_interval hybrid judgment priority vector method in group
decision

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Abstract. The uncertainty and subjectivity of the judgment matrix given by the experts
according to the evaluation will directly affect the rationality of the priority vector. The interval
hybrid judgment matrix can objectively reflect the preference information of experts. Therefore,
combined with the theory of maximum satisfaction, CMIHMSM and CMIHMSM are
established, and a consistency index is proposed to test the consistency of the matrix. On this
basis, group decision making is introduced. Based on the analysis of the consistency
of the matrix, expert weights are divided into inter-class weight and intra-class weight
calculations, so that expert weight under the same consistency of the same type of expert
judgment matrix can be distinguished. Thus, the group priority vector is obtained. Example
shows the feasibility and operability of the model.

1. Introduction
Due to the uncertainty of the problem and the ambiguity of their own thinking, experts may not be
able to express preference information with accurate numerical values, but can only provide judgment
information in the form of intervals[1]. The research on the theory and method of judgment matrix
priority mainly focuses on interval reciprocal, complementary judgment matrix[2-4] or fixed-value
mixed judgment matrix. But there are few theoretical methods to solve the priority vector of interval
hybrid judgment matrix[5-6].

Therefore, combined with the maximum satisfaction method (MSM)[7], the group interval hybrid
judgment matrix model is established, and CMIHMSM and CMIHMSM are used to solve individual
priority vector of experts. Then group decision is introduced. Based on cluster analysis and
consistency of matrix, the expert weights are divided into inter-class weight and intra-class weight
calculations, so that the individual priority vectors are weighted to obtain the group priority vector.

2. Consistency of interval number judgment matrix
Assuming that $\bar{A} = \left(\bar{a}_{ij}\right)_{n \times n}$ is an interval hybrid judgement matrix, the priority vector is
$W = (w_1, w_2, ..., w_n)^T$, with $\sum_{i=1}^{n} w_i = 1$. In the i-th row, the column that belongs to the interval complementary
judgment information is marked as $J_i$, and the column that belongs to the interval reciprocal judgment
information is marked as $I_i$.

According to the definition of interval reciprocal or complementary consistency, the definition of
interval hybrid judgment matrix satisfying consistency is as follows:
Definition 1: When \( l_y \leq \frac{w_i}{w_j} \leq u_y, j \in I_i \) and \( l_y \leq \frac{w_i + u_i}{w_j} \leq u_y, j \in J_i \) are established, it can be considered that \( A \) achieves multiplicative consistency.

Definition 2: When \( l_y \leq \frac{w_i}{w_j} \leq u_y, j \in I_i \) and \( l_y \leq 0.5 \left( w_i - w_j \right) + 0.5 \leq u_y, j \in J_i \) are established, it can be considered that \( A \) achieves additive consistency.

3. Interval hybrid judgment priority vector method in group decision

3.1. MSM model of interval hybrid judgment matrix

3.1.1. CMIHMSM. When \( A \) satisfies multiplicative consistency, formula (1) must hold:

\[
\begin{cases}
-w_i + l_y w_j \leq 0 \\
w_i - u_y w_j \leq 0 \\
(1 \cdot l_y) w_i + l_y w_j \leq 0 \\
1 - u_y w_i - u_y w_j \leq 0 \\
0.5 \leq l_y \leq 0.5 \\
-j \in I_i, i = 1, 2, ..., n - 1, j > i
\end{cases}
\]

(1)

The formula (1) is represented by \( R_y (w) \), where the membership function of \( R_y (w) \) is shown in formula (2):

\[
u_y (w) = \begin{cases}
1 & R_y (w) \leq 0 \\
1 - \frac{R_y (w)}{d_y} & 0 < R_y (w) \leq d_y, i = 1, 2, ..., n - 1; j = 2, 3, ..., n; j > i \\
0 & R_y (w) \geq d_y
\end{cases}
\]

(2)

Where \( d_y \) represents the deviation factor under allowed satisfaction, \( d_y \geq 0 \).

