Chiral corrections and lattice QCD results for $f_{B_s}/f_{B_d}$ and $\Delta m_{B_s}/\Delta m_{B_d}$

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Abstract

It has been argued recently that the inclusion of the chiral logarithms in extrapolation of the lattice data can shift the value of the hadronic parameter $\xi = f_{B_s}\sqrt{B_{B_s}}/f_{B_d}\sqrt{B_{B_d}}$, from 1.16(6) to 1.32(10) and even higher. If true, that would considerably change the theoretical estimate for the ratio of oscillation frequencies in the $B_s^0$- and $B_d^0$-systems, and would affect the standard CKM unitarity triangle analysis. In this letter we show that $f_{B_s}/f_{B_d} \approx f_K/f_\pi$, and thus the uncertainty due to the missing chiral logs is smaller than previously thought. By combining the NLO chiral expansion with the static heavy quark limit we obtain $\xi = 1.22(8)$.

1 Uncertainty in $\Delta m_{B_s}/\Delta m_{B_d}$ comes from $f_{B_s}/f_{B_d}$

The $K^0-\bar{K}^0$ and $B^0-\bar{B}^0$ mixing amplitudes are the main ingredients in the standard unitarity triangle analysis [1]. They overconstrain the apex of the CKM triangle in the $(\bar{\rho}, \bar{\eta})$-plane. Confronting such an overconstrained triangle with the directly measured $\sin(2\beta)$ [2] allows for the consistency check from which we hope to either see the effects of the non-Standard Model physics, or simply to confirm the validity of Standard Model at the accessible accuracy [3]. One of the essential constraints is provided by the ratio of oscillation frequencies

$$\frac{\Delta m_{B_s}}{\Delta m_{B_s}} = \left| \frac{V_{td}}{V_{ts}} \right|^2 \frac{m_{B_s}}{m_{B_d}} \frac{1}{\xi^2} \simeq \lambda^2 \left[(1-\bar{\rho})^2 + \bar{\eta}^2 \right] \frac{m_{B_s}}{m_{B_d}} \frac{1}{\xi^2} + \mathcal{O}(\lambda^4),$$

(1)
where $m_{B_{s(d)}}$ is the mass of $B_{s(d)}^0$-meson, while $V_{ts}$ and $V_{td}$ are the CKM matrix elements. The mass difference $\Delta m_{B_d}$, which measures the oscillation frequency in the $B_d^0 - \bar{B}_d^0$ system, has been determined accurately in the experiments, $\Delta m_{B_d} = 0.503 \pm 0.006$ ps$^{-1}$ \[4\]. On the other hand, $\Delta m_{B_s}$ is currently only bounded from below, $\Delta m_{B_s} > 14.4$ ps$^{-1}$ \[4\], but will hopefully be measured soon at Tevatron \[5\]. On the r.h.s. of eq. (1), one has the hadronic parameter

$$\xi = \frac{f_{B_s} \sqrt{\hat{B}_{B_s}}}{f_{B_d} \sqrt{\hat{B}_{B_d}}} ,$$

with $f_{B_q}$ and $\hat{B}_{B_q}$ being the decay constant and the so called “bag”-parameter, respectively. The value of the parameter $\xi$ is expected to be determined from the QCD simulations on the lattice with a good accuracy since many uncertainties should cancel in the ratio.

Recently, however, it has been argued that the chiral logarithms might be a source of a large, yet unaccounted for, uncertainty in $\xi$ \[6\]. The reason for this lies in the fact that the masses of light quarks that are directly accessible from the lattice studies are those around the strange quark, $m_s / 2 \lesssim m_q \lesssim 3m_s / 2$. In this range $f_{B_q}$ and $\hat{B}_{B_q}$, computed on the lattice, exhibit a linear behaviour under the variation of $m_q$. This linear dependence is then extrapolated in $m_q \rightarrow m_d$, to reach $f_{B_d}$ and $\hat{B}_{B_d}$. The claim of ref. \[6\] is that the inclusion of the chiral logarithms in the mentioned extrapolation produces a shift from the “standard” value $f_{B_s} / f_{B_d} = 1.16(5)$, to $1.32(8)$. In ref. \[7\] the systematic error due to the shift is estimated to be even larger, +0.24. In the following we revise the problem and argue that the shift is indeed present but it is actually smaller than previously claimed. We will show that $f_{B_s} / f_{B_d} \approx f_K / f_\pi$.

