Abelian groups definable in $p$-adically closed fields

Will Johnson and Ningyuan Yao

August 23, 2022

Abstract

Recall that a group $G$ has finitely satisfiable generics ($fsg$) or definable $f$-generics ($dfg$) if there is a global type $p$ on $G$ and a small model $M_0$ such that every left translate of $p$ is finitely satisfiable in $M_0$ or definable over $M_0$, respectively. We show that any abelian group definable in a $p$-adically closed field is an extension of a definably compact $fsg$ definable group by a $dfg$ definable group. We discuss an approach which might prove a similar statement for interpretable abelian groups. In the case where $G$ is an abelian group definable in the standard model $\mathbb{Q}_p$, we show that $G^0 = G^{\infty}$, and that $G$ is an open subgroup of an algebraic group, up to finite factors. This latter result can be seen as a rough classification of abelian definable groups in $\mathbb{Q}_p$.

1 Introduction

In this paper we study abelian groups definable in $p$-adically closed fields. Recall that a definable group $G$ has finitely satisfiable generics ($fsg$) if there is a global type $p$ on $G$, finitely satisfiable in a small model, with boundedly many left translates. Similarly, $G$ has definable $f$-generics ($dfg$) if there is a definable global type on $G$ with boundedly many left translates. The main theorem of this paper is the following decomposition of abelian definable groups into $dfg$ and $fsg$ components:

**Theorem 1.1.** Suppose that $M$ is a $p$-adically closed field and $G$ is an abelian group definable in $M$. Then there is a short exact sequence of definable groups

$$1 \to H \to G \to C \to 1$$

where $H$ has $dfg$ and $C$ is definably compact and has $fsg$.

An analogous decomposition for definably amenable groups in o-minimal structures was proved by Conversano and Pillay [CP12, Propositions 4.6–7] (see also [PY16, Fact 1.18]). Pillay and Yao asked whether such a decomposition exists for any definably amenable group in a distal theory [PY16, Question 1.19]; Theorem 1.1 can be seen as evidence towards a positive answer.

When $M = \mathbb{Q}_p$, we obtain two useful consequences from Theorem 1.1.
Theorem 1.2. Suppose that $G$ is an abelian definable group in $\mathbb{Q}_p$.

1. $G^{00} = G^0$.

2. There is a finite index definable subgroup $E \subseteq G$ and a finite subgroup $F \subseteq E$ such that $E/F$ is isomorphic to an open subgroup of an algebraic group.

This yields a loose “classification” of abelian definable groups in $\mathbb{Q}_p$—up to finite factors, they are exactly the open subgroups of algebraic groups.

Acknowledgments. The first author was supported by the National Natural Science Foundation of China (Grant No. 12101131). The second author was supported by the National Social Fund of China (Grant No. 20CZX050). Section 5 was partially based on joint work with Zhentao Zhang, who declined to be an author on this paper.

1.1 Outline

In Section 2 we review some tools needed in the proof. In Section 3 we prove the decomposition in Theorem 1.1. In Section 4 we obtain the consequences for $\mathbb{Q}_p$-definable groups listed in Theorem 1.2. In Section 5 we discuss our original strategy for Theorem 1.1 which suggests a generalization of Theorem 1.1 to interpretable groups (Conjecture 5.14).

There are also two appendices. Appendix A proves a technical statement about topological properties of ict patterns in interpretable groups, needed in Lemma 5.8. Appendix B is on $dfg$ in short exact sequences, and generalizes some facts in Section 2.1 beyond the context of $pCF$.

1.2 Notation and conventions

“Definable” means “definable with parameters.” We write the monster model as $M$. A “type” is a complete type, and a “partial type” is a partial type. Tuples are finite by default. We usually write tuples as $a, b, x, y$ rather than $\bar{a}, \bar{b}, \bar{x}, \bar{y}$. We distinguish between “real” elements or tuples (in $M$) and “imaginaries” (in $M^{eq}$), and we distinguish between “definable” (in $M$) and “interpretable” (in $M^{eq}$). The exception is Appendix B where we work in $M^{eq}$. If $D$ is a definable set, then $\hat{\gamma} D^{\gamma}$ denotes its code, a tuple in $M^{eq}$. If $p$ is a definable type, then $\hat{\gamma} p^{\gamma}$ denotes its code, an infinite tuple in $M^{eq}$.

Throughout, $pCF$ means the complete theory of $\mathbb{Q}_p$, and a “$p$-adically closed field” is a model of this theory, or equivalently, a field elementarily equivalent to $\mathbb{Q}_p$. We do not consider “$p$-adically closed fields” in the broader sense (fields elementarily equivalent to finite extensions of $\mathbb{Q}_p$), though we strongly suspect that all the results generalize to these theories. We write the language of $pCF$ as $\mathcal{L}$. The language $\mathcal{L}$ should be one-sorted; otherwise the choice of $\mathcal{L}$ is irrelevant.
2 Tools

In this section, we review a few tools that will be needed in the proof of the main theorems. In Section 2.1 we show that certain properties ($G^0 = G^{00}$, $dfg$) behave well in short exact sequences. In Section 2.2 we show that we can take quotients by certain $dfg$ groups without leaving the definable category.

2.1 Extensions

Recall that $G^{00}$ and $G^0$ exist for definable groups $G$ in NIP theories [HPP08, Proposition 6.1].

Lemma 2.1 (Assuming NIP). Let $\pi : G \to X$ be a surjective homomorphism of definable groups. Then $\pi(G^{00}) = X^{00}$.

Proof. There is a surjection $G/G^{00} \to X/\pi(G^{00})$, so $X/\pi(G^{00})$ is bounded and $\pi(G^{00}) \supseteq X^{00}$. There is an bijection $G/\pi^{-1}(X^{00}) \to X/X^{00}$, so $G/\pi^{-1}(X^{00})$ is bounded and $G^{00} \subseteq \pi^{-1}(X^{00})$. This implies $\pi(G^{00}) \subseteq X^{00}$. \qed

Lemma 2.2 (Assuming NIP). Let $1 \to H \to G \xrightarrow{\pi} X \to 1$ be a short exact sequence of definable groups. If $H^0 = H^{00}$ and $X^0 = X^{00}$, then $G^0 = G^{00}$.

Proof. The fact that $H^0 = H^{00}$ and $X^0 = X^{00}$ means that $H/H^{00}$ and $X/X^{00}$ are profinite. The short exact sequence

\[ 1 \to H/(H \cap G^{00}) \to G/G^{00} \to X/X^{00} \to 1 \quad (\ast) \]

shows that $H/(H \cap G^{00})$ is bounded, and then (\ast) is continuous in the logic topology. As $H/(H \cap G^{00})$ is bounded, it must be a quotient of $H/H^{00}$ which is profinite. Therefore $H/(H \cap G^{00})$ is profinite. In the category of compact Hausdorff groups, an extension of a profinite group by a profinite group is profinite. Therefore $G/G^{00}$ is profinite, which implies $G^0 = G^{00}$. \qed

Recall that $p\text{CF}$ has definable Skolem functions.

Lemma 2.3. Suppose that $M$ is a saturated model of $p\text{CF}$. Let

\[ 1 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 1 \]

be a short exact sequence of definable groups. Then $B$ has $dfg$ iff $A$ and $C$ do.

Proof. We prove the following:

1. If $B$ has $dfg$, then $C$ has $dfg$.
2. If $B$ has $dfg$, then $A$ has $dfg$.
3. If $A$ and $C$ have $dfg$, then $B$ has $dfg$. 

3
By definable Skolem functions, there is a definable function $f : C \rightarrow B$ which is a set-theoretic section of $\pi$, in the sense that $\pi(f(c)) = c$ for $c \in C$. Now we proceed with the proofs:

1. If $\text{tp}(b/\mathcal{M})$ is a definable $f$-generic type in $B$, then $\text{tp}(\pi(b)/\mathcal{M})$ is a definable $f$-generic type in $C$.

2. The proof is nearly identical to [PY19, Lemmas 2.24, 2.25]. In an elementary extension $\mathcal{M}' \succeq \mathcal{M}$, take $b_0 \in B(\mathcal{M}')$ realizing a definable $f$-generic type in $B$. Write $b_0$ as $a_0 \cdot f(\pi(b_0))$ for some $a_0 \in A(\mathcal{M}')$. Then $a_0 \in \text{dcl}(\mathcal{M}b_0)$, so $\text{tp}(a_0/\mathcal{M})$ is definable. We claim that $\text{tp}(a_0/\mathcal{M})$ has boundedly many left translates, and is therefore a definable $f$-generic type in $A$. Note that $A^{00} \subseteq B^{00}$ because $A/(A \cap B^{00}) \cong AB^{00}/B^{00}$ is bounded. If $\delta \in A^{00}(\mathcal{M})$, then $\text{tp}(\delta \cdot b_0/\mathcal{M}) = \text{tp}(b_0/\mathcal{M})$, and therefore

$$\text{tp}(\delta \cdot b_0 \cdot f(\pi(\delta \cdot b_0))^{-1}/\mathcal{M}) = \text{tp}(b_0 \cdot f(\pi(b_0))^{-1}/\mathcal{M}) = \text{tp}(a_0/\mathcal{M}).$$

But $\pi(\delta \cdot b_0) = \pi(b_0)$, and so

$$\text{tp}(\delta \cdot b_0 \cdot f(\pi(\delta \cdot b_0))^{-1}/\mathcal{M}) = \text{tp}(\delta \cdot b_0 \cdot f(\pi(b_0))^{-1}/\mathcal{M}) = \text{tp}(\delta \cdot a_0/\mathcal{M}).$$

Therefore $\text{tp}(a_0/\mathcal{M})$ is invariant under left translation by any $\delta \in A^{00}$, and it has boundedly many left translates.

