A $\beta$–STURM LIOUVILLE PROBLEM ASSOCIATED WITH THE GENERAL QUANTUM OPERATOR

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Abstract. Let $I \subseteq \mathbb{R}$ be an interval and $\beta : I \to I$ a strictly increasing and continuous function with a unique fixed point $s_0 \in I$ that satisfies $(s_0 - t)(\beta(t) - t) \geq 0$ for all $t \in I$, where the equality holds only when $t = s_0$.

The general quantum operator defined by Hamza et al.,

$$D_{\beta}[f](t) := \frac{f(\beta(t)) - f(t)}{\beta(t) - t}$$

if $t \neq s_0$ and $D_{\beta}[f](s_0) := f'(s_0)$ if $t = s_0$, generalizes the Jackson $q$-operator $D_q$ and also the Hahn (quantum derivative) operator, $D_{q,\omega}$.

Regarding a $\beta$–Sturm Liouville eigenvalue problem associated with the above operator $D_{\beta}$, we construct the $\beta$–Lagrange’s identity, show that it is self-adjoint in $L^2_\beta([a, b])$, and exhibit some properties for the corresponding eigenvalues and eigenfunctions.

1. Introduction

The $\beta$–operator $D_{\beta}$ described in the abstract was introduced in [33] together with the corresponding general quantum difference calculus. It generalizes the $(q, \omega)$–derivative operator (the Hahn’s quantum operator)

$$D_{q,\omega}[f](x) := \frac{f(qx + \omega) - f(x)}{(q - 1)x + \omega},$$

which, in turn, generalizes both the Jackson $q$–derivative

$$D_q[f](x) := \frac{f(qx) - f(x)}{(q - 1)x},$$

and the (forward difference) $\omega$–derivative

$$\triangle_\omega[f](x) := \frac{f(x + \omega) - f(x)}{\omega},$$

where $0 < q < 1$ and $\omega \geq 0$ are fixed parameters.

Of particular relevance are the corresponding inverse operators, which enable one to define, the following integrals, respectively: the Jackson-Thomae-Nörlund $(q, \omega)$–integral

$$\int_a^b f \, d_{q,\omega} := \int_{\omega_0}^b f \, d_{q,\omega} - \int_{\omega_0}^a f \, d_{q,\omega},$$

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the Jackson $q$-integral

$$\int_a^b f d_q := (1 - q) \sum_{k=0}^{+\infty} [bf(bq^k) - af(aq^k)]q^k$$

and the Nörlund integral

$$\int_a^b f \Delta_\omega := \omega \sum_{k=0}^{+\infty} [f(b + k\omega) - f(a + k\omega)].$$

For more details over the $q$-integrals see, for example, [12].

Those difference operators together with its inverse operators are very important in mathematics investigation and in applications, with a large number of publications and a variety of topics including, but not limited to, the quantum calculus [16], the quantum variational calculus [12, 25, 34, 41], $q$-difference equations properties [6, 7, 38, 14], Sturm-Liouville problems [29, 15, 10, 13, 41, 14], Paley-Wiener results [27, 4, 26]. Sampling theory [4, 27, 2, 26, 13, 11, 32], $q$-exponential, trigonometric, hyperbolic and other important families of functions associated with Fourier expansions and corresponding properties connected and derived from its orthogonality feature [10, 17, 5, 38, 14, 13, 41, 14, 32]. These and many other subjects has attracted many researchers.

In 2015, Hamza et al. [33] introduced a general quantum difference operator, the $\beta$-derivative, generalizing the Hahn’s quantum operator (for certain functions $\beta$), and its inverse operator, the $\beta$-integral. Also in 2015 [34], $\beta$-Hölder, $\beta$-Minkowski, $\beta$-Gronwall, $\beta$-Bernoulli and $\beta$-Lyapunov inequalities were exhibited. In 2016, it was proved the existence and uniqueness of solutions of general quantum difference equations [35]. Later, in [30], some new results on homogeneous second order linear general quantum difference equations were presented and, in [36], the exponential, trigonometric and hyperbolic functions were introduced. The theory of nth-order linear general quantum difference equations was developed in [31] while the general quantum variational calculus was build up in [24]. In [22], properties of the $\beta$-Lebesgue spaces were produced and, recently, in [44], a general quantum Laplace transform was displayed and studied.

All of these publications generalizes previous results and are directly related with the above mentioned general quantum difference operator.

The aim of this work is to obtain a self-adjoint formulation of a $\beta$-Sturm-Liouville problem and to prove properties for the corresponding eigenvalues and eigenfunctions. Its construction follows ideas from [14] and from other previous publications.

In section 2 we recall the definition of the $\beta$-difference operator and its inverse operator, the $\beta$-integral, together with some of its properties. Section 3 is devoted to the $\beta$-Sturm-Liouville problem to be considered.

The outcome of this work can be found in section 3. We believe that Lemmas 3.1 and 3.2, Corollaries 3.3, 3.5, 3.8, 3.9 and 3.10, as well as Theorems 3.5 and 3.7, are original. Subsection 3.4 is also new.

2. THE $\beta$–DIFFERENCE OPERATOR AND THE $\beta$–INTEGRAL

2.1. The $\beta$–difference operator. In the following, $I \subseteq \mathbb{R}$ will denote an interval and $\beta : I \rightarrow I$ a strictly increasing and continuous function with a unique fixed
point $s_0 \in I$ satisfying
\[(t - s_0)(\beta(t) - t) \leq 0\]
for all $t \in I$, where the equality holds only when $t = s_0$.

For each function $f : I \to \mathbb{K}$ where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, Hamza et al. \cite{Ref33} defined the general quantum difference operator
\[
D_{\beta}[f](t) := \begin{cases} \frac{f(\beta(t)) - f(t)}{\beta(t) - t} & \text{if } t \neq s_0, \\ f'(s_0) & \text{if } t = s_0 \end{cases},
\]
provided that $f'(s_0)$ exists. $D_{\beta}[f](t)$ is called the $\beta$–derivative of $f$ at $t \in I$. If $f'(s_0)$ exists then $f$ is said to be $\beta$–differentiable on $I$.

It is obvious that when $\beta(t) = qt + \omega$ one obtains the Hahn operator \cite{Ref1.1}, being the fixed point given by $s_0 = \frac{\omega}{1-q}$.

We remark that it is possible to replace the above condition (2.1) by $(t - s_0)(\beta(t) - t) > 0$ for $t \in I$.

