Frequently hypercyclic translation semigroups

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Abstract

Frequent hypercyclicity for translation $C_0$-semigroups on weighted spaces of continuous functions is investigated. The results are achieved by establishing an analogy between frequent hypercyclicity for the translation semigroup and for weighted pseudo-shifts and by characterizing frequent hypercyclic weighted pseudo-shifts in spaces of vanishing sequences. Frequent hypercyclic translation semigroups in weighted $L^p$-spaces are also characterized.

1 Introduction and preliminaries

A continuous linear operator $T$ on a separable Banach space is called hypercyclic if there is an element $x \in X$, called hypercyclic vector, such that the orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in $X$. The first historically known examples of hypercyclic operators are due to Birkhoff, MacLane and Rolewicz. In particular, the last author studied hypercyclicity in the setting of weighted shift operators on $l^p$ and $c_0$. The interest in the study of linear dynamics of shift operators is nowadays still alive, since many classical operators (e.g. derivative operator in spaces of entire functions) can be viewed as such operators. We refer to the recent monographs [8] and [24] for a complete overview on the subject.

In 2005, motivated by Birkhoff’s ergodic theorem, Bayart and Grivaux introduced in [10] the notion of frequent hypercyclic operators, trying to quantify how “often” an orbit meets non-empty open sets. More precisely, if the lower density of a set $A \subset \mathbb{N}$ is defined as

$$\text{dens}(A) := \lim \inf_{N \to \infty} \frac{\# \{n \leq N : n \in A\}}{N},$$

an operator $T \in L(X)$ is said to be frequently hypercyclic if there exists $x \in X$ (called frequently hypercyclic vector) such that, for every non-empty open subset $U \subset X$,

$$\text{dens}(\{n \in \mathbb{N} : T^n x \in U\}) > 0.$$

This notion has been deeply investigated by various authors, see e.g. [23, 14, 18]. In particular frequently hypercyclic weighted shifts have been investigated in [14, 11], until their behaviour has been completely characterized in $l^p$ and $c_0$ by Bayart and Rusza [9].

In parallel with the theory for linear operators, since the seminal paper by Desch, Schappacher and Webb [20], many researchers turned their attention to the hypercyclic behaviour of strongly continuous semigroups. Actually hypercyclicity appears in solution...
semigroups of evolution problems associated with “birth and death” equations for cell populations, transport equations, first order partial differential equations, Black and Scholes equations, diffusion operators as Ornstein-Uhlenbeck operators [2, 4, 5, 6, 12, 13, 15, 17, 21, 25, 27, 28].

We recall that, if $X$ be a separable infinite-dimensional Banach space, a $C_0$-semigroup $(T_t)_{t \geq 0}$ of linear and continuous operators on $X$ is said to be hypercyclic if there exists $x \in X$ (called hypercyclic vector for the semigroup) such that the set $\{T_t x : t \geq 0\}$ is dense in $X$. An element $x \in X$ is said to be a periodic point for the semigroup if there exists $t > 0$ such that $T_t x = x$. A semigroup $(T_t)_{t \geq 0}$ is called chaotic if it is hypercyclic and the set of periodic points is dense in $X$.

The role of “test” class, which is played by weighted shifts in the setting of discrete linear dynamical systems, is covered by translation semigroups in the setting of continuous linear dynamical systems.

An admissible weight function on $\mathbb{R}$ is a strictly positive measurable function $\rho : \mathbb{R} \to \mathbb{R}$ for which there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(\tau) \leq M e^{\omega \tau} \rho(\tau + t)$ for all $\tau \in [0, +\infty]$ and all $t > 0$.

If $\rho$ is an admissible weight, then for every $l > 0$ there exist $A, B > 0$ such that for every $\sigma \in \mathbb{R}$ and for every $t \in [\sigma, \sigma + l]$, it holds

$$A \rho(\sigma) \leq \rho(t) \leq B \rho(\sigma + l).$$

Consider the following function spaces:

$$L^p_\rho(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R} \mid u \text{ is measurable and } \|u\|_p < \infty\},$$

where $\|u\|_p = (\int_{\mathbb{R}} |u(t)|^p \rho(t) dt)^{\frac{1}{p}}$, and

$$C^p_0(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R} \mid u \text{ is continuous and } \lim_{x \to \pm \infty} u(x) \rho(x) = 0\},$$

with $\|u\|_\infty = \sup_{t \in \mathbb{R}} |u(t)| \rho(t)$.

If $X$ is any of the spaces above and $\rho$ is an admissible weight, the translation semigroup $T = (T_t)_{t \geq 0}$ is defined as usual by

$$T_t f(x) = f(x + t), \quad t \geq 0, \ x \in \mathbb{R},$$

and is a $C_0$-semigroup (see e.g. [20]).

