Generalized Cartan-Kac Matrices inspired from Calabi-Yau spaces

E. TORRENTE-LUJAN,

GFT, Dept. of Physics, Universidad de Murcia,
Spain. email: e.torrente@cern.ch

Abstract
The object of this work is the systematical study of a certain type of generalized Cartan matrices associated with the Dynkin diagrams that characterize Cartan-Lie and affine Kac-Moody algebras. These generalized matrices are associated to graphs which arise in the study and classification of Calabi-Yau spaces through Toric Geometry. We focus in the study of what should be considered the generalization of the affine exceptional series $E^{(1)}_{5,7,8}$ Kac-Moody matrices. It has been conjectured that these generalized simply laced graphs and associated link matrices may characterize generalizations of Cartan-Lie and affine Kac-Moody algebras.

Keywords: Gauge symmetries, Lie Kac-Moody Algebras, Cartan Matrix, Dynkin diagrams, Singularity theory, Calabi-Yau geometry.

1. Introduction
Progress in fundamental physics is dependent on the identification of underlying symmetries such as general coordinate invariance or gauge invariance. The gauge symmetries appearing in the highly successful Standard Model (SM) of
particles and interactions is based on Cartan-Lie Algebras and their direct products. There have been valiant efforts to extend the SM within the framework of Cartan-Lie algebras and with the objective of, for example, reducing the number of free parameters appearing in the theory. However, attempts to formulate Grand Unified theories (GUT) in which the direct product of the symmetries of the SM is embedded in some larger simple Cartan-Lie group have not had the same degree of success as the SM. The alternative possibility of unifying the gauge interactions with gravity in some 'Theory of Everything' based on string theory is very enticing, in particular because this offers novel algebraic structures.

At a very basic level, and without any obvious direct interest for the content of the SM, Cartan-Lie symmetries are closely connected to the geometry of symmetric homogeneous spaces, which were classified by Cartan himself. Subsequently, an alternative geometry of non-symmetric spaces appeared, and their classification was suggested in 1955 by Berger using holonomy theory [1]. There are several infinite series of spaces with holonomy groups \( SO(n), U(n), SU(n), Sp(n) \) and \( Sp(n) \times Sp(1) \), and additionally some exceptional spaces with holonomy groups \( G(2), Spin(7), Spin(16) \).

Beyond the SM, new theories as Superstrings offer new clues how to attack the problem of the nature of symmetries at a very basic geometric level. For example, the compactification of the heterotic string leads to the classification of states in a representation of the Kac-Moody algebra of the gauge group \( E_8 \times E_8 \) or \( Spin(32)/\mathbb{Z}_2 \). These structures arose in compactifications of the heterotic superstring on 6-dimensional Calabi-Yau spaces, non-symmetric spaces with an \( SU(3) \) holonomy group [2]. It has been shown [3] that group theory and algebraic structures play basic roles in the generic two-dimensional conformal field theories (CFTs) that underlie string theory. The basic ingredients here are the central extensions of infinite-dimensional Kac-Moody algebras. There is a clear connection between these algebraic and geometric generalizations. Affine Kac-Moody algebras are realized as the central extensions of loop algebras, namely the sets of mappings on a compact manifold such as \( S^1 \) that take values on a finite-dimensional Lie algebra. Superstring theory contains a number of other infinite-dimensional algebraic symmetries such as the Virasoro algebra associated with conformal invariance and generalizations of Kac-Moody algebras themselves, such as hyperbolic and Borcherd algebras.

In connection with Calabi-Yau spaces, (Coxeter-)Dynkin diagrams which are in one-to-one correspondence with both Cartan-Lie and Kac-Moody algebras have been revealed through the technique of the crepant resolution of specific quotient singular structures such as the Kleinian-Du-Val singularities \( \mathbb{C}^2/G \) [4], where \( G \) is a discrete subgroup of \( SU(2) \). Thus, the rich singularity
structure of some examples of non-symmetrical Calabi-Yau spaces provides
another opportunity to uncover infinite-dimensional affine Kac-Moody sym-
metries. The Cartan matrices of affine Kac-Moody groups are identified with
the intersection matrices of the unions of the complex projective lines result-
ing from the blow-ups of the singularities. For example, the crepant resolution
of the $\mathbb{C}^2/\mathbb{Z}_n$ singularity gives for rational, i.e., genus-zero, (-2)
curves an intersection matrix that coincides with the $A_{n-1}$ Cartan matrix. This is also
the case of $K3 \equiv CY_2$ spaces, where the classification of the degenerations
of their elliptic fibers (which can be written in Weierstrass form) and their
associated singularities leads to a link between $CY_2$ spaces and the infinite
and exceptional series of affine Kac-Moody algebras, $A^{(1)}_r$, $D^{(1)}_r$, $E^{(1)}_6$, $E^{(1)}_7$
and $E^{(1)}_8$ (ADE) [5, 6].

