ALMOST NEWTON, SOMETIMES LATTÈS

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Abstract. Self-maps everywhere defined on the projective space $\mathbb{P}^N$ over a number field or a function field are the basic objects of study in the arithmetic of dynamical systems. One reason is a theorem of Fakkruddin \cite{f} (with complements in \cite{h}) that asserts that a “polarized” self-map of a projective variety is essentially the restriction of a self-map of the projective space given by the polarization. In this paper we study the natural self-maps defined the following way: $F$ is a homogeneous polynomial of degree $d$ in $(N + 1)$ variables $X_i$ defining a smooth hypersurface. Suppose the characteristic of the field does not divide $d$ and define the map of partial derivatives $\phi_F = (F_{X_0}, \ldots, F_{X_N})$. The map $\phi_F$ is defined everywhere due to the following formula of Euler:

$$\sum X_i F_{X_i} = dF,$$

which implies that a point where all the partial derivatives vanish is a non-smooth point of the hypersurface $F = 0$. One can also compose such a map with an element of $\text{PGL}_{N+1}$. In the particular case addressed in this article, $N = 1$, the smoothness condition means that $F$ has only simple zeroes. In this manner, fixed points and their multipliers are easy to describe and, moreover, with a few modifications we recover classical dynamical systems like the Newton method for finding roots of polynomials or the Lattès map corresponding to the multiplication by 2 on an elliptic curve.

1. Introduction

Self-maps everywhere defined on the projective space $\mathbb{P}^N$ over a number field or a function field are the basic objects of study in the arithmetic of dynamical systems. One reason is a theorem of Fakkruddin \cite{f} (with complements in \cite{h}) that asserts that a “polarized” self-map of a projective variety is essentially the restriction of a self-map of the projective space given by the polarization. In this paper we study the natural self-maps defined the following way: $F$ is a homogeneous polynomial of degree $d$ in $(N + 1)$ variables $X_i$ defining a smooth hypersurface. Suppose the characteristic of the field does not divide $d$ and define the map of partial derivatives $\phi_F = (F_{X_0}, \ldots, F_{X_N})$. The map $\phi_F$ is defined everywhere due to the following formula of Euler:

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We begin by recalling some of the definitions and objects we need from dynamical systems before stating the main results.

Given a morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ we can iterate $\phi$ to create a (discrete) dynamical system. We denote the $n$th iterate of $\phi$ as $\phi^n = \phi(\phi^{n-1})$. Calculus students are exposed to dynamical systems

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through the iterated root finding method known as Newton’s Method where, given a differentiable function \( f(x) \) and an initial point \( x_0 \), one constructs the sequence

\[
x_{n+1} = \phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

In general, this sequence converges to a root of \( f(x) \). In terms of dynamics, we would say that the roots of \( f(x) \) are attracting fixed points of \( \phi(x) \). More generally, one says that \( P \) is a periodic point of period \( n \) for \( \phi \) if \( \phi^n(P) = P \).

A common example of a dynamical system with periodic points is to take an endomorphism of an elliptic curve \([m] : E \to E\) and project onto the first coordinate. This construction induces a map on \( \mathbb{P}^1 \) called a Lattès map, and for \( m \in \mathbb{Z} \) its degree is \( m^2 \) and its periodic points are the torsion points of the elliptic curve.

Denote \( \text{Hom}_d \) as the set of degree \( d \) morphisms on \( \mathbb{P}^1 \). There is a natural action on \( \mathbb{P}^1 \) by \( \text{PGL}_2 \) through conjugation that induces an action on \( \text{Hom}_d \). We take the quotient as \( M_d = \text{Hom}_d / \text{PGL}_2 \).

By [9], the moduli space \( M_d \) is a geometric quotient. We say that \( \gamma \in \text{PGL}_2 \) is an automorphism of \( \phi \) if \( \gamma^{-1} \circ \phi \circ \gamma = \phi \). We denote the (finite [7]) group of automorphisms as \( \text{Aut}(\phi) \).

