Yet another version of Mumford’s theorem

Robert Laterveer

Abstract. The aim of this note is to provide a variant statement of Mumford’s theorem. This variant states that for a general variety, all Chow groups are “as large as possible”, in the sense that they cannot be supported on a divisor.

Mathematics Subject Classification (2010). Primary 14C15; Secondary 14C25.

Keywords. Algebraic cycles, Chow groups.

1. Introduction

Mumford’s theorem [9] asserts that for a general variety over $\mathbb{C}$, the Chow group of 0–cycles is very large. One version of Mumford’s theorem states that for a variety $X$ with geometric genus $p_g(X)$ non–zero, the Chow group $A^nX$ is not supported on any closed subvariety:

Theorem 1.1. (Bloch–Srinivas [4]) Let $X$ be a smooth projective variety of dimension $n$, and suppose $A^nX_{\mathbb{Q}}$ is supported on a divisor. Then $p_g(X) = 0$.

Since the seminal paper [4], a plethora of variant statements and generalizations have seen the day (cf. [17] Chapter 3) for a recent and comprehensive overview of the field). The modest aim of this short expository note is to provide yet one more variant statement, showing that for a general variety, all Chow groups are very large. The price to pay for starting out not with 0–cycles but with cycles of arbitrary codimension $i$ is that we need to assume the standard Lefschetz conjecture $B(X)$. Here is the main result of this note:

Theorem 1.2. Let $X$ be a smooth projective variety, and suppose $B(X)$ is true. Suppose there is an $i$ such that the Chow group $A^i(X)_{\mathbb{Q}}$ is supported on a divisor. Then the cohomology group $H^i(X, \mathbb{Q})$ is supported on a divisor.

This can be used to provide instances of varieties for which all Chow groups are very large:
Corollary 1.3. Let \( X \) be an abelian variety. Then no Chow group \( A^i(X)_\mathbb{Q} \) is supported on a divisor.

More examples of this type are given below (Corollary 3.2); the same statement holds for any variety for which one knows \( B(X) \) and whose Hodge diamond is of maximal width. This is not surprising, and probably known to experts, yet we couldn’t find a reference. Closely related results appear in work of Lewis \([7, 8]\) and Schoen \([10]\), yet their statements (as well as the proofs) are slightly different from ours.

The present note was written while preparing for the Strasbourg “groupe de travail” based on the book \([17]\). I’d like to thank the participants of this groupe de travail for a very pleasant and stimulating atmosphere.

Convention. In this note, the word variety refers to a smooth projective algebraic variety over \( \mathbb{C} \).

2. The Lefschetz standard conjecture

Let \( X \) be a smooth projective variety of dimension \( n \), and \( h \in H^2(X, \mathbb{Q}) \) the class of an ample line bundle. The hard Lefschetz theorem asserts that the map

\[
L^{n-i} : H^i(X, \mathbb{Q}) \to H^{2n-i}(X, \mathbb{Q})
\]

obtained by cupping with \( h^{n-i} \) is an isomorphism, for any \( i < n \). One of the standard conjectures asserts that the inverse isomorphism is algebraic.

Definition 2.1. For a given \( i < n \), we say that \( B(X, i) \) holds if for all ample \( h \) the isomorphism

\[
(L^{n-i})^{-1} : H^{2n-i}(X, \mathbb{Q}) \cong H^i(X, \mathbb{Q})
\]

is induced by a correspondence.

Definition 2.2. (Lefschetz standard conjecture \( B(X) \)) Following convention, we say that \( B(X) \) holds if \( B(X, i) \) holds for all \( i = 0, \ldots, n - 1 \).

For later use, we recall the notion of geometric coniveau:

Definition 2.3. (geometric coniveau) The geometric coniveau filtration on cohomology is defined as

\[
N^j H^i(X, \mathbb{Q}) = \sum_{Z \subset X} \text{Im}(H^j_Z(X, \mathbb{Q}) \to H^i(X, \mathbb{Q}))
\]

where \( Z \) runs through all subschemes of \( X \) of codimension \( \geq j \).

