Modular hyperbolas and bilinear forms of Kloosterman sums *

Shkredov I.D.

Annotation.

In this paper we study incidences for hyperbolas in \( \mathbb{F}_p \) and show how linear sum–product methods work for such curves. As an application we give a purely combinatorial proof of a nontrivial upper bound for bilinear forms of Kloosterman sums.

1 Introduction

Let \( p \) be an odd prime number, and \( \mathbb{F}_p \) be the finite field. Given two sets \( A, B \subset \mathbb{F}_p \), define the sumset, the product set and the quotient set of \( A \) and \( B \) as

\[
A + B := \{a + b : a \in A, b \in B\},
\]

\[
AB := \{ab : a \in A, b \in B\},
\]

and

\[
A/B := \{a/b : a \in A, b \in B, b \neq 0\},
\]
correspondingly. This paper is devoted to the so–called sum–product phenomenon, which says that either the sumset or the product set of a set must be large up to some natural algebraic constrains. One of the strongest form of this principle is the Erdős–Szemerédi conjecture [9], which says that for any sufficiently large set \( A \) of reals and an arbitrary \( \epsilon > 0 \) one has

\[
\max \{|A + A|, |AA|\} \gg |A|^{2-\epsilon}.
\]

The best up to date results in the direction can be found in [30] and in [26] for \( \mathbb{R} \) and \( \mathbb{F}_p \), respectively. Basically, in this paper we restrict ourselves to the case of the finite fields only.

It was Elekes [7] who realised that the sum–product phenomenon is connected with Incidence Geometry. Incidence Geometry deals with the incidences among basic geometrical objects such as points, lines, curves, surfaces and so on. After Elekes various results on incidences of different types in \( \mathbb{R} \) were obtained by many authors (see, e.g., [37]). Nevertheless, in \( \mathbb{F}_p \) only linear incidences, i.e., incidences between linear objects as points/planes, points/lines, lines/lines

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were obtained see, e.g., [27], [35], [38]. A remarkable exception is the case of so-called SL\(_2(\mathbb{F}_p)\)–hyperbolas and this exception was suggested by Bourgain [2] who gives, in particular, the first nontrivial upper bound for cardinality of the following set
\[
\{(a + b)(c + d) = \lambda : a \in A, b \in B, c \in C, d \in D\}
\] (1)
for any \(\lambda \neq 0\) and arbitrary sets \(A, B, C, D \subseteq \mathbb{F}_p\). The importance of hyperbolas in Additive Combinatorics and Number Theory was discussed in [32]. Bourgain’s approach was connected with the group actions (the importance of the group actions in Additive Combinatorics was realized by Elekes as well, see [6], [8]) and it was based on Helgott’s result on growth in SL\(_2(\mathbb{F}_p)\), see [12], [13] and on some additional considerations [3].

In this paper we obtain a series of new upper bounds for cardinality of the set from (1). Here are two of our results (other results can be found in Sections 5, 6, see, e.g., Theorem 32 below).

**Theorem 1** Let \(A, B, C, D \subseteq \mathbb{F}_p\) be sets. Then for any \(\lambda \neq 0\), one has
\[
\{|(a + b)(c + d) = \lambda : a \in A, b \in B, c \in C, d \in D\}| - \frac{|A||B||C||D|}{p} \lesssim |A|^{1/4}|B||C||D|^{1/2} + |A|^{3/4}(|B||C|)^{41/48}|D|^{1/2}.
\] (2)

The Theorem above allows to obtain a uniform upper bound for size of hyperbola with elements from a set with the small sumset.

**Corollary 2** Let \(A \subseteq \mathbb{F}_p\) be a set. Suppose that \(|A + A| \ll |A|\) and \(|A| \ll p^{13/23}\). Then for any \(\lambda \neq 0\), one has
\[
|\{a_1a_2 = \lambda : a_1, a_2 \in A\}| \lesssim |A|^{149/156}.
\] (3)

Another rather unusual result (for example, the proof uses the fact that the group SL\(_2(\mathbb{Z})\) contains free subgroups) on incidences (1) is the following (an analogue of this statement in \(\mathbb{F}_p\) is our Theorem 32 from Section 6).

**Theorem 3** Let \(A, D \subset \mathbb{R}, B \subset \mathbb{Z}\) be sets, and \(\lambda \neq 0\) be any number. Then
\[
\{|(a + b)(c + d) = \lambda : a \in A, d \in D, b, c \in B\}| \ll |\lambda| \sqrt{|A||D||B|^2} \cdot \max\{|D|^{-1/2}, |B|^{-1/4}\}.
\] (4)
Rather mysterious part of Helfgott’s proof of $\text{SL}_2(\mathbb{F}_p)$–growth result was that the sum–product phenomenon in $\mathbb{F}_p$, which deals exclusively with linear objects as points/lines, points/planes and so on gives absolutely nontrivial results for completely different curves, namely, for hyperbolas. An explanation in a particular but a transparent case is given in our Lemma 14, where we estimate a certain energy of a subset of matrices from $\text{SL}_2(\mathbb{F}_p)$ via purely linear sum–product quantity. Now it remains to notice that energies of subsets of acting groups are naturally related with the incidences, see, e.g., [23], [24], [28].

It turns out that incidences between hyperbolas and points are connected with bilinear forms of Kloosterman sums, see [1], [10], [14]–[19], [32]–[34] and other papers. We obtain the following result in this direction (see Theorems 33, 34 from Section 7), which we formulate here in a particular case (the main advantage of our method is that it allows to consider rather general sets and weights). Recall that the Kloosterman sum in a finite field $\mathbb{F}$ is

$$K(n, m) = \sum_{x \in \mathbb{F}\{0\}} e(nx + mx^{-1}).$$

We are interested in bilinear forms of Kloosterman sums [15]–[17], that is, the sums of the form

$$S(\alpha, \beta) = \sum_{n,m} \alpha(n)\beta(m)K(n, m),$$

where $\alpha : \mathbb{F} \to \mathbb{C}$, $\beta : \mathbb{F} \to \mathbb{C}$ are rather arbitrary functions.

**Theorem 4** Let $\alpha, \beta : \mathbb{F}_p \to \mathbb{C}$ be functions with supports on $\{1, \ldots, N\} + t_1$ and $\{1, \ldots, M\} + t_2$, respectively, and $N$ or $M$ is at most $p^{1-c}$, $c > 0$. Then

$$S(\alpha, \beta) \lesssim \|\alpha\|_2\|\beta\|_2 p^{1-\delta},$$

where $\delta(c) > 0$ is a positive constant. Besides, if $M^2 < pN$, then

$$S(\{1, \ldots, N\} + t_1, \beta) \lesssim \|\beta\|_2 \left( N^{3/7}M^{1/7}p^{13/14} + N^{3/4}p^{3/4} + N^{1/4}p^{13/12} \right).$$

It is easy to check that the last result is better than [33, Theorem 7], as well as [10, Theorem 1.17(2)] but worse than the current world record from [16] in the case $N = M$. Of course the advantage of our results is that they hold in very general situation. Also, the method of the proof is not analytical but combinatorial one and hence does not require deep tools from Algebraic Geometry as in [16].

In our paper we develop the ideas from [23], where growth in $\text{SL}_2(\mathbb{F}_p)$ was applied to the Zaremba conjecture about continued fractions. Before this paper various analytical tools (as Kloosterman sums) were used in the aforementioned area, see, e.g., [22]. After [23] it is not surprising that sometimes combinatorial methods give results of comparable quality to the ones, which were obtained via deep analytical techniques.

All logarithms are to base 2. The signs $\ll$ and $\gg$ are the usual Vinogradov symbols. For a positive integer $n$, we set $[n] = \{1, \ldots, n\}$. Having a set $A$, we will write $a \lesssim b$ or $b \gtrsim a$ if $a = O(b \cdot \log^c |A|)$, $c > 0$.

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generally, we deal with a higher energy
Plancherel formula and its particular case
where
is, in this case. One can consider
additive energy
denote the Fourier transform of a function
\( f \) for the
\( A \)
\( F \)
\( \hat{f}(\xi) = \sum_{x \in \mathbb{F}_p} f(x)e(-\xi \cdot x) \),
where \( e(x) = e^{2\pi ix/p} \). We rely on the following basic identities. The first one is called the
Plancherel formula and its particular case \( f = g \) is called the Parseval identity
\[ \sum_{x \in \mathbb{F}_p} f(x)\overline{g(x)} = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} \hat{f}(\xi)\overline{\hat{g}(\xi)}. \] (8)
Another particular case of (8) is
\[ \sum_{y \in \mathbb{F}_p} |(f * g)(y)|^2 = \sum_{y \in \mathbb{F}_p} \left| \sum_{x \in \mathbb{F}_p} f(x)g(y - x) \right|^2 = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} |\hat{f}(\xi)|^2|\hat{g}(\xi)|^2. \] (9)
and
\[ f(x) = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} \hat{f}(\xi)e(\xi \cdot x). \] (10)
is called the inversion formula. The (normalized) Wiener norm of \( f(x) \) is defined as
\[ \| \hat{f} \|_{L^1} = \| f \|_W := \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} |\hat{f}(\xi)|. \] (11)
Clearly, by the Parseval identity (8), the inverse formula (10) and the Cauchy–Schwarz inequality, we have
\[ \| f \|_\infty \leq \| f \|_W \leq \| f \|_2 \quad \text{and} \quad \| f \|_1 \leq p\| f \|_W. \] (12)
It is well–known that equipped with the Wiener norm the set of functions on the group forms an algebra relatively pointwise multiplication. In this paper we use the same letter to denote a set \( A \subseteq \mathbb{F} \) and its characteristic function \( A : \mathbb{F} \to \{0, 1\} \). Also, we write \( f_A(x) \) for the balanced
function of a set \( A \subseteq \mathbb{F}_p \), namely, \( f_A(x) = A(x) - |A|/p \). Let \( m \cdot A \) be the set \( \{ma : a \in A\} \).

