On the optimal domain for minimizing the buckling load of a clamped plate

Kathrin Stollenwerk

Abstract. We prove the existence of an optimal domain for minimizing the buckling load among all, possibly unbounded, open subsets of $\mathbb{R}^n$ ($n \geq 2$) with given measure. Our approach is based on the extension of a two-dimensional existence result of Ashbaugh and Bucur and on the idea of Alt and Caffarelli to focus on the eigenfunction.

Mathematics Subject Classification. 49Q10.

Keywords. Buckling load, Clamped plate, Optimization of shapes.

1. Introduction

We consider the following variational problem. Let $\Omega \subset \mathbb{R}^n$ be an open set and define

$$\mathcal{R}(v, \Omega) := \frac{\int_{\Omega} |\Delta v|^2 dx}{\int_{\Omega} |\nabla v|^2 dx}$$

for $v \in W_0^{2,2}(\Omega)$ (see (2)). If the denominator vanishes, we set $\mathcal{R}(v, \Omega) = \infty$. The buckling load of the clamped plate $\Omega$ is defined as

$$\Lambda(\Omega) := \min\{\mathcal{R}(v, \Omega) : v \in W_0^{2,2}(\Omega)\}.$$ 

In 1951, Polya and Szegö conjectured that the ball minimizes the buckling load among all open sets of given measure (see [12]). It is still an open question to confirm their conjecture. Up to now, there are only partial results known.

If there exists a smooth, bounded, connected and simply connected open set $\Omega$ which minimizes the buckling load among all open sets of given measure in $\mathbb{R}^n$, it is known that $\Omega$ is a ball (see [14,15]).

In [13], the existence of an optimal domain for minimizing the buckling load among all open sets of a given measure which are contained in a sufficiently large ball $B \subset \mathbb{R}^n$, $n = 2, 3$, is proven. However, [13] does not provide any information about the regularity of the achieved optimal domain.

Ashbaugh and Bucur proved the existence of a plane optimal domain for minimizing $\Lambda$ in two different settings [3]. On the one hand, they prove the existence of a minimizer in the family of connected and simply connected open sets of given measure in $\mathbb{R}^2$. On the other hand, they find an optimal set $\bar{\Omega}$ for minimizing a relaxed version of the buckling load among all open sets of given measure in $\mathbb{R}^2$.

In the present paper, we will adapt a part of the approach by Ashbaugh and Bucur. Therefore, let us briefly summarize their idea. For $\omega_0 > 0$ let us denote

$$\mathcal{O}_{\omega_0} := \{\Omega \subset \mathbb{R}^2 : \Omega \text{ open, } |\Omega| \leq \omega_0\},$$

where $|\Omega|$ denotes the measure of $\Omega$. The existence of an optimal domain for minimizing $\Lambda$ among all open sets of given measure in $\mathbb{R}^n$ is stated in [3].
where $|Ω|$ denotes the $n$-dimensional Lebesgue measure of $Ω ⊂ \mathbb{R}^n$. Ashbaugh and Bucur start from a minimizing sequence $(Ω_k)_k ⊂ O_{ω_0}$ and the sequence $(u_k)_k$ of corresponding normalized buckling eigen-functions $u_k ∈ W^{2,2}_{0}(Ω_k)$. Applying a concentration-compactness lemma, they deduce the existence of a limit function $u ∈ W^{2,2}(\mathbb{R}^2)$ such that

$$ R(u, \mathbb{R}^2) ≤ \liminf_{k → ∞} R(u_k, Ω_k) = \liminf_{k → ∞} Λ(Ω_k) = \inf_{Ω ∈ O_{ω_0}} Λ(Ω). $$

(1)

Since $u ∈ W^{2,2}(\mathbb{R}^2)$, Sobolev’s embedding theory implies that $u$ is continuous and the set

$$ Ω := \{ x ∈ \mathbb{R}^2 : u(x) ≠ 0 \} $$

is an open set. Moreover, the strong $L^2$-convergence of $u_k$ to $u$ implies that $|Ω| ≤ ω_0$. Hence, $Ω ⊂ O_{ω_0}$. At this point the authors face the difficulty that their ansatz does not provide any further information about $Ω$ and $u$ except that $Ω ⊂ O_{ω_0}$ and $u$ is continuous. In particular, they cannot conclude that $u ∈ W^{2,2}(Ω)$. They circumvent that problem by introducing the relaxed Sobolev space $\tilde{W}^{2,2}(Ω)$ by

$$ \tilde{W}^{2,2}(Ω) := \{ v ∈ W^{2,2}(\mathbb{R}^2) : v = 0 \text{ a.e. in } \mathbb{R}^2 \setminus Ω \} $$

and the relaxed buckling load by

$$ \tilde{Λ}(Ω) := \min_{v ∈ \tilde{W}^{2,2}(Ω)} R(v, Ω). $$

By construction, $u ∈ \tilde{W}^{2,2}(Ω)$ and, consequently, $Ω$ minimizes $\tilde{Λ}$ in $O_{ω_0}$, i.e.,

$$ \tilde{Λ}(Ω) \stackrel{(1)}{=} \inf_{Ω ∈ O_{ω_0}} Λ(Ω), $$

where [3, Theorem 3.1] provides the last equation.

In this paper, we will adapt the idea of Ashbaugh and Bucur in [3] and extend it to arbitrary dimension. Contrary to their construction via $\tilde{W}^{2,2}$ we prove higher regularity of the limit function $u$. Thereby we follow the idea of Alt and Caffarelli in [2]. We will find that the first-order derivatives of $u$ are $α$-Hölder continuous in $\mathbb{R}^n$ for every $α ∈ (0, 1)$.

Recall (c.f. [1, Th. 9.1.3] or [8, Sec. 3.3.5]) that for an open set $Ω ⊂ \mathbb{R}^n$ and $v ∈ W^{2,2}(\mathbb{R}^n)$ there holds

$$ v ∈ W^{2,2}(Ω) \text{ if and only if } v = |∇v| = 0 \text{ quasi-everywhere in } \mathbb{R}^n \setminus Ω. $$

(2)

Thereby, a property $p(x)$ is said to hold ‘quasi-everywhere’ if the set of points in which $p(x)$ does not hold true has got zero capacity. Since the concept of capacity of a set will not play any further role in the sequel of the present paper, we refer to [8] for more details on the concept of capacity. However, if a property holds pointwise, it holds quasi-everywhere, as well.

Consequently, the Hölder continuity of the first-order derivatives of $u$ implies that $u ∈ W^{0,2}(Ω^*)$ for

$$ Ω^* := \{ x ∈ \mathbb{R}^n : u(x) ≠ 0 \text{ and } ∇u(x) ≠ 0 \}. $$

In addition, $Ω^*$ satisfies $|Ω^*| = ω_0$ and we deduce that $Ω^*$ minimizes the buckling load among all open sets of given measure in $\mathbb{R}^n$.

Moreover, we will show that the minimizer $Ω^*$ is connected.

### 2. Existence of a minimizer

For $ω_0 > 0$ we denote the class of admissible sets by

$$ O_{ω_0} := \{ Ω ⊂ \mathbb{R}^n : Ω \text{ open}, |Ω| ≤ ω_0 \}, $$

where $|Ω|$ denotes the $n$-dimensional Lebesgue measure of $Ω ⊂ \mathbb{R}^n$, $n ≥ 2$.

