DECOMPOSING HESSENBERG VARIETIES OVER CLASSICAL GROUPS

JULIANNA S. TYMOCZKO

Abstract. Hessenberg varieties are a family of subvarieties of the flag variety, including the Springer fibers, the Peterson variety, and the entire flag variety itself. The seminal example arises from a problem in numerical analysis and consists for a fixed linear operator $M$ of the full flags $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n$ in $GL_n$ with $MV_i \subseteq V_{i+1}$ for all $i$.

In this paper I show that all Hessenberg varieties in type $A_n$ and semisimple and regular nilpotent Hessenberg varieties in types $B_n, C_n$, and $D_n$ can be paved by affine spaces. Moreover, this paving is the intersection of a particular Bruhat decomposition with the Hessenberg variety. In type $A_n$, an equivalent description of the cells of the paving in terms of certain fillings of a Young diagram can be used to compute the Betti numbers of Hessenberg varieties.

As an example, I show that the Poincare polynomial of the Peterson variety in $A_n$ is $\sum_{i=0}^{n-1} \binom{n-1}{i} x^{2i}$.

Acknowledgements. I would like to thank Gil Kalai, David Kazhdan, Arun Ram, and Eric Sommers for many helpful conversations and suggestions. Discussions at different stages with Jared Anderson, Andrew Booker, Emma Carberry, Henry Cohn, Ketan Delal, Jordan Ellenberg, Miranda Hodgson, Allison Klein, and David Nadler provided me with useful suggestions, insights, and morale. I particularly want to thank Jared Anderson, Henry Cohn, and Eric Sommers for reading early drafts of this thesis and offering corrections and improvements.

I am grateful for the support and entertainment given me continually by the extended Tymoczko clan: Maria Tymoczko, Marlene Wong, Alexei Tymoczko, Dmitri Tymoczko, Elisabeth Camp, Molly Donohue, and Misha Kazhdan. I am grateful also to Marshall Poe.

Robert MacPherson taught me how to think about mathematics. This thesis would have been impossible without him.

1. Introduction

Hessenberg varieties form a large class of subvarieties of the flag variety, many examples of which have been of great importance to geometers, representation theorists, combinatorists, and numerical analysts, among others. In this paper I describe the basic topology of many Hessenberg varieties.

Given a Lie algebra $g$ with a Borel subalgebra $b$, a Hessenberg space $H$ is a $b$-submodule of $g$ which contains $b$. For a fixed element $M$ in $g$, we can consider the elements $g$ in an associated linear algebraic group $G$ such that $\text{Ad} \ g^{-1}(M)$ lies in $H$. This gives a subset $G(M, H)$ of the linear algebraic group. Since $H$ is closed under conjugation by the elements of the Borel subgroup $B$ which corresponds to $b$, the subset $G(M, H)$ is closed under right multiplication by elements of $B$. Thus
the image of $G(M, H)$ in the flag variety $G/B$ is a closed subvariety $\mathcal{H}(M, H)$ of $G/B$. This subvariety $\mathcal{H}(M, H)$ is the Hessenberg variety of $M$ and $H$.

Hessenberg varieties as such were introduced by De Mari, Procesi, and Shayman in [MPS]. De Mari and Shayman were first motivated to study these spaces because of a question in numerical analysis related to efficient computation of the eigenvalues and eigenspaces of the operator $M$. Given certain $H$, the space $\mathcal{H}(M, H)$ parametrizes the bases with respect to which the operator $M$ can be efficiently diagonalized via the QR-algorithm [MS]. In [MPS], the authors provided a cell decomposition of $\mathcal{H}(M, H)$ when $M$ is regular semisimple by using a natural torus action that exists for those $M$. They observed that when $H$ is generated by $b$ as well as the simple negative root spaces then $\mathcal{H}(M, H)$ is the toric variety associated to the decomposition into Weyl chambers. This space is combinatorially interesting as well, since its Betti numbers generalize Eulerian descents of a permutation and can be used to give generating functions for several permutation statistics [F].

For entirely different reasons, several major examples of nilpotent Hessenberg varieties have been intensely studied recently. Springer initiated this research when he discovered an amazing connection between the cohomology of the Springer fibers and the irreducible representations of the Weyl group [S]. These Springer fibers are in fact the nilpotent Hessenberg varieties $\mathcal{H}(N, b)$. Springer’s original proof was algebraic but later work expanded on the geometric nature of the results, including [BM], [CG], [He], [KL], and [L], among others. Spaltenstein identified the components of type-$A_n$ Springer fibers $\mathcal{H}(N, b)$ and proved they were equidimensional and then extended the proof of equidimensionality to general Springer fibers (see [Sp1, Sp2]). Shimomura partitioned type-$A_n$ Springer fibers into affine spaces in a manner similar to that used here [S1]. Spaltenstein further showed that there is a Schubert decomposition whose intersection with the Springer fibers gives a paving by affines for type $A_n$ in [Sp3, section II.5] and Shimomura extended this to apply to partial flag varieties [S2]. Spaltenstein also gave a combinatorial description of the cells used in this paving [Sp3]. Several years later, De Concini, Lusztig, and Procesi provided a paving by affines of the Springer fibers for all classical types by reducing to the case of Springer fibers of distinguished nilpotents [CLP]. In their work, Borho and MacPherson generalized Springer fibers to the larger class of nilpotent Hessenberg varieties given by $\mathcal{H}(N, p)$ for each parabolic subalgebra $p$. They showed that the intersection cohomology of these Hessenberg varieties also could be viewed as representations of the Weyl group [BM].

More recently still, Peterson defined the Peterson variety, which plays a role in quantum cohomology and whose totally positive part has interesting properties. The Peterson variety is the nilpotent Hessenberg variety $\mathcal{H}(N, H)$ when $N$ is regular and $H$ is the Hessenberg space generated by $b$ together with the simple negative root spaces. Kostant showed that the coordinate ring of a particular open affine subvariety of the Peterson variety coincides with the quantum cohomology of the flag variety [Ko]. Rietsch has shown that the totally nonnegative part of the Peterson variety $\mathcal{H}(N, H)$ is homeomorphic to the totally nonnegative part of Givental’s critical point locus for the mirror symmetric family for the flag variety [R]. Research into the Peterson variety is ongoing.

In the rest of this paper I describe general Hessenberg varieties and then give a paving by affine spaces for all Hessenberg varieties in type $A_n$ as well as semisimple and regular nilpotent Hessenberg varieties in the other classical types. The main
Theorem. Fix a regular nilpotent element $N$, let $b$ be the unique Borel subalgebra with $N \in b$, and let $B$ be the Borel subgroup corresponding to $b$. Let $H$ be a Hessenberg space for this Borel subalgebra. The intersection of the Bruhat decomposition with respect to $B$ and the Hessenberg variety $\mathcal{H}(N,H)$ is a paving by affines of $\mathcal{H}(N,H)$ for each $H$. The nonempty cells of this paving are $B\pi B \cap \mathcal{H}(N,H)$ satisfying $\pi^{-1} \cdot N \in H$.

The dimension of each cell is the cardinality of a certain set of positive roots depending on $\pi$, $H$, and $N$. This set is described precisely in Theorem 28. One consequence is that regular nilpotent, semisimple, and all type-$A_n$ Hessenberg varieties have no odd-dimensional cohomology. In type $A_{n-1}$, I offer an alternative description of the paving in terms of certain fillings of certain Young diagrams. In this type, the Hessenberg space $H$ is equivalent to a function $h : \{1,2,\ldots,n\} \rightarrow \{1,2,\ldots,n\}$ such that $h(i) \geq \max\{i,h(i-1)\}$ for all $i$. (The relation between $h$ and $H$ is described in greater detail in Section 3.2.) The theorem for nilpotent Hessenberg varieties follows.

Theorem. Let $N$ be a nilpotent operator. Associate to $N$ the Young diagram whose $i^{th}$ column has the same number of boxes as the dimension of the $i^{th}$ Jordan block for $N$. Assume this Young diagram is left-aligned and bottom-aligned.

The Hessenberg variety $\mathcal{H}(N,H)$ is paved by affine spaces each of which is associated to a permutation $\pi$. The nonempty cells of the paving correspond to those fillings of the Young diagram associated to $N$ for which the configuration

$$\begin{array}{c}
\pi^{-1}k \\
\pi^{-1}j
\end{array}$$

only occurs if $\pi^{-1}j \leq h(\pi^{-1}k)$.

Given a nonempty cell represented as a (filled) Young tableau, the dimension of this cell is the sum of the following two quantities:

1. The number of configurations

$$\begin{array}{c}
\pi^{-1}j \\
\pi^{-1}i
\end{array}$$

where box $i$ is to the right of or below box $j$, there is no box above $j$, and the values filling these boxes satisfy $\pi^{-1}i > \pi^{-1}j$.

2. The number of configurations

$$\begin{array}{c}
\pi^{-1}k \\
\pi^{-1}i \\
\pi^{-1}j
\end{array}$$

where box $i$ is to the right of or below box $j$ and the values filling these boxes satisfy $\pi^{-1}j < \pi^{-1}i \leq h(\pi^{-1}k)$.

This combinatorial method lends itself to computational results, as I demonstrate by providing the Betti numbers of the Peterson variety in type $A_n$.

The strategy of the proof is to use $M$ to choose a Bruhat decomposition so that on each Schubert cell the Hessenberg variety $\mathcal{H}(M,H)$ is an iterated tower of affine fibrations. This procedure is independent of the particular Hessenberg space $H$ so
that if $M$ is fixed, the inclusion of Hessenberg spaces gives rise to a natural inclusion of cells within their respective Hessenberg varieties.

Consider the nilpotent matrix

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the unipotent matrix

$$u = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Note that the conjugate $u^{-1}Nu$ is

$$u^{-1}Nu = \begin{pmatrix} 0 & 1 & a_{23} - a_{12} & a_{24} - a_{12}(a_{34} - a_{23}) - a_{13} \\ 0 & 0 & 1 & a_{34} - a_{23} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Each flag $gB$ can be written as $u\pi B$ for some $u$ and a unique permutation $\pi$. The flag $gB$ is in the Hessenberg variety $H(N, H)$ if and only if $u^{-1}Nu$ is in $\pi H\pi^{-1}$.

Chapter 2 establishes the basic notational conventions of this paper. In type $A_{n-1}$, we can identify the Lie algebra $\mathfrak{g}$ with a subset of $n \times n$ matrices. This has a natural basis of matrix units $E_{ij}$ defined to have value one in the $(i, j)$ entry and zero elsewhere. In terms of the previous example, Chapter 2 shows that $H$ is spanned by certain $E_{ij}$ and so the flag $gB$ is in $H(N, H)$ if and only if the matrix $u^{-1}Nu$ is zero in certain entries, which is equivalent to certain polynomial equations in the entries of the matrix $u$ being zero. Section 2.4 describes these equations in general. These equations are not necessarily linear, as the example shows. However, the equations in the top row are affine functions of the variables $a_{ij}$ in terms of the variables $a_{ij}$ for $i \geq 2$. Theorem 11 makes this claim in more general terms. To prove Theorem 11 we need two main tools: Section 2.1 describes a decomposition of classical Lie algebras that generalizes the rows of a matrix; and Section 2.3 introduces a class of algebraic varieties called sequentially linear varieties whose added structure can be used to identify pavings. In Section 2.6 we use the row decomposition to show that Hessenberg varieties are paved by sequentially linear varieties and provide some conditions under which they are in fact paved by affine spaces. This amounts to partitioning flags $gB$ into Schubert cells and then showing that within each Schubert cell, the affine function $a_{24} - a_{12}(a_{34} - a_{23}) - a_{13}$ of the variables $a_{ij}$ will have the same dimension solution space independent of the choice of $a_{ij}$ for $i \geq 2$. Chapter 3 contains the main theorems of this paper. Section 3.2 includes a detailed analysis of the Peterson variety in type $A_n$ and an explicit description of its cells.

