Forks, Noodles and the Burau Representation
for $n = 4$

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Abstract

The reduced Burau representation is a natural action of the braid group $B_n$ on the first homology group $H_1(\tilde{D}_n;\mathbb{Z})$ of a suitable infinite cyclic covering space $\tilde{D}_n$ of the $n$–punctured disc $D_n$. It is known that the Burau representation is faithful for $n \leq 3$ and that it is not faithful for $n \geq 5$. We use forks and noodles homological techniques and Bokut–Vesnin generators to analyze the problem for $n = 4$. We present a Conjecture implying faithfulness and a Lemma explaining the implication. We give some arguments suggesting why we expect the Conjecture to be true. Also, we give some geometrically calculated examples and information about data gathered using a C++ program.

1 Introduction

Let us recall the definition of the reduced Burau representation in terms of the first homology group $H_1(\tilde{D}_4;\mathbb{Z})$ of a suitable infinite cyclic covering space $\tilde{D}_4$ of the 4-punctured disc $D_4$. Let $D_4$ be the unit closed disc on the plane with center $(0,0)$ and four punctures at: $p_1 = (-\frac{1}{2}, \frac{1}{2})$, $p_2 = (\frac{1}{2}, \frac{1}{2})$, $p_3 = (\frac{1}{2}, -\frac{1}{2})$, $p_4 = (-\frac{1}{2}, -\frac{1}{2})$ (see Figure 1).

The braid group $B_4$ is the group of all equivalence classes of orientation preserving homeomorphisms $\varphi : D_4 \to D_4$ which fix the boundary $\partial D_4$ pointwise, where equivalence relation is isotopy relative to $\partial D_4$. Let $\pi_1(D_4)$ be the fundamental group of the 4–punctured disc $D_4$ with respect to the basepoint $p_0 = (-1,0)$. Consider the map $\varepsilon : \pi_1(D_4) \to \langle t \rangle$ which sends a loop $\gamma \in \pi_1(D_4)$ to $t^{|\gamma|}$, where $|\gamma|$ is the winding number of $\gamma$ around punctured points $p_1, p_2, p_3, p_4$ (meaning: the sum of the four winding numbers for individual points). Let $\pi : \tilde{D}_4 \to D_4$ be the infinite cyclic covering space corresponding to the kernel $\ker(\varepsilon)$ of the map $\varepsilon : \pi_1(D_4) \to \langle t \rangle$. Let $\tilde{p}_0$ be
any fixed basepoint which is a lift of the basepoint $p_0$. In this case $H_1(\tilde{D}_4; \mathbb{Z})$ is free $\mathbb{Z}[t, t^{-1}]$–module of rank 3 (see [3]). Let $\varphi : D_4 \to D_4$ be a homeomorphism representing of an element $\sigma \in B_4$. It can be lifted to a map $\tilde{\varphi} : \tilde{D}_4 \to \tilde{D}_4$ which fixes the fiber over $p_0$. Therefore it induces a $\mathbb{Z}[t, t^{-1}]$–module automorphism $\tilde{\varphi}_* : H_1(\tilde{D}_4; \mathbb{Z}) \to H_1(\tilde{D}_4; \mathbb{Z})$. Consequently, the reduced Burau representation

\begin{equation}
\rho : B_4 \to \text{Aut} \left( H_1(\tilde{D}_4; \mathbb{Z}) \right)
\end{equation}

is given [3] by

\begin{equation}
\rho(\sigma) = \tilde{\varphi}_*, \ \forall \sigma \in B_4.
\end{equation}

It is known that the Burau representation is faithful for $n \leq 3$ [1], [2] and it is not faithful for $n \geq 5$ [2], [6], [7]. Therefore, the problem is open for $n = 4$. In this paper, we use the Bokut-Vesnin generators $a, a^{-1}, b, b^{-1}$ of a certain free subgroup of $B_4$ (see [4]) and a technique developed in [3], to prove the crucial lemma, which gives the opportunity to decompose entries $\rho_{11}(a^n\sigma)$ and $\rho_{13}(a^n\sigma)$ of the Burau matrix $\rho(a^n\sigma)$ as a sum of three uniquely determined polynomials and the formula to calculate $\rho_{13}(a^{n+m}\sigma)$ and $\rho(a^{n+m}\sigma)$ polynomials using the given decomposition. Besides, we formulate Conjecture 4.2, which implies that if a non–trivial braid $\sigma \in \ker \rho$ has a certain additional property, then there exists a sufficiently large $l_0$ with respect to the length of $\sigma$ (to be explained in Section 3, Corollary 3.2) and a sufficiently large $m_0$ such that for each $m > m_0$ and $l > l_0$ the difference of lowest degrees of polynomials $\rho_{13}(a^{n+m}\sigma)$ and $\rho(a^{n+m}\sigma)$ is $-1$. 
We will present arguments and experimental data showing why we expect the conjecture to be true. Also, we will consider several examples calculated geometrically. We will show that the conjecture implies faithfulness of the Burau representation for $n = 4$.

2 The Burau representation, Forks and Noodles

The Burau representation for $n = 4$ was defined by (1.1) and (1.2). On the other hand $H_1(\tilde{D}_4; \mathbb{Z})$ is a free $\mathbb{Z}[t, t^{-1}]$-module of rank 3 and if we take a basis of it, then $\text{Aut}\left(H_1(\tilde{D}_4; \mathbb{Z})\right)$ can be identified with $GL(3, \mathbb{Z}[t, t^{-1}])$. For this reason we will review the definition of the forks.

**Definition 2.1.** A fork is an embedded oriented tree $F$ in the disc $D$ with four vertices $p_0, p_i, p_j$ and $z$, where $i \neq j, i, j \in \{1, 2, 3, 4\}$ such that (see [3]):

1. $F$ meets the puncture points only at $p_i$ and $p_j$;
2. $F$ meets the boundary $\partial D_4$ only at $p_0$;
3. All three edges of $F$ have $z$ as a common vertex.

