Seiberg-Witten Theories on Ellipsoids

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Abstract: We present a set of equations for a 4D Killing spinor which guarantees the Seiberg-Witten theories on a curved background to be supersymmetric. The equations involve an $SU(2)$ gauge field and some auxiliary fields in addition to the metric. Four-dimensional ellipsoids with $U(1) \times U(1)$ isometry are shown to admit a supersymmetry if these additional fields are chosen appropriately. We compute the partition function of general Seiberg-Witten theories on ellipsoids, and the result suggests a correspondence with 2D Liouville or Toda correlators with general coupling constant $b$.

Keywords: Supersymmetric gauge theory
1. Introduction

Supersymmetric gauge theories have a characteristic feature that, due to cancellations of bosonic and fermionic contributions, certain physical quantities can be evaluated beyond perturbation theory. In this area, a number of important exact results have been obtained for the theories realized on deformed or curved backgrounds which admit rigid supersymmetry. For example, in 4D $\mathcal{N} = 2$ supersymmetric gauge theories or Seiberg-Witten (SW) theories, an analytic formula for the partition function on Omega background was given by Nekrasov in the pioneering work [1]. More recently, an exact formula for partition function as well as expectation values of Wilson loops on round four-sphere has been obtained by Pestun [2]. Similar exact results have also been obtained for 3D $\mathcal{N} \geq 2$ gauge theories on round three-sphere [3,4,5], its orbifold [6,7], and 2D theories on sphere [8,9]. These all served as new powerful tools to study the strong coupling behavior of the theories at low energy or other non-perturbative aspects. They also led to a discovery of a surprising connection between SW theories and 2D Liouville or Toda conformal field theories, called AGT relation [10,11].

So far, most of the work in this field has been focusing on theories on round spheres. A natural question would then be what other curved spaces admit rigid supersymmetry. Some systematic analysis has been made in [12,13,14,15,16] to draw conditions on the background geometry from Killing spinor equation. Also, in [17,18,19] another construction of supersymmetric gauge theories in three or five dimensions has been discussed in connection with contact geometry, and moreover some exact results have been worked out for theories on 3D Seifert manifolds. On the other hand, there is also a less systematic approach in which one focuses on a specific class of deformations of round sphere aiming for a particularly interesting physical consequence.

According to the AGT relation, partition functions of certain class of SW theories on round $S^4$ agree with correlation functions of 2D Liouville or Toda Theories at a special
value $b = 1$ of the coupling. The coupling $b$ characterizes uniquely the underlying conformal symmetry of the 2D theory. For example it enters in the Liouville central charge,

$$c_L = 1 + 6Q^2, \quad Q \equiv b + \frac{1}{b}. \quad (1.1)$$

One would therefore naturally imagine there is a deformation of the round sphere which can reproduce the CFT correlators for general values of the coupling $b$. Actually, similar problem has been resolved in the setting of a generalized AGT relation involving 3D $\mathcal{N} = 2$ supersymmetric gauge theories. There one introduces an S-duality domain wall\cite{20, 21} along an $S^3$ inside the $S^4$ which supports a 3D gauge theory on its worldvolume. AGT relation then implies that the partition function of the wall theory on $S^3$ should agree with the matrix element of the corresponding S-duality transformation in the representation theory of the (extended) conformal symmetry at $b = 1$. In this setting, it has been found\cite{22} that by deforming the round $S^3$ into a 3D ellipsoid,

$$\frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1, \quad (1.2)$$

with a suitable background $SO(2)_R$ gauge field to ensure rigid supersymmetry, one can change the value of the coupling to $b = (\ell/\tilde{\ell})^{1/2}$. For other recent work on supersymmetric deformations of the round $S^3$ with additional background fields, see\cite{23, 24, 25, 26, 27, 28}. The above result in three dimensions implies that the correct deformation of $S^4$ should be a fibration of the ellipsoid (1.2) over a line segment, because the S-duality wall can then wrap the 3D fiber anywhere in four dimensions in a supersymmetric manner.

In this paper we show that SW theories on the 4D ellipsoids,

$$\frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1, \quad (1.3)$$

with some additional background fields, reproduce the 2D Liouville or Toda CFTs with the coupling $b = (\ell/\tilde{\ell})^{1/2}$. As can be easily guessed from our previous result, the additional fields include an R-symmetry gauge field which takes values on $SU(2)$ Lie algebra this time. Moreover, it turns out that the relevant off-shell 4D $\mathcal{N} = 2$ supergravity multiplet contains some more auxiliary fields, and they also have to take nonzero values to make the background supersymmetric.

The organization of this paper is as follows. After a brief summary of our notations on 4D spinor calculus, in Section 2 we present the set of Killing spinor equations, and the action and supersymmetry of general SW theories on arbitrary curved backgrounds which support Killing spinors. Then in Section 3 we analyze the Killing spinor equation on ellipsoids. It will be shown that, by assuming that a Killing spinor on round $S^4$ remains after the deformation of the metric, one can solve the Killing spinor equation in favor of the background gauge and auxiliary fields and determine their form up to some arbitrariness. The square of the supersymmetry yields an isometry of the ellipsoid which fixes two special points, i.e. the north and south poles. It is shown that the theory looks near the two poles like the (anti-)topologically twisted theory with Omega deformation parameter
(\epsilon_1, \epsilon_2) = (\ell^{-1}, \ell^{-1})$. In Section 3 we carry out the explicit path integration using the SUSY localization principle. Our argument here follows closely that of Pestun [2]. Finally in Section 3 we conclude with a few remarks, including a quick check of the AGT relation in the simplest examples.

**Notations.** Under the 4D rotation group $SO(4) \simeq SU(2) \times SU(2)$, chiral and anti-chiral spinors transform as doublets of the first and the second $SU(2)$, respectively. We use the indices $\alpha, \beta, \cdots$ and $\dot{\alpha}, \dot{\beta}, \cdots$ for chiral and anti-chiral spinors. These indices are raised and lowered by the antisymmetric invariant tensors $\epsilon^{\alpha \beta}$, $\epsilon^{\dot{\alpha} \dot{\beta}}$, $\epsilon_{\alpha \beta}$, $\epsilon_{\dot{\alpha} \dot{\beta}}$ with nonzero elements

$$
\epsilon^{12} = -\epsilon^{21} = -\epsilon^{12} = \epsilon^{21} = 1.
$$

Following Wess and Bagger, pairs of undotted indices are suppressed when contracted in the up-left, down-right order, and similarly for contracted dotted indices in the down-left, up-right order.

We introduce a set of $2 \times 2$ matrices $(\sigma^a)_{\alpha \dot{\alpha}}$ and $(\tilde{\sigma}^a)^{\dot{\alpha} \alpha}$ with $a = 1, \cdots, 4$ satisfying standard algebras. In terms of Pauli’s matrices $\tau^a$ they are given by

$$
\sigma^a = -i \tau^a, \quad \tilde{\sigma}^a = i \tau^a, \quad (a = 1, 2, 3) \tag{1.5}
$$

We also use $\sigma_{ab} = \frac{1}{2}(\sigma_a \sigma_b - \sigma_b \sigma_a)$ and $\tilde{\sigma}_{ab} = \frac{1}{2}(\tilde{\sigma}_a \sigma_b - \sigma_b \tilde{\sigma}_a)$. Note that $\sigma_{ab}$ is anti self-dual, namely $\sigma_{ab} = -\frac{1}{2} \epsilon_{abcd} \sigma^{cd}$, while $\tilde{\sigma}_{ab}$ is self-dual.

2. Seiberg-Witten Theories on Curved Spaces

Manifolds which can support supersymmetric field theories are characterized by the existence of Killing spinors. In this paper we consider theories which, when realized on a flat $\mathbb{R}^4$, have eight supercharges, i.e. 4D $\mathcal{N} = 2$ supersymmetric theories or Seiberg-Witten (SW) theories. For these theories, supersymmetry is characterized by a pair of a chiral and an anti-chiral Killing spinors $\xi \equiv (\xi_{\alpha A}, \tilde{\xi}^{\dot{\alpha}}_A)$, both with an additional $SU(2)_R$ doublet index $A, B, \cdots$. We use the tensors $\epsilon^{AB}, \epsilon_{AB}$ with nonzero elements (1.4) to raise or lower $SU(2)_R$ indices. We also require the Killing spinors to satisfy the reality condition

$$
(\xi_{\alpha A})^\dagger = \xi^A_{\alpha} = \epsilon^{\alpha \beta} \epsilon^{AB} \xi_{\beta B}, \quad (\tilde{\xi}^{\dot{\alpha}}_A)^\dagger = \tilde{\xi}^{\dot{\alpha}}_A = \epsilon^{\dot{\alpha} \dot{\beta}} \epsilon^{AB} \xi^{\dot{\beta}}_B. \tag{2.1}
$$

Our Killing spinor equation consists of two sets of equations. The first set is called the main equation

\begin{align*}
D_m \xi_A + T^{kl} \sigma_{kl} \sigma_m \xi_A &= -i \sigma_m \xi_A, \\
D_m \xi_A + \tilde{T}^{kl} \sigma_{kl} \sigma_m \xi_A &= -i \sigma_m \xi_A \quad \text{for some } \xi_A, \xi_A. \tag{2.2}
\end{align*}

Here $T^{kl}, \tilde{T}^{kl}$ are a self-dual and an anti-self-dual real tensor background fields, and the covariant derivatives contain a background $SU(2)_R$ gauge field $V_m^A B$ in addition to spin connection $\Omega_m^{ab}$.

\begin{align*}
D_m \xi_A &\equiv \partial_m \xi_A + \frac{1}{4} \Omega_m^{ab} \sigma_{ab} \xi_A + i \xi_B V_m^A A, \\
D_m \xi_A &\equiv \partial_m \xi_A + \frac{1}{4} \Omega_m^{ab} \tilde{\sigma}_{ab} \xi_A + i \tilde{\xi}_B V_m^A A. \tag{2.3}
\end{align*}
The second set is called the **auxiliary equation**:

\[
\sigma^m \bar{\sigma}^n D_m D_n \xi_A + 4 D_l T_{mn} \sigma^m \bar{\sigma}^n \bar{\xi}_A = M \xi_A,
\]

\[
\bar{\sigma}^m \sigma^n D_m D_n \bar{\xi}_A + 4 D_l \bar{T}_{mn} \bar{\sigma}^m \sigma^n \xi_A = M \bar{\xi}_A,
\]

(2.4)

where \(M\) is a scalar background field. We will later show that, if a 4D manifold with possibly nonzero background fields \(T^{kl}, \bar{T}^{kl}, V_m A_B\) and \(M\) admits a Killing spinor satisfying these equations, one can define SW theories on it with a rigid supersymmetry.

The above generalized Killing spinor equation was found following the suggestion of [12] to consider the coupling to off-shell supergravity. The set of background fields and Killing spinor equations can be compared to the auxiliary fields in the supergravity multiplet and BPS equations of [29], but there are some differences due to the change in spacetime signature. As an example, although SW theories are known to have BPS equations of [29], but there are some differences due to the change in spacetime signature. As an example, although SW theories are known to have \(SU(2) \times U(1)\) R-symmetry, we do not consider background \(U(1)_R\) gauge field because the \(U(1)_R\) phase rotation is not compatible with the reality condition of SUSY parameter (2.1). Also, this \(U(1)_R\) will be broken explicitly if the background fields \(T^{kl}, \bar{T}^{kl}\) take nonzero values.

