Incomplete poly-Bernoulli numbers associated with incomplete Stirling numbers

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Abstract. By using the associated and restricted Stirling numbers of the second kind, we give some generalizations of the poly-Bernoulli numbers. We also study their analytic and combinatorial properties. As an application, at the end of the paper we present a new infinite series representation of the Riemann zeta function via the Lambert $W$.

1. Introduction

Let $\mu \geq 1$ be an integer in the whole text. Our goal is to generalize the following relation for the poly-Bernoulli numbers $B_n^{(\mu)}$ ([Kan, Theorem 1]):

$$B_n^{(\mu)} = \sum_{k=0}^{n} (-1)^{n-k} \frac{k!}{(k+1)^\mu} \left\{ \begin{array}{c} n \\ k \end{array} \right\} (n \geq 0, \mu \geq 1),$$

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ are the Stirling numbers of the second kind, determined by

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

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When $\mu = 1$, $B_n^{(1)}$ are the classical Bernoulli numbers, defined by the generating function

$$\frac{x}{1 - e^{-x}} = \sum_{n=0}^{\infty} B_n^{(1)} \frac{x^n}{n!}. \quad (1)$$

Notice that the classical Bernoulli numbers $B_n$ are also defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!},$$

satisfying $B_n^{(1)} = B_n (n \neq 1)$ with $B_1^{(1)} = 1/2 = -B_1$.

The generating function of the poly-Bernoulli numbers $B_n^{(\mu)}$ is given by

$$\frac{\text{Li}_\mu(1 - e^{-x})}{1 - e^{-x}} = \sum_{n=0}^{\infty} B_n^{(\mu)} \frac{x^n}{n!}, \quad (2)$$

where

$$\text{Li}_\mu(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^\mu}$$

is the $\mu$-th polylogarithm function ([Kan, (1)]). The generating function of the poly-Bernoulli numbers can also be written in terms of iterated integrals ([Kan, (2)]):

$$e^x \cdot \underbrace{\frac{1}{e^x - 1} \int_0^x \frac{1}{e^x - 1} \int_0^x \cdots \frac{1}{e^x - 1} \int_0^x \frac{x}{e^x - 1} dx}^{\mu-1} \cdots dx = \sum_{n=0}^{\infty} B_n^{(\mu)} \frac{x^n}{n!}. \quad (3)$$

Several generalizations of the poly-Bernoulli numbers have been considered ([BayHam1], [BayHam2], [CopCan], [Jol], [Sas]). However, most kinds of generalizations are based upon the generating functions of (1) and/or (2). On the contrary, our generalizations are based upon the explicit formula in terms of the Stirling numbers. In [KomMezSza], a similar approach is used to generalize the Cauchy numbers $c_n$, defined by $x/\log(1 + x) = \sum_{n=0}^{\infty} c_n x^n / n!$. In this paper, by using the associated and restricted Stirling numbers of the second kind, we give substantial generalizations of the poly-Bernoulli numbers. One of the main results is to generalize the formula in (2) as

$$\sum_{n=0}^{\infty} B_n^{(\mu)} \frac{t^n}{n!} = \text{Li}_\mu \left( 1 - E_m(-t) \right) \frac{1 - E_m(-t)}{1 - E_m(-t)}$$
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and

$$\sum_{n=0}^{\infty} B_{n}^{(\mu)} \frac{t^{n}}{n!} = \frac{\text{Li}_{\mu}(E_{m-1}(-t) - e^{-t})}{E_{m-1}(-t) - e^{-t}},$$

where $E_{m}(t) = \sum_{k=0}^{m} \frac{t^{k}}{k!}$. See Theorem 1 below.

