Equations of a relative equilibrium in Yang-Mills theory

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Abstract

The equations of a relative equilibrium in a pure Yang–Mills gauge theory with the Coulomb gauge fixing are obtained. They are derived as a direct consequence of the results of our previous work on Wong’s equations in gauge theory. The obtained equations are similar to the equations of a relative equilibrium in reducible finite-dimensional dynamical systems with a symmetry. In all these equations, the description of the reduced motion is performed by making use of the dependent coordinates.

1 Introduction

In our previous paper [1] we have obtained Wong’s equations in a pure Yang–Mills gauge theory with the Coulomb gauge fixing. Our derivation was based on the Marsden-Weinstein reduction theory for the dynamical systems with a symmetry.

As is known, in systems with gauge degrees of freedom it is possible to use only an implicit description of the local dynamics given on the orbit space of the gauge group. This is also valid for the most of finite-dimensional dynamical systems with a symmetry if their motion on the configuration space can be regarded as occurring on the total space of the principal fiber bundle.

In order to apply the reduction theory to such systems, we have developed a method in which the description of the local dynamics was given by means

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of dependent coordinates. It was done both for the finite-dimensional systems \[2-5\] and for the gauge-invariant dynamical systems \[6\].

Our Wong’s equations for a quite certain finite-dimensional dynamical system\footnote{This system describes a motion of a particle on a compact Riemannian manifold with a given free isometric smooth action of a compact semisimple Lie group.} (and for a pure Yang-Mills dynamical system) were obtained as geodesic equations in a special coordinate basis – the horizontal lift basis, in which the original metric is a diagonal one. It is important to note that horizontal Wong’s equation are directly related to the classical Yang-Mills equations on the reduced space. In order to come to the Yang-Mills equation, the only thing one needs to do is to add the potential in the horizontal Wong’s equation.

In this note we will consider the one of the consequences of the reduced Yang–Mills equations. It will be shown that in a particular case, these equations can be used for determination of the relative equilibrium of the reduced system.

In a classical mechanics, the relative equilibrium of the dynamical system with a symmetry is such a motion of the system in which the shape of the system is not changed. In other words, the system performs a steady motion in a group direction without disturbing the form of the whole system. We notice that the study of the set of the points of the relative equilibrium is a first step in an investigation of bifurcation dynamics in the reduced system. Moreover, after determination of the points of the relative equilibrium and establishing their stability, one can make use by the approximate description of the motion of the system in the vicinity of the equilibrium point.

These and other questions connected with the studies of the various aspects of the relative equilibria in the finite-dimensional dynamical systems (and also in some infinite-dimensional systems) were considered by many authors \[7-10\]. In the cited works one can find the additional references on a literature of this direction.

In the first part of our notes we will introduce the notations and definitions by taking them from \[1\]. Also, we briefly recall some facts obtained there. Then it will be shown how to get the reduced equations and the equations for a relative equilibrium in finite-dimensional case and for the case of a pure Yang-Mills theory.

2 Definitions and notations

The material of this section is taken from our previous article where one can find the consideration of the questions that are omitted here.
The description of the motion of the scalar particle on a compact manifold \( \mathcal{P} \) with a given free isometric smooth right action of a semisimple compact Lie group \( \mathcal{G} \) can be done with the help of the fiber bundle picture in which \( \mathcal{P} \) is regarded as the total space of the principal fiber bundle \( \pi : \mathcal{P} \to \mathcal{P}/\mathcal{G} = \mathcal{M} \).

