An adaptive proximal point algorithm framework and application to large-scale optimization

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Abstract. We investigate the proximal point algorithm (PPA) and its inexact extensions under an error bound condition, which guarantees a global linear convergence if the proximal regularization parameter is larger than the error bound condition parameter. We propose an adaptive generalized proximal point algorithm (AGPPA), which adaptively updates the proximal regularization parameters based on some implementable criteria. We show that AGPPA achieves linear convergence without any knowledge of the error bound condition parameter, and the rate only differs from the optimal one by a logarithm term. We apply AGPPA on convex minimization problem and analyze the iteration complexity bound of the resulting algorithm. Our framework and the complexity results apply to arbitrary linearly convergent inner solver and allows a hybrid with any locally fast convergent method. We illustrate the performance of AGPPA by applying it to solve large-scale linear programming (LP) problem. The resulting complexity bound has a weaker dependence on the Hoffman constant and scales with the dimension better than linearized ADMM. In numerical experiments, our algorithm demonstrates improved performance in obtaining solution of medium accuracy on large-scale LP problem.

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1 Introduction

Let $\mathcal{X}$ be a finite dimensional Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $T : \mathcal{X} \rightrightarrows \mathcal{X}$ be a maximal monotone operator. We aim to find a solution $z \in \mathcal{X}$ such that

$$0 \in T(z).$$

We shall assume throughout the paper that the solution set $\Omega := T^{-1}(0)$ is nonempty.

The framework of proximal point algorithm (PPA) plays a highly influential role in optimization history. It has widespread applications in various fields and inspires tremendous creative work in design and analysis of optimization methods. This framework chooses a sequence of proximal regularization parameters $\{\sigma_k\}_{k \geq 0}$ and generates $\{z_k\}_{k \geq 0}$ from an arbitrary initial point $z^0$ by the following rule:

$$z^{k+1} = J_{\sigma_k T}(z^k) := \left( I + \sigma_k T \right)^{-1}(z^k), \quad \forall k \geq 0.$$

Here $I : \mathcal{X} \to \mathcal{X}$ denotes the identity operator. One inexact version of PPA approximates $J_{\sigma_k T}(z^k)$:

$$z^{k+1} \approx J_{\sigma_k T}(z^k), \quad \forall k \geq 0,$$

based on the following conceptual inexactness criteria:

$$\|z^{k+1} - J_{\sigma_k T}(z^k)\| \leq \min \left\{ \eta_k, \delta_k \|z^{k+1} - z^k\| \right\}, \quad \forall k \geq 0.$$

Here, $\{\eta_k\}_{k \geq 0}$ and $\{\delta_k\}_{k \geq 0}$ are error parameters which control the precision of the approximation. Rockafellar analyzed the convergence of PPA in [36], and provided three kinds of applications in convex programming [35], including primal application, dual application (leading to the augmented Lagrangian method (ALM) [19,33]) and minimax application (leading to the proximal ALM [25]).

In this paper, we consider the following generalization of inexact PPA:

$$w^k \approx J_{\sigma_k M^{-1} T}(z^k), \quad z^{k+1} = \gamma w^k + (1 - \gamma) z^k, \quad \forall k \geq 0$$

along with the following inexactness criteria:

$$\|w^k - J_{\sigma_k M^{-1} T}(z^k)\|_M \leq \min \left\{ \eta_k, \delta_k \|w^k - z^k\|_M \right\}, \quad \forall k \geq 0.$$

Here, $\gamma \in (0,2)$ is a relaxation factor and $M$ is a self-adjoint positive definite linear operator over $\mathcal{X}$, known as preconditioner. In the literature, such generalization has been studied in various context and is known under different
names: relaxed PPA ($\mathcal{M} = I$) \cite{9,13}, preconditionned PPA ($\gamma = 1$) \cite{25} and relaxed customized PPA \cite{5,14}. It is observed that using a relaxation factor $\gamma > 1$ can speedup the convergence in practice \cite{11,5}. Besides, a suitable preconditionner allows to exploit specific structure and alleviate the difficulty of solving the inner problems \cite{5,14}.

We shall call (1) the inexact generalized PPA (IGPPA) and study its convergence property under the following error bound condition.

**Assumption 1** For any $r > 0$, there exists $\kappa_r > 0$ such that

$$\operatorname{dist}(z, \Omega) \leq \kappa_r \operatorname{dist}(0, T(z)), \quad \forall \|z\| \leq r. \quad (3)$$

Tao and Yuan \cite{38} computed the linear convergence rate of relaxed PPA (1) with $\mathcal{M} = I$ under the Lipschitz continuity assumption of $T^{-1}$ at 0, which is stronger than Assumption 1. In Theorem 2 we establish an upper bound on the ratio

$$\frac{\operatorname{dist}_{\mathcal{M}}(z^{k+1}, \Omega)}{\operatorname{dist}_{\mathcal{M}}(z^k, \Omega)} \leq \rho_k$$

for IGPPA (1) under Assumption 1. Our results allow to recover the linear convergence rate obtained in \cite{38, Theorem 3.5}, under a weaker assumption. In the special case when $\gamma = 1$, Theorem 2 also recovers \cite{25, Theorem 2}. The bound $\rho_k$ can be made arbitrarily close to $\sqrt{1 - \min\{\gamma, 2\gamma - \gamma^2\}}$, if $\sigma_k$ is sufficiently large and $\delta_k$ is sufficiently small. In particular, when $\gamma = 1$, this corresponds to the well known superlinear convergence property of PPA. However, even to guarantee linear convergence (i.e., to make $\sup_k \rho_k < 1$), it is required to know the value of the error bound condition parameter $\kappa_r$ to choose appropriate $\{\sigma_k\}_{k \geq 0}$ and $\{\delta_k\}_{k \geq 0}$, which is an unrealistic assumption.

To deal with the unknown $\kappa_r$, we propose to adaptively choose the proximal regularization parameters $\{\sigma_k\}_{k \geq 0}$ by verifying if $\|z^{k+1} - z^k\|$ decreases sufficiently fast. The resulting algorithm, called adaptive generalized PPA (AGPPA), is able to find an $\epsilon$-solution within

$$O\left( \ln \kappa_r \ln \frac{\kappa_r}{\epsilon} \right)$$

number of iterations, without requiring any a priori knowledge on $\kappa_r$, see Theorem 3 for more details. Here $r$ is an upper bound on $\sup_{k \geq 0} \|z^k\|$, which is finite as long as $\{\eta_k\}_{k \geq 0}$ is summable, see \cite{13}.

We apply AGPPA to solve the convex optimization model

$$\min_{x \in \mathbb{R}^n} f_0(x) + g(x) + h(Ax), \quad (4)$$

where $f_0$ is a smooth convex function, $A \in \mathbb{R}^{m \times n}$ and $g$ and $h$ are proper, closed and convex functions. We take the standard maximal monotone operator $T_\ell$ associated with the Lagrangian function of (4), see (40). Assuming that $T_\ell$ satisfies Assumption 1 we analyze the complexity of the resulting proximal method of multipliers (a.k.a. proximal ALM) with adaptive parameters. By merely requiring the inner solver to satisfy the so-called homogeneous objective decrease (HOOD) property (see Assumption 3), we deduce in Theorem 4 an
upper bound on the number of inner iterations for reaching (2). This upper bound directly yields a complexity bound for AGPPA with any first-order inner solver satisfying the HOOD property, including randomized methods, see Theorem 3. Our theoretical complexity bound continues to apply if an arbitrary inner solver is used, provided that it is carefully combined with a qualified first-order solver, see Section 4.3.

Examples of (3) with the associated maximal monotone operator $T_\ell$ satisfying Assumption 1 features linear-quadratic programming problems. Conditions on $f_0$, $g$, and $h$ so that $T_\ell$ satisfies Assumption 1 requires further study, which is out of the scope of this paper. We point to [41] for possible other relevant models. In this paper, we illustrate the application of AGPPA and its complexity results to large-scale LP problem. We show in Theorem 7 that the batch complexity (i.e., the number of pass over the data matrix $A$) of AGPPA to obtain an $\epsilon$-KKT solution is

$$O \left( \min \left( \max_{i \in [n]} \|a_i\|, \frac{\|A\|_F}{\sqrt{m}} \right) \frac{\theta r \ln(\theta r) \ln \left( \frac{\theta r \ln \frac{\theta r}{\epsilon}}{\epsilon} \right)}{\epsilon} \right),$$

where $a_i$ is the $i$th column vector of $A$, $r$ is an upper bound on the norm of all the iterates and $\theta$ is the constant satisfying (78), upper bounded by the Hoffman constant associated with the KKT system. In contrast, the best known batch complexity of linearized ADMM (LADMM) to obtain an $\epsilon$-KKT for solving LP is [41]:

$$O \left( \|A\|^2 \theta^2 (r')^2 \ln \frac{1}{\epsilon} \right),$$

where $r'$ is instead an upper bound on the norm of all the iterates of LADMM. The bound in (5) can be understood as an acceleration compared with the bound (6), from the dependence on the constant $\theta$. It is also easy to see that

$$\min \left( \max_{i \in [n]} \|a_i\|, \frac{\|A\|_F}{\sqrt{m}} \right) \leq \|A\|,$$

and can be much smaller if either $n$ or $m$ are large. In Section 5.3, we also provide a detailed comparison with other related solvers based on (proximal) ALM [25,40] or ADMM [41,39] on large-scale LP.

We test the practical performance of our method on different large-scale LP problems, using real and synthetic data. The number of variables $n$ and the number of constraints $m$ are both ranged from $10^5$ to $10^8$. We compare our algorithm with an ALM based method AL\_CD [40], an ADMM based solver SCS [32], and the Gurobi software [17], up to accuracy $10^{-3}$ and $10^{-5}$ for the normalized KKT residual defined as in (83). The results are shown in Table 5, Table 9, and Table 13 for memory usage, in Table 6, Table 10, and Table 14 for time efficiency up to accuracy $10^{-3}$ and in Table 7, Table 11, and Table 15 for time efficiency up to accuracy $10^{-5}$. Based on the experimental results, we see a consistently better performance of AGPPA both in memory usage and time efficiency. Besides, the numerical results also demonstrate that transforming
first into the standard form $(93)$ before applying AGPPA may significantly slow down the convergence. It shows the advantage of AGPPA compared with other closely related algorithms [25,39,32] limited to the standard form $(93)$.

We summarize below our contributions:

1. We develop a new technique guiding the update of proximal regularization parameters in PPA, so that the resulting method AGPPA enjoys a global linear convergence without requiring the knowledge of parameter $\kappa_r$.

2. We establish the iteration complexity bound $\tilde{O}(\log(1/\epsilon))$ of AGPPA combined with linearly convergent first-order inner solvers applied to convex optimization model $(4)$, under the condition that the associated $T_\ell$ satisfies Assumption 1.

3. We illustrate the application of AGPPA to LP problem and obtain an improved complexity bound compared with the up-to-date complexity of LADMM. We observe numerically a better performance of AGPPA, compared with other PPA based solvers and the Gurobi software, on large-scale problems.

The paper is organized as follows. In Section 2, we revisit IGPPA and present some convergence results for preparation. In Section 3, we introduce AGPPA and give an upper bound of the number of IGPPA steps. In Section 4, we apply AGPPA to convex optimization problem and show the overall iteration complexity bound. In Section 5, we apply our method and complexity results to LP problem. In Section 6, we present numerical results. In Section 7, we make some conclusions. Missing proofs can be found in the Appendix.

Notations. The set of self-adjoint positive definite linear operators over $X$ is denoted by $S^{++}$. For $M \in S^{++}$, $M^{-1} : X \to X$ denotes the inverse operator of $M$. For any $z, z' \in X$ and $M \in S^{++}$, denote $(z, z')_M = (z, Mz')$ and $\|z\|_M = \sqrt{(z, Mz)}$. For a closed set $D \subset X$, denote the weighted distance from $z$ to $D$ by $\text{dist}_M(z, D) = \min_{d \in D} \|d - z\|_M$. If $M$ is the identity operator $I$, we omit it from the subscript.

We use $\|\cdot\|$ to denote the standard Euclidean norm for vector and spectral norm for matrix. The set of $n$-by-$n$ positive definite matrices is denoted by $S^{++}_n$. For any $k > 0$, define $[k] := \{1, \ldots, k\}$. For any $x \in \mathbb{R}^n$ and $k \in [n]$, denote by $[x]^k_+$ the projection of $x$ into $\mathbb{R}^k_+ \times \mathbb{R}^{n-k}$. The same, $[x]^k_-$ means the projection of $x$ into $\mathbb{R}^k_- \times \mathbb{R}^{n-k}$. For any $k \geq 1$, $x_1 \in \mathbb{R}^{n_1}, \ldots, x_k \in \mathbb{R}^{n_k}$, we write $[x_1; \ldots; x_k]$ the vector in $\mathbb{R}^{n_1+\cdots+n_k}$ obtained by concatenating $x_1, \ldots, x_k$. Similarly, for any two matrices $A \in \mathbb{R}^{m_1 \times n}$ and $B \in \mathbb{R}^{m_2 \times n}$, $[A; B]$ is the matrix in $\mathbb{R}^{(m_1+m_2) \times n}$ obtained by concatenating $A$ and $B$.

2 Inexact Generalized PPA

In this section, we revisit the inexact generalized PPA. Let $M \in S^{++}$. Note that the operator $M^{-1}T : X \to X$ is a maximal monotone operator in the Hilbert space $X$ endowed with inner product $\langle \cdot, \cdot \rangle_M$. Consider the resolvent operator of $M^{-1}T$:

$$J_{\sigma M^{-1}T} := (I + \sigma M^{-1}T)^{-1},$$
with parameter $\sigma > 0$. Without loss of generality, we assume
$$\lambda_{\text{max}}(M) = 1. \quad (7)$$

An IGPPA step first approximately applies the resolvent operator $J_{\sigma M^{-1}}$ and then makes an affine combination with the current iterate for some relaxation factor $\gamma \in (0, 2)$. A more specific inexactness condition is described in the following procedure.

\[
z^+ = \text{IGPPAstep}(z, \sigma, \eta, \delta, \gamma, M)
\]

1. Compute an approximate solution $w \approx J_{\sigma M^{-1}}(z)$ such that
$$\|w - J_{\sigma M^{-1}}(z)\|_M \leq \min\{\eta, \delta \|w - z\|_M\} \quad (8)$$

2. Compute
$$z^+ = \gamma w + (1 - \gamma)z \quad (9)$$

3. Output $z^+$

The inexactness is controlled by (8) along with two error parameters $\eta$ and $\delta$. When relaxation factor $\gamma = 1$ and preconditioner $M = I$, the above procedure reduces to the classical inexact proximal point algorithm [36]. The general convergence result is established by Eckstein and Bertsekas [9, Theorem 3].

**Theorem 1** ([9]) Let \( \{z^k\}_{k \geq 0} \) be a sequence in $X$ such that
$$z^{k+1} = \text{IGPPAstep}(z^k, \sigma_k, \eta_k, \delta_k, \gamma, M), \quad \forall k \geq 0. \quad (10)$$
where \( \{\sigma_k\}_{k=0}^\infty, \{\eta_k\}_{k=0}^\infty, \{\delta_k\}_{k=0}^\infty \) are nonnegative sequences such that
$$\sum_{k=0}^\infty \eta_k < +\infty, \quad \inf_k \sigma_k > 0, \quad \sup_k \delta_k < 1.$$

Then for any $z^* \in \Omega$, we have
$$\|z^{k+1} - z^*\|_M \leq \|z^k - z^*\|_M + \gamma \eta_k, \quad \forall k \geq 0. \quad (11)$$

In addition, \( \{z^k\}_{k \geq 0} \) converges to a point $z^\infty \in \Omega$.

The fact that (11) holds for any $z^* \in \Omega$ implies
$$\|z^{k+1} - z^0\|_M \leq \|z^k - z^0\|_M + \gamma \eta_k, \quad \forall k \geq 0.$$
where $z^0$ is the projection of the initial point $z^0$ into solution set $\Omega$. It follows that the sequence \( \{z^k\}_{k \geq 0} \) generated by (10) satisfies
$$\sup_k \|z^k - z^0\| \leq \frac{1}{\lambda_{\text{min}}(M)} \left( \text{dist}_M(z^0, \Omega) + \gamma \sum_{k=0}^{\infty} \eta_k \right), \quad (12)$$

\[^1\] There is a slight redundancy in using both the parameter $\sigma$ and the preconditioner $M$. We could set the preconditioner as $M/\lambda_{\text{max}}(M)$ and the proximal regularization parameter as $\sigma \lambda_{\text{max}}(M)$ to have the same resolvent operator.
and hence
\[
\sup_k \|z^k\| \leq \|z^0\| + \frac{1}{\lambda_{\min}(M)} \left( \text{dist}_M(z^0, \Omega) + \gamma \sum_{k=0}^{\infty} \eta_k \right). \tag{13}
\]

**Remark 1** Eckstein and Bertsekas [9] allow the relaxation factor \(\gamma\) to vary with \(k\). They prove the same convergence results under the condition \(0 < \inf_k \gamma_k \leq \sup_k \gamma_k < 2\). For the sake of simplicity, we restrict our discussion to constant relaxation factor \(\gamma\). Similarly, referring to [25], we could also allow \(M\) to vary with \(k\) and all the results can be extended immediately if there is \(\lambda_l I \preceq M_{k+1} \preceq M_k \preceq \lambda_u I\) holds for any \(k \geq 0\).

