O’Grady’s Birational Maps via Wall-hitting

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Abstract
We observe that O’Grady’s birational maps [O’G97] between moduli of sheaves on an elliptic K3 surface can be interpreted as intermediate wall-crossing (wall-hitting) transformations at the so-called totally semistable walls, studied by Bayer and Macrì [BM14b]. As an ingredient to prove this observation, we describe the first totally semistable wall for ideal sheaves of $n$ points on the elliptic K3. We then use this observation to make a remark on Marian and Oprea’s strange duality [MO13].

1 Introduction
In [O’G97], K. O’Grady constructed, in particular, a series of birational maps

$$\psi_r : M_H(v_r) \to M_H(v_{r+1}),$$

where $M_H(v_r)$ is the moduli of Gieseker $H$--semistable rank $r$ sheaves on an elliptic K3 surface and of class $v_r \in H^*(X,\mathbb{Z})$. In [BM14b], A. Bayer and E. Macrì described the wall-crossing behavior of moduli of complexes on a K3 surface $X$ with fixed class $v \in H^*(X,\mathbb{Z})$, which in particular revealed the birational geometry of moduli of sheaves on $X$. The main goal of this note is to show that O’Grady’s birational maps appear as intermediate wall-crossing (also refer to as wall-hitting) transformations at totally semistable walls.

Our motivation comes from the study of strange duality for K3 surfaces: in [MO13], A. Marian and D. Oprea interpret O’Grady’s birational maps, from $M_H(v_r)$ to Hilbert scheme of points $X^a$, as a Fourier-Mukai transform and use that to propagate strange duality isomorphisms on Hilbert schemes of points to a large class of pairs of moduli spaces of higher rank sheaves; on the other hand, work of Bayer and Macrì, [BM14a] and [BM14b], enable one to extend strange duality to Moduli of complexes setting and to study them by wall-crossing. Their work indicates that crossing totally semistable walls is particularly interesting for strange duality. As an application of our main theorem, we obtain new examples of strange duality for elliptic K3 surfaces, based on a result of Marian and Oprea.

1.1 O’Grady’s birational maps as wall-crossing transformations. We recall the construction in [O’G97], see also [MO13]. Assume that $\pi : X \to \mathbb{P}^1$ is an elliptic K3 surface whose Néron-Severi group is $NS(X) = \mathbb{Z}c \oplus \mathbb{Z}f$, where $c$ and $f$ are the classes of a section and a fiber of $\pi$, respectively. Therefore, we have $c^2 = -2$, $c \cdot f = 1$, $f^2 = 0$.

Now fix a Mukai vector $v_r = (r, c + kf, p) \in H^0(X,\mathbb{Z}) \oplus H^{1,1}(X,\mathbb{Z}) \oplus H^1(X,\mathbb{Z})$, choose a polarization $H := c + mf$, where $m$ is sufficiently large. Suppose that $F_r$ is a $H$--stable torsion-free sheaf with class $v(F_r) = v_r$. Twisting by $\mathcal{O}(f)$ if necessary, we may assume $\chi(F_r) = -1$. It can be shown that $\text{Ext}^1(F_r, \mathcal{O}_X) \cong \mathbb{C}$ for $F_r \in M_H(v_r)$ general in moduli. The corresponding nontrivial extension

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produces a rank $r + 1$ sheaf $F_{r+1}$ which is $H$-stable provided that $F_r \in M_H(v_r)$ is away from a particular codimension one locus. Thus we obtain a birational map

$$\psi_r : M_H(v_r) \dasharrow M_H(v_{r+1}).$$

On the other hand, in [BM14b] Bayer and Macrì study systematically a wall-crossing behavior known as totally semistable wall-crossing. (In the case of skyscraper sheaves, this kind of walls had been discovered and exploited by Bridgeland [Bri08].) At a totally semistable wall $W$ for a class $v$, the moduli spaces $M_{\sigma_+}(v)$ and $M_{\sigma_-}(v)$ on the two sides share no common objects, where $\sigma_+$ and $\sigma_-$ are two stability conditions separated by $W$. However, they are linked via an intermediate moduli $M_{\sigma_0}(v_0)$, which parametrizes objects of another class $v_0$ and stable with respect a stability condition $\sigma_0 \in W$. More precisely, there are birational maps

$$M_{\sigma_+}(v) \dasharrow M_{\sigma_-}(v) \dasharrow M_{\sigma_0}(v_0),$$

Moreover, these birational maps $\phi_{\pm}$ are induced by spherical twists or inverse spherical twists, see section 2 for detail. We refer to $\phi_{\pm}$ as wall-hitting transformations at the wall $W$.

Our observation is that the extension $(\star)$ inducing O’Grady’s maps is precisely the defining triangle of an inverse spherical twist:

$$ST^{-1}_{O_X}(F_r) \rightarrow F_r \rightarrow \bigoplus_i R^i \text{Hom}(O_X,F_r) \otimes O_X[2-i] \cong O_X[1].$$

One sees that $F_{r+1} \cong ST^{-1}_{O_X}(F_r)$. Besides, up to twisting by $O(f)$, we have

$$(v(O_X),v_{r+1}) = -\chi(O_X,F_{r+1}) = -1,$$

which satisfies a lattice-theoretic criterion for a totally semistable wall of $v_{r+1}$ to occur, according to Bayer-Macrì’s classification of walls (see proposition 2.19).

With these coincidences, it is natural to conjecture that O’Grady’s birational maps $\psi_r$ are in fact the wall-hitting transformations at some totally semistable walls. We show that this is true. Moreover, one can manage to pass these walls consecutively:

**Theorem 1.1.** For any integer $r \geq 1$, there exists a path $\gamma$ in the space of stability conditions $\text{Stab}^1(X)$, starting from the Gieseker chamber of $v_r$, and passing through a wall $W_i$ for each $v_i$, $1 \leq i \leq r$, in the decreasing order (see fig. 7), such that $W_i$ is the first totally semistable of $v_i$ along $\gamma$. Moreover, one of the wall-hitting transformations at $W_i$, $1 \leq i \leq r$, can be identified with O’Grady’s birational maps $\psi_{r-1}^{-1} : M_H(v_{r-1}) \dasharrow M_H(v_{r-1})$ generically.

Again $H := c + mf$ is an ample class with $m \gg 0$. It determines a two dimensional slice (see fig. 4) of stability conditions $P_H := \{\sigma_{n_H,U} : u \in \mathbb{R}, t > 0\}$ (definition 2.9), on which the walls $W_i$, $1 \leq i \leq r$ are nested semicircles, as we will see in lemma 3.10. (Note that these are walls of different classes, so a priori could intersect each other.) The path $\gamma$
can be taken to be a vertical ray (the red dashed line in fig. 1) on $P_H$, starting from $\infty$ and going downward.

Thus, O’Grady’s classical result that $M_H(v_r)$ is birational to $X^{[n]}$, the Hilbert scheme of $n$ points on $X$, can be obtained by composing the wall-hitting transformations at these totally semistable walls that $\gamma$ crosses.

**Remark 1.2.** Notice that we identify the wall-hitting transformations with O’Grady’s maps $M_H(v_r) \rightarrow M_H(v_{r-1})$ only up to birational equivalences, because: first, we can only expect a general element in the moduli of complexes $M_{\sigma}(v_i)$, where $\sigma$ is above $W_i$, to be a sheaf, but not all; second, on the moduli of complexes side, the spherical twist in fact induces an isomorphism in codimension one, according to [BM14b] (see theorem 2.28), while the O’Grady’s maps are given directly by the extension process only on the complement of a divisor. The isomorphism in codimension one induced uniformly by a spherical twist will be important for our application to strange duality.

As the first step to prove theorem 1.1, we describe the first totally semistable wall for ideal sheaves of $n$ points on the elliptic K3 surface $X$:

**Proposition 1.3.** There exists a path $\gamma_1$ in $\text{Stab}^b(X)$, strating from the Gieseker chamber of ideal sheaves $I_Z$ of $n$ points on $X$, such that the first totally semistable wall for $I_Z$ along $\gamma_1$ is caused by $O(-C) \rightarrow I_Z$, where $C$ is a genus $n$ curve that contains $Z$.

**Remark 1.4.** The geometric intuition of this proposition is that given $n$ points on a K3 surface, there is always a genus $g$ curve that pass through these points, as long as $g \geq n$. As in [ABE13], the ideal sheaf of this curve $O(-C)$, as a subobject of $I_Z$ for all $Z \in X^{[n]}$, can become destabling and so potentially cause a totally semistable wall.

### 1.2 Strange duality for elliptic K3 surfaces.

As an application of theorem 1.1, we give two examples of strange duality isomorphisms for an elliptic K3 (example 1.6 and
example [1.9], based on a result of Marian and Oprea. Along the way, we also generalize an observation by Bayer and Macrì about wall-crossing for strange duality (proposition [1.8]).

We recall very briefly the setup of strange duality for a K3 surface $X$ (see e.g. [MO07] for details). Let $v, w \in H^*_{alg}(X)$ be a pair of Mukai vectors with $(v, w^\vee) = 0$. On the moduli space $M_H(v)$ of stable torsion-free sheaves of class $v$, one has a determinant line bundle $\theta_v(w)$, depending the orthogonal class $w$ (see definition 2.30). Symmetrically, we have another moduli space $M_H(w)$ with a line bundle $\theta_w(v)$. It has been observed that for some choices of $v$ and $w$, we have the following so-called strange duality phenomena:

1. $h^0(M_H(v), \theta_v(w)) = h^0(M_H(w), \theta_w(v))$;

2. Moreover, sometimes there exists a geometric explanation of the above equality: under certain assumptions, the locus $\Theta := \{(E, F) \in M_H(v) \times M_H(w) \mid H^0(X, E \otimes F) \cong Ext^1(F^\vee[1], E) \neq 0\}$,

   is an effective divisor whose associated line bundle is $\theta_v(w) \boxtimes \theta_w(v)$ [LP03]. Thus, it defines a map $SD : H^0(M_H(v), \theta_v(w)) \to H^0(M_H(w), \theta_w(v))^*$,

which in some cases turn out to be an isomorphism.

A well-known example of strange duality isomorphism is between a pair of rank one vectors, that is, the moduli spaces are Hilbert schemes of points. In [MO13], A. Marian and D. Oprea interpret O’Grady’s maps $\Psi_r : M_{H(v_r)} \dashrightarrow X[a]$ as induced by Fourier-Mukai transforms, and use that to propagate Hilbert schemes strange duality to a large family of pairs of vectors. Here is one of their theorems which we will base on.

**Theorem 1.5** ([MO13], Theorem 2). Let $X$ be the elliptic K3 surface as in section 1.1 with a polarization $H := c + mf$, $m >> 0$. Give $v, w$ be Mukai vecotrs with $(v, w^\vee) = 0$ and of ranks $r$ and $s$ respectively, satisfying the following:

1. $r, s \geq 2$;
2. $c_1(v).f = 1$, $c_1(w).f = 1$;
3. $(v, v) + (w, w) \geq 2(r + s)^2$.

