CRITICAL POINTS OF MASTER FUNCTIONS AND THE MKDV HIERARCHY OF TYPE $A_2^{(2)}$

A. VARCHENKO*, T. WOODRUFF, D. WRIGHT

Department of Mathematics, University of North Carolina at Chapel Hill
Chapel Hill, NC 27599-3250, USA

Abstract. We consider the population of critical points generated from the critical point of the master function with no variables, which is associated with the trivial representation of the affine Lie algebra $A_2^{(2)}$. We describe how the critical points of this population define rational solutions of the equations of the mKdV hierarchy associated with $A_2^{(2)}$.

CONTENTS

1. Introduction 2
2. Kac-Moody algebra of types $A_2^{(2)}$ and $A_2^{(1)}$ 3
2.1. Kac-Moody algebra of type $A_2^{(2)}$ 3
2.2. Kac-Moody algebra of type $A_2^{(1)}$ 3
3. mKdV equations 3
3.1. mKdV equations of type $A_2^{(2)}$ 3
3.2. mKdV equations of type $A_2^{(1)}$ 3
3.3. Comparison of mKdV equations of types $A_2^{(2)}$ and $A_2^{(1)}$ 3
3.4. KdV equations of type $A_2^{(1)}$ 3
3.5. Miura maps 3
4. Critical points of master functions and generation of pairs of polynomials 4
4.1. Master function 4
4.2. Polynomials representing critical points 4
4.3. Elementary generation 4
4.4. Degree increasing generation 4
4.5. Degree-transformations and generation of vectors of integers 4
4.6. Multistep generation 4
5. Critical points of master functions and Miura opers 5
5.1. Miura oper associated with a pair of polynomials, [MV2] 5
5.2. Deformations of Miura opers of type $A_2^{(2)}$, [MV2] 5
5.3. Miura opers associated with the generation procedure 5
6. Vector fields 5
6.1. Statement 5
6.2. Proof of Theorem 6.1 for $m = 1$ 5
6.3. Proof of Theorem 6.1 for $m > 1$ 5
6.4. Proof of Theorem 6.4 5
6.5. Critical points and the population generated from $y^\emptyset$ 5
References 5

* Supported in part by NSF grant DMS-1101508
1. Introduction

Let $\mathfrak{g}$ be a Kac-Moody algebra with invariant scalar product $(\cdot, \cdot)$, $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra, $\alpha_0, \ldots, \alpha_N$ simple roots. Let $\Lambda_1, \ldots, \Lambda_n$ be dominant integral weights, $k_0, \ldots, k_N$ nonnegative integers, $k = k_0 + \cdots + k_N$.

Consider $\mathbb{C}^n$ with coordinates $z = (z_1, \ldots, z_n)$. Consider $\mathbb{C}^k$ with coordinates $u$ collected into $N + 1$ groups, the $j$-th group consisting of $k_j$ variables,

$$u = (u^{(0)}, \ldots, u^{(N)}), \quad u^{(j)} = (u_1^{(j)}, \ldots, u_{k_j}^{(j)}).$$

The master function is the multivalued function on $\mathbb{C}^k \times \mathbb{C}^n$ defined by the formula

$$\Phi(u, z) = \sum_{a<b} (\Lambda_a, \Lambda_b) \ln(z_a - z_b) - \sum_{a,i,j} (\alpha_j, \Lambda_a) \ln(u_i^{(j)} - z_a) + \sum_{j<j'} \sum_{i,i'} (\alpha_j, \alpha_{j'}) \ln(u_i^{(j)} - u_{i'}^{(j')}) + \sum_{j < i'} \sum (\alpha_j, \alpha_j') \ln(u_i^{(j)} - u_{i'}^{(j')}),$$

with singularities at the places where the arguments of the logarithms are equal to zero.

Examples of master functions associated with $\mathfrak{g} = \mathfrak{sl}_2$ were considered by Stieltjes and Heine in 19th century, see [Sz]. Master functions were introduced in [SV] to construct integral representations for solutions of the KZ equations, see also [V1, V2]. The critical points of master functions with respect to $u$-variables were used to find eigenvectors in the associated Gaudin models by the Bethe ansatz method, see [BF, RV, V3]. In important cases the algebra of functions on the critical set of master functions is closely related to Schubert calculus, see [MTV].

In [ScV, MV1] it was observed that the critical points of master functions with respect to the $u$-variables can be deformed and form families. Having one critical point, one can construct a family of new critical points. The family is called a population of critical points. A point of the population is a critical point of the same master function or of another master function associated with the same $\mathfrak{g}, \Lambda_1, \ldots, \Lambda_n$ but with a different integer parameters $k_0, \ldots, k_N$. The population is a variety isomorphic to the flag variety of the Kac-Moody algebra $\mathfrak{g}$, Langlands dual to $\mathfrak{g}$, see [MV1, MV2, F].

In [VW], it was discovered that the population originated from the critical point of the master function associated with the affine Lie algebra $\widehat{\mathfrak{sl}}_{N+1}$ and the parameters $n = 0, k_0 = \cdots = k_N = 0$ is connected with the mKdV integrable hierarchy associated with $\widehat{\mathfrak{sl}}_{N+1}$. Namely, that population can be naturally embedded into the space of $\widehat{\mathfrak{sl}}_{N+1}$ Miura opers so that the image of the embedding is invariant with respect to all mKdV flows on the space of Miura opers. For $N = 1$, that result follows from the classical paper by M. Adler and J. Moser [AM].

In this paper we prove the analogous statement for the twisted affine Lie algebra $A_2^{(2)}$.

In Section 2 and 3 we follow the paper [DS] by V. Drinfeld and V. Sokolov and review the Lie algebras of types $A_2^{(2)}, A_2^{(1)}$ and the associated mKdV hierarchies.

In Section 4 formula (4.1), we introduce our master functions associated with $A_2^{(2)}$,

$$\Phi(u; k_0, k_1) = 2 \sum_{i<i'} \ln(u_i^{(0)} - u_{i'}^{(0)}) + 8 \sum_{i<i'} \ln(u_i^{(1)} - u_{i'}^{(1)}) - 4 \sum_{i,i'} \ln(u_i^{(0)} - u_{i'}^{(1)}).$$
Following [MV1, MV2, VW], we describe the generation procedure of new critical points starting from a given one. We define the population of critical points generated from the critical point of the function with no variables, namely, the function corresponding to the parameters $k_0 = k_1 = 0$. That population is partitioned into complex cells $\mathbb{C}^m$ labeled by finite sequences $J = (j_1, \ldots, i_m)$, $m \geq 0$, of the form $(0, 1, 0, 1, \ldots)$ or $(1, 0, 1, 0, \ldots)$. Such sequences are called basic.

In Sections 5 to every basic sequence $J$ we assign a map $\mu^J : \mathbb{C}^m \to \mathcal{M}(A_2^{(2)})$ of that cell to the space $\mathcal{M}(A_2^{(2)})$ of Miura opers of type $A_2^{(2)}$. We describe some properties of that map. In Section 6, we formulate and prove our main result. Theorem 6.1 says that for any basic sequence, the variety $\mu^J(\mathbb{C}^m)$ is invariant with respect to all mKdV flows on $\mathcal{M}(A_2^{(2)})$ and that variety is point-wise fixed by all flows $\frac{\partial}{\partial t_r}$ with index $r$ greater than $3m + 1$.

In the next papers we plan to extend this result to arbitrary affine Lie algebras.

2. Kac-Moody algebra of types $A_2^{(2)}$ and $A_2^{(1)}$

In this section we follow [DS] Section 5.

2.1. Kac-Moody algebra of type $A_2^{(2)}$.

2.1.1. Definition. Consider the Cartan matrix of type $A_2^{(2)}$,

$$A_2^{(2)} = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & \beta \\ 0 & \beta & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}. $$

The diagonal matrix $D = \text{diag}(d_0, d_1) = \text{diag}(4, 1)$ is such that $B = DA = \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}$ is symmetric. The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A_2^{(2)})$ of type $A_2^{(2)}$ is the Lie algebra with canonical generators $e, h, f \in \mathfrak{g}$, $i = 0, 1$, subject to the relations

$$[e, f] = \delta_{i,j} h_i, \quad [h, e] = a_{i,j} e_j, \quad [h, f] = -a_{i,j} f_j, \quad \text{for } i \neq j,$$

$$(\text{ad} e_i)^1 a_{i,j} e_j = 0, \quad (\text{ad} f_i)^1 a_{i,j} f_j = 0, \quad 2h_0 + h_1 = 0,$$

see this definition in [DS] Section 5. More precisely, we have

$$[h_0, e_0] = 2e_0, \quad [h_0, e_1] = -e_1, \quad [h_1, e_0] = -4e_0, \quad [h_1, e_1] = 2e_1,$$

$$[h_0, f_0] = -2f_0, \quad [h_0, f_1] = f_1, \quad [h_1, f_0] = 4f_0, \quad [h_1, f_1] = -2f_1,$$

$$(\text{ad} e_0)^2 e_1 = 0, \quad (\text{ad} e_1)^5 e_0 = 0, \quad (\text{ad} f_0)^2 f_1 = 0, \quad (\text{ad} f_1)^5 f_0 = 0.$$

The Lie algebra $\mathfrak{g}$ is graded with respect to the standard grading, $\deg e = 1, \deg f = -1$, $i = 0, 1$. Let $\mathfrak{g}^i = \{x \in \mathfrak{g} \mid \deg x = j\}$, then $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}^j$.

Notice that $\mathfrak{g}^0$ is the one-dimensional space generated by $h_0, h_1$. Denote $\mathfrak{h} = \mathfrak{g}^0$. Introduce elements $\alpha_0, \alpha_1$ of the dual space $\mathfrak{h}^*$ by the conditions $\langle \alpha_j, h_i \rangle = a_{i,j}$ for $i, j = 0, 1$. More precisely,

$$\langle \alpha_0, h_0 \rangle = 2, \quad \langle \alpha_1, h_1 \rangle = 2, \quad \langle \alpha_0, h_1 \rangle = -4, \quad \langle \alpha_1, h_0 \rangle = -1.$$
2.1.2. **Realizations of** \(\mathfrak{g}\). Consider the complex Lie algebra \(\mathfrak{gl}_3\) with standard basis \(e_{i,j}, i, j = 1, 2, 3\).

Let \(w = e^{\pi i/3}\). Define the *Coxeter automorphism* \(C : \mathfrak{sl}_3 \to \mathfrak{sl}_3\) of order 6 by the formula

\[
e_{1,2} \mapsto w^{-1}e_{2,3}, \quad e_{2,3} \mapsto w^{-1}e_{1,2}, \quad e_{2,1} \mapsto we_{3,2}, \quad e_{3,2} \mapsto we_{2,1}.
\]

Denote \((\mathfrak{sl}_3)_j = \{x \in \mathfrak{sl}_3 \mid Cx = w^j x\}\). Then

\[
(\mathfrak{sl}_3)_0 = \langle e_{1,1} - e_{3,3} \rangle, \quad (\mathfrak{sl}_3)_1 = \langle e_{2,1} + e_{3,2}, e_{1,3} \rangle, \quad (\mathfrak{sl}_3)_2 = \langle e_{1,2} - e_{2,3} \rangle, \\
(\mathfrak{sl}_3)_3 = \langle e_{1,1} - 2e_{2,2} + e_{3,3} \rangle, \quad (\mathfrak{sl}_3)_4 = \langle e_{2,1} - e_{3,2} \rangle, \quad (\mathfrak{sl}_3)_5 = \langle e_{1,2} + e_{2,3}, e_{3,1} \rangle.
\]

The twisted Lie subalgebra \(L(\mathfrak{sl}_3, C) \subset \mathfrak{sl}_3[\xi, \xi^{-1}]\) is the subalgebra

\[
L(\mathfrak{sl}_3, C) = \oplus_{j \in \mathbb{Z}} \xi^j \otimes (\mathfrak{sl}_3)_j \mod 6.
\]

The isomorphism \(\tau_C : \mathfrak{g} \to L(\mathfrak{sl}_3, C)\) is defined by the formula

\[
e_0 \mapsto \xi \otimes e_{1,3}, \quad e_1 \mapsto \xi \otimes (e_{2,1} + e_{3,2}), \\
f_0 \mapsto \xi^{-1} \otimes e_{3,1}, \quad f_1 \mapsto \xi^{-1} \otimes (2e_{1,2} + 2e_{2,3}), \\
h_0 \mapsto 1 \otimes (e_{1,1} - e_{3,3}), \quad h_1 \mapsto 1 \otimes (-2e_{1,1} + 2e_{3,3}).
\]

Under this isomorphism we have \(\mathfrak{g}^j = \xi^j \otimes (\mathfrak{sl}_3)_j\).

