The effects of strong temperature anisotropy on the kinetic structure of collisionless slow shocks and reconnection exhausts. Part II: Theory

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Simulations of collisionless oblique propagating slow shocks have revealed the existence of a transition associated with a critical temperature anisotropy \( \varepsilon = 1 - \mu_0(P_\parallel - P_\perp)/B^2 = 0.25 \) (Liu, Drake and Swisdak (2011)[1]). An explanation for this phenomenon is proposed here based on anisotropic fluid theory, in particular the Anisotropic Derivative Nonlinear-Schrödinger-Burgers equation, with an intuitive model of the energy closure for the downstream counter-streaming ions. The anisotropy value of 0.25 is significant because it is closely related to the degeneracy point of the slow and intermediate modes, and corresponds to the lower bound of the coplanar to non-coplanar transition that occurs inside a compound slow shock (SS)/rotational discontinuity (RD) wave. This work implies that it is a pair of compound SS/RD waves that bound the outflows in magnetic reconnection, instead of a pair of switch-off slow shocks as in Petschek’s model. This fact might explain the rareness of in-situ observations of Petschek-reconnection-associated switch-off slow shocks.

I. INTRODUCTION

Shocks in isotropic MHD have been intensively studied, including the existence of intermediate shocks (IS) [2-5] and references therein), the occurrence of dispersive wavetrains [6-8], and the nested subshocks inside shocks predicted by the Rankine-Hugoniot jump conditions [9-10]. In a collisionless plasma, the effects of temperature anisotropy need to be considered, which can be done for linear waves with the Chew-Goldberger-Low (CGL) framework [11][12]. Hau and Sonnerup have pointed out the abnormal properties of the linear slow mode under the influence of a firehose-sense \( (P_\parallel > P_\perp) \) pressure anisotropy, including a faster phase speed compared to the intermediate mode, a fast-mode-like positive correlation between magnetic field and density, and the steepening of the slow expansion wave. In kinetic theory both anisotropy and high \( \beta \) can greatly alter the linear mode behavior [13][14]. The anisotropic Rankine-Hugoniot jump conditions have been explored while taking the downstream anisotropy as a free parameter [15][16], while Hudson [17] calculated the possible anisotropy jumps across an anisotropic rotational discontinuity. Karimabadi et al. [14] noticed the existence of a slow shock whose upstream and downstream are both super-intermediate. But, a comprehensive nonlinear theory describing the coupling between slow and intermediate shocks under the influence of temperature anisotropy has not yet been presented.

In Petschek’s description of magnetic reconnection, the reconnection exhaust is bounded by a pair of back-to-back standing switch-off slow shocks. Particle-in-cell (PIC) simulations of such shocks [1][18] exhibit large downstream temperature anisotropies. In Liu et al. (2011) [1] (hereafter called Paper I) and Fig. 1 of this paper we show that when the parameter \( \varepsilon = 1 - \mu_0(P_\parallel - P_\perp)/B^2 = 0.25 \), the behavior of the coplanar shock undergoes a transition to non-coplanar rotation. This firehose-sense temperature anisotropy slows the linear intermediate mode and speeds up the linear slow mode enough so that, at some point, their relative velocities can be reversed [11][12]. This reversal is reflected in the structure of the Sagdeev potential (also called the pseudo-potential) [20], which characterizes the nonlinearity of the system. In this work a simplified theoretical model is developed to explore the effect of temperature anisotropy on the structure of the Sagdeev potential and to provide an explanation for the extra transition inside the switch-off slow shock (SSS) predicted by isotropic MHD. The theory suggests that in PIC simulations a compound slow shock (SS)/ rotational discontinuity (RD) is formed instead of a switch-off slow shock. This work may help to explain satellite observations of compound SS/RD waves [21][22], anomalous slow shocks [23] and the trapping of an RD by the internal temperature anisotropy of a slow shock in hybrid simulations [24].

In Sec. II of this paper we introduce our model equations for studying the nonlinear coupling of slow and intermediate waves under the influence of a temperature anisotropy. In Sec. III we calculate the speeds and the eigenmodes of slow and intermediate waves. Sec. IV points out the existence of extra degeneracy points between slow and intermediate modes introduced by the temperature anisotropy, and comments on the consequences (in the context of the Riemann problem) of having the slow wave faster than the intermediate wave. In Sec. V we introduce a simple energy closure. In Sec. VI. A, we calculate the pseudo-potential of stationary solutions, and apply the equal-area rule to identify the existence of compound SS/RD waves and compound SS/IS waves. In Sec. VI. B we demonstrate the significance of \( \varepsilon = 0.25 \) as being the lower bound of the SS to RD transition in compound SS/RD waves. In Sec. VII we discuss the time-dependent dynamics that help keep \( \varepsilon = 0.25 \). In Sec VIII we provide more evidence from PIC simulations to support the existence of compound SS/RD waves at the boundaries of reconnection exhausts. In Sec
II. THE ANISOTROPIC DERIVATIVE NONLINEAR SCHRODINGER-BURGERS EQUATION

Instead of analyzing the anisotropic MHD equations, which have seven characteristics (waves), we simplify the system into a model equation that possesses only two characteristics. This model equation will be ideal for demonstrating the underlying coupling between the nonlinear slow and intermediate modes. Beginning with the anisotropic MHD equations \[15\], we follow the procedure of Kennel et al. \[14, 25\] to derive the Anisotropic Derivative Nonlinear Schrödinger-Burgers equation (ADNLSB) (see Appendix I for details),

\[
\partial_t \mathbf{b}_x + \partial_y [\alpha \mathbf{b}_t (b_y^2 - b_{0t}^2) + \Omega \mathbf{b}_x (\varepsilon - \varepsilon_0)] = \partial_y (R \partial_y \mathbf{b}_x) - \frac{1}{2\sqrt{\varepsilon_0}} \partial_y^2 (\mathbf{e}_x \times \mathbf{b}_t) \tag{1}
\]

This equation describes waves that propagate in the $x$-direction in the upstream intermediate speed frame. In this frame $\eta \equiv x' - \sqrt{\varepsilon_0} C_{An} t$ is the spatial coordinate with $C_{An}^2 \equiv B_0^2/(\mu_0 \rho_0)$, and $\tau \equiv \sqrt{\varepsilon_0} C_{An} t$ is the time used to measure the slow variations (such as steepening processes). $\mathbf{b}_x = B_x / B_z$, where the subscript "t" represents the component tangential to the wave-vector and here will be in the $y$-$z$ plane. The anisotropy parameter $\varepsilon = 1 - \mu_0 (P_b - P_e) / B^2$. The subscript "0" denotes the upstream parameters. The right hand side term proportional to the ion inertial length, $d_i \equiv \sqrt{m_i/\mu_0 ne^2}$, represents dispersion (which can be viewed as the spreading tendency of Fourier decomposed waves of different wavenumbers), while the term containing $R$ describes dissipation from magnetic resistivity. Here $R$ is a constant. The terms proportional to $\alpha$ and $\Omega$ are the nonlinearities of this wave equation, where

\[
\alpha \equiv \frac{[(3\gamma - 1)\varepsilon_0 - (3\gamma - 4)]C_{An}^2}{12[(\varepsilon_0 - A)C_{An}^2 - C_S^2]} \tag{2}
\]

\[
\Omega \equiv \frac{(b_{0t}^2 - 2)C_{An}^2}{6[(\varepsilon_0 - A)C_{An}^2 - C_S^2]} + \frac{1}{2\varepsilon_0} \tag{3}
\]

and

\[
A \equiv \frac{2}{3}(\gamma - 1) (1 - \varepsilon_0) (1 + b_{0t}^2) \tag{4}
\]

Here $C_S^2 \equiv \gamma P_0 / \rho_0$ and $\gamma = 5/3$ for monatomic plasma. Since we are studying reconnection exhausts, $\theta_0$ (the angle between the upstream magnetic field and $\hat{e}_x$) is typically large ($\sim 80^\circ$). Therefore $\beta_n \equiv \beta / \cos^2 \theta_0 = C_S^2 / C_{An}^2 \gg 1$, and $(\varepsilon_0 - A)C_{An}^2 - C_S^2 \sim -C_S^2 < 0$. Hence this equation describes only the slow and intermediate modes \[1\] (this is shown explicitly in the next section). This fact relates to the degeneracy properties of ideal MHD for parallel propagating waves, namely that the fast and intermediate modes degenerate in $\beta < 1$ plasmas, while the slow and intermediate mode degenerate in $\beta > 1$ plasmas. Finally, we note that Eq. \[1\] is applicable in the weak nonlinearity limit.

