Delocalization of Phase Fluctuations and the Stability of AC Electricity Grids

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The energy transition towards an increased supply of renewable energy raises concerns that existing electricity grids, built to connect few centralized large power plants with consumers, may become more difficult to control and stabilized with a rising number of decentralized small scale generators. Here, we aim to study therefore, how local phase fluctuations affect the AC grid stability. To this end, we start from a model of nonlinear dynamic power balance equations. We map them to complex linear wave equations and find stationary solutions for the distribution of phases $\varphi_i$ at the generator and consumer sites $i$. Next, we derive differential equations for deviations from these stationary solutions. Starting with an initially localized phase perturbation, it is found to spread in a periodic grid diffusively throughout the grid. We derive the parametrical dependence of diffusion constant $D$. We apply the same solution strategy to general grid topologies and analyse their stability against local fluctuations. The fluctuation remains either localized or becomes delocalized, depending on grid topology and distribution of consumers and generators $P_i$. Delocalization is found to increase the lifetime of phase fluctuations and thereby their influence on grid stability, while localization results in an exponentially fast decay of phase fluctuations at all grid sites.

The stability of electricity grids requires to protect them against fluctuations\cite{2}. Therefore, the electrical power system must be constructed in such a way that a physical disturbance does not result in exceeding bounds of system variable fluctuations. The energy transition towards an increased supply of decentralized renewable energy necessitates to study the consequences of such structural changes for the stability of electricity grids, and to find efficient ways to modify the grids to ensure their stability \cite{2}. Since this is a highly complex and nonlinear problem the study of its dependence on network topology, operating conditions and forms of disturbances requires to make modelling assumptions\cite{1}. Recently, the synchronization of rotor angles in electricity grids has been modeled by a network of nonlinear oscillators\cite{3}. Here, networks of generators and engines are described by a system of coupled differential equations for the local rotor angles of the generators $\varphi_i$, where $i$ is a node of the grid. The numerical solution of these differential equations on networks of various topologies showed that, on the one hand, networks become more stable with increasing decentralization against perturbations on short time scales with large amplitude. On the other hand, the danger of a blackout, the total disruption of the electricity grid, can be reduced by decentralization\cite{3}. In this article, we study how phase fluctuations evolve in time in AC grids. We start by mapping the nonlinear dynamic power balance equations to complex linear wave equations. This allows us to find stationary solutions for the spatial distribution of phase $\varphi_i$, for given distribution of active and reactive power, $P_i$ and $Q_i$, at the grid nodes $i$. Next, we derive the phase dynamics of deviations of the phase from these stationary solutions. Thereby, we explore how a local phase perturbation spreads with time through the grid diffusively, or it may become localized.

Dynamics of phase fluctuations in AC transmission grids.— Phase dynamics in AC electricity grids has been modeled by AC power balance equations, with additional terms which describe the dynamics of rotating machines.

One term is the first time derivative of the kinetic energy of a rotating machine with rotor angle $\varphi_i$, describing the inertia towards changes of their kinetic energy with inertia $J$. Another term is due to friction in the machine with friction coefficient $\gamma$\cite{3,4}. The resulting power balance equation at node $i$ is given by

$$S_i = \frac{d}{dt} \left( \frac{d\varphi_i}{dt} \right)^2 + \gamma \left( \frac{d\varphi_i}{dt} \right)^2 + \sum_j V_i \left( \frac{V_i - V_j}{Z_{ij}} \right)^2,$$

where $S_i = P_i + iQ_i$. Here $P_i$ is the active power produced at generator nodes (when $P_i > 0$), or the active power consumed at consumer nodes (when $P_i < 0$). $V_i$ is the voltage at node $i$. $Q_i$ is respectively the reactive power at node $i$. The transmission line from node $i$ to $j$ has impedance $Z_{ij}$. Taking the real part of Eq. (1) returns the dynamic equations of Refs. \cite{3,4} when the transmission lines are assumed to be purely inductive $Z_{ij} = \omega L$, with inductance $L$ at grid frequency $\omega$. Here, we will consider the complex balance equation, Eq. (1), describing the balance of both active and reactive power.