2. Definitions and Basic Properties

Throughout this paper we use the notation and language of algebraic groups as in [12]. Let $G$ be a linear algebraic group of classical type over the field $\mathbb{C}$ and denote its Lie algebra by $\mathfrak{g}$. (The results in this paper for nilpotent Hessenberg varieties
generalize to fields of nonzero characteristic. The results for semisimple Hessenberg varieties hold for algebraically closed fields other than \( \mathbb{C} \).) Choose a maximal Cartan subalgebra \( h \) in the Lie algebra and define positive roots \( \Phi^+ \) and simple roots \( \Delta \) with respect to this torus. Write the decomposition of the Lie algebra into root spaces as \( g = h \oplus \bigoplus_{\alpha \in \Phi} g_\alpha \). On occasion, we will fix a nonzero root vector \( E_\alpha \) in \( g \) which spans \( g_\alpha \). Let \( b \) be the Borel subalgebra associated to \( \Phi^+ \) and let \( n \) be its nilradical.

Assume that the simple roots \( \alpha_1, \ldots, \alpha_n \) are indexed according to the conventions of, e.g. \([H1]\). In other words, the bond between \( \alpha_{n-1} \) and \( \alpha_n \) in the Dynkin diagram for \( g \) determine the type of the Lie algebra.

A Hessenberg space \( H \) is a \( b \)-submodule of \( g \) which contains \( b \). Let \( M_H = \{ \alpha \in \Phi : g_\alpha \subseteq H \} \).

Then \( M_H \) is a subset of roots closed under addition of positive roots and containing all positive roots. Conversely, given any such subset of roots, there is a unique Hessenberg space containing the corresponding root spaces \([MPS]\). Consider the subspace \( i \) orthogonal to \( H \) with respect to the Killing form in \( g = H \oplus i \). Note that \( H \) is a Hessenberg space if and only if \( i \) is an ad-nilpotent ideal in the sense of \([CP]\) with respect to the opposite Borel subalgebra \( b^- \). The results of \([CP]\) thus show that the number of Hessenberg spaces in type \( A_{n-1} \) is the \( n^{th} \) Catalan number. They also show that slight variations of Catalan numbers enumerate Hessenberg spaces in the other classical types.

Given a Hessenberg space \( H \) and an element \( M \) in \( g \), consider the subset of \( G \) defined by

\[ G(M, H) = \{ g \in G : \text{Ad} g^{-1}(M) \in H \}. \]

We often denote \( \text{Ad} g(M) \) by \( g \cdot M = gMg^{-1} \).

Since \( H \) is closed under the adjoint action of the Borel subgroup \( B \) corresponding to \( b \), the subset \( G(M, H) \) is closed under right multiplication by \( B \). We may thus look at the image of \( G(M, H) \) under the quotient map

\[
\begin{array}{ccc}
g & \supseteq & G(M, H) \\
\downarrow & & \downarrow \\
G/B & \supseteq & \mathcal{H}(M, H)
\end{array}
\]

The image \( \mathcal{H}(M, H) \) is the Hessenberg variety corresponding to \( M \) and \( H \). The space \( G(M, H) \) is defined by closed conditions and the quotient map is closed, so \( \mathcal{H}(M, H) \) is a closed and hence projective variety.

Unless otherwise stated, we assume that \( M \) has been chosen from the Borel subalgebra \( b \). We use \( N \) to denote an element of the nilradical \( \bigoplus_{\alpha \in \Phi^+} g_\alpha \) and \( S \) to denote an element from the Cartan subalgebra \( h \) in \( b \).

2.1. A Decomposition of the Nilradical of \( b \). In this section we examine a decomposition of the nilradical \( n \) of the fixed Borel algebra \( b \) and prove some basic properties of this decomposition.

The standard partial order on the set of roots is defined by

\[ \alpha \geq \beta \text{ if and only if } \alpha - \beta \text{ is a sum of positive roots.} \]

We define \( \alpha > \beta \) analogously so that \( \alpha > \beta \) if and only if \( \alpha \geq \beta \) and \( \alpha \neq \beta \). We often use the stronger condition that \( \alpha - \beta \in \Phi^+ \).
Recall that the roots associated to the classical groups are described by the strings of simple roots given in this table.

| Root | Parameters | Type |
|------|------------|------|
| $\sum_{i=1}^{n} \alpha_j$ | $1 \leq i \leq k \leq n$ | $A_n, B_n, C_n, D_n$ |
| $\sum_{j=1}^{n} \alpha_j$ except $\alpha_{n-1} + \alpha_n \not\in \Phi$ in type $D_n$ | $A_n, B_n, C_n, D_n$ |
| $\sum_{j=i}^{n} \alpha_j + \sum_{j=k} \alpha_j$ $1 \leq i < k \leq n$ | $B_n$ |
| $\sum_{j=i}^{n} \alpha_j + \sum_{j=k} \alpha_j$ $1 \leq i \leq k < n$ | $C_n$ |
| $\sum_{j=i}^{n} \alpha_j + \alpha_n$ $1 \leq i \leq n - 2$ | $D_n$ |
| $\sum_{j=i}^{n} \alpha_j + \sum_{j=k} \alpha_j$ $1 \leq i < k \leq n - 2$ | $D_n$ |

This follows from the definition of the root systems of classical Lie groups over characteristic zero fields as in [H1, section 12].

We often refer to the extremal simple roots of a root $\alpha$.

**Definition 1.** The extremal simple roots of $\alpha$ are the simple roots $\alpha_i$ such that $\alpha - \alpha_i$ is in $\Phi^+$. The extremal roots of $\alpha$ are the positive roots $\beta$ such that $\alpha - \beta$ is in $\Phi^+$. 

For instance, any non-simple root $\alpha$ of type $A_n$ has exactly two extremal simple roots. Recall that if $\alpha$ is written as a sum of simple roots $\alpha = \sum_{j=1}^{k} \alpha_j$, then the height $ht(\alpha)$ of $\alpha$ is defined to be the number $k$ of simple summands. In type $A_n$, any non-simple root $\alpha$ has $2(\text{ht}(\alpha) - 1)$ extremal roots. By inspection of Table 2.1, we see that in the other classical types a non-simple root can have either one, two, or three extremal simple roots.

We define a partition of the positive roots and a collection of nilpotent subalgebras associated to each part. Let $\Phi^i$ be the subset of roots given by

$$\Phi^i = \{ \alpha \in \Phi^+ : \alpha_i \leq \alpha, \alpha_j \not\in \alpha \text{ for each } j < i \}.$$ 

In type $A_3$ this partition is $\Phi^1 = \{ \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3 \}$, $\Phi^2 = \{ \alpha_2, \alpha_2 + \alpha_3 \}$, and $\Phi^3 = \{ \alpha_3 \}$. By contrast, the partition is $\Phi^1 = \{ \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2 \}$ and $\Phi^2 = \{ \alpha_2 \}$ in type $B_2$.

Let $n_i = \bigoplus_{\alpha \in \Phi^i} g_\alpha$ be the subspace of $n$ spanned by the root spaces corresponding to $\Phi^i$. Note that $n_i$ is a nilpotent subalgebra of $n$. We refer to this subalgebra $n_i$ as the $i^{th}$ row of the Lie algebra $g$. The terminology is inspired by the example of $gl_n$ considered as the collection of $n \times n$ matrices. In this case, the subalgebra $n_i$ is precisely those matrices whose only nonzero entries are in the $i^{th}$ row and above the diagonal.

The next lemma proves that $n_i$ is either abelian or Heisenberg in classical types.

**Lemma 2.** The subalgebras $n_i$ satisfy

$[n_i, n_j] \subseteq n_i$ for all $i \leq j$.

1. In types $A_n, B_n$, and $D_n$, the $n_i$ are abelian Lie algebras.
2. In type $C_n$, the $n_i$ are Heisenberg Lie algebras for $i < n$. The subalgebra $n_n$ is an abelian Lie algebra.

**Proof.** The first claim follows from the definition of $n_i$ as well as the property that

$[g_\alpha, g_\beta] = \begin{cases} g_{\alpha + \beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{if } \alpha + \beta \not\in \Phi. \end{cases}$

(See, e.g., [H1, section 8.4].)
The second follows from the observation that
\[ \{ \alpha + \beta : \alpha, \beta \in \Phi^i; \alpha + \beta \in \Phi \} = \emptyset \]
in types \( A_n, B_n, \) and \( D_n. \) In type \( C_n \) the set
\[ \{ \alpha + \beta : \alpha, \beta \in \Phi^i; \alpha + \beta \in \Phi \} = \left\{ \sum_{j=i}^{n-1} 2\alpha_j + \alpha_n \right\} \subseteq \Phi^i. \]
Moreover, each root \( \alpha \neq \sum_{j=i}^{n-1} 2\alpha_j + \alpha_n \) in \( \Phi^i \) generates a complementary root \( \sum_{j=i}^{n-1} 2\alpha_j + \alpha_n - \alpha \in \Phi^i. \) With Property \( \text{(1)} \), these conditions characterize abelian and Heisenberg Lie algebras, respectively. \( \square \)

In type \( C_n, \) the roots \( 2\sum_{j=i}^{n-1} \alpha_j + \alpha_n \) are the long roots. We remark that there is exactly one long root in each row \( \Phi^i \) when \( i < n \) in \( C_n. \)

The following proposition lists some characteristics of the row partition in classical types.

**Proposition 3.** Partition each row \( \Phi^i \) of a given root system by height and denote the parts
\[ \Phi^i_k = \{ \alpha \in \Phi^i : \text{ht}(\alpha) = k \}. \]
These \( \Phi^i_k \) satisfy the following properties in classical types:

1. If \( \alpha \) is in \( \Phi^i_k \) and \( \beta \) is in \( \Phi^i_{k-1} \) then \( \alpha > \beta. \)
2. For each \( i, \) the cardinality \( |\Phi^i_k| \) is one except for at most one \( k_0, \) for which \( |\Phi^i_{k_0}| = 2. \)
3. For all \( 2 \leq i \leq n-2, \) if \( |\Phi^i_i| = 2 \) then \( \Phi^i_{i+1} = \alpha_{i-1} + \Phi^i_i \) and \( |\Phi^i_{i+1}| = 2. \)

Root systems whose rows satisfy these properties are called *vertical*. We often call the rows themselves vertical. To see that the rows in types \( A_n, B_n, C_n, \) and \( D_n \) are all vertical, we simply inspect the entries in Table 2.1. Indeed, each row in types \( A_n, B_n, \) and \( C_n \) is in fact ordered by height. Conditions (2) and (3) apply only to rows in type \( D_n; \) the conditions seem clumsy but will permit a general approach later in Lemma 20.

We often extend the definition of rows as follows to certain subgroups in the unipotent subgroup of the linear algebraic group \( G \) which corresponds to \( g. \) When \( G \) is of classical type other than \( A_n, \) we may assume that \( G \) has been embedded into \( GL(N, \mathbb{C}) \) so that \( \text{rk}(G) = \lfloor N/2 \rfloor \) and so that the simple roots \( \alpha_i \) for \( G \) are simple roots for \( GL_N(\mathbb{C}) \) when \( i < \lfloor N/2 \rfloor. \) Recall that \( \exp(X) = \sum_{n \geq 0} \frac{X^n}{n!} \) is a formal power series over \( g(\mathbb{C}) \) which is a polynomial whenever \( X \) is nilpotent \([\text{H2}] \) section 15.1]. Define \( U_i \) to be the subgroup generated by \( U_i = \exp(n_i). \) The map \( \exp \) is a homomorphism when \( n_i \) is an abelian Lie algebra. Whether \( n_i \) is abelian or Heisenberg, the subgroup \( U_i \) is the product of the root subgroups \( U_\alpha \) associated to the roots \( \alpha \) in \( \Phi^i. \) Note that the rows \( U_i \) generate the unipotent subgroup \( U = \prod_{i=1}^n U_{n-i+1}. \) We use this ordering to describe \( U \) throughout this paper.

### 2.2. The Bruhat Decomposition.

Here we recall some facts about Bruhat decompositions of the flag variety. Write \( T \) for the torus in \( G \) whose Lie algebra is \( \mathfrak{h} \) and denote the normalizer of \( T \) by \( N(T). \) The Weyl group \( W \) of \( G \) is the quotient \( W = N(T)/T. \) The Schubert cell in \( G \) associated to a Weyl group element \( \pi \) is the double coset \( B\pi B. \) By a slight abuse of notation, we also denote the image of
this double coset under the projection to $G/B$ by $B\pi B$. This is the Schubert cell corresponding to $\pi$ in the flag variety.

**Definition 4.** A paving $\mathcal{P}$ of an algebraic variety $X$ is an ordered partition $\mathcal{P} = (P_1, P_2, \ldots)$ of $X$ into disjoint varieties $P_i$ such that each finite union $\bigcup_{j \leq i} P_j$ is closed in $X$. If each $P_i$ is isomorphic to affine space, then $\mathcal{P}$ is a paving by affines.

