Integral formulas and antisymmetrization relations for the six-vertex model

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Abstract. We study the relationship between various integral formulas for nonlocal correlation functions of the six-vertex model with domain wall boundary conditions. Specifically, we show how the known representation for the emptiness formation probability can be derived from that for the so-called row configuration probability. A crucial ingredient in the proof is a relation expressing the result of antisymmetrization of some given function with respect to permutations in two sets of its variables in terms of the Izergin-Korepin partition function. This relation generalizes another one obtained by Tracy and Widom in the context of the asymmetric simple exclusion process.

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1. Introduction

In the study of correlation functions of the six-vertex model, and of the closely related Heisenberg XXZ spin chain, representations in terms of multiple integrals play an important role [1–4]. Besides allowing for an exact treatment [5,6], and for asymptotic analysis of the correlation functions [7,8], they stimulated the development of novel algebraic approaches [9–11].

For the six-vertex model with domain wall boundary conditions [12–14], which we consider here, some multiple integral representations are available, for example, for the emptiness formation probability [15]. They have already proved useful in the study of phase separation phenomena in the model, in particular, to obtain the arctic curve (frozen boundary of the limit shape) [16,17].
To study this model in more detail, for example, in order to obtain the limit shape (and not just its frozen boundary), more sensitive correlation functions are necessary. In [18], the so-called row configuration probability was introduced, which can be used as a building block for the calculation of various (both local and nonlocal) correlation functions. In particular, it can be related to the emptiness formation probability by certain sum rule. As already observed in [18], to be verified at the level of the multiple integrals, this sum rule requires a rather non-trivial identity to hold.

The purpose of the present paper is to expose in detail how integral formulas for these two correlation functions are connected. In the proof, we use essentially a relation expressing the result of antisymmetrization of some given function with respect to permutations in two sets of its variables in terms of the Izergin-Korepin partition function, see Proposition 6.

It is to be mentioned that similar relations already appeared in the context of the six-vertex model, see, e.g., [3], as well as in the theory of symmetric polynomials [19–22]. The relation we use here does not seem to be a particular case of any of them, even if sharing the property that its right-hand side is expressible in terms of the Izergin-Korepin partition function. Instead, it appears to be a generalization of the antisymmetrization relation given in [23], in the context of the asymmetric simple exclusion process.

2. Preliminaries

We consider the six-vertex model in the standard formulation in terms of arrows on edges (we follow conventions and notations of the monograph [24] and papers [15, 18]). Our aim is to discuss the model with domain wall boundary conditions. This means that, given an $N \times N$ lattice (a square lattice with $N$ horizontal and $N$ vertical lines), all arrows on the external edges are fixed as follows: on the left and right boundaries they are outgoing (left and right arrows, respectively), while on the top and bottom boundaries they are incoming (down and up arrows, respectively).

In the most general setup (compatible with integrability), the weights of the model depend on the two sets of spectral parameters, $\lambda_1, \ldots, \lambda_N$ and $\nu_1, \ldots, \nu_N$, where the parameters are assumed to be all different within each set. The parameters of the first set are assigned to the vertical lines, from right to left, and the parameters of the second set are assigned to the horizontal lines, from top to bottom. The weights of the vertex being at the intersection of $k$-th horizontal line and $j$-th vertical line are $a_{jk} = a(\lambda_j, \nu_k)$, $b_{jk} = b(\lambda_j, \nu_k)$, and $c_{jk} = c(\lambda_j, \nu_k)$, where

\begin{align}
  a(\lambda, \nu) &= \sin(\lambda - \nu + \eta), \\
  b(\lambda, \nu) &= \sin(\lambda - \nu - \eta), \\
  c(\lambda, \nu) &= \sin 2\eta.
\end{align}

Here, $\eta$ is the so-called crossing parameter, $\Delta = \cos 2\eta$, where $\Delta$ is the invariant of the model [24]. The partition function is denoted by $Z_N(\lambda_1, \ldots, \lambda_N; \nu_1, \ldots, \nu_N)$ and it is symmetric under permutations within each set of parameters, due to integrability of the model via the Yang-Baxter relation.

The partition function was originally studied by Korepin [12] and Izergin [13], who proved that it admits the following representation in terms of an $N \times N$
determinant:

\[ Z_N(\lambda_1, \ldots, \lambda_N; \nu_1, \ldots, \nu_N) = \frac{\prod_{j,k=1}^{N} a(\lambda_j, \nu_k) b(\lambda_j, \nu_k) \prod_{1 \leq j < k \leq N} d(\lambda_k, \lambda_j) d(\nu_j, \nu_k)}{\prod_{1 \leq j < k \leq N} \det [\varphi(\lambda_j, \nu_k)]}. \quad (2) \]

Here,

\[ \varphi(\lambda, \nu) = \frac{c(\lambda, \nu)}{a(\lambda, \nu) b(\lambda, \nu)}, \]

and we also denote

\[ d(\lambda, \lambda') = \sin(\lambda - \lambda'). \quad (3) \]

In the present paper we mostly discuss the model in the homogeneous limit, that can be obtained by letting \( \lambda_1, \ldots, \lambda_N \to \lambda \), and \( \nu_1, \ldots, \nu_N \to \nu \), where, furthermore, since the weights then are just functions of the difference \( \lambda - \nu \), we put \( \nu = 0 \). In the limit, the partition function reads

\[ Z_N = \frac{(ab)^N}{\prod_{j=1}^{N-1} (j!)^2} \det_{1 \leq j, k \leq N} [\varphi^{(j+k-2)}], \quad \varphi^{(n)} \equiv \partial^{n}_\lambda \varphi(\lambda, 0). \quad (4) \]

For quantities of the homogeneous model we use the shorthand notation \( Z_N \equiv Z_N(\lambda, \ldots, \lambda; 0, \ldots, 0) \), and \( a \equiv a(\lambda, 0) \), etc. In discussion of the correlation functions, we mostly use, instead of \( \lambda \) and \( \eta \), the parameters \( t \) and \( \Delta \), which in terms of the Boltzmann weights \( a, b, \) and \( c \) read:

\[ t = \frac{b}{a}, \quad \Delta = \frac{a^2 + b^2 - c^2}{2ab}. \quad (5) \]

A detailed exposition of proofs of (2) and (4) can be found in [14]; alternative derivations are also possible, see, e.g., [15,25].