III. THE CONSERVATIVE FORM- WAVE PROPAGATION

In order to explore the structure of the reconnection exhaust, a comprehensive understanding of how waves connect to each other across a transition is required. This is called a Riemann problem. Neglecting the source terms on the right hand side (RHS), the left hand side (LHS) of Eq. \[1\] is a hyperbolic equation in conservative form.

Letting $\varepsilon - \varepsilon_0 = \delta \varepsilon (b_z, b_y)$, and $\mathbf{b}_{i0} = b_{i0} \hat{e}_x$, we obtain,

\[
\partial_t \mathbf{q} + \partial_y \mathbf{f}(\mathbf{q}) = 0, \tag{5}
\]

with

\[
\mathbf{q} \equiv \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} b_z \\ b_y \end{bmatrix}, \quad \mathbf{f} \equiv \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \alpha b_z (b_z^2 + b_y^2 - b_{0t}^2) + \Omega b_x \delta \varepsilon \\ \alpha b_y (b_z^2 + b_y^2 - b_{0t}^2) + \Omega b_y \delta \varepsilon \end{bmatrix} \tag{6}
\]
We can obtain the characteristics (waves) of this equation by analyzing its flux function, \( f \). Its Jacobian is

\[
\partial_q f = \left[ \begin{array}{c} \alpha(3b_z^2 + b_y^2 - b_{0y}^2) + \Omega(\delta \varepsilon + b_z \delta \varepsilon_{b_z}) \
2 \alpha b_z b_y + \Omega b_y \delta \varepsilon_{b_z} \
\alpha(3b_y^2 + b_z^2 - b_{0z}^2 + \Omega(\delta \varepsilon + b_y \delta \varepsilon_{b_y})) \end{array} \right],
\]

where \( \delta \varepsilon_{b_z} = \frac{\partial (\delta \varepsilon)}{\partial b_z}, \delta \varepsilon_{b_y} = \frac{\partial (\delta \varepsilon)}{\partial b_y} \). One eigenvalue (also called the characteristic speed) is

\[
\lambda_{SL} = \alpha(3b_z^2 - b_{0y}^2) + \Omega(\delta \varepsilon + b_z \delta \varepsilon_{b_z} + b_y \delta \varepsilon_{b_y})
\]

with eigenvector

\[
\mathbf{r}_{SL} = \frac{1}{b_t} \left[ \begin{array}{c} b_z \\ b_y \end{array} \right].
\]

In isotropic ideal MHD, where the \((\varepsilon - \varepsilon_0)\) term is dropped, the eigenvalue in the infinitesimal limit \((b_t \to b_{0t})\) is \(\lambda_{SL} = 2ab_{0y}^2\), which is the phase speed of the linear slow mode in the intermediate mode frame. The subscript “SL” means the slow mode. The eigenvector indicates that slow mode is coplanar (i.e., in the radial direction in \(b_z - b_y\) space).

The other mode has eigenvalue

\[
\lambda_I = \alpha(b_t^2 - b_{0t}^2) + \Omega \delta \varepsilon
\]

and eigenvector

\[
\mathbf{r}_I \propto \left[ \begin{array}{c} \alpha b_y + \Omega \delta \varepsilon_{b_y} / 2 \\ -\alpha b_z - \Omega \delta \varepsilon_{b_z} / 2 \end{array} \right].
\]

In isotropic ideal MHD, where the \((\varepsilon - \varepsilon_0)\) term is dropped, the eigenvalue in the infinitesimal limit is \(\lambda_I = 0\), which is the phase speed of the linear intermediate mode in the intermediate mode frame. The subscript “I” means the intermediate mode. The eigenvector indicates that this intermediate mode is non-coplanar (i.e., in a non-radial direction).

It can be shown that \(\nabla_q \lambda_I(q) \cdot \mathbf{r}_I(q) = 0\) for all \(q\). This means that along the eigen-direction of the intermediate mode the characteristic speed is constant, and thus the mode exhibits no steepening or spreading, just as is the case for its counterpart in isotropic MHD. (This behavior is also confirmed by the anisotropic MHD simple wave calculation; see Appendix II). Therefore the intermediate mode is termed “linearly degenerate”. If we are looking for a transition in the \(-\mathbf{r}_{SL}\) direction (toward \(b_t = 0\)), then the portion of the slow mode with \(\nabla_q \lambda_{SL}(q) \cdot \mathbf{r}_{SL}(q) < 0\) will steepen into a slow shock. When \(\lambda_{SL}\) at the downstream of a transition is larger than that at the upstream, the downstream wave will catch up with the upstream wave and thus steepen.

### IV. A NEW DEGENERACY POINT DUE TO THE TEMPERATURE ANISOTROPY

In the Riemann problem for our two mode system, we seek to determine the middle state \(q_m\) that connects the faster “2-wave” from a given state \(q_r\), to the slower “1-wave” from a given state \(q_l\) (see Fig. 2(a); the subscripts “r” and “l” mean right and left respectively). In order to determine the path that connects \(q_r\) to \(q_l\) in the state space \((b_t - b_y\) space, in this case), the Hugoniot locus that connects \(q_l\) or \(q_r\) to a possible asymptotic state by shock waves needs to be calculated, as do the integral curves for possible rarefaction waves (see, for example, [26]). The Hugoniot locus in state space is a curve formed by allowing one of the parameters in the standard Rankine-Hugoniot jump condition to vary. The integral curve is formed by following the eigenvector from a given state in state space.

In order to proceed we further assume a gyrotropic energy closure, \(\delta \varepsilon = \delta \varepsilon(b_t)\), which allows us to write

\[
\mathbf{r}_{SL} = \frac{1}{b_t} \left[ \begin{array}{c} b_z \\ b_y \end{array} \right], \quad \mathbf{r}_I = \frac{1}{b_t} \left[ \begin{array}{c} b_y \\ -b_z \end{array} \right], \quad \lambda_{SL} - \lambda_I = 2\alpha b_t^2 + \Omega \delta \varepsilon_{b_t} b_t
\]

The Hugoniot locus and integral curves of the intermediate mode are identical in our system. This is also the case for the slow mode (we perform the calculation in Appendix III). For a given state, \(q_r = b_{0y} = (b_{0z}, 0)\), the Hugoniot locus and integral curve of the intermediate mode are

\[
b_y^2 = b_{0y}^2 - b_z^2
\]
which is a circle in state space. Note that even though we can calculate the Hugoniot locus and integral curve for the intermediate mode, the solution is the same as for a finite amplitude intermediate mode that does not steepen into a shock or spread into a rarefaction. For the slow mode, the Hugoniot locus and integral curve are

\[ b_y = 0 \]  

which is in the radial direction in state space. This direction implies that the slow shock is coplanar, even in the presence of temperature anisotropy, just as is the case for its counterpart in full anisotropic MHD [15]. In the isotropic case, this curve forms a slow shock if the path is toward the origin, and a slow rarefaction if the path is away from the origin.

The state \( q_r \) can connect to \( q_l = (0, 0) \) by following the Hugoniot locus of a slow mode that starts from \( q_r \), as shown in Fig. 2(b). In isotropic MHD this forms a switch-off slow shock. However, a strong enough temperature anisotropy introduces new degeneracy points (which occur where \( \lambda_{SL} - \lambda_I = 0 \)) when \( 2\alpha b_t + \Omega \delta \varepsilon_{b_t} = 0 \), other than the traditional degeneracy point at \( b_t = 0 \). These points form a band circling the origin as shown in Fig. 2(c). Inside the band, the intermediate mode is slower than the slow mode. Physically, this implies that a rotational intermediate mode can arise downstream of a slow mode, something which is not allowed in a Riemann problem in isotropic MHD. This effect is realized when the path along the Hugoniot locus (\( -r_{SL} \) direction) of the slow mode from \( q_r \) switches to the solution of the intermediate mode (circular direction) somewhere (\( q_m \)) inside the degeneracy band.

This behavior can explain the morphological differences between the shock simulations in the cases \( \theta_{BN} = 30^\circ, 45^\circ \) and those for \( \theta_{BN} = 60^\circ, 75^\circ, 83^\circ \) of Paper I [1]. The latter has an extra transition to the rotational direction that is similar to the path in Fig. 2(c). We now look for a similar effect in state space and a way of determining \( q_m \) in a more detailed analytical model.