Restricting us first to a purely inductive grid, the voltage is given by $V_i = V \exp(-i\varphi_i)$, where $V$ is the nominal voltage. Thus, each transmission line has the power capacity $K = V^2/(\omega L)$. Defining $\psi_i(t) = \exp(-i\varphi_i(t))$, we can rewrite Eq. (1) as a linear wave equation. Defining $\varphi_i(t) = \omega t + \theta_i(t)$, and assuming that $\theta_i \ll \omega$, we find

$$
\left( S_i - 1 \omega (\partial_t + \omega)^2 - 2 \gamma \omega (\partial_t - \omega) - \gamma \omega^2 \right) \psi_i(t)
= \sum_j \psi_j(t) - 2iK_{ij}(\psi_i(t) - \psi_j(t)).
$$

(2)

Our strategy for solving this equation is the following: First, find a stationary solution $\psi_i^0(t) = \exp(-i\Omega t - i\theta_i^0)$,
where the frequency $\Omega$ may in general deviate from $\omega$. Then, study the dynamics of phase fluctuations $\alpha_i(t)$ around those stationary solutions for which $\Omega = \omega$, with $\psi_i(t) = \exp(-i\omega t - \theta_i(t))$.

**Periodic hypercubic grid.**— Let us first consider Eq. (2) on a regular $d$-dimensional grid where all transmission lines have equal length $a$, with a periodic arrangement of generators and consumers as shown for a cubic electricity grid in the upper Fig. 1. We place all generators on sublattice $G$ and assume that all of them generate the same power $\dot{P}_x = \ddot{P}_x - \gamma \omega^2 = +P$, when $x \in G$, which includes the dissipative power $\gamma \omega^2$ consumed by each rotating machine. All consumers on sublattice $C$ are assumed to consume the power $\ddot{P}_x = -P$, when $x \in C$, which includes the dissipated power. Thus, we can make the Ansatz for the stationary solution

$$
\psi_k(x \in G, t) = \psi_{Gk} e^{i k x} \exp(-i \Omega_k t), \\
\psi_k(x \in C, t) = \psi_{Ck} e^{i k x} \exp(-i \Omega_k t),
$$

(3)

where $k$ is the wave number. We assume periodic boundary conditions of the hypercubic lattice of linear size $L$ in all $d$ directions with the unit vectors $\hat{e}_n$, $n = 1, \ldots, d$: $\psi_k(x + L \hat{e}_n) = \psi_k(x)$, for all $n = 1, \ldots, d$. The dispersion of the frequency $\Omega_k$, and the phase factors $\psi_{Gk}, \psi_{Ck}$ are then determined by insertion of Eq. (3) into Eq. (2). Thereby, we do get two coupled algebraic equations. In an electricity grid, the reactive power $Q$ of the consumers is given, while the one of the generators can be adjusted. For the periodic arrangement of consumers and generators, we find that the reactive power $Q$ is constrained by the balance equations Eq. (1) to be the same at all consuming and all generating nodes.

From the condition, that the frequency $\Omega_k$ of the solution of Eq. (2) of the form Eq. (3) is identical to the grid frequency, $\Omega_k = \omega$, we find for given reactive power $Q$ the wave vector $k$, from the condition

$$
f_k^2 = (Q/K - 2d)^2 + P^2/K^2,
$$

(4)

where $f_k = 2 \sum_{n=1}^d \cos(k_n a)$, with $k_n$ the $d$ components of the wavevector $k$. The phase factors on sublattices $G$ and $C$ are then found to be related by

$$
\psi_{Ck} = \exp(i \delta_k) \psi_{Gk},
$$

(5)

with phase difference $\delta_k$, given by

$$
\delta_k = \arcsin \frac{P}{f_k K}.
$$

(6)

**Transmitted power in a stationary state.**— For the homogenous solution $k = 0$ the active power transmitted between all neighbour generator sites $i \in G$ and consumer sites $j \in C$ is the same: $Re F_{ij} = P/(2d)$. This state is shown in Fig. 1 (right) on a square lattice. The arrows indicate the direction of the active power transmission from the generators (red) to their neighbor consumers (blue). There is also a finite reactive power transmission $Im F = K(1 - \sqrt{1 - P^2/(2d)^2 K^2})$, which for $P \ll K$ is much smaller than the active one. If the wave vector $k$ is finite, the transmitted active power between nodes $i \in G$ and $j \in C$ is given by

$$
Re F_{ij}^G C = K \sin(\delta_k + k(x_j - x_i)).
$$

(7)

For $k = k \hat{e}_n$, with unit vector $\hat{e}_n$, $Re F_{ij} = K \sin(\delta_k \pm k_n a)$, when the unit vector between nodes $i$ and $j$ is $\hat{e}_i \times j = \pm \hat{e}_n$ and $Re F_{ij} = K \sin(\delta_k)$, in all other $2(d-1)$ directions of the $d$-dimensional hypercubic lattice.

