ON 2D-4D MOTIVIC WALL-CROSSING FORMULAS

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Abstract. In this paper we propose definitions and examples of categorical enhancements of the data involved in the 2d−4d wall-crossing formulas which generalize both Cecotti-Vafa and Kontsevich-Soibelman motivic wall-crossing formulas.

1. Introduction

In the foundational work [9] the authors proposed a categorical framework for motivic Donaldson-Thomas theory. Among other things they established a multiplicative wall-crossing formula which shows how the Donaldson-Thomas invariants change when the stability condition on the category crosses a real codimension one subvariety called wall.

It was also explained in the loc. cit. that the wall-crossing formulas themselves do not require the categorical framework. They arise when one has a much simpler structure consisting of a Lie algebra over $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_{\gamma}$ over $\mathbb{Q}$ graded by a free abelian group $\Gamma$ (lattice) and endowed with the so-called stability data. The latter consist of the homomorphism of abelian groups $Z : \Gamma \to \mathbb{C}$ (central charge) and a collection of elements $a(\gamma) \in \mathfrak{g}_{\gamma}, \gamma \in \Gamma - \{0\}$. The stability data satisfy one axiom called Support Property. We are going to recall the details in the next section.

Having stability data one can define for each strict sector $V \subset \mathbb{R}^2$ a pronilpotent Lie algebra $\mathfrak{g}_V$ as well as the pronilpotent Lie group $G_V = \exp(\mathfrak{g}_V)$. The element $A_V = \exp(\sum_{Z(\gamma) \in V} a(\gamma))$ is well-defined as an element of $G_V$. For each ray $l \subset V$ one has an element $A_l \in G_l \subset G_V$. The wall-crossing formula is in fact a collection of the following formulas, one for each strict sector $V$:

$$A_V = \prod_{l \subset V} A_l.$$ 

Here the product in the RHS is taken in the clockwise order. It can contain countably many factors.

It was explained in the loc.cit. that the above data appear in many different frameworks. In particular the Donaldson-Thomas theory of a 3-dimensional Calabi-Yau category proposed in the loc. cit. gives an example of a graded Lie algebra endowed with stability data.

In the most abstract setting the wall-crossing formulas appears appear in the following way. Suppose one has a triangulated $A_{\infty}$-category $\mathcal{C}$ which is linear over some ground field. The lattice $\Gamma$ is the $K$-group (or its quotient) and the Lie algebra $\mathfrak{g}$ is the motivic Hall algebra $H(\mathcal{C})$ defined in [9]. Choosing a stability structure on $\mathcal{C}$ one can define for each strict sector $V$ as above a subcategory $\mathcal{C}_V$ generated by semistables with the central charge in $V$. This gives the group $G_V$ as the group of invertible elements in the natural completion of the associative algebra $H(\mathcal{C}_V)$. 

Then the wall-crossing formula can be written as the product

\[ A_{V}^{Hall} = \prod_{l \in V} A_{l}^{Hall}. \]

Each factor in the RHS represents the input of semistable objects of \( C \) with the central charge belonging to a particular ray \( l \). One can change the central charge without changing the LHS. Then we obtain the formula which gives the equality of products of inputs of semistable objects for two different stability structures. Intuitively one can think that in the simplest case those stability structures are separated by a real codimension one “wall”. This explains the term “wall-crossing formula”.

The authors of [9] considered an important class of categories for which the wall-crossing formula can be further simplified. This is the case when \( C \) is a 3-dimensional Calabi-Yau category (3CY category for short). In this case there is an algebra homomorphism from each \( H(C_{V}) \) to the so-called motivic quantum torus. It induces a wall-crossing formula in a much simpler algebra, namely in the motivic quantum torus. Those are wall-crossing formulas for motivic Donaldson-Thomas invariants of the category \( C \). There is a “quantization parameter” in the story, which is the Lefshetz motive \( L \), i.e. the motive of the affine line. One can consider a “quasi-classical limit” as \( L \to 1 \). It is well-defined under some assumptions. The underlying graded Lie algebra is the one of the Hamiltonian vector fields on the Poisson torus (or its quantization) associated with the quotient of \( K_{0}(C) \). It is naturally graded by this quotient, and it is endowed with the integer skew-symmetric form induced by the Euler form. Graded components of this Lie algebra are 1-dimensional. Hence elements \( a(\gamma) \) are rational numbers. It was explained in [9] that the inverse Möbius transform produces a collection of numbers \( \Omega(\gamma) \) which are “tend” to be integers. The number \( \Omega(\gamma) \in Z \) is called the numerical Donaldson-Thomas invariant of class \( \gamma \). (DT-invariant for short). It measures the “number” of semistable objects of \( C \) having a fixed \( K \)-theoretical class \( \gamma \).

The above class of wall-crossing formulas covers a variety of examples which include Calabi-Yau 3-folds and quivers with potential. They are known in gauge theory as 4d wall-crossing formulas for BPS invariants (or refined BPS invariants in the case of quantum tori).

Another class of examples is known as 2d wall-crossing formulas. They are associated with finite-dimensional simple Lie algebras graded by their root lattices. Contrary to the 4d wall-crossing formulas they do not have an obvious categorical or motivic origin.

Finally, there are so-called 2d − 4d wall-crossing formulas (see [3]) which also follow from the formalism of [9], but similarly to the 2d case without categorical or motivic examples.

Despite of various unfinished attempts to find the categorical framework for the 2d and 2d − 4d wall-crossing formulas, the problem is widely open. One of the purposes of this paper is to propose such a categorical framework and to illustrate it in algebraic and geometric examples.

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2. Groupoids and stability data on graded Lie algebras

2.1. Groupoids and associated graded Lie algebras. This section will describe a generalization of the definition of stability data as introduced in [9, Section 2] to groupoids. In fact, this generalization is in some sense a restricted version of the usual notion of stability data. To accomplish this generalization, we begin by recalling the structures considered in [3]. There, the authors introduced the following algebraic data which are used in the 2d-4d wall-crossing formulas discussed in the loc.cit.:

1. A lattice $\Gamma$ with anti-symmetric pairing $\langle \cdot, \cdot \rangle$.
2. A connected groupoid $V$ with objects $V \sqcup \{o\}$ whose automorphism groups are canonically isomorphic to $\Gamma$.
3. A central charge $Z : V \to \mathbb{C}$ which is a homomorphism of groupoids (where $\mathbb{C}$ is regarded as the groupoid with one object).
4. A function $\Omega : \Gamma \to \mathbb{Z}$.
5. A function $\mu : M(V) \to \mathbb{Z}$ where $M(V)$ are the morphisms of $V$ with distinct source and target.
6. A 2-cocycle $\sigma \in C^2(V, \{\pm 1\})$.

The corresponding wall-crossing formulas (WCF for short) incorporate both 2d Cecotti-Vafa WCF and 4d Kontsevich-Soibelman WCF (non-motivic version). For this paper, we will begin with a small groupoid $V$ and its 2-cocycle $\sigma$, building a generalized stability data from this point. Following [3], we will write the objects of $V$ using the notation like $i,j,k$ and the morphism sets $\text{Hom}_V(i,j)$ will be often denoted by $\Gamma_{ij}$. Composition in $V$ will be written additively. If $\gamma_1$ and $\gamma_2$ are not composable, we define $\sigma(\gamma_1, \gamma_2) = 0$.

In our setting, we will allow for disconnected groupoids (i.e. groupoids with potentially more than one isomorphism class). Write $\mathcal{W}$ for the unique groupoid equivalent to $V$ for which, given distinct $i,j \in \mathcal{W}$, $\text{Hom}_\mathcal{W}(i,j) = \emptyset$. Fix equivalences

1. $F : V \to \mathcal{W}$, $G : \mathcal{W} \to V$, $\eta : GF \Rightarrow 1_V$.

Taking $\mathbb{C}$ to be the groupoid with one object, morphisms complex numbers and composition equal to addition, we will frequently consider the additive group of groupoid homomorphisms $\text{Hom}_{\text{grpd}}(V, \mathbb{C})$. We now give an alternative characterization of this group. For every $i \in \text{Ob}(\mathcal{W})$, consider the groups

$\Gamma_i = \text{Hom}_\mathcal{W}(i,i)$,
$W_i = \left\{ \sum a_j e_j \in \mathbb{Z}^{F^{-1}(i)} : \sum a_j = 0 \right\}$,

and define

$\Gamma_V := \bigoplus_{i \in \text{Ob}(\mathcal{W})} (\Gamma_i \oplus W_i)$.

\footnote{In fact the framework of [9, Section 2] covers all types of wall-crossing formulas, including 2d, 4d, and 2d-4d. However the tradition in physics literature is to call 2d wall-crossing formulas “Cecotti-Vafa WCF”, while 4d wall-crossing formulas are called “Kontsevich-Soibelman WCF”.
}
We note that $\Gamma^V_*$ does not depend on the equivalence $F$. We take $\pi^W: \Gamma^V \to \oplus_{i \in \text{Ob}(\mathcal{W})} W_i$ to be the projection.

**Lemma 2.1.** There is an isomorphism of abelian groups

\[ \text{Hom}_{\text{grpd}}(\mathcal{V}, \mathbb{C}) \cong \text{Hom}_{\text{Ab}}(\Gamma^V_*, \mathbb{C}). \]

**Proof.** For $j \in \text{Ob}^V$ let $w_j = e_j - e_{GF_j} \in W_{F_j}$ and note that \{ $w_j : j \in F^{-1}(i)$ \} forms a basis for $W^i$. As $G$ is an equivalence, we have that $G: \Gamma^i_i \cong \Gamma^i_{GiGi}$. Define the map $f: \text{Hom}_{\text{grpd}}(\mathcal{V}, \mathbb{C}) \to \text{Hom}_{\text{Ab}}(\Gamma^V_*, \mathbb{C})$

by

\[ f(Z)(w_j) = Z(\eta_j), \quad f(Z)(\gamma) = Z(G(\gamma)). \]

Here $\mathbb{C}$ means the groupoid with one object and space of morphisms equal to the field of complex numbers. An elementary check shows this to be an isomorphism of groups. □

When $\mathcal{V}$ is finite may extend the elements $w_j$ into a type $A$ root system. Generally, we take $R^i = \{ e_j - e_k | W_i \}, \quad R^V = \oplus_{i \in \mathcal{W}} R^i$.

We can now define a groupoid graded Lie algebra.

**Definition 2.2.** A $\mathcal{V}$-graded Lie algebra $\mathfrak{g}$ is a $\Gamma^V_*$-graded Lie algebra

$\mathfrak{g} = \oplus_{\gamma \in \Gamma^V} \mathfrak{g}_\gamma$

satisfying $\mathfrak{g}_\gamma \neq 0$ implies $\pi^W(\gamma) \in R^V$.

In the case where $\text{Ob}(\mathcal{V})$ is finite, this implies that the grading has a map to a product of type $A$ root systems.

**Example 2.3.** Write $K[\mathcal{V}]$ for twisted groupoid algebra of $\mathcal{V}$ defined by $\sigma$. In particular, $K[\mathcal{V}]$ has a basis \{ $e_\gamma : \gamma \in \Gamma^i_j$ \} with multiplication

\[ e_{\gamma_1} e_{\gamma_2} = \sigma(\gamma_1, \gamma_2) e_{\gamma_1 + \gamma_2}, \]

whenever $\gamma_1$ and $\gamma_2$ are composable and zero otherwise.

Given an anti-symmetric integer pairing $\langle , \rangle$ on $\oplus_{j \in \mathcal{W}} \Gamma^i_j$, orthogonal on distinct summands, we may define compatibility of the pairing and $\sigma$ as

\[ \sigma(\gamma_1, \gamma_2) = (-1)^{\langle F(\gamma_1), F(\gamma_2) \rangle} \]

for all $i \in \text{Ob}(\mathcal{V})$ and $\gamma_1, \gamma_2 \in \Gamma^i$.

