SYMBOLIC POWERS OF SUMS OF IDEALS

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Abstract. Let $I$ and $J$ be nonzero ideals in two Noetherian algebras $A$ and $B$ over a field $k$. Let $I + J$ denote the ideal generated by $I$ and $J$ in $A \otimes_k B$. We prove the following expansion for the symbolic powers:

$$(I + J)^{(n)} = \sum_{i+j=n} I^{(i)} J^{(j)}.$$  

If $A$ and $B$ are polynomial rings and if $\text{char}(k) = 0$ or if $I$ and $J$ are monomial ideals, we give exact formulas for the depth and the Castelnuovo-Mumford regularity of $(I + J)^{(n)}$, which depend on the interplay between the symbolic powers of $I$ and $J$. The proof involves a result of independent interest which states that under the above assumption, the induced map $\text{Tor}_i^A(k, I^{(n)}) \rightarrow \text{Tor}_i^A(k, I^{(n-1)})$ is zero for all $i \geq 0$, $n \geq 0$. We also investigate other properties and invariants of $(I + J)^{(n)}$ such as the equality between ordinary and symbolic powers, the Waldschmidt constant and the Cohen-Macaulayness.

1. Introduction

Let $R$ be a commutative Noetherian ring and let $Q$ be an ideal in $R$. For an integer $n \geq 1$, the $n$-th symbolic power of $Q$ is defined by

$$Q^{(n)} := R \cap \left( \bigcap_{P \in \text{Min}(Q)} Q^n R_P \right).$$

In other words, $Q^{(n)}$ is the intersection of the primary components of $Q^n$ associated to the minimal primes of $Q$.

When $R$ is a polynomial ring over an algebraically closed field $k$ of characteristic zero and $Q$ is a radical ideal, Nagata and Zariski showed that $Q^{(n)}$ consists of polynomials in $R$ whose partial derivatives of orders up to $n - 1$ vanish on the zero set of $Q$ (see e.g. [11]). Therefore, symbolic powers carry more geometric information than ordinary powers of an ideal. In general, it is difficult to study properties of symbolic powers.

Let $A$ and $B$ be commutative Noetherian algebras over an arbitrary field $k$. Let $I \subseteq A$ and $J \subseteq B$ be nonzero proper ideals. For simplicity we also use $I$ and $J$ to denote the extensions of $I$ and $J$ in the algebra $R := A \otimes_k B$. The main aim of this paper is to study the depth and the Castelnuovo-Mumford regularity (or simply regularity) of the symbolic powers of the sum $I + J$ in $R$. Such sums of ideals appear naturally in various contexts:

- Fiber product of affine schemes: Let $X$ and let $Y$ be the affine schemes $\text{Spec}(A/I)$ and $\text{Spec}(B/J)$, then the fiber product $X \times_k Y$ is the affine scheme $\text{Spec}(R/I + J)$.

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• **Join of simplicial complexes:** Let $\Delta$ and $\Gamma$ be simplicial complexes over disjoint vertex sets with Stanley-Reisner ideals $I_\Delta$ and $I_\Gamma$, then $I_\Delta + I_\Gamma$ is the Stanley-Reisner ideal of the join complex $\Delta \ast \Gamma$;

• **Edge ideal of a graph:** Let $I(G)$ denote the edge ideal a simple graph $G$. If $G_1, \ldots, G_n$ are the connected components of $G$, then

$$I(G) = I(G_1) + \cdots + I(G_n).$$

Though symbolic powers have been studied extensively (see e.g. [2, 7, 8, 9, 20, 21, 22, 25, 26, 28, 33, 34, 35, 37]), symbolic powers of such sums of ideals have not been considered in this general setting. We shall see from this paper and our sequential work [18] that studying sums of ideals may indeed provide new insights to many problems on symbolic powers.

Several results on the depth and the regularity of the ordinary powers $(I + J)^n$ have been recently established in [19, 29]. These results have had a number of interesting consequences. It is quite natural to ask whether there are similar results on the symbolic powers $(I + J)^{(n)}$.

The first step is to characterize $(I + J)^{(n)}$ in terms of $I$ and $J$. In general, if $I$ and $J$ are prime ideals, $I + J$ needs not be a primary ideal. This indicates that such a characterization would be complicated. Surprisingly, we can show that there is a binomial expansion for the symbolic power $(I + J)^{(n)}$:

**Theorem 3.4.** $(I + J)^{(n)} = \sum_{i+j=n} I^{(i)} J^{(j)}$.

This formula was not known even in the simple case when $B = k[x]$ is a polynomial ring and $J = (x)$. It was known before only for squarefree monomial ideals by Bocc et al [3]. The proof of Theorem 3.4 is based on a thorough study of associated primes of tensor products of modules over $A$ and $B$, which is of independent interest.

Theorem 3.4 allows us to study several aspects of $(I + J)^{(n)}$. First, we show that when $A$ and $B$ are local rings or domains, $(I + J)^{(n)} = (I + J)^n$ if and only if $I^{(t)} = I^t$ and $J^{(t)} = J^t$ for all $t \leq n$, and that when $I$ and $J$ are homogeneous ideals of polynomial rings, then

$$\hat{\alpha}(I + J) = \min\{\hat{\alpha}(I), \hat{\alpha}(J)\},$$

where $\hat{\alpha}(I)$ denotes the Waldschmidt constant of an ideal, which appears in several areas of mathematics [6, 14, 20, 38]. This formula for $\hat{\alpha}(I + J)$ was known before only for squarefree monomial ideals [3].

Our main results on the depth and the regularity of $(I + J)^{(n)}$ can be summarized as follows.

**Theorem 4.6.** Let $A$ and $B$ be polynomial rings over a field $k$. Let $I \subseteq A$ and $J \subseteq B$ be nonzero proper homogeneous ideals. Then

(i) $\text{depth } R/(I + J)^{(n)} \geq \min_{i \in [1,n-1]} \{\text{depth } A/I^{(n-i)} + \text{depth } B/J^{(i)} + 1, \text{depth } A/I^{(n-j+1)} + \text{depth } B/J^{(j)}\}$,

(ii) $\text{reg } R/(I + J)^{(n)} \leq \max_{i \in [1,n-1]} \{\text{reg } A/I^{(n-i)} + \text{reg } B/J^{(i)} + 1, \text{reg } A/I^{(n-j+1)} + \text{reg } B/J^{(j)}\}.$
Theorems 5.6 and 5.11. The inequalities of Theorem 4.0 are equalities if \( \text{char}(k) = 0 \) or if \( I \) and \( J \) are monomial ideals.

We expect that equalities hold regardless of the characteristic of the field \( k \).

The above results are intricate in the sense that the right-hand sides of the above formulae involve the minimum or maximum value of two sets of different terms, which can be attained separately. It is a distinctive feature of polynomial rings because we can show that they do not hold if one of the rings \( A \) and \( B \) is not a polynomial ring.

Using the same approach we further obtain exact formulas for the depth and the regularity of the quotients \((I + J)^{(n)}/(I + J)^{(n+1)}\), that are independent of the characteristic of the field \( k \).

**Theorem 4.7.** Let \( A \) and \( B \) be polynomial rings over a field \( k \). Let \( I \subseteq A \) and \( J \subseteq B \) be nonzero proper homogeneous ideals. Then

\[
\begin{align*}
(\text{i}) \quad \text{depth}(I + J)^{(n)}/(I + J)^{(n+1)} &= \min_{i+j=n} \{ \text{depth}(I^{(i)}/I^{(i+1)} + \text{depth}(J^{(j)}/J^{(j+1)}) \}, \\
(\text{ii}) \quad \text{reg}(I + J)^{(n)}/(I + J)^{(n+1)} &= \max_{i+j=n} \{ \text{reg}(I^{(i)}/I^{(i+1)} + \text{reg}(J^{(j)}/J^{(j+1)}) \}.
\end{align*}
\]

As a consequence of Theorem 4.7 we show that \( R/(I + J)^{(i)} \) is Cohen-Macaulay for all \( i \leq n \) if and only if \( A/I^{(i)} \) and \( B/J^{(i)} \) are Cohen-Macaulay for all \( i \leq n \).

The above results hold in a more general framework. Given two filtrations of ideals \( \{I_n\}_{n \geq 0} \) in \( A \) and \( \{J_n\}_{n \geq 0} \) in \( B \), we give bounds for the depth and the regularity of the binomial sum

\[ Q_n := \sum_{i+j=n} I_i J_j. \]

For the filtrations of ordinary or symbolic powers of the ideals \( I \) and \( J \), we have \( Q_n = (I + J)^n \) or \( Q_n = (I + J)^{(n)} \), respectively. This approach can be also applied to filtrations of integral closures or of saturations of powers of \( I \) and \( J \).

We say that a filtration of ideals \( \{I_n\}_{n \geq 0} \) is Tor-vanishing if the induced map \( \text{Tor}^A_i(k, I_n) \to \text{Tor}^B_i(k, I_{n-1}) \) is zero for all \( i \geq 0 \) and \( n \geq 0 \). We show that the bounds for the depth and the regularity of the binomial sum \( Q_n \) become equalities if the filtrations \( \{I_n\}_{n \geq 0} \) and \( \{J_n\}_{n \geq 0} \) are Tor-vanishing. Theorems 5.6 and 5.11 follow from this result because Tor-vanishing holds for filtrations of symbolic powers in these cases:

**Propositions 5.5 and 5.10.** Let \( I \) be a homogeneous ideal in a polynomial ring \( A \) over a field \( k \). If \( \text{char}(k) = 0 \) or if \( I \) is a monomial ideal, the filtration \( \{I^{(n)}\}_{n \geq 0} \) is Tor-vanishing.