In the search for new symmetries and from Calabi-Yau geometry as inspira-
tional basis, it has been defined the so called Berger Matrices [7, 8] which
generalize Cartan and Cartan-Kac-Moody matrices. In purely algebraic terms,
a Berger matrix $B$ is a finite integral matrix characterized by the following
data:

$$
B_{ii} = k(i), \ k(i) \in \mathbb{Z} \\
B_{ij} \leq 0, (i \neq j), \ B_{ij} \in \mathbb{Z}, \\
B_{ij} = 0 \Rightarrow B_{ji} = 0, \\
\det B = 0, \ \det B_{\{ij\}} > 0.
$$

The last condition means positivity of all principal proper minors of the ma-
trix, this, together with the determinant equal zero corresponds to the affine
condition. A condition shared by Kac-Moody Cartan matrices.

The concept of Berger matrices is obtained by weakening the conditions
on the generalized Cartan matrix $\hat{A}$ appearing in affine Kac-Moody algebras.
In fact, the only difference with respect to them is that the usual restriction
on the diagonal elements ($B_{ii} = 2$ for CKM matrices) is relaxed. Note that,
more than one type of diagonal entry is allowed now: positive integer 2, 3, ...
diagonal entries can coexist in a given matrix.

“Non-affine” Berger Matrices are also defined: the condition of zero deter-
minal is eliminated. These matrices could play in some way the same role of
basic simple blocks as finite Lie algebras play for the case of affine Kac-Moody
algebras.

The study of properties and the systematic enumeration of the various
possibilities concerning the large family of possible Berger matrices can be fa-
cilitated by the introduction for each matrix of its generalized Dynkin diagram
$\Delta(B)$. A schematic prescription for the most simple cases could be: A) For a
matrix of dimension \( n \), define \( n \) vertexes and draw them as small circles. In case of appearance of vertexes with different diagonal entries, some graphical distinction will be performed. Consider all the element \( i, j \) of the matrix in turn. B) Draw one line from vertex \( i \) to vertex \( j \) if the corresponding element \( A_{ij} \) is non zero.

Some essential properties of Cartan and Berger matrices are easily deduced from the two well known Frobenius-Perron Lemmas which we repeat here for convenience [9]: First lemma. For a real symmetric matrix \( M \) with elements \( M_{ij}, i \neq j \leq 0 \), If the matrix is positive semi-definite, then the smallest eigenvalue of the matrix, eventually a zero eigenvalue, has multiplicity one and the corresponding eigenvector has all positive coordinates. Second lemma, in the same case, let \( N \) be the \((n-1) \times (n-1)\) matrix obtained by deleting the \( i \) row and column from \( M \). Then \( N \) is positive definite.

According to these lemma, if the Berger matrix \( B \) is affine (has determinant equal to zero) then it has one and only one zero eigenvalue and corresponding to it, there is an eigenvector \( c \) with all positive entries. The entries of this vector \( c \) are by definition the Coxeter labels. For a Berger matrix \( B \) of dimension \( n \), the rank is \( r = n - 1 \). Since \( B \) is of rank \( r = n - 1 \), we can find one, and only one, non zero vector \( \mu \) such that \( B\mu = 0 \). The numbers, \( a_i \), components of the vector \( \mu \), are called Coxeter labels. The sums of the Coxeter labels \( h = \sum \mu_i \) is the Coxeter number. For a symmetric generalized Cartan matrix only this type of Coxeter number appear.

The general objective of this work is to study, enumerate and classify all possible matrices of Berger type beyond those already known (Cartan and Kac-Moody matrices) and which are particular cases of the first ones. In concrete we study the most basic, yet non trivial, cases, the generalization of the exceptional Kac-Moody affine matrices \( E^{(1)}_{6,7,8} \). We choose as a guide the following remark: the matrices of these exceptional algebras basically consist of blocks containing standard \( A_r \) Cartan matrices on them. The corresponding graphs consist of a central node from which three legs depart. The number of nodes in each leg is strongly restricted. In fact only three possibilities, the three exceptional cases, exist.

## 2. The Berger generalization of Kac-Moody exceptional matrices

As a starting example (other examples generalizing \( A, D \) series will be the object of some other work) we are interested in Berger matrices which generalize the structure \( E^{(1)}_{6,7,8} \) matrices. Let us consider Berger matrices with the
following block structure:

\[
B_{SL} = \begin{pmatrix}
A_{r_1} & 0 & 0 & 0 & v_1 \\
0 & A_{r_2} & 0 & 0 & v_2 \\
0 & 0 & A_{r_3} & 0 & v_3 \\
0 & 0 & 0 & A_{r_4} & v_4 \\
v_1' & v_2' & v_3' & v_4' & k
\end{pmatrix}
\]

(1)

where \(A_{r_i}\) are Cartan matrices of undefined dimension \(r_i\) and the \(v_i\) are column vectors filled with zeroes except for one negative entry, \(v_i' = (0, \ldots, 0, -1)\). The number \(k\) is an arbitrary positive integer. Obviously these matrices are similar in structure to those matrices of the exceptional Kac-Moody affine algebras. The only difference is the number of blocks (four instead of three) and the value of \(k\).

