Radial disintegration for $\ell_p$-norm symmetric survival functions on the positive orthant

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Abstract

We derive a stochastic representation for the probability distribution on the positive orthant $(0, \infty)^d$ whose association between components is minimal among all probability laws with $\ell_p$-norm symmetric survival functions. It is given by a transformation of a uniform distribution on the standard unit simplex that is multiplied with an independent finite mixture of certain beta distributions and an additional atom at unity. On the one hand, this implies an efficient simulation algorithm for arbitrary probability laws with $\ell_p$-norm symmetric survival function. On the other hand, this result is leveraged to construct an exact simulation algorithm for max-infinitely divisible probability distributions on the positive orthant whose exponent measure has $\ell_p$-norm symmetric survival function. Both applications generalize existing results for the case $p = 1$ to the case of arbitrary $p \geq 1$.

1 Introduction

Fix $p \geq 1$ and write $\theta := 1/p$ throughout to simplify notation. Let $\mu$ be a measure on $[0, \infty)^d$ with the property that its survival function takes the specific form

$$\mu((x, \infty]) = \varphi(||x||_p), \quad x > 0,$$

for some function $\varphi : [0, \infty) \to [0, \infty)$ of one variable. Probability measures $\mu$ of the kind \cite{1}, as studied in \cite{1}, widely appear in many areas of applications including finance, risk management, and environmental sciences; we refer to \cite{2} for background, examples, and statistical inference. Non-exchangeable extensions of \cite{1} are discussed in \cite{4}.

If $\mu$ is a probability measure on $(0, \infty)^d$, it follows from results in \cite{7,1} (this logic being recalled in \cite{11} below) that $\varphi$ is $d$-monotone with $\varphi(0) = 1$ and $\mu$ is the distribution

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where \( R \) is a random variable on \((0, \infty)\) whose distribution \( F_\varphi \) depends solely on \( \varphi \), the vector \( U^{(1)} \) is uniformly distributed over the standard unit simplex \( S_{d,1} := \{ x \in [0,1]^d : \| x \|_1 = 1 \} \), \( V_p \) is a random variable on \([0,1]\) whose distribution \( F_p \) solely depends on \( p \), and \( R, V_p \) and \( U^{(1)} \) are independent. Equivalently, \( \mu \) can be factored as

\[
\mu(A) = \int_{(0,\infty)} \int_{[0,1]} \int_{S_{d,1}} 1_{\{r \cdot v \cdot w^\theta \in A\}} \, du \, dF_p(v) \, dF_\varphi(r), \quad A \subset (0,\infty)^d \text{ a Borel set.} \tag{2}
\]

The distribution function \( F_p \) of \( V_p \) has not been explicitly found to date. We explicitly derive this probability distribution, finding that a random variable \( V_p \sim F_p \) satisfies the distributional equality

\[
V_p \sim W_{(d+1-D_\theta)}, \quad \mathbb{P}(D_\theta = i) = a_i^{(d)}, \quad i = 1, \ldots, d,
\]

where, independently of \( D_\theta \), we denote by \( W_{(1)} \leq \ldots \leq W_{(d-1)} \) the order statistics of independent standard uniform random variables \( W_1, \ldots, W_{d-1} \) and \( W_{(d)} = 1 \), and the mixture probabilities \((a_1^{(d)}, \ldots, a_d^{(d)}) \in S_{d,1} \) can conveniently be computed by the recursive relationship

\[
a_i^{(k)} = a_i^{(k-1)} \theta \frac{k-i}{k-1} + a_{i-1}^{(k-1)} \left( 1 - \theta \frac{k-i+1}{k-1} \right), \quad i = 1, \ldots, k, \quad k = 2, \ldots, d, \tag{3}
\]

with initial value \( a_1^{(1)} = 1 \) and auxiliary notation \( a_0^{(k-1)} = a_k^{(k-1)} = 0 \). This finding implies an efficient simulation algorithm for random vectors \( Z \sim \mu \). Existing simulation algorithms to date are either restricted by the assumption that \( \varphi \) is completely monotone, or rely on computations of partial derivatives as in [1, Proposition 5.3], which is infeasible for large \( d \).

Moreover, if \( \mu \) in (1) is a non-finite Radon measure on \( E := [0,\infty) \setminus \{0\} \), the results in [3] imply that the function \( \varphi \) is \( d \)-monotone and a bijection on \((0,\infty)\). The same factorization (2) of \( \mu \) is valid, only with the probability distribution \( F_\varphi \) being replaced by a non-finite “radial” Radon measure \( \nu_\varphi \) on \((0,\infty)\) that solely depends on \( \varphi \). Thus, our result on the explicit finding of the probability distribution \( F_p \) can be leveraged to derive an exact simulation algorithm for max-infinitely divisible random vectors \( Y \) with exponent measure \( \mu \). To the best of our knowledge, such algorithm is unknown to

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3 Throughout, raising a vector to a power \( \theta \), as well as applying other functions of one variable to a vector, should always be understood component-wise.
The remainder of this article is organized as follows. Section 2 provides background on different concepts of $\ell_p$-norm symmetry in the context of multivariate probability distributions. Section 3 derives the explicit form of the aforementioned distribution $F_p$. Section 4 presents an efficient simulation algorithm for $F_p$ and thus for arbitrary random vectors with $\ell_p$-norm symmetric survival functions. Section 5 proves a stochastic representation for $\max$-infinitely divisible random vectors $Y$ on $(0, \infty)^d$ whose exponent measure has $\ell_p$-norm symmetric survival function, and explains how to simulate $Y$ exactly from it.

