A Closed Class of Hydrodynamical Solutions for the Collective Excitations of a Bose-Einstein Condensate

Pippa Storey\textsuperscript{1,2} and Maxim Olshanii\textsuperscript{1,2}

\textsuperscript{1}Lyman Laboratory, Harvard University, Cambridge, MA 02138, USA
\textsuperscript{2}Department of Physics and Astronomy, University of Southern California, Los Angeles CA 90089-0484 USA
E-mail: p.storey@auckland.ac.nz, maxim@atomsun.harvard.edu

(April 16, 2002)

A trajectory approach is taken to the hydrodynamical treatment of collective excitations of a Bose-Einstein condensate in a harmonic trap. The excitations induced by linear deformations of the trap are shown to constitute a broad class of solutions that can be fully described by a simple nonlinear matrix equation. An exact closed-form expression is obtained for the solution describing the mode \( n = 0, m = +2 \) in a cylindrically symmetric trap, and the calculated amplitude-dependent frequency shift shows good agreement with the experimental results of the JILA group.

The recent realisation of Bose-Einstein condensation in dilute trapped atomic vapours \cite{[1]–[3]} has motivated an extensive theoretical study of these systems (for a review see reference \cite{[1]}). Many of the nonlinear features of the condensates are manifested in their collective excitations \cite{[4]–[5]}. In this letter we consider the collective excitations of a condensed atomic gas in a harmonic trap in the regime in which the number of atoms is sufficiently large that the hydrodynamical treatment of Stringari is valid \cite{[6]}. We apply this treatment by means of a trajectory approach, which constitutes a generalisation of the scaling transformation of Castin and Dum \cite{[10]}. To first order in the excitation amplitude our results for the oscillatory modes in a stationary trap are identical to those of the perturbative treatment of Ohberg \textit{et al.} \cite{[7]}. However for a broad class of excitations, our analysis produces a simple nonlinear matrix equation, which is valid to all orders in the excitation amplitude. The excitations to which this matrix equation applies are those induced by time-varying linear deformations of the trap (that is, perturbations of the trapping potential that maintain its harmonicity). This class includes all the excitations that have to date been studied experimentally.

We suppose that the condensed gas is confined by a harmonic potential, which, in its most general form, is given by

\[ U(r, t) = \frac{1}{2} \sum_{i,j} K_{ij}(t) [r_i - \bar{r}_i(t)] [r_j - \bar{r}_j(t)], \]

where \( \bar{r}(t) \) is the position of the centre of the trap, and \( K(t) \) is a symmetric matrix with components \( K_{ij}(t) \), which represent the spring ‘constants’ of the trap.

In the mean field approximation, the evolution of the condensate wavefunction is determined by the time-dependent Gross-Pitaevski equation, which, for zero temperature, is

\[ i\hbar \partial_t \Phi(r, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + U(r, t) + Ng |\Phi(r, t)|^2 \right] \Phi(r, t) \]

where \( N \) is the number of atoms in the condensate, and \( g \) is a measure of the strength of the atomic interactions. \( g \) is related to the \( s \)-wave scattering length \( a \) through \( g = 4\pi\hbar^2 a/m \), and is assumed to be positive.

Defining the density of the condensate as \( \rho(r, t) = N |\Phi(r, t)|^2 \), and its velocity field as \( v(r, t) = (\hbar/2im) \nabla \log[\Phi(r, t)/\Phi^* \Phi(r, t)] \), it is easily shown that the Gross-Pitaevski equation (2) is equivalent to the following equations for \( \rho \) and \( v \)

\[ \partial_t \rho + \nabla (\rho v) = 0 \]

\[ m \partial_t v + \nabla \left( U_{\text{eff}} + \frac{1}{2} m \omega^2 \right) = 0, \]

where

\[ U_{\text{eff}} = U + g \rho - \frac{\hbar^2}{2m\sqrt{\rho}} \nabla \sqrt{\rho} \]

is an effective potential.