A general matrix can be obtained considering any number of blocks \(A_r\). The dimension of the original matrix \(B\) is given by \(D = 1 + \sum_{1, m} r_i\). For further reference, let us also define another important quantity in algebraic geometry applications is the “squared canonical class” \(K^2\) of the matrix \(B\). This quantity is defined as the sum of all the elements of the matrix \(K^2 = \sum_{i,j} B_{i,j}\). For the type of matrices, \(k\) of interest in this work, is important to remark that for Cartan matrices \(A_r\), \(K(A_r) = 2\). Then

\[
K^2(B_{SL}) = \sum_i K^2(A_{r_i}) - 2m + k = 2m - 2m + k = k.
\]

Finally, let us remark that the trace of the matrix \(B_{SL}\) is given by \(tr B_{SL} = 2 \sum_i r_i + k\).

From this assumed structure it is obviously clear that this family of matrices fulfill nearly all the conditions for a Berger matrix. The objective is to find all the sets of values of \(k, r_1, r_2, r_3, r_4\) which make vanish the determinant of the matrix and then check for its positive-semidefinitness.

To this purpose, we will compute the determinant of a general matrix of the above structure in three different ways.

As a first way, the determinant of a general matrix of the type \(B_{SL}\) can easily be computed by induction following the Laplace rule along the entries of the column of the \(v_i'\)'s. We have the result (for the \(m = 4\) case for keeping notation simple)

\[
\det B_{SL} = k Q - \det A_{r_1} \det A_{r_2} \det A_{r_3} \det A_{r_4} - \det A_{r_1} \det A_{r_2} \det A_{r_3} \det A_{r_4} - \det A_{r_1} \det A_{r_2} \det A_{r_3} \det A_{r_4} - \det A_{r_1} \det A_{r_2} \det A_{r_3} \det A_{r_4} - \det A_{r_1} \det A_{r_2} \det A_{r_3} \det A_{r_4} - \det A_{r_1} \det A_{r_2} \det A_{r_3} \det A_{r_4}
\]

\[
= Q \left( k - \frac{\det A_{r_1}}{\det A_{r_1}} - \frac{\det A_{r_2}}{\det A_{r_2}} - \frac{\det A_{r_3}}{\det A_{r_3}} - \frac{\det A_{r_4}}{\det A_{r_4}} \right)
\]

(2)
where the $A_{r_i-1}$ are matrices of dimension one less than the original matrix, obtained from them by eliminating the last column and row. We have $\det A_r = r + 1$. The formula is in this case:

$$\det B_{SL} = Q \left( k - \frac{r_1}{r_1 + 1} - \frac{r_2}{r_2 + 1} - \frac{r_3}{r_3 + 1} - \frac{r_4}{r_4 + 1} \right)$$

(3)

$$= Q \left( k - 4 + \frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} + \frac{1}{r_3 + 1} + \frac{1}{r_4 + 1} \right),$$

(4)

$$Q = (r_1 + 1)(r_2 + 1)(r_3 + 1)(r_4 + 1).$$

(5)

This is the final formula. From here we can obtain the finite number of combinations of integers $r_i$ which make vanish the determinant.

We can study the different cases according to the value of $k$.

- If $k \geq 4$ then $\det B_{SL}$ is always positive.

- If $k = 3$, then $\det B_{SL}$ can be positive, negative or zero.

  The determinant equals zero if and only if

  $$1 = \frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} + \frac{1}{r_3 + 1} + \frac{1}{r_4 + 1}. \quad (6)$$

  If the right side of the expression is less (greater) than one then the determinant is negative (positive)

- If $k = 2$ the determinant is positive or zero. The only possibility for the determinant to equal zero is that all $r_i = 1$. For $r_i > 1$ the determinant is negative.

- Finally, if $k = 0, 1$ the determinant is always negative.

We can easily obtain the behavior in the general case for a matrix $B_{SL}$ of similar structure as Eq.(1) but with an arbitrary number $m$ of matrices $A_{r_i}$ in the diagonal. It is interesting the particular case $k = m - 1$. In this case the condition for the vanishing of the determinant is:

$$\sum_{i=1,m} \frac{1}{r_i + 1} = 1.$$  

(7)

For all the other values of $k \geq m$ the determinant is always positive.

