Riesz transforms and multipliers for the Grushin operator

K. Jotsaroop, P. K. Sanjay and S. Thangavelu

Abstract. We show that Riesz transforms associated to the Grushin operator $G = -\Delta - |x|^2 \partial_t^2$ are bounded on $L^p(\mathbb{R}^{n+1})$. We also establish an analogue of Hörmander-Mihlin multiplier theorem and study Bochner-Riesz means associated to the Grushin operator. The main tools used are Littlewood-Paley theory and an operator valued Fourier multiplier theorem due to L. Weis.

1. Introduction

The aim of this paper is to study Riesz transforms and multipliers associated to the Grushin operator $G = -\Delta - |x|^2 \partial_t^2$ on $\mathbb{R}^n \times \mathbb{R}$. The spectral decomposition of this non-negative operator is explicitly known:

$$Gf(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \left( \sum_{k=0}^{\infty} (2k + n)|\lambda|P_k(\lambda)f^\lambda(x) \right) d\lambda$$

where $P_k(\lambda)$ are the spectral projections of the scaled Hermite operator $H(\lambda) = -\Delta + \lambda^2 |x|^2$ so that

$$H(\lambda) = \sum_{k=0}^{\infty} (2k + n)|\lambda|P_k(\lambda)$$

and

$$f^\lambda(x) = \int_{-\infty}^{\infty} f(x,t)e^{i\lambda t} dt.$$
Thus, given a bounded function \( m \) on \( \mathbb{R} \) we can define the multiplier transformation \( m(G) \) by setting

\[
m(G)f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \left( \sum_{k=0}^{\infty} m((2k + n)|\lambda|)P_k(\lambda)f^\lambda(x) \right) d\lambda.
\]

It is obvious that \( m(G) \) is bounded on \( L^2(\mathbb{R}^{n+1}) \) but for the boundedness of \( m(G) \) on other \( L^p \) spaces we need to assume more conditions on \( m \).

Recall that \( H(\lambda) \) can be written as

\[
H(\lambda) = \frac{1}{2} \sum_{j=1}^{n} (A_j(\lambda)A_j(\lambda)^* + A_j(\lambda)^*A_j(\lambda))
\]

where

\[
A_j(\lambda) = -\frac{\partial}{\partial x_j} + \lambda x_j, A_j(\lambda)^* = \frac{\partial}{\partial x_j} + \lambda x_j
\]

are the creation and annihilation operators (for \( \lambda > 0 \)). The operators

\[
R_j(\lambda) = A_j(\lambda)H(\lambda)^{-\frac{1}{2}}, R_j^*(\lambda) = A_j(\lambda)^*H(\lambda)^{-\frac{1}{2}}
\]

are Riesz transforms associated to the Hermite operator. It is therefore natural to consider the operators

\[
R_jf(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} R_j(\lambda)f^\lambda(x)d\lambda
\]

and \( R_j^* \) similarly defined in terms of \( R_j^*(\lambda) \) and call them the Riesz transforms for the Grushin operators. Again these Riesz transforms are trivially bounded on \( L^2 \) but their \( L^p \) boundedness is far from trivial.

We prove:

**Theorem 1.1.** For \( 1 < p < \infty \) the Riesz transforms \( R_j, R_j^* \), \( j = 1, 2, ..., n \) are all bounded on \( L^p(\mathbb{R}^{n+1}) \).

We also consider higher order Riesz transforms which are defined as follows. Let \( \mathcal{H}_{p,q} \) stand for the space of bigraded solid harmonics of bidegree \((p, q), p, q \in \mathbb{N} \). Let \( G_\lambda(P) \) stand for the Weyl correspondence, see Section 2.2 for the definition. Then it is known that \( G_\lambda(P)H(\lambda)^{-(p+q)/2} \) are bounded operators on \( L^p(\mathbb{R}^n), 1 < p < \infty \) [6].

We define higher order Riesz transforms for the Grushin operator by setting

\[
R_P f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} G_\lambda(P)H(\lambda)^{-(p+q)/2} f^\lambda(x)d\lambda.
\]

We prove:
Theorem 1.2. For any \( P \in H_{p,q} \) the Riesz transforms \( R_P \) are bounded on \( L^p(\mathbb{R}^{n+1}), 1 < p < \infty \).

Riesz transforms and higher order Riesz transforms associated to the sublaplacian on the Heisenberg group \( \mathbb{H}^n \) has been studied by several authors, see e.g. [1], [2] and [7]. Riesz transforms on \( \mathbb{H}^n \) and more generally on nilpotent Lie groups turn out to be singular integral operators. As we do not have a group structure behind the Grushin operator it is not possible to use their techniques. We use a different method described below. Concerning general multiplier transforms \( m(G) \) we prove the following result.

