Quantum chaos in classically non-chaotic systems

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One of the general goals in the field of quantum chaos is to establish a correspondence between the dynamics of classical chaotic systems and their quantum counterparts. For isolated systems in the absence of decoherence, this correspondence in dynamics usually persists over an extremely short time – the Ehrenfest time, logarithmic in $\hbar$ – because quantum-mechanical interference washes out classical chaos. We demonstrate that a new kind of drastic disagreement can occur between quantum and classical descriptions of the same model even within this early-time window. Remarkably, quantum mechanics appears capable of playing the opposite to its usual role and brings chaos to classically non-chaotic systems. Our calculations employ the out-of-time-ordered correlator (OTOC) – a diagnostic that has recently received a lot of attention as a useful tool to study quantum chaos. Specifically, we show that certain non-convex polygonal billiards, whose classical Lyapunov exponents are always zero, demonstrate a Lyapunov-like exponential growth of OTOC at early times with an $\hbar$-dependent Lyapunov rate. This behavior is sharply contrasted with the oscillatory early-time behavior of OTOC in convex polygonal billiards, which are also classically non-chaotic. These results suggest that in general, classical-to-quantum correspondence in dynamics may be violated even at early stages of quantum evolution before quantum interference comes into play.

Introduction

Quantum mechanics has a general effect that it washes out sharp features of classical dynamics due to the uncertainty principle and wave-like nature of quantum dynamics. This effect becomes crucial for chaotic systems, because sharp features such as sensitive dependence on initial conditions, also known as the butterfly effect, are eventually destroyed. In isolated systems, this suppression of the butterfly effect occurs after a logarithmically short time of semiclassical evolution, once a quantum state spreads across the entire system [1–3]. This time scale is known as the Ehrenfest time $t_E$ or the scrambling time.

It is important to note that although the scrambling time is very short in isolated systems, when decoherence by environment (or by an internal “bath” induced by the system itself) is taken into account, the system oftentimes resets back to this semiclassical regime. So, its relevance becomes significant and even explains why classical chaotic dynamics is observable at all [2–5]. There have also been alternative views on the long Ehrenfest-type “paradox” expressed in Refs. [6]. Regardless of the explanation, the behavior of quantum systems in the Ehrenfest window and the fate of classical-to-quantum correspondence in this regime are clearly of fundamental interest.

In this Letter, we demonstrate that in sharp contrast to conventional wisdom, quantum mechanics can induce exponential instabilities in models, which are classically non-chaotic. While our construction, described below, is specific to billiards, we believe that this surprising phenomenon can exist in a variety of dynamical systems.

A specific model we consider is based on a simple set of observations. Consider a classical “mathematical billiard” – i.e., motion of a point particle reflecting off of hard billiard’s walls, such as the polygonal black shape in Fig. 1, for example. It has been rigorously proven [7] that Kolmogorov-Sinai entropy and the closely related Lyapunov exponents of any polygonal billiard are strictly zero. Next, consider a “physical billiard” – a classical hard disc of radius $r_p$ reflecting off of the same polygonal walls. Clearly, this physical billiard is equivalent to

![FIG. 1. Outer black line: polygonal butterfly-shaped billiard. The area is unit. Inner blue line: effective mathematical billiard hosting a point particle classically equivalent to the polygonal billiard hosting a rigid circular particle of radius $r_p = \sigma \sqrt{\hbar \text{eff}}/2$ and zero moment of inertia. Note that the inward-pointing corners of the polygonal billiard are smoothened into circular arcs or radius $r_p$, making the effective mathematical billiard classically chaotic with positive Lyapunov exponent. Gray shaded region: a close sub-$r_p$ vicinity of the billiard wall: small changes of the billiard geometry within this region do not affect the early-time quantum dynamics. Middle red line: a smoothened billiard used for comparison purposes below.](image-url)
a mathematical billiard of a smaller size, since the particle’s center is not allowed to approach the walls of the physical billiard closer than \( r_p \) – the inner blue shape in Fig. 1. (We assume that the particle’s mass is concentrated in the center, and ignore rotational motion.) A crucial observation [8] is that this redrawing may give rise to a smoothing of sharp features of non-convex polygons, such as the black shape in Fig. 1. The resulting shape is no longer a polygon, and the obstruction for the Kolmogorov-Sinai entropy to vanish is removed. Indeed, the blue shape in our example in Fig. 1 is classically chaotic for small enough, but finite particle’s radius, with a positive Lyapunov exponent. Finally, consider a quantum particle embedded into a non-convex polygonal billiard. Semiclassical early-time dynamics of a quantum wave packet is similar to motion of a finite-size classical particle; i.e., classically chaotic motion in the physical billiard. As shown below, there is indeed the onset of quantum chaos in the classically non-chaotic systems of such kind, hence providing a drastic example of violation of the conventional view on the classical-to-quantum correspondence.

