Stress-Energy-Momentum Tensors in Lagrangian Field Theory.

Part 2. Gravitational Superpotential.

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Abstract

Our investigation of differential conservation laws in Lagrangian field theory is based on the first variational formula which provides the canonical decomposition of the Lie derivative of a Lagrangian density by a projectable vector field on a bundle (Part 1: gr-qc/9510061). If a Lagrangian density is invariant under a certain class of bundle isomorphisms, its Lie derivative by the associated vector fields vanishes and the corresponding differential conservation laws take place. If these vector fields depend on derivatives of parameters of bundle transformations, the conserved current reduces to a superpotential. This Part of the work is devoted to gravitational superpotentials. The invariance of a gravitational Lagrangian density under general covariant transformations leads to the stress-energy-momentum conservation law where the energy-momentum flow of gravity reduces to the corresponding generalized Komar superpotential. The associated energy-momentum (pseudo) tensor can be defined and calculated on solutions of metric and affine-metric gravitational models.
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PART 2

The present Part of the work is devoted to SEM conservation laws and energy-momentum superpotentials in the gravitation theory.

There are different approaches to the description of conservation laws in the gravitation theory; different energy-momentum (pseudo) tensors of gravity have been suggested \[1, 3, 4, 11, 13, 17, 21\]. Our analysis of the energy-momentum conservation law in the gravitation theory is based on the first variational formula in Lagrangian field theory (see Sections 3, 4 of the Part 1 [8]). In accordance with this formula, the invariance of a gravitational Lagrangian density under general covariant transformations implies the differential SEM conservation law where the corresponding energy-momentum flow reduces to the superpotential \[4, 7, 9, 15\].

Let us briefly remind the basic features of the geometric approach to field theory when classical fields are described by global sections of a bundle $Y \to X$ over a world manifold $X$ and their dynamics is phrased in terms of jet manifolds (see Section 3).

We restrict ourselves to the first order Lagrangian formalism, for most of contemporary field models are described by first order Lagrangian densities. This is not the case for the Einstein-Hilbert Lagrangian density of the Einstein’s gravitation theory which belongs to the special class of second order Lagrangian densities. Its Euler-Lagrange equations are however of the order two as like as in the first order theory (see Sections 13, 14).

In the first order Lagrangian formalism, the finite-dimensional configuration space of fields represented by sections $s$ of a bundle $Y \to X$ is the first order jet manifold $J^1Y$ of $Y$. Given fibered coordinates $(x^\mu, y^i)$ of $Y$, the jet manifold $J^1Y$ is endowed with the adapted coordinates $(x^\mu, y^i, y^i_\mu)$. A first order Lagrangian density on $J^1Y$ is defined to be an exterior horizontal density

$$L = \mathcal{L}(x^\mu, y^i, y^i_\mu)\omega, \quad \omega = dx^1 \wedge \ldots \wedge dx^n.$$ 

Let $G_t$ be a 1-parameter group of bundle isomorphisms of a bundle $Y \to X$, and let

$$u = u^\lambda(x)\partial_\lambda + u^i(y)\partial_i$$

be the corresponding projectable vector field on $Y$. A Lagrangian density $L$ on the configuration space $J^1Y$ is invariant under these transformations iff its Lie derivative by the jet lift

$$j^1_0 u = u^\lambda \partial_\lambda + u^i \partial_i + (\partial_\lambda u^i - y^i_\mu \partial_\lambda u^\mu)\partial_i;$$

$$\partial_\lambda = \partial_\lambda + y^j_\lambda \partial_j + y^j_\mu \partial^\mu_j + \cdots$$

of $u$ onto $J^1Y$ is equal to zero, i.e.

$$L_{j^1_0 u} L = 0. \tag{2.1}$$

In accordance with the first variational formula (35), there is the canonical decomposition

$$L_{j^1_0 u} L = u \mathcal{E}_L + dT(u) \tag{2.2}$$
where
\[\mathcal{E}_L = \left[ \partial_i - (\partial_\lambda + y^j_\lambda \partial_j + y^j_\mu \partial_\mu) \partial_i^\lambda \right] \mathcal{L} dy^i \wedge \omega = \delta_i \mathcal{L} dy^i \wedge \omega \] (2.3)
is the Euler-Lagrange operator,
\[T(u) = T^\lambda(u) \omega_\lambda = [\pi^i_\lambda (u^i - u^\mu y^i_\mu) + u^\lambda \mathcal{L}] \omega_\lambda,\]
\[\pi^i_\mu = \partial^i_\mu \mathcal{L}, \quad \omega_\lambda = \partial_\lambda \omega,\]
is the corresponding current, and
\[u_V = (u^i - y^i_\mu u^\mu) \partial_i\]
is the vertical part of the vector field \(u\).

The Euler-Lagrange operator \(\mathcal{E}_L\), by definition, vanishes on the critical sections of the bundle \(Y \to X\), and the equality (2.2) comes to the weak identity
\[0 \approx \partial_\lambda [\pi^i_\lambda (u^i - u^\mu y^i_\mu) + u^\lambda \mathcal{L}],\]
(see Section 4).

The examples of gauge fields (see Section 7) and tensor fields (see Section 8) show that, in case of gauge-type transformations when the corresponding vector fields \(u\) depend on derivatives of the parameters \(\alpha(x)\) of these transformations, the conservation law (2.4) takes the form
\[T = W + dU\]
where \(W \approx 0\) and \(U\) is a superpotential which depends on parameters \(\alpha(x)\) of gauge transformations that provide the gauge invariance of the conservation law (2.4).

The Einstein’s gravitation theory and the affine-metric gravitation theory are field models on the bundles of geometric objects (see Section 8). Gravitational Lagrangian densities are
invariant under general covariant transformations. As a consequence, we get the SEM conservation law with respect to the canonical lift of a vector field $\tau$ on a world manifold $X$ onto the corresponding bundle of gravitational fields.

In the purely metric Einstein’s gravitation theory [13], the SEM conservation law corresponding to the invariance of the Hilbert-Einstein Lagrangian density under general covariant transformations takes the form

$$\frac{d}{dx^\lambda} T^\lambda(\tau) \approx 0, \quad T^\lambda \approx \frac{d}{dx^\lambda} U^\mu{}^\lambda(\tau), \quad (2.6)$$

where

$$U^\mu{}^\lambda(\tau) = \frac{\sqrt{-g}}{2\kappa} (g^\mu{}^\nu \tau^\mu - g^\mu{}^\nu \tau^\lambda), \quad (2.7)$$

is the well-known Komar superpotential [13] associated with a vector field $\tau$ on a world manifold $X$. Here the symbol “; $^\mu$” denotes the covariant derivative with respect to the Levi-Civita connection.

In the recent paper [4], it was shown that (2.8) has a kind of universal property, in the sense that the SEM flow of any Lagrangian density depending nonlinearly on the scalar curvature, constructed from a metric and a torsionless connection, reduces always to the Komar superpotential.

This result has been extended to the affine-metric gravity in case of a general linear connection $K^\alpha{}^\gamma{}^\mu$ and arbitrary Lagrangian density $L$ invariant under general covariant transformations [9]. The corresponding SEM conservation law is brought into the form (2.6) where

$$U^\mu{}^\lambda(\tau) = \frac{\partial L}{\partial K^\alpha{}^\nu{}^\mu{}^\lambda} (D_\gamma \tau^\alpha + \Omega^\alpha{}^\nu{}^\sigma \tau^\sigma) \quad (2.8)$$

is the generalized Komar superpotential. Here $D_\gamma$ is the covariant derivative with respect to the general linear connection $K$ and $\Omega$ is the torsion of this connection. In the particular case of the Hilbert-Einstein Lagrangian density and symmetric connections, we have

$$\frac{\partial L_{\text{HE}}}{\partial K^\alpha{}^\nu{}^\mu{}^\lambda} = \frac{\sqrt{-g}}{2\kappa} (\delta^\mu{}^\gamma g^\nu{}^\lambda - \delta^\lambda{}^\gamma g^\nu{}^\mu),$$

so that the superpotential (2.8) comes to the Komar superpotential (2.7). Also, if the Lagrangian density is of the kind considered in Ref. [1], the superpotential (2.8) recovers the superpotential found in that paper.

In case of also gauge gravitation theory, we show that the covariant derivative of Dirac fermion fields takes the form

$$\bar{D}_\lambda = \partial_\lambda - \frac{1}{2} A^{abc}{}^\mu (\partial_\lambda h^\mu c + K^\mu{}^\nu{}^\lambda h^\nu c) I_{ab}, \quad (2.9)$$

$$A^{abc}{}^\mu = \frac{1}{2} (\eta^{ca} h^b_\mu - \eta^{cb} h^a_\mu),$$
where \( h(x) \) is a tetrad gravitational field, \( K \) is a general linear connection, \( \eta \) is the Minkowski metric, and

\[
I_{ab} = \frac{1}{4} [\gamma_a, \gamma_b]
\]

are the generators of the spinor group \( L_\gamma = SL(2, \mathbb{C}) \) \(^10\).

