AFFINE TRANSLATION HYPERSURFACES IN EUCLIDEAN
AND ISO TROPIC SPACES

MUHITTIN EVREN AYDIN

ABSTRACT. In this paper, we extend the notion of affine translation surfaces introduced by Liu and Yu (Proc. Japan Acad. Ser. A Math. Sci. 89, 111–113, 2013) in a Euclidean space $\mathbb{R}^3$ to higher dimensional ambient spaces. We provide that an affine translation hypersurface of constant Gauss-Kronocker curvature $K_0$ in $\mathbb{R}^{n+1}$ is a cylinder, i.e. $K_0 = 0$. As further applications we describe such hypersurfaces in the isotropic spaces satisfying certain conditions on the isotropic curvatures and the Laplacian.

1. Introduction

Let $\mathbb{R}^{n+1}$ be a Euclidean space and $(x_1, x_2, ..., x_{n+1})$ the orthogonal coordinate system in $\mathbb{R}^{n+1}$. Then a hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$, is called translation hypersurface if it is the graph of the form

$$x_{n+1}(x_1, x_2, ..., x_n) = f_1(x_1) + f_2(x_2) + ... + f_n(x_n),$$

where $f_1, f_2, ..., f_n$ are real-valued smooth functions of one variable (see [2, 7, 30]). These hypersurfaces are obtained by translating the curves (called generating curves) lying in mutually orthogonal planes of $\mathbb{R}^{n+1}$.

Dillen et al. [7] proved that a minimal (vanishing mean curvature) translation hypersurface in $\mathbb{R}^{n+1}$ is either a hyperplane or a product manifold $M^2 \times \mathbb{R}^{n-2}$, where $M^2$ is Scherk’s minimal translation surface in $\mathbb{R}^3$ given in explicit form

$$x_3(x_1, x_2) = \frac{1}{c} \log \left| \frac{\cos (cx_1)}{\cos (cx_2)} \right|, c \in \mathbb{R} - \{0\}.$$ 

In 3-dimensional context, many different generalizations of Scherk’s surface were treated on $\mathbb{A}^3$ [9, 31], $\mathbb{N}^3$ [12], $\mathbb{H}^3$ [16], $\mathbb{S}^3$ [17], $\mathbb{R}^3$ [18, 19].

Constant Gauss-Kronocker curvature (CGKC) and constant mean curvature (CMC) translation hypersurfaces in $\mathbb{R}^{n+1}$ (also in the Lorentz-Minkowski space $\mathbb{E}^{n+1}_1$) were described in [28] by Seo. For lightlike counterparts of such results see [11].

Most recently, Moruz and Munteanu [22] conjectured a new class of translation hypersurfaces in $\mathbb{R}^4$ as the graph of the form

$$x_4(x_1, x_2, x_3) = f_1(x_1) + f_2(x_2, x_3).$$

This one appears as the sum of a curve in $x_1, x_4$–plane and a graph surface in $x_2x_3x_4$–space. Immediately afterwards this new concept was generalized to higher
dimensionals by Munteanu et al. [23] as considering the form
\[(1.2) \quad x_{n+m+1}(x_1, x_2, \ldots, x_{n+m}) = f_1(x_1, x_2, \ldots, x_n) + f_2(x_{n+1}, x_{n+2}, \ldots, x_{n+m}).\]
The graph of the form (1.2) in \(\mathbb{R}^{n+m+1}\) is called translation graph. The authors in [22, 23] obtained new classifications and results by imposing the minimality condition. Due to the above framework, the following problems can be stated:

**Problem 1.** To obtain CMC and CGKC translation hypersurfaces in \(\mathbb{R}^{n+1}\) (as defined by Dillen et al.) whose either
1. the generating curves are planar lying in non-orthogonal planes; or
2. some of them generating curves are planar, others are not; or
3. the generating curves are all non-planar (space curves).

**Problem 2.** To characterize CGKC and CMC translation graphs in \(\mathbb{R}^{n+1}\) (as defined by Moruz et al.) without imposing restrictions.

