A PROPERTY OF THE BIDIMENSIONAL SPHERE

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Abstract. It is natural to ask for a reasonable constant $k$ having the property that any open set of area greater than $k$ on a bidimensional sphere of area 1 always contains the vertices of a regular tetrahedron. We shall prove that it is sufficient to take $k = \frac{3}{4}$. In fact we shall prove a more general result. The interested reader will not have any problem in establishing that $\frac{3}{4}$ is the best constant with this property.

Keywords: area; open set; Haar measure; rotation group of the sphere.

Our result is the following:

Theorem 1. Let $n$ be a positive integer, and let $S$ be a bidimensional sphere of area 1. If $M \subset S$ is an open set of area greater than $\frac{n-1}{n}$ and $X \subset S$ is a finite set with $n$ elements, then there exists a rotation $\rho$ of the sphere such that $\rho(X) \subset M$.

In the proof, we use the following result whose proof we postpone:

Lemma 2. Let $M, M' \subset S$ be open sets such that $A(M) > A(M')$. Then there exists a finite number of mutually disjoint spherical caps $U_\alpha$ and rotations $\rho_\alpha$ such that:

(i) $\bigcup_\alpha U_\alpha \subset M$;
(ii) $M' \subset \bigcup_\alpha \rho_\alpha(U_\alpha)$;
(iii) $M \setminus \bigcup_\alpha U_\alpha$ has non-empty interior.

Proof of the Theorem. Let $\mu$ be a Haar measure on $SO(3)$ such that $\mu(SO(3)) = 1$.

For any $A \subset S$, let $\Phi_A$ be the characteristic function of $A$.

Fix $a \in S$ and let $I^A_a \in \mathbb{R}$ be $I^A_a = \int_{SO(3)} \Phi_A \circ x(a) d\mu(x)$.

Remark 3. Note that if $b$ is an arbitrary point on $S$, then $I^A_b = I^A_a$. Indeed if $\rho \in SO(3)$ is such that $\rho(a) = b$ (and such a $\rho$ always exists), then:

$$I^A_b = \int_{SO(3)} \Phi_A \circ x(\rho(a)) d\mu(x) = \int_{SO(3)} \Phi_A \circ (x \circ \rho)(a) d\mu(x \circ \rho)$$

$$= \int_{SO(3)} \Phi_A \circ x(a) d\mu(x),$$

since $d\mu(x \circ \rho) = d\mu(x)$, the Haar measure being rotation invariant.

\footnote{For any $A \subset S$, $A(A)$ denotes its area.}
Moreover, if $B \subset S$ is an open set such that there exists $\rho_1 \in SO(3)$ with $\rho_1(A) = B$, then again $I_a^B = I_a^A$. Indeed,

$$I_a^B = \int_{SO(3)} \Phi_{\rho(A)} \circ x(a) d\mu(x) = \int_{SO(3)} \Phi_A \circ \rho_1^{-1} \circ x(a) d\mu(x)$$

$$= \int_{SO(3)} \Phi_A \circ (\rho_1^{-1} \circ x)(a) d\mu(\rho_1^{-1} \circ x) = I_a^A.$$

Returning to the problem, if $X = \{a_1, \ldots, a_n\}$, let $f : SO(3) \to \mathbb{R}$, $f(x) = \sum_{i=1}^n \Phi_M \circ x(a_i)$.

Note that it is enough to find an $x \in SO(3)$ with $f(x) > n - 1$. Then, since $f(x)$ is an integer $\leq n$, we obtain $f(x) = n$ and hence $x(a_1), \ldots, x(a_n) \in M$, which proves the Theorem. To find such an $x$, it is enough to show that

$$\int_{SO(3)} f(x) d\mu(x) > n - 1.$$

But this means that

$$\sum_{i=1}^n I_{a_i}^M > n - 1,$$

which is implied by

$$I_{a_i}^M > \frac{n - 1}{n},$$

for each $i$, that is

$$I_a^M > \frac{n - 1}{n}.$$

We divide the sphere $S$ in $n$ spherical lunes $F_1, \ldots, F_n$ of equal areas. Obviously, each $F_i$ can be obtained as a rotation of $F_1$. This implies:

$$1 = I_a^S = \sum_{i=1}^n I_a^{F_i} = nI_a^{F_1}, \quad \text{hence} \quad I_a^{F_1} = \frac{1}{n}.$$

Let now $M' = S \setminus F_n$. Then

$$I_a^{M'} = \sum_{i=1}^{n-1} I_a^{F_i} = \frac{n - 1}{n}.$$

With $U_\alpha$ and $\rho_\alpha$ as in the Lemma, we deduce:

$$I_a^M > I_a^{U_\alpha U_\alpha} = \sum_{\alpha} I_a^{U_\alpha} = \sum_{\alpha} I_a^{\rho_\alpha(U_\alpha)} \geq I_a^{M'} = \frac{n - 1}{n},$$

and the proof is complete. \qed

**Proof of the Lemma.** Let $0 < m < 1$ and let $C_i$, for $i \in \{1, \ldots, k\}$, be spherical caps of diameter $d$ such that

$$\bigcup_{i=1}^k C_i = S,$$
and let \( P_i \) be the plane containing the center of \( S \) and parallel to the circle bounding \( C_i \). If \( \pi_i : S \to P_i \) is the orthogonal projection on \( P_i \), we can choose \( \varepsilon \) small enough such that:

