The Fine-Grained Complexity of Andersen’s Pointer Analysis

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Pointer analysis is one of the fundamental problems in static program analysis. Given a set of pointers, the task is to produce a useful over-approximation of the memory locations that each pointer may point-to at runtime. The most common formulation is Andersen’s Pointer Analysis (APA), defined as an inclusion-based set of \( m \) pointer constraints over a set of \( n \) pointers. Scalability is extremely important, as points-to information is a prerequisite to many other components in the static-analysis pipeline. Existing algorithms solve APA in \( O(n^2 \cdot m) \) time, while it has been conjectured that the problem has no truly sub-cubic algorithm, with a proof so far having remained elusive. It is also well-known that APA can be solved in \( O(n^2) \) time under certain sparsity conditions that hold naturally in some settings. Besides these simple bounds, the complexity of the problem has remained poorly understood.

In this work we draw a rich fine-grained complexity landscape of APA, and present upper and lower bounds. First, we establish an \( O(n^3) \) upper-bound for general APA, improving over \( O(n^2 \cdot m) \) as \( n = O(m) \). Second, we show that sparse instances can be solved in \( O(n^{3/2}) \) time, improving the current \( O(n^2) \) bound. Third, we show that even on-demand APA ("may a specific pointer \( a \) point to a specific location \( b \)?") has an \( \Omega(n^3) \) (combinatorial) lower bound under standard complexity-theoretic hypotheses. This formally establishes the long-conjectured "cubic bottleneck" of APA, and shows that our \( O(n^3) \)-time algorithm is optimal. Fourth, we show that under mild restrictions, APA is solvable in \( \tilde{O}(n^\omega) \) time, where \( \omega < 2.373 \) is the matrix-multiplication coefficient. It is believed that \( \omega = 2 + o(1) \), in which case this bound becomes quadratic. Fifth, we show that even under such restrictions, even the on-demand problem has an \( \Omega(n^2) \) lower bound under standard complexity-theoretic hypotheses, and hence our algorithm is optimal when \( \omega = 2 + o(1) \).

CCS Concepts: • Software and its engineering → Software verification and validation; • Theory of computation → Theory and algorithms for application domains; Program analysis.

Additional Key Words and Phrases: static pointer analysis, inclusion-based pointer analysis, fine-grained complexity, Dyck reachability

ACM Reference Format:
Andreas Pavlogiannis. 2020. The Fine-Grained Complexity of Andersen’s Pointer Analysis. 1, 1 (June 2020), 30 pages. https://doi.org/10.1145/nnnnnnn.nnnnnnn

1 INTRODUCTION

Programs execute by allocating memory for storing data and manipulating pointers to that memory. Pointer analysis takes a static view of a program’s heap and asks the question “given a pointer \( a \), what are the memory locations that \( a \) may point-to at program runtime?” Such information is vital...
to almost all questions addressed by static analyses in general [Ghiya et al. 2001; Hind 2001], hence many static analyzers begin with some form of pointer analysis. In particular, for an analysis to be useful, it needs to be able to determine aliasing, i.e., whether two pointers may be pointing to the same memory location. For example, in the program of Figure 1, the value of \(c\) depends on whether \(a\) and \(b\) are aliases. Naturally, aliasing is decided by (implicitly or explicitly) computing whether the intersection of the points-to space of the two pointers is empty.

As usual in static analyses, points-to information can be modeled at various degrees of precision, which has consequences on the decidability and complexity of the problem. Flow-sensitive formulations, which take into account the order of execution of pointer-manipulation statements, are typically intractable, with results ranging from undecidability [Ramalingam 1994], to PSPACE-completeness [Chakaravarthy 2003] and NP-/co-NP-hardness [Landi and Ryder 1991]. In contrast, flow-insensitive formulations can be viewed as relational approaches that ignore the order of execution, and typically result in more tractable algorithmic problems. Another feature that affects complexity is the level of indirection (i.e., how many nested dereferences can occur in a single statement) [Horwitz 1997], and thus is typically kept small. Flow-insensitive analyses are faster and achieve remarkable precision in practice [Blackshear et al. 2011; Das et al. 2001; Shapiro and Horwitz 1997]. This sweet spot between efficiency and precision has made flow-insensitive analyses dominant over alternatives. Popular approaches in this domain are inclusion-based [Andersen 1994], equality-based [Steensgaard 1996] and unification-based [Das 2000]. We refer to [Smaragdakis and Balatsouras 2015] for an excellent exposition.

**Andersen’s pointer analysis.** The most commonly used and actively studied formulation is Andersen’s Pointer Analysis (APA) [Andersen 1994]. The input is a set of \(n\) pointers and \(m\) statements of the four types shown in Figure 1. The solution to the analysis is the least fixpoint of a set of inclusion constraints between the points-to sets of the pointers (see Section 2.1 for details). APA has been the subject of a truly huge body of work, ranging from adoptions to diverse programming languages [Jang and Choe 2009; Lyde et al. 2015; Sridharan and Fink 2009], extensions to incorporate various features (e.g., context/flow/field-sensitivity) [Hardekopf and Lin 2011; Hirzel et al. 2004; Pearce et al. 2004; Whaley and Lam 2002] and implementations in various frameworks [Wal 2003; Lhoták and Hendren 2003; Vallée-Rai et al. 1999], to name a few.

**Complexity.** The standard statement in the literature with regards to the complexity of APA is that it is cubic. However, the parameter \((n\ or\ m,\ for\ n\ pointers\ and\ m\ statements)\) on which this cubic bound is expressed is often left unspecified, leading to a variety of statements. The standard expression is an \(O(m^3)\) bound [Melski and Reps 2000; Möller and Schwartzbach 2018], by reducing the problem to \(m\) inclusion set constraints [McAllester 1999]. Other works give a more refined bound of \(O(n^2 \cdot m)\) [Kodumal and Aiken 2004; Pearce et al. 2004] which is an improvement over \(O(m^3)\) as \(n = O(m)\). Note that, in general, \(m\) can be as large as \(\Theta(n^2)\), hence both types of statements result in worst-case dependency on \(n\) that is at least quartic, as already observed in [Kodumal and

Fig. 1. A program where analysis depends on aliasing (left) and the four types of statements in APA (right).
Aiken 2004]. Finally, in most literature, the core algorithm constructs incrementally the closure of a flow graph by introducing edges dynamically. However, the complexity analysis often ignores the cost for inserting edges. As already noted by others [Heintze and McAllester 1997; Sridharan and Fink 2009], the cost of edge insertion needs to be accounted for when analyzing the complexity. Under this consideration, the complexity of all these approaches is at least quartic in $n$.

The need for a faster algorithm is apparent from the long literature of heuristics [Aiken et al. 1997; Berndl et al. 2003; Dietrich et al. 2015; Fähndrich et al. 1998; Hardekopf and Lin 2007; Heintze and Tardieu 2001b; Pearce et al. 2004; Pek and Madhusudan 2009; Rountev and Chandra 2000; Su et al. 2019; Vedurada and Nandivada 2019; Xu et al. 2009]. Despite all efforts, no algorithmic breakthrough below the cubic bound has been made for over 25 years. In some cases, the complexity of APA can be reduced to quadratic [Sridharan and Fink 2009]. This reduction holds when the instances adhere to certain sparsity conditions, which hold naturally in some settings.

**Exhaustive vs on-demand.** One popular approach to reducing running time lies on the observation that we are typically interested in on-demand variants of the problem. That is, we want to decide whether $a$ may point to $b$ for a given pointer $a$ and memory location $b$, rather than the exhaustive case that computes the points-to set of every pointer. Under this restriction, many techniques devise analysis algorithms that aim to solve on-demand APA faster [Chatterjee et al. 2018; Heintze and Tardieu 2001a; Lu et al. 2013; Sridharan et al. 2005; Sui and Xue 2016; Vedurada and Nandivada 2019; Zhang et al. 2013; Zheng and Rugina 2008]. On the theoretical side, it is an open question whether on-demand analysis has lower complexity than exhaustive analysis.

**Lower bounds and cubic bottlenecks.** Despite the complete lack of algorithmic improvements for APA for over 25 years, no lower bounds are known. The only observation is that exhaustive APA has output size $\Theta(n^2)$, which leads to a trivial similar lower bound for running time. For the on-demand case, no lower bound is known. APA is often reduced to a specific framework of set constraints [Heintze 1992; Su et al. 2000], which is computationally equivalent to CFL-Reachability [Melski and Reps 2000]. Set constraints and CFL-Reachability are known to have cubic lower bounds [Heintze and McAllester 1997]. Unfortunately, these lower bounds do not imply any lower bound for APA as the reduction is only one way (i.e., from APA to set constraints). The recurrent encounter of cubic complexity is frequently referred to as the “cubic bottleneck in static analysis”, though the bottleneck is only conjectured for APA, with a proof so far having remained elusive.

1.1 Our Contributions

In this work, we draw a rich fine-grained complexity landscape of Andersen’s Pointer Analysis, by resolving open questions and improving existing bounds. We refer to Section 2.3 for a formal presentation of our main results as well as a discussion on their implications to the theory and practice of pointer analysis.

**Main contributions.** Consider as input an APA instance $(A, S)$ of $n = |A|$ pointers and $m = |S|$ statements. Our main contributions are as follows.

1. We show that Exhaustive APA is solvable in $O(n^3)$ time, regardless of $m$. To our knowledge, this gives the sharpest cubic bound on the analysis, as it holds even when $m = \Theta(n^2)$.

2. We show that inputs satisfying the sparsity conditions of [Sridharan and Fink 2009] are solvable in $O(n^{3/2})$ time. This improves the $O(n^2)$ bound developed in that work.
(3) We show that even On-demand APA does not have a (combinatorial) sub-cubic algorithm (i.e., with complexity $O(n^{3-\epsilon})$ for some fixed $\epsilon > 0$) under the combinatorial BMM hypothesis. This formally proves the long-conjectured cubic bottleneck for APA.

(4) We consider a bounded version of Exhaustive APA where points-to information is witnessed by bounding the execution of type-4 statements by a poly-logarithmic bound. We show that bounded Exhaustive APA is solvable in $\tilde{O}(n^\omega)$ time, where $\omega$ hides poly-logarithmic factors and $\omega$ is the matrix-multiplication coefficient. It is known that $\omega < 2.373$ [Le Gall 2014], hence our algorithm is sub-cubic.

(5) It is believed that $\omega = 2 + o(1)$, in which case our previous bound becomes nearly quadratic. We complement this result by showing that even On-demand APA with witnesses that are logarithmically bounded (i.e., a simpler problem than that in Item 4) does not have a sub-quadratic algorithm (i.e., with complexity $O(n^{2-\epsilon})$ for some fixed $\epsilon > 0$) under the Orthogonal Vectors hypothesis [Williams 2019]. Hence, our algorithm for Item 4 is optimal when $\omega = 2 + o(1)$.

**Technical contributions.** Our main theoretical results rely on a number of technical novelties that might be of independent interest.

(1) Virtually all existing algorithms for APA represent the analysis as a flow-graph that captures inclusion constraints between pointers. In contrast, we develop a Dyck-graph representation over the Dyck language of 1 parenthesis type $D_1$, which allows us to develop new insights for the problem.

(2) We show that Dyck-Reachability over $D_1$ can be solved in time $\tilde{O}(n^\omega)$, where $\omega$ is the matrix-multiplication coefficient. This is the first sub-cubic algorithm for the problem.

(3) Our lower bounds are based on fine-grained complexity, an emerging field in complexity theory that establishes relationships between problems in P. We believe that this field can have an important role in understanding and optimizing static program analyses. Our work makes some of the first steps in this direction.

## 2 Preliminaries and Summary of Results

In this section we give a formal presentation of Andersen’s pointer analysis and develop some general notation. We also define Dyck graphs and show how reachability relationships in such graphs can be used to represent points-to relationships between pointers. Finally, we present the main theorems of this paper. Given a number $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$.

### 2.1 Andersen’s Pointer Analysis

We begin with giving the formal definition of Andersen’s pointer analysis, as well as a bounded version of the problem.