3. Let $p(x) \in S_A(\mathcal{M})$ and $q(y) \in S_C(\mathcal{M})$ be $dfg$ types of $A$ and $C$ respectively. Let $\mathcal{M}_0$ be a small model defining the section $f$, the short exact sequence, and all the left translates of $p$ and $q$.

In some elementary extension $\mathcal{M}' \succeq \mathcal{M}$, take $c_0 \models q$ and $a_0 \models p|\mathcal{M}c_0$. Then $\text{tp}(a_0, c_0/\mathcal{M})$ is $\mathcal{M}_0$-definable—it is the Morley product of $p$ and $q$. Let $b_0 = f(c_0) \cdot a_0$. Then $\text{tp}(b_0/\mathcal{M})$ is again $\mathcal{M}_0$-definable. We claim that every left translate of $\text{tp}(b_0/\mathcal{M})$ is $\mathcal{M}_0$-definable. Fix some $\delta \in B(\mathcal{M})$. Let $b_1 = \delta \cdot b_0$. Let $c_1 = \pi(\delta) \cdot c_0$. Let $\delta' = f(c_1)^{-1} \cdot \delta \cdot f(c_0)$. Note

$$\pi(\delta') = \pi(f(c_1))^{-1} \cdot \pi(\delta) \cdot \pi(f(c_0)) = c_1^{-1} \cdot \pi(\delta) \cdot c_0 = 1,$$

so $\delta' \in A(\mathcal{M}')$. Let $a_1 = \delta' \cdot a_0$. Then

$$b_1 = \delta \cdot b_0 = \delta \cdot f(c_0) \cdot a_0 = f(c_1) \cdot \delta' \cdot a_0 = f(c_1) \cdot a_1.$$

Now $\text{tp}(c_1/\mathcal{M}) = \text{tp}(\pi(\delta) \cdot c_0/\mathcal{M})$ is a left-translate of the $dfg$ type $\text{tp}(c_0/\mathcal{M}) = q$, and so $\text{tp}(c_1/\mathcal{M})$ is $\mathcal{M}_0$-definable. If $U$ is $\text{dcl}(\mathcal{M}c_0) = \text{dcl}(\mathcal{M}c_1)$, then $\text{tp}(a_1/U) = \text{tp}(\delta' \cdot a_0/U)$ is a left translate of the $dfg$ type $\text{tp}(a_0/U) = p|U$ (because $\delta' \in U$). Therefore $\text{tp}(a_1/U)$ is again $\mathcal{M}_0$-definable. As $b_1 = f(c_1) \cdot a_1$, we see that $\text{tp}(\delta \cdot b_0/\mathcal{M}) = \text{tp}(b_1/\mathcal{M})$ is $\mathcal{M}_0$-definable for the same reason that $\text{tp}(b_0/\mathcal{M})$ is $\mathcal{M}_0$-definable, essentially because $\text{tp}(c_1/\mathcal{M})$ and $\text{tp}(a_1/Mc_1)$ are $\mathcal{M}_0$-definable.

See Theorem B.6 in the appendix for an alternate proof of (3) not using definable Skolem functions.
2.2 Codes and quotients

Let $G$ be a definable group and $H$ be a normal subgroup. A priori, the quotient group $G/H$ is interpretable, not definable. In this section, we show that for certain dfg groups $H$, the quotient $G/H$ is automatically definable (Corollary 2.9). The key is to show that certain definable types are coded by real tuples (Theorem 2.7). Both of these results will be proved in greater generality in future work [AGJ22, Theorems 3.4, 4.1].

If $D$ is a definable set in a model $M$, let $⌜D⌝$ denote “the” code of $D$ in $M$, which is well-defined up to interdefinability. If $σ ∈ Aut(M)$, then

$$σ(D) = D ⇐⇒ σ(⌜D⌝) = ⌜D⌝,$$

and this property characterizes $⌜D⌝$ when $M$ is sufficiently saturated and homogeneous.

**Lemma 2.4.** Let $K$ be a field and $V ⊆ K^n$ be Zariski closed. Then the definable set $V$ is coded by a tuple in $K$ (rather than $K^eq$). In particular, finite subsets of $K^n$ are coded by tuples in $K$.

**Proof.** Passing to an elementary extension, we may assume $K$ is $\aleph_1$-saturated and strongly $\aleph_1$-homogeneous. Let $M = K^{al}$. Let $\overline{V}$ be the Zariski closure of $V$ in $M^n$. Note $V = \overline{V} ∩ K^n$. By elimination of imaginaries in ACF, there is a tuple $b ∈ M$ which codes $\overline{V}$ in the structure $M^n$. If $σ ∈ Aut(M/K)$ then $σ$ fixes $V$ setwise, so it also fixes the Zariski closure $\overline{V}$. Therefore $σ(b) = b$, for any $σ ∈ Aut(M/K)$. By Galois theory, $b$ is in the perfect closure of $K$. Replacing $b$ with $b^{\sigma}$ if necessary, we may assume $b$ is a tuple in $K$.

We claim that $b$ codes $V$ in the structure $K$. Suppose $σ_0 ∈ Aut(K)$. Extend $σ_0$ to an automorphism $σ ∈ Aut(M)$ arbitrarily. Then $b$ codes $V$ because

$$σ_0(V) = V ⇐⇒ σ(V) = V ⇐⇒ σ(\overline{V}) = \overline{V} ⇐⇒ σ(b) = b ⇐⇒ σ_0(b) = b.$$

The starred $\Leftarrow$ requires some explanation. The direction $⇒$ holds because the formation of Zariski closures is automorphism invariant. The direction $\Leftarrow$ holds because $σ$ fixes $K$ setwise and $V = \overline{V} ∩ K^n$. $\square$

**Lemma 2.5.** Work in a monster model $\mathbb{M}$ of $pCF$.

1. If an imaginary tuple $a$ is algebraic over a real tuple $b$, then $a$ is definable over $b$.

2. If an imaginary tuple $a$ is interalgebraic with a real tuple $b$, then $a$ is interdefinable with some real tuple $b'$.

More generally, both statements hold if we work over a set of real parameters $C ⊆ \mathbb{M}$.

**Proof.** 1. Note that $dcl(b) ≤ \mathbb{M}$ by definable Skolem functions, and so $dcl^{eq}(b) ≤ \mathbb{M}^{eq}$. Submodels are algebraically closed, so $acl^{eq}(b) = dcl^{eq}(b)$ and $a ∈ acl^{eq}(b)$.
2. By part (1), \( a \in \text{dcl}^{eq}(b) \). Write \( a \) as \( f(b) \) for some \( \emptyset \)-definable function \( f \). Let \( S \subseteq M^n \) be the set of realizations of \( \text{tp}(b/a) \). Then \( S \) is finite as \( b \in \text{acl}^{eq}(a) \). Moreover, \( S \) is \( a \)-definable, and so the code \( \sigma(S) \) is in \( \text{dcl}^{eq}(a) \). By Lemma 2.4 we can take the code \( \sigma(S) \) to be a real tuple. For any \( c \in S \), we have \( f(c) = a \), which implies \( a \in \text{dcl}^{eq}(\sigma(S)) \).

Then \( a \) is interdefinable with the real tuple \( \sigma(S) \).

The “more general” statements follow by the same proofs. Indeed, we can name the elements of \( C \) as constants without losing definable Skolem functions or codes for finite sets.

If \( p \) is a definable \( n \)-type over \( M \), let \( \delta p \) denote the infinite tuple \( \langle \delta D_\varphi \rangle : \varphi \in L \rangle \), where

\[
D_\varphi = \{ b \in M^n : \varphi(x,b) \in p(x) \}.
\]

For \( \sigma \in \text{Aut}(M) \), we have

\[
\sigma(p) = p \iff \sigma(\delta p) = \delta p,
\]

and this property determines \( \delta p \) up to interdefinability when \( M \) is sufficiently saturated and homogeneous.

**Lemma 2.6.** If \( q \in S_1(M) \) is definable, then \( \delta q \) is interdefinable with a (finite) real tuple.

**Proof.** By [JY22, Proposition 2.24], the type \( q \) must accumulate at some point \( c \) in the projective line \( \mathbb{P}^1(M) \), because \( \mathbb{P}^1(M) \) is definably compact. If necessary, we can push \( q \) forward along the map \( x \mapsto 1/x \) to ensure \( c \neq \infty \). Then \( c \in M \). Note \( c \in \text{dcl}^{eq}(\delta q) \).

