Confining Bethe–Salpeter equation from scalar QCD

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Abstract

We give a nonperturbative derivation of the Bethe–Salpeter equation based on the Feynman–Schwinger path integral representation of the one–particle propagator in an external field. We apply the method to the quark–antiquark system in scalar QCD and obtain a confining BS equation assuming the Wilson area law in the straight line approximation. The result is strictly related to the relativistic flux tube model and to the $q\bar{q}$ semirelativistic potential.

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I. INTRODUCTION

Various attempts have been done in the literature to apply the Bethe-Salpeter equation to a study of the spectrum and the properties of the mesons. The hope was to obtain an unified and consistent description of the quark-antiquark bound states envolving light quarks as well as heavy ones. In all such attempts, at our knowledge, the choice of the kernel of the equation was purely conjectural and only made in such a way that the successful heavy quarks potential could be recovered in the non relativistic limit. While a kind of derivation of the $q\bar{q}$ potential from QCD can be given in the Wilson loop context, even if at the price of not completely proved assumptions [1]–[5], no similar derivation seems to exist for the BS equation.

In this paper we want to consider this problem for the simplified model of the scalar QCD.

As it is well known a satisfactory semirelativistic potential for the heavy quark–antiquark system can be obtained from the assumption

$$i \ln W = i (\ln W)_{\text{pert}} + \sigma S_{\text{min}}$$  \hspace{1cm} (1.1)

where: \(W\) stands for the Wilson loop integral

$$W = \frac{1}{3} \langle \text{Tr} P \exp ig \{ \oint_{\Gamma} dx^\mu A_\mu \} \rangle ;$$  \hspace{1cm} (1.2)

\(\Gamma\) denotes a closed loop made by a quark world line \((\Gamma_1)\), an antiquark world line \((\Gamma_2)\) followed in the reverse direction and two straight lines connecting the initial and the final points of the two world lines; \((\ln W)_{\text{pert}}\) and \(S_{\text{min}}\) are the perturbative evaluation of \(\ln W\) and the minimum area enclosed by \(\Gamma\) respectively; finally the expectation value in (1.2) stands for the functional integration on the gauge field alone. More sofisticate evaluations of \(i \ln W\) have also been attempted [2], [3], [4], but they shall not be considered here.

In the derivation of the potential, \(S_{\text{min}}\) is further approximated by the surface spanned by the straight lines connecting equal time points on the quark and the antiquark worldlines, i.e. by the surface of equation
\[ x^0 = t, \quad \mathbf{x} = s\mathbf{z}_1(t) + (1 - s)\mathbf{z}_2(t), \]  

(1.3)

\( t \) being the ordinary time, \( \mathbf{z}_1(t) \) and \( \mathbf{z}_2(t) \) the quark and the antiquark positions at the time \( t \), and \( s \) a parameter with \( 0 \leq s \leq 1 \). The result is

\[
S_{\text{min}} \cong \int_{t_i}^{t_f} dt \int_0^1 ds \left[ 1 - (s\dot{z}_{1T} + (1 - s)\dot{z}_{2T})^2 \right]^{\frac{1}{2}} =
\]  

(1.4)

where \( \dot{z}_{2T} \) stands for \( \left( \delta^{hk} - \frac{\delta h}{r^2} \right) \frac{dz}{dt} \) and \( \mathbf{r}(t) = \mathbf{z}_1(t) - \mathbf{z}_2(t) \).

In fact it can be shown that (1.4) is correct up to the second order in the velocities \[2, 3\] and so is perfectly appropriate for a derivation of the semirelativistic potential. For a full relativistic extension, Eq.(1.4) cannot be correct as it stands, due to the privileged role played by the time in it. However one can try to assume (1.4) in the center of mass frame and then obtain implicitly the corresponding equation in a general frame simply by Lorentz transformation. In this way the assumption would become equivalent to the so called relativistic flux tube model \[8, 3\].

In this paper we want to show that in scalar QCD (i.e. neglecting the spin of the quarks) a Bethe Salpeter (BS) equation which include confinement can be derived from (1.1) and (1.4) in the center of mass frame. In the momentum representation and after the factorization of the four–momentum conservation \( \delta \), the resulting kernel turns out in the form

\[
\hat{I}(p_1, p_2; p'_1, p'_2) = \hat{I}_{\text{pert}}(p_1, p_2; p'_1, p'_2) + \hat{I}_{\text{conf}}(p_1, p_2; p'_1, p'_2)
\]  

(1.5)

\( (p'_1 + p'_2 = p_1 + p_2) \), where \( \hat{I}_{\text{pert}} \) is the usual perturbative kernel which can be written, at the lowest order in the strong coupling constant,

\[
\hat{I}_{\text{pert}}(p_1, p_2; p'_1, p'_2) = \frac{16}{3} g^2 \left( \frac{p_1 + p'_1}{2} \right) \mu D_{\mu\nu}(p'_1 - p_1)\left( \frac{p_2 + p'_2}{2} \right)^\nu,
\]  

(1.6)

while in the center of mass system \( \hat{I}_{\text{conf}} \) is given by

\[
\hat{I}_{\text{conf}}(p_1, p_2; p'_1, p'_2) = \int d^3\mathbf{r} e^{i(k' - \mathbf{k}) \cdot \mathbf{r}} J(\mathbf{r}, \frac{p_1 + p'_1}{2}, \frac{p_2 + p'_2}{2})
\]  

(1.7)

\( (p_1 = -p_2 = \mathbf{k}, p'_1 = -p'_2 = \mathbf{k'}) \) with \( J \) expressed as an expansion in \( \frac{1}{m^2} \).
\[ J(r, q_1, q_2) = 4 \sigma r \frac{1}{q_{10} + q_{20}} \left[ q_{20}^2 \sqrt{q_{10}^2 - q_{11}^2} + q_{10}^2 \sqrt{q_{20}^2 - q_{12}^2} + q_{10}^2 q_{20}^2 \left( \text{arcsin} \frac{|q_r|}{|q_{10}|} + \text{arcsin} \frac{|q_r|}{|q_{20}|} \right) \right] + \ldots \]  

(1.8)

Notice that by a usual instantaneous approximation, from the kernel defined by (1.5)-(1.8) one can obtain the following hamiltonian

\[ H(r, q) = \sqrt{m_1^2 + q^2} + \sqrt{m_2^2 + q^2} + \sigma r \frac{1}{2 \sqrt{m_1^2 + q^2} + \sqrt{m_2^2 + q^2}} \left\{ \sqrt{m_1^2 + q^2} \sqrt{m_1^2 + q^2} + \sqrt{m_2^2 + q^2} \sqrt{m_2^2 + q^2} + \left( \frac{m_1^2 + q^2 \sqrt{m_2^2 + q^2}}{q_r} \right) \left( \text{arcsin} \frac{q_r}{m_1^2 + q^2} + \text{arcsin} \frac{q_r}{m_2^2 + q^2} \right) \right\} + \ldots + V_{\text{pert}}(r, q) \]  

(1.9)

with an appropriate ordering prescription. In Eq.(1.9) \( q \) stands now for the momentum in the center of mass frame, \( q_r = (\hat{r} \cdot q) \hat{r} \), \( q_r^k = (\delta^{hk} - \hat{r}^h \hat{r}^k) q^k \), while \( V_{\text{pert}} \) is the ordinary perturbative Salpeter potential. We stress that Eq.(1.9) is identical to the hamiltonian for the already mentioned relativistic flux tube model and consequently is strictly connected to the semirelativistic potential as given in [2], [3] when the spin dependent part is neglected. Notice however that the ordering in (1.9) corresponding to (1.7) is not identical to the Weyl prescription.