Following Stringari \cite{[6]} we consider the limit in which the number of atoms \( N \) is sufficiently large that the density \( \rho \) becomes smooth, and it is valid to neglect the kinetic energy pressure term \( \hbar^2 (\nabla \sqrt{\rho})/2m\sqrt{\rho} \). The ratio between the typical force caused by this term and the force \( -\nabla(g\rho) \) caused by inter-particle interactions can be shown to be of the order of \( (\hbar^2/mR_0^2)/\mu \sim N^{-2/3}(T_c/\mu)^2 \), and for the most of the BEC experiments this ratio is as small as \( 10^{-1} - 10^{-3} \). (Here and below \( R_0 \sim \sqrt{\mu/m\omega^2} \) is the Thomas-Fermi radius of the condensate, \( T_c \sim \hbar\omega N^{1/3} \) is the Bose-Einstein condensation temperature, \( \mu \sim g \rho \) is the chemical potential, and \( \omega \) is a typical trapping frequency.) In this regime the condensate can be modeled as a classical gas in which each particle is subject to a force

\[ \mathbf{F}(r, t) = -\nabla_r [U(r, t) + g \rho(r, t)], \]
and for which the velocity field is irrotational. We include the subscript $r$ on the gradient operator $\nabla$ to indicate explicitly that all derivatives are taken with respect to the components of $r$. This notation is used since we shall later introduce a change of coordinates.

The requirement that $\nabla_r \times \mathbf{v}(t) = 0$ imposes a constraint only on the initial condition, the reason being that since the only force involved is $\rho(t)$ the gradient of a potential, the total derivative of $\nabla_r \times \mathbf{v}(t)$ vanishes. This means that, provided the velocity field is initially irrotational everywhere, it will remain so with time.

Using expression (3) for the classical force, we obtain the following evolution equation for $r(t)$

$$m\ddot{r}(t) = -\mathcal{K}(t) [r(t) - \bar{r}(t)] - g \nabla_r \rho(r, t). \quad (7)$$

In order to solve this equation, it is necessary to determine the density of the condensate at position $r$ and time $t$, which in turn requires a knowledge of the trajectories of all the other particles in the condensate. To this end, we associate with each trajectory $r(t)$ a unique reference point $r_0$, corresponding to the position of a particle in the ground state $\Phi_0$ of the condensate in the hydrodynamical limit for a stationary harmonic trap $U_0(r)$, given by equation (1) with $g = 0$ and $\mathcal{K}(t) = \mathcal{K}_0$. $\Phi_0$ is obtained from the solution $\rho_0$ of equation (4) with $F(r, t) = 0$, through

$$\rho_0(r_0) = N |\Phi_0(r_0)|^2, \quad (8)$$

and is given by

$$\Phi_0(r_0) = \sqrt{\frac{\mu - U_0(r_0)}{Ng}}, \quad (9)$$

where $\mu$ is the chemical potential. Note that this is just the Thomas-Fermi solution.

The density of the condensate at time $t$ can now be written as

$$\rho(r, t) = \det \left[ \frac{\partial r}{\partial r_0} \right]^{-1} \rho_0(r_0). \quad (10)$$

where $\partial r/\partial r_0$ is a matrix whose $ij$ th element is $\partial r_i/\partial r_{0,j}$. Given that $\nabla_r = \left[ (\partial r/\partial r_0)^{-1} \right]^T \nabla_{r_0}$, we can rewrite equation (10) as a partial differential equation for $r(r_0, t)$

$$m\ddot{r}_0^r = -\mathcal{K}(t) [r - \bar{r}(t)]
- g \left[ \left( \frac{\partial r}{\partial r_0} \right)^{-1} \right]^T \nabla_{r_0} \left[ \det \left( \frac{\partial r}{\partial r_0} \right)^{-1} \rho_0(r_0) \right]. \quad (11)$$

