Adjoint-state method for Hybridizable Discontinuous Galerkin discretization: application to the inverse acoustic wave problem

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Abstract

In this paper, we perform non-linear minimization using the Hybridizable Discontinuous Galerkin method (HDG) for the discretization of the forward problem, and implement the adjoint-state method for the efficient computation of the functional derivatives. Compared to continuous and discontinuous Galerkin discretizations, HDG reduces the computational cost by using the numerical traces for the global linear system, hence removing the degrees of freedom that are inside the cells. It is particularly attractive for large-scale time-harmonic quantitative inverse problems which make repeated use of the forward discretization as they rely on an iterative minimization procedure. HDG is based upon two levels of linear problems: a global system to find the solution on the boundaries of the cells, followed by local systems to construct the solution inside. This technicality requires a careful derivation of the adjoint-state method, that we address in this paper. We work with the Euler’s equations in the frequency domain and illustrate with a three-dimensional experiment using partial reflection-data, where we further employ the features of DG-like methods to efficiently handle the topography with p-adaptivity.

1 Introduction

Quantitative inverse wave problems aim to recover the physical medium parameters that characterize the wave propagation from partial observations of the phenomenon. This inverse scattering problem arises, for instance, in geophysics for the identification of Earth’s properties, [51, 70, 61, 73, 35], in medical imaging or in mechanical engineering for the non-destructive testing, see, e.g., [11, 24, 65, 25, 62, 6, 10] and the references therein.

In the framework of quantitative inversion, the measurements of the waves (e.g., mechanical or electromagnetic), \(d\), are used to define a misfit functional \(J\) and the reconstruction of the parameters is recast as a non-linear minimization problem,

\[
\min J(m), \quad \text{with} \quad J(m) = \frac{1}{2} \left\| F(m) - d \right\|^2,
\] (1)

where the forward problem \(F\) is the map from the model parameters \(m\) to the observable (i.e., the quantities measured at the position of the receivers). That is, simulations of the wave phenomenon are compared with the measurements to successively update the model parameters. Our misfit criterion in (1) is the \(L^2\) difference and least-squares problems are further analyzed in [17, 8]. Several alternatives for the misfit have been investigated, we refer to, e.g., [53, 68, 36, 14, 55, 31, 74, 2, 32]. Furthermore, one can incorporate a regularization term in (1) to reduce the ill-posedness, by the means of additional constraints, see, e.g., [64, 27, 48, 45, 44, 34]. Note

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that the use of a different misfit criterion or the incorporation of regularization terms generates only minor modifications of the analysis we provide here.

The resolution of (1) using deterministic optimization techniques makes use of algorithms in the family of the Newton method. It consists in successive updates of the model parameters where, at iteration $k$,

$$m_{k+1} = m_k + \rho_k s_k.$$  

(2)

In the full Newton approach, the search direction for the update, $s$, depends on the gradient and the Hessian of the misfit functional. To reduce the numerical cost, alternatives that avoid the full Hessian computation are often employ, with Quasi or Truncated-Newton methods such that BFGS and L-BFGS [58], conjugate gradient approach [41, 56], approximated pseudo-Hessian [19], or Landweber iterations [54]. We refer to [59] for a review of methods for local optimization. Then, the scalar step $\rho$ is selected to obtain an appropriate amplitude of the updates, using line-search algorithms, e.g., [59, 17]. In our implementation, we also avoid the computation of the Hessian such that, at each iterations of the minimization, one must

1. solve the forward problem using the current model parameters,

2. compute the gradient of the misfit functional,

Non-linear minimization suffers from local minima, which cannot be avoided with the deterministic approach, see, e.g., [15, 69, 8, 33] in the context of seismic. We can mention the use of statistical-based methods but in the large-scale applications we have in mind, such approaches remain unusable at the moment.

**Discretization of the forward problem** The iterative minimization procedure is computationally intensive for large-scale applications because the forward problem must be solved at each iteration, and for all sources that generate the data. For instance, in seismic applications, this easily amounts to solving the problem for several hundreds or thousand of sources, at each iterations. For this reason, in the time-harmonic framework that we consider, we rely on a direct solver to handle the linear system generated by the discretization, as it enables for multi right-hand sides (rhs), that is, once the matrix factorization is obtained, the solution of the many rhs is computationally cheap. In particular, we use the solver Mumps [4, 5]. On the other hand, this factorization possibly requires large amounts of computational memory.

Therefore, the choice of discretization methods plays a crucial role in the numerical efficiency. While Finite Differences (FD) have early been employed for the discretization of wave problems (e.g., [72, 43]), it works with a cartesian grid which makes it difficult to handle complex geometry of parameters and topography, [63]. One can instead rely on methods based upon an unstructured mesh of the domain to have more flexibility, such that the Finite Element (Continuous Galerkin, CG) method (e.g., [18, 1, 29]), or on the Spectral Element Method (SEM), popularized in seismic applications in [50, 49].

Then, the Discontinuous Galerkin (DG) method has been introduced, [26, 42, 13]. As the name indicates, DG works with discontinuous basis functions, independently defined on each cell. It offers additional features compared to CG as it easily handle the $h$ and $p$-adaptivity, that is, the use of cells of different size and the use of different order of polynomial between cells, respectively, [42]. On the other-hand, the DG method, by working with discontinuous basis functions, usually results in an increase in the number of degrees of freedom (dof) as these are not shared between the elements.

The Hybridizable Discontinuous Galerkin method (HDG, also referred to as Local DG, LDG) allows to keep the features of the DG while avoiding the oversized linear systems. It shows a growing interest, e.g., [23, 7, 21, 22, 39, 47, 12, 30, 9], and we refer to the introduction of [21] for
the background on the development of the method, and historical references. Concretely, the
global linear system in HDG only contains the dof that are on the faces of the elements, such
that all interior ones are eliminated. Consequently, the method is shown to be more efficient
(less memory consumption) than CG or usual DG depending on the order of the polynomials [47, 12]. Therefore, it is the perfect candidate to handle the large-scale inverse problem, as we
can account for complex geometry using smaller linear systems. We provide in Section 3 the
steps for the implementation: contrary to the traditional discretization method, it consists in two
stages: first to compute the numerical traces (the global system only with the dof on the faces
of the elements), secondly we solve the local sub-problems to build the volume approximation.

Gradient computation For the iterative reconstruction, the gradient of the misfit func-
tional (1) must be computed: \( \nabla J = DF^* (F - d) \). To avoid the explicit computation of the
Fréchet derivative \( DF \), the adjoint-state method has been designed and instead compute the
action of \( DF \), which is sufficient to extract the gradient, as we detail in Section 4. The method
originates from the work of [52] with early application in [16]. It is now commonly used in applica-
tions and we refer to [60, 17, 46, 10, 8]. With ‘direct’ discretization method (CG, SEM, FD), the adjoint-state method relies on the resolution of a backward problem, which reduces to
the adjoint of the forward problem with the residuals (the difference \( F - d \)) at the right-hand
side. Because of the two-stages in the HDG discretization (the global and local systems), the
adjoint-state method must be appropriately derived, see Section 4.

