SEMIANNIHILATOR SMALL SUBMODULES WITH RESPECT TO AN ARBITRARY SUBMODULE

S. RAJAEE ∗, F. FARZALIPOUR, AND M. POYAN

Abstract. In this paper we introduce a new concept namely $T$-semiannihilator small ($T$-sa-small for short) submodules of an $R$-module $M$ with respect to an arbitrary submodule $T$ of $M$ which is a generalization of the concept of semiannihilator small submodules. We say that a submodule $N$ of $M$ is $T$-sa-small in $M$ provided for each submodule $X$ of $M$ such that $T \subseteq N + X$ implies that $\text{Ann}(X) \ll (T : M)$. We investigate some new results concerning to this class of submodules. Among various results we prove that for a prime module $M$ over a semisimple ring $R$, if $N$ is a sa-small submodule of $M$, then for every submodule $T$ of $M$ such that $N \subsetneq T$, $N$ is also a $T$-sa-small submodule of $M$. For a faithful finitely generated multiplication module $M$, we prove that $N$ is a $T$-sa-small submodule of $M$ if and only if $(N : M)$ is a $(T : M)$-sa-small ideal of $R$.

1. Introduction

Throughout this paper, $R$ will denote a commutative ring with identity $1 \neq 0$ and $M$ a nonzero unital $R$-module. We use the notations $\subseteq$ and $\leq$ to denote inclusion and submodule. A nonempty subset $S$ of $R$ is said to be a multiplicatively closed set (briefly, m.c.s) of $R$ if $0 \notin S$, $1 \in S$ and $st \in S$ for each $s, t \in S$. For an $R$-module $M$, the set of all submodules of $M$, denoted by $\text{L}(M)$ and also $\text{L}^*(M) = \text{L}(M) \setminus \{M\}$. As usual, the rings of integers and integers modulo $n$ will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. A module $M$ on a ring $R$ (not necessarily commutative) is called prime if for every nonzero submodule $K$ of $M$, $\text{Ann}(K) = \text{Ann}(M)$. An $R$-module $M$ is called a multiplication module, if every submodule $N$ of $M$ has the form $N = IM$ for some ideal $I$ of $R$, and in this case, $N = (N :_R M)M$, see [5]. A submodule $N$ of $M$ is called small (superfluous) which is denoted by $N \ll M$, if for every submodule $L$ of $M$, $N + L = M$, implies that $L = M$. Clearly, the

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∗Corresponding author.

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zero submodule of every nonzero module is superfluous. The Jacobson radical of $M$, denoted by $J(M)$, is the intersection of all maximal submodules of $M$ and also it is the sum of all small submodules of $M$, i.e., $J(M) = \cap_{N \in \text{Max}(M)} N = \sum_{N \ll M} N$. If $M$ does not have maximal submodules, we put $J(M) = M$. Consequently, if $J(M)$ is a small submodule of $M$, then $J(M)$ is the largest small submodule of $M$. We refer the reader to [3, 11] for the basic properties and more information on small submodules. We know that if $M$ is a semisimple module, then the zero submodule is the only small submodule of $M$ and $M$ is the only essential submodule of $M$.

Gilmer [7, p.60] has defined the concept of cancellation ideal to be the ideal $I$ of $R$ which satisfies the following: Whenever $AI = BI$ with $A$ and $B$ are ideals of $R$ implies $A = B$. Mijbass in [10] has generalized this concept to modules. An $R$-module $M$ is called a cancellation module whenever $IM = JM$ with $I$ and $J$ are ideals of $R$ implies $I = J$.

We recall that $R$ is a von Neumann regular ring (associative, with 1, not necessarily commutative) if for every element $a$ of $R$, there is an element $b \in R$ with $a = aba$. These rings are characterized by the fact that every left $R$-module is flat.

2. Preliminaries and Notations

In [2], the authors introduced the concept of annihilator-small submodules of any right $R$-module $M$. For an unitary right $R$-module $M$ on an associative ring $R$ with identity they called a submodule $K$ of $M_R$ annihilator-small if $K + T = M$, such that $T$ is a submodule of $M_R$, implies that $\ell_S(T) = 0$, where $\ell_S(T)$ indicates the left annihilator of $T$ over $S = \text{End}(M_R)$.

A submodule $N$ of $M$ is called an $R$-annihilator-small (breifly $R$-a-small) submodule of $M$, if $N + X = M$ for some submodule $X$ of $M$, implies that Ann($X$) = 0, see [1]. We use the notation $N \ll^a M$ to indicate this concept. In [12], the author introduced the concept of a semiannihilator small submodule $N$ of a module $M$ on a commutative ring $R$ with identity $1 \neq 0$ such that $N$ is called semiannihilator small (sa-small for short), denoted by $N \ll^a M$, if for every submodule $L$ of $M$ with $N + L = M$ implies that Ann($L$) $\ll R$. It is clear that every R-a-small submodule is sa-small, but the converse is not true. Also a sa-small submodule need not be small, for example consider $M = \mathbb{Z}$ as a $\mathbb{Z}$-module, every proper submodule of $\mathbb{Z}$ is a sa-small submodule whereas the zero submodule of $\mathbb{Z}$ is the only small submodule of $\mathbb{Z}$. Let $M$ be a faithful $R$-module, then clearly every small submodule is
an R-a-small submodule and hence it is a sa-small submodule. The converse is not true, because consider the \( \mathbb{Z}_8 \)-module \( M = \mathbb{Z}_8 \), then \( \mathbb{Z}_8 = \mathbb{Z}_8 + \langle 2 \rangle \) and \( \text{Ann}(\langle 2 \rangle) = \langle 4 \rangle \neq \langle 0 \rangle \), whereas \( \text{Ann}(\langle 2 \rangle) \ll \mathbb{Z}_8 \).

