Research Article

Analytical Solutions for the Nonlinear Partial Differential Equations Using the Conformable Triple Laplace Transform Decomposition Method

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In this paper, a novel analytical method for solving nonlinear partial differential equations is studied. This method is known as triple Laplace transform decomposition method. This method is generalized in the sense of conformable derivative. Important results and theorems concerning this method are discussed. A new algorithm is proposed to solve linear and nonlinear partial differential equations in three dimensions. Moreover, some examples are provided to verify the performance of the proposed algorithm. This method presents a wide applicability to solve nonlinear partial differential equations in the sense of conformable derivative.

1. Introduction

Fractional calculus has attracted many researchers in the last decades. The impact of this fractional calculus on both pure and applied branches of science and engineering has been increased. Many researchers started to approach with the discrete versions of this fractional of calculus which are summarized into two approaches: nonlocal (classical) and local. Most popular definitions in the area of nonlocal fractional calculus are the Riemann–Liouville, Caputo, and Grunwald–Letnikov definitions. The obtained fractional derivatives lack some basic properties such as chain rule and Leibniz rule for derivatives [1]. However, the semigroup properties of these fractional operators behave well in some cases. In [2], later on, Khalil et al. (2014) presented a new definition of a local fractional derivative, known as conformable derivative, which is well behaved and obeys the Leibniz rule and chain rule for derivatives. While conformable derivative has been criticized in [3, 4], we believe that the new definition deserves to be explored further with its analysis and applications because many research studies have been conducted on this definition and its applications to various phenomena in physics and engineering. Therefore, throughout this paper, we will call this definition as conformable derivative. It is defined as follows.

For a function \( f : (0, \infty) \rightarrow \mathbb{R} \), the conformable derivative of order \( \alpha \in (0, 1] \) of \( f \) at \( x > 0 \) is defined by

\[
T^{\alpha} f(x) = \lim_{h \to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}.
\] (1)

For this derivative, Ataganana et al. (2018) presented new properties [5] which have been analysed for real valued multivariable functions [6] by Gozutok et al. (2018). In [7], conformable gradient vectors are defined, and a conformable sense Clairaut’s theorem has also been proven. In [8–14], the researchers have worked on the linear ordinary and partial differential equations based on the conformable derivatives. Namely, two new results on homogeneous functions involving their conformable partial derivatives are...
introduced, specifically, homogeneity of the conformable partial derivatives of a homogeneous function and the conformable version of Euler’s theorem. The conformable Laplace transform was studied and modified by Jarad et al. (2019) [15]. The conformable double Laplace transform was defined and applied in [16]. The conformable Laplace transform is not only useful to solve local conformable fractional dynamical systems but also it can be employed to solve systems within nonlocal conformable fractional derivatives that were defined and used in [17]. Finally, it is also a remarkable fact that there are a large number of studies in the theory and application of fractional differential equations. Here, the conformable Laplace transform has been rarely defined and coupled with Adomian decomposition method to solve systematic nonlinear partial fractional differential equations by defining a function in 3-dimensional Laplace transform and resolvent kernel methods was introduced, specifically, homogeneity of the conformable partial fractional differential equation is solved using the proposed method. In Section 5, numerical experiment is conducted using the proposed method to validate the obtained results. In Section 6, a conclusion of our research work is provided.

2. Basic Definitions and Tools

In this section, we provide some fundamental definitions on conformable partial derivatives.

Definition 1. Given a function \( f : R^* \times R^* \times R^* \to R \), the conformable partial fractional derivatives (CPFDs) of orders \( \alpha, \beta, \gamma \) of the function \( f(x, y, t) \) are defined as follows:

\[
\begin{align*}
\partial_x^\alpha f &= \frac{\partial^\alpha}{\partial x^\alpha} f(x, y, t) = \lim_{h \to 0} \frac{f(x + h x^{1-\alpha}, y, t) - f(x, y, t)}{h}, \\
\partial_y^\beta f &= \frac{\partial^\beta}{\partial y^\beta} f(x, y, t) = \lim_{k \to 0} \frac{f(x, y + k y^{1-\beta}, t) - f(x, y, t)}{k}, \\
\partial_t^\gamma f &= \frac{\partial^\gamma}{\partial t^\gamma} f(x, y, t) = \lim_{\epsilon \to 0} \frac{f(x, y, t + \epsilon t^{1-\gamma}) - f(x, y, t)}{\epsilon},
\end{align*}
\]

(2)

where \( 0 < \alpha, \beta, \gamma \leq 1, x, y, t > 0 \), and \( \partial_x^\alpha = (\partial^\alpha/\partial x^\alpha), \partial_y^\beta = (\partial^\beta/\partial y^\beta) \) and \( \partial_t^\gamma = (\partial^\gamma/\partial t^\gamma) \) are called the fractional partial derivatives of orders \( \alpha, \beta, \gamma \), respectively.

We prove the basic Theorem 1 and the relation between the CPFDs and partial derivatives as follows.

Theorem 1. Let \( \alpha, \beta, \gamma \in (0, 1] \) and \( f(x, y, t) \) be a differentiable at a point for \( x, y, t > 0 \). Then,

(i) \( \partial_x^\alpha f = (\partial^\alpha/\partial x^\alpha) f(x, y, t) = x^{1-\alpha} (\partial f(x, y, t)/\partial x) = x^{1-\alpha} \partial_x^\alpha f \)

(ii) \( \partial_y^\beta f = (\partial^\beta/\partial y^\beta) f(x, y, t) = y^{1-\beta} (\partial f(x, y, t)/\partial y) = y^{1-\beta} \partial_y^\beta f \)

(iii) \( \partial_t^\gamma f = (\partial^\gamma/\partial t^\gamma) f(x, y, t) = t^{1-\gamma} (\partial f(x, y, t)/\partial t) = t^{1-\gamma} \partial_t^\gamma f \)

Proof. By the definition of CPFD,

\[
\begin{align*}
\frac{\partial^\alpha}{\partial x^\alpha} f(x, y, t) &= \lim_{h \to 0} \frac{f(x + h x^{1-\alpha}, y, t) - f(x, y, t)}{h}, \\
&= \lim_{\phi \to 0} \frac{f(x + \phi x^{1-\alpha}, y, t) - f(x, y, t)}{\phi x^{1-\alpha}} \\
&= x^{1-\alpha} \lim_{\phi \to 0} \frac{f(x + \phi, y, t) - f(x, y, t)}{\phi} \\
&= x^{1-\alpha} \frac{\partial f(x, y, t)}{\partial x}.
\end{align*}
\]

(3)

Similarity, we can prove the results (ii) and (iii).
In the next proposition, we mention the conformable partial fractional derivative of some functions. By using
Theorem 1, it can be verified easily.