To define a $\mathcal{V}$-graded Lie algebra, if $\gamma \in \Gamma^i_j$, we take the degree of $e_\gamma$ to be

\[ |e_\gamma| := F(\gamma) + e_j - e_i \in \Gamma^V. \]

and define the bracket

\[ [e_{\gamma_1}, e_{\gamma_2}] = \sigma(\gamma_1, \gamma_2) (F(\gamma_1), F(\gamma_2)) (e_{\gamma_1 + \gamma_2} + e_{\gamma_2 + \gamma_1}). \]
2.2. \(\mathcal{V}\)-stability data. The generalization of stability data of a \(\Gamma\)-graded Lie algebra (see [9]) to the \(\mathcal{V}\)-graded case is straightforward.

**Definition 2.4.** Let \(\mathcal{V}\) be a groupoid with finitely many objects. Given a \(\mathcal{V}\)-graded Lie algebra \(g\), stability data on \(g\) is a pair \((Z,a)\)

1. \(Z: \mathcal{V} \to \mathbb{C}\) is a homomorphism of groupoids,
2. \(a = (a(\gamma))_{\gamma \in \Gamma}\) is a collection of elements in \(g\) where \(a(\gamma) \in g_{\gamma}\).

This data must satisfy the following **Support Property**

For a norm \(\|\|\) on \(\Gamma \otimes \mathbb{R}\), there exists \(C > 0\) such that for any \(\gamma \in \text{supp}(a)\) we have

\[ \|\gamma\| \leq CZ(\gamma). \]

Equivalently, the last condition means the following: there exists a quadratic form \(Q\) on \(\Gamma \otimes \mathbb{R}\) such that it is non-positive on \(\ker(Z)\) and non-negative on the set of \(\gamma\)'s such that \(a(\gamma) \neq 0\) (sometimes this set is called the **support** of stability data).

In other words, groupoid stability data is precisely stability on the \(\Gamma \mathcal{V}\)-graded Lie algebra \(g\) in the sense of [9, Section 2.1]. Fixing the quadratic form \(Q\) we may define the subset of stability data \(\text{Stab}_Q(g)\) as in [9, Section 2.2]. By applying Theorems 2 and 3 of loc. cit., we may assert the following facts concerning \(\text{Stab}(g)\):

1. The set of stability data \(\text{Stab}(g)\) equals \(\bigcup_Q \text{Stab}_Q(g)\).
2. The set of stability data \(\text{Stab}(g)\) can be endowed with a Hausdorff topology.
3. For a quadratic form \(Q_0\) and \(Z_0: \Gamma \mathcal{V} \to \mathbb{C}\), let
   
   \[ U_{Q_0,Z_0} = \{ Z \in \text{Hom}_{A\infty}(\Gamma \mathcal{V}, \mathbb{C}) : Q_0\text{ is negative definite on } \ker Z_0 \}. \]

Then there exists a local homeomorphism \(U_{Q_0,Z_0} \to \text{Stab}_{Q_0}(g)\) which is a section of the projection to \(\text{Hom}_{A\infty}(\Gamma \mathcal{V}, \mathbb{C})\).

As a result, for any strict sector \(V\) in \(\mathbb{C}\) (i.e. the one which does not contain a straight line) one can define an element \(A_V = \exp(\sum_{Z(\gamma) \in V, Q(\gamma) \geq 0} a(\gamma))\) of the pro-nilpotent group \(G_{V,\mathcal{V},Q}\) corresponding to the pro-nilpotent Lie algebra \(g_{V,\mathcal{V},Q}\). This satisfies the **factorization property**

\[ A_V = A_{V_1}A_{V_2} \]

where \(V = V_1 \sqcup V_2\) in clockwise order.

3. **Categorification of 2d-4d wall-crossing formulas**

3.1. **Motivic correspondences and motivic Hall algebras.** Given an ind-constructible, pre-triangulated, \(A_\infty\)-category \(\mathcal{C}\) over \(\mathbb{K}\), we recall from [9, Section 6.1] the definition of the motivic Hall algebra. We refer to the loc.cit. about a convenient introduction to the motivic functions on algebraic varieties and algebraic stacks.

We will assume all our categories \(\mathcal{C}\) are ind-constructible triangulated \(A_\infty\)-categories with finite dimensional \(\text{Hom}'s\). We first have that \(\text{Ob}(\mathcal{C})\) is an ind-constructible stack which is a countable disjoint union of quotient stacks

\[ \text{Ob}(\mathcal{C}) = \bigcup_{i=1}^\infty Y_i/GL(N_i) \]

for some \(N_i \geq 1\).
The motivic Hall algebra $H(C)$ is then the $\text{Mot}(\text{Spec}(K))[[\mathbb{L}^{-1}]]$-module $\oplus_{i} Mot_{st}(Y_{i}, GL(N_{i}))[\mathbb{L}^{-1}]$, where $Mot_{st}(Y_{i}, GL(N_{i}))$ denote the abelian group of motivic functions on the quotient stack $Y_{i}/GL(N_{i})$. We recall that, given $E, F \in C$, the $N$-truncated Euler characteristic is given as

$$(E, F)_{\leq N} := \sum_{i \leq N} (-1)^{i} \dim \text{Hom}_{C}(E, F[i]).$$

The product structure in $H(C)$ is defined such as follows. For any two constructible families $\pi_{1} : X_{1} \to Ob(C), \pi_{2} : X_{2} \to Ob(C)$ and $n \in \mathbb{Z}$ the authors of [9] define the spaces

$$W_{n} = \{ (x_{1}, x_{2}, \alpha) | x_{i} \in X_{i}, \alpha \in \text{Ext}^{1}(\pi_{2}(x_{2}), \pi_{1}(x_{1})), (\pi_{2}(x_{2}), \pi_{1}(x_{1}))_{\leq 0} = n \}.$$ 

The element $[W_{n} \to Ob(C)] \in H(C)$ is defined as taking the cone of $\alpha : \pi_{2}(x_{2})[-1] \to \pi_{1}(x_{1})$. Then the product on $H(C)$ is given as

$$[\pi_{1} : X_{1} \to Ob(C)] \cdot [\pi_{2} : X_{2} \to Ob(C)] := \sum_{n \in \mathbb{Z}} [W_{n} \to Ob(C)] \mathbb{L}^{-n}$$

It is shown in loc. cit. that it makes $H(C)$ into an associative algebra.

We now introduce a more general version of motivic Hall algebras. Let $\iota : W \to Ob(C) \times Ob(C)$ be a homogeneous ind-constructible affine subbundle of $\text{Ext}^{*}$ (with possibly empty fibers). We will call $W$ a motivic correspondence of $C$. Let $\text{supp}(W)$ be the substack of $Ob(C) \times Ob(C)$ over which the fibers of $W$ are non-empty. Given $(A, B) \in \text{supp}(W)$ define the degree $\deg_{A,B}(W)$ of $W$ over $(A, B)$ to be $k$ if $\iota^{*}(A, B) \subset \text{Ext}^{k}(A, B)$. There may be some ambiguity here if $\iota^{*}(A, B) = 0$, but we will take 0 to be an element of any degree. We will take cone : $W \to Ob(C)$ to be the restriction of the map taking $\alpha \in \text{Ext}^{k}(A, B)$ to the cone of $\alpha : A[-k] \to B$. Let $\nu : \text{supp}(W) \to \mathbb{Z}$ be a constructible function, which we call a weight, so that $\text{supp}(W) = \cup_{n \in \mathbb{Z}} \text{supp}^{n}(W)$ and $W = \cup_{n \in \mathbb{Z}} W_{n}$ where $W_{n}$ is the fiber of $\iota$ over $\text{supp}^{n}(W) := \nu^{-1}(n)$. Then, using the decomposition from $\nu$, one may decompose as before

$$[(\pi_{1} \times \pi_{2})^{*}(W) \to Ob(C) \times Ob(C)] = \sum_{n \in \mathbb{Z}} [(\pi_{1} \times \pi_{2})^{*}(W_{n}) \to Ob(C) \times Ob(C)].$$

Taking cones of both sides gives the decomposition

$$[(\pi_{1} \times \pi_{2})^{*}(W) \to Ob(C)] = \sum_{n \in \mathbb{Z}} [(\pi_{1} \times \pi_{2})^{*}(W_{n}) \to Ob(C)].$$

The pair $(W, \nu)$ induces a product in the motivic Hall algebra as in the case where $W = \mathcal{E} \text{xt}^{1}$. In particular, one takes

$$[\pi_{1} : X_{1} \to Ob(C)] \cdot (W, \nu) [\pi_{2} : X_{2} \to Ob(C)] := \sum_{n \in \mathbb{Z}} [(\pi_{1} \times \pi_{2})^{*}(W_{n}) \to Ob(C)] \mathbb{L}^{-n}$$

**Definition 3.1.** We call the pair $(W, \nu)$ an associative correspondence if $(W, \nu)$ is an associative product in $H(C)$.

Given an associative correspondence $(W, \nu)$, we may consider the algebra $H_{(W, \nu)}(C)$ which is identical to $H(C)$ as a vector space but endowed with product $(W, \nu)$.

**Example 3.2.** Let $Q$ be the $A_{n}$ quiver with vertices $Q_{0} = \{ 1, \cdots, n \}$ and arrows $Q_{1} = \{ i \to (i-1) : 1 < i \leq n \}$ oriented as in Figure [7]. Take $\{ u_{i} : 1 \leq i \leq n \}$ to be the standard basis for the dimension vectors $\mathbb{N}^{Q_{0}}$. Let $A$ be the dg category of
representations of the path algebra \( \mathbb{K}Q \). Let \( \mathcal{B} \) be the full subcategory of indecomposable representations (concentrated in degree zero). By Gabriel’s Theorem, the objects of \( \mathcal{B} \) have dimension vectors
\[
\sum_{k=i+1}^{j} u_k : 0 < i < j \leq n.
\]
These correspond to positive roots of the \( A_n \)-root system. Write \( M_{ij} \) for the corresponding indecomposable module and consider the correspondence
\[
W = \{ (M_{kl}, M_{ij}, \gamma) : \gamma \in \text{Ext}^1(M_{ij}, M_{kl}), \text{cone}(\gamma) \text{ is indecomposable} \}.
\]
Take \( \nu : \text{Ob}(\mathcal{A}) \times \text{Ob}(\mathcal{A}) \to \mathbb{Z} \) to be constant and equal to zero. Then we claim that \((W, \nu)\) is an associative correspondence.

To verify this, first one makes a basic computation to see
\[
\text{Hom}^*(M_{ij}, M_{kl}) \cong \begin{cases} \mathbb{K} \cdot \alpha & i \leq k < j \leq l, \\ \mathbb{K} \cdot \beta \langle -1 & k < i \leq l < j, \\ 0 & \text{otherwise}. \end{cases}
\]
Furthermore, with the indices as in equation (9) and setting \( M_{ii} = 0 \) for any \( i \), we have
\[
\begin{align*}
\text{cone}(\alpha) & \cong M_{jl} \oplus M_{ik}[1] \\
\text{cone}(\beta) & \cong M_{il} \oplus M_{kj}.
\end{align*}
\]
Thus we may rewrite \( W \) as
\[
W = \{ (M_{kl}, M_{ij}, \beta) : 0 \leq k < i < j \leq n \},
\]
where \( \text{cone} \) maps \( (M_{ki}, M_{ij}, \beta) \) to \( M_{kj} \).

For a \( \mathbb{K} \)-algebra \( R \), let \( U_n(R) \) be the algebra of strictly upper triangular matrices with entries in \( R \). Then there is a natural algebra homomorphism
\[
\int : H_{(W,\nu)}(\mathcal{A}) \to U_n(\text{Mot}(\text{Spec}(\mathbb{K})))
\]
taking \( [\pi : X \to M_{ij}] \) to the upper triangular matrix \( [X] \) in the \( i \)-th row and \( j \)-th column.