Pavings have less structure than CW-complexes but can still be used to compute Betti numbers. This motivates us to pave varieties by simple spaces. For instance, the Schubert cells $B\pi B$ form a paving by affines of the flag variety [Fu, section 9.4]. Since Hessenberg varieties $\mathcal{H}(M, H)$ are closed in $G/B$ the Schubert cells form a paving of $\mathcal{H}(M, H)$ as well. The main claim of this paper is that in many cases $B$ can be chosen so that this is in fact a paving by affines.

Define a subgroup $U_\pi$ of the unipotent group $U$ by

$$U_\pi = \{u \in U : \pi^{-1} \cdot u \in U^\circ\}$$

where $U^\circ$ is the opposite unipotent group associated to $U$. The group $U_\pi$ parametrizes the Schubert cell corresponding to $\pi$ in the flag variety. Note that $U_\pi$ is a set of coset representatives of the flags in the Schubert cell associated to $\pi$ [H2, sections 28.1 and 28.4]. Under the natural map

$$U_\pi \longrightarrow G/B$$

$$u \mapsto u\pi B$$

the subgroup $U_\pi$ is isomorphic to the corresponding Schubert cell $B\pi B$ in the flag variety $G/B$ [FH, page 396]. Denote the Lie algebra of $U_\pi$ by $n_\pi$.

**Proposition 5.** Each subgroup $U_\pi$ decomposes into a product of its rows

$$U_\pi = \prod_{i=1}^{n} U_{\pi, n-i+1},$$

where $U_{\pi, i} = U_\pi \cap U_i$. The Lie algebra $n_\pi$ can be written

$$n_\pi = \text{span}(g_\alpha : \alpha > 0, \pi^{-1}\alpha < 0)$$

and decomposes into rows $n_{\pi, i} = n_\pi \cap n_i$ each of which is the Lie algebra of the corresponding subgroup $U_{\pi, i}$.

**Proof.** The subgroup $U$ can be written as a product

$$U = \prod_{i=1}^{n} \prod_{\alpha \in \Phi^{n-i+1}} \exp(g_\alpha)$$

for this fixed ordering of $\Phi^+$ by rows. This follows from repeated application of the Chevalley commutator relations. Thus $U_\pi$ inherits a decomposition into row subgroups.

Moreover,

$$\pi^{-1} \cdot \exp(g_\alpha) = \exp(g_{\pi^{-1}\alpha}).$$

By the Chevalley commutator relations, the product $\prod \exp(X_\alpha)$ is in $U_\pi$ if and only if each $X_\alpha$ is in $n_\pi$ [H2, section 26.3]. So $n_\pi$ is in fact the Lie algebra of $U_\pi$. It follows that $n_{\pi, i}$ is the Lie algebra of $U_{\pi, i}$ [H2, section 13.1]. \qed
2.3. Sequentially Linear Varieties. In this section we define sequentially linear algebraic varieties and give some of their preliminary properties. We will later show that Hessenberg varieties are examples of sequentially linear varieties and use these properties to prove the main claims of this paper.

Let $X$ be an algebraic variety, either affine or projective.

**Definition 6.** A sequentially linear structure on a variety $X$ is a finite sequence of varieties $X^i$ and morphisms $p_i$ so that

$$X = X^n \xrightarrow{p_n} X^{n-1} \xrightarrow{p_{n-1}} \cdots X^1 \xrightarrow{p_1} X^0 = \{\text{point}\}$$

and so that each $p_i$ has affine spaces as fibers.

If in addition each $p_i$ is a trivial affine fibration then $X$ is a constant rank sequentially linear variety, often simply called constant rank.

The following proposition is clear from the definitions.

**Proposition 7.** If $X = X^n \xrightarrow{p_n} \cdots X^1 \xrightarrow{p_1} X^0 = \{\text{point}\}$ is a constant rank sequentially linear variety then $X$ is isomorphic to affine $m$-dimensional space, where $m = \sum_{i=1}^{n} m_i$ and each $m_i = \dim p_i^{-1} x_i$ for $x_i$ in $X^i$.

2.4. The Adjoint Action of Rows. Here we discuss how the adjoint action $\text{Ad}: G \rightarrow \text{End} \, g$ behaves when considered as a map $\text{Ad}: U_i \rightarrow \text{End} \, b$. We also discuss the differential of this map $\text{ad} : \mathfrak{n}_i \rightarrow \text{End} \, b$. A modification of the Chevalley commutator relation and of Equation (1) permits an explicit description of $u^{-1} \cdot M$ and $\text{ad}X(M)$, respectively. In both cases, properties of the $i^{th}$ row simplify this description substantially. Our ultimate goal is to use these properties to show that Hessenberg varieties are paved by sequentially linear varieties.

As mentioned in Section 2.1, the exponential map on a nilpotent subalgebra $\mathfrak{n}$ of $g$ can be written $\exp(X) = \sum_{n \geq 0} \frac{(adX)^n}{n!}$ (see, e.g., [K, section 1.73]). In particular, the operator $adX$ can be viewed as an element of $\mathfrak{gl}(b)$ and so $\exp adX$ is in $GL(b)$. We write this map explicitly as

$$\exp(adX) = \sum_{n \geq 0} \frac{(adX)^n}{n!}$$

For any set $K$ of positive roots, the space $\mathfrak{n}_K = \bigoplus_{\alpha \in K} \mathfrak{g}_\alpha$ is a vector subspace of $b$ whose natural basis of root vectors extends to a basis for $b$. Denote the corresponding quotient map by $\rho_K : \mathfrak{b} \rightarrow \mathfrak{n}_K$. We may push $\rho_K$ forward to obtain the morphism

$$\rho^*_K : \text{End}(b) \rightarrow \text{Hom}(b, \mathfrak{n}_K).$$

Here and subsequently $\text{End}$ and $\text{Hom}$ refer to the underlying vector-space endomorphisms and homomorphisms of the Lie algebras. When $K = \Phi^i$ we abbreviate the projection to the $i^{th}$ row by $\rho_i$ and when $K = \{\alpha\}$ we write the projection to the root space $\mathfrak{g}_\alpha$ by $\rho_\alpha$. We also have occasion to write $\iota_K : \mathfrak{n}_K \hookrightarrow b$ for the natural vector space inclusion.

**Lemma 8.** Let $X$ be an element of $\mathfrak{n}_i$. In classical types the operator $(adX)^k$ in $\text{End}(b)$ is identically zero when $k \geq 3$. When $k \geq 1$,

$$\rho^*_i(adX)^k = 0 \text{ for all } j > i.$$ 

Furthermore,

$$\rho^*_i(adX)^2 = 0 \text{ in types } A_n, B_n, \text{ and } D_n.$$
and
\[ \text{Im}\rho_i^*(\text{ad}X)^2 \subseteq \mathfrak{g}_{\gamma_i} \text{ in type } C_n, \]
where \( \gamma_i \) is the unique long root in \( \Phi^i \).

**Proof.** Fix a set of generators \( \{S_1, \ldots, S_{rk}\} \) for the torus \( \mathfrak{h} \) in \( \mathfrak{b} \). The elements
\[ \{(\text{ad}X)^kE_\alpha : \alpha \in \Phi^+\} \cup \{(\text{ad}X)^kS_i : 1 \leq i \leq rk\} \]
generate the image \( \text{Im}(\text{ad}X)^k \). Using identity (1) repeatedly, we obtain
\[ (\text{ad}X)^kE_\alpha = \sum_{\beta_1 + \cdots + \beta_k + \alpha \in \Phi} \left( \prod_{j=1}^{k} x_{\beta_j} \right) E_{\sum \beta_j + \alpha}. \]

Similarly,
\[ (\text{ad}X)^kS_i = \sum_{\beta_1 + \cdots + \beta_k \in \Phi} \left( \prod_{j=1}^{k} x_{\beta_j} \beta_j(S_i) \right) E_{\sum \beta_j}. \]

Table 2.1 shows that no such \( \beta_1 + \cdots + \beta_k \) exist when \( k \) is at least three, either for Equation (2) or for Equation (3). It follows that \( (\text{ad}X)^k \) is identically zero when \( k \geq 3 \).

Lemma 2 and the definition of \( \mathfrak{h} \) show that \( [\mathfrak{n}_i, \mathfrak{b}] \subseteq \sum_{j \leq i} \mathfrak{n}_j \). Thus, when \( X \) is in \( \mathfrak{n}_i \) and \( j > i \) the operator \( \rho_i^*(\text{ad}X)^k \) is identically zero.

In types \( A_n, B_n, \) and \( D_n \) the Lie algebra \( \mathfrak{n}_i \) is abelian so
\[ \text{ad}X \left( \sum_{j \leq i} \mathfrak{n}_j \right) \subseteq \sum_{j < i} \mathfrak{n}_j \]
and \( \rho_i^*(\text{ad}X)^2 \) is identically zero on \( \mathfrak{b} \).

In type \( C_n \) each row is a Heisenberg Lie algebra so
\[ \text{ad}X \left( \sum_{j \leq i} \mathfrak{n}_j \right) \subseteq \sum_{j < i} \mathfrak{n}_j + \mathfrak{g}_{\gamma_i}. \]

\( \square \)

Several results follow. The following definitions are useful for notational brevity.

**Definition 9.** If \( M \) is an element of \( \mathfrak{g} \) write \( M = S_M + \sum_{\alpha \in \Phi_M} c_\alpha E_\alpha \) with the \( c_\alpha \) nonzero constants and with \( S_M \) in the fixed Cartan subalgebra \( \mathfrak{h} \). The set \( \Phi_M \) is the collection of roots associated to \( M \).

Given a subset \( u \subseteq \mathfrak{g} \), write \( \Phi_u = \bigcup_{M \in u} \Phi_M \).

In our applications \( M \) is in \( \mathfrak{b} \) and so \( \Phi_M \) is a subset of the positive roots. We also write \( \Phi_\pi \) for \( \Phi_{n_\pi} \), the roots associated to the parameterization \( U_\pi \) of the Schubert cell \( B\pi B \). These roots are more concisely defined as \( \Phi_\pi = \Phi^+ \cap \pi \Phi^- \) (see [12] sections 28.1 and 28.4). Similarly \( \Phi_{\pi,i} = \Phi_\pi \cap \Phi^i \) denotes the roots associated to \( n_{\pi,i} \).

**Corollary 10.** Fix \( X \) in \( \mathfrak{n}_i \). The operator \( \rho_j^* \exp \text{ad}X \) in \( \text{Hom}(\mathfrak{b}, \mathfrak{n}_j) \) satisfies the following:

1. If \( j > i \) then \( \rho_j^* \exp \text{ad}X = \rho_j \).
(2) If \( j = i \) then
\[
\rho_j^* \exp \ad X = \begin{cases} 
\rho_j + \rho_j^* \ad X & \text{in types } A_n, B_n, \text{ and } D_n, \\
\rho_i + \rho_i^* \ad X + \frac{(\ad X)^2}{2} & \text{in type } C_n.
\end{cases}
\]

(3) If \( j < i \) and \( M = S_M + \sum_{\alpha \in \Phi_M} c_\alpha E_\alpha \) is in \( b \) then
\[
\rho_j \exp \ad X(M) = \rho_j M + \sum_{\alpha \in \Phi_M \cap \Phi^j, \ad E_\alpha(n_i) \neq \{0\}} c_\alpha \left( \ad X(E_\alpha) + \frac{(\ad X)^2}{2}(E_\alpha) \right).
\]

Proof. Lemma \ref{lemma} and the explicit description of the exponentiation map show that
\[
\rho_j^* \exp \ad X = \begin{cases} 
\rho_j & \text{if } j > i, \\
\rho_j + \rho_j^* \ad X & \text{if } j = i \text{ in types } A_n, B_n, D_n, \text{ and} \\
\rho_j + \rho_j^* \ad X + \frac{\rho_j^* (\ad X)^2}{2} & \text{if } j = i \text{ in type } C_n.
\end{cases}
\]

Equations \ref{eq1} and \ref{eq2} complete the proof. \( \square \)

2.5. The Variety \( U(M, n_K) \). In this section we show that Hessenberg varieties are paved by sequentially linear varieties. The strategy is to intersect a fixed Hessenberg variety with a fixed Schubert cell and study its preimage in \( G \). We then identify a subvariety of the unipotent group in this preimage that is isomorphic to the original intersection of Hessenberg variety with Schubert cell. This subvariety of \( U \) will be sequentially linear.

