# THE MANIN CONJECTURE IN DIMENSION 2

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1. Introduction

The study of integer solutions to Diophantine equations is a topic that is almost as old as mathematics itself. Since its inception at the hands of Diophantus of Alexandria in 250 A.D., it has been found to relate to virtually every mathematical field. The purpose of these lecture notes is to focus attention upon an aspect of Diophantine equations that has only crystallised within the last few decades, and which exhibits a fascinating interplay between the subjects of analytic number theory and algebraic geometry.

Suppose that we are given a polynomial \( f \in \mathbb{Z}[x_1, \ldots, x_n] \), and write

\[ S_f := \{ x = (x_1, \ldots, x_n) \in \mathbb{Z}^n \setminus \{0\} : f(x) = 0 \} \]

for the corresponding locus of non-zero solutions. There are a number of basic questions that can be asked about the set \( S_f \). When is \( S_f \) non-empty? How large is \( S_f \) when it is non-empty? When \( S_f \) is infinite can we describe the set in some way? A lot of the work to date has been driven by trying to understand the situation for equations in only \( n = 2 \) or \( 3 \) variables. The last 50 years in particular has delivered a remarkable level of understanding concerning the arithmetic of curves. In stark contrast to this, the situation for equations in 4 or more variables remains a relatively untamed frontier, with only a scattering of results available.

We will restrict attention to the study of Diophantine equations \( f = 0 \) for which the corresponding zero set \( S_f \) is infinite. The description that we will aim for is quantitative in nature, the main goal being to understand how the counting function

\[ N(f; B) := \#\{ x \in S_f : |x| \leq B \} \]  

behaves, as \( B \to \infty \). Here, as throughout these lecture notes, \( |z| \) denotes the norm \( \max_{1 \leq i \leq n} |z_i| \) for any \( z \in \mathbb{R}^n \). Aside from being intrinsically interesting in their own right, as we will see shortly, the study of functions like \( N(f; B) \) is often an effective means of determining whether or not the equation \( f = 0 \) has any non-trivial integer solutions at all. In many applications of the Hardy–Littlewood circle method, for example, one is able to prove that \( S_f \) is non-empty by showing that \( N(f; B) > 0 \) for large enough values of \( B \). In fact the method usually carries with it a proof of the fact that \( S_f \) is infinite. In the context of the circle method at least, it is useful to have a general idea of which polynomials \( f \) might have an infinite zero locus \( S_f \).

Suppose that \( f \in \mathbb{Z}[x_1, \ldots, x_n] \) has degree \( d \geq 1 \). Then for the vectors \( x \in \mathbb{Z}^n \) counted by \( N(f; B) \), the values of \( f(x) \) will all be of order \( B^d \). In fact a positive proportion of them will have exact order \( B^d \). Thus the probability that a randomly chosen value of \( f(x) \) should vanish might be expected to be of order \( 1/B^d \). Since the number of \( x \) to be considered has order \( B^n \), this leads us to the following general expectation.

**Heuristic.** When \( n \geq d \) we have

\[ B^{n-d} \ll N(f; B) \ll B^{n-d}. \]  

As a crude first approximation, therefore, this heuristic tells us that we might expect polynomials whose degree does not exceed the number of variables to have infinitely many solutions. Unfortunately there are a number
of things that can conspire to upset this heuristic expectation. First and foremost, local conditions will often provide a reason for $N(f; B)$ to be identically zero no matter the values of $d$ and $n$. By local obstructions we mean that the obvious necessary conditions for $S_f$ to be non-empty fail. These are the conditions that the equation $f(x) = 0$ should have a real solution $x \in \mathbb{R}^n$, and secondly, that the congruence

$$f(x) \equiv 0 \pmod{p^k}$$

should be soluble for every prime power $p^k$. When $f$ is homogeneous we must take care to ignore the trivial solution $x = (0, \ldots, 0)$ in both cases.

It is quite easy to construct examples that illustrate the failure of these local conditions. For example, when $d$ is even, the equation

$$x_1^{2d} + \cdots + x_n^{2d} = 0$$

doesn’t have any integer solutions, since it patently doesn’t have any real solutions. Let us now exhibit an example, due to Mordell [50], of a polynomial equation that fails to have integer solutions because it fails to have solutions as a congruence modulo a prime $p$. Let $K$ be an algebraic number field of degree $d$, with ring of integers $O_K$. Write

$$N(y_1, \ldots, y_d) := N_{K/\mathbb{Q}}(y_1 \omega_1 + \cdots + y_d \omega_d)$$

for the corresponding norm form, where $\omega_1, \ldots, \omega_d$ is a basis for $K$ over $\mathbb{Q}$. It is clear that $N$ is a homogeneous polynomial of degree $d$, with coefficients in $\mathbb{Z}$.

**Exercise 1.** Let $y \in \mathbb{Z}^n$ and let $p$ be a rational prime such that the ideal $(p) \subset O_K$ is prime. Show that $p \mid N(y)$ if and only if $p \mid y$.

We define the homogeneous polynomial

$$f_1 := N(x_1, \ldots, x_d) + p N(x_{d+1}, \ldots, x_{2d}) + \cdots + p^{d-1} N(x_{d^2 - d + 1}, \ldots, x_{d^2}),$$

(1.3)

which has degree $d$ and $d^2$ variables. We claim that the only integer solution to the equation $f_1(x) = 0$ is the trivial solution $x = 0$. To see this we argue by contradiction. Thus we suppose there to be a vector $x \in \mathbb{Z}^{d^2}$ such that $f_1(x) = 0$, with $\gcd(x_1, \ldots, x_{d^2}) = 1$. Viewed modulo $p$ we deduce that $p \mid N(x_1, \ldots, x_d)$, whence $p \mid x_1, \ldots, x_d$ by Exercise 1. Writing $x_i = py_i$ for $1 \leq i \leq d$, and substituting into the equation $f_1 = 0$, we deduce that

$$p^{d-1} N(y_1, \ldots, y_d) + N(x_{d+1}, \ldots, x_{2d}) + \cdots + p^{d-2} N(x_{d^2 - d + 1}, \ldots, x_{d^2}) = 0.$$

But then we deduce in a similar fashion that $p \mid x_{d+1}, \ldots, x_{2d}$. We may clearly continue in this fashion, ultimately concluding that $p \mid x_1, \ldots, x_{d^2}$, which is a contradiction. This polynomial illustrates that for any $d$ it is possible to construct examples of homogeneous polynomials in $d^2$ variables that have no non-zero integer solutions. This fits with the facts rather well: when $d = 2$ we know from Meyer’s theorem that an indefinite quadratic form always has non-trivial solutions as soon as its rank is at least 5. Similarly, it is conjectured that 10 variables are always enough to ensure the solubility in integers of an arbitrary homogeneous cubic equation.

So far we have only seen examples of polynomials $f$ for which the zero locus $S_f$ is empty. In this case the corresponding counting function $N(f; B)$
is particularly easy to estimate! There are also examples which show that
$N(f; B)$ may grow in quite unexpected ways, even when $n \geq d$. An equation
that illustrates excessive growth is provided by the polynomial
\[
f_2 := x_1^d - x_2(x_3^{d-1} + \cdots + x_n^{d-1}).
\]
Here there are “trivial” solutions of the type $(0, 0, a_3, \ldots, a_n)$ which already
contribute $\gg B^{n-2}$ to the counting function $N(f; B)$, whereas (1.2) predicts
that we should have exponent $n - d$.

It is also possible to construct examples of varieties which demonstrate
inferior growth. Let $n > d^2$ and choose any $d^2$ linear forms $L_1, \ldots, L_{d^2} \in \mathbb{Z}[x_1, \ldots, x_n]$ that are linearly independent over $\mathbb{Q}$. Consider the form
\[
f_3 := f_1(L_1(x_1, \ldots, x_n), \ldots, L_{d^2}(x_1, \ldots, x_n)),
\]
where $f_1$ is given by (1.3). Then it is clear that $N(f_3; B)$ has the same order
of magnitude as the counting function associated to the system of linear
forms $L_1 = \cdots = L_{d^2} = 0$. Since these forms are linearly independent we
deduce that $N(f_3; B)$ has order of magnitude $B^{n-d^2}$, whereas (1.2) led us
to expect an exponent $n - d$.

We have seen lots of reasons why (1.2) might fail — how about some
evidence supporting it? One of the most outstanding achievements in this
direction is the following very general result due to Birch [4].

**Theorem 1.1.** Suppose $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is a non-singular homogeneous
polynomial of degree $d$ in $n > (d-1)2^d$ variables. Assume that $f(x) = 0$ has
non-trivial solutions in $\mathbb{R}$ and each $p$-adic field $\mathbb{Q}_p$. Then there is a constant
$c_f > 0$ such that
\[
N(f; B) \sim c_f B^{n-d},
\]
as $B \to \infty$.

Birch’s result doesn’t apply to either of the polynomials $f_2, f_3$ that we con-
sidered above, since both of these actually have a rather large singular locus.
Since generic homogeneous polynomials are non-singular, Birch’s result an-
swers our initial questions completely for typical forms with $n > (d-1)2^d$.
It would be of considerable interest to reduce the lower bound for $n$, but ex-
cept for $d \leq 4$ this has not been done. Theorem 1.1 is established using the
circle method, and exhibits a common feature of all Diophantine problems
successfully tackled via this machinery: the number of variables involved
needs to be large compared to the degree. In particular, there is an obvious
disparity between the range for $n$ in Birch’s result and the range for $n$ in
(1.2). The main aim of these lecture notes is to discuss the situation when
$n$ is comparable in size with $d$.

It turns out that phrasing things in terms of single polynomial equations
is far too restrictive. It is much more satisfactory to work with projective
algebraic varieties $V \subseteq \mathbb{P}^{n-1}$. All of the varieties that we will work with
are assumed to be cut out by a finite system of homogeneous equations,
all of which are defined over $\mathbb{Q}$. In line with the above, our main interest
lies with those varieties for which we expect the set $V(\mathbb{Q}) = V \cap \mathbb{P}^{n-1}(\mathbb{Q})$ to be infinite. Let $x = [x] \in \mathbb{P}^{n-1}(\mathbb{Q})$ be a projective rational point, with
\[
x = (x_1, \ldots, x_n) \in \mathbb{Z}^n
\]
chosen so that $\gcd(x_1, \ldots, x_n) = 1$. Then we define
the height of \( x \) to be \( H(x) := |x| \), where as usual \( |z| \) denotes the norm \( \max_{1 \leq i \leq n} |z_i| \). Given any subset \( U \subseteq V \), we may then define the counting function

\[
N_U(B) := \# \{ x \in U(\mathbb{Q}) : H(x) \leq B \},
\]

for each \( B \geq 1 \). The main difference between this counting function and the quantity introduced in (1.1) is that we are now only interested in primitive integer solutions, by which we mean that the components of the vector \( x \in \mathbb{Z}^n \) should share no common prime factors. When the polynomial in (1.1) is homogeneous, this formulation has the advantage of treating all scalar multiples of a given non-zero integer solution as a single point.

Recall the definition of the Möbius function \( \mu : \mathbb{N} \to \{0, 1\} \), which is given by

\[
\mu(n) = \begin{cases} 
0, & \text{if } p^2 \mid n \text{ for some prime } p, \\
1, & \text{if } n = 1, \\
(-1)^r, & \text{if } n = p_1 \cdots p_r \text{ for distinct primes } p_1, \ldots, p_r.
\end{cases}
\]

The Möbius function is a multiplicative arithmetic function, and will play a very useful rôle in our work.

Exercise 2. Let \( S \subseteq \mathbb{Z}^n \) be an arbitrary set. Show that

\[
\# \{ x \in S : \gcd(x_1, \ldots, x_n) = 1 \} = \sum_{k=1}^{\infty} \mu(k) \# \{ x \in S : k \mid x_i, \ (1 \leq i \leq n) \}.
\]

We now have the tools with which to relate the counting function (1.4) to our earlier counting function \( N(f; B) \) in (1.1), when \( U = V \) and \( V \subseteq \mathbb{P}^{n-1} \) is a hypersurface with underlying homogeneous polynomial \( f \in \mathbb{Z}[x_1, \ldots, x_n] \).

On noting that \( x \) and \( -x \) represent the same point in \( \mathbb{P}^{n-1} \), it follows from Exercise 2 that

\[
N_V(B) = \frac{1}{2} \sum_{k=1}^{\infty} \mu(k) N(f; B/k).
\]

When \( f \) is non-singular of degree \( d \), with \( n > (d - 1)2^d \), it can be deduced from Theorem 1.1 that \( N_V(B) \sim \tilde{c}_f B^{n-d} \), where \( \tilde{c}_f = \frac{1}{d} \zeta(n - d)^{-1} c_f \).

Returning to the counting function (1.4), it is easy to check that \( N_U(B) \) is bounded for each \( B \), no matter what the choice of \( U \) and \( V \). This follows on combining the fact that \( N_V(B) \leq N_{\mathbb{P}^{n-1}}(B) \) with the self-evident inequalities \( N_{\mathbb{P}^{n-1}}(B) \leq \# \{ x \in \mathbb{Z}^n : |x| \leq B \} \leq (2B + 1)^n \). In fact it is not so hard to establish an asymptotic formula for \( N_{\mathbb{P}^{n-1}}(B) \).

Exercise 3. Let \( n \geq 2 \). Use Exercise 2 to show that

\[
N_{\mathbb{P}^{n-1}}(B) = \frac{2^{n-1}}{\zeta(n)} B^n + O_n \left( B^{n-1} (\log B)^{b_n} \right),
\]

where \( b_2 = 1 \) and \( b_n = 0 \) for \( n > 2 \).

1.1. Notation. Before embarking on the main thrust of these lecture notes, we take a moment to summarise some of the key pieces of notation that we will make use of.
• $A(x) = O(B(x))$ means that there exists a constant $c > 0$ and $x_0 \in \mathbb{R}$ such that $|A(x)| \leq cB(x)$ for all $x \geq x_0$. Throughout our work we will follow the convention that the implied constant is absolute unless explicitly indicated otherwise by an appropriate subscript. We will often use the alternative notation $A(x) \ll B(x)$ or $B(x) \gg A(x)$.

• $A(x) \asymp B(x)$ means $A(x) \ll B(x) \ll A(x)$.

• $A(x) = o(B(x))$ means $\lim_{x \to \infty} A(x)/B(x) = 0$.

• $A(x) \sim B(x)$ means $\lim_{x \to \infty} A(x)/B(x) = 1$.

• $\mathbb{N} = \{1, 2, 3, \ldots\}$ will denote the set of natural numbers.

• $Z^n$ will denote the set of primitive vectors in $\mathbb{Z}^n$, and $Z^*_n$ will denote the set of $\mathbf{v} \in Z^n$ such that $v_1 \cdots v_n \neq 0$.

• $|\mathbf{z}| := \max_{1 \leq i \leq n} |z_i|$, for any vector $\mathbf{z} \in \mathbb{R}^n$.

1.2. The Manin conjectures. Around 1989 Manin initiated a program to relate the asymptotic behaviour of counting functions to the intrinsic geometry of the underlying variety, for suitable families of algebraic varieties. It is precisely this rich interplay between arithmetic and geometry that this set of lecture notes aims to communicate.

Several of the varieties that we have looked at so far have many rational points, in the sense that $N_V(B)$ grows like a power of $B$. For such varieties it is natural to look at the quantity

$$\beta_V := \lim_{B \to \infty} \frac{\log N_V(B)}{\log B},$$

assuming that this limit exists. In general we may consider $\beta_U$ for any Zariski open subset $U \subseteq V$. It is clear that $\beta_U$ gives a measure of “how large” the set $U(\mathbb{Q})$ is, since we will have

$$B^{\beta_U - \varepsilon} \ll N_U(B) \ll B^{\beta_U + \varepsilon}$$

for sufficiently large values of $B$ and any $\varepsilon > 0$. The insight of Manin was to try and relate $\beta_U$ to the geometry of $V$ via the introduction of a certain quantity $\alpha(V)$. Before defining this quantity we will need some facts from algebraic geometry. The facts that we will need are summarised in more detail in the book of Hindry and Silverman [44, §A].

Assume that $V \subset \mathbb{P}^{n-1}$ is non-singular and let $\text{Div}(V)$ be the free abelian group generated by finite formal sums of the shape $D = \sum n_Y Y$, with $n_Y \in \mathbb{Z}$ and $Y$ running over geometrically irreducible codimension 1 subvarieties of $V$. A divisor $D \in \text{Div}(V)$ is effective if $n_Y \geq 0$ for all $Y$, and $D$ is said to be principal if $D = \sum_Y \text{ord}_Y(f) Y = D_f$, say, for some rational function $f \in \mathbb{C}(V)$. The intuitive idea behind the definition of the ord function for a codimension 1 subvariety $Y$ is that $\text{ord}_Y(f) = k$ if $f$ has a zero of order $k$ along $Y$, while $\text{ord}_Y(f) = -k$ if $f$ has a pole of order $k$ along $Y$. If $f$ has neither a zero nor a pole along $Y$, then $\text{ord}_Y(f) = 0$. Since $D_f + D_g = D_{f+g}$ and $D_{1/f} = -D_f$, the principal divisors form a subgroup $\text{PDiv}(V)$ of $\text{Div}(V)$. We define the geometric Picard group associated to $V$ to be

$$\text{Pic}_g(V) := \text{Div}(V)/\text{PDiv}(V).$$

A divisor class $[D] \in \text{Pic}_g(V)$ is effective if there exists an effective divisor in the class. One may also construct the geometric Néron–Severi
group $\text{NS}_Q(V)$, which is $\text{Div}(V)$ modulo a further equivalence relation called “algebraic equivalence”. When $V$ is covered by curves of genus zero, as in all the cases of interest to us in these lecture notes, it turns out that $\text{NS}_Q(V) = \text{Pic}_Q(V)$. We illustrate the definition of $\text{Pic}_Q(V)$ by calculating it in the simplest possible case $V = \mathbb{P}^{n-1}$.

**Lemma 1.1.** We have $\text{Pic}_Q(\mathbb{P}^{n-1}) = \mathbb{Z}$.

**Proof.** An irreducible divisor on $\mathbb{P}^{n-1}$ has the form $Y = \{F = 0\}$ for some absolutely irreducible form $F \in \mathbb{C}[x_1, \ldots, x_n]$. For such a divisor, define the degree of $Y$ to be $\deg Y = \deg F$. Extend the definition of degree additively, so that

$$\deg \left( \sum_Y n_Y Y \right) = \sum_Y n_Y \deg Y.$$  

The map $\deg : \text{Div}(\mathbb{P}^{n-1}) \rightarrow \mathbb{Z}$ is clearly a homomorphism, and to establish the lemma it will suffice to show that the kernel of this map is precisely the subgroup $\text{PDiv}(\mathbb{P}^{n-1})$. To see this, we note that $\deg D_f = 0$ for any rational function $f = F_1/F_2$. Indeed, the sum of the positive degree terms will be $\deg F_1$, whereas the sum of the negative degree terms will be $\deg F_2$, and this two degrees must coincide in order to have a well-defined rational function. Conversely, if $D = n_1Y_1 + \cdots + n_kY_k$ has degree zero, with $Y_i = \{F_i = 0\}$ for $1 \leq i \leq k$, then $f = F_1^{n_1} \cdots F_k^{n_k}$ is a well-defined rational function on $\mathbb{P}^{n-1}$ with $D_f = D$. This completes the proof of the lemma. 

Returning to the setting of arbitrary non-singular varieties $V \in \mathbb{P}^{n-1}$, let $H \in \text{Div}(V)$ be a divisor corresponding to a hyperplane section. Furthermore, let $K_V \in \text{Div}(V)$ be the canonical divisor. This is a common abuse of notation: really $K_V$ refers to the class of $D_\omega$ in $\text{Pic}_Q(V)$ for any differential $(\dim V)$-form $\omega$ of $V$. It would take us too far afield to include precise definitions of these objects here. We may now define the real number

$$\alpha(V) := \inf\{r \in \mathbb{R} : r[H] + [K_V] \in \Lambda_{\text{eff}}(V)\},$$

where

$$\Lambda_{\text{eff}}(V) := \{c_1[D_1] + \cdots + c_k[D_k] : c_i \in \mathbb{R}_{\geq 0}, [D_i] \in \text{NS}_Q(V) \text{ effective}\}$$

is the so-called effective cone of divisors. It does not matter too much if this definition is currently meaningless: the main thing is that $\alpha(V)$ depends in an explicit way on the geometry of $V$ over $\mathbb{C}$. We now have the following basic conjecture due to Batyrev and Manin [\square Conjecture A].

**Conjecture 1.1.** For all $\varepsilon > 0$ there exists a Zariski open subset $U \subseteq V$ such that $\beta_U \leq \alpha(V) + \varepsilon$.

A non-singular variety $V \subseteq \mathbb{P}^{n-1}$ is said to be Fano if $K_V$ does not lie in the closure of the effective cone $\Lambda_{\text{eff}}(V) \subset \text{NS}_Q(V) \otimes \mathbb{R}$. This is equivalent to $-K_V$ being ample, and implies in particular that $V$ is covered by rational curves. As an example, suppose that $V$ is a complete intersection, with $V = W_1 \cap \cdots \cap W_t$ for hypersurfaces $W_i \subset \mathbb{P}^{n-1}$ of degree $d_i$. Then $V$ is Fano if and only if $d_1 + \cdots + d_t < n$. With this in mind we have the following supplementary prediction.
Conjecture 1.2. Assume that $V$ is Fano and $V(\mathbb{Q})$ is Zariski dense in $V$. Then there exists a Zariski open subset $U \subseteq V$ such that $\beta_U = \alpha(V)$.

We have $\alpha(V) = n - d_1 - \cdots - d_t$ when $V$ is a non-singular complete intersection as above. In particular, when $V$ is a hypersurface of degree $d$ we may deduce from Theorem 1.1 that Conjecture 1.2 holds when $n$ is sufficiently large in terms of $d$. It also holds for $n \geq 3$ when $d = 2$ (see Heath-Brown [37], for example). Finally we remark that Conjecture 1.2 holds for projective space. This follows from Exercise 3 and the fact that $\alpha(f) = \beta(f)$, for example. Finally we remark that Conjecture 1.2 holds for projective space. This follows from Exercise 3 and the fact that $[K_{P^n-1}] = [-nH]$ in $\text{Pic}(\mathbb{P}^{n-1})$, whence $\alpha(\mathbb{P}^{n-1}) = n$.

