SUB-GAUSSIAN BOUND FOR THE ONE-DIMENSIONAL BOUCHAUD TRAP MODEL

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Abstract. We establish an annealed sub-Gaussian lower bound for the transition kernel of the one-dimensional, symmetric Bouchaud trap model using the Ray-Knight description of the local time of a one-dimensional Brownian motion. Using the same ideas we also prove the corresponding result for the FIN diffusion.

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1. Introduction

The Bouchaud trap model (BTM) was introduced by J.-P. Bouchaud in [8] as a toy model for the analysis of the dynamics of some complex disordered systems such as spin glasses. This model is a great simplification of the actual dynamics of such models, nevertheless, it presents some interesting properties which had been observed in the real physical systems. For an account of the physical literature on the BTM we refer to [11].

A basic question is to describe the behavior in space and time of the annealed transition kernel of the one-dimensional, symmetric version of the BTM. In this article we establish a sub-Gaussian bound on the annealed transition kernel of the model which provides a positive answer to the behavior conjectured by E.M. Bertin and J.-P. Bouchaud in [7]. That article contains numerical simulations and non-rigorous arguments which support their claim. A first step on establishing the conjecture was given by J. Černý in [9] where he proved the upper side of the sub-Gaussian bound. In this article we provide the proof for the corresponding lower bound.

The one-dimensional, symmetric BTM is a continuous time random walk \((X_t)_{t \geq 0}\) on \(\mathbb{Z}\) with a random environment. Let \((\tau_z)_{z \in \mathbb{Z}}\) be a family of i.i.d., non-negative random variables defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Those random variables will stand for the environment. For \((\tau_x)_{x \in \mathbb{Z}}\) fixed, we define \(X\) as a homogeneous Markov process with jump rates:

\[
c(x, y) := \begin{cases} 
(2\tau_x)^{-1} & \text{if } |x - y| = 1, \\
0 & \text{otherwise.}
\end{cases}
\]
That is, when $X$ is on site $x$, it waits an exponentially distributed time with mean $\tau_x$ until it jumps to one of its neighbors with equal probabilities. Thus $\tau_x$ should be regarded as the depth of the trap at $x$. The trapping mechanism becomes relevant in the large time behavior of the model only if $E(\tau_0) = \infty$, i.e., when the environment is heavy tailed. Thus, we assume that

$$\lim_{u \to \infty} u^{\alpha} P(\tau_x \geq u) = 1$$

for some $\alpha \in (0, 1)$. Under this assumption the BTM presents a sub-diffusive behavior.

Having defined precisely the one-dimensional, symmetric BTM, we can proceed to state the main result obtained in this article.

**Theorem 1.** There exists positive constants $C_1, c_1, C_2, c_2$ and $\epsilon_1$ such that

$$C_1 \exp \left( -c_1 \left( \frac{x}{t^{1+\alpha}} \right)^{1+\alpha} \right) \leq P(|X_t| \geq x) \leq C_2 \exp \left( -c_2 \left( \frac{x}{t^{1+\alpha}} \right)^{1+\alpha} \right)$$

for all $t \geq 0$ and $x \geq 0$ such that $x/t \leq \epsilon_1$.

As we have previously stated, the lower bound in theorem has been already obtained in [1].

**Observation 2.** We can take $x = at$ with $0 \leq a \leq \epsilon_1$ on theorem. Then we obtain exponential upper and lower bounds on $P(|X_t| \geq at)$. This indicates that a large deviation principle for the BTM might hold.

The proof that we will present for the corresponding lower bound relies heavily on the fact that $(X_t)_{t \geq 0}$ has a clearly identified scaling limit. This scaling limit is called the *Fontes, Isopi, Newman singular diffusion* (FIN). It was discovered by Fontes, Isopi and Newman in [10] and it is a singular diffusion on a random environment. More accurately, this diffusion is a speed measure change of a Brownian motion through a random, purely atomic measure $\rho$, where $\rho$ is the Stieltjes measure associated to an $\alpha$-stable subordinator.

To define the FIN diffusion, first we recall the definition of a speed measure changed Brownian motion. Let $B_t$ be a standard one dimensional Brownian motion defined over $(\Omega, \mathcal{F}, P)$ and starting at zero. Let $l(t, x)$ be a bi-continuous version of his local time. Given any locally finite measure $\mu$ on $\mathbb{R}$, denote

$$\phi_\mu(s) := \int_{\mathbb{R}} l(s, y) \mu(dy),$$

and its right continuous generalized inverse by

$$\psi_\mu(t) := \inf \{ s > 0 : \phi_\mu(s) > t \}.$$  

Then we define the *speed measure change of $B$ with speed measure* $\mu$, $(B[\mu]_t)_{t \geq 0}$ as

$$B[\mu]_t := B_{\psi_\mu(t)}.$$  

(3)
Now, we proceed to define the random measure appearing on the definition of the FIN diffusion. Let \((V_x)_{x \in \mathbb{R}}\) be a two sided, \(\alpha\)-stable subordinator with cadlag paths defined over \((\Omega, \mathcal{F}, \mathbb{P})\) and independent of \(B\). That is, \(V_x\) is the non-decreasing Levy process with cadlag paths and satisfying
\[
V_0 = 0
\]
\[
\mathbb{E}(\exp(-\lambda(V_x+y-V_x))) = \exp(\alpha y \int_0^\infty (e^{-\lambda w} - 1)w^{-1-\alpha}dw)
\]
\[
= \exp(-y\lambda^\alpha \Gamma(1-\alpha))
\]
for all \(x, y \in \mathbb{R}\) and \(\lambda > 0\).

