Bounds on the Pure Point Spectrum of Lattice Schrödinger Operators

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Abstract

In dimension $d \geq 3$, a variational principle for the size of the pure point spectrum of (discrete) Schrödinger operators $H(\epsilon, V)$ on the hypercubic lattice $\mathbb{Z}^d$, with dispersion relation $\epsilon$ and potential $V$, is established. The dispersion relation $\epsilon$ is assumed to be a Morse function and the potential $V(x)$ to decay faster than $|x|^{-2(d+3)}$, but not necessarily to be of definite sign. Our estimate on the size of the pure-point spectrum yields the absence of embedded and threshold eigenvalues of $H(\epsilon, V)$ for a class of potentials of this kind. The proof of the variational principle is based on a limiting absorption principle combined with a positive commutator (Mourre) estimate, and a Virial theorem. A further observation of crucial importance for our argument is that, for any selfadjoint operator $B$ and positive number $\lambda > 0$, the number of negative eigenvalues of $\lambda B$ is independent of $\lambda$.

Key words. discrete Schrödinger operators; embedded eigenvalues.

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1 Introduction

Let $\Gamma = \mathbb{Z}^d$ be the $d$-dimensional hypercubic lattice. Given a bounded potential $V \in \ell^\infty(\Gamma; \mathbb{R})$, the discrete Schrödinger operator corresponding to $V$ is

$$-\Delta_{\Gamma} + V,$$

where $V$ acts as a multiplication operator and $\Delta_{\Gamma}$ is the discrete Laplacian defined by

$$[\Delta_{\Gamma}\psi](x) = \sum_{|v|=1} \{\psi(x + v) - \psi(x)\}.$$ 

More generally, we assume to be given a function $\epsilon \in C^2(\Gamma^*; \mathbb{R})$ on the $d$-dimensional torus (Brillouin zone) $\Gamma^* = (\mathbb{R}/2\pi\mathbb{Z})^d \equiv [-\pi, \pi)^d$, the dual group of $\Gamma$. We refer to $\epsilon$ as a dispersion relation or simply a dispersion. We then consider the self-adjoint operator

$$H(\epsilon, V) = h(\epsilon) + V,$$
on $\ell^2(\Gamma)$, where $h(\epsilon) \in B[\ell^2(\Gamma)]$ is the hopping matrix (convolution operator) corresponding to the dispersion relation $\epsilon$, i.e.,

$$[\mathcal{F}(\epsilon h(\epsilon)\psi)](p) = \epsilon(p) [\mathcal{F}^*(\psi)](p),$$

for all $\psi \in \ell^2(\Gamma)$. Here,

$$\mathcal{F}^*: \ell^2(\Gamma) \to L^2(\Gamma^*), \quad [\mathcal{F}^*(\psi)](p) = \sum_{x \in \Gamma} e^{-i(p,x)}\psi(x),$$

is the usual discrete Fourier transformation with inverse

$$\mathcal{F}: L^2(\Gamma^*) \to \ell^2(\Gamma), \quad [\mathcal{F}(\psi)](x) = \int_{\Gamma^*} e^{i(p,x)}\psi(p) \, d\mu^*(p),$$

where $\mu^*$ is the (normalized) Haar measure on the torus, $d\mu^*(p) = \frac{d^d}{(2\pi)^d}$. Put differently, $h(\epsilon) = \mathcal{F}\epsilon\mathcal{F}^*$ is the Fourier multiplier corresponding to $\epsilon$.

For each $x \in \Gamma$, let $\delta_x \in \ell^2(\Gamma)$ be the normalized vector $\delta_x(y) = \delta_{x,y}$, where $\delta_{x,y}$ is the Kronecker delta. For a dispersion relation $\epsilon$ and a pair $(x, y) \in \Gamma^2$, define the hopping amplitude

$$h(\epsilon)_{xy} = \langle \delta_x | h(\epsilon)\delta_y \rangle.$$

We say that $h(\epsilon)$ has a finite range if, for some $R < \infty$ and all $(x, y) \in \Gamma^2$, $h(\epsilon)_{xy} = 0$ when $|x - y| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_d - y_d)^2} > R$. The smallest number $R(\epsilon) \geq 0$ with this property is the range of the hopping matrix $h(\epsilon)$. Equivalently, $h(\epsilon)$ has a finite range whenever $\epsilon$ is a trigonometric polynomial.

W.l.o.g., the minimum of $\epsilon$ is assumed to be 0, so

$$\epsilon(\Gamma^*) = [0, \epsilon_{\text{max}}(\epsilon)].$$

We will further assume that the dispersion $\epsilon$ satisfies the following condition:

$$(\text{M}) \quad \epsilon \text{ and } |\nabla \epsilon|^2 \equiv \sum_{k=1}^d |\partial_{p_k} \epsilon|^2 \text{ are Morse functions.}$$

Clearly, the condition (M) is stable under small perturbations in the $C^3$-sense, i.e., if $\|\epsilon' - \epsilon\|_{C^3(\Gamma^*)}$ is sufficiently small and $\epsilon$ satisfies (M), then so does $\epsilon'$. Moreover, if the dispersion $\epsilon$ has a finite range then so do $|\nabla \epsilon|^2$, with $R(|\nabla \epsilon|^2) \leq 2R(\epsilon)$. Note that $-\Delta_{\Gamma} = h(\epsilon_{\text{Lapl}})$ and $\min \epsilon_{\text{Lapl}}(\Gamma^*) = 0$, where

$$\epsilon_{\text{Lapl}}(p) = 2 \sum_{i=1}^d (1 - \cos(p_i))$$

is a dispersion fulfilling (M) with $R(\epsilon_{\text{Lapl}}) = 1$. 2
To consider more general dispersions than $\epsilon_{\text{Lapl}}$ is important, for instance, for the analysis of many-body problems on the lattice $\Gamma$ — even in the situation where the dispersion relation for the one-body sector is chosen to be $\epsilon_{\text{Lapl}}$: Let $\epsilon$ be a dispersion relation. For each $K \in \Gamma^*$ define the non-negative function $\epsilon^{(K)} : \Gamma^* \to \mathbb{R}_0^+$ by

$$\epsilon^{(K)}(p) = \epsilon(p) + \epsilon(K - p) - E_0^{(K)}, \quad (1)$$

where

$$E_0^{(K)} = \min_{p' \in \Gamma} \{ \epsilon(p') + \epsilon(K - p') \}.$$

Dispersions of the form (1) come about in the analysis of systems of two particles on the lattice $\Gamma$ both having the same dispersion $\epsilon$ and interacting by a (translation invariant) potential $V(x_1 - x_2)$. Indeed, the two-particle Hamiltonian is unitarily equivalent to the direct integral

$$\int_{\Gamma^*} [H(\epsilon^{(K)}, V) + E_0^{(K)}] \, d\mu^*(K).$$

The function $\epsilon^{(K)}$ is viewed as the (effective) dispersion of a pair of particles travelling through the lattice with total quasi-momentum $K \in \Gamma^*$. Clearly, $\epsilon^{(K)}$ fulfills (M) — at least in a neighborhood of $K = 0$ — if $\epsilon$ does. As soon as $K \neq 0$, however, $\epsilon_{\text{Lapl}}^{(K)}$ is not proportional to $\epsilon_{\text{Lapl}}$. Similar facts hold true for the $N$-body problem, $N > 2$.

Our aim in the current paper is to give bounds on the size $N_{pp}[\epsilon, V]$ of the pure point spectrum of $H(\epsilon, V)$,

$$N_{pp}[\epsilon, V] = \text{Tr} [E_{pp}(H(\epsilon, V))],$$

in dimensions $d \geq 3$. Here, $E_{pp}(H)$ is the spectral projector associated with the pure-point spectrum of $H(\epsilon, V)$, i.e., the range of $E_{pp}(H)$ is the closed linear span of the eigenvectors of the selfadjoint operator $H$. In quantum mechanics, $N_{pp}[\epsilon, V]$ is the number of linearly independent bound states of a particle moving in the $d$-dimensional lattice $\Gamma$, with dispersion $\epsilon$ and in presence of the potential $V$. Another important physical aspect of the quantity $N_{pp}[\epsilon, V]$ concerns the scattering of such a particle on the potential $V$: When the particle is not bound by the potential (i.e., its energy distribution vanishes at all eigenvalues of the Hamiltonian $H(\epsilon, V)$) then $N_{pp}[\epsilon, V]$ is related to the time-delay caused by the scattering process. See, for instance, [BS-B12, Eq. (1)] and references therein.