To diagnose this behavior, we employ the out-of-time-ordered correlator (OTOC). OTOC was introduced by Larkin and Ovchinnikov [9] in the context of quasiclassical approximation in the theory of superconductivity in disordered metals and used recently in the pioneering works by Kitaev [10] and Maldacena et al. [11] to define and describe many-body quantum chaos with an eye on fundamental puzzles in the black-hole physics. In the last few years, OTOC has become a standard tool to describe chaotic features in complex quantum systems (see e.g. Refs. [12]). It was shown in Refs. [13, 14] that the exponential growth of OTOC, although not always equal, is directly connected to the exponential divergences of orbits in the phase space of an effective classical system. In certain cases, such as the celebrated Sinai billiard [15] and Bunimovich stadium [16], it is straightforward to understand this classical limit. Below, we consider non-chaotic polygonal billiards instead. In a polygon, for any pair of trajectories – no matter how close the initial conditions are – one can identify the origin of each trajectory evolving the dynamics backward in time [7].

Note that in the ergodic hierarchy, which, in the order of increasing chaoticity consists of the following systems:

| Ergodic | Weakly Mixing (WM) | Strongly Mixing (SM) | Kolmogorov (K) | Bernoulli (B) |

polygonal billiards fall within the strongly mixing class at most (only K- and B-systems have a non-zero Kolmogorov-Sinai entropy; see e.g. Ref. [17] for a detailed discussion of the hierarchy). Interestingly, however, mixing property at the classical level is sufficient to generate Wigner-Dyson or intermediate energy-level statistics on the quantum side, as was shown, for example, in Ref. [18] for a family of irrational triangular billiards [19].

Model — We perform explicit calculations for a butterfly-shaped polygonal billiard shown in Fig. 1 (outer black line), but the conclusions apply to a large class of non-convex polygonal billiards. In particular, we have checked our results for another test case of the “deformed triangular” (quadrilateral) billiard (Fig. 2).

We launch a wave-packet with the initial wave-function

\[
\Psi_0(\mathbf{r}) \propto \exp \left[-\frac{(\mathbf{r} - \mathbf{r}_0)^2}{2 \hbar_{\text{eff}} \sigma^2} + i \frac{p_0 \cdot \mathbf{r}}{\hbar_{\text{eff}}}ight],
\]

(decompose it into billiard’s energy eigenstates, and evolve it accordingly. This requires numerically solving the Schrödinger equation for the billiard:

\[
-\frac{\hbar_{\text{eff}}^2}{2} \nabla^2 \Psi(x, y) = E \Psi(x, y), \quad \Psi(\mathbf{r}) \bigg|_{\mathbf{r}\in\text{billiard walls}} = 0,
\]

where \( \hbar_{\text{eff}} = h/(p_0 \sqrt{A}) \), \( A \) is the billiard’s area, and \( p_0 = |\mathbf{p}_0| \) is the wave-packet’s average momentum. \( A = 1 \) and \( p_0 = 1 \) are chosen as the units along with the particle’s mass \( m = 1 \). The butterfly-shaped billiard has two reflection symmetries with respect to \( x \to -x \) and \( y \to -y \). Thus, its eigenstates fall into four parity classes. In order to enforce these parities and speed up the calculations, one solves the eigenvalue problem on a quarter of the billiard imposing the Dirichlet and/or Neumann boundary conditions on each cut, which determines the parity class of solutions. We solve these four boundary-value problems for the Laplace operator numerically using the finite-element method, and find all eigenstates of each class up to a certain energy cutoff. The accuracy of the numerical solution generally decreases with the number of found eigenstates [20]. We use the Weyl formula for the number of modes [21] to control it. The Weyl law sets the asymptotic behavior of the average number \( \mathcal{N}(E) \) of eigenstates below energy \( E \) as:

\[
\mathcal{N}(E) \simeq \frac{A}{4\pi} \frac{2}{\hbar_{\text{eff}}^2} E - \frac{P}{4\pi} \sqrt{\frac{2}{\hbar_{\text{eff}}^2} E}, \quad E \to \infty,
\]

FIG. 2. Deformed triangular billiard (unit area). All angles are incommensurate with \( \pi \). For a finite particle, the inward-pointing corner gets smoothened analogous to those in Fig. 1.
where \( P \) is the billiard’s perimeter. For our present purposes, it is sufficient to use around \( N_{\text{max}} = 10^4 \) eigenstates.