If $X$ is one of the spaces $L^p_\rho(\mathbb{R})$ or $C^p_0(\mathbb{R})$ with an admissible weight function $\rho$, the translation semigroup $T$ on $X$ is hypercyclic if and only if $\liminf_{t \to +\infty} \rho(t) = 0$.

If $X = C^0_0(\mathbb{R})$, then the translation semigroup $T$ on $X$ is chaotic if and only if $\lim_{x \to \pm \infty} \rho(x) = 0$.

If $X = L^p_\rho(\mathbb{R})$, $T$ is chaotic if and only if $\sum_{k \in \mathbb{Z}} \rho(k) < \infty$ [19, 20, 27, 28].

The concept of frequent hypercyclicity was extended to $C_0$-semigroups in [14].

The lower density of a measurable set $M \subset \mathbb{R}_+$ is defined by

$$\text{Dens}(M) := \liminf_{N \to \infty} \mu(M \cap [0, N])/N,$$

where $\mu$ is the Lebesgue measure on $\mathbb{R}_+$.

A $C_0$-semigroup $(T_t)_{t \geq 0}$ is said to be frequently hypercyclic if there exists $x \in X$ such that $\text{Dens}(\{t \in \mathbb{R}_+ : T_t x \in U\}) > 0$ for any non-empty open set $U \subset X$. In [16, 20], it was proved that $x \in X$ is a (frequently) hypercyclic vector for $(T_t)_{t \geq 0}$ if and only if $x$ is
a (frequently) hypercyclic vector for each single operator $T_t$, $t > 0$. However, this is not the case in general if we consider the chaos property \cite{[12]}. In \cite{[26]}, it was proved a continuous version of the Frequent Hypercyclicity criterion based on the Pettis integral and that chaotic translation semigroups on weighted spaces of integrable or continuous functions on the real line are frequently hypercyclic.

Moreover, in \cite{[29]}, it is proved that the Frequent Hypercyclicity criterion for semigroups implies the existence of strongly-mixing Borel probability measures with full support.

In this paper we characterize, in the line of \cite{[10]}, frequently hypercyclic translation semigroups on weighted function spaces. The main results are Theorems 5 and 11 that are proved in the last section. In particular, Theorem 5 will be consequence of Theorem 1, which characterizes frequent hypercyclicity of the so-called pseudo-shifts on spaces $c_0(I)$, where $I$ is a countably infinite set.

\section{Frequently hypercyclic weighted pseudo-shift}

We recall the concept of weighted pseudo-shift that was introduced by Grosse-Erdmann in \cite{[22]}. Given $X, Y$ topological sequence spaces over countably infinite sets $I$ and $J$, a continuous linear operator $T : X \to Y$ is called a \textit{weighted pseudo-shift} if there is a sequence $(b_{ij})_{i \in I}$ of non-zero scalars and an injective mapping $\phi : J \to I$ such that $T[(x_i)_{i \in I}] = (b_{ij}x_{\phi(j)})_{j \in J}$ for $(x_i)_{i \in I} \in X$.

We will be interested in weighted pseudo-shifts acting on spaces of vanishing sequences. More precisely, given a countable set $I$, we consider the space

$$c_0(I) = \{(x_i)_{i \in I} : \forall \epsilon > 0 \exists J \subset I, \text{ J finite } \forall i \in I \setminus J \ |x_i| < \epsilon\},$$

endowed with the norm $\|((x_i)_{i \in I})\| = \sup_{i \in I} |x_i|$.

The first result that we prove is a characterization of frequently universal sequences of weighted pseudo-shifts on $c_0(I)$.

We recall that a sequence $(T_n)_{n \in \mathbb{N}}$ of continuous mappings between topological spaces $X$ and $Y$ is called frequently universal if there exists $x \in X$ such that for every non-empty open set $U \subseteq Y$,

$$\text{dens}\{n \in \mathbb{N} : T_nx \in U\} > 0.$$ 

Following the idea of Bayart and Ruzsa in \cite{[9]} for weighted backward shifts on $c_0(\mathbb{Z})$, we first obtain a characterization for weighted pseudo-shifts.

**Theorem 1.** Let $(T_n)_{n}$ be a sequence of pseudo-shifts on $c_0(I)$ defined by $T_n[(x_i)_{i \in I}] = (\rho_{n} b_{ij}^n x_{\phi_{n}(i)})_{i \in I}$, where $\rho_{n}$ are positive real numbers and there exists $\rho > 1$ such that $\frac{1}{\rho^{n-m}} \leq \frac{b_{ij}^n}{b_{ij}^m}$ for all $n, m \in \mathbb{N}, i, j \in I$. Suppose there exists $g : I \to \mathbb{R}$ such that if $\phi_{n}(s) = \phi_{m}(t)$ then $|n - m| \leq |g(s) - g(t)|$ and that $(\phi_{n})_{n}$ is a run-away sequence, i.e. for each pair of finite subsets $I_0, J_0 \subset I$ there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $\phi_n(J_0) \cap I_0 = \emptyset$.

