Computing the exact sign of sums of products with floating point arithmetic

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Abstract

In computational geometry, the construction of essential primitives like convex hulls, Voronoi diagrams and Delaunay triangulations require the evaluation of the signs of determinants, which are sums of products. The same signs are needed for the exact solution of linear programming problems and systems of linear inequalities. Computing these signs exactly with inexact floating point arithmetic is challenging, and we present yet another algorithm for this task. Our algorithm is efficient and uses only of floating point arithmetic, which is much faster than exact arithmetic. We prove that the algorithm is correct and provide efficient and tested C++ code for it.

1 Introduction

From a computational geometry perspective, we address the following problem: given a point \( C \), defined explicitly or implicitly as the intersection of two lines, we want to decide on which side of a line \( AB \) it lies using only floating point arithmetic (or to find whether it is on the line.) This problem is classic. It plays a fundamental role in the computation of convex hulls, Voronoi diagrams and Delaunay triangulations [1,14,16]. The essence of the geometric issues is described in Figure 1.

![Figure 1: Rounding errors may lead to the conclusion that the point C lies in the wrong side of the line AB.](image)

From an algebraic perspective, the determination of the position of the point with respect to the line is equivalent to the computation of the sign of a sum of products

\[
S = \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij}.
\]

The inexactness of floating point arithmetic makes it hard to compute this sign exactly in some cases. There are already several references about the computation sums of floating point numbers [5, 4, 5, 6, 7, 8, 10, 12, 13, 15, 16, 17], with applications in computations geometry. The idea of decomposing products in sums is also as old as
and the fourth chapter of \cite{14} presents an algorithm for obtaining the signs of such sums. However, all of these references differ in one way or another from the present article. In summary, none of them gives a complete solution to the problem we solve here, using IEEE floating point arithmetic and taking underflow into account. We present complete algorithm for this task, in gory detail. In particular, we take as input the factors $a_{ij}$ of the products $p_i$, and before applying most of the other algorithms we would need to obtain an exact representation of the products $p_i$, or assume that such a product does not underflow. The algorithm was implemented in C++, and it was carefully tested, and is supported by a detailed theory. It is quite likely that the theoretical results presented here can be generalized to bases other than two and other rounding modes. We are not interested in such generalizations. The goal of our theory is only to provide a justification to what matter most us: the practical algorithms we present, and our theoretical results are only meant to justify our algorithms.

Finite floating point numbers suffice to solve our problem in the situations usually found in practice, and we only care about such numbers, which are elements of a set $\mathcal{F}$. Deviating from the tradition, we assume that the arithmetic operations $\text{op} \in \{+,-,\times\}$ are performed rounding up. For instance, we assume that the subtraction of floating point numbers is defined as

$$b \ominus a = \text{up}(b - a)$$

where $\text{up}$ is the function from $(\min \mathcal{F}, \max \mathcal{F}) \to \mathcal{F}$ defined by

$$\text{up}(x) := \min \{y \in \mathcal{F} \text{ with } y \geq x\}, \quad (1)$$

and the operations $\oplus$ and $\otimes$ are similar (we have no need for expensive $\oslash$’s.) For brevity, we leave the result of the arithmetic operation $a \text{ op } b$ undefined when $a \text{ op } b \notin (\min \mathcal{F}, \max \mathcal{F})$.

In practice, when the rounding mode is not upwards already, enforcing our assumption requires a function call to change the rounding mode upwards in the beginning of the use our functions, and another function call to restore it when we are done, to be polite. The cost of this two calls is amortized, but cases like this make us believe that always rounding upwards (or always rounding downwards) is a good option for code requiring exact results. Our Moore library \cite{9} is a good example of this. Since it works with rounding mode upwards by default, there would be no need for changes in the rounding mode when executing the algorithms described here.

The motivation for this article is to have a bullet proof algorithm for computing the sign of $S = \sum_{n=1}^{m} \prod_{i=1}^{n} a_{ij}$ with floating point arithmetic. Such an algorithm is needed because sometimes rounding errors may lead to wrong conclusions by naive algorithms. However, floating point arithmetic is an excellent tool: such naive algorithms will be correct most of the time, and we should care about both precision and performance. For this reason, we believe that we should proceed in two steps: we first try to compute the sign using a quick algorithm, which is unable to compute the sign only in rare cases. In such rare cases we resort to a robust algorithm, which can handle all cases but is more expensive. We present both algorithms below. In Section2 we describe a simple algorithm, which is quite efficient but may not find the sign in rare cases. In Section3 we present a robust algorithm, which is more expensive and should be used only after a quick algorithm was unable to find the sign.