The Tor-vanishing of the symbolic powers are of independent interest because they can be used to investigate homological relationships between \( I^{(n-1)} \) and \( I^{(n)} \) for a homogeneous ideal \( I \). They can be also considered as a higher order generalization of the inclusion \( I^{(n)} \subseteq m/(m^{(n-1)} \), where \( m \) is the ideal generated by the variables of \( R \). This inclusion was proved by Eisenbud and Mazur [12] under the same assumption of Propositions 5.5 and 5.10. Using an example of [12] and the polarization trick of McCullough and Peeva [27] we can find homogeneous ideals whose filtration of symbolic powers is not Tor-vanishing if \( \text{char}(k) > 0 \).

Our paper is structured as follows. In Section 2 we study the associated primes of tensor products of modules. In Section 3 we prove the binomial expansion of \((I + J)^{(n)}\). In Section
we present bounds for the depth and the regularity of $R/(I + J)^{(n)}$ and the exact formulas for the depth and the regularity of $(I + J)^{(n-1)}/(I + J)^{(n)}$ in terms of those of $I$ and $J$. In Section 5 we use the technique of Tor-vanishing to study the problem when the obtained bounds for the depth and the regularity of $R/(I + J)^{(n)}$ become exact formulas.

We assume that the reader is familiar with basic properties of associated primes, depth and regularity, which we use without references. For other unexplained notions and terminology, we refer the reader to [5, 10].

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2. ASSOCIATED PRIMES OF TENSOR PRODUCTS

Let $A$ and $B$ be two Noetherian algebras over a field $k$ such that $R := A \otimes_k B$ is Noetherian. For our investigation on the symbolic powers of sums of ideals, we need to know the associated primes of $R$-modules of the form $M \otimes_k N$, where $M$ and $N$ are nonzero finitely generated modules over $A$ and $B$.

Let $\text{Min}_A(M)$ and $\text{Ass}_A(M)$ denote the sets of minimal associated primes and associated primes of $M$ as an $A$-module, respectively. The aim of this section is to describe $\text{Min}_R(M \otimes_k N)$ and $\text{Ass}_R(M \otimes_k N)$ in terms of those of $M$ and $N$.

We begin with the following observations.

Lemma 2.1. Let $P$ be a prime ideal of $R$, $p = P \cap A$, and $q = P \cap B$. Then

(i) $P \in \text{Min}_R(M \otimes_k N)$ if and only if $p \in \text{Min}_A(M)$, $q \in \text{Min}_B(N)$ and $P \in \text{Min}_R(R/p + q)$;

(ii) $P \in \text{Ass}_R(M \otimes_k N)$ if and only if $p \in \text{Ass}_A(M)$, $q \in \text{Ass}_B(N)$ and $P \in \text{Ass}_R(R/p + q)$.

Proof. It is clear that $(M \otimes_k N)_P = M_p \otimes_{A_p} (A \otimes_k N)_P$. Since the map $A \to R$ is flat, the map $A_p \to R_P$ is also flat. Applying [17 Chap. IV, (6.1.2)], we have

$$\dim(M \otimes_k N)_P = \dim M_p + \dim k(p) \otimes_{A_p} (A \otimes_k N)_P,$$

where $k(p)$ denotes the residue field of $A_p$. Since $k(p) = (A/p)_p$, we have $((A/p) \otimes_k N)_P = k(p) \otimes_{A_p} (A \otimes_k N)_P$. Therefore,

$$\dim(M \otimes_k N)_P = \dim M_p + \dim((A/p) \otimes_k N)_P.$$

Since the map $B \to R$ is flat, we can also show that

$$\dim((A/p) \otimes_k N)_P = \dim N_q + \dim((A/p) \otimes_k (B/q))_P.$$

Note that $(A/p) \otimes_k (B/q) = R/p + q$. From the above equalities we get $\dim(M \otimes_k N)_P = 0$ if and only if

$$\dim M_p = \dim N_q = \dim(R/p + q)_P = 0.$$
For an arbitrary finite \( R \)-module \( E \), we know that \( P \in \text{Min}_R(E) \) if and only if \( \dim E_P = 0 \). Therefore, \( P \in \text{Min}_R(M \otimes_k N) \) if and only if \( p \in \text{Min}_A(M) \), \( q \in \text{Min}_B(N) \) and \( P \in \text{Min}_R(R/p + q) \).

Similarly, we can apply \[17\] Chap. IV, (6.3.1)] to show that \( \text{depth}(M \otimes_k N)_P = 0 \) if and only if

\[
\text{depth} M_p = \text{depth} N_q = \text{depth}(R/p + q)_P = 0.
\]

We also know that \( P \in \text{Ass}_R(E) \) if and only if \( \text{depth} E_P = 0 \). Therefore, \( P \in \text{Ass}_R(M \otimes_k N) \) if and only if \( p \in \text{Ass}_A(M) \), \( q \in \text{Ass}_B(N) \) and \( P \in \text{Ass}_R(R/p + q) \).

**Remark 2.2.** We need the assumption on the Noetherian property of \( A \otimes_k B \) for applying \[17\] Chap. IV, (6.1.2) and (6.3.1). In general, \( A \otimes_k B \) is not necessarily Noetherian, even when \( A \) and \( B \) are field extensions of \( k \). For more information on this topic see \[36\].

Notice that \( p + q \) is not necessarily a prime or even a primary ideal as illustrated by the following example.

**Example 2.3.** Let \( p := (x^2 + 1) \subset A = \mathbb{Q}[x] \) and \( q := (y^2 + 1) \subset B = \mathbb{Q}[y] \). Both \( p \) and \( q \) are prime ideals. However, \( p + q = (x^2 + 1, y^2 + 1) \) is not primary in \( R = \mathbb{Q}[x, y] \) because \( x^2 - y^2 = (x + y)(x - y) \in p + q \).

**Lemma 2.4.** Let \( p \) and \( q \) be prime ideals of \( A \) and \( B \), respectively. Let \( P \in \text{Ass}(R/p + q) \). Then

(i) \( P \cap A = p \) and \( P \cap B = q \),

(ii) \( P \in \text{Min}_R(R/p + q) \).

**Proof.** Note that \( R/p + q = (A/p) \otimes_k (B/q) \). Applying Lemma 2.1 (ii) to \( (A/p) \otimes_k (B/q) \) we obtain \( P \cap A \in \text{Ass}_A(A/p) = \{p\} \) and \( P \cap B \in \text{Ass}_B(B/p) = \{q\} \), which implies (i).

Let \( k(p) \) and \( k(q) \) denote the residue fields of \( A_p \) and \( B_q \). Because of (i) we can consider \( ((A/p) \otimes_k (B/q))_P \) as a localization of the algebra \( k(p) \otimes_k k(q) \) at a prime ideal \( P' \). Since \( P \) is an associated prime of \( (A/p) \otimes_k (B/q) \), \( P' \) is an associated prime of \( k(p) \otimes_k k(q) \). By \[30\], Theorem 3], \( (k(p) \otimes_k k(q))_P \) is a primary ring, i.e. any of its zero divisors is a nilpotent element. From this it follows that \( P' \) is a minimal associated prime of \( k(p) \otimes_k k(q) \). Hence, \( P \) must be a minimal associated prime of \( (A/p) \otimes_k (B/q) \), which proves (ii).

By Lemma 2.4 (ii), the ideal \( p + q \) is always unmixed though it may be not a primary ideal.

Now we can describe the associated and the minimal associated primes of \( M \otimes_k N \) in terms of \( M \) and \( N \) as follows. This description gives more precise information on \( \text{Ass}_R(M \otimes_k N) \) than \[31\] Corollary 3.7(1)]

**Theorem 2.5.** Let \( M \) and \( N \) be nonzero modules over \( A \) and \( B \), respectively. Then

(i) \( \text{Min}_R(M \otimes_k N) = \bigcup_{p \in \text{Min}_A(M), q \in \text{Min}_B(N)} \text{Min}_R(R/p + q) \).

(ii) \( \text{Ass}_R(M \otimes_k N) = \bigcup_{p \in \text{Ass}_A(M), q \in \text{Ass}_B(N)} \text{Min}_R(R/p + q) \).

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Proof. By Lemma 2.1, we have
\[ \text{Min}_R(M \otimes_k N) = \bigcup_{p \in \text{Min}_A(M), q \in \text{Min}_B(N)} \{ P \in \text{Min}_R(R/p + q) \mid P \cap A = p, P \cap B = q \}. \]

By Lemma 2.4, we have
\[ \text{Ass}_R(M \otimes_k N) = \bigcup_{p \in \text{Ass}_A(M), q \in \text{Ass}_B(N)} \{ P \in \text{Ass}_R(R/p + q) \mid P \cap A = p, P \cap B = q \}. \]

By Lemma 2.4, we have \( P \cap A = p, P \cap B = q \) for all \( p \in \text{Ass}_R(R/p + q) \) and
\[ \text{Ass}_R(R/p + q) = \text{Min}_R(R/p + q). \]

Hence, we can rewrite the above formulas as in the statement of the theorem. \( \square \)

The following immediate consequence of Theorem 2.5 is a generalization of a classical result of Seidenberg on unmixed polynomial ideals under base field extensions in [32].

**Corollary 2.6.** \( \text{Ass}_R(M \otimes_k N) = \text{Min}_R(M \otimes_k N) \) if and only if \( \text{Ass}_A(M) = \text{Min}_A(M) \) and \( \text{Ass}_B(N) = \text{Min}_B(N) \).

One may ask when is the sum \( p + q \) of two prime ideals \( p \subset A \) and \( q \subset B \) a prime ideal in \( R \)? This question has the following simple answer.