2. Incomplete Stirling numbers of the second kind

In place of the classical Stirling numbers of the second kind $\{n \atop k\}$ we substitute
the restricted Stirling numbers and the associated Stirling numbers

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\}_{\leq m} \quad \text{and} \quad \left\{ \begin{array}{c} n \\ k \end{array} \right\}_{\geq m},$$

respectively. Some combinatorial and modular properties of these numbers can be found in [Mez], and other properties can be found in the cited papers of [Mez]. The generating functions of these numbers are given by

$$\sum_{n=k}^{m} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_{\leq m} \frac{x^{n}}{n!} = \frac{1}{k!}(E_{m}(x) - 1)^{k} \tag{4}$$

and

$$\sum_{n=m}^{\infty} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_{\geq m} \frac{x^{n}}{n!} = \frac{1}{k!}(e^{x} - E_{m-1}(x))^{k} \tag{5}$$

respectively, where

$$E_{m}(t) = \sum_{k=0}^{m} \frac{t^{k}}{k!}$$

is the $m$th partial sum of the exponential function sum. These give the number of the $k$-partitions of an $n$-element set, such that each block contains at most or at least $m$ elements, respectively. Since the generating function of $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is given by

$$\sum_{n=k}^{\infty} \left\{ \begin{array}{c} n \\ k \end{array} \right\} \frac{x^{n}}{n!} = \frac{(e^{x} - 1)^{k}}{k!}$$

(see e.g., [Jor]), by $E_{\infty}(x) = e^{x}$ and $E_{0}(x) = 1$, we have

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\}_{\leq \infty} = \left\{ \begin{array}{c} n \\ k \end{array} \right\}_{\geq 1} = \left\{ \begin{array}{c} n \\ k \end{array} \right\}. $$

These give the number of the $k$-partitions of an $n$-element set, such that each block contains at most or at least $m$ elements, respectively. Notice that these numbers
where \( m = 2 \) have been considered by several authors (e.g., [Com], [How], [Rio], [Zha]).

It is well-known that the Stirling numbers of the second kind satisfy the recurrence relation:

\[
\left\{ \begin{array}{c} n+1 \\ k \end{array} \right\} = k \left\{ \begin{array}{c} n \\ k \end{array} \right\} + \left\{ \begin{array}{c} n \\ k-1 \end{array} \right\}
\]

for \( k > 0 \), with the initial conditions

\[
\left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} = 1 \quad \text{and} \quad \left\{ \begin{array}{c} n \\ 0 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ n \end{array} \right\} = 0
\]

for \( n > 0 \). The restricted and associated Stirling numbers of the second kind satisfy the similar relations. It is easy to see the initial conditions

\[
\left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \leq m = 1 \quad \text{and} \quad \left\{ \begin{array}{c} n \\ 0 \end{array} \right\} \leq m = \left\{ \begin{array}{c} 0 \\ n \end{array} \right\} \leq m = 0
\]

\[
\left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \geq m = 1 \quad \text{and} \quad \left\{ \begin{array}{c} n \\ 0 \end{array} \right\} \geq m = \left\{ \begin{array}{c} 0 \\ n \end{array} \right\} \geq m = 0
\]

for \( n > 0 \).

**Proposition 1.** For \( k > 0 \) we have

\[
\left\{ \begin{array}{c} n+1 \\ k \end{array} \right\} \leq m = \sum_{i=0}^{m-1} \left. \left( \begin{array}{c} n \\ i \end{array} \right\} \left( \begin{array}{c} n-i \\ k-1 \end{array} \right\} \right. \leq m \quad (7)
\]

\[
= k \left\{ \begin{array}{c} n \\ k \end{array} \right\} \leq m + \left\{ \begin{array}{c} n \\ k-1 \end{array} \right\} \leq m - \left( \begin{array}{c} n \\ m \end{array} \right\} \left\{ \begin{array}{c} n-m \\ k-1 \end{array} \right\} \leq m , \quad (8)
\]

\[
\left\{ \begin{array}{c} n+1 \\ k \end{array} \right\} \geq m = \sum_{i=m-1}^{n} \left. \left( \begin{array}{c} n \\ i \end{array} \right\} \left( \begin{array}{c} n-i \\ k-1 \end{array} \right\} \right. \geq m \quad (9)
\]

\[
= k \left\{ \begin{array}{c} n \\ k \end{array} \right\} \geq m + \left( \begin{array}{c} n \\ m-1 \end{array} \right\} \left\{ \begin{array}{c} n-m+1 \\ k-1 \end{array} \right\} \geq m . \quad (10)
\]