An introduction to the $\beta$–calculus related with this general quantum difference operator can be found in \cite{Ref33}.

### 2.2. The $\beta$–integral

Defining $\beta^k(t) := (\underbrace{\beta \circ \beta \circ \ldots \circ \beta}_k)(t)$ for $t \in I$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, with $\beta^0(t) := t$, we can consider the $\beta$–interval with extreme points $a$ and $b$,
\[ [a, b]_\beta := \{ \beta^n(x) | (x, n) \in \{a, b\} \times \mathbb{N}_0 \} . \]

Of course that
\[ a, b \in I \quad \Rightarrow \quad [a, b]_\beta \subset I , \]
whenever $a$ and $b$ are real numbers.

The following proposition can be found in \cite{Ref33} Lemma 2.1, page 3).

**Proposition A.** The sequence of functions $\{\beta^k(t)\}_{k \in \mathbb{N}_0}$ converges uniformly to the constant function $\hat{\beta}(t) := s_0$ on every compact interval $J \subset I$ containing $s_0$.

The quantum difference inverse operator, the $\beta$–integral, with $a, b \in I$, is defined by
\[
\int_a^b f \, d\beta := \int_{s_0}^b f \, d\beta - \int_{s_0}^a f \, d\beta
\]
where
\[
\int_{s_0}^x f \, d\beta := \sum_{k=0}^{+\infty} \left( \beta^k(x) - \beta^{k+1}(x) \right) f(\beta^k(x)) .
\]

Thus,
\[
\int_a^b f \, d\beta = \sum_{k=0}^{+\infty} \left( \beta^k(b) - \beta^{k+1}(b) \right) f(\beta^k(b)) - \sum_{k=0}^{+\infty} \left( \beta^k(a) - \beta^{k+1}(a) \right) f(\beta^k(a)) .
\]

If the infinite sum in the right side of (2.4) is convergent then we say that the function $f$ is $\beta$–integrable in $[s_0, x]$. The $\beta$–integral in the left side of (2.3) is

\footnote{In fact, $\mathbb{K} = \mathbb{X}$ can represent any Banach space \cite{Ref33} p.2}
well defined provided that at least one of the \( \beta \)-integrals in the right side is finite and we say that \( f \) is \( \beta \)-integrable in \([a, b]\) if it is both \( \beta \)-integrable in \([s_0, a]\) and in \([s_0, b]\).

As important particular cases, one obtains the Jackson-Thomae-Nörlund integral when \( \beta(t) = qt + \omega \) with \( 0 < q < 1 \) and \( \omega \geq 0 \). The Jackson \( q \)-integral corresponds to \( \omega = 0 \) in the previous case. Its fixed points are given by \( s_0 = \frac{a}{q} \) and \( s_0 = 0 \), respectively.

2.3. Properties of the \( \beta \)-derivative and of the \( \beta \)-integral. We go back to the the definition of the \( \beta \)-derivative operator (2.2).

Notice that if \( f \) is differentiable at a point \( t \in I \), then

\[
\lim_{\beta(t) \to t} D_\beta[f](t) = f'(t),
\]

hence \( D_\beta \) is a beta-analogue of the standard derivative operator.

The \( \beta \)-derivative satisfies properties which may be regarded as \( \beta \)-analogues of the corresponding properties for the usual derivative. For instance, the quantum operator (2.2) is linear, i.e.,

\[
D_\beta[\alpha f + \beta g](t) = \alpha D_\beta[f](t) + \beta D_\beta[g](t),
\]

where \( \alpha \) and \( \beta \) are any real or complex numbers, and satisfies the following \( \beta \)-product rule: for \( t \in I \),

\[
D_\beta[f \cdot g](t) = D_\beta[f](t) \cdot g(t) + f(\beta(t)) \cdot D_\beta[g](t)
\]

(2.6)

\[
D_\beta[g](t) \cdot f(t) + g(\beta(t)) \cdot D_\beta[f](t)
\]

if \( f \) and \( g \) are \( \beta \)-differentiable in \( I \). Also, \( f \) will be the constant function such that \( f(t) = f(s_0) \) for all \( t \in I \) whenever \( D_\beta[f](t) = 0 \) for all \( t \in I \). For these and other properties of the general quantum difference operator \( D_\beta \) see [33]. These equalities hold for all \( t \neq s_0 \), and also for \( t = s_0 \) whenever \( f'(s_0) \) and \( g'(s_0) \) exist.

2.3.1. The fundamental theorem of \( \beta \)-calculus. The following statement of the \( \beta \)-analogue of the fundamental theorem of calculus can be found in [22].

**Theorem B.** Let \( \beta : I \to I \) be a function satisfying the conditions described in subsection 2.1. Fix \( a, b \in I \) and let \( f : I \to \mathbb{C} \) be a function such that \( D_\beta[f] \in \mathcal{L}_\beta^1[a, b] \). Then:

(i) The equality

\[
\int_a^b D_\beta[f] \, d_\beta = \left[ f(s) - \lim_{k \to +\infty} f(\beta^k(s)) \right]_{s=a}^b
\]

holds, provided the involved limits exist.

(ii) In addition, assuming that \( a < s_0 < b \), if \( f \) has a discontinuity of first kind at \( s_0 \) then

\[
\int_a^b D_\beta[f] \, d_\beta = f(b) - f(a) - \left( f(s_0^+) - f(s_0^-) \right).
\]

Of course, if \( f \) is continuous at \( s_0 \) then

\[
\int_a^b D_\beta[f] \, d_\beta = f(b) - f(a).
\]
2.3.2. The $\beta$–integration by parts formula. Now we state the $\beta$–analogue of the integration by parts formula [22].

Theorem C. Let $\beta : I \to I$ be a function satisfying the conditions described in subsection 2.1. Fix $a, b \in I$ and two functions $f : I \to \mathbb{C}$ and $g : I \to \mathbb{C}$. Then:

$$\int_a^b f \cdot D_\beta[g] \, d_\beta = \left( (f \cdot g)(s) - \lim_{k \to +\infty} (f \cdot g)(\beta^k(s)) \right)_{s=a}^b - \int_a^b (g \circ \beta) \cdot D_\beta[f] \, d_\beta$$

holds, provided $f, g \in L^p_\beta[a,b]$, $D_\beta[f]$ and $D_\beta[g]$ are bounded in $[a,b]$, and the limits exist.

