IRREDUCIBLE MODULES FOR SUPER-VIRASORO ALGEBRAS FROM ALGEBRAIC D-MODULES

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Abstract. In this paper, we introduce a new family of functors from the category of modules for the Weyl algebra to the category of modules for the super-Virasoro algebras. The properties of these functors are investigated, with an emphasis on irreducibility preservation and natural isomorphisms. By utilizing these functors, we recover some old irreducible super-Virasoro modules, including those from the irreducible intermediate series as well as irreducible $U(h)$-free modules. Additionally, we provide several families of new irreducible super-Virasoro modules via our constructed functors.

1. Introduction

The super-Virasoro algebras are crucial infinite-dimensional Lie superalgebras in theoretical physics. They serve as supersymmetric extensions of the ordinary Virasoro algebra and consist of two sectors: the Neveu-Schwarz sector $[19]$ and the Ramond sector $[20]$. The study of string theory and conformal field theory relies heavily on representations of the super-Virasoro algebra, as they describe the quantum states of a supersymmetric theory. The highest weight modules for the super-Virasoro algebras have been extensively studied in literature (see $[8,11,18]$, etc.). In $[9]$, researchers investigated Verma modules’ structure for the super-Virasoro algebras, constructing Bernstein-Gelfand-Gelfand type resolutions for irreducible highest weight modules. Additionally, Fock modules over super-Virasoro algebras were analyzed in $[10]$.

Recently, there have been several advancements in the study of representation theory. For example, the classification of simple Harish-Chandra modules, i.e., irreducible modules with finite-dimensional weight spaces, for the Neveu-Schwarz and Ramond algebras has been achieved in $[3,6,17]$. Every such module is either an irreducible highest or an irreducible lowest weight module, or an irreducible module of the intermediate series. Furthermore, in $[4]$, the concept of Whittaker modules for these algebra was introduced and analyzed. In addition to this, $[5,14]$ explored the study of simple restricted modules that generalize both highest weight and Whittaker modules for the Neveu-Schwarz and Ramond algebras. In $[21]$, the authors presented a family of non-weight modules for the super-Virasoro algebra, called $U(h)$-free modules. They demonstrated that these modules provide a complete

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classification of free $U(b)$-modules of rank 1 over the Ramond algebra. Moreover, they established that the same family of modules also constitutes a complete classification of free $U(b)$-modules of rank 2 over the Neveu-Schwarz algebra.

The theory of Weyl algebras, which are also known as algebras of differential operators, is an interesting and important subject. The study of algebraic D-modules, which are modules over Weyl algebras \([4]\), has a significant impact on various fields. Used the twisting technique, Lü and Zhao in \([15]\) created lots of new irreducible Virasoro modules by twisting irreducible modules over the Weyl algebra. For additional insights and further references, please refer to \([12]\) and \([16]\).

This paper aims to extend the twisting technique introduced in \([15]\) to the super-Virasoro algebra. We present a method for constructing new irreducible modules for the super-Virasoro algebras, based on irreducible modules for the Weyl algebra. We start by defining a functor $S$ that maps modules over the Weyl algebra $\mathcal{D}$ to modules over the Weyl superalgebra $S\mathcal{D}$. We prove that this functor preserves irreducibility and establishes a one-to-one correspondence between irreducible modules over the two algebras. Next, we introduce a family of functors $F_{\epsilon,b}$ that operate on modules for the Weyl superalgebra and yield modules for the centerless super-Virasoro algebra $g[\epsilon]$. These functors depend on parameters $\epsilon$ (which can be either 0 or $\frac{1}{2}$) and $b \in \mathbb{C}$. By composing these functors with $S$, we obtain a new family of functors $H_{\epsilon,b}$ that operate on modules for the Weyl algebra and produce modules for the super-Virasoro algebra. Using Block’s classification of irreducible Weyl modules in \([1]\), together with our newly defined functors, we establish that these functors preserve irreducibility for all values of $b$ except for a few cases which are also clearly described. We also determine necessary and sufficient conditions under which two such functors are naturally isomorphic. Our main results are presented below.

**Main theorem** [Theorem 4.5 and Theorem 4.8] Suppose that $\epsilon \in \{0, \frac{1}{2}\}$, $b, b_1, b_2 \in \mathbb{C}$. Then there exists a functor $H_{\epsilon,b} : \text{Mod} \mathcal{D} \to \text{Mod} g[\epsilon]$ with $M \mapsto H_{\epsilon,b}(M)$, where $H_{\epsilon,b}(M) = M \oplus M$ and is defined by the following actions:

\[
L_m \cdot v = -t^m \left( D + (m - 2\epsilon)b + \frac{m + 2\epsilon}{2} \theta \partial \theta \right) v,
\]

\[
G_p \cdot v = t^{p-\epsilon} \left( \theta D + 2(p - \epsilon) b \theta - t^{2\epsilon} \partial \theta \right) v,
\]

for $m \in \mathbb{Z}$, $p \in \mathbb{Z} + \epsilon$, and $v \in M \oplus M$. Furthermore, we show that

(i) $H_{\epsilon,b}$ preserves irreducible modules if and only if $b \notin \{0, \frac{1}{2}\}$.

(ii) $H_{\epsilon,b_1} \cong H_{\epsilon,b_2}$ if and only if $b_1 = b_2$. 

Applying these results, we recover some old irreducible super-Virasoro modules, including those from the irreducible intermediate series \([\mathbb{N}]\) as well as irreducible \(U(h)\)-free modules \([\mathbb{N}]\). Finally, we provide several examples of new irreducible super-Virasoro modules via our constructed functors.

Our approach for constructing modules for the super-Virasoro algebra can also be applied to the \(N = 2\) superconformal algebras. For more information on these modules, please refer to our forthcoming paper \([\mathbb{N}]\).

This paper is organized as follows: Section 2 provides a brief review of the definitions of super-Virasoro algebras, Weyl algebra, and Weyl superalgebra. In Section 3, we introduce a family of functors that map modules for the Weyl algebra to modules for the super-Virasoro algebras. Section 4 investigates properties of irreducibility-preserving and natural isomorphisms for these functors. Finally, in Section 5, we utilize these functors to construct several new irreducible modules for the super-Virasoro algebras.

Throughout this paper, \(\mathbb{C}, \mathbb{C}^*, \mathbb{Z}, \mathbb{Z}^*, \mathbb{N}\) and \(\mathbb{Z}_+\) will denote the sets of complex numbers, nonzero complex numbers, integers, nonzero integers, nonnegative integers, and positive integers, respectively. All vector spaces, vector algebras, and modules will be considered to be over the field of complex numbers \(\mathbb{C}\), and all modules for Lie superalgebras will be \(\mathbb{Z}_2\)-graded.

2. Preliminaries

In this section, we recall some notations, and collect the known facts about Lie superalgebras, the super-Virasoro algebras, and the Weyl (super)algebra.