### 2.1 Conditional linear convergence of IGPPA

In this subsection, we establish the linear convergence of IGPPA under Assumption 4. The linear convergence of the proximal point algorithm has been extensively studied in the literature under various assumptions for both exact and inexact versions. The linear convergence of the classical inexact PPA \((\gamma = 1 \text{ and } M = I)\) is established under the Lipschitz continuity assumption of \(T^{-1}\) at 0 by Rockafellar in [36] and extended to the local upper Lipschitz continuity case by Luque in [26]. Later it is shown by Leventhal [23] that the metric subregularity assumption, which is a weaker condition, guarantees a (local) linear convergence of the classical PPA. Tao and Yuan [38] computed the linear convergence rate of relaxed PPA \((M = I)\) under the Lipschitz continuity assumption of \(T^{-1}\) at 0. In the exact case, the rate is shown to be optimal only for \(\gamma \in [1, 2]\) and the optimal rate for \(\gamma \in (0, 1)\) was provided by Gu and Yang [15]. Li et al [25] established the linear convergence rate of the inexact preconditioned PPA \((\gamma = 1)\) under the following assumption: for any \(r > 0\), there exists \(\kappa_r > 0\) such that
\[
\text{dist}(z, \Omega) \leq \kappa_r \text{dist}(0, T(z)), \quad \forall z \text{ s.t. dist}(z, \Omega) \leq r. \tag{14}
\]

This assumption is stronger than Assumption 4 and weaker than Luque’s assumption (see [25, Lemma 2]). The linear convergence established in [25] for \(\gamma = 1\) actually also holds under Assumption 4. Furthermore, as we will show below, the results can be extended to \(\gamma \in (0, 2)\).

In the following, without further specification, \(\{z^k\}_{k \geq 0}\) denotes the sequence generated by (10). Let \(r > 0\) be any upper bound on the right-hand side of (13) and \(\kappa_r\) be the constant satisfying (3) in Assumption 4. Then for any sequence of proximal regularization parameters \(\{\sigma_k\}_{k \geq 0}\) and any sequence of error parameters \(\{\delta_k\}_{k \geq 0}\),
\[
\text{dist}(z^k, \Omega) \leq \kappa_r \text{dist}(0, T(z^k)), \quad \forall k \geq 0. \tag{15}
\]

We recall the following critical property for proving the linear convergence of IGPPA. It follows directly from [38, Lemma 5.3] by considering the maximal monotone operator \(M^{-1}T\) in the Hilbert space \(X\) with inner product \(\langle \cdot, \cdot \rangle_M\).
Lemma 1 (38) For any $k \geq 0$, we have

$$\text{dist}_M(J_{\sigma^k}M^{-1}T(z^k), \Omega) \leq \frac{\kappa_r}{\sqrt{\sigma_k^2 + \kappa_r^2}} \text{dist}_M(z^k, \Omega).$$

(16)

Based on Lemma 1, we obtain the following recursive inequality on the distance to the solution set.

Theorem 2 For any $k \geq 0$, we have

$$\text{dist}_M(z^{k+1}, \Omega) \leq \rho_k \text{dist}_M(z^k, \Omega).$$

with

$$\rho_k := \frac{1}{1 - \delta_k} \left( \sqrt{1 - \frac{\min\{\gamma, 2\gamma - \gamma^2\} \sigma_k^2}{\sigma_k^2 + \kappa_r^2}} + \delta_k \left( \frac{\min\{\gamma, 1\} \kappa_r}{\sqrt{\sigma_k^2 + \kappa_r^2}} + 1 \right) \right).$$

(17)

Remark 2 When $\gamma = 1$, the rate (17) reduces to the rate obtained in [25, Theorem 2].

Remark 3 In the exact case, i.e., when $\eta_k \equiv 0$ and $\delta_k \equiv 0$, the rate (17) reduces to the rate given in [38, Theorem 3.5], established for the generalized PPA under the Lipschitz continuity assumption of $T^{-1}$ at 0.

Note that the factor $\rho_k$ given in (17) depends explicitly on the error parameter $\delta_k$ and implicitly on the error parameter $\eta_k$ via $\kappa_r$. In addition, $\rho_k$ increases with $\delta_k$ and for any $\sigma_k > 0$, there is $\delta_k > 0$ such that $\rho_k < 1$ and IGPPA enjoys linear convergence. This corresponds to the commonly known fact that linear convergence is guaranteed if the subproblem $w \approx J_{\sigma^k}M^{-1}T(z)$ is solved with sufficiently high accuracy. On the other hand, when the subproblem accuracy $\delta_k$ is fixed in $[0, 1/2)$, we can also make $\rho_k < 1$ by choosing a sufficiently large parameter $\sigma_k$.

Hereinafter, for simplicity we take constant $\delta_k \equiv \delta \in [0, 1/2)$ for all $k \geq 0$. Let $\alpha > 0$ such that

$$\rho := \frac{1}{1 - \delta} \left( \sqrt{1 - \frac{\min\{\gamma, 2\gamma - \gamma^2\} \alpha^2}{\alpha^2 + 1}} + \delta \left( \frac{\min\{\gamma, 1\} \kappa_r}{\sqrt{\alpha^2 + 1}} + 1 \right) \right) < 1.$$ 

(18)

It is easy to see that for any $k \geq 0$, if $\sigma_k \geq \kappa_r \alpha$, then $\rho_k \leq \rho$. Hence we have the following corollary.

Corollary 1 If

$$\sigma_k \geq \kappa_r \alpha, \ \forall k \geq 0$$

(19)

then

$$\text{dist}_M(z^k, \Omega) \leq \rho^k \text{dist}_M(z^0, \Omega), \ \forall k \geq 0.$$
Corollary 1 establishes the conditional linear convergence of IGPPA. If \( \kappa_r \) is known, then we choose \( \{\sigma_k\}_{k \geq 0} \) satisfying (19) and the algorithm converges linearly with rate \( \rho \). However, in general \( \kappa_r \) is not known. Recall that for each iteration \( k \), one need to find an approximate solution \( w \approx \mathcal{J}_{\kappa_r}^{-1} \mathcal{M}^{-1} T(z) \). In principle, the larger \( \sigma_k \) is, the harder the inner problem is to solve \( \mathcal{J}_{\kappa_r}^{-1} \mathcal{M}^{-1} T(z) \).

We then have the following dilemma to deal with. On the one hand, we tend to choose very large parameters \( \{\sigma_k\}_{k \geq 0} \) so that (19) holds to guarantee linear convergence. On the other hand, we do not want excessively large \( \{\sigma_k\}_{k \geq 0} \) in order to control the inner problem complexity.

### 2.2 Verification of linear convergence

We shall rely on the following property which relates the unknown value \( \text{dist}_M(z^k, \Omega) \) with the computable value \( \|z^{k+1} - z^k\|_M \).

**Proposition 1** For any \( k \geq 0 \), we have
\[
\|z^{k+1} - z^k\|_M \leq \text{dist}_M(z^k, \Omega) \leq \frac{1 + \delta}{\gamma} \left( 1 - \sqrt{\frac{\kappa^2 r}{\sigma_k^2 + \kappa^2 r}} \right) \|z^{k+1} - z^k\|_M.
\]

**Corollary 2** If (19) holds, then
\[
\|z^{k+1} - z^k\|_M \leq C \rho^k \|z^1 - z^0\|_M, \quad \forall k \geq 0.
\] where
\[
C := \frac{1 + \delta}{(1 - \delta) \left( 1 - \sqrt{\frac{1}{\sigma^2 + 1}} \right)}.
\]

Corollary 2 is practically more interesting than Corollary 1 since (20) can always be verified at each iteration.

Let \( \mathcal{E} : \mathcal{X} \to \mathbb{R}^+ \) be a computable error residual function that we use to measure the approximation to \( \Omega \). We shall assume the existence of a constant \( \zeta > 0 \) such that
\[
\mathcal{E}(z) \leq \zeta \text{ dist}(z, \Omega), \quad \forall z \in \mathcal{X}.
\] (21)

If inequality (21) holds for all \( k \geq 0 \), then \( \{\|z^{k+1} - z^k\|_M\}_{k \geq 0} \) and consequently \( \{\mathcal{E}(z^k)\}_{k \geq 0} \) decreases linearly with rate \( \rho \).

**Proposition 2** If (20) holds, then \( \mathcal{E}(z^k) \leq \epsilon \) after
\[
k \geq \log_\frac{1}{\epsilon} \left( \frac{R(\sigma) \zeta \text{ dist}_M(z^0, \Omega)}{\lambda_{\min}(\mathcal{M}) \epsilon} \right)
\] number of IGPPA steps, where
\[
R(\sigma) := \frac{C (1 + \delta)}{(1 - \delta) \left( 1 - \sqrt{\frac{\kappa^2 r}{\sigma^2 + \kappa^2 r}} \right)}.
\] (22)

\footnote{For a concrete example, see (28).}
If instead (20) does not hold, then we know that (19) is false. We then get a certificate of \( \sigma_k \) being too small, which suggests us to increase the next parameter \( \sigma_{k+1} \). Note that Corollary 2 can be strengthened as follows.

**Corollary 3** If (19) holds, then
\[
\| z^{k+1} - z^k \|_M \leq C \min_{0 \leq j \leq k} \rho^{k-j} \| z^{j+1} - z^j \|_M, \quad \forall k \geq 0.
\]

### 3 Adaptive Generalized PPA (AGPPA)

In this section, we apply the results in the previous section to adaptively choose the proximal regularization parameters. We propose a double loop algorithm with \( s \) and \( t \) being respectively the number of outer and inner iterations. Given an accuracy parameter \( \epsilon > 0 \), the objective is to find a solution \( z^\circ \in X \) such that \( \mathcal{E}(z) \leq \epsilon \). Choose some \( \eta_0 > 0, \varsigma > 1, \varrho_\eta \in (0, 1) \) and define the sequence \( \{\eta_{s,t}\}_{s \geq 0, t \geq 0} \) as
\[
\eta_{0,0} = \eta_0, \eta_{s+1,0} = \eta_{s,0} \varrho_0, \quad \eta_{s,t} = \eta_{s,0} (1 + t)^{-\varsigma}, \quad \forall s \geq 0, t \geq 0.
\]
Choose some \( \sigma_0 > 0 \) and \( \rho_\sigma > 1 \) and define the sequence \( \{\sigma_s\}_{s \geq 0} \) as
\[
\sigma_{s+1} = \sigma_s \rho_\sigma, \quad \forall s \geq 0.
\]
We generate the sequence \( \{z^{s,t}\}_{s \geq 0, t \geq 0} \) from an arbitrary initial point \( z^{0,0} \) by
\[
\begin{cases}
    z^{s,t+1} = \text{IGPPAstep}(z^{s,t}, \eta_{s,t}, \sigma_s, \delta, \gamma, \mathcal{M}), \quad \forall s \geq 0, N_s \geq t \geq 0 \\
    z^{s+1,0} = \arg \min \{ \mathcal{E}(z) : z \in \{z^{s,0}, \ldots, z^{s,N_s+1}\} \}
\end{cases}
\]
where \( N_s \) is the smallest \( t \) such that either
\[
\mathcal{E}(z^{s,t}) \leq \epsilon,
\]
or
\[
\| z^{s,t+1} - z^{s,t} \|_M > C \min_{0 \leq j \leq t} \{ \rho^{t-j} \| z^{s,j+1} - z^{s,j} \|_M \}.
\]
At each outer iteration \( s \), we run IGPPA with parameter \( \sigma_s \) until either 26 or 27 holds. The parameters \( \{\sigma_s\}_{s \geq 0} \) is increased by a fixed factor \( \rho_\sigma \) at the end of each outer iteration, or equivalently, when either 26 or 27 holds. Each outer iteration starts from an iterate which minimize the error residual function \( \mathcal{E} \) among all the past iterates. We call the algorithm 25 Adaptive Generalized Proximal Point Algorithm (AGPPA). An equivalent description of AGPPA is given in Algorithm 1.

Algorithm 1 takes two inputs: an accuracy parameter \( \epsilon > 0 \) and an initial point \( z_0 \in X \). It terminates when an approximate solution \( z_\circ \in X \) such that \( \mathcal{E}(z_\circ) \leq \epsilon \) is found. The output reports the solution \( z_\circ \), as well as the total number of IGPPA steps \( N_\circ \), the last parameter \( \sigma_\circ \), the last error parameter \( \eta_\circ \).
**Algorithm 1 AGPPA**

**Input:** $0 > \epsilon, z^0 \in X$

**Parameters:** $\mathcal{M} \in \mathcal{S}^{++}, \eta_0 > 0, \varsigma > 1, \varrho_\eta \in (0, 1), \sigma_0 > 0, \varrho_\sigma > 1, \alpha > 0, \delta \in [0, 1/2)$, $\gamma \in (0, 2)$

**Initialize:** $z^{0,0} = z^0, \eta_{0,0} = \eta_0, s = 0$

1. **while** $\mathcal{E}(z^{s,0}) > \epsilon$ **do**
2. \hspace{0.5cm} $t = t + 1$
3. \hspace{0.5cm} **repeat**
4. \hspace{1cm} $t = t + 1$
5. \hspace{1cm} $\eta_{s,t} = \eta_{s,0}(1 + t)^{-\varsigma}$
6. \hspace{1cm} $z^{s,t+1} = \text{IGPPAstep}(z^{s,t}, \sigma_s, \eta_{s,t}, \delta, \gamma, \mathcal{M})$
7. **until** $\mathcal{E}(z^{s,t}) \leq \epsilon$ or $\|z^{s,t+1} - z^{s,t}\|_\mathcal{M} > C \min_{0 \leq j \leq t} \{\rho^{t-j}\|z^{s,j+1} - z^{s,j}\|_\mathcal{M}\}$
8. \hspace{0.5cm} $N_s = t$
9. \hspace{0.5cm} $z^{s+1,0} = \arg \min \{\mathcal{E}(z) : z \in \{z^{s,0}, \ldots, z^{s,N_s+1}\}\}$
10. \hspace{0.5cm} $\sigma_{s+1} = \sigma_s \varrho_\sigma$
11. \hspace{0.5cm} $\eta_{s+1,0} = \eta_{s,0} \varrho_\eta$
12. \hspace{0.5cm} $s = s + 1$
13. **end while**
14. $z_s = z^{s,0}$
15. **if** $s == 0$ **then**
16. \hspace{0.5cm} $s_0 = 0$
17. \hspace{0.5cm} $N_0 = 0$
18. **else**
19. \hspace{0.5cm} $s_0 = s - 1$
20. \hspace{0.5cm} $N_0 = \sum_{s=0}^{s_0}(N_s + 1)$
21. **end if**
22. $s_\circ = s_0$
23. $\eta_\circ = \eta_{s_0, N_s}$

**Output:** $(z_\circ, N_\circ, \sigma_\circ, \eta_\circ, s_\circ)$

and the total number of outer iterations $s_0$. Note that $\rho$ in Line 7 of Algorithm 1 is defined in [13] using the parameters.

Next we show that without any knowledge on the constant $\kappa_r$ in the error bound condition [15], the total number of IGPPA steps $N_\circ$ required before finding a solution $z$ such that $\mathcal{E}(z) \leq \epsilon$ satisfies

$$N_\circ \leq O \left( \ln \kappa_r \ln \left( \frac{\kappa_r \alpha \sigma_0}{\epsilon} \right) \right).$$

**Theorem 3** Suppose that the parameters required in Algorithm 4 are chosen such that $\rho$ defined in [15] is strictly less than 1. For any initial point $z^0$, there is a constant $\kappa_r > 0$ such that for any $\epsilon > 0$, Algorithm 4 terminates with output $(z_\circ, N_\circ, \sigma_\circ, \eta_\circ, s_\circ)$ satisfying

$$\mathcal{E}(z_\circ) \leq \epsilon,$$

$$s_0 \leq \bar{s} := \left\lceil \max \left( \log_{\varrho_\sigma} \left( \frac{\kappa_r \alpha}{\sigma_0} \right), 0 \right) \right\rceil,$$

$$\sigma_\circ \leq \bar{\sigma} := \kappa_r \varrho_\sigma.$$
\[
N_0 \leq \bar{N} := s \left[ \max \left( \log_2 \left( \frac{R}{\epsilon} \right), 0 \right) + 1 \right],
\]

\[
\eta := \bar{\eta} \left[ \max \left( \log_2 \left( \frac{R}{\epsilon} \right), 0 \right) + 1 \right]^{-s},
\]

where
\[
R := \frac{(1 + \delta)^2 \zeta r}{(1 - \delta^2 \left( 1 - \sqrt{\frac{\alpha}{\alpha + 1}} \right) \left( 1 - \sqrt{\frac{\kappa^2}{\kappa^2 + \kappa^2}} \right) \lambda_{\min}(\mathcal{M})}.
\]

with \( r \) given by
\[
r = \left\| \bar{z} \right\| + \frac{1}{\lambda_{\min}(\mathcal{M})} \left( \text{dist}_{\mathcal{M}}(z^0, \Omega) + \gamma \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \eta_{s,t} \right).
\]

Proof It is easy to verify that \( \{ \eta_{s,t} \}_{s \geq 0, t \geq 0} \) defined by (23) is summable:
\[
\sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \eta_{s,t} \leq \frac{\kappa \gamma \sigma_0}{(\zeta - 1)(1 - \bar{\eta})}.
\]

Therefore the sequence \( \{ z^{s,t} \}_{s \geq 0, t \geq 0} \) generated by AGPPA (23) falls into the general framework as described in Theorem 1 and for any \( s \geq 0, 0 \leq t \leq N_s + 1, \)
\[
\left\| z^{s,t} \right\| \leq \left\| z^{0,0} \right\| + \frac{1}{\lambda_{\min}(\mathcal{M})} \left( \text{dist}_{\mathcal{M}}(z^{0,0}, \Omega) + \gamma \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \eta_{s,t} \right) \leq r.
\]

Similar to (15), it follows from Assumption 1 that there is \( \kappa_r > 0 \) such that for any \( s \geq 0, 0 \leq t \leq N_s + 1, \)
\[
\text{dist}(z^{s,t}, \Omega) \leq \kappa_r \text{dist}(0, T(z^{s,t})).
\]

We then know from Proposition 2 that for any \( s \geq 0, \)
\[
N_s \leq \left[ \log_2 \left( \frac{R(\sigma_s)\zeta \text{dist}_{\mathcal{M}}(z^{s,0}, \Omega)}{\lambda_{\min}(\mathcal{M})\epsilon} \right) \right] \leq \left[ \log_2 \left( \frac{R(\sigma_0)\zeta r}{\lambda_{\min}(\mathcal{M})\epsilon} \right) \right].
\]

Here the second inequality used (12), (7) and the fact that \( R(\sigma_s) \leq R(\sigma_0). \)

Plugging in (36) the definition (22), we obtain the following bound on the number of inner iterations: for any \( s \geq 0 \)
\[
N_s \leq \left[ \max \left( \log_2 \left( \frac{R}{\epsilon} \right), 0 \right) \right],
\]

with \( R \) being defined in (33). Besides, in view of (24), once the outer iteration counter \( s \) satisfies:
\[
s \geq \left[ \max \left( \log_{\bar{\kappa}_0} \left( \frac{\kappa_r \sigma}{\sigma_0} \right), 0 \right) \right],
\]
we will have
\[ \sigma_s \geq \kappa_r \alpha. \]

By Corollary 3, condition (27) will not occur when \[ \sigma_s \geq \kappa_r \alpha \] and the algorithm terminates at this outer iteration. Consequently, we obtain (29) and hence (30). To obtain (32), it suffices to note that
\[ \eta = \eta_0 (1 + \kappa_r \alpha)^{-\varsigma} \geq \eta_0 \theta^\delta (1 + \bar{N})^{-\varsigma}. \]

\textbf{Remark 4} Note that the output \((z_0, N_0, \sigma, \eta, s_0)\) of Algorithm 1 are random variables. Indeed, as we shall see below, the algorithm to solve the IGPPA step may be random and in this case, all the sequences produced of Algorithm 1 are random. However, their bounds \((\bar{N}, \bar{\sigma}, \bar{\eta}, \bar{s})\) given in Theorem 3 are all deterministic.

