Cosmological applications of the Brown-York quasilocal mass

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The Brown-York quasilocal energy is applied to three cosmological problems which have previously been studied with the Hawking-Hayward quasilocal energy (Newtonian simulations of large scale structure formation, turnaround radius in the present accelerating universe, and lensing by the cosmological constant). It is found that, in an appropriate gauge and to first order in the amplitude of the cosmological perturbations describing local structures, the Hawking-Hayward and the Brown-York quasilocal masses predict the same results.

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I. INTRODUCTION

In General Relativity (GR), the notion of total mass-energy of an asymptotically flat system (including its rest mass, stresses, kinetic and gravitational energy) is well understood and is identified with the Arnowitt-Deser-Misner mass [1]. Non-asymptotically flat spacetimes are more difficult to describe. The equivalence principle embodied in GR makes it impossible to localize gravitational energy and, for non-asymptotically flat geometries, one must then resort to a quasilocal definition of energy. Defining the quasilocal energy of a non-asymptotically flat spacetime is highly non-trivial and several quasilocal definitions have been introduced in the literature (see [2] for a review). It seems that the Hawking-Hayward quasilocal energy [7] is more commonly used, while also the Brown-York definition [5] has been popular. Generally speaking, the available definitions of quasilocal energy are quite formal and not practical to use. However, the mass of an astrophysical system is one of its most basic properties and, if a notion of quasilocal mass is to be useful in science, it should not remain confined to an abstract domain but it should be useful for practical calculations in astrophysics and cosmology. With this goal in mind, we have applied the Hawking-Hayward quasilocal energy to cosmology in previous publications [6, 7].

The first problem studied was whether the Newtonian simulations of large scale structure formation (Newtonian simulations of large scale structure formation, turnaround radius in the present accelerating universe, and lensing by the cosmological constant) and especially the meaning of “mass contained in the sphere of critical radius” were clarified by applying the Hawking-Hayward quasilocal energy [7], which reduces to the better known Misner-Sharp-Hernandez mass [35] in spherical symmetry [36]. The Misner-Sharp-Hernandez mass is widely used in relativistic fluid dynamics, in simulations of spherical gravitational collapse to black holes, and in black hole thermodynamics [37] and is related to apparent horizons [38]. Finally, the Hawking-Hayward/Misner-Sharp-Hernandez quasilocal mass was

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applied to the long-standing problem of whether the cosmological constant contributes directly to the deflection angle of light rays caused by a local gravitational lens. Two opposing viewpoints support opposite answers to this question and the decade-long debate is still open [39]. A new approach based on the quasilocal energy [3] shows that the debate exists because of ambiguities in the concept of mass contained in the sphere grazed by the light rays and provides a definite answer to this problem. The scope is also extended, for arbitrary forms of dark energy can be included, not only the cosmological constant to which almost all of the previous literature was limited [39].

Some common physics underlies these three applications of the quasilocal energy to cosmology, namely the competition between local effects due to a localized mass distribution (which is a Newtonian-like perturbation of the underlying FLRW universe), and the effects of the cosmic expansion. The competition between local physics and cosmic expansion is a recurrent theme in cosmology [10]. These two competing effects should be compared to first order in the amplitude of the cosmological perturbations, which is usually sufficient for all practical purposes. The approach using the quasilocal mass is particularly suited to this kind of problem. In fact, to first order in the cosmological perturbations, the Hawking-Hayward quasilocal mass $M_{HH}$ splits into two contributions: the first one, determined by the perturbation, is local and “Newtonian”, while the second contribution is purely cosmological. More precisely, one writes the perturbed FLRW line element in the conformal Newtonian gauge as

$$ds^2 = - (1 + 2\phi) dt^2 + a^2(t) \left( (1 - 2\phi) \left( dr^2 + r^2 d\Omega_2^2 \right) + a^2(\eta) \left[ -(1 + 2\phi) d\eta^2 + (1 - 2\phi) \left( dr^2 + r^2 d\Omega_2^2 \right) \right] \right) \tag{1}$$

where the scale factor $a(\eta)$ is a function of the conformal time $\eta$ (related to the comoving time $t$ by $dt = a d\eta$), $\phi(r)$ is a Newtonian potential describing the local perturbation, and $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the line element on the unit 2-sphere. The two $\phi$s appearing in Eq. (1) should a priori be different, but they turn out to be equal to first order, as implied by the fact that the perturbative energy-momentum tensor is diagonal [11].

