Analytic Continuation of Divergent Integrals

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Abstract

In this work the improper integral of monomial $\mu(s) = \int_1^\infty x^{-s}dx$ is considered as continuous analogy of infinite series in the context of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Both the integral and the sum of monomial functions diverge to infinity for $s \in \mathbb{C}$ with $\text{Re}(s) \leq 1$, while they become convergent otherwise. This paper presents analytic continuation of divergent integral of monomial over the entire complex plane with the exception of the pole at one, similar to analytic continuation of the zeta-function. It is shown that the improper integral can be equivalently written in term of Dirichlet series by making term-by-term integration of the monomial over successive integer intervals and using the Newton’s generalization of the binomial theorem. This allows us to establish an elegant relationship between the $\mu$-function and $\zeta$-function and a functional equation that are exploited to assign meaningful solution to the divergent integral in the sense of analytic continuation. Subsequently, it is proved that the $\mu$-function is holomorphic everywhere except for a simple pole at $s = 1$ and that the analytic continuation solution is consistent with the Abel-Plana formula.

1 Introduction

Analytic continuation of functions can be used in mathematics to assign a meaningful finite value to an infinite series that diverges to infinity. These functions in complex analysis are first defined in a small domain and then they are extended to a much larger domain via analytic continuation. The most well-known application of analytic continuation is the Riemann zeta function. It was Riemann who was the first to extend Euler’s definition of the function to the complex plane in order to establish the analytic continuation of the $\zeta$-function to $\mathbb{C} \setminus \{1\}$,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \quad (1)$$

The analytic continuation of the $\zeta$-function lead to some of the more baffling results in modern mathematics. For example, $\zeta(-1) = -\frac{1}{12}$ implies that somehow the sum of all positive integers is a finite number which is neither positive nor an integer [1]. The divergent series also appears in several calculations of modern physics such as the Casimir force acting on two parallel plates due to the vacuum energy. The most striking fact is that when physicists experimentally characterised the Casimir force in laboratory, they found it to precisely match the results of the

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Analytic continuation of the zeta-function corresponding to $\zeta(-3) = \frac{1}{120}$ [2,3]. This astonishing result reveals that the analytic continuation of the divergent series is not just a math abstract concept but is intrinsically linked to the real physical world. In other words, it appears that even nature prefers to choose analytic continuation as a viable means to avoid divergence to infinity. If this is not enough to make the Riemann zeta-function one of the most interesting and mysterious developments in modern maths, it has been also shown that the zeta-function has also deep connections to the distribution of prime numbers, which are the building block or “DNA” of all integer numbers [4].

By the same token, divergent integrals also appears in mathematics of modern physics such as models describing interacting physical particles in quantum field theory [2,4]. Regularization or renormalization are common techniques to define the finite part of a divergent integral by introducing a cutoff [7,8]. In particular, the notion of renormalization is often used in modern physics for removing the divergent part of the integral of some physical quantity [9]. There are also other methods such as weakly singular and hypersingular integral regularization based on the theory of distributions [10,11]. The end goal of this paper is to extend the domain of certain divergent integrals to give a formal finite solution in the sense of analytic continuation. More specifically, while the improper integral of monomial function $x^{-s}$ converges only for $\text{Re}(s) > 1$, function

$$
\mu(s) = \int_{1}^{\infty} x^{-s} \, dx
$$

is defined all over $\mathbb{C} \setminus \{1\}$.

