CURVATURE IDENTITIES ON ALMOST HERMITIAN MANIFOLDS AND APPLICATIONS

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Abstract. In this paper, we systematically compute the Bianchi identities for the canonical connection on an almost Hermitian manifold. Moreover, we also compute the curvature tensor of the Levi-Civita connection on almost Hermitian manifolds in terms of curvature and torsion of the canonical connection. As applications of the curvature identities, we obtain some results about the integrability of quasi Kähler manifolds and some properties of nearly Kähler manifolds.

1. Introduction

Almost Hermitian manifolds are almost complex manifolds equipped with a Riemannian metric compatible with the almost complex structure. They form the largest class of generalized Kähler manifolds. All the other generalized Kähler manifolds such as almost Kähler, quasi Kähler, nearly Kähler and semi Kähler manifolds are all special almost Hermitian manifolds. Among all these classes of generalized Kähler manifolds, almost Kähler and nearly Kähler manifolds attracted the most attentions, because the former one is related to symplectic geometry and the latter on is nowadays related to theoretical physics.

Integrability is one of the themes for researches on almost Hermitian manifolds. In [11], Goldberg showed that for an almost Kähler manifold, if the curvature tensor commutes with the almost complex structure then it must be Kähler. In [12], Gray introduced some curvature identities on almost Hermitian manifolds and considered their relation with integrability of almost Hermitian manifolds. In [11], Goldberg also proposed a conjecture that any almost Kähler Einstein manifold must be Kähler. The conjecture was solved with a further assumption that the scalar curvature is nonnegative by Sekigawa in [22]. However...
the full conjecture is still open. For other topics and progresses about the integrability of almost Kähler manifolds, one can consult the survey [1] written by Apostolov and Drăghici and references there in.

One should note that the above mentioned results are all with respect to the Levi-Civita connection. However, on a general almost Hermitian manifold, Levi-Civita connection does not compatible with the almost complex structure. In fact, there is a connection more related to the almost complex structure generalized the Chern connection on Hermitian manifolds (see [4]). The connection is called the canonical connection for almost Hermitian manifolds first introduced by Ehresmann and Lebermann[8]. The motivation for studying almost Hermitian, more restrictive speaking, almost Kähler geometry comes from Donaldson [6, 7] program to find canonical metric on symplectic manifolds which naturally generalized Calabi’s program for Kähler manifolds. In [26], Tossati, Weinkove and Yau used the canonical connection to solve the Calabi-Yau equation on symplectic manifolds with certain positivity on a combination of curvature tensor and torsion of the canonical connection. Their work is related to Donaldson’s program. Donaldson’s program or solving the Calabi-Yau equation on symplectic manifolds is another theme in the research of almost Kähler geometry. The canonical connection is also useful for the study of the structure of strictly nearly Kähler manifolds by Nagy([20, 21]).

Nevertheless, almost Hermitian geometry may have its own interests. in [25], Tossati obtained a Laplacian comparison, a Schwartz lemma for almost Hermitian manifolds which is a generalization of Yau’s Schwartz lemma for Hermitian manifolds(See [29]). Moreover, with the help of the generalized Laplacian comparison and Schwartz lemma, Tossati extended a result by Seshadri-Zheng [23] on the nonexistence of complete Hermitian metrics with holomorphic bisectional curvature bounded between two negative constants and bounded torsion on a product of complex manifolds to a product of almost complex manifolds with almost Hermitian metrics. In [9], Fan, Tam and the author further weaker the curvature assumption of the result of Tossati and obtain the same conclusion which is also a generalization of a result of Tam-Yu [24].

In this paper, we first systematically compute the first and second Bianchi identities on almost Hermitian manifolds. Some of the Bianchi identities listed in this paper are hidden in different forms in [26, 18, 19]. Then, with the help of the Bianchi identities and the local pseudo holomorphic normal frame introduced in [30], we compute the curvature of the Levi-Civita connection on an almost Hermitian manifold in terms of curvature and torsion of the canonical connection. In [5], the authors made a converse computation for quasi Kähler manifolds. Indeed, they
compute the curvature of the canonical connection in terms of curvature of the Levi-Civita connection for quasi Kähler manifolds. Hence, the curvature identities for quasi Kähler manifolds we obtained in section 3 are also hidden in a converse form in [5].

For example, for the holomorphic sectional curvature of the Levi-Civita connection and the canonical connection, we obtain that

\[ R^L_{\bar{ii}ii} = R_{\bar{ii}ii} + \tau_{\bar{i}i}^j \tau_{ij}^i - \frac{1}{2} \tau_{\bar{i}i}^j \tau_{ij}^i \]

where \( R^L \) and \( R \) means the curvature tensor of the Levi-Civita connection and the canonical connection with respectively and \( \tau \) means the torsion of the canonical connection. By this identity, we know that when the manifold is Hermitian, that is \( \tau_{ij}^k = 0 \) for all \( i, j \) and \( k \), then the holomorphic sectional curvature of the Levi-Civita connection is not greater than the holomorphic sectional curvature of the canonical connection. Moreover, when the manifold is quasi Kähler, that is \( \tau_{ij}^k = 0 \), the holomorphic sectional curvature of the Levi-Civita connection is not less than the holomorphic sectional curvature of the canonical connection. Also, by this identity, one can find that the curvature assumption for diameter estimate in [3] and [12] coincides.

Finally, with the help of the curvature identities, we obtain the following two integrability results for quasi Kähler manifolds.

**Theorem 1.1.** Let \((M, J, g)\) be a quasi Kähler manifold. Then

\[ S^c \leq S^* \]

all over \( M \). Moreover, if the equality holds all over \( M \), then \( R_{ijkl} = 0 \) for all \( i, j, k \) and \( l \) all over \( M \), and if the manifold is almost Kähler, then it must Kähler when the equality holds all over \( M \).

**Theorem 1.2.** Let \((M, J, g)\) be a compact quasi Kähler manifold with quasi positive second Ricci curvature and parallel (2,0)-part of the curvature tensor for canonical connection. Then, the manifold must be Kähler.

For the first result above, there are some related discussions on almost Kähler manifolds in [5]. For the second result above, one should note that without any curvature assumption, even for almost Kähler manifolds, the vanishing of (2,0)-part of the curvature tensor of the canonical connection does not imply integrability. One can find such kind of examples in [1]. Moreover, the assumption that the second Ricci curvature is quasi positive cannot be relaxed to nonnegative. Indeed, in [5], the authors constructed quasi Kähler structures on the
Iwasawa manifold which can never be Kähler by topological reasons with vanishing curvature tensor for the canonical connection.

Finally, by the help of the curvature identities, we obtain the following two properties of nearly Kähler manifolds.

**Theorem 1.3.** Let \((M, J, g)\) be a nearly Kähler manifold. Then, if the Ricci curvature of the canonical connection is positive definite or negative definite at some point, then the manifold must be Kähler.

**Theorem 1.4.** Let \((M^6, J, g)\) be a non-Kähler nearly Kähler manifold. Then \(R_{ij} = 0\) for all \(i\) and \(j\).

The study of nearly Kähler manifolds was initiated by A. Gray in the 70’s of the last century. In [13, 14, 15], Gray intensively studied the structure of nearly Kähler manifolds. In [15], Gray proposed a conjecture that any homogenous nearly Kähler manifold must be 3-symmetric. The conjecture was reduced to six dimensional case by Nagy [20] and finally solved by Butruille [2]. However, this is not the end of the story. All the known examples of compact strictly nearly Kähler manifolds are 3-symmetric. So, it was conjectured by Butruille [2] that all compact strictly nearly Kähler manifolds are 3-symmetric. It is really an interesting and important problem to find out all the six dimensional strictly nearly Kähler manifolds.

