Exactness of Conventional and Supersymmetric
JWKB Formulae
and
Global Symmetries of Stokes Graphs

Piotr Milczarski

Theoretical Physics Department II, University of Łódź,
Pomorska 149/153, 90-236 Łódź, Poland
e-mail: jezykmil@krysia.uni.lodz.pl

Abstract

It has been shown that the cases of the JWKB formulae in 1–dim QM quantizing the energy levels exactly are results of essentially one global symmetry of both potentials and their corresponding Stokes graphs. Namely, this is the invariance of the latter on translations in the complex plain of the space variable i.e. the potentials and the Stokes graphs have to be periodic. A proliferation of turning points in the basic period strips (parallelograms) is another limitation for the exactness of the JWKB formulae. A systematic analyses of a single-well class of potentials satisfying suitable conditions has been performed. Only ten potentials (with one or two real parameters) quantized exactly by the JWKB formulae have been found all of them coinciding (or being equivalent to) with the well-known ones found previously. It was shown also that the exactness of the supersymmetric JWKB formulae is a consequence of the corresponding exactness of the conventional ones and vice versa. Because of the latter two exactly JWKB quantized potentials have been additionally established. These results show that the exact SUSY JWKB formulae choose the Comtet at al form of them independently of whether the supersymmetry is broken or not. A close relation between the shape invariance property of potentials considered and their meromorphic structure on the x-plane is also demonstrated.

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1 Introduction

It has long been known that for some number of potentials in 1–dim quantization problems (and also in these cases of n-dim problems which can be reduced to 1–dim ones) their corresponding JWKB quantization formulae for energy levels (or some their generalizations or modifications, whilst in most solvable cases (i.e. in those for which their corresponding energy spectrum is known by other means) the JWKB quantization appears to be only approximate.

Evidently, the same solvable potentials which are quantized accurately by the JWKB formulae are also quantized exactly when the supersymmetric (SUSY) modification of the JWKB method is used. But in general the SUSY JWKB quantization formulae similarly to the standard JWKB ones do not provide accurate quantization conditions for energy levels in most solvable cases of potentials.

Of course there have been attempts of proving that these known cases of both the conventional and SUSY JWKB quantization formulae had to be accurate but from our point of view both these attempts suffered from arbitrary and erroneous conventions used to sum divergent series of necessary phases and one can easily convince oneself that some relevant parts of proofs in the theses mentioned are certainly incorrect (see also Sec.3 and Appendix 1).

A way to treat this problem properly is to use the Weierstrass infinite product representations for meromorphic functions which the quantized potentials really are. But then it appears that it is necessary to take into account properly not only the phases coming from the infinite proliferations of (complex) turning points but also the phases coming from other (exponential) factors present in the corresponding Weierstrass products (see Appendix 1).

Also a clear understanding of the exactness of the known (both conventional and supersymmetric) JWKB formulae seems to be still missing and they are considered to some extent as accidental. In particular, unknown are (possibly simple) criteria (different from a trivial direct comparison between JWKB results and exact energy spectrum known by other means) allowing us to conclude for which potentials the corresponding JWKB quantization formulae could be exact and for which certainly not. Having such criteria would be very important for applications of the JWKB method since it would relax us from appealing to other exact methods.

The following are the aims of this thesis:
1. To prove rigorously the exactness of the known JWKB formulae;

2. To clarify why the known JWKB quantization formulae are exact;

3. To clarify the observed relation between the simultaneous (in-)accuracy of the JWKB and the SUSY JWKB quantization formulae; and

4. To provide criteria allowing us to judge whether a given JWKB quantization formula can be accurate or not.

For the beginning let us note however that in general there is no a common meaning of what a JWKB approximation really is.

One of a typical way of considering it is just to get some standard semiclassical solution provided by the Schrödinger equation (SE) and valid in the considered domain. Next to truncate the corresponding infinite series on a given term. Such a truncation is then considered as an approximate (to a given order in $\hbar$) solution to SE in the domain chosen. In this way one gets (up to a $\hbar$-dependent constant) two types of the approximate semiclassical solutions (ASS) to SE correspondingly to two different signs of the classical momenta generating them. Then correspondingly to the domain considered the dominate ASS is chosen. If none of the two ASS’s is dominating a linear combination of both is considered.

Next, the ASS’s constructed in the above way in different domains are smoothly joined and the ASS obtained in this way and covering the total domain of importance is then considered as a given order semiclassical solution to the problem considered. Taking the lowest order of this approximation we get what is then called the JWKB approximation to the problem.

The above way of constructing of ASS has been described by Berry and Mount and by Maslov and Fedoriuk for the 1–dim cases and by an arbitrary but finite number of dimensions. In particular the first paper discusses the difficult (and unsolved satisfactorily) problem of joining smoothly the ASS’s across the so called Stokes lines on which they change their dominating character into a subdominant one (the so called 'connection problem').

From our point of view the above way of constructing ASS’s suffers on a complete ignorance of the exact solutions to SE which are approximated in this way and which in fact are unknown. This causes many confuses in applications of the method since together with the exact solutions also corresponding boundary conditions the solutions have to satisfy are ignored, too.

Because of that and similarly to that it has been done in earlier papers we shall be following here rather oppositely first starting with a particular set of well defined
solutions to SE satisfying a sufficient number of boundary conditions and having well defined semiclassical asymptotics (SA) as well. Since the latter property depends on domains the exact solutions are defined in we shall choose as such a set of them these having this property in a maximal way.

It has been shown [1, 13, 14, 15] that such a set of the solutions to SE really exists and is known as a set of fundamental solutions (FS). In this thesis the descriptions ’ASS’ and ’JWKB approximation’ shall be understood only just as the corresponding approximations to the fundamental solutions. This assumption has serious consequences for the form of the JWKB approximations which can differ seriously from the one described above. In particular, the presence of simple and second order poles in a considered potentials generate unavoidably changes in the corresponding JWKB formulae.

A set of the FS’s are accompanied by the so called Stokes graph (SG). Making use of the latter we get a uniform and systematic way of solving any interesting 1-dim problem both exactly and in the semiclassical limit [1, 13, 14, 15]. The main property of SG is that it takes into account global features of a given problem considered in the complex planes of variables entered the problem (i.e. a position variable, energy, the Planck constant, some potential parameter(s), etc.). It is just these global features determining global structures of corresponding Stokes graphs which allows us to justify all the known cases of exact JWKB formulae as well as to get an insight what decides that a given JWKB formula can be exact or not.

The fundamental solutions which play the main role in our approach have the following three basic properties:

1. They satisfy some definite and desired boundary conditions [13, 14];
2. They possess well defined semiclassical limits [13, 14]; and
3. The corresponding semiclassical series are Borel summed to the solutions themselves [16].

In fact FS’s are the unique solutions to a given SE which have the above three properties altogether [15].

The thesis is organized in the following way.

In the next section we summarized essentialities related to Stokes graphs and fundamental solutions.
In Sec. 3 we establish necessary symmetry conditions for potentials and their SG’s to ensure the corresponding JWKB quantization formulae to be exact. Using these conditions we perform a systematic analyses showing that among all potentials which satisfy them only eight of them provide us with the exact JWKB quantization formulae.

In Sec. 4 we describe the way of generalization of the results of the previous section invoking some our earlier results.

In Sec. 5 we argue that supersymmetric JWKB formulae cannot be exact if the conventional ones are not as such too and demonstrate why in all cases of the potentials of Sec. 3 quantized exactly by the corresponding JWKB formulae the SUSY forms of the latter have to be also exact. By direct calculations we find in all these cases the validity of the SUSY JWKB quantization formula in the form of Comtet et al. independent of that whether the considered superpotentials satisfy or break the supersymmetry conditions. The latter result is not however in a contradiction with that of Inomata et al. since our result concerns the exact JWKB quantization whilst that of the last authors is only the JWKB approximation.

We discuss also in this section the result of Dutt et al. and Barclay et al. that the SUSY JWKB formulae are exact for the shape invariant potentials and notice that all of them known as being shaped invariant under translational transformation are also quantized exactly by the JWKB formulae. According to that two more general theorems on the exactness of the SUSY and conventional JWKB quantizations of the shaped invariant potentials are formulated and proved.

In Sec. 6 we summarize our results and draw our conclusions.

2 Stokes graphs, fundamental solutions and quantization

We shall resume here the basic facts about Stokes graphs and fundamental solutions.

2.1 Stokes graphs

Consider the SE written in the following form:

\[ \Psi''(x, E, \lambda) - \lambda^2 q(x, E, \lambda)\Psi(x, E, \lambda) = 0 \]  

(2.1)
where: $\lambda = 2\hbar^{-2}$, $q(x, E, \lambda) = V(x, \lambda) - E$ and a potential $V(x, \lambda)$ is assumed to be a meromorphic function of $x$ and $\lambda$ with the following asymptotic behaviour for $\lambda \to +\infty$ ($\hbar \to 0$):

$$V(x, \lambda) \sim V_0(x) + \frac{1}{\lambda}V_1(x) + \frac{1}{\lambda^2}V_2(x) + \ldots$$ (2.2)

Together with $q(x, E, \lambda)$ we shall consider a function $\tilde{q}(x, E, \lambda) \equiv q(x, E, \lambda) + \delta(x, E, \lambda)/\lambda^2$ where $\delta(x, E, \lambda)$ behaves according to (2.2) when $\lambda \to +\infty$. The precise form of $\delta(x, E, \lambda)$ depends on types of singularities of $q(x, E, \lambda)$ in particular on whether the latter possesses simple or second order poles (see a discussion below).

Let $E$ be real and let $x_1, x_2, \ldots$, be roots of $\tilde{q}(x, E, \lambda)$ and $y_1, y_2, \ldots$ be its simple poles.

Some of them can therefore be real but the rest ones are complex and conjugated pairwise.

For each point $x_i, y_i, i = 1, 2, \ldots$, let us construct actions:

$$W^r_i(x, E, \lambda) = \int_{x_i}^{x} \sqrt{\tilde{q}(y, E, \lambda)} dy$$

and

$$W^p_i(x, E, \lambda) = \int_{y_i}^{x} \sqrt{\tilde{q}(y, E, \lambda)} dy$$

and associate with them a system of lines defined by the conditions:

$$\Re W^r_p(x, E, \lambda) = 0$$ (2.4)

These are Stokes lines (SL). To be a little bit more precise we call a Stokes line each connected set of points of the $x$-plane satisfying the conditions (2.4). A collection of all Stokes lines is called a Stokes graph.

If the $\Re$-operation in (2.4) is substituted by the $\Im$-one then the corresponding set of lines are called anti-Stokes lines (ASL). The two sets of lines are orthogonal to each other at all the points except the roots or poles of $\tilde{q}(x, E, \lambda)$. 

Let $z_k, k = 1, 2, \ldots$, be infinite points of the actions (2.3) i.e. the points where the integrals (2.3) diverge to infinity. They are created by poles of $\tilde{q}(x, E, \lambda)$ (including these at infinities of the $x$-plane) and, therefore, their positions coincides with these of the poles. A total number of infinite points is assumed to be even infinite.

The roots of $\tilde{q}(x, E, \lambda)$ and its simple poles are starting points for SL’s. Due to our assumption about the root multiplicity only three Stokes lines can emanate from each $x_i$. On the other hand only one SL can emerge from each simple pole. Each of SL’s starting from some root or simple pole: $1^0$ can end at some other root or simple pole, $2^0$ can end at the same root forming a loop around a second order pole, or $3^0$ runs to an infinity point $z_k$.

A domain $S_k$ containing a point $z_k$ and bounded by some Stokes lines emenating from the roots of $\tilde{q}(x, E, \lambda)$ is called a sector of SG. Therefore, there are at least as many sectors as the infinite points $z_k$. Typically the points $z_k$ collect a number of sectors which depends on a rate of increasing of the action when it approaches $z_k$’s. Sectors corresponding to finite $z_k$’s are also finite. The remaining ones extend to infinities of the $x$-plane. There are no roots $x_i$ and no simple poles $y_i$ inside $S_k$, but there are some at its boundary.

For the purposes of the thesis it is enough to know only the proper topology of relevant SG’s i.e. their precise metric structures can be ignored. Therefore to draw the corresponding SG’s it is sufficient for all the cases considered here to apply the following rules:

$1^0$ From each root of $\tilde{q}(x, E, \lambda)$ (all roots are assumed to be simple) emanate three SL’s and three ASL’s;

$2^0$ From each simple pole of $\tilde{q}(x, E, \lambda)$ emanates only one SL and only one ASL;

$3^0$ From each second order pole of $\tilde{q}(x, E, \lambda)$ emanate only SL’s or only ASL’s depending on whether the pole coefficient is real negative or real positive, respectively;

$4^0$ From each higher order pole ($n > 2$) of $\tilde{q}(x, E, \lambda)$ emanate $n - 2$ directions to which SL’s are tangent asymptotically;

$5^0$ Any SL and any ASL can have only single common point;

The above rules together with the asymptotic properties of the actions (2.3) for $x \to \infty$ as well as an analytic behaviour of SG’s on the parameters of $\tilde{q}(x, E, \lambda)$ the latter function can depend on allows us to draw any such SG considered in the thesis.
2.2 Fundamental solutions

To any given SG we can attach to each of its sectors $S_k$ a solution $\Psi_k$ to SE called a fundamental solution and having the following structure [2, 13, 14, 17]:

$$\Psi_k(x) = \tilde{q}^{-\frac{1}{4}}(x)e^{\sigma_k W_i(x)}\chi_k(x)$$  \hspace{1cm} (2.5)

where:

$$\chi_k(x) = 1 + \sum_{n \geq 1} \left[ -\frac{\sigma_k}{2\lambda} \right]^n \int_{z_k}^x dy_1 \int_{z_k}^{y_1} dy_2 \cdots \int_{z_k}^{y_{n-1}} \int_{z_k}^{y_n} \omega(y_1)\omega(y_2)\cdots\omega(y_n) \cdot$$

$$\cdot \left(1 - e^{-2\sigma_k \lambda (W_i(x) - W_i(y_1))} \right) \left(1 - e^{-2\sigma_k \lambda (W_i(y_1) - W_i(y_2))} \right) \cdots$$

$$\cdots \left(1 - e^{-2\sigma_k \lambda (W_i(y_{n-1}) - W_i(y_n))} \right)$$  \hspace{1cm} (2.6)

with

$$\omega(y) = \frac{\delta(y)}{q^4(y)} - \frac{1}{4} \frac{q''(y)}{q^4(y)} + \frac{5}{16} \frac{q''^2(y)}{q^4(y)}$$  \hspace{1cm} (2.7)

In the above formulae $x_i$ is some of the roots lying at the boundary of $S_k$ and $\sigma_k = \pm 1$ is chosen each time so as to ensure a sign of $\Re(\sigma_k W_i(x))$ to be negative for the whole sector $S_k$.

One of the conditions determining the function $\delta(x, E, \lambda)$ introduced earlier is to make all the multiple integrals in (2.6) convergent at their lower limits $z_k$. It appears that for the last reason this function has to be defined as non zero only if $z_k$ is a second order pole of the potential considered. The second reason appears when the FS is to be continued to a point being a simple or double pole for the potential. Namely, for both these cases we have to correct the potential always by the same term $\delta(x, E, \lambda) = (2(x - z_k))^{-2}$ at each simple or double pole of the potential. Of course, in the case of infinite number of these singularities the arising infinite series has to be sum into some function having them as its own simple and double poles. The $\delta$–terms correcting the potentials considered in the above way we shall call the Langer corrections.

We would like to stress at this moment that introducing the Langer corrections are *unavoidable* part of the FS constructions whenever it is necessary to take into account the presence of the simple and double poles in the potential.
However, there are still another reasons for which particular forms of \( \delta(x, E, \lambda) \) have to be considered (see the next sections).

If \( x \in S_k \) then all the integrations in (2.6) can be performed along so-called canonical paths for which the condition \( \Re(\sigma_k W_i(y_j) - \sigma_k W_i(y_{j+1})) \leq 0 \) is fulfilled for any two successive integration variables.

If \( x \) is any point such that the integrations in (2.6) can be performed along some canonical paths then it is called a canonical point. A collection of all canonical points corresponding to the solution \( \Psi_k \) is called a canonical domain. We denote the latter by \( D_k \).