Then the duality map $SD : H^0(M_H(v), \theta_v(w)) \to H^0(M_H(w), \theta_w(v))^*$ is an isomorphism.

With our wall-crossing interpretation of O’Grady’s maps $\psi_r : M_{H(v_r)} \dashrightarrow M_{H(v_{r+1})}$, we obtain the following based on theorem 1.5

**Example 1.6.** Let $X$ and $H$ be as in theorem 1.5, suppose that $v, w$ is a pair Mukai vectors with $(v, w^\vee) = 0$. Denote $r, s$ the rank of $v, w$ respectively. If $v$ and $w$ in addition satisfy:

1. $r, s \geq 0$, $r + s \geq 4$;
2. $c_1(v).f = 1$, $c_1(w).f = 1$;
3. $(v, v) + (w, w) = 2(r + s)^2$.

Then, strange duality holds for the pair $v$ and $w$. 


Note that condition (iii) is much more restrictive, compare to that of theorem \(1.5\). The only gain in this example is that one of the vectors can have rank 0 or 1 while the other have a higher rank.

To state the second example, we need to generalize the setting of strange duality. In \[BM14b\], Bayer and Macrì extend the definition of determinant line bundles (definition 2.30) and therefore strange duality to Moduli of complexes. Let \(X\) be any K3 surface, \(v, w\) be Mukai vectors with \((v, w) = 0\) and \(\sigma \in Stab(X)\) be a stability condition. They consider a strange duality morphism

\[ SD_\sigma : H^0(M_\sigma(v), \theta_v^\ast(w)) \rightarrow H^0(M_\sigma(w), \theta_w^\ast(v))^\ast, \]

defined by the effective divisor \(\Theta_\sigma := \{(E, F) \in M_\sigma(v) \times M_\sigma(w) | Ext^1(F, E) \neq 0\}\).

**Remark 1.7.** To recover the strange duality morphism \(SD\) considered by Marian and Oprea, \(\sigma\) should be chosen in the Gieseker chamber of \(v\) but not in that of \(w\). Indeed, we should have \(M_H(v) = M_\sigma(v)\) and \(M_H(w) \cong M_\sigma(-w^\vee)\).

One can ask how does the morphism \(SD_\sigma\) change as the stability condition \(\sigma\) varies. In the case when \((w, w) = 0, (v, v) \geq 2\), they prove that crossing a totally semistable wall of \(v\) "annihilates" \(SD_\sigma\). More precisely, let \(W\) be a totally semistable wall of \(v, \sigma_\pm\) two stability conditions that are separated by \(W\) and sufficiently close. Suppose that \(SD_\sigma\) is an isomorphism, then \(SD_{\sigma_-} = 0\) (see \[BM14b\], proposition 15.1). We observe that this can be generalized as follow:

**Proposition 1.8.** Let \(X\) be any K3 surface, \(v, w\) be orthogonal Mukai vectors with \((v, v) > 0, (w, w) > 0\). \(W\) denotes a nonisotropic totally semistable wall of \(w\) (resp. \(v\)) and \(\sigma_\pm\) two stability conditions separated by \(W\). Suppose that \(SD_{\sigma_+}\) is well-defined and isomorphic, then \(SD_{\sigma_-}\) is also defined and moreover:

1. if \((v, w_0) \neq 0\) (resp. \((w, v_0) \neq 0\)), then \(SD_{\sigma_-} = 0\).
2. otherwise, \(SD_{\sigma_-}\) is an isomorphism.

Here \(w_0\) is a Mukai vector appearing in the wall-hitting transformations, uniquely determined by \(w\) and the wall \(W\) (see proposition 2.27).

**Example 1.9.** Let \(X\) be the elliptic K3 with a polarization \(H := c + mf, m \gg 0\). Suppose \(v, w\) is a pair of Mukai vectors with \((v, w) = 0\). Denote \(r, s\) the rank of \(v, w\) respectively. Suppose in addition that:

(i) \(r, s \geq 3\);
(ii) \(c_1(v)f = 1, c_1(w)f = 1\);
(iii) \((v, v) + (w, w) = 2(r + s)(r + s - 2)\).

Then, there is a stability condition \(\sigma'\) such that

\[ SD_{\sigma'} : H^0(M_\sigma(v), \theta_v^{\sigma'}(w)) \rightarrow H^0(M_\sigma(w), \theta_w^{\sigma'}(v))^\ast \]

is isomorphic. As we will see, this \(\sigma'\) is separated from the place where \(SD_\sigma = SD\) by a totally semistable wall, and consequently \(SD : H^0(M_H(v), \theta_v(w)) \rightarrow H^0(M_H(w), \theta_w(v))^\ast\) is the zero morphism, by proposition 1.8.
Note that in this example, the quantity \((v, v) + (w, w)\) goes beyond the lower bound in condition \((iii)\) of theorem 1.5. The price we pay is a wall-crossing.

The rest of this paper is organized as following: in section 2 we review some preliminaries, in section 3 we prove theorem 1.1 and in section 4 we justify the remarks about strange duality.

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2 Review on stability conditions and wall-crossing for moduli.

We collect here some preliminaries, and introduce Bayer and Macrì’s work on wall-crossing for moduli of complexes on K3 surfaces \([BM14b]\). Throughout this section, \(X\) is any K3 surface.

2.1 Mukai lattice of \(X\).

**Definition 2.1.** The Mukai lattice of \(X\) is \(H^*_\text{alg}(X) := H^0(X) \oplus H_1(X, \mathbb{Z}) \oplus H^2(X)\) with the Mukai pairing \(((v_0, v_2, v_4), (w_0, w_2, w_4)) := v_2w_2 - v_0w_4 - w_0v_0\). We call an element in this lattice a Mukai vector.

**Definition 2.2.** Given an object \(E \in \mathcal{D}b(X)\) in the derived category of coherent sheaves on \(X\), define its Mukai vector \(v(E) := \text{ch}(E)\sqrt{td(X)} \in H^*_\text{alg}(X)\), where \(\text{ch}(E)\) is the Chern character of \(E\), and \(td(X) = (1, 0, 2)\) is the Todd class of \(X\).

Note that this defines an additive map \(v(-) : \mathcal{D}b(X) \to H^*_\text{alg}(X)\).

**Proposition 2.3** (e.g. [Huy06], Chapter 5). (a) Given \(E, F \in \mathcal{D}b(X)\), then

\[-\chi(E, F) := \sum_i (-1)^{i-1} \text{ext}^i(E, F) = (v(E), v(F)).\]

(b) Suppose that \(\Phi_p : \mathcal{D}b(X) \to \mathcal{D}b(Y)\) is a Fourier-Mukai transformation with kernel \(P \in \mathcal{D}b(X \times Y)\), and \(\Phi_{v(P)} : H^*_\text{alg}(X) \to H^*_\text{alg}(Y)\) is the correspondence, then the following diagram commutes:

\[
\begin{array}{c}
\mathcal{D}b(X) \\ v(-) \downarrow \\
H^*_\text{alg}(X)
\end{array} \xrightarrow{\Phi_p} \begin{array}{c}
\mathcal{D}b(Y) \\ v(-) \downarrow \\
H^*_\text{alg}(Y)
\end{array}
\]

2.2 Bridgeland stability conditions and moduli of complexes.

**Definition 2.4.** A Bridgeland stability condition on \(\mathcal{D}b(X)\) is a pair \((Z, \mathcal{A})\), where \(Z : H^*_\text{alg}(X) \to \mathbb{C}\) is a group homomorphism and \(\mathcal{A}\) is a heart of a bounded t-structure of \(\mathcal{D}b(X)\), satisfying all the following conditions:
(1) For any nonzero object $E \in \mathcal{A}$, $Z(E) := Z(v(E)) = r(E)e^{i\pi \phi(E)}$, then $r(E) > 0$ and 
$\phi(E) \in (0,1]$. ($\phi(E)$ is called the phase of $E$, and it defines a notion of semistability: an object $E \in \mathcal{A}$ is semistable if for any nonzero subobject $F \hookrightarrow E$, $\phi(F) \leq \phi(E)$.)

(2) For any object $E \in \mathcal{A}$, $E$ has a Harder-Narasimhan filtration

$$0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_{n-1} \hookrightarrow E_n = E,$$

such that quotient objects $A_i \cong E_i/E_{i-1}$ are semistable with decreasing phases, i.e. $\phi(A_i) > \phi(A_{i+1})$ for $i = 1, 2, \ldots, n$.

(3) For a given norm $\|\cdot\|$ on $H^*_\text{alg}(X)$, there exists a constant real number $C > 0$ such that

$$|Z(E)| < C\|v(E)\|$$

for every semistable object in $E \in \mathcal{A}$.

**Definition 2.5.** An object $E \in D^b(X)$ is $\sigma$-semistable if $E[i] \in \mathcal{A}$ is semistable.

**Example 2.6 [Bri08, Proposition 7.1].** Given $\beta, \omega \in NS(X) \otimes \mathbb{R}$ with $\omega$ ample, T. Bridgeland shows that a stability condition $\sigma_{\beta, \omega}$ can be constructed as follow: define

$$Z_{\beta, \omega} := (\exp(\beta + i\omega), -),$$

where $(\cdot , , \cdot)$ is the Mukai paring, and $\exp(\beta + i\omega) := (1, \beta, \frac{\beta^2 - \omega^2}{2} + i(0, \omega, \beta))$. Also define two additive subcategories of the abelian category of coherent sheaves $T := \{E \in \text{Coh}(X) : \mu_\omega(E/E_{\text{tor}}) \geq \beta \omega\}$, and $F := \{E \in \text{Coh}(X) \text{ torsion free} : \mu_\omega(E) < \beta \omega\}$, where $\mu_\omega(E) := \frac{c_1(E) \omega}{\text{rk}(E)}$. Then the tilt category $\mathcal{A}_{\beta, \omega}$ with respect to $T, F$, namely,

$$\mathcal{A}_{\beta, \omega} := \{E \in D^b(X) : \mathcal{H}^{-1}(E) \in F, \mathcal{H}^0(E) \in T, \mathcal{H}^{i \neq -1, 0}(E) \cong 0\}$$

is a heart of a bounded $t$-structure of $D^b(X)$, and moreover, the pair $(Z_{\beta, \omega}, \mathcal{A}_{\beta, \omega})$ is Bridgeland stability condition.

Let $\text{Stab}(X)$ denote the set of all Bridgeland stability conditions on $D^b(X)$.

**Theorem 2.7 [Bri07, Theorem 1.2].** $\text{Stab}(X)$ admits a complex manifold structure of dimension $2 + \rho(X)$, where $\rho(X)$ is the Picard rank of $X$.

Let $\text{Stab}^b(X)$ denote the connected component of $\text{Stab}(X)$ containing $\sigma_{\beta, \omega}$ (defined in example 2.6). The following result is well-known.