Based on the principle of maximum and minimum, the membership function of fuzzy feasible region is:

\[
\delta = \mu_y (w) = \max_{\omega \in \Omega} \min \{ \mu_y (w) | i = 1, 2, ..., n - 1; j = 2, 3, ..., n; j > i \}
\]

(3)

Convert formula (3) to a linear optimization problem:

\[
\max \delta \\
\text{s.t. } d_y \delta - w_i + l_y w_j \leq d_y \\
d_y \delta + w_i - u_y w_j \leq d_y, j \in I_i \\
d_y \delta - (1 - l_y) w_i + l_y w_j \leq d_y \\
d_y \delta + (1 - u_y) w_i - u_y w_j \leq d_y, j \in J_i, i = 1, 2, ..., n - 1, j > i \\
\sum_{i=1}^{n} w_i = 1, w_i > 0, i = 1, 2, ..., n; \delta \leq 1
\]

(4)

According to the above formula, the priority vector can be obtained. This method is called consistent multiplicative interval hybrid maximum satisfaction method (CMIHMSM).

3.1.2. CMIHMSM. Interval mixed judgment matrix with additive consistency \( A \), Formula (5) is bound to hold true:

\[
\begin{cases}
\overline{\delta}_y \equiv \frac{w_i}{w_j}, j \in I_i \\
\underline{\delta}_y \equiv 0.5 \left( w_i - w_j \right) + 0.5, j \in J_i
\end{cases}
\]

(5)
Substituting formula (5) into formulas (2-3), the linear programming model of the additive interval mixed judgment matrix can be obtained as follows:

\[
\begin{align*}
\text{max} & \quad \delta \\
\text{s.t.} & \quad d_y \delta - w_i y w_j \leq d_y \\
& \quad d_y \delta + w_i y w_j \leq d_y, j \in I_y \\
& \quad d_y \delta + 0.5 w_i y - 0.5 w_j \leq d_y + u_j - 0.5 \\
& \quad d_y \delta - 0.5 w_i y + 0.5 w_j \leq d_y - l_j + 0.5, j \in J_y, i = 1,2...n, j > i \\
\sum_{i=1}^{n} w_i = 1, w_i > 0, i = 1,2...n \\
\delta & \leq 1
\end{align*}
\]

(6)

According to the above formula, the priority vector can be obtained, this method is called consistent additive interval hybrid maximum satisfaction method (CAIHMSM).

Theorem 1: Assuming that it is the optimal solution of formula (4) or (6), the consistency index of the matrix is defined as:

\[
\delta_{\text{max}} = -(\delta^* - 1)d = \max \left\{ \left[ R_j \left( w^* \right) \right] \right\}, i = 1,2...n, j = 2,3...n, j > i
\]

(7)

The better the consistency of the matrix, the smaller the \(\delta_{\text{max}}\); the larger the \(\delta_{\text{max}}\), the worse the consistency of the matrix.

3.2. Determination of Experts’ Weight in Group Decision

The weight of experts is related to the consistency and similarity of the judgment matrix. Therefore, the cluster analysis is used to classify experts. Generally, experts with similar judgment information are classified into the same category. The more experts belonging to the same category, it indicates that they have a higher consensus. The weight given to experts is called inter-class weight. After dividing the types of experts, the consistency of the same type of expert matrix is analyzed, and the class core and different individual priority vector spacing are considered to determine intra-class weight, and then the expert weight is obtained.

