A PERTURBATION APPROACH FOR PANEITZ ENERGY ON STANDARD THREE SPHERE

FENGBO HANG AND PAUL C. YANG

ABSTRACT. We present another proof of the sharp inequality for Paneitz operator on the standard three sphere, in the spirit of subcritical approximation for the classical Yamabe problem. To solve the perturbed problem, we use a symmetrization process which only works for extremal functions. This gives a new example of symmetrization for higher order variational problems.

1. INTRODUCTION

Fourth order Paneitz operator ([Br, P]) gains interest due to its role in the progress of four dimensional conformal geometry (see [CGY, HY5]). On a three dimensional Riemannian manifold \((M, g)\), it is given by

\[
P'\varphi = \Delta^2 \varphi + 4 \text{div}(Rc(\nabla \varphi, e_i) e_i) - \frac{5}{4} \text{div}(R\nabla \varphi) - \frac{1}{2}Q\varphi.
\]  

Here

\[
Q = -\frac{1}{4} \Delta R - 2 |Rc|^2 + \frac{23}{32} R^2,
\]

\(Rc\) is the Ricci curvature, \(R\) is the scalar curvature and \(e_1, e_2, e_3\) is a local orthonormal frame. The Paneitz operator satisfies the transformational law

\[
P_{\rho^{-4}g} \varphi = \rho^7 P_g (\rho \varphi)
\]

for any smooth positive function \(\rho\).

This is similar to the conformal Laplacian operator \(L\varphi = -\frac{4(n-1)}{n-2} \Delta \varphi + R\varphi\) on a Riemannian manifold with dimension \(n \geq 3\), which satisfies

\[
L_{\rho^{-\frac{4}{n-2}} g} \varphi = \rho^{\frac{n+2}{n-2}} L_g (\rho \varphi)
\]

for any positive smooth function \(\rho\).

On the standard sphere \(S^n\),

\[
L = -\frac{4(n-1)}{n-2} \Delta + n(n-1)
\]

is positive definite. Its associated quadratic form

\[
\int_{S^n} L\varphi \cdot \varphi d\mu = \int_{S^n} \left[ \frac{4(n-1)}{n-2} |\nabla \varphi|^2 + n(n-1) \varphi^2 \right] d\mu
\]

satisfies the sharp Sobolev inequality ([SY, Chapter 5])

\[
\inf_{\varphi \in H^1(S^n) \setminus \{0\}} \frac{\int_{S^n} \left[ \frac{4(n-1)}{n-2} |\nabla \varphi|^2 + n(n-1) \varphi^2 \right] d\mu}{\|\varphi\|_{L^\frac{2n}{n-2}}^2} = n(n-1)|S^n|^{\frac{2}{n}}.
\]  

(1.7)
Here $|S^n|$ is the volume of $S^n$. In another word the the functional minimizes at $\varphi = 1$.

On standard three sphere $S^3$,
\[ P\varphi = \Delta^2 \varphi + \frac{1}{2} \Delta \varphi - \frac{15}{16} \varphi \] (1.8)
and it has a negative first eigenvalue $-\frac{15}{16}$. Denote
\[ E(\varphi) = \int_{S^3} P\varphi \cdot \varphi d\mu \]
\[ = \int_{S^3} \left( (\Delta \varphi)^2 - \frac{1}{2} |\nabla \varphi|^2 - \frac{15}{16} \varphi^2 \right) d\mu, \] (1.9)
and we look at the Paneitz energy (see [HY1, section 1]) \[ \|\varphi^{-1}\|_{L^q}^2 E(\varphi) \] for $\varphi \in H^2(S^3)$ with $\varphi > 0$. Note the condition $\varphi > 0$ makes sense since $H^2(S^3) \subset C^{\frac{1}{2}}(S^3)$ by the Sobolev embedding theorem. It is not clear at all whether Paneitz energy even has a lower bound. Nevertheless it is proven in [YZ] that

**Theorem 1.1 ([YZ]).** For any $\varphi \in H^2(S^3)$ with $\varphi > 0$, we have
\[ \|\varphi^{-1}\|_{L^q}^2 \int_{S^3} \left( (\Delta \varphi)^2 - \frac{1}{2} |\nabla \varphi|^2 - \frac{15}{16} \varphi^2 \right) d\mu \geq -\frac{15}{16} |S^3|^\frac{1}{q}. \] (1.10)

In another word, the Paneitz energy minimizes at $\varphi = 1$.

All minimizers can be identified by the Liouville type theorem in [CX].

To continue, we observe that a classical way to derive (1.7) is by the subcritical approximation. If we try to prove the extremal problem (1.7) indeed has a minimizer, the direct method in calculus of variations meets the problem that the embedding $H^1(S^n) \subset L^{\frac{2n}{n-2}}(S^n)$ is not compact. On the other hand, for $2 < q < \frac{2n}{n-2}$, we have a compact embedding $H^1(S^n) \subset L^q(S^n)$, this together with standard argument shows we can find a positive smooth function $u$ such that
\[ \inf_{\varphi \in H^1(S^n) \setminus \{0\}} \frac{\int_{S^n} \left[ \frac{4(n-1)}{n-2} |\nabla \varphi|^2 + n(n-1) \varphi^2 \right] d\mu}{\|\varphi\|_{L^q}^2} \] (1.11)
is achieved at $u$. Moreover after scaling, $u$ satisfies
\[ Lu = u^{q-1} \text{ on } S^n. \] (1.12)
The method of moving plane ([GNN]) tells us (1.12) has only constant solution. Hence
\[ \inf_{\varphi \in H^1(S^n) \setminus \{0\}} \frac{\int_{S^n} \left[ \frac{4(n-1)}{n-2} |\nabla \varphi|^2 + n(n-1) \varphi^2 \right] d\mu}{\|\varphi\|_{L^q}^2} = n(n-1) |S^n|^{1-\frac{2}{q}}. \] (1.13)
Let $q \uparrow \frac{2n}{n-2}$, we get (1.7). We also note that this subcritical approximation process can be used in the solution to Yamabe problem (see [SY, Chapter 5]).