**Phase fluctuations around the stationary state.**— Having found stationary states for the periodic grid, Eq. (3), we now study the dynamics of phase fluctuations $\alpha_i(t)$. First, we consider fluctuations where all generators fluctuate in synchrony, $\alpha_i(t) = \alpha_G(t) \in G$, as do all consumers with phase $\alpha_i(t) = \alpha_C(t) \in C$. Thus, the $N$ differential equations for $\alpha_i(t)$ reduce to two coupled differential equations for $\alpha_G$ and $\alpha_C$. Defining $\alpha(t) = \alpha_G(t) - \alpha_C(t)$, $\beta(t) = \alpha_G(t) + \alpha_C(t)$, we find

$$
\begin{align*}
\partial_t^2 \alpha(t) - 4 \Gamma \dot{\alpha}(t) \beta(t) + 2 \Gamma \dot{\beta}(t) \\
- 2g \cos(\delta_k - \alpha - \delta_k)
\end{align*}
$$

(8)

$$
\begin{align*}
\partial_t^2 \beta(t) - \frac{1}{2} \dot{\alpha}(t)^2 - \frac{1}{2}(\dot{\beta}(t))^2 + 2 \Gamma \dot{\beta}(t) \\
- 12g \cos(\delta_k - \alpha - \delta_k)
\end{align*}
$$

(9)

where $g = 4dK/4\Omega$. Solutions are obtained for $\partial_t \beta = 0$ and $\alpha$ satisfying

$$
\partial_t^2 \alpha(t) + 2 \Gamma \dot{\beta}(t) = -2g \sin(\delta_k + \sin(\alpha - \delta_k)),
$$

(10)

which is the differential equation of the damped driven nonlinear pendulum. It is in general not integrable, and a well known example of a chaotic system. Demanding also balance of the reactive power, the imaginary part of Eq. (9) simplifies for $\partial_t \beta = 0$ to the equation $\frac{1}{2}(\partial_t \alpha(t))^2 = 2g \cos(\delta_k - \alpha - \delta_k)$. This fixes the relation between $\partial_t \alpha(t)$ and $\alpha(t)$. It turns out that there is no real solution which satisfies both this equation and Eq. (10), besides the trivial solution $\partial_t \alpha = 0 = \alpha$, corresponding to the stationary state.
In previous studies of phase dynamics in AC grids, only the balance of active power has been considered. Such a situation can be realized, if the reactive power $Q = Q(t)$ is changing in time to compensate the reactive power caused by time dependent phase $\alpha(t)$. Taking the real part of Eqs. (6) yields Eq. (10), and $\frac{d^2}{dt^2} \beta(t) + 2D \beta(t) = 0$. It has the solution $\beta(t) = \exp(-2\Gamma t)$, decaying to $\beta(t) = 0$, $\partial_t \beta(t) = 0$ with rate $\Gamma$, so that for $t \gg 1/\Gamma$ the phases at consumer and at generator sites fluctuate synchronously in antiphase, $\alpha_G(t) = -\alpha_C(t)$. We plot exemplary phase curves of Eq. (10) in Fig. 3.

For large times $t \gg 0$, there are two stable solutions:

1. Stability analysis around the stationary solution $\alpha = 0, \partial_t \alpha = 0$, (and equivalent solutions $\alpha = n2\pi, \partial_t \alpha = 0$) yields two eigenvectors in phase space, along which the phase fluctuation is proportional to $\sim \exp(\lambda_k t)$ with eigenvalues $\lambda_k = -\Gamma \pm \sqrt{\Gamma^2 - 2g \sin \delta_k}$. Thus, for $\Gamma^2 > 2g \sin \delta_k$, its real part is still negative and the imaginary part $\Omega_k = \pm \sqrt{\Gamma^2 - 2g \sin \delta_k}$ is finite, so that the phase oscillates with frequency $\Omega_k$, and decays exponentially in time.