We define a pool of examples for which \( B(X) \) is known to hold:

Definition 2.4. Let \( \mathcal{B} \) be the class of varieties defined by the following rules:

1 Both Lewis and Schoen suppose the generalized Hodge conjecture holds true, rather than “only” \( B(X) \). Also, both work with the notion of “representable Chow group”, rather than with the notion of “Chow group supported on a divisor”. 
(1) The following varieties are in $\mathcal{B}$:
   (i) Curves and surfaces;
   (ii) Threefolds not of general type (i.e. having Kodaira dimension $< 3$);
   (iii) Abelian varieties;
   (iv) $n$–dimensional varieties $X$ which have $A_i(X)_\mathbb{Q}$ supported on a subvariety of dimension $i + 2$ for all $i \leq \frac{n-3}{2}$;
   (v) $n$–dimensional varieties $X$ which have $H_i(X, \mathbb{Q}) = N^\star \mathbb{H}_i(X, \mathbb{Q})$ for all $i > n$.

(2) $\mathcal{B}$ is closed under taking products, and under taking smooth hyperplane sections.

(3) $\mathcal{B}$ is closed under blow–up, i.e. if $\tilde{X}$ is the blow–up of $X$ with center $Y$, then $\tilde{X}$ is in $\mathcal{B}$ if and only if $X$ and $Y$ are in $\mathcal{B}$.

Proposition 2.5. For $X$ in $\mathcal{B}$, the Lefschetz standard conjecture $B(X)$ is true.

Proof. For curves, surfaces and abelian varieties, this is proven by Kleiman [5, §2 Appendix]. The case of threefolds not of general type was proven by Tankeev [12]. Case (iv) is [14, Theorem 7.1]. Case (v) follows from [15, Theorem 4.2].

The fact that products and hyperplane sections preserve the truth of $B(X)$ is well–known [6]. The statement for blow–ups is proven in [11].

 Remark 2.6. Point (iv) of Definition 2.4 implies that rationally connected fourfolds are in $\mathcal{B}$. It also implies that all linear varieties (as defined in [13]) are in $\mathcal{B}$; this class includes toric varieties and spherical varieties. Point (v) implies that every threefold with $h^{0,2} = 0$ is in $\mathcal{B}$.

Remark 2.7. Point (3) of Definition 2.4 implies the following: if $X$ is a variety of dimension $\leq 4$, and $X$ is birational to a variety in $\mathcal{B}$, then $X \in \mathcal{B}$.

3. Main result

Theorem 3.1. Let $X$ be a smooth projective variety over $\mathbb{C}$. Suppose there is an $i$ such that the Chow group $A^i(X)_\mathbb{Q}$ is supported on a divisor, and that $B(X, j)$ is true for $j \leq i$. Then the cohomology group $H^i(X, \mathbb{Q})$ is supported on a divisor.

This is useful in showing the following: for a general variety, all the Chow groups are as large as possible. Here a “general variety” means a variety having Hodge diamond of maximal width, and “large Chow group” means not supported on a subvariety.

Corollary 3.2. Let $X$ be a variety in $\mathcal{B}$. Suppose for all $i = 1, \ldots, n$ the Hodge numbers $h^{i,0}(X)$ are $\neq 0$. Then there is no Chow group $A^i(X)_\mathbb{Q}$ supported on a divisor.
Proof. (of Corollary 3.2) This is immediate from Theorem 3.1, plus the fact that $H^i(X, \mathbb{C})$ being supported on a divisor implies (by functoriality of the Hodge filtration) that $H^{i,0}X$ is 0.

By way of example, we present two explicit instances of Corollary 3.2:

**Corollary 3.3.** Let $X$ be an abelian variety. Then no Chow group $A^i(X)_{\mathbb{Q}}$ is supported on a divisor.

**Corollary 3.4.** Let $L$ be any variety in $B$ of dimension $m$. Let $C_1, \ldots, C_r$ be non–rational curves. Let $X \subset (L \times C_1 \times \cdots \times C_r)$ be a complete intersection of codimension $\geq m$. Then no Chow group $A^i(X)_{\mathbb{Q}}$ is supported on a divisor.

