Exactness of Bousfield localizations of simplicial presheaves and local lifting

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Abstract
We show that weak equivalences in a (cofibrantly generated) left Bousfield localization of the projective model category of simplicial presheaves can be characterized by a local lifting property if and only if the localization is exact.

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1 Introduction
Let $\mathcal{S}$ be a small category with Grothendieck topology. Dugger, Hollander and Isaksen \[1\] showed as a byproduct of their proofs that weak equivalences in the Bousfield localization of simplicial presheaves on $\mathcal{S}$ at all hypercovers can be characterized by a local lifting property which itself involves hypercovers of objects in $\mathcal{S}$ and their refinements. This article grew out of an attempt to generalize a similar statement to Čech weak equivalences (i.e. weak equivalences in the localization at the Čech covers) with the hope to get a better understanding of these. It is shown that such a characterization by local lifting is possible more generally — by purely formal reasons — whenever the localization is exact (i.e. as functor of derivators, or (∞, 1)-categories, commutes with homotopically finite homotopy limits).

The main result is the following:
Theorem 4.9. Let $\mathcal{S}$ be a small category. Choose the projective model category structure on $\mathcal{SET}^{\mathcal{S}^{\text{op}} \times \Delta^{\text{op}}}$. Consider a (cofibrantly generated) left Bousfield localization with class $\mathcal{W}_{\text{loc}}$ of weak equivalences. Let $\mathcal{COV}$ be a subcategory of coverings satisfying (C1)–(C4) below. The following are equivalent:

1. $\mathcal{W}_{\text{loc}}$ is stable under pull-back along fibrations;
2. $\mathcal{S}$, a generating set of cofibrations, goes to $\mathcal{W}_{\text{loc}}$ under pull-back along fibrations with cofibrant source;
3. The left Bousfield localization is exact, i.e. the localization functor (left adjoint) commutes with homotopically finite homotopy limits;
4. $\mathcal{W}_{\text{loc}} \subseteq \mathcal{W}_{\text{COV}}$;
5. $\mathcal{W}_{\text{loc}} = \mathcal{W}_{\text{COV}}$.

In the Theorem $\mathcal{COV} \subseteq \mathcal{SET}^{\mathcal{S}^{\text{op}} \times \Delta^{\text{op}}}$ is a subcategory of “coverings” which satisfies the following axioms:

(C1) Each object of $\mathcal{COV}$ is cofibrant;
(C2) Every representable presheaf (considered as constant simplicial presheaf) is in $\mathcal{COV}$;
(C3) Each morphism of $\mathcal{COV}$ is in $\mathcal{W}_{\text{loc}}$;
(C4) If $Y \in \mathcal{COV}$ and $Y' \to Y$ is in $\text{Fib} \cap \mathcal{W}_{\text{loc}}$ with $Y'$ cofibrant then $Y' \to Y$ is in $\mathcal{COV}$.

The class $\mathcal{W}_{\text{COV}}$ consists by definition of those morphisms $f$ for which there is a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f'} & Y'
\end{array}
\]

where $f'$ has the local homotopy lifting property w.r.t. $\mathcal{COV}$ (cf. Definition A.5). $X'$ and $Y'$ are fibrant and the horizontal morphisms are in $\mathcal{W}$.

The $(\infty,1)$-category defined by a cofibrantly generated exact localization of $\mathcal{SET}^{\mathcal{S}^{\text{op}} \times \Delta^{\text{op}}}$ as in Theorem 4.9 is by definition a $(\infty,1)$-topos. If the localization is furthermore topological in the sense of Lurie [4, Definition 6.2.1.4], it can be shown that it corresponds to a Grothendieck topology on $\mathcal{S}$ and is given by the Čech localization considered in Example 3.6. However, also the localization at all hypercovers, which presents the hypercompletion of this $(\infty,1)$-topos is still exact.

Notation

Let $\mathcal{S}$ be a small category. Recall that the category $\mathcal{SET}^{\mathcal{S}^{\text{op}} \times \Delta^{\text{op}}}$ of simplicial presheaves on $\mathcal{S}$ is a simplicial category which is tensored and cotensored. We denote the corresponding functors by

\[
\otimes : \mathcal{SET}^{\Delta^{\text{op}}} \times \mathcal{SET}^{\mathcal{S}^{\text{op}} \times \Delta^{\text{op}}} \to \mathcal{SET}^{\mathcal{S}^{\text{op}} \times \Delta^{\text{op}}},
\]
and

\[ \text{Hom} : (\mathcal{S} \mathcal{E} \mathcal{T}^{\Delta^{op}})^{op} \times \mathcal{S} \mathcal{E} \mathcal{T}^{S^{op} \times \Delta^{op}} \to \mathcal{S} \mathcal{E} \mathcal{T}^{S^{op} \times \Delta^{op}}. \]

For a morphism \( f : X \to Y \) of simplicial sets and a morphism \( g : A \to B \) of simplicial presheaves we denote by

\[ f \uplus g : (X \otimes B) \oplus (X \otimes A) \to (Y \otimes B) \]

the induced morphism and likewise

\[ \square \text{Hom}(f, g) : \text{Hom}(Y, A) \to \text{Hom}(Y, B) \times \text{Hom}(X, B) \text{Hom}(X, A). \]

2 Exactness of left Bousfield localizations

In this section let \((\mathcal{M}, \text{Cof}, \text{Fib}, \mathcal{W})\) be a right proper model category and let \((\mathcal{M}, \text{Cof}, \text{Fib}_{\text{loc}}, \mathcal{W}_{\text{loc}})\) be a left Bousfield localization thereof.

**Definition 2.1.** A morphism \( f \) has property \( P \) if any pull-back of \( f \) is in \( \mathcal{W}_{\text{loc}} \). A morphism \( f \) has property \( P_{\text{fib}} \) if any pull-back of \( f \) along a fibration is in \( \mathcal{W}_{\text{loc}} \).

Obviously we have

\[ f \text{ has } P \Rightarrow f \text{ has } P_{\text{fib}} \Rightarrow f \in \mathcal{W}_{\text{loc}} \]

and also (right properness and \( \mathcal{W} \subset \mathcal{W}_{\text{loc}} \)):

\[ f \in \mathcal{W} \Rightarrow f \text{ has } P_{\text{fib}}. \]

**Lemma 2.2.**

1. If a fibration has property \( P_{\text{fib}} \) then it has property \( P \).

2. If \( f = hg \) and \( g \) has property \( P_{\text{fib}} \) then \( f \) has property \( P_{\text{fib}} \) if and only if \( h \) has property \( P_{\text{fib}} \).

3. If \( f \) is a morphism with property \( P_{\text{fib}} \) and \( f = hg \) with \( g \) trivial cofibration and \( h \) fibration, then \( h \) has property \( P \).

**Proof.** 1. Let \( w : X \to Y \) be a fibration with property \( P_{\text{fib}} \). Let \( f \) be an arbitrary morphism. Factor \( f = pc \) with \( c \in \text{Cof} \cap \mathcal{W} \) and \( p \in \text{Fib} \). Then in the pull-back

\[
\begin{array}{ccc}
\square & \xrightarrow{c'} & \square \\
\downarrow w' & & \downarrow w \\
Z & \xrightarrow{c} & Y \\
\downarrow w' & & \downarrow w \\
Z' & \xrightarrow{p} & Y
\end{array}
\]

the morphism \( w' \) is in \( \mathcal{W}_{\text{loc}} \) by assumption. It is also in \( \text{Fib} \), hence by right properness we get \( c' \in \mathcal{W} \subset \mathcal{W}_{\text{loc}} \). Therefore \( w'' \in \mathcal{W}_{\text{loc}} \) by 2-out-of-3.

