The cohomology ring of the sapphires that admit the Sol geometry

Sérgio Tadao Martins* and Daciberg Lima Gonçalves†

Institute of Mathematics and Statistics, University of São Paulo, Brazil

December 12, 2014

Abstract

Let \(G\) be the fundamental group of a sapphire that admits the Sol geometry and is not a torus bundle. We determine a finite free resolution of \(\mathbb{Z}\) over \(\mathbb{Z}G\) and calculate a partial diagonal approximation for this resolution. We also compute the cohomology rings \(H^*(G; A)\) for \(A = \mathbb{Z}\) and \(A = \mathbb{Z}_p\) for an odd prime \(p\), and indicate how to compute the groups \(H^*(G; A)\) and the multiplicative structure given by the cup product for any system of coefficients \(A\).

Keywords: torus semi-bundles; sapphires; Sol 3-manifolds; finite free resolution; diagonal approximation; cohomology ring

Mathematics Subject Classification 2010: primary: 20J06; secondary: 57M05; 57R19

1 Introduction

In this paper we consider the problem of calculating the cohomology ring of 3-manifolds that admit the Sol geometry. The 3-manifolds that admit this geometry are either torus bundles or torus semi-bundles. In [Mar13] we considered the problem of determining the cohomology ring of the torus bundles, so we now focus on the cohomology of torus semi-bundles.

A torus semi-bundle (also called a sapphire) is an orientable 3-manifold obtained from two twisted \(I\)-bundles \(K_1I\) and \(K_2I\) over the Klein bottle by identifying their boundaries, as described in [Mor85]. Using the notation from that article, let \(X = K(r, s, t, u)\) be the sapphire obtained when we identify the boundaries by a homeomorphism \(h: \partial K_2I \to \partial K_1I\) such that the induced isomorphism \(h_*: \pi_1(\partial K_2I) \to \pi_1(\partial K_1I)\) is given by \(h_*(\alpha_2) = r\alpha_1 + s\beta_1\) and \(h_*(\beta_2) = t\alpha_1 + u\beta_1\), where the meanings of the generators \(\alpha_i\) and \(\beta_i\) for \(i = 1, 2\) are explained in [Mor85]. If we let \(G = \pi_1(X)\), then the space \(X\) is a \(K(G, 1)\) space, so its cohomology coincides with the cohomology of \(G\).

Morimoto proved in [Mor85] that \(X\) is a torus bundle over \(S^1\) if, and only if, \(t = 0\). In [Mar13], a finite free resolution of \(G\) was constructed and also a diagonal approximation for that resolution was given when \(X\) is a torus bundle, and with those ingredients the cohomology rings \(H^*(G; \mathbb{Z})\) and

---

*Electronic address: sergiotm@ime.usp.br. The author has received financial support from FAPESP, processes 2013/07510-4 and 2013/21394-7

†Corresponding author. Electronic address: dligonal@ime.usp.br
$H^*(G; \mathbb{Z}_p)$ for a prime $p$ were computed. Moreover, the results obtained in \cite{Mar13} also allow us to calculate the cup product $H^p(G; M) \otimes H^q(G; N) \to H^{p+q}(G; M \otimes N)$ for nontrivial coefficients $M$ and $N$. As we are interested in the cohomology ring of $G$, we can exclude the case where $X$ is a torus bundle over $S^1$, so from now on we assume $t \neq 0$. Moreover, if the sapphire $X = K(r, s, t, u)$ admits the Sol geometry and is not a torus bundle, then we actually have $rstu \neq 0$ (see \cite{SWW10}), so that’s what we assume from now on.

The rest of the paper is organized as follows: in Section 2 we exhibit some properties of the group $G$, including some normal forms for its elements, and prove that the group ring $\mathbb{Z}G$ has no nontrivial zero divisors. In Section 3, we construct a finite free resolution $F$ of $\mathbb{Z}$ over $\mathbb{Z}G$, determine a partial diagonal approximation for $F$ and discuss how to compute cup products for arbitrary systems of coefficients. In Section 4 we show some examples of computations for the cohomology of $G$ with twisted integer coefficients and also determine the ring $H^*(G; \mathbb{Z}_p)$ when $p$ is an odd prime.

## 2 The group $G$ and the group ring $\mathbb{Z}G$

Morimoto showed \cite{Mor85} that the fundamental group $G$ of the torus semi-bundle $X = K(r, s, t, u)$ is given by the presentation

$$G = \langle a_1, b_1, a_2 \mid a_1 b_1 a_1^{-1} = b_1^{-1}, a_2^2 = a_1^2 b_1^s, a_2 a_1 b_1 a_2^{-1} = b_1^{-u} a_1^{-2t} \rangle \quad (1)$$

and from that he also got

$$H_1(G; \mathbb{Z}) = \begin{cases} \mathbb{Z}_{4t} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{if } s \text{ is even,} \\ \mathbb{Z}_{4t} \oplus \mathbb{Z}_4, & \text{if } s \text{ is odd.} \end{cases}$$

Since we are assuming that $t \neq 0$, $H_1(G; \mathbb{Z})$ is finite and it follows from the universal coefficient theorem for cohomology that $H^1(G; \mathbb{Z}) = 0$. The computation of $H^2(G; \mathbb{Z})$ follows from Poincaré duality (remembering that $X$ is orientable), which also gives us $H^3(G; \mathbb{Z}) = \mathbb{Z}$. We state this result formally in the next theorem for future reference.

**Theorem 2.1** The cohomology groups $H^*(G; \mathbb{Z})$ are given by

$$H^0(G; \mathbb{Z}) = \mathbb{Z},$$

$$H^1(G; \mathbb{Z}) = 0,$$

$$H^2(G; \mathbb{Z}) = \begin{cases} \mathbb{Z}_{4t} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{if } s \text{ is even,} \\ \mathbb{Z}_{4t} \oplus \mathbb{Z}_4, & \text{if } s \text{ is odd,} \end{cases}$$

$$H^3(G; \mathbb{Z}) = \mathbb{Z}.$$  

The above theorem tells us everything about the cohomology of $G$ with trivial integer coefficients, since $H^1(G; \mathbb{Z}) = 0$ and $\text{cd}(G) = 3$ imply that all the potentially nontrivial cup products are actually null. For other coefficients, however, we need to know more about $G$.

First of all we present two exact sequences featuring $G$. The reader can consult \cite{GW12} for more details. If we let $x = a_1^2$ and $y = b_1$, then $xy = yx$ and the subgroup $N$ of $G$ generated by
$x$ and $y$ is normal in $G$ and $Q = G/N = \langle \overline{a_1}, \overline{a_2} | \overline{a_1}^2 = \overline{a_2}^2 = 1 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$, so $G$ fits in an exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow 0.$$ 

The exact sequence above implies that every element $g \in G$ can be uniquely written as $g = wx^iy^j$, where $i$ and $j$ are integers and $w$ is a (possibly empty) word in the alphabet $\{a_1, a_2\}$ with alternating letters. This can be seen as a “normal form” for the elements of $G$. This normal form is used to define the map $s_0$ that appears just before Theorem 3.4.

