Astrometric tests of General Relativity in the Solar System: mathematical and computational scenarios

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Abstract. We review the mathematical models available for relativistic astrometry, discussing the different approaches and their accuracies in the context of the modern experiments from space like Gaia and GAME, and we show how these models can be applied to the real world, and their consequences from the mathematical and numerical point of view, with specific reference to the case of Gaia, whose launch is due before the end of the year.

1. Introduction
Tests of gravity theories within the Solar System are usually analyzed in the framework of the so-called Parametrized Post-Newtonian framework which enables the comparison of several theories through the estimation of the value of a limited number of parameters. Among these parameters, γ and β are of particular importance for astrometry since they are connected with the classical astrometric phenomena of the light deflection and of the excess of perihelion precession in the orbits of massive objects. The same parameters are of capital importance in fundamental physics, for the problem of characterizing the best gravity theory, and for the Dark Energy/Dark Matter [1, 2]. Moreover, precise astrometric measurements are also important in other tests of fundamental physics [3–5] since, e.g., they have the potentiality to improve on the ephemeris of the Solar System bodies. These are the reasons why Solar System astrometric experiments like Gaia and other projects presently under study [6–8] have received a particular attention from the community of fundamental physicists and cosmologists.

Such kind of experiments, however, call for a reliable model applicable to the involved astrometric measurements which has not only to include a correct relativistic treatment of the propagation of light, but a relativistic treatment of the observer and of the measures as well.

At the same time, the large amount of data to be processed, and the complexity of the problem to be solved, call for the use of High-Performance Computing (HPC) environments in the data reduction.

2. Astrometric models
The development of an astrometric model based on a relativistic framework can be dated back to at least 25 years ago [9, 10]. In their seminal work of 1992 Klioner and Kopeikin [11]
described a relativistic astrometric model accurate to the \( \mu \)as level foreseen for the next generation astrometric missions. This model is built in the framework of the post-Newtonian (pN) approximation of GR, where the finite dimensions and angular momentum of the bodies of the Solar System are included and linked to the motion of the observer in order to consider the effects of parallax, aberration, and proper motion, and the light path is solved using a matching technique that links the perturbed internal solution inside the near zone of the Solar System with the assumed flat external one. The light trajectory is solved in a perturbative way, as a straight line plus integrals containing the perturbations which represent, i.e., the effects of the aberrational terms, of the light deflection, etc. An extension of this model accurate to 1 \( \mu \)as called GREM (Gaia RElativistic Model) was published in 2003 [12]. This has been adopted as one of the two model for the Gaia data reduction, and it is formulated according to the PPN (Parametrized Post-Newtonian) formalism in order to include the estimation of the \( \gamma \) parameter.

A similar approach was followed by Kopeikin, Schäfer, and Mashhoon [13, 14] in the post-Minkowskian approximation. In this case, however, the authors used a Liénard-Wiechert representation of the metric tensor to describe a retarded type solution of the gravitational field equations and to avoid the use of matching techniques to solve the geodesic equations.

RAMOD (Relativistic Astrometric MODEL) is another family of models, whose development started in 1995. In this approach the definition of the observable according to the theory of measure [15] and the immediate application to the problem of the astrometric sphere reconstruction was privileged. As a consequence, it started as a simplified model [16] based on a plain Schwarzschild metric. Further enhancement brought to the first realistic estimation of the performances of Gaia for the determination of the PPN \( \gamma \) parameter [17], and to the development of a fully accurate N-Body model of the light propagation and of an observer suitable for application to space missions [18,19]. Since the so-called RAMOD3 [20], this model was built on a complete pM background, and the light propagation was described with the equation of motion of measurable quantities varying all along the geodesic connecting the starting point to the observer. This approach brought to a specific form of the geodesic equations as a set of coupled nonlinear differential equations which could be solved only by numerical integration. This represented a problem for an extensive application of this model to practical astrometric problems, which has been solved only recently for RAMOD3 by Crosta [21] who applied a re-parametrization of these equations of motion to demonstrate their equivalence to the model in [14], thus opening the road to an analytical solution of RAMOD3 [22] and to its full application to astrometry problems.

Finally, another class of models based on the Time Transfer Function (TTF) technique, has been developed since 2004 [23]. The TTF formalism stands as a development of the Synge World Function [24] which, contrary to all the method described so far, is an integral approach based on the principle of least action. In this models one does not solve the system of differential equations of the geodesic equations, and thus does not retrieve the solution of the equations of motion of the photons, but it concentrates on obtaining some essential information about the propagation of these particles between two points at finite distance; the coordinate time of flight, the direction triple of the light ray at either the point of emission (\( A \)) and of reception (\( B \)), and the ratio \( K \equiv (k_0)_B / (k_0)_A \) of the temporal components of the tangent four-covector \( k_\alpha \), which is related to the frequency shift of a signal between two points.