2. Consider a diagram in which the squares are Cartesian

\[
\begin{array}{ccc}
X' & \xrightarrow{\text{Fib}} & X \\
\mathcal{W}_{\text{loc}} & \xrightarrow{g} & \\
Y' & \xrightarrow{\text{Fib}} & Y \\
\downarrow h & & \downarrow h \\
Z' & \xrightarrow{\text{Fib}} & Z
\end{array}
\]
Since \( g \) has property \( P_{fib} \) the upper left morphism is in \( W_{loc} \). Hence the statement follows from 2-out-of-3.

3. \( g \) has property \( P_{fib} \) and thus by 2. the same holds for \( h \). Therefore, we deduce from 1. that \( h \) has property \( P \).

\[ \square \]

**Theorem 2.3.** Let \((\mathcal{M}, \text{Cof}, \text{Fib}, \mathcal{W})\) be a right proper model category and let \((\mathcal{M}, \text{Cof}, \text{Fib}_{loc}, \mathcal{W}_{loc})\) be a left Bousfield localization. Then the following are equivalent:

1. \( W_{loc} \) is stable under pull-back along morphisms in Fib;

2. The localization functor (left adjoint) commutes with homotopy pull-backs (and hence with homotopically finite homotopy limits). We also say that the localization is **exact**.

**Proof** (compare also \([4, 6.2.1.1]\)).

1. \( \Rightarrow \) 2.: Assume that \( W_{loc} \) is stable under pull-back along morphisms in Fib, i.e. all morphisms in \( W_{loc} \) have property \( P_{fib} \). Then by Lemma 2.2 1. the class \( \text{Fib} \cap W_{loc} \) is stable under arbitrary pull-back. Consider a morphism of diagrams

\[
\begin{array}{ccc}
X & \rightarrow & X' \\
\downarrow & & \downarrow \\
Z & \rightarrow & Y \\
\end{array}
\]

in which in both diagrams both morphisms are in Fib, all objects are Fib-fibrant, and \( \mu \) is point-wise in \( W_{loc} \). We claim that then the induced morphism between pull-backs is in \( W_{loc} \). We may factor \( \mu = \mu_2 \mu_1 \) where \( \mu_1 \) is a point-wise trivial cofibration (between diagrams with the same properties) and \( \mu_2 \) is a point-wise fibration and thus still point-wise in \( W_{loc} \). Since the statement is clear for \( \mu_1 \) we may thus assume w.l.o.g. that \( \mu \) is a point-wise fibration.

The morphism between pull-backs might be written as the following composition:

\[
X \times_Y Z \rightarrow X \times_{Y'} Z \rightarrow X \times_Y Z' \rightarrow X' \times_{Y'} Z'.
\]

The last two morphisms are in \( W_{loc} \) because of 1. The first is the following pullback of the diagonal \( Y \rightarrow Y \times_Y Y' \):

\[
\begin{array}{ccc}
X \times_Y Z & \rightarrow & Y \\
\downarrow & & \downarrow \\
X \times_{Y'} Z & \rightarrow & Y \times_{Y'} Y \\
\end{array}
\]

in which the bottom morphism is a composition of pull-backs of morphisms in Fib, hence in Fib. Therefore it suffices to see that the diagonal \( Y \rightarrow Y \times_Y Y' \) is in \( W_{loc} \). But that has a section \( Y \times_{Y'} Y \rightarrow Y \) which is a pullback of the morphism \( Y \rightarrow Y' \) in \( \text{Fib} \cap W_{loc} \) and thus it is in \( W_{loc} \) itself. This shows that fibrant replacement of diagrams in \( \mathcal{M} \) also derives the pull-back functor in the Bousfield localization and hence the localization commutes with homotopy pull-back.

2. \( \Rightarrow \) 1.: Consider a Cartesian square

\[
\begin{array}{ccc}
X \times_Y Z & \rightarrow & X \\
\downarrow & & \downarrow \\
Z & \rightarrow & Y \\
\end{array}
\]

in which \( f \in \text{Fib} \) and \( w \in W_{loc} \). Since \( f \) is a fibration, the pull-back is a homotopy pull-back in \( \mathcal{M} \) by right properness of \( \mathcal{M} \). By assumption it is also a homotopy pull-back in the localization. Since \( w \) is a weak equivalence in the localization, also \( w' \) must be.

\[ \square \]
Theorem 2.4. A morphism of right derivators with domain homotopically finite diagrams commutes with pullbacks and terminal object if and only if it commutes with all homotopically finite limits.

Proof. This is (the dual of) [5, Theorem 7.1].

3 Exact localizations of simplicial presheaf categories

3.1. In their article [1] (cf. in particular [1, Proposition 5.1]) the authors investigate — as a byproduct of their proofs — to which extent weak equivalences in a left Bousfield localization of a model category of simplicial presheaves (the localization at all hypercovers) can be characterized by local lifting properties along hypercovers. In the appendix we subsumed several general facts about the abstract notion of “local lifting”. In this section and the next we show that weak equivalences in a left Bousfield localization of simplicial presheaves can be described by such a local lifting property precisely if the localization is exact. This works therefore also for the Čech local model structure. However, it is more of theoretical interest because the abstract lifting property is quite self-referential.

3.2. In this section we consider a model category structure \((\mathcal{S}ET^{S^\text{op} \times \Delta^\text{op}}, \text{Cof}, \text{Fib}, \mathcal{W})\) where \(\mathcal{W}\) is the class of section-wise weak equivalences, and a left Bousfield localization \((\mathcal{S}ET^{S^\text{op} \times \Delta^\text{op}}, \text{Cof}, \text{Fib}_{\text{loc}}, \mathcal{W}_{\text{loc}})\) thereof. We assume that the first structure is simplicial, left and right proper, and cofibrantly generated, and that the class of cofibrations is contained in the class of monomorphisms. For instance, this holds for the projective or for the injective model category structure.

3.3. Recall that the injective structure is characterized by the fact that the cofibrations are the monomorphisms. Thus fibrations are those morphisms that have the right lifting property w.r.t. all monomorphisms that are also section-wise weak equivalences. In the projective structure the fibrations are the section-wise surjective morphisms and the cofibrations are those morphisms \(X \to Y\) for which the morphism

\[ L_n Y \cup L_n X_n \to Y_n \]

is of the form \(A \to A \cup \bigsqcup B_i\), where the \(B_i\) are retracts of representables. If \(S\) is idempotent complete (for example if it has fiber products) then the \(B_i\) are representable themselves. There is an even more concrete description of the cofibrant objects (cf. e.g. [2, Proposition 4.9]). In particular those are degree-wise coproducts of retracts of representables.

3.4. We assume that also the Bousfield localization is cofibrantly generated or, equivalently, that it is a left Bousfield localization generated by a set \(S\) of cofibrations as in Theorem [4, A.3.7.3.]. It follows that the set of weak equivalences \(\mathcal{W}_{\text{loc}}\) is also part of a left Bousfield localization of the injective structure and we will sometimes use this fact.

3.5. The notions “fibration” and “trivial cofibration” will mean the corresponding notion for the global model structure. Because of the assumption on the existence of a subset \(S \subset \text{Cof} \cap \mathcal{W}_{\text{loc}}\) of generating cofibrations for the left Bousfield localization every trivial cofibration in the localization is a retract of a transfinite composition of trivial cofibrations and of push-outs of morphisms of the form \((\partial \Delta_n \to \Delta_n) \circ f\), where \(f\) is a cofibration in \(S\).

