Quantum-Limited Amplification of Cavity Optomechanics without Resolved Sideband Condition

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(Dated: June 16, 2015)

We propose a scheme to realize the phase-preserving amplification without the restriction of resolved sideband condition. As a result, our gain-bandwidth product is about one magnitude larger than the existing proposals. In our model, an additional cavity is coupled to the cavity-optomechanical system. Therefore our operating frequency is continuously tunable via adjusting the coupling coefficient of the two cavities.

I. INTRODUCTION

In the past few years, significant progress has been achieved in the cavity optomechanics$^{1-4}$. Examples include ground state cooling of a mechanical oscillator which is a prerequisite for its applications in quantum information processing$^{5-10}$ and sensitive measurement$^{11-14}$, and its serving as intermediate transducer of hybrid systems$^{14-17}$.

The paper is organized as follows. Section II presents the model, its Hamiltonian, and the analytical formulas of the amplification process. In Section III we optimize the parameters within and beyond the resolved-sideband regime numerically. Then we discuss the adjustability of the amplifier’s center frequency. We give the conclusion in Sec. IV.

II. MODEL AND FORMULAS

A schematic of our model is shown in Fig. 1. An auxiliary cavity is coupled to a cavity-optomechanical system. The Hamiltonian of the compound cavity system is

\[ H = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + J (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \omega_m \hat{b}^\dagger \hat{b} 
- g_0 (\hat{b}^\dagger + \hat{b}) \hat{a}_1^\dagger \hat{a}_1 + \Omega (\hat{a}_1^\dagger e^{-i\omega_L t} + \hat{a}_1 e^{i\omega_L t}), \]

where $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators of the two cavity modes with $\omega_1$ and $\omega_2$ being their frequencies, $J$ is the coupling coefficient of the two cavity modes, $\hat{b}$ is the annihilation operator of the mechanical oscillator with frequency $\omega_m$, $g_0$ is the parametric coupling strength between the first cavity and the mechanical oscillator, $\Omega$ is the strength of the external driving field for the first cavity, and $\omega_L$ is the frequency of the driving field.

Moving into the frame of the drive, the Hamiltonian reads

\[ \hat{H} = -\Delta_1 \hat{a}_1^\dagger \hat{a}_1 - \Delta_2 \hat{a}_2^\dagger \hat{a}_2 + J (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) 
+ \omega_m \hat{b}^\dagger \hat{b} - g_0 (\hat{b}^\dagger + \hat{b}) \hat{a}_1^\dagger \hat{a}_1 + \Omega (\hat{a}_1^\dagger + \hat{a}_1), \]

where $\Delta_1 = \omega_L - \omega_1$ and $\Delta_2 = \omega_L - \omega_2$ are the detunings between the two cavity modes and the driving field.
We can linearize the dynamical equations of the driven compound cavity system by assuming \( \hat{a}_1 = \hat{a}_1 + \delta \hat{a}_1 \), \( \hat{a}_2 = \hat{a}_2 + \delta \hat{a}_2 \) and \( \hat{b} = \hat{b} + \delta \hat{b} \) where \( \hat{a}_1, \hat{a}_2 \) and \( \hat{b} \) are the respective mean values. Neglecting nonlinear terms we get

\[
\dot{\delta \hat{a}_1} = i \Delta_1 \delta \hat{a}_1 - \frac{\kappa_1}{2} \delta \hat{a}_1 - iJ \delta \hat{a}_2 + iG(\delta \hat{b} + \delta \hat{b}^\dagger) + \sqrt{\kappa_1} \delta \hat{a}_{1,in}, \\
\dot{\delta \hat{a}_2} = i \Delta_2 \delta \hat{a}_2 - \frac{\kappa_2}{2} \delta \hat{a}_2 - iJ \delta \hat{a}_1 + \sqrt{\kappa_2} \delta \hat{a}_{2,in}, \\
\dot{\delta \hat{b}} = -i\omega_m \delta \hat{b} - \frac{\Gamma}{2} \delta \hat{b} + iG(\delta \hat{a}_1 + \delta \hat{a}_1^\dagger) + \sqrt{\Gamma} \delta \hat{b}_{in},
\]

where \( \Delta_1 = \Delta_{10} + g_0(\hat{b} + \hat{b}^\dagger) \) denotes the effective detuning of the first cavity, \( G = g_0 \alpha_1 \Gamma \) is the enhanced optomechanical coupling, \( \kappa_1, \kappa_2 \) and \( \Gamma \) are the decay rates of cavity 1, 2 and the mechanical oscillator.