Our aim is to prove the existence of a set $Ω^* ∈ O_{ω_0}$ which minimizes $Λ$ in $O_{ω_0}$. In the beginning, we follow the idea of [3].
Let \((\Omega_k)_k \in \mathcal{O}_{\omega_0}\) be a minimizing sequence for the buckling load, i.e.,

\[
\lim_{k \to \infty} \Lambda(\Omega_k) = \inf_{\Omega \in \mathcal{O}_{\omega_0}} \Lambda(\Omega) = \Lambda_{\omega_0}.
\]

By \(u_k \in W^{2,2}_{0}(\Omega_k)\) we denote the normalized buckling eigenfunction on \(\Omega_k\). Hence, \(u_k\) satisfies

\[
\int_{\Omega_k} |\nabla u_k|^2 \, dx = 1 \quad \text{and} \quad \Lambda(\Omega_k) = \int_{\Omega_k} |\Delta u_k|^2 \, dx.
\]

We now apply the approach by Ashbaugh and Bucur from [3] to show that \((u_k)_k\) converges weakly to a limit function \(u\) in \(W^{2,2}(\mathbb{R}^n)\).

We will use the following concentration-compactness lemma (see [3,10]) adapted to our setting.

**Lemma 1.** Let \((\Omega_k)_k \subset \mathcal{O}_{\omega_0}\) be a minimizing sequence for the buckling load in \(\mathcal{O}_{\omega_0}\) and \((u_k)_k\) be the sequence of corresponding eigenfunctions. Then there exists a subsequence \((u_k)_k\) such that one of the three following situations occurs.

1. **Compactness** \(\exists (y_k)_k \subset \mathbb{R}^n\) such that \(\forall \varepsilon > 0, \exists R < \infty\) and

   \[
   \forall k \in \mathbb{N} \quad \int_{B_R(y_k)} |\nabla u_k|^2 \, dx \geq 1 - \varepsilon.
   \]

2. **Vanishing** \(\forall R \in (0, \infty)\)

   \[
   \lim_{k \to \infty} \sup_{y \in \mathbb{R}^n} \int_{B_R(y)} |\nabla u_k|^2 \, dx = 0.
   \]

3. **Dichotomy** There exists a \(\beta \in (0,1)\) such that \(\forall \varepsilon > 0\) there exist two bounded sequences \((u^1_k)_k, (u^2_k)_k \subset H^{2,2}(\mathbb{R}^n)\) such that:

   \[
   \|\nabla u_k - \nabla u^1_k - \nabla u^2_k\|_{L^2(\mathbb{R}^2, \mathbb{R}^2)} \leq \delta(\varepsilon) \xrightarrow{k \to \infty} 0^+\quad (a),
   \]

   \[
   |\int_{\mathbb{R}^n} |\nabla u^1_k|^2 \, dx - \beta| \to 0 \quad \text{and} \quad |\int_{\mathbb{R}^n} |\nabla u^2_k|^2 \, dx - (1 - \beta)| \to 0,\quad (b)
   \]

   \[
   \text{dist}(\text{supp}(u^1_k), \text{supp}(u^2_k)) \xrightarrow{k \to \infty} \infty,\quad (c)
   \]

   \[
   \liminf_{k \to \infty} \left[ \int_{\mathbb{R}^n} |\Delta u_k|^2 - |\Delta u^1_k|^2 - |\Delta u^2_k|^2 \, dx \right] \geq 0.\quad (d)
   \]

**Proof.** As mentioned in the proof of [3, Lemma 3.5], the proof is done by considering the concentration function

\[
R \to Q_k(R) := \sup_{y \in \mathbb{R}^n} \int_{B_R(y)} |\nabla u_k|^2 \, dx
\]

for \(R \in [0, \infty)\) and following the same steps as in [10].

We will see that for the sequence of eigenfunctions \((u_k)_k\) the case of vanishing and dichotomy cannot occur. Hence, \((u_k)_k\) contains a subsequence, which we again denote by \((u_k)_k\), for which the case of compactness holds true. This compactness will imply the weak convergence of \(u_k\) to a limit function \(u\) in \(W^{2,2}(\mathbb{R}^2)\). Moreover, the compactness yields that \(u_k\) converges to \(u\) strongly in \(W^{1,2}(\mathbb{R}^n)\).
The case of dichotomy can be disproved in exactly the same way as in [3]. For the sake of brevity, we forgo the repetition of this argument.

In order to disprove the case of vanishing, we slightly differ from [3]. Nevertheless, we adopt the following lemma [4, Lemma 3.3] (or [9, Lemma 6]) which is used in [3] and which we will apply to disprove the vanishing, as well.

**Lemma 2.** Let \((w_k)_k\) be a bounded sequence in \(W^{1,2}(\mathbb{R}^n)\) such that \(\|w_k\|_{L^2(\mathbb{R}^n)} = 1\) and \(w_k \in W^{1,2}_0(D_k)\) for a \(D_k \in \mathcal{O}_{\omega_k}\). There exists a sequence of vectors \((y_k)_k \subset \mathbb{R}^n\) such that the sequence \((w_k(\cdot + y_k))_k\) does not possess a subsequence converging weakly to zero in \(W^{1,2}_0(\mathbb{R}^n)\).

Now let us assume that for a subsequence of \((u_k)_k\), again denoted by \((u_k)_k\), the case of vanishing occurs. Hence, for every \(R > 0\) there holds

\[
\lim_{k \to \infty} \sup_{y \in \mathbb{R}^n} \int_{B_R(y)} |\nabla u_k|^2 dx = 0. \tag{3}
\]

Since there holds \(\|\nabla u\|_{L^2(\mathbb{R}^n)} = 1\) for every \(k \in \mathbb{N}\), we obtain for at least one \(1 \leq l_k \leq n\)

\[
\int_{\mathbb{R}^n} |\partial_{l_k} u_k|^2 dx \geq \frac{1}{n}.
\]

We now consider the sequence \((\partial_{l_k} u_k)_k\). Then \(\partial_{l_k} u_k \in W^{1,2}_0(\Omega_k)\) and

\[
\frac{1}{\sqrt{n}} \leq \|\partial_{l_k} u_k\|_{L^2(\mathbb{R}^n)} := c_k.
\]

The sequence \((v_k)_k\) given by \(v_k := c_k^{-1} \partial_{l_k} u_k\) then satisfies the assumptions of Lemma 2. Consequently, there exists a sequence \((y_k)_k \subset \mathbb{R}^n\) such that the sequence \((v_k(\cdot + y_k))_k \subset W^{1,2}(\mathbb{R}^n)\) does not possess a subsequence which converges weakly to zero in \(W^{1,2}(\mathbb{R}^n)\). However, the sequence \((v_k(\cdot + y_k))_k\) is uniformly bounded in \(W^{1,2}(\mathbb{R}^n)\) because of the normalization. Hence, there exists a \(v \in W^{1,2}(\mathbb{R}^n)\) such that a subsequence of \((v_k(\cdot + y_k))_k\) converges weakly in \(W^{1,2}(\mathbb{R}^n)\) to \(v\). In particular, there holds

\[
v_k(\cdot + y_k) \xrightarrow{k \to \infty} v \text{ in } W^{1,2}(B_R(0)) \text{ for every } R > 0
\]

and

\[
v_k(\cdot + y_k) \xrightarrow{k \to \infty} v \text{ in } L^2(B_R(0)) \text{ for every } R > 0.
\]