There are only boundedly many types concentrating at \( c \) by [Joh18, Corollary 7.5] or [JY22, Fact 2.20], so \( \delta q \) has a small orbit under \( \text{Aut}(M/c) \). Then \( \delta q \in \text{acl}^{eq}(c) \). As in the proof of Lemma 2.5(1), \( \delta q \in \text{dcl}^{eq}(c) \), so \( \delta q \) is interdefinable with \( c \).

**Theorem 2.7.** Suppose \( q \in S_n(M) \) is a definable type, and \( \dim(q) = 1 \). Then \( \delta q \) is interdefinable with a real tuple.

**Proof.** Take an elementary extension \( M' \supseteq M \) containing a realization \( \bar{a} \) of \( q \). Then tr. \( \deg(\bar{a}/M) = \dim(q) = 1 \), so there is some \( i \) such that \( a_i \) is a transcendence basis of \( \bar{a} \) over \( M \), implying that \( \bar{a} \) is field-theoretically algebraic over \( M \) and \( a_i \). Then there is a Zariski-closed set \( V_0 \subseteq M^n \) such that there are only finitely many \( b \in V_0(M'/i) \) with \( b_i = a_i \).

Let \( V \subseteq M^n \) be the smallest Zariski-closed set such that \( \bar{a} \in V(M'/i) \), or equivalently, the smallest Zariski-closed set on which \( q \) concentrates. Any automorphism of \( M \) which fixes \( q \) fixes \( V \), and so

\[
\delta V \in \text{dcl}^{eq}(\delta q).
\]

As \( V \subseteq V_0 \), there are only finitely many \( b \in V(M'/i) \) with \( b_i = a_i \). Therefore \( \bar{a} \in \text{acl}^{eq}(\delta V) \). By Lemma 2.4 we may assume \( \delta V \) is a real tuple in \( M \), and then \( \bar{a} \in \text{dcl}^{eq}(\delta V) \) by Lemma 2.5(1). Therefore \( \bar{a} \) and \( a_i \) are interdefinable over \( \delta V \).

Take a bijection \( f \) defined over \( \delta V \) such that \( \bar{a} = f(a_i) \). Then \( q = \text{tp}(\bar{a}/M) \) is the push-forward of the definable type \( r := \text{tp}(a_i/M) \) along the \( \delta V \)-definable bijection \( f \). Therefore

\[
\delta q \in \text{dcl}^{eq}(\delta V r).
\]
Likewise, $r$ is the pushforward of $q$ along the 0-definable coordinate projection $\pi(x) = x_i$, so

$$^r r \in \text{dcl}^\text{eq}(^r q) \quad (3)$$

Combining equations (1)–(3), we see that $^r q$ is interdefinable with $^r V^\pi r$. But $^r V$ is a real tuple by Lemma 2.4 as noted above, and $^r r$ is a real tuple by Lemma 2.6.

Using a different argument, one can show that Theorem 2.7 holds for any definable n-type, without the assumption $\dim(q) = 1$ [AGJ22, Theorem 3.4]. However, the real tuple may need to be infinite [AGJ22, Proposition 3.7].

**Proposition 2.8.** If a one-dimensional dfg group $G$ acts on a definable set $X$, then the quotient space $X/G$ is definable (not just interpretable).

**Proof.** Take a global definable type $p$ on $G$ with boundedly many right translates. Take a small model $M_0$ over which everything is defined, including the boundedly many right translates of $p$. It suffices to show that every element of the interpretable set $X/G$ is interdefinable over $M_0$ with a real tuple. By Lemma 2.5(2), it suffices to show that every element of $X/G$ is interalgebraic over $M_0$ with a real tuple. Fix some element $e = G \cdot a \in X/G$, where $a \in X$.

Let $p \cdot a$ denote the pushforward of $p$ along the map $x \mapsto x \cdot a$. Note that the global types $p$ and $p \cdot a$ both have dimension 1 (or less). By Theorem 2.7, the code $^r p \cdot a^\gamma$ can be taken to be a real tuple. We claim that $^r p \cdot a^\gamma$ is interalgebraic with $e$ over $M_0$.

In one direction, $p \cdot a$ is contained in the collection

$$S = \{ p \cdot a' : a' \in G \cdot a \} = \{ p \cdot (g \cdot a) : g \in G \} = \{ (p \cdot g) \cdot a : g \in G \},$$

which is $\text{Aut}(M/M_0e)$-invariant by the first line, and small by the second line. It follows that $p \cdot a$ has a small number of conjugates over $M_0e$, and so $^r p \cdot a^\gamma \in \text{acl}^\text{eq}(M_0e)$.

In the other direction, the type $p \cdot a$ concentrates on $G \cdot a$, so its pushforward along the $M_0$-definable map $X \to X/G$ is the constant type $x = e$. Therefore $e \in \text{dcl}^\text{eq}(M_0^r p \cdot a^\gamma)$. This completes the proof that $e$ is interalgebraic with $^r p \cdot a^\gamma$ over $M_0$.

Again, this holds without the assumption $\dim(G) = 1$. See [AGJ22, Theorem 4.1].

**Corollary 2.9.** Let $G$ be a definable group and $H$ be a 1-dimensional definable normal subgroup. If $H$ has dfg, then $G/H$ is definable and $\dim(G/H) = \dim(G) - 1$.

### 3 Proof of Theorem 1.1

Work in a model $M \models p\text{CF}$.

**Theorem 3.1.** Let $M$ be a $p$-adically closed field and $G$ be a definable abelian group in $M$. Then there is a definable short exact sequence

$$1 \to H \to G \to C \to 1$$

such that $H$ has dfg, $C$ has fsg, and $C$ is definably compact.
Proof. For definable groups, $fsq$ is equivalent to definable compactness [Joh21, Theorem 1.2]. Say a subgroup $H \subseteq G$ is “good” if $G/H$ is definable and $H$ has $dfg$. For example, $H = \{1\}$ is good. Take a good subgroup $H$ maximizing $\dim(H)$. If $G/H$ is definably compact then we are done. Otherwise, $G/H$ is not definably compact. By [JY22, Corollary 6.11], there is a 1-dimensional definable $dfg$ subgroup of $G/H$. This subgroup has the form $H'/H$ for some definable subgroup of $H$. The short exact sequence

$$1 \rightarrow H \rightarrow H' \rightarrow H'/H \rightarrow 1$$

shows that $H'$ has $dfg$ by Lemma 2.3 and that

$$\dim(H') = \dim(H) + \dim(H'/H) = \dim(H) + 1 > \dim(H).$$

The quotient $G/H' = (G/H)/(H'/H)$ is definable by Corollary 2.9 and so $H'$ is a good subgroup, contradicting the choice of $H$. 

4 Abelian groups over $\mathbb{Q}_p$

Fact 4.1. Let $G$ be a definably amenable group definable over $\mathbb{Q}_p$. There is an algebraic group $H$ over $\mathbb{Q}_p$ and a definable finite-to-one group homomorphism from $G^{00}$ to $H$.

Proof. This follows from [MOS20, Theorem 2.19] via the proof of [MOS20, Corollary 2.22].

Theorem 4.2. If $G$ is an abelian group definable over $\mathbb{Q}_p$, then $G^0 = G^{00}$.

Proof. Theorem 3.1 gives a short exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow C \rightarrow 1$$

where $H$ has $dfg$ and $C$ is definably compact. Then $C^0 = C^{00}$ because $C$ is definably compact and defined over $\mathbb{Q}_p$ [OP08, Corollary 2.4], and $H^0 = H^{00}$ because $H$ is $dfg$ [PY16, proof of Lemma 1.15]. Then $G^0 = G^{00}$ by Theorem 2.2.

Corollary 4.3. If $G$ is an abelian group definable in $\mathbb{Q}_p$, then there is a finite index definable subgroup $E \subseteq G$ and finite subgroup $F$ such that $E/F$ is isomorphic to an open subgroup of an algebraic group $A$ over $\mathbb{Q}$.

Proof. By Theorem 4.2, $G^0 = G^{00}$. By Fact 4.1 there is an algebraic group $H$ and a finite-to-one definable homomorphism $f : G^0 \rightarrow H$. By compactness there is a finite-index subgroup $E \subseteq G$ such that $f$ extends to a finite-to-one definable homomorphism $f' : E \rightarrow H$. Replacing $H$ with the Zariski closure of the image of $f'$, we may assume the image is an open subgroup of $H$. 

8
5 Interpretable groups

In this section, we discuss our original approach to Theorem 3.1, which yielded a weaker result, only giving an interpretable group. However, this approach is more general in one way—one can start with an interpretable group. Unfortunately, in the interpretable case we don’t know how to prove the termination of the recursive process implicit in the proof of Theorem 3.1.

Proposition 5.1. Let $G$ be an abelian definable group, let $H$ be a definable subgroup, and let $X = G/H$ be the interpretable quotient group. Consider the canonical definable manifold topology on $G$, and the quotient topology on $X$.