Let \( K \) be a number field and \( F \in K[X,Y] \) be a homogeneous polynomial of degree \( d \) with distinct roots. Define

\[
\phi_F(X,Y) = [F_Y, -F_X] : \mathbb{P}^1 \to \mathbb{P}^1.
\]

**Remark.** We can think of \( \phi_F \) as the linear combination

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
F_X \\
F_Y
\end{pmatrix}.
\]

It may also be interesting to study other families of linear combinations of the partial derivatives arising from other elements of \( \text{GL}_2 \), but we do not address them in this article.

In Section 2 we examine the dynamical properties of these maps.

**Theorem (Theorem 5).** The fixed points of \( \phi_F(X,Y) \) are the solutions to \( F(X,Y) = 0 \), and the multipliers of the fixed points are \( 1 - d \).

**Theorem (Theorem 6 and Corollary 7).** The family of maps of the form \( \phi_F = (F_Y, -F_X) : \mathbb{P}^1 \to \mathbb{P}^1 \) is invariant under the conjugation action by \( \text{PGL}_2 \).

We also give a description of the higher order periodic points and a recursive definition of the polynomial whose roots are the \( n \)-periodic points. We examine related, more general Newton-Raphson maps and, finally, recall the connection to invariant theory and maps with automorphisms.

In Section 3 we explore the connection with Lattès maps.

**Theorem (Theorem 17).** Maps of the form

\[
\tilde{\phi}(x) = x - 3 \frac{f(x)}{f'(x)}
\]

are the Lattès maps from multiplication by [2] and \( f(x) = \prod(x - x_i) \) where \( x_i \) are the \( x \)-coordinates of the 3-torsion points.

Finally, when \( E \) has complex multiplication \( (m \notin \mathbb{Z}) \) the associated \( \phi_F \) can have a non-trivial automorphism group.

**Theorem (Theorem 18).** If \( E \) has \( \text{Aut}(E) \supseteq \mathbb{Z}/2\mathbb{Z} \) and the zeros of \( F(X,Y) \) are torsion points of \( E \), then an induced map \( \phi_F \) has a non-trivial automorphism group.
2. Almost Newton Maps

Let $K$ be a field and consider a two variable homogeneous polynomial $F(X,Y) \in K[X,Y]$ of degree $d$ with no multiple roots. Consider the degree $d - 1$ map

$$\phi_F : \mathbb{P}^1 \to \mathbb{P}^1$$

$$(X, Y) \mapsto (F_Y(X,Y), -F_X(X,Y)).$$

In particular, $F_X = F_Y = 0$ has no nonzero solutions and so $\phi_F$ is a morphism. Label $x = \frac{X}{Y}$ and consider

$$f(x) = \frac{F(X,Y)}{Y^d}$$

and notice that

$$f'(x) = \frac{F_X(X,Y)}{Y^{d-1}}.$$

**Lemma 1.** The map induced on affine space by $\phi_F$ is given by

$$\tilde{\phi}_F(x) = x - d \frac{f(x)}{f'(x)}.$$

**Proof.**

$$\tilde{\phi}_F(x) = -\frac{F_Y(X,Y)}{F_X(X,Y)} = -\frac{YF_Y(X,Y)}{YF_X(X,Y)} = \frac{XF_X(X,Y) - dF(X,Y)}{YF_X(X,Y)} = x - d \frac{f(x)}{f'(x)}.$$

□

**Definition 2.** Let $\phi = (\phi_1, \phi_2) : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map on $\mathbb{P}^1$. Define $\text{Res}(\phi) = \text{Res}(\phi_1, \phi_2)$, the resultant of the coordinate functions of $\phi$. For a homogeneous polynomial $F$, denote $\text{Disc}(F)$ for the discriminant of $F$.

**Proposition 3.** Let $F(X,Y)$ be a homogeneous polynomial of degree $d$ with no multiple roots. Then,

$$\text{Res}(\phi_F(X,Y)) = (-1)^{d(d-1)/2} d^{d-2} \text{Disc}(F(X,Y)).$$

**Proof.** Denote $F(X,Y) = a_d X^d + a_{d-1} X^{d-1} Y + \cdots + a_0 Y^d$. Then we have

$$F_X(X,Y) = da_d X^{d-1} + \cdots + a_1 Y^{d-1}$$

$$F_Y(X,Y) = a_{d-1} X^{d-1} + \cdots + da_0 Y^{d-1}.$$

From standard properties of resultants and discriminants we have

$$a_d \text{Disc}(F(X,Y)) = (-1)^{d(d-1)/2} \text{Res}(F(X,Y), F_X(X,Y))$$

$$= (-1)^{d(d-1)/2} \left( -\frac{1}{d^{d-1}} \text{Res}(dF(X,Y), -F_X(X,Y)) \right)$$