## 2 Divergent Integral of Monomial

The improper integral $\int_{1}^{\infty} x^{-s} \, dx$ is convergent in domain $\{s \in \mathbb{C} | \text{Re}(s) > 1\}$, i.e.,

$$
\int_{1}^{\infty} x^{-s} \, dx = \frac{1}{s-1} \quad \text{Re}(s) > 1
$$

Clearly, the improper integral of monomials becomes divergent if $\text{Re}(s) \leq 1$. Our end goal here is that the domain of the above improper integral as a function of complex variable $s$ is extended over the entire complex plane (except the isolated point $s = 1$) in order to assign the divergent integral with a finite value in the sense of analytic continuation, i.e.,

$$
\mu(s) = \int_{1}^{\infty} x^{-s} \, dx \quad s \in \mathbb{C} \setminus \{1\}
$$

---

1. Euler found a second representation of the $\zeta$-function

$$
\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}}
$$

where $\mathbb{P}$ is the set of all prime numbers. The 160-year-old Riemann hypothesis, which asserts that the nontrivial zeros of $\zeta(s) = 0$ all have real part $\text{Re}(s) = \frac{1}{2}$, has deep connections to the distribution of prime numbers and remains one of enigmatic unsolved problems in mathematics.
Notice that $\mu$-function definition (5) can be viewed as the continuous analogy of the discrete Riemann zeta-function

$$
\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad s \in \mathbb{C} \setminus \{1\}
$$

In the following analysis, we will first show that there exists an analytical relationship between the $\mu$-function and $\zeta$-function. Next, the link between the two functions is exploited to achieve analytic continuation of the mu-function. Let us the $\mu$ function represents the term by term integration of the monomial $x^{-s}$ over successive integer intervals, i.e.,

$$
\mu(s) = \int_1^2 x^{-s}dx + \int_2^3 x^{-s}dx + \int_3^4 x^{-s}dx + \cdots
$$

$$
= \sum_{n=2}^{\infty} P_n(s) \quad (6)
$$

where $P_n(s)$ is the integral of the monomial over integer interval $n - 1$ to $n$, i.e.,

$$
P_n(s) = \int_{n-1}^{n} x^{-s}dx = \frac{1}{1-s}\left(\sum_{n=2}^{\infty} n^{1-s} - (n-1)^{1-s}\right) \quad (7)
$$

According to the Newton’s generalization of the binomial theorem [12], we can say

$$
(n - 1)^{1-s} = \sum_{k=0}^{\infty} (-1)^k \binom{1-s}{k} \frac{n^{-s-k}}{n} \quad \forall s \in \mathbb{C}, n > 1 \quad (9)
$$

Here, the binomial coefficient is defined as

$$
\binom{s}{k} = \frac{1}{k!} \prod_{j=0}^{k-1} (s-j) = \frac{(s)_k}{k!} \quad s \in \mathbb{C} \quad (10)
$$

where $(\cdot)_k$ denotes falling factorial. Substituting (9) in (7) yields

$$
P_n(s) = \frac{1}{s-1} \sum_{k=1}^{\infty} (-1)^k \binom{1-s}{k} n^{-s-k} \quad \forall s \in \mathbb{C} \setminus \{1\}, \ n \geq 2 \quad (11)
$$

According to the ratio test, if the ratio of two successive terms of the series in (6) converges to $L$ which is less than one, i.e.,

$$
L = \lim_{k \to \infty} \left| \frac{(1-s)_{k+1}}{n(k+1)(1-s)_k} \right| = \frac{1}{n} \lim_{k \to \infty} \left| \frac{k+s-1}{k+1} \right| = \frac{1}{n}, \quad (12)
$$

\footnote{If $s \in \mathbb{C}$ and $|a| < |b|$, then we have

$$
(a + b)^r = \sum_{k=0}^{\infty} \binom{r}{k} a^k b^{r-k} \quad \forall r \in \mathbb{C}, \ |a| < |b| \quad (8)
$$