**Proposition 2.8.** Given a Mukai vector $v$, there exists a locally finite set of walls (real codimension 1 subsets) of $\text{Stab}^b(X)$, such that

1. when $\sigma$ varies within a chamber, the set of $\sigma$-semistable objects does not change;
2. when $\sigma$ cross a wall, the set of $\sigma$-semistable objects changes.

See e.g. [BM14a, Proposition 2.3].

**Definition 2.9.** Given an ample divisor $H$, we get a two dimensional slice of $\text{Stab}^b(X)$ defined as $P_H := \{\sigma_{uH, tH} : u \in \mathbb{R}, t > 0\}$.
The wall-and-chamber structure on $P_H$ is particularly neat:

**Lemma 2.10 ([ABC13], Section 6).** Given a Mukai vector $v$, then set of walls for $v$ intersects $P_H$ at either semicircles or vertical rays, and they do not intersect each other.

Fix $v$ as above, we say a stability condition $\sigma$ is generic with respect to $v$ if it does not lie on a wall of $v$. The existence of moduli of $\sigma$-semistable objects of class $v$ as a scheme was proved in [BM14a], based on a result of Y. Toda [Tod08].

**Theorem 2.11 ([BM14a], Theorem 1.3).** Assume that $\sigma$ is generic with respect to $v$. The set of $\sigma$-semistable objects in the heart $A_\sigma$ of class $v$ forms a (coarse) moduli space, denoted by $M_\sigma(v)$. It is an irreducible normal projective variety. Moreover, there is an open subset $M^s_\sigma(v)$ (possibly empty) of $M_\sigma(v)$ which parametrizes $\sigma$-stable objects; when $v$ is primitive in $H^*_{alg}(X)$, $M_\sigma(v) = M^s_\sigma(v)$ is smooth and projective.

These moduli of complexes should be viewed as variants of moduli of (twisted) Gieseker stable torsion-free sheaves, according to the following proposition by T. Bridgeland.

**Proposition 2.12 ([B108], Proposition 14.2).** Given $\beta, \omega \in NS(X) \otimes \mathbb{R}$ with $\omega$ ample, and $E \in D^b(X)$ with positive rank and the imaginary part $\text{Im}(Z(E)) > 0$, then $E$ is of $\sigma_{\beta, \omega}$-semistable for all $t$ sufficiently large if and only if $E$ is the shift of a $(\beta, \omega)$-Gieseker semistable torsion-free sheaf.

Such a chamber of $v$ where semistable objects are (shifts of) Gieseker semistable sheaves is called the Gieseker chamber of $v$.

### 2.3 Wall-crossing

Now we review some of the work in [BM14b] about wall-crossing for $M_\sigma(v)$. Throughout this subsection, $v$ is primitive with $(v, v) > 0$. When $W$ is a wall of $v$, $\sigma_0$ denotes a generic element in $W$ (i.e. it does not lie in any other wall), $\sigma_+$, $\sigma_-$ denote generic stability conditions separated by $W$, and sufficiently closed to $\sigma_0$.

**Theorem 2.13 ([BM14b], Theorem 1.1).** $M_{\sigma_+}(v)$ and $M_{\sigma_-}(v)$ are birational. Moreover, there is a derived equivalence $\Phi : D^b(X) \rightarrow D^b(X)$ induces a birational map $\phi : M_{\sigma_+}(v) \rightarrow M_{\sigma_-}(v)$ in the sense that: there exists a big open subset $U \subset M_{\sigma_+}(v)$ such that $\phi$ is an isomorphism from $U$ to its image, and for any $u \in U$, $\Phi(\mathcal{E}_u) = \mathcal{F}_{\phi(u)}$, where $\mathcal{E}_u$ and $\mathcal{F}_{\phi(u)}$ are the semistable objects that parametrized by the points $u$ and $\phi(u)$ respectively.

The main technical tool behind this theorem is their classification of walls. The following sublattice of $H^*_{alg}(X)$ plays a key role:

$$\mathcal{H}_W := \{ u \in H^*_{alg}(X) : Z_\sigma(u) \in \mathbb{R}Z_\sigma(v), \forall \sigma \in W \}.$$ 

**Proposition 2.14 ([BM14b], Proposition 5.1).** $\mathcal{H}_W$ is a primitive hyperbolic sublattice of rank two containing $v$. If $E$ is a $\sigma_+$-stable object of class $v$, then the classes of its $\sigma_0$-Jordan-Hölder factors and its $\sigma_-$-Harder-Narasimhan factors are contained in $\mathcal{H}_W$.

Remarkably, $\mathcal{H}_W$ contains enough information to determine the wall-crossing behavior of $M_\sigma(v)$ at $W$. To give a precise statement, we need some definitions.

**Definition 2.15.** Given an arbitrary primitive hyperbolic rank two sublattice $\mathcal{H}$, define a potential wall $W$ associated to $\mathcal{H}$ as a connected component of the codimension one submanifold $\{ \sigma \in \text{Stab}^1(X) : Z_\sigma(\mathcal{H}) \subset \mathbb{R}e^{i\pi \phi} \}$.
Definition 2.16. Let $W$ be a potential wall associated to $\mathcal{H}$, define the effective cone of $W$

$$C_W := \{ u \in \mathcal{H} \otimes \mathbb{R} : (u, u) \geq -2, Z_\sigma(u) \in \mathcal{R}_{>0}Z_\sigma(v), \forall \sigma \in W \}.$$ 

The following lemma justifies the name of $C_W$:

Lemma 2.17 ([BM14b], Proposition 5.5). If $u \in C_W \cap \mathcal{H}$, then for every $\sigma \in W$ there exist a $\sigma$-semistable object of class $u$. If $u \notin C_W$, then for a generic $\sigma \in W$, there does not exist any $\sigma$-semistable object of class $u$.

Definition 2.18. A class $u \in \mathcal{H}$ is called: effective if $u \in C_W$; isotropic if $(u, u) = 0$; spherical if $(u, u) = -2$.

Proposition 2.19 ([BM14b], Proposition 5.7). Given $v$, $\mathcal{H}$ as above and $W$ a potential wall associated to $\mathcal{H}$, then $W$ is a totally semistable wall of $v$ if and only if in $\mathcal{H}$ there exists either an isotropic class $w$ with $(v, w) = 1$, or an effective spherical class $s$ with $(v, s) < 0$.

Remark 2.20. Their proposition also clarifies when $W$ induce a divisorial contraction, a small contraction, when it is a fake wall (being totally semistable but inducing an isomorphism between moduli) and when it is not a wall.

We will need to further classify totally semistable walls, using proposition 2.19 and the following.

Proposition 2.21 ([BM14b], Lemma 8.3). Suppose that $W$ is a totally semistable wall for $v$ such that $\mathcal{H}_W$ contains an isotropic class, then there exists a unique $\sigma_0$-stable spherical object $S$ with $v(S) \in C_W$. Furthermore, if $E \in M_{\sigma_+}(v)$ generic in moduli, then its HN filtration with at $\sigma_-$ has length two and of the form either $S^{\otimes a} \to E \to F$ or $F \to E \to S^{\otimes a}$, where $a > 0$.

Corollary 2.22. If $W$ is a totally semistable wall of $v$, and $E \in M_{\sigma_+}(v)$ generic in moduli, then there exists either a spherical subobject or spherical quotient object $S$ of $E$ with $(E, S) < 0$, or the class decomposes as $v = s + nw'$, where $s$ spherical, $w'$ primitive and isotropic, and $n$ is the number such that $(v, v) = 2n - 2$.

Proof. Suppose that we have an effective $(s, v) < 0$. Let $S \in M_{\sigma_0(s)}$, then $(s, v) = \text{ext}^1(S, E) - \text{hom}(S, E) - \text{ext}^2(S, E) < 0$ implies that we have either $\text{hom}(S, E) \neq 0$ or $\text{hom}(E, S) \neq 0$. Since $E$ is $\sigma_0$-semistable, $S$ is either a subobject or quotient of $E$. Suppose otherwise, then we have $(v, w) = 1$ where $w$ is isotropic by proposition 2.19. So proposition 2.21 applies and $v = as + v' := av(S) + v(F)$. Furthermore, the proof of [BM14b] proposition 8.4 shows that we must have $a = 1$ or $n - 1$, $(v', v') = 0$, $(s, v') = n$ and $v' = nw'$, where $n$ is the number such that $(v, v) = 2n - 2$ and $w'$ is a primitive isotropic class. We observe that if $a = n - 1$, then $(v, s) = (n - 1)(s, s) + (v', s) = 2 - n$. Since we assume that $(v, s) \geq 0$, so $n = 2$. So in any case $a = 1$. Thus $v = s + nw'$.

Definition 2.23. We refer to the former case in corollary 2.22 as a spherical totally semistable wall, and the latter as a Hilbert-Chow totally semistable wall. Also, we call a total semistable wall $W$ isotropic if $\mathcal{H}_W$ contains an isotropic class, and nonisotropic otherwise.

As we shall see, Hilbert-Chow totally semistable walls are relatively simple for our purpose: they are some vertical rays on a slice $P_H \subset Stab(X)$. On the other hand, spherical
totally semistable walls will be the main character in section 3: we need to sort out the first wall among these. Once that is done, we will see the first wall is in fact nonisotropic. So we will need some of Bayer and Macrì’s analysis on nonisotropic totally semistable walls [BM14b, Section 6]. First, recall an important lemma by Mukai:

**Lemma 2.24** ([Muk87], Corollary 2.8). Suppose that $A$ is a heart of a bounded $t$-structure of $D^b(X)$, and $0 \to A \to E \to B \to 0$ is a short exact sequence in $A$, such that $\text{Hom}(A, B) = 0$. Then $\text{ext}^1(E, E) \geq \text{ext}^1(A, A) + \text{ext}^1(B, B)$.

**Corollary 2.25** (c.f. [HMS08], Section 2). If $S$ is a $\sigma$-semistable spherical object, then all its Jordan-Hölder factors are spherical ($\sigma$-stable object).

**Proposition 2.26** ([BM14b], proposition 6.3). Suppose that $W$ is a nonisotropic totally semistable wall, and $\sigma_0 \in W$ generic. Then $W$ contains infinitely many spherical classes, and two of them, denoted by $s$ and $t$, admits a stable object in their moduli, that is $M_{\sigma_0}(s) = M^s_{\sigma_0}(s) = \{S\}$ and $M_{\sigma_0}(t) = M^t_{\sigma_0}(t) = \{T\}$. Other spherical $\sigma_0$-semistable objects are strictly semistable, whose Jordan-Hölder factors are necessarily $S$ or $T$ (by corollary 2.25).

Consider $G \subset \text{Aut}(H)$ the subgroup generated by spherical twists $\{\rho_s : s \in H \text{ effective and spherical}\}$.