**Lemma 2.7.** Let \( k(p) \) and \( k(q) \) denote the fields of fractions of \( A/p \) and \( B/q \). Then \( p + q \) is a prime ideal if and only if \( k(p) \otimes_k k(q) \) is a domain.

**Proof.** Let \( p + q \) be a prime ideal. Then \( (A/p) \otimes_k (B/q) = R/p + q \) is a domain. Since \( k(p) \otimes_k k(q) \) is a localization of \( (A/p) \otimes_k (B/q) \), it must be a domain, too. The converse is true since we have an injection \( (A/p) \otimes_k (B/q) \rightarrow k(p) \otimes_k k(q) \). \( \square \)

By [39, Corollary 1, p. 198], the tensor product of two field extensions of \( k \) is a domain if \( k \) is algebraically closed. In this case, Theorem 2.5 can be reformulated as follows.

**Corollary 2.8.** Let \( k \) be an algebraically closed field. Then
\[ \text{(i)} \quad \text{Ass}_R(M \otimes_k N) = \{ p + q \mid p \in \text{Ass}_A(M) \text{ and } q \in \text{Ass}_B(N) \}, \]
\[ \text{(ii)} \quad \text{Min}_R(M \otimes_k N) = \{ p + q \mid p \in \text{Min}_A(M) \text{ and } q \in \text{Min}_B(N) \}. \]

### 3. Binomial expansion of symbolic powers

Let \( A \) and \( B \) be two commutative Noetherian algebras over a field \( k \) such that \( R = A \otimes_k B \) is also a Noetherian ring. Let \( I \) and \( J \) be nonzero proper ideals of \( A \) and \( B \), respectively. We will use the same symbols \( I, J \) for the extensions of \( I, J \) in \( R \). The aim of this section is to prove that the symbolic power \( (I + J)^{(n)} \) has a binomial expansion.

We shall need the following observations.

**Lemma 3.1.** \( I \cap J = IJ \).

**Proof.** Let \( V \) and \( W \) be two sets of elements of \( A \) and \( B \). We denote by \( V \otimes W \) the set of the elements \( f \otimes g, f \in V \text{ and } g \in W \). Choose bases \( V \) and \( W \) of the \( k \)-vector spaces \( I \) and \( J \) and extend them to bases \( V^* \) and \( W^* \) of \( A \) and \( B \), respectively. Then \( V \otimes W^* \) and \( V^* \otimes W \) are bases of the \( k \)-vector spaces \( I \otimes_k B \) and \( A \otimes_k J \), respectively. Since \( V \otimes W^* \) and \( V^* \otimes W \) are
subsets of $V^* \otimes W^*$, which is a basis of $A \otimes_k B$, the vector space $I \cap J = (I \otimes_k B) \cap (A \otimes_k J)$ is generated by the set $(V \otimes W^*) \cap (V^* \otimes W) = V \otimes W$. Since $IJ = I \otimes_k J$ is also generated by $V \otimes W$, we conclude that $I \cap J = IJ$. \hfill \square

Lemma 3.1 was known before for polynomial ideals [24, Lemma 1.1].

Lemma 3.2. Let $I'$ and $J'$ be subideals of $I$ and $J$, respectively. Then $$\left(\frac{I}{I'}\right) \otimes_k \left(\frac{J}{J'}\right) \cong IJ/(IJ' + I'J).$$

Proof. We have $$\left(\frac{I}{I'}\right) \otimes_k \left(\frac{J}{J'}\right) \cong \left(\frac{(I \otimes_k J) / (I \otimes_k J')}{(I' \otimes_k J')}ight) \cong \left(\frac{IJ/\left(IJ' + I'J\right)}{IJ'/IJ'}\right).$$

By Lemma 3.1,

$$I'J' = I' \cap J' = I'J \cap J' = I'J \cap IJ'.$$

From this it follows that $$\left(\frac{I}{I'}\right) \otimes_k \left(\frac{J}{J'}\right) \cong \left(\frac{IJ/\left(IJ' + I'J\right)}{IJ'/IJ'}\right) \cong IJ/(IJ' + I'J).$$ \hfill \square

In the following, we will consider binomial sums of filtrations of ideals, which is defined as follows.

For simplicity, we call a sequence of ideals $\{Q_n\}_{n \geq 0}$ in $R$ a filtration if it satisfies the following conditions:

(1) $Q_0 = R$,
(2) $Q_1$ is a nonzero proper ideal,
(3) $Q_n \supseteq Q_{n+1}$ for all $n \geq 0$.

Examples of such filtrations are the ordinary powers $\{Q^n\}_{n \geq 0}$, the symbolic powers $\{Q^{(n)}\}_{n \geq 0}$, and the integral closures of powers $\{\overline{Q^n}\}_{n \geq 0}$, where $Q$ is a nonzero proper ideal.

Let $\{I_i\}_{i \geq 0}$ and $\{J_j\}_{j \geq 0}$ be two filtrations of ideals in $A$ and $B$, respectively. For each $n \geq 0$, we define

$$Q_n := \sum_{i+j=n} I_iJ_j.$$

We call $Q_n$ the $n$-th binomial sum of the filtrations $\{I_i\}_{i \geq 0}$ and $\{J_j\}_{j \geq 0}$.

Our next result shows that quotients of successive binomial sums have a nice decomposition.

Proposition 3.3. For any integer $n \geq 0$, there is an isomorphism

$$Q_n/Q_{n+1} \cong \bigoplus_{i=0}^n \left(\frac{I_i/I_{i+1}}{J_{n-i}/J_{n-i+1}}\right).$$

Proof. First, we will show that

$$Q_n/Q_{n+1} \cong \bigoplus_{i=0}^n \left((I_iJ_{n-i} + Q_{n+1})/Q_{n+1}\right).$$
For that it suffices to show that for $0 \leq i \leq n$,

$$(I_i J_{n-i} + Q_{n+1}) \cap \left( \sum_{j \neq i} I_j J_{n-j} + Q_{n+1} \right) \subseteq Q_{n+1}$$

or, equivalently,

$$I_i J_{n-i} \cap \left( \sum_{j \neq i} I_j J_{n-j} + Q_{n+1} \right) \subseteq Q_{n+1}.$$

We have

$$\sum_{j \neq i} I_j J_{n-j} + Q_{n+1} = \sum_{j \neq i} I_j J_{n-j} + \sum_{0 \leq j \leq n+1} I_j J_{n-j+1} \subseteq J_{n-i+1} + I_{i+1},$$

because $J_{n-j} \subseteq J_{n-i+1}$ for $j < i$ and $J_j \subseteq I_{i+1}$ for $j \geq i + 1$. On the other hand, every element of $J_{n-i+1} + I_{i+1}$ in $I_i J_{n-i}$ is a sum of two elements, one in $J_{n-i+1} \cap (I_{i+1} + I_i J_{n-i})$ and the other in $I_{i+1} \cap (J_{n-i+1} + I_i J_{n-i})$. Therefore,

$$I_i J_{n-i} \cap \left( \sum_{0 \leq j \leq n} I_j J_{n-j} + Q_{n+1} \right) \subseteq J_{n-i+1} \cap (I_{i+1} + I_i J_{n-i}) + I_{i+1} \cap (J_{n-i+1} + I_i J_{n-i})
\subseteq J_{n-i+1} \cap I_i + I_{i+1} \cap J_{n-i} = I_{i+1} J_{n-i} + I_i J_{n-i+1} \subseteq Q_{n+1},$$

where the equality holds thanks to Lemma 3.1

The above inclusions also show that $I_i J_{n-i} \cap Q_{n+1} = I_{i+1} J_{n-i} + I_i J_{n-i+1}$. Hence,

$$(I_i J_{n-i} + Q_{n+1})/Q_{n+1} \simeq I_i J_{n-i}/(Q_{n+1} \cap I_i J_{n-i})
\simeq I_i J_{n-i}/(I_{i+1} J_{n-i} + I_i J_{n-i+1})
\simeq (I_i/I_{i+1}) \otimes_k (J_{n-i}/J_{n-i+1}),$$

where the last isomorphism follows from Lemma 3.2 Therefore,

$$Q_n/Q_{n+1} = \bigoplus_{i=0}^{n} (I_i J_{n-i} + Q_{n+1})/Q_{n+1} \simeq \bigoplus_{i=0}^{n} (I_i/I_{i+1}) \otimes_k (J_{n-i}/J_{n-i+1}).$$

We are now ready to prove the main result of this section.