Theorem 1.3. Let \( N \geq \frac{n}{2} + 1 \) be an integer and let \( m \in C^N(\mathbb{R}^*) \) satisfy the estimates \( |m^{(k)}(\lambda)| \leq C_k|\lambda|^{-k} \) for \( |\lambda| \) large for all \( k = 0, 1, 2, ..., N \). Then \( m(G) \) is bounded on \( L^p(\mathbb{R}^{n+1}) \) for all \( 1 < p < \infty \).

In his thesis [5] R. Meyer has studied the wave equation associated to the Grushin operator in one dimension. There he mentions about the possibility of proving a multiplier theorem (stated as a conjecture) for the Grushin operator. The above theorem gives such a result though the proof is completely different from what he had in mind. We also have results for the wave equation which will be presented in a forthcoming paper.

We can also treat Bochner-Riesz means associated to the Grushin operator. For \( R > 0 \) Bochner-Riesz means \( B_R^\delta f \) of order \( \delta > 0 \) are defined by

\[
B_R^\delta f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \left( 1 - \frac{H(\lambda)}{R} \right)^\delta f^\lambda(x) d\lambda.
\]

Here \( \left( 1 - \frac{H(\lambda)}{R} \right)^\delta f^\lambda \) are the Bochner-Riesz means associated to \( H(\lambda) \). For \( \lambda \) fixed these means have been studied by various authors, see [9]. Concerning \( B_R^\delta f \) for the Grushin operator we prove the following result.

Theorem 1.4. For \( \delta > \frac{n+1}{2} + 1/6 \), the Bochner-Riesz means \( B_R^\delta \) are uniformly bounded on \( L^p(\mathbb{R}^{n+1}), 1 < p < \infty \).

It may not be possible to improve the above result when \( n = 1 \) as the critical index for the Bochner-Riesz summability of one dimensional Hermite expansions is \( 1/6 \), see [9]. However, the critical index for Hermite expansions on \( \mathbb{R}^n, n \geq 2 \) is \( (n-1)/2 \) and hence it should be possible to improve the above result. We conjecture that the above result is true for \( \delta > \frac{n+1}{2} \) for \( n \geq 2 \).
We now briefly describe the methods used to prove these theorems. If

\[ m(H(\lambda)) = \sum_{k=0}^{\infty} (2k + n)|\lambda|P_{k}(\lambda) \]

is the multiplier transform for the Hermite operator then it follows that

\[ (m(G)f)^{\lambda}(x) = m(H(\lambda))f^{\lambda}(x). \]

Therefore, if we identity \( L^{p}(\mathbb{R}^{n+1}) \) with \( L^{p}(\mathbb{R}, X) \) where \( X = L^{p}(\mathbb{R}^{n}) \) then we can view \( m(G) \) as an operator valued Fourier multiplier (for the Fourier transform on \( \mathbb{R} \)) acting on \( L^{p}(\mathbb{R}, X) \). Sufficient conditions on \( m \) are known so that such multipliers are bounded on \( L^{p}(\mathbb{R}, X) \).

Indeed, we make use of the following theorem of L. Weis [12].

**Theorem 1.5.** Let \( X \) and \( Y \) be UMD spaces. Let \( m : \mathbb{R}^{*} \to B(X, Y) \) be a differentiable function such that the families \( \{m(\lambda) : \lambda \in \mathbb{R}^{*}\} \) and \( \{\lambda \frac{d}{d\lambda}m(\lambda) : \lambda \in \mathbb{R}^{*}\} \) are R-bounded. Then \( m \) defines a Fourier multiplier which is bounded from \( L^{p}(\mathbb{R}, X) \) into \( L^{p}(\mathbb{R}, Y) \) for all \( 1 < p < \infty \).

We need this theorem only for \( X = Y = L^{p}(\mathbb{R}^{n}) \) and in this case the R-boundedness is equivalent to a vector-valued inequality for \( m(\lambda) \) and \( \lambda \frac{d}{d\lambda}m(\lambda) \). Indeed, the R-boundedness of a family of operators \( T(\lambda) \) is equivalent to the inequality

\[ \left\| \left( \sum_{j=1}^{\infty} |T(\lambda_{j})f_{j}|^{2} \right)^{\frac{1}{2}} \right\|_{p} \leq C \left\| \left( \sum_{j=1}^{\infty} |f_{j}|^{2} \right)^{\frac{1}{2}} \right\|_{p} \]

for all possible choices of \( \lambda_{j} \in \mathbb{R}^{*} \) and \( f_{j} \in L^{p}(\mathbb{R}^{n}) \). Thus we only need to verify this vector-valued inequality for the two families in the theorem.

