On gravitational Stefan-Boltzmann law and Casimir effect in FRW universe

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Abstract
Both Stefan-Boltzmann law and the Casimir effect, in a universe described by the FRW metric with zero curvature, are calculated. These effects are described by Thermo Field Dynamics (TFD). The gravitational energy-momentum tensor is defined in the context of Teleparallel Equivalent to General Relativity (TEGR). Each of the two effects gives a consistent prediction with what is observed on a cosmological scale. One of the effect establishes a minimum range for the deceleration parameter. While another leads to the conclusion that a possible cosmological constant has a very small order of magnitude.

Keywords  Finite temperature · TFD · Teleparallel gravity · Casimir effect · FRW universe

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1 Introduction

The introduction of temperature in the gravitational field has been successfully implemented recently [1]. Thermo Field Dynamics (TFD) was used for this purpose which is an approach that allows both a temporal evolution of the field at finite temperature. It is an advantage over the historical approach that associates time with temperature [2]. A gravitational field at finite temperature is a theory of quantum gravity since TFD uses creation and annihilation operators. The field propagator is the fundamental entity of the thermalization process. It is interesting to note that such an approach associates the temperature with space such that a universe with zero temperature will not be expected [3]. Absolute zero temperature will not be natural even in flat space. Within the scope of TFD, there is a topological structure that allows treating effects such as diverse as the Stefan-Boltzmann law and the Casimir effect on an equal footing. An area of intense investigation into the implications of various aspects of quantum gravity is black hole thermodynamics. Whether in the investigation of entropy of black holes, or in the understanding of the information paradox. It was recently investigated how the evaporation process of a black hole generates an entanglement between quantum fields and geometry, this yields a modified Page curve that can have implications for several theories of quantum gravity [4]. It has also been shown that the structure of TFD plays a key role in this approach [5]. The TFD appears to be a promising theory of quantum gravity.

It is necessary to thermalize the energy-momentum tensor of the field in addition to a propagator. The standard model of gravitation is problematic. In the construction of gravity at finite temperature, an alternative theory of gravitation is used, Teleparallelism Equivalent to General Relativity (TEGR) [6]. In TEGR the problem of gravitational energy is well established, as well as other conserved quantities. As a result, gravitational entropy is introduced as a direct consequence of Maxwell’s relationships involving gravitational pressure. Normally this gravitational entropy may be seen as a fundamental quantity when made equal to Hawking’s expression induces a temperature of the black hole event horizon different from that commonly accepted. The whole space-time has a finite temperature, not just the event horizon of a black hole. Then there is a smooth transition from singularity to infinity [7]. We must note that TEGR is a formulation of gravitation that takes into account local Lorentz’s symmetry, such a dependence appears in the field equations that are entirely equivalent to Einstein’s equations. On the other hand, recently, proposals have emerged that attribute
the local Lorentz symmetry to the spin connection [8–10], this line of investigation has received some criticism and in our opinion still requires further investigation [11]. In TEGR the conserved quantities are sensitive to the global Lorentz transformations and that is the limit of our approach.

One of the major problems in cosmology is why there is an accelerated expansion of the universe. Usually the explanation given is an exotic energy known as dark energy. On the other hand, instead of looking for candidates for such energy, alternative explanations can be tried. This last chain of thought will be used. The more interesting features of the universe are analyzed. There is a non-zero temperature other than zero even at the most distant point in interstellar space. In addition, it has an observable dynamic horizon increasing with time. Such a horizon works as a causal barrier to events within it. Mainly this system behaves like a spherical Casimir effect. There appears to be two associated phenomena observed in the universe: (i) a thermal radiation like Stefan-Boltzmann’s law and (ii) a force a la Casimir effect responsible for an accelerated expansion of the system. This leads us to consider that gravitation at finite temperature explains such a phenomena. This hypothesis is explored here.

This article is divided as follows. In Sect. 2 TFD is introduced briefly. In Sect. 3 the TEGR is presented and the thermal expressions are calculated. In Sect. 4 the energy-momentum tensor at finite temperature is applied to the FRW universe. With this both the Stefan-Boltzmann law and the Casimir effect for a zero curvature in the metric are calculated. Finally conclusions are presented in the last section.