1. The quotient map $\pi : G \to X$ is an open map.

2. The quotient topology on $X$ is definable.

3. The quotient topology on $X$ is a group topology.

4. The quotient topology on $X$ is Hausdorff.

Proof. 1. If $U \subseteq G$ is open, then $\pi^{-1}(\pi(U)) = U \cdot H = \bigcup_{h \in H}(U \cdot h)$ which is open. By definition of the quotient topology, $\pi(U)$ is open.

2. If $B$ is a definable basis of opens on $G$, then $\{\pi(U) : U \in B\}$ is a definable basis of opens on $X$, because $\pi$ is an open map.

3. We claim $(x, y) \mapsto x \cdot y^{-1}$ is continuous on $X$. Fix $a, b \in X$. Let $U \subseteq X$ be an open neighborhood of $a \cdot b^{-1}$. Take $\tilde{a}, \tilde{b} \in G$ lifting $a$ and $b$. Then $\tilde{a} \cdot \tilde{b}^{-1} \in \pi^{-1}(U)$, which is open. By continuity of the group operations on $G$, there are open neighborhoods $V \ni \tilde{a}$ and $W \ni \tilde{b}$ such that $x \in V$, $y \in W \implies x \cdot y^{-1} \in \pi^{-1}(U)$. Because $\pi$ is an open map, $\pi(V)$ and $\pi(W)$ are open neighborhoods of $a$ and $b$, respectively. If $x \in \pi(V)$ and $y \in \pi(W)$, then $x \cdot y^{-1} \in U$, because we can write $x = \pi(\tilde{x})$, $y = \pi(\tilde{y})$ for $\tilde{x} \in V$, $\tilde{y} \in W$, and then $x \cdot y^{-1} = \pi(\tilde{x} \cdot \tilde{y}^{-1}) \in \pi(\pi^{-1}(U)) = U$. This proves continuity of $x \cdot y^{-1}$ at $(a, b)$.

4. Because the quotient topology is a group topology, it suffices to show that $\{1_X\}$ is closed. By definition of the quotient topology, it suffices to show that $H$ is closed in $G$. On definable manifolds, the frontier of a set is lower-dimensional than the set itself \cite[Theorem 3.5]{CKDL17}:

$$\dim(\overline{H} \setminus H) < \dim(H).$$

But $\overline{H} \setminus H$ is a union of cosets of $H$, and each coset has dimension $\dim(H)$. Therefore $\overline{H} \setminus H$ must be empty, and $H$ is closed.

Definition 5.2. A manifold-dominated group is an interpretable group $X$ with a Hausdorff definable group topology such that there is a definable manifold $\tilde{X}$ and an interpretable surjective continuous open map $f : \tilde{X} \to X$. 9
In the setting of Proposition 5.1, $X$ is manifold dominated via the map $G \to X$.

**Remark 5.3.** If $X$ is any interpretable group, then there is a definable group topology $\tau$ on $X$ making $(X, \tau)$ be manifold-dominated \cite{Joh22}*{Theorem 5.10}. Moreover, $\tau$ is uniquely determined, though the manifold $\tilde{X}$ is not. This motivates working in the more general context of manifold-dominated abelian groups, rather than the special case of quotient groups $G/H$.

**Theorem 5.4.** Let $X$ be a manifold-dominated interpretable abelian group. Suppose $X$ is not definably compact. Then there is an interpretable subgroup $X' \subseteq X$ with the following properties:

1. $X'$ is not definably compact.
2. $\text{dp-rk}(X') = 1$.
3. $X'$ has dfg.

Theorem 5.4 is an analogue of \cite{JY22}*{Theorem 6.8, Corollary 6.11}, and the proof is similar. Nevertheless, we sketch the proof for completeness.

For the rest of the section, work in a monster model $\mathbb{M}$. Fix a definable manifold $\tilde{X}$, an interpretable abelian group $X$ with a Hausdorff definable group topology, and an interpretable continuous surjective open map $\pi : \tilde{X} \to X$. Also fix a small model $K$ over which everything is defined.

**Definition 5.5.** If $S$ is an interpretable topological space (in $p\text{CF}$) and $x_0 \in S$, then a good neighborhood basis of $x_0$ is an interpretable family $\{O_t\}_{t \in \Gamma}$ with the following properties:

1. $\{O_t\}_{t \in \Gamma}$ is a neighborhood basis of $x_0$.
2. $t \leq t' \implies O_t \subseteq O_{t'}$.
3. Each set $O_t$ is clopen and definably compact.
4. $\bigcup_t O_t = S$.

This is more general than the definition in \cite{JY22}*{Definition 2.27}, since we are considering topological spaces rather than topological groups. The definition here is slightly weaker, since we do not require $O_t^{-1} = O_t$ when $S$ is a group.

Fix some element $\tilde{1} \in \tilde{X}$ lifting $1 \in X$. By the proof of \cite{JY22}*{Proposition 2.28}, there is a good neighborhood basis $\{O_t\}_{t \in \Gamma}$ of $\tilde{1}$ in $\tilde{X}$. Let $V_t = \pi(O_t)$. Then $\{V_t\}_{t \in \Gamma}$ is a good neighborhood basis of $1$ in $X$. The analogue of \cite{JY22}*{Proposition 2.29} holds, via the same proof:

1. For any $t \in \Gamma$, there is $t' \in \Gamma$ such that $V_{t'} \cdot V_{t'}^{-1} \subseteq V_t$.
2. For any $t \in \Gamma$, there is $t'' \in \Gamma$ such that $V_t \cdot V_t^{-1} \subseteq V_{t''}$.
Say that a set $S \subseteq X$, not necessarily interpretable, is **bounded** if $S \subseteq V_t$ for some $t \in \Gamma$. As in [JY22, Proposition 2.10], $S$ is bounded if and only if $S$ is contained in a definably compact subset of $X$. If $A, B \subseteq X$, let $A \diamond B$ denote the set
\[ \{ g \in A : gB \cap A = \emptyset \}, \]
as in [JY22] §4.1. Let $A \diamond B \setminus C$ mean $A \diamond (B \setminus C)$.

**Lemma 5.6.** Let $I \subseteq X$ be an unbounded interpretable set. Let $A \subseteq X$ be bounded, but not necessarily interpretable. Then there is $t \in \Gamma_K$ such that $I \diamond V_t \setminus A$ is bounded.

**Proof.** Take $\beta \in \Gamma$. Let $\pi = \pi_s \gamma$.

Recall our assumption that $\pi : \tilde{X} \to X$ is $K$-interpretable for some small model $K$. Fix $|K|^+$-saturated $L$ with $K \preceq L \preceq M$. If $\Sigma$ is a definable type or definable partial type over $K$, then $\Sigma^L$ denotes its canonical extension over $L$. (See [PS17, Definition 2.12] for definability of partial types. When $\Sigma$ is complete, $\Sigma^L$ is the heir of $\Sigma$.)

**Lemma 5.7.** There is a 1-dimensional definable type $p \in S_{\tilde{X}}(K)$ whose pushforward $q = \pi_* p$ has the following properties:

1. $q$ is “unbounded” over $K$, in the sense that $q$ does not concentrate on any $K$-interpretable bounded set, or equivalently, $q$ does not concentrate on $V_t$ for any $t \in \Gamma_K$.

2. Similarly, the heir $q^L$ is unbounded over $L$.

3. If $b \in X$ realizes $q$ and $b \notin V_t$ for any $t \in \Gamma_L$, then $b$ realizes $q^L$.

**Proof.** Take $u \in M$ with $v(u) > \Gamma_K$. In other words, $u$ is infinitesimally close to $0$ over $K$. Then $\text{tp}(u/K)$ is definable. Let $\gamma = v(u)$. As $X$ is not definably compact, $V_\gamma \neq X$. The set $\pi^{-1}(X \setminus V_\gamma)$ is a non-empty $K$-definable subset of $\tilde{X}$. By definable Skolem functions, there is $\beta_0 \in \pi^{-1}(X \setminus V_\gamma)$ with $\beta_0 \in \text{dcl}(Ku)$. Then $\beta_0 = f(u)$ for some $K$-definable function $f$. Let $p = \text{tp}(\beta_0/K)$. Then $p = f_*(\text{tp}(u/K))$, so $p$ is definable. Let $b_0 = \pi(\beta_0)$ and let $q = \pi_* p = \text{tp}(b_0/K)$. By choice of $\beta_0$, $b_0 = \pi(\beta_0) \notin V_\gamma$, which implies $b_0 \notin V_t \subseteq V_\gamma$ for any $t \in \Gamma_K$. Thus $q$ is unbounded over $K$. As $q^L$ is the heir, it is similarly unbounded over $L$.

Finally, suppose that $b$ satisfies the assumptions of (3). Then $\text{tp}(b/K) = q = \text{tp}(b_0/K)$, so there is $\sigma \in \text{Aut}(M/K)$ with $\sigma(b_0) = b$. Let $\beta = \sigma(\beta_0)$. Then $(b, \beta) \equiv_K (b_0, \beta_0)$, and in particular $\beta$ realizes $p$ and $\pi(\beta) = b$. Recall the sets $O_t$ used to define $V_t$. If $\beta \notin O_t$ for some $t \in \Gamma_L$, then $b = \pi(\beta) \notin \pi(O_t) = V_t$, contradicting the assumptions. Therefore, $\beta \notin O_t$ for any $t \in \Gamma_L$. By [JY22, Lemma 2.25], $\beta$ realizes $p^L$. Then $b = \pi(\beta)$ realizes $\pi_*(p^L) = q^L$. 