Due to the lack of narrow outer corners, the butterfly-shaped billiard allows for a relatively long lifetime of the initial minimal-uncertainty wave packet until this packet becomes completely scrambled and loses classical-like dynamics (see Fig. 3). Along with this billiard, we introduce an effective mathematical billiard (Fig. 1, inner blue line) that is obtained by tracing the set of positions available to the center of a circular particle of radius \( \sigma \sqrt{h_{\text{eff}}/2} \) inside the polygonal butterfly billiard. The squeezing parameter \( \sigma \) is defined in Eq. (1).

**Diagnostic tool** — To quantify quantum chaotic dynamics, we use the OTOC \([10–14]\), defined as follows:

\[
C(t) = -\left\langle \left\langle \hat{x}(t), \hat{p}_x(0) \right\rangle^2 \right\rangle, \tag{4}
\]

where \( \hat{x}(t) \) and \( \hat{p}_x(t) \) are the Heisenberg operators of the \( x \)-components of particle’s position and momentum. As was first pointed out by Larkin and Ovchinnikov \([9]\), the OTOC probes the sensitivity of quasiclassical trajectories to initial conditions, because \( \hat{p}_x(0) = -i\hbar \partial / \partial x(0) \), and hence \( C(t) = \hbar^2 \left\langle \left\langle \frac{\partial x(t)}{\partial x(0)} \right\rangle^2 \right\rangle \). Therefore, the classical Lyapunov-like growth is anticipated at early times, \( C(t) \propto \exp(2\lambda t) \), for a chaotic system, with \( \lambda \) related to its Lyapunov exponent in the respective subspace.

As was shown in Ref. \([14]\), whether OTOC actually exponentially grows or not, depends on an initial quantum state and on the existence of a finite time-window between the first collision and the Ehrenfest time. For billiards, a natural choice of the initial state is the minimal-uncertainty wave-packet, Eq. (1). The scrambling (Ehrenfest) time in a chaotic system is logarithmically short: \( t_E = \ln(h_{\text{eff}}^{-1}) / \lambda_{\text{cl}} \), where \( \lambda_{\text{cl}} \) is the positive Lyapunov exponent of the classical counterpart of the system. This estimate is based on the fact that, in contrast to non-chaotic systems where the spreading of wave-packets is algebraic in time, the spreading is typically exponential in chaotic systems (quantum counterparts of K- and B-systems from the ergodic hierarchy). Extending the Ehrenfest window to include the ergodic (long-time) semiclassical behavior (which is required to define the global Lyapunov exponents in chaotic systems) is an exponentially demanding numerical task. The local finite-time Lyapunov exponents can be defined, although they fluctuate at these short times \([14]\).

**Breakdown of classical-to-quantum correspondence** — As shown in Ref. \([14]\), in quantum billiards, which are classically chaotic, the exponential growth of OTOC hinges on the classical Lyapunov instability and extends up until the Ehrenfest time. After that, the packet spreads across the entire system, and no further quantum exponential growth is possible. The classical counterpart of OTOC can be written as:

\[
C_{\text{cl}}(t)/\hbar_{\text{eff}}^2 = \lim_{\Delta x(0) \to 0} \left\langle \left\langle \frac{\Delta x(t)}{\Delta x(0)} \right\rangle^2 \right\rangle, \tag{5}
\]

where \( \left\langle \ldots \right\rangle \) denotes classical phase-space average over a Gaussian Wigner function corresponding to the initial quantum packet in Eq. (1), and \( \Delta x \) is the distance along the \( x \)-axis between a pair of trajectories starting near some point in the phase space. \( C_{\text{cl}}(t) \) agrees with its quantum version all the way up to \( t_E \). After that, they deviate from each other: the quantum OTOC saturates, while \( C_{\text{cl}}(t) \) continues to grow exponentially. In the polygonal billiards, there are no positive classical Lyapunov exponents, and the corresponding classical OTOC does not grow exponentially at any time. However, remarkably, the quantum-mechanical OTOC in non-convex polygonal billiards shows a clear exponential growth at early times that has no origin in the classical polygonal mathematical billiards. For the case of the butterfly-shaped billiard, Fig. 4 demonstrates such a growth at var-
effects of non-smoothness of the polygonal boundary, classically chaotic (see Fig. 1, middle red line). Wave-packet’s size) – the closely related system, which is liard (i.e., smoothened on lengthscales smaller than the region in Fig. 1, and, in particular, compare the behavior of that by changing the boundary within the shaded gray polygonal variations. We check for possible consequences in finite-curvature regions, except for specially chosen slight variation of the wall’s shape almost always results of closed curves corresponding to billiard walls. A tute an everywhere dense, measure-zero set in the space one.