2 Thermo field dynamics (TFD)

A quantum field theory at finite temperature is developed by two distinct, but equivalent, approaches: (i) the imaginary time formalism [2] and (ii) the real time formalism [12–20]. TFD is a real-time finite temperature formalism. The temperature dependent vacuum is defined such that the vacuum expectation value of an arbitrary operator $A$ agrees with the statistical average, i.e.,

$$\langle A \rangle = \langle 0(\beta) | A | 0(\beta) \rangle,$$

(1)

where $| 0(\beta) \rangle$ is the thermal vacuum and $\beta = \frac{1}{k_B T}$, with $T$ being the temperature and $k_B$ the Boltzmann constant. To construct this thermal state two elements are necessary: the doubling of the original Hilbert space and the Bogoliubov transformation. This doubling is defined by $S_T = S \otimes \tilde{S}$, where $S$ is the Hilbert space and $\tilde{S}$ is the dual (tilde) space. The map between the non-tilde $A_i$ and tilde $\tilde{A}_i$ operators is given by tilde (or dual) conjugation rules. These rules are

$$(A_i A_j) \sim = \tilde{A}_i \tilde{A}_j,$$

$$(c A_i + A_j) \sim = c^* \tilde{A}_i + \tilde{A}_j,$$

$$(A_i^\dagger) \sim = \tilde{A}_i^\dagger,$$

$$(\tilde{A}_i) \sim = -\xi A_i,$$

(2)
with \( \xi = -1(\pm 1) \) for bosons (fermions). In addition, the tilde conjugation rules associate each operator in \( S \) to two operators in \( \tilde{S} \). Considering \( a \) as an operator leads to

\[
A = a \otimes 1, \quad \tilde{A} = 1 \otimes a.
\]

TFD and Bogoliubov transformations introduce thermal effects through a rotation between tilde \((\tilde{S})\) and non-tilde \((S)\) operators. With an arbitrary operator \( \mathcal{O} \), the Bogoliubov transformation is defined as

\[
\begin{pmatrix}
\mathcal{O}(k, \alpha)
\xi \tilde{\mathcal{O}}^\dagger(k, \alpha)
\end{pmatrix}
= B(\alpha)
\begin{pmatrix}
\mathcal{O}(k)
\xi \tilde{\mathcal{O}}^\dagger(k)
\end{pmatrix},
\]

where the \( \alpha \) is called the compactification parameter defined by \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{D-1}) \) and \( B(\alpha) \) is

\[
B(\alpha) = \begin{pmatrix}
u(\alpha) & -w(\alpha) \\
\xi w(\alpha) & u(\alpha)
\end{pmatrix},
\]

with \( u^2(\alpha) + \xi w^2(\alpha) = 1 \). These quantities \( u(\alpha) \) and \( w(\alpha) \) are related to the Bose distribution. For the case \( \alpha_0 \equiv \beta \) and \( \alpha_1, \ldots, \alpha_{D-1} = 0 \), the temperature effect is introduced. Using such formalism, a topological quantum field theory is considered.

A topology \( \Gamma^d_D = (S^1)^d \times \mathbb{R}^{D-d} \) with \( 1 \leq d \leq D \) is used. Here \( D \) is the space-time dimensions and \( d \) is the number of compactified dimensions. Any set of dimensions of the manifold \( \mathbb{R}^D \) can be compactified.

In the TFD formalism, all propagators are written in terms of the compactification parameter \( \alpha \). Here the scalar field propagator is defined as

\[
G_{0}^{(AB)}(x - x'; \alpha) = i \langle 0, 0 | \tau [\phi^A(x; \alpha)\phi^B(x'; \alpha)] | 0, 0 \rangle,
\]

where \( \tau \) is the time ordering operator and \( A, B = 1, 2 \). The Bogoliubov transformation is used to write as

\[
\phi(x; \alpha) = B(\alpha)\phi(x)B^{-1}(\alpha).
\]