Fix $p, q$ as in Lemma 5.7. Fix $\beta \in \tilde{X}$ realizing $p^L$ and let $b = \pi(\beta)$. Then $b$ realizes $q^L$.

We will make use of the notation and facts from [JY22, §5], applied to the group $X$ and the definable type $q$. In particular, $\mu$ is the infinitesimal partial type of $X$ over $K$, $\mu^L$ is the
infinitesimal partial type of \( X \) over \( L \), and \( \text{st}_L^M \) is the standard part map, a partial map from \( X \) to \( X(L) \). The domain of \( \text{st}_L^M \) is the subgroup \( \mu^L(M) \cdot X(L) \) of points in \( X \) infinitesimally close to points in \( X(L) \). If \( Y \subseteq X \), then \( \text{st}_L^M(Y) \) denotes the image of \( Y \cap (\mu^L(M) \cdot X(L)) \) under \( \text{st}_L^M \).

The following lemma takes the place of [JY22, Fact 6.3].

**Lemma 5.8.** Suppose \( Y \subseteq X \) is \( \beta \)-interpretable.

1. The set \( \text{st}_L^M(Y) \subseteq X(L) \) is interpretable (in the structure \( L \))
2. \( \text{dp-rk}(\text{st}_L^M(Y)) \leq \text{dp-rk}(Y) \).

See Remark A.1 for the definition of ict pattern and dp-rank.

**Proof.**

1. Fix some interpretable basis of opens for \( X \). Let \( \mathcal{F} \) be the collection of \( L \)-interpretable basic open sets which intersect \( Y \). Then \( \mathcal{F} \) is interpretable in the structure \( L \), because \( \mathcal{F} \) is defined externally using \( \beta \), but \( \text{tp}(\beta/L) \) is definable. Now if \( a \in X(L) \), the following are equivalent:
   
   (a) \( a \in \text{st}_L^M(Y) \).
   
   (b) There is \( a' \in Y \) such that for every \( L \)-interpretable basic open neighborhood \( U \ni a \), we have \( a' \in U \).
   
   (c) For every \( L \)-interpretable basic open neighborhood \( U \ni a \), there is \( a' \in Y \) such that \( a' \in U \).
   
   (d) Every \( L \)-interpretable basic open neighborhood of \( a \) is in \( \mathcal{F} \).

   Indeed, (a) \( \iff \) (b) by definition, (b) \( \iff \) (c) by saturation of \( M \), and (c) \( \iff \) (d) by definition of \( \mathcal{F} \). Condition (d) is definable because \( \mathcal{F} \) is.

2. Let \( r \) be the dp-rank of the interpretable set \( D := \text{st}_L^M(Y) \). It is finite, bounded by \( \text{dp-rk}(X) \). There is an ict-pattern of depth \( r \) in \( D \). That is, there are are uniformly interpretable sets \( S_{i,j} \subseteq D \) for \( i < r \) and \( j < \omega \), and points \( b_\eta \in D \) for \( \eta \in \omega^r \), such that \( b_\eta \in S_{i,j} \iff j = \eta(i) \). By Theorem A.6 in the appendix, we can also ensure that \( S_{i,j} \) is open and \( j \neq \eta(i) \implies b_\eta \notin S_{i,j} \). As \( L \) is \( \aleph_1 \)-saturated, we can arrange for all the data to be \( L \)-interpretable. Then each \( b_\eta \) is \( \text{st}_L^M(b'_\eta) \) for some \( b'_\eta \in Y \). Since \( S_{i,j} \) is open and \( L \)-interpretable, we have \( b'_\eta \in S_{i,j} \) for \( j = \eta(i) \). Since \( \overline{S_{i,j}} \) is closed and \( L \)-interpretable, we have \( b'_\eta \notin \overline{S_{i,j}} \) for \( j \neq \eta(i) \). Then the sets \( S_{i,j} \) and elements \( b'_\eta \) are an ict-pattern of depth \( r \) in \( Y \), showing \( \text{dp-rk}(Y) \geq r = \text{dp-rk}(D) \).

**Lemma 5.9.** The following subsets of \( X(L) \) are equal:

1. \( \text{stab}(\mu^L \cdot q^L) \).
2. \( \bigcap_{\phi \in \mathcal{L}} \text{stab}_\phi(\mu \cdot q)(L) \).
3. \( \text{st}_L^M(q^L(M)b^{-1}) \)
4. \( \bigcap_{\psi \in \mathcal{L}} \text{st}_L^M(\psi(\mathcal{M})b^{-1}) \)

5. \( \bigcap_{\psi \in \mathcal{L}} \text{st}_L^M(\psi(\mathcal{M})b^{-1}) \).

See [JY22] Definition 5.3] for the definition of \( \text{stab}_\varphi(-) \).

**Proof.** The equivalence of (1)–(4) is Remark 5.12 and Lemma 5.13 in [JY22]. The equivalence of (4) and (5) follows by a similar argument to the proof of [JY22, Lemma 6.2], using Lemma 5.7(3) instead of [JY22, Lemma 2.25]. \( \square \)

**Lemma 5.10.** If \( I \subseteq X \) is \( L \)-interpretable and contains \( b \), then \( \text{st}_L^M(Ib^{-1}) \) is unbounded in \( X(L) \).

**Proof.** If not, take \( t \in \Gamma_L \) such that \( \text{st}_L^M(Ib^{-1}) \subseteq V_t \). By Lemma 5.7(2), \( b \) is not in any \( L \)-interpretable bounded sets. Therefore \( I \) is unbounded. By Lemma 5.6 we can find \( t' \in \Gamma_L \) such that \( I \odot V_{t'} \setminus V_t \) is bounded. Then \( b \notin I \odot V_{t'} \setminus V_t \). This means that

\[
b \cdot (V_{t'} \setminus V_t) \cap I \neq \emptyset.
\]

Therefore there is \( a \in V_{t'} \setminus V_t \) such that \( ba \in I \). Then there is \( \alpha \in O_{t'} \) with \( \pi(\alpha) = a \). The conditions on \( \alpha \) and \( a \) are definable over \( \text{dcl}(Lb) \subseteq \text{dcl}(L\beta) \) (where \( \beta \) is the realization of \( p^L \)). By definable Skolem functions, we can assume \( \alpha \in \text{dcl}(L\beta) \). Then \( \text{tp}(\alpha/L) \) is a pushforward of \( \text{tp}(\beta/L) \), so \( \text{tp}(\alpha/L) \) is a 1-dimensional definable type on \( \hat{X} \). This type \( \text{tp}(\alpha/L) \) concentrates on the definably compact set \( O_{t'} \subseteq \hat{X} \), and therefore \( \text{tp}(\alpha/L) \) specializes to some point \( \gamma \in G(L) \) by [JY22, Lemma 2.23]. As the map \( \pi : \hat{X} \rightarrow X \) is continuous, \( \text{tp}(\alpha/L) \) specializes to \( c := \pi(\gamma) \in X(L) \). Thus \( \text{st}_L^M(a) \) exists and equals \( c \). Since \( V_{t'} \setminus V_t \) is closed, \( \text{st}_L^M(a) \in V_{t'} \setminus V_t \). But \( a \in b^{-1}I = Ib^{-1} \), and

\[
\text{st}_L^M(a) \subseteq \text{st}_L^M(Ib^{-1}) \subseteq V_t,
\]

a contradiction. \( \square \)

We can now complete the proof of Theorem 5.4. By Lemma 5.9

\[
\bigcap_{\varphi \in \mathcal{L}} \text{stab}_\varphi(\mu \cdot q)(L) = \bigcap_{\psi \in \mathcal{L}} \text{st}_L^M(\psi(\mathcal{M})b^{-1}).
\]

The groups \( \text{stab}_\varphi(\mu \cdot q) \) are \( K \)-interpretable because \( \mu \cdot q \) is a \( K \)-definable partial type. The sets \( \text{st}_L^M(\psi(\mathcal{M})b^{-1}) \) are interpretable by Lemma 5.8(1). Both intersections involve at most \(|K|\) terms, and both intersections are filtered.

If some \( \text{stab}_\varphi(\mu \cdot q)(L) \) is bounded, then \( |K|^+ \)-saturation of \( L \) we have \( \text{st}_L^M(\psi(\mathcal{M})b^{-1}) \subseteq \text{stab}_\varphi(\mu \cdot q)(L) \) for some \( \psi(x) \in q(x) \), contradicting Lemma 5.10. Therefore, every group \( \text{stab}_\varphi(\mu \cdot q)(L) \) is unbounded. Consequently, no \( \text{stab}_\varphi(\mu \cdot q) \) is definably compact.

