SIC-POVMs from Stark Units

Marcus Appleby,\textsuperscript{1,a) Ingemar Bengtsson,\textsuperscript{2,b) Markus Grassl,\textsuperscript{3,4,c) Michael Harrison\textsuperscript{d) and Gary McConnell\textsuperscript{e)}}

\textsuperscript{1) Centre for Engineered Quantum Systems, School of Physics, University of Sydney, Sydney NSW 2006, Australia}
\textsuperscript{2) Stockholm University, AlbaNova, Fysikum, S-106 91 Stockholm, Sweden}
\textsuperscript{3) International Centre for Theory of Quantum Technologies, University of Gdansk, 80-308 Gdansk, Poland}
\textsuperscript{4) Max Planck Institute for the Science of Light, 91058 Erlangen, Germany}
\textsuperscript{5) Controlled Quantum Dynamics Theory Group, Imperial College, London}

We propose a recipe for constructing a SIC fiducial vector in any complex Hilbert space whose dimension is of the form $d = n^2 + 3$, starting from Stark units in a ray class field that does not contain a complex root of unity. The recipe relies on some number theoretical theorems, a version of the Stark conjectures, and some standard conjectures for SICs. In this paper we focus on the conceptually simplest case, prime dimensions of the form $d = n^2 + 3$, and report that we have constructed SICs in twelve prime dimensions of this kind, the highest being $d = 19603$.

\textit{Dedicated to John Coates.}

\section{INTRODUCTION}

SIC-POVMs, or SICs for short, is a notion from quantum information theory. Stark units come from number theory. The connection between them is the point of departure for this paper, so we should first sketch how this was discovered.

That SICs exist in every finite dimensional complex Hilbert space was first conjectured by Zauner\textsuperscript{1} and Renes et al\textsuperscript{3}. This is relevant to quantum information theory\textsuperscript{4}, quantum foundations\textsuperscript{5,6}, classical information theory\textsuperscript{7}, and frame theory\textsuperscript{8}. For recent reviews see Refs.\textsuperscript{9} and\textsuperscript{10}. The acronym SIC stands for “symmetric informationally complete”, while a POVM corresponds to a

\textsuperscript{a)Electronic mail: marcus.appleby@gmail.com}
\textsuperscript{b)Electronic mail: ingemar@fysik.su.se}
\textsuperscript{c)Electronic mail: markus.grassl@ug.edu.pl}
\textsuperscript{d)Electronic mail: mch1728@gmail.com}
\textsuperscript{e)Electronic mail: g.mcconnell@imperial.ac.uk}
kind of measurement. However, here we will regard SICs more abstractly, as a geometrical structure. Mathematically, they are maximal sets of equiangular lines in complex Hilbert space; or equivalently, maximal regular simplices in complex projective Hilbert space. It might seem that SIC-measurements on systems described by Hilbert spaces whose dimensions are as high as those that we will encounter in this paper are completely unrealistic, but such a conclusion may be too hasty\textsuperscript{11}.

Constructing SICs has proved remarkably difficult, and the subject has proceeded in the spirit of experimental mathematics. Cursory inspection of the first published catalogue of exact SICs\textsuperscript{12} shows that the SIC vectors are built from algebraic numbers. Close inspection of every case examined reveals that these numbers belong to abelian extensions of the real quadratic number field

$$K = \mathbb{Q}(\sqrt{D}),$$

(1)

where $D$ is the square-free part of $(d - 3)(d + 1)$ and $d$ is the dimension\textsuperscript{13}. Moreover every dimension holds a ray class SIC that can be constructed using the ray class field over $K$ with finite modulus $d$ (or $2d$ if $d$ is even)\textsuperscript{14,15}. The minimal modulus defining a particular field extension is the conductor of the extension. A major achievement of mathematics in the first half of the twentieth-century was to classify all ray class fields. The Stark conjectures imply (among other things) that there exists an analytic function from which generators of these ray class fields can be obtained\textsuperscript{16,17}. These generators are known as Stark units. Proving this conjecture would solve an instance of Hilbert’s 12th problem\textsuperscript{18}. That Stark units can be used to construct SICs, at least in certain cases, is a more recent insight\textsuperscript{19}.

Number theorists may be interested in a geometrical problem in which Stark units play a role. Readers who are not number theorists may find it helpful if we describe the familiar roots of unity in similar terms. The $n$th roots of unity are in fact generators of the cyclotomic number fields of modulus $n$, and by the celebrated Kronecker–Weber theorem these number fields house all the abelian extensions of the rational number field. It turns out that the cyclotomic fields $\mathbb{Q}(e^{\frac{2\pi i}{n}})$ are subfields of the ray class fields mentioned in the previous paragraph\textsuperscript{14}. Roots of unity are needed in the geometrical problem of dividing the circle into $n$ equal parts. They are needed in the SIC problem too, because roots of unity are used in the representation of the Weyl–Heisenberg group, and this is the group that generates the SICs we are interested in.

In this paper “SIC” means a SIC generated by the Weyl–Heisenberg group in a dimension $d > 3$. (For SICs not satisfying these conditions see Ref.\textsuperscript{10}.) Thus a SIC is constructed by acting on a fiducial vector $\Psi$ by elements of the Weyl–Heisenberg group. This means that the SIC is fully specified by the single vector $\Psi$, on which we accordingly focus. The necessary and sufficient condition for a $d$ dimensional vector $\Psi$ to be a SIC fiducial vector is

$$|\langle \Psi | D_{j,k} | \Psi \rangle |^2 = \begin{cases} 1 & j = k = 0 \\ \frac{1}{d+1} & \text{otherwise} \end{cases}$$

(2)
for \( j, k \) running from 0 to \( d - 1 \). Here \( D_{j,k} \) is a Weyl–Heisenberg displacement operator in the standard Weyl representation, as defined in Appendix A. We refer to the complex scalar products \( \langle \Psi| D_{j,k}|\Psi \rangle \) as SIC overlaps. The vector \( \Psi \) can be reconstructed if all the overlaps are known, up to an irrelevant phase factor. We define the SIC field to be the number field generated by the ratios of the components of \( \Psi \) together with the root of unity \( e^{i\pi/d} \) (which comes in via the operators \( D_{j,k} \)).

That Stark units appear in the SIC problem was first observed in a careful study of SIC overlaps\(^{20}\), and this was developed in a significant way by Kopp\(^{19}\). For certain dimensions he used Stark units to construct the SIC overlaps from which SICs can be reconstructed. Forthcoming work will present evidence that this approach will work for every SIC in every dimension\(^{21}\). Kopp’s work served as an inspiration for us, and there are several points of contact. Our idea is different in that we focus on a special sequence of dimensions in which we construct a fiducial vector \( \Psi \) directly, without going via the overlaps. The reason is that it has been observed that in some dimensions fiducial vectors can be constructed using only a subfield of the full ray class field\(^{22–24}\). Here we will select a special infinite sequence of dimensions in which we expect that the entire cyclotomic subfield of the full ray class field “decouples” from a suitably chosen fiducial vector. We then use Stark units in a small ray class field to construct that fiducial vector. It is not our immediate aim to find a formula for every SIC; not even in these special dimensions. In this sense our approach is limited in scope. On the other hand we will be able to construct SICs in dimensions that are much higher than achieved by any previous method, whether based on exact solutions or numerical searches. The highest dimension reported in this paper is \( d = 19603 \) (see Table III).

The special sequence of dimensions we are interested in consists of all dimensions of the form \( d = n^2 + 3 \), so that \( D \) is simply the square free part of \( d + 1 \). For these dimensions the standard conjecture is that the ray class SIC has an anti-unitary symmetry in addition to the unitary symmetry which appears to be always present\(^{12,25}\). In fact the unitary symmetry in our case is expected to be of order \( 3\ell \), where \( \ell \) is an odd integer that is known if \( d \) is known (as explained in Section II A). Moreover one can show that all the dimensions in our sequence have a prime decomposition of the form

\[
d = n^2 + 3 = 2^{e_1} \cdot 3^{e_2} \cdot p_1^{r_1} \cdot \ldots \cdot p_s^{r_s},
\]

where \( e_1 \in \{0, 2\} \), \( e_2 \in \{0, 1\} \), and all the primes \( p_j = 1 \) modulo 3 (see Ref. 26). Arguments that we will go into later then suggest that there exists a fiducial vector from which the cyclotomic subfield of the full ray class field decouples. The full ray class field will be generated by the components of the fiducial vector together with the cyclotomic unit that comes from the Weyl–Heisenberg group.

In this paper we make the further restriction to the case that \( d = n^2 + 3 = p \), a prime number. This last restriction is not made for any fundamental reason, but only because it simplifies the presentation at some points. It is then natural to ask whether this sub-sequence of prime dimensions is infinite.
‘Conjecture F’ by Hardy and Littlewood indeed implies that the sequence is infinite, but this remains unproven.

The equation
\[ d = \left( \sqrt{d+1} + 1 \right) \left( \sqrt{d+1} - 1 \right) \] (4)
together with the fact that \( \sqrt{d+1} \) is an algebraic integer in \( K \) means that the prime \( d \) splits into two prime ideals. It follows that the ray class field with finite modulus \( d \) contains two small ray class fields with finite moduli \( \sqrt{d+1} \pm 1 \) as subfields. By class field theory, each of them has trivial intersection with the cyclotomic subfield, which makes them natural candidates for the small ray class field that is to hold the fiducial vector.

Thus, let \( d = n^2 + 3 = p \) and define two vectors: an un-normalized fiducial vector
\[ \hat{\Psi} = (x_0, x_1, \ldots, x_{d-1})^T, \] (5)
all of whose components belong to the SIC field, and the SIC fiducial vector properly speaking which is the unit vector
\[ \Psi = N\hat{\Psi} = (a_0, a_1, \ldots, a_{d-1})^T. \] (6)
The overall phase of the vector \( \Psi \) is arbitrary, but it is required that the \( x_j \) together with the absolute value squared \( |N|^2 \) belong to the SIC field. Because of the symmetries referred to above, the number of independent components, not counting \( x_0 \), is expected to be \( (d-1)/3\ell \). We now state three conjectures, and comment on them afterwards:

Conjecture 1 (Almost flat fiducial). In each dimension in the sequence there exists an un-normalized SIC fiducial \( \hat{\Psi} \) for which
\[ x_j = \begin{cases} 
-2 - \sqrt{d+1}, & j = 0 \\
\sqrt{x_0 e^{i\vartheta_j}}, & j > 0 
\end{cases} \] (7)
for suitable phases \( e^{i\vartheta_j} \). It can be seen that \( \hat{\Psi} \) is almost flat, in the sense that \( |x_1| = \cdots = |x_{d-1}| \).

Conjecture 2 (Small ray class fields). The components of \( \hat{\Psi} \) generate one of the two small ray class fields of \( K \) of finite moduli \( \sqrt{d+1} \pm 1 \).

Conjecture 3 (Stark phase units). We now come to the key hypothesis: the numbers \( e^{i\vartheta_j} \), in eq. (7) are Galois conjugates of Stark units for one of these two small ray class fields. We refer to them as Stark phase units.

To construct a SIC fiducial from these conjectures one needs to order the components correctly. Standard conjectures about SIC symmetries imply that we need \( (d-1)/3\ell \) phase factors in the almost flat fiducial. At the same time, there are \( (d-1)/3\ell \) Stark phase units, cyclically ordered by a Galois group. The action of the Clifford group on fiducial vectors suggests that one should order the non-zero indices \( j = 1, \ldots, d-1 \) in eq. (7) with respect to the
The multiplicative group of invertible elements in $\mathbb{Z}_d^\times$. For prime dimensions, this group is cyclic as well. These orderings can be matched, although we do not know \textit{a priori} in which way. Furthermore, there is an ambiguity in choosing the square roots in (7), which we are able to reduce to the choice of only one sign. Thus the conjectures do not fully specify the vector, and a small amount of trial and error remains in identifying the correct choices.

Conjecture 1 is closely related\textsuperscript{30,31} to anti-unitary symmetry, as will be shown in Section V. Concerning Conjecture 2 note that for the dimensions under consideration $\sqrt{d+1}$ is always in $K$, but it is by no means obvious that $\sqrt{\alpha e^{i\vartheta}}$ is in the small ray class field. Nevertheless this will be made somewhat more plausible in Section IV. Concerning Conjecture 3 is new, and encourages the hope that one might be able to use the properties of Stark units to prove SIC existence. This paper has a more limited aim. Its purpose is to develop techniques that enable us to calculate fiducial vectors with these properties, and to verify that they are indeed SIC fiducials. The conjectures can be modified to cover the case when the dimension is a composite number of the form $n^2 + 3$.

Our constructions rely on algebraic number theory, on the representation theory of the Clifford group, and on advanced calculational recipes. In order to make the logic clear, also for readers that are new to one or more of these topics, we now give a summary of the major features of each section.

Sections II–IV establish the number theoretical framework for our recipe. Section II starts with a review of the dimension towers $\{d_\ell\}_{\ell=1}^\infty$, introduced in Ref. 14. It is shown that whenever $d_1$ is of the form $n^2 + 3$ then so is $d_\ell$ for each odd integer $\ell$. We go on to prove a number of technical lemmas that will be needed in the sequel. Section III begins with a review of some relevant theory of ray class fields and their moduli. We then consider the splitting of the modulus, as in eq. (4). At the end of this section we know the degrees of the resulting small ray class fields. Section IV introduces the key objects of this paper, the Stark units, and sketches how they can be calculated as special values of Dedekind zeta functions (which are generalisations of the Riemann zeta function). We summarise some key points of Stark’s construction. Finally we prove a non-trivial result regarding the square root that appears in eq. (7). At the end of this section we have in hand the number theoretical underpinning for the calculations in Section VII.

In Section V we enter Hilbert space and state the standard conjectures about SIC symmetries on which we rely. To do this we need the representation theory of the Weyl–Heisenberg and Clifford groups. A short summary of the relevant theory, with references, is provided in Appendix A. We show that the standard conjectures come close to implying that a fiducial vector taking the form given in eqs. (5)–(7) exists. In Section VI we discuss the “decoupling” phenomenon: that is to say the expectation that a small ray class field should suffice to construct this fiducial vector. Indeed, despite their apparent ubiquity in defining the geometry via the Heisenberg-Weyl group, the cyclotomic numbers nevertheless seem mysteriously to vanish from the final vector entries.
In Section [VII] we state, in ten precise steps, our recipe for how to construct SIC fiducial vectors from Stark units. We present a combination of numerical and algebraic techniques that allow us to carry out the required calculations for dimensions which are considerably larger than that of any previous example (see Table III). Section VIII illustrates these calculations for two low dimensions ($d = 7$ and $d = 199$) where computational complexity is not an issue.

In Section IX we compare our method of calculating SICs to previous methods. We then raise two open questions. Finally we give a preview of our results in non-prime dimensions. Section X gives our conclusions. At the end of the paper we will be in a position to program a computer to calculate a SIC from scratch; but not in a position to specify in advance the precision that will be needed for the numerical part of the calculations.

Throughout the paper the integer $d \geq 4$ will continue to signify the dimension for the ambient complex Hilbert space in which the SIC-POVMs, or sets of equiangular lines, are supposed to exist. Because of their somewhat technical nature, Sections II B and III A may be left out on a first reading.

II. THE GEOMETRIC DIMENSION $d$ AND PELL’S EQUATION IN $\mathbb{Q}(\sqrt{D})$

A good reference for this section is the book by Cohn\textsuperscript{32}. Fix an integer $d \geq 4$. Write $K$ for the real quadratic field $\mathbb{Q}(\sqrt{D})$, where $D \geq 2$ is the square-free part of $(d + 1)(d - 3)$. Let $\text{Gal}_{K/\mathbb{Q}}$ denote the Galois group of the number field extension $K/\mathbb{Q}$, with unique non-trivial element $\tau$. We shall often use the same symbol, where it will cause no confusion, for a lifting of $\tau$ to Galois groups of which $\text{Gal}_{K/\mathbb{Q}}$ is a quotient. The field $K$ can be embedded into the real numbers $\mathbb{R}$ in one of two ways: either by sending $\sqrt{D}$ to $\sqrt{D} > 0$, or to $-\sqrt{D} < 0$. Fix $j$ to be the embedding $K \hookrightarrow \mathbb{R}$ under which it is sent to $\sqrt{D} > 0$, with $j\tau$ denoting the other.

The unique non-trivial element $\tau \in \text{Gal}_{K/\mathbb{Q}}$ then interchanges these embeddings, or (real) infinite places, $j$ and $j\tau$ of $K$. Note that here we are using the standard notation from Galois theory, that an object $\alpha$ which is acted upon by a field automorphism $\sigma$ is written $\alpha^\sigma = \sigma(\alpha)$. Once these embeddings are fixed, any non-zero algebraic number $\alpha \in K$ has a well-defined signature, being one of the four possibilities ($\pm, \pm$) given by the sign of $\alpha$ under each respective embedding.

For example, let $D = 5$ and consider $u = (1 + \sqrt{5})/2 \in K$. Then $j(u) > 0$ but $j\tau(u) < 0$, so the signature of $u$ would be $(+, -)$, as for example would be that of $1/u$. In contrast $-u$ or $-1/u$ would have signature $(-, +)$; and $u^2$ or $1/u^2$, $(+, +)$. The algebraic number $u^2$ would then be referred to as being totally positive; similarly $-u^2$ would be totally negative.

We shall need the notions\textsuperscript{33} of the norm and the trace of an algebraic number $\alpha \in F$ down to its base field $k$. Namely, in our situation where the extension $F/k$ will always be Galois, the norm and trace of $\alpha$ are respectively given by the product and sum of all of its Galois conjugates $\alpha^\sigma$, for $\sigma \in \text{Gal}_{F/k}$:
\[ N_{F/k} \alpha = \prod_{\sigma \in \text{Gal}_F/k} \alpha^\sigma \text{ and } \text{Tr}_{F/k} \alpha = \sum_{\sigma \in \text{Gal}_F/k} \alpha^\sigma. \]

By \( \mathbb{Z}_F \) we will denote the ring of algebraic integers of a number field \( F \): that is to say, all elements \( \alpha \in F \) whose minimal polynomial \( f_\alpha(X) \in \mathbb{Z}[X] \) is monic. The fact that \( \mathbb{Z}_F \) is a Dedekind domain means in particular that any ideal factorises uniquely into a product of prime ideals. The group of units, or multiplicatively invertible elements, of a ring \( R \) will be denoted by \( R^\times \). Note that \( \mathbb{Z}_F^\times \) consists precisely of those \( \alpha \in \mathbb{Z}_F \) for which \( f_\alpha(X) \) has constant term \( \pm 1 \).