![Diagram of a fork](image)

Figure 2: The line form $p_0$ to $z$ is the handle and the curve from $p_1$ to $p_4$ is the tine $T(F)$ of the fork $F$.

The edge of $F$ which contains $p_0$ is called the handle. The union of the other two edges is denoted by $T(F)$ and it is called tine of $F$. Orient $T(F)$ so that the handle of $F$ lies to the right of $T(F)$ (see Figure 2) [3].

For a given fork $F$, let $h : I \rightarrow D_4$ be the handle of $F$, viewed as a path in $D_4$ and take a lift $\tilde{h} : I \rightarrow \tilde{D}_4$ of $h$ so that $\tilde{h}(0) = \tilde{p}_0$. Let $\tilde{T}(F)$ be the
connected component of \( \pi^{-1}(T(F)) \) which contains the point \( \tilde{h}(1) \). In this case any element of \( H_1(\tilde{D}_4;\mathbb{Z}) \) can be viewed as a homology class of \( \tilde{T}(F) \) and it is denoted by \( F_3 \).

Standard fork \( F_i, \quad i = 1, 2, 3 \) is the fork whose tine edge is the straight arc connecting the \( i \)-th and the \( (i+1) \)-st punctured points and whose handle has the form as in Figure 3. It is known that if \( F_1, F_2 \) and \( F_3 \) are the corresponding homology classes, then they form a basis of \( H_1(\tilde{D}_4;\mathbb{Z}) \) (see [3]).

Using the basis derived from \( F_1, F_2, F_3 \), any automorphism \( \tilde{\varphi}_* : H_1(\tilde{D}_4;\mathbb{Z}) \to H_1(\tilde{D}_4;\mathbb{Z}) \) can be viewed as a 3 \( \times \) 3 matrix with elements in the free \( \mathbb{Z}[t,t^{-1}] \)-module [3]. If \( \varphi : D_4 \to D_4 \) is representing an element \( \sigma \in B_4 \), then we need to write the matrix \( \rho(\sigma) = \tilde{\varphi}_* \) in terms of homology (algebraic) intersection pairing

\[
\langle -,- \rangle : H_1(\tilde{D}_4;\mathbb{Z}) \times H_1(\tilde{D}_4,\partial\tilde{D}_4;\mathbb{Z}) \to \mathbb{Z}[t,t^{-1}].
\]

For this aim we need to define the noodles which represent relative homology classes in \( H_1(\tilde{D}_4,\partial\tilde{D}_4;\mathbb{Z}) \).

**Definition 2.2.** A noodle is an embedded oriented arc in \( D_4 \), which begins at the base point \( p_0 \) and ends at some point of the boundary \( \partial D_4 \) [3].

For each \( a \in H_1(\tilde{D}_4;\mathbb{Z}) \) and \( b \in H_1(\tilde{D}_4,\partial\tilde{D}_4;\mathbb{Z}) \) we should take the corresponding fork \( F \) and noodle \( N \) and define the polynomial \( \langle F,N \rangle \in \mathbb{Z}[t,t^{-1}] \). It does not depend on the choice of representatives of homology classes and so

\[
\langle -,- \rangle : H_1(\tilde{D}_4;\mathbb{Z}) \times H_1(\tilde{D}_4,\partial\tilde{D}_4;\mathbb{Z}) \to \mathbb{Z}[t,t^{-1}]
\]
is well-defined [3]. The map defined by the above formula is called the noodle–fork paring. Note that geometrically it can be computed in the following way: Let \( F \) be a fork and \( N \) be a noodle, such that \( T(F) \) intersects \( N \) transversely. Let \( z_1, z_2, \ldots, z_n \) be the intersection points. For each point \( z_i \) let \( \varepsilon_i \) be the sign of the intersection between \( T(F) \) and \( N \) at \( z_i \) (the intersection is positive if going from tine to noodle according to the chosen directions means turning left) and \( e_i = [\gamma_i] \) be the winding number of the loop \( \gamma_i \) around the puncture points \( p_1, p_2, p_3, p_4 \), where \( \gamma_i \) is the composition of three paths \( h, t_i \) and \( n_i \):

1. \( h \) is a path from \( p_0 \) to \( z \) along the handle of \( F \) (see Figure 4a);
2. \( t_i \) is a path from \( z \) to \( z_i \) along the tine \( T(F) \) (see Figure 4b);
3. \( n_i \) is a path from \( z_i \) to \( p_0 \) along the noodle \( N \) (see Figure 4c).

In such case the noodle-fork pairing of \( F \) and \( N \) is given by (see [3]):

\[
\langle F, N \rangle = \sum_{1 \leq i \leq n} \varepsilon_i t^{e_i} \in \mathbb{Z}[t, t^{-1}].
\]

Let \( N_1, N_2 \) and \( N_3 \) be the noodles given in Figure 5. These are called standard noodles. For each braid \( \sigma \in B_4 \), the corresponding Burau matrix \( \rho(\sigma) \) can be computed using noodle-fork pairing of standard noodles and standard forks. In particular the following is true.

