Liquid Time-constant Recurrent Neural Networks as Universal Approximators

Ramin M. Hasani¹, Mathias Lechner², Alexander Amini³, Daniela Rus,³ and Radu Grosu¹
¹Cyber Physical Systems (CPS), Technische Universität Wien (TU Wien), 1040 Vienna, Austria
²Institute of Science and Technology (IST), 3400 Klosterneuburg, Austria
³Computer Science and Artificial Intelligence Lab (CSAIL), Massachusetts Institute of Technology (MIT), Cambridge, USA

Abstract
In this paper, we introduce the notion of liquid time-constant (LTC) recurrent neural networks (RNN)s, a subclass of continuous-time RNNs, with varying neuronal time-constant realized by their nonlinear synaptic transmission model. This feature is inspired by the communication principles in the nervous system of small species. It enables the model to approximate continuous mapping with a small number of computational units. We show that any finite trajectory of an n-dimensional continuous dynamical system can be approximated by the internal state of the hidden units and n output units of an LTC network. Here, we also theoretically find bounds on their neuronal states and varying time-constant.

1 Introduction
Continuous-time spatiotemporal information processing can be performed by recurrent neural networks (RNN)s. In particular, a subset of RNNs whose hidden and output units are determined by ordinary differential equations (ODE), as in continuous-time recurrent neural networks (CTRNN)s (Funahashi and Nakamura 1993; Mozer, Kazakov, and Lindsey 2017). Typically, in CTRNNs, the time-constant of the neurons’ dynamics is a fixed constant value, and networks are wired by constant synaptic weights. We propose a new CTRNN model, inspired by the nervous system dynamics of small species, such as Ascaris (Davis and Stretton 1989), Leech (Lockery and Sejnowski 1992), and C. elegans (Wicks, Roehrig, and Rankin 1996; Hasani et al. 2017a), in which synapses are nonlinear sigmoidal functions that model the biophysics of synaptic interactions. As a result, state of the postsynaptic neurons are defined by the incoming presynaptic nonlinearities to the cell. This attribute, originates varying time-constant for the cell and strengthens its individual neurons’ expressivity in terms of output dynamics.

Dynamic network simulations based on such models have been deployed in many application domains such as simulations of animals’ locomotion (Wicks, Roehrig, and Rankin 1996), large-scale simulations of nervous systems (Gleeson et al. 2018), neuronal network’s reachability analysis (Islam et al. 2016), model of learning mechanisms (Hasani et al. 2017b) and robotic control in reinforcement learning environments (Hasani et al. 2018).

In this paper, we formalize networks built based on such principles as liquid time-constant (LTC) RNNs (Sec. 2) and theoretically prove their universal approximation capabilities (Sec. 3). We also find bounds over their varying time-constant as well as their neuronal states (Sec. 4).

2 Liquid Time-constant RNNs
Dynamics of a hidden or output neuron i, $V_i(t)$, of an LTC RNN are modeled as a membrane integrator with the following ordinary differential equation (ODE) (Koch and Segev 1998):

$$C_i \frac{dV_i}{dt} = G_{Leak,i} (V_{Leak,i} - V_i(t)) + \sum_{j=1}^{n} I_{in}^{(ij)}, \quad (1)$$

with neuronal parameters: $C_i, G_{Leak,i}$ and $V_{Leak,i}$, $I_{in}^{(ij)}$ represents the external currents to the cell. Hidden nodes are allowed to have recurrent connections while they synapse into motor neurons in a feed-forward setting.

Chemical synapses – Chemical synaptic transmission from neuron j to i, is modeled by a sigmoidal nonlinearity ($\mu_{ij}, \gamma_{ij}$), which is a function of the presynaptic membrane state, $V_j(t)$, and has maximum weight of $w_{ij}$ (Koch and Segev 1998):

$$I_{s_{ij}} = \frac{w_{ij}}{1 + e^{-\gamma_{ij}(V_j(t) - V_i(t))}}. \quad (2)$$

The synaptic current, $I_{s_{ij}}$, is then linearly depends on the state of the neuron i. $E_i$, sets whether the synapse excites or inhibits the succeeding neuron’s state.

