Lelek’s problem is not a metric problem

Conspici Quam Prodesse

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Lelek’s problem is not a metric problem
Two Notions

1. Two Notions

2. The Problem

3. The conversion

4. A better reflection

5. Sources

K. P. Hart

Lelek’s problem is not a metric problem
Chainability

Definition

A continuum, $X$, is **chainable** if every (finite) open cover $\mathcal{U}$ has an open chain-refinement $\mathcal{V}$, i.e.,

$\{V_i : i < n\}$ such that $V_i \cap V_j \neq \emptyset$ iff $|i - j| \leq 1$.

$[0, 1]$ is chainable; the circle $S^1$ is not.
A continuum, $X$, is chainable if every (finite) open cover $\mathcal{U}$ has an open chain-refinement $\mathcal{V}$, i.e., $\mathcal{V}$ can be written as $\{V_i : i < n\}$ such that $V_i \cap V_j \neq \emptyset$ iff $|i - j| \leq 1$. 

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Span zero

**Definition**

A continuum, $X$, has **span zero** if every subcontinuum $Z$ of $X \times X$ that satisfies $yyy$ intersects the diagonal $\{\langle x, x \rangle : x \in X\}$.
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$$\text{xxx}$$  $$\text{yyy}$$
Span zero

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$$\langle x, x \rangle \subseteq \pi_1[Z] \subseteq \pi_2[Z]$$

Lelek’s problem is not a metric problem
Two Notions
The Problem
The conversion
A better reflection
Sources

Span zero

Definition
A continuum, \( X \), has \textbf{span zero} if every subcontinuum \( Z \) of \( X \times X \) that satisfies \textbf{yyy} intersects the diagonal \( \{ \langle x, x \rangle : x \in X \} \).

\[
\begin{align*}
\text{xxx} & \quad \text{yyy} \\
\ldots & \quad \pi_1[Z] = \pi_2[Z] \\
\text{semi} & \quad \pi_1[Z] \subseteq \pi_2[Z]
\end{align*}
\]
Span zero

Definition

A continuum, $X$, has **span zero** if every subcontinuum $Z$ of $X \times X$ that satisfies $y$ intersects the diagonal $\{\langle x, x \rangle : x \in X \}$. 

| $\pi_1[Z]$ | $\pi_2[Z]$ | $\pi_1[Z] \subseteq \pi_2[Z]$ |
| $\pi_1[Z]$ | $\pi_2[Z]$ | $\pi_1[Z] = \pi_2[Z] = X$ |
Span zero

Definition

A continuum, $X$, has **span zero** if every subcontinuum $Z$ of $X \times X$ that satisfies $\ldots$ intersects the diagonal $\{\langle x, x \rangle : x \in X\}$.

| xxx | yyy |
|-----|-----|
| $\ldots$ | $\pi_1[Z] = \pi_2[Z]$ |
| semi | $\pi_1[Z] \subseteq \pi_2[Z]$ |
| surjective | $\pi_1[Z] = \pi_2[Z] = X$ |
| surjective semi | $\pi_2[Z] = X$ |

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Span zero

**Definition**

A continuum, $X$, has **span zero** if every subcontinuum $Z$ of $X \times X$ that satisfies $yyy$ intersects the diagonal $\{\langle x, x \rangle : x \in X\}$.

- $\ldots$
- **semi**
- **surjective**
- **surjective semi**

$$\pi_1[Z] = \pi_2[Z]$$

$$\pi_1[Z] \subseteq \pi_2[Z]$$

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$[0, 1]$ has all spans zero, $S^1$ has all spans non-zero
Lelek’s problem is not a metric problem
The problem

Theorem

In a chainable continuum all spans are zero.
The problem

Theorem

*In a chainable continuum all spans are zero.*

Question (Lelek)

What about the converse?
The problem

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What about the converse?

This is an important problem in metric continuum theory.
The Problem

Theorem

In a chainable continuum all spans are zero.

Question (Lelek)

What about the converse?

