Probing Cosmic Acceleration Using Model-independent Parameterizations and Three Kinds of Supernova Statistics Techniques

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Abstract

In this work, we explore the evolution of the dark energy equation of state $\omega$ using Chevallier–Polarski–Linder parameterization and binned parameterizations. For binned parameterizations, we adopt three methods to choose the redshift interval: (1) ensure that “$\Delta z = \text{const}$,” where $\Delta z$ is the width of each bin; (2) ensure that “$n \Delta z = \text{const}$,” where $n$ is the number of SN Ia in each bin; and (3) treat redshift discontinuity points as model parameters, i.e., “free $\Delta z$. For observational data, we adopt JLA SN Ia samples, SDSS DR12 data, and Planck 2015 distance priors. In particular, for JLA SN Ia samples, we consider three statistic techniques: magnitude statistics, which is the traditional method; flux statistics, which reduces the systematic uncertainties of SN Ia; and improved flux statistics, which can reduce the systematic uncertainties and give tighter constrains at the same time. The results are as follows. For all the cases, $\omega = -1$ is always satisfied at the 1$\sigma$ confidence regions; this means that $\Lambda$CDM is still favored by current observations. For magnitude statistics, the “free $\Delta z$” model will give the smallest error bars. However, this conclusion does not hold true for flux statistics and improved flux statistic. The improved flux statistic yields the largest present fractional density of matter $\Omega_m$, in addition, this technique will give the largest current deceleration parameter $q_0$, which reveals the universe with the slowest cosmic acceleration.

Key words: cosmological parameters – cosmology: observations – dark energy

1. Introduction

Since the discovery of cosmic acceleration in 1988 (Riess et al. 1998; Perlmutter et al. 1999), it has been widely believed that there exists mysterious dark energy (DE) that drives the current accelerating expansion of the universe (Padmanabhan 2003; Frieman et al. 2008; Li et al. 2011a, 2013; Bamba et al. 2012). To explain this strange phenomenon, numerous theoretical models have been proposed, such as $\Lambda$CDM (Einstein 1917), quintessence (Caldwell et al. 1998; Zlatev et al. 1999), phantom (Caldwell 2002; Carroll et al. 2003), k-essence (Armendariz-Picon et al. 1999; Chiba et al. 2000), Chaplygin gas (Kamenshchik et al. 2001; Bento et al. 2002), holographic DE (Li 2004; Li et al. 2009a, 2009b; Wang et al. 2017), agegraphic DE (Cai 2007; Wei & Cai 2008), Ricci DE (Gao et al. 2009), and the Yang–Mills condensate (Zhang et al. 2007; Wang et al. 2008; Wang & Zhang 2008).

In addition to specific DE models, another way of exploring the nature of DE is to adopt model-independent reconstruction (Shapiro et al. 2004; Wang & Tegmark 2004; Wang & Mukherjee 2007; Wang 2009; Wang et al. 2011). A popular approach is the so-called specific ansatz. Here, we consider the most popular Chevallier–Polarski–Linder (CPL) (Chevallier & Polarski 2001; Linder 2003). Another important method is the so-called binned parameterization, which was first proposed by Huterer and Starkman based on principal component analysis (Huterer & Starkman 2003). The basic idea of binned parameterization is to divide the redshift range into different bins and set the equation of state (EOS) $w$ as piecewise constant in redshift $z$.

There are different opinions in the literature regarding the optimal choice of the discontinuity points of redshift. Wang (2010) argued that the width of each redshift bin, i.e., $\Delta z$, should be a constant (hereafter we call it the “$\text{const } \Delta z$” model). Riess et al. (2007) argued that the number of SN Ia in a bin (i.e., $n$), times the width of this bin should be a constant (hereafter we call it the “$n \Delta z^{\text{const}}$” model). In Huang et al. (2009), one of the current authors and his collaborators showed that the discontinuity points of redshift can be treated as model parameters in performing cosmology fits (hereafter we call it the “free $\Delta z$” model. In this work, we will consider all three binned models.

From an observational point of view, we mainly focus on SN Ia that can be regarded as cosmological standard candles to measure directly the expansion history of the universe (Weinberg et al. 2013). In the literature, studies typically use the distance–redshift relation $\mu(z)$ to calculate the $\chi^2$ function of SN Ia. We call this statistical method for measuring SN Ia the “magnitude statistic” (MS) method. However, many recent studies showed that the MS technique suffers from systematic uncertainties of SN Ia (Hu et al. 2016). For example, it has been proven that, in the framework of an MS statistic, SN color-luminosity parameter $\beta$ should evolve along with redshift $z$ at the 5$\sigma$ confidence level (Wang & Tegmark 2005; Wang et al. 2013, 2014a, 2014b, 2015; Mohlabeng & Ralston 2014; Li et al. 2016). To overcome the shortcomings of the MS, Wang (2000) proposed a “flux-averaged” (FA) technique, which averages the observed flux of SN Ia at a series of uniformly divided redshift bins. Hereafter, we call the statistical method based on the FA technique the “flux statistic” (FS) method, which can effectively reduce the systematic uncertainties of SN Ia (Wang & Tegmark 2005; Wang 2009; Wang et al. 2012). However, adopting the FS method will yield larger error bars for various model parameters. Therefore, in 2013, one of the current authors and Wang (Wang & Wang 2013a) developed an improved FA method that only uses the FA technique in high-redshift regions. We call this improved FA method the “improved flux statistics” (IFS), which can reduce systematic
uncertainties of SN Ia and give tighter DE constraints at the same time (Wang et al. 2017; Wen et al. 2018). In this paper, we consider all three SN Ia analysis techniques.