**Andersen’s pointer analysis (APA).** An instance of APA is a pair $(A, S)$, where $A$ is a set of pointers\(^1\) and $S$ is a set of statements. Each statement has one of the four types shown in Table 1. Conceptually, the pointers may reference memory locations during the runtime of a program which uses the statements to manipulate the pointers.

1. A type 1 statement $a = b$ represents pointer assignment.
2. A type 2 statement $a = &b$ represents making $a$ point to the location of $b$.

\(^1\)Although in practice not all variables are pointers, we will use this term liberally for simplicity of presentation. This does not affect the results obtained in this work.
As standard practice, more complex statements like \( *a = *b \) have been normalized by introducing slack pointers [Andersen 1994; Horwitz 1997; Møller and Schwartzbach 2018]. Given some \( i \in [4] \), we denote by \( S_i \) the statements of \( S \) of type \( i \). Given a pointer \( a \), we let \( [a] \subseteq A \) be the points-to set of \( a \). Typically in pointer analysis, \( [a] \) is an over-approximation of the locations that \( a \) can point-to during the lifetime of a program. In Andersen’s inclusion-based pointer analysis, the sets \([a]\) are defined as follows. Each statement generates an inclusion constraint between various points-to sets, as shown in Table 1. The solution to APA is the smallest assignment \( \{[a] \rightarrow 2^A\}_{a \in A} \) that satisfies all constraints.

As standard in the literature, we distinguish between the exhaustive and on-demand versions of the problem. In each case, the input is an instance of APA. The Exhaustive APA problem asks to compute the points-to set of every pointer \( a \in A \). The On-demand APA problem asks to compute whether \( b \in [a] \) for a given pair of pointers \( a, b \in A \). Hence, On-demand APA is a simplification of Exhaustive APA where the size of the output is a single bit, as opposed to \( \Theta(n^3) \) bits required to output the points-to set of every pointer. The two variants can be viewed as analogues to the all-pairs and single-pair formulations of graph problems (e.g., reachability).

### Operational semantics

Operational semantics. Since the statements in APA come out of programs, it is convenient to consider them as executable instructions and assign simple operational semantics to them. The semantics are over a global store \( [] : A \rightarrow 2^A \) that maps every pointer to its points-to set, which is initially empty. Executing one statement corresponds to updating the store as shown in Table 1. This operational view already hints an (albeit inefficient) algorithm for solving APA, namely, by iteratively executing some statement until no execution modifies the store.

### Witnesses

Witnesses. The operational semantics allow us to define witnesses of points-to relations. Given two pointers \( a, b \in A \), a witness program (or simply, witness) for \( b \in [a] \) is a sequence of statements from \( S \) that results in \( b \in [a] \). See Figure 2 for an illustration.

### Bounded APA

Motivated by practical applications, we introduce a bounded version of APA that restricts the length of witnesses. Consider two pointers \( a, b \in A \) and a witness program \( P \) for

| Type | Statement | Inclusion Constraint | Operational Semantics |
|------|-----------|----------------------|-----------------------|
| 1    | \( a = b \) | \( [b] \subseteq [a] \) | \( [a] \leftarrow [a] \cup \{b\} \) |
| 2    | \( a = &b \) | \( b \in [a] \) | \( [a] \leftarrow [a] \cup \{b\} \) |
| 3    | \( a \leftarrow *b \) | \( \forall c \in [b] : [c] \subseteq [a] \) | \( [a] \leftarrow [a] \cup (\bigcup_{c \in [b]} [c]) \) |
| 4    | \( *a = b \) | \( \forall c \in [a] : [b] \subseteq [c] \) | \( \forall c \in [a] : [c] \leftarrow [c] \cup \{b\} \) |

Table 1. The four types of statements of APA, the inclusion constraints they generate and the associated operational semantics.
We often represent \((ii)\) a (resp., \(\&\)) represents all statements in \(E\) of \(P\) such that for any two pointers \(a, b\) with \(b \not\in [a]\), any witness program that results in \(b \in [a]\) executes more than \(j\) statements of type \(i\).

**Remark 1.** The boundedness of \((i, j)\)-bounded APA is only one-way, i.e., for relationships of the form \(b \not\in [a]\) and not of the form \(b \in [a]\). In particular, it is allowed to have \(b \in [a]\) even if this is witnessed only by programs that execute statements of type \(i\) more than \(j\) times.

It is clear that bounded versions of APA do not necessarily have a unique solution, but any solution suffices as long as (i) every points-to relationship \(b \in [a]\) reported has a witness, and (ii) all points-to relationships that have a bounded witness are reported (wrt the given bound). Similar techniques for witness bounding are used widely in practice in order to speed up static analyses.

### 2.2 Dyck Reachability and Representation of Andersen’s Pointer Analysis

Here we develop some notation on Dyck languages and Dyck reachability, and use it to represent instances of APA as Dyck graphs.

**Dyck languages.** Given a non-negative integer \(k \in \mathbb{N}\), we denote by \(\Sigma_k = \{\varepsilon\} \cup \{\alpha_i, \overline{\alpha}_i\}_{i=1}^k\) a finite alphabet of \(k\) parenthesis types, together with a null element \(\varepsilon\). We denote by \(D_k\) the Dyck language over \(\Sigma_k\), defined as the language of strings generated by the following context-free grammar \(G_k:\)

\[
S \rightarrow S S \mid \alpha_1 S \overline{\alpha}_1 \mid \cdots \mid \alpha_k S \overline{\alpha}_k \mid \varepsilon
\]

In words, \(D_k\) contains all strings where parenthesis are properly balanced. In this work we focus on the special case where \(k = 1\), i.e., we have only one parenthesis type. To capture the relationship between \(D_1\) and APA, we will let \(\alpha_1\) be \(\&\) and \(\overline{\alpha}_1\) be \(\ast\). This relationship will become clearer later in this section.

**Dyck graphs.** A Dyck graph \(G = (V, E)\) is a digraph where edges are labeled with elements of \(\Sigma_1\), i.e., \(E \subseteq V \times V \times \Sigma_1\) and edges have the form \(\tau = (a, b, \lambda)\). Often we will only be interested in the endpoints \(a, b\) of an edge, in which case we represent \(\tau = (a, b)\), and we will denote by \(\lambda(\tau)\) the label of \(\tau\). The label \(\lambda(P)\) of a path in \(G\) is the concatenation of the labels along the edges of \(P\).

We often represent \(P\) from \(a\) to \(b\) graphically as \(a \xrightarrow{\lambda(P)} b\), and, given some \(i \in \mathbb{N}\), we write \(a \xrightarrow{i} b\) (resp., \(a \xrightarrow{\lambda} b\)) to denote a path \(P: a \xrightarrow{\lambda} b\) with label \(i\) consecutive symbols \(\&\) (resp., \(\ast\)), possibly interleaved with \(\varepsilon\) symbols. We say that \(b\) is Dyck-reachable (or \(D\)-reachable) from \(a\) if there exists a path \(P: x \xrightarrow{\lambda} y\) with \(\lambda(P) \in D\). We say that \(b\) flows into \(a\) via a node \(c\) if (i) we have \(b \xrightarrow{\&} c\), and (ii) \(a\) is \(D\)-reachable from \(c\). The \(D_1\)-Reachability problem takes as input a Dyck graph and asks to return all pairs of nodes \((b, a)\) such that \(a\) is \(D\)-reachable from \(b\).

**Graph representation of APA.** For convenience, we frequently represent instances of APA using Dyck graphs. In particular, given an instance \((A, S)\) of APA, we use a Dyck graph \(G = (A, E)\) where \(E\) represents all statements in \(S \setminus S_4\). In particular, we have the following edges.

1. For every type 1 statement \(a = b\), we have \(b \xrightarrow{\varepsilon} a\) in \(E\).
2. For every type 2 statement \(a = \& b\), we have \(b \xrightarrow{\&} a\) in \(E\).
3. For every type 3 statement \(a = \ast b\), we have \(b \xrightarrow{\ast} a\) in \(E\).
The instance \((A, S)\) is represented as a pair \((G, S_4)\) where \(G = (A, E)\) is the Dyck graph and \(S_4\) is the type-4 statements of \(S\). See Figure 3 for an illustration. The motivation behind this representation comes from the following lemma, which establishes a correspondence between paths in Dyck graphs and APA without statements of type 4.

**Lemma 2.1.** Consider the Dyck graph representation \((G = (E, V), S_4)\) of \((A, S)\), and the modified APA instance \((A, S \setminus S_4)\). For every two pointers \(a, b \in A\), we have \(b \in \boxed{a}\) in \((A, S \setminus S_4)\) iff \(b\) flows into \(a\) in \(G\).

**Resolved Dyck graphs.** Consider an APA instance \((A, S)\) and the corresponding Dyck-graph representation \((G = (A, E), S_4)\). The resolved Dyck graph \(\overline{G} = (A, \overline{E})\) is a Dyck graph where \(\overline{E}\) is the smallest set that satisfies the following conditions.

1. \(E \subseteq \overline{E}\).
2. For every statement \(*a = b\), for every node \(c\) that flows into \(a\), we have \(b \xrightarrow{\epsilon} c\).

Intuitively, all type 4 statements have been resolved as \(\epsilon\)-edges in \(\overline{G}\). See Figure 3 for an illustration. The following lemma follows directly from Lemma 2.1.

**Lemma 2.2.** For every two pointers \(a, b \in A\), we have \(b \in \boxed{a}\) iff \(b\) flows into \(a\) in \(\overline{G}\).

**Intuition behind the Dyck-graph representation.** Virtually all algorithms for APA in the literature use a flow graph for representing inclusion relationships between pointers. Pointer inclusion occurs when the analysis discovers that, for two pointers \(a, b\) we have \(\boxed{b} \subseteq \boxed{a}\), represented via an edge \(b \rightarrow a\) in the flow graph.

Our Dyck graph \(G\) is a richer structure compared to the standard flow graph. In fact, we obtain the (initial) flow graph if we remove from \(G\) edges representing pointer references (labeled with \&\&) and dereferences of type 3 (labeled with \(*\)). The only information missing from \(G\) is statements of type 4. The analysis can be seen as iteratively discovering type 4 statements \(*a = b\) and inserting an edge \(b \xrightarrow{\epsilon} c\) in \(G\). This process terminates when we have constructed the resolved graph \(\overline{G}\). Lemma 2.2 implies that, at that point, all points-to information can be expressed as flows-into relationships in \(\overline{G}\).

### 2.3 Summary of Main Results

We are now ready to present our main theorems, followed by a discussion on their implications to the theory and practice of pointer analysis. In later sections we develop the proofs.

**Cubic upper-bound of APA.** It is well stated that APA can be solved in cubic time. However, “cubic” refers to the size of the input, and typically has the form \(O(m^3)\) [Melski and Reps 2000;
Møller and Schwartzbach 2018] or $O(n^2 \cdot m)$ [Kodumal and Aiken 2004; Pearce et al. 2004], for $n$ pointers and $m$ statements. Note that $m$ can be as large as $\Theta(n^2)$, which yields the bound $O(n^4)$, as already observed in [Kodumal and Aiken 2004]. Our first theorem shows that in fact, the problem is solvable in $O(n^3)$ time regardless of $m$. Although we do not consider this our major result, we are not aware of a proven $O(n^3)$ bound. We also hope that the theorem will provide a future reference for a formal $O(n^3)$ complexity statement for APA.

**Theorem 2.3.** Exhaustive APA is solvable in $O(n^3)$ time, for any $m$, where $n$ is the number of pointers and $m$ is the number of statements.

One other advantage of the algorithm behind Theorem 2.3 is that it yields a better bound for sparse instances, as we outline below.

**Faster sparse APA.** We next turn our attention to a special version of sparse\(^2\) APA, which is known to arise naturally under standard conventions in some settings [Sridharan and Fink 2009]. In that work, the authors show that Sparse Exhaustive APA can be solved in $O(n^2)$ time. Perhaps surprisingly, we show that the problem can be solved even faster.

