Dual canonical bases, quantum shuffles and $q$-characters

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Abstract
Rosso and Green have shown how to embed the positive part $U_q(n)$ of a quantum enveloping algebra $U_q(g)$ in a quantum shuffle algebra. In this paper we study some properties of the image of the dual canonical basis $B^*$ of $U_q(n)$ under this embedding $\Phi$. This is motivated by the fact that when $g$ is of type $A_r$, the elements of $\Phi(B^*)$ are $q$-analogues of irreducible characters of the affine Iwahori-Hecke algebras attached to the groups $GL(m)$ over a $p$-adic field.

1 Introduction
In [LLT1, LLT2, LT1] some close relationships were observed between the representation theory of type $A$ Hecke algebras and quantized Schur algebras on one side, and the canonical bases of certain quantum groups on the other side. Since then these connections have been studied by several authors and other similar correspondences have been discovered [A, LT2, VV, Gr, BK, LNT, B1, B2].

Roughly speaking the principle is the following: the basis of simple modules in the Grothendieck group of some appropriate category of representations of a certain algebra $A$ (e.g. a Hecke algebra of type $A$ or $B$, an Ariki-Koike algebra, a $q$-Schur algebra, ...) can be identified with the specialization at $q = 1$ of the dual canonical basis of (a representation of) a certain quantum enveloping algebra $U$.

Let $g$ be a complex simple Lie algebra and $n$ its maximal nilpotent subalgebra. In this paper we show that even when there is no apparent connection with the Grothendieck group of some category, the dual canonical basis $B^*$ of $U_q(n)$ exhibits some features resembling certain classical properties of the irreducible characters in Lie theory.

In order to observe this, one has to study $B^*$ in the particular realization of $U_q(n)$ discovered by Rosso [R1, R2] and Green [G] in terms of quantum shuffles. When $g$ is of type $A_r$, the embedding $\Phi$ of $U_q(n)$ in the quantum shuffle algebra $F$ is precisely a $q$-analogue of the map $[M] \mapsto \text{ch } M$ from the Zelevinsky ring of a category of representations of the affine Hecke algebras of type $GL(m)$ to the corresponding character ring. Indeed, as shown in [Gr, GV], the multiplication of characters coming from parabolic induction of Hecke modules is given by the (classical) shuffle product. Moreover, the above principle states in this case that the basis $B^*$ is a $q$-analogue of the basis of the Zelevinsky ring consisting of the classes of simple modules, hence $\Phi(B^*)$ may be regarded as a $q$-analogue of the set of irreducible characters (see section 6 below for more details).

This motivates our investigation of $\Phi(B^*)$ for general $g$. We shall be concerned with three main properties. First, it is easy to describe explicitly the image $\Phi(U_q(n))$ of the embedding of $U_q(n)$ in the quantum shuffle algebra (Theorem 5). We think of this result as an analogue of the classical fact that the characters of the (virtual) integrable $g$-modules are the polynomials invariant under the action of the Weyl group. Secondly, we show that the elements of $\Phi(B^*)$ are parametrized by their maximal word for the lexicographic order (Theorem 40). This is similar to the parametrization of irreducible integrable $g$-modules by their highest weight. Thirdly, it follows
from Lusztig’s geometric construction of the canonical bases that for \( g \) of simply laced type, the coefficients of the elements of \( \Phi(\mathcal{B}^*) \) belong to \( \mathbb{N}[q, q^{-1}] \) (Theorem 42). We conjecture that this positivity property is also true in the non-simply laced case. This is analogous to the fact that the character of a \( g \)-module is a positive sum of weights.

As an application, we describe in section 5.5 an algorithm for calculating the basis \( \mathcal{B}^* \), which allowed us to discover examples of imaginary vectors of \( \mathcal{B}^* \) for \( g \) of type \( A_5, B_3, C_3, D_4, G_2 \), thus disproving a conjecture of Berenstein and Zelevinsky for all types except \( A_n \) (\( n \leq 4 \)) and \( B_2 \) [13].

There are some formal similarities between our results and the theory of \( g \)-characters for finite-dimensional representations of quantum affine algebras developed by Frenkel-Reshetikhin [PR], Frenkel-Mukhin [FM] and Nakajima [N]. Actually, in type \( A_r \), our \( g \)-characters for affine Hecke algebras are interpreted geometrically in terms of the same graded quiver varieties as those used by Nakajima for defining the \( (q, t) \)-characters of \( U_q(\mathfrak{sl}_n) \) [11], so both families of characters contain essentially the same information. For other types though, there is no clear relationship between \( \Phi(\mathcal{B}^*) \) and the \( g \)-characters of \( U_q(\hat{\mathfrak{g}}) \).

The paper is structured as follows. In section 2 we review, following Rosso and Green, the construction of the quantum shuffle embedding of \( U_q(\mathfrak{n}) \). Our presentation is based on the \( q \)-derivations \( e_i' \) of Kashiwara, which in the type \( A \) case have the natural interpretation of \( i \)-restriction operators in terms of affine Hecke algebras. Then we prove Theorem 3. We also describe explicitly the embedding of the algebra of regular functions \( \mathbb{C}[\mathcal{N}] \) in the (classical) shuffle algebra obtained by specializing at \( q = 1 \) the embedding \( \Phi \) (here \( \mathcal{N} \) stands for a maximal unipotent subgroup of a complex simple Lie group \( G \) with Lie algebra \( \mathfrak{g} \)). Sections 3 and 4 are devoted to certain monomial bases and PBW-type bases, respectively, which play an essential role in the proofs of our results on \( \mathcal{B}^* \). These two sections are based on some beautiful theorems of Lalonde and Ram [LR] and Rosso [R3]. In particular, Lalonde and Ram have defined for any root system a set of Lyndon words in one-to-one correspondence with the positive roots. These so-called ‘good Lyndon words’ and their nonincreasing products label in a natural way certain monomial and Lyndon bases. For the convenience of the reader, we have included proofs of most of the statements of [LR] [R3] needed for our purposes. The main new result (Theorem 36) describes the maximal words of the images under \( \Phi \) of the elements of certain Lusztig’s PBW-type bases. It is obtained by relating Lusztig’s PBW-bases to Rosso’s Lyndon bases. In section 5 we derive the above-mentioned properties of \( \Phi(\mathcal{B}^*) \) and we present an algorithm to compute it. Section 6 discusses the case of \( g \) of type \( A_r \) and its relationship with the representation theory of affine Hecke algebras, while section 7 presents a conjectural analogue of this relationship for type \( B_r \) and the affine Hecke-Clifford superalgebras of Jones and Nazarov [JN] whose representation theory was recently studied by Brundan and Kleshchev [BK]. Finally, section 8 describes a family of root vectors of \( \mathcal{B}^* \) for classical and simply-laced types. More precisely, for the classical types \( A_r, B_r, C_r, D_r \) we give a closed \( q \)-shuffle formula and for the simply-laced types \( A_r, D_r, E_r \) a simple combinatorial description. (For type \( G_2 \), the root vectors are calculated in [S]). This last section may serve to illustrate many statements of the paper.

2 Embedding of \( U_q(\mathfrak{n}) \) in a quantum shuffle algebra

2.1 Let \( g \) be a simple Lie algebra of rank \( r \) over \( \mathbb{C} \) and let \( U_q(\mathfrak{g}) \) be the corresponding quantized enveloping algebra over \( \mathbb{Q}(q) \) with Chevalley generators \( e_i, f_i \ (i = 1, \ldots, r) \). The Cartan matrix of \( \mathfrak{g} \) is denoted by \( [a_{ij}]_{i,j=1,\ldots,r} \). Let \( \Delta \) be the root system of \( \mathfrak{g} \), \( \Delta^+ \) the subset of positive roots, \( Q \) the root lattice, \( \Pi = \{\alpha_1, \ldots, \alpha_r\} \) the set of simple roots, \( Q^+ = \oplus_{i=1}^r \mathbb{N} \alpha_i \) the monoid generated
by the simple roots, and $(\cdot, \cdot)$ a symmetric bilinear form on $Q$ such that

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = \frac{(\alpha_i, \alpha_j)}{d_i}, \quad (1 \leq i, j \leq r)$$

where $d_i = (\alpha_i, \alpha_i)/2 \in \{1, 2, 3\}$.

Let $U_q(n)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements $e_i (i = 1, \ldots, r)$. The defining relations of $U_q(n)$ are the so-called $q$-Serre relations:

$$\sum_{k+l=1-a_{ij}} (-1)^k \left[ \frac{1-a_{ij}}{k} \right] e_i^k e_j = 0, \quad (1 \leq i \neq j \leq r). \quad (1)$$

Here we use the standard notation for $q$-integers and $q$-binomial coefficients, namely,

$$q_i = q^{d_i}, \quad [k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}, \quad \left[ \begin{array}{c} m \\ k \end{array} \right]_i = \frac{[m]_i[m-1]_i \cdots [m-k+1]_i}{[k]_i[k-1]_i \cdots [1]_i}.$$ 

The algebra $U_q(n)$ is $Q^+$-graded by assigning to $e_i$ the degree $\alpha_i$. We shall denote by $|u|$ the $Q^+$-degree of a homogeneous element $u$ of $U_q(n)$.

2.2 Kashiwara [K1] has introduced some $q$-derivations $e_i'(i = 1, \ldots, r)$ of $U_q(n)$. These are the elements of $\text{End} U_q(n)$ characterized by

$$e_i'(e_j) = \delta_{ij}, \quad e_i'(uv) = e_i'(u)v + q^{-(\alpha_i, |u|)}ue_i'(v), \quad (2)$$

for all homogeneous elements $u, v$ of $U_q(n)$. It is known [K1] that

$$e_i'(u) = 0, \quad (i = 1, \ldots, r) \iff |u| = 0. \quad (3)$$

It is also known [K1] that these endomorphisms satisfy the $q$-Serre relations, that is,

$$\sum_{k+l=1-a_{ij}} (-1)^k \left[ \frac{1-a_{ij}}{k} \right] (e_i')^k (e_j')^l = 0, \quad (1 \leq i \neq j \leq r). \quad (4)$$

Kashiwara [K1] proves that there is a unique nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $U_q(n)$ such that $(1, 1) = 1$ and

$$(e_i'(u), v) = (u, e_i v), \quad (u, v \in U_q(n), \ i = 1, \ldots, r), \quad (5)$$

that is, $e_i'$ is the endomorphism adjoint to left multiplication by $e_i$.

Note that Lusztig uses a slightly different scalar product $(\cdot, \cdot)_L$ satisfying

$$(e_i'(u), v)_L = \frac{1}{1-q^{2d_i}} (u, e_i v)_L, \quad (u, v \in U_q(n), \ i = 1, \ldots, r), \quad (6)$$

(see [L2], 1.2.3, 1.2.13). It is easy to see that if $u$ and $v$ are homogeneous elements of $U_q(n)$ we have $(u, v)_L = (u, v)_L = 0$ if $|u| \neq |v|$, and

$$(u, v)_L = \prod_{i=1}^r \frac{1}{1-q^{2d_i}c_i} (u, v)$$

if $|u| = |v| = \sum c_i \alpha_i$. It follows that if $\mathcal{B}$ is a basis of $U_q(n)$ consisting of homogeneous vectors, then the adjoint bases of $\mathcal{B}$ with respect to $(\cdot, \cdot)$ and $(\cdot, \cdot)_L$ differ only by some normalization factors. In particular, $\mathcal{B}$ is orthogonal with respect to $(\cdot, \cdot)$ if and only if it is orthogonal with respect to $(\cdot, \cdot)_L$. In this paper we shall use Kashiwara’s form $(\cdot, \cdot)$.
2.3 Let $\mathcal{M}$ (resp. $\mathcal{F}$) be the free monoid (resp. the free associative algebra over $\mathbb{Q}(q)$) generated by the set of letters $I = \{w_1, \ldots, w_r\}$. We will use the notation $w[i_1, \ldots, i_k] := w_{i_1} \cdots w_{i_k}$. The empty word is written $w[]$. The length of a word $w \in \mathcal{M}$ is denoted by $\ell(w)$. The algebra $\mathcal{F}$ is $\mathbb{Q}^+$-graded by assigning to $w_i$ the degree $\alpha_i$. The degree of a homogeneous element $f \in \mathcal{F}$ is denoted by $|f|$. 

In \cite{L2}, Lusztig has endowed $\mathcal{F}$ with a twisted bialgebra structure defined in terms of the bilinear form on $Q$. He has shown that there exists a unique symmetric bilinear form on $\mathcal{F}$ which adjoining the multiplication and the twisted comultiplication. Moreover the radical of this form coincides with the kernel of the homomorphism $\mathcal{F} \to U_q(n)$ mapping $w_i$ to $e_i$, and the form it induces on $U_q(n)$ is nothing else than the form (6) above. Similarly $U_q(n)$ is endowed with a twisted bialgebra structure whose comultiplication is adjoint to the multiplication with respect to (6). Hence by taking graded duals, we obtain a natural embedding of vector spaces 

\[ U_q(n) \cong U_q(n)^* \to \mathcal{F}^* \cong \mathcal{F} \]

in which the multiplication of $U_q(n)$ is sent to the multiplication of $\mathcal{F}^*$ coming from Lusztig’s comultiplication on $\mathcal{F}$, and is a $q$-analogue of the shuffle product as explained very clearly by Green [Gr].

Here we are going to indicate briefly how to recover this result by means of the $q$-derivations $e'_i$. An advantage of this approach is that it shows immediately how this embedding specializes at $q = 1$ to an embedding of $\mathbb{C}[N]$ in the shuffle algebra, given explicitly in terms of differential operators (see \cite{L2}).

2.4 To $w = w[i_1, \ldots, i_k]$ we associate $\partial_w := e'_{i_1} \cdots e'_{i_k} \in \text{End} U_q(n)$. (For $w = w[]$ we set $\partial_w = \text{Id}_{U_q(n)}$.) If $u$ is a homogeneous element of $U_q(n)$ and $|w| = |u|$ then $\partial_w(u)$ is of degree 0, that is, a scalar. We define a $\mathbb{Q}(q)$-linear map $\Phi : U_q(n) \to \mathcal{F}$ by setting

\[ \Phi(u) = \sum_{w \in \mathcal{M}, |w| = |u|} \partial_w(u)w \]  

for a homogeneous element $u \in U_q(n)$. It follows easily from (3) that the map $\Phi$ is injective.

It may be helpful to think of (7) as a formal Taylor expansion of $u$ (see below Proposition 2).

2.5 Define inductively a bilinear map $\ast$ from $\mathcal{F}$ to $\mathcal{F}$ by setting, for $a, b \in I$ and $w, x \in \mathcal{M}$

\[ wa \ast xb = (w \ast x)b + q^{-|wa||b|} (wa \ast x)b, \quad w[] \ast x = x \ast w[] = x. \]  

Iterating (8) we get

\[ w[i_1, \ldots, i_m] \ast w[i_{m+1}, \ldots, i_{m+n}] = \sum_{\sigma} q^{-e(\sigma)} w[i_{\sigma(1)}, \ldots, i_{\sigma(m+n)}] \]

where the sum runs over the $\sigma \in \mathfrak{S}_{m+n}$ such that $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$, and

\[ e(\sigma) = \sum_{k \leq m < l; \sigma(k) < \sigma(l)} (\alpha_{i_{\sigma(k)}}, \alpha_{i_{\sigma(l)}}). \]

Thus, for $q = 1$, $\ast$ is the classical shuffle product $\shuffle$ in $\mathcal{F}$ [Reu], and in particular it is associative and commutative. The next proposition follows easily from the definitions and its proof will be omitted.

4
Proposition 1 The product $*$ is associative, and for $w, x \in \mathcal{M}$
\[ w * x = q^{-(|w|,|x|)} x \overline{w} \]
where $\overline{w}$ is the map obtained by replacing $q$ by $q^{-1}$ in the definition of $*$. \hfill \Box

The following Lemma is a simple rank 2 computation.

Lemma 2 For $i \neq j$,
\[
\sum_{k+l=1-a_{ij}} (-1)^k \left[ \frac{1-a_{ij}}{k} \right] w_i^{a_{ij}} w_j = 0,
\]
where we have put $w_i^{a_{ij}} = w_i \cdots w_i$ ($k$ factors). \hfill \Box

We introduce $e_i' \in \text{End} \mathcal{F}$ ($i = 1, \ldots, r$) by setting
\[
e_i'(w[i_1, \ldots, i_k]) = \delta_{i,i_k} w[i_1, \ldots, i_{k-1}], \quad e_i'(w[\|]) = 0. \quad (11)
\]

Lemma 3 The endomorphisms $e_i'$ satisfy
\[
e_i'(w_j) = \delta_{ij}, \quad e_i'(x * z) = e_i'(x) * z + q^{-\langle \alpha_i, |x| \rangle} x * e_i'(z),
\]
for all homogeneous elements $x, z$ of $\mathcal{F}$.

Proof — Follows immediately from (11). \hfill \Box

Theorem 4 (R1, R2, G) For $u, v \in U_q(n)$ we have $\Phi(uv) = \Phi(u) \Phi(v)$.

