Angular and Linear Momentum in General Relativity: Their Geometric Structure and Interrelation

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Abstract

Generalized definitions for angular and linear momentum are given and shown to reduce to the ADM (at spatial infinity) definitions and the definitions at null infinity in the appropriate limit. These definitions are used to express angular momentum in terms of linear momentum. The formalism allows one to see the connection with the classical and special relativistic notions of momenta. Further, the techniques elucidate, for the first time, the geometric nature of these conserved quantities. The boosted Schwarzschild solution is used to illustrate some aspects. The definitions are useful and give insight in the region far from all masses where gravity waves are detected.

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1 Introduction

The key conserved quantities (energy, linear momentum and angular momentum) are far from intuitive in general relativity. The non-locality of the gravitational field, the lack (in general terms) of symmetries, and the general coordinate invariance of the equations make such definitions non-trivial.

Linear and angular momentum are the most subtle of the three conserved quantities. Linear momentum was not defined at null infinity\(^1\) until the 1960’s ([2],[3],[4]) and angular momentum waited until the late 1990’s to get such a definition (cf. [5],[1],[6],[7]). Previous to these definitions only the spatial (ADM [8],[9],[10],[11],[12]) definition of these quantities were known; that is,

\(^1\)“Null infinity” is the place and time at which gravity waves arrive far from all matter in an asymptotically flat space-time (cf. [1]). The phrase “null infinity” arises out of the “conformal” picture where one “brings” infinity into a finite place by the appropriate division in the limit.
only definitions that give the “total momentum” for the whole space-time were available. The definitions at null infinity are particularly valuable at present because they allow the exchange of momentum due to gravitational radiation, and experimental science, e.g. in the form of LIGO or VIRGO, is rapidly moving to empirically investigate gravitational radiation and its sources.

Here I give the generalized definitions of linear and angular momentum and show that they reduce to the proper form for the ADM (spatial) and null definitions. Furthermore, I give, for the first time also, the geometric interpretation of both these quantities. In this context, it is shown that light rays “at” null infinity can be used to probe the structure of the space-time. Since the space-time structure results from a source at a given retarded time one can thereby obtain the momenta of the source at that time. In addition, this paper connects the definitions in GR with our classical and special relativistic understanding. The calculations are in themselves instructive because they reveal something of the facility and of the mechanics of using the Christodoulou-Klainerman formalism which is so useful for far field problems involving gravity waves.

In section two, we setup the foliation of space-time needed to facilitate the analysis. In section three, we generalize the notion of curvature with an eye toward generalizing the momenta; we take inspiration from the result given in [1]. Section four, gives the decay law for the extrinsic curvature. Section five, generalizes the linear momentum. Section six shows that it is the correct generalization and reminds us that [1] shows that the angular momentum can be written using the generalized curvature in the manner shown there. Finally, the last section (seven) discusses the geometrical insights gained by the process and the interrelation between the linear and angular momentum. This section uses the Schwarzschild space-time to illustrate the point.

2 The Foliation

We begin by deciding on the foliation of a generic asymptotically-flat [13] space-time that will be needed to do the calculations in the subsequent sections. The foliation will give us the surfaces of integration we need for the definitions and facilitate taking the appropriate limits to null and spatial infinity. We will use the foliation defined in [5] (for more foliations confer [13],[14], [15],[1] and [7]). The description of the foliation given in [5] is repeated here for the convenience of the reader. The foliation is defined pictorially in figure 1; it chops the space-time near $C_s^-$ into topologically $S^2$ surfaces. The foliation is created by starting with a maximal (one with traceless extrinsic curvature) hypersurface at some time $t$, $\Sigma_t$, and encircling, far from the matter, all matter in the slice with a surface of constant potential (i.e. use radius coordinate, $r_{\text{laplace}}$, obtained from a level surface of the solution of Laplace’s equation for an electric charge in the spatial hypersurface $\Sigma_t$). As one moves farther and farther away from all mass, the radius coordinate $r_{\text{laplace}}$ defines more and more closely a sphere
until finally, “at infinity,” it is precisely so. Label one of these far away surfaces on $\Sigma_t$, $S_{-k}$. Light rays moving inward from $S_{-k}$ with initial tangent vector $l' = T' - N'$, where $T'$ is the time direction normal to $\Sigma$ and $N'$ is the normal in $\Sigma$ perpendicular to $S_{-k}$, form a cone (topology $S^2 \times R$) called $C^-$. The tangents to the null rays define $l$ on the cone. Define an affine parameter $u$ such that: $lu = 1$ and $u = -k$ on $S_{-k}$ with $k$ large. All the $S^2$ surfaces that foliate the cone are labeled by the same affine parameter, $s$, such that $ls = 1$ and $s = r \equiv \sqrt{\frac{\text{Surface Area of } S_{0,s}}{4\pi}}$ that is, using the areal radius of the $u = 0$ surface, $S_{0,s}$. Next, define the outgoing null vector, $l$, at each point on the cone by requiring $l \cdot l = -2$. For each constant $u$ surface on $C^-_u$, $S_{u,s}$, use $l$ to send light rays into the space-time to generate a new cone, $C^+_u$. The topologically $S^2$ surfaces given by $s$ and $u$ constant constitute leafs of a foliation of space-time near $C^-_u$. $C^-_u$ becomes null infinity as the radius coordinate defined by Laplace’s equation goes to infinity.\(^2\) Hence, the foliation is a “near null infinity” foliation. Finally, on each point of each leaf, we put a set of appropriately normalized null tetrads; two spatial vectors, $e_A$ where $A \in \{1, 2\}$ and we already have two null vectors: $e_3 = l$, $e_4 = l$ (tangent vector of the appropriate geodesic of $C^+_u$). The connection coefficients, $\Gamma_{abc} = e_a \cdot \nabla_c e_b$ where $\nabla$ is the covariant derivative with respect to the space-time metric $g_{\mu\nu}$ and $a, b, c \in \{1, 2, 3, 4\}$, are called the Christoffel symbols when a coordinate basis is used. With a null tetrad system, they are called the null Ricci rotation coefficients $\gamma_{abc}$. For convenience the Ricci Coefficients are defined in appendix I.

In short, the affine parameters, $s$ and $u$ defined above serve to label the topologically $S^2$ surfaces $S_{u,s}$; it is on such surfaces that the energy and momentum integrals are to be taken.

### 3 Generalized Curvature

We can now generalize the extrinsic curvature of a maximal slice $\Sigma_t$ to an extrinsic curvature more appropriate to the $S^2$ slicing of the foliation. We then will evaluate the important curvature components in terms of Ricci rotation coefficients (cf. appendix I of this paper and [1] for summary, see also. [13],[15],[5]) for the general case (arbitrary “lapse function,” $a$) and for two specific cases. We, first, recall the definition of the extrinsic curvature of a maximal slice $\Sigma_t$:

\[
K_{ij} = -(D_i T^l, e_j) \tag{1}
\]

where $i, j \in \{1, 2, 3\}$, the spatial indices

and $D_\mu$ is the covariant derivative in the $\mu$ direction \(^2\)

\(^2\)Taking the limit makes all the previous construction precise. One also lets $S_{-k} \to S^-_{-\infty}$ so that $-\infty < u < \infty$ on null infinity. Further, note the foliation construction picks out an origin on the initial spatial slice.
Next, recall that the standard null pair is: \((13, 1)\) \(l' = T' + N'\) and \(\vec{l}' = T' - N'\), where \(T'\) is the unit vector orthogonal to the maximal slice and \(N'\) is the unit vector perpendicular to the \(S^2\) slice within the maximal slice \(\Sigma_t\).

