Correlational quantum theory and correlation constraints

Ding Jia (贾丁)\textsuperscript{1,2,*}

\textsuperscript{1}Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada
\textsuperscript{2}Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2L 2Y5, Canada

A correlational dialect is introduced within the quantum theory language to give a unified treatment of finite-dimensional informational/operational quantum theories, infinite-dimensional quantum field theories, and quantum gravity. Theories are written in terms of correlation diagrams which specify correlation types and strengths. Feynman diagrams emerge as topological classes of correlation diagrams without any perturbative considerations. The correlational formalism is applied in a study of correlation constraints, revealing new classes of quantum processes that evade previous characterizations of general quantum processes including quantum causal structure.

I. INTRODUCTION

Quantum theory has gone through several phases of evolution. It started as the quantum mechanics of particles. Then came the quantum theory of fields. More recently the quantum theory of information has been on the rise. Will the future reveal yet new phases of quantum theory?

Our vision is that besides quantum particles, fields, and bits (dits), quantum correlations should also be used as a fundamental concept in constructing quantum theories. Here we understand quantum correlation in a both general and specific way – general because we hope to build correlational theories that break the boundaries among theories based on quantum particles, fields, and bits (dits), and specific because we want to offer a concrete prescription for constructing correlational theories.

Generally, we understand quantum correlation as anything that is mediated and has a quantifiable strength in a quantum theory. As such quantum correlations transcend the distinction between particles and fields, which both involve mediated quantifiable correlations, and go beyond qudits, which are limited to finite dimensions.

Specifically, we offer a prescription to construct correlational quantum theories based on “correlation diagrams”, which are graphs labelled by correlation types and correlation strengths. Upon composition, new diagrams form out of old diagrams, correlations mediate selectively according to type (color) matching, and new strengths are calculated according to old strengths. The essence of this correlational formalism is to keep track of the mediation of correlations in a manifest way through the composition of diagrams. Previous works on quantum theory that hold correlations essential include [1–4]. Previous works that express quantum theory as a compositional/diagrammatic theory include [5–9]. The correlation diagrams proposed here are in particular similar to Hardy’s duotensors.

As we show in the paper, the diagrammatic setup considered here allows the construction of both finite-dimensional informational/operational quantum theories and infinite-dimensional quantum field theories. It enable us to transport concepts and tools among theories based on quantum particles, field and bits (dits).

As an interesting example, we show how Feynman diagrams emerge as topological classes of correlation diagrams for both finite and infinite-dimensional theories, with or without perturbative considerations. From this perspective, Feynman diagrams are not merely mathematical bookkeeping devices restricted to field theories or perturbation theories, but arise fundamentally whenever quantum correlational configurations are under investigation.

The correlational formalism also supplies useful technical tools. As an example, we introduce a binary string calculus to study correlation constraints under composition and decomposition. This study yields new classes of processes that go beyond previous characterizations of the most general finite-dimensional quantum processes.

Finally, the correlational perspective leads to a relational approach to quantum gravity, as developed in a separate paper [10].

II. CORRELATIONAL QUANTUM THEORIES

Given a multipartite quantum channel \( N \), e.g., in its Kraus operator description \( N : \rho \mapsto \sum_i K_i \rho K_i^\dagger \), how do we know which parties are correlated?

To make manifest the correlational structure of quantum processes, we introduce correlation diagrams. As illustrated in Figure 1, a correlation diagram is a graph with colored edges representing correlation types, and a weight variable representing correlation strengths. Correlation diagrams compose to represent the mediation of correlations. Old diagrams give rise to new diagrams only if the correlation types match at the composed systems. The new strength variable is obtained by theory-specific rules from the old strength variables. This setting captures correlation as something that can be mediated/blocked (according to types) and quantified (according to strength variables).

A process such as a quantum channel is described by a list of correlation diagrams, and only systems/parties...
sharing diagrams with non-trivial types are correlated (which shall be clear from the study on correlation constraints below).

The correlational formalism allows one to define theories directly in terms of correlation diagrams from the outset, without going through Hilbert space vectors and operators, much like the path integral. The formalism is quite versatile, being applicable to both finite and infinite dimensions, and to both amplitude and density operator formalism.

**A. Finite-dimensional theories**

In operational terms the basic elements of a quantum theory are preparation, evolution, and measurement. These can all be represented as completely positive (CP) maps [12, 13], taking possibly the trivial one-dimensional systems as inputs and/or outputs [8]. Quantum theory can then be formulated in terms of composition of CP maps [14].

A convenient correlation diagram description of a CP map \( A \) is the following.

1. Obtain the Choi operator [15] (Appendix A) of the CP map.
2. Expand the Choi operator in a generalized Pauli operator basis (Appendix B).
3. The basis indices and expansion coefficients correspond to correlation types and correlation strength variables.

The first two steps yields

\[
A = \sum_{i,j,\cdots,k} a_{ij\cdots k} \sigma_i^1 \otimes \sigma_j^2 \otimes \cdots \otimes \sigma_k^n
\]

as Choi’s positive semidefinite operator of the CP map \( A \) associated to \( n \) input and output systems. \( \sigma_i^m \) is the \( i \)-th generalized Pauli operator on the \( m \)-th system, \( a_{ij\cdots k} \in \mathbb{R} \) is a coefficient in the basis expansion, and the sum is over all the basis elements of all systems. The third step yields a list of correlation diagrams, with \( i, j, \cdots, k \) represented as colored legs on the \( 1, 2, \cdots, n \)-th systems, and \( a_{ij\cdots k} \) as the strength variables.

The correlation diagram composition rule follows from that of the Choi operators [14]

\[
A \ast B := \frac{1}{d_{S^*}} \text{Tr}_{S^*}[A^{T_{S^*}} B].
\]

\( S^* \) is the set of composed systems. \( T_{S^*} \) is the partial transpose with respect to a standard basis on \( S^* \). \( \text{Tr}_{S^*} \) and \( d_{S^*} \) are the partial trace on and dimension of \( S^* \). Since on every system \( \text{Tr}[\sigma_i^m \sigma_j^m] = \pm 2 \delta_{ij} \) (Appendix B), all correlation types must match at the composed systems to generate a non-vanishing new diagram, and the new strength variable is the product of the old ones, multiplied by \( (\prod_{m \in S^*} c_m)/d_{S^*} \), where \( c_m = \pm 2 \) originates from \( \text{Tr}[\sigma_i^m \sigma_j^m] = \pm 2 \delta_{ij} \) (in the convention of Appendix B): -2 for \( i = (j, k) \) with \( 1 \leq k < j \leq d \) and +2 otherwise and \( d_{S^*} \) from (2).

