Abstract

In this note we prove an equivariant version of a result of Cartan \cite{Car76} for equivariant simplicial cohomology with local coefficients.

Keywords: Simplicial sets, group action, local coefficients, Cartan cohomology theory, equivariant twisted cohomology, generalized Eilenberg-MacLane complex.

1 Introduction

To generalize Sullivan’s theory of rational de Rham complexes on simplicial sets to cochain complexes over arbitrary ring of coefficients, Cartan \cite{Car76} introduced the notion of ‘Cohomology theory’. Over the coefficient ring \( \mathbb{Z} \), Cartan’s result can be described as follows. Recall that a simplicial differential graded algebra over \( \mathbb{Z} \) is a simplicial object in the category DGA of differential graded algebras over \( \mathbb{Z} \), so that for each \( p \geq 0 \) we have a differential graded algebra

\[
(A^*_p, \delta): \quad A^0_p \xrightarrow{\delta} A^1_p \xrightarrow{\delta} A^2_p \rightarrow \cdots
\]

together with face and degeneracy maps \( \partial_i : A^*_{p+1} \rightarrow A^*_p \) and \( s_i : A^*_p \rightarrow A^*_{p+1} \) which are homomorphism of differential graded algebras satisfying the usual simplicial and differential identities. Then a cohomology theory in the sense of Cartan is a simplicial differential graded algebra \( A \) over \( \mathbb{Z} \) such that

1. each cochain complex \((A^*_p, \delta)\) is exact and \( Z^0 A = Ker(A^0_* \xrightarrow{\delta} A^1_* ) \) is a simplicially trivial algebra over \( \mathbb{Z} \) (here simplicially trivial means that all the face and degeneracy maps are isomorphisms),

2. the homotopy groups \( \pi_i (A^n_p) \) of the simplicial set \( A^n_p = \{ A^p_n \}_{p \geq 0} \) are trivial for all \( i, n \geq 0 \).

A cohomology theory \( A \) determines a contravariant functor from the category of simplicial sets to DGA which assigns to each simplicial set \( K \) the differential...

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graded algebra $A(K) = \{Hom(K, A^n)\}_{n \geq 0}$, where $Hom(K, A^n)$ is the abelian group of simplicial maps $K \rightarrow A^n$ and the differential on $A(K)$ is induced from that of $A$. Then Cartan’s theorem states that there is a natural isomorphism

$$H^*(A(K)) \cong H^*(K; \mathbb{Z}(A)),$$

where $\mathbb{Z}(A)$ is the abelian group $(\mathbb{Z}^0 A)_0$.

In [Hir79], Hirashima generalized Cartan’s result for cohomology with local coefficients. Moreover Cartan’s theorem was generalized in [MN98] for $G$-simplicial sets, i.e for simplicial sets equipped with an action of a group $G$ by simplicial maps. In the equivariant setting, ordinary cohomology of simplicial sets is replaced by Bredon cohomology of $G$-simplicial sets.

Recently, in [MS10] the notion of equivariant cohomology with local coefficients, called Bredon-Illman cohomology with local coefficients, has been formulated for $G$-simplicial sets. This is the simplicial version of the equivariant cohomology with local coefficients for $G$-spaces as introduced in [MM96], which generalizes Bredon-Illman cohomology [Bre67], [Ill75]. It is therefore reasonable to prove a version of Cartan’s theorem on $G$-simplicial sets for equivariant cohomology with local coefficients. In this note we define the notion of equivariant twisted Cartan cohomology theory and prove a version of Cartan’s theorem which reduces to the result of [MN98] when the local coefficients system is simple, and, to that of [Hir79] when $G$ is a trivial group. In the present context Cartan’s cohomology theory appears as a contravariant functor from the category $O_G$ of canonical orbits to the category of cohomology theory in the sense of Cartan [Car76], satisfying some naturality conditions (see Definition 4.1 for details). It may be remarked that the equivariant analogue of a point is the orbit of the point and these orbits are precisely the objects of the category $O_G$.

Throughout the paper, when referring to ‘model’, we will always mean it to be minimal.

This paper is organized as follows. Section 2 is a review of the basic definitions and results that will be used in the sequel. In section 3, we describe the notion of equivariant twisted cohomology and state a classification theorem which is proved in [MS10]. In Section 4, we define the notion of equivariant Cartan cohomology theory and prove our main result.

## 2 Preliminaries

In this section we recall some basic definitions and facts about simplicial sets [May67] and related topics. We denote the category of simplicial sets and simplicial maps by $\mathcal{S}$ and the category of simplicial groups and simplicial group homomorphisms by $SG$. Throughout $G$ will denote a discrete group.

**Definition 2.1.** [Moore56]

Let $B$ be a simplicial set and $\Gamma$ a simplicial group. Then a graded function

$$\tau: B \rightarrow \Gamma, \quad \tau_q: B_q \rightarrow \Gamma_{q-1}$$

is called a graded function on $B$ with coefficients in $\Gamma$. The graded function $\tau$ induces a homomorphism $\tau_*: C_\ast(B) \rightarrow C_\ast(B)$, where $C_\ast(B)$ is the chain complex of $B$.

