A Simple Proof of the Non-Renormalization of the Chern-Simons Coupling

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Abstract

We give a very simple proof that the renormalization of the Chern-Simons coupling in the Wilsonian effective action is exhausted at one-loop. Our proof can apply to arbitrary 2+1-dimensional abelian as well as nonabelian gauge theories without a bare Chern-Simons coupling, including any non-renormalizable interactions and non-minimal couplings. Our proof reveals that small (but not large) gauge invariance is enough to ensure the absence of higher order corrections.

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1 Introduction

Various powerful tools have been developed recently in understanding the dynamics of supersymmetric gauge theories (for a review, see [3]). Holomorphy is one of such tools, and Seiberg [2] used it to greatly simplify the original proof of the non-renormalization theorem [4] and succeed to find the exact vacuum structures of various supersymmetric models. Although holomorphy is inherent in supersymmetric theories, some of the techniques developed in supersymmetric theories seem to be applied to non-supersymmetric ones. Nevertheless, very few are known about examples of such applications. The purpose of this letter is to present an example that techniques developed in supersymmetric theories are applied to non-supersymmetric ones to obtain (perturbative or non-perturbative) exact results.

We consider a general (non-supersymmetric) gauge theory in 2+1-dimensions and give a new simple proof that the renormalization of the Chern-Simons coupling is exhausted at one-loop by using techniques developed in supersymmetric theories. Our proof makes it clear why the one-loop correction to the Chern-Simons coupling is so special. Most of our results have already been known but some of them are new.

Gauge theories in 2+1-dimensions have a special feature to allow a Chern-Simons term [4] (for a review, see [5])

\[ \int d^3 x \mathcal{L}_{CS} = \int d^3 x \kappa \epsilon^{\mu \nu \lambda} \text{tr} \left( A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right). \] (1)

This term can be generated in one-loop order of perturbation theory [4, 6, 7, 8] but the two-loop correction is shown to vanish [9, 10]. For abelian theories, there are no further corrections at higher loops [11, 12, 13]. It is widely believed that this is true even for nonabelian theories, although no rigorous proof has been given. This expectation is based on the topological nature of the Chern-Simons term. In nonabelian theories, this term is not invariant under large gauge transformations, and consequently its coefficient must be quantized to obtain consistent quantum theories. If there were further corrections at higher loops, they would necessarily spoil the quantized nature of the coupling. Another circumstantial evidence of the non-renormalization is that the \( \beta \)-function of the Chern-Simons coupling vanishes in all orders of perturbation theory [14, 15], although this does not mean that there are no finite corrections to it.

The above topological reasoning to the non-renormalization of the Chern-Simons coupling is, however, somewhat mysterious because in perturbation theory we would not \textit{a priori} expect to “know” anything about non-perturbative large gauge transformations and because the argument cannot apply to abelian theories. Our proof given in the next section can apply to both abelian and nonabelian theories and relies only on gauge invariance under small gauge transformations. Furthermore, our proof can apply to a wider
class of gauge theories than those discussed by Coleman and Hill [12]. The authors proved the non-renormalization theorem of the Chern-Simons coupling in a class of minimally coupled gauge theories, in which gauge interactions are introduced by replacing spacetime derivatives by covariant ones, though their proof can allow non-renormalizable interactions. Gauge theories we consider are more general and are not restricted to minimally coupled ones.

The paper is organized as follows: In Section 2, some background of recent developments in supersymmetric theories is explained and then a very simple proof of the non-renormalization theorem is given. In Section 3, several remarks are made.

2 A Proof

We consider a general 2 + 1-dimensional gauge theory involving scalar fields $\Phi_i$, spinor fields $\Psi_a$ and the gauge field $A_\mu$. The general gauge-invariant action we start with is of the form

$$S = \int d^3x \left\{ -\frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} - \Phi^+_i \left( D_\mu D^\mu + m_i^2 \right) \Phi_i + \overline{\Psi}_a (i\not\!D - m_a) \Psi_a + G(A_\mu, \Phi, \Psi) \right\} ,$$

(2)

where $G$ is a general gauge-invariant function of $A_\mu$, $\Phi_i$ and $\Psi_a$. We here assume that there is no bare Chern-Simons coupling. A generalization to the theory with a non-vanishing bare Chern-Simons coupling will be discussed in the next section.

It should be emphasized that we take the normalization that the gauge coupling $g$ does not appear in the covariant derivative $D_\mu$, i.e. $D_\mu \equiv \partial_\mu - iA_\mu$. It is known that in supersymmetric gauge theories this normalization of the gauge coupling preserves holomorphy and is crucial to prove the one-loop exactness of the gauge coupling renormalization [16], as stressed in Ref. [17]. This normalization of the gauge coupling turns out to be crucial also in our proof of the one-loop exactness of the Chern-Simons coupling renormalization, as we will see below.

We shall give a proof of the non-renormalization theorem of the Chern-Simons coupling in terms of the Wilsonian effective action (for a review, see [18]), which is identical to the 1PI effective action when there are no interacting massless particles. In supersymmetric theories, it is crucial to distinguish two effective actions, as pointed out by Shifman and Vainshtein [16]. When interacting massless particles are present, the 1PI effective action suffers from infrared ambiguities and might suffer from holomorphic anomalies, which would lead to the violation of non-renormalization theorems [18, 20, 21, 22]. These are absent in the Wilsonian effective action. Interestingly, a similar situation happens in our problem. It has been reported, in Refs. [14, 23, 24, 25], that non-vanishing radiative

\footnote{We assume that $G$ contains no quadratic terms in the fields.}
corrections to the Chern-Simons coupling will appear in higher loops when massless particles are present. These higher order corrections are caused by infrared singularities. To avoid such infrared problems, it should be understood that our proof is applied to the Chern-Simons coupling defined in the Wilsonian effective action. Another advantage of the use of the Wilsonian effective action is that non-renormalizable terms in the action (2) can be managed because high-momentum modes beyond a cutoff are not included in the functional integration. The momentum cutoff, however, gets into trouble with gauge invariance [26]. This subject is beyond the scope of this letter and we assume that the cutoff respects gauge invariance in the Wilsonian effective action.