The present paper has mainly a pedagogical purpose and we have not done any attempt to explicitly apply the kernel (1.5)–(1.8) to an evaluation of the spectrum and of the properties of the mesons. Notice however that very interesting results have been obtained in this direction by the relativistic flux tube model [9] and in a sense we may consider our paper also as providing a more fundamental justification to that model.

As in potential theory the starting object in our derivation is the gauge invariant quark-antiquark propagator

\[ G_{11}^{q_1}(x_1, x_2; y_1, y_2) = \langle 0 \mid \phi_2^*(x_2) U(x_2, x_1) \phi_1(x_1) \phi_1^*(y_1) U(y_1, y_2) \phi_2(y_2) \mid 0 \rangle = -\frac{1}{3} \text{Tr} \{ U(x_2, x_1) \Delta_F^{(1)}(x_1, y_1, A) U(y_1, y_2) \Delta_F^{(2)}(y_2, x_2, A) \} , \]  

(1.10)
where $U(b,a) = P_{ba} \exp(ig \int_{a}^{b} dx^\mu A_\mu(x))$ is the path ordered gauge string (the integration path is over the straight line joining $a$ to $b$), while $\Delta^{(1)}_F$ and $\Delta^{(2)}_F$ denote the Feynman propagators for the two quarks in the external field $A_\mu$. In contrast with the potential case, however, no semirelativistic expansion is used, but the propagators are treated exactly using the covariant Feynman-Schwinger path integral representation.

Notice that in establishing the BS equation we have to neglect in $(i \ln W)_{\text{pert}}$ the contributions from the two extreme lines $x_1 x_2$ and $y_1 y_2$ and in $S_{\text{min}}$ the border contribution corresponding to $x_{10} \neq x_{20}$ and $y_{10} \neq y_{20}$, as in the potential case. This is correct for $x_1^0 - y_1^0$ and $x_2^0 - y_2^0$ large with respect to $|x_1 - x_2|, |y_1 - y_2|, x_1^0 - x_2^0$ and $y_1^0 - y_2^0$. So, strictly, we obtain a BS equation but for a quantity $G_4$ which coincides with $G_{4}^{\text{gl}}$ in the above limit. This is immaterial for what concerns bound states or asymptotic states. Naturally out of the limit situation, $G_4$ is no longer gauge invariant as it should be talking of a BS equation.

The significance of the Feynman–Schwinger representation in the framework of QCD has already been appreciated in [10] and particularly in [4]. With respect to Ref. [4] we have however a different attitude on the role of the BS equation.

The plan of the paper is the following one: in Sec. 2 we illustrate our method on the example of a one dimensional particle in a velocity dependent potential. In Sec. 3 we apply the method to the derivation of the Bethe–Salpeter equation for a system of two scalar particles interacting via a scalar field and discuss the various complications related to the perturbative kernel. In Sec.4 we derive the BS equation for a quark-antiquark system in scalar QCD under the assumption discussed above and obtain the kernel reported. Finally in Sec.5 we derive the Salpeter potential from the BS equation.

### II. ONE DIMENSION POTENTIAL THEORY

Let us consider the model made by a nonrelativistic particle in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + U(x,p) = H_0 + U \quad (2.1)$$
and the corresponding Schrödinger propagators

\[ K(x, y, t) = \langle x | e^{-iHt} | y \rangle, \quad K_0(x, y, t) = \langle x | e^{-iH_0t} | y \rangle. \]  \hspace{1cm} (2.2)

From the operatorial identity

\[ e^{-iHt} = e^{-iH_0t} - i \int_0^t dt' e^{iH_0(t-t')U} e^{-iHt'}, \]  \hspace{1cm} (2.3)

we obtain the equation

\[ K(x, y, t) = K_0(x, y, t) - i \int_0^t dt' \int d\xi \int d\eta K_0(x, \xi, t-t') \langle \xi | U | \eta \rangle K(\eta, y, t'). \]  \hspace{1cm} (2.4)

which is somewhat analogous to the nonhomogeneous Bethe-Salpeter equation in the configuration space.

We want to derive Eq. (2.4) by the path–integral formalism.

Let us be for definiteness

\[ U = V(x) + (W(x)p^2)_{\text{ord}}, \]  \hspace{1cm} (2.5)

where ( )_{\text{ord}} stands for some ordering prescription. In terms of path–integral we can write in the phase space

\[ K(x, y, t) = \int_y^x Dz Dp \exp \left\{ i \int_0^t dt' [p' \dot{z}' - \frac{p'^2}{2m} - V(z') - W(z')p'^2] \right\} \]  \hspace{1cm} (2.6)

with \( z' = z(t'), \ p' = p(t'), \ \dot{z}' = \frac{dz(t')}{dt} \). In Eq. (2.6) the functional “measures” are supposed to be defined by

\[ Dz = \left( \frac{m}{2\pi i \varepsilon} \right)^{\frac{N}{2}} dz_1 \ldots dz_{N-1}, \quad Dp = \left( \frac{i\varepsilon}{2\pi m} \right)^{\frac{N}{2}} dp_1 \ldots dp_{N-1} dp_N \]

\[ Dz Dp = \left( \frac{1}{2\pi} \right)^{N} dp_1 dz_1 \ldots dp_{N-1} dz_{N-1} dp_N \]  \hspace{1cm} (2.7)

where \( \varepsilon = \frac{1}{N} \), the limit \( N \to \infty \) is understood and the end points \( x \) and \( y \) stand for the condition \( z_0 = y, \ z_N = x \). As well known (see e.g. [6]) the ordering prescription is concealed under the particular discretization adopted in the limit procedure implied in the definition of (2.6).
Let us first consider the case $W = 0$. In this case there is no ordering problem and it is possible to calculate explicitly the integral in $p$ in (2.6) obtaining the path-integral representation in the configuration space

$$K(x, y, t) = \int_y^x Dz e^{i \int_0^t dt' \left( \frac{m \dot{z}'^2}{2} - V(z') \right)}.$$  

(2.8)

Then using the identity

$$e^{-i \int_0^t dt' V(z')} = 1 - i \int_0^t dt' V(z') e^{-i \int_0^{t'} dt'' V(z'')}$$

(2.9)

one obtains

$$K(x, y, t) = K_0(x, y, t) - i \int_0^t dt' \int_y^x Dz V(z') e^{i \int_0^{t'} dt'' \left( \frac{m \dot{z}''^2}{2} + V(z'') \right)} - i \int_0^t dt' \int_y^x Dz V(z') e^{i \int_0^{t'} dt'' \left( \frac{m \dot{z}''^2}{2} - V(z'') \right)}$$

(2.10)

which, taking into account that

$$\int_y^x Dz \ldots = \int d\xi \int_y^x Dz \int_y^\xi Dz \ldots$$

(2.11)

(having identified $\xi = z(t')$), can be rewritten in the form (2.4) with $\langle \xi | U | \eta \rangle = V(\xi) \delta(\xi - \eta)$.