**Proposition 1.** Let $\alpha, \beta, \gamma \in (0, 1]$ and $a, b \in R, l, m, n \in N$. Then, we have the following:

1. $(\partial^\alpha / \partial x^\alpha)(au(x, y, t) + bv(x, y, t)) = a(\partial^\alpha / \partial x^\alpha)u(x, y, t) + b(\partial^\alpha / \partial x^\alpha)v(x, y, t)$
2. $(\partial^\alpha y^\beta / \partial x^\alpha) (x^\alpha y^\beta \partial^\gamma y^\gamma) = lmn x^{-a} y^{-m-\beta} t^{n-\gamma}$
3. $(\partial^\beta y^\gamma / \partial x^\alpha)(x^\alpha y^\beta) = l(\partial^\beta y^\gamma / \partial x^\alpha)(x^\alpha y^\beta)$
4. $(\partial^\alpha / \partial x^\alpha)(\sin(\alpha/y)\cos(t^\gamma/y)) = \cos(x^\alpha/y)\cos(t^\gamma/y)$
5. $(\partial^\beta / \partial y^\beta)(\sin(\alpha/y)\cos(t^\gamma/y)) = -\sin(x^\alpha/y)\sin(y^\beta/y)\cos(t^\gamma/y)$

**3. Some Results and Theorems of the Conformable Triple Laplace Transform**

In this section, we recall some basic definitions on conformable Laplace transform and some results which will be

\[ L^a_\alpha L^b_\beta L^c_\gamma (u(x, y)) = U_{a, b, c}(x, y) = \int_0^\infty \int_0^\infty e^{-p(x^\alpha/y) - q(y^\beta/y)} u(x, y) x^{a-1} y^{b-1} dx dy, \]

where $x, y > 0$, $p, q \in \mathbb{C}$, $\alpha, \beta \in (0, 1]$.

Now, we define the conformable triple Laplace transform, for $\alpha, \beta, \gamma \in (0, 1)$, and $p, q, s \in \mathbb{C}$ are the Laplace variables.

\[ L^a_\alpha L^b_\beta L^c_\gamma (u(x, y, t)) = U_{a, b, c}(p, q, s) = \int_0^\infty \int_0^\infty e^{-p(x^\alpha/y) - q(y^\beta/y) - s(t^e/y)} u(x, y, t) x^{a-1} y^{b-1} t^{c-1} dx dy dt, \]

where $p, q, s \in \mathbb{C}$ are Laplace variables and $\alpha, \beta, \gamma \in (0, 1]$.

\[ u(x, y, t) = L_p^{-1} L_q^{-1} L_s^{-1} (U_{a, b, c}(p, q, s)) \]

\[ = \frac{1}{2\pi i} \int_{a-k\omega} e^{p(x^\alpha/y)} \frac{1}{2\pi i} \int_{b-k\omega} e^{q(y^\beta/y)} \frac{1}{2\pi i} \int_{s-k\omega} e^{s(t^e/y)} U_{a, b, c}(p, q, s) ds dq dp. \]

**Definition 5.** A unit step or Heaviside unit step function is defined as follows:

\[ H\left(\frac{x^a}{a} - a, \frac{y^\beta}{\beta} - b, \frac{t^e}{\gamma} - c\right) = \begin{cases} 1, & x > a, y > b, t > c, \\ 0, & x < a, y < b, t < c. \end{cases} \]
Theorem 2. If \( L^\alpha_x L^\beta_y L^l_t (u(x^\alpha/\alpha, y^\beta/\beta, t^l/y)) = U_{\alpha \beta l}(p, q, s) \), \( L^\alpha_x L^\beta_y L^l_t (v(x^\alpha/\alpha, y^\beta/\beta, t^l/y)) = V_{\alpha \beta l}(p, q, s) \), and \( A, B, \) and \( C \) are constants, then

(a) Linearity property:
\[
L^\alpha_x L^\beta_y L^l_t \left( A u \left( \frac{x^\alpha}{\alpha}, \frac{y^\beta}{\beta}, \frac{t^l}{y} \right) + B v \left( \frac{x^\alpha}{\alpha}, \frac{y^\beta}{\beta}, \frac{t^l}{y} \right) \right) = A U_{\alpha \beta l}(p, q, s) + B V_{\alpha \beta l}(p, q, s).
\]

(b) \( L^\alpha_x L^\beta_y L^l_t (C) = (C/\alpha)^{\alpha/\alpha} (C/\beta)^{\beta/\beta} \) for \( n = 0, 1, 2, 3, \ldots \)

(c) \( L^\alpha_x L^\beta_y L^l_t ((x^\alpha/\alpha)^{\alpha/\alpha} (y^\beta/\beta)^{\beta/\beta} t^{l/y}) = (\Gamma(\alpha + 1))^{\alpha/\alpha} (\Gamma(\beta + 1))^{\beta/\beta} (\Gamma(n + 1))^{\Gamma(n + 1)} p^{l+1} q^{m+1} s^{n+1} \), where \( \Gamma(\cdot) \) is the gamma function. Note that \( \Gamma(1) = 1, \Gamma(1) = 1, \Gamma(1) = 1, \ldots \)

(d) The first shifting theorem for conformable triple Laplace transform:
If \( L^\alpha_x L^\beta_y L^l_t (u(x^\alpha/\alpha, y^\beta/\beta, t^l/y)) = U_{\alpha \beta l}(p, q, s) \), then
\[
L^\alpha_x L^\beta_y L^l_t \left( e^{(x^\alpha/\alpha+b)} (y^\beta/\beta+c) \right) = U_{\alpha \beta l}(p - a, q - b, s - c).
\]

(e) \( L^\alpha_x L^\beta_y L^l_t (u(x^\alpha/\alpha, y^\beta/\beta, t^l/y)) = U_{\alpha \beta l}(p, q, s) \), then
\[
L^\alpha_x L^\beta_y L^l_t \left( \left( \frac{x^\alpha}{\alpha} \right)^m \left( \frac{y^\beta}{\beta} \right)^n \left( \frac{t^l}{y} \right)^d \right) = (-1)^d m+n \frac{\partial^d m+n}{\partial p^d q^m \partial s^n} \left( U_{\alpha \beta l}(p, q, s) \right).
\]

(f) \( L^\alpha_x L^\beta_y L^l_t \left( \sin \left( \frac{A x^\alpha}{\alpha} \right) \sin \left( \frac{B y^\beta}{\beta} \right) \sin \left( \frac{C t^l}{y} \right) \right) = \frac{ABC}{(p^2 + A^2)(q^2 + B^2)(r^2 + C^2)} \).

\[
L^\alpha_x L^\beta_y L^l_t \left( \cos \left( \frac{A x^\alpha}{\alpha} \right) \cos \left( \frac{B y^\beta}{\beta} \right) \cos \left( \frac{C t^l}{y} \right) \right) = \frac{p q s}{(p^2 + A^2)(q^2 + B^2)(r^2 + C^2)}.
\]

Proof. From results (a)–(d) and (f), it can be easily proved by using the definition of conformable triple Laplace transform (CTLT). Here only we see the proof of result (e).