Using quantities more natural to the \(S^2\) slicing, one can rewrite equation 1. One can write the bi-normal to the \(S^2\) surface as \(\mathbf{B}' = \frac{1}{2}(l' + \vec{l}')\) and the vector \(N' = \frac{1}{2}(l' - \vec{l}')\) normal to this and normal to the spatial vectors \(\mathbf{e}_A\) in the tangent space of the \(S^2\) surface. Which allows one to generalize equation 1 as:

\[
k'_{ij} = -(D_i \mathbf{B}', e_j)
\]

where we introduced the prime on the \(k_{ij}\) to indicate the use of the standard pair, \(l', \vec{l}'\). For future reference, \(k^g\) refers to the \(k\) curvature associated with the geodesic pair \(^3(l = a^{-1} l', \vec{l} = a \vec{l}')\) (also could be written \(^3(l', \vec{l}')\)) and \(k^t\) refers to that associated with the \(t\)-null pair \((l^t = \phi l', \vec{l}' = \phi^{-1} \vec{l}')\). For instance, \(\mathbf{B}^t = \frac{1}{2}(l^t + \vec{l}')\) and \(k^t_{ij} = -(D_i \mathbf{B}^t, e_j)\).

A brief aside can give an intuitive understanding of the meaning of the lapse transformation and hence further manifest the need to use the appropriate lapse function. Such insight comes from studying the lowest dimensional special relativistic case. In 2-D Minkowski space-time, pick an inertial frame, and establish the following natural coordinates \(\{t, x\}\) (or equivalently, \(\{t, r\}\)). The standard null pair is then: \(l' = \{1, 1\}, \vec{l}' = \{1, -1\}\). If we now boost to a different frame, labeled by superscript \(\rightarrow\), moving at speed \(\beta\), we have, in the new frame, \(l'^\rightarrow = a^{-1} l', \vec{l}'^\rightarrow = a \vec{l}'\) where \(a = \sqrt{1 - \beta^2}\). In this way, it is clear that use of this null pair in the boosted frame is inappropriate, because it has the wrong lapse function.\(^4\) Its lapse function is the one appropriate for the initial frame, because its binormal is the time direction in that frame. Said another way, the choice of binormal, \(\mathbf{B}\), is a choice rest frame with its time direction pointed along \(\mathbf{B}\), and in this frame, the tangent vectors to the light rays emitted inward and outward at rest in that frame will give the appropriate null pair (respectively, \(l\) and \(\vec{l}\)) for that frame.

Now, we can write the general \(k_{ij}\) in terms of the Ricci rotation coefficients (cf. appendix I and \cite{13}, also cf appendix C of \cite{1}). Here, we use a superscript \("any\" on \(k_{ij}\), \(k^{\text{any}}\), to indicate the general nature of the calculation; i.e. that we have not specified a particular \(l\)-pair.

\[
k^{\text{any}}_{AB} = - \left(D_A \left( \frac{1}{2}(l + \vec{l}) \right), e_B \right)
= -\frac{1}{2} (\chi_{AB} + \Sigma_{AB})
\]

\(^3\)To be complete and unambiguous, this null pair should be called the affine geodesic pair, because it has the "normalization" that makes the tangent vector to the geodesic null ray correspond to an affine parameterization of the geodesic. However, because the affine parameter is the most natural to the null geodesic, we will use the term "geodesic null pair."

\(^4\)A lapse transformation is necessary to get to obtain the null pair appropriate to the boosted frame.
\[ k_{AN}^{\text{any}} = -(D_A B, N) \]
\[ = \frac{1}{4} (D_A (l + \mathbf{l}), (l - \mathbf{l})) \]
\[ = -\frac{1}{4} (- (D_A l, \mathbf{l}) + (D_A \mathbf{l}, l)) \]
\[ = V_A \]

\[ k_{NN}^{\text{any}} = -\frac{1}{8} (D_{(l-l)} (l + \mathbf{l}), (l - \mathbf{l})) \]
\[ = -\frac{1}{8} (-(D_{(l-l)} l, \mathbf{l}) + (D_{(l-l)} \mathbf{l}, l)) \]
\[ = -\frac{1}{8} (- (D_l l, \mathbf{l}) + (D_l \mathbf{l}, l) - (D_{\mathbf{l}} l, l)) \]
\[ = -\frac{1}{8} (-4\Omega - 4\Omega - 4\Omega - 4\Omega) \]
\[ = \Omega + \Omega \]

\[ k_{BB}^{\text{any}} = -(D_B B, N) \]
\[ = -\frac{1}{8} (D_{(l+\mathbf{l})} (l + \mathbf{l}), (l - \mathbf{l})) \]
\[ = -\frac{1}{8} (-(D_{(l+\mathbf{l})} l, \mathbf{l}) + (D_{(l+\mathbf{l})} \mathbf{l}, l)) \]
\[ = -\frac{1}{8} (- (D_l l, \mathbf{l}) - (D_{\mathbf{l}} l, l) + (D_{\mathbf{l}} l, \mathbf{l}) + (D_{\mathbf{l}} \mathbf{l}, l)) \]
\[ = \Omega - \Omega \]
\[ k_{BB}^{\text{any}} = -(D_B B, B) = 0 \]

where: \( A, B \in \{1, 2\} \), the indices for vectors tangent to \( S^2 \)

note: \( B \in \{1, 2\} \) is not bold to distinguish it from the bi-normal.

In the above, use is made of the fact that \( l \cdot l = 0 \) and \( \mathbf{l} \cdot \mathbf{l} = 0 \); i.e., they are null vectors. Also, \( \mathbf{l} \cdot \mathbf{l} = -2 \).

Now, for the case of the \( k'_{ij} \ (l' \cdot \mathbf{l}') \), one obtains (using the Ricci rotation coefficients):

\[ k'_{AB} = -\frac{1}{2} (\chi'_{AB} + \chi'_{AB}) \]
\[ k'_{AN} = \epsilon_A \]
\[ k'_{NN,N'} = \frac{1}{2} (-\nabla_N \ln \phi + \delta) + \frac{1}{2} (\nabla_N \ln \phi + \delta) = \delta \]

These second and last equalities are tautological because \( \epsilon_A \) and \( \delta \) are defined to be \( k_{AN} \) and \( k_{NN} \) respectively. Although, these relations do not give us new values for \( k'_{AN} \) and \( k'_{NN,N'} \), they do confirm the correctness of the Ricci coefficients and the previous calculations. Also, note using

\[ tr\chi' + tr\chi' = 2\delta \]

(cf. [13] page 500), one sees that, in an orthonormal basis: \( trk' = \gamma^{AB}k'_{AB} + g_{NN'}k'_{NN'} = (-\delta + \delta) = 0 \) as expected for a maximal hypersurface.

In the case of \( k^g_{ij} \), (the geodesic null pair) one obtains:

\[ k^g_{AB} = -\frac{1}{2}(\chi_{AB} + \chi_{AB}') \]
\[ k^g_{AN} = W_A = \epsilon_A + \nabla_A \ln a \]
\[ k^g_{NN} = \Omega + \Omega' \]
\[ = 0 + \frac{a}{2}(\nabla_N \ln \phi + \delta) - \frac{1}{2} D'_a \]

4 Fall-off Laws for Curvature Near Null Infinity

To complete our task of generalizing and understanding linear and angular momentum, we will need the expansion for large \( r \) (defined in section two) of the \( k'_{ij} \). To do this expansion, we need, in turn, the decay laws for \( a \), \( \phi \) and \( \delta \) and \( \epsilon_A \). Using the results on page 510 [13] we write:

\[ \lim_{C_{u,t} \to \infty} r^2 \nabla_N \ln \phi = \Omega\phi \]
\[ \lim_{C_{u,t} \to \infty} r \ln a = \Psi' \]

\( \Omega\phi \) is a function on the \( S^2 \) surface and a function of the retarded time, \( u \). As a caution against confusion, note that this symbol, \( \Omega\phi \), is not the Ricci coefficient \( \Omega \). \( \Psi' \) is only a function on the \( S^2 \) surface. Both functions are defined in appendix II, which is taken from page 504 of [13].