**Example.** Consider the simplicial set $B = \Delta^n$ and the simplicial group $\Gamma = \mathbb{Z}$ given by

$$\tau: \Delta^n \rightarrow \mathbb{Z}, \quad \tau_q: \Delta^n_q \rightarrow \mathbb{Z}_{q-1}.$$
is called a twisting function if it satisfies the following identities:

\[
\partial_0(\tau_q(b)) = (\tau_{q-1}(\partial_0 b))^{-1} \tau_{q-1}(\partial_1 b), \quad b \in B_q
\]

\[
\partial_i(\tau_q(b)) = \tau_{q-1}(\partial_{i+1} b) \quad i > 0
\]

\[
s_i(\tau_q(b)) = \tau_{q+1}(s_{i+1} b) \quad i \geq 0
\]

\[
\tau_{q+1}(s_0 b) = e_q, \quad e_q being the identity of the group \Gamma_q.
\]

**Definition 2.2.** Let \(B, F\) be simplicial sets, \(\Gamma\) a simplicial group which operates on \(F\) from the left, and \(\tau: B \to \Gamma\) a twisting function. A twisted cartesian product (TCP), with fibre \(F\), base \(B\) and group \(\Gamma\) is a simplicial set, denoted by \(F \times \tau B\) which satisfies

\[
(F \times \tau B)_n = F_n \times B_n
\]

and has face and degeneracy operators

\[
\partial_i(f, b) = (\partial_i f, \partial_i b), \quad i > 0
\]

\[
\partial_0(f, b) = (\tau(b) \partial_0 f, \partial_0 b)
\]

\[
s_i(f, b) = (s_i f, s_i b) \quad i \geq 0
\]

If \(B, F\) are Kan complexes then \(F \times \tau B\) is also a Kan complex and the canonical projection \(p: F \times \tau B \to B\) is a Kan fibration.

For an abelian group \(A\) and an integer \(n > 1\), let \(K(A, n)\) denote a minimal Eilenberg MacLane complex of type \((A, n)\). There is a canonical model of \(K(A, n)\) for which the \(q\)-simplices are described as follows. Consider the simplicial abelian group \(C(A, n)\) with \(q\)-simplices

\[
C(A, n)_q = C^n(\Delta[q]; A),
\]

the group of normalized \(n\)-cochains of the standard simplicial \(q\)-simplex \(\Delta[q]\) \cite{May67}. The face and degeneracy maps of \(C(A, n)\) are given as follows. For \(\mu \in C(A, n)_q, \alpha \in \Delta[q-1]_n\) and \(\beta \in \Delta[q+1]_n\)

\[
\partial_i \mu(\alpha) = \mu(\delta_i(\alpha)), \quad s_j \mu(\beta) = \mu(\sigma_j(\beta)).
\]

Here \(\delta_i: \Delta[q-1] \to \Delta[q]\) and \(\sigma_j: \Delta[q+1] \to \Delta[q]\) are the simplicial maps defined by \(\delta_i(\Delta[q-1]) = \partial_i \Delta[q], \sigma_j(\Delta[q+1]) = \Delta[q], \Delta[q] = (0, 1, \cdots, q)\) being the unique non-degenerate \(q\)-simplex of \(\Delta[q]\).

We have a simplicial group homomorphism

\[
\delta^n: C(A, n) \to C(A, n+1)
\]

such that \(\delta^n c \in C(A, n+1)_q\) is the usual simplicial coboundary of \(c \in C(A, n)_q\). Then

\[
K(A, n)_q = Ker \delta^n = Z^n(\Delta[q]; A)
\]

the group of normalized \(n\)-cocycles. It may be noted that \(K(A, n)\) is a minimal one vertex Kan complex.
Definition 2.3. Let $\pi$ be a group. A $\pi$-module is a pair $(A, \phi)$ where $A$ is an abelian group and $\phi: \pi \to \text{Aut}(A)$ a group homomorphism. A map of $\pi$-modules $f: (A, \phi) \to (A', \phi')$ is a group homomorphism $f: A \to A'$ such that

$$f(\phi(x)a) = \phi'(x)f(a)$$

for all $x \in \pi$ and $a \in A$. The category of $\pi$-modules is denoted by $\pi$-mod.

Let $(A, \phi) \in \pi$-mod. Then $\pi$ acts on the minimal one vertex Kan complex $K(A, n)$ in the following way:

$$x\mu = \phi(x) \circ \mu$$

where $\mu \in K(A, n)_q = Z^n(\Delta[q]; A), x \in \pi$.

The notion of a generalized Eilenberg Maclane complex appears in [Git63], [Hir79], [BFGM03]. Roughly speaking, a generalized Eilenberg Maclane complex is a one vertex minimal Kan complex having exactly two non-vanishing homotopy groups, one of them being the fundamental group. It appears as the total space of a Kan fibration. Gitler [Git63] used it in the construction of cohomology operations in cohomology with local coefficients. It also plays a crucial role in classifying cohomology with local coefficients [Hir79], [BFGM03]. It may be remarked that a product of Eilenberg Maclane complexes is also sometimes referred to as a generalized Eilenberg Maclane complex.

A generalized Eilenberg Maclane complex can be constructed as follows. Let $W\pi$ denotes the standard $W$ construction [May67] of a group $\pi$. Let $(A, \phi)$ be a $\pi$-module. We have a twisting function

$$\tau(\pi): W\pi \to \pi, \text{ where } \tau(\pi)(x_1, \cdots, x_q) = x_1, x_i \in \pi,$$

and $\pi$ is considered as a simplicial group with each component $\pi$ and all the face and the degeneracy maps are identities. For $n > 1$ let

$$L_\pi(A, n) = K(A, n) \times_{\tau(\pi)} W\pi,$$

where the right hand side is the twisted cartesian product as defined in the Definition 2.2. Then it is a one vertex minimal Kan complex whose fundamental group is $\pi$, $n$-th homotopy group is $A$ and all other homotopy groups are trivial. Moreover the action of the fundamental group $\pi$ on the $n$-th homotopy group $A$ is given by $\phi$ [Thu97]. We have a canonical map $p: L_\pi(A, n) \to W\pi, p(c, x) = x$ for $c \in K, x \in W\pi$, which is a Kan fibration.