**Lemma 2.3.** (see [5]). Let \( \sigma \in B_n \). Then for \( 1 \leq i, j \leq n - 1 \), the entry \( \rho_{ij}(\sigma) \) of its Burau matrix \( \rho(\sigma) \) is given by

\[
\rho_{ij}(\sigma) = \langle F_i \sigma, N_j \rangle.
\]
Figure 5: Standard noodles: $N_1$, $N_2$, $N_3$

Note that under the convention adopted here we have

$$\rho(\sigma_1) = \begin{pmatrix} -t^{-1} & 0 & 0 \\ t^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \rho(\sigma_2) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -t^{-1} & 0 \\ 0 & t^{-1} & 1 \end{pmatrix}$$

and

$$\rho(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -t^{-1} \end{pmatrix}.$$  

For example, to calculate $\rho_{11}(\sigma_1)$ entry of the matrix $\rho(\sigma_1)$ see the corresponding Figure 6. Note that intersection of the tine $T(F_1\sigma_1)$ of the fork $F_1\sigma_1$ and the noodle $N_1$ at point $z_1$ is negative which means that $\varepsilon_1 = -1$ (see Figure 6a). On the other hand the winding number $e_1$ of the loop $\gamma_1$ (see Figure 6b) around puncture points equals $-1$ because the considered loop misses $p_1, p_3$ and $p_4$ and it goes around $p_2$ once in anti-clockwise direction. Therefore

$$\rho_{11}(\sigma_1) = \langle F_1\sigma_1, N_1 \rangle = -t^{-1}$$
Figure 6: $\langle F_1 \sigma_1, N_1 \rangle = -t^{-1}$ is monomial, because the intersection of $F_1 \sigma_1$ and the noodle $N_1$ is just one point $z_1$

3 The Bokut-Vesnin generators and kernel elements of the Burau representation

The braid groups $B_4$ and $B_3$ are defined by the following standard presentations [1]:

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \ \sigma_3 \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \sigma_2, \ \sigma_3 \sigma_1 = \sigma_1 \sigma_3 \rangle,$$

$$B_3 = \langle \sigma_1, \sigma_2 | \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle.$$

Let $\varphi : B_4 \to B_3$ be the homomorphism defined by

$$\varphi(\sigma_1) = \sigma_1, \ \varphi(\sigma_2) = \sigma_2, \ \varphi(\sigma_3) = \sigma_1.$$

The kernel of $\varphi$ is known to be a free group $F(a, b)$ of two generators [4];

$$a = \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_3 \sigma_2^{-1} \sigma_1^{-1}, \ b = \sigma_3 \sigma_1^{-1}.$$

This was proved by L. Bokut and A. Vesnin [4]. We will refer to $a$ and $b$ as the Bokut–Vesnin generators. The generators $a$ and $b$ are in fact much more similar than they look at the first glance. This becomes obvious when we interpret $B_4$ as the mapping class group of the 4-punctured disc. In this well-known approach a braid is an isotopy class of homeomorphisms of the punctured disc fixing the boundary. Figure 7 shows $a$ and $b$ as homeomorphisms of the punctured disc. The punctures are arranged to make the
similarity more visible. Another advantage of this approach is that it gives
natural interpretation to various actions of $B_4$ to be considered later in this
paper.

Figure 7:

The following Proposition is crucial to our considerations.

**Proposition 3.1.** $\ker \rho_4 \subset \ker \varphi$

*Proof.* Let us make a slight detour into the realm of the Temperley-Lieb
algebras $TL_3$ and $TL_4$. The Temperley-Lieb algebra $TL_n$ is defined as an
algebra over $\mathbb{Z}[t, t^{-1}]$. It has $n - 1$ generators $\{U^i\}_{i=1}^{n-1}$, and the following
relations:

(TL1) $U^i U^j = (- t^2 + t^{-2}) U^i$,
(TL2) $U^i U^j U^i = U^j$, for $|i - j| = 1$,
(TL3) $U^i U^j = U^j U^i$, for $|i - j| > 1$.

Let us consider the homomorphism $\psi : TL_4 \rightarrow TL_3$ defined by

$U^1_1 \rightarrow U^1_1$, $U^2_2 \rightarrow U^2_2$, $U^3_3 \rightarrow U^1_1$.

Also, we need to use the Jones’ representation $\theta : B_n \rightarrow TL_n$ defined
by sending $\sigma_i$ to $A + A^{-1} U^i$. It is known (see [3], Proposition 1.5) that
for $n = 3, 4$ we have $\ker \theta_n = \ker \rho_n$. Moreover, the following diagram is
obviously commutative:
On the other hand the representation \( \theta_3 \) is faithful and therefore \( \ker \rho_4 = \ker \theta_4 \subset \ker \varphi \).

**Corollary 3.2.** *All kernel elements of the Burau representation may be written as words in the Bokut–Vesnin generators \( a, b, a^{-1}, b^{-1} \). Moreover, all possible nontrivial elements in the kernel may be written as reduced words of positive length.*

We will use this fact in the next section.

We present for future use the images of \( a, b, a^{-1} \) and \( b^{-1} \) under the Burau representation:

\[
\rho(a) = \begin{pmatrix}
-t^{-1}+1 & -t^{-1} + t & -t^{-1} \\
0 & -t & 0 \\
-1 & 0 & 0
\end{pmatrix},
\]

\[
\rho(b) = \begin{pmatrix}
-t & 0 & 0 \\
1 & 1 & 1 \\
0 & 0 & -t^{-1}
\end{pmatrix},
\]

\[
\rho(a^{-1}) = \begin{pmatrix}
0 & 0 & -1 \\
0 & -t^{-1} & 0 \\
-t & t^{-1} - t & 1 - t
\end{pmatrix},
\]

\[
\rho(b^{-1}) = \begin{pmatrix}
-t^{-1} & 0 & 0 \\
t^{-1} & 1 & t \\
0 & 0 & -t
\end{pmatrix}.
\]

### 4 Faithfulness Problem of the Burau representation

Let us outline the strategy for analyzing \( \ker \rho_4 \) in general terms. Consider a braid \( \sigma \) that is a candidate for a non–trivial kernel element of the Burau representation. Of course we can exclude from our considerations all those
non–trivial braids for which we know for whatever reason that they do not belong to the kernel. Also, we can adjust the remaining candidates in some ways — like replacing $\sigma$ with a suitably chosen conjugate of $\sigma$. For such a suitably chosen braid $\sigma$ we need to give some argument which shows that $\rho_{11}(\sigma)$ and $\rho_{31}(\sigma)$ should be non–zero and that $\deg_{\min}(\rho_{11}) - \deg_{\min}(\rho_{31}) = -1$, where $\deg_{\min}$ denotes the exponent of the lowest degree term in the considered Laurent polynomial.