Only a spatially flat FLRW universe is considered. To first order in the perturbations, and following standard literature, vector and tensor perturbations can be safely omitted from the line element [11] because the mass concentrations described by the perturbations have non-relativistic peculiar velocities [11, 42, 43]. Vector and tensor perturbations should be included in second order calculations due to mode-mode coupling, but in this work second order effects are completely negligible. The first-order splitting of the quasilocal energy is [6–8, 44]

$$M_{HH} = ma + \frac{H^2 R^3}{2} (1 - 2\phi) \approx ma + \frac{H^2 R^3}{2}$$

$$= ma(t) + \frac{4\pi R^3}{3} \rho, \tag{2}$$

where $m$ is the local Newtonian mass responsible for the metric perturbation $\phi$ in Eq. (1) and $\rho$ is the cosmological density of the spatially flat FLRW background, related to the Hubble parameter by the Friedmann equation

$$H^2 = \frac{8\pi G}{3} \rho \tag{3}$$

for a spatially flat FLRW universe.

The competition between local dynamics and cosmological expansion is described by the two contributions (local, $ma$, and cosmological, $H^2 R^3/2$) to the quasilocal energy. For example, the critical turnaround radius of a large spherical structure in a FLRW universe is obtained when these two contributions are equal [7]. In the discussion of whether Newtonian simulations of large scale structure formation are adequate, it was shown that the Newtonian, local contribution to the Hawking-Hayward mass dominates over the cosmological contribution [6]. And, in a study of the direct contribution of the cosmological constant to the deflection angle of light rays by a localized gravitational lens, the splitting of the Misner-Sharp-Hernandez mass determines a splitting of the deflection angle into a contribution by the local lens plus a cosmological contribution—the latter vanishes to first order [8].

The results of Refs. [6–8] show that the quasilocal energy does not need to remain a formal concept relegated to the realm of abstract mathematical physics, but is actually useful in more practical applications. However, while this new approach allows one to draw definite conclusions on this kind of perturbative problem in which local dynamics competes with the cosmological expansion in the early or in the late universe, there remains a doubt. Refs. [6–8] use the Hawking-Hayward mass. The question arises naturally of whether the use of a different quasilocal mass would provide different results. In the present article we set out to investigate this question by exploring the predictions of the Brown-York quasilocal mass [5] in the same problems. Although our main motivation is to clarify the issue of choosing a quasilocal mass in view of cosmological applications, the problem is also interesting in principle. If different quasilocal energy prescriptions provide different outcomes, even at the first-order which is testable with astronomical observations [42], then one would in principle have, besides computer simulations, an experimental way of discriminating between different definitions of quasilocal energy. For the three cosmological problems listed above, the analysis of the following sections leads to the conclusion that the Hawking-Hayward and the Brown-York quasilocal energies provide the same result to first order in the cosmolog-
\[ M_{\text{BY}} = R(1 - f) \] (5)

The Brown-York mass is defined with respect to a reference space which, in this case, is obtained for \( N = f = 1 \), turning Eq. (4) into a Minkowski-like metric. The Hawking-Hayward mass, which reduces to the Misner-Sharp-Hernandez prescription in spherical symmetry \([30]\), is

\[ M_{\text{MSH}} = \frac{R}{2}(1 - f^2) \] (6)

as follows from the general definition \([33]\)

\[ 1 - 2 M_{\text{MSH}} = \nabla_c R \nabla^c R \] (7)

The gauge (4) is used almost universally in the literature on black hole thermodynamics, and also in Refs. [47–49], in which the two quasilocal masses have been investigated and an attempt to interpret them physically has been made.

In general, the Brown-York mass is defined in terms of a \( 3 + 1 \) splitting of spacetime and of the associated 3-metric and extrinsic curvature \([3]\), which makes it clear that this quantity depends on the foliation or gauge chosen. By contrast, the Misner-Sharp-Hernandez mass defined by the scalar equation (2) and by the areal radius \( R \) (which is a geometric, gauge-independent quantity), is gauge-invariant. Therefore, it is meaningless to compare these two quasilocal energy constructs in an arbitrary gauge. In our situation, we want to compare the prediction of Misner-Sharp-Hernandez and Brown-York masses for a (spherically symmetric) perturbed FLRW universe. In order for the comparison to make sense, one should choose a gauge in which the two quasilocal energies coincide for a sphere in the unperturbed universe. This will be done in the following calculation, and shown explicitly at its end by taking the limit of zero perturbations.