### 2. Bianchi identities on almost Hermitian manifolds

In this section, we systematically derive the Bianchi identities on almost Hermitian manifolds.

**Definition 2.1** ([18, 19, 10]). Let \((M, J)\) be an almost complex manifold. A Riemannian metric \(g\) on \(M\) such that \(g(JX, JY) = g(X, Y)\) for any two tangent vectors \(X\) and \(Y\) is called an almost Hermitian metric. The triple \((M, J, g)\) is called an almost Hermitian manifold. The two form \(\omega_g = g(JX, Y)\) is called the fundamental form of the almost Hermitian manifold. A connection \(\nabla\) on an almost Hermitian manifold \((M, J, g)\) such that \(\nabla g = 0\) and \(\nabla J = 0\) is called an almost Hermitian connection.

Let \(\nabla\) be a connection on the manifold \(M\). Recall that the torsion \(\tau\) of the connection is a vector-valued two form defined as

\[
\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].
\]

There are many almost Hermitian connections on an almost Hermitian manifold. However, there is a unique one such that \(\tau(X, Y) = 0\) for any two \((1, 0)\)-vectors \(X\) and \(Y\). Such a notion is first introduced by Ehresman and Libermann [8].
**Definition 2.2** ([18][19]). The unique almost Hermitian connection $\nabla$ on an almost Hermitian manifold $(M, J, g)$ with vanishing $(1, 1)$-part of the torsion is called the canonical connection of the almost Hermitian manifold.

In the remaining part of this paper, we adopt the following conventions:

1) Without further indications, the manifold is of real dimension $2n$;
2) $D$ denotes the Levi-Civita connection and $R^L$ denotes its curvature tensor;
3) $\nabla$ denotes the canonical connection, $R$ denote the curvature tensor of $\nabla$ and “;” means taking covariant derivatives with respect to $\nabla$.
4) Without further indications, capital English letters such as $A, B, C$ denote indices in $\{1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n}\}$;
5) Without further indications, $i, j, k$ etc denote indices in $\{1, 2, \ldots, n\}$.
6) Without further indications, Greek letters such as $\lambda, \mu$ denote summation indices going through $\{1, 2, \ldots, n\}$.

Recall the Nijenhuis tensor for a almost complex manifold is a vector value two form defined as

\[
N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]
\]

for any tangent vectors $X$ and $Y$.

The following relation of Nijenhuis tensor and torsion is well know.

**Lemma 2.1** ([8][18][19]). Let $(M, g, J)$ be an almost Hermitian manifold, $(e_1, e_2, \ldots, e_n)$ be a local $(1,0)$-frame. Then $N^k_{ij} = N^k_{i\bar{j}} = N^k_{i\bar{j}} = 0$ and $N^k_{ij} = 4\tau^k_{ij}$ for all $i, j$ and $k$.

Recall the definition of curvature operator:

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

The curvature tensor is defined as

\[
R(X, Y, Z, W) = \langle R(Z, W)X, Y \rangle.
\]

Fixed a unitary $(1,0)$-frame $(e_1, e_2, \ldots, e_n)$, since $\nabla J = 0$, we have

\[
R_{ijAB} = R^j_{iAB} = 0
\]

for all indices $i, j$ and $A, B$. Moreover, similarly as in the Riemannian case, we have the following symmetries of the curvature tensor:

\[
R_{ABCD} = -R_{BACD} = -R_{ABDC}
\]
for all indices $A, B, C$ and $D$. Recall that $R'_{AB} = g^{\alpha\beta}R_{\alpha\beta AB}$ and $R''_{ij} = g^{\alpha\lambda}R_{ij\alpha}$ are called the first and the second Ricci curvature of the almost Hermitian metric $g$ respectively.

The following general first and second Bianchi identities can be found in [17].

**Lemma 2.2** (First Bianchi identity). Let $M$ a smooth manifold and $\nabla$ be an affine connection on $M$ with torsion $\tau$. Then
\begin{equation}
R(X, Y)Z + R(Y, Z)X + R(Z, X)Y
= (\nabla_X \tau)(Y, Z) + (\nabla_Y \tau)(Z, X) + (\nabla_Z \tau)(X, Y)
- \tau(X, \tau(Y, Z)) - \tau(Y, \tau(Z, X)) - \tau(Z, \tau(X, Y))
\end{equation}
for any tangent vectors $X, Y$ and $Z$.

**Lemma 2.3** (Second Bianchi identity). Let $(M, g)$ be a Riemannian manifold and $\nabla$ be an affine connection compatible with the Riemannian metric $g$ with torsion $\tau$. Then,
\begin{equation}
(\nabla_W R)(X, Y, U, V) + (\nabla_U R)(X, Y, V, W) + (\nabla_V R)(X, Y, W, U)
= - R(X, Y, \tau(U, V), W) - R(X, Y, \tau(V, W), U) - R(X, Y, \tau(W, U), V)
\end{equation}
for any tangent vectors $X, Y, U, V, W$.

By directly applying the Bianchi identities above and (2.5), we have the following identities. Some of them can also be found in different forms in [18, 19] and [20].

**Corollary 2.1.** Let $(M, J, g)$ be an almost Hermitian manifold and and fix a unitary frame. Then
\begin{enumerate}
\item \( \tau^j_{ik;l} + \tau^j_{kl;i} + \tau^j_{li;k} = \tau^j_{i\lambda} \tau^\lambda_{kl} + \tau^j_{k\lambda} \tau^\lambda_{li} + \tau^j_{\lambda i} \tau^\lambda_{lk}; \)
\item \( R_{ijkl} - R_{kijl} = \tau^j_{ik;l} - \tau^j_{ik;l}; \)
\item \( R_{ijkl} - R_{iklj} = \tau^j_{ij;kl} - \tau^j_{kl;jl}; \)
\item \( R_{ijkl} - R_{klij} = \tau^j_{lk;ij} + \tau^j_{li;jk} + \tau^j_{ik;lj} - \tau^j_{il;kj} - \tau^j_{ij;lk} - \tau^j_{ij;lk} - \tau^j_{ij;lk}; \)
\item \( R_{ijkl} - R_{ijkl} = -\tau^j_{ik;l} + \tau^j_{ik;l}; \)
\item \( R_{ijkl} + R_{klij} + R_{lijk} = \tau^j_{ik;l} + \tau^j_{kl;i} + \tau^j_{li;k} - \tau^j_{il;k} + \tau^j_{lj;k} - \tau^j_{lj;k} - \tau^j_{lj;k}; \)
\item \( R_{ijkm;l} + R_{ijlm;k} + R_{ijmk;l} = \tau^j_{i\lambda} \tau^\lambda_{km;l} + \tau^j_{i\lambda} \tau^\lambda_{lm;k} + \tau^j_{i\lambda} \tau^\lambda_{jm;l}; \)
\item \( R_{ijkm;l} - R_{ijml;k} = -\tau^j_{i\lambda} \tau^\lambda_{km;l} - \tau^j_{i\lambda} \tau^\lambda_{lm;k} - \tau^j_{i\lambda} \tau^\lambda_{jm;l}; \)
\item \( R_{ijkm;l} - R_{ijlm;k} = -\tau^j_{i\lambda} \tau^\lambda_{km;l} + \tau^j_{i\lambda} \tau^\lambda_{lm;k} + \tau^j_{i\lambda} \tau^\lambda_{jm;l}; \)
\end{enumerate}
Proof. By Lemma 2.2 we have
\begin{equation}
R_{CAB}^D + R_{BAC}^D + R_{BCA}^D = \tau_{BCA} + \tau_{CAB} + \tau_{ABC} - \tau_{ABC} - \tau_{BAC} - \tau_{CAB}.
\end{equation}
Letting $C = i, A = k, B = l$ and $D = j$, we obtain (1). Letting $C = i, A = k, B = l$ and $D = j$ gives us (2). Taking conjugate of (2) give us (3). Subtracting (2) and (3) give us (4). Letting $C = i, A = k, B = l$ and $D = j$ give us (5). Letting $C = i, A = k, B = l$ and $D = j$ give us (6).