Replace variable $r$ by $1-s$ in (8) and set $a = -1$ and $b = n$, the latter equation can be written as (9).}
then the absolute convergence of the series is ensured. Clearly, (12) implies that

\[ L < 1 \quad \forall n \geq 2 \quad s \in \mathbb{C} \]

and thus the power series in (11) converges absolutely with infinite radius of convergence. Thus, \( P_n(s) \) is always well-defined over the whole complex plane. The absolute convergence of the infinite series in (6) requires

\[
\lim_{n \to \infty} \left| \frac{P_{n+1}}{P_n} \right| = \left( 1 + \frac{1}{n} \right)^{1-\text{Re}(s)} < 1
\]

which implies the condition for absolute convergence \( \text{Re}(s) > 1 \). Therefore, by virtue of (6) and (11), the integral (4) can be written as a Dirichlet series of the following form

\[
\mu(s) = \int_1^\infty x^{-s} dx = \frac{1}{s-1} \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{k} \right) \sum_{n=2}^{\infty} n^{1-s-k}
\]

\[
= \frac{1}{s-1} \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{k} \right) \left( \sum_{n=1}^{\infty} n^{1-s-k} - 1 \right)
\]

The infinite series in \( \text{(14)} \) diverges to infinity for \( \text{Re}(s) \leq 1 \). Nevertheless, the Dirichlet series in the RHS of \( \text{(14)} \) is in the form of the Riemann zeta-function, i.e.,

\[
\sum_{n=1}^{\infty} n^{1-s-k} = \zeta(k+s-1)
\]

Therefore, replacing the equivalent of the Dirichlet from (15) into (14), we arrive at the following expression of the improper integral in terms of the Riemann zeta-function

\[
\mu(s) = \frac{1}{s-1} \left[ \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{k} \right) \left( \zeta(k+s-1) - 1 \right) \right],
\]

which is deemed to be new. Through the functional equation

\[
\zeta(s) = \left( 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \right) \zeta(1-s)
\]

it has been proved that \( \zeta(s) \) is holomorphic in the whole complex plane \( \mathbb{C} \) except for the only simple pole at \( s = 1 \). Thus, the meromorphic continuation of \( \zeta(s) \) implies meromorphic continuation of \( \mu(s) \). Now, we can proceed with simplification of the infinite series in the RHS of (16). For the sake of convenience definition, we split the mu-function (16) in the following form

\[
\mu(s) = \frac{1}{s-1} \lambda(s)
\]

where

\[
\lambda(s) = \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{k} \right) \left( \zeta(k+s-1) - 1 \right)
\]
Since Reimann zeta function \( \zeta(s) \) has a simple pole at \( s = 1 \), the above expression implies that \( \lambda(s) \) has poles at: \( \{1\} \cup \mathbb{Z}^- \), where \( \mathbb{Z}^- = \{0, -1, -2, -3, \cdots\} \) denotes the set of non-positive integers. However, it will be later shown that those poles at \( \mathbb{Z}^- \) will be cancelled out by the zeros of the binomial coefficients. In the followings, we will compute \( \lambda(s) \) for two complementary domains: i) \( s \in \mathbb{C} \setminus \{1\} \cup \mathbb{Z}^- \), and ii) \( s \in \mathbb{Z}^- \).

**Corollary 1** The infinite sum of binomial coefficients and zeta-function over \( \mathbb{C} \setminus \{1\} \cup \mathbb{Z}^- \) holds the following identity

\[
\lambda(s) = \sum_{k=1}^{\infty} (-1)^k \binom{1-s}{k} \left( \zeta(k-s-1) - 1 \right) = 1 \quad \forall s \in \mathbb{C} \setminus \{1\} \cup \mathbb{Z}^- \quad (20)
\]

**Proof:** It was shown in [13] that Golback’s theorem [14] assumes the following elegant form for the Riemann zeta function

\[
\sum_{k=2}^{\infty} (\zeta(k)-1) = 1 \quad (21)
\]

The alternative form of the aforementioned generalization of (21) was presented in [15, 16]

\[
\sum_{k=1}^{\infty} \binom{k+s-2}{k} (\zeta(k+s-1)-1) = 1 \quad \forall s \in \mathbb{C} \setminus \{1\} \cup \mathbb{Z}^- , \quad (22)
\]

which is not quite in the form of equation (20). From the definition of binomial coefficient and falling factorial (10), we can say

\[
(1-s)_k = (1-s)(-s+1)\cdots(-s-k+2) \quad = (-1)^k(s-1)s(s+1)\cdots(s+k-2) \quad = (-1)^k(s+k-2)_k
\]

and hence

\[
(-1)^k \binom{1-s}{k} = \binom{k+s-2}{k} \quad \forall s \in \mathbb{C} \quad (23)
\]

By virtue of (23) and (22) we thus arrive at the alternative form of the latter equation as presented in (20). \( \square \)

In the reminder of this section, we compute \( \lambda(s) \) for the second domain, i.e., \( s \in \mathbb{Z}^- \).