3.2. Motivic enhancements of ind-constructible categories. We will need a relative version of an associative correspondence in the next section. For this, let \( \mathcal{C} \) be an ind-constructible triangulated \( A_\infty \)-category, \( \mathcal{D} \) be a small strict category and \( \mathcal{M}(\mathcal{D}) \) the set of morphisms in \( \mathcal{D} \). We assume that \( \mathcal{M}(\mathcal{D}) \) is an ind-constructible stack.

**Definition 3.3.** Let \( \Phi = \{ (\mathcal{C}_{ij}, \epsilon_{ij})_{(i,j) \in \text{Ob}(\mathcal{D})^2}, \{ (W_{ijk}, \nu_{ijk})_{(i,j,k) \in \text{Ob}(\mathcal{D})^3} \} \) consist of the following structures:

1. A collection of full ind-constructible subcategories \( \{ \mathcal{C}_{ij} \}_{(i,j) \in \text{Ob}(\mathcal{D})^2} \) of \( \mathcal{C} \).
(2) For every \((i, j) \in \text{Ob}(D)^2\), an ind-constructible morphism \(\epsilon_{ij} : \text{Ob}(C_{ij}) \to \text{Hom}_D(i, j)\).

(3) A weighted motivic correspondence \((W_{ijk}, \nu_{ijk})\) of \(C\) for every triple of objects in \(D\).

We say that \(\Phi\) is a motivic enhancement of \(D\) by \(C\) if:

(i) for every triple \(i, j, k \in \text{Ob}(D)\), \(C_{jk} \times C_{ij} \subset \text{supp}(W_{ijk})\),

(ii) the product \([\pi_1 : X_1 \to C_{jk}] : [\pi_2 : X_2 \to C_{ij}]\) induced by \((W_{ijk}, \nu_{ijk})\) yields a motive over \(C_{ik}\).

(iii) the product induced by the correspondences \((W_{ijk}, \nu_{ijk})\) is associative.

Example 3.4. Take \(D\) to be the directed category with objects \(\{0, \ldots, n\}\) and a morphism \(i \to j\) if and only if \(i \geq j\). Let \(A\) be the dg category of representations of the \(\Lambda_n\)-quiver as in Example 3.2 and take \(C_{ij} = \{M_{ji}[r] : r \in \mathbb{Z}\}\). For correspondences, \(W_{ijk}\) take \(\{(\beta[s-r], M_{ij}[r], M_{jk}[s]) : r, s \in \mathbb{Z}\}\). In general, one may take non-constant \(\nu_{ijk}\), but for simplicity consider \(\nu_{ijk} = 0\). This enhancement “spreads out” the associative correspondence from the previous example across \(D\).

The fundamental invariant associated to a motivic enhancement will now be defined.

Definition 3.5. Let \(\Phi = (\{C_{ij}, \epsilon_{ij}\}, \{(W_{ijk}, \nu_{ijk})\})\) be a motivic enhancement of \(D\) by \(C\). The motivic Hall category

\[
H_{\Phi}(D)
\]

of the enhancement \(\Phi\) is the category with objects \(\text{Ob}(D)\) and morphisms

\[
\text{Hom}_{H_{\Phi}(D)}(a, b) = H(C_{a,b}).
\]

We will consider the following main example of motivic enhancements throughout the paper. Let \(\Lambda'\) and \(\Lambda\) be ind-constructible abelian groups and a group homomorphism. Suppose that \(A\) is a subset of \(\Lambda\).

Definition 3.6. The groupoid induced by \(\phi\) and \(A\), which we will denote by \(\Lambda_{\phi,A}\) (or simply \(\Lambda_{\phi}\)), has objects equal \(A\) and, given \(\lambda_1, \lambda_2 \in A\)

\[
\text{Hom}_{\Lambda_{\phi}}(\lambda_1, \lambda_2) = \phi^{-1}(\lambda_2 - \lambda_1).
\]

Composition is addition in \(\Lambda'\).

Note that if, for \(\lambda_1, \lambda_2 \in A\),

\[
\lambda_2 - \lambda_1 \notin \text{im}(\phi)
\]

then \(\text{Hom}_{\Lambda_{\phi}}(\lambda_1, \lambda_2) = \emptyset\).

To obtain a motivic enhancement, suppose \(\chi : K_0(C) \to \Lambda'\) is an additional homomorphism. For any \(\lambda \in \Lambda\) let

\[
C_{\lambda} = \{X \in C : (\phi \circ \chi)([X]) = \lambda\}.
\]

Let \(W\) be the motivic correspondence \(\text{Ext}^1\) and \(\nu\) the \(N\)-truncated Euler characteristic with \(N = 0\).

Definition 3.7. The triple \(\Phi = (\{C_{\lambda_2 - \lambda_1} : (\lambda_1, \lambda_2) \in A^2\}, W, \nu)\) is called the motivic enhancement induced by \(\phi, \chi\) and \(A\).
In cases when $\mathcal{N} = K_0(C)$ and $\chi$ is the identity, we will simply say that $\Phi$ is induced by $\phi$ and $A$.

In this instance the motivic correspondence is independent of the triple of objects. An extreme version of this is taking $\Lambda = \{0\}$ and obtaining a motivic Hall category $H_\phi(A)$ with one object whose endomorphism ring is the motivic Hall algebra $\bar{H}(C)$.

**Example 3.8.** Consider the category $\mathcal{A}$ of representations of the $A_n$ quiver as in Example 3.2. Then $K_0(\mathcal{A}) = \mathbb{Z}^{Q_\mathcal{A}}$ is isomorphic to the root lattice of type $A_n$ with roots $R = R^+ \cup -R^+$. There is a homomorphism

$$\phi : K_0(\mathcal{A}) \rightarrow \mathbb{Z}^{n+1}$$

taking the root $u_{ij}$ to $e_j - e_i$. No two elements in the standard basis $A = \{e_0, \ldots, e_n\}$ satisfies equation (18). Thus the category $\mathbb{Z}^{n+1}_{Q,A}$ is the groupoid that has objects $\{0, \ldots, n\}$ (with the correspondence $i \mapsto e_i$) and exactly one morphism $u_{ij} : i \rightarrow j$. We will refer to such groupoids as trivial groupoids in that they are equivalent to a trivial group. Letting $\mathcal{A}_0$ be the full subcategory of $\mathcal{A}$ whose objects have trivial $K_0(\mathcal{A})$ classes, we note that

$$C_{ij} = \begin{cases} M_{ij} + A_0 & \text{if } i \leq j, \\ M_{ij}[1] + A_0 & \text{otherwise.} \end{cases}$$

To show the flexibility of this construction, we define a motivic enhancement which has correspondences that are highly dependent on the objects. Let $Q = (Q_0, Q_1)$ be any quiver and $D_Q$ the path category of $Q$. By this we mean that the objects of $D_Q$ are the vertices $Q_0$ of $Q$ and, given $i, j \in Q_0$, the morphisms $\text{Hom}_{D_Q}(i, j)$ is the set of directed paths from $i$ to $j$ in $Q$. Let $\{(C_{ij}, \epsilon_{ij})\}_{(i,j)\in Q_0^2}$ be any collection of constructible subcategories of $C$, partitioned by arrows in $Q_1$ from $i$ to $j$. Given such an $a \in Q_1$ let $C_a$ be the subcategory $C^{-1}_a(a)$. Take $\mathcal{W} = \{(W_{ijk}, \nu_{ijk})\}$ to be a collection of motivic correspondences and, given a sequence of composable arrows

$$i_1 \xrightarrow{a_1} i_2 \xrightarrow{a_2} \cdots \xrightarrow{a_n} i_{n+1}$$
in $D_Q$, let $E_r \in C_a$ for $r \in \{1, \ldots, n\}$. We will assume that for any sequence of morphisms $\gamma_i \in W_{i(i+1)(i+2)} \cap \text{Ext}^*(E_i, E_{i+1})$, we have that

$$\mu^\mathcal{W}_n(\gamma_n, \ldots, \gamma_1) = 0.$$ 

Here the map $\mu^\mathcal{W}_n$ is the $n$-th multiplication map in the $A_\infty$-structure of $C$.

**Definition 3.9.** We call motivic enhancement minimal if the correspondence over every pair of composable morphisms has minimal dimension.

**Proposition 3.10.** Let $Q$ be a simply-laced quiver and $\Phi' = (\{C_a\}_{a \in Q_1}, \{(W_{ijk}, \nu_{ijk})\})$ be an assignment satisfying equation (22). Then $\Phi'$ has a minimal motivic enhancement.

**Proof.** An enhancement must be made so that there is a assignment of categories to all paths of arrows (or morphisms in $D_Q$). Let $a_j : i_j \rightarrow i_{j+1}$ be a collection of morphisms for $1 \leq j \leq n$ and $a = a_n \cdots a_1$. Take

$$C_a := \left\{ \oplus_{j=1}^n E_j[a_j - 1], \sum_{j=1}^{n-1} \alpha_j : E_j \in C_{i_j}, \alpha_j \in W_{a_{j+1}, a_j} \cap \text{Ext}^*(E_j, E_{j+1}) \right\}.$$
Here the notation \((E, \alpha)\) is for a twisted complex. If \(b_k : i_{n+k} \to i_{n+k+1}\) for \(1 \leq k \leq m\) is another collection of morphisms and \(b = b_m \cdots b_1\), we take
\[
(W_{i_{n+k+1}}, \nu_{i_{n+k+1}}) := (\iota_n \circ W_{i_{n+k+1}} \circ \pi_1, \iota_n \circ \nu_{i_{n+k+1}} \circ \pi_1).
\]

More explicitly, for all \(\alpha \in W_{i_{n+1+k+2}} \cap \mathrm{Ext}^*(E_n, F_1)\), take the composition
\[
\oplus_{j=1}^n E_j \xrightarrow{\pi_n} E_n \xrightarrow{\alpha} F_1 \xrightarrow{\iota_k} \oplus_{k=1}^m F_k.
\]
to yield a correspondence in \(W_{i_{n+1+k+1}}\). To obtain a precise characterization of the weight, write \(\kappa : W_{i_{n+1+k+2}} \to W_{i_{n+1+k+1}}\) for this composition. Over \((E_n, F_1)\), this is a linear map \(\kappa(E_n, F_1) : \mathrm{Ext}^*(E_n, F_1) \to \mathrm{Ext}^*(E, F)\). Furthermore, owing to the description of the categories \(\kappa\), there are constructible functors from \(G_a : C_a \to C_b\) and \(G_b : C_b \to C_b\). The weight \(\nu_{i_{n+1+k+1}}\) is the pushforward of the weight \(\nu_{i_{n+1+k+1}}\) in the sense that
\[
\nu_{i_{n+1+k+1}}(E, F) := \nu(G_a(b), G_a(a))(G_a(E), G_b(F)) + \dim(\ker(\kappa(G_a(E), G_b(F)))).
\]

It immediately follows that if \([\rho_{E_j} : pt \to C_a]\) are objects taking \(pt\) to \(E_j\), then
\[
(23) \quad \rho_1 \cdots \rho_n = \int_{\alpha_j \in W_{i_{n+1+k+2}}} \mathbb{L} \sum_{j=1}^{n-1} \nu_{j+1}(E_j) \rho(\iota_{E_j}, \sum \alpha_j).
\]

We note that one may prove a version of this theorem purely in the triangulated setting where associativity of compositions is directly tied to the octahedral axiom. We conclude this section with an example illustrating this connection.

