Structure-preserving techniques in accelerator physics

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ABSTRACT

To a very good approximation, particularly for hadron machines, charged-particle trajectories in accelerators obey Hamiltonian mechanics. During routine storage times of eight hours or more, such particles execute some $10^8$ revolutions about the machine, $10^{10}$ oscillations about the design orbit, and $10^{13}$ passages through various bending and focusing elements. Prior to building, or modifying, such a machine, we seek to identify accurately the long-term behavior and stability of particle orbits over such large numbers of interactions. This demanding computational effort does not yield easily to traditional methods of symplectic numerical integration, including both explicit Yoshida-type and implicit Runge-Kutta or Gaussian methods. As an alternative, one may compute an approximate one-turn map and then iterate that map. We describe some of the essential considerations and techniques for constructing such maps to high order and for realistic magnetic field models. Particular attention is given to preserving the symplectic condition characteristic of Hamiltonian mechanics.

KEYWORDS

symplectic map, symplectic jet, geometric integration, Lie algebra, Poincaré generating function, Cremona map, Cremona symplectification

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1. Introduction

Particle accelerators consist of arrays of magnets and radio-frequency (rf) cavities. Their purpose is to produce intense beams of high-energy charged particles including electrons, positrons (the antimatter counterpart of electrons), protons, antiprotons, and various ions. The magnets provide magnetic fields that bend, focus, and exert various nonlinear effects on beam particle orbits; and the rf cavities provide electric fields that accelerate and longitudinally bunch particles.

Successful accelerator design requires the accurate calculation/simulation of particle orbits over long periods of time. The most computationally challenging are orbits in storage rings, essentially circular machines in which particles continually circulate for long periods of time. Electron (or positron) storage rings are used to produce intense high-energy X-rays. Electron (or positron) storage rings, as well as proton (or antiproton or ion) storage rings, are used in pairs to produce colliders. In a collider, the beam from one ring collides head-on with the counter-circulating beam in a second ring. For example, the Large Hadron Collider (LHC) in CERN (near Geneva, Switzerland), which collides protons on protons, has a circumference of 27 km, and each ring has some 19,200 elements (magnets, rf cavities, and intervening drift spaces). Protons moving at essentially the speed of light are stored for about 8 hours, and during this time make approximately $4 \times 10^8$ turns around the ring, approximately $8 \times 10^{12}$ element passages, and approximately $20 \times 10^9$ (betatron) oscillations about the design orbit. Following this number of oscillations is comparable to following the earth’s orbit about the sun from the time of the Big Bang. While the number of betatron oscillations in electron (or positron) storage rings is comparable, they need not be followed for as long because those oscillations are damped by the energy loss associated with X-ray emission. Correspondingly, electron/positron storage-ring orbits are less computationally challenging than proton/antiproton/ion storage-ring orbits. This chapter is devoted to the most challenging problem of calculating/simulating particle orbits over long periods of time in proton (or antiproton or ion) storage rings.

2. Lagrangians and Hamiltonians

In Cartesian coordinates, the relativistic Lagrangian $L$ for the motion of a particle of mass $m$ and charge $q$ in an electromagnetic field is given by the expression

$$L(r, v, t) = -mc^2(1 - v^2/c^2)^{1/2} - q\psi(r, t) + qv \cdot A(r, t).$$  \hspace{1cm} (1)

Here $r$ is the particle position at time $t$, $v = dr/\, dt$ is the particle velocity, and $c$ is the speed of light. The quantities $\psi$ and $A$ are the scalar and vector potentials defined in such a way that the electromagnetic fields $E$ and $B$ are given by the standard relations

$$B = \nabla \times A, \hspace{1cm} (2a)$$

$$E = -\nabla \psi - \partial A/\partial t. \hspace{1cm} (2b)$$

\hspace{1cm}

\textsuperscript{1}But, even with the highest achievable beam densities, beam-beam collisions are sufficiently rare that the stored beams are only partially depleted over the storage time. However, the colliding beams can have significant dynamical (both linear and nonlinear) effects on each other, a complication that must be understood/managed but lies beyond the scope of this chapter.
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This formulation ignores spin, radiation reaction (X-ray emission, also referred to as synchrotron radiation), and quantum effects [1]. These effects may be important over long times for lighter particles such as electrons and positrons, but they are significantly less important for heavier particles such as protons and antiprotons and ions.

For the Lagrangian (1) the canonical momentum in Cartesian coordinates is given by the equation

$$p^{\text{can}} = \frac{\partial L}{\partial \dot{v}} = \frac{mv}{(1 - v^2/c^2)^{1/2}} + qA.$$  \hfill (3)

Here we use the superscript \textit{can} to emphasize that (3) defines the canonical momentum. Note that the first term in (3) is just the relativistic mechanical momentum,

$$p^{\text{mech}} = \frac{mv}{(1 - v^2/c^2)^{1/2}} = \gamma mv,$$  \hfill (4)

where \(\gamma\) is the standard relativistic factor

$$\gamma = \frac{1}{(1 - v^2/c^2)^{1/2}}.$$  \hfill (5)

Consequently, the relation (3) may also be written in the forms

$$p^{\text{can}} = p^{\text{mech}} + qA \quad \text{and} \quad p^{\text{mech}} = p^{\text{can}} - qA.$$  \hfill (6)

Upon implementing the standard procedure that relates Lagrangians and Hamiltonians, one finds that the Hamiltonian \(H\) associated with the Lagrangian \(L\) specified by (1) is given by the expression

$$H = \sqrt{m^2c^4 + c^2(p^{\text{can}} - qA) \cdot (p^{\text{can}} - qA)} + q\psi = \sqrt{m^2c^4 + c^2(p^{\text{can}} - qA)^2} + q\psi.$$  \hfill (7)

In the usual Hamiltonian formulation (as in the usual Lagrangian formulation) the time \(t\) plays the distinguished role of an independent variable, and all the coordinates \(q\) and momenta \(p\) are dependent variables.\footnote{We are embarrassed by the custom of also using the symbol \(q\) to denote the charge of the particle in question.} That is, the canonical variables are viewed as functions \(q(t), p(t)\) of the independent variable \(t\). In some cases, it is more convenient to take some coordinate to be the independent variable rather than the time, in which case the time becomes a dependent variable. So doing may facilitate the use of transfer maps, as described in the next section. For example, consider the passage of a collection of particles through a rectangular-shaped beam-line element such as a magnet or an rf cavity. In such a situation, particles with different initial conditions will require different times to pass through the beam-line element. If the quantities of interest are primarily the locations and momenta of the particles as they leave the exit face of the beam-line element, then it would clearly be more convenient to use for an independent variable a coordinate that measures the progress of a particle through the beam-line element. With such a choice, the relation between entering coordinates and momenta and exiting coordinates and momenta could be treated as a transfer map. Remarkably, this goal can be achieved within a Hamiltonian framework [2].

\textbf{Theorem 2.1.} Suppose \(H(q, p, t)\) is a Hamiltonian for a system having \(n\) degrees of freedom. Suppose further that \(\dot{q}_1 = \partial H/\partial p_1 \neq 0\) for some interval of time \(T\) in some region \(R\) of the
phase space described by the $2n$ variables $(q_1, \ldots, q_n)$ and $(p_1, \ldots, p_n)$. Then, in this region and time interval, $q_1$ can be introduced as an independent variable in place of the time $t$. Moreover, the equations of motion with $q_1$ as an independent variable can be obtained from a Hamiltonian that will be called $K$. To construct $K$, define a quantity $p_t$ by the rule

$$p_t = -H(q, p, t).$$  \hfill (8)

Suppose that this relation is solved for $p_1$ to give a relation of the form

$$p_1 = -K(t, q_2, \ldots, q_n; p_t, p_2, \ldots, p_n; q_1).$$  \hfill (9)

Such an inversion is possible according to the inverse function theorem because $\partial H / \partial p_1 \neq 0$ by assumption. Then, as the notation is intended to suggest, $K$ is the desired new Hamiltonian.

As an example, let us use this construction to find the Hamiltonian $K$ corresponding to the Hamiltonian $H$ given by (7) when the $z$ coordinate is taken to be the independent variable. Assume that $\dot{z} > 0$ for the trajectories in question. Then one finds the result

$$K = -[(p_t + q\psi)^2/c^2 - m^2 c^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2]^{1/2} - qA_z. \hfill (10)$$

Here the quantities $p_x$ and $p_y$ denote canonical momenta. Note that, according to (8), $p_t$ is usually negative. For the example at hand, one finds that

$$p_t = -[m^2 c^4 + c^2 (p^{\text{mech}} \cdot p^{\text{mech}})]^{1/2} - q\psi = -\gamma mc^2 - q\psi. \hfill (11)$$

