Approximate Lie Group Analysis of Finite–difference Equations

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Abstract

Approximate group analysis technique, that is, the technique combining the methodology of group analysis and theory of small perturbations, is applied to finite–difference equations approximating ordinary differential equations.

Finite–difference equations are viewed as a system of algebraic equations with a small parameter, introduced through the definitions of finite–difference derivatives. It is shown that application of the approximate invariance criterion to this algebraic system results in relations that can be viewed as prolongation formulae and the invariance criterion for the differential approximation of these finite–difference equations.

This allows us to study the group properties of the finite–difference equations by analyzing the group properties of their differential approximations, which are the differential equations with a small parameter.

In particular, the question of whether the group, admitted by the original differential equation, can be corrected by adding the first–order perturbation to it, so that the resulting group with a small parameter is approximately admitted by the finite–difference approximation, is studied. It is shown by examples that, for a given differential equation, its finite–difference approximation and the group, such a correction may not always be possible. It is also demonstrated that the finite–difference approximation can be modified in such a way that the correction becomes possible.

1 Introduction

An invariance of the differential equations with respect to one–parameter groups of transformations in the space of independent and dependent variables carries important information about the fundamental properties, such as conservation laws, of a physical system that these equations may describe.

Knowledge of the groups admitted by the system of differential equations may also allow one to reduce the order of the system, to find important particular solutions, or to produce new solutions from a single solution which is known. Preserving the group properties of differential equations
in their discretizations seems to be a natural requirement which would ensure that the fundamental properties of the approximated differential equation are preserved in the discrete model.

A step in the direction of analyzing the properties of purely difference equations was made by Maeda [5], where the questions of linearization and reduction of order in ordinary difference equations were treated. Levi and Winternitz [8] applied a Lie groups technique to differential–difference equations arising in study of lattices. Knowledge of these group properties allowed the authors to obtain some non–trivial invariant solutions.

Dorodnitsyn [4] developed a general theory of transformation groups in the spaces of discrete variables. By properly constructing the space of discrete variables and the finite–difference analogues of the differential operators, he demonstrated that the methodology results in the prolongation formulae for the finite–difference derivatives, which are similar to those formulae in the continuous case. This similarity allowed the author to extend some of the group–analytical methods to the finite–difference equations. In particular, the difference analogy of the Noether theorem is proven.

Unfortunately, in general, replacing the system of differential equations with their finite–difference, finite volume or finite element discretizations introduces noninvariance into the discretization. This noninvariance stems from the fact that performing the discretization requires utilizing a discrete set of nodes in the space of independent variables – a computational grid. Formally this implies that the group of transformations acting on the discrete variables needs to leave invariant not only the approximating finite–difference system, but also a set of algebraic equations defining the grid [4]. Although some of the important groups of transformations arising in applications do satisfy this property, generally speaking, this requirement results in the discrete system losing many of the groups of the original differential equation.

The natural way to avoid this restrictiveness of the above approach is to weaken the requirement of invariancy of the finite–difference system. One of the possibilities, suggested and studied by Shokin [7], is to consider the differential approximation of the finite–difference system. By formally applying a Taylor series expansion to the finite–difference approximation, one can obtain a differential equation which can be treated as a perturbation of the original differential equation due to discretization. The perturbation terms, added to the original equation, depend on the step sizes of the grid.

In [7] the general theory of differential approximations of the discrete models is developed and group properties of the differential approximations are studied. Following this methodology one can find the groups admitted by the first differential approximations and deduce some important properties of the discrete model from analysis of these groups.

Despite the simplification introduced by considering the first differential approximations instead of the system of algebraic equations, the approach [7] still requires answering some important questions.

The step sizes of the grid participating in the differential approximation are the parameters which need to be transformed when a group of transformations acts on the independent variable. Since the step size is defined as the difference between the values of the independent coordinate at the end points of the mesh interval, it can easily be seen that the defining equations [6] for the general symmetry group of the given system will no longer be local in the sense that they will contain not only the derivatives of the infinitesimal operator’s components, but also the values of these components at the neighboring nodes. Although for many important symmetry groups this non–locality can be avoided [8], this fact makes finding the general symmetry group as hard as solving the finite–difference equation in closed form.

Another important question related to the approach [7] follows from noticing that, in general,
the symmetry group admitted by the differential approximation of the difference equation does not leave the original difference equation invariant. Therefore, further formalization is needed in order to study the properties of the discrete model through analyzing the groups of transformations admitted by the differential approximation.

In this work the recently developed technique of approximate group analysis due to Baikov, Gazizov and Ibragimov [2] is used as a basis for such a formalization. The discrete model is viewed as the system of algebraic equations with a small parameter introduced through the definitions of the grid step sizes and finite–difference derivatives. The groups of transformations leaving this system of algebraic equations together with the definitions of step sizes and the difference derivatives approximately invariant [2] are then sought. It is demonstrated that the groups of transformations satisfying these properties can also be found by considering the differential approximation of the discrete model, provided that the action of the group on the step sizes is extended in a special way. This “prolongation” of the group to the step sizes is local and therefore it is easier to solve the determining system analytically.

2 Approximate symmetries

In order to make the presentation complete and to introduce some important concepts to be used in the following, the basic notions of approximate symmetries are outlined in this Section following [2].

2.1 Approximate one–parameter groups of transformations

Let \( z = (z_1, ..., z_N) \in \mathbb{R}^N \) denote an independent variable, \( \varepsilon \) be a small parameter, and \( p \) be a fixed natural number.

Consider the following one–parameter family of approximate transformations acting in \( \mathbb{R}^N \):

\[
z' = f(z, \varepsilon, a) + o(\varepsilon^p),
\]

where \( f = (f_1, ..., f_N) \) and \( a \) is a scalar parameter of this family of transformations.

**Definition 1** We say that transformations (1) form an approximate one–parameter group (a.o.p.g.) with respect to the parameter \( a \) if

\[
f(z, \varepsilon, 0) = z + o(\varepsilon^p),
\]

\[
f(f(z, \varepsilon, a), \varepsilon, b) = f(z, \varepsilon, a + b) + o(\varepsilon^p)
\]

and also \( a = 0 \) if \( f(z, \varepsilon, a) = z + o(\varepsilon^p) \) is true for all \( z \).

**Definition 2** An approximate infinitesimal operator corresponding to an a.o.p.g. (1) is

\[
X = \xi^T \partial_z + o(\varepsilon^p),
\]

where \( \partial_z = (\partial_{z_1}, ..., \partial_{z_N})^T \), \( \xi = (\xi_1, ..., \xi_N)^T \) and \( \xi_i(z, \varepsilon) = \frac{\partial}{\partial a} f_i|_{a=0} \), \( i = 1, ..., N \).

The following analogue of the Lie theorem [6] for a.o.p.g. establishes a relation between an a.o.p.g. (1) and its approximate infinitesimal operator (4). The proof can be found in [2].
Theorem 1 (Lie theorem for a.o.p.g.) If the transformation forms an a.o.p.g. with an approximate infinitesimal operator, then the function $f(z, \varepsilon, a)$ satisfies the equation
\[ \frac{\partial}{\partial a} f(z, \varepsilon, a) = \xi(f(z, \varepsilon, a), \varepsilon) + o(\varepsilon^p). \] (5)

Conversely, for arbitrary smooth function, the solution of the Cauchy problem
\[ \frac{dz'}{da} = \xi(z', \varepsilon) + o(\varepsilon^p), \]
\[ z'|_{a=0} = z + o(\varepsilon^p) \]
yields an a.o.p.g. with group parameter $a$.