\(^{1}\)where \(L_n\) denotes the \(n\)-th latching object w.r.t. the Reedy structure on \(\Delta^\text{op}\)
Example 3.6. Fix a Grothendieck pre-topology on $\mathcal{S}$ and let $S$ be the set of (cofibration replacements of) the Čech covers. Those arise from a covering $\{U_i \to X\}$ of an object $X \in \mathcal{S}$ and are morphisms $U \to h_X$ where $U$ is the simplicial presheaf defined by

$$U_n := \left( \bigsqcup_i h_{U_i} \right) \times_{h_X} \cdots \times_{h_X} \left( \bigsqcup_i h_{U_i} \right).$$

Example 3.7. Fix a Grothendieck topology on $\mathcal{S}$ and let $S$ be the class of (cofibration replacements of) hypercovers. Hypercovers are morphisms of simplicial presheaves

$$Y \to h_X$$

in which $X \in \mathcal{S}$ and $Y$ is degree-wise a coproduct of representables such that the morphism

$$Y_n \to \text{Hom}(\partial \Delta_n, Y) \times_{\text{Hom}(\partial \Delta_n, h_X)} h_X$$

is a local epimorphism for any $n$. The class $S$ might not be a set. In [1, 6.5] it is shown that one replace $S$ by a dense set $S' \subset S$ of hypercovers without changing the localization. In particular all hypercovers are still weak equivalences in the localization.

Remark 3.8. The above definition of hypercover is a special case of the following more general definition. A (generalized) hypercover between two simplicial presheaves is a morphism

$$Y \to X$$

such that

$$Y_n \to \text{Hom}(\partial \Delta_n, Y) \times_{\text{Hom}(\partial \Delta_n, X)} X$$

is a local epimorphism for any $n$. For a general morphism $A \to B$ of presheaves this means that for any $X \in \mathcal{S}$ and section $h_X \to B$ there exists a covering $\{U_i \to X\}$ and an extension to a commutative square as follows:

$$\begin{array}{ccc}
\bigsqcup_i h_{U_i} & \longrightarrow & A \\
\downarrow & & \downarrow \\
h_X & \longrightarrow & B
\end{array}$$

We have the following formal property:

Lemma 3.9. Generalized hypercovers are closed under pull-back.

Proof. By definition a generalized hypercover $X \to Y$ has the property that

$$X_n \to Y_n \times_{\text{Hom}(\partial \Delta_n, Y)} \text{Hom}(\partial \Delta_n, X)$$

are local epimorphisms. For a pull-back diagram

$$\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
Y' & \longrightarrow & Y
\end{array}$$
consider the diagram

\[
\begin{array}{ccc}
X_n' & \rightarrow & X_n \\
\searrow & & \searrow \\
\Box & \rightarrow & \Box \\
\downarrow & & \downarrow \\
\Hom(\partial \Delta_n, X') & \rightarrow & \Hom(\partial \Delta_n, X) \\
\downarrow & & \downarrow \\
Y_n' & \rightarrow & Y_n \\
\downarrow & & \downarrow \\
\Hom(\partial \Delta_n, Y') & \rightarrow & \Hom(\partial \Delta_n, Y)
\end{array}
\]

in which the front and back (total) square are Cartesian and also (by definition) the right and left squares. It follows that the back lower square is Cartesian and hence also the back upper square is Cartesian. Therefore, since the upper right vertical morphism is a local epimorphism so is the left.

The goal of this section and the next is to establish several equivalent conditions for a given localization \(\mathcal{SET}_{\text{loc}}^{S\times \Delta^op}\) to be exact in the sense of Theorem 2.3. The first is:

**Proposition 3.10.** If each \(f \in S\) has property \(P_{\text{fib}}\) then the whole class \(W_{\text{loc}}\) has property \(P_{\text{fib}}\).

By Theorem 2.3 this is equivalent to exactness and hence will hold for (the localization of) any right proper model category structure with the same weak equivalences.

We need a couple of lemmas:

**Lemma 3.11.**
1. If \(I\) is an ordinal and \(F : I \rightarrow C\) a functor mapping each morphism (i.e. relation) to a cofibration with property \(P_{\text{fib}}\) then also the morphism \(F(0) \rightarrow \colim F\) (the transfinite composition) has property \(P_{\text{fib}}\).

2. Consider a push-out diagram

\[
\begin{array}{ccc}
X & \rightarrow & X' \\
\downarrow^g & & \downarrow^F \\
Y & \rightarrow & Y'
\end{array}
\]

in which \(f\) or \(g\) is a monomorphism. If \(f\) has property \(P_{\text{fib}}\) also \(F\) has property \(P_{\text{fib}}\).

3. If \(f\) is a retract of \(g\) and \(g\) has property \(P_{\text{fib}}\) then \(f\) has property \(P_{\text{fib}}\).

**Proof.**
1. Consider a morphism \(X \rightarrow \colim f\) and the corresponding constant diagrams \((X), (\colim f) : I \rightarrow C\). Consider the pull-back

\[
(X) \times_{(\colim F)} F.
\]

It is still a diagram of monomorphisms in \(W_{\text{loc}}\). Hence

\[
X \times_{\colim F} F(0) \rightarrow \colim((X) \times_{(\colim F)} F)
\]
is in $W_{\text{loc}}$ (using the injective structure in which the monomorphisms are the cofibrations). Since pull-back commutes with filtered colimits this morphism is the same as

$$X \times_{\text{colim} F} F(0) \to X.$$ 

The transfinite composition has thus property $P_{fib}$. 

2. We may argue as in 1. using that push-outs along monomorphisms commutes with fiber products. 

4. We may argue as in 1. using that fiber products commute with retracts. 

**Lemma 3.12.** If $f : X \to Y$ has property $P_{fib}$ then also 

$$(\partial \Delta_n \to \Delta_n) \oplus f$$

has property $P_{fib}$. 

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
\partial \Delta_n \otimes X & \to & \Delta_n \otimes X \\
\downarrow & & \downarrow \\
\partial \Delta_n \otimes Y & \to & \Delta_n \otimes Y
\end{array}
$$

Since $\partial \Delta_n \otimes X \to \Delta_n \otimes X$ is a monomorphism, by Lemma 2.2, 2. and Lemma 3.11, 2. it suffices to show that $\partial \Delta_n \otimes X \to \partial \Delta_n \otimes Y$ and $\Delta_n \otimes X \to \Delta_n \otimes Y$ have property $P_{fib}$. Factor $f : X \to Y$ as

$$X \overset{g}{\to} X' \overset{h}{\to} Y$$

with $g \in \text{Cof} \cap W$ and $h \in \text{Fib}$. By 2-out-of-3 we have $h \in W_{\text{loc}}$. In the following diagram all squares are Cartesian and $K$ is an arbitrary simplicial set:

$$
\begin{array}{ccc}
\square & \overset{\square}{\to} & K \otimes X \\
\uparrow \in \mathcal{W} & & \uparrow \in \mathcal{W} \\
Fib & \overset{g \in \text{Cof} \cap \mathcal{W}}{\to} & Fib \\
\uparrow \in \mathcal{W}_{\text{loc}} & & \uparrow \in \mathcal{W}_{\text{loc}} \\
Z \overset{Fib}{\to} K \otimes Y & \to & Y
\end{array}
$$

The lower left vertical morphism is in $W_{\text{loc}}$ because by Lemma 2.2 1. $h : X' \to Y$ has property $P$ and the upper left vertical morphism is in $W_{\text{loc}}$ because of right properness and the fact that the middle top vertical morphism is in $W$. 