Proof — By Lemma 2 there exists a linear map $\Psi : U_q(n) \rightarrow \mathcal{F}$ such that
\[
\Psi(e_i) = w_i \quad (i = 1, \ldots, r), \quad \Psi(uv) = \Psi(u) \star \Psi(v) \quad (u, v \in U_q(n)).
\]
By Lemma 3, this map satisfies: $\Psi e_i' = e_i' \Psi$, ($i = 1, \ldots, r$). Let $u \in U_q(n)$ be homogeneous and let $w = w[i_1, \ldots, i_k] \in \mathcal{M}$ be such that $|w| = |u|$. Let $\gamma_w(u)$ be the coefficient of $w$ in $\Psi(u)$. Then $\gamma_w(u) = e_i' e_i' \cdots e_i' \Psi(u) = \Psi e_i' e_i' \cdots e_i' (u) = \delta_{i,i} (u)$. Hence $\Psi(u) = \Phi(u)$, which proves the theorem. \hfill \Box

2.6 By Theorem 4, the algebra $U_q(n)$ is isomorphic to the subalgebra $\mathcal{U}$ of $(\mathcal{F}, *)$ generated by the letters $w_i \in I$. The next theorem gives a more explicit description of $\mathcal{U}$.

Let $i \neq j$ and $0 \leq k \leq 1 - a_{ij}$. For $z, t \in \mathcal{M}$, we set $w(i, j, k; z, t) = zw_i^k w_j^{1-a_{ij}-k} t$.

Theorem 5 The element $f = \sum_{w \in \mathcal{M}} \gamma(w) w$ of $\mathcal{F}$ belongs to $\mathcal{U}$ if and only if
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1-a_{ij}}{k} \right] \gamma(w(i, j, k; z, t)) = 0 \quad (12)
\]
for all $i \neq j$ and $z, t \in \mathcal{M}$. 

5
Proof — Let \( K \) be the subspace of \( \mathcal{F} \) defined by the system of linear equations (12). Let

\[
  f = \Phi(u) = \sum_{|w| = \nu} \gamma(w) w
\]

for some \( u \in U_q(n) \) of degree \( \nu \in Q^+ \). Then for \( w = w[i_1, \ldots, i_k] \) of degree \( \nu \) we have

\[
  \gamma(w) = e'_{i_1} \cdots e'_{i_k}(u) = (e_{i_1} \cdots e_{i_k} - u).
\]

Hence the fact that the elements \( e_i \) satisfy the \( q \)-Serre relations (1) implies that \( f \in K \). So \( U \subseteq K \).

Let \( \mathcal{F}_\nu \) (resp. \( K_\nu, U_q(n)_\nu, U_\nu \)) be the homogeneous component of degree \( \nu \) of \( \mathcal{F} \) (resp. \( K, U_q(n), U \)). Since (1) is a presentation of \( U_q(n) \), we see that \( \dim K_\nu = \dim U_q(n)_\nu \) for every \( \nu \in Q^+ \). Moreover, \( \Phi \) being injective, \( \dim U_q(n)_\nu = \dim U_\nu \), hence \( K = U \). \( \square \)

2.7 The next proposition shows that some important automorphisms of \( U_q(n) \) can be seen as restrictions of certain simple linear maps defined over \( \mathcal{F} \).

**Proposition 6**

(i) Let \( \tau \) be the \( \mathbb{Q}(q) \)-linear map from \( \mathcal{F} \) to \( \mathcal{F} \) such that

\[
  \tau(w[i_1, \ldots, i_k]) = w[i_1, \ldots, i_1].
\]

Then \( \tau(f \ast g) = \tau(g) \ast \tau(f) \) for all \( f, g \in \mathcal{F} \). Hence \( \tau \) restricts to the \( \mathbb{Q}(q) \)-linear anti-automorphism of \( U \) fixing the generators \( w_i \).

(ii) Let \( f \mapsto \overline{f} \) be the \( \mathbb{Q} \)-linear map from \( \mathcal{F} \) to \( \mathcal{F} \) such that

\[
  \overline{q} = q^{-1}, \quad \overline{w[i_1, \ldots, i_k]} = q^{-\sum_{1 \leq \alpha \leq k} (\alpha_i, \alpha_i)} w[i_1, \ldots, i_1].
\]

Then \( \overline{f \ast g} = \overline{f} \ast \overline{g} \) for all \( f, g \) in \( \mathcal{F} \). Hence \( f \mapsto \overline{f} \) restricts to the \( \mathbb{Q} \)-linear automorphism of \( U \) sending \( q \) to \( q^{-1} \) and fixing the generators \( w_i \).

(iii) Let \( \sigma \) be the \( \mathbb{Q} \)-linear map from \( \mathcal{F} \) to \( \mathcal{F} \) such that

\[
  \sigma(q) = q^{-1}, \quad \sigma(w[i_1, \ldots, i_k]) = q^{-\sum_{1 \leq \alpha \leq k} (\alpha_i, \alpha_i)} w[i_1, \ldots, i_1].
\]

Then \( \sigma(f) = \overline{\tau(f)} \). Hence, \( \sigma \) restricts to the \( \mathbb{Q} \)-linear anti-automorphism of \( U \) sending \( q \) to \( q^{-1} \) and fixing the generators \( w_i \).

**Proof —** It is enough to check (i) and (ii) when \( f \) and \( g \) are two words. Then (i) follows immediately from (2). To prove (ii) we may argue by induction on the length of the words. First note that

\[
  \overline{w[i_1, \ldots, i_k]} = q^{-(\alpha_{i_1}, \alpha_{i_2} + \cdots + \alpha_{i_k})} w[i_2, \ldots, i_k] w_{i_1}.
\]

Assume by induction that (ii) is proved for every pair of words whose sum of lengths is equal to \( n \), and let \( a, b \in I \) and \( w, x \in \mathcal{M} \) with \( \ell(aw) + \ell(bx) = n + 1 \). Using (7), we have

\[
  aw \ast bx = q^{-(|a|, |bx|)} a(aw \ast bx) + b(aw \ast x),
\]

hence

\[
  aw \ast bx = q^{-(|a|, |bx|)} a(aw \ast bx) + b(aw \ast x) = q^{-(|a|, |w|)} (aw \ast bx) + q^{-(|b|, |aw| + |x|)} (aw \ast x) b = q^{-(|a|, |w|) + (|b|, |aw| + |x|)} (aw \ast x) b + q^{-(|a|, |w|) - (|b|, |aw| + |x|)} (aw \ast x) b = q^{-(|a|, |w|) - (|b|, |x|)} (aw \ast x) b = \overline{aw} \ast \overline{bx}.
\]
Finally, (iii) follows from (i) and (ii).

For $\nu = \sum_{i=1}^r c_i \alpha_i \in Q^+$, define

$$N(\nu) = \frac{1}{2} \left( (\nu, \nu) - \sum_{i=1}^r c_i (\alpha_i, \alpha_i) \right).$$

(13)

Then, by Proposition 6, $\sigma(w) = q^{-N(|w|)} w$ for all $w$, from which the next lemma follows.

**Lemma 7** Let $f = \sum_{w \in \mathcal{A}} \gamma_w(q) w \in \mathcal{F}$ be homogeneous. Then

$$\sigma(f) = q^{-N(|f|)} f \iff \gamma_w(q^{-1}) = \gamma_w(q), \quad (w \in \mathcal{M}).$$

2.8 We close this section by discussing the specialization of the $q$-shuffle embedding $\Phi$ at $q = 1$. This will be useful in sections 6 and 7 when we study characters of Hecke algebras.

2.8.1 Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Following Lusztig, we introduce the $\mathcal{A}$-subalgebra $U_\mathcal{A}$ of $U_q(n)$ generated by the divided powers $e_i^{(k)} = e_i^k / [k]_!$ $(1 \leq i \leq r, k \in \mathbb{N})$. We set

$$U_\mathcal{A} = \{ u \in U_q(n) \mid (u, v) \in \mathcal{A} \text{ for all } v \in U_\mathcal{A} \}.$$

For $w = w_{i_1} \cdots w_{i_k} \in \mathcal{M}$ with $i_j \neq i_{j+1}$ $(1 \leq j \leq k - 1)$, we define $c_w = [a_1]_{i_1} \cdots [a_k]_{i_k}$ and we write $e_w = e_{i_1} \cdots e_{i_k}$. Thus $c_w^{-1} e_w$ is a product of divided powers. Consider the free $\mathcal{A}$-module $\mathcal{F}_\mathcal{A} = \bigoplus_{w \in \mathcal{M}} \mathcal{A} e_w$ and define $U_\mathcal{A}^* = U \cap \mathcal{F}_\mathcal{A}$.

**Lemma 8** We have $U_\mathcal{A}^* = \Phi(U_\mathcal{A})$.

**Proof** — An element $u \in U_q(n)$ belongs to $U_\mathcal{A}^*$ if and only if $(u, c_w^{-1} e_w) \in \mathcal{A}$ for all $w \in \mathcal{M}$, that is, if and only if $\Phi(u)$ is an $\mathcal{A}$-linear combination of the elements $e_w$, that is, if and only if $\Phi(u) \in \mathcal{F}_\mathcal{A}$. 

It is easy to see from Proposition 6 that $\mathcal{F}_\mathcal{A}$ is in fact a subalgebra of $\mathcal{F}$. It follows that $U_\mathcal{A}^*$ is a subalgebra of $\mathcal{U}$, and by Lemma 8, $U_\mathcal{A}^*$ is a subalgebra of $U_q(n)$, a well-known fact. Define

$$\mathcal{F}_\mathcal{C} = \mathbb{C} \otimes_\mathcal{A} \mathcal{F}_\mathcal{A}, \quad U_\mathcal{C} = \mathbb{C} \otimes_\mathcal{A} U_\mathcal{A}^*, \quad U_\mathcal{C}^* = \mathbb{C} \otimes_\mathcal{A} U_\mathcal{A}^*,$$

where $\mathbb{C}$ is regarded as an $\mathcal{A}$-module via $q \mapsto 1$. The natural maps $\mathcal{F}_\mathcal{A} \to \mathcal{F}_\mathcal{C}$ and $U_\mathcal{A}^* \to U_\mathcal{C}^*$ will be called ‘specialization at $q = 1$’.

The $\mathbb{C}$-linear map defined by $1 \otimes c_w w \mapsto a_1! \cdots a_k! w$ is an algebra isomorphism from $\mathcal{F}_\mathcal{C}$ endowed with the specialization of $^*$ at $q = 1$ to the classical $\mathbb{C}$-shuffle algebra over $\{ w_1, \ldots, w_r \}$, and from now on these two algebras will be identified. The subalgebra $U_\mathcal{C}^*$ of $\mathcal{F}_\mathcal{C}$ can be described explicitly by specializing in the obvious way Theorem 5 at $q = 1$. Note that $U_\mathcal{C}^*$ is in general strictly bigger than the subalgebra of $(\mathcal{F}_\mathcal{C}, [1])$ generated by the letters $w_i$ $(1 \leq i \leq r)$.
Proposition 9 Let \( f \in \mathbb{C}[N] \) be homogeneous of degree \( \nu \in \mathbb{Q}^+ \). Let \( w = w[i_1, \ldots, i_k] \in \mathcal{M} \) be of degree \( \nu \). The coefficient of the word \( w \) in the \( \mathcal{M} \)-expansion of \( \varphi(f) \) is equal to the coefficient of the monomial \( t_1 \cdots t_k \) in the polynomial function

\[
(t_1, \ldots, t_k) \mapsto f(x_{i_1}(t_1) \cdots x_{i_k}(t_k)).
\]

Equivalently, we have

\[
\varphi(f) = \sum_{w = w[i_1, \ldots, i_k], |w| = \nu} \left( \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} f(x_{i_1}(t_1) \cdots x_{i_k}(t_k)) \right)_{t_1 = \cdots = t_k = 0} w.
\]

Proof — Let \( u \in U^*_A \) be homogeneous of degree \( \nu \), and let \( f \in \mathbb{C}[N] \) be its specialization at \( q = 1 \). The group \( N \) acts on \( \mathbb{C}[N] \) by right translations:

\[
(x_i(t)f)(x) = f(xx_i(t)), \quad (x \in N, t \in \mathbb{C}, 1 \leq i \leq r)).
\]

Accordingly, the Lie algebra acts via the infinitesimal right translation operators

\[
\mathcal{E}_i(f)(x) = \frac{d}{dt} f(xx_i(t))|_{t=0}.
\]

These are the specializations at \( q = 1 \) of the endomorphisms \( \mathcal{E}_i^q \in \text{End}_{\mathbb{C}}(\mathcal{M}) \). In particular, for any \( i, \mathcal{E}_i(f) \) is homogeneous of degree \( \nu - \alpha_i \). It is easy to check that

\[
\mathcal{E}_{i_1} \cdots \mathcal{E}_{i_k}(f)(x) = \left( \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} f(xx_{i_1}(t_1) \cdots x_{i_k}(t_k)) \right)_{t_1 = \cdots = t_k = 0}.
\]

If \( \nu = \alpha_{i_1} + \cdots + \alpha_{i_k} \) this function of \( x \) is a constant, equal to the specialization at \( q = 1 \) of \( \mathcal{E}_{i_1}^q \cdots \mathcal{E}_{i_k}^q(u) \), that is, to the coefficient of \( w \) in \( \varphi(f) \), and the proposition follows. \( \square \)

One can also describe the inverse map \( \varphi^{-1} : U^*_C \rightarrow \mathbb{C}[N] \).

Proposition 10 Let \( u = \sum_{w \in \mathcal{M}} \gamma(w) w \) be a homogeneous element of \( U^*_C \) of degree \( \nu \), and let \( f = \varphi^{-1}(u) \). For any \((i_1, \ldots, i_k) \in [1, r]^k \) and \((t_1, \ldots, t_k) \in \mathbb{C}^k \) we have

\[
f(x_{i_1}(t_1) \cdots x_{i_k}(t_k)) = \sum_w \gamma(w) \frac{t_1^{\alpha_{i_1}} \cdots t_k^{\alpha_{i_k}}}{a_1! \cdots a_k!}
\]

where the sum is over all \( w = a_1^{\alpha_{i_1}} \cdots a_k^{\alpha_{i_k}} \) such that \( a_1 \alpha_{i_1} + \cdots + a_k \alpha_{i_k} = \nu \).

Proof — This follows immediately from the identity

\[
f(xx_i(t)) = (x_i(t)f)(x) = ((\exp t\mathcal{E}_i)f)(x) = \sum_{k \geq 0} \frac{1}{k!} (t^k f)(x)
\]

and the proof of the previous proposition. \( \square \)
3 Good words and monomial bases

From now on, we fix an arbitrary total order on the set $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots of $g$. The alphabet $I = \{w_1, \ldots, w_r\}$ is given the corresponding total order, and $M$ the associated lexicographic order. All these orders will be denoted by $\prec$.

3.1 To $w = w[i_1, \ldots, i_k]$ we associate $D_w := e'_{i_1} \cdots e'_{i_k} \in \text{End} \, F$. (For $w = w[]$ we set $D_w = \text{Id}_F$.) We have

$$D_w(\Phi(u)) = \Phi(\partial_w(u)), \quad (u \in U_q(n)).$$

(14)

For a homogeneous element $f$ of $F$ we denote by $\max(f)$ the maximal word $w \in M$ such that $|w| = |f|$ and $D_w(f) \neq 0$, that is, the largest word occuring in the expansion of $f$.

**Definition 11** A word $w \in M$ is called good if there exists a homogeneous $u \in U$ such that $w = \max(u)$.

The set of good words is denoted by $G$. Good words have been introduced by Lalonde and Ram for Lie algebras and universal enveloping algebras $[LR]$, and used by Rosso in the context of quantum groups $[R3]$. Note that our definition is different from that of $[LR][R3]$. It will be shown in Lemma 21 that the two definitions are equivalent.

**Proposition 12** (i) There is a unique basis of homogeneous vectors $\{m_g \mid g \in G\}$ of $U$ such that

$$D_{g_1}(m_{g_2}) = \delta_{g_1,g_2}, \quad (g_1,g_2 \in G, \ |g_1| = |g_2|).$$

(ii) $\{e_g \mid g \in G\}$ is a basis of $U_q(n)$.

**Proof** — Let $\nu \in Q^+$. Let $U_\nu$ be the homogeneous component of degree $\nu$ of $U$, $B_\nu$ a basis of $U_\nu$, and $\{g_1, \ldots, g_m\}$ the subset of $G$ consisting of all words of weight $\nu$ arranged in increasing order. There exists at least one element of $B_\nu$, say $b_m$, such that $\max(b_m) = g_m$. By rescaling $b_m$ and subtracting appropriate multiples of it from the other elements of $B_\nu$ we can arrange that $g_m$ appears in $b_m$ with coefficient 1 and does not occur in any other vector of $B_\nu$. (Here we abuse notation and still denote by $B_\nu$ the basis obtained after these operations.) Similarly, there exists a vector in $B_\nu \setminus \{b_m\}$, say $b_{m-1}$, such that $\max(b_{m-1}) = g_{m-1}$, and we can modify $B_\nu$ in such a way that $D_{g_{m-1}}(b_{m-1}) = 1$ and $g_{m-1}$ occurs in no other element than $b_{m-1}$. Repeating this process we get a subset $\{b_1, \ldots, b_m\}$ of $B_\nu$ such that $D_{g_i}(b_j) = \delta_{ij}$. Finally, $B_\nu = \{b_1, \ldots, b_m\}$, since otherwise there would be some $b \in B_\nu$ with $\max(b) \neq g_i$ for all $i$, which is impossible. Proceeding in the same way in every weight space of $U$ we obtain a basis $\{m_g \mid g \in G\}$ as in (i). The unicity is clear.