An important quantity in the study of the correlation functions of the model is the one-point correlation function \( H_N^{(r)} \), describing polarization near the boundary. To be more specific, let us consider the top boundary where all external (vertical) edges, due to the domain wall boundary conditions, carry down arrows. Consider now the next horizontal row of the vertical edges, located between the first and second horizontal lines. Here, among the \( N \) arrows there are exactly \( N - 1 \) down arrows and there is just one up arrow. The function \( H_N^{(r)} \) gives the probability that this up arrow is located at the \( r \)-th vertical edge (recall, that we count vertical lines from right to left). Note, that since there is exactly one up arrow between the first and second horizontal lines, the following sum rule is valid:

\[ \sum_{r=1}^{N} H_N^{(r)} = 1. \quad (6) \]

It is convenient to introduce the corresponding generating function:

\[ h_N(z) = \sum_{r=1}^{N} H_N^{(r)} z^{r-1}. \quad (7) \]

Due to (6), \( h_N(1) = 1 \).
Further, we introduce functions $h_{N,s}(z_1, \ldots, z_s)$, where the second subscript, $s = 1, \ldots, N$, refers to the number of arguments. These functions are defined as

$$ h_{N,s}(z_1, \ldots, z_s) = \prod_{1 \leq j < k \leq s} \left( z_k - z_j \right)^{s-j} \det \left[ z_k^{s-j} (z_k - 1)^{j-1} h_{N-j+1}(z_k) \right]. \quad (8) $$

Note that these functions are symmetric polynomials of degree $N - 1$ in each of their variables, and satisfy the reduction relations

$$ h_{N,s}(z_1, \ldots, z_s) \bigg|_{z_s=1} = h_{N,s-1}(z_1, \ldots, z_{s-1}), \quad s = 1, \ldots, N, $$

with $h_{N,0} \equiv 1$.

In what follows we need the following identity expressing the partition function of the partially inhomogeneous model with $\nu_j = 0$, $j = 1, \ldots, N$, in terms of the generating functions of the boundary correlation functions of the homogeneous models on lattices of the sizes $s \times s$, $s = 1, \ldots, N$.

**Proposition 1.** The following representation is valid:

$$ Z_N(\lambda_1, \ldots, \lambda_N; 0, \ldots, 0) = Z_N(\lambda, \ldots, \lambda; 0, \ldots, 0) \prod_{j=1}^{N} \left( a(\lambda_j, 0) \right)^{N-1} \times h_{N,N}(\gamma(\lambda_1 - \lambda), \ldots, \gamma(\lambda_N - \lambda)), \quad (9) $$

where the function $\gamma(\xi)$, which also depends on $\lambda$ (and $\eta$) as a parameter, is given by

$$ \gamma(\xi) \equiv \gamma(\xi; \lambda) = \frac{a(\lambda, 0) b(\lambda + \xi, 0)}{b(\lambda, 0) a(\lambda + \xi, 0)}. \quad (10) $$

The proof can be found in [15].

3. Nonlocal correlation functions

We discuss here two nonlocal correlation functions. The first one is the so-called emptiness formulation probability (EFP) and it is denoted as $F^{(r,s)}_N$ [15]. The name originates from the spin chain context; in the present model, it gives the probability that all vertices in an $s \times (N - r)$ rectangular region at the top-left corner of the $N \times N$ lattice have all the same configuration of arrows around them. Namely, all these vertices have left and down arrows, exactly matching the boundary conditions imposed on the attached boundaries.

The second correlation function is the so-called row configuration probability (RCP) and it is denoted as $H_N^{(r_1, \ldots, r_s)}$ [18]. This function gives the probability of obtaining a given configuration of arrows on all the $N$ vertical edges located between the $s$-th and $(s + 1)$-th horizontal lines, where there are exactly $s$ up arrows and $N - s$ down arrows. The integers $r_1, \ldots, r_s$ label the positions of these up arrows, and we set $1 \leq r_1 < \cdots < r_s \leq N$. The RCP essentially generalizes the function $H_N^{(r)}$ discussed above to the case of an arbitrary row.

As indicated in [18], in dealing with the RCP it is useful to imagine cutting all the vertical edges between the $s$-th and $(s + 1)$-th horizontal lines of the $N \times N$ lattice, thus separating the lattice into two parts: the “top” one, of size $s \times N$, and the “bottom” one, of size $(N - s) \times N$. The partition functions of the six-vertex
model on these two parts can be denoted as $Z_{r_1,\ldots,r_s}^{\text{top}}$ and $Z_{r_1,\ldots,r_s}^{\text{bot}}$, respectively; clearly,

$$H^{(r_1,\ldots,r_s)}_{N,s} = \frac{1}{Z_N} Z_{r_1,\ldots,r_s}^{\text{top}} Z_{r_1,\ldots,r_s}^{\text{bot}}. \quad (11)$$

Note that, although the partition functions $Z_{r_1,\ldots,r_s}^{\text{top}}$ and $Z_{r_1,\ldots,r_s}^{\text{bot}}$ are in fact very similar objects, the integers $r_1,\ldots,r_s$ play two distinct, complementary, roles in their definitions. For $Z_{r_1,\ldots,r_s}^{\text{top}}$ they indicate the positions of the arrows at the bottom boundary which are reversed with respect to those on the top boundary (which are all down arrows). For $Z_{r_1,\ldots,r_s}^{\text{bot}}$ they indicate the positions of the arrows at the top boundary which have the same orientation with respect to those on the bottom boundary (which are all up arrows).

As mentioned in [18], while the EFP is useful, for example, for studying phase separation phenomena, and for establishing the arctic curve [16], the RCP plays a somewhat more fundamental role since it can be viewed as a building block to compute other correlation functions. In particular, the EFP can be expressed in terms of the RCP, by noting that the EFP can be equivalently defined as the probability of observing the last $N - r$ arrows between the $s$-th and $(s + 1)$-th horizontal lines to be all pointing down. Since the positions of the remaining arrows of the row are not fixed, we have

$$F^{(r,s)}_N = \sum_{1 \leq r_1 < r_2 < \ldots < r_s \leq r} H^{(r_1,\ldots,r_s)}_{N,s}. \quad (12)$$

In [18] it was shown that the relation (12) requires certain cumbersome antisymmetrization relation to be useful in the context of multiple integral representations. In the remaining part of the paper, our aim is to provide all the necessary details which make it possible to recover the known representation for the EFP from the those for $Z_{r_1,\ldots,r_s}^{\text{top}}$ and $Z_{r_1,\ldots,r_s}^{\text{bot}}$ entering (11).

