Twisted derivations of Hopf algebras

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Abstract

In the paper we introduce the notion of twisted derivation of a bialgebra. Twisted derivations appear as infinitesimal symmetries of the category of representations. More precisely they are infinitesimal versions of twisted automorphisms of bialgebras \[2\]. Twisted derivations naturally form a Lie algebra (the tangent algebra of the group of twisted automorphisms). Moreover this Lie algebra fits into a crossed module (tangent to the crossed module of twisted automorphisms). Here we calculate this crossed module for universal enveloping algebras and for the Sweedler’s Hopf algebra.

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Introduction

The paper studies infinitesimal symmetries of bialgebras which manifest themselves in representation theory. It is very well-known that categories of representations (modules) of bialgebras are examples of so-called tensor categories. It is less acknowledged that relations between bialgebras (homomorphisms of bialgebras) do not capture all relations (tensor functors) between their representation categories. An algebraic notion which does the job (bi-Galois (co)algebra) is known only to specialists. Fully representing tensor relations between representation categories it is sometimes not very easy to work with. For example, composition of tensor functors corresponds to tensor product of Galois (co)algebras and very often is quite tricky to calculate explicitly. At the same time tensor functors of interest could have some additional properties which put restrictions on the corresponding algebraic objects and allow one to have an alternative and perhaps simpler description.

Note that any representation category is equipped with a natural tensor functor to the category of vector spaces, the functor forgetting the action of the bialgebra (the forgetful functor). In \[2\]
we dealt with tensor functors between representation categories which preserve (not necessarily
tensorly) the forgetful functors. The corresponding algebraic notion is of twisted homomorphism.
Composition of tensor functors correspond to composition operation on twisted homomorphism.
Natural transformations of tensor functors correspond to certain relation which was called in [2] (gauge transformations). The main object of study in [2] was the category (groupoid) of twisted automorphisms of a Hopf algebra and its actions on (certain structures on) categories of representations.

In this paper we define the infinitesimal analog of the notion of twisted automorphism which
we call twisted derivation. A twisted derivation of a bialgebra $H$ is a pair $(d, \phi)$ where $d : H \to H$
is an algebra derivation and $\phi$ is an element of $H \otimes H$, such that

$$(I \otimes d + d \otimes I)(\Delta(x)) - \Delta(d(x)) = [\phi, \Delta(x)], \quad \forall x \in H,$$

$1 \otimes \phi + (I \otimes \Delta)(\phi) = \phi \otimes 1 + (\Delta \otimes I)(\phi),$

$\varepsilon d = 0, \quad (\varepsilon \otimes I)(\phi) = (I \otimes \varepsilon)(\phi) = 0.$

Here $\Delta : H \to H \otimes H$ is the comultilication and $\varepsilon : H \to \mathbb{k}$ is the counit of $H$. Twisted derivations
form a Lie algebra $\text{Der}_{\text{tw}}(H)$ with the bracket

$$[(d, \phi), (d', \phi')] = ([d, d'], (d \otimes I + I \otimes d)(\phi') - (d' \otimes I + I \otimes d')(\phi) - [\phi, \phi']).$$

This is the tangent bracket to the group operation on twisted automorphisms. An infinitesimal
analogue of gauge transformations of twisted automorphisms is the notion of gauge transformation of twisted derivations. A gauge transformation from a twisted derivation $(d, \phi)$ to a twisted derivation $(d', \phi')$ is an element $a$ of $H$ such that

$$(d' - d)(x) = [a, x], \quad \forall x \in H,$$

$$\phi' - \phi = (a \otimes 1 + 1 \otimes a) - \Delta(a).$$

Addition in $H$ gives rise to composition operation on gauge transformations turning twisted derivations
into a crossed module of Lie algebras:

$$\text{Der}_{\text{tw}}(H) \xleftarrow{\partial} H.$$

Here

$$\partial(a) = ([a, ], (a \otimes 1 + 1 \otimes a) - \Delta(a)).$$

Recall (from [3]) that a crossed module of Lie algebras is a homomorphism of Lie algebras $\partial : \frak{n} \to \frak{p}$
together with an action of $\frak{p}$ on $\frak{n}$ by derivations such that

$$\partial(p(n)) = [p, \partial(n)] \quad \forall p \in \frak{p}, n \in \frak{n},$$

$$\partial(n)(m) = [n, m] \quad \forall n, m \in \frak{n}.$$

The equivalence class of a crossed module $\frak{p} \xleftarrow{\partial} \frak{n}$ is controlled by two Lie algebras

$$\pi_0(\frak{p} \xleftarrow{\partial} \frak{n}) = \text{coker}(\partial), \quad \pi_1(\frak{p} \xleftarrow{\partial} \frak{n}) = \text{ker}(\partial)$$

and a (Lie algebra) cohomology class (the Jacobiator)

$$J \in H^3(\pi_0, \pi_1).$$

See [7, 8, 6] for details.

For the crossed module of twisted derivations $\text{Der}_{\text{tw}}(H) \xleftarrow{\partial} H$ we have associated two Lie algebras: the algebra of outer twisted derivations

$$\text{OutDer}_{\text{tw}}(H) = \pi_0(\text{Der}_{\text{tw}}(H) \xleftarrow{\partial} H)$$
and the abelian Lie algebra of central primitive elements

\[ Z(H) \cap Prim(H) = \pi_1(Der_{tw}(H) \overset{\phi}{\leftarrow} H). \]

Here \( Prim(H) = \{ x \in H | \Delta(x) = x \otimes 1 + 1 \otimes x \} \) and \( Z(H) = \{ x \in H | xy = yx \ \forall y \in H \} \). The Jacobiator is a cohomology class in

\[ H^3(OutDer_{tw}(H), Z(H) \cap Prim(H)). \]

As an example we treat the case of the universal enveloping algebra \( U(\mathfrak{g}) \) of a Lie algebra \( \mathfrak{g} \). It turns out that in characteristic zero any twisted derivation is gauge equivalent to a bialgebra derivation together with an invariant infinitesimal twist (\( \text{separated case} \)). Gauge classes of invariant twists form an abelian Lie algebra isomorphic to \((\Lambda^2 \mathfrak{g})^\mathfrak{g}\), the invariant elements of the exterior square of the Lie algebra \( \mathfrak{g} \). The Lie algebra of gauge classes of twisted derivations (the Lie algebra of outer twisted derivations) is a crossed product of the Lie algebra of outer derivations of \( \mathfrak{g} \) and the abelian Lie algebra of invariant twists:

\[ OutDer_{tw}(U(\mathfrak{g})) = OutDer(\mathfrak{g}) \rtimes (\Lambda^2 \mathfrak{g})^\mathfrak{g}. \]

The space of central primitive elements of \( U(\mathfrak{g}) \) coincides with the centre of \( \mathfrak{g} \):

\[ Z(U(\mathfrak{g})) \cap Prim(U(\mathfrak{g})) = Z(\mathfrak{g}). \]

We also compute the Jacobiator of \( Der_{tw}(U(\mathfrak{g})) \). In particular we show that its restriction to the abelian subalgebra of invariant twists \((\Lambda^2 \mathfrak{g})^\mathfrak{g}\) is zero.

We present a construction extending a bialgebra \( H \) with an infinitesimal twist \( \phi \) to a bialgebra \( E(H) \) with a derivation \( d : E(H) \to E(H) \) \((E(H) \) is the free algebra with derivation generated by the algebra \( H \)) such that \((d, \phi)\) is a twisted derivation of \( E(H) \). This construction provides examples of non-separated twisted derivations.

We also look at twisted derivations of a non-commutative, non-cocommutative Sweedler’s Hopf algebra \( H_4 \). Here again any twisted derivation is separated. The Lie algebra of bialgebra derivations and of invariant twists are both 1-dimensional. A non-zero bialgebra derivation acts non-trivially on invariant twists. Thus \( OutDer_{tw}(H_4) \) is the 2-dimensional non-abelian Lie algebra.

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1 Twisted automorphisms and twisted derivations of bialgebras

Throughout the paper \( k \) be a ground field.

1.1 Twisted automorphisms of bialgebras

Here we recall (from \([2]\)) the notions of twisted automorphisms of a bialgebra, their transformations and their relation with tensor autoequivalences of the category of representation.