The correlation diagram description applies to any Hermitian operator description of processes obeying the composition rule (2). In particular, it applies when processes are not given as CP maps but directly in terms of Hermitian operators (e.g., quantum theories without predefined time [16, 17]).

The correlation diagrams represent not just the causal propagation of correlations, but also the acausal mediation of correlations. The correlation diagrams can be used to do (de)composition in the spacelike direction (e.g., decomposing an entangled state into two Hermitian operators), in addition to the timelike direction. Spacelike (de)composition are generically present in quantum field theories (see below), and the current setup provides a finite-dimensional analogue.

**B. Feynman diagrams as correlational topological classes**

Feynman diagrams are usually understood as bookkeeping symbols for a perturbation series. Here we present an alternative understanding of Feynman diagrams as topological classes of correlation configurations. They arise in finite- and infinite-dimensional quantum theories, with or without perturbative considerations.

Consider the composition of multiple processes, such as the case depicted in Figure 2. Type matching leads to a list of correlation diagrams, such as those in the second line. Their topological classes, such as those shown in the third line, are Feynman diagrams for correlational theories. We show next that for field theories they are exactly the familiar Feynman diagrams.
Note that the curves integrated over in (3) can move back and forth in time, so are not ordinary particle trajectories. We think of these curves as configurations of the (possibly spacelike) mediation of correlations in spacetime. Thinking in terms of real correlations is more straightforward than in terms of virtual particles that defy causality. Note also that the matter field \( \phi \) does not appear in the above formulation of matter “quantum field theory”. The essential element in this construction is the correlational configuration, not the field configuration. Although (3) still depends on the gravitational field through the \( \bar{x}^2 = g_{ab}\bar{x}^a\bar{x}^b \) term, this dependence on a pointwise field is further weakened in [10] where the gravitational field is replaced by a relational field of the invariant distance.

III. CORRELATION CONSTRAINTS

We now focus on finite dimensions and conduct a study on characterizing constraints of correlations. We introduce a binary string calculus that turns out to simplify deductions and calculations. As a result, we identify new classes of quantum processes that fall beyond previous characterizations of general quantum processes.

A correlation constraint serves to tell which subsystems are allowed to be correlated. We write a constraint as a list of binary strings. For instance, consider channels from systems \( s_1, s_2 \) to systems \( s_3, s_4 \) constrained by \( \{1010, 0101, 1011, 0000\} \). Correlations are allowed among systems sharing 1’s. In this example correlations are allowed among \( s_1 s_3, s_2 s_4 \), and \( s_1 s_3 s_4 \), meaning \( s_1 \) can signal to \( s_3 \), \( s_2 \) can signal to \( s_4 \), and \( s_1 \) can signal to \( s_3 s_4 \) if there is a bipartite decoding. Technically, the binary strings constrain correlation diagrams by correlation types. In terms of the generalized Pauli basis used in (1), \( \sigma_i \propto \mathbb{1} \) corresponds to 0, and all other \( \sigma_i \) correspond to 1. Then for instance 0000 allows terms of the type \( \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \), while 1010 allows terms of the type \( \sigma^1_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \) for \( \sigma_1, \sigma_2 \neq \mathbb{1} \). A correlation constraint as a list of binary strings constrains the processes to only have terms of the types in the list.

The all zero string is special because it is present in all correlation constraints for physical processes. The Choi operators for physical processes are positive-semidefinite and non-zero, so must have positive trace. Since only terms corresponding to the zero string have non-zero trace (Appendix B), the all zero string must be present. We give it a special symbol

\[
u = 00 \ldots 0.
\]  

Second, processes that preserve probabilities have a fixed term \( \mathbb{1} \) for the type \( \nu \). We introduce a special symbol \( \hat{\nu} \) to correspond to \( \mathbb{1} \). For example, \( \{1010, 0101, 1011, \hat{\nu}\} \) implies the only term of type \( \nu \) the processes can have is the fixed term \( \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \). Since \( \mathbb{1} \) obviously belongs to the type \( \nu \), we assume that for correlation constraints, \( \hat{\nu} \) is implicitly
present whenever \( u \) is present. When \( \hat{u} \in A \) but \( u \not\in A \), we write
\[
\hat{u} \in A.
\] (5)

The binary strings together with \( \hat{u} \) are called correlation type elements, which are referred to using lowercase roman letters in the following. A set of correlation type elements is denoted by a capital roman letter such as \( A \). The set of all processes (they have positive semidefinite Choi operators) constrained by \( A \) is denoted by \( C_A \).

### A. Constraints under composition and decomposition

We are interested in two kinds of questions regarding the constraints and the mediation of correlations:

- \( a) \quad C_A \ast C_B \rightarrow ? \)
- \( b) \quad C_A \ast ? \rightarrow C_B \)

a) asks what we can say about the composition of processes constrained by \( A \) and \( B \), while b) asks what we can say about the decomposition of processes constrained by \( B \) when one decomponent is constrained by \( A \). First consider a).

**Theorem 1.** For the composition on systems \( S_u \),
\[
C_A \ast C_B \subset C_{A \ast B},
\] (6)

The proof is given in Appendix E. Here
\[
A \ast B := \{a+b : a \in A, b \in B, (a+b)_{S_u} \in \{u, \hat{u}\}\},
\] (7)

where \( a+b \) is binary string addition (for \( \hat{u}, \hat{u}+x = x+\hat{u} = x \) for any \( x \) including \( \hat{u} \)), and \( x_{S_u} \) is \( x \) restricted to systems \( S_u (\hat{u}_{S_u} = \hat{u}) \). A Choi operator \( \hat{A} \) defined on systems \( S_1 \) is automatically extended to \( S_1 \cup S_2 \) as \( \hat{A} \otimes I \) with identity acting on the new systems, so that the correlation type elements \( a \) and \( b \) are put on the same systems \( S_1 \cup S_2 \) to carry out the addition and the restriction. As explained in Appendix E, Theorem 1 gives the best general characterization for composition, since \( C_{A \ast B} \) on the right hand side cannot be reduced, and \( \subset \) cannot be strengthened to \( = \).