In the general case $W(x) \neq 0$ it is convenient to work with the original path-integral representation in the phase space (2.6) and it is necessary to use discretized expressions explicitly. For Weyl ordering in Eq. (2.7) the correct discretization is the mid–point one and we can write

$$K(x, y, t) = \frac{1}{(2\pi)^N} \int dp_N dz_{N-1} dp_{N-1} \ldots dz_1 dp_1$$

$$\exp i \sum_{n=1}^N \left\{ p_n (z_n - z_{n-1}) - \varepsilon \left[ \frac{p_n^2}{2m} + V\left( \frac{x_n + x_{n-1}}{2} \right) + W\left( \frac{x_n + x_{n-1}}{2} \right) p_n^2 \right] \right\}.$$  

(2.12)

Then it is convenient to introduce the Fourier Transform of $K(x, y, t)$

$$\tilde{K}(k, q, t) = \int dx \int dy e^{-ikx} K(x, y, t) e^{iqy}$$

(2.13)

and to use the discrete counterpart of (2.9)

$$\exp \left( -i \varepsilon \sum_{n=1}^N \left[ V\left( \frac{x_n + x_{n-1}}{2} \right) + W\left( \frac{x_n + x_{n-1}}{2} \right) p_n^2 \right] \right) = 1 - i \varepsilon \sum_{R=1}^N \left\{ \left[ V\left( \frac{x_R + x_{R-1}}{2} \right) + W\left( \frac{x_R + x_{R-1}}{2} \right) p_R^2 \right] \right\}$$

$$\cdot \exp \left( -i \varepsilon \sum_{r=1}^{R-1} \left[ V\left( \frac{x_r + x_{r-1}}{2} \right) + W\left( \frac{x_r + x_{r-1}}{2} \right) p_r^2 \right] \right)$$

(2.14)
Replacing \((2.14)\) in \((2.12)\) and Fourier transforming we have

\[
\tilde{K}(k, q, t) = \tilde{K}_0(k, q, t) - i\varepsilon \frac{1}{(2\pi)^N} \sum_{R=1}^{N} \int dz_N dp_N \ldots dp_1 dz_0 e^{-ikz_N} \exp i \sum_{n=R}^{N} p_n(z_n - z_{n-1})
\]

\[
[V(\frac{z_R + z_{R-1}}{2}) + W(\frac{z_R + z_{R-1}}{2})p_R^2] \exp i \sum_{n=1}^{R-1} [p_n(z_n - z_{n-1}) - \varepsilon(\frac{p_n^2}{2m} + V(\frac{z_n + z_{n-1}}{2}) + W(\frac{z_n + z_{n-1}}{2})p_n^2)] e^{iqz_n} =
\]

\[
= \tilde{K}_0(k, q, t) - i\varepsilon \sum_{R=1}^{N} \frac{1}{(2\pi)^2} \{ \int dp_{R+1} \int dp_{R-1} \frac{1}{(2\pi)^{N-R-1}} \int dz_N dp_N \ldots dp_{R+2} dz_{R+1} e^{-ikz_N} \exp i \sum_{n=R+2}^{N} p_n(z_n - z_{n-1}) \} e^{ip_{R+1}z_{R+1}}
\]

\[
\{ \frac{1}{(2\pi)^{R-2}} \int dz_{R-1} \ldots dz_0 e^{-ip_{R-1}z_{R-1}} \exp i \sum_{n=1}^{R-2} p_n(z_n - z_{n-1}) - \varepsilon[\frac{p_n^2}{2m} + V(\frac{z_n + z_{n-1}}{2}) + W(\frac{z_n + z_{n-1}}{2})p_n^2)] e^{iqz_n} \}
\]

\[
(2.15)
\]

which in the continuous limit reads

\[
\tilde{K}(k, q, t) = \tilde{K}_0(k, q, t) - i \int_0^t dt' \int \frac{dk'}{2\pi} \int \frac{dq'}{2\pi} \tilde{K}_0(k, k', t - t') \tilde{W}(q', q') \tilde{K}(q', q, t')
\]

\[
(2.16)
\]

with

\[
\tilde{W}(k, q) = \frac{1}{2\pi} \int dz'' dp dz' e^{-ikz''} e^{ipz''} [V(\frac{z'' + z'}{2}) + W(\frac{z'' + z'}{2})p^2] e^{-ipz'} e^{iqz'} =
\]

\[
= \tilde{V}(k - q) + \tilde{W}(k - q) \frac{k + q}{2} \equiv \langle k|(V(x) + \frac{1}{4}(p, \{p, W(x)\})|q) \rangle
\]

\[
(2.17)
\]

Eq.\((2.16)\) is equivalent to Eq.\((2.4)\) and the kernel reduces to the Fourier transform of the case \(W(x) = 0\).

Had we considered the symmetric ordering in Eq.\((2.5)\) we should have replaced in \(2.14\) and \(2.15\) \(V(\frac{x_n + x_{n-1}}{2}) + W(\frac{x_n + x_{n-1}}{2})p_n^2\) with \(\frac{1}{2}(V(x_n) + V(x_{n-1})) + \frac{1}{2}(W(x_n) + W(x_{n-1}))p_n^2\) and the result would be

\[
\tilde{I}_S(k, q) = \frac{1}{2\pi} \int dz'' dp dz' e^{-ikz''} e^{ipz''} \frac{V(z'') + V(z')}{2} + \frac{W(z'') + W(z')}{2} p^2 e^{-ipz'} e^{iqz'} =
\]

\[
= \tilde{V}(k - q) + \tilde{W}(k - q) \frac{k^2 + q^2}{2} \equiv \langle k|(V(x) + \frac{1}{2}(p^2, W(x)))|q\rangle
\]

\[
(2.18)
\]
III. SPINLESS PARTICLES INTERACTING THROUGH A SCALAR FIELD

Let us consider two scalar “material” fields $\phi_1$ and $\phi_2$ interacting through a third scalar field $A$ with the coupling $\frac{1}{2}(g_1\phi_1^2 + g_2\phi_2^2 A)$. Then, after integration over $\phi_1$ and $\phi_2$, the full one particle propagator can be written as

$$G_2^{(j)}(x - y) = \langle 0 | T \phi_j(x) \phi_j(y) | 0 \rangle = \langle i\Delta_F^{(j)}(x, y; A) \rangle$$

where $\Delta_F^{(j)}(x, y; A)$ is the propagator for the particle $j$ in the external field $A$, $S_0(A)$ is the free action for the field $A$ and the determinantal factor $M(A)$ comes from the integration of the fields $\phi_j$

$$M(A) = \prod_{j=1,2} \left[ \det(\partial^\mu \partial_\mu + m_j^2 - g_j A) \right]^{-\frac{1}{2}} = 1 - \frac{1}{2} \sum_{j=1,2} \{ -g_j \int d^4 x A(x) \Delta_F^{(j)}(0) - \frac{1}{2}g_j^2 \int d^4 x d^4 y A(x) \Delta_F^{(j)}(x - y) A(y) \Delta_F^{(j)}(y - x) + \ldots \} \quad (3.2)$$

The covariant Feynman-Schwinger representation for $\Delta_F^{(j)}$ reads

$$\Delta_F^{(j)}(x, y; A) = \frac{-i}{2} \int_0^\infty d\tau \int_y^x Dz Dp \exp \{ i \int_0^\tau d\tau' [-p_\mu z'^\mu + \frac{1}{2}p'_\mu p'^\mu - \frac{1}{2}m_j^2 + \frac{1}{2}g_j A(z')] \}$$

$$= \frac{-i}{2} \int_0^\infty d\tau \int_y^x Dz \exp \{ -i \int_0^\tau d\tau' \frac{1}{2} [(z'^2 + m_j^2) - g_j A(z')] \}, \quad (3.3)$$

where the path integrals are understood to be extended over all paths $z^\mu = z^\mu(\tau')$ connecting $y$ with $x$ expressed in terms of an arbitrary parameter $\tau'$ with $0 \leq \tau' \leq \tau$. In Eq.(3.3) $z'$ stands for $z(\tau')$, $p'$ for $p(\tau')$, $z'^\prime$ for $\frac{dz(\tau')}{d\tau'}$ and the “functional measures” are assumed to be defined as

$$Dz = \left( \frac{1}{2\pi i \varepsilon} \right)^{2N} d^4 z_1 \ldots d^4 z_{N-1}, \quad \quad Dp = \left( \frac{i \varepsilon}{2\pi} \right)^{2N} d^4 p_1 \ldots d^4 p_{N-1} d^4 p_N$$

$$Dz Dp = \left( \frac{1}{2\pi} \right)^{4N} d^4 p_1 d^4 z_1 \ldots d^4 p_{N-1} d^4 z_{N-1} d^4 p_N. \quad (3.4)$$

