Fractional Derivative Approach to the Self-gravitation Equation

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ABSTRACT

A new formalism is presented for finding equilibrium distribution functions for axisymmetric systems. The formalism, obtained by using the concept of fractional derivatives, generalizes the methods of Fricke (1952), Kalnajs (1972) and Jiang & Ossipkov (2007), and has the advantage that can be applied to a wider variety of models. We found that this approach can be applied both to tridimensional systems and to flat systems, without the necessity of dealing with a pseudo-volume mass density. As an application, we obtain the distribution functions of the Binney’s logarithmic model and of the Mestel disc.

Key words: stellar dynamics – galaxies: kinematics and dynamics.

1 INTRODUCTION

The construction of self-consistent models for stellar systems is of great interest in astrophysics. As it was pointed out by some authors, the most straightforward way to perform such models is to start with an assumed potential defining the mass density $\rho$ (via Poisson’s equation) and the families of orbits that can lie within the system (via Newton’s equations of motion). Since $\rho$ is the integration of the distribution function (DF) over the velocity variables in the phase space of the system, the problem of finding the DF is that of solving an integral equation. For that reason, such procedure is called as the “from $\rho$ to $f$” approach for finding a self-consistent DF (Binney & Tremaine [1987], Hunter & Quinn [1993], Jiang & Ossipkov [2007]) and the integral relation connecting $f$ and $\rho$ is known as the self-gravitation equation.

By Jeans’s theorem, an equilibrium DF is a function of the isolating integrals of motion that are conserved in each orbit. It has been shown that, for certain potential-density pairs (PDP), it is possible to find analytically such kind of DFs. The simplest case of physical interest corresponds to spherically symmetric PDP, described by an isotropic DF that only depends on the energy. Eddington (1916) showed that it is possible to obtain such DFs by first expressing the density as a function of the potential, and then solving an Abel integral equation.

Another case of interest in astrophysics is the corresponding to axially symmetric systems, for which a great variety of PDP has been constructed, e.g. Kuzmin (1956); Toomre (1963); Miyamoto (1971); Bagin (1987); Kalnajs (1972); Miyamoto & Nagai (1976); Kutuzov & Ossipkov (1980, 1986, 1988); Evans (1993, 1994); Kutuzov (1995); Jiang (2000); Jiang & Moss (2002); González & Reina (2006); Ossipkov & Jiang (2007).

Recently, Jiang & Ossipkov (2007) presented a new method for the axially symmetric case, where the equilibrium DF depends on the energy $E$ and the angular momentum about the axis of symmetry $L_z$, i.e. the two classical integrals of motion. They developed a formalism that essentially combines both the Eddington formulae and the Fricke (1952) expansion in order to obtain the DF’s even part, starting from a density that can be expressed as a function of the radial coordinate and the gravitational potential. Thus, for a given $\rho$, the corresponding even DF can be obtained by solving an Abel integral equation. Once such even part is determined, one can find the DF’s odd part by introducing some reasonable assumptions about the mean circular velocity or using the maximum entropy principle.

In the present paper we show an extension of the formulae derived earlier by Fricke (1952) and Jiang & Ossipkov (2007), by introducing the fractional derivative concept. The
approach developed here has several advantages over the methods introduced before. In one hand, the mathematical difficulties involved in the formalisms based on transformation techniques can be easily overcome. It is worth to point out that the formalism introduced by Jiang & Ossipkov (2007) demands the definition a volumetric pseudo-density, in order to be applicable to flat systems. On the other hand, our method can be applied directly to the case of flat systems. Assume that $\Phi$ and $E$ are, respectively, the gravitational potential and the energy of a star in a stellar system. It is useful to define a relative potential $\Psi = -\Phi + \Phi_0$ and a relative energy $\varepsilon = -E + \Phi_0$, in such a way that the system has only stars with energy $\varepsilon > 0$ (Binney & Tremaine 1987). For the case of an axially symmetric system, it customary to use cylindrical polar coordinates $(R, \varphi, z)$, where the velocity is denoted by $\mathbf{v} = (v_R, v_\varphi, v_z)$. Such system admits two isolating integrals: the component of the angular momentum about the $z$-axis, $L_z = Rv_\varphi$, and the relative energy $\varepsilon$. Hence, the DF of a steady-state stellar system in an axisymmetric potential can be expressed as a non-negative function of $\varepsilon$ and $L_z$, that vanishes for $\varepsilon < 0$, denoted by $f(\varepsilon, L_z)$ and related to the mass density as