Then $(T_n)_n$ is frequently universal on $c_0(I)$ if and only if there exist (for all) a sequence $(M(p))_{p \in \mathbb{N}}$ of positive real numbers tending to $\infty$, a sequence $(E_p)_{p \in \mathbb{N}}$ of subsets of $\mathbb{N}$ and an increasing sequence $(W_p)_{p \in \mathbb{N}}$ of finite subsets of $I$ with $I = \bigcup_{p=1}^{\infty} W_p$, such that:

(a) For any $p \geq 1$, $\text{dens}(E_p) > 0$. 

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(b) For any \(p,q \geq 1, p \neq q, n \in E_p, m \in E_q, \phi_n(W_p) \cap \phi_m(W_q) = \emptyset\).

(c) \(\lim_{n \to \infty, n \in E_p, s \in W_p} b^n_s = \infty\).

(d) For any \(p,q \geq 1, p \neq q, n \in E_p\) and any \(m \in E_q, t \in W_q\) such that \(\phi_n(s) = \phi_m(t)\) :

\[
\frac{b^n_s}{b^m_t} \leq \frac{1}{M(p)M(q)}.
\]

**Proof.** We first observe that we may replace “there exists a sequence \((M(p))\)” by “for any sequence \((M(p))\)” in the statement of the theorem. Indeed, if properties \((a)\) to \((d)\) are true for some sequence \((M(p))\), then they are also satisfied for any subsequence of it.

Consider a sequence \((\alpha_p)_{p \in \mathbb{N}}\) of positive real numbers such that \(\alpha_1 = 2\) and for all \(p \geq 2, \alpha_p > 4\alpha_{p-1}2^{\Psi(p)}, \) where \(\Psi(p) = \max \{|g(t)| : t \in W_p\}\). Let \(x = (x_i)_{i \in I}\) be a frequently universal vector for \((T_n)_n\) and set

\[
E_p = \left\{ n \in \mathbb{N} : ||T_nx - \alpha_p \sum_{i \in W_p} e_i|| < \frac{1}{p} \right\}.
\]

Clearly \(\text{dens}(E_p) > 0\). Let \(p \neq q, n \in E_p, m \in E_q\) and let us show that \(\phi_n(W_p) \cap \phi_m(W_q) = \emptyset\). Assume that \(p < q\). By contradiction, let us assume that there exist \(s \in W_p\) and \(t \in W_q\) such that \(\phi_n(s) = \phi_m(t)\). The \(s\)-th coefficient of \(T_nx\) is \(b^n_{s,\phi_n(s)}\), then

\[
|b^n_{s,\phi_n(s)}| \leq ||T_nx - \alpha_p \sum_{i \in W_p} e_i|| + \alpha_p \sum_{i \in W_q} e_i|| < \frac{1}{p} + \alpha_p < 2\alpha_p.
\]

The \(t\)-th coefficient of \(T_mx\) is \(b^m_{t,\phi_m(t)}\) and

\[
|b^m_{t,\phi_m(t)}| \geq \alpha_q - \frac{1}{q} \geq \frac{\alpha_q}{2}.
\]

Then

\[
\frac{1}{p^{2\Psi(q)}} \leq \frac{1}{p^{|m|}} \leq \frac{|b^n_{s,\phi_n(s)}|}{|b^m_{t,\phi_m(t)}|} \leq 2\alpha_p \frac{2}{\alpha_q}
\]

But this contradicts the definition of \((\alpha_p)\).

Now let \(n \in E_p\) and \(s \in W_p\), the \(s\)-th coefficient of \(T_nx\) is \(b^n_{s,\phi_n(s)}\) and its modulus cannot be less than \(\frac{\alpha_p}{2}\). Let \(M > 0\). Given \(\epsilon = \frac{\alpha_p}{2M}\), since \(x \in c_0(I)\), there exists \(J \subset I\) finite such that \(|x_i| < \epsilon\) for all \(i \in I \setminus J\). As \(\phi_n\) is a run-away sequence there exists \(n_0 \in \mathbb{N}\) such that for all \(n \in E_p, n > n_0\) and for all \(s \in W_p, \phi_n(s) \notin J\), and then \(|x_{\phi_n(s)}| < \epsilon\). As a result, for all \(n \geq n_0\) and \(s \in W_p\):

\[
|b^n_s| \geq \frac{\alpha_p}{2|x_{\phi_n(s)}|} \geq \frac{\alpha_p}{2\epsilon} = M.
\]

So, we have proved \((c)\).