An electrical synapse (gap-junction), between node j and i, was modeled as a bidirectional junction with weight, $\omega_{ij}$, based on Ohm’s law:

$$\dot{I}_{ij} = \omega_{ij} (v_j(t) - v_i(t)). \quad (3)$$

Internal state dynamics of neuron i, $V_i(t)$, of an LTC network, receiving one chemical synapse from neuron j, can be formulated as:

$$\frac{dV_i}{dt} = \frac{G_{Leak,i}}{C_i} (V_{Leak,i} - V_i(t)) + \frac{w_{ij}}{C_i} \sigma_i(V_j(t))(E_{ij} - V_i), \quad (4)$$

(I)
where \( \sigma_i(V_j(t)) = 1/1 + e^{-\gamma_i(V_j(t)+\mu_{ij})} \). If we set the time-
constant of the neuron \( i \) as \( \tau_i = C_m/\alpha_{Leak} \), we can reform this
equation as follows:
\[
\frac{dV_i}{dt} = -\left(1/\tau_i + \frac{w_{ij}}{C_m} \sigma_i(V_j)\right) V_i + \left(\frac{V_{\text{leak}}}{\tau_i} + \frac{w_{ij}}{C_m} \sigma_i(V_j) E_{ij}\right).
\]
Eq. 5 presents an ODE system with a nonlinearly varying
time-constant defined by \( \tau_{\text{system}} = \frac{1}{\tau_i + \frac{w_{ij}}{C_m} \sigma_i(V_j)} \),
which distinguishes the dynamics of the L TC cells compared
to the CTRNN cells.

The overall network dynamics of the LTC RNNs with
\( u(t) = [u_1(t), \ldots, u_{n+N}(t)]^T \) representing the internal states of
\( N \) interneurons (hidden units) and \( n \) motor neurons (output
units) can be written in matrix format as follows:
\[
\dot{u}(t) = -(1/\tau + W \sigma(u(t)))u(t) + A + W\sigma(u(t))B,
\]
in which \( \sigma(x) \) is a \( C^1 \)-sigmoid functions and is applied
element-wise. \( \tau_{n+N} > 0 \) includes all neuronal time-
constants, \( A \) is an \( n+N \) vector of resting states, \( B \) depicts an
\( n+N \) vector of synaptic reversals, and \( W \) is an \( n+N \) vector
produced by the matrix multiplication of a weight matrix of
shape \( (n+N) \times (n+N) \) and an \( n+N \) vector containing the
reversed value of all \( C_m \),s. Both \( A \) and \( B \) entries are bound
to a range \([-\alpha, \beta]\) for \( 0 < \alpha < +\infty \), and \( 0 \leq \beta < +\infty \). \( A 
\]
contains all \( V_{\text{leak}}/C_m \) and \( B \) presents all \( E_{ij} \).

3 Liquid time-constant RNNs are universal
approximators
In this section, we prove that any given finite trajectory of an
\( n \)-dimensional dynamical system can be approximated
by the internal and output states of an LTC RNN, with
\( n \) outputs, \( N \) interneurons and a proper initial condition. Let
\( x = [x_1, \ldots, x_n]^T \) be the \( n \)-dimensional Euclidean space on
\( \mathbb{R}^n \).