The introduction of the dependent coordinate on this bundle may be done as follows. We first replace the original coordinates \( Q^A \), given on a local chart of the manifold \( \mathcal{P} \), by new coordinates \( (Q^+, a^\alpha) \) \( (A = 1, \ldots, \dim \mathcal{P}; \alpha = 1, \ldots, \dim \mathcal{G}) \) that are related to the fiber bundle \( \pi \). The new coordinates \( Q^+ \) are constrained \( \chi^\alpha(Q^+) = 0 \) in order to have a one-to-one correspondence between the old coordinates \( Q^A \) and \( (Q^+, a^\alpha) \). If these constraints define the submanifolds \( \Sigma \) in the manifold \( \mathcal{P} \), then we can get a trivial principal fiber bundle \( P(\mathcal{M}, \mathcal{G}) \) which is locally isomorphic to the trivial bundle \( \Sigma \times \mathcal{G} \to \Sigma \). In this case, and also in a local consideration, the coordinates \( Q^+ \) are used for description of the evolution given on the manifold \( \mathcal{M} \).

Replacement of the original coordinate basis \( (\frac{\partial}{\partial Q^A}) \) for a new coordinate basis \( (\frac{\partial}{\partial Q^+, \frac{\partial}{\partial a^\alpha}}) \) leads to a new representation for the original metric \( \bar{G}_{AB}(Q^+, a) \) of the manifold \( \mathcal{P} \):

\[
\left( \begin{array}{cc}
G_{CD}(P_{\perp})^A_C(P_{\perp})^D_B & G_{CD}(P_{\perp})^D_B K^C_D u^\mu_a(a) \\
G_{CD}(P_{\perp})^C_B K^D_{\nu} u^\nu_{\beta}(a) & \gamma_{\mu\nu} u^\mu_a(a) u^\nu_{\beta}(a)
\end{array} \right),
\]

(1)

where \( K_\mu \) are the Killing vector fields for the Riemannian metric \( G_{AB}(Q) \). \( (K_\mu, G_{CD}, \gamma_{\mu\nu} \text{ and } P_{\perp} \) are restricted to the submanifold \( \Sigma \equiv \{ \chi^\alpha = 0 \} \).

\( \gamma_{\mu\nu} = K_\mu A G_{AB} K_\nu^B \) is the metric given on the orbit of the group action. \( \bar{u}^\alpha_{\beta}(a) \) (and \( u^\beta_{\alpha}(a) \)) are the auxiliary functions for the group \( \mathcal{G} \).

The projection operator \( P_{\perp}(Q^+) \), defined as

\[
(P_{\perp})^A_B = \delta^A_B - \chi^\gamma_B (\chi^\gamma A)^{-1} \alpha_a (\chi^\gamma A)_a^A,
\]

(\( (\chi^\gamma A)_a^A \)) is a transposed matrix to the matrix \( \chi^\gamma_B \equiv \frac{\partial x^\gamma}{\partial Q^B}, (\chi^\gamma A)_a^A = G^{ABC} \chi^\mu_a \chi^\nu_B \),

is used to projects the vectors onto the tangent space to the surface \( \Sigma \).

The pseudoinverse matrix \( \tilde{G}^{AB}(Q^+, a) \) to the matrix (1), i.e. such a matrix for which

\[
\tilde{G}^{AB} \tilde{G}_{BC} = \left( \begin{array}{cc}
(P_{\perp})^A_C & 0 \\
0 & \delta^a_{\beta}
\end{array} \right),
\]

is given by

\[
\left( \begin{array}{cc}
G^{EF} N^C_E N^D_F & G^{SD} N^C_S \chi^\mu_D (\Phi^{-1})^\nu_{\mu} \tilde{\bar{v}}^\sigma_{\nu} \\
G^{CB} (\Phi^{-1})^D_{\gamma} \chi^\gamma_B (\Phi^{-1})^\mu_{\nu} \tilde{\bar{v}}^\alpha_{\mu} \tilde{\bar{v}}^\beta_{\nu}
\end{array} \right).
\]

(2)
The matrix \((\Phi^{-1})^\beta_\mu\) is inverse to the Faddeev – Popov matrix \(\Phi\),

\[
(\Phi)^\beta_\mu(Q) = K^A_\mu(Q) \frac{\partial \chi^\beta(Q)}{\partial Q^A}.
\]

The projection operator

\[
N_C^A \equiv \delta_C^A - K_\alpha^A (\Phi^{-1})^\alpha_\mu \lambda_C^\mu
\]
has the following properties:

\[
N_B^A N_C^B = N_C^A, \quad N_B^A K_B^\mu = 0, \quad (P_\perp)_B^A N_C^C = (P_\perp)_B^C, \quad N_B^A (P_\perp)_A^C = N_B^C.
\]

The matrix \(\tilde{v}_\alpha^\mu(a)\) is an inverse matrix to matrix \(\bar{u}_\alpha^\beta(a)\).