**Vector multiplets.** Vector multiplet consists of a gauge field \(A_m\), gauginos \(\lambda_{\alpha A}, \bar{\lambda}_{\dot{\alpha} A}\), two scalar fields \(\phi, \bar{\phi}\) and an auxiliary field \(D_{AB} = D_{BA}\) all taking values on the same Lie algebra. Their SUSY transformation rule is given by

\[
Q A_m = i \xi^A \sigma_m \bar{\lambda}_A - i \bar{\xi}^A \bar{\sigma}_m \lambda_A,
\]

\[
Q \phi = -i \xi^A \lambda_A,
\]

\[
Q \bar{\phi} = +i \bar{\xi}^A \bar{\lambda}_A.
\]

\[
Q \lambda_A = \frac{1}{2} \sigma^{mn} \xi_A (F_{mn} + 8 \phi T_{mn}) + 2 \sigma^m \xi_A D_m \phi + \sigma^m D_m \xi_A \phi + 2 i \xi_A [\phi, \bar{\phi}] + D_{AB} \xi^B,
\]

\[
Q \bar{\lambda}_A = \frac{1}{2} \bar{\sigma}^{mn} \bar{\xi}_A (\bar{F}_{mn} + 8 \bar{\phi} \bar{T}_{mn}) + 2 \bar{\sigma}^m \bar{\xi}_A D_m \bar{\phi} + \bar{\sigma}^m D_m \bar{\xi}_A \bar{\phi} - 2 i \bar{\xi}_A [\bar{\phi}, \phi] + D_{AB} \bar{\xi}^B,
\]

\[
Q D_{AB} = -2 [\phi, \xi_A \bar{\lambda}_B + \bar{\xi}_B \lambda_A] + 2 [\bar{\phi}, \xi_A \lambda_B + \bar{\xi}_B \lambda_A].
\]

Here and throughout this paper, we take the Killing spinor \(\xi\) to be Grassmann-even so that \(Q\) is the supercharge which flips the statistics of the fields. The above transformation rule is compatible with the reality condition of SUSY parameter (2.1) if we assume

\[
(A_m)^\dagger = A_m, \quad (\lambda_{\alpha A})^\dagger = \lambda^{\dot{\alpha} A}, \quad (\bar{\lambda}_{\dot{\alpha} A})^\dagger = \bar{\lambda}^{\alpha A},
\]

\[
\phi^\dagger = \phi, \quad (\bar{\phi})^\dagger = \bar{\phi}, \quad (D_{AB})^\dagger = D^{AB}.
\]

(2.6)

Note that \(\phi, \bar{\phi}\) are two independent real scalar fields.

The supersymmetry algebra closes off-shell, i.e. \(\{Q_\xi, Q_\eta\}\) is a sum of bosonic symmetries for arbitrary pair of Killing spinors \(\xi, \eta\). Here we give the formula for the square \(Q^2\)
of the supersymmetry for a Killing spinor $\xi$,

$$
Q^2 A_m = i v^n F_{nm} + D_m \Phi,
Q^2 \phi = i v^n D_m \phi + i[\Phi, \phi] + (w + 2\Theta) \phi,
Q^2 \bar{\phi} = i v^n D_m \bar{\phi} + i[\Phi, \bar{\phi}] + (w - 2\Theta) \bar{\phi},
Q^2 \lambda_A = i v^n D_n \lambda_A + i[\Phi, \lambda_A] + (\frac{3}{2} w + \Theta) \lambda_A + \frac{i}{4} \sigma^{kl} \lambda_A D_k v_l + \Theta_{AB} \lambda^B,
Q^2 \bar{\lambda}_A = i v^n D_n \bar{\lambda}_A + i[\Phi, \bar{\lambda}_A] + (\frac{3}{2} w - \Theta) \bar{\lambda}_A + \frac{i}{4} \sigma^{kl} \bar{\lambda}_A D_k v_l + \Theta_{AB} \bar{\lambda}^B,
Q^2 D_{AB} = i v^n D_n D_{AB} + i[\Phi, D_{AB}] + 2w D_{AB} + \Theta_{AC} D^C_B + \Theta_{BC} D^C_A,
\tag{2.7}
$$

where the various transformation parameters are defined as follows,

$$
v^n = 2\bar{\xi}^A \sigma^m \xi_A,
\Phi = -2i\phi \bar{\xi}^A \bar{\xi}_A + 2i\bar{\phi} \xi^A \xi_A,
w = -\frac{1}{2}(\xi^A \sigma^m D_m \bar{\xi}_A + D_m \xi^A \sigma^m \bar{\xi}_A),
\Theta = -\frac{1}{4}(\xi^A \sigma^m D_m \bar{\xi}_A - D_m \xi^A \sigma^m \bar{\xi}_A),
\Theta_{AB} = -i\xi_{(A} \sigma^m D_m \bar{\xi}_{B)} + i D_m \xi_{(A} \sigma^m \bar{\xi}_{B)},
\tag{2.8}
$$

We note that, if $\xi$ satisfies the main Killing spinor equation (2.2) only, the algebra does not close on $D_{AB}$. The failure term

$$
\Delta_{AB} = -2i\phi (\bar{\xi}_{(A} \sigma^k \bar{\xi}_{B)} D_k D(\bar{\xi}_B) + 4\bar{\xi}_{(A} \sigma^{mn} \sigma^k \xi_{B)} D_k T_{mn}) + 2i\bar{\phi} (\xi_{(A} \sigma^k \bar{\xi}_{B)} D_k D(\xi_B) + 4\xi_{(A} \sigma^{mn} \sigma^k \xi_{B)} D_k T_{mn}),
\tag{2.9}
$$

vanishes if $\xi$ satisfies also the auxiliary equation.

The supersymmetric Yang-Mills Lagrangian is given by

$$
\mathcal{L}_{YM} = \text{Tr} \left[ \frac{1}{2} F_{mn} F^{mn} + 16 F_{mn} (\bar{\phi} T^{mn} + \phi \bar{T}^{mn}) + 64 \phi^2 T_{mn} T^{mn} + 64 \bar{\phi}^2 \bar{T}_{mn} \bar{T}^{mn} - 4 D_m \bar{\phi} D^n \phi + 2M \bar{\phi} \phi - 2i\lambda^A \sigma^m D_m \bar{\lambda}_A - 2\lambda^A [\phi, \lambda_A] + 2\bar{\lambda}^A [\phi, \bar{\lambda}_A]
+ 4[\phi, \bar{\phi}]^2 - \frac{1}{2} D^{AB} D_{AB} \right].
\tag{2.10}
$$

For round $S^4$ of radius $\ell$ with no background $SU(2)_R$ gauge field or auxiliary tensor fields turned on, this Lagrangian reduces to the one found by Pestun [2] with $M = -\frac{1}{3} R = -\frac{4}{\ell^2}$. The action is then defined by combining $\mathcal{L}_{YM}$ with the topological term,

$$
S_{YM} = \frac{1}{g_{YM}^2} \int d^4 x \sqrt{g} \mathcal{L}_{YM} + \frac{i\theta}{8\pi^2} \int \text{Tr} (F \wedge F).
\tag{2.11}
$$

Instantons and anti-instantons are topologically non-trivial configurations of gauge field satisfying $*F = -F$ or $*F = F$, and are characterized by the instanton number $n \in \mathbb{Z}$. The classical action on instanton or anti-instanton backgrounds takes values

$$
\text{instanton (} n > 0 \text{)} : -S_{YM} = 2\pi i n \tau, \quad \tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2},
\text{anti-instanton (} n < 0 \text{)} : -S_{YM} = 2\pi i n \bar{\tau},
\tag{2.12}
$$

- 5 -
The Lagrangian (2.10) is not positive definite and path integral becomes ill-defined if the fields take values according to the reality condition (2.6). The actual path integral should therefore be defined with the modified contours along which

\[ \phi^\dagger = -\bar{\phi}, \quad (D_{AB})^\dagger = -D^{AB}. \]  

(2.13)

For \( U(1) \) gauge group, there is also the Fayet-Iliopoulos type invariant. Let \( w^{AB} = w^{BA} \) be a \( SU(2)_R \) triplet background field satisfying

\[ w^{AB}\xi_B = \frac{1}{2}\sigma^n D_n \xi^A + 2T_{kl}\sigma^{kl}\xi^A, \]

\[ w^{AB}\bar{\xi}_B = \frac{1}{2}\sigma^n D_n \bar{\xi}^A + 2\bar{T}_{kl}\sigma^{kl}\bar{\xi}^A. \]  

(2.14)

Then one can construct the following invariant from a \( U(1) \) vector multiplet,

\[ L_{\Phi} \equiv w^{AB}D_{AB} - M(\phi + \bar{\phi}) - 64\phi T^{kl}T_{kl} - 64\bar{\phi}\bar{T}^{kl}\bar{T}_{kl} - 8F^{kl}(T_{kl} + \bar{T}_{kl}). \]  

(2.15)

**Hypermultiplets.** The system of \( r \) hypermultiplets consists of scalars \( q_{AI} \) and fermions \( \psi_{\alpha I}, \bar{\psi}^\dagger_{\dot{I}} \) satisfying the reality conditions

\[ (q_{IA})^\dagger = q^{AI} = \Omega^{IJ}e^{AB}q_{JB}, \]

\[ (\psi_{\alpha I})^\dagger = \psi^{\dagger\alpha} = e^{\alpha\beta}\Omega^{IJ}\psi_{\beta J}, \]

\[ (\bar{\psi}^{\dot{I}})_{\dot{\alpha}} = \bar{\psi}^{\dagger}_{\dot{\alpha}} = e^{\dot{\alpha}\dot{\beta}}\Omega^{IJ}\bar{\psi}^{J}_{\dot{\beta}}. \]  

(2.16)

Here \( I, J = 1, \cdots, 2r \) are \( Sp(r) \) indices and \( \Omega^{IJ} \) is the real antisymmetric \( Sp(r) \)-invariant tensor satisfying

\[ (\Omega^{IJ})^* = -\Omega_{IJ}, \quad \Omega^{IJ}\Omega_{JK} = \delta_K^I. \]  

(2.17)

Pairs of \( Sp(r) \) indices contracted in the order of top-left, bottom-right will be often suppressed. For example, \( q^A q_A \equiv q^{AI}q_{IA} \). These matter fields can couple to vector multiplets through an embedding of the gauge group into \( Sp(r) \). Namely, when vector multiplet fields such as \( A_m \) are multiplied on hypermultiplet fields, they are thought of as \( 2r \times 2r \) matrices with elements \( (A_m)_{IJ} \). The covariant derivatives of matters therefore take the form

\[ D_m q_{IA} = \partial_m q_{IA} - i(A_m)_{IJ} q_{JA} + i q_{IB}(V_m)^B_A, \]

\[ D_m \psi_{\alpha I} = \partial_m \psi_{\alpha I} - i(A_m)_{IJ} \psi_{\beta J} + \frac{i}{4}q_{ab}^{\alpha\beta}(\sigma_{ab})_{IJ} \psi_{\beta J}, \text{ etc.} \]  

(2.18)

It is straightforward to find on-shell SUSY transformation rule,

\[ Q^{\alpha}_{\alpha\psi} = -i\xi_A \psi + i\bar{\xi}_A \bar{\psi}, \]

\[ Q^{\alpha}_{\alpha\psi} = 2\sigma^m\xi_A D_m q^A + \sigma^m D_m \xi_A q^A - 4i\xi_A \bar{\phi} q^A, \]

\[ Q^{\alpha}_{\alpha\bar{\psi}} = 2\sigma^m\xi_A D_m q^A + \sigma^m D_m \xi_A q^A - 4i\bar{\psi} \bar{\phi} q^A, \]  

(2.19)

and the gauge covariant kinetic Lagrangian

\[ L_{\text{mat}}^{\alpha\psi} = \frac{1}{2} D_m q^A D_m q_A - q^A(\phi, \bar{\phi}) q_A + \frac{i}{2} q^A D_{AB} q^B + \frac{1}{8}(R + M) q^A q_A \]

\[ -\frac{i}{2}\bar{\phi} \sigma^m D_m \psi + \frac{1}{2}\bar{\psi} \sigma^m D_m \bar{\phi} + \frac{i}{2}\bar{\psi} \sigma^{kl} T_{kl} \psi - \frac{i}{2}\sigma^{kl} \psi T_{kl} \bar{\psi} \]

\[ -q^A \lambda_A \psi + \bar{\psi} \bar{\lambda}_A q^A. \]  

(2.20)
It is known that one cannot make the full $\mathcal{N} = 2$ SUSY transformation law closed off-shell with finitely many auxiliary fields. For the application of localization principle, however, one focuses on the supersymmetry $Q$ corresponding to a specific choice of Killing spinor $\xi$. It is then sufficient that $Q^2$ for that specific $\xi$ is a linear sum of bosonic symmetries on all fields off-shell.