Similarly, an ideal \( I \) is a sa-small ideal of a ring \( R \), if it is a sa-small submodule of \( R \) as an \( R \)-module, see [12, Definition 2.1]. We know that an ideal \( I \) of \( R \) is small in \( R \) if and only if \( I \subseteq J(R) \). Therefore a submodule \( N \) of a module \( M \) is sa-small in \( M \) if \( N + X = M \) for a submodule \( X \) of \( M \), implies that \( \text{Ann}(X) \ll J(R) \), see [12, Proposition 2.3]. A non-trivial \( R \)-module \( M \) is called a semiannihilator hollow (sa-hollow for short) module if every proper submodule of \( M \) is sa-small in \( M \), see [12, Definition 3.1]. An \( R \)-epimorphism \( f : M \to N \) is called a sa-small epimorphism whenever \( \text{Ker}(f) \ll_{sa} M \).

An \( R \)-module \( M \) is said to be a comultiplication module if for every submodule \( N \) of \( M \) there exists an ideal \( I \) of \( R \) such that \( N = \text{Ann}_M(I) \). An \( R \)-module \( M \) satisfies the double annihilator condition (DAC for short), if for each ideal \( I \) of \( R \), \( I = \text{Ann}_R(\text{Ann}_M(I)) \). Also \( M \) is said to be a strong comultiplication module, if \( M \) is a comultiplication module which satisfies DAC, see [4].

### 3. Sa-small submodules w.r.t. an arbitrary submodule

In this section we generalize the concept of sa-small submodules to the \( T \)-sa-small submodules of \( M \) with respect to an arbitrary submodule \( T \) of \( M \). Moreover, we investigate some new other properties of the sa-small submodules of an \( R \)-module \( M \) and we will generalize these properties to this new class of submodules of \( M \).

**Definition 3.1.** Let \( M \) be an \( R \)-module and let \( T \) be an arbitrary submodule of \( M \).

(i) We say that a submodule \( N \) of \( M \) is a \( T \)-semiannihilator small (berifly, \( T \)-sa-small) submodule of \( M \), denoted by \( N \ll_{sa} T \ M \), provided for every nonzero submodule \( X \leq M \) such that \( T \subseteq N + X \) implies that \( \text{Ann}(X) \ll (T :_R M) = \text{Ann}_R(M/T) \). Equivalently, if for a submodule \( X \) of \( M \), \( \text{Ann}(X) \) is not small in \( (T :_R M) \), then \( T \nsubseteq N + X \).

(ii) We say that \( M \) is a \( T \)-sa-small hollow module if every submodule \( N \) of \( M \) is a \( T \)-sa-small submodule of \( M \). In particular, for an arbitrary ideal \( A \) of \( R \), we say that an ideal \( I \) of \( R \) is an \( A \)-sa-small ideal of \( R \) if \( I \) is an \( A \)-sa-small submodule of \( R \) as an \( R \)-module. We shall denote the sum of all \( T \)-sa-small submodules of \( M \) by \( J_{T}^{sa}(M) \).
(iii) Let $f : M \to N$ be an $R$-epimorphism and let $T$ be an arbitrary submodule of $M$, we say that $f$ is a $T$-sa-small epimorphism in case $\ker(f) \ll_{T}^{sa} M$.

We denote the set of all small (resp. sa-small, $T$-sa-small) submodules of $M$ by $S(M)$ (resp. $S^{sa}(M)$, $S_{T}^{sa}(M)$).

**Note 3.2.** Let $M$ be an $R$-module and let $T$ be an arbitrary submodule of $M$.

(i) If we take $T = M$, then the notions of $T$-sa-small submodules and sa-small submodules are equal.
(ii) Let $T = 0$, then for submodules $N, X$ of $M$, since $0 \subseteq N + X$ hence $N \ll_{0}^{sa} M$ implies that $\text{Ann}(X) \ll \text{Ann}(M)$. Now since $\text{Ann}(M) \subseteq \text{Ann}(X)$ therefore $\text{Ann}(X) = \text{Ann}(M)$. This is impossible because a nonzero module $M$ is never small in itself. It concludes that a nonzero module $M$ has no any 0-sa-small submodule.
(iii) If $T \neq 0$, then $N \ll_{T}^{sa} M$ implies that $T \not\subseteq N$, otherwise, $T \subseteq N + 0$ conclude that $R = \text{Ann}(0) \ll (T : M)$ which is impossible. Let $M$ be a finitely generated $R$-module and let $T$ be a nonzero proper arbitrary submodule of $M$ then by Zorn’s lemma there exists a maximal submodule $N$ of $M$ such that $T \subseteq N$. This implies that $N$ is not a $T$-sa-small submodule of $M$.

**Theorem 3.3.** Let $M$ be an $R$-module and let $T$ be a submodule of $M$.

The following assertions hold.

(i) Every $T$-sa-small submodule of $M$ is a sa-small submodule of $M$.
(ii) If $M$ is a faithful prime $R$-module, then $M$ is a $T$-sa-small hollow module.
(iii) If $M$ is a prime module on a semisimple ring $R$ and $N \ll^{sa} M$, then for every submodule $T \supseteq N$ of $M$, $N \ll_{T}^{sa} M$.

**Proof.** (i) Assume that $N \ll_{T}^{sa} M$ and $N + K = M$ for some module $K$ of $M$. Since $T \subseteq N + K$ and $N \ll_{T}^{sa} M$ hence $\text{Ann}(K) \ll (T : M) \leq R$. By virtue of [11, Remark 2.8, (2)], $\text{Ann}(K) \ll R$ this implies that $N \ll_{T}^{sa} M$.

(ii) By hypothesis, $T \subseteq N + X$ implies that $0 = \text{Ann}(X) = \text{Ann}(M)$ which is small in $(T : M)$.