In the definitions (29), (30), (32) and (31), the constants \(\rho, \sigma_0, \bar{\sigma}, \alpha, \gamma, \delta, \eta_0, \varsigma, \varrho_0 \) and \(\lambda_{\min}(\mathcal{M})\) are all user defined parameters. The constant \(\kappa_r\) is such that (3) holds with \(r\) defined in (34). In the following, to get a better understanding on the complexity, we ignore the user defined constants to extract out of (29), (30), (32) and (31) the dependence on \(r, \kappa_r, \zeta, \) and \(\epsilon\). In particular, we consider user defined constants \(\rho, \sigma_0, \varrho, \alpha, \gamma, \delta, \eta_0, \varsigma, \varrho_0 \) and \(\lambda_{\min}(\mathcal{M})\) as problem independent constants.

\textbf{Corollary 4} We have:
\[ \bar{s} = O \left( \ln \kappa_r \right) \]
\[ \bar{\sigma} = O \left( \kappa_r \right) \]
\[ - \ln \bar{\eta} = O \left( \ln \kappa_r + \ln \ln \left( \frac{\zeta \kappa_r}{\epsilon} \right) \right) \]
\[ \bar{N} = O \left( \ln \kappa_r \ln \left( \frac{\zeta \kappa_r}{\epsilon} \right) \right), \]

where the big O hides problem independent constants.

\section{Iteration complexity of proximal method of multipliers}

In this section we apply AGPPA to solve the following convex optimization problem:
\[ \min_{x \in \mathbb{R}^n} f_0(x) + g(x) + h(Ax). \]  
(38)

Here, \(A \in \mathbb{R}^{m \times n}, f_0 : \mathbb{R}^n \to \mathbb{R}\) is a convex and differentiable function, and the functions \(g, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) are proper, closed and convex. In addition, we assume that \(h\) is a simple function, in the sense that its proximal operator can be easily exactly computed. We write down the Lagrangian function as follows:
\[ \ell(x, \lambda) := f_0(x) + g(x) + \langle Ax, \lambda \rangle - h^*(\lambda), \quad \forall x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m, \]  
(39)
where $h^*$ denotes the Fenchel conjugate function of $h$. Let $X := \mathbb{R}^{n+m}$ and define the multivalued mapping $T_\ell : X \rightrightarrows X$ associated with the convex-concave function $\ell$ by

$$
T_\ell(x, \lambda) := \{(v, u) \in \mathbb{R}^{n+m} : (v, -u) \in \partial \ell(x, \lambda)\}. \tag{40}
$$

It is known that $T_\ell$ is a maximal monotone operator and the set $T_\ell^{-1}(0, 0)$ is the set of saddle points of the Lagrangian [39] which then yields primal and dual optimal solutions of the optimization problem [38], see [4, Theorem 3.4.1].

To apply AGPPA, we shall make the following assumptions on problem (35).

**Assumption 2**
1. $\Omega := T_\ell^{-1}(0, 0)$ is nonempty.
2. The operator $T_\ell$ satisfies the error bound condition as given in Assumption [4].
3. There is a computable residual function $E : X \to \mathbb{R}_+$ satisfying (21) for some $\zeta > 0$.

If $f_0, g, h$ are piecewise linear-quadratic, then $T_\ell$ is a polyhedral multifunction [37, Theorem 11.14, Proposition 12.30]. Thus $T_\ell^{-1}$ is locally upper Lipschitz continuous at $0$ [41, Proposition 1] and it follows that $T_\ell$ satisfies Assumption [12, Lemma 2]. Therefore, our algorithm is applicable to a wide range of convex optimization problem, such as well-known linear programming problem, quadratic programming problem, LASSO problem and so on. However, we highly expect that our method AGPPA can be applied to solve more examples of convex programming other than piecewise linear-quadratic programs. Verifying the error bound condition (3) for the operator $T_\ell$ associated with problem (35) is out of the scope of this paper, but we point out that the techniques in [41] seem to shed light on this challenging task.

Under Assumption 2, Algorithm 1 can be applied to find a solution in $\Omega$. In the remaining of this section, we concretize the IGPPA step for $T_\ell$ and study the total complexity of AGPPA applied to find an approximate solution of $\Omega$.

### 4.1 Implementation of IGPPA step

Let $M^x \in S_n^{++}$, $M^\lambda \in S_m^{++}$ and

$$
M = \begin{pmatrix} M^x & 0 \\ 0 & M^\lambda \end{pmatrix} \in S_{n+m}^{++}.
$$

Let $\eta > 0$ and $\delta > 0$. We fix the base points $\bar{x} \in \mathbb{R}^n$ and $\bar{\lambda} \in \mathbb{R}^m$ and present an implementable form of the IGPPA step for the maximal monotone operator $T_\ell$. Recall that the IGPPA step amounts to compute $\hat{x} \in \mathbb{R}^n$ and $\hat{\lambda} \in \mathbb{R}^m$ such that

$$
\left\| (\hat{x}, \hat{\lambda}) - J_{M^{-1}}(\bar{x}, \bar{\lambda}) \right\|_M \leq \min \left\{ \eta, \delta \right\} \left\| (\hat{x}, \hat{\lambda}) - (\bar{x}, \bar{\lambda}) \right\|_M \Big\}, \tag{41}
$$
Note that (41) can not be verified. Let
\[ \psi(u, \bar{\lambda}, \sigma) := \max_{\lambda \in \mathbb{R}^m} \left\{ \langle u, \lambda \rangle - h^*(\lambda) - \frac{1}{2\sigma} \| \lambda - \bar{\lambda} \|_{M^\perp}^2 \right\}. \] (42)

Recall that \( \psi(\cdot, \bar{\lambda}, \sigma) : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is known as a smoothing approximation of the possible nonsmooth function \( h \), see [28]. Further, let
\[ \Lambda(x, \bar{\lambda}, \sigma) := \arg \max_{\lambda \in \mathbb{R}^m} \left\{ \ell(x, \lambda) - \frac{1}{2\sigma} \| \lambda - \bar{\lambda} \|_{M^{\perp}}^2 \right\}. \]

Define
\[ F(x) := \max_{\lambda \in \mathbb{R}^m} \left\{ \ell(x, \lambda) - \frac{1}{2\sigma} \| \lambda - \bar{\lambda} \|_{M^\perp}^2 \right\} + \frac{1}{2\sigma} \| x - \bar{x} \|_{M^\perp}^2 \] \[ = f_0(x) + g(x) + \psi(Ax - b, \bar{\lambda}, \sigma) + \frac{1}{2\sigma} \| x - \bar{x} \|_{M^\perp}^2. \] (43)

The function \( F \) is strongly convex and has a unique minimizer \( x^* \). It is known that \( 35 \)
\[ (x^*, \Lambda(x^*, \bar{\lambda}, \sigma)) = J_{\sigma M^{-1}T_\ell}(\bar{x}, \bar{\lambda}), \] (44)
and thus the computation of an inexact minimizer of \( F \) yields an inexact solution of the resolvent operator. The following proposition is an immediate generalization of [35, Proposition 8].

**Proposition 3** ([35]) For any \( \bar{x} \in \text{dom}(F) \), we have
\[ \| (\bar{x}, \Lambda(\bar{x}, \bar{\lambda}, \sigma)) - J_{\sigma M^{-1}T_\ell}(\bar{x}, \bar{\lambda}) \|_M \leq \frac{\sigma}{\sqrt{\lambda_{\min}(M^\perp)}} \text{dist}(0, \partial F(\bar{x})). \]

We obtain directly from Proposition 3 the following verifiable condition which ensures (41):
\[ \text{dist}(0, \partial F(\bar{x})) \leq \frac{\sqrt{\lambda_{\min}(M^\perp)}}{\sigma} \min \left\{ \eta, \delta \| (\bar{x}, \Lambda(\bar{x}, \bar{\lambda}, \sigma)) - (\bar{x}, \bar{\lambda}) \|_M \right\}. \] (45)

Hence the IGPPA step is reduced to find an approximate solution of the following minimization problem:
\[ F^* = \min_{x \in \mathbb{R}^n} F(x) \] (46)
4.2 Iteration complexity of IGPPA step

In this subsection, we investigate the complexity of finding an approximate minimizer of (46) so that the computable stopping criterion (45) is satisfied. In principle, any convergent algorithm can be used for finding such $\hat{x}$ because (45) is a directly verifiable condition. There are many algorithms which have high efficiency in practice to find $\hat{x}$ satisfying (45). For example, second order methods have local superlinear convergence property and if the starting point is sufficiently close to the optimal solution, then the convergence is very fast. However, locally convergent property brings inherent difficulty in the complexity analysis. Our analysis is built on the existence of a globally convergent minimizer of $F$ which satisfies the homogeneous objective decrease (HOOD) property \[2\]. More precisely, we require the following assumption.

**Assumption 3** There is an algorithm $A_F : \text{dom}(g) \rightarrow \text{dom}(g)$ which in a fixed number of operations returns an output satisfying:

$$
\mathbb{E}[F(A_F(x)) - F^*] \leq e^{-1} (F(x) - F^*), \quad \forall x \in \text{dom}(g).
$$

(47)

We also require the following additional assumption.

**Assumption 4** The gradient $\nabla f_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is $L_0$-Lipschitz continuous.

We decompose the function $F$ defined in (43) into two parts:

$$F(x) = f(x) + \phi(x),$$

(48)

where

$$f(x) := f_0(x) + \psi(Ax - b, \bar{\lambda}, \sigma), \quad \phi(x) := g(x) + \frac{1}{2\sigma} \|x - \bar{x}\|_{M^x}^2.$$

Then the gradient $\nabla f$ is $L$-Lipschitz continuous with

$$L := L_0 + \frac{\sigma}{\lambda_{\min}(M^x)} \|A\|^2.$$

This follows from the fact that $\psi(\cdot, \bar{\lambda}, \sigma)$ is differentiable and the gradient is $\sigma/\lambda_{\min}(M^x)$-Lipschitz continuous \[28\] Theorem 1.

Define the proximal gradient operator

$$G_F(x) := \arg \min_{y \in \mathbb{R}^n} \left\{ \frac{L}{2} \|y - \left( x - \frac{1}{L} \nabla f(x) \right) \|_2^2 + \phi(y) \right\}, \quad \forall x \in \mathbb{R}^n.$$

(49)

We propose Algorithm 2 for finding $\hat{x}$ satisfying (45).

Note that (47) only holds in expectation and thus in general we cannot guarantee $F(A_F(y^k)) \leq F(y^k)$. To ensure monotone decrease, we always compare $F(A_F(y^k))$ with $F(y^k)$ and let $y^{k+1}$ be $y^k$ if $F(A_F(y^k)) > F(y^k)$. In Algorithm 2 we also need to apply the operator $G_F$ at each iteration. The motivation comes from the following result.
Algorithm 2

Initialize: $x^0 = \mathcal{G}_F(\bar{x})$, $k = 0$

1: repeat
2: \[ y^k = x^k \]
3: \[ y^{k+1} \leftarrow \mathcal{A}_F(y^k) \]
4: if $F(y^{k+1}) > F(y^k)$ then
5: \[ y^{k+1} = y^k \]
6: end if
7: \[ x^{k+1} = \mathcal{G}_F(y^{k+1}) \]
8: \[ k = k + 1 \]
9: until $\text{dist}(0, \partial F(x^k)) \leq \sqrt{\lambda_{\min}(\mathcal{M}^x)} \min \{ \eta, \delta \| (x^k, A(x^k, \bar{\lambda}, \sigma)) - (\bar{x}, \bar{\lambda}) \|_{\mathcal{M}} \}$.
10: $\bar{x} = x^k$

Output: $\bar{x}$

Proposition 4 Let $y \in \mathbb{R}^n$ be a vector satisfying

\[
F(y) - F^* \leq \frac{\lambda_{\min}(\mathcal{M}^x)}{\sigma^2} \min \left\{ \eta^2, \frac{\delta^2 \lambda_{\min}(\mathcal{M})}{4L^2} \left( F(\mathcal{G}_F(\bar{x})) - F^* \right) \right\}.
\]  

(50)

Then the stopping criteria (15) holds with $\bar{x} = \mathcal{G}_F(y)$.

Proof By (14), for any $(x, \lambda)$ we have,

\[
\| x - x^* \| \leq \| (x, \lambda) - (x^*, A(x^*, \bar{\lambda}, \sigma)) \| = \| (x, \lambda) - J_{\sigma M^{-1} T \ell}(\bar{x}, \bar{\lambda}) \|. 
\]

(51)

We recall two properties about the proximal gradient operator (20) (2.13), (2.21):\[
\frac{L}{2} \| \mathcal{G}_F(y) - y \|^2 \leq F(y) - F^*, \]

(52)

\[
F(\mathcal{G}_F(\bar{x})) - F^* \leq \frac{L}{2} \| \bar{x} - x^* \|^2 \leq \frac{L}{2} \| (\bar{x}, \bar{\lambda}) - J_{\sigma M^{-1} T \ell}(\bar{x}, \bar{\lambda}) \|^2.
\]

(53)

Note that we have $0 \in \nabla f(y) + L(\mathcal{G}_F(y) - y) + \partial \phi(\mathcal{G}_F(y))$ with (16), thus

\[
\nabla f(\mathcal{G}_F(y)) - \nabla f(y) - L(\mathcal{G}_F(y) - y) \in \partial F(\mathcal{G}_F(y)).
\]

Then we have

\[
\text{dist}(0, \partial F(\mathcal{G}_F(y)))^2 \leq \| \nabla f(\mathcal{G}_F(y)) - \nabla f(y) - L(\mathcal{G}_F(y) - y) \|^2 \\
\leq 2L^2 \| \mathcal{G}_F(y) - y \|^2 \leq 4L(F(y) - F^*) \]

(50)

\[
\leq \frac{\lambda_{\min}(\mathcal{M}^x)}{\sigma^2} \min \left\{ \eta^2, \frac{\delta^2 \lambda_{\min}(\mathcal{M})}{(1 + \delta)^2 L^2} \left( F(\mathcal{G}_F(\bar{x})) - F^* \right) \right\}.
\]

(50)

\[
\leq \frac{\lambda_{\min}(\mathcal{M}^x)}{\sigma^2} \min \left\{ \eta^2, \frac{\delta^2 \lambda_{\min}(\mathcal{M})}{(1 + \delta)^2} \| (\bar{x}, \bar{\lambda}) - J_{\sigma M^{-1} T \ell}(\bar{x}, \bar{\lambda}) \|^2 \right\}
\]

\[
\leq \frac{\lambda_{\min}(\mathcal{M}^x)}{\sigma^2} \min \left\{ \eta^2, \frac{\delta^2}{(1 + \delta)^2} \| (\bar{x}, \bar{\lambda}) - J_{\sigma M^{-1} T \ell}(\bar{x}, \bar{\lambda}) \|^2_{\mathcal{M}} \right\},
\]

\[
\frac{L}{2} \| \mathcal{G}_F(y) - y \|^2 \leq F(y) - F^*, \]

(51)
and therefore,
\[
\frac{\sigma \text{dist}(0, \partial F(\tilde{x}))}{\sqrt{\lambda_{\min}(M^2)}} \leq \min \left\{ \eta, \frac{\delta}{1 + \delta} \left\| (\tilde{x}, \check{\lambda}) - J_{\sigma, M^{-1}, T_i}(\tilde{x}, \check{\lambda}) \right\|_M \right\}. \tag{54}
\]

In view of Proposition 3, we have
\[
\left\| (\tilde{x}, A(\tilde{x}, \bar{x}, \sigma)) - J_{\sigma, M^{-1}, T_i}(\tilde{x}, \check{\lambda}) \right\|_M \leq \frac{\sigma \text{dist}(0, \partial F(\tilde{x}))}{\sqrt{\lambda_{\min}(M^2)}} \leq \min \left\{ \eta, \frac{\delta}{1 + \delta} \left\| (\tilde{x}, \check{\lambda}) - J_{\sigma, M^{-1}, T_i}(\tilde{x}, \check{\lambda}) \right\|_M \right\}.
\]

It follows that
\[
\left\| (\tilde{x}, A(\tilde{x}, \bar{x}, \sigma)) - J_{\sigma, M^{-1}, T_i}(\tilde{x}, \check{\lambda}) \right\|_M \leq \frac{\delta}{1 + \delta} \left( \left\| (\tilde{x}, A(\tilde{x}, \bar{x}, \sigma)) - (\tilde{x}, \check{\lambda}) \right\|_M + \left\| (\tilde{x}, A(\tilde{x}, \bar{x}, \sigma)) - J_{\sigma, M^{-1}, T_i}(\tilde{x}, \check{\lambda}) \right\|_M \right),
\]

and thus
\[
\left\| (\tilde{x}, A(\tilde{x}, \bar{x}, \sigma)) - J_{\sigma, M^{-1}, T_i}(\tilde{x}, \check{\lambda}) \right\|_M \leq \frac{\delta}{1 + \delta} \left\| (\tilde{x}, A(\tilde{x}, \bar{x}, \sigma)) - (\tilde{x}, \check{\lambda}) \right\|_M.
\]

Consequently,
\[
\left\| (\tilde{x}, \check{\lambda}) - J_{\sigma, M^{-1}, T_i}(\tilde{x}, \check{\lambda}) \right\|_M \leq (1 + \delta) \left\| (\tilde{x}, A(\tilde{x}, \bar{x}, \sigma)) - (\tilde{x}, \check{\lambda}) \right\|_M. \tag{55}
\]

Plugging (55) into (54), we get
\[
\text{dist}(0, \partial F(\tilde{x})) \leq \frac{\sqrt{\lambda_{\min}(M^2)}}{\sigma} \min \left\{ \eta, \frac{\delta}{1 + \delta} \left\| (\tilde{x}, A(\tilde{x}, \bar{x}, \sigma)) - (\tilde{x}, \check{\lambda}) \right\|_M \right\}.
\]

Now we give an upper bound on the number of iterations before Algorithm 2 terminates.