Proof. (of Theorem 3.1) For $i$ equal to $n = \dim X$, this is Mumford’s theorem. We will now suppose $i < n$.

The fact that $B(X, j)$ is true for $j \leq i$ implies [6, Theorem 4-1] that the Künneth component of the diagonal

$$\pi_i \in \text{Im} \left( H^{2n-i}(X, \mathbb{Q}) \otimes H^i(X, \mathbb{Q}) \to H^{2n}(X \times X, \mathbb{Q}) \right)$$

is algebraic. Let $h \in H^2(X, \mathbb{Q})$ be the class of a very ample line bundle, and let $Y \subset X$ be a general complete intersection of dimension $i$ with class $[Y] = h^{n-i} \in H^{2n-2i}(X, \mathbb{Q})$. The weak Lefschetz theorem gives a surjection

$$H^i(Y, \mathbb{Q}) \to H^{2n-i}(X, \mathbb{Q}),$$

implying that actually $\pi_i$ is in the image of the composite map

$$\pi_i \in \text{Im} \left( H^i(Y, \mathbb{Q}) \otimes H^i(X, \mathbb{Q}) \to H^{2i}(Y \times X, \mathbb{Q}) \to H^{2n}(X \times X, \mathbb{Q}) \right).$$

Using Lemma 3.5 proved below, we can find an algebraic class

$$\pi'_i \in H^{2i}(Y \times X, \mathbb{Q})$$

representing $\pi_i$ (i.e. the push–forward of $\pi'_i$ equals $\pi_i$ in $H^{2n}(X \times X, \mathbb{Q})$). We can thus lift $\pi'_i$ to the Chow group $A^i(Y \times X)_{\mathbb{Q}}$ and, under the assumption of Theorem 3.1, we can apply the Bloch–Srinivas argument [4] (in the form of Proposition 3.7 below) to get a decomposition

$$\pi'_i = \Gamma_1 + \Gamma_2 \in A^i(Y \times X)_{\mathbb{Q}},$$

where $\Gamma_1$ (resp. $\Gamma_2$) is supported on $Y' \times X$ where $Y' \subset Y$ is a divisor (resp. supported on $Y \times D$, where $D \subset X$ is a divisor). Going back to cohomology, this induces a decomposition of the Künneth component

$$\pi_i = \Gamma_1 + \Gamma_2 \in H^{2n}(X \times X, \mathbb{Q}),$$

with $\Gamma_1, \Gamma_2$ as above. We now consider the action of the correspondence $\pi_i$ on $H^i(X, \mathbb{Q})$:

$$\text{id} = (\pi_i)_* = (\Gamma_1)_* + (\Gamma_2)_*: H^i(X, \mathbb{Q}) \to H^i(X, \mathbb{Q}).$$
Because $\Gamma_2$ is supported on $Y \times D$, we obviously have
\[(\Gamma_2)_*H^i(X, \mathbb{Q}) \subset H^j_D(X, \mathbb{Q}) \subset N^1H^i(X, \mathbb{Q}).\]
As for $\Gamma_1$, we have that $\Gamma_1$ is supported on $Y' \times X$. This means that the action of $\Gamma_1$ factors over $H^i(Y', \mathbb{Q})$, where $Y' \to Y'$ is a resolution of singularities. Now, $H^i(Y', \mathbb{Q})$ is supported on a divisor $Y''$ (weak Lefschetz for the $(i - 1)$–dimensional variety $Y'$). It follows that the image of $H^i(Y', \mathbb{Q})$ in $H^i(X, \mathbb{Q})$ via the correspondence $\Gamma_1$ is supported on the divisor
\[pr_2(\text{Supp}(\Gamma_1) \cap (pr_1)^{-1}(Y'')) \subset X,\]
and hence
\[(\Gamma_1)_*H^i(X, \mathbb{Q}) \subset N^1H^i(X, \mathbb{Q}).\]
\[\square\]

**Lemma 3.5.** Let $X$ be a variety for which $B(X, i)$ holds. Let $Y \subset X$ a smooth complete intersection of dimension $i$ and with $[Y] = h^{n-i} \in H^{2n-2i}(X, \mathbb{Q})$, for some $h \in H^2(X, \mathbb{Q})$ the class of an ample line bundle. Suppose a class $c \in \text{Im}(H^i(Y, \mathbb{Q}) \otimes H^i(X, \mathbb{Q}) \to H^{2i}(Y \times X, \mathbb{Q}) \to H^{2n}(X \times X, \mathbb{Q}))$ is algebraic. Then there exists an algebraic class $c' \in H^{2i}(Y \times X, \mathbb{Q})$ representing $c$.