2.1. Suppose that \(L\) is a Lie (super)algebra (or an associative (super)algebra), we shall denote the category of \(L\)-modules by \(\text{Mod} \ L\). We will also denote the parity change functor by \(\Pi\), which is defined as follows: \(\Pi(V) = V_\bar{1} \oplus V_\bar{0}\), where \((\Pi(V))_0 = V_1\) and \((\Pi(V))_1 = V_0\), where \(V = V_0 \oplus V_1\) is an \(L\)-module. For a Lie superalgebra \(L\), we will use \(U(L)\) to denote its universal enveloping algebra, and use \(\delta_{i,j}\) to denote the Kronecker delta.

**Definition 2.1.** Let \(M\) be a module of an associative (super)algebra \(L\). If \(0 \neq v \in M\) and some \(x \in L\), there exists \(n \in \mathbb{Z}_+\) such that \(x^n v = 0\), then we call that the action of \(x\) on \(v\) is nilpotent.

**Definition 2.2.** Suppose that \(L\) is an associative (super)algebra (or a Lie (super)algebra) and \(E\) is a subspace of \(L\). Let \(M\) be an \(L\)-module. If there exists \(0 \neq x \in E\) such that \(xv = 0\) for some \(0 \neq v \in M\), the \(M\) is called \(E\)-torsion; otherwise \(M\) is called \(E\)-torsion-free.

**Definition 2.3.** Let \(L\) be a Lie (super)algebra with a Cartan subalgebra \(H\). An \(L\)-module \(M\) is called a weight module if the action of \(H\) on \(M\) is diagonalizable.
2.2. The following notion of the super-Virasoro algebra is a natural supersymmetric generalization of the Virasoro algebra, which was first discovered by Neveu-Schwarz and Ramond, respectively.

**Definition 2.4** ([19], [20]). For $\epsilon \in \{0, \frac{1}{2}\}$, the super-Virasoro algebra $\hat{g}[\epsilon]$ is an infinite dimensional Lie superalgebra whose even part is spanned by $\{L_n, C \mid n \in \mathbb{Z}\}$ and odd part is spanned by $\{G_p \mid p \in \mathbb{Z} + \epsilon\}$ subject to the following relations

$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C,$

$[G_p, G_q] = 2L_{p+q} + \frac{1}{3}(p^2 - \frac{1}{4})\delta_{p+q,0}C,$

$[L_m, G_p] = \left(\frac{m}{2} - p\right)G_{m+p}, \quad [\hat{g}[\epsilon], C] = 0,$

where $m, n \in \mathbb{Z}$ and $p, q \in \mathbb{Z} + \epsilon$. $\hat{g}[\frac{1}{2}]$ and $\hat{g}[0]$ are called the Neveu-Schwarz algebra and the Ramond algebra, respectively.

The even component of $\hat{g}[\epsilon]$ is isomorphic to the Virasoro algebra, which is denoted by $\text{Vir}$. A $\hat{g}[\epsilon]$-module $W$ is said to have a central charge of $c$ if the operator $C$ acts on $W$ as a complex scalar $c$. In this paper, we exclusively focus on $\hat{g}[\epsilon]$-modules with central charge zero, for $\epsilon = 0, \frac{1}{2}$. Let

$\hat{g}[\epsilon] = \hat{g}[\epsilon]/\mathfrak{h},$

where $\mathfrak{h} = \mathbb{C}C$. The Cartan subalgebra of $\hat{g}[\epsilon]$ can be represented by $\mathfrak{h} = \mathbb{C}L_0$.

2.3. Denote by $\mathcal{A} = \mathbb{C}[t, t^{-1}, \theta]$ the associative superalgebra of Laurent polynomials associated to an even, $t$, and an odd, $\theta$, formal variable. Assume that $\theta^2 = 0$ and $\theta t = t \theta$. Let $\partial_t = \frac{d}{dt}$, $D = t\frac{d}{dt}$ and $\partial_{\theta} = \frac{d}{d\theta}$.

**Definition 2.5.** The Weyl algebra $\mathcal{D}$ is the associative algebra of regular differential operators on the circle $S^1$ generated by $t^{\pm 1}$ and $\partial_t$, subject to the relations $\partial_t t - t \partial_t = 1$.

All irreducible $\mathcal{D}$-modules have been classified by R.Block as follows:

**Theorem 2.6** ([15], [16]). Let $M$ be an irreducible $\mathcal{D}$-module.

1. If $M$ is $\mathbb{C}[t^{\pm 1}]$-torsion-free, then $M \cong \mathcal{D}/(\mathcal{D} \cap (\mathbb{C}(t)[D] \tau))$ for some irreducible element $\tau$ in the associative algebra $\mathbb{C}(t)[D]$;
2. If $M$ is $\mathbb{C}[t^{\pm 1}]$-torsion, then $M \cong \Omega(\lambda)$, where $\Omega(\lambda)$ is an irreducible $\mathcal{D}$-module with $\Omega(\lambda) = \mathbb{C}[D]$, for any $\lambda \in \mathbb{C}^*$, and

$t^n D^n = \lambda^n (D - m)^n, \quad D D^n = D^{n+1}, \quad \forall n \in \mathbb{N}, m \in \mathbb{Z}.$
Definition 2.7 (8)). The Weyl superalgebra $SD$ is the associative superalgebra of regular differential operators on the supercircle $S^{1|1}$ generated by $t^\pm 1$ and $\theta$, $\partial_t$, $\partial_\theta$, subject to the relations

$$\partial_t t - t \partial_t = 1, \quad \partial_\theta \theta + \theta \partial_\theta = 1, \quad \partial_t \theta = \partial_\theta t = 0.$$ 

Note that the following elements

$$t^k D^l, t^k D^l \theta \partial_\theta, t^k D^l \theta, \quad k \in \mathbb{Z}, l \in \mathbb{N}$$

form a linear basis of $SD$, where the odd elements $\theta$ and $\partial_\theta$ generate the four-dimensional Clifford superalgebra. Clearly $SD$ is isomorphic to the tensor algebra of the Weyl algebra $D$ and this Clifford superalgebra.

3. Constructions of functors

In this section, we shall construct a family of functors from the category of modules for the Weyl algebra $D$ to the category of modules for the super-Virasoro algebra $g[\epsilon]$.

3.1. From $D$-modules to $SD$-modules.

Lemma 3.1. Let $M$ be a $D$-module, and let $\overline{M}$ denote a copy of $M$ which we can turn into a $D$-module using the operation

$$x \cdot \overline{v} = \overline{xv}, \quad \forall x \in D, \overline{v} \in \overline{M}. \quad (3.1)$$

We define

$$\partial_\theta \cdot v = 0, \quad \theta \cdot v = \overline{v}, \quad \partial_\theta \cdot \overline{v} = v, \quad \theta \cdot \overline{v} = 0, \quad \forall v \in M. \quad (3.2)$$

Then $S(M) = M \oplus \overline{M}$ is an $SD$-module

Proof. It is easy to verify that $S(M) = M \oplus \overline{M}$ is an $SD$-module. \qed

Proposition 3.2. Let $\text{Mod } D$ denote the category of $D$-modules and $\text{Mod } SD$ denote the category of $SD$-modules. There exists a functor $S : \text{Mod } D \rightarrow \text{Mod } SD$ with $M \mapsto S(M)$, where $S(M) = M \oplus \overline{M}$ is defined by (3.1) and (3.2). Moreover, the functor $S$ preserves irreducible objects.