In the thermal vacuum, which is defined as \( |0(\alpha)\rangle = U(\alpha)|0, 0\rangle \), the propagator becomes

\[
G_{0}^{(AB)}(x - x'; \alpha) = i \langle 0(\alpha) | \tau [\phi^A(x)\phi^B(x')] | 0(\alpha) \rangle,
\]

\[
= i \int \frac{d^4k}{(2\pi)^4} e^{-ik(x - x')} G_{0}^{(AB)}(k; \alpha),
\]

where

\[
G_{0}^{(AB)}(k; \alpha) = B^{-1}(\alpha)G_{0}^{(AB)}(k)B(\alpha),
\]
with
\[ G_0^{(AB)}(k) = \begin{pmatrix} G_0(k) & 0 \\ 0 & \xi G_0^*(k) \end{pmatrix}, \] (10)

and
\[ G_0(k) = \frac{1}{k^2 - m^2 + i\epsilon}, \] (11)

where \( m \) is the mass. The Green function becomes
\[ G_0^{(11)}(k; \alpha) = G_0(k) + \xi w^2(k; \alpha)[G_0^*(k) - G_0(k)]. \] (12)

Here the physical quantities are given by the non-tilde variables, i.e. \( A = B = 1 \). In addition, \( w^2(k; \alpha) \) is the generalized Bogoliubov transformation \[21\] given as
\[ w^2(k; \alpha) = \sum_{s=1}^{d} \sum_{\{\sigma_s\}} 2^{s-1} \sum_{l_{\sigma_1},...l_{\sigma_s}=1}^{\infty} (-\xi)^{s+\sum_{i=1}^{s} l_{\sigma_i}} \exp \left[ -\sum_{j=1}^{s} \alpha_{\sigma_j} l_{\sigma_j} k^{\sigma_j} \right], \] (13)

where \( \{\sigma_s\} \) denotes the set of all combinations with \( s \) elements and \( k \) is the 4-momentum.

### 3 Teleparallel gravity

Teleparallelism Equivalent to General Relativity (TEGR) is dynamically equivalent to the standard theory of gravitation formulated in a Riemann space. However, TEGR is described in terms of torsion in the Weitzenböck space. The connection in such a space is
\[ \Gamma_{\mu\lambda\nu} = e^{a}_{\mu} \partial_{\lambda} e_{a\nu}, \]

where \( e^{a}_{\mu} \) is the tetrad field. It is the dynamical variable of the theory. The relationship between the metric tensor and the tetrad field is \( g_{\mu\nu} = e^{a}_{\mu} e_{a\nu} \). The tetrad contains two important symmetries, that is the bridge between them. Lorentz symmetry (Latin indices) and the transformation of coordinates (Greek indices). It is interesting to note that the Weintzenböck connection is curvature free, while the anti-symmetric part establishes the following torsion tensor
\[ T^{a}_{\lambda\nu} = \partial_{\lambda} e^{a}_{\nu} - \partial_{\nu} e^{a}_{\lambda}. \] (14)

This connection is related to the Christoffel symbols by
\[ \Gamma_{\mu\lambda\nu} = 0 \Gamma_{\mu\lambda\nu} + K_{\mu\lambda\nu}, \] (15)
where the contortion tensor, \( K_{\mu \lambda \nu} \), is given by
\[
K_{\mu \lambda \nu} = \frac{1}{2} (T_{\lambda \mu \nu} + T_{\nu \lambda \mu} + T_{\mu \lambda \nu}),
\] (16)
with \( T_{\mu \lambda \nu} = e_{a \mu} T_{\lambda \nu}^a. \) The above identity leads to the relation
\[
e R(e) \equiv -e \left( \frac{1}{4} T_{abc} T_{abc} + \frac{1}{2} T_{abc} T_{bac} - T_a T_a \right) + 2 \partial_\mu (e T^\mu).
\] (17)