Replacing Eq.(3.3) in (3.1) we obtain

$$G_2^{(j)}(x - y) = \frac{1}{2} \int_0^\infty d\tau \int_y^x Dz \exp \{ -i \int_0^\tau d\tau' \left( z'^2 + m_j^2 \right) \} \exp \frac{i g_j}{2} \int_0^\tau d\tau' A(z') \quad (3.5)$$

where, suppressing the tadpole term in (3.2),
\[
\langle \exp \frac{ig_j}{2} \int_0^\tau d\tau' A(z') \rangle = \exp \frac{ig_j^2}{4} \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' [D_F(z' - z'')] + \\
+ \sum_{i} \frac{g_i^2}{2} \int d^4\xi \int d^4\eta D_F(z' - \xi) (\Delta_F^{(i)}(\xi - \eta))^2 D_F(\eta - z'') + \ldots
\] (3.6)

Now let us consider the two particle propagator

\[
G_4(x_1, x_2; y_1, y_2) = \langle 0 | T\phi_2(x_1)\phi_2(x_2)\phi_2(y_1)\phi_2(y_2) | 0 \rangle = \{ G^{(1)}_4(x_1, y_1; A) G^{(2)}_4(x_2, y_2; A) \} = \\
= \left(\frac{1}{2}\right)^2 \int_0^{\tau_1} d\tau_1 \int_0^{\tau_2} d\tau_2 \int_{y_1}^{x_1} Dz_1 \int_{y_2}^{x_2} Dz_2 \times \\
\exp \frac{-i}{2} \left\{ \int_0^{\tau_1} d\tau_1' (\dot{z}_1'^2 + m_1^2) + \int_0^{\tau_2} d\tau_2' (\dot{z}_2'^2 + m_2^2) \right\} \langle \exp \frac{i}{2} \left( g_1 \int_0^{\tau_1} d\tau_1 A(z'_1) + g_2 \int_0^{\tau_2} d\tau_2 A(z'_2) \right) \rangle. \] (3.7)

In this case

\[
\langle \exp \frac{i}{2} \left( g_1 \int_0^{\tau_1} d\tau_1 A(z'_1) + g_2 \int_0^{\tau_2} d\tau_2 A(z'_2) \right) \rangle = \\
\exp \sum_{j=1,2} \frac{ig_j^2}{4} \int_0^{\tau_1} d\tau'_1 \int_0^{\tau'_1} d\tau''_1 [D_F(z'_1 - z''_1)] + \sum_{i=1,2} \frac{g_i^2}{2} \int d^4\xi \int d^4\eta D_F(z'_1 - \xi) \times \\
(\Delta_F^{(i)}(\xi - \eta))^2 D_F(\eta - z''_1) + \ldots + \frac{ig_1g_2}{4} \int_0^{\tau_1} d\tau_1' \int_0^{\tau_2} d\tau_2' [D_F(z'_1 - z'_2)] + \\
+ \sum_{i=1,2} \frac{g_i^2}{2} \int d^4\xi \int d^4\eta D_F(z'_1 - \xi) (\Delta_F^{(i)}(\xi - \eta))^2 D_F(\eta - z''_1) + \ldots
\] (3.8)

If we replace \( M(A) \) by 1 in (3.4) and (3.7), i.e., if we retain only the lowest order terms in the exponent in (3.8) (quenched approximation), we have no "material" fields loops in the evaluation of the gauge field average (3.6) and (3.8) and then we can write exactly

\[
G_4(x_1, x_2; y_1, y_2) = \left(\frac{1}{2}\right)^2 \int_0^{\tau_1} d\tau_1 \int_0^{\tau_2} d\tau_2 \int_{y_1}^{x_1} Dz_1 \int_{y_2}^{x_2} Dz_2 \times \\
\exp \frac{-i}{2} \left\{ \int_0^{\tau_1} d\tau_1' (\dot{z}_1'^2 + m_1^2) - \frac{g_1^2}{2} \int_0^{\tau_1} d\tau'_1 \int_0^{\tau'_1} d\tau''_1 D_F(z'_1 - z''_1) \right\} \times \\
\exp \frac{-i}{2} \left\{ \int_0^{\tau_2} d\tau_2' (\dot{z}_2'^2 + m_2^2) \right\} - \frac{g_2^2}{2} \int_0^{\tau_2} d\tau'_2 \int_0^{\tau'_2} d\tau''_2 D_F(z'_2 - z''_2) \right\} \times \\
\exp \frac{ig_1g_2}{4} \int_0^{\tau_1} d\tau_1' \int_0^{\tau_2} d\tau_2' [D_F(z'_1 - z'_2)].
\] (3.9)

Proceding as in Sec.2, using the identity

\[
\exp \frac{ig_1g_2}{4} \int_0^{\tau_1} d\tau_1' \int_0^{\tau_2} d\tau_2' D_F(z'_1 - z'_2) = \\
= 1 + \frac{ig_1g_2}{4} \int_0^{\tau_1} d\tau_1' \int_0^{\tau_2} d\tau_2' D_F(z'_1 - z'_2) \exp \left[ \frac{ig_1g_2}{4} \int_0^{\tau_1} d\tau_1'' \int_0^{\tau_2} d\tau_2'' D_F(z''_1 - z''_2) \right]
\] (3.10)
(analogous to (2.9)) and Eq.(3.3), we obtain after some manipulations

\[
G_4(x_1, x_2; y_1, y_2) = G_2(x_1 - y_1)G_2(x_2 - y_2) + \frac{ig_1g_2}{4}
\]

\[
\left(\frac{1}{2}\right)^2 \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_0^\infty d\tau_1' \int_0^\infty d\tau_2' \int d^4 z_1' \int d^4 z_2' \int_{z_1}^{x_1} D_{z_1} \int_{z_2}^{x_2} D_{z_2} \int_{z_1'}^{x_1} D_{z_1'} \int_{z_2'}^{x_2} D_{z_2'} D_F(z_1' - z_2')
\]

\[
\exp \frac{i}{2} \left\{ \int_{\tau_1}^{\tau_2} d\tau'' [\bar{z}'' - m_1^2 + \frac{g_1^2}{2} \int_{\tau_1}^{\tau_2} d\tau''' D_F(z_1'' - z_1'')] + \int_{\tau_1}^{\tau_2} d\tau'' [-\bar{z}'' - m_2^2 + \frac{g_2^2}{2} \int_{\tau_1}^{\tau_2} d\tau''' D_F(z_2'' - z_2'')] \right\} \times \exp \frac{i}{2} \left\{ \int_{\tau_1}^{\tau_2} d\tau'' [\bar{z}'' - m_1^2 + \frac{g_1^2}{2} \int_{\tau_1}^{\tau_2} d\tau''' D_F(z_1'' - z_1'')] + \int_{\tau_1}^{\tau_2} d\tau'' [-\bar{z}'' - m_2^2 + \frac{g_2^2}{2} \int_{\tau_1}^{\tau_2} d\tau''' D_F(z_2'' - z_2'')] \right\}
\]

(3.11)