Regarding the efficiency of our algorithms, we emphasize that the quick one will suffice in the overwhelming majority of the cases found in practice, and the robust

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Regarding the efficiency of our algorithms, we emphasize that the quick one will suffice in the overwhelming majority of the cases found in practice, and the robust
algorithm will be an extreme safety measure. As a result, usually the cost of evaluating
the sign will be twice the cost of evaluating the sum using naive floating point arithmetic,
plus the cost of $\sum_{i=1}^{n} n_i - n$ branches, plus the change of rounding modes when they are
necessary. This is a $O(\sum_{i=1}^{n} n_i)$ cost, with a small constant. However, in rare cases in
which the robust algorithm is necessary, the cost can grow exponentially with the $n_i$.

Finally, the actual code is implemented using the template features of C++, and
exploiting the details of this language it is possible to generate code that will only resort
to more expensive operations in the rare situations in which they are needed.

2 Quick sign

In this section we present a fast algorithm which finds the sign of the sum of products
$S := \sum_{i=1}^{n} \prod_{j=1}^{n_i} a_{ij}$ in most cases, but which may be inconclusive sometimes. The
algorithm returns a sign $s \in \{-1, 0, 1\}$. If $s \neq 0$ then it definitively is the sign of the sum.
However, if $s = 0$ then the sign can be anything. In this case we must resort to a more
expensive algorithm to find the sign, like the one in the next section.

As in the rest of this article, we use floating point arithmetic rounding upwards, with
at most two changes in rounding mode. In essence, for each product $p_i = \prod_{j=1}^{n_i} a_{ij}$ the
algorithm computes numbers $d_i$ and $u_i$ such that

$$-d_i \leq p \leq u_i.$$ 

If $\sum d_i < 0$ then $S$ is positive and if $\sum u_i < 0$ then $S$ is negative, otherwise we cannot
decide, and the algorithm returns 0. The algorithm is described as Algorithm 2 below,
which uses the auxiliary Algorithm 1.

Algorithm 1 Auxiliary Algorithm for bounding $a_{ij} \times x$...

procedure quick_prod(d, u, a_{ij}, x...)
  if $a_{ij} \leq 0$ then
    if $a_{ij} = 0$ then
      return 0, 0
    else
      return quick_prod(u, d - a_{ij}, x...)
  end if
  d ← d * a_{ij} \quad \triangleright\ -d \leq \prod_{k=0}^{i-1} a_{ik} \leq u
  u ← u * a_{ij} \quad \triangleright\ -u \leq -\prod_{k=0}^{i-1} a_{ik} \leq d
  if is_empty(x) then
    return d, u
  else
    return quick_prod(d, u, x...)
end procedure

3 Robust sign

This section describes Algorithm 3 which computes the sign of $S = \sum_{i=1}^{n} \prod_{j=1}^{n_i} a_{ij}$,
using binary floating point arithmetics which have subnormal numbers. The most
relevant arithmetics in this class are the ones covered by the IEEE 754 and IEEE 854
Algorithm 2 A quick algorithm for computing the sign of \( \sum p \cdot \sum q \cdot p \)

\[
\text{procedure } \text{QUICK\_SIGN}(p, q) \\
\text{round(up)} \quad \triangleright \text{round object, resets the rounding mode on exit} \\
su \leftarrow 0 \quad \triangleright \text{su is an upper bound on } S \\
sd \leftarrow 0 \quad \triangleright \text{sd is an upper bound on } -S. \\
\text{for each } p \text{ do} \\
d, u \leftarrow \text{quick\_prod}(-1, 1, p) \\
su \leftarrow su + u \\
sd \leftarrow sd + d \\
\text{return } \text{sd} < 0 \, ? \\ 1 \, : \, (su < 0 \, ? \\ -1 \, : \, 0) \\
\text{end procedure}
\]

standards. The algorithm is presented in the last page of the article. It assumes that there is not overflow in the products \( p_i = \sum_{j=1}^{n_i} a_{ij} \), but underflow is handled correctly. It also assumes that

\[
\sum_{i \text{ with } n_i=1} 1 + \sum_{i \text{ with } n_i>1} 2^{n_i-2} < 1/\epsilon, \quad (2)
\]

where \( \epsilon \) is the machine precision. In practice, the largest \( \epsilon \) we care about corresponds to the type \text{float}. In this case \( 1/\epsilon = 2^{23} \) and the algorithm could be used to compute the signs of determinants of dimensions up to 8, because if \( n = 8! \) and \( n_i = 8 \) then

\[
\sum_{i=1}^{n} 2^{n_i-2} = 8! \times 2^6 = 2580480 < 8388608 = 2^{23}. 
\]

Since \( n_i = 8 \) is already more than enough for the usual applications in computational geometry, we have no motive to make the algorithm more complicated than it already is in order to relax the condition (2).

The algorithm is based upon two lemmas. Lemma 1 is about the exact computation of the difference \( b - a \) of floating point numbers. There are versions of this lemma since the late 1960s \cite{2,7,8}, but we prove it here for the case in which we round upwards for completeness, and because the details are not obvious (as stated, Lemma 1 is false if we round downwards for instance.) In essence, it states that we can represent \( b - a \) exactly as the difference of two floating point numbers \( c \) and \( e \), with the additional feature that \( e \) is much smaller than \( c \). As a result, in most cases we can base our decision regarding signs on \( c \), and \( e \) is used only in the rare cases in which knowing \( c \) is not sufficient.