V. AN ENERGY CLOSURE BASED ON COUNTER-STREAMING IONS

In order to close the ADNLSB equations, we need a energy closure \( \varepsilon(b_t) \). The modeling of the energy closure for a collisionless plasma has historically been difficult. The Chew-Goldberger-Low (CGL) condition [27] is one choice, but it does not work well when streaming ions are present. Since we are here just trying to qualitatively demonstrate the underlying physics, we will assume that we have a \( \varepsilon(b_t) \), where \( \varepsilon \) and \( |B| \) are simply related by

\[ \varepsilon = c_1 - \frac{c_2}{B^2} \]  

with positive constants \( c_1 \) and \( c_2 \) and the condition \( c_1 - c_2/B_t^2 = \varepsilon_0 \) is imposed. This functional form is motivated by the nearly constant parallel pressure maintained by free-streaming ions (\( c_2 \sim P_l \)). Although Eq. (15) is strictly empirical, results from PIC simulations (see Fig. 3) suggest that \( c_2 = 0.5 \) provides a reasonable first approximation and will be used in the following calculations.

We take the variation,

\[ \delta \varepsilon = \varepsilon - \varepsilon_0 \sim \frac{c_2}{B^2 b_t^4} \delta (b_t^2) = \frac{c_2}{B^2 b_t^4} (b_t^2 - b_t^2_0) \]  

where \( b_t^2 = 1 + b_t^2 \). This parameterization will be valid whenever \( \delta \varepsilon(b_t) \ll \varepsilon_0 \). Therefore, an effective nonlinearity in Eq. (11) can be written as,

\[ \alpha_{\text{eff}}(b_t) = \alpha + \Omega \frac{c_2}{B^2 b_t^4} \]  

The most important conclusions in the remainder of this work do not depend on the details of the closure, but only that \( \varepsilon \) decreases as \( |B| \) decreases. Note that \( \Omega \sim 1/2\varepsilon_0 \) is mostly positive in the limit in which we are interested. This fact will be used in the following section.

VI. THE PSEUDO-POTENTIAL: LOOKING FOR A STATIONARY SOLUTION
In order to determine both where the path in the state space of Fig. 2(c) will turn to the intermediate rotation and the nontrivial coupling of the slow and intermediate modes when temperature anisotropies are present, we construct the pseudo-potential of a stationary solution. We will look for an equation that possesses traveling stationary waves, by substituting \( b_i = b_i \left[ \xi (\eta - V_S \tau) \right] \) (where \( V_S \) is the speed of the stationary wave observed in the upstream intermediate frame) into Eq. (1) and integrating over \( \xi \) once. We obtain

\[
R \partial_\xi b_i - \frac{1}{2 \sqrt{\xi_0}} d_i \partial_\xi (\xi x_i \times b_i) = -V_S (b_i - b_0) + \alpha_{\text{eff}} (b_i) b_i (b_i^2 - b_0^2) \equiv -F \equiv \partial_{b_i} \Psi. \tag{18}
\]

In this formulation we can treat \( b_i \) as a spatial coordinate, and \( \xi \) as time. The terms on the LHS of Eq. (18) behave analogously to, respectively, a frictional force and a Coriolis force with rotational frequency \( \frac{d_i}{\sqrt{\xi_0}} \) and rotational axis \( \xi x_i \). A pseudo-potential \( \Psi \) that characterizes the nonlinearity is uniquely defined because \( \partial F_x / \partial b_y = \partial F_y / \partial b_z \). We are only interested in the small \( V_S \) limit, because the upstream values with subscript “0” are expected to be the upstream values of a switch-off slow shock in ideal isotropic MHD, which propagates at the upstream intermediate speed. The anisotropy in our PIC simulations does not seem to significantly change this behavior [1].

Calculating the pseudo-work done on the pseudo-particle, \( \int [\text{Eq. (18)}] \cdot \partial_\xi b_i d\xi \), we obtain

\[
\Psi_{\text{down}}^{\text{up}} = R \int_{\text{up}}^{\text{down}} (\partial_\xi b_i)^2 d\xi < 0 \tag{19}
\]

Note that from upstream to downstream is in the negative \( \xi \) direction. The pseudo-particle will move to a lower potential, while its total energy is dissipated by the resistivity and the rate of the drop depends on the strength of the resistivity. Kennel et al. [4] have shown that when pseudo-particles move toward lower pseudo-potentials, the entropy increases and so the resulting shock is admissible. Note that the Coriolis-like force does not do work. It only drives rotation of the pseudo-particle on the iso-surface of the pseudo-potential and hence causes stable nodes to become unstable nodes, unstable nodes to become unstable spiral nodes, and thus leading to the formation of dispersive wavetrains [2,8]. We will neglect its effect in the following discussion.

The pseudo-potential is shown in Fig. 4(a) for the parameters \( \theta_0 = 42^\circ, \beta_0 = 1, \varepsilon_0 = 1, c_2 = 0.5 \) and \( V_S = 0 \). The temperature anisotropy has turned the origin from a local minimum of the pseudo-potential in the isotropic MHD model to a local maximum. We term this the “reversal behavior”. A pseudo-particle initially at point \( q_r \) will slide down the hill in the slow mode eigen-direction, and then follow the circular valley in the intermediate eigen-direction. Without the reversal behavior (e.g., in isotropic MHD) the pseudo-particle will slide down to the origin and form a switch-off slow shock. The trajectory of the pseudo-particle can also be calculated by numerically integrating Eq. (18) with respect to \( \xi \). In (b) the variation of the temperature anisotropy is shown. Similar reversal behaviors can be found in fully anisotropic MHD with the energy closure used here, or with the CGL closure (see Appendix IV.A).

In Fig. 4(c), we plot a cut of the pseudo-force, \( F_z \), effective \( \alpha_{\text{eff}} \) and the pseudo-potential \( \Psi \) along the \( b_z \) axis \( (b_y = 0) \), which is the eigen-direction of a slow mode beginning at \( q_\varepsilon = (b_0, 0) \). Here

\[
F_z = -\alpha_{\text{eff}} (b_z) b_z (b_z^2 - b_0^2) + V_S (b_z - b_0). \tag{20}
\]

It is clear that \( \Psi_{\text{min}} \) occurs at \( F_z = 0 \) in Fig. 4(c), since \( \Psi \) is constructed by integrating the pseudo-force \( F \). In Fig. 4(d) we plot cuts of the characteristics of the slow and intermediate modes along the \( b_z \) axis.

\[
\lambda_{\text{SL}} = \alpha (3 b_z^2 - b_0^2) + \Omega c_2 \frac{b_z}{B_z^2} \left[ \left( \frac{1}{b_z^4} - 4 \frac{b_z^2}{b_0^2} \right) (b_z^2 - b_0^2) + 2 \frac{b_z^2}{b_0^4} \right] \tag{21}
\]

\[
\lambda_{\text{I}} = \alpha_{\text{eff}} (b_z) (b_z^2 - b_0^2) \tag{22}
\]

The temperature anisotropy has changed the shape of these characteristics. As a result, there are new degeneracy points \( \left( \lambda_{\text{SL}} = \lambda_{\text{I}} \right) \) between slow and intermediate waves such as the point “D”. The slow characteristic shows extra nonconvexity points, where no steepening and spreading occurs \( (i.e., \nabla_\mathbf{q} \lambda_{\text{SL}} (\mathbf{q}) \cdot \mathbf{r}_{\text{SL}} (\mathbf{q}) = 0) \), such as the point \( a \) (the local maximum of \( \lambda_{\text{SL}} \)). This is clearer when we compare the slow characteristic here to that in the isotropic case shown in Fig. 10 where \( b_t = 0 \) is the only degeneracy point and the nonconvexity point of the slow mode.

In order to identify the nonlinear waves determined by the route of the pseudo-particle, we apply the equal-area rule, which tells how shocks are steepened from characteristics. The equal-area rule (see Appendix V for more details) applied to \( \lambda_{\text{SL}} \) shows that the sliding route \( \text{(point } q_r \text{ to } q_m \text{)} \) forms a slow shock. Since \( \nabla_\mathbf{q} \lambda_{\text{SL}} (\mathbf{q}) \cdot \mathbf{r}_{\text{SL}} (\mathbf{q}) < 0 \)
therefore $\lambda_{SL}|_{q_m} > \lambda_{SL}|_{q_r}$, and thus the slow mode will steepen until the red area above the horizontal line $V_S = 0$ equals the red area below $V_S = 0$. The slow shock transition immediately connects to the intermediate mode (point $q_m$ to $b$, which is also from the equal-area rule on $\lambda_{SL}$) in the valley. The fact that both the upstream (point $q_m$) and downstream (point $b$) travel at the local $\lambda_I$ makes the intermediate discontinuity a RD. By comparing (c) and (d), we note that the potential minimum is exactly the location of $q_m$ as expected and it is below the degeneracy point ($b_z|_D > b_z|_{q_m}$). This fact is consistent with the comment in section V, which predicts that $q_m$ will be inside the degeneracy band. The horizontal lines $q_r - q_m$ and $q_m - b$ measure the propagation speed of the SS and the RD, which in this case are both zero in the upstream intermediate frame. They therefore form a compound SS/RD wave. The downstream of the slow shock (point $q_m$) is not able to connect to the slow rarefaction (SR) wave (point $a$) and thus not able to form a compound SS/SR, since the rarefaction is faster than the shock itself. This model gives an theoretical explanation for the possible satellite observations of compound SS/RD \cite{21, 22}, and the “compound SS/RD/SS waves” seen in hybrid simulations \cite{23}.