Fix \( K \subseteq \Phi^+ \) and define \( n_K \) to be the subvariety of \( n \) given by \( n_K = \bigoplus_{\alpha \in K} g_\alpha \). We define \( C_K = \Phi^+ \setminus K \) to be the set of positive roots complementary to \( K \). Let \( M \) be an element of \( b \) and write \( U \) for the unipotent subgroup corresponding to the nilradical \( n \). Define
\[
U(M, n_K) = \{ u \in U : \Ad u^{-1}(M) \in n_K \}.
\]

Recall that \( \gamma_i \) denotes the longest root in \( \Phi^i \) in type \( C_n \).

Theorem 11. Fix \( K \subseteq \Phi^+ \).

The decomposition into rows defines a sequentially linear structure on \( U(M, n_K) \) in types \( A_n, B_n, \text{ and } D_n \).

In type \( C_n \), suppose that whenever \( \gamma_i \notin K \) and \( \rho_{\gamma_i}(u^{-1} \cdot M) \neq 0 \) for at least one \( u \) in \( U \) then either
\[
(1) \quad \gamma_i(S_M) \neq 0 \quad \text{or} \quad (2) \quad \text{both } \rho_{\gamma_i}(M) \neq 0 \quad \text{and} \quad (\gamma_i - \alpha_i)(S_M) = 0.
\]

Let \( P_1^i = \Phi^i - \{ \gamma_i \} \) and \( P_2^i = \Phi^i - \{ \gamma_i - \alpha_i, \gamma_i \} \). The refinement of the decomposition into rows whose \( 2i + 1 \text{th} \) part is \( P_1^i \) and whose \( 2i \text{th} \) part is \( \Phi^i - P_1^i \) defines a sequentially linear structure on \( U(M, n_K) \) when condition \( (j) \) holds, for \( j = 1 \) or \( j = 2 \).

Proof. Define \( U^i = \prod_{j \geq i} U_j \) and \( n_{K_i} = (n_K \cap (\bigoplus_{j \geq i} n_j)) \oplus \bigoplus_{j < i} n_j \). There is a natural projection
\[
p_i : \quad U^i \longrightarrow U^{i+1}
\]
where elements of \( U^i \) and \( U^{i+1} \) are expressed as the ordered product of elements in decreasing rows.
Let $u$ be in $U^i$. The calculations of Corollary 10 show that $\rho_j(u^{-1} \cdot M) = \rho_j(p_i(u)^{-1} \cdot M)$ for all $j > i$. Thus, the map $p_i$ restricts to a projection

$$p_i : U^i \cap U(M, n_{K_i}) \longrightarrow U^{i+1} \cap U(M, n_{K_{i+1}}).$$

We now inspect each fiber of this projection to ensure that it is an affine space. Observe that $u$ is in $U^i \cap U(M, n_{K_i})$ if and only if both $p_i(u)$ is in $U^{i+1} \cap U(M, n_{K_{i+1}})$ and $\rho_i(u^{-1} \cdot M) \in n_K \cap n_i$. Fix $u'$ in $U^{i+1} \cap U(M, n_{K_{i+1}})$ and write $M_i = (u')^{-1} \cdot M$. Also write $u = u'u_i$ so that $u^{-1} \cdot M = u_i^{-1} \cdot M_i$. Note that there exists a unique $X_i \in n_i$ such that $u_i^{-1} = \exp X_i$ as described in Section 2.1. Also note that

$$u_i^{-1} \cdot M_i = \exp X_i(M_i) = \exp \text{ad} X_i(M_i)$$

by [K] section 1.93.

In types $A_n$, $B_n$, and $D_n$, Corollary 10 expands the projection $\rho_i(u_i^{-1} \cdot M_i)$ explicitly as the expression

$$\rho_i(u_i^{-1} \cdot M_i) = \rho_i \exp \text{ad} X_i(M_i) = \rho_i M_i + \rho_i \text{ad} X_i(M_i).$$

Since $\exp \text{ad} X_i(M_i) = -\text{ad} M_i(X_i)$, the second term is a linear function of $X_i$. The first term simply translates by the vector $\rho_i M_i$. In other words, the set $\{u_i \in U_i : \rho_i(u_i^{-1} \cdot M_i) \in n_K \cap n_i\}$ describes an affine subspace of $U_i$ for each fixed $u'$ in $U^{i+1} \cap U(M, n_{K_{i+1}})$.

This proves the claim in those cases.

In type $C_n$, define

$$V_j^i = \left( \prod_{\beta \in P_j^i} U_\beta \right) \cap U(M, n_{K_i} \oplus n_\gamma).$$

In each case we refine the tower of morphisms to include

$$U^i \cap U(M, n_{K_i}) \xrightarrow{p_\alpha} V_j^i \xrightarrow{p_i} U^{i+1} \cap U(M, n_{K_{i+1}}).$$

Then $\rho_\alpha$ gives an affine transformation on $V_j^i$ of the part of the $i^{th}$ row corresponding to $P^i_j$ for each $\alpha$ in $\Phi^+ \setminus \{\gamma_i\}$. Likewise, the function $\rho_\gamma$ is an affine transformation in the entries corresponding to $\Phi^+ \setminus P_j^i$ over the entries already fixed in $V_j^i$. Both follow from Corollary 10 and together prove the claim.

The following lemma relates the varieties $U(M, n_H)$ to Hessenberg varieties.

**Lemma 12.** Let $\mathcal{H}(M, H) \cap B\pi B$ be the intersection of the Schubert cell corresponding to $\pi$ with the Hessenberg variety $\mathcal{H}(M, H)$. Then

$$\mathcal{H}(M, H) \cap B\pi B \cong U_\pi \cap U(M, n_{\pi \cdot H})$$

for $n_{\pi \cdot H} = n \cap (\pi \cdot H)$. Consequently, the intersection of each Hessenberg variety with each Schubert cell is sequentially linear in the classical types.

**Proof.** The Schubert cell $B\pi B$ is isomorphic to $U_\pi$ as discussed in [FH] page 396]. Since the projection from $G(M, H)$ to the Hessenberg variety $\mathcal{H}(M, H)$ is $\text{Ad}(B)$-invariant, this isomorphism restricts to the intersection

$$\mathcal{H}(M, H) \cap B\pi B \cong U_\pi \cap U(M, n_{\pi \cdot H}).$$
Explicitly, if we write $gB$ for the flag corresponding to $g$ we see that

$$gB \in \mathcal{H}(M, H) \iff (gb)^{-1} \cdot M \in H \text{ for all } b \in B \iff (u\pi)^{-1} \cdot M \in H \text{ for } u\pi B = gB, u \in U_{\pi} \iff u \in U(M, n_{\pi-H}) \cap U_{\pi}.$$  

The conclusion follows from Theorem 11 and the definition of the Hessenberg space $H$. □

The following theorem summarizes these results.

**Theorem 13.** Suppose there exists a Borel subgroup $B$ such that $U_{\pi} \cap U(M, n_{\pi-H}) \cong \{ \emptyset \} \text{ for some } d$ for each $\pi$ in $W$. Then the paving of $\mathcal{H}(M, H)$ obtained by intersecting the Hessenberg variety with Schubert cells is a paving by affines such that the cell $\mathcal{H}(M, H) \cap B\pi B$ has dimension $d$.

**Proof.** The Schubert cells $\mathcal{H}(M, H) \cap B\pi B$ pave each Hessenberg variety as per the comments in Section 2.2. Also, $\mathcal{H}(M, H) \cap B\pi B$ is isomorphic to $U_{\pi} \cap U(M, n_{\pi-H})$ by Lemma 12. If the hypotheses hold then the Bruhat decomposition actually gives a paving by affines of $\mathcal{H}(M, H)$ and the dimension of $\mathcal{H}(M, H) \cap B\pi B$ equals that of $U_{\pi} \cap U(M, n_{\pi-H})$ for each $\pi$. □

For convenience, we remark that the Adjoint action of $G$ gives an action of $G$ on Hessenberg varieties defined by $\text{Ad} \ g^{-1} (\mathcal{H}(M, H)) = \mathcal{H}(g^{-1} \cdot M, g^{-1} \cdot H)$. The action $\text{Ad} \ g^{-1}$ is an isomorphism of Hessenberg varieties and so all Hessenberg varieties in a fixed $G$-orbit are isomorphic to each other. We state this as a lemma though the proof is immediate.

**Lemma 14.** If $g \in G$ then $\mathcal{H}(M, H) \cong \mathcal{H}(g^{-1} \cdot M, g^{-1} \cdot H)$. 

Note that $\mathcal{H}(M, H)$ is not independent of the choice of Borel subalgebra $\mathfrak{b} \subseteq H$. Indeed, the definition of the Hessenberg space $H$ requires that $\text{ad} \mathfrak{b}(H) \subseteq H$. This is not generally true of Borel subalgebras contained in $H$.

We interpret the choice of a Borel $B$ in Theorem 13 as fixing a basis for the flag variety $G/B$ with respect to which we consider $\mathcal{H}(M, H)$. We will use this in Chapter 3 to select a computationally convenient form of $M$ from its $G$-orbit.

2.6. **Criteria for $U(M, n_{\pi-H}) \cap U_{\pi}$ to be an affine space.** With certain extra assumptions on $M$, the variety $U(M, n_{\pi-H})$ will be not just sequentially linear but will also intersect the closed subgroup $U_{\pi}$ in an affine space. Theorem 13 will then imply that $\mathcal{H}(M, H)$ is paved by affines whose dimensions we can identify.

Let $M$ be an element of $\mathfrak{b}$ written

$$M = S_M + \sum_{\beta \in \Phi_M} m_\beta E_\beta = S_M + N$$

for nonzero constants $m_\beta$, a semisimple element $S_M$ in $\mathfrak{h}$, and a nilpotent $N$ in the nilradical $n$. 

Definition 15. A collection of roots $P$ is non-overlapping if for no pair $\alpha, \beta$ in $P$ is $\alpha > \beta$.

If $M = S_M + N$ is written as above then $M$ is non-overlapping if both of the following hold:

1. $\Phi_M$ is non-overlapping.
2. For each $\alpha \in \Phi_M$ and each simple root $\alpha_i$ with $\alpha \geq \alpha_i$ the equality $\alpha_i(S_M) = 0$ holds.

By an abuse of notation, we call the roots $\Phi_M$ non-overlapping if $M$ is non-overlapping. Note that the second condition implies that $\beta(S_M) = 0$ for each $\beta \leq \alpha$ and each $\alpha \in \Phi_N$. Consequently, the Lie algebra elements $S_M$ and $N$ commute. However, the requirement that $\text{ad}S_M(N) = 0$ is not sufficient to ensure that the second condition holds. For instance, in $\mathfrak{gl}_3$ the element

$$M = \begin{pmatrix} x & 0 & 1 \\ 0 & y & 0 \\ 0 & 0 & x \end{pmatrix}$$

is not non-overlapping while

$$M' = \begin{pmatrix} x & 1 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}$$

is non-overlapping. Not only are $M$ and $M'$ in the same $G$-orbit but they both have a Jordan decomposition into diagonal and non-diagonal parts. Thus, if $M$ is non-overlapping then $M = S_M + N$ as above is a Jordan decomposition but not vice-versa.

Lemma 16. Let $M$ be in $\mathfrak{b}$ and $u$ be in $U$. If $\Phi_M$ is a non-overlapping set of roots then $\Phi_M \subseteq \Phi_{u^{-1} \cdot M}$ and $\Phi_M \cap \Phi_{u^{-1} \cdot M - M}$ is empty.