Now let us go back to (1.10). There have been several different ways to derive this inequality in [H, HY1, YZ], however none of them is in the spirit of the subcritical approximation approach above for (1.7). The main aim of this note is to provide such a method. As we will see soon, the solution to the approximation problem provides a new way of doing symmetrization for variational problems associated with higher order equations.
To motivate the formulation of the perturbation problem, we recall some observations in [HY1, XY]. Let us consider the extremal problem

$$\inf_{\varphi \in H^2(M), \varphi > 0} \|\varphi^{-1}\|_{L^6(M)}^2 E(\varphi);$$

on a smooth compact three dimensional Riemannian manifold $(M, g)$. We say the Paneitz operator satisfies condition P (see [HY1, Section 5]) if for any $\varphi \in H^2(M) \setminus \{0\}$, $\varphi(p) = 0$ for some $p$ would imply $E(\varphi) > 0$. Under condition P, the extremal problem (1.14) is achieved and the set of minimizers is compact under $C^\infty$ topology. Due to the noncompactness of Mobius transformation group on the standard $S^3$, the Paneitz operator can not satisfy condition P in this singular case. Nevertheless it satisfies the condition NN (see [HY1, Sections 5 and 7]): for any $\varphi \in H^2(M) \setminus \{0\}$, $\varphi(p) = 0$ for some $p$ would imply $E(\varphi) \geq 0$. The Green’s function of $P$ can be written as (see [HY1, HY6])

$$G_P(x, y) = -\frac{\|x - y\|}{8\pi}$$

for $x, y \in S^3$, the unit sphere in $\mathbb{R}^4$. In particular for any $x \in S^3$, $E(G_P(x, \cdot)) = 0$. In fact, the constant multiple of Green’s function are the only function $\varphi \in H^2(S^3)$ satisfying $\varphi(p) = 0$ for some $p$ and $E(\varphi) = 0$. To pass from condition NN to condition P, we replace the Paneitz operator $P$ by $P + \varepsilon$ for $\varepsilon > 0$ small. Denote

$$E_\varepsilon(\varphi) = \int_{S^3} (P(\varphi + \varepsilon \varphi) \varphi d\mu = E(\varphi) + \varepsilon \|\varphi\|_{L^2}^2.$$ 

Then we study the variational problem

$$\inf_{\varphi \in H^2(S^3), \varphi > 0} \|\varphi^{-1}\|_{L^6}^2 E_\varepsilon(\varphi).$$

It is straightforward to show the extremal problem has a minimizer (see Lemma 2.1). If $u$ is a minimizer, then $u$ must be positive smooth functions and after scaling it satisfies

$$Pu + \varepsilon u = -u^{-7} \text{ on } S^3.$$ 

If we can prove the Liouville type theorem that every solution to (1.18) must be constant function, then we know for any $\varphi \in H^2(S^3)$, $\varphi > 0$ and $\varepsilon > 0$ small,

$$\|\varphi^{-1}\|_{L^6}^2 \left(E(\varphi) + \varepsilon \|\varphi\|_{L^2}^2 \right) \geq |S^3|^\frac{3}{2} \left(E(1) + \varepsilon |S^3|\right).$$

Letting $\varepsilon \downarrow 0$, we get (1.10). But unfortunately we can not verify such a Liouville theorem at this stage. There are many evidence such a statement should be correct. We put it as a conjecture.

**Conjecture 1.1.** Let $P = \Delta^2 + \frac{1}{2} \Delta - \frac{15}{16}$ be the Paneitz operator on the standard three sphere, $\varepsilon > 0$ be a small positive number. If $u$ is a positive smooth function on $S^3$ such that $Pu + \varepsilon u = -u^{-7}$, then $u$ must be a constant function.
Without such a Liouville type result, we need another approach to show 1 is a minimizer to the perturbed variational problem (1.17). This consists the main part of the note. Our approach is motivated from [BT, HY4, HWY, R]. At first we would like to show that every minimizer must be radial symmetric and decreasing with respect to some point on $S^3$. This will be achieved by a symmetrization process which only works for minimizers (in contrast, usual symmetrization process works for all test functions (see [PS])). Next it follows from the usual Kazdan-Warner type identity that any minimizer must be a constant function. It is worth pointing out the method of symmetrization usually does not work well for higher order variational problems or higher order Sobolev spaces. The point of our approach lies in that it provides a new way to overcome the difficulty.

2. The perturbation problem

For $\varepsilon > 0$ small, we denote

$$-s_\varepsilon = \inf_{\varphi \in H^2(S^3), \varphi > 0} \|\varphi^{-1}\|_{L^6}^2 E_\varepsilon (\varphi) .$$

(2.1)

Here $E_\varepsilon (\varphi)$ is defined in (1.16). Since $E_\varepsilon (1) < 0$, by our choice of notation we see $s_\varepsilon > 0$ and

$$s_\varepsilon = \sup_{\varphi \in H^2(S^3), \varphi > 0} \|\varphi^{-1}\|_{L^6}^2 (-E_\varepsilon (\varphi)) .$$

(2.2)

Lemma 2.1. For $\varepsilon > 0$ small, there exists a $u \in C^\infty (S^3)$ such that $u > 0$ and $\|u^{-1}\|_{L^6}^2 E_\varepsilon (u) = -s_\varepsilon$. By scaling we can assume $\|u^{-1}\|_{L^6} = 1$, then

$$Pu + \varepsilon u = -s_\varepsilon u^{-7} .$$

(2.3)

Proof. We can find a sequence $u_i \in H^2 (S^3)$ such that $u_i > 0$ and $\|u_i^{-1}\|_{L^6}^2 E_\varepsilon (u_i) \to -s_\varepsilon$. By scaling we can assume $\max_{S^3} u_i = 1$. Since $E_\varepsilon (u_i) < 0$, we see $\|u_i\|_{H^2(S^3)} \leq c$. Hence after passing to a subsequence there exists a $u \in H^2 (S^3)$ with $u_i \rightharpoonup u$ weakly in $H^2 (S^3)$. It follows that $u_i \to u$ uniformly on $S^3$ and hence $\max_{S^3} u = 1$. Note that $E_\varepsilon (u) \leq 0$, hence $u$ must be strictly positive everywhere. Indeed if $u (p) = 0$ for some $p$, it follows from the fact $P$ satisfies condition NN (see [HY1, Sections 5 and 7]) that $E (u) \geq 0$, hence $E_\varepsilon (u) \geq \varepsilon \|u\|_{L^2}^2 > 0$, a contradiction.

Since $u$ is positive everywhere, we see $u_i^{-1} \to u^{-1}$ uniformly on $S^3$. Hence

$$-s_\varepsilon \leq \|u_i^{-1}\|_{L^6}^2 E_\varepsilon (u_i) \leq \liminf_{i \to \infty} \|u_i^{-1}\|_{L^6}^2 E_\varepsilon (u_i) = -s_\varepsilon .$$

In another word, $u$ is a minimizer.

After scaling we can assume $\|u^{-1}\|_{L^6} = 1$, it is clear that $u$ satisfies (2.3) weakly. Standard elliptic theory tells us $u$ must be smooth. 

