Function–valued stochastic convolutions arising in integrodifferential equations

Anna Karczewska

Institute of Mathematics, University of Zielona Góra
ul. Podgórna 50, 65-246 Zielona Góra, Poland
e-mail: A.Karczewska@im.uz.zgora.pl

November 9, 2018

Abstract

We study stochastic convolutions providing by fundamental solutions of a class of integrodifferential equations which interpolate the heat and the wave equations. We give sufficient condition for the existence of function–valued convolutions in terms of the covariance kernel of a noise given by spatially homogeneous Wiener process.

1 Introduction

The paper is concerned with the stochastic integrals of the form

\[ \int_0^t P(t - s) * (b(u(s))dW(s)), \quad t \in [0, T]. \]  \hspace{1cm} (1)

In (1), \( P \) is a fundamental solution of some integrodifferential equation introduced in the next section, \( u \) is a stochastic process specified next, \( W \) is a spatially homogeneous Wiener process and \( b: \mathbb{R} \to \mathbb{R} \) denotes the random field.

Key words and phrases: stochastic convolution, integrodifferential equation, homogeneous Wiener process, generalized random field.

2000 Mathematics Subject Classification: primary: 60H20; secondary: 60G20, 60G60, 60H05, 45D05.
The integral (1) looks out formally like those considered in [PeZa2] where the existence of function–valued solutions of nonlinear stochastic wave and heat equations have been analyzed. This is the well-known fact (see, e.g. [Mi]) that solutions to nonlinear wave and heat equations may be represented in terms of fundamental solutions to the Cauchy problems for equations, wave or heat, respectively. This idea has been used in [PeZa2] for representing solutions of stochastic nonlinear equations. In both obtained formulas, the stochastic integrals of the form (1) have appeared. Unfortunately, no analogous formula (with fundamental solution) exists for the solution to nonlinear Volterra equation.

As we have already mentioned, the paper [PeZa2] is concerned with function–valued solutions to the stochastic nonlinear wave and heat equations and provides necessary and sufficient conditions for the existence of such solutions. The problem of existence of function–valued solutions to linear and nonlinear stochastic wave and heat equations has been investigated by many people. They obtained several conditions in terms of function coefficients, the covariance kernel or spectral measure of the noise \( W \). We refer to some papers only: [Da], [DaFr], [DaMu], [KaZa1], [KaZa2], [Pe], [PeZa1] and [PeZa2], for more information see references therein. But only a few papers are concerned with the stochastic Volterra equation. The paper [KaZa3] deals with linear Volterra equation and is written in the spirit of the above mentioned papers.

The aim of this paper is to provide conditions under which the stochastic integral (1) is well–defined process with values in the space \( L^2_v = L^2(\mathbb{R}, v(x)dx) \), where \( v \) is some test function. We follow the idea used in some part of [PeZa2] for our study on the interval \([-R, R]\). It could be done because the fundamental solutions \( P_\alpha \), \( 1 < \alpha < 2 \), considered in the paper are similar in some sense to fundamental solution of the wave equation. Next part of our studies, particularly the passing to the limit as \( R \to +\infty \), bases mostly on specific properties of solutions \( P_\alpha \). The paper proposes a framework for a study of nonlinear stochastic Volterra equations. Till now, to the best of our knowledge, there has been appeared no paper concerning function–valued solutions to nonlinear Volterra equations. We hope the results obtained in the paper will be the first step towards this direction.

In the paper we study only the one-dimensional case. To treat regularity of the stochastic integral (1) in the cases \( d > 1 \) by this method will require development of the results analogous to Fujita’s ones for the higher dimensions. And, to date, this work has not been done.
2 Deterministic integrodifferential equations

The following integrodifferential equation

\[ u(t,x) = g(x) + \int_0^t h(t-s) \Delta u(s,x) \, ds , \quad (2) \]

t > 0, x \in \mathbb{R}, has been treated by many authors (see, e.g. [FrSh], [Fr], [Fu], [Pr] and references therein). The equation (2) is usually used to describe the heat conduction in materials with memory. Very important results have been obtained (see [Fu] and [ScWy]) in the case \( h(t) = t^{\alpha-1}/\Gamma(\alpha) \), for \( 1 \leq \alpha \leq 2 \), where \( \Gamma \) is the gamma function. With such a kernel the equation (2) reads

\[ u(t,x) = g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Delta u(s,x) \, ds, \quad 1 \leq \alpha \leq 2 . \]

(3)

We can see that the family of equations (3) with \( 1 \leq \alpha \leq 2 \) interpolates, in some sense, the heat equation (when \( \alpha = 1 \)) and the wave equation (when \( \alpha = 2 \)). Both papers [Fu] and [ScWy] provide the representation of the solutions to the equations (3) and they are complementary to one another. Additionally, Fujita has characterized the solutions to (3) in detail. Because we shall use his results in the paper, we first recall some facts from Fujita’s work.