In the general case the sign of the determinant is equal to the sign of the expression:

$$T = k - m + \sum_{i=1,m} \frac{1}{N_i + 1} < k - m + \frac{m}{2} = k - \frac{m}{2}. \quad (8)$$
For example, for \( m = 5 \), we have non trivial zero conditions for \( k = 4,3 \). For non trivial, we mean that in that case both expressions

\[
\sum_{i=1}^{m} \frac{1}{r_i + 1} = 1, \quad (k = 4), \text{or},
\]

\[
\sum_{i=1}^{m} \frac{1}{r_i + 1} = 2 \quad (k = 3)
\]

\[
\sum_{i=1}^{m} \frac{1}{r_i + 1} = k - m \quad (\text{generally})
\]

have solutions with no all the \( r_i \) simultaneously equal one. It is obvious that the combination \( m = 5, k = 2 \) has however no zero solutions.

The solution to any of the previous zero conditions will be denoted as \( E^{(m-k)}_{(r_1,r_2,...r_m)} \).

We will see below that Eq. (11) does not give essentially new solutions. Solutions to this equation for fixed \( k, m \) can be built from the solutions to equations of the same parameter \( k \) and different \( m_1, m_2 \) in an iterative way.

For example, below we will obtain the rule \( E^{(m-k_1)}_{(r)} \Delta E^{(m-k_2)}_{(r')} = E^{(2m-k_1-k_2)}_{(r') \cup (r)} \).

The integer solutions to equations (11), e.g., partitions of unity in terms of “Egyptian fractions” are well known in arithmetic. These equations can be casted in some other way: as partitions of an integer in terms of a fixed number of other integers which divide the original number. This can be seen as follows. First, let us define \( s \equiv l.c.m.(r_1+1, \ldots, r_m+1) \) and \( x_i \equiv s/(r_i+1) \). Then the previous equation (11) is obviously written as

\[
\sum x_i = (k - m)s.
\]

Clearly any of the summands divides the total: \( x_i \mid s \). In this formulation, in the most simple case \( k - m = 1 \), we are interested in numbers \( s \) which can written as sum of their divisors \( x_i \). The number of divisors is fixed, \( m \), but they can appear repeated. This divisor does not need to be prime or relative prime. The different solutions will be presented as \( (x_1, \ldots, x_i, \ldots)[s] \) in the tables which follows.

2.1. The Coxeter labels.

We will re-obtain the same results in some slightly different way. The advantage is that now we will obtain the values for the Coxeter labels and Coxeter number.
According to the Frobenius-Perron lemma, if the Berger-Cartan matrix $B$ is affine (has determinant equal to zero) then it has one and only one zero eigenvalue and corresponding to it, there is an eigenvector $c$ with all positive entries. The entries of this vector $c$ are by definition the Coxeter labels.

Let us take the previous matrix $B$ and write the vector $c$ in block form corresponding to each of the submatrices $A_{ri}$: $c^i = (c_1^i, c_2^i, c_3^i, c_4^i, s)$ where the subvectors $c^i = (c_{i,1}, \ldots, c_{i,j}, \ldots, c_{i,j_{max}})$, $j_{max}(i) = r_i$ and $s$ is a, a priori unknown, constant. We assume that $Bc = 0$, this is equivalent to the set of equations:

\[
A_{ri} c_i + sv_{ri} = 0, \quad (for \ i = 1 - 4); \tag{12}
\]
\[
\sum_i v_i^t c_i + ks = 0. \tag{13}
\]

Due to the special form of the vectors $v_i$, the second equation becomes

\[
k s = \sum_i c_{i,j_{max}(i)}. \tag{14}
\]

Let us consider now the first set of equations. We know already the solution to this kind of equations: for a general $A_r$ Cartan matrix

\[
A_r \begin{pmatrix}
1 \\
2 \\
\vdots \\
r
\end{pmatrix} = (r + 1) \begin{pmatrix}
0 \\
0 \\
\vdots \\
1
\end{pmatrix}, \tag{14}
\]

so, we have

\[
c_i^r = x_i(1, 2, \ldots, r_i), \quad x_i = \frac{s}{r_i + 1}. \tag{15}
\]

We arrive then to an important conclusion, for the Coxeter labels to be integers, the constant $s$ is multiple of all the numbers $r_i + 1$, where the $r_i$ are the dimensions of the matrices $A_{ri}$. Now, we insert this solution in the last equation of the system \[13\] We obtain

\[
c_{i,j_{max}} = \frac{s r_i}{r_i + 1},
\]

and then

\[
k s = s \sum_i \frac{r_i}{r_i + 1}, \quad or, \quad k = m - \sum_i \frac{1}{r_i + 1} \ (if \ s \neq 0). \tag{16}
\]
The constant $s$ has cancelled from the last equation and remains arbitrary. The only condition is that $s \mid r_i + 1$. The minimal choice is given by

$$s = \text{lcm}(r_1 + 1, \cdots, r_i + 1, \cdots).$$

An important parameter is the Coxeter number $h$ which is defined as the sum of all the Coxeter labels. This number is given by