Let us begin with the perturbed FLRW line element (1) and let us transform it to a gauge which uses the areal radius

\[ R(\eta, r) = a(\eta)r\sqrt{1 - 2\phi(r)} \] (8)

as the radial coordinate. In the end, we will retain only the lowest order terms in the metric perturbation \( \phi, r\phi' \), and \( HR \), where \( H = \ddot{a}/a \) (an overdot denoting differentiation with respect to the comoving time \( t \)) is the Hubble parameter and \( HR \) is approximately the size \( R \) of the local perturbation in units of the Hubble radius \( H^{-1} \). Equation (9) yields

\[ dr = \frac{\sqrt{1 - 2\phi}}{a(1 - 2\phi - r\phi')} (dR - a\dot{r}\sqrt{1 - 2\phi} d\eta) \] (9)

which, substituted into the line element (11), gives

\[ ds^2 = a^2 \left\{ \left[ -(1 + 2\phi) + \frac{(1 - 2\phi)^2 H^2 R^2}{a^2 (1 - 2\phi - r\phi')} \right] d\eta^2 \\
+ \frac{(1 - 2\phi)^2 R^2}{a^2 (1 - 2\phi - r\phi')} - \frac{2HR (1 - 2\phi)^2}{a^2 (1 - 2\phi - r\phi')^2} d\eta dR \right\} + R^2 d\Omega^2 \] (10)

where

\[ H \equiv \frac{1}{a} \frac{da}{d\eta} \] (11)

is the Hubble parameter of the conformal time \( \eta \). Since we want to arrive at a diagonal gauge, we need to eliminate the time-radius cross-term by redefining the time coordinate, \( \eta \rightarrow T \), according to

\[ dT = \frac{1}{F} (d\eta + \beta dR) \] (12)
where $\beta(\eta, R)$ is a function to be determined and $F(\eta, R)$ is an integrating factor satisfying the equation
\[
\frac{\partial}{\partial R} \left( \frac{1}{F} \right) = \frac{\partial}{\partial \eta} \left( \frac{\beta}{F} \right)
\] (13)
to guarantee that $dT$ is an exact differential. The use of $d\eta = FdT - \beta dR$ in the line element (10) gives
\[
ds^2 = \left[ -a^2 (1 + 2\phi) + \frac{(1 - 2\phi)^2 \mathcal{H}^2 R^2}{(1 - 2\phi - r\phi')^2} \right] F^2 dT^2 \\
+ \left\{ \left[ -a^2 (1 + 2\phi) + \frac{(1 - 2\phi)^2 \mathcal{H}^2 R^2}{(1 - 2\phi - r\phi')^2} \right] \beta \right. \\
\left. + \frac{(1 - 2\phi)^2}{(1 - 2\phi - r\phi')^2} + \frac{2\mathcal{H}^2 R (1 - 2\phi)^2 \beta}{(1 - 2\phi - r\phi')^2} \right\} dR^2 \\
-2F \left\{ \beta \left[ -a^2 (1 + 2\phi) + \frac{(1 - 2\phi)^2 \mathcal{H}^2 R^2}{(1 - 2\phi - r\phi')^2} \right] \\
+ \frac{\mathcal{H} R (1 - 2\phi)^2}{(1 - 2\phi - r\phi')^2} \right\} dT dR + R^2 d\Omega^2_{(2)}, \tag{14}
\]
By setting
\[
\beta(\eta, R) = \frac{\mathcal{H} R (1 - 2\phi)^2}{(1 - 2\phi - r\phi')^2} A^2 \\
= \frac{\mathcal{H} R (1 - 2\phi)^2}{a^2 (1 + 2\phi) (1 - 2\phi - r\phi')^2 - \mathcal{H}^2 R^2 (1 - 2\phi)^2}, \tag{15}
\]
where
\[
A^2(\eta, R) = a^2 (1 + 2\phi) - \frac{(1 - 2\phi)^2 \mathcal{H}^2 R^2}{(1 - 2\phi - r\phi')^2}, \tag{16}
\]
the $dT dR$ cross-term is eliminated and we are left with the diagonal line element
\[
ds^2 = -A^2 F^2 dT^2 + B^2 dR^2 + R^2 d\Omega^2_{(2)}, \tag{17}
\]
where
\[
B^2 = \frac{\mathcal{H}^2 R^4 (1 - 2\phi)^4}{(1 - 2\phi - r\phi')^4 A^2} + \frac{(1 - 2\phi)^2}{(1 - 2\phi - r\phi')^2}. \tag{18}
\]
Algebraic manipulations bring this metric coefficient to the form
\[
B^2 = \frac{a^2 (1 - 2\phi)^2 (1 + 2\phi)}{a^2 (1 + 2\phi) (1 - 2\phi - r\phi')^2 - (1 - 2\phi)^2 \mathcal{H}^2 R^2}, \tag{19}
\]
and, to first order, we have
\[
f^2 = \frac{1}{B^2} \approx 1 - 2r\phi' - \mathcal{H}^2 R^2 (1 - 2\phi), \tag{20}
\]
where we used the relation between comoving and conformal Hubble parameters $H = \mathcal{H}/a$. To first order, in spherical symmetry, the perturbation potential solves the lowest order field equations, which reduce to the usual Poisson equation and give $\Box \phi = -m/r$, where $m$ is the (constant) Newtonian mass of the spherical perturbation. Using this fact and expanding to first order, one obtains the Brown-York quasilocal mass
\[
M_{BY} = R (1 - f) \approx ma + \frac{H^2 R^3}{2} (1 - 2\phi). \tag{21}
\]
Similarly, the Hawking-Hayward/Misner-Sharp-Hernandez mass is
\[
M_{MSH} = R \left( 1 - f^2 \right) \approx ma + \frac{H^2 R^3}{2} (1 - 2\phi). \tag{22}
\]
We can now take the limit $m \to 0$ to an unperturbed FLRW universe. In this limit, one obtains
\[
M_{BY}^{(0)} = M_{MSH}^{(0)} = \frac{H^2 R^3}{2} = \frac{4\pi G}{3} \rho R^3 \tag{23}
\]
using the Friedmann equation (2). As promised, the two quasilocal masses coincide in this limit. This is not the case if other gauges are used because the Brown-York mass is, by definition, gauge-dependent. Therefore, the equality of $M_{BY}^{(0)}$ and $M_{MSH}^{(0)}$ would not hold in other gauges and the comparison of the zero and first order masses would be meaningless.