Since \( \text{tp}(\beta/K) \) has dimension 1, there is some \( K \)-definable set \( D \ni \beta \) of dimension 1. Then \( \text{dp-rk}(\pi(D)) \leq \text{dp-rk}(D) = \dim(D) = 1 \). If \( \psi(x) \) defines \( \pi(D) \), then \( \psi(x) \in q = \text{tp}(b/K) \), and \( \text{st}_L^M(\psi(\mathcal{M})b^{-1}) \) has dp-rank at most 1 by Lemma 5.8(2). By \( |K|^+ \)-saturation, (*) gives
some \( \varphi \) such that \( \text{stab}_{\varphi}(\mu \cdot q)(L) \subseteq \text{st}_{L}^{M}(\psi(M)b^{-1}) \). Then \( \text{stab}_{\varphi}(\mu \cdot q) \) has dp-rank at most 1. On the other hand, \( \text{stab}_{\varphi}(\mu \cdot q) \) is infinite, since it is not definably compact. Therefore \( X' := \text{stab}_{\varphi}(\mu \cdot q) \) has dp-rank at least 1.

It remains to show that the interpretable subgroup \( X' \subseteq X \) has dfg. The proof of [JY22 Lemma 6.10] works with minor changes. For completeness, we give the details. For abelian groups of dp-rank 1, “not fsg” implies dfg as in the proof of [PY19 Lemma 2.9]. It suffices to show that \( X' \) does not have fsg. Assume for the sake of contradiction that \( X' \) has fsg. By [HPP08 Proposition 4.2], non-generic sets form an ideal, and there is a small model \( M_{0} \) such that every generic set contains an \( M_{0} \)-point. Take \( t \) large enough that \( V_{t} \) contains every point in \( X(M_{0}) \). Then \( X' \setminus V_{t} \) is not generic in \( X' \), so \( X' \cap V_{t} \) is generic, meaning that finitely many translates of \( X' \cap V_{t} \) cover \( X' \). But \( X' \cap V_{t} \) and its translates are bounded (as subsets of \( X \)), so then \( X' \) is bounded, a contradiction. This completes the proof of Theorem 5.4.

Corollary 5.11. Let \( X \) be an abelian interpretable group. Then there is \( \alpha \leq \omega \) and an increasing chain of dfg subgroups \( (Y_{i} : i < \alpha) \) with \( Y_{0} = 0 \) such that the quotients \( Y_{i}/Y_{i+1} \) have dp-rank 1. In the case when \( \alpha < \omega \), the quotient \( X/Y_{\alpha-1} \) is definably compact and has fsg.

Proof. Any interpretable group is manifold-dominated [Joh22 Theorem 5.10], so we can apply Theorem 5.4 to any interpretable group. The first application gives \( Y_{1} \); applying the theorem to \( X/Y_{1} \) gives \( Y_{2} \), and so on. The process terminates if any quotient \( X/Y_{i} \) is definably compact. Definably compact groups have fsg [Joh22 Theorem 7.1]. To prove that the groups \( Y_{i} \) have dfg, we can no longer use Lemma 2.3, as \( p \text{CF}^{eq} \) lacks definable Skolem functions. But Theorem B.6 in the appendix works. \(\square\)

Remark 5.12. If we start with a quotient group \( G/H \), we can replace the use of [Joh22 Theorem 5.10] with Proposition 5.1 above.

Remark 5.13. If \( X \) is definable, then the quotients \( Y_{i}/Y_{j} \) are definable by induction on \( i - j \), using Corollary 2.3. Then \( \dim(Y_{i+1}/Y_{i}) = \text{dp-rk}(Y_{i+1}/Y_{i}) = 1 \), which implies \( \dim(Y_{i+1}) > \dim(Y_{i}) \). Therefore, the sequence must terminate, as we saw in the proof of Theorem 5.1. In the general interpretable case, it’s unclear whether this works, so we make a conjecture:

Conjecture 5.14. In Corollary 5.11 \( \alpha \) is finite. Therefore, any abelian interpretable group \( X \) sits in a short exact sequence \( 1 \rightarrow Y_{\alpha-1} \rightarrow X \rightarrow X/Y_{\alpha-1} \rightarrow 1 \) where \( Y_{\alpha-1} \) has dfg and \( X/Y_{\alpha-1} \) has fsg and is definably compact.

Pillay and Yao asked whether any definably amenable group \( G \) in a distal theory sits in a short exact sequence \( 1 \rightarrow H \rightarrow G \rightarrow C \rightarrow 1 \) with \( C \) having fsg and \( H \) having dfg [PY16 Question 1.19]. If Conjecture 5.14 is true, it would provide further evidence for this.

A Nice ict patterns

Remark A.1. Following [Sim15, Definition 4.21], an ict-pattern of depth \( \kappa \) in a partial type \( \Sigma(x) \) is a sequence of formulas \( \varphi_{i}(x; y_{i}) \) and an array \( (b_{i,j} : i < \kappa, \ j < \omega) \) with \( |b_{i,j}| = |y_{i}| \).
such that for any function $\eta : \kappa \to \omega$, the following partial type is consistent:

$$\Sigma(x) \cup \{\varphi_{i,\eta(i)}(x, b_{i,\eta(i)}) : i < \kappa\} \cup \{-\varphi_{i,j}(x, b_{i,j}) : i < \kappa, \ j \neq \eta(i)\}$$

Abusing notation, we say that $(\varphi_i(x; b_{i,j}) : i < \kappa, \ j < \omega)$ is an ict-pattern to mean that the pair $((\varphi_i : i < \kappa), (b_{i,j} : i < \kappa, \ j < \omega))$ is an ict-pattern. Sometimes we consider ict-patterns where the columns are indexed by an infinite linear order $I$ other than $\omega$. The definition is analogous, and ict-patterns of this sort can be converted to ict-patterns indexed by $\omega$ via a compactness argument.

Finally, the $dp$-rank of $\Sigma(x)$ is the supremum of cardinals $\kappa$ such that there is an ict-pattern of depth $\kappa$ in $\Sigma(x)$, possibly in an elementary extension.

Work in $\mathbb{M}^{eq}$ for some monster model $\mathbb{M} \models p\text{CF}$. There is a well-behaved notion of dimension on $\mathbb{M}^{eq}$ \cite{Gag05}, which gives rise to a notion of independence:

$$\dim_{C} a \downarrow b \iff \dim(a/Cb) = \dim(a/C) \iff \dim(b/Ca) = \dim(b/C).$$

This notion satisfies many of the usual properties \cite{Joh22, §2.1.1}. Say that a sequence $\{a_i : i \in I\}$ is dimensionally independent over a set $B$ if $a_i \downarrow_{\dim} b_{<i}$ for $i \in I$, where $a_{<i} = \{a_j : j < i\}$. As usual, this is independent of the order on $I$.

**Lemma A.2.** If $\text{tp}(a/Cb)$ is finitely satisfiable in $C$, then $a \downarrow_{C} \dim b$.

**Proof.** Suppose not. Let $n = \dim(b/Ca) < \dim(b/C)$. By \cite{Gag05, Proposition 3.7}, there is a $Ca$-interpretable set $X$ containing $b$ with $\dim(X) = n$. Write $X$ as $\varphi(a, \mathbb{M})$ for some $L_{C}^{eq}$-formula $\varphi(x, y)$. By \cite{Joh22, Proposition 2.12}, the set $\{a' \in \mathbb{M} : \dim(\varphi(a', \mathbb{M})) = n\}$ is definable, defined by some $L_{C}^{eq}$-formula $\psi(x)$. Then $\mathbb{M} \models \varphi(a, b) \land \psi(a)$. As $\text{tp}(a/Cb)$ is finitely satisfiable in $C$, there is some $a' \in C$ such that $\mathbb{M} \models \varphi(a', b) \land \psi(a')$. Then $b$ is in the $C$-interpretable set $\varphi(a', \mathbb{M})$ which has dimension $n$ as $\mathbb{M} \models \psi(a')$. Therefore $\dim(b/C) \leq n$, a contradiction. 

**Corollary A.3.** Suppose $\ldots, b_{-1}, b_{0}, b_{1}, \ldots, c_{-1}, c_{0}, c_{1}, \ldots$ is $C_{0}$-indiscernible. Then the sequence $\ldots, b_{-1}, b_{0}, b_{1}, \ldots$ is dimensionally independent over $C = C_{0} \cup \{c_{i} : i \in \mathbb{Z}\}$.

**Proof.** For example, $p = \text{tp}(b_{n}/Cb_{1}b_{2}\cdots b_{n-1})$ is finitely satisfiable in $C$; any formula in $p$ is satisfied by $c_{i}$ for $i \ll 0$. This argument shows that any finite subsequence of $\{b_{i}\}_{i \in \mathbb{Z}}$ is dimensionally independent over $C$. This implies the full sequence is dimensionally independent, by finite character of $\downarrow_{\dim}$. 

**Lemma A.4.** If $\{b_{i} : i \in I\}$ is dimensionally independent over $C$, and $\dim(a/C) = n$, then $a \downarrow_{C} \dim b_{i}$ for all but at most $n$ values of $i$.

