On the mathematical hypothesis of phenomena like the confinement

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The Wick rotation provides the standard technique of computing Feynman diagrams by means of Euclidean propagators. Let us suppose that quantum fields in an interaction zone are really Euclidean. In contrast with the well-known Euclidean field theory dealing with the Wightman and Schwinger functions of free fields, we address complete Green’s functions of interacting fields, i.e., causal forms on the Borchers algebra of quantum fields. They are the Laplace transform of the Euclidean states obeying a certain condition. If Euclidean states of a quantum field system, e.g., quarks do not satisfy this condition, this system fails to possess Green’s functions and, consequently, the S-matrix. One therefore may conclude that it is not observed in the Minkowski space.

As is well known, the Wick rotation enables one to compute the Feynman diagrams of perturbative quantum field theory by means of Euclidean propagators. Let us suppose that it is not a technical trick, but quantum fields in an interaction zone are really Euclidean.

For the sake of simplicity, we here restrict our consideration to real scalar fields. One associates to them the Borchers algebra

$$A_{RS^4} = \mathbb{R} \oplus RS^4 \oplus RS^8 \oplus \cdots,$$

where $RS^4$ is the nuclear space of smooth real functions of rapid decrease on $\mathbb{R}^n$ [1, 2]. It is the real subspace of the space $S(\mathbb{R}^4)$ of smooth complex functions of rapid decrease on $\mathbb{R}^4$. Its topological dual is the space $S'(\mathbb{R}^4)$ of tempered distributions (generalized functions) [3, 4]. Any continuous positive form on the Borchers algebra $A_{RS^4}$ (1) is represented by a collection of tempered distributions $\{W_n \in S'(\mathbb{R}^{4n})\}$ such that

$$f(\psi_n) = \int W_n(x_1, \ldots, x_n)\psi_n(x_1, \ldots, x_n)d^4x_1 \cdots d^4x_n, \quad \psi_n \in RS^{4n}. \quad (2)$$

For instance, these are the Wightman functions describing free scalar quantum fields in the Minkowski space.

We address the causal forms $f^c$ on the Borchers algebra $A_{RS^4}$ (1) characterizing quantum fields created at some instant and annihilated at another one. They are given by the functionals

$$f^c(\psi_n) = \int W^c_n(x_1, \ldots, x_n)\psi_n(x_1, \ldots, x_n)d^4x_1 \cdots d^4x_n, \quad \psi_n \in RS^{4n}, \quad (3)$$

$$W^c_n(x_1, \ldots, x_n) = \sum_{(i_1 \ldots i_n)} \theta(x_{i_1}^0 - x_{i_2}^0) \cdots \theta(x_{i_{n-1}}^0 - x_{i_n}^0)W_n(x_1, \ldots, x_n), \quad (4)$$
where \( W_n \in S'(\mathbb{R}^4) \) are tempered distributions, \( \theta \) is the step function, and the sum runs through all permutations \((i_1 \ldots i_n)\) of the tuple of numbers \(1, \ldots, n\) \([5]\). A problem is that the functionals \( W_n^c \) \((4)\) need not be tempered distributions. For instance, \( W_1^c \in S'(\mathbb{R}) \) iff \( W_1 \in S'(\mathbb{R}_\infty) \), where \( \mathbb{R}_\infty \) is the compactification of \( \mathbb{R} \) by means of the point \( \{+\infty\} = \{-\infty\} \). Moreover, the causal forms are not positive. Therefore, they do not provide states of the Borchers algebra \( A_{RS^4} \) in general. At the same time, the causal forms come from the Laplace transformation of Euclidean states of the Borchers algebra \([6, 7]\). These Euclidean states however are not arbitrary but must take the form \((15)\). If Euclidean states of a quantum field system, e.g., quarks do not satisfy this condition, this system fails to possess Green’s functions and, consequently, the \( S \)-matrix. One therefore may conclude that it is not observed in the Minkowski space.