This implies using equation 9 and appendix III of this paper:

\[ D' a = D a' a \]
\[ = \left(1 + \frac{\Psi'}{r}\right)D_l\left(\frac{\Psi'}{r}\right) \]
\[ = -\frac{\Psi'}{r^2}D_2r + o(r^{-2}) \]
\[ = \frac{\Psi'}{r^2} \]

Also, we have the following two decay laws (pg 508 [13]) taken in an orthonormal basis:

\[
\lim_{C_{u,t} \to \infty} r^2 \delta = \Omega_\phi + \Psi' \equiv \delta^{(2)} \tag{10}
\]

\[
\lim_{C_{u,t} \to \infty} r^2 \epsilon_A \equiv E_A
\]

Also, note:

Using equation 3 one gets:

\[
 \frac{2}{r} + \frac{H'}{r^2} + \left(-\frac{2}{r} + \frac{H'}{r^2}\right) = \frac{2\delta^{(2)}}{r^2}
\]

One gets:

\[
 H' + H' = 2\delta^{(2)}
\]

Hence, for \( k' \) we have:

\[
 k'_{AN} \sim \frac{E_A}{r^2}
\]
\[
 k'_{NN} \sim \frac{\delta^{(2)}}{r^2}
\]
\[
 \text{tr} \ k' \sim k_A + k_N = 0
\]

For \( k^g \) we have:

\[
 k^g_{AN} \sim \frac{E_A}{r^2} + \frac{1}{r^2} \Psi'
\]
\[
 k^g_{NN} \sim \frac{\Omega_\phi}{r^2}
\]
\[
 \text{tr} \ k^g = k^g_A + k^g_N = \frac{\Psi'}{r^2}
\]
With the details of the calculation, the last line above becomes:

\[
tr \ k^g = k^g_A + k^g_N N
\]

\[
\sim - \frac{1}{2} \left( a^{-1} \left( \frac{2}{r} + \frac{H'}{r^2} \right) + a \left( -\frac{2}{r} + \frac{H'}{r^2} \right) \right) + k^g_{NN} N^N N
\]

\[
\sim - \frac{1}{2} \left( (1 - \frac{\Psi'}{r}) \left( \frac{2}{r} + \frac{H'}{r^2} \right) + (1 + \frac{\Psi'}{r}) \left( -\frac{2}{r} + \frac{H'}{r^2} \right) \right) + k^g_{NN} N^N N
\]

\[
\sim - \frac{1}{2} \left( 2 \frac{H'}{r^2} - 2\Psi' \right) + k^g_{NN} N^N N
\]

\[
\sim \left( -\frac{\delta^2}{r^2} + 2\Psi' \right) + k^g_{NN} N^N N
\]

The second line makes use of equation 5 and the expansions for \( tr \chi \) and \( tr \chi' \). The last line is kept in a suggestive form for use in calculating the linear momentum, which is our next step.

## 5 Generalizing the Linear Momentum

The question before us now is how to generalize the standard ADM (spatial) definition of linear momentum (cf. page 11 of [13]):

\[
P_i^N(ADM) = \frac{1}{8\pi} \lim_{r \to \infty} \int \frac{(k_i N^r - tr k^i g_i N^r)}{r} d\mu_\gamma \quad (11)
\]

Here \( \gamma \) is the metric on the \( S^2 \) surface and \( d\mu_\gamma \) is an area element on it.

The definition that takes a limit to null infinity, instead of a spatial limit, is more general, because the null definition includes the spatial as a special case. Before just changing the limit in \( P_i^N(ADM) \), consider that the ADM definition of angular momentum taken to null infinity does not yield the proper null definition [1]. The correct definition is obtained if one uses the geodesic null pair instead of the standard null pair [1]. Intuitively, this is no big surprise, since the geodesic null pair is appropriate to the null region whereas the standard pair is appropriate to the spatial slice.

Hence, by analogy with null angular momentum, one should use the null geodesic pair and obtain:

\[
P_i^G(null) = \frac{1}{8\pi} \lim_{u=const \to \infty} \int (k_i^g - tr k^g g_i N^r) e_i^\sigma d\mu_\gamma \quad (12)
\]

We now must show that this equivalent to the Bondi momentum.
\[ P_{\gamma}^g = \frac{1}{8\pi} \lim_{r \to \infty} \int \left( (\Omega_{\phi} + \Psi') - \nabla_0^0 \cdot E \right) (Y_{1-1} - Y_{11}) d\mu_{\gamma_0} \]
\[ I_1 = \frac{k_1}{8\pi} \int (\Omega_{\phi} + \Omega'_{\phi}) k_1 (Y_{1-1} - Y_{11}) \, d\mu_{\gamma_0} \]
\[ I_2 = k_1 \int \left( 2\Psi' + \mathcal{L}^0 \Psi' - P + \frac{1}{2} \sum \Xi \right) (Y_{1-1} - Y_{11}) \, d\mu_{\gamma_0} \]
\[ I_3 = \int \mathcal{L}^0 \Psi' (Y_{1-1} - Y_{11}) \, d\mu_{\gamma_0} \]

Where \( k_1 = \sqrt{2\pi} \)

In the second from the last equality use is made of \( \triangle_{\gamma_0} = -\triangle_{\gamma} \). The first two terms of the integral are defined to be \( I_1 \), the second two are \( I_2 \), and the third two are \( I_3 \).

Starting with \( I_1 \):

\[ I_1 = \frac{1}{8\pi} \int (\Omega_{\phi} + \Omega'_{\phi}) k_1 (Y_{1-1} - Y_{11}) \, d\mu_{\gamma_0} \]

Using \( \Omega_{\phi} \) and \( \Omega'_{\phi} \) from page 504 17.0.9 (and making the substitution \( u \to -u \) to bring notation in line as per above) gives:

\[ \int_{S^2} \left( \int_{-\infty}^{\infty} du \int_{S^2} \frac{|\Xi|^2(u')}{|\hat{x} - \hat{x}'|} d^2x' + \frac{1}{2} \int_{-\infty}^{\infty} du' \text{sgn}(u + u') \frac{|\Xi|^2(u')}{|\hat{x} - \hat{x}'|} \right) (Y_{1-1} - Y_{11}) \, d\mu_{\gamma_0} \]

The expansion below implies that each of the terms in the large brackets are proportional to \( Y_{00} \).

\[ \frac{1}{|\hat{x} - \hat{x}'|} = 4\pi \Sigma_{\ell=0}^{\infty} \Sigma_{m=-\ell}^{\ell} \frac{1}{2\ell + 1} Y_{\ell,m}^{*}(\theta', \varphi') Y_{\ell,m}(\theta, \varphi) \]

where \( |\hat{x}| = |\hat{x}'| = 1 \)

Hence:

\[ I_1 = 0 \]

Now look at \( I_2 \):

\[ I_2 = k_1 \int \left( 2\Psi' + \mathcal{L}^0 \Psi' \right) (Y_{1-1} - Y_{11}) \, d\mu_{\gamma_0} \]

Note that:

\[ \int \mathcal{L}^0 \Psi' (Y_{1-1} - Y_{11}) \, d\mu_{\gamma_0} = \int \Psi' \mathcal{L}^0 (Y_{1-1} - Y_{11}) \, d\mu_{\gamma_0} \]

\[ = \int -2\Psi' (Y_{1-1} - Y_{11}) \, d\mu_{\gamma_0} \]

Where use was made of: \( \int \mathcal{L}^0 f = -\int \nabla \Psi \nabla f = \int \Psi \nabla f \)

Hence we have:

\[ I_2 = k_1 \int (2\Psi' - 2\Psi') (Y_{1-1} - Y_{11}) \, d\mu = 0 \]
and

\[ P'_x = \lim_{C, t \to \infty} \frac{k_1}{8\pi} \int \left( -P + \frac{1}{2} \Sigma \cdot \Xi \right) (Y_{1-1} - Y_{11}) \, d\mu \]  
(21)

Of course, the remaining components of \( P' \) at null infinity, \( (P'_y, P'_z) \) follow in the same way.