We write Eqs. \((3), (4)\) and \((5)\) in the matrix form

\[
\dot{u} = M u + L u_{in},
\]

where

\[
\begin{align*}
\dot{u} &= (\delta \hat{a}_1, \delta \hat{a}_1^\dagger, \delta \hat{a}_2, \delta \hat{a}_2^\dagger, \delta \hat{b}, \delta \hat{b}^\dagger)^T, \\
u_{in} &= (\hat{a}_{1,in}, \hat{a}_{1,in}^\dagger, \hat{a}_{2,in}, \hat{a}_{2,in}^\dagger, \hat{b}_{in}, \hat{b}_{in}^\dagger)^T,
\end{align*}
\]

We work in frequency space and use the input-output relations \( u_{out} = u_{in} - L u \). Then we get

\[
u_{out}[\omega] = U(\omega) u_{in}[\omega]
\]

with

\[
U = 1 + L[i\omega + M]^{-1} L.
\]

The amplification can be expressed as

\[
\dot{\hat{a}}_{1,out} = A(\omega) \hat{a}_{1,in} + B(\omega) \hat{a}_{1,in}^\dagger + C(\omega) \hat{a}_{2,in} + D(\omega) \hat{a}_{2,in}^\dagger + E(\omega) \hat{b}_{in} + F(\omega) \hat{b}_{in}^\dagger, \]

where \( \hat{a}_{1, out} = A(\omega) \hat{a}_{1,in} + B(\omega) \hat{a}_{1,in}^\dagger + C(\omega) \hat{a}_{2,in} + D(\omega) \hat{a}_{2,in}^\dagger + E(\omega) \hat{b}_{in} + F(\omega) \hat{b}_{in}^\dagger \).

In the case of unresolved sideband for the first cavity where \( \kappa_1 \geq \omega_m \), we cannot make the resolved sideband approximation. To simplify, we first consider the case where \( \kappa_2 = 0 \). The power gain \( G[\omega] = |A(\omega)|^2 \) can be expressed as

\[
G[\omega] = \left| 1 + \frac{2i\kappa_1(\Delta_1 - \omega)}{\alpha(\omega)(4J^2 + (\Delta_2^2 - \omega^2)((\kappa_1 - 2i\omega)^2 + 4\Delta_2^2) - 4iJ^2(\kappa_1 - 2i\omega) - 8\Delta_1 \Delta_2 J^2) + 64\kappa_1^2 \omega_m^2 \beta(\omega)} \right|^2,
\]

where \( \alpha(\omega) = 4\omega_m^2 + (\Gamma - 2i\omega)^2 \) and \( \beta(\omega) = \Delta_1(\Delta_2^2 - \omega^2) - \Delta_2 J^2 \). We discuss on the blue sideband where \( \Delta_1 = \omega_m \). To suppress the amplification in the vicinity of \( \omega_m \), we simply choose \( \Delta_2 = -\omega_m \). Denote the denominator of the second term in the right hand side of Eq. \((11)\) as \( \rho(\omega) \).

The critical points of instability appear at \( \rho(\omega) = 0 \).

According to this, we get two groups of boundary conditions:

\[
G^{(1)} = \frac{\sqrt{\Gamma \kappa_1 \sqrt{(4J^2 + \Gamma^2 + \Gamma \kappa_1)^2 + 16\kappa_1^2 \omega_m^2 (\Gamma + \kappa_1)^2}}}{8(\Gamma + \kappa_1) \omega_m},
\]

\[
\omega^{(1)} = \pm \sqrt{J^2 \Gamma (\Gamma + \kappa_1) + \omega_m^2};
\]

\[
G^{(2)} = \frac{\sqrt{\kappa_1 \sqrt{(4J^2 + \Gamma^2 + \Gamma \kappa_1)^2 + 16\kappa_1^2 \omega_m^2}}}{8\sqrt{\Gamma} \omega_m},
\]

\[
\omega^{(2)} = \pm \sqrt{J^2 + \Gamma \kappa_1 + \omega_m^2}.
\]

We plot the stable and unstable boundary \( G^{(1)}(\Gamma) \) in Fig. 2. The gray area is the unstable regime. Applying the Routh-Hurwitz criterion [24], \( G^{(2)}(\Gamma) \) is the separation of two-negative roots and four negative roots. In
both regimes the system is unstable. So we do not draw it here.