The proof is standard, but we include it for completeness. \footnote{The one unusual property is that “$\dim(a/C) = 0$” is strictly weaker than “$a \in \text{acl}(C)$”.}
Proof. Otherwise, passing to a subsequence, we could arrange for $b_1, \ldots, b_{n+1}$ to be dimensionally independent over $C$, but $a \downarrow_C^{\dim} b_i$ for each $i$. The sequence $(\dim(a/Cb_1, \ldots, b_i) : 0 \leq i \leq n+1)$ cannot decrease $n+1$ times, so there is some $0 \leq i \leq n$ such that $\dim(a/Cb_1, \ldots, b_i) = \dim(a/Cb_1, \ldots, b_{i+1})$, i.e.,

$$a \downarrow_{C^{b_1, \ldots, b_i}}^{\dim} b_{i+1}. $$

As $b_1, \ldots, b_i \downarrow_C^{\dim} b_{i+1}$, left transitivity gives $a \downarrow_C^{\dim} b_{i+1}$, a contradiction. \hfill \square

Lemma A.5. Let $X$ be a $C$-interpretable set of parameters, with dp-rank $r$. Then there is $C' \supseteq C$ and an ict pattern of depth $r$ in $X$ of the form $(\varphi_i(x; b_{i,j}) : i < r, j \in \mathbb{Z})$, such that the array $(b_{i,j} : i < r, j \in \mathbb{Z})$ is mutually $C'$-indiscernible, and for each $i$, the sequence $(b_{i,j} : j \in \mathbb{Z})$ is dimensionally independent over $C'$.

Proof. Let $\mathbb{Z} + \mathbb{Z}'$ denote two copies of $\mathbb{Z}$ laid end to end, with the second copy denoted $\mathbb{Z}'$. Take an ict pattern $(\varphi_i(x; b_{i,j}) : i < r, j < \omega)$ in $X$. Let $(b_{i,j} : i < r, j \in \mathbb{Z} + \mathbb{Z}')$ be a mutually $C$-indiscernible array extracted from $(b_{i,j} : i < r, j < \omega)$. Then $(\varphi_i(x; b_{i,j}) : i < r, j \in \mathbb{Z} + \mathbb{Z}')$ is an ict pattern in $X$. Let $C' = C \cup \{b_{i,j} : i < r, j \in \mathbb{Z}'\}$. Then $(b_{i,j} : i < r, j \in \mathbb{Z})$ is mutually $C'$-indiscernible, and each row is dimensionally independent over $C'$ by Corollary A.3. \hfill \square

Theorem A.6. Let $G$ be a manifold-dominated interpretable group of dp-rank $r$. There is an ict-pattern $(\varphi_i(x; b_{i,j}) : i < r, j < \omega)$ in $G$ such that if $S_{i,j} = \varphi_i(M; b_{i,j})$, then the following properties hold:

1. Each set $S_{i,j}$ is open.

2. For each function $\eta : r \to \omega$, there is an element $a_{\eta} \in G$ such that $$j = \eta(i) \implies a_{\eta} \in S_{i,j}$$

$$j \neq \eta(i) \implies a_{\eta} \notin S_{i,j}$$

Proof. By \cite[Theorem 5.10]{Joh22}, the topology on $G$ is “admissible”, and so

$$\dim(\mathcal{D} \setminus D) < \dim(D)$$

(Small boundaries property)

for any interpretable subset $D \subseteq G$, by \cite[Proposition 4.34]{Joh22}. By Lemma A.3 there is an ict-pattern $(\psi_i(x; b_{i,j}) : i < r, j \in \mathbb{Z})$ and a set of parameters $C$ (over which $G$ is interpretable) such that the $b_{i,j}$ are mutually indiscernible over $C$, and each row is dimensionally independent over $C$. Take some $a$ such that $M \models \psi_i(a; b_{i,j}) \iff j = 0$ for all $i < r$ and $j \in \mathbb{Z}$. By \cite[Proposition 3.7]{Gag05} there is a formula $\theta_i(x; b_{i,0}, c_i)$ in $tp(a/Cb_{i,0})$ such that $\dim(\theta_i(x; b_{i,0}, c_i)) = \dim(a/Cb_{i,0})$. Replacing $b_{i,j}$ with $b_{i,j} c_i$ and replacing $\psi_i(x; b_{i,j})$ with $\psi_i(x; b_{i,j}) \land \theta_i(x; b_{i,j}, c_i)$, we may assume that $\dim(\psi_i(x; b_{i,0})) = \dim(a/Cb_{i,0}) =: k_i$. Let $V_{i,j} = \psi_i(M; b_{i,j})$. Then $\dim(V_{i,j}) = \dim(V_{i,0}) = k_i$ by indiscernibility.
For each \( i \), we have \( a \downarrow \dim b_{i,j} \) for all but finitely many \( j \), by Lemma A.4. Throwing away the finitely many bad values of \( b_{i,j} \) in each row, we may assume \( a \downarrow \dim b_{i,j} \) for all \( j \neq 0 \). Thus \( \dim(a/Cb_{i,j}) = \dim(a/C) \) for \( j \neq 0 \). By the Small Boundaries Property,

\[
\dim(V_{i,j} \setminus V_{i,j}) < \dim(V_{i,j}) = \dim(V_{i,0}) = k_i = \dim(a/Cb_{i,0}) \leq \dim(a/C) = \dim(a/Cb_{i,j}),
\]
for \( j \neq 0 \). Then \( a \) cannot be in the \( Cb_{i,j} \)-interpretable set \( V_{i,j} \setminus V_{i,j} \). By choice of \( a \), we also have \( a \notin V_{i,j} \). So \( a \notin V_{i,j} \) for any \( j \neq 0 \). Thus

\[
\begin{align*}
    j = 0 & \implies a \in V_{i,j} \\
    j \neq 0 & \implies a \notin V_{i,j}.
\end{align*}
\]

By mutual indiscernibility, we can find \( a_\eta \) for any \( \eta : r \to \mathbb{Z} \) such that

\[
\begin{align*}
    j = \eta(i) & \implies a_\eta \in V_{i,j} \\
    j \neq \eta(i) & \implies a_\eta \notin V_{i,j}.
\end{align*}
\]

Recall that the topology on \( G \) is a group topology, so every open neighborhood of \( a_\eta \) has the form \( a_\eta \cdot N \) for some open neighborhood \( N \) of 1. For each \( i, j, \eta \) with \( j \neq \eta(i) \), we can find an open neighborhood \( N_{i,j,\eta} \) \( \ni 1 \) such that \( (a_\eta \cdot N_{i,j,\eta}) \cap V_{i,j} = \emptyset \). By saturation, there is an interpretable open neighborhood \( N_0 \ni 1 \) with \( N_0 \subseteq N_{i,j,\eta} \) for all \( i, j, \eta \). Because the topology is a group topology, there is a smaller interpretable open neighborhood \( N \ni 1 \) such that \( N = N^{-1} \) and \( N \cdot N \subseteq N_0 \).

Let \( U_{i,j} = V_{i,j} \cdot N = \{ x \cdot y : x \in V_{i,j}, y \in N \} \). Note that \( U_{i,j} \) is open. If \( j \neq \eta(i) \), then

\[
(a_\eta \cdot N \cdot N) \cap V_{i,j} \subseteq a_\eta \cdot N_{i,j,\eta} \cap V_{i,j} = \emptyset.
\]

The fact that \( (a_\eta \cdot N \cdot N) \cap V_{i,j} = \emptyset \) implies that

\[
(a_\eta \cdot N) \cap U_{i,j} = (a_\eta \cdot N) \cap (V_{i,j} \cdot N) = \emptyset.
\]

The neighborhood \( a_\eta \cdot N \) then shows that \( a_\eta \notin U_{i,j} \). On the other hand, \( 1 \in N \), so \( V_{i,j} \subseteq U_{i,j} \). Therefore, if \( j = \eta(i) \), then \( a_\eta \in V_{i,j} \subseteq U_{i,j} \). Putting everything together, we get

\[
\begin{align*}
    j = \eta(i) & \implies a_\eta \in U_{i,j} \\
    j \neq \eta(i) & \implies a_\eta \notin U_{i,j}.
\end{align*}
\]

The sets \( U_{i,j} \) are uniformly interpretable, so we can find some formula \( \varphi(x; y) \) such that each \( U_{i,j} \) has the form \( \varphi(M; b_{i,j}) \) for some \( b_{i,j} \) (not the original ones). Then \( (\varphi(M; b_{i,j}) : i < r, j < \omega) \) is the desired ict pattern.

\[\square\]

**B Extensions and dfg**

Work in a highly resplendent monster model \( M \). \( \text{acl}(\cdot) \) always means \( \text{acl}^\text{eq} \). All sets and parameters can come from \( M^\text{eq} \) by default. “Definable” means “interpretable.”
Definition B.1. A definable set $D$ is almost $A$-definable if it is $\text{acl}(A)$-definable, or equivalently, $\{\sigma(D) : \sigma \in \text{Aut}(M/A)\}$ is finite. A global definable type $p$ is almost $A$-definable if it is $\text{acl}(A)$-definable, or equivalently, $\{\sigma(p) : \sigma \in \text{Aut}(M/A)\}$ is small.