When $V_S \gtrsim 0$ the potential tilts down in the negative $b_z$ direction (see Fig. 5). In this case, the slow shock (point $q_r$ to $q_m$ in Fig. 5(d)) with shock speed $V_S$ is connected by an intermediate shock (IS) (point $q_m$ to $b$) with shock speed $V_S$, whose upstream is super-intermediate ($V_S > \lambda_I|_{q_m}$) while the downstream is sub-intermediate ($V_S < \lambda_I|_b$). This forms a compound SS/IS wave. Note that the intermediate shock is not steepened from the intermediate mode (which is consistent with the discussion in Sec. IV), but is steepened from the slow mode. The slow shock is abnormal with both upstream (point $q_r$) and downstream (point $q_m$) being super-intermediate. Karimabadi et al. \cite{14} call this kind of slow shock an anomalous slow shock. When $V_S \lesssim 0$ the potential tilts up in the negative $b_z$ direction and there is no extra transition at the SS downstream, since $\Psi(b) > \Psi(q_m)$ in this case and is therefore not accessible.

These results are independent of the details of $\varepsilon(b_z)$, but only require the reversal behavior somewhere downstream of the slow shock. This fact can be inferred from a simple relation: $V_S - \lambda_I|_{b_z} = (b_{z,0}/b_z)V_S$ for $b_z = b_{z,0}|_{q_m}$ or $b_{z,b}$, regardless of the detail of $\varepsilon(b_z)$ (from Eq. (20) and (22)). When $V_S = 0$, this relation ensures that the SS can always connect to a RD since $V_S - \lambda_I = 0$ at both points $q_m$ and $b$. When $V_S > 0$, the SS can always connect to an IS since $V_S - \lambda_I$ is positive (super-intermediate) at $q_m$ and negative (sub-intermediate) at $b$. We therefore conclude that the abnormal transitions of magnetic field structures seen in the PIC simulations of Paper I are most likely the transitions from the SS to the RD in a compound SS/RD wave or the SS to the IS in an SS/IS wave. We can hardly distinguish between these two compound waves in our PIC simulation, since $V_S$ is small and the time-dependent dynamics add uncertainties in measuring the exact value. We focus on further analyzing the compound SS/RD wave.

### B. The significance of $\varepsilon = 0.25$

For SS/RD waves ($V_S = 0$), the stationary points along $b_{z,s}$ are the roots of $F_z = 0$,

$$\alpha_{eff}(b_{z,s})|_{b_{z,s}}(b_{z,s}^2 - b_{z,0}^2) = 0$$

(23)

Here the subscript “s” represents “stationary”. We have three traditional stationary points, $b_{z,s} = b_{z,0}$ (point $q_r$), $-b_{z,0}$ and 0, as well as a new stationary point due to the temperature anisotropy, $b_{z,m}$ (the transition point $q_m$) determined by $\alpha_{eff}(b_{z,m}) = 0$. The fixed-point analysis of the first three points in isotropic fluid theory can be found in literature \cite{4, 7, 8, 28}.

As shown in Fig. 6, there is no slow mode transition if

$$\varepsilon_0 < \frac{3\gamma - 4}{3\gamma - 1} = 0.25$$

(24)

(with $\gamma = 5/3$ for monatomic plasma; note that this relation is independent of $\theta_{BN}$ and $\beta$), which occurs when the nonlinearity $\alpha$ of Eq. (1) changes sign from $\alpha < 0$ to $\alpha > 0$. A positive $\alpha$ will result in a positive $\alpha_{eff}$ in Eq. (17), and therefore no solution for $b_{z,m}$. Only rotation of the magnetic field is thus allowed. If $\varepsilon_0 > 0.25$, we can further show that $\varepsilon_0 \geq \varepsilon_m(\equiv \varepsilon|_{q_m}) \geq 0.25$ is always true for a slow shock transition from $\varepsilon_0$ to $\varepsilon_m$ in this compound wave by the full jump conditions of anisotropic-MHD (Appendix IV. B). Therefore, the nonlinear fluid theory provides a lower bound of $\varepsilon_m \geq 0.25$ at the SS to RD transition inside these compound waves, regardless of the details of $\varepsilon(b_z)$. In other words, the downstream magnetic field cannot exhibit switch-off behavior if the firehose-sense temperature anisotropy is strong enough. This fact explains the non-switch-off slow shocks often seen in kinetic simulations \cite{24} and satellite crossings \cite{30}. Once it transitions to the intermediate mode, a gyrotropic $\varepsilon(b_z)$ will stay close to $\varepsilon_m$, since the intermediate-rotation nearly preserves the magnitude of the $B$ field and therefore $\varepsilon$. Note the assumption of gyrotropic $\varepsilon(b_z)$ is expected to be valid only in length scale larger than local ion inertial length and ion gyro-radius.

In these demonstrations we use shocks with moderate parameters, such as $\theta_{BN} = 42^\circ$ and $\beta_0 = 1$. In general, larger $\theta_{BN}$, $\beta_0$, and smaller $\varepsilon_0$ will make the ratio $|\Omega|/|\alpha|$ larger, and therefore generate a stronger reversal tendency. An
analysis with full anisotropic MHD (Appendix IV) should be used for strong slow shock transitions, due to the limits of the ADNLSB, although the underlying physical picture will be similar.

VII. TOWARD THE CRITICAL $\varepsilon = 0.25$: TIME-DEPENDENT DYNAMICS

The initial conditions that characterize the exhaust of anti-parallel reconnection (initial $b_y = 0$) require $b_z = 0$ at the symmetry line at later time. This eventually forces the pseudo-particle to climb up the potential hill to the local maximum ($b_z = 0, b_y = 0$), which implies an intrinsic time-dependent process at the symmetry line since Eq. (18) does not yield such a solution. Meanwhile, the fact that $b_z$ needs to go to zero at the symmetry line provides a spatial modulation on the amplitude of the rotational intermediate mode. Note that the transition point from SS to RD in compound SS/RD waves could potentially induce modulation too. As suggested in Paper I, a spatially modulated rotational wave tends to break into $d_i$-scale dispersive waves, which can make the rotational component of the transition very turbulent.

As pointed out in Sec VI.  B, the nonlinear fluid theory of the time-independent stationary solutions only provides a lower bound $\varepsilon_m > 0.25$ for the transition point inside these compound SS/RD waves. Counter-streaming ions, by raising $P_i$, push $\varepsilon_m$ lower. Once $\varepsilon_m$ is lower than 0.25, the magnetic field rotates, generates $d_i$-scale waves, and scatters $P_i$ into $P_L$. This raises $\varepsilon_m$, changing the functional form of $\delta \varepsilon$ and driving it toward 0, which self-consistently results in a transition at the potential minimum where $\alpha_{\text{eff}} = \alpha = 0$, and thus $\varepsilon = 0.25$. This argument explains the $\varepsilon = 0.25$ plateau observed in the PIC simulations for different shock angles (see Fig. 1). With $\delta \varepsilon = 0$, this point is exactly the degenerate point of the slow and intermediate modes.