\( \Psi_k \)'s have the following two properties in their corresponding \( D_k \)'s:

\begin{enumerate}
\item[a.] Their series (2.4) are uniformly convergent;
\item[b.] Their asymptotic expansions when \( \lambda \to +\infty \) are dominated by the first two factors in (2.5) whilst the third one approaches then unity.
\end{enumerate}

Additionally we have:

\begin{enumerate}
\item[c.] Every \( \Psi_k \) is Borel summable in some \( B_k(S_k \subset B_k \subset D_k) \) and \( \Psi_k \)'s are the only solutions to SE with this property.
\end{enumerate}

The two first factors mentioned in the property b. above are the ones which just constitute our JWKB approximation to \( \Psi_k \) we have talked about in the Introduction. But this approximation is valid only in the canonical domain \( D_k \) of \( \Psi_k \).

All the fundamental solutions are pairwise independent. But since they are the solutions of the second order linear ODE (2.1) each three of them are linearly dependent. If for some of such a triad, say, \( \Psi_i, \Psi_j, \Psi_k \), canonical domains corresponding to them have common points pairwise then coefficients of a respective linear relation connected them can all be calculated in the following way [13, 14, 17]:

\[
\Psi_i(x) = \alpha_{\frac{i}{j} \to k} \Psi_j(x) + \alpha_{\frac{i}{k} \to j} \Psi_k(x) \quad (2.8)
\]

where

\[
\alpha_{\frac{i}{j} \to k} = \lim_{x \to z_k} \frac{\Psi_i(x)}{\Psi_j(x)}, \quad \ldots \text{etc} \quad (2.9)
\]
and $x$ runs to $z_k$ or $z_j$ along canonical paths. The latter calculations allows us to get immediately corresponding JWKB approximations for the coefficients $\alpha_{i/j \rightarrow k}$.

### 2.3 Singularities of fundamental solutions

Loci of singularities of the fundamental solutions in the $x$-plane coincide with the ones of $q(x, E, \lambda)$ and their nature is governed by the general rules (see for example [18]). Therefore if $q(x, E, \lambda)$ is holomorphic then such is each fundamental solution. The singularities at zeros of $q(x, E, \lambda)$ as provided by the representations (2.5) - (2.7) are therefore only apparent i.e they mutually cancel when the corresponding sums are performed. Nevertheless, they are real singularities in each factor in (2.5) as well as in each integral in (2.6) where they can cause troubles with taking the integrals (see the next section).

Each simple pole $y_k$ of $q(x, E, \lambda)$ is a source of a logarithmic branch point singularity for each fundamental solution which close to $y_k$ behaves as $a(x - y_k) + b(x - y_k) \ln(x - y_k)$ with $b \neq 0$.

Each second order pole $z_k$ of $q(x, E, \lambda)$ generates in each solution to SE (2.1) a branch point of the form $a(x - z_k)^\alpha + b(x - z_k)^\beta$ if $\alpha - \beta$ is not an integer or the branch point of the form $a(x - z_k)^\alpha + b(x - z_k)^\beta \ln(x - z_k)$ in the opposite case. In the case of these fundamental solutions which have to vanish at $z_k$ a real part of at least one of the numbers $\alpha, \beta$ has to be positive (one of the coefficients $a$ and $b$ can vanish in the case when the real part of the corresponding $\alpha$ or $\beta$ is not positive).

Each higher order pole $z_k$ of $q(x, E, \lambda)$ generates a branch point which is simultaneously an essential singularity for each fundamental solution. The fundamental solutions which are defined in the sectors containing this $z_k$ have to vanish at $z_k$ in the corresponding sectors.

It is easy to verify that the behaviour of the FS’s (corrected if necessary by the Langer terms) around singular points of a potential considered is exactly such as described above sometimes being determined totally by their first two JWKB factors. However, let us stress it once again, that at the simple and second order poles this behaviour appears as a result of the Langer corrections.

On all the figures of SG’s drawn for the purposes of the thesis only cuts corresponding to the real branch points of FS’s are marked whilst the branch points which follows from the F–F representation of FS’s as given by (2.5)–(2.7) are completely ignored.

A set of all zeros $x_k$, simple poles $y_k$ and other poles $z_k$ of $q(x, E, \lambda)$ determine for a real
$E$ (and $\lambda$) in a unique way a possibility of constructing a corresponding set of fundamental solutions among which there should be these two of them which determine the corresponding problem of quantization. If some (or both) of these two solutions cannot be constructed because of some specific properties of $q(x, E, \lambda)$ then the corresponding quantization problem cannot be formulated i.e. supposed bound states do not exist. The latter possibility is uniquely related to a particular pattern of SG which should be drawn in such cases.

2.4 Quantization

A quantization of 1–dim quantum systems with the help of the fundamental solutions has been described in many earlier papers [1, 13, 14, 16]. Here we sketch only the procedure for the case of two real turning points $x_1$, $x_2$ whilst the rest of them are complex and conjugated pairwise (we assume $\tilde{q}(x, E, \lambda)$ and $E$ to be real). We assume also that our physical problem is limited to a segment $z_1 \leq x \leq z_2$ at the ends of which the potential has poles. In particular we can push any of $z_{1,2}$ (or both of them) to $\mp \infty$ respectively.

To write the corresponding quantization condition for energy $E$ and to handle simultaneously the cases of second and higher order poles we assume $z_1$ to be the second order pole and $z_2$ to be the higher ones.

It is also necessary to fix to some extent the closest environment of the real axis of the $x$–plane to draw a piece of SG sufficient to write the quantization condition. To this end we assume $x_3$ and $\bar{x}_3$ as well as $\bar{x}_4$ and $\bar{x}_4$ to be another four turning points and $z_3$ and $\bar{z}_3$ another two second order poles of $V(x, \lambda)$ closest the real axis. Then a possible piece of SG can look as in Fig.1.
Fig. 1 The SG corresponding to general quantization rule (2.10)

There is no a unique way of writing the quantization condition corresponding to the figure. Some possible three forms of this condition can be written as [10]:

$$\exp \left[ -\lambda \oint_{K} \tilde{q}^{\pm}(x, \lambda, E) \, dx \right] = -\frac{\chi_{1 \rightarrow 3}(\lambda, E)\chi_{2 \rightarrow 3}(\lambda, E)}{\chi_{1 \rightarrow 3}(\lambda, E)\chi_{2 \rightarrow 3}(\lambda, E)} = -\frac{\chi_{1 \rightarrow 4}(\lambda, E)\chi_{2 \rightarrow 3}(\lambda, E)}{\chi_{1 \rightarrow 3}(\lambda, E)\chi_{2 \rightarrow 4}(\lambda, E)}$$  \hspace{1cm} (2.10)

and $\chi_{k \rightarrow j}(\lambda, E); j = 1, 2, 3, 4$ are the coefficients (2.6) calculated for $x \rightarrow z_j$. The closed integration path $K$ is shown in Fig. 1. In the figure the paths $\gamma_{1 \rightarrow 3}, \gamma_{2 \rightarrow 3}$, etc., are the integration paths in the formula (2.6) whilst the wavy lines designate corresponding cuts of the $x$–Riemann surface on which all the FS are defined. The same conventions in designations are maintained on the remaining figures 2–17.

The above three forms (2.10) of the quantization condition are equivalent and can be substituted by another equivalent forms if the latter are still admitted by the SG considered. It means that in general we can choose between different forms of the conditions (2.10) according to our needs. In particular depending on the form of SG there are forms of (2.10) completely deprived of the JWKB phase factor on its LHS i.e. such forms are composed only from the $\chi$-factors of FS’s.
The condition \((2.10)\) is \textit{exact}. Its LHS has just the JWKB form. If we substitute each \(\chi_{k \to j}(\lambda, E)\) in \((2.10)\) by unity (which these coefficients approach when \(\lambda \to +\infty\)) we obtain the well-known JWKB quantization rule. But in this way the latter is in general only an approximation to \((2.10)\). It is not in the following three cases only:

1\(^{0}\) All \(\chi_{k \to j}(\lambda, E)\)'s in \((2.10)\) are really equal to (identical with) 1;

2\(^{0}\) They all cancel out mutually by some reasons;

3\(^{0}\) Both the above cases take place i.e. some of \(\chi_{k \to j}(\lambda, E)\)'s satisfy 1\(^{0}\) and some 2\(^{0}\).

The first case is very rare and the only known example of it is just the harmonic oscillator potential (see below). It needs in fact for a given \(\chi_{k \to j}(\lambda, E)\) to have a possibility to deform its integration path properly to make all the integrations in \((2.6)\) vanishing i.e. this condition demands some particular topology of turning points on the \(x\)-plane to happen.

The next one if not happens accidentally can take place due to a possible reality of the coefficients entering the formula \((2.10)\) (where the coefficients can appear in pairs with their complex conjugations dividing them) or due to some possible symmetry of the potential \(V(x, \lambda)\) relating the \(\chi\)-coefficients present in the formulae. We shall show in the next sections that the latter case is the main reason for all the known cases of the JWKB formulae which provide us with the exact quantization conditions. In fact the symmetry properties of the potential as well as a particular topology of its turning point distribution cooperating together are the most frequent way of the realization of the JWKB formula exactness.

### 3 Global symmetries of Stokes graphs and quantization

The case when all \(\chi_{j \to k}\) in the condition \((2.10)\) reduce to unity is exceptional and within the holomorphic potentials can contain only a single potential namely the harmonic one. (Note that for the holomorphic potentials we can always put \(\delta(x, E, \lambda) \equiv 0\)).

To see this we note that for the case to happen it is necessary to have possibilities to deform the integration paths in the formula \((2.6)\) for \(\chi\)'s so as to make all the integrations vanishing. The latter property can happen if the integration paths can be pushed out to infinity i.e. none of roots of \(q(x, E, \lambda)\) (which are branch points for the integrands in \((2.6)\)) can prevent such a deformation. It means that these roots cannot extend to infinity and that the corresponding SG for the case should look as in Fig.2 with the blob containing all the roots of the case. Secondly, a number of sectors has to be limited to four as it is shown in Fig.2.
since only then the condition (2.10) can contain only $\chi$’s with vanishing integrations. Next, a number of roots inside the blob has to be finite since $q(x, E, \lambda)$ would vanish identically in the opposite case. Therefore, $q(x, E, \lambda)$ has to be polynomial. It is just a harmonic one since for the polynomial potentials a number of sectors exceeds by two a degree of a potential.

![Fig.2 The case of SG satisfied only by the harmonic oscillator potential among the holomorphic ones](image)

Therefore, other possibilities for the condition (2.10) to be the pure JWKB one are related to possibilities for $\chi$’s in (2.10) to cancel mutually. Such cases to happen if not accidental have to be related to some symmetries of corresponding SG’s the latter being determined by relevant symmetries of underlying potentials.

Suppose therefore $q(x, E, \lambda)$ to satisfy the following symmetry relation:

$$q(y(x), E, \lambda) = q(x, E, \lambda)$$

(3.1)

for $x \to y(x)$. We shall assume that it is always possible to find (if necessary) $\delta(x, E, \lambda)$ such that (3.1) is satisfied by $\tilde{q}(x, E, \lambda)$ as well under the same transformation. If the corresponding SG is also to be invariant under such a transformation then the full set of the
actions (2.3) which define this SG has to be invariant too, up to multiplicative constants. The latter freedom follows from the conditions (2.4) defining SL’s. But according to (3.1) we have:

$$\int_{x_k}^{x} \sqrt{q(\xi, E, \lambda)} d\xi = \int_{x_k}^{x} \sqrt{q(y(\xi), \lambda, E)} d\xi = \int_{y(x_k)}^{y(x)} \sqrt{q(\xi, E, \lambda)} \frac{dx}{dy}(\xi) d\xi$$ \hspace{1cm} (3.2)

from which we can conclude that the mentioned action set invariance is achieved if \(y'(x) = C\), where \(C\) is real.

Therefore the allowed transformations \(y(x)\) are linear. Since they constitute a group then it is easy to see that if \(|C| \neq 1\) then \(q(x, E, \lambda)\) has to have common accumulation points of their roots and poles. Because of that we shall limit our further considerations to less singular cases of \(q(x, E, \lambda)\) what means that we shall put \(C = \pm 1\). The latter limitation leaves us with only two types of the allowed symmetry transformations: the one which is essentially a reflection \(x \rightarrow -x\) and the other being a complex translation of the \(x\)-plane.

Therefore the two resulting classes of potentials remaining invariant under the above two symmetry transformations are: a class of even potentials and a class of periodic potentials. Of course both the classes are not necessarily disjoint. It is, however, rather clear that the evenness of a potential alone is too week to ensure overall cancellations in (2.10) and it is just rather a periodicity of it which can work effectively to cause the relevant cancellations in (2.10) to happen if it is possible at all. We shall show this below.

### 3.1 Periodic holomorphic (entire) potentials

In general \(q(x, E, \lambda)\) as a meromorphic function of complex \(x\) can be periodic with at most two independent (in general complex) periods \([19]\). However, in the case of being holomorphic \(q(x, E, \lambda)\) can have only one period (being a constant in the presence of the second one). Further, since \(q(x, E, \lambda)\) is assumed to be real for its real arguments then its period can be only real or only pure imaginary. For the obvious reason we shall consider only the last case assuming for simplicity the period to be equal to \(2\pi i\). In this case \(q(x, E, \lambda)\) can be expanded into the following Fourier series \([19]\):

$$q(x, E, \lambda) = \sum_{n=-\infty}^{\infty} q_n(E, \lambda)e^{inx}$$ \hspace{1cm} (3.3)
If the behaviour of $q(x, E, \lambda)$ at the $x$–infinity is to be of a finite type the series (3.3) has to be abbreviated providing us with a finite sum. The latter should contain at least three terms if we want $q(x, E, \lambda)$ to possess bound states. Let $k$ and $l(k > l + 1)$ be therefore the upper and lower limits of this abbreviation respectively. We shall consider just below in details a few cases of such abbreviated $q$’s for which $k - l = 2, 3, 4$. By this we shall convince ourselves that the remaining cases of $q(x, E, \lambda)$ cannot provide us with examples of the exactly JWKB quantized potentials.

In our investigations we shall make intensive use of the Weierstrass product representation for the abbreviated series (3.3) in order to perform necessary calculation of phases of $q(x, E, \lambda)$ alone as well as its functions. This representation is considered in Appendix 1. We have calculated there also explicitly, in order to provide us with an example of such calculations, the relevant total phases of $q(x, E, \lambda)$ for the case $k = 2, l = 0$ considered just below.

Case: $k = 2, l = 0$

We can write $q(x, E, \lambda)$ in this case as:

$$q(x, E, \lambda) = \alpha(E, \lambda)e^{2x} - 2\beta(E, \lambda)e^x + \gamma(E, \lambda) \quad (3.4)$$

where $\alpha(E, \lambda)$, $\beta(E, \lambda)$ and $\gamma(E, \lambda)$ are known functions of $E$ and $\lambda$. In particular, for $\alpha \equiv \beta \equiv 1$ and $\gamma \equiv -E$ we get the well–known Morse potential [20].

![Fig.3 The SG for the Morse type potential (3.4)](image)

With $\alpha, \beta, \gamma > 0$ and $\beta^2 > \alpha\gamma$ we get for $q(x, E, \lambda) = 0$ two real roots (modulo $2\pi i$) and the corresponding SG shown in Fig.3 where $x_\pm = \ln(\beta \pm \sqrt{(\beta^2 - \alpha\gamma)})$. The quantization
condition (2.11) according to the figure looks now as follows:

$$\exp\left[-\lambda \oint_{K} q^\frac{1}{2}(x, \lambda, E)dx\right] = -\frac{\chi_{1\rightarrow 3}(\lambda, E)\chi_{2\rightarrow 3}(\lambda, E)}{\chi_{1\rightarrow 3}(\lambda, E)\chi_{1\rightarrow 3}(\lambda, E)}$$

(3.5)

It follows from the figure that $\chi_{2\rightarrow 3} = \chi_{2\rightarrow 3} \equiv 1$ and $\chi_{1\rightarrow 3} \equiv \chi_{1\rightarrow 3}$. The first of these identities is satisfied because both the paths $\gamma_{2\rightarrow 3}$ and $\chi_{2\rightarrow 3}$ can be pushed out to infinities whilst the second because of the periodicity of the corresponding integrands in the formulae (2.6) for $\chi_{1\rightarrow 3}$ and $\chi_{1\rightarrow 3}$. Therefore, we are left finally with the JWKB formula which gives exact energy levels in this case.

It should be noticed, however, that the equality of the coefficients $\chi_{1\rightarrow 3}$ and $\chi_{1\rightarrow 3}$ is not immediate i.e. it does not follow as a direct result of the periodicity of $q(x, E, \lambda)$. First we have to define the total phase of $q(x, E, \lambda)$ according to the prescriptions of Appendix 1 in order to define uniquely its square roots present in the coefficients $\chi_{1\rightarrow 3}$ and $\chi_{1\rightarrow 3}$. For the case just considered it has been done in Appendix 1 where we have found that the phases of $q(x, E, \lambda)$ on the integration paths of $\chi_{1\rightarrow 3}$ and $\chi_{1\rightarrow 3}$ differ exactly by $4\pi i.e. by the period of the square roots of $q(x, E, \lambda)$ just mentioned.