In this paper, we make use of current cosmological observations, including SN Ia, baryon acoustic oscillations (BAO), and cosmic microwave background (CMB), to constrain various model-independent DE reconstructions. In particular, for a comprehensive study, we consider all three binned models (i.e., const \( \Delta z \), const \( n \Delta z \) and free \( \Delta z \)) and all three SN Ia analysis techniques (i.e., MS, FS, and IFS). This paper is organized as follows. In Section 2, we introduce the theoretical models, including CPL and three kinds of binned parameterization models. In Section 3, we introduce all the observational data, and describe the details of using SN Ia, BAO and CMB. Finally, we present our results in Section 4 and conclude in Section 5.

2. Theoretical Models

In a spatially flat universe, the Friedmann equation can be rewritten as

\[
H(z) = H_{0} \sqrt{\Omega_{m}(1 + z)^{4} + \Omega_{\Lambda}(1 + z)^{3} + \Omega_{de}X(z)},
\]

(1)

where \( \Omega_{m} \), \( \Omega_{\Lambda} \), and \( \Omega_{de} \) are the present fractional densities of matter, radiation, and DE, respectively, and \( H_{0} \) is the present-day value of the Hubble parameter \( H(z) \). And the radiation density parameter \( \Omega_{r} \) is given by (Wang & Wang 2013b)

\[
\Omega_{r} = \Omega_{m}/(1 + z_{eq}),
\]

(2)

where \( z_{eq} = 2.5 \times 10^{7} \Omega_{m}^{1/2}(T_{cmb}/2.7 \text{K})^{-4}, T_{cmb} = 2.7255 \text{K}, \) and \( h \) is the reduced Hubble constant. Note that \( X(z) \) is given by the specific DE models:

\[
X(z) = \frac{\rho_{de}(z)}{\rho_{de}(0)} = \exp \left[ 3 \int_{0}^{z} \frac{1 + w(z')}{{1 + z'}}dz' \right],
\]

(3)

where \( \omega = P_{de}/\rho_{de} \) is the EOS of DE, and \( \rho_{de} \) and \( \rho_{de} \) are the pressure and density of DE, respectively. In this paper, we consider two kinds of DE model-independent parameterization approaches.

2.1. Chevallier–Polarski–Linder Parameterization

First, we consider the Chevallier–Polarski–Linder parameterization, with the EOS parameterized as

\[
w(z) = w_{0} + w_{a} \frac{z}{1 + z},
\]

(4)

where \( w_{0} \) and \( w_{a} \) are constant parameters. The corresponding \( X(z) \) can be expressed as

\[
X(z) = (1 + z)^{3(1 + w_{0} + w_{a})} \exp \left( -\frac{3w_{a}z}{1 + z} \right).
\]

(5)

2.2. Three Kinds of Binned Parameterization

For the case where EOS \( \omega \) is piecewise constant in redshift, we only consider the case of 3 bins. So we can reconstruct it as

\[
w(z) = \begin{cases} 
    w_{1} & 0 < z < z_{1} \\
    w_{2} & z_{1} < z < z_{2}, \\
    w_{3} & z_{2} < z
\end{cases}
\]

(6)

where \( w_{1}, w_{2}, w_{3} \) are free parameters and will be determined by the Markov Chain Monte Carlo (MCMC) method (Lewis & Bridle 2002). \( z_{1}, z_{2} \) will be determined by three kinds of binned parameterization. In addition, the corresponding \( X(z) \) takes the form

\[
X(z_{n-1} < z < z_{n}) = (1 + z)^{3(1 + w_{0})} \prod_{i=0}^{n-1} (1 + z_{i})^{3(w_{i} - w_{i+1})},
\]

(7)

where \( w_{i} \) is the EOS parameter in the \( i \)th redshift bin defined by an upper boundary at \( z_{i} \).

As mentioned above, we consider three binned methods as follows:

1. First, we choose \( z_{1} = 0.5, z_{2} = 1.0 \) to get the \( \Delta z = 0.5 \).

   We will call it “const \( \Delta z \).”

2. Second, we choose \( z_{1} = 0.248, z_{2} = 0.679 \). Now, for each redshift bin we find that \( n \Delta z = 97.7 \). It is called “const \( n \Delta z \).”