2 Preliminaries

Concerning analytical characterizations of multivariate probability distributions, there exist two prominent notions of $\ell_p$-norm symmetry. On the one hand, an absolutely continuous random vector $X$ has an $\ell_p$-norm symmetric density $f(x) = g(\|x\|_p)$ if and only if

$$X \sim RU^{(p)},$$

(4)

where $R$ is a positive (absolutely continuous) random variable and $U^{(p)}$ is an independent random vector that is uniformly distributed on the $\ell_p$-sphere. Since $\|\cdot\|_p$ is orthant-monotonic, this statement holds either for $X$ taking values in $\mathbb{R}^d$ or only $(0, \infty)^d$, in which case $U^{(p)}$ is uniform on the restriction of the $\ell_p$-sphere to the positive orthant (the $\ell_p$-simplex). A stochastic representation for $U^{(p)}$ restricted to the positive orthant can be found in [10] and is given by $U^{(p)} \sim \xi^{(p)}/\|\xi^{(p)}\|_p$, where $\xi^{(p)} = (\xi_1^{(p)}, \ldots, \xi_d^{(p)})$ is a vector with iid components satisfying $\xi_1^{(p)} \sim \Gamma(1/p, 1/p)$, where we denote by $\Gamma(\beta, \eta)$ the Gamma distribution with density proportional to $e^{-\eta x} x^{\beta-1}$. It is intuitively clear that for a non-random $R$, the association between the components of $X$ is minimal and negative; indeed, in this case, $X^p$ is a joint mix [14] and hence it represents a form of extreme negative dependence [9]. Furthermore, [10] explain that if the radial variable satisfies $R \sim M \|\xi^{(p)}\|_p$ with a positive random variable $M$ independent of $\xi^{(p)}$, then $X \sim M \xi^{(p)}$ with $M$ and $\xi^{(p)}$ independent. Intuitively, the denominator in $U^{(p)} \sim \xi^{(p)}/\|\xi^{(p)}\|_p$ “cancels out in distribution” in this case, relying on the Lukacs theorem, and the components exhibit positive association (recall that $\xi_1^{(p)}, \ldots, \xi_d^{(p)}$ are iid) whose strength depends on $M$.

\footnote{See Lemma A.1 in the Appendix.}
On the other hand, it follows from results in [1, 7], this logic being explained in (11) below, that a random vector $Z$ taking values in $(0, \infty)^d$ has $\ell_p$-norm symmetric survival function $\mathbb{P}(Z > z) = \varphi \circ \|z\|_p$ if and only if

$$Z \sim RV_p(U^{(1)})^\theta,$$

where $R$ is a positive random variable uniquely determined in law by its so-called Williamson-d-transform (see [15])

$$\mathbb{E}\left[(1 - \frac{x}{R})^{d-1}\right] = \varphi(x), \quad x \geq 0,$$

$U^{(1)}$ is uniform on $S_{d,1}$, and $V_p$ is a random variable taking values in $[0, 1]$ whose probability law is not explicitly known to date, all three objects mutually independent. Provided absolute continuity of the radial variable $R$, the two representations (4) and (5) imply that the notions of $\ell_p$-norm symmetric densities and $\ell_p$-norm symmetric survival functions are equivalent if and only if $p = 1$, and $V_1 \equiv 1$ in that case. The survival copula of $X$, respectively $Z$, in this case $p = 1$, which equals the distribution function of the random vector $\varphi(X)$, is called an Archimedean copula, and is given by

$$C_\varphi(u) = \varphi(\varphi^{-1}(u_1) + \ldots + \varphi^{-1}(u_d)), \quad u \in [0, 1],$$

and the function $\varphi$ is called Archimedean generator; see [5] Chapter 2 for background on the matter. In the general case $p \geq 1$, we see that $Z^{(p)} \sim R^p V_p^p U^{(1)}$ is a particular instance of an $\ell_1$-norm symmetric distribution (in both considered meanings), which implies from the results in [1] that Kendall’s tau between two components of $Z$ is given by $1 - \theta + \theta \tau_\varphi$, where $\tau_\varphi$ denotes Kendall’s tau between two components of $X$ in [4] in the case $p = 1$. It is thus known from [7] that Kendall’s tau is minimized with the choice $\varphi(x) = (1 - x)^{d-1}$ corresponding to $R \equiv 1$, with $\tau_\varphi = -1/(2d - 3)$. For $p < (2d - 2)/(2d - 3)$ this implies negative association between the components of $Z$ and we obtain a similar intuition as in the case of an $\ell_p$-norm symmetric density. In particular, it is known that if $R \sim M\left(\left\|\xi^{(1)}\right\|_1\right)^\theta$ for some positive random variable $M$ independent of $\xi^{(1)}$ as defined above, then $Z \sim M\left(\xi^{(1)}\right)^\theta$ and the components of $Z$ exhibit positive association whose strength is governed by the choice of $M$. Finally, the survival copula of the vector $Z$ in [5] is an Archimedean copula with Archimedean generator $x \mapsto \varphi(x^\theta)$, and these are sometimes referred to as outer power Archimedean copulas. This parameter-enhancement technique has originally been introduced in [8]. The nomenclature “outer power” might appear surprising, since the power $\theta$ is taken “inside” $\varphi$, but is explained from traditional notation in the context of Archimedean
copulas, where the roles of $\varphi$ and $\varphi^{-1}$ are often interchanged.

Our main contribution is an explicit representation for the random variable $V_p$. It can be inferred from the results in [7] that the random variable $V_p$ is uniquely determined by the identity

$$
E\left[\left(1 - \frac{x}{V_p}\right)^{d-1}\right] = (1 - x^\theta)^{d-1}, \quad x \geq 0.
$$

(6)

Unfortunately, this Williamson-$d$-transform is not easy to invert to obtain the explicit law of $V_p$, hence $V_p$. We prove that $V_p$ equals a finite mixture of certain beta distributions and an atom at unity and derive an efficient simulation algorithm. This does not only imply an efficient simulation algorithm for the random vector $Z$ in (5), but also we show how it can be leveraged to obtain an exact simulation algorithm for max-infinitely divisible random vectors $Y$ on $(0, \infty)^d$ whose exponent measure $\mu$ has $\ell_p$-norm symmetric survival function given by (1). An excellent textbook account on max-infinite divisibility is [11]. Such $Y$ is shown in Lemma 5.2 below to have the stochastic representation

$$
Y \sim \left(\max_{k \geq 1} \{\eta_k Z_1^{(k)}\}, \ldots, \max_{k \geq 1} \{\eta_k Z_d^{(k)}\}\right),
$$