It should be stressed at this moment that to get the last result the phases of the exponential factor in the corresponding Weierstrass product (WP) have had to be taken into account i.e. counting the relevant phases providing by the roots of $q(x, E, \lambda)$ alone would give us incorrect result.

**case $k = -l = 1$**

In this case $q(x, E, \lambda)$ is given as:

$$q(x, E, \lambda) = \alpha e^x + \gamma e^{-x} - 2\beta$$

(3.6)

where $\alpha, \beta, \gamma$ all depend on $E$ and $\lambda$ and are positive so that $q(x, E, \lambda)$ represents the infinite well.
For $\beta^2 > \alpha \gamma (> 0)$ $q(x, E, \lambda) = 0$ has again two real roots (modulo $2\pi i$) and the corresponding SG looks as in Fig.4. For the corresponding quantization condition we can get the following two equivalent forms of it:

$$\exp \left[ -\lambda \oint_k q \frac{x}{2} (x, \lambda, E) \right] = -\frac{\chi_{2 \rightarrow 3}(\lambda, E)\chi_{1 \rightarrow 4}(\lambda, E)}{\chi_{2 \rightarrow 4}(\lambda, E)\chi_{1 \rightarrow 3}(\lambda, E)}$$

or

$$\chi_{2 \rightarrow 3}(\lambda, E)\chi_{1 \rightarrow 4}(\lambda, E) = \chi_{1 \rightarrow 3}(\lambda, E)\chi_{2 \rightarrow 4}(\lambda, E)$$

It follows from Fig.4 that $\chi_{1 \rightarrow 3}(E, \lambda) = \chi_{2 \rightarrow 4}(E, \lambda) = \bar{\chi}_{2 \rightarrow 4}(E, \lambda) \equiv 1$, $\chi_{1 \rightarrow 4}(E, \lambda) = \bar{\chi}_{1 \rightarrow 4}(E, \lambda)$ and $\chi_{2 \rightarrow 3}(E, \lambda) = \chi_{4 \rightarrow 1}(E, \lambda)$. The latter equality follows from the fact the phases of points of the corresponding integration paths differ by $2\pi$ (by periodicity of $q(x, E, \lambda)$) what causes the square roots of $q(x, E, \lambda)$ to differ by their signs on the paths compensated however by the opposite signatures of the coefficients considered. (Note that this time the corresponding phase difference calculations can take only into account the phase contributions of roots of (3.6) alone since the exponential factor is now absent in WP corresponding to (3.6)).

Taking all these into account as well as the following general equality [13]:

$$\chi_{j \rightarrow k}(E, \lambda) = \chi_{k \rightarrow j}(E, \lambda)$$
we get finally for the quantization condition of the considered potential:

\[
\exp \left[ -\lambda \oint_{K} q_{\lambda, E}^{2} dx \right] = -\chi_{2\to 3}^{2}(\lambda, E) \tag{3.8}
\]

together with: \(|\chi_{2\to 3}(\lambda, E)| = 1\).

We have to conclude therefore that in this case the exact condition (3.7) cannot be reduced to the exact pure JWKB ones by the pure symmetry arguments only.

The last potential, however, has been concluded initially by Rosenzweig and Krieger [4] as being exactly JWKB quantized and next corrected by Krieger [5] as to be not. The main argument of the last author to support his conclusion was an observation of the unvanishing first order correction to the pure JWKB condition which follows from (3.7). It is easy to see that in terms of the coefficient \(\chi_{2\to 3}(E, \lambda)\) this argument means that the first integral in its representation (2.6) does not vanish i.e. the coefficient has to differ from unity.

It is still worth to note that the potential (3.6) can be obtained from (3.4) just by multiplying the latter by \(e^{-x}\), so that if the coefficients \(\alpha, \beta, \gamma\) in both of them are chosen to be the same then their sets of roots coincide and their WP representations differ exactly by the above factor \(e^{-x}\). And this is just this factor which introduces the dramatic difference between the corresponding Stokes graphs of figures 3 and 4 and the corresponding quantization conditions.

\underline{case k = 3, l = 0}
In order to satisfy the demand of two real turning points and the reality condition
$q(x, E, \lambda)$ has to have now the following form:

$$q(x, E, \lambda) = \alpha (e^x - \beta_+)(e^x - \beta_-)(e^x + \gamma)$$

(3.9)

with $\alpha, \beta_\pm, \gamma > 0$ and all depending on $E$ and $\lambda$ as usually. The corresponding SG is shown in Fig.5 where $x_\pm = \ln(\beta_\pm)$ and $x_\gamma = \ln(-\gamma)$. The quantization condition is given therefore by:

$$\exp\left[-\lambda \oint_K q^\frac{1}{2}(x, E, \lambda) dx\right] = \frac{\chi_{0 \rightarrow 2}(E, \lambda)\chi_{1 \rightarrow 2}(E, \lambda)}{\chi_{0 \rightarrow 2}(E, \lambda)\chi_{1 \rightarrow 2}(E, \lambda)} = \frac{\chi_{0 \rightarrow 2}(E, \lambda)}{\chi_{0 \rightarrow 2}(E, \lambda)}$$

(3.10)

where the equality of $\chi_{1 \rightarrow 2}$ and $\chi_{1 \rightarrow 2}$ to unity which follow from Fig.5 have already been used. Comparing further the phases of $q(x, E, \lambda)$ on the lines $\Im x = +\pi$ and $\Im x = -\pi$ we find that they differ by $6\pi$.

The condition (3.10) shows therefore that if there are not some accidental cancellations then the JWKB formula can be only an approximation in this case i.e. it cannot be exact.

**case $k = 2, l = -1$**

This is the last case which we consider in details. The corresponding $q(x, E, \lambda)$ differs from (3.9) by the last factor in which $e^x$ is to be substituted by $e^{-x}$. Thus $q(x, E, \lambda)$ represents
now an infinite potential well with the corresponding SG shown in Fig.6. In comparison with Fig.5 the horizontal SL emerging from $x_\gamma$ and its periodic distribution reverse their directions into the opposite ones. This change of SG does not however allow all the $\chi$'s to cancel in the corresponding quantization condition:

$$
exp \left[ -\lambda \oint_k q_2^\gamma (x, E, \lambda) dx \right] = -\frac{\chi_{2\rightarrow3}(E, \lambda)}{\bar{\chi}_{2\rightarrow3}(E, \lambda)}
$$

(3.11)

so that also in this case the JWKB formula cannot be exact.

Fig.6 The SG for the infinite potential well:

$$
V(x, \lambda) = \alpha(e^x - \beta_+)(e^x - \beta_-)(e^{-x} + \gamma) - \alpha \beta_+ \beta_- \gamma
$$

corresponding to the quantization formula (3.11)

Considering higher values of $k - l$ ($>3$) it is easy to note that all the conditions used as far (two real turning points, reality and periodicity) are not sufficient to course the full cancellations of the $\chi$'s in the corresponding quantization conditions. The main reason for that is that for higher values of $k+l$ an increasing number of complex turning points in a basic strip of periodicity causes the $\chi$'s entering the corresponding quantization conditions not to be related any longer by the condition of periodicity, because all the relevant integrations are performed inside the same basic strip.

The above situation does not change even if we make $q(x, E, \lambda)$ to be additionally an even function of $x$. 
3.2 Aperiodic exactly JWKB-quantized potentials

The periodic potential (3.4) which provides us with the exact JWKB quantization formulae (3.5) can serve also as the source of aperiodic exactly JWKB-quantized potentials. The latter can be obtained from the former by a trivial change-of-variable procedure \( x \rightarrow y(x) \) resulting in the following potential transformations:

\[
q(x, E, \lambda) \rightarrow \left[ \frac{q(x, E, \lambda)}{y'^2(x)} - \frac{1}{\lambda^2} \left[ \frac{3}{4} y'^2(x) - \frac{1}{2} y'''(x) \right] \right]_{x=x(y)}
\]  

(3.12)

The only necessary demand for a relevant change is to provide by it in the resulting potentials a free (i.e. \( x \)-independent) term which can play a role of the energy parameter (see [28], for example).

The latter demand when applied to the potential (3.4) permits the following two possibilities: 1\(^0\) \( e^{x/2} \rightarrow x \) and 2\(^0\) \( e^x \rightarrow x \). Adjusting properly \( \alpha, \beta \) and \( \gamma \) in (3.4) we get in this way the 1–dim harmonic oscillator potential and the radial parts of the 3–dim homogeneous harmonic oscillator potential in the first case and the Coulomb potential in the second one.

The same method can be applied to the potentials which are not exactly JWKB quantized providing us with aperiodic potentials with the same property i.e. the method allows us not to make any further estimations of the resulting potentials for their being not quantized exactly by the corresponding JWKB formulae (what on their own could not be a simple task).

Applying the method to the potential well (3.6) for example only the substitution \( e^{x/2} \rightarrow x \) is allowed providing us with \( q(x, E, \lambda) = \alpha x^{-4} - \beta x^{-2} - E \) with \( \alpha, \beta > 0 \) and \( -\beta^2/(4\alpha) < E < 0 \). Of course, this potential can be considered as a radial part of a spherically symmetric potential with the repelling term \( x^{-4} \) and the attractive one \( x^{-2} \) the latter being affected by the centrifugal term contribution. It is seen that in this model a number of levels is finite being limited by the increasing of the angular momentum. According to our earlier result this potential cannot be JWKB exactly quantized.

It is worthwhile to note that the above substitution when considered as the inverse transformation is nothing but the well known Langer change–of–variable procedure applied to the cases of the Coulomb and harmonic 3–dim potentials to obtain the exact JWKB formulae for the latter potentials. It is clear that in view of the above discussion Langer’s substitutions are just what is necessary to be done in order to achieve the latter goal.
3.3 Periodic meromorphic potentials

The reality condition demanded for \( q(x, E, \lambda) \) allows us to choose for its two possible basic periods the one being pure real, and the second - pure imaginary.

Within this class of potentials we can ignore obviously \( q(x, E, \lambda) \) with a real period but without real poles. We have therefore to consider the following possibilities for \( q(x, E, \lambda) \):

a. It is holomorphic in some vicinity of the real axis but meromorphic outside it and being periodic with its unique imaginary period equal to \( 2\pi i \);

b. It is meromorphic on the real axis with the only imaginary period equal to \( 2\pi i \);

c. It is meromorphic on the real axis with the only real period equal to \( 2\pi \);

d. It is meromorphic on the real axis with two periods: a real one equal to \( 2\pi \) and a pure imaginary one equal to \( i\omega \) with \( \omega \) being any positive real number.

case a.

In this case \( q(x, E, \lambda) \) is assumed again to have (for some range of \( E \)) two real roots in its basic period strip defined by \(-\pi < \Im x \leq +\pi\). Let us assume also for a while that it has only four complex pairwise conjugated poles in this strip all lying on the imaginary axis (this position can always be achieved by a simple translation). If the order of these poles amounts to \( n \) then \( q(x, E, \lambda) \) has to have the following form:

\[
q(x, E, \lambda) = \frac{q_1(x, E\lambda)}{\left\{ \sinh \frac{1}{2}(x - ia) \sinh \frac{1}{2}(x + ia) \right\}^n} + q_2(x, E, \lambda) \tag{3.13}
\]

where \( q_i(x, E, \lambda), \ i = 1, 2 \), are holomorphic and periodic with their period equal to \( 2\pi i \) and \( 0 < a \leq \pi \). Therefore roots of \( q(x, E, \lambda) \) are given by the equation:

\[
q_1(x, E, \lambda) + \left\{ \sinh \frac{1}{2}(x - ia) \sinh \frac{1}{2}(x + ia) \right\}^n \cdot q_2(x, E, \lambda) = 0 \tag{3.14}
\]

with \( q_i \)'s having forms of the abbreviated series \( [3.14] \). According to our earlier observations that more than two roots in the basic period streap prevent as a rule the JWKB formula to be exact we have to put in \( [3.14] \): \( n = 1 \), \( q_1(x, E, \lambda) = \alpha e^x + \beta e^{-x} + \gamma \) and \( q_2(x, E, \lambda) = \text{const} \).
However, we have to consider the cases \( a \neq \pi \) and \( a = \pi \) separately because their Stokes graphs differ in their structures. In particular, whilst in the first case \( q(x, E, \lambda) \) will have simple poles at the points \( x = \pm ia \pmod{2\pi i} \) in the second case the second order poles are generated.

Satisfying finally the assumption of two real roots of \( q(x, E, \lambda) \) we get the following two potentials:

\[
V_1(x) = \frac{\alpha_1 e^x + \beta_1}{2 \sinh \frac{1}{2}(x - ia) \sinh \frac{1}{2}(x + ia)} = \frac{\alpha_1 e^x + \beta_1}{\cosh x - \cosh a}
\]

\[
V_2(x) = \frac{\alpha_2 e^x + \beta_2}{\cosh^2 \frac{1}{2} x}
\]

the second of which is essentially the Rosen-Morse one \[21\].

To obtain from (3.15) the potentials which would have bound states some conditions on their parameters have to be satisfied. For the first of them they are:

\[
\alpha_1 > 0 > \beta_1 \quad (3.16)
\]

with the quantized energy \( E \) varying in the following range:

\[
- \frac{2x(\alpha_1 - x)^2}{(|\alpha_1 - x| - |y|)^2 + 2|y| |\alpha_1 - x| (1 - \cos a)} < E < 0
\]

\[
x = \sqrt{\alpha_1^2 + \beta_1^2 + 2\alpha_1 \beta_1 \cos a}, \quad y = \beta_1 + 2\alpha_1 \cos a \quad (3.17)
\]

For the second potential in (3.15) we can put \( \alpha_2 > 0 > \beta_2 \) not losing its generality so that the corresponding energy range is:

\[
- \frac{\beta_2^2}{\alpha_2 - \beta_2} < E < 0 \quad (3.18)
\]
A relevant Stokes graph corresponding to the first of the potentials (3.15) is shown in Fig. 7. It follows immediately from the figure that the JWKB formula:

\[ \exp \left[ -\lambda \oint \frac{\alpha e^x + \beta - E}{\cosh x - \cosh a} \frac{1}{2} dx \right] = -1 \]  

cannot be exact in this case since the coefficients \( \chi_{1 \rightarrow 4} \) and \( \chi_{2 \rightarrow 3} \) are not related by periodicity and do not cancel in the exact condition.

The SG for the second of the potential (3.15) is shown on Fig. 8a. In order however to continue the relevant solutions corresponding to Sectors 1 and 2 to Sectors 3 and \( \bar{3} \) (the latter two containing the second order poles at \( x = \pm \pi i \) respectively) we have to choose properly the \( \delta \)-piece of \( \omega \) as defined by (2.7) to admit the integrals in (2.6) to converge at the poles. One can easily convince oneself that the choice \( \delta = [4 \cosh(x/2)]^{-2} \) is sufficient for such a goal leaving simultaneously the original form of SG of Fig. 8a. unchanged i.e. it redefines the coefficient \( \beta \) into \( \beta' = \beta - 1/(4 \lambda^2) \) only. Then Fig. 8a provides us with the following quantization condition for the case:

\[ \exp \left[ -\lambda \oint \frac{\alpha_2 e^x + \beta_2 - \frac{1}{16 \lambda^2}}{\cosh^2 \frac{1}{2} x} - Edx \right] = -\frac{\chi_{1 \rightarrow 3}(\lambda, E) \chi_{2 \rightarrow 3}(\lambda, E)}{\chi_{1 \rightarrow 3}(\lambda, E) \chi_{2 \rightarrow 3}(\lambda, E)} \]  

(3.20)
According to Appendix 1 the total change of the phase of $\tilde{q}(x, E, \lambda)$ in (3.20) is determined only by the distributions of zeros of its nominator as well as by the corresponding zeros of $\cosh^2(x/2)$ being the denominator. Calculated (according to the rules of Appendix 1) with respect to the points of the lines $\Im x = \pi$ and $\Im x = -\pi$ (shifted by the period $2\pi i$) the nominator phase change amounts to $2\pi$ what is exactly the same as the total phase change of the denominator. Therefore, the total phase change of $\tilde{q}(x, E\lambda)$ is exactly equal to zero in this case what is enough for $\chi_{1 \rightarrow 3}$ and $\chi_{1 \rightarrow \bar{3}}$ as well as for $\chi_{2 \rightarrow 3}$ and $\chi_{2 \rightarrow \bar{3}}$ to coincide. It means that the RHS of (3.20) is equal to $-1$ and the JWKB formula corresponding to (3.20) is exact in the case considered.