Finally, from refs. \[8, 9\] it is known that, w.r.t. the result of the linear extrapolation in the light quark mass, the chiral logs produce a tiny (insignificant) shift in the ratio of “bag”-parameters, $B_{B_s} / B_{B_d}$. This latter ratio is known to be very consistent with unity \[9\], and therefore is not the source of any additional uncertainty in the determination of $\xi$.

## 2 Ratios $f_{B_s} / f_{B_d}$ and $f_K / f_\pi$ from the ChPT

The SU(3) light flavour breaking effects in the ratios of the decay constants have been calculated at the 1-loop level of the chiral perturbation theory (ChPT), resulting in the following expressions \[10, 11\]

$$R_{f_B}^{\text{ChPT}} = \frac{\Phi_{B_s}}{\Phi_{B_d}} = 1 + \frac{1 + 3g^2}{4(4\pi f)^2} \left[3I_1(m_\pi^2) - 2I_1(m_K^2) - I_1(m_\eta^2)\right] + \frac{8K}{f^2} (m_K^2 - m_\pi^2),$$

$$R_{f_\pi}^{\text{ChPT}} = \frac{f_K}{f_\pi} = 1 + \frac{1}{4(4\pi f)^2} \left[5I_1(m_\pi^2) - 2I_1(m_K^2) - 3I_1(m_\eta^2)\right] + \frac{8L_5}{f^2} (m_K^2 - m_\pi^2),$$

with the convention $f = 130$ MeV, and the standard notation $\Phi_{B_q} = f_{B_q} \sqrt{m_{B_q}}$. The counter-term $L_5$ is defined in \[10\], while $K$ stands for a sum of counter-terms discussed in
Note that these counter-terms provide the corrections to \( R_{I_e,f_B}^{\text{ChPT}} \) linear in quark mass and that they eliminate the renormalization scale dependence arising from the chiral loop integral

\[
I_1(m^2) = m^2 \log(m^2/\mu^2) .
\] (5)

The renormalization scheme of ref. [10] has been used. The only new parameter accompanying the chiral logarithm in eq. (3) is \( g \), the coupling of the lowest lying spin doublet of heavy mesons to a soft pion. Its experimental value has been recently established in the charm sector, namely \( g_c = 0.59(8) \) [12]. While a recent (quenched) lattice study suggests that the \( g \)-coupling does not change when increasing the heavy quark mass \( g_b \approx g_c \) [13], the light cone QCD sum rule calculation predicts the suppression by about 25\% (i.e. \( g_b/g_c \approx 0.75 \)) [14]. We will take the average of the two predictions and add the difference in the systematic uncertainty, i.e. \( g \equiv g_b = 0.52(7)(7) \). It is worth mentioning that the use of the tree level values for \( g \) and \( f \) is consistent with the use of the ratios (3) and (4), derived at next-to-leading order in the chiral expansion.

Eqs. (3) and (4) have been obtained in refs. [11] and [10], respectively. Notice that in the former case we neglect the \( O(1/m_b) \)-corrections, which are anyway expected to cancel in the ratio (3) to a large extent [11]. Throughout this letter we assume the isospin symmetry \( m_d = m_u \equiv m_q \), and use the Gell-Mann–Oakes–Renner and Gell-Mann–Okubo formulas to write

\[
\begin{align*}
  m_{\pi}^2 &= 2B_0m_s r, \\
  m_K^2 &= 2B_0m_s \frac{r+1}{2}, \\
  m_\eta^2 &= 2B_0m_s \frac{r+2}{3},
\end{align*}
\] (6)

where \( r = m_q/m_s \), and \( B_0 \) is the common factor related to the quark condensate, \( B_0 = -2\langle \bar{q}q \rangle / f^2 \). When varying the ratio \( r \), we will always keep the strange quark mass fixed to its physical value. We can then use the physical kaon and pion masses to get \( 2B_0m_s = 2(m_{\pi}^{\text{phys}})^2 - (m_\eta^{\text{phys}})^2 = 0.468 \text{ GeV}^2 \).