Thus, we obtain

\[
\|v\|^2_{L^2(B_R(0))} = \lim_{k \to \infty} \|v_k(\cdot + y_k)\|^2_{L^2(B_R(0))} = \lim_{k \to \infty} \frac{1}{c_k^2} \int_{B_R(0)} |\partial_{l_k} u_k(x + y_k)|^2 dx \leq n \lim_{k \to \infty} \int_{B_R(0)} |\nabla u_k|^2 dx \leq n \lim_{k \to \infty} \sup_{y \in \mathbb{R}^n} \int_{B_R(y)} |\nabla u_k|^2 dx \overset{(3)}{=} 0.
\]

Hence, \(v = 0\) in \(L^2(B_R(0))\) and since \(v\) is the weak limit of \(v_k(\cdot + y_k)\) this is a contradiction to Lemma 2. Therefore, the case of vanishing cannot occur.

Consequently, the case of compactness must occur. Following the lines of [3], we find that there exists a sequence \((y_k)_k \subset \mathbb{R}^n\) and an \(u \in W^{2,2}(\mathbb{R}^n)\) such that

\[
u_k(\cdot + y_k) \rightharpoonup u \text{ in } W^{2,2}(\mathbb{R}^n) \tag{4}
\]
and, since we are in the compactness case of Lemma 1,
\[ \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = 1. \] (5)

From now on, we set
\[ u_k = u_k(\cdot + y_k) \quad \text{and} \quad \Omega_k = \Omega_k + y_k, \]
where \((y_k)_k\) is given above. This is possible without loss of generality because of the translational invariance of the buckling load.

We now show that \(u_k\) converges strongly to \(u\) in \(W^{1,2}(\mathbb{R}^n)\). Since this observation will be crucial for constructing an optimal domain in Sect. 2.2, we give a detailed proof although we follow the lines of [3].

**Lemma 3.** There holds
\[ u_k \overset{k \to \infty}{\longrightarrow} u \quad \text{in} \quad W^{1,2}(\mathbb{R}^n). \]

**Proof.** We use the notation above. Recall, that \(u_k = u_k(\cdot + y_k)\) and \(\Omega_k = \Omega_k + y_k\). Then we get from (5)
\[ \int_{\mathbb{R}^n} |\nabla u - \nabla u_k|^2 \, dx = 2 - 2 \int_{\mathbb{R}^n} \nabla u. \nabla u_k \, dx \]
and the weak convergence of \((u_k)_k\) to \(u\) in \(W^{2,2}(\mathbb{R}^n)\) yields
\[ \int_{\mathbb{R}^n} |\nabla u - \nabla u_k|^2 \, dx \overset{k \to \infty}{\longrightarrow} 0. \]
Thus, \((\nabla u_k)_k\) converges to \(\nabla u\) in \(L^2(\mathbb{R}^n)\) and, in particular, \((\nabla u_k)_k\) is a Cauchy sequence in \(L^2(\mathbb{R}^n, \mathbb{R}^n)\). Now let \(l, k \in \mathbb{N}\). Then \(u_l - u_k \in W^{0,2}_{0,0}(\Omega_l \cup \Omega_k)\) and applying Poincaré’s inequality we obtain
\[ \int_{\Omega_l \cup \Omega_k} (u_l - u_k)^2 \, dx \leq \left( \frac{|\Omega_l \cup \Omega_k|}{\omega_n} \right)^{\frac{2}{n}} \int_{\Omega_l \cup \Omega_k} |\nabla (u_l - u_k)|^2 \, dx \]
\[ \leq \left( \frac{2\omega_0}{\omega_n} \right)^{\frac{2}{n}} \int_{\Omega_l \cup \Omega_k} |\nabla (u_l - u_k)|^2 \, dx \overset{k \to \infty}{\longrightarrow} 0. \]
Thus, \((u_k)_k\) is a Cauchy sequence in \(L^2(\mathbb{R}^n)\), which converges weakly in \(L^2(\mathbb{R}^n)\) to \(u\). Consequently, \(u_k \overset{k \to \infty}{\longrightarrow} u\) in \(L^2(\mathbb{R}^n)\). This proves the claim. \(\square\)

As a consequence of (4) and Lemma 3, we obtain that
\[ \mathcal{R}(u, \mathbb{R}^n) \leq \liminf_{k \to \infty} \mathcal{R}(u_k, \Omega_k) = \inf_{\Omega \in \mathcal{O}_{\omega_0}} \Lambda(\Omega). \] (6)

The following proposition summarizes what we have achieved so far.

**Proposition 1.** Let \((\Omega_k)_k \subset \mathcal{O}_{\omega_0}\) be a minimizing sequence for the buckling load in \(\mathcal{O}_{\omega_0}\) and \((u_k)_k\) be the sequence of corresponding normalized eigenfunctions. Then there exists a sequence \((y_k)_k \subset \mathbb{R}^n\) such that \(u_k(\cdot + y_k)\) is a normalized eigenfunction on \(\Omega_k + y_k\) and, denoting \(u_k = u_k(\cdot + y_k)\) and \(\Omega_k = \Omega_k + y_k\), there exists a subsequence, again denoted by \((u_k)_k\), and an \(u \in W^{2,2}(\mathbb{R}^n)\) with

1. \(u\) is normalized by
\[ \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = 1. \]
2. \(u_k \to u\) in \(W^{2,2}(\mathbb{R}^n)\) as \(k\) tends to \(\infty\).
3. \( u_k \to u \) in \( W^{1,2}(\mathbb{R}^n) \) as \( k \) tends to \( \infty \).

4. There holds \( R(u, \mathbb{R}^n) \leq \inf_{\Omega \in \mathcal{O}_{\omega_0}} \Lambda(\Omega) \).

Recall that in [3] only the two-dimensional case is considered. Consequently, the limit function \( u \) is continuous due to Sobolev’s embedding theory. Hence, the set

\[ \tilde{\Omega} := \{ x \in \mathbb{R}^2 : u(x) \neq 0 \} \]

is an open set and the strong \( L^2 \)-convergence of \( u_k \) to \( u \) implies that \( \tilde{\Omega} \in \mathcal{O}_{\omega_0} \).

Here, we consider arbitrary dimension. Hence, we need another method to prove regularity of the function \( u \). Inspired by [2], our approach is based on a careful analysis of the function \( u \). This will be done in the next section.

### 2.1. Regularity of the limit function

Our first aim is to show that \( u \) has got Hölder continuous first-order derivatives. This will be done by using Morrey’s Dirichlet Growth Theorem (see Theorem 1) and a bootstrapping argument based on ideas of Q. Han and F. Lin in [7].

From now on, we consider a minimizing sequence \( (\Omega_k) \subset \mathcal{O}_{\omega_0} \) such that there holds

\[ \Lambda_{\omega_0} := \inf_{\Omega \in \mathcal{O}_{\omega_0}} \Lambda(\Omega) \leq \Lambda(\Omega_k) \leq \Lambda_{\omega_0} + \frac{1}{k} \text{ for every } k \in \mathbb{N}. \tag{7} \]

We want to apply the following version of Morrey’s Dirichlet Growth Theorem to the first-order derivatives of \( u \).