This equation (21) is the same as that given in [16] (also see [17],[18]) for the Bondi linear momentum. The Bondi momentum is a general definition for linear momentum, because it is valid for all of null infinity and hence also for spatial infinity; that is, it reduces to the ADM definition in that limit.

- Equation 13 above for \( P^g \) is important for its geometrical interpretation; it can be written as:

\[ P^g_i = \frac{1}{8\pi} \lim_{r \to \infty} \int \left( k^g_{AN} \nabla_A x_i + \frac{1}{2} (tr \chi + tr \chi) \partial_N x_i \right) \, d\mu \gamma \]

\[ P^g_i = \frac{1}{8\pi} \lim_{r \to \infty} \int (W_A \nabla_A x_i + \frac{1}{2} (tr \chi + tr \chi) \nabla_N x_i) \, d\mu \gamma \]  
(22)

Where \( W_A \) is the Ricci Coefficient for the null geodesic pair (cf appendix C of [1]). The geometrical insight that equation 22 contains will be discussed in section seven.

5.1 Limits to Null and Spatial Infinity

Although the null definition is already a generalized definition, there is a slightly higher level of generality that can be achieved. Specifically, it would be nice to be able to say that the new definition of linear momentum, which uses the geodesic null pair (equation 12, i.e. \( \int (k^a_{\gamma N} - tr k^a_{\gamma N}) e_i^a d\mu \gamma \equiv Q^g \) can be taken to null infinity or to spatial infinity and give respectively the null and ADM spatial definition. This statement is demonstrated using the fact that the limit to spatial infinity of the quantity, \( \int (k^{\text{any}}_{\gamma N} - tr k^{\text{any}}_{\gamma N}) e_i^a d\mu \gamma \equiv Q^{\text{any}} \), is invariant under choice of lapse function \( a \) of the null pair chosen in the generalized curvature. Specifically, we know that the limit of the quantity, \( Q^{\text{standard}} \), using the standard null pair gives the standard spatial (ADM) definition of linear momentum, thus the use of the geodesic pair must yield the same limit because of the invariance under choice of lapse function.

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5 This can be seen fairly directly in the case of linear momentum by just noting that the limits go through essentially unchanged (cf. appendix D of [1]) if one takes limits along \( \Sigma_i^s \) allowing \( r \to \infty \).

6 Consider the invariance under lapse transformation from \( k' \) to \( k^{\text{any}} \) induced by the lapse function written as \( a \sim 1 + \frac{1}{r} \), where \( a1 = \Psi' \) for the transformation to the geodesic null pair. The basic invariance arises because the terms involving \( a1 (\Psi') \) in the first term of \( \int (k^{\text{any}}_{\gamma N} - tr k^{\text{any}}_{\gamma N}) e_i^a d\mu \gamma \) cancel with those that appear in the second term. This cancelation for the case of \( k^g \) is seen by following the \( a \) terms in equations 13-14 until the \( \Psi' \)'s cancel. Invoking the limit direction independence of the decay law (same for null and spatial limits) shown appendix D of [1], one sees the lapse transformation invariance for the spatial limit.
6 Generalized Linear and Angular Momentum

Reference [1] shows that the angular momentum defined by:

\[ L(\Omega_{(i)}) = \frac{1}{8\pi} \lim_{s \to \infty} \int_{S^2} W_A \Omega^A_{(l)} dS_\gamma \]

gives the same result in the spatial and null limit, with the gauge condition given in [1], [5] and [6].

Hence, we have the following generalized definitions for angular and linear momentum:

\[ P_i = \frac{1}{8\pi} \int (k^g_{\sigma N} - tr k^g g_{\sigma N}) \epsilon^g_i d\mu_\gamma \]  
\[ L = \frac{1}{8\pi} \int (k^A_{\sigma N} - tr k^g g_{\sigma N}) \Omega^A_{(l)} d\mu_\gamma \]

Where \( \Omega^A_{(l)} \) is the rotation vector field (see figure 3) around the \( i \)-axis.

As stated these reduce to the appropriate limit at spatial and null infinity. The more universal significance of choosing the null pair associated with the null geodesics is apparent in these equations. The significance can be interpreted as arising from the fact that, at null infinity, it is appropriate to use light rays (which of course travel on these geodesics) to probe the structure of space-time that has been established by the mass in the interior of the space-time at some time in the past. This rich insight can now be explored to understand the physical significance of the linear and angular momentum defined in equations 23 and 24.

7 Geometric Picture and Interrelation between L and P

7.1 Linear Momentum

We start our analysis with the linear momentum in the form of equation 22:

\[ p^g_i = \frac{1}{8\pi} \lim_{r \to \infty} \int \left( W_A \nabla_A x_i + \frac{1}{2}(tr_X + tr_Y) \nabla_N x_i \right) d\mu_\gamma \]

Note that the first term in the integrand is proportional to the torsion (cf. [7]) and the second is proportional to the net area increase (cf. [7]). The probing of the space-time structure by following the path of light rays reveals the meaning of these terms. Equivalently, each of the terms can be understood by thinking of our foliation and how it changes along a given direction.
The torsion, \( W_A = D_A(B^\gamma, N^\gamma) \), quantifies the degree to which the \( S^2 \) foliation surface pops forward ("out of itself") as one moves tangentially from a given point (cf. figure 2). That is, the torsion at point \( p \in S^2 \) quantifies how much the null geodesic "shifts" as one moves in the given direction. This shift can, in turn, be interpreted as the linear momentum carried in the given direction. Hence, \( k_{AN} = W_A = \zeta_A \) gives the two tangential components of linear momentum at a given point on the sphere.

To be more complete, one should say that the torsion measures the rate of twisting (cf. appendix VI) at a given point. For concreteness take the world-line in Euclidean space-time shown in figure 2; it can be written as:

\[
\vec{x}(t) = \cos \omega t \, \hat{x} + \sin \omega t \, \hat{y} + t \, \hat{t}
\]

Using the equation for the torsion given in appendix VI, one gets:

\[
torsion \sim \omega
\]

Therefore, the torsion is indeed the rate of twisting.

Now, return to our particular case of an \( S^2 \) surface. Moving an infinitesimal amount in any direction on it is equivalent to moving on a piece of a circle. Hence, if I move forward in time I am twisting forward; that is, there is a twist forward that can be interpreted locally on the foliation as linear momentum which always arises, because of the constraint, from a certain twist. Looking at the expression of conserved quantities in terms of angular variable will elucidate the issue. \( L \sim I \omega \sim m r^2 \omega \) and \( p \sim m \omega r \). In these terms, the only difference is the exponent with which the radial coordinate enters the expression. In any case, it is clear, as stated above, that the torsion is a measure of the linear momentum in the given tangential direction.

