Method of self-similar factor approximants

V.I. Yukalov\textsuperscript{1,2} and E.P. Yukalova\textsuperscript{3}

\textsuperscript{1}Bogolubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Russia,
\textsuperscript{2}Department of Technology and Economics, Swiss Federal Institute of Technology, Zürich CH-8032, Switzerland,
\textsuperscript{3}Laboratory of Information Technologies, Joint Institute for Nuclear Research, Dubna 141980, Russia

Abstract

The method of self-similar factor approximants is completed by defining the approximants of odd orders, constructed from the power series with the largest term of an odd power. It is shown that the method provides good approximations for transcendental functions. In some cases, just a few terms in a power series make it possible to reconstruct a transcendental function \textit{exactly}. Numerical convergence of the factor approximants is checked for several examples. A special attention is paid to the possibility of extrapolating the behavior of functions, with arguments tending to infinity, from the related asymptotic series at small arguments. Applications of the method are thoroughly illustrated by the examples of several functions, nonlinear differential equations, and anharmonic models.

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1 Introduction

The problem of reconstructing functions from their asymptotic series is widely known to be of great importance for a variety of applications [1]. Probably, the most often used technique allowing for such a reconstruction is that based on Padé approximants [2], though there also exist other more complicated methods [3].

Recently, a novel approach has been advanced for reconstructing functions from the related asymptotic series, called the method of self-similar factor approximants [4–6]. The form of these approximants is derived by means of the self-similar approximation theory [7–15]. The factor approximants were shown [4–6] to be more general and more accurate than Padé approximants. However, several points in the method of factor approximants [4–6] have not been investigated.

The most important point, which has remained unclear, is how to construct the factor approximants of odd orders? The problem is that the standard procedure [4–6] requires that the generic series, the factor approximants are built from, be of even order. So that the large amount of information, contained in the odd orders of the series, could not be properly used. Here we advance a uniform approach for treating both odd as well as even orders of power series and we demonstrate, by a number of examples, good performance of this general method.

Another, rather nontrivial, question is why the factor approximants could provide high accuracy for transcendental functions, obtained from their asymptotic series. We give an explanation for this effect, which is unique for the resummation methods based on asymptotic series. Recall that Padé approximants, having the structure of rational functions, are usually not so good for approximating transcendental functions. Moreover, we show that some transcendental functions can be reconstructed, by means of the factor approximants, exactly from a few terms of an asymptotic series.

In the previous papers [4–6] on the method of self-similar factor approximants, we mainly considered the low-order approximants of even order, because of which their numerical convergence could not be properly analyzed. Now, we possess both even and odd orders of these approximants, and we study their numerical convergence for various cases, calculating high-order approximants, up to 20-th order.

We pay a special attention to the possibility of extrapolating physically motivated functions to the whole range of their variables. We demonstrate that the method of self-similar factor approximants can serve as a tool for defining the values of functions at infinity from their asymptotic expressions near zero.

Summarizing, the main results of the present paper are as follows:

(i) The method of self-similar factor approximants is extended by constructing the factor approximants of odd orders, in addition to those of even orders.

(ii) It is shown that transcendental functions can be well approximated. A unique feature of the method is that some transcendental functions can be reconstructed exactly.

(iii) The method allows for the extrapolation of solutions to nonlinear differential equations from asymptotically small to finite variables.

(iv) The characteristics of quantum anharmonic models can be extrapolated from the
region of asymptotically small coupling parameter to the whole range of the latter, including
the case of the coupling parameter tending to infinity.

(v) For all considered examples, the factor approximants of high orders are calculated,
demonstrating the existence of numerical convergence.

2 Self-similar factor approximants

Let us be interested in defining a real function \( f(x) \) of a real variable \( x \in \mathbb{R} \). The extension
of the method to complex-valued functions of complex variables is also possible, but, first,
let us consider a slightly simpler case of real functions and variables. Suppose, we know only
the behavior of the function at asymptotically small values of \( x \to 0 \), where we can get the
sequence \( \{f_k(x)\} \) of the expansions

\[
f_k(x) = \sum_{n=0}^{k} a_n x^n , \tag{1}
\]

with \( k = 0, 1, 2, \ldots \). The sequence, generally, can be divergent. It is known [1] that a function
\( f(x) \), analytic in the vicinity of \( x = 0 \), uniquely defines its asymptotic series. The converse is
not always true. But in what follows, we consider the situation, when there is a one-to-one
correspondence between the function and its asymptotic series, since solely then the problem
of reconstructing functions from their expansions acquires sense.