There is yet another Hamiltonian formulation that is of interest. In the spirit of relativity, and following the insight of Hermann Minkowski (1864–1909), it is reasonable to try to treat space and time on a similar footing [3]. Let us review some of the mathematical machinery of Special Relativity. Suppose the world-line of a particle through space-time is parameterized in terms of some parameter $\tau$ by specifying four functions $x^\mu(\tau)$ that, taken together, form a 4-vector with four contravariant components $x^\mu$. We adopt the convention that the first three components of $x^\mu$ are the spatial coordinates of the particle, and the fourth (with a factor of $c$) is its temporal coordinate. Specifically (for $\mu = 1, 2, 3, 4$ and with $x^4 = ct$) we write

$$x^\mu = (x, y, z, ct) = (r, ct). \hfill (12)$$

In addition, let $(x')^\mu$ denote the four derivatives defined by the equations

$$(x')^\mu = dx^\mu / d\tau. \hfill (13)$$

Under the assumption that the parameterization is unchanged by a Lorentz transformation, $(x')^\mu$ is evidently also a 4-vector, which will be called the 4-velocity. The 3-velocity, $v$, of a particle is given by the ratio $v = (dr/d\tau)/(dt/d\tau)$. Since the speed of a massive particle must be less than $c$, $||v|| < c$, it follows that (for physical
particles) the 4-velocity must satisfy the condition

\[ x' \cdot x' = (x')^\mu (x')^\nu g_{\mu \nu} > 0. \]  

(14)

Here \( g_{\mu \nu} \) denotes the metric tensor, and we have employed the usual Einstein convention that repeated indices are to be summed over. In Cartesian coordinates and for flat space-time, only the diagonal entries of \( g \) are nonzero, and we take them to have the values

\[ g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = 1. \]  

(15)

That is, the space-time interval \( ds \) is taken to be given by the relation

\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu = c^2 dt^2 - (dr)^2. \]  

(16)

We remark that the notation \( ds^2 \) appearing in (16), although universally employed, can be misleading since, depending on circumstances, \( ds^2 \) can be negative, zero, or positive, and is therefore not necessarily the square of anything. But note that \( ds^2 > 0 \) for time-like displacements. Space-time endowed with the metric (15) is sometimes called Minkowski space.

The metric tensor can be used to raise and lower indices. For example, there are the relations

\[ x_\mu = g_{\mu \nu} x^\nu. \]  

(17)

In particular, \( x_\mu \) has the entries

\[ x_\mu = (-x, -y, -z, ct) = (-r, ct). \]  

(18)

Finally, we define a 4-potential \( A^\mu \) with entries

\[ A^\mu = (A_x, A_y, A_z, \psi / c) = (A, \psi / c). \]  

(19)

We are now ready to employ some of this mathematical machinery. Consider the relativistic Lagrangian \( L_R \) defined by the relation

\[ L_R = \frac{1}{2} mc (x')^\mu (x')^\nu g_{\mu \nu} + q (x')^\mu A^\nu g_{\mu \nu}. \]  

(20)

It has the pleasing property that it is algebraically simple and treats space and time on a similar footing. In particular, \( L_R \) is evidently a Lorentz scalar. That is, it is invariant under Lorentz transformations.\(^3\)

The canonical momentum \( p_\mu \) is given by the relation

\[ p_\mu = \partial L_R / \partial (x')^\mu = mc (x')_\mu + q A_\mu, \]  

(21a)

which can also be written in the form

\[ p_\mu = p^{\text{mech}}_\mu + q A_\mu, \]  

(21b)

\(^3\)The quantity \((x')^\mu A^\nu g_{\mu \nu}\) is a scalar under Lorentz transformations provided the 4-potential \( A^\mu \) actually transforms as a 4-vector. See [4] for a discussion of the contrary case.
where the mechanical momentum is given by

\[ p^\text{mech}_\mu = mc(x')_\mu. \quad (21c) \]

According to (21c), the mechanical momentum transforms like a 4-vector under Lorentz transformations because \((x')_\mu\) transforms like a 4-vector. From (21b) we see that the canonical momentum also transforms like a 4-vector to the extent that the 4-potential does so.\(^4\)

Again implementing the standard procedure that relates Lagrangians and Hamiltonians, one finds that the relativistic Hamiltonian \(H_R\) associated with the Lagrangian \(L_R\) specified by (20) is given by the expression

\[ H_R = \frac{1}{2}mc (x')^\mu (x')^\nu g_{\mu\nu} = \frac{1}{2mc}(p^\mu - qA^\mu)(p^\nu - qA^\nu)g_{\mu\nu} \]

\[ = \frac{1}{2mc}(p^\mu - qA^\mu)(p_\mu - qA_\mu). \quad (22) \]

Observe that \(H_R\), like \(L_R\), is Lorentz invariant. Note also that the phase space associated with world-lines is eight-dimensional with canonical coordinates \(x^\mu\) and \(p_\nu\).

Let us see what can be said about the phase-space trajectories generated by \(H_R\). Evidently \(H_R\), as given by (22), does not depend explicitly on \(\tau\),

\[ \frac{dH_R}{d\tau} = 0. \quad (23) \]

It follows that \(H_R\) is a constant (and integral) of motion. Moreover, from (22), we see that the quantity \(ds^2/(d\tau)^2\) defined by

\[ ds^2/(d\tau)^2 = g_{\mu\nu}(x')^\mu (x')^\nu = (x')^\mu (x')_\mu = x' \cdot x' \]

\[ = \text{constant}. \quad (24a) \]

is a constant (and integral) of motion.

\[ ds^2/(d\tau)^2 = \text{constant.} \quad (24b) \]

Suppose we restrict our attention to those solutions that satisfy the relation

\[ x' \cdot x' = 1. \quad (25) \]

From (21c) and (25) we see that for these solutions \((p^\text{mech})^\mu\) satisfies the mass-shell condition

\[ p^\text{mech}_\mu (p^\text{mech})^\mu = (p^\text{mech}) \cdot (p^\text{mech}) = m^2c^2. \quad (26) \]

From (22) and (25), we find that for these solutions \(H_R\) has the value

\[ H_R = (mc/2). \quad (27) \]

Moreover, for those solutions that satisfy (25), we have the result \(ds^2 > 0\) and may

\(^4\text{Again see [4] for a discussion of the contrary case.}\)
therefore select, in accord with (16), (24a), and (25), a parameterization such that

$$\frac{ds}{d\tau} = 1.$$  \hspace{1em} (28)

We have introduced three Hamiltonians, namely $H$, $K$, and $H_R$. It can be shown that they all describe the same physics, and in this sense are equivalent.\(^5\) Which is to be employed depends on context. In what follows, we will use the Hamiltonian $K$ with some reference to the Hamiltonian $H_R$.

3. Transfer Maps, the Symplectic Condition, and Symplectic Integrators

To proceed, it is convenient to introduce some terminology and definitions. Suppose we are working with a $2n$-dimensional phase space. Let the symbol $z$ denote the collection of canonical phase-space variables arranged in the form

$$z = (z_1, z_2, \ldots, z_{2n}) = (q_1, \ldots, q_n; p_1, \ldots, p_n).$$ \hspace{1em} (29)

Let $f(z, t)$ and $g(z, t)$ be any two (possibly “time-dependent”) functions of $z$.\(^6\) Define their Poisson bracket $[f, g]$, by the rule

$$[f, g] = \sum_j \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j}.$$ \hspace{1em} (30)

From this definition one may compute the fundamental Poisson brackets

$$[z_a, z_b] = J_{ab},$$ \hspace{1em} (31)

where $J$ is the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$ \hspace{1em} (32)

Here 0 and $I$ denote $n \times n$ zero and identity blocks, respectively. The matrix $J$ is sometimes called the Poisson matrix.

There is an existence and uniqueness theorem to the effect that sets of first-order ordinary differential equations have solutions, and each solution is uniquely specified by its initial conditions [5]. Hamilton’s equations of motion are first order. Now suppose a charged particle enters a beam-line element (or collection of beam-line elements) with initial conditions $z^i$ and subsequently exits with final conditions $z^f$. Then, by the existence and uniqueness theorem, $z^f$ is uniquely specified by $z^i$. Thus, there is a map $M$, called a transfer map, that sends $z^i$ to $z^f$, and we write

$$z^f = Mz^i.$$ \hspace{1em} (33)

This relation between $z^i$ and $z^f$ is illustrated by the picture shown in figure 1.

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\(^5\)When using $H_R$, we employ only solutions that obey (25).

\(^6\)Here, by the “time”, we mean whatever has been selected to be the independent variable. Note also that the symbol $z$ now no longer refers to the third component of $r$, but rather to the collection of phase-space variables.
Next suppose small changes $dz^i$ are made in the initial conditions. The result will be associated small changes $dz^f$ in the final conditions. These small changes will be connected by the relations

$$dz^f = M dz^i,$$  \hspace{1cm} (34)

where $M$ is the Jacobian matrix with entries

$$M_{ab} = \frac{\partial z^f_a}{\partial z^i_b}. \hspace{1cm} (35)$$

It can be shown that if $M$ is the result of integrating Hamilton’s equations of motion, then its associated Jacobian matrix $M$ will satisfy the condition

$$M^T JM = J,$$  \hspace{1cm} (36)

where $M^T$ denotes the transpose of $M$. A matrix that satisfies (36) is said to be symplectic; correspondingly $M$ is called a symplectic map.\(^7\) Note that in general $M$ depends on $z^i$. But $J$ does not. Therefore (36), since it must hold for all $z^i$, places strong (linear and nonlinear) restrictions on $M$ [6].

Suppose the final conditions $z^f$ are to be determined by integrating Hamilton’s equations of motion numerically, and we also wish to satisfy (36). A numerical integrator with this property is called a symplectic integrator. In the case of a storage ring, such as the LHC, we must integrate through thousands of beam-line elements to integrate through even a single turn, and we wish to integrate through a large number of turns. Therefore, even if we wish to integrate for only a small number of turns, we would like to be able to use an explicit symplectic integrator because the numerous iterations required for an implicit symplectic integrator would make the computation extremely slow. Here we envision that one must iterate the implicit solve to the point where convergence has been achieved to machine precision in order to achieve symplecticity to machine precision. Our concern is that the effort involved in iterating implicit Runge-Kutta or implicit Gauss to machine precision will exceed the effort required for one step of some explicit symplectic method for $K$ or $H_R$ if such an explicit symplectic method can be found.