Our next step is the change of the obtained basis \((\partial/\partial Q^* A, \partial/\partial a^\alpha)\) for the horizontal lift basis \((H_A, L_\alpha)\) \cite{11}. This nonholonomic basis consists of the horizontal vector fields \(H_A\) and the left-invariant vector fields \(L_\alpha = v^\alpha_\mu(a) \frac{\partial}{\partial a^\mu}\) with the commutation relations

\[
[L_\alpha, L_\beta] = c^\gamma_\alpha_\beta L_\gamma,
\]
where the \(c^\gamma_\alpha_\beta\) are the structure constants of the group \(G\).

The horizontal vector fields \(H_A\) are given as follows

\[
H_A = N_E^A(Q^*) \left( \frac{\partial}{\partial Q^* E} - \hat{A}_E^A L_\alpha \right),
\]

where \(\hat{A}_E^A(Q^*, a) = \tilde{\rho}_\mu^\alpha(a) A_E^\mu(Q^*)\). The matrix \(\tilde{\rho}_\mu^\alpha\) is inverse to the matrix \(\rho_\mu^\alpha\) of the adjoint representation of the group \(G\), and \(A_E^\mu = \gamma^\mu_\nu K^R G_{R\nu}\) is the mechanical connection defined in our principal fiber bundle.

The curvature \(\mathcal{F}_{EP}^\alpha\) of the connection \(\hat{A}\) is given by

\[
\mathcal{F}_{EP}^\alpha = \frac{\partial}{\partial Q^* E} \hat{A}_{EP}^\alpha - \frac{\partial}{\partial Q^* P} \hat{A}_E^\alpha + c^\alpha_\nu_{\sigma} \hat{A}_E^\nu \hat{A}_P^\sigma,
\]

\((\mathcal{F}_{EP}^\alpha(Q^*, a) = \tilde{\rho}_\mu^\alpha(a) \mathcal{F}_{EP}^\mu(Q^*))\).

We notice that

\[
L_\alpha \hat{A}_E^\lambda = -c^\lambda_\alpha_\mu \hat{A}_E^\mu.
\]

This follows from the equation satisfied by \(\rho\): \(L_\alpha \rho_\beta^\mu = c^\mu_\alpha_\beta \rho_\gamma^\gamma\).

It can be shown that in a new basis, in which \([H_A, L_\alpha] = 0\), the metric \(\tilde{G}_{AB}\) has the following representation:

\[
\tilde{G}_{AB} = \begin{pmatrix} G^H_{AB} & 0 \\ 0 & \tilde{\gamma}_{\alpha_\beta} \end{pmatrix},
\]

(3)
\[ G(H_A, H_B) \equiv G^H_{AB}(Q^*), \quad \tilde{G}(L_\alpha, L_\beta) \equiv \tilde{\gamma}_{\alpha\beta}(Q^*, a) = \gamma_{\alpha'\beta'}(Q^*) \rho^\alpha_{\alpha'}(a) \rho^\beta_{\beta'}(a). \]

The “horizontal metric” \( G^H \) is defined with the help of the projection operator \( \Pi_{AB} = \delta^A_B - K^A_\mu \gamma_{\mu \nu} K^D_\nu G_{DB} \) as follows: \( G^H_{DC} = \Pi^D_C \tilde{G}_{DC} \). The operator \( \Pi_B^A \) has the following properties: \( \Pi_B^A \Pi_N^L = \Pi_B^A \) and \( \Pi_B^A \Pi_N^L = \delta_N^B \).

