributing to $\tilde{F}_{\rho\rho}(q, t - t')$ into three distinct categories depending on the number of new vertices in a diagram: (A) no type-$A_0$ vertex, (B) one type-$A_0$ vertex, and (C) two type-$A_0$ vertices. The diagrams in category (A) are identical to the diagrams in (a) for $G_{\rho\rho}$ discussed in the previous section.

For diagrams in category (C), one cannot form a path from an external leg at time $t$ to another one at time $t'$ by using only solid (single or double) lines. The external legs have to be connected by a dashed line, which does not carry the time in it. This means that those in category (C) give a constant contribution independent of $t - t'$ (but are still a function of $q$). We shall explicitly show below that the contributions from the diagrams in category (B) reproduce those from the diagrams in (b) for $G_{\rho\rho}$ and that those in category (C) correspond to the time-independent term, Eq. (36), for $F_{\rho\rho}$.

We investigate diagrams in category (B) in more detail. The diagrams in (B) must contain one noise vertex along with one type-$A_0$ vertex. Figure 12 shows the simplest vertex structure appearing in diagrams that belong to (B). We note that the external legs of this diagram have the same structure as the type-$B_1$ vertex used in the previous section. In the diagram in Fig. 12, the noise and the type-$A_0$ vertices give the factors, $-Dq_1 \cdot q_2$ and $-\rho_0 \Gamma(q_1 + q_2)^2$, respectively. Because one of the time integrals gives

$$
\int_0^\infty \text{d}t \, \text{i} \tilde{F}_{\rho\rho}^0(q_1 + q_2, t) = \frac{1}{D(q_1 + q_2)} \tag{41}
$$

we end up with the overall factor of $\rho_0 \Gamma(q_1 + q_2)$. From Eq. (27), we can see that this is exactly the factor for the renormalized type-$B_1$ vertex. Therefore, we can conclude that the diagram in Fig. 12 plays exactly the same role as the renormalized type-$B_1$ vertex. In general, we can show that the diagram shown in Fig. 13, which consists of one noise vertex, one type-$A_0$ vertex and $n - 1$ type-$A$ vertices, is equivalent to the renormalized type-$B_n$ vertex shown in Fig. 4. The detailed proof for the equivalence is presented in Appendix B. This can be regarded as one of the advantages in using the field theory for $\Delta \rho$. A complicated combination of vertices in the field theory for $\Delta \rho$ can be represented as a single vertex in the corresponding formalism for $\delta \rho$.

We now consider the diagrams obtained by connecting the dashed lines in the diagram in Fig. 13. We recall that for the type-$B_n$ vertex, connecting the dashed lines within a single vertex results in its renormalization. However, as we have seen above, the diagram in Fig. 13 is already equivalent to the renormalized type-$B_n$ vertex. We, therefore, expect that the diagrams obtained by pairing up the dashed lines in Fig. 13 all to vanish, which we will demonstrate below. This means that the perturbation expansion for the $\Delta \rho$-field theory generates numerous unnecessary diagrams that, as a whole, give a vanishing contribution. We first note that when the two rightmost dashed lines are paired as in Fig. 14, the momentum flowing through the $\tilde{F}_{\rho\rho}^0$ propagator right next to the loop must be zero. Then, because the type-$A$ vertex involves the dot product between the zero momentum vector and another one, such a diagram vanishes. When no such loop exists, a diagram does not vanish on its own in general. The diagrams in Fig. 15 show such examples. For these diagrams, when we perform the time integrals for the bare propagators, we obtain, apart from the common factor of $(\rho_0D/T^4)q_1 \cdot q_2$,

$$
\int \frac{d^d q'}{(2\pi)^d} \frac{(q_1 + q_2) \cdot q' \cdot (q_1 + q_2 + q') \cdot q'}{(q_1 + q_2 + q')^2} \Delta(q') \tag{42}
$$

and

$$
\int \frac{d^d q'}{(2\pi)^d} \frac{(q_1 + q_2) \cdot q' \cdot (q_1 + q_2 + q') \cdot (q_1 + q_2)}{(q_1 + q_2 + q')^2} \Delta(q'), \tag{43}
$$
respectively. Here, the momentum $q'$ flows through the dashed line loop. We find that the individual diagrams do not vanish, but the sum does, due to the rotational invariance of the integration over $q'$. We expect that for the higher order vertices equivalent to the type-$B_n$, a similar cancellation must occur when the dashed lines are paired within the vertex. This cancellation is a generic feature of the field theory involving $\Delta \rho$. We stress again that in the field theory for $\delta \rho$, these kinds of unnecessary diagrams do not arise. These self-loops in these vertices are handled by using the simple renormalization of the $B_n$ vertex from the outset.