**Theorem 4** Let Assumption 3 and Assumption 4 hold. For any \(0 < p < 1\), denote
\[
K(p) := \left\lceil \max \left( \ln \left( \frac{\zeta_1 \left( \frac{L \sigma \text{dist}(0, \partial F(\tilde{x}), \Omega)}{\eta} \right)}{\eta} \right), \ln \left( \zeta_2 (L \sigma) \right), 0 \right) \right\rceil. \tag{56}
\]

with functions \(\zeta_1, \zeta_2 : \mathbb{R}^+ \to \mathbb{R}^+\) defined as:
\[
\zeta_1(q) := \frac{2q^2}{p \lambda_{\min}(M^2)}, \quad \zeta_2(q) := \frac{2(1 + \delta)^2 q^2}{p \delta^2 \lambda_{\min}(M^2) \lambda_{\min}(M)}.
\]

Then, with probability at least \(1 - p\), Algorithm 3 terminates within \(K(p)\) iterations.
Proof Since \( F(G_F(y^{k+1})) \leq F(y^{k+1}) \) for any \( k \geq 0 \), it is easy to see that
\[
\mathbb{E}[F(x^k) - F^*] \leq e^{-k}(F(x^0) - F^*) = e^{-k}(F(G_F(\bar{x})) - F^*), \quad \forall k \geq 0.
\]
If
\[
k \geq \ln(\zeta_2(L\sigma)),
\]
then
\[
\mathbb{E}[F(x^k) - F^*] \leq \frac{1}{\zeta_2(L\sigma)}(F(G_F(\bar{x})) - F^*).
\]
(57)
We apply (53) to obtain
\[
\mathbb{E}[F(x^k) - F^*] \leq \frac{L}{2e^k}\|((\bar{x}, \lambda) - \mathcal{J}_{\sigma M^{-1} T}(\bar{x}, \lambda))\|^2, \quad \forall k \geq 0.
\]
By [36, Proposition 1], we have
\[
\|((\bar{x}, \lambda) - \mathcal{J}_{\sigma M^{-1} T}(\bar{x}, \lambda))\| \leq \text{dist}((\bar{x}, \lambda), \Omega).
\]
Therefore,
\[
\mathbb{E}[F(x^k) - F^*] \leq \frac{L}{2e^k}\text{dist}^2((\bar{x}, \lambda), \Omega), \quad \forall k \geq 0.
\]
It follows that if
\[
k \geq \ln \left( \zeta_1 \left( \frac{L\sigma \text{dist}((\bar{x}, \lambda), \Omega)}{\eta} \right) \right) = \ln \frac{2L^2\sigma^2 \text{dist}^2((\bar{x}, \lambda), \Omega)}{\eta^2\sigma^2 L},
\]
then
\[
\mathbb{E}[F(x^k) - F^*] \leq \frac{\eta^2\lambda_{\min}(\mathcal{M}^z)}{4\sigma^2 L}.
\]
(58)
From (57) and (58), we know that for any \( k \geq K(p) \),
\[
\mathbb{E}[F(x^k) - F^*] \leq \min \left( \frac{\eta^2\lambda_{\min}(\mathcal{M}^z)}{4\sigma^2 L}, \frac{1}{\zeta_2(L\sigma)}(F(G_F(\bar{x})) - F^*) \right).
\]
Using Markov’s inequality we deduce that for any \( k \geq K(p) \), with probability at least \( 1 - p \), we have
\[
F(x^k) - F^* \leq \min \left( \frac{\eta^2\lambda_{\min}(\mathcal{M}^z)}{4\sigma^2 L}, \frac{\delta^2\lambda_{\min}(\mathcal{M})\lambda_{\min}(\mathcal{M})}{2(1 + \delta)^2 L^2\sigma^2} (F(G_F(\bar{x})) - F^*) \right).
\]
Now it suffices to apply Proposition 4 and the result is proved.

Remark 5 At each iteration of Algorithm 2 it is required to run the following operations:
1. Computation of \( \mathcal{A}_F(y^k) \);
2. Computation of \( F(y^{k+1}) \);
3. Computation of \( G_F(y^{k+1}) \);
4. Computation of dist\((0, \partial F(x^k))\);
5. Computation of \( \lambda(x^k, \bar{x}, \sigma) \) and of \( \|((x^k, \lambda(x^k, \bar{x}, \sigma)) - (\bar{x}, \lambda))\|_{\mathcal{M}^z} \)
The per-iteration cost of Algorithm 2 depends on the function $F$ defined in (48), the condition number of which increases with the parameter $\sigma$. It is reasonable to assume that the per-iteration cost can be upper bounded by $T(\sigma)$, for some increasing function $T : \mathbb{R}^+ \rightarrow \mathbb{R}^+$.  

**Theorem 5** Let Assumption 3 and Assumption 4 hold and $T : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an increasing function such that the per-iteration cost of Algorithm 2 is bounded by $T(\sigma)$. Then with probability at least $1 - p$, the complexity of Algorithm 2 is bounded by 

$$O(K(p)T(\sigma)), \quad (59)$$

with $K(p)$ being defined in Theorem 4.

### 4.3 Hybrid inner solver

Algorithm $A_F$ satisfying the HOOD property (47) must be first-order method and may suffer from slow convergence for ill-conditioned problems, especially when the proximal regularization parameter $\sigma$ becomes large. We next discuss how to bridge the gap between practical performance and complexity analysis by combining Algorithm 2 with any numerically efficient algorithm for the minimization problem (46).

Let $Q_F$ be any algorithm which is likely to be more efficient than Algorithm 2 for solving (46). When extra computation resource is available, we can run Algorithm 2 in parallel with $Q_F$ and stop when whichever first returns a solution. In this way, the iteration complexity is always upper bounded by (59) but the actual running time is smaller than both that of $Q_F$ and of Algorithm 2. When there is no extra computation resource, we can first run $Q_F$ and switch to Algorithm 2 if no solution satisfying (45) is found within a limited number of elementary operations. We formally describe this idea in Algorithm 3. It should be emphasized that there is no constraint on the choice

**Algorithm 3**

**Parameters**: $\mathcal{J} > 0$

**Initialize**: $x^0 = \mathcal{G}_F(x)$, $k = 0$, $J_0 = \mathcal{J}$

1: repeat
2: $(x^{k+1}, D) = Q_F (x^k, J_k)$
3: $J_{k+1} = J_k - D$
4: $k = k + 1$
5: until $\text{dist}(0, \partial F(x^k)) \leq \frac{\lambda_{\min}(M^T)}{\sigma} \min \left\{ \eta, \delta \| (x^k, A(x^k, \tilde{\lambda}, \sigma)) - (\tilde{x}, \tilde{\lambda}) \|_M \right\}$ or $J_k \leq 0$

6: if $J_k \leq 0$ then
7: Run Algorithm 2 and obtain $\tilde{x}$
8: else
9: $\tilde{x} = x^k$
10: end if

**Output**: $\tilde{x}$
of the algorithm $\mathcal{Q}_F$. In principle, we opt for an algorithm which according
to empirical observation is likely to require less computational time than
Algorithm 2. We set a maximum number of elementary operations $J$ allowed
for running algorithm $\mathcal{Q}_F$. The step in Line 2 of Algorithm 3 corresponds
to one iteration of the algorithm $\mathcal{Q}_F$. The second output $D$ records the number
of elementary operations used in this iteration. It is easy to see that as long
as we set $J \leq O(K(p)T(\sigma))$, the complexity of Algorithm 3 is also upper
bounded by (50) with probability at least $1 - p$. In conclusion, Algorithm 3
may benefit from the fast convergence of $\mathcal{Q}_F$ in practice, while maintaining
the same complexity bound as Algorithm 2.

4.4 Overall complexity of AGPPA

In the previous sections, we showed that the IGPPA step can be reduced to
solving the optimization problem (46) and analyzed the complexity assuming
the existence of an algorithm $\mathcal{A}_F$ satisfying the HOOD property (47) with per
iteration cost of Algorithm 2 bounded by $T(\sigma)$. In this section, we assemble
the previous results to give the overall complexity of AGPPA.

Recall that at outer iteration $s$ and inner iteration $t$ of AGPPA, we run
IGPPA step with base points $z^{s,t} = (x^{s,t}, \lambda^{s,t}) \in \mathbb{R}^{n+m}$, proximal regularization
parameter $\sigma_s$ and error parameter $\eta_{s,t}$. From Section 4.1, this problem can
be reduced to find an approximate solution $\tilde{x}$ of the following minimization
problem:

$$\min_{x \in \mathbb{R}^n} \left[ F_{s,t}(x) = f_0(x) + g(x) + \psi(Ax - b, \lambda_{s,t}, \sigma_s) + \frac{1}{2\sigma_s} \|x - x^{s,t}\|_2^2 \right],$$

so that

$$\text{dist}(0, \partial F_{s,t}(\tilde{x})) \leq \sqrt{\lambda_{\text{min}}(\mathcal{M}^2)} \min \left\{ \eta_{s,t}, \delta \left\| (\tilde{x}, \tilde{\lambda}) - (x^{s,t}, \lambda_{s,t}) \right\|_M \right\}$$

with $\tilde{\lambda} := \lambda(\tilde{x}, \lambda_{s,t}, \sigma_s)$. Denote $F^*_{s,t} := \min_{x \in \mathbb{R}^n} F_{s,t}(x)$. We shall make essen-
tially the same assumptions for the function $F_{s,t}$ as for the function $F$ in
Section 4.2. More precisely, we require Assumption 1 and the following as-
sumption.

**Assumption 5**

1. There exists an algorithm, named as $\mathcal{A}_{F_{s,t}}$, for every inner problem (60), which returns an output satisfying

$$\mathbb{E}[F_{s,t}(\mathcal{A}_{F_{s,t}}(x)) - F^*_{s,t}] \leq e^{-1} (F_{s,t}(x) - F^*_{s,t}), \quad \forall x \in \text{dom}(g).$$

within a fixed number of operations.

2. The per-iteration of Algorithm 2 for every inner problem (60) can be upper
bounded by $T(\sigma_s)$ number of elementary iterations.
Then we can apply Algorithm 2 or Algorithm 3 to realize each IGPPA step, or equivalently to find $\tilde{x}$ satisfying (61). The complexity of each IGPPA step follows directly from Theorem 5. For notational ease denote

$$a := \frac{\|A\|^2}{\lambda_{\text{min}}(M^A)}.$$  

(62)

Proposition 5

Under Assumption 4 and Assumption 5, apply AGPPA (Algorithm 1) to the maximal monotone operator $T_\ell$ defined in (40), with each IGPPA step being solved using Algorithm 2 or Algorithm 3. The $n$ with probability at least $1 - p$, the complexity of the IGPPA step at any iteration of AGPPA is bounded by

$$O \left( \max (\ln \tilde{\zeta}_1, \ln \tilde{\zeta}_2, 0) \right)$$

with

$$\tilde{\zeta}_1 := \frac{2(L_0\sigma + a\sigma^2)^2r^2}{p^2\lambda_{\text{min}}(M^x)} \quad \text{and} \quad \tilde{\zeta}_2 := \frac{2(1 + \delta)^2(L_0\sigma + a\sigma^2)^2}{p^2\lambda_{\text{min}}(M^x)\lambda_{\text{min}}(M)}$$  

(64)

and $\bar{\eta}, \bar{\sigma}$ and $r$ defined in Theorem 3 with $z^0 = (x^{0,0}, \lambda^{0,0})$ and $\Omega = T^{-1}_\ell(0,0)$.

Proof

Let $(s, t)$ be any running iteration of AGPPA. More precisely, let any $0 \leq s \leq s^\circ$ and $0 \leq t \leq N_s$. We decompose $F_{s,t}$ into $f_{s,t}(x) \equiv f_0(x) + \psi(Ax - b, \lambda_{s,t}, \sigma_s)$ and $\phi_{s,t}(x) \equiv g(x) + \frac{1}{2\sigma_s}\|x - x^{s,t}\|_{M^x}^2$. Then $\nabla f_{s,t}$ is $L_{s,t}$-Lipschitz continuous with

$$L_{s,t} := L_0 + a\sigma_s, \quad \forall s \geq 0, t \geq 0.$$  

(65)

The complexity of the IGPPA step at iteration $(s, t)$ of AGPPA is bounded by

$$K_{{s,t}}(p) := \max \left( \frac{\|L_{s,t}\|_{s,t}\sigma_s \text{dist}((x^{s,t}, \lambda^{s,t}), \Omega)}{\eta_{s,t}}, \ln(\zeta_2(L_{s,t}\sigma_s)), 0 \right)$$

which is obtained from (56) with $(\bar{x}, \bar{\lambda}) = (x^{s,t}, \lambda^{s,t})$, $L = L_{s,t}$, $\sigma = \sigma_s$ and $\eta = \eta^{s,t}$. It follows from (12) that

$$\text{dist}((x^{s,t}, \lambda^{s,t}), T^{-1}_\ell(0,0)) \leq r.$$  

(66)

Second, from the update rule (23) we know that

$$\eta_{s,t} \geq \eta_{s,N_s} \geq \bar{\eta}. $$  

(67)

Third, for the proximal regularization parameter we have

$$\sigma_s \leq \sigma \leq \bar{\sigma}. $$  

(68)

By (65), (66), (67) and (68), we have

$$\tilde{\zeta}_1 \left( \frac{L_{s,t}\sigma_s \text{dist}((x^{s,t}, \lambda^{s,t}), \Omega)}{\eta_{s,t}} \right) \leq \tilde{\zeta}_1, \quad \zeta_2(L_{s,t}\sigma_s) \leq \tilde{\zeta}_2,$$  

(69)
and
\[
T(\sigma_s) \leq T(\sigma). \tag{70}
\]
Then
\[
K_{s,t}(p) \leq \left[ \max \left( \ln \tilde{\zeta}_1, \ln \tilde{\zeta}_2, 0 \right) \right]. \tag{71}
\]
The above two bounds \((70)\) and \((71)\) then show that \((68)\) is an upper bound of \((63)\) with \((\tilde{x}, \tilde{\lambda}) = (x^{s,t}, \lambda^{s,t}), L = L_{s,t}, \sigma = \sigma_s, \) and \(\eta = \eta_{s,t}\).

The above proposition gives an upper bound on the complexity of one single IGPPA step of AGPPA. In Theorem \(3\) we have a bound on the total number of IGPPA steps. Now it suffices to combine these two bounds to derive the total complexity of AGPPA.

**Theorem 6** Let Assumption \(2\), Assumption \(4\) and Assumption \(5\) hold. Then with probability at least \(1 - \rho\), AGPPA can find a solution \((x, \lambda)\) satisfying
\[
E(x, \lambda) \leq \epsilon,
\]
within
\[
O \left( \tilde{N} \left[ \ln \tilde{N} + \max \left( \ln \tilde{\zeta}_1, \ln \tilde{\zeta}_2, 0 \right) \right] T(\tilde{\sigma}) \right) \tag{72}
\]
number of elementary operations, where \(\tilde{\zeta}_1, \tilde{\zeta}_2\) are defined in Proposition \(5\) and \(\tilde{N}, \tilde{\sigma}\) are defined in Theorem \(3\) with \(x^0 = (x^{0,0}, \lambda^{0,0})\) and \(\Omega = T^{-1}_k(0, 0)\).