This is an important problem in metric continuum theory. We free it from the metric constraints.
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The Problem
The conversion
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Lelek’s problem is not a metric problem
Given a distributive, separative and normal lattice $L$ there is a compact Hausdorff space $wL$ with a base for its closed sets that is isomorphic to $L$. 
A useful tool

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Many properties of a space $X$ are first-order when expressed in terms of $2^X$, its lattice of (all) closed sets.
Given a distributive, separative and normal lattice $L$ there is a compact Hausdorff space $wL$ with a base for its closed sets that is isomorphic to $L$. $wL$ is the Wallman space of $L$.

Many properties of a space $X$ are first-order when expressed in terms of $2^X$, its lattice of (all) closed sets.

Quite often, in the case of $wL$, it suffices to work in $L$ only.
Reflection

Theorem

Any counterexample to Lelek’s problem can be converted into a metrizable counterexample.
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Proof.

Let $X$ be a counterexample, let $L \prec 2^X$ (an elementary sublattice).
Theorem

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Proof.

Let $X$ be a counterexample, let $L \prec 2^X$ (an elementary sublattice). Then $wL$ is a metrizable counterexample.
Any counterexample to Lelek’s problem can be converted into a metrizable counterexample.

Proof.

Let $X$ be a counterexample, let $L \prec 2^X$ (an elementary sublattice). Then $wL$ is a metrizable counterexample.

Not quite
(Non-)chainability is not a first-order property of the lattice $2^X$. 
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**Chainability:**

$$(\forall u_1)(\forall u_2)(\forall u_3)(\forall u_4)$$

$$((u_1 \cup u_2 \cup u_3 \cup u_4 = X) \rightarrow \bigvee_{n \in \omega} \Phi_n(u_1, u_2, u_3, u_4))$$

where $\Phi_n(u_1, u_2, u_3, u_4)$ expresses that $\{u_1, u_2, u_3, u_4\}$ has an $n$-element chain refinement. It suffices to consider four-element open covers only.
Complications

(Non-)chainability is not a first-order property of the lattice $2^X$. Their natural formulations are $L_{\omega_1,\omega}$-formulas.

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(\big((u_1 \cup u_2 \cup u_3 \cup u_4 = X) \rightarrow \bigvee_{n \in \omega} \Phi_n(u_1, u_2, u_3, u_4)\big))$$

where $\Phi_n(u_1, u_2, u_3, u_4)$ expresses that $\{u_1, u_2, u_3, u_4\}$ has an $n$-element chain refinement.

It suffices to consider four-element open covers only.
Another complication

We have no decent formula, $L_{\omega_1,\omega}$ or otherwise, that describes in terms of $2^X$ that $X$ has span (non-)zero.
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Lelek’s problem is not a metric problem
Solution: Use Set Theory

Let $\theta$ be ‘suitably large’ and let $M \prec H(\theta)$ be a countable elementary substructure
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Let $\theta$ be ‘suitably large’ and let $M \prec H(\theta)$ be a countable elementary substructure and let $L = M \cap 2^X$. 

Theorem

In this situation:

- $wL$ is chainable iff $X$ is chainable
- $wL$ has span zero iff $X$ has span zero (any kind)
Let $\theta$ be ‘suitably large’ and let $M \prec H(\theta)$ be a countable elementary substructure and let $L = M \cap 2^X$.

**Theorem**

*In this situation:*

1. $w_L$ is chainable iff $X$ is chainable
2. $w_L$ has span zero iff $X$ has span zero (any kind)

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Lelek’s problem is not a metric problem
Solution: Use Set Theory

Let $\theta$ be ‘suitably large’ and let $M \prec H(\theta)$ be a countable elementary substructure and let $L = M \cap 2^X$.

**Theorem**

*In this situation:*

- $wL$ is chainable iff $X$ is chainable
Let \( \theta \) be ‘suitably large’ and let \( M \preceq H(\theta) \) be a countable elementary substructure and let \( L = M \cap 2^X \).

**Theorem**

*In this situation:*

- \( wL \text{ is chainable iff } X \text{ is chainable} \)
- \( wL \text{ has span zero iff } X \text{ has span zero (any kind)} \)

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Lelek’s problem is not a metric problem
Chainability is now first-order; we can quantify over the finite subsets of $2^X$ and finite ordinals.
Proof for Chainability

Chainability is now first-order; we can quantify over the finite subsets of $2^X$ and finite ordinals.