Example. Consider the case $p = 1$. Assuming that the function $f(z, \varepsilon, a)$ is analytic, write the a.o.p.g. as:
\[ z' = f_0(z, a) + \varepsilon f_1(z, a) + o(\varepsilon) \] (6)

An approximate infinitesimal operator can then be written as
\[ X = X_0 + \varepsilon X_1 + o(\varepsilon) = \xi_0^T(z) \partial_z + \varepsilon \xi_1^T(z) \partial_z + o(\varepsilon), \] (7)

where
\[ \xi_0(z) = \left. \frac{\partial}{\partial a} f_0(z, a) \right|_{a=0}, \quad \xi_1(z) = \left. \frac{\partial}{\partial a} f_1(z, a) \right|_{a=0}. \]

Using (6) and (7) in Theorem 1 we find that $f_0$ and $f_1$ satisfy the equations
\[ \frac{\partial}{\partial a} f_0 = \xi_0(f_0), \]
\[ \frac{\partial}{\partial a} f_1 = \xi_1(f_0) + \xi'_0(f_0) f_1, \]

where $\xi'_0 = \frac{\partial}{\partial z} \xi$. Conversely, the functions $f_0$ and $f_1$ can be found from the approximate infinitesimal operator by solving the following Cauchy problem:
\[ \frac{df}{da} = \xi(f), \] (8)
\[ \frac{df}{da} = \xi(f) + \xi'(f) f, \] (9)
\[ f_0|_{a=0} = z, \quad f_1|_{a=0} = 0. \] (10)
2.2 Approximate invariance

Let $F_i(z, \varepsilon), \ i = 1, \ldots, M$ be given scalar-valued functions and

$$F'(z, \varepsilon) = (F_1(z, \varepsilon), \ldots, F_M(z, \varepsilon))^T. \quad (11)$$

**Definition 3** The equation

$$F(z, \varepsilon) = o(\varepsilon^p) \quad (12)$$

is said to be invariant with respect to an a.o.p.g. \(\ref{eq:aopg}\), if, for all $z$ satisfying \(\ref{eq:z}\), we have

$$F(f(z, \varepsilon, a), \varepsilon) = o(\varepsilon^p). \quad (13)$$

The following criterion of invariance has been proven in \cite{2}.

**Theorem 2 (Invariance criterion)** Let the function \(\ref{eq:function}\) be analytic with respect to the variables $z$ and $\varepsilon$ and also

$$\text{rank} \left( \frac{\partial F}{\partial z}(z, 0) \right) \bigg|_{F(z, 0) = 0} = M. \quad (14)$$

Then the approximate equation \(\ref{eq:approximate}\) is invariant with respect to an a.o.p.g. \(\ref{eq:aopg}\) with an approximate infinitesimal operator \(\ref{eq:operator}\) if and only if

$$X F(z, \varepsilon)\big|_{\ref{eq:approximate}} = o(\varepsilon^p). \quad (15)$$

**Example** \cite{1, 2}. Let us consider the case $p = 1$ and obtain the necessary and sufficient condition under which the equation

$$F(z, \varepsilon) \equiv F_0(z) + \varepsilon F_1(z) = o(\varepsilon), \quad (15)$$

where $F_0$ and $F_1$ are both analytic functions, is invariant with respect to the a.o.p.g. \(\ref{eq:aopg}\), \(\ref{eq:operator}\). First notice that substituting \(\ref{eq:operator}\) into \(\ref{eq:approximate}\), expanding the result in powers of $\varepsilon$ and taking into account analyticity of the involved functions results in the following set of “exact” equations, which is equivalent to the invariance condition \(\ref{eq:approximate}\) with $p = 1$:

$$F(0, f(z, a)) = 0,$$

$$F(0, f(z, a)) + F'(0, f(z, a))f(z, a) = 0,$$

where $F' = \frac{\partial}{\partial z} F$.

Suppose that $F$ satisfies the nondegeneracy condition following from \(\ref{eq:nondegeneracy}\). Then the infinitesimal criterion of invariance can be written, according to Theorem 2, as follows:

$$X \left( F_0(z) + \varepsilon F_1(z) \right)\big|_{\ref{eq:approximate}} = o(\varepsilon), \quad (16)$$
where \( X \) is given by (7). Following \([1, 2]\), we substitute \( z = z_0 + \varepsilon z_1 + o(\varepsilon) \) into (16) in order to obtain the equivalent set of exact equalities:

\[
\begin{align*}
X F(z) & = 0, \\
X F(z) + X F(z) + z \partial_z (X F(z)) & = 0.
\end{align*}
\]

The above two equalities must be satisfied when

\[
\begin{align*}
F(z) & = 0, \\
F(z) + z \partial_z F(z) & = 0,
\end{align*}
\]

as follows from (15).

In the following we further restrict our consideration to the case \( M = 1 \), that is \( F = F_1(z, \varepsilon) \).

In practice, as suggested in \([1, 2]\), it is convenient to get rid of \( z_1 \) in the above relations by employing the following fact. It can easily be seen from (17) and (19) that \( X_0 \) is an infinitesimal operator corresponding to the exact transformation group which leaves equation (17) invariant.

Furthermore, as follows from Proposition 2.10 from \([3, \text{p.84}]\), the left-hand side of (17) can be written as follows:

\[
X_0 F_0(z_0) = \lambda(z_0) F_0(z_0),
\]

where \( \lambda(z_0) \) is some smooth function. Then, using (20) and substituting (21) into (18), we rewrite the determining system (17), (18) together with the relations (19) and (20) as follows:

\[
\begin{align*}
X_0 F(z) & = \lambda(z) F(z), \\
X_0 F(z) + X_0 F(z) & = \lambda(z) F(z).
\end{align*}
\]

If the invariance group for the (“non–perturbed”) equation \( F = 0 \) is known, the relation (22) can be used to determine the smooth function \( \lambda(z) \). This function is then substituted into (23), which must be satisfied by virtue of (13). Solving (23) together with (13), one finds the “perturbation” \( X_1 \) of the infinitesimal operator.

**Example 2.** Letting \( p = 1, \ N = 2, \ z = (z_1, z_2)^T \), equation (15) reads

\[
F(z, \varepsilon) \equiv F(z) + \varepsilon F(z) \equiv z_2^2 - 1 + \varepsilon(z_2^2 \ln z_2 - z_2^2) = o(\varepsilon),
\]

The above equation can be considered as a small perturbation of the equation \( F(z) = 0 \). The latter equation obviously admits the group \( X_0 = \partial_{z_1} \). One may consider a small perturbation of this group given by a.o.p.g. \( X = X_0 + \varepsilon X + o(\varepsilon) \) and establish the condition on \( X \) ensuring that the a.o.p.g. leaves the equation invariant.

First notice that (22) results in \( \lambda(z) \equiv 0 \). Using it in (23), one arrives to the condition

\[
(z_2(z))^2 = (z_1)^2,
\]

which needs to be satisfied when \( (z_2)^2 = 1 \).
One a.o.p.g. satisfying this condition is \( X = (1 + \varepsilon z^2_1)\partial z_1 + \varepsilon z_1 z_2 \partial z_2 + o(\varepsilon) \). Solving the Cauchy problem (8)-(10), one can find the transformations corresponding to this a.o.p.g.:

\[
\begin{align*}
\hat{z}_1 &= z_1 + a + \varepsilon (az_1 + \frac{a^2}{3}) + o(\varepsilon), \\
\hat{z}_2 &= z_2 + \varepsilon (az_1 z_2 + \frac{a^2}{2}) + o(\varepsilon).
\end{align*}
\]

3 Approximate symmetries of finite–difference equations

3.1 Finite-difference approximation as a system of equations with a small parameter

Consider an ordinary differential equation of order \( n \):

\[
F(x, y, y^{(1)}, \ldots, y^{(n)}) = 0,
\]

\( y = y(x), \quad x \in (a, b). \)

Introduce the grid as the set of points \( x_\alpha \in (a, b), \alpha = 1, \ldots, N \), and the grid function \( u_\alpha, \alpha = 1, \ldots, N \) defined at these points and to be used to approximate the values of \( y = y(x) \) at the nodes of the grid.

Consider also the set of algebraic equations with respect to the unknowns \( x_\alpha \) and \( u_\alpha \):

\[
L_\alpha(x_1, \ldots, x_N, u_1, \ldots, u_N) = 0, \quad \alpha = 1, \ldots, N.
\]

In the cases of interest to our work, this set of equations approximates the differential equation (24) in the sense defined below.