2 Notation

In this paper \( \mathbb{F} \) is a field, and \( p \) is an odd prime number, \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \) and \( \mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\} \). We denote the Fourier transform of a function \( f : \mathbb{F}_p \to \mathbb{C} \) by \( \hat{f} \), namely,

\[ \hat{f}(\xi) = \sum_{x \in \mathbb{F}_p} f(x)e(-\xi \cdot x), \] (7)

Put
\( E \) for the common additive energy of two sets \( A, B \subseteq \mathbb{F}_p \) (see, e.g., [37]), that is,
\[ E(A, B) = |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 + b_1 = a_2 + b_2\}|. \]
If \( A = B \), then we simply write \( E(A) \) instead of \( E(A, A) \) and the quantity \( E(A) \) is called the
additive energy in this case. One can consider \( E(f) \) for any complex function \( f \) as well. More
generally, we deal with a higher energy
\[ T_k^+(A) := |\{(a_1, \ldots, a_k, a'_1, \ldots, a'_k) \in A^{2k} : a_1 + \cdots + a_k = a'_1 + \cdots + a'_k\}| = \frac{1}{p} \sum_\xi |\hat{A}(\xi)|^{2k}. \] (13)
The last identity follows from \([39]\). Another sort of higher energy is \([29]\)
\[
E^+_k(A) = |\{(a_1, \ldots, a_k, a'_1, \ldots, a'_k) \in A^{2k} : a_1 - a'_1 = \cdots = a_k - a'_k\}|.
\]
Sometimes we use representation function notations like \(r_{AB}(x)\) or \(r_{A+B}(x)\), which counts the number of ways \(x \in \mathbb{F}_p\) can be expressed as a product \(ab\) or a sum \(a + b\) with \(a \in A, b \in B\), respectively. Further clearly
\[
E^+(A, B) = \sum_x r^2_{A+B}(x) = \sum_x r^2_{A-B}(x) = \sum_x r_{A-A}(x)r_{B-B}(x)
\]
and by \([39]\),
\[
E^+(A, B) = \frac{1}{p} \sum_\xi |\hat{A}(\xi)|^2|\hat{B}(\xi)|^2.
\]
(14)
Similarly, one can define \(E^\times(A, B)\), \(E^\times(A)\), \(E^\times(f)\) and so on.

3 Preliminaries

We need in a sum–product result from \([31, \text{Theorem 32}]\), as well as \([25, \text{Theorem 35}]\).

**Lemma 5** Let \(A, B \subseteq \mathbb{F}_p\) be sets. Then
\[
\sum_x r^2_{(A-A)(B-B)}(x) - \frac{|A|^4|B|^4}{p} \leq (|A||B|)^{5/2}E^+(A, B)^{1/2}.
\]

**Lemma 6** Suppose that \(A\) is a subset of \(\mathbb{F}_p\) such that \(|A + A| = K|A|\) and \(|A| \leq p^{13/23}K^{25/92}\). Then
\[
E^\times(A) \lesssim K^{51/26}|A|^{32/13}.
\]

In \([5, \text{Theorem 2}]\) it was obtained a very precise result on multiplicative energy of arithmetic progressions.

**Theorem 7** Let \(A\) and \(B\) be arithmetic progressions with the difference equals one. Then
\[
E^\times(A, B) = \frac{|A|^2|B|^2}{p} + O(|A||B|\log^2 p).
\]

The last result about Fourier transform of arithmetic progressions is well–known.

**Lemma 8** Let \(P\) be an arithmetic progression. Then \(\|P\|_W \ll \log p\) and for any \(c > 1\) the following holds
\[
p^{-1} \sum_{\xi \in \mathbb{F}_p} |\hat{P}(\xi)|^c \ll |P|^{c-1}.
\]
4 Some non–abelian results

We will formulate and prove a series of results, which hold in general groups although, of course, our main applications concern SL₂(\mathbb{F}_p) and SL₂(\mathbb{Z}).

Let G be a group and A₁, . . . , A₂k ⊆ G be sets. For k ≥ 2 put

\[ T_k(A₁, . . . , A₂k) = \{(a₁a₂⁻¹ . . . aₖ⁻¹ = aₖ₊₁a_{k+₂⁻¹} . . . a₂k⁻¹a₂k⁻¹ : aⱼ ∈ Aⱼ)\}. \tag{15} \]

More generally, one can define T_k(f₁, . . . , f₂k) for any functions f₁, . . . , f₂k : G → C. Basically, we are interested in the case of the characteristic functions f_j. If k = 2, then we write E instead of T₂ as in Section 2. For any g ∈ G one has

\[ T_k(gA₁, . . . , gA₂k) = T_k(A₁, . . . , A₂k) \quad \text{and} \quad T_k(A₁g, . . . , A₂kg) = T_k(A₁, . . . , A₂k). \tag{16} \]

One of the reasons that we have defined T_k(A₁, . . . , A₂k) as in (15) is that we have property (16). For example, if T_k(A₁, . . . , A₂k) was defined as just the number of the solutions to the equation a₁ . . . aₖ = aₖ₊₁ . . . a₂k, then formula (16) fails. Another reason is that for such defined T_k Lemmas [9, 12] take place. Finally, in terms of eigenvalues of some operators, see the proof of Lemma [12] and Remark [13] we have for such T_k a full analogue with the abelian case, compare formula (13) and formula (25).

Now if A₁ = . . . = A₂k = A, then we write T_k(A) for T_k(A, . . . , A) and similarly T_k(f) for T_k(f, . . . , f). It is convenient to put T₁(A) = |A|^2. Also, denote T₂k(A, B) := T₂k(A, B, . . . , A, B).

Since T_k(A) = T_k(gA) = T_k(Ag) for any g ∈ G, it follows that in any matrix group (as SLₙ) an arbitrary permutation of rows or columns preserves T_k. Also, notice that T_k(A) = T_k(A⁻¹).

Further, T_k(A) ≤ |A|²(k−l)T_l(A), l ≤ k because the operator, which fix any l positions from the left side and from the right side in (15) is, clearly, symmetric and nonnegatively defined (obviously, one has \( \sum_{x,y \in M} T(x, y) \leq |M| \sum_{x \in M} T(x, x) \) for any nonnegative operator T(x, y) defined on a set M). Using the Cauchy–Schwarz inequality, we have

\[ \|r_{(AA⁻¹)²}\|_∞, \|r_{(A⁻¹A)²}\|_∞ \leq T_k(A). \tag{17} \]

Further for k ≥ 1 consider the higher energies [29]

\[ E^R_k(A) = \sum_x r^{k}_{AA⁻¹}(x) = \sum_{x_₁, . . . , x_{k−₁}} |A \cap Ax_₁ \cap \cdots \cap Ax_{k−₁}|² \]

and, similarly, \( E^L_k(A) \).

We need in a lemma about quantities T_k(A₁, . . . , A₂k).

**Lemma 9** Let f₁, . . . , f₂k : G → C be functions. Then

\[ T^k_k(f₁, . . . , f₂k) \leq \prod_{j=1}^{2k} T_k(f_j). \tag{18} \]

In particular, for any A, B, C, D ⊆ G one has

\[ E(A, B, C, D) ≤ E(A)E(B)E(C)E(D). \tag{19} \]
Proof. For typographical reasons we will assume sometimes that \( f_j = A_j \) for some sets \( A_j \subseteq G \). Clearly, by the Cauchy–Schwarz inequality for any \( l \)

\[
T_l^2(A_1, \ldots, A_{2l}) \leq T_l(A_1, \ldots, A_l, A_{l+1}, \ldots, A_l)T_l(A_{l+1}, \ldots, A_{2l}, A_{l+1}, \ldots, A_{2l})
\]

and thus it is enough to have deal with the last quantities. Let us begin with (19) because its simplicity and to have the basis of the induction. From the last bound we see that

\[
E(A, B) = E(A, B, A, B)^2 = |\{ ab^{-1} = \bar{a}\bar{b}^{-1} : a, a \in A, b, b \in B \}|^2 = \\
|\{ \bar{a}^{-1} a = \bar{b}^{-1} b : a, a \in A, b, b \in B \}|^2 \leq E(A)E(B)
\]

as required. Clearly, the same is true for functions. Now let \( k = 2 s \) is even (for odd \( k \) a similar arguments hold). Using induction we obtain

\[
T^k_k(A_1, \ldots, A_{2k}) = \prod_{j=1}^{k} T_{k/2}(r_{A_1A_2^{-1}}, \ldots, r_{A_{2k-1}A_{2k}^{-1}}) \leq \prod_{j=1}^{k} T_{k/2}(r_{A_{2j-1}A_{2j}^{-1}}) = \prod_{j=1}^{k} T_k(A_{2j-1}, A_{2j})
\]

and hence it is enough to prove for any \( A \) and \( B \) that

\[
T^k_k(A, B) \leq T_k(A)T_k(B).
\]

Now rewrite \( T_k(A, B) = T_{2s}(A, B) \) as

\[
(\bar{a}_1^{-1} a_1)b_1^{-1} a_2 b_2^{-1} \ldots a_s = \bar{b}_1^{-1} a_2 \bar{b}_2^{-1} \ldots \bar{a}_s (\bar{b}_s^{-1} b_s)
\]

and using induction again and the arguments as in (20), we obtain

\[
T^k_k(A, B) \leq T_k(A)T_k(B)T_{k-2}^k(B, A)
\]

and, similarly,

\[
T^k_k(B, A) \leq T_k(A)T_k(B)T_{k-2}^k(A, B).
\]

Combining the last two formulae, we get

\[
T^{k^2}(A, B) \leq T^k_k(A)T^k_k(B) \left( T_k(A)T_k(B)T_{k-2}^k(A, B) \right)^{k-2} = T_{k-2}^{2k}(A)T_{k-2}^{2k}(B)T_{k-2}^{(k-2)^2}(A, B)
\]

as required. \( \square \)

Corollary 10 The formula \( \|f\| := T_k(f)^{1/2k} \), \( k \geq 2 \) defines a norm of an arbitrary function \( f : G \to \mathbb{C} \). Also, \( T_k(f)^{1/2k} \geq \|f\|_{2k} \).

We need a well–known lemma, which we prove for the sake of completeness.