So, by the definition of CTLT (equation (6)), we have
\[
L^\alpha_x L^\beta_y L^l_t \left( u \left( \frac{x^\alpha}{\alpha}, \frac{y^\beta}{\beta}, \frac{t^l}{y} \right) \right) = U_{\alpha \beta l}(p, q, s)
\]
\[
= \int_0^\infty \int_0^\infty \int_0^\infty \left( e^{-p(x^\alpha/\alpha) - q(y^\beta/\beta) - s(t^l/y)} u \left( \frac{x^\alpha}{\alpha}, \frac{y^\beta}{\beta}, \frac{t^l}{y} \right) \times x^{-1} y^{-1} t^{l-1} \right) dx dy dt.
\]

By differentiating with respect to \( p, l \)-times, we get
\[
\]
\[
\frac{d^i}{dp} U_{\alpha \beta \gamma}(p, q, s) = \frac{d^i}{dp} \left\{ \int_0^\infty \int_0^\infty \int_0^\infty e^{-p(x^a/\alpha - q(y^\beta/\beta - s(t^1/\gamma))} \times \left( x^\alpha y^\beta t^1 \right) \times x^{a-1}y^{\beta-1}t^{1-1} \right\} dx \, dy \, dt
\]

(14)

It reduces to

\[
\frac{d^i}{dp} U_{\alpha \beta \gamma}(p, q, s) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{d^i}{dp} \left[ e^{-p(x^a/\alpha - q(y^\beta/\beta - s(t^1/\gamma))} \times \left( x^\alpha y^\beta t^1 \right) \times x^{a-1}y^{\beta-1}t^{1-1} \right] dx \, dy \, dt.
\]

(15)

Now, we again differentiate with respect to \(q\) and \(s\), \(m\) and \(n\)-times, respectively, and we obtain the simplification as follows:

\[
\frac{d^{i+m+n}}{dp \, dq \, ds} U_{\alpha \beta \gamma}(p, q, s) = \int_0^\infty \int_0^\infty \int_0^\infty \left( -\left( \frac{x^a}{\alpha} \right)^i \right) \left( -\left( \frac{y^\beta}{\beta} \right)^m \right) \left( -\left( \frac{t^1}{\gamma} \right)^n \right)
\]

\[
\times e^{-p(x^a/\alpha - q(y^\beta/\beta - s(t^1/\gamma))} \times \left( x^\alpha y^\beta t^1 \right) \times x^{a-1}y^{\beta-1}t^{1-1} \right\} dx \, dy \, dt
\]

(16)

which implies

\[
\frac{d^{i+m+n}}{dp \, dq \, ds} U_{\alpha \beta \gamma}(p, q, s) = (-1)^{i+m+n} \times \left( \frac{x^a}{\alpha} \right)^i \left( \frac{y^\beta}{\beta} \right)^m \left( \frac{t^1}{\gamma} \right)^n \times \left( \frac{x^\alpha y^\beta t^1}{\alpha \beta \gamma} \right) \times x^{a-1}y^{\beta-1}t^{1-1} \right\} dx \, dy \, dt.
\]

(17)

Now, we multiply \((-1)^{i+m+n}\), on both sides, and we get the required result:

\[
L_x^a L_y^\beta L_t^1 \left( \frac{x^a}{\alpha} \right)^i \left( \frac{y^\beta}{\beta} \right)^m \left( \frac{t^1}{\gamma} \right)^n \times \left( \frac{x^\alpha y^\beta t^1}{\alpha \beta \gamma} \right) \right) = (-1)^{i+m+n} \frac{d^{i+m+n}}{dp \, dq \, ds} U_{\alpha \beta \gamma}(p, q, s).
\]

(18)

**Theorem 3.** If \( L_x^a L_y^\beta L_t^1 \left( u(x^a/\alpha, y^\beta/\beta, t^1/\gamma) \right) = U_{\alpha \beta \gamma}(p, q, s) \), then
\[
L_{x,y}^{\alpha,\beta}L_{x,y}^{\rho,\eta} \left( u \left( \frac{x^\alpha - \zeta^\alpha y^\beta - \eta^\beta t^\gamma}{\alpha - \beta - \gamma} \right) \times H \left( \frac{x^\alpha - \zeta^\alpha y^\beta - \eta^\beta t^\gamma}{\alpha - \beta - \gamma} \right) \right) \\
= e^{-p (\zeta/\alpha) - q (\eta/\beta) - s (\theta/\gamma)} U_{a,b,y} (p, q, s),
\]

where \( H(x, y, t) \) is a Heaviside unit step function as defined in equation (8).

\[
LHS = L_{x,y}^{\alpha,\beta}L_{x,y}^{\rho,\eta} \left( u \left( \frac{x^\alpha - \zeta^\alpha y^\beta - \eta^\beta t^\gamma}{\alpha - \beta - \gamma} \right) \times H \left( \frac{x^\alpha - \zeta^\alpha y^\beta - \eta^\beta t^\gamma}{\alpha - \beta - \gamma} \right) \right) \\
= \int_0^\infty \int_0^\infty \int_0^\infty e^{-p (x^\alpha/a) - q (y^\beta/b) - s (t^\gamma/\gamma)} u \left( \frac{x^\alpha - \zeta^\alpha y^\beta - \eta^\beta t^\gamma}{\alpha - \beta - \gamma} \right) \times x^{\alpha-1} y^{1-\beta} t^{1-\gamma} dx dy dt.
\]

Theorem 4. The conformable triple Laplace transform of the function \( ((x^\alpha/a) u(x, y, t), (y^\beta/b) u(x, y, t)) \) is given by

\[ L_{x,y}^{\alpha,\beta}L_{x,y}^{\rho,\eta} \left( u \left( \frac{x^\alpha - \zeta^\alpha y^\beta - \eta^\beta t^\gamma}{\alpha - \beta - \gamma} \right) \times H \left( \frac{x^\alpha - \zeta^\alpha y^\beta - \eta^\beta t^\gamma}{\alpha - \beta - \gamma} \right) \right) \\
= e^{-p (x^\alpha/a) - q (y^\beta/b) - s (t^\gamma/\gamma)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-p (x^\alpha/a) - q (y^\beta/b) - s (t^\gamma/\gamma)} u \left( \frac{x^\alpha - \zeta^\alpha y^\beta - \eta^\beta t^\gamma}{\alpha - \beta - \gamma} \right) \times x^{\alpha-1} y^{1-\beta} t^{1-\gamma} dx dy dt.
\]
Remaining results (b)–(d) can be obtained via the same process. □