If we let $v = a_1^{-1}a_2$, then the subgroup $L$ of $G$ generated by $x$, $y$ and $v$ is normal of index 2, for $L$ is the kernel of $\eta: G \to \mathbb{Z}_2 = \langle \overline{a_1} | \overline{a_1}^2 = 1 \rangle$ defined by $\eta(a_1) = \overline{a_1}$, $\eta(a_2) = \overline{a_2}$, $\eta(b_1) = 1$. It follows from the presentation (1) that

$$a_2xa_2^{-1} = x^{ru+st}y^{2su},$$

$$a_2ya_2^{-1} = x^{-2rt}y^{-ru-st},$$

hence

$$vxv^{-1} = x^{ru+st}y^{-2su},$$

$$vyv^{-1} = x^{-2rt}y^{ru+st}.$$ 

Therefore we have $L = \langle x, y \mid xy = yx \rangle \rtimes_\theta \langle v \rangle \cong (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_\theta \mathbb{Z}$, where $\theta: \mathbb{Z} \to \text{Aut}(\mathbb{Z} \oplus \mathbb{Z})$ is given by the matrix

$$\theta(1) = \begin{pmatrix} ru + st & -2rt \\ -2su & ru + st \end{pmatrix},$$

and so there is another exact sequence of groups

$$(\mathbb{Z} \oplus \mathbb{Z}) \rtimes_\theta \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}_2.$$ 

The matrix $\theta(1)$ satisfies $\det \theta(1) = 1$ and $|\text{tr}(\theta(1))| > 2$, which implies that $\theta(1)$ doesn’t admit $\pm 1$ as an eigenvalue.

The last exact sequence implies that every element of $G$ can be uniquely written as $x^my^nvy^k$ or $x^my^nvy^ka_1$ for $m$, $n$, $k \in \mathbb{Z}$, which is a second normal form for the elements of $G$. A third normal form is as follows: each $g \in G$ can be uniquely written either as $x^my^nvy^k$ or $x^my^nvy^ka_2$. To prove that, we note first that

$$a_1va_1^{-1} = a_1(v^2a_1^{-2})a_1$$

$$= a_1(vx^{-1})a_1$$

$$= a_1(x^{-ru-st}y^{2su})a_1$$

$$= (a_1x^{-ru-st}y^{2su})va_1$$

$$= x^{-ru-st}y^{-2su}(a_1v)a_1$$

$$= x^{-ru-st}y^{-2su}a_2a_1$$

$$= x^{-ru-st}y^{-2su}(x^ry^s)a_2^{-1}a_1$$

$$= x^{r-su-st}y^{-2su}v^{-1}.$$
Now, if we also denote by $\theta$ the map $N \to N$ given by $\theta(x) = x^{ru+st}y^{-2su}$, $\theta(y) = x^{-2rt}y^{ru+st}$, an element of the group $G$ of the form $g = x^m y^n v^k a_2$ can be written as

$$
g = x^m y^n v^k (a_1 v)
= x^m y^n v^k (x^{ru-st} y^{-2su} v^{-1} a_1)
= x^m y^n \theta^k (x^{ru-st} y^{-2su} v^{k-1} a_1).$$

Our claim now follows from our known previous normal form for the elements of $G$. Finally, we also note that there is a fourth normal form for the elements of $G$: each $g \in G$ can be uniquely written as $v^k x^m y^n$ or $v^k x^m y^n a_2$. This is consequence of our third normal form and of the fact that the matrix $\theta(1)$ is invertible ($\det \theta(1) = (ru - st)^2 = 1$). This last normal form will be used in the proof of Theorem 3.1.

Now we turn our attention to the group ring $\mathbb{Z}G$. Let the class $\mathcal{E}$ of the elementary amenable groups be the smallest subclass of the class of all groups that satisfies the following conditions:

(i) It contains all finite and all abelian groups;
(ii) If $G_1$ is an object of $\mathcal{E}$ and $G_2 \cong G_1$, then $G_2$ is an object of $\mathcal{E}$;
(iii) $\mathcal{E}$ is closed under the operations of taking subgroups, forming quotients and forming extensions;
(iv) $\mathcal{E}$ is closed under directed unions.

We also define the class $\mathcal{C}$ to be the smallest subclass of the class of all groups that contains the free groups and is closed under directed unions and also closed under forming extensions with elementary amenable groups as quotients. The two following results are found in [Luc02, Chapter 10]:

**Theorem 2.2** Let $\Gamma$ be a group such that $\Gamma$ belongs to $\mathcal{C}$ and there is an upper bound on the order of the finite subgroups of $\Gamma$. Then the strong Atiyah conjecture holds for $\Gamma$.

**Lemma 2.3** Let $F$ be a field with $\mathbb{Z} \subset F \subset \mathbb{C}$, and let $\Gamma$ be a torsion-free group. The group ring $F\Gamma$ has no nontrivial zero divisors if the Atiyah conjecture of order $\Lambda = \mathbb{Z}$ with coefficients in $F$ is true for $\Gamma$.

Since $\mathbb{Z}$ is free, the extensions

\[
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\
& & \longrightarrow \quad \mathbb{Z} \\
\mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{k} & L \\
& \xrightarrow{\phi} & \mathbb{Z} \leftarrow \text{elementary amenable} \\
L & \longrightarrow & G \\
& \longrightarrow & \mathbb{Z}_2 \\
\end{array}
\]

show that $G$ belongs to $\mathcal{C}$, therefore we have the following:

**Theorem 2.4** The group ring $\mathbb{Z}G$ has no nontrivial zero divisors.
3 Free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

In this section we determine a finite free resolution $F$ of $\mathbb{Z}$ over $\mathbb{Z}G$, a partial diagonal approximation $\Delta$ for the resolution $F$, and discuss how to compute the cohomology ring of $G$ for arbitrary coefficients.

**Theorem 3.1** A free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ is given by

$$0 \longrightarrow F_3 \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where $F_0 = \mathbb{Z}G$, $F_1 = \mathbb{Z}G^3$, $F_2 = \mathbb{Z}G^3$ and $F_3 = \mathbb{Z}G$. More precisely, if we call $\alpha_1$, $\beta_1$ and $\alpha_2$ the generators of $F_1$ and $\rho_1$, $\rho_2$ and $\rho_3$ the generators of $F_2$, then the maps $d_1$, $d_2$, and $\varepsilon$ are given by

$$
\begin{align*}
\varepsilon(1) &= 1, \\
&d_1(\alpha_1) = a_1 - 1, \\
&d_1(\beta_1) = b_1 - 1, \\
&d_1(\alpha_2) = a_2 - 1, \\
&d_2(\rho_1) = (1 - b_1^{-1})\alpha_1 + (a_1 + b_1^{-1})\beta_1, \\
&d_2(\rho_2) = \frac{\partial a_1^{2\nu}}{\partial a_1} \alpha_1 + a_1^{-2b_1^{-1}u} \frac{\partial a_1^{2\nu}}{\partial a_1} \beta_1 + (-a_2 - 1)\alpha_2, \\
&d_2(\rho_3) = \left( a_2 \frac{\partial a_1^{2\nu}}{\partial a_1} + a_1^{-2b_1^{-1}u} \frac{\partial a_1^{2\nu}}{\partial a_1} \right) \alpha_1 + \\
&\quad + \left( a_2 a_1^{-2b_1^{-1}u} \frac{\partial b_1^{2\nu}}{\partial b_1} + b_1^{-u} \frac{\partial b_1^{2\nu}}{\partial b_1} \right) \beta_1 + (1 - a_1^{-2b_1^{-1}u})\alpha_2,
\end{align*}
$$

where the partial derivatives are the Fox derivatives.