Lemma 11 Let \( G \) be a group and let \( G \) acts \( k \)-transitively on a set \( X \). Suppose that \( G \subseteq G \) and \( A, B \subseteq X \) are sets. Then

\[
\sum_{g \in G} \sum_{x \in B} A(gx) \leq |G|^{-1/k} |A||B| + |G|.
\]

(21)
Proof. Using the Hölder inequality, we get

\[ \sigma^k := \left( \sum_{g \in G} \sum_{x \in B} A(gx) \right)^k \leq \left| G \right|^{k-1} \sum_{g \in G} \left( \sum_{x \in B} A(gx) \right)^k = \left| G \right|^{k-1} \sum_{x_1, \ldots, x_k \in B} \sum_{g \in G} A(gx_1) \ldots A(gx_k). \]

By the assumption \( G \) acts \( k \)-transitively on \( X \). Hence fixing \( (x_1, \ldots, x_k) \in B^k \) and \( (x_1, \ldots, x_k) \neq (a_1, \ldots, a_k) \in A^k \), we find a unique \( g \in G \) such that \( a_j = gx_j \), \( j \in [k] \). Thus

\[ \sigma^k \leq \left| G \right|^{k-1} |A|^k |B|^k + \left| G \right|^{k-1} \sigma. \]

It gives us

\[ \sigma \leq \left| G \right|^{-\frac{1}{k}} |A||B| + \left| G \right| \]

as required. \( \square \)

The well-known "counting lemma" for general actions was proved many times, see, e.g., [2] or [31, Lemma 53]. We recall the proof for the case of completeness and because we will use some parts of the proofs later. Also, we replace \( 2^k \) in (22) to any even integer \( n \geq 2 \) for an arbitrary finite group.

**Lemma 12** Let \( G \) be a group, which acts on a set \( X \) and let \( f_1, f_2 : X \to \mathbb{C} \) be functions. Also, let \( G \subset G \) be a set. Then for any \( k \geq 1 \), we get

\[ \left| \sum_{g \in G} \sum_{x} f_1(x) f_2(gx) \right|^2 \leq \left\| f_1 \right\|_2^2 \left\| f_2 \right\|_2^2 \cdot \sum_{g} r_{(GG^{-1})}^{2k-1}(g) \sum_{x} f_2(x) f_2(gx). \]  \( (22) \)

The same is true in the case \( |G| < \infty \) if one replaces \( 2^k \) to any nonzero even integer.

Proof. Denote by \( \sigma \) the left-hand side of (22). Using the Cauchy–Schwarz, we obtain

\[ \left| \sum_{x} f_2(gx) \right|^2 \leq \left\| f_2 \right\|_2^2 \cdot \sum_{g \in G} f_2(gx) \]

\[ = \left\| f_1 \right\|_2^2 \sum_{g \in G} r_{GG^{-1}}(g) \sum_{x} f_2(x) f_2(gx). \] \( (23) \)

Continuing this way, we get

\[ \left| \sigma \right|^{2k} \leq \left\| f_1 \right\|_2^2 \left\| f_2 \right\|_2^2 \cdot \sum_{g} r_{GG^{-1} \ldots GG^{-1}}(g) \sum_{x} f_2(x) f_2(gx), \]

where the term \( GG^{-1} \) in \( r_{GG^{-1} \ldots GG^{-1}}(g) \) is taken \( 2k-1 \) times. Thus, (22) follows.

Let us give another proof for even powers and finite group \( G \). Returning to (22), we have

\[ \left| \sigma \right|^2 \leq \left| G \right|^{-1} \left\| f_1 \right\|_2^2 \sum_{g,h} r_{h^{-1}GG^{-1}}(gh^{-1}) \sum_{x} f_2(hx) f_2(gx). \]
Consider a hermitian nonnegatively defined operator

\[ T(g, h) = r_{GG^{-1}}(gh^{-1}) = \sum_{\alpha=1}^{\left|G\right|} \mu_\alpha \varphi_\alpha(g) \overline{\varphi_\alpha(h)}, \tag{24} \]

where \( \mu_\alpha \geq 0 \) are eigenvalues and \( \varphi_\alpha \) are correspondent eigenfunctions. Thus

\[ |\sigma|^2 \leq |G|^{-1} \left\| f_1 \right\|_2^2 \left( \sum_{\alpha=1}^{\left|G\right|} \mu_\alpha \left( \sum_x \left| \sum_h f_2(hx) \overline{\varphi_\alpha(h)} \right|^2 \right)^{k-1} \]

Using the Hölder inequality and the orthogonality of the functions \( \varphi_\alpha(g) \), we obtain

\[ |\sigma|^{2k} \leq |G|^{-k} \left\| f_1 \right\|_2^{2k} \sum_{\alpha=1}^{\left|G\right|} \mu_\alpha^k \left( \sum_x \left| \sum_h f_2(hx) \overline{\varphi_\alpha(h)} \right|^2 \right)^{2k-1} = \]

\[ = \left|G\right|^{-k+1} \left\| f_1 \right\|_2^{2k} \left( \sum_g r_{(GG^{-1})^k}(g) \left( \sum_x f_2(x) f_2(gx) \right) \left( \sum_x \left| f_2(hx) \right|^2 \right)^{k-1} \right) = \]

\[ = \left\| f_1 \right\|_2^{2k} \left\| f_2 \right\|_2^{2k-2} \sum_g r_{(GG^{-1})^k}(g) \left( \sum_x f_2(x) f_2(gx) \right)^{k-1} \]

This completes the proof. \( \square \)

Remark 13 In terms of the eigenfunctions of the operator \( T \) from (24), we have the following formula (let \( |G| < \infty \) for simplicity)

\[ T_k(G) = |G|^{-1} \sum_{\alpha=1}^{\left|G\right|} \mu_\alpha^k = |G|^{-1} \text{tr}(T^k), \tag{25} \]

and, clearly, \( T_k(g, h) = r_{(GG^{-1})^k}(gh^{-1}). \)

5 First results on incidences for hyperbolas

Take any \( \lambda \neq 0 \) and consider our basic equation

\[ (y - a)(b - x) = \lambda \]

or, in other words,

\[ y = a + \frac{\lambda}{b - x} = gx, \tag{26} \]

where

\[ u_a v_b = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ -1 & b \end{pmatrix} = \begin{pmatrix} -a & ab + \lambda \\ -1 & b \end{pmatrix} = g \in G(A, B) = G_{\lambda}(A, B). \]
Clearly, \( \det(g) = \lambda \neq 0 \) and hence in our main case \( \lambda = 1 \) we have \( G_1(A, B) \subseteq \text{SL}_2(F) \). Also, in the next Section we will consider the set

\[ G(A) = \{ uav : a \in A \} \subseteq G(A, A). \]

Notice that \( u_{a_1}u_{a_2} = u_{a_1+a_2} \) ("\( u \)" for a unipotent matrix from \( \text{SL}_2(F) \)) and

\[ v_b^{-1} = \lambda^{-1} \begin{pmatrix} b & -\lambda \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad v_b v_2^{-1} = \begin{pmatrix} 1 & 0 \\ \lambda^{-1}(b_1 - b_2) & 1 \end{pmatrix} = u_1^*(b_1 - b_2) \in \text{SL}_2(F). \]

Lemma below shows the connection between energy of a subset of \( \text{SL}_2(F) \) and the sum–product phenomenon. Formulae \((27), (28)\) say us that any nontrivial upper bound for linear incidences in an arbitrary field \( F \) implies a good upper estimate for \( T_2(G_\lambda(A, B)), T_3(G_\lambda(A, B)) \).

**Lemma 14** For any \( A, B \subseteq F \) and \( \lambda \in F, \lambda \neq 0 \) one has

\[ T_3(G_\lambda(A, B)) \leq |A||B| \sum_x r^2_{(A-A)(B-B)}(x) + |A|^4|B|^4. \tag{27} \]

**Besides**

\[ T_2(G_\lambda(A, B)) \leq |A|^2E^+(B) + |B|^2E^+(A). \tag{28} \]

**Proof.** Take three elements \( g_1 = u_{a_1}v_{b_1}, g_2 = u_{a_2}v_{b_2}, g_3 = u_{a_3}v_{b_3} \) from \( G_\lambda(A, B) \). Putting \( \omega_1 = \lambda^{-1}(b_1 - b_2), \omega_2 = a_3 - a_2 \), we obtain

\[
g_1g_2^{-1}g_3 = u_{a_1}v_{b_1}v_{b_2}^{-1}u_{a_2}^{-1}u_{a_3}v_{b_3} = u_{a_1}u_{\omega_1}^*u_{\omega_2}v_{b_3} = \\
= \begin{pmatrix} -a_1(\omega_1\omega_2 + 1) - \omega_2 & \lambda(1 + a_1\omega_1) + b_3(\omega_2 + a_1(1 + \omega_1\omega_2)) \\ -(\omega_1\omega_2 + 1) & \lambda \omega_1 + b_3(\omega_1\omega_2 + 1) \end{pmatrix}.
\]

If \( \omega_1\omega_2 + 1 \neq 0 \), then we reconstruct \( a_1, b_3 \), having the matrix above fixed. Now it remains to notice that

\[
|\{\omega_1\omega_2 = \omega_1'\omega_2' : \omega_1 = \lambda^{-1}(b_1 - b_2), \omega_1' = \lambda^{-1}(b_1' - b_2'), \omega_2 = a_3 - a_2, \omega_2' = a_3' - a_2'\}| = \\
= \sum_x r^2_{(A-A)(B-B)}(x).
\]

Further if \( \omega_1\omega_2 + 1 = 0 \), then \( \omega_1, \omega_2 \neq 0 \) and we can find, say, \( a_1 \), having \( \omega_1, \omega_2, b_3 \) and the matrix above fixed (see the right–up corner of the matrix above). Hence we need to count an additional term, which is at most

\[
|A||B|^2 \sum_{x \neq 0} r^2_{A-A}(x)r^2_{B-B}(-\lambda x^{-1}) \leq |A|^4E^+(A) \leq |A|^4|B|^4.
\]

Similarly, to calculate \( T_2(G_\lambda(A, B)) \), we see that

\[
g_1g_2^{-1} = u_{a_1}u_{\omega_1}^*u_{-a_2} = \begin{pmatrix} 1 + a_1\omega_1 & a_1 - a_2 - a_1a_2\omega_1 \\ \omega_1 & 1 - a_2\omega_1 \end{pmatrix}.
\]
and hence
\[ T_2(G_\lambda(A, B)) = |A|^2(E^+(B) - |B|^2) + |B|^2E^+(A) \leq |A|^2E^+(B) + |B|^2E^+(A). \]

This completes the proof. \( \square \)

**Remark 15** Similarly, one can calculate higher energies of the set \( E_k^R(G_\lambda(A, B)), E_k^L(G_\lambda(A, B)) \) and prove
\[
E_k^R(G_\lambda(A, B)) = |A|^2(E_k^+(B) - |B|^k) + |B|^kE_k^+(A) \leq |A|^2E_k^+(B) + |B|^kE_k^+(A)
\]
and
\[
E_k^L(G_\lambda(A, B)) \leq |B|^2E_k^+(A) + |A|^kE_k^+(B).
\]

Using these upper bounds for the energy of the set \( G(A, B) \), we obtain our first incidence result. Theorem 16 implies Theorem 1 from the Introduction if one applies a trivial bound \( E^+(B, C) \leq (|B||C|)^{3/2} \). Further, the first bound of Theorem 16 is nontrivial only if \( E^+(C) \leq |C|^{3-\varepsilon} \) and \( E^+(B) \leq |B|^{3-\varepsilon} \), where \( \varepsilon > 0 \) but the second one is always nontrivial. Nevertheless it is interesting that incidences for hyperbolas are connected with the ordinary additive energy of a set. Also, the first bound takes place in any field not only in \( \mathbb{F}_p \).