As mentioned above, the diagrams in category (C) are responsible for the time-independent part given in Eq. (36). This part arises only in this formalism, because $\Delta \rho$ is not a density fluctuation around its own average value, but around the uniform value. We again find that many diagrams in the category (C) cancel each other and do not contribute at all. We note that a generic diagram in (C) can be described schematically as in Fig. 16. A gap is present in the middle between the two type-$A_0$ vertices, so the diagram can naturally be split into two disconnected parts. We then have to connect all possible pairs of the dashed lines. We find that a set of diagrams containing a dashed line connection within a disconnected part gives a vanishing contribution. An example is shown in Fig. 17. We can easily see that these diagrams cancel each other, as the diagrammatic structures are essentially the same as those in Fig. 13. The nonvanishing diagrams are those in which the left part is connected to the right part via dashed line. As we show in detail in Appendix C, such a diagram with $2n$ dashed lines gives

$$\frac{\rho_0^2}{n!T^{2n}} \int \prod_{j=1}^n \left[ \frac{d^4q_j}{(2\pi)^d} \delta(q_j) \right] (2\pi)^d \delta^{(4)}(q - \sum_{i=1}^n q_i),$$

where $q$ is the momentum flowing through the diagram in Fig. 16. This is exactly the Fourier transformation of the right-hand side of Eq. (36) at $O(\Delta^n)$.

In this section, we have shown that, although the field theory for the density fluctuation around the uniform average is equivalent to that for $\delta \rho$ studied in the previous section, its perturbation expansion generates numerous diagrams, most of which vanish or cancel each other. This could make the calculation of the correlation functions unnecessarily complicated if done in this formalism.

IV. PHYSICAL RESPONSE FUNCTION AND FLUCTUATION-DISSIPATION RELATION

In this section, we further explore the usefulness of the field theory for $\delta \rho$ developed in Sec. II in the context of the FDR, which is obeyed by the equilibrium dynamics described by the Langevin equations, Eqs. (1) and (5). The FDR, which comes from the time-reversal symmetry of the equilibrium state, provides the relationship between the correlation and the response functions. The MSRJD formalism is known to be suited for studying the correlation and the response of a system described by Langevin equations as the hatted variable arising in the field theory naturally provides an expression for the response function. However, for the Langevin equation in Eq. (5) given in terms of the density variable, the response function does not take a simple form such as $\langle \rho(x',t)\hat{\rho}(x',t') \rangle$. Rather, due to the multiplicative nature of the noise in Eq. (5), the physical response to an external perturbation coupled to the density variable is well known to take a more complicated form given by [37,38,41,52]

$$R(x - x', t - t') = -\Gamma \langle \rho(x,t)\nabla' \cdot (\rho(x',t')\nabla'\hat{\rho}(x',t')) \rangle,$$

where $\Gamma$ is the momentum flowing through the diagram.

which we refer to as the physical response function in the following. The FDR relates $R$ to the correlation function $C$ via

$$-\frac{\partial}{\partial t} C(x,t) = TR(x,t) - TR(x,-t),$$

where

$$C(x - x', t - t') = \langle \rho(x,t)\rho(x',t') \rangle$$

is the density-density correlation function. If we use $\rho(x,t) = \rho_B(x) + \delta \rho(x,t)$, this is related to our correlation function as

$$C(x - x', t - t') = \rho_B(x)\rho_B(x') + G_{\rho_B}(x - x', t - t').$$

The physical response function in our field theory for $\delta \rho$ is obtained by simply replacing $\rho(x,t)$ in Eq. (45) by $\delta \rho(x,t) + \rho_B(x)$. The average is then given with respect
to the effective action $S$ in Eqs. (17) and (18). We first note that both $\langle \rho_0(x) \nabla \cdot (\delta \rho(x', t') \nabla \cdot \delta \rho(x', t')) \rangle_S$ and $\langle \delta \rho(x, t) \nabla \cdot (\delta \rho(x', t') \nabla \cdot \delta \rho(x', t')) \rangle_S$ are equal to zero in this field theory. The former is essentially $G_{\rho \rho}$ evaluated at the same space time point. By connecting the end points in the generic diagram in Fig. 6, we find that we are left with the loops described in Fig. 5, which vanish due to causality. The latter quantity must involve one noise vertex and can contain an arbitrary number of type-$A$ vertices. However, because there is no vertex with two $\delta \rho$ fields, the loops that appear in the perturbative expansion of this quantity are again all in the form of Fig. 5. Therefore, Eq. (45) reduces to

$$R(x - x', t - t') = -\Gamma \langle \delta \rho(x, t) \nabla \cdot (\rho_0(x') \nabla \cdot \delta \rho(x', t')) \rangle_S.$$