We now recall results for the EFP and RCP in terms of multiple contour integrals. In the notation used below, $C_w$ denotes a simple anticlockwise oriented contour surrounding the point $z = w$ and no other singularity of the integrand, $dz \equiv dz_1 \cdots dz_s$, and $t \in \mathbb{R}^+$ and $\Delta \in \mathbb{R}$ are the parameters of the homogeneous model, see (5).

To elucidate the role of antisymmetrization relations solely on the example of the EFP, we start with the following result established in [15].

**Proposition 2.** The EFP admits the representation

$$F_N^{(r,s)} = (-1)^s \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^s \frac{\left[(t^2 - 2\Delta t)z_j + 1\right]^{s-j}}{z_j^r (z_j - 1)^{s-j+1}} \prod_{1 \leq j < k \leq s} \frac{z_j - z_k}{t^2 z_j z_k - 2\Delta t z_j + 1} h_{N,s}(z_1,\ldots,z_s) \frac{dz}{(2\pi i)^s}, \quad (13)$$

where the function $h_{N,s}(z_1,\ldots,z_s)$ is defined in (8).

The proof is based essentially on the extensive use of the Yang-Baxter relation together with methods from the theory of orthogonal polynomials and properties of the function (7). We refer to [15] for details of the proof.

An apparent drawback of (13) is that the integrand is not symmetric with respect to the permutation of the integration variables. The central result about the EFP is therefore the following, which was also established in [15].
THEOREM 1. For the EFP the following representation is valid:

\[
F_N^{(r,s)} = \frac{(-1)^s}{s!} \frac{Z_s}{a^{s(s-1)}c^s} \int_{C_0} \ldots \int_{C_0} \prod_{j=1}^{s} \frac{1}{z_j} \prod_{1 \leq j < k \leq s} \frac{z_k - z_j}{t^2 z_j z_k - 2\Delta t z_j + 1} \times h_{s,a}(u_1, \ldots, u_s) h_{N,s}(z_1, \ldots, z_s) \frac{dz}{(2\pi i)^s}, \tag{14}
\]

where \( Z_s = Z_s(\lambda_1, \ldots, \lambda; 0, \ldots, 0) \) is the Izergin-Korepin partition function of the homogeneous model on \( s \times s \) lattice, and

\[
u_j = -\frac{z_j - 1}{(t^2 - 2\Delta t)z_j + 1}, \quad j = 1, \ldots, s. \tag{15}\]

The proof is based on the statement of Proposition 2 and certain antisymmetrization relation, which is a special case of the relation established by Kitanine, Maillet, Slavnov and Terras in [3]. We discuss this relation in the next section.

The representation (14) has played a central role in the evaluation of the arctic curve of the model [16,17].

Let us now turn to the integral formulas which determine the RCP, via the relation (11). We first mention that for \( Z_{r_1, \ldots, r_s} \), we have:

PROPOSITION 3. For the partition function \( Z_{r_1, \ldots, r_s}^{bot} \), the following representation is valid:

\[
Z_{r_1, \ldots, r_s}^{bot} = Z_N \prod_{j=1}^{s} \frac{t_j^{-r_j}}{a^{s(s-1)}c^s} \int_{C_0} \ldots \int_{C_0} \prod_{j=1}^{s} \frac{1}{z_j} \prod_{1 \leq j < k \leq s} \frac{z_k - z_j}{t^2 z_j z_k - 2\Delta t z_j + 1} \times h_{N,s}(z_1, \ldots, z_s) \frac{dz}{(2\pi i)^s}. \tag{16}
\]

The proof of this result is again essentially based on the use of the Yang-Baxter relation together with methods from the theory of orthogonal polynomials and properties of the function (7). We refer to [18] for details.

In turn, for \( Z_{r_1, \ldots, r_s}^{top} \), we have:

PROPOSITION 4. For the partition function \( Z_{r_1, \ldots, r_s}^{top} \), the following representation is valid:

\[
Z_{r_1, \ldots, r_s}^{top} = \int_{C_1} \ldots \int_{C_1} \prod_{j=1}^{s} (w_j - 1)^s \prod_{1 \leq j < k \leq s} \left[(w_j - w_k)(t^2 w_j w_k - 2\Delta t w_j + 1)\right] \frac{dw}{(2\pi i)^s}. \tag{17}
\]

This result follows from the interpretation of \( Z_{r_1, \ldots, r_s}^{top} \) as a component of the off-shell Bethe wave-function when the homogeneous limit is taken [18].

It is worth emphasizing that the representations (16) and (17) are not directly related in any simple way. For instance, resorting to the crossing symmetry of the model, one can indeed easily derive two more representations for \( Z_{r_1, \ldots, r_s}^{top} \) and \( Z_{r_1, \ldots, r_s}^{bot} \), but these would be \( (N-s) \)-fold (rather than \( s \)-fold) multiple integrals.

Our aim below is to show how the statement of Theorem 1 may be derived from Propositions 3 and 4 via (11) and (12). It turns out that an essential ingredient in
the derivation is a relation which expresses antisymmetrization of some expression with respect to permutations of two sets of variables in terms of the Izergin-Korepin partition function.

4. Antisymmetrization relations

Given a function $f(z_1, \ldots, z_s)$, introduce the antisymmetrizer

$$A_{z_1, \ldots, z_s}[f(z_1, \ldots, z_s)] = \sum_{\sigma} (-1)^{[\sigma]} f(z_{\sigma_1}, \ldots, z_{\sigma_s}),$$

where the sum is taken over permutations $\sigma : 1, \ldots, s \mapsto \sigma_1, \ldots, \sigma_s$, with $[\sigma]$ denoting the parity of $\sigma$.

Here, we discuss two antisymmetrization relations which appear to be of importance for calculation of the EFP. The first one originates from the following relation established and proved by Kitanine, Maillet, Slavnov, and Terras in [3].