If, in addition, $f$ and $g$ are continuous at $s_0$ and $a < s_0 < b$, then

$$\int_a^{s_0} f \cdot D_\beta[g] \, d_\beta = \left[ f \cdot g \right]_{s=a}^{s_0} - \int_a^{s_0} (g \circ \beta) \cdot D_\beta[f] \, d_\beta.$$

2.4. The spaces $L^p_\beta[a,b]$ and $L^\infty_\beta[a,b]$.  

2.4.1. The space $L^p_\beta[a,b]$. For $a, b \in I$, we will denote by $L^p_\beta[a,b]$ the set of functions $f : I \to \mathbb{C}$ such that $|f|^p$ is $\beta$–integrable in $[a,b]$, i.e.,

$$L^p_\beta[a,b] = \left\{ f : I \to \mathbb{C} \middle| \int_a^b |f|^p \, d_\beta < \infty \right\}.$$

We also set

$$L^\infty_\beta[a,b] = \left\{ f : I \to \mathbb{C} \middle| \sup_{k \in \mathbb{N}} \left( |f(\beta^k(a))|, |f(\beta^k(b))| \right) < \infty \right\}.$$

It was proved in [22, Corollary 3.4] that if $a \leq s_0 \leq b$ and $1 \leq p \leq \infty$, then the set $L^p_\beta[a,b]$, with the usual operations of addition of functions and multiplication of a function by a number (real or complex), becomes a linear space over $\mathbb{K}$.

2.4.2. The space $L^p_\beta[a,b]$. For $f, g \in L^p_\beta[a,b]$, writing $f \sim g$ when

$$f(\beta^k(a)) = g(\beta^k(a)) \quad \text{and} \quad f(\beta^k(b)) = g(\beta^k(b))$$

holds for all $k = 0, 1, 2, \cdots$, i.e., we say that $f \sim g$ if $f = g$ in $[a,b]$. Clearly, $\sim$ defines an equivalence relation in $L^p_\beta[a,b]$. We will represent by $L^p_\beta[a,b]$ the corresponding quotient set:

$$L^p_\beta[a,b] = L^p_\beta[a,b] / \sim.$$

Also in [22] it was proved the following theorems.

Theorem D. If $a \leq s_0 \leq b$ and $1 \leq p \leq \infty$ then $L^p_\beta[a,b]$ is a normed linear space over $\mathbb{R}$ or $\mathbb{C}$ with norm

$$\|f\|_{L^p_\beta[a,b]} := \begin{cases} \left( \int_a^b |f|^p \, d_\beta \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty ; \\ \sup_{k \in \mathbb{N}} \left\{ |f(\beta^k(a))|, |f(\beta^k(b))| \right\} & \text{if } p = \infty . \end{cases}$$

In the usual way, here $f \in L^p_\beta[a,b]$ denotes any representative of the correspondent class in $L^p_\beta[a,b]$ where it belongs. Notice that, in view of (2.5) and (2.7), the definition of the norm $\|f\|_{L^p_\beta[a,b]}$ is independent of the chosen representative.

Theorem E. If $a \leq s_0 \leq b$ and $1 \leq p \leq \infty$, then the following holds:
properties, were introduced in [36].

Analogues of the exponential and trigonometric functions, as well as some of its
functions are defined by

\[ (3.1) \]

It is known that

\[ (2.10) \]

and

\[ (2.11) \]

We notice that, under the assumptions on the functions \( p \) and \( \beta \), the infinite
products of (2.10) and of (2.11) are both convergent [36] page 28.

On the other hand, the trigonometric functions are defined by

\[ (2.12) \]

and

\[ (2.13) \]

It is known that

\[ (2.14) \]

3. The \( \beta \)-Sturm-Liouville problem

3.1. Simple formulation of the classical Sturm-Liouville problem. The simplest
formulation of the classical Sturm-Liouville problem with separate type conditions,
is the following:

\[ (3.1) \]

where \( \nu(\cdot) \) is a real valued continuous function on \([a, b]\), \( \lambda \in \mathbb{C} \) and \( |a_{11}| + |a_{22}| \neq 0 \),
\( i = 1, 2 \). The major properties of the Sturm-Liouville problem (3.1) are that there
are only a countable number of real numbers (eigenvalues) \( \lambda_1 < \lambda_2 < \lambda_3 < \cdots \), with \( \lambda_n \to \infty \), such that (3.1) has nontrivial solutions \( \varphi_1, \varphi_2, \varphi_3 \cdots \) (eigenfunctions). Moreover, to each eigenvalue \( \lambda_n \), up to a multiplicative constant, there corresponds only one solution \( \varphi_n \). Thus, the eigenvalues are geometrically simple and they are also algebraically simple since they are simple zeros of the characteristic function associated with (3.1). Also, the set \( \{\varphi_n\}_{n=0}^\infty \) is an orthonormal basis of \( L^2(a,b) \). Regarding all of these aspects see [16].

Annaby and Mansour [15] [16], after Exton [28] [29], developed a \( q \)-Sturm-Liouville theory in the regular setting. Later, together with Makharesh [31], they make progress in the direction of a Sturm-Liouville theory in the regular setting, with separate-type boundary conditions, associated with the \( D_{q,\omega} \) operator (1.1).

In the following, we will establish conditions to develop a similar Sturm-Liouville theory associated with the general quantum operator \( D_\beta \) defined by (2.2).

### 3.2. Further properties for the \( \beta \)-difference operator and the \( \beta \)-integral.

In the next lemmas, \( \beta \) is a fixed real parameter.

**Lemma 3.1.** Let \( \beta : I \to I \) be a function satisfying the conditions described in subsection 2.1 being \( s_0 \) its unique fixed point, and \( f : I \to \mathbb{C} \) a real or complex function. Then

\[
\int_{s_0}^{b} f(\beta(t)) d\beta t = \int_{s_0}^{\beta(b)} f(u) (D_\beta \beta^{-1}) (u) d\beta u.
\]

**Proof.** The proof is straightforward since, by definition, the right side

\[
\int_{s_0}^{\beta(b)} f(u) (D_\beta \beta^{-1}) (u) d\beta u \text{ equals }
\]

\[
\sum_{k=0}^{\infty} f(\beta^k(b)) \left( \beta^{-1}(\beta^k(b)) - \beta(\beta^k(b)) \right) \left( \beta^k(b) - \beta^{k+1}(b) \right),
\]

which becomes, after simplification,

\[
\sum_{k=0}^{\infty} f(\beta^k(b)) \left( \beta^k(b) - \beta^{k+1}(b) \right) = \int_{s_0}^{b} f(\beta(t)) d\beta t.
\]

In the following, in order to simplify the notation of the inner-product (2.9) we will consider \( \langle \cdot , \cdot \rangle \) rather than \( \langle \cdot , \cdot \rangle_\beta \). Of course that when we write the interval \( (s_0, b) \) we admit that \( s_0 \leq b \). If \( b < s_0 \) then one must replace that interval by \( (b, s_0) \). The corresponding proofs follow exactly the same steps.