To simplify notation we will denote by $S_i(t^{\pm 1})$ the $i^{th}$ partial sum of the geometric series with initial term 1 and quotient $-t$ or $-t^{-1}$ (e.g. $S_2(t^{-1}) = 1 - t^{-1} + t^{-2}$).

**Lemma 4.1.** For each braid $\sigma \in B_4$ there exists $n \in \mathbb{N}$, such that

1. the $\rho_{11}(a^n\sigma)$ and $\rho_{31}(a^n\sigma)$ entries of the Burau matrix $\rho(a^n\sigma)$ can be decomposed as a sum of three uniquely determined polynomials

\[
\begin{align*}
\rho_{11}(a^n\sigma) &= P(t,t^{-1})(1 - t^{-1}) + Q(t,t^{-1})(1 - t), \\
\rho_{31}(a^n\sigma) &= -P(t,t^{-1}) - Q(t,t^{-1}) - R(t,t^{-1}).
\end{align*}
\]

such that

2. for each $m \in \mathbb{N}$ we have

\[
\begin{align*}
\rho_{11}(a^{m+n}\sigma) &= P(t,t^{-1})(S_{m+1}(t^{-1})) + Q(t,t^{-1})(S_{m+1}(t)), \\
\rho_{31}(a^{m+n}\sigma) &= -P(t,t^{-1})(S_m(t^{-1})) - Q(t,t^{-1}) - R(t,t^{-1})(S_m(t)).
\end{align*}
\]

3. Moreover, if $\sigma$ is a pure braid, then the polynomial $P$ is non–zero.

**Proof.** First of all let us observe that uniqueness of $P, Q$ and $R$ follows from properties (1) and (2) and general algebra. This means that we only need to prove existence and property (3). While it is possible to give specific algebraic formulas for $P, Q$ and $R$ we prefer to prove existence using forks and noodles. We will always assume that the fork/noodle configuration considered is irreducible.

Let $\sigma \in B_4$ be any braid. By Lemma 2.3 $\rho_{11}(a^n\sigma) = \langle F_1 a^n \sigma, N_1 \rangle$ and $\rho_{31}(a^n\sigma) = \langle F_3 a^n \sigma, N_1 \rangle$. On the other hand $\langle -,- \rangle$ is a bilinear form, so $\langle F_1 a^n \sigma, N_1 \rangle = \langle F_1 a^n, N_1 \sigma^{-1} \rangle$ and $\langle F_3 a^n \sigma, N_1 \rangle = \langle F_3 a^n, N_1 \sigma^{-1} \rangle$. It follows that

\[
\rho_{11}(a^n\sigma) = \langle F_1 a^n, N_1 \sigma^{-1} \rangle,
\]

10
\[ \rho_{31}(a^n\sigma) = \langle F_3 a^n, N_1\sigma^{-1} \rangle. \]

Let us consider \( N_1\sigma^{-1} \), the image of the standard noodle \( N_1 \) under the action of \( \sigma^{-1} \). \( N_1\sigma^{-1} \) is a path in \( D_4 \) that begins at the base point \( p_0 \) and ends at the point \( (0,1) \in \partial D_4 \). By the definition of the standard noodle \( N_1 \) it is clear that \( N_1\sigma^{-1} \) divides \( D_4 \) into two components, such that there is one puncture point in one component and three puncture points in the other. Let us assume that the single point is \( p_1 \). For example see Figure 8.

![Figure 8: p_1 is in one component and p_2, p_2, p_3 are in the other](image)

We intend to define \( P \) and \( R \) by grouping some terms in the sum originally used to define the representation in terms of fork/noodle pairing. The pairing is defined as a certain sum (2.1) of terms corresponding to crossings between forks and noodles. We will choose some of the crossings to define \( P \) and some other to define \( R \). In order to do this we will need some preparations.

Let \( T \) be the boundary of the square whose vertices are the puncture points. We denote the sides with \( T_1, \ldots, T_4 \), where \( T_i \) connects \( P_i \) with the next crossing (clockwise). We would like to work with a fork/noodle arrangement that has certain special properties. We need the pair (of a fork and a noodle) to be irreducible. We need the fork to be drawn in the standard way. We need the noodle to intersect \( T \) transversally with minimum possible number of intersection points. And finally we need the tine of the fork to intersect all segments at which the noodle intersects \( T_4 \) and \( T_2 \). While general position arguments show that we can take care of the first three conditions, there is no possibility of the fourth being satisfied without some further adjustments. Figure 9 shows an example.
Figure 9: The tine $T(F_1)a$ (shown as the black curve) does not intersect the blue segment at which the noodle $N$ (the union of blue and red segments) intersects $T_2$

However it is automatically corrected if we increase the exponent $n$. The effect is just that we add a number of turns around two pairs of punctures. They do not affect the three properties already dealt with and with sufficient increase of $n$ we obtain the fourth property. So we are interested in strings between puncture points which has transversal intersection with $T$ (see Figure 10).

Figure 10:

Note that it is possible to be no such string between $p_2$ and $p_3$ or $p_3$ and $p_4$, but by our assumption ($p_1$ is in the first component) there is an odd number of strings between $p_1$ and $p_2$ and an odd number between $p_1$ and $p_4$
(this guaranties that $P$ and $Q$ are not zero).

The pictures of $T(F_1) a^n$ and $T(F_3) a^n$ are as given in Figure 11. Therefore, they differ from each other by just one string and by the direction. The number of strings around $p_1$, $p_4$ and $p_2$, $p_3$ is $n$ for $T(F_1) a^n$ and $n - 1$ for $T(F_3) a^n$.