3. Basics for symmetrization on the sphere

We discuss symmetrization on $S^n$ for all $n$ since there is no difference between different $n$’s. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. For $x, y \in \mathbb{R}^{n+1}$, $x \cdot y$ denotes the usual dot product. Let $N = (0, \cdots, 0, 1)$ be the north pole and $S = (0, \cdots, 0, -1)$ be the south pole. We write $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}$. 

Definition 3.1. Fix a unit vector \( u \in S^n \). We say a function \( f : S^n \to \mathbb{R} \) is radial symmetric and decreasing with respect to \( u \) if there exists a function \( \phi : [-1, 1] \to \mathbb{R} \) such that \( \phi \) is increasing and \( f(x) = \phi(x \cdot u) \) for all \( x \in S^n \). Similarly we have the notion of radial symmetric and increasing with respect to \( u \).

Let \( f : S^n \to \mathbb{R} \) be a measurable function, then its symmetrization \( f^* : S^n \to \mathbb{R} \) is the unique function satisfying the following conditions:

- \( f^* \) is radial symmetric and decreasing with respect to the north pole \( N \) and for any \( t \in \mathbb{R} \),
  \[ |f > t| = |f^* > t| . \]
- Here for any set \( E \subset S^n \), \( |E| \) denotes the measure of \( E \).
- \( f^* |_{S^n \setminus \{ \} } \) is lower semicontinuous and \( f^* (S) = \text{essinf}_{S^n} f \). Here \( S \) is the south pole.

We also recall some notations from [HY5, section 3]. If \( K = K(x, y) \) is a function on \( S^n \times S^n \), then we say \( K \) is a kernel (function) on \( S^n \). If \( K \) is a nice kernel on \( S^n \) and \( f \) is a nice function on \( S^n \), then we write

\[
(T_K f)(x) = \int_{S^n} K(x, y) f(y) \, d\mu(y) .
\]  

(3.1)

If \( K' \) is another kernel on \( S^n \), then we define their convolution as

\[
(K * K')(x, z) = \int_{S^n} K(x, y) K'(y, z) \, d\mu(y) .
\]  

(3.2)

A special class of kernel is pointed out in [BT, section 2]: if \( k \) is a function on the interval \([-1, 1]\), then we have a kernel \( K_k \) on \( S^n \) given by

\[
K_k(x, y) = k(x \cdot y) .
\]  

(3.3)

The orthogonal group \( O(n + 1) \) acts naturally on \( S^n \times S^n \) as \( R \cdot (x, y) = (Rx, Ry) \) for any \( R \in O(n + 1) \), \( x, y \in S^n \). The above class of kernels are exactly those kernels invariant under the group action. So we call this kind of kernel as invariant kernel. It is easy to see the convolution of invariant kernels are still invariant. If \( k_1 \) and \( k_2 \) are nice functions on \([-1, 1]\), then we denote the function on \([-1, 1]\) associated with \( K_{k_1} * K_{k_2} \) as \( k_1 * k_2 \). In another word, for any \( t \in [-1, 1] \),

\[
(k_1 * k_2)(t) = \int_{S^n} k_1(x \cdot y) k_2(y \cdot z) \, d\mu(y)
\]  

(3.4)

for any \( x, z \in S^n \) with \( x \cdot z = t \). Note that \( k_1 * k_2 \) depends on \( n \), we will use \( k_1 * k_2 \) when confusion could happen.

Lemma 3.1. If \( k_1 \) and \( k_2 \) are both increasing functions on \([-1, 1]\), then so is \( k_1 * k_2 \).

Proof. Without losing of generality, we can assume both \( k_1 \) and \( k_2 \) are smooth functions. Fix two perpendicular unit vectors \( u \) and \( v \). For \( 0 < \theta < \pi \), we denote

\[
p(\theta) = u \cos \theta + v \sin \theta ,
\]  

(3.5)

then

\[
p' (\theta) = -u \sin \theta + v \cos \theta .
\]  

(3.6)

By definition of \( k_1 * k_2 \),

\[
(k_1 * k_2)(\cos \theta) = \int_{S^n} k_1(u \cdot x) k_2(x \cdot p(\theta)) \, d\mathcal{H}^n(x) .
\]  

(3.7)
Here $\mathcal{H}^n$ denotes the Hausdorff measure. We use this notation instead of $\mu$ to make the application of coarea formula easier. Differentiate with respect to $\theta$, we get

$$- \sin \theta (k_1 * k_2)' (\cos \theta)$$

$$= \int_{S^n} k_1 (u \cdot x) k_2' (x \cdot p) (x \cdot p') d\mathcal{H}^n (x)$$

$$= \int_{-1}^{1} \frac{k_2' (\tau)}{\sqrt{1 - \tau^2}} \int_{x \cdot p = \tau}^{x \cdot p = \tau} k_1 (u \cdot x) (x \cdot p') d\mathcal{H}^{n-1} (x).$$

We have applied the coarea formula in the second equality. Now Lemma 3.1 follows from the following claim.

Claim 3.1.

$$\int_{x \in S^n, x \cdot p = \tau} k_1 (u \cdot x) (x \cdot p') d\mathcal{H}^{n-1} (x) \leq 0.$$  (3.9)

Indeed, we have the reflection

$$y = x - 2 (x \cdot p') p'.$$  (3.10)

Note that

$$\int_{x \in S^n, x \cdot p = \tau} k_1 (u \cdot x) (x \cdot p') d\mathcal{H}^{n-1} (x)$$

$$= \int_{x \in S^n, x \cdot p = \tau, x \cdot p' > 0} k_1 (u \cdot x) (x \cdot p') d\mathcal{H}^{n-1} (x) + \int_{x \in S^n, x \cdot p = \tau, x \cdot p' < 0} k_1 (u \cdot x) (x \cdot p') d\mathcal{H}^{n-1} (x)$$

$$= \int_{x \in S^n, x \cdot p = \tau, x \cdot p' > 0} k_1 (u \cdot x) (x \cdot p') d\mathcal{H}^{n-1} (x)$$

$$- \int_{y \in S^n, y \cdot p = \tau, y \cdot p' < 0} k_1 (u \cdot (y - 2 (y \cdot p') p')) (y \cdot p') d\mathcal{H}^{n-1} (y)$$

$$= \int_{x \in S^n, x \cdot p = \tau, x \cdot p' > 0} (k_1 (u \cdot x) - k_1 (u \cdot (x - 2 (x \cdot p') p'))) (x \cdot p') d\mathcal{H}^{n-1} (x)$$

$$\leq 0$$

because $u \cdot p' = - \sin \theta \leq 0$ and $k_1$ is increasing. \qed

For a function $k$ on $[-1, 1]$, we denote

$$T_k = T_{k_1}.$$  (3.12)

In another word,

$$(T_k f) (x) = \int_{S^n} k (x \cdot y) f (y) d\mu (y).$$  (3.13)

- $T_{k 1} = const$ i.e. $1$ is an eigenfunction of $T_k$.