It is interesting to note that the success of the JWKB formula corresponding to (3.20) to be exact depends completely on the total phase change of $\tilde{q}(x, E\lambda)$ to be equal to an integer multiple of $4\pi$. This condition permits also to produce the forms of the exact JWKB formulae different than the one discussed above. They can be obtained for example by doubling the number of turning points in the basic period strip. This can be done in many ways by choosing $\delta$ properly. One of such choices is $\delta = [4\cosh(x/2)]^{-2} + a\lambda 2\sinh^{-2}(x/2)$ with real but arbitrary $a \neq 0$. This choice introduces to the basic period strip the second order pole at $x = 0$ but also two additional zeros (lying close to the pole for small $a$). Both the zeros are real for $a > 0$ or pure imaginary for $a < 0$. Let us note further that the pole at $x = 0$ does not produce a singularity at this point for $\Psi_1(x)$ and $\Psi_2(x)$ but it can do it for their corresponding $\chi$–factors (since it does it for the corresponding JWKB factors). However, the latter possibility does not affect the procedure of writing the corresponding quantization condition nor estimating periods corresponding to the coefficients $\chi_1(x)$ and $\chi_2(x)$. Choosing therefore for definiteness $a$ to be positive we get the corresponding SG in the form shown in Fig.8.

Fig.8 The SG’s corresponding to the second of the potentials (3.15) and the quantization formulae (3.20) (Fig.8a) and (3.21) (Fig.8b)
Fig. 8b and the following quantization condition:

\[
\exp \left[ -\lambda \oint_K \left[ \frac{\alpha_2 e^x + \beta_2 - \frac{1}{16} a e^x}{\cosh^2 \frac{1}{2} x} - \frac{a}{\sinh^2 \frac{1}{2} x} - E \right] \frac{1}{2} dx \right] = \frac{-\chi_{1\to3}(\lambda, E)\chi_{2\to3}(\lambda, E)}{\chi_{1\to3}(\lambda, E)\chi_{2\to3}(\lambda, E)} \tag{3.21}
\]

Note that \(\chi_{1\to3}\) and \(\chi_{2\to3}\) in (3.21) are calculated on different sides of the (possible) cut emanating from the point \(x = 0\).

Now it is easy to see that the \(\chi\)'s in the RHS of (3.21) cancel mutually pairwise by periodicity and we obtain from (3.21) the exact JWKB quantization formula with \(a\) as an arbitrary real parameter. In particular we can put \(a = 0\) in the formula getting it again in the 'standard' Bailey’s form \[3\].

It follows from the above considerations that it is rather hopeless to look for other potentials of the case considered which could provide us with corresponding exact JWKB formulae. The proliferation of roots unavoidable for these potentials should prevent effectively the exact JWKB quantization conditions to appear closing the relevant quantizations inside the single period strip.

\textit{case b.}

Assuming the presence of simple or second order poles in \(q(x, E, \lambda)\) it is clear that we can allow only one such a pole in the main period strip. We assume its localization at \(x = 0\). In this way the problem of quantization has to be reduced to a half of the real axis which we choose not loosing a generality to be the right one. The allowed classes of potentials should have therefore the forms:

\[
V_1(x, \lambda) = \frac{\alpha_1 e^x + \beta_1}{\sinh x} \quad V_2(x, \lambda) = \frac{\alpha_2 e^x + \beta_2}{\sinh^2 \frac{1}{2} x} \tag{3.22}
\]
If bound states are to exist for the first potential in (3.22) its parameters have to satisfy the following relations:

\[ \beta_1 < 0, \quad -\beta_1 < \alpha_1, \quad \alpha_1 + \sqrt{\beta_1^2 + \beta_1^2} < E < 2\alpha_1 \quad (3.23) \]

A shape of the potential is shown in Fig.9a. (Note that its local maximum is below its local minimum). The corresponding SG has to be modified because of the presence of simple poles in the potential and because of the corresponding quantization condition demanding for the wave function to vanish at \( x = 0 \). A ‘minimal’ choice for \( \delta \) is \( \delta = (2 \sinh x)^{-2} \) so that the corresponding SG looks now as in Fig.9b.

The energy quantization demands now the solution \( \Psi_1(x) \) corresponding to Sector 1 and the solution \( \Psi_0(x) \) from Sector 0 to coincide. This provides us with the following quantization condition:

\[ \exp \left[ -\lambda \int \frac{\alpha_1 e^x + \beta_1}{\sinh x} + \frac{1}{4\lambda^2} \frac{1}{\sinh^2 x} - E \right]^{rac{1}{2}} dx = \frac{\chi_{1\rightarrow 3}(E, \lambda)\chi_{0\rightarrow 3}(E, \lambda)}{\chi_{1\rightarrow 3}(E, \lambda)\chi_{0\rightarrow 3}(E, \lambda)} \quad (3.24) \]

From (3.24) it follows, however, that although the coefficients \( \chi_{1\rightarrow 3} \) and \( \chi_{1\rightarrow 3} \) of the formula mutually cancel (by periodicity) the remaining two coefficients do not (i.e. the latter coefficients are not real) and therefore the corresponding JWKB formula which follows of (3.24) cannot be exact.
Fig. 10 The second of potentials (3.22) and the SG’s corresponding to the quantization formulae (3.20) (Fig.10a) and (3.27) (Fig.10b)

Not losing a generality we can assume the parameters $\alpha_2$ and $\beta_2$ of the second potential in (3.22) to satisfy the following conditions:

$$\beta_2, \alpha_2 + \beta_2 > 0 > 2\alpha_2 + \beta_2 > \alpha_2$$

$$-\frac{\beta_2^2}{\alpha_2 + \beta_2} < E < 4\alpha_2$$

(3.25)

to ensure an existence of bound states in the local potential well. The shape of the potential is shown in Fig.10a. The potential has to be modified by the 'standard' $\delta$–term: $\delta = (4\sinh(x/2))^{-2}$ to allow the construction of the FS at $x = 0$ what results with the change: $\beta_2 \rightarrow \beta_2 + (4\lambda)^{-2}$ in the potential. The quantization condition corresponding to SG
of Fig. 10b reads now:

\[
\exp \left[ -\lambda \oint K \left[ \frac{\alpha_2 e^x + \beta_2 + \frac{1}{16 \lambda^2}}{\sinh^2 \frac{x}{2}} - E \right] \frac{1}{2} dx \right] = -\frac{\chi_{1\rightarrow 2}(E, \lambda) \chi_{3\rightarrow 2}(E, \lambda)}{\chi_{1\rightarrow 2}(E, \lambda) \chi_{3\rightarrow 2}(E, \lambda)}
\] (3.26)

It follows from (3.26) and from Fig. 10b that in this case the coefficients \( \chi_{1\rightarrow 2} \) and \( \chi_{1\rightarrow 2} \) cancel mutually by the periodicity arguments (the phase difference produced by the nominator of \( \tilde{q}(x, E, \lambda) \) and equal to \( 2\pi \) for the two integration paths \( \gamma_{1\rightarrow 2} \) and \( \gamma_{1\rightarrow 2} \) is cancelled by its denominator \( \sinh^2(x/2) \)) whilst the remaining two coefficients cancels by their reality (they are real and complex conjugated to each other). It means that the JWKB formula which follows from (3.26) is exact.

Again it is worth to note that the considered potential can be modified by the \( \delta \)-function in a different way to generate at least four zeros in the basic period strip of the corresponding SG allowing the nominator of the modified \( \tilde{q}(x, E, \lambda) \) to change its phase by \( 4\pi \) between the earlier mentioned paths. This can achieved by putting \( \delta = (4 \sinh(x/2)^{-2} + a\lambda^2 \sinh^{-2}((x - x_0)/2)) \) with real and positive \( a \) but sufficiently small for \( x_0 \) satisfying \( x_- < x_0 < x_+ \) where \( x_\pm \) are the outer two (of the total four) real turning points of the modified potential. The corresponding SG is then shown in Fig. 10c and for the quantization condition we get:

\[
\exp \left[ -\lambda \oint K \left[ \frac{\alpha_2 e^x + \beta_2 + \frac{1}{16 \lambda^2}}{\sinh^2 \frac{x}{2}} + \frac{a}{\sinh^2 \frac{x-x_0}{2}} - E \right] \frac{1}{2} dx \right] = -\frac{\chi_{1\rightarrow 2}(E, \lambda) \chi_{3\rightarrow 2}(E, \lambda)}{\chi_{1\rightarrow 2}(E, \lambda) \chi_{3\rightarrow 2}(E, \lambda)}
\] (3.27)

It follows from (3.27) and from Fig. 10c that in this case the coefficients \( \chi_{1\rightarrow 2} \) and \( \chi_{1\rightarrow 2} \) again cancel mutually by the periodicity arguments whilst the remaining two again by their reality. It means that the JWKB formula which follows from (3.27) is again exact and coincides with the previous one when \( a = 0 \).

The possibility of making the last modification enlarging the number of roots in the basic period strip to four still suggests to complete it differently, namely by adding the term coinciding exactly with the second of the potential (3.15). This of course needs also to add the corresponding standard \( \delta \)-term to the potential obtained in this way. The potential we get in this way does not, however, satisfy the rule of no more than two turning points in the period strip so that a possibility of the exact JWKB quantization condition to appear should mostly
depend on symmetry properties of the relevant \(\chi\)-coefficients. The considered potential can have bound states for the following regime of its parameters (see the formula below for the definition of the parameters): \(\alpha, \alpha'\) real and sufficiently close to zero and \(\beta, \beta' > 0\). Then the SG corresponding to the case is shown in Fig.11 and the quantization condition related to it is:

\[
\exp \left[ -\lambda \oint K \left[ \frac{\alpha e^x + \frac{1}{16\lambda^4} e^{2x}}{\sinh^2 \frac{x}{2}} + \frac{\alpha' e^x - \beta' - \frac{1}{16\lambda^4}}{\cosh^2 \frac{x}{2}} - E \right] \frac{1}{2} \right] = \\
= -\frac{\chi_{1\rightarrow2}(E, \lambda) \chi_{3\rightarrow2}(E, \lambda)}{\chi_{1\rightarrow2}(E, \lambda) \chi_{3\rightarrow2}(E, \lambda)}
\]

(3.28)

Fig.11 The SG corresponding to the formula (3.29) quantizing the potential of Pöschl and Teller

It follows from the figure that the coefficients \(\chi_{3\rightarrow2}\) and \(\chi_{3\rightarrow2}\) have to cancel mutually by periodicity but not the remaining two: none symmetry (except the complex conjugation) relates these two coefficients. However, when \(\alpha = \alpha' = 0\) the potential in (3.28) becomes invariant under the reflection \(x \rightarrow -x\) and then the coefficients \(\chi_{1\rightarrow2}\) and \(\chi_{1\rightarrow2}\) are equal just by the last symmetry. (Note, however, the role played in fulfilling this symmetry by \(4\pi\) of difference between the arguments of \(\tilde{q}(x, E, \lambda)\) corresponding to the case the latter takes on the paths \(\gamma_{1\rightarrow2}\) and \(\gamma_{1\rightarrow2}\).)
Therefore the following quantization condition:

\[
\exp \left[ -\lambda \oint \frac{\beta + \frac{1}{16\lambda^2}}{\sinh^2 \frac{x}{2}} \frac{\beta' + \frac{1}{16\lambda^2}}{\cosh^2 \frac{x}{2}} - E \right]^{\frac{1}{2}} dx = -1
\]  

(3.29)

is exact. The potential in the above formula is of Pöschl and Teller [22].

case c.

The case contains the following four potentials:

\[
V_1(x, \lambda) = \frac{\alpha_1 \sin x + \beta_1}{\cos x}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}
\]

\[
V_2(x, \lambda) = \frac{\alpha_2 \sin x + \beta_2}{\cos^2 \frac{x}{2}}, \quad -\pi < x < \pi
\]

\[
V_3(x, \lambda) = \frac{\alpha'_1 \sin x + \beta'_1}{\cos x} + \frac{\alpha''_1 \cos x + \beta''_1}{\sin x}, \quad 0 < x < \frac{\pi}{2}
\]

\[
V_4(x, \lambda) = \frac{\alpha'_2 \sin x + \beta'_2}{\cos^2 \frac{x}{2}} + \frac{\alpha''_2 \sin x + \beta''_2}{\sin^2 \frac{x}{2}}, \quad 0 < x < \pi
\]

(3.30)

the second of which is essentially another of Pöschl-Teller [22].

According to our earlier experience we cannot expect energy levels of the first potential (where \(\pm \alpha_1 + \beta_1 > 0\)) as well as of the third one to be exactly quantized with its corresponding JWKB formulae. This is because the points \(x = \pm \pi/2\) which are singular for the first potential both lie inside its basic period strip and therefore since the corresponding boundary conditions are formulated just for these points the \(\chi\)-coefficients entering the relevant quantization formula cannot mutually cancel i.e. the periodicity argument should not work in this as well as in the third cases.

The second and the fourth potentials are more promising and as it is well known the case \(\alpha_2 = 0\) of the first one is quantized exactly by the JWKB formula.

Consider therefore the first of them. In order to have the binding potential well we have to assume \(\beta_2 > 0\) but the choice of sign of \(\alpha_2\) is arbitrary since both the cases are equivalent. So we shall put \(\alpha_2 > 0\) for convenience. Next we have to notice however that asymmetry introduced to the potential by \(\alpha_2 \neq 0\) completely eliminates the possibility of using the
periodicity arguments. Therefore we shall put $\alpha_2 = 0$ in (3.30). Once more we have to choose $\delta$ taking it in its 'standard' form $\delta = (4 \cos(x/2))^{-2}$ and getting the SG of Fig.12a. It is seen from the figure that the coefficients $\chi_{1 \rightarrow -2}$ and $\chi_{1 \rightarrow 2}$ as well as $\chi_{\bar{1} \rightarrow -2}$ and $\chi_{\bar{1} \rightarrow 2}$ are now equal by the periodicity arguments. Because of that the following JWKB quantization formula:

$$\exp \left[ -\lambda \oint_{K} \left[ \frac{\beta_2 + \frac{1}{16\lambda^2}}{\cos^2 \frac{x}{2}} - E \right]^\frac{1}{2} \, dx \right] = -1$$

(3.31)

is exact.

Fig.12 The SG’s corresponding to the two variants (3.31) (Fig.12a) and (3.32) (Fig.12b) of the JWKB formula for another Pöschl–Teller potential

Similarly to the hyperbolic cosine case there is again possibility to modify $\tilde{q}(x,E,\lambda)$ differently by putting $\delta = a\lambda^2 \sin^{-2}(x/2) + (4 \cos(x/2))^{-2}$. The first term of $\delta$ (with an arbitrary real $a$) introduces two additional zeros in the basic period strip (necessary for the corresponding $\chi$’s to have period $4\pi$). The standard second term allows to construct the convergent solutions at the points $x = \pm \pi$. Choosing for definitness $a > 0$ we obtain the effective form of the SG corresponding to these modifications as shown in Fig.12b. It is seen from the figure that the coefficients $\chi_{1 \rightarrow -2}$ and $\chi_{1 \rightarrow 2}$ as well as $\chi_{\bar{1} \rightarrow -2}$ and $\chi_{\bar{1} \rightarrow 2}$ are equal by the periodicity arguments. Because of that the following JWKB quantization formula:

$$\exp \left[ -\lambda \oint_{K} \left[ \frac{\beta_2 + \frac{1}{16\lambda^2}}{\cos^2 \frac{x}{2}} + \frac{a}{\sin^2 \frac{x}{2}} - E \right]^\frac{1}{2} \, dx \right] = -1$$

(3.32)

is exact for any $a > 0$ and coincides with (3.31) for $a = 0$. 