**Lemma 3.2.** Let \( \beta : I \to I \) be a function satisfying the conditions described in subsection 2.1 being \( s_0 \) its unique fixed point, and \( f, g \in L^2_{\beta}(s_0, b) \) be both continuous functions at \( s_0 \). Then, for \( t \in (s_0, b] \) we have

(i) \( (D_\beta f) (\beta^{-1}(t)) = (D_{\beta^{-1}} f) (t) \);

(ii) \( (D_\beta f, g) = f(b)g(\beta^{-1}(b)) - f(s_0)g(s_0) + \langle f, -D_{\beta^{-1}} D_{\beta^{-1}} g \rangle \);

(iii) \( \langle -D_{\beta^{-1}} D_{\beta^{-1}} f, g \rangle = f(s_0)g(s_0) - f(\beta^{-1}(b))g(b) + \langle f, D_{\beta} g \rangle \).

**Proof.** The proof of (i) is trivial. Let’s prove (ii):
By the $\beta$-integration by parts theorem (Theorem C) we have

$$\langle D_\beta f, g \rangle = \int_{s_0}^{b} \left( D_\beta f \right) (t) g(t) d_\beta t$$

$$= f(b)g(b) - f(s_0)g(s_0) - \int_{s_0}^{b} f(\beta(t)) \left( D_\beta g \right)(t) d_\beta t.$$ 

Considering $u = \beta(t)$ then, by Lemma 3.1, one obtains

$$\langle D_\beta f, g \rangle = f(b)g(b) - f(s_0)g(s_0) - \int_{s_0}^{\beta(b)} f(u) \left( D_\beta \beta^{-1}(u) \right) \left( D_\beta g \right)(\beta^{-1}(u)) d_\beta u,$n

which, by (i), becomes

$$\langle D_\beta f, g \rangle = f(b)g(b) - f(s_0)g(s_0) - \int_{s_0}^{\beta(b)} f(u) \left( D_\beta \beta^{-1}(u) \right) D_{\beta^{-1}}g(u) d_\beta u.$$

Writing (see (iv) of Lemma 3.5 in [33])

$$\int_{s_0}^{\beta(b)} f(u) D_\beta \beta^{-1}(u) D_{\beta^{-1}}g(u) d_\beta u = \int_{s_0}^{b} f(u) D_\beta \beta^{-1}(u) D_{\beta^{-1}}g(u) d_\beta u$$

$$+ \int_{b}^{\beta(b)} f(u) D_\beta \beta^{-1}(u) D_{\beta^{-1}}g(u) d_\beta u,$n

then, by Corollary 3.7 of [33],

$$\int_{s_0}^{\beta(b)} f(u) D_\beta \beta^{-1}(u) D_{\beta^{-1}}g(u) d_\beta u = \int_{s_0}^{b} f(u) D_\beta \beta^{-1}(u) D_{\beta^{-1}}g(u) d_\beta u$$

$$+ (\beta(b) - \beta) f(b) D_\beta \beta^{-1}(b) D_{\beta^{-1}}g(b).$$

Introducing this last identity in (3.2) one gets

$$\langle D_\beta f, g \rangle = f(b)g(b) - f(s_0)g(s_0) - \int_{s_0}^{b} f(u) D_\beta \beta^{-1}(u) D_{\beta^{-1}}g(u) d_\beta u$$

$$- (\beta(b) - \beta) f(b) D_\beta \beta^{-1}(b) D_{\beta^{-1}}g(b).$$

Since

$$f(b)g(b) = f(s_0)g(s_0) - (\beta(b) - \beta) f(b) \left( D_\beta \beta^{-1} \right)(b) \left( D_{\beta^{-1}}g \right)(b) =$$

$$f(b)g(\beta^{-1}(b)) - f(s_0)g(s_0),$$

we obtain

$$\langle D_\beta f, g \rangle = f(b)g(\beta^{-1}(b)) - f(s_0)g(s_0) - \langle f, D_\beta \beta^{-1}D_{\beta^{-1}}g \rangle,$$

which proves (ii).

Finally, (iii) is a consequence of (ii). $\square$

Taking into account the $\beta$-inner-product (2.9), we thus have the following corollary. It is a direct consequence of (ii), Lemma 3.2, therefore its proof will be omitted. Here $\beta$ satisfies the same conditions admitted in Lemmas 3.1 and 3.2.

**Corollary 3.3.** Let $a \leq s_0 \leq b$ and $f, g \in L_{\beta}^2(a, b)$ be both continuous functions at $s_0$. Then,
(i) \((D_{\beta}f) (\beta^{-1}(t)) = (D_{\beta^{-1}}f) (t)\);

(ii) \(\langle D_{\beta}f, y \rangle = f(b)g(\beta^{-1}(b)) - f(a)g(\beta^{-1}(a)) + \langle f, -D_{\beta^{-1}}D_{\beta^{-1}}g \rangle\).

3.3. The \(\beta\)-eigenvalue problem. Consider the following \(\beta\)-Sturm-Liouville problem (\(\beta\)-SLP) in \(L^2_\beta (s_0, b)\) defined by the \(\beta\)-difference equation

\[
\ell_{\beta} y(t) := -D_{\beta}^{-1} D_{\beta^{-1}} D_{\beta} y(t) + r(t)y(t) = \lambda y(t),
\]

with \(s_0 \leq t \leq b < \infty\), \(\lambda \in \mathbb{C}\), and the boundary conditions

\[
\begin{cases}
a_1 y(s_0) + a_2 D_{\beta^{-1}} y(s_0) = 0 \\
b_1 y(b) + b_2 D_{\beta^{-1}} y(b) = 0.
\end{cases}
\]

We assume that \(\beta : I \rightarrow I\) is a function satisfying the conditions described in subsection 2.1 being \(s_0\) its unique fixed point, and also assume that \(r(t)\) is a real valued continuous function on \([s_0, b]\) and \(|a_1| + |a_2| \neq 0 \neq |b_1| + |b_2|\).