For a group $G$, the category of canonical orbits, denoted by $O_G$, is a category whose objects are cosets $G/H$, as $H$ runs over subgroups of $G$. A morphism from $G/H$ to $G/K$ is a $G$-map. Recall that such a morphism determines and is determined by a subconjugacy relation $g^{-1}Hg \subseteq K$ and is given by $\hat{g}(eH) = gK$. We denote this morphism by $\hat{g}$ [Bre67].

A contravariant functor from $O_G$ to $S$ (resp. the category of groups or the category of abelian groups) is called an $O_G$-simplicial set (resp. $O_G$-group or abelian $O_G$-group). We denote by $O_G S$, the category of $O_G$-simplicial sets with morphisms being natural transformations of functors.
A morphism \( f : T \to S \) of \( O_G \)-simplicial sets is called an \( O_G \)-Kan fibration if \( f(G/H) : T(G/H) \to S(G/H) \) is a Kan fibration for each subgroup \( H \) of \( G \). Similarly, an \( O_G \)-simplicial set \( T \) is called an \( O_G \)-Kan complex if each \( T(G/H) \) is a Kan complex for each subgroup \( H \subseteq G \).

We recall the following definition from [MN98].

**Definition 2.4.** Given an \( O_G \)-group \( \lambda \) and an integer \( n \geq 0 \), an \( O_G \)-Kan complex \( T \) is called an \( O_G \)-Eilenberg MacLane complex of type \((\lambda, n)\) if each \( T(G/H) \) is a \( K(\lambda(G/H), n) \) and \( T(\hat{g}) : T(G/H) \to T(G/K) \) is the unique simplicial homomorphism induced by the linear map \( \lambda(\hat{g}) : \lambda(G/H) \to \lambda(G/K), g^{-1}Hg \subseteq K, \) such that \( T(\hat{g})_n : K(\lambda(G/H), n)_n \to K(\lambda(G/K), n)_n \) is \( \lambda(\hat{g}) \).

It is proved in [MN98] that any two \( O_G \)-Eilenberg MacLane complexes of the same type are naturally isomorphic. We denote an \( O_G \)-Eilenberg MacLane complex of type \((\lambda, n)\) by \( K(\lambda, n) \). Let \( n > 1 \) and \( \lambda \) be an abelian \( O_G \)-group. Using the canonical model of an ordinary Eilenberg MacLane complex as described at the beginning of this section, we have a canonical model of \( K(\lambda, n) \), given by \( K(\lambda, n)(G/H)_q = \mathbb{Z}^n(\Delta^n[q];\lambda(G/H)) \).

Let \( C \) be a category and \( T : O_G \to C \) be a contravariant functor. An \( O_G \)-group \( \pi \) is said to act on \( T \) if we have a group homomorphism \( \phi_H : \pi(G/H) \to Aut_C(T(G/H)) \) for each subgroup \( H \) of \( G \) such that for any subconjugacy relation \( g^{-1}Hg \subseteq K, \)

\[
\phi_H(\pi(\hat{g})v) \circ T(\hat{g}) = T(\hat{g}) \circ \phi_K(v), \quad v \in \pi(G/K).
\]

We denote this action simply by \( \phi \). Thus we can talk of an action of \( \pi \) on an \( O_G \)-simplicial set, an \( O_G \)-group etc. If \( \pi \) acts on an abelian \( O_G \)-group \( T \), then we call \( T \) a \( \pi \)-module.

### 3 G-simplicial set, Equivariant Twisted Cohomology and its Classification

In this section we briefly recall the definition of equivariant twisted cohomology and its homotopy classification from [MS10]. Let \( G \) be a discrete group. Recall that a \( G \)-simplicial set is a simplicial set \( X = \{ X_n \} \) such that each \( X_n \) is a \( G \)-set and the face, degeneracy maps commute with this action. A \( G \)-simplicial set \( X \) is called \( G \)-connected if each fixed point simplicial set \( X^H, H \subseteq G \), is connected.

Let \( X \) be a \( G \)-simplicial set and \( \pi \) be an \( O_G \)-group. Let \( \Phi X \) denote the \( O_G \)-simplicial set defined by \( \Phi X(G/H) = X^H, \Phi X(\hat{g})(x) = gx, x \in X^H, g^{-1}Hg \subseteq K. \)

**Definition 3.1.** Let \( T \) be an \( O_G \)-simplicial set and \( \Gamma \) a simplicial \( O_G \)-group. A natural transformation of functors \( \tau : T \to \Gamma \) is called an \( O_G \)-twisting function if \( \tau(G/H) : T(G/H) \to \Gamma(G/H) \) is an ordinary twisting function for each subgroup \( H \) of \( G \).
Example 3.2. Consider the $O_{G}$-group $\pi$ as a simplicial $O_{G}$-group $\{\pi_{n}\}_{n \geq 0}$ where $\pi_{n} = \pi$ for all $n \geq 0$ and face and degeneracy maps are identity natural transformations. Define

$$\tau(\pi): \bigwedge_{\pi} \rightarrow \pi, \; \tau(\pi)(G/H)([x_1, \cdots, x_q]) = x_1,$$

where $[x_1, \cdots, x_q] \in \bigwedge_{\pi}(G/H)_q$, $x_i \in \pi(G/H)$, $1 \leq i \leq q$. It is routine to check that $\tau(\pi)$ is an $O_{G}$-twisting function.