Then, if we denote by \( L_1 \) the last exponential in this equation and replace it by 1, we obtain immediately the Bethe–Salpeter equation

\[
G_4(x_1, x_2; y_1, y_2) = G_2(x_1 - y_1)G_2(x_2 - y_2) + \frac{ig_2}{2} \int_{\tau_1}^{\tau_2} d\tau'' G(z_1'' - z_1')G(z_2'' - z_2') + \frac{ig_1g_2}{4} \int_{\tau_1}^{\tau_2} d\tau'' G(z_1'' - z_1')G(z_2'' - z_2') + \frac{ig_1g_2}{4} \int_{\tau_1}^{\tau_2} d\tau'' G(z_1'' - z_1')G(z_2'' - z_2') (3.12)
\]

with the ladder approximation kernel

\[
I(z_1, z_2, z_1', z_2') = -g_1g_2 D_F(z_1' - z_2')\delta^4(z_1 - z_1')\delta^4(z_2 - z_2') (3.13)
\]

On the contrary if we introduce in (3.11) the entire expansion of \( L_1 \) in the second line of the equation beside \( D_F(z_1' - z_2') \) we obtain additional terms of the type \( \frac{g_2^2}{4} \int_{\tau_1}^{\tau_2} d\tau'' \int_{\tau_1}^{\tau_2} d\tau''' D_F(z_1'' - z_1')D_F(z_1' - z_2') \), \( g_1g_2 \int_{\tau_1}^{\tau_2} d\tau'' \int_{\tau_1}^{\tau_2} d\tau''' D_F(z_1'' - z_1')D_F(z_1' - z_2') \) etc.. As a consequence (see App.A for details) we reobtain Eq.(3.12) but with the kernel

\[
\begin{align*}
I(z_1, z_2, z_1', z_2') &= -g_1g_2 D_F(z_1' - z_2')\delta^4(z_1' - z_1)\delta^4(z_2 - z_2') + ig_1^3g_2 \int d^4 \xi_1 D_F(z_1 - \xi_1)G_2(z_1 - \xi_1)G_2(\xi_1 - z_1') D_F(\xi_1 - z_2')\delta^4(z_2 - z_2') + ig_1g_2^3 \int d\xi_2 D_F(\xi_1 - \xi_2)G_2(z_2 - \xi_2)G_2(\xi_2 - z_2') D_F(z_2 - z_2') + \ldots
\end{align*}
\]

(3.14)
or, graphically:

\[
\begin{align*}
I &= 
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure1.png}
\end{array}
+ 
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure2.png}
\end{array}
+ 
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure3.png}
\end{array}
+ 
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure4.png}
\end{array}
+ \cdots
\end{align*}
\]

Finally, to go beside the quenched approximation and to take into account additional terms in Eq. (3.1), (3.8) amounts to insert $\phi_1 \phi_1$ and $\phi_2 \phi_2$ loops in all possible way inside the graph.

Had we used Eq. (3.5) in the phase space and a method analogous to that employed starting from Eq. (2.12) we would have obtained the Bethe–Salpeter equation in the momentum space

\[
\tilde{G}_4(p_1, p_2; p'_1, p'_2) = G^{(1)}_2(p_1, p'_1)G^{(2)}_2(p_2, p'_2)
- i \int \frac{d^4k_1'}{(2\pi)^4} \frac{d^4k_2'}{(2\pi)^4} \tilde{G}_2(p_1, k_1)\tilde{G}_2(p_2, k_2)\tilde{I}(k_1, k_2, k'_1, k'_2)\tilde{G}_4(k'_1, k'_2; p'_1, p'_2).
\] (3.15)

with

\[
\tilde{I}(p_1, p_2, p'_1, p'_2) = \int d^4z_1 d^4z_2 \int d^4k_1' d^4k_2' \int d^4z_1' d^4z_2' e^{-i(p_1-k_1)z_1} e^{-i(p_2-k_2)z_2}
4g_1 g_2 D\left(\frac{z_1 + z_1'}{2} - \frac{z_2 + z_2'}{2}\right) e^{i(p'_1-k_1)z_1'} e^{i(p'_2-k_2)z_2'}
\] (3.16)

where for definiteness we have assumed the mid–point prescription in the discretized form of (3.9) even if immaterial in this case. Then, introducing the total momentum $P = p_1 + p_2$ and the relative momentum $q = \frac{m_2}{m_1+m_2}p_1 - \frac{m_1}{m_1+m_2}p_2$, defining

\[
\tilde{G}_2(p, p') = (2\pi)^4 \delta^4(p - p')\tilde{G}_2(p) \quad \tilde{G}_4(p_1, p_2; p'_1, p'_2) = (2\pi)^4 \delta^4(P - P')\tilde{G}_4(q, q'; P)
\]

\[
\tilde{I}(p_1, p_2; p'_1, p'_2) = (2\pi)^4 \delta^4(P - P')\tilde{I}(q, q'; P)
\] (3.17)

and factorizing the total momentum conservation delta, we can write in conclusion

\[
\tilde{G}_4(q, q', P) = (2\pi)^4 \delta^4(q - q')\tilde{G}_2^{(1)}(q)\tilde{G}_2^{(2)}(-q) - i \int \frac{d^4k}{(2\pi)^4} \tilde{G}_2^{(1)}(q)\tilde{G}_2^{(2)}(-q)\tilde{I}(q, k, P)\tilde{G}_4(k, q', P)
\] (3.18)
and in the ladder approximation we would have obtained

\[ \hat{I}(q, q'; P) = -g_1g_2D_F(q - q') \]  

(3.19)

### IV. SCALAR QCD

Let us come to the scalar QCD characterized by the lagrangian

\[ L = (D_\rho \phi)^* D^\rho \phi - m^2 \phi^* \phi - \frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + L_{\text{GF}} \]  

(4.1)

where \( D_\rho = \partial_\rho + ieA_\rho \) and \( L_{\text{GF}} \) is the gauge fixing term. In this case an equation analogous to [3.3] can be obtained

\[ \Delta_F^{(j)}(x, y; A_\mu) = -\frac{i}{2} P_{xy} \int_0^\infty d\tau \int_x^x Dz \exp \{-i \int_0^\tau d\tau' [\frac{1}{2} (\dot{z}'^2 + m_j^2) - \dot{z}' A_\mu(z')]\} \]

\[ \Delta_F^{(j)}(y, x; A_\mu) = -\frac{i}{2} P_{yx} \int_0^\infty d\tau \int_y^y Dz \exp \{-i \int_0^\tau d\tau' [\frac{1}{2} (\dot{z}'^2 + m_j^2) + \dot{z}' A_\mu(z')]\} . \]  

(4.2)

Using (4.2) in (1.10) we have

\[ G_4^{q\bar{q}}(x_1, x_2; y_1, y_2) = \left( \frac{1}{2} \right)^2 \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_{y_1}^{x_1} Dz_1 \int_{y_2}^{x_2} Dz_2 \exp \frac{-i}{2} \left\{ \int_{\tau_1}^{\tau_2} d\tau'_1 (m_1^2 + z_1'^2) + \right. \]

\[ + \left. \int_{\tau_2}^{\infty} d\tau'_2 (m_2^2 + z_2'^2) \right\} \frac{1}{3} \langle \text{TrP} \exp [ig \int \tau \text{d}z^\mu A_\mu(z)] \rangle = \]

\[ = \left( \frac{1}{2} \right)^2 \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_{y_1}^{x_1} Dz_1 \int_{y_2}^{x_2} Dz_2 \exp iS^{q\bar{q}} \]  

(4.3)