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Considering finally the last of the potentials (3.30), first we remove (for the same reason discussed earlier) asymmetry in the latter putting $\alpha_2 = \alpha_2'' = 0$ and next we notice that in the basic period strip $-\pi < \Re x \leq \pi$ the number of the four turning points is sufficient to make the relevant $\chi$’s periodic across the strip. Therefore the only necessary modification of the potential is the ‘standard’ choice for $\delta$, i.e. $\delta = (4 \sin(x/2))^{\frac{1}{2}} + (4 \cos(x/2))^{\frac{1}{2}}$ what gives the SG shown in Fig.13. The following relations come then out from the figure: $\chi_{1 \rightarrow 3} = \chi_{-1 \rightarrow 3} = \chi_{1 \rightarrow 3}$ and $\chi_{0 \rightarrow 3} = \chi_{0 \rightarrow 3}$. The first equalities in both of these equality sequences follows from the parity invariance of the potential considered whilst the second in the first one is satisfied by the periodicity arguments. Therefore the following JWKB quantization condition:

$$\exp \left[ -\lambda \oint_{\gamma} \left[ \frac{\beta_2 + \frac{1}{16 \lambda^2}}{\cos^2 \frac{x}{2}} + \frac{\beta_2'' + \frac{1}{16 \lambda^2}}{\sin^2 \frac{x}{2}} - E \right] \frac{1}{2} dx \right] = -1$$

(3.33)

is exact.

Fig.13 The SG corresponding to the exact JWKB formula (3.33)

**case d.**

Examples of the case are provided by elliptic functions [19]. The simplest candidates are the elliptic functions of the second order i.e. containing two simple poles or a pole of the second order in the corresponding basic parallelograms. In this case there are also two simple roots or a double one in each such a parallelogram of periods. However, in general, for a given elliptic function a number of its roots in each parallelogram of periods is always equal to the order of an elliptic function. Therefore, it is rather hopeless to expect an elliptic
functions with its order higher than two to be a good candidate of a potential to produce the corresponding exact JWKB quantization formulae. Considering, however, \( q(x, E, \lambda) \) to be a second order elliptic function we have to assume that the two of its zeros are its real turning points \( x_{\pm} \) placed between 0 and \( 2\pi \) (the latter being the real period of \( q(x, E, \lambda) \)), i.e. \( 0 < x_- < x_+ < 2\pi \), whilst one of its poles is at \( x = 0 \). According to that we have the following two possibilities:

a. there are two real simple poles of \( q(x, E, \lambda) \): one at \( x_0 = 0 \) and the second at \( x_1, x_+ < x_1 < 2\pi \);

b. there is one double pole of \( q(x, E, \lambda) \) (at \( x=0 \)).

Of course, in each of the above cases \( q(x, E, \lambda) \) has to be completed to \( \tilde{q}(x, E, \lambda) \) by the corresponding \( \delta \)-function. The latter, however, has to be a sum of the second order elliptic functions with double poles at each singular point of \( q(x, E, \lambda) \). Therefore \( \delta \) can be represented by a linear combination of two corresponding Weierstrass elliptic functions (case a) or should be proportional to such a function (case b). The periods of the Weierstrass functions coincide in each case with those of \( q(x, E, \lambda) \). Therefore the SG’s corresponding to the two cases have to look as in Fig.14a,b respectively and the quantization conditions which can be prescribed to each of the graphs are the following:

\[
\exp \left[ -\lambda \oint K \left( F(x; 2\pi, i\omega) + \frac{1}{4\lambda^2} \varphi(x; 2\pi, i\omega) + \frac{1}{4\lambda^2} \varphi(x - x_1, 2\pi, i\omega) - E \right)^{\frac{1}{2}} \right] = \frac{-\chi_{0 \rightarrow 2}(\lambda, E) \chi_{1 \rightarrow 2}(\lambda, E)}{\chi_{0 \rightarrow 2}(\lambda, E) \chi_{1 \rightarrow 2}(\lambda, E)}
\]

(3.34)

\[
\exp \left[ -\lambda \oint \left( (\alpha + \frac{1}{4\lambda^2}) \varphi(x; 2\pi, i\omega) - E \right)^{\frac{1}{2}} \right] = \frac{-\chi_{0 \rightarrow 2}(\lambda, E) \chi_{1 \rightarrow 3}(\lambda, E)}{\chi_{0 \rightarrow 3}(\lambda, E) \chi_{1 \rightarrow 2}(\lambda, E)}
\]

(3.35)
Fig. 14 The SG’s corresponding to the elliptic function $F(x; 2\pi, i\omega)$ (Fig. 14a) and Weierstrass one $\wp(x; 2\pi, i\omega)$ (Fig. 14b) taken as potentials.

Here $F(x; 2\pi, i\omega)$ is an elliptic function with two simple poles at $x_0$ and at $x_1$, whilst $\wp(x; 2\pi, i\omega)$ is the Weierstrass elliptic function.

It follows from Fig. 14a that in the first of the above quantization formulae the periodicity arguments cannot work ($x_1$ is not a shift of $x_0$ by $2\pi$) nor in the form used in (3.34) nor in any other of its mutations.

In the condition (3.35) one can use periodicity and reality arguments to show only that $\chi_{1 \rightarrow 2} \chi_{0 \rightarrow 3} = \chi_{0 \rightarrow 2} \chi_{1 \rightarrow 3} e^{i \delta_{0 \rightarrow 3}}$ (where $\delta_{0 \rightarrow 3}$ is the phase of $\chi_{0 \rightarrow 3}$) so that also in this case the RHS of this condition is not -1 i.e. the corresponding JWKB formula is not exact.

4 More general exactly JWKB quantized potentials

The periodic potentials considered in the previous section are the simplest ones of the potentials quantized exactly by the JWKB formula. A generalization of their forms to the ones which still can keep the exactness of the corresponding JWKB formula can be done in the following way.

Let $V(x)$ means any exactly JWKB quantized periodic potential of the previous section. Let $V(x, p)$ means a real parameter family of periodic (with respect to $x$) potentials with the property that in the limit $p \to 0$, $V(x, p)$ approaches smoothly $V(x)$. Then, for $p$ small enough the Stokes graph corresponding to $V(x, p)$ has to resemble the Stokes graph corresponding
to $V(x)$. By such a resembling we mean the following:

1. To any singular point of SG of $V(x)$ there correspond a set of singular points of SG of $V(x, p)$ which reduce to the former point in the limit $p \to 0$. Each such a set we shall call a singular point blob (SPB);

2. To any turning point of SG of $V(x)$ there correspond a set of turning points of $V(x, p)$ which reduce to the former point in the limit $p \to 0$. We shall call such a set a turning point blob (TPB);

3. There is one to one correspondence between the sectors of the two Stokes graphs such that the sectors of SG of $V(x, p)$ reduce smoothly to the corresponding sectors of SG of $V(x)$ when $p \to 0$. In particular the boundary conditions are formulated for both the potentials $V(x, p)$ and $V(x)$ in sectors satisfying the correspondence just describe;

4. Each set of Stokes lines which emerge from some SPB (TPB) can be mapped into a definite set of Stokes lines of SG corresponding to $V(x)$ emerging from the point to which this SPB (TPB) reduces in the limit $p \to 0$. Each such a set can be divided into disjoint subsets of SL’s each of them transforming smoothly when $p \to 0$ into one particular SL emerging from the limiting point.

Examples of $V(x, p)$ with the properties 1 – 4 above with the Planck constant $\hbar$ as the parameter $p$ can be found in [1]. With these properties $V(x, p)$ provides us with the JWKB formula quantizing exactly the energy levels of $V(x, p)$.

5 Some supersymmetric JWKB formula exactness

In connection with the supersymmetric formulation of quantum mechanics the supersymmetric (SUSY) JWKB approximations have been suggested some of which being different from the conventional ones have appeared to be exact [7, 8]. It has been also noticed however that their exactness have been parallel to the exactness of the conventional ones [7, 8, 11].

In the previous sections we have shown that the exactness of the conventional (i.e. not SUSY) JWKB formulae was rather exceptional and related to the periodicity properties of the corresponding SG’s. Since the SUSY QM quantization problems seem to be governed by the same rules we can expect that the exactness of the SUSY JWKB formulae have to follow in some way from the traditional ones. We are going to show below that indeed this is the case.
Let us note, however, that there is also a common conviction that the SUSY JWKB exact quantization conditions are not only independent of the conventional ones but also their exactness in some cases of potentials is to be in contrast with the approximate character in these cases of the conventional JWKB formulae. As such potentials are considered the ones with the shape invariance property \[30\]. We would like to argue below that also in these cases the parallelness of the exactness of both the kind of the formulae seems to be still maintained.

Leaving the investigation of the latter relation for the later discussion let us examine first the question how the SUSY JWKB exact formulae follow from the conventional ones.

For this goal let us remind that if a potential \(V(x,\lambda)\) can be put in its SUSY form \(V(x,\lambda) \equiv V_-(x\lambda) = \phi^2(x,\lambda) - \phi'(x,\lambda)/\lambda + \epsilon_0\) (\(\epsilon_0\) is the energy of the fundamental level in \(V(x)\) if SUSY is exact) then the conventional JWKB quantization condition:

\[-\lambda \oint K \sqrt{V(x,\lambda)} + \frac{\delta(x,\lambda)}{\lambda} - E \, dx = (2m + 1)\pi i\]  
\[m = 0, 1, 2, \ldots\]  \hspace{1cm} (5.1)

for the exact SUSY is to be substituted by \[7, 8\]:

\[-\lambda \oint K \sqrt{\phi^2 - \tilde{E}} \, dx = 2\pi im\]  
\[m = 0, 1, 2, \ldots\]  \hspace{1cm} (5.2)

If (5.1) is exact then as we have mentioned (5.2) is very frequently also.

Let us analyze how this can happen. We shall perform our analysis also for the cases of broken superpotentials \(\phi\) which can represent \(V(x,\lambda)\). We shall show that in these cases the condition (5.2) remains unchanged if it is exact what is in contrast with its form representing the lowest JWKB approximation only in which case its RHS coincides rather with (5.1) \[27\].

At the beginning, let us note that because \(\lambda\) can vary we can take it sufficiently large to expand the integrand in (5.2) into a series with respect to \(\phi' - \delta/\lambda\). We get:

\[-\lambda \oint K \sqrt{\phi^2 - \tilde{E}} \, dx - \sum_{n \geq 1} \frac{1}{\lambda^{n-1}} \frac{\Gamma \left[ n - \frac{1}{2} \right]}{n! \Gamma \left[ -\frac{1}{2} \right]} \oint K \left( \frac{\phi' - \delta}{ \lambda} \right)^n (\phi^2 - \tilde{E})^{n-\frac{1}{2}} \, dx = (2m + 1)\pi i\]  
\[E = E - \epsilon_0\]  \hspace{1cm} (5.3)

where \(E = E - \epsilon_0\).
Making further a change of variable: \( x \to \phi = \phi(x, \lambda) \) in the integrands of the series in (5.3) and putting \( F_1(\phi, \lambda) \equiv \phi'(x(\phi, \lambda), \lambda) - \delta(x(\phi, \lambda), \lambda) / \lambda \) and \( F_2(\phi, \lambda) \equiv \phi'(x(\phi, \lambda), \lambda) \) we obtain:

\[
- \lambda \oint_K \sqrt{\phi^2 - \tilde{E}} \, dx - \sum_{n \geq 1} \frac{1}{\lambda^{n-1}} \frac{\Gamma\left[ n - \frac{1}{2} \right]}{n!} \oint_{K_\phi} \frac{F_1^n(\phi, \lambda)}{(\phi^2 - \tilde{E})^{n-\frac{1}{2}}} F_2(\phi, \lambda) \, d\phi = (2m + 1)\pi \quad (5.4)
\]

where the integrations under the sum in (5.4) go now in the \( \phi \)-plane.

Next let us observe that for all the exactly JWKB quantized potentials considered above \( F_{1,2}(\phi, \lambda) \) are holomorphic functions of \( \phi \) outside some circles of a sufficiently large radius so that the circles contain the branch points at \( \phi = -\sqrt{\tilde{E}} \) and \( \phi = +\sqrt{\tilde{E}} \) of the integrand denominators in (5.4). Moreover, both the functions \( F_{1,2} \) grow with the same powers of \( \phi \) but not faster than the second ones. It follows then from (5.4) that all the integrands of the series in (5.4) are also holomorphic outside such circles. The integrals can all be easily calculated then by taking the size of the contour \( K \) large enough and expanding their denominators:

\[
- \lambda \oint_K \sqrt{\phi^2 - \tilde{E}} \, dx - \sum_{n \geq 1} \frac{1}{\lambda^{n-1}} \frac{\Gamma\left[ n - \frac{1}{2} \right]}{n!} \times \sum_{k \geq 0} \tilde{E}^k \frac{\Gamma\left[ k + n - \frac{1}{2} \right]}{k!} \oint_{K_\phi} \frac{F_1^n(\phi, \lambda)}{\phi^{2n+2k-1}} F_2(\phi, \lambda) \, d\phi = (2m + 1)i \pi \quad (5.5)
\]

A final result of the integrations in (5.5) depends now of course on the particular forms of the expansions of \( F_{1,2}(\phi, \lambda) \) into their corresponding Laurent series.

It can be easily checked that the series in LHS of (5.5) becomes energy independent only in the case when the Laurent series expansions of \( F_{1,2} \) both abbreviate at least on the second power of \( \phi \). This is just the case of the potentials considered.

Suppose therefore that \( F_{1,2}(\phi, \lambda) = \sum_{k \geq 1} F_{1,2,k}(\lambda) \phi^{-k} + a_{1,2}(\lambda) \phi + b_{1,2}(\lambda) \phi + c_{1,2}(\lambda) \phi^2. \) Then from (5.3) we get:

\[
- \lambda \oint_K \sqrt{\phi^2 - \tilde{E}} \, dx + \pi i \delta_{b,0} \delta_{c,0} + \pi i \delta_{c,0} - 2\pi i \frac{\lambda}{c_2} \left[ \sqrt{1 - \frac{c_1}{\lambda}} - 1 \right] = (2m + 1)\pi i \quad (5.6)
\]
Below we shall do an inspection of the JWKB-quantization exact formulae of the previous section to calculate the corresponding coefficients $a_{1,2}$, $b_{1,2}$ and $c_{1,2}$ as well as to show that all these formulae allow the quantization form (5.2) *independently* of whether the supersymmetry represented by $\phi$ is exact or broken. This result, however, does not contradict the one obtained recently by Inomata et al [27] (see also [29] for further references) who have modified the Comtet et al formula (5.2) with the aim to cover also the cases when supersymmetric potentials $\phi$ represent broken supersymmetry. They have argued that in such cases the RHS of (5.2) had to be transformed again into the conventional form of the RHS of (5.1).

The source of the difference between both the conclusions is that our concerns the exact result whilst this of Inomata et al is only the lowest semiclassical approximation of the *exact* quantization condition. Nevertheless, as we shall see further that (5.2) having the same form gives, however, *different* results for energy levels depending on whether the supersymmetry is exact or broken in the latter case reproducing effectively the result of Inomata et al.