Proof. Suppose $M$ is an irreducible $D$-module. Using (3.1) and (3.2), we can define a functor $S$ such that $S(M) = M \oplus \overline{M}$. It follows that $S(M)$ is an irreducible $SD$-module. \qed

Proposition 3.3. There is a one-to-one correspondence between the irreducible $SD$-modules and irreducible $D$-modules.
Proof. Proposition 3.2 states that if $M$ is an irreducible $\mathcal{D}$-module, then $S(M)$ is an irreducible $\mathcal{SD}$-module. Conversely, if $V$ is an irreducible $\mathcal{SD}$-module, then $V$ is also a $\mathcal{D}$-module. Let $M$ be an irreducible $\mathcal{D}$-submodule of $V$. Using $\mathcal{SD} = \mathbb{C}[\theta, \partial_\theta] \mathcal{D}$, we have $V = \mathbb{C}[\theta, \partial_\theta] M$. Note that $\partial_\theta M$ is also a $\mathcal{D}$-module. If $\partial_\theta M \neq 0$, we can replace $M$ with $\partial_\theta M$ and assume that $\partial_\theta M = 0$. Therefore, $V = M \oplus \overline{M} = S(M)$. \hfill \Box

3.2. From $\mathcal{SD}$-modules to $\mathfrak{g}[\epsilon]$-modules.

Proposition 3.4. Let $V$ be an $\mathcal{SD}$-module and $b \in \mathbb{C}$. Then $V$ is a $\mathfrak{g}[\epsilon]$-module with

\begin{align}
L_m \cdot v &= -t^m \left( D + (m - 2\epsilon)b + \frac{m + 2\epsilon}{2} \theta \partial_\theta \right) v, \\
G_p \cdot v &= t^{p-\epsilon} \left( \theta D + 2(p - \epsilon)b \theta - t^{2\epsilon} \partial_\theta \right) v,
\end{align}

for $m \in \mathbb{Z}$, $p \in \mathbb{Z} + \epsilon$, and $v \in V$. We denote this $\mathfrak{g}[\epsilon]$-module by $F_{\epsilon,b}(V)$.

Proof. Note that $\mathfrak{g}[\epsilon]$ can be embedded into the Lie superalgebra $\mathcal{SD}^-$, which is associated with the associative superalgebra $\mathcal{SD}$, by the following linear map $\Phi : \mathfrak{g}[\epsilon] \hookrightarrow \mathcal{SD}^-$ with

\begin{align}
L_m &\mapsto -t^m D - \frac{m + 2\epsilon}{2} t^m \theta \partial_\theta, \\
G_p &\mapsto t^{p-\epsilon} \theta D - t^{p+\epsilon} \partial_\theta,
\end{align}

where $m \in \mathbb{Z}$, $p \in \mathbb{Z} + \epsilon$. Then $\mathcal{A}$ is a natural $\mathfrak{g}[\epsilon]$-module. Consider the extended super-virasoro algebra $\tilde{\mathfrak{g}}[\epsilon] = \mathfrak{g}[\epsilon] \ltimes \mathcal{A}$ with the brackets

$$[x, f] = -(-1)^{|x||f|}[f, x] = x(f), \quad [f, g] = 0, \quad \forall x \in \mathfrak{g}[\epsilon], f, g \in \mathcal{A}.$$ 

For $\epsilon \in \{0, \frac{1}{2}\}$, $b \in \mathbb{C}$, we define a linear map $\sigma_{\epsilon,b} : \tilde{\mathfrak{g}}[\epsilon] \to \tilde{\mathfrak{g}}[\epsilon]$ by

\begin{align}
\sigma_{\epsilon,b}(L_m) &= -t^m D - \frac{m + 2\epsilon}{2} t^m \theta \partial_\theta - (m - 2\epsilon)b t^m, \\
\sigma_{\epsilon,b}(G_p) &= t^{p-\epsilon} \theta D - t^{p+\epsilon} \partial_\theta + 2b(p - \epsilon)t^{p-\epsilon} \theta, \\
\sigma_{\epsilon,b}(t^m) &= t^m, \\
\sigma_{\epsilon,b}(t^m \theta) &= t^m \theta,
\end{align}

for $m \in \mathbb{Z}$, $p \in \mathbb{Z} + \epsilon$. It is clear that $\sigma_{\epsilon,b}$ is an isomorphism of $\tilde{\mathfrak{g}}[\epsilon]$.

Let $V$ be an $\mathcal{SD}$-module. Then $V$ is a natural $\mathcal{SD}^-$-module. It is clear that $V$ can be seen as a $\tilde{\mathfrak{g}}[\epsilon]$-module via the isomorphism $\Phi$. We can define a new action of $\tilde{\mathfrak{g}}[\epsilon]$ on $V$ as follows

$$x \cdot v = \sigma_{\epsilon,b}(x)v, \quad \forall x \in \tilde{\mathfrak{g}}[\epsilon], v \in V.$$
We denote this new module by $F_{\epsilon,b}(V)$, is called the twisted module of $V$ by $\sigma_{\epsilon,b}$. It is clear that the module $F_{\epsilon,b}(V)$ can be seen as a $\mathfrak{g}[(\theta \partial_{\theta})]$ module by restriction to the subalgebra $\mathfrak{g}[(\theta \partial_{\theta})]$ of $\mathfrak{g}[(\epsilon)]$. It follows that

\[ L_m \cdot v = \sigma_{\epsilon,b}(L_m)v = -t^m \left( D + (m - 2\epsilon)b + \frac{m + 2\epsilon}{2} \theta \partial_{\theta} \right) v, \]

\[ G_p \cdot v = \sigma_{\epsilon,b}(G_p)v = t^p \left( D + 2(p - \epsilon)b\theta - t^2 \epsilon \partial_{\theta} \right) v \]

for $m \in \mathbb{Z}$, $p \in \mathbb{Z} + \epsilon$, $v \in V$. □

**Theorem 3.5.** For $b \in \mathbb{C}$, there exists a functor

\[ F_{\epsilon,b} : \text{Mod} \mathcal{S} \mathcal{D} \rightarrow \text{Mod} \mathfrak{g}[(\epsilon)] \]

with $V \mapsto F_{\epsilon,b}(V)$, where $F_{\epsilon,b}(V)$ is defined by (3.3)-(3.7).