Let \(\rho\) be the Lebesgue-Stieltjes measure associated to \(V\), that is \(\rho([a,b]) = V_b - V_a\). It is a known fact that \((V_t)_{t \geq 0}\) is a pure-jump process. Thus we can write
\[
\rho := \sum v_i \delta_{x_i}.
\]
Moreover, it is also known that \((x_i, v_i)_{i \in \mathbb{N}}\) is an inhomogeneous Poisson point process on \(\mathbb{R} \times \mathbb{R}_+\) with intensity measure \(\alpha v^{-1-\alpha}dx dv\). The diffusion \((Z_t)_{t \geq 0}\) defined as \(Z_t := B[\rho]_t\) is the FIN diffusion.

**Observation 3.** It is easy to see that the measure \(\rho\) has scaling invariance in the sense that \(\lambda^{-1/\alpha} \rho(0, \lambda)\) is distributed as \(\rho(0, 1)\) for all \(\lambda > 0\). The Brownian motion \(B\) is scale invariant in the sense that \((\lambda^{-1/2}B_t)_{t \geq 0}\) is distributed as \((B_t)_{t \geq 0}\). Those two facts imply that \(Z\) is scale invariant in the sense that \((\lambda^{-(\alpha+1)/\alpha}Z_t)_{t \geq 0}\) is distributed as \((Z_t)_{t \geq 0}\) for all \(\lambda > 0\). This fact reflects that the FIN diffusion is subdiffusive.

The techniques used to prove theorem 1 can also be applied to obtain the corresponding result for the FIN singular diffusion, i.e., we will prove

**Theorem 4.** There exists positive constants \(C_3, c_3, C_4\) and \(c_4\) such that
\[
C_3 \exp\left(-c_3\left(\frac{x}{t^{1+\alpha}}\right)^{1+\alpha}\right) \leq \mathbb{P}(|Z_t| \geq x) \leq C_4 \exp\left(-c_4\left(\frac{x}{t^{1+\alpha}}\right)^{1+\alpha}\right)
\]
for all \(t \geq 0\) and \(x \geq 0\).

Again, the upper bound of theorem 4 was obtained by J. Černý in [9].

The main difficulty to obtain the lower bound in theorem 4 is to be able to profit from the independence between the Brownian motion \(B\) and the random measure \(\rho\) which appear in the construction of the FIN diffusion \(Z\). As we will see, the Ray-Knight description of the local time of \(B\) allow us to overcome that difficulty. The proof of theorem 4 follows the same line of reasoning that the proof of theorem 4 but the technical details are slightly more complicated.
We would like to point out some results concerning the BTM on other graphs as state space. In higher dimensions (when the state space is $\mathbb{Z}^d, d \geq 2$), the symmetric BTM has a behavior completely different from the one-dimensional case, as shown by Ben Arous and Černý in [4], and by Ben Arous, Černý and Mountford in [6]. In these papers it is shown that the scaling limit of that model is the fractional kinetic process (FK), which is a time-change of a $d$-dimensional Brownian motion through the inverse of an $\alpha$-stable subordinator. In [1] and [2] Ben Arous, Bovier and Gayrard obtained aging properties of the model on the complete graph. A study of this walk for a wider class of graphs can be found on [5]. For a general account on the mathematical study of the Bouchaud trap model and the FIN diffusion, we refer to [3].

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2. Proofs of theorems 1 and 4

We will first present the proof of theorem 4. Then we adapt the techniques used on that proof to obtain theorem 1.

2.1. Proof of theorem 4. We begin by stating the Ray-Knight theorem. Recall that $B$ is a standard one-dimensional Brownian motion started at the origin and $l(t,x)$ is its local time. For any $b \in \mathbb{R}$ let $\tau_b := \inf\{t \geq 0 : B_t = b\}$. The Ray-Knight theorem ([13] and [12]) states that

**Theorem 5.** (Ray-Knight) For each $a > 0$, the stochastic process $(l(t,\tau-a) : t \geq -a)$ is Markovian. Moreover

$$l(t,\tau-a) : -a \leq t \leq 0$$

is distributed as a squared Bessel process of dimension $d = 2$ started at 0. Further

$$l(t,\tau-a) : t \geq 0$$

is distributed as a squared Bessel process of dimension $d = 0$ started at $l(0,\tau-a)$ and killed at 0.

Thanks to the scaling invariance of $Z$, to prove the lower bound in theorem 4 it is enough to show that there exists positive constants $C_4$ and $c_4$ such that

$$\mathbb{P}(\vert Z_1 \vert \geq x) \geq C_4 \exp(-c_4 x^{1+\alpha}) \text{ for all } x \geq 0. \tag{9}$$

Both $\mathbb{P}(\vert Z_1 \vert \geq x)$ and $C_4 \exp(-c_4 x^{1+\alpha})$ are decreasing in $x$. Hence it will suffice to show that there exists positive constants $C_5$ and $c_5$ such that

$$\mathbb{P}(\vert Z_1 \vert \geq n^{1/(1+\alpha)}) \geq C_5 \exp(-c_5 n) \text{ for all } n \in \mathbb{N}. \tag{10}$$
For any $b \in \mathbb{R}$, we define $H_b := \{t \geq 0 : Z_t = b\}$. Let $n \in \mathbb{N}$ be fixed, we define

$$G := \{Z_{(n+2)/n} \leq -n^{1/(1+\alpha)}\}$$  \hspace{1cm} (11)

$$G_1 := \{H_{n^{1/(1+\alpha)}(n+1)/n} \leq (n+2)/n\}$$  \hspace{1cm} (12)

$$G_2 := \{H_{-2n^{1/(1+\alpha)}} - H_{-n^{1/(1+\alpha)}(n+1)/n} \leq (n+2)/n\}$$  \hspace{1cm} (13)

$$G_3 := \{Z_t \leq -n^{1/(1+\alpha)} \text{ for all } t \in [H_{-n^{1/(1+\alpha)}(n+1)/n}, H_{-2n^{1/(1+\alpha)}})\}.$$  \hspace{1cm} (14)

Note that $G \subset G_1 \cap G_2 \cap G_3$. We will establish a sub-Gaussian lower bound for $\mathbb{P}(G)$. Then it will be easy to deduce display (10) (and hence theorem 4).