**Lemma 3.13.** Consider two morphisms $f, g : X \to Y$ which are left homotopic. Then $f$ has property $P_{fib}$ if and only if $g$ has property $P_{fib}$. 

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Proof. Let $\mu : \Delta_1 \otimes X \to Y$ be the homotopy between $f$ and $g$. Consider for $i \in \{0, 1\}$ the pull-back diagram along a fibration:

$$
\begin{array}{ccc}
Z_i & \longrightarrow & \{i\} \otimes X \\
\downarrow & & \downarrow e_i \circ W \\
W' & \longrightarrow & \Delta_1 \otimes X \\
\downarrow & & \downarrow \mu \\
Z' & \longrightarrow & \Delta_1 \otimes X \\
\downarrow & & \downarrow \mu \\
Z & \longrightarrow & Y \\
\end{array}
$$

Note that $e_i : \{i\} \otimes X \to \Delta_1 \otimes X$ is in $W$ (existence of the injective structure which is simplicial and in which $X$ is cofibrant). Hence the upper left vertical morphism is in $W$ because of right properness. Assuming $Z_0 \to Z$ is in $W_{loc}$, the first pull-back shows therefore that $Z' \to Z$ is in $W_{loc}$ and so is $Z_1 \to Z$.

Proof of Proposition 3.10. Let $f \in W_{loc}$. It is the composition of a cofibration in $W_{loc}$ and a trivial fibration. Since trivial fibrations are closed under pull-back we may assume w.l.o.g. that $f$ is a cofibration. Then $f$ is a retract of a transfinite composition of pushouts of trivial cofibrations or morphisms of the form

$$(\partial \Delta_n \to \Delta_n) \circ f$$

in which $f$ is a cofibration in $S$. The morphism $f$ has property $P_{fib}$ by assumption. Trivial cofibrations have property $P_{fib}$ because the model category is right proper. We conclude by Lemma 3.11 and Lemma 3.12.

Lemma 3.14. Let $f : X \to Y$ be a morphism with property $P_{fib}$ between cofibrant objects. Factor $f$ as

$$
\begin{array}{ccc}
X & \longrightarrow & Y' \\
\downarrow g & & \downarrow h \\
Y & \longrightarrow & Y \\
\end{array}
$$

with $g \in \text{Cof}$ and $h \in \text{Fib} \cap W$. Then $g$ has property $P_{fib}$.

Proof. Since $Y'$ is cofibrant there is a section $\sigma : Y \to Y'$ such that the composition $\sigma h : Y' \to Y'$ is left homotopic to the identity. Therefore the morphisms $g, \sigma f : X \to Y'$ are left homotopic as well. Since $f$ and $\sigma$ have property $P_{fib}$, by Lemma 3.13 the same holds for $g$.

Lemma 3.15. If a morphism $f$ has the property that pull-backs along fibrations (resp. morphisms) with cofibrant source are in $W_{loc}$ then $f$ has property $P_{fib}$ (resp. property $P$).

Proof. Consider a diagram with Cartesian squares in which $W$ is a cofibrant replacement of $Z$:

$$
\begin{array}{ccc}
W' & \longrightarrow & Z' \\
\downarrow \text{Fib} \cap W & & \downarrow \text{Fib} \cap W \\
W & \longrightarrow & Z \\
\end{array}
\longrightarrow
\begin{array}{ccc}
\longrightarrow & \longrightarrow & X \\
\downarrow & \downarrow & \downarrow \\
\longrightarrow & \longrightarrow & Y \\
\end{array}
$$

By assumption $W' \to W$ is in $W_{loc}$. Thus by 2-out-of-3 the same holds for $Z' \to Z$.

Lemma 3.16. If $X \to Y$ is a morphism of bisimplicial presheaves in $\mathcal{SET}^{S^{op} \times \Delta^{op} \times \Delta^{op}}$ such that the horizontal morphisms of simplicial presheaves

$$
X_{i,j} \to Y_{i,j}
$$

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are in \( \mathcal{W}_{\text{loc}} \) then the map of diagonal simplicial presheaves

\[
\delta^* X \to \delta^* Y
\]
is in \( \mathcal{W}_{\text{loc}} \). Here \( \delta : \Delta^{\text{op}} \to \Delta^{\text{op}} \times \Delta^{\text{op}} \) is the diagonal.

**Proof.** This follows from the fact that the homotopy colimit over \( \Delta^{\text{op}} \) can be computed by the diagonal of a horizontally point-wise cofibrant diagram (hence no restriction in the injective model structure). Since the localization commutes with homotopy colimits we conclude. \( \square \)

**Proposition 3.17.** Čech covers (cf. 3.6) of an object \( X \in \mathcal{S} \) and hypercovers (cf. 3.7) have property \( P \) w.r.t. the respective class \( \mathcal{W}_{\text{loc}} \).

**Proof.** We make two preliminary considerations:

1. If \( U \to h_X \) is a Čech cover and \( Z \to h_X \) a morphism in which \( Z \) is a coproduct of retracts of representables then also the pull-back

\[
U \times_{h_X} Z \to Z
\]
is in \( \mathcal{W}_{\text{loc}} \). For, if \( Z = h_{X'} \) is representable, then \( U \times_{h_X} h_{X'} \) is (by definition of a Grothendieck pre-topology) degree-wise representable and the morphism is a Čech-cover again. Since \( \mathcal{W}_{\text{loc}} \) is closed under coproducts\(^2\) this reduces the claim to the case in which \( Z \) is a retract of a representable:

\[
Z \to h_{X'} \to Z
\]
But then \( U \times_{h_X} Z \to Z \) is a retract of \( U \times_{h_X} h_{X'} \to h_{X'} \) and thus in \( \mathcal{W}_{\text{loc}} \) as well.

2. If \( Y \to h_X \) is a hypercover and \( Z \to h_X \) a morphism in which \( Z \) is a coproduct of retracts of representables, we claim that the pull-back

\[
Y \times_{h_X} Z \to Z
\]
is again in \( \mathcal{W}_{\text{loc}} \). Like before this is reduced to the case in which \( Z = h_{X'} \) is representable. By Lemma 3.9 the morphism

\[
Y \times_{h_X} h_{X'} \to h_{X'}
\]
is a generalized hypercover. Choose a projectively cofibrant replacement (this may be taken to be degree-wise a coproduct of representables)

\[
Y' \to Y \times_{h_X} h_{X'} \to h_{X'}.
\]
Then \( Y' \to Y \times_{h_X} h_{X'} \), being a projective fibration, is also a generalized hypercover (in which the local epimorphisms are even split epimorphisms). Therefore \( Y' \to h_{X'} \) is in \( \mathcal{W}_{\text{loc}} \) and so is \( Y \times_{h_X} h_{X'} \to h_{X'} \).

Now we are able to prove the statement. Consider a pull-back diagram

\[
\begin{array}{ccc}
\square & \to & U \\
\downarrow & & \downarrow \\
A & \to & h_X
\end{array}
\]

\(^2\)because of the existence of the injective structure in which all objects are cofibrant
in which \( U \to h_X \) is a Čech cover, or an hypercover, respectively. We may assume that \( A \) is cofibrant w.r.t. the projective model category structure (Lemma 3.15), in particular is degree-wise a coproduct of retracts of representables. Then form the bisimplicial set
\[
X_{i,j} := U_i \times_X A_i.
\]
The pullback is the diagonal of this bisimplicial set. We have a morphism
\[
X \to A
\]
where \( A \) is seen as a bisimplicial presheaf constant in the horizontal direction. Therefore the map on the diagonals (i.e. homotopy colimits) is in \( \mathcal{W}_{loc} \) by Lemma 3.16 if each horizontal morphism
\[
X_{i,\bullet} \to A_i
\]
is in \( \mathcal{W}_{loc} \). This has been shown in the preliminary considerations.