Explicit symplectic integrators are available if the Hamiltonian has the form $T(p) + V(q)$. The Hamiltonian $K$ given by (10) is of this form if we make the approximations

$$\psi = 0, \quad A_x = 0, \quad \text{and} \quad A_y = 0,$$  \hspace{1cm} (37)
for it then takes the form

$$K = -(p_t^2/c^2 - m^2c^2 - p_x^2 - p_y^2)^{1/2} - qA_z. \quad \text{(38)}$$

However, it can be shown that this approximation excludes magnetic fringe-field effects, which, by the Maxwell equations, must occur at entry and exit of every magnetic beamline element. It also excludes transverse electric fringe fields, which must occur at entry and exit of all rf cavities. To make accurate calculations that include fringe-field effects (which can be important when nonlinear and even some linear effects are considered), one would like to have an explicit symplectic integrator that does not make the approximations in (37).

Remarkably, there is an explicit symplectic integrator for the Hamiltonian $H_R$ [7]. But there is a caveat: One of the advertised features of symplectic integrators is that they can be used with a rather large step size (thereby reducing computation time) since they at least exhibit the qualitative nature of solutions exactly. However, there is a theorem to the effect that (for any finite step size) symplectic integrators do not preserve the Hamiltonian, even if it has no explicit dependence on the independent variable ($\tau$ in the case of $H_R$) [8]. This may not be a serious problem in some applications of symplectic integrators. In the case of $H_R$, however, (27) tells us that failure to preserve $H_R$ means the particle mass is not preserved. Therefore, to preserve the particle mass to good accuracy, which would seem highly desirable, it is necessary to employ a sufficiently small step size, thereby making symplectic integration in this situation relatively slow.

The rest of this chapter is devoted to exploring other possible approaches to satisfying the symplectic condition (36) while at the same time achieving improved computational speed and taking into account, through some desired order, all linear and nonlinear effects associated with realistic electromagnetic fields, including fringe fields and high-order multipole fields.

4. Lie Algebraic Concepts and Tools

According to (30), Poisson brackets obey the antisymmetry property

$$[g, f] = -[f, g]. \quad \text{(39)}$$

It can be verified that Poisson brackets also satisfy the Jacobi identity. Let $f$, $g$, and $h$ denote any three functions on phase-space. Then there is the identity

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0. \quad \text{(40)}$$

As a consequence, the Poisson bracket satisfies all the requirements for a Lie product. The set of all phase-space functions therefore constitutes a Lie algebra with the Poisson bracket as the Lie product [9].

Given any function $f(z, t)$, define an associated differential operator, denoted by $:f:$ and called a Lie operator, by the rule

$$:f: = \sum_j \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial}{\partial q_j}. \quad \text{(41)}$$
Then, if \( g(z,t) \) is any other phase-space function, the action of \(:f:\) on \( g \) is defined by writing

\[
:f: g = \sum_j \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} = [f,g].
\] (42)

Thus, a Lie operator may be viewed as a Poisson bracket waiting to happen.

In general, Lie operators do not commute. However, the commutator \( \{ :f:, :g: \} \) of any two Lie operators \( :f: \) and \( :g: \) is again a Lie operator. Indeed, as a consequence of the Jacobi identity (40), the commutator \( \{ :f:, :g: \} \) may be written in terms of the Poisson bracket of the two underlying functions \( f \) and \( g \) according to the relation

\[
\{ :f:, :g: \} = :f: :g: - :g: :f: = [f,g].
\] (43)

The relation (42) defines the action of \( :f: \). Powers of \( :f: \) can be defined by the rules

\[
:f:0 = I \iff :f:0 g = g,
\] (44a)

\[
:f:1 g = [f,g],
\] (44b)

\[
:f:2 g = [f,[f,g]],
\] (44c)

and so on. Now that powers of the Lie operator \( :f: \) have been defined, one may also define power series in \( :f: \). Of particular interest is the power series associated with the exponential function by the rule

\[
\exp( :f:) = e^{ :f:} = \sum_{m=0}^{\infty} \frac{1}{m!} :f: m.
\] (45)

This operator \( e^{ :f:} \), called a Lie transformation, therefore acts on \( g \) according to the relation

\[
\exp( :f:) g = g + [f,g] + \frac{1}{2!} [f,[f,g]] + \cdots .
\] (46)

In this context \( :f: \) (and sometimes \( f \)) is called a Lie generator. Its importance for us lies in the fact that any Lie transformation generates a symplectic map.

Poisson brackets are invariant under symplectic maps [10]. An important consequence of this fact is the extremely useful similarity relation [11]

\[
\mathcal{L} \exp( :f:) \mathcal{L}^{-1} = \exp( :\mathcal{L} f:) ,
\] (47)

where \( \mathcal{L} \) denotes any symplectic map.

Suppose the transfer map \( M \) has the property that it maps the origin into itself. In other words, we assume that \( Mz \) has a Taylor expansion of the form

\[
z^f_a = \sum_b R_{ab} z^f_b + \sum_{bc} T_{abc} z^f_b z^f_c + \sum_{bcd} U_{abcd} z^f_b z^f_c z^f_d + \cdots ,
\] (48)
which has no constant term. This can always be accomplished by the use of deviation variables. If $M$ is symplectic, then $R$ must be a symplectic matrix. Moreover, the Taylor coefficients $T, U, \cdots$ cannot be arbitrary, but are constrained by complicated nonlinear relations that follow from the symplectic condition (36). A truncated Taylor expansion of a symplectic map is called a symplectic jet. Finally, the series (48) cannot in general be truncated without violating the symplectic condition. Therefore a symplectic jet is generally not a symplectic map.

However, there is a factorization theorem [12] to the effect that $M$ can also be written in the Lie product form

$$M = R \exp(:f_3:) \exp(:f_4:) \cdots \exp(:f_1:).$$

Here $R$ is the linear symplectic map associated with $R$, and the $f_m$ are homogeneous polynomials of degree $m$. Translations/deviations from the origin, which correspond to constant terms added if desired to the Taylor series (48), are described by $f_1$; and the $f_3, f_4 \cdots$ describe the nonlinear terms in (48). Unlike the Taylor coefficients, there are no restrictions imposed on the $f_m$ by the symplectic condition.

Any analytic symplectic map is uniquely specified by a first-order polynomial $f_1$, a symplectic matrix $R$, together with a collection of homogeneous polynomials $f_3, f_4 \cdots$ that describe the nonlinear part of the map. And the converse also holds. In addition, the (in principle infinite) product appearing in square brackets in (49) can be truncated at any stage without violating the symplectic condition. It can be shown that each factor in (49) is a symplectic map, and the product of any number of symplectic maps is also a symplectic map. In what follows it is also convenient to write (49) in the form

$$M = RN \exp(:f_1:)$$

where $N$, the nonlinear part of $M$, is given by

$$N = \exp(:f_3:) \exp(:f_4:) \cdots.$$  

5. Symplectic Completion of Symplectic Jets

We know that the Lie algebra of all Lie operators, which we will call ispm$(2n, \mathbb{R})$, is infinite dimensional. Correspondingly ISpM$(2n, \mathbb{R})$, the group of symplectic maps, is infinite dimensional.\(^8\) Indeed, the factorization (50) gives a representation of the general analytic symplectic map. We see that the specification of a symplectic map generally requires an infinite number of parameters. This fact produces an awkward situation for human beings and computers, which can work only with a finite number of quantities (and often only with finite precision).

An optimistic perspective on the experimental and theoretical situation, for example in the field of accelerator physics, might be stated as follows: We know that a beam transport system, accelerator, storage ring, or any portion thereof may be described by a symplectic transfer map. However, because we cannot measure or control electromagnetic fields exactly, we are unsure of and unable to control exactly

\(^8\)The letters $m$ and $M$ that appear in this and the previous sentence are abbreviations for map. The letters $i$ and $I$ are abbreviations for inhomogeneous. By inhomogeneous it is meant that the possibility of constant terms appearing in (48) is included.
what this map is. Also, since it is impossible to perform computations with an infinite
number of parameters/variables and to infinite precision, it is necessary to develop
various approximation schemes. Thus, we are able to study computationally (and
probably theoretically) the detailed properties of only a subset of all symplectic maps.
The hope is that if two symplectic maps are in some sense nearly the same, then their
behavior [including, in some cases, long-term (repeated iteration) behavior] will be
in some important ways nearly the same.9 Were that not true from an experimental
standpoint, it would be impossible to build satisfactory storage rings and the like.
Were that not true from a theoretical standpoint, it would be impossible to design
storage rings and the like with any assurance of satisfactory performance.