This study aims to solve a part of first item of Problem 1, that is, to classify the CGKC translation hypersurfaces whose the generating curves lie in non-orthogonal planes. For this, we are motivated by the notion of affine translation surface introduced by Liu and Yu [14] as a graph of the form
\[x_3(x_1, x_2) = f_1(x_1) + f_2(x_2 + cx_1)\]
for some nonzero constant \(c\). Such surfaces with CMC were classified in [15]. By a change of parameter, its parameterization turns to
\[r(u, v) = (u, v - cu, f_1(u) + f_2(v)),\]
which implies that the generating curves lie in non-orthogonal planes. In order to achieve our purpose, we consider the graph in \(\mathbb{R}^{n+1}\) of the form
\[(1.3) \quad x_{n+1}(x_1, x_2, \ldots, x_n) = f_1(y_1) + f_2(y_2) + \ldots + f_n(y_n),\]
where
\[(1.4) \quad y_i = \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1, 2, \ldots, n.\]
If \(A = (a_{ij})\) in (1.4) is non-orthogonal regular matrix, then we call the graph of the form (1.3) affine translation hypersurface and \((y_1, y_2, \ldots, y_n)\) affine parameter coordinates. Note that the generating curves of an affine translation hypersurface lie in non-orthogonal planes due to the non-orthogonality of \(A\).

In the particular case \(y_1 = x_1, y_2 = x_2, \ldots, y_{n-1} = x_{n-1}\), Yang and Fu [31] proposed to obtain some curvature classifications for such a hypersurface in \(\mathbb{R}^{n+1}\). In more general case, we provide the following:

**Theorem 1.1.** Let \(M^n\) be an affine translation hypersurface in \(\mathbb{R}^{n+1}\) with CGKC \(K_0\). Then it is congruent to a cylinder, i.e. \(K_0 = 0\).

Combining this with the result of Seo [28, Theorem 2.5], we derive:

**Corollary 1.1.** There is no a translation hypersurface in \(\mathbb{R}^{n+1}\) with nonzero CGKC provided the generating curves are all planar.

Further we classify these hypersurfaces in isotropic spaces satisfying certain conditions on the isotropic curvatures and the Laplacian.
2. Preliminaries

2.1. Basics on hypersurfaces in \( \mathbb{R}^{n+1} \). Let \( M^n, \mathbb{S}^n, \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) denote a hypersurface, the standard hypersphere, the Euclidean scalar product and the induced norm of \( \mathbb{R}^{n+1} \), respectively. For further properties of submanifolds in \( \mathbb{R}^{n+1} \) see [3].

The map \( \nu : M^n \rightarrow \mathbb{S}^n \) in \( \mathbb{R}^{n+1} \) is called Gauss map of \( M^n \) and its differential \( d\nu \) is known as the shape operator \( A \) of \( M^n \). Let \( T_pM^n \) be the tangent space at a point \( p \in M^n \), then the following occurs:

\[
\langle A_p(x_p), y_p \rangle = \langle d\nu(x_p), y_p \rangle, \quad x_p, y_p \in T_pM^n,
\]

where the induced metric on \( M^n \) from \( \mathbb{R}^{n+1} \) is denoted by same symbol \( \langle \cdot, \cdot \rangle \).

The real number \( \det (A_p) \) is called the Gauss-Kronocker curvature of \( M^n \) at \( p \in M^n \). A hypersurface in \( \mathbb{R}^{n+1} \) for which the Gauss-Kronocker curvature at each point is zero is called flat.

The graph hypersurface in \( \mathbb{R}^{n+1} \) of a given real-valued smooth function \( z = z(x_1, x_2, ..., x_n) \) is of the form

\[
r : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}, \quad r(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n, z(x_1, x_2, ..., x_n)).
\]

The Gauss-Kronecker curvature \( K \) of such a hypersurface in \( \mathbb{R}^{n+1} \) turns to

\[
K = \frac{\det (\text{Hess} (z))}{\left(1 + \sum_{i=1}^{n} (z_{,i})^2 \right)^{\frac{n+1}{2}}},
\]

where \( z_{,i} = \frac{\partial z}{\partial x_i} \) and \( \text{Hess} (z) \) is the Hessian of \( z \), namely

\[
(2.2) \quad \text{Hess} (z) = \begin{bmatrix}
z_{,x_1x_1} & z_{,x_1x_2} & \cdots & z_{,x_1x_n} \\
z_{,x_2x_1} & z_{,x_2x_2} & \cdots & z_{,x_2x_n} \\
\vdots & \vdots & \ddots & \vdots \\
z_{,x_nx_1} & z_{,x_nx_2} & \cdots & z_{,x_nx_n}
\end{bmatrix}
\]

for \( z_{,x_1x_2} = \frac{\partial^2 z}{\partial x_i \partial x_j}, \quad i, j = 1, 2, ..., n \).