We introduce the auxiliary scalars $F_{AI}$ satisfying the reality condition

$$(F_{IA})^\dagger = F^{AI} = \Omega^{IJ} \epsilon^{AB} F_{JB},$$

and put the full Lagrangian as follows,

$$\mathcal{L}_{\text{mat}} = \mathcal{L}_{\text{mat}}^{\text{os}} - \frac{1}{2} F^A F_A.$$  \hspace{1cm} (2.22)

The supersymmetry transformation laws of fields are extended as follows,

$$Q q_A = -i \xi_A \psi + i \bar{\xi}_A \bar{\psi},$$
$$Q \psi = 2 \sigma^m \xi_A D_m q_A + \sigma^m D_m \bar{\xi}_A q^A - 4i \xi_A \bar{\psi} q^A + 2 \bar{\xi}_A F^A,$$
$$Q \bar{\psi} = 2 \sigma^m \xi_A D_m q^A + \sigma^m D_m \bar{\xi}_A q^A - 4i \xi_A \psi \bar{q}^A + 2 \bar{\xi}_A F^A,$$
$$Q F_A = +i \bar{\xi}_A \sigma^m D_m \bar{\psi} - 2 \bar{\xi}_A \phi \psi - 2 \bar{\xi}_A \lambda B q^B + 2i \bar{\xi}_A (\sigma^{kl} T_{kl}) \bar{\psi}$$
$$-i \bar{\xi}_A \bar{\sigma}^m D_m \psi + 2 \bar{\xi}_A \phi \psi + 2 \bar{\xi}_A \lambda B q^B - 2i \bar{\xi}_A (\bar{\sigma}^{kl} T_{kl}) \psi.$$  \hspace{1cm} (2.23)

Here the new transformation parameters $\xi, \bar{\xi}$ are required to satisfy

$$\xi_A \bar{\xi}_B - \bar{\xi}_A \xi_B = 0,$$
$$\xi_A \xi_A + \bar{\xi}_A \bar{\xi}_A = 0,$$
$$\bar{\xi}_A \xi_A + \bar{\xi}_A \bar{\xi}_A = 0,$$
$$\xi_A \sigma^m \bar{\xi}_A + \xi_A \sigma^m \bar{\xi}_A = 0.$$  \hspace{1cm} (2.24)

Similar off-shell transformation rule which makes use of constrained transformation parameters like $\bar{\xi}, \bar{\xi}$ here has been written down for 4D $\mathcal{N} = 4$ gauge theories on $S^4$ in [3], and for 5D SUSY theories on $S^5$ in [4]. One can then show that $Q$ squares into a linear sum of bosonic symmetries off-shell,

$$Q^2 q_A = iv^m D_m q_A + i \Phi q_A + w q_A + \Theta_{AB} q^B,$$
$$Q^2 \psi = iv^m D_m \psi + i \Phi \psi + \frac{3}{2} w \psi - \Theta \psi + i \frac{\sigma^{kl}}{2} D_k \psi, q^B,$$
$$Q^2 \bar{\psi} = iv^m D_m \bar{\psi} + i \Phi \bar{\psi} + \frac{3}{2} w \bar{\psi} + \Theta \bar{\psi} + i \frac{\bar{\sigma}^{kl}}{2} D_k \bar{\psi}, q^B,$$
$$Q^2 F_A = iv^m D_m F_A + i \Phi F_A + 2 w F_A + \bar{\Theta}_{AB} F^B.$$  \hspace{1cm} (2.25)

Here the parameters $v^m, \Phi, w, \Theta, \Theta_{AB}$ are as in (2.8) and

$$\bar{\Theta}_{AB} = 2i \xi_A (\sigma^m D_m \bar{\xi}_B) - 2i D_m \bar{\xi}_A (\sigma^m \bar{\xi}_B) + 4i \hat{\xi}_A (\sigma^{kl} \bar{T}_{kl} \bar{\xi}_B) - 4i \hat{\xi}_A (\bar{\sigma}^{kl} \bar{T}_{kl} \bar{\xi}_B).$$  \hspace{1cm} (2.26)

Note that $\hat{\xi}_A, \bar{\xi}_A$ and $F_A$ transform as doublets under a local symmetry which we call $SU(2)_R$, reflecting the fact that the choice of $\xi_A, \bar{\xi}_A$ satisfying (2.24) is not unique. This
also means that the covariant derivative of $F_A$ contains the background $SU(2)_R$ gauge field $\bar{V}_m{}^{B\ A}$.

$$D_m F_{IA} \equiv \partial_m F_{IA} - i(A_m)^I J_{JA} + i F_{IB} \bar{V}_m{}^{B\ A}. \quad (2.27)$$

For notational simplicity, we use for their doublet indices the same letters $A, B, \cdots$ as for the $SU(2)_R$ indices.

The off-shell transformation rule (2.23) is compatible with the reality condition of the fields (2.16) and (2.21). However, if we define the theory of hypermultiplets by the kinetic Lagrangian $L_{\text{mat}}$, we have to take the actual path integration contour in such a way that its bosonic part is positive definite. Therefore, we choose the integration contour for $F_{AI}$ differently from its real locus, so that

$$(F_{IA})^\dagger = -F^{AI} \quad (2.28)$$

along the contour.

An important fact which will be used later is that the matter kinetic Lagrangian is supersymmetry exact. Assuming $\xi^A \xi_A - \bar{\xi}^A \bar{\xi}_A = 1$ which will be verified in the next section, one can show that

$$L_{\text{mat}} = Q \mathcal{V}_{\text{mat}},$$

$$2\mathcal{V}_{\text{mat}} = \psi \bar{\xi}^A F_A - \bar{\psi} \xi^A F_A + \psi \sigma^m D_m (\xi_A q^A) - \bar{\psi} \bar{\sigma}^m D_m (\xi_A q^A) + 2(i\psi \phi + \psi \sigma^{kl} T_{kl} + i q^B \lambda_B) \xi_A q^A - 2(i \bar{\psi} \bar{\phi} + \bar{\psi} \bar{\sigma}^{kl} \bar{T}_{kl} + i q^B \bar{\lambda}_B) \bar{\xi}_A q^A. \quad (2.29)$$

3. Supersymmetry on 4D Ellipsoids

It has been known that round spheres in various dimensions admit Killing spinors satisfying

$$D_m \zeta = \Gamma_m \zeta' \quad \text{for some } \zeta'.$$ \quad (3.1)

In [22] it was shown that the 3D ellipsoids with $U(1) \times U(1)$ isometry admit a pair of charged Killing spinors coupled to a suitably chosen background $U(1)_R$ gauge field. The ellipsoid is defined by an embedding equation in flat $\mathbb{R}^4$ with Cartesian coordinates $x_1, \cdots, x_4$,

$$\frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\ell^2} = 1. \quad (3.2)$$

The goal of this section is to show that similar ellipsoids in four dimensions,

$$\frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\ell^2} = 1, \quad (3.3)$$

admit a Killing spinor satisfying (2.2) and (2.4) if the background fields $V_m{}^{A\ B}, T_{kl}, \bar{T}_{kl}, M$ are chosen appropriately. We will restrict to those backgrounds with $U(1) \times U(1)$ isometry, and anticipate that the square of the supersymmetry yield a linear combination of the two $U(1)$ isometries which fix the north and south poles of the ellipsoid. Our ellipsoid (3.3) is thus parametrized by three axis-length parameters.
Introducing a polar coordinate system,
\[ x_0 = r \cos \rho, \]
\[ x_1 = \ell \sin \rho \cos \theta \cos \varphi, \]
\[ x_2 = \ell \sin \rho \cos \theta \sin \varphi, \]
\[ x_3 = \tilde{\ell} \sin \rho \sin \theta \cos \chi, \]
\[ x_4 = \tilde{\ell} \sin \rho \sin \theta \sin \chi, \]  
the vielbein one-forms \( E^a = E^a_m dx^m \) can be chosen as
\[ E^1 = \sin \rho e^1, \quad E^2 = \sin \rho e^2, \quad E^3 = \sin \rho e^3 + h d\rho, \quad E^4 = g d\rho, \]  
where
\[ f \equiv \sqrt{\ell^2 \sin^2 \theta + \tilde{\ell}^2 \cos^2 \theta}, \]
\[ g \equiv \sqrt{\ell^2 \sin^2 \rho + \ell^2 \tilde{\ell}^2 f^{-2} \cos^2 \rho}, \]
\[ h \equiv \frac{\tilde{\ell}^2 - \ell^2}{f} \cos \rho \sin \theta \cos \theta, \]  
and \( e^a \) are vielbein of the 3D ellipsoid (3.3) in polar coordinates \((\varphi, \chi, \theta)\),
\[ e^1 = \ell \cos \theta d\varphi, \quad e^2 = \tilde{\ell} \sin \theta d\chi, \quad e^3 = f d\theta. \]  
The spin connection \( \Omega^{ab} = \Omega^{ab}_m dx^m \) has the following components,
\[ \Omega^{12} = 0, \quad \Omega^{13} = -\frac{\ell}{f} \sin \theta d\varphi, \quad \Omega^{23} = -\frac{\tilde{\ell}}{f} \cos \theta d\chi, \]
\[ \Omega^{14} = \frac{\tilde{\ell}^2 \cos \rho}{gf^2} e^1, \quad \Omega^{24} = \frac{\ell^2 \cos \rho}{gf^2} e^2, \quad \Omega^{34} = \frac{\ell^2 \tilde{\ell}^2 \cos \rho}{gf^4} e^3. \]  
Note that \( \Omega^{12}, \Omega^{13}, \Omega^{23} \) are the spin connection of the 3D ellipsoid with vielbein \( e^a \).

**Killing spinors on round \( S^4 \).** Killing spinor equation has solutions on the round \( S^4 \) of radius \( \ell \) with no background gauge or tensor auxiliary fields turned on. The main equation (2.2) consists of eight equations, and we divide them into two groups. The first six equations are given by
\[ \left( \partial_m + \frac{1}{4} \Gamma^a_m \tau^{ab} - \frac{i \cos \rho}{2\ell} \epsilon^a_m \tau^a \right) \xi_A = -\sin \rho \epsilon_m^a \tau^a \xi_A', \]
\[ \left( \partial_m + \frac{1}{4} \Gamma^a_m \tau^{ab} + \frac{i \cos \rho}{2\ell} \epsilon^a_m \tau^a \right) \xi_A' = +\sin \rho \epsilon_m^a \tau^a \xi_A, \]  
where \( a, b = 1, 2, 3 \) and the index \( m \) runs over \( \varphi, \chi, \theta \). Here \( \tau^a \) are Pauli’s matrices as before and we used \( \tau^{ab} \equiv \frac{1}{2} (\tau^a \tau^b - \tau^b \tau^a) \). The last two equations read
\[ \partial_\rho \xi_A = -i \epsilon^2_A, \]
\[ \partial_\rho \xi_A' = -i \epsilon^2_A. \]  

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The equations (3.9) are solved by Killing spinors \( \kappa_{st} \) \((s,t = \pm 1)\) on round \( S^3 \) of radius \( \ell \) with coordinates \( \theta, \varphi, \chi \), which satisfy

\[
\left( \partial_m + \frac{1}{4} \Omega_m^{ab} \tau_{ab} \right) \kappa_{st} = -\frac{i}{2} t e^a_m \tau^a \kappa_{st}, \quad \kappa_{st} \equiv \frac{1}{2} \left( \begin{array}{c} e^{\frac{i}{2}(s\chi + t\varphi - st\theta)} \\ -se^{\frac{i}{2}(s\chi + t\varphi + st\theta)} \end{array} \right) \tag{3.11}
\]

for \( m = (\varphi, \chi, \theta) \) and \( a, b = 1, 2, 3 \). One can form a Killing vector on the \( S^3 \) as a bilinear of \( \kappa_{st} \), \( \kappa_{st} \)

\[
\kappa_{st}^{\dagger} \tau_a \kappa_{st} \cdot e^{am} \partial_m = -\frac{1}{2\ell} (s \partial_\varphi + t \partial_\chi). \tag{3.12}
\]