**Example 3.11.** Let \(f_0 : A_0 \to A_1\) and \(f_1 : A_1 \to A_2\) be degree 0 morphisms in \(\mathcal{C}\) such that \(f_1 f_0 = 0\). Taking \(\delta : A_1 \to B = \text{cone}(f_0)\) and applying the long exact sequence gives a morphism \(h : B \to A_2\) such that \(f_1 = h \delta\). Thus, \(f_0\) and \(f_1\) will generate a bottom “cap”

\[
\begin{align*}
\begin{figure}
\end{figure}
\end{align*}
\]

The octahedral axiom ensures the existence of the “top” cap

\[
\begin{align*}
\begin{figure}
\end{figure}
\end{align*}
\]
Let $D$ be the category with objects labeled by integers and a single morphism $f_{rs} : r \to s$ if and only if $r \leq s$. In particular, this is the path category associated to $Q = (\mathbb{Z}, Q_1)$ where $Q_1$ consists of arrows from $a_r : r \to r + 1$. For $r \in \mathbb{Z}$, write $\bar{r}$ for the congruence class of $r$ modulo $4$. Define the subcategories

$$C_{\bar{r}} = \{ A_{\bar{r}} \}$$

For the correspondences we simply take

$$W_{(ar-1, ar)} = \{ f_r \}$$

Then the first few terms of the motivic enhancement can be sketched as below:

![Diagram](attachment:image.png)

3.3. $\mathbb{V}$-stability conditions. Let us start by defining a certain collection of subcategories of an ind-constructible, triangulated $A_\infty$-category $C$ (for example, the category of twisted complexes of an ind-constructible $A_\infty$-category). If $C'$ is a subcategory of $C$, we write

$$[C'] = \{ [A] \in K_0(C) : A \in Ob(C') \}$$

for the collection of $K_0(C)$ classes of objects in $C'$. Recalling the definition of $\deg_{A,B}(W_{ijk})$ from Section 3.1, we say that $\Phi = (\{ C_{ij} \}, \{ W_{ijk} \})$ is an odd motivic enhancement of $D$, if $\deg_{A,B}(W_{ijk})$ is odd for all $(A, B) \in \text{supp}(W_{ijk})$. Given an odd enhancement $\Phi$, we may define the Grothendieck category $K_\Phi(D)$ of the enhancement as the category with objects equal to $Ob(D)$ and morphisms

$$\text{Hom}_{K_\Phi(D)}(i, j) := [C_{ij}]$$

To justify that $K_\Phi(D)$ is well defined, assume $[A] : i \to j$ and $[B] : j \to k$ and let $r = \deg_{A,B}(W_{ijk})$. Then, since the support of $W_{ijk}$ contains $C_{jk} \times C_{ij}$, it follows that there is some $\delta \in W_{ijk} \cap \text{Ext}^r(A, B)$ such that $\text{cone}(\delta) \in C_{ik}$. Of course, $[\text{cone}(\delta)] = [B] - [A[r]] = [B] + [A]$ so that composition of morphisms is well defined.

For the following definition, we do not require that $\mathbb{V}$ is connected but will assume that the morphisms of $\mathbb{V}$ form an ind-constructible stack.

**Definition 3.12.** Let $C$ be an ind-constructible, triangulated $A_\infty$-category. A motivic enhancement of $\mathbb{V}$ by $\Phi = (\{ C_{ij} \}, \{ W_{ijk}, \nu_{ijk} \})$ will be called a $\mathbb{V}$-collection if

1. $C_{ij} = C_{ji}[1]$,
2. for any $E \in C_{ij}$, the morphism $\text{Id}[1] : E \to E[1]$ is in the support of $W_{ij}$,
3. the morphisms $\epsilon_{ij}$ induce an isomorphism $\epsilon : K_\Phi(\mathbb{V}) \to \mathbb{V}$.

We will say it is a standard $\mathbb{V}$-collection if $W_{ijk}$ is the pullback of $\text{Ext}^1$ along the inclusion $Ob(C_{jk}) \times Ob(C_{ij})$ to $Ob(C) \times Ob(\mathbb{C})$ and $\nu_{ijk}$ is the $N$-truncated Euler characteristic with $N = 0$. 

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ON 2D-4D MOTIVIC WALL-CROSSING FORMULAS

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Remark 3.13. Property (1) implies that the categories $C_{ij}$ are stable under even shifts. Property (2) asserts that one may think of $E[1] \in C_{ji}$ as the inverse of $E \in C_{ij}$.

Remark 3.14. While one frequently uses the notion of equivalence between categories, and rarely the notion of isomorphism, condition (3) in Definition 3.12 relies on the strict notion of isomorphism. Indeed, the case of a trivial groupoid, i.e. one categorically equivalent to a trivial group, is still of substantial interest as it relates directly to the purely 2d WCF.

We will often shorten the notation of a $V$-collection of subcategories to $\{C_{ij} : i,j \in V\}$. A $V$-collection of subcategories $\{C_{ij} : i,j \in V\}$ will be the central notion behind the categorification of 2d-4d wall-crossing data. To make this precise, we must first give definitions of a central charge, Harder-Narasimhan sequence and slicings in this context.

Definition 3.15. A $V$-central charge is a homomorphism of groupoids $Z : V \to C$ where $C$ is taken as the groupoid with one object whose morphisms are $C$.

As seen in the previous section, one may easily describe central charges in the context of $V$-collections. For example, consider a homomorphism $\phi : K_0(C) \to \Lambda$, $A \subset \Lambda$ and the groupoid $\Lambda_\phi$ defined in equation (17). The motivic enhancement $\Phi$ induced by $\phi$ is a standard $\Lambda_\phi$-collection. A generic homomorphism $\tilde{Z} : \Lambda \to C$ (30) will yield a $V$-central charge on $\Lambda_\phi$ by composing with $\phi$.

Definition 3.16. Let $\Phi := \{C_{ij} : i,j \in V\}$ be a $V$-collection of ind-constructible subcategories of $C$ and $Z : V \to C$ a $V$-central charge. A pre-slicing $P$ of $\Phi$ is a collection of ind-constructible subcategories $P_{ij}(\theta) \subset C_{ij}$ for every $i,j \in V$ and $\theta \in \mathbb{R}$. These must satisfy the following conditions:

(i) For all $\theta \in \mathbb{R}$ and $i,j \in V$, $P_{ij}(\theta + 1) = P_{ji}(\theta)[1]$,
(ii) For $i,j,k,l \in V$, $\theta_1 > \theta_2$, $E_1 \in P_{ij}(\theta_1)$ and $E_2 \in P_{kl}(\theta_2)$, $\text{Ext}^0(E_1, E_2) = 0$.

We will often refer to an object $E \in P_{ij}(\theta)$ as semi-stable.

From this point on, we assume that $V$ is a finite groupoid. For a standard $V$-collection $\Phi = \{C_{ij} : i,j \in V\}$, may use the Hall category to define a subalgebra $\bar{H}_\Phi(V) \subset M_r(H(C))$ (31) of $r \times r$-matrices with entries in the usual motivic Hall algebra of $C$. Here $r$ is the number of objects in $V$. In particular,

$$\bar{H}_\Phi(V) = \left\{ \left( \sum_k [\pi^k_{ij} : X^k_{ij} \to C] \cdot \mathbb{L}^k \right) \in M_r(H(C)) : \pi^k_{ij}(X^k_{ij}) \subseteq C_{ij} \right\}. $$

When the $V$-collection $\Phi$ is not standard, we may continue to define $\bar{H}_\Phi(V)$ as an algebra whose set is given in equation (32), but whose multiplicative structure is different. Write $\mathfrak{g}_\Phi$ for the associated Lie algebra and observe that this is a $V$-graded Lie algebra. Furthermore, a pre-slicing $P$ gives rise to a collection $a_P := (a(\gamma))_{\gamma \in \Gamma_V}$. This is obtained as follows: for $\gamma$ corresponding to a morphism $\bar{\gamma}$ from $i$ to $j$, take $a(\gamma) = e_{ij}^{-1}(\bar{\gamma}) \cap P_{ij}$. 
Definition 3.17. A central charge \( Z \) satisfies the support property relative to the pre-slicing \( \mathcal{P} \) if it \((Z,a_{\mathcal{P}})\) satisfies the support property.

The notion of a Harder-Narasimhan sequence for \( \mathcal{V} \)-collections contains slightly more information than the classical case.

Definition 3.18. Let \( \Phi := \{ C_{ij} : i,j \in \mathcal{V} \} \) be a \( \mathcal{V} \)-collection of subcategories of \( \mathcal{C} \), \( \mathcal{P} \) a pre-slicing and \( E \in C_{ij} \) with \( \epsilon_{ij}(E) = \gamma \). A Harder-Narasimhan sequence for \( E \) is a sequence of objects \( i = i_0, \ldots, i_n = j \) in \( \mathcal{V} \), a sequence of objects \( A_k \in C_{ik_{k-1}}(\theta_k) \) for which

1. \( \theta_1 > \theta_2 > \cdots > \theta_k \),
2. If \( \epsilon_{ik_{k-1}}(A_k) = \gamma_k \) then the composition,

\[
\begin{array}{c c c c c c c}
& \gamma_n & i_n & \gamma_{n-1} & \cdots & \gamma_2 & i_1 & \gamma_1 & i_0 = j \\
\end{array}
\]

is \( \gamma = \gamma_1 + \cdots + \gamma_k \).
3. There are maps \( \delta_k \in \text{Ext}^1(A_k,E_{k-1}) \cap W_{i_0i_{k-1}i_k} \) yielding the convolution

\[
0 = E_0 \xrightarrow{\delta_1} E_1 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{n-1}} E_{n-1} \xrightarrow{\delta_n} E_n = E
\]

Given a Harder-Narasimhan sequence as above, we call the sequence \( i = i_0, \ldots, i_n = j \) the path of the sequence.

We note that in the \( \mathcal{V} \)-collection context, this condition translates to the existence of a special type of the usual Harder-Narasimhan sequence for \( E \) in terms of semi-stable objects lying over the morphisms of \( \mathcal{V} \). Now we may define a stability condition for the \( \mathcal{V} \)-collection.

Definition 3.19. Given a \( \mathcal{V} \)-collection of ind-constructible subcategories \( \Phi := \{ C_{ij} : i,j \in \mathcal{V} \} \) of \( \mathcal{C} \), a \( \mathcal{V} \)-stability condition \((Z,\mathcal{P})\) consists of a \( \mathcal{V} \)-central charge \( Z \) and a pre-slicing \( \mathcal{P} \) such that

1. \( Z \) satisfies the support property relative to \( \mathcal{P} \),
2. \( E \in \mathcal{P}_{ij}(\theta) \) then arg\( (Z(\phi_{ij}([E]))) \equiv \theta \mod 2\pi \),
3. every non-zero \( E \in C_{ij} \) has a Harder-Narasimhan sequence.

Observe that if \( \mathcal{V} \) is the category with one object and one morphism and \( C_{ii} = \mathcal{C} \) along with the standard weight \( \nu \), then a \( \mathcal{V} \)-stability condition reduces to the generalization of the Bridgeland stability conditions defined in [9] (the constructibility and the important Support Property were added to the axiomatics of [1]). Using the topology on stability data for \( \mathcal{g}_\Phi \), we obtain a topology on the space of \( \mathcal{V} \)-stability conditions. We also retain a version of uniqueness of Harder-Narasimhan sequences in this context.

Proposition 3.20. Given a \( \mathcal{V} \)-stability condition on \( \mathcal{V} \), \( E \in C_{ij} \) and a sequence \( i = i_0, \ldots, i_n = j \) of objects in \( \mathcal{V} \), there is at most one Harder-Narasimhan sequence \( A_1, \ldots, A_n \) of \( E \) with \( A_k \in C_{ik_{k-1}} \). Furthermore, if \( B_1, \ldots, B_m \) is another Harder-Narasimhan sequence with \( B_k \in C_{ik_{k-1}} \) then \( m = n \) and \( B_k = A_k \).