2.2. Basics on hypersurfaces in \( \mathbb{I}^{n+1} \). For general references of the isotropic space \( \mathbb{I}^{n+1} \) we refer to [5 8 20 21] and [24-27]. \( \mathbb{I}^{n+1} \) is based on the following group of motions

\[
(2.3) \quad \begin{bmatrix} A & 0 \\ B & 1 \end{bmatrix},
\]

where \( A \in \mathbb{R}^n_+ \) is an orthonomal \( n \times n \) -matrix and \( B \in \mathbb{R}^n_0 \) is a \((1 \times n)\) -matrix.

The isotropic distance of \( \mathbb{I}^{n+1} \) which is an invariant under (2.3) is defined as

\[
(2.4) \quad \|p - q\|_i = \sqrt{\sum_{j=1}^{n} (q_j - p_j)^2}
\]

for \( p = (p_1, p_2, ..., p_{n+1}) \), \( q = (q_1, q_2, ..., q_{n+1}) \) \( \in \mathbb{I}^{n+1} \). Thereby \( \mathbb{I}^{n+1} \) can appear as a real affine space endowed with the metric (2.4).

Let \( (x_1, x_2, ..., x_{n+1}) \) be the standard affine coordinates of \( \mathbb{I}^{n+1} \). The metric (2.4) is degenerate along \( x_{n+1} \)-direction and we call the lines in \( x_{n+1} \)-direction isotropic lines. The \( k \)-plane involving an isotropic line is called isotropic \( k \)-plane.

A hypersurface in \( \mathbb{I}^{n+1} \) is called admissible if nowhere it has isotropic tangent hyperplane.
A graph hypersurface $M^n$ in $\mathbb{R}^{n+1}$ of a given smooth function $z(x_1, x_2, ..., x_n)$ is of the form

$$r : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}, \ r(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n, z(x_1, x_2, ..., x_n)).$$

Note that $M^n$ is admissible since its tangent hyperplane spanned by $\{r, x_1, r, x_2, ..., r, x_n\}$ does not involve an isotropic line.

The induced metric $\langle \cdot, \cdot \rangle$ on $M^n$ from $\mathbb{R}^{n+1}$ is given by

$$\langle \cdot, \cdot \rangle = dx_1^2 + ... + dx_n^2. \quad (2.5)$$

Thus, its Laplacian becomes

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \quad (2.6)$$

Now let us consider a curve on $M^n$ that has the position vector

$$r = r(s) = x(s) + z(s) e_{n+1}, \quad (2.7)$$

where

$$x(s) = (x_1(s), x_2(s), ..., x_n(s), 0), \ e_{n+1} = \begin{pmatrix} 0, 0, ..., 0, 1 \end{pmatrix}. \quad (2.8)$$

Derivating of (2.7) with respect to $s$ leads to

$$r' = x' + (x', \nabla z) e_{n+1}, \quad (2.9)$$

where $\nabla$ denotes the gradient operator in $\mathbb{R}^n$. By again derivating of (2.8) with respect to $s$, we arrange the following

$$r'' = x'' + \langle x'', \nabla z \rangle e_{n+1} + (x')^T Hess(z) \cdot x' e_{n+1}, \quad (2.10)$$

where $x'$ is column matrix associated to $x'$ and $(x')^T$ its transpose. Therefore, in (2.9), the following decomposition occurs:

$$Tan(r'') = x'' + \langle x'', \nabla z \rangle e_{n+1}$$

and

$$Nor(r'') = (x')^T Hess(z) \cdot x' e_{n+1}, \quad (2.11)$$

where $Tan(r'')$ implies the projection of $r''$ onto tangent hyperplane of $M^n$ and $Nor(r'')$ the isotropic component of $r''$ which is normal to $M^n$.