To follow further let us recapitulate all the potentials $V_k(x)$ and the corresponding $\tilde{q}_k(x, E, \lambda)$–functions we have found in the previous section to be quantized exactly by the corresponding JWKB-formulae. They are:

\[
\tilde{q}_1(x, E, \lambda) = V_1(x) - E = \alpha^2 e^{2x} - 2\beta e^x - E,
-\infty < x < +\infty, \beta > 0 > E
\]

\[
\tilde{q}_2(x, E, \lambda) = V_2(x) + \frac{1}{4\lambda^2 x^2} - E = -\frac{\alpha}{x} + \frac{\beta + \frac{1}{4\lambda}}{x^2} - E,
\]

\[
x, \alpha, \beta > 0 > E
\]

\[
\tilde{q}_3(x, E, \lambda) = V_3(x) + \frac{1}{4\lambda^2 x^2} - E = \alpha^2 x^2 + \frac{\beta + \frac{1}{4\lambda}}{x^2} - E,
\]

\[
x, \beta, E > 0
\]

\[
\tilde{q}_4(x, E, \lambda) = V_4(x) - \frac{1}{(4\lambda \cosh \frac{x}{2})^2} - E =
\]
\[
\alpha e^x - \beta - \frac{1}{16\lambda^2} - E,
\]
\[-\infty < x < +\infty, \beta > 0, -\beta < 2\alpha\]

\[
\tilde{q}_5(x, E, \lambda) = V_5(x) + \frac{1}{(4\lambda \sinh \frac{x}{2})^2} - E =
\]
\[
\alpha e^x + \beta + \frac{1}{16\lambda^2} - E,
\]
\[0 < x < +\infty, \beta, \alpha + \beta > 0 > 2\alpha + \beta > \alpha,
\]

\[
\tilde{q}_6(x, E, \lambda) = V_6(x) + \frac{1}{(4\lambda \sinh \frac{x}{2})^2} - \frac{1}{(4\lambda \cosh \frac{x}{2})^2} - E =
\]
\[
\frac{\beta + \frac{1}{16\lambda^2}}{\sinh^2 \frac{x}{2}} - \frac{\alpha + \frac{1}{16\lambda^2}}{\cosh^2 \frac{x}{2}} - E,
\]
\[0 < x < +\infty, \alpha, \beta > 0,
\]

\[
\tilde{q}_7(x, E, \lambda) = V_7(x) + \frac{1}{(4\lambda \cos \frac{x}{2})^2} - E = \frac{\alpha + \frac{1}{16\lambda^2}}{\cos^2 \frac{x}{2}} - E,
\]
\[-\pi < x < \pi, \alpha > 0
\]

\[
\tilde{q}_8(x, E, \lambda) = V_8(x) + \frac{1}{(4\lambda \cos \frac{x}{2})^2} + \frac{1}{(4\lambda \sin \frac{x}{2})^2} - E =
\]
\[
\frac{\alpha + \frac{1}{16\lambda^2}}{\cos^2 \frac{x}{2}} + \frac{\beta + \frac{1}{16\lambda^2}}{\sin^2 \frac{x}{2}} - E,
\]
\[0 < x < \pi, \alpha, \beta > 0
\]

In order to represent the above potentials by their supersymmetric ones one has in principle to solve non uniform Riccati equations with their RHS given by the potentials listed. In general such a task is rather difficult. For most of the above potentials, however, it is possible
to find these representations just by a trivial guess. To each of the potentials listed above one can guess several (at least two) solutions one of which correspond to a superpotential realizing the supersymmetry exactly and the remaining ones corresponding to a broken supersymmetry. The latter means that the supersymmetry breaking can be realized in many ways. The ways considered below take into account only the possibility to define by a superpotential $\phi$ the corresponding ground state solution $\Psi_0$ by the following representation:

$$\Psi_0(x) = \exp \left[ -\lambda \int_a^b \phi(y) dy \right]$$  \hspace{1cm} (5.7)

where $a, b$ ($a < b$) define boundaries of the corresponding quantization problem. Note that $\Psi_0$ as given by (5.7) satisfies the SE (2.1) for $E = \epsilon_0$ with the potentials $V(x, \lambda)(\equiv V_-(x, \lambda))$ listed above. There are four possibilities:

1. $\Psi_0$ vanishes at both the boundaries $a, b$ - the supersymmetry is exact and $\Psi_0$ is the ground state wave function;

2. and 3. $\Psi_0$ vanishes at one of the boundaries only ($a$ or $b$ respectively) - the supersymmetry has to be broken; and

4. $\Psi_0$ blows up at both the boundaries - the supersymmetry seems essentially to be broken but there is still possibility that the ground state $\Psi_0$ has been constructed by the erroneous choice of $\phi$ — there are infinitely many solutions satisfying the SE considered with $E = \epsilon_0$ but blowing up at both the boundaries even if the corresponding ground state exists with this energy.

The latter possibility cannot happen in the cases 2 and 3: blowing up of $\Psi_0$ at one of the boundaries only means that the ground state with $E = \epsilon_0$ cannot exists in these cases.

One can expect therefore that resulting relations between the energy spectra provided by the quantization conditions defined by the allowed superpotentials $\phi_k$, corresponding to each of the potentials $V_k$, $k = 1, \ldots, 8$, listed earlier, and the original spectra of the latter potentials can depend on the way the supersymmetry is broken by each particular $\phi_k$.

Below we have enumerated all the allowed superpotentials $\phi_k$ corresponding to each of the potentials $V_k$, $k = 1, \ldots, 8$, with the properties $1 - 4$ just discussed (attaching to each of them the corresponding category) together with their $F_{1,2}$–functions:
\( \phi_1(x, \lambda) = |\alpha| e^x - \frac{\beta}{|\alpha|} + \frac{1}{2\lambda}, \)

\( F_1(\phi_1) = F_2(\phi_1) = \phi_1 + \frac{\beta}{|\alpha|} + \frac{1}{2\lambda}, \)

\( b_1 = 1; \quad \epsilon_0 = -\left( \frac{\beta}{|\alpha|} - \frac{1}{2\lambda} \right)^2; \)

\( \phi_1(x, \lambda) = -|\alpha| e^x + \frac{\beta}{|\alpha|} \frac{1}{2\lambda}, \)

\( F_1(\phi_1) = F_2(\phi_1) = \phi_1 - \frac{\beta}{|\alpha|} + \frac{1}{2\lambda}, \)

\( b_1 = 1; \quad \epsilon_0 = -\left( \frac{\beta}{|\alpha|} + \frac{1}{2\lambda} \right)^2; \)

\( \phi_2(x, \lambda) = -\frac{|2l+1| + 1}{2\lambda} + \frac{\lambda \alpha}{|2l+1| + 1}, \)

\( F_1(\phi_2) = \lambda(2|2l+1| + 1) \left( \phi_2 - \frac{\alpha \lambda}{|2l+1| + 1} \right)^2 \left( |2l+1| + 1 \right)^2, \)

\( F_2(\phi_2) = 2\lambda \left( \phi_2 - \frac{\alpha \lambda}{|2l+1| + 1} \right)^2 \left( |2l+1| + 1 \right), \)

\( \epsilon_1 = \frac{2|2l+1| + 1}{(|2l+1| + 1)^2}; \quad \epsilon_2 = \frac{2\lambda}{|2l+1| + 1}; \)

\( \epsilon_0 = -\frac{(\lambda \alpha)^2}{(|2l+1| + 1)^2}; \)

\( \beta = \frac{l(l+1)}{\lambda^2}, \quad l < -1, \ l > 0; \)

\( \phi_2(x, \lambda) = \frac{|2l+1| - 1}{2\lambda} - \frac{\lambda \alpha}{|2l+1| - 1}, \)

\( F_1(\phi_2) = -\lambda(2|2l+1| - 1) \left( \phi_2 + \frac{\alpha \lambda}{|2l+1| - 1} \right)^2 \left( |2l+1| - 1 \right)^2, \)

\( F_2(\phi_2) = -2\lambda \left( \phi_2 + \frac{\alpha \lambda}{|2l+1| - 1} \right)^2 \left( |2l+1| - 1 \right), \)
\[ c_1 = -\lambda \frac{2|2l + 1| - 1}{(2|2l + 1| - 1)^2}, \quad c_2 = \frac{2\lambda}{2|2l + 1| - 1}; \]
\[ \epsilon_0 = -\frac{(\lambda\alpha)^2}{(2|2l + 1| - 1)^2}, \]
\[ \beta = \frac{l(l + 1)}{\lambda^2}, \quad l < -1, \ l > 0; \]

\[ \phi_3(x, \lambda) = |\alpha|x - \frac{|2l + 1| + 1}{2\lambda x}, \]
\[ F_1(\phi_3) = \lambda(2|2l + 1| + 1) \left[ \phi_3^2 + 2|\alpha| - \frac{|2l + 1| + 1}{\lambda} \right] \frac{1}{2} \phi_3 + \frac{(\phi_3^2 + 2|\alpha|)^{\frac{3}{2}}}{2(2|2l + 1| + 1)^2} + \frac{|\alpha|}{2|2l + 1| + 2}; \]
\[ F_2(\phi_3) = \lambda \left[ \phi_3^2 + 2|\alpha| - \frac{|2l + 1| + 1}{\lambda} \right] \frac{1}{2} \phi_3 + \frac{(\phi_3^2 + 2|\alpha|)^{\frac{3}{2}}}{|2l + 1| + 1}, \]
\[ \epsilon_1^0 = \frac{2|2l + 1| + 1}{(2|2l + 1| + 1)^2}, \quad \epsilon_2^0 = \frac{2\lambda}{|2l + 1| + 1}; \]
\[ \epsilon_0 = (|2l + 1| + 2)\frac{|\alpha|}{\lambda}, \quad \beta = \frac{l(l + 1)}{\lambda^2}, \quad l < -1, \ l > 0; \]

2. we get the case from 1. substituting there \(|2l + 1|\) by \(-|2l + 1|\);

3. we get the case from 1. by the substitution \(|\alpha| \rightarrow -|\alpha|\);

4. we get the case from 1. substituting there \(|2l + 1|\) by \(-|2l + 1|\) and \(|\alpha|\) by \(-|\alpha|\);

\[ \phi_4 = \frac{|2l + 1| - 1}{4\lambda} \tanh \frac{x}{2} + \frac{4\lambda\alpha}{2|2l + 1| - 1}, \]
\[ F_1(\phi_4) = -\lambda(2|2l + 1| - 1) \left[ \frac{\phi_4 - \frac{4\lambda\alpha}{|2l + 1| - 1}}{2|2l + 1| - 1} \right] \frac{1}{2} \phi_4 + \frac{(\phi_4 - \frac{4\lambda\alpha}{|2l + 1| - 1})^2}{16\lambda} + \frac{2|2l + 1| - 1}{16\lambda}, \]
\[ F_2(\phi_4) = -2\lambda \left( \phi_4 - \frac{4\lambda\alpha}{2|2l + 1| - 1} \right)^2 + \frac{|2l + 1| - 1}{8\lambda}, \]
\[ c_1 = -\lambda \left( \frac{2|2l + 1| - 1}{(2|2l + 1| - 1)^2} \right) \frac{2\lambda}{2|2l + 1| - 1}; \]
\[ \epsilon_0 = -\left[ \frac{2|2l + 1| - 1}{2\lambda} \frac{2\lambda}{|2l + 1| - 1} \right]^2; \]
\[ \alpha + \beta = \frac{l(l + 1)}{(2\lambda)^2}, \quad l < -1, \ l; l > 0; \]
we get the case from 10 by the substitution \(|2l + 1| \to -|2l + 1| \)

\[ \phi_5 = -\frac{|2l + 1| + 1}{4\lambda} \coth \frac{x}{2} - \frac{4\lambda}{4|2l + 1| + 1}, \]
\[ F_1(\phi_5) = \lambda(2|2l + 1| + 1) \phi_5 + \left(\frac{4\lambda}{2|2l + 1| + 1}\right)^2 \frac{2|2l + 1| + 1}{16\lambda}, \]
\[ F_2(\phi_5) = +2\lambda \left(\frac{4\lambda}{2|2l + 1| + 1}\right)^2 - \frac{|2l + 1| + 1}{8\lambda}, \]
\[ c_1 = \lambda \frac{2|2l + 1| + 1}{(2|2l + 1| + 1)^2}, \quad c_2 = \frac{2\lambda}{|2l + 1| + 1}; \]
\[ a_1^\infty = a_2^\infty = |\alpha|; \quad \epsilon_0 = -\left[-\frac{|2l + 1| + 1}{2\lambda} + \frac{2\lambda}{2|2l + 1| + 1}\right]^2; \]
\[ \alpha + \beta = \frac{l(l + 1)}{2(\lambda)^2}, \quad l < -1, \quad l > 0; \]

we get the case from 10 by the substitution \(|2l + 1| \to -|2l + 1| \)

\[ \phi_6 = \frac{|2l + 1| - 1}{4\lambda} \tanh \frac{x}{2} - \frac{|2l' + 1| + 1}{4\lambda} \coth \frac{x}{2}, \]
\[ F_1(\phi_6) = -\frac{\lambda}{2} \left(\frac{2|2l + 1| - 1}{2|2l + 1| + 1} - \frac{2l' + 1| + 1}{(2\lambda)^2}\right)^2 \frac{\phi_6 + \left[\phi_6^2 + (2|2l + 1| - 1)\left(\frac{2l' + 1| + 1}{(2\lambda)^2}\right)^2\right]^{\frac{1}{2}}} + \]
\[ \left[\phi_6 + \left(\frac{2l' + 1| + 1}{(2\lambda)^2}\right)^2\right]^{\frac{1}{2}}, \]
\[ F_2(\phi_6) = -\lambda \left[\phi_6^2 + (|2l + 1| - 1)\left(\frac{2|2l + 1| + 1}{(2\lambda)^2}\right)^2\right]^{\frac{1}{2}} \frac{\phi_6 + \left[\phi_6^2 + (|2l + 1| - 1)\left(\frac{2|2l + 1| + 1}{(2\lambda)^2}\right)^2\right]^{\frac{1}{2}}} + \]
\[ \left[\phi_6^2 + (|2l + 1| - 1)\left(\frac{2|2l + 1| + 1}{(2\lambda)^2}\right)^2\right]^{\frac{1}{2}}, \]
\[ c_1^0 = \lambda \frac{2l' + 1| + 1}{(2l' + 1| + 1)^2}, \quad c_1^\infty = \lambda \frac{2|2l + 1| - 1}{(2|2l + 1| - 1)^2} \]
\[ c_2^0 = \frac{2\lambda}{2l' + 1| + 1}, \quad c_2^\infty = -\frac{2\lambda}{2|2l + 1| - 1}; \]

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\[ \epsilon_0 = \frac{(l - l' - 1)^2}{(2\lambda)^2}, \quad \alpha = \frac{l(l + 1)}{(2\lambda)^2}, \quad \beta = \frac{l'(l' + 1)}{(2\lambda)^2}, \]

\[ |2l + 1| - |2l' + 1| > 2, \quad l, l' < -1, \quad l, l' > 0; \]

2\textsuperscript{0} we get the case from 1\textsuperscript{0} taking \( l, l' \) satisfying \(|2l + 1| - |2l' + 1| < 2 \) or substituting \(|2l + 1| \) by \(-|2l + 1|\) there;

3\textsuperscript{0} we get the case from 1\textsuperscript{0} substituting there \(|2l' + 1| \) by \(-|2l' + 1|\) and next taking \( l, l' \) satisfying \(|2l' + 1| \pm |2l + 1| > 2; \)

4\textsuperscript{0} we get the case from 1\textsuperscript{0} substituting there \(|2l' + 1| \) by \(-|2l' + 1|\) and next taking \( l, l' \) satisfying \(|2l' + 1| \pm |2l + 1| < 2;

1\textsuperscript{0}

\[ \phi_7 = \frac{|2l + 1| - 1}{4\lambda} \tan \frac{x}{2}, \quad F_1(\phi_7) = -\lambda (2|2l + 1| - 1) \frac{\phi_7^2}{(|2l + 1| - 1)^2} - \frac{2|2l + 1| - 1}{16\lambda}, \]

\[ F_2(\phi_7) = -2\lambda \frac{\phi_7^2}{|2l + 1| - 1} - \frac{|2l + 1| - 1}{8\lambda}, \]

\[ c_1 = -\lambda \frac{2|2l + 1| - 1}{(|2l + 1| - 1)^2}, \quad c_2 = \frac{2\lambda}{|2l + 1| - 1}, \]

\[ c_0 = \left( \frac{|2l + 1| - 1}{4\lambda} \right)^2 \]

\[ \alpha = \frac{l(l + 1)}{(2\lambda)^2}, \quad l < -1, \quad l > 0; \]

4\textsuperscript{0} we get the case from 1\textsuperscript{0} substituting there \(|2l + 1| \) by \(-|2l + 1|; \)

\[ \phi_8 = \frac{|2l + 1| + 1}{4\lambda} \tan \frac{x}{2} - \frac{|2l' + 1| - 1}{4\lambda} \cot \frac{x}{2}, \]

\[ F_1(\phi_8) = \frac{\lambda}{2} (2|2l + 1| + 1) \left[ \phi_8^2 + (|2l' + 1| - 1) \frac{|2l + 1| + 1}{(4\lambda)^2} \right]^{\frac{1}{2}} \phi_8 + \left[ \phi_8^2 + (|2l' + 1| - 1) \frac{|2l + 1| + 1}{(4\lambda)^2} \right]^{\frac{1}{2}} \]

\[ - \frac{\lambda}{2} (2|2l' + 1| - 3) \left[ \phi_8^2 + \frac{l'(l + 1)}{\lambda^2} \right]^{\frac{1}{2}} \phi_8 - \left[ \phi_8^2 + (|2l' + 1| - 1) \frac{|2l + 1| + 1}{(4\lambda)^2} \right]^{\frac{1}{2}}, \]

\[ F_2(\phi_8) = \lambda \left[ \phi_8^2 + (|2l' + 1| - 1) \frac{|2l + 1| + 1}{(4\lambda)^2} \right]^{\frac{1}{2}} \phi_8 + \left[ \phi_8^2 + (|2l' + 1| - 1) \frac{|2l + 1| + 1}{(4\lambda)^2} \right]^{\frac{1}{2}} \]

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\[-\lambda \left[ \phi_8^2 + (2l' + 1) - 1 \right] \frac{2l + 1 + 1}{(4\lambda^2)} \phi_8 - \left[ \phi_8^2 + (|2l' + 1| - 1) \frac{2l + 1 + 1}{(4\lambda^2)} \right]^{\frac{1}{2}} \frac{2l + 1 + 1}{(2l' + 1) - 1} \]

\[ c_1^0 = \lambda \frac{4l' - 1}{(2l')^2}, \quad c_1^\infty = \lambda \frac{4l + 3}{(2l + 2)^2}, \quad c_2^0 = \frac{\lambda}{l'}, \quad c_2^\infty = \frac{\lambda}{l + 1}, \]

\[ \epsilon_0 = \frac{(l - l' + 1)^2}{(2\lambda)^2}, \quad \alpha = \frac{l(l + 1)}{(2\lambda)^2}, \quad \beta = \frac{l'(l' + 1)}{(2\lambda)^2}, \quad l, l' < -1, \quad l, l' > 0; \]

2. we get the case from 1. substituting there \(|2l + 1|\) by \(-|2l + 1|\);

3. we get the case from 1. substituting there \(|2l' + 1|\) by \(-|2l' + 1|\);

4. we get the case from 1. substituting there \(|2l + 1|\) by \(-|2l + 1|\) as well as \(|2l' + 1|\) by \(-|2l' + 1|\).

