Resonances in $\Lambda d$ Scattering and the $\Sigma$-hypertriton

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Using separable $NN$ and $\Lambda N-\Sigma N$ potentials in the Faddeev equations, we have demonstrated that the predicted enhancement in the $\Lambda d$ cross section near the $\Sigma d$ threshold is associated with resonance poles in the scattering amplitude. The positions of these poles, on the second Riemann sheet of the complex energy plane, are determined by examining the eigenvalues of the kernel of the Faddeev equations. This suggests that for a certain class of $\Lambda N-\Sigma N$ potentials we can form a $\Sigma$-hypertriton with a width of about 8 MeV.
I. INTRODUCTION

In the realm of nonperturbative Quantum Chromo Dynamics (QCD) our description of nuclear phenomena in terms of the physically observable baryons and mesons, the collective modes of the QCD Lagrangian, has enjoyed considerable success. A nonrelativistic two-body potential model picture of the \(^3\text{H}, ^3\text{He},\) and \(^4\text{He}\) bound states as well as few-nucleon low-energy scattering and reactions accounts amazingly well for much of the data. The addition of the strangeness degree of freedom to the nucleus opens the opportunity to ascertain whether these models have predictive power or are merely vehicles of interpolation. That is, can one use the models which have been developed in the conventional, zero strangeness sector to extrapolate beyond that domain to understand the nuclear physics involving \(\Lambda\)s and \(\Sigma\)s?

Although that question remains open, the strong coupling of the \(\Lambda N-\Sigma N\) system has been seen to lead to the enhancement of certain phenomena which appear in nonstrange nuclei. For example, three-body-force effects in the binding energy of the hypertriton \((^3\Lambda\text{H})\), when one eliminates the \(\Sigma N\) channel from the problem, are significant \([1]\); \(i.e., \Lambda N-\Sigma N\) coupling effects in hypernuclei appear to play a much larger role than do \(NN-\Delta N\) coupling effects in nonstrange nuclei. Furthermore, charge symmetry breaking, which is strongly masked in the \(^3\text{H}-^3\text{He}\) isodoublet by the Coulomb force acting between the two protons in \(^3\text{He}\), is clearly obvious in the \(^4\Lambda\text{H}-^4\Lambda\text{He}\) binding energy difference \([2]\). Thus, extending our nuclear physics investigations to include \(S \neq 0\) can magnify certain physical effects.

While the existence of strangeness \(-1\) \(\Lambda\)-hypernuclei is well established from the observation of many bound states, such has not been the case for \(\Sigma\)-hypernuclei. Although structure in the recoilless production of p-shell hypernuclei did indicate the possible existence of \(\Sigma\)-hypernuclei \([3,4,5]\), this structure corresponded to unbound states. Therefore, it was surprising to many when Hayano et al. \([6]\) reported that the \(\pi^-\) spectrum from the \(^4\text{He}(\text{stopped } K^-, \pi^+)\) reactions exhibited narrow structure below the threshold for \(\Sigma\) emission. It is the interpretation of such spectra that we address in this investigation.
Charge conservation forbids the conversion of $\Sigma^- n$ into any $\Lambda N$ charge state. If a $\Sigma^- n$ bound state were to exist, it would decay to $\Lambda N$ only by the weak interaction. Potential model analyses of hyperon-nucleon ($YN$) scattering indicate a weak repulsion in the spin-triplet state and nonbinding attraction in the spin-singlet state. [The $\Sigma^- n$ system belongs to the same $SU(3)$ multiplet as the $nn$ system, which is almost bound in the $^1S_0$ state.] The absence of binding in the $\Sigma^- n$ system was confirmed by May et al. [7] through investigation of the $^3\text{H}(K^-, \pi^+)\Sigma^- n$ reaction. However, this did not rule out the possibility that the $\Sigma^- nn$ system might be bound. Such a bound state would also be stable against $\Sigma N \to \Lambda N$ conversion. However, in an analysis of the $\Sigma NN$ states, Dover and Gal [8] noted that, if a bound state were to exist, then the ($T = 0, S = 1/2$) configuration should lie lowest while the $T = 2$ state would be the least likely to be bound, because of the spin-isospin dependence of the $\Sigma N$ residual interaction.

An analogous analysis of the $A = 4 \Sigma NNN$ system [8] indicated that the ($T = 1/2, S = 0$) configuration should lie lower in energy than the ($T = 3/2, S = 0$) configuration, although the latter state was expected to be narrower. Thus, the report by Hayano et al. [3] that the $\pi$ spectrum from stopped $K^-$ in the reaction $^4\text{He}(K^-, \pi^-)$ exhibited narrow structure below the threshold for $\Sigma$ production was quite exciting. The ($K^-, \pi^-$) reaction can lead to both $T = 1/2$ and $T = 3/2 \Sigma NNN$ states, while the ($K^-, \pi^+$) reaction leads only to the $T = 3/2$ state. Therefore, because no such structure was observed in the spectrum from the complementary $^4\text{He}(K^-, \pi^+)$ reaction, and because the ($K^-, \pi^-$) spin-flip reaction is small, the structure was interpreted as a bound $^4\Sigma$He state having the quantum numbers $T = 1/2$ and $J^\pi = 0^+$. Hayano has recently reported [9] new results for in-flight $^4\text{He}(K^-, \pi^\pm)$ experiments at BNL, which confirm the structure in the ($K^-, \pi^-$) reaction and lack of structure in the ($K^-, \pi^+$) reaction observed in the stopped $K^-$ absorption experiments. The peak in the $\pi^-$ spectrum appears to be centered at $B_{\Sigma^+} = 4 \pm 1$ MeV, consistent with the earlier result [3]. The width of the peak is about $10 \pm 2$ MeV, again consistent with a more refined analysis of the KEK data [10]. Furthermore, the data are not inconsistent with the earlier bubble
chamber data [11] for the exclusive $K^{-4}\text{He} \to \pi^-\Lambda pd$ measurement, recently reanalyzed by Dalitz et al. [12], which appear to show a cusp-like enhancement near the $\Sigma^+$ production threshold. The inferred $A = 4 \Sigma$ hypernucleus would seem to be more bound (by an MeV) than the $\Lambda$ is bound in $^4\Lambda\text{He}$. Although the $\Sigma$ is 10% more massive than the $\Lambda$ which reduces its kinetic energy, it would appear that the $\Sigma^+$ interaction with $^3\text{H}$ in the $(T = 1/2, S = 0)$ channel must be more attractive than the corresponding $\Lambda$ interaction with $^3\text{He}$ or $^3\text{H}$.

Following the work of Dover and Gal on the ordering of the $A = 4 \Sigma NN$ states, Harada et al. [13,14] predicted the existence of an $A = 4 \Sigma NN$ bound state using their SAP-1 approximation to the Nijmegen $YN$ potential model D [15]: $B_{\Sigma^+} = 4.6$ MeV, $\Gamma = 7.9$ MeV. They predicted no other bound state for $A = 2 - 5$. Nonetheless, we were motivated to examine the $\Sigma NN$ system in an effort to understand the properties of the scattering amplitude with respect to observable structure in the physical cross section. For a sufficiently attractive $\Sigma N$ interaction, one would hope to see evidence of a $(T = 0, S = 1/2) \Sigma NN$ bound state or a low lying resonance in the $\Lambda d$ cross section near the threshold for $\Sigma$ production. (Theoretical models of the $YN$ interaction can exhibit a cusp phenomena in the $\Lambda N$ channel as one crosses the $\Sigma N$ threshold [16], but that cusp dissolves into the continuum in the three-body system, where the lowest threshold is not a two-body system but the $\Lambda d \to \Sigma NN$ reaction channel.) Although such model calculations are not directly applicable to the $T = 1$ inflight $^3\text{He}(K^-,\pi^\pm)$ measurements that have been reported by Hungerford [17] and discussed by Hayano [9], they are relevant to the $T = 0$ $^3\text{H}(K^-,\pi^-)$ reaction as well as to analysis of $\Lambda d$ scattering.