**Theorem 2.4.** Sparse Exhaustive APA is solvable in $O(n^3/2)$ time, where $n$ is the number of pointers.

We note that [Sridharan and Fink 2009] presents an informal argument for why the $O(n^2)$ bound should be tight: as the size of the points-to set of a pointer can be $\Theta(n)$, and there are $n$ pointers, the size of the output might be $\Theta(n^2)$ which matches the running time of their algorithm. Our first observation towards Theorem 2.4 is that, although large points-to sets can be of size $\Theta(n)$, the conditions for sparsity imply that the average points-to set has only constant size, and hence the size of the output is only $\Theta(n)$, as opposed to $\Theta(n^2)$.

**Cubic hardness of APA.** Given the cubic upper-bound of Theorem 2.3, it is natural to ask whether sub-cubic algorithms exist for the problem, i.e., algorithms with running time $O(n^{3-\epsilon})$, for some fixed $\epsilon > 0$. Indeed, the rich literature of heuristics (e.g., [Aiken et al. 1997; Berndl et al. 2003; Dietrich et al. 2015; Fähndrich et al. 1998; Hardekopf and Lin 2007; Heintze and Tardieu 2001b; Pearce et al. 2004; Pek and Madhusudan 2014; Rountev and Chandra 2000; Su et al. 2000; Vedurada and Nandivada 2019; Xu et al. 2009]) is indicative of the need for such an improvement. On the other hand, no lower-bound has been known. In fine-grained complexity, there is a widespread distinction between combinatorial and algebraic algorithms. The most famous combinatorial lower bound is for Boolean Matrix Multiplication (BMM). The respective hypothesis states that there is no combinatorial $O(n^{3-\epsilon})$ algorithm for multiplying two $n \times n$ Boolean matrices, for any fixed $\epsilon > 0$. The BMM hypothesis has formed the basis for many lower bounds in graph algorithms, verification, and static analysis [Abboud and Vassilevska Williams 2014; Bansal and Williams 2009; Chatterjee et al. 2018, 2016; Williams and Williams 2018].

Given the combinatorial nature of APA, we examine whether combinatorial sub-cubic improvements are possible. First, note that the edge set $\{((x_i, x_j))\}$ of a digraph can be represented as a set of pointer assignments $\{(x_j = x_i)\}$. This observation leads us to the following remark.

**Remark 2.** Even without statements of type 3 and type 4 (i.e., without pointer dereferences), Exhaustive APA is at least as hard as computing all-pairs reachability in a digraph. Since, under the BMM hypothesis, all-pairs reachability does not have a combinatorial sub-cubic algorithm, the same lower-bound follows for Exhaustive APA.

\(^2\)We note that sparsity here is a stronger assumption than simply requiring $m = O(n)$. We refer to Section 3.2 for details.
On the other hand, single-pair reachability is solvable in linear-time in the size of the graph, and is thus considerably easier than its all-pairs version. Hence, the relevant question is whether On-demand APA (i.e., given pointers $a$, $b$, is it the case that $b \in \text{[a]}$?) has sub-cubic complexity. We answer this question in negative.

**Theorem 2.5.** On-demand APA has no sub-cubic combinatorial algorithm under the BMM hypothesis.

Note that Theorem 2.5 indeed relates a problem with output size $\Theta(1)$ (On-demand APA) to a problem with output size $\Theta(n^2)$ (BMM). The theorem has four implications. First, it establishes formally the long-conjectured "cubic bottleneck" for APA. Second, it shows that the algorithm of Theorem 2.3 is optimal even for On-demand APA, as far as combinatorial algorithms are concerned. Third, it indicates that the hardness of APA does not come from the requirement to produce large-sized outputs (i.e., of size $\Theta(n^2)$ for the points-to set of each pointer), as sub-cubic complexity is also unlikely for constant-size outputs. Fourth, it shows that all on-demand analyses (e.g., [Chatterjee et al. 2018; Heintze and Tardieu 2001a; Lu et al. 2013; Sridharan et al. 2005; Vedurada and Nandivada 2019; Zhang et al. 2013; Zheng and Rugina 2008]), which attempt to reduce complexity by avoiding the exhaustive computation of all points-to sets, can only provide heuristic improvements without any guarantees.

**Bounded APA.** Given the hardness of APA under Theorem 2.5, we next seek for mild restrictions that allow for algorithmic improvements over the cubic bound. Perhaps surprisingly, we show that bounding the number of times statements of type 4 are executed suffices.

**Theorem 2.6.** For all $j \in \mathbb{N}^+$, $(4, j)$-bounded Exhaustive APA is solvable in $\tilde{O}(n^\omega \cdot j)$ time, where $n$ is the number of pointers and $\omega$ is the matrix-multiplication coefficient. In particular, $(4, \tilde{O}(1))$-bounded Exhaustive APA is solvable in $\tilde{O}(n^\omega)$ time.

Here $\tilde{O}$ hides poly-logarithmic factors (i.e., factors of the form $\log^c n$, for some constant $c$). It is known that $\omega < 2.373$ [Le Gall 2014], hence the bound is sub-cubic. Besides its theoretical interest, Theorem 2.6 also has practical relevance, as it reduces the problem to a small number of matrix multiplications. First, some sub-cubic algorithms for matrix multiplication, like Strassen’s [Strassen 1969], often lead to observable practical speedups over simpler, cubic algorithms [Huang et al. 2016; Huss-Lederman et al. 1996]. Second, this reduction can take advantage of both highly optimized practical implementations for matrix multiplication [Kaporin 1999; Laderman et al. 1992] and specialized hardware [Dave et al. 2007]. Third, the algorithm behind Theorem 2.6 is an “anytime algorithm”, in the spirit of [Boddy 1991; Chatterjee et al. 2015]. The algorithm computes $(4, j)$-bounded Exhaustive APA iteratively for increasing values of $j$. It can be terminated in any iteration $j$ according to the runtime requirements of the analysis. At that point, the algorithm has executed for at most $\tilde{O}(n^\omega \cdot j)$ time, and is guaranteed to have computed all points-to relationships as witnessed by $(4, j)$-bounded programs. Hence, (i) a timeout does not waste analysis time, and (ii) the obtained results provide measurable completeness guarantees.

It is believed that $\omega = 2 + o(1)$, in which case Theorem 2.6 yields a quadratic bound. Given such an improvement, a natural question is whether sub-quadratic algorithms are possible when we restrict our attention to witnesses that are poly-logarithmically bounded. Clearly this is not possible for Exhaustive APA, as the size of the output can be $\Theta(n^2)$, but the question becomes interesting in the case of On-demand APA that has output size $\Theta(1)$. We answer this question in negative.

**Theorem 2.7.** (All, $\tilde{O}(1)$)-bounded On-demand APA has no sub-quadratic algorithm under the Orthogonal Vectors hypothesis.
Orthogonal Vectors is a well-studied problem with a long-standing quadratic worst-case upper bound. The corresponding hypothesis states that there is no sub-quadratic algorithm for the problem [Williams 2019]. It is also known that the strong exponential time hypothesis (SETH) implies the Orthogonal Vectors hypothesis [Williams 2005]. Under Theorem 2.7, our algorithm from Theorem 2.6 is optimal when \( \omega = 2 + o(1) \).

Finally, to establish Theorem 2.6, we solve \( D_1 \)-Reachability in nearly matrix-multiplication time. To our knowledge, this is the first sub-cubic algorithm for the problem and will likely be of independent interest.

**Theorem 2.8.** All-pairs \( D_1 \)-Reachability is solvable in \( \tilde{O}(n^\omega) \) time, where \( n \) is the number of nodes and \( \omega \) is the matrix-multiplication coefficient.

Observe the different regimes of \( D_k \)-Reachability for various values of \( k \). When \( k = 0 \), the problem becomes standard graph reachability, which is solvable in \( O(n^\omega) \) time [Munro 1971]. When \( k \geq 2 \), the problem is solvable in \( O(n^3) \) time, and this bound is believed to be tight (wrt polynomial improvements) [Chatterjee et al. 2018; Heintze and McAllester 1997]. The case of \( k = 1 \) had remained open, and Theorem 2.8 establishes that the problem is almost as easy as for \( k = 0 \).

In the following sections we present details of the above theorems. To improve readability, in the main paper we present algorithms, examples, and proofs of all theorems. To highlight the main steps of the proofs, we also present all intermediate lemmas; many lemma proofs, however, are relegated to the appendix.

## 3 A NEW ALGORITHM FOR ANDERSEN’S POINTER ANALYSIS

In this section, we present a new algorithm for APA, which establishes Theorem 2.3 and Theorem 2.4.

### 3.1 A Cubic Algorithm for Andersen’s Pointer Analysis

We start by sharpening the cubic upper-bound of APA. In this section we prove Theorem 2.3, i.e., present a simple algorithm for solving Exhaustive APA in \( O(n^3) \) time, for \( n \) pointers. To our knowledge, this is the first proof of an \( O(n^3) \) bound even when the number of statements is \( m = \Theta(n^2) \).

**Algorithm AndersenAlgo.** The algorithm performs a form of dynamic transitive closure of a flow graph, in similar spirit to existing algorithms in the literature. The key difference is that instead of just maintaining inclusion relationships \( a \xrightarrow{\epsilon} b \), the flow graph also explicitly captures points-to relationships \( a \xrightarrow{k} b \) in its edges. In each iteration, the algorithm processes a newly inserted edge \( a \xrightarrow{t} b \), where \( t \in \{\epsilon, &\} \), and inserts new edges that represent flows-into and inclusion relationships that are implied by \( a \xrightarrow{t} b \), the current state of the flow graph, and the statements in \( S \). See Algorithm 1 for a detailed description.

**Proof of Theorem 2.3.** The correctness of the algorithm follows by a straightforward induction. Here we argue about the complexity. The initialization clearly runs in time \( O(n + m) \). The main loop in Line 10 is executed at most twice for every pair of pointers \((a, b)\), hence \( O(n^2) \) times in total. For every such pair, the loops in Lines 20, 23, 26 and 34 are executed once for each pointer \( c \). Hence, these loops are executed in \( O(n^3) \) time in total. Finally, each of the loops in Lines 37 and 40
is executed once for every pointer $a$. Summing over all statements of type 3 and type 4, we obtain that these loops are executed $O(n \cdot m)$ times in total. The desired result follows.

\[ \square \]

### 3.2 Sparse Andersen’s Pointer Analysis in $O(n^{3/2})$ Time

We now turn our attention to sparse instances of APA. This is a relevant setting as natural conventions in some programming languages yield sparse instances. We refer to [Sridharan and Fink 2009] for details. Under the sparsity conditions, that work decreases the APA analysis time to $O(n^2)$. Here we show that sparsity allows for even faster algorithms. We start by making the sparsity conditions precise.

**Sparse flow graphs.** We now define in detail the flow graph, which is the standard data-structure used in the literature to perform APA. Recall our definition of the resolved Dyck graph $\overline{G} = (A, \overline{E})$ of an APA instance $(A, S)$. The associated (final) flow graph is defined as $\hat{G} = (A, \hat{E})$, where

$$\hat{E} = \{(a, b) : b \text{ is D-reachable from } a \text{ in } \overline{G}\}$$

Following [Sridharan and Fink 2009], given a constant $c > 0$, the instance $(A, S)$ is $c$-sparse if the following conditions hold.

1. $|\hat{E}| \leq c \cdot n$.
2. For every pointer $a$, there are at most $c$ statements of type 3 and type 4 that dereference $a$.  

\[ \text{The Fine-Grained Complexity of Andersen’s Pointer Analysis} \]
(3) Every pointer $a$ is referenced at most $c$ times\(^3\).

We let Sparse Exhaustive APA be the Exhaustive APA problem restricted to $c$-sparse inputs, for some constant $c$. We show that our previous algorithm AndersenAlgo (Algorithm 1) yields an $O(n^{3/2})$ time bound for Sparse Exhaustive APA.