Furthermore, one needs only consider covers and refinements that belong to a certain base.
Key observation: let $K = M \cap 2^{X \times X}$, then $wK = wL \times wL$. 
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This gives the easy part: if there is a ‘bad’ continuum in $X \times X$ then there is one in $M$ and it is equally bad in $wL \times wL$. 
Key observation: let \( K = M \cap 2^{X \times X} \), then \( wK = wL \times wL \).

This gives the easy part: if there is a ‘bad’ continuum in \( X \times X \) then there is one in \( M \) and it is equally bad in \( wL \times wL \).

For the converse . . .
Span zero, continued

... if \( Z \subseteq wL \times wL \) is ‘bad’ then there is an equally bad continuum in \( X \times X \) that maps onto \( Z \).
... if $Z \subseteq wL \times wL$ is ‘bad’ then there is an equally bad continuum in $X \times X$ that maps onto $Z$.

Easier said than constructed
... if $Z \subseteq wL \times wL$ is ‘bad’ then there is an equally bad continuum in $X \times X$ that maps onto $Z$.

Easier said than constructed: the difficulty lies in the fact that $K$ is not (necessarily) an elementary substructure of $2^{wK}$. 
Apply Shelah’s Ultrapower theorem
Apply Shelah’s Ultrapower theorem: take a cardinal $\kappa$, an ultrafilter $u$ on $\kappa$ and an isomorphism $h : \prod_u (2^X \times X) \to \prod_u wK$ (which can be taken to be the identity on $K$).
Apply Shelah’s Ultrapower theorem: take a cardinal $\kappa$, an ultrafilter $u$ on $\kappa$ and an isomorphism $h: \prod_u (2^X \times X) \to \prod_u wK$ (which can be taken to be the identity on $K$).

How does that help?
Apply Shelah’s Ultrapower theorem: take a cardinal $\kappa$, an ultrafilter $u$ on $\kappa$ and an isomorphism $h : \prod_u (2^X \times X) \to \prod_u wK$ (which can be taken to be the identity on $K$).

How does that help?

For that we need some topology.
Take a compact Hausdorff space $Y$ with a lattice base $B$. Also take a cardinal $\kappa$ and an ultrafilter $u$ on $\kappa$. 

Duality problem

The Wallman space of the ultrapower $\prod u B$ is the fiber $p^\leftarrow \kappa (u)$. Bankston calls this the ultracopower of $Y$; we write $Y_u$. 

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Lelek's problem is not a metric problem
Dualizing ultrapowers

Take a compact Hausdorff space $Y$ with a lattice base $B$. Also take a cardinal $\kappa$ and an ultrafilter $u$ on $\kappa$.

Consider $\beta(\kappa \times Y)$. We have two maps
Dualizing ultrapowers

Take a compact Hausdorff space $Y$ with a lattice base $B$. Also take a cardinal $\kappa$ and an ultrafilter $u$ on $\kappa$.

Consider $\beta(\kappa \times Y)$. We have two maps

- $p_\kappa : \beta(\kappa \times Y) \to \beta_\kappa$ (the extension of $\langle \alpha, y \rangle \mapsto \alpha$).
Take a compact Hausdorff space $Y$ with a lattice base $B$. Also take a cardinal $\kappa$ and an ultrafilter $u$ on $\kappa$.

Consider $\beta(\kappa \times Y)$. We have two maps

- $p_\kappa : \beta(\kappa \times Y) \to \beta\kappa$ (the extension of $\langle \alpha, y \rangle \mapsto \alpha$).
- $p_Y : \beta(\kappa \times Y) \to \beta\kappa$ (the extension of $\langle \alpha, y \rangle \mapsto y$).
Take a compact Hausdorff space $Y$ with a lattice base $B$. Also take a cardinal $\kappa$ and an ultrafilter $u$ on $\kappa$.