Without the loss of generality, we may assume that in expansion (1),

\[
a_0 = f_k(0) = f_0(x) = 1 . \tag{2}
\]

Really, if instead of expansion (1), we would have

\[
f^{(k)}(x) = f^{(0)}(x) \sum_{n=0}^{k} a_n x^n ,
\]

with a given function \( f^{(0)}(x) \), then we could immediately return to Eq. (1) defining

\[
f_k(x) \equiv \frac{f^{(k)}(x)}{f^{(0)}(x) a_0} .
\]

The self-similar factor approximants of even orders \( k = 2p = 2, 4, 6, \ldots \) are given [4–6]
by the form

\[
f^{*}_{2p}(x) = \prod_{i=1}^{p} (1 + A_i x)^{n_i} , \tag{3}
\]

whose parameters \( A_i \) and \( n_i \) are obtained from the re-expansion procedure, that is, by
expanding Eq. (3) in powers of \( x \) and comparing the results with the given expansion
(1), equating the like-order terms. This accuracy-through-order procedure yields the set of
equations

\[
\sum_{i=1}^{p} n_i A_i^n = B_n \quad (n = 1, 2, \ldots, 2p) , \tag{4}
\]
with the right-hand sides

\[ B_n \equiv \frac{(-1)^{n-1}}{(n-1)!} \lim_{x \to 0} \frac{d^n}{dx^n} \ln f_k(x). \]  

(5)

In each approximation, the parameters \( A_i, n_i, \) and \( B_n \), of course, depend on the approximation number \( k = 2p \). However, in order to avoid too cumbersome notation, we do not mark explicitly this dependence, which is assumed to be evident.

Each factor in form (3) contains two parameters, \( A_i \) and \( n_i \). This is why these approximants could be straightforwardly defined only for the even-order expansions \( f_{2p}(x) \). But how should we proceed having odd-order expansions (1) with \( k = 2p + 1 = 1, 3, 5, \ldots \)? For the latter, we could write the form

\[ f^*_{2p+1}(x) = \prod_{i=1}^{p+1} (1 + A_i x)^{n_i}, \]  

(6)

with the parameters satisfying the set of equations

\[ \sum_{i=1}^{p+1} n_i A_i^n = B_n \quad (n = 1, 2, \ldots, 2p + 1). \]  

(7)

But the problem is that form (6) contains \( 2p + 2 \) unknown \( A_i \) and \( n_i \), while only \( 2p + 1 \) equations of set (7) are available. One equation is lacking.

To overcome this problem, let us notice that expression (6) is invariant under the scaling transformation

\[ x \to \lambda x, \quad A_i \to \lambda^{-1} A_i. \]  

(8)

Then, taking \( \lambda \to A_1^{-1} \) and using the renotation \( A_i/A_1 \to A_i \), we come to the same form (6), but with

\[ A_1 = 1 \quad (k = 2p + 1). \]  

(9)

Complementing the set of \( 2p + 1 \) equations (7) by the scaling condition (9), we get \( 2p + 2 \) equations for \( 2p + 2 \) unknowns. In this way, we now can construct the factor approximants of odd orders.

If the sought function \( f(x) \) can be reduced to the factor form \( f^*_k(x) \), then it is obvious that this function can be reconstructed exactly for all approximants, starting with the given order \( k \), since both \( f(x) \) and \( f^*_k(x) \) yield the same series of the \( k \)-th order. Thus, there exists a class of functions \( \{f^*_k(x)\} \), having the form of \( f^*_k(x) \), which can be reconstructed exactly from their asymptotic series by means of the factor approximants. This class of exactly reproducible functions can be noticeably extended by including those functions that can be defined as limits of the form \( f^*_k(x) \) with respect to the values of some \( A_i \) and \( n_i \) tending to either zero or infinity. Keeping in mind such limits, we shall denote them as \( \lim_{\{A_i, n_i\}} f^*_k(x) \). All functions, which can either be reduced to the factor form \( f^*_k(x) \) or obtained from the latter by means of limiting procedures with respect to their parameters, form the factor class

\[ \mathbb{F} \equiv \left\{ f^*_k(x), \lim_{\{A_i, n_i\}} f^*_k(x) \right\}. \]  

(10)
As is clear, a function can be exactly reproduced by the factor approximants if and only if it pertains to the factor class (10). This is a rather wide class, including rational as well as irrational functions. In the following section, we show that some transcendental functions can also be reproduced exactly.