VIII. THE SUPPORTING EVIDENCE FROM NUMERICAL EXPERIMENTS

The evidence for a slow mode connecting to rotational waves can be seen in the PIC simulations that are discussed in detail in Paper I. In previous kinetic simulations, the downstream rotational waves were often identified as slow dispersive wavetrains arising from, for instance, the second term in the RHS of Eq. (1). Here we present further evidence, in addition to the numerical evidence that $\varepsilon = 0.25$, to support the idea that the downstream rotational mode is tied to the intermediate mode. Fig. 7 shows the results from three PIC simulations that were designed to explore the structure of reconnection exhausts in the normal direction. The format is the same as for Fig. 5 of Paper I, with the first column showing $\varepsilon$, the second column the magnetic field components, and the third column a hodogram of the fields. The dashed curves in the second column (from ideal isotropic MHD (31)) indicate that a pair of switch-off slow shocks or a pair of rotational discontinuities will propagate out from the center. All three cases show the correlation between $\varepsilon = 0.25$ and the transition from coplanar to non-coplanar rotation of the downstream magnetic fields. The hodograms are readily comparable to the state space plots such as Fig. 2(c). In Fig. 7(a), the downstream region of a slow shock shows a high wavenumber ($\sim 6d_i$) left-handed (LH) polarized rotational wave, which is difficult to distinguish from the predicted downstream ion inertial scale dispersive slow mode wavetrain (3). Fig. 7(b) shows results from a simulation with a larger initial current sheet width and exhibits a longer wavelength ($\sim 30d_i$) LH rotational wave which can be identified as an intermediate mode. The intermediate mode breaks into smaller ion inertial scale waves, which have been identified as dispersive waves in Paper I. By comparing (a) and (b), we note that the downstream primary rotational wave tends to maintain its spatial scale as an intermediate mode with non-steepening and non-spreading properties. Another way to distinguish the dispersive behavior from the non-dispersive rotation is by including a weak guide field. In Fig. 7(c), the front of the rotational downstream wave turns into a well-defined RD when a weak guide field is included. Its amplitude is about the same as that of the large amplitude rotational waves in (a). Most importantly, there is a clear slow shock ahead of the RD. Because the symmetry of the initial condition is broken by the guide field, the downstream RD does not need to end at $b_y = 0$; instead it ends inside the potential valley at $b_i = (0, b_{z,m})$ (see the hodogram of (c)), as expected.

IX. CONCLUSION AND DISCUSSION

The existence of compound SS/RD, SS/IS waves arising from firehose-sense and $|B|$-correlated $\varepsilon$ (temperature anisotropies) are theoretically demonstrated by analyzing the anisotropy-caused reversal of a pseudo-potential. The pseudo-potential is known to characterize the nonlinearity of hyperbolic wave equations. Extra degeneracy points
between slow and intermediate modes as well as extra non-convexity points in the slow characteristics are introduced by the temperature anisotropy. The slow shock portion of a compound SS/IS wave is an anomalous slow shock with both up and downstream being super-intermediate. The nonlinear fluid theory provides a lower bound of $\varepsilon = 0.25$ for the SS to RD transition, regardless of the details of the energy closure $\varepsilon(b_0)$. The wave generation from the rotational intermediate mode discussed here and in Paper I helps keep $\varepsilon = 0.25$. This explains the critical anisotropy plateau observed in the oblique slow shock PIC simulations documented in Paper I. This study also suggests that it is a pair of compound SS/RD waves that bound the antiparallel reconnection outflow, instead of a pair of switch-off slow shocks as in Petschek’s reconnection model. This fact explains the in-situ observations of non-switch-off slow shocks pair of compound SS/RD waves that bound the antiparallel reconnection outflow, instead of a pair of switch-off slow observed in Lottermoser et al.’s large-scale hybrid reconnection simulation [29]. In previous hybrid and PIC simulations, the downstream sharp rotational waves were often identified as slow dispersive waves of a switch-off slow shock. Instead, we propose that they are the intermediate portion of the compound SS/RD wave. The slow shock portion becomes less steep due to the time-of-flight effect of backstreaming ions.

The singularity of $\varepsilon = 0.25$ was also noticed by P. D. Hudson [17] in his study on the anisotropic rotational discontinuity (A-RD). Unlike the RD in a compound SS/RD wave, an A-RD changes both $\varepsilon$ and thermal states. Through the constraint of the positivity of $P_L$, $P_\perp$ and $B^2$, he derived all of the possible jumps (independent of the energy closure used) of the temperature anisotropy across an A-RD, as shown in Fig. 8(a). (Note that in Fig. 8 $\Delta \varepsilon = \varepsilon_{\text{up}} - \varepsilon_{\text{down}} \to 0$ when $\varepsilon \to 0.25$.) The ADNLSB inherits most of the hyperbolic properties (such as the extra nonconvexity and degeneracy points) of anisotropic MHD and also this singular behavior. We can tell this by searching for stationary solutions where the pseudo-force $F_s = 0$ (the general form of Eq. (20)). Again,

$$V_S(b_2 - b_20) - \alpha b_0 [(b_1^2 - b_120) - \Omega b_2(\varepsilon - \varepsilon_0)] = 0$$  \hspace{1cm} (25)

A stationary A-RD exists at $\varepsilon_0 = 0.25$ (i.e., $V_S = 0$, $\alpha = 0$ and $b_\perp \neq 0$), only if we require $\varepsilon - \varepsilon_0 = 0$. An arbitrary magnetic field magnitude and rotation are then allowed, as shown by Hudson. After further constraining the possible jumps by requiring that entropy increase, the solution above the diagonal line in Fig. 8b is eliminated when $\varepsilon_{\text{up}} > 0.25$ while the solution below the diagonal line is eliminated when $\varepsilon_{\text{up}} < 0.25$. He also noticed that the jump behavior of an A-RD for $\varepsilon_{\text{up}} > 0.25$ is slow-mode-like (i.e. $\delta n$ and $\delta B$ are anti-correlated), while it is fast-mode-like (i.e. $\delta n$ and $\delta B$ are correlated) for $\varepsilon_{\text{up}} < 0.25$. This directly relates to the fact that the jump of an A-RD equals the jump of the compound SS/RD wave (see Appendix IV, C), and a slow mode turns fast-mode-like when $\alpha > 0$ [14].

### X. APPENDICES

#### I - From anisotropic MHD to the Anisotropic DNLSB equation:

From moment integrations of the Vlasov equation that neglect the off-diagonal components of the pressure tensor (the empirical validity of this approximation for our system is shown in Fig. 3(c) of Paper I), we can write down the anisotropic MHD (AMHD) equations [13, 16]. The energy closure is undetermined.

In Lagrangian form,

$$\frac{d}{dt} \rho + \alpha \partial_x V_x = 0$$  \hspace{1cm} (26)

$$\rho \frac{d}{dt} V_x + \partial_x P + \partial_x \left[ \frac{1}{3} \left( \varepsilon + \frac{1}{2} \right) \frac{B^2}{\mu_0} \right] - \frac{2}{3} B^2 \partial_x \varepsilon = 0$$  \hspace{1cm} (27)

$$\rho \frac{d}{dt} V_t - \frac{B_x}{\mu_0} \partial_x (\varepsilon B_t) = 0$$  \hspace{1cm} (28)

$$\frac{d}{dt} B_t - B_x \partial_x V_t + B_t \partial_x V_x = \partial_x (\eta \partial_x B_t) - \partial_x \left( \frac{B_x}{\mu_0 \varepsilon} \varepsilon B_x \right)$$  \hspace{1cm} (29)

$$\frac{d}{dt} P - \frac{\gamma P}{\rho} \frac{d}{dt} \rho + (\gamma - 1) \left[ \frac{1}{3} \left( \varepsilon + 2 \right) \frac{B^2}{\mu_0} - \frac{B^2_0}{\mu_0} \right] \partial_x V_x + (\gamma - 1) \left( 1 - \varepsilon \right) \frac{B_x B_t}{\mu_0} \cdot \partial_x V_t + (\gamma - 1) \partial_x Q_x = 0$$  \hspace{1cm} (30)
\[ \varepsilon \equiv 1 - \frac{P_\parallel - P\perp}{B^2/\mu_0}, \quad P \equiv \frac{P_\parallel + 2P\perp}{3}. \]  
(31)

\( \gamma = 5/3 \) or 7/5 for monatomic or diatomic plasma, respectively \[32\]. \( \varepsilon, P\parallel, P\perp, \rho, V_x, V_t, B_x, B_t, Q_x \) and \( \eta \) are the temperature anisotropy factor, pressure parallel to the local magnetic field, pressure perpendicular to the local magnetic field, mass density, velocity of the bulk flow in the normal direction (\( \hat{e}_x \)), velocity of the bulk flow in the tangential direction (\( y-z \) plane), normal component of the magnetic field, tangential components of the magnetic field, the heat flux in the \( x \)-direction and the magnetic resistivity (assumed constant). The first and the second term on the RHS of Eq. (29) are from the magnetic dissipation and the Hall term respectively.