We obtain the well-known fact:

**Corollary 3.18.** Let \( S \) be a small category with Grothendieck (pre-)topology. The left Bousfield localizations of \( \mathcal{SET}^{S^{op} \times \Delta^{op}} \) at all Čech covers (cf. 3.6), and at all hypercovers (cf. 3.7), respectively, are exact in the sense of Theorem 2.3.

**Proof.** The statement is independent of the choice of model category structure (satisfying the assumptions in the beginning of this section). We may thus work with the injective model category structure. Let \( S \) be a class of cofibration replacements of the Čech covers (resp. hypercovers). Proposition 3.17 and Lemma 3.14 show that the morphisms in \( S \) have property \( P_{fib} \). We conclude by Proposition 3.10.

4 Coverings of simplicial presheaves

4.1. In this section we fix the projective model category structure on \( \mathcal{SET}^{S^{op} \times \Delta^{op}} \).

4.2. Fix a subcategory \( \mathcal{COV} \subset \mathcal{SET}^{S^{op} \times \Delta^{op}} \) of “coverings”. Consider the following axioms on \( \mathcal{COV} \):

(C1) Each object of \( \mathcal{COV} \) is cofibrant.

(C2) Every representable presheaf (considered as constant simplicial presheaf) is in \( \mathcal{COV} \).

(C3) Each morphism of \( \mathcal{COV} \) is in \( \mathcal{W}_{loc} \).

(C4) If \( Y \in \mathcal{COV} \) and \( Y' \to Y \) is in \( \text{Fib} \cap \mathcal{W}_{loc} \) with \( Y' \) cofibrant then \( Y' \to Y \) is in \( \mathcal{COV} \).

There is an obvious smallest and biggest choice of \( \mathcal{COV} \) that satisfies (C1–4), but the choice will not matter.

We now define a class of “local weak equivalences” depending on \( \mathcal{COV} \) as follows:
Definition 4.3. Let $W_{\text{COV}}$ be the class of morphisms $f$ for which there is a diagram

\[
\begin{array}{c}
X \xrightarrow{W} X' \\
\downarrow f \\
Y \xrightarrow{W} Y'
\end{array}
\]

in which $f'$ has the local homotopy lifting property w.r.t. $\text{COV}$ (cf. Definition A.5), $X'$ and $Y'$ are fibrant and the horizontal morphisms are in $W$.

Lemma 4.4. Let $\text{COV}$ satisfy (C1) and (C2) but not necessarily the other axioms.

1. Definition 4.3 is independent of the fibrant replacement.
2. $W_{\text{COV}}$ satisfies 2-out-of-3.
3. The class $W_{\text{COV}}$ is stable under pull-back along fibrations.
4. We have $W \subset W_{\text{COV}}$.

Proof. 1. Consider a refinement of fibrant replacements, i.e. a diagram

\[
\begin{array}{c}
X \xrightarrow{W} X' \xrightarrow{W} X'' \\
\downarrow f \\
Y \xrightarrow{W} Y' \xrightarrow{W} Y''
\end{array}
\]

in which $X', Y', X''$ and $Y''$ are fibrant and the horizontal morphisms are in $W$. A morphism in $W$ between fibrant objects has the homotopy lifting property. Hence $f'$ has the homotopy local lifting property if and only if $f''$ has the homotopy local lifting property by 2-out-of-3 (Lemma A.10). Since any two fibrant replacements can be refined by a third one the statement follows.
2. follows from 1. using any functorial fibrant replacement and 2-out-of-3 for homotopy local lifting properties (Lemma A.10).
3. Let $f : X \rightarrow Y$ be a morphism in $W_{\text{COV}}$. We can find a diagram

\[
\begin{array}{c}
X \xrightarrow{W} X' \\
\downarrow \text{Fib} \\
Y \xrightarrow{W} Y'
\end{array}
\]
in which \( X' \) and \( Y' \) are fibrant. Let \( Z \rightarrow Y \) be a given fibration. We get the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\text{Fib}} & W \\
\downarrow & & \downarrow \\
X \times_Y Z & \xrightarrow{\text{Fib}} & X \\
\downarrow & & \downarrow \\
W & \xrightarrow{\text{Fib}} & W \\
\end{array}
\]

in which \( Z \rightarrow Z' \rightarrow Y' \) is the factorization of the composition \( Z \rightarrow Y \rightarrow Y' \) into trivial cofibration followed by fibration and in which the middle floor is the pullback of the bottom floor under the fibration \( X' \rightarrow Y' \). Hence all 5 upright squares are Cartesian. Thus the so indicated morphisms are weak equivalences by right properness. Note that \( X' \times_Y Z' \rightarrow Z' \) has the local lifting property (Lemma A.9). This shows that \( X \times_Y Z \rightarrow Z \) has a fibrant replacement which has the local lifting property. It is thus in \( \mathcal{W}_{\text{Cov}} \) as well.

4. Let \( f: X \rightarrow Y \) a morphism in \( \mathcal{W} \). As in 3, we may replace \( f \) by a fibration which is then trivial by 2-out-of-3. A trivial fibration has obviously the (local) lifting property.

Proposition 4.5. If \( \text{COV} \) satisfies (C1) and (C2) then a fibration in \( \mathcal{W}_{\text{COV}} \) has the local lifting property itself.

Proof. Let \( f \in \mathcal{W}_{\text{COV}} \cap \text{Fib} \). We find a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{Cof} \cap \mathcal{W}} & X' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\text{Cof} \cap \mathcal{W}} & Y' \\
\end{array}
\]

in which \( f' \) is a fibration between fibrant objects that has the local lifting property. Now form the pull-back and factor the induced morphism as indicated:
Here \( f'' \) has the local lifting property. Thus \( f'' h \) has the local lifting property and thus also \( f \) by Lemma A.11.

\( \mathcal{W}_{\mathcal{C}OV} \) shares the following property with \( \mathcal{W}_{\text{loc}} \):

**Lemma 4.6.** Assume that \( \mathcal{C}OV \) satisfies (C1), (C2), and (C3), then

\[
\text{Fib}_{\text{loc}} \cap \mathcal{W}_{\mathcal{C}OV} \subset \text{Fib} \cap \mathcal{W}.
\]

**Proof.** Let \( f \in \text{Fib}_{\text{loc}} \cap \mathcal{W}_{\mathcal{C}OV} \). Consider a diagram

\[
\begin{array}{ccc}
\partial \Delta_n \otimes h_X & \rightarrow & A \\
\downarrow & & \downarrow f \\
\Delta_n \otimes h_X & \rightarrow & B
\end{array}
\]

in which \( X \) is an object in \( S \) (and thus \( h_X \in \mathcal{C}OV \) by C2). Since \( f \) is in particular in Fib, by Proposition 4.5, it has itself the local lifting property. Thus there is a lift

\[
\begin{array}{ccc}
\partial \Delta_n \otimes X' & \rightarrow & \partial \Delta_n \otimes h_X & \rightarrow & A \\
\downarrow h & & \downarrow & & \downarrow f \\
\Delta_n \otimes X' & \rightarrow & \Delta_n \otimes h_X & \rightarrow & B
\end{array}
\]

in which \( X' \rightarrow h_X \) is in \( \mathcal{C}OV \) and hence in \( \mathcal{W}_{\text{loc}} \) by (C3). Factor \( X' \rightarrow h_X \) as

\[
X' \xrightarrow{\text{Cof}_{\text{loc}}} X'' \xrightarrow{\text{Fib}_{\mathcal{W}}} h_X.
\]