**Theorem 5.** For \( \alpha, \beta, \gamma \in (0, 1] \). Let \( u(x, y, t) = u(x^{\alpha}/x, y^{\beta}/y, t^{\gamma}/y) \) be the real valued piecewise continuous function \( x, y, \) and \( t \) of the domain \( D \) on \((0, \infty) \times(0, \infty) \times (0, \infty) \). The CTLT (conformable triple Laplace transform) of the conformable partial fractional derivatives of order \( \alpha, \beta, \) and \( \gamma \) is given by

\[
\begin{align*}
&a L_x^\alpha L_y^\beta L_t^\gamma \left( \frac{\partial^s}{\partial x^s} (u(x^{\alpha}/x, y^{\beta}/y, t^{\gamma}/y)) \right) = p^s U(p, q, s) - U(p, q, s) - U_x(p, q, s) \\
&b L_x^\alpha L_y^\beta L_t^\gamma \left( \frac{\partial^s}{\partial y^s} (u(x^{\alpha}/x, y^{\beta}/y, t^{\gamma}/y)) \right) = q^s U(p, q, s) - q U(p, q, s) - U_y(p, q, s) \\
&c L_x^\alpha L_y^\beta L_t^\gamma \left( \frac{\partial^s}{\partial t^s} (u(x^{\alpha}/x, y^{\beta}/y, t^{\gamma}/y)) \right) = s^s U(p, q, s) - s U(p, q, s) - U_t(p, q, s)
\end{align*}
\]

Proof. Here, we go for proof of result (a), and the remaining results (b)–(g) can be proved. To obtain conformable triple Laplace transform of the partial fractional derivatives, we use integration by parts and Theorem 1.

By applying the definition of CTLT, we have

\[
\begin{align*}
L_x^\alpha L_y^\beta L_t^\gamma & \left( \frac{\partial^s}{\partial x^s} \left( u \left( \frac{x^{\alpha}}{x}, \frac{y^{\beta}}{y}, \frac{t^{\gamma}}{y} \right) \right) \right) = \int_0^\infty \int_0^\infty \left( e^{-p \left( \frac{\alpha}{\alpha} \right) - q \left( \frac{\beta}{\beta} \right) - s \left( \frac{\gamma}{\gamma} \right)} \frac{\partial^s}{\partial x^s} \left( u \left( \frac{x^{\alpha}}{x}, \frac{y^{\beta}}{y}, \frac{t^{\gamma}}{y} \right) \right) \right) dx dy dr \\
&= \int_0^\infty \int_0^\infty \left( e^{-q \left( \frac{\beta}{\beta} \right) - s \left( \frac{\gamma}{\gamma} \right)} \int_0^\infty \left( e^{-p \left( \frac{\alpha}{\alpha} \right) \frac{\partial^s}{\partial x^s} \left( u \left( \frac{x^{\alpha}}{x}, \frac{y^{\beta}}{y}, \frac{t^{\gamma}}{y} \right) \right) \right) dx \right) \times y^{\beta-1} t^{\gamma-1} dy dt.
\end{align*}
\]

(23)

Since we have Theorem 1, \((\partial^s (u)/\partial x^s) = x^{1-s} (du/\partial x)\). We use this result in equation (23).

Therefore, equation (23) becomes

\[
\begin{align*}
L_x^\alpha L_y^\beta L_t^\gamma & \left( \frac{\partial^s}{\partial x^s} \left( u \left( \frac{x^{\alpha}}{x}, \frac{y^{\beta}}{y}, \frac{t^{\gamma}}{y} \right) \right) \right) = \int_0^\infty \int_0^\infty \left( e^{-q \left( \frac{\beta}{\beta} \right) - s \left( \frac{\gamma}{\gamma} \right)} \int_0^\infty \left( e^{-p \left( \frac{\alpha}{\alpha} \right) \frac{\partial^s}{\partial x^s} \left( u \left( \frac{x^{\alpha}}{x}, \frac{y^{\beta}}{y}, \frac{t^{\gamma}}{y} \right) \right) \right) dx \right) \times x^{\beta-1} y^{\gamma-1} dy dt.
\end{align*}
\]

(24)

The integral inside the bracket is given by

\[
\begin{align*}
L_x^\alpha L_y^\beta L_t^\gamma & \left( \frac{\partial^s}{\partial x^s} \frac{\partial^s}{\partial y^s} \frac{\partial^s}{\partial t^s} \left( u \left( \frac{x^{\alpha}}{x}, \frac{y^{\beta}}{y}, \frac{t^{\gamma}}{y} \right) \right) \right) = (pq^s U(p, q, s) - q U(p, q, s) - U(q, 0, s) - pq^s U(p, q, 0) - p U(p, 0, s) - q U(q, 0, s) - s U(0, 0, s) - pq^s U(p, q, 0) - p U(p, 0, s) - q U(q, 0, s) - s U(0, 0, s) - pq^s U(p, q, 0) - p U(p, 0, s) - q U(q, 0, s) - s U(0, 0, s))
\end{align*}
\]

(27)

(b) If \( L_x^\alpha L_y^\beta L_t^\gamma (u(x^{\alpha}/x, y^{\beta}/y, t^{\gamma}/y)) = U_{x, y, \gamma} (p, q, s) \), then
\[
L_z^\alpha L_y^\beta L_t^\gamma \left( \frac{\partial^m}{\partial t^m} \left( u \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right) \right) = s^m U(p, q, s) - \sum_{k=0}^{m-1} s^{m-1-k} U^{(k)}(p, q, 0).
\] (28)

4. Solving Nonlinear Partial Fractional Differential Equation Using the Conformable Triple Laplace Transform Decomposition Method

We consider a general nonlinear nonhomogeneous partial fractional differential equation:

\[
\frac{\partial^m}{\partial t^m} \left( u \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right) + Ru \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) + Nu \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) = g \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right),
\] (29)

where \( m = 1, 2, 3, \ldots \) and \( \gamma \in (0, 1] \) with the initial conditions

\[
\frac{\partial^m}{\partial t^m} \left( u \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right) = f \left( y^\beta, r^\gamma \right),
\] (30)

where \( R \) is the linear differential operator and \( N \) addresses the nonlinear partial fractional operator, and \( g = g(x^\alpha, y^\beta, t^\gamma) \) is the source term. In order to solve equation (29), we follow the following steps:

**Step 1:** applying the conformable triple Laplace transform to equation (29) on both sides, we have

\[
L_z^\alpha L_y^\beta L_t^\gamma \frac{\partial^m}{\partial t^m} \left( u \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right) + L_z^\alpha L_y^\beta L_t^\gamma \left( Ru \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right) + L_z^\alpha L_y^\beta L_t^\gamma \left( Nu \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right) = L_z^\alpha L_y^\beta L_t^\gamma \left( g \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right).
\] (31)

**Using Theorem 5 and equation (30), in equation (31),**

\[
s^m U(p, q, s) - \sum_{k=0}^{m-1} s^{m-1-k} U^{(k)}(p, q, 0) + L_z^\alpha L_y^\beta L_t^\gamma \left( Ru \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right) \]

\[
+ L_z^\alpha L_y^\beta L_t^\gamma \left( Nu \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right) = L_z^\alpha L_y^\beta L_t^\gamma \left( g \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right).
\] (32)

**Step 2:** divide by \( s^m \), and apply the conformable inverse triple Laplace transform to equation (32); it reduces to

\[
u(x, y, t) = G(x, y, t) - L_z^{-1} L_y^{-1} \left[ s^m \left( \frac{L_z^\alpha L_y^\beta L_t^\gamma \left( Ru \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right) + L_z^\alpha L_y^\beta L_t^\gamma \left( Nu \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) \right)}{\frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma}} \right) \],
\] (33)

where \( G(x, y, t) \) represents the term coming from the source term and prescribed initial conditions.

**Step 3:** considering the conformable triple Laplace transform decomposition method, let the solution of equation (29) be an infinite series

\[
u \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right) = \sum_{n=0}^{\infty} u_n \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha \beta \gamma} \right),
\] (34)

and the nonlinear term can be decomposed as
\[ \sum_{n=0}^{\infty} u_n \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha^\beta} \right) = \sum_{n=0}^{\infty} A_n, \quad \text{(35)} \]

where \( A_n \) is called Adomian polynomials of \( u_1, u_2, u_3, \ldots, u_n \), and it can be calculated by the following formula:

\[ \sum_{n=0}^{\infty} u_n \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha^\beta} \right) = G(x, y, t) - L_p^{-1} L_q^{-1} L_s^{-1} \left\{ s^{-m} \left[ L_x^\alpha L_y^\beta L_t^\gamma \left( R \sum_{n=0}^{\infty} u_n \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha^\beta} \right) \right) + L_x^\alpha L_y^\beta L_t^\gamma \left( N \sum_{n=0}^{\infty} A_n \right) \right] \right\}. \quad \text{(37)} \]

Step 4: now, by comparing on both sides of equation (37), we get

\[ u_0 \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha^\beta} \right) = G(x, y, t), \]
\[ u_1 \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha^\beta} \right) = -L_p^{-1} L_q^{-1} L_s^{-1} \left\{ s^{-m} \left[ L_x^\alpha L_y^\beta L_t^\gamma \left( R u_0 \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha^\beta} \right) \right) + NA_0 \right] \right\}, \quad \text{(38)} \]
\[ u_2 \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha^\beta} \right) = -L_p^{-1} L_q^{-1} L_s^{-1} \left\{ s^{-m} \left[ L_x^\alpha L_y^\beta L_t^\gamma \left( R u_1 \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha^\beta} \right) \right) + NA_1 \right] \right\}, \]

and so on...

In general, we write the following recursive formula:

\[ u_{m+1} \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha^\beta} \right) = -L_p^{-1} L_q^{-1} L_s^{-1} \left\{ s^{-m} \left[ L_x^\alpha L_y^\beta L_t^\gamma \left( R u_m \left( \frac{x^\alpha y^\beta t^\gamma}{\alpha^\beta} \right) \right) + NA_m \right] \right\}, \quad \text{(39)} \]

where \( m = 1, 2, 3 \) and \( n = 0, 1, 2, 3, \ldots \).

5. Applications

In this section, a numerical experiment is done using the conformable triple Laplace decomposition method to solve nonlinear homogeneous and nonhomogeneous partial fractional differential equation in 3-dimensional space.

Example 1. To illustrate the proposed method, let us consider the following nonlinear partial fractional differential equation:

\[ \frac{\partial^\gamma u}{\partial t^\gamma} + 6u \frac{\partial^\alpha y^\beta u}{\partial y^\beta \partial x^\alpha} + \frac{\partial^\beta u}{\partial x^{2\beta}} = 0, \quad \text{(41)} \]

with initial condition \( u(x, y, 0) = xy \), and where \( \alpha, \beta, \gamma \in (0, 1], x, y, t \in [0, \infty) \).

Solution 1. Rewrite equation (41) as

\[ \frac{\partial^\gamma u}{\partial t^\gamma} = -6u \frac{\partial^\alpha y^\beta u}{\partial y^\beta \partial x^\alpha} + \frac{\partial^\beta u}{\partial x^{2\beta}}. \quad \text{(42)} \]
Taking the conformable triple Laplace transform on both sides of equation (42), we have

\[ L^{\alpha}_x L^{\beta}_y L^{\sigma}_z \left( \frac{\partial^n u}{\partial t^n} \right) = -L^{\alpha}_x L^{\beta}_y L^{\sigma}_z \left( 6u \frac{\partial^{n+\beta}}{\partial y \partial x^n} + \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \right) \]  \hspace{1cm} (43)

Recalling Theorem 5 (c), \( L^{\alpha}_x L^{\beta}_y L^{\sigma}_z (\frac{\partial^\gamma}{\partial t^\gamma} (u(x, y, t))) = sU(p, q, s) - U(p, q, 0) \).

So, equation (43) reduces to

\[ sU(p, q, s) - U(p, q, 0) = -L^{\alpha}_x L^{\beta}_y L^{\sigma}_z \left( 6u \frac{\partial^{n+\beta}}{\partial y \partial x^n} + \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \right) \]. \hspace{1cm} (44)

By applying the proposed method, in particular, equations (35)–(37), let \( u_0(x, y, t) = xy \), and the recursive relation is given by

\[ u_{n+1} = -L^{\alpha}_x L^{\beta}_y L^{\sigma}_z \left[ \left( \frac{1}{p^\sigma q^\beta s^\alpha} \right) - L^{-1}_x L^{-1}_y L^{-1}_z \left( 6A_n + \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \right) \right], \quad \text{where} \ n = 0, 1, 2, 3, \ldots \]  \hspace{1cm} (47)

where \( A_n \) is the Adomian polynomial to decompose the nonlinear terms by using the relation

\[ A_n = \frac{1}{n!} \left[ \frac{\partial^n}{\partial x^n} f \left( \sum_{i=0}^{\infty} \sigma^j u_i \right) \right]_{\sigma=0} \] \hspace{1cm} (48)

Let the nonlinear term be represented as

\[ f(u_n(x, y, t)) = u_n \frac{\partial^{n+\beta}}{\partial y \partial x^n} \]  \hspace{1cm} (49)

Substituting equation (49) in equation (48), and also calculating \( \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} u_0 \), the resulting expression of equation (47) reduces to the following.