Let \( S(\mathbb{R}) \) denote the space of rapidly decreasing functions and \( C([0, +\infty); S(\mathbb{R})) \) be the space consisting of \( S(\mathbb{R}) \)–valued continuous functions on \([0, +\infty) \). For \( 1 \leq \alpha \leq 2 \) and \( t > 0 \), we define the function

\[ q_\alpha(t,\xi) := \exp[-t|\xi|^\delta e^{i\pi\gamma\text{sgn}(\xi)/2}] , \quad \text{where} \quad \delta = 2/\alpha \quad \text{and} \quad \gamma = 2 - 2/\alpha. \]

Define \( P_\alpha(t,x) \) as follows

\[ P_\alpha(t,x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} q_\alpha(t,\xi)e^{-ix\xi}d\xi . \]

(4)

The function \( P_\alpha(t,x) \) has the following properties

\[
\begin{align*}
P_\alpha(t,x) &\geq 0 , & t \in (0, +\infty), \ x \in \mathbb{R} \\
\int_{-\infty}^{+\infty} P_\alpha(t,x)dx &\geq 1 , & t \in (0, +\infty)
\end{align*}
\]

(5)

i.e. is a probability density function and

\[ P_\alpha(t,x) = P_\alpha(xt^{-\alpha/2}) t^{-\alpha/2} , \quad t \in (0, +\infty), \ x \in \mathbb{R} , \]

(6)
where $P_{\alpha}(x) = P_{\alpha}(1, x)$.

Let us recall the representation of the solution to the equation (3). We assume that the function $g$ in (3) belongs to the space $S(\mathbb{R})$.

**Theorem 1** (Theorem A, [Fu])

For every $1 \leq \alpha \leq 2$, the equation (3) has a unique solution $u_{\alpha}(t, x)$ given by

$$u_{\alpha}(t, x) = \begin{cases} \frac{1}{\alpha} \int_{-\infty}^{\infty} g(x - y) P_{\alpha}(t, |y|) \, dy, & \text{for } 1 \leq \alpha < 2 \\ \frac{1}{2} [g(x + t) + g(x - t)], & \text{for } \alpha = 2. \end{cases}$$

Hence, $\frac{1}{\alpha} P_{\alpha}(t, |x|)$ with $1 \leq \alpha \leq 2$ is the fundamental solution of the equation (3). This means that this function is the integral kernel of the operator acting from the initial data $g$ to the solution $u_{\alpha}(t, x)$ of (3).

The next theorems provide important properties of the fundamental solution to the equation (3).

Let us define, for $1 \leq \alpha < 2$ and $x \in \mathbb{R}$ the following functions

$$a_{\alpha}(x) = |x|^2 \exp \left[ \frac{\pi i}{\alpha} \text{sgn}(x) \right],$$

$$b_{\alpha}(x) = |x|^2 \exp \left[ -\frac{\pi i}{\alpha} \text{sgn}(x) \right],$$

and

$$f_{\alpha}(x) = \begin{cases} \frac{\sin(\alpha \pi)}{\pi} \int_{0}^{\infty} \frac{x^{2} t^\alpha - 1 e^{-t}}{t^{2\alpha + 2x^2 t^\alpha \cos(\alpha \pi) + x^4}} \, dt, & (x \neq 0) \\ 1 - \frac{2}{\alpha} & (x = 0) \end{cases}.$$  

**Theorem 2** (Lemma 1.4, [Fu])

For every $1 \leq \alpha < 2$

$$\mathcal{F}^{-1} [F_{\alpha}](x) = \frac{1}{\alpha} P_{\alpha}(|x|),$$

where

$$F_{\alpha}(x) = \frac{1}{\alpha} \left\{ \exp(a_{\alpha}(x)) + \exp(b_{\alpha}(x)) \right\} + f_{\alpha}(x) \quad \text{for } x \in \mathbb{R}$$

and functions $a_{\alpha}, b_{\alpha}, f_{\alpha}$ are defined like in (7).

**Corollary 1** From (3) we obtain

$$\mathcal{F}^{-1}[\exp(a_{\alpha})](x) = P_{\alpha}(-x) \quad \text{and} \quad \mathcal{F}^{-1}[\exp(b_{\alpha})](x) = P_{\alpha}(x).$$
Theorem 3 (Theorem B, [Fu])

For $1 < \alpha < 2$ we have:

1. $P_\alpha(t, |x|)$ is continuous for $t \in (0, +\infty)$, $x \in \mathbb{R}$.

2. $P_\alpha(t, |x|)$ takes its extreme values at $x = \pm c_\alpha t^{\alpha/2}$ (maximum) and $x = 0$ (minimum), where $c_\alpha > 0$ is a constant determined by $\alpha$. The solution is monotone elsewhere.

3. $P_\alpha(t, |x|)$ never vanishes for $t \in (0, +\infty)$, $x \in \mathbb{R}$.

Comment: By Theorem 3 we see, that the fundamental solution to (3) has the similar property to that of the wave equation. For both equations, the points, where the fundamental solution takes its maximum, propagate with finite speed.

The below picture illustrates Theorem 3.

The fundamental solution to the equation (3) is well-known for the limiting case $\alpha = 0$, not considered here, $\alpha = 1$, that is for the heat equation and $\alpha = 2$, that is for the wave equation. For some more information we refer to [Fu], [ScWy] and [Pr]. Let us notice that the point of view on the equation (3) represented by the authors is different. While Friedman and Prüss emphasize the representation of the fundamental solution $P_\alpha(t, |x|)$ to (3) for general case $0 < \alpha < 2$ in terms of the Mittag-Leffler functions (Friedman [Fr] was the first who observed that fact), Fujita and Schneider with Wyss join the fundamental solution with $\alpha$-stable (or Lévy) probability distributions. Moreover, the latter authors propose a possible physical application of fractional diffusion ($0 < \alpha \leq 1$) not considered in this paper.
As we have already written, we consider the same family of the equations like Fujita, that is for $1 \leq \alpha \leq 2$.