By a theorem of von Neumann and Weyl [Kato, Chapter X, Theorem 2.1], for any self-adjoint operator $H_0$ on a separable Hilbert space $\mathcal{H}$ and any prescribed upper bound $\epsilon > 0$, there is another self-adjoint operator $H_1$ with $\dim E_{pp}(H_1) = \infty$ and $H_1 - H_0$ smaller than $\epsilon$ in the Hilbert-Schmidt norm. Thus, even arbitrarily small perturbations can change the pure-point spectrum of a self-adjoint operator drastically. The appearance of infinitely many eigenvalues driven by an arbitrarily small perturbation is also known for the special case of lattice Schrödinger operators: If $d = 1$ and $\epsilon = \epsilon_{\text{Lapl}}$ then, for any $\epsilon > 0$,
there is a potential $V_\varepsilon$ such that $|V_\varepsilon(x)| \leq \varepsilon (1 + |x|)^{-1}$ and the eigenvalues of $H(\varepsilon, V_\varepsilon)$ are dense in the interval $[0, \epsilon_{\text{max}}(\varepsilon)] = [0, 4]$. See [Na] Theorem 2.1. In particular, $N_{pp}[\varepsilon, V_\varepsilon] = \infty$, whereas $N_{pp}[\varepsilon, 0] = 0$. Note that $[0, \epsilon_{\text{max}}(\varepsilon)]$ is exactly the essential spectrum of $H(\varepsilon, V_\varepsilon)$ and thus arbitrarily small perturbations of the potential of a discrete Schrödinger operator can even generate infinitely many embedded eigenvalues. The appearance of embedded eigenvalues is related to the slow decay of potentials: As proven in [NaYa92], if $d = 1$, $\epsilon = \epsilon_{\text{Lapl}}$ and, for some $\eta > 0$ and $C < \infty$, $|V(x)| \leq C(1 + |x|)^{-(1+\eta)}$ then $H(\epsilon, V)$ has no eigenvalue in the (open) interval $(0, \epsilon_{\text{max}}(\epsilon))$. However, to our knowledge, the precise relation between the presence of embedded eigenvalues and the slow decay of the potential $V$ has not yet been established for discrete Schrödinger operators in dimension higher than one. Moreover, this property of embedded eigenvalues of discrete Schrödinger operators strongly depends (even in one dimension) on the choice $\epsilon$ of the dispersion, as one sees in the following simple example: For $d \in \mathbb{N}$, define the dispersion

$$\epsilon(p) = \frac{3d}{2} - \sum_{k=1}^{d} \left[2 \cos(p_k) - \cos(2p_k)\right] \geq 0.$$ 

With this particular choice, $\epsilon_{\text{max}}(\epsilon) = 9d/2$. Note, moreover, that $\epsilon$ a dispersion satisfying (M). Further, let $\psi \in \ell^2(\Gamma)$ be defined by $\psi(x) \doteq (1 + |x|)^{-(d+1)/2}$. With this definition one has:

$$[(h(\epsilon) - 3d/2)\psi](x) = O\left((1 + |x|)^{-2-(d+1)/2}\right).$$

Define next the (real-valued) potential $V$ by

$$V(x) \doteq -\frac{[(h(\epsilon) - 3d/2)\psi](x)}{\psi(x)} = O\left((1 + |x|)^{-2}\right).$$

This asymptotics is related to the fact that $\epsilon(p) - 3d/2 = O\left(|p|^2\right)$. By construction,

$$[(h(\epsilon) - 3d/2)\psi](x) = -V(x)\psi(x).$$

In particular, $3d/2$ is an embedded eigenvalue of $H(\epsilon, V)$. Simple variations of this construction permit us to obtain potentials $V$ as small as desired, producing embedded eigenvalues and decaying as $x^{-2}$. Relaxing the condition (M) and admitting dispersions (which are not Morse functions) such that $\epsilon(p) - c = O\left(|p|^m\right)$ for some $c > 0$ and $m > 2$, one can explicitly construct arbitrarily small potentials $V(x) = O\left(|x|^{-m}\right)$ leading to embedded eigenvalues for $H(\epsilon, V)$. 

Note that if the dispersion $\epsilon$ is a trigonometric polynomial (i.e., $h(\epsilon)$ has finite range) and $V$ has a finite support, then $H(\epsilon, V)$ has no embedded eigenvalues, i.e., no eigenvalue in the interval $(0, \epsilon_{\text{max}}(\epsilon))$ [BS-BT2, Proposition 9]. In this particular case, the study of the size of the pure point spectrum reduces to the study of the size of the discrete spectrum (i.e., eigenvalues away from the essential spectrum $[0, \epsilon_{\text{max}}(\epsilon)]$) of $H(\epsilon, V)$ and multiplicities of possible (threshold) eigenvalues $0$ and $\epsilon_{\text{max}}(\epsilon)$. Observe, however, that even for zero-range potentials
threshold eigenvalues cannot be excluded, as shown in the following example: Let $d \geq 5$ and define the potential $V(x) = -\delta_{x,0} \left[ \int_{\Gamma^*} \frac{1}{\epsilon_{\text{Lapl}}(p)} d\mu^*(p) \right]^{-1}$, where $\delta_{x,y}$ is the Kronecker delta. Note that the above integral is finite if $d \geq 3$.

Further, define $\psi \in \ell^2(\Gamma)$ by

$$\psi(x) = \int_{\Gamma^*} \frac{e^{ip \cdot x}}{\epsilon_{\text{Lapl}}(p)} d\mu^*(p),$$

i.e., $\psi$ is the inverse Fourier transform of $\epsilon_{\text{Lapl}}^{-1}$, the latter being an element of $L^2(\Gamma^*)$ for $d \geq 5$. Then, by construction,

$$[h(\epsilon_{\text{Lapl}})\psi](x) = -V(x)\psi(x).$$

In particular 0 is an (threshold) eigenvalue of $H(\epsilon_{\text{Lapl}}, V)$. See also [HSSS12].

From the discussion above one sees that the behavior of the dispersion $\epsilon$ at critical points, as well as, the decay of the potential $V$ have a strong influence on the embedded eigenvalues of $H(\epsilon, V)$. As already explained above, we assume that the dispersion $\epsilon$ satisfies the condition (M). Later on, for technical reasons, we additionally assume that $\epsilon \in C^4(\Gamma^*)$. In order to control the spatial decay of the potential we define the following quantity: For any $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, and potential $V : \Gamma \to \mathbb{R}$,

$$\Phi_{m,n}(V) = \sum_{x \in \Gamma} |V(x)|^m (|x|^1 + 1)^n.$$  \hspace{1cm} (2)

If $\Phi_{1,2}(V) < \infty$ then $H(\epsilon, V)$ has no singular continuous spectrum and its pure point spectrum, is finite on any compact subset of the real line not containing any critical value of the dispersion $\epsilon$. See Corollary 2.2. In particular, the eigenvalues of $H(\epsilon, V)$ cannot be dense in its essential spectrum in this case. Nevertheless, $N_{pp}[\epsilon, V]$ could still be infinite, as its eigenvalues may possibly accumulate at critical values of $\epsilon$.

We prove below that, for a finite constant $c(\epsilon)$ depending only on a few derivatives of $\epsilon$,

$$N_{pp}[\epsilon, V] \leq \inf \left\{ |\text{supp } V'| \mid z \in \mathbb{Z}^d, \ V' \text{ with } \Phi_{2,3}(V'(z) - V') < c(\epsilon) \right\},$$  \hspace{1cm} (3)

where $|\text{supp } V'|$ denotes the cardinality of the support $\text{supp } V' \subset \Gamma$ of the potential $V'$, and $V'(z)$ is the translated potential

$$V'(z)(x) = V(z + x).$$  \hspace{1cm} (4)

This is our main result; see Theorem 4.2. An immediate consequence of this estimate is that $N_{pp}[\epsilon, V]$ is finite whenever $\Phi_{2,3}(V)$ is finite. The bound (3)
follows from resolvent estimates, given in Theorem 3.2, combined with positivity arguments in the form of a Virial theorem, formulated as Lemma 2.3 for $H(\epsilon, V)$.

Note that (3) implies that $H(\epsilon, V)$ has no eigenvalue whenever $\Phi_{2,3}(V) < c(\epsilon)$. This result on the absence of pure-point spectrum is slightly strengthened in Corollary 3.3 (i), which states that $N_{pp}[\epsilon, V] = 0$ when $\Phi_{2,2}(V) < c(\epsilon)$. Moreover, the estimate (3) can be used to prove the absence of eigenvalues of $H(\epsilon, V)$ in its continuous spectrum, even if $\Phi_{2,3}(V)$ is big (but finite): If $H(\epsilon, V)$ has $N$ discrete (isolated) eigenvalues, counting their multiplicities, and $\Phi_{2,3}(V^{(z)} - V') < c(\epsilon)$, for some $z \in \mathbb{Z}^d$ and some potential $V'$ with $|\text{supp } V'| = \mathcal{N}$, then it directly follows from (3) that $H(\epsilon, V)$ has only discrete eigenvalues. See Corollary 4.3. Observe that, by Kato’s perturbation theory for discrete eigenvalues [Kato], one sufficient condition on $V$ in order that $H(\epsilon, V)$ has $N$ discrete eigenvalues is that there exists some potential $V'$, such that $|\text{supp } V'| = N$ and

$$\|H(\epsilon, V - V')\|_{B[\ell^2(\Gamma)]} \left[ \min_{x \in \text{supp } V'} |V'(x)| \right]^{-1}$$

is small enough.

Assume that $V$ is a potential with

$$|V(x)| \leq (1 + |x|)^{-\beta}$$

for some $\beta > 0$. If $\beta > 2(d + 3)$ then $\Phi_{2,3}(\lambda V) < \infty$ and, hence, $N_{pp}[\epsilon, \lambda V]$ is finite for all $\lambda \in \mathbb{R}$. In this case the estimate (3) yields

$$N_{pp}[\epsilon, \lambda V] = \mathcal{O}(\lambda^{-d/2(d+3)}).$$

On the other hand, if

$$V(x) \leq -(1 + |x|)^{-\beta},$$

by estimating the size of the negative part of the pure-point spectrum of $H(\epsilon, V)$ (i.e., its discrete spectrum) via the min-max principle, one concludes that $N_{pp}[\epsilon, \lambda V]$ cannot be $o(\lambda^n)$. See also [BdSPL, RS08, RS09]. Ergo, the estimate (3) is order-sharp for large, fast decaying potentials.

Finally, observe that in [BdSPL] we proved a variational principle similar to (3) for the size $N_{\text{disc}}[\epsilon, V] \leq N_{pp}[\epsilon, V]$ of the discrete spectrum of $H(\epsilon, V)$, in any dimension $d \geq 1$, but for potentials $V$ having a definite sign: In the case of the discrete spectrum, by using the Birman-Schwinger principle, bounds like (3) on $N_d[\epsilon, V]$ can be obtained whereby the quantity $\Phi_{2,3}(V)$ is replaced with $\Phi_{1,n}(V)$, $n$ being a small integer depending on the dimension. For the case of the pure-point spectrum considered in the current paper, however, that method is not applicable, for the Birman-Schwinger principle does not capture embedded eigenvalues (at least not directly).