Proof. Let $c_\alpha$ be nonzero constants so that

$$u^{-1} \cdot M = S_M + \sum_{\alpha \in \Phi_{u^{-1} \cdot M}} c_\alpha E_\alpha.$$

Fix $\alpha$ in $\Phi_M$. Write $u = u_n u_{n-1} \cdots u_1$ for each $u_i \in U_i$ and apply the conclusions of Corollary 10 repeatedly to $u^{-1} \cdot M = (\prod_{i=1}^n \exp X_{u_{n-i+1}})^{-1} \cdot M$. For our purposes, we need only the result that the coefficient $c_\alpha$ is the sum of the following three quantities. The first is $m_\alpha$. The second is a sum of terms of the form $\prod_{i=1}^r u_\beta_i \beta_i(S_M)$ for $r$-tuples of $\beta_i \in \Phi^+$ such that $\beta_1 + \cdots + \beta_r = \alpha$ and constant coefficients $u_{\beta_i}$. Regardless of $r$ or the choice of $\beta_i$, Condition (2) in the definition of non-overlapping ensures that this quantity is zero. The third quantity is a sum of terms of the form $m_{\beta_0} \prod_{i=1}^r u_\beta_i$ for $r$-tuples of $\beta_i \in \Phi^+$ such that $\beta_0 + \beta_1 + \cdots + \beta_r = \alpha$ and constants $u_{\beta_i}$. Again by definition of non-overlapping, this quantity is zero. Consequently the term $c_\alpha$ is simply $m_\alpha$.

Corollary 17. If $\Phi_M$ is non-overlapping and $\Phi_M \not\subseteq K$ then $U(M, n_K) = \emptyset$.

Proof. If $u$ is in $U$ then $\Phi_M \subseteq \Phi_{u^{-1} \cdot M}$ by the previous lemma. Since $\Phi_M \not\subseteq K$ the element $u^{-1} \cdot M$ cannot be in $n_K$. 

\end{document}
With certain hypotheses we can reduce to the study of nilpotent Hessenberg varieties. For each semisimple element $S \in \mathfrak{h}$, write $\Phi^+_S$ for the set of roots $\Phi^+_S = \{ \alpha \in \Phi^+ : \alpha(S) = 0 \}$. Let $\Delta_j$ be the maximal irreducible subsets of $\Delta \cap \Phi^+_S$. Denote the parabolic subalgebra associated to the simple roots $\Delta_j$ by $\mathfrak{p}_{\Delta_j}$ and choose its associated Levi part $\mathfrak{l}_{\Delta_j}$, so that $\mathfrak{l}_{\Delta_j} \supseteq \mathfrak{h}$. Let $\mathfrak{U}_{\Delta_j}$ be $\mathfrak{L}_{\Delta_j} \cap \mathfrak{U}$. Write $\mathfrak{p}_M$ for the parabolic subalgebra associated to $\bigcup \Delta_j$ and write $\mathfrak{l}_M$ and $\mathfrak{n}_M$ for its associated Levi and nilpotent parts. Recall that $\mathcal{C}_{\pi, H}$ is the set of positive roots complementary to $\pi \mathcal{M}_H \cap \Phi^+$.

**Theorem 18.** Let $M = \sum M_j$ be an element of $\mathfrak{b}$ and write the decomposition of each summand as $M_j = S_j + N_j$ for $S_j$ in $\mathfrak{h} \cap \mathfrak{l}_{\Delta_j}$ and $N_j$ in $\mathfrak{n}$. Let $S = \sum S_j$ be the semisimple part of $M$. Assume that the following conditions hold:

1. The set of roots $\Phi^+_S$ is the union $\bigcup \text{span}(\Delta_j)$, each $N_j$ is in the corresponding $\mathfrak{l}_{\Delta_j}$, and $\Phi_{M_j}$ is non-overlapping.
2. The variety $U(N_j, \mathfrak{l}_{\Delta_j} \cap \mathfrak{n}_{\pi, H}) \cap U_{\Delta_j} \cap \mathfrak{U}_{\pi}$ has a sequentially linear structure which refines the decomposition into rows of Theorem 11 and which is constant rank of total dimension $n_j$.

Then the variety $U(M, \mathfrak{n}_{\pi, H}) \cap \mathfrak{U}_{\pi}$ is nonempty if and only if $\pi^{-1} \cdot N_j$ is in $\mathfrak{H}$ for each $j$. If nonempty, it is an affine space of dimension $|\Phi_{n_M} \cap \Phi_{\pi} \cap \pi \mathcal{M}_H| + \sum n_j$.

**Proof.** Choose any $u$ in $\mathfrak{U}$. Consider the operator

$$\rho_{\mathfrak{n}_M \cap \mathfrak{h}, \mathfrak{l}_M \cap \mathfrak{n}, \mathfrak{ad}}(u^{-1} \cdot M) \in \text{End}(\mathfrak{n}_M \cap \mathfrak{n})$$

written with respect to the basis of root vectors in $\mathfrak{n}_M$ ordered by height. In type $D_n$, fix any order among root vectors in $\mathfrak{n}_i$ of the same height. In type $C_n$, use the refinement of the row decomposition described in Case (1) of Theorem 11. We saw that this operator is an affine transformation in Theorem 11. Here we show that it has the same dimension for any $u \in \mathfrak{U}_{\pi}$.

The operator $\text{ad}(u^{-1} \cdot M)$ acts by dilation by the nonzero constant $\alpha(S)$ on each root vector $E_\alpha$ in the nilpotent subalgebra $\mathfrak{n}_M$. This follows since the $\Delta_j$ generate both the roots associated to $\mathfrak{l}_M$ and $\Phi^+_S$. Moreover, the operator $\rho_{\mathfrak{n}_M \cap \mathfrak{n}, \mathfrak{ad}}(u^{-1} \cdot M)$ is identically zero on $\mathfrak{g}_\alpha$ for each root $\beta \not\leq \alpha$ in $\Phi^+$. This shows that the operator $\rho_{\mathfrak{n}_M \cap \mathfrak{n}, \mathfrak{l}_M \cap \mathfrak{n}, \mathfrak{ad}}(u^{-1} \cdot M)$ is a lower triangular matrix with nonzero entries $\alpha(S)$ along the diagonal with respect to the basis defined above.

It follows that $U(M, \mathfrak{n}_{\pi, H}) \cap \exp \mathfrak{n}_M$ is a constant rank sequentially linear variety, which is to say an affine space of dimension

$$|\Phi_{n_M} \cap \Phi_{\pi}| - |\Phi_{n_M} \cap \Phi_{\pi} \cap C_{\pi, H}| = |\Phi_{n_M} \cap \Phi_{\pi} \cap \pi \mathcal{M}_H|.$$

If $E_\alpha$ is in $\mathfrak{l}_{\Delta_j} \cap \mathfrak{n}$ then $\alpha$ is generated by $\Delta_j$ by Equations (2) and (3). Thus, the hypotheses on $S_j$ imply that the operators $\rho_{\mathfrak{l}_{\Delta_j} \cap \mathfrak{n}, \mathfrak{ad}}(u^{-1} \cdot M_j)$ and $\rho_{\mathfrak{l}_{\Delta_j} \cap \mathfrak{n}, \mathfrak{ad}}(u^{-1} \cdot N_j)$ agree on the root space $\mathfrak{g}_\alpha$. Consequently, the variety $U(N_j, \mathfrak{l}_{\Delta_j} \cap \mathfrak{n}_{\pi, H}) \cap U_{\Delta_j} \cap \mathfrak{U}_{\pi}$ is a constant rank sequentially linear variety if and only if the variety $U(M_j, \mathfrak{l}_{\Delta_j} \cap \mathfrak{n}_{\pi, H}) \cap U_{\Delta_j} \cap \mathfrak{U}_{\pi}$ is. If so, their dimensions are the same.

Furthermore,

$$\rho_{\mathfrak{l}_{\Delta_j} \cap \mathfrak{n}, \mathfrak{ad}}(u^{-1} \cdot \left( \sum_j N_j \right)) = \rho_{\mathfrak{l}_{\Delta_j} \cap \mathfrak{n}, \mathfrak{ad}}(u^{-1} \cdot N_j)$$

DECOMPOSING HESSENBURG VARIETIES 15
as operators in $\text{Hom}(n, L_\Delta \cap n)$. It follows that $U(M, L_\Delta \cap n_{\pi \cdot H}) \cap U_\Delta \cap U_\pi$ is a constant rank sequentially linear variety if and only if each $U(N_j, L_\Delta \cap n_{\pi \cdot H}) \cap U_\Delta \cap U_\pi$ is. The dimension of the former is the sum of the dimensions of the latter.

Take the sequentially linear structure on $U(M, n_{\pi \cdot H}) \cap U_\pi$ obtained by refining the row decomposition of Theorem 11 to first $I_M \cap n$, and then $n_M \cap n$. The arguments above show that this variety is constant rank. It follows that the variety is an affine space. The total dimension of $U(M, n_{\pi \cdot H}) \cap U_\pi$ is obtained by summing the dimensions restricted to $n_M$ and each $I_\Delta \cap n$.

We now establish conditions under which a nilpotent Hessenberg variety intersects Schubert cells in affine spaces.

**Lemma 19.** Suppose that $P$ and $P'$ are subsets of $\Phi^+$ and suppose that each root $\alpha$ in $P'$ has an extremal root $\beta$ in $P$.

Let $\pi$ be a Weyl group element and let $H$ be a Hessenberg space in $g$ with roots $M_H$ such that $\pi P \subseteq M_H$.

If $\alpha$ is a root in $P'$ such that

$$\pi \alpha \in \Phi - M_H$$

then there exists a root $\beta$ in $P$ satisfying

$$\alpha - \beta \in \Phi_\pi.$$

**Proof.** If $\beta$ is an extremal root for $\alpha$ then $\alpha - \beta$ is a positive root. Let $\beta$ in $P$ be extremal for $\alpha$. Note that $\pi \alpha = \pi(\alpha - \beta) + \pi \beta$ is not in $M_H$ by hypothesis. Since $\pi \beta$ is in $M_H$ and since $M_H$ is closed under addition by positive roots, the root $\pi(\alpha - \beta)$ must be negative. This means that $\alpha - \beta$ is in $\Phi_\pi$. □

We are building to a lemma that describes one set of conditions under which the variety $U(N, n_{\pi \cdot H}) \cap U_\pi$ is an affine space when $N$ is nilpotent.

Given a set $P \subseteq \Phi^+$ define

$$P^i_{\pm \pi N} = \{\alpha - \beta : \alpha \in P, \beta \in \Phi_N, \alpha - \beta \in \Phi^i\}$$

and define $P^+_{\pi N}$ to be the roots $\alpha + \beta$ satisfying the analogous conditions. We often suppress the superscript and write $P_{\pm \pi N}$ if $P$ is a subset of $\Phi^i$.

**Lemma 20.** Let $\Phi_N$ be a non-overlapping set of roots. Assume that $\Phi^i_{\pm \pi N}$ is either empty or $\Phi^j_i$ for some $j$. When $|\Phi^i_k| = 2$, also assume that $\Phi^i_k = \Phi^i_{k+1, \pi N} = \Phi^i_{k+1, \pi N}$.

Suppose that $\{\alpha\}_{\pi \cdot N}$ is nonempty for each $\alpha$ in $\Phi^i_{\pi \cdot U_{\pi \cdot H} \cap C_{\pi \cdot H}}$ and that if $\alpha = \sum_{j \in \Phi_{\pi \cdot N}} \beta_j$ then each $\beta_j \leq \{\alpha\}_{\pi \cdot N}$. Then $U(N, n_{\pi \cdot H}) \cap U_\pi$ is an affine space of dimension

$$|\Phi_\pi| - |C_{\pi \cdot H} \cap U_{\pi \cdot N}|.$$

**Proof.** Let $P_{\pi \cdot H, \pi \cdot N}$ denote the set $(\Phi^i_{\pi \cdot C_{\pi \cdot H}})^{-}_{\pi \cdot N}$. Consider the map

$$N' = \left( \rho_{\Phi^i_{\pi \cdot U_{\pi \cdot H} \cap C_{\pi \cdot H}}} \right) \left( \iota_{P_{\pi \cdot H, \pi \cdot N}} \right) \text{ad}(u^{-1} \cdot N) \in \text{Hom}(n_{P_{\pi \cdot H, \pi \cdot N}}, n_{\Phi^i_{\pi \cdot U_{\pi \cdot H} \cap C_{\pi \cdot H}}}$$

as it acts with respect to a basis of root vectors ordered by height. We will write the matrix for $N'$ explicitly and show that its rank is independent of the choice of $u$ so long as $u \in U_\pi$. In Theorem 11 we gave a sequentially linear structure for $U(N, n_{\pi \cdot H})$ by showing that certain projections $\rho_i(u_i^{-1} \cdot N_i)$ were affine transformations for each $i$. Since $N'$ is the linear part of this affine transformation, its rank
is constant on $U(N, n_{x, H}) \cap U_{x}$ if and only if $U(N, n_{x, H}) \cap U_{x}$ is a constant rank sequentially linear variety, i.e., an affine space.