The contribution of our work is to derive the methodology to solve the non-linear minimiza-
tion problem using HDG discretization, with the perspective of large-scale applications currently
intractable. We consider the inverse acoustic wave problem for the identification of the medium
parameters associated with the Euler’s equations, see Section 2. The discretization of the wave
equation using the HDG method is given in Section 3, where we emphasize the differences com-
pared to the more traditional ‘direct’ discretization methods. In Section 4, the adjoint-state
method is provided in the framework of HDG, hence with the specificity of working with global
and local sub-problems. Eventually, we illustrate the reconstruction procedure using HDG with
a three-dimensional experiment in Section 5, where we consider a medium with topography to
fully use the features of the method, with \( p \)-adaptivity, and give some additional perspectives in
Section 6.

2 Inverse wave problem in acoustics

2.1 Time-harmonic wave propagation

We consider the propagation of time-harmonic waves in an acoustic medium in three dimensions
such that \( \Omega \in \mathbb{R}^3 \), with boundary \( \Gamma \). We denote by \( \mathbf{x} \) the 3D space coordinates \( \mathbf{x} = \{x, y, z\} \), and an initial (scalar) time-harmonic source \( f \). The scalar pressure field, \( p : \Omega \rightarrow \mathbb{C} \), and the
vectorial velocity, \( \mathbf{v} : \Omega \rightarrow \mathbb{C}^3 \), verify the Euler’s equations,

\[
\begin{align*}
-\sigma \rho(x) \mathbf{v}(x) + \nabla p(x) &= 0, \quad \text{in } \Omega, \\
-\frac{\sigma}{\kappa(x)} p(x) + \nabla \cdot \mathbf{v}(x) &= f(x), \quad \text{in } \Omega, \\
\alpha(x) p(x) + \beta(x) \partial_\nu p(x) &= 0, \quad \text{on } \Gamma.
\end{align*}
\]

(3a) (3b) (3c)

The propagation is governed by the physical properties of the medium: the density \( \rho \in \mathbb{R} \) and
the bulk modulus \( \kappa \in \mathbb{R} \). One can also use the wave speed \( c \):

\[
c(x) = \sqrt{\kappa(x)} \rho(x)^{-1}.
\]

(4)
We work with a complex frequency denoted $\sigma$, such that
\[ \sigma = i\omega - s, \] (5)
with $\omega$ the angular frequency. While the “usual” frequency-domain formulation would take $s = 0$, this notation is useful in order to work in the “Laplace–Fourier” domain for the inverse problem, see [66, 67, 31]. It can also serve to incorporate attenuation or viscous behavior, as in helioseismology, e.g., [40]. In the latest case, $\sigma$ can depend on the spatial coordinate; we disregard this case in the following, but it only implies minimal modification of the below analysis. As an alternative, one can also consider complex-valued wave speed [71]. Similarly, having complex-valued parameters would not change our analysis.

Out of generality, we have considered a Robin boundary condition on the boundary $\Gamma$ of the domain (see also Remark 2), (3c), with coefficients $\alpha$ and $\beta$, using $\partial_{\nu}$ to denote the normal derivative. It is written with respect to the pressure field but using (3), we can equivalently define the Robin condition in terms of the velocity, or using a combination of both, such that
\[
\alpha(x)p(x) + \sigma \rho(x)\beta(x)v_{\nu}(x) = 0, \quad \text{alternative Robin condition, (6)}
\]
where $v_{\nu} = v \cdot \nu$ indicates the normal velocity. In particular, in the case of absorbing boundary conditions (ABC, [28]) for acoustic media, the Robin boundary condition becomes
\[
\begin{align*}
-\sigma c(x)^{-1} p(x) + \partial_{\nu} p(x) &= 0, & \text{ABC in pressure, (7)} \\
-(c(x)\rho(x))^{-1} p(x) + v_{\nu}(x) &= 0, & \text{pressure/velocity ABC. (8)}
\end{align*}
\]

**Remark 1.** One can replace the velocity field in (3b) using (3a) to obtain the second-order formulation which only depends on the pressure field:
\[
\frac{\sigma^2}{\kappa(x)} p(x) - \nabla \cdot \left( \frac{1}{\rho(x)} \nabla p(x) \right) = -\sigma f(x), \quad \text{in } \Omega.
\] (9)

In the context where the density is constant, it further simplifies to the Helmholtz equation.

### 2.2 Quantitative identification of the physical parameters

In the framework of inverse problem, one wants to identify the model parameters that characterize the propagation, $\kappa$ and $\rho$, from the measurements of the waves. In the quantitative approach, the reconstruction follows an iterative minimization procedure, that we describe in this section.

We first define the forward problem $F$ to give the solution of (3) at a restricted set of positions, for the model parameters $m = \{ \kappa, \rho \}$. For a source $f$ and frequency $\sigma$, we have
\[
F(m, \sigma, f) = \{ p(m, \sigma, x_1, f), \ldots, p(m, \sigma, x_{n_{\text{rcv}}}, f) \},
\] (10)
where the $x_k$ are a discrete set of positions, that is, the forward problem gives measurements obtained from $n_{\text{rcv}}$ receivers. Here we have considered measurements of the pressure fields (commonly employed in seismic applications), but we can proceed similarly with the velocity, or with both, see, e.g., [2, 32].

The observed measurements are denoted by $d$; these can be seen as a forward problem associated with a target unknown model with added noise. The identification of parameters follows a minimization of a misfit functional $J$ which, in the least-squares framework, is
\[
J(m) = \frac{1}{2} \sum_{\sigma} \sum_k \| F(m, \sigma, f_k) - d(\sigma, f_k) \|_2^2,
\] (11)
where we consider several sources to generate the data. The sum over the frequencies is usually decomposed into sub-sets, following a progression from low to high contents, [15, 69, 10, 33]. As mentioned in the introduction, several alternatives to the least-squares functional have been studied, and it can also be enriched with regularization terms, see the above references.

The minimization of (11) is conducted following Newton-type algorithm with iterative updates of an initial model. At iteration \( k \), the model is updated according to (2), and we refer to [59] for an extensive review of methods. The search direction is computed from the gradient of the misfit functional and in large-scale optimization, the Hessian is usually too cumbersome and one can use its approximation (e.g., Limited-BFGS method) or only the gradient (non-linear conjugate gradient methods).

For the computation of the gradient, we refer Section 4 where we provide the steps for its computation with the adjoint-state method using HDG discretization. We illustrate experiments of reconstruction in Section 5.

3 Hybridizable Discontinuous Galerkin discretization

For numerical applications, the first step is to discretize the wave equation, and we follow the Hybridizable Discontinuous Galerkin method, that works with the first-order problem. As mentioned in the introduction, this approach has the advantage to reduce the size of the global linear system compared to Continuous Galerkin (depending on the order of the polynomials), hence allowing to solve test-cases of larger scales.

3.1 Notation

3.1.1 Domain discretization

The domain is discretized using a non-overlapping partition of \( \Omega \). The mesh of the domain is denoted \( \mathcal{T}_h \). It is composed of \( N \) elements/cells such that

\[
\mathcal{T}_h = \bigcup_{e=1}^{N} K_e.
\]  

(12)

The set of the \( N_\Sigma \) faces \( \Sigma \) is decomposed into the \( N_\Sigma^I \) interior ones (between two adjacent cells), \( \Sigma^I \), and the \( N_\Sigma^B \) exterior ones (between the medium and the exterior), \( \Sigma^B \):

\[
\Sigma = \bigcup_{k=1}^{N_\Sigma} f_k, \quad \Sigma^B = \Sigma \cap \Gamma, \quad \Sigma^I = \Sigma \setminus \Sigma^B, \quad N_\Sigma = N_\Sigma^I + N_\Sigma^B.
\]  

(13)

In our implementation, we use simplex cells, thus triangles for two-dimensional domains and tetrahedra in three dimensions. Consequently, the face of a cell \( f \) is either a segment (in 2D) or a triangle (in 3D).