3.2.1. Inter-class weight. Suppose the experts participating in the evaluation are \(E = \{E_1, E_2, ..., E_n\}\), \(A^{(i)} = (a_{ij}^{(i)})_{m \times n}\) is the judgment matrix given by the experts, substituting them into CAIHMSM and CAIHMSM, the priority vector selected optimally according to the consistency index \(\delta_{\text{max}}\) are \(W^{(i)} = (w_1, w_2, ..., w_n)^T\), \(\sum_{i=1}^{n} w_i = 1\). The Euclidean distance is used to calculate the similarity measure of \(W^{(i)}\) and \(W^{(j)}\) by Formula (8):

\[
d(i, j) = \sum_{k=1}^{n} (w_{ik} - w_{jk})^2
\]

(8)

When \(d(i, j) \leq T\), it is considered that \(E_i\) and \(E_j\) can be clustered into a class, where \(T\) is the set threshold. The inter-class weights are shown in Equation (9).

\[
\lambda^k = \frac{\psi^k_i}{\sum_{k=1}^{\psi} \psi^k_k}
\]

(9)

Where \(\psi_k\) is the number of experts in the \(G_k\) cluster.
3.2.2. Intra-class weight. Suppose the cluster center of the $G_k$ cluster is $W^k = (w^k_1, w^k_2, ..., w^k_n)^T$, where $w^k_i = \frac{\sum_{l \in G_k} w_{il} \psi_{il}}{\psi_{kk}}$, $d(i, W^k) = \sum_{j=1}^{n} (w_{ij} - w^k_j)^2$. The intra-class weight of experts included in the $G_k$ cluster is represented by $\omega_i$, $\sum_{E_i \in G_k} \omega_i = 1$, the calculation formula is as follows:

$$
\omega_i = \mu f\left(d_{i0}\right) + (1 - \mu)g\left[d(i, W^k)\right]
$$

(10)

$$
f\left(d_{i0}\right) = \frac{1}{\sum_{E_i \in G_k} 1 + ad_{i0}}
$$

(11)

$$
g\left[d(i, W^k)\right] = \frac{1}{\sum_{E_i \in G_k} 1 + bd(i, W^k)}
$$

(12)

Where $\mu$ indicates the proportion of consistency of the matrix and similarity between the vector and the cluster center. When $\mu=0.5$, it means that the two are of equal importance. In formula (11-12), $a, b$ is the artificially assumed scale factor.

Considering the inter-class and intra-class weights, the weight of each expert is:

$$
\lambda_i = \lambda^k \cdot \omega_i
$$

(13)

The final weight is $W = (w_1, w_2, ..., w_n)^T$, $w_i = \sum_{j=1}^{n} (w_{ij} \cdot \lambda_j)$, $i = 1, 2, ..., n$

4. Numerical examples

Seven experts are hired to score the 3 elements of a certain criterion level, and their judgment matrices are:

$$
A' = \begin{bmatrix}
1 & 1 & 3 \\
4 & 3 & 3 \\
0.2 & 0.3 & 1 \\
\end{bmatrix}
$$

$$
A' = \begin{bmatrix}
1 & 1 & 3 \\
4 & 3 & 3 \\
0.2 & 0.3 & 1 \\
\end{bmatrix}
$$

$$
A' = \begin{bmatrix}
1 & 1 & 3 \\
4 & 3 & 3 \\
0.2 & 0.3 & 1 \\
\end{bmatrix}
$$

1) Calculate the priority vector of the interval mixed judgment matrix

Use the CMIMHSM and CAIHMSM models to solve the above interval mixed judgment matrix. The calculation results are shown in Table 1.
Table 1. Priority vector and consistency index.

| $E_i$ | CMIHMSM | CAIHMSM | CMIHMSM | CAIHMSM | CMIHMSM | CAIHMSM | CMIHMSM | CAIHMSM |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 1     | 0.071     | 0.002     | 0.673     | 0.698     | 0.248     | 0.238     | 0.080     | 0.068     |
| 2     | 0.098     | 0.050     | 0.732     | 0.750     | 0.207     | 0.2       | 0.061     | 0.05      |
| 3     | 0         | 0         | 0.571     | 0.580     | 0.286     | 0.288     | 0.143     | 0.132     |
| 4     | 0         | 0         | 0.526     | 0.611     | 0.305     | 0.244     | 0.169     | 0.145     |
| 5     | 0.087     | 0.042     | 0.725     | 0.744     | 0.203     | 0.197     | 0.073     | 0.060     |
| 6     | 0.143     | 0.119     | 0.722     | 0.733     | 0.173     | 0.170     | 0.105     | 0.096     |
| 7     | 0         | 0         | 0.510     | 0.534     | 0.355     | 0.345     | 0.135     | 0.122     |

Compared with the consistency index $d_{\delta_0}$, CMIMHSM is more applicable.