**Complexity analysis.** We first set up some helpful notation. Given a pointer $a$, we let
\[
    p_a = |\{ b \in A : (b, a) \in \hat{E} \}| \cdot |\{ b \in A : (a, b) \in \hat{E} \}|
\]
and
\[
    q_a = |[a]| \cdot |\{ b \in A : (a, b) \in \hat{E} \}|
\]
i.e., $p_a$ is the product of the incoming and outgoing neighbors of $a$ in the flow graph $\hat{G}$, and $q_a$ is the product of the size of the points-to set of $a$ with the outgoing neighbors of $a$ in $\hat{G}$. Our complexity analysis relies on the following key insights.

(1) In general, there exist pointers with points-to sets that have size linear in $n$. At a first glance, this results in a quadratic bound in the total size of all points-to sets. Indeed, this is also stated in [Sridharan and Fink 2009]. However, a tighter analysis shows that, despite some large points-to sets, the average size of points-to sets is only constant.

(2) Assume that a flow edge $(a, b, e)$ has just been extracted from the worklist in Line 10, and consider the loop in Line 20. As the algorithm performs a form of transitive closure, it will iterate over all nodes $c$ for which it is already established that $(b, c, e)$, and will establish that $(a, c, e)$ also holds by transitivity. At a first glance, this results in a quadratic bound, as by sparsity there are $O(n)$ edges $(b, c, e)$, and for each such edge there are $O(n)$ pointers $a$ for which we might discover that $(a, b, e)$ also holds. Note, however, that this counting argument overshoots, as it will result in $\Theta(n^2)$ edges $(a, c, e)$, which violates the sparsity of $(A, S)$. A tighter observation is that for every pointer $b$, the loop will be executed $p_b$ times. In addition, since $\hat{G}$ is transitively closed, $p_b$ counts (asymptotically) the number of triangles of $\hat{G}$ that contain $b$. Summing over all $b$, the number of times that the loop is executed is bounded by the number of triangles in $\hat{G}$, which is only $O(n^{3/2})$ [Suri and Vassilvitskii 2011].

The above intuition is formally captured in the following lemma.

**Lemma 3.1.** We have (i) $\sum_{a \in A} |[a]| = O(n)$, (ii) $\sum_{a \in A} p_a = O(n^{3/2})$, and (iii) $\sum_{a \in A} q_a = O(n^{3/2})$.

Based on Lemma 3.1 we prove Theorem 2.4.

**Proof of Theorem 2.4.** We first consider the initialization phase. The first loop in Line 2 is executed $O(n)$ times, since by sparsity, every pointer is referenced at most $c = O(1)$ times. The second loop in Line 6 is executed $O(n)$ times, since every iteration results in an edge in the flow graph $\hat{G}$, and we have $|\hat{E}| = O(n)$, by sparsity.

We now proceed to the function ProcessEps. It is straightforward to see that, for a fixed pointer $b$, the loop in Line 20 is executed at most $p_b$ times. Similarly, for a fixed pointer $a$, the loop in Line 23 is executed at most $p_a$ times, and the loop in Line 26 is executed at most $q_a$ times. By Lemma 3.1, we have $\sum_{a \in A} p_a = O(n^{3/2})$ and $\sum_{a \in A} q_a = O(n^{3/2})$, hence the total running time for the function ProcessEps is $O(n^{3/2})$.

We proceed to the function ProcessRef. Similarly as before, for a fixed pointer $b$, the loop in Line 34 is executed at most $q_b$ times, and thus the loop takes $O(n^{3/2})$ total time for all pointers. Now we turn our attention to the loops in Line 37 and Line 40. Given a pointer $b$, let $r_b$ be an upper-bound on

---

\(^3\)Although this condition is not made explicit in [Sridharan and Fink 2009], it is directly implied by the condition that each program method has constant size. See [Sridharan and Fink 2009, Section 4.2] for details.
number of statements of type 3 and type 4 that dereference $b$. By sparsity, we have $r_b \leq c = O(1)$. The number of times that each of the loops in Line 37 and Line 40 is executed is bounded by

$$\sum_{b \in A} |\{b\}| \cdot r_b \leq c \cdot \sum_{b \in A} |\{b\}| = c \cdot O(n) = O(n),$$

where the second-to-last equality follows from point (i) of Lemma 3.1. Hence the total running time for the function $\text{ProcessRef}$ is $O(n^{3/2}) + O(n) = O(n^{3/2})$.

Finally, note that in our analysis, we have used constant time for the operations of set insertion and membership. This can be easily achieved by implementing each set as a bit-array. Since some of the sets can be of size $\Theta(n)$ (e.g., for storing the neighbors of a high-degree node), performing a naive initialization to all sets would require $\Theta(n^2)$ time. We avoid this cost by using a known technique for constant-time array initialization, regardless of the size of the set [Aho and Hopcroft 1974]. Hence the total time for initializing the data structures is $O(n)$.

The desired result follows. 

We conclude this section with the following remark.

**Remark 3.** The $O(n^2)$-time complexity analysis of the difference-propagation algorithm in [Sridharan and Fink 2009] is tight. Indeed, even just computing set differences results in a $\Theta(n^2)$ bound. In fact, all algorithms in the literature have an $\Omega(n^2)$ lower-bound for Sparse Exhaustive APA. Thus, Algorithm 1 is the first algorithm to break this quadratic bound for the problem.

### 4 THE CUBIC HARDNESS OF ANDERSEN’S POINTER ANALYSIS

In this section we prove Theorem 2.5, i.e., that even On-demand APA (given pointers $a, b$, is it that $b \in \lfloor a \rfloor$?) does not have a combinatorial sub-cubic algorithm under the BMM hypothesis. For the proof, we establish a fine-grained reduction [Williams 2019] from the problem of deciding whether a graph contains a triangle.

**Reduction from finding triangles.** Consider an undirected graph $H = (V, E)$, of $n'$ nodes, where the task is to determine if $H$ contains a triangle. For notational convenience, we take the node set of $H$ to be the set of integers $[n']$. Hence, the task is to determine if there exist distinct $i, j, k \in [n']$ such that $(i, j), (j, k), (k, i) \in E$. For our reduction, we construct an instance of On-demand APA as follows. See Figure 4 for an illustration.

1. For every node $i \in [n']$, we introduce four pointers $a_i, b_i, c_i, d_i$. We also introduce a distinguished pointer $s$.
2. For every $(i, j) \in E$ with $j < i$, we have (i) $a_i = b_j$, and (ii) $b_j = \&c_j$.
3. For every $(i, j) \in E$ with $j > i$, we have $\star a_i = d_j$.
4. Finally, we have the following sets of assignments.

$$d_1 = \&s \quad d_2 = \&d_1 \quad \cdots \quad d_n = \&d_{n-1} \quad \text{and} \quad c_1 = \star c_2 \quad c_2 = \star c_3 \quad \cdots \quad c_{n-1} = \star c_n.$$ 

The on-demand question is whether $s \in [c_1]$. Observe that the number of pointers of our APA instance is $O(n')$, and the above construction can be easily carried out in time proportional to the size of $H$.

**Correctness.** We now establish the correctness of the above construction. The key idea is as follows. Recall our definition of the resolved Dyck graph $\overline{G}$ from Section 2.2. An edge $d_k \rightarrow c_k$ is
Fig. 4. An undirected graph (left) and the corresponding On-demand APA instance (right).

inserted in $\overline{G}$ iff $k$ is both a distance-1- and distance-2 neighbor of some node $i$. In turn, this implies the existence of a triangle in $H$ that contains $i$ and $k$. We have the following lemma.

**Lemma 4.1.** We have that $s$ flows into $c_1$ in $\overline{G}$ iff $H$ has a triangle.

**Proof.** We prove each direction separately.

$(\Rightarrow)$. Assume that $H$ has a triangle $(i, j, k)$, with $i > j > k$. Then we have $a_i = b_j$ and $b_j = \&c_k$ and thus $c_k \to a_j$ in $\overline{G}$. In addition, we have $\ast a_i = d_k$, and thus $d_k \to c_k$ in $\overline{G}$. Observe that this creates a path $s \to d_k \to c_k \to (k-1) \ast c_1$, which witnesses that $s$ flows into $c_1$ in $\overline{G}$.

$(\Leftarrow)$. Assume that $s$ flows into $c_1$. Observe that for all $a_i$, if some node $x$ flows into $a_i$ then $x$ is a $c$ node. It follows that $\overline{G}$ is identical to $G$ with some additional edges from $d$ nodes to $c$ nodes. Hence, since $s$ flows into $c_1$, there exists some $k \in [n']$ such that $\overline{G}$ has an edge $d_k \to c_k$. This means that (i) there exists an $i$ such that $c_k$ flows into $a_i$ (thus $k$ is a distance-2 neighbor of $i$), and (ii) there is a statement $\ast a_i = d_k$ (thus $k$ is a distance-1 neighbor of $i$). Hence $H$ has a triangle containing $i$ and $k$. The desired result follows. \hfill \Box

We conclude this section with the proof of Theorem 2.5.

**Proof of Theorem 2.5.** Due to Lemma 4.1, we have that $s$ flows into $c_1$ iff $H$ contains a triangle. By Lemma 2.2 we have that $s \in [c_1]$ iff $H$ contains a triangle. By [Williams and Williams 2018], triangle detection has no sub-cubic combinatorial algorithm under the combinatorial BMM-hypothesis.

The desired result follows. \hfill \Box

### 5 A SUB-CUBIC ALGORITHM FOR BOUNDED ANDERSEN’S POINTER ANALYSIS

In this section, we first show Theorem 2.6, i.e., that computing points-to relationships when bounding the number of applications of type 4 statements admits a sub-cubic algorithm. To this end, we first prove in Section 5.1 Theorem 2.8, i.e., that $D_1$-Reachability can be solved in nearly matrix-multiplication time. Afterwards, we use this result to prove Theorem 2.6 in Section 5.2. Finally, in Section 5.3 we show the quadratic lower bound of Theorem 2.7.
5.1 A Sub-cubic Algorithm for $D_1$-Reachability

In this section we show that all-pairs $D_1$-Reachability can be solved in nearly matrix-multiplication time. We first set up some helpful notation, and then present the main algorithm. Consider a Dyck graph $G = (V, E)$.

Path indexing. Consider a path $P = x_1, \ldots, x_l$. Given some $i \in [l]$, we denote by $P[i] = x_i$. Given $i, j \in [l]$, with $i \leq j$, we denote by $P[i : j] = x_i, \ldots, x_j$. For simplicity, we let $P[1 : j] = P[1 : j]$ and $P[i : 1] = P[i : l]$.

Stack heights. Consider a path $P$ of length $l$. We denote by $\#_k(P)$ (resp., $\#_i(P)$) the number of & (resp., *) symbols that appear in the label $\lambda(P)$. The stack height of $P$ is defined as $\text{SH}(P) = \#_k(P) - \#_i(P)$ (note that we can have $\text{SH}(P) < 0$). The maximum stack height of a path is defined as $\text{MSH}(P) = \max_{P'} \text{SH}(P')$, where $P'$ ranges over prefixes of $P$.

Monotonicity and local maxima. Consider a path $P$ of length $l$. We say that $P$ is monotonically increasing (resp., monotonically decreasing) if for all $i$ with $1 \leq i < l$, we have $\text{SH}(P[i : i]) \leq \text{SH}(P[i : i + 1])$ (resp., $\text{SH}(P[i : i]) \geq \text{SH}(P[i : i + 1])$). Given some $i$ with $1 \leq i \leq l$, we say that $P$ has a local maxima in $i$ if the following conditions hold.

1. Either $i = 1$ or $\text{SH}(P[i : i - 1]) < \text{SH}(P[i : i])$.
2. For every $j > i$ such that $\text{SH}(P[i : l]) < \text{SH}(P[i : j])$, there exists some $l$ with $i < l < j$ such that $\text{SH}(P[i : l]) < \text{SH}(P[i : i])$.