**Proof.** The sequence of Lie superalgebras

\[ \mathfrak{g}[(\epsilon)] \hookrightarrow \tilde{\mathfrak{g}}[(\epsilon)] \twoheadrightarrow \mathfrak{g}[(\epsilon)] \hookrightarrow \mathcal{S} \mathcal{D}^- \]

induces a sequence of categories of modules for the Lie superalgebras

\[ \text{Mod} \mathcal{S} \mathcal{D}^- \rightarrow \text{Mod} \tilde{\mathfrak{g}}[(\epsilon)] \rightarrow \text{Mod} \mathfrak{g}[(\epsilon)] \rightarrow \text{Mod} \mathfrak{g}[(\epsilon)]. \]

It is clear that each $\mathcal{S} \mathcal{D}$ serves as a $\mathcal{S} \mathcal{D}^-$-module. It follows from Proposition [3.5.4] that the functor $F_{\epsilon,b}$ can be written as the composition of the following four functors as follows:

\[ F_{\epsilon,b} : \text{Mod} \mathcal{S} \mathcal{D} \rightarrow \text{Mod} \mathcal{S} \mathcal{D}^- \rightarrow \text{Mod} \tilde{\mathfrak{g}}[(\epsilon)] \rightarrow \text{Mod} \mathfrak{g}[(\epsilon)] \rightarrow \text{Mod} \mathfrak{g}[(\epsilon)]. \]

□

The main result in this section can be derived by utilizing Proposition [3.5.4] and Theorem [3.5.3].

**Theorem 3.6.** For $b \in \mathbb{C}$, there exists a functor

\[ H_{\epsilon,b} : \text{Mod} \mathcal{D} \rightarrow \text{Mod} \mathfrak{g}[(\epsilon)] \]

with $M \mapsto H_{\epsilon,b}(M)$, where $H_{\epsilon,b}(M) = (F_{\epsilon,b} \circ \mathcal{S})(M) = F_{\epsilon,b}(\mathcal{S}(M))$.

4. Properties of functors $H_{\epsilon,b}$

This section aims to show that, with the exception of $b \in \{0, \frac{1}{2}\}$, the functor $H_{\epsilon,b}$ (as stated in Theorem [3.5.6]) preserves irreducible objects. Additionally, we will investigate the natural isomorphisms between the functors $H_{\epsilon,b}$ for $b \in \mathbb{C}$.
4.1. Preservation of irreducibility.

Lemma 4.1. Let \( V \) be an \( S\mathfrak{d} \)-module, and \( b \in \mathbb{C}, b \neq 0, 1/2 \). For any \( p \in \mathbb{Z} + \epsilon \) and \( d \in \mathbb{Z}^* \), we define \( T_{p,d} \) as follows:

\[
T_{p,d} = \frac{1}{2d^2}(L_{-d} \cdot G_{p+d} + L_d \cdot G_{p-d} - 2L_0 \cdot G_p)
\]

in \( U(\mathfrak{g}[\epsilon]) \). Then, for all \( v \in V \), we have:

\[
\frac{1}{b(1-2b)} T_{p,d} \cdot v = -t^{p-\epsilon} \theta v.
\]

Proof. For \( n \in \mathbb{Z}, p \in \mathbb{Z} + \epsilon \) and \( v \in V \), we have

\[
L_n \cdot G_{p-n} \cdot v
\]

\[
= -(t^p D + (n - 2\epsilon)bt^n + \frac{n + 2\epsilon}{2} t^p \theta \partial_\theta)((t^{p-n-\epsilon} \theta D + 2(p - n - \epsilon)bt^{p-n-\epsilon} \theta - t^{p-n+\epsilon} \partial_\theta)v
\]

\[
= -(t^{p-\epsilon} \theta D^2 + (p + 2b(p - 2\epsilon))t^{p-\epsilon} \theta D - t^{p+\epsilon} \theta \partial_\theta + (p - 2b(p - 2\epsilon))t^{p-\epsilon} \theta - (p + (1 - 2b)\epsilon)t^{p+\epsilon} \partial_\theta + (1 - 2b)n^2 t^{p-\epsilon} \theta))v
\]

Choosing \( n = -d, 0, d \) in above equation, respectively, we obtain that

\[
\frac{1}{b(1-2b)} T_{p,d} \cdot v = -t^{p-\epsilon} \theta v.
\]

\[\square\]

Lemma 4.2. Let \( V \) be an irreducible module over the Weyl superalgebra \( S\mathfrak{d} \) and \( b \in \mathbb{C} \), then the \( \mathfrak{g}[\epsilon] \)-module \( F_{e,b}(V) \) is irreducible if \( b \notin \{0, \frac{1}{2}\} \).

Proof. Suppose that \( b \neq 0, \frac{1}{2} \). Choose a nonzero element \( v \) from \( V_0 \). For any \( p \in \mathbb{Z} + \epsilon \), and nonzero integer \( d \), we can use the linear operator \( T_{p,d} \) defined in Lemma 4.1 to obtain the following results:

\[
- \frac{1}{b(1-2b)} T_{p,d} \cdot v = t^{p-\epsilon} \theta v \in U(\mathfrak{g}[\epsilon])v, \tag{4.1}
\]

\[
\frac{1}{b(1-2b)} G_{-\epsilon} \cdot T_{p,d} \cdot v = t^{p-\epsilon} \theta v \in U(\mathfrak{g}[\epsilon])v. \tag{4.2}
\]

For \( n \in \mathbb{N} \), one has

\[
(-L_0 + 2be)^n \cdot v = D^n v \in U(\mathfrak{g}[\epsilon])v. \tag{4.3}
\]

From (4.1)-(4.3), we can conclude that \( U(\mathfrak{g}[\epsilon])v = F_{e,b}(V) \). This implies that \( F_{e,b}(V) \) is an irreducible \( \mathfrak{g}[\epsilon] \)-module. \[\square\]
Lemma 4.3. Let $M$ be an irreducible module over the Weyl algebra $\mathcal{D}$.

(i) The $\mathfrak{g}[\epsilon]$-module $H_{\epsilon,0}(M)$ is reducible if and only if $M$ is isomorphic to the $\mathcal{D}$-module $\mathbb{C}[t^{\pm 1}]$.

(ii) If $M = \mathbb{C}[t^{\pm 1}]$, then $H_{\epsilon,0}(\mathbb{C}[t^{\pm 1}])$ has a maximum submodule $\mathbb{C}$. Consequently, the factor module $H_{\epsilon,0}(\mathbb{C}[t^{\pm 1}])/\mathbb{C}$ is an irreducible $\mathfrak{g}[\epsilon]$-module.

Proof. (i) Let $\bar{v}$ be any nonzero element in $\overline{M}$. Now we consider the action of $D$ on $\bar{v}$.

Case 1. The action of $D$ on $\bar{v}$ is nilpotent.

It is clear that there exists a nonzero element $\bar{v} \in \overline{M}$ such that $D\bar{v} = 0$. This implies that $D(t^n\bar{v}) = k t^n\bar{v}$. Therefore, we can conclude that $V$ is isomorphic to the $\mathcal{S}\mathcal{D}$-module $\mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}[t^{\pm 1}]$. It can be easily verified that the $\mathfrak{g}[\epsilon]$-module $H_{\epsilon,0}(M)$ has a one-dimensional submodule $\mathbb{C}$, and hence, it is reducible.