If $\|Tan(r'')\|_s \neq 0$ then it is called the geodesic curvature function $\kappa_G$ of $r$. Otherwise $\kappa_G = 1$ is assumed. Accordingly the following function is called the normal curvature function $\kappa_N$ of $r$:

$$\kappa_N = (x')^T Hess(z) \cdot x'. \quad (2.12)$$

The extremal values $\kappa_1, ..., \kappa_n$ of (2.10) corresponding to the eigenvalue functions of $Hess(z)$ are called principal curvatures of $M^n$. Since $Hess(z)$ is symmetric, all eigenvalue functions are real. Thus one gives rise to define the following certain curvature functions:

$$K_i = \frac{1}{(i)!} \left( \kappa_1 \kappa_2 ... \kappa_i + \kappa_1 \kappa_2 ... \kappa_{i-1} \kappa_{i+1} + ... + \kappa_{n-i+1} ... \kappa_n \right). \quad (2.13)$$

By (2.11), the isotropic mean curvature function $H = K_1$ is

$$H = \frac{1}{n} \text{trace}(Hess(z)) = \frac{1}{n} \Delta z \quad (2.14)$$
and the relative curvature (or isotropic Gaussian curvature) function \( K = K_n \)

\[
(2.13) \quad K = \det (\text{Hess} (z)).
\]

A hypersurface in \( \mathbb{R}^{n+1} \) with vanishing relative curvature (resp. isotropic mean curvature) is called isotropic flat (resp. isotropic minimal).

3. Affine translation hypersurfaces in \( \mathbb{R}^{n+1} \)

Let \( x = (x_1, x_2, \ldots, x_n) \) denote the orthogonal coordinate system in \( \mathbb{R}^n \) and \( z : \mathbb{R}^n \rightarrow \mathbb{R}, \quad z = z(y), \) be a smooth function, where

\[
(3.1) \quad y = (y_1, y_2, \ldots, y_n), \quad y_i = \sum_{j=1}^n a_{ij} x_j, \quad a_{ij} \in \mathbb{R}, \quad i = 1, 2, \ldots, n.
\]

If \( A = (a_{ij}) \) is a non-orthogonal \( n \times n \)-matrix and \( \det (A) \neq 0 \), then we call the graph of \( z(y) \) in \( \mathbb{R}^{n+1} \) affine graph of \( z(x) \) and \( (y_1, y_2, \ldots, y_n) \) affine parameter coordinates.

Hence we provide the following result to use later.

**Lemma 3.1.** Let \( z(y) \) be a smooth real-valued function on \( \mathbb{R}^n \), where \( y \) is the affine parameter coordinates given by (3.1). Then the following relation holds:

\[
(3.2) \quad \det [\text{Hess} (z(x))] = \det [A]^2 \det [\text{Hess} (z(y))]
\]

for \( x = (x_1, x_2, \ldots, x_n) \).

**Proof.** The partial derivatives of \( z \) with respect to \( x_i, 1 \leq i \leq n \), gives

\[
z_{,x_i} = \sum_{k=1}^n a_{ki} z_{,y_k}, \quad z_{,x_i, x_j} = \sum_{k,l=1}^n a_{ki} a_{lj} z_{,y_k y_l}, \quad 1 \leq j \leq n.
\]

Then the Hessian of \( z(x) \) follows

\[
(3.3) \quad \text{Hess} (z(x)) = \begin{bmatrix}
\sum_{k,l=1}^n a_{k1} a_{l1} z_{,y_k y_l} & \sum_{k,l=1}^n a_{k1} a_{l2} z_{,y_k y_l} & \cdots & \sum_{k,l=1}^n a_{k1} a_{ln} z_{,y_k y_l} \\
\sum_{k,l=1}^n a_{k2} a_{l1} z_{,y_k y_l} & \sum_{k,l=1}^n a_{k2} a_{l2} z_{,y_k y_l} & \cdots & \sum_{k,l=1}^n a_{k2} a_{ln} z_{,y_k y_l} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k,l=1}^n a_{kn} a_{l1} z_{,y_k y_l} & \sum_{k,l=1}^n a_{kn} a_{l2} z_{,y_k y_l} & \cdots & \sum_{k,l=1}^n a_{kn} a_{ln} z_{,y_k y_l}
\end{bmatrix}.
\]

By considering matrix multiplication in (3.3) we deduce that

\[
(3.4) \quad \text{Hess} (z(x)) = A^T \cdot \text{Hess} (z(y)) \cdot A,
\]

where \( A^T \) denotes the transpose of \( A \). Thus by (3.4) we obtain (3.2). \( \square \)

If \( \det (A) \neq 0 \), Lemma 3.1 immediately implies the following trivial result

**Corollary 3.1.** A graph of a given smooth real-valued function is flat if and only if so is its affine graph in \( \mathbb{R}^{n+1} \).