(iii) By [11, Proposition 3.5], $M$ is a semisimple module, hence $M = T + T'$ for some submodule $T'$ of $M$. Suppose that $N \ll^{sa} M$ and $T \subseteq N + K$ for some submodule $K$ of $M$. This implies that $M = T + T' \subseteq N + K + T'$ and so $N + K + T' = M$. Since $N \ll^{sa} M$ hence
Ann($K + T'$) $\ll R$. Since $M$ is prime hence

$$\text{Ann}(K + T') = \text{Ann}(K) \subseteq (T : M) \subseteq (T : N) = R.$$ 

By virtue of [11, Remark 2.8, (3)], $\text{Ann}(K) \ll (T : M)$ since $R$ is semisimple and the proof is complete.

The following example shows that in general the concepts of small submodules and sa-small submodules are independent of each other. In (iii) we show that the converse of Theorem 3.3 is not true in general and also in (iv) we show that for $R$-modules $M, M'$, if $f : M \to M'$ is an $R$-epimorphism, then the image of a $T$-sa-small submodule of $M$ need not be an $f(T)$-sa-small submodule in $M'$.

**Example 3.4.** Let $Z$ and $Z_n$ be the rings of integers and integers modulo $n$, respectively.

(i) Consider $M = Z_6$ as a $Z$-module. Since $\text{Ann}(Z_6) = 6Z$ is not a small ideal of $Z$, hence $(0) \notin S^{sa}(Z_6)$ whereas $(0) \in S(Z_6)$. We note that the only nonzero proper submodules of $Z_6$ are $N = (2)$ and $L = (3)$ where $N + L = Z_6$ and both of $\text{Ann}(N) = 3Z$ and $\text{Ann}(L) = 2Z$ are not small ideals of $Z$. It concludes that $S^{sa}(Z_6) = \emptyset$ whereas $S(Z_6) = \{ (0) \}$.

(ii) Take $M = Z$ as a $Z$-module. We know that $kZ + sZ = Z$ if and only if $(k, s) = 1$. Since for every submodule $sZ$ of $Z$, $\text{Ann}_Z(Z) = \text{Ann}_Z(sZ) = 0 \ll Z$ hence $kZ \ll^{sa} Z$ for every proper submodule $kZ$ of $Z$. It concludes that $S^{sa}(Z) = L^*(Z)$ whereas $S(Z) = \{ 0 \}$.

(iii) We take the $Z$-module $M = 2Z \times Z_8$. Then $N = \langle (0, 0) \rangle$ is a sa-small submodule of $M$ but $N$ is not a $T$-sa-small submodule of $M$ for submodule $T = \langle (0, 4) \rangle$ of $M$ since $T \subseteq N + \langle (0, 2) \rangle$ whereas $\text{Ann}(\langle (0, 2) \rangle) = 4Z$ is not small in $(T : M) = 0$.

(iv) Consider the natural $Z$-epimorphism $\pi : Z \to Z_8$ where $\pi(n) = \bar{n}$. Take $N = 0$ and $T = 2Z$. Clearly $0 \ll^{sa}_{2Z} Z$, because we have $2Z \subseteq 0 + 2Z$ and also $2Z \subseteq 0 + Z$ and then

$$0 = \text{Ann}(2Z) \ll (2Z :_Z 2Z) = 2Z,$$

$$0 = \text{Ann}(Z) \ll (2Z :_Z 2Z) = 2Z.$$

But $\pi(N) = \pi(0) = \langle 0 \rangle$ is not $\pi(T)$-sa-small submodule of $Z_8$ since $\pi(T) = \langle \bar{2} \rangle$ and $\langle \bar{2} \rangle \subseteq \langle 0 \rangle + Z_8$ whereas $\text{Ann}(Z_8) = 8Z$ is not small in $(\langle \bar{2} \rangle : Z_8) = 2Z$.

**Note 3.5.** Let $M$ be an $R$-module and let $T$ be an arbitrary submodule of $M$. For every submodule $K$ of $M$, $K \ll^{sa}_T M$ if and only if $R$-epimorphism $p_K : M \to M/K$ is a $T$-sa-small epimorphism.
A submodule $N$ of an $R$-module $M$ is said to be completely irreducible if $N = \bigcap_{i \in I} N_i$ where $\{N_i\}_{i \in I}$ is a family of submodules of $M$, then $N = N_i$ for some $i \in I$. It is easy to see that every submodule of $M$ is an intersection of completely irreducible submodules of $M$.

**Theorem 3.6.** Let $M$ be an $R$-module and $N \leq M$.

(i) Let $T \leq T' \leq M$. If $N \trianglelefteq_{T'}^s M$, then $N \trianglelefteq_{T'}^s M$.

(ii) If $T = T_1 \cap \cdots \cap T_s$ and $N \trianglelefteq_{T_i}^s M$ for any $1 \leq i \leq s$. Conversely, if $T$ is a completely irreducible submodule and for any $1 \leq i \leq s$, $N \trianglelefteq_{T_i}^s M$, then $N \trianglelefteq_{T}^s M$.

(iii) If $T = K$ and $N \trianglelefteq_{T}^s M$, then $N \trianglelefteq_{T}^s K$.

(iv) If $T = T_1 + \cdots + T_n$ for some submodules $T_i$ of $M$ and $N \leq T_i^s M$ for some $1 \leq i \leq n$, then $N \leq T_i^s M$.

**Proof.**

(i) Assume that $T' \subseteq N + X$ for some submodule $X$ of $M$, then $T \subseteq N + X$ and so $\text{Ann}(X) \trianglelefteq (T : M) \leq (T' : M)$. By [11, Remark 2.8, (2)], $\text{Ann}(X) \trianglelefteq (T' : M)$ as we needed.