By Dirichlet’s unit theorem, \( \mathbb{Z}_K^\times \) is isomorphic as an abelian group to \( \{\pm 1\} \times u^\mathbb{Z} \) for some unit \( u \). The four possible choices of generator for the infinite part are \( \{\pm u, \pm 1\} \), any one of which may be chosen to be ‘the’ fundamental unit of \( \mathbb{Z}_K \) (or of \( K \)). Throughout this paper we let \( u_K \in \mathbb{Z}_K^\times \) denote the unique choice which is \( > 1 \) under \( \mathfrak{j} \). Notice that in the particular embedding \( \mathfrak{j} \), this gives an ordering of the fundamental units: \( -u_K < -1 < -\frac{1}{u_K} < 0 < \frac{1}{u_k} < 1 < u_K \); and that in any event, \( \tau \) swaps \( u_K \) with \( (N_{K/Q} u_K)/u_K \), which in all of the cases in this paper will be \( -1/u_K \) (see Lemma 1 below).

Before getting into the technical details, let us give a brief description of what is proven in the next few sections leading up to Theorem 17. Because of Conjecture 2, we are interested in square-roots of Stark units. If the standard conjectural properties of Stark units hold (which our computations indeed verify), we know that the Stark units lie in a certain abelian Galois extension of \( K \) and that their square-roots will lie in a further quadratic extension of that which is still abelian over \( K \). It is important for the construction to know that this extension is always the one obtained by adjoining a particular square-root \( \xi_{\ell} = \sqrt{\alpha_0} \), our geometric scaling factor. We demonstrate this by showing that the restricted nature of the ramification above primes over 2 forces the quadratic extension to be the composite with a unique quadratic extension of \( K \). The restriction can be expressed either in terms of discriminants or conductors, the two giving essentially equivalent information for ramified quadratic extensions at primes over 2. We choose to talk primarily in terms of conductors since this fits naturally into the class field setting.

**A. Towers of dimensions over \( K \) when \( N_{K/Q} u_K = -1 \)**

Fix \( D \) and \( K = \mathbb{Q}(\sqrt{D}) \) as above. As per Ref. 14, we may solve for an infinite series of dimensions \( d_\ell = d_\ell(D) \) for \( \ell = 1, 2, 3, \ldots \):

\[
d_\ell = \begin{cases} 
  u_K^\ell + u_K^{-\ell} + 1 & \text{if } u_K \text{ is totally positive (so } N_{K/Q} u_K = +1 \text{)} \\
  2u_K^\ell + u_K^{-2\ell} + 1 & \text{otherwise (} N_{K/Q} u_K = -1 \text{).} 
\end{cases}
\]

These are exactly the \( d \) which yield the quadratic field \( K = \mathbb{Q}(\sqrt{D}) \), via \( D \) being the square-free part of \( (d_\ell - 1)^2 - 4 = (d_\ell + 1)(d_\ell - 3) \). For example, again where \( D = 5 \), we find \( u_K = (1 + \sqrt{5})/2 \) and so the sequence of dimensions indexed by successive powers of \( u_K^2 = (3 + \sqrt{5})/2 \) is

\[4, 8, 19, 48, 124, 323, 844, 2208, 5779, 15128, 39604, \ldots \]
Any integer \( d = d_\ell(D) \geq 4 \) is valid as a dimension and will correspond to a unique \( D \) and a unique \( \ell \geq 1 \) placing it in the list of dimensions associated to that \( D \).

A fundamental unit \( u_K \) is usually found by searching for a solution to Pell’s equation using continued fractions. Its norm \( N_{K/\mathbb{Q}}u_K = u_K u_K^\ell \) may be either +1 or −1. To give an a priori condition that the norm should be −1 (that is, given \( D \) but without calculating the fundamental unit by some continued fraction-style algorithm), is one of the oldest open problems in Diophantine analysis. However by (8), such fields are precisely those yielding dimensions of the form \( d = n^2 + 3 \), as follows.

**Lemma 1** (Theorem 1 in Ref. [34]). Let \( K = \mathbb{Q}(\sqrt{D}) \) be a real quadratic field with fundamental unit \( u_K \). With notation as in eq. (8), the following statements are equivalent:

(I) \( N_{K/\mathbb{Q}}u_K = -1 \): that is, \( u_K \) and \( u_K^\ell \) have opposite signs and so the first totally positive power of \( u_K \) is \( u_K^2 \).

(II) For every odd positive integer \( \ell \), \( d_\ell \) is of the form \( n_\ell^2 + 3 \) for some integer \( n_\ell \).

(III) There exists an integer \( \ell \) such that \( d_\ell \) is of the form \( n_\ell^2 + 3 \) for some integer \( n_\ell \).

(IV) \( d_1 \) is of the form \( n_1^2 + 3 \) for some integer \( n_1 \).

(V) There exists an integer \( n \) such that the square-free part of \( n^2 + 4 \) is \( D \).

**Corollary 2.** For any odd \( \ell \), writing \( d_\ell = n_\ell^2 + 3 \), the minimal polynomial of \( u_K^\ell \) is \( X^2 - n_\ell X - 1 \), and thus \( u_K^\ell = (n_\ell + \sqrt{d_\ell + 1})/2 \).

Note that the trace of \( u_K^\ell \) in these cases is \( n_\ell = u_K^\ell - u_K^{-\ell} \) rather than \( u_K^\ell + u_K^{-\ell} \) as it would be in the totally positive case when \( N_{K/\mathbb{Q}}u_K = +1 \). The even-numbered terms \( d_2, d_4, d_6, \ldots \) in (8) satisfy the formula (cf. Section 3 of Ref. [14]):

\[
d_{2\ell} = d_\ell(d_\ell - 2).
\]

If this product \( d_{2\ell} \) were itself to have the form \( m^2 + 3 \) for some integer \( m \), then writing \( d_\ell = n_\ell^2 + 3 \):

\[
m^2 + 3 = d_{2\ell} = d_\ell(d_\ell - 2) = (n_\ell^2 + 3)(n_\ell^2 + 1) = n_\ell^4 + 4n_\ell^2 + 3 = n_\ell^2(n_\ell^2 + 4) + 3, \tag{11}
\]

forcing \( n_\ell^2 + 4 \) to be a square, which is impossible for \( n_\ell \geq 1 \). Hence every integer \( \ell \geq 3 \) of the form \( n^2 + 3 \) is a \( d_\ell = d_\ell(D) \) for some odd \( \ell \) and some square-free \( D \geq 2 \) for which the real quadratic field \( \mathbb{Q}(\sqrt{D}) \) contains a unit of norm −1.

On the other hand it is interesting to observe that for any square-free \( D \) and any \( k \geq 1 \), \( d_{2k}(D) \) (which upon rearrangement of (10) is \( (d_k - 1)^2 - 1 \)), is always of the form \( d_{2k}(D) = m^2 D + 3 \), where \( m \) is the integer \( (u_K^\delta - u_K^{-\delta})/\sqrt{D} \); defining \( \delta \) in turn as 2 when \( N_{K/\mathbb{Q}}u_K = -1 \) and 1 otherwise.
Hardy and Littlewood made the conjecture (inter alia) in Conjecture F of Ref. [27] — which is now subsumed within the vast so-called Bateman–Horn Conjecture — that infinitely many integers in the series $n^2 + 3$ are in fact prime. Whether or not this is true, the decomposition of their prime factors inside $\mathbb{Z}_K$ is determined a priori, as follows.

**Lemma 3.** Suppose that $d$ has the form $n^2 + 3$ (so $D \equiv 1, 2$ or $5$ mod $8$).

(I) If $p$ is an odd prime dividing into $d$, then it splits in $K/\mathbb{Q}$.

(II) The prime $2$

- splits iff $D \equiv 1$ mod $8$ (iff $n_1 \equiv 0$ mod $8$),
- ramifies iff $D \equiv 2$ mod $8$ (iff $n_1 \equiv 2$ mod $4$),
- is inert iff $D \equiv 5$ mod $8$ (iff $n_1$ is odd (and so $2|d$); or $n_1 \equiv 4$ mod $8$).

**Proof.** The assertions about which classes of $D$ mod $8$ appear, and the relations between $D$ and $n_1$, follow from basic congruence arguments applied to the definitions. For (I): an odd prime $p$ splits in $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$ if and only if $(D/p) = 1$. Since by definition $(d+1)(d-3) = f^2D$ for some $f \in \mathbb{N}$, when $p \neq 3$, we just unwind the definitions using Legendre symbols:

$$ (D/p) = (f^2p) = (f^2p) = (d+1)(d-3) = (d+1)(d-3) = (d+1) = 1, \quad (12) $$

using the fact that $d \equiv 0$ mod $p$. When $p = 3$ we simply observe that $D \equiv 1$ mod $3$. Part (II) follows from the standard result that $\{1, \sqrt{D}\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}_K$ iff $D \equiv 2, 3$ mod $4$: see for example Ref. [32].

Henceforth in the paper the dimensions will always be of the form $d = d_\ell = n_\ell^2 + 3$, where $\ell \geq 1$ is odd. Conversely, given any positive $n \in \mathbb{Z}$ there is associated a unique value of $D$, being the square-free part of $d + 1 = n^2 + 4$. Hence by Lemma [I] for any fixed $D$ there is a series of odd-numbered (but not necessarily odd!) dimensions $d_\ell = n_\ell^2 + 3$ for $\ell = 1, 3, 5, 7, \ldots$; where we may recover each $n_\ell$ (up to sign) via

$$ n_\ell = n_\ell(D) = \sqrt{d_\ell - 3} = u_\ell^\ell - u_\ell^{-\ell} = \text{Tr}_{K/\mathbb{Q}} u_\ell^\ell. \quad (13) $$

**B. The biquadratic field $K(\sqrt{-u_K})$**

We need to link the geometry of the fiducial vector with the number theory. In particular we need an arithmetic handle on the scaling factor which is the square root of the quantity $x_0$ that appears in the fiducial vector, defined in [4]. We start this by relating the quadratic extension $K(\sqrt{-u_K})/K$ ‘at the bottom of the tower’ to the Stark units. We need to invoke some concepts from the theory of $p$-adic (local) fields, for which the book by Cassels [35] is an ideal reference; with more general information for example in Ref. [36] and [37].
In particular for any prime ideal \( \varphi \) of the ring of integers \( \mathbb{Z}_K \) of a number field \( K \) we shall denote by \( \mathbb{K}_\varphi \) and \( \mathbb{Z}_\varphi \) respectively the field of \( \varphi \)-adic numbers and its ring of integers.

Write \( L = K(\sqrt{-u_K}) \). The minimal polynomial over \( K \) for \( \sqrt{-u_K} \) is \( X^2 + u_K \), with discriminant \(-4u_K \). So \( L/K \) must have discriminant dividing \( 4\mathbb{Z}_K \); in particular \( 2 \) is the only rational prime above which ramification could occur. Because the signature of \( u_K \) is \((+,-) \), ramification occurs just at the one infinite place \( j \). Under any embedding above \( j \) the field \( L \) is complex; whereas above \( j^* \) it is real. In particular, therefore, \( L/\mathbb{Q} \) is never Galois.

By way of background, the polynomial

\[
X^4 + n_1X^2 - 1 = (X - \sqrt{-u_K})(X + \sqrt{-u_K})(X - \frac{1}{\sqrt{u_K}})(X + \frac{1}{\sqrt{u_K}}) \quad (14)
\]

generates the quartic extension \( L/\mathbb{Q} \), by Corollary 2. Multiplying (14) by its conjugate, the polynomial which results from sending \( u_K \) to \(-u_K \) (or equivalently to \( 1/u_K \)) in (14), gives the octic \((X^4 + n_1X^2 - 1)(X^4 - n_1X^2 - 1) = X^8 - (d_1 - 1)X^4 + 1 \), a generating polynomial for the splitting field \( \overline{L} \), the normal closure of \( L \) with \( \text{Gal}_{\overline{L}/\mathbb{Q}} \cong D_4 \).

Because we only consider extensions where \( \ell \) is odd, we are free to replace every instance of \( n_1 \) with \( n_\ell \) and \( u_K \) with \( u_\ell \) in the foregoing. In particular, therefore, we note for future reference that for a fixed value of \( D \),

\[
\text{The towers } L/K/\mathbb{Q} \text{ will be identical for every } d_\ell = d_\ell(D), \quad \text{for every odd } \ell \geq 1. \quad (15)
\]

We shall characterise the quadratic extension \( L/K \) in terms of restricted ramification above \( j \) and the primes of \( K \) over 2 in Proposition 8 in the next section.

### III. Calculating the Size of the Galois Group

Given \( \alpha \in \mathbb{Z}_K \) we shall often write \((\alpha)\) in the standard way for the principal ideal \( \alpha \mathbb{Z}_K \) of \( \mathbb{Z}_K \). If \( \alpha, \eta \in \mathbb{Z}_K \) are given, such that the ideals \((\alpha) = \alpha \mathbb{Z}_K \) and \((\eta) = \eta \mathbb{Z}_K \) are co-prime — which is to say, their (unique) factorisations into prime ideals share no common prime factors; or equivalently the ideal \( \alpha \mathbb{Z}_K + \eta \mathbb{Z}_K \) is equal to \( \mathbb{Z}_K \) — then we let \( \text{ord}_\eta \alpha \) denote the order of \( \alpha \mod (\eta) \). That is, the order of the cyclic group \((\alpha + (\eta))\) generated by the image \( \alpha + (\eta) \) of \( \alpha \) inside \( \mathbb{Z}_K/(\eta) \). The next result is a special case of Lemma 12 of Ref. [14].

**Lemma 4.** If \( d_\ell \) is of the form \( n_\ell^2 + 3 \), then \( \text{ord}_{d_\ell} u_K^2 = 3\ell \).

**Corollary 5.** If \( d_\ell \) is of the form \( n_\ell^2 + 3 \), then \( \text{ord}_{d_\ell} u_K = 6\ell \).

**Proof.** By hypothesis and Lemma 4 \( \text{ord}_{d_\ell} u_K = 3\ell \) or \( 6\ell \). Suppose \( \text{ord}_{d_\ell} u_K = 3\ell \): then since \( u_K^2 + (d_\ell) \in (u_K + (d_\ell)) \), it follows that \( \langle u_K + (d_\ell) \rangle = \langle u_K^2 + (d_\ell) \rangle \); moreover we may map one to the other by squaring. Now \( \langle u_K^2 + (d_\ell) \rangle \) is
closed under taking norms down to \((\mathbb{Z}/d\ell\mathbb{Z})^\times\); hence \(\langle u_K + (d\ell) \rangle\) is as well. But \(N_{K/Q}u_K = -1\) and \(-1 \not\equiv +1 \mod (d\ell)\), as \(d\ell \geq 4\): hence \(#\langle u_K + (d\ell) \rangle\) must be even. So squaring is not a surjective endomorphism of \(\langle u_K + (d\ell) \rangle\), giving the desired contradiction.

\[\Box\]

A. The Exact Sequence of Global Class Field Theory

For a readable treatment of the basic notions from class field theory which we invoke below, see for example Refs. [38, 39] or [36]. For an informal introduction situating ray class fields within the SIC-POVM problem, see Section 4 of Ref. [14]. For a general characterisation of fractional ideals in Dedekind domains see Chapter 9 of Ref. [40].

Let \(m_0\) be any integral ideal of the ring \(\mathbb{Z}_K\), and let \(m_\infty\) denote some — possibly empty — subset of \(\{j, j^\tau\}\). The formal product \(m = m_0 \cdot m_\infty\) is called a modulus, by analogy with modular arithmetic over the integers. There is a natural partial order on such moduli, based on unique factorisation into prime ideals in the finite component and on set inclusion for the infinite part. In the absence of any widely-accepted notation we have chosen to write \(K^m\) for the ray class field of \(K\) for the modulus \(m\), keeping the alternative notations of subscripts for local fields \(F_\wp\), and \(F(\alpha)\) for the extensions of \(F\) by an algebraic number or indeterminate \(\alpha\). The actual construction of a ray class field is implicit via class field theory and will not be discussed here.

In particular \(K^{(1)}\) — which we shall denote in the standard way just by \(H_K\) — is the Hilbert class field of \(K\), where the \((1) = 1 \cdot \mathbb{Z}_K\) signifies the ideal \(\mathbb{Z}_K\) itself, the identity in the (multiplicative) group \(\mathcal{J} = \mathcal{J}(K)\) of non-zero fractional ideals of \(\mathbb{Z}_K\). The principal (fractional) ideals \(\mathcal{P}\) of \(\mathbb{Z}_K\) — those ideals \((\alpha) = \alpha\mathbb{Z}_K\) which are generated by a single element \(\alpha \in K\) — form a distinguished subgroup of \(\mathcal{J}\). The ideal class group of \(\mathbb{Z}_K\) is defined to be the quotient \(\mathcal{C}_K = \mathcal{J}/\mathcal{P}\). By class field theory this is isomorphic to the Galois group \(\text{Gal}_{H_K/K}\) and its order \(h_K = \#\mathcal{C}_K\) is what is referred to as the class number of \(K\) (or of \(\mathbb{Z}_K\)).

An abelian extension \(F\) of a number field \(L\) is a Galois field extension \(F/L\) for which the Galois group \(\text{Gal}_{F/L}\) is abelian; and a finite extension is one which is generated by a polynomial of finite degree. The Kronecker–Weber theorem says that any number field which is a finite abelian extension of \(\mathbb{Q}\) is necessarily a subfield of some cyclotomic field: that is, a field obtained from \(\mathbb{Q}\) by adjoining a finite number of roots of unity. Similarly, global class field theory tells us that any finite abelian extension \(E\) of \(K\) naturally embeds inside some ray class field \(K^m\).

Let \(\mathcal{R}^m\) denote the ray class group modulo \(m\), defined as follows. Take the subgroup \(\mathcal{J}^m\) of \(\mathcal{J}\) comprised of all non-zero fractional ideals of \(\mathbb{Z}_K\) which are co-prime to the finite part \(m_0\) of the modulus. Then define \(\mathcal{P}_1^m\), the ray group modulo \(m\), as the subgroup of \(\mathcal{P}\) of principal fractional ideals \((\alpha) = \alpha\mathbb{Z}_K\) whose generators \(\alpha \in K\) are congruent to 1 modulo \(m_0\), as well as being positive at every real place in \(m_\infty\). We then set \(\mathcal{R}^m = \mathcal{J}^m/\mathcal{P}_1^m\) to be the quotient group.
Note that $J^{(1)} = J$ and that $R^{(1)} = J/P = C_K$.

Artin’s reciprocity map then gives a canonical isomorphism between $R^m$ and $\text{Gal}_{K^m/K}$, induced in our situation by the map taking each prime $p$ of $Z_K$ to the corresponding Frobenius automorphism $\sigma_p \in \text{Gal}_{K^m/K}$. This will be a crucial ingredient in Section VII, enabling us to find a canonical ordering of the entries of the fiducial vector derived from the Stark units: once an embedding is fixed from $K$ into $R$, the Artin map unequivocally associates an element of the Galois group $\text{Gal}_{K^m/K}$ to each $Z_K$-ideal $\wp$ coprime to $m$.