This paper is organized as follows:

- In Section 2 we discuss general facts about the spectrum of $H(\epsilon, V)$ and prove a Virial theorem for $H(\epsilon, V)$, as Lemma 2.3 which is pivotal for the proof of the estimate (3).
• In Section 3 we derive resolvent estimates leading to a limiting absorption principle (Theorem 3.2), which is a central ingredient of the proof of (3). An important technical problem we are facing in these estimates arises from singularities of the type
\[
\frac{1}{p_1^2 + \cdots + p_k^2 - p_{k+1}^2 \cdots - p_d^2}
\]
appearing in integrands. Such singularities are called van Hove singularities in condensed matter physics and have important physical consequences. They cannot be handled by simple power counting, and rather sign cancellations have to be exploited in the bounds. This technical aspect is discussed in detail in Section 5.3 of the Appendix.

• In Section 4 we state and prove our main result, Theorem 4.2. Moreover, a result (Corollary 4.3) on the absence of embedded eigenvalues is derived from this last theorem.

• To simplify the exposition and/or for completeness, some technical results are proven in the Appendix (Section 5).

2 The Spectrum of $H(\varepsilon, V)$ — General Facts

We require that $V$ decays at infinity,
\[
V \in \ell_0^\infty(\Gamma; \mathbb{R}) \doteq \left\{ V : \Gamma \to \mathbb{R} \mid \lim_{|x| \to \infty} V(x) = 0 \right\},
\]
or sometimes even that $V$ has finite support. Note that $V \in \ell_0^\infty(\Gamma; \mathbb{R})$ is compact as a multiplication operator on $\ell^2(\Gamma)$ and by a theorem of Weyl,
\[
\sigma_{\text{ess}}[H(\varepsilon, V)] = \sigma_{\text{ess}}[H(\varepsilon, 0)] = [0, \epsilon_{\text{max}}],
\]
where $\epsilon_{\text{max}} \equiv \epsilon_{\text{max}}(\varepsilon)$ and $\sigma_{\text{ess}}[H] \subset \mathbb{R}$ denotes the essential spectrum of the selfadjoint operator $H$.

Let $\varepsilon \in C^2(\Gamma^*; \mathbb{R})$ be a Morse function. As $\Gamma^*$ is compact, $\varepsilon$ has at most finitely many critical points. We denote the set of all critical points of $\varepsilon$ by
\[
\text{Crit}(\varepsilon) \doteq \{ p \in \Gamma^* \mid \nabla \varepsilon(p) = 0 \}.
\]
The critical values of $\varepsilon$, collected in the set
\[
\text{Thr}(\varepsilon) \doteq \varepsilon(\text{Crit}(\varepsilon)),
\]
are called of thresholds of $\varepsilon$.

Define the symmetric operator $\hat{A} = \hat{A}(\varepsilon)$ on $C^\infty(\Gamma^*; \mathbb{C}) \subset L^2(\Gamma^*)$ by
\[
\hat{A}\hat{\psi}(p) \doteq i \sum_{i=1}^d \left\{ [\partial_{p_i} \varepsilon(p)] [\partial_{p_i} \hat{\psi}(p)] + \frac{1}{2}[\partial_{p_i}^2 \varepsilon(p)] \hat{\psi}(p) \right\}. \quad (5)
\]
We denote by $A = A(e)$ the inverse Fourier transform of $\hat{A}$, i.e., the operator $A = \mathcal{F} \hat{A} \mathcal{F}^*$ on $\text{Dom}(A) = \mathcal{F}(C^\infty(\Gamma^*; \mathbb{C}))$. Observe that, for all $\psi \in C^\infty(\Gamma^*; \mathbb{C})$,

$$i[\epsilon, \hat{A}]\psi(p) = |\nabla \epsilon(p)|^2 \psi(p) \doteq \sum_{i=1}^{d} |\partial_{p_i} \epsilon(p)|^2 \psi(p) \ .$$

(6)

In particular, $i[\epsilon, \hat{A}]$ and $i[h(\epsilon), A]$ extend to positive bounded operators on $L^2(\Gamma^*)$ and $\ell^2(\Gamma)$, which we also denote $a$ by $i[\epsilon, \hat{A}]$ and $i[h(\epsilon), A]$, respectively. Note also that $\nabla \epsilon(p)^2$ is a Morse function, by Assumption (M).

Note that $i[V, A]$ uniquely extends to a bounded self-adjoint operator on $\ell^2(\Gamma)$ (also denoted by $i[V, A]$) whenever $V$ has a finite support. (Below, densely defined bounded operators will be identified with their closures.) More precisely, from straightforward computations one obtains

$$i[V, A] = \sum_{x \in \text{supp} \ V} iV(x) : \left( |\delta_x\rangle \langle g_x| - |g_x\rangle \langle \delta_x| \right) , \quad (7)$$

where $g_x = A\delta_x$ has Fourier transform

$$\hat{g}_x(p) = \frac{i}{2} |\nabla \epsilon(p)|^2 + \langle \nabla \epsilon(p), x \rangle \ e^{-i(p,x)}$$

and obeys thus the norm bound

$$\|g_x\|_{L^2(\Gamma)} = \|\hat{g}_x\|_{L^2(\Gamma^*)} \leq C(1 + |x|) .$$

In particular, it follows from (7) that, if $V(x)(1 + |x|)$ is summable, then $\lim_{R \to \infty} \|1_{|x| > R}V(x)1_{|x| > R}\| = 0$, and hence $i[V, A]$ is a compact operator. For such potentials we have the following estimate for the commutator $i[H(\epsilon, V), A]$:

**Lemma 2.1** (Mourre Estimate for $H(\epsilon, V)$). If $\epsilon \in C^4(\Gamma^*; \mathbb{R})$ is a dispersion relation then $A(\epsilon)$ uniquely extends to a self-adjoint operator (also denoted by $A(\epsilon)$). If the potential $V$ is such that $i[V, A]$ is compact then, for any continuous, compactly supported function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying $\text{dist}(\text{Thr}(\epsilon), \text{supp} \ \chi) > 0$, there is a compact operator $K_\chi \in \mathcal{B}(\ell^2(\Gamma))$ and a constant $c_\chi > 0$ such that

$$\chi[H(\epsilon, V)] i[H(\epsilon, V), A] \chi[H(\epsilon, V)] \geq c_\chi \chi^2 \|H(\epsilon, V)\| + K_\chi . \quad (8)$$

Observe that if $\Delta \subset \mathbb{R}$ is a compact subset with $\text{dist}(\text{Thr}(\epsilon), \Delta) > 0$, then there is a continuous function $\chi : \mathbb{R} \to \mathbb{R}$ with compact support such that $\text{dist}(\text{Thr}(\epsilon), \text{supp} \ \chi) > 0$, and $\chi \equiv 1$ on $\Delta$. Let $E_\Delta$ be the spectral projection of $H(\epsilon, V)$ associated with $\Delta$. Then $\chi E_\Delta = \chi E_\Delta = E_\Delta$ and by multiplying equation (8) with $E_\Delta$ from the left and from the right it follows that, for some $c_\Delta > 0$ and some compact operator $K_\Delta$,

$$E_\Delta i[H(\epsilon, V), A] E_\Delta \geq c_\Delta E_\Delta + K_\Delta . \quad (9)$$

From explicit expressions for $AAV$, $AVA$ and $VAA$, similar to (7), one checks that these three operators are bounded if $\Phi_{1,2}(V) < \infty$. By (6), if $\epsilon \in C^3(\Gamma^*; \mathbb{R})$,
$[[h(\epsilon), A], A]$ is a bounded operator. In particular, if $\Phi_{2,2}(V) < \infty$ and $\epsilon \in C^3(\Gamma^*; \mathbb{R})$, then
\[ \| [[H(\epsilon, V), A], A] \|_{B(\ell^2(\Gamma))] < \infty . \]  \hspace{1cm} (10)

The following corollary is a consequence of \((9)\) and \((10)\); see also \cite{CFKS} Theorems 4.7 and 4.9.

**Corollary 2.2.** Let $\epsilon \in C^4(\Gamma^*; \mathbb{R})$ be a dispersion relation and let $V$ be a potential with $\Phi_{1,2}(V) < \infty$. Then $H(\epsilon, V)$ has no singular continuous spectrum and its eigenvalues can only accumulate in points of $\text{Thr}(\epsilon)$.

The next lemma, along with Theorem 3.2, is a central argument of the proof of our main result (Theorem 4.2):

**Lemma 2.3 (Virial Theorem for $H(\epsilon, V)$).** Let $\epsilon \in C^4(\Gamma^*; \mathbb{R})$ be a dispersion relation and let $V_1, V_2$ be potentials such that $i[V_1, A(\epsilon)]$ and $i[V_2, A(\epsilon)]$ are bounded operators. If $\psi$ is an eigenvector of $H(\epsilon, V_1 + V_2)$, then
\[ \langle \psi | i[V_2, A] \psi \rangle = -\langle \psi | i[H(\epsilon, V_1), A] | \psi \rangle . \]

Note that, in the current section, the restriction $\epsilon \in C^4(\Gamma^*; \mathbb{R})$ is only relevant for Corollary 2.2 and Lemma 2.3 above. The proofs of Lemmata 2.1 and 2.3 use adaptations for the lattice case of known methods used for the continuum and are given in Appendix 5.1–5.2, for completeness. See also \cite{CFKS} Chapter 4 and \cite{GSch97}.