We will start by controlling the probability of $G_1$. Notice that, due to the fact that the set of atoms of $\rho$ is $\mathbb{P}$-a.s. dense, the event $\{H_{-n^{1/(1+\alpha)}(n+1)/n} \leq 1\}$ is equivalent to $\{\min\{Z_t : t \in [0,1]\} \leq -n^{1/(1+\alpha)}(n+1)/n\}$. Let $\theta_1 := \inf\{t > 0 : B_t = -n^{1/(1+\alpha)}(n+1)/n\}$. We can express $H_{-n^{1/(1+\alpha)}(n+1)/n}$ as $\int_{-n^{1/(1+\alpha)}(n+1)/n}^{\infty} l(\theta_1, u)\rho(du)$. Let $H^- := \int_{-n^{1/(1+\alpha)}(n+1)/n}^{0} l(\theta_1, u)\rho(du)$ and $H^+ := \int_{0}^{\infty} l(\theta_1, u)\rho(du)$. Thus

$$H^- := \sum_{i=1}^{n+1} \int_{-n^{1/(1+\alpha)}(i-1)/n}^{-n^{1/(1+\alpha)}(i-1)/n} l(\theta_1, u)\rho(du).$$  \hspace{1cm} (15)

Clearly

$$\bigcup_{i=1}^{n+1} \left\{ \int_{-n^{1/(1+\alpha)}(i-1)/n}^{-n^{1/(1+\alpha)}(i-1)/n} l(\theta_1, u)\rho(du) \leq 1/n \right\} \subset \{H^- \leq (n+1)/n\}.$$  \hspace{1cm} (16)

As the intervals $[-n^{1/(1+\alpha)}i/n, -n^{1/(1+\alpha)}(i-1)/n)$, $i = 1, \ldots, n+1$ are disjoint and the process $V$ has independent increments, we have that the random variables $\rho[-n^{1/(1+\alpha)}i/n, -n^{1/(1+\alpha)}(i-1)/n)$, $i = 1, \ldots, n+1$ are independent between them. Also, the Ray-Knight theorem states that $(l(\theta_1, u) : u \geq -n^{1/(1+\alpha)}(n+1)/n)$ is a process with independent increments. We will profit of those independencies by finding a family of $n+1$ independent events with the same probability, whose intersection is contained in $\{H^- \leq (n+1)/n\}$. Then, using the scaling invariance of the measure $\rho$ and the scaling invariance of squared Bessel processes, we will show that all those events have the same probability for all $n$. Then we would have showed that there exists positive constants $C_6$ and $c_6$ such that

$$\mathbb{P}(H^- \leq (n+1)/n) \geq C_6 \exp^{-c_6(n+1)}.$$  \hspace{1cm} (17)

An similar argument can be used to control the probability of $\{H^+ \leq 1/n\}$. Hence we will obtain

$$\mathbb{P}(H_{-n^{1/(1+\alpha)}(n+1)/n} \leq (n+2)/n) \geq C_7 \exp^{-c_7(n+2)}$$  \hspace{1cm} (18)

where $C_7$ is a positive constant. As $(n+2)/n \leq 3$, we can use use scaling invariance of $Z$ to obtain that there exists positive constants $C_8$ and $c_8$ such that

$$\mathbb{P}(H_{-n^{1/(1+\alpha)}(n+1)/n} \leq 1) \geq C_8 \exp^{-c_8 n}.$$  \hspace{1cm} (19)
To obtain theorem \[\text{(19)}\] is not immediate, because the event \(\{H_{n^{1/(1+\alpha)/(n+1)/n}} \leq (n+2)/n\}\) is not independent of \(\rho\). To overcome that obstacle we can make repeated use of the Ray-Knight theorem using the stopping times \(\theta_1\) and \(\theta_2 := \inf\{t > 0 : B_t = -2n^{1/(1+\alpha)}\}\). That will allow us to control simultaneously the probability of \(G_1, G_1\) and \(G_3\).

Next, we give some definitions needed for the proof of theorem \[\text{(4)}\]. Let \((W_t)_{t \geq 0}\) be a Brownian motion defined over \((\Omega, \mathcal{F}, \mathbb{P})\), independent of \(\rho\) and started at 0. Let \((\bar{Y}_t : t \in [-n^{1/(1+\alpha)}(n+1)/n, \infty))\) be the Bessel process with \(d = 2\) given by

\[
\bar{Y}_t := \int_{-n^{1/(1+\alpha)}(n+1)/n}^{t} \frac{1}{2Y_s} ds + W_t - W_{-n^{1/(1+\alpha)}(n+1)/n}.
\]  

(20)

Let \(a > 0\). We also define \((\bar{Y}^i_t : t \in [-n^{1/(1+\alpha)}(n-i)/n, \infty)), i = -1, 0, .., n-1\) as the Bessel processes with \(d = 2\) given by

\[
\bar{Y}^i_t := (an^{-\alpha/(1+\alpha)})^{1/2} + \int_{-n^{1/(1+\alpha)}(n-i)/n}^{t} \frac{1}{2Y_s} ds + W_t - W_{-n^{1/(1+\alpha)}(n-i)/n}.
\]  