Proposition 5 ([3], Proposition C1). For the functions $a(\lambda, \nu)$, $b(\lambda, \nu)$, and $d(\lambda, \lambda')$ given by (1) and (3), and the function $e(\lambda, \lambda') = \sin(\lambda - \lambda' + 2\eta)$, the following relation holds

$$A_{z_1, \ldots, z_s} \left[ \prod_{j=1}^{s} \frac{1}{u_j} \prod_{1 \leq j < k \leq s} \left( t^2 z_j z_k - 2\Delta t z_k + 1 \right) \right] = (-1)^{s(s-1)} \frac{Z_s}{a^{s(s-1)} e^s} \prod_{1 \leq j < k \leq s} (z_k - z_j) \prod_{j=1}^{s} \frac{1}{u_j} h_{s,s}(u_1, \ldots, u_s).$$

Here (and everywhere below) we assume that $u_j \equiv u(z_j)$, with the function $u(z_j)$ defined by the right-hand side of (15).

Clearly, the relation (21) essentially implies the validity of (14), given (13); for more details we refer to [15].

The second antisymmetrization relation appears to be relevant for obtaining the same result for the EFP from the RCP. It can be formulated as follows.
Proposition 6. For \( \tau \in \mathbb{C} \), the following relation is valid

\[
A_{x_1,\ldots,x_s} A_{y_1,\ldots,y_s} \left[ \prod_{j=1}^{s} \frac{(x_j y_j)^{s-j}}{1 - \prod_{l=1}^{j} x_l y_l} \prod_{1 \leq j < k \leq s} (x_j x_k + \tau x_k + 1)(y_j y_k + \tau y_k + 1) \right] = \prod_{j,k=1}^{s} (x_j + y_k + \tau x_j y_k) \det_{1 \leq j,k \leq s} [\psi(x_j, y_k)], \tag{22}
\]

where

\[
\psi(x, y) = \frac{1}{(1 - xy)(x + y + \tau xy)}.
\]

We prove it in Appendix \( \mathbb{A} \).

As shown below, the right-hand side of the relation \( \text{(22)} \) can be expressed in terms of the Izergin-Korepin partition function. In fact, it essentially coincides with the right-hand side of some Cauchy-like identity for the Hall-Littlewood polynomials considered in \([21]\), see equation \((25)\) therein. As for the left-hand side of the relation \( \text{(25)} \) in \([21]\), it can also be rewritten as the result of double antisymmetrization of some function. However, even in the case \( s = 2 \), the function under the antisymmetrization symbol appears to be different from that standing in \( \text{(22)} \).

By its left-hand side, the relation \( \text{(22)} \) reminds another antisymmetrization relation, first established and proved by Tracy and Widom in \([23]\), see \((1.6)\) therein, in the context of the asymmetric simple exclusion process. Indeed, it appears that the relation \( \text{(22)} \) reduces to that of \([23]\) in the limit where \( y_j \to t^{-1} \), \( j = 1,\ldots,s \), upon the identification of the parameters \( t = \sqrt{q/p} \) and \( \tau = -1/\sqrt{pq} \) in terms of the asymmetric simple exclusion process rates \( p \) and \( q \), \( p + q = 1 \). We give details in Appendix \( \mathbb{B} \).

To discuss the relation of \( \text{(22)} \) with Izergin-Korepin partition function, it is convenient to introduce the notation

\[
W_s(x_1,\ldots,x_s; y_1,\ldots,y_s) = \prod_{j,k=1}^{s} (x_j + y_k + \tau x_j y_k) \det_{1 \leq j,k \leq s} [\psi(x_j, y_k)]. \tag{23}
\]

Let us identify \( \tau = -2\Delta \), and set

\[
x_j = \frac{a(\lambda_j, \zeta + \eta)}{b(\lambda_j, \zeta + \eta)}, \quad y_j = \frac{a(\zeta, \nu_j)}{b(\zeta, \nu_j)}, \quad j = 1,\ldots,s, \tag{24}
\]

where \( \zeta \) is an arbitrary parameter to be fixed later. Then we have

\[
\psi(x_j, y_k) = \frac{1}{c^s} [b(\lambda_j, \zeta + \eta)b(\zeta, \nu_k)]^2 \varphi(\lambda_j, \nu_k)
\]

and therefore

\[
\det_{1 \leq j,k \leq s} [\psi(x_j, y_k)] = \frac{1}{c^{3s}} \prod_{j=1}^{s} [b(\lambda_j, \zeta + \eta)b(\zeta, \nu_j)]^2 \det_{1 \leq j,k \leq s} [\varphi(\lambda_j, \nu_k)].
\]

Note that, in the right-hand side, the parameter \( \zeta \) enters only the prefactor and not the function \( \varphi(\lambda_j, \nu_k) \). Plugging this into \( \text{(23)} \) yields

\[
W_s(x_1,\ldots,x_s; y_1,\ldots,y_s) = (-1)^s \frac{\prod_{j=1}^{s} b(\lambda_j, \zeta + \eta)b(\zeta, \nu_j)}{c^{3s} \prod_{j,k=1}^{s} b(\lambda_j, \nu_k)} \times Z_s(\lambda_1,\ldots,\lambda_s; \nu_1,\ldots,\nu_s). \tag{25}
\]
Let us now consider (25) in the partial homogeneous limit, where \( \nu_j = 0, j = 1, \ldots, s \). To make a contact with our previous discussion let us, furthermore, identify \( \zeta = \lambda \); so that, when comparing (24) with (19), we have in the limit

\[
x_j = tz_j, \quad y_j = t^{-1}, \quad j = 1, \ldots, s.
\]

Using (9), we thus obtain

\[
W_s(tz_1, \ldots, tz_s; t^{-1}, \ldots, t^{-1}) = \frac{(-1)^s Z_s}{c^s b(s - 1)} \prod_{j=1}^{s} \frac{1}{(z_j - 1)u_j^{s-1}} h_{s,s}(u_1, \ldots, u_s). \tag{26}
\]

Here, as usual, \( u_j \)'s and \( z_j \)'s are related by (15).