Recalling that \( \partial_\varphi \) and \( \partial_\chi \) are rotations in the \((x_1, x_2)\)-plane and \((x_3, x_4)\)-plane, we restrict to those with \( s = t \) so that our choice of Killing spinor corresponds to Omega deformations with \( \epsilon_1 = \epsilon_2 \) at the north pole. Assuming \( \xi_A \) and \( \bar{\xi}_A \) are all proportional to \( \kappa_{++} \) or \( \kappa_{--} \), the remaining equations (3.10) become

\[
-i \ell \xi'_A = \partial_\rho \xi_A = \frac{\cos \rho + 1}{2 \sin \rho} \xi_A,
\]

\[
-i \ell \bar{\xi}'_A = \partial_\rho \bar{\xi}_A = \frac{\cos \rho - 1}{2 \sin \rho} \bar{\xi}_A. \tag{3.13}
\]

In the following we take a particular solution of the main equation (2.2) which also satisfies the reality condition (2.1). We also require

\[
\xi^A \xi'_A = \bar{\xi}^A \bar{\xi}'_A = 0, \tag{3.14}
\]

so that the square of the corresponding supersymmetry transformation does not give rise to dilation or \( U(1)_R \) transformation, namely \( w = \Theta = 0 \) in (2.8). It is unique up to the symmetries of the theory.

\[
\xi_A = (\xi_1, \xi_2) = \sin \frac{\rho}{2} (\kappa_{++}, \kappa_{--}),
\]

\[
\bar{\xi}_A = (\bar{\xi}_1, \bar{\xi}_2) = \cos \frac{\rho}{2} (i\kappa_{++}, -i\kappa_{--}). \tag{3.15}
\]

The Killing vector which appears in the square of the supersymmetry transformation is

\[
v^m \partial_m = 2 \xi^A \sigma^m \xi_A \partial_m = \frac{1}{\ell} (\partial_\varphi + \partial_\chi). \tag{3.16}
\]

This solution also satisfies the auxiliary equation (2.4) with the choice \( M = -\frac{1}{3} R = -4 \ell^{-2} \).

**A Killing spinor on ellipsoids.** Next we study the Killing spinor equation on ellipsoids (3.3). Our strategy is to assume that, for a suitable choice of the background gauge and auxiliary fields, the Killing spinor (3.15) on round \( S^4 \) remains a Killing spinor also on ellipsoids. Then we will see that the Killing spinor equation can be turned into a set of linear algebraic equations on the background fields which have nontrivial solutions. A similar approach worked in the case of 3D ellipsoids [22]. Note that under this assumption the Killing vector on the ellipsoid becomes

\[
2 \xi^A \sigma^m \xi_A \partial_m = \frac{1}{\ell} \partial_\varphi + \frac{1}{\ell} \partial_\chi, \tag{3.17}
\]
which can be interpreted as the Omega deformation with \( \epsilon_1 = \ell^{-1}, \epsilon_2 = \tilde{\ell}^{-1} \) near the north and south poles. This point will be explained in more detail later.

In solving the Killing spinor equation to determine the background fields, a useful fact is that the 3D spinors \( \kappa_{st} \) on \( S^3 \) remain Killing spinors after the deformation to 3D ellipsoids if a suitable background \( U(1) \) gauge field is turned on at the same time. More explicitly, one has

\[
\left( \partial_m + \frac{1}{4} \Omega^{ab}_m \tau^{ab} \mp i V^{[3]}_m \right) \kappa_{\pm \pm} = - \frac{i}{2 f} e^a_m \tau^a \kappa_{\pm \pm},
\]

where \( m = \varphi, \chi, \theta \) and \( a, b = 1, 2, 3 \). Another useful fact is that the following 2 \( \times \) 2 matrix,

\[
\tau^1_{\theta} \equiv \cos \theta + \tau^2 \sin \theta,
\]

satisfies \( \tau^1_{\theta} \kappa_{\pm \pm} = \mp \kappa_{\pm \pm} \) and therefore \( \tau^1_{\theta} \xi_A = - \xi_B (\tau^3)^B_A \). At this point we find it convenient to regard \( \xi_A \) and \( \bar{\xi}_A \) as 2 \( \times \) 2 matrices, on which 2 \( \times \) 2 matrices with spinor indices act from the left and those with \( SU(2)_R \) indices act from the right. The latter equation can then be rewritten in the matrix form,

\[
\tau^1_{\theta} \xi = - \xi \tau^3.
\]

Hereafter all the boldface letters can be regarded as 2 \( \times \) 2 matrix quantities. By using the above equation in combination with

\[
\tau^3 \xi = \xi \{ \cos (\chi + \varphi) \tau^1 + \sin (\chi + \varphi) \tau^2 \},
\]

any \( SU(2) \) action from the right of \( \xi \) can be translated into an \( SU(2) \) action from the left, and vice versa. Note also that

\[
\tau^1_{\theta} \xi = i \tan \frac{\rho}{2} \bar{\xi}.
\]

Let us now turn to the analysis of Killing spinor equation. We introduce the notations

\[
V + V^{[3]} \tau^3 \equiv \bar{V} = E^a \bar{V}_a, \quad iT \equiv \sigma_{kl} T^{kl}, \quad i \bar{T} \equiv \bar{\sigma}_{kl} \bar{T}^{kl}.
\]

We also require that (3.14) is still satisfied on ellipsoids, and introduce a pair of antisymmetric tensors \( S_{kl}, \bar{S}_{kl} \) and matrices \( S, \bar{S} \) by the formula

\[
\xi' = S \xi = -i \sigma_{kl} S_{kl} \xi, \quad \bar{\xi}' = \bar{S} \bar{\xi} = -i \bar{\sigma}_{kl} \bar{S}_{kl} \bar{\xi}.
\]

Inserting these together with (3.17) into the main equation (2.2), we obtain

\[
\xi \bar{V}_4 + T \xi + \bar{S} \xi = i \frac{\cos \rho + 1}{2 g \sin \rho} \xi - \frac{h}{2 f g \sin \rho} \tau^3 \xi - \frac{h \Omega^4_3}{2 g} \tau^3 \xi,
\]

\[
\bar{\xi} \bar{V}_4 + T \bar{\xi} + S \bar{\xi} = i \frac{\cos \rho - 1}{2 g \sin \rho} \bar{\xi} - \frac{h}{2 f g \sin \rho} \tau^3 \bar{\xi} + \frac{h \Omega^4_3}{2 g} \tau^3 \bar{\xi},
\]

\[
(3.25)
\]
and

\[ \xi \tilde{V}_a - i T \tau^a \xi - i \tau^a S \xi = \frac{1}{2f \sin \rho} \tau^a \xi + \frac{1}{2} \Omega^{b4}_a \tau^b \xi, \]
\[ \tilde{\xi} \tilde{V}_a + i \tilde{T} \tau^a \xi + i \tau^a S \xi = \frac{1}{2f \sin \rho} \tau^a \tilde{\xi} - \frac{1}{2} \Omega^{b4}_a \tau^b \tilde{\xi}, \]  
(3.26)

where \( a, b = 1, 2, 3 \) and the nonzero components of \( \Omega^{b4}_a \) are

\[ \Omega^{14} = \frac{\ell^2 \cos \rho}{gf^2 \sin \rho}, \quad \Omega^{24} = \frac{\ell^2 \cos \rho}{gf^2 \sin \rho}, \quad \Omega^{34} = \frac{\ell^2 \ell^2 \cos \rho}{gf^4 \sin \rho}. \]  
(3.27)

The equations (3.25) and (3.26) can be regarded as a system of inhomogeneous linear algebraic equations for the unknowns \( \tilde{V}, T, \tilde{T}, S \) and \( \tilde{S} \). We found that these equations have nontrivial solutions, and moreover the solution is not unique. A special solution for which \( T, \tilde{T} \) take particularly simple form is

\[ T = \frac{1}{4} \left( \frac{1}{f} - \frac{1}{g} \right) \tau^1, \quad \tilde{T} = \frac{1}{4} \left( \frac{1}{f} - \frac{1}{g} \right) \tau^1 - \frac{h}{4fg} \tau^2, \]
\[ S = -\frac{1}{4} \left( \frac{1}{f} + \frac{1}{g} \right) \tau^1 - \frac{h}{4fg} \tau^2, \quad \tilde{S} = -\frac{1}{4} \left( \frac{1}{f} + \frac{1}{g} \right) \tau^1 + \frac{h}{4fg} \tau^2, \]
\[ \xi \tilde{V}_1 = \left\{ \cos \frac{\theta}{2} \sin \frac{\rho}{f} \left( \frac{1}{f} - \frac{1}{g} \right) - \sin \frac{\theta}{2} \cos \frac{h}{2} \sin \frac{\rho}{f} \right\} \tau^1 \xi + \sin \frac{\theta}{2} \cos \frac{h}{2} \sin \frac{\rho}{f} \left( 1 - \frac{\ell^2}{gf} \right) \tau^2 \xi, \]
\[ \xi \tilde{V}_2 = \left\{ \frac{\sin \theta}{2} \cos \frac{h}{2} \sin \frac{\rho}{f} \left( \frac{1}{f} - \frac{1}{g} \right) + \frac{\cos \theta}{2} \cos \frac{h}{2} \sin \frac{\rho}{f} \right\} \tau^1 \xi - \cos \frac{\theta}{2} \cos \frac{h}{2} \sin \frac{\rho}{f} \left( 1 - \frac{\ell^2}{gf} \right) \tau^2 \xi, \]
\[ \xi \tilde{V}_3 = -\frac{\cos \frac{\theta}{2} \sin \frac{\rho}{f}}{2f \sin \rho} \left( 1 - \frac{\ell^2}{gf} \right) \tau^3 \xi, \]
\[ \xi \tilde{V}_4 = \frac{h \cos \frac{\theta}{2} \sin \frac{\rho}{f}}{2fg \sin \rho} \left( 1 - \frac{\ell^2}{gf} \right) \tau^3 \xi. \]  
(3.28)

where we introduced \( \tau^2 \equiv i \tau^1 \tau^3 \). This special solution can be shifted by solutions of the homogeneous equation, namely the equations (3.25) and (3.26) with the r.h.s. set to zero. They are parametrized by three arbitrary functions \( c_1, c_2, c_3 \) as follows.

\[ \Delta T = \tan \frac{\theta}{2} \left( +c_1 \tau^1 + c_2 \tau^2 + c_3 \tau^3 \right), \]
\[ \Delta \tilde{T} = \cot \frac{\theta}{2} \left( -c_1 \tau^1 + c_2 \tau^2 + c_3 \tau^3 \right), \]
\[ \Delta S = \cot \frac{\theta}{2} \left( +c_1 \tau^1 + c_2 \tau^2 + c_3 \tau^3 \right), \]
\[ \Delta \tilde{S} = \tan \frac{\theta}{2} \left( -c_1 \tau^1 + c_2 \tau^2 + c_3 \tau^3 \right), \]
\[ \xi \cdot \Delta \tilde{V}_1 = -2 \sin \theta \left( c_2 \tau^1 \xi - c_1 \tau^2 \xi \right), \]
\[ \xi \cdot \Delta \tilde{V}_2 = +2 \cos \theta \left( c_2 \tau^1 \xi - c_1 \tau^2 \xi \right), \]
\[ \xi \cdot \Delta \tilde{V}_3 = -2c_1 \tau^3 \xi + 2c_3 \tau^1 \xi, \]
\[ \xi \cdot \Delta \tilde{V}_4 = +2c_2 \tau^3 \xi - 2c_3 \tau^2 \xi. \]  
(3.29)
In $2 \times 2$ matrix notations, the auxiliary equation (2.4) becomes

\[
-4 \cot \frac{\rho}{2} \left( \sigma_m D_m \tilde{S} - D_m T \sigma^m \right) \tau^1_\theta - 4 \sigma^m \tilde{S} T \sigma_m = 4 \tan \frac{\rho}{2} \left( \tilde{\sigma}^m D_m S - D_m \tilde{\sigma}^m \right) \tau^1_\theta - 4 \sigma^m \tilde{S} T \sigma_m = M \cdot 1. \tag{3.30}
\]

This is satisfied by the above special solution (3.28) with

\[
M = \frac{1}{f^2} - \frac{1}{g^2} + \frac{h^2}{f^2 g^2} - \frac{4}{fg}. \tag{3.31}
\]

We found that the auxiliary equation is still satisfied even after nonzero $c_1, c_2, c_3$ are turned on, as long as they are functions of $\theta$ and $\rho$ only. The shift of $M$ is then given by

\[
\Delta M = 8 \left( \frac{1}{g} \partial_\rho - \frac{h}{gf \sin \rho} \partial_\theta + \frac{\ell^2 \tilde{\ell}^2 \cos \rho}{gf^4 \sin \rho} + \frac{\cos \rho (\ell^2 + \tilde{\ell}^2 - f^2)}{gf^2 \sin \rho} - \frac{\cos \rho}{f \sin \rho} \right) c_1
\]

\[
+ 8 \left( \frac{1}{f \sin \rho} \partial_\theta + \frac{h \ell^2 \tilde{\ell}^2 \cos \rho}{gf^2 f^4 \sin \rho} + \frac{2 \cot 2 \theta}{f \sin \rho} - \frac{h \cos \rho}{fg \sin \rho} \right) c_2 - 16 \left( c_1^2 + c_2^2 + c_3^2 \right). \tag{3.32}
\]

We thus determined the form of all the additional background fields in order for SW theories on the ellipsoid (3.3) to admit a rigid supersymmetry. In the rest of this section we check two more properties of our background. The first is that the square of the supersymmetry is a sum of bosonic transformations which indeed leave all the background fields invariant. The second is that our background is regular and approaches Omega background near the two poles.

