ON SOME FANO–ENRIQUES THREEFOLDS

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ABSTRACT. We give a classification of Fano threefolds $X$ with canonical Gorenstein singularities such that $X$ possess a regular involution, which acts freely on some smooth surface in $|-K_X|$, and the linear system $|-K_X|$ gives a morphism which is not an embedding. From this classification one gets, in particular, a description of some natural class of Fano–Enriques threefolds.

1. INTRODUCTION

In this article we use the following

Definition 1.1. Three-dimensional normal projective variety $W$ with canonical singularities is called a Fano–Enriques threefold if the canonical divisor $K_W$ is not Cartier, but $-K_W \sim QH$ for some ample Cartier divisor $H$. The number $g := \frac{1}{2}H^3 + 1$ is called genus of $W$.

In [7] G. Fano studied three-dimensional normal projective varieties $W$ with general hyperplane section $H$ which is a smooth Enriques surface (see also [8]). Such varieties are always singular (see [5]). Moreover, according to [2], if singularities of $W$ are worse than canonical, then $W$ is a cone over $H$. Hence, from the viewpoint of classification, the case when $W$ has canonical singularities is of the main interest. If this holds, then by [2] $W$ is a Fano–Enriques threefold with isolated singularities such that $-K_W \sim QH$. In [7] G. Fano was able to obtain only partial description of such varieties (see also [4], [5]).

A new approach became possible due to the Minimal Model Program (see [3]). First of all, according to Propositions 3.1 and 3.3 in [15], general element $H_0 \in |H|$ on a Fano–Enriques threefold $W$ has only Du Val singularities and the minimal resolution of $H_0$ is a smooth Enriques surface. From this one can deduce that $2(K_W + H_0) \sim 0$ on $W$ (see [2]). Further, take a global log canonical cover $\pi : X \to W$ with respect to $K_W + H_0$ (see, for example, [11]). Here morphism $\pi$ has degree 2 and $\pi^*(K_W + H_0) \sim 0$. Moreover, $\pi$ is ramified exactly at those points on $W$ where $K_W$ is not Cartier. Since $W$ has canonical singularities, the number of such points is finite. In particular, we obtain that $-K_X \sim \pi^*(H_0)$ and $X$ is a Fano threefold with canonical Gorenstein singularities and degree $-K_X^3 = 4g - 4$. Furthermore, Galois involution of the double cover $\pi$ induces an automorphism $\tau$ on $X$ of order 2 such that $\tau$ acts freely in codimension 2 and $W = X/\tau$.

The above construction has lead to the complete description of Fano–Enriques threefolds with terminal cyclic quotient singularities (see [1], [19]). Now let $W$ be a Fano–Enriques threefold with isolated singularities. According to [15, Corollary 3.7], if $H^3 \neq 2$, then general element $H_0 \in |H|$ is a smooth Enriques surface. In this case on the corresponding Fano threefold $X$ there is a $\tau$-invariant smooth K3 surface $\pi^*(H_0) \in |-K_X|$ with a free action of $\tau$.

The main result of the present paper is the following

Theorem 1.2. Let $X$ be a Fano threefold with canonical Gorenstein singularities and $S \in |-K_X|$ be a smooth K3 surface. Suppose that there is an action of regular involution $\tau$ on $X$ such that $\tau(S) = S$ and $\tau$ does not have fixed points on $S$. Then

- the factor $X/\tau$ is a Fano–Enriques threefold with isolated singularities;
- the linear system $|-K_X|$ does not have base points;

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• if the morphism $\varphi_{|K_X}^{[1]}$ is not an embedding, then one has the following possibilities:
  A) $X$ is the intersection of a quartic and a quadric in $\mathbb{P}(1,1,1,1,2)$, $-K_X^3 = 4$;
  B) $X$ is the image of threefold $V$, which is a double cover of the scroll $\mathbb{F}(d_1,d_2,d_3) := \text{Proj} \left( \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}(d_i) \right)$ with ramification at some divisor in the linear system $|4M - 2(2 - \sum_{i=1}^{3} d_i)L|$, where $M$ is the class of tautological divisor on $\mathbb{F}(d_1,d_2,d_3)$ and $L$ is the class of a fibre of the natural projection $\mathbb{F}(d_1,d_2,d_3) \to \mathbb{P}^1$, under birational morphism, given by multiple anticanonical linear system $|-rK_V|$, $r \gg 0$. Furthermore, for $(d_1,d_2,d_3,-K_X^{(2)})$ only the following values are possible:
  
  $(2,1,1,8), (2,2,2,12), (2,2,0,8), (3,1,0,8), (3,3,0,12), (4,2,0,12), (4,4,0,16), (5,3,0,16), (6,4,0,20), (7,5,0,24), (8,6,0,28),$

  and each of the cases in [A] and in [B] does occur.