5. From RCP to EFP

Let us now consider the derivation of (14) from (16) and (17), basing on (11) and (12). For convenience, we change the integration variables \( z_j \mapsto x_j/t \) in (16), that yields

\[
Z_{r_1, \ldots, r_s}^{\text{bot}} = \frac{Z_N}{a^s(N-1)c^s} \int_{C_0} \cdots \int_{C_0} \prod_{j=1}^{s} \frac{1}{x_j} \prod_{1 \leq j < k \leq s} \frac{x_k - x_j}{x_j x_k - 2 \Delta x_j + 1} \times h_{N,s}(\frac{x_1}{t}, \ldots, \frac{x_s}{t}) \frac{d^s x}{(2\pi)^s}, \tag{27}
\]

and also we change \( w_j \mapsto 1/(ty_j) \) in (17), that yields

\[
Z_{r_1, \ldots, r_s}^{\text{top}} = c^s a^s(N-1) \int_{C_{1/s}} \cdots \int_{C_{1/s}} \prod_{j=1}^{s} \frac{1}{(ty_j - 1)^s y_j^{r_j s - 1}} \times \prod_{1 \leq j < k \leq s} [(y_k - y_j)(y_j y_k - 2 \Delta y_k + 1)] \frac{d^s y}{(2\pi)^s}. \tag{28}
\]

Now, putting all together in (11), and using (12), we have

\[
F^{(r,s)}_{N} = \int_{C_{1/s}} \cdots \int_{C_{1/s}} d^s y = \frac{1}{(2\pi)^s} \int_{C_0} \cdots \int_{C_0} \prod_{j=1}^{s} y_j^{r_j s - 1}(ty_j - 1)^s \times \prod_{1 \leq j < k \leq s} \frac{(y_k - y_j)(y_j y_k - 2 \Delta y_k + 1)(x_k - x_j)}{x_j x_k - 2 \Delta x_j + 1} h_{N,s}(\frac{x_1}{t}, \ldots, \frac{x_s}{t}) \times \sum_{1 \leq r_1 < r_2 < \cdots < r_s \leq r} \prod_{j=1}^{s} \frac{1}{(x_j y_j)^{r_j}(2\pi)^s}. \tag{29}
\]

To prove that this representation indeed simplifies to (14), one has to: 1) make the multiple sum, 2) perform symmetrization of the integrand, and 3) evaluate integrations in one set of variables.

The first task can be accomplished by the following.

**Lemma 1.** Let variables \( z_1, \ldots, z_s \) all take values inside the unit circle in \( \mathbb{C} \), that is \( |z_j| < 1, j = 1, \ldots, s \), then

\[
\sum_{-\infty < r_1 < r_2 < \cdots < r_s \leq r} \prod_{j=1}^{s} \frac{1}{z_j^{r_j}} = \prod_{j=1}^{s} \frac{1}{z_j^{r_j - s + j}(1 - \prod_{l=1}^{s-1} z_l)}.
\]
Proof. By successively performing the summations in the left-hand side of (29) with respect to $r_1, \ldots, r_s$, in that order, one arrives at the expression in the right-hand side.

To apply Lemma 1 we first remark that the integral representation (27) vanishes whenever one of the $r_j$'s is negative. In (28) we may thus replace the sum over the values $1 \leq r_1 < r_2 \cdots < r_s \leq r$ by the sum over the values $-\infty < r_1 < r_2 \cdots < r_s \leq r$. Then we can apply (29) by setting $z_j = x_jy_j$, $j = 1, \ldots, s$. As a result, the expression (28) simplifies to:

$$F_{N}^{(r,s)} = \frac{1}{(s!)^2} \oint_{C_1} \cdots \oint_{C_r} \frac{d^s y}{(2\pi i)^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{1}{(ty_j - 1)^{y_j + 1} x_j^{y_j + s}} \left( y_k - y_j \right)^2 \left( x_k - x_j \right)$$

$$\times \prod_{1 \leq j < k \leq s} \frac{(x_k - x_j)^2 (y_k - y_j)^2}{(x_j x_k - 2\Delta x_j + 1)(x_j x_k - 2\Delta x_k + 1)} \times h_{N,s} \left( \frac{x_1}{t}, \ldots, \frac{x_s}{t} \right) \frac{d^s x}{(2\pi i)^s}.$$  (30)

The second step relies on Proposition 6. Indeed, the symmetrization of the integrand of (30) with respect to permutations in the two sets of the integration variables essentially reduce to the left-hand side of (6) with $\tau = -2\Delta$. Using the notation (23) for the right-hand side of (22), we have

$$F_{N}^{(r,s)} = \frac{1}{(s!)^2} \oint_{C_1} \cdots \oint_{C_r} \frac{d^s y}{(2\pi i)^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{1}{(ty_j - 1)^{y_j + 1} x_j^{y_j + s - 1}} \left( y_k - y_j \right)^2 \left( x_k - x_j \right)$$

$$\times \prod_{1 \leq j < k \leq s} \frac{(x_k - x_j)^2 (y_k - y_j)^2}{(x_j x_k - 2\Delta x_j + 1)(x_j x_k - 2\Delta x_k + 1)} \times W_s(x_1, \ldots, x_s; y_1, \ldots, y_s) h_{N,s} \left( \frac{x_1}{t}, \ldots, \frac{x_s}{t} \right) \frac{d^s x}{(2\pi i)^s}.$$

At the last step we perform integrals over the variables $y_1, \ldots, y_s$. For this purpose we use the following.

**Lemma 2.** For an arbitrary symmetric function $\Phi(y_1, \ldots, y_s)$, regular in each of its variables at the point $y = w$, one has

$$\oint_{C_w} \cdots \oint_{C_w} \prod_{j=1}^{s} \frac{1}{(y_j - w)^s} \prod_{1 \leq j < k \leq s} (y_k - y_j)^2 \Phi(y_1, \ldots, y_s) \frac{d^s y}{(2\pi i)^s} = (-1)^{s(s-1)/2} \frac{d^s \Phi(w, \ldots, w)}{s!}.$$  (30)

Proof. In evaluating the residues recursively in each variable, it is easily seen that only those terms survive where differentiations all act on the squared Vandermonde product, and the result follows.
Applying Lemma 2 with $w = t^{-1}$, we obtain

$$F_N^{(r,s)} = \frac{e^s(r-1)}{s!} \oint_{C_0} \cdots \oint_{C_0} \frac{1}{\prod_{j,k=1}^s x_j x_k - 2\Delta x_j + 1} \frac{x_j - x_k}{x_j} \prod_{j \neq k} x_j x_k - 2\Delta x_j + 1 \times W_s(x_1, \ldots, x_s; t^{-1}, \ldots, t^{-1}) h_{N,s}(\frac{x_1}{t}, \ldots, \frac{x_s}{t}) (2\pi i)^s.$$

Finally, making back the change of the integration variables $x_j \mapsto tz_j$, $j = 1, \ldots, s$, and using (26), we immediately arrive to (14).