Proof. This follows from the uniqueness argument of Harder-Narasimhan sequences for slicings as in [1] Section 3] applied to the pre-slicing \( \mathcal{P}_{ij} \).[\( \square \)

Proposition 3.20 allows us to define the following constants.
Definition 3.21. Given a $\mathcal{V}$-stability condition $(Z, \mathcal{P})$ on $\Phi$, $E \in \mathcal{C}_{ij}$, let $\theta^-(E) = \theta_1$ and $\theta^+(E) = \theta_n$ if there exists a Harder-Narasimhan sequence $A_1, \ldots, A_n$ of $E$ such that $A_1 \in \mathcal{P}_{i_0}(\theta_1)$ and $A_n \in \mathcal{P}_{i_{n-1}}(\theta_n)$.

3.4. Motivic 2d-4d wall-crossing formulas. Let us now describe the wall-crossing formulas associated with the standard $\mathcal{V}$-stability collection.

Let $I = [\theta_1, \theta_2]$ be any half open interval in $\mathbb{R}$. For any sequence of objects $s = (i_0, \ldots, i_r)$ in $\mathcal{V}$, define the ind-constructible category

$$\mathcal{P}_{(i_0, \ldots, i_r)}(I) := \{ E \in \mathcal{C}_{i_{0 i_r}} : \theta^\pm(E) \in I, E \text{ has a HN sequence with path } s \}.$$ 

Let $S_{ij} = \cup_{n \in \mathbb{N}} \{ i \} \times \mathbb{V}^n \times \{ j \}$ be the set of all paths from $i$ to $j$. Using Proposition 3.20 once again along with the finiteness of $\mathcal{V}$, we may define the element

$$A_{ij}(I) = \sum_{s \in S_{ij}} [\mathcal{P}_s(I) \to \mathcal{C}_{ij}].$$

for every $i, j \in \mathcal{V}$. Clearly $A_{ij}(I)$ is an element of the motivic Hall algebra of $\mathcal{C}_{ij}$. Assembling these elements into a matrix gives the element

$$A_I = (A_{ij}(I)) \in \hat{H}_q(\mathcal{V}).$$

This brings us to the motivic Hall category wall-crossing formula.

Theorem 3.22. Given a standard $\mathcal{V}$-collection of subcategories $\Phi := \{ \mathcal{C}_{ij} : i, j \in \mathcal{V} \}$ of $\mathcal{C}$ and a $\mathcal{V}$-stability condition $(Z, \mathcal{P})$, if $\theta_1 < \theta_2 < \theta_3$ then

$$A_{\theta_1, \theta_2} \cdot A_{\theta_2, \theta_3} = A_{\theta_1, \theta_3}.$$
for numerical DT-invariants one should assume the “absence of poles conjecture” from the loc. cit.

For a geometrically minded reader and in order to make a link with \cite{3} let us provide a more geometric intuitive picture of the numerical Donaldson-Thomas invariants in the categorical framework.

For that we assume that, given a generic stability condition \((Z, P)\) on the ind-constructible \(A)\)-triangulated category \(H^0(C)\), and a class \(\lambda \in K_0(C)\) there is a moduli stack \(\mathcal{M}^\lambda_{\text{ss}}\) of isomorphism classes of semi-stable objects with \(K_0\) class \(\lambda\) which is Artin and of finite type. We assume that there are sub-stacks \(\mathcal{M}^\lambda_{\text{ss}}(0)\) which carry a virtual fundamental class of virtual dimension zero. The associated function \(\Omega_{(Z, P)} : K_0(C) \to \mathbb{Z}\) given by \(\Omega_{(Z, P)}(\lambda) = \# \mathcal{M}^\lambda_{\text{ss}}(0)\) will be called the DT invariant of \((Z, P)\). Here \(\Omega_{(Z, P)}(\lambda)\) is meant to count the “number of semi-stable objects whose \(K_0\) class equals \(\lambda\)”.

We do not discuss here the actual meaning of these words, since for now, questions of when and how such a function is defined will be irrelevant. We will say that a sub-category \(i : B \hookrightarrow C\) is algebraic if the moduli stack \(\mathcal{M}^\lambda_{\text{ss}}\) in \(B\) is an algebraic substack of \(\mathcal{M}^\lambda_{\text{ss}}\). For full algebraic subcategories \(B \subset C\), one may restrict the count of \(\mathcal{M}^\lambda_{\text{ss}}\) to the substack of objects in \(B\). We write this function as \(\Omega^B_{(Z, P)}\).

Now assume that a generic stability condition \((Z, P)\) on \(C\) has been chosen. Given an algebraic \(\mathbb{V}\)-collection of subcategories \(\{C_{ij} : i, j \in \mathbb{V}\}\) of \(C\), we define a 2d-4d wall-crossing data as follows. Let \(\Gamma = K_0(C) \otimes \mathbb{Z}\) and the pairing \((,\)\) equal to the anti-symmetrization of the Euler form on \(\Gamma\). Take \(\Gamma_{ij} = K_0(C_{ij}) \otimes \mathbb{Z}\) with the same pairing. Note that in our setting, we need not have that \(\Gamma_{ij}\) is a torsor over \(\Gamma\). Nonetheless, we may restrict \(Z : \Gamma \to \mathbb{C}\) to \(\Gamma_{ij}\) to obtain a central charge \(Z : \mathbb{V} \to \mathbb{C}\). Take \(\Omega = \Omega^Z_{(Z, P)}\) and \(\mu : \Gamma_{ij} \to \mathbb{Z}\) as \(\mu = \Omega^Z_{(Z, P)} \circ \phi^{-1}_{ij}\). Finally, for \(\sigma\) introduced in Section \[3.3\] we take \(\sigma([B], [C]) = (-1)^{(|B|, |C|)}\). This completes the description of numerical DT-invariants.

4. Examples from algebra

We will now give examples of motivic correspondences, enhancements and \(\mathbb{V}\)-collections in the algebraic setting. We will explore \(\mathbb{V}\)-stability conditions arising in an elementary algebraic case.

4.1. Algebraic motivic enhancements. The first set of examples explores a categorification for a pre-triangulated dg-category \(A\).

**Example 4.1.** Let \(A\) be the category of twisted complexes over a given dg-category \(C\) and let \(\mathbb{V}\) be a trivial connected groupoid, i.e. every pair of objects in \(\mathbb{V}\) has a unique invertible morphism between them. Assume the set of objects in \(\mathbb{V}\) is labeled by a set \(\mathbb{V}\) of objects in \(C\). Given any \(X, Y \in \mathbb{V}\), let \(A_{XY}\) be the full subcategory of twisted complexes of the form \(C_g := (Y \oplus X[-1], d_g)\) where

\[
d_g = \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}
\]

for some zero degree cocycle \(g : Y \to X\). In other words, \(A_{XY}\) are all cones of morphisms from \(Y\) to \(X\) (shifted by \(-1\)). Noting that \([A_{XY}] = \{[Y] - [X]\}\) and \(\Gamma_{XY}\) are both one element sets, the maps \(\Phi_{XY} : [A_{XY}] \to \Gamma_{XY}\) are uniquely determined.
Given objects $C_g$ and $C_h$ lying over two composable morphisms

\[(39) \quad X \xrightarrow{C_g} Y \xrightarrow{C_h} Z \]

Take $f_{C_g, C_h}: C_g \to C_h$ to be the map given by

\[(40) \quad X[-1] \quad Y[-1] \xrightarrow{\oplus} \oplus \quad Y \quad Z \]

It is clear that cone($f_{C_g, C_h}$) is quasi-isomorphic to $C_{gh} \in A_{XZ}$. Associativity is also immediate from this observation.

Our next example is a generalization of the last one to non-trivial groupoids.

**Example 4.2.** As in the previous example, let $A$ be the category of twisted complexes over a given dg-category $C$ with a distinguished set of objects $V$ in $C$. Take $B \subseteq A$ to be a full pre-triangulated subcategory, $\Gamma$ its Grothendieck group $K_0(B)$, and $\tilde{\Gamma}$ the Grothendieck group $K_0(A)$ of $A$. Define a formal element $o$ and take $V$ to be the groupoid with objects labeled by $V \cup \{o\}$. Map the additional element $o$ to $\tilde{\Gamma}$ by taking $[o] = 0 \in \tilde{\Gamma}$. For every pair $X, Y \in V \cup \{o\}$ define the set of morphisms in $V$ from $X$ to $Y$ to be the coset

\[(41) \quad \Gamma_{XY} = [Y] - [X] + \Gamma \subset \tilde{\Gamma}. \]

Composition is simply addition in $\tilde{\Gamma}$.

For any $X \in V$, let $B_X$ be the full subcategory of objects which are quasi-equivalent to cones cone($f$)[-1] or cone($f'$) where $f: X \to B$ or $f': B \to X$ for some $B \in B$. Also take $B_o := B$. Then, for any $X, Y \in V \cup \{o\}$, take $A_{XY}$ to be the full subcategory of all objects $C_g := \text{cone}(g)[-1]$ for morphisms $g: Y \to X$ where $Y \in B_Y$ and $X \in B_X$. Noting that any object $X \in B_X$ has $K_0(A)$ class lying in the coset $[X] + \Gamma$, and similarly with $Y$, we have that $[C_g] = [Y] - [X] \in [Y] - [X] + \Gamma$ yielding a well defined map $\phi_{XY}: [A_{XY}] \to \Gamma_{XY}$. The morphism $f_{C_g, C_h}$ is as in Example 4.1.

There are several variants that can be considered of examples 4.1 and 4.2. For instance, one could consider cones of any even degree morphism instead of simply zero degree morphisms. Alternatively, in Example 4.2, one could restrict to shifts of cones from $B$ to $X$ or to cones from $X$ to $B$. We now examine a $V$-subcollection of $C_V$-collection as in Example 4.1.

**Example 4.3.** Consider the $A_n$-category as in Examples 3.2 and 3.8. Take $V = \mathbb{Z}^{n+1}_{\phi, A}$ to be the trivial connected quiver with objects $\{0, 1, \ldots, n\}$ and $\{C_{ij}\}$ the induced $V$-collection of subcategories given in (21). In particular, take the standard motivic correspondence induced by $\phi$ and $A$ as in Definition 3.7.

Now let $\tilde{Z}: \mathbb{Z}^{n+1} \to \mathbb{C}$ be a homomorphism so that, by composing with $\phi$, we obtain the central charge $Z = \phi \circ \tilde{Z}$ which, in this case, simply assigns the unique morphism $i \to j$ to $\lambda_{ij} := \tilde{Z}(e_j - e_i)$. Assume no three points in $\{\tilde{Z}(e_i)\}$ are collinear. As we observe in the next section, for $n = 2$, the space of stability conditions can be described concretely.
4.2. Φ-stability conditions for $A_2$-representations. Next we consider the elementary, but rich, example of a $V$-stability condition for the category of $A_2$-representations. 

Consider the $A_2$-category $\mathcal{A}$ as in Examples 3.2 and 3.8. Then $V$ is a trivial groupoid with objects $\{0, 1, 2\}$. Restricting to the indecomposable objects leaves us with the representation illustrated in Figure 2 of the motivic enhancement over $V$. The labels over the morphisms in $V$ indicate all objects lying in $C_{ij}$ which are even or odd translations of the indicated indecomposable representations. Any function $f : \{0, 1, 2\} \to \mathbb{C}$ induces a homomorphism $\tilde{Z} : \mathbb{R}^3 \to \mathbb{C}$ which, in turn, yields the central charge $Z_f : K_0(\mathcal{A}) \to \mathbb{C}$ induced by $\phi$ as in (30). Note that $Z_f$ is uniquely determined up to a translation of $f$. For $Z_f$ to be generic, we must have that $f(i) \neq f(j)$ for all $i \neq j$. Let 

$$P := \{f : f(0), f(1), f(2) \text{ distinct}\} \text{ translations}$$

be the set of such central charges. One notes that $P$ is isomorphic to $P_1 \times \mathbb{C}^*$ where $P_1$ is a pair of pants (i.e. $C \setminus \{0, 1\}$). To obtain a pre-slicing, we must first choose an argument $\theta_{ij}$ of $f(j) - f(i)$ for $j > i$. Such a choice yields an abelian cover $\tilde{P}$ of $P$ as well as a map $\Theta : \tilde{P} \to \mathbb{R}^3$. For a fixed central charge, the determination of a stability condition is purely dependent on the choice of the arguments (but not necessarily determined uniquely), so it remains to find the possible pre-slicings for any given element $(\theta_{01}, \theta_{12}, \theta_{02}) \in \Theta(\tilde{P})$.
Note that rotating $f$ by $e^{2\pi i \theta}$ results in adding $(\theta, \theta, \theta)$ to $(\theta_{01}, \theta_{12}, \theta_{02})$ and results in no effect on the constraints in the pre-slicing (nor, as we will show in a moment, the existence of Harder-Narasimhan sequences). Thus we may find $\Theta(P)$ by considering its intersection with (or projection to) the plane $V = \{(\theta_{01}, \theta_{12}, \theta_{02}) : \theta_{01} + \theta_{12} + \theta_{02} = 0\}$.