From Theorem 1.2 and the above arguments we obtain

**Corollary 1.3.** Let $W$ be a three-dimensional normal projective variety with general hyperplane section which is a smooth Enriques surface. If $W$ has canonical singularities, then it is a factor of some Fano threefold $X$ with canonical Gorenstein singularities by the action of regular involution on $X$, which acts freely on some smooth surface in $|-K_X|$, so that one of the following holds:

• $X$ is one of the threefolds from Theorem 1.2
• the linear system $|-K_X|$ gives an embedding.

**Remark 1.4.** In that case when Fano–Enriques threefold $W$ has terminal singularities there exists a flat deformation of $W$ to Fano–Enriques threefold with terminal cyclic quotient singularities (see [14]). For Fano threefolds $X$ in case [B] of Theorem 1.2 which correspond to $\mathbb{F}(d_1,d_2,0)$, the same result for $W = X/\tau$ is not known. For such $W$ it is not known also if the linear system $|H|$ is very ample.

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## 2. Preliminaries

We use standard notions and facts from the theory of minimal models and Fano varieties (see [10], [3], [13]). All varieties are assumed to be projective and defined over $\mathbb{C}$. In what follows $X$ is a threefold from Theorem 1.2.

**Lemma 2.1.** Factor $X/\tau$ is a Fano–Enriques threefold with isolated singularities.

**Proof.** Set $W := X/\tau$ and $\pi : X \to W$ to be the factorization morphism. Since $\tau$ acts freely on $S \in |-K_X|$, $H := \pi(S)$ is a smooth Enriques surface and an ample divisor on $W$. In particular, singularities of $W$ are isolated. From this we get $-2K_W \sim 2H$ (see [2] Remark 2.8). Thus, it remains to show that $W$ has canonical singularities.

By the above arguments $W$ is $\mathbb{Q}$-Gorenstein. Then, according to [3], Proposition 6.7, $W$ has log terminal singularities. Suppose that singularities of $W$ are worse than canonical. Then, according to [2], contraction of the negative section $E$ on the $\mathbb{P}^1$-fibration $\mathbb{P} := \text{Proj} (\mathcal{O}_H \oplus \mathcal{O}_H(H|_E))$ gives a birational morphism $g : \mathbb{P} \to W$ such that $K_{\mathbb{P}} = g^*(K_W) - E$. This implies that the discrepancy $a(E,W)$ equals $-1$, which is a contradiction.

**Lemma 2.2.** The degree $-K_X^3$ is divisible by 4.

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$^{[1]}$ for a linear system $\mathcal{L}$ we denote by $\varphi_{\mathcal{L}}$ corresponding rational map.
Proof. In the notation from the proof of Lemma 2.1 for the ample Cartier divisor $H = \pi(S)$ on a Fano–Enriques threefold $W$ we have $-K_W \sim \mathbb{Q}H$. In particular, we get: $-K_X^3 = \pi^*(H)^3 = 2H^3$. On the other hand, according to [13, Lemma 2.2], $H^3$ is divisible by 2. Thus, $-K_X^3$ is divisible by 4.

\begin{lemma}
The linear system $| -K_X|$ does not have base points.
\end{lemma}

Proof. Suppose that $B := Bs| -K_X| \neq \emptyset$. If $\dim B = 0$, then, according to [20], $B$ is a point. We have $B = \tau(B)$ and $B \in S$. On the other hand, $\tau$ acts freely on $S$, a contradiction.

Suppose now that $\dim B = 1$. Then, according to [20], we have $B \cong \mathbb{P}^1$. Thus, since $\tau(B) = B$, there are at least two $\tau$-fixed points on $B$. On the other hand, $B \subset S$ and $\tau$ acts freely on $S$, a contradiction.

Let us consider the anticanonical morphism $\varphi_{| -K_X|} : X \to Y$ and assume that it is not an isomorphism. Then $\varphi_{| -K_X|}$ is a double cover of the threefold $Y := \varphi_{| -K_X|}(X) \subset \mathbb{P}^n$, where $n = -\frac{1}{2}K_X^3 + 2$ (see [10]). Let us denote by $D \subset Y$ the ramification divisor of $\varphi_{| -K_X|}$.

\begin{lemma}
Suppose that $-K_X^3 = 4$. Then $X$ is the intersection of a quartic and a quadric in $\mathbb{P}(1, 1, 1, 1, 2)$.
\end{lemma}

Proof. This follows from [16, Remark 3.2].

\begin{remark}
From the proof of Theorem 1.1 in [19] it follows that there exists a smooth Fano threefold $X$, which is the intersection of a quartic and a quadric in $\mathbb{P}(1, 1, 1, 1, 2)$, with an action of regular involution which acts freely on some smooth surface in $| -K_X|$.
\end{remark}

\begin{lemma}
Threefold $Y$ is not the cone over Veronese surface.
\end{lemma}

Proof. Suppose that $Y$ is the cone over Veronese surface. Then the threefold $X$ is isomorphic to a hypersurface of degree 6 in the weighted projective space $\mathbb{P} := \mathbb{P}(1, 1, 1, 2, 3)$ (see [16, Lemma 3.3]). Since $\text{Pic}(X) \cong \text{Pic}(\mathbb{P}) = \mathbb{Z}$ (see [16], [10, Proposition 1.2.1]) implies that for every $m \in \mathbb{N}$ automorphism $\tau$ naturally lifts to involution acting on the linear system $|O_{\mathbb{P}}(m)|$. This determines the lifting of $\tau$ to involution on $\mathbb{P}$ which we again denote by $\tau$.