## 3 Stochastic integral

Because several papers (cf. [PeZa1], [KaZa2], [KaZa3], [PeZa2]) contain detailed description of the stochastic integral used in this paper, we recall only the most important facts indispensable for understanding the remaining part of the paper.

Let $S(R^d)$ denote the space of all so called rapidly decreasing functions on $R^d$ and $S'(R^d)$ be the space of tempered distributions on $R^d$ (see, e.g. [Yo]). Denote by $S(s)(R^d)$ the space of functions $\varphi \in S(R^d)$ such that $\varphi = \varphi(s)$, where $\varphi(s)(x) = \varphi(-x)$, $x \in R^d$. Analogously, we denote by $S'(s)(R^d)$ the space of all distributions $\xi \in S'(R^d)$ such that $\langle \xi, \varphi \rangle = \langle \xi, \varphi(s) \rangle$ for $\varphi \in S(R^d)$. (In the whole paper, the value of a distribution $\xi$ on a test function $\varphi$ will be denoted by $\langle \xi, \varphi \rangle$.)

Assume that $(\Omega, F, (F_t)_{t \geq 0}, P)$ is a complete probability space. As we have already written, the noise process $W$ is a spatially homogeneous Wiener process. Such process has been used by many authors, see e.g. the papers mentioned above and references therein. Shortly speaking, $W$ is an $S'(R^d)$-valued Wiener process having the following properties:

1. for every $\varphi \in S(R^d)$, $\langle W(t), \varphi \rangle_{t \in [0, +\infty)}$ is a real-valued process;

2. process $W$ has the covariance of the form $E\langle W(t), \varphi \rangle \langle W(t), \psi \rangle = s \wedge t \langle \Gamma, \varphi \ast \psi(s) \rangle$,

where $\varphi \in S(R^d)$, $\psi \in S(s)(R^d)$ and $\Gamma$ is a positive-definite distribution in $S'(R^d)$.

Let us emphasize that the spatially homogeneous Wiener process $W$ for any fixed $t \geq 0$ becomes a stationary, Gaussian, generalized random field.

Since $\Gamma$ is a positive-definite tempered distribution, there exists a positive symmetric so called tempered measure $\mu$ on $R^d$ such that $\Gamma = F(\mu)$. (Let us recall that the measure $\mu$ on $R^d$ is tempered if there exists $r > 0$ such that $\int_{R^d} (1 + |x|^r)^{-1} d\mu(x) < +\infty$.) To underline the fact that the distributions of $W$ are determined by $\Gamma$, we will write $W_\Gamma$. The distribution $\Gamma$ is the space correlation of $W_\Gamma$ and $\mu$ is the spectral measure of $W_\Gamma$ and $\Gamma$.

We denote by $q$, a scalar product on $S(R^d)$ given by the formula: $q(\varphi, \psi) = \langle \Gamma, \varphi \ast \psi(s) \rangle$, where $\varphi, \psi \in S'(R^d)$. In other words, such a process $W$ may be called associated with $q$.

The crucial role in the theory of stochastic integration with respect to $W_\Gamma$ is played by the Hilbert space $H_W \subset S'(R^d)$ called the reproducing kernel Hilbert space of $W_\Gamma$. Namely the space $H_W$ consists of all distributions $\xi \in S'(R^d)$ for which there exists a
constant $C$ such that
\[ |\langle \xi, \psi \rangle| \leq C \sqrt{q(\psi, \psi)}, \quad \psi \in S(\mathbb{R}^d). \]

The norm in $\mathcal{H}_W$ is given by the formula
\[ |\xi|_{\mathcal{H}_W} = \sup_{\psi \in S} \frac{|\langle \xi, \psi \rangle|}{\sqrt{q(\psi, \psi)}}. \]

Let us assume that we require that the stochastic integral should take values in a Hilbert space $H$ continuously imbedded into $S'(\mathbb{R}^d)$. Let $L_{HS}(\mathcal{H}_W, H)$ be the space of Hilbert-Schmidt operators acting from $\mathcal{H}_W$ into $H$. Assume that $\Psi$ is measurable, $\mathcal{F}_t$-adapted, $L_{HS}(\mathcal{H}_W, H)$-valued process such that
\[ E \left( \int_0^t |\Psi(\sigma)|^2_{L_{HS}(\mathcal{H}_W, H)} d\sigma \right) < +\infty \quad \text{for all} \quad t \geq 0. \]

Then the stochastic integral
\[ \int_0^t \Psi(\sigma)dW_T(\sigma), \quad t \geq 0 \]

can be defined in a standard way, see [Ito] or [DaPrZa1].

As in [PeZa1] and [PeZa2], we shall use the characterization of the space $\mathcal{H}_W$.