The title of these lecture notes suggests that we will focus our attention on the situation for varieties of dimension 2. Before doing so, let us consider the situation for curves briefly. For simplicity we will discuss only projective plane curves $V \subseteq \mathbb{P}^2$ of degree $d$. There is a natural trichotomy among such curves, according to the genus $g$ of the curve. For curves with $g = 0$, otherwise known as rational curves, it is possible to show that $N_V(B) \sim c_V B^{2/d}$. This is in complete accordance with the Manin conjecture. It is an amusing exercise to check that such an asymptotic formula holds with $d = 2$ when $V$ is given by the equation $x_1^2 + x_2^2 = x_3^2$, for example. When $g = 1$ and $V(\mathbb{Q}) \neq \emptyset$, the curve is elliptic and it has been shown by Néron [34] Theorem B.6.3] that

$$N_V(B) \sim c_V (\log B)^{r_V/2},$$

where $r_V$ denotes the rank of $V$. Thus although there can be infinitely many points in $V(\mathbb{Q})$, we see that the corresponding counting function grows much more slowly than for rational curves. Elliptic curves are not Fano, and so this is not covered by the Manin conjecture. However it does confirm Conjecture 1.1 since $\alpha(V) = 0$. When $g \geq 2$ the work of Faltings [28] shows that $V(\mathbb{Q})$ is always finite, and so it does not make sense to study $N_V(B)$.

Let us now concern ourselves with Fano varieties of dimension 2. We begin with some simple-minded numerics. Suppose that we are given a Fano variety $V$ of dimension 2 and degree $d$, which is a non-singular complete intersection in $\mathbb{P}^{n-1}$. Thus $V = W_1 \cap \cdots \cap W_t$ for hypersurfaces $W_i \subseteq \mathbb{P}^{n-1}$ of degree $d_i$, and we assume that the intersection is transversal at a generic point of $V$. We are not interested in hyperplane sections of $V$, and so we will assume without loss of generality that $d_i \geq 2$ for each $1 \leq i \leq t$. Then the following inequalities must be satisfied:

1. $d_1 + \cdots + d_t < n$, [Fano]
2. $n - 1 - t = 2$, [complete intersection of dimension 2]
3. $d = d_1 \cdots d_t$, [Bézout]
4. $d_1 \cdots \geq d_i \geq 2$.

It follows that the only possibilities are

$$(d; d_1, \ldots, d_t; n; t) \in \{(2; 2; 4; 1), (3; 3; 4; 1), (4; 2; 2; 5; 2)\}.$$

These surfaces correspond to a quadric in $\mathbb{P}^3$, a cubic surface in $\mathbb{P}^3$, and an intersection of 2 quadrics in $\mathbb{P}^4$, respectively. We have already observed that the Manin conjecture holds for quadrics. Hence one would like to examine the latter two surfaces. In fact these are the most familiar examples of “del Pezzo surfaces”. We will see in [15] that not all del Pezzo surfaces are
complete intersections, and so we have missed out on several surfaces in this
analysis. Nonetheless, a substantial portion of these lecture notes will focus
on cubic surfaces in \( \mathbb{P}^3 \) and intersections of 2 quadrics in \( \mathbb{P}^4 \).

It is now time to give a formal definition of a del Pezzo surface. Let us
begin with a discussion of non-singular del Pezzo surfaces. Let \( d \geq 3 \). Then
a \textit{del Pezzo surface of degree} \( d \) is a non-singular surface \( S \subset \mathbb{P}^d \) of degree
\( d \), with very ample anticanonical divisor \( -K_S \). This latter condition is
equivalent to the equality \( [-K_S] = [H] \) in \( \text{Pic}_{\mathbb{Q}}(S) \), for a hyperplane section
\( H \in \text{Div}(S) \). The facts that we will recall here are all established in the
book of Manin [49], for example. It is well-known that del Pezzo surfaces
\( S \subset \mathbb{P}^d \) arise either as the quadratic Veronese embedding of a quadric in \( \mathbb{P}^3 \),
which is a del Pezzo surface of degree 8 in \( \mathbb{P}^8 \) (isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \)), or
as the blow-up of \( \mathbb{P}^2 \) along 9 – \( d \) points in general position, in which case
the degree of \( S \) satisfies \( 3 \leq d \leq 9 \). We will meet the notion of “general
position” when \( d = 3 \) in \( \S 2.1 \). Since \( [-K_S] = [H] \) in \( \text{Pic}_{\mathbb{Q}}(S) \), we see that
\( \alpha(S) = 1 \) for non-singular del Pezzo surfaces of degree \( d \).

The geometry of del Pezzo surfaces is very beautiful and well-worth study-
ing. However, to avoid straying from the main focus of these lecture notes,
we will content ourselves with simply quoting the facts that are needed.
One of the remarkable features of del Pezzo surfaces of small degree is that
each such surface contains finitely many lines. The precise number of lines
is recorded in Table 1.

| \( d \) | number of lines |
|---|---|
| 3 | 27 |
| 4 | 16 |
| 5 | 10 |
| 6 | 6 |

Table 1. Lines on non-singular del Pezzo surfaces of degree \( d \)

It turns out that dealing with del Pezzo surfaces of degree \( d \) gets easier
as the degree increases. In these lecture notes we will focus our attention
on the del Pezzo surfaces of degree \( d \in \{3, 4, 5, 6\} \). It turns out that for del
Pezzo surfaces of degree \( d \), the geometric Picard group \( \text{Pic}_{\mathbb{Q}}(S) \) is a finitely
generated free \( \mathbb{Z} \)-module, with

\[
\text{Pic}_{\mathbb{Q}}(S) \cong \mathbb{Z}^{10-d}.
\]  

\[ (1.6) \]

This is established in Manin [49], where an explicit basis for the group is
also provided (see \( \S 2.1 \) for a concrete example). Let \( K \) be a splitting field
for the finitely many lines contained in \( S \). The final invariant that we will
need to introduce is the \textit{Picard group}

\[
\text{Pic}(S) := \text{Pic}_{\mathbb{Q}}(S)^{\text{Gal}(K/\mathbb{Q})}
\]  

\[ (1.7) \]

of the surface. This is just the set of elements in \( \text{Pic}_{\mathbb{Q}}(S) \) that are fixed by
the action of the Galois group. Write \( \rho_S \) for the rank of \( \text{Pic}(S) \). Let \( U \subset S \)
be the Zariski open subset formed by deleting the finitely many lines from
\( S \). Then we have the following [49 Conjecture C’].
**Conjecture 1.3.** Suppose that $S \subset \mathbb{P}^d$ is a non-singular del Pezzo surface of degree $d$. Then there exists a non-negative constant $c_{S,H}$ such that
\[ N_U(B) = c_{S,H} B (\log B)^{\rho_S - 1} (1 + o(1)). \] (1.8)

In these lecture notes this is what will commonly be termed as “the Manin conjecture”. Note that the exponent of $B$ agrees with Conjecture 1.2, since $\alpha(S) = 1$. Moreover the exponent of $\log B$ is at most $9 - d$, since the geometric Picard group has rank $10 - d$. We will develop some heuristics to support this power of $\log B$ in §2. The value of the constant $c_{S,H}$ has also received a conjectural interpretation at the hands of Peyre [51], an interpretation that has been extended by Batyrev and Tschinkel [2], and by Salberger [55].

There are a number of refinements to Conjecture 1.3 that are currently emerging, which we will not have space to discuss here. Some of these are discussed in more details in the author’s survey [14, §2], for example. One such refinement is that there should exist a polynomial $P \in \mathbb{R}[x]$ of degree $\rho_S - 1$, and a real number $\delta > 0$, such that
\[ N_U(B) = BP(\log B) + O(B^{1-\delta}). \] (1.9)

One obviously expects the leading coefficient of $P$ to agree with Peyre’s prediction, but there has so far been rather little investigation of the lower order terms. All of the del Pezzo surfaces that we have discussed so far have been non-singular. In the following section we will meet some singular ones.

1.3. **Degree 3 surfaces.** The del Pezzo surfaces $S \subset \mathbb{P}^3$ of degree 3 are the geometrically integral cubic surfaces in $\mathbb{P}^3$, which are not ruled by lines. In particular, this definition covers both singular and non-singular del Pezzo surfaces of degree 3. Given such a surface $S$ defined over $\mathbb{Q}$, we may always find an absolutely irreducible cubic form $C \in \mathbb{Z}[x_1, x_2, x_3, x_4]$ such that $S$ is defined by the equation $C = 0$. In this section we will discuss the Manin conjecture in the context of cubic surfaces. Let us begin by considering the situation for non-singular cubic surfaces, for which one takes $U \subset S$ to be the open subset formed by deleting the famous 27 lines. Peyre and Tschinkel [53, 54] have provided ample numerical evidence for the validity of the Manin conjecture for diagonal cubic surfaces. However we are still rather far away from proving it for any single example. The best upper bound available is
\[ N_U(B) = O_{\varepsilon,S}(B^{4/3 + \varepsilon}), \] (1.10)
due to Heath-Brown [38]. This applies when the surface $S$ contains 3 coplanar lines defined over $\mathbb{Q}$, and in particular to the Fermat cubic surface
\[ x_1^3 + x_2^3 = x_3^3 + x_4^3. \]

Heath-Brown [40] has extended the bound (1.10) to all non-singular cubic surfaces, subject to a natural conjecture concerning the size of the rank of elliptic curves over $\mathbb{Q}$.

The problem of proving lower bounds is somewhat easier. Under the assumption that $S$ contains a pair of skew lines defined over $\mathbb{Q}$, Slater and Swinnerton-Dyer [59] have shown that $N_U(B) \gg_S B (\log B)^{\rho_S - 1}$, as predicted by the Manin conjecture. This does not apply to the Fermat cubic
surface, however, since the only skew lines contained in this surface are defined over \( \mathbb{Q}(\sqrt{-3}) \).

It turns out that much more can be said if one permits \( S \) to contain isolated singularities. For the remainder of this section let \( S \subset \mathbb{P}^3 \) be a geometrically integral cubic surface, which has only isolated singularities and is not a cone. Then there exists a unique “minimal desingularisation” \( \pi : \tilde{S} \to S \) of the surface, which is just a sequence of blow-up maps, and furthermore, that the asymptotic formula (1.8) is still expected to hold, with \( \rho_S \) now taken to be the rank of the Picard group of \( \tilde{S} \). As usual \( U \subset S \) is obtained by deleting all of the lines from \( S \). The classification of singular cubic surfaces \( S \) is a well-established subject, and can be traced back to the work of Cayley [18] and Schl"afli [56] over a century ago. A contemporary classification of singular cubic surfaces has since been given by Bruce and Wall [16], over \( \mathbb{Q} \). Of course, if one is interested in a classification over the ground field \( \mathbb{Q} \), then many more singularity types can occur (see Lipman [48], for example). In Table 2 we have provided a classification table of the 20 singularity types over \( \overline{\mathbb{Q}} \), including the number of lines that each surface contains. We will presently meet some explicit examples of cubic forms \( C \in \mathbb{Z}[x_1, x_2, x_3, x_4] \) that typify some surface types.

| type | # lines | singularity |
|------|---------|-------------|
| i    | 21      | \( A_1 \)   |
| ii   | 16      | \( 2A_1 \)  |
| iii  | 15      | \( A_2 \)   |
| iv   | 12      | \( 3A_1 \)  |
| v    | 11      | \( A_1 + A_2 \) |
| vi   | 10      | \( A_3 \)   |
| vii  | 9       | \( 4A_1 \)  |
| viii | 8       | \( 2A_1 + A_2 \) |
| ix   | 7       | \( A_1 + A_3 \) |
| x    | 7       | \( 2A_2 \)  |
| xi   | 6       | \( A_4 \)   |
| xii  | 6       | \( D_4 \)   |
| xiii | 5       | \( 2A_1 + A_3 \) |
| xiv  | 5       | \( A_1 + 2A_2 \) |
| xv   | 4       | \( A_1 + A_4 \) |
| xvi  | 3       | \( A_5 \)   |
| xvii | 3       | \( D_5 \)   |
| xviii| 3       | \( 3A_2 \)  |
| ix   | 2       | \( A_1 + A_5 \) |
| xx   | 1       | \( E_6 \)   |

Table 2. Classification (over \( \overline{\mathbb{Q}} \)) of singular del Pezzo surfaces of degree 3 in \( \mathbb{P}^3 \)

The labelling of each singularity type corresponds to the “Dynkin diagram” that describes the intersection behaviour of the exceptional divisors obtained by resolving the singularities in the surface. For example, consider
the cubic surface

\[ S_1 = \{ x_1^2 x_3 + x_2 x_3^2 + x_4^3 = 0 \}. \quad (1.11) \]

Up to isomorphism over \( \overline{\mathbb{Q}} \) this is the unique cubic surface of type \( \text{xx} \) in the table, and is discussed further in \cite{34}. The process of resolving the singularity gives 6 exceptional divisors \( E_1, \ldots, E_6 \) and produces the minimal desingularisation \( \tilde{S}_1 \) of the surface \( S_1 \). If \( L \) denotes the strict transform of the unique line on \( S_1 \), then \( L, E_1, \ldots, E_6 \) satisfy the intersection behaviour encoded in the Dynkin diagram

\[
\begin{array}{ccccccc}
E_2 & \; & \; & \; & \; & \; & \; \\
E_1 & \longrightarrow & E_3 & \longrightarrow & E_6 & \longrightarrow & E_5 & \longrightarrow & E_4 & \longrightarrow & L
\end{array}
\]

There is a line connecting two divisors in this diagram if and only if they meet in \( \tilde{S}_1 \). In what follows the reader can simply think of these Dynkin diagrams as a convenient way to label the surface type.

It turns out, as discussed in \cite{16}, that some types of surfaces do not have a single normal form, but an infinite family. This happens precisely for the surfaces of type \( \text{ix}, \text{ii}, \text{iii}, \text{iv}, \text{v}, \text{vi} \) and \( \text{ix} \). By \cite{16} Lemma 4 the type \( \text{xi} \) surface, with a \( D_4 \) singularity, is the only surface that has more than one normal form, but not a family. In fact it has precisely two normal forms, given by

\[ S_2 = \{ x_1 x_2 (x_1 + x_2) + x_4 (x_1 + x_2 + x_3)^2 = 0 \} \quad (1.12) \]

and

\[ S_3 = \{ x_1 x_2 x_3 + x_4 (x_1 + x_2 + x_3)^2 = 0 \}. \quad (1.13) \]

That these equations actually define distinct surfaces can be seen by calculating the corresponding Hessians in each case.

Let \( \tilde{S} \) denote the minimal desingularisation of any surface \( S \) from Table 2, and assume that all of its singularities and lines are defined over \( \mathbb{Q} \). In this case the surface is said to be split, and it follows that the Picard group of \( \tilde{S} \) has maximal rank 7 by (1.6), since \( \text{Pic}(\tilde{S}) = \text{Pic}_{\mathbb{Q}}(\tilde{S}) \). For example, \([L],[E_1],\ldots,[E_6]\) provide a basis for \( \text{Pic}(\tilde{S}) \). One would like to try and establish (1.8) for each such surface \( S \), with \( \rho_S = 7 \). Several del Pezzo surfaces are actually special cases of varieties for which the Manin conjecture is already known to hold. Recall that a variety of dimension \( D \) is said to be toric if it contains the algebraic group variety \( \mathbb{G}_m^D \) as a dense open subset, whose natural action on itself extends to all of the variety. The Manin conjecture has been established for all toric varieties by Batyrev and Tschinkel \cite{3}. It can be checked that the surface representing type \( \text{xviii} \) is toric. In fact this particular surface has been studied by numerous authors, including la Bretèche \cite{5}, la Bretèche and Swinnerton-Dyer \cite{11}, Fouvry \cite{29}, Heath-Brown and Moroz \cite{43}, and Salberger \cite{55}. Of the unconditional asymptotic formulae obtained, the most impressive is the first. This consists of an estimate like (1.9) for any \( \delta \in (0,1/8) \), with \( \deg P = 6 \).

The next surface to have received serious attention is the Cayley cubic surface

\[ S_4 = \{ x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 = 0 \}, \]
which is the type \textit{vii} surface in the table. Heath-Brown [42] has shown that there exist absolute constants $A_1, A_2 > 0$ such that

$$A_1 B \log B \leq N_U(B) \leq A_2 B \log B.$$  

An estimate of precisely the same form has been obtained by the author [13] for the $D_4$ surface $S_3$ in (1.13). In both cases the lines in the surface are all defined over $\mathbb{Q}$, so that the surfaces are split. Thus the corresponding Picard groups have rank 7 and the exponents of $B$ and $\log B$ agree with Manin’s prediction. In this set of lecture notes we will establish an upper bound for the remaining $D_4$ cubic surface $S_2$ in (1.12). This will be carried out in §4 in two basic attacks. First we will give a completely self-contained account of the upper bound $N_U(B) = O(B^{1+\varepsilon})$, for any $\varepsilon > 0$. Next, by making use of the work in [13], we will establish the following finer result.

**Theorem 1.2.** Let $S_2$ be given by (1.12). We have $N_U(B) \ll B \log B$.

The cubic surface $S_2$ contains the unique singular point $[0,0,0,0,1]$, together with the 6 lines

$$x_i = x_4 = 0, \quad x_1 + x_2 = x_j = 0, \quad x_i = x_1 + x_2 + x_3 = 0,$$  

for distinct indices $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Thus the surface is split and it follows that Theorem 1.2 agrees with the Manin conjecture.

**Exercise 4.** Check that (1.14) are all of the lines contained in $S_2$.

The final surface to have been studied extensively is the $E_6$ cubic surface $S_1$ that we discussed above. The figure below, which was constructed by Derenthal, shows all the rational points of height $\leq 1000$ on this surface. Recent joint work of the author with la Bretèche and Derenthal [12] has succeeded in establishing the Manin conjecture for this surface. In fact an asymptotic formula of the shape (1.9) is achieved, with $P$ of degree 6 and any $\delta \in (0, 1/11)$. It should be remarked that Dr. Michael Joyce has also established the Manin conjecture for $S_1$ in his doctoral thesis, albeit with a weaker error term of $O(B \log B^5)$.

1.4. **Degree 4 surfaces.** A quartic del Pezzo surface $S \subset \mathbb{P}^4$, that is defined over $\mathbb{Q}$, can be recognised as the zero locus of a suitable pair of quadratic forms $Q_1, Q_2 \in \mathbb{Z}[x_1, \ldots, x_5]$. Again we do not stipulate that the surface should be non-singular. As usual let $U \subset S$ denote the open subset formed by deleting all of the lines from $S$. Let us begin by discussing the situation for non-singular surfaces, where there are 16 lines to delete. The best result available is the estimate $N_U(B) = O_{\varepsilon,S}(B^{1+\varepsilon})$, valid for any quartic non-singular del Pezzo surface $S \subset \mathbb{P}^4$ containing a conic defined over $\mathbb{Q}$. This
result was established in an unpublished note due to Salberger in 2001. It would be interesting to see whether one could adapt the methods of [40] to show that 
\[ N_{U}(B) = O_{\varepsilon}(B^{5/4+\varepsilon}) \] for any non-singular del Pezzo surface of degree 4, assuming the same hypothesis on the ranks of elliptic curves.

As previously, it emerges that much more can be said if one permits \( S \) to contain isolated singularities. For the remainder of this section let \( S \subset \mathbb{P}^{4} \) be a geometrically integral intersection of two quadric hypersurfaces, which has only isolated singularities and is not a cone, and let \( \tilde{S} \) be the minimal desingularisation of \( S \). Then the asymptotic formula (1.8) is still expected to hold, with \( \rho_{S} \) now taken to be the rank of the Picard group of \( \tilde{S} \), and \( U \subset S \) obtained by deleting all of the lines from \( S \). In particular, when \( S \) is split one always has \( \rho_{S} = 6 \). The classification of singular quartic del Pezzo surfaces can be extracted from the work of Hodge and Pedoe [45, Book IV, §XIII.11], where it is phrased in terms of the so-called Segre symbol. The Segre symbol of a matrix \( M \in M_{5}(\mathbb{C}) \) is defined as follows. If the Jordan form of \( M \) has Jordan blocks of sizes \( a_{1}, \ldots, a_{n} \), with \( a_{1} + \cdots + a_{n} = 5 \), then the Segre symbol is the symbol

\[(a_{1}, \ldots, a_{n})\]

with extra parentheses around the Jordan blocks with equal eigenvalues. Suppose that our quartic del Pezzo surface \( S \) is defined by a pair of quadric hypersurfaces, with underlying symmetric matrices \( A, B \in M_{5}(\mathbb{Q}) \). Then the Segre symbol of \( S \) is defined to be the Segre symbol associated to \( A^{-1}B \). A crucial property of the Segre symbol is that it does not depend on the choice of \( A \) and \( B \) in the pencil of quadrics defining \( S \). Since we are assuming that \( S \) is not a cone, one may always suppose that \( A, B \) are chosen so that \( A \) has full rank.

To illustrate the calculation of the Segre symbol, let us consider the surface \( S \) defined by the pair of equations

\[ x_{1}x_{2} + x_{3}x_{4} = 0, \quad x_{1}x_{4} + x_{2}x_{3} + x_{3}x_{5} + x_{4}x_{5} = 0. \] (1.15)

Let \( A, B \in M_{5}(\mathbb{Q}) \) denote the underlying matrices of the first and second equations, respectively. Then \( A \) has rank 4, and so we replace it with \( A + 2B \), which has full rank. A simple calculation reveals that the matrix \((A + 2B)^{-1}B \) has Jordan form

\[ J = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 1 \\
0 & 0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}. \]

This matrix has 4 Jordan blocks, one of size 2 and the rest of size 1. The eigenvalues associated to the different Jordan blocks are all different, and so it follows that the surface (1.15) has Segre symbol \((2, 1, 1, 1)\).