Choose homogeneous coordinates $x_0, x_1, x_2, x_3, x_4$ on $\mathbb{P}$, where $\deg x_0 = \deg x_1 = \deg x_2 = 1$, $\deg x_3 = 2$, $\deg x_4 = 3$, such that $x_i$ is an eigen function of $\tau$ with an eigen value $\pm 1$. After multiplication by $-1$ and renumbering one can set $x_0$ and $x_1$ to be $\tau$-invariant. Then the action of $\tau$ on $\mathbb{P}$ is

\begin{equation}
[x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [x_0 : x_1 : -x_2 : -x_3 : -x_4].
\end{equation}

Indeed, in any other expression the locus of $\tau$-fixed points on $\mathbb{P}$ contains a surface. But $\text{Pic}(\mathbb{P}) = \mathbb{Z}$ and $X$ is a Cartier divisor. Hence the locus of $\tau$-fixed points on $X$ must contain a curve which is impossible because $\tau$ acts freely on $S \in | -K_X|$.

Further, according to (2.7), the locus of $\tau$-fixed points on $\mathbb{P}$ consists of the curves $C_1 = (x_2 = x_3 = x_4 = 0), C_2 = (x_0 = x_1 = x_3 = 0)$ and the point $O = [0 : 0 : 0 : 1 : 0]$. Since $\tau$ acts freely on $S \in | -K_X|$, we have $C_1, C_2 \not\subset X$. This and (2.7) imply, since $\tau(X) = X$ and $X \in |O_{\mathbb{P}}(6)|$, that the equation of $X$ is

\begin{equation}
F_6(x_0 : x_1) + \alpha_1 x_0^6 + x_1^4 F_2(x_0 : x_1) + \alpha_2 x_0^2 x_1 + x_1^3 x_3 F_1(x_0 : x_1) +
+ x_2^2 F_4(x_0 : x_1) + x_2 x_3 F_3(x_0 : x_1) + x_2 x_4 F_2(x_0 : x_1) +
+ x_3^2 H_2(x_0 : x_1) + x_3 x_4 G_1(x_0 : x_1) + \alpha_3 x_3^4 = 0,
\end{equation}

where $\alpha_i \in \mathbb{C}, F_i, G_i$ are homogeneous polynomials in $x_0, x_1$ of degree $i$. 

\footnote{for a linear system $\mathcal{L}$ we denote by $Bs(\mathcal{L})$ its base locus.}
On the other hand, for the $\tau$-invariant surface $S \in |-K_X|$ we have $S \cap (C_1 \cup C_2 \cup \{O\}) = \emptyset$ by assumption. This and (2.7) imply, since $\text{Pic}(X) \cong \text{Pic}(\mathbb{P}) = \mathbb{Z}$ and $-K_X \sim \mathcal{O}_X(2)$, that the equation of $S$ on $X$ is one of the following:

\[(2.9) \quad \alpha x_3 + x_2 H_1(x_0 : x_1) = 0\]

or

\[(2.10) \quad \beta x_2^2 + H_2(x_0 : x_1) = 0,\]

where $\alpha, \beta \in \mathbb{C}$, $H_i$ are homogeneous polynomials in $x_0, x_1$ of degree $i$. But in case (2.9) one gets $S \cap C_1 \neq \emptyset$ and in case (2.10) we have $S \ni O$. Thus, in both cases $S$ contains a $\tau$-fixed point. The obtained contradiction proves Lemma 2.6.

**Remark 2.11.** From Lemmas 2.2, 2.4, Remark 2.5 and Lemma 2.6 we deduce that to prove Theorem 1.1 it remains to consider the case when $-K_X^3 \geq 8$ and the threefold $Y$ is not the cone over Veronese surface. In what follows we assume these conditions to be satisfied for $X$.

Since the degree of $Y \subset \mathbb{P}^n$ equals $n - 2$, by Remark 2.11 and by Enriques Theorem (see [9, Theorem 3.11]) there is a birational morphism $\varphi_{|M|} : \mathbb{F}(d_1, d_2, d_3) \rightarrow Y$. Here $\mathbb{F}(d_1, d_2, d_3) := \text{Proj} \left( \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^i}(d_i) \right)$ is a rational scroll, $M$ is the class of tautological divisor on $\mathbb{F}(d_1, d_2, d_3)$, $d_1 \geq d_2 \geq d_3 \geq 0$. Let us also denote by $L$ the class of a fibre of the natural projection $\mathbb{F}(d_1, d_2, d_3) \rightarrow \mathbb{P}^1$.

**Lemma 2.12.** The equality $-K_X^3 = 2(d_1 + d_2 + d_3)$ takes place.

**Proof.** We have $-\frac{1}{2}K_X^3 = \text{deg}(Y) = M^3$. On the other hand, $M^3 = d_1 + d_2 + d_3$ by [18, A.4], which implies equality we need.