Moreover, by Lemma 2.3 we have
\begin{equation}
R_{ABCD}^{DE} + R_{ABDE}^{CD} + R_{ABEC}^{CD} = -\tau_{CD}^{DE} R_{ABFE}^{CD} - \tau_{DE}^{ED} R_{ABFC}^{ED} - \tau_{EC}^{EC} R_{ABFD}.
\end{equation}
Letting $A = i, B = j, C = k, D = l$ and $E = m$ give us (6). Letting $A = i, B = j, C = k, D = l$ and $E = m$ give us (7). Finally, letting $A = i, B = j, C = k, D = l$ and $E = m$ give us (8).

By Lemma 2.3 we know that when the complex structure is integrable, we have $\tau_{ij}^k = 0$. Hence, we have the following identities on Hermitian manifolds.

Corollary 2.2. Let $(M, J, g)$ be a Hermitian manifold and fix a unitary frame. Then
\begin{enumerate}
  \item $R_{ijkl} - R_{klij} = \tau_{ik\ell j};$
  \item $R_{ijkl} - R_{ik\ell j} = \tau_{ij\ell k};$
  \item $R_{ijkl} - R_{klij} = \tau_{ij\ell k};$
  \item $R_{ijkl} = 0;$
  \item $\tau_{ik\ell j} + \tau_{kj\ell i} + \tau_{li\ell k} = \tau_{ik\ell j} \lambda + \tau_{kj\ell i} \lambda + \tau_{li\ell k} \lambda;$
  \item $R_{ijkl;m} - R_{ijkm;l} = -\tau_{mk} \lambda R_{ijkl};$
  \item $R_{ijkl;m} - R_{ijkm;l} = \tau_{lm} \lambda R_{ijkl}.$
\end{enumerate}

Recall that an almost Hermitian manifold $(M, J, g)$ is called almost Kähler if $\omega = 0$ and it is called quasi Kähler if $\bar{\partial} \omega = 0$. It was shown in [26] (see also [18, 19]) that quasi Kahlerity is equivalent to $\tau_{ij}^k = 0$ for all $i, j$ and $k$, and almost Kahlerity is equivalent to $\tau_{ij}^k = 0$ and $\tau_{ij}^k + \tau_{ki}^j + \tau_{jk}^i = 0$ for all $i, j$ and $k$. Hence, we have the following corollary.

Corollary 2.3. Let $(M, J, g)$ be a quasi Kähler manifold and fix a unitary frame. Then,
\begin{enumerate}
  \item $\tau_{ik\ell j} + \tau_{kj\ell i} + \tau_{li\ell k} = 0;$
  \item $R_{ijkl} - R_{klij} = -\tau_{ik\ell j} \lambda;$
  \item $R_{ijkl} - R_{ik\ell j} = -\tau_{ik\ell j} \lambda.$
\end{enumerate}
Recall that an almost Hermitian manifold \((M, J, g)\) is said to be nearly Kähler if \((D_X J)X = 0\) for any tangent vector field \(X\). The following criterion for nearly Kähler manifolds is well known, see for example [20, 21].

**Lemma 2.4.** An almost Hermitian manifold \((M, J, g)\) is nearly Kähler if and only if \(\tau^i_{\bar{j}l} = 0\) and \(\tau^k_{ij} = \tau^k_{ij\bar{l}}\) for all \(i, j, k\) and \(l\) when we fix a \((1,0)\)-frame.

It turns out that the torsion for a nearly Kähler manifold must be parallel. This fact was first shown by Kirichenko [16] and was crucial to the study of structure nearly Kähler manifolds in [20, 21]. We give a proof of this fact using the curvature identity we derived in the section.

**Theorem 2.1** (Kirichenko). Let \((M, J, g)\) be a nearly Kähler manifold. Then \(\nabla \tau = 0\).

**Proof.** Fix a unitary frame, we only need to show that \(\tau^k_{ij;l} = \tau^k_{ij;l\bar{l}} = 0\) for all \(i, j, k\) and \(l\).

By Lemma 2.4 and (5) in Corollary 2.3, we know that

\[(2.11) \quad R_{ijkl} = -\tau^k_{ij\bar{l}}.\]

Substituting this into (6) of Corollary 2.3 and using Lemma 2.4, we have

\[(2.12) \quad 3\tau^i_{kl;j} = \tau^i_{kl\bar{j}} + \tau^k_{li;j} + \tau^l_{ik;j} = 0.\]

So \(\tau^k_{ij;l} = 0\) for all \(i, j, k\) and \(l\).

On the other hand, by (1) in Corollary 2.3 and Lemma 2.4

\[(2.13) \quad \tau^i_{ij;k} = -\tau^i_{jk;i} - \tau^i_{kji} = -\tau^k_{ij;i} - \tau^k_{ijl} = -\tau^k_{ij;l}.\]

Therefore

\[(2.14) \quad 2\tau^k_{ij;l} = \tau^k_{ij;l} - \tau^k_{ij;k} = \tau^i_{kji;l} - \tau^i_{jki;l} = -\tau^k_{jki;l} - \tau^k_{ijk;l} = \tau^k_{ikj;l}.

Therefore

\[(2.15) \quad \tau^i_{kl;j} = -2\tau^k_{ij;l} = 4\tau^i_{kl;j}.

and hence \(\tau^i_{kl;j} = 0\) for all \(i, j, k\) and \(l\). \qed
By Lemma 2.4 and Theorem 2.1, we have the following Bianchi identities for nearly Kähler manifolds.

**Corollary 2.4.** Let \((M, J, g)\) be a nearly manifold and fix a unitary frame. Then,

1. \(R_{ij\bar{k}} - R_{kij\bar{\ell}} = -\tau_{ik}^\lambda \tau_{\bar{j}l}^\lambda;\)
2. \(R_{ij\bar{k}} - R_{ik\bar{j}} = -\tau_{ik}^\lambda \tau_{\bar{j}l}^\lambda;\)
3. \(R_{ij\bar{k}} - R_{kij\bar{l}} = 0;\)
4. \(R_{ij} = R_{ij}^\ell;\)
5. \(R_{ij\bar{k}} = 0;\)
6. \(\tau_{ik}^\lambda R_{ijm\lambda} + \tau_{lm}^\lambda R_{ijk\lambda} + \tau_{mk}^\lambda R_{ijl\lambda} = 0;\)
7. \(R_{ij\bar{k}l\bar{m}} - R_{ijml\bar{k}} = 0;\)
8. \(R_{ij\bar{k}l\bar{m}} - R_{ijkm\bar{l}} = 0.\)

**Remark 2.1.** (1), (2), (7), (8) disguised in different forms can also be found in [28].

Since the first and second Ricci curvature tensor coincides for nearly Kähler manifolds, we simply denote them as \(R_{ij}.\)

## 3. CURVATURES OF LEVI-CIVITA AND CANONICAL CONNECTIONS

In this section, we compare the curvature tensor \(R^L\) of the Levi-Civita connection \(\nabla\) and the curvature tensor \(R\) of the canonical connection \(\nabla\) on an almost Hermitian manifold \((M, J, g).\)

Recall the following comparison of Levi-Civita connection and canonical connection on almost Hermitian manifolds. One can find it in [10] and in [9] for a proof.