**Theorem 1.** Let \( v \in W^{1,2}(\mathbb{R}^n) \) and \( 0 < \alpha \leq 1 \) such that for every \( x_0 \in \mathbb{R}^n \) and every \( 0 < r \leq r_0 \) there holds

\[ \int_{B_r(x_0)} |\nabla v|^2 dx \leq M \cdot r^{n-2+2\alpha}. \]

Then \( v \) is \( \alpha \)-Hölder continuous almost everywhere in \( \mathbb{R}^n \) and for almost every \( x_1, x_2 \in \mathbb{R}^n \) there holds

\[ \left| \frac{v(x_1) - v(x_2)}{|x_1 - x_2|^\alpha} \right| \leq C(\alpha) \cdot M. \]

For a proof of this theorem we refer to [11, Theorem 3.5.2], for example. Hence, we need a \( L^2 \)-estimate for the second-order derivatives of \( u \) in every ball \( B_r(x_0) \subset \mathbb{R}^n \).

The following lemmata are preparatory for the proof of Theorem 2, which is the main theorem of this section. Before we start, note that by scaling there holds

\[ \Lambda_{\omega_0} \leq \left( \frac{\omega_n}{\omega_0} \right)^{\frac{2}{n}} \Lambda(B_1) \leq C(n, \omega_0), \tag{8} \]

where \( B_1 \) denotes the unit ball in \( \mathbb{R}^n \).

**Lemma 4.** Let \( u \in W^{2,2}(\mathbb{R}^n) \) be the limit function according to Proposition 1 and \( 0 < R \leq 1 \). There exists a constant \( C = C(n, \omega_0) > 0 \) such that for every \( x_0 \in \mathbb{R}^n \) there holds

\[ \int_{B_R(x_0)} |\Delta(u - v_0)|^2 dx \leq C(n, \omega_0) \left( R^n + \int_{B_R(x_0)} |\nabla u|^2 dx \right), \]

where \( v_0 \in W^{2,2}(B_R(x_0)) \) with \( v_0 - u \in W^{2,2}_0(B_R(x_0)) \) and \( \Delta^2 v_0 = 0 \) in \( B_R(x_0) \).
Proof. The proof is done in three steps.

Step 1 We choose \( x_0 \in \mathbb{R}^n \) arbitrary, but fixed. Let \( v_k \in W^{2,2}(B_R(x_0)) \) with \( v_k - u_k \in W^{2,2}_0(B_R(x_0)) \) and \( \Delta^2 v_k = 0 \) in \( B_R(x_0) \). If \( B_R(x_0) \cap \Omega_k = \emptyset \), \( u_k \) and \( v_k \) vanish in \( B_R(x_0) \). Consequently, we obtain

\[
\int_{B_R(x_0)} |\Delta (u_k - v_k)|^2 \, dx = 0. \tag{9}
\]

If \( B_R(x_0) \cap \Omega_k \neq \emptyset \), we set

\[
\hat{u}_k = \begin{cases} u_k, & \text{in } \mathbb{R}^n \setminus B_R(x_0) \\ v_k, & \text{in } B_R(x_0) \end{cases}.
\]

Note that \( \Omega_k \cup B_R(x_0) \) is an open set and that \( \hat{u}_k \in W^{2,2}_0(\Omega_k \cup B_R(x_0)) \). Let us first consider the case \( |\Omega_k \cup B_R(x_0)| \leq \omega_0 \). Hence, \( \Omega_k \cup B_R(x_0) \in \mathcal{O}_{\omega_0} \) and there holds

\[
\Lambda_{\omega_0} = \inf_{\Omega \in \mathcal{O}_{\omega_0}} \Lambda(\Omega) \leq \Lambda(\Omega_k \cup B_R(x_0)) \leq \mathcal{R}(\hat{u}_k, \mathbb{R}^n)
\]

since \( \hat{u}_k \in W^{2,2}_0(\Omega_k \cup B_R(x_0)) \). Rearranging terms and applying the definition of \( \hat{u}_k \) yields

\[
\Lambda_{\omega_0} \left( 1 - \int_{B_R(x_0)} |\nabla u_k|^2 \, dx \right) \leq \Lambda(\Omega_k) - \int_{B_R(x_0)} |\Delta u_k|^2 - |\Delta v_k|^2 \, dx. \tag{10}
\]

Since \( v_k - u_k \in W^{2,2}_0(B_R(x_0)) \) and \( v_k \) is biharmonic in \( B_R(x_0) \), there holds

\[
\int_{B_R(x_0)} |\Delta u_k|^2 - |\Delta v_k|^2 \, dx = \int_{B_R(x_0)} |\Delta^2 (u_k - v_k)|^2 \, dx,
\]

where \( \Delta^2 v \) denotes the Hessian matrix of a function \( v \). Moreover, we denote by

\[
|\Delta^2 v|^2 := \sum_{i,j=1}^n \partial_{i,j} v \partial_{i,j} v
\]

the Euclidian norm of the matrix \( \Delta^2 v \) as a vector in \( \mathbb{R}^{n^2} \). We rearrange terms in (10) and obtain

\[
\int_{B_R(x_0)} |\Delta^2 (u_k - v_k)|^2 \, dx \leq \Lambda(\Omega_k) - \Lambda_{\omega_0} + \Lambda_{\omega_0} \int_{B_R(x_0)} |\nabla u_k|^2 \, dx. \tag{11}
\]

Let us now assume that \( |\Omega_k \cup B_R(x_0)| > \omega_0 \). Then we set

\[
\mu_k := \left( \frac{|\Omega_k| + |B_R|}{|\Omega_k|} \right)^{\frac{1}{2}}. \tag{12}
\]

and find that \( \mu_k^{-1} \cdot (\Omega_k \cup B_R(x_0)) \in \mathcal{O}_{\omega_0} \). Recall that for every \( M \subset \mathbb{R}^n \) and \( t > 0 \) the buckling load satisfies

\[
\Lambda(M) = t^2 \Lambda(tM). \tag{13}
\]

Hence, we obtain

\[
\Lambda_{\omega_0} \leq \Lambda(\mu_k^{-1}(\Omega_k \cup B_R(x_0))) = \mu_k^2 \Lambda(\Omega_k \cup B_R(x_0)) \leq \mu_k^2 \mathcal{R}(\hat{u}_k, \mathbb{R}^n).
\]

and, subsequently,

\[
\mu_k^2 \int_{B_R(x_0)} |\Delta^2 (u_k - v_k)|^2 \, dx \leq \mu_k^2 \Lambda(\Omega_k) - \Lambda_{\omega_0} + \Lambda_{\omega_0} \int_{B_R(x_0)} |\nabla u_k|^2 \, dx.
\]
Since $\mu_k > 1$, we proceed to

$$
\int_{B_R(x_0)} |D^2(u_k - v_k)|^2 dx \leq \mu_k^2 \Lambda(\Omega_k) - \Lambda_{\omega_0} + \Lambda_{\omega_0} \int_{B_R(x_0)} |\nabla u_k|^2 dx. \quad (14)
$$