In particular, the affine graph of (1.1), so-called affine translation hypersurface, has the form

\[
(3.5) \quad z(x_1, x_2, \ldots, x_n) = f_1(y_1) + f_2(y_2) + \cdots + f_n(y_n), \quad z = x_{n+1},
\]

where \( f_1, f_2, \ldots, f_n \) are arbitrary nonzero smooth functions and \( (y_1, y_2, \ldots, y_n) \) is affine parameter coordinates given by (3.1). Remark that such a hypersurface reduces to the standard translation hypersurface, if \( A \) is an orthogonal matrix.
Denote $A^{-1} = (a_{ij})$ the inverse matrix of $A = (a_{ij})$. Then, by a change of parameter, the affine translation hypersurface $M^n$ has a parameterization

$$r(y_1, y_2, \ldots, y_n) = (\sum_{i=1}^n a_{1i} y_i + \sum_{i=1}^n a_{2i}^2 y_i, \ldots, \sum_{i=1}^n f_i(y_i))$$

with respect to the variables $y_1, y_2, \ldots, y_n$. By (3.7), the Hessian of $z$ is given by

$$Hess(z) = \begin{bmatrix} f''_1 & 0 & \cdots & 0 \\ 0 & f''_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f''_n \end{bmatrix}.$$

Substituting (3.9) into (3.2) leads to

$$\det [Hess(z(x))] = \det [A]^2 f''_1 f''_2 \ldots f''_n,$$

where $x = (x_1, x_2, \ldots, x_n)$.

Now we assume that the affine translation hypersurface $M^n$ in $\mathbb{R}^{n+1}$ has $K = K_0 = \text{const}$. Then (2.1), (3.7) and (3.10) imply that

$$K_0 = \frac{\det (A)^2 (f''_1 f''_2 \ldots f''_n)}{\left(1 + \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} f_j''\right)^2\right)^{\frac{n+2}{2}}}.$$

**Case 1** If $K_0 = 0$ in (3.11), then at least one of $f_1, f_2, \ldots, f_n$ is a linear function with respect to the variables $y_1, y_2, \ldots, y_n$, respectively. Without lose of generality, we may assume that $f_1(y_1) = cy_1 + d, c, d \in \mathbb{R}$. Considering this one into (3.6), we conclude

$$r(y_1, y_2, \ldots, y_n) = y_1 (a_{11}, a_{21}, \ldots, c) + \left(\sum_{i=2}^n a_{1i} y_i, \sum_{i=2}^n a_{2i}^2 y_i, \ldots, d + \sum_{i=2}^n f_i(y_i)\right),$$

which implies that $M^n$ turns to a cylinder.
**Case 2** Otherwise, i.e. $K_0 \neq 0$, the functions $f_1, f_2, ..., f_n$ have to be non-linear.

Put $W := 1 + \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ji} f_j' \right)^2$. Taking partial derivative of (3.11) with respect to $y_p$, $p = 1, 2, ..., n$, gives

$$f_p' n \sum_{i,j=1}^{n} a_{pi} a_{qj} f_j' = 0.$$

Substituting (3.15) into (3.14) leads to either

$$\sum_{i,j=1}^{n} a_{pi} a_{qi} = 0 \quad \text{or} \quad \sum_{i,j=1}^{n} a_{pi} a_{qi} f_j' = 0.$$

Taking partial derivative in the second equality of (3.16) with respect to $y_p$ gives

$$f_p' n \sum_{i=1}^{n} (a_{pi})^2 = 0$$

which implies $a_{p1} = a_{p2} = ... = a_{pn} = 0$. This is a contradiction since $\det(A) \neq 0$, which completes the proof.

### 4. Further Applications

Before introducing the affine translation hypersurfaces in $\mathbb{I}^{n+1}$, let us reconsider the notion of translation hypersurface in $\mathbb{I}^{n+1}$. By means of the isotropic motions given by (2.3), a translation hypersurface in $\mathbb{I}^{n+1}$ generated by translating the curves lying in orthogonal isotropic planes is the graph of the form (3.1). Such hypersurfaces in $\mathbb{I}^{n+1}$ with constant relative curvature (CRC) and constant isotropic mean curvature (CIMC) were provided in [1].