- If we denote $\lambda = T_{k 1}$, then for any nice function $f$ on $S^n$,

$$\int_{S^n} T_k f (x) d\mu (x) = \lambda \int_{S^n} f d\mu.$$  (3.14)
For $R \in O(n + 1)$, we let

$$f_R(x) = f(Rx)$$

(3.15)

for all $x \in S^n$. Then

$$T_k(f_R) = (T_k f)_R.$$  

(3.16)

As a consequence, we see if $u \in S^n$ and $f$ is radial symmetric with respect to $u$, then so is $T_k f$.

**Lemma 3.2.** Assume $k$ is increasing function on $[-1,1]$, $u \in S^n$, $f : S^n \to \mathbb{R}$ is symmetric and decreasing with respect to $u$, then so is $T_k f$.

**Proof.** We assume $f(x) = \phi(x \cdot u)$. Then there exists a function $\psi$ such that $T_k f(x) = \psi(x \cdot u)$. Without losing of generality we can assume $k$ and $f$ are both smooth functions. Choose a unit vector $v$ such that $u \cdot v = 0$, let $p(\theta) = u \cos \theta + v \sin \theta$, then

$$\psi(\cos \theta) = T_k f(p(\theta)) = \int_{S^n} k(p \cdot x) \phi(x \cdot u) dH^n(x).$$

The argument in the proof of Lemma 3.1 tells us $\psi$ is increasing. Lemma 3.2 follows. \qed

Later on we will need an analogue of the classical Riesz rearrangement inequality proved in [BT].

**Proposition 3.1** (Theorem 2 in [BT]). Let $k$ be a bounded increasing function on $[-1,1]$, then for any $f, g \in L^1(S^n)$, we have

$$\int_{S^n} \int_{S^n} k(x \cdot y) f(x) g(y) d\mu(x) d\mu(y) \leq \int_{S^n} \int_{S^n} k(x \cdot y) f^*(x) g^*(y) d\mu(x) d\mu(y).$$

(3.17)

Here we are going to discuss some results on when the equality is achieved. We will not aim for most general statements, but restrict to case sufficient for our purpose.

**Lemma 3.3.** If $k$ is a bounded strictly increasing function on $[-1,1]$ and $f, g \in C(S^n, \mathbb{R})$ such that

$$\int_{S^n} \int_{S^n} k(x \cdot y) f(x) g(y) d\mu(x) d\mu(y) = \int_{S^n} \int_{S^n} k(x \cdot y) f^*(x) g^*(y) d\mu(x) d\mu(y)$$

(3.18)

and both $f$ and $g$ are not constant functions, then $f$ and $g$ must be radial symmetric and decreasing with respect to the same point.

To continue, we recall some basic operations in [BT]. Let $u$ be a unit vector, then we have a half space

$$H = \{x \in \mathbb{R}^{n+1} : x \cdot u \geq 0\}.$$
Note that $u$ is simply the inner normal direction at $\partial H$. Let $R$ be the reflection with respect to $\partial H$. For any function $f$ on $S^n$, we denote
\[
    f_H(x) = \begin{cases} 
        \max \{ f(x), f(Rx) \}, & \text{if } x \in H \cap S^n; \\
        \min \{ f(x), f(Rx) \}, & \text{if } x \in S^n \setminus H.
    \end{cases}
\]
Note that $f_H$ has the same distribution functions as $f$. In particular, $(f_H)^* = f^*$.

**Lemma 3.4.** If $f \in C(S^n, \mathbb{R})$ such that for every half space $H$ and $x, y \in S^n \cap H$ we have
\[
    (f(x) - f(Rx))(f(y) - f(Ry)) \geq 0,
\]
here $R$ is the reflection with respect to $\partial H$, then $f$ must be radial symmetric and decreasing with respect to some point.

**Proof.** At first we note that if $x \in S^n \cap H$ such that $f(x) > f(Rx)$, then for every $y \in S^n \cap H$, $f(y) \geq f(Ry)$. We will use this assertion repeatedly.

The lemma will be proved by induction on $n$. First assume $n = 1$. In this case we identify $\mathbb{R}^2$ with $\mathbb{C}$ as usual. We can assume $f$ is not a constant function, then we fix a number $t$ such that $\min_{S^1} f < t < \max_{S^1} f$. Let $E = \{ f < t \}$ and the connected components of $E$ are open arcs. After rotation, we can assume one connected component is written as $\{ e^{i\theta} : -\delta < \theta < \delta \}$, here $0 < \delta < \pi$. For any $0 < \alpha < \pi$, we claim $f(e^{i\alpha}) = f(e^{-i\alpha})$, that is, $f$ is symmetric with respect to the real axis. Indeed, for $0 < \varepsilon < \min \{ \alpha, \delta \}$, let $H$ be the top half plane cut by line in the direction $e^{i\varepsilon}$. Since $f(e^{i\theta}) = t > f(e^{i(-\delta+\varepsilon)})$, we see $f(e^{i\alpha}) \geq f(e^{i(-\alpha+\varepsilon)})$. Let $\varepsilon \downarrow 0$, we see $f(e^{i\alpha}) \geq f(e^{-i\alpha})$. Repeat the same argument with $e^{-i\varepsilon}$ in place with $e^{i\varepsilon}$, we see $f(e^{-i\alpha}) \geq f(e^{i\alpha})$. The claim follows. Next we want to show if $0 \leq \beta < \alpha \leq \pi$, then $f(e^{i\alpha}) \geq f(e^{i\beta})$ if $\alpha - \beta < \delta$. As a consequence, we know $f$ is radial symmetric and decreasing with respect to $-1$. Indeed let $H$ be the top half plane cut by line in the direction $e^{i\frac{\pi}{2}}$. Since $f(e^{i\varepsilon}) > f(e^{i(-\delta+\alpha-\beta)})$, we see $f(e^{i\alpha}) \geq f(e^{-i\beta}) = f(e^{i\beta})$. The claim follows.