The following upper bound on the size of the pure-point spectrum of $H(\epsilon, V)$, in case that $V$ has finite support and $h(\epsilon)$ is of finite range, is an immediate consequence of the Virial theorem (Lemma 2.3) above:

**Corollary 2.4 (Upper Bound on $N_{pp}[\epsilon, V]$, Finite Range Case).** Let $d \geq 1$. If the potential $V$ has finite support and $\epsilon \in C^4(\Gamma^*; \mathbb{R})$ is a dispersion relation satisfying (M) then
\[ N_{pp}[\epsilon, V] \leq \text{Tr} [E_-(i[V, A(\epsilon)])] , \]
where $E_-(i[V, A])$ is the spectral projector associated with the strictly negative spectrum of the (selfadjoint) commutator $i[V, A]$.

**Proof.** Note that the range of $i[V, A]$ has finite dimension, by \((7)\). From \((9), i[h(\epsilon), A]$ is positive and has purely absolutely continuous spectrum. Thus, for all $\psi \in \ell^2(\Gamma) \setminus \{0\}$, $\langle \psi | i[h(\epsilon), A] | \psi \rangle > 0$. Setting $V_1 = 0$ and $V_2 = V$, it follows from Lemma 2.3 that, for any normalized eigenvector $\psi$ of $H(\epsilon, V)$, we have
\[ \langle \psi | i[V, A] | \psi \rangle = -\langle F^*(\psi) | \nabla \epsilon^2 F^*(\psi) \rangle < 0 . \]

Hence, denoting by $X \subset \ell^2(\Gamma)$ any finite-dimensional subspace of eigenvectors of $H(\epsilon, V)$ we obtain, by compactness of the $n$-sphere, with $n = \dim X - 1$, the estimate
\[ \max \left\{ \langle \psi | i[V, A] | \psi \rangle \mid \psi \in X, \| \psi \|_2 = 1 \right\} < 0 . \]
By the min-max principle, the dimension of $X$ cannot exceed the number of strictly negative eigenvalues (with multiplicities) of the self-adjoint operator $i[V,A]$. □

In the following corollary we show that the quantity $\text{Tr} \left[ E_- (i[V,A(\epsilon)]) \right]$ (appearing in the above estimate on $N_{pp}[\epsilon,V]$) is nothing else than the size of the support of the potential $V$:

**Corollary 2.5.** Let $d \geq 1$. If the potential $V$ has finite support and $\epsilon \in C^4(\Gamma^*;\mathbb{R})$ is a dispersion relation satisfying (M) then

$$\text{Tr} \left[ E_- (i[V,A(\epsilon)]) \right] = |\text{supp} \ V|.$$  

**Proof.** Note that, for all $\lambda > 0$,

$$\text{Tr} \left[ E_- (i[\lambda V,A(\epsilon)]) \right] = \text{Tr} \left[ E_- (i[V,A(\epsilon)]) \right].$$

By Kato’s perturbation theory for discrete eigenvalues [Kato], for sufficiently large $\lambda > 0$, $H(\epsilon,\lambda V)$ has, at least, $|\text{supp} \ V|$ eigenvalues, counting their multiplicities. Hence, Corollary 2.4 implies that

$$|\text{supp} \ V| \leq N_{pp}[\epsilon,\lambda V] \leq \text{Tr} \left[ E_- (i[\lambda V,A(\epsilon)]) \right] = \text{Tr} \left[ E_- (i[V,A(\epsilon)]) \right].$$

Repeating this argument for $-V$ we conclude that:

$$|\text{supp \ V}| \leq \text{Tr} \left[ E_- (\pm i[V,A(\epsilon)]) \right].$$

In other words, the subspaces associated to the strictly negative and strictly positive eigenvalues of the selfadjoint, finite-range operator $i[V,A(\epsilon)]$ have both dimension of at least $|\text{supp \ V}|$. On the other hand, from (7) we conclude that the dimension of the range of this operator, which is

$$\text{Tr} \left[ E_- (i[V,A(\epsilon)]) \right] + \text{Tr} \left[ E_- (-i[V,A(\epsilon)]) \right],$$

cannot exceed $2|\text{supp \ V}|$. Ergo,

$$\text{Tr} \left[ E_- (i[V,A(\epsilon)]) \right] = \text{Tr} \left[ E_- (-i[V,A(\epsilon)]) \right] = |\text{supp \ V}|.$$

Combining the two last corollaries, under the same assumptions on the dispersion $\epsilon$, we arrive at:

$$N_{pp}[\epsilon,V] \leq |\text{supp \ V}|.$$  

In dimension $d \geq 3$, the upper bound on $N_{pp}[\epsilon,V]$ of Corollary 2.4 can be improved in the following sense:

- If $\Phi_{2,2}(V)$ is small enough then $H(\epsilon,V)$ has no bound states cf. Corollary 3.3 (i).

- If $V = V_1 + V_2$ with $V_1$ having finite support and $\Phi_{2,3}(V_2)$ being small enough (but $V$ not necessarily having a finite support), then the bound on $N_{pp}[\epsilon,V]$ in the corollary remains true when $V$ is replaced with $V_1$. See Corollary 4.1.
3 Resolvent Estimates

Let \( \epsilon' \) be the Hessian matrix of the dispersion relation \( \epsilon \in C^2(\Gamma^*; \mathbb{R}) \) at \( p \in \text{Crit}(\epsilon) \). Define the minimal curvature of \( \epsilon \) at \( p \in \text{Crit}(\epsilon) \) by:

\[
K(\epsilon, p) = \min \left\{ |\lambda|^2 : \lambda \text{ is an eigenvalue of } \epsilon''(p) \right\}.
\]

Define also the minimal (critical) curvature of \( \epsilon \) by

\[
K(\epsilon) = \min \{ K(\epsilon, p) \mid p \in \text{Crit}(\epsilon) \}.
\]

Note that \( K(\epsilon) > 0 \) and \( K(|\nabla\epsilon|^2) > 0 \) under Assumption (M).

For \( m \in \mathbb{N}_0 \), we recall the standard definition

\[
\|\epsilon\|_{C^m} = \max_{|p| \leq m} \| \partial^{(2)} \epsilon(p) \|
\]

of the norm on \( C^m(\Gamma^*; \mathbb{C}) \).

**Lemma 3.1.** Let \( \epsilon \) be any dispersion relation from \( C^3(\Gamma^*; \mathbb{R}) \). Let \( K > 0 \) and \( C < \infty \) be constants with \( K(\epsilon) \geq K \), and \( \|\epsilon\|_{C^3} \leq C \). Then there is a constant \( 3.1 < \infty \) depending only on \( K \) and \( C \) such that

\[
\left\| V^{\frac{1}{2}}(z - h(\epsilon))^{-1}V^{\frac{1}{2}} \right\|_{B^2(\Gamma)} \leq 3.1 \Phi_2(V),
\]

\[
\left\| V^{\frac{1}{2}}(z - h(\epsilon))^{-1}AV^{\frac{1}{2}} \right\|_{B^2(\Gamma)} \leq 3.1 \Phi_3(V),
\]

\[
\left\| V^{\frac{1}{2}}A(z - h(\epsilon))^{-1}V^{\frac{1}{2}} \right\|_{B^2(\Gamma)} \leq 3.1 \Phi_2(V),
\]

\[
|\langle \delta_x (z - h(\epsilon))^{-1} \delta_y \rangle| \leq 3.1 (1 + |x|)^2 (1 + |y|)^2,
\]

\[
\left\| V^{\frac{1}{2}}(z - h(\epsilon))^{-1}\varphi_x \right\|_{L^2} \leq 3.1 (1 + |x|)^2 \Phi_2(V),
\]

\[
\left\| V^{\frac{1}{2}}A(z - h(\epsilon))^{-1}\varphi_x \right\|_{L^2} \leq 3.1 (1 + |x|)^2 \Phi_3(V),
\]

for all potentials \( V \), all \( z \in \mathbb{C} \setminus \mathbb{R} \), and all \( x, y \in \Gamma \). Here, \( V^{\frac{1}{2}} \) denotes an arbitrary function \( V : \Gamma \to \mathbb{C} \) with \( (V^{\frac{1}{2}}(x))^2 = V(x) \).

**Proof.** We freely use the equality \((V^{\frac{1}{2}})^* = (V^{\frac{1}{2}})^2 = V\) in the sequel without further mentioning. We write

\[
\langle \delta_x \left( z - h(\epsilon) \right)^{-1} \varphi_y \rangle = (1 + |x|)^2 (1 + |y|)^2 \int_{\Gamma} \frac{E_{\varphi}(p)}{z - e^*(p)} \, d\mu^*(p),
\]

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We note
\[ F_{xy}(p) = e^{i p (x-y)} / \left(1 + |x|^2(1 + |y|^2)\right), \]
and note that \( \sup \{ \|F_{xy}\|_{C^2} \mid x, y \in \Gamma \} < \infty \). Hence, it follows from Lemma 5.1 that there is a constant \( \text{const} < \infty \), such that
\[
\left| \langle \delta_x, (z - h(x))^{-1} \delta_y \rangle \right| \leq \text{const} (1 + |x|^2(1 + |y|^2),
\]
for all \( z \in \mathbb{C} \setminus \mathbb{R} \) and all \( x, y \in \Gamma \).