3 Reconstruction of transcendental functions

Consider such an entire transcendental function as the exponential

\[ f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \]  

(11)

From the expansion \( f_2(x) = 1 + x + \frac{1}{2}x^2 \), we find the factor approximant

\[ f_2^*(x) = \lim_{a \to 0} (1 + ax)^{1/a} = e^x. \]  

(12)

Similarly, from the form \( f_3(x) \) at \( x \to 0 \), we obtain

\[ f_3^*(x) = \lim_{b \to 0} (1 + x)^{b/(1-b)} (1 + bx)^{1/(1-b)} = e^x. \]  

(13)

This procedure can be continued to any order, showing that the transcendental function \( e^x \) is exactly reproduceable by the factor approximants of any finite order \( k \geq 2 \),

\[ f_k^*(x) = e^x \quad (k \geq 2). \]  

(14)

The hyperbolic functions, such as sinh \( x \) and cosh \( x \), can also be reconstructed exactly, if one notices that the series, corresponding to \( 2 \sinh x = e^x - e^{-x} \) and \( 2 \cosh x = e^x + e^{-x} \), can be represented as the sums of two series for the exponentials, which, as is shown above, are reproduceable exactly. The trigonometric functions \( \sin x \) and \( \cos x \) can also be reproduced exactly, if one treats them as the analytic continuations of the related hyperbolic functions to the imaginary axis, when \( x \) is replaced by \( ix \).

The fact that some transcendental functions can be reproduced by factor approximants exactly does not imply that all transcendental functions are exactly reproduceable. However, the very existence of exactly reproduceable transcendental functions explains why other transcendental functions, though not exactly reproduceable, nevertheless, can be well approximated by the factor approximants.

As an example, let us consider the series for \( \sin x \),

\[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n-1)!} x^{2n+1}. \]  

(15)

Note that for the alternating series, it is convenient to shift them by a constant, so that all factor approximants for the shifted function be sign defined, removing this shift afterwards. The actual value of the shifting constant plays no role, provided the considered function becomes sign defined. In the present case, we, first, consider \( \sin x + const \), and remove the constant at the end. It is important to emphasize that the factor approximants, obtained
from the Taylor series, are much more accurate than the latter. For an illustration, we show in Fig. 1 the factor approximant of the order \( k = 18 \), derived from series (15), and the Taylor series of the same order \( k = 18 \), as compared to the exact function \( \sin x \). As is seen, the factor approximant is essentially more accurate.

Constructing the factor approximants for the tangent,

\[
\tan x = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} - 1}{(2n)!} B_{2n} x^{2n-1},
\]

where \( B_n \) is the Bernoulli number, we get such a good accuracy for the approximants of the order \( k \geq 4 \), that they practically coincide with the exact function. Therefore, we concentrate on defining the point of singularity, which for \( \tan x \) occurs at \( x = \pm \pi/2 \). For each factor approximant, we find the point of its singularity \( x_k \) and compare it with the exact value \( \pi/2 \). We consider the positive value \( x > 0 \), since for \( x < 0 \), the behavior is similar, being antisymmetric. The accuracy of the approximants can be characterized by the percentage error

\[
\varepsilon_k \equiv \frac{x_k - \pi/2}{\pi/2} \cdot 100\%.
\]

The error quickly diminishes with the approximation order, demonstrating fast numerical convergence, for instance, \( \varepsilon_2 = -6.8\% \), \( \varepsilon_3 = -5.4\% \), \( \varepsilon_4 = 0.13\% \), \( \varepsilon_5 = 0.096\% \), and \( \varepsilon_6 = -0.00035\% \).

Numerical convergence is well seen for different transcendental functions. We have considered a variety of the latter. For illustration, we show in Fig. 2 how the factor approximants approach the function

\[
f(x) = x + \cos x.
\]

Often transcendental functions appear as solutions of differential equations. Let the function \( y(t) \) of time \( t \geq 0 \) be defined by a differential equation. The simplest way to construct its solutions in the form of the factor approximants is as follows. We may look for the solution at small \( t \to 0 \), representing it as a series

\[
y(t) \simeq \sum_n a_n t^n \quad (t \to 0).
\]

And then, it is straightforward to derive the factor approximants from the given series, as is explained in Sec. 2.