Assume for some $n \geq 2$, the lemma is correct for functions on $S^{n-1}$, we want to show it is also valid on $S^n$. Again we assume $f$ is not a constant function and $\min_{S^n} f < t < \max_{S^n} f$. Let $E = \{ f < t \}$. For $p \in S^n$ and $0 < r \leq \pi$, we denote $B_r^S(p)$ as the open geodesic ball of radius $r$ on $S^n$ with center $p$. We can find a geodesic ball with largest radius among all such balls contained in $E$. By rotation we assume it is given by $B_{\delta}^S(S)$, here $S$ is the south pole. Then there two different points $p_1, p_2$ in $\partial B_{\delta}^S(N) \cap \partial E$. Note that $f(p_1) = f(p_2) = t$. Let $L$ be any hyperplane (which may not pass the origin) in $\mathbb{R}^{n+1}$ containing $p_1$ and $p_2$, then by applying the induction hypothesis on $f|_{L \setminus S^n}$, we conclude that $f|_{S^n \setminus B_{\delta}^S(S)} \geq t$. Hence $f|_{\partial B_{\delta}^S(S)} = t$.

Now we claim $f$ is radial symmetric and decreasing with respect to $N$.

Indeed assume $0 < \alpha < \pi$, $u', u'' \in S^{n-1}$ and $u' \neq u''$, we will show
\[
    f(u' \sin \alpha, \cos \alpha) = f(u'' \sin \alpha, \cos \alpha).
\]
For $\alpha < \beta < \pi$ and $\beta$ close to $\alpha$, we will show
\[
    f(u' \sin \alpha, \cos \alpha) \geq f(u'' \sin \beta, \cos \beta).
\]
Let $\beta \downarrow \alpha$, we get
\[
    f(u' \sin \alpha, \cos \alpha) \geq f(u'' \sin \alpha, \cos \alpha).
\]
Switching $u'$ and $u''$, we get the inequality in the other direction and hence the equality. Let
\begin{equation}
 u = \frac{(u' \sin \alpha, \cos \alpha) - (u'' \sin \beta, \cos \beta)}{||(u' \sin \alpha, \cos \alpha) - (u'' \sin \beta, \cos \beta)||}.
\end{equation}
Since $\beta$ is close to $\alpha$, $u$ is close to $(\frac{u'-u''}{||u'-u'||}, 0)$. Let $H = \{x \in \mathbb{R}^{n+1} : x \cdot u \geq 0\}$, and $R$ be the reflection with respect to $\partial H$. Let
\begin{equation}
 x = \left(\frac{u' - u''}{||u' - u''||} \sin \delta, -\cos \delta\right),
\end{equation}
note that $Rx = x - 2(x \cdot u)u$. Since $x \cdot u$ is close to $\sin \delta$, we know $x \cdot u > 0$. Hence
\begin{equation}
 (Rx)_{n+1} = -\cos \delta - 2(x \cdot u)u_{n+1} < -\cos \delta
\end{equation}
and it follows that $Rx \in B_\delta^S (S)$. In particular $f(x) > f(Rx)$. Since
\begin{equation}
 (u' \sin \alpha, \cos \alpha) \cdot u > 0
\end{equation}
and
\begin{equation}
 R(u' \sin \alpha, \cos \alpha) = (u'' \sin \beta, \cos \beta),
\end{equation}
we see
\begin{equation}
 f(u' \sin \alpha, \cos \alpha) \geq f(u'' \sin \beta, \cos \beta).
\end{equation}
It follows that $f$ is radial symmetric and decreasing with respect to $N$. \hfill \Box

Next we generalize the Lemma 3.4 to two functions setting.

**Lemma 3.5.** If $f, g \in C(S^n, \mathbb{R})$ such that for every half space $H$ and $x, y \in S^n \cap H$ we have
\begin{equation}
 (f(x) - f(Rx))(g(y) - g(Ry)) \geq 0,
\end{equation}
here $R$ is the reflection with respect to $\partial H$, and both $f$ and $g$ are not constant functions, then $f$ and $g$ must be radial symmetric and decreasing with respect to the same point.

To reduce Lemma 3.5 to Lemma 3.4, we need some basic facts about generators of the orthogonal group $O(n + 1)$. Let $u \in S^n$, we have the hyperplane
\begin{equation}
 u^\perp = \{x \in \mathbb{R}^{n+1} : x \cdot u = 0\}.
\end{equation}
The reflection with respect to $u^\perp$ is given by
\begin{equation}
 R_u x = x - 2(x \cdot u)u.
\end{equation}

**Lemma 3.6.** For any $u_0 \in S^n$ and $\varepsilon > 0$, the set
\begin{equation}
 \{R_u : u \in S^n, ||u - u_0|| < \varepsilon\}
\end{equation}
generates $O(n + 1)$.

**Proof.** We prove by induction on $n$. For convenience we denote $G$ as the subgroup generated by the set $\{R_u : u \in S^n, ||u - u_0|| < \varepsilon\}$. We hope to show $G = O(n + 1)$.
Assume $n = 1$. As usual we identify $\mathbb{R}^2$ with $\mathbb{C}$. We can write $u_0 = e^{i\alpha_0}$, then for any $0 < \delta < \varepsilon$, we have
\begin{equation}
 |e^{i(\alpha_0+\delta)} - e^{i\alpha_0}| < \delta < \varepsilon
\end{equation}
and
\begin{equation}
 R_{e^{i(\alpha_0+\delta)}} R_{e^{i\alpha_0}} e^{it} = e^{i(2\delta + \theta)}.
\end{equation}
Hence all rotation lies in $G$. This together with the fact $R_{u_0} \in G$ implies $G = O(2)$.

Assume for some $n \geq 2$, the lemma is true for $O(n)$, we will show it is also true for $O(n+1)$. Give $v \in S^n$ satisfying $v \perp u_0$, if $O \in O(n+1)$ such that $Ov = v$, then $O|_{v^\perp} : v^\perp \to v^\perp$ is an orthogonal transformation too. By induction hypothesis we know $O \in G$. Next for any $u \in S^n$, using $n \geq 2$, we can find $v \in S^n$ such that $v \perp u$ and $v \perp u_0$. It follows that $R_u v = v$ and hence $R_u \in G$. Since every orthogonal transformation is a finite composition of reflections, we see $G = O(n+1)$. \hfill \Box

Now we proceed to prove Lemma 3.5.

**Proof of Lemma 3.5.** We first claim for every half space $H$ and $x, y \in S^n \cap H$ we have

$$(f(x) - f(Rx))(f(y) - f(Ry)) \geq 0.$$  

Indeed if this is not the case, then for some $H$ and $x_0, y_0 \in H \cap S^n$ such that

$$(f(x_0) - f(Rx_0))(f(y_0) - f(Ry_0)) < 0.$$  

We may assume $f(x_0) > f(Rx_0)$ and $f(y_0) < f(Ry_0)$. Then for any $x \in H \cap S^n$,

$$
\begin{align*}
(f(x_0) - f(Rx_0))(g(x) - g(Rx)) &\geq 0; \\
(f(y_0) - f(Ry_0))(g(x) - g(Rx)) &\geq 0.
\end{align*}
$$  

It follows that $g(x) = g(Rx)$. Hence $g = g_R$. Let $u_0$ be the inner normal direction of $H$, then $x_0 \cdot u_0 > 0$, $y_0 \cdot u_0 > 0$. For $\varepsilon > 0$ small and $\|u - u_0\| < \varepsilon$, we still have $x_0 \cdot u > 0$, $y_0 \cdot u > 0$, $f(x_0) > f(R_u x_0)$ and $f(y_0) < f(R_u y_0)$. It follows that $g = g_{R_u}$. By Lemma 3.6 all these $R_u$'s generate $O(n+1)$, hence $g = g_O$ for any $O \in O(n+1)$. This implies $g$ must be a constant function and it contradicts with the assumption.