**Proof:** Let $r_1 = a_1 b_1 a_1^{-1}b_1$, $r_2 = a_1^{2\nu} b_1^{2\nu} a_1^{-2\nu}$ and $r_3 = a_2 a_1^{2\nu} b_1^{2\nu} a_1^{-2\nu} b_1^{2\nu}$. From [II], we can write $G = \langle a_1, b_1, a_2 \mid r_1, r_2, r_3 \rangle$. Following [LS77] chapter 2, section 3, the sequence

$$F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

is exact. But $\text{cd}(G) = 3$ implies that $\ker(d_2)$ is projective (see [Bro82] Lemma VIII.2.1). We are going to prove that $\ker(d_2)$ is actually free and isomorphic to $\mathbb{Z}G$.

The element $X r_1 + Y r_2 + Z r_3$ belongs to $\ker(d_2)$ if, and only if,

$$
\begin{align*}
X(1 - b_1^{-1}) + Y \frac{\partial a_1^{2\nu}}{\partial a_1} + Z \left( a_2 \frac{\partial a_1^{2\nu}}{\partial a_1} + a_1^{-2b_1^{-1}u} \frac{\partial a_1^{2\nu}}{\partial a_1} \right) &= 0, \\
X(a_1 + b_1^{-1}) + Y a_1^{-2b_1^{-1}u} \frac{\partial b_1^{2\nu}}{\partial b_1} + Z \left( a_2 a_1^{-2b_1^{-1}u} \frac{\partial b_1^{2\nu}}{\partial b_1} + b_1^{-u} \frac{\partial b_1^{2\nu}}{\partial b_1} \right) &= 0, \\
Y(-a_2 - 1) + Z(1 - a_1^{-2b_1^{-1}u}) &= 0.
\end{align*}
$$

(6)
Let’s focus on the third equation \(Y(a_2 + 1) = Z(1 - x^{-t}y^{-u})\). Given an element \(g \in G\), write it in the normal form \(g = v^k x^m y^n a_2^\ell\), where \(\ell\) is 0 or 1. We have
\[
\begin{align*}
v^k x^m y^n(a_2 + 1) &= v^k x^m y^n a_2 + v^k x^m y^n, \\
v^k x^m y^n a_2(a_2 + 1) &= v^k x^m y^n + v^k x^m y^n a_2, \\
v^k x^m y^n (1 - x^{-t}y^{-u}) &= v^k x^m y^n - v^k x^m y^n, \\
v^k x^m y^n a_2(1 - x^{-t}y^{-u}) &= v^k x^m y^n a_2 - v^k x^m y^n + a_2^2.
\end{align*}
\]

The important point to notice is that in the four equations above, the exponent of \(v\) is the same on the left side and on both terms of the right side. This motivates the following definition: given \(W = \sum_{k,m,n} \phi_{k,m,n} v^k x^m y^n + \psi_{k,m,n} v^k x^m y^n a_2 \in \mathbb{Z}G\) and \(\ell \in \mathbb{Z}\), the \(\ell\)-component of \(W\) is the sum of all the terms of \(W\) where the exponent of \(v\) is equal to \(\ell\):
\[
\ell\text{-component of } W = \sum_{m,n} \phi_{\ell,m,n} v^\ell x^m y^n + \psi_{\ell,m,n} v^\ell x^m y^n a_2.
\]

Therefore, if \(Y(a_2 + 1) = Z(1 - x^{-t}y^{-u})\), then this equality remains valid if we substitute \(Y\) and \(Z\) by their respective \(\ell\)-components. So let
\[
\sum_{m,n} \alpha_{m,n} v^\ell x^m y^n + \beta_{m,n} v^\ell x^m y^n a_2
\]
and
\[
\sum_{m,n} \gamma_{m,n} v^\ell x^m y^n + \delta_{m,n} v^\ell x^m y^n a_2
\]
be the \(\ell\)-components of \(Y\) and \(Z\), respectively. Now \(Y(a_2 + 1) = Z(1 - x^{-t}y^{-u})\) implies
\[
\sum_{m,n} (\alpha_{m,n} + \beta_{(m-r),(n-s)}) v^\ell x^m y^n + (\alpha_{m,n} + \beta_{m,n}) v^\ell x^m y^n a_2 =
\]
\[
= \sum_{m,n} (\gamma_{m,n} - \gamma_{(m+t),(n+u)}) v^\ell x^m y^n + (\delta_{m,n} - \delta_{(m-t),(n-u)}) v^\ell x^m y^n a_2.
\]

From the above equality it follows that
\[
(\alpha_{m,n} + \beta_{m,n}) - (\alpha_{m,n} + \beta_{(m-r),(n-s)}) = (\delta_{m,n} - \delta_{(m-t),(n-u)}) - (\gamma_{m,n} - \gamma_{(m+t),(n+u)}) \iff \\
\beta_{m,n} - \beta_{(m-r),(n-s)} = (\delta_{m,n} - \delta_{(m-t),(n-u)}) - (\gamma_{m,n} - \gamma_{(m+t),(n+u)}),
\]
and therefore
\[
\sum_{k \in \mathbb{Z}} \beta_{(m+kt),(n+ku)} - \beta_{(m-r+kt),(n-s+ku)} = 0.
\]

The sum on the left side makes sense because only finitely many of the coefficients \(\beta_{m,n}, \gamma_{m,n}\) and \(\delta_{m,n}\) are distinct from zero. If we define \(B(m,n) = \sum_{k \in \mathbb{Z}} \beta_{(m+kt),(n+ku)}\), the above equality means that
\[
B(m,n) = B(m-r,n-s) = B(m-2r,n-2s) = \cdots = B(m-kr,n-ks) = \cdots,
\]
that is, $B(m,n) = B(m-kr,m-ks)$ for all $k \in \mathbb{Z}$. Since only finitely many $\beta_{m,n}$ are distinct from zero and $(r,s)$ and $(t,u)$ are linearly independent, we actually have $B(m,n) = 0$ for all $m$, $n$. A similar argument beginning with the identity

$$\alpha_{m,n} + \beta_{m,n} = \delta_{m,n} - \delta_{(m-t),(n-u)}$$

shows that $A(m,n) = \sum_{k \in \mathbb{Z}} \alpha_{(m+kt,n+ku)} = 0$ for all $m$, $n$. Now we notice that the conditions $A(m,n) = 0$ and $B(m,n) = 0$ for all $m$, $n \in \mathbb{Z}$ imply that the $\ell$-component of $Y$ can be factored as $Y'(1 - x^t y^u)$. The same then can be said for $Y$, so there is a $W \in \mathbb{Z}G$ such that $Y = W(1 - x^t y^u)$ and

$$Y(a_2 + 1) = Z(1 - x^{-t} y^{-u}) \iff W(1 - x^t y^u)(a_2 + 1) = Z(1 - x^{-t} y^{-u}) \iff W(a_2 - x^t y^u) = Z.$$