\( S^{q\bar{q}} \) being a kind of effective \( q\bar{q} \) action. Then, in the quenched approximation, keeping \( (\ln W)_{\text{pert}} \) at the lowest order and rewriting appropriately Eq.(1.4), one has

\[ i \ln W = i \ln \left( \frac{1}{3} \text{TrP} \exp ig \int \text{d}z^\mu A_\mu(z) \right) = \frac{4}{3} g^2 \int_{\tau_1}^{\tau_2} d\tau'_1 \int_0^{\tau_1} d\tau''_1 D_{\mu\nu}(z'_1 - z''_1) \dot{z}'_1^\mu \dot{z}'_1^\nu + \]

\[ + \frac{2}{3} g^2 \int_{\tau_1}^{\tau_2} d\tau'_1 \int_0^{\tau_1} d\tau''_1 D_{\mu\nu}(z'_1 - z''_1) \dot{z}'_1^\mu \dot{z}'_1^\nu + \frac{2}{3} g^2 \int_{\tau_1}^{\tau_2} d\tau'_2 \int_0^{\tau_2} d\tau''_2 D_{\mu\nu}(z'_2 - z''_2) \dot{z}'_2^\mu \dot{z}'_2^\nu \]

\[ + \sigma \int_{\tau_1}^{\tau_2} d\tau'_1 \int_0^{\tau_2} d\tau'_2 \delta(z'_{10} - z'_{20}) |z'_1 - z'_2| \int_0^1 ds \{ \dot{z}'_{20}^{z_{10}} - (s \dot{z}'_{1T}^{z'_{10}} + (1 - s) \dot{z}'_{2T}^{z'_{10}})^2 \}^{\frac{1}{2}} \]  

(4.4)

with \( \dot{z}'_j = \frac{dz'_j}{d\tau'_j} \) as in Sec.3.

Let us now introduce the momenta \( p_{j\mu} = \frac{\delta S^{q\bar{q}}}{\delta z'_j} \).
\[
p'_{\mu_1} = \dot{z}'_{\mu_1} - \frac{4}{3} \int_0^{r_2} d\tau_2 D_{\mu\nu}(z'_1 - z'_2) \dot{z}'_{2\nu} - \frac{4}{3} \int_0^{r_1} d\tau_1 D_{\mu\nu}(z'_1 - z''_1) \dot{z}'_{1\nu} + \sigma f'_{1\mu}
\]
\[
p'_{\mu_2} = \dot{z}'_{\mu_2} - \frac{4}{3} \int_0^{r_2} d\tau_2 D_{\mu\nu}(z'_1 - z'_2) \dot{z}'_{2\nu} - \frac{4}{3} \int_0^{r_2} d\tau_2 D_{\mu\nu}(z'_1 - z'_2) \dot{z}'_{2\nu} + \sigma f'_{2\mu}
\]
(4.5)

with

\[
u' = s \dot{z}'_{20}z_{1T} + (1 - s) \dot{z}'_{10}z_{2T}
\]
\[
f'_1 = |z'_1 - z'_2| \int_0^1 ds s \frac{\nu'_s}{[\dot{z}'_{10}z_{20} - \nu'_s]^2}
\]
\[
f'_2 = |z'_1 - z'_2| \int_0^1 ds (1 - s) \frac{\nu'_s}{[\dot{z}'_{10}z_{20} - \nu'_s]^2}
\]
\[
f'_{10} = - \int_0^{r_2} d\tau_2 \delta(z'_1 - z'_2)|z'_1 - z'_2| \int_0^{r_1} ds \frac{\dot{z}'_{10}z_{20} - (1 - s) \dot{z}'_{2T} \cdot \nu'_s}{[\dot{z}'_{10}z_{20} - \nu'_s]^2}
\]
\[
f'_{20} = - \int_0^{r_2} d\tau_2 \delta(z'_1 - z'_2)|z'_1 - z'_2| \int_0^{r_1} ds \frac{\dot{z}'_{20}z_{20} - s z'_{1T} \cdot \nu'_s}{[\dot{z}'_{10}z_{20} - \nu'_s]^2}
\]
(4.6)

Eq. (4.5) cannot be inverted in a closed form, however, we can invert it by an expansion in 
\[\alpha_s = \frac{g^2}{4\pi} \text{ and } \frac{\sigma}{m^2}.\]
At the lowest order we have

\[
\dot{z}'_{1\mu} = p'_{1\mu} + \frac{4}{3} g^2 \int_0^{r_2} d\tau_2 D_{\mu\nu}(z'_1 - z'_2) p'_{2\nu} + \frac{4}{3} g^2 \int_0^{r_1} d\tau_1 D_{\mu\nu}(z'_1 - z'_1) p'_{1\nu} + \sigma \tilde{f}'_{1\mu}
\]
\[
\dot{z}'_{2\mu} = p'_{2\mu} + \frac{4}{3} g^2 \int_0^{r_2} d\tau_2 D_{\mu\nu}(z'_1 - z'_2) p'_{2\nu} + \frac{4}{3} g^2 \int_0^{r_2} d\tau_2 D_{\mu\nu}(z'_2 - z'_2) p'_{2\nu} + \sigma \tilde{f}'_{2\mu}
\]
(4.7)

with

\[
u'_s = s p'_{20}p_{1T} + (1 - s) p_{10}p'_{2T}
\]
\[
f'_1 = - |z'_1 - z'_2| \int_0^1 ds s \frac{\nu'_s}{[p'_{10}p'_{20} - \nu'_s]^2}
\]
\[
f'_2 = - |z'_1 - z'_2| \int_0^1 ds (1 - s) \frac{\nu'_s}{[p'_{10}p'_{20} - \nu'_s]^2}
\]
\[
f'_{10} = |z'_1 - z'_2| \int_0^1 ds p'_{10}p'_{20} - (1 - s) p'_{2T} \cdot \nu'_s
\]
\[
f'_{20} = |z'_1 - z'_2| \int_0^1 ds p'_{10}p'_{20} - s p'_{1T} \cdot \nu'_s
\]
(4.8)

Then we can perform the Legendre trasformation

\[
\phi^{q\bar{q}} = - \sum_{j=1,2} \int_0^{r_j} d\tau_j p'_j \cdot \dot{z}'_j - S^{q\bar{q}}_0 = \sum_{j=1,2} \int_0^{r_j} d\tau_j [- \frac{1}{2} (p^2_j + m^2_j)]
\]

14
\[
+ \frac{4}{3} g^2 \int_0^{r_1} d\tau_1^\prime \int_0^{r_2} d\tau_2^\prime D_{\mu\nu}(z_1'-z_2') p_1^{\mu'} p_2^{\nu'} \:+ \frac{2}{3} g^2 \int_0^{r_1} d\tau_1^\prime \int_0^{r_1''} D_{\mu\nu}(z_1''-z_2'') p_1^{\mu''} p_2^{\nu''} \\
+ \frac{2}{3} g^2 \int_0^{r_2} d\tau_2^\prime \int_0^{r_2''} D_{\mu\nu}(z_1'-z_2'') p_2^{\mu''} p_2^{\nu''} \\
+ \sigma \int_0^{r_1} d\tau_1^\prime \int_0^{r_2} d\tau_2^\prime \delta(z_{10}'-z_{20}') |z_1'-z_2'| \int_0^{1} ds (p_1^{\tau_2_1} p_2^{r_1} - \bar{u}_s)^{\frac{1}{2}} \] (4.9)

and set

\[
S^{q\bar{q}} = - \sum_{j=1,2} \int_0^{r_j} d\tau_j^\prime p_{j\mu} \bar{z}^\mu_j - \phi^{q\bar{q}}. \] (4.10)