Let us consider the nonlinear ordinary differential equation

\[
\frac{d^2y}{dt^2} + y + y^2 = 0,
\]

(19)

describing a nonlinear oscillator with the cubic potential

\[
U = \frac{1}{2} y^2 + \frac{1}{3} y^3.
\]

The solution to this equation (19) corresponds to the oscillations around the center \( (y = 0, \dot{y} = 0) \), where the overdot, as usual, means time derivative, during the time interval \( 0 \leq t < t_0 \). At the point \( t_0 \), depending on initial conditions, the solution diverges as
\[ y \propto (t - t_0)^{-2} \]

Constructing the solutions to this equation from the related time series at \( t \to 0 \), as is explained above, we obtain the factor approximants \( y_k^*(t) \) providing the approximate solutions for longer times. In Fig. 3, the factor approximants \( y_{14}^*(t) \) and \( y_{16}^*(t) \) are shown in comparison with the exact numerical solution of Eq. (19) with the initial conditions \( y(0) = 0 \) and \( \dot{y}(0) = 1 \). For these initial conditions, the divergence occurs at \( t_0 \approx 6.5 \). We have analyzed the behavior of high-order approximants \( y_k^*(t) \), with \( k \geq 15 \), finding that they provide a good extrapolation from the region of asymptotically small times \( t \to 0 \) to the temporal interval \( 0 \leq t \leq 5 \).

As another example of nonlinear differential equations, let us take the Rayleigh equation

\[
\frac{d^2y}{dt^2} + y = \varepsilon \frac{dy}{dt} - \varepsilon \left( \frac{dy}{dt} \right)^3,
\]

(20)
describing an oscillator perturbed by the right-hand side of Eq. (20), with \( 0 < \varepsilon \ll 1 \). Written in the normal form, Eq. (20) represents a two-dimensional dynamical system having one unstable fixed point \( (y = 0, \dot{y} = 0) \), with the Jacobian eigenvalues \( J^\pm = (\varepsilon \pm \sqrt{\varepsilon^2 - 4})/2 \). By Poincaré-Bendixon theorem [16], there exists a limit cycle around this unstable fixed point. We look for the solution of Eq. (20), with \( \varepsilon = 0.1 \), under the initial conditions \( y(0) = 0, \dot{y}(0) = 1 \). As is described above, we, first, find the power-series solution at \( t \to 0 \), from which we construct the factor approximants \( y_k^*(t) \). In Fig. 4, the approximants \( y_{18}^*(t) \) and \( y_{19}^*(t) \) are compared to the exact numerical solution of Eq. (20). Again we observe that the factor approximants of the order \( k \geq 19 \) well extrapolate the time-series solution at \( t \to 0 \) to the finite region \( 0 \leq t \leq 6 \).

It is worth noting that the problem of extrapolating time series is of high importance for the possibility of predicting the evolution of different complex systems such as markets and societies [17]. The examples, considered above, demonstrate that the self-similar factor approximants provide us with the tool of extrapolating the time series for small times \( t \to 0 \) to the region of finite times, that is, making it possible to give forecasts, at least, for some near future.

4 Extrapolation for anharmonic models

Many interacting physical systems have the structure analogous to anharmonic models. This is why the latter serve as a touchstone for testing different approximation theories. The question to be investigated in the present section is as follows. Suppose, we are able to find a function of interest only by means of perturbation theory with respect to the coupling parameter \( g \). We keep in mind here the simple perturbation theory in powers of \( g \), without invoking any optimization procedures (see review-type articles [12,15]). Then, as is well known, the sought function has the form of an asymptotic series

\[
f(g) \simeq \sum_n a_n g^n \quad (g \to 0),
\]

(21)

which diverges for any finite \( g \). Could we, being based solely on this asymptotic expansion, extrapolate by factor approximants the function to the whole region of the coupling parameter \( g \in [0, \infty) \)? It is especially interesting whether we could predict the behavior of the
function at very large couplings \( g \to \infty \). The latter has, clearly, no sense for the power series derived for \( g \to 0 \). Padé approximants are also not of the help for this purpose, since the limit \( g \to \infty \) for a Padé approximant \( P_{[M/N]}(g) \) is not defined, as
\[
P_{[M/N]}(g) \propto g^{M-N} \quad (g \to \infty)
\]
can tend to infinity, zero, or a constant, depending on \( M \) and \( N \), so that, as \( g \to \infty \),
\[
P_{[M/N]}(g) \to \begin{cases} 
\infty, & M > N \\
\text{const}, & M = N \\
0, & M < N .
\end{cases}
\]