Here we have used the fact that $\mathbb{Z}G$ has no nontrivial zero divisors, so the cancellation law holds. Given $Y = W(1 - x^t y^u)$ and $Z = W(a_2 - x^t y^u)$, let’s show that there is exactly one $X$ that satisfies the first two equations of (6). It is clear that there is at most one such $X$. Let $Y_0 = 1 - x^t y^u$ and $Z_0 = a_2 - x^t y^u$, and suppose that $r > 0$, $t < 0$. In this case,

$$X_0 = \left( \sum_{k=t}^{r-t} a_k^r \frac{\partial b^s_k}{\partial b_1} + \sum_{k=t+1}^{r-t} a_k^r \frac{\partial b^s_k}{\partial b_1} + \sum_{k=t+1}^{r-t+1} a_k^r \frac{\partial b^s_k}{\partial b_1} + \sum_{k=t+1}^{r-t+1} a_k^r \frac{\partial b^s_k}{\partial b_1} \right) b_1 \quad (7)$$

is such that $X = X_0$, $Y = Y_0$ and $Z = Z_0$ is a solution of (6). In all the other cases, we can similarly determine $X_0$, but Theorem 1 of [Mor85] shows that there is no loss of generality in assuming $r > 0$ and $t < 0$. Hence ker$(d_2)$, as a $\mathbb{Z}G$-module, is free, isomorphic to $\mathbb{Z}G$ and generated by the element $X_0 \rho_1 + Y_0 \rho_2 + Z_0 \rho_3$. With this notation, the map $d_3: F_3 \to F_2$ is defined by $d_3(1) = X_0 \rho_1 + Y_0 \rho_2 + Z_0 \rho_3$. 

We now present a partial diagonal approximation for the free resolution of Theorem 3.1. In order to do that, the next two propositions proved by Handel (see [Han93]) tell us how we can calculate a diagonal approximation for a given free resolution, provided that we can find a contracting homotopy for it.

**Proposition 3.2 (Handel)** For a group $\Gamma$, let

$$\cdots \to C_n \to \cdots \to C_1 \to C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$. If $s$ is a contracting homotopy for the resolution $C$, then a contracting homotopy $\tilde{s}$ for the free resolution $C \otimes C$ of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ is given by

$$\tilde{s}_{-1}: C_0 \otimes C_0$$

$$\tilde{s}_{-1}(1) = s_{-1}(1) \otimes s_{-1}(1),$$

$$\tilde{s}_n: (C \otimes C)_n \to (C \otimes C)_{n+1}$$

$$\tilde{s}_n(u_i \otimes v_{n-i}) = s_i(u_i) \otimes v_{n-i} + s_{-1}\varepsilon(u_i) \otimes s_{n-i}(v_{n-i}) \quad \text{if } n \geq 0,$$
where \( s_{-1}\varepsilon : C_0 \to C_0 \) is extended to \( s_{-1}\varepsilon = \{ (s_{-1}\varepsilon)_n : C_n \to C_n \} \) in such a way that \((s_{-1}\varepsilon)_n = 0 \) for \( n \geq 1 \).

**Proposition 3.3 (Handel)** For a group \( \Gamma \), let

\[
\cdots \xrightarrow{d_n} C_n \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{} 0
\]

be a free resolution of \( \mathbb{Z} \) over \( \mathbb{Z}\Gamma \). Let \( B_n \) be a \( \mathbb{Z}\Gamma \)-basis for \( C_n \) for each \( n \geq 0 \) such that \( \varepsilon(b) = 1 \) for all \( b \in B_0 \), and let \( s \) be a contracting homotopy for this resolution \( C \). If \( \tilde{s} \) is the contracting homotopy for the resolution \( C \otimes C \) given by Proposition 3.2, then a diagonal approximation \( \Delta : C \to C \otimes C \) can be defined in the following way: for each \( n \geq 0 \), the map \( \Delta_n : C_n \to (C \otimes C)_n \) is given in each element \( \rho \in B_n \subset C_n \) by

\[
\Delta_0 = s_{-1}\varepsilon \otimes s_{-1}\varepsilon,
\]

\[
\Delta_n(\rho) = \tilde{s}_{n-1}\Delta_{n-1}d_n(\rho) \quad \text{if} \quad n \geq 1.
\]

**Remark:** Tomoda and Zvengrowski have used the propositions above to calculate the cohomology rings of some 4-periodic groups [TZ08].

**Remark:** Using the same notation of the previous propositions and using \([ \quad ]\) to denote cohomology classes, if \( u \in \text{Hom}_{\mathbb{Z}\Gamma}(C_p, M) \) and \( v \in \text{Hom}_{\mathbb{Z}\Gamma}(C_q, N) \), where \( M \) and \( N \) denote arbitrary \( \mathbb{Z}\Gamma \)-modules, the cup product \([u] \smile [v] \in H^{p+q}(\Gamma; M \otimes N)\) is the cohomology class of the homomorphism \((u \otimes v) \circ \Delta_{p,q}\), where \( \Delta_{p,q}\) denotes the composition of \( \Delta_{p+q}\) with the projection \( \pi_{p,q} : (C \otimes C)_{p+q} \to C_p \otimes C_q \). Hence, for the calculation of

\[
H^p(\Gamma; M) \otimes H^q(\Gamma; N) \xrightarrow{\smile} H^{p+q}(\Gamma; M \otimes N),
\]

we don’t need to discover the map \( \Delta_{p+q}\), but only \( \Delta_{p,q}\). We’ll use this fact later.

Notice that if we define the maps of abelian groups \( s_{-1} : \mathbb{Z} \to F_0 \) by \( s_0(g) = \frac{\partial g}{\partial a_1} + \frac{\partial g}{\partial b_1} + \frac{\partial g}{\partial a_2} \) whenever \( g \in G \) is written in the first normal form \( g = wx^iy^j \) mentioned in Section 2 then \( d_1s_0 + s_{-1}\varepsilon = \text{id}_{F_0}\). Now a simple application of the Propositions 3.2 and 3.3 determines a partial diagonal approximation for our projective resolution of \( \mathbb{Z} \) over \( \mathbb{Z}\Gamma \).

**Theorem 3.4** A partial diagonal approximation \( \Delta : F \to (F \otimes F) \) for the free resolution of The-
\[ \Delta_1: F_1 \to (F \otimes F)_1 \]
\[ \Delta_1(\alpha_1) = \alpha_1 \otimes a_1 + 1 \otimes \alpha_1, \]
\[ \Delta_1(\beta_1) = \beta_1 \otimes b_1 + 1 \otimes \beta_1, \]
\[ \Delta_1(\alpha_2) = \alpha_2 \otimes a_2 + 1 \otimes \alpha_2, \]
\[ \Delta_{1,1}: F_2 \to (F_1 \otimes F_1) \]
\[ \Delta_{1,1}(\rho_1) = b_1^{-1}\beta_1 \otimes b_1^{-1}\alpha_1 + \alpha_1 \otimes a_1\beta_1 - b_1^{-1}\beta_1 \otimes \beta_1, \]
\[ \Delta_{1,1}(\rho_2) = s_0 \left( \frac{\partial a_1^{2t}}{\partial a_1} \right) \otimes \alpha_1 + s_0 \left( \frac{a_1^{2t} \partial b_1}{\partial b_1} \right) \otimes \beta_1 - \alpha_2 \otimes a_2, \]
\[ \Delta_{1,1}(\rho_3) = s_0 \left( a_2 \frac{\partial b_1^{2t}}{\partial a_1} + a_1^{-2t}b_1^{-u} \frac{\partial a_1^{2t}}{\partial a_1} \right) \otimes \alpha_1 + \]
\[ + s_0 \left( a_2a_1^{2t} \frac{\partial b_1^{2t}}{\partial b_1} + b_1^{-u} \frac{\partial a_1^{2t}}{\partial b_1} \right) \otimes \beta_1 - s_0(a_1^{-2t}b_1^{-u}) \otimes a_2. \]

**Remark:** In the above expressions, the elements on which we need to compute \( a \) are already in the normal form \( wx^i y^j \) or can be written in that form using the relation \( a_i b_1 a_1^{-1} = b_1^{-1} \).