Now consider b). For compositions on \( S_u \), so that \( S_u \cap S_B = \emptyset \) (since composition eliminates systems), define
\[
C_A \rightarrow C_B := \{ C \in C : C_A \ast C \subset C_B \},
\] (8)
\[
A \rightarrow B := \{ c : A \ast c \subset B \}
\] (9)

where \( C \) is the set of all positive semidefinite operators. \( C_A \ast C = \{ A \ast C : A \in C_A \} \), so that \( C_A \rightarrow C_B \) contains all processes obeying the constraint of b). \( A \ast c \) is understood according to (6) where the second set has one correlation type element \( c \).

**Theorem 2.** For compositions on \( S_u \), so that \( S_u \cap S_B = \emptyset \),
\[
C_A \rightarrow C_B = C_{A \rightarrow B},
\] (10)
\[
A \rightarrow B = \begin{cases} (B^\perp - A)_{S_A \rightarrow B} \cup \{ \hat{u} \}, & A \ast \hat{u} \subset B, \\ (B^\perp - A)_{S_A \rightarrow B}^\perp, & \text{otherwise}, \end{cases}
\] (11)
\[
C_A \rightarrow C_B \neq \emptyset \text{ if and only if } A \ast \hat{u} \subset B.
\] (12)

The proof of the theorem is given in Appendix G. \( S_u \), the support of \( \alpha \), is the set of systems on which \( \alpha \) is not 0. \( S_A := \cup_{\alpha \in A} S_{\hat{u}} \). \( S_{\hat{u}} = S_\emptyset = \emptyset \), and \( \hat{u} \) and \( u \) are understood as supported on an one-dimensional trivial system. \( \tau_S \) is the set of all binary strings (not including \( \hat{u} \)) on \( S \), and
\[
A_S^\perp := \tau_S \setminus A, \quad A^\perp := \tau_S \setminus A,
\] (13)

with the conventions that \( A \) or \( \tau_S \) be extended by joining 0’s to carry out \( \tau_S \setminus A \) if necessary, \( \{ u \}^\perp = \{ u \} \), and \( \{ \hat{u} \}^\perp = \emptyset \) (think of \( \tau_S \) on the trivial system \( S = \emptyset \)). \( A - B = \{ a - b : a \in A, b \in B \} \), where analogous to \( a + b \), \( a - b \) is binary string subtraction under automatic extension. \( a + b \neq a - b \) in general because while \( a - \hat{u} = a \) for all \( a \) including \( a = \hat{u}, \hat{u} - b \) outputs no element for \( b \neq \hat{u} \) (since only \( \hat{u} \) added to \( \hat{u} \) gives \( \hat{u} \)).

The (11) characterization for \( A \rightarrow B \) admits an interpretation as excluding (through \( "S_{\hat{u}}^\perp\) elements \( (B^\perp - A) \) that compose with \( A \) to form elements outside \( B \). Finally \( \hat{u} \) needs special care when \( A \ast \hat{u} \subset B \).

Next we use Theorem 2 to reproduce and generalize previous characterizations of general processes.

### Process matrices

The process matrix framework [20–22] takes an operational approach to study general quantum correlations. Start with \( n \) “local laboratories” inside each an agent performs an arbitrary quantum operation, modelled as a quantum instrument [13] from one input to one output system. The \( n \)-party quantum correlations are encoded in the process matrices as Choi operators that always yield valid probabilities (non-negative probabilities that sum to one) when composed with these arbitrary local operations. The most general such correlations can indicate the presence of quantum indefinite causal structure among the agents, so generalize ordinary quantum theory with definite causal structure. Precisely which of these theoretically defined process matrices are realizable in nature is an open question.

The valid probability requirement imposes correlation constraints. For example, for two parties it imposes the constraint \( W_2 = \{ \hat{u}, 1000, 0010, 1010, 0110, 1001, 1110, 1011 \} \) [20], which can be reproduced by applying Theorem 2 to the following setup.
Example 3 (Two-party process matrices). $S_* = \{s_1, s_2, s_3, s_4\}$. $A = \{u, 01, 11\}_{s_1, s_2} \times \{u, 01, 11\}_{s_3, s_4}$. $B = \{\hat{u}\}.$

The first agent has input $s_1$ and output $s_2$, and the operation is constrained by $\{\hat{u}, 01, 11\}_{s_1, s_2}$ to be a channel. Similarly for the second agent. $A$ denotes the constraint on the joint operations as product channels. $(X \times Y := \{xy : x \in X, y \in Y\}$ denotes elongated elements so that $C_X \otimes C_Y = C_{X \times Y}$. E.g., if $a = 00, b = 11$, then $ab = 0011$. For $\hat{u}$, $\hat{u}_y = uy, x\hat{u} = xu$, and $\hat{u}\hat{u} = \hat{u}$.) $B$ on the trivial system constrains the probability to be 1. $A \rightarrow B$ then defines the process matrices as composing with arbitrary product channels to yield the normalized probability. Since $B^\perp - A = \{u\} - A = (A\{\hat{u}\}) \cup \{u\}$, Theorem 2 implies $A \rightarrow B = (B^\perp - A)^\perp_{S_A \rightarrow B} \cup \{\hat{u}\} = (A^\perp\{u\}) \cup \{\hat{u}\} = W_2$

The $n$-parties constraint [21, 22] can similarly be reproduced by applying Theorem 2 to the following setup.

Example 4 ($n$-party process matrices). $S_i = \{s_{i1-1, s_{i2-1}\}$. $S_i = \bigcup_{i=1}^n S_i = \{s_{i1}, s_{i2}, \ldots, s_{i2n-1, s_{i2n}}\}. A_i = \{u, 01, 11\}$ with support $S_{A_i} = S_i, A = \times_{i} A_i, B = \{\hat{u}\}$.

We have $A \rightarrow B = (B^\perp - A)^\perp_{S_A \rightarrow B} \cup \{\hat{u}\} = (A^\perp\{u\}) \cup \{\hat{u}\} \cup \{a \in \tau_{S_i} : a_{S_i} \in A_i^\perp\text{ for any } 1 \leq i \leq n\} = \{\hat{u}\} \cup \{a \in \tau_{S_i} : a_{S_i} = 10\text{ for any } 1 \leq i \leq n\}$.