**Proof** By Proposition \(5\) each IGPPA step terminates within
\[
O \left( \left[ \ln \tilde{N} + \max \left( \ln \tilde{\zeta}_1, \ln \tilde{\zeta}_2, 0 \right) \right] T(\tilde{\sigma}) \right)
\]
number of elementary operations with probability at least \(1 - \rho/\tilde{N}\). From Theorem \(3\) the total number of IGPPA steps can not exceed \(\tilde{N}\). Then we conclude by applying the union bound property.

We can apply Corollary \(4\) which leads to a better understanding of the complexity by removing problem independent constants. Before that let us add an assumption on \(T\).

**Assumption 6** There are constants \(\vartheta > 0\) and \(\iota > 0\) such that
\[
T(\sigma) = O(\vartheta \sigma^\iota + T), \tag{73}
\]
where the big \(O\) hides problem independent constants.

**Corollary 5** Under Assumption \(6\) the bound \((72)\) is
\[
O \left( \left( \vartheta (\rho \varrho_{\kappa_r} \kappa_r) + T \right) \ln \kappa_r \ln \frac{\zeta r \kappa_r}{\epsilon} \left( \ln \frac{r \kappa_r (L_0 + a)}{p} + \ln \ln \frac{\zeta r \kappa_r}{\epsilon} \right) \right),
\]
where the big \(O\) hides problem independent constants.
Remark 6 The constant $\epsilon$ satisfying (73) is determined by the algorithm class that we employ to solve the subproblem (60). Hence it can be considered as problem independent constant, in the sense that it does not vary for problem instances in the same class. Then by the definition of $\alpha$ as (62), the bound given in Corollary 5 can be further reduced to

$$O\left( (\vartheta \kappa_r + T) \ln \kappa_r \ln \frac{\zeta r \kappa_r}{\epsilon} \left( \ln \frac{r \kappa_r (L_0 + \|A\|)}{p} + \ln \ln \frac{\zeta r \kappa_r}{\epsilon} \right) \right)$$

(74)

In conclusion, we proved that the overall complexity of AGPPA applied on the convex constrained problem (38) is bounded by (74) with probability at least $1 - p$.

5 Application example

We emphasize again that AGPPA and the given complexity results can be applied to convex programming problem (38) satisfying Assumption 2, such as linear-quadratic problem and LASSO problem. We defer to future work the inclusion of other relevant models into the application pool of AGPPA, following the technique of [41]. In this section, we illustrate the application of AGPPA for solving LP problem. Following the same notations as [40], we describe our LP problem as follows:

$$\min_{x \in \mathbb{R}^n} c^T x$$

s.t.  
$$A_I x \leq b_I$$
$$A_E x = b_E$$
$$x_1, \ldots, x_{n_b} \geq 0.$$  

(75)

Here, $c \in \mathbb{R}^n$, $A_I \in \mathbb{R}^{m_I \times n}$, $b_I \in \mathbb{R}^{m_I}$, $A_E \in \mathbb{R}^{m_E \times n}$, $b_E \in \mathbb{R}^{m_E}$ and $n_b \in [n]$. We thus have $m_I$ inequalities, $m_E$ equalities and the first $n_b$ coordinates must be nonnegative. Denote $m := m_I + m_E$, $b := [b_I; b_E] \in \mathbb{R}^m$ and $A := [A_I; A_E] \in \mathbb{R}^{m \times n}$.

5.1 Applicability of AGPPA

5.1.1 Error bound assumption

Problem (75) is a special case of the convex programming problem (38) with:

$$f_0(x) = c^T x$$
$$g(x) = \begin{cases} 0 & \text{if } x_1, \ldots, x_{n_b} \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$
$$h(y) = \begin{cases} b_i & \text{if } y_1, \ldots, y_{m_I} \leq 0, y_{m_I+1}, \ldots, y_m = 0 \\ +\infty & \text{otherwise} \end{cases}$$


In the following, we decompose \( x \in \mathbb{R}^n \) as \( x = [x_b; x_I] \) with \( x_b \in \mathbb{R}^{n_b} \) and \( \lambda \in \mathbb{R}^{m_r} \) as \( \lambda = [\lambda_I; \lambda_E] \) with \( \lambda_I \in \mathbb{R}^{m_I} \). The associated Lagrangian function can then be written as:

\[
\ell(x, \lambda) = c^T x + \langle Ax - b, \lambda \rangle - \delta_{\{\lambda_I \geq 0\}}(\lambda) + \delta_{\{\lambda_E \geq 0\}}(x).
\]

Then \( T_\ell : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m} \) defined in (40) becomes

\[
T_\ell(x, \lambda) = \begin{cases} (v, u) \in \mathbb{R}^{n+m} & v \in c + A^T \lambda + \partial \delta_{\{\lambda_E \geq 0\}}(x), \\ u \in b - Ax + \partial \delta_{\{\lambda_I \geq 0\}}(\lambda). \end{cases}
\]

(76)

It is clear that \( T_\ell \) is a polyhedral multifunction and \( T_\ell^{-1} \) is locally upper Lipschitzian [34]. Hence the error bound condition (Assumption 11) is satisfied. We assume the existence of optimal solution of LP problem (75), which then implies the nonemptiness of the solution set

\[
\Omega = T_\ell^{-1}(0, 0) = \left\{ (x, \lambda) \in \mathbb{R}^{n+m} \mid \begin{array}{l} x_b \geq 0, \\ \lambda_I \geq 0, \\ -A^T \lambda - c \in \partial \delta_{\{\lambda_E \geq 0\}}(x), \\ Ax - b \in \partial \delta_{\{\lambda_I \geq 0\}}. \end{array} \right\}.
\]

(77)

Note that the solution set \( \Omega \) can be represented by the following system of linear inequalities:

\[
\Omega = \left\{ (x, \lambda) \in \mathbb{R}^{n+m} \mid \begin{array}{l} x_b \geq 0, \lambda_I \geq 0, \\ c^T x + b^T \lambda = 0, \\ [A^T \lambda + c]_I^m = 0, \\ [Ax - b]_I^m = 0. \end{array} \right\}.
\]

(77)

Denote by \( \theta \) the smallest constant satisfying

\[
\text{dist} \left( (x, \lambda), \Omega \right) \leq \theta \left\| \left[ c^T x + b^T \lambda; [A^T \lambda + c]_I^m; [Ax - b]_I^m \right] \right\|,
\]

(78)

for all \( x, \lambda \) with \( x_b \geq 0 \) and \( \lambda_I \geq 0 \). It is clear that \( \theta \) is upper bounded by the Hoffman constant [20] associated with the system (77).

**Lemma 2** For any \( x, \lambda \) with \( x_b \geq 0 \) and \( \lambda_I \geq 0 \), we have

\[
\text{dist} \left( (x, \lambda), \Omega \right) \leq \theta \left( \left\| (x, \lambda) \right\|^2 + 1 \right)^{1/2} \text{dist} \left( 0, T_\ell(x, \lambda) \right).
\]

(79)

**Proof** Let any \( (v, u) \in T_\ell(x, \lambda) \). By (76), we have \( v \leq c + A^T \lambda \) and \( u \leq b - Ax \), from which we deduce that

\[
\left\| [A^T \lambda + c]_I^m \right\| \leq \left\| [v]_I^m \right\|,
\]

\[
\left\| [Ax - b]_I^m \right\| \leq \left\| [-u]_I^m \right\|.
\]

(80)
Furthermore, we also have $x^T(-v + c + A^T \lambda) = 0$ and $\lambda^T(u + Ax - b) = 0$, from which we know that

$$c^T x + b^T \lambda = v^T x + u^T \lambda.$$  \hfill (81)

In view of (78), one has

$$\text{dist} ((x, \lambda), \Omega) \leq \theta \left\| [c^T x + b^T \lambda; [A^T \lambda]_{m_1}^n; [c + A^T \lambda]_{n_1}^m] \right\|$$

$$\leq \theta \left\| [v^T x + u^T \lambda; [-u]_{m_1}^n; [v]_{n_1}^m] \right\|$$

$$\leq \theta \left\| [v^T x + u^T \lambda; u; v] \right\|$$

$$\leq \theta \left( \| (x, \lambda) \|^2 + 1 \right)^{1/2} \| (v, u) \|$$

which implies (79).

Lemma 2 asserts that $T_\ell$ satisfies Assumption 1 with $\kappa_r$ bounded by:

$$\kappa_r \leq \theta (r^2 + 1)^{1/2} = O(\theta r).$$  \hfill (82)

5.1.2 Error residual function

With regard to the computable error residual function $E$, we here provide three examples.

**Example 1 (KKT-residual)** For any $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$, define the error residual function $E_1$ as

$$E_1(x, \lambda) := \left\| [c^T x + b^T \lambda; [A^T \lambda]_{m_1}^n; [c + A^T \lambda]_{n_1}^m] \right\|$$

then $E_1$ satisfies (21) with

$$\zeta = \left( \| [b; c] \|^2 + \| A \|^2 \right)^{1/2}$$

**Example 2 ([25])** For any $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$, define the error residual function $E_2$ as

$$E_2(x, \lambda) := \max \left\{ \frac{|c^T x + b^T \lambda|}{1 + |c^T x| + |b^T \lambda|}, \frac{\| [A^T \lambda]_{m_1}^n \|}{1 + \| b \|}, \frac{\| [c + A^T \lambda]_{n_1}^m \|}{1 + \| c \|} \right\},$$  \hfill (83)

then $E_2$ satisfies (21) with

$$\zeta = \max \left\{ \| [b; c] \|, \frac{\| A \|}{1 + \| b \|}, \frac{\| A \|}{1 + \| c \|} \right\}.$$
Example 3 ([39][40]) For any $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$, define the error residual function $\mathcal{E}_3$ as

$$
\mathcal{E}_3(x, \lambda) := \max \left\{ \frac{|c^T x + b^T \lambda|}{\max \{1, |c^T x|\}}, \|[Ax - b]_+\|_\infty, \|[c + A^T \lambda]_-\|_\infty \right\},
$$

then $\mathcal{E}_3$ satisfies (21) with

$$
\zeta = \max \left\{ \|b; c\|, \max_{i \in [m]} \|a_i\|, \max_{i \in [n]} \|A_i\| \right\}.
$$

Remark 7 The above three examples of error residual functions $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ are all computable and satisfy (21) for some $\zeta > 0$ upper bounded by

$$
\zeta \leq \|b; c\| + \|A\|. \quad (84)
$$

We just verified that Assumption 2 holds for LP problem (75) and so we can apply AGPPA to solve it.

5.2 Complexity results

The complexity bound of AGPPA applied to LP problem (75) follows directly from (74), by specifying the constants $\vartheta$, $\iota$, $\Upsilon$, $\zeta$ and $L_0$. Note that Assumption 4 holds with $L_0 = 0$ (85) and the bound for $\zeta$ is already given by (84). It remains to find out the constants $\vartheta$, $\iota$ and $\Upsilon$ so that the per-iteration cost function $T$ satisfies Assumption 6.

5.2.1 Qualified inner solvers

For simplicity, let $\mathcal{M}$ be the identity matrix. Fix base points $\bar{x} \in \mathbb{R}^n$ and $\bar{\lambda} \in \mathbb{R}^m$, then the function $\psi$ defined in (42) is:

$$
\psi(u, \bar{\lambda}, \sigma) = \max_{\lambda} \left\{ \langle u, \lambda \rangle - \delta_{\lambda \geq 0}(\lambda) - \frac{1}{2\sigma} \|\lambda - \bar{\lambda}\|^2 \right\}
= \frac{1}{2\sigma} \left\| \left[ \lambda + \sigma u \right]_+ \right\|^2 - \frac{1}{2\sigma} \|\lambda\|^2.
$$

The function $F$ defined in (43) thus takes the following form:

$$
F(x) = c^T x + \frac{1}{2\sigma} \left\| \left[ \lambda + \sigma (Ax - b) \right]_+ \right\|^2 - \frac{1}{2\sigma} \|\lambda\|^2
+ \frac{1}{2\sigma} \|x - \bar{x}\|^2 + \delta_{\{z_0 \geq 0\}}(x), \quad (86)
$$
which can be written as $F(x) = f(x) + g(x)$ with

$$f(x) \equiv c^T x + \frac{1}{2\sigma} \left[ (\lambda + \sigma(Ax - b))^+ \right]^2 - \frac{1}{2\sigma} \|\lambda\|^2,$$

$$g(x) \equiv \frac{1}{2\sigma} \|x - \bar{x}\|^2 + \delta_{\{x \geq 0\}}(x).$$

The gradient of $f$ is

$$\nabla f(x) \equiv c + A^T \Lambda(x, \bar{\lambda}, \sigma)$$

where

$$\Lambda(x, \bar{\lambda}, \sigma) = \left[ \bar{\lambda} + \sigma(Ax - b) \right]_{+}^{m}. \quad (87)$$

To apply the complexity results, one need to find an algorithm $A_F$ so that the HOOD property (47) holds. Note that $F$ in (86) is a structured problem taking the following form:

$$F(x) \equiv \sum_{i=1}^{m} \phi_i(A_i^T x) + g(x)$$

where $A_i$ is the $i$th row vector of $A$, $\phi_i$ is differentiable with $\sigma$-Lipschitz continuous gradient and $g$ is a sufficiently simple and $1/\sigma$-strongly convex function. In particular, the condition number of $F$ is of order $O(\sigma^2)$ and thereby the problem complexity increases with $\sigma$. We mention two suitable algorithms which appear to be particularly interesting in the large-scale setting.

**Example 4** Restarted APPROX [12] finds $A_F(x)$ satisfying the HOOD property (47) in

$$O \left( 1 + \sigma \max_{i \in [n]} \|a_i\| \right) \text{nnz}(A).$$

number of elementary operations, where $a_i$ denotes the $i$th column vector of $A$.

**Example 5** Katyusha [1] finds $A_F(x)$ satisfying the HOOD property (47) in

$$O \left( 1 + \sigma \frac{\|A\|_F}{\sqrt{m}} \right) \text{nnz}(A).$$

number of elementary operations.

### 5.2.2 Per-iteration cost of Algorithm 2

Recall from Remark 5 the main operations of Algorithm 2. We showed in the last section that the computation cost of $A_F(\cdot)$ can be less than

$$O \left( 1 + \sigma \min \left\{ \max_{i \in [n]} \|a_i\|, \frac{\|A\|_F}{\sqrt{m}} \right\} \right) \text{nnz}(A).$$

Next we discuss the cost of the other four operations. From (87) we see that $A(x, \bar{\lambda}, \sigma)$ and thus $\nabla f(x)$ can be computed in $O(\text{nnz}(A))$ operations.
Then the computation of the function value $F(\cdot)$ defined in (86) is clearly upper bounded by $O(n)$ when $\Lambda(x, \tilde{\lambda}, \sigma)$ is known. In addition, the proximal operator $G_F$ defined in (49) is

$$G_F(x) = \left[ \frac{L\sigma \left( x - \frac{1}{\sigma} \nabla f(x) + \bar{x} \right)}{L\sigma + 1} \right]_+,$$

which can be computed in $O(n)$ when $\nabla f(x)$ is known. Finally dist$(0, \partial F(x))$ can also be computed in no more than $O(n)$ operations because

$$\text{dist}(0, \partial F(x)) = \sum_{i \in B} \left[ (\nabla_i f(x) + \sigma (x_i - \bar{x}_i))^2 \right]_{+} + \sum_{i \notin B} \left[ (\nabla_i f(x) + \sigma (x_i - \bar{x}_i))^2 \right]_{+},$$

with $B := \{i \in [n_b] | x_i = 0\}$ and $\cdot$ denotes the projection into $\mathbb{R}_{-}$. Thereby, the other four operations costs at most $O(\text{nnz}(A))$ and the per-iteration cost of Algorithm 2 for $F$ defined in (86) is

$$T(\sigma) = O \left( 1 + \sigma \min \left\{ \max_{i \in [n]} \| a_i \|, \frac{\| A \|_F}{\sqrt{m}} \right\} \right) \text{nnz}(A).$$

(88)

We deduce that Assumption 6 holds with $\vartheta = \min \left( \max_{i \in [n]} \| a_i \|, \frac{\| A \|}{\sqrt{m}} \right) \text{nnz}(A)$, $\iota = 1$, $\Upsilon = \text{nnz}(A)$. (89)

5.2.3 Overall complexity bound

We are now in the position to deduce the overall complexity of AGPPA applied to LP problem (75). Hereinafter we shall measure the batch complexity, which refers to the number of pass over data, i.e., the number of elementary operations divided by $\text{nnz}(A)$.

**Theorem 7** Apply AGPPA to solve the LP problem (75) with $M = I$ and error residual function $E$ being $E_1, E_2$ or $E_3$ given in Section 5.1.2. For any starting primal dual pair $(x^0, \lambda^0)$, let

$$r = \| (\bar{x}^0, \bar{\lambda}^0) \| + \text{dist}((x^0, \lambda^0), \Omega) + \frac{\gamma \sqrt{\delta_0}}{(s - 1)(1 - \vartheta_0)},$$

(90)

with $(\bar{x}^0, \bar{\lambda}^0)$ being the projection of $(x^0, \lambda^0)$ into $\Omega$. Let $\kappa_F$ be the constant satisfying (3) for $T = T(F)$. With probability at least $1 - p$, AGPPA finds a solution satisfying $E(x, \lambda) \leq \epsilon$ with batch complexity bounded by

$$O \left( \min \left\{ \max_{i \in [n]} \| a_i \|, \frac{\| A \|_F}{\sqrt{m}} \right\} \kappa_F \ln \kappa_F \ln \left( \frac{\zeta \sqrt{\delta_0}}{\epsilon} \right) \left( \ln \frac{r \kappa_F \| A \|}{\sqrt{m}} + \ln \frac{\zeta \sqrt{\delta_0}}{\epsilon} \right) \right).$$

(91)

where $\zeta \leq \| [b; c] \| + \| A \|$.