Theorem 1. Let \( S \) be an open subset of \( \mathbb{R}^n \) and \( F : S \to \mathbb{R}^n \), be an autonomous ordinary differential equation, be a
\( C^1 \)-mapping, and \( x(F) \) determine a dynamical system on \( S \). Let \( D \)
be a compact subset of \( S \) and we consider a
finite trajectory of the system as: \( I = [0, T] \). Then, for a positive \( \varepsilon \), there exist an integer \( N \) and a liquid time-constant recurrent
neural network with \( N \) hidden units, \( n \) output units, such that for any given trajectory \( \{x(t) : t \in I\} \) of the system
with initial value \( x(0) \in D \), and a proper initial condition of the network, the statement below holds:
\[
\max_{t \in I} |x(t) - u(t)| < \varepsilon
\]

We base our proof on the fundamental universal approximation
theorem [Hornik, Stinchcombe, and White 1989] on feed-forward
networks [Funahashi 1989, Cybenko 1989, Hornik, Stinchcombe, and White 1989, Schäfer and Zimmermann 2006] and time-continuous
RNNs [Funahashi and Nakamura 1993]. We first define
Lemma 1 to be used in the proof of Theorem 1.

Lemma 1. for an \( F : \mathbb{R}^n \to \mathbb{R}^n \) which is a bounded \( C^1 \)-
mapping, the differential equation
\[
\dot{x} = -(1/\tau + F(x))x + A + BF(x),
\]
in which \( \tau \) is a positive constant, and \( A \) and \( B \) are constants coefficients bound to a range \([-\alpha, \beta]\) for \( 0 < \alpha < +\infty \), and
\( 0 \leq \beta < +\infty \), has a unique solution on \([0, \infty)\).

Proof. Based on the assumptions, we can take a positive \( M \), such that
\[
0 \leq F_i(x) \leq M (\forall i = 1, \ldots, n)
\]
by looking at the solutions of the following differential equation:
\[
\dot{y} = -(1/\tau + M)y + A + BM,
\]
we can show that
\[
\min{|x_i(0)|, \frac{M + BM}{1 + M}} \leq x_i(t) \leq \max{|x_i(0)|, \frac{M + BM}{1 + M}}.
\]

if we set the output of the max to \( C_{\text{max}} \), and the output of the
min to \( C_{\text{min}} \), and also set \( C_1 = \min\{C_{\text{min}}\} \) and \( C_2 = \max\{C_{\text{max}}\} \), then the solution \( x(t) \) satisfies
\[
\sqrt{n}C_1 \leq x(t) \leq \sqrt{n}C_2.
\]

Based on Lemma 2 and Lemma 3 in
[Funahashi and Nakamura 1993], a unique solution exists on the interval \([0, +\infty)\). \( \square \)

Lemma 1 demonstrates that an LTC network defined by
Eq. 7 has a unique solution on \([0, +\infty)\), since the output function
is bounded and \( C^1 \).

Proof of Theorem 1

Proof. For proving Theorem 1, we adopt similar
steps to that of Funahashi and Nakamura on the
approximation ability of continuous time RNNs
[Funahashi and Nakamura 1993], to approximate a dynamical system with a larger dynamical system given by an
LTC RNN.

Part 1. We choose an \( \eta \) which is in range \([0, \min\{\epsilon, \lambda\}]\),
for \( \epsilon > 0 \), and \( \lambda \) the distance between \( \bar{D} \) and boundary \( \partial S \)
of \( S \). \( D_\eta \) is set:
\[
D_\eta = \{x \in \mathbb{R}^n; \exists z \in \bar{D}, |x - z| \leq \eta\}.
\]

\( D_\eta \) stands for a compact subset of \( S \), because \( \bar{D} \) is compact. Thus, \( F \) is Lipschitz on \( D_\eta \) by Lemma 1 in
[Funahashi and Nakamura 1993]. Let \( L_F \) be the Lipschitz constant of \( F |_{K_\eta} \), then, we can choose an \( \epsilon_t > 0 \), such that
\[
\epsilon_t < \frac{\eta L_F}{2(exp L_F T - 1)}.
\]

Based on the universal approximation theorem, there is an integer \( N \), and an \( n \times N \) matrix \( B \), and an \( N \times n \) matrix \( C \)
an an \( N \)-dimensional vector \( \mu \) such that
\[
\max |F(x) - B \sigma(C x + \mu)| < \frac{\epsilon_t}{2}.
\]