Let \( V \) be a potential with \( \text{Ran}(V^{\frac{1}{2}}) \subset \text{dom}(A) \). For all \( \psi \in \ell^2(\Gamma) \), we define the following functions on \( \Gamma^* \),
\[
F_V^\psi(p) = F_x \circ V^{\frac{1}{2}}(\psi)(p) = \sum_{x \in \Gamma} e^{-ip \cdot x} V^{\frac{1}{2}}(x) \psi(x),
\]
\[
F_{AV}^\psi(p) = \sum_{i=1}^d \langle [\partial_{\mu_i}(\psi)] [\partial_{\mu_i}] F_V^\psi(p) \rangle + \frac{i}{2} \langle \nabla \psi \rangle^2 F_V^\psi(p).
\]

Then, for all \( x \in \Gamma \),
\[
\langle (z - h(x))^{-1} V^{\frac{1}{2}} \psi \mid \delta_x \rangle = \int_{\Gamma^*} \frac{F_V^\psi(p)e^{-ip \cdot x}}{z - \psi(p)} \, d\mu^*(p),
\]
\[
\langle (z - h(x))^{-1} AV^{\frac{1}{2}} \psi \mid \delta_x \rangle = \int_{\Gamma^*} \frac{F_{AV}^\psi(p)e^{-ip \cdot x}}{z - \psi(p)} \, d\mu^*(p).
\]

We note
\[
F_#^\psi(p)e^{-ip \cdot x} = (1 + |x|)^2 \left[ (1 + |x|)^{-1} F_#^\psi(p)e^{-ip \cdot x} \right],
\]
where \( # \) denotes \( V \) or \( AV \), and observe that the \( C^2 \)-norms of the functions
\[
p \mapsto (1 + |x|)^{-2} F_V^\psi(p)e^{-ip \cdot x}, \quad p \mapsto (1 + |x|)^{-2} F_{AV}^\psi(p)e^{-ip \cdot x}
\]
are bounded by \( \text{const} \Phi_2(V)^{\frac{1}{2}} \) and \( \text{const} \Phi_3(V)^{\frac{1}{2}} \), \( \text{const} < \infty \), respectively, uniformly in \( x \in \Gamma \) and \( \psi \in \ell^2(\Gamma) \), \( \|\psi\|_2 \leq 1 \). It follows from Lemma 5.1 that, for some constant \( \text{const} < \infty \), for all \( x \in \Gamma \), \( z \in \mathbb{C} \setminus \mathbb{R} \), and all \( \psi \in \ell^2(\Gamma) \), with \( \|\psi\|_2 \leq 1 \),
\[
\|V^{\frac{1}{2}}(z - h(x))^{-1} \delta_x\|_2 \leq \text{const} (1 + |x|)^2 \Phi_2(V)^{\frac{1}{2}},
\]
\[
\|V^{\frac{1}{2}} AV(z - h(x))^{-1} \delta_x\|_2 \leq \text{const} (1 + |x|)^2 \Phi_3(V)^{\frac{1}{2}},
\]
\[
\sum_{x \in \Gamma} |(z - h(x))^{-1} V^{\frac{1}{2}} \psi | V^{\frac{1}{2}} \delta_x |^2 \leq \text{const}^2 \Phi_2(V)^2,
\]
\[
\sum_{x \in \Gamma} |(z - h(x))^{-1} AV^{\frac{1}{2}} \psi | V^{\frac{1}{2}} \delta_x |^2 \leq \text{const}^2 \Phi_3(V)^2 \Phi_2(V)
\]
\[
\leq \text{const}^2 \Phi_3(V)^2.
\]
Thus,
\[
\| V^\frac{1}{2} (z - h(\epsilon))^{-1} V^\frac{1}{2} \|_{\mathcal{B}(\ell^2(\Gamma))} \leq \text{const} \Phi_2(V),
\]
\[
\| V^\frac{1}{2} (z - h(\epsilon))^{-1} AV^\frac{1}{2} \|_{\mathcal{B}(\ell^2(\Gamma))} \leq \text{const} \Phi_3(V),
\]
for some const \( c < \infty \), all \( z \in \mathbb{C} \setminus \mathbb{R} \) and all \( x \in \Gamma \). By taking adjoints, we further obtain
\[
\| V^\frac{1}{2} A(z - h(\epsilon))^{-1} V^\frac{1}{2} \|_{\mathcal{B}(\ell^2(\Gamma))} \leq \text{const} \Phi_3(V). \tag{11}
\]
Similarly, it follows, for a suitable constant const \( c < \infty \), all \( z \in \mathbb{C} \setminus \mathbb{R} \), all \( x \in \Gamma \), and all \( \psi \in \ell^2(\Gamma) \), \( \| \psi \|_2 \leq 1 \), that
\[
\sum_{x \in \Gamma} |(z - h(\epsilon))^{-1} AV^\frac{1}{2} \psi | AV^\frac{1}{2} \delta_x |^2 \leq \text{const}^2 \Phi_{2,3}(V)^2.
\]
Thus,
\[
\| V^\frac{1}{2} A(z - h(\epsilon))^{-1} AV^\frac{1}{2} \|_{\mathcal{B}(\ell^2(\Gamma))} \leq \text{const} \Phi_{2,3}(V)
\]
for all \( z \in \mathbb{C} \setminus \mathbb{R} \) and all \( x \in \Gamma \). \( \square \)

**Theorem 3.2** (Resolvent Estimates in Dimension \( d \geq 3 \)). Let \( d \geq 3 \) and \( \epsilon \in C^3(\Gamma^*; \mathbb{R}) \) be a dispersion relation satisfying (M).

(i) If \( \Phi_{2,2}(V) < 1 \) then there exists a constant \( \Phi_{3,2} < \infty \) such that, for all \( z \in \mathbb{C} \setminus \mathbb{R} \) and all \( x, y \in \Gamma \),
\[
|\langle \delta_x | (z - H(\epsilon, V))^{-1} \delta_y \rangle| \leq \Phi_{3,2}(1 + |x|)^2(1 + |y|)^2.
\]

(ii) If \( 2\Phi_{2,3}(V) < 1 \) then there exists a constant \( \Phi_{3,2,4} < \infty \) such that, for all \( z \in \mathbb{C} \setminus \mathbb{R} \) and all \( x \in \Gamma \),
\[
|\langle \delta_x | (z - i[H(\epsilon, V), A])^{-1} \delta_y \rangle| \leq \Phi_{3,2,4}(1 + |x|)^2(1 + |y|)^2.
\]

**Proof.** For \( n \in \mathbb{N} \) and \( z \in \mathbb{C} \setminus \mathbb{R} \) let
\[
O_n(z) \doteq [V(z - h(\epsilon))^{-1}]^n = V^\frac{1}{2} \tilde{O}_n(z) V^\frac{1}{2} (z - h(\epsilon))^{-1},
\]
where
\[
\tilde{O}_n(z) \doteq \left[ V^\frac{1}{2} (z - h(\epsilon))^{-1} V^\frac{1}{2} \right]^{n-1}.
\]
Assume that \( \Phi_{3,1}(\Phi_{2,2}(V) < 1. \) Then, by Lemma 5.1, \( \| \tilde{O}_n(z) \| < a^{n-1}, \) for some \( 0 < a < 1. \) Thus we can define the operators
\[
\tilde{O}(z) \doteq \sum_{n=1}^{\infty} \tilde{O}_n(z).
\]
It follows that, for each \( z \in \mathbb{C} \setminus \mathbb{R}, \| \tilde{O}(z) \|_{\mathcal{B}(\ell^2(\Gamma))} \leq (1 - a)^{-1} \) and that \( (z - h(\epsilon) - V) \) has a bounded inverse given by
\[
(z - h(\epsilon) - V)^{-1} = (z - h(\epsilon))^{-1} \left[ 1 + V^\frac{1}{2} \tilde{O}(z) V^\frac{1}{2} (z - h(\epsilon))^{-1} \right].
\]

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This along with Lemma 3.1 imply (i).

To prove (ii), we temporarily ignore questions of convergence and write

\[(z - i[H(\epsilon, V), A])^{-1} = \sum_{n=0}^{\infty} R_0(i[V, A]R_0)^n,\]

where \(R_0 \doteq (z - i[h(\epsilon), A])^{-1}\). Observe that

\[i[V, A] = iV^{1/2}(V^{1/2}A) + i(-AV^{1/2})V^{1/2} = i \sum_{\sigma=0}^{1} (-1)^\sigma (A^{\sigma}V^{1/2})(V^{1/2}A^{1-\sigma}).\]

Hence,

\[(z - i[H(\epsilon, V), A])^{-1} - R_0 = \sum_{n=1}^{\infty} \sum_{\sigma_1, \ldots, \sigma_n=0}^{1} i^n (-1)^{n+1} R_0 A^{\sigma_1} V^{1/2} \left( \prod_{j=1}^{n-1} V^{1/2} A^{1-\sigma_j} R_0 A^{\sigma_j+1} V^{1/2} \right) V^{1/2} A^{1-\sigma_n} R_0.\]

Now, due to Lemma 3.1 we have that

\[\left\| V^{1/2} A^\sigma R_0 \delta_x \right\|_2 \leq g_{3.1} (1 + |x|^2) \max \{ \Phi_{2,2}(V)^{1/2}, \Phi_{2,3}(V)^{1/2} \} = g_{3.1} (1 + |x|^2) \Phi_{2,3}(V)^{1/2},\]

\[\left\| V^{1/2} A^\sigma R_0 A^\nu V^{1/2} \right\|_{B^2[\ell^2(\Gamma)]} \leq g_{3.1} \max \{ \Phi_{2,2}(V), \Phi_{2,3}(V) \} = g_{3.1} \Phi_{2,3}(V).\]

for all \(x \in \Gamma\) and \(\sigma, \eta \in \{0, 1\}\). By assumption, \(2g_{3,1} \Phi_{2,3}(V) < 1\), and the Neumann series evaluated on the vectors \(\delta_x\) and \(\delta_y\), converges. Namely,

\[|\langle \delta_x | (z - i[H(\epsilon, V), A])^{-1} \delta_y \rangle | \leq g_{3.1}^2 (1 + |x|^2)(1 + |y|^2) \sum_{n=0}^{\infty} (2g_{3,1} \Phi_{2,3}(V))^n = (2g_{3.1}^2 (1 + |x|^2)(1 + |y|^2)) / (1 - 2g_{3,1} \Phi_{2,3}(V)).\]

\[\square\]

Corollary 3.3. Let \(d \geq 3\) and \(\epsilon \in C^3(\Gamma^*; \mathbb{R})\) be a dispersion relation satisfying (M).