(21)

Note that we are using the same Brownian motion for the construction of the \(\bar{Y}^i, i = -1, 0, .., n-1\) and \(\bar{Y}\). Let \((\bar{X}_t)_{t \geq 0}\) be a Bessel process with \(d = 0\) given by

\[
\bar{X}_t := \bar{Y}_0 - \int_{0}^{t} \frac{1}{2\bar{X}_s} ds + W_t.
\]  

(22)

We also define

\[
\bar{X}^0_t := (an^{-\alpha/(1+\alpha)})^{1/2} - \int_{0}^{t} \frac{1}{2\bar{X}^0_s} ds + W_t.
\]  

(23)

We aim to use the Ray-Knight theorem, which deals with squared Bessel processes. Thus we define \(Y_t := (\bar{Y}_t)^2; Y^i_t := (\bar{Y}^i_t)^2; X_t := (\bar{X}_t)^2\) and \(X^0_t := (\bar{X}^0_t)^2\).

We have that

\[
G_1 = \left\{ \int_{\mathbb{R}} l(\theta_1, u) \rho(du) \leq (n+2)/n \right\}.
\]  

(24)

Thus, in view of the Ray-Knight theorem

\[
\mathbb{P}(G_1) = \mathbb{P} \left( \int_{-n^{1/(1+\alpha)}(n+1)/n}^{0} Y_t \rho(dt) + \int_{0}^{\infty} X_t \rho(dt) \leq (n+2)/n \right).
\]

Let

\[
A := \left\{ \int_{-n^{1/(1+\alpha)}(n+1)/n}^{0} Y_t \rho(dt) + \int_{0}^{\infty} X_t \rho(dt) \leq (n+2)/n \right\}.
\]  

(25)

For all \(i = -1, .., n-1\), let

\[
A_i := \left\{ \int_{-n^{1/(1+\alpha)}(n-i-1)/n}^{-n^{1/(1+\alpha)}(n-i)/n} Y_t \rho(dt) \leq 1/n \right\}
\]  

(26)

and

\[
B := \left\{ \int_{0}^{\infty} X_t \rho(dt) \leq 1/n \right\}.
\]  

(27)
Thus, it is clear that
\[ \left( \bigcap_{i=1}^{n-1} A_i \right) \cap B \subset A. \]  
(28)

We would like to have independence of the events in the L.H.S. of (28) to compute a lower bound for \( P(A) \). But they are not independent. Thus, for \( i = -1, \ldots, n-1 \) we define
\[ \tilde{A}_i := \left\{ \int_{-n^{1/(1+\alpha)}(n-i)/n}^{0} \gamma_i^i \rho(dt) \leq 1/n; \gamma_i^i \leq an^{-\alpha/(1+\alpha)} \right\} \]  
(29)

and
\[ \tilde{B} := \left\{ \int_0^\infty \gamma_0^0 \rho(dt) \leq 1/n \right\}. \]  
(30)

We can use independence on those events. The fact that all the Bessel processes appearing are defined using the same Brownian motion \( W \) implies that, conditioned on \( \{ \gamma_i^i \leq an^{-\alpha/(1+\alpha)} \}_{i=0}^{n-1} \), we have that \( \gamma_i^i \leq \gamma_i^i \) for all \( t \in [-n^{1/(1+\alpha)}(n-i)/n, \infty) \). Also, conditioned on \( \{ \gamma_0^0 \leq (an^{-\alpha/(1+\alpha)})^{1/2} \} \), we have that \( \gamma_i^i \leq \gamma_i^i \) for all \( t \geq 0 \). Thus, it is clear that
\[ \left( \bigcap_{i=1}^{n-1} \tilde{A}_i \right) \cap \tilde{B} \subset A. \]  
(31)

For any \( b \in \mathbb{R}_+ \), \( P_b \) will denote probability conditioned on \( \gamma_0 = b \). When there is no risk on confusion, \( P_b \) will also denote probability conditioned on \( \gamma_0 = b \). Note that, for all \( 1 = -1, \ldots, n-1 \) we have that
\[ P(\tilde{A}_i) = P_{an^{-\alpha/(1+\alpha)}} \left( \int_0^{-n^{-\alpha/(1+\alpha)}} \gamma_i^i \rho(dt) \leq 1/n; \gamma_i^i \leq an^{-\alpha/(1+\alpha)} \right). \]  
(32)

Now consider
\[ P_a \left( \int_0^1 \gamma_i^i \rho(dt) \leq 1; \gamma_i^i \leq a \right). \]  
(33)

Let us perform a change of variables inside the integral, we obtain that (33) equals
\[ P_a \left( \int_0^{-n^{-\alpha/(1+\alpha)}} \gamma_0^0 \rho(n^{-\alpha/(1+\alpha)} ds) \leq 1; \gamma_1^1 \leq a \right). \]  
(34)

Using the scale invariance of the measure \( \rho \) we obtain
\[ \left( \int_0^{-n^{-\alpha/(1+\alpha)}} \gamma_0^0 \rho(ds) \leq \left( n^{-\alpha/(1+\alpha)} \right)^{1/\alpha} \right); \gamma_1^1 \leq a \right). \]  
(35)

Thus, the last expression equals
\[ P_a \left( \int_0^{-n^{-\alpha/(1+\alpha)}} n^{-\alpha/(1+\alpha)} \gamma_0^0 \rho(ds) \leq \frac{1}{n}; \gamma_1^1 \leq a \right). \]  
(36)