The functions $r$ satisfying this equation can be used to construct hydrodynamical solutions of the Gross-Pitaevski equation (3). Given that the velocity field is irrotational, we can express it as the gradient of some function $\Theta(r, t)$

$$\mathbf{v}(t) = \nabla_r \Theta(r, t), \quad (12)$$

in terms of which we obtain

$$\Phi(r, t) \approx e^{-i\beta(t)} e^{i m \mathcal{K}(r, t)/\hbar} \frac{\Phi_0(r_0)}{\sqrt{\det (\partial r/\partial r_0)}}. \quad (13)$$

where $r_0$ is uniquely defined in terms of $r$ and $t$, and $\beta(t)$ is some function of time.

By applying this analysis to oscillatory modes in the stationary harmonic potential $U(r, t) = U_0(r)$ (that is, $\mathcal{K}(t) = \mathcal{K}_0$, $\bar{r}(t) = 0$) in the limit of small excitation amplitudes, we recover the results of "Ohberg et al. [1], as we now show. For low amplitude oscillations

$$r(r_0, t) \approx r_0 + \epsilon \left[ \delta r(r_0) e^{-i\omega t} + \delta r^*(r_0) e^{i\omega t} \right], \quad (14)$$

where $\epsilon$ is a perturbation parameter. Substituting this expression into equation (11), and keeping only first order terms in $\epsilon$ we obtain the relation

$$m\omega^2 \delta r = \nabla \{ \delta r \cdot (\mathcal{K}_0 r) - [\mu - U_0(r)] \nabla \cdot \delta r \}, \quad (15)$$

in which we have dropped the distinction between $r$ and $r_0$, since this affects only higher orders. $\delta r$ is clearly expressible as a gradient

$$\delta r = \nabla W, \quad (16)$$

where the function $W$ satisfies the equation

$$m\omega^2 W = \nabla W \cdot (\mathcal{K}_0 r) - [\mu - U_0(r)] \Delta W. \quad (17)$$

Expanding the solution (13) to first order, we obtain

$$\Phi(r, t) \approx e^{-i\mu t/\hbar} \left[ \phi_0(r) + \epsilon \left( u e^{-i\omega t} - v^* e^{i\omega t} \right) \right], \quad (18)$$

where

$$u = \frac{f_+ + f_-}{2}, \quad v = \frac{f_+ - f_-}{2} \quad (19)$$

and

$$f_\pm = C_\pm \left[ 1 - \frac{U_0(r)}{\mu} \right]^{\pm 1/2} W, \quad (20)$$

with $C_+ - C_- = \hbar \omega / 2\mu$. The expression (18), with the differential equation (17) for $W$, is precisely the result obtained by "Ohberg et al. [1]. We have thus shown that their solution describes the elementary phonon-like excitations of the nonlinear hydrodynamical model.

The oscillatory modes for which the function $W(r)$ is of third or higher order in the components of $r$ cannot be excited by linear deformations of the harmonic trap. That is, they cannot be produced by applying a time-dependent harmonic potential of the form (4) to
the ground state condensate $\Phi_0(r)$. However, the modes for which $W(r)$ is linear or quadratic can be produced in this way, and for these modes it is possible to go beyond the perturbative limit, to obtain a simple nonlinear matrix equation that fully describes the dynamics of the condensate in the hydrodynamical regime.

If the condensate is initially in the ground state $\Phi_0(r)$, then the effect of a time-dependent harmonic potential $V_{\text{ext}}(r)$ is simply to distort the condensate in a linear fashion

$$r(r_0, t) = R(t) + M(t)r_0.$$  \hfill{(21)}

Here $R(t)$ is the centre of mass of the condensate, and $M(t)$ describes the time-dependent contraction or shearing of the condensate along various directions. The matrix $\partial r/\partial r_0$ is now independent of position

$$\frac{\partial r}{\partial r_0} = M(t) = \text{constant}(r_0),$$  \hfill{(22)}

and equation \(^{(1)}\) can be separated into a part that is independent of position, and a part that depends linearly on position, giving the following two relations

$$m\ddot{R} = -\mathcal{K}(t)[R - r(t)]$$  \hfill{(23)}

$$m\ddot{M} = -\mathcal{K}(t)M + \text{det}(M)^{-1}(M^{-1})^T \mathcal{K}_0.$$  \hfill{(24)}