**2. Riesz transforms for the Grushin operator**

**2.1. On the boundedness of \( R_{j} \) and \( R_{j}^{*} \).** In this subsection we show that the Riesz transforms \( R_{j} \) and \( R_{j}^{*} \) defined in the introduction are all bounded on \( L^{p}(\mathbb{R}^{n+1}) \) as long as \( 1 < p < \infty \). This is done by showing that the operator valued functions \( m_{j}(\lambda) = R_{j}(\lambda) \) and \( m_{j}^{*}(\lambda) = R_{j}^{*}(\lambda) \) satisfy the conditions stated in the theorem of Weis. In view of the theorem of Weis and the equivalent condition for R-boundedness Theorem 1.1 will follow once we prove
Theorem 2.1. Let $T(\lambda)$ be any of the families $R_j(\lambda), R_j^*(\lambda), \lambda \frac{d}{d\lambda} R_j(\lambda)$ or $\lambda \frac{d}{d\lambda} R_j^*(\lambda)$. Then the vector valued inequality is satisfied:

$$\| \left( \sum_{k=1}^{\infty} |T(\lambda_k)f_k|^2 \right)^{\frac{1}{2}} \|_p \leq C \| \left( \sum_{k=1}^{\infty} |f_k|^2 \right)^{\frac{1}{2}} \|_p$$

for any $1 < p < \infty$.

We only treat the cases of $R_j(\lambda)$ and its derivative as the other families are similarly dealt with. Without loss of generality let us assume $\lambda > 0$. Recall that $R_j(\lambda) = A_j(\lambda) H(\lambda)^{-\frac{1}{2}}$ where $H(\lambda)^{-\frac{1}{2}}$ can be written in terms of the Hermite semigroup $e^{-tH(\lambda)}$ as

$$H(\lambda)^{-\frac{1}{2}} = \int_0^\infty t^{-\frac{1}{2}} e^{-tH(\lambda)} dt.$$ 

An orthonormal basis for $L^2(\mathbb{R}^n)$ consisting of eigenfunctions of $H(\lambda)$ are provided by $\Phi_{\alpha}^\lambda(x) = \lambda^{n/4} \Phi_{\alpha}(\lambda^{1/2} x)$ where $\Phi_{\alpha}$ are the Hermite functions on $\mathbb{R}^n$ satisfying $H\Phi_{\alpha} = (2|\alpha| + n)\Phi_{\alpha}$. Here $H = H(1)$ is the Hermite operator. Thus $\Phi_{\alpha}^\lambda$ are eigenfunctions of $H(\lambda)$ with eigenvalues $(2|\alpha| + n)\lambda$. The spectral projections $P_k(\lambda)$ of $H(\lambda)$ are defined by

$$P_k(\lambda)f = \sum_{|\alpha| = k} (f, \Phi_{\alpha}^\lambda) \Phi_{\alpha}^\lambda.$$ 

The kernel $h^\lambda_t(x, y)$ of the semigroup $e^{-tH(\lambda)}$ is explicitly known, thanks to Mehler’s formula, and hence the kernel of $R_j(\lambda)$ is given by $A_j(\lambda) h^\lambda_t(x, y)$. It turns out that $R_j(\lambda)$ are all Calderon-Zygmund singular integrals. Indeed, Stempak and Torrea [7] have shown that

$$|A_j(1)h^1_t(x, y)| \leq C|x - y|^{-n}$$

and

$$|\frac{\partial}{\partial x_i} A_j(1)h^1_t(x, y)| + |\frac{\partial}{\partial y_i} A_j(1)h^1_t(x, y)| \leq C_i|x - y|^{-n-1}$$

(see Theorem 3.3 in [7]). As can be easily verified $A_j(\lambda) h^\lambda_t(x, y) = \lambda^{n/2}(A_j(1) h^1_t)(\lambda^{1/2} x, \lambda^{1/2} y)$ and hence the kernels of $R_j(\lambda)$ are Calderon-Zygmund kernels and CZ constants are independent of $\lambda$. Now we can appeal to the vector valued inequalities for CZ singular integrals proved by Cordoba and Fefferman (see Theorem 1.3 Chapter XII in [11]). Thus the Riesz transforms $R_j(\lambda)$ are R-bounded.

We now turn our attention to the derivative of $R_j(\lambda)$. Denoting the kernel of $R_j(1)$ by $R_j(x, y)$ and noting that the kernel of $R_j(\lambda)$ is $\lambda^{n/2} R_j(\lambda^{1/2} x, \lambda^{1/2} y)$ we can easily prove the following.
Lemma 2.2. $\lambda \frac{d}{dx} R_j(\lambda)$ is a linear combination of $R_j(\lambda)$ and an operator $T_j(\lambda)$ given by

$$T_j(\lambda)f(x) = \lambda^{n/2} \int_{\mathbb{R}^n} ((x \cdot \nabla_x + y \cdot \nabla_y) R_j) (\lambda^{1/2} x, \lambda^{1/2} y) f(y) dy$$

where $x \cdot \nabla_x = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ and $y \cdot \nabla_y = \sum_{i=1}^{n} y_i \frac{\partial}{\partial y_i}$.