**Proposition 1** (Proposition 1.2, [PeZa1])
A distribution $\xi$ belongs to $\mathcal{H}_W$ if and only if $\xi = \mathcal{F}(u\mu)$ for a certain $u \in L^2_{(s)}(\mathbb{R}^d, \mu)$. Moreover, if $\xi = \mathcal{F}(u\mu)$ and $\eta = \mathcal{F}(v\mu)$, then
\[ \langle \xi, \eta \rangle_{\mathcal{H}_W} = \langle u, v \rangle_{L^2_{(s)}(\mathbb{R}^d, \mu)}, \]
where $L^2_{(s)}(\mathbb{R}^d, \mu)$ denotes the subspace of $L^2_{(s)}(\mathbb{R}^d, \mu; \mathbb{C})$ consisting of all functions $u$ such that $u_{(s)} = u$ and $\langle \cdot, \cdot \rangle$ denotes products on particular spaces.

### 4 Estimates on the interval $[-R, R]$

As we have already written, the aim of the paper is to provide conditions under which the stochastic integral of the form (1), where $P = P_\alpha$ is the fundamental solution of the equation (3), is a well-defined stochastic process with values in the space $L^2(\mathbb{R}, e^{-|x|}dx)$.

Let $v$ belong to the space $S(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$ and be a strictly positive even function such that $v(x) = e^{-|x|}$ for $|x| \geq 1$. By $L^2_v$ we denote the space $L^2(\mathbb{R}, vdx)$ which is isomorphic with the space $L^2(\mathbb{R}, e^{-|x|}dx)$.

**Comment:** Results obtained in the paper remain true for the function $v(x) = (1 + |x|^2)^{-\rho}$, with $\rho > \frac{1}{2}$, $|x| \geq 1$. 

7
Using the notation from section 1, the stochastic integral \( I_\alpha(t) := \int_0^t P_\alpha(t-s) \ast (b(u(s)) \, dW(s)) \), \( 9 \)

where the convolution means the convolution with respect to the space variable, that is

\[
P_\alpha(t-s) \ast (b(u(s)) \, dW(s))(x) = \int_\mathbb{R} P_\alpha(t-s, y-x)(b(u(s,x))) \, dW(s,x) \, dx.
\]

We assume that \( u \) is an \( L^2_v \)-valued measurable \( (\mathcal{F}_t) \)-adapted process such that

\[
\sup_{0 \leq t \leq T} E|u(t)|^2_{L^2_v} < +\infty.
\]

Denote by \( X_T \) the space of such processes. In the paper \( b : \mathbb{R} \to \mathbb{R} \) is such a function that the process \( b(u) \) belongs to the space \( X_T \). A quite natural example of \( b \) is any Lipschitz continuous function.

We shall use the following

**Hypothesis (H):** There exists a \( \kappa \geq 0 \) such that \( \Gamma + \kappa \lambda \) is a non-negative measure, where \( \lambda \) denotes Lebesgue measure on \( \mathbb{R} \).

**Comment:** This is known from previous papers (cf [PeZa1], [KaZa2], [KaZa3] or [PeZa2]) that the hypothesis (H) is equivalent to the assumption: there is a constant \( \kappa \) such that the measure \( \mu + \kappa \delta_0 \) is a positive-definite distribution, where \( \mu \) is a spectral measure of the noise \( W_\Gamma \) and \( \delta_0 \) is Dirac function.

We recall from section 3 that \( \Gamma = \mathcal{F}(\mu) \). Next, if \( \Gamma \) is a function bounded from below then the hypothesis (H) holds. Additionally, the hypothesis (H) is equivalent to the condition

\[
\int_\mathbb{R} \frac{d\mu(x)}{1 + |x|^2} < +\infty.
\]

We define the space \( H^0_W \) consisting of all distributions of the form \( \eta = \mathcal{F}(\psi \mu) \), where \( \psi \in S(s)(\mathbb{R}) \).

**Remarks:**

1. The space \( H^0_W \) is a dense subspace of \( H_W \).
2. The following estimate holds \( \int_\mathbb{R} |\psi(x)| \mu(dx) < +\infty \), where \( \psi \in S(s)(\mathbb{R}) \).
3. From Prop. 1.3 [PeZa1] and the inclusion \( H^0_W \subset H \) we obtain \( H^0_W \subset C_b(\mathbb{R}) \).
Define $P^R_\alpha(t)$ as follows

$$
P^R_\alpha(t) := \begin{cases} 
P_\alpha(t, |x|) & \text{for } |x| \leq R \\
0 & \text{for } |x| > R 
\end{cases}
$$

where $R \in \mathbb{R}$ is finite.

Let us define, analogously like in [PeZa2], the following operator

$$
K^R(t,u) \eta \overset{\text{def}}{=} P^R_\alpha(t) * (u\eta)
$$

for $t > 0$, $u \in L^2_v$ and $\eta \in \mathcal{H}_W^0$.

Clearly,

$$
K^R(t,u) \eta(x) = \int_\mathbb{R} u(x-y)\eta(x-y)P^R_\alpha(t)(dy), \quad x \in \mathbb{R}.
$$

In this section we shall show that the operator $K^R(t,\cdot)$ has an extension to Hilbert-Schmidt operator from the space $\mathcal{H}_W$ into $L^2_v$. Additionally, we will give the estimate

$$
|K^R(t,u)|_{L(HS)(\mathcal{H}_W, L^2_v)} \leq Ce^R |u|_{L^2_v},
$$

where $C$ is an appropriate positive constant. In other words we shall prove that $K^R(t,\cdot)$ can be uniquely extended to linear bounded operator acting from the space $L^2_v$ into the space of Hilbert-Schmidt operators $L(HS)(\mathcal{H}_W, L^2_v)$.