The pseudoinverse matrix \( \tilde{G}^{AB} \) to the matrix (3) is defined by the following orthogonality condition:

\[ \tilde{G}^{AB} \tilde{G}^{BC} = (N^A_C 0 0 \delta^\alpha_\beta) , \]

and can be written as

\[ \tilde{G}^{AB} = \left( \begin{array}{cc} G^{EF} N^A_E N^B_F & 0 \\ 0 & \tilde{\gamma}^{\alpha\beta} \end{array} \right) . \]

In new coordinates, the quadratic form of the metric \( \tilde{G}^{AB} \) is

\[ ds^2 = G^H_{AB} d\omega^A d\omega^B + \tilde{\gamma}_{\alpha\beta} d\omega^\alpha d\omega^\beta \]

with the following dual basis:

\[ \omega^A = (P_\perp)_{A}^S dQ^S, \]
\[ \omega^\alpha = u^\alpha_\mu da^\mu + \tilde{\gamma}_{E}^\alpha (P_\perp)_E^S dQ^S, \]

for which \( \omega^A(H_B) = N^A_B, \omega^\alpha(H_A) = 0, \) and \( \omega^\alpha(L_\beta) = \delta^\alpha_\beta \). With a new metric \( \tilde{G}_{BC} \), it is possible to calculate the Christoffel symbols (3) and then to obtain the geodesic equation (1).

### 3 The equations of motion and relative equilibrium

For a simple mechanical system with the Lagrangian \( L = K - V \) where \( K \) is a kinetic energy associated to obtained Riemannian metric \( ds^2 \) and where \( V \) is an invariant potential, one can derive, by using the standard methods, the equations of motion. The horizontal equation is as follows

\[
\frac{d\dot{Q}^A}{dt} + \Gamma_{BC}^{AB} \dot{Q}^B \dot{Q}^C + G^{AB} N^F_S F_{EF} \dot{Q}^E p_F + \frac{1}{2} G^{AB} N^E_S (D_E \gamma^a_{\kappa\sigma}) p_{\kappa} p_{\sigma} + G^{AD} \partial_D V = 0, \tag{4}
\]

where the Christoffel symbols \( \Gamma_{CD}^{AB} \) are defined by the equality

\[ G_{AB}^H \Gamma_{CD}^{AB} = \frac{1}{2} \left( G_{AC,D}^H + G_{AD,C}^H - G_{CD,A}^H \right); \]
\( G^H_{AC,D} \equiv \frac{\partial G^H_{AB}(Q)}{\partial Q^D} \bigg|_{Q=Q^*} \). And the covariant derivative is given by

\[
D_E \gamma_{\alpha\beta} = \left( \frac{\partial}{\partial Q^*} \gamma_{\alpha\beta} - c^\sigma_{\mu\alpha} A^\mu_E \gamma_{\sigma\beta} - c^\sigma_{\mu\beta} A^\mu_E \gamma_{\sigma\alpha} \right).
\]

Notice that the equation (4) is written in the projected form. We have omitted the general multiplier, the projector \( N^{\beta A}_\lambda \), which stands before the equation. This operator performs the projection on the plane which is orthogonal to the orbit. Also, in derivation of the equation, we have used the equality \( N^\lambda E\partial_E V = \partial_S V \). It follows from the invariance of the potential \( V \) under the group action.

It worth to note that this form of the equation is similar to the corresponding equation obtained in [12] for the \( n \)-body problem in mechanics.

The vertical equation of motion (for the internal momentum, or the internal charge) can be written in the following form

\[
\frac{dp_\sigma}{dt} - c^\kappa_{\mu\sigma} A^\mu_E p_\kappa \dot{Q}^* E - c^\kappa_{\mu\nu} A^\mu_E \gamma_{\nu\sigma} p_\nu \gamma_{\sigma\kappa} \dot{Q}^* E - c^\mu_{\sigma\nu} \gamma^{\nu\kappa} p_\mu p_\kappa = 0.
\]

The horizontal equation (4) can be used for description of the motion on the reduced space. If, for example, we set \( p_\mu = 0 \) in this equation, we get the equation for the motion of our system on the orbit space of the principal fiber bundle. This case corresponds to the “zero-momentum reduction” case in the reduction theory. Another case is related with the reduction onto the “non-zero momentum level”. It can be realized when \( p_\mu \) takes the constant value in this equation.