3.1.2 Function and discretization spaces

The space of discretization consists in piecewise polynomial of order less than or equal to \( p \). In dimension three, the space of polynomial for simplexes is given by

\[
P_p = \left\{ g(x) = g(x,y,z) = \sum_{i,j,k=0}^{p} g_{ijk} x^i y^j z^k \mid i + j + k \leq p \right\},
\]  

(14)
where the $g_{ijk}$ are scalar coefficients. Note that in DG methods, the polynomials are defined separately on each cell (i.e., piecewise), allowing discontinuities. We introduce the following function spaces, associated with the polynomial on a cell $K_e$ and the cell faces $\partial K_e$, such that, for all $e \in 1, \ldots, N$:

\begin{align}
W_h &= \{ w_h \in L^2(\Omega), \quad w_h |_{K_e} \in P_{p_e}(K_e), \quad \forall K_e \in T_h \}, \\
W_h &= \{ w_h \in (L^2(\Omega))^3, \quad w \cdot n |_{K_e} \in P_{p_e}(K_e), \quad \forall K_e \in T_h \}, \\
U_h &= \{ u_h \in L^2(\Sigma), \quad u_h |_{f_k} \in P_{q_k}(f_k), \quad \forall f_k \in \Sigma \}.
\end{align}

(15a) (15b) (15c)

Therefore, we allow for a different order of polynomial on each cell and each face.

### 3.1.3 Jump operator

We define the jump of a quantity between two adjacent cells, that we denote by brackets. On the interface $f$ shared by two cells (elements) $K_e^+$ and $K_e^-$, the jump of $w \cdot \nu$ is

$$[w \cdot \nu]_f := w_{K_e^+}^+ \cdot \nu^+_f + w_{K_e^-}^- \cdot \nu^-_f = w_{K_e^+}^+ \cdot \nu^+_f - w_{K_e^-}^- \cdot \nu^-_f,$$

(16)

where, by convention, $\nu^\pm$ points outward of $K_e^\pm$, as illustrated in Figure 1.

![Figure 1: Inward and outward normals at the interface between two triangle cells.](image)

### 3.2 Local problem for the HDG discretization

Upon assuming that the right-hand side (i.e. the source function) $f \in L^2(\Omega)$, we can write the variational formulation for (3a) and (3b), where we use test functions $\phi(x) \in L^2(\Omega)$ and $\psi(x) \in (L^2(\Omega))^3$. Over each cell $K_e$ of the domain mesh, we have:

\begin{align}
\int_{K_e} \left( -\sigma \rho \mathbf{v} \cdot \bar{\psi} + \nabla p \cdot \bar{\psi} \right) \, \mathrm{d}K_e &= 0, \\
\int_{K_e} \left( -\sigma \kappa^{-1} p \bar{\phi} + (\nabla \cdot \mathbf{v}) \bar{\phi} \right) \, \mathrm{d}K_e &= \int_{\partial K_e} f \bar{\phi} \, \mathrm{d}\partial K_e,
\end{align}

(17a) (17b)

where $\bar{\cdot}$ denotes the conjugation and we omit the space dependency for clarity. Integration by part gives

\begin{align}
\int_{K_e} \left( -\sigma \rho \mathbf{v} \cdot \bar{\psi} - p (\nabla \cdot \bar{\psi}) \right) \, \mathrm{d}K_e + \int_{\partial K_e} p \, \bar{\psi} \cdot \mathbf{v} \, \mathrm{d}\partial K_e &= 0, \\
\int_{K_e} \left( -\sigma \kappa^{-1} p \bar{\phi} - \mathbf{v} \cdot \nabla \bar{\phi} - f \bar{\phi} \right) \, \mathrm{d}K_e + \int_{\partial K_e} \bar{\phi} \, \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}\partial K_e &= 0.
\end{align}

(18a) (18b)

Note that the integral over the cell boundary can be substituted by a sum over the faces of the cell such that, e.g.,

$$\int_{\partial K_e} p \, \bar{\psi} \cdot \mathbf{v} \, \mathrm{d}\partial K_e = \sum_{f \in \partial K_e} \int_f p \, \bar{\psi} \cdot \mathbf{v}_f \, \mathrm{d}f.$$

(19)
The solutions $p$ and $\mathbf{v}$ are approximated by polynomials of order $p_e$ on the cell, respectively with the discretization variables

$$p_h \in W_h \quad \text{and} \quad \mathbf{v}_h \in W_h,$$

and we refer to $\hat{p}_h \in U_h$ for the numerical trace of $p$.

By introducing the discretization variables in (18), we obtain,

$$\begin{cases}
\int_{K_e} \left( - \sigma \rho \mathbf{v}_h^{(e)} \cdot \mathbf{w} - p_h^{(e)} (\nabla \cdot \mathbf{w}) \right) \, dK_e + \int_{\partial K_e} \hat{p}_h^{(e)} \mathbf{w} \cdot \mathbf{n} \, d\partial K_e = 0, \\
\int_{K_e} \left( - \sigma \kappa^{-1} p_h^{(e)} \mathbf{w} - \mathbf{v}_h^{(e)} \cdot \nabla \mathbf{w} - f \, \mathbf{w} \right) \, dK_e + \int_{\partial K_e} \hat{\mathbf{v}}_h^{(e)} \mathbf{w} \cdot \mathbf{n} \, d\partial K_e = 0
\end{cases} \quad (21)$$

where $p_h^{(e)} \in \mathbb{P}_{p_e} = p_h|_{K_e}$, similarly for $\mathbf{v}_h^{(e)}$, according to (15). The essence of the HDG method is to formulate the numerical flux $\hat{\mathbf{v}}_h^{(e)}$ such that [(20, 21, 22)]

$$\hat{\mathbf{v}}_h^{(e)} = \mathbf{v}_h^{(e)} + \tau (p_h^{(e)} - \lambda_h^{(e)}) \mathbf{v}^{(e)}, \quad \text{with} \quad \lambda_h^{(e)} = \hat{p}_h^{(e)}, \quad (22)$$

where $\mathbf{v}^{(e)}$ denotes the normal on the boundary of $K_e$. Here, $\tau$ is a penalization/stabilization parameter defined on each face and allowed to have distinct values for the two sides of the faces, that is, as illustrated in Figure 1, on a face $\hat{f}$ we distinguish between $\tau^+ \hat{f}$ and $\tau^- \hat{f}$. It is required that $\tau > 0$ for uniqueness, see [7, 21, 57]. Practically, following the work of [57] for diffusion problems, we use

$$\tau^\pm = \rho^{-1}(K_e^\pm), \quad \text{for a piecewise constant (per cell) density model.} \quad (23)$$

On the one hand, $p_h$ and $\mathbf{v}_h$ are piecewise polynomial on the cells, thus allowing discontinuities between two adjacent cells (Figure 1); on the other hand, $\lambda_h$ is defined on the ‘skeleton’ of the mesh (i.e., on the faces of the cells) and allows discontinuity only at the nodes in two dimensions, and at the edges in three dimensions, see Figure 2.