First, let us consider the so-called partition function of the zero-dimensional \( \varphi^4 \) model,
\[
Z(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\varphi^2 - g\varphi^4} d\varphi . \quad (22)
\]
The power series (21), corresponding to Eq. (22), possess the coefficients
\[
a_n = \frac{(-1)^n}{\sqrt{\pi n!}} \Gamma \left( 2n + \frac{1}{2} \right) ,
\]
which factorially increase at large \( n \), because of which series (21) diverges for any finite \( g \).

Being based on series (21), we construct, according to the general scheme of Sec. 2, the factor approximants \( Z_k^*(g) \) and check their accuracy by calculating percentage errors
\[
\varepsilon_k(g) \equiv \frac{Z_k^*(g) - Z(g)}{Z(g)} \cdot 100% \quad (23)
\]
with respect to the exact value (22). The results for different approximation orders and coupling parameters are presented in Table 1. This shows that the even-order factor approximants demonstrate the uniform numerical convergence for all \( g \), and the errors of the odd-order approximants oscillate in the frame of the errors of the even-order approximants.

At large \( g \), the partition function (22) has the asymptotic behavior
\[
Z(g) \simeq 1.022765 \, g^{-1/4} \quad (g \to \infty) . \quad (24)
\]
It is interesting whether the factor approximants \( Z_k^*(g) \), based solely on the power expansion (21) at small \( g \to 0 \), could, nevertheless, catch the behavior of \( Z(g) \) at large \( g \to \infty \). The most nontrivial would be to predict the noninteger index \(-1/4\) in the asymptotic form (24).

For the found factor approximants, we study the limiting expression
\[
Z_k^*(g) \simeq c_k g^{-\alpha_k} \quad (g \to \infty) \quad (25)
\]
and define the percentage error of the predicted index \( \alpha_k \) as
\[
\varepsilon_k(g) \equiv \frac{\alpha_k - 0.25}{0.25} \cdot 100% . \quad (26)
\]
In Table 2, we present the parameters of the asymptotic form (25), characterizing the behavior of the factor approximants at large coupling $g \to \infty$, together with the percentage error (26) for the index $\alpha_k$. Again, we see that the even-order approximants display numerical convergence, and the odd-order approximants oscillate around the values of the even-order ones. The index $\alpha_k$, as well as the amplitudes $c_k$, predict the large-coupling limit (24) with an error of about 20%. This is not as bad, if we remember that all calculations are based solely on the weak-coupling expansion (21), valid for $g \to 0$, and no additional optimization procedures have been involved.

Another standard touchstone for testing new approximations is the calculation of the ground-state energy for the one-dimensional anharmonic oscillator with the Hamiltonian

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 + gx^4,$$  \hspace{1cm} (27)

in which $x \in (-\infty, \infty)$ and $g \in [0, \infty)$. The weak-coupling expansion (21) for the ground-state energy $E(g)$ is known [18,19]. In the latter references, one can find the corresponding values of the expansion coefficients $a_n$.

Following the standard procedure of Sec. 2, we construct the factor approximants $E_k^*(g)$ for the ground-state energy $E(g)$ of the anharmonic oscillator with the Hamiltonian (27) and compare the obtained $E_k^*(g)$ with the exact numerical data [20]. The percentage errors

$$\varepsilon_k(g) = \frac{E_k^*(g) - E(g)}{E(g)} \cdot 100\%$$  \hspace{1cm} (28)

of the factor approximants are given in Table 3. Our results show that the even-order and odd-order approximants form two sequences $\{E_{2p}^*(g)\}$ and $\{E_{2p+1}^*(g)\}$ each of which uniformly converges for all $g$.