In conclusion at the specified order the \( q\bar{q} \) propagator can be written as

\[
G_{4}^{q\bar{q}}(x_1, x_2; y_1, y_2) = (\frac{i}{2})^2 \int_0^{\infty} d\tau_1 \int_0^{\infty} d\tau_2 \int_0^{x_1} D_{x_1 \bar{D} p_1} \int_0^{x_2} D_{z_2 \bar{D} p_2} \exp i \{ \sum_{j=1,2} \int_0^{r_j} d\tau_j^\prime \partial_j^\prime z_j' + \\
\frac{1}{2} (p_j^{\tau_2} - m_j^2) + \frac{4}{3} g^2 \int_0^{r_1} d\tau_1 \int_0^{r_2} d\tau_2 D_{\mu\nu}(z_1'-z_2') p_1^{\mu'} p_2^{\nu'} \\
- \frac{2}{3} g^2 \int_0^{r_1} d\tau_1 \int_0^{r_1''} D_{\mu\nu}(z_1''-z_2'') p_1^{\mu''} p_2^{\nu''} - \frac{2}{3} g^2 \int_0^{r_2} d\tau_2 \int_0^{r_2''} D_{\mu\nu}(z_2'-z_2'') p_2^{\mu''} p_2^{\nu''} \\
- \sigma \int_0^{r_1} d\tau_1 \int_0^{r_2} d\tau_2 \delta(z_{10}'-z_{20}') |z_1'-z_2'| \int_0^{1} ds (p_1^{\tau_2_1} p_2^{r_1} - \bar{u}_s)^{\frac{1}{2}} + \ldots \}. \] (4.11)

In Eq. (4.11) one can proceed as in Eqs. (2.12)–(2.17) and one arrives to the BS equation (3.13) in the momentum space with a kernel given by

\[
\check{I}(p_1, p_2, p_1', p_2') = \int d^4 z_1 d^4 z_2 \int d^4 k_1 d^4 k_2 \frac{1}{(2\pi)^8} \int d^4 z_1' d^4 z_2' e^{-i(p_1-k_1)z_1} e^{-i(p_2-k_2)z_2} \\
\frac{16}{3} g^2 D_{\mu\nu}(\frac{z_1+z_1'}{2} - \frac{z_2+z_2'}{2}) k_1^{\mu'} k_2^{\nu'} + \\
4 \sigma \delta(\frac{z_1'+z_1}{2} - \frac{z_2+z_2'}{2}) |z_1+z_1'| |z_2+z_2'| \int_0^{1} ds (k_{10}k_{20} - \bar{u}_s)^{\frac{1}{2}} e^{i(p_1'-k_1)z_1'} e^{i(p_2'-k_2)z_2'} \] (4.12)

Calculating the integrals in Eq. (4.11), using the definition (3.17) and factorizing the four-momentum conservation \( \delta \) we arrive to a BS equation of the type (3.18) with a kernel given by (1.6)–(1.8).

The particular Weyl ordering in Eqs. (4.12), (1.7), (1.8) corresponds to have interpreted (1.2) at the discrete level as

\[
W \simeq \langle \text{TrP} \prod_s \exp i(x_\mu - x_{\mu-1}^\mu) A_\mu \frac{x_\mu + x_{\mu-1}}{2} \rangle \] (4.13)

Notice that \( \int x^{s+1}_s d\mu A_\mu(x) \) differs from only \( i(x_\mu - x_{\mu-1}^\mu) A_\mu \frac{x_\mu + x_{\mu-1}}{2} \) by terms of the order \( O[(x_\mu - x_{\mu-1})^3] = O(\varepsilon^3) \).
V. SALPETER POTENTIAL AND RELATIVISTIC FLUX TUBE MODEL

Let us now consider the potential to be used in the Salpeter equation or in the semirelativistic Schrödinger equation and corresponding to the kernel \( \hat{I} \) given in Eqs. (1.5)–(1.8).

In the scalar model and in the center of mass system, the standard relation occurring between the BS kernel and the potential reads

\[
\langle k | V | k' \rangle = \frac{1}{(2\pi)^3} \frac{1}{4\sqrt{w_1(k)w_2(k')w_1(k')w_2(k')}} \hat{I}_{\text{inst}}(k, k'),
\]

where \( w_j(k) = \sqrt{m_j^2 + k^2} \) and \( \hat{I}_{\text{inst}} \) denotes the so-called instantaneous kernel. Precisely \( \hat{I}_{\text{inst}} \) is obtained from \( \hat{I} \) replacing \( p_j(0) \) and \( p'_j(0) \) by appropriate functions of \( p_j \) and \( p'_j \). The simplest prescription corresponds to take

\[
p_j(0) = p'_j(0) = w_j(k) + w_j(k') - \frac{1}{2}.
\]

In the Coulomb gauge the resulting potential is

\[
\langle k | V | k' \rangle = \rho_1 \rho_2 \left\{ -\frac{1}{2\pi^2} \frac{4}{3} \alpha_s \left[ \frac{1}{(k' - k)^2} + \frac{1}{q_{10}q_{20}(k' - k)^2} \left( q_1^2 + \frac{(k - k') \cdot q}{(k - k')^2} \right) \right] + \frac{1}{(2\pi)^3} \int d^3r e^{i(k' - k) \cdot r} \frac{\sigma}{2} \frac{\sqrt{q_{10}^2 - q_T^2}}{q_{10}} - \frac{q_{10}}{q_{20}} \sqrt{q_{20}^2 - q_T^2} + \frac{\sqrt{q_T^2}}{q_{10}} \left( \text{arcsin} \left( \frac{q_T}{q_{10}} \right) + \text{arcsin} \left( \frac{q_T}{q_{20}} \right) \right) \right\} + \ldots
\]

with \( q = \frac{k + k'}{2} \) and \( q_{j0} = \frac{w_j(k) + w_j(k')}{2} \) and \( \rho_j = \frac{q_{j0}}{\sqrt{w_j(k)w_j(k')}} \). The potential (5.3) corresponds to a particular ordering prescription in the Hamiltonian (1.9) which, as already noticed, is identical to the Hamiltonian of the relativistic flux tube model [8,9]. Obviously, by an expansion in \( \frac{q^2}{m^2} \) one obtains also the semirelativistic potential derived in references [2,3]. We notice however that, due to the substitution (5.2) and then to the occurrence of the factors \( \rho_1 \rho_2 \) in (5.3), the ordering prescription does not simply coincide with the Weyl prescription given in [3].
VI. CONCLUSIONS

In conclusion we have established that it is possible to extend the Wilson loop method, used to obtain the semirelativistic potential, to the derivation of a Bethe–Salpeter equation. We stress that this amounts to show that it is possible to obtain the BS kernel from more fundamental arguments than those usually used in the phenomenological application of the BS equation. The result is strictly related to the relativistic flux tube model and to the semirelativistic potential for heavy quarks. Some additional remarks are in order.

1. Due to the difficulty in solving directly the BS–equation the general strategy in the application of such equation should be this: first solve the three–dimensional Salpeter equation for the potential (5.3), i.e. the eigenvalue equation for the Hamiltonian (1.9); then evaluate the “retardation correction” by some kind of iterative method as it is usually done in the positronium case [11] or we have done in [12].

2. In Equation (3.12) or (3.17), $G_2$ stands for the complete one particle propagator which in principle should be given by (3.5) or its counterpart for the QCD case. Due to the absence of a closed path in (3.5), in the present context such quantity should be consistently evaluated simply by its perturbative expansion in contrast with what sometimes supposed.

3. The assumption of the approximation (1.4) could seem unjustified in a relativistic treatment. Notice however that the important point in QCD is to have some zero order estimate of the interesting quantities in a formalism which already provide confinement and then to proceed by subsequent corrections.