Figure 11:

If we take curves $T(F_1) a$ and $N_1 \sigma^{-1}$ in the same $D_4$ and assume that their intersection is transversal, then it is possible that $T(F_1) a$ does not intersect all strings between $p_1$ and $p_4$ or $p_2$ and $p_3$. For example see Figure 9.

Figure 12:

In this case we must take more numbers of $a$’s and finally we will obtain
the curves $T(F_1) a^n$ and $N_1 \sigma^{-1}$ such that we can find neighborhoods $U_1$ and $U_2$ of $T_4$ and $T_2$ respectively, with the following picture, illustrated in Figure 12a.

In this case for Figure 12a the polynomial corresponding to intersections inside $U_1$ and $U_2$ can be written as $P(t, t^{-1})(1 - t^{-1})$ and $R(t, t^{-1})(1 - t)$ respectively. Let $Q(t, t^{-1}) = \rho_{11}(a^n \sigma) - P(t, t^{-1})(1 - t^{-1}) - R(t, t^{-1})(1 - t)$, then we have

$$\rho_{11}(a^n \sigma) = P(t, t^{-1})(1 - t^{-1}) + Q(t, t^{-1}) + R(t, t^{-1})(1 - t).$$

On the other hand if we look at Figure 12b and keep in mind that directions of $T(F_1) a^n$ and $T(F_3) a^n$ are different we can say that

$$\rho_{31}(a^n \sigma) = -P(t, t^{-1}) - Q(t, t^{-1}) - R(t, t^{-1}).$$

After that if we multiply the braid $a^n \sigma$ by $a$ on the left side then we obtain the following picture, illustrated in Figure 13.

![Diagram](image)

**Figure 13:**

Therefore we will have

$$\rho_{11}(a^{n+1} \sigma) = P(t, t^{-1})(1 - t^{-1} + t^{-2}) + Q(t, t^{-1}) + R(t, t^{-1})(1 - t^1 + t^2),$$

$$\rho_{31}(a^{n+1} \sigma) = -P(t, t^{-1})(1 - t^{-1}) - Q(t, t^{-1}) - R(t, t^{-1})(1 - t^1).$$

Note that the same argument it sufficient to complete the proof. \(\square\)
**Example 1.** Let $\sigma = b^{-1}a^{-1}b$, then for $n = 2$ we have

\[
\rho_{11}(a^2b^{-1}a^{-1}b) = -t^{-3} + t^{-2} - 1 + 2t - t^2 - t^3 + 2t^4 - t^5 =
\]

\[
t^{-2} (1 - t^{-1}) + (-1 + t - t^3 + t^4) (1 - t),
\]

\[
\rho_{31}(a^2b^{-1}a^{-1}b) = -t^{-2} + 1 - t + t^3 - t^4 = -t^{-2} - (-1 + t - t^3 + t^4).
\]

See Figure 14. Let $m = 3$, then we can see that

See Figure 14. The polynomial corresponding to the marked intersection point on the left side is $P(t,t^{-1}) = t^2$ and the polynomial corresponding to the marked intersection points on the right side is $R(t,t^{-1}) = -1 + t - t^3 + t^4$

\[
\rho_{11}(a^5b^{-1}a^{-1}b) = t^{-2} (1 - t^{-1} + t^{-2} - t^{-3} + t^{-4}) +
\]

\[
(-1 + t - t^3 + t^4) (1 - t + t^2 - t^3 + t^4) =
\]

\[
t^{-6} - t^{-5} + t^{-4} - t^{-3} + t^{-2} - 1 + 2t - 2t^2 + t^3 - t^5 + 2t^6 - 2t^7 + t^8,
\]

\[
\rho_{31}(a^5b^{-1}a^{-1}b) = -t^{-2} (1 - t^{-1} + t^{-2} - t^{-3}) -
\]

\[
(-1 + t - t^3 + t^4) (1 - t + t^2 - t^3) =
\]

\[
t^{-5} - t^{-4} + t^{-3} - t^{-2} + 1 - 2t + 2t^2 - t^3 - t^4 + 2t^5 - 2t^6 + t^7
\]
Example 2. Let $\sigma = ba^{-2}b^{-1}$, then for $n = 2$ we have

$$\rho_{11}(a^2ba^{-2}b^{-1}) = t^{-6} - 2t^{-5} + t^{-4} + t^{-3} - t^{-2} + t^{-1} =$$

$$(-t^{-5} + t^{-4} - t^{-2})(1 - t^{-1}) + t^{-1},$$

$$\rho_{31}(a^2ba^{-2}b^{-1}) = t^{-5} - t^{-4} + t^{-2} - t^{-1} = -(-t^{-5} + t^{-4} - t^{-2}) - t^{-1}.$$ See Figure 15 Let $m = 4$ then we can see that

Figure 15: The polynomial corresponding to the three marked intersection points on the left side is $P(t,t^{-1}) = -t^{-5} + t^{-4} - t^{-2}$ and the polynomial corresponding to the single marked intersection point on the right side is the monomial $Q(t,t^{-1}) = t^{-1}$

$$\rho_{11}(a^6ba^{-2}b^{-1}) = (-t^{-5} + t^{-4} - t^{-2})(1 - t^{-1} + t^{-2} - t^{-3} + t^{-4} - t^{-5}) + t^{-1} =$$

$$t^{-10} - 2t^{-9} + 2t^{-8} - t^{-7} + t^{-6} - t^{-5} + t^{-3} - t^{-2} + t^{-1},$$

$$\rho_{13}(a^6ba^{-2}b^{-1}) = (-t^{-5} + t^{-4} - t^{-2})(1 - t^{-1} + t^{-2} - t^{-3} + t^{-4}) + t^{-1} =$$