Consequently, we can collect the estimates (9), (11) and (14) in just one estimate: for every $x_0 \in \mathbb{R}^n$ and every $k \in \mathbb{N}$ there holds

$$
\int_{B_R(x_0)} |D^2(u_k - v_k)|^2 dx \leq \mu_k^2 \Lambda(\Omega_k) - \Lambda_{\omega_0} + \Lambda_{\omega_0} \int_{B_R(x_0)} |\nabla u_k|^2 dx. \quad (15)
$$

**Step 2** We want to understand the limit as $k$ tends to $\infty$ on both sides of (15). This needs some preparation. First, recall that we choose a minimizing sequence $(\Omega_k)_k$ such that (7) holds. Then applying (13) yields

$$
\Lambda_{\omega_0} \leq \Lambda \left( \left( \frac{\omega_0}{|\Omega_k|} \right)^{\frac{2}{n}} \Omega_k \right) = \left( \frac{|\Omega_k|}{\omega_0} \right)^{\frac{2}{n}} \Lambda(\Omega_k) \leq \left( \frac{|\Omega_k|}{\omega_0} \right)^{\frac{2}{n}} \left( \Lambda_{\omega_0} + \frac{1}{k} \right).
$$

Rearranging terms yields

$$
0 \leq \Lambda_{\omega_0} \left( 1 - \left( \frac{|\Omega_k|}{\omega_0} \right)^{\frac{2}{n}} \right) \leq \left( \frac{|\Omega_k|}{\omega_0} \right)^{\frac{2}{n}} \frac{1}{k} \leq \frac{1}{k}.
$$

Thus, there holds $|\Omega_k| \to \omega_0$ as $k$ tends to $\infty$. This immediately implies that

$$
\mu_k \xrightarrow{k \to \infty} \left( 1 + \frac{|B_R|}{\omega_0} \right)^{\frac{2}{n}},
$$

where $\mu_k$ is given in (12). In addition, recall that $u_k \rightharpoonup u$ in $W^{2,2}(\mathbb{R}^n)$ and, therefore, $u_k \rightarrow u$ in $W^{2,2}(B_R(x_0))$. In order to prove the weak convergence of the sequence $(v_k)_k$ in $W^{2,2}(B_R(x_0))$, we estimate the $W^{2,2}(B_R(x_0))$-norm of $v_k$ independently of $k$ for every $k \in \mathbb{N}$. We start with the $W^{2,2}(B_R(x_0))$-norm of $u_k - v_k$ and apply Poincaré’s inequality (as stated in [6, Formula (7.44)]) on the first summand in the integral on the right-hand side. This yields

$$
\|u_k - v_k\|_{W^{2,2}(B_R(x_0))}^2 = \int_{B_R(x_0)} |u_k - v_k|^2 + |\nabla(u_k - v_k)|^2 + |D^2(u_k - v_k)|^2 dx
$$

$$
\leq (R^2 + 1) \int_{B_R(x_0)} |\nabla(u_k - v_k)|^2 dx + \int_{B_R(x_0)} |D^2(u_k - v_k)|^2 dx
$$

$$
= (R^2 + 1) \sum_{i=1}^n \int_{B_R(x_0)} |\partial_i(u_k - v_k)|^2 dx + \int_{B_R(x_0)} |D^2(u_k - v_k)|^2 dx.
$$

Applying Poincaré’s inequality on the integrals containing the first-order derivatives of $u_k - v_k$, we obtain

$$
\|u_k - v_k\|_{W^{2,2}(B_R(x_0))}^2 \leq n(R^2 + 1) \int_{B_R(x_0)} |D^2(u_k - v_k)|^2 dx \leq 2n \int_{B_R(x_0)} |D^2(u_k - v_k)|^2 dx,
$$

where we additionally used that $R < 1$. Now Young’s inequality applied in the last integral yields

$$
\|u_k - v_k\|_{W^{2,2}(B_R(x_0))}^2 \leq 4n \left( \int_{B_R(x_0)} |D^2 u_k|^2 + |D^2 v_k|^2 dx \right).
$$
Recall that \( u_k - v_k \in W^{2,2}_0(B_R(x_0)) \) and \( \Delta^2 v_k = 0 \) in \( B_R(x_0) \). Hence, there holds
\[
\int_{B_R(x_0)} |D^2 v_k|^2 \, dx = \min \left\{ \int_{B_R(x_0)} |D^2 w|^2 \, dx : w - v_k \in W^{2,2}_0(B_R(x_0)) \right\} \leq \int_{B_R(x_0)} |D^2 u_k|^2 \, dx
\]
and, consequently,
\[
\|u_k - v_k\|_{W^{2,2}(B_R(x_0))}^2 \leq 8n \int_{B_R(x_0)} |D^2 u_k|^2 \, dx \leq 8n \|u_k\|_{W^{2,2}(B_R(x_0))}^2.
\]
Since \( u_k \rightharpoonup u \) in \( W^{2,2}(B_R(x_0)) \), there exists a constant \( C(n) \), which is only depended on the dimension \( n \), such that
\[
\|u_k - v_k\|_{W^{2,2}(B_R(x_0))}^2 \leq 8n \int_{B_R(x_0)} |D^2 u_k|^2 \, dx \leq 8n \|u_k\|_{W^{2,2}(B_R(x_0))}^2 \leq C(n).
\]
Consequently, for all \( k \in \mathbb{N} \) there holds
\[
\|v_k\|_{W^{2,2}(B_R(x_0))} \leq \|u_k - v_k\|_{W^{2,2}(B_R(x_0))} + \|u_k\|_{W^{2,2}(B_R(x_0))} \leq C(n)
\]
and there exists a \( v_0 \in W^{2,2}(B_R(x_0)) \) such that
\[
v_k \rightharpoonup v_0 \text{ in } W^{2,2}(B_R(x_0)) \text{ and } v_0 - u \in W^{2,2}_0(B_R(x_0)).
\]
Moreover, for every \( \phi \in C_c^\infty(B_R(x_0)) \) there holds
\[
0 = \lim_{k \to \infty} \int_{B_R(x_0)} \Delta v_k \Delta \phi \, dx = \int_{B_R(x_0)} \Delta v_0 \Delta \phi \, dx
\]
because \( v_k \) is biharmonic in \( B_R(x_0) \) for every \( k \in \mathbb{N} \) and the weak convergence of \( v_k \) to \( v_0 \) in \( W^{2,2}(B_R(x_0)) \).
Hence, \( v_0 \) is biharmonic in \( B_R(x_0) \).