$$\rho = \frac{4\pi}{R} \int_0^{\Psi} \int_0^{\Psi/2} f_+(\varepsilon, L_z) dL_z d\varepsilon,$$

where $f_+(\varepsilon, L_z)$ is the even part of $f$ with respect to the angular momentum $L_z$. On the other hand, for the case of flat systems, the surface mass density $\Sigma$ is related to $f$ through

$$\Sigma = 4 \int_0^{\Psi} \int_0^{\Psi/2} f_+(\varepsilon, L_z) dL_z d\varepsilon \sqrt{2R^2(\Psi - \varepsilon) - L_z^2}.$$  

Now, by defining a pseudo-volume density $\hat{\rho}$, according to (Hunter and Quian 1993)

$$\hat{\rho} = \sqrt{2} \int_0^{\Psi} \frac{\Sigma(R^2, \Psi') d\Psi'}{\sqrt{\Psi - \Psi'}}.$$

it is also possible to use equation (1) to deal with these flat systems.

2 TRIDIMENSIONAL SYSTEMS

Most of the methods developed to solve (1) require some kind of dependence between the DF, the relative energy $\varepsilon$ and the angular momentum $L_z$ (see Fricke 1952 and Jiang & Ossipkov 2007, as examples), which will define the corresponding mass density $\rho(R, \Psi)$. Therefore, in order to study the problem, we shall start by assuming some general types of DFs.

2.1 DFs of the form $\sum_n L_z^{2\alpha_n} h_n(\varepsilon)$

To generalize the Jiang & Ossipkov method, we suppose that the DF can be written as

$$f_+(\varepsilon, L_z) = \sum_n L_z^{2\alpha_n} h_n(\varepsilon),$$

where $\alpha_n \in \mathbb{R}$ and the 2 in the exponent of $L_z$ guarantees that $f_+$ is even. Now, performing the integral (1) with respect to $L_z$, we obtain

$$\rho = \sum_n \frac{4\pi 2^{\alpha_n + \frac{1}{2}}}{2\alpha_n + 1} \int_0^{\Psi} h_n(\varepsilon)(\Psi - \varepsilon)^{\alpha_n + \frac{1}{2}} d\varepsilon,$$

for $\alpha_n > -1/2$, while it diverges for $\alpha_n \leq -1/2$. Therefore, we assume that the the mass density is given by

$$\rho(R, \Psi) = \sum_n R^{2\alpha_n} \hat{\rho}_n(\Psi), \quad \text{for} \quad \alpha_n > -\frac{1}{2}.$$  

A comparison between (5) and (6), leads to the relation

$$\hat{\rho}_n(\Psi) = \frac{4\pi 2^{\alpha_n + \frac{1}{2}}}{2\alpha_n + 1} \int_0^{\Psi} h_n(\varepsilon)(\Psi - \varepsilon)^{\alpha_n + \frac{1}{2}} d\varepsilon.$$  

At this point, we introduce the fractional derivative operator $D^\alpha_\Psi$, which represents a $\alpha$-order derivative, with respect to $\Psi$, for any real value of $\alpha$ (see Bologna & Grigolini 2003). Assuming that $(D^\alpha_\Psi \hat{\rho}_n(\Psi))_{\Psi=0} = 0$ for all $\alpha \in (0, \alpha_n + 1/2)$, then

$$D^\alpha_\Psi \hat{\rho}_n(\Psi) = \pi 2^{\alpha_n + \frac{1}{2}} \Gamma(\alpha_n + \frac{1}{2}) \int_0^{\Psi} h_n(\varepsilon) d\varepsilon.$$  

This integral equation is simpler than the first one and can be inverted easily if one takes the derivative once again with respect to $\Psi$.

$$h_n(\varepsilon) = \frac{D^\alpha_\Psi \hat{\rho}_n(\Psi)}{\pi 2^{\alpha_n + \frac{1}{2}} \Gamma(\alpha_n + \frac{1}{2})}.$$  

and the distribution function can be expressed as

$$f_+(\varepsilon, L_z) = \sum_n L_z^{2\alpha_n} D^\alpha_\Psi \hat{\rho}_n(\Psi) \frac{\Psi - \varepsilon}{\pi 2^{\alpha_n + \frac{1}{2}} \Gamma(\alpha_n + \frac{1}{2})}.$$  

When $\alpha_n \in \mathbb{N}$, by the definition of the Riemann-Liouville operator, equation (11) reduces to the formulae obtained by Jiang & Ossipkov (2007).