In the absence of disorder, $\rho_0 = \rho_0$, and the physical response function reduces to $\rho_0 \nabla^2 iF_{\rho \rho}(x - x', t - t')$ [44], where the superscript $f$ denotes the case where $\Phi$ is set to zero. The effect of disorder in our field theory on the physical response function is reflected simply in the appearance of the inhomogeneous average density $\rho_0$.

On the other hand, in the field theory using $\Delta$, the physical response function takes a more involved form. By substituting $\rho$ with $\rho + \Delta \rho$ in Eq. (45), we have

$$R(x - x', t - t') = -\rho_0 \nabla^2 iF_{\rho \rho}(x - x', t - t')$$

$$-\Gamma \langle \Delta \rho(x, t) \nabla \cdot (\Delta \rho(x', t') \nabla \cdot \delta \rho(x', t')) \rangle_{S_D}.$$

Evaluating the physical response function in this case involves the calculation of two correlation functions with respect to the effective action $S_D$, one of which is a three-point function. The three-point function is nonvanishing in this case due to the presence of the type-$A_0$ vertex. In the absence of disorder, the three-point function in Eq. (50) vanishes, and we have only the first term and have the same result as before: $\rho_0 \nabla^2 iF_{\rho \rho}(x - x', t - t')$ (note that $F_{\rho \rho}^f = G_{\rho \rho}^f$). The effect of disorder in this case is encoded in the three-point function as well as in $F_{\rho \rho}$, both of which have to be evaluated explicitly.

The simple structure of the physical response function, Eq. (49), in our field theory for $\delta \rho$ enables us to do a non-perturbative calculation. If we expand $\rho_0$ in Eq. (49) in powers of $\Phi$, we find that a generic diagram contributing to $R$ takes the form depicted in Fig. 18, where all possible pairs of dashed lines are to be connected. On the right end point $(x', t')$, we have multiple $\Phi(x')$'s coming from the expansion of $\rho_0$ along with the $\rho$ field. In the middle, we have dashed lines from a collection of type-$A$ vertices reaching up to $(x, t)$ where $\delta \rho$ lies. The contribution from this kind of diagram with a total of $n$ dashed lines to the Fourier transform $\tilde{R}(q, t - t')$ can be written as

$$\int \prod_{i=1}^n \left[ \frac{d^d k_i}{(2\pi)^d} \right] \tilde{\Xi}_n(q; k_1, k_2, \ldots, k_n; t - t')$$

(51)

for some vertex function $\tilde{\Xi}_n$ with the condition $\sum_{i=1}^n k_i = 0$. We note that, by construction, $\tilde{\Xi}_n$ is symmetric under the permutation of $n$ momenta, $(k_1, k_2, \ldots, k_n)$. The response function $R$ can then be obtained by connecting all possible pairs of $\Phi$'s and by summing over all $n = 0, 1, 2, \ldots$.

Despite the simple diagrammatic structure, finding a general nonperturbative expression for $\Xi_n$ and thus for $\tilde{R}(q, t - t')$ is still a difficult task. However, we can make a progress if we focus on the zero frequency component of the response function,

$$\tilde{R}(q, \omega = 0) = \int_{-\infty}^{\infty} dt \tilde{R}(q, t),$$

(52)

where the time integral is actually from 0 to $\infty$ due to the causal nature of the response function. This quantity provides an important physical insight into the long-time behavior of the correlation function, because it is equal to $(\tilde{C}(q, 0) - \tilde{C}(q, \infty))/T$, as can be obtained from the FDR, Eq. (46). In order to calculate this, we define

$$\tilde{\Xi}_n(q; k_1, k_2, \ldots, k_n; \omega = 0)$$

$$= \int_{-\infty}^{\infty} dt \tilde{\Xi}_n(q; k_1, k_2, \ldots, k_n; t).$$

(53)