**Lemma 1** if \( a, b \in \mathcal{F}, 0 < a < b \) and

\[
c := b \oplus a, \quad d := b \ominus c \quad \text{and} \quad e := a \ominus d
\]

then

\[
b - a = c - e \quad \text{and} \quad 0 \leq e < c\epsilon, \quad (3)
\]

where \( \epsilon \) is the machine precision. ▲

**Lemma 2** is the analogous to Lemma 1 for multiplication, but it is more subtle. It relies on the fused multiply add operation (fma), which is available in most processors and programming language these days. In other words, we assume that given \( a, x, y \in \mathcal{F} \) such that \( ax + y \in (\min \mathcal{F}, \max \mathcal{F}) \) we can compute

\[
fma(a, x, y) := \text{up}(ax + y). \quad (4)
\]
It is well known that using the fma we can represent the product \(ab\) as the difference of two floating point numbers, but we are not aware of proofs (or even statements) of results describing conditions under which this representation is exact when rounding upwards. It is important to notice that such conditions are necessary, because Lemma 2 may not hold if the condition (7) is violated. In order to state the decomposition result for multiplications we need to define some constants that characterize the floating point arithmetic:

- \(v\) is the smallest positive normal element of \(\mathbb{F}\) and \(ve\) is the smallest positive element of \(\mathbb{F}\)
- \(e \in \mathbb{F}\) is the machine precision, that is, \(1 + e\) is the successor of 1 in \(\mathbb{F}\).
- \(\sigma\) is the largest power of two in \(\mathbb{F}\), and we assume that \(1/\sigma \in \mathbb{F}\).

Using the constants above we define the threshold
\[
\tau := 2v/e.
\]

The values of these constants for the usual arithmetics are presented in Table 1. By inspecting this table, readers will notice that the following assumptions used in Lemma 2 are satisfied:
\[
\sigma^2 \geq 2 \quad \text{and} \quad \sigma^2 ve > 2.
\]

|        | float  | double | long double | quad      |
|--------|--------|--------|-------------|-----------|
| \(\nu\)| 1.2e-38| 2.2e-308| 3.4e-4932   | 3.3e-4932 |
| \(e\)  | 1.2e-07| 2.2e-016| 1.1e-0019   | 1.9e-0034 |
| \(\sigma\)| 1.7e+38| 9.0e+307| 5.9e+4931   | 5.9e+4931 |
| \(\tau\)| 2.0e-31| 2.0e-292| 6.2e-4913   | 3.4e-4898 |

Table 1: Values of the constants for the usual arithmetics.

We now state the decomposition lemma for multiplications.

**Lemma 2** Consider \(a, b \in \mathbb{F}\), with \(0 < a \leq b\), for which \(a \otimes b\) is defined. If
\[
c := a \otimes b > \tau,
\]
then
\[
ab = c - d \quad \text{for} \quad d := \text{fma}(\neg a, b, c) \geq 0.
\]
If the condition (7) is not satisfied then \(\hat{a} := \sigma \otimes a = \sigma a \in \mathbb{F}\) and if
\[
\hat{c} := \hat{a} \otimes b > \tau,
\]
then
\[
ab = \sigma^{-1}\hat{c} - \sigma^{-1}\hat{d} \quad \text{for} \quad \hat{d} := \text{fma}(\neg \hat{a}, b, \hat{c}) \geq 0.
\]
Finally, if Equations (7) and (9) are not satisfied and the first inequality in Equation (6) holds then \(\tilde{b} := \sigma \otimes b = \sigma b \in \mathbb{F}\), and if the second inequality in Equation (6) holds then
\[
ab = \sigma^{-2}\tilde{c} - \sigma^{-2}\tilde{d} \quad \text{for} \quad \tilde{d} := \text{fma}(\neg \tilde{a}, b, \tilde{c}) \geq 0.
\]
In summary, if $0 < a \leq b$ and $a \otimes b$ is defined then there exists $e \in \{0, 1, 2\}$ and $c, d \in \mathcal{F}$ such that
\[ ab = \sigma^{-e}c - \sigma^{-e}d. \] (12)

In words, Lemma 2 shows that we may fail twice in trying to represent exactly the product $ab$ as a difference of two floating point numbers, but the third time is a charm: we finally can represent $ab$ exactly as the scaled difference of two floating point numbers. Scaling is essential here in order to deal with underflow. We scale numbers by multiplying them by the constant $\sigma$, which is a power of two. Such scaling does not introduce rounding errors, but requires some book keeping. In C++ we can keep the books using an struct like

```cpp
struct scaled_number {
    scaled_number(T t, int exp)
        : t(t), exp(exp) {}
    T t;
    int exp;
};
```

where $T$ is the type of the floating point numbers. An scaled number $s$ represents
\[ x = \text{value}(s) = \sigma^{-s}\text{exp} \times s. \]