In the following, we omit the cell exponent $^{(e)}$ for the sake of clarity. Let us first incorporate $\lambda_h$ in (21a), we have,

$$\int_{K_e} \left( - \sigma \rho \mathbf{v}_h \cdot \mathbf{w} - p_h (\nabla \cdot \mathbf{w}) \right) \, dK_e + \int_{\partial K_e} \lambda_h \mathbf{w} \cdot \mathbf{n} \, d\partial K_e = 0. \quad (24)$$

Next, for (21b), we replace the numerical flux by (22):

$$\int_{K_e} \left( - \sigma \kappa^{-1} p_h \mathbf{w} - \mathbf{v}_h \cdot \nabla \mathbf{w} - f \, \mathbf{w} \right) \, dK_e + \int_{\partial K_e} \mathbf{v}_h + \tau (p_h - \lambda_h) \mathbf{v} \, d\partial K_e = 0. \quad (25)$$

Using the revert integration by part, it gives

$$\int_{K_e} \left( - \sigma \kappa^{-1} p_h \mathbf{w} + (\nabla \cdot \mathbf{v}_h) \mathbf{w} - f \, \mathbf{w} \right) \, dK_e + \int_{\partial K_e} \tau (p_h - \lambda_h) \mathbf{w} \, d\partial K_e = 0. \quad (26)$$

### 3.3 Continuity condition for the HDG discretization

To complete the problem, the continuity of the discretized velocity field is enforced and we impose that, on all of the cell faces, the jump (16) equates zero. For any test function $\xi \in U_h$, we write

$$\forall \xi \in \Sigma, \quad \int_{\hat{f}} \left[ \hat{\mathbf{v}}_h \cdot \mathbf{n} \right] \xi \, d\hat{f} = 0, \quad (27)$$
with \( \nu_l \) the normal on the face. For an inner boundary, we inject (22) and use the jump definition (16):

\[
\forall \xi \in \Sigma^I, \quad \int_{\xi} [\hat{v}_h \cdot \nu_l] \xi \, df = \int_{\xi} \left( \hat{v}_h^{(+)} \cdot \nu_l^+ + \hat{v}_h^{(-)} \cdot \nu_l^- \right) \xi \, df = \int_{\xi} \left( \nu_l^+ \cdot \nu_l^+ + \tau^+(p_h^- - \lambda_h) + \nu_l^- \cdot \nu_l^- + \tau^-(p_h^- - \lambda_h) \right) \xi \, df,
\]

where the exponent \( \pm \) indicates the adjacent cells, cf. Figure 1. On a face that belongs to the outer boundary, only one side remains, and we have

\[
\forall \xi \in \Sigma^B, \quad \hat{v}_h |_{\xi \in \Sigma^B} = v_h + \tau(p_h - \lambda_h) \nu_l.
\]

Then, using the specified Robin boundary condition, \( \hat{v}_h \) must also verifies (6), such that,

\[
\forall \xi \in \Sigma^B, \quad \hat{v}_h \cdot \nu_l = -\frac{\alpha}{\sigma \rho \beta} \lambda_h = -\frac{\alpha}{\sigma \rho \beta} \lambda_h = v_h \cdot \nu_l + \tau(p_h - \lambda_h),
\]

where we also replace the trace of \( p \) by \( \lambda_h \) in the Robin condition.

Eventually, we summarize the complete HDG problem.

\[
\begin{align*}
&\int_{\xi} \left( -\sigma \rho \nu_l \cdot \bar{\psi} - p_h (\nabla \cdot \bar{\psi}) \right) dK_e + \sum_{\xi \in \partial K_e} \lambda_h \bar{\psi} \cdot \nu_l \, df = 0, \quad \forall K_e \in T_h, \\
&\int_{\xi} \left( -\sigma \kappa^{-1} p_h \bar{\phi} + (\nabla \cdot v_h) \bar{\phi} - f \bar{\phi} \right) dK_e + \sum_{\xi \in \partial K_e} \tau_l (p_h - \lambda_h) \bar{\phi} \, df = 0, \quad \forall K_e \in T_h, \\
&\int_{\xi} \left( v_h^+ \cdot \nu_l^+ + \tau^+_l (p_h^+ - \lambda_h) + v_h^- \cdot \nu_l^- + \tau^-_l (p_h^- - \lambda_h) \right) \xi \, df = 0, \quad \forall \xi \in \Sigma^I, \\
&\int_{\xi} \left( v_h \cdot \nu_l + \tau_l p_h + \left( \frac{\alpha}{\sigma \rho \beta} - \tau_l \right) \lambda_h \right) \xi \, df = 0, \quad \forall \xi \in \Sigma^B.
\end{align*}
\]

The two first equations represent local problems, on each cell, while the last two equations give the conditions on the boundaries of the cells.

### 3.4 Linear systems

We consider that the approximated pressure and velocity are represented in a basis of \( \mathbb{P}_p \) (e.g., with the Lagrange basis functions) such that, on every cell,

\[
p^{(e)}_h = \sum_{k=1}^{N_{\text{dof}}^{(e)}} p_k^{(e)} \phi_k(x), \quad v^{(e)}_h = \sum_{k=1}^{N_{\text{dof}}^{(e)}} v_k^{(e)} \phi_k(x),
\]

The number of degrees of freedom, \( N_{\text{dof}}^{(e)} \), is given from (14) and amounts at a given order \( p_e \) to \( N_{\text{dof}}^{(e)} = (p_e + 1)(p_e + 2)(p_e + 3)/6 \) for three-dimensional simplexes (tetrahedra). Thanks to the discontinuous basis functions which make the solution piecewise polynomial, it is easy to allow for different order of approximation depending on the cell, i.e., \( p \)-adaptivity. This is one
of the advantage of methods in the Discontinuous Galerkin family. The positions of the volume
degrees of freedom are illustrated in Figure 2b. We concatenate all coefficients $p^{(e)}_k$ and $v^{(e)}_{x,k}$ so that the unknowns associated with the cell $K_e$ are represented in a vector by

$$U_e = \left( p^{(e)}_1 \quad p^{(e)}_2 \quad \ldots \quad p^{(e)}_{N_{dof}} \quad v^{(e)}_{x,1} \quad \ldots \quad v^{(e)}_{x,N_{dof}} \right)^T. \quad (33)$$

Similarly, $\lambda_h$ is represented with a polynomial on each face, see Figure 2c, such that

$$\lambda_h |_f = \sum_{k=1}^{N_{dof}^{(f)}} \lambda^{(f)}_{h,k} \xi_k(x), \quad \forall f \in \Sigma. \quad (34)$$

where $N_{dof}^{(f)}$ refers to the number of degrees of freedom for the face $f$. We define the vector $\Lambda$ to assemble all the coefficients such that

$$\Lambda = \left( \lambda^{(f_1)}_{h,1} \quad \lambda^{(f_1)}_{h,2} \quad \ldots \quad \lambda^{(f_{N_{dof}^{(f)}})}_{h,1} \right)^T = \left( \lambda_h,1 \quad \lambda_h,2 \quad \ldots \quad \lambda_h,N_{dof}^{\\Sigma} \right)^T, \quad (35)$$

where $N_{dof}^{\\Sigma}$ indicates the total number of degrees of freedom for the faces. We also introduce the connectivity map $R_e$, which gives, from the global array $\Lambda$, the degrees of freedom on the current cell faces such that

$$R_e \Lambda = \Lambda |_{\partial K_e}. \quad (36)$$

Therefore, we have a volume discretization for $p_h$ and $v_h$ with piecewise polynomials on each cell, while $\Lambda$ is defined on the skeleton (piecewise-polynomial on the faces) of the complete mesh. We illustrate in Figure 2 with the positions of the degrees of freedom.