The scale invariance of the squared Bessel processes implies that, under \( P_a \), we have that \( \gamma_1^1 := n^{-\alpha/(1+\alpha)} \gamma_0^0 \) is distributed as \( \gamma_1^1 \) but starting from \( an^{-\alpha/(1+\alpha)} \). Also \( \{ \gamma_1^1 \geq a \} \) is equivalent
to \( \{ \tilde{Y}_{n-a/(1+\alpha)} \geq an^{-\alpha/(1+\alpha)} \} \). Thus (33) equals
\[
P_{an^{-\alpha/(1+\alpha)}} \left( \int_0^{n-a/(1+\alpha)} Y_s \rho(ds) \leq 1/n; Y_{n-\alpha/(1+\alpha)} \leq an^{-\alpha/(1+\alpha)} \right).
\] (37)
Hence
\[
P(\tilde{A}_1) = P_a \left( \int_0^1 Y_t \rho(dt) \leq 1; Y_1 \leq a \right).
\] (38)
To control the time spent in the negative axis we perform a similar argument to show that
\[
P_{an^{-\alpha/(1+\alpha)}} \left( \int_0^\infty X_s \rho(ds) \leq 1/n \right) = P_a \left( \int_0^1 Y_s \rho(ds) \leq 1 \right).
\] (39)
Which in turn equals \( P(\tilde{B}) \). Thus we have showed that

**Lemma 6.**

\[
P(G_1) \geq P_a \left( \int_0^\infty X_s \rho(ds) \leq 1 \right) P_a \left( \int_0^1 Y_t \rho(dt) \leq 1; Y_1 \leq a \right)^{n+1}.
\]

Our lemma states that the probability of \( G_1 \) is big enough for our purposes. We aim to deduce a upper bound for \( P(\cap_{t=1}^3 G_t) \). Recall that \( \theta_1 = \inf \{ t > 0 : B_t = -n^{1/(1+\alpha)}(n+1)/n \} \) and \( \theta_2 = \inf \{ t > 0 : B_t = -2n^{1/(1+\alpha)} \} \). The strategy will be to make repeated use of the Ray-Knight using the stopping times \( \theta_1 \) and \( \theta_2 \). That will allow us to control the probability of \( G_1, G_2 \) and \( G_3 \) simultaneously.

By the strong Markov property, \( B_{\theta_1+t} + n^{1/(1+\alpha)}(n+1)/n \) is distributed as a Brownian motion starting from the origin and has local time
\[
\tilde{l}(t, u) := l(\theta_1 + t, u + n^{1/(1+\alpha)}(n+1)/n) - l(\theta_1, u + n^{1/(1+\alpha)}(n+1)/n).
\] (40)
Thus, \( \tilde{l}(t, u) \) has the distribution of \( l(t, u) \). Let us apply the Ray-Knight theorem to \( \tilde{l} \) using the stopping time \( \theta_2 \). Then the time that \( Z \) spends between its first visit to \( -n^{1/(1+\alpha)}(n+1)/n \) and its first visit to \( -2n^{1/(1+\alpha)} \) is represented as the integral of a squared Bessel process with respect to \( \rho \). Let \( (W_r^*)_{r \geq 0} \) be a Brownian motion defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \) with \( W_{-n^{1/(1+\alpha)}(n+1)/n}^* = 0 \) and independent of \( W \). Let \( (\tilde{Y}_r^* : t \in [-2n^{1/(1+\alpha)}, \infty)) \) be the Bessel process with \( d = 2 \) given by
\[
\tilde{Y}_t^* = \int_{-2n^{1/(1+\alpha)}}^t \frac{1}{2\tilde{Y}_s^*} ds + W_t^* - W_{-2n^{1/(1+\alpha)}}^*.
\] (41)
We also define \( (\tilde{Y}_t^{(i*,)} : t \in [-2n^{1/(1+\alpha)}(2n-i)/n, \infty)), i = 0, \ldots, n-2 \) as the Bessel process with \( d = 2 \) given by
\[
\tilde{Y}_t^{(i*,)} = (an^{-\alpha/(1+\alpha)})^{1/2} + \int_{-2n^{1/(1+\alpha)}(2n-i)/n}^t \frac{1}{2\tilde{Y}_s^{(i*,)}} ds + W_t^* - W_{-(2n-i)n^{1/(1+\alpha)}/n}^*.
\] (42)
Let \( (\tilde{X}_t^* : t \in [-n^{1/(1+\alpha)}(n+1)/n, \infty)) \) be a Bessel process with \( d = 0 \) given by
\[
\tilde{X}_t^* = \tilde{Y}_{-n^{1/(1+\alpha)}(n+1)/n} - \int_{-n^{1/(1+\alpha)}(n+1)/n}^t \frac{1}{2\tilde{X}_s^*} ds + W_t^*.
\] (43)
We also define
\[\bar{X}^{(0,*)}_t = (an^{-\alpha/(1+\alpha)})^{1/2} - \int_{\bar{X}^{(0,*)}_t}^t \frac{1}{2} \tilde{\mathcal{X}}^{(*)}_s ds + W_t^* .\] (44)

We define \(\mathcal{Y}^{(*)}_t := (\tilde{\mathcal{Z}}^{(*)}_t)^2; \mathcal{Y}^{(i,*)}_t := (\tilde{\mathcal{Z}}^{(i,*)}_t)^2; \mathcal{X}^{(0,*)}_t := (\tilde{\mathcal{X}}^{(*)}_t)^2\) and \(\mathcal{X}^{(i,*)}_t := (\tilde{\mathcal{X}}^{(i,*)}_t)^2\).