3. Third, we determine the values of \( z_{1} \) and \( z_{2} \) by performing a best-fit analysis. It must be emphasized that adopting different SN Ia statistics techniques will give different \( z_{1} \) and \( z_{2} \) (shown in Table 1). We will call this case “free \( \Delta z \).”

For the convenience of the reader, we list the details of the redshift discontinuity points of the three binned methods in Table 1.

3. Observational Data

In our work, we used various cosmological observations, including SN Ia, BAO, and CMB, to perform cosmology fits. Therefore,

\[
\chi^{2} = \chi^{2}_{\text{SN Ia}} + \chi^{2}_{\text{BAO}} + \chi^{2}_{\text{CMB}}.
\]

(8)

In addition, we perform a MCMC likelihood analysis (Lewis & Bridle 2002) to obtain \( O(10^{6}) \) samples for each set of results presented in this paper.

Moreover, the figure of merit (FoM) is a very useful tool to assess the ability of constraining the DE of an experiment. In this paper, we adopt a generalized FoM (Wang 2008) given by

\[
\text{FoM} = \frac{1}{\sqrt{\text{det} \text{Cov}(f_{1}, f_{2}, f_{3}, \ldots)}},
\]

(9)

where \( \text{Cov}(f_{1}, f_{2}, f_{3}, \ldots) \) is the covariance matrix of the chosen set of DE parameters. As is well known, larger FoM indicates better accuracy.

In the following, we will describe how to calculate the \( \chi^{2} \) functions of SN Ia, BAO, and CMB.

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**Table 1:**

| Interval Chosen | Statistics Method | \( z_{1} \) | \( z_{2} \) |
|-----------------|-------------------|-------------|-------------|
| const \( \Delta z \) | All               | 0.5000      | 1.0000      |
| const \( n \Delta z \) | All               | 0.2481      | 0.6787      |
| free \( \Delta z \) | Magnitude statistics | 0.6482      | 1.2808      |
|                  | Flux statistics   | 0.4489      | 1.4330      |
|                  | Improved flux statistics | 0.1410      | 0.7376      |
3.1. Type Ia Supernovae

For the SN Ia data, we make use of a “Joint Light-curve Analysis” (JLA) data set (Betoule et al. 2014). In particular, we introduce in detail the three kinds of SN Ia statistics techniques (i.e., MS, FS, and IFS).

3.1.1. Magnitude Statistics

We describe MS in this section. Theoretically, the distance modulus \( \mu_{\text{th}} \) in a flat universe can be written as

\[
\mu_{\text{th}} = 5 \log_{10} \left( \frac{d_L(z_{\text{hel}}, z_{\text{cmb}})}{\text{Mpc}} \right) + 25,
\]

where \( z_{\text{cmb}} \) and \( z_{\text{hel}} \) are the CMB rest frame and heliocentric redshifts of SN Ia. The luminosity distance \( d_L \) is given by

\[
d_L(z_{\text{hel}}, z_{\text{cmb}}) = (1 + z_{\text{hel}}) r(z_{\text{cmb}}),
\]

where \( r(z) \) is given by

\[
r(z) = \frac{c H_0^{-1} \int_0^z \frac{dz'}{E(z')}}{c},
\]

c is the speed of light, and \( E(z) \equiv H(z)/H_0 \). The observation of distance modulus \( \mu_{\text{obs}} \) is given by an empirical linear relation:

\[
\mu_{\text{obs}} = m_B - M_B + \alpha_0 X_1 - \beta_0 \times C,
\]

where \( m_B \) is the observed peak magnitude in the rest frame of the B band, \( X_1 \) describes the time stretching of the light curve, and \( C \) describes the supernova color at maximum brightness. Note that \( \alpha_0 \) and \( \beta_0 \) are the SN stretch-luminosity parameter and SN color-luminosity parameter, respectively. In addition, \( M_B \) is the absolute B-band magnitude, which relates to the host stellar mass \( M_{\text{stellar}} \) via a simple step function

\[
M_B = \begin{cases} 
M_B^1 & \text{if } M_{\text{stellar}} < 10^{10} M_\odot, \\
M_B^2 & \text{otherwise},
\end{cases}
\]

where \( M_\odot \) is the mass of the Sun. The \( \chi^2 \) of JLA data can be obtained by

\[
\chi^2_{\text{SNLS}} = \Delta \mu^T \cdot \text{Cov}^{-1} \cdot \Delta \mu,
\]

where \( \Delta \mu \equiv \mu_{\text{obs}} - \mu_{\text{th}} \) is the data vector and \( \text{Cov} \) is the total covariance matrix, which can be calculated as

\[
\text{Cov} = D_{\text{stat}} + C_{\text{stat}} + C_{\text{sys}}.
\]