**Theorem 16** Let \( A, B, C, D \subseteq \mathbb{F}_p \) be sets. Then for any \( \lambda \neq 0 \), one has

\[ \|\{(a + b)(c + d) = \lambda : a \in A, b \in B, c \in C, d \in D\}\| - \frac{|A||B||C||D|}{p} \lesssim (29) \]
\[
\lesssim \min\{|D|^{1/2}B||C| + |A||D|^{1/2}(|B||C|)^{1/3}(|B|^{1/3}E^+(C))^{1/6} + |C|^{1/2}E^+(B))^{1/6},
\[
|A|^{1/4}|B||C||D|^{1/2} + |A|^{3/4}(|B||C|)^{19/24}|D|^{1/2}(E^+(B, C))^{1/24}\}.
\]

**Proof.** Rewrite our basic equation \( (a + b)(c + d) = \lambda \) as \( (y - a)(b - x) = \lambda \), where we have new variables \( y \in A, x \in -D, a \in -B, b \in C \). In other words, we need to count the number of the solutions \( \sigma \) to the equation \( gx = y \) with \( g \in G_\lambda(-B, C) := G \) and \( y \in A, x \in -D \). Let \( f_1(x) = D(-x) - |D|/p \). Then

\[
\sigma = \frac{|A||B||C||D|}{p} + \sum_{g \in G} \sum_x f_1(x)A(gx) = \frac{|A||B||C||D|}{p} + \sigma_*.
\]

Here one can consider other balanced functions, e.g., of the set \( A \) or even of the sets \((-B), C \) in our set of actions \( G \) (in other words \( g \) is taken with the correspondent weight in this case). Using Lemma [12] with \( k = 1 \), we get

\[
\sigma_*^2 \leq |D| \sum_g r_{GG^{-1}}(g) \sum_{x \in A} A(gx).
\]
Applying Lemma 11 with $k = 3$, as well as the second part of Lemma 14 we obtain
\[
\sigma_* \ll |D|^{1/2}|B||C| + |A||D|^{1/2}(|B||C|)^{1/3}T_2^{1/2}(G) \leq \]
\[
|D|^{1/2}|B||C| + |A||D|^{1/2}(|B||C|)^{1/3}(|B|^{1/3}E^+(C))^{1/6} + |C|^{1/3}E^+(B)^{1/6}).
\]
Similarly, using Lemma 12 with $k = 2$, we have
\[
\sigma_*^4 \leq |D|^2|A| \sum_g r_{GG^{-1}G^{-1}}(g) \sum_{x \in A} A(gx).
\]
Denote by $w(g) = \sum_{x \in A} A(gx)$.
It gives us
\[
\sigma_*^4 \leq |D|^2|A| \sum_g r_{GG^{-1}G^{-1}}(g) r_{wG}(g),
\]
and by the pigeonhole principle there is $\tau$ such that
\[
\sigma_*^4 \lesssim |D|^2|A| \tau \sum_g r_{GG^{-1}G^{-1}}(g) r_{S_\tau G}(g),
\]
where $S_\tau = \{g \in SL_2(\mathbb{F}_p) : \tau \leq w(g) \leq 2\tau\}$. From the last inequality one can derive $\tau \gg |D|^{-2}|A|^{-1}|G|^{-4}\sigma_*^4$. It follows that if $\tau \ll 1$, then $\sigma_* \ll |G||D|^{1/2}|A|^{1/4}$. Otherwise in view of Lemma 11 we have $|S_\tau| \ll |A|^6/\tau^3$. Now combining (32), the second part of Lemma 5 and the Cauchy–Schwarz inequality, we obtain
\[
\sigma_*^8 \lesssim |D|^4|A|^2\tau^2T_{3}(G)E(G, S_\tau^{-1}, G, S_\tau^{-1}) \leq |D|^4|A|^2\tau^2T_3(G)|G|^2|S_\tau|.
\]
Using $|S_\tau| \ll |A|^6/\tau^3$ and our lower bound $\tau \gg |D|^{-2}|A|^{-1}|G|^{-4}\sigma_*^4$, we get
\[
\sigma_*^{12} \cdot (|D|^{-2}|A|^{-1}|G|^{-4}) \ll \sigma_*^8 \tau \lesssim |D|^4|A|^8|G|^2T_3(G).
\]
Applying Lemma 14 and Lemma 5 to estimate $T_3(G)$, we derive
\[
T_3(G) \leq |B||C| \sum_x r_{(B-B)(C-C)}(x) + |B|^4|C|^4 \lesssim (|B||C|)^{7/2}(E^+(B, C))^{1/2}.
\]
Here we do not need to have deal the term $(|B||C|)^4/p$ in Lemma 5 because one can consider the balanced function of $(-B)$, $C$ in the set of actions $G$ (see details in (31)). Combining the last inequality with (33), we obtain
\[
\sigma_*^{12} \lesssim |D|^6(|B||C|)^6|A|^9(|B||C|)^{7/2}E^+(B, C)^{1/2}.
\]
This completes the proof.

**Remark 17** One can apply general results from [23], [37] to nontrivially estimate $T_4(G)$ via $T_2(G)$ in Theorem 10 (see formula (31)) but we prefer to use $T_3(G)$ because it gives better bounds.
Using a trivial bound \( E^+(B, C) \leq (|B||C|)^{3/2} \), we obtain

**Corollary 18** Let \( A, B, C, D \subseteq \mathbb{F}_p \) be sets. Then for any \( \lambda \neq 0 \), one has

\[
|\{(a + b)(c + d) = \lambda : a \in A, b \in B, c \in C, d \in D\}| - \frac{|A||B||C||D|}{p} \lesssim |A|^{1/4}|B||C||D|^{1/2} + |A|^{3/4}(|B||C|)^{41/48}|D|^{1/2}.
\] (35)

**Remark 19** In [31, Theorem 41] it was proved, in particular, that for any \( c < \frac{1}{192} \) one has

\[
\sum_x r_{(A-A)(A-A)}^2(x) \ll |A|^{13/2-c},
\]

provided \( |A| \leq p^{48/97} \). It gives an improvement of Theorem 16 at least in the symmetric case (in particular, it gives a better bound in formula (36) of Corollary 20 below for small \( A \)).

Now let us obtain an upper bound for size of hyperbola with elements from a set with small sumset. First results of this type were obtained in [23] but our new bound is more "quantitative".

**Corollary 20** Let \( A \subseteq \mathbb{F}_p \) be a set. Suppose that \( |A + A| \leq K|A| \). Then for any \( \lambda \neq 0 \), one has

\[
r_{AA}(\lambda) \lesssim \frac{K^2|A|^2}{p} + K^{5/4}|A|^{23/24}.
\] (36)

Finally, if \( |A - A|^{92} \leq p^{52} \), then

\[
r_{AA}(\lambda) \lesssim \frac{K^2|A|^2}{p} + O(|A|^{149/156}).
\] (37)

**Proof.** Put \( S = A + A \). We have \( A + x \subseteq A + A \) for any \( x \in A \). Hence

\[
r_{AA}(\lambda)|A|^2 \lesssim |\{(a + b)(c + d) = \lambda : a \in A + A, b \in -A, c \in -A, d \in A + A\}|.
\] (38)

Applying the second part of Theorem 16 as well as a trivial estimate \( E^+(-A, -A) \leq |A|^3 \), we get

\[
r_{AA}(\lambda) \lesssim \frac{K^2|A|^2}{p} + K^{3/4}|A|^{3/4} + K^{5/4}|A|^{23/24} \leq \frac{K^2|A|^2}{p} + K^{5/4}|A|^{23/24}.
\]

To obtain the second bound of Corollary 20 we apply estimate (38) and then we use the first part of Lemma 14 directly. It gives us (see formulae (33), (34) from the proof of Theorem 16)

\[
r_{AA}(\lambda) \ll \left(\sum_x r_{(A-A)(A-A)}^2(x)\right)^{1/12}.
\] (39)
To estimate $\sum_x r_{(A-A)(A-A)}^2(x)$, we use the Plünnecke inequality [37], combining with Lemma 6, and obtain
$$\sum_x r_{(A-A)(A-A)}^2(x) \leq |A|^4 E^\times (A - A) \ll_K |A|^{4+32/13},$$
provided $|A - A|_{117} \leq p^{52}|2A - 2A|_{25}$. The last inequality satisfies thanks to our condition $|A - A|_{92} \leq p^{52}$. This completes the proof. 

To compare, using the Szemerédi–Trotter Theorem [36], one can obtain $r_{AA}(\lambda) \ll_K |A|^{2/3}$ for any finite $A \subset \mathbb{R}$ with $|A + A| \leq K|A|$ and $\lambda \neq 0$.

In our final consequence of Theorem 16 we have deal with the case of arithmetic progressions.