**Step 3** We take the lim inf on both sides of (15). Since \( u_k \rightharpoonup u \) in \( W^{2,2}_0(B_r(x_0)) \), this leads to
\[
\int_{B_r(x_0)} |D^2(u - v_0)|^2 \, dx \leq \liminf_{k \to \infty} \int_{B_r(x_0)} |D^2(u_k - v_k)|^2 \, dx
\]
\[
\leq \liminf_{k \to \infty} \left( \mu_k^2 \Lambda_k(\Omega_k) - \Lambda_{\omega_0} + \Lambda_{\omega_0} \int_{B_R(x_0)} |\nabla u_k|^2 \, dx \right)
\]
\[
= \left( 1 + \frac{|B_R|}{\omega_0} \right)^2 \Lambda_{\omega_0} - \Lambda_{\omega_0} + \Lambda_{\omega_0} \int_{B_R(x_0)} |\nabla u|^2 \, dx
\]
\[
\leq C(n, \omega_0) \left( R^n \right) + \int_{B_R(x_0)} |\nabla u|^2 \, dx
\]
This proves the claim. \( \square \)

Now let \( v_0 \in W^{2,2}(B_R(x_0)) \) be the function from Lemma 4 and \( 0 < r \leq R \). Then there obviously holds
\[
\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq \int_{B_r(x_0)} |D^2 v_0|^2 \, dx + 2 \int_{B_R(x_0)} |D^2(u - v_0)|^2 \, dx
\]
and applying Lemma 4 yields

$$\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq 2 \int_{B_r(x_0)} |D^2 v_0|^2 \, dx + C(n, \omega_0) \left( R^n + \int_{B_R(x_0)} |\nabla u|^2 \, dx \right).$$  \hfill (17)

In order to estimate the first summand on the right-hand side of the above inequality, we cite Lemma 2.1 from [13].

**Lemma 5.** Using the notation above, there exists a constant $C = C(n) > 0$ such that for $0 < r \leq R$ there holds

$$\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq C(n) \left( \frac{r}{R} \right)^n \int_{B_R(x_0)} |D^2 u_0|^2 \, dx.$$  

The constant $C$ does not depend on $r, R$ or $x_0$, but on the dimension $n$.

Thus, (17) becomes

$$\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq C(n) \left( \frac{r}{R} \right)^n \int_{B_R(x_0)} |D^2 u|^2 \, dx + C(n, \omega_0) \left( R^n + \int_{B_R(x_0)} |\nabla u|^2 \, dx \right).$$  \hfill (18)

This estimate will be the starting point for the bootstrapping argument which will lead to the Hölder-continuity of the first-order derivatives of $u$.

From [5, Chapter III, Lemma 2.1] we cite the next lemma.

**Lemma 6.** Let $\Phi$ be a nonnegative and non-decreasing function on $[0, R]$. Suppose that there exist positive constants $\gamma, \alpha, \kappa, \beta, \beta < \alpha$, such that for all $0 \leq r \leq R \leq R_0$

$$\Phi(r) \leq \gamma \left[ \left( \frac{r}{R} \right)^\alpha + \delta \right] \Phi(R) + \kappa R^\beta.$$  

Then there exist positive constants $\delta_0 = \delta_0(\gamma, \alpha, \beta)$ and $C = C(\gamma, \alpha, \beta)$ such that if $\delta < \delta_0$, for all $0 \leq r \leq R \leq R_0$ we have

$$\Phi(r) \leq C \left( \frac{r}{R} \right)^\beta \left[ \Phi(R) + \kappa R^\beta \right].$$

The following lemma is based on ideas of [7, Chapter 3]. It will be the crucial observation for the bootstrapping.

**Lemma 7.** Suppose that for each $0 \leq r \leq 1$ there holds

$$\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq M \, r^\mu,$$

where $M > 0$ and $\mu \in [0, n)$. Then there exists a constant $C(n) > 0$ such that for each $0 \leq r \leq 1$

$$\int_{B_r(x_0)} |\nabla u|^2 \, dx \leq C(n, M) \, r^\lambda,$$

where $\lambda = \mu + 2$ if $\mu < n - 2$ and $\lambda$ is arbitrary in $(0, n)$ if $n - 2 \leq \mu < n$.

**Proof.** Let $0 \leq r \leq s \leq 1$. For a function $w \in W^{1,2}(\mathbb{R}^n)$ we set

$$(w)_{r,x_0} := \int_{B_r(x_0)} w \, dx = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} w \, dx.$$
Using this notation, we write
\[
\int_{B_r(x_0)} |\nabla u|^2 \, dx = \sum_{i=1}^n \int_{B_r(x_0)} |\partial_i u - (\partial_i u)_{x,0} + (\partial_i u)_{x,0}|^2 \, dx.
\]

Then Young's inequality implies
\[
\int_{B_r(x_0)} |\nabla u|^2 \, dx \leq 2 \sum_{i=1}^n \left( \int_{B_r(x_0)} (\partial_i u)^2 \, dx + \int_{B_r(x_0)} |\partial_i u - (\partial_i u)_{x,0}|^2 \, dx \right)
\]
\[
\leq 2 \sum_{i=1}^n \left( |B_r| \left( \int_{B_r(x_0)} |\partial_i u| \, dx \right)^2 + \int_{B_r(x_0)} |\partial_i u - (\partial_i u)_{x,0}|^2 \, dx \right).
\]

Applying Hölder’s and a local version of Poincaré’s inequality, we find that
\[
\int_{B_r(x_0)} |\nabla u|^2 \, dx \leq C(n) \left[ \left( \frac{r}{s} \right)^n \int_{B_r(x_0)} |\nabla u|^2 \, dx + s^2 \int_{B_r(x_0)} |D^2 u|^2 \, dx \right],
\]
where the constant \( C \) only depends on \( n \). By assumption, we can proceed to
\[
\int_{B_r(x_0)} |\nabla u|^2 \, dx \leq C(n) \left[ \left( \frac{r}{s} \right)^n \int_{B_r(x_0)} |\nabla u|^2 \, dx + M s^{\mu+2} \right].
\]

Now Lemma 6 implies that for each \( 0 \leq r \leq s \leq 1 \) there holds
\[
\int_{B_r(x_0)} |\nabla u|^2 \, dx \leq C(n) \left[ \left( \frac{r}{s} \right)^\lambda \int_{B_r(x_0)} |\nabla u|^2 \, dx + M s^\lambda \right],
\]
where \( \lambda = \mu + 2 \) if \( \mu < n - 2 \) and \( \lambda \) is arbitrary in \((0, n)\) if \( n - 2 \leq \mu < n \). Choosing \( s = 1 \), we deduce
\[
\int_{B_r(x_0)} |\nabla u|^2 \, dx \leq C(n, M) r^\lambda.
\]

Now we are able to prove the Hölder continuity of the first-order derivatives of \( u \).

**Theorem 2.** Let \( u \) be the limit function according to Proposition 1. The first-order derivatives of \( u \) are \( \alpha \)-Hölder continuous almost everywhere on \( \mathbb{R}^n \) for every \( \alpha \in (0, 1) \).