**Lemma 2.13.** If $d_3 \neq 0$, then $\varphi_{|M|}$ is an isomorphism and $D \in |4M - 2(\sum_{i=1}^{3} d_i - 2)L|$. Moreover, $(d_1, d_2, d_3) = (2, 1, 1)$ or $(2, 2, 2)$.

**Proof.** The fact that $\varphi_{|M|}$ is an isomorphism for $d_3 \neq 0$ follows from [18, Theorem 2.5]. Thus, we have $-K_X \sim \varphi^*_{-K_X}(M)$ and $K_Y \sim -3M + (d_1 + d_2 + d_3 - 2)L$ (see [18, A. 13]). This together with the Hurwitz formula gives $D \in |4M - 2(\sum_{i=1}^{3} d_i - 2)L|$. Finally, since $S \in |-K_X|$ is a smooth surface, the threefold $X$ has isolated singularities. According to Table 1 in the proof of Theorem 1.5 in [16] and Lemmas 2.2, 2.12, this is possible only for $(d_1, d_2, d_3) = (2, 1, 1)$ and $(2, 2, 2)$.

**Remark 2.14.** Let $X$ be a double cover of $\mathbb{F}(d_1, d_2, d_3)$, where $(d_1, d_2, d_3) = (2, 1, 1)$ or $(2, 2, 2)$, with ramification at general divisor in $D := |4M - 2(\sum_{i=1}^{3} d_i - 2)L|$. It is easy to write down the basis of the linear system $D$ (see [18, 2.4] or [3.1] below) and obtain that $D$ does not have base points. This together with the Hurwitz formula implies that $X$ is a smooth Fano threefold, of degree 8 or 12. Moreover, according to [16, Remark 1.8], $X$ belongs to the list from Theorem 1.1 in [19]. Hence there is an action of regular involution $\tau$ on $X$ such that the factor $W := X/\tau$ is a Fano–Enriques threefold with isolated singularities. Since the genus of $W$ equals 3 or 4, it follows from [15, Corollary 3.7] that $\tau$ acts freely on some smooth K3 surface in $|-K_X|$ (see arguments in Introduction). Moreover, by construction the linear system $|-K_X|$ gives a morphism of degree 2.

It follows from Lemma 2.13 and Remark 2.14 that to prove Theorem 1.2 it remains to consider the case when $d_3 = 0$. In what follows we assume this condition to be satisfied for $X$. Set $\mathbb{F} := \mathbb{F}(d_1, d_2, 0)$.

**Lemma 2.15.** In the above notation, morphism $\varphi_{|M|} : \mathbb{F} \rightarrow Y$ is a small birational contraction and $\varphi_{|M|}(D) \in |4M - 2(d_1 + d_2 - 2)L|$. The exceptional locus of $\varphi_{|M|}$ is an irreducible rational curve and the threefold $Y$ is a cone with the unique singularity at the vertex.
Proof. We have $d_2 \neq 0$. Indeed, if $d_2 = 0$, then $Y$ is a cone with a curve of singularities (see the proof of Theorem 2.5 in [18]). The latter implies that the singularities of $X$ are non-isolated, which is impossible because $S \in |-K_X|$ is a smooth surface. Further, as in the proof of Lemma 2.13 we obtain that $\varphi_M^*(D) \in |4M - 2(d_1 + d_2 - 2)L|$. Finally, the fact that the exceptional locus of $\varphi_M^*$ is an irreducible rational curve and $Y$ is a cone with the unique singularity at the vertex follows from the proof of Theorem 2.5 in [18].

Lemma 2.16. In the above notation, let $V$ be the double cover of $\mathbb{F}$ with ramification divisor $\varphi_M^*(D)$. Then $X$ is an image of $V$ under birational morphism, given by the multiple anticanonical linear system $|-rK_V|$, $r \gg 0$.

Proof. This follows from [16] Remark 3.8.

Further, one has the following exact sequence:

$$1 \rightarrow G \rightarrow \text{Aut}(X) \xrightarrow{f} \text{Aut}(Y) \rightarrow 1,$$

where $G$ is the group generated by Galois involution which corresponds to $\varphi_{-K_X}^*$. Set $\sigma := f(\tau)$.

Lemma 2.18. In the above notation, involution $\sigma$ lifts to the regular involution on $\mathbb{F}$.

Proof. We have $K_\mathbb{F} \sim -3M + (d_1 + d_2 - 2)L$ (see [18] A.13). Let $C \simeq \mathbb{P}^1$ be the exceptional locus of $\varphi_M^*$ (see Lemma 2.15). Then, since $C = M_1 \cdot M_2$ for general $M_1 \in |M - d_1L|$ and $M_2 \in |M - d_2L|$ (see the proof of Theorem 2.5 in [18]), we have $K_\mathbb{F} \cdot C = d_1 + d_2 - 2$ (see [18] A.14).