Example 3.3. Consider the simplicial set $Y$ and for a morphism $g: X \rightarrow \pi X$ of fundamental groups induced by the simplicial map $g: X \rightarrow \pi X$. Define

$$\pi X(G/H) = \pi_1(X^H, v)$$

and for a morphism $\tilde{g}: G/H \rightarrow G/K$, $g^{-1}Hg \subseteq K$, $\pi X(\tilde{g})$ is the homomorphism of fundamental groups induced by the simplicial map $\tilde{g}: X^K \rightarrow X^H$. We regard $\pi X$ as an $O_{G}$-group complex in the trivial way, that is, $\pi X(G/H)_n = \pi X(G/H)$ for all $n$. We choose a 0-simplex $x$ on each $G$-orbit of $X_0$ and a 1-simplex $\omega_x \in X^G$ such that $\partial_0 \omega_x = x$, $\partial_1 \omega_x = v$. For any other 0-simplex $y$ on the orbit of $x$ we define $\omega_y = g\omega_x$ if $y = gx$. Then it is an easy check that this is well defined and $\omega_y \in X^G$. For a 0-simplex $x \in X^H$, let $\xi_H(x) = [\pi_x]_n$ be the homotopy class of $\pi_x: \Delta[1] \rightarrow X^H$. Here for any $q$-simplex $\sigma$ of a simplicial set $Y$, $\pi: \Delta[q] \rightarrow Y$ denote the unique simplicial map satisfying $\pi(\Delta[n]) = \sigma$. Define

$$\{\kappa(G/H)_n\}: X^H \rightarrow \pi_1(X^H, v)$$

by

$$\kappa(G/H)_n(y) = \xi_H(\partial(0,2,\cdots,n)y)^{-1} \circ [\partial(2,\cdots,n)y] \circ \xi_H(\partial(1,\cdots,n)y)$$

where $y \in (X^H)_n$ and

$$\partial(0,2,\cdots,n)y = \partial_0 \partial_2 \cdots \partial_n y, \partial(2,\cdots,n)y = \partial_2 \cdots \partial_n y, \partial(1,2,\cdots,n)y = \partial_1 \partial_2 \cdots \partial_n y.$$

It is standard that $\kappa(G/H)$ is a twisting function on $X^H$. We verify that

$$\kappa: \Phi X \rightarrow \pi X, \; G/H \mapsto \kappa(G/H)$$

is natural. Suppose $H$ and $K$ are subgroups such that $g^{-1}Hg \subseteq K$. Then $y = gz \in X^H_n$. Observe that if $x_1, x_2 \in X^K_1$ are 1-simplices such that $\overline{x_1} \simeq \overline{x_2}$, as simplicial maps into $X^K$, then $\overline{y_1} \simeq \overline{y_2}$ as simplicial maps into $X^H$ where $y_i = gx_i$, $i = 1, 2$. Thus

$$\kappa(G/H)_n \circ \Phi X(\tilde{g})(z) = \kappa(G/H)_n(y)$$

$$= \xi_H(\partial(0,2,\cdots,n)y)^{-1} \circ [\partial(2,\cdots,n)y] \circ \xi_H(\partial(1,\cdots,n)y)$$

$$= \xi_H(g\partial(0,2,\cdots,n)z)^{-1} \circ [g\partial(2,\cdots,n)z] \circ \xi_H(g\partial(1,\cdots,n)z)$$

$$= g\kappa_K(\partial(0,2,\cdots,n)z)^{-1} \circ g[\partial(2,\cdots,n)z] \circ g\kappa_K(\partial(1,\cdots,n)z)$$

$$= \pi X(\tilde{g}) \circ \kappa(G/K)_n(z).$$

Thus $\kappa: \Phi X \rightarrow \pi X$ is an $O_{G}$-twisting function.
Let $X$ be a $G$-simplicial set, $\tau: \Phi X \to \pi$ be an $O_G$-twisting function and $M$ a $\pi$-module, given by $\phi$. We define equivariant twisted cohomology of $(X, \tau, \phi)$ as follows.

We denote the category of abelian $O_G$-groups by $C_G$. We have a cochain complex in the abelian category $C_G$ defined by

$$C_n(X): O_G \to Ab, \quad G/H \mapsto C_n(X^H; \mathbb{Z}),$$

where $C_n(X^H; \mathbb{Z})$ is the free abelian group generated by the non-degenerate $n$-simplexes of $X^H$ and for any morphism $\hat{g}: G/H \to G/K$, $g^{-1}Hg \subseteq K$ in $O_G$, $C_n(X)(\hat{g})$ is given by the map $g_*: C_n(X^K; \mathbb{Z}) \to C_n(X^H; \mathbb{Z})$, induced by the simplicial map $g: X^K \to X^H$. The boundary $\partial_n: C_n(X) \to C_{n-1}(X)$ is a natural transformation defined by $\partial_n(G/H): C_n(X^H; \mathbb{Z}) \to C_{n-1}(X^H; \mathbb{Z})$, where $\partial_n(G/H)$ is the ordinary boundary map of the simplicial set $X^H$. Dualising this chain complex in the abelian category $C_G$ we get the cochain complex

$$\{C^n_G(X; M) = \text{Hom}_{C_G}(C_n(X), M), \delta^n\},$$

which defines the ordinary Bredon cohomology of the $G$-simplicial set $X$ with coefficients $M$ [Bre67]. To define the twisted cohomology of the $G$-simplicial set $X$ we modify the coboundary maps as follows

$$\delta^n: C^n_G(X; M) \to C^{n+1}_G(X; M), \quad f \mapsto \delta^n f$$

where

$$\delta^n f(G/H): C_{n+1}(X^H; \mathbb{Z}) \to M(G/H)$$

is given by

$$\delta^n f(G/H)(x) = (\tau(G/H))^{-1} f(G/H)(\partial_0 x) + \sum_{i=1}^{n+1} (-1)^i f(G/H)(\partial_i x)$$

for $x \in X^H_{n+1}$. Note that the first term of the right hand side is obtained by the given action $\phi$. We denote the resulting cochain complex by $C^n_G(X; \tau, \phi)$.