Therefore, as similar to Euclidean case, we can state that an affine translation hypersurface in $\mathbb{I}^{n+1}$ is the graph of a function given via (3.1) and (3.5). Point out that the generating curves for this one lie in non-orthogonal isotropic planes. So, by having in mind that the generating curves may also lie non-isotropic planes, the problems given in the Introduction can be also considered in the isotropic spaces.

By (2.13) and (3.9), for an affine translation hypersurface with CRC $K_0$ in $\mathbb{I}^{n+1}$, we get

$$K_0 = \det (A)^2 f_1' f_2' ... f_n'.$$
where \( f''_i = \frac{\partial^2 f}{\partial x_i^2} \) and \((y_1, y_2, \ldots, y_n)\) the affine parameter coordinates given by (3.1).

Hence (4.1) immediately implies that \( K_0 \) vanishes when at least one \( f_1, f_2, \ldots, f_n \) is a linear function with respect to the variables \( y_1, y_2, \ldots, y_n \), respectively. Suppose that \( K_0 \neq 0 \). Taking partial derivative of (4.1) with respect to \( y_p \) leads to

\[
f''_1 f''_2 \ldots f''_p \ldots f''_n = 0,
\]

namely

\[
f_p(y_p) = c_p y_p^2 + d_p y_p + e_p, \quad p = 1, 2, \ldots, n
\]

for some constants \( c_p, d_p, e_p \in \mathbb{R} \). Consequently, the following result can be expressed:

**Theorem 4.1.** Let \( M^n \) be an affine translation hypersurface in \( \mathbb{R}^{n+1} \) with \( K_0 \). Then, it is either congruent to a cylinder \( (K_0 = 0) \) or given by \( (K_0 \neq 0) \)

\[
\begin{align*}
\{ & z(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n c_i y_i^2 + d_i y_i + e_i, \\
& c_i, d_i, e_i \in \mathbb{R}, \quad c_i \neq 0, \quad c_1 c_2 \ldots c_n = \frac{K_0}{2^n \det(A)^2}, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

where \((y_1, y_2, \ldots, y_n)\) is the affine parameter coordinates given by (3.1).

Next we assume that an affine translation hypersurface \( M^n \) in \( \mathbb{R}^{n+1} \) has CIMC \( H_0 \). Hence we have from (2.12) and (3.7) that

\[
n H_0 = \sum_{i,j=1}^n a_{ij} f''_i.
\]

Taking partial derivative of (4.2) with respect to \( y_p, p = 1, 2, \ldots, n \), gives

\[
\left( \sum_{i=1}^n a_{ii}^2 \right) f''_p = 0
\]

or

\[
f_p(y_p) = \frac{c_p}{2 \sum_{i=1}^n a_{ii}^2} y_p^2 + d_p y_p + e_p
\]

for some constants \( c_p, d_p, e_p \) such that \( \sum_{i=1}^n c_i = n H_0 \).

Therefore we can present the following result.

**Theorem 4.2.** Let \( M^n \) be an affine translation hypersurface in \( \mathbb{R}^{n+1} \) with CIMC \( H_0 \). Then, it is given in explicit form

\[
\begin{align*}
\{ & z(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n \left( \frac{c_i}{\sum_{j=1}^n a_{ij}} \right) y_i^2 + d_i y_i + e_i, \\
& \sum_{i=1}^n c_i = n H_0, c_i, d_i, e_i \in \mathbb{R},
\end{align*}
\]

where \((y_1, y_2, \ldots, y_n)\) is the affine parameter coordinates given by (3.2). In particular, \( M^n \) is isotropic minimal provided \( \sum_{i=1}^n c_i = 0 \).

Finally we aim to observe the affine translation hypersurface \( M^n \) in \( \mathbb{R}^{n+1} \) whose the coordinate functions are eigenfunctions of the Laplacian, i.e., that satisfies the condition

\[
\triangle r_k = \lambda_k r_k, \quad \lambda_k \in \mathbb{R}, \quad k = 1, 2, \ldots, n + 1,
\]

where \( r_k \) is the coordinate function of the position vector of an arbitrary point on \( M^n \) and \( \triangle \) the Laplace operator of \( M^n \) with respect to the induced metric from \( \mathbb{R}^{n+1} \).
In the particular case \( \lambda_1 = \lambda_2 = \ldots = \lambda_{n+1} = \lambda \), the condition (4.3) was firstly treated to Riemannian submanifolds by Tahakashi [29]. Then Garay [10] generalized this condition as follows:

\[ \Delta r = Ar, \quad A \in \mathbb{R}^{n+1}. \]

One is also related to the notion of submanifolds of finite type conjectured by Chen (see [4, 7]).