$$h \equiv s + \sum_{i,j} c_{i,j} = s + \sum_i \frac{s}{r_i + 1} \frac{r_i(r_i + 1)}{2} \quad (17)$$

$$= s \left( 1 + \frac{1}{2} \sum_i r_i \right). \quad (18)$$

The value of $h$ is minimal for the choice of $s$ above. We can write the previous formula in the simple form including the dimension $D$ of the matrix

$$h = \frac{s}{2} (1 + D). \quad (19)$$

### 2.2. The determinant and eigenvalues of $B$

It is convenient to re-obtain the same results with yet another method. As a byproduct of this, the matrix $B$ will be diagonalized. A simple algorithm for this purpose is described in Ref.[10] (see also Ref.[11] for a description of the method) to diagonalize matrices of the form $B_{SL}(\Delta)$ where $\Delta$ is a certain graph (a plumbing tree-like graph) The diagonalized matrix has the form $D = P^t A(\Delta) P$ with $\det P = 1$. Thus $\det D = \det A(\Delta)$.

The diagonalization of the matrix can be translated into a new idea: the diagonalization of the corresponding graph. A diagonal graph is one with isolated nodes, not linked by edges. Clearly the determinant of the matrix corresponding to the diagonalized graph, where the weights has been recalculated in some way, should be the same as the determinant of the initial matrix and graph.

Let us suppose generic trees weighted by arbitrary rational numbers $e_i$. To diagonalize a given such tree $\Delta$, pick some vertex (the central node for example) and direct all edges toward this vertex. Now simplify the graph $\Delta$ by recursively deleting edges according to the procedures presented in table 12.3 of Ref.[11]. Eventually we end up with a collection of isolated points with new weights $d_i$, this is the diagonalized graph $\Delta_D$. Then $D = \text{diag}(d_i)$ is the desired diagonalization of $A(\Delta)$. Thus $\det A(\Delta) = \prod_i d_i$. In particular, all the weights but the last one are guaranteed to be strictly positive numbers.
The last one is given by:
\[ e'_f = e_f - \sum_{i=1}^{m} \frac{1}{e'_i}. \]

For the case of purely linear subgraphs with nodes weighted by \( w_j \), each of the \( e'_i \) are continued fractions of the type
\[ e'_i = 1 - \frac{1}{w_1 - \frac{1}{w_2 - \ldots}}. \] (20)

In the case of interest to us here, obviously we put \( e_f = k \) and the quantities \( e'_i \) are the values of a continued fraction corresponding to a linear graph of weight-2 nodes (any of our \( A_r \) sub-graph). They are easily obtained
\[ e'_i = \frac{r_i}{r_i + 1}. \]

Obviously we reobtain the formula given by Eq.2. In addition we obtain the rest of eigenvalues of the matrix and the explicit confirmation that is positive-semidefinite.

Note that the continued fraction \( e'_i \) correspond to the ratio of two determinants of matrices. The quantity \( e_f \) is a generalized fraction. It can be interpreted as a (generalized) determinant of a set of matrices.

2.3. Results: enumeration of cases and conclusions

We deal first with the cases with \( K^2(B) = k = m - 1 \). These correspond to integer solutions of the equation [7]. We consider small values of \( m \) (\( m \leq 6 \)).

Case \( m=2, k=m-1=1 \). In this case the condition for vanishing determinant is
\[ 1 = \frac{1}{r_1 + 1} + \frac{1}{r_2 + 1}, \] (21)
the only possibility is the trivial one \( r_1 = r_2 = 1 \). This case correspond to the generalized Cartan Matrix:
\[ E^{(1)}_{(2,2)} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}. \] (22)

Case \( m = 3, k = m - 1 = 2 \). In this case the condition for vanishing determinant contains three “Egyptian fractions”
\[ 1 = \frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} + \frac{1}{r_2 + 1}. \] (23)
There are exactly three integer solutions to this equation. They are \( N = (2, 2, 2), N = (1, 3, 3), N = (1, 2, 5) \). They obviously correspond to the graphs and Cartan matrices of the three exceptional affine Kac-Moody algebras \( E_{6,7,8}^{(1)} \). In our notation

\[
E_{(2,2,2)}^{(1)} \equiv E_{6}^{(1)}, \\
E_{(1,3,3)}^{(1)} \equiv E_{7}^{(1)}, \\
E_{(1,2,5)}^{(1)} \equiv E_{8}^{(1)}.
\]

Note that the formula for the determinant in this case has already been given in the classical book of Kac \[9\] where the equivalent quantities \( T_{r,s,p} \) are computed.

Case \( m = 4, k = m - 1 = 3 \). In this case the condition for vanishing determinant contains four Egyptian fractions

\[
1 = \frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} + \frac{1}{r_3 + 1} + \frac{1}{r_4 + 1},
\]

there are exactly fourteen solutions to this equation. They are tabulated in table 2.4.