**Proposition 2.27** ([BM14b], proposition and definition 6.6). Given $v \in H$ with $(v, v) > 0$ and effective, the $G$-orbit of $v$ contains a unique class $v_0$ satisfying $(v_0, s) \geq 0$ for any effective spherical class $s \in H$.

**Theorem 2.28** ([BM14b], proposition 6.8). Let $W$ be a nonisotropic totally semistate wall, $\sigma_+, \sigma_-, \sigma_0$ and $v_0$ as before. Then one has the following diagram:

\[
\begin{array}{ccc}
M_{\sigma_+}(v) & \leftarrow & M_{\sigma_0}(v_0) \\
\phi_+ & \nearrow & \phi_- \\
& M_{\sigma_-}(v) & \\
\end{array}
\]

Here the morphisms $\phi_\pm$ are isomorphic in codimension 1 and induced by derived equivalences $\Phi_\pm$ respectively (in the sense of theorem 2.13), where $\Phi_+$ and $\Phi_-$ are either a composition of finitely many spherical twists or inverse spherical twists, e.g.

\[
\Phi_+ = ST_{S^1_+} \circ \cdots \circ ST_{S^1_+}, \quad \Phi_- = ST_{S^1_-} \circ \cdots \circ ST_{S^1_-},
\]

where $S^\pm_i$ are $\sigma^\pm$-stable spherical objects, and $ST_{S^\pm_i}$ denotes the spherical twist with respect to $S^\pm_i$.

**Definition 2.29.** We call $\phi_\pm$ (resp. $\Phi_\pm$) in the above wall-hitting transformations (resp. wall-hitting derived equivalences) at a spherical totally semistable wall $W$.

**2.4 Line bundles on $M_\sigma(v)$.** We recall the definition of determinant line bundles in various setting:

**Definition 2.30.** Suppose $v, w$ are primitive Mukai vectors with $v^2 \geq 0$ and $(v, w^\vee) = 0$. 

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Assume the Euler characteristic of the extension $\chi$ respectively.

In this section, we prove theorem 1.1. We return to the setting in section 1.1, where $X$ is O'Grady’s birational maps via wall-crossing.

(1) On fine moduli of sheaves $M_H(v)$: with a universal family $\mathcal{U}$, it is defined via the Fourier-Mukai transform with kernel $\mathcal{U}$:

$$\theta_v(w) := \det^{-1}\Phi_{\mathcal{U}}(F) = \det[Rp_{Y*}(\mathcal{U} \otimes Lp_X^*(F))]^{-1},$$

where $F$ is any sheaf on $X$ of class $w$.

(2) On coarse moduli of sheaves $M_H(v)$: one can still do the same construction on the defining Quot scheme over $M_H(v)$, the resulting line bundle descends because of the condition $(v, w^\vee) = 0$. See for example [HL10, Chapter 8].

(3) On moduli of complexes $M_\sigma(v)$: given the above assumption of $v$, and in addition that $\sigma$ is generic with respect to $v$, then there exists a quasi universal family $E$ in the sense that, $E|_{E \times X} \cong E^{\oplus \rho}$. In this case $\theta_v^\sigma(w) \in NS(M_\sigma(v))$ is defined as a numerical divisor class via intersection numbers

$$\theta_v^\sigma(w^\vee).C = -\frac{1}{\rho}(w^\vee, \Phi_{X}(\mathcal{O}_C)),$$

where $C$ is a curve in $M_\sigma(v)$. See [BM14a section 5].

Proposition 2.31 ([BM14a, proposition 4.4]). The definitions are compatible, namely, if $M_\sigma(v) = M_H(v)$, then $\theta_v^\sigma(-w^\vee) = \theta_v(v)$.

Proof.

$$\det^{-1}\Phi_{\mathcal{U}}(F).C = -\chi_1(\Phi_{\mathcal{U}}(F)).C$$

$$= -\chi_1(\Phi_{\mathcal{U}}(F)|_C)$$

$$= -\chi(C, \Phi_{\mathcal{U}}(F)|_C), \text{ as } rk(\Phi_{\mathcal{U}}(F)) = \chi(v \cdot w^\vee) = 0,$$

$$= -\chi(C \times X, q^*F \otimes \mathcal{U}|_C)$$

$$= -\chi(X, F \otimes \Phi_{\mathcal{U}}(\mathcal{O}_C))$$

$$= (w^\vee, v(\Phi_{\mathcal{U}}(\mathcal{O}_C))).$$

\]

\]

3 O’Grady’s birational maps via wall-crossing.

In this section, we prove theorem [1.1]. We return to the setting in section [1.1] where $X$ is an elliptic K3 surface with $NS(X) \cong \mathbb{Z}c \oplus \mathbb{Z}f$. Also, we fix a polarization $H := c + mf$, for $m$ sufficiently large. Such a polarization has the property that being $H$-Gieseker stable is equivalent to being $H$-slope stable, so in particular, twisting by a line bundle preserves $H$-Gieseker stability.

In O’Grady’s construction, we define $F_1 := \mathcal{I}_X \otimes \mathcal{O}_X(c + nf)$ so that $\chi(F_1) = 1$, and $\tilde{F}_1 := F_1 \otimes \mathcal{O}(-2f)$ with $\chi(\tilde{F}_1) = -1$. Let $v_1$ and $\tilde{v}_1$ denote the Mukai vectors of $F_1$ and $\tilde{F}_1$, respectively.

For $\tilde{F}_1 \in M_H(\tilde{v}_1)$ general, one has a unique nontrivial extension, which is also Gieseker $H$-semistable,

$$0 \to \mathcal{O}_X \to F_2 \to \tilde{F}_1 \to 0.$$ 

Assume the Euler characteristic of the extension $\chi(F_2) = 1$, and if we define $\tilde{F}_2 := F_2(-2f)$, then $\chi(\tilde{F}_2) = -1$. So we repeat the extension process and obtain recursively $\tilde{F}_r := F_{r-1} \otimes \mathcal{O}(-2f)$, and $F_r$ as the unique extension of $\tilde{F}_{r-1}$ by $\mathcal{O}_X$. 

\]
By lemma 3.4, we can only encounter spherical totally semistable walls. Recall that this kind of walls is caused by a spherical quotient or subobject $S$ case, the equation of the wall is

$$w = 0,$$

which is impossible unless $\gamma \in \mathbb{N}$ becomes $kH$, where equality holds if and only if $\gamma$.

We claim that in fact

$$\Delta = (n-1)r+p \leq |(n-1)\gamma^2 - 4rp(n-1) = (1+r+p-r-p)^2 = 1,$$

where equality holds if and only if $c_1 = kH$, for some $k \in \mathbb{N}$. Thus, $\Delta > 0$ if and only if $c_1 = kH$, for some $k \in \mathbb{N}$.

We claim that in fact $c_1 = 0$. Indeed, given that $c_1 = kH$, the two conditions above become $k^2(n-1) = rp$ and $1 + p + r = nr$. Note that all the variables here are integers, which is impossible unless $k = 0$. Thus, either $w = 0, 0, -1$ or $n = 2, w = 1, 0, 0$. In any case, the equation of the wall is $u = 0$.

**Proof of proposition 3.3** By lemma 3.4, we can only encounter spherical totally semistable walls. Recall that this kind of walls is caused by a spherical quotient or subobject $S$ with $(S, I_E) < 0$.

We first consider a wall $W_E$ that caused by a spherical subobje $E \hookrightarrow I_E$ with $(E, I_E) \leq 0$. Choose an ample divisor $H$ with $H^2 = 1$, and $G$ orthogonal to $H$ with $G^2 = -1$, then we can write $c_1(E) = (r, d_h H + d_g G, ch_2)$, where $d_h := c_1(E).H$ and $d_g := -c_1(E).G$. We summarize some useful properties of $E$ in the following lemma and prove it below.

**Proposition 3.1.** There exists a path $\gamma$ crossing $W$ in decreasing order, such that each $W$ is the first totally semistable wall of $v_i$ along $\gamma$.

**Proposition 3.2.** At each wall $W_{i+1}$, one of the wall-hitting derived equivalences is exactly

$$ST_{S_i}^{-1}(E_i) \rightarrow E_i \rightarrow \text{RHom}(S_i, E_i) \otimes S_i.$$
Lemma 3.5. E is a torsion-free sheaf, so in particular has rank \( r \geq 1 \). If \( r = 1 \), then \( E = \mathcal{O}(-C) \), where \( C \) is an effective curve on \( X \). Moreover, \( d_h := c_1(E).H < 0 \).

Consider the 2-dimensional slice of stability conditions \( P_H := \{ \sigma_{H,tH} \mid t > 0, u \in \mathbb{R} \} \). The equation of the wall \( W \) on the slice is

\[
d_h(u^2 + t^2) - 2(nr + ch_2)u + 2nd_h = 0.
\]

It defines a semicircle if the discriminant \( \Delta = (nr + ch_2)^2 - 2nd_h^2 > 0 \). Note that we have \( c_1(E)^2 = d_h^2 - d_g^2 \leq d_h^2 \) and \( -2 = (E, E) = c_1(E)^2 - 2r(ch_2 + r) \), therefore,

\[
\Delta = 2n(2rch_2 - d_h^2) + (nr - ch_2)^2 \\
\leq 2n(2rch_2 - c_1(E)^2) + (nr - ch_2)^2 \\
= -4n(r^2 - 1) + ((I_Z, E) + 2r)^2
\]

For the last equality, we also used

\[
(I_Z, E) = ((1, 0, 1 - u), (r, c_1, ch_2 + r)) = nr - ch_2 - 2r.
\]

Recall that by assumption \( (I_Z, E) < 0 \). On the other hand, we have a lower bound of \( (I_Z, E) \):

Lemma 3.6. Suppose that \( E \) is a destabilizing subobject of \( I_Z \) that causes a wall on the 2-dimensional slice \( P_H \), then \( (I_Z, E) > -4r \), where \( r = ch_0(E) \).

We will prove this lemma below. Given the lower bound, we have

\[
\Delta \leq -4n(r^2 - 1) + (2r - 1)^2 = -4(n - 1)r^2 - 4r + (4n + 1).
\]

If \( r \geq 2 \), then \( \Delta \leq -4(n - 1)r^2 - 4r + (4n + 1) < 0 \) and \( E \) does not define a wall on the 2-dimensional slice \( P_H \).

Otherwise \( r = 1 \). Then the only possibilities to have \( \Delta > 0 \) are \( (E, I_Z) = -1 \) or \( -4r + 1 \). By lemma 3.5, \( E = \mathcal{O}(-C) \). Recall that the condition \( (E, I_Z) = -1 \) or \( -4r + 1 \) is equivalent to \( nr - ch_2 = 1 \) or \( -2r + 1 \), and consequently \( n - ch_2 = n - \frac{1}{2}C^2 = n - (g - 1) = \pm 1 \), where \( g \) is the genus of \( C \). This shows that the first totally semistable wall is caused by \( \mathcal{O}(-C) \).