**Theorem 3.4.** $(I + J)^{(n)} = \sum_{i+j=n} I^{(i)} J^{(j)}$.

**Proof.** Consider the symbolic filtrations $\{I^{(i)}\}_{i \geq 0}$ and $\{J^{(j)}\}_{j \geq 0}$. In this case, we have the $n$-th binomial sum

$$Q_n = \sum_{i+j=n} I^{(i)} J^{(j)}.$$

We shall first prove the inclusion $(I + J)^{(n)} \subseteq Q_n$. For that we need to investigate the associated primes of $R/Q_n$. It follows from the short exact sequence

$$0 \to Q_{t-1}/Q_t \to R/Q_t \to R/Q_{t-1} \to 0,$$

where

$$Q_n = \sum_{i+j=n} I^{(i)} J^{(j)}.$$
that $\text{Ass}_R(R/Q_t) \subseteq \text{Ass}_R(Q_{t-1}/Q_t) \cup \text{Ass}_R(R/Q_{t-1})$. Hence,

$$\text{Ass}_R(R/Q_n) \subseteq \bigcup_{t=1}^n \text{Ass}_R(Q_{t-1}/Q_t).$$

By Proposition 3.3, we have

$$\text{Ass}_R(Q_{t-1}/Q_t) = \bigcup_{i+j=t-1} \text{Ass}_R\left(I^{(i)}/I^{(i+1)} \otimes_k J^{(j)}/J^{(j+1)}\right).$$

By Theorem 2.5(ii), we have

$$\text{Ass}_R\left(I^{(i)}/I^{(i+1)} \otimes_k J^{(j)}/J^{(j+1)}\right) = \bigcup_{p \in \text{Ass}_A(I^{(i)}/I^{(i+1)})} \bigcup_{q \in \text{Ass}_B(J^{(j)}/J^{(j+1)})} \text{Min}_R(R/p + q).$$

Since $I^{(i)}/I^{(i+1)}$ and $J^{(j)}/J^{(j+1)}$ are ideals of $A/I^{(i+1)}$ and $B/J^{(j+1)}$, we have

$$\text{Ass}_A(I^{(i)}/I^{(i+1)}) \subseteq \text{Ass}_A(A/I^{(i+1)}) = \text{Min}_A(A/I^{(i+1)}) = \text{Min}_A(A/I),$$

$$\text{Ass}_B(J^{(j)}/J^{(j+1)}) \subseteq \text{Ass}_B(B/J^{(j+1)}) = \text{Min}_B(B/J^{(j+1)}) = \text{Min}_B(B/J).$$

Since $R/I + J = (A/I) \otimes_k (B/J)$, it follows from Theorem 2.5(i) that

$$\text{Min}_R(R/I + J) = \bigcup_{p \in \text{Min}_A(A/I)} \bigcup_{q \in \text{Min}_B(B/J)} \text{Min}_R(R/p + q).$$

Therefore,

$$\text{Ass}_R\left(I^{(i)}/I^{(i+1)} \otimes_k J^{(j)}/J^{(j+1)}\right) \subseteq \text{Min}_R(R/I + J).$$

So we get

$$\text{Ass}_R(R/Q_n) \subseteq \text{Min}_R(R/I + J) = \text{Min}_R(R/(I + J)^n).$$

This shows that every associated prime of $Q_n$ is a minimal associated prime of $(I + J)^n$. Since $Q_n \supseteq \sum_{i,j=0}^n I^i J^j = (I + J)^n$, it follows from the definition of symbolic powers that every primary component of $Q_n$ contains a primary component of $(I + J)^n$. Therefore, $Q_n \supseteq (I + J)^n$.

Now, we shall prove the converse inclusion $(I + J)^n \supseteq Q_n$. Let $P$ be an arbitrary minimal associated prime of $(I + J)^n$. Then $P$ is a minimal associated prime of $R/I + J = (A/I) \otimes_k (B/J)$. Set $p = P \cap A$ and $q = P \cap B$. By Lemma 2.1(i), $p$ and $q$ are minimal associated primes of $I$ and $J$. Therefore, $(I^{(i)})_p = (I^i)_p$ and $(J^{(j)})_q = (J^j)_q$ for all $i, j \geq 0$. This implies that

$$(I^{(i)})_p \otimes_k (J^{(j)})_q = (I^i)_p \otimes_k (J^j)_q.$$ 

Since $(I^{(i)} \otimes J^{(j)})_P$ and $(I^i \otimes J^j)_P$ are localizations of $(I^{(i)})_p \otimes_k (J^{(j)})_q$ and $(I^i)_p \otimes_k (J^j)_q$ at a prime ideal of $A_p \otimes B_q$, we get

$$(I^{(i)} J^{(j)})_P = (I^i \otimes J^j)_P = (I^i \otimes J^j)_P = (I^i J^j)_P$$

for all $i, j \geq 0$. Thus,

$$(Q_n)_P = \sum_{i+j=n} (I^{(i)} J^{(j)})_P = \sum_{i+j=n} (I^i J^j)_P = (I + J)^n_P.$$
This shows that every primary component of \((I + J)^{(n)}\) contains a primary component of \(Q_n\). Hence, \((I + J)^{(n)} \supseteq Q_n\). □

Theorem 3.4 extends a result on squarefree monomial ideals of Bocci et al [3] Theorem 7.8 to arbitrary ideals. It will play a key role in our investigation on invariants of \((I + J)^{(n)}\) in the next sections.

Moreover, Theorem 3.4 yields the following criterion for the equality of symbolic and ordinary powers of \(I + J\).

**Corollary 3.5.** Assume that \(I^{(t)} \neq I^{t+1}\) and \(J^{(t)} \neq J^{t+1}\) for all \(t \leq n - 1\). Then \((I + J)^{(n)} = (I + J)^n\) if and only if \(I^{(t)} = I^t\) and \(J^{(t)} = J^t\) for all \(t \leq n\).

**Proof.** Assume that \(I^{(t)} = I^t\) and \(J^{(t)} = J^t\) for all \(t \leq n\). By Theorem 3.4, we have

\[
(I + J)^{(n)} = \sum_{i+j=n} I^{(i)} J^{(j)} = \sum_{i+j=n} I^i J^j = (I + J)^n.
\]

Conversely, assume that \((I + J)^{(n)} = (I + J)^n\). Since \((I + J)^{n-1}/(I + J)^n \subseteq R/(I + J)^n\), we have

\[
\text{Ass}_R((I + J)^{n-1}/(I + J)^n) \subseteq \text{Ass}_R(R/(I + J)^n) = \text{Min}_R(R/(I + J)^n) = \text{Min}_R(R/(I + J)) = \text{Min}_R ((A/I) \otimes_k (B/J)).
\]

By Proposition 3.3, we have

\[
(I + J)^{n-1}/(I + J)^n = \bigoplus_{i+j=n-1} (I^i/I^{i+1}) \otimes_k (J^j/J^{j+1}).
\]

Hence,

\[
\text{Ass}_R ((I + J)^{n-1}/(I + J)^n) = \bigcup_{i+j=n-1} \text{Ass}_R ((I^i/I^{i+1}) \otimes_k (J^j/J^{j+1})).
\]

Therefore,

\[
\text{Ass}_R ((I^i/I^{i+1}) \otimes_k (J^j/J^{j+1})) \subseteq \text{Min}_R ((A/I) \otimes_k (B/J)).
\]

Since \(I^i \neq I^{i+1}\) and \(J^j \neq J^{j+1}\), we can apply Theorem 2.5 to get \(\text{Ass}_A(I^i/I^{i+1}) \subseteq \text{Min}_A(A/I)\) for \(i \leq n - 1\). Similarly as in the proof of Theorem 3.4, we have

\[
\text{Ass}_A(A/I^t) \subseteq \bigcup_{i=0}^{t-1} \text{Ass}_A(I^i/I^{i+1}) \subseteq \text{Min}_A(A/I).
\]

This implies that \(\text{Ass}_A(A/I^t) = \text{Min}_A(A/I)\). Hence, \(I^{(t)} = I^t\) for all \(t \leq n\). By the same way, we can also show that \(J^{(t)} = J^t\) for all \(t \leq n\). □

It is easy to see that the assumption of Corollary 3.5 is satisfied if \(A\) and \(B\) are local rings or domains. The following example shows that Corollary 3.5 does not hold if we remove the assumption that \(I^t \neq I^{t+1}\) and \(J^t \neq J^{t+1}\) for all \(t \leq n - 1\).

**Example 3.6.** Let \(A = k[x]/(x - x^2)\) and \(I = xA\). Then \(I^2 = I\). Let \(B = k[y, z, t]\) and \(J = (y^4, y^3z, y^2z^3, z^4, y^2z^2t)\). Then \(J^{(1)} = (y, z)^4 \neq J\) and \(J^{(2)} = J^2 = (y, z)^8\). However, \((I + J)^{(2)} = (I + J)^2 = (x, (y, z)^8)R\).
Remark 3.7. Corollary 3.5 shows that if \((I + J)^{(n)} = (I + J)^n\) then \((I + J)^{(t)} = (I + J)^t\) for all \(t \leq n - 1\). However, for an arbitrary ideal \(Q\) in a polynomial ring, \(Q^{(n)} = Q^n\) does not imply \(Q^{(t)} = Q^t\) for all \(t \leq n - 1\). The ideal \(J\) in the above example is such a case.

We end this section by giving another interesting application of Theorem 3.4. Recall that for a nonzero proper homogeneous ideal \(I\), \(\alpha(I) := \min\{d \mid I_d \neq 0\}\) is the smallest degree of a nonzero element in \(I\), and the Waldschmidt constant of \(I\) is defined by

\[
\hat{\alpha}(I) := \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.
\]

This limit exists and was first investigated by Waldschmidt in complex analysis [38]. Since then, it has appeared in different areas of mathematics, e.g., in number theory, algebraic geometry and commutative algebra [6, 14, 20]. The following consequence of Theorem 3.4 extends a result on the Waldschmidt constant of squarefree monomial ideals [3, Corollary 7.10] to arbitrary homogeneous ideals.

**Corollary 3.8.** Let \(A\) and \(B\) be standard graded polynomial rings over \(k\). Let \(I \subset A\) and \(J \subset B\) be nonzero proper homogeneous ideals. Then

\[
\hat{\alpha}(I + J) = \min\{\hat{\alpha}(I), \hat{\alpha}(J)\}.
\]

**Proof.** The proof goes in the same line of arguments as that of [3, Corollary 7.10] replacing [3, Theorem 7.8] by our more general statement in Theorem 3.4. \(\square\)

4. Depth and regularity of binomial sums

Throughout this section, let \(A = k[X]\) and \(B = k[Y]\) be polynomial rings over an arbitrary field \(k\) in different sets of variables. Then \(R := A \otimes_k B = k[X,Y]\). If \(I \subset A\) and \(J \subset B\) are homogeneous ideal, then their extensions in \(R\) are also homogeneous. As before, we shall also denote these ideals by \(I\) and \(J\).