First we introduce on the interval $[-R,R]$ the integral analogous to that defined in (9):

$$
I^R_\alpha(t) := \int_0^t P^R_\alpha(t-s) * (b(u(s))dW(s)) = \int_0^t K^R(t-s, b(u(s)))dW(s). \quad (12)
$$

We can see that the operator $K^R$ has to fulfil the condition (11) which guarantees that the integral (12) is well-defined. In order to obtain it, the operator $K^R$ has to belong to $L(HS)(\mathcal{H}_W, \mathcal{H})$. Indeed, if $u$ is an $L^2_v$-valued measurable $(\mathcal{F}_t)$-adapted process fulfilling condition (11) then for any $t > 0$, the operator-valued process $K^R(t-s, b(u(s)))$, $s \in (0,t)$, will be adapted and will satisfy the required condition

$$
E \left( \int_0^t |K^R(t-s, b(u(s)))|_{L(HS)(\mathcal{H}_W, \mathcal{H})} ds \right) < +\infty.
$$

Because the process $b(u(t))$, $t \in [0,T]$, belongs to the space $X_T$ for any $u \in X_T$, in the sequel we shall consider the operators $K^R(t,u)$ instead of $K^R(t,b(u))$.

Let us emphasize that for any our function $v$ there is a constant $C_v$ such that

$$
v(x-z) \leq C_v e^R v(x), \quad \text{where } x \in \mathbb{R} \text{ and } z \in [-R,R]. \quad (13)
$$
Lemma 1 Assume that $C_v$ fulfills the estimate (13). Then
\[
\left( P^R_\alpha(t) \ast v \right)(x) = \int_{-R}^R (x-y) P^R_\alpha(t) dy \leq C_v e^R v(x), \quad x \in [-R, R].
\] (14)

Moreover, for all $t \geq 0$, $\psi \in S(\mathbb{R}^d)$, the convolution $P^R_\alpha(t) \ast \psi \in L^2_v$ and
\[
|P^R_\alpha(t) \ast \psi|_{L^2_v} \leq C_v e^R |\psi|_{L^2_v}^2.
\] (15)

Proof: From the property (5) we obtain
\[
\left( P^R_\alpha(t) \ast v \right)(x) = \int_{-R}^R (x-y) P^R_\alpha(t) dy \leq C_v e^R \int_{-R}^R P^R_\alpha(t) dy \leq C_v e^R v(x),
\]
which proves (14).

In order to prove (15) it is enough to write explicitly the norm $|P^R_\alpha(t) \ast \psi|_{L^2_v}$ for $\psi \in S(\mathbb{R}^d)$ and then use the estimate (14).

Corollary 2 For any $t \geq 0$, there is a unique operator $P^R_\alpha(t) \in L(L^2_v, L^2_v)$ such that for every $\psi \in S(\mathbb{R})$,
\[
P^R_\alpha(t) \psi = P^R_\alpha(t) \ast \psi.
\]

Additionally, there is a constant $C$ such that
\[
|P^R_\alpha(t)|_{L(L^2_v, L^2_v)} \leq C e^R.
\] (16)

From the definition (11) of the operator $K_R$ and the estimation (16), we obtain
\[
|K_R(t, u) \eta|_{L^2_v} \leq C e^R |u\eta|_{L^2_v}
\]
for all $t \geq 0$, $u \in L^2_v$ and $\eta \in \mathcal{H}_W^0$.

Now, let us assume that the measure $\mu$ satisfies
\[
\int_{\mathbb{R}} \frac{d\mu(x)}{1 + |x|^2} < +\infty.
\] (17)

We want to show that under the condition (17) for any $t > 0$, the operator $K_R(t, \cdot)$ has an extension to a bounded linear operator from the space $L^2_v$ into the space of Hilbert-Schmidt operators $L_{(HS)} (\mathcal{H}_W, L^2_v)$.

Remark: Let us note that the hypothesis (H) and the condition (17) are equivalent. This fact comes from Theorem 2, [KaZa2].
Lemma 2 Assume that \( u \in C_b(\mathbb{R}) \) and \( \{ f_k \} \subset \mathcal{H}_W^0 \) is an orthonormal basis of the space \( \mathcal{H}_W \). Then, for any \( t > 0 \), holds
\[
\sum_{k=1}^{+\infty} |K_R(t,u) f_k|^2_{L^2_v} = \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}(P^R_\alpha(t)(x-\cdot)u)(y)|^2 \mu(dy)v(x)dx . \tag{18}
\]

Comment: Lemma 2 is formulated for \( u \in C_b(\mathbb{R}) \), not for \( u \in L^2_v \). This trick provides the sense of the right hand side of (18). The main result of this section, Lemma 5, will be formulated for \( u \in L^2_v \). It will be possible because \( C_b(\mathbb{R}) \) is a dense subspace of the space \( L^2_v \).

Proof: In order to prove (18) it is enough to rewrite the left hand side of (18), analogously like in [PeZa2], using the definition (11) of the operator \( K_R \). Next, we have to use properties of the Fourier transform and the convolution \( P^R_\alpha(t) * u f_k \). \( \blacksquare \)

Lemma 3 If the spectral measure \( \mu \) of the process \( W_T \) fulfills the condition (17) then the operators \( K_R(s,1) \), \( s \geq 0 \), acting from the space \( \mathcal{H}_W \) into \( L^2_v \), are Hilbert-Schmidt operators and moreover
\[
\int_0^t |K_R(s,1)|^2_{L_{HS}(\mathcal{H}_W,L^2_v)} ds < +\infty
\]
for every \( t \geq 0 \).