With this caveat in mind, we have obtained the result that, to first order and in the gauge chosen, the Brown-York and the Misner-Sharp-Hernandez masses coincide and split nicely (in the same way) into local and cosmological parts in a perturbed FLRW universe. Therefore, the results previously obtained for Newtonian $N$-body simulations of large scale structures, for the turnaround radius, and for lensing by the cosmological constant, which hinge on this decomposition, will hold exactly in the same way as discussed for the Hawking-Hayward quasilocal mass.

III. CONCLUSIONS

In the context of the three cosmological problems considered, the Brown-York quasilocal mass provides the same results previously obtained with the Hawking-Hayward quasilocal mass. This conclusion follows from the clean splitting of the Brown-York mass into a cosmological contribution $H^2 R^3/2$ (where $H$ is the Hubble parameter and $R$ is the size of the system) and a local contribution $ma$ (where $m$ is the Newtonian mass of the local perturbation and $a(t)$ is the scale factor of the FLRW background), to first order. A third contribution $-H^2 R^3 \phi$ (where $\phi$ is the Newtonian potential of the local perturbation) is completely negligible in comparison with the first two. This splitting is analogous to that.
found for the Hawking-Hayward or the Misner-Sharp-Hernandez mass. However, the Brown-York mass is a gauge-dependent quantity even in spherical symmetry, as is clear from its definition involving the extrinsic curvature \( \Omega \) and the 3 + 1 splitting of spacetime into space and time which depends on the observer already in special relativity. By contrast, the Misner-Sharp-Hernandez mass is a true scalar, gauge-independent, quantity, which is shown by its definition (2).

The gauge dependence here means observer-dependence. The Brown-York energy depends on the observer in the sense that it depends on the particular spacetime foliation one chooses. As an example, it can easily be verified that this energy vanishes for the de Sitter metric in the comoving \((t, r)\)-coordinates, such that

\[
ds^2 = -dt^2 + a(t)^2 \left( dr^2 + r^2 d\Omega^2_{(2)} \right),
\]

where the curvature, due to the cosmological constant \( \Lambda \), cannot be detected by those coordinates. However the energy is non-zero, for example, in Schwarzschild-like coordinates such that

\[
ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2_{(2)},
\]

because the curvature due to the cosmological constant becomes then encoded in the spatial component of the metric. Thus, this energy allows one to choose from which perspective (foliation) one wishes to measure the gravitational energy of spacetime. The Brown-York energy corresponds to the energy of spacetime as seen by a specific observer, whereas the Misner-Sharp-Hernandez energy corresponds to the energy encoded in the full metric of that spacetime. The choice of the coordinates would then be dictated by the simple goal to recover the contribution to the energy coming from the cosmological constant. This is actually the motivation that led us to choose the gauge \( \xi \) for extracting such an energy for an expanding universe.