$$= (-1)^{d(d-1)/2} \left( -\frac{1}{d^{d-1}} \text{Res}(XF_X(X,Y) + YF_Y(X,Y), -F_X(X,Y)) \right)$$

$$= (-1)^{d(d-1)/2} \left( -\frac{1}{d^{d-1}} \text{Res}(YF_Y(X,Y), -F_X(X,Y)) \right).$$
Now we see that

\[ \text{Res}(YF_Y, -F_X) = \begin{vmatrix} 0 & a_{d-1} & 2a_{d-2} & \cdots & da_1 & 0 \\ 0 & 0 & a_{d-1} & 2a_{d-2} & \cdots & da_1 \\ \vdots & & \vdots & & \vdots & \vdots \\ -da_d & -(d - 1)a_{d-1} & \cdots & -a_1 & 0 & 0 \\ 0 & -da_d & -(d - 1)a_{d-1} & \cdots & -a_1 & 0 \\ \vdots & & \vdots & & \vdots & \vdots \end{vmatrix}. \]

Expanding down the first column we have

\[ \text{Res}(YF_Y(X,Y), -F_X(X,Y)) = -da_n(-1)^{d+1} R(F_Y(X,Y), -F_X(X,Y)). \]

Thus, we compute

\[ a_d \text{Disc}(F(X,Y)) = (-1)^{d(d-1)/2} \left( -1 \right)^d \frac{(-1)^d}{d^{d-1}} \text{Res}(YF_Y(X,Y), -F_X(X,Y)) \]

\[ = (-1)^{d(d-1)/2} \left( -1 \right)^d \frac{(-1)^{d+2}da_n}{d^{d-2}} \text{Res}(F_Y(X,Y), -F_X(X,Y)) \]

\[ = (-1)^{d(d-1)/2} \frac{a_d}{d^{d-2}} \text{Res}(F_Y(X,Y), -F_X(X,Y)). \]

\[ \square \]

**Remark.** The similar relationship for flexible Lattès maps ([m] for \( m \in \mathbb{Z} \))

\[ \text{Disc}(\Psi_{E,m-1}\Psi_{E,m+1}) = c \text{Res}(\phi_{E,m}), \]

where \( \Psi_{E,m} \) is the \( m \)-division polynomial and \( \phi_{E,m} \) is the Lattès map induced by \([m]\), seems to not be currently known. Using conjectures on \( \text{Disc}(\Psi_{E,m}) \) from [2] and the formula for \( \text{Res}(\phi_{E,m}) \) [10 Exercise 6.23] it appears that the exponent is correct and that constant should be

\[ c = \pm 2^a(m - 1)^b(m + 1)^c \]

for some integers \( a, b, c \). It would be interesting to determine the exact relation.

**Definition 4.** Let \( P \) be a periodic point of period \( n \) for \( \tilde{\phi} \), then the multiplier at \( P \) is the value \( (\tilde{\phi}^n)'(P) \). If \( P \) is the point at infinity, then we can compute the multiplier by first changing coordinates.

**Theorem 5.** The fixed points of \( \phi_F(X,Y) \) are the solutions to \( F(X,Y) = 0 \), and the multipliers of the fixed points are \( 1 - d \).

**Proof.** The projective equality

\[ \phi(X,Y) = (X,Y) \]

is equivalent to

\[ YF_Y(X,Y) = -XF_X(X,Y). \]
Using the formula of Euler for homogeneous polynomials we then have
\[ X F_X(X, Y) + Y F_Y(X, Y) = d F(X, Y) = 0. \]
Since \( d \) is a nonzero integer the fixed points satisfy \( F(X, Y) = 0 \).