An affine translation hypersurface \( M^n \) in \( \mathbb{R}^{n+1} \) is of the form

\[ r (x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, f_1 (y_1) + f_2 (y_2) + \ldots + f_n (y_n)), \]

where \((y_1, y_2, \ldots, y_n)\) is the affine parameter coordinates given by (3.1). Let us put

(4.4) \[ r_1 = x_1, \quad r_2 = x_2, \ldots, \quad r_n = x_n \]

and

(4.5) \[ r_{n+1} = f_1 (y_1) + f_2 (y_2) + \ldots + f_n (y_n). \]

From (2.6), (4.4) and (4.5), we conclude that

(4.6) \[ \Delta r_1 = \Delta r_2 = \ldots = \Delta r_n = 0 \quad \text{and} \quad \Delta r_{n+1} = \sum_{i,j=1}^n a_{ij} f_i''. \]

Now suppose that \( M^n \) holds (4.3). Then (4.6) implies \( \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0 \) and the following system of ordinary differential equations:

(4.7) \[ \sum_{i,j=1}^n a_{ij}^2 f_i'' = \lambda \sum_{i=1}^n f_i, \quad \lambda_{n+1} = \lambda. \]

In the case \( \lambda = 0 \), \( M^n \) becomes isotropic minimal stated already via Theorem 4.2. Hence it is meaningful to assume \( \lambda \neq 0 \). Since \( f_1, f_2, \ldots, f_n \) depend on the variables \( y_1, y_2, \ldots, y_n \), (4.7) turns to

(4.8) \[ \sum_{j=1}^n a_{ij}^2 f_i'' - \lambda f_i = \mu_i, \]

where \( \mu_i \) are some constants such that \( \sum_{i=1}^n \mu_i = 0 \). If \( \lambda > 0 \) in (4.8), then by solving it we obtain

\[ f_i (y_i) = c_i \exp \left( \sqrt{\frac{\lambda}{\sum_{j=1}^n a_{ij}^2}} y_i \right) + d_i \exp \left( -\sqrt{\frac{\lambda}{\sum_{j=1}^n a_{ij}^2}} y_i \right) - \frac{\mu_i}{\lambda}, \]

and if \( \lambda < 0 \)

\[ f_i (y_i) = c_i \cos \left( \sqrt{-\frac{\lambda}{\sum_{j=1}^n a_{ij}^2}} y_i \right) + d_i \sin \left( \sqrt{-\frac{\lambda}{\sum_{j=1}^n a_{ij}^2}} y_i \right) - \frac{\mu_i}{\lambda}, \]

where \( c_i, d_i \) are some constants.

Therefore we have proved next result.

**Theorem 4.3.** Let \( M^n \) be a non isotropic minimal affine translation hypersurface in \( \mathbb{R}^{n+1} \) satisfying \( \Delta r_k = \lambda_k t_k \). Then \((\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) = (0, 0, \ldots, \lambda \neq 0)\) and \( M^n \) is congruent to the graph of the function either

\[ z (x_1, x_2, \ldots, x_n) = \sum_{i=1}^n c_i \exp \left( \sqrt{\frac{\lambda}{\sum_{j=1}^n a_{ij}^2}} y_i \right) + d_i \exp \left( -\sqrt{\frac{\lambda}{\sum_{j=1}^n a_{ij}^2}} y_i \right). \]
or

\[ z(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} c_i \cos \left( \sqrt{-\lambda} \frac{1}{\sum_{j=1}^{n} a_{ij}^2} y_i \right) + d_i \sin \left( \sqrt{-\lambda} \frac{1}{\sum_{j=1}^{n} a_{ij}^2} y_i \right), \]

where \((y_1, y_2, \ldots, y_n)\) is the affine parameter coordinates given by (3.1) and \(c_i, d_i\) some constants.

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**Departments of Mathematics, Faculty of Science, Firat University, Elazig, 23200, Turkey**

*E-mail address: meaydin@firat.edu.tr*