4. Finally we notice that the kernel (1.7) is highly singular for $k' = k$, due to the occurrence of the factor $r$ under the Fourier transform. This may be inconvenient in numerical calculations and makes some equation ill defined. The problem is the same that occurs with the linear potential, if one works in the momentum
representation, and it is obviously related to confinement. Our philosophy is that one should introduce an appropriate infrared regularization (e.g. make the substitution \( r \rightarrow r e^{-\lambda r} = \frac{\partial^2}{\partial x^2} e^{-\lambda x} \)) and take the limit \( \lambda \to 0 \) only at an advanced stage of the calculation.

**APPENDIX A:**

As an example let us derive the contribution in (3.14) corresponding to the crossed diagram in Fig.1. Expanding \( L_1 \) in (3.11) we have

\[
G_4(x_1, x_2; y_1, y_2) = G_2(x_1 - y_1)G_2(x_2 - y_2) + \frac{i g_1 g_2}{4} \left( \frac{1}{2} \right)^2 \int_0^\infty d\tau_1 \int_0^\infty d\tau_1' \int_0^\infty d\tau_2 \int_0^\infty d\tau_2' \int d^4 z \int d^4 z' D_{z_1} \int d^2 z_2 D(z' - z_2)
\]

\[
\exp \frac{i}{2} \left\{ \int_{\tau_1}^{\tau_1'} d\tau_1'' [-\dot{z}_1'' - m_1^2 + \frac{g_1^2}{2} \int_{\tau_1'}^{\tau_1''} d\tau_1''' D(z_1'' - z_1''')] + \int_{\tau_2}^{\tau_2'} d\tau_2'' [-\dot{z}_2'' - m_2^2 + \frac{g_2^2}{2} \int_{\tau_2'}^{\tau_2''} d\tau_2'''] D(z_2'' - z_2''') \right\} \times
\]

\[
\left\{ 1 + \frac{g_1^2}{2} \int_{\tau_1}^{\tau_1'} d\tau_1'' \int_{\tau_2}^{\tau_2'} d\tau_2''' D(z_1'' - z_1''') + \frac{g_2^2}{2} \int_{\tau_2}^{\tau_2'} d\tau_2'' \int_{\tau_1}^{\tau_1'} d\tau_1''' D(z_2'' - z_2''') + \right.\]

\[
\left. + \frac{g_1 g_2}{2} \int_{\tau_1}^{\tau_1'} d\tau_1'' \int_{\tau_2}^{\tau_2'} d\tau_2''' D(z_1'' - z_2''') + \ldots \right\}.
\]

The contribution corresponding to the crossed diagram (CD) is that coming from the last term in inside the curl bracket in (A1). We can write

\[
\text{CD} = \frac{i g_1 g_2}{4} \left( \frac{1}{2} \right)^2 \int_0^\infty d\tau_1 \int_0^\infty d\tau_1' \int_0^\infty d\tau_2 \int_0^\infty d\tau_2' \int d^4 z \int d^4 z' D(z_1 - z_2) D(z_1' - z_2')
\]

\[
\exp \frac{i}{2} \left\{ \int_{\tau_1}^{\tau_1'} d\tau_1'' [-\dot{z}_1'' - m_1^2 + \frac{g_1^2}{2} \int_{\tau_1'}^{\tau_1''} d\tau_1''' D(z_1'' - z_1''')] + \int_{\tau_2}^{\tau_2'} d\tau_2'' [-\dot{z}_2'' - m_2^2 + \frac{g_2^2}{2} \int_{\tau_2'}^{\tau_2''} d\tau_2'''] D(z_2'' - z_2''') \right\} \times
\]

\[
\left\{ 1 + \frac{g_1^2}{2} \int_{\tau_1}^{\tau_1'} d\tau_1'' \int_{\tau_2}^{\tau_2'} d\tau_2''' D(z_1'' - z_1''') + \frac{g_2^2}{2} \int_{\tau_2}^{\tau_2'} d\tau_2'' \int_{\tau_1}^{\tau_1'} d\tau_1''' D(z_2'' - z_2''') + \right.\]

\[
\left. + \frac{g_1 g_2}{2} \int_{\tau_1}^{\tau_1'} d\tau_1'' \int_{\tau_2}^{\tau_2'} d\tau_2''' D(z_1'' - z_2''') \right\} =
\]

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\[
\int_0^\infty d\tau_1'' \int_0^\infty d\tau_1' \int_0^\infty d\tau_2 \int_0^\infty d\tau_2'' \int_0^\infty d\tau_2' \\
\int_{\tau_1'}^{\tau_1} D_{\tau_1} \int_{\tau_2'}^{\tau_2} D_{\tau_2} \int_{\tau_2''}^{\tau_2} D_{\tau_2} \int_{\tau_2''}^{\tau_2} D_{\tau_2} \int_{\tau_2}^{\tau_2} D_{\tau_2} \int_{\tau_2}^{\tau_2} D_{\tau_2} \\
\exp \frac{i}{2} \left\{ \int_{\tau_1'}^{\tau_1} d\tau_1'' [-\dot{z}_1'' - m_1^2 + \frac{g_1^2}{2} \int_{\tau_1'}^{\tau_1} d\tau_1''' D(z_1''' - z_1'')] + \\
+ \int_{\tau_1'}^{\tau_1} d\tau_1'' [-\dot{z}_1'' - m_1^2 + \frac{g_1^2}{2} \int_{\tau_1'}^{\tau_1} d\tau_1''' D(z_1''' - z_1'')] \right\} \times \\
\exp \frac{i}{2} \left\{ \int_{\tau_1''}^{\tau_1'} d\tau_1''' [-\dot{z}_1''' - m_1^2 + \frac{g_1^2}{2} \int_{\tau_1''}^{\tau_1'} d\tau_1''' D(z_1''' - z_1'')] \right\} \times \\
\exp \frac{i}{2} \left\{ \int_{\tau_2'}^{\tau_2} d\tau_2''' [-\dot{z}_2''' - m_2^2 + \frac{g_2^2}{2} \int_{\tau_2'}^{\tau_2} d\tau_2''' D(z_2''' - z_2'')] \right\} \times \\
\exp \frac{i}{2} \left\{ \int_{\tau_2'}^{\tau_2} d\tau_2''' [-\dot{z}_2''' - m_2^2 + \frac{g_2^2}{2} \int_{\tau_2'}^{\tau_2} d\tau_2''' D(z_2''' - z_2'')] \right\} \times \\
\exp \left\{ \frac{g_1}{2} \int_{\tau_1}^{\tau_1'} d\tau_1''' \int_{\tau_2}^{\tau_2} d\tau_2''' D(z_1''' - z_2''') \right\} \times \\
\exp \left\{ \frac{g_2}{2} \int_{\tau_1}^{\tau_1'} d\tau_1''' \int_{\tau_2}^{\tau_2} d\tau_2''' D(z_1''' - z_2''') \right\} \times \\
\exp \left\{ \frac{g_1}{2} \int_{\tau_1}^{\tau_1'} d\tau_1''' \int_{\tau_2}^{\tau_2} d\tau_2''' D(z_1''' - z_2''') \right\}.
\tag{A2}
\]

In conclusion, replacing again the last exponential \(L_2\) in (A2) by 1, we have

\[
CD = ig_1^2 g_2^2 \int d^4 z_1' \int d^4 z_2' \int d^4 z_1'' \int d^4 z_2'' D(z_1' - z_2') D(z_1'' - z_2'') G_2(x_1 - z_1') G_2(z_1' - z_1'') \\
G_2(x_2 - z_2') G_2(z_2'' - z_2') G_4(z_1', z_1'', z_2', y_1, y_2). \tag{A3}
\]

The method can be immediately extended to higher order terms or iterated on (A.2) expanding even \(L_2\).
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