By (5) and (14) we see that the basis of $U_q(n)$ adjoint to $\{\Phi^{-1}(m_g) \mid g \in G\}$ is $\{e_\tau(g) \mid g \in G\}$, where $\tau$ is as in Proposition 6. Finally, applying the anti-automorphism of $U_q(n)$ which fixes the generators $e_i$, we obtain that $\{e_g \mid g \in G\}$ is a basis of $U_q(n)$.

**Lemma 13** ([LR]) Every factor of a good word is good.

**Proof** — Let $w = w[i_1, \ldots, i_k] \in G$ and let $u \in U$ be such that $w = \max(u)$. One checks easily that $w[i_1, \ldots, i_j] = \max(e'_{i_{j+1}} \cdots e'_{i_k}(w)), \ (1 \leq j < k)$. We know that $U$ is stable under the endomorphisms $e'_i$, hence all left factors of $w$ are good. To conclude in the general case, we may
introduce the endomorphisms $e_i^T$ of $F$ defined by $e_i^T(w[i_1, \ldots, i_k]) = \delta_{i,i_1}w[i_2, \ldots, i_k]$. In other words, $e_i^T = \tau \circ e_i \circ \tau$, where $\tau$ is as in Proposition [7]. This shows that $U$ is stable under $e_i^T$. Therefore, $w[i_h, \ldots, i_j] = \max(e_{i_{j+1}}^T \cdots e_{i_k}^T e_{i_{h-1}}^T \cdots e_{i_1}^T(w))$ is good for all $1 < h \leq j < k$. □

We are now going to study the set $G$ of good words, and find an explicit description of it in terms of Lyndon words.

3.2 A word $l = w[i_1, \ldots, i_k] \in M$ is a Lyndon word if it is smaller than all its proper right factors, that is,

$$l < w[i_j, \ldots, i_k], \quad (j = 2, \ldots, k).$$

We shall denote by $L$ the set of Lyndon words in $M$. For properties of Lyndon words which are not proved here, see [Loi] chapter 5, or [Reu].

We have the following inductive characterization of Lyndon words, namely, $l$ is a Lyndon word if and only if $l \in I$ or $l$ has a non trivial factorization $l = l_1l_2$ where $l_1$ and $l_2$ are Lyndon words and $l_1 < l_2$.

For $l \in L \setminus I$, write $l = l_1l_2$ with $l_2$ a Lyndon word of maximal length. It is known that $l_1$ is then also a Lyndon word, and $l = l_1l_2$ is called the standard factorization of $l$.

Similarly, write $l = l_1^*l_2^*$ where $l_1^*$ is a Lyndon word of maximal length. Then $l_2^*$ is also a Lyndon word and we shall call $l = l_1^*l_2^*$ the co-standard factorization of $l$. This follows from the next lemma which gives a description of $l_2^*$.

**Lemma 14** Let $l = l_1^*l_2^*$ be the co-standard factorization of $l \in L$. Then $l_2^*$ is of the form

$$l_2^* = (l_1^*)^kf_x,$$

where $k \in \mathbb{N}$, $f$ is a left factor of $l_1^*$ (possibly empty), and $x$ is a letter such that $fx > l_1^*$.

**Proof** — Let $m$ be a non trivial left factor of $l_2^*$. We want to prove that $m = (l_1^*)^kf$ for some $k \in \mathbb{N}$ and some left factor $f$ of $l_1^*$. We will proceed by induction on $\ell(m)$.

Note first that by definition of the co-standard factorization, $l_1^*m$ is not a Lyndon word. Thus $l_1^*m$ has a right factor $\leq l_1^*m$. This factor cannot be of the form $dm$ for some right factor $d$ of $l_1^*$, since $d > l_1^*$ and $\ell(d) < \ell(l_1^*)$ imply $dm > l_1^*m$. Therefore this factor is a right factor of $m$. In particular, if $\ell(m) = 1$, we obtain that $m$ is less or equal to the first letter $a$ of $l_1^*$, and since $l_2^*$ is a Lyndon word, we must have $m = a$, which proves the claim in this case.

Suppose now that $\ell(m) > 1$ and write $m = m'y$ where $y$ is a letter. By induction we may assume that $m' = (l_1^*)^kf$ for some $k \in \mathbb{N}$ and some non trivial left factor $f$ of $l_1^*$, and we have to prove that $fy$ is a left factor of $l_1^*$ (possibly equal to $l_1^*$). There exists a right factor $d$ of $m$ such that $d \leq l_1^*m$. We have $d = d'(l_1^*)l_2^*fy$ for some $l \leq k$ and some right factor $d'$ of $l_1^*$. In fact $d'$ must be empty, otherwise, since $l_1^*$ is Lyndon we would have $d' > l_1^*$ and $\ell(d') < \ell(l_1^*)$, hence

$$d = d'(l_1^*)l_2^*fy > l_1^*(l_1^*)^kf_y = l_1^*m.$$  

Therefore, $d = (l_1^*)^{k+1}fy \leq (l_1^*)^{k+1}fy$ for some $l \leq k$. It follows that $fy \leq (l_1^*)^i$ for some $i > 0$, and since $\ell(fy) < \ell(l_1^*)$ we have in fact $fy \leq l_1^*$. Now either $fx$ is a left factor of $l_1^*$ or there is a $j \leq \ell(fy)$ such that the $j$th letter of $fy$ is strictly smaller than the $j$th letter of $l_1^*$. The second case is impossible since then $l_1^*l_2^*$ could not be a Lyndon word. Therefore $fy$ is a left factor of $l_1^*$.

Now we can apply this to the longest strict left factor $m$ of $l_2^*$, and we obtain that $l_2^*$ is as we claimed. Moreover, since $l_1^*l_2^*$ is a Lyndon word, the letter $x$ has to be such that $fx > l_1^*$. □
Proposition 18 \( [LR] \) Lyndon words have good factors. The converse follows immediately from Proposition 16.

Proof — By Proposition 12 and Proposition 17 the products
\[
e_{l_1}^{n_1} \cdots e_{l_k}^{n_k}, \quad (n_1, \ldots, n_k \in \mathbb{N}, \ l_1, \ldots, l_k \in \mathcal{G} \mathcal{L}, \ l_1 > \cdots > l_k)
\]
form a basis of \( U_q(n) \). This implies that the generating series of the dimensions of the homogeneous components of \( U_q(n) \) is equal to
\[
\sum_{\nu \in Q^+} \dim U_q(n)_\nu \exp \nu = \prod_{l \in \mathcal{G} \mathcal{L}} \frac{1}{1 - \exp |l|}.
\]
On the other hand it is well-known that
\[
\sum_{\nu \in Q^+} \dim U_q(n)_\nu \exp \nu = \prod_{\beta \in \Delta^+} \frac{1}{1 - \exp \beta},
\]
and by comparing the two expressions the claim follows. \( \square \)

Let \( \mathcal{G} \mathcal{L} \) denote the subset of \( \mathcal{G} \) consisting of all good Lyndon words.

Proposition 16 \( [LR] \) Let \( l \in \mathcal{G} \mathcal{L} \) and \( g \in \mathcal{G} \) with \( l \geq g \). Then \( lg \in \mathcal{G} \).

Proof — Let \( u, v \) be homogeneous elements of \( \mathcal{U} \) such that \( \max(u) = l \) and \( \max(v) = g \). Rescaling \( u \) and \( v \) if necessary we can assume that \( u = l + r \) and \( v = g + s \) where \( r \) (resp. \( s \)) is a linear combination of words \( < l \) (resp. \( < s \)). We have \( u \ast v = l \ast g + l \ast s + r \ast g + r \ast s \). By Lemma 15 \( \max(l \ast g) = lg \). Now if \( w \) and \( w' \) are words such that \( |w| = |l|, \ |w'| = |g|, \ w \leq l \) and \( w' \leq g \), any word occurring in the shuffle of \( w \) and \( w' \) will be less or equal to the corresponding word in the shuffle of \( l \) and \( g \), so \( \max(u \ast v) = lg \).

It is well known \( [Lo, Reu] \) that every word \( w \in \mathcal{M} \) has a unique factorization \( w = l_1 \cdots l_k \) where \( l_1, \ldots, l_k \in \mathcal{L} \) and \( l_1 \geq \cdots \geq l_k \).

Proposition 17 \( [LR] \) A word \( g \) is good if and only if it is of the form
\[
g = l_1 \cdots l_k, \quad l_1 \geq \cdots \geq l_k,
\]
where \( l_1, \ldots, l_k \) are good Lyndon words.

Proof — By Lemma 13 if \( g \) is good, its canonical factorization as a non-increasing product of Lyndon words has good factors. The converse follows immediately from Proposition 16. \( \square \)

Proposition 18 \( [LR] \) The map \( l \mapsto |l| \) is a bijection from \( \mathcal{G} \mathcal{L} \) to \( \Delta^+ \).

Proof — By Proposition 12 and Proposition 17 the products
\[
e_{l_1}^{n_1} \cdots e_{l_k}^{n_k}, \quad (n_1, \ldots, n_k \in \mathbb{N}, \ l_1, \ldots, l_k \in \mathcal{G} \mathcal{L}, \ l_1 > \cdots > l_k)
\]
form a basis of \( U_q(n) \). This implies that the generating series of the dimensions of the homogeneous components of \( U_q(n) \) is equal to
\[
\sum_{\nu \in Q^+} \dim U_q(n)_\nu \exp \nu = \prod_{l \in \mathcal{G} \mathcal{L}} \frac{1}{1 - \exp |l|}.
\]
On the other hand it is well-known that
\[
\sum_{\nu \in Q^+} \dim U_q(n)_\nu \exp \nu = \prod_{\beta \in \Delta^+} \frac{1}{1 - \exp \beta},
\]
and by comparing the two expressions the claim follows. \( \square \)

We shall denote by \( \beta \mapsto l(\beta) \) the inverse of the above bijection. It is an embedding of \( \Delta^+ \) in \( \mathcal{L} \). We will call it a Lyndon covering of \( \Delta^+ \).
4 PBW-type bases

In this section, we introduce following Lalonde-Ram [LR] and Rosso [R3] another basis \( \{ r_g \} \) of \( \U \) labelled by good words, the Lyndon basis. Then we show that this basis is up to normalization the image under the anti-automorphism \( \sigma \) of a basis \( \{ E_g \} \) of PBW-type, as defined by Lusztig [L2]. This allows us to prove that \( \max(E_g) = g \) (Theorem 4.3). In [4.3] we also provide an algorithm for computing explicitly the map \( \beta \mapsto l(\beta) \). This works for any root system and any ordering of the simple roots, and is simpler than the procedure of [LR] which needs some case-by-case discussion. Finally, we prove that the normalization coefficient \( \kappa_g \) between \( \sigma(r_g) \) and \( E_g \) is a bar-invariant Laurent polynomial (Proposition 4.2), which will be used in 5.1.

4.1 For homogeneous elements \( f_1, f_2 \in \mathcal{F} \) we define

\[
[l, f_1 f_2]_q := f_1 f_2 - q(l_1 l_2) f_2 f_1.
\]

Let \( l \in \mathcal{L} \). We define inductively the \( q \)-bracketing \([l] \in \mathcal{F}\) by \([l] = l \) if \( l \) is a letter, and otherwise \([l] = \left[ [l_1], [l_2] \right]_q \) where \( l = l_1 l_2 \) is the co-standard factorization of \( l \).

Proposition 19 \([l] = l + r \) where \( r \) is a linear combination of words \( > l \).

Proof — We argue by induction on the length of \( l \). If \( l \) is a letter, the statement is obvious. Otherwise \( [l] = \left[ [l_1], [l_2] \right]_q \) and we can assume by induction that \( [l_1] = l_1 + r_1 \) and \( [l_2] = l_2 + r_2 \) where \( r_1 \) and \( r_2 \) are linear combinations of words \( > l_1 \) and \( > l_2 \) respectively. Hence,

\[
\left[ [l_1], [l_2] \right]_q = [l_1, l_2]_q + [r_1, l_2]_q + [l_1, r_2]_q + [r_1, r_2]_q.
\]

The first bracket is \( l_1 l_2 - q(l_1 l_2) l_1 l_2 \), and since \( l_1 l_2 \) is a Lyndon word, \( l_2 l_1 > l_1 l_2 \). Clearly, all words occuring in the other brackets are either \( > l_1 l_2 \) or \( > l_2 l_1 > l_1 l_2 \), and the statement follows. \( \square \)

Let \( w = l_1 \cdots l_k \) be the canonical factorisation of \( w \) as a non-increasing product of Lyndon words. We define \([w] := [l_1] \cdots [l_k] \in \mathcal{F}\).

Proposition 20 \( \{ [w] \mid w \in \mathcal{M} \} \) is a basis of \( \mathcal{F} \).

Proof — It follows easily from Proposition 19 that \([w] = w + s \) where \( s \) is a linear combination of words \( > w \). Hence the transition matrix from the basis \([w] \) to the family of vectors \( \{ [w] \} \) is unitriangular. \( \square \)

4.2 Let \( \Xi \) be the algebra homomorphism from \((\mathcal{F}, \cdot)\) to \((\mathcal{F}, \ast)\) such that \( \Xi(w_i) = w_i \) for every letter \( w_i \), that is, each word is mapped by \( \Xi \) to the quantum shuffle product of its letters. Clearly, \( \Xi(\mathcal{F}) = \U \).

Lemma 21 The word \( w \) is good if and only if it cannot be expressed modulo \( \ker \Xi \) as a linear combination of words \( v > w \).
Proposition 22 (R3)

Note that for any word \( v > w \), that is, there exists a relation of the form

\[
    w_{i_1} \cdots w_{i_k} = \sum_{v = w[j_1, \ldots, j_k]} x_v w_{j_1} \cdots w_{j_k}
\]

(15)

for some scalars \( x_v \in \mathbb{Q}(q) \). Using the isomorphism \( \Phi^{-1} : \mathcal{U} \rightarrow U_q(n) \) this is equivalent to

\[
    e_{i_1} \cdots e_{i_k} = \sum_{v = w[j_1, \ldots, j_k]} x_v e_{j_1} \cdots e_{j_k},
\]

and since the algebra generated by the \( e_i' \) is isomorphic to \( U_q(n) \) this is in turn equivalent to

\[
    \partial_w = \sum_{v > w} x_v \partial_v.
\]

Therefore, if for some homogeneous \( u \in \mathcal{U} \) of weight \( |u| = |w| \) one has \( D_w(u) \neq 0 \), then there exists a \( v > w \) such that \( D_v(w) \neq 0 \), and \( w \neq \max(u) \). Hence \( w \) is not good. Let us denote by \( \mathcal{H} \) the set of words \( w \) which satisfy no relation of the form (15). We have proved that \( \mathcal{G} \subset \mathcal{H} \).

Conversely, it is easy to prove that \( \{ e_w \mid w \in \mathcal{H} \} \) is a basis of \( U_q(n) \). Indeed, this set contains the monomial basis of Proposition 12 and it is linearly independent, since if we had a linear relation between words of \( \mathcal{H} \) we could express the smallest one in terms of the others and it would not belong to \( \mathcal{H} \). Hence \( \mathcal{G} = \mathcal{H} \), as required. \( \square \)

Note that in [LK] [R3], Lemma 21 is taken as the definition of a good word.

For \( g \in \mathcal{G} \), let us write \( r_g = \Xi([g]) \).

Proposition 23 (R3) \( \{ r_g \mid g \in \mathcal{G} \} \) is a basis of \( \mathcal{U} \).

Proof — Note that for any word \( w \), we have \( \Xi(w) = \Phi(e_w) \). As in the proof of Proposition 20 for \( g \in \mathcal{G} \) we have \([g] = g + \sum_{w > g} x_{gw} w \). Thus \( r_g = \Phi(e_g) + \sum_{w > g} x_{gw} \Phi(e_w) \). By Lemma 21 this last sum can be rewritten as \( r_g = \Phi(e_g) + \sum_{h > g} y_{gh} \Phi(e_h) \), where the words \( h \) are good. Hence, the transition matrix from the basis \( \Phi(\{ e_g \mid g \in \mathcal{G} \}) \) to \( \{ r_g \mid g \in \mathcal{G} \} \) is unitriangular. \( \square \)

We call \( \{ r_g \mid g \in \mathcal{G} \} \) the Lyndon basis of \( \mathcal{U} \).