Proof: Because lemmas analogous to Lemma 3 has already been formulated in [KaZa2] and [PeZa2], we write here only a sketch of the proof.

We have to obtain the suitable estimate of the right hand side of (18), in the case when the function \( u \equiv 1 \), in terms of the condition (17). Particularly, we have to estimate the term \( |\mathcal{F}(P^R_\alpha(s)(y))|^2 \). In our considerations we shall use formulas (6), (7) and Corollary 1.

For symmetry of \( P^R_\alpha \) let us consider \( y \geq 0 \). Then, we may write
\[
\int_{\mathbb{R}} |\mathcal{F}(P^R_\alpha(s)(y))|^2 \mu(dy) \leq 2 s^{-\alpha/2} \int_0^{+\infty} |\mathcal{F}(P_\alpha(ys^{-\alpha/2}))|^2 \mu(dy) = 2 s^{-\alpha/2} \int_0^{+\infty} \left[ \exp \left( (ys^{-\alpha/2})^{2/\alpha} \cos \frac{\pi}{2} \right) \right] \mu(dy) ,
\]
where \( 1 < \alpha < 2 \).

For every \( 1 < \alpha < 2 \) and fixed \( s \geq 0 \), we may choose a constant \( C_\alpha(s) \) that
\[
\frac{1}{\exp(\frac{\alpha}{s}y^{2/\alpha})} \leq \frac{C_\alpha(s)}{1 + |y|^2} ,
\]
where \( a = |\cos \frac{\pi}{\alpha}|. \)

Additionally, we may choose finite constant \( C(t) \) that

\[
\int_0^t \frac{s^{-\alpha/2}}{\exp\left(\frac{2}{\pi}y^2/s^2\right)} ds \leq \frac{C(t)}{1 + |y|^2}, \quad y \in \mathbb{R}.
\]

Hence, the above estimates give the thesis. \( \blacksquare \)

**Lemma 4** Assume that measure \( \mu \) satisfies condition (17). Then there exists a constant \( C \) such that

\[
|K_R(t, 1)|^2_{L_{HS}(\mathcal{H}_W, L_2^2)} \leq C_\alpha \left( \int_{\mathbb{R}} v(x) dx \right) \int_{\mathbb{R}} \frac{\mu(dy)}{1 + |y|^2}, \quad t \geq 0.
\]  

**Lemma 5** Assume that the hypothesis (H) holds. Then the operator \( K_R(t, u) \), for \( t \geq 0, u \in L_v^2 \), is a Hilbert-Schmidt operator acting from the space \( \mathcal{H}_W \) into \( L_v^2 \).

Additionally, the following estimate is true

\[
|K_R(t, u)|^2_{L_{HS}(\mathcal{H}_W, L_2^2)} \leq C e^R |u|_{L_v^2}^2,
\]  

where \( C \) is an appropriate positive constant.

**Proof:** Because proof follows the proof of Lemma 3.3 of [PeZa2] and is technical, we present only a sketch of it. We shall use the following auxiliary result.

**Proposition 2** (Proposition 3.2, [PeZa2])

Let \( \mu \) be the spectral measure of \( W_T \), and let \( \delta_0 \) be the Dirac distribution. Condition (H) holds if and only if there exists \( \kappa \geq 0 \) such that for \( N \in \mathbb{N} \), the Fourier transforms of the measures

\[
\mu_{N,k}(dy) := e^{-|y|^2/N} (\mu(dy) + \kappa \delta_0(dy)),
\]

are non-negative functions.

Let us note that now the relationship \( \Gamma_{N,k} = \mathcal{F}(\mu_{N,k}) \), analogous to that \( \Gamma = \mathcal{F} (\mu) \) is true.

The proof is conducted for functions \( u \in C_b(\mathbb{R}) \). Because the space \( C_b \) is dense in \( L_v^2 \) and the operator \( K_R(t, u) \) is linear with respect to \( u \), the required estimation obtained for \( u \in C_b(\mathbb{R}) \) will be automatically extended to \( u \in L_v^2 \).

The proof is done in some steps. First, we consider the case \( k = 0 \) in the above proposition. We rewrite the formula (18) from Lemma 2 and express the integral with
respect to the measure $\mu$ like the limit of integrals with respect to the measures $\mu_{N,0}$ from Proposition 2. Using the formula $\Gamma_{N,0} = \mathcal{F}(\mu_{N,0})$ and Fubini’s theorem, there is possible to write the series $\sum_{k=1}^{+\infty} |\mathcal{K}_R(t,u) f_k|_{L^2_v}^2$ like integrals expressed in terms of $\Gamma_{N,0}$. Particularly, it may be done for $u = 1$.

Next, using the estimate (13), we have the following estimate for all $u \in C_b(\mathbb{R})$ and $t \geq 0$:

$$\sum_{k=1}^{+\infty} |\mathcal{K}(t,u) f_k|_{L^2_v}^2 \leq C e^{R} |u|_{L^2_v}^2 \left( \int_{\mathbb{R}} v(x) dx \right)^{-1} \sum_{k=1}^{+\infty} |\mathcal{K}(t,1) f_k|_{L^2_v}^2.$$  

From the estimate (19), we obtain the required result (20).