**Proof** It suffices to plug in (74) the estimations (85), (84) and (89).
5.3 Comparison with related work

In the past decades, numerous research has been devoted to the development of numerical solutions for LP. Commonly used LP solvers ubiquitously implement the interior-point method (IPM) \cite{7,30,22} and the simplex method \cite{21,42,6}. These two classical methods are recognized to be highly efficient for low or medium sized LP problems. However, their complexity are known to be at least quadratic in the number of variables or constraints \cite{31}. With the ever-increasing size of the LP problems to be solved, search of more efficient solvers in the large-scaled setting has attracted a lot of attention.

Previous to our work, many papers studied the application of PPA to large-scale LP, including in particular the augmented Lagrangian method \cite{10,16,10,25} and the alternating directional method of multiplier \cite{39,8,32}. To facilitate the comparison, we shall omit constants other than $\epsilon$ and $\kappa_r$ from the log terms appearing in the batch complexity bound. Namely, we simplify the batch complexity bound of AGPPA given in (91) as follows:

$$O\left(\min\left\{\max \left\{ \|a_i\|, \|A\|_F \right\}, \frac{\kappa_r \ln \kappa_r \ln \left(\kappa_r \ln \frac{\kappa_r}{\epsilon}\right)}{\ln \left(\kappa_r \ln \frac{\kappa_r}{\epsilon}\right)}\right\} \ln \left(\kappa_r \ln \frac{\kappa_r}{\epsilon}\right)\right)$$

Before giving more details we summarize the main contribution of our work in comparison with closely related work.

1. We provide an iteration complexity bound (91) of PPA based LP solver. In contrast, \cite{25} shows the asymptotique superlinear convergence of PPA outer iterations.

2. Our algorithm and theoretical analysis does not require any knowledge on the error bound condition parameter $\kappa_r$ satisfying (3) for $T = T_\ell$. In \cite{40}, the theoretical results were derived by requiring the proximal regularization parameter to be proportional to some error bound condition parameter $\kappa$, which is in general unknown.

3. The inner problem stopping criterion (61) is implementable, which is not the case for the stopping criteria proposed in \cite{39,40}.

4. Compared with existing complexity bounds of related methods (see Table \ref{table1}), the complexity bound of AGPPA has a weaker dependence on the dimension of the problem and on the Hoffman constant of the associated KKT system.

5. Our algorithm is directly applicable to LP problem in the general form of (75), while \cite{25,39} were specifically designed for standard LP problem as (93). It is true that general LP problem can be transformed into standard form. However, numerical experiments suggest a loss of efficiency by applying such transformation first, see Section \ref{comparisonSNIPAL}.

5.3.1 Comparison with SNIPAL

When $n_b = 0$ in (75), the function $F$ defined in (80) becomes semismooth and the inner problem can be solved by semismooth Newton (SSN) method.
Based on this property, Li et al. [25] proposed a semismooth Newton based inexact proximal augmented Lagrangian (SNIPAL) method for LP. They focused particularly on how to exploit the structure of the matrix $A$ and of the generalized Hessian of $F$ to efficiently solve each Newton system in the high dimensional setting ($m \gg n$). Asymptotique superlinear convergence for SNIPAL is obtained by requiring $\{\sigma_k\}_{k \geq 0}$ to tend to infinity along with the local superlinear convergence of semismooth Newton method. In this paper we primarily focus on giving explicit updating formula for $\{\sigma_k\}_{k \geq 0}$ and the overall complexity analysis.

When $m_I = 0$ in (75), the LP problem takes the following form:

$$\min_{x \in \mathbb{R}^n} c^T x \quad \text{s.t.} \quad Ax = b, x_i \geq 0, i \in [n_b],$$

and the associated dual LP becomes

$$\min_{\lambda \in \mathbb{R}^m} b^T \lambda \quad \text{s.t.} \quad -A_b^T \lambda \leq c_b, -A_f^T \lambda = c_f,$$

to which SNIPAL can be applied.

When $n_b > 0$ and $m_I > 0$ in (75), we can first transform the problem into the form of (94) by appropriately enlarging the matrix $A$. However, we find that applying such transformation may slow down the practical performance, see Section 6.

### 5.3.2 Comparison with linearized ADMM

Linearised ADMM [41] (LADMM) is a method of multipliers with inner problems exactly solved, which leads to a larger linear convergence rate and thus a slower outer iteration convergence. To reach an $\epsilon$-KKT solution, i.e., finding a pair of primal and dual points $(x, \lambda)$ with $E_1(x, \lambda) \leq \epsilon$, LADMM has a batch complexity bound

$$O\left(\|A\|^2 \kappa^2 r^2 \ln \frac{1}{\epsilon}\right),$$

which by (82) yields:

$$O\left(\|A\|^2 \theta^2 (r')^2 \ln \frac{1}{r}\right).$$

Here, $r'$ refers to any upper bound on the norm of the iterates of the algorithm LADMM and we know that

$$r' \leq \|(\bar{x}^0, \bar{\lambda}^0)\| + \text{dist}((x^0, \lambda^0), \Omega) \leq r.$$ 

Plugging (82) into (92) we get the following batch complexity bound of AGPPA:

$$O\left(\min_{\{\max_{i \in [n]} \|a_i\|, \|A\|_F^2 \sqrt{m}\} \theta r \ln(\theta r) \ln \left(\frac{\theta r}{\epsilon}\right) \ln \left(\frac{\theta r}{\epsilon}\right)}\right)$$

Comparing (92) with (95), we see that the worst batch complexity bound of AGPPA scales better than that of LADMM for large scale problems and also has a weaker dependence on the Hoffman constant $\theta$ (of the KKT system (77)).
Remark 8 He et al. [18] introduced an ADMM with self-adaptive proximal regularization parameters for monotone variational inequalities. This self-adaptive technique is based on the value of primal and dual residual. It shows superior numerical performance and is frequently used in experiments. However, there is no theoretical bound yet on the number of ADMM steps using this self-adaptive technique.

5.3.3 Comparison with iADMM

In [39], Wang and Shroff proposed to apply inexact ADMM (iADMM) to solve LP problem in the form of (93). They add auxiliary variable $y$ and $n$ equalities $y = x$ to (93):

$$\min_{x \in \mathbb{R}^n} c^T x$$

s.t. $Ax = b,$

$$y = x,$$

$$y_i \geq 0, i \in [n].$$

and then applied the inexact ADMM. They proposed to approximately solve every inner problem with an accelerated coordinate descent method (ACDM) [3] until the function value at the current point is close enough to the optimal value. However, the proposed stopping criterion is not implementable as the optimal value of the subproblem is unknown.

It was shown in [39] that to have an $\epsilon$-dual optimal solution (i.e., a dual solution with distance to the dual optimal solution set bounded by $\epsilon$), the batch complexity of iADMM is

$$O\left(\max_{i \in [n]} \|a_i\| (r_x \|A\| + r_z)^2 \theta_S^2, \frac{1}{\epsilon \ln \frac{\theta_S}{\epsilon}}\right).$$

(99)

Here $r_x$ and $r_z$ are upper bounds respectively on the primal and dual iterates, and $\theta_S$ is the Hoffman constant associated with the KKT system of problem (98). In particular, it can be shown that

$$\theta \leq \theta_S.$$

5.3.4 Comparison with AL_CD

The algorithm in [40], named as AL_CD, solves (75) by combining an inexact augmented Lagrangian method with a randomized coordinate descent method (CDM) [40] for inner problems. This amounts to apply the proximal point method to the dual problem:

$$\min_{\lambda \in \mathbb{R}^m} \{b^T \lambda + \delta F_d(\lambda)\}$$

with

$$F_d = \{\lambda \in \mathbb{R}^m | -A_b^T \lambda \leq c_b, -A_f^T \lambda = c_f, \lambda_j \geq 0, j \in [m_f]\}.$$
Denote by \( \tilde{r} \) an upper bound of the dual iterates of ALCD and \( \kappa \) the error bound condition parameter such that

\[
\text{dist} \left( \lambda, T_d^{-1}(0) \right) \leq \kappa \text{dist} \left( 0, T_d(\lambda) \right), \quad \forall \lambda \text{ s.t. } \|\lambda\| \leq \tilde{r}.
\]

Here \( T_d \) is the polyhedral multifunction defined by \( T_d : \lambda \to b + \partial \delta_{F_d}(\lambda) \). The proximal regularization parameter of ALCD is required to be proportional to the error bound condition parameter \( \kappa \), which is in general unknown.

The inner problems in ALCD are non-strongly convex without the proximal term for the primal variable. ALCD’s inner problem stopping criterion is also conceptual, which requires to know the optimal inner problem solution. To obtain an \( \epsilon \)-dual optimal solution, the batch complexity bound of ALCD is

\[
O \left( \max_{i \in [n]} \|a_i\|^2 \theta_{S\kappa} \ln^2 \left( \frac{\kappa}{\epsilon} \right) \right)
\]

where \( \theta_{S} \) is the Hoffman’s constant that depends on the polyhedron formed by the set of optimal solutions of the subproblems.

We summarize the comparison of batch complexity bounds in Table 1.

### 5.4 Hybrid inner solver

Recall that we proposed in Algorithm 3 to combine an algorithm satisfying the HOOD property with any other algorithm with local fast convergence while keeping the validity of the complexity bound (74). We here propose to combine APPROX or Katyusha with a modified version of the algorithm Projected Newton-Conjugate Gradient (PN-CG), see [40, Section 3.4]. We show the detailed algorithm in Appendix [3].

| Algorithm | Measure | Batch complexity bound |
|-----------|---------|------------------------|
| AGPPA     | \( \epsilon \)-KKT | \( O \left( \min \left\{ \max_{i \in [n]} \|a_i\|, \frac{Lg}{\epsilon} \right\} \theta r \ln(\theta r) \ln \left( \frac{\theta r}{\epsilon} \right) \right) \) |
| LADMM [41] | \( \epsilon \)-KKT | \( O \left( \|A\|^2 \theta^2 r^2 \ln \frac{1}{\epsilon} \right) \) |
| iADMM [29] | \( \epsilon \)-dual | \( O \left( \max_{i \in [n]} \|a_i\| (\|r_x\| + \|r_z\|)^2 \theta_{S}^2, \ln \frac{1}{\epsilon} \ln \frac{2}{\epsilon} \right) \) |
| ALCD [40] | \( \epsilon \)-dual | \( O \left( \max_{i \in [n]} \|a_i\|^2 \theta_{S\kappa} \ln^2 \left( \frac{\kappa}{\epsilon} \right) \right) \) |

Table 1: Comparison of batch complexity bounds. More details on the constants can be found in Section 5.3.2, Section 5.3.3 and Section 5.3.4.
6 Numerical results

When there is at least one coordinate with nonnegative constraint, we first transform the problem into the form of (94), then apply AGPPA to the transformed problem and we call the resulting algorithm AGPPAi. In this section, we compare the numerical performance of AGPPA with AGPPAi, AL_CD [40], SCS [32] and the commercialized LP solver Gurobi [6]. Here, AL_CD is an inexact ALM and SCS is an inexact ADMM and Gurobi includes the interior-point method and the simplex method. We employ the error residual function $E_2$ defined in (83) for accuracy measure. For AGPPA, AGPPAi, AL_CD, SCS and IPM we stop the algorithm when either the error residual $E_2$ is smaller than a threshold or the maximal running time is reached. However, for the simplex method we will show the computation time when Gurobi terminates, since it is not convenient to compute error residual before the termination.

We compare the algorithms in single thread, but note that AGPPA can also be implemented with multi-threads. In the following, we list some specific settings of the algorithms.

1. We apply AL_CD to both the primal and dual LP problems and present the best result.
2. SCS has two versions, the difference of which is the way to solve the linear systems generated by inner problems. One applies an indirect solver based on conjugate gradients and another one uses a direct solver, which uses a cached LDL factorization and may require more memory. For large-scale LP problems, the direct solver works inefficient for the test data sets and we present the best result of SCS with indirect solver to both the primal and dual LP problems.
3. For Gurobi, we set method = 2 to use IPM and method = -1 to use simplex methods. Note that when method = -1 under the setting of single thread, it will automatically choose a method from primal simplex method and dual simplex method. For both IPM and simplex methods, if the presolve phrase consumes too many time and does not reduce any problem size, we turn off the presolve phrase.
4. If $m_l > n_b$, we apply AGPPA to the primal general LP problems (75), otherwise, we apply AGPPA to the dual problems.
5. Recall that the parameters of AGPPA are $\rho$, $\sigma_0$, $\omega$, $\alpha$, $\gamma$, $\delta$, $\eta_0$, $\varsigma$, $\eta$. Let

$$
\delta = g_\delta \left( \rho - \sqrt{1 - \min\{\gamma, 2\gamma - \gamma^2\}} \right) \frac{1}{1 + \rho}.
$$

(102)

---

5 Our solver AGPPA has been released in: https://github.com/lumeng16/APPA
4 The solver AL_CD has been released in: http://ianyen.site/LPsparse/
5 The solver SCS has been released in: https://github.com/cvxgrp/scs
8 The solver Gurobi has been released in: https://www.gurobi.com/
| parameter | value |
|-----------|-------|
| $M$       | 1     |
| $\gamma$  | 0.7   |
| $\rho$    | 0.9   |
| $\varrho_0$ | 0.9 |
| $\varrho_1$ | 5     |
| $\varsigma$ | 1.1   |
| $\eta_0$  | 1e16  |

Table 2: Default settings of input parameters

| parameter | $\delta$ | $\alpha$ | $\sigma_0$ |
|-----------|----------|----------|------------|
| equation  | (102)    | (18)     | (103)      |

Table 3: Default settings of deduced parameters

for some $q_d \in (0, 1)$. We set initial guess for $\kappa_r$ as $1/\|A\|_F$, then initial proximal regularization parameter $\sigma_0$ is equal to

$$\sigma_0 = \frac{\alpha}{\|A\|_F}$$  \hfill (103)

We set the values of $\rho$, $\varrho_1$, $\gamma$, $\eta_0$, $\varsigma$, $\varrho_0$ following Table 2 and the values of $\delta$, $\alpha$ and $\sigma_0$ are deduced from specific equation, see Table 3.

In the following, we compare the numerical performance of the mentioned algorithms in the above mentioned settings on different types of LP problems, including LP problems generated from the L1-regularized multi-class Support Vector Machine problem (L1-SVM), randomly generated sparse LP problems and covering and packing LP problems. We make comparison from three aspects: memory, time to reach low precision ($\mathcal{E}_2(x, \lambda) \leq 1e-3$) and time to reach medium precision ($\mathcal{E}_2(x, \lambda) \leq 1e-5$). For reference purpose we also list the running time of simplex method in the last column. However recall that this is the running time when the simplex method terminates and thus returns a high accuracy solution.

In all the tables, o.m. means out of memory (95GB) and * means reaching of maximal running time, which is set to be 1 day for AGPPA and AGPPAi, and 2e5 seconds (around 2.3 days) for other algorithms.

6.1 L1-regularized multi-class SVM

We consider LP instances generated from L1-SVM, which is a classical machine learning problem. Let $x_i \in \mathbb{R}^{p_d}$ be collected data, where $p_d$ is the number of features. Let $y_i \in [k]$ be labels with $k$ being the number of classes. Set $p_n$ as the number of samples. Then following the same notations in [40], we show L1-SVM as:

$$\min \lambda \sum_{j=1}^k \|w_j\|_1 + \sum_{i=1}^{p_n} \xi_i$$

s.t. $w_{j,\top} x_i - w_{i,\top} x_i \geq e_i^j - \xi_i \quad \forall i \in [p_n], j \in [k]$
where $e^i_j = 0$ if $y_i = j$, $e^i_j = 1$ otherwise. Set $w_j = [w^+_j] + [w^-_j]$, then
\[ \| w_j \|_1 = 1^T([w^+_j] - [w^-_j]) \text{ with } [w^+_j] \geq 0 \text{ and } [w^-_j] \leq 0. \]
Thus, we can transform SVM into a LP problem with
\[ m_I = (k - 1)p_d, \; m_E = 0, \; m_b = 2kp_d + p_n, \; n_f = 0. \]
In the simulation, if the data is dense, we conventionally normalize the data:
\[ x_{ij} = x_{ij} - \frac{\sum_{j=1}^{p_d} x_{ij}}{\sqrt{\sum_{j=1}^{p_d} (x_{ij} - \sum_{j=1}^{p_d} x_{ij})^2}}, \forall j = 1, \ldots, p_d, \]
otherwise, we scale the data:
\[ x_{ij} = \frac{x_{ij}}{\| x_i \|}, \forall j = 1, \ldots, p_d, \]
where $x_{ij}$ are the coordinates of vector $x_i$. We set the penalty parameter $\lambda$ to be 1. All the datas in Table 4 come from LBSVM data sets.