We keep the scaled numbers in two heaps, one for the positive values, called $\mathcal{P}$, and another for the negative values, called $\mathcal{N}$. In the $\mathcal{N}$ heap we store the absolute value of the corresponding numbers, so that the $t$ field in our scaled numbers is always positive, and $\text{exp} \leq 0$. The elements in the heaps are sorted in increasing order according to the following comparison function:

```cpp
bool is_less(scaled_number x, scaled_number y) {
    if (x.exp > y.exp) return true;
    if (x.exp < y.exp) return false;
    return x.t < y.t;
}
```

In order to ensure the consistency of the order above we only push two kinds of scaled numbers in our heaps:

- **Scaled numbers $s$ with** $s.\text{exp} = 0$ and $\tau < s.\text{t}$, (13)
- **Scaled numbers $s$ with** $s.\text{exp} > 0$ and $\tau < s.\text{t} \leq \sigma\tau$. (14)

We assume that these conditions are enforced by the constructor of `scaled_number`, which is only called with positive $\tau$'s. We then have the following Lemma

**Lemma 3** Under the conditions (13) and (14) for scaled numbers $x$ and $y$ we have
\[ \text{value}(x) < \text{value}(y) \iff \text{is_less}(x, y). \]

We have now all the ingredients to describe our algorithm. It uses an auxiliary function split which splits each product $p_i = \prod_{j=1}^{n_i} a_{ij}$ as a sum of $2^{n_i-1}$ scaled numbers using Lemma 2 (given this lemma, writing such a function is trivial.) If $n_i > 1$
then half of the parts in which \( p_i \) is split will be negative and the other half will be positive. Therefore, if \( n_i > 1 \) then \( p_i \) contributes \( 2^{n_i-2} \) scaled numbers to each heap. As a result, the left hand side of Equation (4) is the maximum number of elements which we will have on each heap, and the condition (2) ensures that this number does not exceed \( 1/\varepsilon \).

Once the heaps are filled with products, we start to compare them. While

\[
s_n := \text{size}(N) > 0 \quad \text{and} \quad s_p := \text{size}(P) > 0
\]

we pop the top elements \( n \) and \( p \) of \( N \) and \( P \) and compare them. If \( p > s_n \cdot n \) then the sign of the sum is 1. If \( n > s_p \cdot p \) then the sign of the sum is \(-1\), otherwise, conceptually, we use Lemma 1 to split \( p - n \) and push the parts back into the heaps.

There is a catch in this argument in that \( n \) and \( p \) are scaled numbers, which may have different exponents. If these exponents differ by more than one then the numbers on the heap with the numbers with the largest exponent are negligible and we are done. When the exponents differ by one we multiply the \( t \) field of the one with the largest exponent by \( 1/\varepsilon \), reducing both numbers to the same exponent. This multiplication by \( 1/\varepsilon \) may be inexact, but this inexactness is harmless. For instance, when \( p \) has the largest exponent and the multiplication \( t := p.t \leftarrow p.t \otimes 1/\varepsilon \) is inexact, then Lemma 4 in Section 4 implies that \( z \leq v \) and Equations (13) and (14) yield

\[
n.t > \tau = \frac{2}{\epsilon} v \geq \frac{2}{\epsilon} z \geq 2spz,
\]

and we will reach the correct conclusion that the sign of the sum is \(-1\) even if we use the incorrect \( z \). Once we have both numbers with the same exponent, simply split the difference \( p.t - n.t \) and adjust the exponents of the results consistently.

Finally, the algorithm terminates because we have two possibilities after we reduce \( p \) and \( n \) to the same exponent. When \( n.t \leq p.t \) (and the case \( n.t > p.t \) is analogous):

(i) If \( p.t \odot p.n < p.t \) then the largest \( t \) field decreases, and this can only happen a finite number of times.

(ii) If \( p.t \odot p.n = p.t \) then Lemma 1 implies that \( e = p.n < \epsilon p.t \). Since the number of elements in \( N \) is at most \( 1/\varepsilon \) by bound (2), this implies that \( p.t > s_n \cdot p.n \), and the algorithm returns 1 due to this condition.

This is only a high level description of the algorithm. A reasonably detailed version of it is presented in the last page of this article. The actual code is a bit more involved, due to optimizations which replace scaled_numbers by plain floating point numbers when possible. Readers interested in the implementation details should look at the C++ code available as supplementary material to the arxiv version of this article. This code is distributed under the Mozilla Public License 2.0.

4 Proofs

Here we prove the results stated above and two auxiliary results. Our proofs use the following characteristics shared by the usual binary floating point arithmetics with subnormal numbers, like the ones covered by the IEEE standards 754 and 854. There are three kinds of elements in the set \( \mathcal{F} \) of finite floating point numbers:

- 0 is a floating point number.
• \( x \in \mathcal{F} \) if and only if \(-x \in \mathcal{F}\).