Define the *standard automorphism* \(\sigma_0 : \mathfrak{sl}_3 \to \mathfrak{sl}_3\) of order 2 by the formula

\[
e_{1,2} \mapsto e_{2,3}, \quad e_{2,3} \mapsto e_{1,2}, \quad e_{2,1} \mapsto e_{3,2}, \quad e_{3,2} \mapsto e_{2,1}.
\]

Let \((\mathfrak{sl}_3)_{0,j} = \{x \in \mathfrak{sl}_3 \mid \sigma_0 x = (-1)^j x\}\). Then

\[
(\mathfrak{sl}_3)_{0,0} = \langle e_{1,1} - e_{3,3}, e_{1,2} + e_{2,3}, e_{2,1} + e_{3,2} \rangle, \quad (\mathfrak{sl}_3)_{0,1} = \langle e_{1,2} - e_{2,3}, e_{2,1} - e_{3,2}, e_{1,3}, e_{3,1} \rangle.
\]

The twisted Lie subalgebra \(L(\mathfrak{sl}_3, \sigma_0) \subset \mathfrak{sl}_3[\lambda, \lambda^{-1}]\) is the subalgebra

\[
L(\mathfrak{sl}_3, \sigma_0) = \oplus_{j \in \mathbb{Z}} \lambda^j \otimes (\mathfrak{sl}_3)_{0,j} \mod 2.
\]

The isomorphism \(\tau_0 : \mathfrak{g} \to L(\mathfrak{sl}_3, \sigma_0)\) is defined by the formula

\[
e_0 \mapsto \lambda \otimes e_{1,3}, \quad e_1 \mapsto 1 \otimes (e_{2,1} + e_{3,2}), \\
f_0 \mapsto \lambda^{-1} \otimes e_{3,1}, \quad f_1 \mapsto 1 \otimes (2e_{1,2} + 2e_{2,3}), \\
h_0 \mapsto 1 \otimes (e_{1,1} - e_{3,3}), \quad h_1 \mapsto 1 \otimes (-2e_{1,1} + 2e_{3,3}).
\]

Define the *standard automorphism* \(\sigma_1 : \mathfrak{sl}_3 \to \mathfrak{sl}_3\) of order 4,

\[
e_{1,2} \mapsto i e_{2,3}, \quad e_{2,3} \mapsto i e_{1,2}, \quad e_{2,1} \mapsto i^{-1} e_{3,2}, \quad e_{3,2} \mapsto i^{-1} e_{2,1},
\]

where \(i = \sqrt{-1}\). Let \((\mathfrak{sl}_3)_{1,j} = \{x \in \mathfrak{sl}_3 \mid \sigma_0 x = i^j x\}\). Then

\[
(\mathfrak{sl}_3)_{1,0} = \langle e_{1,3}, e_{3,1}, e_{1,1} - e_{3,3} \rangle, \quad (\mathfrak{sl}_3)_{1,1} = \langle e_{1,2} - e_{2,3}, e_{2,1} + e_{3,2} \rangle, \\
(\mathfrak{sl}_3)_{1,2} = \langle e_{1,1} - 2e_{2,2} + e_{3,3} \rangle, \quad (\mathfrak{sl}_3)_{1,3} = \langle e_{2,1} - e_{3,2}, e_{1,2} + e_{2,3} \rangle.
\]

The twisted Lie subalgebra \(L(\mathfrak{sl}_3, \sigma_1) \subset \mathfrak{sl}_3[\mu, \mu^{-1}]\) is the subalgebra

\[
L(\mathfrak{sl}_3, \sigma_1) = \oplus_{j \in \mathbb{Z}} \mu^j \otimes (\mathfrak{sl}_3)_{1,j} \mod 4.
\]
The isomorphism $\tau_1 : g \to L(\mathfrak{sl}_3, \sigma_1)$ is defined by the formula
\[
e_0 \mapsto 1 \otimes e_{1,3}, \quad e_1 \mapsto \mu \otimes (e_{2,1} + e_{3,2}),
\]
\[
f_0 \mapsto 1 \otimes e_{3,1}, \quad f_1 \mapsto \mu^{-1} \otimes (2e_{1,2} + 2e_{2,3}),
\]
\[
h_0 \mapsto 1 \otimes (e_1,1 - e_{3,3}), \quad h_1 \mapsto 1 \otimes (-2e_{1,1} + 2e_{3,3}).
\]

The composition isomorphism $L(\mathfrak{sl}_3, \sigma_0) \to L(\mathfrak{sl}_3, C)$ is given by the formula $\lambda^m \otimes e_{k,l} \mapsto \xi^{3m+k-l} \otimes e_{k,l}$. The composition isomorphism $L(\mathfrak{sl}_3, \sigma_0) \to L(\mathfrak{sl}_3, \sigma_1)$ is given by the formula $\lambda^m \otimes e_{k,l} \mapsto \mu^{2m+1-k} \otimes e_{k,l}$.

**Remark.** The standard automorphisms $\sigma_0, \sigma_1$ correspond to the two vertices of the Dynkin diagram of type $A_2^{(2)}$, see [DS, Section 5].

2.1.3. *Element $\Lambda$.*** Denote by $\Lambda$ the element $e_0 + e_1 \in g$. Then $\mathfrak{j} = \{x \in g \mid [\Lambda, x] = 0\}$ is an Abelian Lie subalgebra of $g$. Denote $\mathfrak{j}^j = \mathfrak{j} \cap g^j$, then $\mathfrak{j} = \oplus_{j \in \mathbb{Z}} \mathfrak{j}^j$. We have $\dim \mathfrak{j}^j = 1$ if $j = 1, 5 \mod 6$ and $\dim \mathfrak{j}^j = 0$ otherwise.

If $g$ is realized as $L(\mathfrak{sl}_3, C)$, then the element $\xi^{6m+1} \otimes (e_{2,1} + e_{3,2} + e_{1,3})$ generates $\mathfrak{j}^{6m+1}$ and the element $\xi^{6m-1} \otimes (e_{1,2} + e_{2,3} + e_{3,1})$ generates $\mathfrak{j}^{6m-1}$.

If $g$ is realized as $L(\mathfrak{sl}_3, \sigma_0)$, then the element $\lambda^{m} \otimes (e_{2,1} + e_{3,2}) + \xi^{2m+1} \otimes e_{1,3}$ generates $\mathfrak{j}^{6m+1}$ and the element $\lambda^{m} \otimes (e_{2,1} + e_{3,2}) + \xi^{2m-1} \otimes e_{3,1}$ generates $\mathfrak{j}^{6m-1}$.

**Lemma 2.1.** For any $m \in \mathbb{Z}$, the elements
\[
(\tau_C)^{-1}(\xi^{6m+1} \otimes (e_{2,1} + e_{3,2} + e_{1,3})), \quad (\tau_0)^{-1}(\lambda^{2m} \otimes (e_{2,1} + e_{3,2}) + \xi^{2m+1} \otimes e_{1,3})
\]
of $\mathfrak{j}^{6m+1}$ are equal. Similarly, the elements
\[
(\tau_C)^{-1}(\xi^{6m-1} \otimes (e_{1,2} + e_{2,3} + e_{3,1})), \quad (\tau_0)^{-1}(\lambda^{2m} \otimes (e_{2,1} + e_{3,2}) + \lambda^{2m-1} \otimes e_{3,1})
\]
of $\mathfrak{j}^{6m-1}$ are equal. □

Denote the elements $(\tau_C)^{-1}(\xi^{6m+1} \otimes (e_{2,1} + e_{3,2} + e_{1,3})), (\tau_C)^{-1}(\xi^{6m-1} \otimes (e_{1,2} + e_{2,3} + e_{3,1}))$ of $g$ by $\Lambda_{6m+1}$ and $\Lambda_{6m-1}$, respectively. Notice that $\Lambda_1 = e_0 + e_1 = \lambda$.

We set $\Lambda_j = 0$ if $j \neq 1, 5 \mod 6$.

**Lemma 2.2.** Let us consider the elements $\xi^{6m+1} \otimes (e_{2,1} + e_{3,2} + e_{1,3}), \xi^{6m-1} \otimes (e_{1,2} + e_{2,3} + e_{3,1})$ as $3 \times 3$ matrices,
\[
A_{6m+1} = \begin{pmatrix}
0 & 0 & \xi^{6m+1} \\
\xi^{6m+1} & 0 & 0 \\
0 & \xi^{6m+1} & 0
\end{pmatrix}, \quad A_{6m-1} = \begin{pmatrix}
0 & \xi^{6m-1} & 0 \\
\xi^{6m-1} & 0 & 0 \\
0 & 0 & \xi^{6m-1}
\end{pmatrix},
\]
respectively. Then $A_{6m+1} = (A_1)^{6m+1}$ and $A_{6m-1} = (A_1)^{6m-1}$.

Similarly, let us consider the elements $\lambda^{2m} \otimes (e_{2,1} + e_{3,2}) + \xi^{2m+1} \otimes e_{1,3}, \lambda^{2m} \otimes (e_{2,1} + e_{3,2}) + \lambda^{2m-1} \otimes e_{3,1}$ as $3 \times 3$ matrices,
\[
B_{6m+1} = \begin{pmatrix}
0 & 0 & \lambda^{2m+1} \\
\lambda^{2m} & 0 & 0 \\
0 & \lambda^{2m} & 0
\end{pmatrix}, \quad B_{6m-1} = \begin{pmatrix}
0 & \lambda^{2m} & 0 \\
0 & 0 & \lambda^{2m} \\
\lambda^{2m-1} & 0 & 0
\end{pmatrix},
\]
respectively. Then $B_{6m+1} = (B_1)^{6m+1}$ and $B_{6m-1} = (B_1)^{6m-1}$. □

2.2. *Kac-Moody algebra of type $A_2^{(1)}$.***
2.2.1. **Definition.** Consider the Cartan matrix of type $A_2^{(1)}$,

$$A_2^{(1)} = \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ a_{1,0} & a_{1,1} & a_{1,2} \\ a_{2,0} & a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. $$

The Kac-Moody algebra $g(A_2^{(1)})$ of type $A_2^{(1)}$ is the Lie algebra with canonical generators $E_i, H_i, F_i \in g$, $i = 0, 1, 2$, subject to the relations

$$[E_i, F_j] = \delta_{i,j} H_i, $$

$$[H_i, E_j] = a_{i,j} E_j, 
\quad [H_i, F_j] = -a_{i,j} F_j, $$

$$(\text{ad } E_i)^{1-a_{i,j}} E_j = 0, 
\quad (\text{ad } F_i)^{1-a_{i,j}} F_j = 0, $$

$$H_0 + H_1 + H_2 = 0,$$

see this definition in [DS, Section 5]. The Lie algebra $g(A_2^{(1)})$ is graded with respect to the standard grading, $\deg E_i = 1, \deg F_i = -1, i = 0, 1, 2$. Let $g(A_2^{(1)})^j = \{ x \in g(A_2^{(1)}) | \deg x = j \}$, then $g(A_2^{(1)}) = \bigoplus_{j \in \mathbb{Z}} g(A_2^{(1)})^j$.

