Type A $\mathcal{N}$-fold Supersymmetry and Generalized Bender–Dunne Polynomials

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Abstract

We derive the necessary and sufficient condition for type A $\mathcal{N}$-fold supersymmetry by direct calculation of the intertwining relation and show the complete equivalence between this analytic construction and the $\mathfrak{sl}(2)$ construction based on quasi-solvability. An intimate relation between the pair of algebraic Hamiltonians is found. The classification problem on type A $\mathcal{N}$-fold supersymmetric models is investigated by considering the invariance of both the Hamiltonians and $\mathcal{N}$-fold supercharge under the $GL(2, K)$ transformation. We generalize the Bender–Dunne polynomials to all the type A $\mathcal{N}$-fold supersymmetric models without requiring the normalizability of the solvable sector. Although there is a case where weak orthogonality of them is not guaranteed, this fact does not cause any difficulty on the generalization. It is shown that the anti-commutator of the type A $\mathcal{N}$-fold supercharges is expressed as the critical polynomial of them in the original Hamiltonian, from which we establish the complete type A $\mathcal{N}$-fold superalgebra. A novel interpretation of the critical polynomials in view of polynomial invariants is given.

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I. INTRODUCTION

The concept of symmetry has played a central role in modern theoretical physics. A discovery of a new symmetry enlarges our ability and possibility to describe new phenomena both in the physical nature and in mathematical models. Conversely, it may be almost certain that there is an underlying symmetry if a system under consideration exhibits a significant property that is not shared in the generic cases. Actually, $\mathcal{N}$-fold supersymmetry was discovered from the observation of the disappearance of the leading divergence of the perturbation series for the specific energy levels of a quantum mechanical model at the particular values of a parameter involved in the model [1]. This symmetry is a generalization of the ordinary supersymmetry in one-dimensional quantum mechanics [2, 3] and is characterized by $\mathcal{N}$th order derivative supercharges. Similar generalizations were found and investigated in various different contexts [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

A generalization to $\mathcal{N}$th order derivative supercharges for all $\mathcal{N}$ is, however, difficult and most of the investigations by the other authors were limited to the cases of the second-order generalization and to the cases constructed by the factorization method. Recently, we found another $\mathcal{N}$-fold supersymmetric model [19] and further found general forms of an $\mathcal{N}$-fold supersymmetric family which we have called type A [20]. On the other hand, several quasi-solvable models [21, 22] were found to have $\mathcal{N}$-fold supersymmetry in Refs. [23, 24, 25, 26] in addition to those reported in Refs. [1, 19]. Then, it was proved in generic way [27] that $\mathcal{N}$-fold supersymmetry is essentially equivalent to quasi-solvability. Furthermore, it was also proved [28] that the type A $\mathcal{N}$-fold supersymmetric models are essentially equivalent to the quasi-solvable models constructed by $\mathfrak{sl}(2)$ generators [29]. Dynamical properties of the $\mathcal{N}$-fold supersymmetric models were discussed in Ref. [27] and investigated for a couple of models in Ref. [31]. The nonperturbative analyses carried out in Ref. [31] together with those in Ref. [1] revealed several significant properties that the type A $\mathcal{N}$-fold supersymmetric models share. Especially, we clarified the important role of the normalizability of the solvable sector, which is crucial for the dynamical $\mathcal{N}$-fold supersymmetry breaking but has rarely discussed by the other authors. Furthermore, it was argued that $\mathcal{N}$-fold supersymmetry is not only sufficient but may also be necessary for the existence of convergent perturbation series. Up to now, most of the quasi-solvable one-dimensional quantum mechanical systems belong to the $\mathfrak{sl}(2)$ quasi-solvable models. Several new findings have been reported in connection with them. One of the examples is the new orthogonal polynomials firstly found by Bender and Dunne [32]. The idea was soon generalized to all the $\mathfrak{sl}(2)$ quasi-exactly solvable models [33, 34]. Furthermore, several realistic physical systems have been found, which can be reduced to one-dimensional quasi-solvable models [24, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44].

In this article, we will report the general aspects of type A $\mathcal{N}$-fold supersymmetry, most of which remain unsolved yet. In Ref. [28], it was discussed that, in view of the $\mathfrak{sl}(2)$ construction of type A $\mathcal{N}$-fold supersymmetry, the condition for type A $\mathcal{N}$-fold supersymmetry derived in Ref. [20] provides only a sufficient but not necessary conditions. On the other hand, only one of the pair of the type A Hamiltonians was investigated in Ref. [28] and relations between the pair in view of the $\mathfrak{sl}(2)$ structure have not been known. In other words, the equivalence between the analytic construction in Ref. [21] and the algebraic one in Ref. [28] has not been established. Another problem to be solved is the classification problem. In Ref. [30], we attempted to classify the type A $\mathcal{N}$-fold supersymmetric models by considering the invariance under the linear transformations but found to be incomplete. Later we found that the complete classification of the $\mathfrak{sl}(2)$ quasi-solvable models was already
achieved in Ref. [45, 46] by the consideration of more extensive $GL(2, \mathbb{R})$ transformations. Therefore, we have recognized that the complete classification of the type A models should be done by investigating the invariance of both the Hamiltonians and $\mathcal{N}$-fold supercharge under the $GL(2, \mathbb{R})$ transformations. Lastly, the structure of the anti-commutator of the type A $\mathcal{N}$-fold supercharges, which we have called mother Hamiltonian, has not been investigated yet at all and thus we have not established the complete type A $\mathcal{N}$-fold superalgebra. We will give the complete answers to the above problems in this article.

The article is organized as follows. In the next section, we give a brief review on the definition and the general properties of $\mathcal{N}$-fold supersymmetry needed in this article. In Section III, we define type A $\mathcal{N}$-fold supersymmetry and give two different approaches, namely, analytic and algebraic approaches to construct the type A $\mathcal{N}$-fold supersymmetric models. The classification problem is discussed in Section IV. By considering the invariance property of both the Hamiltonians and $\mathcal{N}$-fold supercharge, we obtain the complete answer to the problem. In Section V, we investigate 2-fold supersymmetry with an emphasis on the uniqueness of supercharges and the polynomiality of the 2-fold superalgebras. A novel feature of weak quasi-solvability defined in Section II is discussed. In Section VI, we first show the polynomiality of the type A $\mathcal{N}$-fold superalgebras. Then, by a suitable generalization of the original idea of the Bender–Dunne polynomials to all the type A $\mathcal{N}$-fold supersymmetric models, we show that the anti-commutators of the type A $\mathcal{N}$-fold supercharges are expressed as the critical polynomials of the generalized Bender–Dunne polynomials. We claim that both the normalizability of the solvable sector and the weak orthogonality of the polynomials do not play essential roles on the generalization. A novel interpretation of the critical polynomials is given in view of polynomial invariants. Interesting future problems are discussed in the final section. In Appendix A, we summarize the results of the invariant theory of polynomial systems needed in this article.

II. $\mathcal{N}$-FOLD SUPERSYMMETRY

A. Definition

We start with defining $\mathcal{N}$-fold supersymmetry in one dimensional quantum mechanics. Let us introduce a bosonic coordinate $q$ and fermionic coordinates $\psi$ and $\psi^\dagger$ satisfying,

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0, \quad \{\psi, \psi^\dagger\} = 1.$$  \hspace{1cm} (2.1)

Hamiltonian $H_N$ is given by,

$$H_N = H_N^- \psi^\dagger \psi + H_N^+ \psi \psi^\dagger,$$  \hspace{1cm} (2.2)

where $H_N^\pm$ are ordinary scalar Hamiltonians:

$$H_N^\pm = \frac{1}{2} p^2 + V_N^\pm(q),$$  \hspace{1cm} (2.3)

with $p = -id/dq$. $\mathcal{N}$-fold supercharges $Q_N$ and $Q_N^\dagger$ are introduced by,

$$Q_N = P_N^- \psi, \quad Q_N^\dagger = P_N^+ \psi^\dagger,$$  \hspace{1cm} (2.4)
where $P_N$ is given by a polynomial of $N$th degree in $p$:

$$P_N = p^N + w_{N-1}(q)p^{N-1} + \cdots + w_1(q)p + w_0(q),$$  

(2.5)

that is, $P_N$ is an $N$th-order linear differential operator. Then, the system (2.2) is defined to be $N$-fold supersymmetric if the following algebra holds:

$$\{Q_N, Q_N\} = \{Q^\dagger_N, Q^\dagger_N\} = 0,$$

(2.6)

$$[Q_N, H_N] = [Q^\dagger_N, H_N] = 0.$$  

(2.7)

The former relation is trivial due to Eq. (2.1) while the latter is equivalent to the following intertwining relations:

$$P_N H_N^\dagger - H_N^\dagger P_N = 0, \quad P_N H_N - H_N P_N = 0.$$  

(2.8)

Therefore, the relation (2.8) gives the condition for the system $H_N$ to be $N$-fold supersymmetric. From the definition, it is evident that $N$-fold supersymmetry reduces to the ordinary supersymmetry [2, 3] when $N = 1$.

B. Quasi-solvability

The $N$-fold supersymmetric models defined above have several significant properties similar to those of the ordinary supersymmetric models [27]. One of the most notable ones is quasi-solvability [21, 22]. A linear differential operator $H$ of a single variable $q$ is said to be quasi-solvable if it preserves a finite dimensional functional space $\mathcal{V}_N$ whose basis admits an explicit analytic form:

$$H \mathcal{V}_N \subset \mathcal{V}_N, \quad \dim \mathcal{V}_N = n(N) < \infty, \quad \mathcal{V}_N = \text{span} \{ \phi_1(q), \ldots, \phi_{n(N)}(q) \}.$$  

(2.9)

An immediate consequence of the above definition of quasi-solvability is that, since we can calculate finite dimensional matrix elements $S_{k,l}$ defined by,

$$H \phi_k = \sum_{l=1}^{n(N)} S_{k,l} \phi_l \quad (k = 1, \ldots, n(N)),$$

(2.10)

we can diagonalize the operator $H$ and obtain the spectra of it in the space $\mathcal{V}_N$, at least, algebraically. Furthermore, if the space $\mathcal{V}_N$ is a subspace of a Hilbert space $L^2(S)$ ($S \subset \mathbb{R}$) on which the operator $H$ is naturally defined, the solvable spectra and the corresponding vectors of $\mathcal{V}_N$ give the exact eigenvalues and eigenfunctions of $H$, respectively. In this case, the operator $H$ is said to be quasi-exactly solvable. The role of the normalizability of the solvable sector is investigated in view of dynamical properties in Ref. [1, 31]. To construct a quasi-solvable model, it is convenient to introduce an $N$th-order linear differential operator $P$ and define the vector space $\mathcal{V}_N$ as,

$$\mathcal{V}_N = \ker P.$$  

(2.11)

Now, it is easy to see that an operator $H$ is quasi-solvable with the solvable sector (2.11) if the following quasi-solvability condition holds [20]:

$$PH \mathcal{V}_N = 0.$$  

(2.12)
This formulation enables us to clarify the relation between quasi-solvability and \( \mathcal{N} \)-fold supersymmetry. Indeed, it can be easily shown that all the quasi-solvable models satisfying Eq. (2.12) are \( \mathcal{N} \)-fold supersymmetric if we set \( P_\mathcal{N} = P \), \( H^-_\mathcal{N} = H \), and \( H^+_\mathcal{N} = H + i w'_{\mathcal{N}-1} \), where \( w_{\mathcal{N}-1} \) is defined in Eq. (2.5). The converse is also true. From the intertwining relations (2.8), we find that all the \( \mathcal{N} \)-fold supersymmetric systems are quasi-solvable: the quasi-solvability condition (2.12) holds for \( H = H^-_\mathcal{N}(H^+_\mathcal{N}) \) and \( P = P_\mathcal{N}(P^\dagger_\mathcal{N}) \), respectively. Therefore, if we define,

\[
V^-_\mathcal{N} = \ker P_\mathcal{N} = \text{span} \{ \phi^-_n : n = 1, \ldots, \mathcal{N} \}, \tag{2.13a}
\]

\[
V^+_\mathcal{N} = \ker P^\dagger_\mathcal{N} = \text{span} \{ \phi^+_n : n = 1, \ldots, \mathcal{N} \}, \tag{2.13b}
\]

we have \( H^\pm_\mathcal{N} V^\pm_\mathcal{N} \subset V^\pm_\mathcal{N} \) and the following Schrödinger equations on the subspaces \( V^\pm_\mathcal{N} \):

\[
H^\pm_\mathcal{N} \phi^\pm_k = \sum_{l=1}^\mathcal{N} S^\pm_{k,l} \phi^\pm_l = E^\pm_k \phi^\pm_k \quad (k = 1, \ldots, \mathcal{N}). \tag{2.14}
\]

The spectra \( E^\pm \) are determined from the characteristic equations for the matrices \( S^\pm \):

\[
\det M^\pm_\mathcal{N}(E^\pm) = 0, \quad M^\pm_\mathcal{N}(\lambda) = 2(\lambda I - S^\pm). \tag{2.15}
\]

We should note that, for a given operator \( P \), we cannot always obtain analytic solutions of Eq. (2.11). Therefore, quasi-solvability formulated from Eqs. (2.11) and (2.12) is less restrictive than the one defined by Eq. (2.9). In the situation where this difference is crucial, we may be better to call the less restrictive case \textit{weakly} quasi-solvable. With this terminology, we say more correctly that \( \mathcal{N} \)-fold supersymmetry is equivalent to \textit{weak} quasi-solvability. In Section V we will discuss weak quasi-solvability again.