$$t^{-9} - 2t^{-8} + 2t^{-7} - t^{-6} + t^{-5} - t^{-3} + t^{-2} - t^{-1}.$$ Example 3. Let $\sigma = ab^2ab^{-1}$, then for $n = 2$ we have

$$\rho_{11}(a^2ab^2ab^{-1}) = -t^{-6} + 2t^{-5} - t^{-4} + t^{-3} + 3t^{-2} - 3t^{-1} + 2 - t^{-2} + 2t^3 - t^4 =$$

$$(t^{-5} - t^{-4} + t^{-2} - 2t^{-1} + 1)(1 - t^{-1})$$

$$(-t^{-4} + t^{-3} - 2t^{-1} + 2 - t) + (1 - t^2 + t^3)(1 - t),$$

$$\rho_{11}(a^2ab^2ab^{-1}) = -t^{-5} + 2t^{-4} - t^{-3} - t^{-2} + 4t^{-1} - 4 + t + t^2 - t^3 =$$

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The polynomial corresponding to the intersection points marked on the left side is $P(t, t^{-1}) = t^5 - t^4 + t^2 - 2t^{-1} + 1$. The polynomial corresponding to the marked intersection points above punctuation points is $Q(t, t^{-1}) = -t^4 + t^{-3} - 2t^{-1} + 2 - t$ and the polynomial corresponding to the marked intersection points on the right side is $R(t, t^{-1}) = 1 - t^2 + t^3 - (t^5 - t^4 + t^2 - 2t^{-1} + 1) - (-t^4 + t^{-3} - 2t^{-1} + 2 - t) - (1 - t^2 + t^3)$. See Figure 16

Let $m = 2$ then we can see that

\[ \rho_{11}(a^4ab^2ab^{-1}) = (t^5 - t^4 + t^2 - 2t^{-1} + 1) (1 - t^{-1} + t^{-2} - t^{-3}) \]
\[ - (t^{-4} + t^{-3} - 2t^{-1} + 2 - t) + (1 - t^2 + t^3) (1 - t + t^2 - t^3) = -t^{-8} + 2t^{-7} - 2t^{-6} + t^{-5} + t^{-4} - 3t^{-3} + 4t^{-2} - 5t^{-1} + 4 \]
\[-2t + t^3 - 2t^4 + 2t^5 - t^6, \]

\[ \rho_{13}(a^4ab^2ab^{-1}) = - (t^5 - t^4 + t^2 - 2t^{-1} + 1) (1 - t^{-1} + t^{-2}) - (-t^{-4} + t^{-3} - 2t^{-1} + 2 - t) - (1 - t^2 + t^3) (1 - t + t^2) = -t^{-7} + 2t^{-6} - 2t^{-5} + t^{-4} + 2t^{-3} - 4t^{-2} + 5t^{-1} - 4 + 2t - 2t^3 + 2t^4 - t^5. \]

We formulate a conjecture that describes a certain regularity, experimentally observed for images (matrices) of braids of a special form. For future reference let us state clearly that what we mean by regularity is that the $(1,1)$ and $(3,1)$ entries in the considered matrix are non-zero Laurent polynomials and that the difference of the degrees of lowest degree terms is equal to $-1$. 

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Conjecture 4.2. Let $\sigma \in B_4$ be any non-trivial pure braid which is not equivalent to $\Delta^m$, for some $m \in \mathbb{N}$. We assume that $\sigma$ acts non-trivially on $T_4$. Then there exists a sufficiently large $l_0 \in \mathbb{N}$ with respect to the length of $\sigma$ and a sufficiently large $m_0 \in \mathbb{N}$ such that for each $m > m_0, l \geq l_0$ the difference of the lowest degrees of the polynomials $\rho_{11}(a^m \sigma a^{-l})$ and $\rho_{31}(a^m \sigma a^{-l})$ is equal to $-1$ and the polynomials are non-zero.

While experimental data suggest that the Conjecture is true as formulated, we are really interested in the situation when $\sigma$ is a product of Bokut–Vesnin generators. Therefore we may refer to the length of $\sigma$, meaning the length of $\sigma$ as a reduced word in $a, b, a^{-1}, b^{-1}$. Now, we give some arguments showing why we expect the Conjecture to be true. Take a sufficiently large $l_0 \in \mathbb{N}$ with respect to the length of $\sigma$. Consider the curves $N_1 a^{l_0}$ and neighborhood $U_1$ of $T_4$ inside of which it looks as in Figure 17:

![Figure 17](image)

Apply to the curve $N_1 a^{l_0}$ the transformation corresponding to the braid $\sigma^{-1}$. Note that $l_0$ is sufficiently large with respect to the length of $\sigma^{-1}$ and so in the curve $N_1 a^{l_0} \sigma^{-1}$ almost all parallel lines to line $T_4$ are followed by the curve $T_4 \sigma^{-1}$. On the other hand the transformation corresponding to the braid $\sigma^{-1}$ acts non-trivially on the line $T_4$. Therefore the final image $N_1 a^{l_0} \sigma^{-1}$ does not have problematic strings around $T_4$, as in Figure 18.

In general, if we take any braid $\sigma$, then $F_1 \sigma$ may have the strings in the form illustrated in Figure 18. For example if $\sigma = b^{-1}ab^{-1}$ or $\sigma = a^3$ then the corresponding curves are shown in Figure 19.