The covariant derivative (2.9) has been considered by several authors \(2, 16, 24\). In accordance with the well-known theorem \(12\), every general linear connection being projected onto the Lie algebra of the Lorentz group yields a Lorentz connection.

It follows that the configuration space of metric (or tetrad) gravitational fields and general linear connections may play the role of the universal configuration space of realistic gravitational models. In particular, one can think of the generalized Komar superpotential as being the universal energy-momentum superpotential of gravity. The corresponding energy-momentum (pseudo) tensor reads

\[
T^\lambda_\alpha = \frac{d}{dx^\mu} \left( \frac{\partial L}{\partial K^{\sigma \nu \mu,\lambda}} K^\sigma_{\nu a} \right).
\]

One can calculate it on solutions of metric and affine-metric gravitational models. In particular, the torsion contributes into this energy-momentum (pseudo) tensor.

Note that the dependence of the energy-momentum superpotentials of gravity on the vector field \( \tau \) reflects the fact that the SEM conservation law (2.6) is preserved under general covariant transformations.

Hereafter, the 4-dimensional base manifold \( X \) is required to satisfy the well-known topological condition in order that a pseudo-Riemannian metric can exist. To summarize these conditions, we assume that the manifold \( X \) is not compact and that the tangent bundle of \( X \) is trivial. We call \( X \) the world manifold. Pseudo-Riemannian metrics and general linear connections in tangent and cotangent bundles of \( X \) are called the world metrics and the world connections respectively.

### 13 Reduced second order Lagrangian formalism

Given a bundle \( Y \to X \) coordinatized by \((x^\lambda, y^i)\), let \( L \) be a second order Lagrangian density on the second order jet manifold \( J^2Y \) of \( Y \). Its different Lagrangian equivalents exist on \( J^3Y \), but the associated Poincaré-Cartan form \( \Xi_L \) is uniquely defined and given by the coordinate expression

\[
\Xi_L = L \omega + \left[ (\partial_i^\lambda L - \partial_\mu \partial_i^{\lambda \mu} L) \tilde{y}^j + \partial_i^{\mu \lambda} L \tilde{y}^j_\mu \right] \wedge \omega_\lambda. \tag{2.10}
\]

In the second order case, the first variational formula (23) is written

\[
\pi^4_2 J^3u L = h_0(j^3_0 u \cdot d\rho_L) + h_0 d(j^3_0 u \cdot \rho_L)
\]

for any projectable vector field \( u \) on the bundle \( Y \to X \). When \( \rho_L = \Xi_L \), it takes the form

\[
\pi^4_2 J^3u L = u_\nu [\mathcal{E}_L + d_H h_0(\tau] \Xi_L) \tag{2.11}
\]
where
\[ \mathcal{E}_L = (\partial_i - \partial_{\lambda} \partial_{\lambda}^i + \partial_{\mu} \partial_{\lambda} \partial_{\lambda}^{i\mu}) L dy^i \wedge \omega \] (2.12)
is the 4-order Euler-Lagrange operator and the second order jet lift \( j^2_0 u \) of the vector field \( u \) reads
\[ j^2_0 u = u^\lambda \partial_\lambda + u^i \partial_i + (\partial_\lambda u^i - y^i_\nu \partial_\lambda u^\nu) \partial_\lambda^i + [\partial_{\mu}(\partial_\lambda u^i - y^i_\nu \partial_\lambda u^\nu) - y^i_\nu \partial_\mu \partial_\lambda u^\nu] \partial_{\lambda}^{i\mu}. \]

Being restricted to the kernel of the Euler-Lagrange operator (2.12), the equality (2.11) comes to the weak identity
\[ \pi^{i^{\mu}}_2 L j^2_0 u L \approx \partial_\lambda [u^\lambda L + u^i (\partial_i L - \partial_{\mu} \partial_{\lambda}^{i\mu} L) + \partial_{\mu} u^i \partial_{\lambda}^{i\mu} L] \omega \] (2.13)
and to the corresponding differential transformation law on critical sections of the bundle \( Y \to X \).

Let us consider a second order Lagrangian density \( L \) whose Euler-Lagrange operator \( \mathcal{E} \) (2.12) reduces to the second order differential operator \([14]\). It takes place, if the associated Poincaré-Cartan form \( \Xi_L \) (2.10) is defined on the first order jet manifold \( J^1 Y \) of \( Y \). This is the case iff the Lagrangian density \( L \) obeys the conditions
\[ \partial_j^{\alpha\beta} \partial^{\mu\nu} L = 0, \]
\[ (\partial_j^{\mu\alpha} \partial_j^{\nu\beta} - \partial_j^{\mu\beta} \partial_j^{\nu\alpha}) L = 0. \] (2.14)
This Lagrangian density is linear in the coordinates \( y^i_{\lambda\mu} \) and, in each coordinate chart, it is given by the expression
\[ L = (L' + \pi^{i\mu}_i y^i_{\mu\lambda}) \omega \] (2.15)
where \( L' \) is a local function on \( J^1 Y \) and the Lagrangian momentum \( \pi \) is a section of the Legendre bundle
\[ \wedge^n T^* X \otimes_{J^1 Y} (\vee^2 T^* X) \otimes_{J^1 Y} V^* Y. \]

In virtue of the relation (2.12), there exists a local horizontal form \( \phi = \phi^{\lambda} \omega_\lambda \) on \( J^1 Y \to X \) such that
\[ \pi^{i\lambda}_i = \partial_i^{\lambda} \phi^{\lambda}. \]

Let us consider the local form
\[ \epsilon = \Xi_L - d\phi. \]

It is the Lagrangian equivalent of the local first order Lagrangian density
\[ L_1 = h_0(\epsilon) = L - d_H \phi \] (2.16)
which leads to the same second order Euler-Lagrange operator in a given coordinate chart as the Lagrangian density (2.13) does.

In particular, if the functions \( \pi^{i\mu}_i \) are independent of the coordinates \( y^i_\mu \), we can take
\[ \phi = \pi^{i\mu}_i (y^i_\mu - \Gamma^i_\mu) \omega_\lambda \] (2.17)
where $\Gamma$ is a connection on $Y \rightarrow X$. The form (2.17) is globally defined and we get the first order Lagrangian density

$$L_1 = L - \hat{\partial}_\lambda [\pi_t^{i\lambda}(y^i_\mu - \Gamma^i_\mu)]\omega$$

which leads to the same second order Euler-Lagrange operator as the Lagrangian density (2.15) does.

However, the first order Lagrangian densities $L_1$ (2.16) and (2.18) fail to possess the same symmetries of the second order Lagrangian density $L$ (2.15) in general. Therefore, we do not take advantage of its application in conservation laws as a rule.

### 14 Conservation laws in Einstein’s gravitation theory

In Einstein’s gravitation theory, gravity is described by a pseudo-Riemannian metric whose Lagrangian density is the Hilbert-Einstein Lagrangian density.

Let $\Sigma_g \rightarrow X$ be the bundle of pseudo-Riemannian world metrics. Its 2-fold covering is the bundle

$$\Sigma = LX/\text{SO}(3,1)$$

where $LX$ is the principal bundle of oriented linear frames and $\text{SO}(3,1)$ is the connected Lorentz group. Hereafter, we shall identify $\Sigma_g$ with the open subbundle of the tensor bundle

$$\sqrt{2} \, T^*X \rightarrow X.$$

In induced coordinates of $T^*X$, the bundle $J^2\Sigma_g$ is coordinatized by $(x^\lambda, g_{\alpha\beta}, g_{\alpha\beta\lambda}, g_{\alpha\beta\lambda\mu})$.

The second order Hilbert-Einstein Lagrangian density on the configuration space $J^2\Sigma_g$ reads

$$L_{\text{HE}} = -\frac{1}{2\kappa} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\mu\nu}\sqrt{-g} \omega$$

where

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(g_{\alpha\nu\beta\mu} + g_{\beta\mu\alpha\nu} - g_{\alpha\mu\beta\nu} - g_{\beta\nu\alpha\mu}) + g^{\varepsilon\sigma}(\gamma_{\varepsilon\beta\mu}\gamma_{\sigma\alpha\nu} - \gamma_{\varepsilon\beta\nu}\gamma_{\sigma\alpha\mu});$$

$$\gamma_{\alpha\mu\nu} = \frac{1}{2}(g_{\alpha\mu\nu} + g_{\alpha\nu\mu} - g_{\mu\nu\alpha});$$

$$g^{\alpha\beta} = \frac{1}{g} \frac{\partial g}{\partial g_{\alpha\beta}}, \quad \frac{\partial}{\partial g_{\alpha\beta}} = -g^{\alpha\mu} g^{\beta\nu} \frac{\partial}{\partial g_{\mu\nu}}.$$

This is the reduced second order Lagrangian density like that considered in the previous Section. It leads to the second order Euler-Lagrangian operator.