**Remark.** The fourth relation (10) appeared in a different form in [How]. Since

\[
\sum_{i=1}^{n-k+1} \left( \begin{array}{c} n \\ i \end{array} \right\} \left( \begin{array}{c} n-i \\ k-1 \end{array} \right\} = k \left\{ \begin{array}{c} n \\ k \end{array} \right\} ,
\]

the relation (7) and the relation (9) are both reduced to the relation (6), if \( m \geq n-k+2 \) and if \( m = 1 \), respectively. It is trivial to see that the relations (8) and (10) are also reduced to the original relation (6), if \( m > n \) and if \( m = 1 \), respectively.
PROOF OF PROPOSITION 1. The combinatorial proofs of the previous theorem are given as follows. We shall give combinatorial proofs. First, identity (7). To construct a partition with \( k \) blocks on \( n + 1 \) element we can do the following. The last element in its block can have \( i \) elements by side, where \( i = 0, 1, \ldots, m - 1 \). We have to choose these \( i \) elements from \( n \). This can be done in \( \binom{n}{i} \) ways. The rest of the elements go into \( k - 1 \) blocks in \( \left\{ \binom{n-i}{k-1} \right\}_{i \leq m} \) ways. Summing over the possible values of \( i \) we are done.

The proof of (8). The above construction can be described in another way: the last element we put into a singleton and the other \( n \) elements must form a partition with \( k - 1 \) blocks: \( \left\{ \binom{n}{k-1} \right\}_{i \leq m} \) possibilities. Or we put this element into one existing block after constructing a partition of \( n \) elements into \( k \) blocks. This offers us \( k \left\{ \binom{n}{k} \right\}_{i \leq m} \) possibilities, but we must subtract the possibilities when we exceed the block size limit \( m \). This happens if we put the last element into a block of \( m \) elements. There are \( \binom{n}{m} \left\{ \binom{n-m}{k-1} \right\}_{i \leq m} \) such partitions in total. The proof is done.

The proof of (9) and (10) is similar. \( \square \)

Note that the classical Stirling numbers of the second kind \( \left\{ \binom{n}{k} \right\} \) satisfy the identities:

\[
\left\{ \binom{n}{1} \right\} = \left\{ \binom{n}{n} \right\} = 1, \quad \left\{ \binom{n}{n-1} \right\} = \left( \binom{n}{2} \right),
\]
\[
\left\{ \binom{n}{n-2} \right\} = \frac{3n-5}{4} \left\{ \binom{n}{3} \right\}, \quad \left\{ \binom{n}{n-3} \right\} = \frac{n}{4} \left( \binom{n}{2} - \binom{n-2}{2} \right),
\]
\[
\left\{ \binom{n}{2} \right\} = 2^{n-1} - 1, \quad \left\{ \binom{n}{3} \right\} = \frac{3^{n-1}}{2} - 2^{n-1} + \frac{1}{2},
\]
\[
\left\{ \binom{n}{4} \right\} = \frac{4^{n-1}}{6} - \frac{3^{n-1}}{2} + 2^{n-2} - \frac{1}{6}.
\]

By the definition (4) or Proposition 1 (7), we list several basic properties about the restricted Stirling numbers of the second kind. Some basic properties about the associated Stirling numbers of the second kind can be found in [Com], [How], [Mez], [Zha].

