Defining the local part of a hidden variable model: a comment

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In [Physical Review Letters 101, 050403 (2008)], we showed that quantum theory cannot be explained by a hidden variable model with a non-trivial local part. The purpose of this comment is to clarify our notion of local part, which seems to have caused some confusion in the recent literature.

This notion is based on Bell’s and demands that local hidden variables are physical, the idea being that, if discovered, they would not contradict basic physical principles. We explain why the recent supposed “counterexamples” that have appeared are not counterexamples to our theorem—in fact they are based on a definition of local hidden variables which would allow signalling and is therefore not physical.

**Introduction.**—In a famous paper, John Bell asked the question as to whether there could be hidden (as yet undiscovered) parameters that determine the seemingly random outcomes of quantum experiments [1]. Such parameters he termed local hidden variables. Local hidden variables are physical in the sense that, if they were discovered, they would not contradict the notion of local hidden variables cannot completely determine the experimental outcomes.

In a recent Letter [2], we asked the question as to whether local hidden variables can betray some information about the outcomes. Our main result is that they cannot. For example, for quantum measurements that give equally likely outcomes, there cannot exist undiscovered observables that provide any indication about which outcome is more likely. This result has sparked some controversy (see the articles by Wechsler [3, 4] and Larsson and Cabello [5–8]), at the heart of which is the notion of local hidden variables. The notion we use is based on Bell’s and is physical in the sense above. We give its mathematical definition below.

**The notion of local hidden variables.**—We use the notation of our original Letter [2] which we summarize here (see also Figure 1). Consider a source emitting two particles, which travel to two detectors controlled by Alice and Bob. We assume Alice and Bob are free to choose their measurement settings, which we denote by A and B. The measurement devices generate the outcomes X and Y on Alice’s and Bob’s sides respectively. We introduce local hidden variables, denoted U and V, with an arbitrary joint distribution, P_{UV}. These additional variables are required to be physical in the sense that they would not enable signalling between Alice and Bob, i.e. the relations

\[ P_{X|ABUV} = P_{X|AUV} \quad \text{and} \quad P_{Y|ABUV} = P_{Y|BUV} \quad (1) \]

are satisfied. In addition, U and V are said to be trivial if \( P_{X|AUV} = P_{X|A} \) and \( P_{Y|BUV} = P_{Y|B} \) (i.e. if they do not convey any information about the outcomes, X and Y). By way of illustration, we give an explicit hidden variable model in Appendix A.

Note that hidden variables which are not local according to (1) necessarily violate the assumption that Al-

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1. To quote Bell [1], locality is the requirement that if a theory is supplemented by additional variables then “...the result of a measurement on one system is unaffected by operations on a distant system which it has interacted in the past...”, i.e. operations on separated systems cannot be used to signal. Therefore, local hidden variables might alternatively be called non-signalling hidden variables.

2. Note that the alternative versions of this paper each contain different arguments, to which we respond individually below.

3. The assumption of freedom of choice of a measurement setting, A, implies in particular that, for any pre-existing data, \( \Theta \) (which could include the settings and outcomes of measurements already made), the distribution \( P_{A\theta} \) can be chosen to be product, i.e. of the form \( P_A \times P_\theta \).

4. In [2], we also introduced non-local hidden variables as ones that are not physical in the above sense. These are not needed for our statements, which are only about the marginal distribution \( P_{XY|abuv} \). (NB. the formula for \( P_{XY|abuv} \) given in [2] is for the case where there is a non-local hidden variable, \( W \), which is independent of U and V. More generally, \( P_{XY|abuv} := \sum_w P_W(w)P_{XY|abuvw} \).)

5. Note that the treatment here is more general than that in [2, which is recovered in the case where U contains a copy of V and vice-versa.
show that even classical correlations do not fit into our model. However, this theorem is incorrect: the flaw in the proof is the use of the undeclared (and unphysical) assumption that the hidden variables are completely uncorrelated, i.e., that $P_{BV} = P_U \times P_V$. Moreover, it is easy to see that all classical correlations fit into our model.

In [5], the incorrect theorem of [2] is removed and a fresh criticism is presented. We disagree with the new criticism and point out several problems with it below.

First, a new definition of local is used, which is not physical in the sense described in the introduction. In particular, the equality $P_{AVBUV} = P_A \times P_V$ would not hold, which means that Alice would no longer be able to choose her measurement setting, $A$, freely.

In addition, Larsson and Cabello claim that our definition of local places an additional non-signalling restriction on the non-local part. This is not the case: as we explain in Footnote [3] our result can be formulated without mention of non-local hidden variables, which are hence not restricted in any way.

Furthermore, the authors use their Eqn. (6) (which states that $P_{X|AB} = P_{X|A}$) to justify this claim. We emphasize that this relation does not have to hold to model correlations within our framework (see also Appendix B). However, since all quantum correlations obey this relation, Larsson and Cabello’s reasoning would lead to the conclusion that non-local hidden variables cannot exist for any quantum correlations. This conclusion is not correct: the fact that $P_{X|AB} = P_{X|A}$ does not imply the impossibility of adding a non-local hidden variable $W$ which enables signalling.

Appendix A: A restricted theory with a non-trivial local part.—To illustrate the meaning of a non-trivial local part, we consider a restricted theory in which there are only two devices, each of which can make two possible measurements. This theory can be explained by a hidden variable model with a non-trivial local part. Note that, because it does not contain the full set of quantum correlations, this model is not in contradiction with the main claim in [2].

We label the measurements $A \in \{0, 2\}$ and $B \in \{1, 3\}$ (cf. [2]), and the outputs $X, Y \in \{+1, -1\}$. The correlations are always such that they maximally violate the CHSH inequality [9] and are given in Table I where we define $\alpha := \frac{1}{2} \sin^2 \frac{\pi}{8}$.

In [2], a modified model is proposed. As Eqn. (9) of [2] shows, the hidden variables in this model do not give any information about the measurement outcomes and hence are trivial, in contrast to the authors’ claim. Their model is hence in direct agreement with our result [2].

Appendix B: Signalling correlations.—Theorem 1 of [2] is intended to show that our framework is not general
enough to explain signalling correlations. In fact, the completely signalling correlations $X = B$, $Y = A$ fit trivially into our framework. Since it is impossible to add variables to this model such that the resulting distribution is non-signalling, no model with local hidden variables can exist for any signalling correlations.

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