2. The overwinding pendula solution, where the driving force and the damping term are in balance, so that the phase velocity converges for $t \gg 0$ to $\partial_t \alpha(t) = -\frac{\pi}{2} \sin \delta_k + A \cos(\alpha(t) - \delta_k) + B \sin(\alpha(t) - \delta_k)$, where $A = \frac{2g}{\sin \delta_k} (1 - \frac{4\pi^2}{(2g \sin \delta_k)^2})^{-1}$, and $B = \frac{4\pi^3}{2g \sin \delta_k} (1 - \frac{4\pi^2}{(2g \sin \delta_k)^2})^{-1}$.

The exact solution can be written in terms of periodic elliptic functions. For $t \gg 0$ it can be approximated as

$$\alpha(t) = -\Omega_k (t - t_0) + \frac{C}{\Omega_k} \sin(\Omega_k (t - t_0) - \delta_k + \eta), \quad (11)$$

with frequency $\Omega_k = \frac{\pi}{2} \sin \delta_k = \frac{2dP}{f_k \gamma \omega}$, phase shift $\eta = \arctan(2\Gamma^2 / (g \sin \delta_k))$ and $C = A / \cos \eta$. Thus, the ratio of generator power $P$ and friction $\gamma$ determines the position of the limiting orbit in phase space, oscillating with frequency $\Omega_k$. We note that we assumed $\theta \ll \omega$, so that our analysis is valid for $\Omega_k \ll \omega$

In Fig. 3 the regions of stability are shaded in blue. All phase points outside of these basins of attraction converge to the open orbit solution Eq. (11), plotted in Fig. 3 in red. While the shape of these basins is irregular, we note that all points in phase space satisfying

$$\alpha^2 + \left(\frac{\partial_t \alpha}{\sin \zeta^-}\right)^2 \ll (\pi - 2\delta_k)^2, \quad (12)$$

where $\zeta^- = \arctan(-\Gamma - \sqrt{\Gamma^2 + 2g \sin \delta_k})$, are inside the basin of attraction. Note that for $\delta_k = \pi/2$ its extension shrinks to zero. This corresponds to a critical value of the generator power $P_c = f_k K_c$. For $k = 0$ the basin of attraction has finite extension for $P < P_c = 2dK$. Thus, the stability of the grid increases with the total capacity of all transmission lines connected to a node, $K_{\text{total}} = 2dK$. Outside of that stable region the grid is driven to the limiting orbit, oscillating with frequency $\Omega_0 = \frac{P}{\omega \gamma}$. The transmitted power oscillates then with frequency $\Omega_0$ around the stationary solution Eq. (7).

Local phase perturbations and diffusion.— Next, we consider local perturbations of the phase, $\alpha_i(t)$, around stationary solutions Eq. (3) at time $t = 0$ at site $i = l$. $\alpha_i(0) = 0$ at all other sites $i \neq l$. Initially, $\alpha_i(t)$ follows Eq. (10). However, as the perturbation spreads throughout the grid, phase fluctuations occur at other nodes $i \neq 0$ for $t > 0$ due to the transmission capacity $K_{ij}$. Assuming that the initial perturbation is in the stable basin of attraction, Eq. (12), we can linearize the coupled differential equations, and obtain

$$\frac{\partial^2}{\partial t^2} \alpha_i(t) + 2D \partial_t \alpha_i(t) = -g \cos \delta_k \sum_j A_{ij}(\alpha_i - \alpha_j), \quad (13)$$

where $A_{ij}$ is the adjacency matrix of the grid. Using the spectral representation $\alpha_i(t) = \sum_q c_q e^{i q \omega t} e^{-i \varphi_q t}$, we find by insertion into Eq. (13) the complex frequency

$$\epsilon_q = -i \Gamma \left(1 \pm \sqrt{1 - \frac{g \cos \delta_k}{\Gamma^2} (1 - \frac{f_q}{2d})}\right). \quad (14)$$

For small $q$ the dispersion is quadratic, $\epsilon_q = -\frac{g a^2 \cos \delta_k}{2 \omega} q^2$. Starting from a localized phase $c_q = \text{const.}$ at $t_0 = 0$, the phase $\alpha_i(t)$, becomes for $t > \tau = 1/\Gamma$,

$$\alpha_i(t) = \frac{c_0}{(4\pi D_k t)^{d/2}} \exp\left(-\frac{(r_i - r_0)^2}{4D_k t}\right). \quad (15)$$

and the perturbation spreads diffusively, where

$$D_k = \frac{g a^2 \cos \delta_k}{2d\Gamma} = \frac{2K_a}{\omega^2} \sqrt{1 - \frac{P^2}{f_k^2 K^2}}. \quad (16)$$