We state the exact sequence of global class field theory (see (2.7) in Ref. 38, Theorem 1.7 in Ref. 39, or §0 in Ref. 41). With any number field $K$ as base field, and any modulus $m = m_0 \cdot m_\infty$, the following sequence (defining the map $\psi$) is exact:

$$1 \to U^m_1 \to Z^\times_K \xrightarrow{\psi} (Z_K/m_0)^\times \times \{\pm 1\} \# m_\infty \to R^m \to C_K \to 1,$$

where $\# m_\infty$ is the number of real primes in $m$. That is to say, $m_\infty$ represents the embeddings of $K$ into $R$ which are allowed to ramify (note that this terminology is not universal: see for example the alternative notational set-up with the sets $S$, $T$ in Gras’ book 42, or Ref. 41) in the ray class field extension: namely, to be extended in such a way that they contain non-real complex numbers. The term $\ker \psi = U^m_1$ is the subgroup of the global units $Z^\times_K$ which are simultaneously congruent to 1 modulo $m_0$ and positive at the real places in $m_\infty$. In the case of real quadratic fields this kernel has $Z$-rank one and is torsion-free except possibly when the residue class ring $Z_K/m_0$ has characteristic 2.

We apply (16) to the situation which will become the normal context for the remainder of this paper, from Section IV C onwards. Extracting a short exact sequence from the middle of (16), ending with the cokernel of $\psi$, we obtain:

$$1 \to Z^\times_K/U^m_1 \xrightarrow{\psi} (Z_K/m_0)^\times \times \{\pm 1\} \# m_\infty \to \text{coker } \psi \to 1,$$

where the tail fits back into (16) via

$$1 \to \text{coker } \psi \to R^m \to C_K \to 1.$$

So $R^m$ is an extension of $C_K$ by $\text{coker } \psi$. We need to invoke an independent result from Section III B just below.

**Lemma 6.** Suppose that $K = \mathbb{Q}(\sqrt{D})$ is as in the foregoing, with the additional stipulation that our ‘dimension’ $d_\ell = d_\ell(D)$ is a prime of the form $p = n_\ell^2 + 3$ for some integer $n_\ell = n_\ell(D)$ (and where $\ell$, as always, is odd). By Lemma 3 we know that $p = pp^\tau$ splits in $Z_K$. With our notation $j$, $j^\tau$ for the (real) infinite places of $K$, and referring to the notation of the sequence (16), the ray class field $K^{pj}$ has the property that for a unique cyclic group $\Gamma$ of (odd) order $\frac{p-1}{6\ell}$,

$$\text{coker } \psi \cong \Gamma \times C_2.$$
Proof. Under the assumptions of the lemma, the finite part \( p \) of our conductor \( m = p j \) is a non-rational principal prime ideal whose norm is an odd rational prime \( p \equiv 3 \mod 4 \), and therefore \( \frac{p - 1}{2} = \frac{n^2}{2} + 1 \) is odd and consequently since \( \mathbb{Z}_K / \mathfrak{p} \cong \mathbb{F}_p \),

\[(\mathbb{Z}_K / \mathfrak{p})^\times \cong C_{\frac{p - 1}{2}} \times C_2. \tag{20}\]

By exactness, the image \( \text{im} \psi \) of \( \psi \) in (17) is isomorphic to \( \mathbb{Z}_K^\times / U_m \), where, as we explained in the introduction to Section II, \( \mathbb{Z}_K^\times = \langle u_K \rangle \times \{\pm 1\} \). But the proof of Proposition 11 below shows that \( \text{ord}_p u_K = 3 \ell \), which is odd, and hence:

\[\text{im} \psi \cong C_{3\ell} \times C_2; \tag{21}\]

where \( C_2 \) maps onto the copy of \( \{\pm 1\} \) in (17) resulting from the single infinite place \( j \) (that is, \#\( m_\infty = 1 \)); so finally in turn:

\[\text{coker} \psi \cong C_{\frac{p - 1}{6\ell}}. \tag{22}\]

But \( \frac{p - 1}{6\ell} \) equals 2 times the odd integer \( \frac{p - 1}{6\ell} \), so we may define \( \Gamma \) as in the lemma.

We shall revisit this in Lemma 11 and in the proof of Theorem 17. There, we will identify the cyclic group \( \Gamma \) with a subgroup of the Galois group of \( K^{p_j}/K \).

Having introduced some of the basic exact sequences of global class field theory, we now prove two results that will be key components in the proof of Theorem 17.

Lemma 7. Let \( F \) be a number field and \( \mathfrak{p} \) a prime of \( F \) above 2 of absolute ramification index \( e \). If \( u \in F \) is a \( \mathfrak{p} \)-unit and \( E = F(\sqrt{u}) \) a non-trivial quadratic extension of \( F \), then the exponent of \( \mathfrak{p} \) in the conductor of \( E/F \) is at most \( 2e \).

Proof. Since this is a local question and local conductors are preserved under completion, we can replace \( F \) by the local field \( F_\mathfrak{p} \) and \( u \) by its image in that, which will be a unit. If \( u \) is a square in \( F_\mathfrak{p} \), then \( \mathfrak{p} \) splits in \( E/F \) and the exponent of the conductor is 0. So we are reduced to proving the statement of the lemma for a quadratic extension by the square-root of a unit \( u \), of a 2-adic local field of ramification index \( e \) over \( \mathbb{Q}_2 \). Let \( \mathfrak{q} \) be the prime of \( E \) lying over \( \mathfrak{p} \), so we are considering \( E_\mathfrak{q}/F_\mathfrak{q} \).

We could prove the local version with an explicit case-by-case analysis when \( e = 1 \) or 2, which are the only cases we shall need later; however it is easier to prove the general case using local class field theory and some properties of local Hilbert symbols.

Let \((x, y)\) denote the local Hilbert symbol for \( m = 2 \) (see, e.g., Ex. 2 in Ref. 36). This is a (multiplicative) bilinear pairing from \( F_\mathfrak{p}^\times \times F_\mathfrak{p}^\times \) to \( \langle \pm 1 \rangle \). The basic properties we need are...
(i) \((x, y) = 1\) iff \(y \in \text{Norm}_{M/F}(M^x)\), where \(M = F_p(\sqrt{\alpha})\).

(ii) \((x, y) = (y, x)\).

Notice (ii) can be viewed as a kind of local reciprocity law, saying that \(x\) is a norm from \(F_p(\sqrt{\alpha})\) iff \(y\) is a norm from \(F_p(\sqrt{\alpha})\).

As usual, let \(U^{(0)} = U\), the units of \(F_p\), and, for \(n \geq 1\), let \(U^{(n)} = \{u : u \in U \mid u \equiv 1 \mod p^n\}\). By local class field theory, the conductor of \(E_p/F_p\) is \(p^r\) where \(r\) is the smallest integer \(\geq 0\) such that \(U^{(r)} \subseteq \text{Norm}_{E_p/F_p}(E_p^x)\). So the statement that we have to prove is equivalent to \(U^{(2e)} \subseteq \text{Norm}_{E_p/F_p}(E_p^x)\).

By the definition of \(e\), \(U^{(2e)}\) consists of precisely those units of \(F_p\) which are congruent to 1 modulo \(4\mathbb{Z}_p\). So equivalently, by (i), we have to show that if \(v \in U\), \(v \equiv 1 \mod 4\mathbb{Z}_p\), then \((v, v) = 1\) which by (ii) is true if and only if \((v, u) = 1\).

But (e.g. by Ex. 2.12 of Ref. 36), \(v\) is \(p\)-primary and \(F_p(\sqrt{\alpha})/F_p\) is trivial or unramified: so it has conductor 0. Thus, \(U \subseteq \text{Norm}_{F_p(\sqrt{\alpha})/F_p}(F_p(\sqrt{\alpha})^x)\), and for any unit \(w\), \((v, w) = 1\). In particular \((v, u) = 1\), which is what we had to prove.

**Proposition 8.** With the notation of Section II B, if \(L_1/K\) is a subextension of \(K^{m_0}\) not lying in \(H_K\) with \([L_1 : K]\) a power of two, where \(m_0 = 4\mathbb{Z}_K\), then \(L_1H_K = LH_K = H_K(\sqrt{-u_K})\).

**Proof.** By the second paragraph of Section II B, \(L/K\) is unramified at all primes of \(K\) except possibly those above 2, and is ramified at \(j\) but not at \(j^*\). Thus by Lemma 7, it has conductor dividing \(m_0\), so \(L \subseteq K^{m_0}\). Since \(L/K\) has ramification at \(j\), \(L \not\subseteq H_K\).

The result will then follow from that fact that \([K^{m_0} : H_K]\) equals 2 times an odd number. From the exact sequence (17), \(\text{Gal}_{K^{m_0}/H_K}\) is isomorphic to the quotient of \(G := (\mathbb{Z}_K/4\mathbb{Z}_K)^\times \times \langle \pm 1 \rangle\) by the image of the units of \(K\), \(\langle \pm 1 \rangle \times \langle u_K \rangle\). We must show that the two-primary part of this quotient group has order 2. The proof divides into a number of cases depending on the residue class of \(D \mod 8\) and the form of \(u_K\). In all cases, under the map from the units into \(G\), \(-1 \mapsto (\ast, -1)\) and \(u_K \mapsto (\ast, 1)\), since \(u_K > 0\) in the embedding \(j\) of \(K\) into \(\mathbb{R}\).

We let \(u_K = (n + m\sqrt{D})/2\) where \(n, m\) are positive integers with \(n = n_1 = \text{Tr}_{L/K}(u_K)\). Notice \(Z_K = \mathbb{Z} \oplus \mathbb{Z}\tau_1\) where \(\tau_1 = \sqrt{D}\) if \(D\) is even, and \(\tau_1 = (-1 + \sqrt{D})/2\) if \(D\) is odd. Note also that \(m^2D = n^2 + 4\) as \(\text{Norm}_{L/K}(u_K) = -1\). Now, since \(D\) is square-free, we can reduce to a number of cases for \(n, m\), enabling us to analyse the individual congruence classes for \(D \mod 8\).

(I) \(4 \mid n\). Let \(n = 4n_0\). From \(D\) square-free, we find that \(m = 2m_0\), with \(m_0\) odd, and \(m_0^2D = 1 + 4n_0^2\) as well as \(u_K = 2n_0 + m_0\sqrt{D} = (2n_0 + m_0) + 2m_0\tau_1\).

(a) \(n_0\) odd. Congruences \(\mod 8\) show that \(D \equiv 5 \mod 8\)

(b) \(n_0\) even. Here \(D \equiv 1 \mod 8\)
(II) $2 \parallel n$. Let $n = 2n_0$, $n_0$ odd. $D$ square-free $\Rightarrow m = 2m_0$ with $m_0$ odd and $m_0D = n_0^2 + 1$ with $D \equiv 2 \pmod 8$. $u_K = n_0 + m_0\sqrt{D}$

(III) $n$ odd. Here $m$ must also be odd and $D \equiv 5 \pmod 8$. $u_K = (n_0 + m_0)/2 + m_0\tau_1$.

We now address each possible case which arose in $\Box \Box \Box$ for $D$ modulo 8.

(A) $D \equiv 5 \pmod 8$. 
Here, $(\mathbb{Z}_K/4\mathbb{Z}_K)^\times = (\mathbb{Z}_K/\sqrt{2}\mathbb{Z}_K)^\times$ is of order 12 with Sylow 2-subgroup $\Sigma_2$ consisting of elements $\equiv 1 \pmod {2\mathbb{Z}_K}$ and isomorphic to the additive group of $\mathbb{Z}_K/2\mathbb{Z}_K \cong \mathbb{F}_4$, i.e. to the Klein four-group $V_4$ under $1 + 2x \leftrightarrow x$. The quotient $(\mathbb{Z}_K/4\mathbb{Z}_K)^\times/\Sigma_2$ identifies with $(\mathbb{Z}_K/2\mathbb{Z}_K)^\times \cong \mathbb{F}_4^\times$, a cyclic group of order 3.

The Sylow 2-subgroup $G_2$ of $G$ is of order 8 and it is isomorphic to $\Sigma_2 \times \langle \pm 1 \rangle \cong C_2 \times C_2 \times C_2$. Under the identification of $\Sigma_2$ with $\mathbb{Z}_K/2\mathbb{Z}_K$, $-1 \mapsto 1$ since $-1 \equiv 3 \pmod 4$ and $3 = 1 + 2 \times 1$.

In case $\Box \Box$ above, $u_K \equiv 1 \pmod {2\mathbb{Z}_K}$ and $u_K \mapsto \tau_1$ or $\tau_1 + 1$ in $\Sigma_2$.

In case $\Box \Box \Box$, $u_K$ maps to a generator of $((\mathbb{Z}_K/4\mathbb{Z}_K)^\times/\Sigma_2$ and $u_K^k \mapsto \tau_1$ or $\tau_1 + 1$ in $\Sigma_2$. To see this, note that $u_K^2 = nu_K + 1$ which implies that $u_K^2 - 1 = 2((n^2 + 1)/2)u_K + (n - 1)/2$, the expression in brackets is congruent to $u_K$ or $u_K + 1$ modulo $2\mathbb{Z}_K$, and $u_K \equiv \tau_1$ or $\tau_1 + 1$ modulo $2\mathbb{Z}_K$.

In either case, it follows easily that the image of $\langle -1, u_K \rangle$ in $(\mathbb{Z}_K/4\mathbb{Z}_K)^\times$ quotiented by its Sylow 3-subgroup is of order 4, and this gives the result.

(B) $D \equiv 1 \pmod 8$.
Here, 2 splits into 2 primes $\varphi$ and $\varphi^r$ in $K$. $\sqrt{D} \equiv 1$ modulo the square of one of these primes and $-1$ modulo the square of the other.

$(\mathbb{Z}_K/4\mathbb{Z}_K)^\times = (\mathbb{Z}_K/\sqrt{2}\mathbb{Z}_K)^\times \times (\mathbb{Z}_K/\sqrt{2}\mathbb{Z}_K)^\times \cong C_2 \times C_2$, where $(\mathbb{Z}_K/\sqrt{2}\mathbb{Z}_K)^\times$ is identified with $(\mathbb{Z}_K/\sqrt{2}\mathbb{Z}_K)^\times \cong \mathbb{Z}/2\mathbb{Z}$ under $1 + \varpi x \leftrightarrow x$, where $\varpi$ is any uniformiser — that is, an element of $\mathbb{Z}_K$ in $\varphi\sqrt{2}$ (e.g. in this case, 2) — and similarly for the $(\mathbb{Z}_K/\sqrt{2}\mathbb{Z}_K)^\times$ factor.

We are in case $\Box \Box \Box$ above, which shows that $u_K \equiv \pm \sqrt{2} \pmod {4\mathbb{Z}_K}$. So we can choose $\varphi$ so that $u_K \equiv 1 \pmod {\varphi^2}$ and $\equiv -1 \pmod {\varphi^2}$. This translates into $u_K \mapsto (0, 1)$ in $(\mathbb{Z}_K/4\mathbb{Z}_K)^\times$ identified with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as above. Clearly $-1 \mapsto (1, 1)$. This shows that the image of the units in $(\mathbb{Z}_K/4\mathbb{Z}_K)^\times$ or $G$ is again of order 4 as required.

Note that in this case, if we consider the image of units in just $(\mathbb{Z}_K/\sqrt{2}\mathbb{Z}_K)^\times \times \langle \pm 1 \rangle$ or $(\mathbb{Z}_K/\sqrt{2}\mathbb{Z}_K)^\times \times \langle \pm 1 \rangle$, the above explicit images show that $[K^{\varphi^2}: H_K] = 2$ and $K^{(\varphi^2)^4} = H_K$.

(C) $D \equiv 2 \pmod 8$.
In this case, $G_1 := (\mathbb{Z}_K/4\mathbb{Z}_K)^\times = (\mathbb{Z}_K/\sqrt{2}\mathbb{Z}_K)^\times$ is of order 8. To compute structure and images, we write $\varpi$ for $\sqrt{D}$, which is a local uniformiser for $\varphi$, and note that the elements of $G_1$ have representatives of the form $1 + a_1\varpi + a_2\varpi^2 + a_3\varpi^3$ with $a_i \in \{0, 1\}$. Since $\varpi^2 = 2f$ for some $f \equiv 1 \pmod 4$,
we see that $2 = \varpi^2$ in $G_1$ and that we can reduce the product of two elements of $G_1$ back to representative form in $G_1$ by equating all powers $\varpi^r, r \geq 4$ to 0 and replacing a term $b_i \varpi^r, b_i \in \mathbb{Z}$ by $0, \varpi^r, \varpi^{r+2}, \varpi^r + \varpi^{r+2}$ for $b_i \equiv 0, 1, 2, 3 \mod 4$, respectively. A simple computation then shows that $G_1 = \langle v_1 \rangle \times \langle v_2 \rangle \cong C_4 \times C_2$, where $v_1 = 1 + \varpi$ is of order 4 and $v_2 = 1 + \varpi^2$ is order 2, and further $1 + \varpi^3 = v_1^2 v_2$. The image of $-1$ in $G_1$ is just $v_2$. We are in case (i) above, which shows that $u_K - 1$ is in $\varphi^2$, since this is true of $\sqrt{D}$ and $n_0 - 1 \in 2\mathbb{Z}$. Thus $u_K$ maps to $v_1^j v_2^i$ with $r$ odd and the image of the units in $G_1$ is all of $G_1$. Since the images of $-1$ and $u_K$ in the full group $G$ are also of orders 2 and 4, respectively, the image of the units in $G$ is a subgroup of index 2 which is isomorphic to $C_4 \times C_2$. 

\[ \square \]

\textbf{Remark 9.}

(i) The proof of the last proposition actually shows that $K^{m_0}$ equals $LH_K$, unless $D \equiv 5 \mod 8$ and $u_K \equiv 1 \mod 2\mathbb{Z}_K$ when it is an abelian cubic extension of $LH_K$.

(ii) Further analysis in the various cases shows that $L/K$ is actually ramified above 2 with finite part of the conductor $4\mathbb{Z}_K$ when $D \equiv 2, 5 \mod 8$ and $\varphi^2$ when $D \equiv 1 \mod 8$. Furthermore, in the latter case, $m_0 \equiv 1 \mod 4$, so $\varphi$ is the prime with $\sqrt{D} \equiv 1 \mod \varphi^2$.

(iii) For the proof of Theorem 17, it is not hard to see that we could get by with the weaker result that the 2-rank of $\text{Gal}_{K^{m_0}/H_K}$ is 1 (i.e., its Sylow 2-subgroup is cyclic). For this, we could, for example, make use of a more general result like Theorem 2.2 of Ref. [43], which implies that the 2-rank of $\text{Gal}_{M/H_K}$ is $\leq 2$, resp. 3, when $D \equiv 2, 5$ resp. 1 mod 8, where $M$ is the ray class field of conductor $j^r$ times a large power of 2. In the respective cases, we can easily find $C_2$ and $C_2 \times C_2$ subextensions of $M/H_K$ which are linearly independent of $K^{m_0}$ over $H_K$ (with ramification over $j^r$ and/or higher conductor ramification over 2) and this gives the result. However, we think that the explicit proof of the more precise result is fairly concise and clearer to the non-specialist.