• The subnormal numbers \( x \) have absolute value of the form
  \[ x = v \epsilon m, \quad \text{with} \quad m \in \mathbb{N} \quad \text{and} \quad 1 \leq m < 1/\epsilon. \quad (16) \]

• The normal numbers \( x \) have absolute value of the form
  \[ x = 2^e v \epsilon m \]
  for integers \( e \) and \( m \) such that
  \[ 0 \leq e \leq e_{\text{max}} := \log_2(\sigma) - \log_2(\nu) \quad \text{and} \quad 1 \leq \epsilon m < 2. \quad (18) \]

We use two auxiliary lemmas, and the proofs are presented after the statements of these lemmas. The lemmas are proved in the order in which they were stated.

**Lemma 4** If \( x \in \mathcal{F} \) and \( 2|x| \leq \max \mathcal{F} \), then \( 2x \in \mathcal{F} \). Therefore, if \( e \) is a positive integer and \( 2^e |x| \leq \max \mathcal{F} \), then \( 2^e x \in \mathcal{F} \). Similarly, if \( k \geq 0 \) and \( 2^{-k} |x| \geq \nu \) then \( 2^{-k} \otimes x = 2^{-k} x \) is normal. \( \blacklozenge \)

**Lemma 5** If the integer \( \ell \geq 0 \) and the real number \( x \) are such that
  \[ 2^{\ell} \leq \frac{x}{\nu} \leq 2^{\ell+1} \leq \sigma \]
then
  \[ \up(x) = 2^\ell \nu e \left\lfloor \frac{x}{2^\ell \nu} \right\rfloor. \quad (20) \]
\( \blacklozenge \)

**Proof of Lemma 4** If \( b = v \epsilon m_b \) is either subnormal or normal with a minimum exponent, then \( a \) is also of the form \( a = v \epsilon m_a \).

\( c = v \epsilon (m_b - m_a), \quad d = v \epsilon m_a, \quad e = 0 \quad \text{and} \quad b - a = c = c + e, \)
and Equation (3) holds. We can then assume that
  \[ b = 2^{e_b} v \epsilon m_b \quad \text{with} \quad e_b > 0 \quad \text{and} \quad 1 \leq \epsilon m_b < 2, \quad (21) \]
and
  \[ a = 2^{e_a} v \epsilon m_a \quad \text{with} \quad e_a \geq 0 \quad \text{and} \quad 1 \leq m_a < 2/\epsilon. \]
It follows that
  \[ b - a = v h \quad \text{for} \quad h := 2^{e_b} m_b - 2^{e_a} m_a > 0. \quad (22) \]
If \( h < 1/\epsilon \) then \( b - a \) is subnormal and
  \[ c = b \otimes a = b - a \Rightarrow d = a \Rightarrow e = 0 \Rightarrow b - a = c - e, \]
Equation (3) holds and we are done. Let us then assume that \( h \geq \epsilon \) and let \( \ell \geq 0 \) be the integer such that
  \[ 2^\ell \leq \frac{b-a}{\nu} = \epsilon h \leq 2^{\ell+1}. \quad (23) \]
By Lemma\[5\]

\[c = b \otimes a = 2^f v e \left[2^{-f} h \right] = 2^f v e \overline{\tau}\] (24)

for

\[\overline{\tau} := \left[2^{e_a - f} m_b - 2^{e_a - f} m_a \right].\] (25)

Equation (23) leads to the bound

\[h = 2^{e_a} m_b - 2^{e_a} m_a \geq 2^f / \epsilon\] (26)

and Equation (26) shows that

\[2^{e_a} m_b \geq 2^f / \epsilon \Rightarrow 2^{e_a} \geq \frac{2^f}{\epsilon m_b} > 2^{f-1} \Rightarrow e_b \geq \ell,\]

and \(2^{e_a-f} m_b\) is integer.

If \(2^{e_a-f} m_a < 1\) then

\[\overline{\tau} = 2^{e_a-f} m_b, \quad c = b, \quad d = b \otimes c = 0, \quad e = a \otimes d = a \Rightarrow b - a = c - e,\]

and the first part of Equation (3) holds. Moreover, Equation (26) yields

\[e = a = 2^{e_a} m_a v e < 2^f v e < (2^{e_a} m_b \epsilon) v e = e_b = e \epsilon,\]

and the second part of Equation (3) also holds. Therefore, we can assume that

\[2^{f-e_a} \leq m_a < 2 / \epsilon.\] (27)

If \(e_a \geq \ell\) then \(2^{e_a-f}\) is integer,

\[\overline{\tau} = 2^{e_a-f} m_b - 2^{e_a-f} m_a, \quad c = b-a, \quad d = b \otimes c = a, \quad e = a \otimes d = 0 \Rightarrow b-a = c-e\]

and Equation (3) holds again. Therefore, we can assume that \(e_a < \ell\) and

\[q := \ell - e_a > 0.\] (28)