(a) Degrees of freedom for Continuous Galerkin discretization.
(b) Degrees of freedom for $p_h$ (i.e., Internal Penalty Discontinuous Galerkin discretization).
(c) Degrees of freedom for $\lambda_h$ which give the global matrix size in the HDG discretization.

Figure 2: Illustration of the degrees of freedom in two dimensions.

We inject the discretized representation in the local problem, that is, in (24) and (26). We have on each cell,

$$A_e U_e + C_e R_e \Lambda = S_e, \quad \text{HDG discretization: local system.} \quad (37)$$

These squared matrices are defined by testing with respect to each function in the basis of polynomial. The squared matrix $A_e$ is defined by

$$A_e = \begin{pmatrix}
-(\sigma k^{-1} \phi_i, \phi_j)_{K_e} + \tau (\phi_i, \phi_j)_{\partial K_e} & (\partial_z \phi_i, \phi_j)_{K_e} & (\partial_y \phi_i, \phi_j)_{K_e} & (\partial_x \phi_i, \phi_j)_{K_e} \\
-(\phi_i, \partial_x \phi_j)_{K_e} & -(\sigma \rho \phi_i, \phi_j) & 0 & 0 \\
-(\phi_i, \partial_y \phi_j)_{K_e} & 0 & -(\sigma \rho \phi_i, \phi_j)_{K_e} & 0 \\
-(\phi_i, \partial_z \phi_j)_{K_e} & 0 & 0 & -(\sigma \rho \phi_i, \phi_j)_{K_e}
\end{pmatrix}, \quad (38)$$
where \((\cdot, \cdot)\) denotes the inner product \((\phi_1, \phi_2)_{K_e} = \int_{K_e} \phi_1 \overline{\phi_2} \, dK_e\). Here, the index \(i\) changes with the line while \(j\) changes with the column, both take values from 1 to \(N_{\text{dof}}^{(e)}\). The matrix \(C_e\) and the right-hand side \(S_e\) are given by

\[
C_e = \begin{pmatrix}
-\tau(\xi_k, \phi_j)_{f_1} & \cdots & -\tau(\xi_k, \phi_j)_{f_{N_{\text{face}}}} \\
(\xi_k, \phi_j \nu_x)_{f_1} & \cdots & (\xi_k, \phi_j \nu_x)_{f_{N_{\text{face}}}} \\
(\xi_k, \phi_j \nu_x)_{f_1} & \cdots & (\xi_k, \phi_j \nu_y)_{f_{N_{\text{face}}}} \\
(\xi_k, \phi_j \nu_x)_{f_1} & \cdots & (\xi_k, \phi_j \nu_z)_{f_{N_{\text{face}}}}
\end{pmatrix}, \quad
S_e = \begin{pmatrix}
(f, \phi_j)_{K_e} \\
0 \\
0 \\
0
\end{pmatrix},
\tag{39}
\]

where \(N_{\text{face}}\) indicates the number of faces for the element (i.e., \(N_{\text{face}} = 4\) for tetrahedron). Here, the index \(k\) varies with the column and \(j\) with the line. In Table 1, we review the notation and give the dimensions of the matrices for numerical implementation.

We proceed similarly with the conditions derived on the faces, that is, (28) and (30), and sum over the cells. Using the transposed of the connectivity matrix \(R\) of (36) to convert the local index of the degrees of freedom on the face to their global ones, we have

\[
\sum_e R_e^T \left( B_e U_e + L_e R_e \Lambda \right) = 0. \tag{40}
\]

The matrices \(B_e\) and \(L_e\) are composed of blocks for each face of the cell, such that

\[
B_e = \begin{pmatrix}
B_e^{(f_1)} \\
\vdots \\
B_e^{(N_{\text{face}})}
\end{pmatrix}, \quad
L_e = \begin{pmatrix}
L_e^{(f_1)} & 0 & \cdots & 0 \\
0 & \ddots & \cdots & \cdots \\
0 & \cdots & 0 & L_e^{(N_{\text{face}})}
\end{pmatrix}. \tag{41}
\]

The faces of each cell can either be an interior one or a boundary one, that we respectively denote by \(f_i^I\) and \(f_i^B\). The matrix \(B_e\) remains the same in all cases: for the face \(k\), we have the matrix (with the number of lines given by the number of degrees of freedom on the faces, see Table 1)

\[
B_e^{(f_k)} = \begin{pmatrix}
\tau_{f_k} (\xi_k, \phi_j)_{f_k} & (\xi_k, \phi_j \nu_x)_{f_k} & (\xi_k, \phi_j \nu_y)_{f_k} & (\xi_k, \phi_j \nu_z)_{f_k}
\end{pmatrix}. \tag{42}
\]

For the matrix \(L_e\), each block is squared with, however, a different definition for interior and boundary faces, such that

\[
L_e^{(f_k)} = -\tau_{f_k}^I (\xi_k, \xi_j), \quad
L_e^{(f_k)} = \left( \frac{\alpha}{\sigma \rho \beta} - \tau_{f_k}^B \right) (\xi_k, \xi_j). \tag{43}
\]

We review the quantities in Table 1.

To assemble the global system, we first replace \(U_e\) in (40) using (37), we get

\[
\sum_e R_e^T \left( B_e A_e^{-1} (S_e - C_e R_e \Lambda) + L_e R_e \Lambda \right) = 0. \tag{44}
\]

After rearrangement, we obtain the global system for the HDG discretization:

\[
\sum_e R_e^T \left( L_e - B_e A_e^{-1} C_e \right) R_e \Lambda = -\sum_e R_e^T B_e A_e^{-1} S_e. \tag{45}
\]
Table 1: Summary of numerical quantities to implement the HDG discretization. The (sparse) global linear system is of size \( N_{\Sigma\text{dof}} \) to retrieve the coefficients of \( \Lambda \); it is followed by (dense) local linear systems of smaller size on each cell.

| dim | Problem dimension. |
|-----|-------------------|
| \( N_{\text{face}} \) | Number of face of an element (\( N_{\text{face}} = 4 \) for 3D tetrahedron). |
| \( N_{\Sigma\text{dof}}^{(e)} \) | Number of volume degrees of freedom on a cell \( K_e \) (for \( p_h \) and \( v_{\star, h} \)). |
| \( \hat{N}_{\text{dof}}^{(f)} \) | Number of degrees of freedom on the face \( f \) (for \( \lambda_h \)). |
| \( \hat{N}_{\Sigma\text{dof}} \) | Total number of face degrees of freedom, \( \hat{N}_{\Sigma\text{dof}} = \sum_f \hat{N}_{\text{dof}}^{(f)} \). |

\( \Lambda \) Vector of size \( \hat{N}_{\Sigma\text{dof}} \) encompassing all coefficients of the \( \lambda_h \).

\( U_e \) Vector of size \( N_{\Sigma\text{dof}}^{(e)} \) encompassing the coefficients of \( p_h \) and \( v_{\star, h} \) on the cell \( K_e \).

\( A_e \) Matrix of size \( ((\dim + 1) N_{\Sigma\text{dof}}^{(e)}) \times ((\dim + 1) N_{\Sigma\text{dof}}^{(e)}) \), see (38).

