Mass function and particle creation in Schwarzschild-de Sitter spacetime

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Abstract

We construct a mass or energy function for the non-Nariai class Schwarzschild-de Sitter black hole spacetime in the region between the black hole and the cosmological event horizons. The mass function is local, positive definite, continuous and increases monotonically with the radial distance from the black hole event horizon. We derive the Smarr formula using this mass function, and demonstrate that the mass function reproduces the two-temperature Schwarzschild-de Sitter black hole thermodynamics, along with a term corresponding to the negative pressure due to positive cosmological constant. We further give a field theoretic derivation of the particle creation by both the horizons and discuss its connection with the mass function.

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1 Introduction

In general relativity, an important concept is that of a mass function. It should be regarded as a conserved quantity associated with the spacetime itself. There are several criteria which should be obeyed by a quantity if it is to be regarded as a mass or energy of a given spacetime. First, it must be defined with respect to a timelike translational Killing vector field and second, the mass function must be a positive definite quantity. It is the latter criterion that makes it a difficult problem to define a mass function because one cannot construct a satisfactory notion of conserved gravitational energy-momentum tensor unless one goes to the asymptotic region (see e.g. [1] [2]

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and references therein). Only an approximate notion of gravitational Hamiltonian density can be defined perturbatively and locally but the positivity of this quantity is far from obvious.

For asymptotically flat spacetimes a gravitational mass can be defined in several ways. The simplest is the Komar mass \([1]\). This is proportional to the surface integral of the derivative of the norm of the timelike Killing vector field and thus is related directly to geodesic motion. In general the Komar integral will be positive definite only if the matter energy-momentum tensor \(T_{ab}\) satisfies the strong energy condition (SEC): \((T_{ab} - \frac{1}{2}Tg_{ab})\xi^a\xi^b \geq 0\), for any timelike \(\xi^a\).

The second method of defining mass is via the Arnowitt-Deser-Misner (ADM) [3, 4, 5] formalism. In this approach a gravitational Hamiltonian density is defined with respect to the background timelike Killing vector field in the asymptotic region and the integral of this Hamiltonian density is computed. This integral is interpreted as the gravitational mass.

It is known from the Raychaudhuri equation that geodesics would converge in a mass distribution only if the latter satisfies the SEC [1, 6, 7]. Also, it is known that the SEC usually implies the weak energy condition, i.e. the positivity of the energy density. Using these two facts a third approach to define gravitational mass and to prove its positivity was developed in [8, 9] for asymptotically flat spacetimes, assuming every complete null geodesic congruence in the domain of outer communications admits a pair of conjugate points.

As we mentioned earlier, unlike usual matter fields, the positivity of the gravitational Hamiltonian density or the gravitational mass is far from obvious, and hence it requires a formal proof. The positivity conjecture of the ADM mass was first proved in [10, 11]. Soon afterward, a remarkable proof of the positivity appeared in [12] using a spacelike spinor field on a spacelike non-singular Cauchy surface. This result was generalized for black holes in asymptotically flat or anti-de Sitter spacetimes in [13]. The \(\Lambda \leq 0\) spacetimes usually have well defined asymptotic structure or infinities which are accessible to the geodesic observers. The references mentioned above consider explicit asymptotic structures of such spacetimes at spacelike infinities which are uniquely Minkowskian or anti-de Sitter. In fact the positivity of the ADM mass for \(\Lambda \leq 0\) physically reasonable spacetimes admitting spin structure and well defined asymptotics is well understood so far.

But recent observations suggest that our universe is undergoing a phase of accelerated expansion [14, 15], and hence there is a strong possibility that our universe is endowed with a small but positive cosmological constant. We note that, since a positive \(\Lambda\) violates SEC, it repels geodesics (see e.g. [16, 17, 18, 19, 20]) and hence the first and the third of the methods mentioned above to define gravitational mass do not seem to be applicable in this case. Also, the known exact stationary solutions with \(\Lambda > 0\) (see e.g. [21]) usually exhibits an outer horizon, namely the cosmological event horizon. The tiny observed value of \(\Lambda\) (of the order of \(10^{-52}\)m\(^{-2}\)) sets the length scale of the horizon to be \(O\left(\frac{1}{\sqrt{\Lambda}}\right)\), which is of course very large, but finite. If a black hole is present, it will be located inside the cosmological horizon and the cosmological event horizon acts in such a spacetime as an outer causal boundary [22], beyond which the timelike Killing vector field becomes spacelike and communication is not possible along a future directed causal path thereby ruling out any meaningful use of asymptotics for an observer located inside the cosmological horizon.

To the best of our knowledge, one of the earliest construction of mass in de Sitter black hole spacetimes appeared in [22], where mass function was defined on the black hole and cosmological horizons using the integral of their respective surface gravities. The variation of this mass function gave a Smarr formula. In [23], metric perturbation was considered in a region far away from the black hole but inside the cosmological event horizon where the background spacetime was de Sitter. A local gravitational energy momentum tensor was constructed and with respect to the background de Sitter timelike Killing field the mass of the perturbation was defined. This perturbative approach has much in common with the usual Hamiltonian ADM formulation of general relativity.

