UNIFORM BOUNDEDNESS OF HIGHEST NORM FOR 2D QUASILINEAR WAVE

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ABSTRACT. We consider the two-dimensional quasilinear wave equations with quadratic nonlinearities. We introduce a new class of null forms and prove uniform boundedness of the highest order norm of the solution for all time. This class of null forms include several prototypical strong null conditions as special cases. To handle the critical decay near the light cone we inflate the nonlinearity through a new normal form type transformation which is based on a deep cancelation between the tangential and normal derivatives with respect to the light cone. Our proof does not employ the Lorentz boost and can have promising applications to systems with multiple speeds.

1. INTRODUCTION

We consider the following quasilinear wave equation:

\[ \Box u = g^{kij} \partial_k u \partial_j u, \quad t > 2, \quad x \in \mathbb{R}^2, \]
\[ u|_{t=2} = \varepsilon f_1, \quad \partial_t u|_{t=1} = \varepsilon f_2. \] (1.1)

Here \( \Box = \partial_t^2 - \Delta \) is the wave operator. The functions \( f_1, f_2 \) are real-valued. On the RHS of (1.1) we employ the usual Einstein summation convention with \( \partial_0 = \partial_t \) and \( \partial_i = \partial_{x^i} \) for \( l = 1, 2 \). For simplicity we assume \( g^{kij} \) are constant coefficients, \( g^{kij} = g^{jik} \) for any \( i, j \), and satisfy the standard null condition:

\[ g^{kij} \omega_k \omega_i \omega_j = 0, \] (1.2)

where \( \omega \) is any null vector, namely, \( \omega_0 = -1, \omega_1 = \cos \theta, \omega_2 = \sin \theta \) with \( \theta \in [0, 2\pi] \). In addition, we assume that

\[ (g^{1ij} \omega_2 - g^{2ij} \omega_1) \omega_j = 0, \quad \text{for any null vector } \omega. \] (1.3)

We shall call the condition (1.2)–(1.3) a new type of strong null condition. Although the extra condition (1.3) seems a bit odd-looking at first sight, we shall show later that it serves as a natural generalization of several prototypical null conditions in the literature. Our main result reads as follows.

**Theorem 1.1.** Consider (1.1) with \( g^{kij} \) satisfying the strong null condition (1.2)–(1.3). Let \( m \geq 5 \) and assume \( f_1 \in H^{m+1}(\mathbb{R}^2), f_2 \in H^m(\mathbb{R}^2) \) are compactly supported in the disk \( \{ |x| \leq 1 \} \). There exists \( \varepsilon_0 > 0 \) depending on \( g^{kij} \) and \( \|f_1\|_{H^{m+1}} + \|f_2\|_{H^m} \) such that for all \( 0 \leq \varepsilon < \varepsilon_0 \), the system (1.1) has a unique global solution. Furthermore, the highest norm of the solution remains uniformly bounded, namely

\[ \sup_{t \geq 2} \sum_{|\alpha| \leq m} \| (\partial \Gamma^\alpha u)(t, \cdot) \|_{L^2(\mathbb{R}^2)} < \infty. \] (1.4)

Here \( \Gamma = \{ \partial_t, \partial_{x_1}, \partial_{x_2}, \partial_\theta, t \partial_\theta + r \partial_r \} \) does not include the Lorentz boost (see (1.8) for notation).

In the seminal work \[ \cite{Ali} \], Alinhac showed that under the general null condition (1.2) the system (1.1) has small data global wellposedness with the highest norm polynomially bounded in time. In \[ \cite{ALM} \], by deeply exploiting a type of strong null condition together with Alinhac’s method, Lei established small data global wellposedness for 2D incompressible elastodynamics. In \[ \cite{ALM2} \], developing upon Lei’s strong null condition in \[ \cite{ALM} \], Cai, Lei and Masmoudi considered the following quasilinear wave equations:

\[ \Box u = A_l \partial_l (N_{ij} \partial_i u \partial_j u), \] (1.5)

where \( A_l, N_{ij} \) are constants, and

\[ N_{ij} \omega_i \omega_j = 0, \quad \text{for any null vector } \omega. \] (1.6)

A special case of (1.5) is the following typical quasilinear wave equation

\[ \Box u = \partial_t (|\partial_t u|^2 - |\nabla u|^2). \] (1.7)
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Advantage of the hyperbolic change of variable is that one can gain better control of the conformal energy thanks to the extra integrability in the hyperbolic time \( s = \sqrt{t^2 - r^2} \). One should note, however, that if one works with the advanced coordinate \( s = t - r \), then there is certain degeneracy in the \( \partial_s \) direction which renders (even any generalized) conformal energy out of control. In this connection an interesting further issue is to explore the monotonicity of the conformal energy (and possible generalizations) with respect to different space-time foliations.

As was already mentioned earlier, the main purpose of this work is to develop a new strategy (building upon Alinhac’s ghost weight method) to prove the uniform boundedness of highest norm for general 2D quasilinear wave equations with null conditions. To understand the role of various null conditions a natural first step is to classify the standard null forms. To this end, we define the following:

\[
F^A_i = \partial_i(\partial_t u^2 - |\nabla u|^2), \quad i = 0, 1, 2; \tag{1.8}
\]

\[
F^B_i = \partial_i u \Box u, \quad i = 0, 1, 2; \tag{1.9}
\]

\[
F^C_i = \partial_i \Delta u + \partial_1 u \partial_i u; \quad F^D_i = \partial_i u \partial_{22} u - \partial_2 u \partial_{11} u; \tag{1.10}
\]

\[
F^D_i = \partial_0 u \partial_{11} u - \partial_1 u \partial_{12} u; \quad F^D_2 = \partial_2 u \partial_{11} u - \partial_1 u \partial_{12} u. \tag{1.11}
\]

**Theorem 1.2** (Full classification of null conditions). If \( g^{kij} \) satisfies the standard null condition \( \Box u \), then

\[
g^{kij} \partial_k u \partial_j u = \sum_{l=0}^2 C_{1,l} F^A_l + \sum_{l=0}^2 C_{2,l} F^B_l + \sum_{l=0}^2 C_{3,l} F^C_l + \sum_{l=0}^3 C_{4,l} F^D_l, \tag{1.12}
\]

where \( C_{m,l} \) are constants. If \( g^{kij} \) satisfies the strong null condition \( \Box u \), then

\[
g^{kij} \partial_k u \partial_j u = \sum_{l=0}^2 C_{1,l} F^A_l + \sum_{l=0}^2 C_{2,l} F^B_l. \tag{1.13}
\]

If \( N_{ij} \) satisfies the null condition \( \Box u \), then

\[
A_0 \partial_t (N_{ij} \partial_t u \partial_j u) = \sum_{l=0}^2 C_{1,l} F^A_l. \tag{1.14}
\]

**Remark 1.1.** In yet other words, the null condition in [15] is simply \( \partial(\partial_t u^2 - |\nabla u|^2) \), whereas our strong null condition is \( \partial(\partial_t u^2 - |\nabla u|^2) + \partial_t u \Box u \). In [20], Peng and Zha considered (see formula (1.4) therein) the situation \( g^{kij} = g^{ijk} = g^{ijk} \) for any \( i, j \) (besides the standard null condition). However, such a strong condition apparently does not include the standard nonlinearity \( \partial(\partial_t u^2 - |\nabla u|^2) \).

**Remark 1.2.** Define the standard null forms

\[
G^C_i = \partial_0 u \partial_1 (\partial_i u) - \partial_1 u \partial_0 (\partial_i u), \quad i = 0, 1, 2; \tag{1.15}
\]

\[
G^D_i = \partial_0 u \partial_2 (\partial_i u) - \partial_2 u \partial_0 (\partial_i u), \quad i = 0, 1, 2. \tag{1.16}
\]

It is easy to check that

\[
G^C_0 = \partial_0 u \partial_1 u - \partial_1 u \Delta u - \partial_1 u \Box u = \frac{1}{2} \partial_0(\partial_t u^2 - |\nabla u|^2) - \partial_1 u \partial_{22} u - \partial_1 u \Box u.
\]

Thus \( G^C_0 \) is a linear combination of \( F^A_1, F^B_1 \) and \( F^D_2 \).