Case 2. The action of $D$ on $\bar{v}$ is non-nilpotent.

We can choose a nonzero $\bar{w} = D^n\bar{v} \in \overline{M}$, where $k \in \mathbb{Z}_+$. For $m \in \mathbb{Z}$, $d \in \mathbb{Z}^+$, we define

$$Q_{m,d} = \frac{2}{d^2}(L_{-d} \cdot L_{m+d} + L_d \cdot L_{m-d} - 2L_0 \cdot L_m).$$

By the similar calculation in Lemma [1.1], we can get

$$Q_{m,d} \cdot \bar{w} = t^n\bar{w} \in U(\mathfrak{g}[\epsilon])v,$$

where $m \in \mathbb{Z}$, $d \in \mathbb{Z}^+$. Using this, one gets

$$-G_{-\epsilon} \cdot (t^n\bar{w}) = t^n\bar{w} \in U(\mathfrak{g}[\epsilon])v.$$

For $n \in \mathbb{N}$, one has

$$(L_0 - \epsilon)^n \cdot \bar{w} = D^n\bar{w} \in U(\mathfrak{g}[\epsilon])v.$$

Thus, $H_{\epsilon,0}(M) = U(\mathfrak{g}[\epsilon])v$. In other words, $H_{\epsilon,0}(M)$ is an irreducible $\mathfrak{g}[\epsilon]$-module in this case.

(ii) It is clear that $(\mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}[t^{\pm 1}])/\mathbb{C} = \text{span}_\mathbb{C}\{t^n, t^k | n \in \mathbb{Z}^+, k \in \mathbb{Z}\}$. For $n \in \mathbb{Z}^+, k \in \mathbb{Z}$, $m \in \mathbb{Z}$, we have

$$L_m \cdot t^n = -(t^nD)t^n = -nt^{m+n},$$

$$L_m \cdot t^k = -(t^mD + \frac{m+2\epsilon}{2}t^m \partial_\theta)t^k = -\left(k + \frac{m+2\epsilon}{2}\right)t^{m+k},$$

$$G_p \cdot t^n = (t^{p-\epsilon}D - t^{p+\epsilon} \partial_\theta)t^n = nt^{p-\epsilon+n},$$

$$G_p \cdot t^k = (t^{p-\epsilon}D - t^{p+\epsilon} \partial_\theta)t^k = -t^{p+\epsilon+k}.$$

This shows that $(\mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}[t^{\pm 1}])/\mathbb{C}$ is an irreducible $\mathfrak{g}[\epsilon]$-module. □
Suppose that $M$ is an irreducible $\mathcal{D}$-module, we define

$$H'_{\epsilon,0}(M) = \begin{cases} H_{\epsilon,0}(M)/\mathbb{C}, & \text{if } M \cong \mathbb{C}[t^{\pm 1}]; \\ H_{\epsilon,0}(M), & \text{otherwise.} \end{cases}$$

**Lemma 4.4.** Suppose that $M$ is an irreducible $\mathcal{D}$-module.

(i) The module $H'_{\epsilon,\frac{1}{2}}(M)$ is an irreducible $\mathfrak{g}[\epsilon]$-module if and only if $D(M) = M$.

(ii) If $D(M) = M$, then $H'_{\epsilon,\frac{1}{2}}(M) \cong \Pi \circ H'_{\epsilon,0}(M)$, where $\Pi$ is the parity change functor for $\mathfrak{g}[\epsilon]$-modules.

**Proof.** (i) Consider $M \oplus \overline{M}$ as a $\mathbb{Z}_2$-graded modules over $\mathcal{D}$. Let $v \in M$ and $\overline{v} \in \overline{M}$. For any $m \in \mathbb{Z}$ and $p \in \mathbb{Z} + \epsilon$, $\mathfrak{g}[\epsilon]$-module $H'_{\epsilon,\frac{1}{2}}(M)$ is defined by

$$L_m \cdot v = -\left(t^m D + \frac{m - 2\epsilon}{2} t^m\right)v,$$

$$L_m \cdot \overline{v} = -(t^m D + mt^m) \cdot \overline{v} = -\overline{D t^m v},$$

$$G_p \cdot v = (t^p - \epsilon D \theta + (p - \epsilon)t^p \theta) \cdot v = \overline{D t^p \epsilon v},$$

$$G_p \cdot \overline{v} = -t^{p+\epsilon} v.$$

We see that $(H'_{\epsilon,\frac{1}{2}}(M))_0 \oplus D(H'_{\epsilon,\frac{1}{2}}(M))_1$ is a $\mathfrak{g}[\epsilon]$-submodule, namely, $M = D(M)$. For $k \in \mathbb{Z}$, $p \in \mathbb{Z} + \epsilon$ and $v \in M$, we have

$$(G_p G_{k-p} - G_{p-1} G_{k-p+1}) \cdot v = t^k v \in U(\mathfrak{g}[\epsilon])v, \quad G_p \cdot v = \overline{D t^p \epsilon} v \in U(\mathfrak{g}[\epsilon])v.$$

For any $i \in \mathbb{N}$ and $v \in M$, we have $(-L_0 + \epsilon)^i \cdot v = D^i v \in U(\mathfrak{g}[\epsilon])v$. Evidently, $(H'_{\epsilon,\frac{1}{2}}(M))_0 \oplus D(H'_{\epsilon,\frac{1}{2}}(M))_1$ is an irreducible $\mathfrak{g}[\epsilon]$-submodule of $H'_{\epsilon,\frac{1}{2}}(M)$.

(ii) Now we define the following linear map

$$\phi : (H'_{\epsilon,\frac{1}{2}}(M))_0 \oplus D(H'_{\epsilon,\frac{1}{2}}(M))_1 \longrightarrow \Pi(H_{\epsilon,0}(M))$$

$$t^2 \epsilon v \mapsto \overline{v}, \quad \overline{D v} \mapsto v,$$

where $v \in M, \overline{v} \in \overline{M}$. For $m \in \mathbb{Z}$, one can check that $\phi$ is a $\mathfrak{g}[\epsilon]$-module isomorphism.

If $H'_{\epsilon,0}(M)$ is reducible, from Case 1 in Lemma 4.3(i), one gets

$$H'_{\epsilon,0}(M) = (\mathbb{C}[t^{\pm 1}] \oplus \overline{\mathbb{C}[t^{\pm 1}]})/\mathbb{C}. $$
We see that the following linear map $\psi$ is a $g\lbrack \epsilon \rbrack$-module isomorphism, where
\[
\psi : (H_{\epsilon,1/2}(M))_0 \oplus D(H_{\epsilon,1/2}(M))_1 \longrightarrow H'_{\epsilon,0}(M)
\]
\[
-\frac{m+2\epsilon}{2} \mapsto \frac{1}{\tilde{t}^m}
\]
\[
Dt^n \mapsto t^n,
\]
for $m \in \mathbb{Z}, n \in \mathbb{Z}^*$.

\[\square\]

Based on Proposition 3.3 and Lemmas 4.2-4.4, we can derive the following theorem.