Then we follow the procedure of Kennel et al. [4], using Lagrangian mass spatial coordinates,

\[ \frac{d}{dt} \rightarrow \partial_t, \quad x' \equiv \int \frac{\rho}{\rho_0} dx \]  
(32)

Jumping to the upstream (subscripted by “0”) intermediate frame in order to separate the slow and fast variations, we take

\[ \tau \equiv \sqrt{\varepsilon_0 C_{\text{An}} \tau}, \quad \eta \equiv x' - \sqrt{\varepsilon_0 C_{\text{An}} t} \]  
(33)

where \( C_{\text{An}}^2 \equiv B^2/(\mu_0 \rho_0) \), and with the approximations

\[ \partial_\eta \ll \partial_\tau; \quad \delta b^2 \ll b^2_{\text{00}}; \quad \delta \varepsilon \ll \varepsilon_0; \quad \delta \rho \ll \rho_0; \quad \delta P \ll P_0, \]  
(34)

(where \( \delta \) means variation), we can collapse the original seven equations into two coupled equations

\[ \partial_\tau b_x + \partial_\eta [\alpha b_x (b^2_{\text{00}} - b^2_\tau) + \Omega b_x (\varepsilon - \varepsilon_0) + \Lambda b_x (Q_x - Q_{x0})] = \partial_\eta (R \delta \eta b_x) - \frac{1}{2\sqrt{\varepsilon_0}} d_t \delta^2 (\hat{e}_x \times b_t) \]  
(35)

with

\[ \alpha \equiv \frac{[3(\gamma - 1)\varepsilon_0 - (3\gamma - 4)]C_{\text{An}}^2}{12[(\varepsilon_0 - A)C_{\text{An}}^2 - C_S^2]}; \quad \Omega \equiv \frac{(b^2_{\text{00}} - 2)C_{\text{An}}^2}{6[(\varepsilon_0 - A)C_{\text{An}}^2 - C_S^2]} + \frac{1}{2\varepsilon_0}; \quad \Lambda \equiv \frac{(\gamma - 1)C_{\text{An}}^2}{2\sqrt{\varepsilon_0}(\varepsilon_0 - A)C_{\text{An}}^2 - C_S^2} \]  
(36)

and

\[ A \equiv \frac{2}{3} (\gamma - 1)(1 - \varepsilon_0)(1 + b^2_{\text{00}}) \]  
(37)

where \( C_S^2 \equiv \gamma P_0/\rho_0, R \equiv \eta/\sqrt{\varepsilon_0 C_{\text{An}}}, \) and \( d_t \equiv \sqrt{m_i/(\mu_0 n e^2)} \).

Since the heat flux \( Q_x \) is approximately proportional to \( \nabla |B| \) (as pointed out in Fig. 3(a) of Paper I), it should enter the source term on the RHS as \(-\Lambda \delta^2 b_t |b_t|\). In plasmas with \( \beta_n > 1 \), \( \Lambda \) is negative and hence the heat flux helps shocks dissipate energy. We implicitly incorporate it into the resistivity \( \rho \) (as, for instance, is done for the shear and longitudinal viscosities discussed in [4]). We then arrive at the anisotropic DNLSB equation, Eq. (1). This equation can also be derived from regular reductive perturbation methods with a proper ordering scheme. For instance, a DNLS equation with the CGL condition and more corrections, including finite ion Larmor radius effects and electron pressure, was derived using regular reductive perturbation methods [33].

**II - Non-steepening and non-spreading of the intermediate mode:**

Beginning with the anisotropic MHD equations in Lagrangian form in Appendix I, we neglect the dissipation and the Hall term on the RHS of Eq. (29). Then the simple wave solution can be obtained by substituting \( d/dt \rightarrow -C \delta \), \( \partial_x \rightarrow \hat{e}_x \delta \), where \( C \) is the wave speed and \( \delta \) means variation. [44]

\[ -C \delta \rho + \rho \delta V_x = 0 \]  
(38)

\[ -C \rho \delta V_x + \delta P - \frac{B^2}{\mu_0} \delta \varepsilon + \frac{B_x}{\mu_0} \delta B_x = 0 \]  
(39)

\[ -C \rho \delta V_x - \frac{B_x}{\mu_0} \delta B_x - \frac{B_x B_z}{\mu_0} \delta \varepsilon = 0 \]  
(40)
\[- C \rho \delta V_y - \varepsilon \frac{B_x}{\mu_0} \delta B_y = 0 \quad (41) \]
\[- C \delta B_z + B_z \delta V_x - B_x \delta V_z = 0 \quad (42) \]
\[- C \delta B_y - B_y \delta V_z = 0 \quad (43) \]

Eq. (41) and Eq. (43) give us the intermediate speed

\[ C_I = \sqrt{\varepsilon B_z / \sqrt{\mu_0 \rho}} \]

Combined with Eq. (38), the steepening tendency of an intermediate mode can then be expressed as,

\[ \delta (C_I + V_x) = C_I^2 \left( \frac{\delta \rho}{\rho} + \frac{\delta \varepsilon}{\varepsilon} \right) \quad (44) \]

Using Eqs. (38), (39), (40) and (42), we get \( \delta (C_I + V_x) = 0 \). Therefore, the intermediate mode in anisotropic MHD does not steepen or spread, no matter what energy closure is used. It is linearly degenerate, as is its counterpart in isotropic MHD.

III - The integral curves and Hugoniot Locus:

To find the integral curves, we follow the eigenvector of the slow mode to form a curve,

\[ \frac{db_z}{d\zeta} = b_z, \quad \frac{db_y}{d\zeta} = b_y, \quad (45) \]

where \( \zeta \) is a dummy variable. The integral curve is

\[ b_y = \left( b_y(0)/b_z(0) \right) b_z. \]

For the intermediate mode,

\[ \frac{db_z}{d\zeta} = b_y, \quad \frac{db_y}{d\zeta} = -b_z. \quad (46) \]

Therefore, the integral curve is \( b_z^2 = b_z(0)^2 \).

As to the Hugoniot locus, we need to compute the shock speed

\[ S(b_z - b_z(0)) = \frac{fb_z(b_z^2 - b_z(0)^2)}{\rho V_x - (\varepsilon - 1) \frac{B_z}{\mu_0} \cdot B_t} \]

\[ S(b_y - b_y(0)) = \frac{fb_y(b_y^2 - b_y(0)^2)}{\rho V_x - (\varepsilon - 1) \frac{B_y}{\mu_0} \cdot B_t} \quad (47) \]

These can be combined to give,

\[ (b_z(0)b_y - b_y(0)b_z)|\alpha(b_z^2 - b_z(0)^2) + \Omega \delta \varepsilon| = 0. \quad (48) \]

The first root is the Hugoniot locus of the slow mode: \( b_y = (b_y(0)/b_z(0)) b_z \). For \( \varepsilon(b_1) \), so that \( \delta \varepsilon \simeq \left( \partial (\delta \varepsilon) / \partial (b_z^2) \right)(b_z^2 - b_z(0)^2) \), the second root gives us the Hugoniot locus of the intermediate mode: \( b_z^2 = b_z(0)^2 \). Although these results are the same as derived from the integral curves, this is not generally the case.

IV - The pseudo-potential of Anisotropic MHD (AMHD):

In the de Hoffmann-Teller frame, the jump conditions can be written as (following Hau and Sonnerup’s procedure [7, 28]),

\[ [\rho V_x]_0^0 = 0 \quad (49) \]
\[ \left[ \rho V_x^2 + P + \frac{1}{3} \left( \varepsilon + \frac{1}{2} \right) \frac{B_x^2}{\mu_0} - \varepsilon \frac{B_x^2}{\mu_0} \right]_0^0 = 0 \quad (50) \]
\[ \left[ \rho V_x V_t - \varepsilon \frac{B_x B_t}{\mu_0} \right]_0^0 = 0 \quad (51) \]
\[ \left[ \left( \frac{1}{2} \rho V_x^2 + \frac{\gamma}{\gamma - 1} P + \frac{1}{3} (\varepsilon - 1) \frac{B_x^2}{\mu_0} \right) V_x - (\varepsilon - 1) \frac{B_x B_t}{\mu_0} \cdot V_t - (\varepsilon - 1) \frac{B_x^2}{\mu_0} \cdot V_x \right]_0^0 = 0 \quad (52) \]
where we define a jump relation \( Q^0 = Q_0 - Q \), with \( Q_0 \) the upstream value and \( Q \) the value inside the transition region. From Eq. (49)-(52), we can derive

\[
A_x^2 = \frac{V^2}{B_x^2/(\mu_0 \rho)} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

where

\[
a = 1 - \frac{\gamma - 1}{2\gamma},
\]

\[
b = -A_x^2 - \left[ \frac{\beta_0}{2} + \frac{1}{3} \left( \varepsilon_0 + \frac{1}{2} \right) \right] \sec^2 \theta_0 + \varepsilon_0 + \frac{1}{3} \left( \varepsilon + \frac{1}{2} \right) \frac{B^2}{B_x^2} - \varepsilon + \frac{2\gamma - 2}{3\gamma} (\varepsilon - 1) - \frac{\gamma - 1}{3\gamma} (\varepsilon - 1) \frac{B_i^2}{B_x^2},
\]

\[
c = \frac{\gamma - 1}{2\gamma} \sec^2 \theta_0 A_x^4 + \left[ \frac{\beta_0}{2} - \frac{2\gamma - 2}{3\gamma} (\varepsilon_0 - 1) \right] \sec^2 \theta_0 A_x^2 - \frac{\gamma - 1}{2\gamma} \left[ (A_x^2 - \varepsilon_0) \tan \theta_0 + \varepsilon_0 B_i/B_x \right]^2 + \frac{\gamma - 1}{\gamma} (\varepsilon - 1) \frac{B_i}{B_x} \left[ (A_x^2 - \varepsilon_0) \tan \theta_0 + \varepsilon_0 B_i/B_x \right]
\]

with \( \cos \theta_0 \equiv B_x/B_0 \).