Then \( h \) descends by definition of \( \text{Fib}_{\text{loc}} \) because the morphism

\[
(\partial \Delta_n \rightarrow \Delta_n) \amalg (X' \rightarrow X'')
\]

is a cofibration in \( \mathcal{W}_{\text{loc}} \):

\[
\begin{array}{ccc}
\partial \Delta_n \otimes X' & \rightarrow & \partial \Delta_n \otimes X'' & \rightarrow & \partial \Delta_n \otimes h_X & \rightarrow & A \\
\downarrow & & \downarrow & & \downarrow h & & \downarrow f \\
\Delta_n \otimes X' & \rightarrow & \Delta_n \otimes X'' & \rightarrow & \Delta_n \otimes h_X & \rightarrow & B
\end{array}
\]

The morphisms \( X'' \rightarrow h_X \) are trivial fibrations between cofibrant objects and thus are deformation retractions. This shows that we have finally also a lift

\[
\begin{array}{ccc}
\partial \Delta_n \otimes h_X & \rightarrow & A \\
\downarrow \ & & \downarrow \ \\
\Delta_n \otimes h_X & \rightarrow & B
\end{array}
\]

Hence \( f \) is a trivial projective fibration.

**Lemma 4.7.** Assume that \( \mathcal{C}OV \) satisfies (C1), (C2), and (C3). Then \( \mathcal{W}_{\text{loc}} \subset \mathcal{W}_{\mathcal{C}OV} \) implies \( \mathcal{W}_{\text{loc}} = \mathcal{W}_{\mathcal{C}OV} \).
Proof. This is a standard argument using Lemma 4.6. Let \( f \in W_{\text{COV}} \). Factor \( f \) as cofibration followed by a trivial fibration. The trivial fibration is in \( W_{\text{COV}} \). Thus it suffices to show: Given a fibration \( g \in \text{Fib}_{\text{loc}} \) and \( f \in W_{\text{COV}} \cap \text{Cof} \) then \( f \) has the left lifting property w.r.t. \( g \). Consider the following commutative diagram in which the right hand square is Cartesian and factor \( X \to \Box \) as indicated:

\[
\begin{array}{ccc}
X & \xrightarrow{W_{\text{loc}} \cap W_{\text{COV}}} & A \\
\downarrow f & & \downarrow g \in \text{Fib}_{\text{loc}} \\
Y & \xrightarrow{\text{Fib}_{\text{loc}}} & B \\
\end{array}
\]

Since the morphism \( X \to Y \) is in \( W_{\text{COV}} \) also \( X' \to Y \) must be in \( W_{\text{COV}} \) and hence by Lemma 4.6 in \( \text{Fib} \cap W \). Therefore a lift indicated by the dotted arrow exists using merely that \( f : X \to Y \) is a cofibration.

**Proposition 4.8.** Assume that \( \text{COV} \) satisfies (C1), (C2), and (C3). If \( W_{\text{loc}} \subset W_{\text{COV}} \) then a fibration is in \( W_{\text{COV}} \) if and only if it has the local lifting property itself.

Proof. One direction is Proposition 4.5. For the converse assume that \( f \) is a fibration which has the local lifting property. Factor \( f \) as a morphism in \( \text{Cof} \cap W_{\text{loc}} \) followed by a morphism in \( \text{Fib}_{\text{loc}} \). Then factor the morphism in \( \text{Cof} \cap W_{\text{loc}} \) as trivial cofibration followed by fibration. The fibration is in \( W_{\text{loc}} \) by 2-out-of-3 and thus in \( W_{\text{COV}} \) by assumption:

\[
X \xrightarrow{\text{Cof} \cap W} X' \xrightarrow{\text{Fib} \cap W_{\text{COV}}} X'' \xrightarrow{\text{Fib}_{\text{loc}}} Y.
\]

By Lemma A.11 the fibration (composition of the last 2 morphisms) has the local lifting property. The second morphism has also the local lifting property by Proposition 4.5. Therefore, by 2-out-of-3 (Lemma A.10), also the morphism in \( \text{Fib}_{\text{loc}} \) has the local lifting property and is therefore by the proof of Lemma 4.6 a trivial fibration. In total the morphism \( f \) is in \( W_{\text{COV}} \).

The following theorem summarizes the discussion:

**Theorem 4.9.** Let \( S \) be a small category. Choose the projective model category structure on \( \mathcal{S} \mathcal{E} \mathcal{T}^{\text{Op} \times \Delta^{\text{op}}} \). Consider a (cofibrantly generated) left Bousfield localization with class \( W_{\text{loc}} \) of weak equivalences. Let \( \text{COV} \) be a subcategory of coverings satisfying (C1–4) of 4.2. The following are equivalent:

1. \( W_{\text{loc}} \) is stable under pull-back along fibrations;
2. \( S \), a generating set of cofibrations, goes to \( W_{\text{loc}} \) under pull-back along fibrations with cofibrant source;
3. The left Bousfield localization is exact, i.e. the localization functor (left adjoint) commutes with (homotopically) finite homotopy limits;
4. \( W_{\text{loc}} \subset W_{\text{COV}} \);
5. \( W_{\text{loc}} = W_{\text{COV}} \).
Proof. The implication \( 2 \Rightarrow 1 \) is Proposition 3.10 and \( 1 \Rightarrow 2 \) is clear.
The equivalence \( 1 \Leftrightarrow 3 \) is Theorem 2.3.
5. \( \Rightarrow 1 \). By Lemma 4.4, \( W_{COV} \) is stable under pull-back along fibrations.
1. \( \Rightarrow 4 \). Consider \( Y \in COV \) and \( f \in W_{loc} \). We may choose a fibrant replacement \( f' \) of \( f \) which is also a fibration. Hence we have a commutative square

\[
\begin{array}{ccc}
\partial \Delta_n \otimes Y & \longrightarrow & A \\
\downarrow & & \downarrow f' \\
\Delta_n \otimes Y & \longrightarrow & B
\end{array}
\]

in which \( f' \in Fib \cap W_{loc} \) and \( A \) and \( B \) are fibrant objects. We have to show that a local lifting exists.
Consider the diagram with Cartesian squares

\[
\begin{array}{ccc}
Y' & \xrightarrow{\square} & \text{Hom}(\Delta_n, A) \\
\downarrow W_{\cap Fib} & & \downarrow \text{Hom}(\partial \Delta_n, A) \\
Y & \xrightarrow{\square} & \text{Hom}(\partial \Delta_n, B) \\
\downarrow \text{Hom}(\Delta_n, B) & & \downarrow \text{Hom}(\partial \Delta_n, B)
\end{array}
\]

in which the indicated vertical morphism is in \( W_{loc} \) by Lemma 3.12 and Lemma 2.2, 1., and \( Y' \to \square \) is a cofibrant replacement. Putting things together we get a lift in

\[
\begin{array}{ccc}
\partial \Delta_n \otimes Y' & \longrightarrow & \partial \Delta_n \otimes Y \\
\downarrow & & \downarrow f \\
\Delta_n \otimes Y' & \longrightarrow & \Delta_n \otimes Y \\
\end{array}
\]

and the morphism \( Y' \to Y \) is in \( COV \) by (C4).
4. \( \Rightarrow 5 \). is Lemma 4.7.