For $j = 0, 1, 2$, we denote by $n_j \subset g(A_2^{(1)})$ the Lie subalgebra generated by $F_i, i \in \{0, 1, 2\}, i \neq j$. For example, $n_0$ is generated by $F_1, F_2$.

2.2.2. **Realizations of $g(A_2^{(1)})$**. Consider the Lie algebra $\mathfrak{sl}_3[\lambda, \lambda^{-1}]$. The isomorphism $\tau_0^{(0)} : g(A_2^{(1)}) \to \mathfrak{sl}_3[\lambda, \lambda^{-1}]$ is defined by the formula

$$E_0 \mapsto \lambda \otimes e_{1,3}, 
\quad E_1 \mapsto 1 \otimes e_{2,1}, 
\quad E_2 \mapsto 1 \otimes e_{3,2}, $$

$$F_0 \mapsto \lambda^{-1} \otimes e_{3,1}, 
\quad F_1 \mapsto 1 \otimes e_{1,2}, 
\quad F_2 \mapsto 1 \otimes e_{2,3}, $$

$$H_0 \mapsto 1 \otimes (e_{1,1} - e_{3,3}), 
\quad H_1 \mapsto 1 \otimes (e_{2,2} - e_{1,1}), 
\quad H_2 \mapsto 1 \otimes (e_{3,3} - e_{2,2}).$$

Let $\epsilon = e^{2\pi i / 3}$. Define the Coxeter automorphism of type $A_2^{(1)}$, $C^{(1)} : \mathfrak{sl}_3 \to \mathfrak{sl}_3$ of order 3 by the formula

$$e_{1,2} \mapsto \epsilon^{-1} e_{1,2}, 
\quad e_{2,3} \mapsto \epsilon^{-1} e_{2,3}, 
\quad e_{2,1} \mapsto \epsilon e_{2,1}, 
\quad e_{3,2} \mapsto \epsilon e_{3,2}. $$

Denote $(\mathfrak{sl}_3)^{(1)}_j = \{ x \in \mathfrak{sl}_3 | C^{(1)} x = \epsilon^j x \}$. Then

$$\mathfrak{sl}_3^{(1)} = \langle e_{1,1} - e_{2,2}, e_{2,2} - e_{3,3} \rangle, 
\quad (\mathfrak{sl}_3)^{(1)}_1 = \langle e_{2,1}, e_{3,2}, e_{1,3} \rangle, 
\quad (\mathfrak{sl}_3)^{(1)}_2 = \langle e_{1,2}, e_{2,3}, e_{3,1} \rangle.$$ The twisted Lie subalgebra $L(\mathfrak{sl}_3, C^{(1)}) \subset \mathfrak{sl}_3[\xi, \xi^{-1}]$ is the subalgebra

$$L(\mathfrak{sl}_3, C^{(1)}) = \bigoplus_{j \in \mathbb{Z}} \xi^j \otimes (\mathfrak{sl}_3)^{(1)}_j \mod 3.$$ The isomorphism $\tau_{C^{(1)}} : g(A_2^{(1)}) \to L(\mathfrak{sl}_3, C^{(1)})$ is defined by the formula

$$E_0 \mapsto \xi \otimes e_{1,3}, 
\quad E_1 \mapsto \xi \otimes e_{2,1}, 
\quad E_2 \mapsto \xi \otimes e_{3,2}, $$

$$F_0 \mapsto \xi^{-1} \otimes e_{3,1}, 
\quad F_1 \mapsto \xi^{-1} \otimes e_{1,2}, 
\quad F_2 \mapsto \xi^{-1} \otimes e_{2,3}, $$

$$H_0 \mapsto 1 \otimes (e_{1,1} - e_{3,3}), 
\quad H_1 \mapsto 1 \otimes (e_{2,2} - e_{1,1}), 
\quad H_2 \mapsto 1 \otimes (e_{3,3} - e_{2,2}).$$ Under this isomorphism we have $g(A_2^{(1)})^j = \xi^j \otimes (\mathfrak{sl}_3)^{(1)}_j$. 
The composition isomorphism \( \mathfrak{sl}_3[\lambda, \lambda^{-1}] \rightarrow \mathfrak{L}(\mathfrak{sl}_3, C^{(1)}) \) is given by the formula \( \lambda^m \otimes e_{k,l} \mapsto \xi^{3m+k-l} \otimes e_{k,l} \).

2.2.3. **Element \( \Lambda^{(1)} \)**. Denote by \( \Lambda^{(1)} \) the element \( E_0 + E_1 + E_2 \in \mathfrak{g}(A_2^{(1)}) \). Then \( \mathfrak{z}(A_2^{(1)}) = \{ x \in \mathfrak{g}(A_2^{(1)}) \mid \mathfrak{z}(A_2^{(1)}) \} = \{ x \in \mathfrak{g}(A_2^{(1)}) \mid \mathfrak{z}(A_2^{(1)}) \} = \mathfrak{z}(A_2^{(1)}) \). Denote \( \mathfrak{z}(A_2^{(1)}) \) by \( \mathfrak{z}(A_2^{(1)}) \) and \( \mathfrak{z}(A_2^{(1)}) \) by \( \mathfrak{z}(A_2^{(1)}) \). We have \( \dim \mathfrak{z}(A_2^{(1)}) = 1 \) if \( j \neq 0 \mod 3 \) and \( \dim \mathfrak{z}(A_2^{(1)}) = 0 \) otherwise. If \( \mathfrak{g}(A_2^{(1)}) \) is realized as \( \mathfrak{sl}_3[\lambda, \lambda^{-1}] \) or \( \mathfrak{L}(\mathfrak{sl}_3, C^{(1)}) \), then a basis of \( \mathfrak{z}(A_2^{(1)}) \) is formed by the matrices \( (\Lambda^{(1)})^j \) with \( j \neq 0 \mod 3 \).

Let \( \mathfrak{g}(A_2^{(1)}) \) be realized as \( \mathfrak{sl}_3[\lambda, \lambda^{-1}] \). Consider \( \Lambda^{(1)} = e_{2,1} + e_{3,2} + \lambda \otimes e_{1,3} \) as a \( 3 \times 3 \) matrix depending on the parameter \( \lambda \). Let \( T = \sum_{j=0}^m T_j \) be a formal series with \( T_j \in \mathfrak{g}(A_2^{(1)}) \).

Denote \( T^+ = \sum_{j=0}^n T_j, T^- = \sum_{j<0} T_j. \)

By [DS] Lemma 3.4, we may represent \( T \) uniquely in the form \( T = \sum_{j=-\infty}^m b_j (\Lambda^{(1)})^j, b_j \in \text{Diag}, \) where Diag \( \subset \mathfrak{g} \) is the space of diagonal \( 3 \times 3 \) matrices. Denote \( (T)^+_\Lambda = \sum_{j=0}^n b_j (\Lambda^{(1)})^j, (T)^-\Lambda = \sum_{j<0} b_j (\Lambda^{(1)})^j. \)

**Lemma 2.3.** We have \( (T)^+_\Lambda = T^+, (T)^-\Lambda = T^-, b_0 = T^0. \)

**Proof.** The isomorphism \( \iota : \mathfrak{sl}_3[\lambda, \lambda^{-1}] \rightarrow \mathfrak{L}(\mathfrak{sl}_3, C^{(1)}) \) is given by the formula \( \lambda^m \otimes e_{k,l} \mapsto \xi^{3m+k-l} \).

We have \( \iota(b_0) = \iota(b_1 e_{1,1} + b_2 e_{2,2} + b_3 e_{3,3}) = 1 \otimes (b_1 e_{1,1} + b_2 e_{2,2} + b_3 e_{3,3}) \in \mathfrak{g}(A_2^{(1)})^0, \)

\[ \iota(b_1 \Lambda^{(1)}) = \iota((b_1 e_{1,1} + b_2 e_{2,2} + b_3 e_{3,3})(e_{2,1} + e_{3,2} + \lambda e_{1,3})) = \iota(b_1 \lambda e_{1,3} + b_2 e_{2,1} + b_3 e_{3,2}) = \xi \otimes (b_1 e_{1,3} + b_2 e_{2,1} + b_3 e_{3,2}) \in \mathfrak{g}(A_2^{(1)})^1, \]

\[ \iota(b_1 \Lambda^{(1)} - b_1 \Lambda^{(1)}) = \iota(b_1 e_{1,1} + b_2 e_{2,2} + b_3 e_{3,3})(e_{1,2} + e_{2,3} + \lambda^{-1} e_{3,1}) = \xi \otimes (b_1 e_{1,2} + b_2 e_{2,3} + b_3 e_{3,1}) \in \mathfrak{g}(A_2^{(1)})^{-1}. \]

Similarly one checks that \( \iota(b_j (\Lambda^{(1)})^j) \in \mathfrak{g}(A_2^{(1)})^j \) for any \( j. \)

Let us consider the elements \( F_0, 2F_1 + 2F_2 \) as the \( 3 \times 3 \) matrices \( \lambda^{-1} e_{3,1}, 2e_{1,2} + 2e_{2,3}, \) respectively.

**Lemma 2.4.** Let \( \lambda \in \mathbb{C}. \) Then

\[
\begin{align*}
e^{9F_0} &= 1 + g e_{3,3}(\Lambda^{(1)})^{-1}, & e^{g(2F_1+2F_2)} &= 1 + 2g(e_{1,1} + e_{2,2})(\Lambda^{(1)})^{-1} + 2g^2 e_{1,1}(\Lambda^{(1)})^{-2}. \end{align*}
\]

**Lemma 2.5.** We have \( (\Lambda^{(1)})^{-1} = e_{1,2} + e_{2,3} + \lambda^{-1} e_{3,1}, \) and

\[
e_{i+1,i+1} \Lambda^{(1)} = \Lambda^{(1)} e_{i,i}, & e_{i,i} (\Lambda^{(1)})^{-1} = (\Lambda^{(1)})^{-1} e_{i+1,i+1} \]

for all \( i, \) where we set \( e_{4,4} = e_{1,1}. \)

**2.2.4. Lie algebra \( \mathfrak{g}(A_2^{(2)}) \) as a subalgebra of \( \mathfrak{g}(A_2^{(1)}) \).** The map \( \varrho : \mathfrak{g}(A_2^{(2)}) \rightarrow \mathfrak{g}(A_2^{(1)}) \),

\[
e_0 \mapsto E_0, & e_1 \mapsto E_1 + E_2, & f_0 \mapsto F_0, & f_1 \mapsto 2F_1 + 2F_2, \]

realizes the Lie algebra \( \mathfrak{g}(A_2^{(2)}) \) as a subalgebra of \( \mathfrak{g}(A_2^{(1)}) \). This embedding preserves the standard grading and \( \varrho(\Lambda) = \Lambda^{(1)}. \) We have \( \varrho(\mathfrak{z}(A_2^{(2)})) \subset \mathfrak{z}(A_2^{(1)}). \)

If there is no confusion, we will consider \( \mathfrak{g}(A_2^{(2)}) \) as a subalgebra of \( \mathfrak{g}(A_2^{(1)}). \)
3. mKdV Equations

3.1. mKdV equations of type $A_2^{(2)}$. Denote by $B$ the space of complex-valued functions of one variable $x$. Given a finite dimensional vector space $W$, denote by $B(W)$ the space of $W$-valued functions of $x$. Denote by $\partial$ the differential operator $\frac{d}{dx}$.