**Lemma 3.1.**

\[
\langle D_X Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \frac{1}{2} (\langle \tau(X, Y), Z \rangle + \langle \tau(Y, Z), X \rangle - \langle \tau(Z, X), Y \rangle)
\]

for any tangent vector fields \(X, Y\) and \(Z.\)

**Theorem 3.1.** Let \((M, J, g)\) be an almost Hermitian manifold and fix a unitary frame. Then,

\[
R^L_{ij\bar{k}l} \quad = \frac{1}{2} (R_{ik\bar{j}} + R_{k\bar{j}l}) - \frac{1}{4} (\tau_{ik}^\lambda \tau_{\bar{j}l}^\lambda + \tau_{ik}^\lambda \tau_{\bar{j}l}^\lambda - \tau_{ik}^\lambda \tau_{\bar{j}l}^\lambda)
\]

\[
- \frac{1}{2} (\tau_{ik}^\lambda \tau_{\bar{j}l}^\lambda + \tau_{ik}^\lambda \tau_{\bar{j}l}^\lambda) + \frac{1}{4} (\tau_{ik}^\lambda \tau_{\bar{j}l}^\lambda + \tau_{ik}^\lambda \tau_{\bar{j}l}^\lambda - \tau_{ik}^\lambda \tau_{\bar{j}l}^\lambda + \tau_{ik}^\lambda \tau_{\bar{j}l}^\lambda).
\]
Proof. For the purpose of simplification, for \( p \in M \), let \( (e_1, e_2, \cdots, e_n) \) be a local pseudo holomorphic normal frame at \( p \) (See [30]). Then (1) \( \nabla e_i(p) = 0; (2) \) \( g_{ij}(p) = \delta_{ij} \) and (3) \( [e_k, \overline{e}_i](p) = 0. \)

By Lemma 3.1, we have

\[
D_{\overline{e}_i} e_i(p) = \frac{1}{2} \tau^i_{\bar{i} \lambda}(p) \overline{e}_\lambda + \frac{1}{2} \tau^i_{\bar{\lambda} \bar{j}}(p) e_\lambda
\]

and

\[
D_{e_k} e_i(p) = \frac{1}{2} \tau^k_{\imath \lambda}(p) e_\lambda + \frac{1}{4} \left( \tau^k_{\imath \bar{k}} + \tau^k_{\bar{k} \lambda} - \tau^k_{\lambda \imath} \right) \overline{e}_\lambda.
\]

Hence

\[
\langle D_{e_k} D_{\overline{e}_i} e_i, \overline{e}_j \rangle(p) = \left( \nabla_{\overline{e}_i} \nabla_{e_k} e_i, \overline{e}_j \right)(p) + \frac{1}{2} \left( \langle \tau(D_{\overline{e}_i} e_i, e_k), \overline{e}_j \rangle - \langle \tau(\overline{e}_j, D_{\overline{e}_i} e_i), e_k \rangle \right)(p)
\]

where we have used properties of local pseudo holomorphic normal frame, Lemma 3.1 (3.2) and (3.3).

Similarly, we have

\[
\langle D_{\overline{e}_i} D_{e_k} e_i, \overline{e}_j \rangle(p)
\]

\[
= \frac{1}{2} \left( \langle \nabla_{\overline{e}_i} e_k, e_i \rangle(p) + \langle \nabla_{e_k} e_i, e_i \rangle(p) \right) - \frac{1}{4} \left( \tau^i_{\bar{k} \lambda} + \tau^k_{\bar{k} \lambda} - \tau^k_{\lambda \bar{k}} \right) \overline{e}_\lambda
\]

\[
= \frac{1}{2} \left( \langle \nabla_{e_k} e_i, \overline{e}_j \rangle(p) - \frac{1}{4} \tau^k_{\bar{j} \bar{k}} + \tau^k_{\bar{j} \lambda} - \tau^k_{\lambda \bar{j}} - \frac{1}{4} \tau^i_{\bar{j} \bar{i}} + \frac{1}{4} \tau^i_{\bar{j} \lambda} - \frac{1}{4} \tau^i_{\lambda \bar{j}} \right) \overline{e}_\lambda
\]
which means that the holomorphic sectional curvature of the Levi-Civita canonical connection. Moreover, when the complex structure is integrable,

\[
R^L_{ijkl}(p) = (D_{e_k} D_{e_i} e_j - D_{e_i} D_{e_k} e_j)(p)
\]

\[
= R_{ijkl} - \frac{1}{2}(\tau^i_{j,k} + \tau^i_{k,j}) - \frac{1}{2}(\tau^j_{k,i} + \tau^j_{i,k}) + \frac{1}{4}(\tau^i_{k\lambda} + \tau^i_{\lambda k} - \tau^i_{\lambda j})(\tau^j_{\lambda i} + \tau^j_{i\lambda} - \tau^j_{\lambda k}) + \frac{1}{4}\tau^i_{j,k}\tau^i_{j,k}
\]

\[
= \frac{1}{2}(R_{ikj} + R_{kj i}) - \frac{1}{2}(\tau^i_{k\lambda} + \tau^i_{\lambda k} + \tau^i_{j\lambda} + \tau^i_{\lambda j} - \tau^i_{\lambda k} - \tau^i_{k\lambda} - \tau^i_{\lambda j} - \tau^i_{j\lambda})
\]

where we have used (2),(3) in Corollary 2.1, (3.4) and (3.5). □

**Corollary 3.1.** Let \((M, J, g)\) be an almost Hermitian manifold and fix a unitary frame. Then

\[
R^L_{ii\lambda i\lambda} = R^L_{ii\lambda} + \tau^i_{i\lambda} \tau_{i\lambda} - \frac{1}{2}\tau^i_{i\lambda} \tau_{i\lambda}.
\]

Moreover, when the complex structure is integrable,

\[
R^L_{ii\lambda i\lambda} = R^L_{ii\lambda} - \frac{1}{2}\tau^i_{i\lambda} \tau_{i\lambda},
\]

which means not the holomorphic sectional curvature of the Levi-Civita connection is not greater than then holomorphic sectional curvature of the Chern connection. When the manifold is quasi Kähler,

\[
R^L_{ii\lambda i\lambda} = R^L_{ii\lambda} + \tau^i_{i\lambda} \tau_{i\lambda},
\]

which means that the holomorphic sectional curvature of the Levi-Civita connection is not less than the holomorphic sectional curvature of the canonical connection. Furthermore, when the manifold is nearly Kähler,

\[
R^L_{ii\lambda i\lambda} = R^L_{ii\lambda},
\]

which means that the holomorphic sectional curvature of the Levi-Civita connection is the same as the holomorphic sectional curvature of the canonical connection.