We note here that although this formalism is not applicable to spacetimes where the black hole and cosmological horizons are comparable in size, it is well suited to a universe where black holes and the cosmological constant are of sizes comparable to ours.

We shall adopt this approach explicitly in this paper and from now on call it the Abbott-Deser
(AD) formalism. For asymptotically Schwarzschild-de Sitter spacetimes the mass in this asymptotic region with respect to the de Sitter background but inside the cosmological horizon was found to be \( M \) \(^{23}\), i.e. the mass parameter of the Schwarzschild-de Sitter metric. The spinorial proof of positivity of ADM mass for asymptotically flat spacetimes \(^{12}\) was generalized later in \(^{23, 25}\) to show that the mass thus defined in the sense of \(^{23}\) with respect to the background de Sitter spacetime is indeed a positive definite quantity.

What do we wish to do then and with what purpose? The answer is the following. Since there exists no preferred asymptotic region in between the two horizons, there can be no preferred position for an observer. Accordingly, we shall derive the AD masses in other regions between the two horizons too, where perturbation scheme is valid, keeping in mind that firstly the mass must be a continuous positive definite quantity, and secondly, since positive \( \Lambda \) corresponds to a positive energy density, the \( \Lambda \) part of this mass function should increase monotonically with radial distance from the black hole horizon. We shall not give here a formal proof for the positive energy theorem in de Sitter spacetimes, but shall construct a physically reasonable mass function for the Schwarzschild-de Sitter black hole spacetimes.

We outline the basic scheme now. We shall set \( c = k_B = G = \hbar = 1 \) throughout. Let us consider the metric for the Schwarzschild-de Sitter spacetime written in spherical polar coordinates,

\[
ds^2 = - \left( 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} \right) dt^2 + \left( 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \tag{1}
\]

For \( 3M\sqrt{\Lambda} < 1 \), this spacetime admits three Killing horizons,

\[
r_H = \frac{2}{\sqrt{\Lambda}} \cos \left[ \frac{1}{3} \cos^{-1} \left( 3M\sqrt{\Lambda} \right) + \frac{\pi}{3} \right], \quad r_C = \frac{2}{\sqrt{\Lambda}} \cos \left[ \frac{1}{3} \cos^{-1} \left( 3M\sqrt{\Lambda} \right) - \frac{\pi}{3} \right], \quad r_U = -(r_H + r_C).
\]

\( r_H \) is the black hole event horizon and \( r_C > r_H \) is the cosmological horizon, whereas \( r_U \), being negative, is unphysical.

As we mentioned earlier, the gravitational mass of the perturbation over the de Sitter background has been defined and computed earlier in \(^{23}\) in a region where \( \left( 1 - \frac{\Lambda r^2}{3} \right) \gg 2M \) and it turned out to be \( M \). Instead, we divide the region between the black hole and the cosmological event horizon \( (r_H < r < r_C) \) into three regions of perturbation

\[
1 \gg \frac{2M}{r} + \frac{\Lambda r^2}{3} \quad \text{(Region I)}, \quad \left( 1 - \frac{\Lambda r^2}{3} \right) \gg \frac{2M}{r} \quad \text{(Region II)}, \\
\left( 1 - \frac{2M}{r} \right) \gg \frac{\Lambda r^2}{3} \quad \text{(Region III)},
\]

where in the first inequality each term on the right hand side is much smaller than unity, the term on the right hand side of the second inequality is much smaller than each of the terms on the left hand side and similarly for the third inequality. In this sense the three regions are distinct. These three regions can respectively be interpreted as perturbations over background Schwarzschild, Minkowski and de Sitter spacetimes. For the observed value of \( \Lambda \sim 10^{-52} \text{m}^{-2} \) \(^{14, 15}\), and \( 2M \) ranging between the extremes \( 10^4 \text{m} \) to \( 10^{14} \text{m} \), the above regions exist and are merged smoothly in between. In order to see this explicitly in an example, first note that the cosmological horizon is at \( r \sim 10^{26} \text{m} \). Let us now consider a black hole with \( 2M \sim 10^4 \text{m} \). Let us also agree to use the symbol \( \gg \) to mean that the quantity on the left hand side is more than \( 10^{10} \) times that on the right hand side. Then Region III extends from the black hole event horizon till \( r \sim 10^{30} \text{m} \), the inequality breaking down around Planck distance from the horizon. Region II extends from \( r \sim 10^{14} \text{m} \) till about \( 10^{-12} \text{m} \) of the cosmological horizon. Region I is then from \( r \sim 10^{12} \text{m} \) to \( r \sim 10^{21} \text{m} \). Clearly there is considerable overlap between the three regions. A similar calculation for \( 2M \sim 10^{14} \text{m} \)
leads to similar conclusions. It is clear that Region I in Eq. (3) is located in between the two other regions. Regions II and III are respectively repulsion and attraction dominated, whereas Region I is effectively flat in the sense that repulsion and attraction nearly balance each other there. In other words, starting from the side of the black hole horizon, the consecutive sequence of the above three regions with increasing radial distance are : III, I, II.