**Theorem 4.5.** Suppose that $M$ is an irreducible $D$-module. Then $H_{\epsilon,b}(M)$ is irreducible if and only if one of the following holds:
(i) $b \notin \{0, \frac{1}{2}\}$;
(ii) $b = 0$ and $M \cong \mathbb{C}[t^{\pm 1}]$;
(iii) $b = \frac{1}{2}$ and $M = D(M)$.

4.2. Natural isomorphisms.

**Lemma 4.6.** Let $b_1, b_2 \in \mathbb{C}, b_1 \notin \{0, \frac{1}{2}\}$ and $M_1, M_2$ be irreducible $D$-modules. Then $H_{\epsilon,b_1}(M_1) \cong H_{\epsilon,b_2}(M_2)$, as $g\lbrack \epsilon \rbrack$-modules, if and only if $b_1 = b_2$ and $M_1 \cong M_2$, as $D$-modules.

**Proof.** The sufficiency is evident, so we only need to provide a proof for the necessity. Suppose that $\psi : H_{\epsilon,b_1}(M_1) \rightarrow H_{\epsilon,b_2}(M_2)$ is a $g\lbrack \epsilon \rbrack$-module isomorphism. For any $p \in \mathbb{Z} + \epsilon$, $d \in \mathbb{Z}^*$, and $0 \neq v \in M_1$, we have $\psi(T_{p,d} \cdot v) = T_{p,d} \cdot \psi(v)$ by applying Lemma 4.1. This, together with the same lemma, yields
\[
b_1(1 - 2b_1)\psi(t^{p-\epsilon} \cdot \partial \cdot v) = b_2(1 - 2b_2)t^{p-\epsilon} \cdot \partial \cdot v. \tag{4.4}
\]
Note that $b_1 \neq 0$ and $b_1 \neq \frac{1}{2}$. By Lemma 3.1, we see that the action of $\partial_\theta$ on $v \in M_1$ is trivial. Then, using the fact that
\[
G_{-\epsilon} \cdot \psi \left( \frac{1}{b_2(1 - 2b_2)} t^{p-\epsilon} \right) = \psi \left( \frac{1}{b_2(1 - 2b_2)} G_{-\epsilon} \cdot t^{p-\epsilon} \right),
\]
we get
\[
\frac{1}{b_2(1 - 2b_2)} \psi(t^{p-\epsilon} \cdot v) = \frac{1}{b_1(1 - 2b_1)} t^{p-\epsilon} \psi(v). \tag{4.5}
\]
Setting $p = \epsilon$ in (4.5), we obtain
\[
b_1(1 - 2b_1) = b_2(1 - 2b_2). \tag{4.6}
\]
Substituting (4.6) into (4.4) and (4.5), we immediately obtain
\[
\psi(t^{p-\epsilon} \cdot v) = t^{p-\epsilon} \psi(v) \quad \text{and} \quad \psi(t^{p-\epsilon} \cdot v) = t^{p-\epsilon} \psi(v),
\]
where $p \in \mathbb{Z} + \epsilon$. For any $i \in \mathbb{N}$ and $v \in V$, we have
\[
\psi((-L_0 + 2eb)^i \cdot v) = (-L_0 + 2eb)^i \cdot \psi(v),
\]
which implies $\psi(D^i v) = D^i \psi(v)$. Therefore, we conclude that $M_1 \cong M_2$.

Now, for any $m \in \mathbb{Z}$ and $0 \neq v \in M_1$, from $\psi(L_m \cdot v) = L_m \cdot \psi(v)$, it is easy to check that $b_1 = b_2$. □

**Lemma 4.7.** Suppose that $M_1$ and $M_2$ are irreducible $\mathcal{D}$-modules. Then $H_{\epsilon,0}(M_1) \cong H_{\epsilon,0}(M_2)$, as $g[\epsilon]$-modules, if and only if $M_1 \cong M_2$, as $\mathcal{D}$-modules.

**Proof.** The sufficiency of the conditions is clear. Suppose that $\psi : H_{\epsilon,0}(M_1) \to H_{\epsilon,0}(M_2)$ is an isomorphism. For any $\overline{v} \in M_1$, we note that $(-L_0 - \epsilon) \cdot \overline{v} = D\overline{v}$. We consider the action of $D - k$ on $\overline{v}$.

If $(D - k) \cdot \overline{v} = 0$ for some $k \in \mathbb{Z}$ and a nonzero $\overline{v} \in M_1$, then $D^{t^{-k}v} = 0$ where $t^{-k}v \neq 0$. According to Case 1 in the proof of Lemma 4.3, we know that the $\mathcal{S}\mathcal{D}$-module $M_1 \oplus M_1 \cong C[t^{\pm 1}] \oplus C[t^{\pm 1}]$. Similarly, using $(D - k)\psi(\overline{v}) = 0$ in $M_2$, we deduce that $M_2 \oplus M_2 \cong C[t^{\pm 1}] \oplus C[t^{\pm 1}]$. Thus $M_1 \cong M_2$.

Consider that $D - k$ is injective on both $M_1$ and $M_2$ for all $k \in \mathbb{Z}$. By Lemma 3.4, we see that any $v \in M_1$ can be annihilated by $\partial_0$. For any $v \in M_1$, $p \in \mathbb{Z} + \epsilon$, we have
\[
\psi(G_p \cdot v) = \psi(D - p + \epsilon)\overline{v} = (D - p + \epsilon)\overline{v} = (D - p + \epsilon)\overline{v}.
\]
Inserting (4.7) into $\psi(G_p \cdot v) = G_p \cdot \psi(v)$, it is easy to check $\psi(t^{\pm \epsilon}v) = t^{\pm \epsilon}\psi(v)$ for $p \in \mathbb{Z} + \epsilon$.

Then by $\psi(G_{-p} \cdot (t^{\pm \epsilon}v)) = G_{-p} \cdot (t^{\pm \epsilon}\psi(v))$, we see that $\psi(t^{\pm \epsilon}v) = t^{\pm \epsilon}\psi(v)$ for all $v \in M_1$ and $p \in \mathbb{Z} + \epsilon$. For $i \in \mathbb{N}, v \in M_1$, from $\psi(L_0^i \cdot v) = L_0^i \cdot \psi(v)$, one has $\psi(D^i v) = D^i \psi(v)$. Thus $M_1 \cong M_2$ in this case. □

The isomorphism theorem can be obtained from Lemmas 4.1, 4.4, and 4.7.

**Theorem 4.8.** Suppose that $b_1, b_2 \in \mathbb{C}$, $M_1, M_2$ are irreducible $\mathcal{D}$-modules. Then $H_{\epsilon,b_1}(M_1) \cong H_{\epsilon,b_2}(M_2)$ as $g[\epsilon]$-modules if and only if one of the following holds:

(i) $b_1 = b_2$, $M_1 \cong M_2$;

(ii) $(b_1, b_2) = (\frac{1}{2}, 0)$, $M_1 \cong M_2$, and $M_1 = D(M_1)$;

(iii) $(b_1, b_2) = (0, \frac{1}{2})$, $M_1 \cong M_2$, and $M_2 = D(M_2)$.