**Lemma 1.** For \( 0 \leq n \leq k - 1 \) or \( n \geq mk + 1 \), we have

\[
\left\{ \binom{n}{k} \right\}_{i \leq m} = 0. \tag{11}
\]

For \( k \leq n \leq mk \), we have

\[
\left\{ \binom{n}{k} \right\}_{i \leq m} = \left\{ \binom{n}{k} \right\} \quad (k \leq n \leq m), \tag{12}
\]
\[ \{ \binom{n}{n} \}_{\leq m} = 1 \quad (n \geq 0, \ m \geq 1), \quad (13) \]
\[ \{ \binom{n}{n-1} \}_{\leq m} = \binom{n}{2} \quad (n \geq 2, \ m \geq 2), \quad (14) \]
\[ \{ \binom{n}{n-2} \}_{\leq m} = \binom{3n-5}{n} \binom{n}{3} \quad (n \geq 4, \ m \geq 4); \]
\[ \{ \binom{n}{n-3} \}_{\leq m} = \binom{n}{4} \binom{n-2}{2} + 15(n^4) + 10(n^3) \quad (n \geq 4, \ m = 3); \]
\[ \{ \binom{n}{1} \}_{\leq m} = 1 \quad (1 \leq n \leq m), \quad (17) \]
\[ \{ \binom{n}{2} \}_{\leq m} = 2^{n-1} - 1 \quad (2 \leq n \leq m+1). \quad (18) \]

**Proof.** Some of the above special values are trivial. Some of them can be proven by analyzing the possible block structures.

We take (15) as a concrete example.

Let \( m = 2, \) and the number of blocks be \( k = n - 2. \) Then for the block structure we have the only one possibility

\[
\begin{array}{c}
| . | \cdot | . | \cdot | . | . \\
\hline
n-4
\end{array}
\]

That is, there are \( n-4 \) singletons and two blocks of length 2. There are \( \binom{4}{2} \binom{n}{4} = 3 \binom{n}{4} \) such partitions: we have to choose those four elements going to the non singleton blocks in \( \binom{n}{4} \) ways. Then we put two of four into the first block and the other two goes to the other: \( \binom{4}{2} = 6 \) cases. Finally, we have to divide by two because the order of the blocks does not matter. The last case of (15) follows.

If \( m = 3 \) then we have one more possible distribution of blocks sizes apart from the above:

\[
\begin{array}{c}
| . | \cdot | . | \cdot | . | . \\
\hline
n-5
\end{array}
\]

Into the last block we have \( \binom{n}{4} \) possible option to put 3 elements. So if \( m = 3 \) and \( k = n - 2 \) then we have \( \binom{n}{4} + 3 \binom{n}{4} = \frac{3n-5}{4} \binom{n}{4} \) cases in total.

The rest of the cases can be treated similarly. \( \square \)
3. Incomplete poly-Bernoulli numbers

3.1. Generating function and its integral representation. By using two types of incomplete Stirling numbers, define restricted poly-Bernoulli numbers $B_{n,\leq m}^{(\mu)}$ and associated poly-Bernoulli numbers $B_{n,\geq m}^{(\mu)}$ by

$$B_{n,\leq m}^{(\mu)} = \sum_{k=0}^{n} (-1)^{n-k} \frac{k!}{(k+1)\mu} \binom{n}{k} \mathcal{I}_{\leq m} \quad (n \geq 0),$$

and

$$B_{n,\geq m}^{(\mu)} = \sum_{k=0}^{n} (-1)^{n-k} \frac{k!}{(k+1)\mu} \binom{n}{k} \mathcal{I}_{\geq m} \quad (n \geq 0),$$

respectively. These numbers can be considered as generalizations of the usual poly-Bernoulli numbers $B_n^{(\mu)}$, since

$$B_{n,\leq \infty}^{(\mu)} = B_{n,\geq 1}^{(\mu)} = B_n^{(\mu)}.$$

We call these numbers as incomplete poly-Bernoulli numbers. One can deduce that these numbers have the generating functions.