The process matrices were originally defined under the assumption that the parties have one input and one output, and can apply arbitrary channels across its input and output [20–22]. What if the parties have multiple input and output subsystems, and their operations are constrained (e.g., by the spacetime causal structure in the laboratories)? A similar application of Theorem 2 addresses this question.

Proposition 5 (Generalized $n$-party process matrices). Let $S_i$ be the set of systems of the $i$-th party, $S_* = \bigcup_{i=1}^n S_i, A_i$ constrain the allowed channels in the $i$-th party, $A = \times_{i} A_i$, and $B = \{\hat{u}\}$. Then $A \rightarrow B = (A^\perp\{u\}) \cup \{\hat{u}\} \cup \{a \in \tau_{S_i}: a_{S_i} \in A_i^\perp\text{ for any } 1 \leq i \leq n\}$.

Higher order maps

A higher order map [14, 23–26] is one that maps between lower order maps. E.g., $A \rightarrow B$ characterizes higher order maps between those characterized by $A$ and $B$. This philosophy was used in [14, 23, 24] to iteratively construct a hierarchy of maps specified by what this paper regards as correlation constraints. At the lowest level are states on single systems. Next comes maps between single system states etc. This hierarchy is quite general, incorporating all the process matrices considered above, which already goes beyond ordinary quantum theory with definite causal structure. Does this hierarchy capture all processes of interest?

From the correlational perspective there is no reason to focus on processes belonging to this hierarchy. Theorem 2 democratizes and generalizes the hierarchical order maps: 1) In each application of the theorem, the multi-system correlation constraints $A$ and $B$ can be specified arbitrarily and need not come from any hierarchy. 2) $A$ and $B$ may share any common set of systems, in which case the composition is on a subset of the systems of $A$. 3) The processes that the theorem applies to need not distinguish input and output systems (e.g., as in the case of generalized Choi operators of [17] without predefined time). As a special case, Theorem 2 implies:

**Corollary 6.** Let the constraints $A$ and $B$ be supported on distinct systems, i.e., $S_A \cap S_B = \emptyset$. For $S_* = S_A$ (recall (5))

$$A \rightarrow B = \begin{cases} (A^\perp \times \tau_{S_B}) \cup (A \times B) \cup \{\hat{u}\}, & A \ast \hat{u} \subset B, \\ (A^\perp \times \tau_{S_B}) \cup (A \times B), & \text{otherwise.} \end{cases}$$

(14)

**Proof.** Since $S_A \cap S_B = \emptyset, B^\perp - A = A \times B^\perp$. In addition, $S_{A \rightarrow B} = S_A \cup S_B$. Then $(B^\perp - A)^\perp_{S_{A \rightarrow B}} = (A \times B^\perp)^\perp_{S_A \cup S_B} = (A \times \tau_{S_B}) \cup (A \times B)$. The result follows from Theorem 2.

The set $(A^\perp \times \tau_{S_B}) \cup (A \times B)$ can also be expressed as $(A^\perp \times B) \cup (A^\perp \times \tau_{S_B}) \cup (A \times \tau_{B}) \cup (A \times B)$. If $A$ and $B$ characterize normalized processes, i.e., $\hat{u} \in A, B$, then the condition $A \ast \hat{u} \subset B$ is fulfilled, and the above result reproduces Proposition 1 of [24] (see also Lemma 2 of [23]) that characterizes general hierarchical higher order maps. If $A$ and $B$ characterize process matrices, then the above result characterizes transformations of process matrices, and an iteration reproduces the characterization of [25].

IV. CONCLUSION

We presented a correlational formalism that gives a unified treatment of informational/operational quantum theories and quantum field theories. In a separate paper [10], we show in relational treatment that quantum gravity can also be incorporated. In all these formulations, correlation diagrams play a crucial role as configurations for the mediation of correlations in classical or quantum spacetime. Feynman diagrams emerge as topological classes of correlations for general finite and infinite-dimensional quantum theories, even when no perturbation is performed. Through studying correlation constraints, we found new classes of quantum processes that evade previous characterizations. An interesting topic for future research is to study measures of correlation strength based on the correlation diagrams, which are applicable across quantum information theory, quantum field theory, and quantum gravity.

ACKNOWLEDGEMENT

I am very grateful to Lucien Hardy, Achim Kempf, Nita Sakharwade, Fabio Costa, and Ognyan Oreshkov for...
valuable discussions. I especially thank Rafael Sorkin for his cautions on the informational approach to fundamental physics (partially summarized in the opening part of [27]) – this work is an attempt to reduce the gap between models and frameworks.

Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Economic Development, Job Creation and Trade. This publication was made possible through the support of the grant “Causal Structure in Quantum Theory” from the John Templeton Foundation and the grant “Operationalism, Agency, and Quantum Gravity” from FQXi. The opinions expressed in this publication are those of the author and do not necessarily reflect the views of the funding agencies.

Appendix A: Choi operators

For a Hilbert space \( \mathcal{H} \), we denote by \( \mathcal{L}(\mathcal{H}) \) the space of bounded linear operators on \( \mathcal{H} \). By the Choi isomorphism [15], there is an one-to-one correspondence between completely positive (CP) maps \( \mathcal{M} : \mathcal{L}(\mathcal{H}^{a_1}) \to \mathcal{L}(\mathcal{H}^{a_2}) \) and positive-semidefinite operators \( M \in \mathcal{L}(\mathcal{H}^{a_2} \otimes \mathcal{H}^{a_1}) \),

\[
M := d_{a_2} (\mathcal{M} \otimes I) \sum_{i,j=1}^{d_{a_1}} |ii\rangle\langle jj|,
\]

where \( I \) is the identity channel on system \( a_1 \), \( d_x = \dim \mathcal{H}^x \), and the sums are over an orthonormal basis of \( \mathcal{H}^{a_1} \). The normalization convention is so chosen that \( M = I + X \), where \( X \) is a traceless operator. The positive-semidefinite operator in (A1) is called the Choi operator of the quantum process \( \mathcal{M} \).

The above amounts to sending half of a (unnormalized) maximally entangled state to the original CP map to obtain a (unnormalized) bipartite state. For a CP map with multiple inputs and outputs, the Choi operator is obtained by sending half of a (unnormalized) maximally entangled state to each input.