**Corollary 21** Let $A, B, C, D \subseteq \mathbb{F}_p$ be sets and $B, C$ be arithmetic progressions with the differences equal one. Then for any $\lambda \neq 0$, one has
$$|\{(a + b)(c + d) = \lambda : a \in A, b \in B, c \in C, d \in D\} - \frac{|A||B||C||D|}{p}| \lesssim$$
$$\lesssim |A|^{1/4}|B||C||D|^{1/2} + |A|^{3/4}|D|^{1/2}(|B||C|)^{5/6} \left( 1 + \left( \frac{|B||C|}{p} \right)^{1/12} \right).$$

**Proof.** Indeed, by Theorem 7, we know that
$$E^\times (B, C) = \frac{|B|^2|C|^2}{p} + O(|B||C| \log^2 p).$$
We have
$$\sum_x r_{(B-B)(C-C)}^2(x) \leq (|B||C|)^2 E^\times (B - B, C - C).$$
After that apply the arguments of the proof of Theorem 16 and our upper bound for $E^\times (B, C)$. It gives us, in particular,
$$T_3(G(-B, C)) \leq |B||C| \sum_x r_{(B-B)(C-C)}^2(x) + |B|^4|C|^4 \ll$$
$$(|B||C|)^3 E^\times (B - B, C - C) + |B|^4|C|^4 \ll (|B||C|)^3 E^\times (B - B, C - C) \ll \frac{(|B||C|)^5}{p} + (|B||C|)^4 \log^2 p$$
and after that we substitute this bound into estimate $\sigma_*$ as in Theorem 16. This completes the proof. 

In a natural way, in the case of arithmetic progressions one can try to estimate higher energies $T_k(G_\lambda(B, C))$. It turns out that in this situation “first–stage” methods [3] work rather good and it will be done in the next Section, see Proposition 27 and Theorem 29 below.
6 Asymmetric results

In [23] and [31] the authors obtain a series of upper bounds for equation (11) in asymmetric cases (i.e. when the set of actions is relatively small). Let us recall two results from these papers.

**Theorem 22** Let \( \lambda \in \mathbb{F}_p^* \), \( f_1, f_2 : \mathbb{F}_p \to \mathbb{C} \) be functions, and \( S, T \subseteq \mathbb{F}_p \) be sets. Also, let \( G = G\lambda(S, T) \) and \( |S||T| \geq p^\delta \). Then there is \( \delta = \delta(\varepsilon) > 0 \) such that

\[
\sum_{g \in G} \sum_{x} f_1(x) f_2(gx) - p^{-1} \left( \sum_{x} f_1(x) \right) \cdot \left( \sum_{x} f_2(x) \right) \leq 2 \| f_1 \|_2 \| f_2 \|_2 |S||T| p^{-\delta} \tag{41}
\]

Further, if \( A, B, C, D \subseteq \mathbb{F}_p \) are any sets with \( |B||C| \geq (|A||D|)^\varepsilon \) and \( \lambda \in \mathbb{F}_p^* \), then

\[
|\{(a + b)(c + d) = \lambda : a \in A, b \in B, c \in C, d \in D\}| - \frac{|A||B||C||D|}{p} \ll \frac{(|A||D|)^{1/2}(|B||C|)^{1-\delta}}{p} \tag{42}
\]

The first part of Theorem 22 is Lemma 53 from [31] and the second part follows by the same arguments (with \( S = B, T = C, f_1(x) = A(x) - |A|/p, f_2(x) = D(x), \) say) if one uses, in addition, any rough incidence result in \( \text{SL}_2(\mathbb{F}_p) \) see, e.g., [24] or our Lemma 11.

Now if the sets \( B, C \) in our sets \( G\lambda(B, C), G\lambda(B) \) are special, then one can improve Theorem 22 as was done in [Theorem 11][23] (or see the sketch of the proof of Theorem 32 below).

**Theorem 23** Let \( A, D \subseteq \mathbb{F}_p \) be any sets, and \( B = 2 \cdot [N], N \leq p^\tau, 0 < \tau < 1/8 \). Suppose that \( |D| \leq p^{1-\delta}, \delta = C_1^{-1/\tau} \). Then

\[
|\{(a + b)(b + d) = 1 : a \in A, b \in B, d \in D\}| \ll \sqrt{|A||D|N} \cdot N^{-C_2\delta/\tau} \tag{43}
\]

Here \( C_1, C_2 > 1 \) are absolute constants.

**Corollary 24** Let \( A \subseteq \mathbb{F}_p \) be a set. Suppose that \( |A| < p^{0.99} \) and \( N \leq p^{\tau_0} \), where \( \tau_0 < 1/8 \) is an absolute constant. Then there is \( i \in 2 \cdot [N] \) such that

\[
\left| (A + i) \cap \frac{1}{A + i} \right| \ll \frac{|A|}{N^c} \tag{44}
\]

where \( c > 0 \) is an absolute constant.

The proof of Theorem 23 uses the following result, see [23, Lemma 27].

**Lemma 25** Let \( G = G(2 \cdot [N]) \subset \text{SL}_2(\mathbb{Z}) \). Then there is an absolute constant \( C_\ast > 0 \) such that for an arbitrary integer \( s \) the following holds

\[
T_s(G) \ll C_\ast |G|^s \tag{45}
\]

In \( \mathbb{F}_p \) the same is true for all \( s \) such that

\[
s \leq \frac{1}{4} \log_N p \tag{45}
\]
Remark 26 Actually, one can check from the proof of [23, Lemma 27], see [23, Theorem 25,29] that condition (45) can be replaced by \( s \leq (\frac{1}{2} - \varepsilon) \log N \) \( p \) for any \( \varepsilon > 0 \) and sufficiently large \( p \) and \( N \). This square root condition looks rather natural. Further, in \( \mathbb{R} \) one can consider a more general family \( G = G(\omega \cdot [N]) \subset SL_2(\mathbb{R}) \), where \( |\omega| \geq 2 \), see the proof of Theorem 29.

In the real setting one can easily calculate the constant \( c \) from (44) in a simple way. Surprisingly, that our saving in the asymmetric case of sets of rather different cardinality (when \( |A|, |D| \) are large comparable to \( N \), see below) is better than the famous Szemerédi–Trotter Theorem gives us (of course it is because a pair of our sets are arithmetic progressions). Although the focus of this paper is \( \mathbb{F}_p \) we give a proof of this result here because its simplicity and because we will use some parts of the proof later.

Proposition 27 Let \( \lambda \neq 0 \) be any number, \( A \subset \mathbb{R} \) be a set and let \( N \geq 1 \) be an integer. Then for \( |\omega| \geq 2 \) one has

\[
|\{(a+b)(b+d) = 1 : a \in A, d \in D, b \in \omega \cdot [N]\}| \ll \sqrt{|A||D|N \max \{ |D|^{-1/2}, N^{-1/5} \}}. \tag{46}
\]

More precisely, if for a certain \( l \) the following holds \( |D|^2 \geq N^l \), then

\[
|\{(a+b)(b+d) = 1 : a \in A, d \in D, b \in \omega \cdot [N]\}| \ll \sqrt{|A||D|N^{2/3} \cdot |D|^{1/6l}}. \tag{47}
\]

Proof. Denote by \( \sigma \) the left-hand side of (46) and let \( G = G(\omega \cdot [N]) \). Using Lemma 12, we obtain that for any \( k \geq 1 \) the following holds

\[
\sigma^{2k} \leq |A|^k |D|^{k-1} \sum_g r_{GG^{-1}k}(g) \sum x D(x) D(gx).
\]

Applying the Szemerédi–Trotter Theorem [39], we see that either

\[
\sigma \ll |G| \sqrt{|A||D|} |D|^{-1/2k} \tag{48}
\]

or

\[
\sigma \ll |G|^{1/3} \sqrt{|A||D|} |D|^{1/6k} T_{2k}(G)^{1/6k}. \tag{49}
\]

By Lemma 25(also, see Remark 26) we know that \( T_{s}(G) \ll C_{s} |G|^{s} \), where \( C_{s} > 0 \) is an absolute constant and \( s \) is an arbitrary integer. Hence in the second case, we have

\[
\sigma \ll |G|^{2/3} \sqrt{|A||D|} |D|^{1/6k}. \tag{50}
\]

Now let \( k \) be the first number such that (48) takes place. We can assume that \( k > 1 \) because otherwise we are done. Then (50) holds with \( k - 1 \geq 1 \). Comparing bounds (48), (50), we obtain

\[
\sigma \ll \sqrt{|A||D|} G^{1/k-1/2k} \leq \sqrt{|A||D|} G^{1/s}
\]

as required. To prove (47) suppose that (50) does not hold with \( k = l \). Then by (48), we get

\[
|G|^{2/3} \sqrt{|A||D|} |D|^{1/6l} \ll \sigma \ll |G| \sqrt{|A||D|} |D|^{-1/2l}
\]

or, in other words, \( |D| \ll |G|^{l/2} = N^{l/2} \) and this contradicts with our assumption. This completes the proof. \( \square \)
Example 28 Suppose that \( Q \) is an arithmetic progression on length \( 2M \) and put
\[
A = Q \bigcup \{ \bigcup_{j=1}^{N} (Q^{-1} + j) \}, \quad M \geq N.
\]
Then it is easy to see that for any \( i, j \in [N] \) the set \((A - i) \cap (A - j)^{-1}\) contains \([M]\). Also, choosing \( Q \) in an appropriate way, we can assume that \( |A| \sim N|Q| \). Hence our saving 1/5 in (48) (or analogously the saving 1/3 in (47)) cannot be replaced by any number strictly greater than 1.

Now we formulate an analogue of Proposition 27 for an arbitrary set \( B \subset \mathbb{Z} \).