Suppose we are given a \( ZG \)-module \( A \). The problem of calculating the (co)homology groups \( H_*(G; A) \) and \( H^*(G; A) \) can be solved using our free resolution of \( Z \) over \( ZG \). If \( B \) is also a \( ZG \)-module, another problem is to determine the cap products
\[ H^1(G; A) \otimes H^1(G; B) \to H^2(G; A \otimes B) \]
and
\[ H^1(G; A) \otimes H^2(G; B) \to H^3(G; A \otimes B). \]

As mentioned in the second remark after Proposition 3.3, the product \( H^1(G; A) \otimes H^1(G; B) \to H^2(G; A \otimes B) \) can be computed using the map \( \Delta_{1,1} \) of Theorem 3.3.

In order to compute \( H^1(G; A) \otimes H^2(G; B) \to H^3(G; A \otimes B) \), we recall [Bro82, Chapter V and Section VIII.10] that there is a commutative diagram
\[ H^1(G; A) \otimes H^1(G; B) \to H^3(G; A \otimes B) \]
\[ 1 \otimes (\_ \cdot \_ \cdot z) \]
\[ \cong \]
\[ H^1(G; A) \otimes H^1(G; B) \to H^3(G; A \otimes B) \]
\[ \cong \]
\[ H^3(G; A \otimes B) \]

where \( z \) is a generator of \( H_3(G; Z) \cong Z \). In terms of the free resolution \( F \), the cap product \( H^1(G; A) \otimes H_1(G; B) \to H_0(G; A \otimes B) \) can be calculated from a diagonal approximation \( \Delta: F \to (F \otimes F) \) by the composition
\[ [\text{left}] \_{\text{Hom}_G(F_1, A) \otimes (F_1 \otimes G B)} \rightleftarrows [\text{right}] \text{Hom}_G(F_1, A) \otimes ((F \otimes F)_1 \otimes B) \]
\[ [\text{left}] \gamma \to [\text{right}] F_0 \otimes (A \otimes B) \]
and then taking cohomology classes, where $\gamma$ is given by $\gamma(u \otimes (x \otimes y \otimes n)) = (-1)^{\deg(u)\deg(x)}x \otimes u(y) \otimes n$ (see [Bro82, Section V.3] for more details). All we need to know about $\Delta$ to perform all the above computations is $\Delta_{0,1}$, which is a summand of $\Delta_1$ given by Theorem 3.3.

Therefore, in order to completely understand the multiplicative structure given by the cup product $H^*(G; A) \otimes H^*(G; B) \xrightarrow{\partial} H^*(G; A \otimes B)$, we must understand the isomorphism

$$\varphi_n = \sim z: H^n(G; A) \rightarrow H_{3-n}(G; A). \quad (10)$$

Following [Bro82, Section V.4], $H_*(G; \mathbb{Z}) \cong H_*(F \otimes G F)$ and there is a product

$$\text{Hom}_G(F, A) \otimes (F \otimes G F) \rightarrow F \otimes G A$$

given by

$$u \otimes (x \otimes y) \mapsto (-1)^{\deg(u)\deg(x)}x \otimes u(y) \quad (11)$$

that induces the cap product

$$H^p(G; A) \otimes H_q(G; \mathbb{Z}) \rightarrow H_{q-p}(G; A)$$

In particular, the isomorphism $\varphi_n$ can be calculated if we find an element $\zeta \in (F \otimes G F)_3$ such that $[\zeta] = z \in H_3(G; \mathbb{Z})$.

In the double complex $(F \otimes G F)$, the differential $\partial$ is given by

$$\partial_{p+q}: (F \otimes G F)_{p+q} \rightarrow (F \otimes G F)_{p+q-1}$$

$$\partial_{p+q}(x \otimes y) = \partial_{p,q}(x \otimes y) + \partial'_{p,q}(x \otimes y),$$

where $\partial_{p,q}(x \otimes y) = d_p(x) \otimes y$ and $\partial'_{p,q}(x \otimes y) = (-1)^{p}x \otimes d_q(y)$ for $x \in F_p$ and $y \in F_q$.

Since $F_3 = ZG$, the cokernel of $\partial'_{1,3}$ is isomorphic to $\mathbb{Z}$ via the map $\epsilon': F_0 \otimes G F_3 \rightarrow \mathbb{Z}$ defined by $\epsilon'(x \otimes 1) = \epsilon(x)$, where $\epsilon: ZG \rightarrow \mathbb{Z}$ is the augmentation. Similarly, $\text{coker}(\partial'_{1,2}) \cong \mathbb{Z}^3$, with the isomorphism $\epsilon''_{2}: F_0 \otimes G F_2 \rightarrow \mathbb{Z}^3$ defined in the obvious way using $\epsilon$.

The element $(1 \otimes 1) \in F_0 \otimes G F_3$ is such that $\epsilon'(1 \otimes 1) = 1 \neq 0$, hence $(1 \otimes 1) \notin \text{im}(\partial'_{1,3})$. On the other hand, the element $d_3(1) = X_0p_1 + Y_0p_2 + Z_0p_3 \in F_2$ satisfies $\epsilon(X_0) = \epsilon(Y_0) = \epsilon(Z_0) = 0$, therefore $\epsilon''_{2}(1 \otimes 1) = 0$, which means that $\partial_{0,3}'(1 \otimes 1) \in \text{im}(\partial'_{1,2})$. A simple diagram chasing now shows that there is $\zeta \in (F \otimes F)_3$ such that $\pi_{0,3}(\zeta) = 1 \otimes 1$ (where $\pi_{p,q}: (F \otimes F)_{p+q} \rightarrow F_p \otimes F_q$
denotes the projection) and \( \zeta \in (\ker(\partial_3) - \text{im}(\partial_4)) \), hence \( z = [\zeta] \in H_3(G; \mathbb{Z}) \) is nonzero. If there is an integer \( k \) and \( \omega \in (F \otimes F)_3 \) such that \( w = [\omega] \in H_3(G; \mathbb{Z}) \) satisfies \( z = kw \), then \( \varepsilon'_3 \pi_{0,3}(\zeta - k\omega) = 0 \iff 1 = k \cdot \varepsilon'_3 \pi_{0,3}(\omega) \Rightarrow k = \pm 1 \). Therefore \( z \) generates \( H_3(G; \mathbb{Z}) \) and is the same \( z \) that gives the isomorphism \( \varphi_n \) of (10). Now the formula (11) tells us that, for the computation of \( \varphi_3 \), we need \( \pi_{0,3}(\zeta) \), which is simply \((1 \otimes 1)\). To calculate \( \varphi_2 \), the same formula (11) requires \( \pi_{1,2}(\zeta) \). This can be determined in the following way: let \( \psi : ZG \to ZG \) denote the function \( \psi(\sum \alpha_g g) = \sum \alpha_g g^{-1} \). We have
\[
\partial''_{0,3}(1 \otimes 1) = 1 \otimes X_0 \rho_1 + 1 \otimes Y_0 \rho_2 + 1 \otimes Z_0 \rho_3
\]