**Corollary 2** Sum of binomial coefficients and zeta-function hold the expression below:

\[
\beta(s) = \sum_{k=1}^{1-s} (-1)^k \binom{1-s}{k} \left( \zeta(k+s-1) - 1 \right) = \frac{(-1)^{-s}}{2-s} + 1 \quad s \in \mathbb{Z}^- \quad (24)
\]

**Proof:** One can verify that the successive binomial coefficients for positive integers \( k \) and \( -s \) hold the following useful identity

\[
\binom{1-s}{k} = \frac{2-k-s}{2-s} \binom{2-s}{2-k-s} \quad (25)
\]
Moreover, the partial sums of the binomial coefficients satisfies the following identity \[17\]

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0,
\]

which implies

\[
\sum_{k=1}^{1-s} (-1)^{-k} \binom{1-s}{k} = -1 \tag{26}
\]

Upon substituting (25) and (26) in the LHS of (24), we can equivalently write the expression of the latter equation by

\[
\beta(s) = 1 + \frac{1}{2-s} \sum_{k=1}^{1-s} (-1)^{-k} \binom{2-s}{2-k-s} (2-k-s) \cdot \zeta(k+s-1) \tag{27}
\]

Moreover, for positive integers \(q \geq 0\), the Riemann zeta function is related to the Bernoulli numbers by

\[
\zeta(-q) = (-1)^q \frac{B_{q+1}}{q+1} \tag{28}
\]

Setting \(q = 1 - k - s\) in (28) and then substituting the resultant equation in (27) yields the expression of the \(\beta\) function as follows

\[
\beta(s) = 1 + \frac{(-1)^{1-s}}{2-s} \sum_{k=1}^{1-s} \binom{2-s}{2-k-s} B_{2-k-s} \tag{29}
\]

or equivalently

\[
\beta(s) = 1 + \frac{(-1)^{1-s}}{2-s} \sum_{k=1}^{1-s} \binom{2-s}{k} B_k \tag{29}
\]

On the other hand, the Bernoulli numbers satisfy the following property

\[
\sum_{k=0}^{q-1} \binom{q}{k} B_k = 0 \quad q \in \mathbb{Z},
\]

which implies

\[
\sum_{k=1}^{1-s} \binom{2-s}{k} B_k = -1 \tag{30}
\]

Finally by plunging the equivalent value of the summation from (30) in the second term of the RHS of (29) one can readily conclude identity (24). \(\square\)

Corollary 3  There exists the following relation between zeta function and the binomial coefficient

\[
\alpha(p) = \lim_{\epsilon \to 0} \left( p - 1 + \epsilon \right) \zeta(1 - \epsilon) = -\frac{1}{p} \quad \forall p \in \mathbb{Z}^+ \tag{31}
\]
Proof: Using the identities \((r)_p = \Gamma(r + 1)/\Gamma(r - p + 1)\) and \(\Gamma(a + 1) = a\Gamma(a)\) in the binomial coefficient \((10)\), one can expand the expression in \(\alpha(p)\) as follows

\[
\alpha(p) = \lim_{\epsilon \to 0} \frac{(p - k + \epsilon)p\zeta(1 - \epsilon)}{p!}
\]
\[
= \lim_{\epsilon \to 0} \frac{\Gamma(p + \epsilon)\zeta(1 - \epsilon)}{p(p - 1)\Gamma(\epsilon)}
\]
\[
= \lim_{\epsilon \to 0} \frac{(p - 1)!\zeta(1 - \epsilon)}{p!\Gamma(\epsilon)}
\]
\[
= \frac{1}{p} \lim_{\epsilon \to 0} \frac{\zeta(1 - \epsilon)}{\Gamma(\epsilon)}
\]  
(32)