**Square of SUSY.** The supersymmetry transformation $Q$ acting on fields of SW theory squares into a sum of bosonic symmetries according to (2.7) and (2.25). It can also be expressed as

\[
Q^2 = iL_v + \text{Gauge}(\hat{\Phi}) + \text{Lorentz}(L_{ab}) + \text{Scale}(w) + R_{U(1)}(\Theta) + R_{SU(2)}(\hat{\Theta}_{AB}) + \hat{R}_{SU(2)}(\hat{\Theta}_{AB}), \tag{3.33}
\]

where

\[
\hat{\Phi} \equiv \Phi - iv^n A_n, \\
L_{ab} \equiv D_{[a v^b]} + v^n \Omega_{nab}, \\
\hat{\Theta}_{AB} \equiv \Theta_{AB} + v^n V_{nAB}, \\
\hat{\Theta}_{AB} \equiv \hat{\Theta}_{AB} + v^n \tilde{V}_{nAB}.
\]

Let us compute these transformation parameters for our ellipsoid background. First of all, our condition (3.14) on the Killing spinor guarantees that $w = \Theta = 0$. Moreover one can show

\[
L_{ab} \equiv 0, \quad \hat{\Theta}^A_B = \left( -\frac{1}{2\ell} - \frac{1}{2\ell} \right) \cdot (\tau^3)^A_B. \tag{3.35}
\]

\[\]
using the explicit form of vielbein, spin connection and the background $SU(2)_R$ gauge field obtained above. It follows that our Killing spinor is invariant under $Q^2$.

\begin{align}
Q^2 \xi_A &= i L_v \xi_A - \xi_B \dot{\Theta}^B_A = 0, \\
Q^2 \bar{\xi}_A &= i L_v \bar{\xi}_A - \bar{\xi}_B \dot{\Theta}^B_A = 0.
\end{align}

The background fields $V^A_{m B}, T_{kl}, \bar{T}_{kl}, M$ are also invariant under $Q^2$ since they are constructed from $L_v$-invariant functions and Killing spinor.

To determine the action of $Q^2$ on all the fields, we still need to determine $\check{\xi}_A, \check{\bar{\xi}}_A$ and the background $SU(2)_R$ gauge field $\check{V}^B_m A$ which have been left somewhat ambiguous. Hereafter we take the following solution of (2.24).

\begin{align}
\check{\xi}_A &= \cot \frac{\rho}{2} \xi_A, \\
\check{\bar{\xi}}_A &= - \tan \frac{\rho}{2} \bar{\xi}_A.
\end{align}

Note that this has an effect of gauge fixing the local $SU(2)_R$ symmetry relative to $SU(2)_R$, and the following choice of the $SU(2)_R$ gauge field is consistent with it.

\begin{align}
\check{V}^B_m A &= V^B_m A.
\end{align}

Using (3.37) one can also show

\begin{align}
\check{\Theta}_{AB} = \Theta_{AB}, \text{ therefore } \check{\Theta}_{AB} = \hat{\Theta}_{AB},
\end{align}

and conclude that all the background fields are invariant under $Q^2$.

**Omega-background revisited.** Here we focus on the behavior of our ellipsoid background near the north and south poles.

Near the north pole where $x_0 \simeq r$ in (3.3), the other four coordinates $(x_1, \cdots, x_4)$ can be regarded as the Cartesian coordinates on $\mathbb{R}^4$. The norm of $\check{\xi}$ approaches a constant while that of $\xi$ is proportional to the radial distance from the pole. In a suitable gauge, the Killing spinor should therefore take the form

\begin{align}
\check{\xi}_A \simeq \frac{1}{\sqrt{2}} \delta_A^\alpha, \quad \xi_{\alpha A} \simeq - \frac{1}{2\sqrt{2}\ell} (x_1 \sigma_2 - x_2 \sigma_1)_{\alpha A} - \frac{1}{2\sqrt{2}\ell} (x_3 \sigma_4 - x_4 \sigma_3)_{\alpha A}
\end{align}

so that

\begin{align}
2 \xi^A \sigma^m \xi_A \frac{\partial}{\partial x_m} &= \frac{1}{\ell} \left( x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) + \frac{1}{\ell} \left( x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3} \right) \\
&= \frac{1}{\ell} \partial \varphi + \frac{1}{\ell} \partial \chi.
\end{align}

The first equation in (3.40) indicates that near the north pole our supersymmetry approach that of the topologically twisted theory which identifies the dotted spin $SU(2)$ index with the $SU(2)$ R-symmetry index. From this viewpoint, the second equation in (3.40) tells nothing but the fact that $\ell^{-1}, \tilde{\ell}^{-1}$ play the role of the Omega-deformation parameters
Note that, for the spinor field (3.40) to satisfy Killing spinor equation (2.2) and (2.4) on flat $\mathbb{R}^4$, one has to turn on the background field as follows,

$$T^\Omega \equiv \frac{1}{2} T_{mn} dx^m dx^n = \frac{1}{16} \left( \frac{1}{\ell} - \frac{1}{\tilde{\ell}} \right) \left( dx_1 dx_2 - dx_3 dx_4 \right),$$

$$V^\Omega = \bar{T}^\Omega = M^\Omega = 0.$$  \hspace{1cm} (3.42)

In other words, Omega background with $\epsilon_1 \neq \epsilon_2$ is related to a flat $\mathbb{R}^4$ with constant background field $T_{kl}$.

In much the same way, near the south pole one can choose a gauge in which $\xi^A_\alpha$ is proportional to the identity matrix. There the supersymmetry approaches that of the anti-topologically twisted theory with Omega deformation. One can also relate the flat $\mathbb{R}^4$ with constant $\bar{T}_{mn}$ to Omega background of the anti-topologically twisted theory.

It remains to check whether our ellipsoid background is regular at the two poles. To do this, we rewrite the above regular Omega background (3.42) with the following polar coordinates of $\mathbb{R}^4$,

$$x_1 = \ell \rho \cos \theta \cos \varphi, \quad x_3 = \tilde{\ell} \rho \sin \theta \cos \chi,$n

$$x_2 = \ell \rho \sin \theta \cos \varphi, \quad x_4 = \tilde{\ell} \rho \sin \theta \sin \chi.$$  \hspace{1cm} (3.43)

The auxiliary field $T$ for the Omega background then takes the form,

$$T^\Omega = \frac{1}{16 f} \left( \frac{1}{\ell} - \frac{1}{\tilde{\ell}} \right) \left\{ \ell \sin \theta (E^1 E^3 + E^2 E^4) - \tilde{\ell} \cos \theta (E^1 E^4 - E^2 E^3) \right\},$$  \hspace{1cm} (3.44)

where $E^a$ are the natural vielbein one-forms on $\mathbb{R}^4$ in the polar frame,

$$E^1 = \rho e^1, \quad E^2 = \rho e^2, \quad E^3 = \rho e^3 + h(0) d\rho, \quad E^4 = g(0) d\rho.$$  \hspace{1cm} (3.45)

Here $h(0)$ and $g(0)$ denote the values of the functions $h$ and $g$ in (3.6) at $\rho = 0$. Then one finds

$$T^\Omega \equiv -i T^\Omega_{mn} \sigma^{mn} = \frac{1}{4} \left( \frac{1}{f} - \frac{1}{g(0)} \right) \tau^1_\theta + \frac{h(0)}{4 f g(0)} \tau^2_\theta,$$  \hspace{1cm} (3.46)

which agrees with our special solution (3.28) near the north pole. However, there is a finite mismatch between the value of $\bar{T}$, which is zero on the Omega background (3.42) but nonvanishing near the north pole of (3.28). This indicates that our special solution has singularity at the two poles and a suitable nonzero $c_1, c_2$ has to be chosen so as to cancel it. A simple choice which leads to $\bar{T} = 0$ at the north pole and $T = 0$ at the south pole is given by

$$c_1 = \frac{1}{8} \left( \frac{1}{f} - \frac{1}{g} \right) \sin \rho \cos \rho, \quad c_2 = \frac{h}{8 f g} \sin \rho \cos \rho.$$  \hspace{1cm} (3.47)

One is still left with the freedom to shift the $c$’s by functions which vanish as $\sin^2 \rho$ or faster near the two poles.

### 4. Explicit Path Integration

Here we use the SUSY localization principle and evaluate partition functions of general SW theories on the ellipsoid backgrounds. Our analysis follows closely that of [2]. We first focus on the theories with vector multiplets only, and introduce matter hypermultiplets later.
Saddle points for SYM theories. According to the SUSY localization principle, non-zero contribution to the path integral arises only from saddle points which are characterized by

\[ Q\Psi = 0 \quad \text{for all the fermions } \Psi. \]

The first step in computing partition function is to find out the saddle point locus. Though we have to modify the supercharge \( Q \) upon introducing BRST ghost system, the saddle point locus remain the same.

To find out the saddle point locus for vector multiplets, it is convenient to study the following quantity,

\[ I_{\text{vec}} \equiv \text{Tr} \left[ (Q\lambda_{\alpha A})^\dagger(Q\lambda_{\alpha A}) + (Q\bar{\lambda}_{\dot{A}})^\dagger(Q\bar{\lambda}_{\dot{A}}) \right], \tag{4.1} \]

which is manifestly positive semi-definite and vanishes on saddle points. Using the transformation law and the reality condition (2.13), one can rewrite it as follows,

\[ I_{\text{vec}} = \text{Tr} \left[ D_m\phi_1 D^n\phi_1 - [\phi_1, \phi_2]^2 - \frac{1}{2} (D_{AB} + i\phi_1 w_{AB})(D^{AB} + i\phi_1 w^{AB}) \right. \]
\[ \left. + \xi^A\xi_A \left( F_{mn} - 4\phi_2 T_{mn} - 4\phi_2 S_{mn} + \frac{1}{\xi^A\xi_A} v_{[m} D_{n]} - \phi_2 \right)^2 \right. \]
\[ \left. + \bar{\xi}_{\dot{A}}\bar{\xi}^\dot{A} \left( F_{mn}^+ + 4\phi_2 \bar{T}_{mn} + 4\phi_2 \bar{S}_{mn} - \frac{1}{\xi_{\dot{A}}\xi_{\dot{A}}} v_{[m} D_{n]} + \phi_2 \right)^2 \right. \]
\[ \left. + \frac{1}{4\xi^A\xi_A \cdot \xi_B\xi_B} (v^m D_m \phi_2)^2 \right], \tag{4.2} \]

where the suffix ± for antisymmetric tensors indicates the self-dual or anti-self-dual parts, and we introduced

\[ \phi_1 \equiv i(\phi + \bar{\phi}), \quad \phi_2 \equiv \phi - \bar{\phi}, \]
\[ w_{AB} \equiv 4\xi_A\sigma^{mn}\xi_B (T_{mn} - S_{mn}) = -4\bar{\xi}_A\bar{\sigma}^{mn}\bar{\xi}_B (\bar{T}_{mn} - \bar{S}_{mn}). \tag{4.3} \]

Note that \( w_{AB} \) here satisfies the condition (2.14) and therefore can be used to construct FI Lagrangian.