We again pay a special attention to the ability of factor approximants to predict the behavior of the sought function at infinitely large coupling parameter $g \to \infty$. Then the ground-state energy behaves as

$$E(g) \simeq 0.667986 \, g^{1/3}.$$  \hspace{1cm} (29)

We analyze the factor approximants $E_k^*(g)$, obtained on the basis of the weak-coupling expansion (21) at $g \to 0$, in the strong-coupling limit $g \to \infty$ and compare their asymptotic form

$$E_k^*(g) \simeq b_k g^{\beta_k} \quad (g \to \infty)$$  \hspace{1cm} (30)

with the exact asymptotic behavior (29). The results are presented in Table 4, where also the percentage error

$$\varepsilon_k(\beta) = \frac{\beta_k - 1/3}{1/3} \cdot 100\%$$

of the predicted index $\beta_k$ is shown. The even-order approximants demonstrate numerical convergence, and the errors of the odd-order approximants fluctuate around those of the even-order approximants. The errors of the amplitudes $b_k$ are close to 10% for all $k \geq 2$. 
5 Conclusion

The method of self-similar factor approximants has been completed by defining a general way for constructing both even-order as well as odd-order approximants. It is shown that transcendental functions can be well approximated by factor approximants. Moreover, some transcendental functions can be reconstructed exactly from just a few terms of their asymptotic expansions at small arguments. For all cases considered, the even-order approximants demonstrate uniform numerical convergence. The accuracy of the odd-order approximants is close to that of the even-order ones.

It is shown that the solutions to nonlinear differential equations can be extrapolated from the values of asymptotically small variables to the finite region of the latter.

A special attention has been paid to the possibility of predicting the asymptotic behavior of functions at their arguments tending to infinity from the knowledge of their asymptotic expansions at zero. Such an extreme extrapolation is shown to be feasible by means of the factor approximants.

Also, we have studied the problem of a possible evaluation of the accuracy of the factor approximants, when the exact solution would not be available. For this purpose, we have compared, for the cases analyzed above, the quantities \((f_k^* - f_{k-1}^*)/2\) and \(f_k^* - f\), where \(f\) is the known exact numerical solution. It turned out that these quantities are close to each other. Hence the difference \((f_k^* - f_{k-1}^*)/2\) can serve as a reasonable estimate for the accuracy of the \(k\)-th order factor approximant \(f_k^*\).
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Figure Captions

**Fig. 1.** Factor approximant $f_{18}^*(x)$ (dashed line) and the Taylor series of the same order $k = 18$ (dashed-dotted line), as compared to the exact $\sin x$ (solid line).

**Fig. 2.** Factor approximants $f_8^*(x)$ (dashed-dotted line), $f_9^*(x)$ (dotted line), and $f_{10}^*(x)$ (dashed line), as compared to the exact function (18) (solid line).

**Fig. 3.** Solutions to the nonlinear oscillator equation (19), with the initial conditions $y(0) = 0$ and $\dot{y}(0) = 1$, obtained from the time series at $t \to 0$ in the form of the factor approximants $y_k^*(t)$. Here shown are $y_{14}^*(t)$ (dotted line), $y_{16}^*(t)$ (dashed line), and the exact numerical solution (solid line).

**Fig. 4.** Solutions to the Rayleigh equation (20) in the form of the factor approximants $y_{18}^*(t)$ (dotted line) and $y_{19}^*(t)$ (dashed line), compared to the exact numerical solution (solid line).
Table Captions

**Table 1.** Percentage errors of the factor approximants $Z^*_k(g)$ for the partition function of the anharmonic model (22).

**Table 2.** Amplitudes $c_k$ and indices $\alpha_k$ for the strong-coupling limit (25) of the factor approximants for the partition function (22), together with the percentage errors (26) for the predicted indices.

**Table 3.** Percentage errors of the factor approximants $E^*_k(g)$ for the ground-state energy of the anharmonic oscillator.

**Table 4.** Amplitudes $b_k$ and indices $\beta_k$ for the strong-coupling limit (30) of the factor approximants for the ground-state energy of the anharmonic oscillator, with the percentage errors $\varepsilon_k(\beta)$ of the predicted indices $\beta_k$. 
| $k \setminus g$ | 0.1 | 1   | 5   | 10  | 100 |
|---------------|-----|-----|-----|-----|-----|
| 2             | 0.2 | 5.4 | 19  | 27  | 70  |
| 3             | -0.07 | -4.6 | -17 | -23 | -41 |
| 4             | 0.02 | 2.0 | 9.4 | 15  | 42  |
| 5             | -0.008 | -1.3 | -5.7 | -8.0 | -13 |
| 6             | 0.003 | 0.9 | 5.8 | 9.7 | 30  |
| 7             | -0.001 | -0.5 | -2.0 | -2.5 | -1.6 |
| 8             | 0.0006 | 0.5 | 3.9 | 6.9 | 23  |
| 9             | -0.0002 | -0.2 | -0.5 | -0.2 | 3.8 |
| 10            | 0.0001 | 0.3 | 2.8 | 5.2 | 19  |
| 11            | -0.00004 | -0.05 | 0.2 | 0.8 | 6.4 |
| 12            | 0.00004 | 0.2 | 2.1 | 4.1 | 16  |
Table 2