Let $\delta_\lambda$ stand for the dilation operator $f(x) \rightarrow f(\lambda^{1/2} x)$. Then we have $T_j(\lambda) = \delta_\lambda T_j(1) \delta_\lambda^{-1}$. Integrating by parts in

$$T_j(1)f(x) = \int_{\mathbb{R}^n} \left( \sum_{i=1}^{n} (x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}) R_j \right)(x, y) f(y) dy$$

we see that it is a sum of $-R_j(1)$ and the commutator of $x \cdot \nabla_x$ with $R_j(1)$. Since $2x_i = A_i(1) + A_i^*(1)$ and $2\frac{\partial}{\partial x_i} = A_i^*(1) - A_i(1)$ we see that

$$x \cdot \nabla_x = \frac{1}{4} \sum_{i=1}^{n} (A_i^2 - A_i^2 + [A_i, A_i^*]) = n/2 I + \frac{1}{4} \sum_{i=1}^{n} (A_i^2 - A_i^2)$$

as $[A_i, A_i^*] = 2I$. Thus the commutator of $x \cdot \nabla_x$ with $R_j(1)$ reduces to a sum of commutators of the form $[A_i^2, R_j(1)]$ and $[A_i^2, R_j(1)]$ and we are left with proving

**Proposition 2.3.** The families $\delta_\lambda[A_i^2, R_j(1)]\delta_\lambda^{-1}$ and $\delta_\lambda[A^2, R_j(1)]\delta_\lambda^{-1}$ are $R$-bounded.

**Proof.** We only consider the family $\delta_\lambda[A_i^2, R_j(1)]\delta_\lambda^{-1}$ as the treatment of the other one is similar. We will also assume $i = j$ as the other cases are simpler as $A_i$ commutes with $A_j$ when $i$ is different from $j$. A moment’s thought reveals that

$$[A_j^2, R_j(1)] = A_j[A_j, R_j(1)] + [A_j, R_j(1)]A_j$$

and hence we are left with the families

$$\delta_\lambda A_j[A_j, R_j(1)]\delta_\lambda^{-1}, \quad \delta_\lambda[A_j, R_j(1)]A_j\delta_\lambda^{-1}$$

and again we will consider only the first family.

As

$$A_j[A_j, R_j(1)] = A_j^3 H^{-1/2} - A_j^2 H^{-1/2} A_j = A_j^2[A_j, H^{-1/2}]$$

making use of the fact that $H^{-1/2} A_j = A_j(H + 2)^{-1/2}$ we get

$$A_j[A_j, R_j(1)] = -A_j^3((H + 2)^{-1/2} - H^{-1/2}).$$

An easy calculation shows that

$$\delta_\lambda A_j\delta_\lambda^{-1} = \lambda^{-1/2} A_j(\lambda), \quad \delta_\lambda H^{-1/2}\delta_\lambda^{-1} = \lambda^{1/2} H(\lambda)^{-1/2}.$$
and hence we finally get
\[ \delta_\lambda A_j[A_j, R_j(1)]\delta_\lambda^{-1} = -\lambda^{-1} A_j(\lambda)^3((H(\lambda) + 2\lambda)^{-1/2} - H(\lambda)^{-1/2}). \]
The above can be rewritten as
\[ A_j(\lambda)^3 \int_0^1 (H(\lambda) + 2\lambda s)^{-3/2} ds. \]
Thus everything boils down to showing that the family
\[ A_j(\lambda)^3 (H(\lambda) + 2\lambda s)^{-3/2} \]
is R-bounded uniformly in \( s, 0 < s < 1 \). But these operators
are also CZ singular integrals whose kernels satisfy estimates uniformly
in \( \lambda \) and \( s \). This completes the proof. \( \square \)

### 2.2. Higher order Riesz transforms.

We briefly recall some notations we use referring to [9] for details. Given a function \( f \) on \( \mathbb{C}^n \) we denote by \( W_\lambda(f) \) the Weyl transform defined by
\[ W_\lambda(f) = \int_{\mathbb{C}^n} f(z)\pi_\lambda(z,0)dz \]
where \( \pi_\lambda \) stands for the Schrödinger representation of the Heisenberg
group realised on \( L^2(\mathbb{R}^n) \). For \( P \in \mathcal{H}_{p,q} \) the operator \( G_\lambda(P) = W_\lambda(F_\lambda P) \)
is called its Weyl correspondence. Here \( F_\lambda \) stands for the symplectic
Fourier transform. The unitary group \( U(n) \) acts on \( \mathcal{H}_{p,q} \) by \( \rho(\sigma)P(z) = P(\sigma^{-1}z) \) and the action is irreducible, i.e. we get an irreducible unitary
representation of \( U(n) \) on \( \mathcal{H}_{p,q} \). In view of this, if \( P \in \mathcal{H}_{p,q} \) then the
linear span of its orbit \( \rho(\sigma)P \) under \( U(n) \) is the whole of \( \mathcal{H}_{p,q} \). Therefore, if we let \( P_0(z) = \bar{z}_1^{p}\bar{z}_2^{q} \) then any \( P \in \mathcal{H}_{p,q} \) is a linear combination
of \( \rho(\sigma)P_0, \sigma \in U(n) \).