It is important to note that even though the chiral logarithms in eqs. (3) and (4) enter with different coefficients, they are numerically very similar. To illustrate this point we set \( K = L_5 = 0 \) and plot in fig. 1 only the chiral log contributions to \( R_{f_B} \) and \( R_{f_\pi} \). Indeed, we see that for the central value of the parameter \( g \), the chiral log contributions to \( R_{f_B} \) and \( R_{f_\pi} \) nearly coincide, whereas the relative difference at the physical point \( r \approx 0.04 \) is below 5\% level when \( g \) is varied in the range \( g \in (0.35, 0.7) \). Note that the near coincidence of chiral logarithmic terms in \( R_{f_B} \) and \( R_{f_\pi} \) is not spoiled by the change of the renormalization scale \( \mu \).

From fig. 1 we also observe the well known fact that the chiral logs are not large enough to reproduce the experimental value \( (f_K/f_\pi)_{\text{exp.}} = 1.22(1) \) [15]. The mismatch is patched by fixing the coupling \( L_5(\mu = 1 \text{ GeV}) \) to \( (8.7 \pm 2.5) \times 10^{-4} \) from eq. (4). The remaining entry in eq. (3), the value of \( K \), will be discussed later on, at the end of section 4.

3 Chiral extrapolation of \( f_{B_s}/f_{B_q} \)

Contrary to the ChPT, which is valid for very small quark masses, the lattice QCD provides predictions for not so light quarks corresponding to \( 1/2 \lesssim r \lesssim 3/2 \). In other words, \( f_{B_s} \)
Figure 1: The chiral logarithmic contributions to $R_{f_{\pi}}$ and $R_{f_B}$ for three different values of $g$ spanning the range $g = 0.52(7)(7)$, discussed in the text. Plotted are eqs. (3) and (4) with $K = L_5 = 0$ and $\mu = 1$ GeV. Variation of $r = m_q/m_s$ is made by keeping $m_s$ fixed to its physical value.

$(r = r_s = 1)$ is accessed directly from the lattice studies, while for $f_{B_d}$ an extrapolation in $r \rightarrow r_d$ is needed ($r_d \simeq 0.04$). That extrapolation is typically made linearly as

$$R_{f_{B}}^{\text{latt}} = 1 + \alpha (1 - r) ,$$

with $\alpha = 0.16(5)$ [4, 17]. The form (7) does not include the chiral logarithmic terms shown in eq. (3), which would increase the result of extrapolation, namely $f_{B_s}/f_{B_d}$. It is, however, not clear at what value of $r$ the leading chiral logs need to be included, i.e. we do not know how light the “pions” should be so that the leading chiral logs remain uncompensated by the higher order terms in the chiral expansion.

In the approach adopted in ref. [6] the NLO prediction of ChPT (3) is assumed to be valid up to $r = 1$, while the counter-term $K$ is fixed from the requirement

$$R_{f_{B}}^{\text{ChPT}}|_{r=r_M} = R_{f_{B}}^{\text{latt}}|_{r=r_M} ,$$

at some intermediate point $r_M$. We slightly modify that approach by actually separating 2 regions: (i) above $r_M$, where the extrapolation of the linearly fitted lattice data (7) is a good approximation; (ii) below $r_M$, where we use the ChPT formula (3) with $K$ fixed from the condition (8), and go down to the physical point $r_d$. The effect is shown in fig. 2 where we plot two curves, corresponding to $r_M = 0.25$, and $0.75$. From fig. 2 we see that farther

\footnote{For an easier orientation we convert $r_M$ by using $m_M^2 = 2B_0m_s r_M (2B_0m_s = 0.468 \text{ GeV}^2)$ to obtain $m_M \approx 330$ and $580$ MeV, respectively.}
Chiral extrapolation of the ratio $R_{f_B} = f_{B_s}/f_{B_d}$, in which the chiral logarithms are included from $r_M = 0.25$ and $0.75$ with $K$ fixed from (3) as in ref. [6]. Note that the variation of $r = m_d/m_s$ is made by keeping $m_s$ fixed to its physical value. The illustration is provided for $g = 0.52$. We also show $R_{f_B} = f_{B_s}/f_{B_d}$ with $K$ fixed as $K = L_5$ (dotted line).