From now on we fix our polarization \( H = c + nf \). Then the first totally semistable wall along a ray \( \gamma_1 := \{ \sigma_{uH,tH} \mid t : \infty \to 0 \} \) is caused by \( \mathcal{O}(-C) \) to \( I_Z \) with \( C = c + nf \), as long as \( u \) is suitable such that \( \gamma_1 \) cross it. Let \( \mathcal{W}_{\mathcal{O}(-C)} \) denotes this wall.

Next, we need to consider wall \( \mathcal{W}_Q \) that are caused by a spherical stable quotient object \( I_Z \to Q \) with \( (I_Z, Q) \leq 0 \). By abuse of notation, we set \( ch(Q) := (r, d_hH + d_gG, ch_2) \). The following lemma states some useful properties of \( Q \):

Lemma 3.7. Suppose \( Q \) as above is a spherical stable quotient of \( I_Z \) with \( (I_Z, Q) \leq 0 \). Then \( Q \cong F[1] \) is the shifting of a torsion-free sheaf \( F \), \( d_h > 0 \) and \( r < 0 \).

We claim \( \mathcal{W}_Q \) does not prevent \( \mathcal{W}_{\mathcal{O}(-C)} \) from being the first totally semistable wall. To prove this, note that the equation of \( \mathcal{W}_{\mathcal{O}(-C)} \) is

\[
(u + \sqrt{2(n - 1)} + \frac{1}{\sqrt{2(n - 1)}})^2 + t^2 = \frac{1}{2(n - 1)}.
\]

In particular, the right end point of \( \mathcal{W}_{\mathcal{O}(-C)} \) (which is a semicircle) is \( u = -\sqrt{2(n - 1)} \).
On the other hand, for $Q$ to be in the heart $A_{u,H,IH}$, we should have $\mu_H(Q) = \frac{d_h}{r} \leq u$.

Now choose a vertical ray $\gamma_u := \{\sigma_{u,H,IH} | t : \infty \to 0\}$ that crosses $W_{C(-C)}$. If there is a wall $W_Q$ that prevent $W_{C(-C)}$ from being the first totally semistable wall along $\gamma_u$, then we have $\frac{d_h}{r} \leq u \leq -\sqrt{2(n-1)}$. However, this would implies $d_h^2 \geq 2(n-1)r^2$, and consequently the discriminant of the equation of $W_Q$ would be
\[
\Delta = (nr + ch_2)^2 - 2n d_h^2 \leq 4r^2(n-1)^2 - 4r^2n(n-1) < 0.
\]

Therefore, $W_{C(-C)}$ is the first totally semistable wall along some path $\gamma_1 := \{\sigma_{u,H,IH} | t : \infty \to 0\}$.

**Proof of lemma 3.5**. Let $0 \to E \to I_Z \to Q \to 0$ be the short exact sequence in the heart, then we have an exact sequence of coherent sheaves:
\[
0 \to H^{-1}(E) \to 0 \to H^{-1}(Q) \to E \to I_Z \to H^0(Q) \to 0.
\]

Hence, $E$ is a sheaf. Note that the image of $E \to I_Z$ must be of the form $I_{ Fr(-C)}$, where $I_{ Fr}$ is a 0-dimensional subscheme and $F$ an effective curve on $X$.

On the other hand, $H^{-1}(Q) \in \mathcal{F}$ is torsion-free and $\mu_H(H^{-1}(Q)) < \mu_H(I_Z) = 0$, that is $c_1(H^{-1}(Q)).H < 0$. Thus, being an extension of $I_{ Fr(-C)}$ by $H^{-1}(Q)$, $E$ is torsion-free with $d_H := c_1(E).H < 0$.

If we assume in addition that $r = 1$, then $E = I_{ Fr(-C)}$, but then $(E,E) = -2$ would imply $\Gamma = \emptyset$ and $E = \mathcal{O}(-C)$.

**Proof of lemma 3.6**. We want to show $(E,I_Z) > -4r$, or equivalently $nr - ch_2 > -2r$. Assume for contradiction that we have $nr - ch_2 \leq -2r$, then $ch_2 \geq (n+2)r$, so let’s write $ch_2 = (n+2+p)r$, for some rational number $p \geq 0$. Notice that
\[
d_h^2 \geq c_1^2 = 2rch_2 + 2(r^2 - 1) \geq 2(n+2+p)r^2.
\]

Therefore, $d_h \leq -r \sqrt{2(n+2+p)}$, since $d_h$ is negative by lemma 3.5.

Now we consider the imaginary part of the central charge of $E$ at $\sigma_{u,H,IH}$ : $\text{Im} Z(E) = t(d_h - ru) \geq 0$, as we assume $E$ is in the heart. This implies that
\[
u \leq \frac{d_h}{r} \leq -\sqrt{2(n+2+p)}.
\]

On the other hand, the points $(u = -\sqrt{2(n+2+p)}, t = 0)$ satisfies the equation (4.1) of the potential wall caused by $E$ (to check this, note that if $u = -\sqrt{2(n+2+p)}$, then $d_h = ru$), and the center of the semicircle defined by eq. (4.1) is
\[
\frac{rn + ch_2}{d_h} \geq \frac{r(2n + 2 + p)}{r\sqrt{2(n+2+p)}} > -\sqrt{2(n+2+p)},
\]

we see that on the wall $W_E$, $E$ is not in the heart. This is a contradiction.

**Proof of lemma 3.7**. We consider the following distinguished triangle
\[
H^{-1}(Q)[1] \to Q \to H^0(Q).
\]
Suppose that $H^0(Q) \neq 0$, then we have $Q \to H^0(Q) \to Q_1 \to Q_2$, where $Q_1$ is the most destabilizing quotient of $H^0(Q)$ and $Q_2$ is a stable factor of $Q_1$. Thus, the composition $Q \to Q_2$ is surjective and $Q_2$ is itself spherical by corollary 2.25. Therefore $Q \cong H^0(Q) \cong Q_2$.

This means we have a short exact sequence $0 \to E = I_{\Gamma}(D) \to I_Z \to Q = \mathcal{O}_{\mathbb{V},D}(-Z) \to 0$ in $\text{Coh}(X)$. For $Q$ to be spherical, $D$ must be a $-2$-curve. For $\text{Ext}^1(Q, Q) = 0$, we see that $\Gamma = \emptyset$. But then $(I_Z, Q) = 1$, contradict to our assumption.

Thus, $Q \cong F[1]$ for some torsion-free sheaf with $\mu_H(F) < u < 0$, and therefore, $d_h > 0$.

**Remark 3.8.** In the case when $n = 2$, we choose $H = \frac{c + (2 + \epsilon)f}{\sqrt{2} + 2\epsilon}$ for some $0 < \epsilon << 1$, so that $H$ is ample and $\mathcal{O}(c + 2f)$ still cause the first totally semistable wall.

In any case, our choice of $H$ is suitable to $w = (1, 0, 1 - n)$ in sense of O’Grady [O’G97].

**Corollary 3.9.** $W_1$ is the first totally semistable wall of $E_1 = I_Z(c + nf)$ along some vertical path on the slice $P_H$.

**Proof.** Choose $H = \frac{c + nf}{\sqrt{2n} - 2}$ as above, $W_{\mathcal{O}(-H)}$ causes the first totally semistable wall of $I_Z$.

Twisting by $\mathcal{O}(H)$ induces an action on $\text{Stab}(X) : (\otimes \mathcal{O}(H)).\sigma_{uH, tH} = \sigma_{(u - \sqrt{2n - 2})H, tH}$. 

### 3.2 The first totally semistable wall of $v_1$.

Denote $P_m$ the 2-dimensional slice $\{\sigma_{uH, tH} : t > 0\}$, where $H := \frac{c + mf}{\sqrt{2m} - 2}$. While corollary 3.3 states that when $m = n$, the first totally semistable wall of $E_1 = I_Z(c + nf)$ on $P_n$ is caused by $\mathcal{O}$, the other walls $W_r$ may not appear on $P_n$. Indeed, lemma 3.10 below shows that we need to let $m$ be sufficiently large, in order to have the walls $W_r, 1 \leq r \leq R$, all appear on $P_m$.

**Lemma 3.10.** Given $R \geq 1$, if we choose the polarization $H = \frac{\mathcal{O}(c + mf)}{\sqrt{2m} - 2}$ with $m$ sufficiently large, then the 2-dimensional slice $P_H$ intersects $W_r$, for $1 \leq r \leq R$. Moreover, on this slice, the walls $\{W_r : 1 \leq r \leq R\}$ are nested semicircles: $W_{r-1}$ is inside $W_r$ on $P_H$.

**Proof.** The wall $W_r$ of $v_r$ caused by $s_{r-1}$ is described as:

$$(r^2 - r + 2 - n - m)(u^2 + t^2) - 2\sqrt{2m - 2u} + 4(r - 1) = 0.$$  

(3.3)

Our assumption guarantees that this semicircle has positive radius and its center is on the $\{u \leq 0\}$ half.

From the equations we can see that $W_{r-1}$ and $W_r$ have no intersection. On the other hand, they both intersect $t$-axis, at $t_{r-1} = 2\sqrt{\frac{r-2}{n+m+3r-4-r}}$ and $t_r = 2\sqrt{\frac{r-1}{n+m+r-2-r}}$ respectively. Note that $t_{r-1} < t_r$. Thus, $W_{r-1}$ is contained in $W_r$.

Thus, we deform $P_m$ by varying $m$ from $n$ to $\infty$, to show that $W_1$ is always the first totally semistable wall for $v_1$ on $P_m$, for $m \geq n$.

**Proposition 3.11.** For all $m \geq n$, $W_1$ is the first totally semistable wall of $v_1$ on $P_m$, along a vertical ray starting from the Gieseker chamber of $v_1$.

**Proof.** Assume for contradiction that on $P_{m_1}$, for some $m_1 > n$, $W_1$ is not the first totally semistable wall along any vertical ray. This means that on $P_{m_1}$, there exists a bigger totally semistable wall $W'$ of $v_1$ (semicircle) contains $W_1$, because walls of a fixed Mukai vector are
nested. Then, \( W_1 \) and \( W' \) must coincide on some \( P_{m_0} \), \( m_0 > n \). Suppose \( W' \) is caused by another spherical object \( S' \).

Note that at \( W_1 \cap P_n \), the Jordan-Hölder filtration is

\[
0 \rightarrow \mathcal{O} \rightarrow I_Z(C) \rightarrow \mathcal{O}_C(-Z) \otimes \mathcal{O}(C) \rightarrow 0,
\]

where \( C := c + nf \). By lemma \([3.13]\) \( \mathcal{O} \) is always stable near \( W' \cap P_{m_0} \). So \( S' \) must be contained in \( \mathcal{O}_C(-Z) \otimes \mathcal{O}(C) \). Since \( (S', \mathcal{O}_C(-Z)) \otimes \mathcal{O}(C) \) \( \prec \) \( (S', I_Z(C) \otimes \mathcal{O}(C)) \) \( \prec \) \( \mathcal{O}_C(-Z) \otimes \mathcal{O}(C) \).