To calculate the multipliers, we first examine the affine fixed points. We take a derivative evaluated at a fixed point to see
\[ \phi_F'(x) = 1 - d \frac{f'(x) f''(x) - f(x) f'''(x)}{(f'(x))^2} = 1 - d \frac{f'(x) f''(x)}{(f'(x))^2} = 1 - d. \]
If a fixed point has multiplier one, then it would have multiplicity at least 2 and, hence, would be at least a double root of \( F \). Since \( F \) has no multiple roots, every multiplier is not equal to one. Thus, to see that the multiplier at infinity (when it is fixed) is also \( 1 - d \) we may use the relation [10] Theorem 1.14
\[ \sum_{i=1}^{d} \frac{1}{1 - \lambda_i} = 1. \]
\[ \square \]

**Remark.** If \( \text{char } K \mid d \), then \( \phi_F \) is the identity map. Let \( F(X, Y) = a_d X^d + a_{d-1} X^{d-1} Y + \cdots + a_0 Y^d \). Then we have
\[ F_X(X, Y) = (d - 1) a_{d-1} X^{d-1} Y + \cdots + a_1 Y^{d-1} = Y((d - 1) a_{d-1} X^{d-1} + \cdots a_1 Y^{d-2}) \]
\[ F_Y(X, Y) = a_{d-1} X^{d-1} + \cdots + (d - 1) a_1 Y^{d-2} X = X(a_{d-1} X^{d-1} + \cdots + (d - 1) a_1 Y^{d-2}). \]
Since \( -i \equiv d - i \pmod{d} \) for \( 0 \leq i \leq d \) we have that
\[ \phi_F(X, Y) = (F_Y, -F_X) = (XP(X, Y), YP(X, Y)) = (X, Y), \]
where \( P(X, Y) \) is a homogeneous polynomial.

We next show that maps of the form \( \phi_F \) form a family in the moduli space of dynamical systems. In other words, for every \( \gamma \in \text{PGL}_2 \) and \( \phi_F \), there exists a \( G(X, Y) \) such that \( \gamma^{-1} \circ \phi_F \circ \gamma = \phi_G \). In fact, \( G(X, Y) \) is the polynomial that results from allowing \( \gamma^{-1} \) to act on \( F \).

**Theorem 6.** Every rational map \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) of degree \( d - 1 \) whose fixed points are \( \{(a_1, b_1), \ldots, (a_d, b_d)\} \) all with multiplier \( (1 - d) \) is a map of the form \( \phi_F(X, Y) = (F_Y(X, Y), -F_X(X, Y)) \) for
\[ F(X, Y) = (b_1 X - a_1 Y)(b_2 X - a_2 Y) \cdots (b_d X - a_d Y). \]

**Proof.** Let \( (a_1, b_1), \ldots, (a_d, b_d) \) be the collection of fixed points for the map \( \psi(X, Y) : \mathbb{P}^1 \to \mathbb{P}^1 \) whose multipliers are \( 1 - d \). Then on \( \mathbb{A}^1 \) we may write the map of degree \( d - 1 \) as
\[ \tilde{\psi}(x) = x - \frac{P(x)}{Q(x)} \]
for some pair of polynomials \( P(x) \) and \( Q(x) \) with no common zeros. Let \( \tilde{\phi}_F(x) \) be the affine map associated to \( F(X, Y) = (b_1 X - a_1 Y) \cdots (b_d X - a_d Y) \) and we can write
\[ \tilde{\phi}_F(x) = x - d \frac{f(x)}{f'(x)} \]
where
\[ f(x) = \frac{F(X, Y)}{Y^d}. \]
The fixed points of $\tilde{\psi}(x)$ are the points where $\frac{P(x)}{Q(x)} = 0$ and, hence, where $P(x) = 0$. The fixed points of $\tilde{\psi}(x)$ are the same as for $\tilde{\phi}_F(x)$, so we must have $P(x) = cf(x)$ for some nonzero constant $c$. Using the fact that the multipliers are $1 - d$, we get

$$\tilde{\psi}'(x) = 1 - \frac{cf'Q - cQ'}{(Q')^2} = 1 - \frac{cf'}{Q} = 1 - d.$$ 

