Non-abelian black strings

Betti Hartmann

School of Engineering and Sciences,
International University Bremen (IUB), 28725 Bremen, Germany

(Dated: November 17, 2018)

Abstract

Non-abelian black strings in a 5-dimensional Einstein-Yang-Mills model are considered. The solutions are spherically symmetric non-abelian black holes in 4 dimensions extended into an extra dimension and thus possess horizon topology $S^2 \times \mathbb{R}$. We find that several branches of solutions exist. In addition, we determine the domain of existence of the non-abelian black strings.

PACS numbers: 04.20.Jb, 04.40.Nr, 04.50.+h, 11.10.Kk

*b.hartmann@iu-bremen.de
I. INTRODUCTION

The idea that we live in more than four dimensions stems from models of unification of all known forces. In Kaluza-Klein theories [1, 2] 4-dimensional electromagnetism and gravity are unified within a 5-dimensional gravity theory. The 5th dimension is compactified on a circle of Planck length. Similar ideas appear in (super)string theories, which are only consistent in 10 and 26 dimensions in the case of superstring, respectively bosonic string theory [3]. Recently, so-called brane world models [4] have gained a lot of interest. These assume the standard model fields to be confined on a 3-brane embedded in a higher dimensional manifold.

A large number of higher dimensional black holes has been studied in recent years. The first solutions that have been constructed are the hyperspherical generalisations of well-known black holes solutions such as the Schwarzschild and Reissner-Nordström solutions in more than four dimensions [5] as well as the higher dimensional Kerr solutions [6]. In $d$ dimensions, these solutions have horizon topology $S^{d-2}$.

However, in contrast to 4 dimensions black holes with different horizon topologies should be possible in higher dimensions. An example is a 4-dimensional Schwarzschild black hole extended into one extra dimension, a so-called Schwarzschild black string. These solutions have been discussed extensively especially with view to their stability [7]. A second example, which is important due to its implications for uniqueness conjectures for black holes in higher dimensions, is the black ring solution in 5 dimensions with horizon topology $S^2 \times S^1$ [8].

The by far largest number of higher dimensional black hole solutions constructed so far are solutions of the vacuum Einstein equations, respectively Einstein-Maxwell equations. The first example of black hole solutions containing non-abelian gauge fields have been discussed in [9]. These are non-abelian black holes solutions of a generalised 5-dimensional Einstein-Yang-Mills system with horizon topology $S^3$.

Here we construct black hole solutions of an Einstein-Yang-Mills model discussed previously in [10, 11, 12]. These solutions are non-abelian black hole solutions in 4 dimensions extended into one extra codimension. We give the model including the ansatz, equations of motion and boundary conditions in Section II. The numerical results are discussed in Section III. The summary is given in Section IV.
II. THE MODEL

The Einstein-Yang-Mills Lagrangian in \( d = (4 + 1) \) dimensions is given by:

\[
S = \int \left( \frac{1}{16\pi G_5} R - \frac{1}{4e^2} F_{aMN}^a F^{aMN} \right) \sqrt{g^{(5)}} d^5 x
\]

with the SU(2) Yang-Mills field strengths \( F_{aMN}^a = \partial_M A_N^a - \partial_N A_M^a + \epsilon_{abc} A_M^b A_N^c \), the gauge index \( a = 1, 2, 3 \) and the space-time index \( M = 0, ..., 5 \). \( G_5 \) and \( e \) denote respectively the 5-dimensional Newton’s constant and the coupling constant of the gauge field theory. \( G_5 \) is related to the Planck mass \( M_{pl} \) by \( G_5 = M_{pl}^{-3} \) and \( e^2 \) has the dimension of [length].

If both the metric and matter fields are independent on the extra coordinate \( y \), the fields can be parametrized as follows [10]:

\[
g^{(5)}_{MN} dx^M dx^N = e^{-\xi} g^{(4)}_{\mu\nu} dx^\mu dx^\nu + e^{2\xi} dy^2, \quad \mu, \nu = 0, 1, 2, 3
\]

and

\[
A_M^a dx^M = A_\mu^a dx^\mu + \Phi^a dy.
\]

\( g^{(4)} \) is the 4-dimensional metric tensor.

A. The Ansatz

Our aim is to construct non-abelian black holes, which are spherically symmetric in 4 dimensions and are extended into one extra dimension. The topology of these non-abelian black strings will thus be \( S^2 \times \mathbb{R} \). Note that we could also construct non-abelian black holes with topology \( S^2 \times S^1 \) if we would make the extra coordinate \( y \) periodic.