By that when the complex structure is integrable, \(\tau^i_{ij} = 0\), we have the following corollary.

**Corollary 3.2.** Let \((M, J, g)\) be an Hermitian manifold and fixed a unitary frame. Then

\[
R^L_{ijkl} = \frac{1}{2}(R_{ikj} + R_{kj i}) - \frac{1}{4}\left(\tau^i_{i\lambda} \tau_{i\lambda} + \tau^i_{i\lambda} \tau_{i\lambda} - \tau^i_{i\lambda} \tau_{i\lambda}\right).
\]

By the second equality in (3.6), we have the follows.
Corollary 3.3. Let $(M, J, g)$ be a quasi Kähler manifold and fix a unitary frame. Then

\begin{equation}
R^L_{ijkl} = R_{ijkl} + \frac{1}{4} \left( \tau^\lambda_{ik} + \tau^\lambda_{k\lambda} - \tau^\lambda_{i\lambda} \right) \left( \tau^\lambda_{ij} + \tau^\lambda_{j\lambda} - \tau^\lambda_{ij} \right).
\end{equation}

Remark 3.1. A similar identity was also obtained in [4].

By noting that for an almost Kähler manifold, $\tau^k_{ij} + \tau^j_{ik} + \tau^j_{ki} = 0$. We have the follows.

Corollary 3.4. Let $(M, J, g)$ be an almost Kähler manifold and fix a unitary frame. Then

\begin{equation}
R^L_{ijkl} = R_{ijkl} + \tau^\lambda_{i\lambda} \tau^\lambda_{j\lambda}.
\end{equation}

By Lemma 2.4, we have the following curvature identity for nearly Kähler manifolds.

Corollary 3.5. Let $(M, J, g)$ be a nearly Kähler manifold and fix a unitary frame. Then

\begin{equation}
R^L_{ijkl} = R_{ijkl} + \frac{1}{4} \tau^\lambda_{ik} \tau^\lambda_{j\lambda}.
\end{equation}

Theorem 3.2. Let $(M, J, g)$ be an almost Hermitian manifold and fix a unitary frame. Then

\begin{equation}
R^L_{ijkl} = \frac{1}{2} \left( R_{klji} - R_{lki} + R_{ijkl} - R_{ijlk} \right) + \frac{1}{2} \left( \tau^\lambda_{ijkl} - \tau^\lambda_{ijlk} + \frac{1}{4} \left( \tau^\lambda_{ijkl} - \tau^\lambda_{ijlk} \right) \right) + \frac{1}{4} \tau^\lambda_{ijkl} \tau^\lambda_{j\lambda} - \frac{1}{4} \tau^\lambda_{ijkl} \tau^\lambda_{j\lambda}.
\end{equation}

Proof. For sake of simplicity, let $p \in M$ and $(e_1, e_2, \cdots, e_n)$ be a local pseudo holomorphic normal frame at $p$ as in the proof of of Theorem 3.1. Then

\begin{equation}
\langle D_{e_k} D_{e_i} e_j, e_j \rangle(p)
= \langle \nabla_{e_k} D_{e_i} e_j, e_j \rangle + \frac{1}{2} \langle \tau(D_{e_i} e_j, e_k), e_j \rangle + \langle \tau(e_k, e_j), D_{e_i} e_j \rangle - \langle \tau(e_j, D_{e_i} e_j), e_k \rangle
= e_k \langle D_{e_i} e_j, e_j \rangle - \frac{1}{4} \tau^\lambda_{ijkl} (\tau^\lambda_{jk} + \tau^\lambda_{k\lambda} - \tau^\lambda_{j\lambda}) - \frac{1}{4} \tau^\lambda_{ijkl} \tau^\lambda_{j\lambda}
= e_k \left( \langle \nabla_{e_i} e_j, e_j \rangle - \frac{1}{2} \langle \tau(e_j, e_j), e_i \rangle \right) - \frac{1}{4} \tau^\lambda_{ijkl} (\tau^\lambda_{jk} + \tau^\lambda_{k\lambda} - \tau^\lambda_{j\lambda}) - \frac{1}{4} \tau^\lambda_{ijkl} \tau^\lambda_{j\lambda}
= \frac{1}{2} \tau^\lambda_{j\lambda}(p) - \frac{1}{4} \tau^\lambda_{ijkl} (\tau^\lambda_{jk} + \tau^\lambda_{k\lambda} - \tau^\lambda_{j\lambda}) - \frac{1}{4} \tau^\lambda_{ijkl} \tau^\lambda_{j\lambda}.\]
Curvature identity

where we have used properties of local pseudo holomorphic normal frame and (3.2). Moreover

\[
\langle D_{\bar{\tau}} D_{\bar{e}_k} e_i, e_j \rangle(p) = \langle \nabla_{\bar{\tau}} D_{\bar{e}_k} e_i, e_j \rangle(p) + \frac{1}{2} \left( \langle \tau(D_{\bar{e}_k} e_i, e_j) - \langle \tau(e_j, D_{\bar{e}_k} e_i), e_i \rangle \right)(p)
\]

\[
= e_i \left( \langle \nabla_{\bar{\tau}} e_i, e_j \rangle + \frac{1}{2} \left( \langle \tau(e_i, e_k), e_j \rangle + \langle \tau(e_k, e_j), e_i \rangle - \langle \tau(e_j, e_i), e_k \rangle \right) \right)(p)
\]

\[
- \frac{1}{4} \tau^j_{\bar{i} \bar{l} \bar{\lambda}} \left( \tau^\bar{\lambda}_{ik} + \tau^\bar{\lambda}_{k\lambda} - \tau^k_{\bar{i} \bar{\lambda}} \right)(p) - \frac{1}{4} \tau^j_{\bar{i} \bar{j} \bar{\lambda}} \tau^k_{\bar{i} \bar{\lambda}}(p)
\]

\[
= \frac{1}{2} \left( \tau^j_{\bar{i}k;\bar{j}} + \tau^k_{\bar{j}i;\bar{j}} - \tau^j_{\bar{i}j;\bar{j}} \right)(p) - \frac{1}{4} \tau^j_{\bar{i} \bar{l} \bar{\lambda}} \left( \tau^\bar{\lambda}_{ik} + \tau^\bar{\lambda}_{k\lambda} - \tau^k_{\bar{i} \bar{\lambda}} \right)(p) - \frac{1}{4} \tau^j_{\bar{i} \bar{j} \bar{\lambda}} \tau^k_{\bar{i} \bar{\lambda}}(p)
\]

\[
= - \frac{1}{2} \left( R_{\bar{i}jik} + R_{\bar{i}ikj} - R_{\bar{k}j\bar{i}j} \right)(p) + \frac{1}{2} \left( \tau^j_{\bar{i} \bar{l} \bar{\lambda}} \tau^l_{\bar{i}k} + \tau^j_{\bar{i} \bar{l} \bar{\lambda}} \tau^l_{\bar{k} \bar{i}} - \tau^l_{\bar{i}k} \tau^j_{\bar{k} \bar{i}} \right)(p)
\]

\[
- \frac{1}{4} \tau^j_{\bar{i} \bar{l} \bar{\lambda}} \left( \tau^\bar{\lambda}_{ik} + \tau^\bar{\lambda}_{k\lambda} - \tau^k_{\bar{i} \bar{\lambda}} \right)(p) - \frac{1}{4} \tau^j_{\bar{i} \bar{j} \bar{\lambda}} \tau^k_{\bar{i} \bar{\lambda}}(p)
\]

where we have used properties of local pseudo holomorphic normal frame, Corollary 2.1 and (3.3).