Therefore, we know that

$$\frac{c}{d} f'(x_i) = Q(x_i)$$

where $x_1, \ldots, x_d$ are the fixed points (or $x_1, \ldots, x_{d-1}$ if $(1,0) \in \mathbb{P}^1$ is a fixed point). Since $f'(x)$ and $Q(x)$ are both degree $d - 1$ polynomials (or $d - 2$), this is a system of $d$ (or $d - 1$) equations in the $d$ (or $d - 1$) coefficients of $Q(x)$. Since the values $x_i$ are distinct (since the multipliers are $\not= 1$) the Vandermonde matrix is invertible and we get a unique solution for $Q(x)$. In particular, we must have

$$\frac{c}{d} f'(x) = Q(x)$$

and thus

$$\tilde{\psi}(x) = \tilde{\phi}(x).$$

\[ \square \]

Corollary 7. The family of maps of the form $\phi_F(X, Y) = (F_Y(X, Y), -F_X(X, Y)) : \mathbb{P}^1 \to \mathbb{P}^1$ is invariant under the conjugation action by $\text{PGL}_2$. In particular, the family of $\phi_F$ where $\deg F(X, Y) = d$ is isomorphic to an arbitrary choice of $d - 3$ distinct points in $\mathbb{P}^1 - \{0, 1, \infty\}$.

Proof. Conjugation fixes the multipliers and moves the fixed points, so by Theorem 6 the conjugated map is of the same form.

A map of degree $d - 1$ on $\mathbb{P}^1$ has $d$ fixed points. The action by $\text{PGL}_2$ can move any 3 distinct points to any 3 distinct points. Thus, the choice of the remaining $d - 3$ fixed points determines $\phi_F$. \[ \square \]

2.1. Extended Example.

Proposition 8. Let $F(X, Y)$ be a degree 4 homogeneous polynomial with no multiple roots with associated morphism $\phi_F(X, Y)$. For any $\alpha \in \overline{\mathbb{Q}} - \{0, 1\}$ we have that $\phi_F(X, Y)$ is conjugate to a map of the form

$$\phi_{F,\alpha}(X, Y) = (X^3 - 2(\alpha + 1)X^2Y + 3\alpha XY^2, -3X^2Y + 2(\alpha + 1)XY^2 - \alpha Y^3).$$

Proof. We can move three of the 4 fixed points to $\{0, 1, \infty\}$ with an element of $\text{PGL}_2$ and label the fourth fixed point as $\alpha$. Then we have

$$F(X, Y, \alpha) = (X)(Y)(X - Y)(X - \alpha Y) = X^3Y - (\alpha + 1)X^2Y^2 + \alpha XY^3$$

and

$$\phi_{F,\alpha}(X, Y) = (F_Y(X, Y, \alpha), -F_X(X, Y, \alpha))$$

$$= (X^3 - 2(\alpha + 1)X^2Y + 3\alpha XY^2, -(3X^2Y - 2(\alpha + 1)XY^2 + \alpha Y^3)).$$

\[ \square \]

Proposition 9. Let $F(X, Y)$ be a degree 4 homogeneous polynomial with no multiple roots with associated morphism $\phi_F(X, Y)$. Assume that $\phi_F(X, Y)$ is in the form of Proposition 8. Then, the two periodic points are of the form

$$\{\pm \sqrt{\alpha}, 1 \pm \sqrt{1 - \alpha}, \alpha \pm \sqrt{\alpha^2 - \alpha}\} \cup \{0, 1, \infty, \alpha\}.$$

Proof. Direct computation. \[ \square \]
Proposition 10. Q-Rational affine two periodic points are parameterized by pythagorean triples.

Proof. The values $\alpha$ and $1 - \alpha$ are both squares and $0 < \alpha < 1$. Thus, there are relatively prime integers $p$ and $q$ so that $\alpha = \frac{p^2}{q^2}$ with $p < q$ and $1 - \alpha = \frac{q^2 - p^2}{q^2}$. Therefore, $r^2 + p^2 = q^2$ is a pythagorean triple, with $r^2 = (1 - \alpha)q^2$.

Remark. The 2-periodic points are not the roots of $f(\tilde{\phi}(x))$, see Theorem 12 for the general relation.

For general $F(X, Y), \phi^2_f(X, Y)$ does not come from a homogeneous polynomial $G$.