We define a \( C^1 \)-mapping \( \tilde{F} : \mathbb{R}^n \to \mathbb{R}^n \) as:
\[
\tilde{F}(x) = -(1/\tau + W \sigma(C x + \mu))x + A + W_B \sigma(C x + \mu),
\]
We set the system's time constant, $\tau_{sys}$ to:

$$\tau_{sys} = \frac{1}{1/\tau + W_{i}\sigma(Cx + \mu)}.$$  

(16)

We chose a large $\tau_{sys}$, conditioned with the following:

(a) $\forall x \in D_{\eta}$: $\left| \frac{x}{\tau_{sys}} \right| < \frac{\epsilon_{l}}{2}$  

(17)

(b) $\frac{\mu}{\tau_{sys}} < \frac{\eta L_{G}}{2(e^{\exp L_{G}T} - 1)}$ and $\left| \frac{1}{\tau_{sys}} \right| < \frac{L_{G}}{2}$,  

(18)

where $L_{G}/2$ is a lipschitz constant for the mapping $W_{i}\sigma: \mathbb{R}^{n+N} \rightarrow \mathbb{R}^{n+N}$ which we will determine later. To satisfy conditions (a) and (b), $\tau W_{i} << 1$ should hold true.

Then by Eq. (14) and (15) we can prove:

$$\max_{x \in D_{\eta}} \left| F(x) - \tilde{F}(x) \right| < \epsilon_{l} (19)$$

Let's set $x(t)$ and $\tilde{x}(t)$ with initial state $x(0) = \tilde{x}(0) = x_{0} \in D$, as the solutions of equations below:

$$\dot{x} = F(x), \quad \dot{\tilde{x}} = \tilde{F}(x).$$  

(20)

(21)

Based on Lemma 5 in (Funahashi and Nakamura 1993), for any $\ell \in I$,

$$\left| x(t) - \tilde{x}(t) \right| \leq \frac{\epsilon_{l}}{L_{F}}(e^{\exp L_{F}T} - 1) \quad (22)$$

$$< \frac{\epsilon_{l}}{L_{F}}(e^{\exp L_{F}T} - 1). \quad (23)$$

Thus, based on the conditions on $\epsilon$,

$$\max_{t \ell} \left| x(t) - \tilde{x}(t) \right| < \frac{\eta}{2}. \quad (24)$$

Part 2. Let's Considering the following dynamical system defined by $\tilde{F}$ in Part 1:

$$\dot{\tilde{x}} = -\frac{1}{\tau_{sys}}\tilde{x} + A_{1} + W_{i}B\sigma(C\tilde{x} + \mu). \quad (25)$$

Suppose we set $\tilde{y} = C\tilde{x} + \mu$; then:

$$\dot{\tilde{y}} = C\dot{\tilde{x}} = -\frac{1}{\tau_{sys}}\tilde{y} + E\sigma(\tilde{y}) + A_{2} + \frac{\mu}{\tau_{sys}}, \quad (26)$$

where $E = CW_{i}B$, an $N \times N$ matrix. We define

$$\tilde{z} = [\tilde{x}_{1}, ..., \tilde{x}_{n}, \tilde{y}_{1}, ..., \tilde{y}_{n}]^{T}, \quad (27)$$

and we set a mapping $\tilde{G} : \mathbb{R}^{n+N} \rightarrow \mathbb{R}^{n+N}$ as:

$$\tilde{G}(\tilde{z}) = -\frac{1}{\tau_{sys}}\tilde{z} + W_{i}\sigma(\tilde{z}) + A + \frac{\mu}{\tau_{sys}}, \quad (28)$$

$$W_{i}^{(n+N)\times(n+N)} = \begin{pmatrix} 0 & B \\ 0 & E \end{pmatrix}, \quad (29)$$

$$\mu_{1}^{n+N} = \begin{pmatrix} 0 \\ \mu \end{pmatrix}, \quad A^{n+N} = \begin{pmatrix} A_{1} \\ A_{2} \end{pmatrix}. \quad (30)$$

By using Lemma 2 in (Funahashi and Nakamura 1993), we can show that solutions of the following dynamical system:

$$\dot{z} = \tilde{G}(\tilde{z}), \quad \tilde{y}(0) = C\tilde{x}(0) + \mu, \quad (31)$$

are equivalent to the solutions of the Eq. (25).