In this paper we explore the structure of the $\Lambda d$ cross section in terms of a Hamiltonian model. For Hermitian Hamiltonians the spectrum consists of the eigenvalues for the bound and scattering states. From the scattering state eigenfunctions we can extract the scattering amplitude and, therefore, the cross section. The presence of rapid fluctuations (structure) in the cross section is normally attributed to resonances, which can be viewed as poles in the scattering amplitude on the second Riemann sheet of the complex energy plane. It is possible to establish a direct relation between the Hamiltonian for the system and the reso-
nance energies and widths by realizing that the Hermitian Hamiltonian (and corresponding
eigenvalue problem) is defined on the first Riemann sheet of the complex energy plane, while
the poles of the scattering amplitude are on the second Riemann sheet. Thus, to directly
obtain the desired resonance energies (poles), one must analytically continue the eigenvalue
problem onto that part of the second sheet where the resonance poles reside. This leads to
an eigenvalue problem for a non-Hermitian Hamiltonian which, therefore, admits complex
eigenvalues. These complex eigenvalues specify the energy and width of the resonances.
The corresponding wave functions are normalizable, as we shall see below, provided one
realizes that the solutions of a non-Hermitian eigenvalue problem and the definition of the
normalization must be appropriately modified.

In terms of the specific problem at hand, if the $YN$ interaction produces a pole in the
$\Lambda d$ amplitude below the $\Sigma NN$ threshold (and on the top sheet of the $\Sigma NN$ branch cut
but the bottom sheet of the $\Lambda NN$ branch cut, the $[bt]$ Riemann sheet $\square$), then one would
anticipate narrow structure in $\Lambda d$ scattering below the $\Sigma NN$ threshold. In contrast, if the
$YN$ interaction produces a pole above the threshold in the $\Sigma NN$ system (and on the top
sheet of the $\Sigma NN$ branch cut but on the second sheet of the $\Lambda NN$ branch cut, again the $[bt]$ sheet),
then the effect of this pole will still be to produce structure in the $\Lambda d$ cross section
below the $\Sigma NN$ threshold. This occurs because, for energies above the $\Sigma NN$ threshold,
the pole is screened from the physical region by the branch cut due to the presence of the
threshold. To see structure above the $\Sigma NN$ threshold, there should be a pole on the second
sheet of both the $\Lambda NN$ and $\Sigma NN$ branch cut, i.e. $[bb]$, above the $\Sigma$ production threshold.
That is, any structure seen below the $\Sigma$ production threshold will be due to the poles on the

$^1$We adopt the convention of Ref. [16] for the labeling of the Riemann sheets corresponding to
the $\Lambda NN$ and $\Sigma NN$ threshold. However, in this problem we have additional sheet structures from
the $\Lambda d$ threshold and any additional branch points arising from the resonance poles of the $YN$ t
matrix.
[bt] Riemann sheet. Such a pole might correspond to (i) a bound state of the ΣNN system in the absence of coupling of the ΣN channel to the ΛN channel (a pole shifted into the complex plane resulting in the structure seen in Λd scattering when the ΛN-ΣN coupling is turned on), or (ii) an unbound state of the ΣNN system in the absence of coupling to the ΛN channel (a pole which is moved onto the [bt] sheet when the coupling is turned on). In either case, enhancement in the Λd cross section below the Σ production threshold corresponds to an eigenstate of the YNN system.

To explore this hypothesis, we present a detailed discussion of the equations describing the YNN system in the presence of a YN (ΛN-ΣN coupled-channel) potential in the following section. Formal solution of the three-body equations is outlined in the Appendix. Numerical results for specific YN potential models are presented in Section III. A discussion of the results and summary of our conclusions can be found in Section IV.

II. THEORY

To establish the connection between the enhancement in the cross section for Λd scattering and the formation of a Σ-hypertriton, we must demonstrate that the structure found in the cross section for Λd scattering is due to poles in the scattering amplitude on the second energy sheet, and that these poles correspond to eigenstates of the Hamiltonian for the YNN system in which Y = Λ, or Σ. This connection between the cross section and the eigenstates of the Hamiltonian is achieved by: (i) showing formally that the energy at which the scattering amplitude has a pole on the second energy sheet can be considered an eigenstate of the Hamiltonian, (ii) demonstrating that for the specific models of the ΛN-ΣN interaction considered, there is a correlation between the enhancement in the cross section and the position of the poles of the scattering amplitude, or T-matrix.

To establish the fact that the position of a pole in the scattering amplitude corresponds to an eigenstate of the Hamiltonian, we consider the YNN system in terms of a three-body Hamiltonian given by
\[ H = H_0 + V , \]  

(2.1)

where \( H_0 \) is the kinetic energy of the three-particle system and \( V \) is the sum of pairwise interactions. In spectator particle notation \( V \) is given by

\[ V = \sum_{\alpha=1}^{3} V_\alpha , \]  

(2.2)

with \( V_3 \) being the \( N N \) interaction, while \( V_1 \) and \( V_2 \) are the \( YN \) interactions. The Schrödinger equation for this three-body system can then be written as

\[ (E - H_0) \, |\Psi\rangle = V \, |\Psi\rangle , \]  

(2.3)

or

\[ |\Psi\rangle = G_0(E) \, V \, |\Psi\rangle \]  

\[ = \sum_{\alpha=1}^{3} G_0(E) \, V_\alpha \, |\Psi\rangle \]  

\[ \equiv \sum_{\alpha=1}^{3} |\psi_\alpha\rangle , \]  

(2.4)

where the free Green’s function \( G_0(E) = (E - H_0)^{-1} \). The last line in Eq. (2.4) corresponds to the Faddeev decomposition of the wave function. The Faddeev components of the wave function \( |\psi_\alpha\rangle \), then satisfy the equation

\[ |\psi_\alpha\rangle = \sum_{\beta} G_0(E) \, V_\alpha \, |\psi_\beta\rangle , \]  

(2.5)

or

\[ [1 - G_0(E) \, V_\alpha] \, |\psi_\alpha\rangle = \sum_{\beta} G_0(E) \, V_\alpha \, \bar{\delta}_{\alpha\beta} \, |\psi_\beta\rangle , \]  

(2.6)

where \( \bar{\delta}_{\alpha\beta} = (1 - \delta_{\alpha\beta}) \). If we now multiply Eq. (2.6) by \([1 - G_0(E) \, V_\alpha]^{-1}\), and take into consideration the fact that the \( T \)-matrix for the two-body sub-system, \( T_\alpha(E) \), is given by \( T_\alpha(E) = [1 - V_\alpha \, G_0(E)]^{-1} \, V_\alpha \), we can write the equation for the Faddeev component of the wave function as

\[ |\psi_\alpha\rangle = \sum_{\beta} G_0(E) \, T_\alpha(E) \, \bar{\delta}_{\alpha\beta} \, |\psi_\beta\rangle , \]  

(2.7)
or

$$|\phi_\alpha\rangle = \sum_\beta G_0(E) \bar{\delta}_{\alpha\beta} T_\beta(E) |\phi_\beta\rangle,$$

(2.8)

where

$$|\phi_\alpha\rangle = \sum_\beta \bar{\delta}_{\alpha\beta} |\psi_\beta\rangle.$$

(2.9)

It is the solution of Eq. (2.7) or Eq. (2.8), that gives the bound state of the hypertriton. To that extent, the solution of either Eqs. (2.7) or (2.8) is identical to the solution of the Schrödinger equation. In fact, the energy at which these equations have a solution corresponds to a bound state, and the solution is an eigenstate of the Hamiltonian for the three-body system. In momentum representation this homogeneous integral equation, Eq. (2.8), has the same kernel as the Alt-Grassberger-Sandhas (AGS) equations [18] for three-particle scattering, which we can write as,

$$X_{\alpha\beta}(E) = \bar{\delta}_{\alpha\beta} G_0 + \sum_\gamma G_0(E) \bar{\delta}_{\alpha\gamma} T_\gamma(E) X_{\gamma\beta}.$$

(2.10)