First we characterize the matrix for $N'$. Order the rows by height according to $\Phi_{k}$, fixing an order if $|\Phi_{k}| = 2$. Note that if $\alpha$ is in $\Phi_{i} \cap C_{\pi H}$ then there exists $\alpha'$ in $\Phi_{N}$ such that $\alpha - \alpha'$ is in $\Phi_{x}$ by Lemma 14. Consequently, the columns in the matrix are indexed by elements of $\Phi_{i}$. Order the columns by increasing $k$ in $\Phi_{i}^{k}$ using the order inherited from the associated rows of the matrix if $|\Phi_{k}| = 2$.

We begin by showing that the order on the columns is consistent with ordering the columns by height. Suppose $\alpha$ is in $\Phi_{k}$ and $\beta$ is in $\Phi_{j}$, that $\alpha > \beta$, and that neither $\{\alpha\}_{-N}$ nor $\{\beta\}_{-N}$ is empty. By the verticality of the rows, we know each element of $\Phi_{k}$ is greater than each element of $\Phi_{j}$. Since neither $\Phi_{i}^{k}$ nor $\Phi_{i}^{j}$ is empty, we have $\Phi_{i}^{k}$ and $\Phi_{i}^{j}$ comparable. Let $\alpha'$ and $\beta'$ be in $\Phi_{N}$ with the property that $\alpha - \alpha' \in \Phi_{i}^{k}$ and $\beta - \beta' \in \Phi_{i}^{j}$. The sets $\Phi_{i}^{k}$ and $\Phi_{i}^{j}$ are comparable by definition of verticality. Moreover, 

\[(\alpha - \alpha') - (\beta - \beta') = (\alpha - \beta) + (\beta' - \alpha') > 0\]

since $\Phi_{N}$ is non-overlapping. Thus each element of $\Phi_{i}^{k}_{-N}$ is greater than $\Phi_{i}^{j}_{-N}$, which shows that the order on the columns previously defined is in fact the order by height (when $|\Phi_{k}| \neq 2$).

There is no $\beta < \alpha'$ with $\beta$ in $\Phi_{u, N}$ since that would imply there were a $\beta'$ in $\Phi_{N}$ with $\beta' < \beta < \alpha'$, which contradicts the definition of non-overlapping. The operator $\rho_{u, \alpha}(x(u^{-1} \cdot N))$ scales $g_{\alpha - \beta'}$ by $n_{\alpha'} \neq 0$ as shown in Lemma 16. In sum,

\[\rho_{u, \alpha - \beta'}(x(u^{-1} \cdot N)) = \begin{cases} 0 & \beta < \alpha' \\ n_{\alpha'} & \beta = \alpha'. \end{cases}\]

It follows that the matrix for $N'$ is lower triangular with respect to this basis.

If no $|\Phi_{k}| = 2$ then this matrix is nonzero along its diagonal. This proves the claim in root systems for which $x(u^{-1} \cdot N)X_{i}$ is an affine transformation and no $|\Phi_{k}| = 2$, namely in types $A_{n}$ and $B_{n}$.

In type $C_{n}$, the operators $\rho_{u, \alpha}(x(u^{-1} \cdot N))$ is not an affine transformation. However, the projection $\rho_{u, \alpha}(x(u^{-1} \cdot N))$ is affine for each $\alpha \neq \gamma_{i}$. Moreover, the non-affine function $(x(u^{-1} \cdot X))^{2}$ satisfies $\rho_{u, \alpha}(x(u^{-1} \cdot X))^{2} = \rho_{u, \alpha}(x(u^{-1} \cdot \rho_{\beta\gamma_{i} < \beta < \alpha'})X)$ for all $X$ in $n_{i}$. In other words, the map $\rho_{u, \alpha}(x(u^{-1} \cdot \gamma_{i} \cdot M))$ is linear and a matrix of constant rank. Using the sequentially linear structure of Theorem 11, the claim follows in type $C_{n}$, as well.

When there exists $\Phi_{k} = \{\beta_{1}, \beta_{2}\}$, note that

\[
\left(\frac{1}{n_{\beta_{1}}} \rho_{u, \beta_{1}} - \frac{1}{n_{\beta_{2}}} \rho_{u, \beta_{2}}\right) x(u^{-1} \cdot N)
\]

is an affine transformation of the root vectors corresponding to $\Phi^{k+1}_{k-1}$ that depends on the choices for the root vectors corresponding to $\cup_{j<k} \Phi_{j}^{j} \cup \cup_{j>k} \Phi_{j}^{j}$. This transformation must be identically zero if $\Phi_{k} \subseteq C_{\pi H}$. Moreover, this affine transformation is linearly independent from $\rho_{u, \alpha}x(u^{-1} \cdot N)$ for $\alpha$ in $\Phi^{k+1}_{k}$. The appropriate modification of the decomposition given in Theorem 11 proves the claim in type $D_{n}$.
3. The Main Theorems

We now give a paving by affines for many Hessenberg varieties by intersecting the Hessenberg variety with the cells of an appropriately chosen Schubert decomposition of \(G/B\). The first section describes Hessenberg varieties in type \(A_n\) and uses certain subsets of positive roots to parametrize the cells of the paving and to give their dimension. The second section also discusses the paving of Hessenberg varieties in type \(A_n\) but describes this paving using the combinatorics of Young tableaux rather than root systems. Section 3.2 contains an extended description of the Peterson variety to demonstrate how these results can be used computationally. The third and fourth sections describe semisimple and regular nilpotent Hessenberg varieties in classical types.

We make several comments which apply to all of the following results. First, the choice of Schubert decomposition for \(\mathcal{H}(M, H)\) is independent of \(M\). In particular, the partial order on Hessenberg spaces gives rise to filtrations of each Schubert cell into affine subspaces via \(B\pi B \cap \mathcal{H}(M, H) \subseteq B\pi B \cap \mathcal{H}(M, H')\) if \(H \subseteq H'\).

That each cell \(B\pi B \cap \mathcal{H}(M, H)\) of the paving is an affine space over the base field \(\mathbb{C}\) implies the following result.

**Proposition 21.** There is no odd-dimensional cohomology for Hessenberg varieties in type \(A_n\) and for semisimple and regular nilpotent Hessenberg varieties in the other classical types.

Note that while this paving can help identify the dimension of various Hessenberg varieties, the question of whether all nilpotent Hessenberg varieties are equidimensional remains open.

### 3.1. Hessenberg Varieties in Type \(A_n\).

We begin with nilpotent Hessenberg varieties and build to general Hessenberg varieties in type \(A_n\).

**Theorem 22.** Let \(G = GL_n(\mathbb{C})\) or \(SL_n(\mathbb{C})\) and let \(N\) be a fixed nilpotent in \(n\). Let \(B'\) be the Borel subgroup constructed by considering all upper triangular matrices in \(G\) with respect to a basis which puts \(N\) in Jordan canonical form.

There exists a permutation \(\sigma\) such that the Borel \(B = \sigma^{-1} \cdot B'\) induces a Bruhat decomposition whose Schubert cells intersect each Hessenberg variety \(\mathcal{H}(N, H)\) in a paving by affines. The nonempty cells are \(B\pi B \cap \mathcal{H}(M, H)\) with \(\pi^{-1} \cdot N \in H\) and have dimension \(|\Phi_\pi| - |C_{\pi, H} \cap \Phi_{U^+ \cdot N}|\).

**Proof.** To construct this Bruhat decomposition, first fix a basis with respect to which \(N\) is in Jordan canonical form. Order the Jordan blocks from smallest to largest, fixing an order among equal-dimensional Jordan blocks once and for all. Then permute this basis according to the following rules. Index the basis vectors in \(\ker N\) from \(e_1\) to \(e_{|\ker N|}\) according to the order of the Jordan blocks containing \(e_i\) so that \(e_1\) belongs to the smallest Jordan block and \(e_{|\ker N|}\) to the largest Jordan block. Given an ordering of the basis vectors in \(\ker N^j\), index the basis vectors in \(\ker N^{j+1}/\ker N^j\) from \(e_{|\ker N^j|+1}\) to \(e_{|\ker N^{j+1}|}\) increasing the index according to the dimension of the Jordan block in which each \(e_i\) is contained.

To show that \(\Phi_N\) satisfies the desired conditions, we need to describe more precisely this basis. Suppose that the nilpotent \(N\) lies in the conjugacy class corresponding to the partition \(\mu = (\mu_1, \ldots, \mu_s)\) with the parts ordered so that \(\mu_i \geq \mu_{i+1}\). Let \(\mu' = (\mu'_1, \ldots, \mu'_{s'})\) be the dual partition associated to \(\mu\).
Define a function \( w : \{1, \ldots, n\} \rightarrow \{1, \ldots, s'\} \) by
\[
\sum_{k=1}^{w(i)-1} \mu_k' \leq i < \sum_{k=1}^{w(i)} \mu_k' \quad \text{for all } i \in \{1, \ldots, n\}.
\]
Note that \( w \) is a nonincreasing function, that is \( w(i) \geq w(i+1) \).
Define the roots
\[
\beta_i = \sum_{j=0}^{\mu_{w(i)}'-1} \alpha_{i-j}
\]
for \( \mu_1' \leq i < n \). These roots have the following two properties:
1. Either \( \beta_i - \alpha_i \in \Phi^+ \) or \( \beta_i = \alpha_i \).
2. \( \beta_i \geq \alpha_j \implies j \leq i \).

In addition, the sequence \( \{\text{ht} \beta_i\} \) is nonincreasing. Consequently, the set \( \{\beta_i : \mu_i' \leq i < n\} \) is non-overlapping.

There is a root space decomposition of \( \mathfrak{g} \) so that \( N = \sum_{i=1}^{n-1} E_{\beta_i} \). In fact, writing \( \mathfrak{g} \) with respect to the basis specified at the outset of the proof gives such a root space decomposition. By Lemma 14 the isomorphism class of the Hessenberg variety is invariant under the choice of basis. The set \( \Phi_N \) equals \( \{\beta_i\} \) for this decomposition.

To see that \( \Phi_N \) satisfies the conditions of Corollary 20 note that if \( \text{ht} \beta_{i+k-1} < k \) then \( \Phi_{k-N} = \Phi_j \) for \( j = k - \text{ht} \beta_{i+k-1} \). If not then \( \Phi_{k-N} \) is empty since \( \Phi_N \) is non-overlapping. Moreover, for \( \alpha \) to be in \( \Phi_{U,N} \cap \Phi_{k} \) means that there exists \( \beta_j \) in \( \Phi_N \) with \( \alpha \geq \beta_j \). Since \( \Phi_N \) is non-overlapping, we further conclude that \( \alpha \geq \beta_{i+k-1} \).

It follows that \( \{\alpha\}^{i-N} \) is nonempty if \( \alpha \) is in \( \Phi^i \cap \Phi_{U,N} \cap C_{\pi,H} \).

Either the hypotheses of Corollary 17 hold or those of Lemma 20 hold. The claims then follow from Theorem 13.

This result extends to general linear operators \( M \) by use of Theorem 15.

**Theorem 23.** Given a linear operator \( M \), there exists a Bruhat decomposition whose intersection with each Hessenberg variety \( \mathcal{H}(M,H) \) is a paving by affines of \( \mathcal{H}(M,H) \). Its nonempty cells are \( B_{\pi} B \cap \mathcal{H}(M,H) \) such that \( \pi^{-1} \cdot M \in H \) and have dimension
\[
|\Phi_\pi| - |\Phi_{U_{\pi,M}(M-s)} \cap C_{\pi,H}| - |\Phi_\pi \cap \Phi_{\pi,M} \cap C_{\pi,H}|.
\]

**Proof.** Choose a basis that puts \( M \) in Jordan canonical form. Permute the basis vectors within each generalized eigenspace so that the nilpotent part of \( M \) on each Jordan block is in the form specified by Theorem 22. Write \( M = \sum(S_j + N_j) \) where \( S_j \) is diagonal, \( N_j \) is nilpotent, and \( S_j + N_j \) is the \( j \)th Jordan block of \( M \).

Again using Lemma 14 we see that the isomorphism class of the Hessenberg variety \( \mathcal{H}(M,H) \) is preserved under this change of basis.