In the above calculations only the non vanishing coefficients \(a_{1,2}, b_{1,2}\) and \(c_{1,2}\) at the highest power of \(\phi\) have been given. Now we can use them to calculate the three pieces of the LHS of (5.6) and to convince ourselves that in all the cases considered all the pieces contribute the total value \(\pi i \lambda\) only so that (5.5) always reduces to (5.2).

A comment to these calculations is necessary in the cases of the \(V_3, V_6\) and \(V_8\) potentials. The necessary integrations that lead to (5.6) are performed here on the two sheeted Riemann surfaces of the variable \(\phi\) on which the corresponding functions \(F_{1,2}(\phi)\) have different asymptotic properties for \(\phi \to \infty\). The superscripts '0' and '∞' at the coefficients \(a_{1,2}, b_{1,2}\) and \(c_{1,2}\) indicate that the integrations contributed to them have been performed on the two different sheets. In such cases the coefficients with both superscripts contribute to (5.6) but these contribution should be multiplied by 1/2 each (see Appendix 2 for an example of such calculations).

Let us finally note that comparing the energy levels obtained by the formula (5.1) with those obtained by (5.2) using in the latter the respective superpotentials of the cases 1. – 4. we get the result that the energy levels given by (5.1) are reproduced by (5.2).

i. exactly in the case 1. of the superpotentials;

ii. being shifted up by a half of a unit used to enumerate the levels in the cases 2. – 3. of the superpotentials;

iii. being shifted up by a whole unit used to enumerate the levels in the cases 4. of the superpotentials;
It is clear that the above differences follow as a result of the different enumeration of energy levels in the compared spectra ($m$ in (5.1) starts from zero whilst in (5.2) from unity) as well as due to different choices of the energy levels $\epsilon_0$ with respect to which the levels of the spectra are measured in every of the cases $1^0 - 4^0$.

5.1 SUSY and conventional JWKB quantization of shape invariant potentials

We have shown in the previous section that all the exactly JWKB quantized cases of potentials are also quantized exactly by their SJWKB quantization formulae. However, the latter property of the considered potentials has been established also as being closely related to their common property of being shape invariant [30]. The latter means that each $V_k(x, \lambda) \equiv V_{-k}(x, \lambda), \ k = 1, \ldots , 8$, depends additionally on some parameter $a$ so that for its SUSY partner $V_{+k}(x, \lambda, a)$ we have:

$$V_{k,+}(x, \lambda, a) = V_{-k}(x, \lambda, a) + R_k(a_1)$$

$$k = 1, \ldots , 8$$

with $a_1 = f_k(a)$. In the case of the considered potentials each $f_k$ is simply a translation of the parameter $a$.

The exactness of (5.2) following from (5.8) has been suggested by Dutt et al [31] and established by Barclay et al [32]. It was argued also (see Cooper et al and [29], for example) that the exactness of SJWKB formulae (5.2) takes place even when the exactness of the conventional ones fails. The latter claim, however, needs not be necessarily true and we would like to show below that in the case of the translational shape invariance all the known cases of the potentials are JWKB quantized exactly, too. To this aim let us note that on the list of the eight of them cited above there are two still lacking on the list when compared with the corresponding list of Cooper et al [29]. These two are:

$$V_9(x, \lambda) = \frac{\alpha + \beta \sin x}{\cos^2 x}, \quad -\frac{\pi}{2} < x < +\frac{\pi}{2}, \ \alpha > \beta > 0$$

$$V_9^{\text{min}} = \frac{1}{2}(\sqrt{\alpha^2 - \beta^2} + \alpha)$$

(5.9)
\[ V_{10}(x, \lambda) = \frac{\alpha + \beta \sinh x}{\cosh^2 x}, \quad -\infty < x < +\infty, \quad \beta > 0 \]

\[ V_{10}^{\text{min}} = -\frac{1}{2}(\sqrt{\alpha^2 + \beta^2} - \alpha) \]

One can easily convince oneself, however, that completed by the 'standard' \( \delta \)-terms \( (2 \cos x)^{-2} \) for the first potential and \( -(2 \cosh x)^{-2} \) for the second one) the potentials are exactly JWKB quantized no doubt reflecting the fact that this exactness follows in some although not obvious way from the underlying periodicity of their SG’s shown in Fig.15.

![Fig.15](image) The SG’s corresponding to the potentials \( V_9(x, \lambda) \) (Fig.15a) and \( V_{10}(x, \lambda) \) (Fig.15b) given by the formulae (5.9).

This exactness can be, however, concluded also applying back the procedure described in the previous section i.e. the exact JWKB formulae for the considered potentials follow from the exactness of the corresponding SJWKB ones. To see this let us find the superpotentials \( \phi_{9,10} \) and their \( F_{1,2} \)-functions corresponding to the potentials (5.9). They are:

\[ \phi_{9} = \frac{|2l + 1| + |2l' + 1| + 2}{4\lambda} \tan x + \frac{|2l + 1| - |2l' + 1|}{4\lambda \cos x} \]

\[ F_1(\phi_9) = (\phi_9^2 + a^2) \frac{(\phi_9^2 + a^2) a - ab^2 + b\phi_9 \sqrt{\phi_9^2 + a^2 - b^2}}{(b^2 + a^2) \phi_9^2 + a^2(a^2 - b^2) + 2ab\phi_9 \sqrt{\phi_9^2 + a^2 - b^2}} \]

\[ F_2(\phi_9) = (\phi_9^2 + a^2) \frac{(\phi_9^2 + a^2) a - ab^2 + b\phi_9 \sqrt{\phi_9^2 + a^2 - b^2}}{(b^2 + a^2) \phi_9^2 + a^2(a^2 - b^2) + 2ab\phi_9 \sqrt{\phi_9^2 + a^2 - b^2}} \]

\[ c_1^{\infty} = \frac{1}{a + b} - \frac{1}{4\lambda (a + b)^2}, \quad c_2^{\infty} = \frac{1}{a - b} - \frac{1}{4\lambda (a - b)^2}; \]
\[c_2^\infty = \frac{1}{a+b}, \quad c_2^{-\infty} = \frac{1}{a-b}\]

\[
\alpha + \beta = \frac{l(l+1)}{\lambda^2}, \quad \alpha - \beta = \frac{l'(l'+1)}{\lambda^2},
\]

\[a = \frac{|2l+1| + |2l'| + 1}{4\lambda} + \frac{1}{2\lambda}, \quad a' = a - \frac{1}{4\lambda}, \quad b = \frac{|2l+1| - |2l'| + 1}{4\lambda}, \quad l, l' < -1, \quad l, l' > 0
\]

2° we get the case substituting in \(1^0 \mid 2l+1\mid\) by \(-|2l+1|\);

3° we get the case substituting in \(1^0 \mid 2l'+1\mid\) by \(-|2l'+1|\);

4° we get the case substituting in \(1^0 \mid 2l+1\mid\) by \(-|2l+1|\) and \(\mid 2l'+1\mid\) by \(-|2l'+1|\);

1°

\[\phi_{10} = \frac{|2l+1| - 1}{2\lambda} \tanh x + \frac{b}{\cosh x}\]

\[F_1(\phi_{10}) = \frac{1}{4\lambda} \frac{(\phi_{10}^2 - a^2)^2 + 4\lambda(\phi_{10}^2 - a^2)(a(\phi_{10}^2 - a^2 - b^2) - ib\phi_{10}\sqrt{\phi_{10}^2 - a^2 - b^2})}{(b^2 - a^2)(\phi_{10}^2 - a^2) + 2a^2b^2 + 2ab\phi_{10}\sqrt{\phi_{10}^2 - a^2 - b^2}}\]

\[F_2(\phi_{10}) = (\phi_{10}^2 - a^2) \frac{a(\phi_{10}^2 - a^2 - b^2) - ib\phi_{10}\sqrt{\phi_{10}^2 - a^2 - b^2}}{(b^2 - a^2)(\phi_{10}^2 - a^2) + 2a^2b^2 + 2ab\phi_{10}\sqrt{\phi_{10}^2 - a^2 - b^2}}\]

\[c_1^\infty = -\frac{1}{a - ib} - \frac{1}{4\lambda(a - ib)^2}, \quad c_1^{-\infty} = -\frac{1}{a + ib} - \frac{1}{4\lambda(a + ib)^2};\]

\[c_2^\infty = \frac{1}{(a - ib)}, \quad c_2^{-\infty} = -\frac{1}{a + ib}\]

\[
\alpha = b^2 - \frac{l(l+1)}{\lambda^2}, \quad \beta = |b||2l+1|, \quad a = \frac{|2l+1| - 1}{2\lambda} > 0, \quad b > 0, \quad l > 0, \quad l < -1;
\]

4° we get the case substituting in \(1^0 \mid 2l+1\mid\) by \(-|2l+1|\) or allowing \(l\) to vary in \(1^0\) in the segment \(-1 < l < 0\) (the allowed \(b\) is then negative in both the cases).

Since the behaviour of the functions \(F_{1,2}\) in both of the above cases corresponds exactly to the assumptions we have done about them earlier to relate the conventional and supersymmetric JWKB formulae these relations have to be maintained also in both the cases considered. A
detailed demonstration of this is performed in Appendix 2. The latter illustrates a typical way of getting such relations.

All the results we have obtained in this section are the good illustrations of theorems we are going to formulate and to prove below.

Let \( V(x, \lambda) \equiv V_-(x, \lambda) \) satisfies the following assumptions:

1\(^0\) \( V_-(x, \lambda) \) being shape invariant is meromorphic on the \( x \)-plane having there finite second order poles and diverging at infinity quadratically or exponentially with \( x \) (the two latter possibilities excludes each other of course);

2\(^0\) The map: \( x \rightarrow \phi = \phi(x) \) defines a finitely sheeted Riemann surface \( R_\phi \) of the \( \phi \)-variable;

3\(^0\) All turning points of \( q(x) = V_\pm(x, \lambda) + \frac{\delta(x)}{\lambda} - E \) are transformed into \( R_\phi \) pairwise i.e. one pair into one sheet of \( R_\phi \) and each pair can be used to define equivalently the contour \( K \) in the quantization condition (5.1) (that is each such a contour surrounds the corresponding turning point pair);

4\(^0\) The functions \( F_1(\phi) \) and \( F_2(\phi) \) defined on \( R_\phi \) are holomorphic on \( R_\phi \) outside some circle of sufficiently large radius not branching at infinities of any sheet.

The above assumptions allow us for the following conclusions:

a. \( \phi(x) \) maps the contours \( K \) of the assumption 3\(^0\) into respective contours \( K_\phi \) on \( R_\phi \) which surround the corresponding maps of the turning point pairs on \( R_\phi \);

b. \( R_\phi \) is cut with the branch points \( \phi(x_k) \) satisfying: \( \phi'(x_k) = 0 \) where \( x_k \) are finite regular points of \( \phi(x) \). All these points lie outside all the contours \( K_\phi \) and are the square root branch points for \( F_1 \) and \( F_2 \). (The latter type of branching follows from the equalities: if \( \phi'(x_k) = 0 \) then close to \( x_k \) \( \phi'(x) \approx \alpha(x - x_k) \) and \( \phi \approx \alpha(x - x_k)^2/2 \);

c. If \( \phi'4 \) vanishes on \( R_\phi \) at some of its regular point \( \phi_0 \) linearly i.e. \( \phi'(x) \approx a(\phi - \phi_0) \) then the point is a map of an essential singularity of \( \phi(x) \) lying at infinity i.e. \( \phi \) approaches \( \phi_0 \) exponentially: \( \phi(x) \approx \phi_0 + Ce^{\alpha'x} \);

d. Second order poles of \( V_\pm(x, \lambda) \) as well as their infinite singular ones are transformed into infinities of different sheets of \( R_\phi \). The divergence to infinity of the \( F_{1,2} \)-functions is the following:

i. For \( V_-(x, \lambda) \) diverging as \( e^x \) for \( x \rightarrow \infty \) \( F_{1,2} \) diverge linearly with \( \phi \) when \( \phi \rightarrow \infty \) on a given sheet;
ii. For $V(x, \lambda)$ diverging as $x^2$ for $x \to \infty$ $F_{1,2}$ approaches constant values when $\phi \to \infty$;

iii. For $x$ close to a second order pole of $V(x, \lambda)$ $F_{1,2}$ diverge to infinity as $\phi^2$ when $\phi \to \infty$ on a sheet which vicinity of its infinity is a map of the corresponding vicinity of the pole $x_0$;

Let us add yet to the four above the following one more assumption:

50 The integrand of the following integrals:

\[
\oint_{K_\phi} \left[ \sqrt{\phi^2 \pm \frac{1}{\lambda^2} \phi' + \frac{\delta}{\lambda^2} - \tilde{E}} - \sqrt{\phi^2 - \tilde{E}} \right] \frac{d\phi}{\phi'}
\]  

(5.10)

where $K_\phi$ is any of the contours of the remark a. above do not possess outside the contours $K_\phi$ singularities different than those described in the conclusion b. .

From the assumptions 10 – 50 and from the remarks a.-d. the following two theorems come out:

Theorem 1  The SJWKB formulae with the superpotentials $\phi(x, \lambda)$ corresponding to $V_{\pm}(x, \lambda)$ are exact independently of whether the supersymmetry is exact or broken.

Theorem 2  The conventional JWKB formulae for $V_{\pm}(x, \lambda)$ are exact.

Proof of the Theorem 1.

The theorem follows from the following sequence of equalities:

\[
\oint_K (\phi^2(a) - \tilde{E})^\frac{1}{2} dx = \oint_K (\phi^2(a_1) - \tilde{E} + R(a_1))^\frac{1}{2} dx + \oint_K (f(F_1^{-}(a_1), \tilde{E} - R(a_1)) - f(F_1^{+}(a), \tilde{E})) dx = \ldots
\]

\[
= \oint_K (\phi^2(a_m) - \tilde{E} + R(a_1) + \ldots + R(a_m))^\frac{1}{2} dx +
\]  

(5.11)
\[
+ \sum_{p=1}^{m} \oint (f(F^{-(a_p)}, \tilde{E} - R(a_1) - \ldots - R(a_p)) - \\
- f(F^{+(a_{p-1})}, \tilde{E} - R(a_1) - \ldots R(a_{p-1}))) \, dx,
\]
\[
a_0 = a, \quad R(a_0) = 0
\]

where \(f(F^{\pm}, \tilde{E})\) is defined by:

\[
\oint_K \left[ \phi^2(a) \pm \frac{\phi'(a)}{\lambda} \frac{\delta}{\lambda^2} - \tilde{E} \right] \frac{1}{F_2} \, dx = \oint_K (\phi^2(a) - \tilde{E}) \frac{1}{F_2} \, dx + \oint_K f(F^{\pm}(a, \tilde{E})) \, dx \quad (5.12)
\]

with \(F^{\pm}_1 = \pm \phi' + \delta/\lambda\).

From assumption 3\(^0\) it follows that every of the contour integrals in the sum of the RHS of (5.11) when rewritten to be taken on some sheet of \(R_\phi\) can be taken on each sheet of \(R_\phi\) in the following way:

\[
\oint_K f(F^{\pm}(x, a), \tilde{E}) \, dx = \frac{1}{n} \sum_{r=1}^{n} \oint_{K_{\phi,r}} f(F^{\pm}(\phi, a), \tilde{E}) \frac{d\phi}{F_2(\phi)} \quad (5.13)
\]

where \(F_2(\phi) \equiv \phi'(x(\phi))\).