**Proof.** Our aim is to show that for every \( x_0 \in \mathbb{R}^n \) and every \( 0 < r \leq 1 \) there holds
\[
\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq C(n, \omega_0) r^{n-2+2\alpha}.
\]
(19)

Then Theorem 1 finishes the proof. Let us choose \( x_0 \in \mathbb{R}^n \), \( 0 < r \leq R \leq 1 \) and recall estimate (18):
\[
\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq C(n) \left( \frac{r}{R} \right)^n \int_{B_R(x_0)} |D^2 u|^2 \, dx + C(n, \omega_0) \left( R^n + \int_{B_R(x_0)} |\nabla u|^2 \, dx \right).
\]
We will improve this estimate using a bootstrap argument based on Lemma 7. Note that for every $0 < r \leq 1$ there holds

$$
\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq \|u\|_{W^{2,2}(\mathbb{R}^n)}^2 = \|u\|_{W^{2,2}(\mathbb{R}^n)}^2 r^0. \quad (20)
$$

Then Lemma 7 implies that for every $0 < r \leq 1$ there holds

$$
\int_{B_r(x_0)} |\nabla u|^2 \, dx \leq C(n, \|u\|_{W^{2,2}(\mathbb{R}^n)}) r^{\lambda_0}, \quad (21)
$$

where $\lambda_0 \in (0, n)$ if $n = 2$ and $\lambda_0 = 2$ if $n \geq 3$. We insert this estimate (18). Since $R \leq 1$ we obtain

$$
\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq C(n) \left( \frac{r}{R} \right)^2 \int_{B_R(x_0)} |D^2 u|^2 \, dx + C(n, \omega_0, \|u\|_{W^{2,2}(\mathbb{R}^n)}) R^{\lambda_0}
$$

for every $0 < r \leq R$. Applying Lemma 6, we obtain

$$
\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq C \left( \frac{r}{R} \right)^{\lambda_0} \left( \int_{B_R(x_0)} |D^2 u|^2 \, dx + C(n, \omega_0, \|u\|_{W^{2,2}(\mathbb{R}^n)}) R^{\lambda_0} \right)
$$

for every $0 < r \leq R$. Choosing $R = 1$ leads to

$$
\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq C(n, \omega_0, \|u\|_{W^{2,2}(\mathbb{R}^n)}) r^{\lambda_0} \quad (22)
$$

for every $0 < r \leq 1$. If $n = 2$, this is (19).

If $n \geq 3$, (22) is an improvement of estimate (20). Recall that here holds $\lambda_0 = 2$. We again apply Lemma 7 and obtain for every $0 < r \leq 1$

$$
\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq C(n, \omega_0, \|u\|_{W^{2,2}(\mathbb{R}^n)}) r^{\lambda_1},
$$

where $\lambda_1 \in (0, n)$ if $n \in \{3, 4\}$ and $\lambda_1 = 4$ if $n \geq 5$. Together with estimate (18) we find that

$$
\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq C(n) \left( \frac{r}{R} \right)^n \int_{B_R(x_0)} |D^2 u|^2 \, dx + C(n, \omega_0, \|u\|_{W^{2,2}(\mathbb{R}^n)}) R^{\lambda_1}
$$

for every $0 < r \leq R \leq 1$. Then Lemma 6 implies

$$
\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq C \left( \frac{r}{R} \right)^{\lambda_1} \left( \int_{B_R(x_0)} |D^2 u|^2 \, dx + C(n, \omega_0, \|u\|_{W^{2,2}(\mathbb{R}^n)}) R^{\lambda_1} \right)
$$

and choosing $R = 1$ there holds

$$
\int_{B_r(x_0)} |D^2 u|^2 \, dx \leq C(n, \omega_0, \|u\|_{W^{2,2}(\mathbb{R}^n)}) r^{\lambda_1} \quad (23)
$$

for every $0 < r \leq 1$. For $n \in \{3, 4\}$, estimate (23) and Theorem 1 proves the claim.

If $n \geq 6$, we repeat the argumentation since (23) is an improvement of (22). Repeating this process proves the claim after finite many steps for every $n \geq 2$. \qed
Due to Theorem 2 the limit function $u$ has a unique representative in $W^{2,2}(\mathbb{R}^n)$ which is continuous in $\mathbb{R}^n$ and which has $\alpha$-Hölder continuous first-order derivatives in $\mathbb{R}^n$ for every $\alpha \in (0, 1)$. From now on, we rename this representative as $u$ and focus on this function.

2.2. The minimizing domain

The regularity of $u$, which we achieved in the previous section, enables us to construct an optimal domain for minimizing the buckling load in $\mathcal{O}_\omega$. Recall that (see (6))

$$\mathcal{R}(u, \mathbb{R}^n) \leq \inf_{\Omega \in \mathcal{O}_\omega} \Lambda(\Omega).$$

If $u \in W^{2,2}_0(\Omega^*)$ for a suitable set $\Omega^* \in \mathcal{O}_\omega$, this set $\Omega^*$ is the desired minimizer. Thus, the challenge is to construct a suitable $\Omega^*$.

Let us define

$$\tilde{\Omega} := \{x \in \mathbb{R}^n : u(x) \neq 0\}$$

and let $(\Omega_k)_k \subset \mathcal{O}_\omega$ be a minimizing sequence and $(u_k)_k \subset W^{2,2}(\mathbb{R}^n)$ the corresponding sequence of eigenfunctions according to Proposition 1. Since $u_k$ converges strongly to $u$ in $L^2(\mathbb{R}^n)$, $u_k$ converges locally in measure to $u$, i.e., for every compact set $C \subset \mathbb{R}^n$ and every $\varepsilon > 0$ there holds

$$|\{x \in C : |u_k(x) - u(x)| \geq \varepsilon\}| \xrightarrow{k \to \infty} 0.$$

Now let $C$ be any compact subset of $\tilde{\Omega}$. Since $u$ is a continuous function and cannot vanish in $\tilde{\Omega}$, there exists a constant $m_C > 0$ such that

$$|u(x)| \geq m_C \text{ for every } x \in C.$$

Recall that $u_k$ vanishes pointwise in $\Omega_k^c$. Thus, there holds

$$|u(x)| = |u_k(x) - u(x)| \text{ for every } x \in \Omega_k^c.$$

Hence, we obtain

$$|\Omega_k^c \cap C| \leq |\{x \in C : |u_k(x) - u(x)| \geq m_C\}| \xrightarrow{k \to \infty} 0.$$

Consequently, for every compact $C \subset \tilde{\Omega}$ and every $k \in \mathbb{N}$ there holds

$$|C| = |C \cap \Omega_k| + |C \cap \Omega_k^c| \leq |\Omega_k| + |C \cap \Omega_k^c| \leq \omega_0 + |C \cap \Omega_k^c|$$

and letting $k$ tend to infinity we find that $|C| \leq \omega_0$ for every compact subset $C$ of $\tilde{\Omega}$. Hence,

$$\tilde{\Omega} := \sup\{|C| : C \subset \tilde{\Omega}, C \text{ compact}\} \leq \omega_0.$$

We now denote

$$\hat{\Omega} = \{x \in \mathbb{R}^n : u(x) = 0 \text{ and } |\nabla u(x)| \neq 0\}.$$

Note that $\hat{\Omega}$ is part of a nodal line of $u$. Since the first-order derivatives of $u$ are continuous, the Implicit Function Theorem implies that the $n$-dimensional Lebesgue measure of $\hat{\Omega}$ is zero. Consequently, the set

$$\Omega^* := \Omega \cup \hat{\Omega} = \{x \in \mathbb{R}^n : u(x) \neq 0 \text{ or } |\nabla u(x)| \neq 0\}$$

is an open set and there holds

$$|\Omega^*| = |\Omega \cup \hat{\Omega}| \leq |\hat{\Omega}| + |\hat{\Omega}| \leq \omega_0.$$

Thus, $\Omega^* \in \mathcal{O}_\omega$ and by construction, $u$ and $\nabla u$ vanish in every point in $\mathbb{R}^n \setminus \Omega^*$.