From equation \(^{(23)}\) it is clear that for a stationary trap ($\mathcal{K}(t) = \mathcal{K}_0$) the centre of mass motion is separable along the three axial directions, and the frequencies of oscillation along these directions are independent of amplitude, in agreement with the Kohn Theorem (see \cite{12} and references therein).

The velocity can be written in terms of $r$ as

$$v = \dot{R} + \dot{M}M^{-1}(r - R),$$  \hfill{(25)}

and hence the requirement that the velocity field have zero curl is equivalent to the constraint that the matrix $\dot{M}M^{-1}$ be symmetric

$$\left[\dot{M}M^{-1}\right]^T = \dot{M}M^{-1}.$$  \hfill{(26)}

Using equation \(^{(24)}\) it is easy to verify that provided condition \(^{(23)}\) is satisfied at the initial time it will be satisfied at all later times.

Given this constraint it is clear that for a stationary potential in the limit of small excitations, there must be exactly six linearly independent modes for which the function $W(r)$ is quadratic in the components of $r$. For cylindrically symmetric traps ($\omega_x = \omega_y \equiv \omega_\perp$) they are (in the notation of reference \cite{11}) $\{n = 2, m = 0, (\pm)\}$, $\{n = 0, m = \pm 2\}$ and $\{n = 1, m = \pm 1\}$ \cite{13}. In this small amplitude limit we can write

$$\mathcal{M}(t) = 1 + \epsilon \left[\delta \mathcal{M}e^{-i\omega t} + \delta \mathcal{M}e^{i\omega t}\right],$$  \hfill{(27)}

where $\delta \mathcal{M}$ is a constant matrix, independent of both position and time. It completely characterises the modes and their linear combinations in the small-amplitude limit. For the scaling modes described by Castin and Dum \cite{10} (the modes $\{n = 2, m = 0, (\pm)\}$, and a symmetric superposition of the modes $\{n = 0, m = \pm 2\}$) $\delta \mathcal{M}$ is diagonal. For the mode $\{n = 0, m = +2\}$, which we shall consider next, it is

$$\delta \mathcal{M} = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  \hfill{(28)}

An exact solution for this mode can be found by setting $\mathcal{K}(t) = \mathcal{K}_0$ in equation \(^{(22)}\), and is given by

$$\mathcal{M}(t) = \exp[\lambda \mathcal{Q}(t)] \mathcal{R}_\lambda(t),$$  \hfill{(29)}

where

$$\mathcal{Q}(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ \sin(\omega t) & -\cos(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$  \hfill{(30)}

and $\omega$ is the mode frequency. From expression \(^{(30)}\) for $\mathcal{Q}(t)$ it can be seen that the transformation $\exp[\lambda \mathcal{Q}(t)]$ describes the distortion of the condensate into an ellipse, and the rotation of that ellipse about the $z$ axis. The parameter $\lambda$ is a measure of the amplitude of excitation, and is precisely the quantity we labelled $\epsilon$ in the small amplitude limit.

The matrix $\mathcal{R}_\lambda(t)$ describes a slow rotation about the $z$ axis, together with a centrifugal stretching in the $x-y$ plane, and a corresponding contraction along $z$

$$\mathcal{R}_\lambda(t) = \begin{pmatrix} \beta \cos(\omega_\perp t) & -\beta \sin(\omega_\perp t) & 0 \\ \beta \sin(\omega_\perp t) & \beta \cos(\omega_\perp t) & 0 \\ 0 & 0 & \beta^{-2/3} \end{pmatrix}.$$  \hfill{(31)}

This rotation is required in order for $\mathcal{M}(t)$ to satisfy condition \(^{(26)}\). $\mathcal{R}_\lambda(t)$ depends on $\lambda$ through both the parameter $\beta$ and the slow frequency $\omega_\perp$.