The relative equilibria of the dynamical system with a symmetry is defined as a motion for which \( \dot{Q}^* A = 0 \). In this motion the system may be “regarded” as a rigid body. Taking \( \dot{Q}^* A = 0 \) in the equation (4) one can conclude from it and the equation (5) for the vertical motion [12] that \( p_\mu \) is a constant. Thus we get the system of two equation for definition of the relative equilibria:

\[
\frac{1}{2} G^{AE} N^E_S (D_E \gamma_{\kappa\sigma}) p_\sigma p_\kappa = G^{AE} \partial_E V,
\]

\[
c^\mu_{\sigma\nu} \gamma^{\nu\kappa} p_\mu p_\kappa = 0.
\]

The second equation of this system is easy solved if we assume that \( p_\mu \) is an eigenfunction \( e_\kappa \) defined by the matrix equation

\[
(k_{\nu\mu} \gamma^{\mu\kappa}) e_\kappa = \lambda e_\varphi,
\]

with \( k_{\alpha\beta} = c^\mu_{\alpha\sigma} c^\mu_{\nu\beta} \).

Since \( \gamma^{\mu\kappa} e_\kappa = \lambda k^{\nu\kappa} e_\nu \), we will have \( \lambda c^\mu_{\sigma\nu} k^{\nu\kappa} e_\nu e_\mu \) in the second equation of the system. Using the identity \( c^\mu_{\sigma\nu} k^{\nu\kappa} = -c^\nu_{\sigma\mu} k^{\nu\kappa} \), we get \( c^\mu_{\sigma\nu} k^{\nu\kappa} e_\nu e_\mu = \)
−e_\sigma^\mu k^{\nu\mu} e_\nu. But this means that e_\sigma^\mu k^{\nu\mu} e_\nu = 0, and therefore the eigenfunction e_\kappa gives the solution to the second equation of the system. After finding the solution, we substitute it into the first equation of \(6\). The obtained equation determines the variable \(Q^{*A}\) at the relative equilibrium of the dynamical system.

4 The relative equilibrium in Yang–Mills theory

In Yang-Mills theory, the original evolution of the dynamical system is considered on the function space of the gauge fields. The gauge transformations form a group which acts on this space. We assume that this group is the gauge group of time-independent transformations (the gauge condition \(A_0 = 0\) is already imposed)

\[
\hat{A}_i^\alpha(x) = \rho_\alpha^\beta(g^{-1}(x))A_i^\beta(x) + u_\alpha^\mu(g(x))\frac{\partial g^\mu(x)}{\partial x_i},
\]

where \(\rho_\alpha^\beta(g) = \bar{u}_\alpha^\nu(g) v^\nu_\beta(g)\) is the matrix of the adjoint representation of the group \(G\).

To fix the gauge symmetry, we use the Coulomb condition \(\partial^k A^\mu_k(x) = 0, \nu = 1, \ldots, N_G\), (or \(\chi^\nu(A) = 0\)). This defines the gauge surface \(\Sigma \equiv \{\chi^\nu(A) = 0\}\).

That is, the gauge fields \(A_i^\alpha(x)\) now play the role of the original coordinates \(Q^A\) of a point \(p \in P\).