We now explain the key steps of the proof of Theorem 1.1 (see section 2 for the relevant notation). To elucidate the idea, we fix any multi-index \( \alpha \) with \( |\alpha| = m \), and denote \( v = \Gamma^\alpha u \). By Lemma \[23\] we have

\[
\Box v = g^{kij} \partial_k v \partial_j u + \cdots, \tag{1.18}
\]
where “…” denotes harmless terms which do not contribute to the main term.

Step 1. Weighted energy estimate. We choose \( p(r, t) = q(r - t) \) with \( q'(s) \) nearly scales as \( (s)^{-1} \) to derive

\[
\frac{1}{2} \frac{d}{dt} \left( \| e^{\frac{1}{2} r \partial_t} \|_2^2 \right) + \frac{1}{2} \int e^{\frac{1}{2} \partial_t} |v|_2^2 \, dx = \int g^{kij} \omega_i \omega_j T_k v \partial_t v \partial_t u e^p \, dx + \cdots. \tag{1.19}
\]

The usual strategy is to use Cauchy-Schwartz to derive

\[
\left| \int g^{kij} \omega_i \omega_j T_k v \partial_t v \partial_t u e^p \, dx \right| \leq \frac{1}{10} \int |T v|^2 q e^p \, dx + \text{const} \cdot \int |\partial v|^2 \frac{|\partial u|^2}{q} e^p \, dx
\]

which yields polynomial growth in time. To resolve this we shall proceed differently.

Step 2. Refined decomposition. By using \( T_1 = \omega_1 \partial_r - \frac{\omega_1}{r} \partial_\theta \) and \( T_2 = \omega_2 \partial_r + \frac{\omega_1}{r} \partial_\theta \), we have

\[
g^{kij} \omega_i \omega_j T_k v = g^{kij} \omega_i \omega_j (\omega_1 \partial_r v - \frac{\omega_1}{r} \partial_\theta v) + g^{kij} \omega_i \omega_j (\omega_2 \partial_r v + \frac{\omega_1}{r} \partial_\theta v)
\]

\[
= h(\theta) \partial_r v + \omega_i \omega_j (g^{kij} \omega_1 - g^{kij} \omega_2) \frac{1}{r} \partial_\theta v = h(\theta) \partial_r v,
\]

where \( h(\theta) = g^{kij} \omega_i \omega_j + g^{kij} \omega_2 \omega_j \omega_j \) and we used \( \frac{1}{r} \) in the last step.

Step 3. Localization, further decomposition and normal form transformation. We use a bump function \( \phi \) which is localized to \( r \sim t \) such that the main piece becomes

\[
\int h(\theta) \partial_r v \partial_t v \partial_t u e^p \phi.
\tag{1.21}
\]

The contribution of the regimes \( r \leq \frac{1}{t} \) and \( r > 2t \) can be shown to be negligible. We further use the decomposition \( \partial_t = \frac{\partial_r + \partial_\theta}{2} \) to transform the main piece as (below we drop the harmless factor 1/2)

\[
\int h(\theta) \partial_r v \partial_t v \partial_t u e^p \phi + \text{Negligible.}
\tag{1.22}
\]

At this point, the crucial observation is to use the fundamental identity

\[\partial_r \partial_\theta = \square + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta,\]

to transform \( \frac{1}{r} \) into an expression which contains an “inflated” nonlinearity. It is this novel normal form type transformation which makes the problem subcritical.

It should be pointed out that we do not employ the usual Lorentz boost vector field in the whole proof. Additionally we developed several new decay estimates for the regime \( r \leq t/2 \) which was previously unavailable due to the lack of Lorentz boost (cf. Lemma \( 2.6 \)). Thus this new strategy could have promising applications in systems with multiple speeds.

Remark 1.3. It is worthwhile pointing out how the symmetry condition \( g^{kij} = g^{kji} \) for all \( k, i, j \) was needed in \( 2.6 \). When bounding the quasilinear piece \( \alpha_2 = \alpha \), we have

\[
\int g^{kij} \partial_k u \partial_i v \partial_t v e^p \, dx = \int g^{kij} \partial_j (\partial_k u \partial_t v e^p) - \int g^{kij} \partial_k u \partial_i v \partial_t v e^p
\]

\[
- \int g^{kij} \partial_k u \partial_i v \partial_t v e^p - \int g^{kij} \partial_k u \partial_i v \partial_j (e^p).
\tag{1.23}
\]

By using the symmetry \( g^{kij} = g^{kji} \) (this is harmless), we have

\[
- \int g^{kij} \partial_k u \partial_i v \partial_t v e^p = - \frac{1}{2} \int g^{kij} \partial_i (\partial_k u \partial_t v e^p) + \frac{1}{2} \int g^{kij} \partial_k u \partial_i v \partial_t v e^p + \frac{1}{2} \int g^{kij} \partial_k u \partial_i v \partial_j (e^p).
\tag{1.24}
\]

Thus

\[
\int g^{kij} \partial_k u \partial_i v \partial_t v e^p \, dx = - \frac{1}{2} \int g^{kij} \partial_k u \partial_i v \partial_t v e^p + \frac{1}{2} \int g^{kij} \partial_k u \partial_i v \partial_j (e^p) + \cdots.
\tag{1.25}
\]

Note that the second term on the RHS of \( 1.25 \) is not a problem due to the good decay of \( \partial_t^2 u \). On the other hand, in \( 2.6 \), the decay of \( \partial^2 u \) in the regime \( r \leq t/2 \) was not sufficient to treat the first term on the
Proof. We focus on the regime $|x| > 1$. For a one-variable function $h \in C_c\infty([0,\infty))$, we have
\[
\rho|h(r)|^2 \leq \int_0^\infty |h(r)|^2 r dr + \int_0^\infty |\partial_r h|^2 r dr, \quad \forall \rho > 0.
\]
(2.7)

It follows that (below we slightly abuse the notation and denote $v(\rho, \theta) = v(x)$ for $x = (\rho \cos \theta, \rho \sin \theta)$)
\[
\rho|\partial_\theta v(\rho, \theta)|^2_{L^2_\theta} \leq \rho \int_0^{2\pi} |\partial_\theta v(\rho, \theta)|^2 d\theta \leq \|\partial_\theta v\|_{L^2_\theta}^2 + \|\partial_\rho v\|_{L^2_\theta}^2.
\]
(2.8)

Denote $\overline{\rho}(\rho)$ as the average of $\rho(v, \theta)$ over $\theta$. By (2.7), we have
\[
\rho|\overline{\rho}(\rho)|^2 = \rho \left(\frac{1}{2\pi} \int_0^{2\pi} v(\rho, \theta) d\theta\right)^2 \leq \rho \int_0^{2\pi} v(\rho, \theta)^2 d\theta \leq \|v\|_{L^2_\theta}^2 + \|\partial_\theta v\|_{L^2_\theta}^2.
\]
(2.9)

Note that $|v(\rho, \theta) - \overline{\rho}(\rho)|^2 \leq |\partial_\theta v|_{L^2_\theta}^2$ by the Poincaré inequality. Thus
\[
|x| |v(x)|^2 \lesssim \|v\|_{L^2_\theta}^2 + \|\partial_\theta v\|_{L^2_\theta}^2 + \|\partial_\rho v\|_{L^2_\theta}^2.
\]
\[\square\]
Lemma 2.2 (Refined Hardy’s inequality). For any real-valued \( h \in C^\infty_0([0, M + 1]) \) with \( M > 0 \), we have
\[
\int_0^{M+1} \frac{h(\rho)^2}{2 + M - \rho} \rho \, dp \leq 4 \int_0^\infty (h'(\rho))^2 \rho \, dp.
\] (2.8)

For \( u \in C^\infty([0, T] \times \mathbb{R}^2) \) with support in \( \{(t, x) : |x| \leq 1 + t\} \), we have
\[
\| \langle |x| - t \rangle^{-1} u \|_{L^2_\rho(\mathbb{R}^2)} \lesssim \| \partial_t u \|_{L^2_\rho(\mathbb{R}^2)}, \quad \langle |x| - t \rangle^{-1} |u(t, x)| \lesssim \langle x \rangle^{-\frac{1}{4}} \| \Gamma^{\leq 1} u \|_{L^2_\rho(\mathbb{R}^2)}.
\]

Proof. The inequality (2.8) follows from integrating by parts:
\[
\text{LHS of (2.8)} = - \int_0^{M+1} \frac{h^2}{2 + M - \rho} \rho \, dp + \int_0^{M+1} \frac{2hh'}{2 + M - \rho} \rho \, dp.
\] (2.9)

The second inequality follows from (2.8) and the fact that \( \langle |x| - t \rangle^{-1} \sim (2 + t - |x|)^{-2} \) for \( |x| \leq 1 + t \).