We evaluate this in detail in Appendix D. The result is quite simple as it gives just a constant

$$\tilde{\Xi}_n(q; k_1, k_2, \ldots, k_n; \omega = 0) = -\rho_0^2 \frac{1}{(2\pi)^d n!}.$$  

(54)

We now connect all possible pairs of $\Phi$'s in Eq. (51) and sum these contributions over $n = 0, 1, 2, \ldots$. Clearly, only the terms with even $n = 2p$ survive. We, therefore, have

$$\tilde{R}(q, \omega = 0) = \sum_{p=0}^{\infty} \frac{(2p)!}{2^p p!} \int \prod_{i=1}^p \frac{d^d k_i}{(2\pi)^d} \tilde{\Delta}(k_i)$$

$$\times \tilde{\Xi}_{2p}(q, k_1, -k_1, k_2, -k_2, \ldots, k_p, -k_p, \omega = 0)$$

(55)

with the understanding that the $p = 0$ term is given just by $\Xi_0 = \rho_0^2/T$. The factor $(2p)!/(2^p p!)$ accounts for the number of possible ways to form all possible pairs of $\Phi$'s. Using Eq. (54), we finally have

$$\tilde{R}(q, \omega) = \frac{\rho_0^2}{T} \sum_{p=0}^{\infty} \frac{1}{2^p p!} \left[ \int \frac{d^d k \tilde{\Delta}(k)}{(2\pi)^d 2T^2} \right]^p$$

$$= \frac{\rho_0^2}{T} \int \frac{d^d k \tilde{\Delta}(k)}{(2\pi)^d 2T^2} = \frac{\rho_0}{T}.$$  

(56)
As mentioned above, this nonperturbative result for the response function has an implication for the long-time behavior of the density-density correlation function. At time \( t = 0 \), the particles are at equilibrium with respect to a given realization of the external potential \( \Phi(x) \) at temperature \( T \). The density-density correlation function must be equal to the static one given by

\[
(\rho(x)\rho(x'))_{\text{stat}} = \delta(x-x')\rho_{\Phi}(x) + \rho_{\Phi}(x)\rho_{\Phi}(x').
\]

with \( \rho_{\Phi} \) given in Eq. (10). Averaging over the disorder configurations, we, therefore, have

\[
C(x-x',0) = \delta(x-x')\rho_{0} + \rho_{\Phi}(x)\rho_{\Phi}(x').
\]

The nonperturbative result, Eq. (56), together with the FDR, Eq. (46), implies that

\[
C(x-x',\infty) = \rho_{\Phi}(x)\rho_{\Phi}(x')
\]

or \( G_{\rho_{\Phi}}(x-x',t) \) goes to zero in the long-time limit. This means that the system remains ergodic and that no ergodicity breaking transition takes place for any value of the disorder strength.

V. DISCUSSION AND CONCLUSION

We have constructed the MSRJD dynamical field theory for noninteracting Brownian particles in a quenched Gaussian random potential. We have set up the diagrammatic perturbation scheme for the connected density-density correlation function. The main variable for our field theory is the density fluctuation around the nonuniform average value. We have studied the diagrammatic perturbation theory in detail. In particular, the role played by the noise and type-\( B_{n} \) vertices, which originate from the multiplicative nature of the DK equation, has been elucidated and treated nonperturbatively. The diagrammatic structures are compared in detail with those in the field theory using the density fluctuation around the uniform average value, which is used in Ref. 52. The latter is shown to generate many unnecessary diagrams that either vanish or cancel among themselves. Using our field theory, we were able to evaluate the zero-frequency component of the response function exactly by summing all the diagrams. According to the FDR, our result implies that the connected density-density correlation function decays to zero in the long-time limit and that the system remains ergodic for all values of the disorder strength.

Our finding is in contrast to the finding in Ref. 52. In that paper, various renormalized perturbation schemes for the density correlation function, which are consistent with the FDR, are presented. In one of the schemes, the connected density correlation function does not decay to zero in the long-time limit, but approaches a finite value when the disorder strength exceeds some critical value. This signals an ergodic-nonergodic transition. This is, however, in contrast to our exact result, because if the transition exists, the right hand side of Eq. (56) will give a different value at the transition. These renormalized perturbation theories basically correspond to replacing the bare correlation functions in some perturbation expansion scheme with the renormalized ones. These in turn produce various types of self-consistent equations for the renormalized correlation function. In terms of the Feynman diagrams, a renormalized perturbation theory corresponds to the partial resummation of a particular infinite subset of diagrams contributing to the density correlation function. Our exact result suggests that the ergodic-nonergodic transition seen in Ref. 52 might be an artifact of the partial resummation of the diagrams. The situation is very reminiscent of the dynamical transition predicted by the MCT of supercooled liquids [49], which can also be regarded as a partial resummation of diagrams for the full theory of supercooled liquids. We expected the sharp transition to be smeared out when other effects such as activated hopping are included and the system to remain ergodic [34,35,53].