**Exercise 5.** Find any matrices \( A, B \in M_{5}(\mathbb{Q}) \) so that the corresponding surface \( x^{t}Ax = x^{t}Bx = 0 \) is non-singular. Show that the surface has Segre symbol \((1, 1, 1, 1, 1)\).
So far we have given a very easy way to check the isomorphism type of a given singular del Pezzo surface of degree 4. How do we match this up with a classification according to the singularity type, as in our discussion of cubic surfaces in Table 2? It turns out that up to isomorphism over \( \mathbb{Q} \), there are 15 possible singularity types for \( S \). Over \( \mathbb{Q} \), Coray and Tsfasman \[23, Proposition 6.1\] have calculated the extended Dynkin diagrams for all of the 15 types, and Knörrer \[47\] has determined the precise correspondence between the singularity type and the Segre symbol. Table 3 is extracted from this body of work, and matches each possible singularity type with the Segre symbol, and the number of lines that the surfaces contains.

| type | Segre symbol | # lines | singularity |
|------|--------------|--------|-------------|
| i    | (2,1,1,1)    | 12     | \( A_1 \)   |
| ii   | (2,2,1)      | 9      | 2\( A_1 \)  |
| iii  | ((1,1),1,1,1)| 8      | 2\( A_1 \)  |
| iv   | (3,1,1)      | 8      | \( A_2 \)   |
| v    | ((1,1),2,1)  | 6      | 3\( A_1 \)  |
| vi   | (3,2)        | 6      | \( A_1 + A_2 \) |
| vii  | (4,1)        | 5      | \( A_3 \)   |
| viii | ((2,1),1,1)  | 4      | \( A_3 \)   |
| ix   | ((1,1),(1,1),1)| 4     | 4\( A_1 \) |
| x    | ((1,1),3)    | 4      | 2\( A_1 + A_2 \) |
| xi   | ((2,1),2)    | 3      | \( A_1 + A_3 \) |
| xii  | (5)          | 3      | \( A_4 \)   |
| xiii | ((3,1),1)    | 2      | \( D_4 \)   |
| xiv  | ((2,1),(1,1))| 2      | 2\( A_1 + A_3 \) |
| xv   | ((4,1))      | 1      | \( D_5 \)   |

Table 3. Classification (over \( \mathbb{Q} \)) of singular del Pezzo surfaces of degree 4 in \( \mathbb{P}^4 \)

In general, given a particular Segre symbol, it’s not entirely straightforward to determine explicit equations that define a singular del Pezzo surface of degree 4 having this symbol. Nonetheless in Table 4 we have done precisely this for each Segre symbol that occurs. In doing so we have retrieved some of the calculations carried out by Derenthal \[26\]. An important feature of the table is that the surfaces recorded are split over \( \mathbb{Q} \). It remains a significant open challenge to establish the Manin conjecture for the 15 surfaces given in Table 4. This will furnish a proof of the Manin conjecture for the class of split singular del Pezzo surfaces of degree 4 that are defined over \( \mathbb{Q} \), and is undoubtedly a key stepping stone on the way towards a resolution of the conjecture for all del Pezzo surfaces. There is huge potential for further work in this area, and I hope that these lecture notes succeed in showing that analytic number theorists are well placed to make an important contribution.

**Exercise 6.** Calculate the Segre symbol for each of the surfaces in Table 4, and check they match up with the correct singularity type in Table 3.
Whereas they share the same singularity type, the surfaces of type \( \text{vii} \) and \( \text{viii} \) differ because in the former there are 5 lines, 4 of which pass through the singularity, whereas in the latter all 4 lines pass through the singularity. Similarly, an important difference between the surfaces of type \( \text{ii} \) and \( \text{iii} \) in Tables 3 and 4 is that for the surface of type \( \text{ii} \), the line joining the two singularities is contained in the surface, whereas for the surface of type \( \text{iii} \) it is not. When the 2 singular points are defined over a quadratic extension of \( \mathbb{Q} \), the latter surface is called an *Iskovskih surface*. There is ample evidence available (see Coray and Tsfasman [23], for example) to the effect that Iskovskih surfaces are the most arithmetically interesting surfaces among the singular del Pezzo surfaces of degree 4. In fact they are the only such surfaces for which the Hasse principle can fail to hold. The main focus of these lecture notes is upon the situation for split singular del Pezzo surfaces, and so we will say no more about Iskovskih surfaces here.

As usual, let \( \tilde{S} \) denote the minimal desingularisation of any surface \( S \) from Table 4. Then the Picard group of \( \tilde{S} \) has rank \( \rho_S = 6 \). The goal recorded above is to try and establish \( \text{I.S.} \) for each \( S \). As in the case of singular cubic surfaces several of the surfaces are actually special cases of varieties for which the Manin conjecture is already known to hold. Thus it can be shown that the surfaces representing types \( \text{ix}, \text{x}, \text{xiv} \) are all toric, so that \( \text{I.S.} \) already holds in these cases by the work of Batyrev and Tschinkel [2]. In a very real sense these surfaces are the “easiest” to deal with in our list.

**Exercise 7.** Show that \( N_U(B) = O_{\epsilon}(B^{1+\epsilon}) \) for the surfaces of type \( \text{ix}, \text{x} \) and \( \text{xiv} \).

It has also been shown by Chambert-Loir and Tschinkel [19] that the Manin conjecture is true for equivariant compactifications of the algebraic group \( \mathbb{G}_m^3 \). Although identifying such surfaces in the table is not entirely routine, it transpires that the type \( \text{xv} \) surface (with a \( D_5 \) singularity) is

| Type | \( Q_1(x) \) | \( Q_2(x) \) |
|------|---------------|---------------|
| i    | \( x_1x_2 - x_3x_4 \) | \( x_1x_4 - x_2x_3 + x_3x_5 + x_4x_5 \) |
| ii   | \( x_1x_2 - x_3x_4 \) | \( x_1x_4 - x_2x_3 + x_3x_5 + \frac{x_2^2}{x_5} \) |
| iii  | \( x_1x_2 - x_3^2 \) | \( x_1x_3 - x_2x_3 + x_4x_5 \) |
| iv   | \( x_1x_2 - x_3x_4 \) | \( (x_1 + x_2 + x_3 + x_4)x_5 - x_3x_4 \) |
| v    | \( x_1x_2 - x_3^2 \) | \( x_2x_3 + x_3^2 + x_4x_5 \) |
| vi   | \( x_1x_2 - x_3x_4 \) | \( x_1x_5 + x_2x_3 + x_4x_5 \) |
| vii  | \( x_1x_2 - x_3x_4 \) | \( x_1x_4 + x_2x_4 + x_3x_5 \) |
| viii | \( x_1x_4 - (x_2 - x_3)x_5 \) | \( (x_1 + x_4)(x_2 + x_3) + x_2x_3 \) |
| ix   | \( x_1x_2 - x_3^2 \) | \( x_2^2 - x_4x_5 \) |
| x    | \( x_1x_2 - x_3^2 \) | \( x_2x_3 - x_4x_5 \) |
| xi   | \( x_1x_4 - x_3x_5 \) | \( x_1x_2 + x_2x_4 + \frac{x_2^2}{x_3} \) |
| xii  | \( x_1x_2 - x_3x_4 \) | \( x_1x_5 + x_2x_3 + \frac{x_2^2}{x_3} \) |
| xiii | \( x_1x_4 - x_2x_5 \) | \( x_1x_2 + x_2x_4 + x_3^2 \) |
| xiv  | \( x_1x_2 - x_3^2 \) | \( x_1^2 - x_4x_5 \) |
| xv   | \( x_1x_2 - x_3^2 \) | \( x_1x_5 + x_2x_3 + x_4^2 \) |

**Table 4.** Split surfaces representing the 15 singularity types
covered by this work. In joint work with la Bretèche, the author [8] has provided an independent proof of the Manin conjecture for this particular surface. In addition to obtaining a finer asymptotic formula of the shape given in (1.9), this work has provided a useful line of attack for several other singular del Pezzo surfaces.

In Table 5, we have recorded a list of progress towards the final resolution of the Manin conjecture for the split singular del Pezzo surfaces of degree 4. We have included the relevant reference in the literature, and whether the result attained amounts to an asymptotic formula for the counting function, or an upper bound. We will not pay attention here to the quality of the error term in the asymptotic formula, but each upper bound is of the correct order of magnitude $B(\log B)^5$. There is still plenty left to do!

| type | type of estimate achieved |
|------|--------------------------|
| v    | upper bound [14]         |
| ix   | asymptotic formula [3]   |
| x    | asymptotic formula [3]   |
| xiii | asymptotic formula [27]  |
| xiv  | asymptotic formula [3]   |
| xv   | asymptotic formula [8]   |

Table 5. Summary of progress for the split singular del Pezzo surfaces of degree 4

It is also interesting to try and establish the Manin conjecture for singular del Pezzo surfaces of degree 4 that are not split over the ground field. In further joint work of the author with la Bretèche [9], the Manin conjecture is established for the surface

$$x_1x_2 - x_3^2 = 0, \quad x_1^2 + x_2x_5 + x_4^2 = 0.$$  

This surface has a $D_4$ singularity and is isomorphic over $\mathbb{Q}(i)$ to the surface of type xiii in Table 4. The Picard group of $\tilde{S}$ has rank 4 in this case, and an asymptotic formula of the shape (1.9) is obtained for any $\delta \in (0, 3/32)$, with $P$ a polynomial of degree 3.

1.5. Degree $\geq 5$ surfaces. It turns out that all del Pezzo surfaces of degree $d \geq 7$ are toric [26, Proposition 8], and that all non-singular del Pezzo surfaces of degree $d \geq 6$ are toric. Thus (1.8) already holds in these cases by the work of Batyrev and Tschinkel [3]. For non-singular del Pezzo surfaces $S \subset \mathbb{P}^5$ of degree 5, the situation is rather less satisfactory. In fact there are very few instances for which the Manin conjecture has been established. The most significant of these is due to la Bretèche [7], who has proved the conjecture for the split non-singular del Pezzo surface $S$ of degree 5, in which the 10 lines are all defined over $\mathbb{Q}$. To be precise, if $U \subset S$ denotes the open subset formed by deleting the lines from $S$, then la Bretèche shows that

$$N_U(B) = c_0 B(\log B)^4 \left(1 + O\left(\frac{1}{\log \log B}\right)\right),$$

for a certain constant $c_0 > 0$. This confirms Conjecture 1.3, since we have seen in (1.6) that $\text{Pic}(S) \cong \mathbb{Z}^5$ for split non-singular del Pezzo surfaces.
of degree 5. The other major achievement in the setting of quintic del Pezzo surfaces is a result of la Bretèche and Fouvry [10], where the Manin conjecture is established for a surface that is not split, but contains lines defined over \( \mathbb{Q}(i) \).

So far we have only discussed the situation for non-singular del Pezzo surfaces of degree \( d \geq 5 \). Let us now turn to the singular setting. When \( d = 6 \) it emerges that there exist such surfaces that are not toric, and so are not covered by [3]. We will focus attention on the situation for del Pezzo surfaces of degree 6, following the investigation of Derenthal [26], where the degree 5 surfaces are also considered. In view of [23, Proposition 8.3], Table 6 lists all possible types of singular del Pezzo surfaces of degree 6.

| type | # lines | singularity |
|------|---------|-------------|
| i    | 4       | A_1         |
| ii   | 3       | A_1         |
| iii  | 2       | 2A_1        |
| iv   | 2       | A_2         |
| v    | 1       | A_1 + A_2   |

Table 6. Classification (over \( \mathbb{Q} \)) of singular del Pezzo surfaces of degree 6

As noted in [26, §5], the surfaces of type i, iii and v are all toric and so do not interest us here. Any singular del Pezzo surface of degree 6 can be realised as the intersection of 9 quadrics in \( \mathbb{P}^6 \). For example, the type iv surface is cut out by the system of equations

\[
\begin{align*}
  x_1 x_6 - x_4 x_5 &= x_1 x_7 - x_2 x_5 = x_1 x_7 - x_3 x_4 = x_3 x_7 + x_4 x_5 + x_5^2 \\
  &= x_5 x_7 - x_3 x_4 = x_2 x_7 + x_4^2 + x_4 x_5 = x_4 x_7 - x_2 x_6 \\
  &= x_4 x_6 + x_5 x_6 + x_2^2 = x_2 x_3 - x_1 x_4 + x_1 x_5 = 0.
\end{align*}
\]

(1.16)

In this set of lecture notes we will establish the Manin conjecture for the type ii surface, which has the simplest possible singularity. When \( S \subset \mathbb{P}^6 \) is a split surface of type ii, then there is unimodular change of variables that takes \( S \) into the surface with equations

\[
\begin{align*}
  x_1^2 - x_2 x_4 &= x_1 x_5 - x_3 x_4 = x_1 x_3 - x_2 x_5 = x_1 x_6 - x_3 x_5 \\
  &= x_2 x_6 - x_3^2 = x_4 x_6 - x_2^2 = x_1^2 + x_4 x_5 + x_5 x_7 \\
  &= x_1 x_2 + x_3 x_7 = x_1 x_3 + x_1 x_5 + x_6 x_7 = 0.
\end{align*}
\]

(1.17)

Let \( \tilde{S} \) denote the minimal desingularisation of \( S \). It follows from (1.6) that \( \text{Pic}(\tilde{S}) \approx \mathbb{Z}^4 \) since \( S \) is split. We will establish the following result in [33]

**Theorem 1.3.** Let \( S \subset \mathbb{P}^6 \) be the \( A_1 \) surface given by (1.17). Then there exist constants \( c_1, c_2 \geq 0 \) such that

\[
N_U(B) = c_1 B (\log B)^3 + c_2 B (\log B)^2 + O(B \log B),
\]

where

\[
c_1 = \frac{\sigma}{144} \prod_p \left( 1 - \frac{1}{p} \right)^4 \left( 1 + \frac{4}{p} + \frac{1}{p^2} \right)
\]
and
\[ \sigma_\infty = 6 \int_{\{u, t, v \in \mathbb{R} : 0 < u, ut^2, uv^2, |tv(t-v)| \leq 1\}} dtdudv. \tag{1.18} \]

Since \( \text{Pic}(\tilde{S}) \) has rank 4, the exponents of \( B \) and \( \log B \) in this asymptotic formula are in complete agreement with Conjecture 1.3. Although we will not give details here, it turns out that the value of the constant \( c_1 \) also confirms the prediction of Peyre [51] in this case. It is hoped that our proof of Theorem 1.3 will encourage other researchers to try their hand at proving asymptotic formulae for \( N_U(B) \). With this in mind Exercise 8 is more of a research problem, and its resolution will therefore conclude the proof of the Manin conjecture for all split (non-singular or singular) del Pezzo surfaces of degree 6.

**Exercise 8.** Establish an asymptotic formula for the type iv surface in Table 6 with underlying equations (1.16).

1.6. **Universal torsors.** Universal torsors were originally introduced by Colliot-Thélène and Sansuc [20, 21] to aid in the study of the Hasse principle and weak approximation for rational varieties. Since their inception it is now well-recognised that they also have a central rôle to play in proofs of the Manin conjecture for Fano varieties, and in particular, for del Pezzo surfaces. Let \( S \subset \mathbb{P}^d \) be a del Pezzo surface of degree \( d \in \{3, 4, 5, 6\} \), and let \( \tilde{S} \) denote the minimal desingularisation of \( S \) if it is singular, and \( \tilde{S} = S \) otherwise. Let \( E_1, \ldots, E_{10-d} \in \text{Div}(\tilde{S}) \) be generators for \( \text{Pic}(\tilde{S}) \), and let \( E_i^\times = E_i \setminus \{\text{zero section}\} \). Working over \( \overline{\mathbb{Q}} \), a universal torsor above \( \tilde{S} \) is given by the action of \( \mathbb{G}_m^{10-d} \) on the map
\[ \pi : E_1^\times \times \cdots \times E_{10-d}^\times \to \tilde{S}. \]

A proper discussion of universal torsors would take us too far afield, and the reader may consult the survey of Peyre [52] for further details, or indeed the construction of Hassett and Tschinkel [34]. The latter outlines an alternative approach to universal torsors via the Cox ring. Given the usual open subset \( U \subset S \), the general theory of universal torsors ensures that there is a partition of \( U(\mathbb{Q}) \) into a disjoint union of patches, each of which is in bijection with a suitable set of integral points on a universal torsor above \( \tilde{S} \).

The guiding principle behind the use of universal torsors is simply that they ought to be arithmetically simpler than the original variety. In our work it will suffice to think of universal torsors as “particularly nice parametrisations” of rational points on the surface. The universal torsors that we encounter in these lecture notes all have embeddings as affine hypersurfaces of high dimension. Moreover, in each case we will show how the underlying equation of the universal torsor can be deduced in a completely elementary fashion, without any recourse to geometry whatsoever. The torsor equations we will meet all take the shape
\[ A + B + C = 0, \]
for monomials \( A, B, C \) of various degrees in the appropriate variables. As in many examples of counting problems for higher dimensional varieties, one can occasionally gain leverage by fixing some of the variables at the
outset, in order to be left with a counting problem for a family of small dimensional varieties. If one is sufficiently clever about which variables to fix first, one is sometimes left with a quantity that we know how to estimate — and crucially — whose error term we can control once summed over the remaining variables.

As a concrete example, we note that Hassett and Tschinkel [34] have calculated the universal torsor for the cubic surface (1.11). It is shown that there is a unique universal torsor above \( \widetilde{S}_1 \), and that it is given by the equation

\[
y_\ell s_\ell^3 s_5^2 s_\ell + y_2 s_2 + y_1 s_1^2 s_3 = 0,
\]

for variables \( y_1, y_2, y_\ell, s_1, s_2, s_3, s_\ell, s_4, s_5, s_6 \). One of the variables does not explicitly appear in this equation, and the torsor should be thought of as being embedded in \( \mathbb{A}^{10} \). It turns out that the way to proceed here is to fix all of the variables apart from \( y_1, y_2, y_\ell \). One may then view the equation as a congruence

\[
y_2^2 s_2 \equiv -y_1^3 s_1^2 s_3 \pmod{s_3^3 s_\ell s_2^4 s_5},
\]

in order to take care of the summation over \( y_\ell \). This is the approach taken in [12], the next step being to employ very standard facts about the number of integer solutions to polynomial congruences that are restricted to lie in certain regions. One if left with a main term and an error term, which the remaining variables need to be summed over. While the treatment of the main term is relatively routine, the treatment of the error term presents a much more serious obstacle.

The universal torsors that turn up in the proofs of Theorems 1.2 and 1.3 can also be embedded in affine space as hypersurfaces. We will see in §3 that the approach discussed above also produces results for the del Pezzo surface of degree 6 considered in Theorem 1.3. In the proof of Theorem 1.2 in §4 our approach will be more obviously geometric, and we will actually view the equation as a family of projective lines, and also as a family of conics. We will then call upon techniques from the geometry of numbers to count the relevant solutions.

2. Further heuristics

We have seen in [1.2] and in particular in the statement of Conjecture 1.3 that for any del Pezzo surface \( S \subset \mathbb{P}^d \) one expects a growth rate like \( c_S B (\log B)^3 \) for the counting function \( N_U(B) \). We have already given some motivation for the exponent of \( B \) in [1.2]. The focus of the present section is to produce much more sophisticated heuristics than we have previously met. In particular we will gain an insight into the exponent of \( \log B \) that appears in the Manin conjecture.

For ease of presentation we restrict attention to non-singular diagonal cubic surfaces \( S \subset \mathbb{P}^3 \). Thus

\[
S = \{ a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + a_4 x_4^3 = 0 \},
\]

for \( a = (a_1, \ldots, a_4) \in \mathbb{N}^4 \) such that \( \gcd(a_1, \ldots, a_4) = 1 \). Define

\[
P := \{3\} \cup \{ p : p \mid a_1 a_2 a_3 a_4 \}.
\]
2.1. The lines on a cubic surface. The facts that we will need in this section are explained in detail in the books of Hartshorne \[33\] and Manin \[40\]. In general, a non-singular cubic surface $S \subset \mathbb{P}^3$ is obtained by blowing up $\mathbb{P}^2$ along a collection of 6 points $P_1, \ldots, P_6$ in general position. By general position we mean that no 3 of them are collinear and they do not all lie on a conic. The 27 lines on the surface arise in the following way. There are 6 exceptional divisors $E_i$ above $P_i$, for $1 \leq i \leq 6$, and 15 strict transforms $L_{i,j}$ of the lines going through precisely 2 points $P_i, P_j$, for $1 \leq i < j \leq 6$. Finally there are the 6 strict transforms of the conics going through all but one of the 6 points. If $\Lambda$ is the strict transform of a line in $\mathbb{P}^2$ that doesn’t go through any of the $P_i$, then a basis of the geometric Picard group $\text{Pic}_\mathbb{Q}(S)$ is given by

$$[\Lambda], [E_1], \ldots, [E_6].$$

The remaining divisors may be expressed in terms of these elements via the relations

$$[L_{i,j}] = [\Lambda] - [E_i] - [E_j], \quad [Q_i] = 2[\Lambda] - \sum_{j \neq i} [E_j]. \quad (2.3)$$

The class of the anti-canonical divisor $-K_S$ is given by $[-K_S] = 3[\Lambda] - \sum_{i=1}^6 [E_j]$, although we will not need this fact in our work. One can check that the hyperplane section has class $-3[\Lambda] + \sum_{i=1}^6 [E_j]$ in $\text{Pic}_\mathbb{Q}(S)$, so that the cubic surface has very ample anticanonical divisor, as claimed in \[1.2\].

When $S$ takes the shape \[1.1\] it is not hard to write down the 27 lines explicitly. The calculations that we present below are based on those carried out by Peyre and Tschinkel \[54, \S 2\]. Fix a cubic root $\alpha$ (resp. $\alpha'$, $\alpha''$) of $a_2/a_1$ (resp. $a_3/a_1$, $a_4/a_1$). We will assume that $\alpha \in \mathbb{Q}$ if $a_2/a_1$ (resp. $a_3/a_1$, $a_4/a_1$) is a cube in $\mathbb{Q}$. Put

$$\beta = \frac{\alpha''}{\alpha'}, \quad \beta' = \frac{\alpha}{\alpha''}, \quad \beta'' = \frac{\alpha'}{\alpha}.$$

We denote by $\theta$ a primitive cube root of one. Let $i$ run over elements of $\mathbb{Z}/3\mathbb{Z}$. Then the 27 lines on the cubic surface \[1.1\] are given by the equations

$$L_i: \begin{cases} x_1 + \theta^i \alpha x_2 = 0, \\ x_3 + \theta^i \beta x_4 = 0, \end{cases} \quad L'_i: \begin{cases} x_1 + \theta^i \alpha x_2 = 0, \\ x_3 + \theta^i + 1 \beta x_4 = 0, \end{cases} \quad L''_i: \begin{cases} x_1 + \theta^i \alpha x_2 = 0, \\ x_3 + \theta^{i+2} \beta x_4 = 0, \end{cases}$$

$$M_i: \begin{cases} x_1 + \theta^i \alpha' x_3 = 0, \\ x_4 + \theta^i \beta' x_2 = 0, \end{cases} \quad M'_i: \begin{cases} x_1 + \theta^i \alpha' x_3 = 0, \\ x_4 + \theta^i + 1 \beta' x_2 = 0, \end{cases} \quad M''_i: \begin{cases} x_1 + \theta^i \alpha' x_3 = 0, \\ x_4 + \theta^{i+2} \beta' x_2 = 0, \end{cases}$$

$$N_i: \begin{cases} x_1 + \theta^i \alpha'' x_4 = 0, \\ x_2 + \theta^i \beta'' x_3 = 0, \end{cases} \quad N'_i: \begin{cases} x_1 + \theta^i \alpha'' x_4 = 0, \\ x_2 + \theta^i + 1 \beta'' x_3 = 0, \end{cases} \quad N''_i: \begin{cases} x_1 + \theta^i \alpha'' x_4 = 0, \\ x_2 + \theta^{i+2} \beta'' x_3 = 0. \end{cases}$$

Let $K = \mathbb{Q}(\theta, \alpha, \alpha', \alpha'')$. It is a Galois extension of $\mathbb{Q}$, and in the generic case has degree 54 with Galois group $\text{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/3\mathbb{Z})^3 \rtimes \mathbb{Z}/2\mathbb{Z}$.