For the metric the Ansatz reads:

\[
g^{(5)}_{MN} dx^M dx^N = e^{-\xi} \left[ -A^2(r) N(r) dt^2 + N^{-1}(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right] + e^{2\xi} dy^2,
\]

with

\[
N(r) = 1 - \frac{2m(r)}{r} \quad \text{and} \quad \xi = \xi(r).
\]

In these coordinates, \( m(\infty) \) denotes the (dimensionful) mass per unit length of the field configuration.
For the gauge fields, we use the purely magnetic hedgehog ansatz [13]:

\[ A_r^a = A_t^a = 0 , \]  
\[ A_\theta^a = (1 - K(r))e_\theta^a , \quad A_\varphi^a = -(1 - K(r)) \sin \theta e_\theta^a , \]  
\[ \Phi^a = vH(r)e_r^a , \]

where \( v \) is a mass scale.

**B. Equations of motion**

With the following rescalings:

\[ x = evr , \quad \mu = evm \]  

the resulting set of ordinary differential equations only depends on the fundamental coupling \( \alpha = 4\pi \sqrt{G_5}v \).

The Einstein equations for the metric functions \( N, A \) and \( \xi \) then read:

\[ \mu' = \alpha^2 \left( e^\xi N(K')^2 + \frac{1}{2} N x^2 (H')^2 e^{-2\xi} + \frac{1}{2x^2} (K^2 - 1)^2 e^\xi + K^2 H^2 e^{-2\xi} \right) + \frac{3}{8} N x^2 (\xi')^2 , \]
\[ A' = \alpha^2 x A \left( \frac{2(K')^2}{x^2} e^\xi + e^{-2\xi} (H')^2 \right) + \frac{3}{4} x A (\xi')^2 , \]
\[ (x^2 AN\xi')' = \frac{4}{3} \alpha^2 A \left[ e^\xi \left( N(K')^2 + \frac{(K^2 - 1)^2}{2x^2} \right) - 2e^{-2\xi} \left( \frac{1}{2} N (H')^2 x^2 + H^2 K^2 \right) \right] , \]

while the Euler-Lagrange equations for the matter functions \( K \) and \( H \) are given by:

\[ (e^\xi ANK')' = A \left( e^\xi K(K^2 - 1) x^2 + e^{-2\xi} H^2 K \right) , \]
\[ (e^{-2\xi} x^2 ANH')' = 2e^{-2\xi} K^2 AH , \]

where the prime denotes the derivative with respect to \( x \).
C. Boundary conditions

The boundary conditions at the regular horizon \( x = x_h \) read:

\[ N(x_h) = 0 \Rightarrow \mu(x_h) = \frac{x_h}{2} \]

(15)

and

\[ (N'K')|_{x=x_h} = \left[ \frac{K(K^2 - 1)}{x^2} + e^{-3\xi}H^2K \right]|_{x=x_h}, \]

(16)

\[ (N'H')|_{x=x_h} = \left( \frac{2}{x^2}K^2H \right)|_{x=x_h}, \]

(17)

\[ (N'\xi')|_{x=x_h} = \left\{ \frac{4}{3x^2}\alpha^2 \left[ e^{\xi} \left( N(K')^2 + \frac{(K^2 - 1)^2}{2x^2} \right) - 2e^{-2\xi} \left( \frac{1}{2}N(H')^2x^2 + H^2K^2 \right) \right] \right\}|_{x=x_h} \]

(18)

with \( A(x_h) < \infty \).

At infinity, finiteness of the energy and asymptotic flatness requires:

\[ A(\infty) = 1 \ , \ K(\infty) = 0 \ , \ H(\infty) = 1 \ , \ \xi(\infty) = 0 \ . \]

(19)

III. NUMERICAL RESULTS

The set of equations has been solved numerically [14]. These black hole solutions can be interpreted as 4-dimensional black holes sitting in the center of particle-like solutions extended into one extra dimension. Due to the non-triviality of the matter fields outside the regular horizon, these solutions possess “hair”.

First, \( x_h \) was fixed and the dependence of the solutions on the gravitational coupling \( \alpha \) was studied. The results are shown for \( x_h = 0.1 \) in Fig. 1 and 2. The behaviour for fixed \( x_h \) is very similar to that of the globally regular solutions [10, 11]. \( \alpha \) was increased from small values and the solutions exist up to a maximal value of the gravitational coupling, \( \alpha_{\text{max}}(x_h) \). For \( x_h = 0.1 \), we find that \( \alpha_{\text{max}}(x_h = 0.1) \approx 1.223 \). On this branch of solutions, the dimensionless mass \( \mu_{\infty}/\alpha^2 \) per unit length of the extra dimension decreases (see Fig 2).