The Hamiltonian of the pure Yang-Mills theory, which is used in the Schrödinger functional approach is given by

\[
H = \frac{1}{2} \mu^2 \kappa \Delta_P[A_a] + \frac{1}{\mu^2 \kappa} V[A_a],
\]

where

\[
\Delta_P[A] = \int d^3x \, \frac{\delta^2}{\delta A_i^\alpha(x) \, \delta A_j^\beta(x)},
\]

\[
V[A] = \int d^3x \, \frac{1}{2} k_{\alpha\beta} F_i^{\alpha}(x) F_j^{\beta}(x).
\]

\(k_{\alpha\beta} = e_\alpha^\mu e_\beta^\mu\) is the Cartan–Killing metric on the group \(G\), \(\mu^2 = \hbar g_0^2\), and \(\kappa\) is a real positive parameter. Since the quadratic part of the Hamiltonian is as follows

\[
G^{(\alpha,i,x)}_{(\beta,j,x')} \frac{\delta^2}{\delta A^{(\alpha,i,x)} A^{(\beta,j,x')}}.
\]
we have a flat metric $G^{(α, i, x)}(β, j, x') = k^{αβ} δ^{ij} \delta^3(x - x')$. The quadratic form of this metric is written as

$$ds^2 = G_{(α, i, x)(β, j, y)} \delta A^{(α, i, x)} \delta A^{(β, j, y)},$$

with

$$G_{(α, i, x)(β, j, y)} = G \left( \frac{δ}{δA^{α}_i(x)} , \frac{δ}{δA^{β}_j(y)} \right) = k^{αβ} \delta_{ij} \delta^3(x - y).$$

We must change the original coordinates $A^α_µ(x)$ for new coordinates $(A^*_µ(x), g^µ(x))$. And $A^*$ must satisfy the gauge condition: $χ^α(A^*) = 0$.

We should perform the same steps as in the finite-dimensional case in order to come to the functional equation of motion for the Yang–Mills fields. The same also may be done with the help of our finite-dimensional dynamical system, if we replace the values of the corresponding equations by their functional counterparts. Instead of the finite-dimensional variables, we introduce their counterparts taken from the gauge field theory. This means that now we regard the indices of the variables in the finite-dimensional equations as a compact notation of the corresponding extended indices:

$$A \rightarrow (α, i, x); \quad µ \rightarrow (µ, u); \quad \text{etc.}$$

Then for the time derivative of the basic variable $Q^{*B}(t)$ we have the following correspondence:

$$\dot{Q}^{*B}(t) \rightarrow \frac{d}{dt} A_{σj}^{σj}(y, t) \equiv \dot{A}_{σj}^{σj}(y, t).$$

A similar replacements must be done in all variables of the finite-dimensional equations. The functional counterparts of the finite-dimensional expressions can be defined in the standard way.

Thus the Killing vectors is given by $K_{(α, y)} = K^{(µ, i, x)}_{(α, y) δA^{α}_i(x)}$ with

$$K^{(µ, i, x)}_{(α, y)}(A) = \left[ (δ^µ_α \partial^i(x) + c^µ_α A^{*i}(x)) \delta^3(x - y) \right] \equiv \left[ D^µ_α A(x) \right] δ^3(x - y).$$

(here $∂^i(x)$ is a partial derivative with respect to $x^i$). The Killing vectors are used for definition of the orbit metric

$$γ^{(µ, x)(ν, y)} = K^{(α, i, z)}_{(µ, x)} G^{(α, i, z)(β, j, u)} K^{(β, j, u)}_{(ν, y)}.$$

It is given by

$$γ^{(µ, x)(ν, y)} = k^{αβ} δ^{kl} \left[ (-δ^µ_ν \partial^k(x) + c^µ_ν A^{*k}(x)) (δ^α_κ \partial_l(x) + c^α_κ A^{*l}(x)) \delta^3(x - y) \right].$$
An "inverse matrix" $\gamma^{(m,y)}(\mu,z)$ to the matrix $\gamma^{(\mu,x)}(\nu,y)$ can be defined by the following equation:

$$\gamma^{(\mu,x)}(\nu,y) \gamma^{(\nu,y)}(\sigma,z) = \delta^{(\sigma,z)}_{(\mu,x)} \equiv \delta^\sigma_{\mu} \delta^z_x.$$ 

That is,

$$k_{\phi \alpha} \delta^{kl} D_{\mu k}^{\mu} (A^*(x)) D_{\nu l}^{\nu} (A^*(x)) \gamma^{(\nu,x)}(\sigma,z) \gamma^{(\sigma,z)}(\mu,x) = \delta_{\mu}^\sigma \delta^z_x.$$ 