A twisted automorphism of a bialgebra \( H \) is a pair \((f, F)\), where \( f : H \to H \) is an algebra automorphism and \( F \) is an invertible element of \( H \otimes H \) (an \( f \)-twist or simply twist) such that

\[ F \Delta(f(x)) = (f \otimes f)(\Delta(x))F, \quad \forall x \in H, \quad (1) \]

\[ (F \otimes 1)(\Delta \otimes I)(F) = (1 \otimes F)(I \otimes \Delta)(F), \quad 2 \text{-cocycle condition} \quad (2) \]
\[\varepsilon f = \varepsilon, \quad (\varepsilon \otimes I)(F) = (I \otimes \varepsilon)(F) = 1, \quad \text{normalisation.} \quad (3)\]

Here and later on \(\Delta : H \to H \otimes H\) is the coproduct and \(\varepsilon : H \to k\) is the counit of \(H\).

For example, a bialgebra automorphism \(f : H \to H\) is a twisted homomorphism with the identity twist \((f, 1)\). A twisted automorphism is \textit{separated} if the first component \(f : H \to H\) is a bialgebra automorphism. For a separated twisted homomorphism the condition \((3)\) amounts to the invariance of the twist with respect to the diagonal sub-bialgebra \(\Delta(H) \subset H \otimes H:\)

\[\Delta(f(x))F = F\Delta(f(x)), \quad \forall x \in H.\]

The \textit{composition} of twisted automorphisms \((f, F)\) and \((f', F')\) is

\[(f, F) \circ (f', F') = (ff', f(F')F). \quad (4)\]

Here \(f(F') = (f \otimes f)(F')\). It is not hard to verify that the result is a twisted automorphism and that the composition is associative. Note that separated twisted automorphisms are closed under composition.

By a \textit{gauge transformation} \((f, F) \to (f', F')\) of twisted automorphisms \((f, F), (f', F')\) of \(H\) we will mean an invertible element \(a\) of \(H\) such that

\[af(x) = f'(x)a, \quad \forall x \in H, \quad (5)\]

\[F'\Delta(a) = (a \otimes a)F. \quad (6)\]

We will depict it graphically as follows:

\[
\begin{array}{c}
\xymatrix{ H \ar@/^/[rr]^{(f,F)} \ar@/_/[rr]_{(f',F')} \ar@{=}[d]^a & \ar@{=}[d]^a \ar@{=}[d]^a & H \\
\xymatrix{ H \ar@/^/[rr]^{(g,G)} \ar@/_/[rr]_{(j,J)} } & & \ar@{=}[d]^b \ar@{=}[d]^b \ar@{=}[d]^b \ar@{=}[d]^b }
\end{array}
\]

Note that the condition \((6)\) together with normalisation conditions for \(F\) and \(G\) implies \(\varepsilon(a) = 1\). Note also that a twisted homomorphism gauge isomorphic to a separated twisted homomorphism is not necessarily separated.

For successive gauge transformations

\[
\begin{array}{c}
\xymatrix{ H \ar@/^/[rr]^{(f,F)} \ar@/_/[rr]_{(f',F')} \ar@{=}[d]^a & \ar@{=}[d]^a \ar@{=}[d]^a & H \\
\xymatrix{ H \ar@/^/[rr]^{(g,G)} \ar@/_/[rr]_{(j,J)} } & & \ar@{=}[d]^b \ar@{=}[d]^b \ar@{=}[d]^b \ar@{=}[d]^b }
\end{array}
\]

the composition \(b.a : (f, F) \to (j, J)\) is simply the product \(ba\) in \(H'\). Again it is quite straightforward to check that this is a transformation.

We can also define compositions of transformations and twisted automorphisms in the following two situations:

\[
\begin{array}{c}
\xymatrix{ H \ar@/^/[rr]^{(f,F)} \ar@/_/[rr]_{(f',F')} \ar@{=}[d]^a & \ar@{=}[d]^a \ar@{=}[d]^a & H \\
\xymatrix{ H \ar@/^/[rr]^{(g,G)} \ar@/_/[rr]_{(j,J)} } & & \ar@{=}[d]^b \ar@{=}[d]^b \ar@{=}[d]^b \ar@{=}[d]^b }
\end{array}
\]
we define it to be \((g, G) \circ a = g(a)\) in the first case and \(b \circ (f, F) = b\) in the second. The following properties intertwining compositions of twisted homomorphisms and gauge transformations are quite straightforward consequences of the definitions:

\[
(a \cdot b) \circ (f, F) = (a \circ (f, F)) \cdot (b \circ (f, F)),
\]

\[
(g, G) \circ (a \cdot b) = ((g, G) \circ a) \cdot ((g, G) \circ b),
\]

\[
(a \circ (f, F)) \cdot ((g, G) \circ b) = ((g, G) \circ b) \cdot (a \circ (f, F)).
\]

Note (see also [2] for details) that the structures described above turn twisted automorphisms and their gauge transformations into a categorical group \(\text{Aut}_{\text{tw}}(H)\).

Recall (see [1] for the history) that Cat-groups are the same as crossed modules of Whitehead ([9]). A crossed module of groups is a pair of groups \(P, C\) with a (left) action of \(P\) on \(C\) (by group automorphisms):

\[
P \times C \to C, \quad (p, c) \mapsto p^c c
\]

and a homomorphism of groups

\[
P \xleftarrow{\partial} C
\]

such that

\[
\partial(p^c c) = p \partial(c) p^{-1}, \quad \partial(c) c' = c c' c^{-1}.
\]

A map of crossed modules \((P, C) \to (E, N)\) is a triple \((\tau, \nu, \theta)\) where \(\tau\) and \(\nu\) are maps making the diagram

\[
P \xleftarrow{\partial} C \xrightarrow{\nu} N
\]

commutative, and \(\theta : P \times P \to N\) is such that

\[
\tau(pq) = \partial(\theta(p,q)) \tau(p) \tau(q), \quad p, q \in P;
\]

\[
\nu(ab) = \theta(\partial(a), \partial(b)) \nu(a) \nu(b), \quad a, b \in C;
\]

\[
\theta(p, qr)^{\tau(p)} \theta(q, r) = \theta(p, qr) \theta(p, q), \quad p, q, r \in P;
\]

\[
\theta(1, q) = 1 = \theta(p, 1), \quad p, q \in P;
\]

\[
\nu(pq) = \theta(p, \partial(a))^{\tau(p)} \nu(a).
\]

Complete invariants of a categorical-group \(\mathcal{G}\) with respect to monoidal equivalences are

\[
\pi_0(\mathcal{G}), \quad \pi_1(\mathcal{G}), \quad \phi \in H^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G})),
\]

where the first is the group of isomorphism classes of objects, the second is the abelian group \((\pi_0(\mathcal{G})\text{-module}) \text{Aut}_\mathcal{G}(I)\) of automorphisms of the unit object and the third is a cohomology class (the associator). In the crossed module setting

\[
\pi_0 = \text{coker}(\partial), \quad \pi_1 = \text{ker}(\partial).
\]

Note that the image of \(\partial\) is normal so the cokernel has sense. The class \(\phi\) is defined as follows: choose a section \(\sigma : \text{coker}(\partial) \to P\) and a map \(a : \text{coker}(\partial) \times \text{coker}(\partial) \to C\) such that

\[
\sigma(fg) = \partial(a(f, g)) \sigma(f) \sigma(g), \quad f, g \in \text{coker}(\partial).
\]
Then for any $f, g, h \in \text{coker}(\partial)$ the expression

$$a(f, gh)^{\phi(f)}a(g, h)a(f, g)^{-1}a(f, h)$$

is always in the kernel of $\partial$ and is a group 3-cocycle of $\text{coker}(\partial)$ with coefficients in $\ker(\partial)$. The cohomology class $\phi$ of this 3-cocycle does not depend on the choices made.

The crossed module of groups corresponding to the categorical group of twisted automorphisms has the form

$$\text{Aut}_{tw}(H) \xleftarrow{\phi} H^\epsilon.$$  \hfill (7)

Here $\text{Aut}_{tw}(H)$ is the group of twisted automorphisms of $H$ with respect to the composition, $H^\epsilon$ is the group of invertible elements $x$ of $H$ such that $\varepsilon(x) = 1$, and $\phi$ sends $x$ into the pair (an inner twisted automorphism) $(x(\cdot), (x \otimes x)\Delta(x)^{-1})$ where the first component is the conjugation automorphism:

$$x(\cdot) : H \to H, \quad x(y) = yx^{-1}.$$  

The action of $\text{Aut}_{tw}(H)$ on $H^\epsilon$ is given by the action of the first component $(f, F)(y) = f(y)$.