We need to equate these lines to the divisors $E_i, L_{i,j}, Q_i$ that we met earlier. There is a certain degree of freedom in doing this, as discussed in...
§V.4, but it turns out that the choice

\[
\begin{align*}
    E_1 &= L_0, & E_2 &= L_1, & E_3 &= L_2, \\
    E_4 &= M_1, & E_5 &= M'_2, & E_6 &= M''_0, \\
    Q_1 &= L'_1, & Q_2 &= L'_2, & Q_3 &= L'_0, \\
    Q_4 &= M_0, & Q_5 &= M'_1, & Q_6 &= M''_2, \\
    L_{1,2} &= L''_1, & L_{2,3} &= L''_2, & L_{1,3} &= L''_0, \\
    L_{4,5} &= M''_1, & L_{5,6} &= M_2, & L_{4,6} &= M'_0, \\
    L_{1,4} &= N_0, & L_{1,5} &= N_1, & L_{1,6} &= N_2, \\
    L_{2,4} &= N'_1, & L_{2,5} &= N'_2, & L_{2,6} &= N'_0, \\
    L_{3,4} &= N''_2, & L_{3,5} &= N''_0, & L_{3,6} &= N''_1,
\end{align*}
\]  

(2.4)

is satisfactory. In assigning lines to \( E_1, \ldots, E_6 \), all that is required is that they should all be mutually skew. Given any cubic surface of the shape (2.1), we now have the tools with which to compute the Picard group (1.7). In fact, from this point forwards the process requires little more than basic linear algebra.

Let us illustrate the procedure by calculating the Picard group for a special case. Consider the Fermat surface

\[
S_1 = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\}. 
\]  

(2.5)

In this case \( \alpha = \alpha' = \alpha'' = \beta = \beta' = \beta'' = 1 \), in the notation above, and \( K = \mathbb{Q}(\theta) \) is a quadratic field extension. We wish to find elements of the geometric Picard group \( \text{Pic}_\mathbb{Q}(S_1) \) that are fixed by the action of \( \text{Gal}(K/\mathbb{Q}) \). Thus we want vectors \( c = (c_0, \ldots, c_6) \in \mathbb{Z}^7 \) such that

\[
(c_0[\Lambda] + c_1[E_1] + \cdots + c_6[E_6])^\sigma = c_0[\Lambda] + c_1[E_1] + \cdots + c_6[E_6],
\]  

(2.6)

for every \( \sigma \in \text{Gal}(K/\mathbb{Q}) \). Under the action of \( \text{Gal}(K/\mathbb{Q}) \) it is not hard to check that \( \Lambda \) and \( E_1 \) are fixed, that \( E_2 \) and \( E_3 \) are swapped, and that \( E_4 \) (resp. \( E_5, E_6 \)) is taken to \( L_{5,6} \) (resp. \( L_{4,5}, L_{4,6} \)). Using (2.3) one sees that the left hand side of (2.6) is equal to

\[
(c_0 + c_4 + c_5 + c_6)[\Lambda] + c_1[E_1] + c_2[E_3] + c_3[E_2] - (c_5 + c_6)[E_4] - (c_4 + c_5)[E_5] - (c_4 + c_6)[E_6],
\]

in \( \text{Pic}_\mathbb{Q}(S_1) \). Thus we are interested in the space of \( c \in \mathbb{Z}^7 \) for which

\[
c_4 + c_5 + c_6 = 0, \quad c_2 - c_3 = 0, \quad c_4 + 2c_5 = 0, \quad c_4 + 2c_6 = 0.
\]

This system of homogeneous linear equations in 7 variables has underlying matrix of rank 3. Thus the space of solutions has rank \( 7 - 3 = 4 \), and so we may conclude that \( \text{Pic}(S_1) \cong \mathbb{Z}^4 \). In fact a little thought reveals that the 4 elements

\[
[\Lambda], [E_1], [E_2] + [E_3], -2[E_4] + [E_5] + [E_6],
\]

provide a basis for \( \text{Pic}(S_1) \).

Next, consider the surface

\[
S_2 = \{x_1^3 + x_2^3 + x_3^3 + px_4^3 = 0\}
\]  

(2.7)
for a prime $p$. When $p = 2$ or $3$, the arithmetic of this surface has been considered in some detail by Heath-Brown [36], who provides some numerical evidence to the effect that the corresponding counting function $N_U(B)$ should grow like $c_p B$ for a certain constant $c_p > 0$.

**Exercise 9.** Show that $\text{Pic}(S_2) \cong \mathbb{Z}$.

This calculation has also been carried out by Colliot-Thélène, Kanevsky and Sansuc [22, p. 12]. More generally, it is known that the Picard group of the surface (2.1) has rank 1 if and only if the ratio

$$\frac{a_{\sigma(1)}a_{\sigma(2)}}{a_{\sigma(3)}a_{\sigma(4)}}$$

is not a cube in $\mathbb{Q}$, for each permutation $\sigma$ of $(1, 2, 3, 4)$. This result is due to Segre [57].

### 2.2. Cubic characters and Jacobi sums.

Throughout this section let $p$ be a prime. Recall that a (multiplicative) character on $\mathbb{F}_p^* = \mathbb{Z}/p\mathbb{Z}$ is a map $\chi : \mathbb{F}_p^* \to \mathbb{C}^*$ such that

$$\chi(ab) = \chi(a)\chi(b)$$

for all $a, b \in \mathbb{F}_p^*$. The *trivial character* $\varepsilon$ is defined by the relation $\varepsilon(a) = 1$ for all $a \in \mathbb{F}_p^*$. It is convenient to extend the domain of definition to all of $\mathbb{F}_p$ by assigning $\chi(0) = 0$ if $\chi \neq \varepsilon$ and $\varepsilon(0) = 1$.

We begin by collecting together a few basic facts, all of which are established in [46, §8].

**Lemma 2.1.** Let $p$ be a prime. Then the following hold:

1. Let $\chi$ be a character on $\mathbb{F}_p^*$ and let $a \in \mathbb{F}_p^*$. Then $\chi(1) = 1$, $\chi(a)$ is a $(p - 1)$-th root of unity, and $\chi(a^{-1}) = \overline{\chi}(a) = \chi(a)^{-1}$.
2. For any character $\chi$ on $\mathbb{F}_p$ we have

$$\sum_{a \in \mathbb{F}_p} \chi(a) = \begin{cases} 0, & \text{if } \chi \neq \varepsilon, \\ p, & \text{if } \chi = \varepsilon. \end{cases}$$

3. The set of characters on $\mathbb{F}_p$ forms a cyclic group of order $p - 1$.

It follows from part (3) of Lemma 2.1 that $\chi^{p-1} = \varepsilon$ for any character on $\mathbb{F}_p$. We define the *order* of a character to be the least positive integer $n$ such that $\chi^n = \varepsilon$. In our work we will mainly be concerned with the characters of order 3. Let us turn briefly to the topic of generalised Jacobi sums. Given any characters $\chi_1, \ldots, \chi_r$ on $\mathbb{F}_p$, a *Jacobi sum* is a sum of the shape

$$J_0(\chi_1, \ldots, \chi_r) := \sum_{\substack{t = (t_1, \ldots, t_r) \in \mathbb{F}_p^r \\ t_1 + \cdots + t_r \equiv 0 \mod p}} \chi_1(t_1) \cdots \chi_r(t_r).$$

The key fact that we will need concerning these sums is that

$$|J_0(\chi_1, \ldots, \chi_r)| = \begin{cases} 0, & \text{if } \chi_1 \cdots \chi_r \neq \varepsilon, \\ (p - 1)p^{r/2 - 1}, & \text{if } \chi_1 \cdots \chi_r = \varepsilon. \end{cases}$$

(2.8)

This is established in [46, §8.5].

Let $p$ be a rational prime. We proceed to consider $p$ as an element of the ring of integers $\mathbb{Z}[\theta]$ associated to the quadratic field $\mathbb{Q}(\theta)$ obtained by
adjoining a primitive cube root of unity \( \theta \). It follows from basic algebraic number theory that \( p \) is a prime in \( \mathbb{Z}[\theta] \) if \( p \equiv 2 \mod 3 \), whereas it splits as \( p = \pi \overline{\pi} \) if \( p \equiv 1 \mod 3 \), where \( \pi \) is a prime in \( \mathbb{Z}[\theta] \). When \( p \equiv 2 \mod 3 \) the only cubic character on \( \mathbb{F}_p \) is the trivial character \( \varepsilon \). On the other hand, when \( p = \pi \overline{\pi} \equiv 1 \mod 3 \) then there are precisely two non-trivial cubic characters \( \chi_\pi, \chi_{\overline{\pi}} \) on \( \mathbb{F}_p \), where

\[
\chi_\omega(\cdot) = \left( \frac{\cdot}{\omega} \right)_3
\]

is the cubic residue symbol for any prime \( \omega \) in \( \mathbb{Z}[\theta] \). All of these facts are established in [46, §9].

It turns out that Jacobi sums can be used to give formulae for the number of solutions to appropriate equations over finite fields. Given any \( q \in \mathbb{N} \), let

\[
N(q) := \# \{ x \mod q : a_1x_1^3 + \cdots + a_4x_4^3 \equiv 0 \pmod{q} \},
\]

and

\[
N^*(q) := \# \{ x \mod q : \gcd(q, x_1, \ldots, x_4) = 1 \}.
\]

When \( p \) is a prime not belonging to the finite set of primes \( \mathcal{P} \) defined in (2.2), we can write down a very precise expression for \( N(p) \). Thus it follows from [46, §8.7] that

\[
N(p) = p^3 + \sum_{\chi_1, \chi_2, \chi_3, \chi_4} \chi_1(a_1^{-1})\chi_2(a_2^{-1})\chi_3(a_3^{-1})\chi_4(a_4^{-1})J_0(\chi_1, \chi_2, \chi_3, \chi_4)
\]

where the summation is over all non-trivial cubic characters \( \chi_i : \mathbb{F}_p^* \to \mathbb{C} \) such that \( \chi_1\chi_2\chi_3\chi_4 = \varepsilon \).

**Exercise 10.** Let \( p \not\in \mathcal{P} \) be a prime, and let \( \chi_1, \chi_2, \chi_3, \chi_4 \) be non-trivial cubic characters on \( \mathbb{F}_p^* \) such that \( \chi_1\chi_2\chi_3\chi_4 = \varepsilon \). Deduce from (2.8) that

\[
J_0(\chi_1, \chi_2, \chi_3, \chi_4) = p(p - 1).
\]

It follows from Exercise [10] that

\[
N^*(p) = N(p) - 1 = p^3 + p(p - 1)\delta_p(a) - 1.
\]

for any prime \( p \not\in \mathcal{P} \), where

\[
\delta_p(a) := \sum_{\chi_1, \chi_2, \chi_3, \chi_4} \chi_1(a_1^{-1})\chi_2(a_2^{-1})\chi_3(a_3^{-1})\chi_4(a_4^{-1}).
\]

Let \( a \in \mathbb{F}_p^* \) and suppose that \( p \) splits as \( \pi \overline{\pi} \). Then it will be useful to observe that

\[
\chi_\pi(a) + \chi_{\overline{\pi}}(a) = \begin{cases} 
2, & \text{if } a \text{ is a cubic residue modulo } \pi, \\
-1, & \text{otherwise}.
\end{cases}
\]

We have \( \delta_p(a) = 0 \) when \( p \equiv 2 \mod 3 \), since there are then no non-trivial cubic characters modulo \( p \). When \( p \equiv 1 \mod 3 \), with \( p = \pi \overline{\pi} \not\in \mathcal{P} \), we have

\[
\delta_p(a) = \chi_\pi \left( \frac{a_1a_2}{a_3a_4} \right) + \chi_{\overline{\pi}} \left( \frac{a_1a_2}{a_3a_4} \right) + \chi_\pi \left( \frac{a_1a_3}{a_2a_4} \right) + \chi_{\overline{\pi}} \left( \frac{a_1a_3}{a_2a_4} \right) + \chi_\pi \left( \frac{a_1a_4}{a_2a_3} \right) + \chi_{\overline{\pi}} \left( \frac{a_1a_4}{a_2a_3} \right).
\]
Still with this choice of prime $p$, let $\nu_p(a)$ denote the number of indices $i \in \{2, 3, 4\}$ for which the cubic character $\chi_\pi(\frac{a_1a_i}{a_jak})$ is equal to 1, with $\{i, j, k\}$ a permutation of $\{2, 3, 4\}$. Then we may deduce from (2.12) that

$$\delta_p(a) = \begin{cases} 0, & \text{if } p \equiv 2 \mod 3, \\ 3\nu_p(a) - 3, & \text{if } p \equiv 1 \mod 3, \end{cases}$$

(2.13)

when $p \notin P$.

2.3. The Hardy–Littlewood circle method. We are now ready to consider the counting function $N_U(B)$ that is associated to the diagonal cubic surface $S \subset \mathbb{P}^3$ given in (2.1). Our aim is to provide heuristic evidence in support of Manin’s original conjecture, and we will say rather little about the predicted value of the constant. The Hardy–Littlewood circle method is an extremely effective means of estimating counting functions associated to projective algebraic varieties, but it only works when the dimension of the variety is substantially larger than the degree. We have already seen evidence of this in the statement of Theorem 1.1, which is based on an application of the circle method. Although it has not been made to produce asymptotic formulae for the counting functions associated to del Pezzo surfaces, in this section we will see how the Hardy–Littlewood method can still be used as a useful heuristic tool. The key idea is to consider only the contribution from the major arcs.

We will simplify matters by applying the heuristic to count all of the rational points on $S$, rather than restricting attention to the open subset $U$. Although the details are formidable, it is in fact possible to obtain upper bounds for $N_U(B)$ using the circle method. Thus Heath-Brown [39] has shown that $N_U(B) = O_{\varepsilon,S}(B^{3/2+\varepsilon})$ under a certain hypothesis concerning the Hasse–Weil $L$-function associated to the surface. An interesting feature of this work is that the contribution from the rational points lying on rational lines in the surface is successfully separated out. When the surface contains no lines defined over $\mathbb{Q}$, such as the surface given by (2.7) for example, one obviously has

$$N_U(B) = N_S(B) + O(1).$$

When $S$ contains lines defined over $\mathbb{Q}$ there is a general consensus among people working on the circle method that the dominant contribution (ie. the contribution from the points on rational lines) should come from the minor arc integral.

In what follows let $e(z) := e^{2\pi iz}$ for any $z \in \mathbb{R}$. As usual, $Z^4$ denotes the set of primitive vectors in $\mathbb{Z}^4$. The igniting spark in the Hardy–Littlewood circle method is the simple identity

$$\int_0^1 e(\alpha n) d\alpha = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \in \mathbb{Z} \setminus \{0\}. \end{cases}$$
On taking into account the fact that $x$ and $-x$ represent the same point in projective space, and applying Exercise 2, we deduce that

$$N_S(B) = \frac{1}{2} \int_0^1 \sum_{x \in \mathbb{Z}^4, |x| \leq B} e(\alpha (a_1 x_1^3 + \cdots + a_4 x_4^3)) d\alpha$$

$$= \frac{1}{2} \sum_{k=1}^\infty \mu(k) \int_0^1 \sum_{x \in \mathbb{Z}^4, |x| \leq B/k} e(\alpha (a_1 x_1^3 + \cdots + a_4 x_4^3)) d\alpha$$

$$= \frac{1}{2} \sum_{k=1}^\infty \mu(k) \int_0^1 S(\alpha) d\alpha,$$

say. Let us write $P = B/k$ and $I_A(P) := \int_A S(\alpha) d\alpha$, for any bounded subset $A \subset \mathbb{R}$. The cubic exponential sum $S(\alpha)$ can actually be rather large when $\alpha$ is well-approximated by a rational number with small denominator. For example, we clearly have $S(0) = 2^4 P^4 + O(P^3)$. The philosophy that underpins the Hardy–Littlewood method is that one expects $S(\alpha)$ to be small for values of $\alpha \in [0,1]$ that are not well-approximated by rational numbers with small denominator. This is notoriously difficult to prove in general, and as indicated above, is expected to be false in the present setting!

Our heuristic will be based on analysing $I_M(P)$ for a suitable choice of “major arcs” $M$. We will not give full details here, the gaps being easily filled by consulting the relevant techniques in Davenport [25]. Let $\varepsilon > 0$ be a small parameter. Given $a, q \in \mathbb{Z}$ such that

$$1 \leq a \leq q \leq P^\varepsilon, \quad \gcd(a, q) = 1, \quad (2.14)$$

we define the interval

$$M(a, q) := \left[ \frac{a}{q} - P^{-3+\varepsilon}, \frac{a}{q} + P^{-3+\varepsilon} \right].$$

We take as major arcs the union

$$M := \bigcup_{q \leq P^\varepsilon} \bigcup_{1 \leq a \leq q, \gcd(a, q) = 1} M(a, q).$$

It is clear that $M$ contains all the points in the interval $[0,1]$ that are well-approximated by rational numbers with small denominator.

**Exercise 11.** Show that $M$ is a disjoint union for $\varepsilon < 1$.

The “minor arcs” are defined to be $m := [0,1] \setminus M$, and we will proceed under the assumption that the minor arc integral $I_m(P)$ can be ignored. Actually we will also ignore the contribution that this term makes once it is summed up over values of $k$. In truth there will be several points in the argument where we will simply ignore subsidiary contributions. We will indicate all of these by an appearance of the word “error”. Thus, to begin with, we have

$$N_U(B) = \frac{1}{2} \sum_{k=1}^\infty \mu(k) I_M(P) + \text{error}. $$
Here we have made the further assumption that the contribution \( N_{S \setminus U}(B) \) from the points lying on lines in \( S \) arises in the minor arc integral.

Let \( a, q \in \mathbb{Z} \) such that \( (2.14) \) holds, and let \( \alpha = a/q + z \in \mathcal{M}(a, q) \). Let \( C(x) \) denote the diagonal cubic form in \( (2.1) \). We now break the sum into congruence classes modulo \( q \), giving

\[
S(a/q + z) = \sum_{r \mod q} e(aC(r)/q) \sum_{x \in \mathbb{Z}^4 \cap [-P, P]^4 \mod q} e(zC(x)).
\]

We would like to replace the discrete variable \( x \) by a continuous one in the inner sum, and the summation over \( x \) by an integral. For this we will appeal to the following general result.

**Lemma 2.2.** Let \( P \geq 1 \), let \( a \in \mathbb{Z}^n \) and let \( r \in \mathbb{N} \) such that \( r \leq P \). Let \( F \) be a function on \( \mathbb{R}^n \) all of whose first order partial derivatives exist and are continuous on \( \mathbb{R}^n := [-P, P]^n \). Define

\[
M_F := \sup_{x \in \mathbb{R}^n} \max_{1 \leq i \leq n} |\frac{\partial F}{\partial x_i}(x)|.
\]

Then we have

\[
\sum_{x \in \mathbb{Z}^n \cap \mathbb{R}^n} e(F(x)) = \frac{1}{r^n} \int_{\mathbb{R}^n} e(F(t)) \, dt + O\left(\frac{P^{n-1}(1 + PM_F)}{r^{n-1}}\right).
\]

**Proof.** Our proof of Lemma 2.2 is based on the Euler–Maclaurin summation formula [60, §I.0]. Let \( B_k(x) \) denote the \( k \)-th Bernoulli polynomial, for \( k \in \mathbb{Z} \geq 0 \), and let \( s \in \mathbb{Z} \geq 0 \). Let \( A, B \in \mathbb{Z} \), with \( A < B \). For any function \( f: \mathbb{R} \to \mathbb{C} \) whose \( (s + 1) \)-th derivative \( f^{(s+1)} \) exists and is continuous on the interval \( [A, B] \), the Euler–Maclaurin summation formula states that

\[
\sum_{A < n \leq B} f(n) = \int_A^B f(t) \, dt + \sum_{k=0}^s \frac{(-1)^{k+1}B_{k+1}(0)}{(k+1)!} (f^{(k)}(B) - f^{(k)}(A))
\]

\[
+ \frac{(-1)^s}{(s+1)!} \int_A^B B_{s+1}(t) f^{(s+1)}(t) \, dt.
\]

(2.16)

Let \( a \in \mathbb{Z} \) and \( r \in \mathbb{N} \). We will apply this result with \( f_0(x) = f(a + rx) \) and

\[
A_0 = \frac{A - a}{r}, \quad B_0 = \frac{B - a}{r}.
\]

Taking \( s = 0 \) in the Euler–Maclaurin formula, we therefore deduce that

\[
\sum_{A < n \leq B \atop n \equiv a \mod r} f(n) = \frac{1}{r} \int_A^B f(t) \, dt - \frac{f(B_0) - f(A_0)}{2} + \int_A^B B_1 \left(\frac{t - a}{r}\right) f'(t) \, dt,
\]

(2.17)

since \( B_1(x) = x - [x] - \frac{1}{2} \).

We are now ready to establish Lemma 2.2, which we will do by induction on \( n \). Write \( S_n \) for the \( n \)-dimensional sum that is to be estimated. The case
$n = 1$ of Lemma 2.2 follows from (2.17) with $f(x) = e(F(x))$. Assuming now that $n \geq 2$, we have

$$S_n = \sum_{y \in \mathbb{Z} \cap [-P,P]} \sum_{y \equiv a_1 \mod r} e(G(x_2, \ldots, x_n)),$$

where $G(x_2, \ldots, x_n) = F(y, x_2, \ldots, x_n)$, and the sum over $x_2, \ldots, x_n$ is over all integers in $[-P, P]$ such that $x_i \equiv a_i \mod r$ for $2 \leq i \leq n$. We may employ the induction hypothesis to estimate the inner sum in $n - 1$ variables. It therefore follows that

$$S_n = \sum_{y \in \mathbb{Z} \cap [-P,P]} \frac{1}{r^{n-1}} \int_{[-P,P]^{n-1}} e(G(t_2, \ldots, t_n))dt_2 \cdots dt_n$$

$$+ O\left(\frac{P^{(n-2)}(1 + PM_G)}{r^{n-2}}\right).$$

Now there are $O(P/r)$ integers $y$ in the interval $[-P, P]$ that are congruent to $a_1$ modulo $r$, since $r \leq P$ by assumption. Moreover, it is clear that $M_G \leq M_F$. Hence

$$S_n = \frac{1}{r^{n-1}} \sum_{y \in \mathbb{Z} \cap [-P,P]} f(y) + O\left(\frac{P^{(n-1)}(1 + PM_F)}{r^{n-1}}\right),$$

where

$$f(y) = \int_{[-P,P]^{n-1}} e(F(y, t_2, \ldots, t_n))dt_2 \cdots dt_n.$$

The statement of Lemma 2.2 is now an easy consequence of (2.17). \qed

It is clear from the proof of Lemma 2.2 that when $F$ has partial derivatives to a higher order, one may obtain a much sharper estimate by including higher order terms in the Euler–Maclaurin summation formula. The present bound is satisfactory for our purposes, however.