**Theorem 1.** We have

$$\sum_{n=0}^{\infty} B_{n,\leq m}^{(\mu)} \frac{t^n}{n!} = Li_{\mu}(1 - E_m(-t)) \frac{1}{1 - E_m(-t)},$$

and

$$\sum_{n=0}^{\infty} B_{n,\geq m}^{(\mu)} \frac{t^n}{n!} = Li_{\mu}(E_{m-1}(-t) - e^{-t}) \frac{E_m(-1) - e^{-t}}{E_{m-1}(-t) - e^{-t}}.$$

**Remark.** In the first formula $m \to \infty$ gives back the poly-Bernoulli numbers (2) since $E_\infty(-t) = e^{-t}$ and $Li_1(z) = -\log(1 - z)$, while in the second we must take $m = 1$ since $E_0(-t) = 1$.

**Proof of Theorem 1.** By the definition of (20) and using (4), we get

$$\sum_{n=0}^{\infty} B_{n,\leq m}^{(\mu)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k}k!}{(k+1)\mu} \binom{n}{k} \mathcal{I}_{\leq m} \frac{t^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(k+1)\mu} \sum_{n=k}^{\infty} \binom{n}{k} \mathcal{I}_{\leq m} \frac{(-t)^n}{n!}.$$
Similarly, by the definition of (19) and using (5), we get
\[
\sum_{k=0}^{\infty} \frac{(1 - E_m(-t))^k}{(k + 1)^\mu} = \frac{\text{Li}_\mu(1 - E_m(-t))}{1 - E_m(-t)}.
\]

For \( \mu \geq 1 \), the generating functions can be written in the form of iterated integrals. We set \( E_{-1}(-t) = 0 \) for convenience.

Theorem 2.
\[
\frac{1}{1 - E_m(-t)} \int_0^t E_{m-1}(-t) \int_0^t \cdots \int_0^t E_{m-n}(t) \int_0^t E_{m-1}(t) \frac{x^n}{n!} dt \cdots dt = \sum_{n=0}^{\infty} B_{n, \geq m}^{(\mu)} \frac{x^n}{n!}, \tag{23}
\]
\[
\frac{1}{E_{m-1}(-t) - e^{-t}} \int_0^t e^{-t - E_{m-2}(-t)} \int_0^t \cdots \int_0^t e^{-t - E_{m-1}(-t) - e^{-t}} \int_0^t e^{-t - E_{m-2}(-t)} \frac{x^n}{n!} dt \cdots dt = \sum_{n=0}^{\infty} B_{n, \geq m}^{(\mu)} \frac{x^n}{n!}. \tag{24}
\]

Remark. If \( m \to \infty \) in (23), by \( E_\infty(-t) = e^{-t} \), and if \( m = 1 \) in (24), by \( E_0(-t) = 1 \) and \( E_{-1}(-t) = 0 \), both of them are reduced to (3).
Proof of Theorem 2. Since for \( \mu \geq 1 \)
\[
\frac{d}{dt} \text{Li}_\mu (1 - E_m(-t)) = \frac{E_{m-1}(-t)}{1 - E_m(-t)} \text{Li}_{\mu - 1} (1 - E_m(-t)),
\]
we have
\[
\text{Li}_\mu (1 - E_m(-t)) = \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \text{Li}_{\mu - 1} (1 - E_m(-t)) dt
\]
\[
= \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \text{Li}_{\mu - 2} (1 - E_m(-t)) dt dt
\]
\[
= \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \cdots \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \text{Li}_1 (1 - E_m(-t)) \frac{dt}{\mu - 1} \text{Li}_{\mu - 1} (1 - E_m(-t)) dt \cdots dt
\]
Therefore, we obtain (23). Similarly, by
\[
\frac{d}{dt} \text{Li}_\mu (E_m(-t) - e^{-t}) = \frac{e^{-t} - E_{m-2}(-t)}{E_{m-1}(-t) - e^{-t}} \text{Li}_{\mu - 1} (E_{m-1}(-t) - e^{-t}),
\]
we obtain (24). \( \square \)

If \( \mu = 1 \) in Theorem 1 or in Theorem 2, the generating functions of the restricted Bernoulli numbers \( B^{(1)}_{n, \leq m} \) and associated Bernoulli numbers \( B^{(1)}_{n, \geq m} \) are given. Both functions below are reduced to the generating function (1) of the Bernoulli numbers \( B^{(1)}_n \) if \( m \to \infty \) and \( m = 1 \), respectively.