Next, such mixing, but non-chaotic systems constitute an everywhere dense, measure-zero set in the space of closed curves corresponding to billiard walls. A slight variation of the wall’s shape almost always results in finite-curvature regions, except for specially chosen polygonal variations. We check for possible consequences of that by changing the boundary within the shaded gray region in Fig. 1, and, in particular, compare the behavior of OTOC in the polygonal and in a weakly rounded billiard (i.e., smoothened on length scales smaller than the wave-packet’s size) – the closely related system, which is classically chaotic (see Fig 1, middle red line).

Finally, via this comparison, we also check for possible effects of non-smoothness of the polygonal boundary, such as diffraction, as in the case in the quantum baker’s map [22]. Note, however, a major difference between the Lyapunov behaviors of this dynamical system and our billiards: the latter do not have a classical Lyapunov exponent at all and the exponential growth is a purely quantum effect.

We find that the early-time quantum-mechanical evolution of the wave-packet cannot distinguish between the wall-shape variations well below the size of the wave-packet, and displays the same OTOC growth rate in the polygonal and the smoothened billiard (see Fig. 5). The smaller the wave-packet, the smaller the effective radius of the corner becomes. For the non-convex billiards, it makes the inner caps more and more defocusing, increasing the instability around them and causing the quantum “Lyapunov exponent” $\lambda$ to grow as $h_{\text{eff}}$ decreases, as shown in Fig. 4. Note that our choice of initial condition specifically picks the contribution of the quasiclassical trajectories in the $r_p\text{-vicinity}$ of an inner corner in order to amplify the effect. In contrast, the outer corners, stay focusing after rounding. Therefore, they cause a more limited and delayed effect on the dynamics, such as the case in Fig 6, which shows no exponential instability in the initial evolution, with slow algebraic growth appearing at later times.

**Non-quantum-chaotic dynamics in convex polygonal billiards** — Non-convexity, which is crucial to finite-size renormalization of non-chaotic billiards into chaotic ones, is indeed the main effect causing the emergence of quantum Lyapunov growth in classically non-chaotic systems. In order to verify it, we consider an irrational triangular billiard formed by two bottom and the topmost corners of the billiard in Fig. 2. Classical finite-size renormalization of the triangle keeps it a triangle, and does not change its ergodicity status, although small variations of the boundary well within the wave-packet size can still make this billiard weakly chaotic by smoothing the outer corners. Note here that upon quantization, the level statistics of irrational triangular billiards – the most widely used “quantum-chaotic” diagnostic – can still be close to Wigner-Dyson surmise [18], putting it outside of the Bohigas-Giannoni-Schmit conjecture [23]. We have calculated the OTOC for the irrational triangle and have been unable to find signatures of an exponential growth. As one can see from Fig. 6, the early-time behavior is dominated by oscillations, which at later times do turn into an algebraic growth. The late-time algebraic growth is likely related to the moderate effects of smoothing of the outer corners as well as to the mixing dynamics itself in the triangular billiard. Regardless, there is no quantum exponential Lyapunov instability in contrast to the non-convex billiard in Fig. 1, which supports our hypothesis that quantum-mechanical finite-size effects push the non-convex polygonal billiards up the ergodic hierarchy.
FIG. 6. Logarithm of OTOC in an irrational triangular billiard (upper blue line and open circles). OTOC oscillates and slowly grows at short times. No exponential Lyapunov-like growth is present. The related quantity $L(t)$ introduced in Fig. 5 is shown for comparison (lower red line and asterisks).

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[1] M. Toda and K. Ikeda, Phys. Lett. A 124, 165 (1987); G. Berman and G. Zaslavsky, Physica A 91, 450 (1978); Y. Gu, Physics Letters A 149, 95 (1990); W. H. Zurek and J. P. Paz, Phys. Rev. Lett. 72, 2508 (1994); Physica D: Nonlinear Phenomena 83, 300 (1995).