Bell-shape-reachability. We call a path $P$ bell-shaped if it has exactly one local maxima. If $P$ is bell-shaped, it can be decomposed as $P = P_1 \circ P_2$ where $P_1$ (resp., $P_2$) is a monotonically increasing (resp., monotonically decreasing) path. Consider two nodes $x$, $y$. We say that $y$ is $i$&-reachable (resp., $i*$-reachable) from $x$, for some $i \in \mathbb{N}$, if there is a path $x \rightsquigarrow_{i\&}^i y$ (resp., $x \rightsquigarrow_{i*}^i y$). We say that $y$ is bell-shape-reachable from $x$ if there exists a bell-shaped path $x \rightsquigarrow^i y$.

Node distances. Given two nodes $x$, $y$, we define the distance $\delta(x, y)$ from $x$ to $y$ as the length of the shortest path $P : x \rightsquigarrow y$ with $\lambda(P) \in D_1$, if such a path exists, otherwise $\delta(x, y) = \infty$. The maxima-distance $\gamma(x, y)$ is the smallest number of local maxima among all shortest paths $x \rightsquigarrow y$. The following known lemma states that the distances between two reachable nodes is at most quadratic. Note that $\gamma(x, y) \leq \delta(x, y)$, hence the same bound holds for the maxima-distance.

Lemma 5.1 ([Deleage and Pierre 1986]). For every $x, y \in V$, if $\delta(x, y) < \infty$ then $\delta(x, y) = O(n^2)$.

Routine BellReachAlgo. The main component of our algorithm for $D_1$-Reachability is a routine BellReachAlgo that computes bell-shape-reachability. Given an input Dyck graph $G = (V, E)$, BellReachAlgo computes all pairs of nodes $(x, y)$ such that $y$ is bell-shape-reachable from $x$. The algorithm constructs a sequence of $O(\log L)$ plain (i.e., not Dyck) digraphs $(G_i = (K, R_i))_i$, where $L$ is an upper bound on the distance $\delta(x, y)$ of every pair of nodes $x, y \in V$, given by Lemma 5.1. The node set $K$ is common to all $G_i$ and consists of three copies $x_1, x_2, x_3$ for every node $x \in V$. In iteration $i$, the algorithm performs all-pairs reachability in $G_i$, and using this reachability information, constructs the edge set $R_{i+1}$. In high level, $G_i$ consists of three copies of the graph $G$, where bell-shaped paths of maximum stack height at most $2^i - 1$ are summarized as $\epsilon$ edges in the first and second copy. Paths between the nodes in the first and third copy are used to summarize monotonically increasing and (resp., decreasing) paths in $G$ with labels of the form $2i$& (resp., $2i*$). We refer to Algorithm 2 for a detailed description and to Figure 5 for an illustration.
Algorithm 2: BellReachAlgo

Input: A Dyck graph $G = (V,E)$
Output: A set $\{(x,y) | x, y \in V\}$ such that $y$ is D-reachable from $x$.

// Initialization
1. Construct a node set $K = \{x_1, x_2, x_3 : x \in V\}$
2. Construct an edge set $R_1$, initially $R_1 \leftarrow \emptyset$
3. foreach $j \in [2]$
   4. Insert $(x_j, y_j) \in R_1$ iff $(x, y, e) \in E$
   5. Insert $(x_j, y_{j+1}) \in R_1$ iff $(x, y, \& \) \in E$
   6. Insert $(y_{j+1}, x_j) \in R_1$ iff $(x, y, \epsilon ) \in E$
4. Construct the graph $G_1 = (K, R_1)$
5. Let $L \leftarrow$ an upper bound on $\delta(x,y)$ for all $x, y \in V$

// Computation
6. foreach $i \in [\lceil \log L \rceil ]$
   7. Compute all-pairs reachability in $G_i$
   8. Construct an edge $R_{i+1}$, initially $R_{i+1} \leftarrow \emptyset$
   9. foreach $j \in [2]$
      10. Insert $(x_j, y_i) \in R_{i+1}$ iff $x_1 \leadsto x_1$ in $G_i$
      11. Insert $(x_i, y_{j+1}) \in R_{i+1}$ iff $x_1 \leadsto y_3$ in $G_i$
      12. Insert $(y_{j+1}, x_j) \in R_{i+1}$ iff $y_3 \leadsto x_1$ in $G_i$
   13. end
   14. Construct the graph $G_{i+1} = (K, R_{i+1})$
   15. end
   16. return $R_{\lceil \log L \rceil + 1}$

Fig. 5. Illustration of BellReachAlgo on the Dyck graph $G$ (left). Bell-shape-reachability in $G$ as witnessed by paths $P: x \leadsto y$ with $\text{MSH}(P) \leq 2^i - 1$ is captured in graph $G_i$ (right) by the path $x_1 \leadsto y_1$. Dashed edges in $G_i$ represent the summarization of the path, which is carried over to $G_{i+1}$ as a single edge.

Correctness of BellReachAlgo. It is straightforward that for each iteration $i$, if $x_1 \leadsto y_1$ in $G_i$, then $y$ is D-reachable from $x$ in $G$. The following lemma captures the inverse direction restricted to bell-shaped paths, i.e., if $x \leadsto y$ via a bell-shaped path $P$ in $G$ with $\text{MSH}(P) \leq 2^i - 1$, then $x_1 \leadsto y_1$ in $G_i$. The key invariants are stated in the following lemma.

**Lemma 5.2.** Consider an execution of the routine BellReachAlgo. For each $i \in [\lceil \log L \rceil ]$, the following assertions hold.

1. If $y$ is D-reachable from $x$ via a bell-shaped path $P$ in $G$ with $\text{MSH}(P) \leq 2^i - 1$, then $x_1 \leadsto y_1$ in $G_i$.
2. If $x \leadsto y$ via a monotonically increasing path in $G$, then $x_1 \leadsto y_3$ in $G_i$.
3. If $y \leadsto x$ via a monotonically decreasing path in $G$, then $y_3 \leadsto x_1$ in $G_i$.

Algorithm D$_1$-ReachAlgo. We are now ready to describe our algorithm D$_1$-ReachAlgo for D$_1$-Reachability. The algorithm performs $\lceil \log L \rceil$ iterations of BellReachAlgo, where $L$ is an upper bound on the distance between any two reachable nodes in $G$. See Algorithm 3 for a detailed description.
Algorithm 3: $D_1$-ReachAlgo

Input: A Dyck graph $G = (V, E)$

Output: A set $\{(x, y)\}_{x, y \in V}$ such that $y$ is bell-shape-reachable from $x$.

// Initialization
1 Let $G_1 = (V, E_1)$ be a Dyck graph with $E_1 = E$

// Computation
2 foreach $i \in [(\log L) + 1]$ do
3 Let $X = BellReachAlgo$ on input $G_i$
4 Let $E_{i+1} = E_i \cup \{(x, y, e) : (x, y) \in X\}$
5 Construct the graph $G_{i+1} = (V, E_{i+1})$
6 end
7 return $E^{[\log L]+1}$

---

**Correctness of $D_1$-ReachAlgo.** We now establish the correctness of $D_1$-ReachAlgo. We start with an intuitive description of the correctness, and afterwards we make the argument formal (see Figure 6 for an illustration).

*Intuitive argument of correctness.* Consider an iteration $i$, and let $x, y \in V$ such that $y$ is $D$-reachable from $x$ via a path $P: x \rightsquigarrow y$. At the end of the iteration, due to the execution of routine BellReachAlgo, all bell-shaped paths $u \rightsquigarrow v$ in $G_i$ are summarized as $\epsilon$-edges $u \xrightarrow{\epsilon} v$ in $G_{i+1}$. Hence, $P$ is summarized by a path $P'$ in $G_{i+1}$, where the bell-shaped sub-paths of $P$ are replaced by $\epsilon$-edges in $P'$. How many times do we need to perform this iteration until the whole of $P$ is summarized by a single $\epsilon$-labeled edge? The key insight is that the number of local maxima in $P'$ is at most half of that in $P$. Hence, it suffices to compute bell-shape-reachability a number of times that is logarithmic in the maxima-distance $\gamma(x, y)$. Since $\gamma(x, y) \leq \delta(x, y)$ and $\delta(x, y) = O(n^2)$ (by Lemma 5.1), $[\log L] = O(\log n)$ iterations suffice.

*Formal correctness.* We now proceed to make the above argument formal. Given some iteration $i$ of $D_1$-ReachAlgo, consider the graphs $G_i$ and $G_{i+1}$. Let $P: x \rightsquigarrow y$ be any path that witnesses $D$-reachability of $y$ from $x$ in $G_i$. Let $(j_\ell, l_\ell)_{\ell}$ be the index pairs that mark bell-shaped sub-paths in $P$. We require that each $(j_\ell, l_\ell)_{\ell}$ is maximal, in the following way.

1. None of $P[j_{l-1}, l_{l}]$, $P[j_{l+1}, l_{l} + 1]$ and $P[j_{l-1}, l_{l} + 1]$ is bell-shaped.
2. If $\ell > 1$, then (i) $P[j_\ell, l_\ell + 1]$ is not bell-shaped, and if $P[j_{\ell-1}, l_{\ell}]$ or $P[j_{\ell-1}, l_{\ell} + 1]$ is bell-shaped, then $j_{\ell} - 1 \leq l_{\ell}$.
Intuitively, the first bell-shaped sub-path of $P$ is as long as possible, and every following bell-shaped sub-path is as wide as possible as long as it does not overlap with the previous bell-shaped sub-path. We decompose $P$ as

$$P = P_{1}^{↓} \circ P_{1}^{↑} \circ P[j_{1}: l_{1}] \circ P_{2}^{↓} \circ P_{2}^{↑} \circ P[j_{2}: l_{2}] \circ \cdots \circ P[j_{k}, l_{k}] \circ P_{k+1}^{↓},$$

where each $P_{\ell}^{↓}$ (resp., $P_{\ell}^{↑}$) is a monotonically decreasing (resp., monotonically increasing) path. Note that $P_{1}^{↓} = e$. Observe that $P$ has $k$ local maxima, one in each bell-shaped sub-path $P[j_{\ell}: l_{\ell}]$. In $G_{i+1}$, the path $P$ is summarized by a path $P'$ identical to $P$, but with all the bell-shaped sub-paths $P[j_{\ell}: l_{\ell}]$ replaced by edges $x_{j_{\ell}} \rightarrow y_{j_{\ell}}$ (see Figure 6 for an illustration). Given some index $1 \leq h \leq |P|$ with $h \not\in [j_{\ell} + 1, l_{\ell} - 1]$ for each $\ell \in [k]$, we denote by $f(h)$ the corresponding index in $P'$.

**Remark 4.** For every index $h$ of $P'$, we have $SH(P'[: h]) = SH(P[ : f(h)])$.

We first have two technical lemmas. The first lemma states that all local maxima in $P'$ appear on the first node of bell-shaped sub-paths of $P$.

**Lemma 5.3.** Assume that $P'$ has a local maxima at some $h$. Then $f^{-1}(h) = j_{\ell}$ for some $\ell \in [k]$.

The following lemma formalizes the following observation: if the beginning of a bell-shaped sub-path of $P$ marks a local maxima for $P'$, then the beginning of the next bell-shaped sub-path of $P$ cannot mark a local maxima for $P'$. This is shown by arguing that the two bell-shaped sub-paths of $P$ are next to each other, i.e., there are no monotonically decreasing and increasing paths separating them.

**Lemma 5.4.** Assume that $P'$ has a local maxima at some $h$. Then $P_{j_{\ell}+1}^{↓} = P_{j_{\ell}+1}^{↑} = e$, where $\ell = f^{-1}(h)$.

With Lemma 5.3 and Lemma 5.4, we can now formalize the insight that the maxima-distance between any two nodes halves in each iteration of $D_{1}$-ReachAlgo. Given some iteration $i$ of the algorithm, we denote by $\gamma_{i}(x, y)$ the maxima-distance from $x$ to $y$ in the graph $G_{i}$. We have the following lemma.

**Lemma 5.5.** For each $i \in [\lceil \log L \rceil]$, for any two nodes $x, y \in V$ such that $y$ is reachable from $x$ in $G$, we have that $\gamma_{i+1}(x, y) \leq \gamma_{i}(x, y)/2$.