4.10. Consider the case of the left Bousfield localization at the Čech covers (cf. 3.6). We may define the smallest class \( COV \) that contains all representable constant presheaves and inductively all morphisms in \( Fib \cap W_{loc} \) between cofibrants object whose target is already an object in \( COV \), i.e. the smallest class satisfying (C1–4). Theorem 4.9, together with Corollary 3.18, shows that \( W_{loc} = W_{COV} \) in this case. However, the characterization is self-referential and thus gives no concrete description of \( W_{loc} \).

4.11. Consider the case of the left Bousfield localization at all hypercovers (cf. 3.7). In this case, we may define the smallest class \( COV \) that contains all representable constant presheaves and inductively all morphisms in \( Fib \cap W_{loc} \) between cofibrants object whose target is already an object in \( COV \), i.e. the smallest class satisfying (C1–4). It other words, \( COV \) contains the objects in \( S \), their hypercovers (which are also global fibrations) and refinements between those. Theorem 4.9, together with Corollary 3.18, shows that \( W_{loc} = W_{COV} \) in this case. Alternatively one can enlarge \( COV \) to contain all hypercovers because this class still satisfies (C3) and (C4).
However, $\mathcal{W}_{\text{loc}}$ may be also be described simply as $\mathcal{W}_C$ (still by Definition 4.3) where $C$ is the subcategory with morphisms $\coprod h_{V_i} \to \coprod h_{U_j}$ induced by refinements of coverings $\{U_i \to X\}$ and $\{V_j \to X\}$ of $X \in S$ (of course this class does not satisfy (C3–4)). This is well-known (cf. [3]) and will not be reproven here.

## A Local homotopy lifting

**A.1.** Let $S$ be a small category and fix the projective model structure on $\mathcal{S}ET^{S^{\text{op}} \times \Delta^{\text{op}}}$ (simplicial presheaves).

**A.2.** Fix a (non-full) subcategory $\mathcal{C}OV$ in $\mathcal{S}ET^{S^{\text{op}} \times \Delta^{\text{op}}}$ such that the objects are cofibrant in the projective model structure (which are in particular degree-wise coproducts of retracts of representables). Furthermore assume that $\mathcal{C}OV$ contains all constant representable presheaves, i.e. assume axioms (C1) and (C2) of 4.2. Axioms (C3) and (C4) will not play any role in this appendix.

**Example A.3.** Examples (for $S$ being equipped with a Grothendieck pre-topology):

1. Hypercovers/bounded hypercovers of varying $X \in S$ and their refinements;
2. Hypercovers of varying $X \in S$ which are also Čech weak equivalences and their refinements;
3. Morphisms of the form $\coprod h_{V_i} \to \coprod h_{U_j}$ for usual refinements of coverings $\{V_i\} \to \{U_j\} \to X$.

**Definition A.4.** We say that a morphism $f$ is a local fibration if in every square

$$
\begin{array}{ccc}
\Lambda_{n,k} \otimes X & \longrightarrow & A \\
\downarrow & & \downarrow f \\
\Delta_n \otimes X & \longrightarrow & B
\end{array}
$$

in which $X$ is an object in $\mathcal{C}OV$ there is a morphism $X' \to X$ in $\mathcal{C}OV$ and a morphism $h$ in the diagram

$$
\begin{array}{ccc}
\Lambda_{n,k} \otimes X' & \longrightarrow & \Lambda_{n,k} \otimes X \\
\downarrow h & & \downarrow f \\
\Delta_n \otimes X' & \longrightarrow & \Delta_n \otimes X
\end{array}
$$

making the upper and lower triangle commute.

Note that a global fibration is in particular a local fibration.

**Definition A.5.** We say that a morphism $f$ has the local (homotopy) lifting property if in every square

$$
\begin{array}{ccc}
\partial \Delta_n \otimes X & \longrightarrow & A \\
\downarrow & & \downarrow f \\
\Delta_n \otimes X & \longrightarrow & B
\end{array}
$$

where $X$ is an object in $\mathcal{C}OV$ there is a morphism $X' \to X$ in $\mathcal{C}OV$ and a morphism $h$ in the diagram

$$
\begin{array}{ccc}
\partial \Delta_n \otimes X' & \longrightarrow & \partial \Delta_n \otimes X \\
\downarrow & & \downarrow h \\
\Delta_n \otimes X' & \longrightarrow & \Delta_n \otimes X
\end{array}
$$
making the upper triangle commute and making the lower triangle commute (resp. commute up to left homotopy).

**Lemma A.6.** If \( f \) has the (homotopy) local lifting property and \( K \rightarrow L \) is an inclusion of finite simplicial sets then also each square

\[
\begin{array}{ccc}
K \otimes Y & \rightarrow & A \\
\downarrow & & \downarrow f \\
L \otimes Y & \rightarrow & B
\end{array}
\]

with \( Y \in \text{COV} \) has a local (homotopy) lifting in the obvious sense. For the homotopy case assume that \( A \) and \( B \) are locally fibrant.

If \( f \) has the local lifting property then also

\[ \boxempty \text{Hom}(K \rightarrow L, f) \]

has it.

**Proof.** Since \( K \rightarrow L \) is a finite composition of push-outs of the form \( \partial \Delta_n \rightarrow \Delta_n \), the first assertion follows by induction.

For the second assertion note that the local lifting property for \( \boxempty \text{Hom}(K \rightarrow L, f) \) is equivalent to the existence of a local lifting in the diagram

\[
\begin{array}{ccc}
(L \times \partial \Delta_n \cup K \times \Delta_n) \otimes Y & \rightarrow & A \\
\downarrow & & \downarrow f \\
(L \times \Delta_n) \otimes Y & \rightarrow & B
\end{array}
\]

If \( K \rightarrow L \) is an inclusion of finite simplicial sets then also \((K \rightarrow L) \boxempty (\partial \Delta_n \rightarrow \Delta_n)\) is.

**Lemma A.7.** If \( f \) is a local fibration and \( K \rightarrow L \) is a strong anodyne extension of finite simplicial sets then each square

\[
\begin{array}{ccc}
K \otimes Y & \rightarrow & A \\
\downarrow & & \downarrow f \\
L \otimes Y & \rightarrow & B
\end{array}
\]

with \( Y \in \text{COV} \) has a local lifting.

If \( f \) is a local fibration and \( K \rightarrow L \) is a strong anodyne extension of finite simplicial sets

\[ \boxempty \text{Hom}(K \rightarrow L, f) \]

is a local fibration.

**Proof.** The first assertion follows by the same proof as [3, 1.4]. Furthermore, if \( K \rightarrow L \) is a a strong anodyne extension of finite simplicial sets extension then also \((K \rightarrow L) \boxempty (\Lambda_{n,k} \rightarrow \Delta_n)\) is by [3, 1.3].