**Proof.** Let $\gamma$ denote the correspondence inducing the isomorphism
\[H^{2n-i}(X, \mathbb{Q}) \xrightarrow{\cong} H^i(X, \mathbb{Q})\]
inverse to the cup–product with $h^{n-i}$, and let $\Delta \in H^{2n}(X \times X, \mathbb{Q})$ denote the class of the diagonal. Then $\gamma \times \Delta$ is a correspondence that acts
\[(\gamma \times \Delta)_* : \text{Im}(H^{2n-i}X \otimes H^iX \to H^{2n}(X \times X)) \to \text{Im}(H^iX \otimes H^iX \to H^{2i}(X \times X)).\]
It follows that
\[c'' := (\gamma \times \Delta)_*c \in H^{2i}(X \times X)\]
is algebraic. But then, denoting $j : Y \to X$ the inclusion, the restriction
\[c' := (j \times \text{id})^*(c'') \in H^{2i}(Y \times X)\]
is algebraic as well. But the composition $j_*j^*$ is cup–product with $h^{n-i}$ on cohomology, so that
\[j_*j^*\gamma_* = \text{id} : H^{2n-i}(X, \mathbb{Q}) \to H^{2n-i}(X, \mathbb{Q}).\]
This implies that
\[(j \times \text{id})_*(c') = (j \times \text{id})_*(j \times \text{id})^*(\gamma \times \text{id})_*c\]
\[= (j_*j^*\gamma_* \times \text{id})c\]
\[= c \quad \in H^{2n}(X \times X, \mathbb{Q}).\]
\[\square\]
Lemma 3.5 is a particular case of Voisin’s standard conjecture \cite[Conjecture 2.29]{17}, \cite{16}. That it can be proven here is because we suppose $B(X, i)$; this is a particular instance of the fact that the Lefschetz standard conjecture implies Voisin’s standard conjecture \cite[Proposition 2.32]{17}.

**Proposition 3.7.** (Bloch–Srinivas-style) Let $X$ be a smooth projective variety of dimension $n$. Suppose that for some $i = 1, \ldots, n$, the Chow group $A^i(X)_{\mathbb{Q}}$ is supported on a subvariety $Z \subset X$. Then for any variety $Y$, and any cycle $\pi \in A^i(X \times Y)_{\mathbb{Q}}$, there is a decomposition

$$\pi = \Gamma_1 + \Gamma_2 \in A^i(X \times Y)_{\mathbb{Q}},$$

where $\Gamma_1$ is supported on $Z \times Y$, and $\Gamma_2$ is supported on $X \times Y'$, where $Y'$ is a divisor on $Y$.

**Proof.** We may suppose everything ($X$, $Z$, $Y$ and $\pi$) is defined over a field $k \subset \mathbb{C}$ which is finitely generated over its prime subfield. Let $k(Y)$ denote the function field of $Y$. Since $k(Y) \subset \mathbb{C}$, we have a map

$$A^i(X_{k(Y)})_{\mathbb{Q}} \to A^i(X_{\mathbb{C}})_{\mathbb{Q}},$$

which is injective \cite[Appendix to Lecture 1]{3}. Hence the hypothesis on $A^i(X_{\mathbb{C}})_{\mathbb{Q}}$ implies that $A^i(X_{k(Y)})_{\mathbb{Q}}$ is supported on the subvariety $Z$. On the other hand,

$$A^i(X_{k(Y)})_{\mathbb{Q}} = \lim_{\rightarrow} A^i(X \times U)_{\mathbb{Q}},$$