In order to introduce a small parameter into (25), extend the system of equations (25) by introducing the new unknowns and adding the following equations to it:

\[
\begin{align*}
x_{\alpha+1} - x_\alpha &= \varepsilon h_\alpha + o(\varepsilon^p), \\
u^{(J)}_{\alpha+1} - u^{(J)}_\alpha &= \sum_{k=1}^{p} \frac{\varepsilon^k}{k!} h_\alpha^{(J+k)} + o(\varepsilon^p), \\
u^{(0)}_{\alpha'} &= u^{(0)}_\alpha', \quad \alpha = 1, \ldots, N-1, \quad \alpha' = 1, \ldots, N, \quad p \geq n, \quad J = 0, \ldots, p,
\end{align*}
\]

where \( \varepsilon \) is a small parameter and the added unknowns, \( h_\alpha \) and \( u^{(J)}_\alpha, \) \( (J = 0, \ldots, 2p) \) will be referred to as, respectively, step sizes and finite–difference derivatives of \( J \)-th order.

Although this system is comprised of algebraic equations, equations (27) can be viewed as a result of a formal expansion in the Taylor series at the points \( x = x_\alpha \), which is used in order to obtain the values of \( u^{(J)}_{\alpha+\beta}, \) where \( \beta = 1 \). The following two Lemmas demonstrate that, by virtue of the equations (24), (27) and (28), this analogy with the Taylor expansion can be extended to the cases when \( x_\alpha \) and \( x_{\alpha+\beta} \) are two arbitrary nodes of the grid, and that it can be extended to the smooth functions of the variables \( x_\alpha \) and \( u_\alpha \).

Lemma 1 If \( 1 \leq \alpha \leq N, \beta \neq 0, \quad 1 \leq \alpha + \beta \leq N \) and \( 0 \leq J \leq p - 1 \), then, by virtue of (24), (27) and (28)

\[
u^{(J)}_{\alpha+\beta} - u^{(J)}_\alpha = \sum_{k=1}^{p-J} \frac{\varepsilon^k}{k!} h_{\alpha+\beta,\alpha}^{(J+k)} + o(\varepsilon^{p-J}), \quad (29)
\]
where

\[ h_{\alpha+\beta,\alpha} = \begin{cases} 
\sum_{i=\alpha}^{\alpha+\beta-1} h_i & \text{if } \beta \geq 1 \\
-\sum_{i=\alpha+\beta} h_i & \text{if } \beta \leq -1 
\end{cases} \]

Proof. Consider first the case \( \beta \geq 1 \) and use induction by \( \beta \). If \( \beta = 1 \), (23) is true because of (27).

Assume (24) is true for some \( \beta = l \geq 1 \). Then, using (26), (27) and (28), we get by direct calculation (see Appendix A for details):

\[ u^{(J)}_{\alpha+l+1} = u^{(J)}_{\alpha} + (u^{(J)}_{\alpha+l+1} - u^{(J)}_{\alpha+l}) + (u^{(J)}_{\alpha+l} - u^{(J)}_{\alpha}) = u^{(J)}_{\alpha} + \sum_{i=1}^{p-l} \frac{\varepsilon i}{i!} h_{\alpha+l+1,\alpha} u^{(J+i)}_{\alpha} + o(\varepsilon^{p-J}), \]  

(30)

which proves the validity of (29) for \( \beta = l + 1 \). Therefore, by induction, the Lemma is true for \( \beta \geq 1 \).

Consider now the case \( \beta \leq -1 \). We again perform induction by \( \beta \) starting from \( \beta = -1 \) and going in the direction of decreasing \( \beta \). To prove the statements on each of the steps of this induction, induction by \( J \) will be used.

To prove (29) for \( \beta = -1 \), consider induction by \( J \) from \( J = p \) to \( J = 0 \). First, for \( J = p \) (27) implies \( u^{(p)}_{\alpha-1} = u^{(p)}_{\alpha} + o(1) \), which means that (29) is valid when \( \beta = -1 \) and \( J = p \). Assume that (29) is true for \( \beta = -1 \) and \( J = J_0 + 1, \ldots, p \). Prove it for \( J = J_0 \). Using (26), (27), (28) and induction hypothesis one obtains by direct calculation (see Appendix A for details):

\[ u^{(J_0)}_{\alpha-1} = u^{(J_0)}_{\alpha} - \sum_{k=1}^{p-J_0} \frac{\varepsilon k h_{\alpha-k-1}}{k!} u^{(J_0-k)}_{\alpha-1} + o(\varepsilon^{p-J_0}) \]

\[ = u^{(J_0)}_{\alpha} - \sum_{k=1}^{p-J_0} \frac{\varepsilon k h_{\alpha-k-1}}{k!} \left[ \sum_{m=0}^{J_0-k} \frac{\varepsilon^m (-h_{\alpha-1})^m}{m!} u^{(J_0-k)+m)}_{\alpha-1} \right] + o(\varepsilon^{p-J_0}) \]

(31)

which proves the validity of (29) for \( \beta = J_0 - 1 \) and all possible values of \( J \).

Now, assume (23) is true for all \( \beta \geq \beta_0 \), where \( \beta_0 \leq -1 \). Prove it for \( \beta = \beta_0 - 1 \). Again use induction by \( J \). For \( J = p \) (29) immediately follows from (27). Suppose (29) is valid for \( J = J_0 + 1, \ldots, p \). Using (26), (27), (28) and induction hypothesis prove that it is true for \( J = J_0 \) (see Appendix A):

\[ u^{(J_0)}_{\alpha+\beta_0-1} = \sum_{k=0}^{p-J_0} \frac{\varepsilon^k (-h_{\alpha+\beta_0-1})^k}{k!} u^{(J_0+k)}_{\alpha+\beta_0} + o(\varepsilon^{p-J_0}) \]

\[ = \sum_{k=0}^{p-J_0} \frac{\varepsilon^k (-h_{\alpha+\beta_0-1})^k}{k!} \left[ \sum_{m=0}^{p-J_0-k} \frac{\varepsilon^m h_{\alpha+\beta_0-1}^m}{m!} u^{(J_0+k)+m)}_{\alpha+\beta_0} \right] + o(\varepsilon^{p-J_0}) \]

(32)
which proves (29) for all possible $\beta \leq -1$ and $J$.

Lemma 2 If $1 \leq \alpha \leq N$, and $\xi(x, u)$ is a $p$ times continuously differentiable function, then, by virtue of (26), (27) and (28)

$$
\xi(x_{\alpha+1}, u_{\alpha+1}) - \xi(x_{\alpha}, u_{\alpha}) = \sum_{m=1}^{p} \frac{(\varepsilon h_{\alpha})^{m}}{m!} D^{m} \xi(x_{\alpha}, u_{\alpha}) + o(\varepsilon^{p}),
$$

(33)

where $D = \partial_{x} + u^{(1)} \partial_{u} + u^{(2)} \partial_{u^{(1)}} + ...$

Proof. Using smoothness of the function $\xi(x, u)$ and rearranging terms in a Taylor expansion (see Appendix B) write

$$
\xi(x_{\alpha+1}, u_{\alpha+1}) = \sum_{m=0}^{p} \frac{(\varepsilon h_{\alpha})^{m}}{m!} \left[ \sum_{i=0}^{m} \frac{(m-i)!}{n!} L_{(n)}(m-i) \partial_{u} \xi(x_{\alpha}, u_{\alpha}) \right] + o(\varepsilon^{p}),
$$

(34)

where

$$
L_{(n)}^{(l)} = \sum_{l_{1}+...+l_{n}=l} L_{l_{1}}...L_{l_{n}}, \quad n \neq 0,
$$

$L_{l} = u^{(l)}/l!$ and $L_{(0)}^{(l)} = \delta_{l,0}$. To prove the Lemma, show that the expression in square brackets in (34) is equal to

$$
D^{m} \xi(x_{\alpha}, u_{\alpha}) = \left[ \partial_{x} + \sum_{j} u^{(j+1)} \partial_{u^{(j)}} \right]^{m} \xi(x_{\alpha}, u_{\alpha}) = \sum_{i=0}^{m} C^{i}_{m} \left( \sum_{j} u^{(j+1)} \partial_{u^{(j)}} \right)^{m-i} \partial_{x}^{i} \xi(x_{\alpha}, u_{\alpha})
$$

(35)

Comparing equations (35) and (34), one can see that it is sufficient to prove that the expressions in round brackets in these equations are equal for all values of $l = m - i$ varying between 0 and $m$, ie

$$
\sum_{n=0}^{l} \frac{n!}{n!} L_{(n)}^{(l)} \partial_{u}^{n} \xi(x_{\alpha}, u_{\alpha}) = \left( \sum_{j} u^{(j+1)} \partial_{u^{(j)}} \right)^{l} \xi(x_{\alpha}, u_{\alpha})
$$

(36)

This equality is proven using induction by $l$. When $l = 0$, the equality becomes an identity. Assume that the equality is true for $l = k < m$. By applying the differential operator $\sum_{j} u^{(j+1)} \partial_{u^{(j)}}$ to both sides of (36) written for $l = k$ and manipulating the terms in the left–hand side of the resulting equation (see Appendix B), one obtains equation (36) with $l = k + 1$. This completes the proof of the Lemma.