In the group ring \( ZG \), the identities
\[
(gh - 1) = g(h - 1) + (g - 1),
\]
\[
(g^n - 1) = \frac{\partial g^n}{\partial g}(g - 1)
\]
for \( g, h \in G \) allow us to write each of the elements \( \psi(X_0) \), \( \psi(Y_0) \) and \( \psi(Z_0) \) as
\[
\psi(X_0) = X_{a_1}(a_1 - 1) + X_{b_1}(b_1 - 1) + X_{a_2}(a_2 - 1),
\]
\[
\psi(Y_0) = Y_{a_1}(a_1 - 1) + Y_{b_1}(b_1 - 1) + Y_{a_2}(a_2 - 1),
\]
\[
\psi(Z_0) = Z_{a_1}(a_1 - 1) + Z_{b_1}(b_1 - 1) + Z_{a_2}(a_2 - 1),
\]
and then
\[
\psi(X_0) \otimes \rho_1 + \psi(Y_0) \otimes \rho_2 + \psi(Z_0) \otimes \rho_3 = \partial'_{1,2}((X_{a_1} b_1 + X_{b_1} a_2) \otimes \rho_1 + (Y_{a_1} b_1 + Y_{b_1} a_2) \otimes \rho_2 + (Z_{a_1} b_1 + Z_{b_1} a_2) \otimes \rho_3).
\]

This means we can take
\[
\pi_{1,2}(\zeta) = (X_{a_1} b_1 + X_{b_1} a_2) \otimes \rho_1 + (Y_{a_1} b_1 + Y_{b_1} a_2) \otimes \rho_2 + (Z_{a_1} b_1 + Z_{b_1} a_2) \otimes \rho_3.
\]

(12)

4 Examples of computations

In this section we compute the cohomology groups \( H^*(G; \tilde{\mathbb{Z}}) \), where \( \tilde{\mathbb{Z}} \) represents a non-trivial \( G \)-module with underlying abelian group \( \mathbb{Z} \), show some examples of computation of the cup products for these twisted integer coefficients, and also determine the cohomology ring \( H^*(G; \mathbb{Z}_p) \) for an odd prime \( p \).

First we determine the groups \( H^*(G; \mathbb{Z}_n) \), where \( \mathbb{Z}_n \) stands for the \( G \)-module \( \mathbb{Z} \) determined by the homomorphism \( \eta : G \to \text{Aut}(\mathbb{Z}) = \{1, -1\} \). We begin the analysis with the maps \( \eta \) such that \( \eta(b_1) = -1 \) (notice that this implies that \( s \) is even).
**Theorem 4.1** Suppose that $\eta: G \to \text{Aut}(\mathbb{Z})$ is such that $\eta(b_1) = -1$. Then

\[
\begin{align*}
H^0(G; \mathbb{Z}_\eta) &= 0, \\
H^1(G; \mathbb{Z}_\eta) &= \mathbb{Z}_2, \\
H^2(G; \mathbb{Z}_\eta) &= \mathbb{Z}_2 \oplus \mathbb{Z}_2, \\
H^3(G; \mathbb{Z}_\eta) &= \mathbb{Z}_2.
\end{align*}
\]

**Proof:** We could compute the cohomology groups using the projective resolution of Theorem 3.1, but a spectral sequence argument readily computes them when the action $\eta$ is such that $\eta(b_1) = -1$. Let $N$ and $Q$ be the subgroups of $G$ mentioned in Section 2. As a $N$-module, $\mathbb{Z}_\eta$ is such that $x \cdot k = k$ and $y \cdot k = -k$ for all $k \in \mathbb{Z}_\eta$. A direct calculation using, for example, the free resolution of $thm{3.1}$, we have

\[
H^0(N; \mathbb{Z}_\eta) = 0, \quad H^1(N; \mathbb{Z}_\eta) = \mathbb{Z}_2 \quad \text{and} \quad H^2(N; \mathbb{Z}_\eta) = \mathbb{Z}_2.
\]

Hence the Lyndon-Hochschild-Serre spectral sequence associated with the extension

\[
N \longrightarrow G \longrightarrow Q
\]

is such that $E_2^{p,q} = H^p(Q; H^q(N; \mathbb{Z}_\eta))$ is zero for $q = 0$ and, if $q \in \{1, 2\}$,

\[
E_2^{p,q} = \begin{cases} 
\mathbb{Z}_2, & \text{if } p = 0, \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{if } p \geq 1.
\end{cases}
\]

Let’s plot the lines corresponding to $q = 1$ and $q = 2$ of the term $E_2$:

\[
\begin{array}{cccccccc}
(q = 2) & \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \cdots \\
\downarrow d_2^{0,2} & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow \\
(q = 1) & \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \cdots
\end{array}
\]

Since $E_3 = E_\infty$ and $H^n(G; \mathbb{Z}_\eta) = 0$ for $n \geq 4$, the maps $d_2^{p,0}: E_2^{p,0} \to E_2^{p+1,0}$ must be isomorphisms for $p \geq 1$. Also, using Poincaré duality, we see that $H^3(G; \mathbb{Z}_\eta) = H_0(G; \mathbb{Z}_\eta) \cong \mathbb{Z}_2$, since the action of $b_1 \in G$ on $\mathbb{Z}_\eta$ is non-trivial. Therefore the map $d_2^{0,2}$ must be injective and the theorem follows.

\[
\square
\]

The actions $\eta: G \to \text{Aut}(\mathbb{Z})$ that remain satisfy $\eta(b_1) = 1$. Hence, as we’ve already computed the cohomology groups of $G$ with trivial $\mathbb{Z}$ coefficients, we are left with three actions to consider. For all non-trivial actions $\eta: G \to \text{Aut}(\mathbb{Z})$, we have $H^0(G; \mathbb{Z}_\eta) = 0$ and $H^3(G; \mathbb{Z}_\eta) \cong H_0(G; \mathbb{Z}_\eta) \cong \mathbb{Z}_2$. Also, the groups $H^1(G; \mathbb{Z}_\eta)$ and $H^2(G; \mathbb{Z}_\eta)$ can be readily computed from the resolution of $thm{3.1}$ and we get the following result:

**Theorem 4.2** If $\eta_1: G \to \text{Aut}(\mathbb{Z})$ is the action given by $\eta_1(a_1) = \eta_1(b_1) = 1$, $\eta_1(a_2) = -1$, we get $H^1(G; \mathbb{Z}_{\eta_1}) = \mathbb{Z}_2$ and

\[
H^2(G; \mathbb{Z}_{\eta_1}) = \begin{cases} 
\mathbb{Z}_{2r} \oplus \mathbb{Z}_2, & \text{if } s \text{ is even}, \\
\mathbb{Z}_{4r}, & \text{if } s \text{ is odd}.
\end{cases}
\]
For the action $\eta_2 : G \to \text{Aut}(\mathbb{Z})$ defined by $\eta_2(a_1) = -1$, $\eta_2(b_1) = \eta_2(a_2) = 1$, we get $H^1(G; \mathbb{Z}_{\eta_2}) = \mathbb{Z}_2$ and

$$H^2(G; \mathbb{Z}_{\eta_2}) = \begin{cases} \mathbb{Z}_{2^n} \oplus \mathbb{Z}_2, & \text{if } s \text{ is even}, \\ \mathbb{Z}_{4u}, & \text{if } s \text{ is odd.} \end{cases}$$

Finally, if $\eta_3 : G \to \text{Aut}(\mathbb{Z})$ is the action given by $\eta_3(a_1) = -1$, $\eta_3(b_1) = 1$, $\eta_3(a_2) = -1$, we get $H^1(G; \mathbb{Z}_{\eta_3}) = \mathbb{Z} \oplus \mathbb{Z}_2$ and $H^2(G; \mathbb{Z}_{\eta_3}) = \mathbb{Z} \oplus \mathbb{Z}_s$.