Theorem 23 (R3) The Lyndon basis has the following form

\[
    \{ r_{l_1} \cdots r_{l_k} \mid k \in \mathbb{N}, \ l_1, \ldots, l_k \in \mathcal{G} \mathcal{L}, \ l_1 \geq \cdots \geq l_k \}.
\]

Proof — By definition of \([g]\), if \( g = l_1 \cdots l_k \) is the canonical factorization of \( g \) as a non-increasing product of Lyndon words, we have \( r_g = r_{l_1} \cdots r_{l_k} \), and by Lemma 14 each factor \( l_k \) is good. Conversely, if \( l_1, \ldots, l_k \) are good Lyndon words and \( l_1 \geq \cdots \geq l_k \) then by Proposition 17 \( g = l_1 \cdots l_k \) is good. \( \square \)

Proposition 24 Let \( \beta_1, \beta_2 \in \Delta^+ \) be such that \( \beta_1 + \beta_2 = \beta \in \Delta^+ \) and \( l(\beta_1) < l(\beta_2) \). Then \( l(\beta_1)l(\beta_2) \leq l(\beta) \).
Proof — As seen in the proof of Proposition 22, the transition matrix from the basis \( \{ \Xi(g) \mid g \in G \} \) to the basis \( \{ r_g \mid g \in G \} \) is unitriangular. Hence, writing \( l_1 = l(\beta_1) \) and \( l_2 = l(\beta_2) \), we have

\[
    r_{l_1} * r_{l_2} = \left( \Xi(l_1) + \sum_{h_1 > l_1; h_1 \in G} y_{l_1 h_1} \Xi(h_1) \right) * \left( \Xi(l_2) + \sum_{h_2 > l_2; h_2 \in G} y_{l_2 h_2} \Xi(h_2) \right)
\]

for some \( z_g \in \mathbb{Z}[q, q^{-1}] \). Recall from Proposition 2.8.1 the \( \mathcal{A} \)-subalgebra \( U_{\mathcal{A}} \) of \( U_q(\mathfrak{n}) \). Let \( x \mapsto x \) denote the specialization \( q \mapsto 1 \) from \( U_{\mathcal{A}} \) to \( U(\mathfrak{n}) \). For \( l \in GL \) set \( s_l = \Phi^{-1}(r_l) \). Then \( s_l \in \mathfrak{n} \) (this is an iterated bracket of Chevalley generators \( e_1 \)), and \( [s_{l_1}, s_{l_2}] \) belongs to the weight space of weight \( \beta \) of \( \mathfrak{n} \). By hypothesis, this weight space is 1-dimensional and spanned by \( s_{l(\beta)} \). Hence,

\[
    s_{l_1} s_{l_2} = s_{l_2} s_{l_1} + c s_{l(\beta)}
\]

for some \( c \in \mathbb{Z}^* \). It follows that \( z_{l(\beta)} \neq 0 \), which implies that \( l(\beta) \geq l_1 l_2 \).

4.3 Proposition 24 implies the following simple inductive rule for determining the set \( GL \) of good Lyndon words. If \( \beta = \alpha_i \) is a simple root, then \( l(\beta) = w_1 \). If \( \beta \) is not a simple root there exists a factorization \( l(\beta) = l_1 l_2 \) with \( l_1 \) and \( l_2 \) Lyndon. By Lemma 13 \( l_1 = l(\beta_1) \) and \( l_2 = l(\beta_2) \) for some \( \beta_1, \beta_2 \in \Delta^+ \). By induction we may assume that we know \( l(\gamma) \) for any \( \gamma \in \Delta^+ \) of height smaller than the height of \( \beta \). Let

\[
    C(\beta) = \{ (\beta_1, \beta_2) \in \Delta^+ \times \Delta^+ \mid \beta_1 + \beta_2 = \beta, \ l(\beta_1) < l(\beta_2) \}.
\]

Then, by Proposition 24 we get

**Proposition 25** \( l(\beta) = \max \{ l(\beta_1) l(\beta_2) \mid (\beta_1, \beta_2) \in C(\beta) \} \). \( \square \)

Note that the sets \( \mathcal{L}, \mathcal{G}, \mathcal{GL} \) depend on the choice of a total order on \( \Pi \), and that we have \( r! \) possible choices. In [LR] the sets \( \mathcal{GL} \) are calculated for all root systems and for a particular total order on \( \Pi \) (see also section 8).

By Proposition 17 we can calculate the set \( \mathcal{G} \) of good words by taking the non-increasing products of elements of \( \mathcal{GL} \). Note that, by Proposition 12 we have thus obtained for each total order on \( \Pi \) a simple and explicit monomial basis of \( U_q(\mathfrak{n}) \). This basis seems to be different from the monomial bases of Chari and Xi [CX] and Reineke [Rc1].

4.4 Since \( \mathcal{GL} \) is totally ordered (lexicographically) we obtain a total order (still denoted by \( < \)) on \( \Delta^+ \). In [R3] the following key fact is stated.

**Proposition 26 (R3)** The order \( < \) on \( \Delta^+ \) is convex, that is, if \( \beta_1 \) and \( \beta_2 \) are elements of \( \Delta^+ \) such that \( \beta_1 + \beta_2 = \beta \) belongs to \( \Delta^+ \), then \( \beta_1 < \beta < \beta_2 \) or \( \beta_2 < \beta < \beta_1 \).

Note that by Proposition 24 we have that if \( l(\beta_1) < l(\beta_2) \) and \( \beta_1 + \beta_2 = \beta \) then

\[
    l(\beta) \geq l(\beta_1) l(\beta_2) > l(\beta_1).
\]

On the other hand, if \( l(\beta) = l(\beta_1) l(\beta_2) \), since \( l(\beta) \) is a Lyndon word, \( l(\beta) < l(\beta_2) \). It only remains to prove that, even when \( l(\beta) > l(\beta_1) l(\beta_2) \) we have \( l(\beta) < l(\beta_2) \).
Corollary 27 Let $\beta \in \Delta^+$. The good Lyndon word $l$ of weight $\beta$ is the smallest good word of weight $\beta$.

Proof — Let $g \neq l$ be a good word of weight $\beta$ and let $g = l_1 \cdots l_k$ be its unique expression as a non-increasing product of good Lyndon words. Let $\beta_i = |l_i|$ $(1 \leq i \leq k)$. If $g < l$ then $l_1 < l$. Indeed, Melançon has shown that if $w = m_1 \cdots m_r$ and $w' = m'_1 \cdots m'_s$ are the factorizations into non-increasing products of good Lyndon words of $w$ and $w'$, we have $w > w'$ if and only if there exist $j$ such that $m_i = m'_j$ for $i < j$ and $m_j > m'_j$ \[\text{(M)}.\] Therefore $l_i < l$ for all $i = 1, \ldots, k$, hence we have $\beta_1 + \cdots + \beta_k = \beta$ with all $\beta_i < \beta$, contrary to the fact that $< \beta$ is convex. \[\square\]

4.5 It is well-known \[\text{\cite{P}}\] that any convex ordering $\beta_1 < \cdots < \beta_n$ of $\Delta^+$ arises from a unique reduced decomposition $w_0 = s_{i_1} \cdots s_{i_n}$ of the longest element of the Weyl group $W$ in the following way:

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \ldots, \quad \beta_n = s_{i_1} \cdots s_{i_{n-1}}(\alpha_{i_n}).$$

To this data Lusztig associates a PBW-type basis of $U_q(n)$

$$E(a_1)(\beta_1) \cdots E(a_n)(\beta_n), \quad (a_1, \ldots, a_n) \in \mathbb{N}^n,$$

defined using the braid group action on $U_q(n)$ \[\text{(L})\] \[\text{2.20.2)\}.\] (We choose the action via the operators $T_{i,1}^{-1}$ of \[\text{L2} \] 37.1.3, with $q = v^{-1}$.) Let us fix from now on the PBW-type basis associated with the convex ordering on $\Delta^+$ coming from its Lyndon covering $GL$.

Theorem 28 For all $\beta \in \Delta^+$, the vectors $\Phi(E(\beta))$ and $r_l(\beta) = \Xi([l(\beta)])$ are proportional.

Proof — We argue by induction on the height $k$ of $\beta$. If $\beta$ is a simple root, the claim is trivial. Suppose that $k > 1$ and that the result is proved for all roots of height $\leq k - 1$. We can write $\beta$ as a sum $\beta_1 + \beta_2$ of two positive roots $\beta_1 < \beta_2$ \[\text{(Bo). Prop. 19,}\] and clearly these roots both have height $\leq k - 1$. Among all such decompositions, pick up the one for which $\beta_1$ is maximum and denote it by $\beta = \beta_1^+ + \beta_2^+$. By a result of Levendorskii and Soibelman \[\text{[LS]}\] (see also \[\text{[CP]. 9.3,}\]

$$E(\beta_1^+)^t E(\beta_2^+) - q(\beta_1^+, \beta_2^+) E(\beta_2^+) E(\beta_1^+)$$

is a linear combination of products $E(\beta_i) \cdots E(\beta_j)$ where $\beta_{i_1} + \cdots + \beta_{i_s} = \beta$ and $\beta_{i_j}^+ < \beta_{i_j} < \beta_{i_j}^+$ for every $j$. (Note that the $\beta_{i_j}$ are not necessarily distinct.) Suppose there occurs in this linear combination a term other than $E(\beta)$, that is, a term with $s > 1$, and let us consider it. By \[\text{[Bo], Prop. 19,}\] we have (after renumbering the $\beta_{i_j}$ if necessary) that $\beta_{i_1} + \cdots + \beta_{i_s} = \beta$ and $\beta_{i_1}^+ < \beta_{i_2} < \beta_{i_2}^+$ for every $j$. In particular $\beta' = \beta_{i_1} + \cdots + \beta_{i_{s-1}}$ and $\beta'' = \beta_{i_s}$ are two positive roots with $\beta' + \beta'' = \beta$. Therefore, by definition of $\beta_1^+$ and because $< \beta$ is convex, either $\beta' < \beta_1^+ < \beta < \beta''$ or $\beta'' < \beta_1^+ < \beta < \beta'$. In the second case we would have $\beta_1 < \beta_1^+$ which is impossible. The first case is also impossible since if all $\beta_{i_j} > \beta_1^+$ then $\beta' > \beta_1^+$ by convexity. Hence,

$$E(\beta_1^+)^t E(\beta_2^+) - q(\beta_1^+, \beta_2^+) E(\beta_2^+) E(\beta_1^+)$$

is proportional to $E(\beta)$.

On the other hand, let us consider the element

$$r = r_l(\beta_1^+) r_l(\beta_2^+) - q(\beta_1^+, \beta_2^+) r_l(\beta_2^+) r_l(\beta_1^+).$$
Let \( l = l_1^t l_2^t \) be the co-standard factorization of \( l = l(\beta) \). We have \( l(\beta_1^t) \geq l_1^t \) by definition of \( \beta_1^t \), and \( l(\beta_2^t) l(\beta_2^t) \leq l_1^t l_2^t \) by Proposition 24.

This implies that \( l(\beta_1^t) = l_1^t \) and \( l(\beta_2^t) = l_2^t \). Indeed, the two inequalities imply that \( l_1^t \) is a left factor of \( l(\beta_1^t) \), that is, \( l(\beta_1^t) = l_1^t m \). Suppose that \( m \) is not the empty word. If \( l(\beta_1^t) l(\beta_2^t) = l_1^t l_2^t \), then \( l_1^t l_2^t \) would not be the co-standard factorization, so we must have \( l(\beta_1^t) l(\beta_2^t) < l_1^t l_2^t \). By Lemma 14, \( l_2^t = (l_1^t)^k f x \) for some \( k \in \mathbb{N} \), some left factor \( f \) of \( l_1^t \) and some letter \( x \) such that \( f x > l_1^t \). Since \( l(\beta_1^t) \) is a Lyndon word, \( m > l_1^t \). On the other hand we must have \( l_1^t m = l(\beta_1^t) < l_1^t l_2^t \), hence \( m < l_2^t \), and this implies that \( l_1^t \) is a left factor of \( m \). Since for any \( k' \) and any left factor \( f' \) of \( l_1^t \), \( (l_1^t)^k f' \) is not a Lyndon word, we see that the only possibility is \( m = l_2^t \), but then \( l(\beta_2^t) \) would be empty, a contradiction. Therefore \( m \) is empty, and \( l(\beta_1^t) = l_1^t \). Finally, since there is only one good Lyndon word of weight \( \beta_2^t \), we also have \( l_2^t = l(\beta_2^t) \).

Hence \( r = r_l(\beta) \) and since by induction \( r_l(\beta_1^t) \) and \( r_l(\beta_2^t) \) are proportional to \( \Phi(E(\beta_1^t)) \) and \( \Phi(E(\beta_2^t)) \) respectively, we conclude that \( r_l(\beta) \) is proportional to \( \Phi(E(\beta)) \).

In [RI], Ringel has proved a result similar to the above theorem for the PBW-bases of Lusztig associated to reduced words for \( w_0 \) adapted to an orientation of the Dynkin diagram of \( \mathfrak{g} \), that is, for those bases coming from the theory of Hall algebras. Note that the convex orderings of \( \Delta^+ \) coming from Lyndon coverings are in general different from those coming from Hall algebras, as shown by the next example. Hence Theorem 28 is different from Ringel’s result, since the PBW-bases involved are not the same.

**Example 29** Let \( \mathfrak{g} \) be of type \( D_4 \), with Dynkin diagram numbered as in 8.4. The Lyndon covering associated to the order \( \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \) corresponds to the reduced decomposition

\[
w_0 = s_1 s_3 s_2 s_4 s_3 s_1 s_4 s_3 s_2 s_4 s_3 s_4
\]

which is adapted to no orientation of the Dynkin diagram of \( \mathfrak{g} \).

**4.6** For a good word \( g = l(\beta_1)^{a_1} \cdots l(\beta_k)^{a_k} \), where \( \beta_1 > \cdots > \beta_k \) and \( a_1, \ldots, a_k \in \mathbb{N}^* \), we will denote by

\[
E_g := \Phi(E(\beta_k)^{a_k}) \cdots E(\beta_1)^{a_1}),
\]

the corresponding vector of the PBW-type basis of \( \mathcal{U} \), and we will write

\[
r_l = \lambda_l E_l \quad (l \in \mathcal{G} \mathcal{L}).
\]

Note that in (16), the factors are taken in the order opposite to the order used for defining \( r_g \). Recall from 27 the anti-automorphism \( \sigma \).

**Proposition 30** (i) Let \( l = w \in \mathcal{G} \mathcal{L} \). We have \( \sigma(r_l) = (-1)^{\ell(l) - 1} q^{-|\mathcal{N}(l)|} r_l \).

(ii) The vectors \( E_g \) and \( \sigma(r_g) \) are proportional for all \( g \in \mathcal{G} \).

**Proof** — Since \( \sigma \) is an anti-automorphism and \( E_l \) is proportional to \( r_l \) for \( l \in \mathcal{G} \mathcal{L} \), we see that (ii) follows immediately from (i). Let \( l = l_1^t l_2^t \) be the co-standard factorisation of \( l \), so that

\[
r_l = r_{l_1}^t * r_{l_2}^t - q^{(|l_1|,|l_2|)} \ r_{l_2}^t * r_{l_1}^t.
\]

Let us assume that (i) holds for \( l_1 \) and \( l_2 \). Then

\[
\sigma(r_l) = \sigma(r_{l_2}^t)^* \sigma(r_{l_1}^t) - q^{(|l_1|,|l_2|)} \ \sigma(r_{l_1}^t)^* \sigma(r_{l_2}^t) = (-1)^{\ell(l_1) + \ell(l_2) - 1} q^{-|\mathcal{N}(l_1)| - |\mathcal{N}(l_2)|} r_l,
\]

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On the other hand, an elementary calculation gives

\[-(|l_1|, |l_2|) - N(|l_1|) - N(|l_2|) = - \sum_{1 \leq s < t \leq k} (\alpha_i, \alpha_i) = -N(|l|).\]

\(\square\)

In the sequel, we will write

\[\sigma(r_g) = \kappa_g E_g \quad (g \in \mathcal{G}).\]  \hspace{1cm} (18)

Write \(g = l_1^{a_1} \cdots l_k^{a_k}\) where \(l_1 > \cdots > l_k \in \mathcal{GL}\) and \(a_1, \ldots, a_k \in \mathbb{N}^*\). Then \(r_g = r_{l_1}^{a_1} \cdots r_{l_k}^{a_k}\), while \(E_g = E_{l_1}^{(a_1)} \cdots E_{l_k}^{(a_k)}\), where for \(l = l(\beta) \in \mathcal{GL}\) we set \(E_l^{(a)} = E_l^a / [a]!\) if \((\beta, \beta) = (\alpha_i, \alpha_i)\). It follows that, writing \([a]! := [a],\]

\[\kappa_g = \prod_{j=1}^k \kappa_{l_j}^{a_j} [a_j]!.\]  \hspace{1cm} (19)

**Proposition 31** For \(l \in \mathcal{GL}\) we have \(\sigma(E_l) = (-1)^{\ell(l) - 1} q^{N(|l|)} E_l\).

**Proof** — Let \(\beta_1 < \beta_2 < \cdots < \beta_n\) be the convex ordering of \(\Delta^+\) associated with \(\mathcal{GL}\), and let \(w_0 = s_{i_1} \cdots s_{i_n}\) be the corresponding reduced decomposition of \(w_0\). Suppose that \(|l| = \beta_k\). Then we have \(E_l = T_{i_1} \cdots T_{i_{k-1}}(e_{i_k})\). Using \((1.2)\) \((7.2)\) \((4.4)\), we see that \(\sigma(E_l) = (-1)^{A(l)} q^{B(l)} E_l\) where

\[A(l) = \frac{1}{d_{i_1}} (\alpha_{i_1}, s_{i_2} \cdots s_{i_{k-1}} (\alpha_{i_k})) + \frac{1}{d_{i_2}} (\alpha_{i_2}, s_{i_3} \cdots s_{i_{k-1}} (\alpha_{i_k})) + \cdots + \frac{1}{d_{i_{k-1}}} (\alpha_{i_{k-1}}, \alpha_{i_k}),\]

and

\[B(l) = (\alpha_{i_1}, s_{i_2} \cdots s_{i_{k-1}} (\alpha_{i_k})) + (\alpha_{i_2}, s_{i_3} \cdots s_{i_{k-1}} (\alpha_{i_k})) + \cdots + (\alpha_{i_{k-1}}, \alpha_{i_k}).\]

On the other hand, an elementary calculation gives

\[\beta_k = \alpha_{i_k} - \frac{1}{d_{i_{k-1}}} (\alpha_{i_{k-1}}, \alpha_{i_k}) \alpha_{i_{k-1}} - \frac{1}{d_{i_{k-2}}} (\alpha_{i_{k-2}}, s_{i_{k-1}} (\alpha_{i_k})) \alpha_{i_{k-2}} - \cdots - \frac{1}{d_{i_1}} (\alpha_{i_1}, s_{i_2} \cdots s_{i_{k-1}} (\alpha_{i_k})) \alpha_{i_1}.\]

Therefore, writing \(\beta_k = \sum_{i=1}^r c_i \alpha_i\), we see that \(A(l) = 1 - \sum_{i=1}^r c_i = 1 - \ell(l)\). Finally,

\[N(|l|) = N(\beta_k) = \frac{1}{2} \left( (\beta_k, \beta_k) - \sum_{i=1}^r c_i (\alpha_i, \alpha_i) \right) = \frac{1}{2} \left( (\alpha_{i_k}, \alpha_{i_k}) - \sum_{i=1}^r 2 c_i d_i \right) = B(l).\]

**Proposition 32** (i) For \(l \in \mathcal{GL}\) we have \(\kappa_l = \kappa_l^{\mathcal{GL}} = (-1)^{\ell(l) - 1} q^{-N(|l|)} \lambda_l\).