This suggests that if we convert the homogeneous integral equation Eq. (2.8) to an eigenvalue problem of the form

$$\lambda_n(E) |\phi_{n,\alpha}\rangle = \sum_\beta G_0(E) \bar{\delta}_{\alpha\beta} T_\beta(E) |\phi_{n,\beta}\rangle,$$

(2.11)

where the $\lambda_n(E)$ are the eigenvalues and $|\phi_{n,\alpha}\rangle$ are the eigenstates, then the solution of the inhomogeneous integral equation, Eq. (2.10), for the amplitude $X_{\alpha\beta}$ can be written in terms of the eigenvalues and eigenstates of the homogeneous equation, Eq. (2.11), as (see Appendix) [19]:

$$X_{\alpha\beta}(E) = \sum_n |\phi_{n,\alpha}(E)\rangle \frac{[\bar{\lambda}_n(E^*)]^*}{1 - \lambda_n(E)} \langle \bar{\phi}_{n,\beta}(E^*) |.$$

(2.12)

Here, $|\bar{\phi}_{n,\beta}\rangle$ and $\bar{\lambda}_n$ are the eigenstates and eigenvalues of the adjoint kernel. It is clear from Eq. (2.12) that for energies at which $\lambda_n(E) = 1$, the scattering amplitude $X_{\alpha\beta}(E)$ has a pole. Thus, the positions of the poles of $X_{\alpha\beta}(E)$ on the second Riemann sheet of the energy plane can be determined by examining the eigenvalues of Eq. (2.11) for complex energies.
Since resonance poles reside on the second Riemann sheet of the complex energy plane, we deform our contour of integration in momentum space in order to analytically continue our eigenvalue equation, Eq. (2.11), onto the second sheet. However, the deformation of the contour of integration requires a knowledge of the position of the singularities of the kernel in the energy variable. In fact, as we will demonstrate, these singularities constrain the energy domain onto which we can analytically continue our equations. The singularities of the kernel of the AGS equation are determined by the dynamics of the two-body interaction we include in our analysis. Since we will restrict our calculations to separable two-body potentials that include $\Lambda N-\Sigma N$ coupling, we can rewrite Eqs. (2.10) and (2.11) for this class of interactions. These separable potentials can be written in matrix form, after partial wave expansion, as

$$ V_\alpha = |g_{\kappa_\alpha}\rangle C_{\kappa_\alpha} \langle g_{\kappa_\alpha}|, \quad (2.13) $$

where $C_{\kappa_\alpha}$ is the strength of the interaction in the $\kappa_\alpha$ partial wave, while $|g_{\kappa_\alpha}\rangle$ is the corresponding form factor. The $t$ matrix, in two-body Hilbert space, for this potential is then given by

$$ t_{\kappa_\alpha}(\varepsilon_\alpha) = |g_{\kappa_\alpha}\rangle \tau_{\kappa_\alpha}(\varepsilon_\alpha) \langle g_{\kappa_\alpha}|, \quad (2.14) $$

where the “quasi-particle” propagator, $\tau_{\kappa_\alpha}$, takes the form

$$ \tau_{\kappa_\alpha}(\varepsilon_\alpha) = \left[C_{\kappa_\alpha}^{-1} - \langle g_{\kappa_\alpha}|g_0(\varepsilon_\alpha)\rangle\langle g_{\kappa_\alpha}|\right]^{-1}. \quad (2.15) $$

Here, $g_0(\varepsilon_\alpha)$ is the two-body free Green’s function for the pair $(\beta\gamma)$.

With the above results for the two-body $t$ matrix, we can proceed to write the AGS equations, and the corresponding homogeneous equation for a given total angular momentum $J$ and isospin $T$ as

$$ X^{JT}_{k_\alpha;k_\beta}(q,q';E^+) = Z^{JT}_{k_\alpha;k_\beta}(q,q';E^+) $$$$ + \sum_{k_\gamma} \int_0^\infty dq'' K^{JT}_{k_\alpha;k_\gamma}(q,q'';E^+) X^{JT}_{k_\gamma;k_\beta}(q'',q';E^+), \quad (2.16) $$
and

$$\lambda_n(E) \phi_{n,\alpha}(q;E) = \sum_{k\beta} \int_0^\infty dq' K_{k\alpha;k\beta}^{JT}(q,q';E) \times \phi_{n,k\beta}(q';E),$$  \hspace{1cm} (2.17)

where the kernel of the integral equations is given by

$$K_{k\alpha;k\beta}^{JT}(q,q';E) = Z_{k\alpha;k\beta}^{JT}(q,q';E) \times \tau_{k\beta} [E - \varepsilon_\beta(q')] q'^2.$$  \hspace{1cm} (2.18)

Here, $k_\alpha$ refers to the set of quantum numbers that define the partial wave three-body channel with particle $\alpha$ the spectator, while $\varepsilon_\alpha$ is the energy of the spectator particle. The partial wave Born amplitude, $Z_{k\alpha;k\beta}^{JT}$, is given by

$$Z_{k\alpha;k\beta}^{JT}(q,q';E) = \bar{\delta}_{\alpha\beta} \langle TJk\alpha q; g_{k\alpha} | G_0(E) \times |g_{k\beta} q' k\beta JT \rangle.$$  \hspace{1cm} (2.19)

An explicit expression for this Born amplitude has been given previously [1].

To analytically continue Eq. (2.17) onto the second Riemann sheet of the complex energy plane, we rotate the contour of integration; i.e., we make the transformation

$$q \to q e^{-i\theta}, \quad q' \to q' e^{-i\theta} \quad \text{with} \quad \theta > 0.$$  \hspace{1cm} (2.20)

This should, in principle, extend the energy domain over which Eq. (2.17) is defined to that part of the second energy plane for which $|\arg E| < 2\theta$. However, the singularities of the kernel put a constraint on the range of values $\theta$ can assume. Since both $q$ and $q'$ in Eq. (2.17) are rotated by the same angle, the singularities of the Born amplitude are such that the only constraint they place on $\theta$ is that $\theta < \frac{\pi}{2}$ \[20,19\]. This, for all practical purposes, imposes no serious constraint on the energy domain to which we can extend our equation in order to search for resonance poles. This leaves us with the singularities of the “quasi-particle” propagator $\tau_\kappa$, which are of two kinds: (i) simple poles due to two-body bound or resonance.
states, and (ii) square root branch points which give rise to the unitarity cuts in the two-body subsystem. The class (i) poles lead to branch points in the three-body amplitude, which correspond to the thresholds for the production of a bound or resonant pair. The class (ii) branch points give rise to thresholds in the three-body amplitude. Both types of singularities can be exhibited by writing the quasi-particle propagator as

\[ \tau_{\kappa\alpha} [E - \varepsilon_{\kappa\alpha}(q')] = \frac{S_{\kappa\alpha} [E - \varepsilon_{\kappa\alpha}(q')]}{E - \varepsilon_{\kappa\alpha}(q') - \epsilon_{r\alpha}}, \tag{2.21} \]

where

\[ \epsilon_{r\alpha} = \begin{cases} M_{b\alpha} & \text{for two-body bound states} \\ M_{r\alpha} - \frac{i}{2} \Gamma_{r\alpha} & \text{for two-body resonances} \end{cases}. \tag{2.22} \]

Here, \( M_{b\alpha} \) is the mass of the two-body bound state, \( i.e. \ M_{b\alpha} = m_\beta + m_\gamma - B \), with \( B \) the two-body binding energy, while \( M_{r\alpha} \) and \( \Gamma_{r\alpha} \) are the mass and full width of the resonance in the two-body subsystem. In Eq. (2.21) the function \( S_{\kappa\alpha}(\varepsilon) \) has square root branch points at \( \varepsilon_{\kappa\alpha} = m_\beta + m_\gamma \), while the energy denominator has the poles of the quasi-particle propagator.

