High-energy head-on collisions of particles and hoop conjecture

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(Dated: April 19, 2002)

We investigate the apparent horizon formation for high-energy head-on collisions of particles in multi-dimensional spacetime. The apparent horizons formed before the instance of particle collision are obtained analytically. Using these solutions, we discuss the feature of the apparent horizon formation in the multi-dimensional spacetime from the viewpoint of the hoop conjecture.

PACS numbers: 04.50.+h, 04.20.Cv, 04.70.Bw, 11.10.Kk

Introduction. — Brane world scenario has been discussed by many authors recently. The scenario suggests that the Planck energy can be as low as $O(\text{TeV})$ scale [1]. If the Planck energy is TeV scale, it is possible to create black holes using accelerators, such as LHC [2]. Further, the collisions of cosmic rays with our atmosphere have energy reach beyond that of LHC and their observation will find the existence of the large extra-dimension or will place improved bounds on the fundamental Planck scale. Hence we would like to better understand the process of the black hole formation via particle collisions. For this purpose, we investigate the formation of the apparent horizon for the system of the head-on collisions of high-energy particles.

To simplify the analysis, we follows the method adopted by Eardley and Giddings [3]. First, the tension of the brane which is expected to be the Planck scale can be negligible if the center of mass energy is substantially larger than the Planck scale. Second, the geometry of the extra dimensions plays no essential role if the geometrical scales of the extra dimensions are large compared to the horizon radius for the center of mass energy. Thus we consider the head-on collisions in $D$-dimensional Einstein gravity. The metric with a high-energy point particle is obtained by infinitely boosting the Schwarzschild black hole metric with the fixed total energy $\mu$. The resulting metric represents a massless particle moving in the $+z$ direction with the speed of light:

$$ds^2 = -d\bar{u}d\bar{v} + \sum_{i=1}^{D-2} d\bar{x}_i^2 + \Phi(\bar{x},)\delta(\bar{u})d\bar{u}^2,$$

where $\bar{u} = \bar{t} - \bar{z}$ and $\bar{v} = \bar{t} + \bar{z}$. $\Phi$ depends on only the transverse radius $\bar{\rho} = \sqrt{\bar{x}_1^2 + \ldots + \bar{x}_{D-2}^2}$ and takes the form

$$\Phi = \frac{-8G_M \log \bar{\rho}}{\Omega_{D-3}(D-4) \bar{\rho}^{D-4}}, \quad \text{for } D > 4. \quad (4)$$

A delta function appeared in (2) shows that two coordinate systems are discontinuously connected on $\bar{u} = 0$. The continuous coordinate system can be introduced by

$$\bar{u} = u,$$

$$\bar{v} = v + \Phi \theta(u) + \frac{u}{4} \delta(u) \left( \nabla_i \Phi \nabla^i \Phi \right), \quad (5)$$

$$\bar{x}_i = x_i + \frac{u}{2} \nabla_i \Phi (x_i) \theta(u),$$

where $\theta$ is the Heaviside step function and $\nabla_i$ is the $(D-2)$-dimensional flat-space derivative. We can superpose the two solutions to obtain the exact geometry outside the future light cone of the collision of the shocks:

$$ds^2 = -du dv + \left( H^{(1)}_{ik} H^{(1)}_{jk} + H^{(2)}_{ik} H^{(2)}_{jk} - \delta_{ij} \right) dx^i dx^j, \quad (6)$$
where
\[ H^{(1)}_{ij} = \delta_{ij} + \frac{u}{2} \theta(u) \nabla_i \nabla_j \Phi^{(1)}(x), \]
\[ H^{(2)}_{ij} = \delta_{ij} + \frac{v}{2} \theta(v) \nabla_i \nabla_j \Phi^{(2)}(x). \] (7)

Here \( x \equiv (x^i) \) is the point in flat \( D-2 \)-space that is transverse to the direction of particle motion.

The apparent horizon is defined as a closed spacelike \( D-2 \)-surface on which the outer null geodesic congruence have zero convergence. It was shown that the apparent horizon exists in the union of the two shock waves, \( u = 0 > v \) and \( v = 0 > u \). This apparent horizon consists of two flat discs with radii
\[ r_0 \equiv \left( \frac{8\pi\mu G_D}{\Omega_{D-3}} \right)^{1/(D-3)}, \] (8)
and \( r_0 \) gives a characteristic scale for each dimension \( D \).