Note that, since the causal forms \((4)\) are symmetric, the Euclidean states of the Borchers algebra \( A_{RS^4} \) can be obtained as states of the corresponding commutative tensor algebra \( B_{RS^4} \) \([6, 7]\). Provided with the direct sum topology, \( B_{RS^4} \) becomes a topological involution algebra. It coincides with the enveloping algebra of the Lie algebra of the additive Lie group \( T_{RS^4} \) of translations in \( RS^4 \). Therefore, one can obtain the states of the algebra \( B_{RS^4} \) by constructing cyclic strongly continuous unitary representations of the nuclear Abelian group \( T_{RS^4} \). Such a representation is characterized by a positive-definite continuous generating function \( Z \) on \( RS^4 \) which is the Fourier transform of a bounded positive measure of total mass 1 on the space \( S'(\mathbb{R}^4) \) of generalized functions \([8]\). This generating function \( Z \) plays the role of a generating functional of Euclidean Green’s functions represented by a functional integral on \( S'(\mathbb{R}^4) \).

It should be emphasized that the above mentioned Euclidean states differ from the well-known Schwinger functions in the so called Euclidean field theory \([4, 9, 10]\). They are the Laplace transform of Wightman functions, and describe free Euclidean quantum fields (see Appendix). Note that the Euclidean counterpart of time ordered correlation functions is also considered in the Euclidean quantum field theory, but not by means of the Wick rotation \([11]\).

In order to describe the Wick rotation of Euclidean states, we start with the basic formulas of the Fourier–Laplace (henceforth FL) transformation \([4]\). It is defined on Schwartz distributions, but we focus on the tempered ones.

Recall that functions of rapid decrease on \( \mathbb{R}^n \) are complex smooth functions \( \psi(x) \) such that the quantities
\[
\|\psi\|_{k,m} = \max_{|\alpha| \leq k} \sup_x (1 + x^2)^m |D^\alpha \psi(x)|
\]
are finite for all \( k, m \in \mathbb{N} \). Here, we follow the standard notation
\[
D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n,
\]
for an \( n \)-tuple of natural numbers \( \alpha = (\alpha_1, \ldots, \alpha_n) \). The functions of rapid decrease constitute the nuclear space \( S(\mathbb{R}^n) \) with respect to the topology determined by the seminorms.
Its dual is the space $S'({\mathbb R}^n)$ of tempered distributions. The corresponding contraction form is written as

$$\langle \psi, h \rangle = \int \psi(x) h(x) d^n x,$$

$\psi \in S({\mathbb R}^n), \quad h \in S'({\mathbb R}^n).$

The space $S({\mathbb R}^n)$ is provided with the nondegenerate separately continuous Hermitian form

$$\langle \psi | \psi \rangle = \int \psi(x) \overline{\psi(x)} d^n x.$$

The completion of $S({\mathbb R}^n)$ with respect to this form is the space $L^2_{C}({\mathbb R}^n)$ of square integrable complex functions on $\mathbb{R}^n$. We have the rigged Hilbert space

$$S({\mathbb R}^n) \subset L^2_{C}({\mathbb R}^n) \subset S'({\mathbb R}^n).$$

Let $\mathbb{R}_+^n$ and $\mathbb{R}_+^n$ further denote the subset of points of $\mathbb{R}^n$ with strictly positive Cartesian coordinates and its closure, respectively. Let $f \in S'({\mathbb R}^n)$ be a tempered distribution and $\Gamma(f)$ the convex subset of points $q \in \mathbb{R}^n$ such that

$$e^{-qx} f(x) \in S'({\mathbb R}^n).$$

In particular, $0 \in \Gamma(f)$. Let $\text{Int} \, \Gamma(f)$ and $\partial \Gamma(f)$ denote the interior and boundary of $\Gamma(f)$, respectively. The FL transform of a tempered distribution $f \in S'({\mathbb R}^n)$ is defined as the tempered distribution

$$f^{FL}(p + iq) = (e^{-qx} f(x))^F(p) = \int f(x) e^{i(p + iq)x} d^n x \in S'({\mathbb R}^n),$$

which is the Fourier transform of the distribution (8) depending on $q$ as parameters. One can think of the FL transform (9) as being the Fourier transform with respect to the complex arguments $k = p + iq$.