Finally, we prove Theorem 2.8, i.e., that all-pairs $D_{1}$-Reachability is solvable in $\tilde{O}(n^{o})$ time.

**Proof of Theorem 2.8.** We first argue about the correctness of $D_{1}$-ReachAlgo. It follows immediately from the correctness of the routine BellReachAlgo that if $D_{1}$-ReachAlgo returns that $y$ is $D$-reachable from $x$ then there is a path $y$ is $D$-reachable from $x$ in $G$. Here we focus on the inverse direction, i.e., assume that there is a path $P: x \leadsto y$ in $G$ with $\lambda(P) \in D_{1}$, and we argue that $D_{1}$-ReachAlgo returns that $y$ is $D$-reachable from $x$. Recall that $\gamma_{i}(x, y)$ is the maxima-distance from $x$ to $y$ in the graph $G_{i}$ constructed by the algorithm in the $i$-th iteration. We have

$$\gamma_{i}(x, y) = y(x, y) \leq \delta(x, y) \leq L,$$

where the last inequality follows from our choice of $L$ as an upper-bound on $\delta(x, y)$. By Lemma 5.5, we have $\gamma_{i+1}(x, y) \leq \gamma_{i}(x, y)/2$ for each $i \in [\lceil \log L \rceil]$, hence after $i = [\log L]$ iterations, we have $\gamma_{i}(x, y) = 1$. Thus, in the last iteration of the algorithm, $y$ is bell-shape reachable from $x$, and by the correctness of the routine BellReachAlgo (Lemma 5.2), BellReachAlgo will return that $y$ is $D$-reachable from $x$. Thus $D_{1}$-ReachAlgo will return that $y$ is $D$-reachable from $x$ in $G$, as desired.
We now turn our attention to the complexity of BellReachAlgo. The algorithm performs $O(\log L)$ invocations to the routine BellReachAlgo. In each invocation, BellReachAlgo performs $O(\log n)$ transitive closure operations on graphs with $O(n)$ nodes. Using fast BMM [Munro 1971], each transitive closure takes $O(n^\omega)$ time. The total running time of $D_1$-ReachAlgo is $O(n^\omega \cdot \log^2 L) = \tilde{O}(n^\omega)$, as by Lemma 5.1, we have $L = O(n^2)$.

The desired result follows. □

5.2 Bounded Andersen’s Pointer Analysis in Sub-cubic Time

In the previous section we saw that $D_1$-Reachability can be solved in nearly BMM time, i.e., $\tilde{O}(n^\omega)$. In this section we show how we can use this result to speed-up bounded Exhaustive APA, towards Theorem 2.6. Recall that, given some $i \in [4]$ and $j \in \mathbb{N}$, the $(i, j)$-bounded APA asks to compute all memory locations $b$ that a pointer $a$ may point to, as witnessed by straight-line programs (under the operational semantics of Table 1) that use statements of type $i$ at most $j$ times. We start with a simple lemma that allows us to consider only instances of APA which contain only linearly many statements of type 4.

**Lemma 5.6.** Wlog, we have $|S_4| \leq n$.

**Algorithm BoundedAPAAlgo.** We now present our algorithm BoundedAPAAlgo which solves $(4, j)$ Exhaustive APA for an instance $(A, S)$ and some given $j \geq 0$. The algorithm performs $j + 1$ iterations of $D_1$-Reachability on graphs $G_i = (A, E_i)$, for $i \in [j + 1]$, where initially $G_1$ is the Dyck graph in the representation $(G_1, S_1)$ of the APA instance $(A, S)$. In iteration $i$, the algorithm solves $D_1$-Reachability in $G_i$, and then computes all pointers $c$ that flows into some pointer $a$ in $G_i$ for which there is a statement $*a = b$ in $S_4$. Then the algorithm resolves the statement by inserting an edge $(b, c, e)$ in $G_{i+1}$. See Algorithm 4 for a detailed description. We conclude this section with the proof of Theorem 2.6.

**Proof of Theorem 2.6.** The correctness follows directly from the correctness of $D_1$-ReachAlgo (Theorem 2.8). By induction, at the end of iteration $i$, BoundedAPAAlgo has solved $(4, i - 1)$-bounded Exhaustive APA, hence at the end of iteration $j + 1$ the algorithm has solved $(4, j)$-bounded Exhaustive APA.

We now turn our attention to complexity. In each iteration of the main loop in Line 3, we have an invocation to $D_1$-ReachAlgo in Line 4, which, by Theorem 2.8, takes $\tilde{O}(n^\omega)$ time. In addition, the all-pairs reachability in Line 7 can be performed using fast BMM [Munro 1971] in $O(n^\omega)$ time. Finally, by Lemma 5.6, the loop in Line 9 is executed at most $n$ times, while the inner loop in Line 10 is clearly executed at most $n$ times as well. Hence, in each iteration, the running time is dominated by the invocation to $D_1$-ReachAlgo, and thus the total time for all iterations is $\tilde{O}(n^\omega \cdot j)$.

The desired result follows. □

5.3 The Quadratic Hardness of Bounded Andersen Pointer Analysis

Finally, in this section we prove Theorem 2.7, i.e., if we restrict our attention to points-to relationships as witnessed by programs of length $\tilde{O}(1)$, even the on-demand problem has a quadratic (conditional) lower bound. Our reduction is from the problem of Orthogonal Vectors.
Algorithm 4: BoundedAPAAlgo

**Input:** An instance \((A, S)\) of APA, a bound \(j\) on statements of type 4

**Output:** A set \((a, b)\) of all points-to relationships \(b \in \{a\}\) witnessed by \((4, j)\)-bounded programs.

// Initialization
// \(|S_i| \leq n\) wlog
1. Let \((G_1 = (A, E_1), S_1)\) be the Dyck-graph representation of \((A, S)\)
2. Let \(V = \{a_1, a_2 : a \in A\}\) be a node set

// Computation
3. foreach \(i \in [j + 1]\) do
4. Solve \(D_i\)-Reachability in \(G_i\) using \(D_i\)-ReachAlgo
5. Let \(Z_i^1 = \{(a_1, b_2) : b = &a\} \text{ is a statement in } S\)
6. Let \(Z_i^2 = \{(a_2, b_2) : b \text{ is } D\text{-reachable from } a \text{ in } G_i\}\)
7. Solve all-pairs reachability in \(H_i = (V, Z_i^1 \cup Z_i^2)\)
8. Let \(E_{i+1} = E_i\)
9. foreach statement \(*a = b\) in \(S_i\) do
10. foreach \(c \in A\) with \(c_{1} \rightsquigarrow a_2\) in \(H_i\) do
11. Insert \((b, c, \epsilon)\) in \(E_{i+1}\)
12. end
13. end
14. Construct the graph \(G_{i+1} = (A, E_{i+1})\)
15. end
16. return \((a, b) : b_1 \rightsquigarrow a_2 \text{ in } H_{j+1}\)

![Algorithm 4](attachment:image)

**Orthogonal Vectors (OV).** The input to the problem is two sets \(X, Y\), each containing \(n'\) vectors in \(\{0, 1\}^D\), for some dimension \(D = \Theta(\log n')\). The task is to determine if there exists a pair \((x, y) \in (X \times Y)\) that is orthogonal, i.e., for each \(j \in D\), we have \(x[j] \cdot y[j] = 0\). The respective hypothesis states that the problem cannot be solved in time \(O(n'^{2 - \epsilon})\), for any fixed \(\epsilon > 0\).

**Reduction from OV.** Consider an instance \(X, Y\) of OV. We assume wlog that \(D\) is even. We construct an instance of On-demand APA as follows. See Figure 7 for an illustration. First, we introduce a pointer \(z\).

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Fig. 7. Reduction from OV with vector sets \(A = \{x_1, x_2\}\) and \(B = \{y_1, y_2\}\) to (All, \(O(\log n)\))-bounded APA on the pair \(s \in [t]\).
For every vector $x^j \in X$, we introduce pointers $a^j_1, \ldots, a^j_D$ and $b^j_2, b^j_4, \ldots, b^j_D$.

1. We have $z = &a^j_1$. If $x^j[1] = 0$, we also introduce a new pointer $&a^j_1$ and two assignments $z = &a^j_1$.
2. For every even $j \in [D]$, we have $*a^j_j = b^j_j$ and $a^j_j = *a^j_{j-1}$. If $x^j[j] = 0$, we also have $a^j_j = a^j_{j-1}$.
3. For every odd $j \in [D]$ with $j > 1$, we have $b^j_{j-1} = &a^j_j$. If $x^j[j] = 0$, we also introduce a new pointer $&a^j_j$ and two assignments $b^j_{j-1} = &a^j_j$ and $&a^j_j = &a^j_j$.

For every vector $y^i \in Y$, we introduce pointers $u^i_D, \ldots, u^i_1$ and $v^i_D, v^i_{D-1}, \ldots, v^i_1$.

1. We have $u^i_1 = *z$. If $y^i[1] = 0$, we also have $u^i_1 = z$.
2. For every odd $j \in [D]$, we have $*u^i_j = v^i_j$ and $u^i_j = *u^i_{j-1}$. If $y^i[j] = 0$, we also have $u^i_j = u^i_{j-1}$.
3. For every even $j \in [D]$, we have $v^i_{j-1} = &u^i_j$. If $y^i[j] = 0$, we also introduce a new pointer $&u^i_j$ and two assignments $v^i_{j-1} = &u^i_j$ and $&u^i_j = &u^i_j$.

Finally, we introduce two pointers $s$ and $t$. For every $i \in [n']$, we have $b^i_D =$ &s and $t = u^i_D$. The on-demand question is whether $s \in [t]$. Observe that we have used $n = O(n' \cdot D)$ pointers, and the above construction can be easily carried out in $O(n)$ time.

**Correctness.** We now establish the correctness of the above construction. The key idea is as follows. Recall our definition of the Dyck-graph representation $(G = (A, E), S_i)$ of the APA instance, and the resolved Dyck graph $\bar{G}$ (see Section 2.2). The resolved graph $\bar{G}$ is constructed from $G$ by iteratively (i) finding three pointers $a, b, c$ such that $a$ flows into $b$ and we have a type 4 statement $*b = c$, and (ii) inserting an edge $c \xrightarrow{\epsilon} a$ in $G$. The above construction guarantees that, for two integers $i_1, i_2 \in [n']$, the following hold by induction on $j \in [D]$.

1. If $j$ is odd, we have $v^i_j \xrightarrow{\epsilon} a^j_j$ iff $\sum_{j' \leq j} x^{i_j}[j'] \cdot y^{i_j}[j'] = 0$.
2. If $j$ is even, we have $b^i_j \xrightarrow{\epsilon} u^i_j$ iff $\sum_{j' \leq j} x^{i_j}[j'] \cdot y^{i_j}[j'] = 0$.

Once such an $\epsilon$-labeled edge is inserted for some $j$, it creates a path that leads to a flows-into relationship that leads to inserting the next $\epsilon$-labeled edge for $j + 1$ iff $x^{i_j}[j + 1] \cdot y^{i_j}[j + 1] = 0$. Note that $s$ flows into $t$ iff there exist $i_1, i_2 \in [n']$ such that $b^i_1 \xrightarrow{\epsilon} u^i_1$ which, by the above, holds if $x^{i_1}$ and $y^{i_2}$ are orthogonal. Finally, since $D = \Theta(\log n)$, the witness program for $s \in [t]$ has length $\tilde{O}(1)$.

The above idea is formally captured in the following two lemmas.

**Lemma 5.7.** If there exist $i_1, i_2 \in [n']$ such that $x^{i_1}$ and $y^{i_2}$ are orthogonal, then $s$ flows into $t$ in $\bar{G}$. Moreover, there exists a witness program $\mathcal{P}$ of length $O(\log n)$ that results in $s \in [t]$.

**Lemma 5.8.** If $s$ flows into $t$ in $\bar{G}$, there exist $i_1, i_2 \in [n']$ such that $x^{i_1}$ and $y^{i_2}$ are orthogonal.