2.2. Higher order periodic points. We set the following notation

$$f(x) = \frac{F(X, Y)}{Y^d} = \sum_{i=0}^{d-1} a_i x^i$$

$$\tilde{\phi}^n(x) = \frac{A_n(x)}{B_n(x)}$$

$$c_n = -\frac{B_{n+1}(x)}{F_X(A_n(x), B_n(x))}$$

where $A_n(x)$ and $B_n(x)$ are polynomials and $c_n$ is a constant.

Definition 11. Let $\Psi_n(x)$ be the polynomial whose zeros are affine $n$-periodic points.

The polynomial $\Psi_n(x)$ is the equivalent of the $n$-division polynomial for elliptic curves, see [6, Chapter 2] for information on division polynomials.

While it is possible, to define $\Psi_n(x)$ recursively, the relation is not as simple as for elliptic curves. If we let $\Psi_{E, m}$ be the $m$-division polynomial for an elliptic curve $E$, then

$$\Psi_{E, 2m+1} = \Psi_{E, m+2}\Psi_{E, m}^3 - \Psi_{E, m-1}\Psi_{E, m+1}^3 \quad \text{for } m \geq 2$$

$$\Psi_{E, 2m} = \left( \frac{\Psi_{E, m}}{2y} \right) (\Psi_{E, m+2}\Psi_{E, m-1}^2 - \Psi_{E, m-2}\Psi_{E, m+1}^2) \quad \text{for } m \geq 3.$$ 

Notice that these relations depend only on $\Psi_{E, m}$ for various $m$, whereas the formula in the following theorem also involves iterates of the map.

Theorem 12. We have the following formulas

$$\tilde{\phi}^n(x) = x + d\frac{\Psi_n(x)}{B_n(x)}$$

and

$$\Psi_{n+1}(x) = \frac{F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x))}{B_n(x)c_n}$$

with multipliers

$$\prod_{i=0}^{n-1} \left( 1 - d + d\frac{f(\tilde{\phi}^i(x))f''(\tilde{\phi}^i(x))}{f'(\tilde{\phi}^i(x))^2} \right).$$

Proof. We proceed inductively. For $n = 1$ we know that the fixed points are the zeros of $f(x)$.

$$\tilde{\phi}(x) = x - d\frac{f(x)}{f'(x)} = x - d\frac{f(x)}{F_X(A_0(x), B_0(x))} = x - d\frac{f(x)}{-B_1(x)} = x + d\frac{\Psi_1(x)}{B_1(x)}.$$

Now assume that

$$\tilde{\phi}^n(x) = x + d\frac{\Psi_n(x)}{B_n(x)}.$$
Computing
\[ \tilde{\phi}^{n+1}(x) = x + d \frac{\Psi_n(x)}{B_n(x)} - d \frac{f'(\tilde{\phi}^n(x))}{f'(\tilde{\phi}^n(x))} \]
= \[ x + d \frac{\Psi_n(x)}{B_n(x)} - d \frac{F(A_n(x), B_n(x))}{B_n(x)F_X(A_n(x), B_n(x))} \]
= \[ x - \frac{d}{F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x))} \]
= \[ x - \frac{d}{F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x))} \]
\[ \frac{c_n B_n(x) B_{n+1}(x)}{c_n B_n(x) B_{n+1}(x)} \] .

So we have to show that \( B_n(x) \) divides \( F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x)) \). Working modulo \( B_n(x) \) we see that
\[ F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x)) \equiv A_n(x)^d - (A_n(x)/d)dA_n(x)^{d-1} \equiv 0 \pmod{B_n(x)} \]
where we used the induction assumption for \( \Psi_n(x) \). Thus, the \( n \)-periodic points are among the roots of \( \Psi_n(x) \).