The \( \frac{1}{2}(\text{tr} \chi + \text{tr} \chi) \) term is the radial term. The first (respectively, second) term in this expression tells the degree to which the outgoing (respectively ingoing) null geodesics are expanding. Hence the sum gives the net momentum carried in the radial direction at a given point.\(^7\)

To get the Cartesian component, one must do the correct projections; for the case considered above (i.e., \( P_x \)), these projections are done with derivatives of \( x \). Finally, we must sum all the contributions to the linear momentum in the \( \hat{x} \) direction from each small patch of the sphere (\( S^2 \) surface) to get the total linear momentum of the system at the retarded time \( u \).

An example will help manifest the meaning of the tangential and radial terms described above. Further, doing the specific calculation, as is often the case, reveals and elucidates much of the meaning of the terms. We take the case of a Schwarzschild solution in isotropic coordinates.

\(^7\)Note that, because of the terms that add asymmetrically when one does a lapse transformation, the tangential and radial terms will not be separately invariant under such transformations.
\[ ds^2 = -\left(1 - \frac{m^2}{2r^2}\right)^2 dt^2 + \left(1 + \frac{m^2}{2r}\right)^4 \left(dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right) \]  

(25)

\[ = -\alpha^2 dt^2 + \alpha_2(dx^2 + dy^2 + dz^2) \]  

(26)

\[ = \left(-1 + \frac{2M}{r} - \frac{2M^2}{r^2}\right) dt^2 \]  

(27)

where: \( \alpha = \frac{1}{1 + \frac{m^2}{2r}} = \frac{\alpha_2}{\alpha_1} \); \( \alpha_2 = 1 + \frac{m^2}{2r} \); \( r = \sqrt{x^2 + y^2 + z^2} \). The last line is the metric expanded out to order \( O \left(\frac{1}{r}\right)^2 \), which is the requisite order for linear momentum.

In the Schwarzschild case, one sees that both the tangential and radial terms are zero. Note, in these coordinates, the constant time, \( t \), slice is maximal (tr \( k = 0 \)).

Now, let us boost to a frame at spatial infinity that is moving with velocity \( \vec{\beta} = \frac{\beta}{\gamma} \hat{z} \) with respect to the background field. The intuitive picture is clear; in this new frame the Schwarzschild black hole is moving at speed \( -\beta \) in the \( \hat{z} \) direction. However, the choice of coordinate transformation corresponding to the boost is ambiguous because of the gauge freedom available with the boost.

For simplicity, we will analyze the propagation of the null geodesics of the boosted Schwarzschild linear momentum on a spatial slice far from all mass (i.e., near spatial infinity). On such a slice, the natural null pair to use is the standard null pair \( (l', \tilde{l}') \). Thus, the tangential and radial terms of the linear momentum are: \( \epsilon_A \nabla_M x_i \) and \( \frac{1}{2}(tr \chi' + tr \tilde{\chi}') \nabla_M x_i \), respectively.

Calculating even in this simple case is very tricky and complicated. In fact, I have not found this calculation or even one of this type done in any detail anywhere in the literature. Another paper will remedy this deficiency in the literature ([19]). Here I will outline the method. We will consider the lowest order in \( \beta \). First, one cannot use a simple boost in the \( \hat{z} \) direction of the form:

\[ t' = t - \beta z \]  

and

\[ z' = -\beta t + \gamma z \]  

(8)

This simple boosted Schwarzschild gives, in terms of unboosted radius coordinate:

\[
  g_{\mu\nu} = \begin{pmatrix}
    1 + \frac{2M}{r} + \frac{3M^2}{r^2} & 0 & 0 & 0 \\
    0 & 1 + \frac{2M}{r} + \frac{3M^2}{r^2} & 0 & 0 \\
    0 & 0 & 1 + \frac{2M}{r} + \frac{3M^2}{r^2} & \frac{4M\beta}{r^2} + \frac{M^2}{r^2} \\
    0 & 0 & \frac{4M\beta}{r^2} + \frac{M^2}{r^2} & -1 + \frac{2M}{r} - \frac{2M^2}{r^2}
  \end{pmatrix}
\]

where \( r = \sqrt{x'^2 + y'^2 + (z' + \gamma \beta t')^2} \)

\[ ^8 \text{In lowest order in \( \beta \), these equations become: } t' = t - \beta z \text{ and } z' = -\beta t + z, \text{ because we can take } \gamma = 1. \]
One must alter the boost transformation with a gauge transformation to make resulting metric diagonal. The diagonal metric allows one to view the new time coordinate, \( t \), as generating spatial slices \( (t = \text{constant}) \), \( \Sigma_t \), that foliate the space-time with boosted slices. Finding such a transformation is non-trivial (cf. [19]). The transformation needed is:

\[
\begin{align*}
x &= xp \\
y &= yp \\
z &= zp + \beta t p \\
t &= \left(1 + \frac{4M}{rp} + \frac{9M^2}{rp^2}\right) tp + \beta zp
\end{align*}
\]

where \( rp = \sqrt{xp^2 + yp^2 + zp^2} \). Note here that we use \( rp \) to label the gauge boosted “radius” coordinate and distinguish it from the simple boosted (no gauge terms) “radius” \( r' \). To insure the metric is diagonal, we consider and calculate the momentum on the surface \( tp = 0 \).

In order to calculate the extrinsic curvature \( k_{\mu\nu} = \text{Proj}_{\Sigma}(D_{\mu}T_{\nu}) \), \( T \) must be the normal to the leaf of the foliation \( (t = \text{constant}) \) of which one wants to know the curvature.\(^9\) Hence, to calculate, for example, the \( k \) of a boosted slice in the boosted frame one must use: \( k'_{\mu'\nu'} = \text{Proj}_{\Sigma'}(D_{\mu'}T'_{\nu'}) \), where the prime on the \( T' \) means use the normal to the boosted slice and the primes on the indices mean evaluate in components in the boosted frame.\(^10\) The same issue, of course, arises for the related quantities \( \{ t r\chi, tr\chi, \text{and the torsion} \} \) which we want to calculate. In particular, as above, one must use the correct \( T \) in the definition of \( l' = T' - N' \). The extrinsic curvature is a helpful pedagogical starting point\(^11\), because its geometric meaning as the curvature of a surface as it appears embedded in the larger space-time facilitates grasping the importance of the correct choice of \( T \). This issue is very murky in the literature and so needs to be emphasized. Often the literature reads as if one can just do a simple coordinate transformation to get the \( k_{\mu\nu} \) on a second surface from the \( k_{\mu\nu} \) on a first surface. This is not true even when the surfaces are related by coordinate transformations as in the above case where one is moving from a rest surface to a boosted surface.

\(^9\) The curvature, of course, is related to the \( tr\chi \) and torsion terms above, and a similar statement applies to the calculation of them. They require the correct \( T' \) in the definition of \( l' = T' - N' \). The extrinsic curvature is used as starting point because its geometric meaning as the curvature as it appears imbedded in the larger space-time facilitates grasping the importance of the correct choice of \( T \). This issue is very murky in the literature and so needs to be emphasized.

\(^10\) It turns out to be easier in many ways to calculate the quantities directly in the gauge boosted coordinates where we will evaluate the momentum integral.