(7)

where $\{Z^{(k)}\}_{k \geq 1}$ is a sequence of iid copies of $Z$ in (5) with $R \equiv 1$ and, independently, $\{\eta_k\}_{k \geq 1}$ denoting the decreasing enumeration of the points of a Poisson random measure, whose mean measure $\nu = \nu_\varphi$ is Radon on $(0, \infty]$ satisfying $\nu_\varphi\{\infty\} = 0$ and $\nu_\varphi((0, \infty]) = \infty$. Since our main result implies an exact simulation algorithm for the involved $Z^{(k)}$, this stochastic representation serves as basis to derive an exact simulation algorithm for $Y$. Its idea enhances an algorithm viable for the case $p = 1$ that was presented in [6]. The copula of $Y$ is called reciprocal Archimedean copula with generator $\varphi$ in [3]. This nomenclature is justified by some “reciprocal” analogies with Archimedean copulas, e.g., like Archimedean copulas also reciprocal Archimedean copulas can be written in terms of their generating function $\varphi$. Our algorithm shows how to simulate reciprocal Archimedean copulas whose generator is given by $y \mapsto \varphi(y^\theta)$. In analogy to the aforementioned Archimedean case, we refer to the copula of $Y$ as outer power reciprocal Archimedean copula.

3 Explicit representation for the law of $V_p$

Concerning notations, we denote by $\beta_{m,n}$ for $m, n \geq 1$ the cdf of a beta distribution with density proportional to $x^{m-1}(1-x)^{n-1}$. For the sake of a convenient notation,
we further denote by $\beta_{m,0}(x) = 1_{\{x \geq 1\}}$ the cdf of a random variable that is identically constant equal to one, for $m \geq 1$ arbitrary.

Our goal is to find the random variable $V_p$ satisfying $[6]$. The solution will be given in Theorem 3.4 below, where it is shown that $V_p$ is a (convex) mixture of certain beta distributions. Before presenting it, some auxiliary steps are carried out. First of all, for the sake of completeness, we formally prove that $V_p$ satisfying $[6]$ exists and is unique in law. To this end, recall that a function $\varphi : (0, \infty) \to [0, \infty)$ is called $d$-monotone if the derivatives $\varphi^{(k)}$ exist for $k = 0, \ldots, d - 2$, and $(-1)^k \varphi^{(k)}$ is non-negative, non-increasing and convex. A result of [15], lying at the heart of the results in [7], shows that functions $\varphi : [0, \infty) \to [0, 1]$ which are $d$-monotone on $(0, \infty)$ and satisfy $\varphi(0) = 1$ form a simplex with extremal boundary given by the functions $\varphi_v(x) := (1 - x/v)^{d-1}$, $v > 0$. In probabilistic terms, this means that for any such function $\varphi$ there is a random variable $V_\varphi$, uniquely determined in distribution, such that $\varphi(x) = E[\varphi_{V_\varphi}(x)]$, $x \geq 0$. Applied to our situation, in order to formally prove that $V_p$ exists and its law is unique it is sufficient to verify that $x \mapsto (1 - x^\theta)^{d-1}_+$ is $d$-monotone.

**Lemma 3.1** ($x \mapsto (1 - x^\theta)^{d-1}_+$ is $d$-monotone)  
The function $x \mapsto (1 - x^\theta)^{d-1}_+$ is $d$-monotone on $(0, \infty)$.

**Proof**  
Denote $\varphi(x) = (1 - x^\theta)^{d-1}_+$ within this proof. We apply [12] Theorem 12], which states that $f \circ g$ is $d$-monotone if both $f$ and $g$ are. Applying this statement with $f(x) = (1 + x)^{d-1}_+$ and $g(x) = -(x^\theta)$ on $(-1, 0)$, which are both easily seen to be $d$-monotone, then implies that $\varphi = f \circ g(\cdot)$ is $d$-monotone on $(0, 1)$. Since $\varphi^{(k)}$ is identically zero on $[1, \infty)$ and $\varphi^{(k)}(1) = 0$ for $k = 0, \ldots, d - 2$, we obtain that $\varphi^{(k)}$ is actually convex on all of $(0, \infty)$, hence $\varphi$ is $d$-monotone on $(0, \infty)$.

By definition of $d$-monotonicity we also know that $x \mapsto (1 - x^\theta)^{d-1}_+$ is $k$-monotone on $(0, \infty)$ for each $k = 1, \ldots, d$. Consequently, for each $k = 1, \ldots, d$ there exists a positive random variable $V^{(k)}_d$, which is unique in law, such that

$$(1 - x^\theta)^{d-1}_+ = E\left[\left(1 - \frac{x}{(V^{(k)}_d)^{k-1}}\right)^{k-1}\right], \quad x > 0.$$  

Our goal is to determine the probability law of $V_p = V^{(d)}_d$, in fact we even determine the law of all $V^{(k)}_d$ for $k = 1, \ldots, d$ in the following. We denote the cdf of $V^{(k)}_d$ by $F^k_d$ and, as a first step, we derive a recursion for $F^k_d$. To this end, we note that for $k = 1, \ldots, d,$
$F^k_d$ is the unique distribution which satisfies the equation

$$
\int_{c^\theta}^1 \left(1 - \frac{c}{x^p}\right)^{k-1} dF^k_d(x) = (1 - c^\theta)^{d-1}, \quad c \in [0, 1]. \tag{8}
$$

**Lemma 3.2 (A recursion for $F^k_d$)**

Let the (a priori possibly signed) measure $F^k_d$ be given by the following recursive formulas: $F^1_d = \beta_{1,d-1}$, and for $k = 2, 3, \ldots, d$,

$$
F^k_d = \frac{d - 1}{k - 1} \theta F^{k-1}_d + \left(1 - \frac{d - 1}{k - 1} \theta\right) F^{k-1}_d.
$$

Then, $F^k_d$ satisfies (8).