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- $p_\kappa : \beta(\kappa \times Y) \to \beta\kappa$ (the extension of $\langle \alpha, y \rangle \mapsto \alpha$).
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The Wallman space of the ultrapower $\prod_u B$ is the fiber $p_\kappa^\leftarrow(u)$. 
Dualizing ultrapowers

Take a compact Hausdorff space $Y$ with a lattice base $B$. Also take a cardinal $\kappa$ and an ultrafilter $u$ on $\kappa$.

Consider $\beta(\kappa \times Y)$. We have two maps

- $p_\kappa : \beta(\kappa \times Y) \to \beta\kappa$ (the extension of $\langle \alpha, y \rangle \mapsto \alpha$).
- $p_Y : \beta(\kappa \times Y) \to \beta\kappa$ (the extension of $\langle \alpha, y \rangle \mapsto y$).

The Wallman space of the ultrapower $\prod_u B$ is the fiber $p_{\kappa_u}^{-1}(u)$. Bankston calls this the ultracopower of $Y$; we write $Y_u$. 
Back to $Z \subseteq wK$. 

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Span zero, the real argument

Back to $Z \subseteq wK$.

- Let $Z_u = \text{cl}(\kappa \times Z) \cap p^{-1}_\kappa(u)$.
Span zero, the real argument

Back to $Z \subseteq wK$.

- Let $Z_u = \operatorname{cl}(\kappa \times Z) \cap p_\kappa^{-1}(u)$.
- $Z_u$ is a continuum
Back to $Z \subseteq wK$.

- Let $Z_u = \text{cl}(\kappa \times Z) \cap p_\kappa^{-1}(u)$.
- $Z_u$ is a continuum
- $\text{wh}[Z_u]$ is a continuum in $(X \times X)_u$ ($\text{wh}$ is dual to $h$).
Back to $Z \subseteq wK$.

- Let $Z_u = \text{cl}(\kappa \times Z) \cap p_{\kappa}^{-1}(u)$.
- $Z_u$ is a continuum
- $wh[Z_u]$ is a continuum in $(X \times X)_u$ ($wh$ is dual to $h$).
- $Z_X = p_{X \times X}[wh[Z_u]]$ is a continuum in $X \times X$. 

Lelek’s problem is not a metric problem.
Back to $Z \subseteq wK$.

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- $Z_u$ is a continuum
- $wh[Z_u]$ is a continuum in $(X \times X)_u$ ($wh$ is dual to $h$).
- $Z_X = p_{X \times X}[wh[Z_u]]$ is a continuum in $X \times X$.
- And

\[
q_K[Z_X] = q_K[p_{X \times X}[wh[Z_u]]] = p_{wK} [(wh)^{-1}[wh[Z_u]]] = Z
\]
Span zero, the real argument

Back to $Z \subseteq wK$.

- Let $Z_u = \text{cl}(\kappa \times Z) \cap p_\kappa^-(u)$.
- $Z_u$ is a continuum
- $wh[Z_u]$ is a continuum in $(X \times X)_u$ ($wh$ is dual to $h$).
- $Z_X = p_{X \times X}[wh[Z_u]]$ is a continuum in $X \times X$.
- And

$$q_K[Z_X] = q_K[p_{X \times X}[wh[Z_u]]] = p_{wK}[(wh)^{-1}[wh[Z_u]]] = Z$$

So, that’s it!?
Back to $Z \subseteq wK$.

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$$q_K[Z_X] = q_K[p_{X \times X}[wh[Z_u]]] = p_{wK}[(wh)^{-1}[wh[Z_u]]] = Z$$

So, that’s it!? Almost.
First expand the language of lattice with two function symbols $\pi_1$ and $\pi_2$. 
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Apply Shelah’s theorem with this extended language. Then \( Z_X \) will inherit the mapping properties that \( Z \) has.
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Apply Shelah’s theorem with this extended language. Then $Z_X$ will inherit the mapping properties that $Z$ has.

Finally then: if $X$ is a non-chainable continuum that has span zero (of one of the four kinds) than so is $wL$. 
Comment from Piotr Minc

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*Lelek’s problem is not a metric problem*, to appear.