**Proposition 2.4.** If the higher order Riesz transform \( R_{P_0} \) is bounded
on \( L^p(\mathbb{R}^{n+1}) \) then so is \( R_P \) for any \( P \in \mathcal{H}_{p,q} \).

**Proof.** In order to prove this proposition we need to use several
facts about the symplectic group \( Sp(n, \mathbb{R}) \) and the metaplectic repre-
sentations \( \mu_\lambda(\sigma) \) of \( Sp(n, \mathbb{R}) \). A good source for the material we use
here is Chapter 4 of Folland [3]. The action of \( U(n) \) on the Heisenberg
group leads to certain unitary operators, denoted by \( \mu_\lambda(\sigma) \) so that
\[ \mu_\lambda(\sigma)\pi_\lambda(z,t)\mu_\lambda(\sigma)^* = \pi_\lambda(\sigma z, t). \]
Recalling the definition of \( R_P \) we see that this means
\[ R_{\rho(\sigma)P}f(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \mu_\lambda(\sigma)G_\lambda(P)\mu_\lambda(\sigma)^* f^\lambda(x)d\lambda. \]
Therefore, our proposition will be proved if we show that the operators
\[ f \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \mu_\lambda(\sigma)f^\lambda(x)d\lambda \]
are all bounded on $L^p(\mathbb{R}^{n+1})$.

If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ the group $U(n)$ corresponds to $Sp(n, \mathbb{R}) \cap O(2n, \mathbb{R})$. By Proposition 4.10 in [3] the group $Sp(n, \mathbb{R})$ is generated by certain subgroups $N = \left\{ \left( \begin{array}{cc} I & 0 \\ C & I \end{array} \right) : C = C^* \right\}$ and

$$D = \left\{ \left( \begin{array}{cc} A & 0 \\ 0 & A^{-1} \end{array} \right) : A \in GL(n, \mathbb{R}) \right\}$$

The matrix $J = \left( \begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right)$.

For elements of these subgroups $\mu_\lambda(\sigma)$ are explicitly known. In fact we have:

$$\mu_\lambda(\sigma)\varphi(\xi) = |A|^{-1/2} \varphi(A^{-1} \xi)$$

if $\sigma \in D$ and

$$\mu_\lambda(\sigma)\varphi(\xi) = e^{i\lambda \xi \cdot C \xi} \varphi(\xi)$$

if $\sigma \in N$. Moreover, $\mu_\lambda(J)\varphi(\xi) = \hat{\varphi}(\lambda \xi)$

is the Fourier transform on $\mathbb{R}^n$ followed by a dilation. Clearly, the operator

$$f \to \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} |A|^{-1/2} f^\lambda(A^{-1} x) d\lambda = |A|^{-1/2} f(A^{-1} x, t)$$

is bounded on $L^p(\mathbb{R}^{n+1})$. And so is the operator

$$f \to \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} e^{i\lambda \xi \cdot C \xi} f^\lambda(x) d\lambda = f(x, t - \frac{1}{2} x \cdot C x).$$

Though Fourier transform does not define a bounded operator on $L^p$ unless $p = 2$ we can take care of the operators involving $\mu_\lambda(J)$ in the following way. Observe that we need to prove the boundedness of

$$f \to \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \mu_\lambda(J) G_\lambda(P) \mu_\lambda(J)^* f^\lambda(x) d\lambda$$

whenever the operator

$$f \to \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} G_\lambda(P) f^\lambda(x) d\lambda$$

is bounded. But as $P \in \mathcal{H}_{p,q}$ is homogeneous of bidegree $(p, q)$ and the action of $J$ is $\rho(J)P(z) = P(-iz) = (-i)^{p-q} P(z)$ the former operator is just a scalar multiple of the latter and hence bounded. As $Sp(n, \mathbb{R})$ is generated by $N, D$ and $J$ the proposition is proved. □

In view of the above proposition Theorem 1.2 will be proved once we obtain the following result.

**Theorem 2.5.** Let $P_0(z) = z_1^p z_2^q$. Then $R_{P_0}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$. 

Theorem 3.1. Assume that \( m \in C^k(\mathbb{R}^n) \) satisfies \( |m^{(j)}(\lambda)| \leq C_j |\lambda|^{-j} \) for large \(|\lambda|\) and for all \( j = 0, 1, 2, \ldots, k \) where \( k \geq n/2 \). Then for any choice of \( \lambda_j \in \mathbb{R}^* \) and \( f_j \in L^p(\mathbb{R}^n), 1 < p < \infty \) we have the uniform estimate

\[
\left\| \sum_{j=0}^{\infty} \left| m(H(\lambda_j)) f_j \right|^2 \right\|^{1/2} \leq C \left\| \left( \sum_{j=0}^{\infty} \left| f_j \right|^2 \right)^{1/2} \right\|^{1/2}_{p}.
\]