Now it follows further from assumption 5\(^0\) that every contour \(K_{\phi,r}, r = 1, \ldots, n\), can be deformed on a sheet which it is defined on to a circle of sufficiently large radius and to pieces of this contour which cancel mutually with analogous pieces of other contours. The net result of these deformations are the integrations performed on every sheet along the circle with sufficiently large radius. Outside the circle the integrated \(f\)'s (divided by \(F_2\)) are holomorphic and diverging to infinity not faster than the second power of \(\phi\). This guarantees that all these integrals can be calculated in the way similar to that used by us earlier. It is easy to check that independently of the type of the divergencies listed in the points i.-iii. above each infinity contributes the same to the sum (5.13) namely \(\mp i\pi/\lambda\) for the \(F^{\pm}_1\)–cases respectively. Therefore, the total value of the integral in the LHS of (5.13) is also \(\mp i\pi/\lambda\) accordingly. Finally the formula (5.11) becomes:

\[
\oint_K (\phi^2(a) - \tilde{E}) \frac{1}{F_2} \, dx = \oint_K (\phi^2(a_m) - \tilde{E} + R(a_1) + \ldots + R(a_m)) \frac{1}{F_2} \, dx + 2\pi im
\]
\[
a_0 = a, \quad R(a_0) = 0 \quad (5.14)
\]
Putting now in (5.14) $E = R(a_1) + \ldots + R(a_m) \equiv \tilde{E}_m$ we get the result (5.2) where for the broken supersymmetry the integer $m$ starts rather from $m = 1$. QED.

Proof of Theorem 2.

The claim of the theorem follows immediately from the formula (5.12) and from the above proof of the theorem 1. Namely, from (5.12) we get:

$$\oint_K \left[ \frac{\phi^2(a) \pm \frac{1}{\lambda} \phi'(a) + \frac{\delta}{\lambda^2} - R(a_1) - \ldots - R(a_m)}{2} \right] \, dx = (2m \mp 1)\pi \tag{5.15}$$

QED.

Some remarks are in order.

First if $F_{1,2}(\phi)$ diverged to infinity faster than $\phi^2$ then every integral of $f(F_{1,2})$ in (6.11) would contain $E$-dependent infinite series not reducing of course to simple values $\pm i\pi$ i.e. the relation (5.14) as well as (5.15) could not be valid any longer.

Second one can easily check that if $F_{1,2}(\phi)$ do not diverge to infinity faster than $\phi^2$ and the potentials $V_{\pm}(x, \lambda)$ are holomorphic then they have to satisfy the assumptions $1^0 - 5^0$ above.

Therefore we can draw a conclusion that our assumptions about the potentials $V_{\pm}$ and the functions $F_{1,2}$ fit in some way in with the property of $V_{\pm}$ being shape invariant. But as we have checked they are not determined in some unique way by the shape invariance condition (5.8). (For example, others than the second power rate of growth of $F_{1,2}(\phi)$ with $\phi$ are allowed by (5.8), see Appendix 3).

6 Discussion and conclusions

In this thesis we have demonstrated that there are two basic symmetries, a reflection: $x \rightarrow -x$ and a translation: $x \rightarrow x + a$, of potentials and of their corresponding Stokes graphs which decide whether the JWKB quantization formulae are only approximations to the exact formulae (2.10) or they are exact by themselves.

We have established also that despite the above two symmetries for the latter case to happen an additional property of the considered potentials and the corresponding Stokes
graphs has to be present. Namely, this is the simplicity of SG’s generated by the original potential expressing itself in no more than two turning points and in no more than two singular points in the basic period strips to appear. In the opposite case a proliferation of additional sectors in the basic period strip prevents the periodicity properties of the corresponding quantization conditions (2.10) to be used to reduce the conditions to the pure JWKB ones. The possible relaxation of these conditions has been described in Sec.4 and the corresponding examples were given in [1].

Altogether, the above two symmetries and the simplicity condition reduce effectively a number of exactly JWKB-quantized potentials to only eight of them. All of them have long been known. But due to our investigations we have given them the property of being rather exceptional.

We have also shown that the SUSY JWKB exact quantization formulae seem to be only different formulations of the exact conventional ones at least in the case of the translationally shape invariant potentials. We have supported the validity of this conclusion showing the exactness of the JWKB formulae for the two cases of the shape invariant potentials ($V_9$ and $V_{10}$ of Sec.5.1) not found by our earlier analysis of Sec.3. Additionally, our two theorems of Sec.5.1 suggest also that there is close relation between the property of being translationally shape invariant and the meromorphic structure of the considered potentials on the $x$–plane which is constrained to contain a limited number of second order poles (in the whole $x$–plane or in the basic period strip if the potential is periodic) and to have a particular behaviour at the infinity.

We would like also to stress that the earlier proofs of the exactness of some JWKB quantization formulae as done by Rosenzweig and Krieger [4, 5] and in the case of their SUSY forms by Crescimanno [11] are incorrect by erroneous calculations of necessary phases.

We have to note also that the results obtained by Inomata et al [27] for the form of the SUSY JWKB formulae in the cases of the broken supersymmetric potentials do not contradict ours since the latter concern their exact, not approximated forms which appear to coincide rather with those of Comtet et al [7].
Appendix 1

Here we show that for the following holomorphic \(2\pi i\)-periodic function:

\[ q(x, E, \lambda) = \sum_{n=l}^{k} q_n(E, \lambda)e^{nx} \quad (A1.1) \]

with even \(k-l\) and having only simple zeros its Weierstrass product representation is the following:

\[ q(x, E, \lambda) = Ce^{k-l \over 2} x \prod_{n \geq 1} \left[ 1 - {x \over x_n} \right] \quad (A1.2) \]

where \(C = q(x, E, \lambda)/x|_{x=0}\) or \(C = q(0, E, \lambda)\) if \(x = 0\) is not a root of \(q(x, E, \lambda)\).

The above formula follows from the observation that \(Q(x, E, \lambda) = q(x, E, \lambda) \cdot \exp(-k/2 - l/2)\) is also \(2\pi i\)-periodic and holomorphic with the same roots as \(q(x, E, \lambda)\) and therefore its WP representation should be:

\[ Q(x, E, \lambda) = Ce^{\alpha x} x \prod_{n \geq 1} \left[ 1 - {x \over x_n} \right] \quad (A1.3) \]

where \(\alpha\) is an integer by periodicity of \(Q\). On the other hand the representation \((A1.3)\) depends analytically on the coefficients \(q_n\) of \((A1.1)\) and we can always choose them in such a way to make \(Q\) symmetric under the reflection: \(x \rightarrow -x\). This operation does not change in \((A1.3)\) the product itself (the distribution of roots are then invariant under the operation) but changes \(e^{\alpha x}\) into \(e^{-\alpha x}\). However, \(\alpha\) being integer cannot change with analytic continuation od \(q_n\)’s and therefore it has to be zero from the very beginning.

As an example consider \(q(x, E, \lambda)\) given by \((3.4)\) for which its distribution of roots is shown in Fig.3. We have for it:

\[ \alpha e^{2x} - 2\beta e^x + \gamma = (\alpha - 2\beta + \gamma)e^x \prod_{n \geq 1} \left[ 1 - {x \over x_n} \right] \quad (A1.4) \]

We want to calculate with the help of \((A1.4)\) a change of phase of \(q(x, E, \lambda)\) when transporting it from a point \(x_0\) of the line \(\Im x = \pi\) to the point \(x_0 - 2\pi i\) of the line \(\Im x = -\pi\). To
this goal we note that as it follows from (A1.4) the roots of \( q(x, E, \lambda) \) lying in large distances from the points considered almost do not contribute to the values of \( q(x, E, \lambda) \) in the considered strip (their product in (A1.4) is close to 1). Therefore we can take a sufficiently large but finite number of roots around the considered points to perform the calculations needed (eventually we can take the limit of the infinite number of roots).

Starting from the point \( x_0 \) we can consider \( n \) pairs of roots lying above the line \( \Im x = \pi \) (\( n \) is large) and \( n \) pairs of roots lying below the line. The arguments of \( x_0 - x_k \) we take to be positive for \( x_k \) lying below the line \( \Im x = \pi \) and negative in the opposite case. It is clear that the net result of summing the corresponding arguments of the product in (A1.4) is zero. But there is still non zero contribution to the argument of \( q(x_0, E, \lambda) \) coming from the factor \( e^x \) of (A1.4). It amounts of course to \( \pi \) and this is the total argument of \( q(x_0, E, \lambda) \).

At the points \( x_0 - \pi i \) our calculations are similar. Keeping the same set of roots as chosen previously we see that to the total phase of the product at \( x_0 - \pi i \) contribute only the two most distant pairs of roots lying above the line \( \Im x = \pi \) so according to our convention this contribution amounts to \( 4(-\pi/2) = -2\pi \) (in the limit of the root number going to infinity). Together with the argument \(-\pi\) provided by the factor \( e^x \) we get the argument of \( q(x_0 - \pi i, E, \lambda) \) to be equal to \(-3\pi\). Therefore the total change of the argument of \( q(x, E, \lambda) \) between the lines considered is equal to \(-4\pi\).

**Appendix 2**

We demonstrate here particularities of our statement that the exactness of the conventional JWKB formula for the potential \( V_9(x, \lambda) \equiv V_{9,-}(x, \lambda) \), the first one of those in (5.9), follows from its SJWKB one. For the potential \( V_{10}(x, \lambda) \) our considerations would be similar.
To this end let us consider the relation (5.12) using the superpotentials \( \phi_9(x, \lambda) \) given above. First consider the case 1\(^0\) of the exact supersymmetry. The corresponding Riemann surface \( R_{\phi_9} \) is depicted on Fig.16. This is two sheeted surface with the branch points at \( \phi_9 = \pm i(a^2 - b^2)^{1/2} \). The latter are the unique singularities of the integrand of the following integral:

\[
\oint_{K} \phi_9 \left[ \sqrt{\phi_9^2 - \frac{1}{\lambda} F_1(\phi_9) - \tilde{E}} - \sqrt{\phi_9^2 - \tilde{E}} \right] \frac{d\phi_9}{F_2(\phi_9)}
\]

(A2.1)

(since roots of \( F_2 \) at \( \phi = \pm ia \) are also the roots of \( F_1 \)).

\( R_{\phi_9} \) is, as it can be easily noticed, a map of the basic period strip \(-\pi \leq x \leq \pi\) of the \( x \)-plane (see Fig.15), so that the four turning points of \( q_{9,-}(x, \lambda, E) \) from this strip are mapped pairwise into \( R_{\phi_9} \): the two from the segment \((-\pi/2, \pi/2)\) into the sheet a) of Fig.16 and the other two into the second one. It is also easy to note that in the quantization formulæ (5.1) and (5.2) the contour \( K_1 \) on Fig.15 can be substituted by the contour \( K_2 \) of the figure by the periodicity. The contours are mapped into \( R_{\phi_9} \) as \( K_{1,\phi} \) and \( K_{2,\phi} \) respectively, the latter surrounding the respective pairs of the turning point pictures on \( R_{\phi_9} \) (see Fig.16). Therefore for the quantization formulæ (5.1) and (5.2) we can write:

\[
-\lambda \oint_{K_1} \sqrt{\phi_9^2 - \frac{1}{\lambda} \phi_9' + \delta - \tilde{E}} dx = -\lambda \left[ \oint_{K_1} + \oint_{K_2} \right] \sqrt{\phi_9^2 - \frac{1}{\lambda} \phi_9' + \delta - \tilde{E}} dx \quad (A2.2)
\]
\[ = -\lambda \oint_{K_1} \sqrt{\phi_9^2 - \tilde{E}} dx - \frac{\lambda}{2} \left[ \oint_{K_{1,\phi}} + \oint_{K_{2,\phi}} \right] \left[ \sqrt{\phi_9^2 - \frac{1}{\lambda} F_1(\phi_9) - \tilde{E}} - \sqrt{\phi_9^2 - \tilde{E}} \right] \frac{d\phi_9}{F_2(\phi_9)} = \]

\[ = -\lambda \oint_{K_1} \sqrt{\phi_9^2 - \tilde{E}} dx - \frac{\lambda}{2} \left[ \oint_{K_{\infty_1,\phi}} + \oint_{K_{\infty_2,\phi}} \right] \left[ \sqrt{\phi_9^2 - \frac{1}{\lambda} F_1(\phi_9) - \tilde{E}} - \sqrt{\phi_9^2 - \tilde{E}} \right] \frac{d\phi_9}{F_2(\phi_9)} \]

where \( K_{\infty_1,\phi} \) and \( K_{\infty_2,\phi} \) are the contours obtained by an obvious deformations of the contours \( K_{1,\phi} \) and \( K_{2,\phi} \) which contain all the singularities of \( F_{1,2}(\phi_9) \). Making use of the explicite forms of \( F_{1,2}(\phi_9) \) as given in Sec.5 we can calculate the last integral in (A2.2) getting for it the value \(+i\pi\). Altogether with (5.2) this gives the result (5.1).

Consider now the broken case \( 2^0 \) of the superpotential \( \phi_9 \). The corresponding basic period strip of \( q_{0,-}(x,\lambda,E) \) and the quantization contours \( K_1 \) and \( K_2 \) transform into \( R_{\phi_9} \) as it is shown in Fig.17. Once again we can write the sequence analogous to (A2.2) deforming the contour \( K_{1,\phi} \) and \( K_{2,\phi} \) of Fig.17 into \( K_{\infty_1,\phi} \) and \( K_{\infty_2,\phi} \) respectively to perform the final integration getting again \(+i\pi\) and consequently the exact formula (5.1). Of course, the starting value of \( m \) can be now zero.

Fig.17 The two sheeted \( \phi_9 \)-Riemann surface for broken superpotential \( \phi_9 \) (the case \( 2^0 \))

Appendix 3

We shall show here that the shape invariance condition (5.8) does not prevent in some obvious way for \( F_{1,2}(\phi) \) to diverge with any power of \( \phi \) when \( \phi \to \infty \). To this end let us
rewrite (5.8) in terms of superpotentials. We get:

\[ \phi^2(x, \lambda, a) + \frac{1}{\lambda} \phi'(x, \lambda, a) = \phi^2(x, \lambda, a_1) - \frac{1}{\lambda} \phi'(x, \lambda, a_1) + R(a_1) \]  \hspace{1cm} (A3.1)

\[ a_1 = f(a) \]

Introducing farther to (A3.1) the function \( F_2(\phi, a) \) (\( \equiv \phi'(x(\phi, a), a) \)) we obtain:

\[ F_2(\phi, a) = \lambda \frac{2\phi \Delta(\phi, a) + \Delta^2(\phi, a) + R(f(a))}{2 + \Delta'_{\phi}(\phi, a)} \]  \hspace{1cm} (A3.2)

where \( \Delta(\phi, a) \) is defined as:

\[ \Delta(\phi, a) = \tilde{\Delta}(x(\phi, a), a) \]  \hspace{1cm} (A3.3)

\[ \phi(x, a_1) = \phi(x, a) + \tilde{\Delta}(x, a) \]

It follows from (A3.2) that the behaviour of \( F_2(\phi, a) \) when \( \phi \to \infty \) comes out from the corresponding behaviour of \( \Delta(\phi, a) \). The latter, however, under the assumption that \( \phi = \infty \) is at most a pole for it has to be following:

\[ \Delta(\phi, a) = \sum_{k \geq 0} b_k(a) \phi^{-k+1} \]  \hspace{1cm} (A3.4)

i.e. this pole has to be simple at most.

The last equation is a conclusion of the condition:

\[ x(\phi + \Delta(\phi, a), a_1) = x(\phi, a) \]  \hspace{1cm} (A3.5)

under which the shape invariance property (A3.1) is satisfied. Note also that due to equality: \( x'_{\phi}(\phi, a) = 1/F_2(\phi, a) \), the following relation is coming out from (A3.5):

\[ F_2(x(\phi + \Delta(\phi, a), f(a)) = (1 + \Delta'_{\phi}(\phi, a)) F_2(\phi, a) \]  \hspace{1cm} (A3.6)
It is now easy to conclude from (A3.2) that if $b_0 \neq 0, -2$ then $F_2(\phi, a)$ grows as $\phi^2$ when $\phi \to \infty$. But for example if $b_0 = -2$ and $b_1, b_n \neq 0$ with $b_2, \ldots, b_{n-1} = 0$, $n \geq 2$, then $F_2(\phi, a)$ has to grow as $\phi^{n+1}$ when $\phi \to \infty$.

Of course, whether $\Delta(\phi, a)$ can really behave in the above ways depends totally on the properties of the superpotentials considered which on their own are constrained by (A3.6).

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