For \( n = 0 \),

\[ A_1 = \frac{1}{1!} \left[ \frac{\partial}{\partial \sigma} f(u_0 + \sigma u_1) \right]_{\sigma=0} = \frac{1}{1!} \left[ \frac{\partial}{\partial \sigma} \left( u_0 + \sigma u_1 \right) \left( \frac{\partial^{n+\beta}}{\partial y \partial x^n} (u_0 + \sigma u_1) \right) \right]_{\sigma=0} \]  \hspace{1cm} (52)

Using equation (51), \( u_0 \), and \( u_1 \) and simplifying, equation (52) becomes

Since \( u(x, y, 0) = xy \), we have

\[ U(p, q, 0) = \frac{1}{p^\beta q^\alpha} \] \hspace{1cm} (45)

Now, by applying the conformable inverse triple Laplace transform to equation (44) and using initial condition equation (45), we obtain from equation (44)

\[ u(x, y, t) = L^{-1}_x L^{-1}_y L^{-1}_z \left( \frac{1}{p^\beta q^\alpha} \right) - \frac{1}{s^\alpha} \left( 6A_n + \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \right), \quad \text{where} \ n = 0, 1, 2, 3, \ldots \] \hspace{1cm} (46)

We have

\[ u_1 = -6x^{2-a} y^{2-\beta} t - (1 - \alpha)x^{1-2a} y. \] \hspace{1cm} (51)
\[ A_1 = u_t \frac{\partial^{\alpha+\beta} (u_0)}{\partial y^\alpha \partial x^\beta} + u_0 \frac{\partial^{\alpha+\beta} (u_1)}{\partial y^\alpha \partial x^\beta} \]

\[ = \left(-6x^{2-a}y^{2-\beta}t - (1 - \alpha)x^{1-2a}yt\right)x^{1-a}y^{1-\beta} \]

\[ + \left(xy \left[-6(2 - \alpha)(2 - \beta)x^{3-2a}y^{3-2\beta}t - (1 - \alpha)(1 - 2\alpha)x^{1-3a}y^{1-\beta}t\right]\right) \]

\[ \frac{\partial^{2\alpha} u_1}{\partial x^{2\alpha}} = \left[-6(2 - \alpha)(2 - 2\alpha)x^{2-3a}y^{2-\beta}t - (1 - \alpha)(1 - 2\alpha)(1 - 3\alpha)x^{1-4a}yt\right]. \]

We have

\[ u_2 = -L_p^{-1}L_q^{-1}L_t^{-1}\left[ \frac{1}{s} L_s^{-1} L_t^{-1} \left[ 6\left(-6x^{2-a}y^{2-\beta}t - (1 - \alpha)x^{1-2a}yt\right)x^{1-a}y^{1-\beta} \right. \right. \]

\[ + 6xy \left[-6(2 - \alpha)(2 - \beta)x^{2-2a}y^{2-2\beta}t - (1 - \alpha)(1 - 2\alpha)x^{1-3a}y^{1-\beta}t\right] \]

\[ \left. + \left[-6(2 - \alpha)(2 - 2\alpha)x^{2-3a}y^{2-\beta}t - (1 - \alpha)(1 - 2\alpha)(1 - 3\alpha)x^{1-4a}yt\right] \right]. \]

Simplifying equation (54), we have

\[ u_2 = 18x^{3-2a}y^{3-2\beta} + 3(1 - \alpha)x^{2-3a}y^{2-\beta} \]

\[ + 18(2 - \alpha)(2 - \beta)x^{3-2a}y^{3-2\beta} + 3(1 - \alpha)(1 - 2\alpha)x^{2-3a}y^{2-\beta} \]

\[ + 3(2 - \alpha)(2 - 2\alpha)x^{2-3a}y^{2-\beta} + \frac{1}{2} \left[(1 - \alpha)(1 - 2\alpha)(1 - 3\alpha)x^{1-4a}y\right]t^2 \]

\[ = \left\{ 18[1 + (2 - \alpha)(2 - \beta)]x^{3-2a}y^{3-2\beta} + 3(2 - \alpha)(2 - 2\alpha)x^{2-3a}y^{2-\beta} + \frac{1}{2} (1 - \alpha)(1 - 2\alpha)(1 - 3\alpha)x^{1-4a}yt \right\} t^2. \]

For \( n = 2, \)

\[ A_2 = \frac{1}{2!} \left[ \frac{\partial^2}{\partial \sigma^2} f(u_0 + \sigma u_1 + 6^2 u_2) \right]_{\sigma=0} \]

\[ = \frac{1}{2!} \left[ \frac{\partial^2}{\partial \sigma^2} \left( u_0 + \sigma u_1 + 6^2 u_2 \left( \frac{\partial^{\alpha+\beta}}{\partial y^\alpha \partial x^\beta} (u_0 + \sigma u_1 + 6^2 u_2) \right) \right) \right]_{\sigma=0} \]

\[ = \frac{1}{2!} \left[ 2u_0 \frac{\partial^{\alpha+\beta}}{\partial y^\alpha \partial x^\beta} (u_0) + 2u_1 \frac{\partial^{\alpha+\beta}}{\partial y^\alpha \partial x^\beta} (u_1) + 2u_2 \frac{\partial^{\alpha+\beta}}{\partial y^\alpha \partial x^\beta} (u_0) \right]. \]
Substituting \( u_0, u_1, \) and \( u_2 \) in equation (56) and simplifying, we obtain

\[
A_2 = \left[ (18[1 + (2 - \alpha)(2 - \beta)] + 18[1 + (2 - \alpha)(2 - \beta)](3 - 2\alpha)(3 - 2\beta) + 36(2 - \alpha)(2 - \beta))x^{4 - 3\alpha}y^{4 - 3\beta}l^2 \\
+ (3(2 - 2\alpha)(3 - 2\alpha) + 3(2 - 2\alpha)(3 - 2\alpha)(2 - \beta) + 6(1 - \alpha)(1 - 2\alpha) + 6(1 - \alpha)(2 - \alpha)(2 - \beta))x^{3 - 4\alpha}y^{3 - 3\beta}l^2 \\
+ \left( \frac{1}{2}(1 - 3\alpha) + \frac{1}{2}(1 - 3\alpha)(1 - 4\alpha) + (1 - \alpha) \right)(1 - \alpha) + 2(2 - \alpha))x^{2 - 5\alpha}y^{2 - \beta}l^2 \right].
\]