If \( \Delta_j \) are the simple roots whose root vectors generate the \( j \)th Jordan block then \( \Phi^+_\sum S_j = \text{span}(\bigcup \Delta_j) \). In particular, each \( S_j + N_j \) is non-overlapping since \( \Phi_{N_j} \) is contained in span \( \Delta_j \). The conditions of Part (1) in Theorem 15 are thus met.

As usual, use \( p_{\Delta_j} \) to denote the parabolic subalgebra associated to \( \Delta_j \) and \( l_{\Delta_j} \) to denote its Levi subalgebra. Write \( p_M \) for the parabolic associated to \( \bigcup \Delta_j \) and \( n_M \) (respectively \( l_M \)) for its nilpotent (respectively Levi) part. All Levi subalgebras are assumed to contain the diagonal matrices.

By Theorem 22 the variety \( U(N_j, l_{\Delta_j} \cap n_{\pi,H}) \cap U_\pi \cap U_{\Delta_j} \) is nonempty only if \( \pi^{-1} \cdot N_j \) is in \( H \). In that case the variety is affine space of dimension \( |\Phi_\pi \cap \text{span} \Delta_j| - \)

This ensures the conditions of Part (2) of Theorem \ref{thm:main}. The claim follows by summing over $j$. \hfill $\square$

3.2. Combinatorial Description of the Paving. Let $E_{ij}$ be the standard basis for $\mathfrak{gl}_n(\mathbb{C})$ in which $E_{i,i+1}$ corresponds to the root vector $E_{\alpha_i}$. This basis can be used to construct a bijection between the set of Hessenberg spaces $H$ and functions $h : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ such that

$$h(i) \geq \max\{i, h(i-1)\}.$$ Explicitly, the element $E_{ij}$ is in $H$ if and only if $i \leq h(j)$. We use this basis throughout this section to establish a bijection between the dimension of the cells from Section \ref{sec:dimensions} and the number of certain configurations in specific Young tableaux.

Let $N$ be a nilpotent whose conjugacy class corresponds to the partition $\mu = (\mu_1, \ldots, \mu_s)$ with the parts ordered so that $\mu_i \geq \mu_{i+1}$. Associate to $N$ the Young diagram whose $i$th column has $\mu_i$ blocks. We use the convention that the blocks in this Young diagram are both left-aligned and bottom-aligned as in this example.

We refer to this as the Young diagram associated to $N$.

**Theorem 24.** The nonempty cells of the paving of $\mathcal{H}(N, H)$ given in Theorem \ref{thm:paving} correspond to those fillings of the Young diagram associated to $N$ for which the configuration

$$\begin{array}{c}
\pi^{-1}k \\
\pi^{-1}j
\end{array}$$
only occurs if $\pi^{-1}j \leq h(\pi^{-1}k)$.

Given a nonempty cell represented as a (filled) Young tableau, the dimension of this cell is the sum of the following two quantities.

(1) The number of configurations

$$\begin{array}{c}
\pi^{-1}j \\
\pi^{-1}i
\end{array}$$
where box $i$ is to the right of or below box $j$, there is no box above $j$, and the values filling these boxes satisfy $\pi^{-1}i > \pi^{-1}j$.

(2) The number of configurations

$$\begin{array}{c}
\pi^{-1}k \\
\pi^{-1}j \\
\pi^{-1}i
\end{array}$$
where box $i$ is to the right of or below box $j$ and the values filling these boxes satisfy $\pi^{-1}j < \pi^{-1}i \leq h(\pi^{-1}k)$.

Proof: We begin by fixing a number in $\{1, \ldots, n\}$ to each box in the Young diagram associated to $N$. The blocks are indexed from the bottom rightmost box to the top leftmost box by incrementing leftwards along each row then going to the rightmost
block on the next higher row and repeating as needed. For instance, the Young diagram shown previously is indexed as follows:

```
6
5 4
3 2 1
```

With respect to the standard matrix basis, the expression for $N$ given in Theorem 22 is equivalent to

$$N = \sum_{\{j,k\}: \text{box } k \text{ above box } j} E_{jk}.$$ 

In other words, $N$ sends the basis vector $e_k$ to $e_j$ if and only if the $k$th box lies above the $j$th box. 

Given this Young diagram, we can describe the Bruhat cells as (filled) Young tableaux. In particular, we associate a Young tableau to each permutation $\pi$ by filling the $i$th block with $\pi^{-1}i$. The roots in $\Phi_\pi$ are indexed by the set

$$\Phi_\pi = \bigcup_{1 \leq j \leq n} \{(i,j) : i < j, \pi^{-1}i > \pi^{-1}j\}.$$ 

This is the number of boxes to the right of or below the $j$th box that are filled with numbers greater than that filling the $j$th box.

A basis vector $E_{ij}$ is in $H$ if and only if $h(j) \geq i$. Consequently, the set $C_{\pi \cdot H}$ can be characterized as

$$C_{\pi \cdot H} = \{(\pi i, \pi k) : i > h(k)\}.$$ 

Lemma 19 showed that the roots in $\Phi_{U_{\pi^{-1}N}} \cap C_{\pi \cdot H}$ correspond bijectively to the elements of the set

$$\{(i,k) : i < j, \pi^{-1}i > \pi^{-1}j, \text{box } j \text{ is below box } k \} \cap C_{\pi \cdot H}.$$ 

This is because each root in $\Phi_{U_{\pi^{-1}N}}$ can be written as a sum $\alpha + \beta$ with $\beta$ in a higher row than $\alpha$ and with $\beta$ in $\Phi_N$. The cardinality of this set is the same as that of

$$\{(i,k) : i < j, \pi^{-1}i > \pi^{-1}j, \text{box } j \text{ is below box } k, \pi^{-1}i > h(\pi^{-1}k)\}.$$ 

It follows that the quantity $|\Phi_\pi| - |\Phi_{U_{\pi^{-1}N}} \cap C_{\pi \cdot H}|$ is the cardinality of the set

$$\{(i,k) : i < j, \pi^{-1}i > \pi^{-1}j, \text{box } j \text{ is below no box}\} \cup \{(i,k) : i < j, \pi^{-1}i > \pi^{-1}j, \text{box } j \text{ is below box } k, \pi^{-1}i \leq h(\pi^{-1}k)\}.$$ 

Given a Young tableau, the cardinality of the first set is the same as the number of configurations

```
\pi^{-1}j
\pi^{-1}k
```

where box $i$ is to the right of or below box $j$, there is no box above $j$, and the values filling these boxes satisfy $\pi^{-1}i > \pi^{-1}j$. The cardinality of the second set is the number of configurations

```
\pi^{-1}k
\pi^{-1}j
\pi^{-1}i
```

where box $i$ is to the right of or below box $j$ and the values filling these boxes satisfy $\pi^{-1}j < \pi^{-1}i \leq h(\pi^{-1}k)$. 

Finally, the Schubert cell corresponding to $\pi$ is nonempty if and only if $\pi^{-1} \cdot N$ is in $H$. This is equivalent to the statement that $\pi^{-1} j \leq h(\pi^{-1} k)$ for each instance when the $j^{th}$ box is below the $k^{th}$ box. □

The following result demonstrates how this theorem can be used computationally. It gives the Betti numbers for the Peterson variety. The Peterson variety was defined generally in the Introduction. In type $A_n$, it can be more simply described. Let $N$ be a regular nilpotent operator on $\mathbb{C}^n$, that is a nilpotent with a single Jordan block. Let $V = (V_1, \ldots, V_n)$ denote a full flag in $\mathbb{C}^n$, which is to say that each $V_i$ is an $i$-dimensional complex vector space and $V_i \subseteq V_{i+1}$ for $i = 1, \ldots, n - 1$. The Peterson variety $\mathcal{H}(N, H)$ is defined to be

$$\mathcal{H}(N, H) = \{ V : V \text{ is a full flag, } NV_i \subseteq V_{i+1} \text{ for } i = 1, \ldots, n - 1 \}.$$ 

This corresponds to choosing

$$H = \text{span}(E_{\alpha} : -\alpha \in \Phi^- \cup \Delta) = \text{span}(E_{ij} : 1 \leq i, j \leq n, i \leq j + 1)$$

or, equivalently, choosing $H$ to be the set of $n \times n$ matrices which are zero below the subdiagonal.

If the basis for $\mathbb{C}^n$ is chosen so that $N$ is in Jordan canonical form, the flags in the Peterson variety can be described completely as follows. First consider matrices of the form

$$\begin{pmatrix}
  c & b & a & 1 \\
  b & a & 1 & 0 \\
  a & 1 & 0 & 0 \\
  1 & 0 & 0 & 0
\end{pmatrix}$$

with ones along the cross-diagonal, zeroes below, and constant values along each line above and parallel to the cross-diagonal. Write $J_i$ for an $i \times i$ matrix of this form. The flags in $\mathcal{H}(N, H)$ are in bijective correspondence to matrices of the form

$$\begin{pmatrix}
  J_{i_1} & 0 & 0 & 0 \\
  0 & J_{i_2} & 0 & 0 \\
  0 & 0 & \ddots & 0 \\
  0 & 0 & 0 & J_{i_k}
\end{pmatrix}.$$ 

To obtain a flag from a matrix, let $V_i$ be the span of the first $i$ columns.

**Theorem 25.** Let $N$ be a regular nilpotent. Let $H$ be the subspace of $\mathfrak{g}$ given by

$$H = b \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha}.$$ 

There is a natural bijection between the cells of the Peterson variety $\mathcal{H}(N, H)$ and the ordered partitions of $n$. If $C = (c_1, \ldots, c_k)$ is the cell whose associated partition has parts $c_i$, then the complex dimension of $C$ is $n - k$. The number of cells with complex dimension $k$ is $\binom{n-1}{k}$.

**Proof.** A regular nilpotent has only one Jordan block so the associated Young diagram consists of a single column with $n$ boxes. Since $h(i) = i + 1$, Theorem 24 tells us that each adjacent pair of boxes

$$\begin{array}{c}
\pi^{-1}(i+1) \\
\pi^{-1}(i)
\end{array}$$
in a Young tableau for a nonempty cell satisfies either \( \pi^{-1}(i + 1) = \pi^{-1}(i) - 1 \) or \( \pi^{-1}(i + 1) > \pi^{-1}(i) \).

In fact, we can characterize the Young tableaux completely by selecting boxes of the column to initiate increasing runs

\[(\pi^{-1}(i), \pi^{-1}(i) + 1, \ldots, \pi^{-1}(i) + c - 1)\]

of the numbers filling the boxes. Suppose that \( (i_1, \ldots, i_1 + c_1 - 1) \) and \( (i_2, \ldots, i_2 + c_2 - 1) \) index the boxes in two increasing runs. If the Young tableau represents a nonempty cell, then \( \pi^{-1}(i_1) < \pi^{-1}(i_2) \) implies that the first run must fill lower boxes of the Young diagram than the second. The sizes of the increasing runs give an ordered partition of \( n \).

In addition, given an arrangement like

\[
\begin{array}{c}
\pi^{-1}(j + 1) \\
\pi^{-1}(j) \\
\vdots \\
\pi^{-1}(i) \\
\end{array}
\]

the quantity \( \pi^{-1}(i) \) is greater than \( \pi^{-1}(j) \) only if \( \pi^{-1}(j) \) is in the same increasing run as \( \pi^{-1}(i) \). If \( \pi^{-1}(j + 1) \) is also in this increasing run then \( h(\pi^{-1}(j + 1)) = \pi^{-1}(j + 1) + 1 = \pi^{-1}(j) \) so the arrangement does not contribute to the dimension of the cell, as per Theorem 24. By comparing each lower box in the run to the topmost in that run, we see that each increasing run of length \( c \) contributes exactly \( c - 1 \) to the total dimension of the cell.

Thus, if \( C = (c_1, \ldots, c_k) \) is the cell whose associated partition has parts \( c_i \) then the dimension of \( C \) is

\[\dim C = \sum_{i=1}^{k} (c_i - 1) = n - k.\]

The number of cells with \( n - k \) parts is the same as the number of the ways to choose \( n - k - 1 \) out of the \( n - 1 \) lower boxes of the Young diagram, since the top box always initiates an increasing run.

In [ST] we give further analyses of the Poincare polynomials of regular nilpotent Hessenberg varieties.

Semisimple and general Hessenberg varieties can also be described combinatorially. Associate to a linear operator \( M \) the multidiagram with one Young diagram for each generalized eigenspace of \( M \). The operator \( M \) acts on each generalized eigenspace by the sum of a nilpotent operator \( N_j \) and a semisimple operator constant on the generalized eigenspace. The Young diagram corresponding to the \( j^{th} \) generalized eigenspace is simply that associated to \( N_j \).