A basic operation of CP maps is composition. Eventually the probabilistic predictions of the theory comes from composing processes. For example, composing a single measurement with a bipartite state leads to a reduced single system state, which when composed with another measurement leads to a list of probabilities for the measurement outcomes. For writing down the composition formula of Choi operators, it is convenient to automatically extend operators to larger sets of systems such that \( A \) acting on \( \mathcal{H}^1 \) (which may be a tensor product of Hilbert spaces) is freely viewed as \( A \otimes I \) acting on \( \mathcal{H}^1 \otimes \mathcal{H}^2 \) for arbitrary \( \mathcal{H}^2 \).

The composition on systems \( S_+ \) of two operators \( A \) on systems \( S_A \) and \( B \) on systems \( S_B \) is given by the composition formula (2) [14]

\[
A \ast B := \frac{1}{d_{S_+}} Tr_{S_+}[A T_{S_+} B],
\]

where \( T_{S_+} \) is the partial transpose on \( S_+ \) in the basis of the maximally entangled states used to obtain the Choi operator, and \( Tr_{S_+} \) is the partial trace on \( S_+ \). The normalization is so chosen to match ordinary probabilistic predictions. The composition symbol can be extended to sets of operators, so that \( A \ast B := \{ A \ast B : A \in \mathcal{A}, B \in \mathcal{B} \} \).

The Choi operator has been generalized to processes that do not distinguish input and output systems [17]. A processes can be specified directly in terms of a positive semidefinite operator instead of a CP map. The correlational formalism and the characterization of correlation constraints in this work apply to the generalized Choi operators as well, since we work directly with the positive semidefinite operators.

Appendix B: Generalized Pauli operators

On a \( d \)-dimensional Hilbert space \( \mathcal{H} \), a set of \( d^2 \) many generalized Pauli operators \( \sigma_i \) with \( i = (m, n) \) for \( m, n = 1, \ldots , d \) forms a basis for the Hermitian operators [28]:

\[
\sigma_i = \sigma_{(m,n)} = \begin{cases} 
\sum_{l=1}^{m} \left( \frac{2}{\sqrt{m(m+1)}} \right)^{1/2} (E_l - mE_{m+1}), & 1 \leq m < n \leq d, \\
\frac{2}{\sqrt{m(m+1)}} \sum_{l=1}^{m} E_l, & 1 \leq n < m \leq d, \\
\frac{2}{\sqrt{m(m+1)}} + E_m, & 1 \leq m = n \leq d - 1, \\
\frac{2}{\sqrt{m(m+1)}} + E_m, & m = n = d,
\end{cases}
\]

where \( E_{mn} = |m\rangle\langle n| \), and \( E_m = |m\rangle\langle m| \). \( \text{Tr} \sigma_i = 0 \) for all \( i \) except \( i = (d, d) \).

It is easy to check that

\[
\text{Tr}[\sigma_i \sigma_j] = 2 \delta_{i,j}
\]

so this is an orthogonal basis under the Hilbert-Schmidt norm for the real space of the Hermitian operators. In
addition,
\[
\text{Tr}[\sigma_i \sigma_j^T] = \text{Tr}[\sigma_j^T \sigma_i] = \begin{cases} 
-2\delta_{i,j}, & i = (m,n), 1 \leq n < m \leq d \\
2\delta_{i,j}, & \text{otherwise.}
\end{cases}
\]

Hence orthogonality is preserved if one operator is transposed, as in the composition formula (2). In the context of correlation diagrams, this means a color mismatch at any system of composition eliminates the diagrams.

### Appendix C: Correlation type expansion

For a Hilbert space \( \mathcal{H} \) with \( \mathcal{L}(\mathcal{H}) \) as the space of bounded linear operators on \( \mathcal{H} \), denote by \( \mathcal{L}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H}) \) the real subspace of hermitian operators on \( \mathcal{H} \). When \( \mathcal{H} \) is clear from the context, we sometimes omit it and write \( L \) for \( L(\mathcal{H}) \).

\( L \) can be expanded into a traceful and a traceless part as
\[
L = L_0 \oplus L_1.
\]

The traceful part \( L_0 \subset L \) is the (one-dimensional) subspace generated by the identity operator and the traceless part \( L_1 \subset L \) is the subspace of the traceless operators. The generalized Pauli operators of the previous section form a basis for \( L \), with \( \sigma_{(d,d)} \) spanning \( L_0 \) and the rest \( \sigma_{(m,n)} \) spanning \( L_1 \).

On a tensor product space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_m \), define
\[
L_a := L_{a_1} \otimes L_{a_2} \otimes \cdots \otimes L_{a_m} \subset L(\mathcal{H}),
\]
e.g., \( L_{0100} = L_0 \otimes L_1 \otimes L_0 \otimes L_0 \). Then
\[
L(\mathcal{H}) = \oplus_{a \in A} L_a,
\]
where the sum is over all binary strings of length \( m \).

The binary strings together with \( \hat{u} \) form the correlation type elements. With \( L_a := \{ 1 \} \), each correlation type element now has a corresponding set of operators \( L_a \subset L \). A set \( A \) of correlation type elements corresponds to \( L_A := \oplus_{a \in A} L_a \subset L \). We set \( L_{\emptyset} = \{0\} \).

It follows that any operator \( A \in L(\mathcal{H}) \) has a correlation type expansion
\[
A = \sum_{a \in A} A_a, \quad 0 \neq A_a \in L_a,
\]
where \( A_a \) are obtained by projecting \( A \) to \( L_a \) and keeping non-zero elements. We use \( \hat{u} \) instead of \( u \) in the set \( A \) whenever possible. \( A \) is called \( A \)'s correlation type.

### Appendix D: Correlation type calculus

The correlation type/binary string calculus and related conventions are summarized here.

For the composition on systems \( S_a \), define
\[
A \ast B := \{ a + b : a \in A, b \in B, (a + b)_{S_a} \in \{ u, \hat{u} \} \}.
\]

Here \( x_S \) is \( x \) restricted to systems \( S \), with \( \hat{u}_S = \hat{u} \).

**Convention 7.** Define
\[
A + B = \{ a + b : a \in A, b \in B \}
\]
according to
\[
a + b = \begin{cases} 
\text{a + b as binary addition,} & a, b \neq \hat{u}, \\
\hat{u}, & a = \hat{u}, b \neq \hat{u}, \\
\{ a \} & b = \hat{u}.
\end{cases}
\]

\( \{ a \} \ast B \) and \( \{ a \} \ast \{ b \} \) are often abbreviated as \( a \ast B \) and \( a \ast b \), respectively. Sometimes we abuse notation to treat \( a \ast b \) as an element rather than a set and use expressions such as \( a \ast b \in C \).