Note that
\[G_2 = \left\{ \int_{\mathbb{R}} l(\theta_2, u) - l(\theta_1, u) \rho(du) \geq (n + 2)/n \right\}.\] (45)

Thus, in view of the Ray-Knight theorem
\[\mathbb{P}(G_2) = \mathbb{P} \left( \int_{-2n^{1/(1+\alpha)}}^{n^{1/(1+\alpha)}} \mathcal{Y}^{(*)}_t \rho(dt) + \int_{-n^{1/(1+\alpha)}}^{\infty} \mathcal{X}^{(*)}_t \rho(dt) \geq (n + 2)/n \right).\] (46)

Let
\[C := \left\{ \int_{-2n^{1/(1+\alpha)}}^{n^{1/(1+\alpha)}} \mathcal{Y}^{(*)}_t \rho(dt) + \int_{-n^{1/(1+\alpha)}}^{\infty} \mathcal{X}^{(*)}_t \rho(dt) \geq (n + 2)/n \right\}.\] (47)

Let
\[C_0 := \left\{ \int_{-2n^{1/(1+\alpha)}}^{n^{1/(1+\alpha)}} \mathcal{Y}^{(0,*)}_t \rho(dt) \geq 4/n \right\}.\] (48)

For \(i = 1, \ldots, n - 2\), let
\[C_i := \left\{ \int_{-n^{1/(1+\alpha)}}^{n^{1/(1+\alpha)}} \mathcal{Y}^{(i,*)}_t \rho(dt) \geq 1/n \right\}.\] (49)

and \(D := \{ \mathcal{X}^{(0,*)}_{-n^{1/(1+\alpha)}} = 0 \}\). Thus, it is clear that
\[\left( \bigcap_{i=0}^{n-1} C_i \right) \subset C .\] (50)

We can apply here the same argument leading to lemma [7] to obtain
\[\mathbb{P}(C_0) = \mathbb{P}_a \left( \int_0^1 \mathcal{Y}_t \rho(dt) \geq 4; \mathcal{Y}_1 \geq a \right).\] (51)

Similarly, for \(i = 1, \ldots, n - 2\), we have that
\[\mathbb{P}(C_i) = \mathbb{P}_a \left( \int_0^1 \mathcal{Y}_t \rho(dt) \geq 1; \mathcal{Y}_1 \geq a \right).\] (52)

Thus we have proved
\[\mathbb{P}(G_2) \geq \mathbb{P}_a \left( \int_0^1 \mathcal{Y}_t \rho(dt) \geq 4; \mathcal{Y}_1 \geq a \right) \mathbb{P}_a \left( \int_0^1 \mathcal{Y}_t \rho(dt) \geq 1; \mathcal{Y}_1 \geq a \right)^{n-2} .\] (53)

The argument leading to lemma [7] can be applied once more to get
\[\mathbb{P}(D) = \mathbb{P}_a(\bar{X}_1 = 0).\] (54)

The event \(G_3\) is equivalent to \(\{ B_t \leq n^{1/(1+\alpha)} \} \) for all \(t \in [\theta_1, \theta_2] \). Which, in turn, is equivalent to \(\{ l(-1/n, \theta_2) = 0 \} \). This, in turn is equivalent to \(D\). Moreover \(G_1\) is equivalent to \(A\) and \(G_2\) is
equivalent to \( C \). Thus \( G \) is equivalent to \( A \cap C \cap D \). But
\[
\bigcap_{i=1}^{n-1} \tilde{A}_i \cap \tilde{B} \cap \bigcap_{i=0}^{n-2} C_i \cap D \subset A \cap C \cap D.
\] (55)

Furthermore, the L.H.S. of the inclusion (55) is a intersection of independent events. The independence of the events can easily be seen because they are defined in terms of disjoint intervals of \( \rho \) and independent processes. Note that the events \( D \) and \( \tilde{A}_{-1} \) are defined in terms of events that occur on the same interval \([-n^{1/(1+\alpha)}(n+1), -n^{1/(1+\alpha)})\), but the event \( D \) does not depend upon \( \rho \) so that independence holds. Thus we have deduced that
\[
P(Z_{(n+2)/n} \leq -n^{1/(1+\alpha)}) \geq \mathbb{P}(\tilde{A}_{-1})^{n+1}\mathbb{P}(\tilde{B})\mathbb{P}(C_1)^{n-2}\mathbb{P}(D)\mathbb{P}(C_0).
\] (56)

To check that \( \mathbb{P}(\tilde{A}_{-1})^{n+1}\mathbb{P}(\tilde{B})\mathbb{P}(C_1)^{n-2}\mathbb{P}(D)\mathbb{P}(C_0) > 0 \) we recall (54), (52), (51), (38) and (39). Thus the fact that those probabilities are non zero can be easily checked easily using the facts that, for each \( \epsilon > 0 \), \( \mathbb{P}(\rho(0, 1) \leq \epsilon) > 0 \), for each \( M > 0 \), \( \mathbb{P}(\rho(0, 1) \geq M) > 0 \) and that the Bessel processes can be bounded below and above with positive probability. We need also to use the fact that a 0-dimensional Bessel process hits the origin before time 1 with positive probability. Thus we find that there exists positive constants \( C_9 \) and \( c_9 \) such that
\[
P(Z_{(n+2)/n} \leq -n^{1/(1+\alpha)}) \geq C_9 \exp(-c_9n).
\] (57)

The scaling invariance of \( Z \) can be used to deduce theorem 4.