By Lemma 3.4 we know $f$ must be radial symmetric and decreasing with respect to some point on $S^n$. Without losing of generality we assume that point in the north pole $N$. Since $f$ is not a constant function, we can find $t$ such that $f(S) < t < f(N)$. Then

$$\{f < t\} = B_\delta^S (S)$$

for some $\delta \in (0, \pi)$. The argument at the end of proof of Lemma 3.4 tells us $g$ must be radial symmetric and decreasing with respect to $N$ too. \hfill \Box

We are ready to prove Lemma 3.3.

**Proof of Lemma 3.3.** We prove it by a contradiction argument. If the conclusion is not true, then by Lemma 3.5 we can find a half space $H$ and $x_0, y_0 \in H \cap S^n$ such that

$$(f(x_0) - f(Rx_0))(g(y_0) - g(Ry_0)) < 0.$$  

Here $R$ denotes the reflection with respect to $\partial H$. Without losing of generality we assume $f(x_0) > f(Rx_0)$ and $g(y_0) < g(Ry_0)$. Note that $x_0, y_0 \notin \partial H$. By continuity of $f$ and $g$ we can find a neighborhood of $x_0$ in $S^n$, namely $U(x_0)$, such that $U(x_0) \subset H$ and for every $x \in U(x_0)$, $f(x) > f(Rx)$. Similarly, we find $U(y_0) \subset H$ such that for every $y \in U(y_0)$, $g(y) < g(Ry)$. For function $F$ and $G$
on \( S^n \), we have
\[
\int_{S^n} \int_{S^n} k(x \cdot y) F(x) G(y) \, d\mu(x) \, d\mu(y)
= \int_{H \cap S^n} \int_{H \cap S^n} [k(x \cdot y)(F(x) G(y) + F(Rx) G(Ry))
+ k(Rx \cdot y)(F(Rx) G(y) + F(x) G(Ry))] \, d\mu(x) \, d\mu(y). 
\]
A careful but elementary calculation shows (see [BT, proof of lemma 1]) for \( x, y \in H \cap S^n \),
\[
k(x \cdot y)(f(x) g(y) + f(Rx) g(Ry)) + k(Rx \cdot y)(f(Rx) g(y) + f(x) g(Ry))
\leq k(x \cdot y)(f_H(x) g_H(y) + f_H(Rx) g_H(Ry))
+ k(Rx \cdot y)(f_H(Rx) g_H(y) + f_H(x) g_H(Ry)).
\]
For \( x \in U(x_0) \) and \( y \in U(y_0) \), this inequality is strict. Indeed in this case,
\[
\text{RHS} - \text{LHS} = (k(x \cdot y) - k(Rx \cdot y))(f(x) - f(Rx))(g(Ry) - g(y)) > 0.
\]
Here RHS and LHS mean the right hand side and left hand side respectively of the above inequality. It follows that
\[
\int_{S^n} \int_{S^n} k(x \cdot y) f(x) g(y) \, d\mu(x) \, d\mu(y)
< \int_{S^n} \int_{S^n} k(x \cdot y) f_H(x) g_H(y) \, d\mu(x) \, d\mu(y)
\leq \int_{S^n} \int_{S^n} k(x \cdot y) (f_H)^*(x) (g_H)^*(y) \, d\mu(x) \, d\mu(y)
= \int_{S^n} \int_{S^n} k(x \cdot y) f^*(x) g^*(y) \, d\mu(x) \, d\mu(y).
\]
This gives us a contradiction with the equality (3.18).

4. Every minimizer of perturbation problem must be radial symmetric and decreasing

Recall that for any \( O \in O(4) \) and function \( f \) on \( S^3 \), we write \( f_O(x) = f(0x) \) for \( x \in S^3 \). The Paneitz operator \( P \) is invariant under the orthogonal group i.e. for any \( f \in C^\infty(S^3) \) and \( O \in O(4) \), \( (Pf)_O = Pf_O \). Hence the Green’s function \( G_P \) is invariant too. For convenience we denote
\[
K = -G_P.
\]
Then
\[
K(x, y) = \frac{\|x - y\|}{8\pi} \quad \text{for } x, y \in S^3.
\]
The associated single variable function \( k \) is given by
\[
k(t) = \frac{\sqrt{1-t^2}}{4\sqrt{2\pi}} \quad \text{for } t \in [-1, 1].
\]
Similarly for $\varepsilon > 0$ small, $G_{P+i\varepsilon}$ is also invariant. Let $K_\varepsilon = -G_{P+i\varepsilon}$ and the associated single variable function is denoted as $k_\varepsilon$. Since
\[ (P + \varepsilon)^{-1} = \sum_{j=0}^{\infty} (-\varepsilon)^j (P^{-1})^{j+1}, \] we see
\[ G_{P+i\varepsilon} = \sum_{j=0}^{\infty} (-\varepsilon)^j G_{P} \cdots \cdots G_{P}. \] It follows that
\[ K_\varepsilon = \sum_{j=0}^{\infty} \varepsilon^j K \cdots \cdots K \] and
\[ k_\varepsilon = \sum_{j=0}^{\infty} \varepsilon^j k \cdots \cdots k. \]