The operator \(\ell_{\beta}\) \((3.3)\) satisfies the following \(\beta\)-Lagrange’s identity.

**Theorem 3.4.** If \(y, z \in L^2_\beta (s_0, b)\) then

\[
\int_{s_0}^b \left[ (\ell_{\beta}y(t))z(t) - y(t)(\ell_{\beta}z(t)) \right] d_{\beta}u = [y, z](b) - [y, z](s_0),
\]

where

\[
y, z(t) = y(t)D_{\beta^{-1}} z(t) - z(t)D_{\beta^{-1}} y(t)
\]

**Proof.** Consider \(y, z \in L^2_\beta (s_0, b)\).

On one hand, using (iii), Lemma 3.2 with \(f(t) = D_{\beta}y(t)\) and \(g(t) = z(t)\), we have

\[
\langle -D_{\beta}^{-1} D_{\beta^{-1}} D_{\beta} y, z \rangle = -D_{\beta} y (\beta^{-1}(b)) z(b) + D_{\beta} y (s_0) z(s_0) + \langle D_{\beta} y, D_{\beta} z \rangle.
\]

By (i) of Lemma 3.2 we then obtain

\[
\langle -D_{\beta}^{-1} D_{\beta^{-1}} D_{\beta} y, z \rangle = -D_{\beta^{-1}} y(b) z(b) + D_{\beta^{-1}} y (s_0) z(s_0) + \langle D_{\beta} y, D_{\beta} z \rangle.
\]

On the other hand, using (ii), Lemma 3.2 with \(f(t) = y(t)\) and \(g(t) = D_{\beta} z(t)\), we get

\[
\langle D_{\beta} y, D_{\beta} z \rangle = y(b) D_{\beta} z (\beta^{-1}(b)) - y(s_0) D_{\beta} z (s_0) + \langle y, -D_{\beta}^{-1} D_{\beta^{-1}} D_{\beta} z \rangle,
\]

which becomes, by (i) of Lemma 3.2

\[
\langle D_{\beta} y, D_{\beta} z \rangle = y(b) D_{\beta^{-1}} z(b) - y(s_0) D_{\beta^{-1}} z(s_0) + \langle y, -D_{\beta}^{-1} D_{\beta^{-1}} D_{\beta} z \rangle.
\]

Combining \((3.7)\) with \((3.8)\) it results

\[
\langle -D_{\beta}^{-1} D_{\beta^{-1}} D_{\beta} y, z \rangle = y(b) \left(D_{\beta^{-1}} z \right)(b) - (D_{\beta^{-1}} y)(b) z(b) - y(s_0) \left(D_{\beta^{-1}} z \right)(s_0) + D_{\beta^{-1}} y (s_0) z(s_0) + \langle y, -D_{\beta}^{-1} D_{\beta^{-1}} D_{\beta} z \rangle
\]

which is equivalent to

\[
\langle -D_{\beta}^{-1} D_{\beta^{-1}} D_{\beta} y, z \rangle = [y, z](b) - [y, z](s_0) + \langle y, -D_{\beta}^{-1} D_{\beta^{-1}} D_{\beta} z \rangle.
\]
Now we easily derive the \( \beta \)-Lagrange’s identity (3.3): using (3.9) and the fact that \( r(t) \) is real we have

\[
\langle \ell \beta y, z \rangle = \langle D \beta^{-1} D \beta^{-1} D \beta y(t) + r(t)y(t), z(t) \rangle \\
= \langle D \beta^{-1} D \beta^{-1} D \beta y(t), z(t) \rangle + \langle r(t)y(t), z(t) \rangle \\
= [y, z](b) - [y, z](s_0) + \langle y(t), -D \beta^{-1} D \beta z(t) \rangle \\
+ \langle r(t), r(t)z(t) \rangle \\
= [y, z](b) - [y, z](s_0) + \langle y(t), -D \beta^{-1} D \beta z(t) + r(t)z(t) \rangle \\
= [y, z](b) - [y, z](s_0) + \langle y, \ell \beta z \rangle
\]

\( \Box \)

The following corollary follows now easily.

**Corollary 3.5.** The \( \beta \)-Sturm Liouville eigenvalue problem (3.3)-(3.4) is self-adjoint in \( L_2^\beta (s_0, b) \) (i.e., \( \ell \beta \) is self-adj. in \( \{ y \in L_2^\beta (s_0, b) : y \text{ satisfies (3.4)} \} \)).

**Proof.** Let \( y \) and \( z \) satisfy the boundary conditions (3.4).

(i) If \( a_2 \neq 0 \) then, from (3.6),

\[
[y, z](s_0) = y(s_0) \left( \frac{a_1}{a_2} z(s_0) + \frac{a_1}{a_2} y(s_0) \right) = 0.
\]

(ii) If \( a_2 = 0 \) then, since \( |a_1| + |a_2| \neq 0 \), one must have \( a_1 \neq 0 \) which, by (3.3), implies that \( y(s_0) = 0 \).

By similar arguments it follows that \( z(s_0) = 0 \). Thus, \( [y, z](s_0) = 0 \).

Arguing as above one proves also that \( [y, z](b) = 0 \) thus,

\[
\langle \ell \beta y, z \rangle = \langle y, \ell \beta z \rangle.
\]

\( \Box \)

**Remark 3.6.** As a consequence of (3.9), under the boundary conditions (3.4), of course we also have

\[
\langle -D \beta^{-1} D \beta^{-1} D \beta y, z \rangle = \langle y, -D \beta^{-1} D \beta z \rangle.
\]

**Theorem 3.7.** All eigenvalues of the problem (3.3)-(3.4) are real. Eigenfunctions corresponding to different eigenvalues are orthogonal.

**Proof.** We separate the proof in two parts: (i) and (ii).

(i) First we show that the eigenfunctions are real.

Let \( \lambda_0 \) be an eigenvalue and \( y_0(t) \) be a corresponding eigenfunction. Since

\[
\ell \beta y_0(t) = \lambda_0 y_0(t), \quad \ell \beta y_0(t) = \lambda_0 y_0(t)
\]

and, by Corollary 3.3

\[
\int_{s_0}^{b} \ell \beta y_0(t) y_0(t) d\beta t = \int_{s_0}^{b} y_0(t) \ell \beta y_0(t) d\beta t,
\]

then,

\[
\lambda_0 \int_{s_0}^{b} |y_0(t)|^2 d\beta t = \lambda_0 \int_{s_0}^{b} |y_0(t)|^2 d\beta t,
\]


Thus
\[(\lambda_0 - \lambda_0) \int_{s_0}^{b} |y_0(t)|^2 \, d\beta t = 0.\]

Since \(y_0(t)\) is an eigenfunction then \(\lambda_0 = \lambda_0\).