The case $k > 0$ deals with extending the results obtained in the case $k = 0$. Now, the measure $\nu = \mu + k\delta_0$ satisfies the hypothesis (H). Because the new measure $\nu$ fulfills the condition (17), we may write the estimate

$$|\mathcal{K}_R(t,u)|_{L^{HS}(\mathcal{H}_V,L^2_v)}^2 \leq C e^{R} |u|_{L^2_v}^2$$

for all $t \geq 0$, $u \in C_b(\mathbb{R})$. Here $\mathcal{H}_V$ denotes the reproducing kernel of the spatially homogeneous Wiener process $V$ with the spectral measure $\nu$.

Basing on the estimate (18) and the inequality

$$\int_{\mathbb{R}} |\mathcal{F}(P^R_\alpha(t)(x - \cdot))(y)|^2 \mu(dy) \leq \int_{\mathbb{R}} |\mathcal{F}(P^R_\alpha(t)(x - \cdot))(y)|^2 \nu(dy),$$

we have

$$\sum_{k=1}^{+\infty} |\mathcal{K}_R(t,u) f_k|_{L^2_v}^2 \leq |\mathcal{K}_R(t,u)|_{L^{HS}(\mathcal{H}_V,L^2_v)}^2.$$  

But from (21) we obtain $\mathcal{K}_R(t,u) \in L^{HS}(\mathcal{H}_W,L^2_v)$ and the estimate (20), what finishes the proof.

We have the following consequence of Lemma 5.

**Corollary 3** Assume that the hypothesis (H) is satisfied. It is possible to choose a positive constant $C$ such that if $u \in L^2_v$ and $u(x) \geq 1$ for almost all $x \in \mathbb{R}$, then

$$\sum_{k=1}^{+\infty} |\mathcal{K}(t,1) f_k|_{L^2_v}^2 \leq \sum_{k=1}^{+\infty} |\mathcal{K}(t,u) f_k|_{L^2_v}^2 + C e^{R} |u|_{L^2_v}^2, \quad t \geq 0.$$  

In the above estimate the set $\{f_k\} \subset \mathcal{H}_W^0$ is an arbitrary orthonormal basis of the space $\mathcal{H}_W$.  

13
5 Passing to the limit \( R \to +\infty \)

As we have already written, the aim of the paper is to give conditions under which the integral (9) is well-defined process with values in the space \( L^2_v \). Till now we have done it on the integral \([-R, R]\) for any finite \( R \). Now, we have to extend our results for \( R \to +\infty \).

**Theorem 4** Assume that \( b \) is Lipschitz continuous function, \( P_\alpha \) is defined by (4) and the hypothesis (H) holds. Then the integral \( I_\alpha(t) \) given by (9) is \( L^2_v \)-valued.

**Proof:** We will show that there exists enough large finite \( \tilde{R} > 0 \) such that for any \( R \geq \tilde{R} \) (even \( R \to +\infty \)) the following estimate holds

\[
|K_R(t, u)|_{L^2(H^0, L_v^2)} \leq |K_{\tilde{R}}(t, u)|_{L^2(H^0, L_v^2)} + M_{\tilde{R}},
\]

where \( M_{\tilde{R}} \) is finite.

In the proof of the estimate (22) we shall use the definition (11) of the operator \( K_R \) and the properties of the function \( P_\alpha(t, x) \) defined by (4). Let us recall that for any \( R > 0 \), \( t > 0 \), \( x \in \mathbb{R} \), \( u \in L^2_v \) and \( \eta \in H^0_w 

\[
K_R(t, u) \eta(x) = \int_{\mathbb{R}} u(x - y) \eta(x - y) P_\alpha(t, y) \, dy.
\]

Let us notice that \( P_\alpha^R(t) = P_\alpha(t, |x|) \) for \( |x| \leq R \), i.e. is a fundamental solution to the equation (3). Additionally, for \( x \geq 0 \), \( P_\alpha(t, |x|) = P_\alpha(t, x) \), where \( P_\alpha(t, x) \) is a probability density function (recall (4),(5) and (6)). Because of symmetry of \( P_\alpha(t, |x|) \), we will study only the case when \( x \geq 0 \).

In order to prove the estimate (22) it is enough to show that for "large" \( \tilde{R} \)

\[
\lim_{R \to +\infty} \int_{\tilde{R}}^R u(x - y) \eta(x - y) P_\alpha(t, y) \, dy = M_{\tilde{R}},
\]

where \( M_{\tilde{R}} < +\infty \), for \( x \in \mathbb{R} \) and \( t \in (0, +\infty) \).

For any \( u \in C_b(\mathbb{R}) \) and \( \eta \in H^0_w \subset C_b(\mathbb{R}) \), we may write

\[
\int_{\tilde{R}}^R u(x - y) \eta(x - y) P_\alpha(t, y) \, dy \leq M_u M_\eta \int_{\tilde{R}}^R P_\alpha(t, y) \, dy,
\]

where \( M_u \) and \( M_\eta \) are appropriate constants.