**Proof**

We know $\beta_{1,d-1}(x) = 1 - (1 - x)^{d-1}$, which implies that

$$
\int_{c^\theta}^1 \left(1 - \frac{c}{x^p}\right)^0 d\beta_{1,d-1}(x) = \int_{c^\theta}^1 d\beta_{1,d-1}(x) = (1 - c^\theta)^{d-1},
$$

as claimed for $k = 1$. Regarding the induction step, for $k \geq 2$ define two functions $F(c) := \int_0^1 (1 - \frac{c}{x^p})^{k-1} dF^k_d(x)$ and $G(c) := (1 - c^\theta)^{d-1}$ for $c \in [0, 1]$. Note that

$$
F'(c) = \frac{d}{dc} \int_{c^\theta}^1 \left(1 - \frac{c}{x^p}\right)^{k-1} dF^k_d(x) = -(k - 1) \int_{c^\theta}^1 \frac{1}{x^p} \left(1 - \frac{c}{x^p}\right)^{k-2} dF^k_d(x).
$$

Moreover, using (8),

$$
cF'(c) = -(k - 1) \int_{c^\theta}^1 \frac{c}{x^p} \left(1 - \frac{c}{x^p}\right)^{k-2} dF^k_d(x)
$$

$$
= -(k - 1) \left(\int_{c^\theta}^1 \left(1 - \frac{c}{x^p}\right)^{k-2} dF^k_d(x) - F(c)\right)
$$

$$
= - \int_{c^\theta}^1 \left(1 - \frac{c}{x^p}\right)^{k-2} \left((d - 1) \theta dF^{k-1}_d(x) + (k - 1 - (d - 1) \theta) dF^{k-1}_d(x)\right)
$$

$$
+ (k - 1)F(c)
$$

$$
= -(d - 1) \theta (1 - c^\theta)^{d-2} - (k - 1 - (d - 1) \theta) (1 - c^\theta)^{d-1} + (k - 1)F(c)
$$

$$
= -(d - 1) \theta (1 - c^\theta)^{d-2} c^\theta + (k - 1) (F(c) - G(c)).
$$

We know $F(0) = G(0)$. If $c > 0$, we divide both sides of the above equality by $c$, and get

$$
F'(c) = G'(c) + \frac{k - 1}{c}(F(c) - G(c)).
$$
Since $F(1) = G(1) = 0$, we know that $F \equiv G$ on $[0, 1]$. Thus, (8) holds. □

The term $\theta (d - 1)/(k - 1)$ in (9) may be greater than one, so that we do not obtain convex combinations of beta distributions directly. Indeed, if this term is no larger than one (i.e., $p \geq d - 1$), then applying (9) repeatedly gives rise to $F_d$ as a mixture of $\beta_{1,k}$ for $k = 1, \ldots, d - 1$. In general, this is not the case: we will see that $F_d$ is a mixture of beta distributions, but not all of the form $\beta_{1,k}$ for $k = 1, \ldots, d - 1$. The following auxiliary lemma is helpful to solve the recursion in (9).

**Lemma 3.3 (Auxiliary identities on the beta distribution)**
The following two identities hold for the beta distribution, for integers $m, n \geq 1$:

$$
\beta_{m+1,n-1}(x) - \beta_{m,n}(x) = -\binom{m+n-1}{m} x^m (1 - x)^{n-1},
$$

$$
\beta_{m,n-1}(x) - \beta_{m,n}(x) = -\binom{m+n-2}{m-1} x^m (1 - x)^{n-1}.
$$

**Proof**
Straightforward, for the sake of completeness sketched in the Appendix. □

**Theorem 3.4 (Solving the recursion)**
For each $k = 1, \ldots, d$, there exists $(a_1^{(k)}, \ldots, a_k^{(k)}) \in S_{k,1}$ such that

$$
F_k = \sum_{i=1}^k a_i^{(k)} \beta_{k+1-i,d-1+i}.
$$

Furthermore, the $a_i^{(k)}$ satisfy the recursive relationship (3).

**Proof**
If we fix $i \in \{1, \ldots, k - 1\}$, then the second identity in Lemma 3.3 gives

$$
\frac{d-1}{k-1} (\beta_{k-i,d-k+i-1}(x) - \beta_{k-i,d-k+i}(x)) = -\frac{d-1}{k-1} \binom{d-2}{k-i-1} x^{k-i} (1 - x)^{d-k+i-1}
$$

$$
= -\binom{d-1}{k-i} x^{k-i} (1 - x)^{d-k+i-1} \leq \frac{k-i}{k-1}.
$$

Consequently, we observe with the help of the first identity in Lemma 3.3 that

$$
\frac{d-1}{k-1} (\beta_{k-i,d-k+i-1} - \beta_{k-i,d-k+i}) + \beta_{k-i,d-k+i}
$$

$$
= \frac{k-i}{k-1} \beta_{k-i+1,d-k+i-1} + \left(1 - \frac{k-i}{k-1}\right) \beta_{k-i,d-k+i}.
$$

(10)
Now, inductively, we proceed as follows to compute $F^k_d$ via the recursion of Lemma 3.2:

$$F^k_d = \frac{d-1}{k-1} \theta F^{k-1}_{d-1} + \left(1 - \frac{d-1}{k-1} \theta\right) F^k_d$$

$$= \theta \left(\frac{d-1}{k-1} \left(F^{k-1}_{d-1} - F^k_d\right) + F^k_d\right) + (1 - \theta) F^k_d.$$

We know by induction that there exist $a^{(k-1)}_1, \ldots, a^{(k-1)}_{k-1} \geq 0$ that sum up to one and

$$F^k_{d-1} = \sum_{i=1}^{k-1} a^{(k-1)}_i \beta_{k-i,d-1-k+i}, \quad F^k_{d-1} = \sum_{i=1}^{k-1} a^{(k-1)}_i \beta_{k-i,d-1-k+i}.$$

Notice that we have used here that the $a^{(k-1)}_i$ are independent of $d$, which is important.