\section*{B. Splitting of the modulus ($d$)}

Define $\vartheta = 1 + \sqrt{d + 1}$: this is a key invariant. We will often just write the symbol $\vartheta$ for the principal ideal $(\vartheta) = \vartheta \mathbb{Z}_K$. Notice $\vartheta^r = 1 - \sqrt{d + 1} = 2 - \vartheta$ and $\vartheta \mathbb{Z}_K = \vartheta \vartheta^r$. In particular, when $d = p$ is prime, $\vartheta = p$ and $\vartheta^r = p^r$ are principal prime ideals which split completely in the Hilbert class field extension $H_K/K$ (for definitions, see Section III.A). When it is important to distinguish different levels $\ell$ over the same field $K$, we shall write $\vartheta_\ell = 1 + \sqrt{d_\ell + 1}$. 


Let \( d = n^2 + 3 \) be as above: then \( d \equiv 0 \) or \( 3 \) mod \( 4 \) and moreover if \( d \) is even then \( d \equiv 4 \) mod \( 8 \). Hence \( d \) is of the form \( 4^r s \) for some odd \( r \in \mathbb{N} \) and \( s \in \{0, 1\} \). By Lemma \[3\] writing \( r \) in terms of its distinct prime powers as \( r = \prod q^e \), each prime \( q \) appearing in the factorisation of \( r \) inside \( \mathbb{Z} \) splits into co-prime ideals as \( q\mathbb{Z}_K = qq' \). (In general, when one of the primes \( q \) equals 3, it is possible to have \( \gcd(d, D) = 3 \) whence 3 would be ramified with \( q = q' \). It is easy to see however that this cannot occur when \( d \) has the form \( n^2 + 3 \).) In other words \( q\mathbb{Z}_K = q^t(q')^s \). We remark that it is shown in Ref. \[26\] that \( q = 1 \) mod \( 3 \) for every one of these primes \( q > 3 \).

Consider the ideal \( \partial = \partial\mathbb{Z}_K \): no odd rational prime can divide it, and its norm is \( d\mathbb{Z}_K \); so it follows that one but not both of the factors \( q^u \) and \( (q')^v \) must divide it. We thus define a unique \( \mathbb{Z}_K \)-ideal factor \( q^u \) dividing into \( \partial \) for each distinct rational odd prime power factor \( q^u \) of \( r \). Let \( \mathfrak{f} = \prod q^u \), and similarly for \( \mathfrak{f}' \). The factor 4, if it appears, may be regarded for these purposes as splitting into \( 2 \times 2 \) so finally we see that \( \partial = 2^s\mathfrak{f} \).

The individual prime divisors of \( \partial \) and \( \partial' \) are not, in general, principal ideals themselves, unless \( d \) is prime. For example in the case of \( d = 6^2 + 3 = 39 \), each of \( 3\mathbb{Z}_K \) and \( 13\mathbb{Z}_K \) splits into a pair of non-principal ideals; but since the class number is 2 the product of any pair of them is principal.

Incidentally, these quantities \( 1 \pm \sqrt{d / 2} \) form a series of intermediate “irrational dimensions” which arise from solving eq. \((10)\) for \( d_\ell \) whenever \( \ell \) is odd:

\[
0 = X^2 - 2X - d_\ell = (X - 1)^2 - (1 + d_\ell).
\] (23)

Namely, the case of norm \(-1\) yields a series “\( d_\ell \)” corresponding to odd powers of \( u_K \), using eq. \((8)\) with \( u_K \) in place of \( u^2 \), viz.:

\[
\partial_\ell = 1 + \sqrt{d_\ell + 1} = u_K^\ell + u_{-K}^\ell + 1 = “d_\ell”.
\] (24)

When \( \ell \) is even, they align with the series \((8)\) defined in terms of \( u_K^2 \): so “\( \partial_{2k} = d_k \)”.

We now come to the main result of Section \[III\]. Write \( \varphi : \mathbb{N} \to \mathbb{N} \) for the Euler phi-function, and CRT for the Chinese remainder theorem.

**Proposition 10.** Let \( d_\ell \geq 7 \) be of the form \( d_\ell = n^2 + 3 \) as above.

(I) When \( d_\ell \) is odd, \( \text{Gal}_{K^{\sqrt{d_\ell}}/K} \) has order \( h_K \varphi(d_\ell)/3\ell \). In particular therefore in our case where \( d_\ell = p \) is a prime, \( \text{Gal}_{K^{\sqrt{d_\ell}}/K} \) has order \( (p - 1)h_K/3\ell \).

(II) When \( d_\ell \) is even, so \( d_\ell/4 \) is odd, \( \text{Gal}_{K^{\sqrt{d_\ell}}/K} \) has order \( h_K \varphi(d_\ell^2)/\ell \).

**Proof.** \[4\] Expansion of the divisors \( \partial_\ell, \partial_\ell' \) above in terms of the various \( q \) affords, via the CRT, a split of \((\mathbb{Z}_K/d_\ell)^\times\) into two isomorphic groups each of order \( \varphi(d_\ell) \):

\[
(\mathbb{Z}_K/d_\ell)^\times \cong (\mathbb{Z}_K/\partial_\ell)^\times \times (\mathbb{Z}_K/\partial_\ell')^\times.
\] (25)

Define \( w = \text{ord}_K u_K \) and \( w_v = \text{ord}_{\partial_\ell'} u_K \). By Corollary \[5\] the order of the image of \( u_K \) inside \((\mathbb{Z}_K/d_\ell)^\times\) is \( 6\ell \), which must therefore be equal to \( \text{lcm}(w, w_v) \). The
exact sequence (16), evaluated in our case where \( m = \partial j \) and so \( \# m_\infty = 1 \) then shows that the order of the Galois group \( \text{Gal}_{K^w/J} \) is \( 2h_K \varphi(d_\ell) \) divided by the order of the image of the global units \( \langle u_K \rangle \times \langle -1 \rangle \) inside \( (\mathbb{Z}_K/\partial_\ell)^\times \times \{\pm 1\} \), under the map denoted \( \psi \) in (16).

We chose \( u_K \) to have signature \((+,-)\), so since we are only allowing ramification at \( j \), its image is \( \psi(u_K) = (u_K + \partial_\ell, 1) \). The torsion component \( \langle -1 \rangle \) injects diagonally into the right-hand side as the characteristic of the ring \( \mathbb{Z}/\partial_\ell \) is not 2. So the kernel of \( \psi \) is torsion-free and of rank one. Indeed \( u_K^w \) is a generator; hence the image of \( \psi \) has order 2\( w \). In summary, we have shown that the Galois group \( \text{Gal}_{K^w/J} \) has order \( h_K \varphi(d_\ell)/w \). We must now show that \( w = 3\ell \).

In order to make use of the analogy between (8) and the Chebyshev polynomials of the first kind \( T_n \), a shifted version has been defined in Ref. [14]:

\[
T_n^*(x) = 1 + 2T_n^*(\frac{x-\tau}{2}),
\]

(26)

which also satisfies the fundamental composition relation \( T_m^*(T_n^*(x)) = T_{mn}^*(x) \) for every \( m,n \geq 0 \). The first few polynomials are \( T_0^*(x) = 3 \), \( T_1^*(x) = x \), \( T_2^*(x) = x^2 - 2x \), \( T_3^*(x) = x^3 - 3x^2 + 3 \). The \( T_n^* \) are independent of \( D \) so given \( d_0 = d_0(D) = 3 \) and \( d_1 = d_1(D) \), we know all \( d_k = d_k(D) = T_k^*(d_1) \). The defining recursion \( T_n^*(x) = 2xT_{n-1}^*(x) - T_{n-2}^*(x) \) for the \( T_n^* \) yields for the \( T_n^* \):

\[
T_n^*(x) = xT_{n-1}^*(x) - xT_{n-2}^*(x) + T_{n-3}^*(x).
\]

(27)

The definition above at (8) of the ‘dimensions’ \( d_\ell \) is in terms of powers of \( u_K^\ell \); but it could equally well be made in terms of \( u_K^\ell \) as in eq. (24) above. Consequently for any \( \ell \geq 1 \) (see equation C0 in Ref. [14]):

\[
T_3^*(\partial_\ell) - T_0^*(\partial_\ell) = \partial_{3\ell} - 3 \quad \text{is a } \mathbb{Z}_K\text{-multiple of } \partial_\ell,
\]

(28)

hence by (24):

\[
\partial_{3\ell} - 3 = u_K^3 + u_K^{3\ell} + 1 - 3 \equiv 0 \mod \partial_\ell.
\]

(29)

But since \( \text{ord}_{u_K} u_K = 6\ell \), it follows that \( u_K^{3\ell} \equiv 1 \mod \partial_\ell \) and thus in particular, \( u_K^{3\ell} \equiv u_K^{-3\ell} \mod \partial_\ell \). (In fact more is true: it is possible using Lemmas 1 and 7 of Ref. [44] to show that for every odd \( \ell \), \( u_K^{3\ell} \) is actually congruent to 1 or \(-1\) mod \( \partial_\ell \) (and so \( \partial_\ell^f \)). This is important for non-prime \( d \); although we do not need it here). So, from (29), \( 2u_K^{3\ell} \equiv 2 \mod \partial_\ell \). As 2 is invertible modulo \( d_\ell \), it follows that \( u_K^{3\ell} \equiv 1 \mod \partial_\ell \).

Suppose \( w \leq 3\ell \): since \( u_K^w = -1/u_K \) it follows by taking \( \tau \)-conjugates of the congruence \( u_K^w \equiv 1 \mod \partial_\ell \), that \( u_K^w \equiv (-1)^w \mod \partial_\ell^f \) and therefore that \( u_K^{2w} \equiv 1 \mod \partial_\ell^f \). On the other hand, \( w_\tau \) is defined to be minimal with the property that:

\[
u_K^{w_\tau} \equiv 1 \mod \partial_\ell^f.
\]

(30)

So in fact \( w_\tau | 2w \). But \( \text{lcm}(w, w_\tau) = 6\ell \), so by our assumption that \( w < 3\ell \):

\[
6\ell = \text{lcm}(w, w_\tau) \leq \text{lcm}(2w, w_\tau) = 2w < 6\ell,
\]

(31)
a contradiction. Incidentally, that \( w_r = 6 \ell \) follows immediately from this.

$(\text{II})$ This is an easy modification of the argument for $(\text{I})$, using the fact that $D \equiv 5 \mod 8$ and so $(\mathbb{Z}_K/4)^\times \cong C_2 \times C_2 \times C_3$.

So we have established the promised order of the main Galois group (the degree of the ray class field extension $K^{(\ell)} / K$). Finally, for future reference, we extend Lemma $6$ as follows. For any prime number $p$ and any group $G$ let $Syl_p G$ denote the Sylow $p$-subgroup of $G$.

**Lemma 11.** We work once more with the notation and restrictive hypotheses of Lemma $6$ in particular, $d = p$ is a prime, $p\mathbb{Z}_K = pp^\tau$, and $p = \partial\mathbb{Z}_K$.

The quadratic subextension of $K^{(p)}/H_K$ — that is, the field $F_1 = (K^{(p)})^{\Gamma}$ fixed by $\Gamma$ — is generated over $H_K$ by the polynomial $X^2 + \partial$, where $\partial = 1 + \sqrt{p} + 1$ as defined above, with $\partial \partial^\tau = -p$.

Furthermore, the $2$-primary part of the extension in $(\text{II})$ is split. In other words, recalling that $\Re^{(p)} \cong \text{Gal}_{K^{(p)}/K}$:

\[
Syl_2 \text{Gal}_{K^{(p)}/K} \cong C_2 \times Syl_2 C_K.
\]

**Proof.** Let $E = H_K(\sqrt{-\partial})$. The discriminant of this extension divides into $-4p$; and it will be complex only above the first infinite place $j$, as $j(\sqrt{d} + 1) > 1$. So the first assertion follows from the fact that primes of $K$ above $2$ do not ramify in $E/K$; whereas $p$ clearly does, so $E$ is indeed a subfield of $K^{(p)}$ properly containing $H_K$.

The normal closure of $F_1$ over $\mathbb{Q}$ is the compositum of $F_1$ and $(K^{(p)\tau})^{\tau \Gamma \tau^{-1}}$. This compositum is abelian over $K$ since it is contained in the ray class field with conductor $pjj^\tau$; so its Galois group over $K$ is a direct product of $C_K$ by the Klein four-group $V_4$. The result follows.

**IV. STARK UNITS AND ZETA FUNCTIONS**

**A. Dedekind ray class zeta functions**

By taking the classical harmonic series $\sum_{n \geq 1} n^{-1}$ and making it naturally into a function of a complex variable $s$, initially with real part $\Re(s) > 1$, the Riemann zeta function $\zeta(s) = \sum_{n \geq 1} n^{-s}$ encodes arithmetic information about the distribution of the prime integers within $\mathbb{Z}$. This is particularly evident upon rewriting it as an Euler product, with one Euler factor at each prime:

\[
\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.
\]

Both of these expressions are valid as written for $\Re(s) > 1$; moreover $\zeta(s)$ has an analytic continuation to the whole complex plane, which renders it a meromorphic function on $\mathbb{C}$ with a single pole of order one at $s = 1$.

By analogy we define a Dedekind zeta function attached to any number field $F$ by taking the natural analogue of a ‘harmonic series’ for $F$, this time over
the non-zero ideals of the ring of integers \( \mathbb{Z}_F \), and using the same approach to convert it into a meromorphic function of \( s \). The norm of an ideal \( \mathfrak{a} \) of \( \mathbb{Z}_F \), denoted by \( N_{\mathbb{F}/\mathbb{Q}}\mathfrak{a} \), is defined to be the cardinality of the finite quotient ring \( \mathbb{Z}_F/\mathfrak{a} \). As is the convention we shall simply write \( N_{\mathbb{F}/\mathbb{Q}}\mathfrak{a} \) for \( (N_{\mathbb{F}/\mathbb{Q}}\mathfrak{a})^z \) for a complex exponent \( z \). The Dedekind zeta function attached to \( \mathbb{F} \) is then:

\[
\zeta_{\mathbb{F}}(s) = \sum_{(0) \neq \mathfrak{a} \text{ ideals of } \mathbb{Z}_F} N_{\mathbb{F}/\mathbb{Q}}\mathfrak{a}^{-s}.
\] (34)

Once again this expression defines a function for \( \Re(s) > 1 \) which may be analytically continued to the whole of \( \mathbb{C} \); it is also meromorphic with a single simple pole at \( s = 1 \). By the unique factorisation into prime ideals we also obtain an expression for \( \zeta_{\mathbb{F}}(s) \) as an Euler product in the region \( \Re(s) > 1 \):

\[
\zeta_{\mathbb{F}}(s) = \prod_{(0) \neq \mathfrak{p} \text{ prime ideals of } \mathbb{Z}_F} \frac{1}{1 - N_{\mathbb{F}/\mathbb{Q}}\mathfrak{p}^{-s}}.
\] (35)

Now let \( \mathbb{F} \) be the ray class field \( \mathbb{K}^m \) of our base real quadratic field \( \mathbb{K} \) corresponding to the modulus \( m = m_0m_\infty \). To each element \( \sigma \in G = \text{Gal}_{\mathbb{K}^m/\mathbb{K}} \) there corresponds under the inverse of the Artin reciprocity map, an infinite coset of fractional ideals \( \mathfrak{A}_\sigma \) of \( \mathbb{Z}_K \), characterised by their multiplicative class modulo the ray group \( \mathcal{P}_m \) defined above. This is known as the ray class \( [\mathfrak{A}_\sigma] \) of each \( \mathfrak{A}_\sigma \). Each ray class contains a set of prime ideals of Dirichlet density \( \frac{1}{\#G} \). Let us fix for each such \( \sigma \) a representative integral ideal \( \mathfrak{A}_\sigma \).

We may then define for each \( \sigma \in G \) a partial zeta function:

\[
\zeta(s, \sigma) = \sum_{a \in [\mathfrak{A}_\sigma]} N_{\mathbb{K}/\mathbb{Q}}a^{-s},
\] (36)

and then observe from (34), (35) and (36):

\[
\zeta_{\mathbb{K}}(s) = \prod_{p | m} \frac{1}{1 - N_{\mathbb{F}/\mathbb{Q}}\mathfrak{p}^{-s}} \cdot \sum_{\sigma \in G} \zeta(s, \sigma).
\] (37)

In other words, once we adjust for the Euler factors from the ‘bad primes’ \( p | m \), we recover the original Dedekind zeta function.

**Remark 12.** It is important in what follows to bear in mind that \( m \) — and ipso facto \( \ell \) — is implicit in each \( \sigma \), by virtue of \( [\mathfrak{A}_\sigma] \) being a member of a specific ray class group of that modulus; even though this is not explicit in the abbreviated notation we use for the ‘Stark units’ \( \epsilon_\sigma \) below.

### B. Stark units as special values of zeta functions

Aside from References [16,17,41], the material in this section is explained in a down-to-earth manner in our context in Refs. [19] and [46].
From now on our modulus will always be \( m = \partial j \). Each of the partial zeta functions has an analytic continuation which is meromorphic on the whole of \( \mathbb{C} \), with just a single simple pole at \( s = 1 \), the residue of which is independent of \( \sigma \). So the differences \( \zeta(s, \sigma) - \zeta(s, \sigma') \) are entire functions (cf. Theorem 3.4 in Ref.\[19\]). The Stark units attached to the extension \( K^{\partial}/K \) are defined from a particular set of these, as we now explain.

We use Stark’s symbol \( c_0 \) for a new ideal subgroup of the ray group \( P_1^0 \), of index 1 or 2. Namely, \( c_0 \leq P_1^0 \) consists of the principal ideals \( \langle \alpha \rangle \) where \( \alpha \equiv 1 \mod \partial \) and \( j(\alpha) > 0 \). If the index \( \mid P_1^0 : c_0 \mid \) is 1, then the conjectures in Ref.\[16\] are not able to be formulated for the modulus \( \partial j \). However, as we shall now show, this index is always 2 in our situation and so there exists a coset of \( c_0 \) in \( P_1^0 \), referred to as \( T \) in Ref.\[16\] and as \( R \) in the equivalent adaptation in Section 3.2 of Ref.\[19\], in which each principal ideal \( \langle \beta \rangle \in P_1^0 \) satisfies \( j(\beta) < 0 \).

Indeed, as Stark points out, being of index 2 is equivalent to requiring that \( j(u) > 0 \) for any unit \( u \in \mathbb{Z}_K^* \) satisfying \( u \equiv 1 \mod \partial \). By our choice of \( j \), any power of \( u_K \) will in fact be positive under \( j \). Since the characteristic of \( \mathbb{Z}_K/\partial \) is strictly larger than 2, we know from Proposition \[10\] that the global units which are congruent to 1 modulo \( \partial = \partial_K \) are just generated by \( (1, u_K^j) \), and so Stark’s condition is fulfilled. Incidentally, the same line of reasoning also implies that \( K^\partial = K^{\partial^*} \) and \( K^{\partial} = K^{\partial^*} \), because \( j^*(u_K) < 0 \) and so the order at the other place is \( 6\ell \) rather than \( 3\ell \).