**Definition 3.4.** The $n^{th}$ equivariant twisted cohomology of $(X, \tau, \phi)$ is defined as

$$H^n_G(X; \tau, \phi) = H_n(C^n_G(X; \tau, \phi)).$$

Suppose that $B, F$ are $O_G$-Kan complexes and $\Gamma$ an $O_G$-group complex. Also assume that $B$ is a $\Gamma$-module and $\kappa: B \to \Gamma$ an $O_G$-twisting function. Then we have the $O_G$-Kan complex $F \times_\kappa B$, defined as

$$(F \times_\kappa B)(G/H) = F(G/H) \times_{\tau(G/H)} B(G/H), \quad (F \times_\kappa B)(\hat{g}) = (F(\hat{g}), B(\hat{g}))$$

for each object $G/H$ and morphism $\hat{g}: G/H \to G/K$ of the category $O_G$. We call this $O_G$-Kan complex the $O_G$-twisted cartesian product (TCP), with fibre $F$, base $B$, group $\Gamma$ and twisting $\kappa$. Observe that the second factor projection gives an $O_G$-Kan fibration $p: (F \times_\kappa B) \to B$. We view $(F \times_\kappa B, p)$ as an object in the slice category (cf. [CJ99]) $O_G S/B$. 

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Let $M$ be a $\pi$-module with module structure given by $\phi$. For each subgroup $H$ of $G$, define a group homomorphism
\[ \psi_H: \pi(G/H) \rightarrow \text{Aut}_S(K(M(G/H), n)) \]
as follows. For $u \in \pi(G/H)$, let $\psi_H(u)$ be the unique simplicial automorphism of $K(M(G/H), n)$ such that
\[ \phi_H(u) = \psi_H(u)_n: K(M(G/H), n) \rightarrow K(M(G/H), n), \ u \in \pi(G/H). \]
This defines an action of the $O_G$-group $\pi$ on the $O_G$-Kan complex $K(M, n)$.

Therefore we can form the $O_G$-Kan fibration $p: K(M, n) \times_{\pi(G/H)} \overline{W}_\pi \rightarrow \overline{W}_\pi,$
where $\tau(\pi)$ is the $O_G$-twisting function as described in the Example [4.2]. If we use the canonical model of $K(M, n)$, the total complex of the resulting $O_G$-Kan fibration is denoted by $L_\phi(M, n)$. Since any two models of $K(M, n)$ are naturally isomorphic, $K(M, n) \times_{\pi(G/H)} \overline{W}_\pi$ is isomorphic to $L_\phi(M, n)$ for any model of $K(M, n)$. We call $L_\phi(M, n)$ a generalized $O_G$-Eilenberg Maclane complex. Note that $L_\phi(M, n)(G/H)$ is the generalized Eilenberg Maclane complex
\[ L_{\pi(G/H)}(M(G/H), n) = Z^n(\Delta[-]; M(G/H)) \times_{\tau(\pi(G/H))} \overline{W}_\pi(G/H). \]

The equivariant twisted cohomology $H^*_G(X; \tau, \phi)$ has been classified by the $O_G$-Kan complex $L_\phi(M, n)$ in [MS10]. This classification result can be described as follows.

Let $X$ be a $G$-simplicial set and $\tau: \Phi X \rightarrow \overline{W}_\pi$ be an $O_G$-twisting function. It determines an $O_G$-simplicial map $\theta(\tau): \Phi X \rightarrow \overline{W}_\pi$ defined as,
\[ \theta(\tau)(G/H): X^H_q \rightarrow \overline{W}_\pi(G/H)_q, \]
\[ x \mapsto [\tau(G/H)_q(x), \tau(G/H)_{q-1}(\partial_0 x), \ldots, \tau(G/H)_1(\partial_0^{q-1} x)]. \]

Let $(\Phi X, L_\phi(M, n))_{\overline{W}_\pi}$ denote the set of liftings of the map $\theta(\tau)$ with respect to $p$: $L_\phi(M, n) \rightarrow \overline{W}_\pi, \ p(c, g) = g.$

**Definition 3.5.** Let $f, g \in (\Phi X, L_\phi(M, n))_{\overline{W}_\pi}$. Then $f$ and $g$ are said to be vertically homotopic, written $f \sim_v g$, if there is a map $F: \Phi X \times \Delta[1] \rightarrow L_\phi(M, n)$ of $O_G$-simplicial sets such that for every object $G/H$ of $O_G$, $F(G/H)$ is a homotopy of the simplicial maps $f(G/H), g(G/H)$ and $p \circ F = \theta(\tau) \circ p_1$, where $p_1: \Phi X \times \Delta[1] \rightarrow \Phi X$ is the projection onto the first factor.