There are two important Cat-subgroups in $\text{Aut}_{tw}(H)$. The first is the full Cat-subgroup $\text{Aut}_{tw}^1(H)$ of twisted automorphisms with the identity as the first component. Its crossed module is

$$\text{Aut}_{tw}^1(H) \xleftarrow{\phi} (Z(H)_\epsilon).$$

Here $\text{Aut}_{tw}^1(H)$ is the group of invariant twists on $H$ (invertible elements of $H \otimes H$ commuting with the image $\Delta(H)$ and satisfying the 2-cocycle condition), $(Z(H)_\epsilon)$ is the group of invertible elements of the centre of counit 1: $\varepsilon(x) = 1$. Again $\phi$ assigns to $x$ the invariant twist $(x \otimes x)\Delta(x)^{-1}$.

The action of $\text{Aut}_{tw}^1(H)$ on $(Z(H)_\epsilon)$ is trivial.

The second is the full Cat-subgroup $\text{Aut}_{bialg}(H)$ of bialgebra automorphisms of $H$. Here the crossed module is

$$\text{Aut}_{bialg}(H) \xleftarrow{\phi} G(H),$$

where $\text{Aut}_{bialg}(H)$ is the group of automorphisms of $H$ as a bialgebra, $G(H) = \{x \in H, \Delta(x) = x \otimes x\}$ is the group of group-like elements of $H$ and $\phi$ sends $x$ into the conjugation automorphism. The action of $\text{Aut}_{bialg}(H)$ on $G(H)$ is obvious.

Note that the Cat-subgroup $\text{Aut}_{tw}^1(H)$ is what might be called normal: the components of its crossed module are normal subgroups in the components of the crossed module for $\text{Aut}_{tw}(H)$ and the action of $\text{Aut}_{tw}(H)$ on $H^\epsilon$ preserves the subgroup $Z(H)^\epsilon$.

The Cat-subgroup $\text{Aut}_{bialg}(H)$ is not in general normal. In [2] it was characterised as the stabiliser of a certain action.

Remark 1.1. Recall from [2] the following categorical interpretation of twisted automorphisms and their gauge transformations. For a homomorphism of algebras $f : H \to H'$ there is defined the inverse image functor $f^* : H' - \text{Mod} \to H - \text{Mod}$, which turns an $H'$-module $M$ into an $H$-module $f^*(M)$. As a vector space, $f^*(M)$ is the same as $M$ but with a new module structure $x.m = f(x)m$ for $x \in H$ and $m \in M$.

For a twisted automorphism $(f, F) : H \to H$ the inverse image functor $f^* : H - \text{Mod} \to H - \text{Mod}$ becomes tensor, with the tensor structure given by multiplication by the twist:

$$F_{M,N} : f^*(M \otimes H) \to f^*(M) \otimes f^*(N), \quad m \otimes n \mapsto F(m \otimes n).$$

Compositions of twisted homomorphisms and corresponding functors are related as follows:

$$((f, F) \circ (g, G))^* = (g, G)^* \circ (f, F)^*.$$

A gauge transformation $a : (f, F) \to (g, G)$ defines a tensor natural transformation $a : (f, F)^* \to (g, G)^*$:

$$a_M : f^*(M) \to g^*(M), \quad m \mapsto am.$$

Compositions of gauge transformations correspond to compositions of natural transformations.
It is straightforward to see that the condition (1) guarantees $H$-linearity of the tensor structure $F_{M,N}$ while the 2-cocycle condition for $F$ is equivalent to the coherence axiom for the tensor structure. Similarly, the condition (2) for a gauge transformation $a$ says that $a_M$ is a morphism of $H$-modules and the condition (3) is equivalent to the monoidality of $a_M$.

### 1.2 Infinitesimal twisted automorphisms

Here we look at infinitesimal twisted automorphisms and their gauge transformations. This leads us to the notion of twisted derivation.

Let $h$ be the dual number, i.e. $h^2 = 0$. Let $(f,F)$ be a twisted automorphism of a bialgebra $H$, defined over the algebra of dual numbers $k[h]$, such that

\[ f = I + hd, \quad F = 1 \otimes 1 + h\phi, \]  \hspace{1cm} (8)

for a (multiplicative) derivation $d : H \to H$ and an element $\phi \in H^{\otimes 2}$.

**Lemma 1.2.** The defining conditions (1,2,3) of a twisted automorphism are equivalent to the following equations on a derivation $d$ and an element $\phi$:

\[ (I \otimes d + d \otimes I)(\Delta(x)) - \Delta(d(x)) = [\phi, \Delta(x)], \quad \forall x \in H, \]  \hspace{1cm} (9)

\[ 1 \otimes \phi + (I \otimes \Delta)(\phi) = \phi \otimes 1 + (\Delta \otimes I)(\phi), \]  \hspace{1cm} (10)

\[ \varepsilon d = 0, \quad (\varepsilon \otimes I)(\phi) = (I \otimes \varepsilon)(\phi) = 0. \]  \hspace{1cm} (11)

**Proof.** The equations (9,10,11) follow from the comparison of coefficients for $h$ in the equations (1,2,3) respectively. \(\Box\)

We call a pair $(d,\phi)$ satisfying (9,10,11) a **twisted derivation** of the bialgebra $H$. Denote by $Der_{tw}(H)$ the set of twisted derivations of bialgebra $H$.

Call a twisted automorphism (3) the **infinitesimal twisted automorphism** corresponding to a twisted derivation $(d,\phi)$. Composition of infinitesimal twisted automorphisms corresponds to addition of twisted derivations:

\[ (I + hd, 1 \otimes 1 + h\phi) \circ (I + hd', 1 \otimes 1 + h\phi') = (I + h(d + d'), 1 \otimes 1 + h(\phi + \phi')). \]

The infinitesimal twisted automorphism corresponding to a twisted derivation $(d,\phi)$ is **separated** if $d$ is a bialgebra derivation:

\[ (I \otimes d + d \otimes I)(\Delta(x)) = \Delta(d(x)). \]

Note that $\phi$ for a separated twisted derivation is invariant with respect to $\Delta(H)$ and that the pair $(0,\phi)$ is a twisted derivation. Note also that the notion of separated twisted derivation is not gauge invariant. We call a twisted derivation **separable** if it is gauge equivalent to a separated one.

Now suppose that $h^3 = 0$ and consider the next order infinitesimal twisted automorphisms of $H$, i.e. twisted automorphisms of the form

\[ f = I + hd + h^2\delta, \quad F = 1 \otimes 1 + h\phi + h^2\psi. \]

The commutator of two such twisted automorphisms has a form

\[ [(I + hd + h^2\delta, 1 \otimes 1 + h\phi + h^2\psi), (I + hd' + h^2\delta', 1 \otimes 1 + h\phi' + h^2\psi')] = (h^2[d,d'], h^2(I \otimes I + I \otimes d)(\phi') - (d' \otimes I + I \otimes d')(\phi) - [\phi, \phi']). \]

This in particular shows that the operation

\[ [(d,\phi),(d',\phi')] = ([d,d'], (d \otimes I + I \otimes d)(\phi') - (d' \otimes I + I \otimes d')(\phi) - [\phi, \phi']) \]  \hspace{1cm} (12)
is a Lie bracket on the vector space $\text{Der}_{\text{tw}}(H)$ of twisted derivations. Of course this fact can be checked directly.

Now we go back to first order infinitesimal twisted automorphisms (assuming again that $h^2 = 0$). Consider two such twisted automorphisms $(f, F), (f', F')$, corresponding to twisted derivations $(d, \phi)$ and $(d', \phi')$. Let $a : (f, F) \to (f', F')$ be a gauge transformation of the form

$$a = 1 + ha, \quad a \in H.$$ 

**Lemma 1.3.** The conditions (5,6) are equivalent to the following

$$(d' - d)(x) = [a, x], \quad \forall x \in H,$$

$$\phi' - \phi = (a \otimes 1 + 1 \otimes a) - \Delta(a). \quad (13)$$

**Proof.** Follow from the comparison of coefficients for $h$ in the equations (5,6) respectively. \qed

We call $a \in H$ satisfying the conditions (13) a gauge transformation between twisted derivations $(d, \phi)$ and $(d', \phi')$.

Define a map $\partial : H_\varepsilon \to \text{Der}_{\text{tw}}(H)$ by

$$\partial(a) = ([a, ], (a \otimes 1 + 1 \otimes a) - \Delta(a)).$$

Here $H_\varepsilon = \ker(\varepsilon)$ is the kernel of the counit (the augmentation ideal). The above calculations show that this map is a crossed module of Lie algebras with respect to the $\text{Der}_{\text{tw}}(H)$-action on $H$:

$$(d, \phi)(\alpha) = d(\alpha), \quad \alpha \in H.$$ 

More precisely the following statement is established by the calculations of this section.