In conclusion, in the present paper we have studied the relationship between various integral formulas for nonlocal correlation functions of the six-vertex model with domain wall boundary conditions. Specifically, we have shown how the known result for the EFP can be derived from that for the RCP. A crucial role in the proof has been played by the relation (22). This relation may appear useful in the study of other correlation functions of the model, and it is definitely not purely specific of the domain wall boundary conditions. On the other hand, calculation of other correlation functions than EFP may require the use of different but similar relations, and establishing them can be the subject of further study.

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Appendix A. Proof of Proposition 6

Here we prove Proposition 6. We apply induction in $s$, using the symmetries of (22) in the involved variables and comparing singularities of both sides. Our proof goes in many respects along the lines of that of Proposition 5, given in (3) (see Appendix C therein).

Denote by $L_s = L_s(x_1, \ldots, x_s; y_1, \ldots, y_s)$ and $R_s = R_s(x_1, \ldots, x_s; y_1, \ldots, y_s)$ the left- and right-hand sides of (22), respectively. For $s = 1$, we immediately verify that $L_1 = R_1$. Our aim is, assuming $L_{s-1} = R_{s-1}$, to show that $L_s = R_s$.

First we show that $L_s$ can be written in the form

$$L_s = \frac{\prod_{1 \leq j < k \leq s}(x_k - x_j)(y_k - y_j)}{(1 - \prod_{i=1}^s x_i y_i) \prod_{j,k=1}^s (1 - x_j y_k)} P_s(x_1, \ldots, x_s; y_1, \ldots, y_s), \quad (A.1)$$

where $P_s(x_1, \ldots, x_s; y_1, \ldots, y_s)$ is a polynomial of degree $s$ in each variable, separately symmetric under permutations of the variables within each set. For this, we
split the sums over permutations in the left-hand side of (22) as follows

\[
\left(1 - \prod_{i=1}^{s} x_i y_i\right) L_s = \sum_{j,k=1}^{s} \sum_{\sigma = \sigma_1 \cdots \sigma_s, \sigma_j = k} (-1)^{\sigma} \prod_{l=1}^{s-1} \frac{(x_{\sigma_l} y_{\rho_l})^{s-1}}{1 - \prod_{l=1}^{s-1} x_{\sigma_l} y_{\rho_l}} \times \prod_{1 \leq m < n \leq s} (1 + x_{\sigma_m} x_{\sigma_n} + \tau x_{\sigma_n})(1 + y_{\rho_m} y_{\rho_n} + \tau y_{\rho_n}) \\
= \sum_{j,k=1}^{s} (-1)^{j+k} L_{s-1}(\{x_j\} \setminus y_k) \times \prod_{m=1}^{s} x_m(1 + x_m x_j + \tau x_j) \prod_{n=1}^{s} y_n(1 + y_n y_k + \tau y_k). \tag{A.2}
\]

Here, \(L_{s-1}(\{x_j\} \setminus y_k)\) denotes the quantity \(L_{s-1}\) which is constructed from the sets \(x_1, \ldots, x_s\) and \(y_1, \ldots, y_s\) with \(x_j\) and \(y_k\) removed, respectively. Noticing that can be written in the form

\[
R_s = \prod_{1 \leq j < k \leq s} (x_k - x_j)(y_k - y_j) \prod_{j,k=1}^{s} (1 - x_j y_k) Q_{s-1}(x_1, \ldots, x_s; y_1, \ldots, y_s), \tag{A.3}
\]

where \(Q_{s-1}(x_1, \ldots, x_s; y_1, \ldots, y_s)\) is a polynomial of degree \(s - 1\) in each variable, separately symmetric under permutations of the variables within each set, and applying the inductive step \(L_{s-1}(\{x_j\} \setminus y_k) = R_{s-1}(\{x_j\} \setminus y_k)\), we conclude that the expression in (A.2) has poles only at the points \(x_j = y_k^{-1}\) and therefore \(L_s\) can be written in the form (A.1).

Next we must show that the expressions (A.1) for \(L_s\) and (A.3) for \(R_s\) are equal; to this purpose, it suffices to show that they coincide at \(s + 1\) distinct values of \(x_1\). We show below that both expressions have equal residues at \(x_1 = y_j^{-1}, j = 1, \ldots, s\), and that they coincide for \(x_1 = 0\).

Let us evaluate the residue at \(x_1 = y_1^{-1}\). For \(R_s\), we find

\[
\text{Res}_{x_1=y_1^{-1}} R_s = -y_1^{-1} \prod_{i=2}^{s} (y_1^{-1} + y_i + \tau y_1^{-1} y_i)(x_i + y_1 + \tau x_i y_1) R_{s-1}(\{x_1\} \setminus y_1). \tag{A.4}
\]

In the case of \(L_s\), it follows from the antisymmetrization that the pole at \(x_1 = y_1^{-1}\) is present only in the terms of the sum in which \(x_1\) and \(y_1\) are not permuted. Restricting to these terms and setting \(x_1 = y_1^{-1}\), we find

\[
\text{Res}_{x_1=y_1^{-1}} L_s = -y_1^{-1} \prod_{i=2}^{s} (y_1^{-1} + y_i + \tau y_1^{-1} y_i)(x_i + y_1 + \tau x_i y_1) L_{s-1}(\{x_1\} \setminus y_1). \tag{A.4}
\]

Using the inductive step, we conclude that

\[
\text{Res}_{x_1=y_1^{-1}} L_s = \text{Res}_{x_1=y_1^{-1}} R_s.
\]

By symmetry, the residues at \(x_1 = y_j^{-1}, j = 2, \ldots, s\), coincide as well.