Away from $r_d$ lies the point $r_M$, the larger is the shift in $R_{f_B}$. By using $g = 0.52$, we obtain

$$R_{f_B} = 1.22(6)_{r_M=0.25}, \quad 1.30(7)_{r_M=0.75},$$

(9)

to be compared with the result of the linear extrapolation $R_{f_B} = 1.16(5)$. To estimate this shift, the authors of ref. [6] take $r_M = 0.75 \pm 0.25$. Although the relevance of the leading chiral logs for the matching scale at $m_M \simeq (580 \pm 100)$ MeV may be doubted, at this point there is no legitimate argument to exclude the possibility that the central value for $R_{f_B}$ gets as high as $1.3$.5

4 Double ratio $(f_{B_s}/f_{B_d})/(f_K/f_\pi)$

To get around the above ambiguity we construct the double ratio (10)

$$R = \frac{\Phi_{B_s}/\Phi_{B_d}}{f_K/f_\pi} = 1 - \frac{1}{4(4\pi f)^2} \left[ (2 - 9g^2)I_1(m_\pi^2) + 6g^2I_1(m_K^2) - (2 - 3g^2)I_1(m_\eta^2) \right]$$

$$+ \frac{8(K - L_5)}{f_2} (m_K^2 - m_\pi^2).$$

Had we used the approach similar to the one in ref. [18] with both the size and the derivative of $R_{f_B}$ matched to the lattice data at $r_M$, the corresponding shifts would be smaller than the ones presented in (9). Note that this procedure amounts to introducing an additional constant term $K'$ in (3) (arising from NNLO contributions) with both $K$ and $K'$ then fixed from lattice.
As shown at the end of section 2 the chiral log contributions in \( R = R_{f_B}/R_{f_\pi} \) almost completely cancel so that the double ratio \( R \), to a very good approximation, depends on \( r \) only linearly. One can then envisage a future lattice analysis in which the chiral extrapolation of \( R \) is made linearly and then the result for \( f_{B_s}/f_{B_d} \) is deduced by using the experimental value \((f_K/f_\pi)_{\text{exp.}} = 1.22(1)\).

In this letter, we will use the formula (10) to estimate the value of \( f_{B_s}/f_{B_d} \). To do so we have to fix the couplings \( L_5 \) and \( K \). We already mentioned that \( L_5 = (8.7 \pm 2.5) \times 10^{-4} \) (at \( \mu = 1 \) GeV). However, the coupling \( K \) is unknown. To get its value, but only relative to \( L_5 \), we may rely on the experience with the QCD simulations on the lattice in which the heavy-light (\( B_q \)) and light-light (\( P_{qq} \)) pseudoscalar meson decay constants are fitted well to the linear forms

\[
   f_{B_q} = a_0 + a_1 m_{P_{qq}}^2, \quad \text{and} \quad f_{P_{qq'}} = b_0 + b_1 m_{P_{qq'}}^2.
\]

By using eq. (6), these forms can be rewritten as

\[
   R_{f_B} \simeq 1 + \frac{a_1}{a_0} 2B_0 m_s (1 - r), \quad \text{and} \quad R_{f_\pi} \simeq 1 + \frac{b_1}{2b_0} 2B_0 m_s (1 - r).
\]

The values of parameters \( \alpha \) and \( \beta \) can be obtained from the linearly fitted lattice data for \( R_{f_B} \) and \( R_{f_\pi} \), which at \( r = 0 \) simply read, \( R_{f_B} = 1 + \alpha \) and \( R_{f_\pi} = 1 + \beta \). From the recent compilation of unquenched calculations (with \( n_f = 2 \)) of the heavy-light decay constants, which are obtained through the linear extrapolation, we take \( f_{B_s}/f_{B_d} = 1.16(5) \) \cite{7,17}. The MILC collaboration has recently verified that the ratio \( f_{B_s}/f_{B_d} \) remains stable when the number of dynamical flavors \( n_f \) and/or the values of their masses are changed \cite{19}. We therefore take \( \alpha = 0.16(5) \). From an extensive study of the light hadrons with two flavors of dynamical quarks by the JLQCD and the CP-PACS collaborations \cite{21}, one has \( f_K/f_\pi = 1.16(2), \) i.e. \( \beta = 0.16(2) \). \(^3\) In other words, \( \alpha/\beta \approx 1 \). That conclusion is also in agreement with the observation made by the MILC collaboration that the slope \( b_1 \) is about 30\% larger than the slope \( a_1 \) \cite{22}. \(^4\) Moreover, by fitting, linearly in \( r \), the unquenched data for both the light-light and the heavy-light decay constants obtained by the JLQCD collaboration and presented in ref. \cite{23}, one arrives to the same conclusion, namely \( \alpha/\beta \approx 1 \).