**Theorem 1.4.** The crossed module of Lie algebras $\partial : H_\varepsilon \to \text{Der}_{\text{tw}}(H)$ is tangent to the crossed module of groups (7):

$$\text{Aut}_{\text{tw}}(H) \overset{\partial}{\leftarrow} H_\varepsilon.$$ 

Note the $\ker(\partial)$ coincides with the space $Z(H) \cap \text{Prim}(H)$ of central primitive elements of $H$.

We call the Lie algebra $\text{OutDer}_{\text{tw}}(H) = \text{coker}(\partial)$ the algebra of outer twisted derivations.

There are two important sub-crossed modules of Lie algebras in $\text{Der}_{\text{tw}}(H)$. The first is the sub-crossed module $\text{Der}^0_{\text{tw}}(H) \overset{\partial}{\leftarrow} Z(H)_\varepsilon$.

Here $\text{Der}^0_{\text{tw}}(H)$ is the Lie algebra (with respect to commutator) of invariant infinitesimal twists on $H$ (elements of $H_\varepsilon \otimes H_\varepsilon$ commuting with the image $\Delta(H)$ and satisfying the 2-cocycle condition (ref)), and $Z(H)_\varepsilon = Z(H) \cap H_\varepsilon$. The map $\partial$ assigns to $x$ the invariant infinitesimal twist $(x \otimes x) - \Delta(x)$. The action of $\text{Der}^0_{\text{tw}}(H)$ on $Z(H)_\varepsilon$ is trivial. We denote by

$$\text{OutDer}^0_{\text{tw}}(H) = \text{coker}(\partial : Z(H)_\varepsilon \to \text{Der}^0_{\text{tw}}(H))$$

the Lie algebra of outer invariant infinitesimal twists.

The second is the sub-crossed module of bialgebra derivations of $H$:

$$\text{Der}_{\text{bialg}}(H) \overset{\partial}{\leftarrow} \text{Prim}(H),$$

where $\text{Der}_{\text{bialg}}(H)$ is the Lie algebra of derivations of $H$ as a bialgebra. The map $\partial$ sends $x$ into the conjugation derivation $[x, -]$. The action of $\text{Der}_{\text{bialg}}(H)$ on $\text{Prim}(H)$ is the obvious one. We denote by

$$\text{OutDer}_{\text{bialg}}(H) = \text{coker}(\partial : \text{Prim}(H) \to \text{Der}_{\text{bialg}}(H))$$
the Lie algebra of outer bialgebra derivations.

Note that the crossed submodule \( \text{Der}^0_{\text{tw}}(H) \) is what might be called normal: the components are Lie ideals in the components of the crossed module \( \text{Der}_{\text{tw}}(H) \) and the action of \( \text{Der}_{\text{tw}}(H) \) on \( H \) preserve the subspace \( Z(H) \).

In the case when any twisted derivation is gauge equivalent to a separated one (the separated case) the Lie algebra of outer twisted derivations is the semi-direct product:

\[
\text{OutDer}_{\text{tw}}(H) = \text{OutDer}_{\text{bialg}}(H) \ltimes \text{OutDer}^0_{\text{tw}}(H).
\]

### 1.3 Twisted derivations of quasi-triangular bialgebras

Recall that a quasi-triangular structure on a bialgebra \( H \) is an invertible element \( R \in H \otimes H \) (a universal \( R \)-matrix) satisfying

\[
Rt\Delta(x) = \Delta(x)R \quad \forall x \in H,
\]

along with triangle equations:

\[
(I \otimes \Delta)(R) = R_{23}R_{13}, \quad (\Delta \otimes I)(R) = R_{12}R_{13}.
\]

Here \( R_{12} = R \otimes 1 \), \( R_{13} = (I \otimes t)(R_{12}) \) etc., where \( t : H \otimes H \to H \otimes H \) denotes the transposition of tensor factors.

Denote by \( \mathcal{T}r(H) \) the set of universal \( R \)-matrices.

**Remark 1.5.** Recall that a triangular structure \( R \) on a bialgebra \( H \) defines a braiding:

\[
c_{M,N} : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto R(n \otimes m)
\]

on the category \( H \to \text{Mod} \).

Indeed the condition ([13]) implies that \( c_{M,N} \) is a morphism of \( H \)-modules:

\[
c_{M,N}(\Delta(x)(m \otimes n)) = Rt\Delta(x)(n \otimes m) = \Delta(x)R(n \otimes m) = \Delta(x)c_{M,N}(m \otimes n).
\]

The triangle equations are equivalent to the hexagon axioms for the braiding.

The interpretation of quasi-triangular structures as braidings allows us to define an action of twisted automorphisms on quasi-triangular structures. For a twisted automorphism \( (f, F) \) and a quasi-triangular structure \( R \) on \( H \) the twisted quasi-triangular structure is defined as follows:

\[
R^{(f,F)} = F^{-1}(f \otimes f)(R)F_{21}.
\]

It is straightforward to verify that the properties of the \( R \)-matrix are preserved. Moreover, gauge isomorphic twisted automorphisms act equally. Indeed, for \( g(x) = af(x)a^{-1} \) and \( G = (a \otimes a)F\Delta(a)^{-1} \),

\[
R^{(g,G)} = \Delta(a)F^{-1}(a \otimes a)^{-1}(a \otimes a)(f \otimes f)(R)(a \otimes a)^{-1}(a \otimes a)F_{21}t\Delta(a)^{-1} = \\
= \Delta(a)F^{-1}(f \otimes f)(R)F_{21}t\Delta(a)^{-1} = R^{(f,F)}.
\]

Thus an action of the group \( \text{Out}_{\text{tw}}(H) \) on the set \( \mathcal{T}r(H) \) of universal \( R \)-matrices is defined. For \( R \in \mathcal{T}r(H) \) denote by \( \text{Out}_{\text{tw}}(H, R) \) the stabiliser \( St_{\text{Out}_{\text{tw}}(H)}(R) \):

\[
\text{Out}_{\text{tw}}(H, R) = \{(f, F) \in \text{Out}_{\text{tw}}(H) \mid (f \otimes f)(R)F_{21} = FR\}.
\]

**Remark 1.6.** Note that the group \( \text{Out}_{\text{tw}}(H, R) \) naturally embeds in to the group of isomorphism classes of braided tensor autoequivalences of \( H \to \text{Mod} \) with the braiding given by \( R \).
Indeed, writing an R-matrix as 

\[ R = r_{23}R_{13} + r_{23}r_{13}, \quad (\Delta \otimes I)(r) = r_{12}R_{13} + R_{12}r_{13}, \quad \Delta(x)r = rt\Delta(x) \quad \forall x \in H, \]

conditions in (18). Similarly the tangent Lie algebra of the stabiliser (1 7) is

\[ \text{Proposition 1.7.} \]

The above implies to the following 

\[ \text{R-matrices:} \]

\[ \text{OutDer}_{tw}(H, R) = \{ (d, \phi) \in \text{OutDer}_{tw}(H) | (I \otimes d + d \otimes I)(R) = \phi_{23}R - R\phi \}. \]

Expanding the ingredients of (16) in powers of \( h \) with \( h^3 = 0 \) and looking at the change in the coefficient for \( h^2 \) we get the following formula for the action of twisted derivations on infinitesimal R-matrices:

\[ (d, \phi)(r) = (I \otimes d + d \otimes I)(r) - \phi r + r\phi_{21}. \quad (19) \]

The above implies to the following

\[ \text{Proposition 1.7. The formula (19) defines the structure of an OutDer}_{tw}(H, R)-module on} \]

\[ T_R(T_r(H)). \]

\[ \text{Remark 1.8. Let} \ H \ \text{be a cocommutative bialgebra. It is known that infinitesimal quasi-triangular} \]

\[ \text{structures (i.e. R-matrices of the form} \ R = 1 + hr) \ \text{on} \ H \ \text{correspond to invariant classical R-matrices, i.e. elements} \ r \in H^\otimes 2 \ \text{satisfying} \]

\[ r\Delta(x) = \Delta(x)r \quad \forall x \in H \]

and

\[ (I \otimes \Delta)(r) = r_{13} + r_{12}, \quad (\Delta \otimes I)(r) = r_{13} + r_{23}, \]

Note that the last two equations imply that \( r \) belongs to \( \text{Prim}(H)^\otimes 2 \). Thus the tangent space to the space of universal R-matrices at a point 1 \( \in T_r(H) \) is \( T_{r_{inf}}(H) = (\text{Prim}(H)^\otimes 2)^H \) the vector space of invariant classical R-matrices of \( H \). The tangent Lie algebra of the stabiliser of the canonical quasi-triangular structure on \( H \) (the one corresponding to 1 \( \in H^\otimes 2 \)) is

\[ \text{OutDer}_{tw}(H, 1) = \{ (d, \phi) \in \text{OutDer}_{tw}(H) | \phi = \phi_{21} \}. \]

The \( \text{OutDer}_{tw}(H, 1) \)-action on \( (\text{Prim}(H)^\otimes 2)^H \) has the form

\[ (d, \phi)(r) = (I \otimes d + d \otimes I)(r) - [\phi, r]. \]

\section{Examples}

\subsection{Twisted derivations of universal enveloping algebras}

In this section we assume that the ground field \( k \) is of characteristic zero.