(ii) It is clear by (i). Conversely, since $T$ is a completely irreducible submodule hence there exists $1 \leq i \leq s$ such that $T_i = T$ and the proof is complete again using by (i).

(iii) Assume that $T \subseteq N + L$ for some submodule $L$ of $K$. Then $\text{Ann}(L) \trianglelefteq (T : M) \leq (T : K)$ and this implies that $\text{Ann}(L) \trianglelefteq (T : K)$ and therefore $N \trianglelefteq_{T}^s K$.

(iv) The proof is straightforward by (i). \qed

**Theorem 3.7.** The following assertions hold.

(i) $S(R) \subseteq S^{sa}(R)$.

(ii) If $R$ is an Artinian ring and $I \trianglelefteq_{J(R)}^s R$, then for every maximal ideal $m$ of $R$, $I \trianglelefteq_{m}^s R$.

(iii) Let $M = N \oplus K$ be a multiplication module such that $N, K$ are finitely generated submodules of $M$. Then $N$ and $K$ are not sa-small in $M$.

**Proof.**

(i) Let $I \in S(R)$ and $I + J = R$ for some ideal $J$ of $R$. Then $J = R$ and hence $\text{Ann}(J) = \text{Ann}(R) = 0$ which is a small ideal of $R$. Therefore $I \in S^{sa}(R)$.

(ii) Assume that $\text{Max}(R) = \{m_1, \ldots, m_s\}$, then $J(R) = \cap_{i=1}^s m_i$. The proof is clear by Theorem 3.6 (i).

(iii) By [6, Corollary 2.3], $\text{Ann}(N) + \text{Ann}(K) = R$. If $N \trianglelefteq_{M}^s$, then since $M = N + K$ hence $\text{Ann}(K) \trianglelefteq R$ and so $\text{Ann}(N) = R$. It concludes that $N = 0$ which is impossible. \qed

**Theorem 3.8.** Let $M$ be an $R$-module. The following assertions hold.

(i) $M \notin S^{sa}(M)$. 

(ii) If $M$ is a completely irreducible submodule of $M$, then $M = R$.
(ii) $0 \in S^{sa}(M)$ if and only if $\text{Ann}(M) \ll R$.

(iii) If $R$ is a simple ring, then $L^*(M) = S^{sa}(M)$.

(iv) Let $m \in \text{Max}(R) \cap S^{sa}(R)$. If $x \notin m$, then $\text{Ann}(Rx) \subseteq J(R)$.

(v) If $M$ is a prime module with $\text{Ann}(M) \ll R$, then $L^*(M) = S^{sa}(M)$. Moreover, in this case, $S(M) \subseteq S^{sa}(M)$.

**Proof.** (i) Assume that $M \in S^{sa}(M)$, then the equality $M + 0 = M$ implies that $\text{Ann}(0) = R$ is a small ideal of $R$ which is impossible.

(ii) The proof is straightforward.

(iii) Let $R$ be a simple ring and let $N$ be a proper submodule of $M$ such that $N + X = M$ for some submodule $X$ of $M$. If $\text{Ann}(X) = 0$, then $\text{Ann}(X) \ll R$ and the proof is complete. If $\text{Ann}(X) = R$, then $X = 0$ and hence $N = M$ which is contradiction.

(iv) Suppose that $x \in R - m$, then $m + Rx = R$ hence $\text{Ann}(Rx) \ll R$ since $m \in S^{sa}(R)$. This implies that $\text{Ann}(Rx) \subseteq J(R)$.

(v) It is clear. □

**Corollary 3.9.** Every proper submodule of a faithful prime module $M$ is a sa-small submodule of $M$.

**Theorem 3.10.** Let $R$ be a commutative ring. The following statements are true.

(i) If $R$ has a nonzero comaximal sa-small ideal, then $R$ is not semisimple.

(ii) Every sa-small hollow semisimple ring is simple.

(iii) Let $R$ be a sa-small hollow ring. If $x \in Z(R)$, then $R \neq Rx + Ry$ for some element $y \in R$ and also $1$ is the only nonzero idempotent element of $R$.

(iv) If $R$ is a von Neumann regular ring, then none of the finitely generated ideals of $R$ is a sa-small ideal of $R$.

(v) Let $R$ be an integral domain and let $M$ be a faithful multiplication module, then $L^*(M) = S^{sa}(M)$.

**Proof.** (i) Suppose that $I$ is a nonzero comaximal sa-small ideal of $R$, then there exists an ideal $J$ of $R$ with $I + J = R$. We claim that $I \cap J \neq 0$. Assume $I \cap J = 0$ and so $IJ = 0$. Hence $R = I + J \subseteq \text{Ann}(J) + \text{Ann}(I)$, then $\text{Ann}(I) + \text{Ann}(J) = R$. Now $I \ll^{sa} R$, implies that $\text{Ann}(J) \ll R$ and thus $\text{Ann}(I) = R$. It concludes that $I = 0$ which is a contradiction. We infer that $I$ is not a direct summand of $R$ and so $R$ is not semisimple.

(ii) Suppose that $I$ is a nonzero ideal of $R$, then $I$ is a direct summand of $R$ since $R$ is semisimple. By (i), $I$ is not a sa-small ideal of $R$ and this is contradiction because $R$ is a sa-small hollow ring. It concludes that $R$ has no any nonzero ideal and so $R$ is a simple ring.
(iii) Assume that \( x \in \mathbb{Z}(R) \), then there exists an element \( 0 \neq y \in R \) such that \( xy = 0 \). Let \( R = Rx + Ry \), then \( Ry \subseteq \text{Ann}(Rx) \ll R \). This implies that \( Ry \ll R \) hence \( Rx = R \) and so \( x \in U(R) \) which is contradiction.