We shall need the following results of Hoa and Tam in [24].

**Lemma 4.1.** [24, Lemmas 2.2 and 3.2] Let \(I \subseteq A\) and \(J \subseteq B\) be nonzero proper homogeneous ideals. Then

\[
\text{(i) } \text{depth } R/IJ = \text{depth } A/I + \text{depth } B/J + 1.
\]

\[
\text{(ii) } \text{reg } R/IJ = \text{reg } A/I + \text{reg } B/J + 1.
\]

Let \(\{I_i\}_{i \geq 0}\) and \(\{J_j\}_{j \geq 0}\) be filtrations of homogeneous ideals in \(A\) and \(B\), respectively. Recall that the ideal

\[
Q_n := \sum_{i+j=n} I_i J_j
\]

is called the \(n\)-th binomial sum of these filtrations. The aim of this section is to give bounds for the depth and the regularity of \(R/Q_n\) in terms of those of \(I_i\) and \(J_j\).

**Theorem 4.2.** For all \(n \geq 1\), we have

\[
\text{(i) } \text{depth } R/Q_n \geq \min_{i \in [1,n-1], j \in [1,n]} \{\text{depth } A/I_{n-i} + \text{depth } B/J_i + 1, \text{depth } A/I_{n-j+1} + \text{depth } B/J_j\},
\]

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(ii) $\text{reg} R/Q_n \leq \max_{i \in [1,n-1]} \{ \text{reg} A/I_{n-i} + \text{reg} B/J_i + 1, \text{reg} A/I_{n-j+1} + \text{reg} B/J_j \}.$

Proof. We shall only prove the bound for depth. The bound for regularity can be obtained in the same fashion.

For $t = 0, \ldots, n$, set $P_{n,t} := I_n J_0 + I_{n-1} J_1 + \cdots + I_{n-t} J_t.$ Then $P_{n,t} = P_{n,t-1} + I_{n-t} J_t$ for $1 \leq t \leq n.$ Since $P_{n,t-1} \subseteq I_{n-t+1}$, we have $P_{n,t-1} \cap I_{n-t} J_t \subseteq I_{n-t+1} \cap J_t = I_{n-t+1} J_t$ by Lemma 3.1. On the other hand, $I_{n-t+1} J_t \subseteq I_{n-t+1} J_{t-1} \subseteq P_{n,t-1}$ and $I_{n-t+1} J_t \subseteq I_{n-t} J_t.$ This implies that $P_{n,t-1} \cap I_{n-t} J_t = I_{n-t+1} J_t.$

Hence, there is a short exact sequence

$$0 \rightarrow R/I_{n-t+1} J_t \rightarrow (R/P_{n,t-1}) \oplus (R/I_{n-t} J_t) \rightarrow R/P_{n,t} \rightarrow 0.$$ 

Therefore, we have

$$\text{depth } R/P_{n,t} \geq \min \{ \text{depth } R/P_{n,t-1}, \text{depth } R/I_{n-t} J_t, \text{depth } R/I_{n-t+1} J_t - 1 \}.$$ 

We will use these bounds to successively estimate depth $R/Q_n$ as follows.

For $t = n$, we have $P_{n,n} = Q_n.$ Since $I_0 J_n = J_n$, we have depth $R/I_0 J_n = \dim A + \text{depth } B/J_n.$ Applying Lemma 4.1 to the product $I_1 J_n$, we get

$$\text{depth } R/Q_n \geq \min \{ \text{depth } R/P_{n,n-1}, \dim A + \text{depth } B/J_n, \text{depth } A/I_1 + \text{depth } B/J_n \}.$$ 

For $t = n - 1, \ldots, 2$, by applying Lemma 4.1 to the products $I_{n-t} J_t$ and $I_{n-t+1} J_t$, we get

$$\text{depth } R/P_{n,t} \geq \min \{ \text{depth } R/P_{n,t-1}, \text{depth } A/I_{n-t} + \text{depth } B/J_t + 1, \text{depth } A/I_{n-t+1} + \text{depth } B/J_t \}.$$ 

For $t = 1$, we have depth $R/P_{n,0} = \text{depth } A/I_n + \dim B$ because $P_{n,0} = I_n J_0 = I_n$. Applying Lemma 4.1 to the product $I_n J_1$ now yields

$$\text{depth } R/P_{n,1} \geq \min \{ \text{depth } A/I_n + \dim B, \text{depth } A/I_{n-1} + \text{depth } B/J_1 + 1, \text{depth } A/I_n + \text{depth } B/J_1 \}.$$ 

Putting all these estimates for depth $R/P_{n,t}$ together, we obtain

$$\text{depth } R/Q_n \geq \min \{ \dim A + \text{depth } B/J_n, \text{depth } A/I_n + \dim B,$$

$$\min_{i \in [1,n-1]} \{ \text{depth } A/I_{n-i} + \text{depth } B/J_i + 1, \text{depth } A/I_{n-j+1} + \text{depth } B/J_j \} \}.$$ 

Since $I_1 \neq (0)$, we have

$$\dim A + \text{depth } B/J_n > \text{depth } A/I_1 + \text{depth } B/J_n.$$ 

Since $J_1 \neq (0)$, we have

$$\text{depth } A/I_n + \dim B > \text{depth } A/I_n + \text{depth } B/J_1.$$ 

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The right-hand sides of the above inequalities are depth $A/I_{n-j+1} + \text{depth } B/J_j$ for $j = n, 1$. Hence, we can remove the terms $\dim A + \text{depth } B/J_n$ and $\text{depth } A/I_n + \dim B$ from the estimate for depth $R/Q_n$ to obtain that

$$\text{depth } R/Q_n \geq \min_{i \in [1, n-1], j \in [1, n]} \{\text{depth } A/I_{n-i} + \text{depth } B/J_i + 1, \text{depth } A/I_{n-j+1} + \text{depth } B/J_j\}.$$ 

□

**Remark 4.3.** Let $I \subset A$ and $J \subset B$ be nonzero homogeneous ideals. If $I_i = I^i$ and $J_j = J^j$ for all $i, j \geq 0$, we have $Q_n = (I + J)^n$. In this case, Theorem 4.2 recovers a main result of our previous work on depth and regularity of ordinary powers [19, Theorem 2.4]. As pointed out in [19, Propositions 2.6 and 2.7], both terms on the right-hand side of these bounds are essential (i.e., are attainable). Hence, this is also the case for the two terms on the right-hand side of the bounds of Theorem 4.2.

If we consider the quotients $Q_n/Q_{n+1}$ instead of the quotient rings $R/Q_n$, we can compute their depth and regularity explicitly in terms of those of quotients of successive $I_i$’s and $J_j$’s.

**Theorem 4.4.** For all $n \geq 1$, we have

(i) $\text{depth } Q_n/Q_{n+1} = \min_{i+j=n} \{\text{depth } I_i/I_{i+1} + \text{depth } J_j/J_{j+1}\}$,

(ii) $\text{reg } Q_n/Q_{n+1} = \max_{i+j=n} \{\text{reg } I_i/I_{i+1} + \text{reg } J_j/J_{j+1}\}$.

**Proof.** Proposition 3.3 gives

$$Q_n/Q_{n+1} = \bigoplus_{i+j=n} (I_i/I_{i+1}) \otimes_k (J_j/J_{j+1}).$$

The desired conclusion now follows from [19, Lemma 2.5], which expresses the depth and the regularity of a tensor product over $k$ in terms of those of the components. □

As a consequence of Theorem 4.4, we obtain bounds for the depth and the regularity of $R/Q_n$ in terms of those of quotients of successive $I_i$’s and $J_j$’s.

**Corollary 4.5.** For all $n \geq 1$, we have

(i) $\text{depth } R/Q_n \geq \min_{i+j \leq n-1} \{\text{depth } I_i/I_{i+1} + \text{depth } J_j/J_{j+1}\}$,

(ii) $\text{reg } R/Q_n \leq \max_{i+j \leq n-1} \{\text{reg } I_i/I_{i+1} + \text{reg } J_j/J_{j+1}\}$.

**Proof.** Using the short exact sequences

$$0 \rightarrow Q_t/Q_{t+1} \rightarrow R/Q_{t+1} \rightarrow R/Q_t \rightarrow 0$$

for $t \leq n-1$ we deduce that

$$\text{depth } R/Q_n \geq \min_{t \leq n-1} \text{depth } Q_t/Q_{t+1},$$

$$\text{reg } R/Q_n \leq \max_{t \leq n-1} \text{reg } Q_t/Q_{t+1}.$$ 

Hence, the assertions follow from Theorem 4.4. □
If \( \{I_i\}_{i \geq 0} \) and \( \{J_j\}_{j \geq 0} \) are the filtrations of symbolic powers of two nonzero homogeneous ideals \( I \subset A \) and \( J \subset B \), then \( Q_n = (I + J)^{(n)} \) by Theorem 3.4. For the sake of applications, we reformulate Theorem 4.2 and Theorem 4.4 in this case.

**Theorem 4.6.** For all \( n \geq 1 \), we have

(i) \( \text{depth } R/(I + J)^{(n)} \geq \min_{i \in [1, n-1], \ j \in [1, n]} \{ \text{depth } A/I^{(n-i)} + \text{depth } B/J^{(i)} + 1, \text{depth } A/I^{(n-j+1)} + \text{depth } B/J^{(j)} \} \),

(ii) \( \text{reg } R/(I + J)^{(n)} \leq \max_{i \in [1, n-1], \ j \in [1, n]} \{ \text{reg } A/I^{(n-i)} + \text{reg } B/J^{(i)} + 1, \text{reg } A/I^{(n-j+1)} + \text{reg } B/J^{(j)} \} \).