$\square$
Consider one of the equations (24) corresponding to \( \alpha = \alpha_0 \). Using equations (26), (27) and Lemma 1 equation (25) may be rewritten as follows:

\[
L_{\alpha_0}(x_1, \ldots, x_N, u_1, \ldots, u_N) = \sum_{i=1}^{\alpha_0-1} h_i, \ldots, x_0, \ldots, x_{\alpha_0} + \sum_{i=\alpha_0}^{N-1} h_i,
\]

\[
u_{\alpha_0} + \frac{p}{k!} \left( - \sum_{i=1}^{\alpha_0-1} h_i \right)^k u_0^{(k)} + \ldots, u_{\alpha_0}, \ldots, u_{\alpha_0} + \frac{p}{k!} \sum_{i=\alpha_0}^{N-1} h_i u_0^{(k)} + o(\varepsilon^p) \tag{37}
\]

\[
= L_{\alpha_0}^{(0)}(x_0, u_0^{(0)}, u_0^{(1)}, \ldots, u_0^{(p)}, h_1, \ldots, h_{N-1}) + \sum_{i=1}^{r} \varepsilon^i L_{\alpha_0}^{(i)}(x_0, u_0^{(0)}, u_0^{(1)}, \ldots, u_0^{(p)}, h_1, \ldots, h_{N-1}) + o(\varepsilon^r),
\]

where \( 1 \leq r \leq p \) is some integer number.

**Definition 4** The system of algebraic equations (24), (26), (27) and (28) is said to approximate equation (24) at the node \( x_{\alpha_0} \), if, for all values of \( x_0, u_0, u_1, \ldots, u_p, h_1, \ldots, h_{N-1}, \) we get

\[
L_{\alpha_0}^{(0)}(x_0, u_0^{(0)}, u_0^{(1)}, \ldots, u_0^{(p)}, h_1, \ldots, h_{N-1}) = F(x_0, u_0^{(0)}, u_0^{(1)}, \ldots, u_0^{(n)}),
\]

where \( F \) is the function from (24).

**Definition 5** A differential equation

\[
F(x, y, y^{(1)}, \ldots, y^{(n)}) + \sum_{i=1}^{r} \varepsilon^i L_{\alpha_0}^{(i)}(x, y, y^{(1)}, \ldots, y^{(p)}, h_1, \ldots, h_{N-1}) = 0, \tag{38}
\]

where \( 1 \leq r \leq p \), \( \varepsilon \) is a small parameter and \( h_1, \ldots, h_{N-1} \) are parameters (not small) is called a differential approximation of \( r \)-th order at the node \( x_{\alpha_0} \) for the system of equations (24), (26), (27) and (28). If \( r = 1 \), the differential equation (38) is called the first differential approximation at the node \( x_{\alpha_0} \).

Assuming that the set of equations (25), (26), (27) and (28) approximates the differential equation (24), one can rewrite the former set of algebraic equations as follows:

\[
F(x_{\alpha'}, u_{\alpha'}^{(0)}, u_{\alpha'}^{(1)}, \ldots, u_{\alpha'}^{(n)}) + \sum_{i=1}^{r} \varepsilon^i L_{\alpha'}^{(i)}(x_{\alpha'}, u_{\alpha'}^{(0)}, u_{\alpha'}^{(1)}, \ldots, u_{\alpha'}^{(p)}, h_1, \ldots, h_{N-1}) = o(\varepsilon^r), \tag{39}
\]

\[
x_{\alpha+1} - x_{\alpha} - \varepsilon h_{\alpha} = o(\varepsilon^r), \tag{40}
\]

\[
u_{\alpha+1}^{(J)} - u_{\alpha}^{(J)} - \sum_{k=1}^{p} \varepsilon^k h_{\alpha}^{k(J+k)} = o(\varepsilon^r), \tag{41}
\]

\[
u_{\alpha+1}^{(0)} - u_{\alpha}^{(0)} = 0, \tag{42}
\]

where \( \alpha = 1, \ldots, N - 1, \quad \alpha' = 1, \ldots, N, \quad 1 \leq r \leq p, \quad p \geq n, \quad J = 0, \ldots, p. \)

As a result of the above, we obtain the set of equations (39), (40), (41), (42) with a small parameter \( \varepsilon \). In the following, approximate symmetries of this system and their relation to the symmetries of the differential equation (24) will be studied.
Example. Consider a general first order differential equation resolved with respect to the first derivative:

\[ F(x, y, y^{(1)}) \equiv y^{(1)} - f(x, y) = 0, \quad x \in (0, 1) \] (43)

and its finite-difference approximation based upon forward differencing:

\[ L_\alpha(x_\alpha, x_{\alpha+1}, u_\alpha, u_{\alpha+1}) \equiv \frac{u_{\alpha+1} - u_\alpha}{x_{\alpha+1} - x_\alpha} - f(x_\alpha, u_\alpha) = 0, \quad \alpha = 1, ..., N - 1. \] (44)

Set \( p = 2 \) and obtain the system (39), (40), (41), (42) corresponding to the finite-difference equation (44). Using (40) and (41) in (44), we get:

\[ u^{(1)}_\alpha - f(x_\alpha, u^{(0)}_\alpha) + \frac{\varepsilon h_\alpha}{2} u^{(2)}_\alpha = o(\varepsilon), \]

\[ x_{\alpha+1} - x_\alpha - \varepsilon h_\alpha = o(\varepsilon), \]

\[ u^{(j)}_{\alpha+1} - u^{(j)}_\alpha - \varepsilon h_\alpha u^{(j+1)}_\alpha = o(\varepsilon), \]

\[ u^{(0)}_\alpha - u^{(\alpha')} = 0, \]

where \( \alpha = 1, ..., N - 1, \quad \alpha' = 1, ..., N, \quad J = 0, 1, 2. \)

According to the definitions given, the above system of finite-difference equations approximates a differential equation (43) at all nodes \( x_\alpha, \alpha = 1, ..., N - 1. \) At any of these nodes the first differential approximation corresponding to (44) is

\[ y^{(1)} - f(x, y) + \frac{\varepsilon h_\alpha}{2} y^{(2)} = 0. \]

As can be seen from this Example, the term with a small parameter, appearing in the first differential approximation, is linear with respect to \( h_\alpha. \) The following Theorem gives the necessary and sufficient condition that equation (38) with \( r = 1 \) needs to satisfy in order to be a first differential approximation of some finite-difference approximation of equation (24).

Theorem 3 Let \( N > p > n, \) then the equation

\[ F(x, y, y^{(1)}, ..., y^{(n)}) + \varepsilon L^{(1)}_{\alpha_0}(x, y, y^{(1)}, ..., y^{(p)}, h_1, ..., h_{N-1}) = 0, \] (45)

is a first differential approximation of some finite-difference approximation of equation (24) at the node \( x_{\alpha_0} \) if and only if

\[ X(\varepsilon L^{(1)}_{\alpha_0}) \equiv 0, \] (46)

where

\[ X = -\varepsilon \partial_x + \sum_{i=1}^{n} h_\alpha \partial_{\alpha_i}. \] (47)

Proof. 1) Let the condition (46) be satisfied. The following describes the procedure resulting in a finite-difference approximation which has (45) as its first differential approximation.