To remain within the framework of bundles of geometric objects, we utilize the Proca fields as the matter source of gravitational fields. Their Lagrangian density (90) depends on a world metric, but not on the symmetric part of the world connection.
The Proca fields are described by sections of the cotangent bundle $T^*X$ (see Section 11). The total configuration space of metric gravitational fields and Proca fields is $J^2T$ where

$$T = \sqrt{g} T^*X \times T^*X.$$  

On this configuration space, the Lagrangian density of Proca fields is given by the expression

$$L_P = -\frac{1}{4\gamma} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} F_{\mu\nu} - \frac{1}{2} m^2 g^{\mu\lambda} k_{\mu} k_{\lambda} \sqrt{-g} \omega,$$  \hspace{1cm} (2.21)

where

$$F_{\mu\nu} = k_{\nu\mu} - k_{\mu\nu}$$

and $g^{\nu\beta}$ is the inverse matrix of $g_{\nu\beta}$.

The total Lagrangian density $L$ is the sum of the Hilbert-Einstein Lagrangian density (2.20) and the Lagrangian density of Proca fields (2.21).

The associated Poincaré-Cartan form on the jet manifold $J^1T$ reads

$$\Xi_L = \Xi_{HE} + \Xi_P$$

where

$$\Xi_{HE} = -\frac{1}{2\kappa} \sqrt{-g} [g^{\mu\alpha} g^{\nu\beta} g^{\xi\sigma} (\gamma_{\xi\mu\nu} \gamma_{\alpha\beta\sigma} - \gamma_{\xi\beta\alpha} \gamma_{\mu\sigma\nu}) \omega + (g^{\mu\alpha} g^{\nu\beta} - g^{\alpha\beta} g^{\mu\lambda}) (dg_{\alpha\beta\mu} + g^{\nu\sigma} \gamma_{\sigma\alpha\beta} dg_{\mu\nu}) \wedge \omega],$$

[14] and $\Xi_P$ is given by the expression

$$\Xi_P = (\mathcal{L}_P - \pi^{\mu\lambda} k_{\mu\lambda}) \omega + \pi^{\mu\lambda} dk_{\mu} \wedge \omega_{\lambda},$$  \hspace{1cm} (2.22)

$$\pi^{\mu\lambda} = -\frac{1}{\gamma} g^{\mu\alpha} g^{\nu\beta} F_{\beta\alpha} \sqrt{-g}.$$  

The total Euler-Lagrange operator is

$$\mathcal{E}_L = \mathcal{E}_{HE} + \mathcal{E}_P,$$

$$\mathcal{E}_{HE} = -\frac{1}{2} g^{\mu\alpha} g^{\nu\beta} \sqrt{-g} [\frac{1}{\kappa} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + t_{\mu\nu}] dg_{\alpha\beta} \wedge \omega = \delta^{\alpha\beta} \mathcal{L} dg_{\alpha\beta} \wedge \omega$$

$$\mathcal{E}_P = [-\sqrt{-g} m^2 g^{\alpha\beta} k_{\alpha} + \frac{1}{\gamma} \partial_{\alpha} (g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} \sqrt{-g})] dk_{\beta} \wedge \omega = \delta^{\beta\beta} \mathcal{L} dk_{\beta} \wedge \omega,$$  \hspace{1cm} (2.23)

where $t$ is the metric energy-momentum tensor (94) of the Proca fields.

The Lagrangian densities $L_{HE}$ (2.20) and $L_P$ (2.21) are invariant separately under general covariant transformations of the bundle of geometric objects $T = \Sigma_g \times T^*X$. Therefore, the Lie derivative of their sum $L$ by the jet lift $j^2 \bar{\tau}$ of the vector field on $T$

$$\bar{\tau} = \tau^\lambda \partial_\lambda - (g_{\nu\beta} \partial_{\alpha} \tau^\nu + g_{\alpha\nu} \partial_{\beta} \tau^\nu) \frac{\partial}{\partial g_{\alpha\beta}} - \partial_{\alpha} \tau^\nu k_{\nu} \frac{\partial}{k_{\alpha}}$$
naturally induced by the vector field $\tau$ on $X$ vanishes. Hence, we get the weak conservation law which takes the form

$$0 \approx \hat{\partial}_\lambda [-2g_{\mu\alpha}\tau^\mu \delta^{\lambda\alpha} L + \tau^\nu k_\nu \delta^\lambda L + \frac{1}{2\kappa}\sqrt{-g}(g^{\mu\nu} \tau^\lambda - g^{\lambda\nu} \tau^\mu)_{,\mu} - \hat{\partial}_\mu (\pi^{\mu\lambda} \tau^\nu k_\nu)]$$ (2.24)

A glance at the conservation law (2.24) shows that the SEM flow of a metric gravitational field with respect to the vector field $\tilde{\tau}$ reduces to the Komar superpotential (2.7). The total superpotential contains also the superpotential

$$Q^{\mu\lambda}(\tau) = \pi^{\mu\lambda} \tau^\nu k_\nu$$ (2.25)

of the Proca fields. We observe that, in case of exact general covariant transformations, the energy-momentum flow (95) of the Proca fields comes to the superpotential term (see Section 11).

15 First order Palatini formalism

This Section is devoted to the SEM conservation laws of gravitational theory in the first order (Palatini) variables when a world metric and a symmetric world connection are considered on the same footing. To compare the SEM flows in this model with those in the Einstein’s gravitational theory, we restrict our attention the Hilbert-Einstein Lagrangian density in the Palatini variables. The corresponding Euler-Lagrange equations are well-known to be equivalent to the Einstein’s equations [4].

Let $LX \to X$ be the principal bundle of linear frames in the tangent spaces to $X$. Its structure group is $GL^+(4, \mathbb{R})$. The world connections are associated with the principal connections on the principal bundle $LX \to X$. Hence, there is the 1:1 correspondence between the world connections and the global sections of the quotient bundle

$$C_w = J^1 LX/GL^+(4, \mathbb{R}).$$ (2.26)

With respect to a holonomic atlas, the bundle $C_w$ is coordinatized by $(x^\lambda, k^\alpha_{\beta\lambda})$ so that, for any section $K$ of $C_w$,

$$K^\alpha_{\beta\lambda} = k^\alpha_{\beta\lambda} \circ K$$

are the coefficients of the linear connection

$$K = dx^\lambda \otimes \left( \frac{\partial}{\partial x^\lambda} + K^\alpha_{\beta\lambda} \dot{x}_\alpha \frac{\partial}{\partial \dot{x}_\beta} \right)$$

on $T^*X$. In this Section, we restrict ourselves to symmetric world connections.

The bundle $C_w$ (2.26) admits the canonical splitting

$$C_w = C_- \oplus C_+$$
\[ k_{\beta \lambda} = k_{[\beta \lambda]} + k_{(\beta \lambda)}, \]

where
\[ C_- = \frac{2}{\sqrt{\gamma}} T^* X \otimes T X \]
is the bundle of torsion soldering forms and \( C_+ \rightarrow X \) is the affine bundle modelled on the vector bundle
\[ \frac{2}{\sqrt{\gamma}} T^* X \otimes T X. \]
Sections of the bundle \( C_+ \rightarrow X \) are symmetric world connections. This bundle is coordinatized by \((x^\lambda, k_{\alpha \beta \lambda})\) where \( k_{\alpha \beta \lambda} = k_{\alpha \lambda \beta} \).

The total configuration space of the Palatini gravitational model is
\[ J^1(\Sigma \times C_+). \] (2.27)

It is coordinatized by
\((x^\lambda, g_{\alpha \beta}, k_{\alpha \beta \lambda}, g_{\alpha \beta \mu}, k_{\alpha \beta \lambda \mu})\).

On the configuration space (2.27), the Hilbert-Einstein Lagrangian density reads
\[ L_{\text{HE}} = -\frac{1}{2\kappa} g^{\beta \lambda} R_{\beta \alpha \lambda \omega} \sqrt{-\gamma}, \]

\[ R_{\beta \nu \lambda} = k_{\beta \lambda \nu} - k_{\beta \nu \lambda} + k_{\epsilon \nu \lambda} k_{\beta \epsilon} - k_{\epsilon \nu \epsilon} k_{\beta \lambda \epsilon}. \]
It is of the order zero with respect to the metric fields \( g^{\alpha \beta} \) and of the first order with respect to the coordinates \( k_{\alpha \beta \lambda} \).