Finally, let \(n \in E_p, m \in E_q, t \in W_q\) such that \(\phi_n(s) = \phi_m(t)\), then \(s \notin W_p\) and

\[
\frac{b^n_s}{b^m_t} = \frac{|b^n_{s,\phi_n(s)}|}{|b^m_{t,\phi_m(t)}|} \leq \frac{1}{p} \frac{2}{\alpha_q} \leq \frac{11}{pq}.
\]
This shows (d) with $M(p) = p$.

We now show that the condition is sufficient. We may assume that, for any $p \geq 1$, $M(p) \geq \rho^p$. We set

$$E'_p = E_p \setminus \bigcup_{s \in W_p} \{ n \in \mathbb{N} : b^n_s \leq \rho^p \}$$

Clearly, $E'_p$ is a cofinite subset of $E_p$ due to (c), hence $\text{dens}(E'_p) > 0$. Let $(y^p)_{p \geq 0}$ be a dense sequence in $c_0(I)$ such that $\text{supp}(y^p) \subseteq W_p$ and $\| y^p \| < \rho^p$. We define $x \in \mathbb{R}^I$ by setting

$$x_i = \begin{cases} \frac{1}{b^n_s} y^p(s), & \text{if } i = \phi_n(s), n \in E'_p, s \in W_p \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

This definition is not ambiguous because given $n \in E'_p, m \in E'_p$ we have $\phi_n(W_p) \cap \phi_m(W_q) = \emptyset$. If $s_1, s_2 \in W_p, s_1 \neq s_2$, then $\phi_n(s_1) \neq \phi_n(s_2)$ because $\phi_n$ is injective. Moreover, if $n, m \in E'_p$ are such that $\phi_n(s) = \phi_m(s)$, by hypothesis $|n - m| \leq |g(s) - g(s)| = 0$, then $n = m$.

We claim that $x \in c_0(I)$. Indeed, given $\epsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that for $p \geq p_0$ and $n \in E'_p, s \in W_p$: $|x_i| \leq \frac{\rho^p}{\rho^p} \leq \epsilon$.

If $p \leq p_0$:

$$|x_i| \leq \frac{\rho^p}{b^n_s} \to 0, n \to \infty.$$  

We finally show that $x$ is a frequently hypercyclic vector by proving that for all $p \geq 1, n \in E'_p$, $\| T_n x - y^p \| < \epsilon(p)$ with $\epsilon(p) \to 0$ as $p \to \infty$. Observe that

$$\| T_n x - y^p \| = \sup_{s \in W_p} \| b^n_s x_{\phi_n(s)} \|.$$  

The terms which appear in the norm are nonzero if and only if $\phi_n(s) = \phi_m(t), m \in E'_q, t \in W_q$ and for these terms it holds that

$$|b^n_s x_{\phi_n(s)}| = \frac{b^n_s}{\rho^q} g(t) \leq \frac{\rho^q}{M(p)M(q)} \leq \rho^q \rho^q = \frac{1}{\rho^p}.$$

\[\square\]

As a corollary, we obtain a characterization of frequent hypercyclicity for weighted backward shifts operators defined on $c_0(I)$, in the case that $I \subseteq \mathbb{R}$.

**Corollary 2.** Let $I$ be a countably infinite subset of $\mathbb{R}$ such that $I + Z \subseteq I$, $I = \bigcup_{p=1}^\infty W_p$, where $(W_p)_p$ is an increasing sequence of finite subsets. Let $(w_i)_{i \in I}$ be a bounded and bounded below sequence of positive integers. The operator $T : c_0(I) \to c_0(I)$ defined by $T(x_i)_{i \in I} = (w_{i+1}x_{i+1})_{i \in I}$ is frequently hypercyclic on $c_0(I)$ if and only if there exist (for all) a sequence $(M(p))$ of positive real numbers tending to $\infty$ and a sequence $(E_p)$ of subsets of $\mathbb{N}$ such that

(a) For any $p \geq 1$, $\text{dens}(E_p) > 0$. 

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(b) For any \( p, q \geq 1, p \neq q, \) \( (E_p + W_p) \cap (E_q + W_q) = \emptyset. \)

(c) \( \lim_{n \to \infty, n \in E_p, s \in W_p} w_{s+1} \cdots w_{s+n} = \infty. \)

(d) For any \( p, q \geq 1, n \in E_p, m \in E_q, n \neq m \) and \( t \in W_q; \)

\[
\frac{w_{m-n+t+1} \cdots w_{m+t}}{w_{t+1} \cdots w_{t+m}} \leq \frac{1}{M(p)M(q)}.
\]

**Proof.** This corollary is a particular case of Theorem 1 when we consider \( T_n = T^n \) with \( T[(x_i)_{i \in I}] = (w_{i+1}x_{i+1})_{i \in I}, \) \( b^a_s = w_{s+1}w_{s+2} \cdots w_{s+n}, \phi(s) = s + 1, \phi_n = \phi \circ \cdots \circ \phi \) and \( g : I \to \mathbb{R} \) defined by \( g(s) = s. \)

**Remark 3.** Observe that condition (d) is equivalent to say that for any \( p, q \geq 1, n \in E_p, m \in E_q, n \neq m \) and \( t \in W_q; \)

\[
\begin{cases}
  w_{t+1} \cdots w_{t+m-n} \geq M(p)M(q), & \text{if } m > n \\
  w_{t+(m-n)+1} \cdots w_{t-1}w_t \leq \frac{1}{M(p)M(q)}, & \text{if } m < n.
\end{cases}
\]

and we obtain the conditions of Theorems 12 in [9].