In proving this theorem we closely follow the proof of Theorem 4.2.1 in [9] and hence we briefly recall the proof. Fixing \( \lambda > 0 \) for the sake of definiteness we consider the boundedness of \( m(H(\lambda)) \) on \( L^p(\mathbb{R}^n) \). In [9] this is achieved by means of Littlewood-Paley g-functions. We define,
for each positive integer $k$, the function

$$(g_k^\lambda(f, x))^2 = \int_0^\infty |\partial_t^k T_t^\lambda f(x)|^2 t^{2k-1} dt$$

where $T_t^\lambda = e^{-tH(\lambda)}$ is the semigroup generated by $H(\lambda)$. In [9] it is shown that (see Theorem 4.1.2) $g_1^\lambda(f)$ can be considered as a singular integral operator whose kernel is taking values in the Hilbert space $L^2(\mathbb{R}^+, tdt)$ and hence bounded on $L^p(\mathbb{R}^n)$. Moreover, it is also shown that $g_k^\lambda(f)$ is a constant multiple of isometry on $L^2(\mathbb{R}^n)$ and hence we have the equivalence of norms:

$$C_1 \|f\|_p \leq \|g_1^\lambda(f)\|_p \leq C_2 \|f\|_p$$

for all $1 < p < \infty$. Here, it can be verified that the constants $C_j, j = 1, 2$ can be taken independent of $\lambda$. Moreover, we also have the pointiest estimate $g_k^\lambda(f, x) \leq C_k g_{k+1}^\lambda(f, x)$.

A version of Cordoba-Fefferman theorem for CZ singular integrals whose kernels are taking values in a Hilbert space is true. In fact an easy modification of the proof given in Torchinsky [11] substantiates this claim. By treating the $g$-functions as singular integral operators with kernels taking values in $L^2(\mathbb{R}^+, tdt)$ we can deduce the following result.

**Theorem 3.2.** For any choice of $\lambda_j \in \mathbb{R}^*$ and $f_j \in L^p(\mathbb{R}^n)$ we have

$$\|(\sum_{j=0}^\infty |g_j^\lambda(f_j)|^2)^{1/2}\|_p \leq C\|(\sum_{j=0}^\infty |f_j|^2)^{1/2}\|_p$$

for all $1 < p < \infty$. The reverse inequality also holds.

In order to prove the multiplier theorem we also need to consider $g_k^{*\lambda}$ functions which are defined by

$$(g_k^{*\lambda}(f, x))^2 = \int_{\mathbb{R}^n} \int_0^\infty t^{-n/2}(1 + t^{-1}|x - y|^2)^{-k} |\partial_t T_t^\lambda f(y)|^2 t dt dy$$

If $k > n/2$ it is proved in [9] (see Theorem 4.1.3) that

$$\|g_k^{*\lambda}(f)\|_p \leq C\|f\|_p, \ p > 2.$$  

The proof of this depends on two facts: the boundedness of $g_1^\lambda$ functions and the Hardy-Littlewood maximal functions. By the celebrated theorem of Fefferman-Stein the Hardy-Littlewood maximal function satisfies a vector valued inequality, see Theorem 1.1, Chapter XII in [11]. Hence by combining this with Theorem 3.2 above we get
Theorem 3.3. For any choice of \( \lambda_j \in \mathbb{R}^* \) and \( f_j \in L^p(\mathbb{R}^n) \) we have

\[
\| \sum_{j=0}^{\infty} |g_k^*\lambda_j(f_j)|^{1/2} \|^p \leq C \| \sum_{j=0}^{\infty} |f_j|^2 \|^{1/2}
\]

for all \( p > 2 \) provided \( k > n/2 \).

Under the hypothesis on \( m \) it has been proved in [9] (see Section 4.2) that the pointwise inequality

\[
g_{k+1}^\lambda(m(H(\lambda))f, x) \leq C_k g_k^*\lambda(f, x)
\]

holds. It can be checked, by following the proof carefully, that the constant \( C_k \) can be taken independent of \( \lambda \). It is then clear that a vector valued analogue of the above estimate is valid. Hence, by appealing to Theorems 3.2 and 3.3 we obtain Theorem 3.1. This takes care of the R-boundedness of \( m(H(\lambda)) \).

3.2. R-boundedness of the derivative of \( m(H(\lambda)) \). We begin with the observation that the kernel of \( m(H(\lambda)) \) is given by

\[
\lambda^{n/2} \sum_{k=0}^{\infty} m((2k + n)\lambda)\Phi_k(\lambda^{1/2}x, \lambda^{1/2}y)
\]

where \( \Phi_k(x, y) \) is the kernel of \( P_k \). If we let \( m_\lambda(t) = m(t\lambda) \) then, with obvious notations, we have

\[
m(H(\lambda))(x, y) = \lambda^{n/2}m_\lambda(H)(\lambda^{1/2}x, \lambda^{1/2}y).
\]

Using this we can easily prove

Lemma 3.4. \( \lambda \frac{d}{d\lambda} m(H(\lambda)) \) is a linear combination of operators of the form \( m(H(\lambda)) \), \( H(\lambda)m'(H(\lambda)) \) and the commutators \( \delta_\lambda[A_j^2, m_\lambda(H)]\delta_\lambda^{-1} \) and \( \delta_\lambda[A_j^2, m_\lambda(H)]\delta_\lambda^{-1} \).