Here, \( D_{\text{stat}} \) is the diagonal part of the statistical uncertainty, which is given by

\[
D_{\text{stat},i} = \frac{5}{z_i \ln 10} \sum_{j=1}^{N} \sigma^2_{\zeta,i} + \sigma^2_{\text{int}} + \sigma^2_{\text{lensing}} + \sigma^2_{m_B,i} + \sigma^2_{X_1,i} + \sigma^2_{C,i} + \sigma^2_{\Delta \mu_{0,i}},
\]

where the first three terms account for the uncertainty in redshift due to peculiar velocities, the intrinsic variation in SN magnitude, and the variation of magnitudes caused by gravitational lensing. \( \sigma^2_{m_B,i} \), \( \sigma^2_{X_1,i} \), and \( \sigma^2_{C,i} \) denote the uncertainties of \( m_B \), \( X_1 \), and \( C \) for the \( i \)th SN. In addition, \( C_{m_BX_1,i} \), \( C_{m_BC,i} \), and \( C_{X_1C,i} \) are the covariances between \( m_B \), \( X_1 \), and \( C \) for the \( i \)th SN. Moreover, \( C_{\text{stat}} \) and \( C_{\text{sys}} \) are the statistical and the systematic covariance matrices given by

\[
C_{\text{stat}} + C_{\text{sys}} = V_0 + \alpha_0^2 V_a + \beta_0^2 V_b + 2 \alpha_0 \beta_0 V_{ab} + 2 \alpha_0 V_{ba} - 2 \beta_0 V_{ab} - 2 \alpha_0 \beta_0 V_{ab},
\]

where \( V_0 \), \( V_a \), \( V_b \), \( V_{ab} \), \( V_{ba} \) are six matrices that will be given in Betoule et al. (2014).

3.1.2. Flux Statistics

FS is based on the FA technique, which is very useful to reduce the systematic uncertainties of SN Ia (Wang & Mukherjee 2004; Wang & Tegmark 2005; Wang 2009). The original FA method divides the whole redshift region of SN Ia into multiple bins, where the redshift interval of each bin is \( \delta z \). In the following, we will introduce the specific steps of FA (Wang et al. 2012):

1. Convert the distance modulus of SN Ia into “fluxes,”

\[
F(z_i) = 10^{-(\mu_{\text{obs}}(z_i) - 25)/2.5} = \left( \frac{d_L^{\text{obs}}(z_i)}{\text{Mpc}} \right)^2.
\]

Here, \( z_i \) represents the CMB rest-frame redshift of an SN.

2. For a given set of cosmological parameters \( \{s\} \), calculate “absolute luminosities,” \( \{L(z_i)\} \),

\[
L(z_i) = d_L^2(z_b) F(z_i).
\]

3. Flux-average the “absolute luminosities,” \( \{L'_i\} \) in each redshift bin \( i \) to obtain \( \{\mathcal{L}'\} \):

\[
\mathcal{L}'_i = \frac{1}{N_i} \sum_{l=1}^{N_i} L'_i(z_l), \quad \mathcal{T} = \frac{1}{N_i} \sum_{l=1}^{N_i} z_l.
\]

4. Place \( \mathcal{L}' \) at the mean redshift \( \mathcal{T} \) of the \( i \)th redshift bin; now the binned flux is

\[
F(z_i) = L'_i / d_L^2(z_b). \quad \mathcal{L}(z_i) = \mathcal{T}' / d_L^2(z_b). \quad \mathcal{T} = \frac{1}{N_i} \sum_{l=1}^{N_i} z_l.
\]

5. Calculate the covariance matrix of \( \{\mathcal{p}(z_i)\} \) and \( \{\mathcal{p}(z)\} \):

\[
\text{Cov}[\{\mathcal{p}(z_i)\}, \{\mathcal{p}(z_j)\}] = \frac{1}{N_i N_j \mathcal{T}' \mathcal{T}} \sum_{l=1}^{N_i} \mathcal{L}(z_l) \mathcal{L}(z_j) \mathcal{L}^2(z_l) \mathcal{L}^2(z_j)
\]

where \( \mathcal{D} \) is the covariance of the measured distance moduli of the \( i \)th SN Ia in the \( i \)th redshift bin, and the \( m \)th SN Ia in the \( m \)th redshift bin. \( \mathcal{L}(z) \) is defined by Equations (19) and (20).

6. For the flux-averaged data, \( \{\mathcal{p}(z)\} \), calculate

\[
\chi^2_{\text{SNLS}} = \sum_{i} \Delta \mathcal{p}(z_i) \text{Cov}^{-1}[\{\mathcal{p}(z_i), \mathcal{p}(z)\}] \Delta \mathcal{p}(z_i),
\]

where

\[
\Delta \mathcal{p}(z_i) = \mathcal{p}(z_i) - \mu(z_i), \quad \mathcal{p}(z_i) = -2.5 \log_{10} F_{\text{p}}(z_i) + 25,
\]

and

\[
F_{\text{p}}(z_i) = (d_L(z_b)/\text{Mpc})^2.
\]
3.1.3. Improved Flux Statistics