Now let \( (d, \phi) \) be a twisted derivation of \( U(g) \). In particular \( \phi \in U(g)^\otimes 2 \) is an infinitesimal (or co-Hochschild) 2-cocycle (see Appendix 3), i.e.:

\[ 1 \otimes \phi + (I \otimes \Delta)(\phi) = \phi \otimes 1 + (\Delta \otimes I)(\phi). \]

Proposition 3.3 implies that there is \( a \in U(g) \) such that

\[ \phi = \bar{\phi} + a \otimes 1 + 1 \otimes a - \Delta(a), \quad \bar{\phi} = \text{Alt}_2(\phi) = \frac{1}{2}(\phi - \phi_{21}). \quad (20) \]

Thus the twisted derivation \( (d, \phi) \) is gauge equivalent to a twisted derivation \( (d', \bar{\phi}) \) (with \( \bar{\phi} \in \Lambda^2 g \)):

\[ (d, \phi) = (d + [a, -] - [a, -], \bar{\phi} + a \otimes 1 + 1 \otimes a - \Delta(a)) = (d', \bar{\phi}) - \partial(a). \]
Now look at the equation (9) for the twisted derivation \((d', \phi)\):

\[
(I \otimes d' + d' \otimes I)(\Delta(x)) - \Delta(d'(x)) = [\phi, \Delta(x)],
\]

The left hand side is a symmetric element of \(U(\mathfrak{g})^\otimes 2\) (due to cocommutativity of \(U(\mathfrak{g})\)), while the right hand side is anti-symmetric:

\[
[\phi, \Delta(x)] \in \Lambda^2 \mathfrak{g}
\]

Thus they both must be zero:

\[
(I \otimes d' + d' \otimes I)(\Delta(x)) - \Delta(d'(x)) = 0,
\]

\[
[\phi, \Delta(x)] = 0,
\]

i.e. \((d', \phi)\) is a separated twisted derivation.

Thus we have the following.

**Theorem 2.1.** Let \(\mathfrak{g}\) be a Lie algebra over a field of characteristic zero. Then the Lie algebra of outer twisted derivations of \(U(\mathfrak{g})\) has a form:

\[
\text{OutDer}_\text{tw}(U(\mathfrak{g})) \simeq \text{OutDer}(\mathfrak{g}) \ltimes (\Lambda^2 \mathfrak{g})^\mathfrak{g},
\]

where the crossed product is taken with respect to the natural action of \(\text{OutDer}(\mathfrak{g})\) on the abelian Lie algebra \((\Lambda^2 \mathfrak{g})^\mathfrak{g}\).

**Proof.** We have established that any twisted derivation of \(U(\mathfrak{g})\) is gauge equivalent to \((d, \phi)\), where \(d\) is a bialgebra derivation and \(\phi \in (\Lambda^2 \mathfrak{g})^\mathfrak{g}\). A bialgebra derivation of \(U(\mathfrak{g})\) is (induced by) a Lie algebra derivation of \(\mathfrak{g}\). Now all we need to show is that the gauge class of \((d, \phi)\) depends only on the class of \(d\) in the Lie algebra of outer derivations \(\text{OutDer}(\mathfrak{g})\). This can be seen by applying gauge equivalences corresponding to \(x \in \mathfrak{g} \subset U(\mathfrak{g})\):

\[
(d, \phi) + \partial(x) = (d + [x, -], \phi).
\]

The commutator \([X, Y]\) (in \(U(\mathfrak{g})^\otimes 2\)) of two skew-symmetric elements is necessarily symmetric. Thus the commutator on \((\Lambda^2 \mathfrak{g})^\mathfrak{g}\) is trivial. It follows from (12) that the action of \(\text{OutDer}(\mathfrak{g})\) on \((\Lambda^2 \mathfrak{g})^\mathfrak{g}\) is

\[
d(X) = (d \otimes 1 + 1 \otimes d)(X).
\]

In particular the theorem implies that the Lie algebra of invariant infinitesimal twists coincides with \((\Lambda^2 \mathfrak{g})^\mathfrak{g}\). This of course can be seen directly. Indeed, if \(\phi \in U(\mathfrak{g})^\otimes 2\) is a \(\mathfrak{g}\)-invariant infinitesimal twist then \(\phi\) is also \(\mathfrak{g}\)-invariant which altogether makes

\[
\partial(a) = a \otimes 1 + 1 \otimes a - \Delta(a) = \phi - \phi
\]

\(\mathfrak{g}\)-invariant. According to lemma 3.8 the last implies that \(a\) can be chosen to be \(\mathfrak{g}\)-invariant (central).

**Remark 2.2.** Here we say a few words about the Jacobiator of the crossed module of Lie algebras

\[
\text{Der}_\text{tw}(U(\mathfrak{g})) \xleftarrow{\partial} U(\mathfrak{g})
\]

as a cohomology class in

\[
H^3(\text{OutDer}_\text{tw}(U(\mathfrak{g})), Z(\mathfrak{g})) = H^3(\text{OutDer}(\mathfrak{g}) \ltimes (\Lambda^2 \mathfrak{g})^\mathfrak{g}, Z(\mathfrak{g})).
\]

We start by showing that its restriction to \(H^3((\Lambda^2 \mathfrak{g})^\mathfrak{g}, Z(\mathfrak{g}))\), which coincides with the Jacobiator of the crossed module of Lie algebras

\[
Z^2(U(\mathfrak{g}))^\mathfrak{g} \xleftarrow{\partial} C^1(U(\mathfrak{g}))^\mathfrak{g}
\]

(21)
is trivial. Here $Z^2$ is the Lie algebra (with respect to the commutator in $U(\mathfrak{g})^{\otimes 2}$) of 2-cocycles of the subcomplex of $\mathfrak{g}$-invariants of $\mathfrak{Z}$ and $C^1(U(\mathfrak{g}))^g = Z(U(\mathfrak{g}))$ is the abelian Lie algebra of 1-cochains of the same subcomplex. Note that the action of $Z^2(U(\mathfrak{g}))^g$ on $C^1(U(\mathfrak{g}))^g$ is trivial and the commutator $[X, \partial(a)]$ is zero for any $X \in Z^2(U(\mathfrak{g}))^g$ and $a \in Z(U(\mathfrak{g}))$ (thus in particular fulfilling the axioms of crossed module of Lie algebras). Thus for any $X, Y \in Z^2(U(\mathfrak{g}))^g$ we have $[X, Y] = [\text{Alt}_2(X), \text{Alt}_2(Y)]$. As was noted in the proof of theorem 2.1, the commutator $[\text{Alt}_2(X), \text{Alt}_2(Y)]$ is a symmetric 2-cocycle and hence a coboundary. So we can write it as $\partial(a(\text{Alt}_2(X), \text{Alt}_2(Y)))$ for some function $a : (\Lambda^2 \mathfrak{g})^g \otimes (\Lambda^2 \mathfrak{g})^g \to Z(U(\mathfrak{g}))$. The function $a$ can be extended to a function $a : Z^2(U(\mathfrak{g}))^g \otimes Z^2(U(\mathfrak{g}))^g \to Z(U(\mathfrak{g}))$ by $a(X, Y) = a(\text{Alt}_2(X), \text{Alt}_2(Y))$. Now we have $a(X, [Y, Z]) = 0$, so the Jacobiator of the crossed module of Lie algebras $\mathfrak{Z}$ is trivial.