For the third inequality, consider first the case \( |x| > 1 \). By Lemma 2.2, we have
\[
\langle |x| - t \rangle^{-1} |u(t, x)| \lesssim \langle x \rangle^{-\frac{1}{4}} \| \partial^\Sigma_1 u \|_{L^2_\rho(\mathbb{R}^2)}.
\]
(2.10)

On the other hand, for \( |x| \leq 1 \), we have
\[
\langle |x| - t \rangle^{-1} |u(t, x)| \lesssim \langle t \rangle^{-1} (\|u\|_{L^2_\rho(|x| \leq 1)} + \|\partial^2 u\|_{L^2_\rho(|x| \leq 1)}) \lesssim \|\langle |x| - t \rangle^{-1} u\|_{L^2_\rho(\mathbb{R}^2)} + \|\Delta u\|_{L^2_\rho(\mathbb{R}^2)} \lesssim \|\nabla u\|_2 + \|\Delta u\|_2.
\]

Note that we actually proved the inequality:
\[
\langle x \rangle^{-\frac{1}{4}} \langle |x| - t \rangle^{-1} |u(t, x)| \lesssim \begin{cases} \|\partial_t \partial^\Sigma_1 u\|_2, & |x| \geq 1; \\ \|\nabla u\|_2 + \|\Delta u\|_2, & |x| < 1. \end{cases}
\]

Lemma 2.3. If \( g^{kij} \) satisfies the null condition, then for \( t > 0 \) we have
\[
g^{kij} \partial_k f \partial_j h = g^{kij} (T_k f \partial_j h - \omega_k \partial_t f \partial_j h + \omega_k \omega_j \partial_t f \partial_j h),
\] (2.11)
where \( T = (T_1, T_2) \) is defined in (2.3). It follows that
\[
\| g^{kij} \partial_k f \partial_j h \|_2 \lesssim (T f) \| \partial^2 h \| + |\partial f| |T \partial h| \lesssim 1 \langle t \rangle \langle f \rangle \langle \Gamma \partial h \rangle \lesssim 1 \langle r \rangle \langle f \rangle \langle \Gamma \partial h \rangle \lesssim 1 \langle r \rangle \langle f \rangle \langle \Gamma \partial h \rangle |r - t|.
\] (2.12)

Suppose \( g^{kij} \) satisfies the null condition and \( \Box u = g^{kij} \partial_k u \partial_j u \). Then for any multi-index \( \alpha \), we have
\[
\Box \Gamma^\alpha u = \sum_{\alpha_1 + \alpha_2 \leq \alpha} g^{kij}_{\alpha_1 \alpha_2} \partial_k \Gamma^{\alpha_1} u \partial_j \Gamma^{\alpha_2} u,
\] (2.14)
where for each \( (\alpha_1, \alpha_2) \), \( g^{kij}_{\alpha_1 \alpha_2} \) also satisfies the null condition. In addition, we have \( g^{kij}_{00 \alpha} = g^{kij}_{\alpha 0 0} = g^{kij} \).

Proof. The identity (2.11) follows by applying repeatedly the identity \( \partial_t = T_1 - \omega \partial_\rho \) and using the null condition at the last step. The inequality (2.13) is obvious if \( r \leq \frac{t}{2} \) or \( r \geq 2t \), or \( r \sim t \lesssim 1 \) since \( (r + t) \sim (r - t) \) in these regimes. On the other hand, if \( r \sim t \gtrsim 1 \), then one can use the identities
\[
T_1 = \omega_1 \partial_\rho - \frac{\omega_2}{r} \partial_\theta, \quad T_2 = \omega_2 \partial_\rho + \frac{\omega_1}{r} \partial_\theta; \quad \partial_+ = \frac{1}{t + r} (2L_0 - (t-r) \partial_-).
\] (2.15)

The identity (2.14) follows from Hörmander [8].

Lemma 2.4. Suppose \( \bar{u} = \bar{u}(t, x) \) has continuous second order derivatives. Then
\[
\| (r-t) \partial_t \bar{u}(t, x) \| \lesssim \| \partial^\Sigma \bar{u}(t, x) \| + \| (r-t) \Delta \bar{u}(t, x) \|,
\]
and
\[
\| (r-t) \partial^2 \bar{u}(t, x) \| \lesssim \| \partial^{\Sigma^2} \bar{u}(t, x) \| + \| (r-t) \Box \bar{u}(t, x) \|, \quad \forall r \geq t/10, \ t \geq 1.
\] (2.17)
Suppose $T_0 \geq 1$ and $u \in C^\infty([1, T_0] \times \mathbb{R}^2)$ solves $\text{(1.1)}$ with support in $|x| \leq t + 1, 1 \leq t \leq T_0$. For any integer $l_0 \geq 2$, there exists $\epsilon_1 > 0$ depending only on $l_0$, such that if at some $1 \leq t \leq T_0$,
\begin{equation}
\|(\partial_t^l \nabla^{1, t} u)(t, \cdot)\|_{L^2_x(\mathbb{R}^2)} \leq \epsilon_1, \quad (\text{here } [z] = \min\{n \in \mathbb{N} : n \geq z\})
\end{equation}
then for the same $t$, we have the $L^2$ estimate:
\begin{equation}
\|(r-t)^\epsilon \partial_t \nabla^l u(t, \cdot)\|_{L^2_x(\mathbb{R}^2)} \lesssim \|(\partial_t^l \nabla^{l+1} u)(t, \cdot)\|_{L^2_x(\mathbb{R}^2)}.
\end{equation}
For any integer $l_1 \geq 0$, there exists $\epsilon_2 > 0$ depending only on $l_1$, such that if at some $1 \leq t \leq T_0$,
\begin{equation}
\|(\partial_t^l \nabla^{l+1} u)(t, \cdot)\|_{L^2_x(\mathbb{R}^2)} \leq \epsilon_2,
\end{equation}
then for the same $t$, we have the point-wise estimate:
\begin{equation}
|(r-t)^\epsilon \partial_t \nabla^l u(t, x)| \lesssim |(\partial_t^l \nabla^{l+1} u)(t, x)|, \quad \forall r \geq t/10.
\end{equation}