(i) If \(2g_{3,1} \Phi_{2,2}(V) < 1\) then \(H(\epsilon, V)\) has purely absolutely continuous spectrum and

\[\sigma_{ac}(H(\epsilon, V)) = [0, \epsilon_{\max}].\]
(ii) If \( 2 \Phi_{2,3}(V) < 1 \) then \( i[H(\epsilon, V), A] \) is positive and has purely absolutely continuous spectrum.

Proof. Assume that \( c \Phi_{2,2}(V) < 1 \). From Theorem 3.2(i), for all \( z \in \mathbb{C} \setminus \mathbb{R} \) and all vectors \( \psi \in \text{span}\{\delta_x \mid x \in \Gamma\} \), i.e. \( \psi \) of finite support, we have that

\[
|\langle \psi | (z - H(\epsilon, V)^{-1}) \psi \rangle| \leq c(\psi) < \infty
\]

with \( c(\psi) \) depending only on \( \psi \). As \( \text{span}\{\delta_x \mid x \in \Gamma\} \) is dense in \( \ell^2(\Gamma) \), this last estimate implies the absolute continuity of the spectrum of \( H(\epsilon, V) \). See, for instance, [CFKS] Proposition 4.1. Analogously, by Theorem 3.2(ii), \( i[H(\epsilon, V), A] \) has only absolutely continuous spectrum whenever \( 2 \Phi_{2,2}(V) < 1 \). If \( \Phi_{2,2}(V) < \infty \) then \( V \) and \( i[V, A] \) define trace class operators. By the Kato-Rosenblum theorem,

\[
\sigma_{ac}(H(\epsilon, V)) = \sigma_{ac}(h(\epsilon)) = [0, \epsilon_{\max}],
\]

\[
\sigma_{ac}(i[H(\epsilon, V), A]) = \sigma_{ac}(i[h(\epsilon), A]) \subset \mathbb{R}_0^+.
\]

\( \square \)

4 Bound on \( N_{pp}[\epsilon, V] \)

The positivity of the commutator \( i[H(\epsilon, V'), A] \) at small \( \Phi_{2,3}(V') \), stated in Corollary 3.3, yields upper bounds on \( N_{pp}[\epsilon, V] \) for \( d \geq 3 \), without assuming that \( V \) has a finite support:

Corollary 4.1 (Upper Bound on \( N_{pp}[\epsilon, V] \), Infinite Range Case, \( d \geq 3 \)). Let \( d \geq 3 \) and \( \epsilon \) be a dispersion relation from \( C^4(\Gamma^*; \mathbb{R}) \) satisfying (M). Let \( V \) be a potential with \( \Phi_{2,3}(V) < \infty \) and choose \( V_1, V_2 \) such that \( 2 \Phi_{2,3}(V_1) < 1 \) and\( V_2 \) has finite support. Then

\[
N_{pp}[\epsilon, V] \leq \text{Tr}[E_-(i[V_2, A])].
\]

Proof. If \( 2 \Phi_{2,3}(V_1) < 1 \) then, by Corollary 3.3, \( i[H(\epsilon, V_1), A] \geq 0 \) and has purely absolutely continuous spectrum. Thus, by Lemma 2.3 if \( \psi \) is an eigenvector of \( H(\epsilon, V) \) then \( \langle \psi | i[V_2, A] \psi \rangle < 0 \). Ergo, \( N_{pp}[\epsilon, V] \leq \text{Tr}[E_-(i[V_2, A])] \). See the proof of Corollary 2.4 for more details.

From the corollary,

\[
N_{pp}[\epsilon, V] \leq \min \left\{ \text{Tr}[E_-(i[V', A])] \mid |\text{supp } V'| < \infty, \Phi_{2,3}(V - V') < \frac{1}{2} \right\}.
\]

Recall that \( E_- (i[V', A]) \) is the spectral projector associated with the negative spectrum of the (selfadjoint) commutator \( i[V', A] \). As the operator \( h(\epsilon) \) (the hopping matrix the Schrödinger operator \( H(\epsilon, V) \)) is invariant with respect to translations, for all \( z \in \mathbb{Z}^d \),

\[
N_{pp}[\epsilon, V] = N_{pp}[\epsilon, V(z)] .
\]
where \( V^{(z)} \) is the translation \( (14) \) of the potential \( V \). From this remark, Corollary \( 2.5 \) and the estimate \( (12) \), we arrive at our main result:

**Theorem 4.2** (Bound on \( N_{pp} (\epsilon, V) , \ d \geq 3 \), infinite range case). Let \( d \geq 3 \) and \( \epsilon \) be a dispersion relation from \( C^4 (\Gamma^*; \mathbb{R}) \) satisfying (M). Then, for the finite constant \( (4.2) = (2 (3.1))^{-1} \),

\[
N_{pp} (\epsilon, V) \leq \inf \left\{ \| \text{supp } V' \| \ | z \in \mathbb{Z}^d , \ V' \text{ with } \Phi_{2,3} (V^{(z)} - V') < \left( \frac{16}{\epsilon^2} \right) \right\}.
\]

The above estimate on \( N_{pp} (\epsilon, V) \) implies the absence of embedded eigenvalues of \( H (\epsilon, V) \) for a class of potentials \( V \):

**Corollary 4.3** (Absence of embedded eigenvalues). Let \( d \geq 3 \) and \( \epsilon \) be a dispersion relation from \( C^4 (\Gamma^*; \mathbb{R}) \) satisfying (M). Let \( V \) be a potential with \( \Phi_{2,3} (V) < \infty \). Assume that \( V = V_1 + V_2 \), where \( \Phi_{2,3} (V_2^{(z)}) < \left( \frac{16}{\epsilon^2} \right) \) for some translation \( z \in \mathbb{Z}^d \). \( V_1 \) has a finite support and \( H (\epsilon, V) \) has exactly \( | \text{supp } V_1 | \) discrete eigenvalues, counting their multiplicities. Then all eigenvalues of \( H (\epsilon, V) \) are discrete.

5 Appendix

5.1 Proof of Lemma 2.1

Let \( N \) be the unique self-adjoint extension of the operator \( \tilde{N} \) defined on \( C^\infty (\Gamma^*; \mathbb{C}) \subset L^2 (\Gamma^*) \) by

\[
\tilde{N} \psi (p) = \sum_{i=1}^d (1 - \partial_{p_i}^2) \psi (p).
\]

Observe that for some const \( \epsilon < \infty \) and all \( \hat{\psi} , \hat{\psi}' \in C^\infty (\Gamma^*; \mathbb{C}), \)

\[
\langle \hat{\psi}' | A \hat{\psi} \rangle \leq \text{const} \ \| \hat{\psi}' \|_2 \| \hat{\psi} \|_2 \leq \text{const} \ \| N \frac{1}{\epsilon} \hat{\psi}' \|_2 \| N \frac{1}{\epsilon} \hat{\psi}_2 \|_2.
\]

For all \( \hat{\psi} \in C^\infty (\Gamma^*; \mathbb{C}), \)

\[
(NA - AN) \hat{\psi} (p) = -i \sum_{k,k'=1}^d \left\{ 2 (2 \partial^2_{p_k} \partial_{p_{k'}} \epsilon (p)) (\partial_{p_v} \psi (p)) + \frac{1}{2} \partial^2_{p_k} \partial^2_{p_{k'}} \epsilon (p) \right\} \psi (p)
\]

\[
+2 \partial_{p_k} \partial_{p_{k'}} \epsilon (p) (\partial_{p_k} \partial_{p_{k'}} \psi (p)).
\]

An integration of the terms with second derivatives of \( \hat{\psi} \) by parts yields, for some \( 0 < \text{const} < \infty \) and all \( \hat{\psi} , \hat{\psi}' \in C^\infty (\Gamma^*; \mathbb{C}), \)

\[
\left| \langle N \hat{\psi}' | A \hat{\psi} \rangle - \langle A \hat{\psi}' | N \hat{\psi} \rangle \right| \leq \text{const} \ \| N \frac{1}{\epsilon} \hat{\psi}' \|_2 \| N \frac{1}{\epsilon} \hat{\psi}_2 \|_2.
\]

Thus, by Nelson’s commutator theorem (see [RS2, Theorem X.36]), \( A \) is essentially self-adjoint on \( C^\infty (\Gamma^*; \mathbb{C}) \).
Clearly, as \( \chi \) is continuous and has compact support,

\[
\chi(H(\epsilon, V)) - \chi(H(\epsilon, 0)) = \lim_{\eta \downarrow 0} \frac{1}{\sqrt{\pi \eta}} \int_0^\infty \chi(t) \left[ \exp \left( -\frac{(H(\epsilon, V) - t)^2}{\eta} \right) - \exp \left( -\frac{(H(\epsilon, 0) - t)^2}{\eta} \right) \right] dt
\]

in norm sense. Observe that

\[
-\eta \int_0^\infty \chi(t) \left[ \exp \left( -\frac{(H(\epsilon, V) - t)^2}{\eta} \right) - \exp \left( -\frac{(H(\epsilon, 0) - t)^2}{\eta} \right) \right] dt = \int_0^1 \left[ \int_0^\infty \chi(t) \exp \left( -\frac{s(H(\epsilon, V) - t)^2}{\eta} \right) \exp \left( -\frac{(1 - s)(H(\epsilon, 0) - t)^2}{\eta} \right) dt \right] ds .
\]

As \( V \) is a compact operator, it follows from (13) that \( \chi(H(\epsilon, V)) - \chi(H(\epsilon, 0)) \) is compact.