By definition, \( T \) is a ray ideal class in its own right, which for consistency we could denote by \([t] \in \mathcal{R}^\partial \) for some ideal \( t \in T \). We shall denote the automorphism corresponding to this under the Artin map by \( \sigma_T \in G \); this is the non-trivial element of \( \text{Gal}_{K^\omega/K^\partial} \). Multiplication by \([t] = [\mathcal{A}_{\sigma_T}] \) represents this same involution inside \( \mathcal{R}^\partial \). In the first equation on page 65 of Ref.\[16\], Stark effectively converts this action into a character which splits the \( \zeta(s, \sigma) \) into a positive and a negative part, yielding a holomorphic difference we denote by \( \delta \):

\[
\delta(s, \sigma) = \zeta(s, \sigma) - \zeta(s, \sigma_T \sigma).
\]

This function can be analytically continued to the entire complex plane, and for every \( \sigma \in G \) it satisfies:

\[
\delta(s, \sigma) = -\delta(s, \sigma_T \sigma).
\]

We finally define the Stark units attached to \( K^{\partial}/K \) as the quantities

\[
\epsilon_{\sigma} = \exp(\delta'(0, \sigma)).
\]

There are \( \#G \) of these, arranged into \( \#G/2 \) inverse pairs \( \{\epsilon_{\sigma}, \epsilon_{\sigma}^{-1}\} \) by virtue of \(39\). The torsion-free \( \mathbb{Z} \)-rank of the group of units \( \mathbb{Z}^*_{K^{\partial}} \) of the maximal totally real subfield \( K^\partial \) of \( K^{\partial} \) is equal to \( \#G - 1 \) by Dirichlet’s theorem, since the degree of the field extension \( K^\partial/\mathbb{Q} \) is \( \#G \). On the other hand the rank of \( \mathbb{Z}^*_{K^{\partial}} \) is equal to \( \#G/2 - 1 \) as half of the places are now complex: the extra units therefore have rank exactly \( \#G/2 \). For an explanation in a similar context see the final part of Section 6 of Ref.\[14\].
We summarise some key points of Stark’s remarkable construction, assuming his Conjectures 1 and 2 in Ref. [16] to be true for our base field $K$ and modulus $\partial_j$. In addition for part [IV] below we assume Conjecture 1 in Ref. [17], together with what seems to be a widely-accepted slight strengthening thereof, as in Conjecture 4.2 in Ref. [47], for example. As a dictionary for Stark’s notation in Ref. [16] in terms of ours: his $k$ is our $K$; his $F$ is our $K^{\partial}$; his modulus $\mathfrak{p}^{(2)}_{\infty}$ is our $\partial_j$; and his $K$ is our $K^{\partial}$.

**Lemma 13** (Consequences of Stark’s Conjectures).

(I) (Theorem 1 in Ref. [16]) Let $\Sigma$ denote a set of fundamental units for $\mathbb{Z}_K^{\times}$. The set $\Sigma \cup \{\epsilon_\sigma : \sigma \in G\}$ generates a set of units of $K^{\partial}$ of finite index in $\mathbb{Z}_K^{\times}$. Notice per the foregoing discussion, that the rank of the additional units provided by the Stark units (in pairs) is exactly the ‘deficit’ of $\frac{|G|}{2}$.

(II) (§4, p. 74 in Ref. [16]) The Galois element $\sigma_T$ induces complex conjugation in the complex embeddings of $K^{\partial}$. Since it is also algebraic inversion, it forces the Stark units in $K^{\partial}$ to lie on the unit circle in their complex embeddings.

(III) In their real embeddings, the $\epsilon_\sigma$ are all positive.

(IV) (Stark/Tate ‘over-$\mathbb{Z}$’) The extension $K^{\partial}(\sqrt{\epsilon_\sigma})$ of $K^{\partial}$ obtained by adjoining the square root of any one of the $\epsilon_\sigma$ itself an abelian extension of $K$.

**Remark 14.** Note that Theorem 1 in Ref. [16] applies in our case since $K^1 = H_K$ (which follows easily from [16]). Namely, the characters $\phi$ appearing in Theorem 1, considered as Galois group characters, are ones on $\text{Gal}_{K^{\partial}/K}$ that don’t factor through $\text{Gal}_{K^\partial/K}$. Since $K^1 = H_K \subseteq K^{\partial}$, all of these characters must have prime ideal $\partial$ as the finite part of their conductor. We shall make use of this in Remark 18 below, linking Conjecture 4 below to Conjectures 2 and 3.

**Remark 15.** The third part of the Lemma is true for the chosen real embedding of $K^{\partial}$ in which the $\epsilon_\sigma$ as defined above lie, since they are all given by the exponential of a real value. Since the $\epsilon_\sigma$ are all $\text{Gal}_{K^{\partial}/K}$ conjugates of each other, and this Galois group permutes the real embeddings of $K^{\partial}$ transitively, the positivity extends to all of the real embeddings.

We should note here that part [II] of the above justifies the notation in the introduction, whereby the complex-valued Galois conjugates of the Stark units $\epsilon_\sigma$ were referred to by the notation $e^{\sigma}$. We will refer to these complex numbers as Stark phase units.

**C. Scaling the Stark phase units**

We now need to impose the restriction that our $d_\ell = n_\ell^2 + 3$ be a prime, and we write $\partial = p$ as before.
The geometric scaling factor $\xi_\ell = \sqrt{x_0}$:

As explained in the introduction to Section 11.3 we are on a quest to establish certain number-theoretical properties of the scaling factor $\xi_\ell = \sqrt{x_0}$ which arises from eq. (12). For $j > 0$, the components $x_j$ of the un-normalized SIC fiducial vector are $x_j = \sqrt{x_0} e^{i\theta}$. The complex numbers $\sqrt{e^{i\theta}}$ of modulus one are obtained via the complex embeddings of the square roots $\sqrt{\sigma}$ referred to in Lemma 13 (IV). With our fixed embedding $j : K \to \mathbb{R}$ under which $\sqrt{D} > 0$, for $d_\ell = n_\ell^2 + 3$ for odd $\ell > 0$, define a scaling factor $\xi_\ell \in \mathbb{R} \cdot \sqrt{-1}$ by:

$$\xi_\ell = \sqrt{x_0} = \sqrt{-2 - \sqrt{d_\ell + 1}}. \quad (41)$$

The absolute minimal polynomial of $\xi_\ell$ is $X^4 + 4X^2 - n_\ell^2$. Note that $\xi_\ell^2 = -(\partial + 1)$ and $(\xi_\ell^2)^r = -(\partial - 3)$, mimicking the $(d + 1)(d - 3)$ configuration. Moreover $-\xi_\ell^2 (\xi_\ell^2)^r = d - 3 = n_\ell^2$.

**Lemma 16.** For every odd $\ell \geq 1$: $K(\xi_\ell) = K(\sqrt{-u_K})$; and this is a quadratic extension of $K$. Moreover the prime $p$ is inert in $L/K$; while the prime $p^r$ splits.

**Proof.** That $K(\sqrt{-u_K})/K$ is quadratic follows from the definition of $u_K$. So it suffices to express $\sqrt{-u_K}$ $K$-linearly in terms of $\xi_\ell$, for any odd positive integer $\ell$, using the definitions and Corollary 2 When $\ell = 1$, $\sqrt{-u_K} = (1 - u_K)\xi_1/n_1$; or equivalently $\xi_1 = n_1 \sqrt{-u_K} / (1 - u_K)$. Replacing every instance of $u_K$ with $u_K^\ell$ (and so $n_1$ with $n_\ell$) in this formula gives similar expressions for $\xi_\ell$.

The second part follows from the proof of Proposition 10 bearing in mind that $p \equiv 3 \pmod{4}$: $-u_K$ has order $6\ell$ and so is not a square in the cyclic multiplicative group mod $p$ of order $p - 1$; conversely it is a square mod $p^r$ because its order is $3\ell$.

**Theorem 17.** Fix any $\epsilon_\sigma$. Then the field $K^{p^r}(\sqrt{\epsilon_\sigma}) \subseteq K^{p^r}(\xi_\ell)$ for every odd $\ell$.

**Proof.** See Remark 12 for the tacit link between $\ell$ and $\sigma$.

Since $K^{p^r}(\sqrt{\epsilon_\sigma})$ is Galois (and even abelian) over $K$ by Lemma 13 (IV), Kummer theory tells us that any two of the $\epsilon_\sigma$, which are Galois conjugates over $K$, differ multiplicatively by a square in $K^{p^r}$. Thus, $K_\sigma := K^{p^r}(\sqrt{\epsilon_\sigma})$ is independent of $\sigma$.

If the $\epsilon_\sigma$ are squares in $K^{p^r}$ then $K_\sigma = K^{p^r}$ and the theorem is trivial. So, we assume that $K_\sigma$ is a quadratic extension of $K$.

We consider the ramification of $K_\sigma/K^{p^r}$. Since $\epsilon_\sigma$ is a unit, $K_\sigma/K^{p^r}$ is unramified at all primes of $K^{p^r}$ which do not lie over 2 by the same argument as in the second paragraph of Section 11.3 The real places of $K^{p^r}$ are those above $j^r$ and the $\epsilon_\sigma$ have positive images under these real embeddings by Lemma 13 (III), so $K_\sigma/K^{p^r}$ is unramified at these places.

Since $K_\sigma$ is an abelian extension of $K$ containing $K^{p^r}$, and no finite or infinite places outside of those over 2 can ramify in $K_\sigma/K^{p^r}$, the conductor of $K_\sigma/K$ must be $p^n m_0$ where $m_0$ is some power product of primes of $K$ over 2. As
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\( K_\sigma \not\subset K^p \), \( m_0 \) is non-trivial. If we take the fixed field \( F \) of the inertia subgroup at \( \mathfrak{p} \) of \( \text{Gal}_{K_\sigma/K} \), then \( F/K \) is an abelian extension of conductor dividing \( m_0 \) with \( K_\sigma = K^\mathfrak{p}F \). As all extensions are abelian, and \( [K_\sigma : K^\mathfrak{p}] = 2 \), we easily see that we can replace \( F \) by its maximal subfield over \( K \) of 2-power degree over \( K \), and so can assume that \( [F : K] \) is a power of 2.

We now show that \( m_0 \mid 4\mathbb{Z}_K \).

Let \( \wp_1 \) be any prime of \( K \) lying over 2 and \( \mathcal{P} \) any prime of \( K^\mathfrak{p} \) lying over \( \wp_1 \).

From the fact that \( K^\mathfrak{p}/K \) is unramified at \( \wp_1 \), standard properties of higher ramification groups and the conductor show that the exponent of \( \wp_1 \) in \( m_0 \) is the exponent, \( r \) say, of \( \mathcal{P} \) in the conductor of the abelian extension \( K_\sigma/K^\mathfrak{p} \). Since \( \epsilon_\sigma \) is a unit in \( K^\mathfrak{p} \), Lemma 7 shows that \( r \leq 2e \), where \( e \) is the absolute ramification index of \( \mathcal{P} \), which equals the absolute ramification index of \( \wp_1 \). So \( r \) does not exceed the exponent of \( \wp_1 \) in \( 4\mathbb{Z}_K \). Since this is true for all \( \wp_1 \), we venture to make the following conjecture, which would make the geometry of the subsequent sections of the paper align neatly with the number theory we have just been studying. Strictly speaking, this arithmetic conjecture is necessary for the three main conjectures in Section III to hold as stated. However, should it turn out not to be true, those three conjectures would simply take on a somewhat less concise form.

Conjecture 4 (Stark units are non-squares). The Stark unit \( \epsilon_\sigma \) of Theorem 17 is not a square. So the statement in Theorem 17 is an equality, viz.:

\[
K^\mathfrak{p}(\sqrt{\epsilon_\sigma}) = K^\mathfrak{p}(\xi_\ell);
\]

and hence:

\[
\sqrt[\mathfrak{p}]{\epsilon_\sigma} \xi_\ell = \sqrt{x_0} \epsilon_\sigma \in K^\mathfrak{p}.
\]

The fact that (43) follows from (12) is simply that \( \xi_\ell^2 \) and \( \epsilon_\sigma \) are numbers whose square roots generate the same quadratic extension of \( K^\mathfrak{p} \); hence their ratio is an element of \( (K^\mathfrak{p})^2 \), i.e., a square in \( K^\mathfrak{p} \).

As we explain in Section VII by taking generators of the (abelian) Galois group we are given a canonical ordering on these \( \sqrt{\epsilon_\sigma} \). This is the starting-point for the calculations in this paper.
Remark 18. As noted in remark 14, Theorem 1 in Ref. 16 applies in our case. Hence the \( \text{Gal}_{K^{0}/K} \)-conjugates of any one of our Stark units are distinct, which in turn is equivalent to any one of them generating the ‘small’ ray class field \( K^{0} \) over \( K \). Conjecture 4 says that the components of our fiducial vector all lie in that same field — and so generate it — since their squares give the Stark units up to an element of \( K \). Therefore Conjecture 2 is a consequence of Conjectures 3 and 4 combined.

On the other hand, when Stark’s conjecture is true and allows our construction of a fiducial — i.e. when Conjecture 3 is true — it is clear that Conjecture 2 implies Conjecture 4, because if the Stark units were squares in \( K^{0} \), the fiducial vector components would not lie in that field. Thus, given Conjecture 3, Conjectures 2 and 4 are essentially equivalent.

V. SIC COMPONENTS

We now go to a Hilbert space of dimension \( d = n^2 + 3 = p \) where \( p \) is a prime number. As mentioned above, there may well be an infinite number of such dimensions\(^ {27} \). In this section our task is to explain why we expect all these Hilbert spaces to contain a SIC fiducial vector of the general form given in eqs. (6)–(7). The argument will hinge entirely on the Clifford group, and not at all on number theory. The fact that the dimension is a prime means that we will be able to avoid a number of complications, and the fact that this prime equals 1 modulo 3 leads to further simplifications. The background we need is given in Appendix A.

A battle-tested conjecture by Zauner\(^ {1} \) implies that there exists a SIC fiducial vector invariant under a symplectic unitary of order three. Zauner’s conjecture has been refined and extended over the years\(^ {12,22,25,29} \). In particular we expect that every dimension \( d_\ell \) houses a SIC that has a unitary symmetry of order \( 3\ell \), where \( \ell \) refers to the position of \( d_\ell \) in the sequence of dimensions described in (8) of Section IV A. These SICs were called minimal SICs in Ref. 14, and in this paper our sole concern is with them. What is special about dimensions equal to 1 modulo 3 is that the conjugacy class of symplectic unitaries that leave some SIC vector invariant contains a representative that derives from a diagonal symplectic matrix. This is important because, in the standard Weyl representation, a diagonal symplectic matrix is represented by a permutation matrix\(^ {29} \). Accordingly, we choose an integer \( \alpha \) such that \( \alpha^{3\ell} \equiv 1 \mod d \), and a symplectic matrix

\[
F_S = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \quad \Rightarrow \quad (U_S)_{r,s} = \delta_{\alpha r,s}.
\]

As advertised the unitary matrix \( U_S \) is a permutation matrix, and the by now standard conjecture says that there exists a SIC fiducial vector \( \Psi \) left invariant by \( U_S \).

The centralizer of \( F_S \) within the symplectic group is an abelian group gen-
erated by a matrix

\[ F_G = \begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{pmatrix} \implies (U_G)_{r,s} = \delta_{br,s}, \quad (45) \]

where \( \theta \) is a generator of the multiplicative group \( \mathbb{Z}_d^\times \), so that the list \( \theta, \theta^2, \ldots, \theta^{d-1} = 1 \) runs through all the non-zero integers counted modulo \( d \) (recall that \( d \) is prime). In particular

\[ \alpha = \theta^{\frac{d-1}{3}}. \quad (46) \]

The unitary \( U_G \) is a permutation matrix leaving the first (or ‘zeroth’) component of our vector invariant, and permuting the others according to

\[ \Psi_{\theta j} \rightarrow \Psi_{\theta j+1}. \quad (47) \]

Because of the symmetry that we have postulated

\[ \Psi_{\theta j+(d-1)/3\ell} = \Psi_{\alpha \theta j} = \Psi_{\theta j}. \quad (48) \]

The SIC fiducials have \( d \) components. The first (zeroth) component is \( x_0 = \xi^2 \) in the notation used in Section 4.3. The remaining \( d-1 \) components can be expressed in terms of a smaller set of independent numbers \( z_0, z_1, \ldots, z_{(d-1)/3\ell-1} \) by

\[ \hat{\Psi}_{\theta j \mod d} = z_j \mod (d-1)/3\ell. \quad (49) \]

(Here we find it convenient to work with the un-normalized vector \( \hat{\Psi} \). See eq. (6). The point is that the normalizing factor \( N \) does not belong to the ray class field, although its square does.) Thus the fiducial vector is made up from \( 3\ell \) copies of an ordered set of \( (d-1)/3\ell \) complex numbers. The generator of the symplectic centralizer, \( U_G \), permutes these numbers cyclically and gives rise to Clifford equivalent SICs.

But more can be said. The extended centralizer contains an anti-unitary represented by pure complex conjugation\(^{29}\). A SIC fiducial vector left invariant by this anti-unitary must be real. Numerical evidence suggests that every odd dimension of the form \( d = n^2 + 3 \) contains such a real fiducial vector (and sometimes more than one inequivalent real fiducial vector)\(^{25}\). From there we can reach a fiducial vector having the same symmetry if we can find a symplectic matrix \( F \) such that \( FF_S F^{-1} = F_S^{-1} \). It turns out that there is a solution, and that the resulting unitary transformation is a discrete Fourier transform, because

\[ F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \implies (U_F)_{r,s} = \frac{1}{\sqrt{d}} e^{\frac{2\pi irs}{d}}, \quad r, s \in \{0, \ldots, d-1\}, \quad (50) \]

(where the global phase factor has been chosen so that the matrix elements belong to the cyclotomic field).