We thus obtain

$$F^k_d = \theta \sum_{i=1}^{k-1} a^{(k-1)}_i \left(\frac{d-1}{k-1} \left(\beta_{k-i,d-1-k+i} - \beta_{k-i,d-k+i}\right) + \beta_{k-i,d-k+i}\right) + (1 - \theta) F^k_d$$

$$= \theta \sum_{i=1}^{k-1} a^{(k-1)}_i \left(\frac{k-i}{k-1} \beta_{k-i+1,d-k+i-1} + \left(1 - \frac{k-i}{k-1}\right) \beta_{k-i,d-k+i}\right) + (1 - \theta) F^k_d$$

$$= \sum_{i=1}^{k-1} a^{(k-1)}_i \left\{ \theta \left(\frac{k-i}{k-1} \beta_{k-i+1,d-k+i-1} + \left(1 - \frac{k-i}{k-1}\right) \beta_{k-i,d-k+i}\right) + (1 - \theta) \beta_{k-i,d-k+i}\right\}$$

$$= \sum_{i=1}^{k-1} a^{(k-1)}_i \theta \frac{k-i}{k-1} \beta_{k-i+1,d-k+i-1} + \sum_{i=1}^{k-1} a^{(k-1)}_i \left(1 - \theta \frac{k-i}{k-1}\right) \beta_{k-i,d-k+i}$$

$$= \sum_{i=0}^{k-2} a^{(k-1)}_{i+1} \theta \frac{k-i}{k-1} \beta_{k-i,d-k+i} + \sum_{i=1}^{k-1} a^{(k-1)}_i \left(1 - \theta \frac{k-i}{k-1}\right) \beta_{k-i,d-k+i}$$

$$= \sum_{i=2}^{k-1} \left(a^{(k-1)}_i \theta \frac{k-i}{k-1} + a^{(k-1)}_{i-1} \left(1 - \theta \frac{k-i+1}{k-1}\right) \right) \beta_{k-i+1,d-k+i-1}$$

$$+ a^{(k-1)}_1 \theta \beta_{k,d-k} + a^{(k-1)}_{k-1} \left(1 - \theta \frac{1}{k-1}\right) \beta_{1,d-1}.$$

This implies the claim. \(\square\)

Apparently, $a^{(d)}_1 = p^{-(d-1)}$. Since $\beta_{d,0} = \delta_1$ by our convenient notation, this implies that $V_p = V^{(d)}_d$ is equal to one with probability $a^{(d)}_1$ and with complementary probability $1 - a^{(d)}_1$ follows an absolutely continuous distribution with support $[0, 1]$. 

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4 Simulation of \( \ell_p \)-norm symmetric survival functions

Based on Theorem 3.4, we first derive a convenient method to simulate \( V_p \) exactly.

**Lemma 4.1 (Simulating \( V_p \))**

Let \( W_1, \ldots, W_{d-1} \) be iid from \( \mathcal{U}[0,1] \), and \( W(i) \) be the \( i \)-th order statistics, from the smallest to the largest, and \( W(d) := 1 \). Define a counting process \( (N_k)_{k=1}^d \) independent of \( (W_1, \ldots, W_{d-1}) \) via \( N_k = \sum_{j=1}^k B_j \) where \( B_1 = 1 \) and for \( j = 2, \ldots, d \),

\[
P(B_j = 1 | N_{j-1}) = 1 - P(B_j = 0 | N_{j-1}) = \frac{N_{j-1}}{j-1} \theta.
\]

Then \( W(N_k) \sim F_k^d \).

**Proof**

Let \( T_k = k+1-N_k = 1+\sum_{j=1}^k (1-B_j), k = 1, \ldots, d \). Note that \( T_1 = 1 \). For \( k = 2, \ldots, d \) and \( i = 1, \ldots, k \),

\[
P(T_k = i) = P(N_k = k-i+1) = P(N_{k-1} = k-i, B_k = 1) + P(N_{k-1} = k-i+1, B_k = 0)
\]

\[
= P(N_{k-1} = k-i) \frac{k-i}{k-1} \theta + P(N_{k-1} = k-i+1) \left( 1 - \frac{k-i+1}{k-1} \theta \right)
\]

\[
= P(T_{k-1} = i) \frac{k-i}{k-1} \theta + P(T_{k-1} = i-1) \left( 1 - \frac{k-i+1}{k-1} \theta \right).
\]

Hence, the sequence \( (P(T_k = i) : 1 \leq i \leq k \leq d) \) satisfies the recursive relation (3) and has the same initial element as \( (a_i^{(k)} : 1 \leq i \leq k \leq d) \). As a consequence, \( a_i^{(k)} = P(T_k = i) \) for each \( i \) and \( k \). Note that \( W(k+1-i) \sim \beta_{k+1-i,d-k-1+i} \) for \( i = 1, \ldots, k \). The law of total probability implies

\[
W(k+1-T_k) \sim \sum_{i=1}^k P(T_k = i) \beta_{k+1-i,d-k-1+i} = \sum_{i=1}^k a_i^{(k)} \beta_{k+1-i,d-k-1+i} = F_d^k.
\]

Thus, \( W(N_k) \sim F_d^k \). \( \square \)

Algorithm [1] summarizes our simulation algorithm for \( V_p \). The required sort algorithm is standard in most programming languages, like the sub-routine \( \text{SimulateU}[0,1](n) \), which denotes a simulation algorithm for a list of \( n \) iid uniform variates on \([0,1]\).

Now denote by \( V_p \) a random variable satisfying \( V_p^p \sim F_d^d \), for instance simulated via Algorithm [1] and denote by \( U^{(1)} \) an independent random vector that is uniformly distributed.
Algorithm 1 Simulation of $V_p$

1: procedure SimulateVp($p, d$)
2: $W = (W_1, \ldots, W_{d-1}) \leftarrow \text{SimulateU}[0, 1](d - 1)$
3: $W = (W, 1)$
4: $W \leftarrow \text{Sort}(W)$
5: $N \leftarrow 1$
6: for $j = 2, \ldots, d$ do
7: \hspace{1em} $B \leftarrow 0$
8: \hspace{1em} $U \leftarrow \text{SimulateU}[0, 1](1)$
9: \hspace{1em} if $U < \theta \frac{S}{j - 1}$ then
10: \hspace{2em} $B \leftarrow 1$
11: \hspace{2em} $N \leftarrow N + B$
12: return $V_p \leftarrow W_N$

on the standard unit simplex in $[0, 1]^d$, for instance simulated using the stochastic representation $U^{(1)} \sim \xi^{(1)}/\|\xi^{(1)}\|_1$, relying on a simulation of $d$ iid unit exponentials. We consider the random vector $Z = V_p(U^{(1)})^\theta$, and observe that by construction

$$P(Z > z) = P(U^{(1)} > \frac{z^p}{V_p^p}) = E\left[\left(1 - \frac{\|z\|^p_{V_p^p}}{1}\right)^{d-1}\right] = E\left[\left(1 - \frac{\|z\|^p_{V_p^p}}{1}\right)^{d-1}\right] = \int_0^1 \left(1 - \frac{\|z\|^p_{V_p^p}}{1}\right)^{d-1} dF_d(v) = (1 - \|z\|_p)^{d-1}. $$