Moreover, since the Reimann zeta-function has a pole of first order at \(s = 1\), then it has a complex residue

\[
\lim_{\epsilon \to 0} \epsilon \zeta(1 + \epsilon) = 1
\]  
(33)

Similarly the Gamma function has a pole at \(s = 0\) and it is well know that

\[
\lim_{\epsilon \to 0} \epsilon \Gamma(\epsilon) = 1
\]  
(34)

Equations (33) and (34) indicate that zeta and Gamma functions approach equally fast towards their corresponding singular values at one and zero, respectively. Using (33) and (34) in (32) concludes the proof of the Corollary, i.e., equation (31). □

Now from the results of the above Corollaries in (29) and (31) and knowing that

\[
\binom{1 - s}{k} = 0 \quad \forall k > 2 - s,
\]  
(35)

we are ready to compute \(\lambda(s)\) as a function of non-positive integers by dividing the infinite sum \((19)\) into three intervals \(1 \leq k \leq 1 - s, k = 2 - s\) and \(k \geq 3 - s\). That is

\[
\lambda(s) = \sum_{k=1}^{\infty} (-1)^k \binom{1 - s}{k} \left( \zeta(k + s - 1) - 1 \right) \quad \forall s \in \mathbb{Z}^- \\
= \sum_{k=1}^{1-s} (-1)^k \binom{1 - s}{k} \left( \zeta(k + s - 1) - 1 \right) + \underbrace{(-1)^{-s} \alpha(2 - s) + 0 + 0 + \cdots}_{k=2-s} \\
= \beta(s) + (-1)^{-s} \alpha(2 - s) + 0 \\
= \left( 1 + \frac{(-1)^{-s}}{2 - s} \right) - \frac{(-1)^{-s}}{2 - s} = 1 \quad \forall s \in \mathbb{Z}^- 
\]  
(36)

We have now built up everything we need to show that the \(\mu\)-function can be extended analytically to the whole complex plane (aside from the simple pole at \(s = 1\) that is.). From \((18)\) together with \((20)\) and \((36)\), one can conclude that the analytic continuation of divergent integral of monomial \(\int_{1}^{\infty} x^{-s} dx\) is represented by

\[
\mu(s) = \frac{1}{s - 1} \quad \forall s \in \mathbb{C} \setminus \{1\}.
\]  
(37)
2.1 Changing lower limit of the integral to zero

We will show that a consequence of changing the lower limit of the divergent integral to zero is that the value of the analytic continuation of the improper integral becomes zero. To this end, consider the following definite integral

\[ \int_0^1 x^{-s} dx, \quad (38) \]

which converges to \(-1/(s - 1)\) for \(\text{Re}(s) < 1\) and diverges to infinity for \(\text{Re}(s) \geq 1\). It’s quite simple to show that the above integral as a function of \(s\) can be extended to the entire complex plane excluding \(s = 1\). By changing of variable \(x = y^{-1}\) and the corresponding differential of the variable \(dx = -y^{-2} dy\), one can equivalently write the above integral as

\[
\int_0^1 x^{-s} dx = \lim_{\epsilon \to 0} \int_{1/\epsilon}^1 y^{s-2} dy \\
= \int_1^\infty y^{s-2} dy = \mu(-s + 2),
\]

and hence

\[
\int_0^1 x^{-s} dx = \frac{1}{-s + 1} \quad \forall s \in \mathbb{C} \setminus \{1\} \quad (40)
\]