If $\text{Int} \, \Gamma(f) \neq \emptyset$, the FL transform $f^{FL}(k)$ is a holomorphic function $h(k)$ of complex arguments $k = p + iq$ on the open tube $\mathbb{R}_+^n + i\text{Int} \, \Gamma(f) \subset \mathbb{C}_n$ over $\text{Int} \, \Gamma(f)$. Moreover, for
any compact subset $Q \subset \text{Int } \Gamma(f)$, there exist strictly positive numbers $A$ and $m$, depending on $Q$ and $f$, such that

$$|f^{FL}(p + iq)| \leq A(1 + |p|)^m, \quad p \in \mathbb{R}_n, \quad q \in Q.$$  \hspace{1cm} (10)

The evaluation (10) is equivalent to the fact that the function $h(p + iq)$ defines a family of tempered distributions $h_q(p) \in S'(\mathbb{R}_n)$ of the variables $p$ depending continuously on parameters $q \in S$. If $0 \in \text{Int } \Gamma(f)$, then

$$f^{FL}(p + i0) = \lim_{q \to 0} f^{FL}(p + iq)$$

coincides with the Fourier transform $f^F(p)$ of $f$. The case of $0 \not\in \text{Int } \Gamma(f)$ is more intricate. Let $S$ be a convex domain in $\mathbb{R}^n$ such that $0 \in \partial S$, and let $h(p + iq)$ be a holomorphic function on the tube $\mathbb{R}_n + iS$ which defines a family of tempered distributions $h_q(p) \in S'(\mathbb{R}_n)$, depending on parameters $q$. One says that $h(p + iq)$ has a generalized boundary value $h_0(p) \in S'(\mathbb{R}_n)$ if, for any frustum $K_r \subset S \cup \{0\}$ of the cone $K \subset \mathbb{R}_n$ (i.e., $K_r = \{q \in K : |q| \leq r\}$), one has

$$h_0(\psi(p)) = \lim_{|q| \to 0, q \in K_r \setminus \{0\}} h_q(\psi(p))$$

for all functions $\psi \in S(\mathbb{R}_n)$ of rapid decrease. Then the following holds [4].

**Theorem 1**: Let $f \in S'(\mathbb{R}^n)$, $\text{Int } \Gamma(f) \neq \emptyset$ and $0 \not\in \text{Int } \Gamma(f)$. A generalized boundary value of the FL transform $f^{FL}(k)$ in $S'(\mathbb{R}_n)$ exists and coincides with the Fourier transform $f^F(p)$ of the distribution $f$.

Let us apply this result to the following important case. The support of a tempered distribution $f$ is defined as the complement of the maximal open subset $U$ where $f$ vanishes, i.e., $f(\psi) = 0$ for all $\psi \in S(\mathbb{R}^n)$ of support in $U$. Let $f \in S'(\mathbb{R}^n)$ be of support in $\mathbb{R}^n_+$. Then $\mathbb{R}^n_+ \subset \Gamma(f)$, and the FL transform $f^{FL}$ is a holomorphic function on the tube over $\mathbb{R}^n_+$, while its generalized boundary value in $S'(\mathbb{R}^n_+)$ is given by the equality

$$h_0(\psi(p)) = \lim_{|q| \to 0, q \in \mathbb{R}^n_+} f^{FL}_q(\psi(p)) = f^F(\psi(p)), \quad \forall \psi \in S(\mathbb{R}_n).$$

Conversely, one can restore a tempered distribution $f$ of support in $\mathbb{R}^n_+$ from its FL transform $h(k) = f^{FL}(k)$ even if this function is known only on $i \mathbb{R}^n_+$. Indeed, the formulas

$$\tilde{h}(\phi) = \int_{\mathbb{R}^n_+} h(iq)\phi(q)d_+q = \int_{\mathbb{R}^n_+} d_+q \int_{\mathbb{R}^n_+} e^{-qx} f(x)\phi(q)d_+x =$$

$$\left. \int_{\mathbb{R}^n_+} f(x)\hat{\phi}(x)d_+x, \quad \phi \in S(\mathbb{R}^n_+), \right.$$

$$\hat{\phi}(x) = \int_{\mathbb{R}^n_+} e^{-qx}\phi(q)d_+q, \quad x \in \mathbb{R}^n_+, \quad \hat{\phi} \in S(\mathbb{R}^n_+),$$

\hspace{1cm} (11) \hspace{1cm} (12)
define a linear continuous functional \( \tilde{h}(q) = h(iq) \) on the space \( S(\mathbb{R}_{n+}) \). It is called the Laplace transform \( f^L(q) = f^{FL}(iq) \) of a tempered distribution \( f \).