More generally, let now $\varphi$ be an arbitrary, non-negative $d$-monotone function with $\varphi(0) = 1$, and denote by $R_{\varphi}$ a random variable, unique in law, satisfying

$$E\left[\left(1 - \frac{X}{R_{\varphi}}\right)^{d-1}\right] = \varphi(x), \quad x \geq 0,$$

independent of $V_p$ and $U^{(1)}$. Then the random vector

$$Z = R_{\varphi} V_p U^\theta$$

satisfies

$$P(Z > z) = P(V_p(U^{(1)})^\theta > \frac{z}{R_{\varphi}}) = E\left[\left(1 - \frac{\|z\|_p}{\|R_{\varphi}\|_p}\right)^{d-1}\right] = \varphi(\|z\|_p), \quad \text{(11)}$$

as desired.

Example 4.2 (Simulation of strict outer power Clayton copulas)

Consider the Archimedean generator $\varphi(x) = (1 - x/a)^a$ for a parameter $a \geq d - 1$, \ldots
which is known as a *strict Clayton generator*. In [7, Example 3.3] this is shown to be $d$-monotone and it is also shown that the distribution function of $R_\varphi$ is given by

$$
P(R_\varphi \leq x) = 1 - \sum_{k=0}^{d-1} \frac{a(a-1) \cdots (a-k+1)}{k!} \left( \frac{x}{a} \right)^k \left( 1 - \frac{x}{a} \right)^{a-k}, \quad x \in [0, a].$$

Taking the derivative, it is not difficult to compute from this expression that for $a > d-1$ the random variable $R_\varphi$ satisfies the distributional equality $R_\varphi/a \sim \beta_{d,a-d+1}$. Our results imply that the random vector $Z \sim R_\varphi V_p(U(1))^\theta$ has survival function $(1 - \|\|_p/a)^a$. Consequently, the distribution function of the random vector $(\varphi(Z_1), \ldots, \varphi(Z_d))$ is the Archimedean copula with $x \mapsto \varphi(x^\theta) = (1 - x^\theta/a)^a$ as Archimedean generator. This is a strict outer power Clayton copula. Figure 1 visualizes scatter plots for this copula in the case $d = 2$ (because larger $d$ are difficult to visualize), which have been produced making use of Algorithm 1.

![Figure 1](image_url)

**Fig. 1** Scatter plot of 2,500 samples from the strict outer power Clayton copula in Example 4.2. Left: $p = 1$ (so proper Clayton) and $a = 1.75$. Right: $p = 2.5$ and $a = 1.75$.

**Remark 4.3 (Relation to positive stable distribution)**

If $\varphi(x) = \exp(-x)$, it is well-known and easy to verify that $Z$ with survival function $\exp(-\|\|_p)$ satisfies $Z \sim M_{\vartheta}^\theta (\xi^{(1)})^\theta$, where $M_{\vartheta}$ is a positive stable random variable with Laplace transform $x \mapsto \exp(-x^\vartheta)$. Since $\varphi$ equals the Williamson-$d$-transform of an Erlang distributed random variable $E$ with $d$ degrees of freedom, our results thus imply the distributional identity

$$M_{\vartheta}^{-\theta} (\xi^{(1)})^\theta \sim EV_p \left( \frac{\xi^{(1)}}{\|\xi^{(1)}\|_1} \right)^\theta.$$

Since $V_p$ is a finite mixture of beta distributions, this resembles a distributional equality found in [13, Theorem 1], representing the positive stable distribution with rational $\vartheta$. 
in terms of beta distributions.

5 Max-infinitely divisible laws with \( \ell_p \)-norm symmetric exponent measures

A random vector \( \mathbf{Y} \) taking values in \((0, \infty)^d\) is called \emph{max-infinitely divisible} if for arbitrary \( n \geq 1 \) there exist iid random vectors \( \mathbf{Y}^{(1,n)}, \ldots, \mathbf{Y}^{(n,n)} \) such that

\[
\mathbf{Y} \sim \left( \max_{i=1,\ldots,n} \{ Y_1^{(i,n)} \}, \ldots, \max_{i=1,\ldots,n} \{ Y_d^{(i,n)} \} \right).
\]

It is well known (see [11]) that this is equivalent to \( \mathbf{Y} \) having a distribution function given by

\[
P(\mathbf{Y} \leq \mathbf{y}) = \exp \left\{ -\mu \left( E \setminus [0, \mathbf{y}] \right) \right\},
\]

where \( \mu \) is a measure on \( E := [0, \infty] \setminus \{0\} \) subject to the properties

\[
\mu \left( E \setminus [0, \mathbf{y}] \right) < \infty \quad \forall \mathbf{y} > 0, \quad \lim_{y \to \infty} \mu \left( E \setminus [0, \mathbf{y}] \right) = 0.
\]

The measure \( \mu \) is called the \emph{exponent measure} of \( \mathbf{Y} \) and [3] investigate exponent measures \( \mu \) with \( \ell_1 \)-norm symmetric survival function. We generalize this investigation to \( \ell_p \)-norm symmetric survival functions in the following. To wit, we say that \( \mu \) has an \( \ell_p \)-norm symmetric survival function if there is a function \( \varphi : (0, \infty) \to [0, \infty) \) in one variable, called \emph{generator}, such that

\[
\mu \left( (\mathbf{y}, \infty] \right) = \varphi(\|\mathbf{y}\|_p), \quad \mathbf{y} \in E.
\]

As [3] explain, with Poincaré’s inclusion exclusion identity we may write

\[
\mu \left( E \setminus [0, \mathbf{y}] \right) = \sum_{\emptyset \neq I \subset \{1,\ldots,d\}} (-1)^{|I|+1} \mu \left( \mathbf{y}_I, \infty] \right),
\]

where \( \mathbf{y}_I \in [0, \infty) \) denotes a point whose \( j \)-th coordinate equals \( y_j 1_{\{j \in I\}} \). If now \( \mu \) has \( \ell_p \)-norm symmetric survival function, then we obtain

\[
\mu \left( E \setminus [0, \mathbf{y}] \right) = \sum_{\emptyset \neq I \subset \{1,\ldots,d\}} (-1)^{|I|+1} \varphi(\|\mathbf{y}_I\|_p),
\]
so that the distribution function of $Y$ is given in terms of the univariate function $\varphi$. Furthermore, it is immediately clear from this computation that $Y$ is max-infinitely divisible with $\ell_p$-norm symmetric survival function and generator $\varphi$ if and only if the random vector $Y^p$ is max-infinitely divisible with $\ell_1$-norm symmetric survival function and generator $x \mapsto \varphi(x^p)$. The following lemma gives a concise recap of the results in [3].