It follows from the spectral sequence of the extension

$$(\Lambda^2 \mathfrak{g})^g \to \text{OutDer}_{tw}(U(\mathfrak{g})) \to \text{OutDer}(\mathfrak{g})$$

that the cohomology $E^3_{\infty} = H^3(\text{OutDer}_{tw}(U(\mathfrak{g})), Z(\mathfrak{g}))$ has a filtration

$$E^3_{\infty} = F^0 E^3_{\infty} \supset F^1 E^3_{\infty} \supset F^2 E^3_{\infty} \supset F^3 E^3_{\infty}$$

with the first associated quotient

$$F^0 E^3_{\infty}/F^1 E^3_{\infty} = E^{0,3}_{\infty} \subset E^{0,3}_2 = H^3((\Lambda^2 \mathfrak{g})^g, Z(\mathfrak{g})).$$

Since the induced map

$$H^3(\text{OutDer}_{tw}(U(\mathfrak{g})), Z(\mathfrak{g})) = E^3_{\infty} \to E^{0,3}_2 = H^3((\Lambda^2 \mathfrak{g})^g, Z(\mathfrak{g}))$$

is the restriction we have shown that the Jacobiator belongs to $F^1 E^3_{\infty}$. The second quotient is

$$F^1 E^3_{\infty}/F^2 E^3_{\infty} = E^{1,2}_{\infty} \subset E^{1,2}_2 = H^1(\text{OutDer}(\mathfrak{g}), H^2((\Lambda^2 \mathfrak{g})^g, Z(\mathfrak{g}))).$$

The image of the Jacobiator under the homomorphism

$$F^1 E^3_{\infty} \to H^1(\text{OutDer}(\mathfrak{g}), H^2((\Lambda^2 \mathfrak{g})^g, Z(\mathfrak{g})))$$

can be described as follows. Let $d \in \text{Der}(\mathfrak{g})$, $X, Y \in (\Lambda^2 \mathfrak{g})^g$. Since

$$(d \otimes I + I \otimes d)([X, Y]) = (d \otimes I + I \otimes d)(\partial(a(X, Y))) = \partial(d(a(X, Y)))$$

coincides with

$$[(d \otimes I + I \otimes d)(X), Y] + [X, (d \otimes I + I \otimes d)(Y)] = \partial(a((d \otimes I + I \otimes d)(X), Y) + a(X, (d \otimes I + I \otimes d)(Y)))$$

the difference

$$a(d, X, Y) = a((d \otimes I + I \otimes d)(X), Y) + a(X, (d \otimes I + I \otimes d)(Y)) - d(a(X, Y))$$

belongs to $Z(\mathfrak{g})$. The assignment

$$d \mapsto (X, Y \mapsto a(d, X, Y))$$

defines a class in $H^1(\text{OutDer}(\mathfrak{g}), H^2((\Lambda^2 \mathfrak{g})^g, Z(\mathfrak{g})))$, which is the class of Jacobiator.

There is a choice of $a(X, Y)$ which makes $a(d, X, Y)$ identically zero. Indeed $a$ is the second component of the co-chain homotopy from remark 3.5. The $\mathfrak{g}$-invariance of the homotopy implies that $a(d, X, Y)$ is zero.

In general if the class of $a(d, X, Y)$ in $H^1(\text{OutDer}(\mathfrak{g}), H^2((\Lambda^2 \mathfrak{g})^g, Z(\mathfrak{g})))$ is trivial, the Jacobiator belongs to $F^2 E^3_{\infty}$. The obstruction for the Jacobiator to be in $F^3 E^3_{\infty}$ is measured by its image in

$$F^2 E^3_{\infty}/F^3 E^3_{\infty} = E^{2,1}_{\infty} \subset E^{2,1}_2 = H^2(\text{OutDer}(\mathfrak{g}), H^1((\Lambda^2 \mathfrak{g})^g, Z(\mathfrak{g}))).$$

Since $a(d, X, Y)$ is identically zero the class of the Jacobiator in $H^2(\text{OutDer}(\mathfrak{g}), H^1((\Lambda^2 \mathfrak{g})^g, Z(\mathfrak{g})))$ is trivial. Thus the Jacobiator belongs to $F^2 E^3_{\infty} = H^3(\text{OutDer}(\mathfrak{g}), Z(\mathfrak{g}))$ and we have the following statement.
Proposition 2.3. The Jacobiator of the crossed module of Lie algebras

\[ \text{Der}_{tw}(U(\mathfrak{g})) \xrightarrow{\partial} U(\mathfrak{g}) \]

as a cohomology class in

\[ H^3(\text{OutDer}_{tw}(U(\mathfrak{g})), Z(\mathfrak{g})) = H^3(\text{OutDer}(\mathfrak{g}) \ltimes (\Lambda^2 \mathfrak{g})^0, Z(\mathfrak{g})) \]

coincides with the image of the canonical class of the crossed module of Lie algebras

\[ Z(\mathfrak{g}) \xrightarrow{\partial} \mathfrak{g} \xrightarrow{\partial} \text{Der}(\mathfrak{g}) \xrightarrow{\partial} \text{OutDer}(\mathfrak{g}) \]

in \( H^3(\text{OutDer}(\mathfrak{g}), Z(\mathfrak{g})) \).

2.2 Non-separated twisted derivations

Here we present a construction providing examples of non-separated twisted derivations. Let \( H \) be an algebra. Denote by \( \mathcal{T}(H[t]) \) the tensor algebra of the vector space \( H[t] \) of polynomials with coefficients in \( H \). Consider the quotient algebra

\[ E(H) = \mathcal{T}(H[t]) / \langle \sum_{i=0}^{n} C_i^n x t^i \otimes y t^{n-i}, 1 t^n, \forall x, y \in H, n \geq 0 \rangle. \]

Note that

\[ H \to E(H), \ x \mapsto x t^0 \]

is an embedding of algebras.

Lemma 2.4. The assignment

\[ d : H[t] \to E(H), \ d(x t^n) = x t^{n+1} \]

extends to an algebra derivation.

Proof. All we need to check is that \( d \) preserves the defining relations of \( E(H) \). This is a straightforward consequence of the property of binomial coefficients \( C_i^{n+1} = C_{i-1}^n + C_i^n \). Indeed

\[ d(\sum_{i=0}^{n} C_i^n x t^i \otimes y t^{n-i}) = \sum_{i=0}^{n} C_i^n (d(x t^i) \otimes y t^{n-i} + x t^i \otimes d(y t^{n-i})) = \sum_{i=0}^{n} C_i^n (x t^{i+1} \otimes y t^{n-i} + x t^i \otimes y t^{n-i+1}) \]

coincides with

\[ d(x t^n) = x t^{n+1} = \sum_{i=0}^{n+1} C_{i+1}^{n+1} x t^i \otimes y t^{n-i+1}. \]

Remark 2.5. The assignment \( H \mapsto E(H) \) is a functor from the category of algebras to the category of algebras with a derivation (differential algebras), which is a left adjoint to the forgetful functor. In other words \( E(H) \) is the free differential algebra on an algebra \( H \).

From now on we will suppress the tensor sign and will use \( d'(x) \) instead of \( x t^i \) when working with elements of \( E(H) \).

Let now \( H \) be a bialgebra and let \( \phi \in H^\otimes 2 \) be a co-Hochschild 2-cocycle:

\[ 1 \otimes \phi + (I \otimes \Delta)(\phi) = \phi \otimes 1 + (\Delta \otimes I)(\phi). \]
Lemma 2.6. The inductively defined assignment
\[ \Delta(d^n(x)) = (d \otimes I + I \otimes d)(\Delta(d^{n-1}(x))) - [\phi, \Delta(d^{n-1}(x))], \quad x \in H \] (22)
extends to a homomorphism of algebras \( \Delta : E(H) \rightarrow E(H) \otimes E(H) \), making \( E(H) \) a bialgebra with
the counit defined by
\[ \varepsilon(d^n(x)) = 0, \quad x \in H, \quad n > 0. \]

Proof. We will use induction to prove that \( \Delta \) preserves the defining relations of \( E(H) \):
\[
\Delta(d^n(xy)) - \sum_{i=0}^{n} C_i^n \Delta(d^i(x))\Delta(d^{n-i}(y)) = (d \otimes I + I \otimes d)(\Delta(d^{n-1}(xy))) - [\phi, \Delta(d^{n-1}(xy))] - \\
- \sum_{i=0}^{n} C_i^n \Delta(d^i(x))((d \otimes I + I \otimes d)(\Delta(d^{n-i-1}(y)))) - [\phi, \Delta(d^{n-i-1}(y))]
\]
\[
= (d \otimes I + I \otimes d)(\Delta(d^{n-1}(xy)) - \sum_{i=0}^{n-1} C_i^n \Delta(d^i(x))d^{n-i-1}(y)) - \\
- [\phi, \Delta(d^{n-1}(xy)) - \sum_{i=0}^{n-1} C_i^n d^i(x)d^{n-i-1}(y))] = 0.
\]

The coassociativity of \( \Delta \) can also be proved by induction:
\[
(\Delta \otimes I - I \otimes \Delta)\Delta(d^n(x)) = (\Delta \otimes I - I \otimes \Delta)\((d \otimes I + I \otimes d)(\Delta(d^{n-1}(x))\))- (\Delta \otimes I - I \otimes \Delta)\([\phi, \Delta(d^{n-1}(x))]) = \\
(\Delta\otimes I + \Delta \otimes d - d \otimes \Delta - I \otimes \Delta d)\Delta(d^{n-1}(x)) - \\
- [(\Delta \otimes I)(\phi), (\Delta \otimes I)(\Delta(d^{n-1}(x)))] + [(I \otimes \Delta)(\phi), (I \otimes \Delta)(\Delta(d^{n-1}(x)))] = \\
(\Delta \otimes I + I \otimes d I + I \otimes \Delta d)(\Delta \otimes I - I \otimes \Delta)\Delta(d^{n-1}(x)) - \\
- [\phi \otimes 1 + (\Delta \otimes I)(\phi), (\Delta \otimes I)(\Delta(d^{n-1}(x)))] + [1 \otimes \phi + (I \otimes \Delta)(\phi), (I \otimes \Delta)(\Delta(d^{n-1}(x)))] = 0.
\]

Note that the embedding \( H \rightarrow E(H) \) is a homomorphism of bialgebras.