As shown in Fig. 1, both the value of \( A(x_h) \) as well as the minimal value of the metric function \( N(x) \) in the interval \([x_h : \infty]\), \( N_{\text{min}} \), decrease from the flat space values \( A = 1 \), respectively \( N = 1 \) (see Fig. 1), while the value of \( \xi(x_h) \) increases from \( \xi = 0 \) (see Fig. 2). Starting from the maximal value \( \alpha_{\text{max}} \), a second branch of solutions exists up to a critical value of \( \alpha \), \( \alpha_{\text{cr}}(x_h = 0.1) \approx 0.3513 \). On this second branch \( A(x_h) \) and \( N_{\text{min}} \) decrease
further, while $\xi(x_h)$ now decreases, crosses zero and then becomes negative. The mass per unit length on this branch is higher than that on the first branch of solutions, see Fig. 2. Starting from this second branch of solutions, a third branch of solutions exists and extends to $\alpha^{(2)}_x(x_h = 0.1) \approx 0.408$. On this third branch, $A(x_h)$ decreases further to $\approx 0$, while $\xi(x_h)$ tends to $-\infty$. However, $N_{\text{min}}$ stays finite. The mass on this third branch is lower than that on the second, but differs so marginally that it cannot be seen in the plot.

As is clearly seen, the extend of the branches in $\alpha$ decreases, which is very similar to the globally regular case. It is likely that an infinite number of branches exists which extend around $\alpha \approx 0.4$.

Note that on the third branch of solutions, the function $K(x)$ starts to develop oscillations. In Fig. 3, the matter and metric functions close to the horizon $x_h = 0.1$ are shown for the solutions on the third branch for $\alpha = 0.408$. $N(x)$ has developed a local minimum. Clearly, the solution is non-trivial outside the regular horizon and thus represents a non-abelian black string.

The oscillations of the function $K(x)$ indicate that -like in the globally regular case- the solutions converge to the fixed point described in [10].

The fact that the solutions exist only up to a maximal value of the gravitational coupling was already observed in other gravitating systems containing non-abelian gauge fields, e.g. in the 4-dimensional Einstein-Yang-Mills-Higgs system [16]. The first branch of solutions can be interpreted as a branch on which the variation of $\alpha = 4\pi\sqrt{G_5}v$ corresponds to the variation of the gravitational coupling $G_5$, while $v$ is kept fixed. This can be concluded from the fact that the $\alpha = 0$ limit corresponds to the flat limit $G_5 = 0$. In contrast, on the further branches $G_5$ is kept fixed and $v$ varied. Since $v = 0$ would correspond effectively to a 4-dimensional Einstein-Yang-Mills-dilaton system, which has Bartnik-McKinnon-type solutions [17], with zeros of the gauge field functions and thus different boundary conditions, the $\alpha = 0$ limit doesn’t exist for these additional branches. The existence of a maximal value of $\alpha$ is thus related to the existence of a maximal possible value of the gravitational coupling $G_5$.

In Fig. 4, the domain of existence of the non-abelian black strings is shown. The maximal possible value $\alpha_{\text{max}}$ of the globally regular solutions is $\alpha_{\text{max}} = 1.268$ [10, 11]. From this value $\alpha_{\text{max}}(x_h)$ decreases with increasing $x_h$ and finally reaches $\alpha_{\text{max}}(x_h) = 0$ at $x_h \approx 0.654$. Non-abelian black strings thus exist only in a limited domain of the $x_h$-$\alpha$-plane.
IV. SUMMARY

Black holes in more than 4 dimensions have gained renewed interest in recent years due to theories of unification of all known forces. Since these theories assume that the extra dimensions are compactified on a circle of the Planck length, black hole solutions which are hyperspherically symmetric in the full dimensions are important at early stages in the universe. A large number of these solutions has been constructed and extensively discussed. For smaller energies, i.e. later stages in the evolution of the universe, black hole solutions which are spherically symmetric in 4 dimensions and extended trivially or non-trivially into the extra dimensions are certainly of more interest. Here, we have constructed the first example of a 5-dimensional black string including non-abelian gauge fields. These solutions are spherically symmetric black holes sitting in the center of particle-like solutions in 4 dimensions, which are trivially extended into a fifth dimension. We have shown that these solutions exist only in a limited domain in the $x_h$-$\alpha$-plane and that several branches of solutions (likely infinitely many) exist. In the limit of critical coupling these solutions converge -like the globally regular ones- to the fixed point described in [10].