\( C_e \) Matrix of size \( ((\dim + 1) N_{\Sigma\text{dof}}^{(e)}) \times (\sum_{f \in \partial K_e} \hat{N}_{\text{dof}}^{(f)}) \), see (39).

\( S_e \) Right-hand side vector of size \( (\dim + 1) N_{\Sigma\text{dof}}^{(e)} \), see (39).

\( B_e \) Matrix of size \( (\sum_{f \in \partial K_e} \hat{N}_{\text{dof}}^{(f)}) \times ((\dim + 1) N_{\Sigma\text{dof}}^{(e)}) \), see (41) and (42).

\( L_e \) Matrix of size \( (\sum_{f \in \partial K_e} \hat{N}_{\text{dof}}^{(f)}) \times (\sum_{f \in \partial K_e} \hat{N}_{\text{dof}}^{(f)}) \), see (41) and (43).

**Remark 2 (Boundary conditions.).** If one considers Neumann boundary conditions for (3c), it amounts to taking \( \alpha = 0 \) such that there is no more distinction between interior and exterior boundaries in (43). However, for Dirichlet boundary condition (\( \beta = 0 \)), one cannot use formula (43) due to the singularity. In this case, the solution is to consider an identity block for \( L_e^{(f_{\Sigma})} \), which in turn enforces that the values of the trace is zero, while we also impose \( B_e^{(f_{\Sigma})} = 0 \).

**Remark 3 (Numerical approximation of the integrals).** In the numerical implementations, there are commonly two ways to approximate the value of the integrals that are needed to form the matrices. On the one hand, one can use the quadrature rules for the integration of polynomial functions. On the other hand, one can work with a reference element, that is, find the geometrical transformation from an arbitrary simplex to a regular one, and then explicitly obtain the integral of the polynomials. We believe that the use of quadrature rules is more appropriate, because it is more stable for high order polynomials, and also because it is easier to consider model parameters that vary within a cell (while this cannot be supported with the reference element method).

### 3.5 Numerical features

The numerical discretization using HDG follows two layers, with a global and local linear systems to be solved, respectively (37) and (45). The global linear system is written with respect to the degrees of freedom (dof) of \( \lambda_h \), such that its size is the total number face dof only (dimension of \( \Lambda \)). Therefore, contrary to CG or DG, we avoid the inner cell dof for the global problem, hence reducing the size of the linear system to be solved, upon taking a sufficiently high order (e.g., higher than three in 2D and four in 3D). This is particularly useful for applications such as seismic or helioseismology, where the time-harmonic approach remains overwhelming for large domain.

Next, the local systems are a specificity of HDG. While it might appear as an overhead burden...
compared to other discretization scheme, it is important to note that the local problems (37) are independent by cell. That means it is ‘embarrassingly parallelizable’, i.e., it does not need any communication between the working processors. In addition, these local systems are usually composed of small matrices, as the size is the number of dof in the current cells. Note also that, similarly to the other methods in the DG-family, we have independent contributions of each cell to the global matrix, which makes it convenient in a parallel implementation.

In addition, the HDG discretization works with the first-order formulation, which means that here, both the scalar pressure field and the vectorial velocity are computed with the same accuracy while the global matrix is only assembled for one, scalar unknown (Λ). This is another advantage of HDG: in the case of other discretizations (such as CG, FD or DG), the global matrix works directly with the fields of interest such that, if one wants to discretize the first-order formulation, the size of the system contains both the scalar and vectorial unknowns (i.e., in three-dimensions, it means four set of global unknowns instead of one for HDG). As an alternative, one can solve for the scalar unknown only (the pressure field), using the second-order formulation (9) and then deduce the vectorial velocity. However this means that the numerical approximation for the velocity looses one order of accuracy compared to the pressure, because of the derivative in (3). Therefore, one would need more advanced techniques to obtain both fields with similar accuracy, while this is natural with the HDG discretization. It has motivated its used in application of inverse problem where both the velocity and the pressure field are employed, e.g., [32].

Remark 4 (p-adaptivity). Similarly to other discretization method in the DG family, HDG can easily account for different polynomials among the cells. Indeed, one simply has to carefully compute the size of the local matrices. Regarding the edges, it is convenient to take the polynomial order on a face as the maximum order between the corresponding two adjacent cells.

4 Adjoint-state method for HDG discretization

To perform the iterative minimization for the quantitative reconstruction of the model parameters, the derivative of the misfit functional (11) must be evaluated. For large-scale applications, the adjoint-state method ([16]) is the natural choice as it avoids the explicit formation of the Fréchet derivative $D F$. The method is well-known, e.g., [60, 17, 10, 33] but requires some careful steps in the context of HDG, because we have two levels of discretization (local and global), contrary to the other discretization approaches.

In the following, for simplicity, we assume that there is only one source and one frequency for the misfit functional (11). By linearity, they can be reintroduced later on. Furthermore, we write in the discretized settings,

$$J(m) = \frac{1}{2} \| \mathcal{R} U(m) - d \|_2^2,$$

where $\mathcal{R}$ is a restriction operator (linear) that maps the numerical solution to the values at the receivers location.

We follow the steps of the adjoint-state method, and first consider the following minimization problem with constraints,

$$\min_m J(m), \quad \text{subject to (37) and (40)}.$$

We write the formulation with Lagrangians ([52, 38]) and explicit the constraints:

$$\mathcal{L}(m, \hat{U}, \hat{\Lambda}, \hat{\gamma}_1, \hat{\gamma}_2) = J(m) + \sum_e \langle A_e U_e + C_e R_e \hat{\Lambda} - S_e, \hat{\gamma}_1 \rangle \tag{48}$$

$$+ \langle B_e U_e + L_e R_e \hat{\Lambda}, R_e \hat{\gamma}_2 \rangle.$$
Here, $(\cdot, \cdot)$ denotes the complex inner product in $L^2$ such that $\langle u, v \rangle = u^* v$, and $^*$ is the adjoint. The formulation contains two Lagrange multipliers: $\tilde{\gamma}_{1,e}$ has the same dimension as $U_e$ and $\tilde{\gamma}_2$ has the same dimension as $\Lambda$.

The derivative of $\mathcal{L}$ with respect to the model parameter $m = \{\kappa, \rho\}$ is

$$
\partial_m \left( \mathcal{L}(m, \tilde{U}, \tilde{\Lambda}, \tilde{\gamma}_{1,e}, \tilde{\gamma}_2) \right) = \text{Re} \left( \frac{d \mathcal{L}}{d m} + \frac{\partial \mathcal{L}}{\partial U} \frac{\partial U}{\partial m} + \frac{\partial \mathcal{L}}{\partial \Lambda} \frac{\partial \Lambda}{\partial m} \right),
$$

(49)

where we follow [8, Appendix A] for the specificity of the derivative with complex-variables. Upon taking $U$ and $\Lambda$ solutions of (37) and (40), we have

$$
\frac{\partial_m \mathcal{L}(\kappa, \rho, \tilde{U} = U, \tilde{\Lambda} = \Lambda, \tilde{\gamma}_{1,e}, \tilde{\gamma}_2)}{= \nabla_m J, \quad \text{for} \quad m = \{\kappa, \rho\}.}
$$

(50)

The adjoint states $\gamma_{1,e}$ and $\gamma_2$ are then selected such that the differential of the Lagrangian $\mathcal{L}$ with respect to $\tilde{U}$ and $\tilde{\Lambda}$ equates zero, that is:

$$
\begin{align}
\sum_e A_e^* \gamma_{1,e} + B_e^* \mathcal{R}_e \gamma_2 &= -\partial_U J = -\text{Re}(\Re(\tilde{U} - \mathbf{d})), \\
\sum_e \mathcal{R}_e^T C_e^* \gamma_{1,e} + \mathcal{R}_e^T L_e^* \mathcal{R}_e \gamma_2 &= 0,
\end{align}
$$

(51a) (51b)

where, because $\mathcal{R}_e$ is real, $\mathcal{R}_e^* = \mathcal{R}_e^T$, with $^T$ the transposed (indeed, $\mathcal{R}_e$ only converts the local indexes to the global ones, (36)).