**Theorem 29** Let \( A, D \subset \mathbb{R} \), \( B, C \subset \mathbb{Z} \) be sets, and \( \lambda \neq 0 \) be any number. Then
\[
|\{(a + b)(c + d) = \lambda : a \in A, d \in D, b \in B, c \in C\}| \ll_{|\lambda|} |A||D||B||C| \cdot \max\{|D|^{-1/2}, \rho(B, C)\},
\]
where
\[
\rho(B, C) := \max_{k \geq 2} \left( \frac{E^+(C)^{k-1}E^+(B)^{k-2}}{|B|^{4k-6}|C|^{4k-4}} \right)^{1/(8k-6)} \leq (|B|^k|C|^k)^{-1} - \frac{1}{8k-6}.
\]
More precisely, if for a certain \( l \) the following holds \(|D|^4 \geq |B|^{4l-2} |C|^{4l} E^+(C)^{-l} E^+(B)^{-l+1} \), then
\[
|\{(a + b)(c + d) = \lambda : a \in A, d \in D, b \in B, c \in C\}| \ll_{|\lambda|} (|B||C|)^{1/3} \sqrt{|A||D|} \cdot |D|^{1/6l} \left( |B|^2 E^+(C)^l E^+(B)^{-l+1} \right)^{1/6l}.
\]

**Proof.** We use the arguments from the proof of Proposition 27. In particular, applying Lemma 12 we see that our aim is to estimate the quantity \( \mathcal{T}_{2k}(G_{\lambda}(-B, C)) \) as in (49) but before we need some preparations. Put \( m = 2|\lambda| \) and \( G = G_{\lambda}(-B, C) \). Split \( B \) onto odd/even numbers \( B_0, B_1 \), further, split \( C \subset \mathbb{Z} \) onto congruence classes \( C_j \) modulo \( m \) and use Lemma 9 or its consequence Corollary 10 to estimate \( \mathcal{T}_{2k}(G) \) via \( \mathcal{T}_{2k} \) on sets \( G_{\lambda}(B_i, C_j) \). In the notation of the beginning of Section 5 we get
\[
g_1g_2^{-1} \cdots g_{2k-1}g_2^{-1} = u_{a_1}^{*}u_{\lambda-1}(b_{2-b_1})u_{a_3-a_2} \cdots u_{\lambda-1}(b_{2k-b_{2k-1}})u_{a_{2k}}^{-1} = \bar{y}_1\bar{y}_2^{-1} \cdots \bar{y}_{2k-1}\bar{y}_{2k}^{-1} = u_{a_1'}^{*}u_{\lambda-1}(b_{2'_{2-b_1'}})u_{a_3'-a_2'} \cdots u_{\lambda-1}(b_{2'_{2k-b_{2k-1}}})u_{a_{2k}'}^{-1},
\]
where variables \( a_j, a_j' \) and \( b_j, b_j' \) are from \((-B), C\), correspondingly. Further, it is well–known (see, e.g., [20]) that the matrices
\[
\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}
\]
generate a free subgroup of \( \text{SL}_2(\mathbb{Z}) \), provided \(|s|, |t| \geq 2 \) or even when \(|st| \geq 4 \) (it easily follows from the ping–pong lemma). Rewriting (53) as
\[
(u_{\lambda-1}(b_{2k-b_{2k-1}}))^{-1} \cdots (u_{\lambda-1}(b_{2-b_1}))^{-1}u_{a_1-a_1'}u_{\lambda-1}(b_{2-b_1})u_{a_3-a_2} \cdots u_{\lambda-1}(b_{2k-b_{2k-1}})u_{a_{2k}-a_{2k}}^{-1}
\]

(55)
we see that the number solutions to the last equation is \(|B|^2(\mathcal{E}^+(C))^k(\mathcal{E}^+(B))^{k-1}\) (for \(k = 1\) it coincides with the second bound of Lemma 14 in the symmetric case). Indeed, since \(B_i - B_i \subset 2 \cdot \mathbb{Z}, C_j - C_j \subset m \cdot \mathbb{Z}\), it follows that all \(u_h\) and \(u_h^\prime\) are powers of matrices from \((54)\) with \(s = 2\) and \(|t| = |m/\lambda| \geq 2\), correspondingly, and hence equation \((55)\) has no nontrivial solutions. Thus, as in \((48), (49)\), we have either

\[
\sigma \ll |B||C|\sqrt{|A||D||D|^{-1/2k}}
\]

or

\[
\sigma \ll (|B||C|)^{1/3}\sqrt{|A||D||D|^{1/6k}}(|B|^2\mathcal{E}^+(C)^k\mathcal{E}^+(B)^{k-1})^{1/6k}
\]

and hence as before

\[
\sigma \ll \sqrt{|A||D||B||C|} \cdot \left(\frac{\mathcal{E}^+(C)^{k-1}\mathcal{E}^+(B)^{k-2}}{|B|^{4k-6}(\mathcal{E}^+(C)^{4k-4})^{1/6k-6}}\right)^{1/(8k-6)} = \sqrt{|A||D||B||C|\rho(B, C)}
\]

as required. To obtain \((52)\) we use the same calculations as in Proposition \(27\). This completes the proof.

As we have seen the proof of Theorem 29 gives us an analogue of Lemma 25, which we formulate in \(\mathbb{Z}\) and in \(\mathbb{F}_p\). Write \(a^+ := \max\{a, 1\}\).

Lemma 30 Let \(B, C \subseteq \mathbb{Z}, \lambda \neq 0\) be a real number. Put \(G = G_\lambda(B, C)\). Then for an arbitrary integer \(s\) the following holds

\[
T_{2k}(G) \leq (8|\lambda|^+)^{4k}|C|^{3k}|B|^{3k-1}.
\]

Now, let \(B, C \subseteq [N] \subset \mathbb{F}_p\) and \(\lambda \in \mathbb{F}_p^\ast\), \(|\lambda| \leq N^2\). Then

\[
T_{2k}(G) \leq (8|\lambda|)^{4k} \left(\frac{2(2N^2)^{2k}}{p} + 1\right)^4 |B|^{3k-1}|C|^{3k}.
\]

Proof. Take \(m = 2||\lambda|| \leq 4|\lambda|^+\) and split \(B\) onto odd/even numbers and \(C\) onto congruence classes modulo \(m\). Using Corollary \(10\) and calculations in \((53), (55)\), we obtain

\[
T_{2k}(G) \leq (2m)^{4k}|B|^2(\mathcal{E}^+(C))^k(\mathcal{E}^+(B))^{k-1} \leq (8|\lambda|^+)^{4k}|B|^2(\mathcal{E}^+(C))^k(\mathcal{E}^+(B))^{k-1} \leq (8|\lambda|^+)^{4k}|C|^{3k}|B|^{3k-1}
\]

as required.

Now let us obtain bound \((58)\) and again we split \(B, C\) modulo two and \(m\), correspondingly, but before let us remark that by the definition of the set \(G\) the operator \(l^2(\mathbb{R}^2)\)–norm of any element of \(g \in G\) is

\[
\|g\| := \sup_{\|x\|=1} \|gx\|_2 \leq \sqrt{1 + |a|^2 + |b|^2 + (|ab| + |\lambda|)^2} \leq \sqrt{1 + 2N^2 + (N^2 + N^2)^2} \leq 2N^2.
\]

Hence \(\|g_1 \ldots g_{2k}\| \leq (2N^2)^{2k}\) and \(g_1 \ldots g_s \equiv g_1' \ldots g_{2k}' \pmod{p}\) implies that

\[
g_1 \ldots g_{2k} = g_1' \ldots g_{2k}' + sp,
\]

(59)
where for a matrix $s$ one has $p\|s\| \in [-2N^2 - 2, 2N^2 - 2]$ (see similar arguments in [21], [11], [4]). Clearly, there are at most $(2(2N^2 - 2k)p - 1)^3$ of such matrices $s$. Fixing $s$ and $g_1, \ldots, g_{2k} \in G$ in (59), we need to solve this equation in $g_1, \ldots, g_{2k}$. Thanks to (55) there are at most $|B|^{4 - 1} |C|^k$ choices for $g_1, \ldots, g_{2k}$. This completes the proof.

\[ \square \]

Remark 31 Notice that there is a universal way to estimate the energy $T_{2k}(G \lambda (B, C))$ in any field, namely, by Lemma 9 we always have $T_{2k}^2(G \lambda (B, C)) \leq T_{4k}^+(B) T_{4k}^+(C)$. Again it connects the problem about incidences for hyperbolas with ordinary additive energies of sets.

The same arguments as in the proof of Theorem 23, Theorem 29 and Lemma 30 (or see the proof of [23] Theorem 24) give us an analogous result for subsets of $[N]$ and for our two-parametric family of transformations from $SL_2(\mathbb{F}_p)$.

Theorem 32 Let $\lambda \in \mathbb{F}_p^*$, $A, D \subseteq \mathbb{F}_p$ be any sets, and $B, C \subseteq [N]$, $|B| |C| \leq N^\varepsilon$, $N \preceq p^\tau$, $0 < \tau < 1/8$. Suppose that $|D| \leq p^{1 - \delta}$, $\delta = C_1^{-1/\varepsilon}$ and $|\lambda| \leq (|B| |C|)^{1/8}/8$. Then

\[ |\{(a + b)(c + d) = \lambda : a \in A, b \in B, c \in C, d \in D\}| \ll \sqrt{|A| |D| |B| |C|} \cdot (|B| |C|)^{-C_2\delta/\varepsilon}. \quad (60) \]

Here $C_1, C_2 > 1$ are absolute constants.

Sketch of the proof. Let $\sigma$ be the left-hand side of (60) and $G = G \lambda (-B, C)$. We take $m$ such that $N^{4m} \sim p$ and thus by Lemma 30 one has $T_{2m}(G) \ll (8|\lambda|)^{4m} |G|^{3m}$. Considering $\nu(g) = r_{(G \lambda(-1))^{2m}}(g)$, we have by estimate (17) that $\|\nu\| \ll (8|\lambda|)^{4m} |G|^{3m}$ and similar for the intersection of $\nu$ with any proper subgroup of $SL_2(\mathbb{F}_p)$, see [23], [23]. By Lemma 9 one has

\[ \sigma^{4m} \leq |A|^{2m} |D|^{2m - 1} \sum_g r_{(G \lambda(-1))^{2m}}(g) \sum_x D(x) D(gx) = |A|^{2m} |D|^{2m - 1} \sum_g \nu(g) \sum_x D(x) D(gx). \]

Put $K = |G|^{m/2}$. Thanks to our condition $|\lambda| \leq (|B| |C|)^{1/8}/8$, we have $\|\nu\| \ll (8|\lambda|)^{4m} |G|^{3m} \leq |G|^{4m}/K$ and similar for the intersection of $\nu$ with any proper subgroup of $SL_2(\mathbb{F}_p)$. Then by general expansion result in $SL_2(\mathbb{F}_p)$, see [23] Theorem 9 or just formula of Theorem 22 we get

\[ \sigma^{4m} \ll \frac{|A|^{2m} |D|^{2m + 1} |G|^{4m}}{p} + |A|^{2m} |D|^{2m} |G|^{4m} p^{-\eta}, \]

where $\eta \ll 2^{-k}$ and $k \leq \log p/\log K \sim \log N/\log |G| \leq \varepsilon^{-1}$. The first term in the last formula is negligible because our assumption $|D| \leq p^{1 - \delta}$. Thus, our saving is $p^{\eta/4m} \gg |G|^{C_2\delta/\varepsilon}$. This completes the proof.