**Remark 1:** Let us illustrate the restoration of a tempered distribution from the functional (11) in the case of \( n = 1 \). Let \( f \in S'(\mathbb{R}_+) \). Its FL transform reads

\[
\tilde{h}(p + iq) = \infty \int_0^\infty e^{i(p+iq)x} f(x) dx, \quad q > 0. \tag{13}
\]

Since \( \tilde{h}(p + iq) \) (13) has the generalized boundary value \( \tilde{h}(p + i0) \), the \( f \) is reconstructed from \( \tilde{h}(z) = h(iz) \) by the formulas

\[
f(x) = \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} e^{zx} \tilde{h}(z) dz, \quad \tilde{h}(0 - ip) = f^F(p), \tag{14}
\]

where

\[
\tilde{h}(q) = h(iq) = \infty \int_0^\infty e^{-qx} f(x) dx, \quad q > 0,
\]

is the Laplace transform \( f^L \) of \( f \).

The image of the space \( S(\mathbb{R}_{n+}) \) with respect to the mapping \( \phi(q) \mapsto \hat{\phi}(x) \) (12) is dense in \( S(\mathbb{R}_+) \). Then the family of seminorms \( \|\phi\|_{k,m} = \|\hat{\phi}\|_{k,m} \), where \( \|\cdot\|_{k,m} \) are seminorms (5) on \( S(\mathbb{R}^n) \), determines the new coarsen topology on \( S(\mathbb{R}_{n+}) \) such that the functional (11) remains continuous with respect to this topology. Then the following is proved [4].

**Theorem 2:** The mappings (11) and (12) provide one-to-one correspondence between the Laplace transforms \( f^L(q) = f^{FL}(iq) \) of tempered distributions \( f \in S'(\mathbb{R}_+) \) and the elements of \( S'(\mathbb{R}_{n+}) \) which are continuous with respect to the coarsen topology on \( S(\mathbb{R}_{n+}) \).

With Theorem 2, the above mentioned Wick rotation of Green’s functions of Euclidean quantum fields to causal forms in the Minkowski space is described as follows.

Let us denote by \( X \) the space \( \mathbb{R}^4 \) associated to the real subspace of \( \mathbb{C}^4 \) and by \( Y \) the space \( \mathbb{R}^4 \), coordinated by \((y^0, y^{1,2,3})\) and associated to the subspace \( \tilde{Y} \) of \( \mathbb{C}^4 \) whose points possess the coordinates \((iy^0, y^{1,2,3})\). If \( X \) is the Minkowski space, then one can think of \( Y \) as being its Euclidean partner. Since \( X \) and \( Y \) in \( \mathbb{C}^4 \) have the same spatial subspace, we further omit the dependence on spatial coordinates. Therefore, let us consider the complex plane \( \mathbb{C}^1 = X \oplus i\mathbb{Z} \) of the time \( x \) and the Euclidean time \( z \) and the complex plane \( \mathbb{C}_1 = P \oplus iQ \) of the associated momentum coordinates \( p \) and \( q \).

Let \( W(q) \in S'(Q) \) be a tempered distribution such that

\[
W = W_+ + W_-, \quad W_+ \in S'(\overline{Q}_+), \quad W_- \in S'(\overline{Q}_-). \tag{15}
\]
For instance, \( W(q) \) is an ordinary function at 0. For every test function \( \psi_+ \in S(X_+) \), we have

\[
\frac{1}{2\pi} \int_{Q_+} W(q) \hat{\psi}_+(q) dq = \frac{1}{2\pi} \int_{X_+} dx \left[ W(q) \exp(-qx) \psi_+(x) \right] =
\]

\[
\frac{1}{(2\pi)^2} \int_{Q_+} dq \int_{X_+} dp \int dx \left[ W(q) \psi_+(p) \exp\left(-ipx - qx\right) \right] =
\]

\[
\frac{-i}{(2\pi)^2} \int_{Q_+} dq \int_{X_+} dp \left[ W(q) \frac{\psi_+(p)}{p - i} \right] = \frac{1}{2\pi} \int_{Q_+} W(q) \psi_+^{FL}(iq) dq,
\]

(16)
due to the fact that the FL transform \( \psi_+^{FL}(p + iq) \) of the function \( \psi_+ \in S(X_+) \subset S'(X_+) \) exists and that it is holomorphic on the tube \( P + iQ_+, Q_+ \). Moreover, \( \psi_+^{FL}(p + i0) = \psi_+^{F}(p) \), and the function \( \hat{\psi}_+(q) = \psi_+^{FL}(iq) \) can be regarded as the Wick rotation of the test function \( \psi_+(x) \). The equality (16) can be brought into the form

\[
\frac{1}{2\pi} \int_{Q_+} W(q) \hat{\psi}_+(q) dq = \int_{X_+} \hat{W}_+(x) \psi_+(x) dx,
\]