For the computation of the adjoint states, we derive the global system by replacing $\gamma_{1,e}$ in (51b) by its expression from (51a):

$$
\sum_e \mathcal{R}_e^T C_e^* \left( -A_e^* B_e^* \mathcal{R}_e \gamma_2 \right) + \mathcal{R}_e^T L_e^* \mathcal{R}_e \gamma_2 = \sum_e \mathcal{R}_e^T C_e^* A_e^{-*} \left( \text{Re}(\Re(\tilde{U} - \mathbf{d})) \right),
$$

(52)

that is,

$$
\sum_e \mathcal{R}_e^T \left( L_e^* - C_e^* A_e^{-*} B_e^* \right) \mathcal{R}_e \gamma_2 = \sum_e \mathcal{R}_e^T C_e^* A_e^{-*} \left( \text{Re}(\Re(\tilde{U} - \mathbf{d})) \right). (53)
$$

We recognize on the left-hand side the adjoint of the global problem forward one (45), where $\mathcal{R}$ is only to convert from local to global indexing. Then the local problems verify (51a), that we recall for convenience:

$$
A_e^* \gamma_{1,e} = -B_e^* \mathcal{R}_e \gamma_2 - (\text{Re}(\Re(\tilde{U} - \mathbf{d}))), \quad \forall K_e \in \mathcal{T}_h.
$$

(54)

It is crucial that the global problem is the adjoint of the forward (with different right-hand side) to avoid the re-factorization of the matrix. The time (and memory) consuming part in time-harmonic applications is the matrix factorization which, however, allows for the fast resolution of multiple rhs problems. Here, the factorization of the forward problem can be reused, using direct solvers such as Mumps [4, 5], to compute the gradient avoiding an additional matrix factorization (this is standard with the adjoint-state method for time-harmonic equations). However, we see that the local problems for the forward problem are not similar to the ones for the adjoint-state computation. Therefore, the code must be adapted with the appropriate operations with the matrix $\mathbf{B}$, but it remains cheap thanks to the parallelizability of the local problems, as discussed in Subsection 3.5.
The gradient of the functional is obtained from (49) and (50), injecting the adjoint states which simplify the equation by imposing $\partial_U L = \partial_A L = 0$, following (51). We obtain,

$$\nabla_{\mu J} = \text{Re} \left( \sum_e \langle \partial_{m} A_e U_e + \partial_{m} C_e R_e \Lambda, \gamma_{1,e} \rangle + \langle \partial_{m} B_e U_e + \partial_{m} L_e R_e \Lambda, R_e \gamma_{2} \rangle \right).$$  

(55)

In fact, for the Euler’s equation, only $\Lambda$ depends on the medium parameter if we ignore the Robin boundary conditions which involves $\rho$ (note that in elasticity using the same convention, the matrix $C$ also depends on the medium parameters, [12]). Then, the derivative of $\Lambda$ is straightforward from its definition in (38).

Compared to the adjoint-state method derived for the usual discretization (e.g., [60, 8]), the right-hand side is not simply made of the residuals (the difference between the observations and the simulations), but includes the discretization matrices $A$ and $C$, see (53). Furthermore, note that this modification of the rhs is different from the one applied in the forward problem (which uses $B$, see (45)). Then the local problem also involves the residuals, and the local matrix $B$ instead of $C$ for the forward problem, see respectively (54) and (37).

5 Three-dimensional experiment

In this section, we illustrate the performance of the HDG discretization in the context of iterative reconstruction, and design a three-dimensional synthetic experiments of size $2 \times 2 \times 1$ km$^3$. The surface is not flat, with a topography made of two large craters and smaller variations, it is illustrated Figure 3b. Firstly, it is necessary to accurately capture the topography to account for the reflections from the surface, therefore, we require a very fine mesh of the surface, that we illustrate in Figure 3a. In particular, we rely on the software mmg\textsuperscript{1} to create meshes, and it allows to fix the surface cells while possibly coarsening or refining the deeper area. Namely, we use different meshes (to generate the data or for the iterations at high-frequency), but we have the guarantee that the surface remains the same. For the numerical implementation of the method, we use a combination of mpi and OpenMP for parallelism while the code is written in Fortran90.

For the numerical computations, the use of the HDG discretization is appropriate and allow for a flexible framework using $p$-adaptivity. Indeed, the surface is finely mesh, such that low order polynomials can be used in the area. On the other hand, in the deepest part where the cells are larger, we use higher order polynomials and benefit from the HDG discretization which disregards the inner dof for the global linear system.

In this experiment, the wave speed model contains layers of high-contrast velocities, and vary from 2000 to 5500 m s$^{-1}$, it is show in Figure 4 where we extract vertical and horizontal sections for visualization. Per simplicity, we consider a constant density with $\rho = 1000$ kg m$^{-3}$.

Following a seismic context, a Dirichlet boundary condition is imposed at the surface (with the topography), while absorbing boundary conditions are implemented on the other boundaries.

5.1 Synthetic partial reflection-data

We work with reflection data acquired from the surface only, with receivers positioned just underneath it to measure the pressure field. They are positioned to follow the topography in a lattice with about 75 m between each receivers, along the $x$ and $y$ directions. In total, we have 625 receivers. For the data, we consider a set of 100 point-sources (i.e., delta-Dirac function for the right-hand side of (3)) that are independently excited. For each of the sources, the

\footnote{https://www.mmgtools.org/}.
(a) Three-dimensional domain meshed with about 100,000 tetrahedron cells.

(b) Cartography of the topography.

Figure 3: Domain of interest of size \(2 \times 2 \times 1 \text{ km}^3\), where the surface must be finely mesh to accurately capture the topography. Per convention, the positive values indicate the depth.

Figure 4: Target wave speed model for the inverse problem, for visualization, we extract vertical sections in \(x = 1 \text{ km}\) (top left) and \(y = 1 \text{ km}\) (bottom right), and a horizontal section in \(z = 500 \text{ m}\) (bottom left).

625 receivers measure the resulting pressure field. The sources are also positioned to follow the topography, at the surface, in a lattice with about 190 m between them, for a total of 100 sources. This further motivates the use of direct solvers which, as mentioned, allow for the fast resolution of all sources once the matrix is factorized. As all of the acquisition devices are restricted to the surface area, the partial data available only consist in reflection data, generated from only one sided illumination.

We work with synthetic data but include white noise to make our experiments more realistic. The noise is incorporated in the data with a signal-to-noise ratio of 10 dB for each measurement.
In addition, the mesh and order of discretization differs between the generation of the data and the inversion procedure. Following the geophysical setup of this experiment, the available frequencies for the reconstruction are limited between 5 and 15 Hz. In particular, the absence of low-frequency content in the data is an unavoidable difficulty of seismic applications [15, 69, 33].