Thus we have a destabilizing sequence \( E' \rightarrow \mathcal{O}_C(-Z) \otimes \mathcal{O}(C) \rightarrow Q' \) with either \( E' \) or \( Q' \) being the stable spherical object \( S' \). Now consider

\[
0 \rightarrow H^{-1}(Q') \rightarrow E' \rightarrow \mathcal{O}_C(-Z) \otimes \mathcal{O}(C) \rightarrow H^0(Q') \rightarrow 0,
\]

here \( H^0(Q') \) is supported on points at most, since we can assume \( \mathcal{O}_C(-Z) \otimes \mathcal{O}(C) \) is generic in moduli and therefore \( C \) is irreducible. Also \( H^{-1}(Q') \) is not zero, because otherwise \( Q' \) and \( E' \) cannot possibly be spherical.

Write \( ch(E') = (r, d_hH + d_gG, ch_2) \), where \( H = \frac{c+mf}{\sqrt{2m-n}} \) and \( G = \frac{c+(2-m)f}{\sqrt{2m-n}} \). Then the equation of \( W' \) on \( P_H \) is

\[
(2 - m - n)r(u^2 + t^2) - 2r\sqrt{2m-n}u + 2((m + n - 2)ch_2 + \sqrt{2m-n}d_h) = 0.
\]

Compare it to the equation of \( W_1 \) (see eq. (3.3)), one sees that a necessary and sufficient condition for two walls to overlap is

\[
(m + n - 2)ch_2 + \sqrt{2m-n}d_h = 0. \tag{3.4}
\]

For each \((u, t) \in W' \cap P_H\), we have \( d_h(E') \geq u \). Note that \( r(E') > 0 \) because \( H^{-1}(Q') \neq 0 \). Since \( u \) can be sufficiently close to 0, we see that \( d_h(E') \geq 0 \). On the other hand, \( d_h(H^{-1}(Q')) < ur(H^{-1}(Q')) < 0 \) and therefore

\[
0 \leq d_h(E') < d_h(\mathcal{O}_C(-Z) \otimes \mathcal{O}(C)) = \frac{m + n - 2}{\sqrt{2m-n}}.
\]

By section \(4.2\), we then have \( 0 \leq ch_2(E') < 1 \). Because \( X \) is a K3 surface, \( ch_2(E') \) is integral. Thus \( ch_2(E') = 0 \) and \( d_h(E') = 0 \).

Consequently, the Mukai vector \( v(E') = (r, k(c + (2-m)f), r) \) and \( (E', E') = -2k^2(m-1)-2r^2 \), so it cannot be spherical unless it is \( \mathcal{O}_X \), but that would contradict our assumption. Thus, \( Q' \) has to be spherical. Note that by Mukai’s lemma, \( Ext^1(H^0(Q'), H^0(Q')) = 0 \) and therefore \( H^0(Q') = 0 \). The equality \((Q', Q') = -2\) gives

\[
-(m-1)k^2 + n + k(m-n) = r(r+1). \tag{3.5}
\]

Given that \( r \geq 1 \) and \( m \geq n \), section \(4.2\) holds only when \( k = 0 \) and \( n = r(r+1) \). In particular, \( E' = \mathcal{O}^r \).

Therefore we have a surjective map \( \mathcal{O}_X \rightarrow \mathcal{O}_C(-Z) \otimes \mathcal{O}(C) \), which must factor through \( \mathcal{O}_C \rightarrow \mathcal{O}_C(-Z) \otimes \mathcal{O}(C) \). Via the Abel-Jacobi map

\[
Sym^n(C) \rightarrow Pic^{-n}(C) \cong Pic^{n-2}(C),
\]

we see that \( \mathcal{O}_C(-Z) \otimes \mathcal{O}(C) \) is generic in \( Pic^{n-2} \), provided that \( Z \subset C \) is generic. However, a generic element in \( Pic^{n-2}(C) \) is not effective since \( Sym^{n-2}(C) \rightarrow Pic^{n-2}(C) \) is not surjective. This is a contradiction.
Remark 3.12. $W_1$ is actually a wall for $O_C(-Z) \otimes O(C)$, although not totally semistable. Indeed, the destabilized objects are precisely those in the Brill-Noether variety $W^{-1}_{n-2}(C)$, namely, degree $n - 2$ line bundles on $C$ that are globally generated with $r$ sections. Note that its dimension $\dim W^{-1}_{n-2}(C) = n - r(r + 1) = 0$. And the destabilizing sequence is

$$O^r \rightarrow O_C(-Z) \otimes O(C) \rightarrow F'[1],$$

where $F'$ is the Lazasfeld-Mukai bundle. See [Bay16].

3.3 Proofs of proposition 3.1 & proposition 3.2 In this subsection we first prove proposition 3.2 and then use it together with proposition 3.1 and O’Grady’s result to prove proposition 3.1. Recall $W_r, S_r, E_r, s_r$, and $v_r$ from the beginning of this section.

Lemma 3.13. Choose a polarization $H = \frac{O(c + mf)}{\sqrt{2m-2}}$ with $m$ sufficiently large, then the spherical object $S_{r-1}$ is stable at the wall $W_r$ on $P_H$.

Proof. By a result of Arcara and Miles [AM14] theorem 1.1, $S_{r-1}$ can only be possibly destabilized by $S_{r-1}(-c)$, where $c$ is the section of the elliptic fibration. Such a wall has the following equation:

$$(2 - m)(u^2 + t^2) + 2\sqrt{2m-2}(2r - 1)u - 4(2r - 1)(r - 1) = 0.$$  \hspace{1cm} (3.6)

In particular, it lies on the $\{u \geq 0\}$ half.

An elementary computation shows that this semicircle [3.6] has no intersection with that of [3.3], and since the latter centers at the $\{u \leq 0\}$ half, these two walls do not contain each other. As $S_{r-1}$ is stable at large volume limit, it is stable at the wall $W_r$ on $P_H$. \sqcap

Proof of proposition 3.2. We want to prove that at the wall $W_r$ of $v_r$ caused by $s_{r-1}$, the wall-hitting derived equivalences $\Phi_k$ consist of one spherical twist (or inverse) respectively. Based on Bayer-Macrì’s analysis, this is amount to show that $v_{r-1}$ pairs with all effective spherical classes non-negatively, in the rank two hyperbolic sublattice $H_r$ of the wall.

Note that by lemma 3.13 $S_{r-1}$ is stable at the wall. If $H_r$ is isotropic, then by proposition 2.21 $s_{r-1}$ is the unique effective spherical class in $H_r$ and $(s_{r-1}, v_{r-1}) = 1$. If $H_r$ is non-isotropic, $s_{r-1}$ is one of the two spherical classes that have a stable object in their moduli. Suppose that $t_0$ is the other one, then $(t_0, s_{r-1}) \geq 3$. Now let’s use $\{v_r, s_{r-1}\}$ as a (rational) basis of $H_r$ and write $t = xs_{r-1} + yv_r$. All spherical classes $t$ lie on a hyperbola:

$$-2 = (t, t) = -2x^2 + (2n - 2)y^2 - xy.$$  \hspace{1cm} (3.7)

$(t, v_{r-1}) = 0$ defines a line:

$$y = -\frac{1}{2n - 1}x.$$  \hspace{1cm} (3.8)

Also we have the constraint $(t, s_{r-1}) \geq 3$, for all $t$ that are effective spherical and does not lie on the same branch with $s_{r-1}$. This gives

$$2x + y \leq -3.$$  \hspace{1cm} (3.9)

The graphs of these equations (see fig. 2) show that all effective spherical classes pair with $t_0$ positively. Indeed, on the right branch of the hyperbola, effective spherical classes are all above $s_{r-1}$ and therefore above the line $(v_{r-1}, t) = 0$. On the left branch, effective
classes are to the left of the line \((s_{r-1}, t) = 3\) and on the upper half-plane, and consequently also above the line \((v_{r-1}, t) = 0\).

Thus, \(v_{r-1}\) is the minimal class in the orbit of \(v_r\) and \(\Phi_{\pm}\) both consist of one spherical twist.

Proof of proposition \[3.1\]
We shall let \(m\) be sufficiently large, then by lemma \[3.10\] we have a path \(\gamma := \{\sigma_{uH,tH} \mid t : \infty \to 0\}\) for \(H = c + mf\) and some suitable \(u\) such that it passes through \(W_i, 1 \leq i \leq r\) in decreasing order and at generic points of the walls. Let \(\gamma(t_i)\) denotes the point when \(\gamma\) meets \(W_i\).

By proposition \[3.11\] \(W_1\) is the first totally semistable wall for \(v_1\). Thus a generic stable object in the moduli \(M_{\gamma(t_i)}(v_1)\), for \(t > t_1\), is of the form \(I_Z(c + nf)\), since it is stable at large volume limit.

Then according to O'Grady's result and proposition \[3.2\] the wall-hitting derived equivalence at \(W_i\) is the unique extension

\[0 \to S_{i-1} \to E_i \to E_{i-1} \to 0.\]

In particular, a generic \(E_i\) is stable for \(t_i < t < \infty\). That is, \(W_i\) is the first totally semistable wall of \(v_i\) along \(\gamma\), for all \(i \geq 1\).

4 Remarks on strange duality for elliptic K3.

In section \[4.1\] we prove proposition \[1.8\]. In section \[4.2\] we use wall-crossing and theorem \[1.5\] to give two examples of strange duality for an elliptic K3 surface.

4.1 Wall-crossing behavior of \(SD_{\sigma}\). Throughout this subsection, \(X\) is any K3 surface, \(v, w\) are orthogonal Mukai vectors. Let \(W\) be a nonisotropic totally semistable wall of \(v\) but not of \(w, \sigma_0 \in W\) be generic at the wall, \(\sigma_+\) and \(\sigma_-\) are separated by \(W\) and sufficiently close to \(\sigma_0\). To simplify notation, denote \(SD_{\sigma}\) the maps \(SD_{\sigma} : H^0(M_{\sigma_0}(v), \theta_v(w)) \to H^0(M_{\sigma_0}(w), \theta_w(v))^*\) described in section \[1.2\].
Lemma 4.1. If $SD_+$ is defined, then so is $SD_-$.

Proof. First we note that $\phi_0(v) \neq \phi_0(w)$, where $\phi_0$ is the phase function at $\sigma_0$: suppose otherwise, then $v, w \in \mathcal{H}_W$, the rank 2 hyperbolic sublattice associate to $W$, however the assumptions $(v, w) = 0, v^2 > 0, w^2 > 0$ imply that $v$ and $w$ are linearly independent but does not span a hyperbolic lattice, contradiction.