Recall that a Choi operator \( A \) defined on systems \( S_1 \) is automatically extended to \( S_1 \cup S_2 \) as \( A \otimes I \). Therefore the correlation type elements \( a \) and \( b \) can always be put on the same systems to carry out the addition, the restriction, and the following subtraction.

**Convention 8.** Define
\[
A - B = \{ a - b : a \in A, b \in B \}
\]
according to
\[
a - b = \begin{cases} 
\text{a - b as binary subtraction,} & a, b \neq \hat{u}, \\
\emptyset, & a = \hat{u}, b \neq \hat{u}, \\
\{ a \} & b = \hat{u}.
\end{cases}
\]

Here \( \emptyset \) means \( a - b \) outputs no element, since nothing added to \( b \neq \hat{u} \) gives \( a = \hat{u} \). This \( \emptyset \) symbol is used under the rule that \( \emptyset + a = a + \emptyset = \emptyset - a = a - \emptyset = \emptyset \) for all \( a \), and \( \{ \emptyset \} = \emptyset \).

### Appendix E: Proof of Theorem 1

**Lemma 9.** For \( S_a \supset S_a \cap S_b \),
\[
L_a \ast L_b = L_{a+b}.
\]
Proof. The composition formula $A \ast B = \frac{1}{d_x d_y} \text{Tr}_{S_y} [A^T \ast_x B]$ (2) has three steps: 1) Partial transpose $A$ at $S_x$. 2) Multiply $A^T \ast_x$ with $B$. 3) Take the partial trace and multiply by $\frac{1}{d_x d_y}$. The first step leaves $L_a$ invariant, the second step forms product operators, and the third step with the partial trace projects out operators traceless on $S_x$.

Since the first step leaves $L_a$ invariant, we move to the second step and introduce the notation $L_a \cdot L_b := \{AB : A \in L_a, B \in L_b\}$ for operator products. We claim that $L_{a+b} \subset L_a \cdot L_b \subset L_{a+b} \oplus L_C$, where $L_C$ is a space that will be projected out in the third step. First consider $A$ and $B$ on an individual system. Unless $a = b = 1$, $L_a \cdot L_b = L_{a+b}$. When $a = b = 1$, this system belongs to $S_a \cap S_b$ and hence also to $S_x$. We have $L_{a+b} = L_0 \oplus L_a \cdot L_b \subset L_{a+b} \oplus L_C = L_0 \oplus L_1$, where $L_{a+b}$ and $L_C$ will be projected out by the partial trace on $S_x$. The generalization to multiple systems amounts to singling out the systems on which both $a$ and $b$ are 1 and apply the same reasoning as above.

Now apply the third step to $L_{a+b} \subset L_a \cdot L_b \subset L_{a+b} \oplus L_C$ to get $L_{a+b} \subset L_a \ast L_b \subset L_{a+b}$, and hence the result.

Lemma 10. For $S_x \supset S_A \cap S_B,$

$$L_A \ast L_B \subset L_{A \ast B}.$$  \hfill (E2)

Proof. $L_A \ast L_B = (\oplus_{a \in A} L_a) \ast (\oplus_{b \in B} L_b) \subset (\oplus_{a \in A, b \in B} L_a \ast L_b = (\oplus_{a \in A, b \in B} L_{a+b}) = L_{A \ast B},$ where we used Lemma 9, knowing that $S_a \cap S_b \subset S_A \cap S_B \subset S_x$ for arbitrary $a \in A$ and $b \in B$.

Theorem 1. For the composition on systems $S_x,$

$$C_A \ast C_B \subset C_{A \ast B}.$$  \hfill (E3)

Proof. By (E2), $C_A \ast C_B \subset L_A \ast L_B = L_{A \ast B}.$ Intersecting with the set of positive semidefinite operators yields the result.

Theorem 1 gives the best general characterization at the level of correlation constraints. The set $C_{A \ast B}$ on the right hand side cannot be reduced, since there are $A$ and $B$ so that $C_A \ast C_B = C_{A \ast B}.$ For example, if $C_A$ are channels from $s_1$ to $s_2$, and $C_B$ are channels from $s_2$ to $s_3$, then $C_A \ast C_B$ composed on system $s_2$ contains all channels from $s_1$ to $s_3$, which equals $C_{A \ast B}$ (because $A \ast B = \{\hat{u}, 010, 110\} \ast \{\hat{u}, 001, 011\} \ast \{\hat{u}, 011, 11\} = \{\hat{u}, 01, 01\}$).

On the other hand, the $\subset$ cannot be replaced by $=,$ either. For example, if $C_A$ are states on $s_1$ and $C_B$ are states on $s_2,$ then $C_A \ast C_B$ composed on the trivial system are just the product states on $s_1 s_2,$ which is a proper subset of the all the bipartite states $C_{A \ast B}$ (because $A \ast B = \{\hat{u}, 10\} \ast \{\hat{u}, 01\} = \{\hat{u}, 10, 01, 11\}$).

Appendix F: First part proof of Theorem 2

For composition on $S_x$ so that $S_x \cap S_B = \emptyset,$

$$C_A \rightarrow C_B := \{C \in C : C_A \ast C \subset C_B\},$$  \hfill (F1)

$$L_A \rightarrow L_B := \{C \in L : L_A \ast C \subset L_B\},$$  \hfill (F2)

$$A \rightarrow B := \{C : A \ast C \subset B\}.$$  \hfill (F3)

Lemma 11. Suppose $a \ast b \notin C,$ $B \in L_b,$ and $B \neq 0.$ Then $L_a \ast B \notin L_C.$

Proof. Since $a \ast b \notin C,$ $L_{a+b} \cap L_C = \{0\}.$ By Theorem 1, $L_a \ast B \subset L_a \ast L_b \subset L_{a+b},$ so $L_a \ast B \subset L_C,$ unless $L_a \ast B \subset \{0\}.$

Since $a \ast b \notin C,$ $a \ast b \neq \emptyset.$ By the definition of $a \ast b$ (Convention 7), this implies that $(a + b)_S \in \{\hat{u}, u\},$ which means either $a_S = b_S,$ or one of them is $\hat{u}$ and the other is $u.$ In either case it is possible to pick $A \in L_a$ so that $A \ast B \neq 0.$ This implies $L_a \ast B \notin \{0\}.$