Now let \( e \) be an idempotent element of \( R \). Since \( R(1 - e) + Re = R \) therefore \( R(1 - e) \subseteq \text{Ann}(Re) \ll R \). This implies that \( R(1 - e) \ll R \) and so \( Re = R \). It concludes that \( e = 1 \).

(iv) The proof follows from the fact that since \( R \) is a von Neumann regular ring hence every finitely generated ideal \( I \) of \( R \) is a direct summand of \( R \) such that \( I \) is generated by an idempotent. Suppose that \( R = I \oplus J \), then \( IJ = 0 \) and by (i) \( I \) can not be a \( sa \)-small ideal.

(v) Take \( N = 0 \), then \( 0 + M = M \) and so \( \text{Ann}(M) = 0 \) is small in \( R \). Now assume that \( N \) is a nonzero submodule of \( M \), then \( N = IM \) for some nonzero ideal \( I \) of \( R \). Let \( r \in \text{Ann}(N) \), then \( r(IM) = 0 \) and so \( rI = 0 \). It concludes that \( r = 0 \) since \( R \) is an integral domain and so \( \text{Ann}(N) = 0 \) which is a small ideal of \( R \).

**Proposition 3.11.** Let \( M \) be an \( R \)-module with \( N \leq K \leq M \) and let \( T \) be an arbitrary submodule of \( M \). The following statements are true.

(i) Let \( M \) be a strong comultiplication module and \( N \ll^{sa} M \). Then for every nonzero submodule \( L \) of \( M \) with \( N + L = M \), \( L \leq_{e} M \).

(ii) If \( K \ll^{sa} T \ M \), then \( N \ll^{sa} T \ M \).

(iii) Assume \( \{N_{\lambda}\}_{\lambda \in \Lambda} \) be a family of submodules of \( M \). If \( N_{t} \ll^{sa} T \ M \) for some \( t \in \Lambda \), then \( \cap_{\lambda \in \Lambda} N_{\lambda} \ll^{sa} T \ M \).

(iv) Let \( T \subseteq K \) and \( N \ll^{sa} T \ K \), then \( N \ll^{sa} T \ M \).

(v) Suppose that \( M \) is a multiplication module. If \( N \ll^{sa} M \) (resp. \( N \ll^{sa} T \ M \)), then \( (N : M) \ll^{sa} R \) (resp. \( (N : M) \ll^{sa} (T : M) \ R \)). The converse is true if \( M \) is also a finitely generated faithful module. Furthermore, in this case, \( J^{sa}_{T}(M) = J^{sa}_{T(M)}(R)M \).

(vi) Assume that \( M \) and \( M' \) are \( R \)-modules and \( f : M \to M' \) is an \( R \)-epimorphism. If \( N' \ll^{sa} T \ M' \) for some submodule \( T' \) of \( M' \), then \( f^{-1}(N') \ll^{sa} T^{-1}(T') \ M \).

(vii) Assume that \( T \supseteq N \) is an arbitrary submodule of \( M \). If \( K/N \ll^{sa} T/N \ M/N \), then \( K \ll^{sa} T M \) and \( N \ll^{sa} T M \).

(viii) Let \( M \) be a Noetherian \( R \)-module. If \( S \) is a m.c.s. of \( R \) and \( S^{-1}N \) is an \( S^{-1}T \)-sa-small submodule of \( S^{-1}R \)-module \( S^{-1}M \), then \( N \) is a \( T \)-sa-small submodule of \( M \).

**Proof.** (i) Since \( N \ll^{sa} M \) and \( N + L = M \) hence \( \text{Ann}(L) \ll R \). By virtue of \([13, \text{Theorem 2.5}]\), \( L = (0 :_{M} \text{Ann}_{R}(L)) \) is an essential submodule of \( M \).

(ii) Assume that \( T \subseteq N + X \) for some submodule \( X \) of \( M \). Therefore
$T \subseteq K + X$ and since $K \ll_{T}^{sa} M$ hence $\text{Ann}(X) \ll (T : M)$ and the proof is complete.

(iii) It is clear by (ii).

(iv) Let $L \subseteq M$ and $T \subseteq N + L$, then by the modular law $T \subseteq (N + L) \cap K = N + (L \cap K)$. Since $N \ll_{T}^{sa} K$, hence $\text{Ann}(L) \subseteq \text{Ann}(L \cap K) \ll (T :_R M)$ and this implies that $\text{Ann}(L) \ll (T :_R M)$ and the proof is complete.

(v) $(\Rightarrow)$ Let $(N : M) + J = R$ for some ideal $J$ of $R$, then $(N : M)M + JM = M$. Since $M$ is multiplication hence $N + JM = M$ and so $\text{Ann}(JM) \ll R$. From $\text{Ann}(J) \subseteq \text{Ann}(JM)$ we infer that $\text{Ann}(J) \ll R$ and so $(N : M) \ll^{sa} R$.

$(\Leftarrow)$ Suppose that $(N : M) \ll^{sa} R$ and $N + K = M$ for some submodule $K$ of $M$. Thus $(N : M)M + (K : M)M = RM$ and since $M$ is a finitely generated faithful multiplication module hence $M$ is a cancellation module and so $(N : M) + (K : M) = R$. It concludes that $\text{Ann}(K : M) \ll R$. By hypothesis, since $M$ is a faithful module hence $\text{Ann}(K : M) = \text{Ann}(K) \ll R$ and the proof is complete.