## 3 Frequently hypercyclic translation semigroups

The purpose of this section is to obtain a characterization of frequent hypercyclicity for translation semigroups on \( C_0^\infty(\mathbb{R}) \) and \( L^0_c(\mathbb{R}). \)

To treat the case of continuous functions, we will first need to recall some known results about the construction of a Schauder basis in \( C_0(\mathbb{R}), \) referring for more details to [30].

Let \( \tilde{D} \) be the set of dyadic numbers, that is \( \tilde{D} = \bigcup_{n=0}^\infty D_n \) where \( D_0 = \{0, 1\} \) and, if \( n \geq 1, \)

\[
D_n = \left\{ \frac{2k-1}{2^n} : k = 1, \ldots, 2^n-1 \right\}.
\]

For any \( \tau \in D_n, \) set \( \tau^- = \tau - 2^{-n} \) and \( \tau^+ = \tau + 2^{-n}. \)

Let \( \varphi(x) = \max(0, 1 - |x|), \) \( x \in \mathbb{R} \) and define \( \varphi_{k+\tau}(x) = \varphi(2^n(x - k - \tau)) \) where \( k \in \mathbb{Z}, \tau \in D_n, \tau \neq 1. \) Observe that \( \varphi_{k+\tau}(x) = \varphi_{\tau}(x - k) \) where \( \varphi_{\tau} \) is the Faber-Schauder dyadic function with peak at \( \tau. \)

Set \( I = \mathbb{Z} + \tilde{D} \) and consider the partition \( I = \bigcup_{n \geq 0} V_n \) where \( V_0 = \{0, 1\}, \) and

\[
V_n = \{-n+h+D_h \mid h = 1, \ldots, n\} \cup \{h+D_{n-h} \mid h = 0, 1, \ldots, n\}.
\]

We define an order on \( I \) assuming that the elements of \( V_k \) are earlier than the elements of \( V_n \) if \( 0 \leq k < n, \) and within each \( V_n \) we keep the usual order.

The system \( (\varphi_i)_{i \in I}, \) is a Schauder basis in \( C_0(\mathbb{R}). \) More precisely, if \( f \in C_0(\mathbb{R}), \) then

\[
f = \sum_{k+\tau \in \mathbb{Z}+\tilde{D}} a_{k+\tau} \varphi_{k+\tau}
\]

\[
a_{k+\tau} = \begin{cases}
  f(k) & k \in \mathbb{Z}, \tau = 0 \\
  f(k + \tau) - \frac{1}{2}(f(k + \tau^-) + f(k + \tau^+)) & k \in \mathbb{Z}, \tau \in \tilde{D}.
\end{cases}
\]
Lemma 4. Let \( \rho \) be an admissible weight function on \( \mathbb{R} \) such that \( \rho(x) = \rho([x]) \) for any \( x \in \mathbb{R} \) and let \( T_1 : C^0_0(\mathbb{R}) \to C^0_0(\mathbb{R}) \) be the translation operator defined as \( T_1 f(x) = f(x + 1) \). Then \( T_1 \) is quasi conjugated to the weighted backward shift operator \( B_w : c_0(\mathbb{Z} + \tilde{D}) \to c_0(\mathbb{Z} + \tilde{D}) \), defined by

\[
B_w[(x_{k+\tau})_{k+\tau \in \mathbb{Z} + \tilde{D}}] = (w_{k+\tau}x_{k+\tau})_{k+\tau \in \mathbb{Z} + \tilde{D}},
\]

where \( w_{k+\tau} = \frac{\rho(k)}{\rho(k + 1)}, \) \( k + \tau \in \mathbb{Z} + \tilde{D} \).