C. Mother Hamiltonian

In the ordinary supersymmetry, the anti-commutator of the supercharges corresponds to the Hamiltonian. However, it is not the case in \( \mathcal{N} \)-fold supersymmetry. This is because \( \{ Q^\dagger_\mathcal{N}, Q_\mathcal{N} \} \) is now a \( 2\mathcal{N} \)th-order differential operator. The half of the anti-commutator is called \textit{mother Hamiltonian} and is denoted by \( \mathcal{H}_\mathcal{N} \):

\[
\mathcal{H}_\mathcal{N} = \frac{1}{2} \{ Q^\dagger_\mathcal{N}, Q_\mathcal{N} \}. \tag{2.16}
\]

An immediate consequence of the above definition is that the mother Hamiltonian always commutes with the \( \mathcal{N} \)-fold supercharges, that is, it is \( \mathcal{N} \)-fold supersymmetric:

\[
[Q_\mathcal{N}, \mathcal{H}_\mathcal{N}] = [Q^\dagger_\mathcal{N}, \mathcal{H}_\mathcal{N}] = 0. \tag{2.17}
\]

Furthermore, if the original Hamiltonian \( H_\mathcal{N} \) is \( \mathcal{N} \)-fold supersymmetric, the mother Hamiltonian also commutes with \( H_\mathcal{N} \) due to the relation (2.7):

\[
[H_\mathcal{N}, \mathcal{H}_\mathcal{N}] = 0. \tag{2.18}
\]

From the above relations, it is expected that the mother Hamiltonian \( \mathcal{H}_\mathcal{N} \) has an intimate relation with the original Hamiltonian \( H_\mathcal{N} \). Indeed, it was shown \cite{27} that, if the \( \mathcal{N} \)-fold
supercharges $Q_N$ and $Q_N^\dagger$ are uniquely determined and $H^+_N \neq H^-_N$. $\mathcal{H}_N$ is expressed as the characteristic polynomial of $N$-th degree for $S^\pm$ appeared in Eq. (2.15) with the argument replaced by $H_N$:

$$\mathcal{H}_N = \frac{1}{2} \det M^+_N(H_N) = \frac{1}{2} \det M^-_N(H_N).$$

(2.19)

III. TYPE A $\mathcal{N}$-FOLD SUPERSYMMETRIC MODELS

In contrast to the ordinary supersymmetric quantum mechanics, the construction of an $\mathcal{N}$-fold supersymmetric model is a non-trivial problem. In the case of ordinary supersymmetry, the mother Hamiltonian (2.16) defined by the anti-commutator of the supercharges is a desirable Schrödinger operator of the form (2.2) and thus can be identified as a supersymmetric Hamiltonian due to the relation (2.17). In the case of 2-fold supersymmetry, the condition (2.8) can be completely solved and thus the most general form of the 2-fold supersymmetric Hamiltonian and the 2-fold supercharge have been known [5, 6, 27]. However, we can hardly solve the condition (2.8) for $\mathcal{N} \geq 3$ in general. Nevertheless, we have found a special case where the condition (2.8) can be solved for arbitrary $\mathcal{N}$. We have called this case type A [20]. The type A $\mathcal{N}$-fold supercharge is defined as the following special form of $\mathcal{N}$th-order linear differential operator:

$$P_N = \prod_{k=-(\mathcal{N}-1)/2}^{(\mathcal{N}-1)/2} (p - iW(q) + ikE(q))$$

$$= \left( p - iW(q) + i\frac{\mathcal{N}-1}{2}E(q) \right) \left( p - iW(q) + i\frac{\mathcal{N}-3}{2}E(q) \right) \times \cdots$$

$$\cdots \times \left( p - iW(q) - i\frac{\mathcal{N}-3}{2}E(q) \right) \left( p - iW(q) - i\frac{\mathcal{N}-1}{2}E(q) \right).$$

(3.1)

In the above definition, we note that $E(q)$ and $W(q)$ correspond to $\tilde{E}(q)$ and $\tilde{W}(q)$, respectively, in the previous articles [20, 27, 28, 30]. We will use tildes for another particular purpose (see Eqs. (3.3) and below) and thus we do not follow the old notation anymore. A system (2.2) is said to be type A $\mathcal{N}$-fold supersymmetric if the condition (2.8) is fulfilled with this type A $\mathcal{N}$-fold supercharge. There are two ways to construct a type A model, namely, analytic and algebraic constructions. The former is to solve the intertwining relation (2.8) directly while the latter is to solve the quasi-solvability condition (2.12). In the following sections, we will first show the analytic construction and next the algebraic one.

A. Analytic Construction

The condition for type A $\mathcal{N}$-fold supersymmetry was firstly investigated with the aid of induction in Ref. [20]. Later, it was reexamined in the context of the algebraic construction [28]. Then, it has turned out that the set of the conditions derived in Ref. [20] only gives a sufficient one. The origin of the defect was explained in Ref. [28]. In the following, we will give an improved direct proof of the necessary and sufficient condition for type A $\mathcal{N}$-fold supersymmetry.
The necessary and sufficient condition for the system \eqref{eq:SUSY} to be $\mathcal{N}$-fold supersymmetry with respect to the type A $\mathcal{N}$-fold supercharge \eqref{eq:Supercharge} is the following:

\[
V^\pm_N(q) = \frac{1}{2}W(q)^2 + \frac{1}{2}v^\pm_N(q), \quad v^\pm_N(q) = \frac{N^2 - 1}{12} \left( (E(q)^2 - 2E'(q)) \pm \mathcal{N}W'(q) - 2R \right),
\]

\[
\left( \frac{d}{dq} - E(q) \right) \left( \frac{d}{dq} + E(q) \right) W(q) = 0 \quad \text{for} \quad \mathcal{N} \geq 2,
\]

\[
\left( \frac{d}{dq} - 2E(q) \right) \left( \frac{d}{dq} - E(q) \right) \frac{d}{dq} + E(q) E(q) = 0 \quad \text{for} \quad \mathcal{N} \geq 3,
\]

where $R$ is an arbitrary constant.

**Proof.** The proposition will be proved by induction. At first, we make gauge transformations on $P_N$ and $H^\pm_N$ to facilitate the calculations, as follows:

\[
\tilde{P}_N = i^N(G_N U)P_N(G_N U)^{-1} = \prod_{k=0}^{N-1}(\partial - kE),
\]

\[
\tilde{H}^\pm_N = (G_N U)H^\pm_N(G_N U)^{-1},
\]

where $G_N$ and $U$ are defined by,

\[
G_N = \exp \left( \frac{N-1}{2} \int dqE(q) \right), \quad U = \exp \left( \int dqW(q) \right).
\]

In the above and hereafter, we attach tildes to operators, vectors and vector spaces to indicate that they are quantities gauge-transformed with the gauge factor $G_N U$. We set,

\[
\tilde{I}_N = 2(\tilde{P}_N \tilde{H}^-_N - \tilde{H}^+_N \tilde{P}_N).
\]

We assume that $\tilde{I}_N = 0$ for a natural number $N$ if the set of the conditions \eqref{eq:SUSY} holds for this $N$. Then, we will prove that $\tilde{I}_{N+1} = 0$ if and only if the set of the conditions \eqref{eq:SUSY} holds for $\mathcal{N}$ replaced by $\mathcal{N} + 1$. From the following relation:

\[
2(G_{N+1} U)H^\pm_N(G_{N+1} U)^{-1} = 2\tilde{H}^\pm_N + E\partial + \frac{1}{2}E' - \frac{2N-1}{4}E^2 - EW,
\]

we obtain,

\[
2\tilde{H}^\pm_{N+1} = (G_{N+1} U)(2\tilde{H}^\pm_N - v^\pm_N + v^\pm_{N+1})(G_{N+1} U)^{-1}
\]

\[
= 2\tilde{H}^\pm_N + E\partial + \frac{1}{2}E' - \frac{2N-1}{4}E^2 - EW - v^\pm_N + v^\pm_{N+1}
\]

\[
= 2\tilde{H}^\pm_N + E\partial + u^\pm_{N+1} - u^\pm_N,
\]

where $u^\pm_N$ are defined by,

\[
u^\pm_N = v^\pm_N + W' + \frac{N-1}{2}E' - \frac{(N-1)^2}{4}E^2 - (N-1)EW.
\]
In the above, we note that $v^\pm_N$ are given by Eq. (3.2a) from the inductive assumption while $v^\pm_{N+1}$ are unknown functions to be determined. From Eq. (3.7), we obtain,

$$I_{N+1} = \tilde{P}_{N+1} \left( 2\tilde{H}_N + E\partial + u^-_{N+1} - u^+_N \right) - \left( 2\tilde{H}_N^+ + E\partial + u^+_{N+1} - u^-_N \right) \tilde{P}_{N+1}$$

$$= (u^-_{N+1} - u^-_N - u^+_{N+1} + u^+_N) \tilde{P}_{N+1} - 2[\tilde{H}_N, \partial - NE] \tilde{P}_N + \tilde{P}_{N+1} E\partial - E\partial \tilde{P}_{N+1} + [\tilde{P}_{N+1}, u^-_{N+1} - u^-_N].$$

(3.9)

The last four terms in the r.h.s. of Eq. (3.9) are calculated as follows:

$$2[\tilde{H}_N, \partial - NE] = ((N + 1)E' - 2W')(\partial - NE')$$

$$- (u^+_N + 2NWE - NE^2 - NE'),$$

(3.10a)

$$\tilde{P}_{N+1} E\partial = E\tilde{P}_{N+2} + \sum_{n=1}^{N+1} \prod_{k=n}^{N} (\partial - kE) E_{(-1)} \tilde{P}_n,$$

$$= E\tilde{P}_{N+2} + (N + 1)E_{(-1)} \tilde{P}_{N+1} + \sum_{n=1}^{N} \prod_{k=n+1}^{N} (\partial - kE) E_{(0,-1)} \tilde{P}_n,$$

(3.10b)

$$E\partial \tilde{P}_{N+1} = E\tilde{P}_{N+2} + (N + 1)E^2 \tilde{P}_{N+1},$$

(3.10c)

$$[\tilde{P}_{N+1}, u^-_{N+1} - u^-_N] = \sum_{n=0}^{N} \prod_{k=n+1}^{N} (\partial - kE)(u^-_{N+1} - u^-_N)(0) \tilde{P}_n,$$

(3.10d)

In the above and hereafter, we employ the following abbreviations:

$$f(k) = (\partial - kE)f, \quad f(k,...) = (\partial - kE)f(...).$$

(3.11)

The following formula is useful for the calculations:

$$\prod_{k=n}^{N} (\partial - kE)f = f \prod_{k=n}^{N-1} (\partial - kE) + \sum_{n'=n}^{N-1} \prod_{k=n+1}^{n'} (\partial - kE)f(0) \prod_{k'=n}^{n'-1} (\partial - k'E).$$

(3.12)

Substituting Eqs. (3.10) for Eq. (3.9), we have,

$$\tilde{I}_{N+1} = \left( 2W' + u^-_{N+1} - u^-_N - u^+_{N+1} + u^+_N \right) \tilde{P}_{N+1} + (u^+_N + 2NWE - NE^2 - NE') \tilde{P}_N + \sum_{n=0}^{N} \prod_{k=n+1}^{N} (\partial - kE) \left[ (u^-_{N+1} - u^-_N)(0) + nE_{(0,-1)} \right] \tilde{P}_n.$$

(3.13)

From Eq. (3.13), we see that $\tilde{I}_{N+1}$ is at most $(N+1)$th-order differential operator. Therefore, $\tilde{I}_{N+1} = 0$ if and only if all the coefficients of $\partial^k (k = 0, 1, \ldots, N+1)$ vanish. Only the first term in the r.h.s. of Eq. (3.13) contains the $\partial^{N+1}$ term. Therefore, one of the conditions for $\tilde{I}_{N+1} = 0$ reads,

$$2W' + u^-_{N+1} - u^-_N - u^+_{N+1} + u^+_N = 0.$$  

(3.14)