The resulting complex fiducial vector consists of one real component and \( (d-1)/2 \) complex conjugate pairs. It has also been noticed that it is almost
flat, in the sense that the absolute value of every component takes one out of only two possible values. Let us go through the argument and add one assumption. Denoting the real fiducial vector by $|Ψ_R⟩$ and the complex one by $|Ψ_C⟩ = U_F |Ψ_R⟩$ we note that

$$\langle Ψ_R | X^i | Ψ_R \rangle = \langle Ψ_C | U_F X^i U_F^{-1} | Ψ_C \rangle = \langle Ψ_C | Z^i | Ψ_C \rangle = \sum_{k=0}^{d-1} \omega^{ik} |a_k|^2,$$

(51)

where $\{a_k\}_{k=0}^{d-1}$ are the components of the complex fiducial vector. It follows that

$$|a_k|^2 = \frac{1}{d} \sum_{i=0}^{d-1} \omega^{-ki} \langle Ψ_R | X^i | Ψ_R \rangle.$$

(52)

In effect, keeping the explicit form of the permutation matrix $X$ in mind, we have used the Wiener–Khinchin theorem for the autocorrelation in a time series. For a SIC vector all the overlaps $\langle Ψ_R | X^i | Ψ_R \rangle$ have the same modulus when $i \neq 0$, and they are real numbers because the matrix elements of $X$ are real. We now add the assumption that all of them have the same positive sign (because this is what we observe in the known examples). Then it must hold that

$$\sqrt{d+1} \langle Ψ_R | X^i | Ψ_R \rangle = \begin{cases} \sqrt{d+1} & \text{if } i = 0, \\ 1 & \text{if } i \neq 0. \end{cases}$$

(53)

From eq. (52) it then follows that

$$|a_0|^2 = \frac{\sqrt{d+1} + d - 1}{d\sqrt{d+1}}, \quad |a_1|^2 = \cdots = |a_{d-1}|^2 = \frac{\sqrt{d+1} - 1}{d\sqrt{d+1}}.$$

(54)

A short calculation shows that this implies equations (5)–(7) provided we choose

$$N^2 = \frac{d + 3 - 3\sqrt{d+1}}{d(d-3)\sqrt{d+1}}.$$

(55)

This choice is dictated by our insistence that the components of the vector belong to the ray class field, up to an overall factor whose square must also be in the field.

One more remark is useful. In the notation of Appendix A it holds that

$$FJF^{-1} = PJ.$$

(56)

Given that the complex fiducial vector $Ψ$ is related to a real vector by a Fourier transformation, given that $J$ is represented by complex conjugation, and given the representation (A5) of the parity operator $P$, it follows that the components of $Ψ$ obey

$$\bar{a}_r = a_{-r},$$

(57)

where the bar denotes complex conjugation and the indexing, as always, is modulo $d$. 
To avoid any misunderstanding: At the present time the only dimensions for which it has been proved that a SIC exists are those for which exact solutions have been found. All we have done in this section is to give a plausibility argument suggesting that a SIC fiducial vector of the form given in eqs. (5)–(7) should exist whenever \( d = n^2 + 3 \). Before the end of the paper we will have used this special form to prove that SICs exist in a number of dimensions where, previously, they were not even known in numerical form.

VI. DECOUPLING OF THE CYCLOTOMIC FIELD

The fact that a small subfield of the full ray class field sometimes suffices to write down a SIC fiducial is not exclusive to the dimensions we consider here. It was in fact discovered when the exact solution for \( d = 323 \) was constructed\(^{22}\), and it has been noticed in many dimensions since then\(^{23,24}\). It is however especially pronounced\(^{26}\) when \( d = n^2 + 3 \). An assumption that underlies our recipe is that the fiducial vector in eqs. (6)–(7) lies in the small ray class field \( K^m \) that was identified in Section III B. This means that the cyclotomic field and the fiducial vector decouple completely.

We will now explore some consequences of that assumption. The symplectic group \( SL(2) \) can be extended to the general linear group \( GL(2) \) in a natural way. Consider the Galois transformation that takes the root of unity \( \zeta^{2d} = e^{i \pi d} \) to \( \zeta^{k d} \). We can represent this by the \( GL(2) \) matrix

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}.
\]

This is a natural extension of the definition (A6) for complex conjugation\(^{14}\). It has led to a conjecture, well supported by evidence, stating that the Galois group that acts on the SIC overlaps is isomorphic to the quotient of two matrix groups\(^{13,14,19,51,52}\). We allow us to state it in the somewhat simplified form that it assumes in our case (where we deal with ray class SICs and, since the dimension is a prime, with centred\(^{14}\) fiducial vectors of type \( F_z \)). Then the statement is that

\[
\text{Gal}_{K^p/H_K} \cong C/S,
\]

where \( K^p \) is the field generated by the SIC overlaps, \( H_K \) is the Hilbert class field, \( S \) is the symplectic stabilizer of the centred fiducial, and \( C \) is the centralizer of \( S \) within \( GL(2, \mathbb{Z}_d) \). The ‘overlap field’ \( K^p \) was denoted by \( \mathfrak{R}_1 \) when it was first introduced\(^{14}\). The Galois transformations of the overlaps take the form

\[
\sigma(\langle \Psi|D_p|\Psi \rangle) = \langle \Psi|D_{Gp}|\Psi \rangle,
\]

where \( G \in C/S \) (i.e., \( G \) is a representative of a coset of \( S \) in \( C \)).

There is another way of looking at this. When \( d \) is a prime equal to 2 modulo 3 the non-trivial overlaps form a single orbit under this Galois group. This is a key ingredient in the work published by Kopp\(^{19}\). A key ingredient
in our work is that for the dimensions we consider there exists a fiducial for which the stabilizer $S$ consists of diagonal matrices. It follows that the centralizer also consists of diagonal symplectic matrices. This has two important consequences. In the first place it means the set of overlaps of the form $\langle \Psi | X^j | \Psi \rangle$ form an orbit by themselves. In the second place it means that the Galois group permutes the components of the fiducial vector, and in the prime dimensional case considered in this paper it permutes them cyclically. The fact that the matrix elements of the displacement operator $X$ are natural numbers (either 1 or 0) means that the overlaps $\langle \Psi | X^j | \Psi \rangle$ are in the same field as the fiducial components. The fact that the Galois group cycles through these comparatively small sets of numbers means that this field is of much lower degree than the full ray class field—which takes us back to the point with which we began this section.

Let us change the subject slightly. If we are given a vector in a definite number field it is possible to check whether it is indeed a SIC fiducial without venturing outside that number field. This is the content of an interesting observation that we repeat here because it is not widely known. Let the components of the vector $\Psi$ be denoted by $\{a_r\}_{r=0}^{d-1}$. A short calculation using the representation of the Weyl–Heisenberg group given in Appendix A suffices to verify that

$$G(i, k) \equiv \frac{1}{d} \sum_j \omega^{kj} |\langle \Psi | X^i Z^j | \Psi \rangle|^2 = \sum_{r=0}^{d-1} \bar{a}_r \bar{a}_{r+k} a_{r-i} a_{r+k},$$

(61)

where $X$ and $Z$ are the group generators. Thus, if

$$G(i, k) = \sum_{r=0}^{d-1} \bar{a}_{r+i} \bar{a}_{r+k} a_r a_{r+i+k},$$

(62)

then it follows from the invertibility of the discrete Fourier transform that

$$|\langle \Psi | X^i Z^j | \Psi \rangle|^2 = \frac{d \delta_{i,0} \delta_{j,0} + 1}{d+1} \iff G(i, k) = \frac{\delta_{i,0} + \delta_{k,0}}{d+1}.$$  \hspace{0.01cm} (63)

Every reference to the roots of unity has been made to disappear.

For the form of the fiducial vector that we are using it is the case that $\bar{a}_r = a_{-r}$, with labels counted modulo $d$ (see eq. (57)). Hence the conditions become

$$G(i, k) = \sum_{r=0}^{d-1} a_{-r-i} a_{-r-k} a_r a_{r+i+k} = 0, \quad 0 < i \leq k < d.$$  \hspace{0.01cm} (64)

This defines an algebraic variety over $\mathbb{C}$. By changing the range of the summation index we find that

$$G(i, k) = G(i, -k) = G(-i, k) = G(-i, -k).$$  \hspace{0.01cm} (65)
Moreover the conditions on $G(0, i)$ and $G(i, 0)$ are automatically obeyed for our almost flat fiducial vectors. This allows us to decrease the number of equations to be checked. More may be true\footnote{[64]}. Be that as it may, the alternative form of the SIC conditions offers a significant computational speed-up when checking the SIC property, in particular for exact solutions.

VII. OUR RECIPE

With this preparation we are ready to introduce our recipe for how to construct a SIC in a prime dimension of the form $d = n^2 + 3$. As before, $D$ is the square free part of $(d + 1)(d - 3)$ and $K = \mathbb{Q}(\sqrt{D})$. The Hilbert class field over $K$ is denoted by $H_K$, and its degree over $K$ equals the class number $h = h_K$. The degree of the ray class field $K_{\mathfrak{p}}$ over the Hilbert class field equals $m = (d - 1)/3\ell$, and the Galois group $K_{\mathfrak{p}}/H_K$ is cyclic of order $m$.

Recall that the entries of the fiducial vector are complex numbers which are, with the exception of $x_0$, proportional to square roots of Stark units with respect to their complex embeddings. By the Stark conjectures, special values of zeta functions provide us a way to compute real approximations of Stark units, i.e., with respect to their real embeddings. In order to directly switch between the real and the complex embeddings, we would need exact, algebraic expressions. For this, we would first have to compute the exact minimal polynomial of the Stark units to define the corresponding number field. Then we would have to compute the roots of the defining polynomial in that number field. While there is no theoretical obstacle, these calculations are only feasible for small examples such as those in Section VIII. When the dimension and the degree of the ray class fields grow, these direct calculations quickly become infeasible. Instead, we are using a combination of numerical and algebraic techniques that allow us to carry out the calculations for much larger cases. Additional refinements might allow to push the computational limits even further.

In all numerical steps, we fix the embedding $j: K \to \mathbb{R}$ with $j(\sqrt{D}) > 0$. Our recipe to compute a fiducial vector in prime dimension $d = n^2 + 3$ may then be summarized as follows:

1. compute the sequence of numerical Stark units and their exact minimal polynomial $p_1(t) \in K[t]$

2. compute the exact Galois polynomial $g_1(t) \in K[t]$ corresponding to an automorphism $\sigma_m$ of order $m$ that fixes the Hilbert class field

3. apply the automorphism $\tau: \sqrt{D} \mapsto -\sqrt{D}$ to obtain the polynomials $p_2(t) = p_1(t)^\tau$ and $g_2(t) = g_1(t)^\tau$

4. factorise the polynomial $p_2(t)$ over the Hilbert class field and pick a factor $p_3(t)$
5. compute numerical approximations $\tilde{y}_j$ of the roots $y_j$ of $p_3(t)$ and order them using repeated iterations of the Galois polynomial $g_2(t)$

6. factorise the polynomial $p_3(t^2/x_0)$ over the Hilbert class field and pick a factor $p_4(t)$

7. compute the square roots $z_j = \pm \sqrt{x_0 y_j}$ and choose the sign such that $p_4(z_j) = 0$

8. search for a primitive element $\theta$ of $\mathbb{Z}_d$ together with a global sign for the square roots such that eq. (49) yields a fiducial vector

9. compute the exact Galois polynomial $g_4(t) \in H_K[t]$ from the numerical square roots $z_j$ and compute an exact fiducial vector over the field $L \cong H_K[t]/(p_4(t))$

10. check the SIC-POVM conditions (63).

The roots $y_j$ are in fact the sought-after Stark phase units, $y_j = e^{i\vartheta_j}$, but we use a different notation because here we treat them strictly as algebraic numbers. Clearly, step (9) can be omitted if one is only interested in a numerical solution.

In the following, we will discuss the steps of this recipe in some detail.

A. Numerical Stark units and their exact minimal polynomial

The first and currently most time-consuming step in our recipe is the numerical calculation of the Stark units to sufficient precision. As in Section III B, let $K^{pj}/K$ denote the ray class field over $K = \mathbb{Q}(\sqrt{D})$ and $G = \text{Gal}_{K^{pj}/K}$. For every automorphism $\sigma \in G$, the corresponding Stark unit (40) is given by

$$\epsilon_\sigma = \exp(\delta'(0, \sigma)),$$

(66)

where $\delta'(s, \sigma)$ is the first derivative with respect to $s$ of the difference of partial zeta functions

$$\delta(s, \sigma) = \zeta(s, \sigma) - \zeta(s, \sigma_T \sigma) = 2\zeta(s, \sigma).$$

(67)

For the last equality, we have used that the involution $\sigma_T$ changes the sign of the zeta function (see (39)), allowing us to remove the explicit dependence on $\sigma_T$. In order to compute the values $\delta'(0, \sigma)$, we are using built-in functions in Magma (or PARI/GP) for Hecke $L$-functions (see also Definition 3.6 in Ref. 19). The partial zeta functions $\delta(s, \sigma)$ are obtained via finite Fourier transformation with respect to the abelian group $G$ (i.e., using the orthogonal characters of $G$) from the Hecke $L$-functions. The sign symmetry of the partial zeta function with respect of the action of $\sigma_T$ mentioned above implies that half of the first derivatives of the $L$-functions vanish. Moreover, the fact that the derivatives of the zeta function are real implies that the derivatives of the $L$-functions come in complex conjugate pairs (or have real values). Using those symmetries, it suffices to compute only about one quarter of the derivatives.
of the $L$-functions. We have implemented functions in Magma that make use of those symmetries. First, we compute the Stark units with, say, 20 digits of precision, allowing us to identify the complex conjugate pairs. Zeros of the derivative of the $L$-functions can be easily calculated from the corresponding Hecke character.

The numerical approximation of the minimal polynomial of the Stark units is given as

$$\tilde{p}_1(t) = \prod_{\sigma \in G} (t - \tilde{\epsilon}_\sigma) = \sum_{i=0}^{|G|} \tilde{c}_i t^i,$$

where the tilde indicates numerical approximation. The coefficients $c_i$ of the exact minimal polynomial are integers in the field $K = \mathbb{Q}(\sqrt{D})$. They can be expressed as $c_i = c_{i,0} + c_{i,1} u_K$ with $c_{i,j} \in \mathbb{Z}$, where $u_K$ is a fundamental unit of $K$. In order to obtain the coefficients $c_{i,j}$, we apply an integer relation algorithm to $(1, u_K, \tilde{c}_i)$. More precisely, we fix an embedding $j: K \to \mathbb{R}$ and consider $(1, j(u_K), \tilde{c}_i)$. Note that a priori, we do not know which precision of the numerical Stark units is sufficient to find the correct coefficients $c_{i,j}$.

It turns out that the absolute error of the coefficients $\tilde{c}_i$ is approximately bounded by the absolute error of the numerical Stark units $\tilde{\epsilon}_\sigma$ multiplied by the maximal absolute value of the coefficients $\tilde{c}_i$: see Theorem 2.12 in Ref. 56.

Hence the required precision appears to be at least twice the size of the largest coefficient, which is the height of the polynomial. Moreover, as we are looking for an integer relation with three components, one would expect that the required precision would have to be tripled. Initially, we calculated the numerical Stark units with higher precision and checked whether the integer relations were stable when reducing the precision. Luckily, we observed that a precision requirement of a bit more than twice the size of the largest coefficient $\tilde{c}_i$ was sufficient for our examples.

This aspect is important, as for high precision the run-time of the algorithm we use to compute the numerical Stark units appears to scale like $(\text{#digits})^{3.3}$. Hence doubling the precision increases the run-time by about a factor of ten. In Table I we indicate the CPU time for computing the numerical derivatives of the $L$-functions. For larger examples, we used separate processes for the individual values. This allowed us, for example, to compute the numerical Stark units for $d = 2707$ in about one calendar month instead of about 2.5 years. Eventually, we used the implementation of $L$-functions in PARI/GP, which appears to be more efficient than our implementation in Magma and which moreover allows the use of multiple CPU cores on an HPC cluster.

**B. Exact Galois polynomial of the Stark units**

The numerical Stark units $\tilde{\epsilon}_\sigma$ are indexed by the elements of the Galois group $G$ of $K^{p_1}/K$. Hence we know how the Galois group permutes them. Moreover, our implementation in Magma allows us to identify an automorphism $\sigma_m$ of order $m$ that fixes the Hilbert class field. We choose an indexing of the
TABLE I. Run-time to compute the derivatives of the $L$-functions. Note that we have used different computers and different versions of our Magma programme so that the values are just indicators for the complexity. For the last two dimensions we have additionally used PARI/GP.

| $d$  | $\deg(K^{p}/K)$ | $\log \text{height}$ | precision | CPU time   |
|------|-----------------|-----------------------|-----------|------------|
| 487  | 324             | 424                   | 1000 digits | 251 hours |
| 787  | 262             | 299                   | 1000 digits | 118 hours |
| 2707 | 902             | 1861                  | 3800 digits | 900 days   |
| 4099 | 1366            | 974                   | 2000 digits | 170 days   |
| 5779 | 214             | 127                   | 300 digits  | 18 min     |
| 1447 | 964             | 2158                  | 4600 digits | 111 days   |
| 19603| 2178            | 1754                  | 4000 digits | 82 days    |

$h m$ Stark units $\epsilon_{\sigma} = \epsilon_{j}$ such that the action of $\sigma_{m}$ on them is given by the permutation

$$\pi_{m} = (0 \ldots m-1)(m m+1 \ldots 2m-1)\ldots(m(h-1) m(h-1)+1 \ldots mh-1)$$

consisting of $h$ cycles of length $m$ each. With this indexing,

$$\epsilon_{j}^{\sigma_{m}} = \epsilon_{j^{\pi_{m}}}.$$  \hspace{1cm} (70)

The mapping of (70) can also be realized as a polynomial function, i.e., we are looking for a polynomial $g_1(t)$ such that

$$g_1(\epsilon_j) = \epsilon_{j^{\pi_{m}}} \quad \text{for } j = 0, \ldots, mh - 1.$$ \hspace{1cm} (71)

We refer to the unique polynomial $g_1(t)$ of degree at most $mh - 1$ as Galois polynomial. The conditions (71) are invariant under the action of the whole Galois group $G$. Therefore, $g_1(t)$ is a polynomial with coefficients in $K$, which is fixed by $G$. Note that a similar approach is discussed in Ref. 19 where the coefficients of the Galois polynomial are computed by solving a linear system with a Vandermonde matrix. Using standard algorithms, this has cubic complexity in the number $mh$ of Stark units. We compute a numerical approximation $\tilde{g}_1(t)$ via polynomial interpolation, which has quadratic complexity in the implementation in Magma. There are more sophisticated methods using fast Fourier transformations with complexity $O(mh \log^{2}(mh))$ which we have not implemented.

The coefficients of the exact polynomial $g_1(t) \in K[t]$ are again determined via an integer relation algorithm. Unlike the situation of the exact minimal polynomial of the numerical Stark units, we do not yet have a good heuristic for the required precision. This is not necessarily a severe problem, as we can increase the precision of the numerical Stark units using Newton’s method, which has quadratic convergence. For Newton’s method, we use numerical
approximations of the exact polynomial \( p_1(t) \) and its derivative to arbitrary precision. However, it turns out that, in general, the coefficients of \( g_1(t) \) have very large numerators/denominators. For \( d = 2707 \), we used a precision of about two million digits. Computing \( \tilde{g}_1(t) \) of degree 901 using numerical polynomial interpolation took about 18.5 hours. Determining the 902 exact coefficients took about 420 CPU hours in total, which was done in parallel on multiple CPU cores. For dimension \( d = 19603 \), the coefficients of the exact Galois polynomial have denominators that are integers with more than one million digits.