Take $\alpha_1 = \theta_{02} - \theta_{01}$, $\alpha_2 = \theta_{12} - \theta_{02}$ and $\alpha_3 = \theta_{12} - \theta_{01}$ to be elements of $V$ and note that these form the positive roots of the $A_2$-root system. To describe possible stability conditions, we first identify the regions in the plane $V$ where there exists an $f$ yielding $(\theta_{01}, \theta_{12}, \theta_{02})$. First note that in the ambient $\mathbb{R}^3$, we may restrict to the region $[-1, 1]^3$ (as translating by an even integral lattice element results in a distinct choice of arguments for the same $f$) and that, upon projecting to $V$, we obtain a hexagonal region bounded by the inequalities $|\alpha_i| \leq 1$ for all $i$. The existence of an $f$ then implies that either $\alpha_1 > 0$ and $\alpha_2 > 0$ or $\alpha_1 < 0$ and $\alpha_2 < 0$. These regions are depicted in Figure 3.

The image is known as the coamoeba of a pair of pants when considered in $V/(\mathbb{Z}^3 \cap V)$. For a rigorous proof of this description and its relationship to the affine $A_2$ alcove decomposition, see [7, 11]. Thus the regions in $V$ for which there exists a $V$-central charge are translates of this by $2\mathbb{Z}\{\alpha_1, \alpha_2\}$.

\[ M_{02} \rightarrow M_{12} \]
\[ M_{01}[1] \]
\[ \alpha_1 > 1 \]

\[ M_{12}[-1] \rightarrow M_{01} \]
\[ M_{02} \]
\[ \alpha_2 > 1 \]

\[ M_{01} \rightarrow M_{02} \]
\[ M_{12} \]
\[ 0 > \alpha_3 \]

**Figure 4.** Inequalities for pre-slicings.

**Figure 5.** Inequalities associated to Harder-Narasimhan sequences.
Continuing, we must now describe the stability conditions, i.e. pre-slicings which satisfy Definitions 3.16, 3.19. By Definition 3.16(i) we must have that either all translations $M_{ij}[n]$ are semi-stable or none. Also, in order to satisfy Definition 3.16(ii), equation (9) implies that the restrictions illustrated in Figure 4 must be made if the corresponding pairs of indicated objects are semi-stable.

Finally, we must examine when, in any of the cases from Figure 4, the remaining indecomposable has a Harder-Narasimhan sequence in terms of the semi-stable indecomposables. Such sequences arise as rotations of the single exact triangle between indecomposables, each yielding a corresponding inequality illustrated in Figure 5.

Putting these inequalities together yields a picture of the space of stability conditions $\text{Stab}_\Phi(A)$ for $A_2$ as shown in Figure 6. To understand this picture, we note that there is a local homeomorphism $\pi : \text{Stab}_\Phi(A) \to \tilde{P}$ which (after quotienting by rotation) maps to $V$ in such a way that over the grey, yellow, blue and red regions is one to one and over the purple, green and orange regions is two to one. Over the grey central region, we must have that all indecomposables are stable. We note that the only points on $V$ over which the set of stable objects changes (locally in $\text{Stab}_\Phi(A)$) is at the vertices of the grey triangle.

The space of stability conditions for $A_n$ appears to have a similar description in terms of alcove decomposition of its affine root system. It is an intriguing question to explore analogous structures for the $D$ and $E$ root systems.
5. Examples from symplectic geometry

5.1. Fukaya and matching path categories. Let \((X, \omega)\) be a Kähler manifold, \(Y\) a complex curve and \(f : X \to Y\) a holomorphic function with complex Morse singularities. Let \(\text{Crit}_p(f)\) and \(\text{Crit}_v(f)\) denote the critical points and values of \(f\) respectively. We assume that the critical values of two distinct critical points are distinct. For \(y \in Y\) we write \(F_y\) for the fiber \(f^{-1}(y)\). Let \(X^\circ = X \setminus \bigcup_{y \in \text{Crit}_v(f)} F_y\) be the complement of the singular fibers of \(f\), \(Y^\circ = Y \setminus \text{Crit}_v(f)\) and denote \(f^\circ : X^\circ \to Y^\circ\) for the restriction of \(f\). As described in [2] the symplectic orthogonal to each fiber \(F_y\) defines a connection on the fiber bundle \(f^\circ\). Assuming reasonable behavior of \(f\) at infinity (for example, one may assume that \(f\) arises as the a pencil with smooth base loci and \(X\) is the complement of the base loci), we then obtain a symplectic parallel transport map

\[ P : \Pi(Y^\circ) \to \text{Symp/Ham}. \]

This is a functor taking the fundamental groupoid \(\Pi(Y^\circ)\) of \(Y^\circ\) to the category of symplectic manifolds with symplectomorphisms up to Hamiltonian isotopy. Here \(P(y) = F_y\) and, given a path \(\delta : [0, 1] \to Y\) from \(y_0\) to \(y_1\) representing a morphism in \(\Pi(Y^\circ)\), the associated symplectomorphism \(P(\delta)\) is obtained by symplectic parallel transport.

**Definition 5.1.** An immersed path \(\delta : [0, 1] \to Y\) is called \(f\)-admissible if

1. \(\delta([0, 1]) \subset Y^\circ\),
2. \(\delta(1) \in \text{Crit}_v(f)\).

Given an \(f\)-admissible path \(\delta\) for which \(\delta(1)\) is the image \(f(x)\) of the critical point \(x\), we define the Lagrangian vanishing cycle \(L_\delta\) over \(\delta\) to be the submanifold

\[ L_\delta := \left\{ p \in F_{\delta(0)} : \lim_{t \to 1} P(\delta_{[0,t]})(p) = x \right\}. \]

We also recall that the vanishing thimble over \(\delta\) is defined as

\[ T_\delta = \cup_{t \in [0,1]} L_{\delta_{[t,1]}} \cup \{x\}. \]

In other words, the thimble is the union of the vanishing cycles over \(\delta\) along with the critical point \(x\). It is well known that \(L_\delta\) is an exact Lagrangian sphere in \(F_{\delta(0)}\) and that \(T_\delta\) is an immersed Lagrangian ball (possibly after a suitable perturbation).

For every \(y \in \text{Crit}_v(f)\), we fix a small embedded disc \(D_y \subset Y\) so that \(D_y \cap \text{Crit}_v(f) = y\). We also choose a point \(\tilde{y} \in \partial D_y\). Then there is an unambiguous way...
to define the Hamiltonian isotopy class of the vanishing cycle of $y$ in $F_\tilde{y}$. Indeed any two embedded paths from $\tilde{y}$ to $y$ in $D_\tilde{y}$ yield Hamiltonian isotopic spheres.

Now suppose $y_0, y_1 \in \text{Crit}_v(f)$ and $\tilde{y}_0, \tilde{y}_1$ are their neighboring points. An immersed path $\delta : [0, 1] \to Y \setminus (\cup D_{\tilde{y}})$ with $\delta(0) = \tilde{y}_0$ and $\delta(1) = \tilde{y}_1$ will be called a matching path if $P(\delta)(L_0)$ is Hamiltonian isotopic to $L_1$. Extending such a matching path to the critical values by concatenating with $\delta_{y_0}$ and $\delta_{y_1}$, one has an immersed path in $Y$ over which lies an immersed Lagrangian sphere $M_\delta$ called the matching cycle of $\delta$.

**Example 5.2.** Take $X, Y$ and $f$ as above. Let $\tilde{Y} \subset Y$ be a sub-surface with boundary $\partial Y \cong S^1$ and $\{y_1, \ldots, y_{r+1}\} = \partial Y \cap \text{Crit}_v(f)$ the critical values of $f$ on the boundary oriented clockwise. Also assume that $\tilde{Y} = \cup_{i=1}^r \tilde{Y}_i$ where

1. $y_{i-1}, y_i \in \tilde{Y}_i$,
2. $\tilde{Y}_i \cap \tilde{Y}_j = \emptyset$ for $j \neq i + 1$ and,
3. $\tilde{Y}_i \cap \tilde{Y}_{i+1}$ is a closed subset of the boundary of both.

Let $D$ be the subcategory of the fundamental groupoid $\Pi(\tilde{Y})$ defined as follows. The objects of $D$ are the critical values $\{y_1, \ldots, y_r\}$ of $f$ on the boundary of $\tilde{Y}$. The morphisms are defined via

$$
\text{Hom}_D(y_i, y_j) = \begin{cases} 
\{ \delta \in \Pi(\cup_{i=1}^{j-1} \tilde{Y}_i) : \delta(0) = y, \delta(1) = y_j \} & \text{if } j > i, \\
\{1\} & \text{if } j = i, \\
\emptyset & \text{if } j < i.
\end{cases}
$$

In general, it could occur that the groupoid $V$ may not be connected.

Take $C$ to be the Fukaya category of $X$. For $y_i, y_j \in V$, we take $C_{y_i y_j}$ to be the full subcategory of $C$ with objects

$$
\{M_\delta : \delta \in \text{Hom}_V(y_i, y_j)\}.
$$

For two composable morphisms $[\delta] \in \text{Hom}_D(y_i, y_j)$ and $[\delta'] \in \text{Hom}_D(y_j, y_k)$, the critical point $y_j$ is a distinguished intersection point $M_\delta \cap M_{\delta'}$. Thus for the motivic correspondence we simply take

$$
W_{y_i y_j y_k} = \{y_j\} \times C_{y_i y_j} \times C_{y_j y_k}
$$

along with the trivial weight $\nu = 0$.

By the result of [10, Theorem 17.16], we have that $\text{cone}(y_j)$ is a matching cycle over the matching path which is the concatenation $\delta' \ast \delta \in \text{Hom}_D(y_i, y_k)$.

![Figure 8. The regions $\tilde{Y}_i$.](image-url)
We follow this general construction with a particular case.

Example 5.3. Take $X$ to be the exact symplectic manifold which is the smooth fiber of an $A_r$-singularity and $Y = \mathbb{C}$. For purposes of illustration, consider $X$ to be the Riemann surface $\{(x, y) : x^2 + y^2 + x^{n+1} = 1\}$ and take $f : X \to \mathbb{C}$ to be a Lefschetz fibration which is a double branched cover. We may assume the critical values $\{y_0, \ldots, y_n\}$ of $f$ are $\{0, \ldots, n\} \subset \mathbb{R}$. Letting $y_{-1} = -\infty$ and $y_{n+1} = \infty$, we define the regions $\tilde{Y}_i = \{z \in \mathbb{C} : \text{Im}(z) \leq 0, y_{i-1} \leq \text{Re}(z) \leq y_i\}$ and take $\tilde{Y}$ to be the negative imaginary half-plane. Then it is clear that for $i < j \text{ Hom}_D(y_i, y_j)$ consists of one path $\delta_{ij}$, over which there is a matching cycle which is a circle in $X$. Either through a direct appeal \[8\] or through an elementary computation, one exhibits an isomorphism between the motivic correspondence in this case and that in Example $3.4$.