Combining Eq. (3.14) with Eqs. (3.2a) and (3.8), we have,

$$v^+_N - v^-_{N+1} = 2(N + 1)W'.$$

(3.15)
Applying the formula (3.12) and the condition (3.14) to Eq. (3.13), we obtain,

\[ \tilde{I}_{N+1} = \left[ \frac{N(N+1)}{2} E_{(0,-1)} + (N+1)(u_{N+1}^- - u_N^-) + (u_N^+ + 2NWE - NE^2 - NE')' \right] \tilde{P}_N + \sum_{n=0}^{N-1} \sum_{n'=n}^{N} \prod_{k=n'+2}^{N} (\partial - kE) \left[ (u_{N+1}^- - u_N^-) + nE_{(1,0,-1)} \right] \tilde{P}_{n'} + \sum_{n=0}^{N} \sum_{n'=n}^{N} \prod_{k=n'+2}^{N} (\partial - kE) \left[ (u_{N+1}^- - u_N^-)_{(1,0)} + nE_{(1,0,-1)} \right] \tilde{P}_{n'} \]  

(3.16)

Only the first term in the r.h.s. of Eq. (3.16) contains the \( \partial^N \) term. Therefore, we obtain another condition for \( \tilde{I}_{N+1} = 0 \):

\[ \frac{N(N+1)}{2} E_{(0,-1)} + (N+1)(u_{N+1}^- - u_N^-) + (u_N^+ + 2NWE - NE^2 - NE')' = 0. \]  

(3.17)

Combining the assumption (3.2a) with Eqs. (3.15) and (3.17), we finally have,

\[ v_{N+1}^\pm = \frac{(N+1)^2 - 1}{12} (E^2 - 2E') \pm (N+1)W' + \text{const.} \]  

(3.18)

The resulting \( v_{N+1}^\pm \) is nothing but the assumed form (3.2a) with \( N \) replaced by \( N + 1 \). Therefore, the first condition (3.2a) has been proved inductively. Under the above conditions satisfied, we have from Eqs. (3.15) and (3.18),

\[ u_{N+1}^- - u_N^- = v_{N+1}^- - v_N^- + \frac{1}{2} E' - \frac{2N-1}{4} E^2 - EW \]

\[ = -\frac{N-1}{3} E_{(-1)} - W_{(-1)}. \]

(3.19)

Substituting the above for Eq. (3.15), we obtain,

\[ \tilde{I}_{N+1} = -\sum_{n=0}^{N} \sum_{n'=n}^{N} \prod_{k=n'+2}^{N} (\partial - kE) \left[ W_{(1,0,-1)} - \frac{3n-N+1}{3} E_{(1,0,-1)} \right] \tilde{P}_{n'}. \]  

(3.20)

The above expression for \( \tilde{I}_{N+1} \) is further arranged with the aid of the formula (3.12) as follows:

\[ \tilde{I}_{N+1} = -\sum_{n=0}^{N} \sum_{n'=n}^{N-1} \left( W_{(1,0,-1)} - \frac{3n-N+1}{3} E_{(1,0,-1)} \right) \tilde{P}_{n'-1} \]

\[ -\sum_{n=0}^{N-1} \sum_{n'=n}^{N-2} \sum_{n''=n'}^{N-3} \prod_{k=n''+3}^{N} (\partial - kE) \left( W_{(2,1,0,-1)} - \frac{3n-N+1}{3} E_{(2,1,0,-1)} \right) \tilde{P}_{n''} \]

\[ = -\frac{N(N+1)}{2} W_{(1,0,-1)} \tilde{P}_{N-1} \]

\[ -\frac{1}{2} \sum_{n=0}^{N-2} (n+1)(n+2) \prod_{k=n+3}^{N} (\partial - kE) \left( W_{(2,1,0,-1)} - \frac{n-N+1}{3} E_{(2,1,0,-1)} \right) \tilde{P}_{n}. \]

(3.21)
When $N = 1$, $\tilde{I}_2 = -W_{(1,0,-1)}$ and thus we obtain,

$$\tilde{I}_2 = 0 \iff W_{(1,0,-1)} = \left(\frac{d}{dq} - E\right)\left(\frac{d}{dq} + E\right)W = 0, \quad (3.22)$$

i.e., the condition (3.2b) in addition to Eq. (3.2a). When $N \geq 2$, the condition (3.2b) has been already assumed and thus,

$$\tilde{I}_{N+1} = 0 \iff W_{(1,0,-1)} = \left(\frac{d}{dq} - 2E\right)\left(\frac{d}{dq} - E\right)\left(\frac{d}{dq} + E\right)E = 0, \quad (3.24)$$

i.e., the condition (3.2c) in addition to Eqs. (3.2a) and (3.2b). When $N \geq 3$, the condition (3.2c) has been already assumed too and thus,

$$\tilde{I}_{N+1} = 0. \quad (3.25)$$

Therefore, no additional condition is required any more.

From the set of the conditions (3.2), we find a procedure to construct a type A $N$-fold supersymmetric model as follows:

1. Find out a particular solution $E(q)$ of the nonlinear differential equation (3.2c).
2. Substitute the above $E(q)$ for the general solution of the linear differential equation (3.2b) given by,

$$W(q) = C_1 e^{-\int dq E(q)} \int dq \left(e^{\int dq E(q)} \int dq e^{\int dq E(q)}\right) + C_2 e^{-\int dq E(q)} \int dq e^{\int dq E(q)} + C_3 e^{-\int dq E(q)} \quad (C_i = \text{arbitrary constants}), \quad (3.26)$$

$$\int dq$$

3. Substitute the above $E(q)$ and $W(q)$ for Eq. (3.2a) to obtain the pair of potentials $V^\pm_N(q)$.

B. $\mathfrak{sl}(2)$ Construction

As previously noted, $N$-fold supersymmetry is essentially equivalent to quasi-solvability. Recently, some special quasi-solvable models which can be constructed from the $\mathfrak{sl}(2)$ generators were found to be type A $N$-fold supersymmetric \[1, 19, 20, 23, 24, 25, 26\]. Then, it was shown in Ref. \[28\] that the type A $N$-fold supersymmetric model is essentially equivalent to the $\mathfrak{sl}(2)$ quasi-solvable model. In this section, we will review the equivalence
and then clarify the relation between the pair of the Hamiltonian $H_N^\pm$ in the framework of the \(\mathfrak{sl}(2)\) quasi-solvable models, which has not been discussed yet in the previous articles. Let us first construct $H_N^-$ so that it is quasi-solvable with respect to the type A operator \((3.1)\). The quasi-solvability condition \((2.12)\) in this case is,

\[
P_N H_N^- \mathcal{V}_N^- = 0,
\]

where $\mathcal{V}_N^-$ is given by Eq. \((2.13a)\). On the gauge-transformed space, the condition \((3.27)\) is equivalent to,

\[
\tilde{P}_N \tilde{H}_N^- \tilde{V}_N^- = 0,
\]

where $\tilde{P}_N$ and $\tilde{H}_N^-$ are given by Eqs. \((3.3)\) and $\tilde{V}_N^-$ is defined by,

\[
\tilde{V}_N^- = \ker \tilde{P}_N = \text{span} \{G_N U \phi : \phi \in \ker P_N \}.
\]

Introducing a function $h(q)$ defined as a solution of the following differential equation:

\[
h''(q) - E(q) h'(q) = 0,
\]

we find that $\tilde{P}_N$ is expressed in terms of $h$ as,

\[
\tilde{P}_N = (h')^N \frac{d^N}{dh^N}.
\]

Thus, we easily have,

\[
\tilde{V}_N^- = \text{span} \{1, h(q), \ldots, h(q)^{N-1} \}.
\]

From Eqs. \((3.28)\) and \((3.32)\), the quasi-solvability condition for $\tilde{H}_N^-$ reads,

\[
\frac{d^N}{dh^N} \tilde{H}_N^- h^{k-1} = 0 \quad \text{for} \quad \forall \ k = 1, \ldots, N.
\]

Any constant is a trivial solution of \((3.33)\). For $N = 1$, any first-order differential operator of the following form,

\[
f_1(h) \frac{d}{dh},
\]

is a solution, while for $N \geq 2$ there are three independent first-order differential operators as solutions of \((3.33)\):

\[
\frac{d}{dh} = J^-,
\]

\[
h \frac{d}{dh} = J^0 + \frac{N - 1}{2},
\]

\[
h^2 \frac{d}{dh} - (N - 1) h = J^+.
\]

The $J^{+,0,-}$ defined above satisfy the \(\mathfrak{sl}(2)\) algebra:

\[
[J^+, J^-] = -2J^0, \quad [J^\pm, J^0] = \mp J^\pm.
\]
In the same way, we find that for $N = 1, 2$, any second-order differential operator of the following form,

$$f_2(h) \frac{d^2}{dh^2},$$

is a solution, while for $N \geq 3$ there are five independent second-order differential operators as solutions of (3.33):

$$\begin{align*}
\frac{d^2}{dh^2} &= (J^-)^2, \\
h \frac{d^2}{dh^2} &= J^0 J^- + \frac{N-1}{2} J^-,
\end{align*}$$

$$\begin{align*}
h^2 \frac{d^2}{dh^2} &= (J^0)^2 + (N-2) J^0 + \frac{(N-1)(N-3)}{4}, \\
h^3 \frac{d^2}{dh^2} - (N-1)(N-2) h &= J^+ J^0 + \frac{3N-5}{2} J^+, \\
h^4 \frac{d^2}{dh^2} - 2(N-2) h^3 \frac{d}{dh} + (N-1)(N-2) h^2 &= (J^+)^2.
\end{align*}$$

Therefore, the general solution of (3.33) for $N \geq 3$ can be expressed as,

$$\tilde{H}_N = - \sum_{i,j=+,-, \atop i \geq j} a_{ij}^{(-)} J^i J^j + \sum_{i=+,-} b_i^{(-)} J^i - C^{(-)},$$

where $a_{ij}^{(-)}$, $b_i^{(-)}$ and $C^{(-)}$ are constants. This gauged Hamiltonian is nothing but the $\mathfrak{sl}(2)$ quasi-solvable model [29]. We can set $a_{++}^{(-)} = 0$ without any loss of generality due to the relation:

$$\frac{1}{2} (J^+ J^- + J^- J^+) - (J^0)^2 = -\frac{1}{4} (N^2 - 1).$$

For convenience, we introduce new parameters as follows:

$$\begin{align*}
a_4 &= a_{++}^{(-)}, \\
a_3 &= a_{-+}^{(-)}, \\
a_2 &= a_{00}^{(-)}, \\
a_1 &= a_{0-}^{(-)}, \\
a_0 &= a_{-+}^{(-)},
\end{align*}$$

$$\begin{align*}
b_2 &= -b_+^{(-)} - \frac{a_{+0}^{(-)}}{2}, \\
b_1 &= -b_0^{(-)}, \\
b_0 &= -b_-^{(-)} - \frac{a_{0-}^{(-)}}{2}.
\end{align*}$$

Then, the gauged Hamiltonian (3.38) is expressed in terms of $h$ as,

$$\tilde{H}_N = -P(h) \frac{d^2}{dh^2} - \left[ Q(h) - \frac{N-2}{2} P'(h) \right] \frac{d}{dh}$$

$$\quad - \left[ R \frac{N-1}{2} Q'(h) + \frac{(N-1)(N-2)}{12} P''(h) \right],$$

where $P(h)$ and $Q(h)$ are given by,

$$\begin{align*}
P(h) &= a_4 h^4 + a_3 h^3 + a_2 h^2 + a_1 h + a_0, \\
Q(h) &= b_2 h^2 + b_1 h + b_0.
\end{align*}$$
while $R = C^{(-)} + (N^2 - 1)a_2/12$ is a constant. If the Hamiltonian (3.41) is gauge-transformed back to the original Hamiltonian $H_{\tilde{N}}$, with the gauge factor $G_N U$, it is in general not of the canonical form of the Schrödinger operator like Eq. (2.3). We can find that the original Hamiltonian $H_{\tilde{N}}$ becomes of the canonical form if and only if the following conditions hold:

$$P(h) = \frac{1}{2}(h'(q))^2, \quad (3.44a)$$
$$Q(h) = -W(q)h'(q). \quad (3.44b)$$

Under the above conditions satisfied, we have,

$$H_{\tilde{N}} = (G_N U)^{-1}\tilde{H}_{\tilde{N}}(G_N U) = -\frac{1}{2} \frac{d^2}{dq^2} + V_{\tilde{N}}(q), \quad (3.45)$$

where the potential $V_{\tilde{N}}$ is given by,

$$V_{\tilde{N}} = -\frac{1}{12} P \left[ (N^2 - 1) \left( PP'' - \frac{3}{4}(P')^2 \right) + 3N(P'Q - 2PQ') - 3Q^2 \right] - R. \quad (3.46)$$

Substituting Eqs. (3.30) and (3.44) for Eq. (3.46), we can check that the above expression for $V_{\tilde{N}}$ is identical with the one in Eq. (3.2a).