(ii) Finally, we show that eigenfunctions corresponding to different eigenvalues are orthogonal.

Let \(\lambda_1 \neq \lambda_2\) be distinct eigenvalues with eigenfunctions \(\phi_1\) and \(\phi_2\), respectively. From Corollary 3.3 and because the eigenvalues are real one have
\[
\lambda_1 \int_{s_0}^{b} \phi_1(t)\phi_2(t) \, d\beta t = \lambda_2 \int_{s_0}^{b} \phi_1(t)\phi_2(t) \, d\beta t.
\]

Since \(\lambda_1 \neq \lambda_2\) the orthogonality follows.

\[\square\]

Now we generalize these results in the following way, where \(\beta\) is a function satisfying the conditions introduced in the very beginning of subsection 2.1.

If \(a \leq s_0 \leq b\) then we may consider the \(\beta\)-SLP in \(L^2_{\beta}(a, b)\) defined by the \(\beta\)-difference equation
\[
(\ell_{a,b}^{\beta} y)(t) := -D_{\beta}\beta^{-1}D_{\beta-1}D_{\beta}y(t) + r(t)y(t) = \lambda y(t),
\]
with \(-\infty < a \leq s_0 \leq b < \infty\), \(\lambda \in \mathbb{C}\), and the boundary conditions
\[
\begin{align*}
\{ & a_1 y(a) + a_2 D_{\beta-1} y(a) = 0 \\
& b_1 y(b) + b_2 D_{\beta-1} y(b) = 0.
\end{align*}
\]

We also assume that \(r(t)\) is a real valued continuous function on \([a, b]\) and \(|a_1| + |a_2| \neq 0 \neq |b_1| + |b_2|\).

This operator \(\ell_{a,b}^{\beta}\) defined by (3.10)-(3.11) satisfies the following Corollaries.

**Corollary 3.8.** If \(y, z \in L^2_{\beta}(a, b)\) then
\[
\int_{a}^{b} \left[ (\ell_{a,b}^{\beta} y)(t)\overline{z}(t) - y(t)(\ell_{a,b}^{\beta} \overline{z}(t)) \right] \, d\beta u = [y, \overline{z}](b) - [y, \overline{z}](a),
\]
with \([y, \overline{z}]\) defined by (3.6).

**Proof.** To prove this Corollary one follows the procedure of the proof of Theorem 3.4 and make use of Corollary 3.3. \(\square\)

**Corollary 3.9.** The \(\beta\)-Sturm Liouville eigenvalue problem (3.10)-(3.11) is self-adjoint in \(L^2_{\beta}(a, b)\) (i.e., \(\ell_{a,b}^{\beta}\) is self-adj. in \(\{ y \in L^2_{\beta}(a, b): y \text{satisfies (3.11)}\}\)).

**Corollary 3.10.** All eigenvalues of the problem (3.10)-(3.11) are real. Eigenfunctions corresponding to different eigenvalues are orthogonal.

3.4. A particular case. The generality of the \(\beta\)-SLP (3.10)-(3.11) implies the difficulty on finding explicit non-trivial solutions. The aim of this subsection is to consider some particular cases, although very comprehensive, in such a way that enables one to exhibit solutions of equation (3.10) or (3.11).

In the following, \(I \subseteq \mathbb{R}\) is an interval and the function \(\beta : I \to I\) satisfies the conditions introduced in subsection 2.1. Furthermore, the function \(p\) satisfies the same assumptions assumed in the beginning of subsection 2.5. We have the following Proposition relative to the \(\beta\)-exponential and \(\beta\)-trigonometric functions (2.10)-(2.13).
Proposition 3.11. The following identities hold on $I$:

\[
D_{\beta^{-1}}e_{p,\beta}(t) = p(\beta^{-1}(t))e_{p,\beta}(t), \quad D_{\beta^{-1}}E_{p,\beta}(t) = p(\beta^{-1}(t))E_{p,\beta}(\beta(t)),
\]
\[
D_{\beta^{-1}}\sin_{p,\beta}(t) = p(\beta^{-1}(t))\cos_{p,\beta}(t), \quad D_{\beta^{-1}}\cos_{p,\beta}(t) = -p(\beta^{-1}(t))\sin_{p,\beta}(t),
\]
\[
D_{\beta^{-1}}\cos_{p,\beta}(t) = -p(\beta^{-1}(t))\sin_{p,\beta}(t).
\]

Proof. We prove the first identity.

\[
D_{\beta^{-1}}e_{p,\beta}(t) = \frac{e_{p,\beta}(t) - e_{p,\beta}(\beta^{-1}(t))}{t - \beta^{-1}(t)} = \frac{1}{1 - p(\beta^{-1}(t))(\beta^{-1}(t) - t)} \left( t - \beta^{-1}(t) \right) \prod_{k=0}^{\infty} \left[ 1 - p(\beta^{k})(\beta^{k}(t) - \beta^{k+1}(t)) \right] = -\frac{p(\beta^{-1}(t))(\beta^{-1}(t) - t)}{t - \beta^{-1}(t)} e_{p,\beta}(t) = p(\beta^{-1}(t))e_{p,\beta}(t).
\]

In a similar way to the $\beta$-product rule (2.6) we have

\[
D_{\beta^{-1}}[f \cdot g](t) = D_{\beta^{-1}}[f](t) \cdot g(t) + f(\beta^{-1}(t)) \cdot D_{\beta^{-1}}[g](t)
\]

(3.12)

Using the corresponding definitions and this latter formula we obtain the following properties.