5.2. Fukaya-Seidel categories. Other examples that arise in symplectic geometry require additional background. In this section we adapt a notion of the Fukaya-Seidel category to one over an arbitrary base curve $Y$ (as opposed to the conventional definition over $\mathbb{C}$). We will eschew some open questions regarding generators of this category and simply define it using a basic class of generators. The classical construction which we refer to is given in [10]. One aim is to apply matching path techniques to give some geometrically motivated examples of 2d-4d categorified stability structures.

First we define the category. As a preliminary step, we consider a particular flow on the disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Let $g : D \setminus \mathbb{R}_{\leq 0} \to \mathbb{R} \times \mathbb{R}_{\geq 0}$ be an orientation preserving diffeomorphism taking $re^{i\theta}$ to $(-\ln(r), \tilde{g}(\theta))$ for a monotonic $\tilde{g}$. Let $H : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a function that is zero on $\mathbb{R}_{\leq 0} \times \mathbb{R}$ and, for some constant $C > 0$, equal to $x$ for $x > C$. Taking the standard symplectic form $\omega_{st}$ on $\mathbb{R}^2$, consider the time $1$ Hamiltonian flow $\tilde{\phi}_H : \mathbb{R} \times \mathbb{R}_{\geq 0}$ and let $\phi$ be its pullback $g^*\left(\tilde{\phi}_H\right)$. It is clear that $\phi$ is the identity on $\partial D$. Now, let $f : X \to Y$ be as before and choose a regular basepoint $z \in Y$ along with a real tangent vector $\partial_z \in T_{\text{st}}Y$. Consider a disc $D_z$ centered at $z$ with $-\partial_x = \partial_z$. Identifying $D_z$ with $D$, we may extend $\phi$ on $D_z$ to all of $Y$.

To define the Fukaya-Seidel category $\mathcal{F}_z(f)$, we start with an initial $A_\infty$-category $\mathcal{A}_z(f)$. Let $X_z = X \setminus F_z$ and define an admissible Lagrangian thimble to be an open thimble $T_\delta \subset X_z$ of an $f$-admissible path $\delta : [0, 1] \to Y$ such that $\delta(0) = z$ and $\delta'(0) \neq \partial_z$. Using the quadratic holomorphic form on $X$, one may equip $T_\delta$ with a grading $\alpha$ and decorate it with a pin structure $\beta$. Take the triple $L_\delta = (T_\delta, \alpha, \beta)$ to be a Lagrangian brane.

Morphisms in $\mathcal{A}_z(f)$ are given by Floer complexes

$$\text{Hom}_{\mathcal{A}_z(f)}(L_0, L_1) := CF^*(L_0, \phi(L_1)).$$

The grading in this complex is determined by the respective gradings of the Lagrangian branes and the signs of the differential, composition and higher compositions are determined by pin structures. This definition assumes a universal choice of perturbation datum in order to ensure transversality.

Definition 5.4. The Fukaya-Seidel category $\mathcal{F}_z(f)$ of $f$ at $z$ is the category of twisted complexes $\text{Tw}(\mathcal{A}_z(f))$. 

When \( Y = \mathbb{C} \), there is an alternative and equivalent description of \( \mathcal{F}_z(f) \). Let \( \tilde{X} \) be defined as the pullback

\[
\begin{array}{ccc}
\tilde{X} & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\mathbb{C} & \longrightarrow & \mathbb{C}
\end{array}
\]

The \( \mathbb{Z}/2 \)-action on \( \mathbb{C} \) given by taking \( (u-z) \) to \( -(u-z) \) lifts to a \( \mathbb{Z}/2 \) symplectic action on \( \tilde{X} \). This induces an action on the Fukaya category of compact Lagrangians \( \mathcal{F}(\tilde{X}) \). The theorem of [10, Theorem 18.24] gives an equivalence between the \( \mathbb{Z}/2 \)-invariant subcategory of a Fukaya category \( \mathcal{F}(\tilde{X})^{\mathbb{Z}/2} \). When \( Y \neq \mathbb{C} \), this alternative route is not as readily available. In the first place, there is not a unique branched double cover \( \tilde{Y} \to Y \) with sole ramification point \( z \in Y \), and indeed, such a cover may not exist at all.

The Grothendieck group of a Fukaya-Seidel category admits a homomorphism to the relative homology

\[
\phi : K_0(\mathcal{F}_z(f)) \to H_n(X, f^{-1}(z))
\]

which takes a Lagrangian to its homology class. Composing with the connecting homomorphism gives the map

\[
\phi^\partial : K_0(\mathcal{F}_z(f)) \to H_{n-1}(f^{-1}(z)).
\]

Using Definition 3.7, one may choose to study the motivic enhancements of the groupoids \( H_n(X, f^{-1}(z))_{\phi} \) (respectively \( H_{n-1}(f^{-1}(z))_{\phi^\partial} \)) by \( \mathcal{F}_z(f) \). While interesting, the standard \( V \)-collection arising from this setup is difficult to access. Instead, we will reduce the dimension of our problem by considering potentials which factor through surfaces. Before discussing this approach, we consider a basic motivic enhancement in the Fukaya-Seidel setting.

### 5.3. Surface factorizations

The next construction is inspired by the approach in [5] to 2d-4d wall-crossings in Hitchin systems, which will be commented on more thoroughly in section 5.6. In this case, \( Y \) is the base curve \( C \), \( X \) is a conic bundle over cotangent bundle \( T^*C \) ramified over the spectral curve \( \Sigma \). As it turns out, there is a much more general framework which may be employed to provide a categorification of this setup. Given any \( f : X \to Y \) we write \( \Delta_f \subset X \) for the discriminant of \( f \); namely, the set of points in \( X \) for which the total derivative of \( f \) does not have maximal rank. We take \( \Delta_f \) to be the image \( f(\Delta_f) \).

**Definition 5.5.** A surface factorization \( S = (f, g, h, X, S, Y) \) of a holomorphic function

\[
f : X \to Y
\]

is a the commutative diagram of holomorphic functions

\[
\begin{array}{ccc}
X & \xrightarrow{g} & S \\
\downarrow f & & \downarrow h \\
Y & & 
\end{array}
\]

where \( S \) is a complex surface such that
(1) $h$ has no critical points,
(2) $\Delta_g$ is a smooth complex curve and $g|_{\Delta_g}$ is an isomorphism.
(3) $h|_{\Delta_g}$ has at worst Morse critical points.

This definition ensures that $f$ also has at worst Morse critical points. The symplectic version of this definition is that of a Lefschetz bifibration and can be found in [10, Section 15e].

![Diagram](image)

**Figure 9.** Vanishing thimbles, cycles and matching paths of a surface factorization.

Given a surface factorization of $f$, an $f$-admissible path $\delta$ will also be $h|_{\Delta_g}$-admissible. Thus there are two vanishing thimbles, $T^f_\delta$ an exact Lagrangian ball in $X$ and an interval $T^h_\delta \subset \Delta_g$, associated to such a path. The thimble $T^h_\delta \subset X$ has another feature. Namely, for any regular value $q \in Y$, the map $g$ restricted to the fiber $F_q = f^{-1}(q)$ is a complex Morse function with values in $h^{-1}(q)$ and critical values in $\Delta_g \cap h^{-1}(q)$. Furthermore, for $q = \delta(t)$ and $t \neq 1$, the vanishing cycle $\mathcal{L}^f_\delta$ of $\delta$ over $q$ is a matching cycle for a matching path $\bar{\delta}^q : [0,1] \rightarrow h^{-1}(q)$. Letting $D^+$ be the upper half disc $\{z \in \mathbb{C} : |z| \leq 1, \text{Im}(z) \geq 0\}$, this implies that to any $f$-admissible path $\delta$ from a basepoint $z \in Y$, there is a function

(50) \[ u_\delta : (D^+, \partial D^+) \rightarrow (S, \Delta_g \cup h^{-1}(z)) \]
which takes the interval \( I_t = \{ z \in D^+ : \text{Im}(z) = t \} \) to the matching path \( \delta^t \).

Such a disc is determined up to isotopy relative the boundary and results in a homomorphism

\[
\chi_S : K_0(\mathcal{F}_z(f)) \to H_2(S, \Delta_g \cup h^{-1}(z); \mathbb{Z}).
\]

This setup is illustrated in Figure 9.

Fixing a surface factorization \( S \) of \( f \) and choice of a basepoint \( z \in Y \) over which both \( f \) and \( h|_{\Delta_g} \) are regular, we define a groupoid \( \mathcal{V}_S \). The objects of \( \mathcal{V}_S \) will be defined as

\[
\text{Ob}(\mathcal{V}_S) = h^{-1}(z) \cap \Delta_g.
\]

Let \( \Gamma \subset H_1(\Delta_g; \mathbb{Z}) \) be the subgroup generated by all matching cycles of \( h|_{\Delta_g} \). To describe the morphisms in \( \mathcal{V}_S \), let \( \Gamma^\partial \subset H_1(\Delta_g, \mathcal{V}_S; \mathbb{Z}) \) be the subgroup generated by vanishing thimbles of \( h|_{\Delta_g} \). Consider the connecting homomorphism from long exact sequence of relative homology

\[
\delta : H_1(\Delta_g, \mathcal{V}_S; \mathbb{Z}) \to H_0(\mathcal{V}_S; \mathbb{Z}).
\]

For \( i \in \mathcal{V}_S \) take \( \bar{i} \) to be the class in \( H_0(\mathcal{V}_S; \mathbb{Z}) \) and define

\[
\text{Hom}_{\mathcal{V}_S}(i, j) := \delta^{-1}(j - \bar{i}) \cap \Gamma^\partial.
\]

We note that \( \mathcal{V}_S \) is a connected groupoid if and only if \( \Delta_g \) is connected. Using the language of Definition 3.12, \( \mathcal{V}_S \) is the groupoid induced by \( \delta \) and \( A = \{ [p] : p \in h^{-1}(z) \cap \Delta_g \} \). A standard \( \mathcal{V}_S \)-collection associated to this groupoid will be examined in 5.5. First we will consider a more restrictive enhancement in this context.

5.4. Motivic enhancements by \( \mathcal{F}_z(f) \). We now describe a collection of subcategories \( \{ \mathcal{C}_{ij} \} \) of \( \mathcal{F}_z(f) \) lying over the category \( \mathcal{D} \). If the resulting collection yields a \( \mathcal{V}_S \)-collection, by Definition 3.12, we must provide an odd motivic enhancement \( \Phi = (\{ \mathcal{C}_{ij}, \epsilon_{ij} \}, \{ (W_{ijk}, \nu_{ijk}) \} ) \) of \( \mathcal{V}_S \) for which \( K_\Phi(\mathcal{V}_S) \cong \mathcal{V}_S \).

We begin by taking the ambient ind-constructible \( A_\infty \)-category to be \( \mathcal{F}_z(f) \).

Given an \( f \)-admissible path \( \delta \), recall that the vanishing cycle \( L^f_\delta \) is Hamiltonian isotopic to a matching cycle \( M_\delta \) over a matching path \( \delta : [0, 1] \to F_z \). Note that \( \delta \) is determined up to isotopy in \( F_z \setminus \mathcal{V}_S \). Using this fact, we will choose tangent vectors in \( T_I(F_z) \) for each \( i \in \mathcal{V}_S \) assume that \( \delta'(0) \) and \( \delta'(1) \) agree with these choices. This enables a grading of \( T^f_\delta \) to induce one on \( T_h^\delta \). Such a grading is part of a brane structure on \( L^f_\delta = (T^f_\delta, \alpha, \beta) \) (inducing one on \( T_h^\delta \)). Of course, the grading on \( T_h^\delta \) induces an orientation yielding an oriented path in \( \Delta_g \) that connects two points in \( \mathcal{V} \). Write \( [L^f_\delta] \) for the associated homology class in \( H_1(\Delta_g, \mathcal{V}_S; \mathbb{Z}) \) and define \( \mathcal{C}_{ij} \) to be the full subcategory of \( \mathcal{F}_z(f) \) with objects

\[
\text{Ob}(\mathcal{C}_{ij}) = \left\{ \begin{array}{c} L^f_\delta : [L^f_\delta] \in \Gamma_{ij} \end{array} \right\}.
\]

The maps \( \epsilon_{ij} \) are simply given by \( \epsilon_{ij}(L^f_\delta) = [L^f_\delta] \).