**Proof.** In the 3D case, the estimate (2.11) is an elementary but deep observation of Sideris (cf. 11). Note that for 2D by using $\partial_b L_0 \tilde{u} = (t-r)\partial_b \partial_t \tilde{u} + r(\partial_t + \partial_b) \partial_b \tilde{u}$, we have $|t-r| \cdot \frac{1}{2} \partial_t \partial_b \tilde{u} \lesssim |\partial_t \tilde{u}|$ which (together with the estimate of $|\partial_t \partial_b \tilde{u}|$) settles the estimate for $\nabla \partial_t \tilde{u}$. The estimate (2.17) can be derived along similar lines since $r \geq t/10$ (note that $\partial^2$ includes $\partial_t \partial_t$!). For (2.10), by using a simple integration-by-parts argument, one has (below $k_0 \geq 0$ is a running parameter)
\begin{equation}
\sum_{i, j = 1}^{\epsilon, \epsilon} \|(r-t)\partial_t \partial_j \Gamma \nabla^{\leq k_0} u\|_2 \lesssim \|\partial_t \nabla^{\leq k_0} u\|_2 + \|\partial_t \nabla^{k_0} \Gamma \|_2.
\end{equation}
By using (2.16) and (2.13) we have
\begin{equation}
\|(r-t)^{\epsilon} \partial_t \nabla^{\leq k_0} u\|_2 \lesssim \|\partial_t \nabla^{k_0+1} u\|_2 + \|\partial_t \nabla^{k_0} \Gamma \|_2.
\end{equation}
By (2.22), we obtain
\begin{equation}
\|(r-t)^{\epsilon} \partial_t \nabla^{\leq k_0} u\|_2 \lesssim \|\partial_t \nabla^{k_0+1} u\|_2 + \sum_{m+1 \leq k_0} \|\partial_t \nabla^{m+1} u\|_2 + \|\partial_t \nabla^{m} u\| \|\partial_t \nabla^{l+1} u\|_2 + \|\partial_t \nabla^{m} u\| \|\partial_t \nabla^{l} u\| \|r-t\|_2.
\end{equation}
If $m \leq l + 1$, then we use the estimates (note that $m+2 \leq \lfloor \frac{m+1}{2} \rfloor + 2 \leq \lfloor \frac{m}{2} \rfloor + 2$)
\begin{equation}
(r-t)^{-1}|(\nabla^{m+2} \Gamma u)(t, x)| \lesssim \|\partial_t \nabla^{m+2} u\|_2, \quad \|\partial_t \nabla^{m} u\|_\infty \lesssim \|\partial_t \nabla^{m+2} u\|_2.
\end{equation}
If $m \geq l + 2$, then $l \leq \frac{k_0-1}{2}$ and we use the estimates (see (2.30) for the second estimate)
\begin{equation}
\|\nabla^{m+1} u\|_2 \lesssim \|\partial_t \nabla^{m+1} u\|_2, \quad \|\partial_t \nabla^{m+1} \Gamma u(t, x)| \lesssim \|\partial_t \nabla^{m+2} \Gamma u\|_2.
\end{equation}
Thus if $\|\partial_t \nabla^{k_0+1} u\|_2 \lesssim 1$, we obtain
\begin{equation}
\|\partial_t \nabla^{k_0+1} \Gamma u(t, \cdot)\|_2 \lesssim \|\partial_t \nabla^{k_0+1} u\|_2 \lesssim 1.
\end{equation}
To prove (2.19) under the assumption (2.18) we first take $k_0 = \lfloor \frac{m+1}{2} \rfloor + 1$ and show that
\begin{equation}
\|(r-t)^{\epsilon} \partial_t \nabla^{\leq \lfloor \frac{m+1}{2} \rfloor + 1} u\|_2 \lesssim \|\partial_t \nabla^{\lfloor \frac{m+1}{2} \rfloor + 2} u\|_2.
\end{equation}
We then use this smallness in (2.25) and obtain the desired result for $k_0 = l_0$ (Note that $\lfloor \frac{m+2}{2} \rfloor + 2 \leq \lfloor \frac{m}{2} \rfloor + 1$). The estimate of (2.21) follows from (2.17). \qed

**Lemma 2.5.** For any $f \in C_c^{\infty}(\mathbb{R}^2)$, we have
\begin{equation}
\|f(x_0) - t \partial_x f(x_0)\| \lesssim \|f\|_2 + \|\langle |x| - t \rangle \nabla f\|_2 + \|\langle |x| - t \rangle \partial_t \partial_x f\|_2, \quad \forall x_0 \in \mathbb{R}^2, t \geq 0;
\end{equation}
\begin{equation}
\|\langle |x| - t \rangle \partial_t f\|_\infty \lesssim \|\langle |x| - t \rangle \partial_t f\|_2 + \|\langle |x| - t \rangle \partial_t^2 f\|_2 + \|\langle |x| - t \rangle \partial^2 f\|_2, \quad \forall t \geq 0.
\end{equation}
It follows that
\begin{equation}
\|f\|_{L_t^\infty \nabla_x^{1, \epsilon}(\mathbb{R}^2)} \lesssim (t)^{-\frac{\epsilon}{2}}(\|f\|_2 + \|\langle |x| - t \rangle \nabla \nabla^{\leq 1} f\|_2), \quad \forall t \geq 0,
\end{equation}
where \( \tilde{t} = (\partial_1, \partial_2, \partial_3) \).

\[ \text{Proof.} \] The case \(|x_0| - t| \leq 2 \) follows from the inequality \(|f(x_0)|^2 \leq \int |\partial_1 \partial_2 (f(x))| dx_1 dx_2 \). For \(|x_0| - t| > 2 \), we note that \(|x_0| - t| \leq \frac{|x_0| - t^2}{(x_0 + t)} \). We need to work with the latter weight since it will be smooth near the spatial origin. Note that \( \frac{|x_0|^2 - t^2}{(x_0 + t)} \leq (|x| - t) \) for all \( x \in \mathbb{R}^2, t \geq 0 \). This will be used in the computation below.

Since \( f \in C^w \), by using the Fundamental Theorem of Calculus we have

\[
(|x_0| - t)|f(x_0)|^2 \lesssim \int \left| \partial_3 (\frac{|x_0|^2 - t^2}{(x_0 + t)} f(x_0)) \right| dx_1 dx_2
\]

\[
\lesssim \|f\|_2^2 + \|\nabla f\|_2^2 + \|f\|_2 \|\nabla f\|_2 + \|\|f\|_2 - t| \partial_1 f\|_2 \|f\|_2
\]

\[
\lesssim \|f\|_2^2 + \|\|f\|_2 - t\| \partial_1 f\|_2^2.
\]

(2.32)

Thus (2.29) follows. The proof of (2.30) is similar. Let \( W(x) = \frac{|x|^2 - t^2}{(x + t)} \) and observe that

\[
\sum_{1 \leq i \leq 2} \|\partial_i W\|_\infty + \sum_{1 \leq i,j \leq 2} \|\partial_i \partial_j W\|_\infty \lesssim 1.
\]

(2.33)

One can then work with the expression \( W(x_0)|^2 \int f(x)|^2 \) for the case \(|x_0| - t| > 2 \) and derive the result. For \(|x_0| - t| \leq 2 \), we may assume \( t \geq 2 \). The case \(|x_0| \leq t/2 \) follows from (2.29). The case \(|x_0| > t/2 \) follows from Lemma 2.1.

\[ \square \]

Lemma 2.6 (Decay estimates). Suppose \( T_0 \geq 2 \) and \( u \in C^w([2, T_0] \times \mathbb{R}^2) \) solves \( (2.11) \) with support in \(|x| \leq t + 1, 2 \leq t \leq T_0 \). Suppose \( m \geq 4 \) and

\[
E_m(u(t, \cdot)) = \|\Delta \Gamma \leq m u(t, \cdot)\|_2 \lesssim \tilde{\epsilon},
\]

(2.34)

where \( \tilde{\epsilon} > 0 \) is sufficiently small. Then we have the following decay estimates:

\[
\|\Delta \Gamma \leq m u\|_{L^\infty_t} + \|\partial_1 \partial_2 \Gamma \leq m u\|_{L^\infty_t} \lesssim 1;
\]

(2.35)

\[
\|\partial_1 \partial_2 \Delta \Gamma \leq m u\|_{L^\infty_t} \lesssim t^{-\frac{\tilde{\epsilon}}{2}} \log t;
\]

(2.36)

\[
\|\partial_1 \partial_2 \partial_i \partial_j \Delta \Gamma \leq m u\|_{L^\infty_t} \lesssim t^{-\frac{\tilde{\epsilon}}{2}} \log t;
\]

(2.37)

\[
\|\partial_1 \partial_2 \partial_i \partial_j \Delta \Gamma \leq m u\|_{L^2_t} \lesssim (t) \log t;
\]

(2.38)

\[
\|\partial_1 \partial_2 \partial_i \partial_j \Delta \Gamma \leq m u\|_{L^2_t} \lesssim (t) \log t;
\]