- For any open \( C \subset C_i \), we have \( \mathcal{A}(\pi_i(C)) > m \mathcal{A}(C) \).
- For any \( A \neq B \in C_i \), we have the inequality of segment lengths:
  \[
  |\pi_i(A)\pi_i(B)| > m \cdot |AB|.
  \]

Define now \( M_1 = C_1 \cap M, M_2 = C_2 \cap (M \setminus M_1), M_3 = C_3 \cap (M \setminus M_1 \cup M_2), \ldots, M_k = C_k \cap (M \setminus M_1 \cup \cdots \cup M_{k-1}) \), and similarly construct \( M'_1, M'_2, \ldots, M'_k \).

Let \( N_i = \pi_i(M_i), N'_i = \pi_i(M'_i) \). For \( 1 - m \) close enough to 0, we have:

\[
\sum_{i=1}^{k} \mathcal{A}(N_i) > \sum_{i=1}^{k} \mathcal{A}(N'_i).
\]

In each plane \( P_i \), we fix a side length \( \varepsilon \) square lattice. It can be proven (see [1] pag. 315,327) that the number \( n_i \) of squares contained in \( N_i \) is

\[
\frac{1}{\varepsilon^2} \mathcal{A}(N_i) + O\left(\frac{1}{\varepsilon}\right),
\]

and analogously we have an approximation for the number \( n'_i \) of squares contained in \( N'_i \). Hence, for small enough \( \varepsilon \), we get

\[
\sum_{i=1}^{k} n_i > \sum_{i=1}^{k} n'_i.
\]

Therefore, we can choose an injection \( u \) from the set \( \mathcal{P}' \) of squares contained in \( \bigcup_{i=1}^{k} N'_i \) into the set \( \mathcal{P} \) of squares contained in \( \bigcup_{i=1}^{k} N_i \).

Let \( P \in \mathcal{P}' \) (and hence \( P \subset N'_i \) for some \( i \)), let \( Q \in C_i \) be the point whose projection on \( P_i \) is the center of \( P \), and let \( D_P \) be the spherical cap defined as the intersection of \( S \) with the ball centered in \( Q \) and of radius \( \varepsilon/2 \).

Similarly, define \( D_{u(P)} \), corresponding to \( u(P) \). Clearly, \( D_P = \rho_P(D_{u(P)}) \) for some \( \rho_P \in SO(3) \). We remove from \( M \) all the caps \( D_{u(P)} \) and from \( M' \) all the caps \( D_P \), for \( P \in \mathcal{P}' \).

Define now \( s = \mathcal{A}(M), s' = \mathcal{A}(M') \). Since \( \sum n'_i \varepsilon^2 \to \sum \mathcal{A}(N'_i) \), when \( \varepsilon \to 0 \), we can choose \( \varepsilon \) and \( 1 - m \) small enough such the above procedure removes from \( M \) and \( M' \) the sets \( M_1 \) and \( M'_1 \) of area greater than \( \frac{1}{2} s' \).

Inductively, define \( S_i, S'_i \) as follows: \( S_1 = M \setminus M_1 \) and \( S'_1 = M' \setminus M'_1 \). By repeating the above process, obtain the sets \( S_2, S'_2 \) and so on.

Obviously, \( \mathcal{A}(S'_i) < \left(\frac{1}{2}\right)^t \to 0 \) as \( t \) grows to infinity. Since \( \mathcal{A}(S_i) > s - s' > 0 \), there exists some \( t \) such that

\[
\mathcal{A}(S_i) > 4 \cdot \mathcal{A}(S'_i).
\]

Once again, we go through the first step of the above construction applied to the sets \( S_i, S'_i \) with the difference that \( \mathcal{P}' \) will be the minimal set of all squares of lattices in \( P_1 \) which cover \( \bigcup_{i=1}^{k} N'_i \), and \( \mathcal{P} \) will contain all the squares of lattices with side length \( 2\varepsilon \) that are included in \( \bigcup_{i=1}^{k} N_i \). Also, \( D_P \) will be the intersection of \( S \) with the ball centered at \( Q \) and of radius
and $D_u(P)$ is constructed analogously. The circle with the same center as $u(P)$, of radius $\frac{\varepsilon}{\sqrt{2}}$, is included in $u(P)$.

Letting the set of $U_{\alpha}$ be the set of all $D_u(P)$, the conditions (i) – (iii) in the Lemma are satisfied and the proof is complete. \hfill \Box

References

[1] M. R. Murty, J. Esmonde, *Problems in algebraic number theory*, Springer-Verlag (2005)

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