**Example:** Let’s calculate the cup products

$$H^p(G; \mathbb{Z}_{\eta_1}) \otimes H^q(G; \mathbb{Z}_{\eta_2}) \xrightarrow{\cup} H^{p+q}(G; \mathbb{Z}_{\eta_1} \otimes \mathbb{Z}_{\eta_2}) \cong H^{p+q}(G; \mathbb{Z}_{\eta_3}).$$

Since the cohomology groups depend on the parity of $s$, let’s assume that $s$ is odd. In this case, we have $H^1(G; \mathbb{Z}_{\eta_1}) \cong H^1(G; \mathbb{Z}_{\eta_2}) \cong \mathbb{Z}_2$ and $H^2(G; \mathbb{Z}_{\eta_3}) = \mathbb{Z} \oplus \mathbb{Z}_s$, so the product

$$H^1(G; \mathbb{Z}_{\eta_1}) \otimes H^1(G; \mathbb{Z}_{\eta_2}) \xrightarrow{\cup} H^2(G; \mathbb{Z}_{\eta_3})$$

is zero. In order to compute $H^1(G; \mathbb{Z}_{\eta_1}) \otimes H^2(G; \mathbb{Z}_{\eta_2}) \xrightarrow{\cup} H^3(G; \mathbb{Z}_{\eta_3})$, we use $\mathbb{S}$ with $A = \mathbb{Z}_{\eta_1}$ and $B = \mathbb{Z}_{\eta_2}$:

$$\begin{array}{c}
H^1(G; \mathbb{Z}_{\eta_1}) \otimes H^2(G; \mathbb{Z}_{\eta_2}) \xrightarrow{\cup} H^3(G; \mathbb{Z}_{\eta_3}) \\
1 \otimes (\mathbb{S} \circ \varphi) \cong \\
H^1(G; \mathbb{Z}_{\eta_1}) \otimes H_1(G; \mathbb{Z}_{\eta_2}) \xrightarrow{\cup} H_0(G; \mathbb{Z}_{\eta_3})
\end{array}$$

One generator of $H^1(G; \mathbb{Z}_{\eta_1}) \cong \mathbb{Z}_2$ is the class of the map $\alpha_2^* \in \text{Hom}_G(F_1, \mathbb{Z}_{\eta_1})$ defined by $\alpha_2^*(a_1) = 0$, $\alpha_2^*(b_1) = 0$, and $\alpha_2^*(a_2) = 1$, and one generator of $H_1(G; \mathbb{Z}_{\eta_2}) \cong \mathbb{Z}_{4u}$ is the class of $(\beta_1 \otimes 1)$. Using $\mathbb{S}$, we get

$$[\varphi] \cup [\beta_1 \otimes 1] = 0,$$

hence the product $H^1(G; \mathbb{Z}_{\eta_1}) \otimes H^2(G; \mathbb{Z}_{\eta_2}) \xrightarrow{\cup} H^3(G; \mathbb{Z}_{\eta_3})$ is also null.

**Example:** Assuming $s$ odd, let’s compute the cup products

$$H^p(G; \mathbb{Z}_{\eta_3}) \otimes H^q(G; \mathbb{Z}_{\eta_3}) \xrightarrow{\cup} H^{p+q}(G; \mathbb{Z}_{\eta_3} \otimes \mathbb{Z}_{\eta_3}) \cong H^{p+q}(G; \mathbb{Z}).$$

In order to do that, we need representatives for the generating classes of the cohomology groups $H^1(G; \mathbb{Z}_{\eta_3})$, $H^2(G; \mathbb{Z}_{\eta_3})$, and $H^2(G; \mathbb{Z})$, which can be obtained using the free resolution $\mathbb{F}$. The group $H^1(G; \mathbb{Z}_{\eta_3}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ is generated by $[\alpha^*_2] \oplus [\alpha_2^*]$, where $[\alpha_2^*]$ generates a subgroup isomorphic to $\mathbb{Z}$ and $[\alpha_2^* + \alpha_2^*]$ generates a subgroup isomorphic to $\mathbb{Z}_2$. The group $H^2(G; \mathbb{Z}_{\eta_3}) \cong \mathbb{Z} \oplus \mathbb{Z}_s$ is generated by $[\rho^*_1 + \rho^*_3]$ and $[\rho^*_2]$, where $[\rho^*_1 + \rho^*_3]$ generates a subgroup isomorphic to $\mathbb{Z}$ and $[\rho^*_2]$ generates a subgroup isomorphic to $\mathbb{Z}_s$. The group $H^2(G; \mathbb{Z}) = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is generated by $[\rho^*_3]$ and $[\rho^*_1 + u\rho^*_3]$, where $[\rho^*_3]$ generates a subgroup isomorphic to $\mathbb{Z}_{4u}$ and $[\rho^*_1 + u\rho^*_3]$ generates a subgroup isomorphic to $\mathbb{Z}_4$. We also note that, in $H^2(G; \mathbb{Z})$, $[2\rho^*_1 + \rho^*_2 + 2u\rho^*_3] = 0$.

Using the map $\Delta_{1,1}$ of Theorem 3.4, we determine the cup products $H^1(G; \mathbb{Z}_{\eta_3}) \otimes H^1(G; \mathbb{Z}_{\eta_3}) \xrightarrow{\cup} H^2(G; \mathbb{Z})$:

$$[\alpha_2^*]^2 = 2[\rho^*_1 + u\rho^*_3],$$

$$[\alpha_2^*] \cup [\alpha_1^* + \alpha_2^*] = 2t[\rho^*_3] + 2[\rho^*_1 + u\rho^*_3],$$

$$[\alpha_1^* + \alpha_2^*]^2 = 2t[\rho^*_3] - 2(r - 1)[\rho^*_1 + u\rho^*_3].$$
Moreover, if \( \alpha \in \text{generator of } \mathbb{Z} \) according to (12). But the expression for \( Z \) hence \( \dim(\mathbb{Z}) \)
Theorem 4.3 Hillman [Hil14], but could also be recovered from our techniques.
\[
\begin{align*}
\alpha_1^* + \alpha_2^* & \sim [\rho_1^* + \rho_3^*] = 0, \\
[\alpha_1^* + \alpha_2^*] & \sim [\rho_2^*] = 0, \\
[\alpha_2^*] & \sim [\rho_3^*] = 0.
\end{align*}
\]
It remains to compute \([\alpha_2^*] \sim [\rho_1^* + \rho_3^*] \). The isomorphism \( \varphi_2 \) of (10) maps \([\rho_1^* + \rho_3^*] \) to
\[
\xi = [(X_{a_1} \alpha_1 + X_{b_1} \beta_1 + X_{a_2} \alpha_2) \otimes 1 + (Z_{a_1} \alpha_1 + Z_{b_1} \beta_1 + Z_{a_2} \alpha_2) \otimes 1],
\]
according to (12). But the expression for \( X_0 \) of (7) shows that \( X_{a_2} = 0 \). We also have
\[
\psi(Z_0) = (a_2^{-1} - 1) - (x^{-t} y^{-u} - 1)
= -\frac{\partial a_2^{-2t}}{\partial a_1} (a_1 - 1) - a_2^{-2t} \frac{\partial b_1^{-u}}{\partial b_1} (b_1 - 1) - a_2^{-1} (a_2 - 1),
\]
hence \( Z_{a_2} = -a_2^{-1} \). Now (11) shows that \([\alpha_2^*] \sim \xi \) is a generator of \( H_0(\mathbb{G}; \mathbb{Z}) \), hence \([\alpha_2^*] \sim [\rho_1^* + \rho_3^*] \) generates \( H^3(\mathbb{G}; \mathbb{Z}) \).