Suppose, as an approximation, the product appearing in the square brackets of
(49), namely the map \( N \), is \textit{truncated} at \( m = \maxm \) to produce the map \( N^{\text{trunc}} \) given
by

\[
N^{\text{trunc}} = \exp(f_3) \exp(f_4) \cdots \exp(f_{\maxm}).
\]  

(52)

In analogy with (50) we also make the definition

\[
M^{\text{trunc}} = RN^{\text{trunc}} \exp(f_1).
\]  

(53)

The map \( M^{\text{trunc}} \), while exactly symplectic, requires only a finite number of parameters
for its specification. We only need store a first-order polynomial \( f_1 \), a symplectic
matrix \( R \), and a collection of homogeneous polynomials \( f_3, f_4, \ldots f_{\maxm} \). For example,
in the case of a 6-dimensional phase space, 3002 parameters are required when
\( \maxm = 8 \).

What can be said about the \textit{accuracy} of \( M^{\text{trunc}} \)? Suppose the Taylor expansion (48)
is terminated by retaining only terms through degree \(( \maxm - 1 )\), thereby forming
a symplectic jet that we will call \( J \). An examination of the proof of the factorization
theorem (49) shows that a knowledge of the coefficients in \( J \) supplies just enough
information to determine the ingredients of \( M^{\text{trunc}} \), and vice versa. In other words, a
knowledge of \( J \) supplies just enough information to determine \( R \), the polynomial \( f_1 \),
and the polynomials \( f_3 \) through \( f_{\maxm} \), and vice versa. (In particular, \( f_3 \) contributes
only to \( T \) terms and to terms beyond second order, \( f_4 \) contributes only to \( U \) terms
and to terms beyond third order, \textit{etc.}) Thus, while exactly symplectic, the map \( N^{\text{trunc}} \)
is guaranteed accurate only through terms of degree \(( \maxm - 1 )\). With regard to
memory requirements, the storage of a jet requires more locations because it does
not exploit the symplectic condition. For example, 10296 locations are required in
the case of 6-dimensional phase space when \( \maxm = 8 \).

At present there are Lie-algebraic results and Truncated Power Series (TPSA)
methods that make it possible to compute in principle the ingredients in \( M^{\text{trunc}} \) as
given by (53), with \( \maxm = 8 \), for any beam-line element or collection of beam-
line elements (including a full ring) based on field data provided numerically on a
grid [13].

What has been accomplished here? Given a symplectic jet \( J \), we have found a
map \( N^{\text{trunc}} \) that is guaranteed symplectic and whose Taylor expansion agrees with \( J \)
through terms of degree \(( \maxm - 1 )\). This Taylor expansion will in general contain
terms of degrees beyond \(( \maxm - 1 )\). It will be called the \textit{Lie symplectic completion}
of \( J \).

---

9Note that a similar optimism is shared by practitioners of symplectic integration.
But there is a problem: Suppose we wish to evaluate $M^{\text{trunc}}_{i} z^i$ for some initial condition $z^i$. Because of the infinite series that appears in the definition (45) of a Lie transformation, the symplectic completion of $J$ provided by $M^{\text{trunc}}$ will in general contain an infinite number of terms. As a consequence, the evaluation of $M^{\text{trunc}}_{i} z^i$ will generally involve the summation of infinite series, a task that generally lies beyond numerical methods, or is at best numerically intensive unless the series converges rapidly.

We have seen that in principle the symplectic completion of a symplectic jet is possible. What we would like are other symplectic jet completions whose actions on $z^i$ can be computed rapidly and to machine precision. Two such methods will be described in the next two sections of this chapter.

6. Symplectic Completion Using Generating Functions

We have described how in general the computation of the action of a Lie transformation on phase space involves the summation of an infinite series if the symplectic condition is to be honored. Before exploring a particular method to deal with this problem, we begin this section by studying a simple example of what happens if only a finite number of terms in the series expansion are employed.

As such an example consider, for a two-dimensional phase space, the symplectic map $M$ given by the relation

$$M = RN$$

with linear part

$$R = \exp(-\frac{\theta}{2} p^2 + q^2)$$

and nonlinear part

$$N = \exp(q p^2).$$

This map may be viewed as a toy model for the one-turn map of a storage ring. In this case the infinite series for $R$ can be summed exactly to give the results

$$Q = Rq = q \cos \theta + p \sin \theta, \quad P = Rp = -q \sin \theta + p \cos \theta.$$ (57)

And the map $N$ can also be evaluated exactly, since the exponential of any monomial Lie operator can be evaluated exactly [14]. For the case at hand there is the result

$$Q = Nq = q(1-p)^2, \quad P = Np = p/(1-p).$$ (58)

Therefore $M$ can also be evaluated exactly. Note that the result (58) for $P$, and therefore the map $N$, has a pole on the phase-space surface $p = 1$.

Figure 2 shows the result of applying $M$ repeatedly to seven initial conditions for the case $\theta/2\pi = 0.22$. In other words, seven initial conditions have been selected, and their orbits have been found under the repeated action of $M$. One initial condition lies very near the origin, and its orbit appears to lie on a closed curve that is nearly
elliptical. (It would be nearly circular had the horizontal and vertical scales been equal.) This is to be expected because the effect of the nonlinear part $N$ is so small on such orbits that they are essentially those of the rotation map $\mathcal{R}$. By contrast, the other initial conditions lie successively farther from the origin, and the effect of $N$ becomes ever more significant. Their orbits appear to lie on closed curves that, the farther they lie from the origin, the more noticeably do nonlinearities distort them from being circular. The origin itself is an elliptic fixed point corresponding to a one-turn closed orbit.

Now suppose the nonlinear map $N$ is truncated to form the map $N_{tr}^2$ by retaining only the first two terms in its Taylor expansion. In Lie form we have the result

$$N_{tr}^2 = J + qp^2.$$  

This truncated map $N_{tr}^2$ has the effect

$$Q = N_{tr}^2 q = (J + qp^2:)q = q + [qp^2, q] = q - 2qp \Leftrightarrow Q - q = -2qp,$$

$$P = N_{tr}^2 p = (J + qp^2:)p = p + [qp^2, p] = p + p^2 \Leftrightarrow P - p = p^2.$$  

Evidently, $N_{tr}^2$ is a symplectic jet map that retains only terms through degree 2. Indeed, one finds using (60) the result

$$[Q, P] = 1 - 4p^2 \neq 1,$$

and therefore $N_{tr}^2$, while a symplectic jet map, is as expected not a symplectic map.\textsuperscript{10}

Next define a corresponding map $M_{tr}^2$ by writing

$$M_{tr}^2 = \mathcal{R}N_{tr}^2.$$  

The left-hand graphic in Figure 3 shows the orbits of $M_{tr}^2$ for two initial conditions: one near the origin, and one quite far away. Inspection of the figure shows that orbits are no longer distorted circles, but instead appear to spiral into the origin. This motion toward the origin occurs because $N_{tr}^2$, and consequently $M_{tr}^2$, is not symplectic.

\textsuperscript{10}It follows from (36) that a symplectic map must preserve Poisson brackets, and vice versa.
We could also retain the next term in the Taylor series for \( \exp(\cdot q p^2 : \cdot) \) to form the symplectic jet map \( N^{tr_3} \),

\[
N^{tr_3} = J + :q p^2 : + \frac{1}{2} :q p^2 :^2 .
\]

(63)

It retains terms through degree 3. It is still nonsymplectic, but more nearly symplectic than \( N^{tr_2} \). Again define a corresponding map \( M^{tr_3} \) by writing

\[
M^{tr_3} = RN^{tr_3} .
\]

(64)

The phase-space portrait for \( M^{tr_3} \) is found to be somewhat more like that for \( M \) than that provided by \( M^{tr_2} \), because \( M^{tr_3} \) is more nearly symplectic. However, there is still substantial/disastrous nonsymplectic spiraling, in this case out of the origin. We see that violation of the symplectic condition can convert the origin, initially an elliptic fixed point, into a nonlinear attracting or a nonlinear repelling fixed point.

Suppose we know that the behavior of some dynamical system is describable by a symplectic map. This system may be a beam-line element, some collection of beam-line elements, or even the one-turn map for a storage ring. And suppose that a truncated Taylor expansion (symplectic jet \( J \)) is known for this map through terms of some order. As stated earlier, such knowledge is in fact computable using Lie algebraic and TPSA algorithms [15]. According to the factorization theorem, as far as nonlinear effects are concerned, such knowledge is equivalent to the knowledge of a set of homogeneous polynomials \( f_3, \ldots , f_n \). What we would like to find is a map that is symplectic, has the jet \( J \) through terms of degree \((n - 1)\), and is relatively easy to compute. Because it is symplectic, its Taylor expansion must in general also have terms beyond degree \((n - 1)\). In some way that is not yet clearly described, we would like these additional terms to be as small as possible while remaining consistent with the symplectic condition. For example, their effect over the phase-space region of interest should not be appreciably larger than the extent to which \( J \) violates the symplectic condition.

How can we find such symplectic maps? We need a supply of relatively easily computed symplectic maps. It is known that such maps can be produced with the aid of generating functions.\(^ {11}\) Four types of generating functions, commonly called \( F_1(q, Q) \), \( F_2(q, P) \), \( F_3(p, Q) \), and \( F_4(p, P) \), are usually presented in graduate Classical Mechanics textbooks. What is less familiar is that, for a \( 2n \) dimensional phase space, there is in fact a \( 2n(4n + 1) \) dimensional family of types of generating functions: There is a type for every \( 4n \times 4n \) symplectic matrix [16]. Among these types, we have found the so-called Poincaré generating function, which we denote as \( F_+ \), to be particularly attractive [17].