2.2. **Proof of theorem [1]**. The strategy to prove theorem [1] will be to mimic the arguments leading to theorem [4] using the fact that the FIN diffusion is the scaling limit of the one-dimensional, symmetric version of the BTM. The main tool used in [10] to prove that the FIN diffusion is the scaling limit of the BTM is a coupling between different time scales of the BTM. We will make use of this coupling for the proof of theorem [1] so we proceed to recall it.

For each \( \epsilon > 0 \), we define a family of random variables \( (\tau^\epsilon_z)_{z \in \mathbb{Z}} \) as follows. Let \( G : [0, \infty) \to [0, \infty) \) be the function defined by the relation
\[
P(V_1 > G(u)) = \mathbb{P}(\tau_0 > u).
\] (58)

The function \( G \) is well defined since \( V_1 \) has a continuous distribution function. Moreover, \( G \) is non-decreasing and right continuous. Thus \( G \) has a right continuous generalized inverse \( G^{-1}(s) := \inf\{t : G_t \geq s\} \). Now, for all \( \epsilon > 0 \) and \( z \in \mathbb{Z} \), we define the random variables \( \tau^\epsilon_z \) as
\[
\tau^\epsilon_z := G^{-1}(\epsilon^{-1/\alpha} \rho(\epsilon z, \epsilon(z+1)))
\] (59)

For all \( \epsilon > 0 \), we have that \( (\tau^\epsilon_z)_{z \in \mathbb{Z}} \) is an i.i.d. family of random variables distributed according to \( \tau_0 \). For a proof of that fact we refer to [10].
We define a coupled family of random measures as
\[
\rho^\epsilon := \sum_{z \in \mathbb{Z}} \epsilon^{1/\alpha} t^{1/\alpha} \delta_z.
\]
for all $\epsilon > 0$. Using that measures we can express the rescalings of $X$ as speed measure changed Brownian motions. That is

**Lemma 7.** For all $\epsilon > 0$ the process $(tX_{t(1-\epsilon)/\alpha})_{t \geq 0}$ has the same distribution that $(B[\rho^\epsilon]_t)_{t \geq 0}$.

Moreover, we have that
\[
\rho^\epsilon \overset{\text{w}}{\longrightarrow} \rho \text{ \ P-a.s. as } \epsilon \to 0
\]
where $\overset{\text{w}}{\longrightarrow}$ denotes vague convergence of measures.

For the proof of this statement we refer to [10]. Lemma 7 implies in particular that $(X_t)_{t \geq 0}$ is distributed as $(B[\rho^1]_t)_{t \geq 0}$. Thus
\[
\mathbb{P}(|X_t| \geq x) = \mathbb{P}(|B[\rho^1]_t| \geq x)
\]
for all $x \geq 0$ and $t \geq 0$.

We will proceed as in the proof of lemma 7. Let $t \geq 0$ be fixed. It will suffice to establish the lower bound of theorem 7 for $x = m^{1/(1+\alpha)}t^{\alpha/(1+\alpha)}$, where $m \in \mathbb{N}$ (and $x/t \leq \epsilon_1$). Then, using the fact that for fixed $t \geq 0$, $\mathbb{P}(|X_t| \geq x)$ is decreasing on $x$ we can extend our result to all $x \geq 0$ (with $x/t \leq \epsilon_1$).

Let $H^0_t := \inf\{t \geq 0 : B[\rho^1]_t = b\}$. We define a collection of events analogous to $G, G_1, G_2$ and $G_3$ defined on the displays (11), (12), (13) and (14). Let
\[
G^0 := \{B[\rho^1]_{t(m+2)/m} \leq -x\}
\]
(63)
\[
G^0_1 := \{H^1_{x(m+1)/m} \leq t(m+2)/m\}
\]
(64)
\[
G^0_2 := \{H^1_{2x} - H^1_{x(m+1)/m} \geq t(m+2)/m\}
\]
(65)
\[
G^0_3 := \{B[\rho^1]_t \leq -x \text{ for all } t \in [H^1_{x(m+1)/m}, H^1_{2x}]\}
\]
(66)
Note that $G^0 \subset G^0_1 \cap G^0_2 \cap G^0_3$.