We have just shown that the families \( m(H(\lambda)) \) and \( H(\lambda)m'(H(\lambda)) \) are R-bounded under the hypothesis of Theorem 1.3. To handle the commutators we proceed as in the case of Riesz transforms. We need to consider the operators \( \delta_\lambda[A_j, m_\lambda(H)]\delta_\lambda^{-1} \) and \( \delta_\lambda[A_j, m_\lambda(H)]A_j\delta_\lambda^{-1} \). As before using \( m_\lambda(H)A_j = A_j m_\lambda(H + 2) \) we get

\[
\delta_\lambda[A_j, m_\lambda(H)]\delta_\lambda^{-1} = -\delta_\lambda[A_j^2, (m_\lambda(H + 2) - m_\lambda(H))]\delta_\lambda^{-1}.
\]

Finally, the above can be written in the form

\[
A_j(\lambda)^2 \int_0^1 m'(H(\lambda) + 2\lambda s)ds.
\]
Since $A_j(\lambda)^2H(\lambda)^{-1}$ are singular integral operators we need to check the R-boundedness of
\[ \int_0^1 H(\lambda) m'(H(\lambda) + 2\lambda s))ds. \]

The hypothesis on $m$ allows us to handle this family with estimates uniformly in $s$, $0 < s < 1$. This completes the proof of the R-boundedness of $\lambda \frac{d}{d\lambda} M(H(\lambda))$ and hence Theorem 1.3 is proved.

### 3.3. Boundedness of the Bochner-Riesz means.

Recall that the kernel of $P_k(\lambda)$ is of the form $|\lambda|^{n/2} \Phi_k(|\lambda|^{1/2}x, |\lambda|^{1/2}y)$ and hence
\[ B_R^\delta f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} e^{-iM|\lambda|^{n/2} s_R^\delta \ell R/|\lambda|(|\lambda|^{1/2}x, |\lambda|^{1/2}y)f^\lambda(y)dyd\lambda. \]

where
\[ s_R^\delta(x, y) = \sum_{(2k+n)\leq R} (1 - (2k + n)/R)^\delta \Phi_k(x, y). \]

If $D_r f(x, t) = r^{n+2} f(rx, r^2t)$ stands for the nonisotropic dilation, then it is easy to see that
\[ B_R^\delta D_r f(x, t) = D_r B_R^\delta f(x, t) \]
and hence it is enough to prove the boundedness of $B_1^\delta$. Note that
\[ B_1^\delta f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iMm_\delta(\lambda)} f^\lambda(x) d\lambda \]
where
\[ m_\delta(\lambda) = \sum_{(2k+n)|\lambda|\leq 1} (1 - (2k + n)|\lambda|)^\delta P_k(\lambda). \]

Thus we can view $B_1^\delta$ as a Fourier multiplier corresponding to the operator valued multiplier $m_\delta(\lambda)$. Therefore, in order to prove Theorem 1.4 we only need to show that $m_\delta(\lambda)$ and $\lambda \frac{d}{d\lambda} m_\delta(\lambda)$ are both R-bounded families of operators.

Let
\[ S_R^\delta f(x) = \sum_{(2k+n)\leq R} (1 - (2k + n)/R)^\delta P_k f \]
stand for the Bochner-Riesz means associated to the Hermite operator $H$. Then it follows that
\[ m_\delta(\lambda) = \delta_{|\lambda|} \delta_{|\lambda|}^{-1} \delta_{|\lambda|}^{-1}. \]

We make use of the following result in order to prove the R-boundedness of the family $m_\delta(\lambda)$. Let $Mf$ stand for the Hardy-Littlewood maximal function of $f$. 

Theorem 3.5. Assume that $f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty$ and $\delta > (n - 1)/2 + 1/6$. Then $\sup_{R > 0} |S_R^\delta f(x)| \leq C \left( Mf(x) + Mf(-x) \right)$.

This theorem can be proved using the estimates given in [9] (see Theorem 3.3.5). Details can be found in [8]. Using this result it is not difficult to prove

Theorem 3.6. The family $m_\delta(\lambda)$ is R-bounded on $L^p(\mathbb{R}^n)$ for all $\delta > (n - 1)/2 + 1/6$ and $1 < p < \infty$.