Now assume that $N \ll_{T}^{sa} M$ and $(T : M) \subseteq (N : M) + J$ for some ideal $J$ of $R$. We show that $\text{Ann}(J) \ll ((T :_R M) :_R R) = (T :_R M)$. Since $M$ is a multiplication module hence $T = (T : M)M \subseteq (N : M)M + JM = N + JM$. Now the hypothesis $N \ll_{T}^{sa} M$ implies that $\text{Ann}(J) \subseteq \text{Ann}(JM) \ll (T :_R M)$ and so $\text{Ann}(J) \ll (T :_R M)$ as we needed. Conversely, let $(N : M) \ll^{sa}_{(T,M)} R$ and $T \subseteq N + K$ for some submodules $N, K$ of $M$. Since $M$ is multiplication there exist ideals $I, J$ of $M$ such that $N = IM = (N : M)M$ and $K = JM = (K : M)M$. Therefore $T = (T : M)M \subseteq (N : M)M + JM = ((N : M) + J)M$ and so $(T : M) \subseteq (N : M) + J$, because $M$ is a cancellation module. From the hypothesis $(N : M) \ll^{sa}_{(T,M)} R$, we infer that $\text{Ann}(J) \ll (T : M)$. Since $M$ is faithful hence $\text{Ann}(J) = \text{Ann}(JM) \ll (T : M)$ and the proof is complete. For the second part we note that

$$J_{T}^{sa}(M) := \sum_{N \ll_{T}^{sa} M} N = \sum_{N \ll_{T}^{sa} M} (N : M)M$$

$$:= \left( \sum_{(N: M) \ll^{sa}_{(T,M)} R} (N : M) \right) M = J_{(T,M)}^{sa}(R)M.$$

(vi) Suppose that $f^{-1}(T') \subseteq f^{-1}(N') + L$ for some submodule $L$ of $M$. Then since $f$ is an $R$-epimorphism hence

$$T' = f(f^{-1}(T')) \subseteq f(f^{-1}(N') + L) \subseteq N' + f(L).$$
Since $N' \ll^{sa} T'$, $M'$ hence $\text{Ann}(f(L)) \ll (T' : M')$ and therefore

$$\text{Ann}(L) \subseteq \text{Ann}(f(L)) \ll (T' : R M') = (f^{-1}(T') : R M).$$

It implies that $\text{Ann}(L) \ll (f^{-1}(T') : R M)$ and the proof is complete.

(vii) Assume that $K/N \ll^{sa} T/N M/N$ and also $T \subseteq K + L$ for some submodule $L$ of $M$. Then $T/N \subseteq (K + L)/N = K/N + L/N$ implies that $\text{Ann}(L/N) \ll (T/N : M/N) = (T : M)$. Since $\text{Ann}(L) \subseteq \text{Ann}(L/N) \ll (T : M)$ hence $\text{Ann}(L) \ll (T : M)$. It conclude that $K \leq^{sa} M$ and by (ii), $N \leq^{sa} M$.

(viii) We recall that if $X$ is a finitely generated submodule of $M$, then $S^{-1}(0 : R X) = (S^{-1}0 :_{S^{-1}R} S^{-1}X)$. Suppose that $T \subseteq N + X$ we show that $\text{Ann}_R(X) \ll (T : R M)$. Then we have

$$S^{-1}T \subseteq S^{-1}(N + X) = S^{-1}N + S^{-1}X.$$

By hypothesis $\text{Ann}_{S^{-1}R}(S^{-1}X) \ll (S^{-1}T :_{S^{-1}R} S^{-1}M)$. Therefore

$$\text{Ann}_{S^{-1}R}(S^{-1}X) : = (S^{-1}0 :_{S^{-1}R} S^{-1}X) = S^{-1}(0 : R X) \ll S^{-1}(T : R M) = (S^{-1}T :_{S^{-1}R} S^{-1}M).$$

This implies that $\text{Ann}_R(X) \ll (T : R M)$ and the proof is complete.

**Corollary 3.12.** Let $M$ be a faithful finitely generated multiplication $R$-module and let $T$ be an arbitrary submodule of $M$. Then $M$ is a $T$-sa-hollow module if and only if $R$ is a $(T : M)$-sa-hollow ring.

**Proof.** The proof is straightforward by Proposition 3.11 (v). □

**Corollary 3.13.** Let $(R, m)$ be a local ring and let $A$ be an arbitrary ideal of $R$. If $m \in S_0^{sa}(R)$, then $I \in S_0^{sa}(R)$ for every ideal $I$ of $R$.

**Proof.** The proof is straightforward by Proposition 3.11 (ii). □

**Theorem 3.14.** Let $f : M \rightarrow M'$ be an $R$-epimorphism and let $T'$ be a submodule of $M'$. If $M'$ is a $T'$-sa-hollow module, then $M$ is an $f^{-1}(T')$-sa-hollow module.

**Proof.** Assume that $K$ is a submodule of $M$. Then $f(K) \leq_{T'}^{sa} M'$ since $M'$ is a $T'$-sa-hollow module. By Proposition 3.11 (vi), $f^{-1}(f(K))$ is an $f^{-1}(T')$-sa-small submodule of $M$. Since $K \subseteq f^{-1}(f(K))$, so by Proposition 3.11 (ii), $K$ is also an $f^{-1}(T')$-sa-small submodule of $M$. □