Now, given \( L^f_\delta \in \mathcal{C}_{ij} \) and \( L^f_\delta \in \mathcal{C}_{jk} \) there is a unique intersection point \( q \in L^f_\delta \cap L^f_\delta \) at \( j \) whose Maslov index is difference of the orientations of \( T_h^\delta \) and \( T_h^\delta \) and therefore odd. We take \( W_{ijk} \) to be the union of these intersections over such objects. It is known that concatenation of vanishing thimbles in \( \Delta_g \) corresponds to a mutation of underlying paths and that the categorical incarnation of this is to
take the cone of a morphism between the two thimbles of \( f \). It is now not difficult to define the functor \( \phi \). In particular, \( \phi_{ij} \) simply takes \([L^f_{ij}] = [T^h_{ij}]\).

**Remark 5.6.** While it is unclear whether the above collection of categories can be assembled into a non-standard \( \mathbb{V}_S \)-collection \( \Phi \), it is certain that any such collection will not have a \( \Phi \)-stability condition. This is due to the fact that such stability conditions require \( \mathcal{C}_{ij} \) to be stable under even translations, which would force the zero morphism to be contained in the support of the motivic correspondence between two such categories. This suggests again that a more flexible framework of stability conditions in this context may be more useful in the non-standard setting.

### 5.5. Standard \( \mathbb{V}_S \)-collections

The motivic enhancement considered in the previous section of a surface factorization is not standard. In other words, the motivic correspondence does not contain the full \( \text{Ext}^1 \) group. This does obstruct a direct application of the wall-crossing formula in Theorem 3.22 to the symplectic case. However, an analogous formula is expected to apply in certain non-standard contexts (see Remark 5.6). Alternatively, we may consider a larger \( \mathbb{V}_S \)-collection which is standard. To obtain this collection, recall that equation (51) gave the homomorphism

\[
\chi_S : K_0(\mathcal{F}_z(f)) \to H_2(S, \Delta_y \cup h^{-1}(z); \mathbb{Z}).
\]

Consider the triple \( h^{-1}(z) \subset \Delta_y \cup h^{-1}(z) \subset S \) and the connecting homomorphism \( \delta \) of the long exact sequence of the homology of this triple. Along with excision, we obtain the map

\[
\chi'_S : K_0(\mathcal{F}_z(f)) \to H_1(\Delta_y, h^{-1}(z) \cap \Delta_y; \mathbb{Z}) = H_1(\Delta_y, \mathbb{V}_S; \mathbb{Z}).
\]

As the image of \( \chi'_S \) is generated by of a set of surfaces whose boundary along \( \Delta_y \) is a vanishing thimble \( T^h \), it follows that the image of \( \chi'_S \) is \( \Gamma \) as defined after equation (52). Using this data with Definition 3.7 we obtain a standard \( \mathbb{V}_S \)-collection.

**Definition 5.7.** Given a surface factorization \( S \), the standard \( \mathbb{V}_S \)-collection is the motivic enhancement induced by \( \delta, \chi'_S \) and \( \mathbb{V}_S \).

Let us examine the examples of an \( A_n \) singularity in general.

**Example 5.8.** The \( A_n \)-singularity \( f : \mathbb{C}^3 \to \mathbb{C} \) is given by

\[
f(x_1, x_2, x_3) = x_1^{n+1} + x_2^2 + x_3^3.
\]

The Fukaya-Seidel category of this singularity has been thoroughly studied in [?], so we will summarize briefly and put it into the context of a 2d-4d categorification. To obtain Morse singularities, we choose a perturbation

\[
f(x_1, x_2, x_3) = x_1^{n+1} - (n + 1)\varepsilon x_1 + x_2^2 + x_3^2
\]

which admits the factorization \( f = hg \) where \( g : \mathbb{C}^3 \to \mathbb{C}^2 \) via \( g(x_1, x_2, x_3) = (x_1, x_2^2 + x_3^3) \) and \( h : \mathbb{C}^2 \to \mathbb{C} \) via \( h(u_1, u_2) = u_1^{n+1} - (n + 1)\varepsilon u_1 + u_2 \). One obtains

\[
\Delta_g = \{(x, 0, 0) : x \in \mathbb{C}\},
\]

\[
\Delta_y = \{(x, 0) : x \in \mathbb{C}\} \cong \mathbb{C}.
\]

So \( h|\Delta_y : \mathbb{C} \to \mathbb{C} \) is simply the polynomial \( x^{n+1} - \varepsilon x \). The critical points and values are then

\[
\text{Crit}_p(f) = \{ (\zeta, \sqrt[n]{\varepsilon}, 0) : \zeta^n = 1 \},
\]

\[
\text{Crit}_v(f) = \text{Crit}_v(h|\Delta_y) = \{-n\zeta \sqrt[n]{\varepsilon} : \zeta^n = 1 \}.
\]
Choose a base point \( z \notin \text{Crit}(f) \).

In this case, there are no matching paths for \( h : \Delta_g \to C \) so that \( \Gamma = \{0\} \) (which is also clear from the fact that \( \Gamma \subset H_1(C; \mathbb{Z}) \)). Thus \( \mathcal{V}_S \) is a connected trivial groupoid with \((n + 1) = |h^{-1}\Delta_g(z)|\) objects. Recalling the equation (20) from Example 3.2 dealing with representations of the \( A_n \)-quiver \( Q = (Q_0, Q_1) \) and labeling \( \text{Ob}(\mathcal{V}_S) = h^{-1}(z) \cap \Delta_g \) as \( \{0, \ldots, n\} \), we see that there is a commutative diagram realizing the connecting homomorphism of the homology sequence with \( \phi \):

\[
\begin{array}{cccc}
H_1(\Delta_g, \mathcal{V}_S; \mathbb{Z}) & \rightarrow & H_0(\mathcal{V}_S; \mathbb{Z}) & \rightarrow & H_0(\Delta_g; \mathbb{Z}) \\
\delta & \downarrow & \delta & \downarrow & \delta \\
\mathbb{Z}^{Q_0} & \rightarrow & \mathbb{Z}^{n+1} & \rightarrow &
\end{array}
\]

Thus in this case we recover the \( \mathcal{V} \)-collection from Example 4.3.

5.6. Categorical framework for 2d-4d wall-crossing formulas and Hitchin integrable systems. We briefly recall the well-known setup for the Hitchin integrable system. One can interpret the data for such a system as a surface factorization defined in the following way. Let \( Y = C \) be the base curve of the Hitchin system and \( S = T^*C \) is its cotangent bundle. Take \( K \) to be the canonical bundle on \( C \) and \( B = \oplus_{i=2}^n H^0(C, K^i) \) the set of tuples of differentials and \( u = (\phi_2, \ldots, \phi_n) \in B \). The spectral curve \( \Sigma_u \) equals the set of solutions in \( T^*C \) to the equation \( \sum_{i=2}^n \phi_i \lambda^{n-i} = 0 \). The local solutions to the equation yield \( n \)-differential forms \( \lambda_i \) on \( C \) which pullback to a single form \( \lambda \in \Omega^1(C) \).

Suppose \( L \) is a line bundle for which \( L^\otimes 2 = K^\otimes n \) and define

\[
X = \left\{ (x_1, x_2, \lambda) \in L \oplus L \oplus K : \sum_{i=2}^n \phi_i \lambda^{n-i} = x_1^2 + x_2^2 \right\}
\]

When the \( \phi_i \) are meromorphic forms there are modifications of the above construction see e.g. [9, Section 8] or [12, Section 3]. Letting \( g \) be the projection to \( \lambda \), \( h \) the bundle projection and \( f \) their composition, we obtain a surface factorization of \( f \) for generic choices of \( u \).

This surface factorization leads to a strictly 4d set-up, and to obtain the full 2d-4d system, one must add a basepoint \( z \in C \) which gives us a reference fiber for the Fukaya-Seidel category \( \mathcal{F}_z(f) \). Using the choice of basepoint gives us the standard \( \mathcal{V}_S \)-collection. Integrating the 1-form \( \lambda \) then yields a central charge

\[
Z_\gamma := \frac{1}{\pi} \int_\gamma \lambda.
\]

To define the slicing, one utilizes WKB, or spectral networks as defined in [3, 4, 5]. Recall that a spectral network is a collection of curves on \( C \), and in this case we assume they are WKB curves. Such a WKB curve of phase \( \theta \) comes with a pair of lifts to \( \Sigma_u \) such that \( \text{arg}(\langle h_*(\lambda_i), h_*(\lambda_j), \partial_i \rangle) = \theta \) along the curve where \( \lambda_i \) and \( \lambda_j \) are the two local forms for \( \lambda \). A finite spectral network consists of a finite trivalent graph on \( C \) whose edges are WKB curves satisfying the conditions:

1. at any trivalent vertex, the three adjacent edges are labeled by \( ij, jk \) and \( ki \),
2. any leaf must be a critical value or \( z \)
any critical value leaf $q$ must be labeled by $ij$ where, near the critical point $p$ over $q$, the vanishing cycle of the curve is $ij$.

The networks we will consider will also satisfy the following additional admissibility conditions:

(4) for any point $c$ in the interior WKB curve labeled by $ij$, there exists no $k \in \Sigma_u$ lying on the line segment between $i$ and $j$ in $T^*_c C$.

(5) over any trivalent vertex, the triangle $ijk$ in the cotangent fiber does not contain any $l \in \Sigma_u$ lying in its interior.

Note that these conditions are vacuous for a type $A_1$-system. To any such admissible network, one may define an embedded Lagrangian in $X$ as follows. For each WKB curve $\delta$ in a given network which has an endpoint on a critical value, we take the vanishing thimble $T_\delta$ of $f$. Note that by condition (4), we may assume that the vanishing cycle over any point on the interior of such a curve is the matching cycle over a matching path isotopic to the straight line segment from $i$ to $j$. Consider a trivalent vertex $\tilde{z}$ of WKB curves $\delta_1, \delta_2$ and $\delta_3$ of types $ij$, $jk$ and $ik$. Note that, by property (4), the vanishing cycles $L_\delta_1, L_\delta_2$ are matching cycles over straight line curves connecting $i$ to $j$ and $j$ to $k$ respectively. Concatenating these paths gives the Lagrangian sum $L_\delta_1 \# L_\delta_2$. By property (5), after performing an isotopy near the trivalent vertex, this yields a matching cycle over the line segment connecting $i$ to $k$. It is thus clear that over every point on a spectral network there is a well defined matching cycle. Assembling these Lagrangians over the spectral network yields a Lagrangian submanifold of $X$.

Then we may define the following collection of subcategories of $\mathcal{F}_f(z)$. For distinct objects $i, j \in Ob(V_S)$ and $\theta \in \mathbb{R}$, let $\mathcal{P}_{ij}(\theta)$ consist of all such Lagrangian branes obtained from spectral networks of phase $\theta$ whose edge incident to $z$ is labeled by $ij$. It is not hard to see that $\mathcal{P}_{ij}$ is a subcategory of $\mathcal{C}_{ij}$.

**Conjecture 5.9.** The collection $\mathcal{P} = \{\mathcal{P}_{ij}(\theta)\}$ form a pre-slicing of the standard $V_S$-collection and $(Z, \mathcal{P})$ a $V_S$-stability condition.

In particular, the Conjecture means that the Lagrangian submanifolds associated with spectral networks are special.
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