To describe the use of \( F_+ \), it is useful to make some additional definitions. In addition to the definition of \( z \) given by (29), which describes the collection of what we might call old variables, we introduce the symbol \( Z \) to denote a collection of what we may call new variables:

\[
Z = (Z_1, Z_2, \ldots , Z_{2n}) = (Q_1, \ldots , Q_n; P_1, \ldots , P_n) .
\]

(65)

\(^{11}\)Symplectic completion of symplectic jets using a generating function was first implemented—in the context of Accelerator Physics—in the Lie-algebra based accelerator design code MaryLie.
We shall also need the sums and differences defined by

$$\Sigma = Z + z \quad \text{and} \quad \Delta = Z - z. \quad \iff \quad z = \frac{1}{2}(\Sigma - \Delta) \quad \text{and} \quad Z = \frac{1}{2}(\Sigma + \Delta). \tag{66}$$

Specifically, for future use and in the case of a two-dimensional phase space, the relations (66) take the equivalent forms

$$\Sigma_1 = Q + q \quad \text{and} \quad \Delta_1 = Q - q \quad \iff \quad q = \frac{1}{2}(\Sigma_1 - \Delta_1) \quad \text{and} \quad Q = \frac{1}{2}(\Sigma_1 + \Delta_1), \tag{67a}$$

$$\Sigma_2 = P + p \quad \text{and} \quad \Delta_2 = P - p \quad \iff \quad p = \frac{1}{2}(\Sigma_2 - \Delta_2) \quad \text{and} \quad P = \frac{1}{2}(\Sigma_2 + \Delta_2). \tag{67b}$$

Finally, we will need a collection of $2n$ auxiliary variables, which we will call $u$:

$$u = (u_1, u_2, \ldots, u_{2n}). \tag{68}$$

Now let $F_+ (u, t)$ be any function of $u$ and perhaps the independent/time variable $t$. We define its action on phase space by the rule

$$\Delta = J\partial_u F_+ |_{u = \Sigma}. \tag{69}$$

Note that this definition specifies a relation between $\Delta$ and $\Sigma$, which in turn, when the relations on the right-hand side of (66) are taken into account, specifies a relation between $Z$ and $z$. By the general theory of generating function machinery, this relation between $Z$ and $z$ is guaranteed to be a symplectic map for any choice of $F_+ (u, t)$.

To see how this mathematical machinery works in some detail, let us apply it to a simple example in two-dimensional phase-space. Suppose $F_+$ is the cubic monomial

$$F_+ (u) = F_+^3 (u) = -\frac{1}{4}u_1u_2^2. \tag{70}$$

We then compute

$$J\partial_u F_+^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{4}u_2^2 \\ -\frac{1}{2}u_1u_2 \end{pmatrix} = \frac{1}{2}u_1u_2, \tag{71}$$

and hence

$$J\partial_u F_+^3 |_{u = \Sigma} = \begin{pmatrix} -\frac{1}{4}\Sigma_1 \Sigma_2 \\ \frac{1}{4}\Sigma_2^2 \end{pmatrix}. \tag{72}$$

From (69) and (72) it follows that

$$\Delta_1 = -\frac{1}{2}\Sigma_1 \Sigma_2, \quad \text{and} \quad \Delta_2 = \frac{1}{4}\Sigma_2^2. \tag{73}$$

Finally, employ the relations (67) in (73) to obtain the relations

$$Q - q = -\frac{1}{2}(Q + q)(P + p) \quad \iff \quad Q = q - \frac{1}{2}(Q + q)(P + p), \tag{74a}$$

$$P - p = \frac{1}{4}(P + p)^2 \quad \iff \quad P = p + \frac{1}{4}(P + p)^2. \tag{74b}$$
Figure 3. Phase-space portraits, in the case $\theta/2\pi = 0.22$. Left: The resulting of applying the map $N^{tr2}$ repeatedly (1000 times) to the two initial conditions $(q, p) = (0.01, 0)$ and $(0.4, 0)$. These orbits spiral into the origin. Right: The result of applying the map $N^{psc2}$ repeatedly (2000 times) to the seven initial conditions $(q, p) = (0.01, 0), (0.1, 0), (0.15, 0), (0.2, 0), (0.25, 0), (0.3, 0),$ and $(0.35, 0)$. Light gray curves in the background indicate the exact result.

The two equations on the right-hand sides of (74) specify a relation between $Z$ and $z$.

As is the case with the use of any generating function, the above relation between $Z$ and $z$ is **implicit**. We can **begin** to make it **explicit** by seeking a Taylor expansion using iteration: For the zeroth iteration, make the Ansatz

$$Q = q \quad \text{and} \quad P = p. \quad (75a)$$

Now substitute this Ansatz into the right-hand sides of (74) to yield for the first iteration the results

$$Q = q - 2qp \quad \text{and} \quad P = p + p^2. \quad (75b)$$

Observe that the results (75b) agree with the jet results (60). We have found a symplectic map whose jet through terms of second order agrees with $N^{tr2}$.

It happens that for this example the implicit equations on the right-hand sides of (74) can be made explicit by algebraic manipulation. The equation on the right-hand side of (74b) is quadratic in $P$, and on choosing the negative square root (the solution for which $p$ vanishing implies that $P$ also vanishes), we obtain the relation

$$P = -(p - 2) - 2\sqrt{1 - 2p}. \quad (76a)$$

Once $P$ is known, the equation on the right-hand side of (74a) is linear in $Q$ and has the immediate solution

$$Q = \frac{q\sqrt{1 - 2p}}{2 - \sqrt{1 - 2p}}. \quad (76b)$$

We see that, in this case, the use of $F_+$ produces a map that has a square-root branch point on the phase-space surface $p = 1/2$ and a pole at $p = -3/2$. Note that the branch-point singularity lies closer to the origin than does the pole of the exact map, which we have noted lies on the surface $p = 1$.

Let $N^{psc2}$ be the map given by (76). One can verify by direct computation that it has the property

$$[Q, P] = 1, \quad (77)$$
and hence the map \( N^{psc2} \) is exactly symplectic, as desired and expected. The reader can also verify that the terms through degree 2 in the Taylor expansions of (76) agree with the terms in (60). This is just the result (75b) that we have already found by iteration.\(^\text{12}\) We may therefore say that \( N^{psc2} \) is the Poincaré symplectic completion of the degree-two symplectic jet map \( N^{tr2} \) given by (60). Correspondingly, suppose we define the associated map \( M^{psc2} \) by the relation

\[
M^{psc2} = \mathcal{R}N^{psc2}.
\]

We expect it to be symplectic because it is the product of two symplectic maps.

The right-hand graphic in Figure 3 shows the result of applying \( M^{psc2} \) repeatedly to seven initial conditions for the case \( \theta/2\pi = 0.22 \). Note that orbits generated by \( M^{psc2} \) exhibit no spurious spiraling towards or away from the origin. Moreover, comparison with the background light gray curves showing the exact result reveals that the orbits closely agree in shape, but there is some difference in phase advance. It seems remarkable that the relatively meager information about \( N \) present in \( N^{tr2} \) suffices, after Poincaré symplectic completion has been performed, to give such good agreement.

The reader may wonder how we knew to make the inspired choice (70) for \( F^3_+(u) \). Suppose we make for \( F_+(u) \) the expansion

\[
F_+(u) = \sum_{m=3}^{\text{maxm}} F^m_+(u),
\]

where the \( F^m_+(u) \) are homogeneous polynomials of degree \( m \). Then there are formulas that determine the \( F^m_+(u) \) in terms of the \( f_m(u) \). For example, there are the relations

\[
F^3_+(u) = -\frac{1}{4} f_3(u) \quad \text{and} \quad F^4_+(u) = -\frac{1}{8} f_4(u).
\]

Observe that if we assume \( F_+(u) \) has only a \( F^3_+(u) \) component, then the associated map will have an \( f_3 \) given by (80) and a vanishing \( f_4 \). (This is one of the virtues of the Poincaré generating function.) In general the \( f_{>4} \) will not vanish. Results for the \( F^m_+(u) \) in terms of the \( f_m(u) \) are known through order \( m = 8 \), but become increasingly complicated as \( m \) increases beyond \( m = 4 \) \([18]\). However, it is also possible to proceed without these formulas, thereby bypassing their complications. We will next illustrate how to do so for our simple example.