For equivalence, we count degrees. Again, proceeding inductively it is clear for \( n = 1 \). For \( n + 1 \) we have that
\[ \deg(F(A_n(x), B_n(x)) = d(d - 1)^n = (d - 1)^{n+1} + (d - 1)^n \]
and
\[ \deg(\Psi_n(x)F_X(A_n(x), B_n(x))) \leq (d - 1)^n + 1 + (d - 1)^{n+1} \]
depending on whether the point at infinity is periodic or not. Thus,
\[ \deg(\Psi_{n+1}(x)) \leq (d - 1)^n + 1 + (d - 1)^{n+1} - (d - 1)^n = (d - 1)^{n+1} + 1. \]

Since the number of (projective) periodic points of \( \tilde{\phi}^n \) is \( (d - 1)^n + 1 \), every affine fixed point must be a zero of \( \Psi_n(x) \).

We compute the multipliers as
\[ \tilde{\phi}'(x) = 1 - d \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = 1 - d \frac{f(x)f''(x)}{f'(x)^2} \]
\[ (\tilde{\phi}^n(x))' = \prod_{i=0}^{n-1} \tilde{\phi}'(\tilde{\phi}^i(x)) = \prod_{i=0}^{n-1} \left( 1 + d \frac{\tilde{\phi}'(x)f''(\tilde{\phi}^i(x))}{f'(\tilde{\phi}^i(x))^2} \right). \]

\[ \square \]

2.3. Replace \( d \) with \( r \): Modified Newton-Raphson Iteration. We have considered maps of the form
\[ \tilde{\phi}_F(x) = x - d \frac{f(x)}{f'(x)} \]
where \( d = \deg(F(X, Y)) \). However, we could also consider affine maps of the form
\[ \tilde{\phi}(x) = x - r \frac{f(x)}{f'(x)} \]
for some \( r \neq 0 \) and polynomial \( f(x) \). When used for iterated root finding, such maps are often called the modified Newton-Raphson method. The fixed points are again the zeros of \( f(x) \) and are
all distinct with multipliers $1 - r$. Thus, if $\deg f \neq r$, then the point at infinity must also be a fixed point by (1) with multiplier

$$\sum_{i=1}^{d+1} \frac{1}{1 - \lambda_i} = \frac{\deg f(x)}{r} + \frac{1}{1 - \lambda_\infty} = 1$$

$$\lambda_\infty = \frac{\deg f(x)}{\deg f(x) - r}.$$ 

These maps also form a family in the moduli space of dynamical systems and are determined by their fixed points.

**Theorem 13.** Let $r$ be a non-zero integer. Every rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d - 1$ which has $d - 1$ affine fixed points all with multiplier $(1 - r)$ and fixes $(1, 0)$ with multiplier $\frac{d-1}{d-r-1}$ is a map of the form (2).

**Proof.** The method of proof is identical to the proof of Theorem 6, so is omitted. □

**Remark.** Note that if we choose $r = 1$, then all of the affine fixed points are also critical points ($\dot{\phi}(x) = 0$) as noted in [3, Corollary 1].

2.4. Connection to Maps with Automorphisms. Let $\Gamma \subset \text{PGL}_2$ be a finite group.

**Definition 14.** We say that a homogeneous polynomial $F$ is an invariant of $\Gamma$ if $F \circ \gamma = \chi(\gamma)F$ for all $\gamma \in \Gamma$ and some character $\chi$ of $\Gamma$. The invariant ring of $\Gamma$ denoted $K[X,Y]^\Gamma$ is the set of all invariants.

The following was known as early as [5, footnote p.345].

**Theorem 15.** If $F(X,Y)$ is a homogeneous invariant of a finite group $\Gamma \subset \text{PGL}_2$, then $\Gamma \subset \text{Aut} (\phi_F)$.

**Proof.** Easy application of the chain rule. □

3. Connection to Lattès Maps

Consider an elliptic curve with Weierstrass equation $E : y^2 = g(x)$ for $g(x) = x^3 + ax^2 + bx + c$. The solutions $g(x) = 0$ are the 2-torsion points. If we integrate $g(x)$ we get $G(x) = x^4/4 + a/3x^3 + b/2x^2 + cx + C$ for some constant $C$. If we let $C = -(b^2 - 4ac)/12$, then the solutions $G(x) = 0$ are the 3-torsion points.