The Lagrangian density for TEGR is
\[
\mathcal{L}(e_{a \mu}) = -\kappa e \left( \frac{1}{4} T_{abc} T_{abc} + \frac{1}{2} T_{abc} T_{bac} - T_a T_a \right) - \mathcal{L}_M
\] \(
\equiv -\kappa e \Sigma_{abc} T_{abc} - \mathcal{L}_M,
\) (18)
where \( \kappa = 1/(16\pi) \), \( \mathcal{L}_M \) is the Lagrangian density of matter fields and \( \Sigma_{abc} \) is given by
\[
\Sigma_{abc} = \frac{1}{4} (T_{abc} + T_{bac} - T_{cab}) + \frac{1}{2} (\eta^{ac} T^{b} - \eta^{ab} T^{c}),
\] (19)
with \( T^a = e_{a \mu} T^\mu. \) If a derivative of Eq. (18) with respect to the tetrad field is performed, the field equation reads
\[
\partial_\nu (e \Sigma_{\lambda \nu}^{a \lambda \nu}) = \frac{1}{4\kappa} ee_{a \mu} (\tau_{\lambda \mu} + T^{\lambda \mu}),
\] (20)
where
\[
\tau_{\lambda \mu} = \kappa \left[ 4 \Sigma^{bc \lambda \nu} T_{\mu}^{bc} - g_{\lambda \mu} \Sigma^{abc} T_{abc} \right],
\] (21)
is the gravitational energy-momentum tensor. Such an expression is frame dependent and to calculate its average a class of observers must be chosen, that is, certain conditions must be imposed on the tetrad field. It is to be noted that the skew-symmetry in \( \Sigma_{a \lambda \nu} \) leads to
\[
\partial_\lambda \partial_\nu (e \Sigma_{a \lambda \nu}^{a \lambda \nu}) \equiv 0.
\] (22)
This is the conservation law. It is then possible to establish the energy-momentum vector as
\[
P^a = \int_V d^3 x e e^{a \mu} (T^{0 \mu} + T^{0 \mu}),
\] (23)
or with the help of Eq. (20), it reads
\[
P^a = 4k \int_V d^3 x \partial_\nu (e \Sigma^{a \nu}_a).
\] (24)

This is the total energy vector. It is interesting to note that it is a vector under global Lorentz transformation which implies that energy, as the zero component of this 4-vector, is not an invariant. In fact, it depends on the choice of tetrad, which determines
the very choice of the observer. On the other hand the quantity is not dependent on the coordinate choice. These are indeed desirable features for any definition of energy-momentum.

With the well-established definition of an energy-momentum tensor, the first element necessary for the application of TFD is defined. It is still necessary to obtain a propagator for the field. Using the weak field approximation

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \]  

which in Eq. (18) leads to

\[ \langle e_{b\lambda}, e_{d\gamma} \rangle = \Delta_{bd\lambda\gamma} = \frac{\eta_{bd}}{k q^b q^d}. \]  

This is the graviton propagator [22]. Then the Green function is

\[ G_0(x, x') = -i \Delta_{bd\lambda\gamma} g^{\lambda\gamma} \eta_{bd}. \]  

Explicitly it is

\[ G_0(x, x') = -i \frac{64\pi}{q^2}, \]  

with \( q = x - x' \), where \( x \) and \( x' \) are four vectors. With the weak field approximation the gravitational energy-momentum tensor \( t^{\lambda\mu} \) becomes

\[ t^{\lambda\mu}(x) = k \left[ g^{\mu\alpha} \partial^\gamma e^{b\lambda} \partial_{\gamma} e_{b\alpha} - g^{\mu\gamma} \partial^\alpha e^{b\lambda} \partial_{\gamma} e_{b\alpha} - g^{\mu\alpha} (\partial^\lambda e^{b\gamma} \partial_{\gamma} e_{b\alpha} - \partial^\lambda e^{b\gamma} \partial_{\alpha} e_{b\gamma}) 
- 2g^{\lambda\mu} \partial^\gamma e^{ba}(\partial_{\gamma} e_{b\alpha} - \partial_{\alpha} e_{b\gamma}) \right]. \]  

For dealing with the mean of the energy-momentum tensor the standard procedure is to consider it at different points of the space and then take the limit. This avoids divergences. Hence

\[ \langle t^{\lambda\mu}(x) \rangle = \langle 0 | t^{\lambda\mu}(x) | 0 \rangle, \]
\[ = \lim_{x^\mu \to x'^\mu} 4i k \left( -5g^{\lambda\mu} \partial^\gamma \partial_{\gamma} + 2g^{\mu\alpha} \partial^\alpha \partial_{\gamma} \right) G_0(x - x'), \]  

where \( \langle e^\lambda_c(x), e_{ba}(x') \rangle = i \eta_{cb} \delta^\lambda_\alpha G_0(x - x') \). This average applies to any metric that is related to the linearized Einstein’s equations. On the other hand, the validity of this expression is restricted to stationary observers.
4 Stefan-Boltzmann law and Casimir effect in FRW universe