As mentioned above, the improved FA method (Wang & Wang 2013a) introduces a new quantity: the redshift cutoff $z_{\text{cut}}$. For the SN samples at $z < z_{\text{cut}}$, the $\chi^2$ is computed using the usual MS (i.e., Equation (15)); for the SN samples at $z \geq z_{\text{cut}}$, the $\chi^2$ is computed by using the “flux statistics” (i.e., Equation (25)). Therefore, the total $\chi^2$ can be written as

$$\chi^2 = \begin{cases} \chi^2_{\text{MS}} & \text{if } z < z_{\text{cut}}, \\ \chi^2_{\text{FS}} & \text{otherwise}. \end{cases}$$

(28)

This new method includes the advantages of MS and FS, and thus can reduce systematic uncertainties and give tighter DE constraints at the same time.

In previous works, Wang & Dai (2016) applied this improved FA method to explore the JLA data, and found that it can give tighter constraints on DE. But in Wang & Dai (2016), only one kind of FA recipe, $(z_{\text{cut}} = 0.5, \delta z = 0.04)$, was considered. In a recent paper (Wang et al. 2017), we scanned the whole $(z_{\text{cut}}, \delta z)$ plane, and found that adopting the FA recipe, $(z_{\text{cut}} = 0.6, \delta z = 0.06)$, yielded the tightest DE constraints. So in this paper, we will use the IFS technique with the best FA recipe $(z_{\text{cut}} = 0.6, \delta z = 0.06)$.

The details of these three statistic methods of SN Ia are listed in Table 2.

### 3.2. Other Observational Data

#### 3.2.1. Baryon Acoustic Oscillations

The BAO matter clustering provides a “standard ruler” for length scale in cosmology. And the signals can be used to measure the Hubble parameter $H(z)$ and angular diameter distance $D_{\text{A}}(z) = r(z)/(1 + z)$ in the radial and tangential directions, respectively. Here, the $r(z)$ is given by Equation (12). In this paper we use the data of BOSS DR12 (Alam et al. 2017), which includes the combinations $H(z)r(z)/r_{\text{fid}}$ and $D_{\text{A}}(z)r(z)/r_{\text{fid}}$. Here $r_{\text{fid}} = 147.8$ Mpc is the sound horizon of the fiducial model, and $D_{\text{A}}(z) = (1 + z)D_h(z)$ is the comoving angular diameter distance. $r(z)/r_{\text{fid}}$ is the sound horizon at the drag epoch $z_d$, given by

$$r(z_d) = \int_{z_d}^{\infty} \frac{c_s(z)}{H(z)} dz,$$

(29)

where $c_s(z) = z^{1/2}c \left[ 1 + \frac{3}{4} \rho_b(z)/\rho_c(z) \right]^{-1/2}$ is the sound speed in the photon-baryon fluid. In Alam et al. (2017), $r(z_d)$ is approximated by Aubourg et al. (2014),

$$r(z_d) = \frac{55.154}{\sqrt{\frac{0.1287}{0.2531}}} \exp[-72.3(\omega_{\nu} + 0.0006)^2] \frac{\Omega_b^{0.1287}}{\Omega_b^{0.2531}},$$

(30)

where $\omega_{\nu} = 0.0107(\Sigma m_e/1.0 \text{ eV})$ is the density parameter of neutrinos; $\omega_b = \Omega_b h^2$ is the density parameter of baryons, and $\omega_c = \Omega_c h^2 - \omega_\nu$ are the density parameters of baryons and (cold) dark matter. Following the Aubourg et al. (2014), we set $\Sigma m_e = 0.06$ for all the models we considered. There are six BAO data points given in Table 7 of Alam et al. (2017):

$$\begin{align*}
p_1 &= H(0.38)r_{\text{fid}}/r(z_d), \quad p_1^{\text{data}} = 1512, \\
p_2 &= H(0.38)r_{\text{fid}}/r(z_d), \quad p_2^{\text{data}} = 81.2, \\
p_3 &= H(0.51)r_{\text{fid}}/r(z_d), \quad p_3^{\text{data}} = 1975, \\
p_4 &= H(0.51)r_{\text{fid}}/r(z_d), \quad p_4^{\text{data}} = 90.9, \\
p_5 &= H(0.61)r_{\text{fid}}/r(z_d), \quad p_5^{\text{data}} = 2307, \\
p_6 &= H(0.61)r_{\text{fid}}/r(z_d), \quad p_6^{\text{data}} = 99.0 .
\end{align*}$$

(31)

Therefore, the $\chi^2$ function for current BAO data can be expressed as

$$\chi^2_{\text{BAO}} = \Delta p_1 \left[ \text{Cov}_{\text{BAO}}(p_1, p_2) \right] \Delta p_2,$$

(32)

The covariance matrix $\text{Cov}_{\text{BAO}}$ is given by the online files of Alam et al. (2017).