Returning to (2.15) we apply Lemma 2.2 with

$$F(x) = zC(x), \quad n = 4, \quad a = r, \quad r = q.$$

In particular we have $q \leq P^{\varepsilon} \leq P$, as required for the lemma. Furthermore, $M_F = M_{zC} \ll |z|^2P \leq P^{-1+\varepsilon}$ for any $\alpha = a/q + z \in \mathfrak{M}(a, q)$. It follows that

$$S(a/q + z) = q^{-4}T(a, q)V_P(z) + O(P^{3+2\varepsilon})$$

on the major arcs, where

$$T(a, q) := \sum_{r \mod q} e(aC(r)/q), \quad V_R(z) := \int_{[-R,R]^4} e(zC(x))dx. \quad (2.18)$$

The set of major arcs has $\text{meas}(\mathfrak{M}) = O(P^{3+3\varepsilon})$. On carrying out the integration over $z$ and the summation over $a$ and $q$ one is therefore led to
the conclusion that
\[ I_{20}(P) = P^4 \sum_{q \leq P^\varepsilon} \sum_{1 \leq a \leq q \atop \gcd(a,q) = 1} q^{-4}T(a,q) \int_{|z| \leq P^{-3+\varepsilon}} V_1(z P^3) \, dz + O(P^{5\varepsilon}). \]
\[ = P \sum_{q \leq P^\varepsilon} \sum_{1 \leq a \leq q \atop \gcd(a,q) = 1} q^{-4}T(a,q) \int_{|z| \leq P^\varepsilon} V_1(z) \, dz + O(P^{5\varepsilon}). \]

Define
\[ \mathcal{I}(R) := \int_{|z| \leq R} V_1(z) \, dz = \int_{|z| \leq R} \int_{[-1,1]^4} e(zC(x)) \, dx \, dz. \]
It can be shown that \( \mathcal{I}(R) \) is a bounded function of \( R \), and furthermore, \( \mathcal{I}(R) \to \mathcal{I}_0 > 0 \) as \( R \to \infty \). A standard calculation reveals that the limit \( \mathcal{I}_0 \) is equal to \( 2\sigma_\infty \), where
\[ \sigma_\infty := \frac{1}{6a_1^{4/3}} \int \frac{dx_1 dx_2 dx_3}{(a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3)^{2/3}} \]
is the real density of solutions. Here the integral is over \( x_1, x_2, x_3 \in [-1,1] \) such that \( |(a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3)/a_4| \leq 1 \). We now define
\[ \mathfrak{S}(R) := \sum_{q \leq R} q^{-4} \sum_{1 \leq a \leq q \atop \gcd(a,q) = 1} T(a,q). \]
On bringing everything together, our investigation has so far succeeded in showing that
\[ N_U(B) = \sigma_\infty \sum_{k=1}^\infty \mu(k) P \mathfrak{S}(P^\varepsilon) + \text{error}, \quad (2.19) \]
for a suitably small value of \( \varepsilon > 0 \), where \( P = B/k \).

Our task is now to examine the sum \( \mathfrak{S}(R) \), as \( R \to \infty \). Let us write
\[ S_q := \sum_{1 \leq a \leq q \atop \gcd(a,q) = 1} T(a,q), \]
where \( T(a,q) \) is the complete exponential sum defined in \( (2.18) \). Then \( \mathfrak{S}(R) = \sum_{q \leq R} q^{-4} S_q \). It turns out that \( S_q \) is a multiplicative function of \( q \). This can be established along the lines of \( [23, \text{Lemma 5.1}] \). Recall the definition \( (2.9) \) of \( N(q) \). We now come to the key relation between \( S_q \) and \( N(q) \) at prime power values of \( q \).

**Exercise 12.** Let \( p \) be a prime and let \( \varepsilon \geq 1 \). Use Lemma \( (2.1) \) to show that
\[ S_{p^\varepsilon} = p^\varepsilon N(p^\varepsilon) - p^{3+\varepsilon} N(p^{\varepsilon - 1}). \]

Let us for the moment ignore considerations of convergence, and consider the local factors \( \sum_{e=0}^\infty p^{-4e} S_{p^e} \) in the infinite product formula for \( \mathfrak{S}(\infty) \). Now it follows from Exercise \( 12 \) that
\[ \sum_{e=0}^E p^{-4e} S_{p^e} = 1 + \sum_{e=1}^E (p^{-3e} N(p^e) - p^{-3-3e} N(p^{e-1})) = p^{-3E} N(p^E), \]
for any $E \geq 1$. Hence, formally speaking, we have $\mathcal{G}(\infty) = \prod_p \tau_p$, where
\[
\tau_p := \lim_{e \to \infty} p^{-3e} N(p^e).
\]
If $\mathcal{G}(R)$ was convergent, which it certainly is not in general, we could then conclude from (2.19) that
\[
N_U(B) = B \sigma_\infty \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \prod_p \tau_p + \text{error}.
\]
(2.20)

Arguing formally, we now replace the summation over $k$ by its Euler product, concluding that
\[
N_U(B) = B \sigma_\infty \prod_p \sigma_p + \text{error},
\]
(2.21)
with $\sigma_p := (1 - 1/p) \tau_p$. Note that
\[
\sigma_p = \lim_{e \to \infty} p^{-3e} N^*(p^e),
\]
(2.22)
in the notation of (2.10), which clearly follows from the observation that
\[
N^*(p^e) = N(p^e) - 8N(p^{e-3}),
\]
for any $e > 3$. The estimate in (2.21) therefore gives a heuristic asymptotic formula for $N_U(B)$ which is visibly a product of local densities. Among other things, we have assumed that $\mathcal{G}(R)$ is convergent in formulating this heuristic. This is expected to be true for diagonal cubic surfaces whose Picard group has rank 1, but not in general.

One can make the transition from (2.20) to (2.21) completely rigorous if the Hardy–Littlewood heuristic produces a leading term involving $P^\alpha$ with exponent $\alpha > 1$. The outcome is that to go from counting points on the affine cone to counting projective points, one merely replaces $N(p^e)$ by $N^*(p^e)$. For $\alpha = 1$, however, the “renormalization” procedure remains heuristic. Let
\[
\mathcal{G}^*(R) := \sum_{q \leq R} q^{-4} S_q^*;
\]
where
\[
S_q^* := \sum_{1 \leq a \leq q \atop \gcd(a,q) = 1} \sum_{r \mod q \atop \gcd(r,q) = 1} e(aC(r)/q).
\]
It is easily checked that $S_q^*$ is a multiplicative function of $q$, and furthermore, that the corresponding version of Exercise 12 holds, relating $S_p^*$ to $N^*(p^e)$. In fact, formally speaking, one has
\[
\prod_p \sum_{e=0}^{\infty} p^{-4e} S_p^* = \prod_p \sigma_p,
\]
with $\sigma_p$ given by (2.22). Bearing all of this in mind we will proceed under the bold assumption that (2.19) can be replaced by
\[
N_U(B) = B \sigma_\infty \mathcal{G}^*(B) + \text{error},
\]
(2.23)
where we have taken $\varepsilon = 1$ in the expressions for $\mathcal{G}(B^\varepsilon)$ and $\mathcal{G}^*(B^\varepsilon)$. 

We now turn to a finer analysis of $\mathcal{G}^*(B)$, as $B \to \infty$. Our task is to determine the analytic properties of the corresponding Dirichlet series

$$F(s) := \sum_{q=1}^{\infty} \frac{S_q^*}{q^s}$$

for $s = \sigma + it \in \mathbb{C}$. Armed with this analysis we will ultimately apply Perron’s formula to obtain an estimate for $\mathcal{G}^*(R)$. Using the multiplicativity of $q^{-s}S_q^*$ we deduce that

$$F(s) = \prod_p \sigma_p(s), \quad \sigma_p(s) := \sum_{e=0}^{\infty} p^{-es} S_{pe}^*.$$  

(2.25)

In examining $F(s)$ it clearly suffices to ignore the value of the factors $\sigma_p(s)$ at any finite collection of primes $p$. With this in mind we will try and determine $\sigma_p(s)$ for $p \notin \mathcal{P}$, where $\mathcal{P}$ is given by (2.2). Recall the definition (2.10) of $N^*(q)$.

**Exercise 13.** Let $e \geq 1$ and let $p \notin \mathcal{P}$ be a prime. Use Hensel’s lemma to show that $N^*(p^e) = p^{3e-3}N^*(p)$.

It therefore follows from Exercise 13 that

$$\sigma_p(s) = 1 + \sum_{e=1}^{\infty} p^{-es} (p^eN^*(p^e) - p^{3+e}N^*(p^{e-1})) = 1 - \frac{1}{p^{s-4}} + \frac{N^*(p)}{p^{s-1}},$$

for any $p \notin \mathcal{P}$. Hence (2.11) yields

$$\sigma_p(s) = 1 + \frac{\delta_p(a)}{p^{s-3}} - \frac{\delta_p(a)}{p^{s-2}} - \frac{1}{p^{s-1}},$$

where $\delta_p(a)$ is given by (2.13).

We now pursue our analysis in the special case $a = (1, 1, 1, 1)$ of the Fermat cubic surface (2.5). Now it is clear from (2.13) that $\delta_p(1, 1, 1, 1) = 0$ if $p \equiv 2 \bmod 3$ and

$$\delta_p(1, 1, 1, 1) = 3\nu_p(1, 1, 1, 1) - 3 = 6$$

if $p \equiv 1 \bmod 3$. Let $\lambda : \mathbb{Z} \to \mathbb{C}$ be the real Dirichlet character of order 2 defined by

$$\lambda(n) := \begin{cases} \left( \frac{n}{3} \right), & \text{if } 3 \nmid n, \\ 0, & \text{otherwise}, \end{cases}$$

where $\left( \frac{n}{3} \right)$ is the Legendre symbol. Then we may write

$$\sigma_p(s) = 1 + \frac{3(1 + \lambda(p))}{p^{s-3}} - \frac{3(1 + \lambda(p))}{p^{s-2}} - \frac{1}{p^{s-1}}$$

$$= \left( 1 - \frac{1}{p^{s-3}} \right)^{-3} \left( 1 - \frac{\lambda(p)}{p^{s-3}} \right)^{-3} \left( 1 + O\left( \frac{1}{p^{\min\{\sigma-2, 2\sigma-2\}}} \right) \right),$$

for any $p \notin \mathcal{P}$. Let $L(s, \lambda)$ denote the usual Dirichlet $L$-function associated to $\lambda$. When $a = (1, 1, 1, 1)$ we have therefore succeeded in showing that

$$F(s) = \zeta(s-3)^3L(s-3, \lambda)^3G(s),$$

(2.26)
where \( G(s) \) is a function that is holomorphic and bounded on the half-plane \( \sigma \geq 7/2 + \delta \), for any \( \delta > 0 \). For future reference we note that \( G(4) \) has local factors

\[
G_p(4) = \begin{cases} 
(1 - \frac{1}{p})^7(1 + \frac{7}{p} + \frac{1}{p^2}), & \text{if } p \equiv 1 \mod 3, \\
(1 - \frac{1}{p})^4(1 + \frac{1}{p})^3(1 + \frac{1}{p} + \frac{1}{p^2}), & \text{if } p \equiv 2 \mod 3.
\end{cases}
\]  

(2.27)

Although we will not prove it here, it can be deduced from (2.22) and Hensel’s lemma that

\[
G_3(4) = \frac{8}{27} \lim_{\epsilon \to \infty} 3^{-3\epsilon} N^*(3^\epsilon) = \frac{16}{27}.
\]  

(2.28)

We are now ready for our application of Perron’s formula, which we will apply in the following form.

**Lemma 2.3.** Let \( F(s) = \sum_{n=1}^{\infty} a_n n^{-s} \) be a Dirichlet series with abscissa of absolute convergence \( \sigma_a \). Suppose that \( x \not\in \mathbb{Z} \) and let \( c > \sigma_a \). Then we have

\[
\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s} \, ds + O\left( \frac{x^c}{T} \sum_{n=1}^{\infty} \frac{|a_n| n^{-c}}{|\log(x/n)|} \right),
\]

for any \( T \geq 1 \).

**Proof.** Let \( c > 0 \). The lemma follows from the identity

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \, ds = \begin{cases} 
1 + O(x^c(T |\log x|)^{-1}), & \text{if } x > 1, \\
\frac{1}{x} + O(c^{-1}), & \text{if } x = 1, \\
O(x^c(T |\log x|)^{-1}), & \text{if } 0 < x < 1,
\end{cases}
\]

which is a straightforward exercise in contour integration. \( \square \)

In our case we have \( a_q = S_q^* \) and we are interested in the Dirichlet series \( F(s + 4) \), in the notation of (2.21). In order to apply Lemma 2.3 we will need an upper bound for this quantity. For our purposes the trivial upper bound \( a_q \ll q^3 \) is sufficient. Thus the Dirichlet series \( F(s + 4) \) is absolutely convergent for \( \sigma > 2 \). Taking \( c = 2 + \epsilon \) for any \( \epsilon > 0 \), we may deduce from Lemma 2.3 that

\[
S^*(B) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s + 4) \frac{B^s}{s} \, ds + O\left( \frac{B^c}{T} \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon} |\log(B/n)|} \right),
\]

for any \( T \geq 1 \) and any \( B \not\in \mathbb{Z} \). It is not hard to see that the error term here is \( \ll T^{-1}B^c \). We now apply Cauchy’s residue theorem to the rectangular contour \( C \) joining \( c' - iT, c' + iT, c + iT \) and \( c - iT \), where \( c' = -1/2 + \epsilon \). The relation (2.26) implies that in this region \( F(s + 4)B^s/s \) has a unique pole at \( s = 0 \), and it is a pole of order 4. It has residue

\[
\text{Res}_{s=0} \frac{F(s + 4)B^s}{s} = \frac{L(1, \lambda)^3 G(4) P(\log B)}{3!},
\]

where \( P \in \mathbb{R}[x] \) is a monic polynomial of degree 3. Putting all of this together we have therefore shown that

\[
S^*(B) = \frac{L(1, \lambda)^3 G(4) P(\log B)}{3!} + O(E(B)),
\]  

(2.29)
where
\[
E(B) = \frac{B^c}{T} + \left( \int_{c-iT}^{c+iT} + \int_{c-iT}^{c+iT} + \int_{c-iT}^{c+iT} \right) |H(s)^3\frac{B^s}{s}| ds,
\]
for any \( T \geq 1 \), and where \( H(s) = \zeta(s + 1)L(s + 1, \lambda) \). Here we have used the fact that \( G(s + 4) \) is bounded on the half-plane \( \Re(s) \geq c' \).

To make the analysis simpler, it will be convenient to proceed under the assumption that the Lindelöf hypothesis holds for \( \zeta(s) \), and also for the Dirichlet \( L \)-function \( L(s, \lambda) \). This could be avoided at the cost of extra effort, but there seems no harm in supposing it here. Thus we may assume the bounds
\[
\zeta(\sigma + it) \ll_{\varepsilon} |t|^\varepsilon, \quad L(\sigma + it, \lambda) \ll_{\varepsilon} |t|^\varepsilon,
\]
for any \( \sigma \in [1/2, 1] \) and any \( |t| \geq 1 \). It therefore follows that
\[
H(\sigma + it) \ll_{\varepsilon} \begin{cases} |t|^\varepsilon, & \text{if } -1/2 \leq \sigma \leq 0, \\ 1, & \text{if } \sigma > 0, \end{cases}
\]
for any \( |t| \geq 1 \), which gives
\[
\int_{c-iT}^{c+iT} \left| H(s)^3\frac{B^s}{s} \right| ds \ll_{\varepsilon} \int_c^\infty B^{\sigma T - 1 + 3\varepsilon} d\sigma \ll_{\varepsilon} B^c T^{-1 + 3\varepsilon}.
\]

One obtains the same estimate for the contribution from the remaining horizontal contour. Turning to vertical integral, we find that
\[
\int_{c-iT}^{c+iT} \left| H(s)^3\frac{B^s}{s} \right| ds \ll_{\varepsilon} B^c \int_{-T}^{T} \frac{|H(1/2 + \varepsilon + it)|^3}{1 + |t|^3} dt
\]
\[
\ll_{\varepsilon} B^c \int_{-T}^{T} (1 + |t|)^{3\varepsilon - 1} dt
\]
\[
\ll_{\varepsilon} B^c T^{3\varepsilon},
\]
under the assumption of Lindelöf hypothesis. This shows that
\[
E(B) \ll_{\varepsilon} B^c T^{3\varepsilon} \left( \frac{B^2}{T} + \frac{1}{B^{1/2}} \right),
\]
for any \( T \geq 1 \). Taking \( T \) sufficiently large, we therefore conclude from (2.29) that
\[
\mathfrak{S}^e(B) = \frac{L(1, \lambda)^3 G(4) P(\log B)}{3!} + O(B^{-\Delta}),
\]
for some \( \Delta > 0 \).

We are now ready to return to the Hardy–Littlewood major arc analysis which led us to (2.23). Substituting in our estimate for \( \mathfrak{S}^e(B) \), we conclude that
\[
N_{U_1}(B) \sim c_1 B (\log B)^3
\]
(2.30)
where \( U_1 \subset S_1 \) is the usual open subset of the Fermat surface (2.5), and
\[
c_1 = \sigma_{\infty} L(1, \lambda)^3 G(4) \frac{1}{3!}.
\]
Now it follows from the class number formula that \( L(1, \lambda) = \pi \sqrt{3}/9 \). Hence, on combining this with (2.27) and (2.28), our heuristic argument has led us
to the expectation that \( (2.30) \) holds, with

\[
c_1 = \frac{\sigma_\infty 2^4 \pi^3 \sqrt{3}}{3! 3^8} \prod_{p \equiv 1 \text{ mod } 3} \left(1 - \frac{1}{p}\right)^7 \left(1 + \frac{7}{p} + \frac{1}{p^2}\right) \prod_{p \equiv 2 \text{ mod } 3} \left(1 - \frac{1}{p}\right)^3 \left(1 - \frac{1}{p^2}\right)^3.
\]

The exponents of \( B \) and \( \log B \) in \( (2.30) \) agree with the Manin conjecture, since we have already seen in \( (2.21) \) that the Picard group of \( S_1 \) has rank 4.

It is interesting to compare our analysis with the work of Peyre and Tschinkel \cite{34}, who calculate the leading constant \( c_{\text{Peyre}} \) in Peyre’s refinement \cite{31} of the conjectured asymptotic formula for \( N_{U_1}(B) \). It turns out that

\[
c_{\text{Peyre}} = \gamma(S_1)c_1,
\]

with \( \gamma(S_1) = 7/3 \). For a general non-singular cubic surface \( S \subset \mathbb{P}^3 \), the constant \( \gamma(S) \) is defined to be the volume

\[
\gamma(S) := \int_{\Lambda_{\text{eff}}(S)} e^{-(S.t)} dt.
\]

Thus, in general terms, \( \gamma(S) \) measures the volume of the polytope obtained by intersecting the dual of \( \Lambda_{\text{eff}}(S) \) with a certain affine hyperplane. In particular \( \gamma(S) \in \mathbb{Q} \) for any non-singular cubic surface \( S \), and \( \gamma(S) = 1 \) if and only if the corresponding Picard group has rank 1.

**Exercise 14.** Let \( S_2 \) denote the surface \( (2.4) \). Using a similar argument, show that one expects an asymptotic formula of the shape \( N_{S_2}(B) \sim c_2B \) for some constant \( c_2 \geq 0 \). Check your answer with the heuristic formula obtained by Heath-Brown \cite{36}.

### 3. The \( A_1 \) del Pezzo surface of degree 6

In this section we will establish Theorem 1.3. Any line in \( \mathbb{P}^6 \) is defined by the intersection of 5 hyperplanes. It is not hard to see that the equations

\[
\begin{align*}
  x_1 &= x_2 = x_3 = x_5 = x_6 = 0, \\
  x_3 &= x_5 = x_6 = x_1 + x_4 = x_1 + x_2 = 0,
\end{align*}
\]

all define lines contained in the singular del Pezzo surface \( S \) given by \((1.17)\). Table \ref{table} ensures that these are the only lines contained in \( S \). By definition \( U \) is the open subset of \( S \) on which none of these equations hold. We begin by establishing the following result.

**Lemma 3.1.** We have

\[
N_U(B) = 2M(B) + O(B),
\]

where \( M(B) \) denotes the number of \( x \in \mathbb{Z}^7 \) such that

\[
\begin{align*}
  x_1^2 - x_2 x_4 &= x_1 x_5 - x_3 x_4 = x_1 x_3 - x_2 x_5 = x_1 x_6 - x_3 x_5 \\
  &= x_2 x_6 - x_3^2 = x_4 x_6 - x_5^2 = x_1^2 - x_1 x_4 + x_5 x_7 \\
  &= x_1^2 - x_1 x_2 - x_3 x_7 = x_1 x_3 - x_1 x_5 + x_6 x_7 = 0,
\end{align*}
\]

with \( \gcd(x_1, \ldots, x_7) = 1 \), \( 0 < |x_1|, |x_2|, |x_3|, |x_4|, |x_5|, |x_6| \leq B \) and \( |x_7| \leq B \).
Proof. In view of the fact that $x$ and $-x$ represent the same point in $\mathbb{P}^6$, we have

\[ N_U(B) = \frac{1}{2} \# \{ x \in \mathbb{Z}^7 : |x| \leq B, \ (1.17) \text{ holds, but } (3.1) \text{ does not} \}, \]

where $\mathbb{Z}^7$ denotes the set of primitive vectors in $\mathbb{Z}^7$. We need to consider the contribution to the right hand side from points such that $x_i = 0$, for some $1 \leq i \leq 7$. Let us begin by considering the contribution from vectors $x \in \mathbb{Z}^7$ for which $x_1 = 0$. But then the equations in (1.17) imply that $x_2x_4 = 0$. If $x_2 = 0$, it is straightforward to check that either $x$ satisfies the first system of equations in (3.1), or else

\[ x_0 = x_2 = x_3 = x_7 = 0, \quad x_4x_6 = x_5^2. \]

Such points are therefore confined to a plane conic. We therefore obtain $O(B)$ points overall with $x_1 = x_2 = 0$. If on the other hand $x_1 = x_4 = 0$, then a similar analysis shows that there are $O(B)$ points in this case too. In view of the first equation in (1.17), the contribution from vectors $x$ such that $x_2x_4 = 0$ is also $O(B)$. Let us now consider the contribution from vectors $x$ such that $x_3 = 0$ and $x_1x_2x_4 \neq 0$. It is easily checked that the only such vectors have $x_5 = x_6 = 0$ and $x_1 + x_4 = x_1 + x_2 = 0$, and so must lie on a line contained in $S$. Finally, arguing in a similar fashion, we see that there are no points contained in $S$ with $x_5x_6 = 0$ and $x_1x_2x_3x_4 \neq 0$. We have therefore shown that

\[ N_U(B) = \frac{1}{2} \# \{ x \in \mathbb{Z}^7 : x_1 \cdots x_6 \neq 0, \ |x| \leq B, \ (3.2) \text{ holds} \} + O(B). \]

Here we have noted that there is an obvious unimodular transformation that takes the set of equations in (1.17) into (3.2).