**Proof.** Given \( f \in C^0_0(\mathbb{R}) \), we define \( Q(f(x)) = (a_{k+\tau})_{k+\tau \in \mathbb{Z} + \tilde{D}} \) where

\[
f(x)\rho(x) = \sum_{k+\tau \in \mathbb{Z} + \tilde{D}} a_{k+\tau} \phi_{k+\tau}(x).
\]

Clearly \( Q : C^0_0(\mathbb{R}) \to c_0(\mathbb{Z} + \tilde{D}) \) is a continuous linear operator and

\[
B_w \circ Q(f) = B_w(a_{k+\tau})_{k+\tau \in \mathbb{Z} + \tilde{D}} = \left( \frac{\rho(k)}{\rho(k + 1)} a_{k+\tau+1} \right)_{k+\tau \in \mathbb{Z} + \tilde{D}}.
\]

On the other hand, \( Q \circ T_1 (f(x)) = Q(f(x + 1)) = (b_{k+\tau})_{k+\tau \in \mathbb{Z} + \tilde{D}} \), where \( f(x + 1)\rho(x) = \sum_{k+\tau \in \mathbb{Z} + \tilde{D}} b_{k+\tau} \phi_{k+\tau}(x) \). We have that:

\[
b_{k+\tau} = \begin{cases} 
  f(k + 1)\rho(k) = a_{k+1} \frac{\rho(k)}{\rho(k + 1)}, & \text{if } \tau = 0 \\
  (f(k + 1 + \tau) - \frac{1}{2}(f(k + 1 + \tau^-) + f(k + 1 + \tau^+))\rho(k) = a_{k+\tau+1} \frac{\rho(k)}{\rho(k + 1)}, & \text{if } \tau \neq 0.
\end{cases}
\]

for \( k \in \mathbb{Z}, \tau \in \tilde{D} \), taking into account that \( \rho(k + \tau) = \rho(k + \tau^-) = \rho(k + \tau^+) = \rho(k) \) for all \( k \in \mathbb{Z}, \tau \in \tilde{D} \). Then

\[
Q \circ T_1(f(x)) = \left( \frac{\rho(k)}{\rho(k + 1)} a_{k+\tau+1} \right)_{k+\tau \in \mathbb{Z} + \tilde{D}} = B_w \circ Q(f).
\]

So the diagram

\[
\begin{array}{ccc}
C^0_0(\mathbb{R}) & \xrightarrow{T_1} & C^0_0(\mathbb{R}) \\
\downarrow Q & & \downarrow Q \\
c_0(\mathbb{Z} + \tilde{D}) & \xrightarrow{B_w} & c_0(\mathbb{Z} + \tilde{D})
\end{array}
\]

is commutative and we conclude the result.

**Theorem 5.** Let \( \mathcal{T} \) be the translation semigroup on \( C^0_0(\mathbb{R}) \), where \( \rho \) is an admissible function and \( \sup_{k \in \mathbb{Z}} \frac{\rho(k + 1)}{\rho(k)} < \infty \). \( \mathcal{T} \) is frequently hypercyclic on \( C^0_0(\mathbb{R}) \) if and only if there exist a sequence \( (M(p)) \) of positive real numbers tending to \( \infty \) and a sequence \( (E_p) \) of subsets of \( \mathbb{N} \) such that:

(a) For any \( p \geq 1 \), \( \mathsf{dens}(E_p) > 0 \).

(b) For any \( p, q \geq 1, p \neq q \), \( (E_p + W_p) \cap (E_q + W_q) = \emptyset \).

(c) \( \lim_{n \to \infty, n \in E_p, k \in [-p, p+1]} \rho(k + n) = 0 \).
(d) For any \( p, q \geq 1 \), for any \( n \in E_p \) and any \( m \in E_q \), \( n \neq m \) and for all \( k \in [-q, q+1] \):

\[
\rho(k + m - n + 1) \leq \frac{1}{M(p)M(q)},
\]

where \( W_p = \bigcup_{k=0}^{p} V_k \), where \( V_k \) is defined in (4).

Proof. Let us point out that if \( \rho \) is an admissible weight function then \( \sup \frac{\rho(k)}{\rho(k+1)} < \infty \). By hypothesis we have \( \sup \frac{\rho(k+1)}{\rho(k)} = M < \infty \), then there exist constants \( 0 < A < B \) such that

\[ A\rho(k) \leq \rho(x) \leq B\rho(k + 1) \leq BM\rho(k). \]

Then if we define \( \tilde{\rho}(x) = \rho(x) \) for \( x \in [k, k+1) \), there exist constants \( M_1, M_2 > 0 \) such that

\[ M_1||f||_{\infty}^p \leq ||f||_{\infty}^\rho \leq M_2||f||_{\infty}^\rho. \]

We conclude the result combining Corollary 2 and Lemma 4.

Remark 6. Let \( (E_p) \) be a sequence of subsets of \( \mathbb{N} \) such that for any \( p, q \geq 1 \), \( p \neq q \), \( (E_p + [-p, p]) \cap (E_q + [-q, q]) = \emptyset \).