Therefore, we have

\[
\frac{\partial^2 u_2}{\partial x^2} = \left[ (18[1 + (2 - \alpha)(2 - \beta)](3 - 2\alpha)(3 - 2\beta)x^{3 - 4\alpha}y^{3 - 2\beta}l^2 \\
+ 3(2 - 2\alpha)(3 - 2\alpha)(2 - 4\alpha)x^{2 - 5\alpha}y^{2 - \beta}l^2 \\
+ \frac{1}{2}(1 - \alpha)(1 - 2\alpha)(1 - 3\alpha)(1 - 4\alpha)(1 - 5\alpha)x^{1 - 6\alpha}y^2l^2 \right].
\]

Simplifying equation (58), we have

\[
u_3 = -\left[ \left\{ 2(18[1 + (2 - \alpha)(2 - \beta)] + 18[1 + (2 - \alpha)(2 - \beta)](3 - 2\alpha)(3 - 2\beta) + 36(2 - \alpha)(2 - \beta))x^{4 - 3\alpha}y^{4 - 3\beta} \\
+ (3(2 - 2\alpha)(3 - 2\alpha) + 3(2 - 2\alpha)(3 - 2\alpha)(2 - \beta) + 6(1 - \alpha)(1 - 2\alpha) + 6(1 - \alpha)(2 - \alpha)(2 - \beta))x^{3 - 4\alpha}y^{3 - 2\beta} \\
+ \left( \frac{1}{2}(1 - 3\alpha) + \frac{1}{2}(1 - 3\alpha)(1 - 4\alpha) + (1 - \alpha) \right)(1 - \alpha) + \left\{ 6[1 + (2 - \alpha)(2 - \beta)](3 - 2\alpha)(3 - 2\alpha)x^{3 - 4\alpha}y^{3 - 2\beta} \\
+ (2 - 2\alpha)(3 - 2\alpha)(2 - 3\alpha)(2 - 4\alpha)x^{2 - 5\alpha}y^{2 - \beta} \\
+ \frac{1}{6}(1 - \alpha)(1 - 2\alpha)(1 - 3\alpha)(1 - 4\alpha)(1 - 5\alpha)x^{1 - 6\alpha}y \right\} l^3 \right],
\]
and so on... The approximate series solution is expressed as

\[
u(x, y, t) = [xy - \{6x^{2-a}y^{2-\beta} + (1-a)x^{1-2a}y\}]t
\]

\[+\left\{18[1 + (2-a)(2-\beta)]x^{3-2a}y^{3-2\beta} + 3(3-2a)(2-2a)x^{2-3a}y^{2-\beta} + \frac{1}{2}(1-a)(1-2a)(1-3a)x^{1-4a}y\right\}t^2
\]

\[-\left\{2(18[1 + (2-a)(2-\beta)]) + 18[1 + (2-a)(2-\beta)](2-a)(2-\beta)(3-2a)(3-2\beta)x^{4-3a}y^{4-3\beta}
\]

\[+ 36(2-a)(2-\beta) + (3-2a)(3-2a)
\]

\[+ 3(2-a)(3-2a)(2-3a)(2-\beta) + 6(1-a)(1-2a) + 6(1-a)(2-a)(2-\beta)x^{3-4a}y^{3-2\beta}
\]

\[+ \left\{\frac{1}{2}(1-3a) + \frac{1}{2}(1-3a)(1-4a) + (1-a)(1-2a)x^{2-5a}y^{2-\beta}\right\}
\]

\[+ \left\{6[1 + (2-a)(2-\beta)](3-2a)(3-2a)x^{3-4a}y^{3-2\beta} + (2-2a)(3-2a)(2-3a)
\]

\[(2-4a)x^{2-5a}y^{2-\beta} + \frac{1}{6}(1-a)(1-2a)(1-3a)(1-4a)(1-5a)x^{1-6a}y\right\}t^3\].

(60)

From equation (60), if we consider \(\alpha = \beta = \gamma = 1\), then the solution of equation (41) reduces to

\[u(x, y, t) = xy - 6xyt + 36xyt^2 - 216xyt^3 + \cdots = \frac{xy}{1 + 6t}
\]

(61)

Figures 1 and 2 show the 3D graphical representations of equation (60) with various values of \(\alpha\) and \(\beta\).

**Example 2.** Consider a nonlinear nonhomogeneous partial fractional differential equation, for \(a, b, \gamma \in (0, 1], x, y, t \in [0, \infty)\).

\[
\frac{\partial^\alpha u}{\partial y^\beta} \frac{\partial^\gamma u}{\partial t^\gamma} - \frac{\partial^\alpha u}{\partial x^\alpha} = u(x, y, t),
\]

(62)

with initial conditions

\[
L_x^\alpha L_y^\beta L_t^\gamma \left(\frac{\partial^\alpha u}{\partial x^\alpha} (u(x, y, t))\right) = p^2 U(p, q, s) - pU(0, q, s) - U_x(0, q, s).
\]

(66)

Equation (66) becomes

\[
\frac{\partial^\alpha u}{\partial x^\alpha} (u(0, y, t)) = yt,
\]

(63)

\[
u_x(0, y, t) = -1.
\]

(64)
\[ p^3 U(p,q,s) = pU(0,q,s) + U_x(0,q,s) + L_x^a L_y^b L_t^c \left( \frac{\partial^\beta u}{\partial y^\beta} \frac{\partial^\gamma u}{\partial t^\gamma} - u(x,y,t) \right). \] (67)