First we will control the probability of $G^0_1$ in the same way that we controlled the probability of $G_1$ in the proof of lemma 7. Recall that $W_t$ is a Brownian motion defined over $(\Omega, \mathcal{F}, \mathbb{P})$, independent of $\rho$ and started at 0. Let $(\hat{Y}^0_t : t \in [-x(m+1)/m, \infty))$ be the Bessel process with $d = 2$ given by
\[
\hat{Y}^0_t := \int_{-x(m+1)/m}^t \frac{1}{2\hat{Y}^0_s} ds + W_t - W_{-x(m+1)/m}.
\]
(67)
Let $a > 0$. We also define $(\hat{Y}^{(0,i)}_t : t \in [-x(m-i)/m, \infty)), i = -1, 0, ..., m-1$ as the Bessel processes with $d = 2$ given by
\[
\hat{Y}^{(0,i)}_t := (ax/m)^{1/2} + \int_{-x(m-i)/m}^t \frac{1}{2\hat{Y}^{(0,i)}_s} ds + W_t - W_{-x(m-i)/m}.
\]
(68)
But (\(X_0^t\))_{t \geq 0} be a Bessel process with \(d = 0\) given by
\[
X_0^t := \tilde{X}_0^t - \int_0^t \frac{1}{2X_s^0} ds + W_t. \tag{69}
\]
We also define
\[
\tilde{X}_t^{(0,0)} := (ax/m)^{1/2} - \int_0^t \frac{1}{2\tilde{X}_s^{(0,0)}} ds + W_t. \tag{70}
\]
We aim to use the Ray-Knight theorem, which deals with squared Bessel processes. Thus we define
\[
Y_t^0 := (\tilde{Y}_t^0)^2; Y_t^{0,i} := (\tilde{Y}_t^{0,i})^2; \tilde{X}_t^0 := (\tilde{X}_t^0)^2 \text{ and } \tilde{X}_t^{(0,0)} := (\tilde{X}_t^{(0,0)})^2.
\]
Using the squared Bessel processes constructed above, we define a family of events analogous to the events \(A, \tilde{A}_i\) and \(\tilde{B}\) appearing on displays \((25),(29)\) and \((30)\). Let
\[
A^0 := \left\{ \int_{-x(m+1)/m}^0 Y_t^0 \rho^1(dt) + \int_0^\infty \tilde{X}_t^0 \rho^1(dt) \leq t(m + 2)/m \right\}. \tag{71}
\]
For \(i = -1, \ldots, m - 1\) let
\[
\tilde{A}_i^0 := \left\{ \int_{-x(m-i-1)/m}^{-x(m-i)/m} Y_t^{0,i} \rho^1(dt) \leq t/m; Y_{-x(m-i-1)/m}^{0,i} \leq ax/m \right\}. \tag{72}
\]
Also let
\[
\tilde{B}^0 := \left\{ \int_0^\infty \tilde{X}_t^{(0,0)} \rho^1(dt) \leq t/m \right\}. \tag{73}
\]
Thus, it is clear that
\[
\left( \bigcap_{i=-1}^{m-1} \tilde{A}_i^0 \right) \cap \tilde{B}^0 \subseteq A^0. \tag{74}
\]
Using the Ray-Knight theorem we see that \(G^0_t\) is equivalent to \(A^0\).

From now on, \(P_b\) will denote probability conditioned on \(Y_0^0 = b\), and when there is no risk of confusion, it will also denote probability conditioned on \(X_0^0 = b\). Let \(d = x/m\). As in the proof of theorem \(E\) we have that, for \(i = -1, \ldots, m - 1\)
\[
P(\tilde{A}_i^0) = P_{ad} \left( \int_0^d \mathcal{Y}_s \rho^1(ds) \leq \frac{t}{m}; \mathcal{Y}_d \leq ad \right) \tag{75}
\]
We define
\[
Sc(r)(\mu) := r^{1/\mu} \mu(r^{-1}A) \tag{76}
\]
Performing the change of variables \(u = sd^{-1}\) inside the integral, we obtain
\[
P(\tilde{A}_i^0) = P_{ad} \left( \int_0^{1} \mathcal{Y}_u d^{1/\mu} Sc(d^{-1}) \rho^1(du) \leq \frac{t}{m}; \mathcal{Y}_d \leq ad \right) \tag{77}
\]
Using the scaling invariance of the squared Bessel process \(\mathcal{Y}\) we obtain
\[
P(\tilde{A}_i^0) = P_a \left( \int_0^{1} \mathcal{Y}_u Sc(d^{-1}) (\rho^1)(du) \leq 1; \mathcal{Y}_1 \leq a \right) \tag{78}
\]
But \(Sc(d^{-1})(\rho^1)\) is distributed as \(\rho^{d^{-1}}\). Thus, we can replace to obtain
\[
P(\tilde{A}_i^0) = P_a \left( \int_0^{1} \mathcal{Y}_u \rho^{d^{-1}}(du) \leq 1; \mathcal{Y}_1 \leq a \right) \tag{79}
\]
Similar arguments can be applied to get
\[ P(\tilde{B}^0) = P_a \left( \int_0^\infty X_u \rho^{-1}(du) \leq 1 \right) \] (80)

On the other hand, using display (61) in lemma 7 we can prove that
\[ \int_1^0 Y_t \rho^\epsilon(dt) \rightarrow 0 \quad \epsilon \rightarrow 0 \quad \text{P-almost surely.} \] (81)

and
\[ \int_0^\infty X_t \rho^\epsilon(dt) \rightarrow 0 \quad \text{P-almost surely.} \] (82)

Thus, there exists \( \epsilon_0 \) small enough such that, for \( m \in \mathbb{N} \) and \( t \geq 0 \) such that \( (m/t)^\alpha \leq \epsilon_0 \) we have that
\[ P(\tilde{A}^0_i) \geq \frac{1}{2} P_a \left( \int_0^1 Y_u \rho(du) \leq 1; Y_1 \leq a \right) \] (83)

and
\[ P(\tilde{B}^0) \geq \frac{1}{2} P_a \left( \int_0^\infty X_u \rho(du) \leq 1 \right) \] (84)

Hence, we have showed that

**Lemma 8.** There exists \( \epsilon_0 \) small enough such that, for \( m \in \mathbb{N}, t \geq 0 \) and \( x = m^{1/(1+\alpha)} t^{\alpha/(1+\alpha)} \) such that \( (x/t)^\alpha \leq \epsilon_0 \) we have that
\[ P(G_0^1) \geq \frac{1}{2} P_a \left( \int_0^\infty X_u \rho(ds) \leq 1 \right) \left( \frac{1}{2} P_a \left( \int_0^1 Y_t \rho(dt) \leq 1; Y_1 \leq a \right) \right)^{m+1}. \]

This lemma states that the probability that \( X \) passes through \(-x(m+1)/m\) before time \( t(m+2)/m \) is big enough for our purposes. Now we aim to deduce a lower bound for \( P(|X_t| \geq x) \). But we can use displays (81) and (82) to adapt the proof for the FIN diffusion to the case of the BTM, in the same way we adapted the proof of lemma 7 to obtain lemma 8.

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