We consider the Palatini gravitational model in the presence of the Proca fields. The total Lagrangian density \( L \) on the jet manifold \( J^1 T \) where
\[ T = \Sigma \times C_+ \times T^* X \]
is the sum
\[ L = L_{\text{HE}} + L_{\text{P}} \] (2.29)
of the Hilbert-Einstein Lagrangian density (2.28) and the Lagrangian density of Proca fields (2.21).

The associated Poincaré-Cartan form on the jet manifold \( J^1 T \) is the sum
\[ \Xi_L = \Xi_{\text{HE}} + \Xi_{\text{P}} \]
of the form
\[ \Xi_{\text{HE}} = -\frac{1}{2\kappa} \sqrt{-\gamma} g^{\beta \lambda} R_{\beta \alpha \lambda \omega} + \pi_{\alpha \beta \nu \lambda} \frac{\partial k_{\gamma \beta \nu}}{\partial \omega_{\lambda}}, \]

\[ \pi_{\alpha \beta \nu \lambda} = \frac{1}{2\kappa} (\delta_{\nu \alpha} g_{\beta \lambda} - \delta_{\alpha \nu} g_{\beta \lambda}) \sqrt{-\gamma}, \] (2.30)
and the form $\Xi_P$ (2.22). The Euler-Lagrange operator corresponding to the Lagrangian density (2.29) is the sum
\[ E_L = E_P + E_K + E_g \]
of the Euler-Lagrange operator for the Proca field $E_P$ (2.23), for the symmetric connection
\[ E_K = \frac{1}{2\kappa}[(\sqrt{-g}g^{\beta\nu}),_\alpha - (\sqrt{-g}\delta^{\alpha}_{\mu}g^{\beta\lambda})_\lambda]dk^{\alpha}_{\beta\nu} \wedge \omega = \delta^{\alpha}_{\beta\mu}Ldk^{\alpha}_{\beta\nu} \wedge \omega \] (2.31)
where
\[ g^{\alpha\beta;\lambda} = \hat{\partial}_{\lambda}g^{\alpha\beta} + k^{\alpha}_{\mu\lambda}g^{\mu\beta} + k^{\beta}_{\mu\lambda}g^{\alpha\mu}, \]
and for the metric field
\[ E_g = \frac{1}{2}\sqrt{-g}(T_{\alpha\beta} + t_{\alpha\beta})dg^{\alpha\beta} \wedge \omega = \delta_{\alpha\beta}Ldg^{\alpha\beta} \wedge \omega \] (2.32)
where
\[ T_{\alpha\beta} = -\frac{1}{\kappa}(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) \] (2.33)
and $t_{\alpha\beta}$ is the metric energy-momentum tensor (94) of Proca fields. One can think of $T_{\alpha\beta}$ (2.33) as being the metric energy-momentum tensor of symmetric world connections.

The total Lagrangian density $L$ (2.29) of the Palatini gravitational model is invariant under general covariant transformations of the bundle of geometric objects
\[ T = \Sigma_g \times C_+ \times T^*X. \]

Therefore, the Lie derivative of $L$ by the jet lift $j^{j}_{1}\bar{\tau}$ of the vector field on $T$
\[ \bar{\tau} = \tau^{\lambda}\partial_{\lambda} + (g^{\nu\beta}\partial_{\nu}^{\alpha} + g^{\alpha\nu}\partial_{\nu}^{\beta})\frac{\partial}{\partial g^{\alpha\beta}} \]
\[ +[\partial_{\nu}\tau^{\alpha}k^{\nu}_{\beta\mu} - \partial_{\beta}\tau^{\nu}k^{\alpha}_{\nu\mu} - \partial_{\mu}\tau^{\nu}k^{\alpha}_{\beta\nu} - \partial_{\beta\mu}\tau^{\alpha}]\frac{\partial}{\partial k^{\alpha}_{\beta\mu}} \]
\[ -\partial_{\alpha}\tau^{\nu}k^{\nu}_{\beta\mu}\frac{\partial}{\partial k^{\alpha}_{\beta\mu}} \] (2.34)
naturally induced by a vector field $\tau$ on $X$ vanishes. Hence, we get the weak conservation law (2.4) in the form
\[ 0 \approx \hat{\partial}_{\lambda}[\pi^{\mu\lambda}_{\alpha}(\tau^{\nu}k^{\alpha}_{\beta\mu} + \partial_{\nu}\tau^{\alpha}k^{\nu}_{\beta\mu} - \partial_{\beta}\tau^{\nu}k^{\alpha}_{\nu\mu} - \partial_{\mu}\tau^{\nu}k^{\alpha}_{\beta\nu} - \partial_{\beta\mu}\tau^{\alpha}) \]
\[ +\pi^{\nu\lambda}(\partial_{\nu}\tau^{\mu}k^{\mu}_{\beta\nu} - \tau^{\nu}k^{\mu}_{\nu\mu}) + \tau^{\lambda}L]. \]

This can be brought into the form (2.6) where as like as in the previous case the total superpotential is the sum of the gravitational Komar superpotential (2.7) and the superpotential (2.25) of Proca fields.
16 Energy-momentum superpotential of affine-metric gravity

Let us consider the affine-metric gravitational model where dynamic variables are pseudo-Riemannian metrics and general linear connections on $X$.

Note that, since the world connections are the principal connections, one may apply the standard procedure of gauge theory in order to obtain the SEM conservation law (see Section 10). However in this case, the nonholonomic gauge isomorphisms of the linear frame bundle $LX$ and the associated bundles have to be considered [11]. The canonical lift $\tilde{\tau}$ of a vector field $\tau$ on the base $X$ onto the bundles $\Sigma_g$ and $C_w$ does not correspond to these isomorphisms. One must use a horizontal lift of $\tau$ by means of some connection on these bundles. At the same time, we observe that the configuration space of gauge gravitation potentials in the gauge gravitation theory itself reduces to the configuration space of general linear connections [10].

The total configuration space of the affine-metric gravity is

$$J^1Y = J^1(\Sigma_g \times C_w)$$

(2.35)

coordinatized by

$$(x^\lambda, g^{\alpha\beta}, k^{\alpha\beta\lambda}, g'_{\alpha\beta\mu}, k^{\alpha\beta\lambda\mu}).$$

We assume that a Lagrangian density $L_{am}$ of the affine-metric gravitation theory on the configuration space (2.35) depends on a metric $g^{\alpha\beta}$ and the curvature

$$R^{\alpha\beta\nu\lambda} = k^{\alpha\beta\nu\lambda} - k^{\alpha\beta\nu\lambda} + k^{\alpha\nu\lambda}k^{\beta\nu} - k^{\alpha\lambda}k^{\beta\nu}.$$ 

In this case, we have the relations

$$\frac{\partial L_{am}}{k^{\alpha\beta\nu}} = \pi^{\beta\nu\lambda}k^{\sigma\alpha\lambda} - \pi^{\alpha\nu\lambda}k^{\beta\sigma\lambda},$$

$$\pi^{\alpha\beta\nu\lambda} = \partial^{\alpha\beta\nu\lambda}L_{am} = -\pi^{\alpha\beta\lambda\nu}.$$ (2.36)

Let the Lagrangian density $L_{am}$ be invariant under general covariant transformations. Given a vector field $\tau$ on $X$, its canonical lift onto the bundle $\Sigma_g \times C_w$ reads

$$\tilde{\tau} = \tau^\lambda \partial_\lambda + (g^{\nu\beta} \partial_\nu \tau^\alpha + g^{\alpha\nu} \partial_\nu \tau^\beta) \frac{\partial}{\partial g^{\alpha\beta}}$$

$$+ [\partial_\nu \tau^\alpha k^{\nu \beta\mu} - \partial_{\beta \nu} k^{\alpha \nu\mu} - \partial_{\mu \nu} k^{\alpha \nu\beta} - \partial_{\beta \mu} k^{\alpha \nu\nu}] \frac{\partial}{\partial k^{\alpha\beta\mu}}.$$ (2.37)

For the sake of simplicity, let us employ the compact notation

$$\tilde{\tau} = \tau^\lambda \partial_\lambda + (g^{\nu\beta} \partial_\nu \tau^\alpha + g^{\alpha\nu} \partial_\nu \tau^\beta)\partial_{\alpha\beta} + (u^{A\beta}_{\alpha} \partial_{\beta} \tau^\alpha - u^{A\beta\varepsilon}_{\alpha} \partial_{\varepsilon} \tau^\alpha)\partial_A.$$
Since the Lie derivative of $L_{am}$ by the jet lift $j^1_0\tilde{\tau}$ of the field $\tilde{\tau}$ (2.37) is equal to zero, i.e.