We shall see in the next section that the inequalities of Theorem 4.6 are in fact equalities if \( \text{char}(k) = 0 \) or if \( I \) and \( J \) are monomial ideals.

**Theorem 4.7.** For all \( n \geq 1 \), we have

(i) \( \text{depth} (I + J)^{(n)}/(I + J)^{(n+1)} = \min_{i+j=n} \{ \text{depth } I^{(i)}/I^{(i+1)} + \text{depth } J^{(j)}/J^{(j+1)} \} \),

(ii) \( \text{reg} (I + J)^{(n)}/(I + J)^{(n+1)} = \max_{i+j=n} \{ \text{reg } I^{(i)}/I^{(i+1)} + \text{reg } J^{(j)}/J^{(j+1)} \} \).

Using Theorem 4.7 we obtain the following criterion for the Cohen-Macaulayness of \( R/(I + J)^{(n)} \). Recall that a finite graded \( R \)-module \( M \) is Cohen-Macaulay if \( \text{depth } M = \text{dim } M \).

**Corollary 4.8.** The following conditions are equivalent:

(i) \( (I + J)^{(n-1)}/(I + J)^{(n)} \) is Cohen-Macaulay,

(ii) \( R/(I + J)^{(i)} \) is Cohen-Macaulay for all \( i \leq n \),

(iii) \( A/I^{(i)} \) and \( B/J^{(i)} \) are Cohen-Macaulay for all \( i \leq n \),

(iv) \( I^{(i)}/I^{(i+1)} \) and \( J^{(j)}/J^{(j+1)} \) are Cohen-Macaulay for all \( i \leq n-1 \).

**Proof.** It is clear that

\[ \text{dim} (I + J)^{(n-1)}/(I + J)^{(n)} \leq \text{dim } R/(I + J)^{(n)} = \text{dim } R/(I + J). \]

For any prime \( P \in \text{Min}_R(R/(I + J)) \), we have

\[ ((I + J)^{(n-1)}/(I + J)^{(n)})_P = ((I + J)^{n-1}/(I + J)^n)_P = 0 \]

if and only if \( ((I + J)^{n-1})_P = 0 \) by Nakayama’s lemma. But this could not happen because \( R \) is a domain and \( I + J \neq 0 \). Thus,

\[ \text{dim } R/(I + J) \leq \text{dim} (I + J)^{(n-1)}/(I + J)^{(n)}. \]

From this it follows that

\[ \text{dim} (I + J)^{(n-1)}/(I + J)^{(n)} = \text{dim } R/(I + J) = \text{dim } A/I + \text{dim } B/J. \]

Similarly, we also have \( \text{dim } I^{(i-1)}/I^{(i)} = \text{dim } A/I \) and \( \text{dim } J^{(j-1)}/I^{(j)} = \text{dim } B/J \) for all \( i, j \geq 1 \).

The above formulas imply

\[ \text{dim} (I + J)^{(n-1)}/(I + J)^{(n)} = \text{dim } I^{(i)}/I^{(i+1)} + \text{dim } J^{(j)}/J^{(j+1)} \]
for all $n, i, j \geq 1$. Using Theorem 4.7(i) we can show that
\[
\text{depth}(I + J)^{(n-1)}/(I + J)^{(n)} = \dim(I + J)^{(n-1)}/(I + J)^{(n)}
\]
if and only if depth $I^i/J^{i+1} = \dim(I^i/J^{i+1})$ and depth $J^i/J^{i+1} = \dim(J^i/J^{i+1})$ for all $i \leq n - 1$. Thus, $(I + J)^{(n-1)}/(I + J)^{(n)}$ is Cohen-Macaulay if and only if $I^i/J^{i+1}$ and $J^i/J^{i+1}$ are Cohen-Macaulay for all $i \leq n - 1$. From this it follows that (i) $\iff$ (iv). In particular, (i) implies that $(I + J)^{(i)}/(I + J)^{(i+1)}$ is Cohen-Macaulay for all $i \leq n - 1$.

Note that a finite graded $R$-module $M$ is Cohen-Macaulay if and only if $H^t_M(M) = 0$ for $t < \dim M$, where $H^t_M(M)$ denotes the $t$-th local cohomology module of $M$ with respect to the maximal homogeneous ideal $m$ of $R$. Using this characterization and the short exact sequence
\[
0 \to (I + J)^{(i)}/(I + J)^{(i+1)} \to R/(I + J)^{(i+1)} \to R/(I + J)^{(i)} \to 0,
\]
we deduce that $R/(I + J)^{(i)}$ is Cohen-Macaulay for all $i \leq n$ if and only if $(I + J)^{(i)}/(I + J)^{(i+1)}$ is Cohen-Macaulay for all $i \leq n - 1$. This proves the implication (i) $\implies$ (ii).

Similarly, $A/I^{(i)}$ and $B/J^{(i)}$ are Cohen-Macaulay for all $i \leq n$ if and only if $I^{(i)}/I^{(i+1)}$ and $J^{(i)}/J^{(i+1)}$ are Cohen-Macaulay for all $i \leq n - 1$. This proves the equivalence (iii) $\iff$ (iv).

The implication (i) $\implies$ (ii) is remarkable in the sense that the Cohen-Macaulay property of $(I + J)^{(n-1)}/(I + J)^{(n)}$ alone implies that of $R/(I + J)^{(t)}$ for all $t \leq n$. The following example shows that this implication does not hold if we replace $I + J$ by an arbitrary homogeneous ideal.

**Example 4.9.** Take $A = k[x, y, z, t]$ and $I = (x^5, x^4y, xy^4, y^5, x^2y^2(xz + yt), x^3y^3)$. Then $I$ is a $(x, y)$-primary ideal, $\dim A/I = 2$ and $I^{(1)} = I$. It is clear that $z$ is a regular element of $A/I$. Since the socle of $A/(I, z)$ contains the residue class of $x^2y^3$, we have depth $A/I = 1$. Therefore, $A/I^{(1)} = A/I$ is not Cohen-Macaulay. On the other hand, we have $I^2 = (x, y)^{10} \cap (I^2 + (z, t))$. Hence, $I^{(2)} = (x, y)^{10}$. Now, we can see that $z, t$ is a regular sequence of $I^{(1)}/I^{(2)}$. Therefore, $I^{(1)}/I^{(2)}$ is Cohen-Macaulay.

We end this section with the following formulas for the case where one of the ideals $I$ and $J$ is generated by linear forms. Though this case appears to be simple, these formulas had not been known before. The proof is based on the binomial expansion of $(I + J)^{(n)}$.

**Proposition 4.10.** Assume that $J$ is generated by linear forms. Then

(i) $\text{depth } R/(I + J)^{(n)} = \min_{i \leq n} \{\text{depth } A/I^{(i)} + \dim B/J\}$,

(ii) $\text{reg } R/(I + J)^{(n)} = \max_{i \leq n} \{\text{reg } A/I^{(i)} - i\} + n$.

**Proof.** Assume that $B = k[y_1, \ldots, y_s]$. Without restriction we may assume that $J = (y_1, \ldots, y_t)$, $t \leq s$. Then $\dim B/J = s - t$. Set $B' = k[y_1, \ldots, y_t]$, $J' = (y_1, \ldots, y_t)B'$, and $R' = A \otimes_k B'$. It is clear that
\[
\text{depth } R/(I + J)^{(n)} = \text{depth } R'/(I + J')^{(n)} + s - t,
\]
\[
\text{reg } R/(I + J)^{(n)} = \text{reg } R'/(I + J')^{(n)}.
\]
Therefore, we only need to prove the case $t = s$. 

\[15\]
If \( t = s = 1 \), we set \( y = y_1 \). Then \( B = k[y] \) and \( J = (y) \). By Theorem 3.4, we have \((I, y)^{(n)} = \sum_{i=0}^{n} I^{(i)} y^{n-i}\). If we write \( R = \bigoplus_{i \geq 0} A y^i \), then \( R/(I, y)^{(n)} = \bigoplus_{i \leq n} (A/I^{(i)}) y^{n-i} \). From this it follows that

\[
\begin{align*}
\text{depth } R/(I, y)^{(n)} &= \min_{i \leq n} \text{depth } A/I^{(i)}, \\
\text{reg } R/(I, y)^{(n)} &= \max_{i \leq n} \{ \text{reg } A/I^{(i)} - i \} + n.
\end{align*}
\]

If \( t = s > 1 \), we set \( A' = A[y_1, \ldots, y_{s-1}] \) and \( I' = (I, y_1, \ldots, y_{s-1}) A' \). Using induction we may assume that

\[
\begin{align*}
\text{depth } A'/I' &= \min_{i \leq n} \text{depth } A/I^{(i)}, \\
\text{reg } A'/I' &= \max_{i \leq n} \{ \text{reg } A/I^{(i)} - i \} + n.
\end{align*}
\]

Since \( I + J = (I', y_s) \), we have

\[
\begin{align*}
\text{depth } R/(I + J)^{(n)} &= \min_{i \leq n} \text{depth } A'/I'^{(i)} = \min_{i \leq n} \text{depth } A/I^{(i)}, \\
\text{reg } R/(I + J)^{(n)} &= \max_{i \leq n} \{ \text{reg } A'/I'^{(i)} - i \} + n = \max_{i \leq n} \{ \text{reg } A/I^{(i)} - i \} + n.
\end{align*}
\]

\[\blacksquare\]

### 5. Splitting conditions

The aim of this section is to show that the inequalities of Theorem 4.6 are equalities if \( \text{char}(k) = 0 \) or if \( I \) and \( J \) are monomial ideals. Our main tool is the following notion which allows us to compute the depth and the regularity of a sum of ideals in terms of those of the summands and their intersection.