The saddle point condition for \( \phi_2 \) and \( A_m \) is to be derived from the last three terms in the r.h.s. of (4.2). We argue that it is given by

\[ \phi_2 = A_m = 0 \quad \text{up to gauge choice.} \tag{4.4} \]

For round sphere with \( T_{mn} \equiv \bar{T}_{mn} \equiv 0 \), one finds that the last three terms can be reorganized into a different “sum of squares” up to total derivatives,

\[ I_{\text{vec}} = \text{Tr} \left[ \cdots + (D_m \phi_2)^2 + \xi^A\xi_A (F_{mn} - 4\phi_2 S_{mn})^2 + \bar{\xi}_{\dot{A}}\bar{\xi}^\dot{A} (F_{mn}^+ - 4\phi_2 \bar{S}_{mn})^2 \right]. \tag{4.5} \]

This gives a much simpler saddle point condition which immediately leads to (4.4) when combined with Bianchi identity \( D_{[n} F_{mn]} = 0 \). However, as soon as the sphere is deformed, this reorganizing is no longer possible and one has to deal with more complicated saddle points.
point condition which follows from (4.2). But if there are nontrivial solutions to the original saddle point condition on some deformed sphere, they should be continuously connected to nontrivial solutions on round sphere. Such solutions would have to be singular, since they do not minimize $\mathcal{I}_{\text{vec}}$ of (1.5) which differs from (1.2) only by total derivatives. Thus we believe that (4.4) is the only solution to the saddle-point condition. It would be nice to prove this claim rigorously, though we will base our subsequent analysis on this claim and obtain the most natural generalization of the result for round sphere.

Once (4.4) is settled, then the condition for the remaining fields are easily solved. The saddle points are thus labeled by a Lie algebra valued constant $a_0$, and are given by the equations

$$A_m = 0, \quad \phi = \dot{\phi} = -i a_0, \quad D_{AB} = -i a_0 w_{AB}.$$  \hspace{1cm} (4.6)

The values of super-Yang-Mills action (2.10) and FI term (2.15) on this saddle point are

$$\frac{1}{g_{\text{YM}}} \int d^4 x \sqrt{g} L_{\text{YM}} \big|_{\text{saddle point}} = \frac{8 \pi^2}{g_{\text{YM}}} \ell \tilde{\ell} \text{Tr}(a_0^2),$$

$$\zeta \int d^4 x \sqrt{g} L_{\text{FI}} \big|_{\text{saddle point}} = -16i \pi^2 \ell \tilde{\ell} \zeta a_0.$$  \hspace{1cm} (4.7)

They are independent of the precise choices of $c_1, c_2, c_3$ as long as they are smooth.

**Ghosts and BRST symmetry.** For gauge fixing, we proceed in the same way as [2]. Let us introduce the Faddeev-Popov ghost field $c$ and define the BRST transformation by

$$Q_B A_m = D_m c, \quad Q_B \lambda_A = i \{ c, \lambda_A \},
Q_B \phi = i [c, \phi], \quad Q_B \bar{\phi} = i [c, \bar{\phi}],
Q_B D_{AB} = i [c, D_{AB}].$$  \hspace{1cm} (4.8)

We require the square of $Q_B$ to be a constant gauge rotation with parameter $a_0$, so we set

$$Q_B c = icc + a_0.$$  \hspace{1cm} (4.9)

The sum of the SUSY and the BRST transformations, $\hat{Q} = Q + Q_B$, will be the relevant fermionic symmetry in the application of localization principle later on. Requiring its square to act on all the fields as

$$\hat{Q}^2 = i L_v + \text{Gauge}(a_0) + R_{SU(2)}(\hat{\Theta}_{AB}),$$  \hspace{1cm} (4.10)

one finds that the supersymmetry transformation of $c$ has to be,

$$Q_c = -\dot{\Phi} = -\phi_1 - i \cos \rho \phi_2 + iv^n A_n.$$  \hspace{1cm} (4.11)

One also finds that the constant variable $a_0$ has to be invariant,

$$Q a_0 = Q_B a_0 = 0.$$  \hspace{1cm} (4.12)

We furthermore introduce the antighost multiplet with the transformation rules,

$$Q_B \bar{c} = B, \quad Q_B B = i [a_0, \bar{c}],
Q \bar{c} = 0, \quad Q B = i L_v \bar{c},$$  \hspace{1cm} (4.13)
and the multiplets of constant fields which will be used to freeze the constant modes of $c$ and $\bar{c}$.

$$
\begin{align*}
Q_B \bar{a}_0 &= \bar{c}_0, & Q_B c_0 &= i[a_0, \bar{a}_0], & Q_B B_0 &= c_0, & Q_B c_0 &= i[a_0, B_0], \\
Q \bar{a}_0 &= 0, & Q \bar{c}_0 &= 0, & Q B_0 &= 0, & Q c_0 &= 0.
\end{align*}
$$

(4.14)

To fix a gauge correctly, the standard way is to choose a set of conditions $G[A_m, \phi, \cdots]$ and shift the Lagrangian by the gauge-fixing term

$$
\mathcal{L}_{GF} = Q_B V_{GF}, \quad V_{GF} \equiv \text{Tr}(\bar{c}G + \bar{c}B_0 + c\bar{a}_0).
$$

(4.15)

We will later find it convenient to choose

$$
G = i \partial_m A_m + i \mathcal{L}_v (\cos \rho \phi \lambda - v A_m).
$$

(4.16)

For the computation of partition function using localization principle, it is more convenient to replace $Q_B$ in (4.15) by $\hat{Q} = Q + Q_B$. As explained in [2] this replacement does not change the value of partition function.

Now that the gauge-fixed system has the fermionic symmetry $\hat{Q} \equiv Q + Q_B$, we need to revisit the condition for the saddle points

$$
\hat{Q} \Psi = Q \Psi + Q_B \Psi = 0 \quad \text{for all the fermions } \Psi.
$$

(4.17)

For the fermions in vector multiplets, the added term $Q_B \Psi$ is always bilinear in fermions so that the condition for saddle points does not change. For the ghost $c$, the saddle point condition gives

$$
\hat{Q} c = icc + a_0 - \phi_1 - i \cos \rho \phi \lambda + iv^m A_n = 0.
$$

(4.18)

Thus $a_0$ is to be identified with the constant value of $\phi_1$ at saddle points.

**One-loop determinant.** The value of path integral does not change under the shifts of the original Lagrangian by any $\hat{Q}$-exact quantities, $\mathcal{L} \to \mathcal{L} + t \hat{Q} \mathcal{V}$. We take the regulator $\hat{Q} \mathcal{V}$ so that its bosonic part is positive definite and is strictly positive anywhere away from saddle points. Since $t$ can be taken arbitrarily large, Gaussian approximation is exact for the path integration over the fluctuations away from saddle points.

We begin by introducing some new notations for later convenience.

$$
\begin{align*}
\Psi &\equiv Q \phi_2 = -i \xi^A \lambda_A - i \check{\xi}^A \check{\lambda}_A, \\
\Psi_m &\equiv Q A_m = i \xi^A \sigma_m \check{\lambda}_A - i \check{\xi}^A \sigma_m \lambda_A, \\
\Xi_{AB} &\equiv 2 \xi_{(A} \check{\lambda}_{B)} - 2 \check{\xi}_{(A} \lambda_{B)}.
\end{align*}
$$

(4.19)

The inverse of this relation is

$$
\begin{align*}
\lambda_A &= +i \xi_A \Psi - i \sigma^m \check{\xi}_A \Psi_m + \xi^B \Xi_{BA}, \\
\check{\lambda}_A &= -i \check{\xi}_A \Psi - i \sigma^m \xi_A \Psi_m + \check{\xi}^B \Xi_{BA}.
\end{align*}
$$

(4.20)
As the regulator, we take the $\tilde{Q}$-transform of the following quantity which has manifestly positive semi-definite bosonic part $I_{vec}$,

$$V = \text{Tr}\left[ (\hat{Q}\lambda_{\alpha A})^\dagger\lambda_{\alpha A} + (\hat{Q}\bar{\lambda}^{A}_\alpha)^\dagger\bar{\lambda}^{A}_\alpha \right].$$  \tag{4.21}

Inserting (4.20) into this and combining with the gauge fixing term, one finds

$$V + V_{GF} = \text{Tr}\left[ (\hat{Q}\Psi)^\dagger\Psi + (\hat{Q}\Psi^{m})^\dagger\Psi_{m} + \frac{1}{2}(\hat{Q}\Xi^{AB})^\dagger\Xi^{AB} + \bar{c}G + \bar{c}B_{0} + c_{0} \right].$$  \tag{4.22}

The integration over all the variables except for the constant $a_{0}$ will be carried out under the (exact) Gaussian approximation, with the weight given by $\hat{Q}(V + V_{GF})$ truncated up to quadratic order. In doing this, we move to a new set of path integration variables which consists of

$$X \equiv (\phi_{a}, A_{m} ; \bar{a}_{0}, B_{0}), \quad \Xi \equiv (\Xi^{AB}, \bar{c}, c)$$  \tag{4.23}

and their superpartners $\hat{Q}X, \hat{Q}\Xi$. In terms of these variables one can write

$$V + V_{GF}\bigg|_{\text{quad.}} = (\hat{Q}X, \Xi) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X \\ \hat{Q}\Xi \end{pmatrix}. \tag{4.24}
$$

The Gaussian integration gives the square root of the ratio of determinants of kinetic operators for boson and fermions. Using the fact that the operators $D_{ij}$ commute with $H \equiv \hat{Q}^{2}$, one finds after some algebra that

$$\frac{\det K_{\text{fermion}}}{\det K_{\text{boson}}} = \frac{\det \bar{H}}{\det \bar{X}H} = \frac{\det \text{Coker}D_{10}H}{\det \text{Ker}D_{10}H}. \tag{4.25}
$$

Thus the ratio of determinants can be determined from the spectrum of the operator $H$ on the kernel and cokernel of a differential operator $D_{10}$, which is encoded in the index

$$\text{ind}D_{10} \equiv \text{Tr}_{\text{Ker}D_{10}}(e^{-iHt}) - \text{Tr}_{\text{Coker}D_{10}}(e^{-iHt}). \tag{4.26}
$$

**Index of transversally elliptic operators.** In computing this index, we first drop the terms containing constant fields $B_{0}, \bar{a}_{0}$ from $V_{GF}$. These constant fields are thus regarded as sitting in the kernel of $D_{10}$ and making a contribution 2 to the index. To obtain the remaining contribution, we read off the differential operator $D_{10}$ from

$$\Xi D_{10}X + \Xi D_{11}\hat{Q}\Xi = \text{Tr}\left[ \bar{c}G - D_{m}c(\hat{Q}\Psi^{m})^\dagger + \frac{1}{2}\Xi^{AB}(\hat{Q}\Xi^{AB})^\dagger \right] \bigg|_{\text{quad}}, \tag{4.27}
$$

where we have, up to non-linear terms,

$$(\hat{Q}\Psi^{m})^\dagger = -iLvA_{m} + D_{m}(\hat{\Phi} - 2i\cos \rho \phi_{2} + 2iv^{n}A_{n}),$$

$$(\hat{Q}\Xi^{AB})^\dagger = -\xi A\sigma^{kl}\xi^{B}(F_{kl} - 8\phi T_{kl} + 8\phi S_{kl})$$

$$+ \xi A\sigma^{kl}\xi^{B}(F_{kl} - 8\phi \bar{T}_{kl} + 8\phi \bar{S}_{kl}) - 4\xi A\sigma^{n}\xi^{B}D_{n}\phi_{2} - D^{AB}. \tag{4.28}
$$

It turns out that the operator $D_{10}$ is not elliptic but transversally elliptic with respect to the isometry $L_{v}$ of the ellipsoid. Let us show this by computing explicitly its symbol.
We identify the fields $X$ and $\Xi$ with sections of bundles $E_0$ and $E_1$ over the ellipsoid $\mathcal{X}$, and therefore $D_{10} : \Gamma(E_0) \rightarrow \Gamma(E_1)$. Its symbol $\sigma(D_{10})$ is then obtained by retaining only the terms with highest order of derivatives and making the replacement $\partial_x \rightarrow i \rho \partial_x$. Thus $\sigma(D_{10})$ is a homomorphism between two vector bundles $\pi^*E_0, \pi^*E_1$ over the cotangent bundle $\pi : T^*\mathcal{X} \rightarrow \mathcal{X}$. The index of transversally elliptic operators is known to be uniquely determined by their symbols.