Choosing \( F_p = E_{p+1} \), we get that \( (F_p + W_p) \cap (F_q + W_q) = \emptyset \) if \( p \neq q \), where the sets \( W_p \) are defined as in the assumptions of Theorem 5. Indeed, if \( n \in F_p \), \( s \in [-p, p+1] \), \( \sigma \in \overline{D} \), \( \sigma = \frac{2u-1}{2v} \), \( m \in F_q \), \( t \in [-q, q+1] \), \( \tau \in \overline{D} \), \( \tau = \frac{2u-1}{2v} \) are such that

\[ n + s + \sigma = m + t + \tau, \]

we have that \( \tau - \sigma \in \mathbb{Z} \). Thus straightforward calculations give that \( h = k \) and \( |u - v| = a2^{h-1} \) with \( a \in \mathbb{Z}_+ \). On the other hand, \( |u - v| < 2^{h-1} \), hence \( a = 0 \). Therefore \( \tau = \sigma \) and so \( n + s = m + t \). Now the assertion follows by the properties of the set \( E_p \).

As an immediate consequence, we get that if \( \rho \) be an admissible weight function on \( \mathbb{R} \) such that \( \sup_{k \in \mathbb{Z}} \frac{\rho(k+1)}{\rho(k)} < \infty \) and we set \( w_k = \frac{\rho(k)}{\rho(k+1)} \), \( k \in \mathbb{Z} \), then, by \( \mathbb{R} \) [Theorem 9] and by Theorem 7, if \( B_m \) is frequently hypercyclic on \( c_0(\mathbb{Z}) \), also the translation semigroup is frequently hypercyclic on \( C^0_0(\mathbb{R}) \).

Proposition 7. Let \( T \) be a mixing (equivalently chaotic) translation \( C_0 \)-semigroup on \( C^0_0(\mathbb{R}) \). Then \( T \) is frequently hypercyclic.

Proof. As it is proved in \([13, 27, 28]\), chaos and mixing are equivalent properties for the translation \( C_0 \)-semigroup on \( C^0_0(\mathbb{R}) \), and this happens if and only if \( \lim_{x \to \pm \infty} \rho(x) = 0 \). Consider a sequence \( (E_p) \) of subsets of \( \mathbb{N} \) such that for any \( p \geq 1 \), \( \text{dens}(E_p) > 0 \) and for any \( p, q \geq 1 \), \( p \neq q \), \( (E_p + [-p, p]) \cap (E_q + [-q, q]) = \emptyset \). (see e.g. the constructions in [5]). Assumption (c) of Theorem 5 is clearly verified, while (b) is satisfied by Remark 6. Moreover, given \( n \in E_p, m \in E_q \) and \( k \in [-q, q+1] \), we can define for each \( k \in [-q, q+1], i \in \mathbb{N} \):

\[ M(i) = \min_{k \in [-q, q+1]} \left\{ \frac{1}{\sup_{|n| \geq i} \{\rho(k + n + 1)\}} \right\}. \]

It is clear that for \( n \in E_p \), \( m \in E_q \), \( |m - n| \geq \max(p, q) \), and

\[ \rho(k + m - n + 1) \leq \sup_{|k| \geq i} \rho(k + s + 1) \leq \frac{1}{M(i)}, i = p, q, \]

(7)
\begin{equation}
\rho(k + m - n + 1) \leq \frac{1}{\sqrt{M(p)}} \frac{1}{\sqrt{M(q)}},
\end{equation}
and hypothesis (d) is satisfied by the sequence $M'(p) = \sqrt{M(p)}$ and therefore $T$ is frequently hypercyclic. \hfill \Box

Remark 8. The converse of the previous proposition does not hold. Indeed, let $(w_k)_{k \in \mathbb{Z}}$ be one of the sequence of weights constructed in \cite{9,11} such that $B_w$ is frequently hypercyclic on $c_0(\mathbb{Z})$ and $w_1, \ldots, w_k = 1$ for infinitely many $k$. Define $\rho(k) = w_1 \cdot \cdots \cdot w_k$ if $k \geq 1$, $\rho(k) = (w_k \cdot w_{k+1} \cdot \cdots \cdot w_0)^{-1}$ if $k \leq 0$, and $\rho(x) = \rho([x])$ for any $x \in \mathbb{R}$. Then, by Remark 6 the translation semigroup is frequently hypercyclic on $C_0^\rho(\mathbb{R})$, while clearly it is not mixing, since $\rho(k) = 1$ for infinitely many $k$.

The final part of the paper will be devoted to the proof of Theorem 2. To this end we establish a relation between the discrete and the continuous case. We recall that the relation between the discrete and continuous case for Devaney chaos was studied in \cite{12} and for distributional chaos in \cite{7}.