\(^11\) In fact, because one is working on a spatial slice, the first way I did the calculation of the momentum is by calculating the \( k_{ab} \) (where \( a, b \in \{e_A, N\} \)) and then using the expression for \( tr\chi + tr\chi \) and \( e_A \) in terms of \( k \) to obtain them.
One now carries out the transformation and does the appropriate calculations. As a result, for a black hole as viewed from a frame moving at $\beta \hat{z}$ (i.e., in the gauge boosted Schwarzschild metric) one gets, to first order in $\beta$:

\[
\frac{1}{2} (tr \chi' + tr \chi') = -\frac{2 M \beta \cos \theta}{r^2} \quad (28)
\]

\[
\epsilon_\theta = \frac{2 M \beta \sin \theta}{r^2} \quad (29)
\]

\[
\nabla_{N'} z = \cos \theta
\]

\[
\hat{N}_b z \sim -\sin \theta
\]

This gives $(\epsilon_\theta \nabla_{N'} z + \frac{1}{2} (tr \chi' + tr \chi') \nabla_{N'} z) r^2 = -2 M \beta \cos^2 \theta - 2 M \beta \sin^2 \theta$.\(^{12}\)

Hence, after doing the surface integral and taking the limit, one gets the following linear momentum as expected:

\[ P_z = -M \beta \]

Note that the radial component (equation 28) is zero at $\theta = \frac{\pi}{2}$ as one expects, because the radial motion at those points on the sphere is in the x-y plane, and hence should not be effected by the z-motion. By an exactly parallel argument, the tangential component (equation 29) is zero at $\theta = 0$, because, at this point, the tangential component is perpendicular to the boost.

### 7.2 Angular Momentum

Next the angular momentum is:

\[
L_{(i)} = \frac{1}{8\pi} \int (k^g_{\sigma N} - tr k^g g_{\sigma N}) \Omega^\sigma_{(i)} d\mu_\gamma
\]

\[
= \frac{1}{8\pi} \int (k^g_{AN} - tr k^g g_{AN}) \Omega^A d\mu_\gamma
\]

\[
= \frac{1}{8\pi} \int (W_A - \left( -\frac{1}{2} (tr \chi + tr \chi') + k_{NN} \right) g_{AN}) \Omega^A d\mu_\gamma
\]

There are several ways to approach this expression; I will choose the most physical. First, look at the $\hat{r}$ ($\hat{N}$) component of the motion of the null geodesics; in the integrand, it is $\frac{1}{2} (tr \chi + tr \chi')$. We decided that this term was the component of linear momentum at the given location on the $S^2$ foliation that is

\(^{12}\)These calculations confirm the statement above, which was based on heuristic arguments using the motion of the geodesics, that the tangential and radial components are zero.
directed radial outward (i.e. component parallel to $\hat{r}$); hence we do not expect this to contribute to the angular momentum.

This expectation comes from our classical understanding of angular momentum. Recall classically that $\vec{L} = \vec{r} \times \vec{p}$ or special relativistically $L^\mu = x^\mu P^\nu - x^\nu P^\mu$ which gives the spatial angular momentum is in the $z$-direction as: $L^z \equiv L^{zy} = xP^y - yP^x$ and so is the same as the classical in this case and in this formal sense. In these terms, when $\vec{p} \sim \hat{r}$, $\vec{L} = 0$.

As expected the radial component does not contribute because the rotation vector fields only have components tangent to the given location on $S^2$. From the integrand, $g_{AN} \Omega^A = \Omega \cdot N = 0$. From figure 3, it is obvious that the $\hat{r}$ is perpendicular to the rotation vector fields.

The $k_{NN}$ term is a spurious term that arises because of the irrelevancies introduced by the spatial slice that is not necessary for the probing done by the light rays at null infinity. What is important is the spherical foliation induced by the light rays. The rotation vector fields $\Omega^A$ take care of this spurious term because they limit us to the spheres of the foliation.\footnote{Recall in the case of linear momentum a cancelation occurred because a $k_{NN}$ term appeared in each of the two terms ($k_{iN}$ and $trk$).}

Finally, the term involving $k_{AN}$ ($= W_A$) in the integrand above are the components of the linear momentum in the $e_A$ direction, that is the tangential direction. Hence, we can write the following fashion we can write the differential contribution of the angular momentum in the integrand as $W_A \Omega^A$ in analogy to the relation of the classical linear momentum and angular momentum:

$$L_{(i)} = \vec{r} \times \vec{p} = p_A \Omega^A_{(i)}$$

8 Conclusion

We have introduced the generalized linear momentum (equation 23) and the generalized angular momentum (equation 24). We have seen that these give (using the foliation described in section two):

$$P_i(Bondi) = \lim_{s \to \infty} \frac{1}{8\pi} \int (k^\sigma_{\sigma N} - tr k^\sigma g_{\sigma N}) e_i^\sigma d\mu_\gamma$$

$$P_i(ADM) = \lim_{r_{laplace} \to \infty \text{ on } \Sigma_t} \frac{1}{8\pi} \int (k^\sigma_{\sigma N} - tr k^\sigma g_{\sigma N}) e_i^\sigma d\mu_\gamma$$

and

$$L(null) = \lim_{s \to \infty} \frac{1}{8\pi} \int W_A \Omega^A_{(i)} d\mu_\gamma$$

$$L(ADM) = \lim_{r_{laplace} \to \infty \text{ on } \Sigma_t} \frac{1}{8\pi} \int W_A \Omega^A_{(i)} d\mu_\gamma$$
Using these definitions, we explained that the linear and angular momentum are found by following the behavior of light rays near null infinity. The geometry of the ray behavior showed the meaning of the components that naturally arise in a foliation of the space-time based on the light rays. The boosted Schwarzschild solution reveals this component behavior in the way expected. The analysis also elucidated the interrelationship between the linear and angular momentum allowing one to see the angular momentum can be understood as an $\vec{r} \times \vec{p}$.

Hence, we have a precise as well as an intuitive understanding momentum in the far field case where gravity waves are studied. The ideas are relevant to understanding of signals received by gravitational wave detection projects such as LIGO, LISA and VIRGO.

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9 Appendix I: Ricci Rotation Coefficients

The Ricci Rotation Coefficients are defined below with respect to a null pair, $e_3, e_4$ (where $e_3 \cdot e_4 = -2$ and $e_3 = \underline{l}$, $e_4 = \underline{L}$), and the spatial vectors, $e_1, e_2$, orthogonal to these two vectors and tangent to the topologically $S^2$ spheres which make up the foliation. The $e_\mu$ are called the null tetrads.

$$
\chi_{AB} = H_{AB} = \langle D_A e_4, e_B \rangle
$$

$$
ZZ_A = \frac{1}{2} \langle D_3 e_4, e_A \rangle
$$

$$
Y_A = \frac{1}{2} \langle D_4 e_4, e_A \rangle
$$

$$
\Omega = \frac{1}{4} \langle D_4 e_4, e_3 \rangle
$$

$$
\Omega^t = \frac{1}{4} \langle D_4 e_4, e_A \rangle
$$

$$
V_A = \frac{1}{2} \langle D_A e_4, e_3 \rangle
$$

Note that the quantity $\Omega$ is not related to the quantity labeled $\Omega^t$ in equation 8.

The un-superscripted null pair, $l, L$ refer to the (affine) geodesic pair (so called because $l$ is tangent to a null geodesic): un-superscripted Ricci rotation coefficients refer to those with respect to the geodesic null pairs $l, L$. Superscript $t$ on the null pair refers to $l^t = \phi l', L^t = \phi^{-1} L'$ (the $S^2$ foliation is propagated to be on a maximal hypersurface $t$). The primed null pair refers to the “standard” null pair: $l' = T + N = al, L' = T - N = a^{-1}L$ where $T$ is the unit normal to the maximal ( spatial) hypersurface, and $N$ is the unit normal to $S_{t,u}$ in the maximal hypersurface. Ricci rotation coefficients associated with these null tetrads are distinguished respectively by superscript $t$ and primes.