To write the symbol explicitly, it is convenient to introduce four unit vector fields $u_a^m$ ($a = 1, \cdots, 4$) by the formula

$$-2i(\tau^a)^A_B \bar{\xi}^B \bar{\sigma}^m \xi_A = \sin \rho \ u_a^m \quad (a = 1, 2, 3),$$

$$2\bar{\xi}^A \bar{\sigma}^m \xi_A = \sin \rho \ u_4^m \quad (a = 1, 2, 3),$$

and parametrize the momenta in the local orthonormal frame defined by the vielbein $u_a^m$.

For example, by a slight abuse of the notation, one can write

$$4\bar{\xi}^A \bar{\sigma}^m \xi_B \partial_m = \sin \rho (\bar{\sigma}^a)^A_B \ u_a^m \partial_m,$$

$$-4\bar{\xi}^A \bar{\sigma}^m \xi_B \partial_m = \sin \rho (\bar{\sigma}^a)^A_B \ u_a^m \partial_m,$$

and in particular

$$\mathcal{L}_v \equiv v^m \partial_m = \sin \rho u_4^m \cdot \rho p_m = i \sin \rho \cdot p_4.$$

Using this notation together with $\Xi_a \equiv \frac{1}{2} \Xi^B \xi_a^A (\tau_a)^B_A$, one finds

$$\Xi \sigma(D_{10}) X = (\Xi_1, \Xi_2, \Xi_3, -\bar{c}, ic) \begin{pmatrix} c_\rho p_4 & p_3 & -p_2 & -c_\rho p_1 & -s_\rho p_1 \\ -p_3 & c_\rho p_4 & p_1 & -c_\rho p_2 & -s_\rho p_2 \\ p_2 & -p_1 & c_\rho p_4 & -c_\rho p_3 & -s_\rho p_3 \\ p_1 & p_2 & p_3 & c_\rho p_4 & c_\rho s_\rho p_4 \\ p_1 p_4 & p_2 p_4 & p_3 p_4 & p_4^2 & -2s_\rho p_4^2 + 2c_\rho p_4^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ \phi_2 \end{pmatrix},$$

where we denoted $s_\rho \equiv \sin \rho$, $c_\rho \equiv \cos \rho$. The $5 \times 5$ matrix in the middle can be block diagonalized by a suitable change of variables within $X$ and $\Xi$.

$$\sigma(D_{10}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\rho p_4 & p_3 & -p_2 & -p_1 \\ -p_3 & c_\rho p_4 & p_1 & -p_2 \\ p_2 & -p_1 & c_\rho p_4 & -p_3 \\ p_1 & p_2 & p_3 & c_\rho p_4 \\ p_1 p_4 & p_2 p_4 & p_3 p_4 & p_4^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The lower-right $1 \times 1$ block should give a trivial contribution to the index, since the corresponding differential operator should have just one-dimensional kernel and cokernel of constant functions. So the nontrivial contribution to the index arises from the upper-left $4 \times 4$ block of the matrix in the middle,

$$\sigma(D'_{10}) = \begin{pmatrix} c_\rho p_4 & p_3 & -p_2 & -p_1 \\ -p_3 & c_\rho p_4 & p_1 & -p_2 \\ p_2 & -p_1 & c_\rho p_4 & -p_3 \\ p_1 & p_2 & p_3 & c_\rho p_4 \end{pmatrix}.$$

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Near the two poles, the symbol is that of the standard self-dual or anti-self-dual complex on $\mathbb{R}^4$,

\[
\begin{align*}
(\cos \rho = +1) & \quad \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^*} \Omega^{2+}, \\
(\cos \rho = -1) & \quad \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^*} \Omega^{2-}.
\end{align*}
\tag{4.35}
\]

A differential operator is called elliptic if its symbol is invertible for nonzero $p_4$. The above symbol $\sigma$ is not invertible at the equator $\cos \rho = 0$ since $\sigma \sigma^T = (p_1^2 + p_2^2 + p_3^2 + \cos^2 p_4^2) \cdot \text{id}$ as one can easily check. But if we restrict the momentum to be orthogonal to the vector $v$, namely $p_4 \equiv 0$, then $\sigma$ is invertible as long as $(p_1, p_2, p_3)$ are not all zero. The corresponding differential operator is then called transversally elliptic with respect to the symmetry $L_v$. The kernel and cokernel of transversally elliptic operators are generally infinite dimensional, though they are both decomposed into finite dimensional eigenspaces of $H$. Therefore, there is a bit more difficulty in the computation of index for transversally elliptic operators as compared to elliptic ones.

The operator $e^{-iHt}$ is a combination of a finite rotation of the ellipsoid, gauge rotation and $SU(2)_R$ rotation. Its action on an adjoint-valued field $\mathcal{O}$ takes the form

\[
e^{-iHt} \mathcal{O}(x^m) = \gamma_{[\mathcal{O}]} \cdot e^{a_{01}t} \mathcal{O}(\tilde{x}^m) e^{-a_{01}t}, \quad \left(\tilde{\varphi} = \varphi + \frac{t}{\xi}, \quad \tilde{\chi} = \chi + \frac{1}{\xi}\right)
\tag{4.36}
\]

where the coefficient $\gamma_{[\mathcal{O}]}$ encodes the action on the vector and $SU(2)_R$ indices of the field $\mathcal{O}$. For simplicity, let us temporarily take the gauge group to be abelian.

Regarding the index as the difference of the trace of $e^{-iHt}$ over $\Gamma(E_0)$ and $\Gamma(E_1)$, it should be written as a sum of contributions from the two fixed points where $\tilde{x}^m = x^m$. According to the Atiyah-Bott formula, the index is given by

\[
\text{ind}(D'_{10}) = \sum_{x: \text{fixed point}} \frac{\text{Tr}E_0(\gamma) - \text{Tr}E_1(\gamma)}{\det(1 - \partial \tilde{x}/\partial x)},
\tag{4.37}
\]

where the determinant factor is understood to arise from $d^4x \delta^4(\tilde{x}(x) - x)$. Near the north pole, the operator $e^{-iHt}$ acts on the local coordinates $z_1 = x_1 + ix_2, z_2 = x_3 + ix_4$ as

\[
\tilde{z}_1 = e^{it} z_1 \equiv q_1 z_1, \quad \tilde{z}_2 = e^{it} z_2 \equiv q_2 z_2.
\tag{4.38}
\]

Therefore

\[
\det(1 - \partial \tilde{x}/\partial x) = (1 - q_1)(1 - q_1)(1 - q_2)(1 - q_2),
\tag{4.39}
\]

where $q_1 q_1 = q_2 q_2 = 1$. The value of $\gamma$ for various fields reads

\[
\begin{align*}
\gamma[A_{z_1}] &= q_1, & \gamma[\Xi_{11}] &= \bar{q}_1 \bar{q}_2, \\
\gamma[A_{z_2}] &= q_2, & \gamma[\Xi_{12}] &= 1, \\
\gamma[A_{\bar{z}_1}] &= \bar{q}_1, & \gamma[\Xi_{22}] &= q_1 q_2, \\
\gamma[A_{\bar{z}_2}] &= \bar{q}_2, & \gamma[\bar{c}] &= 1.
\end{align*}
\tag{4.40}
\]

These are enough to compute the contribution from the north pole. Combining it with the similar contribution from the south pole and 2 from constant modes, one obtains

\[
\text{ind}(D'_{10}) = \left[-\frac{1 + q_1 q_2}{(1 - q_1)(1 - q_2)}\right] + \left[-\frac{1 + q_1 q_2}{(1 - q_1)(1 - q_2)}\right] + 2.
\tag{4.41}
\]
To extract the information on the multiplicity of eigenvalues of $H$, one needs to expand this expression into power series in $q_1, q_2$. The expansion does not seem to be unique, and the correct way should be found by investigating a suitable deformation of the symbol to make it non-degenerate everywhere away from the two poles. As was explained in [32] and reviewed in [2], this is the main point of difficulty in computing the index of transversally elliptic operators. At the end of the day, the correct way is to expand the first term in positive series and the second term in negative series. Thus we arrive at

$$\text{ind}(D_{10}) = 2 - \sum_{m,n \geq 0} \left( q_1^mq_2^n + q_1^{m+1}q_2^{n+1} + q_1^{-m}q_2^{-n} + q_1^{-m-1}q_2^{-n-1} \right).$$

(4.42)

For non-abelian gauge group $G$, we take $a_0$ to be in the Cartan subalgebra. Then each term in the above is multiplied by

$$\text{rk}G + \sum_{\alpha \in \Delta} \exp (ta_0 \cdot \alpha)$$

(4.43)

where the sum runs over all roots. This finishes the computation of the index $\text{ind}(D_{10})$.

**Infinite-product formula.** The one-loop determinant can be easily computed by extracting the spectrum of eigenvalues of $H$ from the index. Up to normalization factors depending only on $\ell$ and $\tilde{\ell}$, it is given by

$$Z_{\text{vec}}^{1\text{-loop}} = \left[ \frac{\det K_{\text{fermion}}}{\det K_{\text{boson}}} \right]^\frac{1}{2} = \prod_{\alpha \in \Delta_+} \frac{1}{(\hat{a}_0 \cdot \alpha)^2} \prod_{m,n \geq 0} \frac{(mb + nb^{-1} + Q + i\hat{a}_0 \cdot \alpha)(mb + nb^{-1} + i\hat{a}_0 \cdot \alpha)}{(mb + nb^{-1} + Q - i\hat{a}_0 \cdot \alpha)(mb + nb^{-1} - i\hat{a}_0 \cdot \alpha)}$$

(4.44)

where we introduced $b \equiv (\ell/\tilde{\ell})^{1/2}, Q \equiv b + \frac{1}{b}$ and $\hat{a}_0 \equiv \sqrt{\ell\tilde{\ell}}a_0$. We also used the function $\Upsilon(x)$ which has zeroes at $x = Q + mb + \frac{n}{m}, -mb - \frac{n}{m}$ $(m, n \in \mathbb{Z}_{\geq 0})$ to express the appropriately regularized infinite products. It is characterized by

$$\Upsilon(x) = \Upsilon(Q - x), \quad \Upsilon(Q/2) = 1,$$

(4.45)

as well as the shift relations

$$\Upsilon(x + b) = \Upsilon(x)\gamma(bx)b^{1-2bx},$$

$$\Upsilon(x + \frac{1}{2}) = \Upsilon(x)\gamma(x/b)b^{-2x-1}. \quad \left( \gamma(x) \equiv \Gamma(x)/\Gamma(1 - x) \right)$$

(4.46)

The function $\Upsilon(x)$ was used in [33] to write down the three-point structure constant in Liouville CFT with coupling $b$. Thus our result suggests that the ellipsoid is the correct background for SW theories to reproduce Liouville or Toda correlators for general value of the coupling $b$. 

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The final expression for partition function involves an integral with respect to the saddle point parameter $a_0$ over the Lie algebra, but one can restrict its integration domain to Cartan subalgebra. It gives rise to the usual Vandermonde determinant factor which cancels with the factor $(a_0 \cdot \alpha)^2$ in the denominator of (4.44). 