The difference

\[
i[H(\epsilon, V), A] - i[H(\epsilon, 0), A] = i[V, A]
\]

is also a compact operator, by assumption. To finish the proof observe that \( i[H(\epsilon, 0), A] \) is unitarily equivalent to the multiplication operator \( |\nabla \epsilon|^2 \). Moreover,

\[
|\nabla \epsilon|^2 \cdot \chi^2(H(\epsilon, 0)) \geq c \cdot \chi^2(H(\epsilon, 0))
\]

is bounded below on the range of \( \chi^2(H(\epsilon, 0)) \) by a positive multiple of the identity, since \( \chi^2(H(\epsilon, 0)) \) is supported away from the thresholds. Thus, there is a constant \( c_0 > 0 \) such that

\[
\chi(H(\epsilon, 0)) i[H(\epsilon, 0), A] \chi(H(\epsilon, 0)) \geq c_0 \chi^2(H(\epsilon, 0)).
\]

\[\Box\]

### 5.2 Proof of Lemma 2.3

Let \( \psi \) be an eigenvector of \( H(\epsilon, V_1 + V_2) \) and define, for each \( n \in \mathbb{Z} \setminus \{0\} \), the vector

\[
\psi_n = \frac{i n}{i n + A} \psi.
\]

Since \( i[H(\epsilon, V_1), A] \) and \( i[V_2, A] \) are bounded operators, by assumption, we have that

\[
\lim_{n \to \infty} \langle \psi_n | i[H(\epsilon, V_1), A] \psi_n \rangle = \langle \psi | i[H(\epsilon, V_1), A] \psi \rangle, \quad \lim_{n \to \infty} \langle \psi_n | i[V_2, A] \psi_n \rangle = \langle \psi | i[V_2, A] \psi \rangle.
\]
Note that
\[
\langle \psi_n | i[H(\epsilon, V_1 + V_2), A] \psi_n \rangle 
= \langle \psi_n | i[H(\epsilon, V_1), A] \psi_n \rangle + \langle \psi_n | i[V_2, A] \psi_n \rangle .
\]
Hence it suffices to prove, for all \( n \in \mathbb{N} \), that
\[
\langle \psi_n | i[H(\epsilon, V_1 + V_2), A] \psi_n \rangle = 0 .
\]
This is easily seen, however, as for all \( n \in \mathbb{N} \),
\[
\langle \psi_n | i[H(\epsilon, V_1 + V_2), A] \psi_n \rangle = \langle \psi_n | i[H(\epsilon, V_1), A + V_2] \psi_n \rangle = 0 .
\]
□

5.3 Proof of Lemma 3.1

In order to prove Lemma 3.1 we need the following estimate:

**Lemma 5.1.** Assume that \( d \geq 3 \) and let \( \epsilon \) be a dispersion relation with \( K(\epsilon) > 0 \) and \( \| \epsilon \|_{C^3} < \infty \). Suppose that \( \chi \in C^2(\Gamma^*; \mathbb{R}) \). Then there exists a constant \( c_1 < \infty \) depending only on \( K(\epsilon) \), \( \| \epsilon \|_{C^3} \) and \( \| \chi \|_{C^2} \) such that
\[
\int_{\Gamma^*} \frac{\chi(p)}{z - \epsilon(p)} d\mu^*(p) \leq c_1 \text{ for all } z \in \mathbb{C} \setminus \mathbb{R}.
\]

**Proof.** We assume w.l.o.g. that \( z \) is bounded by \( |z| \leq \epsilon_{\text{max}} + 1 \), say. We further note that \( \epsilon \) has only finitely many critical points, \( \#Q < \infty \), abbreviating \( Q = \text{Crit}(\epsilon) \), since \( \Gamma^* \) is compact and \( \epsilon \) is a Morse function. The latter is also the reason that, for each \( q \in Q \), there exist an index \( m_q \in \{0, \ldots, d\} \) and a \( C^2 \)-coordinate chart \( \xi_q \in C^2(B_{d-m} \times B_m; U_q) \), for
\[
B_n = B_{\mathbb{R}^n}(0, r) = \{ x \in \mathbb{R}^n : |x| < r \}, r > 0 ,
\]
denoting the Euclidean open ball in \( \mathbb{R}^n \) of radius \( r \) and \( U_q \subset \Gamma^* \) being an open neighborhood of \( q \) such that, for all \( x \in B_{d-m} \), \( y \in B_m \),
\[
c_1 \leq | \det \text{Jac} \xi_q(x, y) | \leq c_2 ,
\]
\[
\epsilon \circ \xi_q(x, y) = \epsilon(q) + x^2 - y^2 ,
\]
\[
U_q \supseteq B_{\Gamma^*}(q, \delta) ,
\]
for suitable constants \( c_1, \delta > 0 \), \( r \in (0, 1) \) and \( c_2 < \infty \). \( \delta > 0 \) can be chosen such that away from the critical points we can find a finite set
\[
\bar{Q} \subseteq \mathcal{N} = \{ q \in \Gamma^* \mid \epsilon(q) = \text{Re}(z) \}.
\]
and, for each \( q \in \tilde{Q} \), a \( C^2 \)-coordinate chart \( \tilde{\xi}_q \in C^2((-r, r) \times B_{d-1}; \tilde{U}_q) \), with \( \tilde{U}_q \subset \Gamma^* \) being an open neighborhood of \( q \), such that, for all \( x \in (-r, r), y \in B_{d-1} \),

\[
c_1 \leq |\det \tilde{\xi}_q(x, y)| \leq c_2 ,
\]

\[
\mathfrak{c} \tilde{\xi}_q(x, y) = \epsilon(q) + x ,
\]

\[
\bigcup_{q \in \tilde{Q}} \tilde{U}_q \supseteq \{ p \in \Gamma^* : |\epsilon(q) - z| < \delta, \text{dist}(p, \mathcal{Q}) \geq \delta \} .
\]

Let

\[
\hat{\mathcal{N}} := \left\{ p \in \Gamma^* : |\epsilon(p) - z| > \frac{\delta}{2} \right\} .
\]

Then \( \{ \hat{\mathcal{N}} \} \cup \{ \mathcal{U}_q \}_{q \in \mathcal{Q}} \cup \{ \tilde{U}_q \}_{q \in \tilde{Q}^\circ} \) is a finite open covering of \( \Gamma^* \) and there exists a subordinate partition of unity,

\[
\{ \hat{\eta} \} \cup \{ \eta_q \}_{q \in \mathcal{Q}} \cup \{ \tilde{\eta}_q \}_{q \in \tilde{Q}} \subseteq C^\infty(\Gamma^*; [0, 1]) ,
\]

such that

\[
\text{supp} \hat{\eta} \subset \hat{\mathcal{N}}, \quad \text{supp} \eta_q \subset \mathcal{U}_q, \quad \text{supp} \tilde{\eta}_q \subset \tilde{U}_q ,
\]

for \( q \in \mathcal{Q} \cup \tilde{Q} \), and

\[
\hat{\eta} + \sum_{q \in \mathcal{Q}} \eta_q + \sum_{q \in \tilde{Q}} \tilde{\eta}_q \equiv 1.
\]

It follows that

\[
\int_{\Gamma^*} \frac{\chi(p)}{z - \epsilon(p)} d\mu^*(p) = \hat{T} + \sum_{q \in \mathcal{Q}} I_q + \sum_{q \in \tilde{Q}} \tilde{I}_q ,
\]

where

\[
\hat{T} := \int_{\Gamma^*} \frac{\hat{\eta}(p) \chi(p)}{z - \epsilon(p)} d\mu^*(p) ,
\]

\[
\tilde{I}_q := \int_{B_{d-1}} d^{d-1}y \int_{-r}^r dx \frac{\tilde{f}_q(x, y)}{ib - x} ,
\]

\[
I_q := \int_{B_{d-m_q}} d^{d-m_q}x \int_{B_{m_q}} d^{m_q}y \frac{f_q(x, y)}{a_q + ib - x^2 + y^2} ,
\]

where \( b = \text{Im}\{z\}, a_q = \text{Re}\{z\} - \epsilon(q) \), and

\[
\tilde{f}_q = (\tilde{\eta}_q \circ \tilde{\xi}_q)(\chi \circ \tilde{\xi}_q)|\det \tilde{\xi}_q| ,
\]

\[
f_q = (\eta_q \circ \xi_q)(\chi \circ \xi_q)|\det \xi_q| .
\]

Note that \( \tilde{f}_q \in C^2_0((-r, r) \times B_{d-1}; \mathbb{R}) \) and \( f_q \in C^2(B_{d-m_q} \times B_{m_q}; \mathbb{R}) \), due to (14). Moreover, \( \|f_q\|_{C^2}, \|f_q\|_{C^2} < \infty \). The asserted estimate now follows from Lemmata 5.2, 5.4 and the trivial estimate

\[
|\hat{T}| \leq \frac{2}{\delta} \int_{\Gamma^*} |\chi(p)|d\mu^*(p) .
\]
Observe that the constants $r, \delta, c_1, c_2$ and $\# Q, \# \tilde{Q}$ only depend on $K(\epsilon), \|\epsilon\|_{C^3}$ and $\|\chi\|_{C^2}$.

\[ \square \]

**Lemma 5.2.** Assume that $d \geq 1$ and $0 < r < 1$. There is a constant $\hat{C}_1 < \infty$ such that, for all $f \in C^1(\R^d \setminus \{0\})$ and all $b \in \R \setminus \{0\}$,

\[
\left| \int_{B_{d-1}} \int_{-r}^r \frac{f(x,y) - f(0,0)}{ib - x} \right| \leq \hat{C}_1 \|f\|_{C^1}.
\]

**Proof.** For all $x \in (-r, r)$ and all $y \in B_{d-1}$, the fundamental theorem of calculus gives

\[
\left| \frac{f(x,y) - f(0,0)}{ib - x} \right| \leq \left| \frac{x}{ib - x} \right| \|\partial_x f\|_{\infty} \leq \|f\|_{C^1},
\]

and thus

\[
\left| \int_{B_{d-1}} \int_{-r}^r \frac{f(x,y)}{ib - x} \right| \leq 2 \left| B_{d-1} \right| \|f\|_{C^1} \left( 1 + \frac{\int_{-r}^r \frac{dx}{ib - x}}{\left| B_{d-1} \right|} \right).
\]