Observe that $(\Phi X, \theta(\tau))$ and $(L_\phi(M, n), p)$ are objects in the slice category $O_G S_{/\overline{W}_\pi}$ and $(\Phi X, L_\phi(M, n))_{\overline{W}_\pi}$ is the set of morphisms in $O_G S_{/\overline{W}_\pi}$ from $(\Phi X, \theta(\tau))$ to $(L_\phi(M, n), p)$. The category $O_G S$ is a closed model category [DKS3] in the sense of Quillen [Qui67] and hence $O_G S_{/\overline{W}_\pi}$ is also a closed model category [GJ99], where $(L_\phi(M, n), p)$ is a fibrant object and the above notion of vertical homotopy coincides with the abstract homotopy. Therefore $\sim_v$ is an equivalence relation on the set $(\Phi X, L_\phi(M, n))_{\overline{W}_\pi}$. Let $[\Phi X, L_\phi(M, n)]_{\overline{W}_\pi}$ denote the set of equivalence classes. Then the homotopy classification of equivariant twisted cohomology can be stated as follows.
Theorem 3.6. Suppose \( X \) is a \( G \)-simplicial set and \( \tau: \Phi X \to \mathbb{Z} \) is an \( O_G \)-twisting function. Then
\[
H^0_G(X; \tau, \phi) \cong [\Phi X, L_0(M, n)]_{O_G},
\]
for each \( n \geq 0 \).

4 Equivariant Twisted Cartan Cohomology Theory

In this final section we formulate an equivariant version of Cartan’s Cohomology theory \([Car76]\) and prove that Bredon-Illman cohomology with local coefficients of a \( G \)-simplicial set can be computed by the cohomology of a differential graded algebra determined by a given cohomology theory.

We begin with the following equivariant generalization of Cartan Cohomology theory suitable for our purpose.

Definition 4.1. An equivariant twisted Cartan cohomology theory is a sequence \( A = \{A^i\}_{i \geq 0} \) of simplicial abelian \( O_G \)-groups \( A_i \), together with simplicial differentials \( \delta^i: A^i \to A^{i+1} \) such that
1. For each subgroup \( H \subseteq G \), \( A(G/H) = (A^*(G/H)_*, \delta^*(G/H)) \) is a simplicial differential graded algebra over \( \mathbb{Z} \).
2. For each \( p \geq 0 \),
\[
A^0_p \xrightarrow{\delta^0} A^1_p \xrightarrow{\delta^1} A^2_p \to \cdots
\]
is an exact sequence in the abelian category \( C_G \) of abelian \( O_G \)-groups.
3. The \( O_G \)-group \( \pi_n \circ A^i \) is the zero \( O_G \)-group, for all \( n, i \geq 0 \).
4. The simplicial abelian \( O_G \)-group \( Z^0_A = \ker(A^0 \xrightarrow{\delta^0} A^1) \) is simplicially trivial.
5. For each subgroup \( H \subseteq G \) and an integer \( i \geq 0 \) there is a group homomorphism
\[
\psi^i_H: \text{Aut}((Z^0_A)_0(G/H)) \to \text{Aut}_{O_G}(A^i(G/H))
\]
satisfying
- \( \delta^i \circ \psi^i_H(\alpha) = \psi^{i+1}_H(\alpha) \circ \delta^i, \alpha \in \text{Aut}((Z^0_A)_0(G/H)) i \geq 0 \).
- If \( g^{-1}Hg \subseteq K \), \( \alpha \in \text{Aut}((Z^0_A)_0(G/H)) \), \( \beta \in \text{Aut}((Z^0_A)_0(G/K)) \) such that \( \alpha \circ (Z^0_A)_0(\hat{g}) = (Z^0_A)_0(\hat{g}) \circ \beta \) then
\[
\psi_H(\alpha) \circ A^i(\hat{g}) = A^i(\hat{g}) \circ \psi_K(\beta).
\]

Example 4.2. For an abelian group \( B \) and an integer \( n \geq 0 \), let \( C(B, n) \) denote the simplicial abelian group and \( \delta^n: C(B, n) \to C(B, n+1) \) be the simplicial homomorphism as introduced in the Section 1. Then, for an abelian \( O_G \)-group \( M \), \( A = \{A^i\}_{i \geq 0} \) where \( A^n(G/H) = C(M(G/H), n) \) together with the differential \( \delta^n \), defines an equivariant twisted Cartan cohomology theory such that \((Z^0_A)_0 = M\).
Lemma 4.3. Let \( A^0 \xrightarrow{\delta} A^1 \xrightarrow{\delta} \cdots \) be an equivariant twisted Cartan cohomology theory. Then each \( A^n \) is contractible as an object of \( O_G S \).

Proof. Consider the abelian \( O_G \)-simplicial group \( Z^n A \) defined by \( Z^n A(G/H) = Ker(\delta^n(G/H)) \colon A^n(G/H) \to A^{n+1}(G/H) \), \( Z^n A(\hat{g}) = A^n(\hat{g}) \mid_{Z^n A(G/H)} \). For an integer \( n \geq 0 \) and a subgroup \( H \) of \( G \), we have a short exact sequence

\[
0 \to Z^n A(G/H) \to A^n(G/H) \to Z^{n+1} A(G/H) \to 0
\]

of simplicial abelian groups. Therefore \( A^n(G/H) \to Z^{n+1} A(G/H) \) is a principal fibration with fibre \( Z^n A(G/H) \) in the category of simplicial sets, and hence a principal twisted cartesian product (PTCP) of type \( (W) \) with group complex \( Z^n A(G/H) \) \([\text{May67}]\). This PTCP of type \( (W) \) is naturally isomorphic to the universal PTCP of type \( (W) \), \( W(Z^n A(G/H)) \to W(Z^n A(G/H)) \). But \( W(Z^n A(G/H)) \) is contractible. The functions