\[ \square \]

We write Theorem 32 similar to Theorem 23 for compare these two results. Of course constants $C_1, C_2$ in (60) are worse than in (63).
7 On bilinear forms of Kloosterman sums

Let $\mathbb{F}$ be a finite field, $\alpha : \mathbb{F} \to \mathbb{C}$, $\beta : \mathbb{F} \to \mathbb{C}$ be two weights and let

\[ K(n, m) = \sum_{x \in \mathbb{F}^{*}} e(nx + mx^{-1}) = K(mn, 1) \]

be the Kloosterman sum. We are interested in bilinear forms of Kloosterman sums [15]–[17], that is, expressions

\[ S(\alpha, \beta) = \sum_{n, m} \alpha(n) \beta(m) K(n, m). \]

Using the definition of the Fourier transform (17), we see that

\[ S(\alpha, \beta) = \sum_{x} \hat{\alpha}(x) \hat{\beta}(x^{-1}). \]

From the Parseval identity (8) and the Cauchy–Schwarz inequality, we obtain

\[ S(\alpha, \beta) \leq p \| \alpha \|_2 \| \beta \|_2 \]  

(62)

and applying usual upper bound for Kloosterman sum, as well as the Cauchy–Schwarz inequality again, we get

\[ S(\alpha, \beta) \leq 2\sqrt{p} \| \alpha \|_1 \| \beta \|_1 \leq 2\sqrt{p} \| \alpha \|_2 \| \beta \|_2 \sqrt{\text{supp} \alpha \| \text{supp} \beta \|}. \]

(63)

Both basic bounds (62), (63) give $p^{3/2}$ for, say, $\alpha$ and $\beta$ equal the characteristic function of some sets of sizes $\sqrt{p}$. This $p^{3/2}$ estimate is a kind of barrier and our task is to beat it for wide range of functions $\alpha$, $\beta$.

The next general result demonstrates that the quantity $S(\alpha, \beta)$ is connected with a sum–product question, namely, with the counting of incidences for some hyperbolas. Actually, even simple formula (61) shows that this problem has the sum–product flavour. Indeed, suppose for simplicity that $\alpha$ is the characteristic function of a progression, then the question about estimation of bilinear sums is equivalent to the problem how the inverse of a progression correlates with the set of large Fourier coefficients of $\beta$. In other words, it is a question about how additive and multiplicative structure agree.

**Theorem 33** Let $\alpha_1, \alpha_2, \beta_1, \beta_2 : \mathbb{F}_p \to \mathbb{C}$ be functions, $\varepsilon > 0$. Then either

\[ S(\alpha_1 \alpha_2, \beta_1 \beta_2) \leq p^{1/2+\varepsilon} \min\{\|\alpha_1\|_2 \|\alpha_2\|_2 \|\beta_1\|_2 \|\beta_2\|_2 \|\alpha_1 \alpha_2\|_2\} \]  

(64)

or

\[ S(\alpha_1 \alpha_2, \beta_1 \beta_2) \leq \frac{\|\alpha_1\|_2 \|\beta_1\|_2 \|\alpha_2\|_2 \|\beta_2\|_2}{p} + \|\alpha_1\|_2 \|\beta_1\|_2 \|\alpha_2\|_W \|\beta_2\|_W p^{1-\delta}, \]

(65)

where $\delta(\varepsilon) > 0$ depends on $\varepsilon$ only. In particular, if $\|\alpha_2\|_W, \|\beta_2\|_W \leq 1$ and if $\|\alpha_2\|_2^2$ or $\|\beta_2\|_2^2$ is at most $p^{1-c}$, $c > 0$, then

\[ S(\alpha_1 \alpha_2, \beta_1 \beta_2) \lesssim \|\alpha_1\|_2 \|\beta_1\|_2 p^{1-\delta}, \]

(66)

where $\delta(c) > 0$ is a positive constant. Here the sign $\lesssim$ depends on $\log(\|\alpha_1\|_\infty \|\alpha_2\|_\infty \|\beta_1\|_\infty \|\beta_2\|_\infty)$. 

Proof. By (61) and (10), applied for the convolution, we have

\[ S(\alpha_1\alpha_2, \beta_1\beta_2) = |F|^{-2} \sum_x (\hat{\alpha}_1 * \hat{\alpha}_2)(x)(\hat{\beta}_1 * \hat{\beta}_2)(x^{-1}). \]

Let \( A_j = \{ x : 2^{j-1} < |\hat{\alpha}_1(x)| \leq 2^j \}, A'_j = \{ x : 2^{j-1} < |\hat{\alpha}_2(x)| \leq 2^j \}, B_j = \{ x : 2^{j-1} < |\hat{\beta}_1(x)| \leq 2^j \}, B'_j = \{ x : 2^{j-1} < |\hat{\beta}_2(x)| \leq 2^j \}. \) The number of such sets is at most \( L = 2 \log(\|\hat{\alpha}_1\|_\infty\|\hat{\alpha}_2\|_\infty\|\hat{\beta}_1\|_\infty\|\hat{\beta}_2\|_\infty) \) and our sing \( \lesssim \) below depends on this quantity. By the pigeon–hole principle there are \( j_1, j_2, j_3, j_4 \) and sets \( A_{j_1}, A'_{j_2}, B_{j_3}, B'_{j_4}, \) which we denote as \( A, A', B, B' \) and numbers \( \Delta = 2^{j_1}, \Delta' = 2^{j_2}, p = 2^{j_3}, \rho = 2^{j_4} \) such that

\[ S(\alpha_1\alpha_2, \beta_1\beta_2) \lesssim \Delta'\rho'|F|^{-1} \sum_x (A*\bar{A})(x)|\hat{\beta}_1\hat{\beta}_2(x^{-1})| \lesssim \Delta'\rho'|F|^{-1} \sum_x (A*\bar{A})(x)(B*\bar{B}')(x^{-1}). \]

If \(|A'| \) (or \(|B'| \)) is at most \(|F|^\epsilon \), then by (12), the Parseval and the previous formulae, we have

\[ S^2(\alpha_1\alpha_2, \beta_1\beta_2) \lesssim (\Delta'\rho'|F|^{-1})^2|F|^2\|\beta_1\beta_2\|^2|A||A'| \lesssim |F|^1+\epsilon\|\beta_1\beta_2\|^2\|\alpha_1\|^2\|\alpha_2\|^2 \]

and thus we obtain (64). Now we use the fact that \( F \) equals \( F_p \) to apply our incidences results for hyperbolas. Using bound (11) from Theorem 22 as well as formula (12), we get

\[ S(\alpha_1\alpha_2, \beta_1\beta_2) \lesssim \frac{\Delta'\rho'|A||B|}{p^1} + \|\alpha_1\|_2\|\beta_1\|_2|A'||B'|-\delta \ll \]

\[ \ll \frac{\|\alpha_1\|_1\|\beta_1\|_1\|\alpha_2\|_1\|\beta_2\|_1}{p} + \|\alpha_1\|_2\|\beta_1\|_2\|\alpha_2\|_1\|\beta_2\|_1 \ll \frac{\|\alpha_1\|_2\|\beta_1\|_2\|\alpha_2\|_1\|\beta_2\|_1}{p} + \|\alpha_1\|_2\|\beta_1\|_2\|\alpha_2\|_w\|\beta_2\|_w p^{1-\delta} \ll \]

as required.

Finally, inequality (66) follows from (64), (65) by the inverse formula (10), which gives

\[ \|\beta_1\beta_2\|_2 \leq \|\beta_2\|_\infty\|\beta_1\|_2 \leq \|\beta_2\|_w\|\beta_1\|_2 \ll \|\beta_1\|_2 \]

and hence bound (64) is negligible. The first term in (65) is less than the second one, again because (12) (which gives \( \|\alpha_2\|_1, \|\beta_2\|_1 \leq p \)) and our assumption that \( \|\alpha_2\|_2^2 \) or \( \|\beta_2\|_2^2 \) is at most \( p^{1-\epsilon} \). This completes the proof. \( \square \)

Once again the main advantage of our result is its generality. For example, one can easily consider more general sets than arithmetic progressions in (66), say, Bohr sets of bounded dimension (37).

It is easy to check that the last result is better than (33, Theorem 7], as well as (10, Theorem 1.17(2)]. Thus, using relatively simple methods from Additive Combinatorics we break \( p^{3/2} \) barrier in this problem.

Now we obtain a result on bilinear Kloosterman sums in a specific situation when the supports of the weights belong to arithmetical progressions.
Theorem 34 Let $\alpha : \mathbb{F}_p \to \mathbb{C}$, $\beta : \mathbb{F}_p \to \mathbb{C}$ be a function, $\text{supp}\alpha \subseteq \{N\} + t_1$, $\text{supp}\beta \subseteq \{M\} + t_2$ and $t_1, t_2 \in \mathbb{F}_p$ be some shifts. Then

$$S(\alpha, \beta) \lesssim \|\beta\|_2 \left(\|\hat{\alpha}\|_{L^{4/3}} N^{7/48} M^{7/48} p^{23/24} + (\|\alpha\|_2^2\|\alpha\|_1^2)^{1/2} p^{3/4} + \|\alpha\|_W p\right),$$

(67)

and if $M^2 N^2 \|\hat{\alpha}\|_{L^{4/3}}^2 < p \|\alpha\|_2^2$, then

$$S(\alpha, \beta) \lesssim \|\beta\|_2 \left(\|\hat{\alpha}\|_{L^{4/3}}^6/N^{1/7} M^{1/7} p^{13/14} + (\|\alpha\|_2^2\|\alpha\|_1)^{1/2} p^{3/4} + \|\alpha\|_{L^{4/3}} p^{13/12}\right).$$

(68)

Here the sign $\lesssim$ depends on $\log(MN \|\hat{\alpha}\|_\infty \|\hat{\beta}\|_\infty)$.