When $N = 2$, any second-order operator (3.37) can be added to the gauged Hamiltonian $H_{\tilde{N}}$ and thus all the $a_{ij}$ in Eq. (3.38) can be arbitrary functions of $h$. As a consequence, $P(h)$ in Eq. (3.41) can be an arbitrary function of $h$.

When $N = 1$, any first-order operator (3.34) in addition to any second-order one (3.37) can be added to the gauged Hamiltonian (3.38) and thus all the $b_i$ in addition to all the $a_{ij}$ in Eq. (3.38) can be arbitrary functions of $h$. As a consequence, both $P(h)$ and $Q(h)$ in Eq. (3.41) can be arbitrary functions of $h$.

Combining the above considerations for $N = 1, 2$ with the results (3.42) and (3.43) for $N \geq 3$, we obtain the following conditions for $P(h)$ and $Q(h)$:

$$\frac{d^3}{dh^3}Q(h) = 0 \quad \text{for} \quad N \geq 2, \quad (3.47)$$
$$\frac{d^5}{dh^5}P(h) = 0 \quad \text{for} \quad N \geq 3. \quad (3.48)$$

If we rewrite the above conditions in terms of $q$ with the aid of Eqs. (3.30) and (3.44), we find that the condition (3.47) is equivalent to Eq. (3.2c) while the condition (3.48) is equivalent to Eq. (3.2b). Therefore, the complete correspondence between type A $N$-fold supersymmetric models and the $\mathfrak{sl}(2)$ quasi-solvable models has been established except for the other Hamiltonian $H_{\tilde{N}}$, which will be discussed in the next.

The partner Hamiltonian $H_{\tilde{N}}^+$ should be constructed such that it is quasi-solvable with respect to $P_N^\dagger$. It is easy to see that the operator $P_N^\dagger$ can be converted into the form identical with $\tilde{P}_N$ by a gauge transformation with another gauge factor $G_N U^{-1}$:

$$\tilde{P}_N^\dagger = \iota^N (G_N U^{-1})^N P_N^\dagger (G_N U^{-1})^{-1} = (h')^N \frac{d^N}{dh^N}. \quad (3.49)$$

where $h$ is the same function of $q$ as defined previously by Eq. (3.30). In the above and hereafter, we attach bars to operators, vectors and vector spaces to indicate that they are
Then, the partner Hamiltonian

\[ \tilde{H}_N^+ = - \sum_{i,j=+,0,-} a_{ij}^{(+)} j^i j^j + \sum_{i=+,0,-} b_i^{(+)} j^i - C^{(+)} \]  

(3.50a)

Combining the above with Eq. (3.44), we yield the following relations:

\[ \tilde{H}_N^+ = -P^+(h) \frac{d^2}{dh^2} - \left[ Q^+(h) - \frac{N - 2}{2} P'^+(h) \right] \frac{d}{dh} \]

\[ - \left[ R^+ - \frac{N-1}{2} Q'^+(h) + \frac{(N-1)(N-2)}{12} P'''^+(h) \right], \]  

(3.50b)

where \( a_{ij}^{(+)} \), \( b_i^{(+)} \), \( C^{(+)} \) and \( R^+ \) are constants, and \( P^+(h) \) and \( Q^+(h) \) is a polynomial of fourth- and second-degree, respectively. Up to now, there is no relation between \( \tilde{H}_N^- \) and \( \tilde{H}_N^+ \).

To establish the relation between \( R^+ \) and \( R^+ \), we should recall the formula [27]: \( H_N^+ - H_N^- = iw_{N-1}' \), where \( w_{N-1}' \) is defined in Eq. (2.55). For the type A N-fold supercharge [31], we have \( w_{N-1}' = -i\sqrt{N} \). From Eqs. (3.45), (3.46) and (3.54) we finally obtain,

\[ R^+ = R. \]  

(3.55)

To establish the relation between \( R^+ \) and \( R^+ \), we refer to an interesting relation between the algebraic Hamiltonians \( \tilde{H}_N^- \) and \( \tilde{H}_N^+ \). From Eqs. (3.52) and (3.55), the relation between the parameters in \( \tilde{H}_N^- \) and the ones in \( \tilde{H}_N^+ \) reads,

\[ a_{ij}^{(+)} = a_{ij}^{(-)} \quad (\forall i, j = +, 0, -), \quad C^{(+)} = C^{(-)}, \]  

(3.56a)

\[ b_i^{(+)} + b_i^{(-)} + a_{i0}^{(-)} = 0, \quad b_0^{(+)} + b_0^{(-)} = 0, \quad b_i^{(+)} + b_i^{(-)} + a_{i0}^{(-)} = 0. \]  

(3.56b)
Substituting the above for Eq. (3.50a), we can rewrite $\bar{H}_N^+$ in terms of $a_{ij}^{(-)}$ and $b_i^{(-)}$:

$$\bar{H}_N^+ = - \sum_{i,j=+,-} a_{ij}^{(-)} J^i J^j - \sum_{i=+,-} b_i^{(-)} J^i - C^{(-)}. \quad (3.57)$$

Comparing this expression with Eq. (3.58), we see that $\bar{H}_N^+$ is obtained from $\tilde{H}_N^-$ by interchanging the order of the quadratic terms and by interchanging the sign of the coefficients of the linear terms.

IV. CLASSIFICATION OF TYPE A MODELS

It was shown that the $\mathfrak{sl}(2)$ quasi-solvable models can be classified using the shape invariance of the Hamiltonian under the action of $GL(2, K) \ (K = \mathbb{R} \ or \ \mathbb{C})$ of linear fractional transformations [43, 44]. The equivalence established in the previous section ensures that the type A $\mathcal{N}$-fold supersymmetric Hamiltonians can be fit into the same classification scheme. Then, a natural question arises whether the type A $\mathcal{N}$-fold supercharges can be also classified in the same scheme or not. This question was left as an open problem in the previous paper [30] and will be completely answered in this section.

A. $GL(2, K)$ Shape Invariance

The linear fractional transformation of $h$ is introduced by,

$$h \mapsto \hat{h} = \frac{\alpha h + \beta}{\gamma h + \delta} \quad (\alpha, \beta, \gamma, \delta \in K; \ \Delta = \alpha \delta - \beta \gamma \neq 0). \quad (4.1)$$

Then, it turns out that the gauged Hamiltonian (3.41) is shape invariant under the following transformation induced by Eq. (4.1):

$$\tilde{H}_N^-(h) \mapsto \tilde{H}_N^-(\hat{h}) = (\gamma h + \delta)^{N-1} \tilde{H}_N^-\hat{h}(\gamma h + \delta)^{-N-1}, \quad (4.2)$$

where $P(h)$ and $Q(h)$ in Eq. (3.41) are transformed according to,

$$P(h) \mapsto \hat{P}(h) = \Delta^{-2}(\gamma h + \delta)^4 P(\hat{h}), \quad (4.3a)$$

$$Q(h) \mapsto \hat{Q}(h) = \Delta^{-1}(\gamma h + \delta)^2 Q(\hat{h}). \quad (4.3b)$$

B. Invariance of the Hamiltonian

In the next, we will examine the transformation of the original Hamiltonian $H_N^\pm$. Since the function $h(q)$ is determined by Eq. (3.44a), we have,

$$|q| = \int \frac{dh}{\sqrt{2P(h)}} \mapsto |\hat{q}| = \int \frac{d\hat{h}}{\sqrt{2\hat{P}(\hat{h})}} = \int dh \frac{\Delta}{(\gamma h + \delta)^2} \frac{1}{\sqrt{2P(h)}}$$

$$= \int \frac{d\hat{h}}{\sqrt{2\hat{P}(\hat{h})}}. \quad (4.4)$$
Therefore, $h(q)$ before the transformation is identical with $\hat{h}(q)$ as a function of $q$. In other words, the relation between $h$ before the transformation (denoted by $h_{\text{old}}$) and $h$ after the transformation (denoted by $h_{\text{new}}$) is, as a function of $q$, consistent with Eq. (4.1):

$$h_{\text{old}}(q) = \hat{h}(q) = \frac{\alpha h_{\text{new}}(q) + \beta}{\gamma h_{\text{new}}(q) + \delta}.$$ (4.5)

On the other hand, the potentials $V_N^\pm$ can be rewritten as

$$V_N^\pm = -\frac{1}{24P} \left[ 2(N^2 - 1)H[P] + 3N(P,Q)^{(1)} - 6Q^2 \right] - R,$$ (4.6)

where $H[P]$ is the Hessian of $P$ and $(P,Q)^{(1)}$ are the first transvectant of $P$ and $Q$, given by (see also Appendix A),

$$H(P) = PP'' - \frac{3}{4}(P')^2, \quad (P,Q)^{(1)} = 2P'Q - 4PQ'.$$ (4.7)

All the objects $O(h)$ in both the numerator and denominator of Eq. (4.6), namely, $H[P]$, $(P,Q)^{(1)}$, $Q^2$ and $P$ transform according to,

$$O(h_{\text{old}}) \mapsto \hat{O}(h_{\text{new}}) = \Delta^{-2}(\gamma h_{\text{new}} + \delta)^4O(\hat{h}),$$ (4.8)

that is, they belong to the multiplier representation $\rho_{4,-2}$ of $GL(2,K)$ defined in Eq. (A1). As a consequence, the functional form of $V_N^\pm$ are preserved under the transformation:

$$V_N^\pm(h_{\text{old}}) \mapsto \hat{V}_N^\pm(h_{\text{new}}) = V_N^\pm(\hat{h}).$$ (4.9)

Finally, since the functional form of $h_{\text{old}}(q)$ and $\hat{h}(q)$ is identical with each other, Eq. (4.9), the potential $V_N^\pm$ is invariant as functions of $q$:

$$V_N^\pm(h_{\text{old}}(q)) \mapsto \hat{V}_N^\pm(h_{\text{new}}(q)) = V_N^\pm(\hat{h}(q)) = V_N^\pm(h_{\text{old}}(q)).$$ (4.10)

Therefore, the type A $N$-fold supersymmetric Hamiltonians $H_N^\pm$ are invariant under the $GL(2,K)$ transformation.

C. Invariance of the Supercharge

In the next, we will examine the transformation of the type A $N$-fold supercharge $P_N$. The gauged $N$-fold supercharge (3.31) is transformed according to,

$$\hat{P}_N = (h_{\text{old}}')^N \frac{d^N}{dh_{\text{old}}^N} \mapsto \hat{P}_N = (\gamma h_{\text{new}} + \delta)^N \frac{d^N}{dh_{\text{new}}^N}(\gamma h_{\text{new}} + \delta)^{-N-1}.$$ (4.11)

On the other hand, the gauge factors $G_N U^{\pm 1}$ are transformed as,

$$G_N U^{\pm 1}(h_{\text{old}}) = \exp \left[ \int dq \left( \frac{N-1}{2} E \pm W \right) \right] = \exp \left[ \int dh_{\text{old}} \frac{(N-1)P'(h_{\text{old}}) \mp 2Q(h_{\text{old}})}{4P(h_{\text{old}})} \right] \mapsto G_N U^{\pm 1}(h_{\text{new}}) = \exp \left[ \int dh_{\text{new}} \frac{(N-1)\hat{P}'(h_{\text{new}}) \mp 2\hat{Q}(h_{\text{new}})}{4\hat{P}(h_{\text{new}})} \right].$$ (4.12)
From Eqs. (4.11) and (4.3), the r.h.s. of Eq. (4.12) is calculated as,

$$G_N U^{\pm 1}(h_{\text{new}}) = \exp \left[ \int d\hat{h} \frac{(N - 1)P'(\hat{h}) \mp 2Q(\hat{h})}{4P(\hat{h})} + \int dh_{\text{new}} \frac{(N - 1)\gamma}{\gamma h_{\text{new}} + \delta} \right]$$

$$= (\gamma h_{\text{new}} + \delta)^{N-1} G_N U^{\pm 1}(\hat{h}).$$  (4.13)

The $N$-fold supercharge $P_N$ is expressed as,

$$P_N = (-i)^N (G_N U(h_{\text{old}}))^{-1} \tilde{P}_N(G_N U(h_{\text{old}}))$$

$$= (-i \tilde{h}'_{\text{old}})^N (G_N U(h_{\text{old}}))^{-1} \frac{d^N}{dh_{\text{old}}^N} (G_N U(h_{\text{old}})).$$  (4.14)

Combining Eqs. (4.11)–(4.13), we obtain,

$$P_N \mapsto \tilde{P}_N = (-i)^N (G_N U(h_{\text{new}}))^{-1} \tilde{P}_N(G_N U(h_{\text{new}}))$$

$$= (-i \tilde{h}'(\hat{h}))^N (G_N U(\hat{h}))^{-1} \frac{d^N}{dh^N} (G_N U(\hat{h})).$$  (4.15)