For the \( YNN \) system, the deuteron quasi-particle propagator \( \tau_d[E - \varepsilon_d(q)] \) has a pole at the deuteron mass, while for the \( \Lambda N-\Sigma N \) interactions, \( \tau_{\kappa\alpha} \) has the \( \Lambda N \) and \( \Sigma N \) threshold. In addition, some \( YN \) potentials have a resonance pole in the \( ^3S_1 \) channel near the \( \Sigma N \) threshold. In Fig. [4], we illustrate the position of these branch cuts in the three-body energy plane when the contour of rotation is \( \theta \).

To determine how far we can analytically continue Eq. (2.17) into the complex plane, we must examine how the singularities of \( \tau_\kappa \) effect the rotation of the contour of integration. Taking

\[ \varepsilon_{\kappa\alpha}(q') = m_\alpha + \frac{q'^2}{2\mu_\alpha}, \]

where \( m_\alpha \) is the mass of the spectator particle \( \alpha \) and \( \mu_\alpha \) is the reduced mass of the spectator \( \alpha \) with the pair \( \beta\gamma \), we see that the branch point from \( S_{\kappa\alpha} \) is at

\(^2\)We have defined our energy \( E \) to include the mass of the two nucleons and the \( \Lambda \).
\[ q' = \pm \sqrt{2\mu_{\alpha} \left( E - \sum_i m_i \right)} . \]  

(2.23)

For a three-body resonance with energy \( E \) (i.e., \( E = E_r - iE_i, \ E_i > 0 \)), these branch points are in the fourth quadrant of the \( q' \)-plane and at an angle of \( \varphi_u \), where

\[ \tan 2\varphi_u = \frac{E_i}{E_r - \sum_i m_i} . \]  

(2.24)

Thus, in as far as these branch points are concerned, we need to take \( \theta > \varphi_u \) to avoid the singularities. On the other hand the poles of \( \tau_{\kappa_{\alpha}} \) are at

\[ q' = \pm \sqrt{2\mu_{\alpha} \left( E - m_{\alpha} - \varepsilon_{r_{\alpha}} \right)} . \]  

(2.25)

For the case of the deuteron bound state (i.e. \( \varepsilon_{r_{\alpha}} = m_\beta + m_\gamma - B \)) the quasi-particle propagator has poles in the fourth quadrant of the \( q' \)-plane at an angle \( \varphi_d \), where

\[ \tan 2\varphi_d = \frac{E_i}{E_r + B - \sum_i m_i} . \]  

(2.26)

Since \( \varphi_d < \varphi_u \), we need not worry about this pole putting any constraint on the contour rotation. That leaves us with the resonance poles in the quasi-particle propagator for the \( \Lambda N - \Sigma N \) interaction. In this case the angle of the resonance pole in the \( q' \)-plane is \( \varphi_r \), where

\[ \tan 2\varphi_{r_{\alpha}} = -\frac{E_i - \frac{1}{2}\Gamma_{r_{\alpha}}}{E_r - M_{r_{\alpha}} - m_{\alpha}} . \]  

(2.27)

For \( E_r < (M_{r_{\alpha}} + m_{\alpha}) \) and \( E_i < \frac{1}{2}\Gamma_{r_{\alpha}} \), the angle \( 2\varphi_r \) is in the second quadrant, and therefore, \( \frac{\pi}{4} < \varphi_r < \frac{\pi}{2} \). As we proceed along the real axis to the point \( E = (M_{r_{\alpha}} + m_{\alpha}) \), \( \varphi_r \) attains a value of \( \frac{\pi}{4} \), while proceeding parallel to the imaginary axis to the point \( E_i = \frac{1}{2}\Gamma_{r_{\alpha}} \), \( \varphi_r \) attains a value of \( \frac{\pi}{2} \). If we carry this analysis through, we find that as we analytically continue our equation in the energy variable from the real axis through region \( I \) to region \( III \) and then to region \( IV \) (see Fig. \[ \text{Fig. 2} \]), one of the resonance poles in the \( q' \)-plane moves into the region \( -\frac{\pi}{4} < \varphi_r < 0 \) approaching from \( \varphi_r = -\frac{\pi}{4} \). At this stage the two-body unitarity branch point is moving towards \( \varphi_u = \frac{\pi}{4} \). These two singularities could force the contour to deviate from the path along the ray, and this in turn will introduce logarithmic branch points from the
Born amplitude $Z_{k_\alpha;k_\beta}^{IJ}$. Thus, the energy domain on the second Riemann sheet, to which we can analytically continue Eq. (2.17) without introducing elaborate contours of integration, is shown as the shaded area in Fig. 3 [21]. In addition to the above energy domain, we can analytically continue Eq. (2.17) onto the third Riemann sheet through the branch cut generated by the resonance pole in $\tau_{\kappa_\alpha}$; i.e., we start on the real axis in region $II$, then proceed through the branch cut to region $IV$ onto the third Riemann sheet, and then to region $III$ on the third Riemann sheet (see Fig. 2). In this case as we proceed from region $II$ to region $IV$, the resonance pole in the $q'$-plane crosses the real axis into the fourth quadrant, and we can analytically continue the equation into region $IV$ and then $III$ of the third Riemann sheet. However, if we attempt to go to region $I$ of the third energy sheet, we find that the contour of integration is forced onto the negative imaginary $q'$-axis by the two-body resonance pole, and here we encounter the singularities of the Born amplitude. Thus, the only part of the third Riemann sheet of the complex energy plane that we can access is the shaded region in Fig. 4. In the next section we will use the above results to explore the region near the $\Sigma NN$ threshold for possible resonances that might explain the structure we see in the cross section for $\Lambda d$ scattering.

III. NUMERICAL RESULTS

To examine the possible existence of $\Sigma$-hypermultiplet states below the $\Sigma$-production threshold in the $A = 3$ system, we must first consider two-body interactions that could generate such resonances. In particular, we need to know what features of the two-body interaction would produce a resonance in the $YN$ system. This is particularly important as the hyperon-nucleon ($YN$) interactions we use are of separable potential form, and with the limited data available such separable potentials are not uniquely determined. Ideally, we would like to carry out the computational work for the more realistic $YN$ interaction such as the One Boson Exchange (OBE) potentials in which the extensive $NN$ and limited $YN$ data are considered within the unified framework of $SU(3)$. However, this is not justifiable
at this stage, considering the lack of information about the correlation between the results of the two-body $YN$ and three-body $YNN$ systems. Therefore, as a first calculation we utilize several separable potentials previously employed in light hypernuclei investigations. We then vary the strength of the coupling in the $^3S_1$ partial wave between the $\Lambda N$ and $\Sigma N$ channel to explore the variation in the two-body and three-body results.

A. The Two-Body Input

For the present calculations we restrict our $YN$ two-body interactions to $S$-wave. For the $NN$ interaction we use the same potentials previously used in our study of the role of $\Lambda N - \Sigma N$ coupling in the hypertriton \[1\]. In particular, we use a Yamaguchi potential for the $^1S_0$, and the Phillips [22] potential with $P_D = 4\%$ for the $^3S_1 - ^3D_1$ partial wave. The parameters of these potentials, in the present notation, are given in Ref. [1].