The generalized Ohm’s law is

\[
E + V \times B = \frac{\eta_e}{\mu_0} J + \frac{1}{\mu_0 \rho \varepsilon} (J \times B),
\]

where the first and the second term on the RHS are the magnetic dissipation and the Hall term respectively. With the final jump condition \( [E_1]^0 = 0 \), we obtain

\[
\frac{A_{x_0}}{h_0} d_0 (1 + h^2) \frac{dB_y}{dx} = (A_x^2 - \varepsilon)(B_y - h B_z) + (A_{x_0}^2 - \varepsilon_0)(h B_{z_0} - B_{y_0}) \equiv F_{y,AMHD} \equiv \frac{\partial \Psi_{AMHD}}{\partial B_y}
\]

\[
\frac{A_{x_0}}{h_0} d_0 (1 + h^2) \frac{dB_z}{dx} = (A_x^2 - \varepsilon)(B_z + h B_y) - (A_{x_0}^2 - \varepsilon_0)(h B_{y_0} + B_{z_0}) \equiv F_{z,AMHD} \equiv \frac{\partial \Psi_{AMHD}}{\partial B_z}
\]

where \( h = B_z/n \eta_e \). measures the ratio of the dispersion to the resistivity. The pseudo-potential \( \Psi_{AMHD} \) is uniquely defined since \( \partial F_{x,AMHD}/\partial B_y = \partial F_{y,AMHD}/\partial B_z \).

A: Fig. [9](a) shows the pseudo-potential of AMHD for the same parameters as Fig. [4](a) (which was calculated based on the reduced ADNLSB formulation). If \( V_e = 0 \) then \( A_{x_0}^2 = \varepsilon_0 \), and thus the potential minimum (where \( F_{y,AMHD} = 0 \)) occurs at \( A_x^2 = \varepsilon \). This implies that in the shock frame (also the upstream intermediate frame), \( \lambda_{t,AMHD} = -V_z + C_t = -(A_x^2 + \sqrt{\varepsilon}) B_x/\sqrt{\mu_0 \rho} = 0 \) at the potential minimum. This is essentially the same point \( q_{on} \) (where \( \lambda_t = 0 \)) of Fig. [4](d) with ADNLSB. Fig. [9](b) shows a similar reversal with the CGL closure. We note that the CGL closure exhibits an even stronger tendency to reverse the pseudo-potential. In Fig. [9](c), the pseudo-potential for \( \theta_{BN} = 75^\circ \), \( \beta_0 = 0.4 \) and \( c_2 = 0.2 \) is shown, these parameters are more similar to those seen in our PIC simulation.

B: Now we consider the jump conditions to an asymptotic downstream by neglecting the LHS and terms with \( h \) of Eqs. (58) and (59). The relation \( B_{l,d}/B_x = \tan \theta_0 (A_{z_0}^2 - \varepsilon_0)/(A_{z,d}^2 - \varepsilon_d) \) can be derived where we label quantities \( Q \rightarrow Q_d \) (“d” for downstream). We can eventually invert \( A_{z_0}^2 \) as a function of \( A_{z,d}^2 \) from Eq. (53). The result is plotted in Fig. [9](d) which shows possible shock solutions as functions of the downstream intermediate Mach number \( M_{t,d}^2 = V_{x,d}^2/C_{t,d}^2 = A_{z,d}^2/\varepsilon_d \). In the green curve (\( \varepsilon_d = \varepsilon_0 \) case), the portion from A-RD (anisotropic-RD) to SSS is the IS branch, from SSS to LS (linear slow mode) is the SS branch. When \( \varepsilon_d < \varepsilon_0 \), a new slow shock transition from \( A_{z_0}^2 = \varepsilon_0 > 0.25 \) to \( A_{z,d}^2 = \varepsilon_d \) is noted at the point (1,1). This new SS constitutes the slow shock portion of a compound SS/RD wave. For a given \( \varepsilon_0 \), the smallest possible \( \varepsilon_{d,\mathrm{min}} \) shrinks the SS and IS branches to the point (1,1). It can be shown that \( \varepsilon_{d,\mathrm{min}} > 0.25 \) is always true for \( \varepsilon_0 > 0.25 \). Therefore the existence of the new slow shock requires \( \varepsilon_d > 0.25 \). In other words, \( \varepsilon_0 > \varepsilon_m (= \varepsilon_d) > 0.25 \) is always true for a SS/RD compound wave in full anisotropic MHD.

C: From Fig. [9](d) and further investigations, it can be shown that the anisotropic-RD(A-RD) at (1,1) has the same jump as that of the new SS at (1,1) plus a RD that does not change \( \varepsilon \) and thermal states. Therefore an A-RD and the corresponding compound SS/RD wave have the same jump relations.

V - The Equal-Area Rule and Intermediate Shocks:
The equal-area rule applies to conserved quantities in hyperbolic equations, which in our case is $b_z$. From Eq. (6) and the general form of Eq. (20), we find a simple relation between the pseudo-force and the flux function,

$$F_z|_{b_y=0} = -\alpha b_z(b_z^2 - b_{z0}^2) - \Omega b_z \delta\varepsilon(b_z, b_y) + V_S(b_z - b_{z0}) = -f_1|_{b_y=0} + V_S(b_z - b_{z0})$$

(60)

From Eq. (6) and Eq. (3), a simple relation between the slow characteristic and the flux function is

$$\lambda_{SL}|_{b_y=0} = \frac{\partial f_1}{\partial b_z}|_{b_y=0}$$

(61)

It is then easy to show that,

$$\int_{b_{z0}}^{b_z} (\lambda_{SL}|_{b_y=0} - V_S)db_z = -F_z|_{b_y=0}$$

(62)

This indicates that a stationary point $b_z$, where $F_z = 0$, will be located where the integral on the LHS is zero. This is called the equal-area rule. From this relation, with a given $b_{z0}$ and $b_z$, we can determine the shock speed $V_S$ that causes the integral to vanish. Or for a given $b_{z0}$ and $V_S$, we can determine the possible downstream state $b_z$. We apply it to the following examples to demonstrate the formation of intermediate shocks (which have a super-intermediate to sub-intermediate transition) in isotropic MHD.

When the upstream $q_i = (b_{z0}, 0)$ is given and fixed, we can vary $q_f = (b_z, 0)$ to see the effect on possible shock solutions. In Fig. 10(a), when $q_f$ is chosen above $b_z = 0$, a slow shock solution is found by determining a proper horizontal line ($q_i - q_f$; note that the vertical position measures the shock speed $V_S$), which makes the red area below the line $q_i - q_f$ equal the red area above. The shock speed is slower than the upstream intermediate speed (black horizontal line across 0), the upstream (point $q_f$) is super-slow and sub-intermediate ($V_S > \lambda_{SL}|_{q_f}$, $V_S < \lambda_{SL}|_{q_f}$). Since $\lambda$ equivalents to $C - u$ where $C$ is the phase speed and $u$ is the bulk flow speed measured in upstream intermediate frame, $V_S \geq \lambda$ implies that the mach number measured in shock frame $M \equiv (V_S + u)/C \geq 1$ and the downstream (point $q_f$) is sub-slow ($V_S < \lambda_{SL}|_{q_f}$). Traditionally in isotropic MHD, the super-fast state is termed number 1, sub-fast and super-intermediate is 2, sub-intermediate and super-slow is 3, and sub-slow is 4. Therefore a slow shock is also called a 3-4 SS.