Suppose only the functions \( f_3, f_4, \ldots, f_{\text{maxm}} \) are known. For our example we know from (56) that \( f_3 = q{p}^2 \) and \( (\text{maxm} - 1) = 2 \). Next compute the set of Taylor series through terms of degree \( (\text{maxm} - 1) \) for the jet \( N^{tr (\text{maxm} - 1)} \). For our example this set is given by (60). In this set, replace \( Q, q \) and \( P, p \) by their representations in terms of \( \Sigma \) and \( \Delta \) using, for this example, the relations given on the right-hand sides of (67). So doing, for the relations appearing on the far-right sides of (60), yields the results

\[
\Delta_1 = -2 \left[ \frac{1}{2}(\Sigma_1 - \Delta_1) \left( \frac{1}{2}(\Sigma_2 - \Delta_2) \right) \right] = -\frac{1}{2}(\Sigma_1 - \Delta_1)(\Sigma_2 - \Delta_2),
\]

\[
\Delta_2 = \left[ \frac{1}{2}(\Sigma_2 - \Delta_2) \right]^2 = \frac{1}{4}(\Sigma_2 - \Delta_2)^2.
\]

\(^\text{12}\)Surprisingly, the terms through degree 3 in the Taylor expansions of (76) agree with the terms generated by applying (63) to \((q, p)\). This happens due to the second relation in (80) and our tacit assumption that \( f_4 = 0 \).
Approximately solve these equations for the quantities $\Delta$ in terms of the quantities $\Sigma$. Do so in the form of a Taylor series in $\Sigma$ truncated beyond terms of degree $(\text{maxm} - 1)$, which can be done by iteration: For the zeroth iteration make the Ansatz

$$\Delta_1 = 0 \text{ and } \Delta_2 = 0. \quad (82a)$$

Now substitute this Ansatz into the right-hand sides of $(81)$ to obtain, through terms of degree 2, the Taylor expansion

$$\Delta_1 = -\frac{1}{2} \Sigma_1 \Sigma_2 \text{ and } \Delta_2 = \frac{1}{4} \Sigma_2^2. \quad (82b)$$

In this case the iteration process is finished because $(\text{maxm} - 1) = 3 - 1 = 2$. Observe that the relations $(82b)$ agree with the relations $(73)$! The transition from $(73)$ to the right-hand sides of $(74)$ now proceeds as before. We have found, directly from the jet $N^{\text{tr(maxm-1)}}$, the relations that would have flowed from the use of the related $F_i$. And we have therefore identified (in implicit form) the desired symplectic map. Note also that the operations we have just performed involve only well-defined polynomial manipulations, and therefore can be performed on a computer using TPSA routines.

There remains the problem of converting the implicit results given by $(69)$ to explicit results for $Z$ in terms of $z$. For our simple example we were able to do so by solving a quadratic equation. For most applications, however, the equations to be solved are much more complicated, and must be handled numerically. That is, given $z$ as a collection of numbers, we would like to find the associated collection of numbers $Z$. This can be done by simple iteration or by use of Newton’s method. In either case the process can be started using jet results.

As an example of the use of simple iteration, the equations on the right-hand sides of $(74)$ may be converted into the iteration rule

$$Q^{[n+1]} = q - \frac{1}{2} (Q^{[n]} + q)(P^{[n]} + p), \quad (83a)$$

$$P^{[n+1]} = p + \frac{1}{4} (P^{[n]} + p)^2. \quad (83b)$$

Using the jet results $(60)$, one may begin the iteration optimally with the values

$$Q^{[0]} = q - 2qp \text{ and } P^{[0]} = p + p^2. \quad (84)$$

Observe that in general the quantities to be evaluated numerically at each step are polynomials, and therefore this evaluation is quite fast.

For example, consider the case

$$q = -0.3 \text{ and } p = -0.2, \quad (85)$$

which is a point near the boundary of figure 2. Then use of $(76)$ shows that in this case we hope to find the results

$$Q^\infty = -0.434 588 297 681 520 63 \cdots \text{ and } P^\infty = -0.166 431 913 239 846 35 \cdots. \quad (86)$$

For the case $(85)$, table 1 displays simple iteration results near the beginning and end of the iteration process. We see that the point $Q^\infty, P^\infty$ is, as desired, an attractor
with the number of iterations $J$ problem at hand are shown in table 2.

| $n$  | $Q[n]$            | $P[n]$            | $\Delta Q[n]$       | $\Delta P[n]$       |
|------|-------------------|-------------------|----------------------|----------------------|
| 0    | $-0.42000000000000000$  | $-0.16000000000000000$  | $-1.46 \times 10^{-2}$  | $-6.43 \times 10^{-3}$  |
| 1    | $-0.42959999999999999$  | $-0.16760000000000000$  | $-4.99 \times 10^{-3}$  | $1.17 \times 10^{-3}$  |
| 2    | $-0.43410048000000000$  | $-0.16621756000000000$  | $-4.88 \times 10^{-4}$  | $2.14 \times 10^{-4}$  |
| 3    | $-0.43442024329021444$  | $-0.16647117468691164$  | $-1.68 \times 10^{-4}$  | $3.93 \times 10^{-5}$  |
| ...  | ...               | ...               | ...                 | ...                 |
| 17   | $-0.43458829768151264$  | $-0.16643191323984834$  | $-7.99 \times 10^{-15}$  | $1.94 \times 10^{-15}$  |
| 18   | $-0.43458829768152004$  | $-0.16643191323984614$  | $-6.66 \times 10^{-16}$  | $2.78 \times 10^{-16}$  |
| 19   | $-0.43458829768152044$  | $-0.16643191323984654$  | $-1.67 \times 10^{-16}$  | $1.39 \times 10^{-16}$  |
| 20   | $-0.43458829768152074$  | $-0.16643191323984644$  | $5.55 \times 10^{-17}$  | $5.55 \times 10^{-17}$  |

Table 2. Convergence of $Q[n], P[n]$ and $\Delta Q[n], \Delta P[n]$ as a function of $n$ using Newton’s method.

| $n$  | $Q[n]$            | $P[n]$            | $\Delta Q[n]$       | $\Delta P[n]$       |
|------|-------------------|-------------------|----------------------|----------------------|
| 0    | $-0.42000000000000000$  | $-0.16000000000000000$  | $-1.46 \times 10^{-2}$  | $-6.43 \times 10^{-3}$  |
| 1    | $-0.43453493178999588$  | $-0.16446067796610174$  | $-5.34 \times 10^{-5}$  | $8.77 \times 10^{-6}$  |
| 2    | $-0.43458829797514944$  | $-0.16643191325607764$  | $2.94 \times 10^{-10}$  | $1.62 \times 10^{-11}$  |
| 3    | $-0.43458829768152084$  | $-0.16643191323984644$  | $1.11 \times 10^{-16}$  | $8.33 \times 10^{-17}$  |
| 4    | $-0.43458829768152084$  | $-0.16643191323984644$  | $1.11 \times 10^{-16}$  | $5.55 \times 10^{-17}$  |

Convergence of $Q[n], P[n]$ and $\Delta Q[n], \Delta P[n]$ as a function of $n$ using simple iteration. Convergence to machine precision has been achieved after about 20 iterations. However, the simple iteration process converges rather slowly. Let us examine the convergence rate. Define errors $\Delta Q[n], \Delta P[n]$ by the rules $\Delta Q[n] = Q[n] - Q^\infty$ and $\Delta P[n] = P[n] - P^\infty$. Examination of the error columns in table 1 shows that there are the results $|\Delta Q[n]|/|Q^\infty| \sim 0.2$ and $|\Delta P[n]|/|P^\infty| \sim 0.2$, and therefore the errors decrease geometrically with each iteration by factors that are not significantly less than 1. The number of correct digits increases roughly linearly with the number of iterations.

We would like an iteration process that is more rapidly convergent. Solution of the equations on the right-hand sides of (74) can be converted into a fixed-point problem, and this problem can be solved by Newton’s method. The results of so doing for the problem at hand are shown in table 2. Evidently with Newton’s method convergence to machine precision has been achieved with 3 iterations. And for the errors we find the results $|\Delta Q[n+1]/(\Delta Q[n])^2| \sim 0.2$ and $|\Delta P[n+1]/(\Delta P[n])^2| \sim 0.1$. Therefore the convergence is quadratic as expected for Newton’s method. The number of correct digits roughly doubles from one iteration to the next.

7. Symplectic Completion Using Cremona Maps

The previous section illustrated how one may begin with a symplectic jet $J$ truncated at terms beyond degree $(\maxm - 1)$ and then, by use of a suitable Poincaré generating
function, add to $J$ terms of degree $\text{maxm}$ and higher to produce an exactly symplectic map. In effect, that approach usually adds an infinite number of terms beyond degree $(\text{maxm} - 1)$, because the resulting map generally contains singularities. However, as seen for phase-space regions of physical interest, these additional terms seem to have little (no deleterious) effect beyond achieving symplectification.

In this section we describe how one may achieve symplectification by adding only a finite number of terms beyond degree $(\text{maxm} - 1)$. The result is a map that is both polynomial and exactly symplectic. We call such maps Cremona maps. For simplicity, as was done in the previous section, we restrict our discussion to the case of one degree of freedom $[19]$.

As Cremona maps are symplectic, one can use them to approximate the behavior of Hamiltonian systems; as they are polynomial, one can compute them rapidly and exactly. But how might one construct such maps? To answer this question, we consider first the Lie transformation of a polynomial function of $q$ alone; in other words, a map of the form $\exp(\cdot g(q)\cdot)$ with $g$ an arbitrary polynomial in $q$. Using $(46)$ and $(30)$, one may compute the action of such a map to be

$$
\exp(\cdot g(q)\cdot) \left( \begin{array}{c} q \\ p \end{array} \right) = \left( \begin{array}{c} q \\ p + \frac{\partial g}{\partial q} \end{array} \right),
$$

which is necessarily a polynomial symplectic map. Because this map changes only the momentum, we call it a kick map and refer to the corresponding Lie generator $g$ as a kick. Now suppose we choose any linear symplectic map $L$ and use it to form a more general map $\exp(\cdot Lg(q)\cdot)$. With the aid of $(47)$ one may write

$$
\exp(\cdot Lg(q)\cdot) = L \exp(\cdot g(q)\cdot) L^{-1}.
$$

Then the linearity of $L$ assures us that maps of the form $(88)$ are also necessarily polynomial symplectic maps. We call such a generalized kick map a jolt map and its Lie generator, $Lg$, a jolt. What we have learned here is that jolt maps—defined by an $L$ and a $g(q)$—can supply us with an endless stream of Cremona maps.