Time slicing and apparent horizons. —To treat the collision of particles as time evolutional process, we consider the following slice of spacetime:

region I : \( t = z, t \leq T \),
region II : \( z = -t, t \leq T \),
region III : \( t = T, -T \leq z \leq T \), (9)

where \( T \leq 0 \) and particles collide at \( T = 0 \). In order to find apparent horizon on the above slice, we first prepare surfaces with zero expansion in region I and III, then connect them smoothly by requiring that the null normals coincides at the junction of region I and III, \( t = z = T \). In region I, the surface which have zero expansion is given by
\[ v = -\Phi + \text{const.}, \] (10)
and its null normal \( k^i_1 \) is
\[ k^0_1 = (\rho_0/\rho)^{-(D-3)}, \]
\[ k^i_1 = (\rho_0/\rho)^{D-3}, \]
\[ k^D_1 = 1. \] (11)

In region III, the surface which have zero expansion is given by
\[ az = \pm f(\rho), \] (12)
where \( a \) is a constant of integration determined by the matching condition at the junction. For \( D = 4 \), the function \( f(x) \) is given by
\[ f(x) = \cosh^{-1} x, \] (13)
and for \( D > 4 \),
\[ f(x) = -x^{-D+4}/D-4 \cdot _2F_1 \left( \frac{1}{2}, \frac{D-4}{2(D-3)}, \frac{3D-10}{2(D-3)}; x^{2(D-3)} \right) \]
\[ - \sqrt{x} \frac{\Gamma \left( \frac{D-4}{2(D-3)} \right)}{\Gamma \left( \frac{1}{2(D-3)} \right)}. \] (14)

Matching these surfaces and null normals at the junction \( t = z = T \), we have
\[ f(\rho_0) = -aT, \] (16)
\[ (\rho_0/\rho)^{D-3} = (\rho_0/\rho)^{D-3} + (\rho_0/\rho)^{2(D-3)} - 1. \] (17)

where \( \rho_0 \) is the radius of the surface at the junction. From this, the relation between \( T \) and \( \rho_0 \) can be given parametrically as
\[ \frac{T}{\rho_0} = -\xi f \left( \frac{1}{\xi} (2\xi^{3-D}-1)^{1/2(3-D)} \right), \] (18)
\[ \frac{\rho_0}{\rho_0} = (2\xi^{3-D}-1)^{1/2(3-D)}. \] (19)

where \( 0 \leq \xi \leq 1 \). FIG. 1 shows the relation between \( T \) and \( \rho_0 \) for each \( D \). We denote the time when the apparent horizon appears as \( T = T_c \). The value of \( |T_c/\rho_0| \) becomes small as \( D \) increases. For large \( D \), we have
\[ \rho_0/\rho_0 \approx D^{-1/2D}, \quad T_c/\rho_0 \approx -1/D \] (20)
at \( T = T_c \). The intersection of the \( z = \) const. plane and the surface in region III is a \( D-3 \)-dimensional sphere, of which expansion is positive and proportional to \( D-3 \). Thus the surface has negative expansion on \( (\rho/\rho_0, z/\rho_0) \)-plane and its curvature on this plane increases with the increase of space-time dimension \( D \). This leads to the
decrease in the distance of two particles at the horizon formation. The shape of apparent horizons for $D = 4$ and $D = 5$ are shown in FIG. 2 and FIG. 3.

**Hoop conjecture.** Now we examine the difference of the horizon formation for various spacetime dimension using the hoop conjecture. The hoop conjecture gives the criterion of black hole formation in the 4-dimensional general relativity\(^2\). It states that an apparent horizon forms when and only when the mass $M$ of the system gets compacted into a region of which circumference $C$ satisfies

$$H_4 \equiv C/4\pi G_4 M \lesssim 1. \quad (21)$$

As $4\pi G_4 M$ is the circumference of the 4-dimensional Schwarzschild horizon, we can expect that the criterion of black hole formation in the $D$-dimensional Einstein gravity is given by

$$H_D \equiv C/2\pi r_h(M) \lesssim 1, \quad (22)$$

where $r_h(M)$ is the Schwarzschild radius of $D$-dimensional spacetime. This criterion was implicitly used to estimate the total cross section for black hole production via non-head-on collisions\(^2\).

To calculate the ratio $H_D$ and $H_4$, we must specify the definition of the mass of the system. In this paper, we use total energy $E = 2\mu$ as the mass of the system. The circumference $C$ is defined as minimum length which encloses two particles. We take the loop as shown in FIG. 4 and calculate $C$ by taking the limit $c \to 0$. $C$ reduces to $4|T|$ which is the twice the distance of two particles. The value of $H_D$ at $T = T_c$ is shown in TABLE I. As $D$ increases, the value of $H_D$ decreases and the mass $M$ must be compacted into the region with smaller circumference $C$ than $2\pi r_h$ to produce a black hole. This reflects the decrease in $|T_c|/\rho_0$ with increase in $D$.

The value of $H_4$ at $T = T_c$ is also shown in TABLE I. This result can be written as

$$H_4 = F(D)\left(\frac{G_D E^{1/(D-3)}}{G_4 E}\right), \quad (23)$$

where $F(D) = 0.03 \sim 0.2$. The $D$-dimensional gravitational constant is related to the Planck energy as

$$M_p^{D-2} = \frac{(2\pi)^{D-4}}{4\pi G_D}. \quad (24)$$

FIG. 2: The apparent horizon for $D = 4$ at $T/\rho_0 = -0.278, -0.225, 0$. The dark line is the horizon at $T = T_c = -0.278\rho_0$, and light line is the horizon at $T = 0$. The unit of the axis is $\rho_0$.