[2] W. H. Zurek, Physica Scripta T76, 186 (1998).

[3] M. Berry, in Quantum Mechanics, Scientific Perspectives on Divine Action, edited by R. J. Russell, P. Clayton, K. Wegter-McNelly, and J. Polkinghorne (CTNS/Vatican Observatory, 2001) pp. 41–54.

[4] W. H. Zurek and J. P. Paz, “Why we don’t need quantum planetary dynamics: Decoherence and the correspondence principle for chaotic systems,” in Epistemological and Experimental Perspectives on Quantum Physics, edited by D. Greenberger, W. L. Reiter, and A. Zeilinger (Springer Netherlands, Dordrecht, 1999) pp. 167–177.

[5] M. Schlosshauer, Found. Phys. 38, 796 (2008).

[6] N. Wiebe and L. E. Ballentine, Phys. Rev. A 72, 022109 (2005); L. Ballentine, Found. Phys. 38, 916 (2008).

[7] A. N. Zemlyakov and A. B. Katok, Math. notes Acad. Sci. USSR 18, 760 (1975), [Mat. Zametki 18, 291 (1975)]; C. Boldrighini, M. Keane, and F. Marchetti, Ann. Probab. 6, 532 (1978).

[8] M. Bolding, Topics in Dynamics: First Passage Probabilities and Chaotic Properties of the Physical Wind-Tree Model, Ph.D. thesis, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA. (2018).

[9] A. Larkin and Yu. N. Ovchinnikov, Sov. Phys. – JETP 28, 1200 (1969), [Zh. Eksp. Teor. Fiz. 55, 2262 (1969)].

[10] A. Kitaev, KITP talk, online/entangled15/kitaev/ (Nov 10, 2014).

[11] J. Maldacena, S. H. Shenker, and D. Stanford, J. High Energy Phys. 2016:106, 1 (2016).

[12] D. A. Roberts, D. Stanford, and L. Susskind, J. High Energy Phys. 2015, 51 (2015); D. A. Roberts and B. Swingle, Phys. Rev. Lett. 117, 091602 (2016); Y. Huang, Y.-L. Zhang, and X. Chen, Ann. Phys. 529, 1600318 (2017); R. Fan, P. Zhang, H. Shen, and H. Zhai, Science Bulletin 62, 707 (2017); Y. Chen, arXiv:1608.02765 (2016); B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Phys. Rev. A 94, 040302 (2016); B. Swingle and D. Chowdhury, Phys. Rev. B 95, 060201 (2017); N. Y. Yao, F. Grusdt, B. Swingle, M. D. Lukin, D. M. Stamper-Kurn, J. E. Moore, and E. A. Demler, arXiv:1607.01801 (2016); M. Mezei and D. Stanford, J. High Energy Phys. 2017, 65 (2017); S. V. Syzranov, A. V. Gorshkov, and V. M. Galitski, arXiv:1709.09296 (2017); V. Khemani, D. A. Huse, and A. Nahum, Phys. Rev. B 98, 144304 (2018); S. Xu and B. Swingle, arXiv:1802.00801 (2018).

[13] E. B. Rozenbaum, S. Ganeshan, and V. Galitski, Phys. Rev. Lett. 118, 086801 (2017).

[14] E. B. Rozenbaum, S. Ganeshan, and V. Galitski, arXiv:1801.10591 (2018).

[15] V. G. Sinai, Russian Math. Surv. 25, 137 (1970).

[16] L. A. Bunimovich, Functional Analysis and Its Applications 8, 254 (1974); Commun. Math. Phys. 65, 295 (1979).

[17] V. G. Sinai, Introduction to Ergodic Theory, Mathematical Notes (Book 18) (Princeton University Press, 1977).

[18] T. Araújo Lima, S. Rodríguez-Pérez, and F. M. de Aguiar, Phys. Rev. E 87, 062902 (2013).

[19] G. Casati and T. Prosen, Phys. Rev. Lett. 83, 4729 (1999).

[20] V. Veuveline, J. Comput. Phys. 184, 321 (2003).

[21] H. P. Baltes and E. R. Hilf, Spectra of finite systems (Mannheim: Bibliographisches Institut, 1976).

[22] A. Lakshminarayan, Phys. Rev. E 99, 012201 (2019).

[23] G. Casati, F. Valz-Gris, and I. Guarnieri, Lett. Nuovo Cimento 28, 279 (1980); O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. 52, 1 (1984).