Later in this paper, we shall make use of jolt maps to construct a Cremona symplectification for our example map $(54)$. To do so, we first need to develop some concepts and tools.

In the vector space of dynamical polynomials (i.e. all polynomials on phase space), we define a set of general basis monomials of degree $l$ by the rule

$$
G_{r}^{(l)}(z) = \frac{q^{l-r}p^{r}}{\sqrt{(l-r)!r!}}.
$$

And for the basis monomial in $q$ alone, we write $Q^{(l)} = G_{0}^{(l)}$; thus,

$$
Q^{(l)}(q) = \frac{q^{l}}{\sqrt{l!}}.
$$

In addition, we introduce a (very special) inner product $\langle , \rangle$ defined by the rule

$$
\langle G_{r}^{(l)}, G_{r'}^{(l')} \rangle = \delta_{l,l'}\delta_{r,r'}.
$$
We see that with respect to this inner product, the \( G_r^{(l)} \) constitute an orthonormal basis for the space of dynamical polynomials. Now suppose we have dynamical polynomials \( f = \sum_l f_{lr} G_r^{(l)} \) and \( g = \sum_l g_{lr} G_r^{(l)} \). We extend the inner product (91a), to the entire vector space of dynamical polynomials by defining

\[
\langle f, g \rangle = \sum_l f_{lr} g_{lr}.
\]  

(91b)

For our purposes, the essential feature of what we call the invariant scalar product, (91), is that any transformation belonging to the \( U(1) \) subgroup of \( Sp(2, \mathbb{R}) \) leaves this inner product unchanged [20]. That subgroup is the group of plane rotations, and hence

\[
\langle \mathcal{L} f, \mathcal{L} g \rangle = \langle f, g \rangle
\]

for any \( \mathcal{L} \) having the form, (55), of a plane rotation. Because the jolt \( \mathcal{L} g(q) \) is generally a polynomial in both \( q \) and \( p \) (recall (57)), we may hope that a set of jolts \( \{ \mathcal{L}_j Q^{(l)}(q) \} \) can span the space of relevant dynamical polynomials.

We have two other concepts to introduce. The first is that of sensitivity vectors \( \sigma^r \), which have components

\[
\sigma^r_l = \langle G_r^{(l)}, \mathcal{L}_j Q^{(l)} \rangle. \tag{92}
\]

These components measure the content of each jolt \( \mathcal{L}_j Q^{(l)} \) within the given monomial \( G_r^{(l)} \). The second concept is that of the Gram matrix \( \Gamma(l) \), which has components

\[
\Gamma(l)_{rs} = \frac{1}{N} \sum_{j=1}^N \sigma^r_j \sigma^s_j \equiv \{ \sigma^r, \sigma^s \}, \tag{93}
\]

where we have introduced a weighted scalar product denoted \( \{ , \} \). This symmetric matrix, which measures the uniqueness, or linear independence, of the different sensitivity vectors, depends only on one’s choice of \( \mathcal{L}_j \).

With the above concepts and tools in hand, let us return to the example nonlinear Lie generator \( qp^2 \) of (56). We ask ourselves, “How can one express this generator as a linear combination of jolts?” Or the slightly more general question: How do we determine \( N \) jolts \( \mathcal{L}_j Q^{(3)} \) together with associated jolt strengths \( a_j^{(3)} \) so as to obtain a jolt decomposition,

\[
f_3(q, p) = \frac{1}{N} \sum_{j=1}^N a_j^{(3)} \mathcal{L}_j Q^{(3)}, \tag{94}
\]

for any homogeneous dynamical polynomial \( f_3 \) of degree three? Because the mono-
mials $G_r^{(3)}$ form a basis for such polynomials, we may, with the use of (91a), write

$$f_3 = \sum_{r=0}^{3} c_r^{(3)} G_r^{(3)}, \text{ where } c_r^{(3)} = \langle G_r^{(3)}, f_3 \rangle .$$

(95)

Now insert the jolt decomposition (94) into the latter equality. We find that

$$c_r^{(3)} = \frac{1}{N} \sum_{j=1}^{N} a_j^{(3)} \langle G_j^{(3)}, L_j Q^{(3)} \rangle = \frac{1}{N} \sum_{j=1}^{N} a_j^{(3)} \sigma_j^r, = \{ a^{(3)}, \sigma^r \} .$$

(96)

Since no component of $a^{(3)}$ that lies orthogonal to $\sigma^r$ can contribute to $c_r^{(3)}$, we make the Ansatz that the vector of jolt strengths, $a^{(3)}$, must be a linear combination of the sensitivity vectors; thus $a^{(3)} = \sum_s a_s^{(3)} \sigma^s$. Inserting this expansion into (96), we obtain

$$c_r^{(3)} = \sum_{s=0}^{3} a_s^{(3)} \{ \sigma^s, \sigma^r \} = \sum_{s=0}^{3} \Gamma(3)_{rs} a_s^{(3)} ; \text{ hence } c^{(3)} = \Gamma(3) a^{(3)} .$$

(97)

This result tells us in order to compute $a^{(3)}$—and hence the jolt strengths $a^{(3)}$—we require a non-singular Gram matrix $\Gamma(3)$. We may then compute $a^{(3)} = \Gamma(3)^{-1} c^{(3)}$.

One may parameterize the $L_j$ by rotation angles $\theta_j$. It is then possible to compute the sensitivity vectors analytically, with result

$$\sigma_j^r = \binom{3}{r}^{1/2} c_j^{3-r} s_j^r ,$$

(98)

where $c_j = \cos \theta_j$ and $s_j = \sin \theta_j$. One then obtains the Gram matrix elements in the explicit form [21]

$$\Gamma(3)_{rs} = \left[ \binom{3}{r} \binom{3}{s} \right]^{1/2} \frac{1}{N} \sum_j c_j^{6-(r+s)} s_j^{r+s} .$$

(99)

The questions that remain are (i) what is the best choice of angles $\theta_j$, and (ii) how many do we need? It seems reasonable to be democratic about our choice of angles, and analysis has indeed shown evenly-spaced angles to be optimal [22]. A simple dimension-counting argument tells us that for the case $l = 3$, the number of jolts cannot be less than 4. However, an analysis of (99) [23], or direct numerical computation, shows that four evenly-spaced angles yield a singular $\Gamma(3)$; but use of five evenly-spaced angles does not.

Let us review our progress so far: We have the third-order Lie generator $qp^2$, which we want to decompose into a linear combination of jolts, as in (94). Using the evenly-spaced angles $\theta_j = 2\pi j/5$, we construct, cf. (55), the five rotation maps $L_j = R(\theta_j)$, with $j \in \{ 0, \ldots, 4 \}$. We also construct the Gram matrix (99), sensitivity vectors (98), and the jolt strengths $a^{(3)} = \sum_r a_r^{(3)} \sigma^r$, where $a^{(3)} = \Gamma(3)^{-1} c^{(3)}$. This allows us, finally, to construct the jolt decomposition

$$qp^2 \approx -0.6532 L_0 Q^{(3)} + 0.8936 L_1 Q^{(3)} - 0.5670 L_2 Q^{(3)} - 0.5670 L_3 Q^{(3)} + 0.8936 L_4 Q^{(3)} .$$
Figure 4. Phase-space portrait, in the case $\theta/2\pi = 0.22$, resulting from applying the map $M^{cr} = RN^{cr}$ repeatedly (2000 times) to the seven initial conditions $(q, p) = (0.01, 0), (0.1, 0), (0.15, 0), (0.2, 0), (0.25, 0), (0.3, 0)$, and $(0.35, 0)$ to find their orbits. Light gray curves in the background indicate the exact result. In the left-hand graphic, the jolt maps are applied in numerical order, i.e. $\{0, 1, 2, 3, 4\}$. In the right-hand graphic, the jolt maps are instead applied in the order $\{3, 4, 0, 1, 2\}$.

The last step is to split this decomposition into five individual jolt maps. We thereby achieve an approximation to $N$ in the form of a Cremona map,

$$N^{cr} = L_0 e^{-0.6532Q^{(3)}}; L_0^{-1} L_1 e^{0.8936Q^{(3)}}; L_1^{-1} L_2 e^{-0.5670Q^{(3)}}; L_2^{-1} \times L_3 e^{-0.5670Q^{(3)}}; L_3^{-1} L_4 e^{0.8936Q^{(3)}}; L_4^{-1}.$$  \hspace{1cm} (100)

We make two observations about this form. First, the error made by this approximation is, according to the Baker-Campbell-Hausdorff theorem [24], determined by commutators and multiple commutators of the different jolts, which in this case have degree 4 and higher, i.e., starting at one degree higher than the jolts themselves. The hope is that those higher-degree terms effectively added to $N$ by our Cremona factorization do not damage the dynamics. Second, the order of the factors in (100) is not prescribed. We may reorder them without changing the degree of approximation.

Figure 4 shows the result of applying $M^{cr} = RN^{cr}$ repeatedly to seven initial conditions for the case $\theta/2\pi = 0.22$. The two graphics in that figure correspond to different orderings of the factors in (100). To facilitate comparison, light gray curves in the background indicate the corresponding exact results. Evidently, the order of factors can have a dramatic impact on the result’s absolute accuracy. In the left-hand graphic, the original outer curve is replaced by five islands, and the next curve inwards has angular corners. While this effect remains a topic of research, we point out how one may—absent knowledge of the correct result—address the associated uncertainty as to which is the more accurate result.