Finally, it remains to check that \(L_s\) and \(R_s\) coincide at \(x_1 = 0\). For \(R_s\), we have

\[
R_s \big|_{x_1=0} = \prod_{i=1}^{s} y_i \prod_{j=2}^{s} \prod_{k=1}^{s} (x_j + y_k + \tau x_j y_k) \prod_{1 \leq j, k \leq s} \left\{ \begin{array}{c}
\begin{cases}
y_k^{-1} y_j & j = 1 \\
\psi(x_j, y_k) & j \geq 2
\end{cases}
\end{array} \right\}. \tag{A.4}
\]
For $L_s$ we use (A.2), where, when $x_1 = 0$, the sum over $k$ reduces only to the term with $j = 1$, because all the other terms have the factor $x_1$, and we obtain

$$
L_s \Big|_{x_1=0} = \prod_{i=2}^{n} x_i \sum_{k=1}^{s} (-1)^{k+1} \prod_{j=1 \atop j \neq k}^{s} y_j (1 + y_j y_k + \tau y_k) R_{s-1}(\setminus x_1; \setminus y_k),
$$

which can be written in determinantal form

$$
L_s \Big|_{x_1=0} = \prod_{i=1}^{s} y_i \prod_{j=2}^{s} (x_j + y_j + \tau x_j y_k) \det_{1 \leq j, k \leq s} \begin{cases} v_k & j = 1 \\ \psi(x_j, y_k) & j \geq 2 \end{cases} \quad (A.5)
$$

with

$$
v_k = y_k^{-1} \prod_{i=2}^{s} x_i \prod_{j=1 \atop j \neq k}^{s} (1 + y_j y_k + \tau y_k).$$

As a function of $y_k$, the quantity $v_k$ vanishes as $y_k \to \infty$ and has simple poles at $y_k = 0$ and $y_k = -x_j/(1 + \tau x_j)$, $j = 2, \ldots, s$. Therefore, the following pole expansion is valid

$$
v_k = y_k^{-1} \prod_{j=2}^{s} x_i \prod_{i=2 \atop i \neq j}^{s} x_i \prod_{j=1 \atop j \neq k}^{s} (1 - x_j y_k) \frac{1}{(1 - x_j y_k)(x_j + y_k + \tau x_j y_k)} \quad (A.6)
$$

The last factor in (A.6) can be recognized as the function $\psi(x_j, y_k)$. This implies equality of the determinants in (A.4) and (A.5), and therefore

$$
L_s \Big|_{x_1=0} = R_s \Big|_{x_1=0},
$$

that completes the proof of Proposition 6.

**Appendix B. The Tracy-Widom relation**

In [23], Tracy and Widom obtained the following antisymmetrization relation (see [23], equation (1.6)):

$$
A_{z_1,\ldots,z_s} \left[ \prod_{j=1}^{s} \frac{z_j^{s-j}}{1 - \prod_{i=1}^{s} z_i^{1 \leq i < j \leq s}} \prod_{i=1}^{s} (q z_j z_k - z_k + p) \right] = p^{s(s-1)/2} \prod_{j=1}^{s} \frac{1}{1 - z_j} \prod_{1 \leq i < j \leq s} (z_j - z_k), \quad (B.1)
$$

where the parameters $p$ and $q$ obeying the condition $p+q = 1$. In the context of the asymmetric simple exclusion process $p$ and $q$ are the transition rates of the model.

Here we show that the antisymmetrization relation (22) reduces to (B.1), by setting

$$
x_j = t z_j, \quad y_j \to t^{-1}, \quad j = 1, \ldots, s, \quad (B.2)
$$

and

$$
t = \sqrt{q/p}, \quad \tau = -\frac{1}{\sqrt{pq}}.
$$

The condition $p + q = 1$, required for (B.1) to hold, now reads

$$
t^2 + \tau t + 1 = 0. \quad (B.3)
$$

If we take $t$ as an independent parameter, then $\tau = -t - t^{-1}$. 

Before turning to the calculation of both sides of (22) under the condition (B.3) and in the limit (B.2), it is useful to note that these operations commute, as it follows from the formula (A.1). It turns out that in evaluating the left-hand side it is convenient first to assume the condition (B.3) and next perform the limit (B.2), but vice versa for the right-hand side.

Considering the relation (22), we divide it by the Vandermonde product in y’s and perform the change of the variables $x_j = tz_j$, $y_j = t^{-1}(1 + \epsilon_j)$, $j = 1, \ldots, s$. In the $\epsilon_j \to 0$ limit, $j = 1, \ldots, s$, the relation reads

$$
\lim_{t \to 0, \epsilon_1, \ldots, \epsilon_s \to 0} \prod_{1 \leq j < k \leq s} \frac{1}{\epsilon_j - \epsilon_k} A_{z_1, \ldots, z_s} A_{\epsilon_1, \ldots, \epsilon_s} \left[ \prod_{j=1}^{s} \frac{[z_j(1 + \epsilon_j)]^{s-j}}{1 - \prod_{l=1}^{j} z_l(1 + \epsilon_l)} \prod_{1 \leq j < k \leq s} \left( t^2 z_j z_k + \tau t z_k + 1 \right) \left[ 1 + \tau t^{-1} + t^{-2} + t^{-2} \epsilon_j + \left( \tau t^{-1} + t^{-2} \right) \epsilon_k + t^{-2} \epsilon_j \epsilon_k \right] \right] = W_s(tz_1, \ldots, tz_s; t^{-1}, \ldots, t^{-1}) \prod_{1 \leq j < k \leq s} (z_j - z_k). \quad (B.4)
$$

Our aim is to evaluate both sides of this relation under the condition (B.3), that is, in the case where $\tau = -t - t^{-1}$.

As for the left-hand side of (B.4), the evaluation of the limit is in general a cumbersome task, because all terms under the antisymmetrization symbol contribute to the leading order. If the condition (B.3) is imposed, then only the last factor in the double product contributes. Therefore, setting $\tau = -t - t^{-1}$, at leading order we have

$$
\lim_{t \to 0, \epsilon_1, \ldots, \epsilon_s \to 0} \prod_{1 \leq j < k \leq s} \frac{1}{\epsilon_k - \epsilon_j} A_{z_1, \ldots, z_s} A_{\epsilon_1, \ldots, \epsilon_s} \left[ \prod_{j=1}^{s} \frac{[z_j(1 + \epsilon_j)]^{s-j}}{1 - \prod_{l=1}^{j} z_l} \prod_{1 \leq j < k \leq s} \left( t^2 z_j z_k - (1 + t^2) z_k + 1 \right) \left( t^{-2} \epsilon_j - \epsilon_k \right) \right].
$$

Now, the antisymmetrization over $\epsilon_1, \ldots, \epsilon_s$ can be done explicitly.