In the end this step has turned out to be a potential computational bottleneck. For dimension \( d = 2503 \), we have been able to compute the (conjectural) exact minimal polynomial \( p_1(t) \) of degree 3336 using a precision of 13100 digits. This took in total about 60 CPU years on the HPC cluster. Using a precision of ten million digits appears to be insufficient to obtain the exact Galois polynomial \( p_1(t) \).

C. Flipping the sign

The calculation of the exact minimal polynomial \( p_1(t) \) of the Stark units starts with their real approximations, corresponding to the real embeddings of the ray class field \( K^{p_1} \). For the fiducial vector we need the complex Stark phase units, i.e., their values on the unit circle in the complex embeddings. The embeddings of \( K^{p_1} \) are defined by the roots of the absolute minimal polynomial of the Stark units over \( \mathbb{Q} \), which lie in the normal closure of \( K^{p_1} \). Instead of computing the exact roots in the extension field, we apply the automorphism \( \tau: \sqrt{D} \mapsto -\sqrt{D} \) to the exact minimal polynomial \( p_1(t) \) to obtain the polynomial \( p_2(t) = p_1(t)^\tau \). Note that the polynomial \( p_1(t)p_2(t) \in \mathbb{Q}[t] \) is the absolute minimal polynomial of the Stark units over \( \mathbb{Q} \). When computing the coefficients of the exact polynomial \( p_1(t) \), we have used the embedding \( j \). Now, with respect to the same embedding, the roots of the polynomial \( p_2(t) \) will be the desired complex Stark phase units. The exact Galois polynomial \( g_2(t) = g_1(t)^\tau \) has the property that it permutes the Stark phase units in the same way as the polynomial \( g_1(t) \) permutes the real Stark units.

D. Minimal polynomial over the Hilbert class field

When the class number \( h \) is larger than one, the Hilbert class field is a proper extension of degree \( h \) of \( K \). Then the exact minimal polynomial \( p_2(t) \in K[t] \) factorises as

\[
p_2(t) = p_2^{(1)}(t) \cdot p_2^{(2)}(t) \cdot \ldots \cdot p_2^{(h)}(t),
\]

with \( h \) factors \( p_2^{(j)} \in H_K[t] \) of degree \( m \) each. In our examples, the class number is small, and hence exact factorisation of \( p_2(t) \) over the Hilbert class field does not take much time. We pick any of the factors, e.g., \( p_3(t) = p_2^{(1)}(t) \).
Other factors yield fiducial vectors that are related by a Galois automorphism fixing $K$, but acting non-trivially on $H_K$. When $h = 1$, we have $p_3(t) = p_2(t)$.

E. Numerical Stark phase units

The next step is to compute $m$ numerical Stark phase units $\tilde{y}_j$ as roots of the exact polynomial $p_3(t)$ with respect to the fixed embedding $j$. At the same time, we want to order them according to the action of the Galois automorphism $\sigma_m$. For this, we compute one complex root $\tilde{y}_0$ to moderate precision, which again can be increased using Newton’s method. The other roots are computed using the exact Galois polynomial $g_2(t)$ via

$$\tilde{y}_{j+1} = g_2(\tilde{y}_j).$$

(73)

To compensate for precision loss in (73), we apply Newton’s method to $\tilde{y}_{j+1}$ before computing $\tilde{y}_{j+2}$. Note that at this point, we can check whether $\tilde{y}_j$ is an approximate root of $p_3(t)$ and whether it lies on the unit circle. Should this not be the case, then either the precision in one of the previous steps must have been too low; or else one of the conjectures on which our recipe is based must be false.

Note that it does not matter which of the roots gets the label $y_0$, as long as the labelling is consistent with the cyclic permutation via the Galois automorphism $\sigma_m$. This is ensured by using the Galois polynomial $g_2(t)$ in (73) to order the roots. Different choices for $y_0$ lead to fiducial vectors that are related by Clifford transformations.

F. Minimal polynomial of the square roots

Recall from (7) that the components of our non-normalized fiducial vector are $x_0 = -2 - \sqrt{d + 1}$ or of the form $z_j = \sqrt{x_0 y_j}$. When $y_j$ is a root of $p_3(t)$, then $z_j$ is a root of the polynomial $p_3(t^2/x_0)$. For all our examples, it turns out that with the particular choice of $x_0$, the latter polynomial factorises in $H_K[t]$ as

$$x_0^m p_3(t^2/x_0) = p_4(t) \cdot p_4(-t).$$

(74)

We pick $p_4(t)$ as the minimal polynomial of the square roots. The factorisation (74) supports Conjecture 4 as the roots $z_j = \sqrt{x_0 y_j}$ of $p_4(t)$ lie in the same field as the roots $y_j$ of $p_3(t)$. This also implies that the action of the Galois automorphism $\sigma_m$ on the roots $z_j$ is cyclic.

G. Fixing the signs

Instead of directly computing the roots of the polynomial $p_4(t)$, we compute the numerical square roots $\tilde{z}_j = \sqrt{x_0 \tilde{y}_j}$. This has the advantage that we automatically get the correct indices corresponding to the cyclic action of $\sigma_m$. 
There is, however, an ambiguity concerning the sign of the square root. This can be fixed by the observation that each of the square roots is a factor of only one of the factors in (74). So we computed \( |p_t(\pm \sqrt{x_0y_j})| \) for both possibilities and chose the sign which gave the smaller absolute value. It turned out that a moderate precision was sufficient for all examples. As the choice between \( p_t(t) \) and \( p_t(-t) \) was arbitrary, we are permitted to flip the sign of all roots \( \tilde{z}_j \). This yields two alternatives: \( z = (\tilde{z}_0, \ldots, \tilde{z}_{m-1}) \) and \(-z\).

H. Combinatorial search

According to (7), the first component of the non-normalized fiducial vector is set to \( x_0 = -2 - \sqrt{d+1} \), while by (49), the other components are computed as \( \tilde{\Psi}_\theta \mod d = \tilde{z}_j \mod m \). Here \( \theta \) is a primitive element of \( \mathbb{Z}_d \), i.e., it generates the group \( \mathbb{Z}_d^\times \) of invertible elements in \( \mathbb{Z}_d \). When \( d \) is prime, \( \mathbb{Z}_d^\times \) is cyclic of order \( d - 1 \). As we do not know the exact correspondence (59) between elements of the Galois group and matrices in \( GL(2, \mathbb{Z}_d) \), we do not know which primitive element \( \theta \) is the correct one. Hence we try all possibilities, together with the choice between \( z \) and \(-z\). Instead of checking all SIC-POVM conditions, it appears to be sufficient to compute only \( |\langle \tilde{\Psi}|X|\tilde{\Psi} \rangle|^2 \) and check whether it is close to \( 1/(d+1) \). There are \( \Phi(d-1) < d/2 \) candidates for \( \theta \) (as \( d \) is odd) and two choices for the sign, so that we have to test no more than \( d \) candidates for the fiducial vector.

I. Computing an exact fiducial vector

It has to be noted that all exact calculations in the previous steps of our recipe are carried out in the Hilbert class field, which in our cases all have sufficiently small degree. We do not have to compute exact roots of the polynomials or explicit Galois groups of number fields. Nonetheless, we can obtain an exact fiducial vector. The first component is \( -2 - \sqrt{d+1} \in K \). The other components are roots of the exact polynomial \( p_4(t) \). This allows us to define the number field \( \mathbb{L} = H_K(\gamma) \), where \( p_4(\gamma) = 0 \). (75) In all examples, the field \( \mathbb{L} \) is the same as the field \( (K^\mathbb{P})^\gamma \) which is defined via \( p_2(t) = p_1(t)^\gamma \), where \( p_1(t) \) is the minimal polynomial of the Stark units over \( K \). The mapping \( t: \mathbb{L} \rightarrow \mathbb{C}, \gamma \mapsto \tilde{z}_0 \) (76) extends the chosen embedding of the Hilbert class field \( H_K \) to an embedding of \( \mathbb{L} \).

Similar to our treatment of the Galois polynomial \( g_1(t) \) for the real Stark units, we compute a Galois polynomial \( g_4(t) \in H_K[t] \) from the ordered numerical roots of \( p_4(t) \in H_K[t] \). In contrast to the real Stark units, the numerical values are now complex numbers so that, for example, the multiplication operation is about three times slower than that of real numbers. As the coefficients
of $g_4(t)$ are in the Hilbert class field, we have to find integer relations among the $2h$ elements of a basis of $H_K/Q$, and the approximation of the coefficient. This requires that we increase the precision; but at the same time, there are only $m$ (instead of $mh$) coefficients.

Setting $z_0 = \gamma$, we use the exact Galois polynomial $g_4(t)$ to compute the exact values

$$z_{j+1} = g_4(z_j)$$  \hfill (77)

which are components of the fiducial vector. This yields all exact roots of $p_4(t)$ without directly factorising the polynomial over the number field $L$ of absolute degree $2mh$ over $Q$. At the same time, the Galois automorphism $\sigma_m$ is fully specified by

$$\sigma_m : \gamma \mapsto g_4(\gamma).$$  \hfill (78)

As the Galois group of the extension $L$ of the Hilbert class field $H_K$ is cyclic (see also Lemma 6) and $H_K$ is totally real, $\sigma_m^{m/2}$ is the unique automorphism of order 2 that corresponds to complex conjugation with respect to the embedding (76).

When the ray class field has subfields of small degree, one might be able to obtain a somewhat nicer representation of the field. However, note that the degrees of the fields in Table III have relatively large prime power factors. When working with the field $L$ defined in (75) via $p_4(t)$ and the Galois automorphism defined in (78), the arithmetic gets very slow as the degree of the field increases. That is the reason why we have not yet computed exact solutions for dimensions $d = 1447, 2707, 4099$ or 19603. For dimension 5779, computing the exact fiducial vector based on (78) takes about two hours.

J. Verifying the solution

Rewriting the SIC-POVM conditions (2) in the form (63) avoids complex $d$-th roots of unity which would require an additional field extension of degree $\varphi(d) = d - 1$ when $d$ is prime. Furthermore, we do not have to normalize the fiducial vector, which in general requires an additional field extension of degree two for the square root of the squared norm $|\langle \hat{\Psi} \vert \hat{\Psi} \rangle|^2$ of the un-normalized vector. Instead, we work with the un-normalized vector when computing the sum for $G(i, k)$ in (62) and divide by $|\langle \hat{\Psi} \vert \hat{\Psi} \rangle|^2$. When checking (64) to be zero, we do not need any normalization.

Nonetheless, full verification of the exact fiducial vector is only feasible for moderate dimensions, as the number of arithmetic operations scales with the cube of the dimension. As already mentioned, the complexity of the arithmetic in the ray class field $K^p$ depends on the degree and the representation of the number field. When the ray class group decomposes into small cyclic factors of prime power order, one can find somewhat nicer representations of the field as a compositum of extensions of the Hilbert class field.

For all dimensions for which we have computed an exact solution, we have computed at least a few of the values $G(i, k)$ in (62) or (64) using exact
arithmetic. Timings for a single such evaluation are listed in Table II. When full exact verification was not feasible, we carried out a numerical check to the precision given in Table II.

TABLE II. Run-time for the verification of the solutions. We list the degree of the ray class field \( K^{p_j} \) over \( K \) as a product of prime powers corresponding to the structure of the ray class group. The additional factor 2 for \( d = 103, 487, 1447 \) is due to the class number \( h \) being 2. The column \( G(i, k) \) provides information on the time required to compute one value \( G(i, k) \) using exact arithmetic.

| \( d \) | \( \deg(K^{p_j}/K) \) | precision | CPU time | \( G(i, k) \) |
|---|---|---|---|---|
| 103 | \( 2 \times 2 \times 17 \) | exact | 440 s | 1.3 s |
| 199 | \( 2 \times 11 \) | exact | 310 s | 0.3 s |
| 487 | \( 2 \times 2 \times 3^4 \) | exact | 31 days | 315 s |
| 787 | \( 2 \times 13 \) | 10000 digits | 3 hours | 65 min |
| 1447 | \( 2 \times 2 \times 241 \) | 10000 digits | 17.0 hours | |
| 2707 | \( 2 \times 11 \times 41 \) | 2000 digits | 11.2 hours | |
| 4099 | \( 2 \times 683 \) | 2000 digits | 36.5 hours | |
| 5779 | \( 2 \times 107 \) | 2000 digits | 100 hours | 88 min |
| 19603 | \( 2 \times 3^2 \times 11^2 \) | 1000 digits | 1367 days | |

VIII. TWO DETAILED EXAMPLES

We will take a closer look at dimensions \( d = 7 \) and \( d = 199 \). For the first example \( \ell = 1 \), for the second example \( \ell = 3 \), and they have the same quadratic base field \( K = \mathbb{Q}(\sqrt{2}) \) with class number \( h = 1 \). Hence the degree of the small ray class field \( K^m \) over \( K \) is

\[
m = \frac{h(d - 1)}{3\ell} = \begin{cases} 
2 & \text{for } d = 7, \\
22 & \text{for } d = 199.
\end{cases}
\]  

(79)

They are small enough that we can present the calculations in full detail for \( d = 7 \) and considerable detail for \( d = 199 \). No serious computational difficulties arise. On the other hand all the complications in principle show up for \( d = 199 \); except that the class number \( h = 1 \).

We begin with \( d = 7 \). Our starting point is a set of numerical Stark units, ordered by the ray class group as described in Section IV B. The numerical precision must be made high enough so that we can determine their exact minimal polynomial, which we call \( p_1(t) \). Its coefficients are integers in the quadratic base field \( K \), and it has degree \( m \). For \( d = 7 \)

\[
p_1(t) = t^2 - (1 + \sqrt{2})t + 1.
\]  

(80)
We obtain a polynomial \( p_2(t) \) whose roots are the Stark phase units \( e^{i\vartheta_j} \) by changing \( \sqrt{2} \) to \(-\sqrt{2}\). For \( d = 7 \)

\[
p_2(t) = t^2 - (1 - \sqrt{2})t + 1. \tag{81}
\]

In this example (and the next) the Hilbert class field is identical to the quadratic field, so step (4) of the recipe is trivial, \( p_3(t) = p_2(t) \). For the fiducial vector we need to take the square roots of the Stark phase units, but doing so directly would take us out of the ray class field. To remedy this we introduce the algebraic integer \( x_0 = -2 - 2\sqrt{d} + 1 \), and we obtain the minimal polynomial of the scaled Stark phase units. For \( d = 7 \) it reads

\[
x_0^2 p_3(t/x_0) = t^2 - 2t + 12 + 8\sqrt{2}. \tag{82}
\]

The roots of this polynomial are of the form \( x_0^2 e^{i\vartheta_j} \). We need the square roots of these numbers. For this purpose we make the substitution \( t \to t^2 \). If the square roots of the roots of \( p_3(t) \) do lie in the small ray class field as predicted by Conjecture 4, the resulting polynomial will factorise over the base field, and indeed it does. For \( d = 7 \)

\[
x_0^2 p_3(t^2/x_0) = \left(t^2 + (2 + \sqrt{2})t + 2 + 2\sqrt{2}\right)\left(t^2 - (2 + \sqrt{2})t + 2 + 2\sqrt{2}\right). \tag{83}
\]

To proceed we just pick one of the factors, say the first, and call it \( p_4(t) \) (the other factor is \( p_4(-t) \)). Our sequence of polynomials has now reached its end. For \( d = 7 \)

\[
p_4(t) = t^2 + (2 + \sqrt{2})t + 2 + 2\sqrt{2}. \tag{84}
\]

The roots of \( p_4(t) \) are of the form \( z_j = x_0 e^{i\vartheta_j} \). The only sign ambiguity which is left is the overall sign ambiguity that arises when we choose a factor in eq. (83).

Because \( p_4(t) \) has only two roots we can write down the SIC fiducial vector directly. According to eq. (49) its components are distributed using a generator \( \theta \) of \( \mathbb{Z}_d^* \), the multiplicative group of integers modulo \( d \). For \( d = 7 \) there are two possible choices, \( \theta = 3 \) or \( \theta = 5 \), but due to the Zauner symmetry they lead to the same vector. We pick one of the roots of \( p_4(t) \) and call it \( z_0 \). The two possible choices lead to Clifford-equivalent SICs. Following Section V we define

\[
\hat{\Psi}_{3^0 \mod 7} = \hat{\Psi}_{3^2 \mod 7} = \hat{\Psi}_{3^4 \mod 7} = z_0, \tag{85}
\]

\[
\hat{\Psi}_{3^1 \mod 7} = \hat{\Psi}_{3^3 \mod 7} = \hat{\Psi}_{3^5 \mod 7} = z_1, \tag{86}
\]

where Zauner symmetry was used. The fiducial vector is then

\[
\Psi = N(\pm x_0, z_0, z_1, z_0, z_1, z_1)^T. \tag{87}
\]

The sign ambiguity is the result of our choice of a factor of \( p_3(t) \). (Instead of choosing the overall sign of the \( z_i \), we choose the sign of the first component \( \pm x_0 \).) We resolve it by calculating \( \langle \Psi | X | \Psi \rangle \) numerically, and find that the
choice that works in this case is the plus sign. (We have not yet found a way around this final somewhat inelegant step.) Because \( z_0 \) is the complex conjugate of \( z_1 \) it is straightforward to check that we now have a SIC.

The case \( d = 7 \) (and \( d = 19 \)) is exceptional because the Galois group is cyclic of order two, so that no ordering problem arises. Although we do not need them we can easily calculate the Galois polynomials. In particular, by solving the quadratic equation \( p_4(t) = 0 \) and denoting the roots by \( z_0 \) and \( z_1 \) we see that the polynomial

\[
g_4(t) = 2 + \sqrt{2} - t. \tag{88}
\]

has the properties that \( g_4(z_0) = z_1 \) and \( g_4(z_1) = z_0 \), so this is the desired Galois polynomial. Incidentally the SIC fiducial vector \((87)\) itself was first derived by hand\(^{29}\), by solving the defining eqs. \((2)\).