(17)

\[
\hat{W}_+(x) = \frac{1}{2\pi} \int_{Q_+} \exp(-qx) W(q) dq, \quad x \in X_+.
\]

By virtue of Theorem 2, it associates to a distribution \( W(q) \in S'(Q) \) the distribution \( \hat{W}_+(x) \in S'(X_+) \), continuous with respect to the coarsen topology on \( S(X_+) \).

For every test function \( \psi_- \in S(X_-) \), the similar relations

\[
\frac{1}{2\pi} \int_{Q_-} W(q) \hat{\psi}_-(q) dq = \int_{X_-} \hat{W}_-(x) \psi_-(x) dx,
\]

(18)

\[
\hat{W}_-(x) = \frac{1}{2\pi} \int_{Q_-} \exp(-qx) W(q) dq, \quad x \in X_-,
\]

hold. Combining (17) and (18), we obtain

\[
\frac{1}{2\pi} \int_{Q} W(q) \hat{\psi}(q) dq = \int_{X} \hat{W}(x) \psi(x) dx,
\]

(19)

\[
\hat{\psi} = \hat{\psi}_+ + \hat{\psi}_-, \quad \psi = \psi_+ + \psi_-, 
\]

where \( \hat{W}(x) \) is a linear functional on functions \( \psi \in S(X) \), which together with all derivatives vanish at \( x = 0 \). One can think of \( \hat{W}(x) \) as being the Wick rotation of the distribution (15). One should additionally define \( \hat{W} \) at the point \( x = 0 \) in order to make it to a functional on
the whole space \( S(X) \). This is the well-known ambiguity of chronological forms in quantum field theory.

In particular, let a tempered distribution

\[
M(\phi_1, \phi_2) = \int W_2(x_1, x_2)\phi_1(x_1)\phi_2(x_2)d^n x_1 d^n x_2.
\]

(20)

be the Green’s function of some positive elliptic differential operator \( \mathcal{E} \), i.e.,

\[
\mathcal{E}_{y_1}W_2(y_1, y_2) = \delta(y_1 - y_2),
\]

where \( \delta \) is Dirac’s \( \delta \)-function. Then the distribution \( W_2 \) reads

\[
W_2(y_1, y_2) = w(y_1 - y_2),
\]

(21)

and we obtain the form

\[
F_2(\phi_1, \phi_2) = M(\phi_1, \phi_2) = \int w(y_1 - y_2)\phi_1(y_1)\phi_2(y_2)d^4 y_1 d^4 y_2 =
\]

\[
\int w(y)\phi_1(y_1)\phi_2(y_1 - y)d^4 y_1 = \int w(y)\varphi(y)d^4 y = \int w^F(q)\varphi^F(-q)d_4 q,
\]

\[
y = y_1 - y_2, \quad \varphi(y) = \int \phi_1(y_1)\phi_2(y_1 - y)d^4 y_1.
\]

For instance, if

\[
\mathcal{E}_{y_1} = -\Delta y_1 + m^2,
\]

where \( \Delta \) is the Laplacian, then

\[
w(y_1 - y_2) = \int \frac{\exp(-iq(y_1 - y_2))}{q^2 + m^2}d_4 q,
\]

(22)

where \( q^2 \) is the Euclidean square, is the propagator of a massive Euclidean scalar field. Let the Fourier transform \( w^F \) of the distribution \( w \) (21) satisfy the condition (15). Then its Wick rotation (19) is the functional

\[
\hat{w}(x) = \theta(x) \int_{Q_+} w^F(q)\exp(-qx)dq + \theta(-x) \int_{Q_-} w^F(q)\exp(-qx)dq
\]

on scalar fields in the Minkowski space. For instance, let \( w(y) \) be the Euclidean propagator (22) of a massive scalar field. Then due to the analyticity of

\[
w^F(q) = (q^2 + m^2)^{-1}
\]

on the domain \( \text{Im } q \cdot \text{Re } q > 0 \), one can show that \( \hat{w}(x) = -iD^c(x) \) where \( D^c(x) \) is a familiar causal Green’s function.
Appendix

Let us apply Theorem 2 to the relation between the Wightman and Schwinger functions in the Euclidean field theory in order to show the difference between Schwinger functions and the above mentioned states of Euclidean fields.