In Table 4, nnz ($X$) means the nonzero number of matrix $X$, whose $i$th column is $x_i$. And $A$ is the matrix of LP problem transformed from L1-SVM with nnz ($A$) as the nonzero number of $A$. Note that LP problem generated from L1-SVM has some special structures. When we apply Gurobi to such LP problems, during the presolve phrase, it will reduce a number of rows or columns due to the dependence, which may substantially improve the performance of Gurobi. However, for other algorithms, including our algorithm AGPPA, we use the default settings without presolve phrase. Note that AGPPA still has a satisfactory advantage over Gurobi even without the acceleration from presolve phrase. The numerical results are shown in Table 5, 6, 7.

where $e^i_j = 0$ if $y_i = j$, $e^i_j = 1$ otherwise. Set $w_j = [w^+_j] + [w^-_j]$, then
\[ \| w_j \|_1 = 1^T([w^+_j] - [w^-_j]) \text{ with } [w^+_j] \geq 0 \text{ and } [w^-_j] \leq 0. \]
Thus, we can transform SVM into a LP problem with
\[ m_I = (k - 1)p_d, \; m_E = 0, \; m_b = 2kp_d + p_n, \; n_f = 0. \]
In the simulation, if the data is dense, we conventionally normalize the data:
\[ x_{ij} = x_{ij} - \frac{\sum_{j=1}^{p_d} x_{ij}}{\sqrt{\sum_{j=1}^{p_d} (x_{ij} - \sum_{j=1}^{p_d} x_{ij})^2}}, \forall j = 1, \ldots, p_d, \]
otherwise, we scale the data:
\[ x_{ij} = \frac{x_{ij}}{\| x_i \|}, \forall j = 1, \ldots, p_d, \]
where $x_{ij}$ are the coordinates of vector $x_i$. We set the penalty parameter $\lambda$ to be 1. All the datas in Table 4 come from LBSVM data sets.

In Table 4, nnz ($X$) means the nonzero number of matrix $X$, whose $i$th column is $x_i$. And $A$ is the matrix of LP problem transformed from L1-SVM with nnz ($A$) as the nonzero number of $A$. Note that LP problem generated from L1-SVM has some special structures. When we apply Gurobi to such LP problems, during the presolve phrase, it will reduce a number of rows or columns due to the dependence, which may substantially improve the performance of Gurobi. However, for other algorithms, including our algorithm AGPPA, we use the default settings without presolve phrase. Note that AGPPA still has a satisfactory advantage over Gurobi even without the acceleration from presolve phrase. The numerical results are shown in Table 5, 6, 7.

7 LBSVM website: https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/
Table 6: Time for precision 1e-3 (s). * means reaching of maximal running time. The fifth column uses the default output format in the SCS package.

| Data  | AGPPA | AGPPAi | ALCD | SCS    | IPM    | Simplex |
|-------|-------|--------|------|--------|--------|---------|
| real-sim | 29    | 248    | 72   | 1.57e3 | 16286  | 4166    |
| rcv1   | 811   | *      | 1002 | *      | o.m.   | 44650   |
| news20 | 462   | *      | 595  | *      | o.m.   | 132082  |
| avazu  | 10293 | 10343  | 19485*|        |        |         |

Table 7: Time for precision 1e-5 (s). * means reaching of maximal running time. The fifth column uses the default output format in the SCS package.

| Data  | AGPPA | AGPPAi | ALCD | SCS    | IPM    | Simplex |
|-------|-------|--------|------|--------|--------|---------|
| real-sim | 204   | 3373   | 815  | 7.73e4 | 23178  | 4166    |
| rcv1   | 3636  | *      | 6787 | *      | o.m.   | 44650   |
| news20 | 12407 | *      | 13481| *      | o.m.   | 132082  |
| avazu  | 15581 | 19301  | *    |        | *      | *       |

Table 8: Data Statistics for randomly generated sparse LP

| Data  | m   | n   | sparsity |
|-------|-----|-----|----------|
| M1    | 5e5 | 1e5 | 1e-3     |
| M2    | 1e6 | 1e5 | 1e-3     |
| M3    | 1e6 | 2e5 | 1e-3     |
| M4    | 1e7 | 1e5 | 1e-4     |
| M5    | 1e7 | 2e5 | 1e-4     |
| M6    | 1e7 | 5e5 | 1e-4     |

6.2 Randomly generated sparse LP

We apply AGPPA to LP problem with form as

$$\min_{x \in \mathbb{R}^n} c^\top x$$

s. t. $Ax \leq b$  \hspace{1cm} (104)

where $c \in \mathbb{R}^n, b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. Referring to \cite{25,27}, we generate large synthetic matrix $A$ by

$$A = \text{sprand}(m, n, \text{sparsity}); A = 100 \ast (A - 0.5 \ast \text{spones}(A));$$

where sparsity means the percentage of nonzero numbers of matrix $A$.

Here, LP problem (104) can be transformed into the general LP form \cite{26} with $m_I = m, n_{FE} = 0, n_h = 0$ and $n_f = n$. Then all data sets have $m_I > n_h$ and therefore AGPPA is automatically applied to the primal form. Thanks to the special case where there is no nonnegative constraints for the coordinates in the considered LP problem, AGPPAi is equivalent to AGPPA. The data set is given in Table 5 and the numerical results are shown in Table 6, 7, 8.
Data AGPPA AL\_CD SCS IPM Simplex
M1 3.1 2.5 2.1 40.1 92.3
M2 5.9 4.6 4.1 42 41
M3 11.3 9.5 7.5 10.1 41.8 17.5
M4 6.1 7.3 10.1 41.8 52.2
M5 11.2 12.7 13.3 o.m. 92.0
M6 27.3 30.2 23.8 o.m. 92.0

Table 9: RAM (GB). o.m. means out of memory (95GB).

Data AGPPA AL\_CD SCS IPM Simplex
M1 1492 12944 1.72e3 73687 *
M2 1769 3251 1.50e4 73517 *
M3 6382 12475 8.20e3 o.m. *
M4 1569 9388 2.51e4 129034 *
M5 3922 24673 * o.m. *
M6 14004 42359 9.20e4 o.m. *

Table 10: Time for precision 1e-3 (s). * means reaching of maximal running time. The fourth column uses the default output format in the SCS package.

Data AGPPA AL\_CD SCS IPM Simplex
M1 8191 14789 3.31e4 92433 *
M2 7317 18048 * 100646 *
M3 33713 67460 1.41e5 o.m. *
M4 4676 13087 * 170938 *
M5 10667 38521 * o.m. *
M6 45294 134146 * o.m. *

Table 11: Time for precision 1e-5 (s). * means reaching of maximal running time. The fourth column uses the default output format in the SCS package.

6.3 Covering and packing LPs

Following the same notations as in [25], we define the covering LP problem as:

$$\min \, c^\top x \quad \text{s.t.} \quad Ax \geq e, \quad x \geq 0,$$

(105)

where $e \in \mathbb{R}^m$ is a unit vector, cost vector $c \in \mathbb{R}^n_+$ and matrix $A \in \mathbb{R}^{m \times n}$ are all nonnegative. Same as in [25], we generate large synthetic matrix $A$ by

$$A = \text{sprand}(m, n, \text{sparsity}); A = \text{round}(A);$$

We test covering LP problems [105] with $m < n$ and note that the dual of covering LP problem is packing LP problem.

Here, covering LP problems can be transformed into the general LP form [75] with $m_I = m, m_K = 0, n_b = n, n_f = 0$. Then all data sets have $m_I < n_b$ and therefore AGPPA is automatically applied to the dual form. The data set is given in Table [12] and the numerical results are shown in Table [13, 14, 15].
Table 12: Data Statistics for Convering LPs

| Data | AGPPA | AGPPAi | AL_\text{CD} | SCS | IPM | Simplex |
|------|-------|--------|--------------|-----|-----|---------|
| C1   | 1.4   | 1.5    | 1.1          | 1.3 | 37.6 | 1.2     |
| C2   | 2.6   | 2.6    | 2.2          | 2.5 | 38.3 | 4.1     |
| C3   | 5.6   | 5.6    | 4.1          | 4.3 | o.m. | 5.5     |
| C4   | 3.2   | 3.5    | 3.1          | 8.5 | 20.8 | 6.1     |
| C5   | 6.2   | 6.1    | 5.2          | 10.1| o.m. | 15.5    |
| C6   | 13.4  | 13.6   | 11.5         | 15.4| o.m. | 29.6    |

Table 13: RAM (GB). o.m. means out of memory (95GB).

| Data | AGPPA | AGPPAi | AL_\text{CD} | SCS | IPM | Simplex |
|------|-------|--------|--------------|-----|-----|---------|
| C1   | 161   | 20506  | 440          | 1.40e5| 76578 | 36602 |
| C2   | 428   | 49188  | 764          | *   | 71718 | 20401 |
| C3   | 569   | 2099   | *            | *   | o.m. | *       |
| C4   | 1592  | 9824   | 8.35e4      | 24723| 49422 |
| C5   | 2114  | 13985  | 2426         | *   | o.m. | *       |
| C6   | 4003  | 7836   | *            | *   | o.m. | *       |

Table 14: Time for precision 1e-3 (s). * means reaching of maximal running time. The fifth column uses the default output format in the SCS package.

| Data | AGPPA | AGPPAi | AL_\text{CD} | SCS | IPM | Simplex |
|------|-------|--------|--------------|-----|-----|---------|
| C1   | 398   | 53519  | 653          | *   | 95332| 36602 |
| C2   | 819   | 2659   | *            | 89.04 | 20401 |
| C3   | 857   | 3035   | *            | o.m. | *   |
| C4   | 7029  | 38835  | 10696        | 34281| 49422 |
| C5   | 7261  | 7522   | *            | o.m. | *   |
| C6   | 8459  | 17224  | *            | o.m. | *   |

Table 15: Time for precision 1e-5 (s). * means reaching of maximal running time.

6.4 Conclusions about the numerical results

We draw some conclusions about the numerical results of the above three different types of LP problems, including memory usage based on Table 5, Table 9 and Table 13, time efficiency up to low precision 1e-3 based on Table 6, Table 10 and Table 14, and time efficiency up to medium precision 1e-5 based on Table 7, Table 11 and Table 15. We emphasize that we focus on large-scale
LP problems, see the information on the data sets in Table 4, Table 8 and Table 12.

1. In terms of memory usage, AGPPA, AL_{CD} and SCS are comparable while IPM and simplex methods typically require high memory and can fail due to out of memory problem.
2. AGPPAI shows significantly worse performance than AGPPA, which proves the necessity of directly dealing with the LP problem in the general form of (75).
3. AGPPA V.S. AL_{CD}: for both low and medium precision, we observe that AGPPA performs consistently better than AL_{CD}. For most cases, AL_{CD} is approximately two to four times slower than AGPPA. For the largest dataset avazu, AGPPA shows prominent advantage over AL_{CD}, see Table 7.
4. AGPPA V.S. SCS: for many instances, SCS fails to give a solution within the maximal running time (2e5 seconds). Otherwise, it can be at least four times slower than AGPPA to reach medium precision.
5. AGPPA V.S. IPM: IPM encounters the memory problem and fails on some instances. Otherwise, IPM can be at least ten times slower than AGPPA.
6. AGPPA V.S. simplex: the simplex method may fail to return a solution (of high-accuracy) within the maximal running time. For those instances that the simplex method can solve, the running time of simplex is approximately five to ten times higher than the time before AGPPA returns a solution of medium accuracy.

7 Conclusion

In this paper, we propose a new self-adaptive technique to update the proximal regularization parameters in the proximal point method, for a maximal monotone operator satisfying an error bound condition. The proposed adaptive proximal point algorithm (AGPPA) is proved to have a linear convergence rate without any knowledge on the error bound condition parameter. We apply AGPPA on a class of convex programming problem and analyze the iteration complexity bound if a linearly convergent inner solver can be applied to the subproblems. Our approach allows to have a hybrid inner solver and thus can benefit from local fast convergence of second-order methods while keeping the same complexity bound. We illustrate the application to the LP problem and obtain a complexity bound with a weaker dependence on the problem dimension and on the Hoffman constant associated with the KKT system. Finally we demonstrate the numerical efficiency of our method on various large-scale LP problems.

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A Supplementary proofs

Denote

\[ Q_{\sigma M^{-1} T}(z) := z - J_{\sigma M^{-1} T}(z). \]

We present the properties of \( J_{\sigma M^{-1} T} \) and \( Q_{\sigma M^{-1} T} \) in the following proposition, which is summarized in [25] Proposition 1.

**Proposition 6** ([25]) It holds for all positive real numbers \( \sigma \) and all self-adjoint positive definite linear operators \( \mathcal{M} \) that: for all \( z, z' \in \mathcal{X} \),

(a) \( z = J_{\sigma M^{-1} T}(z) + Q_{\sigma M^{-1} T}(z) \) and \( \sigma^{-1} M Q_{\sigma M^{-1} T}(z) \in T(J_{\sigma M^{-1} T}(z)). \)

(b) \( \langle J_{\sigma M^{-1} T}(z) - J_{\sigma M^{-1} T}(z'), Q_{\sigma M^{-1} T}(z) - Q_{\sigma M^{-1} T}(z') \rangle_{\mathcal{M}} \geq 0. \)

(c) \( \| J_{\sigma M^{-1} T}(z) - J_{\sigma M^{-1} T}(z') \|_{\mathcal{M}}^2 + \| Q_{\sigma M^{-1} T}(z) - Q_{\sigma M^{-1} T}(z') \|_{\mathcal{M}}^2 \leq \| z - z' \|_{\mathcal{M}}^2 . \)

**Proof (Proof of Theorem 2)** Define

\[ z_{k+1}^* := \gamma J_{\sigma M^{-1} T}(z_k) + (1 - \gamma) z_k, \]

then for any \( z^* \in \Omega \), we have

\[
\| z^{k+1} - z^* \|_{\mathcal{M}} \leq \| z^{k+1} - z^* \|_{\mathcal{M}} + \gamma \| w^k - J_{\sigma M^{-1} T}(z_k) \|_{\mathcal{M}} \leq \| z^{k+1} - z^* \|_{\mathcal{M}} + \delta_k \gamma \| w^k - z^k \|_{\mathcal{M}} \leq \| z^{k+1} - z^* \|_{\mathcal{M}} + \delta_k \| z^{k+1} - z^k \|_{\mathcal{M}} + \delta_k \| z^k - z^* \|_{\mathcal{M}},
\]

which implies

\[
\| z^{k+1} - z^* \|_{\mathcal{M}} \leq \frac{1}{1 - \delta_k} \left( \| z^{k+1} - z^* \|_{\mathcal{M}} + \delta_k \| z^k - z^* \|_{\mathcal{M}} \right). \]

Denote

\[
\mu_k := \frac{K_r}{\sqrt{\sigma_k} + \kappa_2},
\]

and

\[
\Pi_{\Omega}^M (J_{\sigma M^{-1} T}(z^k)) := \text{arg min}_{z \in \Omega} \| z - J_{\sigma M^{-1} T}(z^k) \|_{\mathcal{M}},
\]

\[
\Pi_{\Omega}^M (z^k) := \text{arg min}_{z \in \Omega} \| z - z^k \|_{\mathcal{M}},
\]