We would now like to restrict our attention to positive values of $x_1, \ldots, x_6$. The equations for $S$ imply that $x_2, x_4, x_6$ all share the same sign. On absorbing the minus sign into $x_1$ there is a clear bijection between solutions to (3.2) with $x_2, x_4, x_6 < 0$ and solutions with $x_2, x_4, x_6 > 0$. We choose to count the former. Arguing similarly, by absorbing the minus signs into $x_1$ and $x_7$, we see that there is a bijection between the solutions to (3.2) with $x_3 < 0$ and $x_2, x_4, x_6 > 0$, and the solutions with $x_2, x_3, x_4, x_6 > 0$. Fixing our attention on the latter set of points, we therefore complete the proof of Lemma 3.1. \qed

Let $\tilde{S}$ denote the minimal desingularisation of the surface $S$. By determining the Cox ring associated to $\tilde{S}$, Derenthal \cite{26} has calculated the universal torsor above $\tilde{S}$. In this setting it is defined by a single equation

\[ s_1y_1 - s_2y_2 + s_3y_3 = 0, \quad (3.3) \]

embedded in $\mathbb{A}^7$. In particular one of the variables does not appear explicitly in the equation.

3.1. Elementary considerations. As promised in [1.6] we proceed to show how $N_U(B)$ can be related to a count of the integer points on the corresponding universal torsor, which in this case is given by (3.3). Our deduction of this fact is completely elementary, and is based on an analysis of the integer solutions to the system of equations (3.2). It is still somewhat
mysterious as to how or why this rather low-brow process should ultimately lead to the same outcome! Typical of the facts that we will employ is the following.

**Exercise 15.** Show that the general solution of the equation \( xy = z^2 \) is
\[
\begin{align*}
  x &= a^2c, \\
  y &= b^2c, \\
  z &= abc,
\end{align*}
\]
with \(|\mu(c)| = 1\).

Given \( s_0 \in \mathbb{R} \) and \( s = (s_1, s_2, s_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3 \), define
\[
\Psi(s_0, s, y) := \max \{|s_0 s_1^2 s_2^2 s_3^2|, |y_1 y_2 y_3|, |s_0 s_1^2 y_1^2|, |s_0 s_2^2 y_2^2|\}. \tag{3.4}
\]
We are now ready to record our translation of the problem to the universal torsor.

**Lemma 3.2.** We have
\[
N_U(B) = 2 \# \left\{ (s_0, s, y) \in \mathbb{Z}^7 : \begin{gathered}
  \Psi(s_0, s, y) \leq B, \tag{3.3} \\
  s_0, s_1, s_2, s_3, y_1 > 0, \tag{3.5}
  \gcd(y_1, s_0 s_1 s_3) = 1, \tag{3.6}
  \gcd(s_1, s_3) = 1
\end{gathered} \right\} + O(B),
\]
with \( i, j, k \) a permutation of \( 1, 2, 3 \) in the coprimality conditions.

**Proof.** Let \( x \in \mathbb{Z}^7 \) be a primitive vector counted by \( M(B) \), as defined in the statement of Lemma 3.1. Combining the first equation in (3.2) with Exercise 15 we see that
\[
x_1 = a_1 a_2 a_4, \quad x_2 = a_2^2 a_1, \quad x_4 = a_2^2 a_1,
\]
for integers \( a_1, a_2, a_4 \) such that \( a_1, a_2 > 0 \) and
\[
|\mu(a_1)| = \gcd(a_1 \gcd(a_2, a_4)^2, x_3, x_5, x_6, x_7) = 1.
\]
Inserting this into the equation \( x_2 x_6 = x_3^2 \) we deduce that \( a_1 a_2 | x_3 \), whence
\[
x_3 = a_1 a_2 a_3, \quad x_6 = a_1 a_3^2,
\]
for a positive integer \( a_3 \) such that
\[
|\mu(a_1)| = \gcd(a_1 \gcd(a_2, a_4)^2, a_1 a_3 \gcd(a_2, a_3), x_5, x_7) = 1.
\]
Substituting this into the equation \( x_4 x_6 = x_5 \), we deduce that
\[
x_5 = a_1 a_3 a_4,
\]
with
\[
|\mu(a_1)| = \gcd(a_1, x_7) = \gcd(a_2, a_3, a_4, x_7) = 1. \tag{3.5}
\]
Note that the second equation in (3.2) implies that \( x_1, x_5 \) must share the same sign, which here is the sign of \( a_4 \). The equations \( x_1 x_5 = x_3 x_4, x_1 x_3 = x_2 x_5 \) and \( x_1 x_6 = x_3 x_5 \) reveal no new information. Turning instead to the equation \( x_1^2 = x_1 x_4 - x_5 x_7 \), we obtain
\[
a_1 a_2^2 a_4 = a_1 a_2 a_4^2 - a_3 x_7. \tag{3.6}
\]
The coprimality conditions imply that \( a_1 | a_3 \). Moreover, we deduce from this equation that \( a_2 a_4 | a_3 x_7 / a_1 \). We may therefore write
\[
a_2 = a_2 a_7 a_2, \quad a_4 = a_4 a_7.
\]
for integers $a_{2i}, a_{4i}$, with $i = 3, 7$, such that $a_{2i}, a_{43}, |a_{47}| > 0$, and

$$a_1a_{23}a_{43} | a_3, \quad a_27a_{47} | x_7.$$  

Thus there exist further integers $b_3, a_7$ with $b_3 > 0$, such that

$$a_3 = a_1a_{23}a_{43}b_3, \quad x_7 = a_27a_{47}a_7,$$

with (3.5) and (3.6) becoming

$$|\mu(a_1)| = \gcd(a_1, a_27a_{47}a_7) = \gcd(a_{23}a_{27}, a_{23}a_{43}b_3, a_{43}a_{47}, a_27a_{47}a_7) = 1,$$

and

$$a_{23}a_{27} = a_{43}a_{47} - b_3a_7,$$

respectively. The final two equations are redundant. Let us write $d$ for the highest common factor of $a_{23}, a_{43}, b_3$. Thus

$$a_{23} = da'_{23}, \quad a_{43} = da'_{43}, \quad b_3 = db'_3,$$

for positive integers $d, a'_{23}, a'_{43}, b'_3$. On making these substitutions the equation remains the same, but with appropriate accents added, whereas the coprimality conditions become

$$|\mu(a_1)| = \gcd(da_1, a_27a_{47}a_7) = \gcd(a'_{23}a_{27}, a'_{23}a'_{43}b'_3, a'_{43}a_{47}, a_27a_{47}a_7)$$

$$\quad = \gcd(a'_{23}, a'_{43}, b'_3) = 1.$$

Now any $n \in \mathbb{N}$ can be written uniquely in the form $n = ab^2$ for $a, b \in \mathbb{N}$ such that $|\mu(a)| = 1$. We may therefore make the change of variables

$$(s_0; s_1, s_2, s_3; y_1, y_2, y_3) = (a_1d^2; a'_{23}, a'_{43}, b'_3; a_{27}, a_{47}, a_7).$$

Bringing everything together, we have therefore established the existence of $(s_0, s, y) \in \mathbb{Z}^7$ such that (3.3) holds, with

$$s_0, s_1, s_2, s_3, y_1 > 0, \quad (3.7)$$

and

$$\gcd(s_0, y_1y_2y_3) = \gcd(s_1, s_2, s_3) = \gcd(s_1y_1, s_2y_2, s_1s_2s_3, y_1y_2y_3) = 1.$$  

Note that $y_2$ is automatically non-zero for $s_0, s, y$ satisfying the remaining conditions. Once combined with (3.3), it is easy to check that the latter coprimality conditions are equivalent to the conditions

$$\begin{cases}  
\gcd(y_1, s_0s_2s_3) = \gcd(y_2, s_0s_1s_3) = \gcd(y_3, s_0s_1s_2) = 1, \\
\gcd(s_1, s_2) = \gcd(s_1, s_3) = \gcd(s_2, s_3) = 1,  
\end{cases} \quad (3.8)$$

that appear in the statement of the lemma.

At this point we may summarise our argument as follows. Let $T \subset \mathbb{Z}^7$ denote the set of $(s_0, s, y) \in \mathbb{Z}^7$ such that (3.3), (3.7) and (3.8) hold. Then for any primitive vector $x$ counted by $M(B)$, we have shown that there
exists \((s_0, s, y) \in T\) such that
\[
\begin{align*}
x_1 &= s_0 s_1 s_2 y_1 y_2, \\
x_2 &= s_0 s_1^2 y_1^2, \\
x_3 &= s_0 s_1^2 s_2 s_3 y_1, \\
x_4 &= s_0 s_2 y_2^2, \\
x_5 &= s_0^2 s_1^2 s_2 s_3 y_2, \\
x_6 &= s_0^2 s_1^2 s_2 s_3, \\
x_7 &= y_1 y_2 y_3.
\end{align*}
\]

Conversely, we leave it as an exercise to check that any \((s_0, s, y) \in T\) produces a primitive point \(x \in \mathbb{Z}^7\) such that \((3.2)\) holds, with
\[
|x_1|, x_2, x_3, x_4, |x_5|, x_6 > 0.
\]

We may now conclude that \(M(B)\) is equal to the number of \((s_0, s, y) \in T\) such that
\[
\max_{i=1,2} \{|s_0 s_1 s_2 y_1 y_2|, |s_0 s_1^2 y_1^2|, |s_0^2 s_1 s_2 s_3 y_1|, |s_0^2 s_1^2 s_2 s_3|, |y_1 y_2 y_3|\} \leq B.
\]

In view of the fact that \(|s_0 s_1 s_2 y_1 y_2| = \sqrt{s_0 s_1^2 y_1^2} \sqrt{s_0^2 s_2 s_3 y_1^2}|, and furthermore,\(|s_0^2 s_1 s_2 s_3 y_1| = \sqrt{s_0^2 s_1 s_2 s_3} \sqrt{s_0^2 s_2 s_3 y_1^2}|, it follows that this height condition is equivalent to \(\Psi(s_0, s, y) \leq B\) for any \((s_0, s, y) \in T\), where \(\Psi\) is given by \((3.4)\). In summary we have therefore shown that \(M(B)\) is equal to the number of \((s_0, s, y) \in T\) such that \(\Psi(s_0, s, y) \leq B\). Once inserted into Lemma \((3.1)\) this completes the proof of Lemma \((3.2)\) \(\Box\)

At first glance it might seem a little odd that the height restriction \(|s_0 s_2 y_3| \leq B\) doesn’t explicitly appear in the lemma. However, \((3.3)\) implies that \(0 < s_1 y_1 = s_2 y_2 - s_3 y_3\) for any \((s_0, s, y) \in T\), whence the restriction \(\Psi(s_0, s, y) \leq B\) is plainly equivalent to \(\max\{|s_0 s_2 y_3|, \Psi(s_0, s, y)| \leq B\). We have preferred not to include it explicitly in the statement of Lemma \((3.2)\) however.

3.2. The asymptotic formula. Our starting point is Lemma \((3.2)\). Let \(T(B)\) denote the quantity on the right hand side that is to be estimated. Once taken together with \((3.3)\), the height condition \(\Psi(s_0, s, y) \leq B\) is equivalent to
\[
\max_{i=1,2} \{|s_0^3 s_1^2 s_2 s_3|, |s_0 s_1^2 y_1^2|, |y_1 y_2 (s_1 y_1 - s_2 y_2)/ s_3|\} \leq B.
\]

Define
\[
X_0 := \left(\frac{s_0^3 s_1^2 s_2 s_3}{B}\right)^{1/3}, \quad X_i := \left(\frac{s_1 s_2 s_3 B}{s_i^3}\right)^{1/3},
\]
for \(i = 1, 2\). Then the height conditions above can be rewritten as
\[
|X_0| \leq 1, \quad |f_1(y_1)| \leq 1, \quad |f_2(y_2)| \leq 1, \quad |g(y_1, y_2)| \leq 1,
\]
where
\[
f_i(y) := X_0 \left(\frac{y}{X_i}\right)^2, \quad g(y_1, y_2) := \frac{y_1 y_2}{X_1 X_2} \left(\frac{y_1}{X_1} - \frac{y_2}{X_2}\right)
\]
for \(i = 1, 2\). In order to count solutions to the equation \((3.3)\), our plan will be to view the equation as a congruence
\[
s_1 y_1 - s_2 y_2 \equiv 0 \pmod{s_3},\]
which has the effect of automatically taking care of the summation over \( y_3 \). In order to make this approach viable we will need to first extract the coprimality conditions on the \( y_3 \) variable.

Define the set
\[
S := \{(s_0, s) \in \mathbb{N}^4 : \gcd(s_i, s_j) = 1, \; X_0 \leq 1\},
\]
with \( i, j \) generic indices from the set \{1, 2, 3\}. We now apply Möbius inversion, as in Exercise 2, in order to remove the coprimality condition \( \gcd(y_3, s_0s_1s_2) = 1 \). Thus we find that
\[
T(B) = \sum_{(s_0, s) \in S} \sum_{k_3|s_0} \mu(k_3) \# \begin{cases}
\gcd(y_1, s_0s_2s_3) = 1, \\
\gcd(y_2, s_0s_1s_3) = 1,
\end{cases}
\]
\[
y \in \mathbb{Z}^3 : \begin{cases}
y_1 > 0, \\
s_1y_1 - s_2y_2 + k_3s_3y_3 = 0, \\
|f_i(y_i)| \leq 1, \; |g(y_1, y_2)| \leq 1
\end{cases}
\]

Now it is clear that the summand vanishes unless \( \gcd(k_3, s_1s_2) = 1 \). Hence
\[
T(B) = \sum_{(s_0, s) \in S} \sum_{k_3|s_0 \atop \gcd(k_3, s_1s_2) = 1} \mu(k_3)S_{k_3}(B),
\]
where
\[
S_{k_3}(B) := \# \begin{cases}
\gcd(y_1, s_0s_2s_3) = 1, \\
\gcd(y_2, s_0s_1s_3) = 1,
\end{cases}
\]
\[
y_1, y_2 \in \mathbb{Z} : \begin{cases}
y_1 > 0, \\
s_1y_1 \equiv s_2y_2 \mod k_3s_3, \\
|f_i(y_i)| \leq 1, \; |g(y_1, y_2)| \leq 1
\end{cases}
\]

Clearly \( S_{k_3}(B) \) depends on the parameters \( s_0 \) and \( s \), in addition to \( k_3 \) and \( B \).

We now turn to the estimation of \( S_{k_3}(B) \), for which we need the following basic result.

**Exercise 16.** Let \( b \geq a \) and \( q > 0 \). Show that
\[
\#\{n \in \mathbb{Z} \cap (a, b) : n \equiv n_0 \mod q\} = \frac{b - a}{q} + O(1).
\]

We will fix \( y_2 \) and apply Exercise 16 to handle the summation over \( y_1 \). Before this we must use Möbius inversion to remove the coprimality condition \( \gcd(y_1, s_0s_2s_3) = 1 \) from the summand. Thus we find that
\[
S_{k_3}(B) = \sum_{k_3|s_0s_2s_3} \mu(k_3) \# \begin{cases}
\gcd(y_2, s_0s_1s_3) = 1, \\
k_3s_1y_1 \equiv s_2y_2 \mod k_3s_3, \\
|f_1(k_1y_1)| \leq 1, \; |f_2(y_2)| \leq 1, \\
|g(k_1y_1, y_2)| \leq 1, \; y_1 > 0
\end{cases}
\]

In view of the other coprimality conditions, the summand plainly vanishes unless \( \gcd(k_1, k_3s_3) = 1 \). We may therefore write \( \rho \in \mathbb{Z} \) for the (unique) inverse of \( k_1s_1 \) modulo \( k_3s_3 \), whence
\[
S_{k_3}(B) = \sum_{k_1|s_0s_2 \atop \gcd(k_1, k_3s_3) = 1} \mu(k_1)S_{k_1,k_3}(B),
\]

(3.11)
An application of Exercise 16 now reveals that

\[
S_{k_1,k_3}(B) = \sum_{\substack{y_2 \in \mathbb{Z}: |f_2(y_2)| \leq 1 \\ \gcd(y_2, s_0 s_3) = 1}} \left( \frac{X_1 F_1(X_0, y_2/X_2)}{k_1 k_3 s_3} + O(1) \right),
\]

where

\[
F_1(u, v) := \int_{\{t \in \mathbb{R}_{\geq 0}: |ut^2|, |uv(t-v)| \leq 1\}} dt.
\]

We close this section by showing that once summed over all \((s_0, s, y_2) \in \mathbb{N}^5\), the error term in (3.12) makes a satisfactory overall contribution to the error term in Theorem 1.3. Using the fact that \(\sum_{k|n} |\mu(k)| = 2^{\omega(n)}\), we find that this contribution is

\[
\ll \sum_{(s_0, s) \in S} \frac{4^{\omega(s_0)} 2^{\omega(s_2)}}{s_0^{1/2} s_2^{1/2}} X_2 = B^{1/2} \sum_{(s_0, s) \in S} \frac{4^{\omega(s_0)} 2^{\omega(s_2)}}{s_0^{1/2} s_2^{1/2}} \ll B \sum_{s_0, s_2 \in \mathbb{N}} \frac{4^{\omega(s_0)} 2^{\omega(s_2)}}{s_0^{1/2} s_2^{1/2}} \ll B \log B.
\]

This is satisfactory for Theorem 1.3 and so we may henceforth ignore the error term in the above estimate for \(S_{k_1,k_3}(B)\).

Define the arithmetic function

\[
\phi^*(n) := \prod_{p|n} \left(1 - \frac{1}{p}\right),
\]

where as is common convention the product is over distinct prime divisors of \(n\). It will be useful to note that

\[
\phi^*(mn) = \frac{\phi^*(m) \phi^*(n)}{\phi^*(\gcd(m, n))},
\]

for any \(m, n \in \mathbb{N}\). We must now sum over the variable \(y_2\), for which we will employ the following basic result.

**Exercise 17.** Let \(I \subset \mathbb{R}\) be an interval, let \(a \in \mathbb{N}\) and let \(f : \mathbb{R} \to \mathbb{R}_{\geq 0}\) be a function that is continuously differentiable on \(I\). Use (2.16) to show that

\[
\sum_{\substack{n \in \mathbb{Z}/I \\ \gcd(n,a)=1}} f(n) = \phi^*(a) \int_I f(t)dt + O\left(2^{\omega(a)} \sup_{t \in I} |f(t)| \right).
\]

We may now return to (3.12). Using Exercise 17, we deduce that

\[
S_{k_1,k_3}(B) = \frac{\phi^*(s_0 s_1 s_3)}{k_1 k_3 s_3} X_1 X_2 F_2(X_0) + O\left(2^{\omega(s_0 s_1 s_3)} X_1/k_1 k_3 s_3 \right),
\]

(3.14)
where
\[ F_2(u) := \int_{\{t,v \in \mathbb{R} : t > 0, |tu^2|,|u^2v|,|tv(t-v)| \leq 1\}} dt dv. \]

We must now estimate the overall contribution to \( N_U(B) \) from the error term in this estimate, once summed up over the remaining variables. This gives
\[
\leq \sum_{(s_0,s) \in S} 4^{\omega(s_0)} 2^{\omega(s_2)} 2^{\omega(s_0s_1s_3)} X_1 \leq B^{1/3} \sum_{s_0,s_1,s_2,s_3 \in \mathbb{N}} \frac{4^{\omega(s_0)} 2^{\omega(s_1s_2s_3)} s_2^{1/3}}{s_1^{2/3} s_3^{2/3}} \leq B \log B,
\]
by summing over \( s_2 \leq \sqrt{B/(s_0^2s_1^2s_3^2)} \). This is satisfactory for Theorem 1.3 and so we may henceforth ignore the error term in (3.14). As pointed out to the author by Régis de la Bretèche, it is easy to sharpen this error term to \( O(B) \) using the fact that \( \phi^* \) has constant average order.

Now it is trivial to check that
\[
\sum_{d|n \atop \gcd(d,a)=1} \frac{\mu(d)}{d} = \frac{\phi^*(n)}{\phi^*(\gcd(a,n))},
\]
for any \( a,n \in \mathbb{N} \). Bringing together (3.10), (3.11) and (3.14), we conclude that
\[
T(B) = \sum_{(s_0,s) \in S} \sum_{k_3|s_0} \frac{\mu(k_3)}{k_3} \frac{\phi^*(s_0s_2)\phi^*(s_0s_1s_3)}{\phi^*(\gcd(k_3s_3,s_0s_2))} \frac{X_1 X_2 F_2(X_0)}{s_3},
\]
where \( S \) is given by (3.9). It is clear that \( \gcd(k_3s_3,s_0s_2) = \gcd(k_3s_3,s_0) \). Let us define the arithmetic function
\[
\vartheta(s_0,s) = \frac{\phi^*(s_0s_2)\phi^*(s_0s_1s_3)}{\phi^*(\gcd(s_0,s_3))} \sum_{k_3|s_0 \atop \gcd(k_3,s_1s_2)=1} \frac{\mu(k_3)}{k_3} \frac{\phi^*(\gcd(k_3,s_0,s_3))}{\phi^*(\gcd(k_3,s_0))}
\]
when \( \gcd(s_i,s_j) = 1 \) for \( 1 \leq i < j \leq 3 \), and \( \vartheta(s_0,s) = 0 \) otherwise. It follows from (3.13) that
\[
\vartheta(s_0,s) = \phi^*(s_0s_2)\phi^*(s_0s_1s_3) \prod_{p|\gcd(s_0,s_3) \atop p|s_0} \left( 1 - \frac{1}{p} \right) \prod_{p|s_0 \atop p|s_1s_2s_3} \left( 1 - \frac{2}{p} \right)
\]
\[
= \phi^*(s_0s_2)\phi^*(s_0s_1s_3) \prod_{p|s_0 \atop p|s_1s_2s_3} \left( 1 - \frac{2}{p} \right),
\]
\[
= \phi^*(s_0)\phi^*(s_1s_2s_3) \prod_{p|s_0 \atop p|s_1s_2s_3} \left( 1 - \frac{2}{p} \right),
\]
Lemma 2.3 reveals that $\mathfrak{R}$ continuation to all of $C$ $\mathfrak{R}$ on the half-plane since for any $n \in \mathbb{N}$.