**Lemma 5.1 (Genest et al. (2018) [3])**

Fix $p \geq 1$. The following are equivalent for a function $\varphi : (0, \infty) \to [0, \infty)$:

(a) There exists a non-finite Radon measure $\nu$ on $(0, \infty]$ with $\nu(\{\infty\}) = 0$ such that

$$\varphi(t) = \varphi_\nu(t) := \int_t^\infty \left(1 - \frac{t}{r}\right)^{d-1} \nu(dr).$$

(b) $\varphi$ is $d$-monotone and satisfies $\lim_{t \to \infty} \varphi(t) = 0$, $\lim_{t \downarrow 0} \varphi(t) = \infty$.

(c) $\varphi$ is the generator of a max-infinitely divisible law on $(0, \infty)^d$ whose exponent measure has $\ell_1$-norm symmetric survival function.

(d) $\varphi$ is the generator of a max-infinitely divisible law on $(0, \infty)^d$ whose exponent measure has $\ell_p$-norm symmetric survival function.

**Proof**

The equivalences of (a) - (c) have been established in [3], and that (d) is equivalent as well has been explained in the text preceding this lemma. □

Making use of Theorem 3.4, we are able to derive an exact simulation algorithm for max-infinitely divisible $Y$ whose exponent measure satisfies (1). The basis for this algorithm is the following lemma.

**Lemma 5.2 (Stochastic representation for $Y$)**

If one (hence all) of the conditions in Lemma 5.1 is satisfied, a max-infinitely divisible random vector $Y$ whose exponent measure $\mu$ is given by (1) satisfies the distributional equality

$$Y \sim \left( \max_{k \geq 1} \left\{G_{\nu}^{-1}(\xi_1 + \ldots + \xi_k) \, Z_1^{(k)} \right\}, \ldots, \max_{k \geq 1} \left\{G_{\nu}^{-1}(\xi_1 + \ldots + \xi_k) \, Z_d^{(k)} \right\} \right), \quad (12)$$

where $G_{\nu}(x) := \nu((x, \infty])$ denotes the survival function of the Radon measure $\nu$ in Lemma 5.1(a) and $G_{\nu}^{-1}$ its generalized inverse, $\xi_1, \xi_2, \ldots$ is a sequence of iid standard exponential random variables and, independently, $Z^{(1)}, Z^{(2)}, \ldots$ is a sequence of iid copies of $Z \sim V_p \left(U^{(1)}\right)^\theta$. 

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Proof
Notice that
\[ P := \sum_{k \geq 1} \delta_{(\xi_1 + \cdots + \xi_k, Z^{(k)})} \]
is Poisson random measure on \([0, \infty) \times [0, 1]\) with mean measure \(dx \times \mathbb{P}(Z \leq dz)\). We denote by \(\tilde{Y}\) the random vector on the right-hand side of (12), and we compute with the exponential functional formula for the Poisson random measure in \((*)\), see [11], and with the help of inclusion exclusion in \((**)\) that
\[ -\log \left( \mathbb{P}(\tilde{Y} \leq y) \right) = -\log \left( \mathbb{E} \left[ \exp \left\{ - \int \log \left( \prod_{i=1}^d 1_{\left\{ G^{-1}_\nu(x) z_i \leq y_i \right\}} \right) P(dx, dz) \right\} \right] \right) \]
\[ \overset{(*)}{=} \int_{(0, \infty)} \int_{[0, 1]} \left( 1 - \prod_{i=1}^d 1_{\left\{ G^{-1}_\nu(x) z_i \leq y_i \right\}} \right) dx \mathbb{P}(Z \leq dz) \]
\[ = \mathbb{E} \left[ \max_{i=1, \ldots, d} \left\{ G_\nu \left( \frac{y_i}{Z_i} \right) \right\} \right] \]
\[ = \int_0^\infty \mathbb{P} \left( \max_{i=1, \ldots, d} \left\{ G_\nu \left( \frac{y_i}{Z_i} \right) \right\} > x \right) dx \]
\[ \overset{(**)}{=} \sum_{\emptyset \neq I \subset \{1, \ldots, d\}} (-1)^{|I|+1} \int_0^\infty \mathbb{P} \left( Z > \frac{y_I}{G^{-1}_\nu(x)} \right) dx \]
\[ = \sum_{\emptyset \neq I \subset \{1, \ldots, d\}} (-1)^{|I|+1} \int_0^\infty \left( 1 - \frac{\|y_I\|_p}{G^{-1}_\nu(x)} \right)^{d-1} \frac{dx}{x} \nu(dx) = \sum_{\emptyset \neq I \subset \{1, \ldots, d\}} (-1)^{|I|+1} \varphi(\|y_I\|_p), \]
establishing the claim. \hfill \Box

Finally, it remains to be explained how to simulate random vectors \(Y\) with stochastic representation (7) exactly, because it involves a maximum over infinitely many numbers. To this end, the decisive aspect is that the \(Z^{(k)}\) are bounded in \([0, 1]\), due to our disintegration result. This allows to generalize the algorithm of [6] for the case \(p = 1\) to the general case \(p \geq 1\), which is briefly explained. If we denote
\[ M_k := \min_{j=1, \ldots, d} \left\{ \max_{k=1, \ldots, n} \left\{ G^{-1}_\nu(\xi_1 + \cdots + \xi_k) Z_j^{(k)} \right\} \right\}, \quad k \geq 1, \]
the $j$-th component of $\mathbf{Y}$ in (7) is actually equal to

$$Y_j = \max_{k \geq 1} \{ G_{\nu}^{-1}(\xi_1 + \ldots + \xi_k) Z_j^{(k)} \} = \max_{k=1, \ldots, N} \{ G_{\nu}^{-1}(\xi_1 + \ldots + \xi_k) Z_j^{(k)} \},$$

$$N = \min\{ k \geq 1 : G_{\nu}^{-1}(\xi_1 + \ldots + \xi_{k+1}) \leq M_k \},$$

and the random variable $N$ is independent of $j$ and almost surely finite, since $\mathbf{Z}$ is bounded. Summarizing, Algorithm 2 is an exact simulation algorithm for $\mathbf{Y}$, with $\text{SIMULATEExp}(n)$ denoting a sub-routine that generates $n$ iid standard exponentials.