The TEGR expression in the weak field approximation leads to the TFD framework. The mean value of the energy-moment tensor becomes

$$\langle t^{\lambda\mu}(AB)(x; \alpha) \rangle = \lim_{x \rightarrow x'} 4i\kappa \left( -5g^{\lambda\mu} \partial^{\nu} \partial_{\nu} + 2g^{\mu\alpha} \partial^{\lambda} \partial_\alpha \right) G_0^{(AB)}(x - x'; \alpha). \quad (31)$$

If we use the Casimir prescription,

$$T^{\lambda\mu}(AB)(x; \alpha) = \langle t^{\lambda\mu}(AB)(x; \alpha) \rangle - \langle t^{\lambda\mu}(AB)(x) \rangle, \quad (32)$$

then

$$T^{\lambda\mu}(AB)(x; \alpha) = \lim_{x \rightarrow x'} \Gamma^{\lambda\nu}(x, x') G_0^{(AB)}(x - x'; \alpha), \quad (33)$$

where

$$\Gamma^{\lambda\nu} = 4i\kappa \left( -5g^{\lambda\mu} \partial^{\nu} \partial_{\nu} + 2g^{\mu\alpha} \partial^{\lambda} \partial_\alpha \right), \quad (34)$$

and

$$G_0^{(AB)}(x - x'; \alpha) = G_0^{(AB)}(x - x'; \alpha) - G_0^{(AB)}(x - x'). \quad (35)$$

It is necessary to establish the appropriate space-time geometry i.e., analysing the result of such expressions on cosmological scales. A homogeneous and isotropic universe is chosen. The suitable line element is

$$ds^2 = -dt^2 + a(t) \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \quad (36)$$

which is the FRW line element of zero curvature. This metric respects the approach used, as well as the constraints arising from the experiments. If Eq. (34) is used together with Eq. (36), then

$$\Gamma^{00} = \frac{i}{4\pi} \left[ -3\partial_0' \partial_0 + \frac{5}{a^2} \left( \partial_1' \partial_1 + \frac{1}{r^2} \partial_2' \partial_2 + \frac{1}{r^2 \sin^2 \theta} \partial_3' \partial_3 \right) \right] \quad (37)$$

and

$$\Gamma^{11} = \frac{i}{4\pi a^2} \left[ 5\partial_0' \partial_0 - \frac{5}{a^2} \left( \frac{3}{5} \partial_1' \partial_1 + \frac{1}{r^2} \partial_2' \partial_2 + \frac{1}{r^2 \sin^2 \theta} \partial_3' \partial_3 \right) \right]. \quad (38)$$

Using these relations to calculate the energy and pressure for Stefan-Boltzmann law and the Casimir effect according to the Bogoliubov transformation is desired.
4.1 Gravitational Stefan-Boltzmann law

To calculate the Stefan-Boltzmann law, $\alpha = (\beta, 0, 0, 0)$ is chosen, which leads to the Bogoliubov transformation

$$v^2(\beta) = \sum_{j_0=1}^{\infty} e^{-\beta k^0 j_0}, \quad (39)$$

where $\beta = \frac{T}{t}$. Then the Green function is

$$G^{(11)}_0(x - x'; \beta) = 2 \sum_{j_0=1}^{\infty} G^{(11)}_0(x - x' - i \beta j_0 n_0), \quad (40)$$

where $n_0 = (1, 0, 0, 0)$ and the physical component $(AB) = (11)$ is chosen, then

$$E = \frac{32\pi^4}{15} T^4, \quad (41)$$

and

$$P = \frac{32\pi^4}{45a^2} T^4, \quad (42)$$

with $E = \langle t^{00(11)}(x; \beta) \rangle$ and $P = \langle t^{11(11)}(x; \beta) \rangle$. It is interesting to note that the pressure is dependent on the scale factor which in turn is expanded as

$$a = 1 + H_0 (t - t_0) - \frac{q_0 H_0^2}{2} (t - t_0)^2, \quad (43)$$

where $H_0$ and $q_0$ refer to the Hubble constant and the deceleration parameter respectively. So when $a = 1$, the state equation becomes $P = \frac{E}{3}$, which is to be the expected state equation for the graviton. Taking into account the relation $(\frac{\partial P}{\partial T})_V = (\frac{\partial S}{\partial V})_T$, the entropy density is