Consider the Lie algebra $\mathfrak{g}$ of the formal differential operators of the form $c\partial + \sum_{i=-\infty}^{n} p_i$, $c \in \mathbb{C}$, $p_i \in B(g^i)$. Let $U = \sum_{i=0}^{\infty} U_i$, $U_i \in B(g^i)$. If $L \in \mathfrak{g}$, define

$$e^{adU}(L) = L + [U, L] + \frac{1}{2!}[U, [U, L]] + \ldots .$$

The operator $e^{adU}(L)$ belongs to $\mathfrak{g}$. The map $e^{adU}$ is an automorphism of the Lie algebra $\mathfrak{g}$. The automorphisms of this type form a group.

If elements of $\mathfrak{g}$ are realized as matrices depending on a parameter as in Section 2.1.2 then $e^{adU}(L) = e^{U}Le^{-U}$.

A Miura oper of type $A_2^{(2)}$ is a differential operator of the form

$$L = \partial + \Lambda + V$$

where $\Lambda = \epsilon_0 + \epsilon_1 \in \mathfrak{g}$ and $V \in B(g^0)$. Any Miura oper of type $A_2^{(2)}$ is an element of $\mathfrak{g}$. Denote by $\mathcal{M}(A_2^{(2)})$ the space of all Miura opers of type $A_2^{(2)}$.

**Proposition 3.1** ([DS, Proposition 6.2]). For any Miura oper $L$ of type $A_2^{(2)}$ there exists an element $U = \sum_{i=0}^{\infty} U_i$, $U_i \in B(g^i)$, such that the operator $L_0 = e^{adU}(L)$ has the form

$$L_0 = \partial + \Lambda + H,$$

where $H = \sum_{j<0} H_j$, $H_j \in B(g^j)$. If $U, \tilde{U}$ are two such elements, then $e^{adU}e^{-ad\tilde{U}} = e^{adT}$, where $T = \sum_{j<0} T_j$, $T_j \in g^j$.

Let $L, U$ be as in Proposition 3.1. Let $r = 1, 5 \text{ mod } 6$. The element $\phi(\Lambda_r) = e^{-adU}(\Lambda_r)$ does not depend on the choice of $U$ in Proposition 3.1.

The element $\phi(\Lambda_r)$ is of the form $\sum_{i=-\infty}^{n} \phi(\Lambda_r)^i$, $\phi(\Lambda_r)^i \in g^i$. We set $\phi(\Lambda_r)^+ = \sum_{i=0}^{n} \phi(\Lambda_r)^i$, $\phi(\Lambda_r)^- = \sum_{i<0} \phi(\Lambda_r)^i$.

Let $r \in \mathbb{Z}_{>0}$ and $r = 1, 5 \text{ mod } 6$. The differential equation

$$\frac{\partial L}{\partial t_r} = [\phi(\Lambda_r)^+, L]$$

is called the $r$-th mKdV equation of type $A_2^{(2)}$.

Equation (3.2) defines a vector field $\frac{\partial}{\partial t_r}$ on the space $\mathcal{M}(A_2^{(2)})$ of Miura opers. For all $r, s$, the vector fields $\frac{\partial}{\partial t_r}$, $\frac{\partial}{\partial t_s}$ commute, see [DS, Section 6].

**Lemma 3.2** ([DS]). We have

$$\frac{\partial L}{\partial t_r} = -\frac{d}{dx} \phi(\Lambda_r)^0.$$
3.2. mKdV equations of type $A_2^{(1)}$. A Miura oper of type $A_2^{(1)}$ is a differential operator of the form

$$\mathcal{L} = \partial + \Lambda + V$$

where $\Lambda = E_0 + E_1 + E_2 \in \mathfrak{g}(A_2^{(1)})$ and $V \in \mathcal{B}(\mathfrak{g}(A_2^{(1)}))^0$. Denote by $\mathcal{M}(A_2^{(1)})$ the space of all Miura ops of type $A_2^{(1)}$.

**Proposition 3.3** ([DS Proposition 6.2]). For any Miura oper $\mathcal{L}$ of type $A_2^{(1)}$ there exists an element $U = \sum_{i<0} U_i, U_i \in \mathcal{B}(\mathfrak{g}(A_2^{(1)}))^i$, such that the operator $\mathcal{L}_0 = e^{adU}(\mathcal{L})$ has the form

$$\mathcal{L}_0 = \partial + \Lambda + H,$$

where $H = \sum_{j<0} H_j, H_j \in \mathcal{B}(\mathfrak{g}(A_2^{(1)}))^j$. If $U, \bar{U}$ are two such elements, then $e^{adU} e^{-ad\bar{U}} = e^{adT}$, where $T = \sum_{j<0} T_j, T_j \in \mathfrak{g}(A_2^{(1)})^j$.

Let $\mathcal{L}, U$ be as in Proposition 3.3. Let $r \neq 0 \mod 3$. The element $\phi((\Lambda^{(1)})^r) = e^{-adU}((\Lambda^{(1)})^r)$ does not depend on the choice of $U$ in Proposition 3.3.

The element $\phi((\Lambda^{(1)})^r)$ is of the form $\sum_{i=-\infty}^n \phi((\Lambda^{(1)})^r)^i, \phi((\Lambda^{(1)})^r)^i \in \mathfrak{g}(A_2^{(1)})^i$. We set $\phi((\Lambda^{(1)})^r)^+ = \sum_{i=0}^n \phi((\Lambda^{(1)})^r)^i, \phi((\Lambda^{(1)})^r)^- = \sum_{i<0} \phi((\Lambda^{(1)})^r)^i$.

Let $r \in \mathbb{Z}_{>0}$ and $r \neq 0 \mod 3$. The differential equation

$$(3.5) \quad \frac{\partial \mathcal{L}}{\partial r} = [\phi((\Lambda^{(1)})^r)^+, \mathcal{L}]$$

is called the $r$-th mKdV equation of type $A_2^{(1)}$.

Equation (3.5) defines a vector field $\frac{\partial}{\partial r}$ on the space $\mathcal{M}(A_2^{(1)})$ of Miura ops of type $A_2^{(1)}$. For all $r, s$, the vector fields $\frac{\partial}{\partial r}, \frac{\partial}{\partial s}$ commute, see [DS Section 6].

3.3. Comparison of mKdV equations of types $A_2^{(2)}$ and $A_2^{(1)}$. Consider $\mathfrak{g}(A_2^{(2)})$ as a Lie subalgebra of $\mathfrak{g}(A_2^{(1)})$, see Section 2.2.4. Let $\mathcal{L}$ be a Miura oper of type $A_2^{(2)}$. Then $\mathcal{L}$ is a Miura oper of type $A_2^{(1)}$.

**Lemma 3.4.** Let $r = 1, 5 \mod 6, r > 0$. Let $\mathcal{L}^{A_2^{(2)}}(t_r)$ be the solution of the $r$-th mKdV equation of type $A_2^{(2)}$ with initial condition $\mathcal{L}^{A_2^{(2)}}(0) = \mathcal{L}$. Let $\mathcal{L}^{A_2^{(1)}}(t_r)$ be the solution of the $r$-th mKdV equation of type $A_2^{(1)}$ with initial condition $\mathcal{L}^{A_2^{(1)}}(0) = \mathcal{L}$. Then $\mathcal{L}^{A_2^{(2)}}(t_r) = \mathcal{L}^{A_2^{(1)}}(t_r)$ for all values of $t_r$. \(\square\)

**Proof.** The element $U$ in Proposition 3.3 which is used to construct the mKdV equation of type $A_2^{(2)}$ can be used also to construct the mKdV equation of type $A_2^{(1)}$. \(\square\)

3.4. KdV equations of type $A_2^{(1)}$. Let $\mathcal{B}((\partial^{-1}))$ be the algebra of formal pseudodifferential operators of the form $a = \sum_{i \in \mathbb{Z}} a_i \partial^i$, with $a_i \in \mathcal{B}$ and finitely many terms with $i > 0$. The relations in this algebra are

$$\partial^k u - u \partial^k = \sum_{i=1}^\infty k(k-1) \cdots (k-i+1) \frac{d^i u}{dx^i} \partial^{k-i}$$
for any $k \in \mathbb{Z}$ and $u \in \mathcal{B}$. For $a = \sum_{i \in \mathbb{Z}} a_i \partial^i \in \mathcal{B}((\partial^{-1}))$, define $a^+ = \sum_{i > 0} a_i \partial^i$.

Denote $\mathcal{B}[\partial] \subset \mathcal{B}((\partial^{-1}))$ the subalgebra of differential operators $a = \sum_{i = 0}^{m} a_i \partial^i$ with $m \in \mathbb{Z}_{\geq 0}$. Denote $\mathcal{D} \subset \mathcal{B}[\partial]$ the affine subspace of the differential operators of the form $L = \partial^3 + u_1 \partial + u_0$.

For $L \in \mathcal{D}$, there exists a unique $L^3 = \partial + \sum_{i \leq 0} a_i \partial^i \in \mathcal{B}((\partial^{-1}))$ such that $(L^3)^3 = L$. For $r \in \mathbb{N}$, we have $L^3 = \partial^r + \sum_{i = -\infty}^{r-1} b_i \partial^i$, $b_i \in \mathcal{B}$. We set $(L^3)^+ = \partial^r + \sum_{i = 0}^{r-1} b_i \partial^i$.

For $r \in \mathbb{Z}_{> 0}$, the differential equation

$$\frac{\partial L}{\partial t_r} = [L, (L^3)^+]$$

is called the $r$-th KdV equation of type $A_2^{(1)}$.

Equation (3.6) defines flows $\frac{\partial}{\partial t_r}$ on the space $\mathcal{D}$. For all $r, s$ the flows $\frac{\partial}{\partial t_r}$ and $\frac{\partial}{\partial t_s}$ commute, see [DS].

3.5. Miura maps. Let $\mathcal{L} = \partial + \Lambda + V$ be a Miura oper of type $A_2^{(1)}$ with $V = \sum_{k=1}^{3} v_k e_k, k = 1, 2, 3 \ v_k = 0$. For $i = 0, 1, 2$, define the scalar differential operator $L_i = \partial^3 + u_{i,1} \partial + u_{0,i} \in \mathcal{D}$ by the formula

$$L_0 = (\partial - v_3)(\partial - v_2)(\partial - v_1), \ L_1 = (\partial - v_1)(\partial - v_3)(\partial - v_2), \ L_2 = (\partial - v_2)(\partial - v_1)(\partial - v_3).$$

Theorem 3.5 ([DS Proposition 3.18]). Let a Miura oper $\mathcal{L}$ satisfy the $mKdV$ equation (3.5) for some $j$. Then for every $i = 0, 1, 2$, the differential operator $L_i$ satisfies the KdV equation (3.6).

We define the $i$-th Miura map by the formula

$$m_i : \mathcal{M}(A_2^{(1)}) \to \mathcal{D}, \ \mathcal{L} \mapsto L_i,$$

see (3.7).

For $i = 0, 1, 2$, an $i$-oper is a differential operator of the form

$$\mathcal{L} = \partial + \Lambda + V + W$$

with $V \in \mathcal{B}(g(A_2^{(1)})^0)$ and $W \in \mathcal{B}(n^{-})$. For $w \in \mathcal{B}(n^{-})$ and an $i$-oper $\mathcal{L}$, the differential operator $e^{\text{ad}_w}(\mathcal{L})$ is an $i$-oper. The $i$-opers $\mathcal{L}$ and $e^{\text{ad}_w}(\mathcal{L})$ are called $i$-gauge equivalent. A Miura oper is an $i$-oper for any $i$.