**Corollary 4.9.** Let $b_1, b_2 \in \mathbb{C}$. Then $H_{\epsilon,b_1} \cong H_{\epsilon,b_2}$ if and only if $b_1 = b_2$.

5. Applications and examples

In this section, we utilize the functors $H_{\epsilon,b}$ to recover several known irreducible $g[\epsilon]$-modules. Furthermore, we construct lots of new irreducible $g[\epsilon]$-modules.
5.1. Intermediate series modules. Let $\alpha \in \mathbb{C}[t^{\pm 1}]$. Set $\tau = D - \alpha$ in Theorem 4.2.1, we get the irreducible $D$-module $M_\alpha = D / D_\tau$ with a basis $\{t^k \mid k \in \mathbb{Z}\}$, with
\[ D \cdot t^n = t^n(\alpha + n), \quad t^n \cdot t^m = t^{m+n}, \quad \forall m, n \in \mathbb{Z}. \]

For $\epsilon \in \{0, \frac{1}{2}\}, b \in \mathbb{C}$, we obtain $\mathfrak{g}[\epsilon]$-module $M_{\epsilon, \alpha, b} = H_{\epsilon, b}(\mathbb{C}[t^{\pm 1}]) = \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}[t^{\pm 1}]$ with the following actions:
\[ L_m \cdot t^n = - (\alpha + n + (m - 2\epsilon)b)t^{m+n}, \]
\[ L_m \cdot \overline{t^n} = - \left(\alpha + n + m(b + \frac{1}{2}) + \epsilon(1 - 2b)\right)t^{m+n}, \]
\[ G_\rho \cdot t^n = (\alpha + n + 2(p - \epsilon)b)t^{p+n}, \]
\[ G_\rho \cdot \overline{t^n} = - t^{p+n}, \]
for $m, n \in \mathbb{Z}, p \in \mathbb{Z} + \epsilon$.

Especially, if $\alpha \in \mathbb{C}$, then $M_{\epsilon, \alpha, b}$ is isomorphic to the intermediate series weight modules of $\mathfrak{g}[\epsilon]$ in 4.7. Moreover, we have

**Corollary 5.1** ([4.7]). $M_{\epsilon, \alpha, b}$ is irreducible except that $\alpha \in \mathbb{Z} + \epsilon, b = 0$ or $\alpha \in \mathbb{Z}, b = \frac{1}{2}$.

**Remark 5.2.** Suppose that $\alpha \in \mathbb{C}[t^{\pm 1}] \setminus \mathbb{C}$. It is clear that $M_{\epsilon, \alpha, b}$ is neither a weight module nor a restricted modules for $\mathfrak{g}[\epsilon]$. In particular, $M_{\epsilon, \alpha, b}$ is irreducible if and only if $b \neq \frac{1}{2}$.

5.2. $U(\mathfrak{h})$-free modules. For $\lambda \in \mathbb{C}^*$, we introduce the irreducible $D$-module $\Omega(\lambda)$ defined in Theorem 4.2.1, which has a basis $\{D^n \mid n \in \mathbb{N}\}$, and the $D$-actions are defined as follows
\[ t^n \cdot (-D)^m = \lambda^n(-D + m)^n, \quad D \cdot D^n = D^{n+1} \]
for $m \in \mathbb{Z}, n \in \mathbb{N}$.

For $b \in \mathbb{C}$, we have the $\mathfrak{g}[\epsilon]$-module $\Omega_\epsilon(\lambda, b) = H_{\epsilon, b}(\Omega(\lambda))$ with the actions:
\[ L_m \cdot (-D)^n = \lambda^n(-D + m(1 - b) + 2b\epsilon)(-D + m)^n, \]
\[ L_m \cdot \overline{(-D)^n} = \lambda^n(-\overline{D} + m(\frac{1}{2} - b) + (2b - 1)\epsilon)(-\overline{D} + m)^n, \]
\[ G_\rho \cdot (-D)^n = -\lambda^{n-\epsilon}(-\overline{D} + 2p(\frac{1}{2} - b) + (2b - 1)\epsilon)(-\overline{D} + p - \epsilon)^n, \]
\[ G_\rho \cdot \overline{(-D)^n} = -\lambda^{n+\epsilon}(-D + p + \epsilon)^n, \]
for $m \in \mathbb{Z}, p \in \mathbb{Z} + \epsilon, n \in \mathbb{N}$.

It can be seen that the $U(\mathfrak{h})$-module, which was constructed and analyzed in 4.7 through direct calculation, is isomorphic to $\Omega_\epsilon(\lambda, b)$. It follows from Theorem 4.2.1 that

**Corollary 5.3** ([4.7]). The module $\Omega_\epsilon(\lambda, b)$ is irreducible if and only if $b \neq \frac{1}{2}$. 
5.3. **Degree two modules.** Let \( f(t) \in \mathbb{C}[t^{\pm 1}] \) be such that \( D^2 - f(t) \in \mathbb{C}(t)[D] \) is an irreducible element. Set \( \tau = D^2 - f(t) \) in Theorem 4.5. Then we obtain the irreducible \( \mathcal{D} \)-module \( M = \mathcal{D}/(\mathcal{D} \cap (\mathbb{C}(t)[D]\tau)) \) with a basis \( \{t^k, t^kD, | k \in \mathbb{Z}\} \). The \( \mathcal{D} \)-actions on \( M \) are presented as

\[
t^m \cdot t^n = t^{m+n}, \quad t^m \cdot (t^n D) = t^{m+n}D, \quad D \cdot t^n = t^n(D + n), \quad D \cdot (t^n D) = t^n(f(t) + nD)
\]

for \( m, n \in \mathbb{Z} \).

For \( b \in \mathbb{C} \), we have \( g[\epsilon] \)-module \( H_{\epsilon,b}(M) \) with the following actions

\[
L_m \cdot t^n = -t^{m+n}(D + n + (m - 2\epsilon)b), \\
L_m \cdot (t^n D) = -t^{m+n}(f(t) + (n + (m - 2\epsilon)b)D), \\
L_m \cdot (\overline{t^n}) = -\overline{t^{m+n}(D + n + m(b + \frac{1}{2}) + \epsilon(1 - 2b))}, \\
L_m \cdot (\overline{t^n D}) = -\overline{t^{m+n}(f(t) + (n + m(b + \frac{1}{2}) + (1 - 2b)\epsilon)D}), \\
G_p \cdot t^n = t^{p+\epsilon+n}(D + n + 2(p - \epsilon)b), \\
G_p \cdot (t^n D) = t^{p+\epsilon+n}(f(t) + (n + 2(p - \epsilon)b)D), \\
G_p \cdot (\overline{t^n}) = -\overline{t^{p+\epsilon+n}}, \\
G_p \cdot (\overline{t^n D}) = -\overline{t^{p+\epsilon+n} D},
\]

where \( m, n \in \mathbb{Z}, p \in \mathbb{Z} + \epsilon \). From Theorems 4.5 we see that the new degree two modules \( H_{\epsilon,b}(M) \) is irreducible if and only if \( b \neq \frac{1}{2} \).