Recall that a Lattès map is the induced rational function on the first coordinate of the multiplication map $[m] \in \text{End}(E)$ on the rational points of an elliptic curve $E$; $\phi_{E,m}(x(P)) = x([m])$. For integers $m \geq 3$ we have

$$[m](x,y) = \left(x - \frac{\Psi_{E,m-1}\Psi_{E,m+1}}{\Psi_{E,m}^2}, \frac{\Psi_{E,m+2}\Psi_{E,m-1} - \Psi_{E,m-2}\Psi_{E,m+1}}{4y}\Psi_{E,m}^3\right).$$

In other words, the induced Lattès map is given by

$$\phi_{E,m}(x) = x - \frac{\Psi_{E,m-1}\Psi_{E,m+1}}{\Psi_m^2}.$$ 

Hence the fixed points of the Lattès maps are the $x$-coordinates of the $m - 1$ and $m + 1$ torsion points. For $m = 2$, the fixed points of $\phi_{E,2}$ are the 3 torsion points.
Example 16. Given an elliptic curve of the form $y^2 = g(x) = x^3 + ax^2 + bx + c$. The 2-torsion points satisfy $y^2 = 0$, so are fixed points of the map derived from homogenizing $g(x)$.

$$F(X, Y) = X^3 + aX^2Y + bXY^2 + XY^3$$
$$\phi_F(X, Y) = (aX^2 + 2bXY + 3XY^2, -(2aX + bY^2))$$

The fixed points of the doubling map are the points where $x([2]P) = x(P)$, in other words, the points of order 3. They are the points which satisfy the equation

$$\Psi_{E,3}(x) = 3x^4 + 4ax^3 + 6bx^2 + 12cx + (4ac - b^2) = 2g(x)g''(x) - (g'(x))^2$$

So we have

$$F(X, Y) = 3x^4 + 4aX^3Y + 6bX^2Y^2 + 12cXY^3 + (4ac - b^2)Y^4$$
$$\phi_F(X, Y) = (4aX^3 + 12bX^2Y + 36cXY^2 + 4(4ac - b^2)Y^3,$n
$$- (12X^3 + 12aX^2Y + 12bXY^2 + 12cY^3)).$$

For $m = 2$ we get the following stronger result, connecting generalized $\phi_F$ and Lattès maps.

Theorem 17. Maps of the form

$$\tilde{\phi}(x) = x - 3\frac{f(x)}{f'(x)}$$

are the Lattès maps from multiplication by [2] and $f(x) = \prod (x - x_i)$ where $x_i$ are the x-coordinates of the 3-torsion points.

Proof. From [10, Proposition 6.52] we have the multiplies are all $-2$ except at $\infty$ where it is 4 and the fixed points are the 3 torsion points (plus $\infty$). Now apply Theorem 13. □

3.1. Complex Multiplication and Automorphisms. For an elliptic curve $E$, every automorphism is of the form $(x, y) \mapsto (u^2x, u^3y)$ for some $u \in \mathbb{C}^*$ [S III.10]. In general, the only possibilities

are $u = \pm 1$ and $\text{Aut}(E) \cong \mathbb{Z}/2\mathbb{Z}$. However, in the case of complex multiplication $\text{End}(E) \supset \mathbb{Z}$ and it is possible to contain additional roots of unity, thus having $\text{Aut}(E) \supset \mathbb{Z}/2\mathbb{Z}$. The two cases are $j(E) = 0, 1728$ having $\text{Aut}(E) \cong \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$ respectively [S III.10]. These additional automorphisms induce a linear action $x \mapsto u^2x$ which fixes a polynomial whose roots are torsion points. Thus, the corresponding map $\phi_F$ has a non-trivial automorphism of the form

$$\begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix} \in \text{PGL}_2.$$ 

Thus we have shown the following theorem.

Theorem 18. If $E$ has $\text{Aut}(E) \supset \mathbb{Z}/2\mathbb{Z}$ and the zeros of $F(X, Y)$ are torsion points of $E$, then an induced map $\phi_F$ has a non-trivial automorphism group.

Example 19. Let $E = y^2 = x^3 + ax$, for $a \in \mathbb{Z}$, then $j(E) = 1728$ and $\text{End}(E)$ contains the map $(x, y) \mapsto (-x, iy)$. Thus, the automorphism group of every $\phi_F$ coming from torsion points satisfies

$$\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rangle \subset \text{Aut}(\phi_F).$$
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