Consider a set of parameters in the stable regime where $G$ deviates by a small quantity $-\delta(\delta > 0)$ from $G^{(1)}$, which is calculated from other parameters. We can always write $G[\omega]$ in the vicinity of $-|\omega^{(1)}|$ as

$$G[\omega] = \left| 1 + \frac{\kappa}{x + a} \right|^2,$$

(14)

where $x$ is the deviation of $\omega$ from $-|\omega^{(1)}|$. We can easily get that

$$\kappa = \frac{ix_1 J^2 \Gamma (\xi - 4i\omega_m)(\Gamma \xi + 8\omega_m^2 + 8\omega_m \sigma)}{4\omega_m \xi \sigma \left[ (\Gamma + \kappa)^2 (4i\sigma + \Gamma) - 4J^2 (\Gamma - \kappa) \right]},$$

(15)

and

$$a = \frac{-8J^2 \omega_m \sqrt{\kappa_1} \sqrt{\xi^2 + 16\omega_m^2 \delta}}{\xi \sigma \left[ (\Gamma + \kappa)^2 (4i\sigma + \Gamma) - 4J^2 (\Gamma - \kappa) \right]},$$

(16)

where $\xi = \Gamma + 4J^2/(\Gamma + \kappa_1)$, $\sigma = \sqrt{\omega_m^2 + J^2 \Gamma/(\Gamma + \kappa_1)}$. Here $\text{Re}[a]$ is the deviation of the peak from $-|\omega^{(1)}|$. The bandwidth of the gain is given by $2\text{Im}[a]$ and the gain-bandwidth product is $|\kappa|$.

III. DISCUSSION

A. resolved sideband regime

In Fig. 2 we plot the gain-bandwidth product $|\kappa|/\omega_m$ in the $J - \kappa_1$ plane where $\Gamma = 0.1\omega_m$ (colored image). The optimal $\kappa$ is found at $\kappa_1 = \Gamma$. $|B|/|A| = 0.1$ (yellow line) and $0.2$ (red line) are shown in this figure. We choose $J = 1.0\omega_m$, $\kappa_1 = 0.1\omega_m$, $\Gamma = 0.1\omega_m$ and $G = G^{(1)} - 0.0001\omega_m = 0.2561\omega_m$. In this case the maximum power gain appears at $\omega_p = -1.2247\omega_m$, the bandwidth is $5.2 \times 10^{-4}\omega_m$ and the gain-bandwidth product is $|\kappa| = 0.21\omega_m$. We plot the power gain $G[\omega]$ as a function of the frequency $\omega$ in Fig. 2. In Fig. 3 we plot $|B(\omega)|/|A(\omega)|, |C(\omega)|/|A(\omega)|, |D(\omega)|/|A(\omega)|$ and $|E(\omega)|/|A(\omega)|$ as functions of the frequency $\omega$ in the vicinity of the peak. where we can make an approximation that $|B(\omega)|, |C(\omega)|, |D(\omega)|, |E(\omega)| \ll |A(\omega)|$. As a consequence, the amplifier is phase insensitive. We plot $|F(\omega)|/|A(\omega)|$ as a function of the frequency $\omega$ in Fig. 4. It can be proved that at the peak of amplification, we have $|F| = |A|$. According to this, the number of added noise photons

$$N(\omega_p) = \frac{|F|^2}{|A|^2} (\bar{n}_{eff} + \frac{1}{2}) \approx \bar{n}_{eff} + \frac{1}{2},$$

(17)

where $\bar{n}_{eff}$ is the effective phonon number of the reservoir of the oscillator. We can use the method guided by Ref. 12 to cool the mechanical resonator to the quantum ground state. As $|F|^2/|A|^2 \approx 1$ and $\bar{n}_{eff} \rightarrow 0$, the amplifier is quantum limited.

Now we take the non-zero dissipation of cavity $2(\kappa_2 \neq 0)$ into consideration where $\kappa_2 = 0.01\omega_m$, $J = 1.0\omega_m$, $\kappa_1 = 0.1\omega_m$, $\Gamma = 0.1\omega_m$ and $G = 0.2561\omega_m$. We can also get a quantum-limited and phase-preserving amplifier. the maximum gain power appears at $\omega_p = -1.22\omega_m$, the bandwidth is $2.1 \times 10^{-4}\omega_m$ and the gain-bandwidth prod-
\section*{B. beyond resolved sideband condition}