Proof. In view of the relation between $m_\delta(\lambda)$ and $S_R^\delta$ we need to show that

$$\left\| \left( \sum_{j=1}^{\infty} \left| \delta_{R_j}^{-1} S_{R_j}^\delta \delta_{R_j} f_j(x) \right|^2 \right) \right\|_{L^p}^{1/2} \leq C \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right) \right\|_{L^p}^{1/2}$$

for all sequences $R_j > 0$ and $f_j \in L^p(\mathbb{R}^n)$. By the result of Theorem 3.5

$$|\delta_{R_j}^{-1} S_{R_j}^\delta \delta_{R_j} f_j(x)| \leq C \left( M(\delta_{R_j} f_j)(R_j^{-1/2} x) + M(\delta_{R_j} f_j)(-R_j^{-1/2} x) \right)$$

and a simple calculation, recalling the definition of $Mf$, shows that

$$|\delta_{R_j}^{-1} S_{R_j}^\delta \delta_{R_j} f_j(x)| \leq CM f_j(x).$$

Thus we are required to prove

$$\left\| \left( \sum_{j=1}^{\infty} |M f_j|^2 \right) \right\|_{L^p}^{1/2} \leq C \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right) \right\|_{L^p}^{1/2}$$

which is the Fefferman-Stein inequality for the maximal function. □

It remains to be proved that the family $\lambda^{\geq \delta}_{\alpha} m_\delta(\lambda)$ is also R-bounded. This is done in the following theorem.

Theorem 3.7. For $\delta > (n + 1)/2 + 1/6$ the family $\lambda^{\geq \delta}_{\alpha} m_\delta(\lambda)$ is R-bounded on $L^p(\mathbb{R}^{n+1}), 1 < p < \infty$.

Proof. Applying Lemma 3.4 with $m(t) = (1 - t)^\delta_+$ we see that $\lambda^{\geq \delta}_{\alpha} m_\delta(\lambda)$ is a linear combination of $(1 - H(\lambda))^\delta_+, H(\lambda)((1 - H(\lambda))^\delta_+ - 1)$ and commutators of the form $[A_j(\lambda)^2, (1 - H(\lambda))^\delta_+]$ and $[A_j(\lambda)^2, (1 - H(\lambda))^\delta_+]$. The first two families are handled as in the previous theorem. We prove the R-boundedness of $[A_j(\lambda)^2, (1 - H(\lambda))^\delta_+]$ and the other family can be similarly handled. It is enough to look at $A_j(\lambda)^2 (1 - H(\lambda))^\delta_+$ which we write as

$$A_j(\lambda)^2 (1 - H(\lambda))^\delta_+ = A_j(\lambda)^2 H(\lambda)^{-1} H(\lambda)(1 - H(\lambda))^\delta_+.$$
But $A_j(\lambda)^2 H(\lambda)^{-1}$ are singular integral operators and
\[
H(\lambda)(1 - H(\lambda))^\delta_+ = (1 - H(\lambda))^\delta_+ - (1 - H(\lambda))^\delta_+^{+1}.
\]
Thus $A_j(\lambda)^2(1 - H(\lambda))^\delta_+$ is R-bounded. \hfill \Box

Acknowledgments
The work of the last author is supported by J. C. Bose Fellowship from the Department of Science and Technology (DST).

References
[1] T. Coulhon, D. Mueller and J. Zienkiewicz, About Riesz transforms on Heisenberg groups, Math. Annalen 312(1996), 369-379.
[2] A. F. M. ter Elst, D. W. Robinson and A. Sikora, Heat kernels and Riesz transforms on nilpotent Lie groups, Colloq. Math. 74 (1997), 191-218.
[3] G. B. Folland, Harmonic analysis in phase space, Annals of Mathematics Studies, 122. Princeton University Press, Princeton, NJ, 1989.
[4] D. Geller, Spherical harmonics, the Weyl transform and the Fourier transform on the Heisenberg group, Can. J. Math. 36(1984), 615-684.
[5] R. Meyer, $L^p$ estimates for the Grushin operator, arXiv:0709.2188 (2007)
[6] P. K. Sanjay and S. Thangavelu, Revisiting Riesz transforms on Heisenberg groups, preprint.
[7] K. Stempak and J. L. Torrea, Poisson integrals and Riesz transforms for the Hermite function expansions with weights, J. Funct. Anal. 202 (2003), 443-472.
[8] S. Thangavelu, Summability of Hermite expansions I,II, Trans. Amer. Math. Soc. 314 (1989), 119-170.
[9] S. Thangavelu, Lectures on Hermite and Laguerre expansions, Math. Notes No. 42, Princeton Univ. Press (1993).
[10] S. Thangavelu, An introduction to the uncertainty principle, Prog. Math. 217(2004), Birkhauser-Boston.
[11] A. Torchinsky, Real variable methods in harmonic analysis, Dover publications, Inc., Mineola, New York (2004)
[12] L. Weis, Operator valued Fourier multiplier theorems and maximal $L^p$ regularity, Math. Ann. 319 (2001), 735-758.

Department of Mathematics, Indian Institute of Science, Bangalore-560 012
E-mail address: jyoti@math.iisc.ernet.in, sanjay@math.iisc.ernet.in, veluma@math.iisc.ernet.in