If $d_1 + d_2 - 2 \leq 0$, then by Lemma 2.12 we have $-K_X^3 = 2(d_1 + d_2) \leq 4$. This contradicts the assumption for $-K_X^3$ (see Remark 2.11).

Now let $d_1 + d_2 - 2 > 0$. Then the divisor $K_\mathbb{F}$ is ample over $Y$. Hence $\mathbb{F}$ is a relatively minimal model over $Y$. But every such model, which is birational to $\mathbb{F}$, is either isomorphic to $\mathbb{F}$ or connected with $\mathbb{F}$ by a sequence of flops over $Y$ (see [12] Theorem 4.3). Thus, in the present case all such relatively minimal models over $Y$ are isomorphic to $\mathbb{F}$. In particular, this holds for the $\sigma$-equivariant canonical model of a $\sigma$-equivariant resolution of $Y$ (see [13]).

Let us again denote by $\sigma$ the lifting of involution $\sigma$ on $\mathbb{F}$.

Lemma 2.19. In the above notation, linear system $|aM + bL|$ is $\sigma$-invariant on $\mathbb{F}$ for every $a$, $b \in \mathbb{Z}$.

Proof. It follows from [18] Lemma 2.7] that every divisor $B$ on $\mathbb{F}$ is linearly equivalent to divisor $aM + bL$ for some $a$, $b \in \mathbb{Z}$. If $B$ is numerically effective, then we have $a \geq 0$, since otherwise $B$ has negative intersection with every curve in $L$. Moreover, for such $B$ we have $b \geq 0$, since $M \cdot C = 0$ and $L \cdot C = 1$ in the notation from the proof of Lemma 2.18. Thus, divisors $L$ and $M$ generate the cone of numerically effective divisors on $\mathbb{F}$. Since $\sigma$ preserves this cone, $L^3 = 0$ and $M^3 > 0$ (see [18] A.4), we obtain that the linear systems $|L|$ and $|M|$ are $\sigma$-invariant. This implies the result we need.

Remark 2.20. Since $\sigma^*|L| = |L|$ and $|L|$ is a pencil, there exist at least two $\sigma$-invariant fibres $L_0$, $L_1 \in |L|$ on $\mathbb{F}$.

Set $M_S := \varphi_M^*(\varphi_{-K_X}^*(S))$ for the smooth $\tau$-invariant $K3$ surface $S \in |-K_X|$ on $X$. This is a $\sigma$-invariant divisor in $|M|$. Set also $D' := \varphi_M^*(D)$. This is a $\sigma$-invariant divisor in $|4M - 2(d_1 + d_2 - 2)L|$ (see Lemma 2.15 and (2.17)).

Lemma 2.21. In the above notation, the set $D' \cap M_S$ does not contain $\sigma$-fixed points.

Proof. If $D' \cap M_S$ contains a $\sigma$-fixed point, then the surface $S$ contains a $\tau$-fixed point (see (2.17)), which is impossible by assumption.
We use notation and conventions from Section 2. Threefold $F$ is the factor of $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^3 \setminus \{0\})$ by an action of the group $(\mathbb{C}^*)^2$ (see [18, 2.2]). Let us denote by $[x_0 : x_1 : x_2]$ the projective coordinates on a fibre $L \simeq \mathbb{P}^2$ of the natural projection $F \longrightarrow \mathbb{P}^1$. Let also $[t_0 : t_1]$ be projective coordinates on the base $\mathbb{P}^1$. The functions $t_i$, $x_j$ are restrictions of the coordinate functions on $\mathbb{C}^2 \setminus \{0\}$ and $\mathbb{C}^3 \setminus \{0\}$, respectively. For every $a$, $b \in \mathbb{Z}$ it then follows that linear system $|aM + bL|$ is generated by polynomials of the form

\[ \frac{\partial}{\partial t_0} = x_0, \quad \frac{\partial}{\partial t_1} = x_1, \quad \frac{\partial}{\partial t_2} = x_2, \]

where $i_1 + i_2 + i_3 = a$, $i_j \geq 0$, $g_{i_1,i_2,i_3}(t_0 : t_1)$ is a homogeneous polynomial of degree $b + 2i_1 + 2i_2 \geq 0$ (see [18, 2.4]).

**Lemma 3.2.** General element in the linear system $|4M - 2(d_1 + d_2 - 2)L|$ is irreducible.

**Proof.** Let general element in $\mathcal{R} := |4M - 2(d_1 + d_2 - 2)L|$ be reducible. Then, according to Table 1 in the proof of Theorem 1.5 in [16], we have $d_1 > d_2$, and $\mathcal{R}$ is generated by polynomials in $(3.1)$ with $a = 4$, $b = 2(2 - d_1 - d_2)$ and $i_1 > 0$. In particular, divisor $D' = \varphi_{M}(D)$ contains the surface $R \in |M - d_1L|$, given by equation $x_0 = 0$.