**Lemma 4.1.** For $\varepsilon > 0$ small, $k_\varepsilon$ is strictly decreasing on $[-1, 1]$.

**Proof.** We have
\[ k_\varepsilon = k + \varepsilon k \ast k + \sum_{j=1}^{\infty} \varepsilon^{2j} k \ast \cdots \ast k \ast (k + \varepsilon k \ast k). \] By Lemma 3.1 we only need to know $k + \varepsilon k \ast k$ is strictly decreasing. By the formula of $k$ in (4.3) we only need to show $k \ast k \in C^1([-1, 1])$. As in the proof of Lemma 3.1 we fix two perpendicular unit vectors $u$ and $v$ in $\mathbb{R}^4$. For $0 < \theta < \pi$,
\[ p(\theta) = u \cos \theta + v \sin \theta, \] then
\[ p'(\theta) = -u \sin \theta + v \cos \theta. \] We have
\[ (k \ast k)(\cos \theta) = \int_{S^3} k(u \cdot x) k(x \cdot p(\theta)) d\mathcal{H}^3(x). \] Here $k$ is given in (4.3) and
\[ k'(t) = -\frac{1}{8\sqrt{2\pi}} \frac{1}{\sqrt{1 - t}} \text{ for } -1 \leq t < 1. \] Hence for $0 < \theta < \pi$,
\[ -\sin \theta (k \ast k)'(\cos \theta) = \int_{S^3} k(u \cdot x) k'(x \cdot p) \cdot (x \cdot p') d\mathcal{H}^3(x). \] For convenience we denote
\[ \phi(\theta) = \int_{S^3} k(u \cdot x) k'(x \cdot p) \cdot (x \cdot p') d\mathcal{H}^3(x) \] for $0 \leq \theta \leq \pi$. It is clear that $\phi \in C([0, \pi])$. We need to show $\frac{\phi(\theta)}{\sin \theta}$ has a limit at both 0 and $\pi$. Let $O$ be an orthogonal matrix with $Ou = p$ and $Ov = p'$. Then
\[ O^{-1}u = u \cos \theta - v \sin \theta. \]
We make a change of variable $x = Oy$ in (4.14) and get

$$\phi(\theta) = \int_{S^3} k\left(y \cdot (u\cos \theta - v \sin \theta)\right) k'(y \cdot u) \cdot (y \cdot v) \, d\mathcal{H}^3(y).$$  \tag{4.16}$$

It follows from this formula that $\phi \in C^1([0, \pi])$. Moreover

$$\phi(0) = \int_{S^3} k(x \cdot u) k'(x \cdot u) \cdot (x \cdot v) \, d\mathcal{H}^3(x) = 0,$$  \tag{4.17}

$$\phi(\pi) = \int_{S^3} k(-x \cdot u) k'(x \cdot u) \cdot (x \cdot v) \, d\mathcal{H}^3(x) = 0,$$  \tag{4.18}

hence $\frac{\phi}{\sin \theta} \in C([0, \pi])$. Lemma 4.1 follows.

**Remark 4.1.** Here is another way to prove Lemma 4.1. First we observe that $K_\varepsilon(x, y) - K(x, y)$ is $C^2$ is variable ($\varepsilon, x, y$). We have

$$\psi(\theta) = K\left((\sin \theta, 0, 0, \cos \theta), (0, 0, 0, 1)\right) = k(\cos \theta)$$ \tag{4.19}$$

and

$$\psi_\varepsilon(\theta) = K_\varepsilon\left((\sin \theta, 0, 0, \cos \theta), (0, 0, 0, 1)\right) = k_\varepsilon(\cos \theta).$$ \tag{4.20}$$

Note that $\psi_\varepsilon(\theta) - \psi(\theta)$ in $C^2$ in $(\varepsilon, \theta)$. Since for $t \geq 0$ small, $\psi_\varepsilon(\pi - t) = \psi_\varepsilon(\pi + t)$, we see $\psi'_\varepsilon(\pi) = 0$. On the other hand, for $0 \leq \theta \leq \pi$,

$$\psi(\theta) = \frac{1}{4\pi} \sin \frac{\theta}{2}. \tag{4.21}$$

We have

$$\psi'(\theta) = \frac{1}{8\pi} \cos \frac{\theta}{2} > 0$$ \tag{4.22}$$

for $0 \leq \theta < \pi$ and

$$\psi''(\theta) = -\frac{1}{16\pi} \sin \frac{\theta}{2} < 0$$ \tag{4.23}$$

for $0 < \theta \leq \pi$. These together with the fact $\psi'_\varepsilon(\pi) = 0$ implies $\psi'_\varepsilon(\theta) > 0$ for $0 \leq \theta < \pi$ if $\varepsilon$ is small enough. Hence $\psi_\varepsilon$ is strictly increasing on $[0, \pi]$ and $k_\varepsilon$ is strictly decreasing.

Let $u$ be a minimizer for (2.1) with $\|u^{-1}\|_{L^6} = 1$, then

$$Pu + \varepsilon u = -s_\varepsilon u^{-7}. \tag{4.24}$$

We want to show $u$ must be radial symmetric and decreasing with respect to some point on $S^3$. Let $v$ be the smooth function on $S^3$ solving

$$Pv + \varepsilon v = -s_\varepsilon (u^*)^{-7}. \tag{4.25}$$

Then

$$v = s_\varepsilon T_{k_\varepsilon} (u^*)^{-7}. \tag{4.26}$$

Denote

$$\alpha = -E_\varepsilon(v), \tag{4.27}$$
then
\[
\alpha = - \int_{S^3} (P + \varepsilon) v \cdot vd\mu
= s_\varepsilon^2 \int_{S^3} T_{k_\varepsilon} \left((u^*)^{-7}\right) \cdot (u^*)^{-7} d\mu
= s_\varepsilon^2 \int_{S^3} T_{k_\varepsilon} \left(\left(\left(u^{-7}\right)^*\right)_R\right) \cdot \left(\left(u^{-7}\right)^*\right)_R d\mu
= s_\varepsilon^2 \int_{S^3} T_{k_\varepsilon} \left(\left(u^{-7}\right)^*\right)_R \cdot \left(u^{-7}\right)^* d\mu
\leq s_\varepsilon^2 \int_{S^3} T_{k_\varepsilon} \left(u^{-7}\right) \cdot u^{-7} d\mu
= s_\varepsilon \int_{S^3} u^{-6} d\mu
= s_\varepsilon.
\]

Here \(R\) is the reflection given by
\[
R(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4) \quad \text{for } x \in S^3.
\]

We have used Proposition 3.1 in between.

On the other hand, we have
\[
\alpha = s_\varepsilon \int_{S^3} (u^*)^{-7} vd\mu,
\]
hence
\[
\alpha \|v^{-1}\|_{L^6} = s_\varepsilon \left\| (u^*)^{-7} v \right\|_{L^1} \|v^{-1}\|_{L^6} \geq s_\varepsilon \left\| (u^*)^{-7} \right\|_{L^6}^7
= s_\varepsilon \left\| (u^*)^{-1} \right\|_{L^6}^7
= s_\varepsilon \left\| u^{-1} \right\|_{L^6}^7
= s_\varepsilon.
\]

This together with the fact \(\alpha \leq s_\varepsilon\) tells us \(\|v^{-1}\|_{L^6} \geq 1\). We have
\[
s_\varepsilon \geq -E_\varepsilon(v) \|v^{-1}\|_{L^6}^2 = \alpha \|v^{-1}\|_{L^6} \cdot \|v^{-1}\|_{L^6} \geq s_\varepsilon.
\]
Hence \(\|v^{-1}\|_{L^6} = 1\) and \(\alpha = s_\varepsilon\). It follows from Lemma 3.3 and (4.28) that \(u^{-7}\) must be radial and decreasing with respect to some point on \(S^3\) and so is \(u\).