The following example shows that the converse of Proposition 3.11 (vii), is not true.
Proof. (i) It is clear by Proposition 3.16 that if $2\mathbb{Z}$ is a submodule of $\mathbb{Z}$, then either $(k, 8) = 2$ or $(k, 8) = 1$. In any case, $0 = \text{Ann}(\mathbb{Z}) \leq (2\mathbb{Z} : \mathbb{Z}) = 2\mathbb{Z}$, but $K/N = 4\mathbb{Z}/8\mathbb{Z}$ is not a submodule of $\mathbb{Z}/8\mathbb{Z}$, because if $2\mathbb{Z} \subseteq 8\mathbb{Z} + k\mathbb{Z}$ for some submodule $k\mathbb{Z}$ of $\mathbb{Z}$, then either $(k, 8) = 2$ or $(k, 8) = 1$. Let $K/N$ be a submodule of $\mathbb{Z}/8\mathbb{Z}$, then $2\mathbb{Z} \subseteq 4\mathbb{Z} + k\mathbb{Z} = (4, k)\mathbb{Z}$. If $k = 2$, then $\text{Ann}(2\mathbb{Z}/8\mathbb{Z}) = 4\mathbb{Z}$ is not small in $(2\mathbb{Z}/8\mathbb{Z} : \mathbb{Z}/8\mathbb{Z}) = 2\mathbb{Z}$. If $k = 1$, then $\text{Ann}(\mathbb{Z}/8\mathbb{Z}) = 8\mathbb{Z}$ is not small in $(2\mathbb{Z}/8\mathbb{Z} : \mathbb{Z}/8\mathbb{Z}) = 2\mathbb{Z}$.

We recall that for an ideal $I$ of a ring $R$ the radical of $I$ is defined by $\text{rad}(I) = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$. Let $N$ be a proper submodule of $M$. Then, the prime radical of $N$, denoted by $\text{rad}(N)$ is defined to be the intersection of all prime submodules of $M$ containing $N$, and in case $N$ is not contained in any prime submodule then $\text{rad}(N)$ is defined to be $M$.

Lemma 3.16. If $I \llsa R$, then $\text{rad}(I) \llsa R$.

Proof. Let $\text{rad}(I) + J = R$ for some ideal $J$ of $R$. Since $\text{rad}(I) + J \subseteq \text{rad}(I) + \text{rad}(J)$, so $\text{rad}(I) + \text{rad}(J) = R$. This implies that $\text{rad}(I + J) = R$ and so $I + J = R$. Hence $\text{Ann}(J) \ll R$ since $I \llsa R$ and so $\text{rad}(I) \llsa R$.

Proposition 3.17. Let $M$ be a finitely generated faithful multiplication $R$-module. If $N \llsa M$, then $\text{rad}(N) \llsa M$.

Proof. By virtue of [9, Theorem 4], $\text{rad}(N) = \text{rad}(N : M) M$ and so $(\text{rad}(N) : M) = \text{rad}(N : M)$. Since $N \llsa M$, then by Proposition 3.11 (v), $(N : M) \llsa R$ and by Lemma 3.16, $\text{rad}(N : M) \llsa R$. It concludes that $(\text{rad}(N) : M) \llsa R$. Again using Proposition 3.11 (v) we find that $\text{rad}(N) \llsa M$.

Theorem 3.18. Let $N$ be a nonzero sa-small submodule of $M$ and $K \leq M$ with $(N : K) + (K : N) = R$, then $N \cap K \neq 0$.

Proof. Suppose that $N \cap K = 0$, then $(N : K) + (K : N) = \text{Ann}(K) + \text{Ann}(N) = R$. Since $N \in S^{sa}(M)$ hence $\text{Ann}(K) \ll R$ and so $\text{Ann}(N) = R$ which is contradiction.

Theorem 3.19. Let $K, H$ be submodules of $M$.

(i) If $K + H \llsa T M$, then $K \llsa T M$ and $H \llsa T M$.

(ii) If $M$ is a prime module and $K \llsa T M$ and $K + H \neq M$, then $K + H \llsa T M$.

Proof. (i) It is clear by Proposition 3.11 (ii). (ii) Let $T \subseteq K + H + X$ for some submodule $X$ of $M$. Since $K \llsa T M$ hence $\text{Ann}(X) = \text{Ann}(H + X) \ll (T : M)$.
Theorem 3.20. Let $R$ be a semisimple hollow ring and let $M_1, M_2$ be $R$-modules. Suppose that $N_1 \ll_{\tau_1} M_1$ and $N_2 \ll_{\tau_2} M_2$ for submodules $T_1 \leq M_1$ and $T_2 \leq M_2$, then $N_1 \oplus N_2 \ll_{\tau_1 \oplus \tau_2} M_1 \oplus M_2$.

Proof. Suppose that

$$T_1 \oplus T_2 \subseteq (N_1 \oplus N_2) + (L_1 \oplus L_2) = (N_1 + L_1) \oplus (N_2 + L_2).$$

Then $T_1 \subseteq N_1 + L_1$ and $T_2 \subseteq N_2 + L_2$ hence $\text{Ann}(L_1) \ll (T_1 : M_1)$ and $\text{Ann}(L_2) \ll (T_2 : M_2)$. By [8, Lemma 2], if $S \subseteq E \subseteq F$ and $S \ll F$ such that $E$ is a direct summand of $F$, then $S \ll E$. Take $S = \text{Ann}(L_1) \cap \text{Ann}(L_2)$ and $F = (T_1 : M_1) \cap (T_2 : M_2)$, then

$$\text{Ann}(L_1 \oplus L_2) = \text{Ann}(L_1) \cap \text{Ann}(L_2) \ll (T_1 : M_1) \cap (T_2 : M_2)$$

$$= (T_1 \oplus T_2 : M_1 \oplus M_2).$$

\[ \square \]

Theorem 3.21. Let $f : M \rightarrow N$ be a monomorphism and let $T$ be an arbitrary submodule of $M$. If $K \ll_{f(T)} M$, then $f(K) \ll_{f(T)} f(M)$.