Theorem 2.7. The pair \((d, \phi)\) is a twisted derivation of the bialgebra \( E(H) \).

Proof. Clearly, the conditions \( [U, U] \) are obvious. The condition \( [U] \) follows from the definition
of the coproduct (22).

Example 2.8. Let \( a = (x, y) \) be an abelian 2-dimensional Lie algebra. Let \( H = U(a) = k[x, y] \) be
its universal enveloping algebra. The algebra \( E(H) \) is the quotient of the free associative algebra
\( k[x, y, d(x), d(y), d^2(x), d^2(y), \ldots] \) by the ideal generated by \( \sum_{n=0}^{\infty} C_n^n [d^i(x), d^{n-i}(y)] \) for all \( n \). Let
\( \phi = x \otimes y \in H \otimes \) be an infinitesimal twist. The corresponding coproduct on \( E(H) \) is defined by
\[
\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + 1 \otimes y,
\]
\[
\Delta(d(x)) = d(x) \otimes 1 + 1 \otimes d(x), \quad \Delta(d(y)) = d(y) \otimes 1 + 1 \otimes d(y),
\]
\[
\Delta(d^2(x)) = d^2(x) \otimes 1 + 1 \otimes d^2(x) - [x \otimes y, d(x) \otimes 1 + 1 \otimes d(x)],
\]
\[
\Delta(d^2(y)) = d^2(y) \otimes 1 + 1 \otimes d^2(y) - [x \otimes y, d(y) \otimes 1 + 1 \otimes d(y)].
\]
Proof. We will use induction to prove that $\Delta$ preserves the defining relations of $A(H)$. This is straightforward

$$f(xy^n - xt^n \otimes y^n) = xy^{n+1} - xt^{n+1} \otimes y^{n+1} = 0, \quad f(t^n - 1) = t^{n+1} - 1 = 0.$$ \hfill \square

The assignment $H \mapsto A(H)$ is a functor from the category of algebras to the category of algebras with an automorphism, which is a left adjoint to the forgetful functor. In other words $A(H)$ is the free algebra with an automorphism on an algebra $H$.

From now on we will suppress the tensor sign and will use $f'(x)$ instead of $xt$ when working with elements of $A(H)$.

Let now $H$ be a bialgebra and let $F \in H^{\otimes 2}$ be a twist:

$$(1 \otimes F)(I \otimes \Delta)(F) = (F \otimes 1)(\Delta \otimes I)(F).$$

Lemma 2.11. The inductively defined assignment

$$\Delta(f^n(x)) = F^{-1}((f \otimes f)(\Delta(f^{n-1}(x))))F, \quad x \in H$$ (23)

extends to a homomorphism of algebras $\Delta : A(H) \to A(H) \otimes A(H)$, making $A(H)$ a bialgebra with the counit defined by

$$\varepsilon(f^n(x)) = \varepsilon(x), \quad x \in H, n \in \mathbb{Z}.$$  

Proof. We will use induction to prove that $\Delta$ preserves the defining relations of $A(H)$.

$$\Delta(f^n(xy) - f^n(x)f^n(y)) - (1 \otimes \Delta(f^n(x)))(f \otimes f)(\Delta(f^{n-1}(x)))F =$$

$$F^{-1}((f \otimes f)(\Delta(f^{n-1}(xy))))F - F^{-1}((f \otimes f)(\Delta(f^{n-1}(x))))F F^{-1}((f \otimes f)(\Delta(f^{n-1}(y))))F =$$

$$F^{-1}((f \otimes f)(\Delta(f^{n-1}(xy) - f^{n-1}(x)f^{n-1}(y))))F = 0.$$  

The coassociativity of $\Delta$ can also be proved by induction:

$$(\Delta \otimes I - I \otimes \Delta)(f^n(x)) = (\Delta \otimes I - I \otimes \Delta)(F^{-1}((f \otimes f)(\Delta(f^{n-1}(x))))F) =$$

$$(\Delta \otimes I)(F^{-1}(f \otimes f)(\Delta(f^{n-1}(x))))(\Delta \otimes I)(F) - (I \otimes \Delta)(F)^{-1}(f \otimes f)(\Delta(f^{n-1}(x)))(I \otimes \Delta)(F) =$$

$$(\Delta \otimes I)(F^{-1}(f \otimes f)(\Delta(f^{n-1}(x))))(\Delta \otimes I)(F) - (I \otimes \Delta)(F)^{-1}(f \otimes f)(\Delta(f^{n-1}(x)))(I \otimes \Delta)(F) =$$

$$(\Delta \otimes I)(F^{-1}(f \otimes f)(\Delta(f^{n-1}(x))))(\Delta \otimes I)(F) = 0.$$  

Note that the embedding $H \to A(H)$ is a homomorphism of bialgebras.

Theorem 2.12. The pair $(f, F)$ is a twisted automorphism of the bialgebra $A(H)$.

Proof. Clearly, the conditions (23) are obvious. The condition (1) follows from the definition of the coproduct (23).  

\hfill \square
2.3 Sweedler’s Hopf algebra

Here we describe twisted derivations of a non-commutative and non-cocommutative 4-dimensional Hopf algebra defined by Sweedler. Recall that the Sweedler’s Hopf algebra $H_4$ is the algebra

$$H_4 = k\langle g, x | g^2 = 1, x^2 = 0, gx + xg = 0 \rangle.$$ 

It is a Hopf algebra with the comultiplication

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g,$$

the counit

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$

and the antipode

$$S(g) = g^{-1}, \quad S(x) = -x.$$

**Lemma 2.13.** The second cohomology $H^2(H_4)$ of the Sweedler’s Hopf algebra is one-dimensional and is generated by the element $\psi = x \otimes gx \in H_4 \otimes^2$.

*Proof.* Lemma 3.10 implies that $\psi$ is a 2-cocycle.

Let now $(d, \phi)$ be a twisted derivation of $H_4$. According to lemma 2.13, $\phi$ must be a multiple of $\psi = x \otimes gx$. Note that the element $\psi$ is $\Delta(H_4)$-invariant:

$$[\psi, \Delta(g)] = [x \otimes gx, g \otimes g] = xg \otimes gxg - gx \otimes x = 0,$$

$$[\psi, \Delta(x)] = [x \otimes gx, 1 \otimes x + x \otimes g] = 0.$$

Thus up to gauge equivalences twisted derivations are separated:

$$\text{OutDer}_{tw}(H_4) = \text{OutDer}_{bialg}(H_4) \rtimes \text{OutDer}_0^{tw}(H_4).$$

Now all what remains to analyse to have a full control of twisted derivations is the Lie algebra of outer bialgebra derivations. Since the space $\text{Prim}(H_4)$ of primitive elements is zero, $H_4$ does not have inner bialgebra derivations:

$$\text{OutDer}_{bialg}(H_4) = \text{Der}_{bialg}(H_4).$$

For a bialgebra derivation $d : H_4 \to H_4$ the element $g^{-1}d(g)$ has to be primitive. Indeed,

$$\Delta(d(g)) = d(g) \otimes g + g \otimes d(g)$$

and

$$\Delta(gd(g)) = g^{-1}d(g) \otimes 1 + 1 \otimes g^{-1}d(g).$$

Thus for a bialgebra derivation $d : H_4 \to H_4$ we have $d(g) = 0$. The value $d(x)$ is $g$-primitive:

$$\Delta(d(x)) = 1 \otimes d(x) + d(x) \otimes g + x \otimes d(g) = 1 \otimes d(x) + d(x) \otimes g.$$