In the present work, we have only considered the zero-frequency component of the response function. If more useful information on the transport properties of the system is to be found, a reliable scheme to calculate the full time dependence of the correlation and response functions must be found. We have tried to implement various renormalized perturbation schemes including those presented in Ref. 52 for the density correlation function. We have also encountered the same problems as in Ref. 52 such as spurious instabilities when we try to find solutions to self-consistent equations. We believe that in order to further improve the renormalized perturbation theory for this system, one needs to consider the renormalization of the vertices, as well as the propagators, and to find self-consistent equations for these quantities. This is left for future work. For these kinds of calculations and for an eventual generalization to the system of interacting Brownian particles in a quenched random potential, we believe that the perturbation scheme presented in this work will provide a convenient starting point.

ACKNOWLEDGMENTS

We would like to thank Bongsoo Kim for a helpful discussion. This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. 2020R1F1A1062833), and by Konkuk University’s research support program for its faculty on sabbatical leave in 2020.
APPENDIX A: PROPERTIES OF THE GAUSSIAN DISORDER AVERAGE

We consider the disorder average of the denominator on the right hand side of Eq. (9). We have

\[
\int d^nx e^{-\Phi(x)/T} = \int d^nx e^{\Delta(0)/(2T^2)} = V e^{\Delta(0)/(2T^2)},
\]

where \(V\) is the volume. We calculate the variance of this quantity as

\[
\left[\int d^nx e^{-\Phi(x)/T} - V e^{\Delta(0)/(2T^2)}\right]^2 = \int d^nx \int d^nx' e^{(\Delta(0)+\Delta(x-x'))/T^2} - V^2 e^{\Delta(0)/T^2}
\]

\[
= V e^{\Delta(0)/T^2} \int d^nx (e^{\Delta(x)/T^2} - 1).
\]

If we assume \(\Delta(x)\) is a short-ranged function, then the integral gives a finite quantity. Therefore the disorder average and the variance of \(\int d^nx e^{-\Phi(x)/T}\) are both proportional to \(V\), and the denominator on the right hand side of Eq. (9) can be replaced by its average value in the thermodynamic limit.

APPENDIX B: EVALUATION OF THE DIAGRAM IN FIG. 13

We consider the case of \(n\) dashed lines in Fig. 13 carrying momenta \(k_1, k_2, \cdots, k_n\). Figure 13 can be represented in the action as

\[
\int dt \int d^dq_1 \left(\frac{2\pi}{j} \right)^d \int d^dq_2 \left(\frac{2\pi}{j} \right)^d \int \prod_{i=1}^{n} \left(\frac{2\pi}{j} \right)^d \delta(d)(q_1 + q_2 - \sum_{i=1}^{n} k_i) \times (-Dq_1 \cdot q_2) \Lambda_n(k_1, k_2, \cdots, k_n) i\hat{\rho}(-q_1, t) i\hat{\rho}(-q_2, t) \prod_{i=1}^{n} \hat{\Phi}(k_i)
\]

for some vertex function \(\Lambda_n\). In order to evaluate \(\Lambda_n\), we consider the case where the momenta going in through the dashed lines from right to left in Fig. 13 are given by \(k_1, k_2, \cdots, k_n\). For convenience, we use the shorthand notation for the bare propagator as

\[
g(q, t) \equiv i\mathcal{G}^{(0)}_{\rho\rho}(q, t) = i\mathcal{F}^{(0)}_{\rho\rho}(q, t) = \Theta(t)e^{-Dq^2t}.
\]