The difference between the Brown-York and the Misner-Sharp-Hernandez masses is analogous to the difference between the relativistic invariant \( m^2 = -p_\mu p^\mu = E^2 - p^2 \) and the four-vector \( p^\mu = (E, p^i) \). The components of the latter depend on the observer. However, unlike this relativistic example where the \( p^0 \) component of the four-vector coincides with the invariant \( m \) in the reference frame in which \( p^i = 0 \), the Brown-York mass cannot coincide with the Misner-Sharp-Hernandez mass within the same coordinate system, except in Minkowski spacetime in which both vanish. They can only coincide at the first order, as we found above. Therefore, the higher orders are, in this sense, very important for a more complete and a deeper comparison between the two quasilocal concepts. This fact is actually what would make one able to argue for one definition or the other and be able to decide which one is the most adequate for a given problem. However, for the three cosmological problems considered, the second order contributions are extremely small and unobservable.

As for our comparison between the Brown-York and the Hawking-Hayward formulations, we have based our conclusion on the fact that, for spherical symmetry, Hawking-Hayward coincides at the first order with Misner-Sharp-Hernandez which, in turn, coincides with Brown-York for the specific spacetime foliation chosen for the latter. It remains true, however, that just as for Brown-York, the quasilocal Hawking-Hayward concept also depends on the 2-surface chosen, on which the various geometrical quantities, such as the induced Ricci scalar, the expansion tensor, the shear tensor, and the twist vector, are defined. The latter all appear as contracted scalars, and hence as invariants, inside the defining integral of the energy, but remain nonetheless surface-related quantities that depend on the particular 2-surface selected. Our result shows the coincidence of both concepts only for the particular foliation \( \xi \). As such foliation is sufficient for the Brown-York energy to detect the cosmological contribution, we have not investigated here whether the two concepts would still be in agreement for other foliations.

As a consequence, the comparison of the Brown-York and the Misner-Sharp-Hernandez masses and of their effects for a certain spacetime (in our case, the perturbed post-Friedmannian space \( \mathbb{R}^4 \)) is, in general, meaningless. It acquires some physical meaning only when a gauge is found in which the Brown-York mass of the unperturbed FLRW universe reduces to the Misner-Sharp-Hernandez mass of the same geometry. This gauge is identified as one in which the metric is explicitly spherically symmetric, diagonal, and uses the areal radius \( R \) as the radial coordinate. (This constraint restricting to spherical symmetric and diagonal metrics is only due to the fact that, unlike the Brown-York and the Hawking-Hayward masses, the Misner-Sharp-Hernandez mass is defined only for spherically symmetric metrics.) This gauge is the one used in black hole thermodynamics \( \mathbb{R}^4 \) and was also used in Refs. \( \mathbb{R}^4 \) attempting to provide physical interpretations of both quasilocal masses. The gauge-dependence of the Brown-York mass should be kept in mind at all times in our claim that it reproduces the results obtained with the Misner-Sharp-Hernandez mass in \( \mathbb{R}^4 \). Indeed, this is true only in the particular gauge adopted and the comparison would be meaningless if no gauge existed in which the two results can actually be compared.

Also, without knowledge of the previous results obtained with the Hawking-Hayward mass \( \mathbb{R}^4 \), a parallel calculation using the Brown-York mass would be meaningless as it would only give again results which are completely dependent on the gauge choice.

That said, let us look at the interpretation of our result. The local contribution \( ma \) to the Brown-York mass may look puzzling at first sight, but it can be understood by keeping in mind that, in the geometrized units used, a mass is also a length (in the same way that \( 2m \) is the Schwarzschild radius in the Schwarzschild solution of GR) and the length scale \( m \) determined by the local perturbation becomes the (physical) comoving scale \( ma \).
when embedded in a FLRW background.