The aspect ratio $\alpha$ of the condensate is defined as

$$\alpha \equiv \frac{L_l - L_s}{L_l + L_s},$$  \hfill{(32)}

where $L_l/L_s$ is the ratio of the lengths of the long and short axes of the condensate in the $x-y$ plane. It is related to the parameter $\lambda$ by $\alpha = \tan \lambda$. The constant $\beta$ depends in turn on $\alpha$ through $\beta = [(1 + \alpha^4)/(1 - \alpha^4)]^{3/10}$. The slow frequency $\omega_\perp$ is given by $\omega_\perp = \sqrt{2\alpha^2/\sqrt{1 + \alpha^4}} \omega_\perp$, and the mode frequency $\omega$ scales with the aspect ratio as

$$\omega = \sqrt{2 \frac{1 + \alpha^2}{\sqrt{1 + \alpha^4}}} \omega_\perp.$$  \hfill{(33)}

$\omega$ thus depends on the amplitude of excitation. For very low amplitudes, $\lambda \to 0$, $\alpha \to 0$, and the mode frequency
tends to $\omega \to \sqrt{2}\omega_\perp$. At very high amplitudes, $\lambda \to \infty$, $\alpha \to 1$ and the mode frequency approaches the limit $\omega \to 2\omega_\perp$, which coincides with the frequency of oscillation of a non-interacting gas.

This mode has been observed experimentally by Jin et al. [1], who measured its frequency as a function of response amplitude (aspect ratio) after opening the trap and allowing the condensate to expand. Due to the presence of atomic interactions, the aspect ratio will change during the expansion. In order to compare our results with the experimental data it is therefore necessary to model the evolution of the condensate after the trap has been opened. We assume that the opening of the trap occurs instantaneously. Taking the solution (24) for the trapped condensate as an initial condition, we insert it into (24), setting $\mathcal{K}(t) = 0$, and integrate numerically. The aspect ratio of the condensate is calculated from the elements of $\mathcal{M}$ after the appropriate expansion time.

![Graph](image)

**FIG. 1.** The normalised mode frequency $\omega/\omega(0)$ is plotted against response amplitude (aspect ratio $\alpha$) after expansion. Our theoretical predictions (curve) are compared with the experimental data of Jin et al. [1].

Shown in Figure 1 is the experimental data obtained by Jin et al. [1] for the mode frequency as a function of aspect ratio. These points were obtained with a radial trap frequency $\omega_\perp/2\pi$ of 132 Hz and an expansion time of 7 ms, and the frequencies were normalised to the small amplitude value $\omega(0)$ [1]. The ratio of the axial to radial trap frequencies was $\omega_\perp/\omega_\perp = \sqrt{8}$. The number of condensed atoms was approximately 4500, and temperature was $T = (0.50 \pm 0.08)T_c$, where $T_c$ is the critical temperature. The theoretical curve, calculated for the same parameters, shows good agreement with the experimental data.

Note that in our treatment we totally neglected the presence of the non-condensed particles at a finite temperature $T$. Within the mean-field framework the force caused by the non-condensed particles can be shown to be $(T/T_c)^{3/2}(\mu/T_c)^{3/2}$ times smaller than the one related to the condensate, which makes this assumption valid up to a fraction of the transition temperature.

In conclusion, we have considered the collective excitations of a Bose-Einstein condensate in the hydrodynamical regime. We have shown that the excitations induced by arbitrary time-dependent linear deformations of the trapping potential form a broad class of solutions of the hydrodynamical equations, which remains closed under evolution. The dynamics of the condensate for this class of excitations can be described by a nonlinear equation involving a $3 \times 3$ matrix. We have obtained an exact solution of this equation for the $\{n = 0, m = +2\}$ mode, which shows good agreement with experimental results.