We conclude this section with the proof of Theorem 2.7.

**Proof of Theorem 2.7.** Lemma 5.7 and Lemma 5.8, together with Lemma 2.2, state the correctness of the reduction. Note that the APA instance we constructed has $n = O(n' \cdot D)$ pointers and $m = O(n' \cdot D)$ assignments, hence it is a sparse instance. Moreover the time for the construction is $O(n' \cdot D)$. Assume that there exists some fixed $\varepsilon > 0$ such that On-demand APA can be solved in $O(n^{2-\varepsilon})$ time. Then we have a solution for the OV instance in time $O((n' \cdot D)^{2-\varepsilon})$ time, which violates the Orthogonal-Vectors hypothesis.

The desired result follows. □
6 CONCLUSION

Andersen Pointer Analysis is the standard approach to static, flow-insensitive pointer analysis, and it also forms the basis of many on-demand may/must alias analysis techniques. Despite its long history and practical importance, the complexity of the analysis had remained illusive. In this work, we have drawn a rich fine-grained complexity landscape based on various aspects of the problem. We have shown that even deciding whether a single pointer may point to a specific heap location is unlikely to have sub-cubic complexity, and we established a similar quadratic lower bound for sparse instances. These results strongly characterize the hardness of the problem. On the positive side, we have shown an improved bound for sparse instances, and have presented a bounded version of the problem that becomes solvable in nearly matrix-multiplication time. Finally, the theoretical insights we developed while analyzing the complexity of the analysis have allowed us to develop better algorithms that improve the performance in practice.
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A PROOFS

A.1 Proofs of Section 2

**Lemma 2.1.** Consider the Dyck graph representation \((G = (E, V), S_4)\) of \((A, S)\), and the modified APA instance \((A, S \setminus S_4)\). For every two pointers \(a, b \in A\), we have \(b \in [a]\) in \((A, S \setminus S_4)\) iff \(b\) flows into \(a\) in \(G\).

**Proof.** We prove each direction separately.

\((\Rightarrow)\). Assume that \(b \in [a]\), and we argue that there exists a pointer \(c\) such that (i) \(b \xrightarrow{k} c\) and (ii) \(a\) is \(D\)-reachable from \(c\), and hence \(b\) flows into \(a\). We employ the operational semantics of APA. Let \(\mathcal{P}\) be a minimal program that results in \(b \in [a]\). The proof is by induction on the length of \(\mathcal{P}\). For the base case, we have \(|\mathcal{P}| = 1\). Then \(\mathcal{P}\) consists of a single statement \(a = \&b\), hence the lemma holds for \(a = c\). Now let \(|\mathcal{P}| = \ell + 1\), and by the induction hypothesis the statement holds for all points-to relationships witnessed by programs \(\mathcal{P}'\) with length \(|\mathcal{P}'| = \ell\). We distinguish the last statement \(s\) of \(\mathcal{P}\). Note that \(s\) is either a type 1 or a type 3 statement.

1. \(s\) is of the form \(a = d\). By the induction hypothesis on \(\mathcal{P}'\), we have a node \(c'\) such that (i) \(b \xrightarrow{k} c'\) and (ii) \(d\) is \(D\)-reachable from \(c'\). But then \(a\) is also \(D\)-reachable from \(c\), and hence the lemma holds for \(c = c'\).

2. \(s\) is of the form \(a = \ast d\). Then there exists a pointer \(e\) such that \(\mathcal{P}'\) witnesses \(e \in [d]\) and \(b \in [e]\). By the induction hypothesis on \(e \in [d]\), there exists a node \(c_1\) such that (i) \(e \xrightarrow{k} c_1\) and (ii) \(d\) is \(D\)-reachable from \(c_1\). Note that this implies that \(a\) is \(D\)-reachable from \(e\). Similarly, by the induction hypothesis on \(b \in [e]\), there exists a node \(c_2\) such that (i) \(b \xrightarrow{} c_2\) and (ii) \(e\) is \(D\)-reachable from \(c_2\). It follows that \(a\) is \(D\)-reachable from \(c_2\), hence the lemma holds for \(c = c_2\).

\((\Leftarrow)\). Assume that \(b\) flows into \(a\), hence, there exists a pointer \(c\) such that (i) \(b \xrightarrow{} c\), and (ii) \(a\) is \(D\)-reachable from \(c\), and we argue that \(b \in [a]\). The proof is by induction on the label \(\lambda(P)\) of the path \(P: c \leadsto a\) that witnesses the Dyck-reachability. Note that, by construction, we have a statement \(c = \&b\), and the statement holds if \(P\) has no edges.

1. \(\lambda(P) = \epsilon\). Then we have a statement \(a = c\), which, together with \(c = \&b\), implies that \(c \in [a]\).

2. \(\lambda(P) = \&S\ast\). Then there exist intermediate nodes \(d_1, d_2\), such that \(P\) can be decomposed as \(c \xrightarrow{k} d_1 \xrightarrow{S} d_2 \xrightarrow{\ast} a\). Let \(P'\) be the intermediate \(d_1 \xrightarrow{S} d_2\) path, and by the induction hypothesis, we have \(c \in [d_2]\). By construction, we have a statement \(a = \ast d_2\), and thus \(b \in [b]\).

3. \(\lambda(P) = SS\). Then there exists an intermediate node \(d\) such that \(P\) can be decomposed as \(P: P_1 \circ P_2\), where \(P_1: c \xrightarrow{S} d\) and \(P_2: d \xrightarrow{\ast} a\). By the induction hypothesis on \(P_1\), we have \(b \in [d]\). Then, by the induction hypothesis on \(P_2\), we have \(b \in [a]\).

The desired result follows. \(\square\)

**Lemma 2.2.** For every two pointers \(a, b \in A\), we have \(b \in [a]\) iff \(b\) flows into \(a\) in \(\overline{G}\).

**Proof.** Let \(G_1, \ldots, G_\ell\) be a sequence of Dyck graphs where \(G_1 = G\) and \((G = (V, E), S_4)\) is the Dyck-graph representation of \((A, S)\), and \(G_{\ell+1}\) is constructed from \(G_\ell\) by

1. identifying all nodes \(c\) that flow into some node \(a\) for which there is a statement \(\ast a = b\), and

2. inserting an edge \((b, c, e)\) in \(G_{\ell+1}\).
Clearly this sequence is finite, and $G_\ell = \overline{G}$. It is straightforward to establish by induction that, in any $G_i$, if $b$ flows into $a$ then $b \in \llbracket a \rrbracket$. For the inverse direction, a similar induction establishes that for each $i$, if there is a $(4, i)$-bounded program $P$ that witnesses $b \in \llbracket a \rrbracket$ then $b$ flows into $a$ in $G_i$.

The desired result follows.

\section*{A.2 \ Proof of Section 3}

\begin{lemma} \label{lemma:flow-graph-count}
We have (i) $\sum_{a \in A} |\llbracket a \rrbracket| = O(n)$, (ii) $\sum_{a \in A} p_a = O(n^{3/2})$, and (iii) $\sum_{a \in A} q_a = O(n^{3/2})$.
\end{lemma}

\begin{proof}
Consider the flow graph $\hat{G} = (A, \hat{E})$. Given a pointer $a \in A$, let $n_a = \{b \in A: a \& b\}$, i.e., $n_a$ is the number of pointers referenced by $a$ in the input.

For (i), by the sparsity of the input, we have $n_a \leq c$ for each $a \in A$. Thus,

$$\sum_{a \in A} |\llbracket a \rrbracket| = \sum_{a \in A} \left(n_a + \sum_{b: (b, a) \in \hat{E}} n_b\right) \leq \sum_{a \in A} c + \sum_{(b, a) \in \hat{E}} c \leq |A| \cdot c + |\hat{E}| \cdot c \leq n \cdot c + n \cdot c^2 = O(n).$$

For (ii) and (iii), we argue as follows. Consider a Dyck graph $H = (A, R)$, where

$$R = \{(a, b, c): (a, b) \in \hat{E}\} \cup \{(a, b, \&): a \in \llbracket b \rrbracket\}$$

and note that

$$|R| = |\hat{E}| + \sum_{a \in A} |\llbracket a \rrbracket| \leq c \cdot n \cdot O(n) = O(n),$$

as by (i), we have $\sum_{a \in A} |\llbracket a \rrbracket| = O(n)$.

Since $\hat{G}$ is transitively closed, for every triplet of pointers $(a, b, c)$ such that $(b, a), (a, c) \in \hat{E}$, we have $(b, c) \in \hat{E}$. Hence the sum $\sum_{a \in A} p_a$ is bounded by the number of triangles in $H$ (in particular, the ones formed by $\epsilon$-labeled edges). Similarly, by the definition of points-to sets, for every triplet of pointers $(a, b, c)$ such that $a \in \llbracket b \rrbracket$ and $(b, c) \in \hat{E}$, we have $a \in \llbracket c \rrbracket$. Hence, the sum $\sum_{a \in A} q_a$ is bounded by the number of triangles in $H$ (in particular, the ones formed by two &-labeled edges and one $\epsilon$-labeled edge).

It is known that the number of triangles in a graph with $k$ edges is $O(k^{3/2})$ [Suri and Vassilvitskii 2011]. Hence the number of triangles in $H$ is bounded by $O(|R|^{3/2}) = O(n^{3/2})$.

The desired result follows.

\section*{A.3 \ Proofs of Section 5}

\begin{lemma} \label{lemma:bell-reach-algo}
Consider an execution of the routine BellReachAlgo. For each $i \in \{\lfloor \log L \rfloor\}$, the following assertions hold.

1. If $y$ is $D$-reachable from $x$ via a bell-shaped path $P$ in $G$ with $\text{MSH}(P) \leq 2^i - 1$, then $x_1 \leadsto y_3$ in $G_i$.
2. If $x \leadsto y$ via a monotonically increasing path in $G$, then $x_1 \leadsto y_3$ in $G_i$.
3. If $y \leadsto x$ via a monotonically decreasing path in $G$, then $y_3 \leadsto x_1$ in $G_i$.
\end{lemma}
Proof. The proof is by induction on $i$. For $i = 1$, the claim holds directly by construction. Now assume that the claim holds for some $i$, and we show it holds for $i + 1$.

We start with Item 1. Consider any bell-shaped path $P: x \rightsquigarrow y$ with $\text{MSH}(P) \leq 2^{i+1} - 1$. If $\text{MSH}(P) \leq 2^i - 1$, the claim holds by the induction hypothesis and the edge $x_1 \rightarrow y_1$ in $G_{i+1}$. Otherwise, let $j_1, j_2$ be the first and last index of $P$ such that $\text{SH}(P[j_1]) = \text{SH}(P[j_2]) = 2^i$, and $x' = P[j_1]$ and $y' = P[j_2]$. We have

$$\text{SH}(P[j_1 : j_2]) \leq \text{MSH}(P) - 2^i = 2^{i+1} - 1 - 2^i = 2 \cdot 2^i - 1 - 2^i = 2^i - 1.$$ 

In addition, since $P$ is bell-shaped, $P[j_1 : j_2]$ is also bell-shaped. By the induction hypothesis, we have $x' \rightsquigarrow y'$ in $G_i$, and by construction, $x'_2 \rightarrow y'_2$ in $G_{i+1}$. Moreover, note that $P[j_1 : j_2]$ is monotonically increasing and $P[j_2 : i]$ is monotonically decreasing, thus, by the induction hypothesis, we have $x_1 \rightarrow x'_2$ and $y'_2 \rightarrow y_1$ in $G$. By construction, we have $x_1 \rightarrow x'_2$, and $y'_2 \rightarrow y_1$ in $G_{i+1}$. Thus, we have a path $x_1 \rightarrow x'_2 \rightarrow y'_2 \rightarrow y_1$ in $G_{i+1}$, hence $x_1 \rightsquigarrow y_1$, as desired.