Proof. Assume that $f(T) \subseteq f(K) + L$ for some submodule $L$ of $f(M)$. We show that $\text{Ann}(L) \ll (f(T) : R f(M))$. We have

$$T = f^{-1}(f(T)) \subseteq f^{-1}(f(K) + L) = K + f^{-1}(L) \leq M.$$ 

Then $\text{Ann}(L) \subseteq \text{Ann}(f^{-1}(L)) \ll (T : R M) = (f(T) : R f(M))$. \[ \square \]

Theorem 3.22. Let $f : N \rightarrow K$ be a monomorphism and let $T$ be an arbitrary submodule of $N$. If $g : K \rightarrow M$ is an $f(T)$-sa-small epimorphism, then $g \circ f : N \rightarrow M$ is also a $T$-sa-small epimorphism.

Proof. Assume that $T \subseteq \text{Ker}(g \circ f) + X$ for some submodule $X$ of $N$. We show that $\text{Ann}(X) \ll (T : N)$. Since $\text{Ker}(g \circ f) = f^{-1}(\text{Ker}g)$ hence

$$f(T) \subseteq f(\text{Ker}(g \circ f)) + f(X) \subseteq \text{Ker}g + f(X).$$

Since $\text{Ker}g \ll_{f(T)} K$, hence

$$\text{Ann}(X) \subseteq \text{Ann}(f(X)) \ll (f(T) : R K).$$

We have $(f(T) : R K) \subseteq (T : R N)$, because if $r \in (f(T) : R K)$ and $x \in N$, then $rf(x) = f(rx) \in f(T)$. Since $f$ is monomorphism hence $r x \in f^{-1}(f(T)) = T$. It concludes that $r \in (T : N)$. Therefore $\text{Ann}(X) \subseteq \text{Ann}(f(X)) \ll (T : R N)$ and the proof is complete. \[ \square \]

We recall that an $R$-module $F$ is called flat if whenever $N \rightarrow K \rightarrow L$ is an exact sequence of $R$-modules, then $F \otimes N \rightarrow F \otimes K \rightarrow F \otimes L$ is an exact sequence as well. An $R$-module $F$ is called faithfully flat, whenever $N \rightarrow K \rightarrow L$ is an exact sequence of $R$-modules if and only if $F \otimes N \rightarrow F \otimes K \rightarrow F \otimes L$ is an exact sequence.
Theorem 3.23. Let $F$ be a faithfully flat $R$-module and let $M$ be an $R$-module. Assume that $N \leq M$ and $T$ is an arbitrary submodule of $M$. Then the following statements hold.

(i) $N$ is a sa-small submodule of $M$ if and only if $F \otimes N$ is a sa-small submodule of $F \otimes M$.

(ii) $N$ is a $T$-sa-small submodule of $M$ if and only if $F \otimes N$ is a $F \otimes T$-sa-small submodule of $F \otimes M$.

Proof. (i) ($\Rightarrow$) Let $N \leq^sa M$ and $F \otimes N + F \otimes K = F \otimes M$ for some submodule $F \otimes K$ of $F \otimes M$. Then $F \otimes (N + K) = F \otimes N + F \otimes K = F \otimes M$. Thus $0 \to F \otimes (N + K) \to F \otimes M \to 0$ is an exact sequence. Since $F$ is faithfully flat, so $0 \to N + K \to M \to 0$ is an exact sequence. Therefore $N + K = M$ and this implies that $\text{Ann}(K) \ll R$ since $N \leq^sa M$. We have $\text{Ann}(K) = \text{Ann}(F \otimes K)$, because if $r \in \text{Ann}(K)$, then $rK = 0$. Thus $r(F \otimes K) = F \otimes rK = 0$ and so $r \in \text{Ann}(F \otimes K)$. If $r \in \text{Ann}(F \otimes K)$, then $0 \to F \otimes rK \to 0$ is an exact sequence and so $0 \to rK \to 0$ is exact since $F$ is a faithfully flat $R$-module. Therefore $rK = 0$ and so $r \in \text{Ann}(K)$. It concludes that $\text{Ann}(F \otimes K) \ll R$, so $F \otimes N \leq^sa F \otimes M$.

($\Leftarrow$) Let $N + K = M$ for some submodule $K$ of $M$. Then $F \otimes (N + K) = F \otimes N + F \otimes K = F \otimes M$. Hence $\text{Ann}(K) = \text{Ann}(F \otimes K) \ll R$. Therefore $N \leq^sa M$, as needed.

(ii) ($\Rightarrow$) Let $F \otimes T \subseteq F \otimes N + F \otimes K$ for some submodule $F \otimes K$ of $F \otimes M$. Thus $F \otimes T \subseteq F \otimes (N + K)$ and so $0 \to F \otimes T \to F \otimes (N + K)$ is exact. Therefore $0 \to T \to N + K$ is also exact since $F$ is a faithfully flat $R$-module. This implies that $T \subseteq N + K$ and since $N \leq^sa T M$ hence $\text{Ann}(K) \ll (T :_R M)$. It is easy to see that $(T :_R M) = (F \otimes T :_R F \otimes M)$ since $F$ is faithfully flat. Hence $\text{Ann}(F \otimes K) \ll (F \otimes T :_R F \otimes M)$ and so $F \otimes N \leq^sa F \otimes T$. ($\Leftarrow$) Let $T \subseteq N + K$. Then $F \otimes T \subseteq F \otimes (N + K) = F \otimes N + F \otimes K$. Now since $F \otimes N \leq^sa F \otimes T M$ hence $\text{Ann}(F \otimes K) \ll (F \otimes T :_R F \otimes M)$. It concludes that $\text{Ann}(K) \ll (T :_R M)$ and so $N \leq^sa T M$. □

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Saeed Rajaee  
Department of Mathematics, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran.  
Email: saeed_rajaee@pnu.ac.ir

Farkhondeh Farzalipour  
Department of Mathematics, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran.  
Email: f_farzalipour@pnu.ac.ir

Marzieh Poyan  
Department of Mathematics, PhD student of Mathematical Sciences, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran.  
Email: r_poyan@pnu.ac.ir