Hence any bialgebra derivation of $H_4$ is proportional to

$$d(g) = 0, \quad d(x) = x.$$

**Theorem 2.14.** The algebra $\text{OutDer}_{tw}(H_4)$ of twisted derivations of the Sweedler’s Hopf algebra is a two-dimensional non-abelian Lie algebra.
Proposition 3.2. The cup product on co-Hochschild cohomology \( H^*(H) \) is graded commutative:
\[
[X] \cup [Y] = (-1)^{mn} [Y] \cup [X], \quad [X] \in H^m(H), \quad Y \in H^n(H).
\]
Proof. The construction is similar to the one for Hochschild complex from [3]. For \( X \in H^\otimes m \), \( Y \in H^\otimes n \) and \( i = 1, \ldots, m \) define
\[
X \circ_i Y = (I^{n-i-1} \otimes \Delta^{i-1} \otimes I^{m-i}) (X) (1_{H^\otimes i-1} \otimes \partial Y \otimes 1_{H^\otimes m-i}),
\]
where \( \Delta^{n-1} : H \to H^\otimes n \) is the iterated coproduct. The operations \( \circ_i \) satisfy to the axioms of a pre-Lie system of [3]:
\[
(X \circ_i Y) \circ_j Z = \begin{cases} (X \circ_j Z) \circ_{i+p} Y, & j < i \\ X \circ_i (Y \circ_j Z), & i \leq j \end{cases}
\]
for \( X \in H^\otimes m \), \( Y \in H^\otimes n \) and \( Z \in H^\otimes p \). Thus, according to [3], the operation
\[
X \circ Y = \sum_{i=1}^m (-1)^{ni} X \circ_i Y
\]
is a homotopy for commutativity of the cup product:
\[
Y \cup X - (-1)^{nm} X \cup Y = \partial(X) \circ Y + (-1)^{n-1}X \circ \partial(Y) - (-1)^{n-1}(X \circ Y).
\]
\[\square\]

Remark 3.3. The bracket
\[
[[X, Y]] = X \circ Y - (-1)^{mn} Y \circ X
\]
ends \( H^*(H) \) with a graded Lie bracket
\[
[[\ ,\ ]] : H^m(H) \otimes H^n(H) \to H^{m+n-1}(H),
\]
which together with the cup product turn \( H^*(H) \) into a Gerstenhaber algebra.

The following gives a description of co-Hochschild cohomology of a universal enveloping algebra in characteristic zero.

**Proposition 3.4.** For a universal enveloping algebra \( H = U(\mathfrak{g}) \) the alternation map \( \text{Alt}_n : H^\otimes n \to \Lambda^n H \) induces an isomorphism of the \( n \)-th co-Hochschild cohomology and \( \Lambda^n \mathfrak{g} \).

Proof. (Sketch of, for details see [3]):
By Poincaré-Birkhoff-Witt theorem, the map
\[
S^*(\mathfrak{g}) \to U(\mathfrak{g}), \quad x_1 \otimes \ldots \otimes x_n \mapsto \text{Sym}_n(x_1 \ldots x_n), \quad x_i \in \mathfrak{g}
\]
is an isomorphism of coalgebras. The complex [24] for \( H = S^*(\mathfrak{g}) \) breaks into (tensor products of) pieces:
\[
S^n(\mathfrak{g}) \to \bigoplus_{i_1 + i_2 = n} S^{i_1}(\mathfrak{g}) \otimes S^{i_2}(\mathfrak{g}) \to \ldots \to \bigoplus_{i_1 + \ldots + i_s = n} \otimes_{j=1}^s S^{i_j}(\mathfrak{g}) \to \ldots \mapsto \mathfrak{g}^\otimes n
\]
The \( n \)-th piece is isomorphic to the cochain complex of the simplicial \( n \)-cube tensored (over the symmetric group \( S_n \)) with \( \mathfrak{g}^\otimes n \).

Remark 3.5. The map \( \text{Alt}_n : H^\otimes n \to H^\otimes n \) from proposition [24] is a map of co-Hochschild complex(es). Moreover \( \text{Alt}_n \) is homotopic to the identity, i.e. there is a collection of maps \( a_n : H^\otimes n \to H^\otimes n-1 \) such that
\[
I - \text{Alt}_n = \partial \circ a_n + a_{n+1} \circ \partial.
\]

Remark 3.6. Under the isomorphism \( H^*(U(\mathfrak{g})) \to \Lambda^*(\mathfrak{g}) \) the Gerstenhaber bracket on \( H^*(U(\mathfrak{g})) \) corresponds to the Schouten bracket on \( \Lambda^*(\mathfrak{g}) \):
\[
[[x_1 \land \ldots \land x_m, y_1 \land \ldots \land y_n]] = \sum_{i,j} (-1)^{i+j} [x_i, y_j] \land x_1 \land \ldots \land \hat{x}_i \land \ldots \land x_m \land y_1 \land \ldots \land \hat{y}_j \land \ldots \land y_n.
\]
Here \( \hat{x} \) means that \( x \) is omitted in the exterior product.
Remark 3.7. Note that the first cohomology $H^1(H)$ of a bialgebra coincides with the space $\text{Prim}(H)$ of its primitive elements, which forms a Lie algebra with respect to the commutator. The embedding $\text{Prim}(H) \subseteq H$ induces a homomorphism of bialgebras $U(\text{Prim}(H)) \to H$, which in its turn gives rise to a map

$$\Lambda^*(\text{Prim}(H)) \to H^*(H).$$

The following describes the cohomology of the the subcomplex of $H$-invariant elements of $\Lambda^n$. In the case of universal enveloping algebra $H$.

Lemma 3.8. For a universal enveloping algebra $H = U(g)$ the alternation map $\text{Alt}_n : (H \otimes^n)H \to (\Lambda^n H)^H$ induces an isomorphism of the $n$-th cohomology of the subcomplex of $H$-invariant elements of $\Lambda^n$ and the space of $g$-invariant skew-symmetric tensors $(\Lambda^n g)^g$.

Proof. The coalgebra isomorphism between $U(g)$ and $S^*(g)$ is $g$-invariant. The isomorphism between the degree $n$ component of $(\Lambda^n)$ and the cochain complex of the simplicial $n$-cube tensored with $g^{\otimes n}$ is natural in $g$ and in particular $g$-invariant.

Remark 3.9. Note that the Schouten bracket on the $(\Lambda^n g)^g$ is zero.

Now let $K \subseteq H$ be an embedding of Hopf algebras. Clearly $C^*(K) \subseteq C^*(H)$ is a subcomplex. More generally define

$$C^*(H)_m = \left\{ K^{\otimes i}, K^{\otimes m} \otimes H^{\otimes i-m}, \begin{array}{ll} & i \leq m \\text{or} \end{array} i > m \right\}.$$

Then

$$C^*(H) = C^*(H)_0 \supset C^*(H)_1 \supset \ldots$$

is a decreasing filtration by subcomplexes with $\cap_{m=0}^{\infty} C^*(H)_m = C^*(K)$.

Let $g$ be a group-like element of a Hopf algebra $H$. An element $x \in H$ is called $g$-primitive if

$$\Delta(x) = 1 \otimes x + x \otimes g.$$

Denote by $\text{Prim}_g(H)$ the space of $g$-primitive elements of $H$.

Lemma 3.10. Let $g_i, i = 1, \ldots, m$ be a collection of group-like elements in $H$ such that $g_1 \ldots g_m = 1$. Then for any $x_i \in \text{Prim}_{g_i}(H)$

$$F = x_1 \otimes g_1 x_2 \otimes g_2 x_3 \otimes \ldots \otimes g_1 \ldots g_m - 1 x_m \in H^{\otimes m}$$

is a co-Hochschild cocycle.

Proof. Let’s first drop the assumption $g_1 \ldots g_m = 1$. We can prove by induction that $\partial(F) = F \otimes (g_1 \ldots g_m - 1)$. Indeed,

$$\partial(F) = 1 \otimes x_1 \otimes g_1 x_2 \otimes g_2 x_3 \otimes \ldots \otimes g_1 \ldots g_{m-1} x_m +$$

$$\sum_{i=1}^{n} (-1)^{i} x_1 \otimes g_1 x_2 \otimes \ldots \otimes (g_1 \ldots g_{i-1} \otimes g_{i+1} \ldots g_m - 1 x_i + g_1 \ldots g_{i-1} x_i \otimes g_1 \otimes g_1 \otimes \ldots \otimes g_1 \otimes g_{m-1} x_m +$$

$$+ (-1)^{n+1} (x_1 \otimes g_1 x_2 \otimes \ldots \otimes g_1 \ldots g_m - 1 x_m \otimes 1) = F \otimes (g_1 \ldots g_m - 1).$$

Now the lemma follows.
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