5.2 Iterative reconstruction

We perform the iterative reconstruction using data with frequencies from 5 Hz to 15 Hz, following a sequential progression, as advocated in [33]. The initial model is pictured in Figure 5: it consists in a one-dimensional variation (in depth only) where none of the sub-surface layers is initially known and with an inaccurate background velocity. In the inverse procedure, the model is represented as a piecewise-constant for simplicity, that is, we have one value of the parameter per cell.

![Initial wave speed model](image)

Figure 5: Initial wave speed model for the inverse problem, for visualization, we extract a vertical section in $y = 1$ km (right), and a horizontal section in $z = 500$ m (bottom left).

The search direction for the update of the wave speed model is computed using the nonlinear conjugate gradient method, and only depends on the gradient of the misfit functional, cf. [59]. We perform 30 iterations per frequency, for a total of 300 iterations. The order of the polynomials for the basis functions changes with frequency (increases), while we use two meshes for the iterations: a coarse mesh for the first frequencies, from 5 to 10 Hz, and a more refined one for higher frequencies, to capture more details. None of the two meshes is similar to the one used to generate the data, but we guarantee that the surface cells maintain the same topography. In Figure 6, we picture the reconstructed wave speed.

We observe that the layers with high velocities are appropriately recovered: their positions and the values are accurate, except near the boundaries, due to the limited illumination. On the other hand, the lower values are not retrieved, and remain almost similar to the starting model. This is most likely due to the lack of background information in the starting model, which can only be recovered using low-frequency content in the data, cf. [37, 53, 33].
Figure 6: Reconstruction after iterative minimization with frequency up to 15 Hz, for visualization, we extract vertical sections in $x = 1$ km (top left) and $y = 1$ km (bottom right), and a horizontal section in $z = 500$ m (bottom left).

6 Perspectives

We have illustrated the use of HDG discretization in the context of time-harmonic inverse wave problems. The perspective is to handle larger-scale applications, using the smaller linear systems provided by the method, which only accounts for the degrees of freedom (dof) on the faces of the cells. Precisely, we have in mind problems made of multiple right-hand sides (e.g., with seismic acquisition) where the bottleneck is usually the memory needed for the matrix factorization. For an efficient implementation of the HDG discretization, we have the following requirements.

0. One needs to use polynomials of relatively high orders, because the inner-cell dof are omitted in the global linear system, i.e., the mesh must be composed of large cells.

1. On the other hand, large cells cannot be allowed on the whole domain (in order to appropriately represent the geometry) and we rely on $p$-adaptivity to adapt the polynomial orders to the size of the cells.

2. Because we use large cells, the model parameters must be carefully represented.

We provide here a preliminary experiment to illustrate, where we multiply by ten the size of the previous model, hence with a $20 \times 20 \times 10$ km$^3$ domain. We generate a mesh of about 220 000 cells where, similarly as above, the surface is refined to correctly represent the topography.

Step 1: $p$-adaptivity The first step is to select the order of the polynomials on each cells. We rely on the wavelength (that is, the ratio between the frequency and the wave speed on the cell) to select the order, and we illustrate in Figure 7a the resulting orders at 10 Hz frequency. Here, the order of the polynomials varies from 3 to 7, such that the cells near surface, which are smaller to handle the topography, consequently use a low-order polynomial while the sub-surface layer, made of larger cells with low-velocity, need a high order. We also observe the sub-surface layer of increasing velocity, where the order is allowed to be reduced.
Step 2: model representation  Because of the large cells, the use of a piecewise-constant model parameters with one value per cell can lead to a coarse representation, that we illustrate in Figure 7b. Therefore, we can rely on a piecewise-polynomial model representation and illustrate in Figure 7c with polynomials of order 2 per cells. Clearly, the latter allows for a precise representation, removing the inaccuracies between cells. In this experiment where the above layer is mostly constant, we can even envision to use one polynomial on a group of cells instead of one per cell. Such models can be easily accounted for using the quadrature rules to evaluate the integrals, see Remark 3.

![Figure 7: Extension of the model of Figure 3 to a domain of size 20×20×10 km³. (a) Illustration of the p-adaptivity where the order of the polynomial varies per cell depending on the wavelength at 10 Hz. (b) Wave speed model using a piecewise-constant representation (one value per cell). (c) Wave speed model using piecewise-polynomial representation with functions of order 2 per cell.](image)

Numerical cost  In Figure 8, we illustrate the solution of the forward problem with the pressure field for a single source at 6 Hz, where we follow the setup prescribed in Figure 7. The computation is performed using 360 cores (using 10 nodes with 4 processors and 9 threads per node). It corresponds to the following dimensions,

- The size of the linear system is $N_{\text{dof}} = 10\,993\,236$.

- The number of volume dof per unknown is $\sum_e N_{\text{dof}}^{(e)} = 14\,812\,171$, and we have four unknowns, i.e., the pressure and the vectorial velocity.

We see that the size of the linear system reduces by 25 % the number of volume dof for one unknown, and by 80 % if we consider the total size with the four unknowns. In this experiment, the matrix factorization requires 440 GiB using the direct solver Mumps\(^2\). The time to factorize the global matrix is of about 6 min, while the time to solve the local problems to assemble the volume solution is less than 1 s. It highlights that the second step of the method where one has to solve the local problems is very cheap compared to the resolution of the global linear system, in particular as it does not need any communication between the cells.

\(^2\)Note that the memory cost can be reduced by using the block low-rank feature recently implemented in the solver [3].
Remark 5. Comparing with other discretization methods, where the linear system is made of the volume dof (see Figure 2), we can draw the following remarks,

- Standard DG results in a linear system of size the number of volume dof (see Figure 2b). To solve the first-order system with both the pressure field and velocity, it represents an increase of 539% compared to HDG. When solving the second-order system to only recover for the pressure field, it is an increase of 135%.

- Using Continuous Galerkin, let us first remind that it is much harder to employ p-adaptivity because the dof on the faces are shared (see Figure 2a). Then, assuming a constant order 6, the resulting linear system is increased by 590% and 147% compared to HDG, respectively for the first- and second-order systems. Using the maximal order, 7, it amounts to 851% and 212% respectively.

7 Conclusion

We have derived the adjoint-state method for the computation of the gradient of a functional in the framework of the Hybridizable Discontinuous Galerkin discretization in order to handle large-scale time-harmonic inverse problems using direct solvers to account for multiple right-hand sides. HDG reduces the size of the global linear system compared to other discretization methods, and it works with the first-order formulation of the forward problem. We have illustrated with the acoustic wave equation, where it gives the computational approximation for the pressure and the velocity fields at the same accuracy. The HDG method relies on two levels, a global system and local ones, which must be carefully addressed to obtain the adjoint-state where the matrix factorization of the forward problem can still be used for the backward problem. HDG allows to easily account for p-adaptivity, which is useful when some part of the mesh must be particularly refined to take into consideration the specificity of the problem, such that the geometry of the parameters or, as we have illustrated in our experiment, with the topography. It have given preliminary insight to work with larger scale problem, and we need to continue to investigate if HDG can help to fill the gap between the largest time and frequency-domain problems. Extension to elasticity is straightforward, and requires a few modifications of the steps we have developed, it is part of ongoing research.
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