Recall that Wightman functions are defined as tempered distributions \( W_n \subset S'(\mathbb{R}^{4n}) \) in the Minkowski space which obey the Garding–Wightman axioms of axiomatic field theory [4, 9, 10]. Let us mention the Poincaré covariance axiom, the spectrum condition and the locality condition. Due to the translation covariance of Wightman functions \([4, 9, 10]\), let us mention the Poincaré covariance axiom, the spectrum condition and the locality condition. Due to the translation covariance of Wightman functions \(\tilde{W}_n\), there exist tempered distributions \(w_n \in S'(\mathbb{R}^{4n-4})\) such that

\[
W_n(x_1, \ldots, x_n) = w_n(x_1 - x_2, \ldots, x_{n-1} - x_n).
\] (23)

The spectrum condition implies that the Fourier transform \(w_n^F\) of the distributions \(w_n\) (23) is of support in the closed forward light cone \(\mathbf{\nabla}_+\) in the momentum Minkowski space \(\mathbb{R}_4\). It follows that the Wightman function \(w_n\) is a generalized boundary value in \(S'(\mathbb{R}^{4n-4})\) of the function \((w_n^F)^{FL}\), which is the FL transform of the function \(w_n^F\) with respect to variables \(p^0_k\) and which is holomorphic on the tube \((\mathbb{R}^4 + iV_-)^{n-1} \subset \mathbb{C}^4\). Accordingly, \(W_n(x_1, \ldots, x_n)\) is a generalized boundary value in \(S'(\mathbb{R}^4)\) of a function \(W_\eta(z_1, \ldots, z_n)\), holomorphic on the tube

\[
\{z_i : \text{Im} (z_{i+1} - z_i) \in V_-, \ \text{Re} z_i \in \mathbb{R}^4\}.
\]

In accordance with the Lorentz covariance, the Wightman functions admit an analytic continuation onto a wider domain in \(\mathbb{C}^4\), called the extended forward tube. Furthermore, the locality condition implies that they are symmetric on this domain.

Let \(X\) and \(Y\) be the Minkowski subspace and its Euclidean partner in \(\mathbb{C}^4\), respectively. Let us consider the subset \(\tilde{Y}_n^\# \subset \tilde{Y} \subset \mathbb{C}^4\) which consists of the points \((z_1, \ldots, z_n)\) such that \(z_i \neq z_j\). It belongs to the domain of analyticity of the Wightman function \(W_n(z_1, \ldots, z_n)\), whose restriction to \(\tilde{Y}_n^\#\) defines the symmetric function

\[
S_n(y_1, \ldots, y_n) = W_n(z_1, \ldots, z_n), \quad z_i = (iy^0_i, y^{1,2,3}_i),
\]
on \(Y_n^\#\). It is called the Schwinger function. On the domain \(Y_n^\#\) of points \((y_1, \ldots, y_n)\) such that \(0 < y^0_1 < \cdots < y^0_n\), the Schwinger function takes the form

\[
S_n(y_1, \ldots, y_n) = s_n(y_1 - y_2, \ldots, y_{n-1} - y_n),
\] (24)

where \(s_n\) is an element of the space \(S'(Y_n^{n-1})\) which is continuous with respect to the coarsened topology on \(S(Y_n^{n-1})\). Consequently, in accordance with Theorem 2 and by virtue of the formula (11), the Schwinger function \(s_n\) (24) can be represented as

\[
s_n(y_1 - y_2, \ldots, y_{n-1} - y_n) = \int \exp[p^0_j(y^0_j - y^0_{j+1}) - i \sum_{k=1}^3 p^k_j(y^k_j - y^k_{j+1})]w_n^F(p^1, \ldots, p^n)d_4p^1 \cdots d_4p^{n-1},
\] (25)
where \( w_n^F \in S'(\mathbb{R}_n^+) \) is the Fourier transform of the Wightman function \( w_n \), seen as an element of \( S'(\mathbb{R}_n^+) \) of support in the subset \( p_0^i \geq 0 \). The formula (25) enables one to restore the Wightman functions on the Minkowski from the Schwinger functions on the Euclidean space \([9, 10]\).

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