$$L_{j^1_0\tilde{\tau}}L_{am} = 0,$$

we have the weak conservation law

$$0 \approx \hat{\partial}_\lambda [\partial^A_\lambda L_{am}(u^{A\beta}_\alpha \partial_\beta \tau^\alpha - u^{A\epsilon\beta}_\alpha \partial_\epsilon \tau^\alpha - y^A_\alpha \tau^\alpha) + \tau^\lambda L_{am}]$$

(2.39)

where

$$\partial^A_\lambda L_{am} u^{A\epsilon\beta}_\alpha = \pi_\alpha^{\epsilon\beta\lambda},$$

$$\partial^A_\lambda L_{am} u^{A\beta}_\alpha = \pi_\alpha^{\epsilon\mu\beta} k^{\beta}_{\epsilon \gamma} - \pi_\sigma^{\epsilon\mu\beta} k^{\gamma}_{\epsilon \sigma} - \pi_\sigma^{\epsilon\beta\mu} k^{\lambda}_{\epsilon \sigma} = \partial_\alpha \beta \epsilon \lambda L_{am} - \pi_\sigma^{\epsilon\beta\mu} k^{\lambda}_{\epsilon \sigma} \gamma.$$ 

Due to the arbitrariness of the functions $\tau^\alpha$, the equality (2.38) implies the strong equality

$$\delta^\beta_\alpha L_{am} + \sqrt{-g} T^\beta_\alpha = u^{A\beta}_\alpha \partial_A L_{am} + \hat{\partial}_\mu (u^{A\beta}_\alpha \partial_A L_{am} - y^A_\alpha \partial_A L_{am}) = 0.$$ 

(2.40)

One can think of

$$\sqrt{-g} T^\beta_\alpha = 2 g^{\alpha\nu} \partial_\nu L_{am}$$

as being the metric energy-momentum tensor of general linear connections.

Substituting the term $\gamma_0^A \partial^A_\lambda L_{am}$ from the expression (2.40) into the conservation law (2.39), we bring the latter into the form

$$0 \approx \hat{\partial}_\lambda [\partial^A_\lambda L_{am} u^{A\beta}_\alpha \partial_\beta \tau^\alpha - u^{A\epsilon\beta}_\alpha \partial_\epsilon \tau^\alpha - \partial_A L_{am} u^{A\lambda}_\alpha \tau^\alpha - \partial^A_\lambda L_{am} \hat{\partial}_\mu (u^{A\lambda}_\alpha \tau^\alpha)].$$

(2.41)

Let us separate the components of the Euler-Lagrange operator

$$\mathcal{E}_L = (\delta_{\alpha\beta} L_{am} dg^{\alpha\beta} + \delta_\alpha^{\gamma\mu} L_{am} dh^{\gamma}_{\epsilon \mu}) \wedge \omega$$

in the expression (2.41). We get

$$0 \approx \hat{\partial}_\lambda [\partial^A_\lambda L_{am} u^{A\mu}_\alpha \partial_\mu \tau^\alpha - \hat{\partial}_\mu (\partial^A_\mu L_{am} u^{A\lambda}_\alpha \tau^\alpha) + \hat{\partial}_\mu (\partial^\lambda_\sigma L_{am} \tau^\sigma) + \hat{\partial}_\lambda (\hat{\partial}_\mu (\delta_{\alpha\mu \lambda} L_{am}) \tau^\alpha)] +$$

$$\hat{\partial}_\lambda [-2 g^{\alpha\nu} \tau^\alpha \delta_{\alpha\mu} L_{am} - u^{A\lambda}_\alpha \tau^\alpha \delta_{A} L_{am}] - \hat{\partial}_\lambda [\delta_{\alpha\mu \lambda} \tau^\alpha]$$

and then

$$0 \approx \hat{\partial}_\lambda [-2 g^{\alpha\nu} \tau^\alpha \delta_{\alpha\mu} L_{am} - (k^{\lambda}_{\gamma \epsilon} \delta_{\alpha \gamma} L_{am} - k^{\sigma}_{\alpha \mu \lambda} \delta_{\alpha \sigma} \lambda L_{am} - k^{\sigma}_{\gamma \alpha \delta} \gamma L_{am}) \tau^\alpha + \delta_\alpha^{\epsilon \lambda} L_{am} \partial_\epsilon \tau^\alpha]$$

$$-\hat{\partial}_\lambda [\delta_{\alpha \mu \lambda} (D_{\nu} \tau^\alpha + \Omega^{\alpha \nu \sigma} \tau^\sigma)].$$

The final form of the conservation law (2.33) is

$$0 \approx \hat{\partial}_\lambda [-2 g^{\alpha\nu} \tau^\alpha \delta_{\alpha\mu} L_{am} - (k^{\lambda}_{\gamma \epsilon} \delta_{\alpha \gamma} L_{am} - k^{\sigma}_{\alpha \mu \lambda} \delta_{\alpha \sigma} \lambda L_{am} - k^{\sigma}_{\gamma \alpha \delta} \gamma L_{am}) \tau^\alpha + \delta_\alpha^{\epsilon \lambda} L_{am} \partial_\epsilon \tau^\alpha$$

$$-\hat{\partial}_\mu (\delta_{\alpha \mu \lambda} L_{am}) \tau^\alpha] - \hat{\partial}_\lambda [\delta_{\alpha \mu \lambda} (D_{\nu} \tau^\alpha + \Omega^{\alpha \nu \sigma} \tau^\sigma)].$$

(2.42)
It follows that the SEM conservation law in the affine-metric gravity is reduced to the form (2.6) where \( U \) is the generalized Komar superpotential (2.8).

Let us now examine the total system of the affine-metric gravity and the tensor fields described in Section 8, e.g., the Proca fields. In the presence of a general linear connection, the Lagrangian density \( L_P \) (2.22) is naturally generalized through covariant derivatives and depends on the torsion:

\[
F_{\mu
u} = k_{\nu\mu} - k_{\mu\nu} - Q^\sigma_{\nu\mu}k_\sigma.
\]

It is readily observed that, in this case, the superpotential term (2.25) in the energy-momentum flow of the Proca fields (95) is eliminated due to the additional contribution

\[
-\hat{\partial}_\mu (\partial_\alpha \lambda^\mu L_P \tau^\alpha).
\]

Thus, the energy-momentum flow of the Proca fields comes to zero, and the total energy-momentum flow of affine-metric gravity and Proca fields reduces to the generalized Komar superpotential.

One can consider general linear connections and Proca fields in the presence of a background world metric \( g \) when the general covariant transformations are not exact. In this case, we have the weak transformation law (2.3) where the variational derivatives \( \delta_{\alpha\beta} L \) by the metric field fail to vanish. Then, the total SEM flow takes the form

\[
T^\lambda = \sqrt{-g}(T^\lambda_\alpha + t^\lambda_\alpha \tau^\alpha + \hat{\partial}_\mu (\pi_\alpha^{\nu\mu\lambda}(D_\nu \tau^\alpha + \Omega^\alpha_{\nu\sigma} \tau^\sigma))
\]

and the SEM transformation law comes to the form of the covariant conservation law

\[
(T^\lambda_\mu + t^\lambda_\mu)_\lambda \approx 0
\]

of the metric energy-momentum tensors of general linear connections and of the Proca fields. Thus, we observe that the "hidden" non-superpotential part of the energy-momentum flow appears if the invariance under general covariant transformations is broken.

17 Lagrangian systems on composite bundles

The gauge gravitation theory exemplifies the model with spontaneous breaking of space-time symmetries where the matter fermion fields admit only Lorentz transformations. The geometric formulation of the gauge gravitation theory calls into play the composite bundle picture. As a consequence, we get the modified covariant differential of fermion fields which depends on derivatives of gravitational fields [10].

In the gauge gravitation theory, gravity is represented by pairs \((h, A_h)\) of gravitational fields \( h \) and associated Lorentz connections \( A_h \) [11, 18]. The Lorentz connection \( A_h \) is usually identified with both a connection on a world manifold \( X \) and a spinor connection on the the spinor bundle \( S_h \to X \) whose sections describe Dirac fermion fields \( \psi_h \) in the presence of the gravitational field \( h \). The problem arises when Dirac fermion fields are described in the
framework of the affine-metric gravitation theory. In this case, the fact that a world connection is some Lorentz connection may result from the field equations, but it can not be assumed in advance. There are models where the world connection is not a Lorentz connection [11]. Moreover, it may happen that a world connection is the Lorentz connection with respect to different gravitational fields [23]. At the same time, a Dirac fermion field can be regarded only in a pair \((h, \psi_h)\) with a certain gravitational field \(h\).