Thus, $\gamma^{(\nu,x)}(\sigma,z)$ is the Green function of the operator $(D D)^{\mu \nu}$. The mechanical connection, known in the Yang-Mills fields as the “Coulomb connection" $A^{\alpha \beta}$, is given by

$$A^{(\alpha,x)}(\beta,j,y) = [D_{\nu i}^{\mu} (A^*(x)) A^{(\nu,x)}(\sigma,j,y)] A^{(\sigma,y)}(\tau,k,z) \delta_{\tau}^{\tau} \delta^z_x.$$ 

Since the Riemannian metric on the original manifold of the gauge fields is flat, we should rewrite the equation (4), by taking this fact into account, i.e. we should consider the case when $G_{AB} = \delta_{AB}$ in our previous equation. As a result we come to the following horizontal equation:

$$\frac{d}{dt} \frac{dQ^{\star A}}{dt} + H^A_{BC} \dot{Q}^{\star B} \dot{Q}^{\star C} + G^{AS} N_S^F F_{EF} \dot{Q}^{\star E} p_{\nu} + \frac{1}{2} G^{AE} (D E^\gamma_{\kappa \sigma}) p_{\sigma} p_{\kappa}$$

$$+ G^{AD} \partial D V = 0,$$ 

Making use of the functional representations for the members of this equation and performing necessary generalized summation we get the horizontal equation of motion in our Yang-Mills dynamical system.

$$\frac{d}{dt} \frac{d\gamma^{(\alpha,e)}(\mu,x,t)}{dt} + \left( -2 e_{\phi \beta}^{\alpha} \dot{A}^{(\beta,x)}(\sigma,j,y) A^{(\sigma,y)}(\nu,z) \right)$$

$$+ e_{\mu \beta}^{\alpha} \int d \gamma d \sigma d z A^{(\beta,x)}(\gamma,y,t) A^{(\nu,x)}(\sigma,j,y) A^{(\sigma,z)}(\mu,x) + \text{"F - terms"}$$

$$\int d u d z \left( - e_{\phi \mu}^{\alpha} \gamma^{(\sigma,z)}(\mu,x) \left[ D_{\gamma}^{\alpha} (A^*(x)) \gamma^{(\gamma,z)} (\sigma,x) \right] \right)$$

$$+ e_{\mu \beta}^{\alpha} \gamma^{(\mu,x)}(\nu,z) \left[ D_{\gamma}^{\alpha} (A^*(x)) \gamma^{(\gamma,z)} (\mu,x) \right] p_{\kappa}(u,t) p_{\sigma}(z,t)$$

$$+ G^{(\alpha,i,x)}(\gamma,m,y) \frac{\delta}{\delta A^{(\gamma,m,y)}} V[A] = 0.$$ 

where the “"F - terms" corresponds to the terms of eq.(7). They are linear in $p$ and explicitly given as
1. 
\[-2c^{\sigma}_{\varphi\mu}k^{\beta\alpha}\int dy dz A^{(\nu,z)}_{(\sigma,k,y)} A^{(\mu,i,y)}_{(\beta,x)} \dot{A}^{*\varphi k}(y,t) p_{\nu}(z,t) \]

2. 
\[c^{\sigma}_{\mu\epsilon}\int dy dz \left\{ k_{\sigma\varphi} \left( \partial_{\varphi k}(y) \gamma^{(\nu,z)}_{(\epsilon,x)} \right) k^{\beta\alpha}_{\mu,i,y} A^{(\mu,i,y)}_{(\beta,x)} + \gamma^{(\nu,z)}_{(\epsilon,x)} k_{\sigma\varphi} \left[ D_{\sigma k}(A^{*}(y,t)) A^{(\mu,i,y)}_{(\beta,x)} \right] k^{\beta\alpha}_{\mu,i,y} \right\} \dot{A}^{*\varphi k}(y,t) p_{\nu}(z,t) \]