The differential operator \( X \) in (47) can be treated as an infinitesimal operator of the o.p.g. acting in the space of \( p + N + 2 \) variables \( \varepsilon, x, y, y^{(1)}, ..., y^{(p)}, h_1, ..., h_{N-1}. \) As one can easily see,
Taking the limit

\[ \varepsilon L^{(1)}_{\alpha_0} = \tilde{f}_1(x, y, y^{(1)}, ..., y^{(p)}, \varepsilon h_1, ..., \varepsilon h_{N-1}) = \tilde{f}_1(x_{\alpha_0}, y, y^{(1)}, ..., y^{(p)}, x_2 - x_1, ..., x_N - x_{N-1}) = \tilde{f}_1(y, y^{(1)}, ..., y^{(p)}, x_1, ..., x_N) \]

for some smooth functions \( \tilde{f} \) and \( \tilde{f}_1 \).

Using Theorem 2.17 (\[1\], p.88) and formally substituting \( x_{\alpha + 1} - x_\alpha \) for each expression \( \varepsilon h_\alpha \) and \( x_{\alpha_0} \) for \( x \), we establish that

\[ \varepsilon L^{(1)}_{\alpha_0} = \tilde{f}_1(x, y, y^{(1)}, ..., y^{(p)}, \varepsilon h_1, ..., \varepsilon h_{N-1}) = \tilde{f}_1(x_{\alpha_0}, y, y^{(1)}, ..., y^{(p)}, x_2 - x_1, ..., x_N - x_{N-1}) = \tilde{f}_1(y, y^{(1)}, ..., y^{(p)}, x_1, ..., x_N) \]

for some function \( \tilde{f}_1 \). Acting in a similar way, substitute the arguments \( y, y^{(1)}, ..., y^{(n)} \) with their finite-difference approximations of the order higher than the first in the expression for \( F(x, y, y^{(1)}, ..., y^{(n)}) \), and also substitute \( x \) with \( x_{\alpha_0} \) in the same expression:

\[ F(x, y, y^{(1)}, ..., y^{(n)}) = \hat{f}_0(u_1, ..., u_N, x_1, ..., x_N) + o(\varepsilon) \]

for some function \( \hat{f}_0 \).

Now chose the finite-difference approximation at the node \( x_{\alpha_0} \) to be

\[ L(u_1, ..., u_N, x_1, ..., x_N) \equiv \hat{f}_0(u_1, ..., u_N, x_1, ..., x_N) + \tilde{f}_1(u_1, ..., u_N, x_1, ..., x_N) = 0, \quad (48) \]

where \( \hat{f}_0 \) and \( \tilde{f}_1 \) are the previously constructed functions.

Reversing the above speculation, we can see that, by virtue of (29) and (26):

\[ L(u_1, ..., u_N, x_1, ..., x_N) = F(x_{\alpha_0}, u_{\alpha_0}, u_0^{(1)}, ..., u_0^{(n)}) + \varepsilon L^{(1)}_{\alpha_0}(x_{\alpha_0}, u_{\alpha_0}, u_0^{(1)}, ..., u_0^{(n)}, h_1, ..., h_{N-1}) + o(\varepsilon), \]

that is, (45) is a first differential approximation of (48).

2) Let (23) with \( \alpha = \alpha_0 \) approximate (24). Introduce the following notation:

\[ \tilde{L}_{\alpha_0}(x_{\alpha_0}, u_{\alpha_0}, u_0^{(1)}, ..., u_0^{(n)}) = L_{\alpha_0}(x_{\alpha_0} - \varepsilon \sum_{i=1}^{\alpha_0-1} h_i, ..., x_{\alpha_0}, ..., x_{\alpha_0} + \varepsilon \sum_{i=\alpha_0}^{N-1} h_i, u_{\alpha_0} + \sum_{k=1}^{p} \frac{\varepsilon k}{k!} \left( \sum_{i=1}^{\alpha_0-1} h_i \right)^{k} u_{\alpha_0}^{(k)}, ..., u_{\alpha_0}, ..., u_{\alpha_0} + \sum_{k=1}^{p} \frac{\varepsilon k}{k!} \left( \sum_{i=\alpha_0}^{N-1} h_i \right)^{k} u_{\alpha_0}^{(k)} \). \]

Notice that it follows from this definition and (37), that \( X \tilde{L}_{\alpha_0} \equiv 0 \) and also that \( \tilde{L}_{\alpha_0}^{(1)}(\varepsilon) = (\partial_\varepsilon \tilde{L}_{\alpha_0}|_{\varepsilon=0} \).

Using these relations, write

\[ \sum_{i=1}^{n} h_{\alpha_0} \partial_{\alpha_i} (\partial_\varepsilon \tilde{L}_{\alpha_0}) = \partial_\varepsilon \left( \sum_{i=1}^{n} h_{\alpha_0} \partial_{\alpha_i} \tilde{L}_{\alpha_0} \right) = \partial_\varepsilon (\varepsilon \partial_\varepsilon \tilde{L}_{\alpha_0}) = \partial_\varepsilon \tilde{L}_{\alpha_0} + \varepsilon \partial_\varepsilon^2 \tilde{L}_{\alpha_0}. \]

Taking the limit \( \varepsilon \to 0 \) in the latter equation results in (47).

\( \square \)
3.2 Approximate groups leaving finite–difference approximation invariant

Consider a one–parameter group of transformations acting on the variables \( x \) and \( y \) and represented by its infinitesimal operator \( X = \xi(x,y)\partial_x + \varphi(x,y)\partial_y \). Action of this group is extended to the derivatives \( y^{(1)}, \ldots, y^{(n)} \) using prolongation formulae [6]:

\[
X = \xi(x,y)\partial_x + \varphi(x,y)\partial_y + \sum_{i=1}^{n} \varphi^{(i)}(\partial_{y^{(i)}}),
\]

where

\[
\varphi^{(i)} = D\varphi^{(i-1)} - y^{(i)}D\xi, \quad \varphi^{(0)} = \varphi.
\]

Action of this group of transformations can also be naturally extended to the “finite–difference” variables \( x_\alpha \) and \( u_\alpha \), \((\alpha = 1, \ldots, N)\) as

\[
X^{(\Delta)} = \sum_{\alpha=1}^{N} \xi(x_\alpha, u_\alpha)\partial_{x_\alpha} + \varphi(x_\alpha, u_\alpha)\partial_{u_\alpha}.
\]

In order to extend action of this group to the variables \( u^{(J)}_1, \ldots, u^{(J)}_N \), \((J = 1, \ldots, 2p)\) and \( h_1, \ldots, h_{N-1} \) while preserving “locality” of the prolongation formulae, consider an a.o.p.g. represented by its infinitesimal operator

\[
X^{(\Delta)} = \sum_{k=0}^{r} \varepsilon^k X^{(\Delta)}_k + o(\varepsilon^r),
\]

\[
X^{(\Delta)}_k = \sum_{\alpha=1}^{N} \left[ \xi(x_\alpha, u_\alpha)\partial_{x_\alpha} + \varphi(x_\alpha, u_\alpha)\partial_{u_\alpha} + \sum_{i=1}^{2r} \varphi^{(i)}(\partial_{u^{(i)}_\alpha}) \right] + \sum_{\alpha=1}^{N-1} \sigma_\alpha \partial_{h_\alpha}.
\]

In the following, we will be interested in those of the groups (52), which leave invariant the set of approximating finite–difference equations (39), (40) and (41). The following Theorem establishes the analogues of the prolongation formulae (50) for the finite–difference equations:

**Theorem 4** Equations (40) and (41) are invariant with respect to the a.o.p.g. (52) if and only if

\[
\sigma_\alpha = \sum_{l=1}^{k+1} \frac{h_l^l}{l!} D^l \xi_{k+1-l}(x_\alpha, u_\alpha),
\]

\[
\varphi^{(J+1)}_k = D\varphi^{(J)}_k - u^{(J+1)}_\alpha D\xi_k(x_\alpha, u_\alpha),
\]

where \( 0 \leq k \leq r - 1 \).