Indeed, one must define the representation of cotangent vectors to \(X\) by the Dirac’s \(\gamma\)-matrices in order to construct the Dirac operator. Given a tetrad gravitational field \(h(x)\), we have the representation

\[
\gamma_h : dx^\mu \mapsto \hat{dx}^\mu = h_\mu^a \gamma^a.
\]

However, different gravitational fields \(h\) and \(h'\) yield the nonequivalent representations \(\gamma_h\) and \(\gamma_{h'}\).

It follows that, fermion-gravitation pairs \((h, \psi_h)\) are described by sections of the composite spinor bundle

\[
S \rightarrow \Sigma \rightarrow X \quad (2.43)
\]

where \(\Sigma \rightarrow X\) is the bundle of gravitational fields \(h\); the components \(h_\mu^a\) of \(h\) play the role of parameter coordinates of \(\Sigma\), besides the familiar world coordinates. In particular, every spinor bundle \(S_h \rightarrow X\) is isomorphic to the restriction of \(S \rightarrow \Sigma\) to \(h(X) \subset \Sigma\). Performing this restriction, we arrive at the familiar case of a field model in the presence of a gravitational field \(h(x)\).

By a composite bundle is meant the composition

\[
Y \rightarrow \Sigma \rightarrow X \quad (2.44)
\]

of a bundle \(Y \rightarrow X\) denoted by \(Y_\Sigma\) and a bundle \(\Sigma \rightarrow X\). It is coordinatized by \((x^\lambda, \sigma^m, y^i)\) where \((x^\mu, \sigma^m)\) are coordinates of \(\Sigma\) and \(y^i\) are the fiber coordinates of \(Y_\Sigma\). We further assume that \(\Sigma\) has global sections.

The application of composite bundles to field theory is founded on the following [19]. Given a global section \(h\) of \(\Sigma\), the restriction \(Y_h\) of \(Y_\Sigma\) to \(h(X)\) is a subbundle of \(Y \rightarrow X\). There is the 1:1 correspondence between the global sections \(s_h\) of \(Y_h\) and the global sections of the composite bundle \(2.44\) which cover \(h\). Therefore, one can think of sections \(s_h\) of \(Y_h\) as describing fermion fields in the presence of a background parameter field \(h\), whereas sections of the composite bundle \(Y\) describe all the pairs \((s_h, h)\). The configuration space of these pairs is the first order jet manifold \(J^1Y\) of the composite bundle \(Y\).

The feature of the dynamics of field systems on composite bundles consists in the following. Every connection

\[
A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \tilde{A}_\lambda^i \partial_i) + d\sigma^m \otimes (\partial_m + A_m^i \partial_i)
\]

on the bundle \(Y_\Sigma\) yields the horizontal splitting

\[
VY = VY_\Sigma \oplus (Y \times V\Sigma),
\]

\[
\dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\dot{y}^i - A_m^i \dot{\sigma}^m)\partial_i + \dot{\sigma}^m (\partial_m + A_m^i \partial_i).
\]
Using this splitting, one can construct the first order differential operator 
\[ \tilde{D} : J^1 Y \to T^* X \otimes VY_\Sigma, \]
\[ \tilde{D} = dx^\lambda \otimes (y^{\lambda}_i - \tilde{A}^{\lambda}_i - A^i_m \sigma^m) \partial_i, \quad (2.45) \]
on the composite bundle \( Y \). This operator possesses the following property.

Given a global section \( h \) of \( \Sigma \), let \( \Gamma \) be a connection on \( \Sigma \) whose integral section is \( h \), that is, \( \Gamma \circ h = J^1 h \). It is readily observed that the differential (2.45) restricted to \( J^1 Y_h \subset J^1 Y \) comes to the familiar covariant differential relative to the connection
\[ A_h = dx^\lambda \otimes (\partial_\lambda + (A^i_m \partial_m + \tilde{A}^{\lambda}_i) \partial_i) \]
on \( Y_h \). Thus, we may utilize \( \tilde{D} \) in order to construct a Lagrangian density
\[ L : J^1 Y \to T^* X \otimes VY_\Sigma \to \wedge^n T^* X \]
for sections of the composite bundle \( Y \).

18 Composite spinor bundles in gravitation theory

Let us consider the gauge theory of gravity and fermion fields.

Let \( LX \) be the principal bundle of oriented linear frames in tangent spaces to \( X \). In
gravitation theory, its structure group \( GL^+(4, \mathbb{R}) \) is reduced to the connected Lorentz group \( L = SO(1,3) \). It means that there exists a reduced subbundle \( L^h X \) of \( LX \) whose structure group is \( L \). In accordance with the well-known theorem [12], there is the 1:1 correspondence between the reduced \( L \) subbundles \( L^h X \) of \( LX \) and the global sections \( h \) of the quotient bundle (2.19).

Given a section \( h \) of \( \Sigma \), let \( \Psi^h \) be an atlas of \( LX \) such that the corresponding local sections \( z^h_\xi \) of \( LX \) take their values into \( L^h X \). With respect to \( \Psi^h \) and a holonomic atlas \( \Psi^T = \{ \psi^T_\xi \} \) of \( LX \), a gravitational field \( h \) can be represented by a family of \( GL_4 \)-valued tetrad functions
\[ h_\xi = \psi^T_\xi \circ z^h_\xi, \quad dx^\lambda = h^\lambda_a(x) h^a. \quad (2.46) \]

By the Lorentz connections \( A_h \) associated with a gravitational field \( h \) are meant the principal connections on the reduced subbundle \( L^h X \) of \( LX \). They give rise to principal connections on \( LX \) and to spinor connections on the \( L^s \)-lift \( P_h \) of \( L^h X \).

There are different ways to introduce Dirac fermion fields. Here, we follow the algebraic
approach. Given a Minkowski space \( M \), let \( Cl_{1,3} \) be the complex Clifford algebra generated by
elements of \( M \). A spinor space \( V \) is defined to be a minimal left ideal of \( Cl_{1,3} \) on which this algebra acts on the left. We have the representation
\[ \gamma : M \otimes V \to V \]
of elements of the Minkowski space $M \subset Cl_{1,3}$ by Dirac’s matrices $\gamma$ on $V$.

Let us consider a bundle of complex Clifford algebras $Cl_{1,3}$ over $X$ whose structure group is the Clifford group of invertible elements of $Cl_{1,3}$. Its subbundles are both a spinor bundle $S_M \to X$ and the bundle $Y_M \to X$ of Minkowski spaces of generating elements of $Cl_{1,3}$. To describe Dirac fermion fields on a world manifold $X$, one must require $Y_M$ to be isomorphic to the cotangent bundle $T^*X$ of $X$. It takes place if there exists a reduced $L$ subbundle $L^hX$ of $LX$ such that

$$Y_M = (L^hX \times M)/L.$$ 

Then, the spinor bundle

$$S_M = S_h = (P_h \times V)/L_s$$

is associated with the $L_s$-lift $P_h$ of $L^hX$. In this case, there exists the representation

$$\gamma_h : T^*X \otimes S_h = (P_h \times (M \otimes V))/L_s \to (P_h \times \gamma(M \times V))/L_s = S_h$$

of cotangent vectors to a world manifold $X$ by Dirac’s $\gamma$-matrices on elements of the spinor bundle $S_h$. As a shorthand, one can write

$$\tilde{dx}^\lambda = \gamma_h(dx^\lambda) = h^\lambda_a(x)\gamma^a.$$ 

Given the representation (2.48), we shall say that sections of the spinor bundle $S_h$ describe Dirac fermion fields in the presence of the gravitational field $h$. Indeed, let

$$A_h = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2}A^{ab}_\lambda I_{ab}A^B\psi^B\partial_A)$$

be a principal connection on $S_h$. Given the corresponding covariant differential $D$ and the representation $\gamma_h$ (2.48), one can construct the Dirac operator

$$D_h = \gamma_h \circ D : J^1S_h \to T^*X \otimes VS_h \to VS_h,$$

$$y^A \circ D_h = h^\lambda_a\gamma^a_B(y^B - \frac{1}{2}A^{ab}_\lambda I_{ab}A^B\psi^B)$$

on the spinor bundle $S_h$.

Different gravitational fields $h$ and $h'$ define nonequivalent representations $\gamma_h$ and $\gamma_{h'}$. It follows that a Dirac fermion field must be regarded only in a pair with a certain gravitational field. There is the 1:1 correspondence between these pairs and sections of the composite spinor bundle (2.43) defined as follows.