where the limit is taken over opens $U \subset Y$ \cite[Appendix to lecture 1]{3}. Given the cycle $\pi$, we can thus find an open $j: U_0 \subset Y$ such that the restriction $j^* \pi$ equals some cycle $\gamma$ on $Z \times U_0$:

$$j^* \pi = \gamma \in A^i(X \times U_0)_{\mathbb{Q}}.$$

Now defining $\Gamma_1 \in A^i(X \times Y)_{\mathbb{Q}}$ to be any cycle supported on $Z \times Y$ that restricts to $\gamma$, this means we have

$$j^*(\pi - \Gamma_1) = 0 \in A^i(X \times U_0)_{\mathbb{Q}};$$

i.e. the difference $\Gamma_2 := \pi - \Gamma_1$ is supported on $X \times (Y \setminus U_0)$. \hfill $\Box$

**Remark 3.8.** The Bloch–Srinivas style Proposition\cite[3.7]{} can also be deduced as a special case of Voisin’s presentation \cite[Theorem 3.1]{17} of the decomposition principle.

**Remark 3.9.** It is established in \cite{2} that correspondences act on graded of the geometric coniveau filtration. This gives another way of concluding, at the end of the proof of Theorem\cite{3.1}{} that

$$\left(\Gamma_1\right)_* H^i(X, \mathbb{Q}) \subset N^1 H^i(X, \mathbb{Q}).$$

The more direct and explicit argument presented above was suggested by the anonymous referee, to whom we are grateful.
Yet another version of Mumford’s theorem

References

[1] D. Arapura, Motivation for Hodge cycles, Advances in Math. (2006),
[2] D. Arapura and S.-J. Kang, Functoriality of the coniveau filtration, Canad.
    Math. Bull. (2007),
[3] S. Bloch, Lectures on algebraic cycles, Duke Univ. Math. Series, Vol. IV,
[4] S. Bloch and V. Srinivas, Remarks on correspondences and algebraic cycles,
    American Journal of Mathematics Vol. 105, No 5 (1983), 1235—1253,
[5] S. Kleiman, Algebraic cycles and the Weil conjectures, in: Dix exposés sur la
    cohomologie des schémas, North–Holland Amsterdam, 1968, 359—386,
[6] S. Kleiman, The standard conjectures, in: Motives (U. Jannsen et alii, eds.),
    Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1,
[7] J. Lewis, Towards a generalization of Mumford’s theorem, J. Math. Kyoto Univ.
    29 (1989), 267—272,
[8] J. Lewis, A generalization of Mumford’s theorem, II, Illinois Journal of Mathemat-
    ics Vol. 39 No 2 (1995), 288—304,
[9] D. Mumford, Rational equivalence of 0–cycles on surfaces, J. Math. Kyoto Univ.
    Vol. 9 No 2 (1969), 195—204,
[10] C. Schoen, On Hodge structures and non–representability of Chow groups,
    Comp. Math. 88 (1993), 285—316,
[11] S. Tankeev, Monoidal transformations and conjectures on algebraic cycles,
    Izvestiya Math. 71 (2007), no. 3, 629—655,
[12] S. Tankeev, On the standard conjecture of Lefschetz type for complex projec-
    tive threefolds. II, Izvestiya Math. 75:5 (2011), 1047—1062,
[13] B. Totaro, Chow groups, Chow cohomology, and linear varieties, Forum of
    Mathematics, Sigma (2014), vol. 1, e1,
[14] C. Vial, Algebraic cycles and fibrations, Documenta Math. 18 (2013), 1521—
    1553,
[15] C. Vial, Projectors on the intermediate algebraic Jacobians, New York J. Math.
    19 (2013), 793—822,
[16] C. Voisin, The generalized Hodge and Bloch conjectures are equivalent for
    general complete intersections, Annales scientifiques de l’ENS 46, fascicule 3
    (2013), 449-475,
[17] C. Voisin, Chow Rings, Decomposition of the Diagonal, and the Topology of
    Families, Princeton University Press, Princeton and Oxford,

Robert Laterveer
CNRS, Institut de Recherche Mathématiques Avancées
Université de Strasbourg
7 rue René Descartes
F–67084 Strasbourg Cedex
France
e-mail: laterv@math.unistra.fr