Comparing Eq. (4.14) with (4.15), we see that $\tilde{P}_N$ is obtained from $P_N$ with $h_{\text{old}}(q)$ replaced by $\tilde{h}(q)$. From the fact that $\tilde{h}(q)$ is identical with $h_{\text{old}}(q)$ as a function of $q$, Eq. (4.15), we finally conclude,

$$P_N = \tilde{P}_N,$$  (4.16)

that is, the $N$-fold supercharge is also invariant under the $GL(2, K)$ transformation. We note that the manifest invariance of the $N$-fold supercharge is lost if we express it in terms of $E(q)$ and $W(q)$ as Eq. (3.1). This is because $E(q)$ is not an invariant function under the $GL(2, K)$ transformation, as we will see below. From Eq. (4.15), we have,

$$\frac{h''_{\text{old}}(q)}{h'_{\text{old}}(q)} = \frac{\tilde{h}''(q)}{\tilde{h}'(q)} = \frac{h''_{\text{new}}(q)}{h'_{\text{new}}(q)} - \frac{2\gamma h'_{\text{new}}(q)}{\gamma h_{\text{new}}(q) + \delta}.$$  (4.17)

The function $\tilde{h}(q)$ is defined so that Eq. (3.30) is fulfilled and thus the relation between $E(q)$ before and after the transformation reads,

$$E(q) = \tilde{E}(q) - \frac{2\gamma h'_{\text{new}}(q)}{\gamma h_{\text{new}}(q) + \delta}.$$  (4.18)

On the other hand, $W(q)$ is an invariant function:

$$\tilde{W}(q) = -\frac{\tilde{Q}(h_{\text{new}}(q))}{h'_{\text{new}}(q)} = -\frac{Q(\tilde{h}(q))}{\tilde{h}'(q)} = W(q).$$  (4.19)

Therefore, the $N$-fold supercharge of the form (3.1) is expressed as,

$$P_N = (-i)^N \prod_{k=-(N-1)/2}^{(N-1)/2} \left( \frac{d}{dq} + W(q) - kE(q) \right)$$

$$= (-i)^N \prod_{k=-(N-1)/2}^{(N-1)/2} \left( \frac{d}{dq} + \tilde{W}(q) - k\tilde{E}(q) + k\frac{2\gamma h'_{\text{new}}(q)}{\gamma h_{\text{new}}(q) + \delta} \right).$$  (4.20)
D. An example

As an example, we will demonstrate the equivalence between the case \( P(h) = 2h \) and \( P(h) = 2h^3 \). In the previous paper [30], we classified them as different cases, namely, case (1) for the former and case (3a) for the latter. However, it is easy to see that the latter is obtained from the former by the following \( GL(2,K) \) transformation:

\[
h_{\text{old}} \mapsto \hat{h} = \frac{1}{h_{\text{new}}}. \tag{4.21}\]

In this case, the transformation of \( P \) and \( Q \) defined by Eqs. (4.3) reads,

\[
P(h_{\text{old}}) = 2h_{\text{old}}, \quad Q(h_{\text{old}}) = b_2 h_{\text{old}}^2 + b_1 h_{\text{old}} + b_0
\]
\[\mapsto \quad \hat{P}(h_{\text{new}}) = 2h_{\text{new}}^3, \quad \hat{Q}(h_{\text{new}}) = -b_0 h_{\text{new}}^2 - b_1 h_{\text{new}} - b_2. \tag{4.22}\]

From Eq. (4.4), the functions \( h_{\text{old}}(q) \) and \( h_{\text{new}}(q) \) are calculated as,

\[
h_{\text{old}}(q) = q^2, \quad h_{\text{new}}(q) = \frac{1}{q^2}, \tag{4.23}\]

which is consistent with Eq. (4.21). The transvectants appeared in Eq. (4.6) become,

\[
H(P) = -3, \quad (P,Q)^{(1)} = -4 \left(3b_2 h_{\text{old}}^2 + b_1 h_{\text{old}} - b_0 \right)
\]
\[\mapsto \quad H(\hat{P}) = -3h_{\text{new}}^4, \quad (\hat{P},\hat{Q})^{(1)} = 4 \left(b_0 h_{\text{new}}^4 - b_1 h_{\text{new}}^3 - 3b_2 h_{\text{new}}^2 \right). \tag{4.24}\]

Substituting Eqs. (4.22) and (4.24) for Eq. (4.6), we have the potentials \( V_{\hat{N}}^\pm \) and \( \hat{V}_{\hat{N}}^\pm \) as functions of \( \hat{h} \):

\[
V_{\hat{N}}^\pm(h_{\text{old}}) = \frac{1}{8} h_{\text{old}}(b_2 h_{\text{old}} + b_1)^2 + \frac{b_2}{4} (b_0 \mp 3\hat{N}) h_{\text{old}}
\]
\[\quad + \frac{b_0^2 \mp 2\hat{N} b_0 + \hat{N}^2 - 1}{8 h_{\text{old}}} + \frac{b_1}{4} (b_0 \mp \hat{N}) - R, \tag{4.25a}\]

\[
\hat{V}_{\hat{N}}^\pm(h_{\text{new}}) = \frac{1}{8 h_{\text{new}}} \left(\frac{b_2}{h_{\text{new}}^2} + b_1\right)^2 + \frac{b_2}{4} (b_0 \mp 3\hat{N})
\]
\[\quad + \frac{b_0^2 \mp 2\hat{N} b_0 + \hat{N}^2 - 1}{8 h_{\text{new}}} + \frac{b_1}{4} (b_0 \mp \hat{N}) - R. \tag{4.25b}\]

Finally, we confirm the invariance of the potentials as functions of \( q \) by substituting Eq. (4.23) for the above (4.25):

\[
V_{\hat{N}}^\pm(h_{\text{old}}(q)) = \hat{V}_{\hat{N}}^\pm(h_{\text{new}}(q)) = \frac{1}{8} q^2 (b_2 q^2 + b_1)^2 + \frac{b_2}{4} (b_0 \mp 3\hat{N}) q^2
\]
\[\quad + \frac{b_0^2 \mp 2\hat{N} b_0 + \hat{N}^2 - 1}{8 q^2} + \frac{b_1}{4} (b_0 \mp \hat{N}) - R. \tag{4.26}\]

The functions \( E(q) \) and \( W(q) \) are calculated as,

\[
E(q) = \frac{h_{\text{old}}''(q)}{h_{\text{old}}'(q)} = \frac{1}{q}, \quad \hat{E}(q) = \frac{h_{\text{new}}''(q)}{h_{\text{new}}'(q)} = -\frac{3}{q}, \tag{4.27}\]

\[
W(q) = \hat{W}(q) = -\frac{1}{2} \left(b_2 q^3 + b_1 q + \frac{b_0}{q}\right). \tag{4.28}\]

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Thus, the type A $N$-fold supercharge of the form (3.1) reads,

$$P_N = \hat{P}_N = (-i)^N \prod_{k=-(N-1)/2}^{(N-1)/2} \left( \frac{d}{dq} - \frac{b_2}{2} q^3 - \frac{b_1}{2} q - \frac{b_0}{2} + \frac{k}{q} \right).$$

(4.29)

E. Classification of Models

For a given $P(h)$, the function $h(q)$ is determined by Eq. (4.4) and a particular type A $N$-fold supersymmetric model is obtained by substituting this $h(q)$ for Eq. (4.6). Since the potentials $V_{X_N}^\pm(q)$ and the $N$-fold supercharge are invariant under the $GL(2, K)$ transformation, the type A models can be classified according to the inequivalent elliptic integral (4.4) under the transformation. The elliptic integral (4.4) can be classified according to the distribution of the zeros of $P(h)\), e.g., multiplicity of the zeros. This idea was first introduced in Ref. [45] to classify the $\mathfrak{sl}(2)$ quasi-solvable models. Under the transformation (4.3a) of $GL(2, \mathbb{R})$ or $GL(2, \mathbb{C})$, every quartic polynomial $P(h)$ with real or complex coefficients is equivalent to one of the eight or five forms, respectively, as shown in Table I.

| Case | $GL(2, \mathbb{R})$ | $GL(2, \mathbb{C})$ |
|------|-------------------|-------------------|
| I    | 1/2               | 1/2               |
| II   | $2h$              | $2h$              |
| III  | $2\nu h^2$       | $2\nu h^2$       |
| III' | $\nu (h^2 + 1)^2/2$ | $2\nu h^2$       |
| IV   | $2\nu (h^2 - 1)$ | $2\nu (h^2 - 1)$ |
| IV'  | $2\nu (h^2 + 1)$ | $2\nu (h^2 + 1)$ |
| V    | $2h^3 - g_2 h/2 - g_3/2$ | $2h^3 - g_2 h/2 - g_3/2$ |
| V'   | $\nu (h^2 + 1)[(1 - k^2)h^2 + 1]/2$ | $2h^3 - g_2 h/2 - g_3/2$ |

TABLE I: The representatives of $P(h)$ under the $GL(2, \mathbb{R})$ and $GL(2, \mathbb{C})$ transformations.

$\nu, g_2, g_3 \in K$ according to the transformation group $GL(2, K)$, and $\nu \neq 0, 0 < k < 1$, $g_2^3 - 27 g_3^2 \neq 0$. In Ref. [47-48], more general quasi-solvable $M$-body systems constructed by $\mathfrak{sl}(M + 1)$ generators are classified according to the above scheme and the explicit form of the potential for each the cases is shown. The type A models in this article correspond to the models for $M = 1$ (with $b_i \rightarrow \pm b_i$ for $H_{X_N}^\pm$) in Ref. [47-48]. So, we do not repeat the exhibition of the potentials in this article.

V. 2-FOLD AND 2ND-DERIVATIVE POLYNOMIAL SUPERSYMMETRY

All that we have not investigated yet on type A $N$-fold supersymmetry is the anti-commutator of $Q_{X_N}^\dagger$ and $Q_N$, namely, the mother Hamiltonian. For this purpose, it is quite instructive to analyze 2-fold supersymmetry. Let us first consider general 2-fold supersymmetry, in which 2-fold supercharge is given by,

$$P_2 = p^2 + w_1(q)p + w_0(q).$$

(5.1)
In this case, a pair of Hamiltonians \( H^\pm_2 = p^2/2 + V^\pm_2(q) \) satisfies the intertwining relation

\[
P^+_2 H^-_2 - H^+_2 P^-_2 = 0 \text{ if and only if,}
\]

\[
V^\pm_2(q) = -\frac{1}{8}w_1(q)^2 + \frac{1}{4} \left[ \frac{w''_1(q)}{w_1(q)} - \frac{w'_1(q)^2}{2w_1(q)^2} + \frac{2C_1}{w_1(q)^2} \right] \pm \frac{i}{2}w'_1(q) - C_2,
\]

\[
w_0(q) = \frac{1}{4}w_1(q)^2 + \frac{1}{2} \left[ \frac{w''_1(q)}{w_1(q)} - \frac{w'_1(q)^2}{2w_1(q)^2} + \frac{2C_1}{w_1(q)^2} \right] - \frac{i}{2}w'_1(q).
\]

where \( C_i \) are arbitrary constants. From the form of \( V^\pm_2(q) \), it is indicated that \( w^*_1(q) = -w_1(q) \), that is, \( w_1(q) \) is pure imaginary in order that \( V^\pm_2(q) \) be real. The above result was first reported in Refs. \([5, 6]\) and later reexamined in Ref. \([27]\). In Refs. \([5, 6]\), the anti-commutator of the 2-fold supercharges was given by the following form:

\[
2H_2 = \{Q^\dagger_2, Q_2\} = 4(H_2 + C_2)^2 + C_1,
\]

that is, it is a polynomial of degree 2 in \( H_2 \). Later, it was proved in Ref. \([20]\) that, for all \( N \), the anti-commutator of \( Q^\dagger_N \) and \( Q_N \) becomes a polynomial of degree \( N \) in the original Hamiltonian \( H_N \) if \( Q^\dagger_N \) and \( Q_N \) satisfying Eq. \((5.7)\) are uniquely determined for the given \( H_N \). Furthermore, it was also proved in Ref. \([20]\) that, for the above 2-fold supersymmetric systems, 2-fold supercharges are unique unless there is a constant \( C_3 \) which satisfies,

\[
w''_1(q) - 2i w_1(q)w'_1(q) - 2i V^-_2(q) = 2C_3 w'_1(q),
\]

or equivalently,

\[
w'_1(q) - i(w_1(q))^2 - 2i V^-_2(q) = 2C_3 w_1(q),
\]

where the integral constant is omitted since it can be absorbed in \( V^-_2(q) \). If Eq. \((5.6)\) is the case, the 2-fold supersymmetric Hamiltonians \( H^\pm_2 \) satisfy another intertwining relation \( \Delta P^+_2 H^-_2 - H^+_2 \Delta P^-_2 = 0 \) and its conjugation with the following 1-fold supercharges:

\[
\Delta P_2 = p + w_1(q) - iC_3, \quad \Delta P_2^\dagger = p - w_1(q) + iC_3.
\]