Since $d_1 > d_2$, it follows from $(3.1)$ that the linear system $|M - d_1L|$ is generated by $x_0$. Then by Lemma 2.19, we obtain that $R = \sigma(R)$. Let $L_0, L_1 \subset |L|$ be two $\sigma$-invariant fibres on $F$ (see Remark 2.20). Then $R|_{L_0}$ and $M|_{L_0}$ are $\sigma$-invariant lines on $L_0 \simeq \mathbb{P}^2$, $i \in \{0, 1\}$. In particular, the sets $R \cap M \cap L_0$ contain at least one $\sigma$-fixed point each. But $R \cap M \cap L_0 \subset D' \cap M_0$. Thus, we get a contradiction with Lemma 2.21.

According to Table 1 in the proof of Theorem 1.5 in [16] and Lemmas 2.2, 2.12, 3.2, one gets only the following possibilities for $(d_1, d_2)$:

\[ (3.3) \quad (2, 2), (3, 1), (3, 3), (4, 2), (4, 4), (5, 3), (6, 4), (7, 5), (8, 6). \]

This and Lemmas 2.2, 2.10 imply that to prove Theorem 1.2 it remains to show that for every pair $(d_1, d_2)$ in $(3.3)$ there is a Fano threefold $X$ with canonical Gorenstein singularities such that $X$ possess a regular involution, which acts freely on some smooth K3 surface in $| - K_X|$, and the linear system $| - K_X|$ gives a morphism which is not an embedding.

Set $F := F(d_1, d_2, 0)$ for $(d_1, d_2)$ in $(3.3)$. Let us use previous notation for coordinates on the base $\mathbb{P}^1$ and on a fibre $L \simeq \mathbb{P}^2$ of the natural projection $F \longrightarrow \mathbb{P}^1$. We define regular involution $\sigma$ on $F$ by the following relations:

\[ \sigma^*(t_0) = t_0, \quad \sigma^*(t_1) = -t_1 \]

and

\[ \sigma^*(x_0) = -x_0, \quad \sigma^*(x_1) = x_1, \quad \sigma^*(x_2) = -x_2. \]

**Remark 3.6.** Since $t_i$, $x_j$ are restrictions of the coordinate functions on $\mathbb{C}^2 \setminus \{0\}$ and $\mathbb{C}^3 \setminus \{0\}$, respectively, $(3.4)$ and $(3.5)$ commute with the action of the group $(\mathbb{C}^*)^2$, the action of $\sigma$ on $F$ is completely determined by relations $(3.4)$ and $(3.5)$. On the other hand, from Lemma 2.19 it is easy to see that up to the sign change every regular involution on $F$ is determined by relations of the form $(3.4)$ and $(3.5)$.

Let us denote by $C$ the curve on $F$, given by equations $x_0 = x_1 = 0$. We prove the following

**Proposition 3.7.** In the above notation, there are linear systems $D \subseteq |4M - 2(d_1 + d_2 - 2)L|$, $M \subseteq |M|$, where $M$ is the class of tautological divisor on $F$, such that

- $\dim D, \dim M > 0$;
- $D$ consists of $\sigma$-invariant divisors, $Bs(D) = C$ and $\text{mult}_C(D) \leq 3$;
- $M$ consists of $\sigma$-invariant divisors and $Bs(M) \cap C = \emptyset$;
- double cover of $F$ with ramification at general divisor in $D$ has canonical singularities;
• for general divisors $D_0 \in \mathcal{D}$, $M_0 \in \mathcal{M}$ and the set of $\sigma$-fixed points $\mathbb{F}^\sigma$ on $\mathbb{F}$ we have $M_0 \cap D_0 \cap \mathbb{F}^\sigma = \emptyset$.

Proof. The conditions $\sigma(D_0) = D_0$, $\text{Bs}(\mathcal{D}) = C$, $\text{mult}_C(\mathcal{D}) \leq 3$ and \cite{3.1}, \cite{3.4}, \cite{3.5} imply that the equation of general divisor $D_0 \in \mathcal{D}$ for $(d_1, d_2)$ in \cite{3.3} must be one of the following:

| $(d_1, d_2)$ | equation of $D_0$ |
|--------------|--------------------|
| $(2, 2)$     | $\alpha x_0^2 x_2^2 + \beta x_1^2 x_2^2 + \gamma t_0^4 x_0^4 + \gamma' t_1^4 x_0^4 + \delta t_0^4 x_1^4 + \delta' t_1^4 x_1^4 + P_1 = 0$ |
| $(3, 1)$     | $\alpha t_0^2 x_0^2 x_2^2 + \alpha' t_1^2 x_0^2 x_2^2 + \beta x_1^4 + \gamma t_0^8 x_0^4 + \gamma' t_1^8 x_0^4 + P_2 = 0$ |
| $(3, 3)$     | $\alpha t_0^3 x_0^2 x_2 + 2 \beta t_1 x_1^3 x_2 + \gamma t_0^4 x_0^4 + \gamma' t_1^4 x_0^4 + \delta t_0^4 x_1^4 + \delta' t_1^4 x_1^4 + P_3 = 0$ |
| $(4, 2)$     | $\alpha x_0^2 x_2^2 + \beta x_1^4 + \gamma t_0^8 x_0^4 + \gamma' t_1^8 x_0^4 + P_4 = 0$ |
| $(4, 4)$     | $\alpha x_0^4 x_2 + \beta t_0^4 x_0^4 + \beta' t_1^4 x_0^4 + \gamma t_0^4 x_1^4 + \gamma' t_1^4 x_1^4 + P_5 = 0$ |
| $(5, 3)$     | $\alpha t_0^3 x_0^2 x_2 + \beta t_1 x_0^2 x_1 x_2 + \gamma x_1^4 + \delta t_0^8 x_0^4 + \delta' t_1^8 x_0^4 + P_6 = 0$ |
| $(6, 4)$     | $\alpha t_0^3 x_0^2 x_2 + \alpha' t_1^2 x_0^3 x_2 + \beta x_1^4 + \gamma t_0^8 x_0^4 + \gamma' t_1^8 x_0^4 + P_7 = 0$ |
| $(7, 5)$     | $\alpha t_0 x_0^4 x_2 + \beta x_1^4 + \gamma t_0^8 x_0^4 + \gamma' t_1^8 x_0^4 + P_8 = 0$ |
| $(8, 6)$     | $\alpha x_0^3 x_2 + \beta x_1^4 + \gamma t_0^8 x_0^4 + \gamma' t_1^8 x_0^4 + P_9 = 0$ |

Table 1.

Throughout the Table 1 $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' \in \mathbb{C}$, $P_i := P_i(t_0, t_1, x_0, x_1, x_2)$ is a polynomial of degree $\geq 3$ in $x_0, x_1$ such that $\sigma^*(P_i) = P_i$ and $P_i(t_0, t_1, 0, 0, x_2) = 0$ for $1 \leq i \leq 9$.

Lemma 3.8. Double cover of $\mathbb{F}$ with ramification at general divisor in $\mathcal{D}$ has canonical singularities.

Proof. According to \cite{16} Corollary 2.7 and condition $\text{Bs}(\mathcal{D}) = C$, it is enough to show that for every point $p$ on the curve $C$ there is a divisor $D_0 \in \mathcal{D}$ such that the double cover $\varphi : V \to \mathbb{F}$ of $\mathbb{F}$ with ramification at $D_0$ has canonical singularity at the point $o := \varphi^{-1}(p)$.

Put $x_0 = y$, $x_1 = z$, $x_2 = 1$ in equations from Table 1. We obtain:
Table 2.

Throughout the Table 2 $Q_i := P_i(t_0, t_1, x, y, 1)$. It follows that for $(d_1, d_2) \neq (7, 5)$ for every point $p = [t_0 : t_1]$ on the curve $C$ there is a divisor $D_0 \in D$ such that $o = \varphi^{-1}(p) \in V$ is a cDV singularity and hence canonical (see [17]).

For $(d_1, d_2) = (7, 5)$ in the neighborhood of $o$ with local coordinates $x, y, z$ threefold $V$ is given by equation (see Table 1):

\[
(3.9) \quad x^2 + \alpha_0 y^3 + \beta z^3 + \gamma t_0^8 y^4 + \gamma' t_1^8 y^4 + Q_8 = 0,
\]

where $Q_8 := P_8(t_0, t_1, x, y, 1)$. If $p = [t_0 : t_1]$ is a point on the curve $C$ with $t_0 \neq 0$, then one may put $t_0 = 1, t_1 = t$ and find the equation of $V$ in the neighborhood of $o$ with local coordinates $x, y, z, t$:

\[
x^2 + \alpha y^3 + \beta z^3 + \gamma y^4 + \gamma' t^8 y^4 + Q'_8 = 0,
\]

where $Q'_8 := Q_8(1, t, x, y, 1)$. Then [16, Theorem 2.10] implies that for general divisor $D_0$ singularity $o \in V$ is cE_6.

Now let $p = [0 : 1]$. Then in (3.9) one may put $t_0 = t, t_1 = 1$ and find the equation of $V$ in the neighborhood of $o$ with local coordinates $x, y, z, t$:

\[
x^2 + \alpha y^3 + \beta z^3 + \gamma t^8 y^4 + \gamma' y^4 + Q'_8 = 0,
\]

where $Q'_8 := Q_8(t, 1, x, y, 1)$. It is easy to see that the weighted blow up $\tilde{V} \longrightarrow V$ at the point $o$ with weights $(2, 1, 1, 1)$ is crepant (see the proof of Theorem 2.11 in [17]) and the threefold $\tilde{V}$ is smooth. Hence for general divisor $D_0$ singularity $o \in V$ is canonical. Lemma 3.8 is completely proved.