Our proof of the non-renormalization of the Chern-Simons term is an application of the works by Dine [27] and Weinberg [28], in which a simple proof of the non-renormalization [29] of the Fayet-Iliopoulos $U(1)D$-term [30] has been given in general supersymmetric gauge theories. The key technical tool used in their proof is the Seiberg trick [2] of regarding coupling parameters as the scalar components of external superfields. Following the approach by Dine and Weinberg, we regard all the couplings in the action (2) as the external scalar fields. Suppose that the Chern-Simons term is generated in the Wilsonian effective action. Then, the induced Chern-Simons coupling $\kappa$ would depend on the gauge coupling and other couplings appearing as coefficients of interaction terms in $G$. However, if we regard the couplings as the scalar fields, the expression (1) is not gauge-invariant even under small gauge transformations unless $\kappa$ is a constant, i.e. independent of all coupling parameters. The graphs independent of all couplings are just the one-loop ones generated from the action without $G$ in Eq.(2).

3 Remarks

We have presented a very simple (rather trivial) proof of the non-renormalization theorem of the Chern-Simons coupling in 2 + 1-dimensional gauge theories without a bare Chern-Simons coupling. Most of the results given in this letter have already been known but some of them are new: For abelian theories, our proof can apply to a wider class of gauge-invariant theories than those considered by Coleman-Hill [12], in which gauge interactions have been restricted to the minimal coupling. For nonabelian theories, our results are new and our proof has revealed that only small gauge invariance is enough to ensure the absence of higher order corrections. We do not need to require large gauge invariance at all.

We have started with general gauge theories without a bare Chern-Simons coupling.

4 Precisely speaking, we regard the gauge coupling and all coefficients of interaction terms in $G$ as the external scalar fields. The mass parameters remain to be constants.
When a bare Chern-Simons term is present, our proof could not apply to this case. This is because the Chern-Simons term (1) would not be gauge-invariant if the bare Chern-Simons coupling is regarded as a scalar field. For abelian theories, this problem can be remedied by still keeping it a constant and by absorbing it into the gauge field propagator. This does not, however, work for nonabelian theories because the nonabelian Chern-Simons term includes an interaction term. Although we do not know whether our proof can be generalized to this case, it may not be unreasonable to expect that a similar proof could work for Yang-Mills-Chern-Simons gauge theories because no higher order corrections to the Chern-Simons coupling have been proved for pure nonabelian Chern-Simons gauge theories [31, 32, 33].

As mentioned in Section 2, it is crucial, in our proof, to take the convention that the gauge coupling does not appear in the covariant derivative. One might expect that the proof given in the previous section could hold in other conventions of the gauge coupling, for example, the canonical normalization, in which $A_\mu$ in Eq.(2) may be replaced by $gA_\mu$. This is not, however, the case because in the canonical normalization the gauge kinetic term would break gauge invariance if we think of the gauge coupling as a scalar field. Furthermore, the rescaling of the field, $A_\mu \rightarrow gA_\mu$, will cause another problem that the functional measure may not be invariant under the rescaling and a Chern-Simons term could be generated from the integration measure [17]. It would be of interest to clarify the relation between different conventions of the gauge coupling.

It may be instructive to make a comment that our proof tells us that the Chern-Simons term (1) can be generated at most at one-loop but does not forbid the appearance of “would be” Chern-Simons terms in the effective action [34, 35, 36]. For example, a term $\epsilon^{\mu\nu\lambda}\Phi^\dagger F_{\mu\nu} D_\lambda \Phi$, which is manifestly gauge-invariant and may be generated at higher loops, may reduce to a Chern-Simons term after replacing $\Phi$ by its expectation value $<\Phi>$ in the Higgs phase.

The final remark is as follows: In the previous section, we have insisted that the induced Chern-Simons coupling must be independent of all coupling parameters. This will be true even non-perturbatively as long as gauge symmetry is preserved in the Wilsonian effective action. We have then concluded, from this fact, that the radiative corrections to the Chern-Simons coupling come from only the one-loop diagrams generated from the action with $G = 0$ in Eq.(2). This conclusion is true as long as the couplings are treated perturbatively but is this true even non-perturbatively? At the moment, we have no definite answer to this question. This is because the requirement of gauge invariance would not forbid the induced Chern-Simons coupling, for example, to be a function of $\lambda/|\lambda|$ for some coupling parameter $\lambda$, since $\lambda/|\lambda|$ looks like a “constant” as long as $\lambda$ is restricted to be positive (or negative). In fact, such dependence is known to appear at one-loop for
$\lambda =$ the fermion masses and the bare Chern-Simons coupling. Since the mass parameters are incorporated into the propagators and are not treated perturbatively, this may suggest that a locally-constant function of the couplings, which does not contradict gauge invariance, could appear non-perturbatively. It would be of great interest to generalize the perturbative non-renormalization theorem to non-perturbative one.

**Acknowledgement**

We would like to thank for T. Matsuyama and H. So for useful comments and discussions.

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