We turn to \( d = 199 \). The minimal polynomial of the Stark units, \( p_1(t) \), now has degree 22, and it can be calculated using Magma in less than a minute. It is straightforward to go from there to the polynomial \( p_4(t) \) that has the components of the SIC fiducial as roots. It is

\[
p_4(t) = t^{22} + (16 - 18a)t^{21} + (146 - 54a)t^{20} - (5848 - 3884a)t^{19} - (6938 - 49868a)t^{18} + (450480 - 320688a)t^{17} + (9121956 - 6553664a)t^{16} - (1298295036 - 9182919584a)t^{15} - (511130784 - 361362336a)t^{14} - (354768560a)t^{13} + (1298295036 - 9182919584a)t^{12} + (14458284800 - 10213792864a)t^{11} - (157692490944 - 111463664512a)t^{10} - (73845898496 - 52328201280a)t^9 + (914730317568 - 647071560192a)t^8 - (136112007424 - 96919409792a)t^7 - (2444614507008 - 1731374423552a)t^6 + (1421567308800 - 1043262626432a)t^5 + (2891624580096 - 1803172501736a)t^4 - (3497710163968 - 1315439504384a)t^3 + (1952962604032 + 2282964641792a)t^2 - (423531088216 + 5036656351232a)t + 9271967234048 + 7154311792640a,
\]

(89)

where \( a = \sqrt{2} \). To build the fiducial vector we need to order its roots into a cycle of length 22. Testing all possibilities is out of the question, so at this point we need the Galois polynomial \( g_2(t) \). In the recipe that we gave in Section \( \text{VII} \), steps (2), (5), and (9), are concerned with this question. However, for \( d = 199 \) we can try a shortcut and directly compute the action of the Galois group for the number field defined by \( p_4(t) \) using standard Magma routines. For the higher dimensions this would take a prohibitive amount of time, but in the present case we obtain the answer in a few minutes. Since the Galois group is cyclic of order 22 it has 10 generators of order 22. We simply pick one of them. Magma then gives us a polynomial \( g_4(t) \) having the properties that

\[
z_{j+1} = g_4(z_j), \quad z_{j+11} = g_4^{[11]}(z_j) = 1/z_j, \quad z_{j+22} = g_4^{[22]}(z_j) = z_j, \tag{90}
\]
where \( g_4^{[k]}(t) = g_4^{[k-1]}(g_4(t)) \) denotes the \( k \)-fold composition of the polynomial \( g_4(t) \) with itself. The explicit expression for the polynomial is somewhat unwieldy, and we have relegated it to Appendix B.

We now have to relate this ordering of the roots to the ordering of the components provided by the Clifford group. The group \( \mathbb{Z}_d^\times \) has altogether \( \varphi(d - 1) = 60 \) generators \( \theta \). For any choice of generator of the Galois group there will be 6 choices of \( \theta \) that result in a SIC fiducial vector with \( \hat{\Psi}_{\theta j} = z_j \). There is also a sign ambiguity in \( \hat{\Psi}_0 \). We resolve these ambiguities by a numerical calculation of a single overlap \( \langle \Psi | X | \Psi \rangle \). When using \( p_4(t) \) and the Galois polynomial \( g_4(t) \) one finds that we have to choose a negative sign for the zeroth component, while the appropriate choices of \( \theta \) are \( \theta = 41, 75, 134, 167, 189, 190 \). Because of the symmetry these choices result in identical vectors.

We now have a candidate exact SIC fiducial vector for \( d = 199 \). In this case the exact calculation that verifies that we really have a SIC is quick, as can be seen in Table II.

The computational complexity of some of the steps described in Section VII grows with the dimension, but the logic remains simple. All our solutions can be described simply in words: The components of the fiducial vector are square roots of scaled Stark phase units ordered by the Galois group of the small ray class field over the Hilbert class field.

IX. REMARKS, OBSERVATIONS, AND EXTENSIONS

It is time to compare our SIC recipe with other methods that have been used to construct them. The first extensive collection of exact solutions was obtained by solving the defining equations (2) using Gröbner bases. After that the majority of new solutions have been obtained by applying integer relation algorithms to high precision numerical solutions, making use of (and supporting) the conjectures about the relevant number fields. By now exact solutions in more than one hundred different dimensions have been obtained using this method and variations, including many dimensions of the form \( d = n^2 + 3 \), namely every such dimension with \( n \leq 18 \) together with \( d = 403, 844, \) and \( 1299 \). But this method encounters two bottlenecks. One is that it becomes necessary to factorise polynomials of high degree over number fields of high degree. The other is the very time consuming search for a numerical solution to start with. Prior to the work reported here, \( d = 1299 \) was the highest dimension in which a SIC was known in exact form. The highest dimension for which a numerical search has been successful is \( d = 2208 \). For \( d = 5779 \) there has been an attempt to find a numerical solution with the standard routines. After 55065 trials and 17.69 CPU years the search came up empty-handed. Both bottlenecks are avoided by our recipe, which is why we have been able to reach higher dimensions than ever before.

We add two observations that lead to open questions. The first concerns a
theorem that we were unable to prove. Consider the overlap phases

\[ e^{i\nu_j} = \sqrt{d + 1} \langle \Psi | X^j | \Psi \rangle. \]  

(91)

They are phase factors because \( |\Psi\rangle \) is a SIC fiducial vector, and they belong to the small ray class field because the operator \( X \) contains only real integers (in fact, its entries are 1 or 0). Because of the symmetry of the fiducial vector the number of distinct phase factors equals \((d - 1)/3\ell\). Hence one may suspect that these overlap phases are identical to the Stark phase units. Let \( a_j \) be the \( d - 1 \) non-trivial components of the fiducial vector, as in eq. (6). In our examples we have observed that those overlaps are indeed Stark phase units, and moreover

\[ \sqrt{d + 1} \langle \Psi | X^{-2j} | \Psi \rangle = \frac{a_j^2}{|a_j|^2}. \]  

(92)

A proof that this formula must hold would strengthen the motivation for our recipe, and might offer insight into the structure underlying the Stark units.

The second observation concerns the real fiducial vectors that can be obtained by a discrete Fourier transformation from the ones we have constructed. By inspection one can see that real fiducial vectors are largely built from real units\(^2\), but not in any obvious way from Stark units. Still it may in the end prove advantageous to work with real rather than complex fiducials, especially since the ordering of the real Stark units is directly computationally accessible.

Finally, let us comment on the cases when \( d \) is not a prime number. When \( d \) is odd and the multiplicities of its prime factors equal one, the Clifford group splits into a direct product, and each prime factor in \( d \) splits into prime ideals. We now have to compute Stark units for an entire lattice of subfields. For each subfield we have to relate the action of the Galois group to the permutation action on the fiducial vector provided by the Clifford group, and the choices have to be correlated. There is also an issue of principle that arises when the dimension is divisible by 3, which is that we encounter a Zauner symmetry of a different type (of type \( F_a \) in the terminology of Scott and Grassl\(^1\)). We feel that we have not yet found the best way to deal with these problems, and we have decided to postpone a full presentation of these results. When the dimension \( d \) is even it will contain a factor 4, and this factor has to be given special treatment. Having done this we find that there is—after a change of basis—again an almost flat fiducial vector whose components can be obtained from Stark units; but this time there are some significant differences from the case of odd \( d \). As it happens the degrees of the small ray class fields that occur are in some ways more manageable than those that occur for odd \( d \). Again we postpone a full presentation of these results. We can, however, report already now that we have used the connection to the Stark units to write down exact or numerical SIC fiducial vectors in dimension \( d = 7, 12, 19, 28, 39, 52, 67, 84, 103, 124, 147, 172, 199, 259, 292, 327, 403, 487, 628, 787, 844, 964, 1027, 1228, 1299, 1447, 1684, 1852, 2404, 2707, 3028, 4099, 5779, 19603, \) and 39604.
X. CONCLUSIONS

We have proposed a recipe for constructing SICs in prime dimensions of the form $d = n^2 + 3$. The key ingredients of the construction are the (conjectural) Stark units in precisely specified abelian extensions of real quadratic number fields.

The remarkable interplay between the geometry and the number theory began with the fact that a search for SICs in complex dimension $d$ led directly to a ray class field of modulus $d$. It is tempting to make the analogy with the cyclotomic case (a polygon with $d$ sides leads to a ray class field over $\mathbb{Q}$ of conductor $d$); but as yet we know of no abelian variety whose $d$-torsion points would lead to this structure.

In the prime dimensions of concern in this paper we are able to find ‘minimal’ fiducial vectors having components that lie inside a much smaller ray class field, whose conductor has finite part $p$ being just one of the factors $\sqrt{p+1} \pm 1$ of $p$ inside the ring of integers $\mathbb{Z}_K$ of the base field $K$. Among other things, this facilitates calculations as the degree is significantly smaller than that of the full ‘overlap field’ explored in Refs. [14,19,21].

But possibly the most remarkable feature of this construction is that we produce a fiducial vector whose non-trivial entries are precisely the set of Galois conjugates of the units described in the most ambitious form of Stark’s original conjectures. Namely, those entries are the square roots of the units derived directly from Stark’s original predicted $L$-function values, about which Tate — combining this with work of Brumer — later broadened the conjecture, to predict that these square roots would still live in an abelian extension of $K$. It is an additional strange quirk that in every case we have considered, this top field is identical to that which would be created by taking the original ray class field of conductor $p$ and adjoining the geometric scaling factor. (see Conjecture [1].)

Once the Stark units have been computed, the outline of the recipe, as given in Section I, is very simple. A precise version of the recipe, including all the calculational steps, was given in Section VII. As can be seen there it leads to very complex calculations. It should not surprise anyone that working out explicit formulas in dimension 4099 (say) leads to complex calculations, but it is easy to see why it is nevertheless important to do so.

One reason is that we have at present no suggestions for how to turn our recipe into a general existence proof for the infinite sequence of dimensions we consider (unless the open questions that we formulated in Section XI count as such). If an existence proof requires a proof of a version of the Stark conjectures it does indeed look like a tall order; although see Refs. [59,60] for huge recent progress on the $p$-adic version of the conjecture, which may be of some help here. At present existence is proved by explicit construction, dimension by dimension. It then becomes important to guard against low dimensional accidents. The problem is that we do not know what “low” means in this context. We feel, however, that once we get to the four-digit dimensions we are probably on the safe side. See Table III for a summary.
of the results we have achieved in prime dimensions. Additional data can be found online\textsuperscript{61}.

**TABLE III.** Prime dimensions in the sequence $d = n^2 + 3$, including all primes among the first 100 entries. The columns detail the class number $h$, the position $\ell$ in the sequence of dimensions [8], the factorised absolute degree of the small ray class field, the height (absolute value of the largest coefficient) of the minimal polynomial of the Stark units (represented by the $\log_{10}$, when known or estimated), and a note telling if the full verification of the SIC property is numerical or exact. The dimension is in boldface if the relevant SIC has been constructed using the recipe.

| $n$ | $d$ | $h$ | $\ell$ | degree | log height | check |
|-----|-----|-----|--------|--------|------------|-------|
| 2   | 7   | 1   | 1      | $2^2$  | 1          | exact |
| 4   | 19  | 1   | 3      | $2^2$  | 1          | exact |
| 8   | 67  | 1   | 1      | $2^2 \times 11$ | 10      | exact |
| 10  | 103 | 2   | 1      | $2^3 \times 17$ | 44      | exact |
| 14  | 199 | 1   | 3      | $2^2 \times 11$ | 11      | exact |
| 22  | 487 | 2   | 1      | $2^2 \times 3^4$ | 424     | exact |
| 28  | 787 | 1   | 1      | $2^2 \times 131$ | 299     | 10000 digits |
| 38  | 1447| 2   | 1      | $2^3 \times 241$ | 2158    | 10000 digits |
| 50  | 2503| 4   | 1      | $2^4 \times 3 \times 139$ | 6464    |       |
| 52  | 2707| 1   | 1      | $2^2 \times 11 \times 41$ | 1861    | 2000 digits |
| 62  | 3847| 4   | 1      | $2^4 \times 641$ | 11133   |       |
| 64  | 4099| 1   | 1      | $2^2 \times 683$ | 974     | 2000 digits |
| 70  | 4903| 10  | 1      | $2^3 \times 5 \times 19 \times 43$ | 25224   |       |
| 74  | 5479| 4   | 1      | $2^4 \times 11 \times 83$ | 19618   |       |
| 76  | 5779| 1   | 9      | $2^2 \times 107$ | 127     | 2000 digits |
| 92  | 8467| 2   | 1      | $2^3 \times 17 \times 83$ | 12133   |       |
| 94  | 8839| 8   | 1      | $2^5 \times 3 \times 491$ | 48203   |       |
| 140 | 19603| 1   | 3      | $2^2 \times 3^2 \times 11^2$ | 1754    | 1000 digits |

We note that there is only one case ($d = 487$) in which we have carried out a complete exact verification of the SIC condition for a SIC that was not previously known. In the remaining cases the exact verification was at best partial, but complete numerical checks with high precision have been performed in all cases (see Table II). However, it seems to us more important to say that we have found 12 prime dimensions (and 35 dimensions altogether, as listed at the end of Section IX) of the form $d = n^2 + 3$ for which a SIC
fiducial vector can be built from Stark units, and none where the recipe for doing this fails.

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Appendix A:

Here we give a brief summary of the Weyl–Heisenberg and Clifford groups. For a complete account see, e.g., Ref. [29]. Given a positive integer $d$ there is a Weyl–Heisenberg group generated by $X$, $Z$, and $\omega$, subject to

$$\omega^d = X^d = Z^d = 1, \quad ZX = \omega XZ, \quad \omega X = X \omega, \quad \omega Z = Z \omega. \quad (A1)$$

This is an example of a Heisenberg group in the sense of Chapter I of Ref. [62]. It has an essentially unique unitary representation in dimension $d$, such that $\omega$ is represented by $e^{2\pi i / d}$ times the unit matrix, and such that $Z$ is a diagonal $d \times d$ matrix. The basis vectors are labelled by integers modulo $d$ and are ordered according to

$$Z|r\rangle = \omega^r |r\rangle, \quad X|r\rangle = |r + 1\rangle. \quad (A2)$$

Throughout this paper we assume that this basis has been fixed. Note that this means that the labeling of the $d$ components of our vectors goes from 0 to $d - 1$.

The Clifford group is defined as the group of automorphisms of the Weyl–Heisenberg group within the unitary group. To discuss this it is convenient to collect pairs of integers modulo $d$ (or $2d$ if $d$ is even) into a ‘vector’ $p = (a, b)$, and to define the displacement operators

$$D_p = D_{a,b} = \left(-e^{\pi i / d}\right)^{ab} X^a Z^b. \quad (A3)$$

When $d$ is odd the phase factor is a $d$-th root of unity. It can now be shown that the Clifford group contains a symplectic group. More precisely, for every
symplectic two by two matrix $M$ with entries that are integers modulo $d$ (or $2d$ if $d$ is even) there is a unitary representative $U_M$ determined up to an overall phase such that

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \implies \quad U_M D_p U_M^{-1} = D_{MP}. \quad (A4)$$

The matrix $M$ is symplectic if $\det M = \alpha \delta - \beta \gamma = 1$ modulo $d$ (or $2d$ if $d$ is even). Importantly, the representation of this symplectic group is determined up to overall phase factors by the representation of the Weyl–Heisenberg group. Moreover the matrix elements of the symplectic unitaries $U_M$ are $d$-th (or $2d$-th) roots of unity up to an overall factor which also belongs to the cyclotomic field.

An example of a symplectic unitary that is given in the text is the Fourier matrix $U_F$ in eq. (50). The square of the Fourier matrix is a permutation matrix. For the corresponding symplectic matrices it holds that

$$F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad P = F^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (U_P)_{r,s} = \delta_{r+s,0}. \quad (A5)$$

The $\delta$ is a Kronecker delta modulo $d$. The operator $U_P$ whose matrix elements are defined by it is known as the parity operator. Importantly, every diagonal symplectic matrix is represented by a permutation matrix that leaves the first component $\Psi_0$ of any vector invariant. An example is the order $3\ell$ symplectic matrix $F_S$ given in eq. (44). The corresponding symplectic unitary $U_S$ is a permutation matrix leaving the fiducial vector $|\Psi\rangle$ invariant.

A strengthened version of Zauner’s conjecture states that for every SIC that is an orbit under the Weyl–Heisenberg group there exists a transformation in the Clifford group so that one of the resulting vectors is invariant under a symplectic unitary of order three, having the additional property that the trace of its symplectic matrix equals $-1$ modulo $d$ (or $2d$ if $d$ is even). The group of symplectic matrices whose unitary representatives stabilize a given SIC fiducial vector is denoted $S$. Unitary symmetries of order $3\ell$ occur in dimensions that sit higher up in the dimension sequences described in eq. (8) of Section 1.1.

The extended Clifford group is the automorphism group of the Weyl–Heisenberg group within the group of all unitary and anti-unitary transformations in the Hilbert space. Recall that, given a fixed basis, an anti-unitary transformation is represented by complex conjugation followed by a unitary transformation. It turns out that complex conjugation is the representative of the anti-symplectic matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A6)$$

In its defining $2 \times 2$ representation the extended symplectic group is obtained by adding $J$ as a generator to the symplectic group. Such a matrix is represented by an anti-unitary transformation if its determinant equals $-1$ modulo $d$ (or $2d$ if $d$ is even).
Appendix B:

The Galois polynomial that is needed in eq. (90) to specify the exact solution in \( d = 199 \) is, explicitly,

\[
g_4(t) = \frac{1}{2^{10} \cdot 7^{20} \cdot 97 \cdot 137 \cdot 353 \cdot 11777} \times \left( 2^{10}7^{20}(-17078833449878756 + 12080287801165569a) + 2^{10}7^{20}(4028796684934067 - 28489432173813000t) + 2^{8}7^{19}(23468955797498196 - 165942513813988969a)t^2 - 2^{9}7^{19}(1267244797948364094 - 896075533706946133a)t^3 + 2^{8}7^{17}(9850430584095374136 - 6965297266085092019a)t^4 + 2^{8}7^{16}(7074457433286037312 - 5002396790988649349a)t^5 - 2^{7}7^{15}(2833346680504939765576 - 2003478630481275710517a)t^6 - 2^{8}7^{14}(13062826273186848404584 - 9236813105616767488891a)t^7 + 2^{7}7^{13}(50446588512588102272829 - 35671125113271678931917a)t^8 + 2^{7}7^{12}(469995705600596099309418 - 323337149050655917780371a)t^9 - 2^{6}7^{11}(63926156375425158705288 - 452026182486419826134005a)t^{10} - 2^{5}7^{10}(41695217491028775689537 - 2948297111467613829154913a)t^{11} + 2^{4}7^{9}(47611865526525810725476246 - 3366673170591044794098941a)t^{12} + 2^{4}7^{9}(2462129766016039342770763 - 1740988649521184466334163a)t^{13} - 2^{4}7^{7}(674236002730880441782142 - 4767568277271704087503883a)t^{14} - 2^{3}7^{6}(31996193193391918129178252 - 22624725168026626035121663a)t^{15} - 2^{3}7^{5}(8855058917437803643379503 - 626147275361666998085918a)t^{16} + 2^{4}7^{4}(6119378614605756530630451 - 4327055148053001825770557a)t^{17} + 2^{3}7^{3}(12639797088162032274244852 - 8937678964369046837867715a)t^{18} - 2^{2}7^{2}(2094260756589813076610143 - 1480856690531237031689199a)t^{19} - 7(9470731732147123181642606 - 669684429749782828833133a)t^{20} - (1563447700882137315845205 - 1105520045182155035071680a)t^{21} \right), \quad (B1)
\]

where \( a = \sqrt{2} \).

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