In other words, a more general form of the \(\mu\)-function can be transcribed by

\[
\mu(s) = \int_1^\infty x^{-s} dx = -\int_0^1 x^{-s} dx = \frac{1}{s - 1} \quad \forall s \in \mathbb{C} \setminus \{1\} \quad (41)
\]

Let us split the following improper integral that yields:

\[
\int_0^\infty x^{-s} dx = \left( \int_0^1 + \int_1^\infty \right) x^{-s} dx \\
= \mu(-s + 2) + \mu(s) \\
= \frac{1}{-s + 1} + \frac{1}{s - 1} = 0 \quad \forall s \in \mathbb{C} \setminus \{1\} \quad (43)
\]

It can be inferred from the above equation that the \(\mu\)-function holds the following functional equation:

\[
\mu(s) = -\mu(-s + 2) \quad (44)
\]

2.2 Abel-Plana formula

The Abel-Plana formula gives a simple expression of the difference between a sum over discrete values and an integral of an entire function \(f(\cdot)\) as follows [18]:

\[
\sum_{n=0}^{\infty} f(n) - \int_0^\infty f(x) dx = \frac{1}{2} f(0) + i \int_0^\infty \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt \quad (45)
\]
In this section, we will show that the result of the analytic continuation of divergent integral of monomial is consistent with the Abel-Plana formula (45). Setting \( f(x) = x^{-s} \) and assuming \( \Re(s) < 0 \) lead to \( f(0) = 0 \) and hence the Abel-Plana formula boils down to the following simple form

\[
\zeta(s) - \int_0^\infty x^{-s} dx = \left[ i^{1-s} + (-i)^{1-s} \right] \int_0^\infty \frac{t^{-s}}{e^{2\pi t} - 1} dt
\]

(46)

On the other hand, one can show the following identity

\[
\left[ i^{1-s} + (-i)^{1-s} \right] = 2 \sin \left( \frac{\pi s}{2} \right),
\]

(47)

and thus (46) can be further simplified to:

\[
\zeta(s) - \int_0^\infty x^{-s} dx = 2 \sin \left( \frac{\pi s}{2} \right) \int_0^\infty \frac{t^{-s}}{e^{2\pi t} - 1} dt
\]

\[
= 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \int_0^\infty \frac{t^{-s}}{e^t - 1} dt
\]

(48)

Furthermore, the Riemann zeta-function in the domain \( \Re(s) < 0 \) can be also defined by the following integral [19]

\[
\zeta(1-s) = \frac{1}{\Gamma(1-s)} \int_0^\infty \frac{t^{-s}}{e^t - 1} dt
\]

(49)

Finally, using (49) in the RHS of (48), we arrive at

\[
\zeta(s) - \int_0^\infty x^{-s} dx = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \zeta(1-s)
\]

\[
= \zeta(s) \quad \forall \Re(s) < 0
\]

(50)

where (50) is derived by virtue of the fundamental functional equation of the Riemann zeta function (17). The only possibility for (50) to happen is that \( \int_0^\infty x^{-s} dx = 0 \) for \( \Re(s) < 0 \). This represents a special case of the generalized analytic continuation of integral of monomial in (43).

### 3 Conclusions

In this paper, we have used the notion of analytic continuation to derive a unique finite solution to divergent integral of monomial over entire complex \( s \) with the exception of a single pole at \( s = 1 \). The improper integral of monomial \( \mu(s) = \int_1^\infty x^{-s} dx \) has been considered as continuous analogy of infinite series in the context of the Riemann zeta function \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) as both of them diverge to infinity for \( \Re(s) \leq 1 \), and they become convergent otherwise. It has been shown that the improper integral of monomial could be equivalently written in term of Dirichlet series by making term-by-term integration of the monomial over successive integer intervals and using the Newton’s generalization of the binomial theorem. Subsequently, an elegant relationship between the \( \mu \)-function and \( \zeta \)-function has been established that allowed us to assign meaningful finite value to the divergent integral in the sense of analytic continuation. It has been also proved that the analytic continuation solution is consistent with the Abel-Plana formula.
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