5.4. **Fraction modules.** In this subsection, we will define a class of new irreducible modules called fraction modules over \( g[\epsilon] \).

Let \( n \in \mathbb{N}, \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{C}^{n+1}, (b_0, b_1, \ldots, b_n) \in \mathbb{C}^{n+1} \) with \( b_0 = 0 \) and \( b_i \neq b_j \) for all \( i \neq j \). Set \( \tau = \frac{d}{dt} - \sum_{i=0}^{n} \frac{\alpha_i}{t - b_i} \) in Theorem 5.6. Then we get the irreducible \( \mathcal{D} \)-module

\[
M = \mathcal{D}/(\mathcal{D} \cap (\mathbb{C}(t)[D]\tau)) \subset \mathbb{C}[t, (t - b_i)^{-1} | i = 0, 1, \ldots, n].
\]

The actions of \( \mathcal{D} \) on \( M \) are defined as follows

\[
\frac{d}{dt} \cdot f(t) = \frac{d}{dt}(f(t)) + f(t) \sum_{i=0}^{n} \frac{\alpha_i}{t - b_i}, \quad t^m \cdot f(t) = t^m f(t), \quad \partial_\theta \cdot f(t) = 0,
\]

where \( f \in M, m \in \mathbb{Z} \).
For any $b \in \mathbb{C}$, we get the actions of $\mathfrak{g}[\epsilon]$ on $H_{\epsilon,b}(M)$ as follows:

$$L_m \cdot f(t) = -t^{m+1} \frac{d}{dt}(f(t)) - t^{m+1} f(t) \sum_{i=0}^{n} \frac{\alpha_i}{t - b_i} - (m - 2\epsilon)bt^m f(t),$$

$$L_m \cdot \overline{f(t)} = -t^{m+1} \frac{d}{dt}(f(t)) - t^{m+1} f(t) \sum_{i=0}^{n} \frac{\alpha_i}{t - b_i} - (m(b + \frac{1}{2}) + \epsilon(1 - 2b))bt^m f(t),$$

$$G_p \cdot f(t) = \frac{p^e+1}{p^e} d \frac{d}{dt}(f(t)) + \frac{p^e+1}{p^e} f(t) \sum_{i=0}^{n} \frac{\alpha_i}{t - b_i} + 2(p - \epsilon)bt^{p-\epsilon} f(t),$$

$$G_p \cdot \overline{f(t)} = -t^{p+\epsilon} \overline{f(t)}.$$

By Theorems 4.5, we know that the fraction module $H_{\epsilon,b}(M)$ is irreducible if and only if $b \neq \frac{1}{2}$.

5.5. Special degree $n$ modules. For any $n \in \mathbb{N}$, some degree $n$ irreducible elements in $\mathbb{C}(t)[D_t]$ were determined in Lemma 16 of [15].

For any $n \in \mathbb{N}$, choose $\tau = \left(\frac{d}{dt}\right)^n - t$ in Theorem 2.6 (1). Thus, it is easy to get the irreducible $D$-module $M = D/(D \cap (\mathbb{C}(t)[D_t] \tau))$ with a basis

$$\left\{ t^k, \left(\frac{d}{dt}\right)^m \mid k \in \mathbb{Z}, m = 0, 1, \ldots, n-1 \right\}.$$

The actions of $D$ are defined as

$$t^k \cdot \left( t^i \left(\frac{d}{dt}\right)^m \right) = t^{k+i} \left(\frac{d}{dt}\right)^m \quad \text{for } k, i \in \mathbb{Z}, 0 \leq m \leq n-1,$$

$$\frac{d}{dt} \cdot \left( t^i \left(\frac{d}{dt}\right)^m \right) = it^{i-1} \left(\frac{d}{dt}\right)^m + t^i \left(\frac{d}{dt}\right)^{m+1} \quad \text{for } i \in \mathbb{Z}, 0 \leq m < n-1,$$

$$\frac{d}{dt} \cdot \left( t^i \left(\frac{d}{dt}\right)^{n-1} \right) = it^{i-1} \left(\frac{d}{dt}\right)^{n-1} + t^{i+1} \quad \text{for } i \in \mathbb{Z}.$$
For $b \in \mathbb{C}$, we obtain the $\mathfrak{g}[\epsilon]$-module $H_{\epsilon,b}(M)$ with

$$L_k \cdot \left(t \left( \frac{d}{dt} \right)^m \right) = -(i + b(k - 2\epsilon))t^{k+i} \left( \frac{d}{dt} \right)^{m+1}$$

$$L_k \cdot \left(t \left( \frac{d}{dt} \right)^{n-1} \right) = -(i + b(k - 2\epsilon))t^{k+i} \left( \frac{d}{dt} \right)^{n-1} - t^{k+i+2}$$

$$L_k \cdot \left(t \left( \frac{d}{dt} \right)^m \right) = -(i + b(k - 2\epsilon) + \frac{k + 2\epsilon}{2})t^{k+i} \left( \frac{d}{dt} \right)^{m} - t^{k+i+1} \left( \frac{d}{dt} \right)^{m+1}$$

$$L_k \cdot \left(t \left( \frac{d}{dt} \right)^{n-1} \right) = -(i + b(k - 2\epsilon) + \frac{k + 2\epsilon}{2})t^{k+i} \left( \frac{d}{dt} \right)^{n-1} - t^{k+i+2}$$

$$G_p \cdot \left(t \left( \frac{d}{dt} \right)^m \right) = (i + 2(p - \epsilon)b)t^{p+i-\epsilon} \left( \frac{d}{dt} \right)^{m} + t^{p+i+1-\epsilon} \left( \frac{d}{dt} \right)^{m+1}$$

$$G_p \cdot \left(t \left( \frac{d}{dt} \right)^{n-1} \right) = (i + 2(p - \epsilon)b)t^{p+i-\epsilon} \left( \frac{d}{dt} \right)^{n-1} + t^{p+i+2-\epsilon}$$

$$G_p \cdot \left(t \left( \frac{d}{dt} \right)^m \right) = -t^{p+i+\epsilon} \left( \frac{d}{dt} \right)^m$$

$$G_p \cdot \left(t \left( \frac{d}{dt} \right)^{n-1} \right) = -t^{p+i+\epsilon} \left( \frac{d}{dt} \right)^{n-1}$$

where $k, i \in \mathbb{Z}$, $p \in \mathbb{Z} + \epsilon$, $0 \leq m < n - 1$. By Theorems 4, 5, we know that the special degree $n$ modules $H_{\epsilon,b}(M)$ is irreducible if and only if $b \neq \frac{1}{2}$.

**Remark 5.4.** What we want to emphasize is that the degree two module, fractional module, and special degree $n$ module discussed in sections 5.3, 5.4, and 5.5 are novel, and to the best of our knowledge, no similar constructions have been found in the existing literature.

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