The decomposition of the Brown–York quasilocal mass obtained here in a suitable gauge coincides, to first order, with the decomposition of the Hawking–Hayward quasilocal mass previously obtained in Refs. \[13, 34, 39\]. The solution of the three problems studied in these references (namely Newtonian large-scale structure formation in the early universe, the turnaround radius in the present accelerated universe dominated by dark energy, and the direct contribution of a cosmological constant or dark energy to light deflection by a gravitational lens) hinges on the splitting of the quasilocal mass into local and cosmological contributions. Therefore, obtaining exactly the same splitting (to first order) for both notions of quasilocal mass is sufficient to state that the results obtained for these problems coincide using both quasilocal energy notions. On the positive side, this fact testifies of the usefulness and power of the quasilocal mass in GR and encourages further exploration of its uses in cosmology and in astrophysics. On the negative side, it is not possible, at the level of current and foreseeable experiments, to discriminate between the Hawking–Hayward and the Brown–York quasilocal energies by applying them to concrete problems in gravity, as it was hoped for, with the extra complication that the comparison of the results is meaningful only in a certain gauge due to the gauge-dependence of the Brown–York mass.

Here we have restricted ourselves to spherical symmetry. Indeed, the problems of the turnaround radius and of lensing by the cosmological constant have, thus far, been studied only in spherical symmetry \[13, 34, 39\]. The restriction to spherical symmetry is certainly inappropriate to describe large-scale structure formation in the early universe. However, the physical argument provided in Ref. \[13\] as to why Newtonian simulations are ultimately correct, obtained without assuming spherical symmetry, was essentially the same that was previously derived in the same reference using a much simpler spherical toy model. The structure of the non-spherical calculation and the underlying physics are naturally expected to be the same when a general (i.e., non-spherical) analysis is performed using the Brown–York instead of the Hawking–Hayward quasilocal energy. A detailed non-spherical analysis, which will be reported elsewhere, is, however, still lacking in this sense before one could achieve completeness within such an investigation.

As a final remark, we would like to mention here that, although our investigation has been carried out within the perturbed flat FLRW spacetime, taking a more general perturbed FLRW metric would not change our conclusions about the way the various mass definitions split into cosmological and local contributions. The only difference is that they would not agree on the local part anymore.

In fact, a spatially curved FLRW metric would only display the extra factor \((1 - kr^2)^{-1}\) in front of the \(dr^2\) term in \(\text{(1)}\), where \(k\) is the usual spatial curvature parameter that takes on the possible values \(-1, 0,\) or \(+1\). The only effect of this extra factor, however, is the modification of \(\text{(1)}\) into \(f^2 \simeq (1 - 2r\phi')(1 - kr^2) - H^2 R^2 (1 - 2\phi)\). This, in turn, will transform \(\text{(21)}\) and \(\text{(22)}\) into, respectively,

\[
M_{\text{BY}} \simeq R + (ma - R)\left(1 - \frac{kR^2}{a^2} + \frac{H^2 R^3}{2} (1 - 2\phi)\right), \quad (25)
\]

and

\[
M_{\text{MSH}} \simeq \frac{kR^3}{2a^2} + ma \left(1 - \frac{kR^2}{a^2}\right) + \frac{H^2 R^3}{2} (1 - 2\phi). \quad (26)
\]

This clearly shows that the two results do indeed agree on the non-local cosmological contribution to the total mass but differ at the level of the local contributions. Furthermore, this difference is not due to the gauge-dependence of the Brown–York mass, as no specific gauge would make the two local contributions coincide.

This local contribution difference can actually easily be understood as stemming from the fact that the geometric definitions of the Misner-Sharp-Hernandez and the Brown-York masses are different. The former is an observer-independent measure of the “geometric equivalent” of the total mass enclosed within an areal radius \(R\) \[50\] whereas the latter is an observer-dependent measure of the \textit{extrinsic curvature} caused by the total mass within a specific spacetime. As spatially flat, closed, and open universes necessarily possess different spatial extrinsic curvatures, the two masses can only agree in the flat space case.

On the other hand, the agreement the two masses display at the level of the cosmological contribution to the total mass stems from the fact that the overall Hubble expansion of space does not distinguish between the flat, closed, and open universes. In fact, although the Hubble parameter \(H\) itself differs from one spacetime to another, depending on the spatial curvature parameter \(k\) via the Friedmann equation, the expansion of space is everywhere the same for a given spatial curvature.

Note that here we have only based our analysis on the Brown-York and the Misner-Sharp-Hernandez quasilocal masses. The conclusion drawn about the agreement on the cosmological contribution and the disagreement on the local part between the two masses, however, remains valid also for the case of the Hawking-Hayward quasi-local mass. This might easily be seen from Eqs. (32) and (38) of Ref. \[13\], in which the relation between the Misner-Sharp-Hernandez and the Hawking-Hayward masses clearly indicates that the two masses will agree again on the cosmological contribution due to spherical symmetry.

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