For the $YN$ interaction in the $^1S_0$ partial wave we use the potential of Stepien-Rudzka and Wycech (SRW) [23]. Here again the parameters of this potential, in the present notation, were given previously in Ref. [1]. Since the coupling between the $\Lambda N$ and $\Sigma N$ channel has a one pion exchange contribution, we expect the $^3S_1$ channel to be stronger in its long range behaviour than the corresponding $^1S_0$. We therefore have chosen to vary the interaction in this partial wave only. The potentials we have used are the coupled-channel SRW potential, and the potentials constructed by Toker, Gal, and Eisenberg [24] (TGE). The latter potentials where constructed to investigate the question of the possible existence of resonances in $K^-d \to \pi N\Lambda$ near the $\Sigma$ threshold. In particular, potentials B and C, to which we will refer as TGE-B and TGE-C respectively, support a $\Sigma N$ bound state in the absence of coupling between the $\Lambda N$ and $\Sigma N$ channels (see Table [1]), while potential A, referred to here as TGE-A, has a virtual state in the absence of coupling. In Table [1] we present the parameters of these potentials, while in Table [1] we give the positions of the poles in the complex energy plane with and without the coupling between the $\Lambda N$ and $\Sigma N$ channels. Included in the tables are also the parameters of the $^1S_0$ potential of SRW, and the
position of the poles for this potential. In Table I we have used the notation of Pearce and Gibson \cite{16} for specifying the sheet on which the pole resides. Thus, \([tt]\) corresponds to the top sheet of both the \(\Lambda N\) and \(\Sigma N\) branch cuts, while \([bt]\) corresponds to the bottom sheet of the \(\Lambda N\) branch cut and the top sheet of the \(\Sigma N\) branch cut. In Fig. 3 we illustrate the sheet labeling system for the \(YN\) problem, with two square root branch cuts corresponding to the \(\Lambda N\) and \(\Sigma N\) thresholds. From Table II, we observe that potentials TGE-B and TGE-C have poles on the \([bt]\) sheet and in the absence of coupling between the \(\Lambda N\) and \(\Sigma N\) channels these poles become \(\Sigma N\) bound states. In fact, as the coupling between the two channels changes these poles move continuously, tracing a path on the \([bt]\) sheet. On the other hand, for the potentials TGE-A and SRW the pole near the \(\Sigma N\) threshold resides on the \([tb]\) sheet. In this case, turning off the coupling brings the pole to the real energy axis and on the second sheet of the \(\Sigma N\) branch cut. This corresponds to a virtual state of the \(\Sigma N\) system. In particular, we should note that for the SRW potential we have a zero energy bound state in the absence of coupling.

Because we are considering two classes of potentials, those with a bound \(\Sigma N\) and those with a virtual, or unbound, \(\Sigma N\) in the absence of coupling between the two channels, one might like to compare at the same time the effective ranges parameters for these potentials, and possibly compare them to the more “realistic” OBE potentials. For that we would like to calculate the effective range parameters, and particularly the effective range parameters in the \(\Sigma N\) channel. These effective range parameters, which will be complex for the \(\Sigma N\) system, are defined in terms of the two-body diagonal partial-wave \(T\)-matrix in channel \(\alpha\) as

\[
-\frac{1}{\delta_\alpha} + \frac{1}{2}k^2T_\alpha = -\frac{1 - i\pi\mu_\alpha k_\alpha T_{\alpha\alpha}}{\pi\mu_\alpha T_{\alpha\alpha}},
\]

where the \(T\)-matrix is given in terms of the phase shifts by the relation

\[
T_{\alpha\alpha} = \frac{1}{\pi\mu_\alpha k_\alpha} e^{i\delta_\alpha} \sin(\delta_\alpha).
\]

Here, \(k_\alpha\) is the on-shell momentum in a given channel, while \(\mu_\alpha\) and \(\delta_\alpha\) are the reduced mass and phase shift in channel \(\alpha\) respectively. In Table III we present the effective range
parameters for the four $^3S_1$ potentials under consideration, and the $^1S_0$ SRW potential. From this table we observe that potentials TGE-B and TGE-C have a $\Sigma N$ scattering length with a positive real part, while for potential TGE-A, which has a virtual state, the real part of the $\Sigma N$ scattering length is negative. For potential SRW this simple one-channel interpretation of the sign of the scattering length does not work. This suggests that we need to examine the position of the poles of the scattering amplitude, which in general are in the complex energy plane, before we can make any statement about whether the $\Sigma N$ interaction supports a bound state.

Since we will examine the cross section for $\Lambda d$ scattering as a means of determining the presence or absence of resonances, we should study at the same time the cross section for $\Lambda N$ scattering in the $^3S_1$ channel, to investigate whether there are correlations between the results for the two- and three-body systems. In particular, we would like to compare the case when the two-body $\Sigma N$ system supports a bound state in the absence of $\Lambda N - \Sigma N$ coupling, verses the case when there is a virtual state for the uncoupled $\Sigma N$ system. Finally, we would like to investigate whether the shape of the cross section provides any indication as to where the resonance pole resides, and to investigate how this shape carries through to the three-body system. In Fig. 6 we give the cross section for the potentials TGE-B and SRW as examples of a potential supporting a “bound state” and a “virtual state”, respectively. We observe that for TGE-B we have a classic resonance shape from which we might be able to estimate the width of the resonance to be $\approx 5$ MeV. However, for the SRW potential we have a sharp spike which could be interpreted as a threshold effect.

To investigate how the shape of the cross section changes as the resonance pole moves below the $\Sigma N$ threshold and approaches the real energy axis, we have considered the potential TGE-B and varied the strength of the coupling between the $\Lambda N$ and $\Sigma N$ channels.

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3Here, we should remind the reader that a bound state corresponds to a pole on the first sheet, while a virtual state correspond to a pole on the second sheet of the energy plane.
We know the position of this resonance pole, when the coupling is included, to be on the \([bt]\) sheet at an energy of \((2131.7 - 5.4i)\), which is just above the \(\Sigma N\) threshold. This pole moves to \((2126.7 - 0i)\) when the coupling is turned off. This corresponds to a \(\Sigma N\) bound state with a binding energy of 4.3 MeV. We introduce a new parameter \(R\) in the coupling

\[ R \times C_{\Lambda \Sigma} \]

and consider values of \(R = 1.25, 1.0, 0.75, \text{ and } 0.5\), so that we can move the position of the resonance pole from a point above the \(\Sigma N\) threshold to a point below the threshold and closer to the real axis. In Fig. 7, we present the cross section for \(\Lambda N\) scattering for the above values of \(R\). We find, as expected, that as we move the pole closer to the real axis \((R \to 0)\) the width of the resonance is reduced. More important is the fact that, as we move below the threshold and reduce the width, the shape of the resonance in the cross section becomes more symmetric. This suggests that the \(\Sigma N\) branch cut has a shadowing effect on the cross section. A similar effect will be observed for the three-body system.

**B. The Three-Body System**

We now turn to the \(YNN\) system with the aim of examining the possible formation of \(\frac{3}{2}^+\) states near the \(\Sigma NN\) threshold. If such states exist for the two-body potentials under consideration, we expect to find them as poles of the scattering amplitude in the complex energy plane, or as solutions of the Schrödinger equation for complex energies. However, before we proceed to explore the complex energy plane we should examine the quantum numbers such a state would have. Considering the results of Dover and Gal [8], we expect such states to have the lowest isospin possible for the \(YNN\) system, \(T = 0\). This suggests that we could observe these states in \(\Lambda d\) scattering near the \(\Sigma\) production threshold. Furthermore, because the resonance in the \(YN\) system occurs in the \(S\)-wave, we might expect the \(YNN\) resonance to be in the \(J^\pi = \frac{1}{2}^+\) channel. Thus, as a first step in determining the possible existence of \(\Sigma\)-hypernuclear states, we examine the total \(S\)-wave
cross section in the $J^\pi = \frac{1}{2}^+$ partial wave. We should remind the reader at this stage that a resonance will appear in just one partial wave, which will determine the quantum numbers of the resonant state.