Theorem 3.6 ([DS Proposition 3.10]). If Miura oper $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are $i$-gauge equivalent, then $m_i(\mathcal{L}) = m_i(\tilde{\mathcal{L}})$.

4. Critical points of master functions and generation of pairs of polynomials

In this section we follow [MV1, MV2, VW]. For functions $f(x), g(x)$, we denote

$$\text{Wr}(f, g) = f(x)g'(x) - f'(x)g(x)$$

the Wronskian determinant.
4.1. Master function. Choose a pair of nonnegative integers \( k = (k_0, k_1) \). Consider variables \( u = (u_i^j) \), where \( j = 0, 1 \), and \( i = 1, \ldots, k_j \). The master function \( \Phi(u; k) \) is defined by the formula

\[
\Phi(u, k) = 2 \sum_{i < i'} \ln(u_i^0 - u_{i'}^0) + 8 \sum_{i < i'} \ln(u_i^1 - u_{i'}^1) - 4 \sum_{i < i'} \ln(u_i^0 - u_{i'}^1).
\]

The product of symmetric groups \( \Sigma_k = \Sigma_{k_0} \times \Sigma_{k_1} \) acts on the set of variables by permuting the coordinates with the same upper index. The function \( \Phi \) is symmetric with respect to the \( \Sigma_k \)-action.

A point \( u \) is a critical point if \( d\Phi = 0 \) at \( u \). In other words, \( u \) is a critical point if

\[
\begin{align*}
\sum_{i' \neq i} 2 \frac{u_i^0 - u_{i'}^0}{u_i^0 - u_{i'}^1} - \sum_{i' = 1}^{k_1} 4 \frac{u_i^0 - u_i^{1(1)}}{u_i^0 - u_i^{1(1)}}, & = 0, \quad i = 1, \ldots, k_0, \\
\sum_{i' \neq i} 8 \frac{u_i^{1(1)} - u_{i'}^{1(1)}}{u_i^{1(1)} - u_{i'}^0} - \sum_{i' = 1}^{k_0} 4 \frac{u_i^{1(1)} - u_i^0}{u_i^{1(1)} - u_i^0}, & = 0, \quad i = 1, \ldots, k_1.
\end{align*}
\]

The critical set is \( \Sigma_k \)-invariant. All orbits have the same cardinality \( k_0!k_1! \). We do not make distinction between critical points in the same orbit.

Remark. The master functions \( \Phi(u, k) \) for all vectors \( k \) are associated with the Kac-Moody algebra with Cartan matrix \( \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \), which is dual to the Cartan matrix \( A_2^{(2)} \), see [SV, MV1, MV2].

4.2. Polynomials representing critical points. Let \( u = (u_i^j) \) be a critical point of the master function \( \Phi \). Introduce the pair of polynomials \( y = (y_0(x), y_1(x)) \),

\[
y_j(x) = \prod_{i=1}^{k_j} (x - u_i^j).
\]

Each polynomial is considered up to multiplication by a nonzero number. The pair defines a point in the direct product \((\mathbb{C}[x])^2\). We say that the pair \( y = (y_0(x), y_1(x)) \) represents the critical point \( u = (u_i^j) \).

It is convenient to think that the pair \( y^0 = (1, 1) \) of constant polynomials represents in \((\mathbb{C}[x])^2\) the critical point of the master function with no variables. This corresponds to the case \( k = (0, 0) \).

We say that a given pair \( y \in (\mathbb{C}[x])^2 \) is generic if each polynomial \( y_j(x) \) of the pair has no multiple roots and the polynomials \( y_0(x) \) and \( y_1(x) \) have no common roots. If a pair represents a critical point, then it is generic, see (4.2). For example, the pair \( y^0 \) is generic.

4.3. Elementary generation. The pair is called fertile if there exist polynomials \( \tilde{y}_0, \tilde{y}_1 \in \mathbb{C}[x] \) such that the following two equations are satisfied,

\[
\mathrm{Wr}(y_0, \tilde{y}_0) = y_1^4, \quad \mathrm{Wr}(y_1, \tilde{y}_1) = y_0.
\]
These equations can be written as

\[(4.5) \quad \text{Wr}(y_j, \bar{y}_j) = \prod_{i \neq j} y_i^{-a_{i,j}}, \quad j = 0, 1.\]

For example, the pair \(y^0\) is fertile and \(\bar{y}_0 = x + c_1, \bar{y}_1 = x + c_2\), where \(c_1, c_2\) are arbitrary numbers.

Assume that a pair of polynomials \(y = (y_0, y_1)\) is fertile. Equation \(\text{Wr}(y_0, \bar{y}_0) = y_1^4\) is a first order inhomogeneous differential equation with respect to \(\bar{y}_0\). Its solutions are

\[(4.6) \quad \bar{y}_0 = y_0 \int \frac{y_1^4}{y_0^2} \, dx + cy_0,\]

where \(c\) is any number. The pairs

\[(4.7) \quad y^{(0)}(x, c) = (y_0(x, c), y_1(x)) \in (\mathbb{C}[x])^2\]

form a one-parameter family. This family is called the \emph{generation of pairs from \(y\) in the 0-th direction}. A pair of this family is called an \emph{immediate descendant} of \(y\) in the 0-th direction.

Similarly, equation \(\text{Wr}(y_1, \bar{y}_1) = y_0\) is a first order inhomogeneous differential equation with respect to \(\bar{y}_1\). Its solutions are

\[(4.8) \quad \bar{y}_1 = y_1 \int \frac{y_0}{y_1^2} \, dx + cy_1,\]

where \(c\) is any number. The pairs

\[(4.9) \quad y^{(1)}(x, c) = (y_0(x), \bar{y}_1(x, c)) \in (\mathbb{C}[x])^2\]

form a one-parameter family. This family is called the \emph{generation of pairs from \(y\) in the 1-st direction}. A pair of this family is called an \emph{immediate descendant} of \(y\) in the 1-st direction.

**Theorem 4.1 \([\text{MV1}]\).**

(i) A generic pair \(y = (y_0, y_1)\) represents a critical point of a master function if and only if \(y\) is fertile.

(ii) If \(y\) represents a critical point, then for any \(c \in \mathbb{C}\) the pairs \(y^{(0)}(x, c)\) and \(y^{(1)}(x, c)\) are fertile.

(iii) If \(y\) is generic and fertile, then for almost all values of the parameter \(c \in \mathbb{C}\) both pairs \(y^{(0)}(x, c)\) and \(y^{(1)}(x, c)\) are generic. The exceptions form a finite set in \(\mathbb{C}\).

(iv) Assume that a sequence \(y_i, i = 1, 2, \ldots\) of fertile pairs has a limit \(y_\infty\) in \((\mathbb{C}[x])^2\) as \(i\) tends to infinity.

(a) Then the limiting pair \(y_\infty\) is fertile.

(b) For \(j = 0, 1\), let \(y^{(j)}_\infty\) be an immediate descendant of \(y_\infty\). Then for \(j = 0, 1\), there exist immediate descendants \(y^{(j)}_i\) of \(y_i\) such that \(y^{(j)}_\infty\) is the limit of \(y^{(j)}_i\) as \(i\) tends to infinity.

4.4. Degree increasing generation. Let \(y = (y_0, y_1)\) be a generic fertile pair of polynomials. For \(j = 0, 1\), define \(k_j = \deg y_j\).

The polynomial \(\bar{y}_0\) in (4.4) is of degree \(k_0\) or \(\bar{k}_0 = 4k_1 + 1 - k_0\). We say that the generation \((y_0, y_1) \rightarrow (\bar{y}_0, y_1)\) is \emph{degree increasing} in the 0-th direction if \(\bar{k}_0 > k_0\). In that case \(\deg \bar{y}_0 = \bar{k}_0\) for all \(c\).
We say that the formations to \(J = (0, 1, \ldots, j_0)\) of type \(A_2^{(2)}\). We start with the vector \(k^0 = (0, 0)\) and a sequence \(J = (j_1, j_2, \ldots, j_m)\) of integers, where \(J = (0, 1, 0, 1, 0, 1, \ldots)\) or \(J = (1, 0, 1, 0, 1, 0, \ldots)\). We apply the corresponding degree transformations to \(k^0\) and obtain the sequence of vectors \(k^0, k^{(j_1)} = w_{j_1} k^0, k^{(j_1, j_2)} = w_{j_2} w_{j_1} k^0, \ldots\)

\[ k^j = w_{j_m} \ldots w_{j_2} w_{j_1} k^0. \]

We say that the vector \(k^j\) is generated from \((0, \ldots, 0)\) in the direction of \(J\).
4.6. Multistep generation. Let \( J = (j_1, \ldots, j_m) \) be a basic sequence. Starting from \( y^0 = (1, 1) \) and \( J \), we construct by induction on \( m \) a map

\[
Y^J : \mathbb{C}^m \to (\mathbb{C}[x])^2.
\]

If \( J = \emptyset \), the map \( Y^0 \) is the map \( \mathbb{C}^0 = (pt) \to y^0 \). If \( m = 1 \) and \( J = (j_1) \), the map \( Y^{(j_1)} : \mathbb{C} \to (\mathbb{C}[x])^N \) is given by one of the formulas \((4.11)\) or \((4.13)\) for \( y = y^0 \) and \( j = j_1 \).

More precisely, equation \( \text{Wr}(y_0, \tilde{y}_0) = y^4_1 \) takes the form \( \text{Wr}(1, \tilde{y}_0) = 1 \). Then \( \tilde{y}_{0,0} = x \) and

\[
Y^{(0)} : \mathbb{C} \to (\mathbb{C}[x])^2, \quad c \mapsto (x + c, 1).
\]

By Theorem \([4.1]\) all pairs in the image are fertile and almost all pairs are generic (in this example all pairs are generic). Similarly, equation \( \text{Wr}(y_1, \tilde{y}_1) = y_0 \) takes the form \( \text{Wr}(1, \tilde{y}_1) = 1 \). Then \( \tilde{y}_{1,0} = x \) and

\[
Y^{(1)} : \mathbb{C} \to (\mathbb{C}[x])^2, \quad c \mapsto (1, x + c).
\]

Assume that for \( \bar{J} = (j_1, \ldots, j_{m-1}) \), the map \( Y^{\bar{J}} \) is constructed. To obtain \( Y^J \) we apply the generation procedure in the \( j_m \)-th direction to every pair of the image of \( Y^{\bar{J}} \). More precisely, if

\[
Y^{\bar{J}} : \bar{c} = (c_1, \ldots, c_{m-1}) \mapsto (y_0(x, \bar{c}), y_1(x, \bar{c})) \tag{4.17}
\]

then

\[
Y^J : C^m \to (\mathbb{C}[x])^2, \quad (\bar{c}, c_m) \mapsto (y_{0,0}(x, \bar{c}) + c_my_0(x, \bar{c}), y_{1}(x, \bar{c})) \quad \text{if } j_m = 0, \tag{4.18}
\]

\[
Y^J : C^m \to (\mathbb{C}[x])^2, \quad (\bar{c}, c_m) \mapsto (y_0(x, \bar{c}) + c_my_0(x, \bar{c}), y_{1,0}(x, \bar{c}) + c_my_1(x, \bar{c})) \quad \text{if } j_m = 1,
\]

see formulas \((4.10)\), \((4.12)\). The map \( Y^J \) is called the generation of pairs from \( y^0 \) in the \( J \)-th direction.

**Lemma 4.4.** All pairs in the image of \( Y^J \) are fertile and almost all pairs are generic. For any \( c \in \mathbb{C}^m \) the pair \( Y^J(c) \) consists of monic polynomials. The degree vector of this pair equals \( k^J \), see \((4.16)\).