**Inclusion of matter.** Let us next study the case with hypermultiplet matters. The first thing to do is to solve the saddle point condition. For round $S^4$ it was shown in [2] that all the bosonic fields in hypermultiplets have to vanish at saddle points; in other words there is no Higgs branch. We claim this remains true on ellipsoids. The simplest way to see this is to consider the zero locus of the bosonic part of

$$\hat{L}_{\text{mat}} = \hat{Q} V_{\text{mat}},$$

(4.47)

which is the same as the bosonic part of (2.22). The auxiliary field $F_A$ simply has to vanish. The scalar $q_A$ has mass term which is smallest at the origin of the Coulomb branch where $\phi = \bar{\phi} = -\frac{i}{2} a_0 = 0$, and its value $\frac{1}{4}(R + M)$ is strictly positive anywhere on the ellipsoid at least when the deformation from the round sphere with $T_{mn} = \bar{T}_{mn} = 0$ is not large.

The one-loop determinant can be computed in the same way as for the vector multiplets. We define new Grassmann-odd scalar fields which are doublets under $SU(2)_R$ or $SU(2)_{\tilde{R}}$ by the formula,

$$\Psi_A \equiv -i \xi_A \psi + i \bar{\xi}_A \bar{\psi} = Q q_A, \quad \Xi_A \equiv \bar{\xi}_A \psi - \bar{\bar{\xi}}_A \bar{\psi}. \quad (4.48)$$

The inverse of this is

$$\psi = -2i \xi^A \Psi_A - 2\xi^A \Xi_A, \quad \bar{\psi} = -2i \bar{\bar{\xi}}^A \Psi_A - 2\bar{\bar{\bar{\xi}}}^A \Xi_A. \quad (4.49)$$

We then rewrite the regulator Lagrangian (4.47) truncated up to quadratic order in terms of the variables $(q_A, \hat{Q} q_A)$ and $(\Xi_A, \hat{Q} \Xi_A)$. The computation of the one-loop determinant thus reduces to that of the index of an operator $D_{10}^{\text{mat}}$ which can be read from the terms bilinear in $\Xi_A$ and $q_A$ in $V_{\text{mat}}$. Its symbol is given by

$$\Xi_A \left[ \sigma (D_{10}^{\text{mat}}) \right]^A_B q^B = i \cos^2 \frac{\rho}{2} \Xi_A (\sigma^a p_a)^A_B q^B - i \sin^2 \frac{\rho}{2} \Xi_A (\sigma^a p_a)^A_B q^B, \quad (4.50)$$

where we used the notation introduced in (4.30). The ellipticity of $D_{10}^{\text{mat}}$ is violated at $\rho = \frac{\pi}{2}$ but it is transversally elliptic with respect to the isometry generated by $L_v$.

Using Atiyah-Bott formula again, we compute the index from the action of $H$ on fields at the two poles. At the north pole it is most convenient to work with the Cartesian local coordinates $x_1, \cdots, x_4$, in terms of which the metric is flat and the Killing spinor takes the form (3.40). Here one can regard $q_A$ as dotted spinor and identify $\Xi_A$ as undotted spinor $\psi$. Thus we find, for example for $r$ free hypermultiplets,

$$\gamma[q_A^{A=1}] = q_1^\uparrow \bar{q}_2^\downarrow, \quad \gamma[\psi_{A=1}^l] = q_1^\uparrow \bar{q}_2^\downarrow,$$

$$\gamma[q_A^{A=2}] = \bar{q}_1^\downarrow q_2^\uparrow, \quad \gamma[\psi_{A=2}^l] = \bar{q}_1^\downarrow q_2^\uparrow. \quad (I = 1, \cdots, 2r) \quad (4.51)$$
Combining the contribution from the two poles one finds the index

\[
\text{ind}(D_{10}^{\text{mat}}) = 2r \left[ \frac{q_1^{1/2} q_2^{1/2}}{(1 - q_1)(1 - q_2)} \right] + 2r \left[ \frac{q_1^{1/2} q_2^{1/2}}{(1 - q_1)(1 - q_2)} \right] = 2r \sum_{m,n \geq 0} \left( q_1^{m+\frac{3}{2}} q_2^{n+\frac{3}{2}} + q_1^{-m-\frac{3}{2}} q_2^{-n-\frac{3}{2}} \right), \tag{4.52}
\]

where we assumed that a regularization procedure similar to the case with vector multiplet determines how to expand the first line into power series. For hypermultiplets coupled to gauge symmetry, the factor \(2r\) is replaced by a sum over the weight vectors of the corresponding representation. For example, the hypermultiplet is said to be in a representation \(R\) of the gauge group if the index \(I\) furnishes the representation \(R = R \oplus \bar{R}\). Then the index is given by the replacement

\[
2r \rightarrow \sum_{\rho \in R} e^{ta_0 \cdot \rho} = \sum_{\rho \in R} (e^{ta_0 \cdot \rho} + e^{-ta_0 \cdot \rho}), \tag{4.53}
\]

where \(\rho\) runs over all the weight vectors in a given representation \(R\) or \(\bar{R}\). This completes the computation of the index. It is straightforward to translate this result into the matter one-loop determinant,

\[
Z_{1\text{-loop}}^{\text{hyp}} = \prod_{\rho \in R} \prod_{m,n \geq 0} \left( mb + nb^{-1} + \frac{Q}{2} + i\hat{a}_0 \cdot \rho \right)^{-1} \left( mb + nb^{-1} + \frac{Q}{2} - i\hat{a}_0 \cdot \rho \right)^{-1} = \prod_{\rho \in R} \Upsilon(i\hat{a}_0 \cdot \rho + \frac{Q}{2})^{-1}. \tag{4.54}
\]

**Instanton contribution.** In solving the saddle point condition for vector multiplet, the gauge field was assumed to be smooth. Relaxing this assumption, one finds from (4.3) that the gauge field strength can have nonzero anti-self-dual components at the north pole where \(\xi^A \xi_A = \sin^2 \frac{\theta}{2} = 0\), or nonzero self-dual components at the south pole where \(\xi^A \xi_A = \cos^2 \frac{\theta}{2} = 0\).

The system near the north pole approaches the topologically twisted theory with Omega deformation \(\epsilon_1 = \ell^{-1}, \epsilon_2 = \tilde{\ell}^{-1}\), and the contribution of localized instantons is described by Nekrasov’s instanton partition function \(Z_{\text{inst}}(a_0, \epsilon_1, \epsilon_2, \tau)\). Similarly the contribution of anti-instantons localized to the south pole is evaluated by an anti-topologically twisted theory, which leads to Nekrasov’s partition function with the argument \(\bar{\tau}\).

So, our final result for the ellipsoid partition function is

\[
Z = \int d\hat{a}_0 e^{-\frac{8g^2}{\kappa_M} \text{Tr}(i\hat{a}_0^2)} |Z_{\text{inst}}|^2 \prod_{\alpha \in \Delta_+} \Upsilon(i\hat{a}_0 \cdot \alpha) \Upsilon(-i\hat{a}_0 \cdot \alpha) \prod_{\rho \in R} \Upsilon(i\hat{a}_0 \cdot \rho + \frac{Q}{2})^{-1}. \tag{4.55}
\]

**Wilson loops.** The generalization of the above result to expectation values of supersymmetric observables is straightforward. Of particular interest are the Wilson loops. Supersymmetry requires the loops to be aligned with the direction of \(v\). When \(\ell, \tilde{\ell}\) are
incommensurable, there are only two classes of closed loops. One of them winds along the \(\varphi\)-direction and the other along \(\chi\)-direction, and they are both labeled by \(\rho\).

\[
S^1_\varphi(\rho) : (x_0, x_1, x_2, x_3, x_4) = (r \cos \rho, \ell \sin \rho \cos \varphi, \ell \sin \rho \sin \varphi, 0, 0),
\]
\[
S^1_\chi(\rho) : (x_0, x_1, x_2, x_3, x_4) = (r \cos \rho, 0, 0, \tilde{\ell} \sin \rho \cos \chi, \tilde{\ell} \sin \rho \sin \chi).
\] (4.56)

The corresponding supersymmetric Wilson loops are given by

\[
W_\varphi(R) \equiv \text{Tr}_R \text{P} \exp i \int_{S^1_\varphi(\rho)} d\varphi \left( A_\varphi - 2\ell(\phi \cos^2 \frac{\varphi}{2} + \bar{\phi} \sin^2 \frac{\varphi}{2}) \right),
\]
\[
W_\chi(R) \equiv \text{Tr}_R \text{P} \exp i \int_{S^1_\chi(\rho)} d\chi \left( A_\chi - 2\tilde{\ell}(\phi \cos^2 \frac{\chi}{2} + \bar{\phi} \sin^2 \frac{\chi}{2}) \right).
\] (4.57)

At saddle points they take the classical values

\[
W_\varphi(R) = \text{Tr}_R \exp (-2\pi b \hat{a}_0),
\]
\[
W_\chi(R) \equiv \text{Tr}_R \exp (-2\pi b^{-1} \hat{a}_0).
\] (4.58)

The expectation values of Wilson loops can thus be computed by inserting these expressions into the integral formula (4.55).

5. Concluding Remarks

In this paper we have found an interesting deformation of the round \(S^4\) which supports SW theories with a rigid supersymmetry. In the light of the AGT correspondence, our result for partition functions should be related to the 2D Liouville or Toda correlators with the coupling \(b = (\ell/\tilde{\ell})^{1/2}\). Let us quickly check this in the simplest example of \(SU(2)\) SQCD with four fundamental hypermultiplets, which should correspond to Liouville four-point function on sphere. We focus only on the one-loop part of the correspondence, since the other parts, such as the coincidence between Nekrasov’s instanton partition function and Liouville conformal block, have already been extensively investigated for general \(b\).

For \(SU(2)\) SQCD with four fundamental flavors, the saddle points are labeled by a single parameter \(p\), and the mass of the four hypermultiplets \(\mu_1, \cdots, \mu_4\) can be introduced via suitable gauging of the \(U(1)^4\) subgroup of the flavor group \(SO(8)\). The one-loop part of the partition function then reads

\[
Z_{1\text{-loop}} = \frac{\Upsilon(2ip)\Upsilon(-2ip)}{\prod_{i=1}^4 \Upsilon\left(\frac{Q}{2} + ip + i\mu_i\right)\Upsilon\left(\frac{Q}{2} - ip + i\mu_i\right)}.
\] (5.1)

To make correspondence with Liouville theory, we divide the four hypermultiplets into two pairs, and associate each pair with the flavor subgroup \(SO(4) \simeq SU(2) \times SU(2)\). We thus get four copies of \(SU(2)\) flavor groups, and denote by \(p_a\) the mass parameter associated to the \(a\)-th \(SU(2)\). The parameters \(\mu_i\) and \(p_a\) are related by

\[
\mu_1 = p_1 + p_2, \quad \mu_2 = p_1 - p_2, \quad \mu_3 = p_3 + p_4, \quad \mu_4 = p_3 - p_4.
\] (5.2)
Under this identification $Z_{1\text{-loop}}$ agrees, up to some $p$-independent factors, with the product of two Liouville 3-point structure constants $C(p_1,p_2,p_3)$.

$$Z_{1\text{-loop}} \sim C(p_1,p_2,p)C(p_1,p_2,-p) = C(p_1,p_2,p)C(p_1,p_2,p)R(p)^{-1}. \quad (5.3)$$

Here $R(p) \equiv \Upsilon(Q+2ip)/\Upsilon(Q-2ip)$ is the reflection coefficient of Liouville primary operator with momentum $\alpha = Q^2/2 + ip$, and $C(p_1,p_2,p_3)$ is given by

$$C(p_1,p_2,p_3) = \frac{\text{const} \cdot \Upsilon(Q + 2ip_1)\Upsilon(Q + 2ip_2)\Upsilon(Q + 2ip_3)}{\Upsilon(Q^2/2 + ip_{1+2+3})\Upsilon(Q^2/2 + ip_{1+2-3})\Upsilon(Q^2/2 + ip_{1-2+3})\Upsilon(Q^2/2 + ip_{1-2-3})}. \quad (5.4)$$

Thus we found the agreement for general values of the coupling $b$.

We expect that, in comparison to $\mathcal{N}=1$ SUSY theories, the theories with extended SUSY such as SW theories can be put on wider class of backgrounds preserving supersymmetry, since the corresponding off-shell supergravity multiplet contains more fields. It will be an interesting problem to find and classify other 4D manifolds which have solutions to our Killing spinor equation. Studying SW theories on such manifolds may lead to yet another interesting generalization of the AGT relation.

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