From unitarity we can write the total cross section for $\Lambda d$ scattering as

$$\sigma_T = \sum_{J^\pi} \sigma_T^{J^\pi},$$

(3.3)

where the total cross section for the partial wave with total angular momentum and parity $J^\pi$ is given in terms of the imaginary part of the partial wave $T$-matrix by

$$\sigma_T^{J^\pi} = -\frac{4\pi^2 \mu_{\Lambda d}(2J + 1)}{k_0(2s_\Lambda + 1)(2s_d + 1)} \sum_{LS} \text{Im}[T_{LS,LS}^{J^\pi}].$$

(3.4)

Here, $S$ and $L$ are the channel spin and orbital angular momentum of the $\Lambda$ respectively, while $s_\Lambda$ and $s_d$ are the spin of the $\Lambda$ and deuteron. The on-shell momentum is taken to be $k_0$, and $\mu_{\Lambda d}$ is the reduced mass for the $\Lambda d$ system. For $S$-wave we take $L = 0$ and therefore $S = J$. In this case the total elastic cross section can be written in terms of the $S$-wave amplitude as

$$\sigma_T^{el} = \frac{4\pi^3 \mu_{\Lambda d}^2}{(2s_\Lambda + 1)(2s_d + 1)}(2J + 1) |T_{0,0,0}^{J^\pi}|^2,$$

(3.5)

while the inelastic total cross section is given by the difference between the total cross section and the total elastic cross section; i.e.,

$$\sigma_T^{in} = \sigma_T - \sigma_T^{el}.$$  

(3.6)

In Figs. 8 - 11 we give the total cross section for the $^3S_1$ potentials SRW, TGE-A, TGE-B, and TGE-C. The solid curve corresponds to the total elastic cross section, while the dotted curve corresponds to the total inelastic cross section. In general, any structure is more pronounced in the inelastic cross section. Comparing the results for the different potentials, we may conclude that the potential TGE-C has marginal structure if any, while the others have more pronounced structure just below the $\Sigma$ production threshold. The other general conclusion we may draw is that the inelastic total cross section for potentials SRW and TGE-B has a more symmetric shape than that for potential TGE-A. The important question now
is: does any of this structure in the cross section correspond to a resonant state? That is, is it an eigenstate of the Hamiltonian for the $YNN$ system.

In Table IV we present the position of the poles of the amplitude for the $YN$ and $YNN$ systems near the threshold for $\Sigma$ production. Here we observe that potential TGE-C, which produced a very wide resonance shape in the total inelastic cross section, does have a pole in the amplitude on the $[bt]$ sheet; the half width of this resonance is 11 MeV which is consistent with what we would deduce from the cross section. From an experimental point of view such a wide resonance would be hard to observe, and to that extent will give little information on the structure of the Hamiltonian that generated the eigenstate. We also note that for potential TGE-C the two-body $YN$ resonance also lies far from the physical region. Although the half width of the two-body resonance is only 5.3 MeV, the pole lies well above the threshold for $\Sigma$ production. The distance from the pole to the physical region is more than the half width, due to the presence of the branch cut which separates the resonance position and the physical region. This branch cut actually shields the resonance from view in the physical region.

We now turn to the potentials SRW (Fig. 8) and TGE-B (Fig. 10). In this case the two-body system supports either a “bound” state or a “zero energy” bound state when the coupling between the $\Lambda N$ and $\Sigma N$ is set to zero. Here we get a true resonance for potential TGE-B with a half width of 8.9 MeV, which is similar to the result for potential TGE-C, but because of the smaller half width we observe more pronounced structure in the total cross section. This width is comparable to that observed experimentally in the $A = 4$ system. On the other hand, potential SRW gives a resonance with a half width of 1.2 MeV and very pronounced structure. In this case the resonance is slightly above the threshold for $\Sigma$ production, but because of its proximity to the physical region it has considerable influence on the cross section. In this case, unlike the cross section for potential TGE-B, the elastic total cross section falls sharply at threshold, which suggests that for resonances on the $[bt]$ sheet and above the $\Sigma$ production threshold the branch cut produces some shadowing effect, similar to that seen in the $YN$ system for potential TGE-C.
Finally, for potential TGE-A we had some difficulty in determining the actual position of the resonance pole. This, we think, was due to the fact that the pole is very close to the \( \Sigma \) production threshold, and as a result our numerical procedures failed. (We performed the search with 64 point Gauss Legendre points to convert the coupled integral equations to a set of algebraic equations with no satisfactory convergence.) This numerical problem is primarily due to the fact that no contour rotation can move the \( \Sigma NN \) branch point away from the integration path. The resonance position that we list in Table [V] is presented just to show that (i) the resonance is very close to the threshold and (ii) as a result it produces a rapid variation in the cross section over a small energy region near the threshold. Here again, the structure in the cross section is not symmetric – a reflection of the fact that the resonance pole lies above the threshold for \( \Sigma \) production and the branch cut due to the threshold produces a shadowing effect.

From a comparison of the results for the four potentials we may draw the following conclusions regarding the correlation between the two- and three-body systems (see Table [V]). As the pole in the two-body system moves from the \([tb]\) sheet to the \([bt]\) sheet, the width of the resonance in the \(YNN\) system increases. In other words, the presence of the third baryon enhances the overall attraction in the system, effectively “binding” the \(\Sigma NN\) system. When the situation is such that the strength in the two-body interaction produces a pole close to the \(\Sigma\) production threshold, then the pole in the three-body problem lies a little farther from the corresponding \(\Sigma\) production threshold. To illustrate this point, we examine what happens when the interaction is generated from potential TGE-B by modifying the coupling between the \(\Lambda N\) and \(\Sigma N\) as described in the previous section. In Fig. [12] we present the total cross section for the \(J^\pi = \frac{1}{2}^+\) partial wave, as defined in Eq. (3.4), for \(R = 0.5, 0.75, 1.0,\) and 1.25. By comparing the results in Figs. [7] and [12] we illustrate that, as the pole in the \(YN\) system moves closer to the real axis, the pole in the \(YNN\) also moves closer to the physical region. However, the width of the resonance in the \(YNN\) system, as reflected in the total cross section, is in all cases larger than that in the \(YN\) system. The close relation between the result for the \(YN\) and \(YNN\) systems indicates that we need to investigate,
experimentally, the cross section for $\Lambda p$ scattering near the $\Sigma$ production threshold. This need, for more experimental information about the $\Lambda p$ cross section, is further bolstered by the fact that some of the OBE potential models (which are fitted to the existing $\Lambda p$ data) exhibit resonance type structure near or below the $\Sigma$ production threshold.

To demonstrate that the observed structure in the cross section is not a threshold effect, we present in Fig. 13 the partial-wave total cross section $\sigma^{J^\pi}_T$ for the first four partial waves. Two important conclusions can be drawn from the curves in this figure. First, the resonance structure below the $\Sigma$ production threshold is observed only in the $J^\pi = \frac{1}{2}^+$ partial wave. The other partial cross sections exhibit a broad bump above the $\Sigma$ production threshold, which is due to the opening of a new channel. In fact, this enhancement in the $J^\pi \neq \frac{1}{2}^+$ cross sections is a threshold effect that can be seen in all the non-resonant partial waves, while the structure below the $\Sigma$ production lies only in one partial wave allowing us to assign a definite quantum number to that structure. The second interesting feature is that the $S$-wave cross section is not the dominant contribution. The $P$-wave ($J^\pi = \frac{3}{2}^-$) total cross section is larger. This is not unexpected considering the size of the deuteron and the momentum of the incident $\Lambda$ at these energies. Unfortunately, this will make it difficult to observe such $\Sigma$-hypernuclear resonances in $\Lambda d$ scattering, because the total cross section will be dominated by non-resonant partial waves. We should recall that for $\Lambda p$ scattering it is the $S$-wave scattering that provides the main contribution to the overall total cross section. Thus, to observe a $\Sigma$-hypernuclear state in the $A = 3$ system, one must consider reactions that can select, or enhance, the $(T = 0, J^\pi = \frac{1}{2}^+)$ channel.

IV. CONCLUSIONS

Using separable $NN$ and $YN$ potentials in the Faddeev equations for the YNN system, we have demonstrated that the structure in the model $\Lambda d$ cross section near the $\Sigma NN$ threshold is associated with resonance poles in the scattering amplitude. The positions of these poles on the second Riemann sheet of the complex energy plane are, in fact, the eigenvalues of
the analytic continuation of the kernel of the Faddeev equations. Perhaps surprisingly, the cut starting at the $\Sigma NN$ threshold appears to shield from view in the physical region those resonance (pole) singularities lying above that threshold. Therefore, whether the resonance pole, corresponding to a $\Sigma NN$ eigenstate, lies above or below the $\Sigma NN$ threshold, the structure appearing in the $\Lambda d$ cross section lies below the $\Sigma NN$ threshold. If the pole resides below the $\Sigma NN$ threshold, then the structure in the cross section takes the shape of a classic resonance, symmetric about the real part of the resonance eigenvalue. In contrast, for a pole that lies in the shadow of the $\Sigma NN$ cut, the structure can be quite distorted, falling sharply at threshold and producing a more cusp-like shape. In such a case, the position of the peak in the structure does not necessarily correspond to the real part of the resonance eigenvalue, because the pole position is shielded from view in the physical region. Clearly, any shape intermediate between these two extremes is possible, so that one cannot necessarily determine whether a pole lies above or below the $\Sigma NN$ threshold from the shape of the resonance structure in the $\Lambda d$ cross section. Nonetheless, structure below the $\Sigma NN$ threshold in the $\Lambda d$ cross section, like that which has been observed in the $^4\text{He}(K^-,\pi^-)$ reaction, does imply the existence of a resonance (an eigenstate of the Hamiltonian in a particular partial wave) in the $\Sigma NN$ system.