Now we determine \( H^*(\mathbb{G}; \mathbb{Z}_p) \) when \( p \) is an odd prime. The case \( p = 2 \) has been solved by Hillman [Hill1], but could also be recovered from our techniques.

**Theorem 4.3** The cohomology groups \( H^*(\mathbb{G}; \mathbb{Z}_p) \), for an odd prime \( p \), are given by
\[
\begin{align*}
H^0(\mathbb{G}; \mathbb{Z}_p) & \cong \mathbb{Z}_p, \\
H^1(\mathbb{G}; \mathbb{Z}_p) & \cong \begin{cases} 
\mathbb{Z}_p, & \text{if } p \mid t, \\
0, & \text{if } p \nmid t,
\end{cases} \\
H^2(\mathbb{G}; \mathbb{Z}_p) & \cong \begin{cases} 
\mathbb{Z}_p, & \text{if } p \mid t, \\
0, & \text{if } p \nmid t,
\end{cases} \\
H^3(\mathbb{G}; \mathbb{Z}_p) & \cong \mathbb{Z}_p.
\end{align*}
\]
Moreover, if \( p \mid t \), there is a generator \( \alpha \in H^1(\mathbb{G}; \mathbb{Z}_p) \) and a generator \( \beta \in H^2(\mathbb{G}; \mathbb{Z}_p) \) such that \( \alpha \sim \beta \) is a generator of \( H^3(\mathbb{G}; \mathbb{Z}_p) \), so
\[
H^*(\mathbb{G}; \mathbb{Z}_p) \cong \mathbb{Z}_p[\alpha, \beta] / (\alpha^2 = 0, \beta^2 = 0),
\]
where \( \dim(\alpha) = 1 \) and \( \dim(\beta) = 2 \).

**Proof:** The computation of \( H^0(\mathbb{G}; \mathbb{Z}_p) \) is immediate. Using Theorem 2.1 and the universal coefficient theorems for homology and cohomology, we can compute the other groups \( H^*(\mathbb{G}; \mathbb{Z}_p) \): from the short exact sequence
\[
\begin{align*}
&0 \rightarrow H_1(\mathbb{G}; \mathbb{Z}) \otimes \mathbb{Z}_p \rightarrow H_1(\mathbb{G}; \mathbb{Z}_p) \rightarrow \text{Tor}(\mathbb{Z}, \mathbb{Z}_p) \rightarrow 0,
&= 0.
\end{align*}
\]
we get \( H^2(G;\mathbb{Z}_p) \cong H_1(G;\mathbb{Z}_p) \cong H_1(G;\mathbb{Z}) \otimes \mathbb{Z}_p \cong \mathbb{Z}_{\gcd(p,t)} \) by Poincaré duality, and from

\[
0 \rightarrow \text{Ext}(H_0(G;\mathbb{Z}),\mathbb{Z}_p) \rightarrow H^1(G;\mathbb{Z}_p) \rightarrow \text{Hom}(H_1(G;\mathbb{Z}),\mathbb{Z}_p) \rightarrow 0
\]

we get \( H^1(G;\mathbb{Z}_p) \cong \mathbb{Z}_{\gcd(p,t)} \). Finally, Poincaré duality also gives us \( H^3(G;\mathbb{Z}_p) \cong H_0(G;\mathbb{Z}_p) \cong \mathbb{Z}_p \).

Thus, if \( p \nmid t \), there is no cup product to consider and we are done.

Suppose then that \( p \mid t \). From Theorem 3.1, it is easy to check that \( v = [\alpha_1^* + r\alpha_2^*] \) generates \( H^1(G;\mathbb{Z}_p) \cong \mathbb{Z}_p \) and \( w = [\alpha_2 \otimes 1] \) generates \( H_1(G;\mathbb{Z}_p) \cong \mathbb{Z}_p \). But now (9) and the fact that \( p \nmid r \) imply that \( v \smile w \) generates \( H_0(G;\mathbb{Z}_p) \). We get the statement of the theorem defining \( \alpha = v \), \( \beta = \varphi_2^{-1}(w) \) and observing that \( \alpha^2 = 0 \) since \( p \) is odd.

References

[Bro82] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York-Berlin, 1982. MR 672956 (83k:20002)

[GM] Daciberg Lima Gonçalves and Sérgio Tadao Martins, *Diagonal approximation and the cohomology ring of the fundamental groups of surfaces*, To appear in the European Journal of Mathematics, http://arxiv.org/abs/1404.0666, doi:10.1007/s40879-014-0031-3.

[GW12] Daciberg Gonçalves and Peter Wong, *Nielsen numbers of selfmaps of Sol 3-manifolds*, Topology Appl. 159 (2012), no. 18, 3729–3737. MR 2991947

[Han93] David Handel, *On products in the cohomology of the dihedral groups*, Tohoku Math. J. (2) 45 (1993), no. 1, 13–42. MR 1200878 (93m:20072)

[Hil14] J. A. Hillman, *The \( \mathbb{F}_2 \)-cohomology rings of \( \text{Sol}^3 \)-manifolds*, Bull. Aust. Math. Soc. 89 (2014), no. 2, 191–201. MR 3182653

[HSt97] Peter J. Hilton and Urs Stammbach, *A course in homological algebra*, second ed., Graduate Texts in Mathematics, vol. 4, Springer-Verlag, New York, 1997. MR 1438546 (97k:18001)

[LS77] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin-New York, 1977, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89. MR 0577064 (58 #28182)

[Lüc02] Wolfgang Lück, *\( L^2 \)-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002. MR 1926649 (2003m:58033)

[Mar13] Sérgio Tadao Martins, *Diagonal approximation and the cohomology ring of torus fiber bundles*, http://arxiv.org/abs/1307.0518 (2013).
[Mor85] Kanji Morimoto, *Some orientable 3-manifolds containing Klein bottles*, Kobe J. Math. 2 (1985), no. 1, 37–44. MR 811801 (87c:57011)

[SWW10] Hongbin Sun, Shicheng Wang, and Jianchun Wu, *Self-mapping degrees of torus bundles and torus semi-bundles*, Osaka J. Math. 47 (2010), no. 1, 131–155. MR 2666128 (2011f:57002)

[TZ08] Satoshi Tomoda and Peter Zvengrowski, *Remarks on the cohomology of finite fundamental groups of 3-manifolds*, 519–556. MR 2484716 (2009m:57037)

[Wal61] C. T. C. Wall, *Resolutions for extensions of groups*, Proc. Cambridge Philos. Soc. 57 (1961), 251–255. MR 0178046 (31 #2304)