As a particular case, suppose that $\hat{\rho}_n(\Psi)$ can be written in the form

$$\hat{\rho}_n(\Psi) = \sum_k A_{nk} \Psi^{\beta_k},$$

So, applying the fractional derivative operator to (11)

$$D^\alpha_\Psi \hat{\rho}_n(\Psi) = \sum_k A_{nk} \frac{\Gamma(\beta_k + 1) \Psi^{\beta_k - \alpha_n - \frac{1}{2}}}{\Gamma(\beta_k - \alpha_n - \frac{1}{2})},$$  

for $\beta_k > \alpha_n + 1/2$, and the corresponding DF is

$$f_+ = \sum_{n,k} \frac{A_{nk} \Gamma(\beta_k + 1) L_z^{2\alpha_n} \varepsilon^{\beta_k - \alpha_n - \frac{1}{2}}}{\pi 2^{\alpha_n + \frac{1}{2}} \Gamma(\alpha_n + \frac{1}{2}) \Gamma(\beta_k - \alpha_n - \frac{1}{2})}.$$  

This result is totally equivalent to the Fricke solution, for real values of $\alpha_n$, and therefore can be considered as a generalization.

2.2 DFs of the form $\sum_n L_z^{2\alpha_n} g_n(Q)$

It is possible to derive a more general expression for the DF if we assume that it depends on $\varepsilon$ through $Q = \varepsilon - L_z^2/(2R_z^2)$,
where $R_a$ is a scaling radius. Suppose that the system has only stars with $Q > 0$, so $f(Q, L_z) = 0$ for $Q < 0$. Here, $Q \to \varepsilon$ as $R_a \to \infty$. The fundamental equation can be written, in terms of $Q$, as

$$\rho = \frac{4\pi}{R} \int_0^\infty \int_0^R \sqrt{2(\Psi - Q)/(1 + R^2)} f_+(Q, L_z) dL_z dQ,$$

where $f_+(Q, L_z)$ is the even part of $f(Q, L_z)$. So, following a similar procedure than in section 2.1, one can find that a DF of the form

$$f_+(Q, L_z) = \sum_n \frac{L_z^{2\alpha_n} D_\Psi^{\alpha_n + 2} \tilde{\rho}_n(\Psi)}{2^{\alpha_n + 2} \Gamma(\alpha_n + \frac{1}{2})},$$

(15)

corresponds to a mass density of the form

$$\rho(R, \Psi) = \sum_n \frac{R^{2\alpha_n} \tilde{\rho}_n(\Psi)}{2^{\alpha_n + 2} \Gamma(\alpha_n + \frac{1}{2})},$$

(16)

for $\alpha_n > -1/2$, where $\alpha_n \in \mathbb{R}$. Now, if we sum over all possible values of $R_a$ we obtain the general expression

$$f_+(Q, L_z) = \sum_{\alpha_n} \frac{L_z^{2\alpha_n} D_\Psi^{\alpha_n + 2} \tilde{\rho}_n(\Psi)}{2^{\alpha_n + 2} \Gamma(\alpha_n + \frac{1}{2})},$$

(17)

corresponding to a density of the form

$$\rho(R, \Psi) = \sum_{\alpha_n} \frac{R^{2\alpha_n} \tilde{\rho}_n(\Psi)}{2^{\alpha_n + 2} \Gamma(\alpha_n + \frac{1}{2})},$$

(18)

with $R_a > 0$ and $\alpha_n > -1/2$.

2.3 Models with divergent gravitational potential

In a system in which the gravitational potential has no upper bound, it is not possible to define correctly the relative potential $\Psi$ and the relative energy $\varepsilon$, because the escape energy of the system is $\infty$. For this reason, we shall write the fundamental equation in terms of $E$ and $\Phi$,

$$\rho(R, \Phi) = \frac{4\pi}{R} \int_0^\infty \int_0^R R^{2(E-\Phi)} f_+(E, L_z) dL_z dE,$$