Equation (27) shows that \(2^{-q} m_a \geq 2^{e_a-f} \times 2^{f-e_a} = 1\) and the integers

\[\overline{\alpha} := [2^{-q} m_a] \quad \text{and} \quad a = m_a - 2^q \overline{\alpha}\]

are such that

\[m_a = 2^q \overline{\alpha} + a, \quad 1 \leq \overline{\alpha} \leq m_a / 2 < 1 / \epsilon \quad \text{and} \quad 0 \leq a < 2^q \leq m_a < 2 / \epsilon.\] (29)

It follows that

\[2^{e_a-f} m_b - 2^{e_a-f} m_a = 2^{e_a-f} m_b - 2^{-q} m_a = 2^{e_a-f} m_b - \overline{\alpha} - 2^{-q} a\]

and the bound \(0 \leq a < 2^q\) leads to

\[\overline{\tau} = [2^{e_a-f} m_b - 2^{e_a-f} m_a] = 2^{e_a-f} m_b - \overline{\alpha}\]

and

\[c = b \otimes a = b - 2^f v e \overline{\alpha} = b - \hat{a} \quad \text{for} \quad \hat{a} := 2^f v e \overline{\alpha}.\] (30)
The second bound in Equation (29) shows that \( v \varepsilon a \) is subnormal and Lemma 4 shows that \( \hat{a} \in \mathcal{F} \). This implies that

\[
d = b \odot c = b - c = \hat{a}.
\]  \hspace{1cm} (31)

Additionally,

\[
a - d = 2^{e_a} v \varepsilon a - 2^{e} v \varepsilon a = 2^{e_a} v \varepsilon (m_a - 2^{e} a) = 2^{e_a} v \varepsilon a = \hat{a}.
\]  \hspace{1cm} (32)

If \( a = 0 \) then \( \hat{a} \in \mathcal{F} \). If \( a < 1/e \) then the same argument used for \( \hat{a} \) shows that \( \hat{a} \in \mathcal{F} \), and if \( 1/e \leq a < 2/e \) then \( v \varepsilon a \) is normal and Lemma 4 shows that \( \hat{a} \in \mathcal{F} \). Therefore, in all cases for \( a \) in Equation (29) we have that \( \hat{a} \in \mathcal{F} \) and

\[
e = a \odot d = a - d = \hat{a}.
\]  \hspace{1cm} (33)

Equations (31) and (32) show that \( \hat{a} + \hat{a} = a \) and Equations (30) and (33) yield

\[
c - e = b - \hat{a} - \hat{a} = b - a,
\]

and the first part of Equation (3) holds. Finally, Equations (24), (25), (26), (28) and (29) imply that

\[
\epsilon c \geq 2^{f} v \varepsilon = 2^{e_a} \varepsilon v \varepsilon > 2^{e_a} a \varepsilon v \varepsilon = \hat{a}.
\]

and the second part of Equation (3) holds.

\[\text{Proof of Lemma 2}\] Let us start with the case in which \( c = a \odot b > \tau = 2v/\epsilon \). In this case \( b > \sqrt{2v/\epsilon} > \nu \) and \( b \) is normal. Therefore,

\[
b = 2^{e_b} m_b \quad \text{with} \quad 1 \leq \epsilon m_b < 2.
\]  \hspace{1cm} (34)

\( a \) can be normal or subnormal, but in both cases there exist integers \( e_a, \ell_a \) and \( m_a \) with

\[
a = 2^{e_a} m_a, \quad 2^{e_a} \leq m_a < 2^{e_a+1} \quad \text{and} \quad 1 \leq 2^{e_a} < 1/\epsilon.
\]  \hspace{1cm} (35)

If \( \ell > 0 \) is the integer such that

\[
2^{\ell} \leq \frac{a b}{v} < 2^{\ell+1}
\]  \hspace{1cm} (36)

then

\[
2^{\ell+1} \geq \frac{a b}{v} > 1/\epsilon \Rightarrow 2^{\ell} > 1/(2\epsilon) > 1 \Rightarrow \ell > 0,
\]

and Lemma 5 and Equations (34) and (35) show that

\[
a \odot b = 2^{\ell} v \varepsilon \left[ \frac{2^{e_a+e_b} m_a m_b}{2^{\ell} v \varepsilon} \right] = \left[ \frac{m_a m_b}{p} \right]
\]  \hspace{1cm} (37)

for

\[
p := 2^{e_a-e_b} v \varepsilon.
\]  \hspace{1cm} (38)

Since

\[
\frac{a b}{2^{\ell} v \varepsilon} = \frac{m_a m_b}{p},
\]