Further, the conditions $\sigma(M_0) = M_0, \text{Bs}(\mathcal{M}) \cap C = \emptyset$ and $(3.1), (3.4), (3.5)$ imply that the equation of general divisor $M_0 \in \mathcal{M}$ for $(d_1, d_2)$ in (3.3) must be one of the following:


| \((d_1, d_2)\) | equation of \(M_0\) |
|-----------------|------------------|
| \((2, 2)\) | \(at_0^2x_0 + bt_1^2x_0 + cx_2 + F_1 = 0\) |
| \((3, 1)\) | \(at_0^3x_0 + bt_1x_1 + cx_2 + F_2 = 0\) |
| \((3, 3)\) | \(at_0^3x_0 + bt_1^3x_1 + cx_2 + F_3 = 0\) |
| \((4, 2)\) | \(at_0^4x_0 + bt_1^4x_0 + cx_2 + F_4 = 0\) |
| \((4, 4)\) | \(at_0^4x_0 + bt_1^4x_0 + cx_2 + F_5 = 0\) |
| \((5, 3)\) | \(at_0^5x_0 + bt_1^3x_1 + cx_2 + F_6 = 0\) |
| \((6, 4)\) | \(at_0^6x_0 + bt_1^6x_0 + cx_2 + F_7 = 0\) |
| \((7, 5)\) | \(at_0^7x_0 + bt_1^5x_1 + cx_2 + F_8 = 0\) |
| \((8, 6)\) | \(at_0^8x_0 + bt_1^8x_0 + cx_2 + F_9 = 0\) |

Table 3.

Throughout the Table 3 \(a, b, c \in \mathbb{C}, F_i := F_i(t_0, t_1, x_0, x_1)\) is a polynomial of degree 1 in \(x_0, x_1\) such that \(\sigma^*(F_i) = -F_i\) and \(F_i(t_0, t_1, 0, 0) = 0\) for \(1 \leq i \leq 9\).

**Lemma 3.10.** For general divisors \(D_0 \in \mathcal{D}, M_0 \in \mathcal{M}\) and the set of \(\sigma\)-fixed points \(\mathbb{F}^{\sigma}\) on \(\mathbb{F}\) we have \(M_0 \cap D_0 \cap \mathbb{F}^{\sigma} = \emptyset\).

**Proof.** From (3.4), (3.5) we obtain the equations for \(\mathbb{F}^{\sigma}\):

\[t_0t_1 = x_1x_0 = x_1x_2 = 0.\]

This implies that \(\mathbb{F}^{\sigma} = l_1 \cup l_2 \cup O_1 \cup O_2\), where \(l_i = (t_i = x_1 = 0)\) and \(l_i \not\subset O_i = (t_i = x_0 = x_2)\) are a curve and a point on the fibre \(L_i = (t_i = 0), i \in \{0, 1\}\), respectively.

It follows from equations in Tables 1 and 3 that \(O_i = \text{Bs}(\mathcal{M}|_{L_i})\), \(O_i \not\subset D_0\) and the set \(D_0 \cap l_i\) is finite, \(i \in \{0, 1\}\). Then, since \(\mathcal{M}|_{L_i}\) is a pencil of lines on \(L_i \simeq \mathbb{P}^2\), we obtain that \(M_0 \cap D_0 \cap \mathbb{F}^{\sigma} = \emptyset\).

From Lemmas 3.8, 3.10 and Tables 1, 3 we obtain the assertion of Proposition 3.7. \(\square\)

Let \(\mathcal{D}, \mathcal{M}\) be the linear systems from Proposition 3.7 and \(D_0 \in \mathcal{D}, M_0 \in \mathcal{M}\) be general divisors. Let us denote by \(\varphi : V \longrightarrow \mathbb{F}\) the double cover of \(\mathbb{F}\) with ramification at \(D_0\). By Proposition 3.7 threefold \(V\) has canonical singularities. Moreover, from the Hurwitz formula we obtain

\[(3.11)\]

\[-K_V \sim \varphi^*(M).\]

Thus, \(V\) is a weak Fano threefold with canonical Gorenstein singularities. Furthermore, by construction, \(V\) possess a regular involution \(\theta\), which acts non trivially on the fibres of \(\varphi\), such that the restriction of \(\theta\) on \(\mathbb{F}\) coincides with \(\sigma\).

Further, [13 Theorem 3.3], (3.11) and Lemma 2.15 imply that the linear system \(|-rK_V|\), \(r \gg 0\), gives a birational morphism \(\psi : V \longrightarrow X\) such that \(\psi\)-exceptional locus is the curve \(\varphi^{-1}(C)\) and \(X\) is a Fano threefold with canonical Gorenstein singularities which possesses a regular involution \(\tau\), the restriction of \(\theta\).

It follows from (3.11) and Proposition 3.7 that \(S := \varphi^*(M_0) \subset |-K_V|\) is a smooth K3 surface with a free action of involution \(\theta\) such that \(S \cap \varphi^{-1}(C) = \emptyset\). This implies that \(\psi(S) \subset |-K_X|\),
is a smooth K3 surface with a free action of involution $\tau$. Finally, according to [16], the linear system $|-K_X|$ gives a morphism which is not an embedding. This completes the construction of Fano threefolds, which satisfy the conditions of Theorem 1.2 for $(d_1, d_2)$ in (3.3).

Theorem 1.2 is completely proved.

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