| $k$ | $c_k$ | $\alpha_k$ | $\varepsilon_k(\alpha)$ |
|-----|-------|-------------|--------------------------|
| 2   | 0.823 | 0.090       | -64                      |
| 3   | 0.917 | 0.346       | 38                       |
| 4   | 0.806 | 0.129       | -48                      |
| 5   | 0.878 | 0.255       | 2.0                      |
| 6   | 0.806 | 0.148       | -41                      |
| 7   | 0.860 | 0.223       | -11                      |
| 8   | 0.810 | 0.161       | -36                      |
| 9   | 0.850 | 0.209       | -16                      |
| 10  | 0.814 | 0.170       | -32                      |
| 11  | 0.845 | 0.202       | -19                      |
| 12  | 0.819 | 0.178       | -29                      |
| 13  | 0.842 | 0.199       | -20                      |
| 14  | 0.824 | 0.182       | -27                      |
| 15  | 0.841 | 0.197       | -21                      |
| 16  | 0.828 | 0.187       | -25                      |
| 17  | 0.840 | 0.196       | -22                      |
Table 3

| $k \setminus g$ | 0.01 | 0.3 | 1  | 200 |
|-----------------|------|-----|----|-----|
| 2               | -0.07| -2.0| -7.4| -53 |
| 3               | -0.07| 1.4 | 9.0 | 256 |
| 4               | -0.07| -0.4| -2.4| -35 |
| 5               | -0.07| 0.2 | 2.2 | 49  |
| 6               | -0.07| -0.1| -1.0| -25 |
| 7               | -0.07| 0.06| 0.7 | 14  |
| 8               | -0.07| -0.03| -0.5| -19 |
| 9               | -0.07| 0.02| 0.3 | 2.6 |
| 10              | -0.07| -0.01| -0.3| -15 |
Table 4

| $k$ | $b_k$ | $\beta_k$ | $\varepsilon_k(\beta)$ |
|-----|-------|-----------|--------------------------|
| 2   | 0.729 | 0.176     | -47                      |
| 3   | 0.611 | 0.590     | 77                       |
| 4   | 0.755 | 0.231     | -30                      |
| 5   | 0.669 | 0.409     | 22                       |
| 6   | 0.756 | 0.257     | -23                      |
| 7   | 0.696 | 0.351     | 5.3                      |
| 8   | 0.752 | 0.272     | -18                      |
| 9   | 0.710 | 0.328     | -1.7                     |
| 10  | 0.748 | 0.282     | -16                      |
| 11  | 0.718 | 0.317     | -4.8                     |
| 12  | 0.743 | 0.289     | -13                      |
| 13  | 0.721 | 0.312     | -6.3                     |
| 14  | 0.739 | 0.294     | -12                      |
| 15  | 0.723 | 0.309     | -7.2                     |
| 16  | 0.736 | 0.298     | -11                      |
| 17  | 0.725 | 0.308     | -7.5                     |
Figure 1: Factor approximant $f_{18}(x)$ (dashed line) and the Taylor series of the same order $k = 18$ (dashed-dotted line), as compared to the exact $\sin x$ (solid line).
Figure 2: Factor approximants $f_8^r(x)$ (dashed-dotted line), $f_9^r(x)$ (dotted line), and $f_{10}^r(x)$ (dashed line), as compared to the exact function (18) (solid line).
Figure 3: Solutions to the nonlinear oscillator equation (19), with the initial conditions $y(0) = 0$ and $\dot{y}(0) = 1$, obtained from the time series at $t \to 0$ in the form of the factor approximants $y_k^*(t)$. Here shown are $y_{14}^*(t)$ (dotted line), $y_{16}^*(t)$ (dashed line), and the exact numerical solution (solid line).
Figure 4: Solutions to the Rayleigh equation (20) in the form of the factor approximants $y_{18}^*(t)$ (dotted line) and $y_{19}^*(t)$ (dashed line), compared to the exact numerical solution (solid line).