Combining (3.16) and (3.17), we get

\[
R^L_{\bar{i}jkl}(p) = \frac{1}{2} \left( R_{\bar{i}jik} + R_{\bar{i}ikj} - R_{\bar{k}j\bar{i}j} \right) + \frac{1}{2} \left( \tau^j_{\bar{i} \bar{l} \bar{\lambda}} \tau^l_{\bar{i}k} + \tau^j_{\bar{i} \bar{l} \bar{\lambda}} \tau^l_{\bar{k} \bar{i}} - \tau^l_{\bar{i}k} \tau^j_{\bar{k} \bar{i}} \right)
\]

\[
- \frac{1}{4} \tau^j_{\bar{i} \bar{l} \bar{\lambda}} \left( \tau^\bar{\lambda}_{ik} + \tau^\bar{\lambda}_{k\lambda} - \tau^k_{\bar{i} \bar{\lambda}} \right)(p) + \frac{1}{4} \tau^j_{\bar{i} \bar{j} \bar{\lambda}} \left( -\tau^\bar{\lambda}_{ik} + \tau^\bar{\lambda}_{k\lambda} - \tau^k_{\bar{i} \bar{\lambda}} \right)(p)
\]

\[
- \frac{1}{4} \tau^j_{\bar{i} \bar{j} \bar{\lambda}} \tau^k_{\bar{i} \bar{\lambda}}(p)
\]

\[
\square
\]

**Corollary 3.6.** Let \((M, J, g)\) be a Hermitian manifold and fix a unitary frame. Then

\[
R^L_{\bar{i}jkl} = \frac{1}{2} \tau^j_{\bar{i} \bar{\lambda} k} + \frac{1}{4} \left( \tau^j_{\bar{i} \bar{l} \bar{\lambda}} \tau^l_{\bar{i}k} - \tau^l_{\bar{i}k} \tau^j_{\bar{l} \bar{\lambda}} \right).
\]

Applying the properties of quasi Kähler manifolds that \(\tau^k_{\bar{i}j} = 0\) and (6) in Corollary 2.3, we obtain the follows.

**Corollary 3.7.** Let \((M, J, g)\) be a quasi Kähler manifold and fix a unitary frame. Then

\[
R^L_{\bar{i}jkl} = R_{\bar{k}l\bar{i}j}.
\]

**Remark 3.2.** The same identity was also obtained in [5].
By (5) in Corollary 2.3, we have the following corollary for nearly Kähler manifolds.

**Corollary 3.8.** Let \((M, J, g)\) be a nearly Kähler manifold and fix a unitary frame. Then

\[
(3.21) \quad R_{ijkl}^L = 0.
\]

**Theorem 3.3.** Let \((M, J, g)\) be an almost Hermitian manifold and fix a unitary frame. Then,

\[
(3.22) \quad R_{ijkl}^L = \frac{1}{2} \left( \tau_{kl;i}^j - \tau_{kl;j}^i \right) + \frac{1}{2} \left( \tau_{ij;k}^l - \tau_{ij;l}^k \right) + \frac{1}{2} \left( \tau_{ij;k}^l - \tau_{ij;l}^k \right)
\]

\[
= \frac{1}{4} \tau_{ik}^l \left( \tau_{jl}^k - \tau_{jl}^k - \tau_{jl}^l \right) + \frac{1}{4} \tau_{jk}^l \left( \tau_{li}^k - \tau_{li}^k - \tau_{li}^l \right)
\]

\[
= \frac{1}{4} \tau_{il}^k \left( \tau_{jk}^l - \tau_{jk}^k - \tau_{jk}^l \right) - \frac{1}{4} \tau_{ik}^j \left( \tau_{lj}^k - \tau_{lj}^k - \tau_{lj}^l \right)
\]

**Proof.** We proceed similarly as before. Let \(p \in M\) and \((e_1, e_2, \ldots, e_n)\) be an local pseudo holomorphic normal frame at \(p\). Then

\[
(3.23) \quad \langle D_{e_k} D_{e_i} e_j, e_j \rangle (p)
\]

\[
= \langle \nabla_{e_k} D_{e_i} e_j, e_j \rangle + \frac{1}{2} \left( \langle \tau(D_{e_i} e_j, e_k), e_j \rangle + \langle \tau(e_k, e_j), D_{e_i} e_j \rangle - \langle \tau(e_j, D_{e_i} e_k), e_j \rangle \right)
\]

\[
= e_k \langle D_{e_i} e_j, e_j \rangle + \frac{1}{4} \tau_{ik}^j \left( \tau_{jl}^k + \tau_{jl}^k - \tau_{jl}^l \right) + \frac{1}{4} \tau_{jk}^l \left( \tau_{li}^k + \tau_{li}^k - \tau_{li}^l \right)
\]

\[
= e_k \left( \langle \nabla_{e_i} e_j, e_j \rangle + \frac{1}{2} \left( \langle \tau(e_i, e_j), e_j \rangle + \langle \tau(e_j, e_i), e_i \rangle - \langle \tau(e_j, e_i), e_i \rangle \right) \right)
\]

\[
= \frac{1}{2} \left( \tau_{ik;\dot{l}}^j + \tau_{ik;\dot{j}}^l - \tau_{ik;jl}^l \right) - \frac{1}{4} \tau_{ik}^j \left( \tau_{lj}^k + \tau_{lj}^k - \tau_{lj}^l \right) - \frac{1}{4} \tau_{ij}^k \left( \tau_{kl}^l + \tau_{kl}^l - \tau_{kl}^k \right)
\]

where we have used (3.3). Similarly,

\[
(3.24) \quad \langle D_{e_i} D_{e_k} e_j, e_j \rangle (p)
\]

\[
= \frac{1}{2} \left( \tau_{ik;\dot{l}}^j + \tau_{ik;\dot{j}}^l - \tau_{ik;jl}^l \right) - \frac{1}{4} \tau_{ik}^j \left( \tau_{lj}^k + \tau_{lj}^k - \tau_{lj}^l \right) - \frac{1}{4} \tau_{ij}^k \left( \tau_{kl}^l + \tau_{kl}^l - \tau_{kl}^k \right).
\]

Moreover, note that

\[
[e_k, e_l](p) = \nabla_{e_k} e_l (p) - \nabla_{e_l} e_k (p) - \tau(e_k, e_l)(p) = -\tau_{kl}^i e_i e_k - \tau_{kl}^i e_{kl}.
\]
manifolds, we have the following corollaries. where we have used (1) in Corollary 2.2.

\[(3.26)\]

\[
\langle D_{e_k e_l} e_i, e_j \rangle (p) = -\alpha_{kl}^{ij} \langle D_{e_k e_l} e_i, e_j \rangle
\]

so

\[
= -\frac{1}{2} \alpha_{kl}^{ij} (\tau_{ij}^l + \tau_{ij}^l - \tau_{ij}^l) - \frac{1}{2} \alpha_{ij}^{kl} \tau_{kl}^{ij}
\]

\[\text{Hence}\]

\[
(3.26) \quad R_{ijkl}^L(p)
\]

\[
= \langle D_{e_k} D_{e_i} e_i - D_{e_i} D_{e_k} e_i - D_{e_k e_l} e_i, e_j \rangle (p)
\]

\[
= \frac{1}{2} \left( \tau_{ij}^{kl} - \tau_{ij}^{kl} \right) + \frac{1}{2} \left( \tau_{ij}^{kl} - \tau_{ij}^{kl} \right) + \frac{1}{2} \left( \tau_{ij}^{kl} - \tau_{ij}^{kl} \right)
\]

\[
= \frac{1}{2} \left( \tau_{ij}^{kl} - \tau_{ij}^{kl} \right) + \frac{1}{2} \left( \tau_{ij}^{kl} - \tau_{ij}^{kl} \right) + \frac{1}{2} \left( \tau_{ij}^{kl} - \tau_{ij}^{kl} \right)
\]

where we have used (3.2) and (3.3). As before, using properties of Hermitian manifolds and quasi Kähler manifolds, we have the following corollaries.