The assertion follows then from

\[
\left| \frac{\int_{-r}^r \frac{dx}{ib - x}}{\left| B_{d-1} \right|} \right| = 2 \arctan(|b|) \leq \pi.
\]

\[ \square \]

**Lemma 5.3.** Assume that $d \geq 3$ and $0 < r < 1$. There is a constant $\hat{C}_2 < \infty$ such that, for all $f \in C^1(B_d; \R)$, all $a \in \R$ and all $b \in \R \setminus \{0\}$,

\[
\left| \int_{B_d} \frac{f(x)}{a + ib - x^2} d^d x \right| \leq \hat{C}_2 \|f\|_{C^1} (1 + a^2 + b^2).
\]

**Proof.** Introducing spherical coordinates, we observe that

\[
\mathcal{J} := \int_{B_d} \frac{f(x)}{a + ib - x^2} d^d x = \int_{0}^{r} \frac{g(s)s^{d-1}}{a + ib - s^2} ds,
\]

where $g \in C^1([0, 1]; \R)$ is the spherical average of $f$, $\|g\|_{C^1} \leq \|f\|_{C^1}$, defined by

\[
g(s) := \int_{S_{d-1}} f(s\theta) s^{d-1} \sigma(\theta).
\]

An integration by parts gives

\[
\Re\{\mathcal{J}\} = \int_{0}^{r} g(s)s^{d-1} \frac{a - s^2}{(a - s^2)^2 + b^2} ds = \frac{-1}{4} \int_{0}^{r} g(s)s^{d-2} \left( \frac{d}{ds} \ln \left( (a - s^2)^2 + b^2 \right) \right) ds
\]

\[
= \frac{1}{4} \int_{0}^{r} \left( g'(s)s^{d-2} + (d-2)g(s)s^{d-3} \right) \ln \left( (a - s^2)^2 + b^2 \right) ds.
\]

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We used above that $g(r) = 0$. Now, use the elementary estimate

$$|\ln(\lambda)| \leq \frac{1}{2\alpha}(\lambda^\alpha + \lambda^{-\alpha}) ,$$

which holds true for all $\lambda, \alpha > 0$. Choosing $\alpha = \frac{1}{8}$, (15) yields

$$|\text{Re}\{J\}| \leq \frac{d-1}{4} \|g\|_{C^1} \int_0^1 |\ln [(a-s^2)^2 + b^2]| \, ds \leq C \|g\|_{C^1} \left( (1 + a^2 + b^2) \frac{1}{16} + \int_0^1 \frac{2}{s} \, ds \right) \leq C' \|g\|_{C^1} (1 + a^2 + b^2) ,$$

for some universal constants $C, C' < \infty$. Similarly,

$$|\text{Im}\{J\}| = \frac{1}{|b|} \left| \int_0^r \frac{g(s) s^{d-1}}{1 + b^{-2}(a - s^2)^2} \, ds \right| \leq \frac{1}{|b|} \left| \int_0^r \frac{g(s) s^{d-2} (\frac{d}{ds} \arctan \left( \frac{a - s^2}{|b|} \right)) \, ds \right| \leq \frac{1}{2} \left| \int_0^r (g'(s) s^{d-2} + (d-2)g(s) s^{d-3}) \arctan \left( \frac{a - s^2}{|b|} \right) \, ds \right| \leq C \|g\|_{C^1} .$$

\[\square\]

Lemma 5.4. Assume that $d \geq 3$, $0 < r < 1$, and $1 \leq m \leq d - 1$. There is a constant $\tilde{C}_3 < \infty$ such that, for all $f \in C^1_0(B_{d-m} \times B_m; \mathbb{R})$, all $a \in \mathbb{R}$ and all $b \in \mathbb{R} \setminus \{0\}$,

$$\left| \int_{B_{d-m}} \int_{B_m} \frac{f(x,y)}{a + ib - x^2 + y^2} q^{d-m} x \, dx \, dy \right| \leq \tilde{C}_3 \|f\|_{C^1} (1 + a^2 + b^2) .$$

**Proof.** As in Lemma 5.3, we introduce spherical coordinates on $B_{d-m}$ and $B_m$ and define $g \in C^1([0,1] \times [0,1]; \mathbb{R})$, with $\|g\|_{C^1} \leq \|f\|_{C^1}$, by

$$g(x,y) = \int_{S_{d-m-1}} \int_{S_{m-1}} f(x,\vartheta,y,\kappa) \, q^{d-m-1} \sigma(\vartheta) \, d\vartheta \, d\kappa ,$$

so that

$$K \equiv \int_{B_{d-m}} \int_{B_m} \frac{f(x,y)}{a + ib - x^2 + y^2} q^{d-m} x \, dx \, dy = \int_0^r \int_0^r g(x,y) x^{d-m-1} y^{m-1} \, dx \, dy .$$

We perform yet another smooth coordinate change by

$$\phi : (0,r) \times (-1,1) \to (0,2r) \times (0,2r) ,$$




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\[ x = \phi_1(s, u) = s(1 + u), \quad y = \phi_2(s, u) = s(1 - u) . \]

With this definition, \(|\det \text{Jac} \phi(s, u)| = 2s\) and we obtain
\[
\mathcal{K} = 2 \int_0^r \int_{-1}^1 s^{d-1} \overline{g}(s, u) \frac{a - (2s)^2 u}{(a - (2s)^2 u)^2 + b^2} \, du \, dr ,
\]
where
\[ \overline{g}(s, u) \doteq (1 + u)^{d-m-1} (1 - u)^{m-1} g(s(1 + u), s(1 - u)) . \]

Note that \(\overline{g}(s, u) = 0\) whenever \(s(1 \pm u) \geq r\). Following a similar strategy as in the proof of Lemma 5.3, we first observe that
\[
\text{Re}\{\mathcal{K}\} = 2 \int_0^r \int_{-1}^1 s^{d-1} \overline{g}(s, u) \frac{a - (2s)^2 u}{(a - (2s)^2 u)^2 + b^2} \, du \, ds
\]
\[
= -\frac{1}{4} \int_0^r s^{d-3} \left[ \int_{-1}^1 \overline{g}(s, u) \left( \frac{d}{du} \ln [(a - (2s)^2 u)^2 + b^2] \right) du \right] ds
\]
\[
= -\frac{1}{4} \int_0^r s^{d-3} \overline{g}(s, 1) \ln [(a - (2s)^2)^2 + b^2] ds
\]
\[
+ \frac{1}{4} \int_0^r s^{d-3} \overline{g}(s, -1) \ln [(a + (2s)^2)^2 + b^2] ds
\]
\[
+ \frac{1}{4} \int_0^r s^{d-3} \left[ \int_{-1}^1 (\partial_u \overline{g})(s, u) \ln [(a - (2s)^2 u)^2 + b^2] du \right] ds .
\]

We use (15) again to bound
\[
\ln [(a - (2s)^2 u)^2 + b^2] \leq 8(8 + 2a^2 + b^2)^{\frac{1}{2}} + 8|(2s)^2|u| - |a|^{-\frac{1}{2}}
\]
for \(u = \pm 1\) and \(u \in [-1, 1]\), respectively, and hence
\[
|\text{Re}\{\mathcal{K}\}| \leq C \|f\|_{C^1} \left( 1 + a^2 + b^2 + \int_0^2 ds \int_0^{ \frac{ds}{|s^2 - |a||^\frac{1}{2}} } \frac{du}{|s^2 u - |a||^\frac{1}{2}} \right)
\]
for a suitable constant \(C < \infty\). Since
\[
\int_0^2 \frac{ds}{|s^2 - |a||^\frac{1}{2}} = \int_0^2 \frac{ds}{(s + |a|^\frac{3}{2})^\frac{1}{2}} \leq \int_0^2 \frac{ds}{s - |a|^\frac{3}{2}} \leq 4
\]
and
\[
\int_0^2 \int_0^2 ds du \frac{du}{|s^2 u - |a||^\frac{1}{2}} = \int_0^2 \frac{1}{s^{\frac{1}{2}}} \left( \int_0^2 \frac{du}{u - |a|^\frac{3}{2}} \right) ds \leq 8 ,
\]
we obtain that
\[
|\text{Re}\{\mathcal{K}\}| \leq 2^4 C \|f\|_{C^1} \left( 1 + a^2 + b^2 \right) .
\]

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Similarly,

$$\text{Im}\{\mathcal{K}\} = -2 \int_0^{r} \int_{-1}^{1} s^{d-1} \tilde{g}(s,u) \frac{b s^{d-1} \tilde{g}(s,u)}{(a-(2s)^2u+b^2)} \, du \, ds$$

$$= \frac{1}{2} \int_0^{r} s^{d-3} \left[ \int_{-1}^{1} \tilde{g}(s,u) \left( \frac{d}{du} \arctan \left[ \frac{a-(2s)^2u}{b} \right] \right) \, du \right] \, ds$$

$$= \frac{1}{2} \int_0^{r} s^{d-3} \tilde{g}(s,1) \arctan \left[ \frac{a-(2s)^2}{b} \right] \, ds$$

$$- \frac{1}{2} \int_0^{r} s^{d-3} \tilde{g}(s,-1) \arctan \left[ \frac{a+(2s)^2}{b} \right] \, ds$$

$$- \frac{1}{2} \int_0^{r} s^{d-3} \left[ \int_{-1}^{1} (\partial_u \tilde{g})(s,u) \arctan \left[ \frac{a-u(2s)^2}{b} \right] \, du \right] \, ds$$

and $|\arctan(\gamma)| \leq \frac{\pi}{2}$ immediately implies that

$$|\text{Im}\{\mathcal{K}\}| \leq C \|f\|_{C^1}$$

for a suitable constant $C < \infty$.

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