(2.39)

\[
\|\partial_1 \partial_2 \partial_i \partial_j \Delta \Gamma \leq m u\|_{L^2_t} \lesssim (t) \log t;
\]

(2.40)

\[ \square \]

\[ \square \]

Proof. We shall take \( \tilde{\epsilon} \) sufficiently small so that Lemma 2.4 can be applied. The estimate (2.35) follows from Lemma 2.5 and Lemma 2.4. The estimate (2.36) follows from (2.23). Note that the term \( \|\Gamma \leq m u\|_{L^\infty_t} \) therein can be bounded using a dispersive estimate which yields \( t^{\frac{\tilde{\epsilon}}{2}} \) growth. The estimate (2.37) follows from a harmonic analysis estimate using (2.26) and separation of supports. For (2.38), we note that the case \(|x| \leq \frac{t}{2} \) follows from (2.35)–(2.36):

\[
\|\partial_1 \partial_2 \partial_i \partial_j \Delta \Gamma \leq m u\|_{L^2_t} \lesssim (t) \log t.
\]

(2.41)

On the other hand, for \(|x| > \frac{t}{2} \), we denote \( \tilde{u} = \Gamma \leq m^{-2} u \) and estimate \( \|\partial_1 \partial_2 \partial_i \partial_j \Delta \Gamma \leq m u\|_{L^2_t} \) (the estimate for \( T_2 \) is similar). Recall that

\[
T_1 \tilde{u} = \omega_1 \partial_1 \tilde{u} + \partial_3 \tilde{u} = \omega_1 (\partial_1 + \partial_r) \tilde{u} - \frac{\omega_2}{r} \partial_1 \tilde{u}
\]

(2.42)

(2.43)
Clearly for $r = |x| \geq \frac{1}{2}$,
\[
\left| \frac{T_1 \tilde{u}}{\langle r - t \rangle} \right| \lesssim \frac{1}{r} \left( \left| \frac{L_0 \tilde{u}}{r - t} \right| + \left| \partial \tilde{u} \right| + \left| \frac{\partial_y \tilde{u}}{r(r - t)} \right| \right) \\
\lesssim t^{-1} \cdot t^2 \cdot \|\partial \Gamma \|_2 \lesssim t^{-\frac{1}{2}},
\]  
(2.44)
where in the second last step we used Lemma 2.2 (for the term $|\partial \tilde{u}|$ we use (2.33)). The estimates for other terms in (2.36)–(2.39) are similar. We now sketch how to prove (2.40). By using (2.17) (applied to $\tilde{u} = \partial u$), we obtain
\[
\| (r - t) \partial^3 u \| \lesssim |\partial^2 \Gamma \|_1 u + (r + t) \| \partial u |.
\]  
(2.45)
The contribution of the term $|\partial^2 \Gamma \|_1 u$ is clearly OK for us since it can absorb a factor of $(r - t)$. By Lemma 2.3 we have
\[
(r + t) \| \partial u | \lesssim |\Gamma \partial u | | \partial^2 u | | + |\partial u | | \partial^2 \Gamma u | + |\partial u | | \partial^3 u | | + |r - t |.
\]  
(2.46)
The desired estimate then clearly follows by using smallness of the pre-factors. 

3. PROOF OF THEOREM 1.2

Denote (below $w_0 = -1$, $w_1 = \cos \theta$, $w_2 = \sin \theta$)
\[
\alpha_k = g^{kij} w_i w_j, \quad (3.1)
\]  
\[
= g^{k00} + g^{k01} \cos \theta + g^{k02} \sin \theta + g^{k11} \cos^2 \theta + g^{k22} \sin^2 \theta + g^{k12} \cos \theta \sin \theta,
\]  
(3.2)
\[
= (g^{k00} + \frac{g^{k11} + g^{k22}}{2} - 2g^{k01} \cos \theta - 2g^{k02} \sin \theta + \frac{1}{2}(g^{k11} - g^{k22}) \cos 2\theta + g^{k12} \sin 2\theta),
\]  
(3.3)
We compute
\[
\alpha_1 w_1
\]
\[
= (g^{k00} + \frac{g^{k11} + g^{k22}}{2}) \cos \theta - 2g^{k01} \cos \theta - 2g^{k02} \sin \theta + \frac{g^{k11} - g^{k22}}{2} \cos 2\theta + g^{k12} \sin 2\theta \cos \theta + g^{k12} \sin 2\theta \cos \theta
\]  
(3.4)
\[
= - g^{k01} + \frac{g^{k11} + g^{k22}}{2} + \frac{g^{k11} - g^{k22}}{4} \cos \theta + \frac{g^{k12}}{2} \sin \theta - g^{k101} \cos 2\theta - g^{k102} \sin 2\theta
\]  
(3.5)
\[
= (g^{k00} + \frac{g^{k11} + g^{k22}}{2}) \sin \theta - g^{k01} \sin 2\theta - 2g^{k02} \sin^2 \theta + \frac{1}{2}(g^{k11} - g^{k22}) \sin \theta \cos 2\theta + g^{k12} \sin 2\theta \sin \theta
\]  
(3.6)
From the identity $\alpha_0 = \alpha_1 w_1 + \alpha_2 w_2$, we obtain
\[
g^{k01} = - \frac{g^{k00} + g^{k11}}{2}, \quad g^{k02} = \frac{g^{k00} + g^{k22}}{2},
\]  
(3.7)
\[
-2g^{k01} = g^{k00} + g^{k11}, \quad -2g^{k02} = g^{k00} + g^{k22},
\]  
(3.8)
\[
g^{k12} = - \frac{g^{k01}}{2} - \frac{g^{k02}}{2}; \quad g^{k12} = \frac{g^{k11} - g^{k22}}{2}; \quad g^{k12} = - \frac{g^{k11} - g^{k22}}{2}.
\]  
(3.9)
We now set (below we shall treat $(a_i)_{i=1}^{11}$ as free parameters)
\[
g^{k00} = a_1, \ g^{k01} = a_2, \ g^{k02} = a_3, \ g^{k11} = a_4, \ g^{k01} = a_5, \ g^{k02} = a_6,
\]  
(3.10)
\[
g^{k10} = a_7, \ g^{k22} = a_8, \ g^{k12} = a_9, \ g^{k12} = a_{10}, \ g^{k12} = a_{11}.
\]  
(3.11)
Then we obtain the rest of the coefficients as follows:

\[
\begin{align*}
g^{101} & = \frac{a_1 + a_4}{2}, \quad g^{202} = -\frac{a_1 + a_6}{2}, \quad g^{111} = -2a_2 - a_7, \\
g^{200} & = -2a_4 - a_8, \quad g^{201} = -a_5 - a_9, \quad g^{212} = -\frac{2a_2 + a_7 + a_{10}}{2}, \\
g^{211} & = a_8 - 2a_{11}.
\end{align*}
\]  

(3.12) -- (3.14)

We then rearrange the nonlinearity \(f^{ijk} \partial_k u \partial_{ij}u\) as 11 terms:

- \(a_1: \partial_0 u \partial_{00} u - \partial_1 u \partial_{01} u - \partial_2 u \partial_{02} u.\)
- \(\frac{1}{2} a_2: \partial_0 u \partial_{01} u - \partial_1 u \partial_{11} u - \partial_2 u \partial_{12} u.\)
- \(\frac{1}{2} a_3: \partial_0 u \partial_{02} u - \partial_2 u \partial_{03} u.\)
- \(a_4: \partial_0 \partial_1 u - \partial_1 u \partial_{01} u.\)
- \(\frac{1}{2} a_5: \partial_0 \partial_{12} u - \partial_2 \partial_{02} u.\)
- \(a_6: \partial_0 \partial_{22} u - \partial_2 \partial_{03} u.\)
- \(a_7: \partial_1 \partial_{01} u - \partial_1 u \partial_{11} u - \partial_2 \partial_{12} u.\)
- \(a_8: \partial_2 u \partial_{22} u - \partial_2 u \partial_{03} u + \partial_2 u \partial_{11} u.\)
- \(\frac{1}{2} a_{10}: \partial_1 \partial_{22} u - \partial_2 \partial_{12} u.\)
- \(\frac{1}{2} a_{11}: \partial_1 \partial_{12} u - \partial_2 \partial_{13} u.\)

It is not difficult to check that these 11 terms are in one-to-one correspondence of (1.8)–(1.11) (after suitable linear combinations).