Proposition 3.12. The following identities hold on $I$:

\[
D_{\beta^{-1}}D_{\beta}e_{p,\beta}(t) = \left[ p^{2}(\beta^{-1}(t)) + D_{\beta^{-1}}p(t) \right] e_{p,\beta}(t),
\]
\[
D_{\beta^{-1}}D_{\beta}E_{p,\beta}(t) = p(t)p(\beta^{-1}(t))E_{p,\beta}(t) + D_{\beta^{-1}}p(t)E_{p,\beta}(\beta(t)),
\]
\[
D_{\beta^{-1}}D_{\beta}\sin_{p,\beta}(t) = -p^{2}(\beta^{-1}(t))\sin_{p,\beta}(t) + D_{\beta^{-1}}p(t)\cos_{p,\beta}(t),
\]
\[
D_{\beta^{-1}}D_{\beta}\cos_{p,\beta}(t) = -p^{2}(\beta^{-1}(t))\cos_{p,\beta}(t) - D_{\beta^{-1}}p(t)\sin_{p,\beta}(t),
\]
\[
D_{\beta^{-1}}D_{\beta}\sin_{p,\beta}(t) = -p^{2}(\beta^{-1}(t))\sin_{p,\beta}(t) + D_{\beta^{-1}}p(t)\cos_{p,\beta}(t),
\]
\[
D_{\beta^{-1}}D_{\beta}\cos_{p,\beta}(t) = -p^{2}(\beta^{-1}(t))\cos_{p,\beta}(t) - D_{\beta^{-1}}p(t)\sin_{p,\beta}(t).
\]

Proof. The proofs are straightforward. Just to illustrate it we give the proof of the second identity.

By (2.14) and (3.12) we obtain

\[
D_{\beta^{-1}}D_{\beta}E_{p,\beta}(t) = D_{\beta^{-1}}\left[ p(t)E_{p,\beta}(\beta(t)) \right] = D_{\beta^{-1}}p(t)E_{p,\beta}(\beta(t)) + p(\beta^{-1}(t))D_{\beta^{-1}}E_{p,\beta}(\beta(t)).
\]
hence, by (i) of Lemma \[3.2\]
\[
D_{\beta^{-1}}D_{\beta}E_{p,\beta}(t) = D_{\beta^{-1}}p(t)E_{p,\beta}(\beta(t)) + p(\beta^{-1}(t))D_{\beta}E_{p,\beta}(t)
\]
\[
= p(t)p(\beta^{-1}(t))E_{p,\beta}(t) + D_{\beta^{-1}}p(t)E_{p,\beta}(\beta(t)).
\]

For the particular case where \( p \) is the constant function \( p(t) = z \), the following Corollaries hold.

**Corollary 3.13.** If \( p \) is the constant function \( p(t) = z \) on \( I \) then the following identities hold:
\[
D_{\beta^{-1}}D_{\beta}e_{z,\beta}(t) = z^2e_{z,\beta}(t), \quad D_{\beta^{-1}}D_{\beta}E_{z,\beta}(t) = z^2E_{z,\beta}(t),
\]
\[
D_{\beta^{-1}}D_{\beta}\sin_{z,\beta}(t) = -z^2\sin_{z,\beta}(t), \quad D_{\beta^{-1}}D_{\beta}\cos_{z,\beta}(t) = -z^2\cos_{z,\beta}(t),
\]
\[
D_{\beta^{-1}}D_{\beta}\sin_{z,\beta}(t) = -z^2\sin_{z,\beta}(t), \quad D_{\beta^{-1}}D_{\beta}\cos_{z,\beta}(t) = -z^2\cos_{z,\beta}(t).
\]

**Corollary 3.14.** Let \( p \) be the constant function \( p(t) = z \) on \( I \). If the function \( \beta \) satisfies \( D_{\beta}\beta^{-1}(t) = k \) where \( k \) is independent of \( t \) then, the \( \beta \)-exponentials functions \([2.10]-[2.17]\) satisfy \([3.3]\) or \([3.10]\) with \( r(t) = 0 \) in \( I \) and \( \lambda = kz^2 \) while each of the \( \beta \)-trigonometric functions \([2.12]-[2.13]\) satisfy \([3.3]\) or \([3.10]\) with \( r(t) = 0 \) in \( I \) and \( \lambda = -kz^2 \).

**Remark 3.15.**
(1) If \( \beta(t) = qt + \omega \), which corresponds to the \((q, \omega)\)-derivative operator \([1.1]\), the condition on the function \( \beta \) of Corollary \[3.14\] is satisfied since \( D_{\beta}\beta^{-1}(t) = 1/q \) in \( I \).

(2) Notice that, under the conditions of Corollary \[3.14\] the \( \beta \)-exponentials and \( \beta \)-trigonometric functions, for appropriate choices of the constants \( a_1, a_2, b_1, b_2 \), and \( a, b \) are solutions of both the \( \beta \)-Sturm Liouville problems \([3.9]-[3.11]\) or \([3.10]-[3.11]\).

(3) An interesting feature of this setting would be to obtain an explicit function \( \beta \) under the assumptions of Corollary \[3.14\] but not coincident with the one for the Hahn operator \([1.1]\), and specific values of \( a_1, a_2, b_1, b_2 \), and \( a, b \) in order to obtain a fundamental set of solutions of the corresponding \( \beta \)-SLP. This could pave the way to pursue investigations in other directions.

**Final conclusions.** We established conditions that make possible a self-adjoint formulation of the Sturm-Liouville problem \([3.3]-[3.4]\) in the space \( L_2(\beta)(s_0, b) \) and we were able to extend it to the problem \([3.10]-[3.11]\) in \( L_2(\beta)(a, b) \). Its construction is based in the general \( \beta \)-difference quantum operator \([2.2]\). Then, it was proved that all the corresponding eigenvalues are real and the relative eigenfunctions are orthogonal. We believe that the establishment of this frame will allow other researchers to get interested in it and push further to other directions and results.

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CORRIGENDUM

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The definitions of \( \sin_{p,\beta}(t) \) and \( \sin_{p,\beta}(t) \) in (2.12) and (2.13), page 6, should be
\[
\sin_{p,\beta}(t) = \frac{e_{ip,\beta}(t) - e_{-ip,\beta}(t)}{2i}, \quad \sin_{p,\beta}(t) = \frac{E_{ip,\beta}(t) - E_{-ip,\beta}(t)}{2i}.
\]

Also at page 6, the last two identities of formulae (2.14) must be replaced by
\[
D_\beta \sin_{p,\beta}(t) = p(t) \cos_{p,\beta}(\beta(t)), \quad D_\beta \cos_{p,\beta}(t) = -p(t) \sin_{p,\beta}(\beta(t)).
\]

With respect to Proposition 3.11, page 12, the first four identities aren’t correct. Its full statement is:

**Proposition 3.11.** The following identities hold on \( I \):
\[
D_{\beta^{-1}} e_{p,\beta}(t) = p(\beta^{-1}(t)) e_{p,\beta}(\beta^{-1}(t)), \quad D_{\beta^{-1}} E_{p,\beta}(t) = p(\beta^{-1}(t)) E_{p,\beta}(t),
\]
\[
D_{\beta^{-1}} \sin_{p,\beta}(t) = p(\beta^{-1}(t)) \cos_{p,\beta}(\beta^{-1}(t)), \quad D_{\beta^{-1}} \cos_{p,\beta}(t) = -p(\beta^{-1}(t)) \sin_{p,\beta}(\beta^{-1}(t)),
\]
\[
D_{\beta^{-1}} \sin_{p,\beta}(t) = p(\beta^{-1}(t)) \cos_{p,\beta}(\beta^{-1}(t)), \quad D_{\beta^{-1}} \cos_{p,\beta}(t) = -p(\beta^{-1}(t)) \sin_{p,\beta}(\beta^{-1}(t)).
\]

Also regarding Proposition 3.11, in the last line of the first identity’s proof, one should read:
\[
= - \frac{p(t) (\beta^{-1}(t) - t)}{t - \beta^{-1}(t)} e_{p,\beta}(\beta^{-1}(t)) = p(\beta^{-1}(t)) e_{p,\beta}(\beta^{-1}(t)).
\]

Along with this, one finds a new statement of Proposition 3.12 at page 12.

**Proposition 3.12.** The following identities hold on \( I \):
\[
D_{\beta^{-1}} D_{\beta} e_{p,\beta}(t) = [p(t)p(\beta^{-1}(t)) + D_{\beta^{-1}} p(t)] e_{p,\beta}(\beta^{-1}(t)),
\]
\[
D_{\beta^{-1}} D_{\beta} E_{p,\beta}(t) = [p(t)p(\beta^{-1}(t)) D_{\beta^{-1}} \beta(t) + D_{\beta^{-1}} p(t)] E_{p,\beta}(\beta(t)),
\]
\[
D_{\beta^{-1}} D_{\beta} \sin_{p,\beta}(t) = -p^2(\beta^{-1}(t)) \sin_{p,\beta}(\beta^{-1}(t)) + D_{\beta^{-1}} p(t) \cos_{p,\beta}(t),
\]
\[
D_{\beta^{-1}} D_{\beta} \cos_{p,\beta}(t) = -p^2(\beta^{-1}(t)) \cos_{p,\beta}(\beta^{-1}(t)) - D_{\beta^{-1}} p(t) \sin_{p,\beta}(t),
\]
\[
D_{\beta^{-1}} D_{\beta} \sin_{p,\beta}(t) = -p^2(t) D_{\beta^{-1}} \beta(t) \sin_{p,\beta}(\beta(t)) + D_{\beta^{-1}} p(t) \cos_{p,\beta}(t),
\]
\[
D_{\beta^{-1}} D_{\beta} \cos_{p,\beta}(t) = -p^2(t) D_{\beta^{-1}} \beta(t) \cos_{p,\beta}(\beta(t)) - D_{\beta^{-1}} p(t) \sin_{p,\beta}(t).
\]

Regarding the proof of the second identity of Proposition 3.12, its last three lines must be replaced by:
hence, since
\[
D_{\beta^{-1}} E_{p,\beta}(\beta(t)) = \frac{E_{p,\beta}(\beta(t)) - E_{p,\beta}(t)}{t - \beta^{-1}(t)} = \frac{E_{p,\beta}(t) - E_{p,\beta}(\beta(t))}{t - \beta(t)} \frac{\beta(t) - t}{t - \beta^{-1}(t)} = D_{\beta} E_{p,\beta}(t) D_{\beta^{-1}}(t),
\]
then,
\[
D_{\beta^{-1}} D_{\beta} E_{p,\beta}(t) = D_{\beta^{-1}} p(t) E_{p,\beta}(\beta(t)) + p(\beta^{-1}(t)) D_{\beta^{-1}}(t) D_{\beta} E_{p,\beta}(t)
= p(t) p(\beta^{-1}(t)) D_{\beta^{-1}}(t) E_{p,\beta}(\beta(t)) + D_{\beta^{-1}} p(t) E_{p,\beta}(\beta(t)).
\]

Following the previous amendments comes Corollary 3.13.

**Corollary 3.13.** If \( p \) is the constant function \( p(t) = z \) on \( I \) then the following identities hold:

\[
D_{\beta^{-1}} D_{\beta} e_{z,\beta}(t) = z^2 e_{z,\beta}(\beta^{-1}(t)), \quad D_{\beta^{-1}} D_{\beta} E_{z,\beta}(t) = z^2 D_{\beta^{-1}}(t) E_{z,\beta}(\beta(t)),
\]
\[
D_{\beta^{-1}} D_{\beta} \sin_{z,\beta}(t) = -z^2 \sin_{z,\beta}(\beta^{-1}(t)), \quad D_{\beta^{-1}} D_{\beta} \cos_{z,\beta}(t) = -z^2 \cos_{z,\beta}(\beta^{-1}(t)),
\]
\[
D_{\beta^{-1}} D_{\beta} \sin_{z,\beta}(t) = -z^2 D_{\beta^{-1}}(t) \sin_{z,\beta}(\beta(t)), \quad D_{\beta^{-1}} D_{\beta} \cos_{z,\beta}(t) = -z^2 D_{\beta^{-1}}(t) \cos_{z,\beta}(\beta(t)).
\]

Remarking that \( 1 + z(t - \beta(t)) \neq 0 \) holds on \( I \) whenever \( z \) is a constant on \( I \) then, adjusting Corollary 3.14 to Corollary 3.13, we obtain:

**Corollary 3.14.** Let \( p \) be the constant function \( p(t) = z \) on \( I \). Then, the \( \beta \)-exponential function (2.11) satisfy equation

\[
-D_{\beta} \beta^{-1}(t) D_{\beta^{-1}} D_{\beta} E_{z,\beta}(t) = -z^2 E_{z,\beta}(\beta(t)),
\]
or, equivalently, satisfy

\[
-D_{\beta} \beta^{-1}(t) D_{\beta^{-1}} D_{\beta} E_{z,\beta}(t) = -z^2 \frac{E_{z,\beta}(t)}{1 + z(t - \beta(t))},
\]
which shows that \( E_{z,\beta}(t) \) satisfy (3.3) or (3.10) with \( r(t) = \frac{z^2}{1 + z(t - \beta(t))} \) in \( I \) and \( \lambda = 0 \).

As a consequence of this new wording, the item (2) of Remark 3.15 is no longer valid.