FIG. 3: The apparent horizon for $D = 5$ at $T/\rho_0 = -0.227, -0.2, 0$. The dark line is the horizon at $T = T_c = -0.227\rho_0$, and light line is the horizon at $T = 0$. The unit of the axis is $\rho_0$.

FIG. 4: The closed loop to calculate the circumference. We calculate $C$ by taking $c \to 0$. 

\(^2\) References are omitted for brevity.
TABLE I: The value of $H_D$ and $H_4$ at $T = T_c$ for $D = 4 \sim 11$.

| $D$ | $H_D$ | $H_4$ |
|-----|-------|-------|
| 4   | 0.1773 | 0.1773 |
| 5   | 0.1567 | 0.0722 $\left(\frac{G_5 E}{G_4 E}\right)^{1/2}/(G_4 E)$ |
| 6   | 0.1348 | 0.0527 $\left(\frac{G_6 E}{G_4 E}\right)^{1/3}/(G_4 E)$ |
| 7   | 0.1176 | 0.0444 $\left(\frac{G_7 E}{G_4 E}\right)^{1/4}/(G_4 E)$ |
| 8   | 0.1042 | 0.0396 $\left(\frac{G_8 E}{G_4 E}\right)^{1/5}/(G_4 E)$ |
| 9   | 0.0936 | 0.0364 $\left(\frac{G_9 E}{G_4 E}\right)^{1/6}/(G_4 E)$ |
| 10  | 0.0849 | 0.0340 $\left(\frac{G_{10} E}{G_4 E}\right)^{1/7}/(G_4 E)$ |
| 11  | 0.0777 | 0.0321 $\left(\frac{G_{11} E}{G_4 E}\right)^{1/8}/(G_4 E)$ |

Using this formula, Eq. (23) becomes

$$H_4 = F(D) \left(\frac{M_4}{M_p}\right)^2 \left(\frac{8\pi^2 M_p}{E}\right)^{\frac{D-4}{4}}. \quad (25)$$

If the Planck energy is TeV scale, $M_4/M_p \approx 10^{16}$ and $H_4$ becomes $\sim 10^{32}$. Thus the mass does not need to be compacted into a small region of which circumference is $C \lesssim 4\pi G_4 M$ to produce a black hole.

**Summary and discussion.** We have investigated the temporal evolution of the apparent horizon for high energy particle collisions. The apparent horizon which encloses the two particles appears at $T = T_c$. Its radius increases in time and reaches $r_0$ at $T = 0$. We calculated $H_D$ and found that $H_D$ decreases as $D$ increases. This means that if we increase the space-time dimension, the size of the hoop which enclose the system should be much smaller than $2\pi r_h$. Therefore, the formation of the apparent horizon becomes more difficult for larger $D$. On the other hand, $H_4 = H_D \cdot r_h/2G_4 M$ gives a large value $\sim 10^{32}$ regardless of the decrease in $H_D$. This is because the horizon radius $r_h$ becomes far larger than $2G_4 M$. As the horizon radius corresponds to the length scale which enclose the system, this leads to the conclusion that a black hole is easily formed in the TeV scale scenario.

Finally we discuss the validity of the hoop conjecture. Obviously, $H_4$ does not give the picture of the hoop conjecture because its value at the horizon formation is far larger than unity. The ratio $H_D$ also does not give the picture of the hoop conjecture because its value at the horizon formation is much smaller than unity. However, we used the rough estimated values of the circumference $C$ and the mass $M$ to evaluate $H_D$ and $H_4$. The energy of shock wave with a high-energy particle is distributed in the transverse direction of the motion, and our estimation of the circumference $C$ is too small because the region surrounded by this circumference does not enclose much of the gravitational energy. In our previous paper [3], we stated that $H_4$ with Hawking’s quasi-local mass $M_H(S)$ becomes a better parameter to judge the horizon formation for the system with motions. We must calculate $H_4^{(H)}(S) = C(S)/4\pi G_4 M_H(S)$ for all surfaces $S$ and then take the minimum value of them. Even if the Hawking mass in multi-dimensional space-time has not been calculated in this paper, we expect that $H_D^{(H)} \lesssim 1$ becomes a condition for the horizon formation. The value $H_D$ would decrease as $D$ increases even if we use the quasi-local mass because $H_D$ should reflect the decrease in $|T_c|/\rho_0$.

Although we can regard $C/2\pi r_h(M) \lesssim 1$ as the condition for the horizon formation in $D$-dimensional gravity, it does not give a unique condition. The topology of apparent horizon is not restricted to be $S^{D-2}$ surface in a multi-dimensional space-time. Emparan and Reall derived the solution of rotating black ring in $D = 5$ [9]. For apparent horizon which does not have $S^{D-2}$ topology, the criterion for its formation may take another form. Our criterion $C/2\pi r_h(M) \lesssim 1$ is applicable only to the horizon with $S^{D-2}$ topology.

The authors would like to thank Akira Tomimatsu and Masaru Shibata for helpful discussions.

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