According to Fujita’s considerations (see also [Mj] or [Sk]), the asymptotic behaviour of the density \( P_\alpha(1, x) \) is as follows:

\[
P_\alpha(1, |x|) \xrightarrow{|x| \to +\infty} \frac{B_\alpha |x|^{(\alpha - 1)/(2 - \alpha)}}{\exp[A_\alpha |x|^{2/(2 - \alpha)}]},
\]

(24)
where $A_\alpha$ and $B_\alpha$ are positive constants determined by $\alpha$ with $1 \leq \alpha < 2$.

Now, let us recall and then use the property (6), that is $P_\alpha(t, x) = P_\alpha(1, xt^{-\alpha/2}) t^{-\alpha/2}$ for $t \in (0, +\infty), x \in \mathbb{R}$.

Then (24) reads

\[
P_\alpha(t, x) \xrightarrow{|x| \to +\infty} \frac{B_\alpha |xt^{-\alpha/2}|^{(\alpha-1)/(2-\alpha)}t^{-\alpha/2}}{\exp[A_\alpha |xt^{-\alpha/2}|^{2/(2-\alpha)}]} \xrightarrow{|x| \to +\infty} 0^+ \]

for $t \in (0, +\infty), x \in \mathbb{R}_+$.

Hence, the right hand side of the estimate (23) may be written like

\[
\mathcal{M} \int_\mathbb{R} P_\alpha(t, y) dy \xrightarrow{|x| \to +\infty} \mathcal{M}_\mathbb{R},
\]

where $\mathcal{M}_\mathbb{R} < +\infty$, what proves (21).

Since the space $C_b(\mathbb{R})$ is dense in $L^2_v$ and the operator $\mathcal{K}_R$ is linear with respect to $u$, we obtain the estimate (21) for all $t \geq 0$ and $u \in L^2_v$. □

Acknowledgements

The author thanks Professors Y. Fujita and S.-O. Londen for valuable remarks concerning integrodifferential equations.

References

[Da] Dalang R., *Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.*’s Electronic J. Probab. 4 (1999) 1–29.

[DaFr] Dalang R. and Frangos N., *The stochastic wave equation in two spatial dimensions*, The Annals of Probability, No. 1, 26 (1998) 187–212.

[DaMu] Dalang R. and Mueller C., *Some non-linear s.p.d.e.’s that are second order in time*, Electronic J. Probab. 8 (2003) 1–21.

[DaPrZa1] Da Prato G. and Zabczyk J., *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge, 1992.

[Fr] Friedman A., *Monotonicity of solutions of Volterra integral equations in Banach space*, Trans. Am. Math. Soc. 198 (1969) 129–148.

[FrSh] Friedman A. and Shinbrot M., *Volterra integral equations in Banach space*, Trans. Am. Math. Soc. 126 (1967) 131–179.

[Fu] Fujita Y., *Integrodifferential equation which interpolates the heat equation and the wave equation*, Osaka J. Math. 27 (1990) 309–321.
[Ito] Itô K., *Foundations of stochastic differential equations in infinite dimensional spaces*, SIAM, Philadelphia, 1984.

[KaZa1] Karczewska A. and Zabczyk J., *A note on stochastic wave equations*, in: *Evolution Equations and their Applications in Physical and Life Sciences*, G. Lumer and L. Weis, eds., Lecture notes in pure and applied mathematics, Vol. **215**, 501-511, Marcel Dekker, New York 2001.

[KaZa2] Karczewska A. and Zabczyk J., *Stochastic PDE’s with function-valued solutions*, in: *Infinite Dimensional Stochastic Analysis*, P. Clement, F. den Hollander, J. van Neerven and B. de Pagter, eds., Proceedings of the Colloquium of the Royal Netherlands Academy of Arts and Sciences 1999, North Holland, Amsterdam, 2000.

[KaZa3] Karczewska A. and Zabczyk J., *Regularity of solutions to stochastic Volterra equations*, Rend. Math. Acc. Lincei. s. 9, **11** No.3 (2001) 141–154.

[Mi] Mizohata S., *The theory of partial differential equations*, Cambridge Univ. Press, Cambridge, 1973.

[Mj] Mijnheer J.L., *Sample path properties of stable processes*, Mathematical Centre Tracts **59**, Mathematical Centrum, Amsterdam, 1975.

[Pe] Peszat S., *Existence and uniqueness of the solution for stochastic equations on Banach spaces*, Stochastic Stochastic Rep. **55** (1995) 167–193.

[PeZa1] Peszat S. and Zabczyk J., *Stochastic evolution equations with a spatially homogeneous Wiener process*, Stochastic Processes Appl. **72** (1997) 187–204.

[PeZa2] Peszat S. and Zabczyk J., *Nonlinear stochastic wave and heat equations*, Prob. Theory Relat. Fields **116** (2000) 421–443.

[Pr] Prüss J., *Evolutionary integral equations and applications*, Birkhäuser, Basel, 1993.

[ScWy] Schneider W.R. and Wyss W., *Fractional diffusion and wave equations*, J. Math. Phys. **30** (1989), pp. 134-144.

[Sk] Skorohod A.V., *Asymptotic formulas for stable distribution laws*, Selected Translations in Mathematical Statistics and Probability **1**, pp. 157-161, Am. Math. Soc., Providence, 1961.

[Yo] Yosida K., *Functional analysis*, (6-th ed.), Springer-Verlag, New York, 1980.