The following is folklore; see [Joh20, Lemma 3.13] for a proof.

Fact B.2. Suppose $b$ realizes $p|A$ for some almost $A$-definable global type $p$. Suppose $c$ realizes $q|(Ab)$ for some almost $Ab$-definable global type $q$. Then $c$ realizes $r|A$ for some almost $A$-definable global type $r$.

Definition B.3. Let $G$ be an $A$-definable group. Say that $G$ has $\text{dfg over } A$ if there is a global definable type $p$ on $G$ such that $p$ and all its left-translates are almost $A$-definable.

Lemma B.4. Let $G$ be a definable $\text{dfg}$ group and $S$ be a definable set with a regular right action of $G$. Suppose everything is $A$-definable, and $G$ has $\text{dfg over } A$. Then there is a global type on $S$ that is almost $A$-definable.

Proof. For $b \in S$, let $b \cdot p$ denote the pushforward of the $A$-definable type $p$ along the map $x \mapsto b \cdot x$ from $G$ to $S$. Note that $b \cdot p$ is a definable type on $S$.

The set $\mathcal{G} = \{b \cdot p : b \in S\}$ is small, because it is $\{b_0 \cdot g \cdot p : g \in G\}$ for any fixed $b_0 \in S$. If $\sigma \in \text{Aut}(M/\text{acl}(A))$, then $\sigma$ fixes $p$ and $\sigma$ fixes $\mathcal{G}$ setwise, since $\mathcal{G}$ was defined in an invariant way. Therefore any $b \cdot p$ has small orbit under $\text{Aut}(M/\text{acl}(A))$, implying that $b \cdot p$ is almost $A$-definable.

If $G$ is a $\mathcal{O}$-definable group, let $M \ltimes G$ be the new structure obtained by adding a copy of $G$ as a new sort $S$, and putting no structure on $S$ other than the regular right action of $G$. For any $g \in G$, there is an automorphism of $M \ltimes G$ fixing $M$ and acting as left translation by $g$ on the new sort $S$. In fact, $\text{Aut}(M \ltimes G) \cong \text{Aut}(M) \ltimes G$.

This construction is called “Construction C” in [HP11, §1], where it is attributed to Hrushovski’s thesis. It also appears in [Sim15] above Lemma 8.19. As mentioned in [Sim15], $M \ltimes G$ is a conservative extension of $M$, in the sense that it introduces no new $\mathcal{O}$-definable or definable sets on $M$. After naming the element $1 \in S$, the two structures are bi-interpretable. Since we assumed $M$ was very resplendent, $M \ltimes G$ will be too.

Lemma B.5. Let $A \subseteq M$ be a small set of parameters. Suppose that in $M \ltimes G$, there is a global type $p$ on $S$ that is almost $A$-definable. Then $G$ has $\text{dfg over } A$.

Proof. For $b, s \in S$, let $b^{-1} \cdot s$ denote the unique $x \in G$ such that $s = b \cdot x$. Let $b^{-1} \cdot p$ denote the pushforward of $p$ along the map $x \mapsto b^{-1} \cdot x$ from $S$ to $G$. Then $b^{-1} \cdot p$ is a definable type on $G$. If $\sigma \in \text{Aut}(M/A)$, we can extend $\sigma$ to $\tilde{\sigma} \in \text{Aut}((M \ltimes G)/A)$ fixing $b$. Then

$$\sigma(b^{-1} \cdot p) = \tilde{\sigma}(b^{-1} \cdot p) = b^{-1} \cdot \tilde{\sigma}(p).$$

There are only a small number of possibilities for $\tilde{\sigma}(p)$, and so $b^{-1} \cdot p =: q$ is almost $A$-definable.
If \( g \in G \), then \( g \cdot b^{-1} \cdot x = (b \cdot g^{-1})^{-1} \cdot x \) for \( x \in S \), and so
\[
g \cdot q = g \cdot b^{-1} \cdot p = (b \cdot g^{-1})^{-1} \cdot p = (b')^{-1} \cdot p
\]
for \( b' = b \cdot g^{-1} \). Replacing \( b \) with \( b' \) in the argument above, we see that \((b')^{-1} \cdot p = g \cdot q\) is almost \( A \)-definable. In other words, every translate \( g \cdot q \) of \( q \) is almost \( A \)-definable, showing \( G \) has dfg over \( A \).

Lemmas \[B.4\] and \[B.5\] are formally analogous to \[Sim15\, Lemma 8.19\], replacing “non-forking over \( A \)” with “almost \( A \)-definable.”

**Theorem B.6.** If \( 1 \to N \to G \to H \to 1 \) is a short exact sequence of definable groups, and \( N, H \) have dfg, then \( G \) has dfg.

**Proof.** Naming parameters, we may assume the whole sequence is \( \emptyset \)-definable, and that \( N \) and \( H \) have dfg over \( \emptyset \). Construct \( \mathbb{M} \ltimes G \). Let \( S \) be the new sort with a regular right action of \( G \). Let \( S' \) be the quotient \( S/N \). Then \( S' \) has a regular right action by \( H \). By Lemma \[B.4\] there is an almost \( \emptyset \)-definable global type \( p \) on \( S' \). Take \( b \) realizing \( p|\emptyset \). Let \( S'' \) be the fiber of \( S \to S' \) over \( b \in S' \). Then \( S'' \) is a \( b \)-definable set with a \( b \)-definable regular right action by \( N \). By Lemma \[B.4\] there is an almost \( b \)-definable global type \( q \) on \( S'' \). Let \( c \) realize \( q|b \). Note \( c \in S \). By Fact \[B.2\] there is an almost \( \emptyset \)-definable global type \( r \) on \( S \) such that \( c \) realizes \( r|\emptyset \). By Lemma \[B.5\], \( G \) has dfg.

Theorem \[B.6\] generalizes one direction of Lemma \[2.3\]. We cannot expect the reverse direction to hold (if \( G \) has dfg, then \( N \) and \( H \) have dfg). For example, in \( p \text{CF}^{eq} \), the short exact sequence
\[
0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0
\]
is a counterexample: \( \mathbb{Q}_p \) has dfg but \( \mathbb{Z}_p \) does not. So the use of definable Skolem functions in Lemma \[2.3\] is essential.

**References**

[AGJ22] Pablo Andújar Guerrero and Will Johnson. Around definable types in \( p \)-adically closed fields. arXiv:2208.05815v1 [math.LO], 2022.

[CKDL17] Pablo Cubides-Kovacsics, Luck Darnière, and Eva Leenknegt. Topological cell decomposition and dimension theory in \( P \)-minimal fields. *Journal of Symbolic Logic*, 82(1):347–358, 2017.

[CP12] Annalisa Conversano and Anand Pillay. Connected components of definable groups and \( o \)-minimality I. *Advances in Mathematics*, 231:605–623, 2012.

[Gag05] Jerry Gagelman. Stability in geometric theories. *Annals of Pure and Applied Logic*, 132:313–326, 2005.
[HP11] Ehud Hrushovski and Anand Pillay. On NIP and invariant measures. *J. Eur. Math. Soc.*, 13(4):1005–1061, 2011.

[HPP08] Ehud Hrushovski, Ya’acov Peterzil, and Anand Pillay. Groups, measures, and the NIP. *J. Amer. Math. Soc.*, 21(2):563–596, April 2008.

[Joh18] Will Johnson. The canonical topology on dp-minimal fields. *J. Math. Logic*, 18(2):1850007, 2018.

[Joh20] Will Johnson. On the proof of elimination of imaginaries in algebraically closed valued fields. *Notre Dame Journal of Formal Logic*, 61(3):363–381, 2020.

[Joh21] Will Johnson. A note on fsg groups in $p$-adically closed fields. arXiv:2108.06092v1 [math.LO], 2021.

[Joh22] Will Johnson. Topologizing interpretable groups in $p$-adically closed fields. arXiv:2205.00749v1 [math.LO], 2022.

[JY22] Will Johnson and Ningyuan Yao. On non-compact $p$-adic definable groups. *J. Symbolic Logic*, 87(1):188–213, 2022.

[MOS20] Samaria Montenegro, Alf Onshuus, and Pierre Simon. Stabilizers, NTP2, groups with f-generics, and PRC fields. *J. Inst. Math. Jussieu*, 19(3):821–853, 2020.

[OP08] A. Onshuus and A. Pillay. Definable groups and compact $p$-adic Lie groups. *Journal of the London Mathematical Society*, 78(1):233–247, 2008.

[PS17] Y. Peterzil and S. Starchenko. Topological groups, $\mu$-types and their stabilizers. *J. European Math. Soc.*, 19(10):2965–2995, 2017.

[PY16] Anand Pillay and Ningyuan Yao. On minimal flows, definably amenable groups, and o-minimality. *Adv. in Mathematics*, 290:483–502, February 2016.

[PY19] Anand Pillay and Ningyuan Yao. Definable $f$-generic groups over $p$-adic numbers. arXiv:1911.01833v1 [math.LO], 2019.

[Sim15] Pierre Simon. *A guide to NIP theories*. Lecture Notes in Logic. Cambridge University Press, July 2015.