3.1. Our new null condition. Now consider the expression

\[
\begin{align*}
&= (-\sin \theta)a_{11} + (\cos \theta)a_{22} \\
&= (-\sin \theta)g^{100} + \frac{g^{111} + g^{122}}{2} - 2g^{101} \cos \theta - 2g^{102} \sin \theta + \frac{g^{111} - g^{122}}{2} \cos 2\theta + \frac{g^{112}}{2} \sin 2\theta \\
&+ (\cos \theta)(g^{200} + \frac{g^{211} + g^{222}}{2} - 2g^{201} \cos \theta - 2g^{202} \sin \theta + \frac{g^{211} - g^{222}}{2} \cos 2\theta + \frac{g^{212}}{2} \sin 2\theta) \\
&= g^{102} - g^{201} + \left(-\frac{g^{112}}{2} + g^{200} + \frac{g^{211} + g^{222}}{2} + \frac{g^{211} - g^{222}}{4}\right) \cos \theta \\
&+ \left(-\frac{g^{111} + g^{122}}{2} + \frac{g^{211} - g^{122}}{4} + \frac{g^{212}}{2}\right) \sin \theta \\
&+ (-g^{102} - g^{201}) \cos 2\theta + (g^{101} - g^{202}) \sin 2\theta \\
&+ \left(-\frac{g^{112}}{2} + \frac{g^{211} - g^{222}}{4}\right) \cos 3\theta + \left(-\frac{g^{111} - g^{122}}{4} + \frac{g^{212}}{2}\right) \sin 3\theta.
\end{align*}
\]  

(3.15)

Simplifying a bit using the standard null condition, we obtain

\[
\begin{align*}
&= g^{102} - g^{201} + \left(-\frac{g^{112}}{2} + g^{200} + \frac{g^{211} + g^{222}}{2} + \frac{g^{211} - g^{222}}{4}\right) \cos \theta \\
&+ \left(-\frac{g^{111} + g^{122}}{2} + \frac{g^{211} - g^{122}}{4} + \frac{g^{212}}{2}\right) \sin \theta \\
&+ (-g^{102} - g^{201}) \cos 2\theta + (g^{101} - g^{202}) \sin 2\theta.
\end{align*}
\]  

(3.16)

Forcing \((-\sin \theta)a_{11} + (\cos \theta)a_{22} = 0\) and using the standard null condition gives us

\[
\begin{align*}
g^{102} & = g^{201} = 0, \quad g^{101} = g^{202}, \\
g^{102} & = -g^{112}, \quad g^{201} = -g^{212}.
\end{align*}
\]  

(3.17) -- (3.18)

It is then not difficult to check that these lead to (1.13).
3.2. The null condition in \[15\]. Expanding (1.4) in more details, we obtain

\[
N_{00} - (N_{01} + N_{10}) \cos \theta - (N_{02} + N_{20}) \sin \theta + N_{11} \cos^2 \theta + N_{22} \sin^2 \theta + (N_{12} + N_{21}) \cos \theta \sin \theta = 0. 
\]  
(3.19)

Thus

\[
N_{00} + \frac{1}{2} N_{11} + \frac{1}{2} N_{22} = 0; 
\]

\[
N_{01} + N_{10} = 0, \quad N_{02} + N_{20} = 0; 
\]

\[
N_{12} + N_{21} = 0, \quad N_{11} - N_{22} = 0. 
\]  
(3.20), (3.21), (3.22)

Therefore \( N_{11} = N_{22} = -N_{00} \), and

\[
N_{ij} a_i b_j = N_{00}(a_0 b_0 - a_1 b_1 - a_2 b_2) + N_{01}(a_0 b_1 - a_1 b_0) + N_{02}(a_0 b_2 - a_2 b_0). 
\]  
(3.23)

If \( a = b \), then clearly \( N_{ij} a_i a_j = N_{00}(a_0^2 - a_1^2 - a_2^2) \). In particular,

\[
N_{ij} \partial_x v \partial_x v = N_{00} \left( (\partial_t v)^2 - |\nabla v|^2 \right). 
\]  
(3.24)

4. PROOF OF THEOREM 1.1

In this section and Section 5, we carry out the proof of Theorem 1.1. Write \( v = \Gamma^\alpha u \), by Lemma 2.3 we have

\[
\Box v = \sum_{\alpha_1 + \alpha_2 \leq \alpha} g^{kij}_{\alpha_1 \alpha_2} \partial_k \Gamma^{\alpha_1} u \partial_j \Gamma^{\alpha_2} u 
\]

\[
= g^{kij} \partial_k v \partial_j u + g^{kij} \partial_k u \partial_j v + \sum_{\alpha_1 + \alpha_2 \leq \alpha; \alpha_1 + \alpha_2 \leq \alpha} g^{kij}_{\alpha_1 \alpha_2} \partial_k \Gamma^{\alpha_1} u \partial_j \Gamma^{\alpha_2} u. 
\]  
(4.1), (4.2)

Choose \( p(t, r) = q(r - t) \), where

\[
q(s) = \int_0^s (r)^{-(1/2 + \tau^2)} d\tau, \quad s \in \mathbb{R}. 
\]  
(4.3)

Clearly

\[
-\partial_t p = \partial_r p = q'(r - t) = (r - t)^{-1}\left( \log(2 + (r - t)^2) \right)^{-2}. 
\]  
(4.4)

Multiplying both sides of (4.1) by \( e^p \partial_t v \), we obtain

\[
\text{LHS} = \int e^p \partial_t v \partial_t v - \int e^p \Delta v \partial_t v = \int e^p \partial_t v \partial_t v + \int e^p \nabla v \cdot \nabla \partial_t v + \int e^p \nabla v \cdot \nabla p \partial_t v 
\]

\[
= \frac{1}{2} \frac{d}{dt} \int e^p |\partial_t v|^2 - \frac{1}{2} \int e^p |\partial_t v|^2 p_t + \int e^p \nabla v \cdot \nabla p \partial_t v 
\]

\[
= \frac{1}{2} \frac{d}{dt} \|e^p \partial_t v\|_{L^2}^2 + \frac{1}{2} \int e^p q' \cdot \left( |\partial_x v|^2 + \frac{|\partial_\theta v|^2}{r^2} \right) = \frac{1}{2} \frac{d}{dt} \|e^p \partial_t v\|_{L^2}^2 + \frac{1}{2} \int e^p q'|Tv|^2. 
\]  
(4.5)

To simplify the notation in the subsequent nonlinear estimates, we introduce the following terminology. **Notation.** For a quantity \( X(t) \), we shall write \( X(t) = 0 \) OK if \( X(t) \) can be written as

\[
X(t) = \frac{d}{dt} X_1(t) + X_2(t) + X_3(t), 
\]  
(4.6)

where (below \( \alpha_0 > 0 \) is some constant)

\[
|X_1(t)| \ll \left\| (\partial_t^{\leq m} u) (t, \cdot) \right\|_{L^2}^2, \quad |X_2(t)| \ll \sum_{|\alpha| \leq m} \int e^{p \tau'} |(TT^\alpha u)(t, x)|^2 dx, \quad |X_3(t)| \lesssim \langle t \rangle^{-1 - \alpha_0}. 
\]  
(4.7)

In yet other words, the quantity \( X \) will be controllable if either it can be absorbed into the energy, or can be controlled by the weighted \( L^2 \)-norm of the good unknowns from the Alinhac weight, or it is integrable in time.