In this subsection we discard the resolved sideband condition and further optimize the parameters. For discussion simplicity, here we fix \( \kappa_1 = 2.0 \omega_m \). In Fig. 7 we plot the contours of \( |B|/|A| = 0.2 \) (blue solid line) and \( |\kappa| = 0.5 \omega_m \) (red dashed line) in the \( \omega - \Gamma \) plane. We roughly choose \( \kappa_1 = 2.0 \omega_m, \kappa_2 = 0, J = 2.0 \omega_m, \Gamma = 0.6 \omega_m \) and \( G = G^0 - 0.0001 \omega_m = 1.0747 \omega_m \). The maximum gain power appears at \( \omega_p = -1.39 \omega_m \), the bandwidth is \( 1.0 \times 10^{-4} \omega_m \) and the gain-bandwidth product is \( |\kappa| = 0.51 \omega_m \).

The power gain \( G[\omega] \) as a function of the frequency \( \omega \) is shown in Fig. 8. In Fig. 9 we plot \( |B|/|A| \), \( |C|/|A| \), \( |D|/|A| \), \( |E|/|A| \) as functions of the frequency \( \omega \). An approximation \( |B|, |C|, |D|, |E| \ll |A| \) is appropriate. In Fig. 10 we plot \( |F|/|A| \) as a function of \( \omega \). The number of added noise photons \( N = (\pi \Sigma)^{1/2} |F|^2/|A|^2 \approx \Sigma^{1/2} \). Since plenty of schemes have been proposed to cool the mechanical resonator to ground state in the unresolved sideband regime recently [22, 23], our amplifier can reach the quantum limit and is phase insensitive.

Now we discuss the influence of non-zero \( \kappa_2 \). First we consider \( \kappa_2 = 0.01 \omega_m \) and other parameters the same as those in Fig. 8. A quantum limited and phase-preserving amplifier can be obtained. The maximum gain power appears at \( \omega_p = -1.39 \omega_m \), the bandwidth is \( 2.6 \times 10^{-3} \omega_m \) and the gain-bandwidth product is \( |\kappa| = 0.51 \omega_m \).

In the situation where \( \kappa_2 < \omega_m \) but not \( \kappa_2 \ll \omega_m \), we give a set of parameters where \( \kappa_2 = 0.5 \omega_m, \kappa_1 = 3.0 \omega_m, J = 1.49 \omega_m, \Gamma = 0.2 \omega_m \) and \( G = 0.5568 \omega_m \). In this case the maximum gain power appears at \( \omega_p = -1.08 \omega_m \), the bandwidth is \( 6.0 \times 10^{-5} \omega_m \) and the gain-bandwidth product is \( |\kappa| = 0.21 \omega_m \).

\section*{C. tunable frequency window}

The coupling coefficient \( J \) of the compound cavity is tunable. The device features an amplifier with continuously tunable operating frequency. According to Eq. (12) the center frequency appears at \( \omega = -|\omega| \). In Fig. 11 we plot the center frequency as a function of the two cavity coupling coefficient \( J \).
FIG. 8. Power gain $G[\omega] = |A(\omega)|^2$ as a function of the frequency $\omega$ where $\kappa_1 = 2.0\omega_m, \kappa_2 = 0, J = 2.0\omega_m, G = 1.0747\omega_m$ and $\Gamma_m = 0.6\omega_m$. In the inset we plot the power gain $G$ as a function of $\omega$ around peak value.

FIG. 9. $|B(\omega)|/|A(\omega)|$ (blue solid line), $|C(\omega)|/|A(\omega)|$ (red dashed line), $|D(\omega)|/|A(\omega)|$ (green dotted line) and $|E(\omega)|/|A(\omega)|$ (black dash dot line) as functions of $\omega$. The parameters are the same as in Fig. 8.

FIG. 10. $|F(\omega)|/|A(\omega)|$ as a function of $\omega$. The parameters are the same as in Fig. 8.

IV. CONCLUSION

We propose a scheme to realize the phase-preserving amplification beyond the resolved sideband regime. Hence we can optimize the amplification in a large parameter range and achieve an enhancement of one magnitude in gain-bandwidth product. Also the frequency window of our proposed amplifier can be tuned continuously by the adjustment of the cavity coupling.

ACKNOWLEDGMENTS

We acknowledge W. J. Zou, F. C. Lei, M. Gao, Q. L. Jing and Y. H. Zhou for helpful discussion. We acknowledge the financial support in part by the 10000-Plan of Shandong province (Taishan Scholars), NSFC grant No. 11174177 and 11474182, and the National High-Tech Program of China grant No. 2011AA010800 and 2011AA010803.
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