This information can be used to estimate the value of \( K - L_5 \) by matching at some intermediate point \( r_M \) the chiral expression for \( R \) given in eq. (10) to the linear fits of lattice data given in eq. (12)

\[
   \left. R \right|_{r = r_M} = \left( \frac{R_{f_B}^{\text{latt}}}{R_{f_\pi}^{\text{latt}}} \right)_{r = r_M}.
\]

In further discussion we will take \( r_M \) to be in the range \( r_M \in (0.5, 1) \), where lattice simulations are performed. From requirement (13) it follows trivially

\[
   8(m_K^{\text{phys}})^2 f^2 (K - L_5) = (\alpha - \beta) - \frac{1}{1 - r_M} \delta R^{\text{loop}}\bigg|_{r = r_M},
\]

\(^3\)This result corresponds to the average of the results obtained by using \( K^- \) and \( \phi^- \) inputs in the recent paper by the JLQCD Collaboration in which the results of the improved unquenched study were reported.

\(^4\)We thank Claude Bernard for this information.
where $\delta R^{\text{loop}}$ is the chiral log correction to the $R$ ratio (i.e. the first correction term in eq. (10)), while $\alpha$ and $\beta$ are the slopes of the fits to the lattice data points as defined in eq. (12). Finally, the expression for the double ratio (10) is

$$R = 1 + \delta R^{\text{loop}} + \left( (\alpha - \beta) - \frac{1}{1 - r_M} \delta R^{\text{loop}} \big|_{r=r_M} \right) (1 - r)$$

As we already mentioned, the available lattice data suggest $\alpha \approx \beta$. Furthermore, the size of chiral corrections to the matching condition, $\delta R^{\text{loop}} \big|_{r=r_M}/(1 - r_M)$, is below 1% for $r_M \in (0.5, 1)$ and $g \in (0.35, 0.7)$. As such it is much smaller than the present errors on the value of $\alpha - \beta$ and can be safely neglected in the following.

As a first estimate we can thus set the last term in eq. (15) to zero. At the physical point $r = r_d = 0.04$, and by using $g = 0.52(7)(7)$, we get

$$R = 1.011 \pm 0.024 .$$

where the errors reflect the variation of the $g$-parameter only. From this result, we conclude that $f_{B_s}/f_{B_d} \approx f_K/f_{\pi}$.

5 Our estimate of $f_{B_s}/f_{B_d}$ and $\xi$

Finally, to make the numerical estimate we take

$$f = 0.13 \text{ GeV}, \quad g = 0.52(7)(7), \quad K/L_5 = 1.0 \pm 0.6, \quad L_5 = (8.7 \pm 2.5) \times 10^{-4},$$

where the first error in $g$ is considered to be gaussian (experimental) and all the others to be flat. The error in $\langle f_K/f_{\pi}\rangle_{\exp} = 1.22(1)$ is also gaussian. Notice in particular that the value of $K/L_5$ ratio follows from the present value of $\alpha - \beta = 0.00 \pm 0.06$, discussed in the previous section, through the use of eq. (14). That uncertainty may easily be reduced by the direct lattice calculation of the ratio $R$.

With these input values and by using eq. (10), we produced the histogram of $10^6$ Monte Carlo events for $f_{B_s}/f_{B_d} = R \sqrt{m_{B_d}/m_{B_s}} \times \langle f_K/f_{\pi}\rangle_{\exp}$, which is shown in fig. 8. We finally obtain

$$f_{B_s}/f_{B_d} = 1.22 \pm 0.08 ,$$

which is larger than the commonly quoted values, yet it is considerably smaller than the results obtained in refs. [6, 7].