As far as the stability of these solutions is concerned, a detailed analysis is certainly out of the scope of this paper. However, from Morse theory, we can assume that the second branch has one unstable mode more than the first (since it has higher energy) and one less than the third etc. Since for $\alpha = 0$, the globally regular counterparts of the solutions here are effectively the solutions of a 4-dimensional Yang-Mills-Higgs system in the BPS limit (which are known to be stable) it is likely that the globally regular counterparts on the first branch of solutions are stable, while they have $n - 1$ unstable modes on the $n$th branch. Furthermore, since for the 4-dimensional Einstein-Yang-Mills model the number of unstable modes is equal for the globally regular and the non-abelian black hole solutions [18], we expect the same to be true in the 5-dimensional analogue. This leads us to the assumption that the non-abelian black strings are stable on the first branch and unstable (with $n - 1$ unstable modes on the $n$th branch) on the other branches. Surely, this point has to be investigated in detail.

Further directions of investigation of the model studied here are the construction of 4-dimensional axially symmetric non-abelian black holes extended into one extra dimension or the non-trivial extension of these solutions into the extra dimension, i.e. the construction
of non-uniform, non-abelian black strings.

[1] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin (1921).
[2] O. Klein, Z. Phys. 37 (1926), 895.
[3] see e.g. J. Polchinski, String Theory, Cambridge University press (1998).
[4] K. Akama, “Pregeometry”, in Lecture Notes in Physics, 176, Gauge Theory and Gravitation, Proceedings, Nara, 1982, edited by K. Kikkawa, N. Nakanishi and H. Nariai, 267-271 (Springer-Verlag, 1983) \[\text{hep-th/0001113}\]; V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. 125 B (1983), 136; 125 B (1983), 139; G. Davli and M. Shifman, Phys. Lett. B 396 (1997), 64; 407 (1997), 452; I. Antoniadis, Phys. Lett. B 246 (1990), 377; N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B 429 (1998), 263; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B 436 (1998), 257; L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999), 3370; 83 (1999), 4690.
[5] F. R. Tangherlini, Nuovo Cimento 27 (1963), 636.
[6] R. C. Myers and M. J. Perry, Ann. Phys. (NY) 172 (1986), 304.
[7] R. Gregory and R. Laflamme, Phys. Rev. D 37 (1988), 305.
[8] R. Emparan and H. Reall, Phys. Rev. Lett. 88 (2002), 101101.
[9] Y. Brihaye, A. Chakrabarti, B. Hartmann and D. H. Tchrakian, Phys. Lett. B 561 (2003), 161.
[10] M. S. Volkov, Phys. Lett. B 524 (2002), 369.
[11] Y. Brihaye and B. Hartmann, Phys. Lett. B 534 (2002), 137.
[12] Y. Brihaye, F. Clement and B. Hartmann, Phys. Rev. D, in press \[\text{hep-th/0403041}\].
[13] G. ‘t Hooft, Nucl. Phys. B 79 (1974), 276; A. M. Polyakov, JETP Lett. 20 (1974), 194.
[14] To integrate the equations, we used the differential equation solver COLSYS which involves a Newton-Raphson method \[\text{15}\].
[15] U. Ascher, J. Christiansen and R. D. Russell, Math. Comput. 33 (1979), 659; ACM Trans. Math. Softw. 7 (1981), 209.
[16] P. Breitenlohner, P. Forgacs and D. Maison, Nucl. Phys. B 383 (1992),357; Nucl. Phys. B
442 (1999), 126.

[17] R. Bartnik and J. McKinnon, Phys. Rev. Lett. 61 (1988), 141.

[18] see e.g. M. S. Volkov and D. V. Gal’tsov, Phys. Rept. 319 (1999), 1.
FIG. 1: The value of the metric functions $A(x)$ at the horizon, $A(x_h)$, and the minimal value of the metric function $N(x)$ in the interval $[x_h : \infty]$, $N_{\text{min}}$, are shown in dependence on $\alpha$ for $x_h = 0.1$. Note that the figure on the right is an amplification of the domain indicated by the box in the figure on the left.
FIG. 2: The value of the metric function $\xi(x)$ at the horizon, $\xi(x_h)$, is shown in dependence on $\alpha$ for $x_h = 0.1$ (left). Also shown is the dimensionless mass $\mu_\infty/\alpha^2$ in dependence on $\alpha$ for $x_h = 0.1$ (right).
FIG. 3: The functions $N(x)$ (top left), $A(x)$ (top right), $K(x)$ and $H(x)$ (bottom left) and $\xi(x)$ (bottom right) are shown close to the horizon $x_h = 0.1$ for $\alpha = 0.408$. Note that this solution belongs to the third branch of solutions.
FIG. 4: The domain of existence of the non-abelian black strings is shown in the $x_h$-$\alpha$-plane.