In Fig. 10(b), if $q_f$ is chosen below the point $b_z = 0$, a 2-4 intermediate shock $(q_f - q_m)$ is formed, with upstream being super-intermediate and downstream being sub-intermediate and sub-slow. In Fig. 10(c), with the same shock speed, a 2-3 IS transitions to a $q_f$ with a more negative value is also possible. Note that the jump cross a compound 2-3 IS/3-4 SS (from this $q_f$ to the $q_f$ in (b) ) equals to that of the 2-4 IS in (b). In Fig. 10(d), with the same $q_f$ of Fig. 10(c), a 2-3=4 IS $(q_r - q_m)$ with the maximum IS speed could be formed and attached by a slow rarefaction $(q_m - q_f)$. This is a compound IS/SR wave, with $b_z|_{q_m} = -b_{z0}/2$ which can also be determined by $\lambda_{SL}|_{q_m} = \langle f_1(b_{z0}) - f_1(b_{z0}|_{q_m})/(b_{z0} - b_{z0}|_{q_m}) \rangle$, as shown by Brio and Wu [2]. Similar arguments can be made in a system with fast and intermediate modes.

Therefore, an intermediate shock is not directly associated with an intermediate mode. It is steepened by magneto-sonic waves (slow or fast modes), not by intermediate mode itself. This was first justified by Wu’s (1987) coplanar simulations [3] (i.e., no out-of-plane magnetic field is allowed), where the intermediate shock forms even though the intermediate mode is not included (since the out-of-plane $\delta B_y$ is necessary for nontrivial solutions of the intermediate mode, as shown in Appendix II). The coupling of intermediate and magneto-sonic waves and the admissibility of intermediate shocks in the ideal MHD system was discussed by Kennel et al. [4].

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**FIG. 1**: [From Liu et al., (2011)] The $\varepsilon$ distributions of runs with $\theta_{BN} = 60^\circ$ at $500/\Omega_{ci}$, $75^\circ$ at $200/\Omega_{ci}$, $83^\circ$ at $700/\Omega_{ci}$. $\theta_{BN}$ is the angle between the magnetic field and the shock normal direction ($\hat{e}_x$) far upstream in these shock simulations. $\Omega_{ci}$ is the ion cyclotron frequency based on the upstream magnetic field. These simulations are designed to study the structure of reconnection exhausts in the normal direction. The downstream $\varepsilon$ tends to plateau at 0.25. When $\varepsilon < 0$, the plasma is susceptible to the firehose instability.

**FIG. 2**: Panel (a): An initial discontinuity between $q_r$ and $q_l$ results in two waves, the “1-wave” and “2-wave”, that propagate in the $\eta$ direction along time $\tau$. The middle state $q_m$ needs to be determined; Panel (b): The state space plot in the $(b_z, b_y)$ plane. The value $q_r = (b_{z0}, 0)$ is chosen since there is no out-of-plane $B_y$ upstream of the slow shocks in Paper I. $q_r$ straightly connects to $q_l$ and forms a switch-off slow shock; Panel (c): In order to connect $q_r$ to $q_l$, it is necessary to cross the degeneracy band into the reversal region, which could cause the path to rotate at $q_m$.
FIG. 3: The $\varepsilon$ distribution vs. $1/B^2$ for the cases $\theta_{BN} = 30^\circ$ (yellow), $45^\circ$ (magenta), $52^\circ$ (green), $60^\circ$ (blue), $75^\circ$ (red) and $83^\circ$ (black) from Fig. 5 of Paper I [1]. The dashed line has slope -0.5. In comparison, the diamond curve is the theoretical prediction with the CGL condition for the $\theta_{BN} = 75^\circ$ case.

FIG. 4: Panel (a): A pseudo-potential $\Psi$ with $V_S = 0$. Upstream (point $q_r$), $\theta_0 = 42^\circ$, $\beta_0 = 1$, $\varepsilon_0 = 1$, $c_2 = 0.5$ and we choose $\Psi(q_r) = 0$. Since the transition occurs within the radius $b_t = b_{z0}$, we set $\Psi = 0$ for $b_t > b_{z0}$ for a better visualization. The potential for negative $B_y$ is mirror symmetric to the part shown here; Panel (b): $\varepsilon(h_t)$; Panel (c): Cuts of $\Psi$, $F_z$, and $\alpha_{\text{eff}}$ along the $B_z$ axis with $B_y = 0$; Panel (d): $\lambda_I$ and $\lambda_{\text{SL}}$ along the $B_z$ axis with $B_y = 0$. The vertical axis measures speed (normalized to $C_A$). “D” stands for degeneracy. The red area above $V_S$ (zero here) equals the red area below $V_S$, and the same rule applies to the blue area.
FIG. 5: Same format as Fig. 4 but with $V_S = 0.015 \geq 0$. 

FIG. 6: The pseudo-potential with $\varepsilon_0 = 0.24 < 0.25$, while other parameters are the same as Fig. 4.
FIG. 7: Results of PIC simulations (runs f, g and h of Paper I \[1\]). Row (a): A case with $\theta_{BN} = 75^\circ$, $\beta_0 = 0.4$ and initial width $w_i = 1d_i$ at time $200/\Omega_{ci}$. $\varepsilon$ is shown on the left, different magnetic components in the middle, $B_z-B_y$ hodogram on the right; Row (b): A similar case with a wider initial width $w_i = 10d_i$ at time $450/\Omega_{ci}$; Row (c): A similar case to (a), but with a weak guide field $B_y = 0.2B_0$ at time $200/\Omega_{ci}$. The dotted curves in the center column are the predicted $B_z$ magnitudes and positions of switch-off slow shocks (SSS) or rotational discontinuities (RD) from isotropic MHD theory \[31\].

FIG. 8: In panel (a), the possible jumps of $\varepsilon$ of an anisotropic-RD, are constrained by requiring positive $P_{\perp d}$, $P_{\parallel d}$ and $B_z^2$. The red region is forbidden. In panel (b), the plot is further constrained by the requirement that entropy increases (with an entropy from the H-theorem defined as $\ln(P_{\parallel}^{1/3}P_{\perp}^{1/3}/\rho^{5/3})$ for a bi-Maxwellian distribution). The constraint of increasing entropy has eliminated the region above the diagonal line when $\varepsilon_{up} > 0.25$, and the region below the diagonal line when $\varepsilon_{up} < 0.25$. Here $\beta_0 = 1$; a higher $\beta_0$ would collapse the valid region into a narrower region along the diagonal line.
FIG. 9: Results with fully anisotropic MHD. Panel (a) is the pseudo-potential $\Psi$ with our closure, Eq. (15). Upstream (point $q_r$), $V_S = 0$, $\theta_0 = 42^\circ$, $\beta_0 = 1$, $\varepsilon_0 = 1$ and $c_2 = 0.5$; Panel (b) is $\Psi$ with the CGL closure. Upstream (point $q_r$), $V_S = 0$, $\theta_0 = 42^\circ$, $\beta_0 = 1$ and $\varepsilon_0 = 1.5$; Panel (c) is $\Psi$ for $V_S = 0$, $\theta_0 = 75^\circ$, $\beta_0 = 0.4$, $\varepsilon_0 = 1$, and $c_2 = 0.2$ with our closure; Panel (d) is the shock curve with upstream parameters $\theta_0 = 42^\circ$, $\beta_0 = 1$ and $\varepsilon_0 = 1$. In the green curve ($\varepsilon_d = \varepsilon_0$ case), the portion from A-RD (anisotropic-RD) to SSS is the IS branch, from SSS to LS (linear slow mode) is the SS branch. Different curves represent cases with different $\varepsilon_d$ of values 1, 0.95, 0.9, 0.85 (from outer curve to inner curve). Other than the A-RD, a new SS exists at (1,1) when $\varepsilon_d < \varepsilon_0$. Both the IS and SS branches shrink toward the point (1,1) as $\varepsilon_d$ decreases.

FIG. 10: Application of equal-area rules with cases in ideal (isotropic) MHD. The $\lambda_I$ and $\lambda_{SL}$ along $b_z$ are measured in the upstream intermediate frame with upstream (point $q_r$) parameters, $\theta_0 = 42^\circ$, $\beta_0 = 1$, $\varepsilon_0 = 1$. The vertical axis measures speed (normalized to $C_{As}$). Once the $q_r$ is chosen, the shock speed $V_S$ (measured by the red horizontal line) can be determined by equating area (between the $\lambda_{SL}$ and the red line) above the red line to area below the red line. This rule results in shock speeds (a) $V_S = -0.1$; (b) $V_S = 0.05$; (c) $V_S = 0.05$; (d) $V_S = 0.0958$. 