That the cross section structure in the model $\Lambda d$ scattering calculation is a resonance and not just a threshold effect was established by demonstrating that the structure lies only in the $\frac{1}{2}^+$ partial wave, and not in the neighboring channels. Unfortunately, the $L = 0$ partial wave does not dominate the $\Lambda d$ cross section, as is the case in $\Lambda p$ scattering. Therefore, to observe a $\Sigma$-hypermolecular state in the $A=3$ system, one must consider reactions that can select, or enhance, the $\frac{1}{2}^+$ channel.

Finally, in the hypertriton the presence of three baryons enhances the attraction in the unbound $\Lambda N$ system, such that the $\Lambda NN$ system is bound with respect to separation of the $\Lambda$ from the deuteron. Similarly, the presence of the second nucleon enhances the overall attraction in the $\Sigma NN$ system, effectively “binding” that system to produce a resonance pole. Furthermore, we found that, as the pole in the $YN$ system moves closer to the real
axis, the pole in the $YNN$ system moves closer to the physical region. However, the width of the resonance in the $YNN$ system is always larger than that in the $YN$ subsystem.

ACKNOWLEDGMENTS

The work of I. R. Afnan was supported by the Australian Research Council. That of B. F. Gibson was performed under the auspices of the U. S. Department of Energy. The authors thank B. C. Pearce for assistance in determining the positions of the poles of the $YN$ amplitudes and S. B. Carr for help in calculating the $\Lambda N$ cross sections.
FORMAL SOLUTION OF THE AGS EQUATIONS

In this appendix we present a formal solution of the integral equation for the three-particle scattering amplitude in terms of the eigenstates of the kernel of the corresponding homogeneous integral equation. In this way we establish the relation between the poles of the scattering amplitude on the second Riemann sheet of the energy plane, and the eigenstates of the Hamiltonian for the three-body system.

Let us consider the AGS equation for the amplitude $X_{\alpha\beta}$ as given in Eq. (2.10),

$$X_{\alpha\beta} = G_0(E) \delta_{\alpha\beta} + \sum_{\gamma} G_0(E) \delta_{\alpha\gamma} T_\gamma(E) X_{\gamma\beta} .$$  \hspace{1cm} (1)

The corresponding homogeneous equation is given by

$$|\phi_\alpha\rangle = \sum_{\beta} G_0(E) \delta_{\alpha\beta} T_\beta |\phi_\beta\rangle .$$  \hspace{1cm} (2)

This equation is basically the Schrödinger equation for the three-body system, and the determination of the energies at which this equation is satisfied gives us the spectrum of our three-body Hamiltonian. Thus, any solutions of this equation for negative real energies correspond to bound states. To determine the position of the resonance poles which are not on the first Riemann sheet of the complex energy plane, we need to extend the energy domain of Eq. (2). This can be achieved in momentum space by deforming the contour of integration such that $q \rightarrow q e^{-i\theta}$, where $\theta$ is the angle of rotation of the integration variables, in this case the momentum. In this way we have extended the energy domain over which Eq. (2) is defined to that part of the second Riemann sheet where resonances are normally located. The resulting equation is denoted by

$$|\phi_\alpha^\theta\rangle = \sum_{\beta} G_0^\theta(E) \delta_{\alpha\beta} T_\beta^\theta(E) |\phi_\beta^\theta\rangle .$$  \hspace{1cm} (3)

Here, the energy $E$ can be in that part of the second Riemann sheet where the arg$(E) > -2\theta$. In general, there are limitations on this deformation of the contour imposed by the singularities of the kernel. This limitation puts a constrain on the resonances that can
be studied using this approach. To solve Eq. (3), we need to consider the corresponding eigenvalue problem,

$$\lambda_n(E) |\phi_{n,\alpha}^\theta \rangle = \sum_\beta G_0^\theta(E) \delta_{\alpha\beta} T_\beta^\theta(E) |\phi_{n,\beta}^\theta \rangle ,$$  

(4)

where $\lambda_n$ is the eigenvalue of the kernel of the three-body integral equation. For those energies for which there is an eigenvalues, $\lambda_n(E)$, whose value is one, Eq. (3) is said to have a solution. This solution is an eigenstate of the full three-body Hamiltonian, even when the energy $E$, is complex provided it is on the second Riemann sheet.

To expand the three-particle scattering amplitude $X_{\alpha\beta}(E)$ in terms of the solutions of Eq. (4) (i.e., the eigenvectors of the kernel of the integral equation) we must determine the orthonormality condition for the eigenstates $|\phi_{n,\alpha}^\theta \rangle$. For this we need to introduce the eigenvalue equation for the case when the rotation of the contour of integration is taken to be $q \rightarrow q e^{i\theta}$, and the resultant equation is

$$\tilde{\lambda}_n(E) |\tilde{\phi}_{n,\alpha}^\theta \rangle = \sum_\beta G_0^{-\theta}(E) \delta_{\alpha\beta} T_{\beta}^{-\theta}(E) |\tilde{\phi}_{n,\beta}^\theta \rangle .$$  

(5)

This equation extends the energy domain of our eigenvalue problem to that part of the second Riemann sheet where the solutions of the adjoint kernel reside. Making use of the fact that the kernels of Eqs. (4) and (5) are related by

$$[T_{\alpha}^{-\theta}(E^*) G_0^{-\theta}(E^*)]^* = T_{\alpha}^{\theta}(E) G_0^{\theta}(E) ,$$  

(6)

we can show that the eigenstates of the homogeneous equation satisfy the orthonormality condition

$$\sum_\alpha \langle \tilde{\phi}_{m,\alpha}^\theta(E^*) | T_{\alpha}^{\theta}(E) | \phi_{n,\alpha}^\theta(E) \rangle = \delta_{nm} .$$  

(7)

Here we note that unlike bound state solutions, the normalization involves the eigenstates of the kernel and the adjoint kernel. Because the kernel is not Hermitian, which was the case for bound state, we state the orthonormality of the resonance wave function in terms of two eigenvalue equations.
We are now in a position to expand the scattering amplitude \( X_{\alpha\beta}(E) \) in terms of the eigenstates \( |\phi_{n,\alpha}(E)\rangle \). In particular, if we want the amplitude on that part of the second Riemann sheet where the resonance poles reside, we must write the expansion in terms of the eigenstates of the rotated kernel; i.e.,

\[
X_{\theta\alpha\beta}(E) = \sum_n |\phi_{n,\alpha}^{\theta}(E)\rangle C_n^{\beta}(E). 
\] (8)

The constants \( C_n^{\beta}(E) \) can be determined by substituting the expansion in Eq. (8) in the integral equation for the scattering amplitude on the rotated contour, Eq. (1) on the rotated contour. This gives us an expansion for the scattering amplitude in terms of the eigenstates and eigenvalues of the kernel of the integral equation of the form

\[
X_{\alpha\beta}(E) = \sum_n |\phi_{n,\alpha}^{\theta}(E)\rangle \frac{[\tilde{\lambda}_n(E^*)]^*}{1 - \lambda_n(E)} \langle \tilde{\phi}_{n,\beta}^{\theta}(E^*) | , 
\] (9)

In writing Eq. (9), we have established the fact that the energy at which one of the eigenvalues \( \lambda_n(E) \) is one, the amplitude \( X_{\alpha\beta}(E) \) has a pole. However, the energies at which the eigenvalues are one, correspond to solutions of the homogeneous Eq. (3), which correspond to eigenstates of the Hamiltonian when the energy domain on which this Hamiltonian is defined is extended onto the second Riemann sheet. In this way we have established the fact that; poles of the scattering amplitude on the second Riemann sheet of the energy plane, which correspond to resonances, are also the positions of the eigenstates of the Hamiltonian when the energy domain is extended to the second Riemann sheet.
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FIGURES

FIG. 1. The branch cuts and thresholds in the complex energy plane.