Then, for this particular configuration of the momenta, the contribution to \(\Lambda_n\) is given by

\[
\int_{-\infty}^{\infty} \prod_{i=1}^{n} dt_i \left\{ -\rho_0 \Gamma k_1 \cdot k_1 \right\} g(k_1, t_2 - t_1) \left\{ -\Gamma k_2 \cdot (k_1 + k_2) \right\} g(k_1 + k_2, t_3 - t_2)
\]

\[
\times \cdots \times \left\{ -\Gamma k_{n-1} \cdot (k_1 + \cdots + k_{n-1}) \right\} g(k_1 + \cdots + k_{n-1}, t_n - t_{n-1})
\]

\[
\times \left\{ -\Gamma k_n \cdot (k_1 + \cdots + k_n) \right\} g(k_1 + \cdots + k_n, t - t_n),
\]

where \(t_i\) denotes the time when the momentum \(k_i\) is coming in for \(i = 1, 2, \cdots, n\). We make the change of variables, \(s_i = t_{i+1} - t_i\) for \(i = 1, \cdots, n\), with the unit Jacobian. Then the time integral over \(s_i\) is from 0 to \(\infty\) due to the step function in the bare propagator and can be evaluated explicitly as

\[
\rho_0 \prod_{i=1}^{n} \left\{ -\Gamma k_i \cdot (k_1 + k_2 + \cdots + k_i) \int_{0}^{\infty} ds_i e^{-D(k_1 + k_2 + \cdots + k_i)^2 s_i} \right\} = \frac{\rho_0}{(-T)^n} \prod_{i=1}^{n} \frac{k_i \cdot (k_1 + k_2 + \cdots + k_i)}{(k_1 + k_2 + \cdots + k_i)^2}.
\]

As can be seen from Eq. (B1), \(\Lambda_n\) can be symmetrized with respect to the permutation of its arguments. Therefore, we have

\[
\Lambda_n(k_1, k_2, \cdots, k_n) = \frac{1}{n!} \rho_0 (-T)^n O_n(k_1, k_2, \cdots, k_n),
\]

where

\[
O_n(k_1, k_2, \cdots, k_n) = \frac{1}{n!} \rho_0 (-T)^n O_n(k_1, k_2, \cdots, k_n).
\]
where

\[ O_n(k_1, k_2, \ldots, k_n) \equiv \sum_{P} \prod_{i=1}^{n} \frac{k_i \cdot (k_1 + k_2 + \ldots + k_i)}{(k_1 + k_2 + \ldots + k_i)^2} \]  

(B6)

with the summation over the permutations \( P \) of \( (k_1, k_2, \ldots, k_n) \).

We now prove by induction that \( O_n = 1 \) for all \( n = 1, 2, \ldots \). Seeing that \( O_1 = 1 \) and that

\[ O_2(k_1, k_2) = \frac{k_2 \cdot (k_1 + k_2)}{(k_1 + k_2)^2} \frac{k_1 \cdot (k_1 + k_2)}{(k_1 + k_2)^2} = 1 \]  

(B7)

is trivial. Now, we suppose that \( O_{n-1} = 1 \). We note that, compared to \( O_{n-1} \), \( O_n \) contains one additional factor that contains all the \( k_i \)'s. Therefore, we can write

\[ O_n(k_1, k_2, \ldots, k_n) = \sum_{i=1}^{n} \frac{k_i \cdot (k_1 + k_2 + \ldots + k_n)}{(k_1 + k_2 + \ldots + k_n)^2} O_{n-1}(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n), \]  

(B8)

and from the assumption we conclude that

\[ O_n(k_1, k_2, \ldots, k_n) = \sum_{i=1}^{n} \frac{k_i \cdot (k_1 + k_2 + \ldots + k_n)}{(k_1 + k_2 + \ldots + k_n)^2} = 1. \]  

(B9)

Therefore, from Eq. (B5), we have \( \Lambda_n = \rho_0/((n!)(-T)^n) \). By using this value in Eq. (B1) and comparing it with the last term in Eq. (18), we find that the diagram in Fig. 13 is equal to the renormalized type-\( B_n \) vertex, which carries \( \rho_0 \) instead of \( \rho_0' \).