Now $SD_+$ is defined, in particular the locus $\Theta_+$ is nonempty, i.e. $\text{Hom}(E, F) \neq 0$ for some $(E, F) \in M_{\sigma_+}(v) \times M_{\sigma_+}(w)$, thus we should have $\phi^+(w) > \phi^+(v)$. Then $\phi_-(w) \approx \phi_+(w) > \phi_+(v) \approx \phi^-(v)$ as $\sigma_+, \sigma_-$ are sufficiently close. Thus $\text{Ext}^2(E', F') \cong \text{Hom}(F', E'^*) = 0$ for all $(E', F') \in M_{\sigma_+}(v) \times M_{\sigma_+}(w)$. Then by [LP03] proposition 9, $SD_-$ are well-defined.

Recall that at the totally semistable wall $W$, if $\phi_+^+(v) > \phi_+^+(v_0)$, then $\psi_+ : M_{\sigma_0}(v_0) \rightarrow M_{\sigma_+}(v)$ is induced by a series of spherical twists. Let $E_0 \in M_{\sigma_0}(v_0)$, $E_i+1 := ST_{S_{i+1}}^{-1}(E_i)$ as in section 2

Lemma 4.2. For $i \geq 0$, then we have an surjection $\text{Ext}^1(ST_{S_{i+1}}(E_i), F) \rightarrow \text{Ext}^1(E_i, F)$, for any $F \in M_\sigma(w)$.

Proof. To simplify notation, let $E := E_i$, $S := S_{i+1}$, so that $E_{i+1} = ST_S(E)$. By definition of spherical twist, the defining distinguished triangles

$$\bigoplus_j \text{Hom}(S[j], E) \otimes S[j] \rightarrow E \rightarrow ST_S(E)$$

gives an exact sequence:

$$\text{Ext}^1(ST_S(E), F) \rightarrow \text{Ext}^1(E, F) \rightarrow \text{Ext}^1(\bigoplus_j \text{Hom}(S[j], E)) \otimes S[j], F).$$

According to [BM14b], Proposition 6.8], $E$ and $S$ lie in a heart of $D^b(X)$, by Serre duality

$$\text{Ext}^1(\bigoplus_j \text{Hom}(S[j], E)) \otimes S[j], F) = \bigoplus_{j=0}^2 \text{Hom}(S, E[j]) \otimes \text{Hom}(S, F[j+1]).$$

Note that $\text{Hom}(S, F[2]) = \text{Hom}(F, S) = 0$, because $\phi_+(S) \approx \phi_+(v) < \phi_+(w)$. By the "Induction Claim" in the proof of [BM14b] Proposition 6.8], $S$ and $E$ are both simple in a certain additive subcategory, and therefore $\text{Hom}(S, E) = \text{Hom}(E, S) = 0$. Hence, $\text{Ext}^1(\bigoplus_j \text{Hom}(S[j], E)) \otimes S[j], F) = 0$.

Still assume that $\phi_+(v) > \phi_+(v_0)$, the other map $\psi_- : M_{\sigma_0}(v_0) \rightarrow M_{\sigma_-}(v)$ is induced by a series of inverse spherical twists. Let $E_0 \in M_{\sigma_0}(v_0)$ as before, but reset $E_{i-1} := ST_S^{-1}(E_i)$.

Lemma 4.3. For $i \leq 0$, we have an injection $\text{Hom}(E_i, F) \hookrightarrow \text{Hom}(ST_S^{-1}(E_i), F)$, for any $F \in M_\sigma(w)$.

Proof. This is dual to lemma 4.2.

Lemma 4.4. Suppose that $(w, v_0) \neq 0$. If $SD_+ \neq 0$ (resp. $SD_- \neq 0$), then $SD_- = 0$ (resp. $SD_+ = 0$).
Proof. Because $\text{Hom}(E_+,F) \neq 0$ for some $E_+ \in M_{\sigma_+}(v)$, and both $E_+$ and $F$ are $\sigma_0$-semistable, so $\phi_0(v) < \phi_0(w)$. And therefore we have $\phi_0(v_0) = \phi_0(v) < \phi_0(w)$, which implies $\text{Ext}^2(E_0,F) = 0$ for all $E_0 \in M_{\sigma_0}(v_0)$ and all $F \in M_{\sigma_0}(w)$.

Without loss of generality, we may assume $\phi_+(v) > \phi_+(v_0).$ Thus $\Phi_+ : M_{\sigma_0}(v_0) \longrightarrow M_{\sigma_+}(v)$ is induced by a series of spherical twists. If $0 < (v_0, w) = \text{ext}^1(v_0, w) - \text{hom}(v_0, w)$, then $\text{Ext}^1(E_0, F) \neq 0$ for all $(E_0, F) \in M_{\sigma_0}(v_0) \times M_{\sigma_+}(w)$, and by lemma 4.2 this implies $\text{Ext}^1(E^+, F) \neq 0$ for all $(E_+, F) \in M_{\sigma_+}(v) \times M_{\sigma_+}(w)$, which contradicts to our assumption that $SD_+$ is zero. Hence, $(v_0, w) < 0$ and $\text{Hom}(E_0, F) \neq 0$. By lemma 4.3 $SD_-$ is a zero map.

Now we turn to the case when $(v_0, w) = 0$. In this case, the rank two hyperbolic lattice $\mathcal{H}_W$ is perpendicular to $w$. Recall that $v = v_i = \sum_{i=1}^l v_0 + (v_{i-1}, s_i) s_i$, where $(v_{i-1}, s_i) = \text{ext}^1(v_{i-1}, s_i) > 0$. Thus,

$$\theta_+(v) \equiv \theta_+(v_0) \otimes [\otimes_{i=1}^l \theta_+(s_i)]^{(v_{i-1}, s_i)}].$$

Denote $\Theta_{\pm} \subset M_{\sigma_+}(v) \times M_{\sigma}(w)$ the theta divisors, and define $\Theta_0 := \{ (E_0, F) \in M_{\sigma_0}(v_0) \times M_{\sigma}(w) : \text{Ext}^1(E_0, F) \neq 0 \}$. Then

$$\psi_+^*(\theta_+(w) \otimes \theta_+(v)) \equiv [\theta_+(w) \otimes \theta_+(v_0)] \otimes q^* (\otimes_{i=1} (\theta_+(s_i))^{(v_{i-1}, s_i)}),$$

where $q : M_{\sigma_0}(v_0) \times M_{\sigma}(w) \to M_{\sigma}(w)$ is the projection, and $s := \sum_{i=1}^l (v_{i-1}, s_i)$. Correspondingly,

$$\psi_-^*(\Theta_+) = \Theta_0 + \sum_{i=1}^l (v_{i-1}, s_i) q^{-1} D_i^+,$$

where $D_i^+ := \{ (S^+_i, F) \in M_{\sigma_+}(s_1) \times M_{\sigma}(w) : \text{Hom}(S^+_i, F) \neq 0 \}$ is a divisor with associated line bundle $\theta_+(s_i)$.

Similarly, $\psi_-^*(\Theta_-) = \Theta_0 + \sum_{i=1}^l (v_{i-1}, s_i) q^{-1} D_i^-$, where $D_i^- := \{ (S^-_i, F) \in M_{\sigma_-}(s_1) \times M_{\sigma}(w) : \text{Hom}(S^-_i, F) \neq 0 \}$.

Lemma 4.5. Assume that $W$ is a totally semistable wall of $v$ with $(v_0, w) = 0$, then $\psi_+^*(\Theta_+) = \psi_-^*(\Theta_-)$. Consequently, $SD_+ = SD_-.$

Proof. It suffices to show $D_i^+ = D_i^-$, which is amout to show $\text{Hom}(S^+_i, F) \neq 0$ if and only if $\text{Hom}(S^-_i, F) \neq 0$, for any $F \in M_\sigma(w)$.

$S^+_i$ is $\sigma_+$-stable, hence $\sigma_0$-semistable. If $S^+_i$ is $\sigma_0$-stable then $S^-_i = S^+_i$ and the claim follows. So we can assume $S^+_i$ is strictly semistable. By Mukai’s lemma, all its $\sigma_0$-stable factors are also spherical. According to [BMT43], Proposition 6.3, there are exactly two $\sigma_0$-stable spherical objects have the same phase with $s_i$, denoted by $T_1$ and $T_2$. Both of them should appear as stable factors of $S^+_i$, because otherwise $s_i$ would be a multiple of a spherical class, contradicting the assumption that it is spherical. For the same reason, $S^-_i$ should also contain both stable objects.

Now suppose $\text{Hom}(S^+_i, F) \neq 0$, then either $\text{Hom}(T_1, F) \neq 0$ or $\text{Hom}(T_2, F) \neq 0$. As $T_1, T_2 \in W \subset \tilde{w}$ and $\text{Ext}^2(T_i, F) = 0$, $\text{hom}(T_i, F) = \text{ext}^1(T_i, F)$, for $i = 1, 2$, so that fact that either $\text{Hom}(T_1, F) \neq 0$ or $\text{Hom}(T_2, F) \neq 0$ implies $\text{Hom}(S^-_i, F) \neq 0$. The other direction is also true for the same reason. Thus, $D_i^+ = D_i^-.$

Proof of proposition 1.8 By lemma 4.1, lemma 4.4 and lemma 4.5.
4.2 Strange duality via wall-hitting. Let $v_r, w_s$ be Mukai vectors of ranks $r, s$ respectively and satisfy $(v_s, w_r^\vee) = 0$. Write $v_r := (r, c + (a + rp)f, p)$, $w_s := (s, c + (b + rp)f, q)$, so that $(v_r, v_r) = 2a - 2$, $(w_s, w_s) = 2b - 2$. Note that the assumption $(v_r, w_s^\vee) = 0$ becomes $a + b - 2 = -(r + s)(p + q)$. And the condition (iii) in theorem 1.5 is equivalent to $p + q + s + r \leq 0$. Recall that the strange duality morphism

$$SD_{(r,s)} : H^0(M_H(v_r), \theta(w_s)) \to H^0(M_H(w_s), \theta(v_r))^*$$

is defined by the theta locus $\Theta_{(r,s)} := \{(E_r, F_s) \in M_H(v_r) \times M_H(w_s) \mid Ext^1(F_s^\vee[1], E_r) \neq 0\}$.

In section 3.3 we see that the first totally semistable walls $V_r$ of $v_r$, $1 \leq r \leq R$, are nested on a slice $P_m := \{\sigma_{w_H, tH} : t > 0\}$ where $H = c + mf$, for some $m \gg 0$. Now we define $W_s$ to be the potential wall for $v(F_s^\vee[1])$ caused by the class $S_{s-1}^\vee[1]$, where $F_s \in M_H(w_s)$ and $S_{s-1}$ is a spherical object that causes the first totally semistable wall for $w_s$.

**Lemma 4.6.** $W_s$ is the first totally semistable wall for $-w_s^\vee$ along some suitable rays on $P_H$. Given $r, s \geq 0$, $V_r$ intersects $W_s$ on the slice $P_m$.