Because the curve $N_1 a^{l_0} \sigma^{-1}$ does not have any problematic strings for $n = 3$ the intersection of curves $F_1 a^3$ and $N_1 a^{l_0} \sigma^{-1}$ inside the neighbourhoods $U_1$ and $U_2$ of $T_4$ and $T_2$ looks as in Figure 13. So the $\rho_{11}(a^3 \sigma a^{-l_0})$
and \( \rho_{31}(a^3\sigma_{a^-l_0}) \) entries of the Burau matrix \( \rho(a^n\sigma_{a^-l_0}) \) can be written as

\[
\rho_{11}(a^3\sigma_{a^-l_0}) = P(t,t^{-1})(1-t^{-1}) + Q(t,t^{-1}) + R(t,t^{-1})(1-t),
\]

\[
\rho_{31}(a^3\sigma_{a^-l_0}) = -P(t,t^{-1}) - Q(t,t^{-1}) - R(t,t^{-1}),
\]

where polynomial \( P(t,t^{-1}) \) is not zero and for each \( m' \in \mathbb{N} \) we have

\[
\rho_{11}(a^{3+m'}\sigma_{a^-l_0}) = P(t,t^{-1})(S_{m'+1}(t^{-1})) + Q(t,t^{-1}) + R(t,t^{-1})(S_{m'+1}(t)),
\]

\[
\rho_{31}(a^{3+m'}\sigma_{a^-l_0}) = -P(t,t^{-1})(S_{m'}(t^{-1})) - Q(t,t^{-1}) - R(t,t^{-1})(S_{m'}(t)).
\]

On the other hand if we compare the curves \( N_1a^{l_0+1} \) and \( N_1a^{l_0} \) (see Figure 20), it is clear that they differ only by strings around \( T_4 \).
Moreover, if we consider the intersections of curves $N_1a^{l_0+1}$ and $N_1a^{l_0}$ with the strings between the puncture points $p_1$ and $p_4$ as in Figure 21, then corresponding polynomials up to sign $\epsilon$ and multiplications $t^\alpha$ have the forms:

\[
(\epsilon t^\alpha + S(t, t^{-1})(1 - t^{-1})) \left(1 - t + \cdots + (-1)^{l-1}t^{l-1}\right), (*)
\]

\[
(\epsilon t^\alpha + S(t, t^{-1})(1 - t^{-1})) \left(1 - t + \cdots + (-1)^{l+1}t^{l-2}\right). (**)
\]

Therefore their lowest degrees are equal. Note that, $l_0$ is sufficiently large with respect to the length of $\sigma$ and so the pictures of curves $N_1a^{l_0+1}\sigma^{-1}$ and $N_1a^{l_0}\sigma^{-1}$ ‘globally’ are the same. That means that in some ‘local’ pictures there are just different numbers of strings. Now we must look at pictures $N_1a^{l_0+1}\sigma^{-1}$ and $N_1a^{l_0}\sigma^{-1}$ inside a neighborhood of $T$ as it was done in the previous proof (see Figure 10). Note that by the arguments in the proof of Lemma 4.1 and same ‘global’ picture of the curves $N_1a^{l_0+1}\sigma^{-1}$ and $N_1a^{l_0}\sigma^{-1}$ we have

\[
\rho_{11} \left(a^3\sigma a^{-l_0-1}\right) = P'(t, t^{-1})(1 - t^{-1}) + Q'(t, t^{-1}) + R'(t, t^{-1})(1 - t),
\]

\[
\rho_{31} \left(a^3\sigma a^{-l_0-1}\right) = -P'(t, t^{-1}) - Q'(t, t^{-1}) - R'(t, t^{-1}),
\]

and for each $m' \in \mathbb{N}$ we have:
\[
\rho_{11} \left( a^{3+m'} \sigma a^{-l_0-1} \right) = P' \left( t, t^{-1} \right) \left( S_{m'+1}(t^{-1}) \right) + Q' \left( t, t^{-1} \right) \\
+ R' \left( t, t^{-1} \right) \left( S_{m'}(t) \right),
\]
\[
\rho_{31} \left( a^{3+m'} \sigma a^{-l_0-1} \right) = -P' \left( t, t^{-1} \right) \left( S_{m'}(t^{-1}) \right) - Q' \left( t, t^{-1} \right) \\
- R' \left( t, t^{-1} \right) \left( S_{m'}(t) \right).
\]

Figure 21:

Take a large \( m_0 = 3 + m' \) such that the difference of lowest degrees of polynomials \( \rho_{11} \left( a^{m_0} \sigma a^{-l_0} \right) \) and \( \rho_{31} \left( a^{m_0} \sigma a^{-l_0} \right) \) is equal to \(-1\) and these lowest degrees come from the lowest degree of the polynomial \( P(t, t^{-1}) \).

By (*) and (**) the polynomials \( P \left( t, t^{-1} \right) \) and \( P' \left( t, t^{-1} \right) \), \( Q \left( t, t^{-1} \right) \) and \( Q' \left( t, t^{-1} \right) \) and also \( R \left( t, t^{-1} \right) \) and \( R' \left( t, t^{-1} \right) \) have the same lowest degrees and so the same regularity will be true for the polynomials \( \rho_{11} \left( a^{m} \sigma a^{-l_0-1} \right) \) and \( \rho_{31} \left( a^{m} \sigma a^{-l_0-1} \right) \). By induction on the length of \( \sigma \) it will be true for the braid \( a^{m} \sigma a^{-l_0-1}, l > l_0 \) as well.

**Example 4.** Let \( \sigma = b^6a b^{-1}a^{-1}b^{-6}a^{-6} \). Our aim is to find \( n \) which satisfies the conditions of Lemma 4.1 and to calculate the corresponding polynomials \( P, Q \) and \( R \). Then we will take any \( l > 6 \) (In our case we consider \( l = 9 \)) and will show that lowest degrees of the corresponding polynomials do not change. For the given braid it is difficult to see the picture and write down the polynomials \( P, Q \) and \( R \). Therefore we will use the following method: If \( n \) (in our case \( n = 2 \)) is a number as in Lemma 4.1, then we have

\[
\rho_{11} (\sigma) + \rho_{31} (\sigma) = -t^{-1}P(t, t^{-1}) - tR(t, t^{-1}),
\]
\[
\rho_{11} (a\sigma) + \rho_{31} (a\sigma) = t^{-2}P(t, t^{-1}) + t^2R(t, t^{-1}).
\]
Therefore
\[ P(t, t^{-1}) = \frac{t(r_{11}(\sigma) + r_{31}(\sigma)) + (r_{11}(a\sigma) + r_{31}(a\sigma))}{t^{-2} - 1}, \]
\[ R(t, t^{-1}) = \frac{t^{-1}(r_{11}(\sigma) + r_{31}(\sigma)) + (r_{11}(a\sigma) + r_{31}(a\sigma))}{t^2 - 1}. \]