Equation (36) implies that

\[
\frac{a b}{v} = 2^{\ell} e m_a m_b < 2^{\ell+1}
\]  \hspace{1cm} (39)
and Equations (34) and (35) yield
\[ p > \epsilon m_a m_b / 2 = (m_a / 2) (\epsilon m_b) \geq 2^{\ell_a - 1} \geq 1 / 2 \Rightarrow p > 1 / 2. \]
Since \( p \) is a power of two, this implies that \( p \geq 1 \) is integer. On the other hand, Equation (36) implies that
\[ \frac{2^\ell \epsilon m_a m_b}{p} \geq 2^\ell \]
and Equations (34) and (35) lead to
\[ p \leq \epsilon m_a m_b < 4 / \epsilon \Rightarrow p \leq 2 / \epsilon. \]
Euclid’s division algorithm yields integers \( q \) and \( r \) such that
\[ m_a m_b = qp - r \quad \text{and} \quad 0 \leq r < 2 / \epsilon, \] (39)
and Equation (37) shows that
\[ c = a \otimes b = 2^\ell \veq [q - p^{-1} r] = 2^\ell \veq = ab + 2^\ell p^{-1} \ve. \] (40)
Equations (34) and (35) imply that
\[ m_a m_b < 2^\ell + 1 / \epsilon < \frac{4}{\epsilon^2} \Rightarrow \frac{1}{m_a m_b} > rac{\epsilon^2}{4}, \]
the bound \( a \otimes b > 2 \nu / \epsilon \) yields \( ab \epsilon > 2 \nu \) and
\[ 2^\ell \nu^{-1} = \frac{2^\ell \epsilon a \otimes b}{\nu} = \frac{1}{\epsilon^2 m_a m_b} \frac{2^\ell \epsilon a \otimes b}{\nu} = \frac{1}{\epsilon^2 m_a m_b} \frac{ab \epsilon}{\nu} > \frac{2}{\epsilon^2 m_a m_b} > \frac{1}{2}. \]
Since \( 2^\ell \nu^{-1} \) is a power of 2, this implies that \( 2^\ell \nu^{-1} \) is integer, and the last inequality in Equation (39) and Lemma 4 imply that
\[ d := 2^\ell \nu^{-1} \ve \in \mathbb{F}. \]
Combining this with Equations (40) we obtain that
\[ ab = 2^\ell \veq - d = c - d \]
and
\[ (-a) \nu + c = d \in \mathbb{F} \Rightarrow \fma(-a, b, c) = d. \]
This completes the proof of Equation (8).
If \( a \otimes b \leq \tau \) then
\[ a^2 \leq ab \leq \tau \Rightarrow a \leq \sqrt{\frac{2 \nu}{\epsilon}} < 1 \Rightarrow \sigma a < \sigma \Rightarrow \tilde{a} = \sigma a = \sigma \otimes a \in \mathbb{F}, \]
and using the same argument used to prove Equation (8) we can prove Equation (10).
Finally, the smallest value possible for a positive floating point number is \( \ve \), and if the conditions (7) and (9) are not satisfied then
\[ \tilde{a} = \sigma a \geq \sigma \ve. \]
The violation of condition (5) and the first inequality in Equation (6) yield
\[ b \leq \frac{2\nu}{e\delta} \leq \frac{2\nu}{e\sigma\nu} \Rightarrow \sigma b \leq \frac{2}{e^2} \leq \sigma. \]
This bound implies that \( \hat{b} = \sigma \otimes b = \sigma b \in \mathcal{F}. \) We also have that \( \hat{b} \geq \sigma v \) and the second inequality in Equation (6) leads to
\[ \hat{a}\hat{b} \geq \sigma^2 v^2 e^2 = \frac{\sigma^2 v^2}{2} > \frac{2\nu}{e} = \tau. \]
This condition allows us to use the same argument used to prove the validity of Equations (8) and (10) to prove Equations (11), and this proof is complete. \( \blacksquare \)

**Proof of Lemma 3.** If \( x.\exp > y.\exp \) then \( x.\exp > 0 \) and item (ii) implies that \( x.\exp \geq y.\exp + 1 \) and using item (i) we derive that
\[ \text{value}(x) = \sigma^{-x.\exp}x.\exp \leq \sigma^{-y.\exp}y.\exp < \sigma^{-y.\exp}y.\exp \Rightarrow y.\exp = \text{value}(y), \]
and the function \texttt{is_less} returns the correct value in this branch. The branch \( x.\exp < y.\exp \) is analogous. Finally, in the last branch the exponents cancel out and we are left with the correct comparison of the \( t \) fields. \( \blacksquare \)

**Proof of Lemma 4.** If \( x = 0 \) then \( 2x = 0 \in \mathcal{F}. \) If \( x \) is normal, then Lemma 4 follows directly from Equation (17). If \( x = \text{ve}m \) is subnormal then there are two possibilities: If \( |2m| < 1/e \) then \( 2x = \nu \text{e}(2m) \) is also subnormal. If \( |2m| \geq 1/e \) then \( 2x = \nu \text{e}(2m) \) is normal, because \( 1/e \leq |m| < 2/e. \) The part of the Lemma regarding division follows directly from the definition of normal number. \( \blacksquare \)