This result is almost evident from the fact that the condition \((5.5)\) together with Eq. \((5.2a)\) implies,

\[
V^\pm_2(q) = \frac{1}{2}(iw_1(q) + C_3)^2 \pm \frac{i}{2}w'_1(q) - \frac{C_3^2}{2},
\]

that is, the 2-fold supersymmetric Hamiltonian \( H_2 \) in this case is simultaneously ordinary supersymmetric (except for the irrelevant constant term),

\[
[\Delta Q_2, H_2] = [\Delta Q_2^\dagger, H_2] = 0, \quad \{\Delta Q_2^\dagger, \Delta Q_2\} = 2H_2 + C_3^2,
\]

with respect to the supercharges defined by,

\[
\Delta Q_2 = \Delta P_2^\dagger \psi, \quad \Delta Q_2^\dagger = \Delta P_2 \psi^\dagger.
\]
Therefore, if we define new 2-fold supercharges by,

\[ Q_2(\lambda) = Q_2 + \lambda^* \Delta Q_2 = (P_2^\dagger + \lambda^* \Delta P_2^\dagger) \psi, \]  
\[ Q_2^I(\lambda) = Q_2^I + \lambda \Delta Q_2^I = (P_2 + \lambda \Delta P_2) \psi^I, \]  

the 2-fold supersymmetric Hamiltonian \( \mathbf{H}_2 \) with \( w_1(q) \) satisfying Eq. (5.5) commutes with \( Q_2(\lambda) \) and \( Q_2^I(\lambda) \) for arbitrary \( \lambda \):

\[ [Q_2(\lambda), \mathbf{H}_2] = [Q_2^I(\lambda), \mathbf{H}_2] = 0. \]  

(5.11)

On the other hand, the anti-commutator of these new 2-fold supercharges becomes,

\[ \{Q_2^I(\lambda), Q_2(\lambda)\} = 4 (\mathbf{H}_2 + C_2)^2 + |\lambda|^2 (2\mathbf{H}_2 + C_2^Q) + C_1 \]
\[ + \lambda \{Q_2, \Delta Q_2^I\} + \lambda^* \{Q_2^I, \Delta Q_2\}. \]  

(5.12)

It is now evident that the above anti-commutator cannot be, in general, a polynomial in \( \mathbf{H}_2 \) since it contains the term as \((\lambda + \lambda^*)p^3\). Therefore, 2-fold supersymmetry does not always correspond to 2nd-derivative polynomial supersymmetry.

Next, let us return to the case of type A 2-fold supersymmetry. From the definition of the type A \( \mathcal{N} \)-fold supercharge (3.1), type A 2-fold supersymmetry is a special case of 2-fold supersymmetry where \( w_1(q) \) and \( w_0(q) \) in Eq. (3.1) are given by,

\[ w_1(q) = -2iW(q), \quad w_0(q) = -\left( W'(q) + W(q)^2 + \frac{E'(q)}{2} - \frac{E(q)^2}{4} \right). \]

(5.13)

Substituting the above for the condition (5.2), we see that Eq. (5.2a) is equivalent to Eq. (5.2a) while Eq. (5.2b) to Eq. (3.2b). Furthermore, if we substitute Eq. (5.13) for Eq. (5.5), we find that the uniqueness of the type A 2-fold supercharge is guaranteed unless there is a constant \( C_3 \) which satisfies,

\[ 12W(q)^2 - E(q)^2 + 2E'(q) = -16C_3W(q). \]  

(5.14)

In terms of \( P(h) \) and \( Q(h) \), the above condition is rewritten as,

\[ (3Q(h)^2 + H[P])^2 = 32C_3^2P(h)Q(h)^2. \]  

(5.15)

When \( P(h) \) is (at most) a polynomial of fourth degree (remember that \( P(h) \) can be an arbitrary function in the 2-fold supersymmetric case), both side of Eq. (5.15) are (at most) polynomials of eighth degree belonging to the multiple representation \( \rho_{8, -4} \) defined in Eq. (A1). Therefore, Eq. (5.15) is a polynomial identity and the set of its solution constitutes a hyperplane \( \Gamma \) in the parameter space \( \mathbb{R}^{5+3} \) or \( \mathbb{C}^{5+3} \) spanned by \( \{a_i, b_i\} \). On this hyperplane \( \Gamma \), 2-fold supercharges are not determined uniquely and the type A 2-fold supercharges can be deformed as Eq. (5.10) without destroying 2-fold supersymmetry. However, it should be noted that the polynomiality of the anti-commutator of the undeformed type A 2-fold supercharges is preserved on \( \Gamma \). The uniqueness of \( \mathcal{N} \)-fold supercharges is only a sufficient but not a necessary condition for the polynomiality. Conversely, we can choose \( \mathcal{N} \)-fold supercharges such that the anti-commutator of them becomes a polynomial in the Hamiltonian.

We note that the general 2-fold supersymmetry is weakly quasi-solvable since we cannot generally obtain two independent analytic elements of \( \ker P_2 \) where \( P_2 \) is given by Eq. (5.1).
with (5.2b). Nevertheless, we can know the two spectra in the solvable sector. The mother Hamiltonian (5.3) is always a polynomial in the original Hamiltonian. Since it corresponds to the characteristic polynomial which determines the spectra in the solvable sector, they are given by the solutions of $4(E + C_2)^2 + C_1 = 0$. This example shows a novel feature of weak quasi-solvability. Even if, for a given operator $P$, there is no analytic element of $V_N$ defined by Eq. (2.11), we can know the spectra in the solvable sector if the anti-commutator of $N$-fold supercharges constructed from $P$ can be arranged as a polynomial in the Hamiltonian $H$ satisfying the weak quasi-solvability condition (2.12).

VI. MOTHER HAMILTONIANS AND GENERALIZED BENDER–DUNNE POLYNOMIALS

A. Polynorniality of Type A Mother Hamiltonians

Let us come back to further investigation into type A $N$-fold supersymmetry. Suppose the pair of type A $N$-fold supersymmetric Hamiltonians $H_N^\pm = p^2/2 + V_N^\pm(q)$ with the potentials (3.2a) also satisfies the intertwining relation with respect to an $M$-fold supercharge ($N > M$) given by,

$$P_M = p^M - iNW(q)p^{M-1} + \sum_{n=0}^{M-2} w_n(q)p^n. \quad (6.1)$$

From a direct calculation, we have,

$$2i^M(P_M H_N^+ - H_N^- P_M) = \left( NW'' - 2N^2W'W' + 2MV_N' - 2w_0' \right) \partial^{M-1}$$

$$+ \sum_{k=0}^{M-2} \left[ -2(-i)^{k-M}NW'w_k + 2\left(\frac{M}{k}\right)V_N^{-(M-k)} + 2N\left(\frac{M-1}{k}\right)WV_N^{-(M-1-k)} \right] \partial^k.$$  \quad (6.2)

Therefore, $w_k(q)$ ($k = 0, \ldots, M - 2$) must satisfy,

$$w_{M-2} = \frac{N}{2} W'' - N^2W'W' + MV_N', \quad (6.3a)$$

$$w_k = -\sum_{n=k+1}^{M-2} (-i)^{n+1-k} \binom{n}{k} V_N^{-(n-k)} w_n + i \left( \frac{w''_0}{2} - NW'w_k \right)$$

$$- i^{k-1-M} \left[ \binom{M}{k} V_N^{-(M-k)} + \binom{M-1}{k} NWV_N^{-(M-1-k)} \right] (k = 1, \ldots, M - 2), \quad (6.3b)$$

in addition to,

$$w_0'' - 2NW'w_0 + 2\sum_{n=1}^{M-2} (-i)^n nV_N^{-(n)} w_n$$

$$+ 2(-i)^M \left[ MV_N^M + (M-1)NWV_N^{-(M-1)} \right] = 0. \quad (6.4)$$
The set of the first-order differential equations (6.3) can be integrated out in the order from \( w_{N-2}(q) \) to \( w_0(q) \) since all the terms appeared in the r.h.s. of Eqs. (6.3) becomes known functions depending only on \( E(q) \) and \( W(q) \) in the order. Thus, we obtain,
\[
w_k(q) = f_k(E(q), W(q)) + C_k \quad (k = 0, \ldots, M - 2),
\]
where \( C_k \) are integral constants. We can put \( C_M = 0 \) without any loss of generality; if \( C_{M'} \neq 0 \) for an \( M' \), we can always split the \( M' \)-fold supercharge as,
\[
P_{M'}(C_{M'}) = P_{M'} + C_{M'}P_{M'},
\]
which means the system is also \( M' \)-fold supersymmetric. Therefore, \( w_k(q) \) and hence \( P_M \) are uniquely determined. On the other hand, if the type A \( N \)-fold supersymmetric potentials (3.2a) can be rewritten as the type A \( M \)-fold supersymmetric ones with \( W(q) \) replaced by \( \frac{NW(q)}{M} \), that is, if the following relations,
\[
V_\frac{N}{M}[E(q), W(q)] = V_\frac{N}{M}[E(q), \frac{NW(q)}{M}],
\]
are satisfied, it is evident that the type A \( N \)-fold supersymmetric system is also type A \( M \)-fold supersymmetric with respect to the following type A \( M \)-fold supercharge:
\[
P_M = \prod_{k=-\frac{(M-1)}{2}}^{\frac{(M-1)}{2}} \left( p - i\frac{N}{M}W(q) + iE(q) \right).
\]
As we have just shown, however, \( P_M \) is unique and thus the \( M \)-fold supercharge determined by the solutions of Eqs. (6.3) must be the type A \( M \)-fold supercharge Eq. (6.8). Furthermore, the constraint (6.4) must be equivalent to the relation (6.7). Summarizing the result, we have shown that type A \( N \)-fold supercharge is unique unless the system satisfies Eq. (6.7) for an \( M \) (0 < \( M < N \)). The relation (6.7) is equivalent to,
\[
12W(q)^2 - \mathcal{M}^2 \left( E(q)^2 - 2E'(q) \right) = \frac{4}{(\hbar')^2} \left( 3Q(h)^2 + \mathcal{M}^2H[P] \right) = 0,
\]
and its solutions again constitute a hyperplane \( \Gamma \) in the parameter space spanned by \( \{a_i, b_i\} \). Outside the hyperplane, the \( N \)-fold supercharge is unique and the mother Hamiltonian is expressed as a polynomial \( P_N \) in the original Hamiltonian \( \mathcal{H}_N \):
\[
\mathcal{H}_N = \frac{1}{2}\{Q^1_N, Q_N \} = P_N(\mathcal{H}_N) \quad \text{for} \quad \{a_i, b_i\} \in \mathbb{R}^{5+3} \setminus \Gamma \quad \text{or} \quad \mathbb{C}^{5+3} \setminus \Gamma.
\]
On the other hand, as has been shown in Eq. (2.19), this polynomial \( P_N \) corresponds to the characteristic polynomial which determines the spectra in the solvable sector [27]. Since the characteristic polynomial itself is constructed from a finite algebraic operation, it has no discontinuity in the parameter space. Therefore, Eq. (6.10) must be held on \( \Gamma \) and thus the type A mother Hamiltonian must be a polynomial in the original Hamiltonian \( \mathcal{H}_N \) in the whole parameter space.
B. Generalized Bender–Dunne polynomials

In 1996, Bender and Dunne introduced a set of polynomials which determines the spectra in the solvable sector of the quasi-exactly solvable model of case II, namely, the sextic anharmonic oscillator [32]. Soon after, Finkel et al. generalized the idea to all the \( \mathfrak{sl}(2) \) quasi-exactly solvable models [34]. In the following, we will make a generalization without imposing the normalizability of the solvable sector. Let the bases \( \{ \phi^\pm \} \) of the solvable subspaces \( \mathcal{V}_N^\pm \) be,

\[
\phi^\pm_n = \hbar^{n-1} (G_N U^\mp 1)^{-1} \quad (n = 1, \ldots, N),
\]

and the gauged Hamiltonians be,

\[
(G_N U^\mp 1) H^\pm_N (G_N U^\mp 1)^{-1} = -P(h) \frac{d^2}{dh^2} - P^\pm_3(h) \frac{d}{dh} - P^\pm_2(h).
\]

Then we have,

\[
H^\pm_N \phi^\pm_n = -(n-1)(n-2)P(h)\phi^\pm_{n-2} - (n-1)P^\pm_3(h)\phi^\pm_{n-1} - P^\pm_2(h)\phi^\pm_n.
\]

From Eqs. (3.41), and (3.50b) with (3.52) and (3.55), the \( P^\pm_i(h) \) are expressed as,

\[
P^\pm_3(h) = -\frac{N-2}{2} P'(h) \mp Q(h), \tag{6.14a}
\]

\[
P^\pm_2(h) = \frac{(N-1)(N-2)}{12} P''(h) \mp \frac{N-1}{2} Q'(h) + R. \tag{6.14b}
\]