**Lemma 4.5.** The map \( Y^J \) sends distinct points of \( \mathbb{C}^m \) to distinct points of \((\mathbb{C}[x])^2\).

**Proof.** The lemma is easily proved by induction on \( m \).
Example. If \( J = (0,1) \), then
\[
Y^{(0)}(c_1) = (x + c_1, 1), \quad Y^{(0,1)}(c_1, c_2) = (x + c_1, (x + c_1)^2 + c_2 - c_1^2).
\]
If \( J = (1,0) \), then
\[
Y^{(1)}(c_1) = (1, x + c_1), \quad Y^{(1,0)}(c_1, c_2) = ((x + c_1)^4 + c_2 - c_1^2, x + c_1).
\]

The set of all pairs \((y_0, y_1) \in (\mathbb{C}[x])^2\) obtained from \(y^0 = (1, 1)\) by generations in all degree increasing directions is called the \emph{population of pairs} generated from \(y^0\), c.f. [MV1].

5. Critical points of master functions and Miura opers

5.1. Miura oper associated with a pair of polynomials, [MV2]. Define a map
\[
\mu : (\mathbb{C}[x])^2 \to \mathcal{M}(A_2^{(2)}),
\]
which sends a pair \( y = (y_0, y_1) \) to the Miura oper \( \mathcal{L} = \partial + \Lambda + V \) with
\[
V = \ln'(\frac{y_1^2}{y_0}) h_0,
\]
where for a function \( f(x) \) we denote \( \ln'(f(x)) = \frac{f'(x)}{f(x)} \). We say that the Miura oper \( \mu(y) \) is \emph{associated to the pair of polynomials} \( y \). For example,
\[
\mathcal{L}^0 = \partial + \Lambda
\]
is associated to the pair \( y^0 = (1, 1) \).

We have
\[
\langle \alpha_0, V \rangle = \ln'(\frac{y_1^2}{y_0}), \quad \langle \alpha_1, V \rangle = \ln'(\frac{y_0}{y_1^2}).
\]
Equations (5.1) can be written as
\[
\langle \alpha_j, V \rangle = \ln'\left(\prod_{i=0}^{1} y_i^{-\alpha_{i,j}}\right),
\]
see [MV2].

5.2. Deformations of Miura opers of type \( A_2^{(2)} \), [MV2].

Lemma 5.1 ([MV2]). Let \( \mathcal{L} = \partial + \Lambda + V \) be a Miura oper of type \( A_2^{(2)} \). Let \( g \in \mathcal{B} \). Let \( f_j, j \in \{0,1\} \), be one of canonical generators of \( g(A_2^{(2)}) \), see Section 2.1.4. Then
\[
e^{\text{ad } g f_j(\mathcal{L})} = \partial + \Lambda + V - gh_j - (g' - \langle \alpha_j, V \rangle g + g^2)f_j.
\]

Corollary 5.2 ([MV2]). Let \( \mathcal{L} = \partial + \Lambda + V \) be a Miura oper. Then \( e^{\text{ad } g f_j(\mathcal{L})} \) is a Miura oper if and only if the scalar function \( g \) satisfies the Ricatti equation
\[
g' - \langle \alpha_j, V \rangle g + g^2 = 0.
\]

Let \( \mathcal{L} = \partial + \Lambda + V \) be a Miura oper with \( V = vh_0 \). Assume that the functions \( v \) is a rational functions of \( x \). For \( j \in \{0,1\} \), we say that \( \mathcal{L} \) is \emph{deformable in the} \( j \)-th direction if equation (5.4) has a nonzero solution \( g \), which is a rational function.
Theorem 5.3 ([MV2]). Let the Miura oper $\mathcal{L} = \partial + \Lambda + V$ be associated with a pair of polynomials $y = (y_0, y_1)$. Let $j \in \{0, 1\}$. Then $\mathcal{L}$ is deformable in the $j$-th direction if and only if there exists a polynomial $\tilde{y}_j$ satisfying equation (4.5). Moreover, in that case any nonzero rational solution $g$ of the Ricatti equation (5.4) has the form $g = \ln'(\tilde{y}_j/y_j)$ where $\tilde{y}_j$ is a solution of equation (4.5). If $g = \ln'(\tilde{y}_i/y_i)$, then the Miura oper

\begin{equation}
  e^{\text{ad} y_i}(\mathcal{L}) = \partial + \Lambda + V - gh_j
\end{equation}

is associated the pair $y^{(j)}$, which is obtained from the pair $y$ by replacing $y_j$ with $\tilde{y}_j$.

Proof. Write (5.4) as

\begin{equation}
  g'/g + g = \ln'\left(\prod_{j=0}^1 y_i^{-a_{i,j}}\right).
\end{equation}

If $g$ is a rational function, then $g \to 0$ as $x \to \infty$ and all poles of $g$ are simple. Moreover, the residue of $g$ at any point is an integer. Hence $g = c'/c$ for a suitable rational function $c(x)$. Then

\begin{equation}
  c = \int \prod_{j=0}^1 y_j(x)^{-a_{i,j}} dx
\end{equation}

and equation (4.5) has a polynomial solution $\tilde{y}_j = -cy_j$. Conversely if equation (4.5) has a polynomial solution $\tilde{y}_i$, then the function $c$ in (5.7) is rational. Then $g = c'/c$ is a rational solution of equation (5.4).

Let $g = \ln' c = \ln'(\tilde{y}_i/y_i)$, where $\tilde{y}_i$ is a solution of (4.5). Then

\begin{equation}
  e^{\text{ad} y_i}(\mathcal{L}) = \partial + \Lambda + V - \ln'(\tilde{y}_j/y_j)h_j
\end{equation}

and

\begin{equation}
  \langle \alpha_k, V \rangle - \langle \alpha_k, h_j \rangle \ln'(\tilde{y}_j/y_j) = \ln'\left(\prod_{i=0}^1 y_i^{-a_{i,k}}\right) - a_{j,k} \ln'(\tilde{y}_j/y_j) = \ln'\left(y_j^{-a_{j,k}} \prod_{i=0, i \neq j} y_i^{-a_{i,k}}\right).
\end{equation}

Note that if equation (5.4) has one nonzero rational solution $g = c'/c$ with rational $c(x)$, then other nonzero (rational) solutions have the form $g = c'/(c + \text{const})$.

5.3. Miura opers associated with the generation procedure. Let $J = (j_1, \ldots, j_m)$ be a basic sequence, see Section 4.5. Let $Y^J : \mathbb{C}^m \to (\mathbb{C}[x])^2$ be the generation of pairs from $y^0$ in the $J$-th direction. We define the associated family of Miura opers by the formula:

\begin{equation}
  \mu^J : \mathbb{C}^m \to \mathcal{M}(A_2^{(2)}), \quad c \mapsto \mu(Y^J(c)).
\end{equation}

The map $\mu^J$ is called the generation of Miura opers from $L^0$ in the $J$-th direction.

For $\ell = 1, \ldots, m$, denote $J_{\ell} = (j_1, \ldots, j_\ell)$ the beginning $\ell$-interval of the sequence $J$. Consider the associated map $Y^{J_{\ell}} : \mathbb{C}^\ell \to (\mathbb{C}[x])^2$. Denote

\begin{equation}
  Y^{J_{\ell}}(c_1, \ldots, c_\ell) = (y_0(x, c_1, \ldots, c_\ell, \ell), y_1(x, c_1, \ldots, c_\ell, \ell)).
\end{equation}
Introduce
\begin{align}
(5.8) \quad g_1(x, c_1, \ldots, c_m) &= \ln'(y_{j_1}(x, c_1, 1)), \\
\quad g_\ell(x, c_1, \ldots, c_m) &= \ln'(y_{j_\ell}(x, c_1, \ldots, c_\ell, \ell)) - \ln'(y_{j_\ell}(x, c_1, \ldots, c_{\ell-1}, \ell - 1)),
\end{align}
for \( \ell = 2, \ldots, m \). For \( c \in C^m \), define \( U'^{(\ell)}(c) = \sum_{i<0} U'^{(\ell)}(c)_i, (U'^{(\ell)}(c))_i \in B(g^i) \), depending on \( c \in C^m \), by the formula
\begin{align}
(5.9) \quad e^{-a U'^{(\ell)}(c)} = e^{a \text{ad}_{g_m}(x,c)f_{\ell m}} \cdots e^{a \text{ad}_{g_1}(x,c)f_{11}}.
\end{align}

**Lemma 5.4.** For \( c \in C^m \), we have
\begin{align}
(5.10) \quad \mu'^{(\ell)}(c) &= e^{-a U'^{(\ell)}(c)}(L^0)
\end{align}
and
\begin{align}
(5.11) \quad \mu'^{(\ell)}(c) &= \partial + \Lambda - \sum_{\ell=1}^m g_\ell(x,c)h_{j_\ell}.
\end{align}

**Proof.** The lemma follows from Theorem 5.3.

**Corollary 5.5.** Let \( r > 0 \) and \( r = 1, 5 \mod 6 \). Let \( c \in C^m \). Let \( \frac{\partial}{\partial t_r}|_{\mu'^{(\ell)}(c)} \) be the value at \( \mu'^{(\ell)}(c) \) of the vector field of the \( r \)-th mKdV flow on the space \( \mathcal{M}(A_2^{(2)}) \), see (3.2). Then
\begin{align}
(5.12) \quad \frac{\partial}{\partial t_r}|_{\mu'^{(\ell)}(c)} = -\frac{\partial}{\partial x}(e^{-a U'^{(\ell)}(c)}(\Lambda_r))_0.
\end{align}

**Proof.** The corollary follows from (3.2) and (5.10).

We have a natural embedding \( \mathcal{M}(A_2^{(2)}) \hookrightarrow \mathcal{M}(A_2^{(1)}) \). Let \( m_i : \mathcal{M}(A_2^{(1)}) \rightarrow D, L \mapsto L_i \), be the Miura maps defined in Section 3.5 for \( i = 0, 1 \). Below we consider the composition of the embedding and a Miura map.

Denote \( \tilde{j} = (j_1, \ldots, j_{m-1}) \). Consider the associated family \( \mu'^{\tilde{j}} : C^{m-1} \rightarrow \mathcal{M}(A_2^{(2)}) \). Denote \( \tilde{c} = (c_1, \ldots, c_{m-1}) \).

**Lemma 5.6.** For all \( (\tilde{c}, c_m) \in C^m \), we have \( m_1 \circ \mu'^{\tilde{j}}(\tilde{c}, c_m) = m_1 \circ \mu'^{\tilde{j}}(\tilde{c}) \) if \( j_m = 0 \) and we have \( m_0 \circ \mu'^{\tilde{j}}(\tilde{c}, c_m) = m_0 \circ \mu'^{\tilde{j}}(\tilde{c}) \) if \( j_m = 1 \).

**Proof.** The lemma follows from formula (5.10) and Theorem 8.6.

**Lemma 5.7.** If \( j_m = 0 \), then
\begin{align}
(5.13) \quad \frac{\partial \mu'^{\tilde{j}}}{\partial c_m}(\tilde{c}, c_m) &= -a \frac{y_1(x, \tilde{c}, m - 1)^4}{y_0(x, \tilde{c}, c_m, m)^2} h_0
\end{align}
for some \( a \in C^\infty \). If \( j_m = 1 \), then
\begin{align}
(5.14) \quad \frac{\partial \mu'^{\tilde{j}}}{\partial c_m}(\tilde{c}, c_m) &= -a \frac{y_0(x, \tilde{c}, m - 1)}{y_1(x, \tilde{c}, c_m, m)^2} h_1
\end{align}
for some \( a \in C^\infty \).
Proof. Let \( j_m = 0 \). Then \( y_0(x, \tilde{c}, c_m, m) = y_{0,0}(x, \tilde{c}) + c_m y_0(x, \tilde{c}, m - 1) \), where \( y_{0,0}(x, \tilde{c}) \) is such that

\[
\text{Wr}(y_0(x, \tilde{c}, m - 1), y_{0,0}(x, \tilde{c})) = a \ y_1(x, \tilde{c}, m - 1)^4,
\]

for some \( a \in \mathbb{C}^\times \), see (4.10). We have \( g_m = \ln'(y_0(x, \tilde{c}, c_m, m)) - \ln'(y_0(x, \tilde{c}, m - 1)) \).