Let's define a new dynamical system $G : \mathbb{R}^{n+N} \rightarrow \mathbb{R}^{n+N}$ as follows:

$$G(z) = -\frac{1}{\tau_{sys}}z + W\sigma(z) + A, \quad (32)$$

where $z = [x_{1}, ..., x_{n}, y_{1}, ..., y_{n}]^{T}$. Then the dynamical system below

$$\dot{z} = -\frac{1}{\tau_{sys}}z + W\sigma(z) + A, \quad (33)$$

can be realized by an LTC RNN, if we set $h(t) = [h_{1}(t), ..., h_{N}(t)]^{T}$ as the hidden states, and $u(t) = [U_{1}(t), ..., U_{n}(t)]^{T}$ as the output states of the system. Since $\tilde{G}$ and $G$ are both $C^{1}$-mapping and $\sigma(x)$ is bound, therefore, the mapping $\tilde{z} \rightarrow W\sigma(\tilde{z}) + A$ is Lipschitz on $\mathbb{R}^{n+N}$, with a Lipschitz constant $L_{G}/2$. As $L_{G}/2$ is Lipschitz constant for $-\tilde{z}/\tau_{sys}$ by condition (b) on $\tau_{sys}$, $L_{G}$ is a Lipschitz constant of $G$.

From Eq. (28), Eq. (32) and condition (b) of $\tau_{sys}$, we can derive the following:

$$\left| \tilde{G}(z) - G(z) \right| = \left| \frac{\mu}{\tau_{sys}} \right| \leq \frac{\eta L_{G}}{2(e^{\exp L_{G}T} - 1)}. \quad (34)$$

Accordingly, we can set $\tilde{z}(t)$ and $z(t)$, solutions of the dynamical systems:

$$\dot{\tilde{z}} = \tilde{G}(z), \quad \left\{ \begin{aligned} \tilde{x}(0) &= x_{0} \in D \\ \tilde{y}(0) &= C\tilde{x}(0) + \mu \end{aligned} \right. \quad (35)$$

$$\dot{z} = G(z), \quad \left\{ \begin{aligned} u(0) &= x_{0} \in D \\ h(0) &= Cx_{0} + \mu \end{aligned} \right. \quad (36)$$

By Lemma 5 of (Funahashi and Nakamura 1993), we achieve

$$\max_{t \ell} \left| \tilde{z}(t) - z(t) \right| < \frac{\eta}{2}, \quad (37)$$

and therefore we have:

$$\max_{t \ell} \left| \tilde{z}(t) - u(t) \right| < \frac{\eta}{2}, \quad (38)$$

Part 3. Now by using Eq. (24) and Eq. (38) for a positive $\epsilon$, we can design an LTC network with internal dynamical state $z(t)$, with $\tau_{sys}$ and $W$. For $x(t)$ satisfying $\dot{x} = F(x)$, if we initialize the network by $u(0) = x(0)$ and $h(0) = Cx(0) + \mu$, we obtain:

$$\max_{t \ell} \left| x(t) - u(t) \right| < \frac{\eta}{2} + \frac{\eta}{2} = \eta < \epsilon. \quad (39)$$

Remarks. The LTC’s network architecture allows interneurons (hidden layer) to have recurrent connections to each other, however it assumes a feed forward connection.
stream from hidden nodes to the motor neuron units (output units). We assumed no inputs to the system and principally showed that the interneurons’ network together with motor neurons can approximate any finite trajectory of an autonomous dynamical system. The proof subjected an LTC RNN with only chemical synapses. It is easy to extend the proof for a network which includes gap junctions as well, since their contribution to the network dynamics is by adding a linear term to the time-constant of the system ($\tau_{sys}$), and to the equilibrium state of a neuron, $A$ in Eq. 33.