**Proof of Lemma 5.** If \( x \) satisfies the condition (19) then
\[ 1/e \leq q := \frac{x}{2^\nu \nu \epsilon} \leq 2/e. \]
Therefore,
\[ 1/e \leq [q] \leq 2/e. \]
The number
\[ \overline{x} := 2^\nu \nu \epsilon [q] \]
belongs to \( \mathcal{F} \) because if \( [q] < 2/e \) then it fits on definition (17) with \( e = \ell \) and \( m = [q], \) and if \( [q] = 2/e \) then
\[ \overline{x} = 2^{\ell+1} \nu \epsilon / \epsilon \]
and definition (17) holds with \( e = \ell + 1 \) and \( m = 1/e. \) We have that
\[ 2^\nu \nu \epsilon q = x \leq \overline{x}, \]
and the definition of \( \text{up}(x) \) in (11) implies that \( \text{up}(x) \leq \overline{x}. \) To prove that \( \text{up}(x) = \overline{x}, \) we now show that if \( y \in \mathcal{F} \) is such that \( y \geq x \) then \( y \geq \overline{x}. \) In fact, if \( y \geq x \) then \( y \geq 2^\nu \nu \epsilon \) and this implies that \( y \) is normal, that is,
\[ y = 2^\nu \nu \epsilon m_y \quad \text{with} \quad 1/e \leq m_y < 2/e \quad \text{and} \quad 2^\nu \nu \epsilon m_y \geq x = 2^\nu \nu \epsilon q. \]
This leads to
\[ 2^\nu m_y \geq 2^\nu q \quad \Rightarrow \quad 2^\nu m_y \geq q/m_y \geq (1/e)/(2/e) = 1/2 \Rightarrow e_y \geq \ell, \]
and \( 2^\nu m_y \) is integer. As a result
\[ 2^\nu m_y \geq 2^\nu q \Rightarrow 2^\nu m_y \geq q \Rightarrow 2^\nu m_y \geq [q] \Rightarrow y = 2^\nu \nu \epsilon m_y \geq 2^\nu \nu \epsilon [q] = \overline{x}, \]
and we are done. \( \blacksquare \)
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Algorithm 3 Sign of Sums of Products

1: procedure \texttt{sign}(p...) \hfill \triangleright \text{sign of the sum } p[0] + p[1] + ...
2: \hspace{1em} \texttt{round}(r(up)) \hfill \triangleright \text{Rounding object, resets the mode on exit}
3: \hspace{1em} heap neg, pos
4: \hspace{1em} for each p do
5: \hspace{2em} \texttt{split}(neg, pos, p) \hfill \triangleright \text{Each } p \text{ is a product}
6: \hspace{1em} while true do
7: \hspace{2em} sn, sp ← size(neg), size(pos)
8: \hspace{3em} if sn = 0 then
9: \hspace{4em} return (sp = 0) ? 0 : 1
10: \hspace{3em} if sp = 0 then
11: \hspace{4em} return -1
12: \hspace{3em} n, p ← pop(neg), pop(pos)
13: \hspace{4em} if n.exp ≤ p.exp then \hfill \triangleright \text{small } n.\exp \Rightarrow \text{large } n = 2^{-n.\exp} n.t
14: \hspace{4em} if n.exp < p.exp then
15: \hspace{5em} return -1 \hfill \triangleright \text{p is too small}
16: \hspace{4em} if n.t > p.t then
17: \hspace{5em} pos ← scaled_number(e, n.exp)
18: \hspace{4em} c, e ← lemma_1(p.t, n.t) \hfill \triangleright \text{split } n.t - p.t \text{ as in Lemma[1]}
19: \hspace{5em} \text{neg ← scaled_number}(c, n.exp)
20: \hspace{3em} else
21: \hspace{4em} if p.t > n.t then
22: \hspace{5em} c, e ← lemma_1(n.t, p.t) \hfill \triangleright \text{split } p.t - n.t \text{ as in Lemma[1]}
23: \hspace{5em} pos ← scaled_number(c, n.exp)
24: \hspace{4em} if e > 0 then
25: \hspace{5em} \text{neg ← scaled_number}(e, n.exp)
26: \hspace{3em} else
27: \hspace{4em} if n.exp ≥ p.exp + 1 then
28: \hspace{5em} return 1 \hfill \triangleright n \text{ is too small}
29: \hspace{5em} n.t ← n.t * 1/\sigma \hfill \triangleright n.t/\sigma \text{ is exact or } 1 \text{ is returned in line 42}
30: \hspace{4em} if p.t > sn * n.t then
31: \hspace{5em} return 1
32: \hspace{5em} c, e ← lemma_1(n.t, p.t) \hfill \triangleright \text{split } p.t - n.t \text{ as in Lemma[1]}
33: \hspace{5em} pos ← scaled_number(c, p.exp)
34: \hspace{4em} if e > 0 then
35: \hspace{5em} \text{neg ← scaled_number}(e, p.exp)
36: \hspace{3em} end while
37: \hspace{1em} end procedure

\[\]