The following corollary guarantees that $u \in W^{2,2}_0(\Omega^*)$. For the proof of this corollary, we refer to [1, Th. 9.1.3] or [8, Sec. 3.3.5].
Corollary 1. Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set and $v \in W^{2,2}(\mathbb{R}^n)$. If $v$ and its first-order derivatives vanish pointwise in $\mathbb{R}^n \setminus \Omega$, then $u \in W^{2,2}_0(\Omega)$.

Now we can prove our main theorem.

Theorem 3. The set $\Omega^*$ given by (24) minimizes the buckling load $\Lambda$ in $\mathcal{O}_{\omega_0}$.

Proof. Recall that there holds
\[
\mathcal{R}(u, \mathbb{R}^n) \overset{(6)}{=} \liminf_{k \to \infty} \mathcal{R}(u_k, \Omega_k) = \inf_{\Omega \in \mathcal{O}_{\omega_0}} \Lambda(\Omega).
\]
Since $\Omega^* \in \mathcal{O}_{\omega_0}$ and $u \in W^{2,2}_0(\Omega^*)$, there holds
\[
\inf_{\Omega \in \mathcal{O}_{\omega_0}} \Lambda(\Omega) \leq \Lambda(\Omega^*) \leq \mathcal{R}(u, \mathbb{R}^n).
\]
Obviously, this means that
\[
\inf_{\Omega \in \mathcal{O}_{\omega_0}} \Lambda(\Omega) = \Lambda(\Omega^*) = \mathcal{R}(u, \mathbb{R}^n).
\]
\[\square\]

Due to the scaling property of the buckling load, the following corollary holds true.

Corollary 2. Let $\Omega^* \in \mathcal{O}_{\omega_0}$ minimize the buckling load $\Lambda$ in $\mathcal{O}_{\omega_0}$. Then $\Omega^*$ satisfies $|\Omega| = \omega_0$.

As a consequence of Corollary 2, the set $\Omega^*$ is connected.

Corollary 3. The set $\Omega^*$ given by (24) is connected.

Proof. Let us assume that $\Omega^*$ consists of the two connected components $\Omega_1$ and $\Omega_2$ with $|\Omega_k| > 0$ for $k = 1, 2$. By $u_k$ we denote the eigenfunction $u$ restricted to $\Omega_k$, i.e.,
\[
 u_k := \begin{cases} u, & \text{in } \Omega_k \\ 0, & \text{otherwise} \end{cases}.
\]
Since $\Omega_k \in \mathcal{O}_{\omega_0}$, the minimality of $\Omega^*$ for $\Lambda$ implies
\[
\Lambda(\Omega^*) = \mathcal{R}(u, \Omega^*) \leq \Lambda(\Omega_1) \leq \mathcal{R}(u_1, \Omega_1).
\]
Rearranging terms and using that $\|\nabla u\|_{L^2(\Omega^*)} = 1$ we obtain
\[
\left( \int_{\Omega_1} |\Delta u_1|^2 dx + \int_{\Omega_2} |\Delta u_2|^2 dx \right) \left( 1 - \int_{\Omega_2} |\nabla u_2|^2 dx \right) \leq \int_{\Omega_1} |\Delta u_1|^2 dx.
\]
Hence,
\[
\int_{\Omega_2} |\Delta u_2|^2 \leq \Lambda(\Omega^*) \int_{\Omega_2} |\nabla u_2|^2 dx \iff \mathcal{R}(u_2, \Omega_2) \leq \Lambda(\Omega^*).
\]
Then there holds
\[
\Lambda(\Omega_2) \leq \mathcal{R}(u_2, \Omega_2) \leq \Lambda(\Omega^*)
\]
and $\Omega_2$ is a minimizer of $\Lambda$ in $\mathcal{O}_{\omega_0}$. However, since $|\Omega_2| < \omega_0$, this is a contradiction to Corollary 2. \[\square\]

Summing up, we found an optimal domain $\Omega^* \in \mathcal{O}_{\omega_0}$ for minimizing the buckling load in $\mathcal{O}_{\omega_0}$. The set $\Omega^*$ is open, connected and satisfies $|\Omega^*| = \omega_0$. Classical variational arguments show that $u$ solves
\[
\Delta^2 u + \Lambda(\Omega^*) \Delta u = 0 \text{ in } \Omega^*.
\]
Acknowledgements

While preparing this work, the author was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project Number 396521072.

Funding Open Access funding enabled and organized by Projekt DEAL. The author was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project Number 396521072.

Availability of data and material Not applicable

Declarations

Conflict of interest Not applicable

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Adams, D.R., Hedberg, L.I.: Function spaces and potential theory. In: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 314. Springer-Verlag, Berlin (1996)
[2] Alt, H.W., Caffarelli, L.A.: Existence and regularity for a minimum problem with free boundary. J. Reine Angew. Math. 325, 105–144 (1981)
[3] Ashbaugh, M. S., and Bucur, D.: On the isoperimetric inequality for the buckling of a clamped plate. Z. Angew. Math. Phys. 54, 5 (2003), 756–770. Special issue dedicated to Lawrence E. Payne
[4] Bucur, D., and Varchon, N.: Global minimizing domains for the first eigenvalue of an elliptic operator with non-constant coefficients. Electron. J. Differential Equations (2000), No. 36, 10
[5] Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. Annals of Mathematics Studies, vol. 105. Princeton University Press, Princeton, NJ (1983)
[6] Gilbarg, D., and Trudinger, N. S.: Elliptic partial differential equations of second order. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition
[7] Han, Q., and Lin, F.: Elliptic Partial Differential Equations, vol. 1 of Courant Lecture Notes in Mathematics. American Mathematical Society, 1997
[8] Henrot, A., and Pierre, M.: Shape variation and optimization, vol. 28 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2018
[9] Lieb, E.H.: On the lowest eigenvalue of the Laplacian for the intersection of two domains. Invent. Math. 74(3), 441–448 (1983)
[10] Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The locally compact case. I. Ann. Inst. H. Poincaré Anal. Non Linéaire 1, 2 (1984), 109–145
[11] Morrey, C.B.: Multiple integrals in the calculus of variations. Springer, Berlin (1966)
[12] Polya, G., and Szegö, G.: Isoperimetric Inequalities in Mathematical Physics. Annals of Mathematics Studies, no. 27. Princeton University Press, Princeton, N. J., 1951
[13] Stollenwerk, K.: Optimal shape of a domain which minimizes the first buckling eigenvalue. Calc. Var. Partial Differential Equations 55, 1 (2016), Art. 5, 29
[14] Stollenwerk, K., Wagner, A.: Optimality conditions for the buckling of a clamped plate. J. Math. Anal. Appl. 432(1), 254–273 (2015)
[15] Willms, B.: An isoperimetric inequality for the buckling of a clamped plate. Lecture at the Oberwolfach meeting on 'Qualitative properties of PDE' (organized by H. Berestycki, B. Kawohl and G. Talenti) (Feb. 1995)

Kathrin Stollenwerk
Institut für Mathematik, RWTH Aachen
Templergraben 55
D-52062 Aachen
Germany
e-mail: stollenwerk@instmath.rwth-aachen.de

(Received: March 19, 2021; revised: October 23, 2022; accepted: November 3, 2022)