then by Proposition 6(c) and (10), we have

\[
\begin{align*}
\| z^k - \Pi_{\Omega}^M (J_{\sigma M^{-1} T}(z^k)) \|_{\mathcal{M}} & \leq \| J_{\sigma M^{-1} T}(z^k) - \Pi_{\Omega}^M (J_{\sigma M^{-1} T}(z^k)) \|_{\mathcal{M}} + \| Q_{\sigma M^{-1} T}(z^k) \|_{\mathcal{M}} \\
& \leq \text{dist}_{\mathcal{M}} (J_{\sigma M^{-1} T}(z^k), \Omega) + \text{dist}_{\mathcal{M}} (z^k, \Omega) \\
& \leq (\mu_k + 1) \text{dist}_{\mathcal{M}} (z^k, \Omega).
\end{align*}
\]
Case I: $\gamma \in [1, 2)$, and let
\[ z^* = B^M_M^\gamma \left( J_{\sigma_k M^{-1} T} (z^k) \right). \]
Let $a = J_{\sigma_k M^{-1} T} (z^k), b = z^k, c = z^*$ for equality
\[ (a - c, b - c) = \frac{1}{2} \left( \|a - c\|_M^2 - \|a - b\|_M^2 + \|b - c\|_M^2 \right), \]
then we have
\[
\begin{align*}
\|z^{k+1} - z^*\|_M^2 &= \|\gamma (J_{\sigma_k M^{-1} T}(z^k)) - (z^k - z^*)\|_M^2 \\
&= \gamma^2 \text{dist}_{\text{M}}^2 \left( J_{\sigma_k M^{-1} T}(z^k), \Omega \right) + (1 - \gamma^2) \|z^k - z^*\|_M^2 \\
&+ 2\gamma (1 - \gamma) \left( J_{\sigma_k M^{-1} T}(z^k) - z^*, z^k - z^* \right)_M \\
&= \frac{\gamma}{2} \left( J_{\sigma_k M^{-1} T}(z^k), \Omega \right) + (1 - \gamma) \|z^k - z^*\|_M^2 \\
&+ (\gamma^2 - \gamma) \|z^k - J_{\sigma_k M^{-1} T}(z^k)\|_M^2.
\end{align*}
\]
By Proposition 6 (c) we have
\[
\begin{align*}
\|z^k - J_{\sigma_k M^{-1} T}(z^k)\|_M^2 &\leq \text{dist}_{\text{M}}^2 (z^k, \Omega) - \|J_{\sigma_k M^{-1} T}(z^k) - B^M_M^\gamma (z^k)\|_M^2.
\end{align*}
\]
Plugging (111) into (110) leads to
\[
\begin{align*}
\|z^{k+1} - z^*\|_M^2 &\leq \gamma \text{dist}_{\text{M}}^2 \left( J_{\sigma_k M^{-1} T}(z^k), \Omega \right) + (1 - \gamma) \|z^k - z^*\|_M^2 \\
&+ (\gamma^2 - \gamma) \left( \text{dist}_{\text{M}}^2 (z^k, \Omega) - \|J_{\sigma_k M^{-1} T}(z^k) - B^M_M^\gamma (z^k)\|_M^2 \right),
\end{align*}
\]
with
\[ \gamma \geq 1, \|z^k - z^*\|_M \geq \text{dist}_{\text{M}} (z^k, \Omega), \]
and
\[ \|J_{\sigma_k M^{-1} T}(z^k) - B^M_M^\gamma (z^k)\| \geq \text{dist}_{\text{M}} \left( J_{\sigma_k M^{-1} T}(z^k), \Omega \right), \]
we have
\[
\begin{align*}
\|z^{k+1} - z^*\|_M^2 &\leq \gamma \|z^k - z^*\|_M^2 + (\gamma - 1) \|z^k - z^*\|_M^2 \\
&+ (\gamma^2 - \gamma) \left( \|z^k - z^*\|_M^2 - \|J_{\sigma_k M^{-1} T}(z^k) - B^M_M^\gamma (z^k)\|_M^2 \right) \\
&\leq \frac{1}{1 - \gamma} \left( \frac{1 - \gamma^2}{\sigma_k^2 + \kappa^2} \right) \text{dist}_{\text{M}} (z^k, \Omega),
\end{align*}
\]
Therefore, combine (112), (109) and (108), we have
\[
\begin{align*}
\text{dist}_{\text{M}_{k+1}} (z^{k+1}, \Omega) &\leq \sqrt{\frac{1 - \frac{2(\gamma - 1)}{\sigma_k^2 + \kappa^2} + \delta_k \mu_k + 1}{1 - \delta_k}} \text{dist}_{\text{M}} (z^k, \Omega)
\end{align*}
\]
Case II: $\gamma \in (0, 1]$, let 
\[ z^* = \gamma \Pi_{\Omega}^M (J_{s_k} \lambda_{M-1}T (z^k)) + (1 - \gamma) \Pi_{\Omega}^M (z^k), \]
then $z^* \in \Omega$. With Proposition 11 and $\gamma \in (0, 1]$, we have
\[
\begin{align*}
\|z_{k+1}^* - z^*\|_M^2 &= \|\gamma (J_{s_k} \lambda_{M-1}T (z^k)) - \Pi_{\Omega}^M (J_{s_k} \lambda_{M-1}T (z^k))\|_M^2 + (1 - \gamma)^2 \|z^k - \Pi_{\Omega}^M (z^k)\|_M^2 \\
&\quad + 2\gamma (1 - \gamma) \langle J_{s_k} \lambda_{M-1}T (z^k), \Pi_{\Omega}^M (J_{s_k} \lambda_{M-1}T (z^k)) - z^k - \Pi_{\Omega}^M (z^k)\rangle_M \\
&\leq (1 - \gamma) \|z^k - \Pi_{\Omega}^M (z^k)\|_M^2 + \gamma \|J_{s_k} \lambda_{M-1}T (z^k) - \Pi_{\Omega}^M (J_{s_k} \lambda_{M-1}T (z^k))\|_M^2 \\
&\quad - \gamma (1 - \gamma) \|J_{s_k} \lambda_{M-1}T (z^k) - \Pi_{\Omega}^M (J_{s_k} \lambda_{M-1}T (z^k)) - (z^k - \Pi_{\Omega}^M (z^k))\|_M^2 \\
&\leq (1 - \gamma) \text{dist}_M^2 (z^k, \Omega) + \gamma \text{dist}_M (J_{s_k} \lambda_{M-1}T (z^k), \Omega) \\
&\leq \left(1 - \frac{\gamma \sigma_k^2}{\sigma_k^2 + \kappa_\gamma^2}\right) \text{dist}_M^2 (z^k, \Omega),
\end{align*}
\] which implies
\[
\|z_{k+1}^* - z^*\|_M \leq \sqrt{1 - \frac{\gamma \sigma_k^2}{\sigma_k^2 + \kappa_\gamma^2}} \text{dist}_M (z^k, \Omega). \tag{114}
\]
In addition, with 109, we have 
\[
\|z_k - z^*\|_M \leq \gamma \|z^k - \Pi_{\Omega}^M (J_{s_k} \lambda_{M-1}T (z^k))\|_M + (1 - \gamma) \text{dist}_M (z^k, \Omega) \\
\leq (\gamma \mu_k + 1) \text{dist}_M (z^k, \Omega), \tag{115}
\]
Combine 107, 114 and 115, we have 
\[
\text{dist}_M (z^{k+1}, \Omega) \leq \sqrt{1 - \frac{\gamma \sigma_k^2}{\sigma_k^2 + \kappa_\gamma^2} + \delta_k (\gamma \mu_k + 1)} \text{dist}_M (z^k, \Omega). \tag{116}
\]
The linear rate is proved by 113 and 116 with 108.

**Proof (Proof of Proposition 11)** By Proposition 11, we have 
\[
\|Q_{\lambda_{M-1}T} (z^k)\|_M \leq \text{dist}_M (z^k, \Omega). \tag{117}
\]
Then 
\[
\begin{align*}
\|z^{k+1} - z^k\|_M &= \gamma \|w^k - J_{s_k} \lambda_{M-1}T (z^k) + J_{s_k} \lambda_{M-1}T (z^k) - z^k\|_M \\
&\leq \gamma \|w^k - J_{s_k} \lambda_{M-1}T (z^k)\|_M + \gamma \|Q_{\lambda_{M-1}T} (z^k)\|_M \\
&\leq \delta \gamma \|w^k - z^k\|_M + \gamma \|Q_{\lambda_{M-1}T} (z^k)\|_M \\
&\leq \delta \|z^{k+1} - z^k\|_M + \gamma \text{dist}_M (z^k, \Omega),
\end{align*}
\]
leading to
\[
\frac{1 - \delta}{\gamma} \| z^{k+1} - z^k \|_M \leq \text{dist}_M (z^k, \Omega).
\]
Thus, the left part is proved. Now we aim to prove the right part of the inequality. Since
\[
\| z^{k+1} - z^k \|_M = \gamma \| w^k - J_{\sigma_k, M^{-1}T}(z^k) + J_{\sigma_k, M^{-1}T}(z^k) - z^k \|_M
\]
\[
\geq \gamma \| Q_{\sigma_k, M^{-1}T}(z^k) \|_M - \gamma \| w^k - z^k \|_M
\]
\[
\geq \gamma \| Q_{\sigma_k, M^{-1}T}(z^k) \|_M - \delta \gamma \| z^k - z^k \|_M
\]
\[
= \gamma \| Q_{\sigma_k, M^{-1}T}(z^k) \|_M - \delta \| z^{k+1} - z^k \|_M,
\]
we have
\[
\| Q_{\sigma_k, M^{-1}T}(z^k) \|_M \leq \frac{1 + \delta}{\gamma} \| z^{k+1} - z^k \|_M.
\]
In addition, by Lemma 11, we have
\[
\text{dist}_M (z^k, \Omega) \leq \| J_{\sigma_k, M^{-1}T}(z^k) - z^k \|_M + \text{dist}_M (J_{\sigma_k, M^{-1}T}(z^k), \Omega)
\]
\[
\leq \| Q_{\sigma_k, M^{-1}T}(z^k) \|_M + \frac{\kappa_r}{\sqrt{\sigma_k^2 + \kappa_r^2}} \text{dist}_M (z^k, \Omega),
\]
which implies
\[
\left(1 - \frac{\kappa_r}{\sqrt{\sigma_k^2 + \kappa_r^2}} \right) \text{dist}_M (z^k, \Omega) \leq \| Q_{\sigma_k, M^{-1}T}(z^k) \|_M.
\]
Combining (119) and (115), we have
\[
\text{dist}_M (z^k, \Omega) \leq \gamma \left(1 - \frac{\kappa_r}{\sqrt{\sigma_k^2 + \kappa_r^2}} \right) \| z^{k+1} - z^k \|_M,
\]
and the right part is proved.

**Proof (proof of Proposition 2)** If (20) holds and
\[
k \geq \log_\frac{1}{\rho} \left( \frac{R(\sigma) \zeta \text{dist}_M (z^0, \Omega)}{\lambda_{\text{min}}(M) \epsilon} \right),
\]
then we have
\[
\| z^{k+1} - z^k \|_M \leq \frac{C \lambda_{\text{min}}(M) \epsilon}{R(\sigma) \zeta \text{dist}_M (z^0, \Omega)} \| z^1 - z^0 \|_M
\]
\[
\leq \frac{C \lambda_{\text{min}}(M) \epsilon}{R(\sigma) \zeta \text{dist}_M (z^0, \Omega)} \frac{\gamma}{1 - \delta} \text{dist}_M (z^0, \Omega)
\]
\[
\leq \frac{\gamma \left(1 - \sqrt{\frac{\kappa_r^2}{\sigma_k^2 + \kappa_r^2}} \right) \lambda_{\text{min}}(M) \epsilon}{(1 + \delta) \zeta}
\]
(120)
where the second inequality derives from the first inequality in Proposition 1 and the last inequality is from the definition of \( R(\sigma) \) as in (22). In addition,

\[
E(z^k) \leq \zeta \text{dist}(z^k, \Omega) \leq \frac{\zeta}{\lambda_{\min}(\mathcal{M})} \text{dist}_\mathcal{M}(z^k, \Omega).
\]

Then we have

\[
E(z^k) \leq \frac{\zeta}{\lambda_{\min}(\mathcal{M})} \text{dist}_\mathcal{M}(z^k, \Omega) \leq \frac{\zeta}{\lambda_{\min}(\mathcal{M})} \left(1 + \frac{\gamma (1 - \sqrt{1 - \kappa^2 r/\sigma^2_0})}{\sigma^2_1 + \kappa^2 r} \right) \|z^{k+1} - z^k\|_\mathcal{M} \leq \epsilon,
\]

where the second inequality derives from the second inequality in Proposition 1.

Proof (Proof of Corollary 4) First we note that

\[
1 - \frac{r^2}{\sigma^2_0 + \kappa^2 r} \leq 2 \left(1 + \frac{\kappa^2}{\sigma^2_0}\right),
\]

from which we deduce

\[
R = O \left(\zeta \kappa^2 r\right).
\]

It follows that

\[
\left[\max \left(\log \frac{R}{\epsilon}, 0\right) + 1\right] = O \left(\ln \left(\frac{\zeta \kappa^2 r}{\epsilon}\right)\right).
\]

It is also easy to see that

\[
\tilde{s} = \left[\max \left(\log \frac{\kappa r \sigma}{\sigma_0}, 0\right)\right] = O \left(\ln(\kappa r)\right),
\]

Hence,

\[
- \ln \tilde{\eta} = - \ln \eta_0 - \tilde{s} \ln \vartheta_0 + \zeta \ln \left[\max \left(\log \frac{R}{\epsilon}, 0\right) + 1\right] = O \left(\ln \kappa r\right) + O \left(\ln \ln \left(\frac{\zeta \kappa^2 r}{\epsilon}\right)\right),
\]

and

\[
\bar{N} = \tilde{s} \left[\max \left(\log \frac{R}{\epsilon}, 0\right) + 1\right] = O \left(\ln \kappa r \ln \left(\frac{\zeta \kappa^2 r}{\epsilon}\right)\right).
\]
\textbf{Proof (Proof of Corollary 5)} First, ignoring problem independent constants, from definitions in (64) we get
\[
\ln \bar{\zeta}_1 = O \left( \ln \frac{r}{\eta} + \ln \frac{1}{\eta} + \ln \frac{1}{\bar{\eta}} + \ln \frac{1}{p} + \ln \frac{1}{\bar{\eta}} + \ln \left( L_0 + a \right) + \ln \frac{1}{p} \right),
\]
\[
\ln \bar{\zeta}_2 = O \left( \ln \left( L_0 + a \right) + \ln \frac{1}{\eta} + \ln \frac{1}{\bar{\eta}} + \ln \left( L_0 + a \right) + \ln \frac{1}{p} \right),
\]
and therefore
\[
\left[ \ln \bar{N} + \max \left( \max (\ln \bar{\zeta}_1, \ln \bar{\zeta}_2), 0 \right) \right]
= O \left( \ln \bar{N} + \ln \frac{r}{\eta} + \ln \frac{1}{\eta} + \ln \frac{1}{\bar{\eta}} + \ln \left( L_0 + a \right) + \ln \frac{1}{p} \right).
\]
Hence the bound (72) is of order
\[
O \left( (\vartheta \bar{\sigma} + \Upsilon) \bar{N} \left( \ln \bar{N} + \ln \frac{r}{\eta} + \ln \frac{1}{\eta} + \ln \left( L_0 + a \right) + \ln \frac{1}{p} \right) \right).
\]
By (30) we have
\[
O(\bar{\sigma}) = O((\alpha g_{\sigma})^\kappa_{\sigma}),
\]
with $\alpha$ and $g_{\sigma}$ being user defined parameters. Then the result directly follows Corollary 4.

\textbf{B Acceleration with PN\_CG}

The inner problem (86) can be reformulated as:
\[
\min_{x \geq 0} \left\{ \tilde{f}(x) := c^T x + \frac{1}{2\sigma} \left\| \bar{\lambda} + \sigma (Ax - b) + \frac{1}{\sigma} \left( \bar{\lambda} + \sigma (Ax - b) \right) \right\|_+^2 + \frac{1}{2\sigma} \left\| x - \bar{x} \right\|^2 \right\}.
\]
Notice that $\nabla \tilde{f}$ is a semismooth function \cite{Definition 3.5}. Then the gradient and generalized Hessian of $f$ can respectively be expressed as
\[
\nabla \tilde{f}(x) = A^T A(x, \bar{\lambda}, \sigma) + \frac{1}{\sigma} \left( x - \bar{x} \right)
\]
\[
\nabla^2 \tilde{f}(x) = \sigma A^T D \left( A(x, \bar{\lambda}, \sigma) \right) A + \frac{1}{\sigma} I_n,
\]
where $I_n \in \mathbb{R}^{n \times n}$ is an identity matrix and $D(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$ maps to a diagonal matrix with
\[
D_{ii}(w) = \begin{cases} 
0, & \text{if } i \leq m_I \text{ and } w_i < 0 \\
1, & \text{otherwise.}
\end{cases}
\]
PN-CG use the active strategy. Concretely, it only updates the coordinates of $x$ from the active set defined by:

$$\mathcal{A} := \{i \in [n] \mid x_i = 0, \frac{\partial \tilde{f}(x)}{\partial x_i} > 0\}. \tag{125}$$

Use $[\cdot]_{\mathcal{A}}$ for vector with coordinates to be retained from $\mathcal{A}$ and to be zero otherwise or matrix with the entries to be retained for both rows and columns from $\mathcal{A}$ and to be zero otherwise. Then we show the process of PN-CG as Algorithm 4. Different from PN-CG in [40, Section 3.4], we use stopping

**Algorithm 4**

**Parameters:** $\mu \in (0, 1/2), \nu \in (0, 1), \varrho \in (0, 1), \tau \in (0, 1]$

1. Compute the active set as (125).
2. Compute $\nabla_{\mathcal{A}} \tilde{f}(x)$ and $\nabla_{\mathcal{A}}^2 \tilde{f}(x)$ based on (124).
3. Solve the linear system

$$\nabla_{\mathcal{A}}^2 \tilde{f}(x)y = -\nabla_{\mathcal{A}} \tilde{f}(x) \tag{126}$$

exactly or by the conjugate gradient (CG) algorithm to find $y$ such that

$$\left\| \nabla_{\mathcal{A}}^2 \tilde{f}(x)y + \nabla_{\mathcal{A}} \tilde{f}(x) \right\| \leq \min \left\{ \nu \left\| \nabla_{\mathcal{A}} \tilde{f}(x) \right\|, \left\| \nabla_{\mathcal{A}} \tilde{f}(x) \right\|^{1+\tau} \right\} \tag{127}$$

4. (Line search) Compute $\varrho^j$ with $j$ to be the first nonnegative integer satisfying

$$\tilde{f}(x + \varrho^j y_{\mathcal{A}}^n) \leq \tilde{f}(x) + \mu \varrho^j \langle \nabla_{\mathcal{A}} \tilde{f}(x), y_{\mathcal{A}} \rangle$$

5. Output $x^+ = [x + \varrho^j y_{\mathcal{A}}^n]_{\mathcal{A}}$

 criterion (127) for linear system (126) instead of

$$\left\| \nabla_{\mathcal{A}}^2 \tilde{f}(x)y + \nabla_{\mathcal{A}} \tilde{f}(x) \right\| \leq \nu \left\| \nabla_{\mathcal{A}} \tilde{f}(x) \right\|. \tag{128}$$

With this modification, we find that when $n_b = 0$, then Algorithm 4 reduces to SSN method in [25] with slight modification, i.e., we apply stopping criterion (127) for linear system (126) instead of

$$\left\| \nabla_{\mathcal{A}}^2 \tilde{f}(x)y + \nabla_{\mathcal{A}} \tilde{f}(x) \right\| \leq \min \left\{ \nu \left\| \nabla_{\mathcal{A}} \tilde{f}(x) \right\|^{1+\tau} \right\}. \tag{129}$$

Note that when $n_b = 0$, same as the classical SSN, it is trivial to check that Algorithm 4 still globally converges and keeps the local superlinear convergence, which shows the advantage of our stopping criterion over (128). In addition, the form (127) is more amicable for the choice of $\nu$ than (129). Thus we have an adequate motivation to study the convergence of Algorithm 4 for the case where $n_b > 0$ and we will do further research in the subsequent paper. Again, we emphasize that there is no specific requirement on the theoretic convergence of Algorithm 4 when it is used in Algorithm 3.
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