In this way we can see that for the braid \( \sigma = a^2b^6ab^{-1}a^{-1}b^{-6}a^{-6} \) we have
\[
P(t, t^{-1}) = t^{-8} - 3t^{-7} + 6t^{-6} - 9t^{-5} + 11t^{-4} - 11t^{-3} + 8t^{-2} - 2t^{-1} -
-7 + 16t^{-1} - 22t^2 + 23t^3 - 20t^4 + 14t^5 - 5t^6 - 4t^7 +
+10t^8 - 12t^9 + 11t^{10} - 9t^{11} + 6t^{12} - 3t^{13} + t^{14}
\]
\[
Q(t, t^{-1}) = -t^{-1} + 2 - 2t^1 + t^2 - 2t^4 + 4t^5 - 6t^6 + 6t^7 - 4t^8 + t^9 +
+t^{10} - 2t^{11} + 3t^{12} - 3t^{13} + 2t^{14} - t^{15},
\]
\[
R(t, t^{-1}) = -t^{-6} + 3t^{-5} - 6t^{-4} + 9t^{-3} - 11t^{-2} + 11t^{-1} -
-7 - t^1 + 10t^2 - 18t^3 + 23t^4 - 22t^5 + 17t^6 - 8t^7 - t^8 + 8t^9 - 11t^{10} +
+11t^{11} - 9t^{12} + 6t^{13} - 3t^{14} + t^{15}.
\]

Similarly, for the braid \( \sigma' = a^2b^6ab^{-1}a^{-1}b^{-6}a^{-9} \) we obtain
\[
P'(t, t^{-1}) = t^{-8} - 3t^{-7} + 6t^{-6} - 9t^{-5} + 11t^{-4} - 12t^{-3} + 11t^{-2} - 8t^{-1} +
+1 + 8t^1 - 16t^2 + 21t^3 - 23t^4 + 23t^5 - 19t^6 + 12t^7 -
-4t^8 - 3t^9 + 8t^{10} - 11t^{11} + 12t^{12} - 11t^{13} + 9t^{14} - 6t^{15} + 3t^{16} - t^{17}
\]
\[
Q'(t, t^{-1}) = -t^{-1} + 2 - 2t^1 + t^2 - t^4 + 2t^5 - 4t^6 + 6t^7 - 6t^8 + 4t^9 -
-2t^{10} + t^{11} - t^{13} + 2t^{14} - 3t^{15} + 3t^{16} - 2t^{17} + t^{18},
\]
\[
R'(t, t^{-1}) = -t^{-6} + 3t^{-5} - 6t^{-4} + 9t^{-3} - 11t^{-2} + 12t^{-1} -
-10 + 5t^1 + 2t^2 - 10t^3 + 17t^4 - 21t^5 + 22t^6 - 20t^7 + 14t^8 - 7t^9 +
+5t^{10} - 9t^{12} + 11t^{13} - 11t^{14} + 9t^{15} - 6t^{16} + 3t^{17} - t^{18}.
\]

Therefore the lowest degrees of polynomials \( P(t, t^{-1}) \) and \( P'(t, t^{-1}) \) (same situation is with the polynomials \( Q(t, t^{-1}) \) and \( Q'(t, t^{-1}) \) or \( R(t, t^{-1}) \) and \( R'(t, t^{-1}) \)) are equal.
Theorem 4.3. Conjecture 4.2 implies faithfulness of the Burau representation for $n = 4$.

Proof. Let us consider a nontrivial braid $\sigma \in B_4$ written as a reduced word in the Bokut–Vesnin generators. We may assume that it begins and ends with $a$ or $a^{-1}$ (otherwise we will conjugate by a suitable power of $a$).

If we interpret the braid group as the mapping class group, then there is a natural induced action on the set of isotopy classes of forks. Let $\sigma$ act non-trivially on $T_4$. Then by Lemma 4.1 it is possible to find sufficiently large $l_0$ with respect to the length of $\sigma$ and sufficiently large $m_0$, such that for each $m > m_0$ and $l > l_0$ the difference of lowest degrees of the polynomials $\rho_{11}(a^m \sigma a^{-l})$ and $\rho_{31}(a^m \sigma a^{-l})$ is equal to $-1$ and the polynomials are both non-zero. In particular, we can assume that $m = l$ and so $\rho_{11}(a^m \sigma a^{-m})$ and $\rho_{31}(a^m \sigma a^{-m})$ are both non-zero which contradicts the assumption that $a^m \sigma a^{-m} \in \ker \rho$.

The general case (when we do not assume that $\sigma$ acts non-trivially on $T_4$) is easily reduced to the one discussed above. The reason is that if $\sigma$ acts trivially on all four segments, then $\sigma$ is a power of $\Delta$ which is not possible if $\sigma$ is a product of the Bokut–Vesnin generators. And if $\sigma$ acts non-trivially on at least one of the four segments, then we can rotate the whole disc to make the action non-trivial for $T_4$.

\[\square\]

Remark 4.4. We have a C++ program checking whether our regularity works or not for randomly generated examples. We calculated millions of examples and the regularity was always confirmed. In fact we considered examples of type $a^3b^3wb^{-3}a^{-3}$, where $a^3b^3wb^{-3}a^{-3}$ is a reduced word in the Bokut–Vesnin generators. Such a version of Proposition 3.1 is sufficient for the Burau representation faithfulness problem.

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