**Lemma A.8.** For a local fibration local homotopy lifting and local lifting are equivalent
Proof. Consider a homotopy local lifting such that the lower triangle commutes via the homotopy $\mu: fh \Rightarrow a\iota_1$:

\[
\begin{array}{ccc}
\partial \Delta_n \otimes W' & \rightarrow & \partial \Delta_n \otimes W \\
\downarrow & & \downarrow f \\
\Delta_n \otimes W' & \xrightarrow{\iota_1} & \Delta_n \otimes W \\
\downarrow h & & \downarrow a \\
\Delta_n \otimes W & \xrightarrow{a} & Y
\end{array}
\]

Then consider

\[
\begin{array}{ccc}
(\Delta_n \times \{0\} \cup \partial \Delta_n \times \Delta_1) \otimes W' & \xrightarrow{h, c_\iota_1} & X \\
\downarrow & & \downarrow f \\
(\Delta_n \times \Delta_1) \otimes W' & \xrightarrow{\mu} & Y
\end{array}
\]

Since $f$ is a local fibration, by Lemma A.7 there is $\iota_2: W'' \rightarrow W'$ in COV and a lift

\[
\begin{array}{ccc}
(\Delta_n \times \{0\} \cup \partial \Delta_n \times \Delta_1) \otimes W'' & \xrightarrow{h, c_\iota_1 \iota_2} & X \\
\downarrow & & \downarrow f \\
(\Delta_n \times \Delta_1) \otimes W'' & \xrightarrow{\mu \iota_2} & Y
\end{array}
\]

The composition with the left horizontal morphism $e_0$ is then the lift which makes everything commute on the nose. \qed

Lemma A.9. Consider a pull-back square

\[
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow f' & & \downarrow f \\
Y' & \rightarrow & Y
\end{array}
\]

If $f$ has the local lifting property (respectively is a local fibration) then also $f'$ has the local lifting property (respectively is a local fibration).

Proof. Obvious. \qed

Lemma A.10. 1. If $f$ and $g$ have the (homotopy) lifting property then also $gf$ has it.

2. if $gf$ and $f$ have the (homotopy) lifting property then also $g$ has it.

3. if $gf$ and $g$ have the (homotopy) lifting property then also $f$ has it.

For the statements involving “homotopy” assume that $X, Y$ and $Z$ are locally fibrant. For assertion 3. without “homotopy” assume that $f$ is a local fibration.

Proof. 1. Consider a diagram

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow f & & \downarrow g \\
\partial \Delta_n \otimes W & \rightarrow & Z
\end{array}
\]
Applying the assumption, we get a local lift

\[
\begin{array}{c}
X \\
\downarrow \quad f \\
\partial \Delta_n \otimes W' \\
\downarrow \quad h \\
\Delta_n \otimes W' \\
\downarrow \quad \iota_1 \\
\Delta_n \otimes W \\
\downarrow \quad \alpha \\
Z
\end{array}
\]

and a local lift

\[
\begin{array}{c}
\partial \Delta_n \otimes W'' \\
\downarrow \quad h' \\
\partial \Delta_n \otimes W' \\
\downarrow \quad \iota_2 \\
\Delta_n \otimes W'' \\
\downarrow \quad h \\
\partial \Delta_n \otimes W' \\
\downarrow \quad \alpha \\
\Delta_n \otimes W \\
\downarrow \quad \alpha \\
Z
\end{array}
\]

with homotopies \( gh' \Rightarrow gh \iota_2 \Rightarrow a_1 \iota_2 \). The homotopies may be composed (refining the cover if necessary) because \( Z \) is locally fibrant. If the homotopies are equalities then we do not have to assume anything.

2. Consider a diagram

\[
\begin{array}{c}
X \\
\downarrow \quad f \\
\partial \Delta_n \otimes W \\
\downarrow \quad g \\
\Delta_n \otimes W \\
\downarrow \quad \alpha \\
Z
\end{array}
\]

By Lemma [A.6] applied to the morphism \( \varnothing \rightarrow \partial \Delta_n \), we get a lifting

\[
\begin{array}{c}
\partial \Delta_n \otimes W' \\
\downarrow \quad h \\
\partial \Delta_n \otimes W \\
\downarrow \quad g \\
\Delta_n \otimes W \\
\downarrow \quad \alpha \\
Z
\end{array}
\]

and a homotopy \( \mu : gh \Rightarrow a_1 \iota_1 \) defined on \( \partial \Delta_n \otimes W' \).

If this homotopy is not trivial, and \( Z \) is locally fibrant, consider:

\[
(\Delta_n \times \{1\} \cup \partial \Delta_n \times \Delta_1) \otimes W' \xrightarrow{a_{11}, \mu} Z
\]

By Lemma [A.7] there is \( \iota_2 : W'' \rightarrow W' \) in \( \mathcal{COV} \) and a lift

\[
(\Delta_n \times \{1\} \cup \partial \Delta_n \times \Delta_1) \otimes W'' \xrightarrow{a_{112}, \mu \iota_2} Z
\]

\[
(\Delta_n \times \{0\}) \otimes W'' \xrightarrow{\epsilon_0} (\Delta_n \times \Delta_1) \otimes W''
\]
Define $a' := h'e_0$. We have a commutative diagram and a local homotopy lift

\[
\begin{array}{ccc}
\partial\Delta_n \otimes W'' & \rightarrow & \partial\Delta_n \otimes W'' \\
& h & \downarrow \quad \rightarrow \\
& & X
\end{array}
\]

In total we have homotopies:

\[
gfh' \Rightarrow a'i_3 \Rightarrow a'i_1i_2i_3.
\]

Those may be composed refining the cover because $Z$ is locally fibrant. In case that they are identities we do not have to assume anything.

3. Consider a diagram

\[
\begin{array}{ccc}
\partial\Delta_n \otimes W & \rightarrow & X \\
& f & \downarrow \\
\Delta_n \otimes W & \rightarrow & Y \\
& g & \downarrow \\
& Z
\end{array}
\]

We get a lifting

\[
\begin{array}{ccc}
\partial\Delta_n \otimes W' & \rightarrow & \partial\Delta_n \otimes W \\
& h' & \downarrow \quad \rightarrow \\
\Delta_n \otimes W' & \rightarrow & \Delta_n \otimes W \\
& f & \downarrow \\
& Z
\end{array}
\]

and a homotopy $\mu: gfh \Rightarrow gai_1$. Then from the diagram

\[
\begin{array}{ccc}
(\partial_1 \times \Delta_n) \otimes W'' & \rightarrow & (\partial_1 \times \Delta_n) \otimes W' \\
& f_{h,i_1i_2} & \downarrow \\
(\Delta_1 \times \Delta_n) \otimes W'' & \rightarrow & (\Delta_1 \times \Delta_n) \otimes W' \\
& g & \downarrow \\
& Z
\end{array}
\]

we get a homotopy on $\Delta_n \otimes W''$:

\[
f_{h,i_1i_2} \Rightarrow ai_1i_2.
\]

If $f$ is a local fibration, then as in Lemma A.8 — up to refining the cover — this homotopy may be used to change $h$ such that the diagram commutes on the nose.

\[\square\]

**Lemma A.11.** Let $f, f'$ be trivial cofibrations and $g, g'$ be (global) fibrations satisfying $gf = g'f'$. Then $g$ has the local lifting property if and only if $g'$ has the local lifting property.
Proof. Let $g$ have the local lifting property and form the pull-back

\[
\begin{array}{c}
\text{Cof nW} \\
\downarrow f \\
\downarrow h \in \text{Fib} \\
\downarrow f' \\
\tilde{g}' \\
\downarrow g \\
\downarrow g'
\end{array}
\]

Since the local lifting property is preserved under pull-back (Lemma A.9), $\tilde{g}$ has the local lifting property. Now $\tilde{g}h$ and $\tilde{g}'h$ are trivial fibrations and thus have the lifting property. Therefore by 2-out-of-3 also $h$ has the local lifting property. Therefore also $\tilde{g}'$ has the lifting property. Again by 2-out-of-3 $g'$ has it as well because all 3 other fibrations in the square have. \hfill \Box
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