Corollary 3.9. Let \((M, g, J)\) be an Hermitian manifold and fix an arbitrary frame. Then, \(R_{ijkl}^L = 0\).
Corollary 3.10. Let \((M, J, g)\) be an quasi Kähler manifold and fix a unitary frame. Then

\[
R^L_{ijkl} = \frac{1}{2} (\tau^j_{kl} - \tau^j_{kl}) + \frac{1}{2} (\tau^i_{ij} - \tau^i_{ij}).
\]

By Theorem 2.1, we have the following corollary for nearly Kähler manifolds.

Corollary 3.11. Let \((M, J, g)\) be a nearly Kähler manifold and fix a unitary frame. Then \(R^L_{ijkl} = 0\).

At the end of this section, we compute the Ricci curvature of the Levi-Civita connection of an almost Kähler manifold and a nearly Kähler manifold.

Theorem 3.4. Let \((M, J, g)\) be an almost Kähler manifold fixed a unitary frame. Then,

\[
R^L_{ij} = R^L_{i\lambda j\lambda} + R^L_{j\lambda i\lambda};
\]

and

\[
R^L_{ij} = R^L_{ij} - 2\tau^\lambda_{ij\mu\lambda} - 2\tau^\lambda_{ij\mu\lambda}.
\]

Proof. \(R^L_{ij} = R^L_{i\lambda j\lambda} + R^L_{j\lambda i\lambda} = R^L_{i\lambda j\lambda} + R^L_{j\lambda i\lambda} = R^L_{i\lambda j\lambda} + R^L_{j\lambda i\lambda}
\]

where we have used the symmetries for the curvature tensor of Levi-Civita connection and Corollary 3.7. Moreover

\[
R^L_{ij} = R^L_{i\lambda j\lambda} + R^L_{j\lambda i\lambda} = R^L_{i\lambda j\lambda} + R^L_{j\lambda i\lambda} = R^L_{i\lambda j\lambda} + R^L_{j\lambda i\lambda}
\]

(3.31)

where we have used Corollary 3.4 and Corollary 2.3. \(\square\)
Theorem 3.5. Let \((M, J, g)\) be a nearly Kähler manifold and fix a unitary frame. Then

\[(3.32) \quad R^L_{ij} = 0 \]

and

\[(3.33) \quad R^L_{ij} = R_{ij} + \frac{5}{4} \tau^\mu_i \tau^\nu_j. \]

Proof. The proof is the same as the proof of Theorem 3.4 using Corollary 3.5 and Corollary 3.8.

4. Integrability of quasi Kähler manifolds

In this section, with the help of the curvature identities derived in the last two sections, we obtain some results about the integrability of quasi Kähler manifolds.

First, recall that the \(\ast\)-scalar curvature \(S^*\) for the Levi-Civita connection of an almost Hermitian manifold is defined as (see for example [3])

\[(4.1) \quad S^* = R^L_{\lambda\mu\bar{\lambda}\bar{\mu}}. \]

Let \(S^c\) be the scalar curvature of the canonical connection. That is

\[(4.2) \quad S^c = R_{\lambda\bar{\lambda}\mu\bar{\mu}}. \]

Theorem 4.1. Let \((M, J, g)\) be a quasi Kähler manifold. Then

\[(4.3) \quad S^c \leq S^* \]

all over \(M\). Moreover, if the equality holds all over \(M\), then \(R_{ijkl} = 0\) for all \(i, j, k\) and \(l\) all over \(M\), and if the manifold is almost Kähler, then it must Kähler when the equality holds all over \(M\).

Proof. By Corollary 3.3 we know that

\[(4.4) \quad S^* = S^c + \frac{1}{4} \sum_{\lambda, \mu, \nu = 1}^n \left| \tau^\rho_{\lambda\mu} + \tau^\lambda_{\mu\nu} - \tau^\rho_{\nu\lambda} \right|^2 \geq S^c. \]

When the equality holds all over \(M\), we have

\[(4.5) \quad \tau^k_{ij} + \tau^j_{jk} - \tau^j_{ki} = 0 \]

for all \(i, j\) and \(k\), all over \(M\). Then

\[(4.6) \quad \tau^k_{ij;\bar{i}} + \tau^j_{jk;\bar{i}} - \tau^j_{ki;\bar{i}} = 0 \]

which means that

\[(4.7) \quad R_{k\bar{i}j} + R_{i\bar{j}k} - R_{j\bar{k}i} = 0 \]
by (5) in Corollary 2.3. Combining this with (6) in Corollary 2.3 we know that $R_{ijkl} = 0$ for all $i, j, k$ and $l$.

When the manifold is almost Kähler and the equality holds all over $M$. It is clear that $\tau_{ij}^k = 0$ for all $i, j$ and $k$ by combining (1.5) and $\tau_{ij}^k + \tau_{jk}^i + \tau_{ki}^j = 0$ for all $i, j$ and $k$.

\[ \Box \]

**Remark 4.1.** In [1], there is a similar inequality as in the theorem above in integration form for almost Kähler manifolds.

We know that when the complex structure is integrable, then the (2,0)-part of the curvature tensor for the canonical connection vanishes. One may wonder if the converse is true. It turns out that the converse is not true even when the manifold is almost Kähler. Indeed, there are examples of strictly almost Kähler manifolds (almost Kähler but not Kähler) with vanishing (2,0)-part of the curvature tensor for the canonical connection which is equivalent to that the curvature tensor for the Levi-Civita connection satisfies the third Gray identity by Corollary 3.7 (see for example [1]). However, when some curvature conditions are imposed, the answer turns out to be affirmative.

**Theorem 4.2.** Let $(M, J, g)$ be a compact quasi Kähler manifold with quasi positive second Ricci curvature and parallel (2,0)-part of the curvature tensor for canonical connection. Then, the manifold must be Kähler.

**Proof.** Note that for a quasi Kähler manifold $\tau_{ij}^k = 0$, so the Laplacian of operators on functions for the canonical connection and the Levi-Civita connection coincides (See for example [20, 9]). We apply the Bochner technique to $\|\tau\|^2 = \tau_{ij}^k \tau_{ij}^k$ to draw the conclusion.