Substituting Eqs. (3.42), (3.43) and (6.14) for Eq. (6.13), we obtain the matrix elements \( S^\pm_{n,m} \) defined in Eq. (2.14):

\[
H^\pm_N \phi^\pm_n = -(n-N)(n-N+1)a_4 \phi^\pm_{n+2} - (n-N) \left[ \left( n - \frac{N}{2} \right) a_3 \mp b_2 \right] \phi^\pm_{n+1}
- \left\{ \left[ (n-1)(n-N) + \frac{1}{6}(N-1)(N-2) \right] a_2 \mp \left( n - \frac{N+1}{2} \right) b_1 + R \right\} \phi^\pm_n
- (n-1) \left[ \left( n - \frac{N+2}{2} \right) a_1 \mp b_0 \right] \phi^\pm_{n-1} - (n-1)(n-2) a_0 \phi^\pm_{n-2}. \tag{6.15}
\]

As we have discussed in Section IV, there are five independent type \( A \) \( N \)-fold supersymmetric models under the \( GL(2, \mathbb{C}) \) transformation. By a suitable \( GL(2, \mathbb{C}) \) transformation, we can always transform \( P(h) \) so that \( a_0 = 0 \) in all the five cases, see Table III. In Table III, \( e_i \) \((i = 1, 2, 3)\) denote the three different single roots of the algebraic equation \( 2x^3 - g_2x^2 - g_3/2 = 0 \) and \( h_i \) \((i = 2, 3)\) are given by \( h_i = e_i - e_1 \). So, we set \( a_0 = 0 \) hereafter. We note that case I corresponds to the case where \( P(h) \) has a quadruple root and thus \( P(h) = \text{constant} \) (quadruple root at infinity) or \( P(h) \propto (h-h_0)^4 \) (quadruple root at finite \( h_0 \)). Therefore, in contrast to in all the other cases, case II–V, we cannot simultaneously set \( a_4 = 0 \) and \( a_0 = 0 \) by any \( GL(2, \mathbb{C}) \) transformation in case I. We also drop another irrelevant constant by putting \( R = 0 \).
In the next, we introduce a set of functions $P_n^{[N]}(E)$ by putting a solution of the Schrödinger equation as follows:

$$
\psi^\pm = \begin{cases} 
\sum_{n=0}^{\infty} \frac{P_n^{[N]}(E)}{(\pm b_0)^n n!} \phi^\pm_{n+1} & \text{if } a_1 = 0, \\
\sum_{n=0}^{\infty} \frac{P_n^{[N]}(E)}{(-a_1)^n n! \Gamma \left( n - \frac{N+2}{2} \mp \frac{b_0}{a_1} \right)} \phi^\pm_{n+1} & \text{if } a_1 \neq 0.
\end{cases}
$$

(6.16)

From the requirement that the above $\psi^\pm$ satisfies $H_N^{[N]} \psi^\pm = E \psi^\pm$, we obtain a four-term recursion relation for $P_n^{[N]}(E)$:

$$
P_{n+1}^{[N]}(E) = (E + A_n^{[N]}) P_n^{[N]}(E) - n(n-N) B_n^{[N]} P_{n-1}^{[N]}(E) + n(n-1)(n-N)(n-N-1) C_n^{[N]} P_{n-2}^{[N]}(E),
$$

(6.17)

where $A_n^{[N]}$, $B_n^{[N]}$ and $C_n^{[N]}$ are given by,

$$
A_n^{[N]} = \left[ n(n-N+1) + \frac{1}{6} (N-1)(N-2) \right] a_2 \mp \left( n - \frac{N-1}{2} \right) b_1,
$$

(6.18a)

$$
B_n^{[N]} = \left[ \left( n - \frac{N}{2} \right) a_1 \mp b_0 \right] \left[ \left( n - \frac{N}{2} \right) a_3 \mp b_2 \right],
$$

(6.18b)

$$
C_n^{[N]} = a_4 \left[ \left( n - \frac{N}{2} \right) a_1 \mp b_0 \right] \left[ \left( n - \frac{N+2}{2} \right) a_1 \mp b_0 \right].
$$

(6.18c)

We can set $P_0^{[N]}(E) = 1$ without any loss of generality. Then, each $P_n^{[N]}(E)$ generated by Eq. (6.17) becomes a monic polynomial of $n$th degree. We call it a generalized Bender–Dunne polynomial (GBDP). In the case of $a_4 = 0$, Eq. (6.17) reduces to a three-term recursion relation with (in general) $n(n-N)B_n^{[N]} \neq 0$ for $n \neq N$ and thus the set of $P_n^{[N]}(E)$ forms a weakly orthogonal polynomial system [49]. Therefore, case I is special in the sense that the set of polynomials $P_n^{[N]}$ does not form a weakly orthogonal polynomial system. One of the most notable properties of the GBDP is the factorization property; due to the structure of Eq. (6.17), all the polynomials $P_n^{[N]}(E)$ for $n \geq N$ are factorized as,

$$
P_{N+n}^{[N]}(E) = Q_n^{[N]}(E) P_{N}^{[N]}(E) \quad (n \geq 0),
$$

(6.19)
where \( Q_n^{[\mathcal{N}]}(E) \) is a polynomial of degree \( n \) satisfying another four-term recursion relation:

\[
Q_{n+1}^{[\mathcal{N}]}(E) = (E + A_n^{[\mathcal{N}]})Q_n^{[\mathcal{N}]}(E) - n(n + \mathcal{N})B_n^{[\mathcal{N}]}Q_n^{[\mathcal{N}]}(E) + n(n-1)(n+\mathcal{N})(n+\mathcal{N}-1)C_n^{[\mathcal{N}]}Q_n^{[\mathcal{N}]}(E).
\]

In the case of \( a_4 = 0 \), Eq. (6.20) reduces to a three-term recursion relation with (in general) \( n(n+\mathcal{N})B_n^{[\mathcal{N}] \neq 0} \) for all \( n > 0 \) and thus the set of \( Q_n^{[\mathcal{N}]}(E) \) forms an orthogonal polynomial system. Therefore, case I is again special in the sense that the set of polynomials \( Q_n^{[\mathcal{N}]} \) does not form an orthogonal polynomial system. Due to the factorization property (6.19), \( P_N^{[\mathcal{N}]}(E) \) reserves special status among the GBDPs. We call \( P_N^{[\mathcal{N}]}(E) \) \( \mathcal{N} \)-th critical GBDP after the terminology in Ref. [32].

When \( E \) takes one of the spectral values \( E_n (n = 1, \ldots, \mathcal{N}) \) in the solvable sector, \( \psi^\pm \) must be an element of \( \mathcal{V}_N^\pm \). From the factorization property (6.19), the condition \( \psi^\pm \in \mathcal{V}_N^\pm \) is fulfilled if and only if,

\[
P_N^{[\mathcal{N}]}(E_n) = 0 \quad (n = 1, \ldots, \mathcal{N}).
\]

This means that all the zeros of the \( \mathcal{N} \)-th critical GBDP must correspond to the spectra in the solvable sector of the quasi-solvable Hamiltonians \( H_N^\pm \). On the other hand, they are also given by solutions of the characteristic equation (2.15). Therefore, each of the critical GBDP is proportional to the corresponding characteristic polynomial. Comparing the coefficients of the highest-degree term with each other, we have,

\[
\det M_N^\pm(E) = 2^\mathcal{N} P_N^{[\mathcal{N}]}(E).
\]

As we have proved before, the type A mother Hamiltonian is expressed solely by the characteristic polynomial of the original Hamiltonian as Eq. (2.19). Combining Eq. (2.19) with (6.22), we finally obtain an intriguing relation:

\[
\mathcal{H}_\mathcal{N} = \frac{1}{2} \det M_N^\pm(\mathcal{H}_\mathcal{N}) = 2^{\mathcal{N}-1} P_N^{[\mathcal{N}]}(\mathcal{H}_\mathcal{N}).
\]

Furthermore, remembering that the mother Hamiltonian is defined by the anti-commutator of the \( \mathcal{N} \)-fold supercharges (2.16), we get the complete type A \( \mathcal{N} \)-fold superalgebra:

\[
\{Q_{\mathcal{N}}, Q_{\mathcal{N}}\} = 2^{\mathcal{N}} P_N^{[\mathcal{N}]}(\mathcal{H}_\mathcal{N}).
\]

\[
C. \quad \text{Examples}
\]

In order to confirm the previous argument, we will show the explicit results for \( \mathcal{N} = 1, 2, 3 \). By solving the recursion relation (6.17), we obtain \( P_n^{[\mathcal{N}]}(E) (n \leq \mathcal{N}) \) as follows:

1) \( \mathcal{N} = 1 \):

\[
P_1^{[1]}(E) = E,
\]
2) $N = 2$:

\[
P_1^2(E) = E \pm \frac{b_1}{2},
\]

\[
P_2^2(E) = E^2 + b_0b_2 - \frac{b_1^2}{4},
\]

(6.26a, 6.26b)

3) $N = 3$:

\[
P_1^3(E) = E + \frac{a_2}{3} \pm b_1,
\]

(6.27a)

\[
P_2^3(E) = E^2 - \frac{1}{3} (a_2 \mp 3b_1) E
\]

\[-\frac{1}{18} (4a_2^2 - 9a_1a_3 \mp 18a_1b_1 \mp 12a_2b_1 \mp 18a_3b_0 - 36b_0b_2),
\]

(6.27b)

\[
P_3^3(E) = E^3 + \frac{1}{3} (3a_1a_3 - a_2^2 + 12b_0b_2 - 3b_1^2) E - \frac{1}{27} (2a_2^3 - 9a_1a_2a_3
\]

\[+ 27a_1^2a_4 - 108a_4b_0^2 + 54a_3b_0b_1 - 18a_2b_1^2 - 36a_2b_0b_2 + 54a_1b_1b_2),
\]

(6.27c)

On the other hand, the direct calculation of the mother Hamiltonians reads as follows:

1) $N = 1$:

\[2\mathcal{H}_1 = 2\mathcal{H}_1,
\]

(6.28)

2) $N = 2$:

\[2\mathcal{H}_2 = 4(\mathcal{H}_2)^2 + D_2[Q],
\]

(6.29)

3) $N = 3$:

\[2\mathcal{H}_3 = 8(\mathcal{H}_3)^3 - \frac{8}{3} (i_2[P] - 3D_2[Q]) \mathcal{H}_3 + \frac{16}{27} (j_3[P] + 9I_{1,2}[P, Q]).
\]

(6.30)

In the above, $D_2$, $i_2$, $j_3$ and $I_{1,2}$ are the absolute invariants expressed in terms of $E(q)$ and $W(q)$, see Eqs. (A14)–(A17). Comparing the critical GBDPs, Eqs. (6.25), (6.26), and (6.27), obtained by solving the recursion relation (6.17) with the mother Hamiltonians (6.28)–(6.30), and noting that we have put $a_0 = 0$, we confirm the relation (6.23) for $N = 1, 2, 3$. We also note that since the mother Hamiltonians are invariant under the $GL(2, K)$ transformation, all the coefficients of the critical GBDPs should be expressed solely in terms of the absolute invariants listed in Eq. (A6), as the above examples indicate. From the facts that all the GBDPs are homogeneous polynomials in $E$, $a_i$ and $b_i$ due to the structure of the recursion relation (6.17), and that all the critical GBDPs are symmetric under the transformation $b_i \rightarrow -b_i$, the general form of the critical GBDPs should be,

\[
P_N^N(E) = E^N + \sum_{k=0}^{N-2} E^k \left[ \sum_{n_1, \ldots, n_6} \delta_{2n_1+2n_2+3n_3+3n_4+4n_5+12n_6, N-k} \times
\]

\[
C_{k; n_1, \ldots, n_6}^{[N]} (D_2)^{n_1} (i_2)^{n_2} (j_3)^{n_3} (I_{1,2})^{n_4} (I_{2,2})^{n_5} (I_{3,3})^{2n_6} \right].
\]

(6.31)
where $C_{k,n_1,...,n_6}^{[\mathcal{A}]}$ are constants. For example, the first five are,

\begin{align}
P_{1}^{[1]}(E) &= E, \quad (6.32a) \\
P_{2}^{[2]}(E) &= E^2 + \frac{1}{4}D_2, \quad (6.32b) \\
P_{3}^{[3]}(E) &= E^3 - \frac{1}{3}(i_2 - 3D_2)E + \frac{2}{27}(j_3 + 9I_{1,2}), \quad (6.32c) \\
P_{4}^{[4]}(E) &= E^4 - \frac{1}{2}(4i_2 - 5D_2)E^2 + 4I_{1,2}E + (i_2)^2 + 2I_{2,2} + i_2D_2 + \frac{9}{16}(D_2)^2, \quad (6.32d) \\
P_{5}^{[5]}(E) &= E^5 - (7i_2 - 5D_2)E^3 - 2(j_3 - 7I_{1,2})E^2 \\
&\quad + 4 \left(3(i_2)^2 + 4I_{2,2}\right)E + 8(i_2 - D_2)(j_3 + I_{1,2}). \quad (6.32e)
\end{align}

Since the zeros of the critical GBDPs correspond to the spectra of $H_N^\pm$ in the solvable sector, these spectra can be regarded as functions of the six absolute invariants. Furthermore, as an interpretation of Eq. (6.31), the critical GBDPs, and equivalently the type A mother Hamiltonians, can be regarded as generating functions of the absolute invariants composed of $P(h)$ and $Q(h)$.