Let \( R \) be a polynomial ring over a field \( k \). Let \( P, I, J \) be nonzero homogeneous ideals of \( R \) such that \( P = I + J \). The sum \( P = I + J \) is called a **Betti splitting** if the Betti numbers of \( P, I, J, I \cap J \) satisfy the relation

\[
\beta_{i,j}(P) = \beta_{i,j}(I) + \beta_{i,j}(J) + \beta_{i-1,j}(I \cap J)
\]

for all \( i \geq 0 \) and \( j \in \mathbb{Z} \).

This notion was introduced by Francisco, Hà and Van Tuyl [15]. It generalizes the notion of splittable monomial ideals of Eliahou and Kervaire [13]. The following result explains why it is useful to have a Betti splitting.

**Lemma 5.1.** [15] Corollary 2.2] Let \( P = I + J \) be a Betti splitting. Then

(i) \( \text{depth } R/P = \min\{ \text{depth } R/I, \text{depth } R/J, \text{depth } R/I \cap J - 1 \} \),

(ii) \( \text{reg } R/P = \max\{ \text{reg } R/I, \text{reg } R/J, \text{reg } R/I \cap J - 1 \} \).

A Betti splitting can be characterized by the following property. We say that a homomorphism \( \phi : M \to N \) of graded \( R \)-modules is **Tor-vanishing** if \( \text{Tor}_i^R(k, \phi) = 0 \) for all \( i \geq 0 \). This notion was due to Nguyen and Vu [29].

**Lemma 5.2.** [15] Proposition 2.1] The following conditions are equivalent:

(i) The decomposition \( P = I + J \) is a Betti splitting.
(ii) The inclusion maps \( I \cap J \to I \) and \( I \cap J \to J \) are Tor-vanishing.

We say that a filtration of ideals \( \{P_n\}_{n \geq 0} \) in \( R \) is a Tor-vanishing filtration if for all \( n \geq 1 \), the inclusion map \( P_n \to P_{n-1} \) is Tor-vanishing, i.e. \( \operatorname{Tor}_i^R(k, P_n) \to \operatorname{Tor}_i^R(k, P_{n-1}) \) is the zero map for all \( i \geq 0 \).

The following result shows that the inequalities of Theorem 4.2 are equalities if the given filtrations are Tor-vanishing.

**Theorem 5.3.** Let \( \{I_i\}_{i \geq 0} \) and \( \{J_j\}_{j \geq 0} \) be Tor-vanishing filtrations in \( A \) and \( B \). Let \( Q_n = \sum_{i+j=n} I_i J_j \). Then

(i) \( \operatorname{depth} R/Q_n = \min_{i \in [1,n-1]} \{ \operatorname{depth} A/I_{n-i} + \operatorname{depth} B/J_i + 1, \operatorname{depth} A/I_{n-j+1} + \operatorname{depth} B/J_j \} \),

(ii) \( \operatorname{reg} R/Q_n = \max_{i \in [1,n-1]} \{ \operatorname{reg} A/I_{n-i} + \operatorname{reg} B/J_i + 1, \operatorname{reg} A/I_{n-j+1} + \operatorname{reg} B/J_j \} \).

**Proof.** We shall only prove the formula for depth. The formula for regularity can be proved similarly.

For \( t = 0, \ldots, n \), set

\[ P_{n,t} := I_n J_0 + I_{n-1} J_1 + \cdots + I_{n-t} J_t. \]

Then \( P_{n,0} = I_n, P_{n,n} = Q_n \), and \( P_{n,t} = P_{n,t-1} + I_{n-t} J_t \) for \( t \geq 1 \). We will compute \( \operatorname{depth} R/Q_n \) by successively computing \( \operatorname{depth} R/P_{n,t} \). To do that we will first show that \( P_{n,t} = P_{n,t-1} + I_{n-t} J_t \) is a Betti splitting for \( 1 \leq t \leq n \).

We have seen in the proof of Theorem 4.2 that

\[ P_{n,t-1} \cap I_{n-t} J_t = I_{n-t+1} J_t. \]

Note that \( I_{n-t+1} J_t = I_{n-t-1} \otimes_k J_t \) and \( I_{n-t} J_t = I_{n-t} \otimes_k J_t \). By the hypothesis, the inclusion map \( I_{n-t+1} J_t \to I_{n-t} \) is Tor-vanishing. Since the tensor product with \( J_t \) is an exact functor, the inclusion map \( I_{n-t+1} J_t \to I_{n-t} J_t \) is also Tor-vanishing. Similarly, the inclusion map \( I_{n-t+1} J_t \to I_{n-t+1} J_{t-1} \) is Tor-vanishing. Since \( I_{n-t+1} J_t \subseteq I_{n-t+1} J_{t-1} \subseteq P_{n,t-1} \), the inclusion map \( I_{n-t+1} J_t \to P_{n,t-1} \) is also Tor-vanishing. By Lemma 5.2, it follows that \( P_{n,t} = P_{n,t-1} + I_{n-t} J_t \) is a Betti splitting.

Now we can apply Lemma 5.1 to obtain

\[ \operatorname{depth} R/P_{n,t} = \min \{ \operatorname{depth} R/P_{n,t-1}, \operatorname{depth} R/I_{n-t} J_t, \operatorname{depth} R/I_{n-t+1} J_{t-1} \}. \]

Proceeding as in the proof of Theorem 4.2 we will get the desired conclusion. \( \square \)

Tor-vanishing filtrations can be found by the following result of Ahangari Maleki [11] in the case \( \operatorname{char}(k) = 0 \). Let \( \partial(Q) \) denote the the ideal generated by the partial derivatives of the generators of an ideal \( Q \). Using the chain rule for taking partial derivatives, it is not hard to see that \( \partial(Q) \) does not depend on the chosen generators of \( Q \).

**Lemma 5.4.** [11, Propostion 3.5] Assume that \( \operatorname{char}(k) = 0 \). Let \( Q \subseteq Q' \) be homogeneous ideals such that \( \partial(Q) \subseteq Q' \). Then the inclusion map \( Q \to Q' \) is Tor-vanishing.
Proposition 5.5. Let I be a nonzero proper homogeneous ideal in A. If char(k) = 0 then
∂(I(n)) ⊆ I(n−1) for all n ≥ 1. Therefore, the filtration \{I(n)\}_{n≥0} is Tor-vanishing.

Proof. Let Ass°(I) := \bigcup_{n=1}^{∞} Ass\_A(I^n). This is a finite set due to a classical result of Brodmann [4]. Set

\[ L = \bigcap_{P ∈ Ass°(I) \setminus \text{Min}(I)} P, \]

where L = A if Ass°(I) = Min(I). It follows from the definition of symbolic powers that
\[ I^{(m)} = \bigcup_{t≥0} I^m : L^t \text{ for all } m ≥ 0. \]

Let t ≥ 1 be an integer such that \( I^{(n)} L^t \subseteq I^n \). For any \( f ∈ I^{(n)} \) and \( g ∈ L^{t+1} \), we have \( fg ∈ I^n \). Let \( \partial_x \) denote the partial derivative with respect to an arbitrary variable \( x \) of \( A \). We have
\[ \partial_x(fg) + f\partial_x(g) = \partial_x(fg) ∈ ∂(I^n) ⊆ I^{n−1}. \]
Since \( ∂_x(g) ∈ ∂(L^{t+1}) ⊆ L^t \), we have
\[ f\partial_x(g) ∈ fL^t ⊆ I^n. \]
Therefore, \( ∂_x(fg) = ∂_x(fg) − f\partial_x(g) ∈ I^{n−1} \). Hence, \( ∂_x(f) ∈ I^{n−1} : L^{t+1} ⊆ I^{(n−1)} \). So we can conclude that \( ∂(I^{(n)}) ⊆ I^{(n−1)} \). □

Now, we can show that the bounds in Theorem 4.6 are actually the exact values of the
depth and the regularity of \( R/(I + J)^{(n)} \) if char(k) = 0.

Theorem 5.6. Let I and J be nonzero proper homogeneous ideals in A and B. Assume that
char(k) = 0. Then for all \( n ≥ 1 \), we have

(i) depth \( R/(I + J)^{(n)} = \min_{i∈[1,n−1]} \{ \text{depth } A/I^{(n−i)} + \text{depth } B/J^{(i)} + 1, \text{depth } A/I^{(n−j+1)} + \text{depth } B/J^{(j)} \} , \]

(ii) reg \( R/(I + J)^{(n)} = \max_{i∈[1,n−1]} \{ \text{reg } A/I^{(n−i)} + \text{reg } B/J^{(i)} + 1, \text{reg } A/I^{(n−j+1)} + \text{reg } B/J^{(j)} \} . \]

Proof. By Proposition 5.5, the filtrations \{I(i)\}_{i≥0} and \{J(j)\}_{j≥0} are Tor-vanishing. Therefore, the conclusion follows from Theorem 3.4 and Theorem 5.3. □

The following example shows that both equalities of Theorem 5.6 may fail if one of the
base rings is not regular, even if char(k) = 0.