**APPENDIX C: EVALUATION OF THE DIAGRAM IN FIG. 16**

We first note that both disconnected parts in Fig. 16 have the same structure as the diagram in Fig. 13 treated in Appendix B except that at the end points we have the \( \Delta q \) fields instead of the noise vertex. For the disconnected part on the left-hand side, if we denote the external momentum coming out of the left end point by \( q \) and the momenta coming in the \( n \)-dashed lines by \( k_i, i = 1, 2, \ldots, n \), we can write this part as

\[ \int \prod_{i=1}^{n} \frac{d^d k_i}{(2\pi)^d} (2\pi)^d \delta^{(d)}(q - \sum_{i=1}^{n} k_i) \Lambda_n(k_1, k_2, \ldots, k_n) \prod_{i=1}^{n} \tilde{\Phi}(k_i) \]  

(C1)

with the same vertex function \( \Lambda_n \) as considered in Eq. (B1). As expected, there is no time dependence. We represent the part on the right-hand side in the same way using the momenta \( q' \) and \( k'_i, i = 1, 2, \ldots, n \). We then connect all possible pairs of the dashed lines. This produces the disorder correlation \( \Delta(k_i) \) with the delta function enforcing \( k_i + k'_i = 0 \). Therefore, along with the overall delta function \( (2\pi)^d \delta^{(d)}(q + q') \), we have

\[ n! \int \prod_{i=1}^{n} \left( \frac{d^d k_i}{(2\pi)^d} \right) \Lambda_n^2(k_1, k_2, \ldots, k_n) (2\pi)^d \delta^{(d)}(q - \sum_{i=1}^{n} k_i), \]  

(C2)

where \( n! \) denotes the number of different ways of connecting the dashed lines. If we use the result \( \Lambda_n = \rho_0/((n!)(-T)^n) \) obtained in Appendix B, we find that this is exactly Eq. (44).
We consider the diagram in Fig. 18 with a total of $n$ dashed lines. Among the $n$ dashed lines, we consider the situation described in Fig. A1, where $m$ ($m \leq n$) dashed lines exist at the right external source at time $t'$ and $n - m$ dashed lines coming from a series of the type-$\tau$ and $-732$- Journal of the Korean Physical Society, Vol. 77, No. 9, November 2020

the condition that that response function from Eq. (53), we can make a progress. By changing the integration variables from $O_{(B8)}$, we obtain

with the bare propagator $g$ given in Eq. (B2). In general evaluating the time integrals to get a closed form is difficult. However, if we focus on the time integral over $\tau = t - t'$ of the above quantity to get the zero-frequency limit of the response function from Eq. (53), we can make a progress. By changing the integration variables from $t_1, \cdots, t_{n-m}$ and $\tau$ to $s_1 = t - t'$, $s_2 = t - t'$, $\cdots$, $s_{n-m} = t \cdots t - t_{n-m}$ and $s_{n-m} = t - t_{n-m}$, with the unit Jacobian, we can explicitly evaluate all the time integrals as we have done in Appendix B. The result is

As can be seen from Eq. (51), we have to symmetrize this quantity over all permutations of $(k_1, \cdots, k_n)$. We also have to consider all possible cases of $m = 0, 1, \cdots, n$ for given $n$. Therefore, we can express the quantity in Eq. (53) as

where

with $\sum_p$ indicating the sum over all possible permutations of $(k_1, k_2, \cdots, k_n)$. We will prove below that $\sum_{m=0}^{n} O_{nm} = 1$ for all $n = 0, 1, 2, \cdots$ by mathematical induction. First, we note that $O_{00} = 1$ and

thus $O_{10} + O_{11} = 1$. Now we suppose that $\sum_{m=0}^{n-1} O_{n-1,m} = 1$. Using the method similar to the one used to derive Eq. (B8), we obtain

Appendix D: Evaluation of the Vertex Function $\tilde{\Xi}_n(q; k_1, k_2, \cdots, k_n; \omega = 0)$
for $m \leq n - 1$. We, therefore, find that

$$
\sum_{m=0}^{n} O_{nm}(q; k_1, \ldots, k_n) = \sum_{m=0}^{n-1} O_{nm}(q; k_1, \ldots, k_n) + O_{nn}(q; k_1, \ldots, k_n)
$$

$$
= \sum_{m=0}^{n-1} \sum_{i=1}^{n} \left\{ \frac{k_i \cdot (q + k_1 + \cdots + k_n)}{(q + k_1 + \cdots + k_n)^2} \right\} O_{n-1,m}(q; k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n)
$$

$$
+ \frac{k_n \cdot (q + k_1 + \cdots + k_n)}{(q + k_1 + \cdots + k_n)^2} = 1.
$$

From Eq. (D3), we finally have

$$
\tilde{\Xi}_n(q; k_1, \ldots, k_n; \omega = 0) = -\frac{\rho_0^n}{(-T)^{n+1}} \frac{1}{n!}
$$

regardless of its arguments.

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