\[
h^H_{q-i} : W(Z^n A(G/H))_q \to W(Z^n A(G/H))_{q+1}, \quad 0 \leq i \leq q, \quad q \geq 0,
\]

\[
h^H_{q-i}(x_q, \ldots, x_0) = (0^H_{q+1}, \ldots, 0^H_{i+1}, \partial_0^{q-i} x_q \cdots \partial_0 x_{i+1} x_i \cdots x_0),
\]

where \( x_j \in Z^n A(G/H)_j \), \( 0 \leq j \leq q \) and \( 0^H_{q+1-\tau} = 0^H_{q+1} \) is the zero elements of the abelian group \( Z^n A(G/H)_{q+1} \) \( 0 \leq \tau \leq q - i \), defines a contraction of \( W(Z^n A(G/H)) \) which is natural with respect to morphisms of \( O_G \). Hence \( A^n(G/H) \) is also contractible and the contraction is natural. Consequently \( A^n \) is contractible as object of \( O_G S \).

Consider an equivariant twisted Cartan cohomology theory \( A = \{ A^n \}_{n \geq 0} \). It determines an abelian \( O_G \)-group \( (Z^n A)_0 \). We denote it by \( M \). Given a \( G \)-simplicial set \( X \), an \( O_G \)-group \( \pi \), an \( O_G \)-twisting function \( \tau : \Phi X \to \pi \), and a \( \pi \)-module structure \( \phi \) on \( M \), we shall construct a differential graded algebra over \( Z \) whose cohomology will compute the equivariant twisted cohomology of \( (X, \phi, \tau) \).

Note that, by the second condition of the fifth axiom in the Definition 4.1, \( A^n \) becomes a \( \pi \)-module by \( (\psi \phi)_H = \psi_H \phi_H : \pi(G/H) \to Aut_{SG}(A^n(G/H)) \). To see this, observe that for \( g^{-1} H g \subseteq K \), \( v \in \pi(G/K) \) we have

\[
\phi_H(\pi(\hat{g})v) = M(\hat{g}) \circ M(v).
\]

Therefore taking \( \alpha = \phi_H(\pi(\hat{g})v), \beta = \phi_K(v) \) in the second condition of the fifth axiom in the Definition 4.1 we get

\[
\psi_H \phi_H(\pi(\hat{g})v) \circ A^n(\hat{g}) = A^n(\hat{g}) \circ \psi_K \phi_K(v).
\]

Consider the \( O_G \)-twisting function as introduced in the Example 3.2. We form the \( O_G \)-Kan fibration \( p : A^n \times_{\tau(\pi)} W_\pi \to W_\pi \) by taking the \( O_G \)-twisted cartesian product as described in Section 3.

The \( O_G \)-twisting function \( \tau : \Phi X \to W_\pi \) determines a map \( \theta(\tau) : \Phi X \to W_\pi \) defined by

\[
\theta(\tau)(G/H)_q(x) = [\tau(G/H)(x), \tau(G/H)(\partial_0 x), \cdots, \tau(G/H)(\partial_0^{q-1} x)], \quad x \in X_q^H.
\]
Let $A^n_\phi(X; \tau) = \{ f : \Phi X \to A^n \times_{\tau(\underline{\pi})} \underline{W\underline{\pi}} \mid pf = \theta(\tau) \}$. This set has an abelian group structure by fibrewise addition, fibrewise inversion and the zero section. We define a differential $\delta^n : A^n_\phi(X; \tau) \to A^{n+1}_\phi(X; \tau)$ by

$$\delta^n f(G/H)(x) = (\delta^n(G/H)c, b), \ f \in A^n_\phi(X; \tau), x \in X^H, f(x) = (c, b).$$

It is straightforward to check that $\{ A^n_\phi(X; \tau), \delta \}$ is a cochain complex. Furthermore $A^n_\phi(X; \tau)$ admits a graded algebra structure induced from the differential graded algebra $A$. The zero element of this algebra is given by the trivial lift $0$, defined by

$$0(G/H)_q(x) = (0^H_q, \theta(\tau)(G/H)_q(x)),$$

where $x \in X^H_q$ and $0^H_q$ is the zero of the abelian group $A(G/H)_q$. As before we use the notation $[\Phi X, Z^nA \times_{\tau(\underline{\pi})} \underline{W\underline{\pi}}]_{\underline{\pi}}$ to denote the set of vertical homotopy classes of liftings of $\theta(\tau)$.

**Proposition 4.4.** With the above notations, we have

$$H^n(A^n_\phi(X; \tau)) = [\Phi X, Z^nA \times_{\tau(\underline{\pi})} \underline{W\underline{\pi}}]_{\underline{\pi}}.$$

**Proof.** Clearly $\text{Ker}(\overline{\delta}^1) = (\Phi X, Z^nA \times_{\tau(\underline{\pi})} \underline{W\underline{\pi}})_{\underline{\pi}}$. We now show that

$$\text{Im}(\overline{\delta}^{n-1}) = \{ f \in ([\Phi X, Z^nA \times_{\tau(\underline{\pi})} \underline{W\underline{\pi}}]_{\underline{\pi}} \mid f \sim_0 0 \}.$$

Let $F : f \sim_0 0$. Consider the following left lifting problem in the closed model category $O_G S / \underline{W\underline{\pi}}$ ([DK93], [GJ99]).