We will use Perron’s formula to estimate $\sum_{n \leq B} \Delta(n)$, before combining it with partial summation to estimate (3.15). Consider the Dirichlet series

$$D(s) := \sum_{n=1}^\infty \frac{\vartheta(n, s)}{n^s},$$

for any $n \in \mathbb{N}$.

Hence $D(s + 1/3) = E_1(s)E_2(s)$, where $E_1(s) = \zeta(2s + 1)^3\zeta(3s + 1)$ and

$$E_2(s) = \frac{D(s + 1/3)}{\zeta(2s + 1)^3\zeta(3s + 1)} = \prod_p \left(1 + O\left(\frac{1}{p^{4s+2}}\right)\right)$$

on the half-plane $\Re(s) > -1/2$. In particular, $E_1(s)$ has a meromorphic continuation to all of $\mathbb{C}$ with a pole of order 4 at $s = 0$, and $E_2(s)$ is holomorphic and bounded on the half-plane $\Re(s) > -1/4$.

Let $c = 1/3 + \varepsilon$ for any $\varepsilon > 0$, and let $T \geq 1$. Then an application of Lemma 2.3 reveals that

$$\sum_{n \leq B} \Delta(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} E_1(s - 1/3)E_2(s - 1/3) \frac{B^s}{s} ds + O_\varepsilon\left(\frac{B^{1/3+\varepsilon}}{T}\right),$$

provided that $B \notin \mathbb{Z}$. We apply Cauchy’s residue theorem to the rectangular contour $C$ joining the points $1/6 - iT, 1/6 + iT, c + iT$ and $c - iT$. Now the residue of $E_1(s - 1/3)E_2(s - 1/3)B^s/s$ at $s = 1/3$ is clearly

$$\text{Res}_{s=1/3}\left\{E_1(s - 1/3)E_2(s - 1/3)\frac{B^s}{s}\right\} = \frac{E_2(0)}{48}B^{1/3}P(\log B),$$

for some monic polynomial $P \in \mathbb{R}[x]$ of degree 3. Define the difference

$$\mathcal{E}(B) = \sum_{n \leq B} \Delta(n) - \frac{E_2(0)}{48}B^{1/3}P(\log B).$$

Then it follows that

$$\mathcal{E}(B) \ll_\varepsilon \frac{B^{1/3+\varepsilon}}{T} + \left(\int_{1/6-iT}^{c+iT} + \int_{c-iT}^{1/6+iT} + \int_{c+iT}^{1/6+iT}\right)\left|E_1(s - 1/3)\frac{B^s}{s}\right| ds,$$

since $E_2(s - 1/3)$ is holomorphic and bounded on the half-plane $\Re(s) \geq 1/6$. 

when $\gcd(s_i, s_j) = 1$ for $1 \leq i < j \leq 3$. On recalling the definitions of $X_1, X_2$, we deduce that

$$T(B) = B^{2/3} \sum_{n \leq B} \Delta(n)F_2((n/B)^{1/3}),$$

(3.15)
We proceed to estimate the contribution from the horizontal contours. Recall the well-known convexity bounds
\[
\zeta(\sigma + it) \ll_{\epsilon} \begin{cases} 
|t|^{(1-\sigma)/3+\epsilon}, & \text{if } \sigma \in [1/2, 1], \\
|t|^{(3-4\sigma)/6+\epsilon}, & \text{if } \sigma \in [0, 1/2],
\end{cases}
\]
for any $|t| \geq 1$. A proof of these can be found in [60, §II.3.4], for example. It therefore follows that
\[
E_1(\sigma - 1/3 + it) \ll_{\epsilon} |t|^{1-3\sigma+\epsilon}
\]
for any $\sigma \in [1/6, 1/3)$ and any $|t| \geq 1$. We may now deduce that
\[
\int_{1/6-iT}^{c-iT} \left| E_1(s - 1/3) \frac{B^s}{s} \right| ds \ll_{\epsilon} \int_{1/6}^{c} B^{\sigma T^{-3\sigma+\epsilon}} d\sigma
\ll_{\epsilon} \frac{B^{1/3+\epsilon} T^\epsilon}{T} + \frac{B^{1/6} T^\epsilon}{T^{1/2}}.
\]
One obtains the same estimate for the contribution from the remaining horizontal contour. Turning to the vertical contour, (3.18) gives
\[
\int_{1/6+iT}^{1/6+iT} \left| E_1(s - 1/3) \frac{B^s}{s} \right| ds \ll B^{1/6} \int_{-T}^{T} \frac{|E_1(-1/6 + it)|}{1 + |t|} dt \ll B^{1/6} \int_{-T}^{T} \frac{|t|^{1/2+\epsilon}}{1 + |t|} dt \ll B^{1/6} T^{1/2+\epsilon}.
\]
Once combined with (3.19), we conclude that
\[
E(B) \ll_{\epsilon} B^{1/3+\epsilon} T^{-1+\epsilon} + B^{1/6} T^{1/2+\epsilon},
\]
for any $T \geq 1$. Taking $T = B^{1/9}$ we obtain
\[
\sum_{n \leq B} \Delta(n) = \frac{E_2(0)}{48} B^{1/3} P(\log B) + O(\epsilon B^{2/9+\epsilon}),
\]
for any $\epsilon > 0$.

We are now ready to complete the proof of Theorem 1.3. For this it suffices to combine the latter estimate with partial summation in (3.15), and then apply Lemma 3.2. In this way we deduce that
\[
N_U(B) = 2T(B) + O(B \log B)
= \frac{\sigma_\infty E_2(0)}{144} BQ(\log B) + O(\epsilon B^{8/9+\epsilon}) + O(B \log B),
\]
for a further cubic monic polynomial $Q \in \mathbb{R}[x]$. Here $\sigma_\infty = 6 \int_0^1 F_2(u) du$ is given by (1.18), and it follows from (3.17) that
\[
E_2(0) = \prod_p \left( 1 - \frac{1}{p} \right)^4 \left( 1 + \frac{4}{p} + \frac{1}{p^2} \right).
\]
This therefore completes the proof of Theorem 1.3.
4. The $D_4$ del Pezzo surface of degree 3

In this section we consider Manin’s conjecture for the cubic surface

$$S_2 = \{ [x_1, x_2, x_3, x_4] \in \mathbb{P}^3 \colon x_1x_2(x_1 + x_2) + x_4(x_1 + x_2 + x_3)^2 = 0 \},$$

considered in [1.12] Let $U_2 \subset S_2$ be the open subset formed by deleting the lines (1.14) from $S_2$. Our task is to estimate $N_{U_2}(B)$. In doing so it will clearly suffice to establish the estimate for any surface that is obtained from $S_2$ via a unimodular transformation. In view of this we will make the change of variables

$$t_1 = x_1, \quad t_2 = x_2, \quad t_3 = x_1 + x_2 + x_3, \quad t_4 = -x_4,$$

which brings $S_2$ into the shape

$$t_1t_2(t_1 + t_2) = t_3^2t_4,$$

(4.1)

and which we henceforth denote by $S$. The 6 lines on the surface (4.1) take the shape

$$t_i = t_j = 0, \quad t_j = t_1 + t_2 = 0,$$

where $i$ denotes a generic element of the set $\{1, 2\}$, and $j$ an element of $\{3, 4\}$. If $U \subset S$ denotes the open subset formed by deleting these lines from the surface, then we have $t_3t_4 = 0$ for any $[t] \not\in U$. It now follows that

$$N_U(B) = \frac{1}{2} \# \{ t \in \mathbb{Z}^4 : (4.1) \text{ holds, } |t| \leq B, t_3t_4 \neq 0 \}.$$

As in the argument of Lemma 3.1 the factor $\frac{1}{2}$ reflects the fact that $t$ and $-t$ represent the same point in $\mathbb{P}^3$. There is a clear symmetry between solutions such that $t_3$ is positive and negative. Similarly, (4.1) is invariant under the transformation $t_1 = -z_1, t_2 = -z_2, t_3 = z_3$ and $t_4 = -z_4$. Thus we have

$$N_U(B) = 2 \# \{ t \in \mathbb{Z}^4 : (4.1) \text{ holds, } |t| \leq B, t_3, t_4 \geq 1 \},$$

(4.2)

for any $B \geq 1$.

In the following section we will explicate the relation between (4.2) and counting integral points on the corresponding universal torsor. When it comes to the latter task, we will be led to consider the counting problem for rational points on plane curves of degree 1 and 2. The estimates that we require will need to be completely uniform in the coefficients of the equations defining the curves.

Given any plane curve $C \subset \mathbb{P}^2$ of degree $d \geq 1$, that is defined over $\mathbb{Q}$, let

$$N_C(B) := \# \{ x \in C(\mathbb{Q}) : H(x) \leq B \}.$$

As usual we write $\mathbb{Z}^n$ for the set of primitive vectors in $\mathbb{Z}^n$, and $\mathbb{Z}_n^* \subset \mathbb{Z}^n$ for the set of vectors in $\mathbb{Z}^n$ with no components equal to zero. We will restrict our attention to curves that are defined by diagonal ternary forms. We clearly have

$$N_C(B) = \frac{1}{2} M_d(a; B, B, B)$$

for certain non-zero integers $a_1, a_2, a_3$, where

$$M_d(a; B) := \# \{ x \in \mathbb{Z}^3 : a_1x_1^d + a_2x_2^d + a_3x_3^d, |x_i| \leq B_i \},$$

(4.3)

and $B = (B_1, B_2, B_3)$. Let us begin with the situation for projective lines. The following result is due to Heath-Brown [35, Lemma 3].
Lemma 4.1. Let \( a \in \mathbb{Z}_3^* \) and let \( B_i > 0 \). Then we have
\[
M_1(a; B) \ll 1 + \frac{B_1 B_2 B_3}{\max |a_i| B_i}.
\]

Lemma 4.1 shows that there are only \( O(1) \) rational points on lines of sufficiently large height. If one has a line \( L \subset \mathbb{P}^2 \) given by the equation \( a \cdot x = 0 \), for \( a \in \mathbb{Z}_3^* \), then the height of \( L \) is simply defined to be \( H(L) := |a| \).

When \( L \) is an arbitrary line in \( \mathbb{P}^n \), which is defined over \( \mathbb{Q} \), there is still a very natural way of defining its height. The height of \( L \) is just the height of the rational point in the Grassmannian \( G(1, n) \) that corresponds to the line. We will not need this fact in our work. Let \( L \subset \mathbb{P}^2 \) be an arbitrary line defined over \( \mathbb{Q} \). Then it follows from Lemma 4.1 that
\[
N_L(B) \ll 1 + \frac{B^2}{H(L)} \ll B^2.
\]

This is essentially best possible, as can be seen by taking \( n = 2 \) in Exercise 3.

Turning to curves of higher degree, we have the following result, which is a special case of a result due to the author and Heath-Brown [15, Corollary 2]

Lemma 4.2. Let \( a \in \mathbb{Z}_3^* \) such that \( \gcd(a_i, a_j) = 1 \), and let \( B_i > 0 \). Then we have
\[
M_2(a; B) \ll \left(1 + \frac{B_1 B_2 B_3}{|a_1 a_2 a_3|}\right)^{1/3} \tau(a_1 a_2 a_3),
\]
where \( \tau(n) := \sum_{d|n} 1 \) denotes the usual divisor function.

In keeping with our discussion of lines, let us consider to what extent our estimate reflects the true growth rate of \( N_C(B) \), for a quadratic curve \( C \subset \mathbb{P}^2 \) that is defined by a diagonal equation with pairwise coprime coefficients. Recall the estimate \( \tau(n) = O_{\epsilon}(n^\epsilon) \), that holds for any \( \epsilon > 0 \). Writing \( ||C|| \) for the maximum modulus of the coefficients defining \( C \), we deduce from Lemma 4.2 that
\[
N_C(B) \ll_{\epsilon} ||C||^\epsilon B.
\]

This should be compared with the work of Heath-Brown [41, Theorem 3] that shows \( N_C(B) \ll_{d, \epsilon} B^{2/d+\epsilon} \), for any irreducible plane curve \( C \subset \mathbb{P}^2 \) of degree \( d \).

Both Lemma 4.1 and Lemma 4.2 are established using the geometry of numbers.

4.1. Elementary considerations. We proceed to show how \( N_U(B) \) can be related to a count of the integer points on the corresponding universal torsor. Our argument is in complete analogy to that presented in 3.1 although the individual steps differ somewhat. If \( \tilde{S} \) denotes the minimal desingularisation of the surface \( S \), then Derenthal [26] has calculated the universal torsor over \( \tilde{S} \), it being embedded in \( \mathbb{A}^{10} \) by a single equation
\[
s_1 u_1 y_1^2 + s_2 u_2 y_2^2 + s_3 u_3 y_3^2 = 0.
\]

Note that one of the variables does not appear explicitly in the equation. We will need the following basic fact.
Exercise 18. Let \( a, b \in \mathbb{N} \). Show that \( a \mid b^2 \) if and only if \( a = uv^2 \) for \( u, v \in \mathbb{N} \) such that \( u \) is square-free and \( uv \mid b \).

Given \( v \in \mathbb{R} \) and \( s, u, y \in \mathbb{R}^3 \), define

\[
\Psi(v, s, u, y) := \max \left\{ \frac{|s_1 s_2 s_3|}{s_1 u_1^2 u_2^3 u_3^2 v^2 y_1^2}, \frac{|s_1 u_1^2 u_2^3 u_3^2 y_2 y_3|}{s_1 u_1^2 u_2^3 u_3^2 v^2 y_2^2} \right\}. \tag{4.5}
\]

We are now ready to record our translation of the problem to the universal torsor.

**Lemma 4.3.** We have

\[
N_U(B) = 2^# \left\{ (v, s, u, y) \in \mathbb{N}^4 \times \mathbb{Z}^3 \times \mathbb{N}^3 : \begin{array}{l}
\text{\( u_3 > 0 \),} \\
\text{\( \Psi(v, s, u, y) \leq B \)} \tag{4.4}
\end{array} \right\}
\]

where \( i, j, k \) denote distinct elements from the set \( \{1, 2, 3\} \).

**Proof.** Let \( t \in \mathbb{Z}^4 \) be a vector such that (4.4) holds, with \( t_3, t_4 \geq 1 \). Write

\[
\eta_{14} = \gcd(t_1, t_4), \quad \eta_{24} = \gcd(t_2, t_4/\eta_{14}), \quad \eta_{12} = \gcd(t_1/\eta_{14}, t_2/\eta_{24}).
\]

Then \( \eta_{12}, \eta_{14}, \eta_{24} \in \mathbb{N} \) and there exists \( z_4 \in \mathbb{N} \) and \( z_1, z_2 \in \mathbb{Z} \) such that

\[
t_1 = \eta_{12}\eta_{14}z_1, \quad t_2 = \eta_{12}\eta_{24}z_2, \quad t_4 = \eta_{14}\eta_{24}z_4.
\]

Moreover, it is not hard to deduce that

\[
\gcd(\eta_{12}z_1, \eta_{24}z_4) = \gcd(\eta_{12}z_2, z_4) = \gcd(z_1, z_2) = 1,
\]

and

\[
\gcd(t_3, \eta_{14}, \eta_{12}\eta_{24}z_2) = 1.
\]

Under this substitution the equation (4.1) becomes

\[
\eta_{12}^3z_1z_2(\eta_{14}z_1 + \eta_{24}z_2) = t_3^2z_4.
\]

It follows that \( \eta_{12}^3 \mid t_3^2 \) in any given integer solution. Exercise 18 therefore implies that there exist \( u, v, z_3 \in \mathbb{N} \) such that \( |\mu(u)| = 1 \) and

\[
\eta_{12} = uv^2, \quad t_3 = u^2v^3z_3,
\]

with

\[
z_1z_2(\eta_{14}z_1 + \eta_{24}z_2) = uz_3^2z_4.
\]

We proceed to consider the effect of the divisibility condition \( z_1z_2 \mid uz_3^2 \) that this equation entails.

Recall that \( \gcd(z_1, z_2) = \gcd(z_1, z_4) = \gcd(z_2, z_4) = 1 \). Since \( z_1z_2 \mid uz_3^2 \), there must exist \( u_1, u_2, u_3, w_1, w_2, w_3, u_3 > 0 \) and

\[
u = u_1u_2w_3, \quad z_1 = u_1w_1^2, \quad z_2 = u_2w_2^2, \quad z_3 = w_1w_2w_3.
\]

Here we have used the fact that if \( p \) is a prime such that \( p \mid u \) and \( p \mid z_1z_2 \), then \( p \) must divide \( z_1 \) or \( z_2 \) to even order. Under these substitutions our equation becomes

\[
\eta_{14}u_1w_1^2 + \eta_{24}u_2w_2^2 = uz_3^2z_4.
\]
Moreover, we will have the corresponding coprimality conditions
\[ \gcd(u_1 u_2 u_3 v w_1, \eta_2 \eta_4 z_4) = \gcd(u_1 u_2 u_3 v w_2, z_4) = \gcd(u_1 w_1, u_2 w_2) = 1, \]
and
\[ |\mu(u_1 u_2 u_3)| = 1, \quad \gcd(u_1^2 u_2^2 u_3^2 v^3 w_1 w_2 w_3, \eta_1 \eta_4 \eta_2 \eta_4 u_1 u_2 u_3 v^2 w_2^2) = 1. \]
We now set \( s = (\eta_1 \eta_2, \eta_4, z_4) \) and \( y = w \), and replace \((u_1, u_2, u_3)\) by \((-u_1, -u_2, u_3)\). Tracing through our argument, one sees that we have made the transformation
\[
\begin{aligned}
t_1 &= -s_1 u_1^2 u_2^2 u_3^2 v^2 y_1^2, \\
t_2 &= -s_2 u_1^2 u_3^2 v^2 y_2^2, \\
t_3 &= u_1^2 u_2^2 u_3^2 v^3 y_1 y_2 y_3, \\
t_4 &= s_1 s_2 s_3. 
\end{aligned}
\]
In particular, it is clear that the height condition \(|x| \leq B\) is equivalent to \(\Psi(v, s, u, y) \leq B\), in the notation of (4.3). We now observe that under this transformation the equation (4.1) becomes (4.4), and the coprimality relations (4.6) and (4.7) can be rewritten
\[ \gcd(s_2 s_3, u_1 u_2 u_3 v y_1) = \gcd(s_3, u_1 u_2 u_3 v y_2) = \gcd(u_1 y_1, u_2 y_2) = 1, \]
and
\[ |\mu(u_1 u_2 u_3)| = 1, \quad \gcd(s_1, u_1 u_2 u_3 v y_2 \gcd(y_3, s_2)) = 1. \]
We can combine these relations with (4.4) to simplify them still further. In fact, once combined with (4.4), we claim that they are equivalent to the conditions
\[ |\mu(u_1 u_2 u_3)| = 1, \quad \gcd(s_1 s_2 s_3, u_1 u_2 u_3 v) = \gcd(y_i, y_j) = \gcd(y_i, s_j, s_k) = 1, \]
appearing in the statement of the lemma. To establish the forward implication, it suffices to show that \(\gcd(y_1, y_3) = \gcd(y_2, y_3) = 1\), the remaining conditions being immediate. But these two conditions follow on combining (4.4) with the fact that \(\gcd(y_1, s_2 u_2 y_2) = 1\). To see the reverse implication, the conditions are all immediate apart from
\[ \gcd(y_1, s_3 s_3) = \gcd(y_2, s_1 s_3) = \gcd(u_1, y_2) = \gcd(u_2, y_1) = 1. \]
But each of these is an easy consequence of the assumed coprimality relations, and (4.4). Finally, we leave it as an exercise to the reader to check that each \((v, s, u, y)\) counted in the right hand side of Lemma 4.3 produces a primitive solution of (4.1) with \(t_3, t_4 \geq 1\). This completes the proof of Lemma 4.3. \(\square\)

In what follows let us write \(i\) for a generic element of the set \(\{1, 2, 3\}\). Fix a choice of \(v \in \mathbb{N}\) and \(S_i, U_i, Y_i > 0\), and write
\[ \mathcal{N} = \mathcal{N}_v(S; U; Y) \]
for the total contribution to \(\mathcal{N}(B)\) in Lemma 4.3 from \(s, u, y\) contained in the intervals
\[ S_i / 2 < s_i \leq S_i, \quad U_i / 2 < \left| u_i \right| \leq U_i, \quad Y_i / 2 < y_i \leq Y_i. \]
Write
\[ \begin{aligned}
S &= S_1 S_2 S_3, \\
U &= U_1 U_2 U_3, \\
Y &= Y_1 Y_2 Y_3.
\end{aligned} \]
If \( \mathcal{N} = 0 \) there is nothing to prove, and so we assume henceforth that the dyadic ranges in (4.9) produce a non-zero value of \( \mathcal{N} \). In particular we must have
\[
S \ll B, \quad U^2 Y \ll B/v^3, \quad S_i U_i Y_i^2 \ll B/v^2. \tag{4.10}
\]

In this set of lecture notes we will provide two upper bounds for \( N_U(B) \). The object of our first bound is to merely establish linear growth, without worrying about the factor involving \( \log B \) that we expect to see. By ignoring some of the technical machinery needed to get better bounds it is hoped that the overall methodology will be brought into focus. Later we will indicate how the expected upper bound can be retrieved with a little more work.

### 4.2. A crude upper bound

We begin by establishing linear growth for \( N_U(B) \). Note that (4.10) forces the inequalities \( S_i, U_i, Y_i \ll B \). We proceed to establish the following upper bound.

**Lemma 4.4.** We have
\[
N_U(B) \ll (\log B)^9 \sum_{v \leq B^{1/3}} \max_{S_i, U_i, Y_i > 0} \mathcal{N}_v(S; U; Y),
\]
where the maximum is over \( S_i, U_i, Y_i > 0 \) such that (4.10) holds.