For the standard null pair one gets (rearranging [13] pg. 171):

I. Standard null pair $(l', L')$
\[
\begin{array}{|c|c|}
\hline
\chi'_{AB} = \theta_{AB} - k'_{AB} & \chi'_{AB} = -\theta_{AB} - k'_{AB} \\
ZZ'_{A} = \mathcal{N}_{A} \ln a + \epsilon_{A} & ZZ'_{A} = \mathcal{N}_{A} \ln \varphi - \epsilon_{A} \\
Y'_{A} = 0 & Y'_{A} = \mathcal{N}_{A} \ln \varphi \\
\Omega' = \frac{1}{2}(\nabla_{N} \ln \varphi + \delta) & \Omega' = \frac{1}{2}(\nabla_{N} \ln \varphi + \delta) \\
V'_{A} = \epsilon_{A} = k'_{AN} & \\
\hline
\end{array}
\]

where:

\[a = \frac{1}{|\nabla u|} = \{\text{the lapse function of the foliation induced by } u \text{ on each } \Sigma_{t}\}\]

\[
\theta_{AB} = \{\text{the extrinsic curvature of the surfaces } S_{t,u} \text{ relative to } \Sigma_{t}\}
\]

\[
k'_{ij} = \{\text{extrinsic curvature of the maximal slice}\}
\]

\[
= \frac{1}{2} \text{ Spatial Components}(\mathcal{L}_{\mathcal{T}} \tilde{g}_{ij}) = -(2\varphi)^{-1} \partial_{t} \tilde{g}_{ij}
\]

Decomposition of \( k' \):

\[\eta_{AB} = k'_{AB}, \quad \epsilon_{A} = k'_{AN}, \quad \delta = k'_{NN}\]

One can transforms to a different null pair; this is called a lapse transformation:\(^{14}\)

and has the form:

\[
l^{\text{Trans}} = a^{-1}l, \quad \tilde{l}^{\text{Trans}} = a\tilde{l}
\]

where:

\[a \text{ is called the lapse function and is any function on the } S^{2} \text{ surface.}\]

Note that the normalization \( l \cdot \tilde{l} = -2 \) is preserved under a lapse transformation. Under the lapse transformation above, the Ricci Coefficients transform as:

\[
\begin{align*}
\chi^{\text{Trans}} &= a^{-1} \chi \\
ZZ^{\text{Trans}} &= ZZ \\
Y^{\text{Trans}} &= a^{-2}Y \\
\Omega^{\text{Trans}} &= \frac{1}{2} a^{-2} D_{t}a + a^{-1} \Omega \\
V^{\text{Trans}} &= V + \mathcal{N}_{A} \ln a \\
\end{align*}
\]

Hence, in terms of the geodesic null pair, \( l = a^{-1}l', \tilde{l} = a\tilde{l}' \) one obtains:

\[II. \quad \text{Geodesic null pair } (l, \tilde{l})\]

\(^{14}\)The physical meaning of this lapse function is discussed in [7]. In general, the lapse function, together with its counter part, the shift vector may be described as the non-dynamical variable that tells one how to move forward in time (cf. e.g. [20][11]).
\[
\begin{array}{|c|c|}
\hline
\chi_{AB} = a^{-1}(\theta_{AB} - k_{AB}) & \chi_{AB} = a(-\theta_{AB} - k_{AB}) \\
\hline
ZZ_A = N_A \ln a + \epsilon_A & ZZ_A = N_A \ln \phi - \epsilon_A \\
Y_A = 0 & Y_A = a^2(N_A \ln \phi) \\
\Omega = -a^{-1} \Omega' + a^{-1} \Omega = 0 & \Omega = \frac{1}{2}(\nabla_N \ln \phi + \delta) - \frac{1}{2} D_{\nu} \phi \\
V_A = \epsilon_A + \ln a & V_A = \epsilon_A + \frac{3}{2}\ln a \\
\hline
\end{array}
\]

where: use is made of \(D_t a = D_{a^{-1}t} a = -2 \Omega' \implies D_{\nu} a = -2 a \Omega'\) (derived using relations from [13] pg 264)

In terms of the \(t\)-null pair, \(l^t = \phi l', \underline{l}^t = \phi^{-1} \underline{l}'\), one gets:

\[III. \quad t\text{-null pair} \quad (l^t, \underline{l}^t)\]

\[
\begin{array}{|c|c|}
\hline
\chi^t_{AB} = \phi(\theta_{AB} - k_{AB}) & \chi^t_{AB} = \phi^{-1}(-\theta_{AB} - k_{AB}) \\
ZZ^t_A = N_A \ln a + \epsilon_A & ZZ^t_A = N_A \ln \phi - \epsilon_A \\
Y^t_A = 0 & Y^t_A = \phi^{-2}(N_A \ln \phi) \\
\Omega^t = \frac{1}{2} \phi^2 D_{\nu} \phi^{-1} + \frac{1}{2} (-\nabla_N \ln \phi + \delta) & \Omega^t = \frac{1}{2} \phi^{-1} \nabla_N \ln \phi - \frac{1}{2} D_{\nu} \phi^{-1} \\
V^t_A = \epsilon_A + \frac{3}{2} \ln \phi = W_A - \frac{3}{2} \ln (a \phi) & \\
\hline
\end{array}
\]

### 10 Appendix II: Definition of Functions, \(\Psi\) and \(\Omega_\phi\)

The below functions are taken from [13], but they are modified to take into account our current notational definitions. Recall above we use a definition of \(u\) consistent with [15],[5],[7] and thus different from [13] as mentioned in the text of the current article. The change is made by taking \(\Xi \to -\frac{1}{2} \Xi\) and \(u \to -u\) as needed.

\[
\Psi = -\frac{1}{4 \times 4 \pi} \int_{-\infty}^{\infty} du' \left( \int_{S^2} |\Xi|^2 (u', \hat{x}') d^2x' \right) \\
\Psi' = \frac{1}{4 \times 4 \pi} \int_{-\infty}^{\infty} du' \left( \int_{S^2} \frac{|\Xi|^2 (u', \hat{x}') - |\Xi|^2 (u')}{|\hat{x} - \hat{x}'|} d^2x' \right)
\]

\[
\Omega_\phi = \frac{1}{4 \times 8 \pi} \int_{-\infty}^{\infty} du' \int_{S^2} |\Xi|^2 (u', \hat{x}') d^2x' + \frac{1}{4} \cdot \frac{1}{2} \int_{-\infty}^{\infty} du' \text{sgn}(-u + u') |\Xi|^2 (u', \hat{x})
\]

\[
\Omega'_\phi = -\frac{1}{4 \times 8 \pi} \int_{-\infty}^{\infty} du' \int_{S^2} \frac{|\Xi|^2 (u', \hat{x}') - |\Xi|^2 (u')}{|\hat{x} - \hat{x}'|} d^2x' \\
- \frac{1}{4} \cdot \frac{1}{2} \int_{-\infty}^{\infty} du' \text{sgn}(-u + u') \left( |\Xi|^2 (u', \hat{x}) - |\Xi|^2 (u') \right)
\]
11 Appendix III: Derivative of Areal Radius

Take as the $S^2$ surfaces the ones propagated along $l$ (\(\mathcal{L}\)) by $l (\mathcal{L})$ (the geodesic null pair) so that one has:

$$\int_{S^2} d\mu_\gamma = 4\pi r^2$$

Taking $D_{\mathcal{L}}$ of each side yields:

$$8\pi r D_{\mathcal{L}} r = \int_{S^2} D_{\mathcal{L}} d\mu_\gamma$$
$$= \int_{S^2} tr \chi d\mu_\gamma$$
$$= 4\pi r^2 tr \chi$$  \hspace{1cm} (32)