(19)

and we will suppose that the DF can be written as

$$f_+(E, L_z) = \sum_n L_z^{2\alpha_n} h_n(E),$$

(20)

for $\alpha_n > -1/2$, and that the density is given by

$$\rho(R, \Phi) = \sum_n R^{2\alpha_n} \tilde{\rho}_n(\Phi),$$

(21)

for $\alpha_n > -1/2$. Thus then, by integrating with respect to $L_z$, follows that

$$\tilde{\rho}_n(\Phi) = \frac{4\pi 2^{\alpha_n + 2}}{2\alpha_n + 1} \int_0^\infty h_n(E)(E - \Phi)^{\alpha_n + \frac{1}{2}} dE.$$ 

(22)

Now, if we assume that

$$\lim_{\Phi \to -\infty} D_\Psi^j \tilde{\rho}_n(\Phi) = 0$$

(23)

for all $j \in (0, \alpha_n + 1/2)$, then

$$D_\Psi^{\alpha_n + \frac{1}{2}} \tilde{\rho}_n = i\pi(-2)^{\alpha_n} 2^{2\alpha_n} \Gamma(\alpha_n + \frac{1}{2}) \int_0^\infty h_n(E) dE,$$

(24)

and so we obtain

$$h_n(E) = \frac{D_\Psi^{\alpha_n + \frac{1}{2}} \tilde{\rho}_n(\Phi)}{\pi(-2)^{\alpha_n} 2^{2\alpha_n} \Gamma(\alpha_n + \frac{1}{2})}.$$  

(25)

Therefore, the distribution function is

$$f_+(E, L_z) = \sum_n \frac{L_z^{2\alpha_n} D_\Psi^{\alpha_n + \frac{1}{2}} \tilde{\rho}_n(\Phi)}{\pi(-2)^{\alpha_n} 2^{2\alpha_n} \Gamma(\alpha_n + \frac{1}{2})},$$

(26)

for $\alpha_n > -1/2$.

On the other hand, we can assume that the density has the more general form

$$\rho(R, \Phi) = \sum_n \frac{R^{2\alpha_n} \tilde{\rho}_n(\Phi)}{2^{\alpha_n + 2} \Gamma(\alpha_n + \frac{1}{2})},$$

(27)

for $\alpha_n > -1/2$. So, the corresponding DF will be

$$f_+(Q, L_z) = \sum_n \frac{L_z^{2\alpha_n} D_\Psi^{\alpha_n + \frac{1}{2}} \tilde{\rho}_n(\Phi)}{\pi(-2)^{\alpha_n} 2^{2\alpha_n} \Gamma(\alpha_n + \frac{1}{2})},$$

(28)

for $\alpha_n > -1/2$ and $Q$ defined as $Q = E + L_z^2/(2R_a^2)$. Now, if we sum over all the possible values of $R_a$, we can obtain the generalization

$$f_+(Q, L_z) = \sum_{\alpha_n} \frac{L_z^{2\alpha_n} D_\Psi^{\alpha_n + \frac{1}{2}} \tilde{\rho}_n(\Phi)}{2^{\alpha_n + 2} \Gamma(\alpha_n + \frac{1}{2})},$$

(29)

corresponding to a density of the form

$$\rho(R, \Phi) = \sum_{\alpha_n} \frac{R^{2\alpha_n} \tilde{\rho}_n(\Phi)}{2^{\alpha_n + 2} \Gamma(\alpha_n + \frac{1}{2})},$$

(30)

with $R_a > 0$ and $\alpha_n > -1/2$.

3 FLAT SYSTEMS

The formalism sketched above can also be used directly in the case of flat systems. Note that in the method introduced by Jiang & Ossipkov (2007) it was not possible, since the fundamental equation could not be solved using the Abel integral equation. Now, we will proceed similarly to the tridimensional case, finding the DFs for different kinds of densities. Then, we will also study the case of models with divergent gravitational potential.

3.1 DFs of the form $\sum_n L_z^{2\alpha_n} h_n(\varepsilon)$

As in the tridimensional case, first we suppose that

$$f_+(\varepsilon, L_z) = \sum_n L_z^{2\alpha_n} h_n(\varepsilon),$$

(31)

So, by integrating (22) with respect to $L_z$, we obtain

$$\Sigma = \sum_n \frac{\sqrt{\pi} R^{2\alpha_n} \Gamma(\alpha_n + \frac{1}{2})}{2^{\alpha_n + 2} \Gamma(\alpha_n + 1)} \int_0^\infty (\Psi - \varepsilon)^{\alpha_n} h_n(\varepsilon) d\varepsilon,$$