(ii) For \(g \in \mathcal{G}\) we have \(\kappa_g = \kappa_g^{\mathcal{G}} \in \mathbb{Z}[q, q^{-1}]\).

**Proof** — Since \(r_l = \lambda_l E_l\) we have

\[\sigma(r_l) = (-1)^{\ell(l) - 1} q^{-N(|l|)} r_l = (-1)^{\ell(l) - 1} q^{-N(|l|)} \lambda_l E_l.\]
On the other hand
\[ \sigma(r_l) = \sum \sigma(e_i) = (-1)^{l(l)} q^{N(||l||)} \sum \lambda_i E_i. \]

Hence
\[ \kappa_l = (-1)^{l(l)} q^{-N(||l||)} \lambda_l = (-1)^{l(l)} q^{-N(||l||)} \lambda_l = \kappa_l, \]

which proves (i).

Let us prove (ii). As before, write \( A = \mathbb{Z}[q, q^{-1}] \) and consider the \( A \)-subalgebra \( U_A \) of \( U \) generated by the elements \( \Phi(e_i^{(k)}) \). It is known that \( \{E_g\} \) is an \( A \)-basis of \( U_A \) (1.2, 41.1.4). By construction, \( r_l \) is an iterated \( q \)-commutator of generators \( e_i \), thus it belongs to \( U_A \), and since \( \sigma \) preserves \( U_A \), we have \( \kappa_l \in A \). Finally, it follows from (i) and Equation (19) that \( \overline{\kappa_l} = \kappa_l \in A \).

**Lemma 33** For \( g \in G \) we have \( E_g = \sum_{h \geq g} \alpha_{gh}(q) \Phi(e_{\tau(h)}) \), where \( \alpha_{gg} = \kappa_g^{-1} \).

**Proof** — We have \( r_g = \sum_{h \geq g, h \in G} y_{gh} \Phi(e_h) \), with \( y_{gg} = 1 \). The result then follows from the relation \( \sigma(r_g) = \kappa_g E_g \). \( \square \)

4.7 We endow \( U \) with the nondegenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) obtained by transporting Kashiwara’s form on \( U_q(n) \) to \( U \) via \( \Phi \). It is known that the PBW-type bases of Lusztig are orthogonal (12, 38.2.3). More precisely we have for \( g = l(\beta_1)^{a_1} \ldots l(\beta_n)^{a_n} \) and \( h = l(\beta_1)^{b_1} \ldots l(\beta_n)^{b_n} \) where \( a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{N} \),
\[
(E_g, E_h) = \delta_{gh} \prod_{j=1}^{n} \frac{(E(\beta_j), E(\beta_j))^{a_j}}{\{a_j\}_{(\beta_j, \beta_j)^{a_j}!}},
\]
where for \( \beta = \sum_{i=1}^{r} c_i \alpha_i \in \Delta^+ \),
\[
(E(\beta), E(\beta)) = \prod_{i=1}^{r} \frac{(1 - q^{(\alpha_i, \alpha_i)}) c_i}{1 - q^{(\beta, \beta)}}
\]
and for \( m, p \in \mathbb{N} \),
\[
\{m\}_p! = \prod_{j=1}^{m} \frac{1 - q^{jp}}{1 - q^p}.
\]

Following Lusztig (10, 1.2.10) let us define another symmetric bilinear form \( \{\cdot, \cdot\} \) by setting
\[
\{u, v\} = \langle \overline{u}, \overline{v} \rangle, \quad (u, v \in U).
\]

**Lemma 34** For homogeneous elements \( u, v \in U \) of weight \( \nu \) we have \( \{u, v\} = q^{N(\nu)} (u, \tau(v)) \).

**Proof** — It is enough to check the lemma when \( u, v \) run through two bases of \( U \). Let us take \( u = m_g, v = \Phi(e_h), (g, h \in G) \). We have \( \overline{v} = v \) and writing \( m_g = g + \sum_{w < g, w \not\in G} \gamma(w(q)) w \), by Proposition 6
\[
\overline{u} = q^{-N(|g|)} \left( \frac{\tau(g)}{\gamma(w(q^{-1}) \tau(w))} \right).
\]

Hence, \( \{u, v\} = q^{N(|g|)} \delta_{gh} = q^{N(|g|)} (u, \tau(v)) \). \( \square \)
Proposition 35  The basis \( \{ r_g \} \) is orthogonal with respect to \( \langle \cdot, \cdot \rangle \).

Proof — It is known (L2 1.2.8) that \( \langle \tau(u), \tau(v) \rangle = \langle u, v \rangle \) for all \( u, v \in \mathcal{U} \). Hence we have
\[
\{ r_g, r_h \} = \langle \tau(r_g), \tau(r_h) \rangle = \langle \sigma(r_g), \sigma(r_h) \rangle = \kappa_g \kappa_h \langle E_g, E_h \rangle, \quad (g, h \in \mathcal{G}),
\]
and the result follows from the orthogonality of \( \{ E_g \} \) with respect to \( \langle \cdot, \cdot \rangle \).

Theorem 36  For \( g \in \mathcal{G} \), we have \( \max(r_g) = \max(E_g) = g \).

Proof — By the proof of Proposition 22 we have \( r_g = \sum_{k \geq g, k \in \mathcal{G}} y_{hk} \Phi(e_k), (g \in \mathcal{G}) \), where \( y_{gg} = 1 \). On the other hand \( \{ \Phi(e_k), m_h \} = q^{N(|k|)} \langle \Phi(e_{\tau(k)}), m_h \rangle = q^{N(|k|)} \delta_{kh} \). Hence, by Proposition 35 we have
\[
m_h = \sum_{g \in \mathcal{G}} \{ r_g, m_h \} r_g = \sum_{g \leq h, g \in \mathcal{G}} q^{N(|g|)} y_{gh} \frac{r_g}{\{ r_g, r_g \}},
\]
therefore
\[
r_h = q^{-N(|h|)} \{ r_h, r_h \} m_h + \sum_{g < h, g \in \mathcal{G}} z_{gh} m_g,
\]
for some \( z_{gh} \in \mathbb{Q}(q) \), and \( \max(r_h) = \max(m_h) = h \) for all \( h \in \mathcal{G} \). Finally, by Proposition 6, \( \max(u) = \max(\sigma(u)) \) for all \( u \in \mathcal{U} \), hence using Proposition 30 \( \max(E_g) = \max(r_g) = g \) for all \( g \in \mathcal{G} \).

5  Canonical bases

Kashiwara [K1] and Lusztig [L2] have introduced independently and by different methods a canonical basis \( B \) of \( U_q(n) \). Let \( B^* \) be the basis dual to \( B \) with respect to the scalar product of 2.2. In this section we study the image of \( B^* \) in the embedding \( \Phi : U_q(n) \to \mathcal{F} \).

5.1 The results of section 4 give an easy alternative proof of the existence of \( B \), as we shall now see. For \( g \in \mathcal{G} \) put \( M_g := \Phi(e_{\tau(g)}) \). Inverting the formula of Lemma 33 we get
\[
M_g = \sum_{h \in \mathcal{G}, h \geq g} \beta_{gh}(q) E_h,
\]
where \( \beta_{gg}(q) = \alpha_{gg}(q)^{-1} = \kappa_g \). Write
\[
\overline{E}_g = \sum_{h \in \mathcal{G}} a_{gh}(q) E_h, \quad (g \in \mathcal{G}).
\]

Since \( U_A = U_A \) and \( \{ E_g \} \) is an \( A \)-basis of \( U_A \), the coefficients \( a_{gh}(q) \) belong to \( A \).

Lemma 37  \( a_{gg}(q) = 1 \) for all \( g \in \mathcal{G} \), and \( a_{gh}(q) = 0 \) if \( g > h \).
Proposition 39

\[ b^*(g) = \sum_{h \geq k \geq g; k \in \mathcal{G}} \alpha_{gk}(q^{-1})\beta_{kh}(q). \]

Hence \( a_{gh}(q) = 0 \) if \( g > h \), and \( a_{gg}(q) = \alpha_{gg}(q^{-1})\beta_{gg}(q) = \alpha_{gg}(q)\beta_{gg}(q) = 1 \) by Lemma 33.

\[ \square \]

Let \( L \) be the \( \mathbb{Z}[q] \)-lattice spanned by \( \{ E_g \} \). It is well-known that Lemma 37 implies for all \( g \in \mathcal{G} \) the existence of a unique \( b_g \in L \) of the form

\[ b_g = E_g + \sum_{h \in \mathcal{G}, h > g} \gamma_{gh}(q) E_h, \tag{24} \]

such that \( \gamma_{gh}(q) \in q\mathbb{Z}[q] \) and

\[ \overline{b_g} = b_g, \tag{25} \]

(see for example [1, 1] 7.10). Clearly, \( \{ b_g \mid g \in \mathcal{G} \} \) is a \( \mathbb{Q}(q) \)-basis of \( \mathcal{U} \), a \( \mathbb{Z}[q, q^{-1}] \)-basis of \( \mathcal{U}_A \), and a \( \mathbb{Z}[q] \)-basis of \( L \).

By Equations (20), (21) we have \( (E_g, E_g)_{(q=0)} = 1 \), hence by (24)

\[ (b_g, b_h)_{(q=0)} = \delta_{gh}. \tag{26} \]

It is easy to see that (up to sign) there is a unique \( \mathcal{A} \)-basis of \( \mathcal{U}_A \) satisfying (25) and (26) (see [2] 14.2). Therefore, although the basis \( \{ E_g \} \) depends on the choice of a total order on the set \( \Pi \) of simple roots of \( g \), the basis \( \{ b_g \} \) is independent of this order. This is the image under \( \Phi \) of the canonical basis \( \mathcal{B} \).

Equation (24) yields the next proposition, which is needed for the proof of Theorem 40.

**Proposition 38** For any total order on \( \Pi \), the transition matrix from \( \{ E_g \} \) to \( \{ b_g \} \) is unitriangular, if one arranges its rows and columns in lexicographic order. \( \square \)

For types \( A, D, E \), a similar unitriangularity result was proved by Lusztig [1] for the PBW-type bases coming from the theory of Hall algebras. As noted in 4.5, the convex orderings of \( \Delta^+ \) coming from Lyndon coverings are in general different from those coming from Hall algebras. Hence, even for simply laced type, Proposition 38 is different from Lusztig’s result.

5.2 Denote by \( \{ E^*_g \} \) the basis of \( \mathcal{U} \) adjoint to \( \{ E_g \} \), and by \( \{ b^*_g \} \) the basis adjoint to \( \{ b_g \} \). These are the images under \( \Phi \) of the dual PBW-type basis and the dual canonical basis, respectively.

**Proposition 39** The vector \( b^*_g \) is characterized by the two following properties:

(i) \( b^*_g - E^*_g \) is a linear combination of vectors \( E^*_h \) with coefficients in \( q\mathbb{Z}[q] \);

(ii) The coefficients of the expansion of \( b^*_g \) on the basis of words are symmetric in \( q \) and \( q^{-1} \).

Proof — Clearly, \( b^*_g \) satisfies (i). For \( g, h \in \mathcal{G} \) we have

\[ (\sigma(b^*_g), b_h) = (\tau(b^*_g), b_h) = q^{-N(|g|)} \{ b^*_g, b_h \} = q^{-N(|g|)} \overline{b^*_g, b_h} = q^{-N(|g|)} \delta_{gh}. \]

Hence \( \sigma(b^*_g) = q^{-N(|g|)} b^*_g \) and, by Lemma 7, \( b^*_g \) satisfies (ii). Now if \( v \) is another element of \( \mathcal{U} \) satisfying (i), then \( v = b^*_g + \sum_{h \neq g} \gamma_h(q) b^*_h \) for some \( \gamma_h(q) \in q\mathbb{Z}[q] \). If moreover \( v \) satisfies (ii) then \( \gamma_h(q^{-1}) = \gamma_h(q) \), hence \( \gamma_h(q) = 0 \) for all \( h \in \mathcal{G} \), and \( v = b^*_g. \) \( \square \)
Theorem 40  We have \( \max(b^*_g) = g \) for all \( g \in \mathcal{G} \). Moreover, the coefficient of the word \( g \) in \( b^*_g \) is equal to \( \kappa_g \).

Proof — We have \( E_g = \sum_{h \in \mathcal{G}} (E_g, b^*_h) b_h \). By Proposition 38, \((E_g, b^*_h) = 0\) if \( g > h \), and \((E_g, b^*_g) = 1\). Hence
\[
b^*_h = E^*_h + \sum_{g \in \mathcal{G}; g < h} (E_g, b^*_h) E^*_g, \tag{27}
\]
and the results follow from
\[
E^*_g = \sum_{k \leq g} \beta_k(q)m_k = \kappa_g \kappa_g + \sum_{k < g} \beta_k(q)m_k.
\]
\(\square\)

Note that Theorem 40 holds for any of the \( r! \) different total orders on the set \( \Pi \) of simple roots of \( \mathfrak{g} \). The words \( w \) such that \( \beta^w = \max\beta^w \) for some total order on \( \Pi \) are similar to the extremal weights of the irreducible characters of a \( \mathfrak{g} \)-module.

Finally, we note the following obvious consequence of Proposition 38 and Corollary 27:

Corollary 41  For each \( l \in \mathcal{G} \) we have \( E^*_l = b^*_l \), that is, the root vector \( E^*_l \) belongs to the dual canonical basis.

\[\square\]

5.3  We have the following important positivity property of \( \Phi(B^*) \).

Theorem 42  Assume that \( \mathfrak{g} \) is of type \( A, D, \) or \( E \). For all \( g \in \mathcal{G} \), the coefficients \( D_w(b^*_g) \) of the expansion of \( b^*_g \) on the basis \( \{w \in \mathcal{M}\} \) of \( \mathcal{F} \) belong to \( \mathbb{N}[q,q^{-1}] \).

Proof — Let \( b = \Phi^{-1}(b_g) \in \mathbf{B} \) and \( b^* = \Phi^{-1}(b^*_g) \in \mathbf{B}^* \). For \( w = w[i_1, \ldots, i_k] \) with \( |w| = |g| \) we have
\[
D_w(b^*_g) = (\Phi(e_{\tau(w)}), b^*_g) = (\tau(e_{\tau(w)}), b^*).
\]
This is the coefficient of \( b \) in the \( \mathbf{B} \)-expansion of \( e_{i_k} \cdots e_{i_1} \). If \( \mathfrak{g} \) is of type \( A, D, E \), Lusztig has shown that these coefficients belong to \( \mathbb{N}[q,q^{-1}] \) (L2, 14.4.13).

\(\square\)

In the non-simply laced case, the structure constants of the multiplication on the canonical basis of \( U_q(n) \) need not be positive in general (see L2, 14.4.14). Nevertheless, the following conjecture is supported by extensive calculations.

Conjecture 43  For \( \mathfrak{g} \) of type \( B, C, F_4 \) or \( G_2 \) the coefficients \( D_w(b^*_g) \) of the expansion of \( b^*_g \) on the basis \( \{w \in \mathcal{M}\} \) of \( \mathcal{F} \) belong to \( \mathbb{N}[q,q^{-1}] \).

Note that the conjecture can easily be proved in type \( B_2 \), using the known fact that in this case all elements of \( \mathbf{B}^* \) are \( q \)-commutative monomials in 8 prime elements [RZ, C]. The images of these elements under \( \Phi \) are
\[
w[1], \ w[2], \ w[1,2], \ w[2,1], \ [2] w[1,1,2], \ [2] w[2,1,1], \ w[1,2,1], \ [2] w[2,1,1,2].
\]
Clearly, all \( q \)-shuffle monomials in these words have positive coefficients.

Following Lusztig [L3], define the variety \( N_{\geq 0} \) of totally nonnegative elements in \( N \) as the monoid generated by the \( x_i(t) \) (\( 1 \leq i \leq r, \ t \in \mathbb{R}_{\geq 0} \)). It follows from Proposition 10 that if all
the coefficients of the $M$-expansion of $\varphi(f)$ belong to $\mathbb{R}_{\geq 0}$, then $f$ takes nonnegative values on $N_{\geq 0}$. In particular, for simply-laced type, we recover by means of Theorem 42 the known fact that the functions of $\mathbb{C}[N]$ obtained by specializing at $q = 1$ the elements of $B^*$ take nonnegative values on $N_{\geq 0}$. This property is also true for non simply-laced type. Note however that one can easily find examples of functions $f$ which are nonnegative on $N_{\geq 0}$ while $\varphi(f)$ has some negative coefficients.