$$s = \frac{S}{V} = \frac{128\pi^4}{45a^2} T^3. \quad (44)$$

Using the expansion for the scale factor above, the second time derivative of the entropy density is

$$\ddot{s} = -\frac{256\pi^4 T^3}{45a^2} \left[ \frac{\ddot{a}}{a} - 3 \left( \frac{\dot{a}}{a} \right)^2 \right], \quad (45)$$

where dot means a time derivative. Here the Landau theory of second order phase transition is involved. A divergence in the second derivative of the entropy determines
a critical quantity that characterizes the phase transition. Here it is assumed that time is the dynamic variable. Hence $s \to \infty$ implies $a = 0$. If $\tau = H_0 (t - t_0)$ is defined as an auxiliary variable, then it follows that

$$\frac{q_0}{2} \tau^2 - \tau - 1 = 0. \quad (46)$$

This imposes a constraint on the current deceleration parameter, such that $q_0 \geq -\frac{1}{2}$. This is an interesting result considering that the deceleration parameter is written in terms of the main cosmological parameters. Results obtained in [23] showed that using the broad (truncated) Gaussian $q_0 = -0.5 \pm 1$, it is indeed possible to obtain a competitive constraint on the Hubble constant. These results are consistent with phenomenological models of the interaction rates [24] using the latest microwave background observations from Planck 2018 and baryon acoustic oscillations measurements.

### 4.2 Casimir effect

The Casimir effect is described in TFD with the choice $\alpha = (0, i2d, 0, 0)$, where $d$ is the radius of the outer spherical surface. This leads to the Bogoliubov transformation

$$v^2(d) = \sum_{l_1=1}^{\infty} e^{-i2d k_1 l_1}. \quad (47)$$

If the Green function is given by

$$\overline{G_0^{(11)}(x - x'; d)} = 2 \sum_{l_1=1}^{\infty} G_0^{(11)}(x - x' - 2dl_1 n_1), \quad (48)$$

with $n_1 = (0, 1, 0, 0)$, then

$$E_c = -\frac{2\pi^4}{45d^4a^4}, \quad (49)$$

and

$$P_c = -\frac{2\pi^4}{15d^4a^6}. \quad (50)$$

This result is obtained by choosing the physical component of Green’s function $(AB) = (11)$. The same identification for the average energy-momentum tensor, $E_c = \langle t^{00(11)}(x; d) \rangle$ and $P_c = \langle t^{11(11)}(x; d) \rangle$. Two important features need to be highlighted. The first is the Casimir energy and pressure are obtained in a vacuum. A time-dependent negative pressure is consistent with an accelerating expanding
universe. The second is that Casimir pressure is associated with the cosmological constant $\Lambda$. In natural units, the pressure and the cosmological constant have the same dimension. Thus the cosmological constant is understood as a fluid with the following pressure

$$p = -\frac{c^4 \Lambda}{G},$$

thus this given by, in units of the international system,

$$\Lambda = \frac{2\pi^4 G \hbar}{15d^4 c^5},$$  \hspace{1cm} (51)

in the present time. In this estimate the outer surface is used as the observable radius of the universe, this determines $d$ as $10^{10}$ light years. The cosmological constant is of the order of $10^{-180}$ m$^{-2}$. It is interesting to note that in an incipient universe $\Lambda$ was much larger than it is today.

5 Conclusion

The Stefan-Boltzmann law and the Casimir effect are analyzed in a homogeneous and isotropic universe. The FRW metric for zero curvature is used. From the Stefan-Boltzmann law it is possible to understand that there is an energy and pressure from strictly gravitational thermal radiation. The entropy density provides for a phase transition that limits the range of the deceleration parameter. The Casimir effect establishes a negative pressure consistent with an accelerated expanding universe. Such a quantity when interpreted as the observable radius of the universe leads to the conclusion that the cosmological constant is small. It is important to note that due to its temporal evolution, the cosmological constant played a more relevant role in a primordial universe. When Casimir effect is established at finite temperature, imaginary quantities are obtained. This leads to interpret that the temperature effect in the universe is independent of the pressure exerted by the vacuum. Perhaps both effects are linked on a smaller scale when quantum effects are more relevant.

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