By formula (5.11), we have

\[
\frac{\partial \mu'}{\partial c_m}(\tilde{c}, c_m) = - \frac{\partial g_m}{\partial c_m}(\tilde{c}, c_m) \ h_0
\]

and

\[
\frac{\partial g_m}{\partial c_m}(\tilde{c}, c_m) = \frac{\partial}{\partial c_m} \left( \frac{y_{0,0}'(x, \tilde{c}) + c_m y_0'(x, \tilde{c}, m - 1)}{(y_{0,0}(x, \tilde{c}) + c_m y_0(x, \tilde{c}, m - 1))^2} \right) = \frac{\text{Wr}(y_{0,0}(x, \tilde{c}), y_0(x, \tilde{c}, m - 1))}{(y_{0,0}(x, \tilde{c}) + c_m y_0(x, \tilde{c}, m - 1))^2} = \frac{a y_1(x, \tilde{c}, m - 1)^4}{y_0(x, \tilde{c}, c_m, m)^2}.
\]

This proves formula (5.13). Formula (5.14) is proved similarly.

Let us describe the kernels of the differentials of the Miura maps \( m_i, i = 0, 1 \), restricted to Miura opers of type \( A_2(2) \). A Miura oper \( \mathcal{L} = \partial + \Lambda + v h_0 \) of type \( A_2(2) \) is mapped to the differential operator \((\partial + v)\partial(\partial - v) = \partial^3 - (2v' + v^2)\partial - (v'' + vv')\) by the Miura map \( m_0 \) and to the differential operator \((\partial - v)(\partial + v)\partial = \partial^3 + (v' - v^2)\partial\) by the Miura map \( m_1 \). The derivative maps are

\[
d m_0 : \quad X h_0 \mapsto -(2X' + 2vX)\partial - (X'' + vX' + v'X),
\]

\[
d m_1 : \quad X h_0 \mapsto (X' - 2vX)\partial,
\]

where \( x \in \mathcal{B} \).

Lemma 5.8. Assume that \( \mathcal{L} \) is associated to a pair \((y_0, y_1)\), that is, \( v = \ln' \left( \frac{y_1}{y_0} \right) \). Then the kernel of \( d m_0 \) at \( \mathcal{L} \) is one-dimensional and is generated by the function \( \frac{y_1}{y_0} h_0 \). Also the kernel of \( d m_1 \) at \( \mathcal{L} \) is one-dimensional and is generated by the function \( \frac{y_1}{y_0} h_0 \).

Proof. We have \( X \in \ker d m_0 \) if and only if \( X' + vX = 0 \). This implies the first statement. Similarly, \( X \in \ker d m_1 \) if and only if \( X' - 2vX = 0 \). This implies the second statement. \( \square \)

6. Vector fields

6.1. Statement. Let \( r > 0 \) and \( r = 1, 5 \mod 6 \). Recall that we denote by \( \frac{\partial}{\partial x_r} \) the \( r \)-th mKdV vector field on the space \( \mathcal{M}(A_2(2)) \) of Miura opers of type \( A_2(2) \). We also denote by \( \frac{\partial}{\partial x_r} \) the \( r \)-th mKdV vector field of type \( A_2(1) \) on the space \( \mathcal{M}(A_2(1)) \) of Miura opers of type \( A_2(1) \). We have a natural embedding \( \mathcal{M}(A_2(2)) \hookrightarrow \mathcal{M}(A_2(1)) \). Under this embedding the vector \( \frac{\partial}{\partial x_r} \) on \( \mathcal{M}(A_2(2)) \) equals the vector field \( \frac{\partial}{\partial x_r} \) on \( \mathcal{M}(A_2(1)) \) restricted to \( \mathcal{M}(A_2(2)) \), see Section 3.3.

We also denote by \( \frac{\partial}{\partial x_r} \) the \( j \)-th KdV vector field on the space \( \mathcal{D} \), see Section 3.4.

For a Miura map \( m_i : \mathcal{M} \to \mathcal{D}, \mathcal{L} \mapsto L_i \), denote by \( d m_i \) the associated derivative map \( T \mathcal{M}(A_2(1)) \to T \mathcal{D} \). By Theorem 3.3 we have \( d m_i : \frac{\partial}{\partial x_r} \big|_\mathcal{L} \mapsto \frac{\partial}{\partial x_r} \big|_{L_i} \).
Fix a basic sequence \( J = (j_1, \ldots, j_m) \). Consider the associated family \( \mu^J : \mathbb{C}^m \to \mathcal{M}(A_2^{(2)}) \) of Miura opers. For a vector field \( \Gamma \) on \( \mathbb{C}^m \), we denote by \( \frac{\partial \mu^J}{\partial t_r} \) the derivative of \( \mu^J \) along the vector field. The derivative is well-defined since \( \mathcal{M}(A_2^{(2)}) \) is an affine space.

**Theorem 6.1.** Let \( r > 0 \) and \( r = 1, 5 \mod 6 \). Then there exists a polynomial vector field \( \Gamma_r \) on \( \mathbb{C}^m \) such that

\[
\frac{\partial}{\partial t_r}\big|_{\mu^J(c)} = \frac{\partial \mu^J}{\partial \Gamma_r}(c)
\]

for all \( c \in \mathbb{C}^m \). If \( m \) is even and \( r > 3m \), then \( \frac{\partial}{\partial t_r}\big|_{\mu^J(c)} = 0 \) for all \( c \in \mathbb{C}^m \) and, hence, \( \Gamma_r = 0 \). If \( m \) is odd and \( j_1 = j_m = 0 \), then for \( r > 3m - 2 \) we have \( \frac{\partial}{\partial t_r}\big|_{\mu^J(c)} = 0 \) for all \( c \in \mathbb{C}^m \) and, hence, \( \Gamma_r = 0 \). If \( m \) is odd and \( j_1 = j_m = 1 \), then for \( r > 3m + 1 \) we have \( \frac{\partial}{\partial t_r}\big|_{\mu^J(c)} = 0 \) for all \( c \in \mathbb{C}^m \) and, hence, \( \Gamma_r = 0 \).

**Corollary 6.2.** The family \( \mu^J \) of Miura opers is invariant with respect to all mKdV flows of type \( A_2^{(2)} \) and is point-wise fixed by flows with \( r > 3m + 1 \).

6.2. **Proof of Theorem 6.1 for** \( m = 1 \). If \( m = 1 \), then \( J = (0) \) of \( J = (1) \).

Let \( J = (0) \). Then

\[
\mu^J(c_1) = e^{g_1F_0}L^0e^{-g_1F_0} = (1 + g_1e_{3,3}L^{-1})L^0(1 - g_1e_{3,3}L^{-1}) = \partial + \Lambda + g_1(e_{3,3} - e_{1,1}) = \partial + \Lambda - g_1h_0,
\]

where \( g_1 = \frac{1}{x+c_1} \). By formula (5.12),

\[
\frac{\partial}{\partial t_r}\big|_{\mu^J(c_1)} = -\frac{\partial}{\partial x}(1 + g_1e_{3,3}L^{-1})\Lambda(1 - g_1e_{3,3}L^{-1})^0.
\]

It follows from Lemma 2.5 that \( \frac{\partial}{\partial t_r}\big|_{\mu^J(c_1)} = 0 \) for \( r > 1 \) and hence \( \Gamma_r = 0 \). For \( r = 1 \), we have \( \frac{\partial}{\partial t_1}\big|_{\mu^J(c_1)} = -\frac{\partial}{\partial x}g_1h_0 = \frac{1}{(x+c_1)^2}h_0 \).

Hence \( \Gamma_1 = -\frac{\partial}{\partial c_1} \). Theorem 6.1 is proved for \( m = 1 \), \( J = (0) \).

Let \( J = (1) \). Then

\[
\mu^J(c_1) = e^{ad_{g_1}(2F_1+2F_2)}(L^0) = \partial + \Lambda - g_1h_1,
\]

where \( g_1 = \frac{1}{x+c_1} \). By formula (5.12),

\[
\frac{\partial}{\partial t_r}\big|_{\mu^J(c_1)} = -\frac{\partial}{\partial x}((1 + g_1(e_{1,1} + e_{2,2})L^{-1} + 2g_1^2e_{1,1}L^{-2}) \times \\
\Lambda(1 - g_1(e_{1,1} + e_{2,2})L^{-1} + 2g_1^2e_{1,1}L^{-2})^0.
\]

It follows from Lemma 2.5 that \( \frac{\partial}{\partial t_1}\big|_{\mu^J(c_1)} = 0 \) for \( r > 4 \) and hence \( \Gamma_r = 0 \). For \( r = 1 \), we have \( \frac{\partial}{\partial t_1}\big|_{\mu^J(c_1)} = \frac{1}{(x+c_1)^2}e_{1,1} - e_{3,3} \). On the other hand, \( \frac{\partial}{\partial c_1}\mu^J(c_1) = -\frac{\partial}{\partial c_1}h_1 = \frac{1}{(x+c_1)^2}h_1 \). Hence \( \Gamma_1 = -\frac{1}{2}\frac{\partial}{\partial c_1} \). Theorem 6.1 is proved for \( m = 1 \), \( J = (1) \).
6.3. Proof of Theorem 6.1 for \( m > 1 \).

Lemma 6.3. If \( m \) is even and \( r > 3m \), then \( \frac{\partial}{\partial r} \bigg|_{\mu_j'(c)} = 0 \) for all \( c \in \mathbb{C}^m \) and, hence, \( \Gamma_r = 0 \).
If \( m \) is odd and \( j_1 = j_m = 0 \), then for \( r > 3m - 2 \) we have \( \frac{\partial}{\partial r} \bigg|_{\mu_j'(c)} = 0 \) for all \( c \in \mathbb{C}^m \) and, hence, \( \Gamma_r = 0 \). If \( m \) is odd and \( j_1 = j_m = 1 \), then for \( r > 3m + 1 \) we have \( \frac{\partial}{\partial r} \bigg|_{\mu_j'(c)} = 0 \) for all \( c \in \mathbb{C}^m \) and, hence, \( \Gamma_r = 0 \).

Proof. The vector \( \frac{\partial}{\partial r} \bigg|_{\mu_j'(c)} \) equals the right hand side of formula (5.12). By Lemmas 2.4 and 2.5 the right hand side of (5.12) is zero if \( r \) is as described in the lemma.

We prove the first statement of Theorem 6.1 by induction on \( m \). Assume that the statement is proved for \( J = \{j_1, \ldots, j_{m-1}\} \). Let

\[
Y^J : \tilde{c} = (c_1, \ldots, c_{m-1}) \mapsto (y_0(x, \tilde{c}), y_1(x, \tilde{c}))
\]
be the generation of pairs in the \( J \)-th direction. Then the generation of pairs in the \( J \)-th direction is

\[
Y^J : \mathbb{C}^m \mapsto (\mathbb{C}[x])^2, \quad (\tilde{c}, c_m) \mapsto (\ldots, y_{j_m,0}(x, \tilde{c}) + c_m y_{j_m}(x, \tilde{c}), \ldots),
\]
see (1.17) and (4.18). We have \( g_m = \ln'(y_{j_m,0}(x, \tilde{c}) + c_m y_{j_m}(x, \tilde{c})) - \ln'(y_{j_m}(x, \tilde{c})) \), see (5.8).