In fact, Proposition 42 and 43 show that all the coefficients of $\varphi(f)$ are nonnegative if and only if all the coefficients of the polynomial function $(t_1, \ldots, t_k) \mapsto f(x_{i_1}(t_1) \cdots x_{i_k}(t_k))$ are nonnegative for any sequence $(i_1, \ldots, i_k)$. Thus Conjecture 43 at $q = 1$ can be formulated in the following way: let $G$ be of non-simply laced type and let $f_b \in \mathbb{C}[N]$ denote the specialization at $q = 1$ of $b \in B^*$; then $f_b(x_{i_1}(t_1) \cdots x_{i_k}(t_k)) \in \mathbb{N}[t_1, \ldots, t_k]$ for all sequences $(i_1, \ldots, i_k)$. This has been proved by Berenstein and Zelevinsky in the case where $f_b$ is a generalized minor ([BZ] Th. 5.8).

5.4 Recall from 2.8.1 that $c_w^{-1}e_w$ denotes a monomial in the divided powers of the Chevalley generators. In this section, we shall say for short that $c_w^{-1}e_w$ is ‘a monomial’.

Lemma 44 (i) $\Phi(B^*)$ is an $A$-basis of $U^*_A$.
(ii) Suppose that the monomial $c_w^{-1}e_{r(w)}$ belongs to $B$, and let $b_g = \Phi(c_w^{-1}e_{r(w)})$. Then $w$ occurs in the $M$-expansion of $b_g^*$ with coefficient $c_w w$, and for all $h \neq g$, $w \notin \text{Supp}(b_h^*)$.

Proof — (i) It is known that $B$ is an $A$-basis of $U_A$. It follows that $B^*$ is an $A$-basis of $U^*_A$ and $\Phi(B^*)$ is an $A$-basis of $U^*_A = \Phi(U^*_A)$ (see Lemma 8).

(ii) We have $\delta_{g h} = (b_g, b_h^*) = c_w^{-1}(e_{r(w)}, b_h^*)$, and this is the coefficient of $w$ in the $M$-expansion of $c_w b_h^*$.

For $\nu \in Q^+$ set $B_\nu = \{ b \in B \mid |b| = \nu \}$, $B_\nu^* = \{ b^* \in B^* \mid |b^*| = \nu \}$, and

$$F_{\{|w| = \nu \}} = \bigoplus_{w \in M, |w| = \nu} \mathbb{N}[q, q^{-1}]c_w w, \quad U^*_\nu = \bigoplus_{g \in G, |g| = \nu} \mathbb{N}[q, q^{-1}]b_g^*.$$

We shall assume until the end of section 5.4 that $U^*_\nu \subset F_{\{|w| = \nu \}}$. (By Theorem 42 this holds for all $\nu$ in type $A$, $D$, $E$.)

Proposition 45 The following statements are equivalent:
(i) all elements of $B_\nu$, are monomials;
(ii) for every $b_g^* \in \Phi(B^*_\nu)$ there exists $w \in M$ such that $w$ occurs in the $M$-expansion of $b_g^*$ with coefficient $c_w$ and $w$ does not occur in any other $b_h^*$;
(iii) $U^*_\nu = U \cap F_{\{|w| = \nu \}}$.

Proof — By Lemma 44(ii), we have that (i) implies (ii). Conversely, if (ii) holds, the monomials $c_w^{-1}e_{r(w)}$ associated with each $g$ form a family of vectors adjoint to $B_\nu^*$, and (i) follows.

Suppose that (ii) holds, and let $v = \sum_{w \in M, |w| = \nu} \sigma_{vw} c_w w = \sum_{g \in G, |g| = \nu} \tau_{vg} b_g^*$ be an element of $U \cap F_{\{|w| = \nu \}}$. Then for each $g$ there exists a $w$ such that $\tau_{vg} = \sigma_{vw} \in \mathbb{N}[q, q^{-1}]$, and (iii) holds.
Suppose that (ii) does not hold. Then there exists \( g_0 \) such that every \( w \) occurring in the \( \mathcal{M} \)-expansion of \( b_{g_0}^* \) occurs also in the expansion of some \( b_{g_0}^* \) for some \( g_0 \neq g_0 \). Write

\[
b_g^* = \sum_{w \in \mathcal{M}} \varphi_{g_0w} w, \quad (g \in \mathcal{G}, |g| = \nu).
\]

Then, denoting by \( \delta \) the l.c.m. of all \( \varphi_{g_0w} \) for \( w \in \text{Supp}(b_{g_0}^*) \), and setting \( \delta_w := \delta \varphi_{g_0w} \varphi_w^{-1} \) we have that

\[
-\delta b_{g_0}^* + \sum_{w \in \text{Supp}(b_{g_0})} \delta_w b_{g_0}^*\]

belongs to \( \mathcal{U} \cap \mathcal{F}_\mathcal{N}(\nu) \) but not to \( \mathcal{U}^*_\mathcal{N}(\nu) \).

\[\Box\]

**Definition 46** We say that \( v \in \mathcal{U} \cap \mathcal{F}_\mathcal{N}(\nu) \) is indecomposable if (a) there exists no decomposition \( v = v_1 + v_2 \) with nonzero \( v_1, v_2 \in \mathcal{U} \cap \mathcal{F}_\mathcal{N}(\nu) \), and (b) there exists \( w \in \text{Supp}(v) \) whose coefficient is equal to \( c_w \).

Concretely, what Proposition 45 means is that \( \mathcal{B}_\nu \) consists only of monomials if and only if \( \Phi(\mathcal{B}_\nu^*) \) consists of all the indecomposable elements of \( \mathcal{U} \cap \mathcal{F}_\mathcal{N}(\nu) \).

It is well known that all elements of \( \mathcal{B} \) are monomials when \( g \) is of type \( A_2 \), hence in this case \( \Phi(\mathcal{B}^*) \) is precisely the set of all indecomposable elements of \( \mathcal{U} \cap \mathcal{F}_\mathcal{N}(\nu) \). In general, there are indecomposable elements which do not belong to \( \Phi(\mathcal{B}^*) \). It may also happen that some elements of \( \Phi(\mathcal{B}^*) \) are not indecomposable. It seems to be an interesting problem to understand which elements of \( \Phi(\mathcal{B}^*) \) are indecomposable.

**5.5** We now describe an algorithm to compute the basis \( \{b_g^*\} \). All calculations take place in the \( q \)-shuffle algebra \( (\mathcal{F}, \ast) \) and all vectors are expressed on the basis \( \mathcal{M} \) of words. We fix an arbitrary total order on \( \Pi \).

**5.5.1** The first step is to calculate the set \( \mathcal{GL} \) of good Lyndon words. For this we use 4.3

**5.5.2** For each \( l \in \mathcal{GL} \) we calculate \( r_l \) as an iterated \( q \)-bracket given by the co-standard factorization of \( l \). Then we obtain \( E_l^* \) by an appropriate normalization of \( r_l \). Namely, we have

\[
\kappa_l E_l^* = \frac{(-1)^{\ell(l)-1}}{q^{N(l)(l)}} (E_l, E_l) r_l, \quad (28)
\]

where \( (E_l, E_l) \) is given by (21). It remains to calculate \( \kappa_l \). By Theorem 40 we know that the coefficient of \( l \) in \( E_l^* \) is equal to \( \kappa_l^2 \). Hence the coefficient of \( l \) in (28) is equal to \( \kappa_l^2 \), and to get \( E_l^* \) we just need to divide (28) by the square root of its coefficient of \( l \).

**5.5.3** Let us fix a weight \( \nu \in Q^+ \). By Proposition 17 we can easily calculate the ordered list \( \{g_1 < \ldots < g_s\} \) of all good words of weight \( \nu \). Note that for a good word \( g = l_1^{a_1} \cdots l_k^{a_k} \) with \( l_1 > \cdots > l_k \in \mathcal{GL} \) we have

\[
E_g^* = q^{c_g} (E_{l_1}^*)^{*a_k} \cdots (E_{l_k}^*)^{*a_1}
\]

where \( c_g = \sum_{i=1}^k \binom{a_i}{2} d_i \) (this follows easily from (20) (21)). So we can compute \( E_{g_1}^*, \ldots, E_{g_s}^* \). By (27), we have \( b_{g_1}^* = E_{g_1}^* \). Suppose that for some \( t \leq s \) we have calculated \( b_{g_1}^*, \ldots, b_{g_{t-1}}^* \).
If all the coefficients of the expansion of $E_{g_{j}}^{s}$ on the basis of words are symmetric in $q$ and $q^{-1}$ then $b_{g_{j}}^{s} = E_{g_{j}}^{s}$. Otherwise let $g_{j}$ be the largest good word occurring in $E_{g_{j}}^{s}$ with a non-symmetric coefficient $\alpha \in \mathbb{Z}[q, q^{-1}]$. We know that the coefficient of $g_{j}$ in $b_{g_{j}}^{s}$ is $\kappa_{g_{j}}$, which is symmetric in $q$ and $q^{-1}$. The existence of $b_{g_{j}}^{s}$ implies that there exists $\gamma \in \mathbb{Q}[q]$ such that the coefficient $\rho$ of $g_{j}$ in $E_{g_{j}}^{s} - \gamma b_{g_{j}}^{s}$ is symmetric in $q$ and $q^{-1}$. Moreover if there were other coefficients $\gamma'$ and $\rho'$ satisfying the same properties, we would have $\alpha = \kappa_{g_{j}} \gamma + \rho = \kappa_{g_{j}} \gamma' + \rho'$, hence $\kappa_{g_{j}} (\gamma - \gamma') = \rho' - \rho$, with $\kappa_{g_{j}}, \rho' - \rho$ symmetric in $q$ and $q^{-1}$ and $\gamma - \gamma' \in \mathbb{Z}[q]$. This forces $\rho' - \rho = \gamma - \gamma' = 0$, therefore $\gamma$ is uniquely determined. If now all the coefficients of the expansion of $E_{g_{j}}^{s} - \gamma b_{g_{j}}^{s}$ on the basis of words are symmetric in $q$ and $q^{-1}$ then $b_{g_{j}}^{s} = E_{g_{j}}^{s} - \gamma b_{g_{j}}^{s}$, otherwise we apply the same procedure as above to $E_{g_{j}}^{s} - \gamma b_{g_{j}}^{s}$. After a finite number of steps we will obtain $b_{g_{j}}^{s}$.

5.5.4 Let us demonstrate the algorithm on an example. We choose $g$ of type $G_2$. Then

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$ Let us decide that $w_1 < w_2$. Then, the procedure of 4.3 gives immediately

$$\mathcal{GL} = \{w[1], w[1, 1, 1, 2], w[1, 1, 2, 1, 2], w[1, 1, 2, 1, 2], w[1, 2], w[2]\}.$$ Let us calculate for example the vector $E_{w[1, 1, 2, 2]}^{s}$. We have

$$r_{w[1, 1, 2, 2]} = r_{w[1, 2]} * r_{w[1, 2]} - q r_{w[1, 2]} * r_{w[1, 2]},$$

and by induction, we may assume that $r_{w[1, 2]}$ and $r_{w[1, 2]}$ are already known. Rescaling as indicated above we get

$$\kappa_{w[1, 1, 2, 2]} E_{w[1, 1, 2, 2]}^{s} = [2]_1^2 [3]_1^2 w[1, 1, 2, 1, 2] + [2]_1^2 [3]_1^2 [2]_3 w[1, 1, 1, 2, 2],$$

hence $\kappa_{w[1, 1, 2, 2]} = [2]_1 [3]_1$ and

$$E_{w[1, 1, 2, 2]}^{s} = [2]_1 [3]_1 w[1, 1, 2, 1, 2] + [2]_1 [3]_1 [2]_3 w[1, 1, 1, 2, 2].$$

The other root vectors are calculated similarly and one finds

$$E_{w[1, 1, 1, 2]}^{s} = [2]_1 [3]_1 w[1, 1, 1, 2], \quad E_{w[1, 1, 2]}^{s} = [2]_1 w[1, 2], \quad E_{w[1, 2]}^{s} = w[1, 2].$$

Let us calculate the dual canonical basis of the weight space corresponding to the highest root $\beta = 3\alpha_1 + 2\alpha_2$. The list of good words of weight $\beta$ in increasing order is

$$\mathcal{G}_{\beta} = \{w[1, 1, 2, 1, 2], w[1, 2, 1, 1, 2], w[1, 2, 1, 2, 1], w[2, 1, 1, 1, 2], w[2, 1, 1, 2, 1], w[2, 1, 2, 1, 1], w[2, 1, 2, 2]\}.$$ We have already calculated $b_{w[1, 1, 2, 1, 2]}^{s} = E_{w[1, 1, 2, 1, 2]}^{s}$. Next, we have

$$E_{w[1, 2]}^{s} = E_{w[1, 1, 2]}^{s} * E_{w[1, 2]}^{s}$$

$$= [2]_1 w[1, 2, 1, 1, 2] + q [2]_1 [3]_1 w[1, 1, 2, 1, 2] + q [2]_1 [3]_1 [2]_3 w[1, 1, 1, 2, 2].$$

Hence $b_{w[1, 2, 1, 2]}^{s} = E_{w[1, 2, 1, 2, 2]}^{s} - q b_{w[1, 1, 2, 1, 2], 2}^{s} = [2]_1 w[1, 2, 1, 1, 2].$ Next, we have

$$E_{w[1, 2, 1, 2, 1]}^{s} = q E_{w[1]}^{s} * E_{w[1, 2]}^{s} * E_{w[1, 2]}^{s}$$

$$= [2]_1 w[1, 2, 1, 2, 1] + q^2 [2]_1^2 w[1, 2, 1, 1, 2] + [2]_1 [2]_3 w[1, 1, 2, 1, 2]$$

$$+ ([2]_1 + q^4 [2]_1 [2]_3) w[1, 1, 2, 1, 2] + q^4 [2]_1 [2]_3 [3]_1 w[1, 1, 1, 2, 2].$$
hence
\[ b^*_{w[1,2,1,2,1]} = E^*_{w[1,2,1,2,1]} - q^2 \begin{bmatrix} 2 \end{bmatrix} b^*_{w[1,2,1,1,2]} - q^4 b^*_{w[1,1,2,1,2]} = \begin{bmatrix} 2 \end{bmatrix} w[1,2,1,2,1] + \begin{bmatrix} 2 \end{bmatrix} w[1,1,2,1,2] + \begin{bmatrix} 2 \end{bmatrix} [2]_3 w[1,1,2,1,2]. \]

In the same way one calculates
\[ b^*_{w[2,1,1,1,2]} = \begin{bmatrix} 2 \end{bmatrix} [3]_1 w[2,1,1,1,2], \]
\[ b^*_{w[2,1,1,2,1]} = \begin{bmatrix} 2 \end{bmatrix} w[2,1,1,2,1], \]
\[ b^*_{w[2,1,2,1,1]} = \begin{bmatrix} 2 \end{bmatrix} w[2,1,2,1,1] + \begin{bmatrix} 2 \end{bmatrix} w[1,1,2,1,2] + \begin{bmatrix} 2 \end{bmatrix} [2]_3 [1,2,2,1,1], \]
\[ b^*_{w[2,2,1,1,1]} = \begin{bmatrix} 2 \end{bmatrix} [3]_1 w[2,2,1,1,1] + \begin{bmatrix} 2 \end{bmatrix} [3]_1 w[2,1,2,1,1]. \]

6 Type A and q-characters of affine Hecke algebras

6.1 Let \( t \in \mathbb{C}^* \) be of infinite multiplicative order. Let \( H_m = H_m(t) \) be the algebra over \( \mathbb{C} \) generated by invertible elements \( T_1, \ldots, T_{m-1}, y_1, \ldots, y_m \) subject to the following relations:
\[
\begin{align*}
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, & (1 \leq i \leq m-2), \\
T_iT_j &= T_jT_i, & (|i-j| > 1), \\
(T_i - t)(T_i + 1) &= 0, & (1 \leq i \leq m-1), \\
y_iy_j &= y_jy_i, & (1 \leq i, j \leq m), \\
y_jT_i &= T_jy_i, & (j \neq i, i+1), \\
T_iy_iT_i &= ty_{i+1}, & (1 \leq i \leq m-1).
\end{align*}
\]

This is the Bernstein presentation of the affine Hecke algebra of \( GL(m) \).

6.2 Let \( M \) be a finite-dimensional \( H_m \)-module. Since the elements \( y_i \) are pairwise commutative, \( M \) decomposes as a sum of generalized eigenspaces
\[
M = \bigoplus_{\gamma} M[\gamma],
\]
where for \( \gamma \in \mathbb{C}^m \), we put
\[
M[\gamma] = \{ m \in M \mid \text{for all } i, (y_i - \gamma_i)^{n_i}m = 0 \text{ for some } n_i \in \mathbb{N}^* \}.
\]
The \( \gamma \) such that \( M[\gamma] \neq 0 \) are called the weights of \( M \). We will say that \( M \) is integral if all its weights are of the form \( \gamma = (t^{i_1}, \ldots, t^{i_m}) \) for some \( i_1, \ldots, i_m \in \mathbb{Z} \). In that case we shall write \( M[i_1, \ldots, i_m] \) in place of \( M[\gamma] \).