Corollary 1. We have
\[
\sum_{n=0}^\infty B^{(1)}_{n, \leq m} \frac{t^n}{n!} = \frac{\log E_m(-t)}{E_m(-t) - 1}
\]
and
\[
\sum_{n=0}^\infty B^{(1)}_{n, \geq m} \frac{t^n}{n!} = \frac{\log (1 + e^{-t} - E_{m-1}(-t))}{e^{-t} - E_{m-1}(-t)}.
\]
3.2. Basic divisibility for non-positive $\mu$. In this short subsection we deduce a basic divisibility property for both the restricted and associated poly-Bernoulli numbers.

It is known [GraKnuPat] that

\[ \binom{p}{k} \equiv 0 \pmod{p} \quad (1 < k < p) \]

for any prime $p$. The proof of this basic divisibility is the same for the restricted and associated Stirling numbers, so we can state that

\[ \binom{p}{k} \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots) \]

and

\[ \binom{p}{k} \equiv 0 \pmod{p} \quad (k \geq 2). \]

(Note that $\binom{p}{1} \equiv 1 \pmod{p}$.) These immediately lead to the next statement.

**Theorem 3.** For any $\mu \leq 0$ we have that

\[ B^{(\mu)}_{p, \leq m} \equiv 0 \pmod{p}, \]

\[ B^{(\mu)}_{p, \geq m} \equiv 2|\mu| \pmod{p} \]

hold for any prime $p$.

4. A new series representation for the Riemann zeta function

To present our result, we need to recall the definition of the Lambert $W$ function. $W(a)$ is the solution of the equation

\[ xe^x = a, \]

that is, $W(a)e^{W(a)} = a$. Since this equation, in general, has infinitely many solutions, the $W$ function has infinitely many complex branches denoted by $W_k(a)$ where $k \in \mathbb{Z}$. What we prove is the following:

**Theorem 4.** For any $\mu \in \mathbb{C}$ with $\Re(\mu) > 1$ we have that

\[ \zeta(\mu) = \sum_{n=0}^{\infty} B^{(\mu)}_{n, \geq 2} \frac{(W_k(-1))^n}{n!} \]

for $k = 0, -1$, where $\zeta$ is the Riemann zeta function.
Proof. Let us recall the generating function of $B_{n, \geq m}^{(\mu)}$ in the particular case when $m = 2$:

$$\sum_{n=0}^{\infty} B_{n, \geq 2}^{(\mu)} \frac{(-t)^n}{n!} = \text{Li}_\mu(1 + t - e^t).$$

(25)

By a simple transformation it can be seen that the equation $1 + t - e^t = 1$ is solvable in terms of the Lambert $W$ function, and that the solution is $-W_k(-1)$ for any branch $k \in \mathbb{Z}$. However, (25) is valid only for $t$ such that $|1 + t - e^t| \leq 1$, at least when $\Re(\mu) > 1$. (This comes from the proof of Theorem 1.) Since the absolute value of $-W_k(-1)$ grows with $k$, the only two branches which belong the convergence domain of (25) is $k = -1, 0$. Hence, substituting one of these in place of $t$ we have that

$$\sum_{n=0}^{\infty} B_{n, \geq 2}^{(\mu)} \frac{(W_k(-1))^n}{n!} = \text{Li}_\mu(1 - W_k(-1) - e^{-W_k(-1)}) = \text{Li}_\mu(1) = \zeta(\mu). \quad \Box$$

Note that

$$W'_0(-1) = W_{-1}(-1) \approx -0.318132 + 1.33724i,$$

so all the terms in the incomplete Bernoulli sum are complex, but the sum itself always converges to the real number $\zeta(\mu)$.

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