**Proof.** Using the infinitesimal criterion of approximate invariance (Theorem 2), apply the operator (52) to equation (40) to obtain:

\[
\xi_{\alpha+1} - \xi_\alpha + \sum_{k=1}^{r} \varepsilon^k (\xi_{\alpha+1} - \xi_\alpha - \sigma_{\alpha-1}) = o(\varepsilon^r).
\]
Using Lemma 2, rewrite the left–hand side of the above equation as follows:

\[
\sum_{k=0}^{r} \varepsilon^{k} (\xi_{k+1} - \xi_{k}) - \sum_{k=1}^{r} \varepsilon^{k} \sigma_{k-1} = \sum_{k=0}^{r} \sum_{l=1}^{r} \varepsilon^{k+l} \frac{h_{l}^{k}}{l!} D_{l}^{i} \xi_{k} - \sum_{k=1}^{r} \varepsilon^{k} \sigma_{k-1} + o(\varepsilon^{r})
\]

\[= \sum_{i=1}^{r} \varepsilon^{i} \left( \sum_{l=1}^{i} \frac{h_{l}^{k}}{l!} D_{l}^{i} \xi_{i-l} - \sigma_{i-1} \right) + o(\varepsilon^{r}),\]

which immediately results in (53).

Similarly, having applied Theorem 2 to equation (41), one obtains

\[
\sum_{k=0}^{r} \varepsilon^{k} \left( \varphi_{k+1}^{(J)} - \varphi_{k}^{(J)} \right) - \sum_{i=1}^{r} \varepsilon^{i} \left[ \frac{h_{i}^{k}}{i!} \varphi_{k}^{(J+i)} + \frac{h_{i}^{k-1}}{(i-1)!} \sigma_{k} u_{\alpha}^{(J+i)} \right] = o(\varepsilon^{r}),
\]

which must hold by virtue of (41) and (11). Applying Lemma 2 and (53), transform the left–hand side of the above equation as follows:

\[
\sum_{i=1}^{r} \sum_{k=0}^{r} \varepsilon^{i+k} \left\{ \frac{h_{i}^{k}}{i!} D_{i}^{j} \varphi_{k}^{(J)} - \frac{h_{i}^{k-1}}{(i-1)!} \sigma_{k} u_{\alpha}^{(J+i)} \right\}
\]

\[= \sum_{i=1}^{r} \varepsilon^{i} \sum_{k=0}^{r} \frac{h_{i}^{k}}{i!} D_{i}^{j} \varphi_{k}^{(J)} - \varphi_{k}^{(J+i)} - \frac{h_{i}^{k-1}}{(i-1)!} \sigma_{k} u_{\alpha}^{(J+i)} + o(\varepsilon^{r})
\]

\[= \sum_{i=1}^{r} \varepsilon^{i} \frac{h_{i}^{k}}{i!} \left\{ D_{i}^{j} \varphi_{k}^{(J)} - \varphi_{k}^{(J+i)} - \sum_{n=1}^{r} C_{i}^{n-1} u_{\alpha}^{(J+n)} D_{i}^{n-1} \xi_{l} \right\} + o(\varepsilon^{r}).\]

Equating the obtained expression to \(o(\varepsilon^{r})\) results in the following set of equalities:

\[
\sum_{i=1}^{r} \frac{h_{i}^{k}}{i!} \left\{ D_{i}^{j} \varphi_{k}^{(J)} - \varphi_{k}^{(J+i)} - \sum_{n=1}^{r} C_{i}^{n-1} u_{\alpha}^{(J+n)} D_{i}^{n-1} \xi_{l} \right\} = 0, \tag{55}
\]

where \(l = 1, 2, ..., r\).

Use induction by \(l\) to prove (54) for all \(0 \leq k \leq r - 1\). Setting \(l = 1\) in (53) immediately results in (54) with \(k = 0\). Assume (54) holds for all \(k = 0, 1, ..., k_{1}\) and prove it for \(k = k_{1} + 1 \leq r - 1\). Consider (53) with \(l = k_{1} + 2\). It is easy to see that the terms in the brackets, corresponding to \(i = 2, ..., k_{1} + 2\) nullify because of the induction hypothesis, while the remaining term, being equated to 0, gives (54) with \(k = k_{1} + 1\).

One can immediately notice the similarity between the prolongation formulae for the finite–
difference derivatives (54) and their continuous analogues (53). This similarity allows one to analyze and to establish some important group properties of the finite–difference approximations in terms of the group properties of their differential approximations. Really, consider also the system of algebraic equations with a small parameter (39), (10), (11) and (12) resulting from some discretization of the equation (24). According to Theorem 4, the necessary and sufficient condition for the subsystem (10), (11) to remain invariant under the action of the group (52) is that \(\varphi_{k}^{(J)}\) and \(\sigma_{k} \alpha\) are given by (53) and (53). Using these expressions and applying the operator (52) to equation
(39), one can see that the obtained system of determining equations is the same as the one resulting from the application of the infinitesimal criterion of invariance to the differential approximation (38) provided that action of the group is extended on $h_\alpha$ according to (53).

**Example.** Consider again an ordinary differential equation (24) and the approximating set of algebraic equations (39)–(42) with $r = 1$. The first differential approximations corresponding to the nodes $x_\alpha, \alpha = 1, ..., N$ read:

$$F(x, y, y^{(1)}, ..., y^{(n)}) + \varepsilon L^{(1)}_\alpha(x, y, y^{(1)}, ..., y^{(p)}, h_1, ..., h_{N-1}) = 0. \quad (56)$$

Consider the following a.o.p.g.

$$X = \sum_{k=0}^{1} \varepsilon^k \left( X_k + \sum_{\alpha=1}^{N-1} \sigma_{k \alpha} \partial h_\alpha \right) + o(\varepsilon), \quad (57)$$

where

$$X_k = \xi_k(x, y) \partial_x + \varphi_k(x, y) \partial_y + \sum_{i=1}^{2p} \varphi_k^{(i)}(i) \partial y^{(i)},$$

$$\varphi_k^{(i)} = D \varphi_k^{(i-1)} - y^{(i)} D \xi_k,$$

$$\sigma_{k \alpha} = \sum_{l=1}^{k+1} h_{\alpha l} \frac{D \xi_{k+1-l}}{k+1-l}, \quad (58)$$

that is, in addition to conventional prolongation formulae (58), the relation (53) is used to extend the action of the group on the parameters $h_\alpha$.

Using Theorem 2 and the methodology outlined in the Example following this theorem, write the necessary and sufficient conditions, under which equation (56) is invariant with respect to the a.o.p.g. (57), in the following way:

$$X_0 F - \lambda F = 0, \quad (59)$$

$$\left( X_1 F + \left( X_0 + \sum_{\alpha=1}^{N-1} \sigma_{k \alpha} \partial h_\alpha \right) L^{(1)}_\alpha - \lambda L^{(1)}_\alpha \right) \Bigr|_{F=0} = 0, \quad (60)$$

where $\lambda = \lambda(x, y^{(1)}, ..., y^{(n)})$ is some smooth function and the fact that $F$ does not depend on $h_\alpha$ has been taken into account. It follows from Theorem 3 that $L^{(1)}_\alpha$ must be a linear function of the parameters $h_\alpha$. Using this fact and also noticing that the first term in (60) does not depend on $h_\alpha$ and therefore should be equated to zero separately, rewrite (59) and (60) as follows:

$$X_0 F - \lambda F = 0, \quad (61)$$

$$\left( X_1 L^{(1)}_\alpha + (D \xi_0 - \lambda) L^{(1)}_\alpha \right) \Bigr|_{F=0} = 0, \quad (62)$$

$$\left( X_0 F \right) \Bigr|_{F=0} = 0, \quad (63)$$

The condition (61) implies that $X_0$ must be admitted exactly by the original differential equation (24). This condition also allows one to determine the function $\lambda$. If the additional condition (62)
is satisfied, the group $X_0$ can be corrected by adding a small perturbation $\varepsilon X_1$ to it, so that the resulting corrected group leaves the first differential equation (50) invariant. Finally, (53) shows that the correction $X_1$ may be any group admitted by the original differential equation (24), or it can also be an operator with all zero components.

**Definition 6** A group (49), leaving the differential equation (24) invariant, is called stable with respect to the approximation resulting in (39), (40), (41) and (42) if and only if there exists an infinitesimal operator $X_1$, such that the corresponding a.o.p.g. $X_0^{(\Delta)} + \varepsilon X_1^{(\Delta)} + o(\varepsilon)$ leaves the system of equations (39), (40), (41) and (42) invariant up to the $o(\varepsilon)$ terms, i.e. the a.o.p.g. $X_0^{(\Delta)} + \varepsilon X_1^{(\Delta)} + o(\varepsilon)$ leaves invariant the system (39), (40), (41) and (42), in which $r$ is set to be 1.