4 Bounds on $\tau_{sys}$ and state of an LTC RNN

In this section, we prove that the time constant and the state of neuronal activities in an LTC RNN is bound to a finite range, as depicted in lemmas 2 and 3, respectively.

Lemma 2. Let $v_i$ denote the state of a neuron $i$, receiving $N$ synaptic connections of the form Eq. 2 and $P$ gap junctions of the form Eq. 3 from the other neurons of a LTC network $G$, if dynamics of each neuron’s state is determined by Eq. 1, then the time constant of the activity of the neuron, $\tau_i$, is bound to a range:

$$C_i/(g_i + \sum_{j=1}^{N} w_{ij} + \sum_{j=1}^{P} \hat{w}_{ij}) \leq \tau_i \leq C_i/(g_i + \sum_{j=1}^{P} \hat{w}_{ij}),$$

(40)

Proof. The sigmoidal nonlinearity in Eq. 2 is a monotonically increasing function, bound to a range 0 and 1:

$$0 < S(Y_i, \sigma_{ij}, \mu_{ij}, E_{ij}) < 1$$

(41)

By replacing the upper-bound of $S$, in Eq. 2, and substituting the synaptic current in Eq. 1, we will have:

$$C_i \frac{dv_i}{dt} = g_i (V_{leak} - v_i) + \sum_{j=1}^{N} w_{ij} (E_{ij} - v_i) + \sum_{j=1}^{P} \hat{w}_{ij} (v_j - v_i),$$

(42)

$$C_i \frac{dv_i}{dt} = (g_i V_{leak} + \sum_{j=1}^{N} w_{ij} E_{ij}) + \sum_{j=1}^{P} \hat{w}_{ij} v_j,$$

(43)

$$C_i \frac{dv_i}{dt} = A - BV_i.$$  

(44)

By assuming a fixed $v_j$, Eq. 45 is an ordinary differential equation with solution of the form:

$$v_i(t) = k_1 e^{-\frac{t}{\tau_i}} + \frac{A}{B}.$$  

(46)

From this solution, one can derive the lower bound of the system’s time constant, $\tau_{i_{min}}$:

$$\tau_{i_{min}} = \frac{C_i}{B} = \frac{C_i}{g_i + \sum_{j=1}^{N} w_{ij} + \sum_{j=1}^{P} \hat{w}_{ij}}.$$  

(47)

By replacing the lower-bound of $S$, in Eq. 42, the term $\sum_{j=1}^{N} w_{ij} (E_{ij} - v_i)$ becomes zero, therefore:

$$C_i \frac{dv_i}{dt} = (g_i V_{leak} + \sum_{j=1}^{P} \hat{w}_{ij} v_j) - (g_i + \sum_{j=1}^{P} \hat{w}_{ij}) v_i.$$  

(48)

Thus, we can derive the upper-bound of the time constant, $\tau_{i_{max}}$:

$$\tau_{i_{max}} = \frac{C_i}{g_i + \sum_{j=1}^{P} \hat{w}_{ij}}.$$  

(49)

Lemma 3. Let $v_i$ denote the state of a neuron $i$, receiving $N$ synaptic connections of form Eq. 2 from the other nodes of a network $G$, if dynamics of each neuron is determined by Eq. 1, then the hidden state of the neurons on a finite trajectory, $I = [0, T]$ ($0 < T < +\infty)$, is bound as follows:

$$\min_{t \in I} (V_{leak}, E_{ij}) \leq v_i(t) \leq \max_{t \in I} (V_{leak}, E_{ij}).$$

(50)

Proof. Let us insert $M = \max\{V_{leak}, E_{ij}\}$ as the membrane potential $v_i(t)$ into Eq. 42:

$$C_i \frac{dv_i}{dt} = g_i (V_{leak} - M) + \sum_{j=1}^{N} w_{ij} \sigma(v_j)(E_{ij} - M).$$

(51)