By the induction assumption, there exists a polynomial vector field \( \Gamma_{r,J} = \sum_{i=1}^{m-1} \gamma_i(\tilde{c}) \frac{\partial}{\partial c_i} \) on \( \mathbb{C}^{m-1} \) such that

\[
(6.2) \quad \frac{\partial}{\partial t_r} \bigg|_{\mu_j'(\tilde{c})} = \frac{\partial \mu_j^J}{\partial \Gamma_{r,J}}(\tilde{c})
\]
for all \( \tilde{c} \in \mathbb{C}^{m-1} \).

Theorem 6.4. There exists a scalar polynomial \( \gamma_m(\tilde{c}, c_m) \) on \( \mathbb{C}^m \) such that the vector field \( \Gamma_r = \Gamma_{r,J} + \gamma_m(\tilde{c}, c_m) \frac{\partial}{\partial c_m} \) satisfies (6.1) for all \( (\tilde{c}, c_m) \in \mathbb{C}^m \).

The first statement of Theorem 6.1 follows from Theorem 6.4.

6.4. Proof of Theorem 6.4.

Lemma 6.5. If \( j_m = 1 \), then for all \( (\tilde{c}, c_m) \in \mathbb{C}^m \), we have

\[
(6.3) \quad dm_0 \bigg|_{\mu_j'(\tilde{c}, c_m)} \left( \frac{\partial}{\partial t_r} \bigg|_{\mu_j'(\tilde{c}, c_m)} - \frac{\partial \mu_j^J}{\partial \Gamma_{r,J}}(\tilde{c}, c_m) \right) = 0,
\]
If \( j_m = 0 \), then for all \( (\tilde{c}, c_m) \in \mathbb{C}^m \), we have

\[
(6.4) \quad dm_1 \bigg|_{\mu_j'(\tilde{c}, c_m)} \left( \frac{\partial}{\partial t_r} \bigg|_{\mu_j'(\tilde{c}, c_m)} - \frac{\partial \mu_j^J}{\partial \Gamma_{r,J}}(\tilde{c}, c_m) \right) = 0.
\]

Proof. The proof of this lemma is the same as the proof of Lemma 5.5 in [VW].

Let \( j_m = 1 \). By Lemma 6.5 the vector \( \frac{\partial}{\partial t_r} \bigg|_{\mu_j'(\tilde{c}, c_m)} - \frac{\partial \mu_j^J}{\partial \Gamma_{r,J}}(\tilde{c}, c_m) \) lies in the kernel of the map \( dm_0 \bigg|_{\mu_j'(\tilde{c}, c_m)} \). By Lemma 5.8 this kernel is generated by \( \frac{h_0(\tilde{x}, \tilde{c}, c_m-1)}{h(\tilde{x}, \tilde{c}, c_m)} \). By Lemma 5.7 we
have $\frac{\partial \mu^j}{\partial \delta_{\nu}}(\bar{c},c_m) = a \frac{y_0(x,\bar{c},m-1)}{y_1(x,\bar{c},c_m,m^2)} h_0$ for some $a \in \mathbb{C}^\times$. Hence there exists a number $\gamma_m(\bar{c},c_m)$ such that $\frac{\partial}{\partial \nu} \mu^j(\bar{c},c_m) = \Gamma_{r,j} \gamma_m(\bar{c},c_m) \frac{\partial}{\partial \delta_{\nu}}$.

Let $j_m = 0$. By Lemma 5.5, the vector $\frac{\partial}{\partial \nu} \mu^j(\bar{c},c_m) = \frac{\partial \mu^j}{\partial \Gamma_{r,j}}(\bar{c},c_m)$ lies in the kernel of the map $d\Gamma_{1}(\mu^j(\bar{c},c_m))$. By Lemma 5.8, this kernel is generated by the polynomial $\frac{y_1(x,\bar{c},m-1)}{y_0(x,\bar{c},c_m,m^2)} h_0$.

By Lemma 5.7, we have $\frac{\partial \mu^j}{\partial \delta_{\nu}}(\bar{c},c_m) = a \frac{y_1(x,\bar{c},c_m,m^2)}{y_0(x,\bar{c},c_m,m^2)} h_0$ for some $a \in \mathbb{C}^\times$. Hence there exists a number $\gamma_m(\bar{c},c_m)$ such that $\frac{\partial}{\partial \delta_{\nu}} \mu^j(\bar{c},c_m) = \Gamma_{r,j} \gamma_m(\bar{c},c_m) \frac{\partial}{\partial \delta_{\nu}}$.

**Proposition 6.6.** The function $\gamma_m(\bar{c},c_m)$ is a polynomial on $\mathbb{C}^m$.

**Proof.** The proof is similar to the proof of Proposition 5.9 in [VW].

Let $g = x^d + \sum_{i=0}^{d-1} A_i(c_1,\ldots,c_m)x^i$ be a polynomial in $x, c_1,\ldots,c_m$. Denote $h = \ln g$ the logarithmic derivative of $g$ with respect to $x$. Consider the Laurent expansion of $h$ at $x = \infty$, $h = \sum_{i=1}^{\infty} B_i(c_1,\ldots,c_m)x^{-i}$.

**Lemma 6.7.** All coefficients $B_i$ are polynomials in $c_1,\ldots,c_m$.

The vector $Y = \frac{\partial}{\partial \nu} \mu^j(\bar{c},c_m)$ is a $3 \times 3$ diagonal matrix depending on $x, c_1,\ldots,c_m$, $Y = Y_1(\nu_{1,1} - \nu_{3,3})$ where $Y_1$ is a scalar function.

**Lemma 6.8.** The function $Y_1$ is a rational function in $x, c_1,\ldots,c_m$ which has a Laurent expansion of the form $Y_1 = \sum_{i=1}^{\infty} B_i(c_1,\ldots,c_m)x^{-i}$ where all coefficients $B_i$ are polynomials in $c_1,\ldots,c_m$.

The vector $Y = \frac{\partial \mu^j}{\partial \Gamma_{r,j}}(\bar{c},c_m)$ is a $3 \times 3$ diagonal matrix depending on $x, c_1,\ldots,c_m$, $Y = Y_1(\nu_{1,1} - \nu_{3,3})$ where $Y_1$ is a scalar function.

**Lemma 6.9.** The function $Y_1$ is a rational function of $x, c_1,\ldots,c_m$ which has a Laurent expansion of the form $Y_1 = \sum_{i=1}^{\infty} B_i(c_1,\ldots,c_m)x^{-i}$ where all coefficients $B_i$ are polynomials in $c_1,\ldots,c_m$.

Let us finish the proof of Proposition 6.6. The function $\gamma_m(\bar{c},c_m)$ is determined from the equation

$$\frac{\partial}{\partial \nu} \mu^j(\bar{c},c_m) - \frac{\partial \mu^j}{\partial \Gamma_{r,j}}(\bar{c},c_m) = a_1 \gamma_m(\bar{c},c_m) \frac{y_0(x,\bar{c},m-1)}{y_1(x,\bar{c},c_m,m^2)} h_0$$

if $j_m = 1$ and from the equation

$$\frac{\partial}{\partial \nu} \mu^j(\bar{c},c_m) - \frac{\partial \mu^j}{\partial \Gamma_{r,j}}(\bar{c},c_m) = a_2 \gamma_m(\bar{c},c_m) \frac{y_1(x,\bar{c},c_m,m^2)}{y_0(x,\bar{c},c_m,m^2)} h_0$$

if $j_m = 0$. Here $a_1, a_2$ are nonzero complex numbers independent of $\bar{c},c_m$.

The function $\frac{y_0(x,\bar{c},m-1)}{y_1(x,\bar{c},c_m,m^2)}$ has the Laurent expansion of the form $\sum_{i=1}^{\infty} B_i(c_1,\ldots,c_m)x^{-i}$ and the first nonzero coefficient $B_i$ of this expansion is 1 since the polynomials $y_0, y_1$ are monic polynomials. Hence $\gamma_m$ is a polynomial if $j_m = 1$. Similarly, the function $\frac{y_1(x,\bar{c},c_m,m^2)}{y_0(x,\bar{c},c_m,m^2)}$ has the Laurent expansion of the form $\sum_{i=1}^{\infty} B_i(c_1,\ldots,c_m)x^{-i}$ and the first nonzero coefficient $B_i$ of this expansion is 1 since the polynomials $y_0, y_1$ are monic polynomials. Hence $\gamma_m$ is a polynomial if $j_m = 0$.

**Theorem 6.1** is proved.
6.5. Critical points and the population generated from $y^0$.

**Theorem 6.10 (MV3).** If a pair of polynomials $(y_0, y_2)$ represents a critical point of the master function (4.1) for some parameters $k = (k_0, k_1)$, then $(y_0, y_1)$ is a point of the population of pairs generated from $y^0$.

**References**

[AM] M. Adler, J. Moser, *On a class of polynomials connected with the Korteweg-de Vries equation*, Comm. Math. Phys. **61** (1978), 1–30

[BF] H. Babujian, R. Flume, *Off-shell Bethe ansatz equation for Gaudin magnets and solutions of Knizhnik-Zamolodchikov equations*, Modern Phys. Lett. A **9** (1994), 2029-2039

[DS] V. G. Drinfel’d and V. V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, Current problems in mathematics 24, Itogi Nauki i Tekhniki, pages 81–180, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984

[F] E. Frenkel, *Opers on the projective line, flag manifolds and Bethe ansatz*, Mosc. Math. J. **4** (2004), no. 3, 655-705, 783

[MTV] E. Mukhin, V. Tarasov, A. Varchenko, *Schubert calculus and representations of general linear group*, J. Amer. Math. Soc. **22** (2009), no. 4, 909–940

[MV1] E. Mukhin, A. Varchenko, *Critical points of master functions and flag varieties*, Commun. Contemp. Math. **6** (2004), 111–163

[MV2] E. Mukhin and A. Varchenko, *Miura opers and critical points of master functions*, Cent. Eur. J. Math. **3** (2005), 155–182 (electronic)

[MV3] E. Mukhin and A. Varchenko, *On critical points of master functions associated with affine Lie algebras*, in preparation

[RV] N. Reshetikhin and A. Varchenko, *Quasiclassical asymptotics of solutions to the KZ equations*, Geometry, Topology and Physics for R. Bott, Intern. Press, 1995, 293–322

[SV] V. Schechtman and A. Varchenko, *Arrangements of Hyperplanes and Lie Algebra Homology*, Invent. Math. Vol. 106 (1991), 139–194

[ScV] I. Scherbak and A. Varchenko, *Critical points of functions, $sl_2$ representations, and Fuchsian differential equations with only univalued solutions*, Moscow Math. J. **3**, n. 2 (2003), 621–645

[Sz] G. Szego, *Orthogonal polynomials*, AMS, 1939

[V1] A. Varchenko, *Multidimensional Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups*, Advanced Series in Mathematical Physics, Vol. **21**, World Scientific, 1995

[V2] A. Varchenko, *Special functions, KZ type equations, and Representation theory*, CBMS, Regional Conference Series in Math., n. **98** (2003), AMS

[V3] A. Varchenko, *Quantum integrable model of an arrangement of hyperplanes*, SIGMA Symmetry Integrability Geom. Methods Appl. **7** (2011), Paper 032, 55 pp.

[VW] A. Varchenko, D. Wright, *Critical points of master functions and integrable hierarchies*, [arXiv:1207.2274](http://arxiv.org/abs/1207.2274), 1–42