Order the Young diagrams from largest to smallest, right to left. Index the boxes of each Young diagram as if for \( N_j \). Finally, increment the indices of the \( j^{th} \) Young diagram by the number of boxes in the Young diagrams to the right of it. For example,

As before, describe the permutation \( \pi \) by filling box \( i \) with the value \( \pi^{-1}i \).
Theorem 26. Given $M$ and its associated multidiagram $\mathcal{Y}$, there is a bijective correspondence between nonempty affine cells of $\mathcal{H}(M,H)$ and fillings of $\mathcal{Y}$ such that the configuration \[\begin{array}{c|c|c}
\pi^{-1}i \\
\hline
\pi^{-1}j \\
\hline
\pi^{-1}k \\
\end{array}\] occurs only if $\pi^{-1}j \leq h(\pi^{-1}i)$. The dimension of the affine corresponding to a permissible filling of $\mathcal{Y}$ is the sum of the following quantities.

1. The number of configurations of type
\[\begin{array}{c|c|c}
\pi^{-1}k \\
\hline
\pi^{-1}j \\
\hline
\pi^{-1}i \\
\end{array}\]
where $i$ is less than $j$, box $i$ and box $j$ are in the same Young diagram, and the values filling these boxes satisfy $\pi^{-1}j < \pi^{-1}i \leq h(\pi^{-1}k)$. (If box $k$ does not exist, the latter inequality is considered vacuously satisfied.)

2. The number of configurations of type
\[\begin{array}{c|c|c}
\pi^{-1}j \\
\hline
\pi^{-1}i \\
\end{array}\]
with $h(\pi^{-1}j) \geq \pi^{-1}i > \pi^{-1}j$ and boxes $i$ and $j$ in different Young diagrams.

Proof. The root $\alpha = \alpha_i + \ldots + \alpha_{i+j}$ is in $\Phi^+_M$ if and only if the $i^{th}$ and $(i+j+1)^{th}$ boxes are in the same Young diagram. This root contributes one affine dimension to the total dimension of the cell if and only if the conditions of Theorem 24 are satisfied.

By contrast, $\alpha$ is in $\Phi_\pi$ exactly when the $i^{th}$ and $(i+j+1)^{th}$ boxes are in different Young diagrams. By Theorem 15, this root contributes one dimension to the total dimension of the cell only if $\alpha$ is in $\Phi_\pi \cap \pi M_H$, that is if $\pi^{-1}i > \pi^{-1}j$ and $h(\pi^{-1}j) \geq \pi^{-1}i$.

\[\square\]

3.3. Semisimple Hessenberg Varieties in Classical Type. Let $S$ be any semisimple element in $\mathfrak{g}$, a Lie algebra of classical type. Then the corresponding Hessenberg variety $\mathcal{H}(S,H)$ can be paved by affines. Once a root decomposition is fixed, we establish the notation that $\mathfrak{p}$ denotes the parabolic subalgebra generated by $\Phi^+_S$ and that $\mathfrak{n}$ (respectively $\mathfrak{l}$) denotes its nilpotent (respectively Levi) part. We assume that $\mathfrak{l} \supseteq \mathfrak{h}$.

Theorem 27. If $S$ is a semisimple element in $\mathfrak{g}$ a Lie algebra of classical type then there exists a Bruhat decomposition whose intersection with each Hessenberg variety $\mathcal{H}(S,H)$ is a paving by affines of $\mathcal{H}(S,H)$. Each cell $B\pi B \cap \mathcal{H}(S,H)$ is nonempty and has dimension
\[|\Phi_\pi \cap \pi M_H \cap \Phi_\mathfrak{a}| + |\Phi_\pi \cap \Phi_\mathfrak{l}|.\]

Proof. Embed the corresponding group $G$ into $GL_N(\mathbb{C})$ in the natural way so that $\text{rk} \ G = \lfloor N/2 \rfloor$ and so that the simple roots $\alpha_i$ for $G$ correspond to those for $GL_N(\mathbb{C})$ when $i < \lfloor N/2 \rfloor$.

There exists a basis for $\mathbb{C}^N$ with respect to which $S$ is diagonal and in Jordan canonical form in $gl_N(\mathbb{C})$. In other words, the diagonal entries of $S$ are grouped by eigenvalue. (If zero is an eigenvalue of $S$, we further assume that the diagonal of $S$ is zero in row $\lfloor N/2 \rfloor$.) Write $S_0$ for this element.
Since $G \cdot S = GL_N(\mathbb{C}) \cdot S \cap g$, there exists an element $g$ in $G$ such that $g \cdot S = S_0$ (see [SS, section IV.2.19]). By Lemma 14, the Hessenberg variety $\mathcal{H}(S, H)$ is isomorphic to $\mathcal{H}(S_0, g \cdot H)$. Note that $\{\alpha_i : \alpha_i(S_0) = 0\}$ generate a parabolic subalgebra $p$. The operator $\text{ad}S_0$ is zero on the Levi subalgebra $l \subseteq p$ and is nonzero on each $E_{\alpha}$ in the nilpotent subalgebra $n \subseteq p$. Thus, this decomposition satisfies Theorem 18. The conclusions follow.

3.4. Hessenberg Varieties of Regular Nilpotent Elements in Classical Type. One class of nilpotent Hessenberg varieties can be paved by affines using the same methods. For this result, $G$ is a linear algebraic group of classical type.

**Theorem 28.** Fix a regular nilpotent element $N$, let $b$ be the unique Borel subalgebra with $N \in b$, and let $B$ be the Borel subgroup corresponding to $b$. The intersection of the Bruhat decomposition with respect to $B$ and the Hessenberg variety $\mathcal{H}(N, H)$ is a paving by affines of $\mathcal{H}(N, H)$ for each $H$. The nonempty cells of this paving are $B \pi B \cap \mathcal{H}(N, H)$ satisfying $\pi^{-1} \cdot N \in H$ and have dimension

$$|\Phi_{\pi}| - |C_{\pi \cdot H} \cap \Phi_{U_{\pi \cdot N}}|.$$

**Proof.** Given any regular nilpotent $N$ of classical type, there exists a root space decomposition for $g$ such that

$$N = \sum_{\alpha \in \Delta} E_{\alpha}$$

by [CM] sections 5.2 and 5.4. The Borel subalgebra $b$ corresponding to the positive roots in this decomposition is the unique Borel subalgebra containing $N$ as shown in [CG, 3.2.13 and 3.2.14]. By Lemma 14, the choice of root space decomposition preserves the isomorphism class of the Hessenberg variety. Note that $\Phi_N = \Delta$ is trivially non-overlapping and that the rows $\Phi^i$ in classical type are vertical.

Since $\Phi_N = \Delta$, each $\Phi^i_{k,N}$ is $\Phi^i_{k-1}$ unless $k = 1$. Only in type $D_n$ is there a set $\Phi^i_k$ of size two. In this case, $\Phi^i_k = \Phi^i_{k+1,N} = \Phi^i_{k-1,N}$. Furthermore, if $\alpha = \sum_{\beta \in \Phi^i_k} \beta_j$ is in $\Phi^i_k$ then $\{\alpha\}_{-N} = \Phi^i_{k-1}$ which by row-verticality must be at least as large as each $\beta_j$.

By Corollary 17, if $N$ is not in $\pi \cdot H$ then $U(N, n_{\pi \cdot H})$ is empty. Otherwise, all the hypotheses of Lemma 20 are satisfied. Finally, by Theorem 18, the dimension of the cell $\mathcal{H}(N, H) \cap B \pi B$ is precisely that of $U_{\pi} \cap U(N, n_{\pi \cdot H})$. The conclusions follow.

For a discussion of related results for regular nilpotent Hessenberg varieties including a simpler dimension formula, the Euler characteristic, and some properties of the Poincare polynomials, see [ST].
4. Appendix: List of Symbols

\[
\begin{array}{ll}
G & \Phi_i \\
\mathfrak{g} & U_i \\
\mathfrak{h} & U_\pi \\
\Phi^+ & \rho_K \\
\Delta & \rho_i \\
\mathfrak{g}_\alpha & \rho_\alpha \\
E_\alpha & \iota_K \\
b & \gamma_i \\
\mathcal{M}_H & \Phi_M \\
H & \Phi_u \\
\mathcal{O}_K & \mathcal{O}_K \\
\geq & U(M, n_K) \\
\Phi^i & \Phi^i_{\mathcal{N}} \\
n_i & P_{\pm N} \\
\end{array}
\]

References

[BM] W. Borho and R. MacPherson, Partial resolutions of nilpotent varieties, Asterisque 101–102, Soc. Math. France, Paris, 1983.

[CP] P. Cellini and P. Papi, ad-Nilpotent ideals of a Borel subalgebra, J. of Algebra 225 2000, 130–141.

[CG] N. Chriss and V. Ginzburg, Representation Theory and Complex Geometry, Birkhäuser, Boston, 1997.

[CM] D. Collingwood and W. McGovern, Nilpotent orbits in semisimple Lie algebras, Van Nostrand Reinhold Co., New York, 1993.

[CLP] C. de Concini, G. Lusztig, and C. Procesi, Homology of the zero-set of a nilpotent vector field on a flag manifold, J. Amer. Math. Soc. 11988, 15–34.

[F] J. Fulman, Descent identities, Hessenberg varieties, and the Weil conjectures, J. Combin. Theory Ser. A 87 1999, 390–397.

[Fu] W. Fulton, Young Tableaux, London Mathematical Society Student Texts 35, Cambridge University Press, Cambridge, 1997.

[FH] W. Fulton and J. Harris, Representation Theory, Springer-Verlag, New York, 1991.

[H1] J. Humphreys, Introduction to Lie Algebras and Representation Theory, Grad. Texts in Math. 9, Springer-Verlag, New York, 1972.

[H2] J. Humphreys, Linear Algebraic Groups, Grad. Texts in Math. 21, Springer-Verlag, New York, 1964.

[Ho] R. Hotta, On Springer’s Representations, Jour. Fac. Sci. Univ. of Tokyo IA 28 1982, 836–876.

[KL] D. Kazhdan and G. Lusztig, A topological approach to Springer’s representations, Advances in Math. 38 1980, 222–228.

[K] A. Knapp, Lie Groups Beyond an Introduction, Progress in Math. 40, Birkhäuser, Boston, 1996.

[Ko] B. Kostant, Flag Manifold Quantum Cohomology, the Toda Lattice, and the Representation with Highest Weight \( \rho \), Selecta Math. (N. S.) 2 1996, 43–91.

[L] G. Lusztig, Intersection cohomology complexes on a reductive group, Invent. Math. 75 1984, 205–272.

[MPS] F. de Mari, C. Procesi, and M. A. Shayman, Hessenberg varieties, Trans. Amer. Math. Soc. 332 1992, 529–534.

[MS] F. de Mari and M. A. Shayman, Generalized Eulerian numbers and the topology of the Hessenberg variety of a matrix, Acta Appl. Math. 12 1988, 213–235.

[R] K. Rietsch, Givental’s mirror family for the flag variety and the total positive part of the Peterson variety, preprint 2002.
[S1] N. Shimomura, A theorem on the fixed point set of a unipotent transformation on the flag manifold, J. Math. Soc. Japan 32 1980, 55–64.

[S2] N. Shimomura, The fixed point subvarieties of unipotent transformations on the flag varieties, J. Math. Soc. Japan 37 1985, 537–556.

[ST] E. Sommers and J. Tymoczko, Generalized exponents, in preparation.

[Sp1] N. Spaltenstein, The fixed point set of a unipotent transformation on the flag manifold, Nederl. Akad. Wetensch. Proc. Ser. A 79 1976, 452–456.

[Sp2] N. Spaltenstein, On the fixed point set of a unipotent element on the variety of Borel subgroups, Topology 16 1977, 203–204.

[Sp3] N. Spaltenstein, Classes unipotentes et sous-groupes de Borel, Lecture Notes in Mathematics 946, Springer-Verlag, New York, 1982.

[S] T. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, Invent. Math. 36 1976, 173–207.

[SS] T. Springer and R. Steinberg, Conjugacy classes; Seminar on algebraic groups and related finite groups, Lecture Notes in Mathematics 131, Springer-Verlag, New York, 1970.

E-mail address: tymoczko@umich.edu