**Proof.** First, note that $W_s$ is symmetric to the first totally wall for $w_s$ caused by $S_{s-1}$, about the vertical ray $u = 0$ on $P_m$. The first claim then follows from [BM14b, proposition 2.11]. Recall that the equation of $V_r$ on $P_m$ is (eq. (3.3))

$$(r^2 - r + 2 - a - m)(u^2 + t^2) - 2\sqrt{2m - 2u + 4(r - 1)} = 0.$$ 

Similarly, the equation of $W_s$ is

$$(s^2 - s + 2 - b - m)(u^2 + t^2) + 2\sqrt{2m - 2u + 4(s - 1)} = 0.$$ 

For $m$ sufficiently large, these two equations have a common solution. $\square$

Now choose a sufficiently large $m$, then on $P_m$, $V_r$ and $W_s$ cut out four chambers near their intersection point. Denote $\sigma_0$ their intersection point.

**Lemma 4.7.** Suppose that both $\theta_+(w)$ and $\theta_+(v)$ are movable line bundles. Choose a stability condition $\sigma$ above $V_r$ and $W_s$ on $P_m$, and sufficiently close to $\sigma_0$. Then the strange duality morphism

$$SD_{(r,s)}^\sigma : H^0(M_\sigma(v_r), \theta^\sigma(-w_s^\vee)) \to H^0(M_\sigma(-w_s^\vee), \theta^\sigma(v_r))^*$$

is the same as $SD_{(r,s)}$.

**Proof.** There exists a large volume limit $\sigma_\infty$ such that $M_{\sigma_\infty}(v_r) = M_H(v_r)$ by proposition 2.12 and moreover $\theta^\sigma(-w_s^\vee) = \theta(w_s)$ by proposition 2.31.

Also, by [BM14b, proposition 2.11], we can choose the large volume limit $\sigma_\infty$ such that $M_{\sigma_\infty}(-w_s^\vee) \cong M_H(w_s)$, induced by the derived equivalence $(-)^\vee[1]$. So $\theta^\sigma(v_r)$ is identified with $\theta(v_r)$ under the isomorphism. Also, the theta divisors $\Theta_{(r,s)}^\sigma$ and $\Theta_{(r,s)}$ also get identified. Therefore, $SD_{(r,s)}^\sigma = SD_{(r,s)}$.

Since $\sigma$ is above both $V_r$ and $W_s$, there exists a path $\gamma'$ in $\text{Stab}^1(X)$, going from $\sigma_\infty$ to $\sigma$ and crossing no divisorial nor totally semistable walls. Indeed, this is an argument in the proof of [BM14b, proposition 15.1]: consider the chamber structure on $\text{Stab}^1(X)$ cut out by all totally semistable walls for $v_r$ and $-w_s^\vee$. Note that theorem 1.1 shows that $\sigma$ and $\sigma_\infty$ lie in a common chamber $C_\text{tot}$. Also, under the map $l_v : U(X) \to NS(M_\sigma(v_r))$ (resp. $l_w : \text{Stab}^1(X) \to NS(M_\sigma(-w_s^\vee))$) defined in [BM14b, theorem 10.2], where $U(X) \subset \text{Stab}^1(X)$
is the geometric chamber, $\sigma_{\infty}$ is mapped to an interior point of the movable cone of $M_{\sigma}(v_r)$ (resp. $M_{\sigma}(-w^*_s)$), thus there exist an open subset $V$ containing $\sigma, \sigma_{\infty}$, contained in both $U(X)$ and the totally semistable chamber $C_{tot}$, and maps to the interior of the movable cones of $M_{\sigma}(v_r)$ and $M_{\sigma}(-w^*_s)$ under $l_u$ and $l_w$, respectively. Choose a path in this open subset $V$, then it crosses no totally semistable nor divisorial walls, because divisorial walls of $v$ (resp. $w$) are send to the boundary of the image of $l_u$ (resp. $l_w$) ([BM14B lemma 10.1]). Thus, $SD_{(r,s)}^\sigma = SD_{(r,s)}$.

**Remark 4.8.** Under the assumption of theorem 1.5, the line bundles $L$ are movable. Indeed, via O’Grady’s birational maps, they are identified with tautological line bundles $L$ under the assumption of theorem 1.5, part (a) yields example 1.6 and part (b) gives example 1.9.

For (a), if $p+q+r+s = 0$, then it crosses no totally semistable nor divisorial walls, because divisorial walls of $v$ (resp. $w$) are send to the boundary of the image of $l_u$ (resp. $l_w$) ([BM14B lemma 10.1]). Thus, $SD_{(r,s)}^\sigma = SD_{(r,s)}$.

**Remark 4.8.** Under the assumption of theorem 1.5, the line bundles $\theta_v(w)$ and $\theta_w(v)$ are movable. Indeed, via O’Grady’s birational maps, they are identified with tautological line bundles $L$ on the Hilbert schemes $X[a]$ and $X[b]$ respectively, where $L = O_X(-(p+q+r+s)f)$, which is nef by condition (iii) in theorem 1.5. Thus, the tautological line bundles are nef. Since O’Grady’s birational maps are isomorphic in codimension one, $\theta_v(w)$ and $\theta_w(v)$ are movable.

Now we can compare the strange duality morphisms. To facilitate computation, we “normalize” our sheaves. Given $E_r$ of class $v_r = (r,c + (a + rp)f, p)$, define $\tilde{E}_r := E_r \otimes O((-p + r + 1)f)$ such that $\chi(\tilde{E}_r) = 1$. Then $\tilde{V}$ is a totally semistable wall of $\tilde{E}_r$ caused by $O$. Note that $H^0(E_r \otimes F_s) = H^0(\tilde{E}_r \otimes F_s((p + q + r + s - 2)f))$.

**Lemma 4.9.** Suppose that $\tilde{E}_r$ is generic in moduli, then

(a) $STO(\tilde{E}_r) = E_{r-1}(-2f)$.

(b) $STO(2f)(\tilde{E}_r) = E_{r+1}(2f)$.

**Proof.** These are precisely O’Grady’s construction.

**Proposition 4.10.** (a) If $p + q + r + s = 0$, then $SD_{(r,s)} = SD_{(r+1,s-1)}$.

(b) If $p + q + r + s = -2$, then $SD_{(r,s)} = SD_{(r+1,s+1)}^\sigma$, where $\sigma'$ is the strange duality morphism defined at some $\sigma' \in Stab(X)$. Consequently, assume in addition that $SD_{(r,s)}$ is an isomorphism, then $SD_{(r+1,s+1)} = 0$.

Combine with theorem 1.5, part (a) yields example 1.6 and part (b) gives example 1.9.

**Proof of proposition 4.10.** For (a), if $p + q + r + s = 0$, then

$$Hom(F_s^\vee, E_r) = Hom(\tilde{F}_s^\vee, \tilde{E}_r(-2f))$$
$$= Hom(STO^{-1}(\tilde{F}_s^\vee), STO^{-1}(\tilde{E}_r(-2f)))$$
$$= Hom(STO(\tilde{F}_s^\vee), E_{r+1})$$
$$= Hom((F_{s-1}^\vee, E_{r+1}(-2f))$$
$$= Hom(F_{s-1}^\vee, E_{r+1}).$$

Thus, $SD_{(r,s)} = SD_{(r+1,s-1)}$. For (b), given $p + q + r + s = -2$, then

$$Hom(F_s^\vee, E_r) = Hom(\tilde{F}_s^\vee, \tilde{E}_r(-4f))$$
$$= Hom(STO((\tilde{F}_s(-2f))^\vee), STO(\tilde{E}_r(-2f)))$$
$$= Hom(STO^{-1}(\tilde{F}_s(-2f))^\vee, STO(2f)(\tilde{E}_r)(-2f))$$
$$= Hom(F_{s+1}^\vee, STO(2f)(\tilde{E}_r)(-2f)).$$
As $ST_{O(2f)}^{-1}$ inducing an wall-hitting transformation at the totally semistable wall $\mathcal{V}_r$, $ST_{O(2f)}$ induces the other wall-hitting transformation to the other side of the wall. Thus $E' := ST_{O(2f)}(\tilde{E}_r)(-2f)$ is $\sigma'$-stable, for some $\sigma'$ on the other side of $\mathcal{V}_r$. The equalities shows that

$$SD_{(r,s)} = SD_{\sigma'_{r+1,s+1}}.$$  

Note that this wall is caused by $O(2f)$ and $(F_{s+1}^+ \vee O(2f)) = -3 \neq 0$, thus by proposition [L8] $SD_{(r+1,s+1)} = 0$.

References

[AB13] Daniele Arcara and Aaron Bertram. Bridgeland-stable moduli spaces for K-trivial surfaces. Journal of the European Mathematical Society, 15(1):1–38, 2013.

[ABCH13] Daniele Arcara, Aaron Bertram, Izzet Coskun, and Jack Huizenga. The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability. Advances in Mathematics, 235:580–626, 3 2013.

[AM14] D. Arcara and E. Miles. Bridgeland Stability of Line Bundles on Surfaces. ArXiv e-prints, January 2014.

[Bay16] A. Bayer. Wall-Crossing implies Brill-Noether. Applications of stability conditions on surfaces. ArXiv e-prints, April 2016.

[BM14a] Arend Bayer and Emanuele Macrì. Projectivity and birational geometry of Bridgeland moduli space. J. Amer. Math. Soc., 27(2014):707–752, 2014.

[BM14b] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. Inventiones Mathematicae, 198(3):505–590, 2014.

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. Annals of Mathematics, 166(2):317–345, 2007.

[Bri08] Tom Bridgeland. Stability conditions on K3 surfaces. Duke Math. J., 141(2):241–291, 2008.

[HL10] Daniel Huybrechts and Manfred Lehn. The geometry of moduli spaces of sheaves. Springer, 2010.

[HMS08] Daniel Huybrechts, Emanuele Macrì, and Paolo Stellari. Stability conditions for generic K3 categories. Compositio Mathematica, 144(1):134–162, 2008.

[Huy06] Daniel Huybrechts. Fourier-Mukai transforms in algebraic geometry. Oxford Univ. Press, 2006.

[LP03] Joseph Le Potier. Dualité étrange sur les surfaces. preprint, preliminary version, pages 09–06, 2003.

[MO07] Alina Marian and Dragos Oprea. A tour of theta dualities on moduli spaces of sheaves. arXiv 07102908, 2007.
[MO13] Alina Marian and Dragos Oprea. Generic strange duality for K3 surfaces, with an appendix by Kota Yoshioka. Duke Mathematical Journal, 162(8):1463–1501, 2013.

[Muk87] On the moduli space of bundles on K3 surfaces. i. in vector bundles on algebraic varieties (bombay, 1984). Tata Inst. Fund. Res. Stud. Math., 11:341–413, 1987.

[O’G97] Kieran G O’Grady. The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface. Journal of Algebraic Geometry, 6(4):599–644, 1997.

[Tod08] Yukinobu Toda. Moduli stacks and invariants of semistable objects on K3 surfaces. Advances in Mathematics, 217(6):2736–2781, 2008.