VII. CONCLUDING REMARKS

In this article, we have fully investigated general aspects of type A $\mathcal{N}$-fold supersymmetry. The two different approaches in Section III reveal both the analytic and algebraic structures of the systems. The intimate relation between the algebraic forms of the pair Hamiltonians, Eqs. (3.38) and (3.57), provides an interesting problem. Suppose we have a quasi-solvable system $\tilde{H}$ which is represented by a quadratic polynomial of a set of first-order differential operators constituting a finite dimensional representation of a Lie algebra. If we construct another system $\bar{H}$ from $\tilde{H}$ by interchanging the order of the quadratic terms and interchanging the sign of the coefficients of the linear terms, do $\tilde{H}$ and $\bar{H}$ (with suitable gauge transformations) always form an $\mathcal{N}$-fold supersymmetric pair? This problem may be extended to quasi-solvable many-body systems [47, 48].

The invariance of the Hamiltonians as well as the type A $\mathcal{N}$-fold supercharge under the $GL(2,K)$ transformation enable us to obtain the complete classification of the type A models. This invariance also plays an essential role in generalizing the Bender–Dunne polynomials to for all the type A models. With regard to the issue, we should stress that, up to now, we have found no reason that a set of the polynomials should be weakly orthogonal, although all the existing papers concerning about the issue [32, 33, 34, 50, 51, 52, 53], as far as we know, have required weak orthogonality by imposing, for instance, the normalizability, or restricted the considerations in which weak orthogonality is fulfilled. One of the most essential properties that sets of polynomials associated with quasi-solvable systems should share is the factorization property (6.19). The most general form of a recursion relation for a set of polynomials $P_n^{[\mathcal{A}]}$ which guarantee the factorization property may be the following:

\begin{equation}
P_{n+1}^{[\mathcal{A}]}(E) = (E + A_0^{[\mathcal{A}]})P_n^{[\mathcal{A}]}(E) + \sum_{k=1}^{K} \left[ (-)^k \prod_{l=0}^{k-1} (n-l)(n-N-l)A_{k,n}^{[\mathcal{A}]} P_{n-k}^{[\mathcal{A}]}(E) \right]. \quad (7.1)
\end{equation}

Then, an interesting question is, what conditions a set of the coefficients $\{A_{k,n}^{[\mathcal{A}]}\}$ should satisfy for the existence of a quasi-solvable system whose spectra in its solvable sector are given by
the zeros of the critical polynomial $P_N^{[N]}(E)$ obtained by the above recursion relation. This question may provide an alternative way to find out a new quasi-solvable and also a new $\mathcal{N}$-fold supersymmetric model.

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APPENDIX A: INVARIANTS OF A SYSTEM OF POLYNOMIALS

Since the type $A\mathcal{N}$-fold supersymmetric models have the underlying $GL(2,K)$ invariance discussed in Section 4, all the relevant quantities should be expressed in terms of the covariants and invariants under the transformation. In this appendix, we summarize the covariants and invariants of a system of polynomials needed in this article. For more details, see Ref. [14] and references cited therein.

The multiplier representation $\rho_{m,i}$ of $GL(2,K)$ on the space of polynomials of degree at most $m$ is defined by,

$$F(h) \mapsto \hat{F}(h) = \Delta^i (\gamma h + \delta)^m F(\hat{h}),$$

(A1)

where $\hat{h}$ and $\Delta$ are defined by Eq. (4.1). For example, we see from Eqs. (4.3) that $P(h)$ belongs to the representation $\rho_{4,-2}$ while $Q(h)$ belongs to the representation $\rho_{2,-1}$.

In order to construct a complete system of covariants and invariants, the process of transvection is essential. Let $F(h)$ be a polynomial of degree at most $m$ belonging to the representation $\rho_{m,i}$ and $G(h)$ be a polynomial of degree at most $n$ belonging to the representation $\rho_{n,j}$. The $r$th transvectant of $F$ and $G$ is defined by,

$$\langle F, G \rangle^{(r)} = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \binom{m-r+k}{m-r} \binom{n-k}{n-r} F^{(r-k)} G^{(k)}.$$

(A2)

Then, $\langle F, G \rangle^{(r)}$ is a polynomial of degree at most $m+n-2r$ belonging to the representation $\rho_{m+n-2r,i+j+r}$.

For a single quadratic polynomial $Q$, there are two independent covariants, namely, $Q$ itself and its discriminant $D_2$ given by,

$$D_2[Q] = \frac{1}{2} (Q, Q)^{(2)}.$$

(A3)

For a single quartic polynomial $P$, there are five independent covariants, namely, $P$ itself, the Hessian $H$ and the Jacobian $J$ given by,

$$H[P] = \frac{1}{24} (P, P)^{(2)} = PP'' - \frac{3}{4} (P')^2,$$

(A4a)

$$J[P] = \frac{1}{24} (P, H)^{(1)} = -\frac{1}{24} \left[ 4P^2 P^{(3)} - 6PP'P'' + 3 (P')^3 \right].$$

(A4b)
and the two invariants given by,
\[ i_2[P] = \frac{1}{96}(P, P)^{(4)} , \quad j_3[P] = \frac{1}{96}(P, H)^{(4)}. \] (A5)

The discriminant of \( P \) is expressed in terms of the above invariants as \( D_6[P] = i_2[P]^3 - j_3[P]^2 \). According to the invariant theory of polynomials, the complete list of independent covariants for the system consisting of a quartic and a quadratic polynomial is the following:

1) Absolute invariants \( \rho_{0,0} \):
\[ D_2[Q], \ i_2[P], \ j_3[P], \ I_{1,2}[P, Q], \ I_{2,2}[P, Q], \ I_{3,3}[P, Q], \] (A6)

2) Quadratic covariants \( \rho_{2,-1} \):
\[ Q, \ (P, Q)^{(2)}, \ (H, Q)^{(2)}, \ (P, Q^2)^{(3)}, \ (H, Q^2)^{(3)}, \ (J, Q^2)^{(4)}, \] (A7)

3) Quartic covariants \( \rho_{4,-2} \):
\[ P, \ H[P], \ (P, Q)^{(1)}, \ (H, Q)^{(1)}, \ (J, Q)^{(2)}, \] (A8)

4) Sextic covariant \( \rho_{6,-3} \):
\[ J[P]. \] (A9)

In the above, the absolute invariants \( I_{1,2}, I_{2,2} \) and \( I_{3,3} \) are given by,
\[ I_{1,2}[P, Q] = \frac{1}{96}(P, Q^2)^{(4)}, \quad I_{2,2}[P, Q] = \frac{1}{96}(H, Q^2)^{(4)}, \quad I_{3,3}[P, Q] = \frac{1}{64800}(J, Q^3)^{(6)}. \] (A10)

We can express the above quantities in terms of \( h(q), E(q) \) and \( W(q) \) with the aid of the following relations derived from Eqs. (3.30) and (3.44):
\[ P = \frac{1}{2}(h')^2, \quad P' = h'' = Eh', \quad P'' = E' + E^2, \] (A11a)
\[ P^{(3)} = \frac{1}{h'}(E'' + 2EE'), \quad P^{(4)} = \frac{1}{(h')^2} \left[ E^{(3)} + EE'' + 2(E')^2 - 2E^2E' \right], \] (A11b)
\[ Q = -Wh', \quad Q' = -(W' + EW), \] (A12a)
\[ Q'' = -\frac{1}{h'}(W'' + EW' + E'W). \] (A12b)

For example, the Hessian \( H \) and the Jacobian \( J \) are expressed as,
\[ H[P] = \frac{(h')^2}{4}(2E' - E^2), \quad J[P] = -\frac{(h')^3}{24}(E'' - EE'). \] (A13)

Among the covariants \( (A6)-(A9) \), the absolute invariants play an essential role in Section VI. In the following, we show the explicit forms of them in terms of the coefficients of the
polynomials \( P(h) \) and \( Q(h) \), as well as in terms of \( E(q) \) and \( W(q) \) (for the first four invariants because the expressions for the last two are lengthy):

\[
D_2[Q] = 2QQ'' - (Q')^2 \\
= 2WW'' - (W')^2 + (2E' - E^2)W^2 \\
= 4b_0b_2 - b_1^2, \quad (A14)
\]

\[
i_2[P] = \frac{1}{4} \left[ 2PP^{(4)} - 2P'P^{(3)} + (P'')^2 \right] \\
= \frac{1}{4} \left[ E^{(3)} - EE'' + 3(E')^2 - 4E^2E' + E^4 \right] \\
= 12a_0a_4 - 3a_1a_3 + a_2^2, \quad (A15)
\]

\[
2j_3[P] = \frac{1}{8} \left[ 12PP''P^{(4)} - 6P (P^{(3)})^2 - 9 (P'')^2 P^{(4)} + 6P'P''P^{(3)} - 2 (P'')^3 \right] \\
= \frac{1}{8} \left[ 6E'\dot{E}^{(3)} - 3 (E'')^2 - 3E^2E^{(3)} \right. \\
\left. + 10 (E')^3 + 3E^3E'' - 24 E^2(E')^2 + 12E^4E' - 2E^6 \right] \\
= 72a_0a_2a_4 - 27a_0a_2^2 - 27a_1a_4 + 9a_1a_2a_3 - 2a_3^2, \quad (A16)
\]

\[
4I_{1,2}[P, Q] = P^{(4)}Q^2 - 2P^{(3)}QQ' + 2P'' \left[ QQ'' + (Q')^2 \right] - 6P'Q'Q'' + 6P (Q'')^2 \\
= \left[ E^{(3)} - EE'' + 7 (E')^2 - 8E^2E' + 2E^4 \right]W^2 \\
- 2 (E'' - EE')WW' + (2E' - E^2) \left[ 4WW'' + (W')^2 \right] + 3 (W'')^2 \\
= 4 \left( 6a_4b_0^2 - 3a_3b_0b_1 + 2a_2b_0b_2 + a_2b_1^2 - 3a_1b_1b_2 + 6a_0b_2^2 \right), \quad (A17)
\]

\[
2I_{2,2}[P, Q] = \frac{1}{2} \left\{ H^{(4)}Q^2 - 2H^{(3)}QQ' + 2H'' \left[ QQ'' + (Q')^2 \right] - 6H'Q'Q'' + 6H (Q'')^2 \right\} \\
= 3 \left( 8a_2a_4 - 3a_3^2 \right) b_0^2 - 6 \left( 6a_1a_4 - a_2a_3 \right) b_0b_1 \\
+ \left( 24a_0a_4 + 3a_1a_3 - 2a_2^2 \right) \left( 2b_0b_2 + b_1^2 \right) - 6 \left( 6a_0a_3 - a_1a_2 \right) b_1b_2 \\
+ 3 \left( 8a_0a_2 - 3a_1^2 \right) b_2^2, \quad (A18)
\]

\[
I_{3,3}[P, Q] = \frac{1}{90} \left\{ J^{(6)}Q^3 - 3J^{(5)}Q^2Q' + 3J^{(4)}Q \left[ QQ'' + 2 (Q')^2 \right] - 6J^{(3)}Q' \left[ 3QQ'' + (Q')^2 \right] \right\} \\
+ 18J'' \left[ QQ'' + 2 (Q')^2 \right] - 90J'Q' \left( Q'' \right)^2 + 90J \left( Q' \right)^3 \\
= \left( 8a_1a_4 - 4a_2a_3a_4 + a_3^2 \right) b_0^2 - \left( 16a_0a_4 + 2a_1a_3a_4 - 4a_2^2a_4 + a_2a_3^2 \right) b_0b_1 \\
+ \left( 8a_0a_3a_4 - 4a_1a_2a_4 + a_1a_2^2 \right) \left( b_0b_2 + b_0b_1^2 \right) - 6 \left( a_0a_3^2 - a_1a_2^2 \right) b_0b_1b_2 \\
- \left( a_0a_3^2 - a_1^2a_4 \right) b_1^2 - \left( 8a_0a_1a_4 - 4a_0a_2a_3 + a_1a_2^2 \right) \left( b_0b_2^2 + b_1^2b_2 \right) \\
+ \left( 16a_0^2a_4 + 2a_0a_1a_3 - 4a_0a_2^2 + a_1^2a_2 \right) b_1b_2^2 - \left( 8a_0^2a_3 - 4a_0a_1a_2 + a_1^3 \right) b_2^3. \quad (A19)
\]
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