In the cases \( m = 5 \) and \( m = 6 \) with \( k = m - 1 \) the formulas for vanishing determinant contain summatories of five or six Egyptian fractions respectively. For \( m = 5 \) there are exactly 147 solutions to this equation while for \( m = 6 \) there are exactly 3240 solutions to this equation. In both case, part of the solutions are tabulated in tables 2.4 and 2.4 where we present those solutions with \( D \leq 40 \). Some of them (those with \( D \leq 40 \)). For a full list consult Ref. [12].

For \( m > 6 \), the condition of vanishing is similar, however in these cases the total number of solutions is not known in general. However one can easily obtain a large number of solutions by numerical methods. It is also obvious that solutions for a given \( m \) can be obtained from those solutions of an smaller \( m \).

2.4. The cases \( k = m - 2 \)

Let us study some of the solutions obtained for \( k \neq m - 1 \) with \( m \leq 6 \).

The only non-trivial cases are obtained for \( m = 5, k = 3 \) and \( m = 6, k = 4 \). For \( m = 5, k = 3 \) there are exactly 3 solutions which are presented in table 2.4.

For \( m = 6, k = 4 \) there are exactly 17 solutions which are presented in table 2.4.

These solutions are not new in some sense: they can be obtained from the solutions of \( k = m - 1 \). It is important to remark that apart from them there are not any other really new solutions.
From this we can define a binary operation on graphs, the “τ-product” $C = A \boxtimes B$. This operation has an important property: if $\det A = 0$ and $\det B = 0$ then also $\det C = 0$.

In conclusion, the interest to look for new algebras beyond Lie algebras started from the $SU(2)$-conformal theories (see for example [13, 14]). One can think that geometrical concepts, in particular algebraic geometry, could be a natural and more promising way to do this. This marriage of algebra and geometry has been useful in both ways.

It is very well known, by the Serre theorem, that Dynkin diagrams defines one-to-one Cartan matrices and these ones Lie or Kac-Moody algebras. In this work, we have generalized some of the properties of Cartan matrices for Cartan-Lie and Kac-Moody algebras into a new class of affine, and non-affine Berger matrices. We arrive then to the obvious conclusion that any algebraic structure emerging from this can not be a CLA or KMA algebra. The main
Kac-Moody exceptional algebras (cf. with Table 3: List of all the 14 solutions for the case $m = 3$).

| $E_{(r_a, r_b, r_c)}^{(1)}$ | Dim | $h$ | $(...)[s]$ |
|---------------------------|-----|-----|------------|
| 1 $E_{(2,2,2)}^{(1)}$    | 7   | 12  | (1,1,1)[3] |
| 2 $E_{(1,3,3)}^{(1)}$    | 8   | 18  | (1,1,2)[4] |
| 3 $E_{(1,2,5)}^{(1)}$    | 9   | 30  | (1,2,3)[6] |

Table 2: List of cases with $m = 3, k = 2$. They correspond to the standard Kac-Moody exceptional algebras (cf. with $T_{qr s}$ in table 2.3 in Ref.[9]).

| $E_{(r_a, r_b, r_c, r_d)}^{(1)}$ | Dim | $h$ | $(...)[s]$ |
|---------------------------------|-----|-----|------------|
| 1 $E_{(3,3,3,3)}^{(1)}$         | 13  | 28  | (1,1,1,1)[4] |
| 2 $E_{(2,3,3,5)}^{(1)}$         | 14  | 90  | (2,3,3,4)[12] |
| 3 $E_{(1,5,5,5)}^{(1)}$         | 17  | 54  | (1,1,1,3)[6] |
| 4 $E_{(2,2,5,5)}^{(1)}$         | 15  | 48  | (1,1,2,2)[6] |
| 5 $E_{(1,4,7,7)}^{(1)}$         | 19  | 80  | (1,1,2,4)[8] |
| 6 $E_{(1,4,4,9)}^{(1)}$         | 19  | 100 | (1,2,2,5)[10] |
| 7 $E_{(1,3,5,11)}^{(1)}$        | 21  | 132 | (1,2,3,6)[12] |
| 8 $E_{(2,2,3,11)}^{(1)}$        | 19  | 120 | (1,3,4,4)[12] |
| 9 $E_{(1,3,4,19)}^{(1)}$        | 28  | 290 | (1,4,5,10)[20] |
| 10 $E_{(1,2,11,11)}^{(1)}$      | 26  | 162 | (1,1,4,6)[12] |
| 11 $E_{(1,2,8,17)}^{(1)}$       | 29  | 270 | (1,2,6,9)[18] |
| 12 $E_{(1,2,7,23)}^{(1)}$       | 34  | 420 | (1,3,8,12)[24] |
| 13 $E_{(1,2,9,14)}^{(1)}$       | 27  | 420 | (2,3,10,15)[30] |
| 14 $E_{(1,2,6,41)}^{(1)}$       | 51  | 1092 | (1,6,14,21)[42] |