Lemma 12. For any non-zero $B \in L_b,$ $L_A \ast B \subset L_C \iff A \ast b \subset C.$

Proof. Suppose $A \ast b \subset C.$ Then by Theorem 1, $L_A \ast L_b \subset L_{A \ast b} \subset L_C,$ which implies $L_A \ast B \subset L_C.$ Conversely, suppose $A \ast b \not\subset C.$ Then there is an $a \in A$ so that $a \ast b \notin C.$ Let $B \in L_b$ be an arbitrary non-zero element. By Lemma 11, $L_a \ast B \notin L_C.$ Hence $L_A \ast B \notin L_C.$

Proposition 13. $L_A \rightarrow L_B = L_{A \rightarrow B}.$

Proof. Let $C = A \rightarrow B.$ First, $L_A \rightarrow L_B \subset L_C,$ because $L_A \ast L_C \subset L_{A \ast C} = \oplus_{C \in L} L_{A \ast C} \subset L_C,$ where Theorem 1 is used in the first step and the definition of $A \rightarrow B$ is used in the last step.

Next, we show that $L_A \rightarrow L_B \subset L_C.$ Let $D \in L_A \rightarrow L_B$ be arbitrary, with correlation type $D$ and expansion $D = \sum_{D \in D} D_d$ with $0 \neq D_d \in L_d.$ Let $A \ast D = \sum_{D \in D} A \ast D_d,$ which implies $L_A \ast D_d \subset L_B$ for all $D \in D.$ By Lemma 12, $A \ast d \subset B$ for all $d \in D,$ whence $D \subset C$ and $D \in L_C.$ Since $D$ is arbitrary, $L_A \rightarrow L_B \subset L_C.$

Lemma 14. $(L_A \rightarrow L_B) \cap C = C_D \rightarrow C_B.$
Proof. $LHS = \{C \in C : L_A \ast C \subseteq L_B\}$, and $RHS = \{C \in C : C \ast C \subseteq L_B\}$ by (F4). Since $C_A \subseteq L_A$, $LHS \subseteq RHS$.

We show $LHS \supseteq RHS$ by contradiction. Suppose $LHS \supsetneq RHS$, i.e., there exists $C \in RHS$ so that $C \not\subseteq LHS$. Since $C \in C$, this implies that $C \not\subseteq L_A \to L_B = L_D$, where by Proposition 13, $D = \{d : A \ast d \subseteq C\}$. Denoting the correlation type of $C$ by $c$, we can find $c \in C$ so that $c \not\subseteq D$, which implies there is $a \in A$ so that $a \ast c \not\subseteq B$. By Lemma 11, $L_a \ast C \not\subseteq L_B$, which implies we can find $A \in L_a$ so that $A \ast C \not\subseteq L_B$. By adding a multiple of 1, we can make $A \in C_A$ so that $A \ast C \not\subseteq C_B$. This contradicts the assumption that $C \in RHS$. 

Proposition 15. For composition on $S$, so that $S_a \cap S_b = \emptyset$,

$$C_A \to C_B = C_{A\to B}.$$  \hfill (F5)

Proof. By Lemma 14 and Proposition 13, $LHS = (L_A \to L_B) \cap C = L_{A\to B} \cap C = RHS$.

This establishes the first part (10) of Theorem 2.

Appendix G: Second part proof of Theorem 2

Suppose $S_A \subseteq S$. By the definition of $A_{\frac{1}{2}}$ (13) and the convention that $\hat{u} \in A$ implicitly whenever $u \in A$,

$$A_{\frac{1}{2}} \setminus \frac{1}{2} = \begin{cases} \{u\} \setminus \hat{u} \in A, & \hat{u} \not\subseteq A, \text{ otherwise.} \\ A, & \text{otherwise.} \end{cases}$$  \hfill (G1)

By (D3) and (D5) we have $a + b = a - b$, except when $a = \hat{u}, b \not\subseteq \hat{u}$. Moreover (recall Convention 8 for $\emptyset$),

$$a + b = \begin{cases} a, & a = \hat{u}, b \not\subseteq \hat{u}, \text{ otherwise.} \\ 0, & \text{otherwise.} \end{cases}$$  \hfill (G2)

Lemma 16. Suppose $S_B \cap S_* = \emptyset$. Then

$$(A \to B) \setminus \frac{1}{2} = B \setminus A$$  \hfill (G4)

Proof. For simplicity denote $A \to B = \{c : A \ast c \subseteq B\}$ by $C$. First we show that $C \subseteq B \setminus A$. If $C = \emptyset$ the statement clearly holds. Otherwise let $c' \in C$ be arbitrary. It suffices to show that $c' \subseteq B \setminus A$. By the definition of $C$, there exists $a \in A$ so that $a \ast c' \not\subseteq B$. This implies that $b := a + c' = a \ast c'$. Since $c' \subseteq C \subseteq B \setminus A$, whence $b \not\subseteq \hat{u}$. This implies $b \subseteq B \setminus A$. Now $c' \not\subseteq \hat{u}$ and (G2) imply that $c' = (a + c') = a = b' - a \subseteq B \setminus A$.

Next we show that $C \subseteq B \setminus A$. It suffices to show that $b' - a \subseteq C$ for arbitrary $b' \subseteq B$ and $a \subseteq A$. Since $b' \subseteq B \setminus A$, $b' \not\subseteq \hat{u}$. By (G3), $(b' - a) + a = b' \subseteq B \setminus A$. Since $b' \subseteq B \setminus A$ has support within $S_B$ and $S_B \cap S_* = \emptyset$, $b'_{S_B} = (b' - a) + a_{S_B} \subseteq \{\hat{u}, a\}$. Then $(b' - a) \ast a = (b' - a) + a = b' \subseteq B \setminus A$, whence $b' - a \subseteq C$. Since $b' \not\subseteq \hat{u}$, $b' - a \not\subseteq \hat{u}$, which means $b' - a \subseteq C$.

Proposition 17.

$$A \to B = \begin{cases} (B \setminus A)_{A\to B} \setminus \{\hat{u}\}, & A \ast \hat{u} \subseteq B, \\ (B \setminus A)_{A\to B}, & \text{otherwise}. \end{cases}$$  \hfill (G5)

Proof. This is a direct consequence of (G1) and (G4).