Suppose we rewrite the map $RN$ in the form

$$RN = RN^{1/2}N^{1/2}.$$  \hspace{1cm} (101)

The reduced nonlinear content of the square root, $N^{1/2}$, will make a Cremona symplectification of this map more accurate. In words, this process means we (i) scale the Lie generator $qp^2$ by $1/2$; (ii) split $N^{1/2}$, as we did $N$ in (100), to obtain $N^{rtcr} = (N^{cr})^{1/2}$; and (iii) square this map to obtain an approximation to $M$, which we call $M^{rtcr}$, in the form

$$M^{rtcr} = RN^{rtcr}N^{rtcr}.$$  \hspace{1cm} (102)
Figure 5. Cremona symplectification using five-plus-five jolts with the root trick. The left-hand graphic shows the result of applying the root trick to the map used for the right-hand graphic in Figure 4. And the right-hand graphic here shows the result of applying the symmetrized root trick to that same map.

(Note that in our example we now must apply ten successive jolt maps.) This process, which we call the root trick, can also improve results obtained by use of generating functions [25]. The general technique of scaling, splitting, and squaring has application to a broad range of problems in computational physics [26].

The left-hand graphic in Figure 5 shows the result of applying the root trick to the ordering \( \{3, 4, 0, 1, 2\} \). Note the improved accuracy as compared to the right-hand graphic in Figure 4. Not shown is the result of applying the root trick to the ordering \( \{0, 1, 2, 3, 4\} \), which exhibits greatly improved accuracy. In particular, gone are the offensive islands and angular corners present in the left-hand graphic of Figure 4.

Moreover, we can improve accuracy still further by symmetrizing the factors: By this we mean that one obtains an improved result by approximating \( \mathcal{N} \) as

\[
\mathcal{N}^\text{rev} = \mathcal{N}_{\text{rev}}^\text{rev}
\]

where the subscript ‘rev’ means that the given map’s factors should be applied in the reverse order. It can be shown that this symmetrization of the nonlinear factors leads to an automatic reduction of many spurious higher-order terms including the canceling of all \( f_4 \) terms. (This is the same desirable feature found for the use of a Poincaré generating function described in the previous section.) The full map \( \mathcal{M} \) now has the approximation

\[
\mathcal{M}^\text{rev} = \mathcal{R}\mathcal{N}^\text{rev}.
\]

The right-hand graphic in Figure 5 shows the result of using \( \mathcal{M}^\text{rev} \), with the ordering \( \{3, 4, 0, 1, 2\} \) for \( \mathcal{N}^\text{rev} \), to track particle trajectories. Evidently, there is now near perfect agreement with exact results, agreement that is comparable to that found with the use of a Poincaré generating function.

We have illustrated how to find, over a substantial region of phase space, a Cremona approximation for a nonlinear map (in one degree of freedom) having the form \( \mathcal{N} = \exp(\cdot; f_3 \cdot) \). In an analogous (but substantially more complicated) manner, with the use of more jolt maps, one may find, in two and three degrees of freedom, suitable Cremona approximations to maps of the form (52). Key to the construction of jolt maps for an \( \mathcal{N} \) having \( f > 2 \) in its factored product form, and acting on phase spaces corresponding to two or three degrees of freedom, is an optimal choice of the linear maps \( \mathcal{L}_j \) used in constructing the jolt maps. It is known that an optimal choice of linear maps is related to the construction of suitable cubature formulas for various
manifolds.\textsuperscript{16} How to do so in the case of one degree of freedom is well understood for all \(f > 2\); and the case of two degrees of freedom is reasonably well understood for all \(f < 15\). Much work remains for the case of three degrees of freedom. In particular, one would like to have, for that case, cubature formulas for the manifold \(SU(3)/SO(3)\). Using this approach, a suitable set of 108 \(L_j\) has been found for all \(f < 7\) \cite{27}.

8. Concluding Discussion

By design a storage ring has a (one-time-around) closed orbit. Even if the design is not perfectly executed, there is a fixed-point theorem to the effect that there is still a closed orbit that is near the design closed orbit. Consider the passage of particles near the closed orbit through individual beam-line elements or through collections of successive elements (called lumps) or once around the entire ring. Each such passage is described by a symplectic map whose jet is computable/known to some order \((\text{maxm} - 1)\). Each of these jets can be symplectified to produce symplectic maps that can be used to propagate particles around the ring by letting them act in succession thereby producing in effect a net one-turn symplectic map. This operation is called tracking. The slowest, but presumably most accurate, procedure would be to track element-by-element. A faster procedure would be to track lump-by-lump. Its accuracy could be checked by comparing its result to element-by-element tracking results. Even faster and more daring would be to perform full turn-by-turn tracking using the symplectic map produced by symplectifying the jet for the full one-turn map. Whatever method is selected, it can be applied repeatedly a large number of times to simulate the effect of a large number of turns while being exactly symplectic (to machine precision) and having accuracy through order \((\text{maxm} - 1)\). Since evaluation of the action of these symplectified maps (obtained either by generating function or Cremona symplectification) on phase space is fairly fast, it is possible to track in these ways for a relatively large number of turns with relatively modest use of computer time.

For example, let us consider the case of the LHC, for which a particle passes through approximately 19,000 elements per turn. One approach that has been used in the past is to track particles element by element using the approximation given by (37) and (38). That is, all fringe-field effects are neglected. Moreover, the maps for drift spaces, bending magnets (dipoles), and focusing/defocusing magnets (quadrupoles) are approximated by linear (matrix) maps. Finally, the nonlinear effects of higher multipole magnets (sextupoles, octupoles, etc.), as well as multipole errors in dipoles and quadrupoles, are simply treated as kicks. This approach is often referred to as direct/"exact" tracking, although what it actually does is equivalent to implementing a relatively crude, but exactly symplectic, one-turn map.

Since generating function or Cremona tracking begins with symplectic jets, and these jets can in principle be computed for realistic electromagnetic fields, generating function or Cremona tracking can in principle be expected to give more accurate results for realistic machines. However, in order to assure the Accelerator Physics community of their reliability, generating function or Cremona tracking should also be able to reproduce the results of direct tracking. That is, based on the assumptions made for direct tracking, relatively crude but symplectic jets can be computed for each beam line element, and these jets can be concatenated to form symplectic jets.

\textsuperscript{16}Cubature formulas are higher-dimensional analogs of quadrature formulas.
for lumps or one-turn maps. These jets can then be symplectified using generating
function or Cremona methods, and their tracking results compared with those
obtained by direct tracking. Preliminary studies/comparisons of this kind for the
LHC, prior to its completed construction, were carried out for various nonlinear
imperfection models. They show that, even for relatively large betatron amplitudes
where nonlinear effects are expected to be important, there is good agreement in
dynamic aperture (phase-space region of long-term storage) between direct and one-
turn Cremona map tracking results using jets containing generators $f_{<8}$. Moreover,
in these studies Cremona tracking is approximately 20 times faster than direct
tracking. And, if one wishes to simulate orbits with smaller betatron amplitudes,
amplitudes associated with normal LHC operation, then use of generators with
$f_{<6}$ appears to be adequate, in which case Cremona tracking is about 60 times
faster than direct tracking. Finally, the same Cremona map tracking speeds can be
achieved to produce accurate results for realistic machines. Generating function and
Cremona symplectification methods are therefore worth further study, development,
and implementation.

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us (DTA) during the preparation of this paper.

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[1] Much of the background material for this chapter is most easily found on the Web
in a draft book: A. J. Dragt, Lie Methods for Nonlinear Dynamics with applications to
Accelerator Physics, URL http://www.physics.umd.edu/dsat/dsatliemethods.html. It, in turn,
provides numerous additional references. In subsequent citations it will be referred to
as LM. For a discussion of Lagrangians and Hamiltonians for charged-particle motion
in electromagnetic fields, see LM, Sections 1.5–1.7.
[2] LM, Section 1.6.
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[15] LM, Section 10.5 and Chapter 39. Some authors refer to TPSA as Differential Algebra
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[16] LM, Section 6.7.
Currently we do not find persuasive their invocation of the Hofer metric, but do agree (for other reasons) with their conclusion that use of the Poincaré generating function has several desirable features.

Detailed background material for this section is most easily found on the Web: D. T. Abell, *Analytic Properties and Cremona Approximation of Transfer Maps for Hamiltonian Systems*, PhD dissertation, University of Maryland, College Park, 1995. URL https://www.radiasoft.net/wp-content/uploads/2021/01/thesis_2e.pdf. In subsequent citations it will be referred to as TMCA. See also the chapter “Symplectic maps and computation of orbits in particle accelerators” by A. J. Dragt and D. T. Abell, in the book *Integration Algorithms and Classical Mechanics* (J. E. Marsden, G. W. Patrick, and W. F. Shadwick, eds.), vol. 10 of *Fields Inst. Comm.*, (Providence, Rhode Island), pp. 59–85, American Mathematical Society, 1996. See further S. Blanes, “Symplectic maps for approximating polynomial Hamiltonian systems,” *Phys. Rev. E*, vol. 65, 056703, May 2002. For a history of the term *Cremona maps*, and for a fuller discussion of the concepts of *kicks* and *jolts* than we present here (including the extension to two and three degrees of freedom), see TMCA, Chapters 10 and 11.