FIG. 2: (color online) Examples of phase curves of Eq. (10). Flow directions are indicated by arrows. Stable fixed points are $\alpha = n2\pi, \partial_t \alpha = 0$. Separatrices are $\alpha_l = 2\delta_k + (2l + 1)\pi, \partial_t \alpha = 0$. Blue shaded areas are basins of attraction. All other phase points converge to Eq. (11) (red curve).
is the diffusion constant. Since $\alpha_i$ decays in time, the grid remains stable. The transmitted power converges to the diffusion constant. Since $\delta \alpha$ is time independent, the stationary state is defined by the differential equation

$$\frac{d \theta^t}{dt} = -\Gamma \theta^t$$

which determines the local phase distribution $\theta^t_i$. Perturbing this stationary state by a phase $\alpha_i(t)$, it is governed by the differential equation

$$\partial^2_t \alpha_i + 2\Gamma \partial_t \alpha_i = \frac{P_i}{J\omega} \sum_j \frac{K_{ij}}{J\omega} \sin(\theta^t_i - \theta^t_j + \alpha_i - \alpha_j)$$

Considering a perturbation within the basin of attraction, we can expand Eq. (19) in $\alpha_i - \alpha_j$, which yields

$$\partial^2_t \alpha_i + 2\Gamma \partial_t \alpha_i = -\sum_j t_{ij} (\alpha_i - \alpha_j)$$

with $t_{ij} = \frac{K_{ij}}{J\omega} \cos(\theta^0_i - \theta^0_j)$. Inserting the spectral representation $\alpha_i(t) = \sum_n c_{n_{\tau \sigma}} \exp(-\omega_n t)$ into Eq. (20) we get for $\Gamma = 0$

$$\omega_n^2 c_{n_{\tau \sigma}} = \sum_j t_{ij} (c_{n_{\tau \sigma}} - c_{n_{\tau \sigma}}),$$

yielding Eigenenergies $\omega_n$ and Eigenmodes $c_{n_{\tau \sigma}}$. For $\Gamma \neq 0$ we get from Eq. (20) the same Eigenmodes with complex Eigenenergy $\Omega_n = -\Gamma + i\sqrt{\Gamma^2 - \omega_n^2}$. Eq. (21) arises also in the problem of randomly coupled atoms in harmonic approximation. For chains it has been solved for various random distributions of $t_{ij}$ [10,13]. If $t_{ij}$ is taken from a box distribution, the density of Eigenmodes is constant, and all Eigenstates are Anderson localized with a localization length diverging at $\omega_n = 0$ like $\xi(\omega_n) \sim 1/\omega_n$. In higher dimensions, there can be a critical value $\omega_c$, such that for $\omega_n \geq \omega_c$ all modes are localized, while they are extended for $\omega_n \leq \omega_c$. Thus, if the phase perturbation is in a state localized around site $r_0$ with localization length $\xi_n$, it decays in time as $\alpha_i(t) = \alpha_0 \exp(-\xi_0 \omega_n) \exp(-\Gamma_n t + \sqrt{\Gamma^2 - \omega_n^2} t)$, where $\Gamma_n = \Gamma - \sqrt{\Gamma^2 - \omega_n^2}$. The average phase and frequency shift is then found to be

$$\delta \alpha = \alpha_0 \sqrt{\xi_n} \exp(-2\Gamma_n t), \quad \delta \omega = \Gamma_n \delta \alpha.$$ (22)

Thus, localization causes exponentially fast decay of phase perturbations.

Conclusions. — Local phase perturbations spread diffusively in a periodic grid, while in a grid with random distribution of generators and consumers, the phase perturbation can either remain localized or become delocalized. Delocalization is found to increase the lifetime of phase fluctuations at all sites, as seen in Fig. 3. The amplitude of delocalized phase and frequency fluctuations are found to decay slowly, with a power of time, Eq. (17), while they decrease exponentially when the fluctuation is localized, Eq. (22). Thus, even when local phase perturbations are initially small enough to be within the stable basin of attraction, they may, due to their slow decay in time, add up at some nodes in the grid to large perturbations and push the system outside of the region of stability, Eq. (12). Thus, we may conclude that it is favorable for a stable grid operation to distribute and connect generators and consumers such that phase perturbations stay localized since it results in an exponentially fast decay of phase perturbations at all sites. The consequences of these results for real electricity grid topologies will be studied with numerical simulations in Ref. [14].

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