We would also like to note that described in the literature 3-fold class of nonlinear solutions [10] covers two $\{n = 2, m = 0\}$ modes and any linear combination of $\{n = 0, m = \pm 2\}$ modes with (generally complex) equal by the absolute value coefficients: the observed in the experiment $\{n = 0, m = +2\}$ mode does not belong to this class. The 6-fold closed class of nonlinear solutions obtained in our work extends the treatment to two other modes $\{n = 0, m = \pm 2\}$ respectively as well as to an arbitrary superposition of $\{n = 0, m = \pm 2\}$ modes, including $\{n = 0, m = +2\}$.

In the other words, the existing treatment [10] can describe any excitation obtained by a modulation of the matrix elements of the spring-constant tensor $\mathcal{K}_{ij}(t)$ (see [10]) provided it always remains diagonal. This model can not be applied to describe the rotation-like modulations such as the “$m = 2$” JILA experiment [1]. Instead, our paper presents an extension for this model which can be applied to any kind of the time dependence of the spring-constant tensor, including the one used in the above experiment.

M.O. was supported by the National Science Foundation grant for light force dynamics #PHY-93-12572. P.S. was supported by the University of Auckland. She is grateful to Professor R. Glauber for making possible her visit to Harvard. This work was also partially supported by the NSF through a grant for the Institute for Theoretical Atomic and Molecular Physics at Harvard University and the Smithsonian Astrophysical Observatory. The authors are grateful to Professor E. Cornell for helpful comments.

---

1. M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, Science 269, 198 (1995)
2. C. C. Bradley, C. A. Sackett, J. J. Tollett and R. G. Hulet, Phys. Rev. Lett. 75, 1687 (1995)
3. K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van...
Druten, D. S. Durfee, D. M. Kurn and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995)

[4] F. Dalfovo, S. Giorgini, L. Pitaevskii and S. Stringari, cond-mat/9806033, to be published in Rev. Mod. Phys.

[5] D. S. Jin, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, Phys. Rev. Lett. 77, 420 (1996)

[6] D. S. Jin, M. R. Matthews, J. R. Ensher, C. E. Wieman and E. A. Cornell, Phys. Rev. Lett. 78, 764 (1997)

[7] M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. M. Kurn, D. S. Durfee, C. G. Townsend, and W. Ketterle, Phys. Rev. Lett. 77, 988 (1996)

[8] D. M. Stamper-Kurn, H.-J. Miesner, S. Inouye, M. R. Andrews, and W. Ketterle, Phys. Rev. Lett. 81, 500 (1998)

[9] S. Stringari, Phys. Rev. Lett. 77, 2360 (1996)

[10] Y. Castin and R. Dum, Phys. Rev. Lett. 77, 5315 (1996); Yu. Kagan, E. L. Surkov, and G. V. Shlyapnikov, Phys. Rev. A 54, R1753 (1996); F. Dalfovo, C. Minniti, and L. P. Pitaevskii, Phys. Rev. A 56, 4855 (1997)

[11] P. Öhberg, E. L. Surkov, I. Tittonen, S. Stenholm, M. Wilkens and G. V. Shlyapnikov, Phys. Rev. A 56, R3346 (1997)

[12] J. F. Dobson, Phys. Rev. Lett. 73, 2244 (1994)

[13] Note that in the limit of a spherically symmetric trap, the ‘fast mode’ \(\{n = 2, m = 0, (+)\}\) reduces to the \(l = 0\) mode with frequency \(\sqrt{5} \omega_{\text{trap}}\), and the remaining five modes form an \(l = 2\) quintuplet with frequency \(\sqrt{2} \omega_{\text{trap}}\).

[14] E. A. Cornell and D. S. Jin, private communication