#### 3.2.2. Cosmic Microwave Background

CMB gives us the comoving distance to the photon-decoupling surface $r(z_s)$ and the comoving sound horizon at photon-decoupling epoch $r_s(z_s)$. In this paper, we use the distance priors data extracted from Planck 2015 (Planck Collaboration et al. 2015). This includes the “shift parameter” $R$, the “acoustic scale” $l_A$, and the redshift of the decoupling epoch of photons $z_s$.

The shift parameter $R$ is given by Wang & Mukherjee (2007):

$$R = \frac{\sqrt{\Omega_b h^2}}{r(z_s)/c},$$

(33)

where $r(z_s)$ is the comoving distance given in Equation (12). $z_s$ is the redshift of the photon-decoupling epoch estimated by Hu & Sugiyama (1996):

$$z_s = 1048[1 + 0.00124(\Omega_b h^2) - 0.738] \left[ 1 + g_1(\Omega_m h^2) \delta_1 \right],$$

(34)

where $\Omega_b$ is the present fractional density of baryon, and

$$g_1 = \frac{0.0738(\Omega_b h^2)^{-0.238}}{1 + 39.5(\Omega_b h^2)^{0.763}}, \quad g_2 = \frac{0.560}{1 + 21.1(\Omega_b h^2)^{0.81}}.$$

(35)

The acoustic scale $l_A$ is defined as

$$l_A = \pi r(z_s)/r_s(z_s),$$

(36)

where $r_s(z_s)$ is the comoving sound horizon at $z_s$. The $r_s(z)$ is given by

$$r_s(z) = c H_0^{-1} \int_0^a \frac{da'}{\sqrt{3(1 + \Omega_b a') a^4 E^2(z')}}.$$

(37)
where $R_b = 315000 h^2 (T_{\text{ cmb}}/2.7 K)^{-4}$. These two distance priors, together with $\omega_b \equiv \Omega_b h^2$, provide an efficient summary of CMB data.

The $\chi^2$ function for the CMB distance prior data can be expressed as

$$
\chi^2_{\text{CMB}} = \sum (q_i - \sigma(q_i)) \sigma(q_i) \sigma(q_i) \text{NormCov}_{\text{CMB}}(q_i, q_i),
$$

where $\sigma(q_i)$ is the 1$\sigma$ error of observed quantity $q_i$, and $\text{NormCov}_{\text{CMB}}(q_i, q_i)$ is the corresponding normalized covariance matrix, which are listed in Table 4 of Planck Collaboration et al. (2015).

The Planck 2015 data are

$q_1^\text{data} = 1.7382 \pm 0.0088$,

$q_2^\text{data} = 301.63 \pm 0.15$,

$q_3^\text{data} = 0.02262 \pm 0.00029$. (40)

4. Results

4.1. Probing Cosmic Acceleration with the Model-independent Parameterizations

In this subsection, we will show the fitting results of CPL parameterization and the three binned parameterizations. It must be mentioned that all three SN Ia analysis techniques, MS, FS, and IFS, are taken into account.

4.1.1. CPL Parameterization

First, we consider the case of CPL parameterization. In Figure 1, we plot the evolution of $\omega(z)$ for the CPL parameterization. For comparing, all the three SN Ia statistics techniques are used. The solid blue lines represent the best-fit value of $\omega(z)$, and the dotted cyan lines represent the 1$\sigma$ confidence region of $\omega(z)$. In addition, we also present the results of $\Lambda$CDM which are represented by dashed black lines.

From this figure, we find that the 1$\sigma$ confidence region of CPL's $\omega(z)$ is consistent with the result of $\Lambda$CDM. More importantly, this conclusion is insensitive to the SN Ia statistics techniques.

Moreover, this conclusion also holds true for the cases of SNLS3 (Li et al. 2011b), Union2 (Holsclaw et al. 2010), Union2.1 (Shi et al. 2012), and Constitution (Wei 2010). This implies that this conclusion comes into existence for all the SN Ia data.

4.1.2. Three Binned Parameterizations

Next, we will show the results of three binned parameterizations. For a more systematic and more comprehensive study, all the SN Ia statistics techniques are taken into account. Moreover, we consider three kinds of binned models, i.e., “$\text{ const } \Delta z$,” “$\text{ const } n \Delta z$,” and “$\text{ free } \Delta z$.” Different binned models have different redshift discontinuity points. For the case of “$\text{ const } \Delta z$,” $z_1 = 0.5$, $z_2 = 1.0$; for the case of “$\text{ const } n \Delta z$,” $z_1 = 0.2481$ and $z_2 = 0.6787$. Moreover, for the case of “$\text{ free } \Delta z$,” different SN Ia statistics techniques will give different discontinuity points, as shown in Table 1.

In Figure 2, we plot the evolution of $\omega(z)$ for the “$\text{ const } \Delta z$” case. Note that all three SN Ia statistics techniques are taken into account. The cyan rectangular areas of each panel represent the $1\sigma$ confidence region of $\omega(z)$, and the solid blue lines represent the $2\sigma$ confidence region. In addition, we use dashed black lines to represent the result of $\Lambda$CDM. This figure shows that different SN Ia statistics techniques will give the same evolutionary trend of EoS, i.e., both the first segment and the second segment are smaller than the third one. Moreover, all the results of EoS given by the three SN Ia statistics techniques are consistent with the prediction of $\Lambda$CDM.