Let us consider the $L$-principal bundle

$$LX_\Sigma := LX \to \Sigma$$

where $\Sigma$ is the quotient bundle (2.19). Let $P_\Sigma$ be the $L_s$-principal lift of $LX_\Sigma$ such that

$$P_\Sigma/L_s = \Sigma, \quad LX_\Sigma = r(P_\Sigma).$$
In particular, there is the imbedding of the $L_s$-lift $P_s$ of $L^h X$ onto the restriction of $P_\Sigma$ to $h(X)$.

We define the composite spinor bundle (2.43) where

$$S_\Sigma = (P_\Sigma \times V)/L_s$$

is associated with the $L_s$-principal bundle $P_\Sigma$. It is readily observed that, given a global section $h$ of $\Sigma$, the restriction of $S_\Sigma$ to $h(X)$ is the spinor bundle $S_h$ (2.47) whose sections describe Dirac fermion fields in the presence of the gravitational field $h$.

Let us provide the principal bundle $LX$ with a holonomic atlas $\{\psi^T_\xi, U_\xi\}$ and the principal bundles $P_\Sigma$ and $LX_\Sigma$ with associated atlases $\{z_\xi^V, U_\xi\}$ and $\{z_\xi = r \circ z_\xi^V\}$. With respect to these atlases, the composite spinor bundle is endowed with the bundle coordinates $(x^\lambda, \sigma^a_\mu, \psi^A)$ where $(x^\lambda, \sigma^a_\mu)$ are coordinates of the bundle $\Sigma$ such that $\sigma^a_\mu$ are the matrix components of the group element $(\psi^T_\xi \circ z_\xi^V)(\sigma), \sigma \in U_\xi, \pi_{\Sigma X}(\sigma) \in U_\xi$. Given a section $h$ of $\Sigma$, we have

$$(\sigma^a_\mu \circ h)(x) = h^a_\mu(x),$$

where $h^a_\mu(x)$ are the tetrad functions (2.46).

Let us consider the bundle of Minkowski spaces

$$(LX \times M)/L \rightarrow \Sigma$$

associated with the $L$-principal bundle $LX_\Sigma$. Since $LX_\Sigma$ is trivial, it is isomorphic to the pullback $\Sigma \times T^*X$ which we denote by the same symbol $T^*X$. Then, one can define the bundle morphism

$$\tilde{\gamma}_\Sigma : T^*X \otimes_{\Sigma} S_\Sigma = (P_\Sigma \times (M \otimes V))/L_s \rightarrow (P_\Sigma \times \gamma(M \otimes V))/L_s = S_\Sigma,$$

(2.50)

$$\tilde{dx}^\lambda = \gamma_\Sigma(dx^\lambda) = \sigma^a_\mu \gamma^a,$$

over $\Sigma$. When restricted to $h(X) \subset \Sigma$, the morphism (2.50) comes to the morphism $\gamma_h$ (2.48).

We use this morphism in order to construct the total Dirac operator on the composite spinor bundle $S$ (2.43).

Let

$$\tilde{A} = dx^\lambda \otimes (\partial_\lambda + \tilde{A}^B_\lambda \partial_B) + d\sigma^a_\mu \otimes (\partial^a_\mu + A^{Ba}_\mu \partial_B)$$

be a principal connection on the bundle $S_\Sigma$ and $\tilde{D}$ the corresponding differential (2.43). We have the first order differential operator

$$\mathcal{D} = \gamma_\Sigma \circ \tilde{D} : J^1S \rightarrow T^*X \otimes_{\Sigma} VS_\Sigma \rightarrow VS_\Sigma,$$

$$\psi^A \circ \mathcal{D} = \sigma^a_\mu \gamma^a_B (\psi^B_\lambda - \tilde{A}^B_\lambda - A^{Ba}_\mu \partial_B)$$

on $S$. One can think of it as being the total Dirac operator since, for every section $h$, the restriction of $\mathcal{D}$ to $J^1S_h \subset J^1S$ comes to the Dirac operator $\mathcal{D}_h$ (2.49) relative to the connection

$$A_h = dx^\lambda \otimes [\partial_\lambda + (\tilde{A}^B_\lambda + A^{Ba}_\mu \partial_B)h^a_\mu]$$
on the bundle $S_h$.

In order to construct the differential $\widetilde{D}$ (2.44) on $J^1S$ in explicit form, let us consider the principal connection on the bundle $LX_S$ given by the local connection form

$$\tilde{A} = (\tilde{A}_{ab}^\mu dx^\mu + A_{abc}^\mu d\sigma^\mu_c) \otimes I_{ab},$$

$$\tilde{A}_{ab}^\mu = \frac{1}{2} K^\nu_{ab} \sigma^\nu_c (\eta^{ca} \sigma^b_\nu - \eta^{cb} \sigma^a_\nu),$$

$$A_{abc}^\mu = \frac{1}{2} (\eta^{ca} \sigma^b_\mu - \eta^{cb} \sigma^a_\mu),$$

where $K$ is a general linear connection on $TX$ and (2.52) corresponds to the canonical left-invariant free-curvature connection on the bundle $GL^+(4, \mathbb{R}) \to GL^+(4, \mathbb{R})/L$.

Accordingly, the differential $\widetilde{D}$ relative to the connection (2.51) reads

$$\widetilde{D} = dx^\lambda \otimes [\partial_\lambda - \frac{1}{2} A_{ab}^\mu (\sigma^a_\lambda + K^\nu_{ab} \sigma^\nu_c) I_{ab}^A B \psi^B \partial_\lambda],$$

Given a section $h$, the connection $\tilde{A}$ (2.51) is reduced to the Lorentz connection

$$\tilde{K}_{ab}^\mu = A_{ab}^\mu (\partial_\lambda h^\mu_c + K^\nu_{ab} h^\nu_c)$$

on $L^hX$, and the differential (2.53) leads to the covariant derivative of fermion fields (2.3).

Let us emphasize that the connection (2.54) is not the connection

$$K^k_{m\lambda} = h^k_\mu (\partial_\lambda h^\mu_m + K^\mu_{\nu\lambda} h^\nu_m) = K^k_{\lambda}(\eta_{am} \delta^{k}_b - \eta_{bm} \delta^{k}_a)$$

written with respect to the reference frame $h^a = h^a_\lambda dx^\lambda$, but there is the relation

$$\tilde{K}_{ab}^\lambda = \frac{1}{2} (K_{ab}^\lambda - K_{ba}^\lambda).$$

If $K$ is a Lorentz connection $A_h$, then the connection $\tilde{K}$ (2.54) consists with $K$ itself.

We utilize the differential (2.53) in order to construct a Lagrangian density of Dirac fermion fields. This Lagrangian density is defined on the configuration space $J^1(S \oplus S^+) \otimes \Sigma$ coordinatized by

$$(x^\mu, \sigma^\mu, \psi^A, \psi^+_A, \sigma_{a\lambda}, \psi^+_A, \psi^+_A, \psi^{\pm}_A).$$

It reads

$$L_\psi = \frac{i}{2} \{ \psi^+_A (\gamma^0 \gamma^\lambda)^A_B (\psi^+_A - \frac{1}{2} A_{ab}^\mu (\sigma^a_\lambda + K^\mu_{ab} \sigma^\nu_c) I_{ab}^C \psi^C) -$$

$$(\psi^{+\lambda}_A - \frac{1}{2} A_{ab}^\mu (\sigma^a_\lambda + K^\mu_{ab} \sigma^\nu_c) I_{ab}^C ) (\gamma^0 \gamma^\lambda)^A_B \psi^B \} - m \psi^{+\lambda}_A (\gamma^0)^A_B \psi^B \} \sigma^{-1} \omega,$$ (2.56)
\[ \gamma^\mu = \sigma^a \gamma^a, \quad \sigma = \det(\sigma^a), \]

where

\[ \psi^A_A(\gamma^0) B^B \]

is the Lorentz invariant fiber metric in the bundle \( S \oplus S^+ \)

One can easily verify that

\[ \frac{\partial L}{\partial K_{\mu \nu \lambda}} + \frac{\partial L}{\partial K_{\mu \lambda \nu}} = 0. \]

Hence, the Lagrangian density \( (2.56) \) depends on the torsion of the general linear connection \( K \) only. In particular, it follows that, if \( K \) is the Levi-Civita connection of a gravitational field \( h(x) \), after the substitution \( \sigma^\nu_c = h^\nu_c(x) \), the Lagrangian density \( (2.56) \) comes to the familiar Lagrangian density of fermion fields in the Einstein’s gravitation theory.