2) Calculate inter-class weight of experts

Bring the above priority vector into formula (8) to analyze the similarity between the vectors, and $d_{\max} = d(2,5)$ is obtained, experts $E_2$ and $E_5$ are classified into same category.

Compare the value of the distance between the priority vectors of other experts and the priority vectors of experts $E_2$, $E_5$, and take the threshold $\tau = 0.02$, experts are finally divided into three categories, namely: $G_1 = \{E_1, E_2, E_5\}$, $G_2 = \{E_3, E_4, E_7\}$, $G_3 = \{E_6\}$.

The inter-class weight corresponding to each type of expert is:

$\lambda^1 = \frac{9}{19}$, $\lambda^2 = \frac{9}{19}$, $\lambda^3 = \frac{1}{19}$

3) Calculate intra-class weight of experts

The core of the class $G_i$ expert priority vector is: $W^{\chi^i} = (0.731, 0.212, 0.058)^T$; the core of the class $G_2$ expert priority vector is: $W^{\chi^2} = (0.575, 0.292, 0.133)^T$. Take the artificially assumed scale factor as: $a = 10, b = 10, \mu = 0.5$. Putting the values of the priority vector and the consistency index calculated above into formula (10-12), the result is as follows:

$$d\left(d_{\delta^1_0}\right) = 0.384$$
$$d\left(d_{\delta^2_0}\right) = 0.3$$
$$d\left(d_{\delta^1_0}\right) = 0.316$$
$$f(1,W^{\chi^1}) = 0.002$$
$$f(2,W^{\chi^1}) = 0.01$$
$$f(5,W^{\chi^1}) = 0.0004$$

$$g\left[d(1,W^{\chi^1})\right] = 0.33$$
$$g\left[d(2,W^{\chi^1})\right] = 0.335$$
$$g\left[d(5,W^{\chi^1})\right] = 0.335$$

$\omega_1 = 0.357$ $\omega_2 = 0.317$ $\omega_3 = 0.337$ $f(d_{\delta^1_0}) = \frac{1}{3}$ $f(d_{\delta^2_0}) = \frac{1}{3}$ $f(d_{\delta^1_0}) = \frac{1}{3}$

$$d(3,W^{\chi^2}) = 0.00001$$
$$d(4,W^{\chi^2}) = 0.0037$$
$$d(7,W^{\chi^2}) = 0.0046$$

$$g\left[d(3,W^{\chi^2})\right] = 0.342$$
$$g\left[d(4,W^{\chi^2})\right] = 0.330$$
$$g\left[d(7,W^{\chi^2})\right] = 0.328$$

$\omega_1 = 0.339$ $\omega_4 = 0.332$ $\omega_3 = 0.330$

4) Expert's final weight

$\lambda_1 = 0.169$ $\lambda_2 = 0.150$ $\lambda_3 = 0.160$ $\lambda_4 = 0.157$ $\lambda_5 = 0.053$ $\lambda_7 = 0.157$

5) In summary, the comprehensive priority vector is $W = (0.660, 0.249, 0.096)^T$.

5. Conclusion

The model can not only solve the priority vector of the interval hybrid judgment matrix, but also can check the consistency of the matrix through the consistency index. This model can also be applied to judgment matrices with unreasonable assignments of some elements, incomplete elements or fixed values. Experts are divided into inter-class and intra-class weights for calculation, which can distinguish the weights of the same type of expert judgment matrix under the same consistency, but the threshold of the consistency index can be further studied.
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