We now proceed with the nonlinear estimates. We shall discuss several cases.
4.1. **The case** \( \alpha_1 < \alpha \) and \( \alpha_2 < \alpha \). Since \( g^{kij}_{\alpha_1, \alpha} \) still satisfies the null condition, by (2.11) we have

\[
\sum_{\alpha_1 < \alpha_2 < \alpha_1 + \alpha_2 \leq \alpha} g^{kij}_{\alpha_1, \alpha_2} \partial_k \Gamma^{\alpha_1} u \partial_{ij} \Gamma^{\alpha_2} u
\]

\[
= \sum_{\alpha_1 < \alpha_2 < \alpha_1 + \alpha_2 \leq \alpha} g^{kij}_{\alpha_1, \alpha_2} (T_k \Gamma^{\alpha_1} u \partial_{ij} \Gamma^{\alpha_2} u - \omega_k \partial_t \Gamma^{\alpha_1} u T_i \partial_j \Gamma^{\alpha_2} u + \omega_k \omega_i \partial_i \Gamma^{\alpha_1} u T_j \partial_t \Gamma^{\alpha_2} u). \tag{4.7}
\]

**Estimate of** \( \| T_k \Gamma^{\alpha_1} u \partial^2 \Gamma^{\alpha_2} u \|_2 \). If \( |\alpha_1| \leq |\alpha_2| \), then by Lemma 2.6 we have

\[
\| T_k \Gamma^{\alpha_1} u \partial^2 \Gamma^{\alpha_2} u \|_2 \lesssim \frac{T_k \Gamma^{\alpha_1} u}{(r-t)} \| \partial_t \Gamma^{\alpha_2} u \|_\infty \lesssim (r-t)^{-\frac{1}{2}} \log t \lesssim t^{-\frac{5}{2}} \log t. \tag{4.8}
\]

If \( |\alpha_1| > |\alpha_2| \), then we have

\[
\| T_k \Gamma^{\alpha_1} u \partial^2 \Gamma^{\alpha_2} u \|_2 \lesssim \frac{T_k \Gamma^{\alpha_1} u}{(r-t)} \| \partial_t \Gamma^{\alpha_2} u \|_\infty \lesssim t^{-1} \cdot t^{-\frac{1}{2}} \log t \lesssim t^{-\frac{5}{2}} \log t. \tag{4.9}
\]

**Estimate of** \( \| \partial \Gamma^{\alpha_1} u T \partial \Gamma^{\alpha_2} u \|_2 \). If \( |\alpha_1| \leq |\alpha_2| \) we have

\[
\| \partial \Gamma^{\alpha_1} u T \partial \Gamma^{\alpha_2} u \|_2 \lesssim \| \partial \Gamma^{\alpha_1} u \|_\infty \cdot \| T \partial \Gamma^{\alpha_2} u \|_2 \lesssim t^{-\frac{5}{2}}. \tag{4.10}
\]

If \( |\alpha_1| > |\alpha_2| \) we have

\[
\| \partial \Gamma^{\alpha_1} u T \partial \Gamma^{\alpha_2} u \|_2 \lesssim \| \partial \Gamma^{\alpha_1} u \|_2 \cdot \| T \partial \Gamma^{\alpha_2} u \|_2 \lesssim t^{-\frac{5}{2}} \log t. \tag{4.11}
\]

Collecting the estimates, we have proved

\[
\| \sum_{\alpha_1 < \alpha_2 < \alpha_1 + \alpha_2 \leq \alpha} g^{kij}_{\alpha_1, \alpha_2} \partial_k \Gamma^{\alpha_1} u \partial_{ij} \Gamma^{\alpha_2} u \|_2 \lesssim t^{-\frac{5}{2}} \log t. \tag{4.12}
\]

4.2. **The case** \( \alpha_2 = \alpha \). Noting that \( g^{kij}_{0, \alpha} = g^{kij} \), we have

\[
\int g^{kij} \partial_k u \partial_{ij} v \partial_t v \partial^p = \text{OK} - \int g^{kij} \partial_k u \partial_{ij} v \partial_t v \partial^p - \int g^{kij} \partial_k u \partial_{ij} v \partial_t v \partial^p + \int g^{kij} \partial_k u \partial_{ij} v \partial_t v \partial^p. \tag{4.13}
\]

Here in the above, the term “OK” is zero if \( \partial_j = \partial_1 \) or \( \partial_2 \). This term is nonzero when \( \partial_j = \partial_i \), i.e. we should absorb it into the energy when integrating by parts in the time variable.

Further integration by parts gives

\[
- \int g^{kij} \partial_k u \partial_{ij} v \partial_j v \partial^p = \text{OK} + \int g^{kij} \partial_k u \partial_{ij} v \partial_j v \partial^p + \int g^{kij} \partial_k u \partial_{ij} v \partial_j v \partial^p + \int g^{kij} \partial_k u \partial_{ij} v \partial_j v \partial^p. \tag{4.14}
\]

\[
\int g^{kij} \partial_k u \partial_{ij} v \partial_j v \partial^p = \text{OK} - \int g^{kij} \partial_k u \partial_{ij} v \partial_j v \partial^p - \int g^{kij} \partial_k u \partial_{ij} v \partial_j v \partial^p - \int g^{kij} \partial_k u \partial_{ij} v \partial_j v \partial^p. \tag{4.15}
\]

It follows that

\[
2 \int g^{kij} \partial_k u \partial_{ij} v \partial_j v \partial^p = (I_1 + I_3 + I_5) + (I_2 + I_4 + I_6) + \text{OK}. \]

Observe that if \( \varphi = \partial_k u \) or \( \varphi = \partial_t v \), then

\[
- \partial_j \varphi \partial_k v \partial_t v + \partial_t \varphi \partial_k v \partial_t v - \partial_t \varphi \partial_j v \partial_t v
\]

\[
= - T_j \varphi \partial_k v \partial_t v + \omega_j \varphi \partial_k v \partial_t v + \partial_t \varphi \partial_k v \partial_t v - T_j \varphi \partial_k v \partial_j v + \omega_j \varphi \partial_k v \partial_t v - \omega_j \varphi \partial_k \varphi \partial_j v
\]

\[
= - T_j \varphi \partial_k v \partial_t v + \partial_t \varphi \partial_k v \partial_t v - T_j \varphi \partial_k v \partial_t v + \omega_j \varphi \partial_k v \partial_t v - \omega_j \varphi \partial_k \varphi \partial_t v
\]

\[
= T_j \varphi \partial_k v \partial_t v + \partial_t \varphi \partial_k v \partial_t v - \omega_j \varphi \partial_k \varphi \partial_t v. \tag{4.16}
\]

By (4.16) and rewriting \( \partial_t \varphi = \partial_k \partial_k u = T_k \partial_t u - \omega_k \partial_t u \), we have