Using this notion, the results of this Section can be summarized in the following Theorem.

**Theorem 5** A group (49) leaving the differential equation (24) invariant is stable with respect to the approximation which has the first differential approximation (45) if and only if

$$\left( X_0 L_\alpha^{(1)} + (D(\xi) - \lambda) L_\alpha^{(1)} \right)\big|_{F(x,y,y^{(1)},...,y^{(n)})=0} = 0$$

for all $1 \leq \alpha \leq N$.

### 3.3 Examples

**Example.** Consider the following ordinary differential equation:

$$F(x, y, y') \equiv (y - x)y' + x + y = 0,$$

$$y = y(x), \quad x \in (a, b)$$

and the group

$$X = y\partial_x - x\partial_y - (1 - (y')^2)\partial_{y'} - 3y''y'\partial_{y''}.$$  

Since $XF = -y'F$, the above group leaves the equation invariant and $\lambda = -y'$.

Consider also the following finite–difference approximation of this differential equation:

$$(y_\alpha - x_\alpha)\frac{y_{\alpha+1} - y_\alpha}{x_{\alpha+1} - x_\alpha} + x_\alpha + y_\alpha = 0,$$

where $x_\alpha \in (a, b)$ and $\alpha = 1, ..., N$.

The first differential approximation corresponding to this finite–difference approximation reads:

$$(y - x)y' + x + y + \varepsilon \frac{h_\alpha}{2} y''(y - x) = o(\varepsilon),$$

and therefore, $L_\alpha^{(1)} = \frac{h_\alpha}{2} y''(y - x)$.

From the above we get:

$$XL_\alpha^{(1)} + (D(\xi) - \lambda)L_\alpha^{(1)} = -\frac{h_\alpha}{2} y''((y - x)y' + x + y) = 0,$$

16
if \((y - x)y' + x + y = 0\).
Therefore the considered group is stable.

**Example.** The ordinary differential equation

\[
F(x, y, y') \equiv y' - (x + 1)y^2 - y = 0,
\]

\[
y = y(x), \quad x \in (a, b)
\]
is invariant with respect to the group

\[
X = (y + xy^2)\partial_y + (y' + y^2 + 2xyy')\partial_{y'} + (y'' + 4yy' + 2x(y')^2 + 2xyy'')\partial_{y''},
\]
because \(XF = (1 + 2xy)F\), and therefore, \(\lambda = 1 + 2xy\).

Consider also the following finite–difference approximation of this differential equation:

\[
y_{\alpha+1} - y_\alpha x_{\alpha+1} - x_\alpha - (x + 1)y_\alpha^2 - y_\alpha = 0,
\]

where \(x_\alpha \in (a, b)\) and \(\alpha = 1, \ldots, N\).

Applying the procedure described above, one obtains the following first differential approximation corresponding to this finite–difference approximation:

\[
y' - (x + 1)y^2 - y + \varepsilon \frac{h_\alpha}{2} y'' = o(\varepsilon),
\]

and \(L^{(1)}_\alpha = \frac{h_\alpha}{2} y''(y - x)\).

From the above we get:

\[
XL^{(1)}_\alpha + (D(\xi) - \lambda)L^{(1)}_\alpha = h_\alpha^2 (1 + (x + 1)y)(2 + x + xy(x + 1)),
\]

and the expression on the right–hand side is not identically zero when \(y' - (x + 1)y^2 - y = 0\).

Therefore the considered group is not stable.

**Example.** Build a finite–difference approximation for the differential equation in the previous example, such that the group considered in the same example becomes stable.

First, find a first differential approximation of such a finite–difference approximation:

\[
y' - (x + 1)y^2 - y + \varepsilon L^{(1)}_\alpha(x, y, y', y'', h_\alpha) = o(\varepsilon)
\]

\(L^{(1)}_\alpha\) can be sought as

\[
L^{(1)}_\alpha(x, y, y', y'', h_\alpha) = \frac{h_\alpha}{2} y'' + h_\alpha L^*(x, y).
\]

The stability condition (64) then results in the following equation for \(L^*(x, y)\):

\[
(1 + xy)^2 \left(\frac{L^*}{y(1 + xy)}\right)_y + (1 + (x + 1)y)(2 + x + xy(x + 1)) = 0,
\]

which gives

\[
L^* = -\frac{(x + 1)^2}{x} y(1 + xy) - \frac{y}{x^2} + C(x)y(1 + xy),
\]
where $C(x)$ is an arbitrary function.

Therefore, the first differential approximation of the stable finite–difference approximation must read:

$$y' - (x + 1)y^2 - y + \varepsilon h x \left(\frac{y''}{2} - \frac{(x + 1)^2}{x} y^2 (1 + xy) - \frac{y}{x^2} + C(x)y(1 + xy)\right) = o(\varepsilon).$$

In the previous example we observed that the term with the second derivative in brackets appears as a result of approximation of the first derivative using forward differencing. This observation leads us to the following finite–difference approximation of the given equation, which has the above first differential approximation:

$$\frac{y_{\alpha+1} - y_{\alpha}}{x_{\alpha+1} - x_{\alpha}} - (x_{\alpha} + 1)y_{\alpha}^2 - y_{\alpha} + (x_{\alpha+1} - x_{\alpha}) \left(\frac{(x_{\alpha} + 1)^2}{x_{\alpha}} y_{\alpha}^2 (1 + x_{\alpha}y_{\alpha}) - \frac{y_{\alpha}}{x_{\alpha}^2} + C(x_{\alpha})y_{\alpha}(1 + x_{\alpha}y_{\alpha})\right) = 0.$$

As the above consideration demonstrates, adding the last term to the finite–difference approximation results in the considered group becoming stable.

A. Calculations from the proof of Lemma 1

Derivation of equation (30):

$$u^{(J)}_{\alpha+1} = u^{(J)}_{\alpha} + (u^{(J)}_{\alpha+1} - u^{(J)}_{\alpha}) + (u^{(J)}_{\alpha+1} - u^{(J)}_{\alpha+1})$$

$$= u^{(J)}_{\alpha} + \sum_{k=1}^{p} \frac{\varepsilon k h_{\alpha+1} - (J+k)}{k!} u^{(J+k)}_{\alpha+1} + \sum_{k=1}^{p-J} \frac{\varepsilon k h_{\alpha+1} - (J+k)}{k!} u^{(J+k)}_{\alpha+1} + o(\varepsilon^{p-J})$$

$$= u^{(J)}_{\alpha} + \sum_{k=1}^{p-J} \frac{\varepsilon k h_{\alpha+1} - (J+k)}{k!} \left[ h_{\alpha+1}^{(J+k)} + h_{\alpha+1}^{(J+k)} u^{(J+k)}_{\alpha} + o(\varepsilon^{p-J}) \right]$$

$$= u^{(J)}_{\alpha} + \sum_{k=1}^{p-J} \frac{\varepsilon k}{k!} \left[ h_{\alpha+1}^{(J+k)} \sum_{m=0}^{\infty} \frac{\varepsilon m h_{\alpha+1} - (J+k+m)}{m!} u^{(J+k+m)}_{\alpha} + h_{\alpha+1}^{(J+k)} \right] + o(\varepsilon^{p-J})$$

$$= \sum_{k=1}^{p-J} \sum_{m=0}^{\infty} \frac{\varepsilon k}{k!} m! h_{\alpha+1}^{(J+k+m)} \left[ \sum_{m=0}^{\infty} \frac{\varepsilon m h_{\alpha+1} - (J+k+m)}{m!} u^{(J+k+m)}_{\alpha} + h_{\alpha+1}^{(J+k)} \right] + o(\varepsilon^{p-J})$$

$$= \sum_{k=1}^{p-J} \sum_{i=0}^{\infty} \frac{\varepsilon i}{i!} h_{\alpha+1}^{(J+i)} u^{(J+i)}_{\alpha} + \sum_{i=0}^{\infty} \frac{\varepsilon i}{i!} h_{\alpha+1}^{(J+i)} u^{(J+i)}_{\alpha} + o(\varepsilon^{p-J})$$

$$= \sum_{i=0}^{p-J} \frac{\varepsilon i}{i!} u^{(J+i)}_{\alpha} + \sum_{i=0}^{p-J} \frac{\varepsilon i}{i!} h_{\alpha+1}^{(J+i)} u^{(J+i)}_{\alpha} + o(\varepsilon^{p-J}) = u^{(J)}_{\alpha} + \sum_{i=1}^{p-J} \frac{\varepsilon i}{i!} h_{\alpha+1}^{(J+i)} u^{(J+i)}_{\alpha} + o(\varepsilon^{p-J}).$$