Table 3: List of all the 14 solutions for the case $m = 4, k = 3$. The integers $(r_a, r_b, r_c, r_d)$ define both, the number of nodes in each of the four legs of the graph and the dimension of the each of the block matrices $A_r$. The dimension of the matrix $(D = rank + 1 = 1 + \sum r_i)$. The Coxeter number $h = s/2(1+D)$. In the last column appears the form of the solution $\sum_i x_i = (k - m) s$.

| $E_{(r_1, r_2, r_3, r_4)}^{(2)}$ | Dim | $h$ | $(...)[s]$ |
|----------------------------------|-----|-----|------------|
| 1 $E_{(1,1,2,2,2)}^{(2)}$       | 9   | 30  | (2,2,2,3,3)[6] |
| 2 $E_{(1,1,1,3,3)}^{(2)}$       | 10  | 20  | (1,1,4,4,4)[4] |
| 3 $E_{(1,1,1,2,5)}^{(2)}$       | 11  | 36  | (1,2,3,3,3)[6] |

Table 4: List of $m = 5, k = m - 2 = 3$. 
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$E^{(1)}_{\cdots,r,\cdots}$ & Dim & $h$ & $\cdots[s]$ \\
\hline
1 & $E^{(1)}_{4,4,4,4,4}$ & 21 & 55 & (1, 1, 1, 1, 1)[5] \\
2 & $E^{(1)}_{3,3,5,5,5}$ & 22 & 138 & (2, 2, 2, 3, 3)[12] \\
3 & $E^{(1)}_{2,5,5,5,5}$ & 23 & 72 & (1, 1, 1, 1, 2)[6] \\
4 & $E^{(1)}_{1,3,3,7,7}$ & 24 & 100 & (1, 1, 2, 2, 2)[8] \\
5 & $E^{(1)}_{3,3,4,4,9}$ & 25 & 250 & (2, 2, 1, 4, 5)[20] \\
6 & $E^{(1)}_{2,3,5,7,7}$ & 26 & 312 & (3, 3, 4, 6, 8)[24] \\
7 & $E^{(1)}_{2,4,4,5,9}$ & 27 & 390 & (3, 5, 6, 6, 10)[30] \\
8 & $E^{(1)}_{1,3,3,5,11}$ & 28 & 162 & (1, 2, 3, 3, 3)[12] \\
9 & $E^{(1)}_{2,3,5,5,11}$ & 29 & 168 & (1, 2, 3, 4, 2)[12] \\
10 & $E^{(1)}_{2,2,8,8,8}$ & 30 & 135 & (1, 1, 1, 3, 2)[9] \\
11 & $E^{(1)}_{2,4,4,4,14}$ & 31 & 225 & (1, 3, 3, 3, 5)[15] \\
12 & $E^{(1)}_{3,7,7,7}$ & 32 & 302 & (1, 1, 1, 4, 3)[8] \\
13 & $E^{(1)}_{3,3,7,7}$ & 33 & 372 & (2, 3, 3, 8, 8)[24] \\
14 & $E^{(1)}_{2,3,5,8,8}$ & 34 & 288 & (2, 2, 3, 9, 9)[18] \\
15 & $E^{(1)}_{2,3,3,11,11}$ & 35 & 312 & (1, 1, 3, 3, 4)[12] \\
16 & $E^{(1)}_{2,5,7,7,7}$ & 36 & 396 & (2, 3, 3, 4, 2)[12] \\
17 & $E^{(1)}_{2,2,5,11,11}$ & 37 & 198 & (1, 1, 2, 4, 2)[12] \\
18 & $E^{(1)}_{2,3,3,9,14}$ & 38 & 495 & (4, 6, 15, 15, 20)[60] \\
19 & $E^{(1)}_{1,4,9,9,9}$ & 39 & 170 & (1, 1, 1, 2, 5)[10] \\
\hline
\end{tabular}
\caption{List of the $dim \leq 40$ cases of the total of 147 cases of dimension five $m = 5, k = 4$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$E^{(1)}_{\cdots,r,\cdots}$ & Dim & $h$ & $\cdots[s]$ \\
\hline
1 & $E^{(1)}_{5,5,5,5,5}$ & 31 & 96 & (1, 1, 1, 1, 1)[6] \\
2 & $E^{(1)}_{3,5,5,5,5,7}$ & 32 & 408 & (3, 3, 4, 4, 4, 6)[24] \\
3 & $E^{(1)}_{4,4,5,5,5,9}$ & 33 & 510 & (3, 5, 5, 5, 6, 30) \\
4 & $E^{(1)}_{3,3,7,7}$ & 34 & 144 & (1, 1, 1, 2, 2)[8] \\
5 & $E^{(1)}_{3,4,4,7,7}$ & 35 & 720 & (4, 5, 5, 8, 8, 10)[40] \\
6 & $E^{(1)}_{3,5,5,5,5,11}$ & 36 & 216 & (1, 2, 2, 2, 2, 12)[12] \\
7 & $E^{(1)}_{4,4,4,4,9,9}$ & 37 & 180 & (1, 1, 2, 2, 2, 10) \\
8 & $E^{(1)}_{2,5,7,7,7}$ & 38 & 444 & (3, 3, 3, 3, 8, 8)[24] \\
9 & $E^{(1)}_{3,3,5,8,8,8}$ & 39 & 666 & (4, 4, 4, 6, 9, 9)[36] \\
10 & $E^{(1)}_{2,5,5,8,8,8}$ & 40 & 342 & (2, 2, 2, 3, 3, 6)[18] \\
11 & $E^{(1)}_{3,3,5,7,7,11}$ & 41 & 456 & (2, 3, 3, 4, 6, 6)[24] \\
\hline
\end{tabular}
\caption{List of all the $dim \leq 40$ cases of a total of 3120 cases.}
\end{table}
Table 7: the $m = 6, k = m - 2 = 4$. List of all the distinct 17 cases.

| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | Dim | $h$ | $(...) [s]$ |
|-------------------------------------|-----|-----|-------------|
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 15  | 32  | $(1, 1, 1, 1, 2, 2)[4]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 16  | 102 | $(2, 3, 3, 4, 6, 6)[12]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 19  | 60  | $(1, 1, 1, 3, 3, 3)[6]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 17  | 54  | $(1, 1, 2, 2, 3, 3)[6]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 21  | 88  | $(1, 1, 2, 4, 4, 4)[8]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 21  | 110 | $(1, 2, 2, 5, 5, 5)[10]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 23  | 144 | $(1, 2, 3, 6, 6, 6)[12]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 21  | 132 | $(1, 3, 4, 6, 6)[12]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 30  | 310 | $(1, 4, 5, 10, 10, 10)[20]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 28  | 174 | $(1, 1, 4, 6, 6, 6)[12]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 31  | 288 | $(1, 2, 6, 9, 9, 9)[18]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 36  | 444 | $(1, 3, 8, 12, 12, 12)[24]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 29  | 450 | $(2, 3, 10, 15, 15, 15)[30]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 53  | 1134| $(1, 6, 14, 21, 21, 21)[42]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 13  | 21  | $(1, 1, 1, 1, 1, 1)[3]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 14  | 42  | $(2, 2, 3, 3, 3, 6)[6]$ |
| $E_{\{i_0,i_{1\ldots},i_{m-2}\}}^{(1)}$ | 15  | 96  | $(2, 4, 4, 4, 6)[6]$ |
difference of these matrices with respect previous definitions being in the values that diagonal elements of the matrices can take.

In this work we have enumerated in a systematic way all the possibilities of a special class of Berger matrices which includes as a subclass the ordinary matrices of Kac-Moody affine exceptional algebras. **Acknowledgments.** We acknowledge the financial support of the Spanish CYCIT funding agency (Ministerio de Ciencia y Tecnologia) and the CERN Theoretical Division. We also acknowledge the kind hospitality of the Dept. of Theoretical Physics, C-XI of the U. Autonoma de Madrid.

Figure 2: all 14 figures
References

[1] M. Berger, *Sur les groupes d’holonomie des variétés a connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France 83 (1955), 279-330.

[2] P. Candelas, G. Horowitz, A. Strominger and E. Witten, *Nucl. Phys.* B258 (1985) 46;

[3] A.A. Belavin, A.M. poliakov, A.B. Zomolodchikov, Nucl. Phys. B241,333 (1984).

[4] P. Du Val, *Homographies, Quaternions and Rotations* (Clarendon Press, Oxford 1964).

[5] K. Kodaira, Annals of Mathematics, v77, (1963), 563-626.

[6] M. Bershadsky, K. Intriligator, S. Kachru, D.R. Morrison, V. Sadov, and C. Vafa, Nucl. Phys. B481 (1996) 215.

[7] E. Torrente-Lujan and G. G. Volkov, arXiv:hep-th/0406035

[8] G.G. Volkov, hep-th/0402042

[9] M. Kac, *Infinite dimensional Lie Algebras*, Cambridge University Press, London 1995.

[10] Dissertation, N. Duchon (Maryland, 1982).

[11] D. Eisenbud, W. Neumann, “Three-dimensional link theory and invariants of plane curve singularities”. Princeton University Press, 1985.

[12] E. Torrente-Lujan, http://www.um.es/torrente/GeneralizedKacMoody.html

[13] A. Capelli, C. Itzykson and J.-B. Zuber, Commun. Math. Phys. 184 (1987), 1-26, MR 89b, 81178.

[14] P. Di Francesco and J.-B. Zuber, Nucl. Phys. B 338 (1990), no 3, 602-646.