Where overbar means averaged over the solid angle.
Hence, we obtain:

$$D_{\mathcal{L}} r = \frac{r}{2} tr \chi$$

Expanding for large $r$ yields:

$$D_{\mathcal{L}} r \sim \frac{r}{2} \left( -\frac{2}{r} + \frac{H}{r^2} \right) = -1 + \frac{H}{2r}$$

12 Appendix IV: Basis Vectors and Forms

In this article, I make use of the following set of orthonormal basis vector set for space portion of the space-time:

$$e_r, e_\theta, e_\phi$$

$$g_{ij} = diag(1, 1, 1)$$

This gives the following vectors basis and dual vector basis sets:

$$\hat{e}_r = \frac{\partial}{\partial r}$$
$$\hat{e}_\theta = \frac{1}{r} \frac{\partial}{\partial \theta}$$
$$\hat{e}_\phi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$
\[ \omega^r = dr \]
\[ \omega^\theta = r \, d\theta \]
\[ \omega^\phi = r \sin \theta \, d\phi \]

In coordinates, one uses the following notation in spherically symmetric systems:

\[ ds^2 = dr^2 + r^2 d\theta + r^2 \sin^2 \theta \, d\phi \]

with the basis vectors:

\[ \partial_r, \partial_\theta, \partial_\phi \]

If one has a contravariant vector, \( V \), in Cartesian coordinates:

\[ V = V^\mu \, \hat{x}^\mu \]

In spherical coordinates:

\[ V = V^\mu \, x^\mu \]

In an orthonormal basis:

\[ V = V^\mu \, \hat{e}_\mu \tag{34} \]

The subscript on \( e_\mu \) indicates which the basis vector of the orthonormal set to use. To write the \( x \)-component of \( V \) in terms of the orthonormal components, one can dot equation 34 with \( \hat{x} \); this is the same as contracting with \( dx \).

\[ V^\mu_{\text{cart}} = V^\mu \, \hat{e}_\mu, dx > \]
\[ = V^\mu \, dx(\hat{e}_\mu) \]
\[ = V^\mu \, \nabla_{\hat{e}_\mu} x \]
\[ = V^\mu \, \nabla_\mu x \tag{35} \]

In the last line, \( \nabla_\mu \) is defined to mean \( \nabla_{\hat{e}_\mu} \).

Equation 35 remains the same if one uses any orthonormal basis:

\[ \hat{B} = \hat{T}^i (\frac{\partial}{\partial t}) + \text{const} * \hat{N}^i (\frac{\partial}{\partial r}) \]
\[ \hat{N} = \hat{N}^i (\frac{\partial}{\partial r}) + \text{const} * \hat{T}^i (\frac{\partial}{\partial t}) \]
\[ \hat{e}_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} \]
\[ \hat{e}_\phi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \]

Also, for reference, the decomposition of Cartesian coordinates into \( Y'_{lm} \)'s is given below:
\(x : x = r \sin \theta \cos \varphi = r k_1 (Y_{11} - Y_{11})\)

where \(k_1 = \sqrt{\frac{2\pi}{3}}\)

\(y : y = r \sin \theta \sin \varphi = r k_2 (Y_{11} + Y_{11})\)

where \(k_2 = -i \sqrt{\frac{2\pi}{3}}\)

\(z : z = r \cos \theta = r k_3 Y_{10}\)

where \(k_3 = \sqrt{\frac{4\pi}{3}}\)

13 **Appendix V: The Geodesic Null Pair**

Several issues to note on the frame based on the geodesic null pair:

- It is handy to use the following decomposition of \(B^g\) and \(N^g\) into \(B'(= T')\) and \(N'(= \frac{\partial}{\partial r})\):
  
  * \(N^g \sim N' + \frac{\psi'}{r} T'\); 
  * \(B^g \equiv T^g \sim B' - \frac{\psi'}{r} N'\). 

  * Hence, for example, \(\partial_N x_i \sim \partial_{N'} x_i + \frac{\psi'}{r} \partial_{T'} x_i = \partial_{N'} x_i\).

- Second, using the geodesic null pair:

  * \(e_A, B^g\) are surface forming.
  * \(e_A, N^g\) are surface forming.

  * However, \(e_A, N^g\) are surface forming to order \(O((\frac{1}{r})^0)\).

  * Note: \(g_{N^g N^g} = 1; g_{B^g B^g} = -1\).

14 **Appendix VI: The geometric meaning of a few Ricci Rotation Coefficients**

14.0.1 **Shear**

The shear, \(\hat{H}_{AB} = \hat{\chi}_{AB} (\hat{H} = \hat{\chi}_{AB})\), corresponds to the traceless part of the extrinsic curvature of the \(S^2\) surface, \(\chi_{AB}\). It is the shear of the outgoing (respectively, ingoing) null rays. Geometrically, the shear tells one how the shape a small bundle of null rays changes during a short time period.
14.0.2 Expansion

The expansion, \( tr(\chi) \), \( (tr(\chi)) \) gives the “trace” part of the extrinsic curvature. It tells the amount of expansion of a small bundle of outgoing (respectively, ingoing) null rays in a short time period.

14.0.3 Torsion

For a curve in \( \mathbb{R}^3 \), with tangent vector \( \hat{v} \), and principle normal \( \hat{n} = \frac{\partial \hat{v}}{\partial s} / \| \frac{\partial \hat{v}}{\partial s} \| \), \( \hat{b} = \hat{v} \times \hat{n} \), the binormal, \( \{ \hat{v}, \hat{n}, \hat{b} \} \) form an orthonormal set at each point (see figure 2). Using the Serret-Frenet Formulae, one defines the torsion as:

\[
\frac{\partial \hat{b}}{\partial s} = \text{Torsion} * \hat{n} \quad \text{or} \quad \text{Torsion} = - \frac{\partial \hat{n}}{\partial s} \cdot \hat{b}
\]

where \( s \) is the natural parameter of the curve (arc length).

Here “torsion” measures the degree of twisting out of the initial plane of motion. In “2+1” space-time, this would correspond to the null rays twisting, like the grooves of a screw, as they move forward in time.

In a similar way, given a spacelike surface in \( (g, M) \), with normals \( l \) and \( \underline{l} \), with \( \underline{l} \) chosen to be a null geodesic field \( (D_{\underline{l}}l = 0) \) and \( l \) defined such that \( l \cdot \underline{l} = -2 \) and given \( b = \frac{1}{2}(l + \underline{l}) \), the binormal defined in reference[14], and the principal normal \( n = (\underline{l} - l)/2 \), one defines the torsion, \( \zeta : \)

\[
\zeta_A = -(D_{e,A}n, b) = (D_{e,A}b, n) = \frac{1}{2}(D_{e,A}l, \underline{l})
\]

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Figure 1: The construction of the affine foliation. Let $k, s \to \infty$. $C_\infty$ is null infinity. $S_{-\infty}$ is spatial infinity. Actual $S^2$ surfaces $S_{u,\infty}$ foliate null infinity.

Figure 2: A Euclidean worldline shown in one time and two spatial dimensions. Time is the vertical dimension. The curve is twisting forward in time, so it has non-zero torsion $= -\frac{\partial \hat{n}}{\partial s} \cdot \hat{b}$.

Figure 3: The three rotation vector fields on a two-sphere
This figure "image002.gif" is available in "gif" format from:

http://arxiv.org/ps/gr-qc/0210017v1