To make the discussion complete and get an estimate of the parameter $\xi$, we will also use the result of ref. [8]:

$$\frac{\hat{B}_{B_s}}{B_{B_d}} = 1 + \frac{1 - 3g^2}{2(4\pi f)^2} \left[ I_1(m_{\pi}^2) - I_1(m_{\eta}^2) \right] + \frac{16\tilde{K}}{f^2} (m_{K}^2 - m_{\pi}^2) ,$$

where $\tilde{K}$ denotes the sum of corresponding counter-terms. Combining eq. (19) with eq. (10) gives

$$\mathcal{R}_\xi = \frac{\xi \sqrt{m_{B_s}/m_{B_d}}}{f_K/f_{\pi}} = 1 - \frac{1}{4(4\pi f)^2} \left[ (1 - 6g^2)I_1(m_{\pi}^2) + 6g^2I_1(m_{K}^2) - I_1(m_{\eta}^2) \right] + \frac{8(K + \tilde{K} - L_5)}{f^2} (m_{K}^2 - m_{\pi}^2) .$$
As mentioned at the beginning of this letter, the results of several lattice studies suggest that $\tilde{K}$ can be safely set to zero \cite{9}. With the same input values, given in eq. (17), we compute $R_\xi \sqrt{m_{B_d}/m_{B_s}} \times (f_{K}/f_{\pi})_{\exp}$ from eq. (20), and obtain

$$\xi = 1.22 \pm 0.08.$$  

(21)

It is important to stress that our results for $f_{B_s}/f_{B_d}$ and $\xi$ are obtained by neglecting the $\mathcal{O}(1/m_{b>0}^3)$ corrections that survive the cancellation in the ratio \cite{13}. One may worry about the possibility that those corrections may further increase our results. The experience with lattice QCD and with the QCD sum rule (QSR) calculations suggests that the SU(3) breaking ratio of the decay constants is not decreasing when increasing the heavy meson mass. Indeed, from ref. \cite{17} we read the lattice QCD estimates, $f_{B_s}/f_B = 1.16(5), f_{D_s}/f_D = 1.12(4)$, while from the last paper of ref. \cite{9} we see that the QSR calculation gives, $f_{B_s}/f_B = 1.16(5), f_{D_s}/f_D = 1.15(4)$. Therefore the uncancelled $\mathcal{O}(1/m_{b}^3)$ corrections are likely to shift our result \cite{18} towards the smaller values rather than the larger ones.

Notice that a conclusion similar to ours' has been also reached in the recent reanalysis of the lattice data produced by the MILC collaboration \cite{22}. The authors show that the consistent inclusion of the chiral logarithms in all the chiral extrapolations involved in their data analysis, produces a shift of their central result, $f_{B_s}/f_{B_d} = 1.16(1)(2)(2)$, by only +0.04, which is much smaller than +0.24, as claimed in ref. \cite{7}, or +0.16, as argued in ref. \cite{6}. Moreover, by actually shifting their result, one gets $f_{B_s}/f_{B_d} = 1.20(1)(2)(2)$, which is completely consistent with our main claim, namely $f_{B_s}/f_{B_d} \approx f_K/f_{\pi}$.

Finally we mention, that instead of the double ratio $(f_{B_s}/f_{B_d})/(f_{K}/f_{\pi})$ discussed in the present letter, the double ratio of the heavy meson decay constants $R_1 = (f_{B_s}/f_{B_d})/(f_{D_s}/f_{D_d})$ could be used for the determination of $(f_{B_s}/f_{B_d})$ and correspondingly of $\xi$ as suggested by
Grinstein in [24]. The ratio of charm meson decay constants \((f_{D_s}/f_{D_d})\) is expected to be measured by CLEO-c, while preliminary \(N_f = 2\) dynamical lattice calculations of \(R_1\) are already available [7].

6 Conclusions

In this letter we have discussed the effect of the inclusion of the chiral logarithms on the lattice determination of the ratio \(f_{B_s}/f_{B_d}\). We have shown that in the double ratio

\[
R = \frac{f_{B_s}/f_{B_d}}{f_K/f_\pi},
\]

the chiral logarithms almost completely cancel because of the fact that the numerical value of the coupling \(g\) is close to \(~0.5\). To a very good approximation the double ratio \(R\) then depends linearly on the light quark mass all the way down to the chiral limit.

We have also made a numerical estimate of the hadronic parameter \(\xi\), by using the expressions derived in the ChPT and by fixing the unknown couplings from the available lattice results. Namely, we obtain

\[
\xi = 1.22 \pm 0.08.
\]

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