(32)

for $\alpha_n > -1/2$. Therefore, if we assume that

$$\Sigma(R, \Psi) = \sum_n R^{2\alpha_n} \sigma_n(\Psi),$$

(33)
Then, by taking the fractional derivative we obtain
\[
\sigma_n(\Psi) = \frac{\sqrt{\Gamma(\alpha_n + \frac{1}{2})}}{2^{\alpha_n+1}\Gamma(\alpha_n + \frac{1}{2})} \int_0^\Psi (\Psi - \varepsilon)^{\alpha_n} h_n(\varepsilon)d\varepsilon. \tag{34}
\]

Now, if \((D^\alpha_{\Psi}\sigma_n(\Psi))_{\Psi=0} = 0\) for all \(j \in (0, \alpha_n)\), then equation (34) leads to
\[
D^\alpha_{\Psi}\sigma_n(\Psi) = \sqrt{\pi} 2^{\alpha_n+1}\Gamma(\alpha_n + \frac{1}{2}) \int_0^\Psi h_n(\varepsilon)d\varepsilon. \tag{35}
\]
Consequently,
\[
h_n(\varepsilon) = \frac{D^{\alpha_n+1}_{\Psi}\sigma_n(\Psi)|_{\Psi=\varepsilon}}{\sqrt{\pi} 2^{\alpha_n+1}\Gamma(\alpha_n + \frac{1}{2})}, \tag{36}
\]
and the DF is
\[
f_+(\varepsilon, L_x) = \sum_n \frac{L^{2\alpha_n}_{\Psi} D^{\alpha_n+1}_{\Psi}\sigma_n(\Psi)|_{\Psi=\varepsilon}}{\sqrt{\pi} 2^{\alpha_n+1}\Gamma(\alpha_n + \frac{1}{2})}. \tag{37}
\]
This equation, for \(\alpha_n = 0\), corresponds to the method developed in \cite{Kalnins1970}, working in an adequate rotating frame.

As a particular case, suppose that
\[
\sigma_n(\Psi) = \sum_k A_n \Psi^{\alpha_k}. \tag{38}
\]
Then, by taking the fractional derivative we obtain
\[
D^{\alpha_n+1}_{\Psi}\sigma_n(\Psi) = \sum_k A_n \Gamma(\beta_k + 1) \Psi^{\beta_k - \alpha_n - 1}, \tag{39}
\]
and the DF is
\[
f_+(\varepsilon, L_x) = \sum_{n,k} \frac{A_n \Gamma(\beta_k + 1) L^{2\alpha_n}_{\Psi} \Psi^{\beta_k - \alpha_n - 1}}{\sqrt{\pi} 2^{\alpha_n+1}\Gamma(\alpha_n + \frac{1}{2}) \Gamma(\beta_k + \alpha_n)}. \tag{40}
\]
This relation can be interpreted as the analogous case of the Fricke expansion, when we are dealing with flat systems. It can be verified performing the pseudo-volume density of \(R^{2\alpha_n}\Psi^{\beta_k}\) and taking the Fricke component corresponding to the tridimensional case.

### 3.2 Other DFs for flat systems

We can generalize the result (37) if we express the DF in terms of \(Q = \varepsilon - L^2_x/(2R^2_{\alpha})\). In this way, if the density has the form
\[
\Sigma(R, \Psi) = \sum_{\alpha, n} \frac{R^{2\alpha_n}\sigma_n(\Psi)}{(1 + \frac{R^2_{\alpha}}{2\Psi})^{\alpha_n}}, \tag{41}
\]
the corresponding DF is
\[
f_+(Q, L_x) = \sum_{\alpha, n} \frac{L^{2\alpha_n}_{\Psi} D^{\alpha_n+1}_{\Psi}\sigma_n(\Psi)|_{\Psi=Q}}{\sqrt{\pi} 2^{\alpha_n+1}\Gamma(\alpha_n + \frac{1}{2}) \Gamma(\alpha_n + \frac{1}{2})}, \tag{42}
\]
for \(R_{\alpha} > 0\) and \(\alpha_n > -1/2\).