\[
I_1 + I_3 + I_5 = \int g^{kij} (\partial_k \partial_k u \partial_{ij} v \partial_j v + \partial_t \partial_k u T_i \partial_j v \partial_t v - T_i \partial_k u \partial_t u \partial_j v - \omega_j \partial_k \partial_k u \partial_t u) e^p \text{d}x. \tag{4.17}
\]
By Lemma 2.6, we have \( \|T \partial u\|_\infty \lesssim t^{-1/2} \log t \) and \( \|(r-t)\partial^2 u\|_\infty \lesssim t^{-3/2} \log t \). Clearly then
\[
\int_{r < \frac{1}{4} \text{ or } r > 2t} |\partial^2 u||Tv|^2 \, dx \lesssim t^{-3/2} \log t, \quad \int_{r = t} |\partial^2 u||Tv|^2 \, dx \ll \int e^{\alpha p} |Tv|^2 \, dx. \tag{4.18}
\]
It follows that
\[
I_1 + I_3 + I_5 = \text{OK}. \tag{4.19}
\]
Plugging \( \varphi = e^p \) in (1.16) and noting that \( T_j(e^p) = 0 \), we have
\[
I_2 + I_4 + I_6 = \int g^{kij} \partial_k v \left( (T_k v \partial_j u - \omega_k \partial_j vt \partial_k u + \omega_k \omega_j \partial_j v \partial_k u) - v \partial_j \nabla \partial_k u \right) dx.
\]
By using the identity
\[
g^{kij} \omega_j T_k v = g^{1ij} \omega_1 v \partial_1 + g^{2ij} \omega_2 v \partial_2 + \omega_3 \partial_3 \partial_3 v = h(\theta) \partial_3 v,
\]
we choose nonnegative \( \phi \in C^\infty_c(\mathbb{R}) \) such that \( \phi(z) = 1 \) for \( \frac{1}{3} \leq z \leq \frac{2}{3} \) and \( \phi(z) = 0 \) for \( z \leq \frac{1}{6} \) or \( z \geq 2 \). Then
\[
\int g^{kij} \omega_j T_k v \partial_j u \partial_k v e^p \lesssim \int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p \cdot \left( 1 - \phi \left( \frac{r}{t} \right) \right) + \int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p \cdot \left( \frac{r}{t} \right). \tag{5.2}
\]
By Lemma 2.6, we have
\[
\int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p \cdot (1 - \phi) \lesssim t^{-1} \int |\partial v|^2 |(r-t) \partial_t u| \lesssim t^{-3/2} \log t = \text{OK}. \tag{5.3}
\]
By using the identity \( \partial_t = \frac{\partial_r + \theta}{r} \partial_\theta \) and the fact that \( \|(r-t)\partial^2 u\|_\infty \lesssim t^{-3/2} \log t \), we get
\[
2 \int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p = \int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p + \int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p \phi = \text{OK} + \int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p \phi. \tag{5.4}
\]
Integrating by parts, we have
\[
\int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p \cdot r dr d\theta = \frac{d}{dt} \int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p dx - \int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p dx
\]
\[
- \int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p dx - \int h(\theta) \partial_3 v \partial_3 u \partial_3 v e^p dx. \tag{5.5}
\]
In the above computation, one should note that when integrating by parts in \( r \) we should take into consideration the metric \( rdr \). The fourth term exactly corresponds to the derivative of the metric factor. The first and fourth terms are clearly acceptable by using Hardy and the decay of \( \langle r-t \rangle \partial_t u \). For the second term we have
\[
|\langle r-t \rangle \partial_+ (\partial_{tt}u e^p \phi)| \lesssim |\langle r-t \rangle \partial_+ \partial_t u \phi| + |\langle r-t \rangle \partial_{tt} u \phi| \\
\lesssim t^{-\frac{2}{3}} \|2\langle r-t \rangle L_0 \partial_t u - \langle r-t \rangle (t-r) \partial_\pm \partial_t u \partial_t u \|_{L^p ([1, +\infty]} + t^{-\frac{2}{3}} \lesssim t^{-\frac{2}{3}}. \tag{5.6}
\]
Here in the derivation of (5.6), we used Lemma 2.6 and the inequalities
\[
|\langle r-t \rangle L_0 \partial_t u| \lesssim |\langle r-t \rangle \partial_t \Gamma_{\leq 1} u| \lesssim t^{-\frac{2}{3}}, \quad \text{for } r \geq t/10. \tag{5.7}
\]
For the third term we use the identity \( \partial_+ \partial_\pm v = \Box v + \frac{\partial v}{\partial r} + \frac{\partial_{tt} u}{r^2} \) and compute it as
\[
\int h(\theta) e^{\partial_t u} \partial_t u \partial_+ \partial_\pm v e^p \phi = \int h(\theta) e^{\partial_t u} \left( \frac{\partial_v v}{r} + \frac{\partial_{tt} u}{r^2} \right) e^p \phi + \sum_{\beta_1 + \beta_2 \leq \alpha} \int h(\theta) e^{\partial_t u} g_{ij}^{kij} \Gamma^{ij} \partial_t u \partial_t u \Gamma^{ij} e^p \phi. \tag{5.8}
\]
Integrating by parts (for the term \( \partial_{tt} v \)), we have
\[
\int h(\theta) e^{\partial_t u} \left( \frac{\partial_v v}{r} + \frac{\partial_{tt} u}{r^2} \right) e^p \phi = \int h(\theta) e^{\partial_t u} \left( \frac{\partial_v v}{r} + \frac{1}{r} \right) e^p - \int h(\theta) e^{\partial_t u} \left( \frac{\partial_{tt} u}{r^2} \right) e^p - \int \partial_{t\theta}(h(\theta) e^{\partial_t u}) v \frac{\partial_{tt} u}{r^2} e^p \phi = \text{OK}.
\]
By (4.12), we have
\[
\sum_{\beta_1 < \alpha, \beta_2 \leq \alpha, \beta_1 + \beta_2 \leq \alpha} \int h(\theta) e^{\partial_t u} g_{ij}^{kij} \Gamma^{ij} \partial_t u \partial_t u \Gamma^{ij} e^p \phi \lesssim t^{-2} = \text{OK}.
\]
For the term \( \beta_1 = \alpha, \beta_2 = 0 \) in (5.8), it follows from (2.12) that
\[
\int g_{ij}^{kij} h(\theta) v e^{\partial_t u} \partial_\theta v \partial_t u e^p \phi \lesssim \int |v e^{\partial_t u}| |T e^p \phi| + \int |v e^{\partial_t u}| |T e^p \phi| + \int |T e^p \phi| + t^{-\frac{2}{3}} \|e^p \phi\|_{L^2([1, \infty)} = \text{OK}.
\]
For the term \( \beta_1 = 0, \beta_2 = \alpha \) in (5.8), we apply (2.11) to obtain
\[
\int g_{ij}^{kij} h(\theta) v e^{\partial_t u} \partial_\theta v \partial_t u e^p \phi = \int g_{ij}^{kij} h(\theta) v e^{\partial_t u} \partial_\theta v \partial_t u e^p \phi = \int g_{ij}^{kij} h(\theta) v e^{\partial_t u} \partial_\theta v \partial_t u e^p \phi - \int g_{ij}^{kij} h(\theta) v e^{\partial_t u} \partial_\theta v \partial_t u e^p \phi - \int g_{ij}^{kij} h(\theta) v e^{\partial_t u} \partial_\theta v \partial_t u e^p \phi.
\]
We rewrite it as
\[
\int g_{ij}^{kij} h(\theta) v e^{\partial_t u} \partial_\theta v \partial_t u e^p \phi = \int g_{ij}^{kij} \partial_t v e^{\partial_t u} e^p \phi.
\]
The term \( \int g_{ij}^{kij} h(\theta) v e^{\partial_t u} T_k e^p \phi \) is zero for \( i \neq 0 \). For \( i = 0 \) it is clearly acceptable since it can be absorbed into the time derivative of the energy due to its smallness. By Lemma 2.3 and 2.6 we have
\[
|\partial_t (\theta T_k e^p \phi)| \lesssim |\theta (\theta T_k e^p \phi)| + |h(\theta) u T_k e^p \phi| + |h(\theta) T_k u \partial_t e^p \phi| + |h(\theta) T_k u \partial_t e^p \phi| \lesssim t^{-\frac{2}{3}} + |h(\theta) \partial_\theta \omega_k \partial_t u e^p \phi| + |h(\theta) T_k \partial_t u e^p \phi| + \left| \frac{T_k u}{r-t} \right| \lesssim t^{-\frac{2}{3}}.
\]
The term containing \( v \partial_\theta \partial_t u \) can be handled by (2.30). Thus
\[
\int g_{ij}^{kij} h(\theta) v e^{\partial_t u} T_k e^p \phi = \text{OK}.
\]
Similarly, we have
\[ \int g^{ab} \omega_a h(\theta) v \partial_t u \partial_t v T_j \partial_j v e^p \phi = \int g^{ab} \omega_a h(\theta) v \partial_t u \partial_t u (\partial_j T_i v - \partial_j \omega_i \partial_t v) e^p \phi = \text{OK}, \]
\[ \int g^{ab} \omega_a h(\theta) v \partial_t u \partial_t v T_j \partial_j v e^p \phi = \int g^{ab} \omega_a h(\theta) v \partial_t u \partial_t u \partial_t T_j e^p \phi = \text{OK}. \]
This concludes the estimate of the main piece.

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