For the case of “$\text{ const } n \Delta z$,” we plot the evolution of $\omega(z)$ in Figure 3. Compared to Figure 2, the third segments of $\omega(z)$ have smaller error bars. The reason for this is that for the “$\text{ const } n \Delta z$” case, the third redshift interval contains more SN Ia samples. Moreover, all the 1$\sigma$ regions of $\omega(z)$ given by the three SN Ia statistics techniques contain black dashed line, $\omega = -1$, which correspond to the prediction of $\Lambda$CDM.

Finally, we discuss the case of the “$\text{ free } \Delta z$” model. As shown in Table 1, different models will give different redshift discontinuity points. For this case, the evolution trends of $\omega(z)$ given by different SN Ia statistics techniques have significant differences (see Figure 4). However, all the curves of $\omega(z)$ shown in this figure are consistent with the result $\Lambda$CDM.

From Figures 1 to 4, one can see that all the curves of $\omega(z)$ given by the different theoretical models and different SN Ia statistics techniques always contain the black dashed line of $\omega = -1$ at the 1$\sigma$ confidence region. This proves that $\Lambda$CDM is favored by current observations.

4.2. The Effects of Adopting Different Binned Parameterizations on Parameter Estimation

In this subsection, we discuss the effects of adopting different binned parameterizations on parameter estimation.

In Figure 5, we plot the evolution curves of $\omega(z)$ when adopting MS. For comparison, three kinds of binned parameterization models are used. It is found that, for each bin, the free $\Delta z$ model always yields the smallest error bars. This conclusion is the same as that of Li et al. (2011b).

In Figure 6, we also plot the evolution curves of $\omega(z)$ for the cases of adopting FS and adopting IFS. Note that in previous studies the effects of adopting different binned parameterizations was not discussed in the framework of adopting FS or adopting IFS. From this figure, one can see that the error bars of $\omega(z)$ given by the three binned models are very close. In other words, the conclusion of Figure 5 does not hold true for the cases of FS and IFS.

In addition, we also calculate the corresponding FoM in Table 3. For MS, the free $\Delta z$ model yields the largest value of FoM, which corresponds to the best observational constraints. This result is consistent with Figure 5. However, the free $\Delta z$ model cannot yield the largest FoM value for FS and IFS (see the second and third rows in Table 3). Therefore, we conclude that the conclusion of Figure 5 does not hold true for the cases of FS and IFS. This conclusion is consistent in different studies.

In summary, it is found that, for MS, the free $\Delta z$ will give the smallest error bars among the three binned parameterizations. However, this conclusion does not hold true for FS and IFS.
Figure 1. Evolution of $w(z)$ for the CPL model. Three SN Ia statistics techniques are used: MS (upper left panel), FS (upper right panel), and IFS (lower panel). The solid blue lines represent the best-fit value of $\omega(z)$, and the dotted cyan lines represent the $1\sigma$ confidence region of $\omega(z)$. Moreover, the dashed black lines represent the result of $\Lambda CDM$. The result of CPL is always consistent with the result of $\Lambda CDM$ at the $1\sigma$ confidence region.

Figure 2. Evolution of $w(z)$ for the “const $\Delta z$” case. Three SN Ia statistics techniques are used: MS (upper left panel), FS (upper right panel), and IFS (lower panel). For the three different statistics, the $1\sigma$ and $2\sigma$ confidence regions have the same evolutionary trend. We find that the largest one is FS and the MS has minimum confidence regions. In addition, the line $\omega = -1$ is always contained in the $1\sigma$ confidence region.
Figure 3. Evolution of $w(z)$ for the case of “const $n \Delta z$.” Three SN Ia statistics techniques are used: MS (upper left panel), FS (upper right panel), and IFS (lower panel). Compared to Figure 2, though we adopt different binned methods, the same result is found: the same confidence region size order. Moreover, all the cases in this figure contain the EOS $\omega$ of $\Lambda$CDM in the $1\sigma$ confidence region.

Figure 4. Evolution of $w(z)$ for the case of “free $\Delta z$.” Three SN Ia statistics techniques are used: MS (upper left panel), FS (upper right panel), and IFS (lower panel). Because of the chosen free interval, the three different statistics of SN Ia have different shapes for the $1\sigma$ and $2\sigma$ confidence regions. But even as the shapes of all the confidence regions become different, they are still consistent with the $\Lambda$CDM, as seen in Figures 2 and 3.
4.3. The Effect of Adopting Different SN Ia Analysis Techniques

In this subsection, we discuss the effects of adopting different SN Ia statistics techniques on parameter estimation. In order to present the result more clearly, we perform a cosmology-fit using only the SN Ia data. For simplicity, here we only consider the CPL parameterization and “const $\Delta z$” binned parameterization.