19 Conservation laws in gauge gravitation theory

In accordance with the previous Sections, the total configuration space of fermion fields and affine-metric gravity can be described by the jet manifold \( J^1 Y \) of the product

\[ Y = S \oplus S^+ \times C_w \]

where \( C_w \) is the bundle of general linear connections \( (2.26) \) coordinatized by \( (x^\lambda, k_{\mu \nu \lambda}) \).

The total Lagrangian density \( L \) on this configuration space is the sum of the Lagrangian density \( L_{am} \) of affine-metric gravity where variables \( g^{\alpha \beta} \) are replaced with \( \sigma^a \sigma^b \eta^{ab} \) and the Lagrangian density of fermion fields \( L_\psi \) \( (2.59) \) where the background general linear connection \( K_{\mu \nu \lambda} \) is replaced with the corresponding coordinates \( k_{\mu \nu \lambda} \) of the bundle \( C_w \).

The Lagrangian density \( L \) is constructed to be invariant under the 1-parameter groups of gauge isomorphisms of the \( L \)-principal bundle \( LX \rightarrow \Sigma \). The corresponding vector fields on the bundle \( Y \) \( (2.58) \) read

\[ u = \frac{1}{2} \alpha^{ab}(x)[(\eta_{ac} \delta^d_b - \eta_{bc} \delta^d_a) \sigma^\mu c + I_{ab} A B \psi^B \partial_A + I_{ab}^+ A B \psi^+_A \partial^B] \]

where \( \alpha^{ab}(x) \) are local parameters of gauge transformations. The Lie derivative of the Lagrangian density \( L \) by the jet lift \( j_0 u \) of the vector field \( (2.59) \) is equal to zero and the corresponding weak conservation law

\[ 0 \approx \hat{\partial}_\lambda \left[ \frac{1}{2} \alpha^{ab}(\partial^\lambda A B \psi^B \partial_A + I_{ab}^+ A B \psi^+_A \partial^B \mathcal{L}_\psi) \right]. \]

\( (2.4) \) takes place. However, it is easy to verify that

\[ \partial^\mu \mathcal{L}_\psi(\eta_{ac} \delta^d_b - \eta_{bc} \delta^d_a) \sigma^\mu c + \partial^\lambda \mathcal{L}_\psi I_{ab} A B \psi^B + I_{ab}^+ A B \psi^+_A \partial^B \mathcal{L}_\psi = 0, \]

\( (2.60) \)
and so the conserved current is equal to zero.

Now, we investigate the SEM conservation law of Dirac fermion fields and affine-metric gravity. Let $\tau$ be a vector field on $X$. Both the Lagrangian density $L_\psi$ of fermion fields and the Lagrangian density $L_{am}$ of affine-metric gravity are invariant under the (local) 1-parameter groups of transformations associated with the local vector fields

$$\tilde{\tau} = \tau^\mu \partial_\mu + \partial_\nu \tau^\mu \sigma^\nu_a \frac{\partial}{\partial \sigma_a} + (\partial_\nu \tau^\mu k^{\nu\beta}_\mu - \partial_\beta \tau^\nu k^{\nu\mu}_\beta - \partial_\mu \tau^\alpha k^{\alpha}_\beta - \partial_\beta \tau^\alpha) \frac{\partial}{\partial k^\alpha_{\beta\mu}}. \quad (2.61)$$

Note that, under a gauge (Lorentz) transformation, the field (2.61) is changed as

$$\tilde{\tau}' = \tilde{\tau} + \frac{1}{2} \tau^\mu \partial_\mu (\alpha^{ab}) \left[ (\eta_{ac} \delta^d_b - \eta_{bc} \delta^d_a) \sigma^\mu_d \partial_\mu + I^{AB}_B \psi^A \partial_A + I^{+AB}_A \psi^B \partial_B \right],$$

but in virtue of the relation (2.60), the additional term in $\tilde{\tau}'$ does not contribute in the SEM conservation law.

For the sake of simplicity, let us employ the same compact notation as in Section 16:

$$\tilde{\tau} = \tau^\mu \partial_\mu + \partial_\nu \tau^\mu \sigma^\nu_a \frac{\partial}{\partial \sigma_a} + (u^{A\beta}_\alpha \partial_\beta \tau^\alpha - u^{A\varepsilon\beta}_\alpha \partial_\varepsilon \tau^\alpha - y^{A}_\alpha \tau^\alpha) \partial_A.$$

Since the Lie derivative of $L$ by the jet lift $j^1_0 \tilde{\tau}$ of the field $\tilde{\tau}$ (2.61) is equal to zero, i.e.

$$L_{j^1_0 \tilde{\tau}} L = 0, \quad (2.62)$$

the weak conservation law

$$0 \approx \tilde{\delta}_\lambda [\delta^\lambda_\alpha L_{am} (u^{A\beta}_\alpha \partial_\beta \tau^\alpha - u^{A\varepsilon\beta}_\alpha \partial_\varepsilon \tau^\alpha - y^{A}_\alpha \tau^\alpha)$$

$$+ \frac{\partial L_\psi}{\partial \sigma^\alpha_{c\lambda}} (\partial_\beta \tau^\alpha \sigma^\beta_c - \sigma^\alpha_{c\mu} \tau^\mu) - \frac{\partial L_\psi}{\partial \psi^A} \psi^A \partial_\alpha - \frac{\partial L_\psi}{\partial \psi^+_A} \psi^+_A \partial_\alpha + \tau^\lambda \mathcal{L}] \quad (2.63)$$

takes place. We have the relations (2.30) and the relation

$$\frac{\partial L_\psi}{\partial k^\mu_{\nu\lambda}} = \frac{\partial L_\psi}{\partial \sigma^\nu_{c\lambda}} \sigma^\mu_c.$$

Due to the arbitrariness of the functions $\tau^\alpha$, the equality (2.62) implies the strong equality (2.40) where $\sqrt{-g}$ is replaced by $2\sigma$ and in addition the strong equality

$$\delta^\beta_\alpha \mathcal{L}_\psi + 2\sigma t^\beta_\alpha + \frac{\partial L_\psi}{\partial \sigma^\alpha_{c\lambda}} \sigma^\beta_c \delta_\lambda^\alpha - \frac{\partial L_\psi}{\partial \sigma^\beta_{c\beta}} \sigma^\alpha_c + \partial_\lambda \mathcal{L}_\psi u^{A\beta}_\alpha - \frac{\partial L_\psi}{\partial \psi^A} \psi^A \partial_\alpha - \frac{\partial L_\psi}{\partial \psi^+_A} \psi^+_A \partial_\alpha \quad (2.64)$$

where

$$2\sigma t^\beta_\alpha = \sigma^\beta_a \frac{\partial L_\psi}{\partial \sigma^a_a}.$$
Substituting the term $A^\alpha A_\beta A L$ from the expression (2.40) and the term
\[ \frac{\partial \mathcal{L}_\psi}{\partial \sigma^\mu \sigma_{ca}} + \frac{\partial \mathcal{L}_\psi}{\partial \psi_{\alpha}} \psi^A_{\alpha} + \frac{\partial \mathcal{L}_\psi}{\partial \psi^A_{\alpha}} \psi^+_{A\alpha} \]
from the expression (2.64) into the conservation law (2.63), we bring the latter into the form
\[ 0 \approx \hat{\partial}_\lambda \left[ -\sigma^\alpha \tau^\alpha \delta^a \mathcal{L} - (k)^\gamma \delta^\alpha \psi^A_\alpha \mathcal{L}_{am} - k^\sigma \gamma^\alpha \delta^\alpha \gamma^\lambda \mathcal{L}_{am} \right] \tau^\alpha + \delta^\alpha \mathcal{L}_{am} \partial_e \tau^\alpha \\
- \hat{\partial}^\mu (\delta^\alpha \mathcal{L}_{am} \tau^\alpha) \right] - \hat{\partial}_\lambda \left[ \hat{\partial}^\mu (\pi^\alpha \nu^\mu \mathcal{D} \nu \tau^\alpha + \Omega^\alpha \nu^\sigma \tau^\sigma) \right] \\
+ \hat{\partial}_\lambda \left[ (\frac{\partial \mathcal{L}_\psi}{\partial \sigma^\alpha}_{\alpha a} + \frac{\partial \mathcal{L}_\psi}{\partial \sigma^\alpha}_{\alpha a} \sigma^\mu_a) \partial_\mu \tau^\alpha \right]. \tag{2.65} \]

In accordance with the relation (2.57), the last term in the expression (2.65) is equal to zero, i.e. fermion fields do not contribute to the superpotential. The SEM conservation law (2.63) comes to the form (2.4) where $U$ is the generalized Komar superpotential (2.8).

We can thus conclude that the generalized Komar superpotential (2.8) occurs rather universally in different gravitational models.

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