Here the $O_G$-simplicial set $\Phi X \times \Delta[n]$ is defined by

$$(\Phi X \times \Delta[n])(G/H) = X^H \times \Delta[n], \ (\Phi X \times \Delta[n])(\hat{g}) = (g, id), n \geq 0.$$

We identify $\Phi X$ with $\Phi X \times \Delta[0]$. The canonical inclusions $\delta_0, \delta_1 : \Delta[0] \to \Delta[1]$ (see Section 1) induce natural inclusions $i_0, i_1 : \Phi X \to \Phi X \times \Delta[1]$. Note that $i_1$ is a trivial cofibration and $\overline{\delta}^{n-1}$ is a fibration in $O_G S / \underline{W\underline{\pi}}$. Hence the above left lifting problem has a solution $\tilde{F}$. Then $\tilde{F}i_0 \in A^{n-1}_\phi(X; \tau)$ such that $\overline{\delta}^{n-1}(\tilde{F}i_0) = f$. Therefore $f \in \text{Im}(\overline{\delta}^{n-1})$.

On the other hand, suppose that $f = \overline{\delta}^{n-1}h$ for $f \in A^n_\phi(X; \tau)$ and $h \in A^{n-1}_\phi(X; \tau)$. Then clearly $f \in ([\Phi X, Z^nA \times_{\tau(\underline{\pi})} \underline{W\underline{\pi}}]_{\underline{\pi}}$. Composing $h$ with
first factor projection map, we get a map $h' : \Phi X \to A^{n-1}$ of $O_G S$. But by the Lemma 4.3 $A^{n-1}$ is contractible. Let $H : \Phi X \times \Delta[1] \to A^{n-1}$ be a contracting homotopy for the $O_G$-simplicial set $A^{n-1}$. Then define $\bar{H} : \Phi X \times \Delta[1] \to A^{n-1}(X;\tau)$ by $\bar{H}(x,t) = (H(x,t),\theta(\tau)x)$. Clearly $\bar{H} : h \sim_\varnothing 0$ in $O_G S/W$. Hence $\delta^{-1} \circ \bar{H} : f \sim_\varnothing 0$. This proves the proposition for $n > 0$.

For $n = 0$, we note that $H^0(A^\phi_\circ(X;\tau)) = (\Phi X, Z^0A \times_{\tau,\phi} W_{\pi} W_{\pi})$ and two elements in the right hand side are homotopic if and only if they are equal. □

Observe that the fourth axiom of Definition 4.1 implies $Z^0A$ is an $O_G$-Eilenberg Maclane complex of type $(M,0)$ and hence by induction $Z^nA$ is an $O_G$-Eilenberg Maclane complex of type $(M,n)$. To justify this, consider the fibration

$$A^n(G/H) \to Z^{n+1}A(G/H)$$

with fiber $Z^nA(G/H)$, $H \leq G$. As noted in the Lemma 4.3 this is a PTCP with fibre $Z^nA(G/H)$. Therefore if $Z^nA(G/H)$ is minimal then so is $Z^{n+1}A(G/H)$. But $Z^0A(G/H)$, being simplicially trivial, is minimal. Hence by induction it follows that $Z^nA(G/H)$ is minimal for all $n$. Now applying the homotopy long exact sequence to the above fibration, and using the third axiom of the Definition 4.1 together with induction on $n$, we see that $Z^nA$ is an $O_G$-Eilenberg Maclane complex of type $(M,n)$. Hence it is isomorphic to the canonical model of $K(M,n)$. Therefore $(Z^nA \times_{\tau,\phi} W_{\pi}, p)$ is isomorphic to $(L_\phi(M,n), p)$ as objects in the slice category $O_G S/W$. So we have,

$$H^n(A^\phi_\circ(X;\tau)) = (\Phi X, Z^nA \times_{\tau,\phi} W_{\pi} W_{\pi})$$

$$\cong (\Phi X, L_\phi(M,n))W_{\pi}.$$ 

It follows from the Theorem 3.6 that

$$H^n(A^\phi_\circ(X;\tau)) \cong H^0_G(X;\phi,\tau).$$

Thus we have proved the following theorem.

**Theorem 4.5.** Suppose $A$ is an equivariant twisted Cartan cohomology theory. Then for every $G$-simplicial set $X$ together with an $O_G$-group $\pi$, an $O_G$-twisting function $\tau : \Phi X \to \pi$ and an action $\phi$ of $\pi$ on the abelian $O_G$-group $(Z^0A)_0$ there is an isomorphism of graded algebras

$$H^*_G(X;\tau,\phi) \cong H^*(A(X;\tau,\phi)),$$

where $A(X;\tau,\phi)$ denote the graded algebra $(A^\phi_\circ(X;\tau),\bar{\delta})$.

It has been shown in [MS10] that for a $G$-connected $G$-simplicial set $X$ with a $G$-fixed 0-simplex, the simplicial version of Bredon Illman cohomology with local coefficients can be interpreted as an equivariant twisted cohomology for $\pi = \pi X$ and the $O_G$-twisting function $\kappa$ as described in the Example 3.3 (cf. Theorem 4.7 of [MS10]). Combining it with the Theorem 4.5, we have the following result.
Theorem 4.6. Suppose $\mathcal{A}$ is an equivariant twisted Cartan cohomology theory. Given any $G$-connected $G$-simplicial set $X$ with a $G$-fixed $0$-simplex and an action $\phi$ of $\pi_X$ on $(\mathcal{Z}^0 \mathcal{A})_0$, let $L$ be the equivariant local coefficients (cf. Definition 3.1 and page 1020, section 3 [MS10]) determined by the $\pi_X$-module $(\mathcal{Z}^0 \mathcal{A})_0$ on $X$. Then

$$H^*_G(X; L) \cong H^*(\mathcal{A}(X; \kappa, \phi)).$$

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