Furthermore, if we consider models with gravitational potential having no upper bound, we can deduce that for a density
\[
\Sigma(R, \Phi) = \sum_{\alpha, n, \Phi} \frac{R^{2\alpha_n}\sigma_n(\Phi)}{(1 + \frac{R^2_{\alpha}}{2\Phi})^{\alpha_n + \frac{1}{2}}}, \tag{43}
\]
and assuming that \(\lim_{\Psi \to \infty} D^\alpha_{\Psi}\sigma_n(\Phi) = 0\) for all \(j \in (0, \alpha_n)\), then
\[
f_+(Q, L_x) = \sum_{\alpha, n} \frac{L^{2\alpha_n}_{\Psi} D^{\alpha_n+1}_{\Psi}\sigma_n(\Phi)|_{\Psi=Q}}{\sqrt{\pi} 2^{\alpha_n+1}\Gamma(\alpha_n + \frac{1}{2}) \Gamma(\alpha_n + \frac{1}{2})}, \tag{44}
\]
for \(R_{\alpha} > 0\), \(\alpha_n > -1/2\) and \(Q = E + L^2_x/(2R^2_{\alpha})\).

### 4 SOME APPLICATIONS

In this section we will use the formulae introduced above to the Binney’s logarithmic model and the Mestel disc, and we will see that their corresponding DFs match exactly with those that were found through the application of other methods. Binney’s logarithmic model has a gravitational potential of the form
\[
\Phi(R, z) = \frac{1}{2} v_0^2 \ln \left(1 + \frac{R^2}{2} + \frac{z^2}{q^2}\right), \tag{45}
\]
whereas its mass density is
\[
\rho(R, z) = \frac{v_0^2(1 + 2q^2 + R^2 + (2 - q^{-2})z^2)}{4\pi G q^2 (1 + R^2 + z^2 q^{-2})^2}, \tag{46}
\]
that can be written as
\[
\rho = \frac{v_0^2 ((1 - q^2) R^2 + 1 + (q^2 - \frac{1}{2}) z^2)^2}{2\pi G q^2 e^{2E/\sigma^2}}. \tag{47}
\]
So, as \(D^\alpha_{\Psi}e^{ax} = a^\alpha e^{ax}\) for any \(a \in \mathbb{R}\), we obtain
\[
f_+(E, L_z) = A L^2_x e^{-2E/\sigma^2} + Be^{-4E/\sigma^2} + C e^{-2E/\sigma^2}, \tag{48}
\]
where
\[
A = \frac{2^{\frac{3}{2}} (1 - q^2)}{\pi G q^2 v_0^3}, \quad B = \frac{\sqrt{2}}{\pi G q^2 v_0^3}, \quad C = \frac{2q^2 - 1}{4\pi^{3/2} G q^2 v_0^3},
\]
the same DF founded in \cite{Jiang&Ossipkov2007}, using the Abel’s integral equation, and in \cite{Evans1993b} using Lynden-Bell’s method.

Another case of interest is the Mestel disc, characterized by a gravitational potential of the form
\[
\Phi(R) = v_c^2 \ln \left(\frac{R}{R_0}\right), \tag{49}
\]
and a surface mass density given by
\[
\Sigma(R) = \frac{R_0}{R}, \tag{50}
\]
where \(\Sigma_0 = v_c^2/(2\pi GR_0)\). Now, we can write (50) as
\[
\Sigma(R) = R^{2m} \frac{\Sigma_0}{R_0^{2m}} e^{-(2m+1)\Phi/\sigma^2}, \tag{51}
\]
for any \(m \in \mathbb{R}\). So, equation (44) for \(R_\alpha \to \infty\) leads to
\[
f_+(E, L_z) = FL^2_x e^{-E/\sigma^2}, \tag{52}
\]
where \(F\) and \(\sigma\) are the constants given by
\[
\sigma^2 = \frac{v_c}{2m + 1}, \quad F = \frac{\Sigma_0}{\Gamma(m + 1/2) R_0^{2m} 2^{m+2}}. \tag{53}
\]
This solution was obtained as well by \cite{Evans1993b}, and was proposed earlier by \cite{Tonmer1977}.
5 DISCUSSION

In contrast with the methods based on integral transformation techniques, the formalism developed here does not require that the mass density has an analytic continuation to complex arguments. Indeed, this is the principal disadvantage involved in such methods. Moreover, our fractional derivative approach can be regarded as a general method that contains, as particular cases, the results obtained by Frickel (1952), Kalnajs (1976) and Jiang & Ossipkov (2007). The method developed here can be applied to a wider variety of axisymmetric models, due to the generic form of the density as a function. Another advantage of this formalism is that it can be applied directly both to tridimensional systems and to flat systems, without the implementation of a pseudo-volume density. Therefore, taking into account all the above statements, the present formalism represents a powerful tool on the construction of self-consistent stellar models.

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