### 3. Analytical and numerical comparison

All these models are conceived to be used at least at the \( \mu \)as level, suitable for the accuracy foreseen by future astrometric experiments like Gaia and GAME [6,8]. Nonetheless it has to be considered that, because of the unprecedented level of accuracy which is going to be reached, both the astrometric models and the data processing software will be applied for the first time to a real case. Moreover, in the case of Gaia this problem is even more delicate since here the
satellite is self-calibrating and will perform absolute measurements which is equivalent to the definition of a unit of measure. These are some of the reasons why extensive analytical and numerical comparisons among the different models are being conducted.

From the theoretical and analytical point of view, a first comparison was conducted in [25] showing that GREM and RAMOD3 have an equivalent treatment of the aberration. Later the equivalence of RAMOD3 and the model in [13, 14] at the level of the (differential) equations of motion has then been shown in [21], while the explicit formulae for the light deflection and the flight time of GREM, RAMOD3, and TTF was compared in [26] where it is demonstrated the equivalence of TTF and GREM at 1PN in a time-dependent gravitational field and that of TTF and RAMOD in the static case.

Numerical comparisons between the GREM and the pM models showed that they give the same results at the sub-µas level [27]. On the other side, GREM has been compared with a low-accuracy ((v/c)^2) version of RAMOD proving that even a relatively unsophisticated modeling of the planetary contributions can take into account of the light deflection up to the µas level almost everywhere in the sky. This means that, in principle, some experiments like the reconstruction of the global astrometric sphere of Gaia could initially be done by (v/c)^2 models.

Both the analytical and the numerical comparison, however, showed that the correct computation of the retarded distance of the (moving) perturbing bodies is fundamental to achieve the required accuracy.

4. Data reduction algorithms: the case of Gaia

The reduction of the data coming from astrometric missions bring to the attention of the scientific community another kind of new problems, i.e. those connected to the need of reducing a huge amount of astrometric data in ways that were never experienced before. A significant example is given by the problem of the reconstruction of the global astrometric sphere in the Gaia mission.

From a mathematical point of view, the satellite observations translate into a large number of equations, linearized with respect to the unknown parameters around known initial values, which constitute an overdetermined and sparse equations system that is solved in the least-squares sense to obtain the astrometric catalog with its errors. In the Gaia mission these tasks are done by the Astrometric Global Iterative Solution (AGIS) but the international consortium which is in charge of the reduction of the Gaia data decided to produce also an independent sphere reconstruction named AVU-GSR.

This was motivated by the absolute character of these results, and by uniqueness of the problem which comes from several factors, the main being represented by the dimensions of the system which are of the order of 10^{10} × 10^{8}. A brute-force solution of such system would require about 10^{27} FLOPs, a requirement which cannot be decreased at acceptable levels even considering that the sparsity rate of the reduced normal matrix is of the order of 10^{-6}. It is therefore necessary to resort to iterative algorithms.

AGIS uses additional hypotheses on the correlations among the unknowns which are reflected on the convergence properties of the system and permit a separate adjustment of the astrometric, attitude, instrument calibration, and global parameters, allowing the use of an embarrassingly parallel algorithm [28]. The starting hypotheses, however, can hardly be proved rigorously, and have only be verified “a posteriori” by comparing the results with simulated true values, a situation which cannot hold in the operational phase with real data. Moreover, this method by definition prevents the estimation of the correlations between the different types of unknown parameters, which constitute the other unique characteristic of this problem. These considerations about the AGIS module lead to the solution followed by AVU-GSR, which uses a modified LSQR algorithm [29]) to solve the system of equations which, however, cannot be solved without resorting to HPC parallel programming techniques as explained in [30].
5. Conclusions
The increasing precision in the modern astrometric measurements from space makes high-accuracy tests of the DM/DE vs. Gravity theory debate a target accessible to future space-born astrometric missions. To this aim, viable relativistic astrometric models are needed, and three classes of models have been developed during the last two decades. Work is still ongoing to cross-check their mutual compatibility at their full extent, but what have been done so far demonstrated that they are equivalent at least at the level of accuracy required for the Gaia measurements. At the same time these missions put new challenges to the efforts of data reduction. We have briefly shown how the problem was faced in Gaia, in the limited contest of the reconstruction of the global astrometric sphere, where an additional constraint is put by the absolute character of its main product.

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