The following lemma follows immediately by the conjugacy of the backward shift on $\ell_p^v = \{(x_k)_{k \in \mathbb{Z}} : \sum_{k \in \mathbb{Z}} |x_k| v_k < \infty\}$ and the weighted backward shift $B_w$ on $\ell_p$ where $w_k = \left(\frac{v_k}{v_k}ight)^{\frac{1}{p}}$, $k \in \mathbb{Z}$ and the characterization of frequently hypercyclic weighted backward shifts on $\ell_p$ proved in Theorem 3 in \cite{9}.

Lemma 9. Let $v = (v_k)_{k \in \mathbb{Z}}$ be a sequence of strictly positive weights such that $(\frac{v_k}{v_k})$ is bounded. Then the backward shift operator $B$ is frequently hypercyclic on $\ell_p^v$ if and only if $\sum_{k \in \mathbb{Z}} v_k < \infty$.

Theorem 10. Let $\rho$ be an admissible function on $\mathbb{R}$. If the translation semigroup $T$ is frequently hypercyclic in $L_p^\rho(\mathbb{R})$, then the backward shift operator $B$ is frequently hypercyclic on $\ell_p^v$, where $v_k = \rho(k)$ for all $k \in \mathbb{Z}$.

Proof. Since $\rho$ is an admissible function by \cite{1} there exists $A, B \geq 0$ such that for all $t \in [k, k + 1]$, $A \rho(k) \leq \rho(t) \leq B \rho(k + 1)$. If $(T_t)_{t \geq 0}$ is frequently hypercyclic, then $T_1$ is frequently hypercyclic \cite{10}. Hence there exists $f \in L_p^\rho$ such that for all $g \in L_p^\rho$ and for all $\epsilon > 0$, dens$_\epsilon$($n \in \mathbb{N} : ||T_1^n f - g|| < \epsilon > 0$). Since $f \in L_p^\rho$ we have that $||f|\rho| \in L_p([k, k + 1]) \subset L_1([k, k + 1])$ for every $k \in \mathbb{Z}$. Being $\rho$ a strictly positive continuous function we get that $f \in L_1([k, k + 1])$ for all $k \in \mathbb{Z}$. Therefore we can define $x_k = \int_k^{k+1} f(t) dt$ for all $k \in \mathbb{Z}$. We have that:

$$\sum_{k \in \mathbb{Z}} |x_k|^p \rho(k) = \sum_{k \in \mathbb{Z}} \left| \int_k^{k+1} f(t) dt \right|^p \rho(k) \leq \sum_{k \in \mathbb{Z}} \int_k^{k+1} |f(t)|^p \rho(k) dt \leq \frac{1}{A} \sum_{k \in \mathbb{Z}} \int_k^{k+1} |f(t)|^p \rho(t) dt = \frac{1}{A} ||f||_p \rho(k) < \infty.$$  

So $x = (x_k)_{k \in \mathbb{Z}} \in \ell_p^v$ with $v_k = \rho(k)$. Let $y = (0, \ldots, y_{-N}, \ldots, y_0, \ldots, y_M, 0, \ldots, 0)$ and let $\epsilon > 0$. Set $g = \sum_{k=-N}^M y_k \chi_{[k,k+1]} \in L_p^\rho(\mathbb{R})$. We show that:

$$\{n \in \mathbb{N} : ||T_1^n f - g|| < A^p \epsilon \} \subset \{n \in \mathbb{N} : ||B^n x - y|| < \epsilon \}$$
and therefore
\[ \text{dens}\{n \in \mathbb{N} : ||B^n x - y|| < \epsilon\} > 0 \]
because \( f \) is a frequently hypercyclic vector. We have:
\[ ||B^n x - y||^p = \sum_{k \in \mathbb{Z}} |x_{n+k} - y_k|^p \rho(k) \leq \frac{1}{A} \sum_{k \in \mathbb{Z}} \int_{k}^{k+1} |f(t+n) - g(t)|^p \rho(t) dt \leq \frac{1}{A} A \epsilon^p. \]

By the density of finite sequences in \( \ell_p^n \) we get that \( B \) is frequently hypercyclic. \( \square \)

Finally we are able to characterize frequently hypercyclic translation semigroups in \( L_p^p(\mathbb{R}) \).

**Theorem 11.** Let \( \rho \) be an admissible function on \( \mathbb{R} \). The following assertions are equivalent:

1. the translation semigroup \( T \) is frequently hypercyclic.
2. \( \sum_{k \in \mathbb{Z}} \rho(k) < \infty \).
3. \( \int_{-\infty}^{\infty} \rho(t) dt < \infty \).
4. \( T \) is chaotic.
5. \( T \) satisfies the Frequently Hypercyclicity Criterion.

**Proof.** Observe that \( \left( \frac{a(k)}{p(k)} \right)_k \) is bounded by the admissibility of the function \( \rho \). By Theorem 10 and Lemma 9, (1) \( \Rightarrow \) (2), while (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) \( \Rightarrow \) (5) \( \Rightarrow \) (1) are proved in [20]. \( \square \)

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