**Algorithm 2** Simulation of $\mathbf{Y}$ in (7) with radial measure $\nu$

1: procedure $\text{SIMULATEY}(p, d, \nu)$
2: $\mathbf{Y} = (Y_1, \ldots, Y_d) \leftarrow (0, \ldots, 0)$
3: $T \leftarrow \text{SIMULATEExp}(1)$
4: $\eta \leftarrow G_{\nu}^{-1}(T)$
5: while $\eta > \min\{Y_1, \ldots, Y_d\}$ do
6: $\xi = (\xi_1, \ldots, \xi_d) \leftarrow \text{SIMULATEExp}(d)$
7: $\mathbf{V}_p \leftarrow \text{SIMULATEV}(p, d)$
8: $\mathbf{Z} = (Z_1, \ldots, Z_d) \leftarrow \mathbf{V}_p \left( \frac{\xi}{\xi_1 + \ldots + \xi_d} \right)^{\theta}$
9: for $j = 1, \ldots, d$ do
10: $Y_j \leftarrow \max\{Y_j, \eta Z_j\}$
11: $T \leftarrow T + \text{SIMULATEExp}(1)$
12: $\eta \leftarrow G_{\nu}^{-1}(T)$
13: return $\mathbf{Y}$

**Example 5.3 (An example with singular component)**

Consider the radial measure $\nu = \nu_a = \eta \sum_{k \geq 1} \delta_{1/k}$ for a parameter $a > 0$. As pointed out in [6, Example 2.3] the associated generator $\varphi_{\nu}$ and required inverse $G_{\nu}^{-1}$ are

$$\varphi_{\nu}(t) = a \sum_{k=1}^{[1/t]} (1 - k t)^{\theta-1}, \quad G_{\nu}^{-1}(t) = 1/\left( \frac{t}{a} \right).$$

The scatter plots in Figure 2 depict samples of $\exp(-\varphi(\mathbf{Y}))$ for $d = 2$ (because larger dimensions are difficult to visualize), illustrating that $\mathbf{Y}$ is not absolutely continuous, and demonstrating the effect of introducing $p$ in comparison to the known case $p = 1$.  

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Fig. 2 Scatter plots from the reciprocal Archimedean copula in Example 5.3. Left: $p = 1$ (so regular reciprocal Archimedean copula) and $a = 1.125$. Right: $p = 4$ and $a = 1.125$.

A Appendix

Lemma A.1 ($\ell_p$-norm symmetric density)
The density $f$ of an absolutely continuous random vector $X$ on $(0, \infty)^d$ is $\ell_p$-norm symmetric if and only if $X \sim R \mathcal{U}^{(p)}$, where $\mathcal{U}^{(p)}$ is uniform on the $\ell_p$-sphere (restricted to the positive orthant w.l.o.g.), which we denote $S_{d,p}$, and $R$ is an independent positive random variable.

Proof
We slightly generalize the computation on page 78 in [5], considering the mapping $h : (0, \infty)^d \to S_{d,p} \times (0, \infty), x \mapsto (x/\|x\|_p, \|x\|_p)$. We observe that $(h^{-1})'(y, s) = ps^{p+d-2}$ is independent of the first $d - 1$ components of $h^{-1}$. Clearly, if $X \sim R \mathcal{U}^{(p)}$ then the density is $\ell_p$-norm symmetric. Now assume that $f(x) = g(\|x\|_p)$ for some function $g$ of one variable. For an arbitrary bounded and continuous function $b$ multivariate change of variables implies

$$
\mathbb{E}[b(X)] = \int b(x) f(x) \, dx = \int_{(0,\infty)} \int_{S_{d,p}} b(s \, y) \, dy \, g(s) \, p \, s^{p+d-2} \, ds
$$

$$
= \int_{(0,\infty)} \mathbb{E}[b(R \mathcal{U}^{(p)}) \mid R = s] \, g(s) \, p \, s^{p+d-2} \, ds.
$$

This implies the claim. \qed

Proof (of Lemma 3.3)
To verify the first identity, one may first prove via induction and integration by parts
that
\[
\beta_{m,n}(x) = (m+n-1)! \sum_{k=0}^{n-1} \frac{x^{m+k}(1-x)^{n-1-k}}{(m+k)!(n-1-k)!}.
\] (13)

Using (13), the first identity is readily established. To verify the second identity, we make use of the first in (\ast) below and observe
\[
\begin{align*}
\beta_{m,n-1}(x) - \beta_{m,n}(x) &= \int_0^x \frac{(m+n-2)!}{(m-1)!(n-2)!} y^{m-1}(1-y)^{n-2} \, dy - \frac{(m+n-1)!}{(m-1)!(n-1)!} y^{m-1}(1-y)^{n-1} \, dy \\
&= \int_0^x \frac{(m+n-2)!}{(m-1)!(n-2)!} y^{m-1}(1-y)^{n-2} \left( y - \frac{m}{n-1} (1-y) \right) \, dy \\
&= \frac{m}{m+n-1} \left( \beta_{m+1,n-1}(x) - \beta_{m,n}(x) \right) \overset{(\ast)}{=} - \frac{m}{m+n-1} \left( \frac{m+n-1}{m} \right) x^m (1-x)^{n-1} \\
&= - \left( \frac{m+n-2}{m-1} \right) x^m (1-x)^{n-1}. \quad \square
\end{align*}
\]

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