**Proof.** Our starting point is Lemma 4.3. It follows from (4.5) that \( v \ll B^{1/3} \) for any \( v, s, u, y \) that contributes to the right hand side. Let us fix a choice of \( v \in \mathbb{N} \) such that \( v \ll B^{1/3} \), and cover the ranges for \( s, u, y \) with dyadic intervals. Thus for fixed integers \( \sigma_i, \nu_i, \eta_i \geq 0 \), we write
\[
S_i = 2^{\sigma_i}, \quad U_i = 2^{\nu_i}, \quad Y_i = 2^{\eta_i},
\]
and consider the contribution from \( s, u, y \) in the range (4.9). But this is just \( \mathcal{N} = \mathcal{N}_v(S; U; Y) \). Now we have already seen that \( \mathcal{N} = 0 \) unless (4.10) holds. Finally, since each \( S_i, U_i, Y_i \) is \( O(B) \), it follows that the number of dyadic intervals needed is \( O((\log B)^9) \). This completes the proof of the lemma.

We may now restrict our attention to bounding \( \mathcal{N}_v(S; U; Y) \) for fixed values of \( S_i, U_i, Y_i > 0 \) such that (4.10) holds, and fixed \( v \ll B^{1/3} \). In the arguments that follow it will be necessary to focus attention on primitive vectors \( s \in \mathbb{N}^3 \). To enable this we draw out possible common factors between \( s_1, s_2, s_3 \), obtaining
\[
\mathcal{N}_v(S; U; Y) = \sum_{k=1}^{\infty} \mathcal{N}_v(k^{-1}S; U; Y). \tag{4.11}
\]

Here \( \mathcal{N}_v^\prime(S; U; Y) \) is defined as for \( \mathcal{N}_v(S; U; Y) \) but with the extra condition that \( \gcd(s_1, s_2, s_3) = 1 \). Let us write \( S'_i = S_i/k \) and \( S' = k^{-1}S \).

Recall the equation (4.3) that we must count solutions to, which it will be convenient to denote by \( T \), and which we will think of as defining a variety in \( \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \), with homogeneous coordinates \( s, u, y \). The key idea will be to count points on the fibres of projections \( \pi : T \to \mathbb{P}^2 \times \mathbb{P}^2 \). This amounts to fixing six of the variables and estimating the number of points on the resulting plane curve. Since this family of curves will vary with \( B \), so it is vital to obtain bounds that are completely uniform in the coefficients of the defining equation.
Let us begin by fixing the variables $u, y$, and estimating the corresponding number of vectors $s$. Now it follows from the coprimality conditions in Lemma 4.3 that
\[ \gcd(u_1 y_1^2, u_2 y_2^2, u_3 y_3^2) = 1. \]
For fixed $u, y$, (4.4) defines a line in $\mathbb{P}^2$. We clearly have
\[ \mathcal{N}_v^*(S'; U; Y) \leq \sum_{u,y} M_1(a; S'), \]
in the notation of (4.3), with $a_i = u_i y_i^2$. Since $a$ is primitive, it therefore follows from Lemma 4.1 that
\[ \mathcal{N}_v^*(S'; U; Y) \ll \sum_{u,y} \left( 1 + \frac{S}{k^2 \max S_i U_i Y_i^2} \right) \ll UY + k^{-2} S^{2/3} U^{2/3} Y^{1/3}. \]
Here we have used the trivial lower bound $\max\{a, b, c\} \geq (abc)^{1/3}$, valid for any $a, b, c > 0$. Using (4.10) we conclude that
\[ \mathcal{N}_v^*(S'; U; Y) \ll UY + \frac{B}{k^2 y}. \quad (4.12) \]
The second term here will be satisfactory from our point of view, but the first is disastrous, since we will run into trouble when it comes to summing over $k$ in (4.11).

It turns out that an altogether different bound is required to handle the contribution from really small values of $S'$. For this we will fix values of $s, u$ in (4.3), and count points on the resulting family of conics. First we need to record the coprimality relation
\[ \gcd(s_i u_i, s_j u_j) = 1, \]
which we claim holds for any of the vectors $s, u, y$ in which we are interested. But this follows on noting that $\gcd(s_i, s_j) = 1$ for any primitive vector $s \in \mathbb{Z}^3$ such that (4.4) holds and $\gcd(s_i, u_1 u_2 u_3) = \gcd(y_i, s_j, s_k) = 1$. We now have
\[ \mathcal{N}_v^*(S'; U; Y) \leq \sum_{s,u} M_2(a; Y), \]
in the notation of (4.3), with $a_i = s_i u_i$. In particular $a_i$ is non-zero and $\gcd(a_i, a_j) = 1$ in the statement of Lemma 4.2 whence
\[ \mathcal{N}_v^*(S'; U; Y) \ll \sum_{s,u} \left( 1 + \frac{k^{3/2} Y^{1/2}}{S^{1/2} U^{1/2}} \right)^{2^\omega(s_1 s_2 s_3 u_1 u_2 u_3)}. \]
In view of the bounds $S, U \ll B$, we clearly have
\[ 2^\omega(s_1 s_2 s_3 u_1 u_2 u_3) \ll \varepsilon (s_1 s_2 s_3 u_1 u_2 u_3)^\varepsilon \ll \varepsilon (SU)^\varepsilon \ll \varepsilon B^{2\varepsilon}. \]
Once inserted into our bound for $\mathcal{N}_v^*(S'; U; Y)$, and combined with (4.10), we deduce that
\[ \mathcal{N}_v^*(S'; U; Y) \ll \varepsilon B^{2\varepsilon} \left( \frac{SU}{k^3} + \frac{S^{1/2} U^{1/2} Y^{1/2}}{k^{3/2}} \right) \ll \varepsilon \frac{SU B^{2\varepsilon}}{k^3} + \frac{B^{1+2\varepsilon}}{k^{3/2} y^{3/2}}. \quad (4.13) \]
Here the second term will provide a satisfactory contribution, and we will balance the first term with our earlier estimate (4.12).
Note that
\[ \min \left\{ \frac{SU}{k^3}, UY \right\} \leq \frac{U\sqrt{SY}}{k^{3/2}} \ll \frac{B}{k^{3/2}v^{3/2}}, \]
by (4.10). It therefore follows from (4.13) and (4.12) that
\[ N_v^*(S'; U; Y) \ll \frac{B^{1+2\varepsilon}}{k^{3/2}v}. \]

Once inserted into (4.11), and then into the statement of Lemma 4.4, we may conclude that
\[ N_U(B) \ll \varepsilon (\log B)^9 \sum_{v \leq B^{1/3}} \sum_{k=1}^{\infty} \frac{B^{1+2\varepsilon}}{k^{3/2}v} \ll \varepsilon B^{1+3\varepsilon}. \]

Recall that \( N_{U_2}(B) \leq N_U(B) \), where \( U \subset S \) is the open subset associated to the surface (4.1), and \( N_{U_2}(B) \) is the counting function associated to (1.12).

On redefining the choice of parameter \( \varepsilon > 0 \), we have therefore established the following result.

**Theorem 4.1.** We have \( N_{U_2}(B) \ll \varepsilon B^{1+\varepsilon} \), for any \( \varepsilon > 0 \).

The reader will note that there many places in our argument where we have been wasteful. The most damaging has been in our use of the trivial bound \( 2^{\omega(n)} = O(n^{\varepsilon}) \), in the deduction of (4.13). Using the fact that \( 2^{\omega(n)} \) has average order \( \zeta(2)^{-1} \log n \), it is not particularly difficult to replace the \( B^{\varepsilon} \) in Theorem 4.1 with a large power of \( \log B \).

**Exercise 19.** By analysing the proof of Theorem 4.1, find an explicit value of \( A \geq 6 \) such that \( N_{U_2}(B) \ll B(\log B)^A \).

In the next section we will be able to show that the value \( A = 6 \) is an admissible exponent, as claimed in Theorem 1.2.

### 4.3. A better upper bound.

Crucial to the proof of Theorem 4.1 was an investigation of the density of integer solutions to the equation (4.4). It is in our treatment of this equation that we will hope to gain some saving. Let’s put the problem on a more general footing. For any \( A, B, C \in \mathbb{R}^3_{>1} \), let \( \mathcal{M}(A, B, C) \) denote the number of \( a, b, c \in \mathbb{Z}_*^3 \) such that
\[ a_1b_1c_1^2 + a_2b_2c_2^2 + a_3b_3c_3^2 = 0 \] (4.14)
and
\[ |a_i| \leq A_i, \quad |b_i| \leq B_i, \quad |c_i| \leq C_i, \]
with
\[ \gcd(a_i, c_j) = \gcd(c_i, c_j) = 1 \] (4.15)
and
\[ |\mu(a_1a_2a_3)| = 1, \quad \gcd(a_i, b_j, b_k) = 1. \] (4.16)
Here, we recall that \( \mathbb{Z}_*^3 \) denotes the set of primitive vectors in \( \mathbb{Z}^3 \) with all components non-zero. It will be convenient to set
\[ A = A_1A_2A_3, \quad B = B_1B_2B_3, \quad C = C_1C_2C_3. \]
Arguing exactly as in the previous section, it is not difficult to deduce from Lemmas 4.1 and 4.2 that
\[ M(A, B, C) \ll A \min\{C, A^{1+\varepsilon}\} + A^{2/3} B^{2/3} C^{1/3} + A^{1/2+\varepsilon} B^{1/2+\varepsilon} C^{1/2} \]
for any \( \varepsilon > 0 \), whence
\[ M(A, B, C) \ll A^{2/3} B^{2/3} C^{1/3} + A^{1+\varepsilon} B^{1/2+\varepsilon} C^{1/2}. \] (4.17)

By working a little harder, we would like to replace the terms \( A^{\varepsilon}, B^{\varepsilon} \) with something rather smaller.

The main problem to be faced emerges in the application of Lemma 4.2, which gives
\[ M(A, B, C) \ll \sum a, b \left( 1 + \frac{C^{1/2}}{|a_1 a_2 a_3 b_1 b_2 b_3|^{1/2}} \right) 2^{\omega(a_1 a_2 a_3 b_1 b_2 b_3)}. \]

Rather than using the trivial bound \( 2^{\omega(n)} = O(n^{\varepsilon}) \), as above, we can try to make use of the fact that \( 2^{\omega(n)} \) has average order \( \zeta(2)^{-1} \log n \) in order to get some saving. Following this line of thought it is fairly straightforward to show that \( A^{\varepsilon} B^{\varepsilon} \) can be replaced by \( (\log A) \frac{3}{2} (\log B)^3 \) in (4.17). However this would still not be enough to deduce the best possible upper bound for \( N_U(B) \) that we would like. Let us simplify matters by considering only the contribution
\[ S(A, B) = \sum_{a, b} 2^{\omega(a_1 a_2 a_3 b_1 b_2 b_3)}, \]
to the above estimate for \( M(A, B, C) \). Then \( S(A, B) \) has exact order of magnitude
\[ AB \prod_{i=1}^{3} (\log A_i)(\log B_i), \]
so how can we hope to do better than this? The crucial observation comes in noting that we are only interested in summing over values of \( a, b \) for which the corresponding conic (4.14) has a non-zero solution \( c \in \mathbb{Z}^3 \), with \( \gcd(c_i, c_j) = 1 \). If we denote this finer quantity by \( S^*(A, B) \), then it is actually possible to show that
\[ S^*(A, B) \ll AB. \] (4.18)

This is established in [13, Lemma 1], and is simply a facet of the well-known fact that a random plane conic doesn’t have a rational point. This should be compared with the work of Serre [58]. Using the large sieve inequality, Serre has shown
\[ \#\{y \in \mathbb{Z}^3 : |y| \leq Y, \ (-y_1 y_3, -y_2 y_3)_Q = 1 \} \ll \frac{Y^3}{(\log Y)^{3/2}}, \]
where
\[ (a, b)_Q = \begin{cases} 1, & \text{if } ax^2 + by^2 = z^2 \text{ has a solution } (x, y, z) \neq 0 \text{ in } \mathbb{Q}^3, \\ -1, & \text{otherwise,} \end{cases} \]
denotes the Hilbert symbol. Guo [32] has established an asymptotic formula for the corresponding quantity in which one counts only odd values of \( y_1, y_2, y_3 \) such that the product \( y_1 y_2 y_3 \) is square-free.
Thus in addition to considering the density of integer solutions to diagonal quadratic equations, as in the previous section, we also need to consider how often such an equation has at least one non-trivial integer solution in order to derive sufficiently sharp bounds. The outcome of this investigation is the following result, which is established in \[13\] Lemma 2.

**Lemma 4.5.** For any $\varepsilon > 0$, we have

$$\mathcal{M}(A, B, C) \ll \varepsilon A^{2/3}B^{2/3}C^{1/3} + a\tau AB^{1/2}C^{1/2},$$

where

$$\sigma = 1 + \frac{\min\{A, B\}^\varepsilon}{\min\{B_i, B_j\}^{1/16}}, \quad \tau = 1 + \frac{\log B}{\min\{B_i, B_j\}^{1/16}}.$$

It is clear that this constitutes a substantial sharpening over our earlier estimate \[(4.17)\] for $\mathcal{M}(A, B, C)$. Nonetheless this is still not enough on its own, and we will need an alternative estimate when $B_1, B_2, B_3$ have particularly awkward sizes. The following result is rather easy to establish.

**Lemma 4.6.** We have

$$\mathcal{M}(A, B, C) \ll AB_iB_j(C_k + C_iC_jA_k^{-1})(\log AC)^2,$$

for any permutation $\{i, j, k\}$ of the set $\{1, 2, 3\}$.

**Proof.** For fixed integers $a, b, q$, let $\rho(q; a, b)$ denote the number of solutions to the congruence $at^2 + b \equiv 0 \pmod{q}$. We then have

$$\rho(q; a, b) \leq \sum_{d|q} |\mu(d)|\left(\frac{-ab}{d}\right). \quad (4.19)$$

It will clearly suffice to establish Lemma \[4.6\] in the case $(i, j, k) = (1, 2, 3)$, say. Now it follows from \[(4.14)\] that for given $a_i, b_1, b_2, c_3$, and each corresponding solution $t$ of the congruence

$$a_1b_1t^2 + a_2b_2 \equiv 0 \pmod{a_3c_3^2},$$

we must have $c_1 \equiv tc_2 \pmod{a_3c_3^2}$ in any solution to be counted. This gives rise to an equation of the form $h.w = 0$, with $h = (1, -t, a_3c_3^2)$ and $w = (c_1, c_2, k)$. Upon recalling that $\gcd(c_1, c_2) = 1$ from \[(4.15)\], an application of Lemma \[4.1\] therefore yields the bound

$$\ll \rho(a_3c_3^2; a_1b_2, a_2b_2)\left(1 + \frac{C_1C_2}{a_3c_3^2}\right),$$

for the number of possible $b_3, c_1, c_2$ given fixed choices of $a_i, b_1, b_2$ and $c_3$. It now follows from \[(4.19)\] that

$$\mathcal{M}(A, B, C) \ll \sum_{a_i, b_1, b_2, c_3} \rho(a_3c_3^2; a_1b_2, a_2b_2)\left(1 + \frac{C_1C_2}{a_3c_3^2}\right)$$

$$\ll \sum_{a_i, b_1, b_2, c_3} \sum_{d|a_3c_3} |\mu(d)|\left(\frac{-a_1a_2b_1b_2}{d}\right)\left(1 + \frac{C_1C_2}{a_3c_3^2}\right)$$

$$\ll \sum_{a_i, b_1, b_2, c_3} \tau(a_3)\tau(c_3) + C_1C_2 \sum_{a_i, b_1, b_2, c_3} \frac{\tau(a_3)\tau(c_3)}{|a_3c_3^2|}.$$
A simple application of partial summation now reveals that
\[ \mathcal{M}(\mathbf{A}, \mathbf{B}, \mathbf{C}) \ll (AB_1B_2C_3 + A_1A_2B_1B_2C_1C_2)(\log AC)^2, \]
as required to complete the proof of Lemma 4.6. \( \Box \)

We are now ready to combine Lemmas 4.5 and 4.6 to get a sharper upper bound for \( N_U(B) \). Taking Lemma 4.3 as our starting point we need to bound the quantity \( N = N_u(\mathbf{S}; \mathbf{U}, \mathbf{Y}) \) defined in (4.8), for fixed choices of \( v \in \mathbb{N} \) and \( S_i, U_i, Y_i > 0 \). As previously we will need to extract common factors from \( s_1, s_2, s_3 \), leading to the equality (4.11). Writing \( S'_i = S_i/k \) and \( S' = k^{-1}S \), as before, it is a simple matter to check that we have
\[ N'_u(S'; U; Y) \leq \mathcal{M}(U, S', Y), \]
with \((a, b, c) = (u, s, y)\). Indeed we plainly have
\[ \gcd(u_i, v_j) = \gcd(y_i, v_j) = 1, \quad |\mu(u_1u_2v_3)| = \gcd(u_i, s_j, v_k) = 1, \]
and \( u, s, y \in \mathbb{Z}_*^4 \), for any vectors counted by \( N'_u(S'; U; Y) \), as required for \( \mathcal{M}(U, S', Y) \). It now follows from (4.11) and Lemma 4.5 that
\[ N'_u(S; U; Y) \ll \varepsilon \sum_{k=1}^{\infty} \left( \frac{U^{2/3}S^{2/3}Y^{1/3}}{k^2} + k^{2/16}\sigma \frac{U S^{1/2}Y^{1/2}}{k^{3/2}} \right) \ll \varepsilon U^{2/3}S^{2/3}Y^{1/3} + \sigma U S^{1/2}Y^{1/2}, \]
for any \( \varepsilon > 0 \), where
\[ \sigma = 1 + \frac{\min\{S, U\}^{\varepsilon}}{\min\{S, S_j\}^{1/16}}, \quad \tau = 1 + \frac{\log B}{\min\{S, S_j\}^{1/16}}. \]

In order to obtain our final estimate for \( N_U(B) \) we need to sum this bound over all positive integers \( v \leq B^{1/3} \), as in Lemma 4.4 and over all possible dyadic intervals for \( S_i, U_i, Y_i \), subject to (4.10).

Suppose for the moment that we want to sum over all possible dyadic intervals \( X \leq |x| < 2X \), for which \( |x| \leq \mathcal{X} \). Then in deducing Lemma 4.4 we employed the basic bound \( O(\log \mathcal{X}) \) for the number of possible choices for \( X \). In the present investigation we will be more efficient and take advantage of the easily established estimates
\[ \sum_{X} X^\delta \ll \begin{cases} 1, & \text{if } \delta < 0, \\ \mathcal{X}^\delta, & \text{if } \delta > 0, \end{cases} \]
where the sum is over dyadic intervals for \( X \leq \mathcal{X} \). We will make frequent use of these bounds without further mention.

Returning to our estimate for \( N_u(S; U; Y) \), we may now conclude from the bound \( Y_i \leq B^{1/2}/(u^2S_iS_i) \) in (4.10) that
\[ N_U(B) \ll \varepsilon \sum_{v \leq B^{1/3}} \sum_{S, U, Y} \left( U^{2/3}S^{2/3}Y^{1/3} + \sigma \tau U S^{1/2}Y^{1/2} \right) \ll \varepsilon B^{1/2} \sum_{v \leq B^{1/3}} \sum_{S, U} \frac{S^{1/2}}{v} + \sum_{v \leq B^{1/3}} \sum_{S, U, Y} \sigma \tau U S^{1/2}Y^{1/2} \ll \varepsilon B(\log B)^6 + \sum_{v \leq B^{1/3}} \sum_{S, U, Y} \sigma \tau U S^{1/2}Y^{1/2}. \]
The first term on the right-hand side is clearly satisfactory, and it remains
to deal with the second term, which we denote by $R$ for convenience. We
would like to show that $R \ll B(\log B)^6$.

Suppose without loss of generality that $S_1 \leq S_2 \leq S_3$, so that in particular
$\min\{S_i S_j\} = S_1 S_2$ in $\sigma$ and $\tau$. If there is a constant $A > 0$ such that
$S_3 \leq (S_1 S_2)^A$, then it follows that

$$\sigma \leq (S_1 S_2)^{e-1/16} S_3^e \leq (S_1 S_2)^{(1+A)\varepsilon-1/16} \ll 1,$$

provided that $\varepsilon$ is sufficiently small. Taking $\tau \ll \log B$, we may then argue
as above to conclude that there is a contribution of $O(B(\log B)^6)$ to $R$ from
this case. Suppose now that there exists $A' > 0$ such that $U \leq (S_1 S_2)^{A'}$.
Then we have $\sigma \ll 1$ and $\tau \ll \log B$, so that there is a contribution of
$O(B(\log B)^6)$ to $R$ in this case too.

Finally it remains to consider the contribution to $N_{U}$ from $S_i, U_i, Y_i$
such that

$$S_1 S_2 \leq \min\{S_3, U\}^\delta, \quad (4.20)$$

for some small value of $\delta > 0$, with $S_1 \leq S_2 \leq S_3$. Let us denote this
contribution $N_0$, say. To estimate $N_0$ we will return to the task of estimat-
ing $N_v(S; U; Y)$ for fixed $v, S, U_1, Y_1$, but this time apply Lemma [1.6] with
$(i, j, k) = (1, 2, 3)$. This gives

$$N_v(S; U; Y) \ll (\log B)^2(U S_1 S_2 Y_3 + U_1 U_2 S_1 S_2 Y_1 Y_2).$$

We must now sum over dyadic intervals for $S_i, U_i, Y_i$. Thus it follows from
the bound $Y_i \leq B^{1/2}/(v^2 S_i U_i)\nu^{1/2}$ in (4.10) that

$$N_0 \ll (\log B)^2 \sum_{v \leq B^{1/3}} \sum_{S_i, U_i, Y_i} (U S_1 S_2 Y_3 + U_1 U_2 S_1 S_2 Y_1 Y_2)$$

$$\ll (\log B)^2 \sum_{v \leq B^{1/3}} \left( \sum_{S_i, U_i, Y_i} B^{1/2} U_1 S_1 S_2/v S_3^{1/2} U_3^{1/2} \right) + \sum_{S_i, U_i, Y_3} B(1/S_1 U_2)^{1/2}/v^2 U).$$

Since $U^2 \ll B/(v^2 Y_1 Y_2)$ in (4.10), and $S_1 S_2 \ll S_3^\delta$ by (4.20), we therefore
deduce that the overall contribution from the first inner sum is

$$\ll (\log B)^2 \sum_{v \leq B^{1/3}} \sum_{S_i, Y_1, Y_2, U_2, U_3} B^{3/4} S_1 S_2/v^{7/4} S_3^{1/2} U_3^{1/2} Y_1^{1/4} Y_2^{1/4} Y_1^{1/4}$$

$$\ll \sum_{S_2, S_3, Y_1, Y_2, U_2, U_3} B^{3/4} (\log B)^2/v^{7/4} U_3^{1/2} Y_1^{1/4} Y_2^{1/4} \ll B.$$
Once combined with our earlier work, this therefore concludes the proof of Theorem 1.2.

Exercise 20. By mimicking the argument in [13, §5], establish the lower bound $N_{U_2}(B) \gg B(\log B)^6$.

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