Example 5.7. By abuse of notations, we will denote the residue class of an element \( f \) by \( f \) itself. Let \( S = \mathbb{Q}[a, b, c, d] \). Put \( A = S/(a^4, a^3b, ab^3, b^4, a^3c − b^3d) \) and \( I = (a^2 − b^2)^2 \_A \). Computations with Macaulay2 [16] show that \( I^{(1)} = I, I^{(2)} = I^2 = (a^2b^2)^2 \_A \). Hence, there are isomorphisms of \( S \)-modules
\[ A/I^{(1)} \cong S/(a^2 − b^2, a^4, a^3b, a^3c − b^3d), \]
\[ A/I^{(2)} \cong S/(a^4, a^3b, a^2b^2, ab^3, b^4, a^3c − b^3d). \]
This implies that
\[
\text{depth}(A/I^{(1)}) = \text{depth}(A/I^{(2)}) = 1, \\
\text{reg}(A/I^{(1)}) = \text{reg}(A/I^{(2)}) = 3.
\]

Let \( B = \mathbb{Q}[x, y, z, t] \) and \( J = (x^5, x^4y, xy^4, y^5, x^2y^2(xz^6 - yt^6), x^3y^3) \subseteq B \). Computations with Macaulay2 give \( J^{(1)} = J = (x^5, x^4y, xy^4, y^5, x^2y^2(xz^6 - yt^6), x^3y^3) \), \( J^{(2)} = (x, y)^{10} \), and
\[
\text{depth}(B/J^{(1)}) = 1, \quad \text{depth}(B/J^{(2)}) = 2, \\
\text{reg}(B/J^{(1)}) = 10, \quad \text{reg}(B/J^{(2)}) = 9.
\]

Let \( R = A \otimes_k B \) and \( Q = I + J \subseteq R \). Then
\[
R = \mathbb{Q}[a, b, c, d, x, y, z, t]/(a^4, a^3b, ab^3, b^4, a^3c - b^3d), \\
Q = (a^2 - b^2, x^5, x^4y, xy^4, y^5, x^2y^2(xz^6 - yt^6), x^3y^3)R.
\]

By Theorem 3.4, we have
\[
Q^{(2)} = I^{(2)} + J^{(1)} + J^{(2)} = (a^2b^2, (a^2 - b^2)(x^5, x^4y, xy^4, y^5, x^2y^2(xz^6 - yt^6), x^3y^3, (x, y)^{10})R.
\]

It can be checked that
\[
\text{depth}(R/Q^{(2)}) = 3 > 2 = \text{depth}(A/I^{(2)}) + \text{depth}(B/J^{(1)}), \\
\text{reg}(R/Q^{(2)}) = 13 < 14 = \text{reg}(A/I^{(1)}) + \text{reg}(B/J^{(1)}) + 1.
\]

Hence, both equalities of Theorem 5.6 fail in this example.

The Tor-vanishing of the symbolic powers in Proposition 5.5 can be considered as a higher order generalization of the inclusion \( I^{(n)} \subseteq mI^{(n-1)} \), which was proved by Eisenbud and Mazur for \( \text{char}(k) = 0 \) [12, Proposition 13] and for \( I \) being a monomial ideal [12, Proposition 9].

Eisenbud and Mazur [12] conjectured that \( I^{(2)} \subseteq mI^{(1)} \) if \( I \) is a prime ideal in a power series ring over a field of characteristic zero (see [20] for similar conjectures on higher symbolic powers). They also showed that this conjecture has a negative answer in positive characteristic. Using their result we can construct the following example, which shows that Proposition 5.5 does not hold in positive characteristic.

**Example 5.8.** Let \( R = (\mathbb{Z}/2)[x, y, z, t] \) and \( S = (\mathbb{Z}/2)[u] \). Consider the kernel \( L \) of the map \( R \to S \) given by
\[
x \mapsto u^4, y \mapsto u^6, z \mapsto u^7, t \mapsto u^9.
\]

Clearly, \( L \) is a prime ideal. Hence, \( L^{(1)} = L \). Computations with Macaulay2 [16] show that \( L \) is minimally generated by the following polynomials:
\[
g_1 = x^3 + y^2, g_2 = yz + xt, g_3 = x^2y + z^2, xz^2 + t^2, x^2z + yt, xy^2 + zt.
\]

The map \( \text{Tor}_0^R(k, L^{(2)}) \to \text{Tor}_0^R(k, L) \) is not zero. In fact, for \( f = x^3y - y^3 - xz^2 + t^2 \), we have \( x^2f = g_1g_3 + g_2^2 \in L^2 \), which shows that \( f \in L^{(2)} \). On the other hand, we have \( f \notin m_R L \), where \( m_R = (x, y, z, t) \).

While the ideal \( L \) is not homogeneous in the standard grading, it is weighted homogeneous by setting \( \text{deg } x = 4, \text{deg } y = 6, \text{deg } z = 7, \text{deg } t = 9 \). Now we use the polarization trick of
J. McCullough and I. Peeva [27] to transform $L$ into a homogeneous ideal in the standard grading. Let $A = (\mathbb{Z}/2)[x_1, x_2, x_3, y_1, y_2, y_3, y_4]$. Consider the homogeneous map $\phi : R \to A$ given by

$$x \mapsto x_1^3y_1, \quad y \mapsto x_2^5y_2, \quad z \mapsto x_3^5y_3, \quad t \mapsto x_4^8y_4.$$ 

Let $I$ be the extension of $L$ to $A$. Then one can check with Macaulay2 that $I$ is a prime ideal of $A$, and more importantly, $I$ is homogeneous in the standard grading of $A$. Let $m_A = (x_1, \ldots, x_4, y_1, \ldots, y_4)$. Then the relation $x^2f = g_1g_3 + g_2^2$ shows that $\phi(f) \in I^{(2)} \setminus m_AI$. Hence, the map $I^{(2)} \to I^{(1)}$ is not Tor-vanishing. Using similar arguments and [12, Example, p. 200], we even have similar examples in any positive characteristic.

We could not use Example 5.8 to construct any counterexample to the conclusion of Theorem 5.6 in positive characteristics. We expect that Theorem 5.6 holds regardless of the characteristic of $k$.

Our next main result shows that this is the case for monomial ideals. We shall first collect alternatives for Lemma 5.4 and Proposition 5.5 in this case, and then present the result in Theorem 5.11.

For a monomial ideal $Q$ we denote by $\partial^*(Q)$ the ideal generated by elements of the form $f/x$, where $f$ is a minimal monomial generator of $Q$ and $x$ is a variable dividing $f$.

**Lemma 5.9.** [29, Proposition 4.4 and Lemma 4.2] Let $Q \subseteq Q'$ be monomial ideals such that $\partial^*(Q) \subseteq Q'$. Then the inclusion map $Q \to Q'$ is Tor-vanishing.

**Proposition 5.10.** Let $I$ be a nonzero proper monomial ideal in $A$. Then $\partial^*(I^{(n)}) \subseteq I^{(n-1)}$ for all $n \geq 1$. Therefore, the filtration $\{I^{(n)}\}_{n \geq 0}$ is Tor-vanishing.

**Proof.** Let $Q_1, \ldots, Q_s$ be the primary components of $I$ associated to the minimal primes of $I$. Note that $Q_1, \ldots, Q_s$ are monomial ideals. By [22, Lemma 3.1], we have

$$I^{(n)} = Q_1^n \cap \cdots \cap Q_s^n.$$ 

It is clear that $\partial^*(Q^n) = \partial^*(Q)Q^{n-1} \subseteq Q^{n-1}$ for any monomial ideal $Q$. Therefore,

$$\partial^*(I^{(n)}) \subseteq \partial^*(Q_1^n) \cap \cdots \cap \partial^*(Q_s^n) \subseteq Q_1^{n-1} \cap \cdots \cap Q_s^{n-1} = I^{(n-1)}.$$ 

By Lemma 5.9 this implies that $\{I^{(n)}\}_{n \geq 0}$ is Tor-vanishing. □

**Theorem 5.11.** Let $I$ and $J$ be nonzero proper monomial ideals in $A$ and $B$. Then the equalities of Theorem 5.6 hold regardless of the characteristic of $k$.

**Proof.** By Proposition 5.10 both the filtrations $\{I^{(i)}\}_{i \geq 0}$ and $\{J^{(j)}\}_{j \geq 0}$ are Tor-vanishing. Therefore, the conclusion follows from Theorem 5.4 and Theorem 5.3. □

Our results on Tor-vanishing have some interesting consequences on the relationship between the depth and the regularity of $A/I^{(n-1)}$, $A/I^{(n)}$, and $I^{(n-1)}/I^{(n)}$.

**Proposition 5.12.** Let $I$ be a nonzero proper homogeneous ideal in $A$. Assume that $\text{char}(k) = 0$ or $I$ is a monomial ideal. Then for any $n \geq 1$,

\begin{enumerate}
  \item $\text{depth} I^{(n-1)}/I^{(n)} = \min\{\text{depth} A/I^{(n-1)} + 1, \text{depth} A/I^{(n)}\}$,
  \item $\text{reg} I^{(n-1)}/I^{(n)} = \max\{\text{reg} I^{(n-1)}, \text{reg} I^{(n)} - 1\}$.
\end{enumerate}
Proof. By Proposition 5.5 and Proposition 5.10, the inclusion map $I^{(n)} \to I^{(n-1)}$ is Tor-vanishing, i.e. Tor$_i^A(k, I^{(n)}) \to$ Tor$_i^A(k, I^{(n-1)})$ is the zero map for all $i$. From the exact sequence

$$0 \to I^{(n)} \to I^{(n-1)} \to I^{(n-1)}/I^{(n)} \to 0,$$

it follows that the long exact sequence of Tor splits into short exact sequences

$$0 \to \text{Tor}^A_{i+1}(k, I^{(n-1)}) \to \text{Tor}^A_i(k, I^{(n-1)}/I^{(n)}) \to \text{Tor}^A_i(k, I^{(n)}) \to 0.$$

Using the characterization of the depth and the regularity by Tor we can easily deduce the conclusion. □

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