FIG. 2. Region of the energy plane that can be accessed via contour rotation. The point $A$ at $E = M_r - \frac{i \Gamma}{2} + m_\alpha$ corresponds to the branch point resulting from the resonance in the two-body subsystem.

FIG. 3. The shaded area is the domain of the second Riemann sheet of the energy plane to which we can analytically continue Eq. (2.17), while maintaining the contour deformation along a ray in the fourth quadrant of the $q'$-plane.

FIG. 4. The shaded area is the domain of the third Riemann sheet of the energy plane to which we can analytically continue Eq. (2.17), while maintaining the contour deformation along a ray in the fourth quadrant of the $q'$-plane. Access to this sheet is via the square root branch cut resulting from the resonance pole in $\tau_\kappa$.

FIG. 5. The labeling of the different Riemann sheets for the $\Lambda N$-$\Sigma N$ coupled-channel problem. This labeling scheme is identical to that used in Ref. [16].

FIG. 6. The total cross section for $\Lambda N$ scattering in the $^3S_1$ channel for the two potentials SRW and TGE-B.

FIG. 7. The total cross section for $\Lambda N$ scattering in the $^3S_1$ channel for the TGE-B potential with the coupling between the $\Lambda n$ and $\Sigma N$ channels $C_{\Lambda \Sigma}$ replaced by $R \times C_{\Lambda \Sigma}$.

FIG. 8. The total elastic (solid line) and inelastic (dotted line) $S$-wave $J^\pi = \frac{1}{2}^+$ cross section for $\Lambda d$ scattering as a function of the three-body energy for the potential SRW. The $\Sigma NN$ Threshold is at $E_{cm} = 77$ MeV.
FIG. 9. The total elastic (solid line) and inelastic (dotted line) $S$-wave $J^\pi = \frac{1}{2}^+$ cross section for $\Lambda d$ scattering as a function of the three-body energy for the potential TGE-A. The $\Sigma NN$ threshold is at $E_{cm} = 77$ MeV.

FIG. 10. The total elastic (solid line) and inelastic (dotted line) $S$-wave $J^\pi = \frac{1}{2}^+$ cross section for $\Lambda d$ scattering as a function of the three-body energy for the potential TGE-B. The $\Sigma NN$ threshold is at $E_{cm} = 77$ MeV.

FIG. 11. The total elastic (solid line) and inelastic (dotted line) $S$-wave $J^\pi = \frac{1}{2}^+$ cross section for $\Lambda d$ scattering as a function of the three-body energy for the potential TGE-C. The $\Sigma NN$ threshold is at $E_{cm} = 77$ MeV.

FIG. 12. The total cross section in the $J^\pi = \frac{1}{2}^+$ partial wave for potential TGE-B with the coupling strength between the $\Lambda N$ and $\Sigma N$ scaled by the factor $R$; i.e., $C_{\Lambda \Sigma} \rightarrow R \times C_{\Lambda \Sigma}$.

FIG. 13. The total cross section for partial waves $J^\pi = \frac{1}{2}^+, \frac{1}{2}^-, \frac{3}{2}^+, \frac{3}{2}^-$. 

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TABLE I. The parameters of the $^3S_1 \Lambda N-\Sigma N$ coupled-channel potentials, and the $^1S_0$ SRW potential.

| Potential | $C_{\Lambda\Lambda}$ | $\beta_\Lambda$ | $C_{\Sigma\Sigma}$ | $\beta_\Sigma$ | $C_{\Lambda\Sigma}$ |
|-----------|---------------------|-----------------|-------------------|----------------|-------------------|
| SRW $^3S_1$ | -0.42824            | 1.6             | -1.88913          | 2.0            | 0.84289           |
| TGE-A      | -0.11729            | 1.1069          | -4.33140          | 2.702          | 0.71399           |
| TGE-B      | 0.03569             | 0.9518          | -0.80233          | 1.2789         | 0.43692           |
| TGE-C      | 0.05726             | 0.8752          | -0.07434          | 0.5335         | 0.23226           |
| SRW $^1S_0$ | -0.17339            | 1.18            | 0.45856           | 1.44           | -0.38471          |

TABLE II. The position of the poles of the $\Lambda N - \Sigma N$ amplitude that lie close to the $\Sigma N$ threshold for the four different $^3S_1 Y N$ interaction being considered.$^a$

| Potential | Sheet | Pole with $C_{\Lambda\Sigma} \neq 0$ | Pole with $C_{\Lambda\Sigma} = 0$ |
|-----------|-------|-------------------------------------|---------------------------------|
| SRW       | [tb]  | 2132.5 - 0.4i                       | 2131.0 + 0.0i                   |
| TGE-A     | [tb]  | 2130.9 - 1.9i                       | 2130.5 + 0.0i                   |
| TGE-B     | [bt]  | 2131.7 - 5.4i                       | 2126.7 + 0.0i                   |
| TGE-C     | [bt]  | 2138.0 - 5.3i                       | 2129.0 + 0.0i                   |

$^a$Here, and throughout this paper we have taken our masses to be $m_N = 939$ MeV, $m_\Lambda = 1115$ MeV, and $m_\Sigma = 1192$ MeV. As a result the threshold for $\Sigma$ production is 2131 MeV.
TABLE III. The effective range parameters for the $\Lambda N - \Sigma N$ coupled-channels in the $^3S_1$ partial wave. We have included both the $\Lambda N$ and $\Sigma N$ effective range parameters.

| Potential | $a_{\Lambda N}$ | $r_{\Lambda N}$ | $a_{\Sigma N}$ | $r_{\Sigma N}$ |
|-----------|-----------------|-----------------|----------------|----------------|
| SRW $^3S_1$ | -1.96           | 2.44            | 0.14$ - 4.72i$ | 1.67$ - 0.20i$ |
| TGE-A     | -2.46           | 3.94            | -2.60$ - 2.97i$ | 1.30$ - 0.04i$ |
| TGE-B     | -1.70           | 4.55            | 2.97$ - 1.83i$ | 1.97$ - 0.38i$ |
| TGE-C     | -1.69           | 4.88            | 3.81$ - 1.56i$ | 2.80$ - 1.88i$ |
| SRW $^1S_0$ | -1.98           | 4.03            | 0.59$ - 0.09i$ | -1.30$ - 0.39i$ |

TABLE IV. The position of the poles of the $YNN$ amplitude near the $\Sigma NN$ threshold. Included are also the position of the resonance pole in the $YN$ amplitude for comparison.$^a$

| Potential | Two-Body | Three-Body |
|-----------|----------|------------|
|           | Sheet    | Position   | Sheet | Position |
| SRW       | $[tb]$   | 78.5$ - 0.4i$ | $[bt]$ | 79.5$ - 1.2i$ |
| TGE-A     | $[tb]$   | 76.9$ - 1.9i$ | $[bt]$ | 78$ - 0.5i$ $^b$ |
| TGE-B     | $[bt]$   | 77.7$ - 5.4i$ | $[bt]$ | 75.5$ - 8.9i$ |
| TGE-C     | $[bt]$   | 84.0$ - 5.3i$ | $[bt]$ | 84.0$ - 11.0i$ |

$^a$The energy of the two-body resonance is taken relative to the $\Lambda N$ threshold, 2054 MeV.

$^b$Because the pole position is close to the $\Sigma NN$ threshold, we found it difficult to determine the position of the pole with a high degree of accuracy.