Right hand side of Eq. 51 is negative based on the conditions on $M$, positive weights and conductances, and the fact that $\sigma(v_i)$ is also positive in $\mathbb{R}^N$. Therefore, the left hand-side must also be negative and if we conduct an approximation on the derivative term:

$$C_i \frac{dv_i}{dt} \leq 0, \quad \frac{dv_i}{dt} \approx \frac{v(t + \delta t) - v(t)}{\delta t} \leq 0,$$

(52)

holds. By substituting $v(t)$ with $M$, we have the following:

$$\frac{v(t + \delta t) - M}{\delta t} \leq 0 \implies v(t + \delta t) \leq M$$

(53)

and therefore:

$$v_i(t) \leq \max_{t \in I} (V_{leak}, E_{ij}).$$

(54)

Now if we substitute the membrane potential, $V_i(t)$ with $m = \min\{V_{leak}, E_{ij}\}$, following the same methodology used for the proof of the upper bound, we can derive

$$\frac{v(t + \delta t) - m}{\delta t} \leq 0 \implies v(t + \delta t) \leq m,$$

(55)

and therefore:

$$v_i(t) \geq \min_{t \in I} (V_{leak}, E_{ij}).$$

(56)
5 Conclusions

We proved the universal approximation capability of liquid time-constant (LTC) RNNs, and showed how their varying dynamics are bound in a finite range. We believe that our work builds up the preliminary theoretical bases for investigating the capabilities of LTC networks.

Acknowledgments

R.M.H. and R.G. are partially supported by Horizon-2020 ECSEL Project grant No. 783163 (iDev40), and the Austrian Research Promotion Agency (FFG), Project No. 860424. A.A. is supported by the National Science Foundation (NSF) Graduate Research Fellowship Program.

References

[Cybenko 1989] Cybenko, G. 1989. Approximation by superpositions of a sigmoidal function. Mathematics of control, signals and systems 2(4):303–314.

[Davis and Stretton 1989] Davis, R. E., and Stretton, A. 1989. Signaling properties of ascaris motorneurons: graded active responses, graded synaptic transmission, and tonic transmitter release. Journal of Neuroscience 9(2):415–425.

[Funahashi and Nakamura 1993] Funahashi, K.-i., and Nakamura, Y. 1993. Approximation of dynamical systems by continuous time recurrent neural networks. Neural networks 6(6):801–806.

[Funahashi 1989] Funahashi, K.-I. 1989. On the approximate realization of continuous mappings by neural networks. Neural networks 2(3):183–192.

[Mozer et al. 2018] Mozer, M. C.; Kazakov, D.; Lindsey, R. V. 2018. Re-purposing compact neuronal circuit policies to govern reinforcement learning tasks. arXiv preprint arXiv:1809.04423.

[Koch and Segev 1998] Koch, C., and Segev, K. 1998. Methods in Neuronal Modeling - From Ions to Networks. MIT press, second edition.

[Lockery and Sejnowski 1992] Lockery, S., and Sejnowski, T. 1992. Distributed processing of sensory information in the leech. iii. a dynamical neural network model of the local bending reflex. Journal of Neuroscience 12(10):3877–3895.

[Hasani et al. 2018] Hasani, R. M.; Lechner, M.; Amini, A.; Rus, D.; and Grosu, R. 2018. Re-purposing compact neuronal circuit policies to govern reinforcement learning tasks. arXiv preprint arXiv:1809.04423.

[Hornik, Stinchcombe, and White 1989] Hornik, K.; Stinchcombe, M.; and White, H. 1989. Multilayer feedforward networks are universal approximators. Neural networks 2(5):359–366.

[Islam et al. 2016] Islam, M. A.; Wang, Q.; Hasani, R. M.; Balúñ, O.; Clarke, E. M.; Grosu, R.; and Smolka, S. A. 2016. Probabilistic reachability analysis of the tap withdrawal circuit in caenorhabditis elegans. In High Level Design Validation and Test Workshop (HLDVT), 2016 IEEE International, 170–177. IEEE.