Fix a unitary frame, we have

\begin{equation}
\Delta (\tau_{ij}^k \tau_{ij}^k)
= (\tau_{ij}^k \tau_{ij}^k)_{ll}
= \tau_{ij}^k \tau_{ij, l l}^k + \tau_{ij, l}^k \tau_{ij, l}^k + \tau_{ij}^k \tau_{ij}^k_{ll} + \tau_{ij}^k \tau_{ij}^k_{ll}
\end{equation}

\begin{equation}
= \tau_{ij}^k \tau_{ij, l l}^k + \tau_{ij, l}^k \tau_{ij, l}^k + (\tau_{ij}^k + R_{ijkl} \tau_{ij}^l + R_{ijkl} \tau_{ij}^l + R_{ijkl} \tau_{ij}^l) \tau_{ij}^k + \tau_{ij}^k \tau_{ij}^k_{ll}
\end{equation}

\begin{equation}
= \tau_{ij}^k \tau_{ij, l l}^k + \tau_{ij, l}^k \tau_{ij, l}^k + (-R_{klij} + R_{ijkl} \tau_{ij}^l + R_{ijkl} \tau_{ij}^l + R_{ijkl} \tau_{ij}^l) \tau_{ij}^k - \tau_{ij}^k R_{klij}
\end{equation}

\begin{equation}
= \tau_{ij}^k \tau_{ij, l l}^k + \tau_{ij, l}^k \tau_{ij, l}^k + (R_{ijkl} \tau_{ij}^l + R_{ijkl} \tau_{ij}^l + R_{ijkl} \tau_{ij}^l) \tau_{ij}^k \geq 0
\end{equation}

where we have used Corollary 2.3 and the Ricci identity for commuting covariant derivatives with respect to the canonical connection (See for
By maximum principle, we know that $\|\tau\|^2$ is constant and that $\tau$ is parallel. Since the second Ricci curvature is positive at some point, $\tau$ vanishes at some point. Therefore $\tau$ vanishes all over $M$ and the metric is Kähler.

Remark 4.2. In [3], the authors construct a quasi Kähler structure on the Iwasawa manifold with vanishing curvature tensor of the canonical connection. We cannot obtain integrability by just assuming the second Ricci is nonnegative and vanishing $(2,0)$-part of curvature tensor for a quasi Kähler manifold.

Furthermore, we have the following integral inequality by taking integration on (4.8).

**Theorem 4.3.** Let $(M, J, g)$ be a compact quasi Kähler manifold. Then

$$\int_M \sum_{i,j,k,l=1} (R''_{ki}^{\tau_k} + R''_{kj}^{\tau_k} + R''_{kl}^{\tau_k}) dV \leq \int_M \sum_{i,j,k,l=1} |R_{ijkl}|^2 dV.$$  

**Proof.** By Lemma 3.1, it is not hard to check that the divergence operators on vector fields for the canonical connection and the Levi-Civita connection coincides on quasi Kähler manifolds. Moreover, by (4.8), we have the follows.

$$\Delta(\tau_{ij}^k \tau_{ij}^k) = \tau_{ij}^k \tau_{ij}^k + (R''_{ki}^{\tau_k} + R''_{kj}^{\tau_k} + R''_{kl}^{\tau_k}) \tau_{ij}^k + \tau_{ij}^k \tau_{ij}^k + \tau_{ij}^k \tau_{ij}^k$$

$$= \tau_{ij}^k \tau_{ij}^k - \tau_{ij}^k \tau_{ij}^k + (R''_{ki}^{\tau_k} + R''_{kj}^{\tau_k} + R''_{kl}^{\tau_k}) \tau_{ij}^k + (\tau_{ij}^k \tau_{ij}^k) + (\tau_{ij}^k \tau_{ij}^k)$$

$$\geq - \sum_{i,j,k,l=1} |R_{ijkl}|^2 + R''_{ki}^{\tau_k} + R''_{kj}^{\tau_k} + R''_{kl}^{\tau_k} \tau_{ij}^k + (\tau_{ij}^k \tau_{ij}^k) + (\tau_{ij}^k \tau_{ij}^k)$$

where we have used Corollary 2.3. Taking integration on both sides of the last inequality and applying the divergence theorem, we obtain the conclusion.

5. **Some properties of nearly Kähler manifolds**

In this section, we derive some properties of nearly Kähler manifolds. One should note that some properties of nearly Kähler manifolds are derived by Nagy [21] in terms of curvature of the Levi-Civita connection. In contrast, the properties we derive here are in terms of curvature of the canonical connection.
Proposition 5.1. Let \((M, J, g)\) be a nearly Kähler manifold and fix a unitary frame. Then

\[
\sum_{k,l,\lambda,\mu=1}^{n} R_{ijkl} \tau_{\lambda\mu}^k \tau_{\lambda\mu}^l = 0
\]

for all \(i\) and \(j\).

Proof. By Corollary 2.4 we know that

\[
\sum_{\lambda=1}^{n} \left( \tau_{kl}^\lambda R_{ijkl \lambda} + \tau_{lm}^\lambda R_{ijkl \lambda} + \tau_{mk}^\lambda R_{ijkl \lambda} \right) = 0
\]

for all \(i, j, k, l\) and \(m\). Then

\[
\sum_{k,l,m,\lambda=1}^{n} \left( \tau_{kl}^\lambda R_{ijkl \lambda} + \tau_{lm}^\lambda R_{ijkl \lambda} + \tau_{mk}^\lambda R_{ijkl \lambda} \right) \tau_{kl}^m = 0.
\]

By Lemma 2.4 we have

\[
3 \sum_{k,l,\lambda,\mu=1}^{n} R_{ijkl} \tau_{\lambda\mu}^k \tau_{\lambda\mu}^l = \sum_{k,l,m,\lambda=1}^{n} \left( R_{ijkl \lambda} \tau_{kl}^m + R_{ijkl \lambda} \tau_{lm}^k + R_{ijkl \lambda} \tau_{mk}^l \right) = 0.
\]

Hence

\[
\sum_{k,l,\lambda,\mu=1}^{n} R_{ijkl} \tau_{\lambda\mu}^k \tau_{\lambda\mu}^l = 0.
\]

□

Theorem 5.1. Let \((M, J, g)\) be a nearly Kähler manifold. Then, if the Ricci curvature of the canonical connection is positive definite or negative definite at some point, then the manifold must be Kähler.

Proof. By Proposition 5.1, we have

\[
\sum_{k,l,\lambda,\mu=1}^{n} R_{kl} \tau_{\lambda\mu}^k \tau_{\lambda\mu}^l = 0.
\]

If the \(R_{kl}\) is positive or negative at some point \(p \in M\), then

\[
\tau_{\lambda\mu}^l (p) = 0
\]

for all \(\lambda, \mu\) and \(l\). Note that \(\tau\) is parallel on \(M\) by Theorem 2.1. So \(\tau = 0\) all over \(M\) and hence \((M, J, g)\) is Kähler. □

Theorem 5.2. Let \((M^6, J, g)\) be a non-Kähler nearly Kähler manifold. Then \(R_{ij} = 0\) for all \(i\) and \(j\).
Proof. Let $\Phi(X, Y, Z) = \langle \tau(X, Y), Z \rangle$ for any $(1,0)$-vectors $X, Y$ and $Z$. Then $\Phi$ is a $(3,0)$-form on $M$ by Lemma 2.4. Let $e_1, e_2, e_3$ be a unitary frame and $\omega^1, \omega^2, \omega^3$ be its dual frame. Suppose that $\Phi = c \omega^1 \wedge \omega^2 \wedge \omega^3$. Since the manifold is non-Kähler, $c \neq 0$. Moreover, it is clear that

\[
\tilde{t}^k_{ij} = c \cdot \text{sgn} \left( \begin{array}{ccc} 1 & 2 & 3 \\ i & j & k \end{array} \right).
\]

Substituting the above in to Proposition 5.1, we have

\[
|c|^2 R_{ij} = 0.
\]

This completes the proof. \qed

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