**Lemma 3.** The following antisymmetrization relation is valid

$$
A_{\epsilon_1, \ldots, \epsilon_s} \left[ \prod_{1 \leq j < k \leq s} (\epsilon_j - t^2 \epsilon_k) \right] = \prod_{j=1}^{s} \frac{1 - t^2 j}{1 - t^2} \prod_{1 \leq j < k \leq s} (\epsilon_j - \epsilon_k). \quad (B.5)
$$

**Proof.** The result of the antisymmetrization of a polynomial in $s$-variables of degree at most $s - 1$ is proportional to a Vandermonde product, and thus one just needs to determine the overall constant. Denoting the left-hand side by $D_s = D_s(\epsilon_1, \ldots, \epsilon_s)$, and considering its value as $\epsilon_s \to 0$, we have

$$
D_s \big|_{\epsilon_s=0} = \sum_{l=1}^{s} (-1)^{s-l} \prod_{\sigma=(1)}^{l} (-1)^{\sigma} \prod_{1 \leq j < k \leq s} (\epsilon_{\sigma_j} - t^2 \epsilon_{\sigma_k}) \prod_{j=1}^{l-1} \epsilon_{\sigma_j} \prod_{j=l+1}^{s} (t^{-2} \epsilon_{\sigma_j})
$$

$$
= D_{s-1} \prod_{j=1}^{s} \epsilon_j \sum_{l=1}^{s-1} t^{2(l-1)}.
$$
Denoting $C_s$ the overall constant, $C_1 = 1$, we thus get
\[ C_s = C_{s-1} \sum_{l=1}^{s} t^{2(l-1)} = C_{s-1} \frac{1 - t^{2s}}{1 - t^2} = \prod_{j=1}^{s} \frac{1 - t^{2j}}{1 - t^2}, \]
and (B.5) follows. □

As a result, we get the following expression
\[ \frac{1}{t^{s(s-1)}} \prod_{j=1}^{s} \frac{1 - t^{2j}}{1 - t^2} A_{z_1, \ldots, z_s} \left[ \prod_{j=1}^{s} \frac{z_j^{s-j}}{1 - \prod_{l=1}^{j} z_l^{1 \leq j < l \leq s}} \prod_{1 \leq j < k \leq s} (t^2 z_j z_k - (1 + t^2) z_k + 1) \right] \]
for the left-hand side of (B.4) under the condition (B.3).

As for the right-hand side of (B.4), it turns out that the determinant in (23) can be evaluated explicitly, provided that (B.2) and (B.3) are fulfilled. Consider first the function $W_s(x_1, \ldots, x_s, y_1, \ldots, y_s)$ in the limit where all $y_1, \ldots, y_s$ tend to the same value $y$, while $x_1, \ldots, x_s$ remain different from each other. A standard calculation gives
\[ W_s(x_1, \ldots, x_s; y, \ldots, y) = \prod_{j=1}^{s} (x_j + y + \tau x_j y^s) \frac{\det}{\prod_{1 \leq j < k \leq s} (x_k - x_j)} \left[ \frac{1}{(k-1)!} \partial^{k-1}_y \psi(x_j, y) \right]. \]

Using for the function $\psi(x, y)$ the identity
\[ \frac{1}{1 - xy} = \frac{1}{1 + \tau x + x^2} \left\{ \frac{1}{y + x(1 + \tau x)^{-1}} - \frac{1}{y - x^{-1}} \right\}, \]
we get
\[ \frac{1}{(k-1)!} \partial^{k-1}_y \psi(x, y) = \frac{(-1)^{k-1}}{1 + \tau x + x^2} \left\{ \frac{1}{(y + x(1 + \tau x)^{-1})^k} - \frac{1}{(y - x^{-1})^k} \right\}. \]

Now, setting in the last expression $x = tz$, $y = t^{-1}$, and $\tau = -t - t^{-1}$, we obtain
\[ \left. \frac{1}{(k-1)!} \partial^{k-1}_y \psi(tz, y) \right|_{y=t^{-1}, \tau=-t^{-1}} = \frac{(-1)^{k}}{(1 - z)(1 - t^2 z)} \left\{ \frac{t^k [1 + (1 + t^2) z]^k}{(1 - z)^k} - \frac{(tz)^k}{(z - 1)^k} \right\} = \frac{t^k \{ z^k - [(1 + t^2) z - 1]^k \}}{(1 - z)^{k+1}(1 - t^2 z)}. \]

Therefore, applying (B.2) and (B.3) to (B.7) yields
\[ W_s(tz_1, \ldots, tz_s; t^{-1}, \ldots, t^{-1}) \bigg|_{\tau=-t^{-1}} = \frac{1}{t^{s(s-1)}} \prod_{1 \leq j < k \leq s} \frac{1}{z_k - z_j} \prod_{j=1}^{s} \frac{1}{1 - z_j} \times \det_{1 \leq j, k \leq s} \left[ (1 - z_j)^{s-k} \frac{z_j^k - [(1 + t^2) z_j - 1]^k}{1 - t^2 z_j} \right]. \]

The determinant in (B.8) is proportional to the Vandermonde product, because the entries of the matrix have the form $P_k(z_j)$, where $P_k(z)$, $k = 1, \ldots, s$, are all polynomials of degree $s - 1$. The overall constant can be fixed by considering the
determinant, for example, at $z_s = 1$, along the lines of Lemma 3. In this way, we obtain

$$
\det_{1 \leq j, k \leq s} \left[ (1 - z_j)^{s-k} \frac{z_j^k}{1 - t^2 z_j} - \frac{(1 + t^2) z_j - 1}{1 - t^2} \right] = \prod_{j=1}^{s} \frac{1 - t^{2j}}{1 - t^2} \prod_{1 \leq j < k \leq s} (z_k - z_j),
$$

and hence

$$
W_s(tz_1, \ldots, tz_s; t^{-1}, \ldots, t^{-1}) \big|_{\rho = -t^{-1}} = \frac{1}{t^{s(s-1)}} \prod_{j=1}^{s} \frac{1 - t^{2j}}{1 - t^2} \prod_{1 \leq j < k \leq s} \frac{1}{1 - z_j}. 
$$

As a result, we get the following expression

$$
\frac{1}{t^s} \prod_{j=1}^{s} \frac{1 - t^{2j}}{1 - t^2} \prod_{1 \leq j < k \leq s} \frac{1}{1 - z_j} \prod_{1 \leq j \leq s} (z_j - z_k) \quad \text{(B.9)}
$$

for the right-hand side of (B.4) under the condition (B.3).

Finally, equating (B.6) and (B.9) and setting $t = \sqrt{q/p}$, with $p + q = 1$, we arrive at the relation (B.1).

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