Using initial condition (equation (63)) and taking the inverse triple Laplace transform on equation (67), we obtain

\[ u(x,y,t) = \left[ L_p^{-1} L_q^{-1} L_s^{-1} \left( \frac{1}{pq^2 s^2} - \frac{1}{p^3 qs} \right) + L_p^{-1} L_q^{-1} L_s^{-1} \left( \frac{1}{p^2 L_x^a L_y^b L_t^c} \left( \frac{\partial^\beta u}{\partial y^\beta} \frac{\partial^\gamma u}{\partial t^\gamma} - u(x,y,t) \right) \right) \right]. \] (68)

\[ u(x,y,t) = yt - x + L_p^{-1} L_q^{-1} L_s^{-1} \left( \frac{1}{p^2 L_x^a L_y^b L_t^c} \left( \frac{\partial^\beta u}{\partial y^\beta} \frac{\partial^\gamma u}{\partial t^\gamma} - u(x,y,t) \right) \right). \]
By applying the proposed method, we have the following. Let \( u_0 = yt - x \), and the recursive relation is

\[
\begin{align*}
u_{n+1} &= L_p^{-1} L_q^{-1} L_x^{-1} \left( \frac{1}{p^2} L_x \frac{\partial^\gamma}{\partial t^\gamma} \right) \left( \frac{\partial^\beta}{\partial y^\beta} u_n \frac{\partial^\gamma}{\partial t^\gamma} u_n - u_n \right) \\
&= L_p^{-1} L_q^{-1} L_x^{-1} \left( \frac{1}{p^2} L_x \frac{\partial^\gamma}{\partial t^\gamma} \right) \left( A_n - u_n \right),
\end{align*}
\]

(69)

where \( A_n \) is the Adomian polynomial to decompose the nonlinear terms by using the following relation:

\[
A_n = \frac{1}{n!} \left[ \frac{\partial^n}{\partial \sigma^n} f \left( \sum_{i=0}^{\infty} \sigma^i u_i \right) \right]_{\sigma=0}, \quad \text{where } n = 0, 1, 2, 3, 4, \ldots
\]

(70)

Figure 3: 3D plot of equation (73) for \( y = 0.50; \beta = 0.25; y = 1 \).

Figure 4: 3D plot of equation (73) for \( y = 0.75; \beta = 0.90; y = 1 \).
Let the nonlinear term be represented as

\[ f(u) = \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 u}{\partial t^2}. \]  \hfill (71)

Note that, in Adomian relation, the linear term can be considered, in place of nonlinear. So, you may take \( f(u) = u \), and it leads to the same answer. So, this method is also valid for linear partial fractional differential equation. For \( n = 0 \),

\[ A_0 = f(u_0) \]

\[ u_1 = L_p L_q^{-1} L_s^{-1} \left( \frac{1}{\beta} L_x L_y L_t \left( y^{2-\beta} t^{2-\gamma} - (yt - x) \right) \right) \]

\[ = \frac{x^2}{2} y^{2-\beta} t^{2-\gamma} - \frac{x^2}{2} yt + \frac{x^3}{3!}, \]

\( n = 1, \)

\[ A_1 = \frac{\partial}{\partial \sigma} [ f(u_0 + \sigma u_1) ] \]

\[ = \frac{\partial}{\partial \sigma} \left( \frac{\partial^2 (u_0 + \sigma u_1)}{\partial y^2} \frac{\partial^2 (u_0 + \sigma u_1)}{\partial t^2} \right) \]

\[ = \frac{\partial^2 u_0}{\partial y^2} \frac{\partial^2 u_1}{\partial t^2} + \frac{\partial^2 u_1}{\partial y^2} \frac{\partial^2 u_0}{\partial t^2}, \]  \hfill (72)

\[ A_1 = \left[ \left( 2 - \gamma \right) \frac{x^2}{2} y^{3-2\beta} t^{3-2\gamma} - \frac{x^2}{2} y^{2-\beta} t^{2-\gamma} \right] + \left( 2 - \beta \right) \frac{x^2}{2} y^{3-2\beta} t^{3-2\gamma} - \frac{x^2}{2} y^{2-\beta} t^{2-\gamma} \right), \]

\[ u_2 = L_p L_q^{-1} L_s^{-1} \left( \frac{1}{\beta} L_x L_y L_t \left( \left( 2 - \gamma \right) \frac{x^2}{2} y^{3-2\beta} t^{3-2\gamma} - \frac{x^2}{2} y^{2-\beta} t^{2-\gamma} \right) + \left( 2 - \beta \right) \frac{x^2}{2} y^{3-2\beta} t^{3-2\gamma} - \frac{x^2}{2} y^{2-\beta} t^{2-\gamma} \right) \]

\[ + \frac{x^2}{2} y^{2-\beta} t^{2-\gamma} - \frac{x^2}{2} yt + \frac{x^3}{3!} \right) \right], \]

\[ = \left[ \left( 2 - \gamma \right) \frac{x^4}{4!} y^{3-2\beta} t^{3-2\gamma} - \frac{x^4}{4!} y^{2-\beta} t^{2-\gamma} \right] + \left( 2 - \beta \right) \frac{x^4}{4!} y^{3-2\beta} t^{3-2\gamma} - \frac{x^4}{4!} y^{2-\beta} t^{2-\gamma} \right) - \left( \frac{x^4}{4!} y^{2-\beta} t^{2-\gamma} - \frac{x^4}{4!} yt + \frac{x^5}{5!} \right) \]

\[ u_2 = \left( 4 - \gamma - \beta \right) \frac{x^4}{4!} y^{3-2\beta} t^{3-2\gamma} - \frac{x^4}{8} y^{2-\beta} t^{2-\gamma} + \frac{x^4}{4!} yt - \frac{x^5}{5!}, \]

and so on...

The approximate series solution is written as

\[ u(x, y, t) = (yt - x) + \left( \frac{x^2}{2} y^{2-\beta} t^{2-\gamma} - \frac{x^2}{2} yt + \frac{x^3}{3!} \right) + \left( 4 - \gamma - \beta \right) \frac{x^4}{4!} y^{3-2\beta} t^{3-2\gamma} - \frac{x^4}{8} y^{2-\beta} t^{2-\gamma} + \frac{x^4}{4!} yt - \frac{x^5}{5!} + \cdots. \]  \hfill (73)
For $\alpha, \beta, \gamma \in (0, 1], (x, y, t \in ]0, \infty [)$. 

From equation (73), note that for $\alpha, \beta, \gamma = 1$, the solution of equation (63) reduces to

$$u(x, y, t) = yt - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right) = yt - \sin x. \quad (74)$$

Figures 3 and 4 show the 3D graphical representations of equation (74) with various values of $\gamma$ and $\beta$.

6. Conclusion

In this work, the conformable triple Laplace transform has been investigated using all our obtained novel results and theorems. The new conformable triple Laplace transform decomposition method is applied to find the solution of linear and nonlinear homogeneous and nonhomogeneous partial fractional differential equations. A numerical experiment has been conducted using this proposed method. This proposed method can be applied for simultaneous two or more than two linear and nonlinear partial fractional differential equations. Note that, if we take $\alpha, \beta, \gamma = 1$, in Examples 1 and 2, we obtain an exact solution which was considered in [27]. Our results shed the light on the significance of exploring new generalized methods for solving partial differential equations, particularly nonlinear ones, due to the essential need to explore new analytical solutions to understand the dynamics of solutions for such important equations in physics and engineering.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this study.

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