6.3 Let \( C_{m,r} \) denote the category of integral \( H_m \)-module with weights \((t^{i_1}, \ldots, t^{i_m})\) such that \( 1 \leq i_k \leq r \) for all \( k = 1, \ldots, m \). The character of \( M \) is defined by
\[
\text{ch } M = \sum_{1 \leq i_1, \ldots, i_m \leq r} \dim M[i_1, \ldots, i_m] w[i_1, \ldots, i_m].
\]
This is an element of \( \mathcal{F}_C \) (see [28]).
6.4 Let \( m = m_1 + m_2 \). The parabolic subalgebra \( H_{m_1,m_2} \) of \( H_m \) generated by
\[
T_1, \ldots, T_{m_1-1}, T_{m_1+1}, \ldots, T_{m-1}, y_1, \ldots, y_m,
\]
is isomorphic to \( H_{m_1} \otimes H_{m_2} \). Let \( M_1 \) and \( M_2 \) be a \( H_{m_1} \)-module and a \( H_{m_2} \)-module, respectively. The induction product \( M_1 \odot M_2 \) is the \( H_m \)-module defined by
\[
M_1 \odot M_2 = \text{Ind}_{H_{m_1,m_2}}^{H_m} M_1 \otimes M_2.
\]
If \( M_1 \) and \( M_2 \) are objects of \( C_{m_1,r} \) and \( C_{m_2,r} \), then \( M_1 \odot M_2 \) is an object of \( C_{m,r} \) and we have
\[
\text{ch} M_1 \odot M_2 = \text{ch} M_1 \bullet \text{ch} M_2,
\]
(29)
where \( \bullet \) is the classical shuffle product. This follows from a Mackey-type theorem for \( H_m \).

Let \( \mathcal{R} = \bigoplus_{m \in \mathbb{N}} \mathcal{R}_{m,r} \), where \( \mathcal{R}_{m,r} \) is the complexified Grothendieck group of \( C_{m,r} \) (by convention, we put \( \mathcal{R}_{0,r} = \mathbb{C} \)). The class in \( \mathcal{R} \) of a module \( M \) is denoted by \([M]\). The operation \( \odot \) induces in \( \mathcal{R} \) a multiplication \( \times \) that makes it into a \( \mathbb{C} \)-algebra. Note that \( \times \) is commutative: although \( M_1 \odot M_2 \) is in general not isomorphic to \( M_2 \odot M_1 \), their classes in \( \mathcal{R} \) coincide. Then
\[
\text{ch} : (\mathcal{R}, \times) \rightarrow (F_\mathbb{C}, \bullet)
\]
is a ring homomorphism.

6.5 For \( 1 \leq i \leq j \leq r \), let \( M_{[i,j]} \) be the 1-dimensional \( H_{j-i+1} \)-module on which the \( T_k \)'s act by multiplication by \( t \), and the \( y_k \)'s by multiplication by \( t^{k+i-1} \). It is known that \( \mathcal{R} \) is the polynomial ring over \( \mathbb{C} \) in the variables \([M_{[i,j]}] \) \((1 \leq i \leq j \leq r) \) [Z]. Now, \( \text{ch} M_{[i,j]} = w[i, \ldots, j] \). Therefore, \( \text{ch} \mathcal{R} \) is the subring of \((F_\mathbb{C}, \bullet)\) generated by the words \( w[i, \ldots, j] \) \((1 \leq i \leq j \leq r) \).

6.6 A multi-segment \( \mathbf{m} \) is a list of segments \( \mathbf{m} = ([i_1, j_1], \ldots, [i_k, j_k]) \) written in increasing order with respect to the following total order on segments:
\[
[i, j] < [k, l] \iff (i < k \text{ or } (i = k \text{ and } j < l)).
\]
Following Zelevinsky [Z], to \( \mathbf{m} \) we associate a standard induced module
\[
M_{\mathbf{m}} = M_{[i_1, j_1]} \odot \cdots \odot M_{[i_k, j_k]}
\]
and a simple module \( L_{\mathbf{m}} \) (see for example [Ro] or [LNT]).

Note that the words \( w[i, \ldots, j] \) \((1 \leq i \leq j \leq r) \) are the good Lyndon words for the root system \( A_r \), corresponding to the natural order \( w_1 < \cdots < w_r \), and the multi-segments \( \mathbf{m} \) are in one-to-one correspondence with the good words \( g \) by
\[
\mathbf{m} = ([i_1, j_1], \ldots, [i_k, j_k]) \iff g = w[i, \ldots, j, i_{k-1}, \ldots, j_k-1, \ldots, i_1, \ldots, j_1]. \quad (30)
\]

6.7 Let \( g = sl_{r+1} \) be the Lie algebra of type \( A_r \), and let \( U_q(n) \) be the corresponding quantum algebra. Choose the convex ordering \( \beta_1 < \cdots < \beta_n \) of \( \Delta^+ \) associated with the reduced decomposition
\[
w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_{r-1} \cdots s_2 s_1.
\]
It is easy to check that this is the same as the convex ordering coming from the good Lyndon words above, namely
\[
\alpha_i + \cdots + \alpha_j < \alpha_k + \cdots + \alpha_l \iff w[i, \ldots, j] < w[k, \ldots, l].
\]
The PBW-type basis of $U_q(\mathfrak{g})$ associated with this choice is thus conveniently labelled by multisegments $m = \sum_{1 \leq i, j \leq r} m_{ij} [i, j]$ where $m_{ij}$ denotes the multiplicity of the segment $[i, j]$. We shall write

$$E_m := \prod_{1 \leq i \leq j \leq r} E(\alpha_i + \ldots + \alpha_j)^{m_{ij}}$$

where the product is taken in the order given by the convex ordering above. We denote accordingly \{ $E^*_m$ \} and \{ $b^*_m$ \} the dual PBW-basis and dual canonical basis respectively. We have $\Phi(E^*_m) = E^*_g$ and $\Phi(b^*_m) = b^*_g$ where the correspondence between multisegments $m$ and good words $g$ is given by (30). Moreover, it is easy to check that $\Phi(E^*_{[i,j]}) = w[i, \ldots, j]$.

**6.8** Recall the setup of 2.8. Let $E^*_m \in \mathbb{C}[N]$ and $b^*_m \in \mathbb{C}[N]$ denote the specializations of $E^*_m$ and $b^*_m$ at $q = 1$. Then $E^*_{[i,j]} (1 \leq i \leq j \leq r)$ is just the coordinate function $t_{i,j+1}$ mapping a matrix $g$ to its entry $g_{i,j+1}$. It follows from 6.5 that $\mathbb{C}[N]$ is isomorphic as an algebra to $(R, \times)$. Let $\theta : \mathbb{C}[N] \longrightarrow R$ denote this isomorphism. By a dual version of Ariki’s theorem ($\mathbb{A}$, see also [LNT]), we have more precisely

$$\theta(E^*_m) = [M_m], \quad \theta(b^*_m) = [L_m]. \quad (31)$$

Consider the diagram

$$\begin{array}{ccc}
U^*_A & \xrightarrow{\Phi} & \mathcal{F}_A \\
\downarrow & & \downarrow \\
\mathbb{C}[N] & \xrightarrow{\varphi} & \mathcal{F}_C \\
\theta \swarrow & & \searrow ch \\
& & R
\end{array}$$

where the two vertical arrows denote specialization $q \mapsto 1$. For $m = ([i_1, j_1], \ldots, [i_k, j_k])$, we have

$$\varphi(E^*_m) = w[i_1, \ldots, j_1] \cup \cdots \cup w[i_k, \ldots, j_k] = chM_m,$$

hence the diagram is commutative. Therefore, for all multi-segments $m$,

$$chM_m = \Phi(E^*_m)_{\{q=1\}}, \quad chL_m = \Phi(b^*_m)_{\{q=1\}}.$$
and column $j$ of $\lambda/\mu$ to be $c = j - i$. Let $m = \sum_{i=1}^{j} \lambda_i - \sum_{i=1}^{k} \mu_i$. A standard Young tableau $T$ of shape $\lambda/\mu$ is a filling of the cells of $\lambda/\mu$ by the integers $1, 2, \ldots, m$, increasing on rows and columns. To $T$ and an integer $s$ we associate the word $w[T, s] := w[c_1 + s, \ldots, c_m + s]$, where $c_i$ denotes the content of the cell numbered $i$ in $T$.

These definitions are illustrated in Figure 1. Assume that $\lambda$ is such that $1 - \lambda_1 + r \geq j$ and let $s \in [j, 1 - \lambda_1 + r]$. To $(\lambda/\mu; s)$ we associate the good word $g(\lambda/\mu; s) = w[\mu_1 + s, \ldots, \lambda_1 - 1 + s, \mu_2 - 1 + s, \ldots, \lambda_2 - 2 + s, \ldots, \mu_j - j + 1 + s, \ldots, \lambda_j - j + s]$. This is the word obtained by reading the rows of $\lambda/\mu$ from left to right and bottom to top, the cells being filled by the contents shifted by $s$. (We assume that $\mu$ is made into a sequence of length $j$ by appending a tail of $j - k$ digits 0.)

**Corollary 48** We have

$$b_g^s(\lambda/\mu; s) = \sum_T w[T, s],$$

where $T$ runs through the set of all standard Young tableaux of shape $\lambda/\mu$.

**Proof** — To each choice of $\lambda/\mu$ and $s$ as above corresponds an irreducible $H_m$-module $L_m(\lambda/\mu; s)$ on which the generators $y_1, \ldots, y_m$ act semi-simply. The multi-segment $m(\lambda/\mu; s)$ is obtained from the good word $g(\lambda/\mu; s)$ by the correspondence (32). The character of $L_m(\lambda/\mu; s)$ is known to be given by the right-hand side of (32). Moreover, the generalized eigenspaces of $L_m(\lambda/\mu; s)$ are all 1-dimensional. Hence the $q$-character of $L_m(\lambda/\mu; s)$ coincides with its ordinary character, and the result follows from Theorem 47. \[\square\]

Corollary 48 may also be proved directly (i.e. without using the representation theory of $H_m$) by arguing as in Proposition 50 and Proposition 31 below.

## 7 Type $B$ and $q$-characters of affine Hecke-Clifford superalgebras

### 7.1 Let us take $g$ of type $B_r$. We choose the following numbering of the simple roots

$$1 \prec 2 \prec \cdots \prec r$$

and the standard ordering $w[1] < w[2] < \cdots < w[r]$. The set of good Lyndon words is calculated using (3.3) and we find

$$\mathcal{GL} = \{w[i, \ldots, j], 1 \leq i \leq j \leq r\} \cup \{w[1, \ldots, j, 1, \ldots, k], 1 \leq j < k \leq r\}.$$
Proof — These are straightforward calculations using the algorithm of 5.5.

7.2 Let \( \lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0) \) be a strict partition with \( \lambda_1 \leq r \). We set

\[
\nu_\lambda = \sum_{i=1}^{k} \alpha_1 + \cdots + \alpha_{\lambda_i} \in \mathbb{Q}^+ .
\]

We represent \( \lambda \) graphically by a shifted Young diagram. We define the content of the cell on row \( i \) and column \( j \) of a shifted Young diagram to be \( c = j - i + 1 \). A standard shifted Young tableau \( T \) of shape \( \lambda \) is a filling of the cells of the shifted diagram of \( \lambda \) by the integers \( 1, 2, \ldots, m = \sum_i \lambda_i \), increasing on rows and columns. To \( T \) we associate the word \( w[T] := w[c_1, \ldots, c_m] \), where \( c_i \) denotes the content of the cell numbered \( i \) in \( T \). These definitions are illustrated in Figure 2.

Finally, we associate to \( \lambda \) the good word \( g(\lambda) := w[1, \ldots, \lambda_1, 1, \ldots, \lambda_2, \ldots, 1, \ldots, \lambda_k] \).

**Proposition 50** We have

\[
b_{g(\lambda)}^* = \sum_T w[T],
\]  

where \( T \) runs through the set of all standard shifted Young tableaux of shape \( \lambda \).

Proof — Let \( \Lambda \in \mathfrak{h}^* \). Introduce the adjoint action twisted by \( \Lambda \) from \( U_q(g) \) to \( \text{End}U_q(n) \). It is defined by

\[
\text{Ad}_\Lambda(f_i)(x) = \frac{1}{q^{d_i} - q^{-d_i}} \left( q^{(\Lambda, \alpha_i)} x e_i - q^{-r(\Lambda, \alpha_i) + r(x, \alpha_i)} e_i x \right),
\]

\[
\text{Ad}_\Lambda(e_i)(x) = e'_i(x),
\]
for a homogeneous \( x \in U_q(n) \). It is well-known that \( Ad_\Lambda \) endows \( U_q(n) \) with the structure of a dual Verma module \( M(\Lambda)^* \) with highest weight \( \Lambda \). Moreover the dual canonical basis \( B^*_\Lambda \) of the irreducible submodule \( V(\Lambda) \) generated by the highest weight vector of \( M(\Lambda) \) becomes in this realization a subset of the dual canonical basis \( B^* \) of \( U_q(n) \). Let \( \Lambda_1 \) be the first fundamental weight, so that \( V(\Lambda_1) \) is the spin representation. This is a minuscule representation of dimension \( 2^n \) for which the canonical basis and the dual canonical basis coincide and are given by

\[
B_{\Lambda_1} = B^*_{\Lambda_1} = \{ Ad_{\Lambda_1}(f_{i_1} \cdots f_{i_k})(U_q(n)) \mid 1 \leq i_1, \ldots, i_k \leq r, 1 \leq k \leq r(r+1)/2 \} \setminus \{0\}.
\]

We are going to prove that \( \{ b^*_\nu(\lambda) \} = \Phi(B^*_{\Lambda_1}) = \{ S_\lambda \} \), where \( S_\lambda \) denotes the tableau sum of \( \lambda \).

It is well known that \( B^*_{\Lambda_1} \) has a natural indexation by the strict partitions \( \lambda \) as above, namely we write \( b^*_\lambda \) for the unique element of \( B^*_{\Lambda_1} \) of weight \( \lambda_1 - \nu_\lambda \). Then we have

\[
Ad_{\Lambda_1}(e_i)(b^*_\lambda) = \begin{cases} b^*_{(\lambda_1, \ldots, \lambda_j - 1, \ldots, \lambda_k)} & \text{if } \lambda_j = i \text{ and } \lambda_{j+1} \neq i - 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Using Theorem 5 it is easy to check that \( S_\lambda \in \Phi(U_q(n)) \). Indeed, due to the definition of a standard tableau, no factor of a word \( w[T] \) can be of the form

\[
w[i, i, i + 1], w[i, i + 1, i], w[i + 1, i, i], \quad (2 \leq i \leq r - 1)
\]

\[
w[i, i, i - 1], w[i, i - 1, i], w[i - 1, i, i], \quad (2 \leq i \leq r)
\]

\[
w[1, 1, 1, 2], w[1, 1, 2, 1], w[1, 2, 1, 1], w[2, 1, 1, 1],
\]

hence the only relations to check are those involving subwords of the type \( w[i, j] \) with \( |i - j| \geq 2 \), that is \( a_{ij} = 0 \). These relations are trivially satisfied, since they correspond to the exchange in a standard tableau \( T \) of two consecutive integers located in two cells which are neither in the same row nor in the same column.

Let us prove that \( S_\lambda = \Phi(b^*_\nu(\lambda)) \) by induction on \( |\lambda| = \lambda_1 + \cdots + \lambda_k \). This holds trivially for \( |\lambda| = 0 \). Now, it is clear from the definition of \( e'_i(\lambda) \) that we have

\[
e'_i(S_\lambda) = \begin{cases} S_{(\lambda_1, \ldots, \lambda_j - 1, \ldots, \lambda_k)} & \text{if } \lambda_j = i \text{ and } \lambda_{j+1} \neq i - 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Therefore, by induction, \( e'_i(S_\lambda) = \Phi(e'_i(b^*_\nu(\lambda))) \) for all \( i = 1, \ldots, r \). Hence \( \Phi^{-1}(S_\lambda) = b^*_\nu(\lambda) \) by 5. Finally, since \( \max(S_\lambda) = g(\lambda) \) we have \( S_\lambda = b^*_\nu(g(\lambda)) \).

More generally, we can consider skew shifted Young diagrams \( \lambda/\mu \), where \( \lambda = (\lambda_1 > \cdots > \lambda_j > 0) \) and \( \mu = (\mu_1 > \cdots > \mu_k > 0) \) are strict partitions with \( j \geq k \) and \( \lambda_i \geq \mu_i \) for \( i = 1, \ldots, k \). Then as above we define shifted standard Young tableaux \( T \) of shape \( \lambda/\mu \) and we associate to them a word \( w(T) \) obtained by reading the contents of \( \lambda/\mu \) in the order specified by \( T \). Finally we set \( g(\lambda/\mu) = w[\mu_1 + 1, \ldots, \lambda_1, \mu_2 + 1, \ldots, \lambda_2, \ldots, \mu_j + 1, \ldots, \lambda_j] \). Here we understand that \( \mu_i + 1, \ldots, \lambda_i \) is the empty string if \( \mu_i = 0 \) and \( \mu_i = 0 \) for \( i > k \).

The next proposition generalizes Proposition 50 to skew shifted Young diagrams.