To prove (31) use (24), (27), (28) and the induction hypothesis:

$$v^{(J)}_{\alpha-1} = u^{(J)}_{\alpha} - \sum_{k=1}^{p-j} \frac{\varepsilon k h_{\alpha-1}^{(J+k)}}{k!} u^{(J+k)}_{\alpha-1} + o(\varepsilon^{p-J}).$$
To obtain equation (34), use the notations introduced in the proof of Lemma 2 and write:

\[
B \sum_{i=1}^{\infty} u_{\alpha}^{(J_{i})} = u_{\alpha}^{(J_{0})} - \sum_{k=1}^{p-J_{0}} \frac{\varepsilon^{k}}{k!} \left( \sum_{m=0}^{p-J_{0}} \frac{\varepsilon^{m}(-h_{\alpha-1})^{m}}{m!} u_{\alpha}^{(J_{0}+k+m)} \right) + o(\varepsilon^{p-J_{0}})
\]

Similarly, utilizing (24), (27), (28) and the induction hypothesis, we derive (32):

\[
u_{\alpha+\beta_{0}-1}^{(J_{0})} = \sum_{k=0}^{p-J_{0}} \frac{\varepsilon^{k}(-h_{\alpha+\beta_{0}-1})^{k}}{k!} u_{\alpha+\beta_{0}}^{(J_{0}+k)} + o(\varepsilon^{p-J_{0}})
\]

B Calculations from the proof of Lemma 2

To obtain equation (24), use the notations introduced in the proof of Lemma 2 and write:

\[
\left( \sum_{l=1}^{p} (\varepsilon h_{\alpha})^{l} L_{l} \right)^{n} = \sum_{l=n}^{p} (\varepsilon h_{\alpha})^{l} L_{(n)} + o(\varepsilon^{p}).
\]

Using this relation and changing the order of summation, write

\[
\xi(x_{\alpha+1}, u_{\alpha+1}) = \sum_{k=0}^{p} \sum_{i=0}^{k} \frac{1}{k!} \left( C^{i}_{k} \frac{\partial^{k}}{\partial x^{i} \partial u^{k-i}} (x_{\alpha+1} - x_{\alpha})^{i} (u_{\alpha+1} - u_{\alpha})^{k-i} \right) + o(\varepsilon^{p})
\]

19
\begin{align*}
&= \sum_{k=0}^{p} \sum_{l=0}^{k} \sum_{i=0}^{p} \frac{1}{k!} \left\{ C_i \frac{\partial^{k+i} \xi(x, u)}{\partial x^i \partial u^{k-i}} (\varepsilon h_\alpha)^i \left( \sum_{l=1}^{p} (\varepsilon h_\alpha)^i \frac{u_a}{l!} \right)^{k-i} \right\} + o(\varepsilon^p) \\
&= \sum_{m=0}^{p} \frac{\varepsilon h_\alpha}{m!} \left\{ \sum_{l=0}^{p} \sum_{i=k}^{m} \frac{1}{k!} \sum_{l=0}^{p} \frac{C_i \frac{\partial^{k+i} \xi(x, u)}{\partial x^i \partial u^{k-i}} (\varepsilon h_\alpha)^i L_{(l)}^{(m-i)} (\varepsilon h_\alpha)^i \left( \sum_{l=1}^{p} (\varepsilon h_\alpha)^i \frac{u_a}{l!} \right)^{k-i} \right\} + o(\varepsilon^p) \\
&= \sum_{m=0}^{p} \frac{\varepsilon h_\alpha}{m!} \left\{ \sum_{l=0}^{p} \sum_{i=k}^{m} \frac{(m-i)!}{n!} L_{(n)} \frac{\partial^i \xi(x, u)}{\partial x^i \partial u^{k-i}} \right\} + o(\varepsilon^p).
\end{align*}

Prove now equation (33) with \( l = k + 1 \) using the fact that the same equation is valid for \( l = k \). First notice that, as follows from (33), \( L_{(n)}^{(l)} \) can be expressed as follows:

\[ L_{(n)}^{(l)} = \left( \frac{1}{n!} \frac{d^l}{da^l} \left( \sum_{i=1}^{p} \frac{a^i}{i!} u(i) \right)^n \right) \bigg|_{a=0}. \]  

(66)

Having applied the operator \( \sum_J u^{(j+1)} \partial_u^{(j)} \) to both sides of equation (33) with \( l = k \), use (66) and the expressions for \( L_{(k)}^{(k)} \) and \( L_{(0)}^{(k)} \) to transform the left-hand side as follows (the arguments of \( \xi \) are omitted for brevity):

\[
\sum_J u^{(j+1)} \partial_u^{(j)} \left\{ \sum_{n=0}^{k} \frac{k!}{n!} L_{(n)}^{(k)} \partial_u^a \xi \right\} = \sum_{n=0}^{k} \frac{k!}{n!} L_{(n)}^{(k)} \partial_u^{a+1} \xi + \sum_{n=0}^{k} \frac{k!}{n!} \left( \sum_J u^{(j+1)} \partial_u^{(j)} L_{(n)}^{(k)} \right) \partial_u^a \xi
\]

\[
= \sum_{n=0}^{k} \frac{k!}{n!} L_{(n-1)}^{(k)} n u^{(1)} + \sum_J u^{(j+1)} \partial_u^{(j)} L_{(n)}^{(k)} \partial_u^a \xi + (u^{(1)})^k \partial_u^{a+1} \xi
\]

\[
= \sum_{n=1}^{k} \frac{k!}{n!} \frac{d^k}{da^k} \left( \sum_{i=1}^{p} \frac{a^i}{i!} u(i) \right)^n \sum_{i=0}^{n-1} \frac{a^i}{i!} u(i+1) \bigg|_{a=0} \partial_u^a \xi + (u^{(1)})^k \partial_u^{a+1} \xi
\]

\[
= \sum_{n=1}^{k} \frac{k!}{n!} \frac{d^k}{da^k} \left( \sum_{i=1}^{p} \frac{a^i}{i!} u(i) \right)^n \sum_{i=0}^{n-1} \frac{a^{i-1}}{(i-1)!} u(i) \bigg|_{a=0} \partial_u^a \xi + (u^{(1)})^k \partial_u^{a+1} \xi
\]

\[
= \sum_{n=1}^{k} \frac{k!}{n!} \frac{d^k}{da^k} \left( \sum_{i=1}^{p} \frac{a^i}{i!} u(i) \right)^n \sum_{i=0}^{n-1} \frac{a^{i-1}}{(i-1)!} u(i) \bigg|_{a=0} \partial_u^a \xi + (u^{(1)})^k \partial_u^{a+1} \xi
\]

\[
= \sum_{n=1}^{k} \frac{k!}{n!} \frac{d^k}{da^k} \left( \sum_{i=1}^{p} \frac{a^i}{i!} u(i) \right)^n \bigg|_{a=0} \partial_u^a \xi + (u^{(1)})^k \partial_u^{a+1} \xi
\]

\[
= \frac{1}{n!} \frac{d^k}{da^k} \left( \sum_{i=1}^{p} \frac{a^i}{i!} u(i) \right)^n \bigg|_{a=0} \partial_u^a \xi + (u^{(1)})^k \partial_u^{a+1} \xi
\]

\[
= \frac{1}{n!} \frac{d^k}{da^k} \left( \sum_{i=1}^{p} \frac{a^i}{i!} u(i) \right)^n \bigg|_{a=0} \partial_u^a \xi + (u^{(1)})^k \partial_u^{a+1} \xi
\]

and therefore, equation (33) with \( l = k + 1 \) holds.

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