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**Figure 5.** Evolution of $w(z)$ when adopting MS. Three binned methods are used: const $\Delta z$ (upper left panel), const $n \Delta z$ (upper right panel), and free $\Delta z$ (lower panel). Differing from the result before, the three binned methods have the different evolutionary trend. But it is clear that the free $\Delta z$ model can find the tightest confidence region.

**Figure 6.** Evolution of $w(z)$ for FS (upper panel) and IFS (lower panel). Three binned methods are used: const $\Delta z$ (left panel), const $n \Delta z$ (middle panel), and free $\Delta z$ (right panel). The free $\Delta z$ model clearly cannot find the tightest confidence region.

**Table 3**
The Value of FoM for Three Kinds of Binned Parameterizations (All Three SN Ia Statistics Techniques Are Used)

| FoM | const $\Delta z$ | const $n \Delta z$ | free $\Delta z$ |
|-----|-----------------|---------------------|-----------------|
| MS  | 1226.42         | 1396.20             | 2956.18         |
| FS  | 779.39          | 678.99              | 110.08          |
| IFS | 780.79          | 1123.70             | 653.02          |

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To compare different SN Ia statistics techniques, we mainly focus on the parameter estimation of the present fractional density of matter $\Omega_m$, as well as the current deceleration parameter $q_0$, which is given by

$$q_0 = \frac{\ddot{a}(t_0)}{\dot{a}(t_0)} = \frac{1}{2} \Omega_m + \Omega_r - \Omega_{de}. \quad (41)$$

Here $\Omega_r$ and $\Omega_{de}$ are the present fractional densities of radiation and DE, which can be found in Equation (1). In addition, $t_0$ represents the current time.

In Figure 7, we plot the $1\sigma$ confidence intervals of $\Omega_m$ for the CPL parameterization and the binned parameterization, where all three SN Ia statistics techniques are used. From the bottom to the top of each panel, the three regions represent MS, FS, and IFS. In previous studies, only one binned parameterization and one SN Ia analysis technique were taken into account. In this work, we consider all three binned parameterizations (i.e., const $\Delta z$, const $n\Delta z$, and free $\Delta z$) and all three SN Ia analysis techniques (i.e., MS, FS, and IFS). Therefore, we can present a more comprehensive and more systematic study.

In Figure 8, we present the results of $q_0$ in Figure 8. The current deceleration parameter $q_0$ also rapidly varies for different SN Ia statistics techniques. For CPL parameterization, the best-fit results of $q_0$ are $-0.966$, $-0.776$, and $-0.279$ for MS, FS, and IFS, respectively. For the binned parameterization, the best-fit values of $q_0$ are $-0.581$, $-0.350$, and $-0.211$ for MS, FS, and IFS, respectively. We find that IFS yields the largest $q_0$, which reveals the universe with the slowest cosmic acceleration.

As mentioned above, we conclude that IFS yields the largest present fractional density of matter $\Omega_m$ and current deceleration parameter $q_0$. In other words, this SN Ia statistics technique favors a universe that contains more matter and has slow cosmic acceleration. Note that this conclusion does not rely on a specific model.

5. Conclusions and Discussions

In our work, we explore the evolution of the DE EOS $\omega(z)$ using model-independent parameterizations. The CPL parameterization and three kinds of binned parameterizations (“const $\Delta z$,” “const $n\Delta z$,” and “free $\Delta z$”) are taken into account in this work. To perform cosmology fits, we adopt observation data including SN Ia observations from JLA samples, BAO observations from SDSS DR12, and CMB observations from Planck 2015 distance priors. In particular, for the SN Ia data, we make use of three statistics techniques, i.e., MS, FS, and IFS.

In previous studies, only one binned parameterization and one SN Ia analysis technique were taken into account. In this work, we consider all three binned parameterizations (i.e., const $\Delta z$, const $n\Delta z$, and free $\Delta z$) and all three SN Ia analysis techniques (i.e., MS, FS, and IFS). Therefore, we can present a more comprehensive and more systematic study.
Our results are as follows:

1. For all the cases, $\omega = -1$ is always satisfied at the 1σ confidence regions, which means that ΛCDM is still favored by current observations (from Figures 1 to 4).

2. For magnitude statistics, the “free $\Delta z$” model will give the smallest error bars; this conclusion does not hold true for flux statistics and improved flux statistics (see Figures 5 and 6 and Table 3).

3. The improved flux statistic yields the largest present fractional density of matter $\Omega_m$; in addition, this technique will give the largest current deceleration parameter $q_0$, which reveals the universe with the slowest cosmic acceleration (see Figures 7 and 8).

Recently, the Dark Energy Survey Supernova Program (DES-SN) published the latest SN Ia samples. It would be interesting to analyze the systematic uncertainties of these latest SN Ia samples and then study the corresponding cosmological consequences. This will be done in future works.

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