We proceed with Item 2, Consider any monotonically increasing path $P: x \hookrightarrow y$. Let $j$ be any index of $P$ such that $\text{SH}(P) = 2^{i+1}$, and $z = P[j]$. Note that $P[j : j]$ and $P[j : i]$ are monotonically increasing with $\text{SH}(P[j]) = \text{SH}(P[j]) = 2^{i+1}$. By the induction hypothesis, we have $x_1 \rightsquigarrow z_1$ and $z_1 \rightsquigarrow y_2$ in $G$. By construction, we have $x_1 \rightarrow z_2$ and $z_2 \rightarrow y_3$ in $G_{i+1}$, and thus $x_1 \rightsquigarrow y_3$, as desired.

Finally, Item 3 is similar to Item 2 and is omitted for brevity. The desired result follows.

Lemma 5.3. Assume that $P'$ has a local maxima at some $h$. Then $f^{-1}(h) = j_\ell$ for some $\ell \in [k]$.

Proof. Clearly, $P'[h]$ cannot be a node of some monotonically decreasing path $P_\ell \downarrow$. This is because $\text{SH}(P[l_{\ell-1} : 1])$ is larger than the stack height of $P$ in all nodes of $P_\ell \downarrow$. Hence, $P'[h]$ is a node of some non-monotonically increasing path $P_\ell \uparrow$. Due to the monotonicity of $P_\ell \uparrow$, $P'[h]$ has to be the last node of $P_\ell \uparrow$, and thus the first node of $P[j_{\ell+1} : l_\ell + 1]$. Hence, $f^{-1}(h) = j_{\ell+1}$.

The desired result follows.

Lemma 5.4. Assume that $P'$ has a local maxima at some $h$. Then $P_{j_\ell+1} \downarrow = P_{j_{\ell+1}} \uparrow = \epsilon$, where $\ell = f^{-1}(h)$.

Proof. We first argue that $P_{j_\ell+1} \downarrow = \epsilon$. Assume towards contradiction otherwise. By maximality of the bell-shaped sub-paths of $P$, we have that $\text{SH}(P[l_\ell + 1 : 1]) < \text{SH}(P[l_\ell : 1])$ (note that $P[l_\ell + 1]$ is the second node of $P_{j_\ell+1} \uparrow$). It suffices to argue that $P_{j_\ell} \uparrow \neq \epsilon$, which will violate the maximality of the bell-shaped path $P[j_\ell : l_\ell]$. Indeed, if $P_{j_\ell} \uparrow = \epsilon$, this would violate the fact that $P'$ has a maxima in $h$. Hence $P_{j_\ell+1} \downarrow = \epsilon$.

We now argue that $P_{j_\ell+1} \uparrow = \epsilon$. Indeed, given that $P_{j_\ell+1} \downarrow = \epsilon$, if $P_{j_\ell+1} \uparrow \neq \epsilon$ we would have $\text{SH}(P[j_\ell]) < \text{SH}(P[j_\ell] + 1)$, which contradicts the fact that $P'$ has a local maxima in $h$.

The desired result follows.

Lemma 5.5. For each $i \in [\lceil \log L \rceil]$, for any two nodes $x, y \in V$ such that $y$ is reachable from $x$ in $G$, we have that $y_{i+1}(x, y) \leq y_i(x, y)/2$.

Proof. Consider that $P'$ has a local maxima at some $h$. By Lemma 5.3, we have that $f^{-1}(h) = j_\ell$ for some $\ell \in [k]$. By Lemma 5.4, we have that $P_{j_\ell+1} \downarrow = P_{j_{\ell+1}} \uparrow = \epsilon$. Thus $\text{SH}(P[l_\ell]) = \text{SH}(P[j_\ell + 1])$. 


Since $P[j_{\ell'}, l_{\ell'}]$ is bell-shaped, we have $\text{SH}(P[j_{\ell'}, j_{\ell'+1}]) = \text{SH}(P[j_{\ell'+1}, j_{\ell'+1}])$ and thus $P'$ does not have a local maxima in $f(j_{\ell'} + 1)$. Hence, we can associate with the local maxima of $P'$ at $h$ the two unique local maxima of $P$ that appear in the bell-shaped paths $P[j_{\ell'}, l_{\ell'}]$ and $P[j_{\ell'+1}, l_{\ell'+1}]$.

The desired result follows. □

**Lemma 5.6.** Wlog, we have $|S_4| \leq n$.

**Proof.** Consider any pointer $a \in A$ such that we have statements $\{*a = b_i\}$ in $S_4$. We introduce a new pointer $c$, and (i) we insert a new type-4 statement $*a = c$, and (ii) we replace each $*a = b_i$ statement with $c = b_i$. Performing the above process for each $a$, we create a new instance $(A', S')$ such that for every $a, b \in A$, we have $b \in [a]$ in $(A, S)$ iff the same holds in $(A', S')$. Finally note that $|A'| \leq 2 \cdot |A|$ and $|S'| \leq |S| + |A|$, while $|S'_4| \leq |A'|$ as now every pointer appears in the left-hand side of a type 4 statement at most once.

The desired result follows. □

### A.4 Proofs of Section 5.3

**Lemma 5.7.** If there exist $i_1, i_2 \in [n']$ such that $x^{i_1}$ and $y^{i_2}$ are orthogonal, then $s$ flows into $t$ in $\overline{G}$. Moreover, there exists a witness program $P$ of length $O(\log n)$ that results in $s \in [t]$.

**Proof.** We first argue that in the solution graph $\overline{G}$, we have $b^{i_1} \xrightarrow{\epsilon} u^{i_2}$, which implies that $s$ flows into $t$. We prove by induction the following statement. Consider any $j \in [D]$.

(1) If $j$ is odd, we have $v^{i_2} \xrightarrow{u} u^{i_2}$ in $\overline{G}$.

(2) If $j$ is even, we have $b^{i_1} \xrightarrow{\epsilon} u^{i_2}$ in $\overline{G}$.

For the base case, let $j = 1$. By construction, we have $a^{i_1} \xrightarrow{\epsilon} z$ and $z \xrightarrow{\epsilon} u^{i_2}$. In addition, since $x^{i_1}[j] = y^{i_2}[j] = 0$, we have $a^{i_1} \xrightarrow{\epsilon} z$ or $z \xrightarrow{\epsilon} u^{i_2}$. The above imply that $a^{i_1}$ flows into $u^{i_2}$, hence because of the statement $*u^{i_2} = v^{i_2}$, we have $v^{i_2} \xrightarrow{\epsilon} u^{i_2}$ in $\overline{G}$, as required.

Now assume that the statement holds for $j - 1$, and we argue that it holds for $j$. First, assume that $j$ is odd. By the induction hypothesis, we have $b^{i_1} \xrightarrow{\epsilon} u^{i_2}_{j-1}$ in $\overline{G}$. By construction, we have $a^{i_1} \xrightarrow{\epsilon} b^{i_1}_{j-1}$ and $u^{i_2}_{j-1} \xrightarrow{\epsilon} u^{i_2}_j$. In addition, since $x^{i_1}[j] = y^{i_2}[j] = 0$, we have $a^{i_1} \xrightarrow{\epsilon} b^{i_1}_{j-1}$ or $u^{i_2}_{j-1} \xrightarrow{\epsilon} u^{i_2}_j$.

The reasoning then is similar to the case of $j = 1$.

Finally, assume that $j$ is even. By the induction hypothesis, we have $v^{i_2}_{j-1} \xrightarrow{\epsilon} a^{i_1}_{j-1}$ in $\overline{G}$. By construction, we have $u^{i_2}_{j-1} \xrightarrow{\epsilon} v^{i_2}_{j-1}$ and $a^{i_1}_{j-1} \xrightarrow{\epsilon} a^{i_1}_j$. In addition, since $x^{i_1}[j] = y^{i_2}[j] = 0$, we have $u^{i_2}_{j-1} \xrightarrow{\epsilon} v^{i_2}_{j-1}$ or $a^{i_1}_{j-1} \xrightarrow{\epsilon} a^{i_1}_j$. The above imply that $u^{i_2}_j$ flows into $a^{i_1}_j$, hence because of the statement $*a^{i_1}_j = b^{i_1}_j$, we have $b^{i_1}_j \xrightarrow{\epsilon} u^{i_2}_j$, as required.

Finally, note that our above analysis concerns $O(D) = O(\log n)$ nodes. Hence there is a witness $P$ for $s \in [t]$ that has length $\tilde{O}(1)$.

The desired result follows. □

**Lemma 5.8.** If $s$ flows into $t$ in $\overline{G}$, there exist $i_1, i_2 \in [n']$ such that $x^{i_1}$ and $y^{i_2}$ are orthogonal.
PROOF. First, observe that if \( s \in [1] \), there exist \( i_1, i_2 \in [n'] \) such that \( u_{D}^{i_2} \) is D-reachable from \( b_{D}^{i_1} \). Note that, in fact, \( b_{D}^{i_1} \xrightarrow{e} u_{D}^{i_2} \), as \( \epsilon \)-labeled edges enter nodes that have no outgoing edges in \( \overline{G} \).

We prove the following statement: For any \( i_1, i_2 \in [n'] \), the desired result follows. \( \square \)

First, observe that if \( j \) is odd and \( a_{j}^{i_1} \) is D-reachable from \( v_{j}^{i_2} \) in \( \overline{G} \), then \( \sum_{j' \leq j} x^{i_1}[j'] \cdot y^{i_2}[j'] = 0 \).

For the base case, let \( j = 1 \), and assume that \( a_{1}^{i_1} \) is D-reachable from \( v_{1}^{i_2} \), hence \( a_{1}^{i_1} \) flows into \( u_{1}^{i_2} \). Note that all paths starting from \( a_{1}^{i_1} \) with a \&-labeled edge go through \( z \). Hence, \( a_{1}^{i_1} \) can only flow into \( u_{1}^{i_2} \) via a path \( P: P_1 \circ P_2 \), where \( P_1: a_{1}^{i_1} \xrightarrow{\epsilon} z \) and \( P_2: z \xrightarrow{\epsilon} u_{1}^{i_2} \). Moreover, \( \lambda(P_1) = \& \) or \( \lambda(P_1) = \&\& \), and \( \lambda(P_2) = \epsilon \) or \( \lambda(P_2) = \star \). It follows easily by construction that \( x^{i_1}[1] = 0 \) or \( y^{i_2}[1] = 0 \), as otherwise \( \lambda(P_1) = \& \) and \( \lambda(P_2) = \star \), which would contradict the fact that \( a \) flows into \( u_{1}^{i_2} \) via \( P \).

Now assume that the statement holds for \( j - 1 \), and we argue that it holds for \( j \). First assume that \( j \) is odd. By the induction hypothesis, we have that if \( u_{j-1}^{i_2} \) is D-reachable from \( b_{j-1}^{i_1} \), then \( a_{j-1}^{i_1} \cdot b_{j-1}^{i_2} = 0 \). Note that \( a_{j}^{i_1} \) flows into \( u_{j}^{i_2} \). In addition, all paths starting from \( a_{j}^{i_1} \) with a \&-labeled edge go through \( b_{j-1}^{i_1} \), and thus we indeed have that \( u_{j}^{i_2} \) is D-reachable from \( b_{j-1}^{i_1} \). The proof is similar to the base case, where \( z \) is replaced by \( b_{j-1}^{i_1} \).

Finally, assume that \( j \) is even. By the induction hypothesis, we have that if \( a_{j-1}^{i_1} \) is D-reachable from \( v_{j-1}^{i_2} \) then \( a_{j}^{i_1} \cdot b_{j}^{i_2} = 0 \). Note that \( u_{j}^{i_2} \) flows into \( a_{j}^{i_1} \). In addition, all paths starting from \( u_{j}^{i_2} \) with a \&-labeled edge go through \( v_{j-1}^{i_2} \), and thus we indeed have that \( a_{j}^{i_1} \) is D-reachable from \( v_{j-1}^{i_2} \). The proof is similar to the previous case, where \( a_{j}^{i_1} \) is replaced by \( u_{j}^{i_2} \) and \( b_{j-1}^{i_2} i_1 \) is replaced by \( v_{j}^{i_2} \).

The desired result follows. \( \square \)