**Proposition 51** We have

\[
b^*_g(\lambda/\mu) = \sum_T w[T],
\]

where \( T \) runs through the set of all standard shifted Young tableaux of shape \( \lambda/\mu \).
Proof — Recall the anti-automorphism $\tau$ of $\mathcal{F}$ defined in Proposition 6. It induces on $U_q(\mathfrak{n})$ the anti-automorphism (also denoted by $\tau$) which fixes the Chevalley generators $e_i$. We shall use the following known properties of $B^*$:

(a) $\tau(B^*) = B^*$;
(b) given $b^* \in B^*$ if we have $(e'_1)^k(b^*) \neq 0$ and $(e'_1)^{k+1}(b^*) = 0$, then $(e'_1)^{(k)}(b^*) \in B^*$.

Let $e'_1 = \tau \circ e'_1 \circ \tau$. Combining (a) and (b) we get:

(c) given $b^* \in B^*$ if we have $(e'_1)^k(b^*) \neq 0$ and $(e'_1)^{k+1}(b^*) = 0$, then $(e'_1)^{(k)}(b^*) \in B^*$.

Let us argue by induction on $|\mu| = \sum_{i} \mu_i$. If $|\mu| = 0$, the result is true by Proposition 50. Suppose now that the result holds for all $\lambda/\nu$ with $|\nu| = p$, and choose $\mu$ with $|\mu| = p + 1$. There exists a strict partition $\nu$ with $|\nu| = p$ contained in $\lambda$ such that we pass from $\lambda/\nu$ to $\lambda/\mu$ by erasing one cell situated at the left end of its row. Let $i$ be the content of this cell. Recall from the proof of Lemma 13 that in the shuffle realization $e'_1$ act as $e'_1$, that is by removing the first letter if it is equal to $w[i]$ and by zero otherwise. It is then easy to check that $(e'_1)^k$ applied to $b^*_{p(\lambda/\nu)}$ is zero for $k > 1$ and is equal to the right-hand side of (53) for $k = 1$. Thus the statement follows from (c).

7.3 In this section we propose a conjectural type $B$ analogue of Theorem 27.

7.3.1 Let $\mathcal{H}_m(t)$ denote the affine Hecke-Clifford superalgebra defined by Jones and Nazarov [1N] and further studied by Brundan and Kleshchev [BK]. We assume that the ground field is $\mathbb{C}$ and that the quantum parameter $t \in \mathbb{C}^*$ is not a root of 1.

We shall not write the full presentation of $\mathcal{H}_m(t)$, but only recall that it consists of even generators $T_1, \ldots, T_{m-1}, X_1, \ldots, X_m$ together with odd generators $C_1, \ldots, C_m$, and that the $X_i$ are invertible and pairwise commutative. Brundan and Kleshchev have introduced a class of finite-dimensional $\mathcal{H}_m(t)$-modules, called integral. These are the modules on which all eigenvalues of $X_1 + X_1^{-1}, \ldots, X_m + X_m^{-1}$ are of the form

$$t(i) = \frac{2i^{2i+1} + t^{-2i+1}}{t + t^{-1}}, \quad (i \in \mathbb{N}^+) .$$

Fix $r \geq 2$ and let $C_{m,r}$ denote the category of integral $\mathcal{H}_m(t)$-modules for which these eigenvalues belong to the finite subset $\{t(1), \ldots, t(r)\}$. Let $\mathcal{R} = \bigoplus_{m \in \mathbb{N}} \mathcal{R}_{m,r}$, where $\mathcal{R}_{m,r}$ is the complexified Grothendieck group of $C_{m,r}$. As in 6.4 $\mathcal{R}$ is endowed with a multiplication $\times$ coming from a modification $\otimes$ of parabolic induction appropriate to the superalgebra setting [BK].

There are $r$ irreducible modules $L(1), \ldots, L(r)$ in $C_{1,r}$, and they are all of dimension 2.

7.3.2 Recall the discussion of 22. Let $\xi_1, \ldots, \xi_r$ be the elements of $\mathbb{C}[N]$ obtained by specializing at $q = 1$ the Chevalley generators $e_1, \ldots, e_r$. Brundan and Kleshchev have proved that there exists an algebra isomorphism from $\mathbb{C}[N]$ to $\mathcal{R}$ which maps $\xi_i$ to $[L(i)]$. Moreover there is a natural labelling of the basis of $\mathcal{R}$ consisting of the classes of simple modules by the vertices of the crystal graph of $U_q(\mathfrak{n})$. Here, ‘natural’ means that the Kashiwara operators on the crystal correspond to taking the socle of the $i$-restriction of a simple module.

Brundan and Kleshchev have also introduced a notion of character for the integral modules. Let $M$ be a module in $C_{m,r}$, and let $M[i_1, \ldots, i_m]$ denote the generalized eigenspace of the pairwise commuting operators $X_1 + X_1^{-1}, \ldots, X_m + X_m^{-1}$ corresponding to the eigenvalues
Write $\delta(i_1, \ldots, i_m)$ for the number of occurrences of 1 in the list $(i_1, \ldots, i_m)$. Then
\[\text{ch } M = \sum_{1 \leq i_1, \ldots, i_m \leq r} 2^{|\delta(i_1, \ldots, i_m)/2| - m} \dim M[i_1, \ldots, i_m] w[i_1, \ldots, i_m].\]
This is an element of $F_C$. Moreover, as in 6.4, there holds $BK\left(\text{ch } (M_1 \otimes M_2) = \text{ch } M_1 \text{ ch } M_2.\right.$

**Conjecture 52** Let $b^* \in B^*$ be an element of principal degree $m$. The specialization at $q = 1$ of $\Phi(b^*)$ is the character of an irreducible integral $H_m(t)$-module.

This conjecture is supported by the calculations of Lemma 49, which agree with the character calculations of Brundan and Kleshchev (BK, 5-f), and by Proposition 50, which agrees with the known characters of the finite Hecke-Clifford superalgebras introduced by Olshanski (O). We believe that the $b^*_{\lambda/\mu}$ for $|\lambda/\mu| = m$ give the complete list of irreducible integral ‘tame’ characters of $H_m(t)$, i.e. the characters of the integral simple modules on which $X_1 + X_1^{-1}, \ldots, X_m + X_m^{-1}$ act semi-simply. Finally, (33) suggests that the representations of the affine Hecke-Clifford superalgebras corresponding to the vectors $b^*_w[i, \ldots, j]$ $(1 \leq i \leq j)$ and $b^*_w[1, \ldots, j, 1, \ldots, k]$ $(1 \leq j < k)$ should play the role of the ‘segment’ representations in the Zelevinsky classification of irreducible representations of affine Hecke algebras.

### 8 Good Lyndon words and root vectors

We give below the description of the root vectors $b^*_l = E^*_l$ for all root systems except $F_4$ and $G_2$, for the standard total ordering of $I$, that is, $w[1] < w[2] < \cdots < w[r]$. (For type $G_2$, see 5.5.4.) For types $A, B, C, D$ we provide a closed $q$-shuffle formula for the root vectors, and for types $A, D, E$ we give a simple combinatorial formula (Proposition 56).

For $l = w[i_1, \ldots, i_k] \in GL$, we write $b^*[i_1, \ldots, i_k]$ rather than $b^*_l$.

#### 8.1 Type $A_r$.

The simple roots are numbered as shown on the following Dynkin diagram:

```
1 2 3 4 5 6 7 8 9 10 11 12
```

The set of good Lyndon words is $GL = \{w[i, i+1, \ldots, j], 1 \leq i \leq j \leq r\}$, and the corresponding root vectors are
\[b^*[i, i+1, \ldots, j] = w[i, i+1, \ldots, j], \quad (1 \leq i \leq j \leq r),\]
as can be checked easily by induction on $j - i$, using formula (28) for $b^*_l = E^*_l$.

#### 8.2 Type $B_r$.

We choose the numbering

```
1 2 3 4 5 6 7 8 9 10 11 12
```

32
of the Dynkin diagram. The set of good Lyndon words is
\[ GL = \{ w[i, \ldots, j], \ 1 \leq i \leq j \leq r \} \cup \{ w[1, \ldots, j, 1, \ldots, k], \ 1 \leq j < k \leq r \}. \]

As in 8.1 we have
\[ b^*[i, \ldots, j] = w[i, \ldots, j], \quad (1 \leq i \leq j \leq r). \quad (36) \]

The other root vectors are given by

Lemma 53 For \( 1 \leq j < k \leq r \), one has
\[ b^*[1, \ldots, j, 1, \ldots, k] = [2]_1 w[1] \cdot (w[2, \ldots, j] \ast w[1, \ldots, k]), \]
where \( \ast \) denotes the concatenation product. If \( j = 1 \) we understand \( w[2, \ldots, j] = w[\].

Proof — By (36), \( w[1, \ldots, k] \) and \( w[2, \ldots, j] \) belong to \( U \). It follows that \( w[2, \ldots, j] \ast w[1, \ldots, k] \) also belongs to \( U \). By Theorem 5 we can see now that \( f = w[1] \cdot (w[2, \ldots, j] \ast w[1, \ldots, k]) \) belongs to \( U \). Indeed, we only have to check those equations (12) involving the first letter \( w[1] \) of all words occurring in \( f \), that is, those equations for which \( z = w[1], i = 1 \) and \( j = 2 \). Since there are only 2 occurrences of \( w[1] \) in each word and \( 2 < 1 - a_{12} = 3 \), there are in fact no new relations to check. It is easy to see that \( \max(f) = w[1, \ldots, j, 1, \ldots, k] \), hence, by Proposition 27, \( f \) is proportional to \( b^*[1, \ldots, j, 1, \ldots, k] \). Finally, we have to show that the proportionality factor \( \gamma \) is equal to \( [2]_1 \). Write \( l = w[1, \ldots, j, 1, \ldots, k] \) and let \( l = l_1 l_2 \) be the co-standard factorization. If, \( k = j + 1 \) then \( l_1 = w[1, \ldots, j] \) and \( l_2 = w[1, \ldots, k] \). Combining (28) with (36), we can calculate \( \gamma = [2]_1 \). If \( k > j + 1 \) then \( l_1 = w[1, \ldots, j, 1, \ldots, k - 1] \) and \( l_2 = w[k] \), so we can show by induction on \( k - j \) that \( \gamma = [2]_1 \).

8.3 Type \( C_r \). We choose the numbering

1 \[ \longrightarrow \quad 2 \quad \longrightarrow \quad 3 \quad \longrightarrow \quad \cdots \quad \longrightarrow \quad r \]

The set of good Lyndon words is
\[ GL = \{ w[i, \ldots, j], \ 1 \leq i \leq j \leq r \} \cup \{ w[1, \ldots, k, 2, \ldots, j], \ 1 \leq j \leq k \leq r \}. \]

As in 8.1 we have \( b^*[i, \ldots, j] = w[i, \ldots, j] \) (1 \( \leq i \leq j \leq r \)). The other root vectors are given by

the following lemma, whose proof is similar to that of Lemma 53 and will be omitted.

Lemma 54 For \( 2 \leq j \leq k \leq r \), one has \( b^*[1, \ldots, k, 2, \ldots, j] = w[1] \cdot (w[2, \ldots, j] \ast w[2, \ldots, k]) \).

8.4 Type \( D_r \). We choose the numbering

1
\[ \quad \longrightarrow \quad 3 \quad \longrightarrow \quad 4 \quad \longrightarrow \quad \cdots \quad \longrightarrow \quad r \]

2
The set of good Lyndon words is

\[ \mathcal{GL} = \{w[1]\} \cup \{w[1, 3, \ldots, i], \ 3 \leq i \leq r\} \cup \{w[i, \ldots, j], \ 2 \leq i \leq j \leq r\} \cup \{w[1, 3, \ldots, k, 2, \ldots, j], \ 2 \leq j < k \leq r\}. \]

As in 8.1, we have

\[ b^*[1, 3, \ldots, i] = w[1, 3, \ldots, i], \ (3 \leq i \leq r), \quad b^*[i, \ldots, j] = w[i, \ldots, j], \ (2 \leq i \leq j \leq r). \]

The other root vectors are given by

**Lemma 55** For \( 2 \leq j < k \leq r \),

\[ b^*[1, 3, \ldots, k, 2, \ldots, j] = w[1] \cdot (w[2, \ldots, j] \ast w[3, \ldots, k] - \lambda \ w[2, \ldots, k] \ast w[3, \ldots, j]), \]

where for \( j = 2 \), we understand \( w[3, \ldots, j] = w[\cdot] \).

**Proof** — The proof is similar to that of Lemma 53. First, we see as above that

\[ u = w[2, \ldots, j] \ast w[3, \ldots, k] - \lambda \ w[2, \ldots, k] \ast w[3, \ldots, j] \]

belongs to \( \mathcal{U} \). Secondly, one can check that all words occurring in \( u \) start with the letter \( w[3] \), since all words starting with \( w[2] \) cancel out. Moreover, no word starts with \( w[3, 3] \). Therefore, \( f = w[1] \cdot u \) also belongs to \( \mathcal{U} \) (no new relations to be checked) and the proof is concluded as above.

\[ \square \]

**8.5 Type E.** We choose the same numbering as in [LR]. The set \( \mathcal{GL} \) can then be read from the Lyndon paths in the \( E_8 \)-tree of [LR]. For example the good Lyndon word associated to the highest root of \( E_8 \) is

\[ w[1, 3, 4, 5, 6, 7, 8, 2, 4, 5, 6, 3, 4, 5, 2, 4, 3, 1, 3, 4, 5, 6, 7, 8, 2, 4, 5, 6, 7]. \]

**8.6** For types \( A, D, E \) we have the following combinatorial description of the root vectors attached to the good Lyndon words above.

Let \( \sim \) be the equivalence relation in \( \mathcal{M} \) defined by \( w \sim w' \) if only if \( w' \) can be obtained from \( w \) by a sequence of commutations of two adjacent letters \( w[i] \) and \( w[j] \) with \( a_{ij} = 0 \).

**Proposition 56** Let \( g \) be of type \( A_r, D_r \) or \( E_r \). For any \( l \in \mathcal{GL} \), we have \( b^*_l = \sum_{w \sim l} w \). In particular, all words occurring in \( b^*_l \) have coefficient 1, and \( \kappa_l = 1 \).

**Proof** — We see by inspection of the sets \( \mathcal{GL} \) given above for types \( A, D, E \) that no \( l \in \mathcal{GL} \) has a factor of the form

\[ w[i, i, j], \ w[i, j, i], \ w[j, i, i] \] (37)

with \( a_{ij} = -1 \). Moreover, for any \( i \) occurring more than once in \( l \), and any factor \( x = w[i] \cdot y \cdot w[i] \) of \( l \), we can check that \( y \) contains at least 2 letters \( w[j] \) and \( w[k] \) with \( a_{ij} = a_{ik} = -1 \). This implies that no \( w \) equivalent to \( l \) has a factor of the form (37). It follows that \( f_l = \sum_{w \sim l} w \) satisfies the equations of Theorem 5 and therefore belongs to \( \mathcal{U} \). Moreover, again by Theorem 5, any \( u \in \mathcal{U} \) in which the word \( l \) occurs with coefficient \( \gamma \) contains all words of \( f_l \) with the same
coefficient $\gamma$. Hence, $f_t$ is equal to the element $m_l$ of the basis \{m$_g$\} introduced in the proof of Proposition\[12\] and by Proposition\[27\] $f_t$ is proportional to $b^*_l$.

It remains to prove that the coefficient of proportionality is equal to 1. By \[23\], this amounts to prove that the coefficient of $l$ in

$$\frac{(-1)^{\ell(l) - 1}}{q^{N(l)}(E_l, E_l)} r_t = \frac{1}{(q - q^{-1})^{\ell(l) - 1}} r_t$$

is equal to 1 (in the simply laced case, we have $(E_l, E_l) = (1 - q^2)^{\ell(l) - 1}$ and $N(l) = 1 - \ell(l)$). To see this we proceed by induction on $\ell(l)$ and consider the co-standard factorization $l = l_1 l_2$ of $l$. Since $r_t = r_{l_1} * r_{l_2} - q^{\ell(l_1) \ell(l_2)} r_{l_2} * r_{l_1}$, we are reduced to prove that the coefficient of $l$ in $l_1 * l_2 - q^{\ell(l_1) \ell(l_2)} l_2 * l_1$ is equal to $q - q^{-1}$. To show this, it is enough by Proposition\[1\] to show that the coefficient of $l$ in $l_1 * l_2$ is equal to $q$. By \[9\] \[10\] this coefficient is equal to $q^{-\ell(l_1) \ell(l_2)}$, so all we have to prove is that $(l_1, l_2) = -1$.

For types $A$ and $D$ we see immediately from \[8.1\] and \[8.4\] that $l_2$ is always reduced to the last letter of $l$, and we can easily check that $(l_1, l_2) = -1$. For example in type $D_r$, if $l = w[1,3,\ldots,k,2,\ldots,j]$ with $2 \leq j < k \leq r$, we have

$$(l_1, l_2) = (\alpha_{j-1} + \alpha_j + \alpha_{j+1} + \alpha_{j-1}, \alpha_j) = -1 + 2 - 1 - 1 = -1.$$  

For type $E$, the equality $(l_1, l_2) = -1$ can be checked from the lists of good Lyndon words given in \[LR\]. (In most of the cases $l_2$ is reduced to the last letter of $l$ and the calculation of $(l_1, l_2)$ is very easy.) \hfill $\square$

We believe that Proposition\[56\] also holds for all other total orderings of the set of simple roots.

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