3. 
\[2c^{\alpha}_{\beta\mu} \left( \int dz \gamma^{(\nu,z)}_{(\mu,x)} p_{\nu}(z,t) \right) \dot{A}^{*\varphi i}(x,t) \]

4. 
\[2c^{\sigma}_{\mu\epsilon}k^{\beta\alpha}\int dy dz A^{(\nu,i,z)}_{(\sigma,x)} A^{(\mu,x)}_{(\varphi,k,y)} \dot{A}^{*\varphi k}(y,t) p_{\nu}(z,t) \]

5. 
\[-c^{\sigma}_{\mu\epsilon}\int dy dz \left\{ \delta^{\alpha}_{\sigma} \left( \partial_{\varphi}(x) \gamma^{(\mu,z)}_{(\epsilon,x)} \right) A^{(\mu,x)}_{(\varphi,k,y)} + \gamma^{(\nu,z)}_{(\epsilon,x)} k^{\beta\alpha}_{\mu,i,y} \right\} \dot{A}^{*\varphi k}(y,t) p_{\nu}(z,t) \]

6. 
\[c^{\sigma}_{\beta\sigma}k^{\mu\alpha}\int dy dz A^{(\beta,z)}_{(\varphi,k,y)} A^{(\sigma,i,z)}_{(\mu,x)} \dot{A}^{*\varphi k}(y,t) p_{\nu}(z,t) \]

The last term of (8) is given by 
\[G^{(\alpha,i,x)(\gamma,m,y)} \delta \frac{\delta}{\delta A^{(\gamma,m,y)}} V[A] = 2D^{\alpha}_{\beta j}(A^{*}(x))(F^{\beta})^{ij}(x). \]

The vertical equation of motion is 
\[\frac{d}{dt} p_{\sigma}(x,t) - c^{\alpha}_{\varphi\alpha} p_{\epsilon}(x,t) \int dy A^{(\varphi,x)}_{(\beta,j,y)} \dot{A}^{*\beta j}(y,t) \]
\[-c^{\epsilon}_{\varphi\alpha} \int dv dr \gamma^{(\alpha,v)}_{(\mu,r)} p_{\mu}(r,t) \gamma^{(\sigma,x)}_{(\epsilon,v)} \int dy A^{(\varphi,v)}_{(\beta,j,y)} \dot{A}^{*\beta j}(y,t) \]
\[-c^{\epsilon}_{\sigma\epsilon} p_{\varphi}(x,t) \int dy \gamma^{(\epsilon,x)}_{(\mu,y)} p_{\mu}(y,t) = 0. \]

As in the finite-dymensional case, the obtained equations may be used for studies the motion given on the orbit space of the gauge group and also for
investigation of the relative equilibrium in the reduced Yang--Mills dynamical system. But now we have the integro-differential equations. Thus for for definition of the relative equilibrium we obtain the following equations:

\[
\int du \, dz \left( -c_{\mu \nu}^\alpha \gamma^{(\sigma, z)} (\mu, x) \left[ D_{\nu}^\alpha (A^*(x, t)) \gamma^{(\mu, u)} (\epsilon, z) \right] \\
+ c_\nu^\gamma \gamma^{(\mu, z)} (\kappa, u) \left[ D_{\nu}^\alpha (A^*(x, t)) \gamma^{(\rho, z)} (\epsilon, x) \right] \right) p_{\kappa} (u, t) p_\sigma (z, t) = \\
-2D_{\beta j}^\alpha (A^*(x)) (F^{\beta})_{ij} (x) \quad (10)
\]

and

\[
- c_\sigma^\epsilon p_\sigma (x, t) \int dy \, \gamma^{(\epsilon, x)} (\mu, y) p_\mu (y, t) = 0. \quad (11)
\]

The solution of (11) are the eigenfunction of the Green’s function \( \gamma^{(\epsilon, x)} (\mu, y) \). They must be substituted into (10) for finding of the gauge field used for characterisation of the equilibrium.

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