5. A NEW PROOF OF THE SHARP SOBOLEV INEQUALITY

Following [HWY], we will first derive a Kazdan-Warner type condition and then use it to show the minimizer \(u\) must be a constant function.

As in [HWY, section 3], we first introduce the weighted total \(Q\) curvature functional. If \((M^3, g)\) is a smooth compact Riemannian manifold, and \(\chi\) is a positive
smooth function on $M$, we define
\[ I(M, g, \chi) = \left( \int_M \chi d\mu \right)^\frac{1}{2} \int_M Q d\mu. \] (5.1)

For $\tilde{g} \in [g]$, we write $\tilde{g} = \rho^{-4} g$, then
\[ I(M, \tilde{g}, \chi) = -2 \left( \int_M \chi \rho^{-6} d\mu \right)^\frac{1}{2} \int_M P \rho \cdot \rho d\mu. \] (5.2)

The Euler-Lagrange equation of this functional reads as
\[ P \rho = \text{const} \cdot \chi \rho^{-7}. \] (5.3)

**Lemma 5.1** (Kazdan-Warner type condition). *Assume $(M^3, g)$ is a smooth compact Riemannian manifold, and $\chi$ and $\rho$ are positive smooth functions on $M$ satisfying*
\[ P \rho = -\chi \rho^{-7}. \] (5.4)

*Let $X$ be a conformal vector field on $(M, g)$, then*
\[ \int_M X \chi \cdot \rho^{-6} d\mu = 0. \] (5.5)

**Proof.** The proof is exactly the same as for [HWY, Lemma 3.1]. Indeed let $\phi_t$ be the 1-parameter group generated by $X$, then
\[ \frac{d}{dt} \Bigg|_{t=0} I(M, \phi_t^* (\rho^{-4} g), \chi) = 0. \] (5.6)

On the other hand,
\[ I(M, \phi_t^* (\rho^{-4} g), \chi) = I(M, \rho^{-4} g, \chi \circ \phi_{-t}) \]
\[ = -2 \left( \int_M \chi \circ \phi_{-t} \rho^{-6} d\mu \right)^\frac{1}{2} \int_M P \rho \cdot \rho d\mu. \] (5.7)

Since
\[ \int_M P \rho \cdot \rho d\mu = - \int_M \chi \rho^{-6} d\mu < 0, \] (5.8)
we see $\int_M X \chi \cdot \rho^{-6} d\mu = 0$. $\square$

**Corollary 5.1.** *Assume $\chi$ and $\rho$ are positive smooth functions on $S^3$ such that*
\[ P \rho = -\chi \rho^{-7}, \] (5.9)
then
\[ \int_{S^3} (\nabla \chi (x), \nabla x_i) \rho (x)^{-6} d\mu(x) = 0 \] (5.10)
for $i = 1, 2, 3, 4$.

We are ready to give a perturbation proof of Theorem 1.1.

**Proof of Theorem 1.1.** For $\varepsilon > 0$ small let $u$ be a minimizer of the perturbation problem (2.1) with $\|u^{-1}\|_{L^6} = 1$, then
\[ Pu + \varepsilon u = -s_\varepsilon u^{-7}. \] (5.11)

In another word,
\[ Pu = - (s_\varepsilon + \varepsilon u^8) u^{-7}. \] (5.12)
It follows from Section 4 that \( u \) must be radial symmetric and decreasing with respect to some point on \( S^3 \). By rotation we can assume that point is the north pole \( N \). By Corollary 5.1 we have

\[
\int_{S^3} \langle \nabla (s_\varepsilon + \varepsilon u^8), \nabla x_4 \rangle u (x)^{-6} d\mu (x) = 0. \tag{5.13}
\]

In another word,

\[
\int_{S^3} \langle \nabla_4 u, \nabla x_4 \rangle u (x) d\mu (x) = 0. \tag{5.14}
\]

Since \( \langle \nabla u, \nabla x_4 \rangle \geq 0 \), we see \( \langle \nabla u, \nabla x_4 \rangle = 0 \) and \( u \) must be constant function. It follows that \( s_\varepsilon = \left( \frac{15}{16} - \varepsilon \right) |S^3|^\frac{1}{2} \).

Hence for any \( \varphi \in H^2 (S^3) \), \( \varphi > 0 \) and \( \varepsilon > 0 \) small,

\[
\|\varphi^{-1}\|_{L^6}^2 (E (\varphi) + \varepsilon \|\varphi\|_{L^2}^2) \geq - \left( \frac{15}{16} - \varepsilon \right) |S^3|^\frac{1}{2}. \tag{5.15}
\]

Let \( \varepsilon \downarrow 0 \), we get

\[
\|\varphi^{-1}\|_{L^6}^2 (E (\varphi) + \varepsilon \|\varphi\|_{L^2}^2) \geq - \frac{15}{16} |S^3|^\frac{1}{2}. \tag{5.16}
\]

At last we want to point out that the study of extremal problem (1.14) is motivated by the question of finding conformal metrics with constant \( Q \)-curvature (see [XY]). One of the crucial ingredient in our approach to (1.10) is the condition NN. The validity of condition NN on the standard \( S^3 \) is in some sense straightforward and had been observed in [HY1]. For general metrics, the understanding of condition NN is more recent and depends heavily on an identity found in [HY2, section 2]. More precisely, if \((M, g)\) is a smooth compact Riemannian manifold with \( Y (g) > 0 \) and \( Q > 0 \), then the following three statements are equivalent (see [HY6, section 4]):

- Extremal problem (1.14) is achieved;
- The second eigenvalue of Paneitz operator \( \lambda_2 (P) > 0 \);
- The Paneitz operator \( P \) satisfies the condition NN.

On the other hand, one can solve the constant \( Q \) curvature problem without the solution to extremal problem (1.14) (see [HY3, HY5]). It is still not known whether we can find a smooth compact Riemannian manifold \((M^3, g)\) with \( Y (g) > 0 \), \( Q > 0 \) and the Paneitz operator has two or more negative eigenvalues. For such kind of metrics, the value of (1.14) would be \(-\infty \) and the Paneitz operator does not satisfy condition NN.

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Courant Institute, New York University, 251 Mercer Street, New York NY 10012
E-mail address: fengbo@cims.nyu.edu

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton NJ 08544
E-mail address: yang@math.princeton.edu