Proof. We can suppose that $N, M$ and $p$ are sufficiently large because otherwise the result is trivial. Let us begin with (67). Let $B$ and $C$ be the characteristic functions of the arithmetic progressions $\{N\} + t_1$ and $\{M\} + t_2$, respectively. Then for any weights $\alpha \subseteq B$, $\beta \subseteq C$ we can write $\alpha = \alpha B$ and $\beta = \beta C$. After that we repeat the arguments of the proof of Theorem 33. Namely, splitting the level sets of the functions $\hat{\alpha}, \hat{B}, \hat{C}, \hat{\beta}$, we obtain sets $A, B', C', D$ and positive numbers $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ such that

$$S(\alpha, \beta)(p^2 \Delta_1 \Delta_2 \Delta_3 \Delta_4)^{-1} \lesssim \frac{|A||B'||C'||D|}{p} + |A|^{1/4}|B'||C'| |D|^{1/2} +$$

$$+ |A|^{3/4}|D|^{1/2}(|B'||C'|)^{41/48}.$$  

(69)

(70)

Here we have used Corollary 18 and again in (69) and below our sign $\lesssim$ depends on $L = \log(MN \|\hat{\alpha}\|_\infty \|\hat{\beta}\|_\infty)$. We will show later that the first two terms in (69) give the last two terms in (67) and now let us consider the third term in (70). From Parseval identity (8), we have

$$(\Delta_3^2 |D|)^{1/2} \leq \sqrt{p} \|\beta\|_2.$$  

(71)

In view of Lemma 8 we get

$$(\Delta_4^{48/41}|B'|)^{41/48} \ll p^{41/48} N^{7/48}$$

(72)

and the same for $A$ and $C'$. Thus, we obtain

$$S(\alpha, \beta) \lesssim p^{23/24} \|\beta\|_2 \|\hat{\alpha}\|_{L^{4/3}} (NM)^{7/48}$$

as required. It remains to show that two terms in (69) give the last two terms in (67). In view of Lemma 8 and inequality (12) the first one gives

$$(\Delta_1 \Delta_2 \Delta_3 \Delta_4) \frac{|A||B'||C'| |D|}{p^2} \lesssim p \|\alpha\|_W \|\beta\|_W \ll p \|\alpha\|_W \|\beta\|_W 2$$

and by the same lemma and the Parseval identity, as well as (14), (71), (72), we have for the second term

$$(\Delta_1 \Delta_2 \Delta_3 \Delta_4) \frac{|A|^{1/4}|B'||C'| |D|^{1/2}}{p^2} \lesssim p^{3/4} E^+(\alpha)^{1/4} \|\beta\|_2 \lesssim p^{3/4}(\|\alpha\|_2^2\|\alpha\|_1)^{1/2} \|\beta\|_2.$$
Now let us prove (68). Let $Q_1, Q_2$ be symmetric arithmetic progressions with steps equal one such that

$$|Q_1|N \ll p, \quad |Q_2|M \ll p.$$  \hfill (73)

Let $Z \subseteq \mathbb{F}_p$ be any set and write $\hat{\alpha}_Z(r) := \hat{\alpha}(r)Z(r)$ and similar for $\hat{\beta}$ (in this part of the proof one can put, simply, $Z = \mathbb{F}_p$). If $t_1 = 0$, then we have

$$\|\hat{\alpha}_Z(r) - |Q_1|^{-1}(\hat{\alpha}_Z * Q_1)(r)\|^2 = |Q_1|^{-2} \sum_{r \in Z} \left( \sum_{s \in Q_1} (\hat{\alpha}_Z(r) - \hat{\alpha}_Z(r + s)) \right)^2 = \|Q_1|^{-2} \sum_{r \in Z} \left( \sum_{s \in Q_1} \alpha(x)e(-rx) \sum_{s \in Q_1} (1 - e(-sx)) \right)^2 \leq \|Q_1|^{-2} \sum_{r \in \mathbb{F}_p} \left( \sum_{x=1}^N \alpha(x)e(rx) \sum_{s \in Q_1} (1 - e(sx)) \right)^2 = \|Q_1|^{-2} p \sum_{x=1}^N |\alpha(x)|^2 \sum_{s \in Q_1} (1 - e(sx))^2 \leq \|\alpha\|^2 \frac{N^4 Q_1^4}{p^3}.$$

and because $Q_1$ is a symmetric set, as well as conditions (73), we obtain

$$\|\hat{\alpha}_Z(r) - |Q_1|^{-1}(\hat{\alpha}_Z * Q_1)(r)\|^2 \ll p|Q_1|^{-2} \sum_{x=1}^N |\alpha(x)|^2 \sum_{s \in Q_1} \frac{|s|^2 |x|^2}{p^2} \ll \|\alpha\|_2 \frac{N^4 Q_1^4}{p^3}.$$  \hfill (74)

The same holds for $\beta$ and below we will write $\hat{\alpha}(r) = \hat{\alpha}_0(r)e(-t_1r)$, where $\operatorname{supp} \alpha_0 \subseteq [N]$ and similar for $\hat{\beta}$. Clearly, $\|\alpha\|_L^p = \|\alpha_0\|_L^p$, $\|\alpha\|_q = \|\alpha_0\|_q$ and $\|\beta\|_L^p = \|\beta_0\|_L^p$, $\|\beta\|_q = \|\beta_0\|_q$ for any $q$. Hence by (64), the Cauchy–Schwarz inequality and formula (8), we get

$$S(\alpha, \beta) = \sum_r \hat{\alpha}_0(r)e(-rt_1)\hat{\beta}(r^{-1}) = |Q_1|^{-1} \sum_r (\hat{\alpha}_0 * Q_1)(r)e(-rt_1)\hat{\beta}(r^{-1}) + \|\alpha\|_2 \|\beta\|_2 \cdot O\left( \frac{N^2 Q_1^2}{p} \right) \hfill (74)$$

$$= |Q_1|^{-1} \sum_r (\hat{\alpha}_0 * Q_1)(r^{-1})\hat{\beta}_0(r)e(-t_2r - t_1r^{-1}) + \|\alpha\|_2 \|\beta\|_2 \cdot O\left( \frac{N^2 Q_1^2}{p} \right) = \left( |Q_1||Q_2| \right)^{-1} \sum_r (\hat{\alpha}_0 * Q_1)(r)(\hat{\beta}_0 * Q_2)(r^{-1})e(-t_2r^{-1} - t_1r) + \|\alpha\|_2 \|\beta\|_2 \cdot O\left( \frac{N^2 Q_1^2}{p} + \frac{M^2 Q_2^2}{p} \right).$$

Now our task is to estimate the first sum in (74), which we denote as $\sigma$. As before splitting the level sets of the functions $\hat{\alpha}_0, \hat{\beta}_0$, we obtain sets $A, D$ and numbers $\Delta_1, \Delta_2$ such that

$$\sigma \lesssim \Delta_1 \Delta_2 (|Q_1||Q_2|)^{-1} \sum_r (A * Q_1)(r)(D * Q_2)(r^{-1}).$$  \hfill (75)

The trick with $\hat{\alpha}_Z(r), \hat{\beta}_Z(r)$ allows us to choose $Q_1, Q_2$ not depending on the sets $A, D$ (but here, actually, we do not need in this additional information). Applying Corollary 21 we have

$$\sigma(\Delta_1 \Delta_2)^{-1} \lesssim \frac{|A||D|}{p} + |A|^{1/4} |D|^{1/2} + |A|^{3/4} |D|^{1/2} (|Q_1||Q_2|)^{-1/6} \left( 1 + \left( \frac{|Q_1||Q_2|}{p} \right)^{1/12} \right).$$  \hfill (76)
Again two first terms in the last formula do not exceed two last terms in (68). Additionally, we consider the term \( \frac{\left| Q_1 \right| \left| Q_2 \right|}{p} \) later. As before, one has

\[
\Delta_1 \Delta_2 |A|^{3/4} |D|^{1/2} (|Q_1||Q_2|)^{-1/6} \ll p^{5/4} \parallel \hat{\alpha} \parallel_{L^4/3} \parallel \beta \parallel_2 (|Q_1||Q_2|)^{-1/6}.
\]

Using the last formula and recalling (74), we see that the optimal choice of the parameters \( Q_1, Q_2 \) is \( NQ_1 = MQ_2 \) and hence \( Q_1 = p^{27/28} M^{1/14} N^{-13/14} (\parallel \hat{\alpha} \parallel_{L^4/3} \parallel \alpha \parallel_2^{-1})^{3/7} \), \( Q_2 = p^{27/28} M^{-13/14} N^{1/14} (\parallel \hat{\alpha} \parallel_{L^4/3} \parallel \alpha \parallel_2^{-1})^{3/7} \). The assumption \( M^2 N^2 \parallel \hat{\alpha} \parallel_{L^4/3}^{12} < p \parallel \alpha \parallel_2^{12} \) guarantees that conditions (73) hold. Thus

\[
S(\alpha, \beta) \lesssim \parallel \beta \parallel_2 N^{1/7} M^{1/7} \parallel \hat{\alpha} \parallel_{L^4/3}^{6/7} \parallel \alpha \parallel_2^{1/7} p^{13/14}.
\]

Finally, let us consider the situation when the term \( \frac{\left| Q_1 \right| \left| Q_2 \right|}{p} \) in (76) dominates. In this case \( |Q_1||Q_2| > p \) and

\[
\Delta_1 \Delta_2 |A|^{3/4} |D|^{1/2} (|Q_1||Q_2|)^{-1/6} \left( \frac{|Q_1||Q_2|}{p} \right)^{1/12} \leq p^{5/4} \parallel \hat{\alpha} \parallel_{L^4/3} \parallel \beta \parallel_2 p^{-1/6} = p^{13/12} \parallel \hat{\alpha} \parallel_{L^4/3} \parallel \beta \parallel_2
\]
as required. \( \square \)

It was proved in [1] Theorem 6.1 that under some mild assumptions on \( N, M \) and the case of the initial interval \( [N] \) one has

\[
S([N], \beta) \ll (\parallel \beta \parallel_1 \parallel \beta \parallel_2)^{1/2} p^{3/4+o(1)} M^{1/12} N^{7/12}.
\]

Our bound (67) is better (let \( \beta(x) = [M](x) \) for simplicity) in the case when \( p^{10} \ll M^9 N^9 \) and \( M^2 \gg N, M^4 N^7 \gg p^3 \). Obviously, more precise estimate (68) is even better.

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I.D. Shkredov
Steklov Mathematical Institute,
ul. Gubkina, 8, Moscow, Russia, 119991
and
IITP RAS,
Bolshoy Karetny per. 19, Moscow, Russia, 127994
and
MIPT,
Institutskii per. 9, Dolgoprudnii, Russia, 141701
ilya.shkredov@gmail.com