This establishes the second part (11) of Theorem 2.

Appendix H: Third part proof of Theorem 2

Given $A$, $B$, and $S$, disjoint from $S_B$, it is not guaranteed that there are non-zero elements in $C_A \to C_B$, even when $A \to B \not= \emptyset$.

Example 18. Let $S = \{s_1, s_2, s_3\}$, $S_A = \{s_2, s_3\}$, $S_B = \{s_1, s_3\}$, $S_* = \{s_2\}$, $A = \{\hat{u}, 001, 011\}$, and $B = \{\hat{u}, 101\}$. Then $C_A \to C_B = \{0\}$.

By (11), $A \to B = \{110\}$. Yet $C_A \to C_B = C_{A\to B} = L_{A\to B} \cap C = \emptyset$. The reason that $A \to B = \{110\}$ does not yield non-zero processes is the following.

Lemma 19. Let $A \neq 0$ be a positive semidefinite operator with correlation type $A$. Either $u \in A$ (implying $\hat{u} \in A$ by our convention) or $\hat{u} \subseteq A$.

Proof. By assumption, $A > 0$, so $\text{Tr} A > 0$. Among all the correlation type elements, only $u$ and $\hat{u}$ supply positive trace, so the result follows.

Proposition 20. $C_A \to C_B \not= \emptyset$ if and only if $A \ast \hat{u} \subseteq B$.

Proof. $C_A \to C_B = C_{A\to B}$ by Proposition 15. Suppose $C_{A\to B} = C_A \to C_B \not= \emptyset$. By Lemma 19, $\hat{u} \in A \to B$, and $A \ast \hat{u} \subseteq B$. Now suppose $A \ast \hat{u} \subseteq B$. Then $\hat{u} \in A \to B$, and $\mathbb{1} \in A \to B = C_A \to C_B$.

This establishes the third part (12) of Theorem 2.

---

[1] Carlo Rovelli. Relational quantum mechanics. *International Journal of Theoretical Physics*, 35(8):1637–1678, 8 1996.

[2] N. David Mermin. The Ithaca interpretation of quantum mechanics. *Pramana*, 51(5):549–565, 11 1998.

N. David Mermin. What is quantum mechanics trying to tell us? *American Journal of Physics*, 66(9):753–767, 9 1998.
[3] Lucien Hardy. Probability Theories with Dynamic Causal Structure: A New Framework for Quantum Gravity. arXiv:gr-qc/0509120.

[4] Achim Kempf. The Planck length as the regime of poor statistics. PIRSA: 18080002.

[5] R. P. Feynman. Space-Time Approach to Quantum Electrodynamics. Physical Review, 76(6):769–789, 1949.

[6] Bob Coecke. The logic of entanglement. arXiv:quant-ph/0402014, 2004.

[7] Robert Oeckl. A “general boundary” formulation for quantum mechanics and quantum gravity. Physics Letters, Section B: Nuclear, Elementary Particle and High-Energy Physics, 575(3-4):318–324, 2003.

[8] Giulio Chiribella, Giacomo Mauro D’Ariano, and Paolo Perinotti. Probabilistic theories with purification. Physical Review A, 81(6):062348, 2010.

[9] Lucien Hardy. Foliable operational structures for general probabilistic theories. Deep Beauty: Understanding the Quantum World Through Mathematical Innovation, pages 409–442, 2011.

[10] Ding Jia. World quantum gravity: An approach based on Syng’s world function. arXiv:1909.05322.

[11] Indeed, the correlational formalism may be viewed as a compositional path integral approach where correlational configurations obeying certain composition rules are summed over.

[12] Rudolf Haag and Daniel Kastler. An algebraic approach to quantum field theory. Journal of Mathematical Physics, 5(7):848–861, 1964.

[13] E. B. Davies and J. T. Lewis. An operational approach to quantum probability. Communications in Mathematical Physics, 17(3):239–260, 1970.

[14] Giulio Chiribella, Giacomo Mauro D’Ariano, and Paolo Perinotti. Theoretical framework for quantum networks. Physical Review A - Atomic, Molecular, and Optical Physics, 80(2):22339, 2009.

[15] Man Duen Choi. Completely positive linear maps on complex matrices. Linear Algebra and Its Applications, 10(3):285–290, 1975.

[16] Ognyan Oreshkov and Nicolas J. Cerf. Operational formulation of time reversal in quantum theory. Nature Physics, 11(10):853–858, 2015.

[17] Ognyan Oreshkov and Nicolas J. Cerf. Operational quantum theory without predefined time. New Journal of Physics, 18(7):073037, 2016.

[18] R. P. Feynman. Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction. Physical Review, 80(3):440–457, 1950.

[19] Julian Schwinger. On Gauge Invariance and Vacuum Polarization. Physical Review, 82(5):664–679, 1951.

[20] Ognyan Oreshkov, Fabio Costa, and Časlav Brukner. Efficient calculation of one-loop QCD amplitudes. Physical Review Letters, 113(13):16669–1672, 2019.

[21] Matthew J. Strassler. Field theory without Feynman diagrams: One-loop effective actions. Nuclear Physics B, 385(1-2):145–184, 1992.

[22] Michael G Schmidt and Christian Schubert. The Worldline Path Integral Approach to Feynman Graphs. arXiv:hep-ph/9412358.

[23] Olindo Corradiani and Christian Schubert. Spinning Particles in Quantum Mechanics and Quantum Field Theory. arXiv:1512.08694.

[24] A. M. Polyakov. Gauge Fields and Strings. Harwood Academic Publishers, Chur, Switzerland, 1987.

[25] Zvi Bern and David A. Kosower. Efficient calculation of one-loop QCD amplitudes. Physical Review Letters, 113:16669–1672, 2019.
[26] Aleks Kissinger and Sander Uijlen. A categorical semantics for causal structure. *Logical Methods in Computer Science*, 15(3), 2019.

[27] Prospects and limitations of information theoretic approaches. *PIRSA: 15050101*. https://perimeterinstitute.ca/videos/prospects-and-limitations-information-theoretic-approaches.

[28] F. T. Hioe and J. H. Eberly. N-Level Coherence Vector and Higher Conservation Laws in Quantum Optics and Quantum Mechanics. *Physical Review Letters*, 47(12):838–841, 9 1981.