Of course, the above constructions and the AD formalism are not valid for de Sitter black hole spacetimes for which the black hole and the cosmological horizons are of comparable sizes. But as we have seen, they apply well to the astrophysical black holes, and for the observed value of the cosmological constant. Therefore, in order to do physics in the observed or physical universe, the perturbation scheme described above should be sufficient.

In particular, we shall compute the gravitational or AD mass function for each of these regions following [23] in the next Section. We shall further see in Section 3 that how the continuity of the masses in different perturbation regions leads to the proposal of a new ‘total’ mass function. We shall further present a derivation of the Smarr formula by varying this mass function and relate this to the spectra of particles created in this spacetime for a massless quantum scalar field in Section 4.

2 The mass functions in different perturbation regions

We start by considering the Λ-vacuum Einstein equations

\[ R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0, \]

where \( R_{ab} \) and \( R \) are respectively the Ricci tensor and scalar computed from the metric. Let us assume we can find a region in between the black hole event horizon and the cosmological horizon where the metric \( g_{ab} \) can be decomposed in a background \( g_{ab}^{(0)} \) and a perturbation \( \gamma_{ab} \) over it

\[ g_{ab} = g_{ab}^{(0)} + \gamma_{ab}, \]

where \( |\gamma_{ab}| \ll |g_{ab}^{(0)}| \). The basic scheme described in [23] can be outlined as follows: define a local ‘gravitational energy-momentum tensor’ \( T^{(G)}_{ab} \) which consists of quadratic and higher order terms of the \( \gamma \)'s, whereas the Einstein tensor consists of \( g_{ab}^{(0)} \) and terms linear in \( \gamma_{ab} \). Let \( \nabla_{a}^{(0)} \) denotes the metric compatible connection on the background \( g_{ab}^{(0)} \). Then the ‘energy current’ \( T^{(G)}_{ab} \xi^{(0)}_a \) is conserved with respect to the background, i.e.

\[ \nabla_{b}^{(0)} (T^{(G)}_{ab} \xi^{(0)}_a) \approx 0, \]

where \( \xi^{(0)}_a \) is the timelike Killing field corresponding to the background \( g_{ab}^{(0)} \).

\[ \nabla_{a}^{(0)} s_{b}^{(0)} + \nabla_{b}^{(0)} s_{a}^{(0)} \approx 0. \]

Then one computes the flux of the energy current over a closed spacelike hypersurface \( \Sigma \), orthogonal to \( \xi^{(0)}_a \). We shall apply this scheme to compute the gravitational mass of the perturbations in different regions of Eq. (3).

2.1 Region I

Let us start by considering Region I of Eq. (3), i.e. linear or leading perturbation over the Minkowski spacetime

\[ g^{(0)}_{tt} = -1, \quad g^{(0)}_{rr} = 1, \quad g^{(0)}_{\theta\theta} = r^2, \quad g^{(0)}_{\phi\phi} = r^2 \sin^2 \theta, \]

\[ \gamma_{tt} = \left( \frac{2M}{r} + \frac{\Lambda r^2}{3} \right), \quad \gamma_{rr} = \left( \frac{2M}{r} + \frac{\Lambda r^2}{3} \right), \quad \gamma_{\theta\theta} = 0 = \gamma_{\phi\phi}, \]

where
and the components of the background Killing vector field,
\[\xi^{(0)\mu} = \{1, 0, 0, 0\}, \quad \xi^{(0)}_\mu = \{-1, 0, 0, 0\}.\]  

The Ricci tensor and scalar reads
\[R_{ac} = R^{(0)}_{ac} + \frac{1}{2} \left[ \nabla^{(0)e} \left( \nabla^{(0)}_a \gamma_{ce} + \nabla^{(0)}_c \gamma_{ae} - \nabla^{(0)}_e \gamma_{ac} \right) - \nabla^{(0)}_a \left( \nabla^{(0)} f_{\gamma cf} + \nabla^{(0)}_c \gamma - \nabla^{(0)} d \gamma_{cd} \right) \right] + O(\gamma^2) + \ldots,\]
\[R = R^{(0)} + \left[ \nabla^{(0)e} \nabla^{(0)} \gamma_{ce} - \nabla^{(0)e} \nabla^{(0)} \gamma \right] + O(\gamma^2) + \ldots,\]  

where the trace is defined with respect to \(g^{(0)}_{ab}\), and \(\gamma = \gamma_{ab} g^{(0)ab}\). Also, for the Minkowski background which is a \(\Lambda = 0\) vacuum, the Einstein equations become identities
\[R^{(0)}_{ab} - \frac{1}{2} R^{(0)} g^{(0)}_{ab} = 0.\]  

Now we use Eqs. (4) and (10) to expand Einstein’s equations (11). We shift the \(O(\gamma^2)\) and other higher order terms to the right hand side of Eq. (4) which define the gravitational energy-momentum tensor \(8\pi T^{(G)}_{ab} - \Lambda g_{ab}\) to get
\[\frac{1}{2} \left[ \nabla^{(0)e} \nabla^{(0)} t_{bd} + \nabla^{(0)d} \nabla^{(0)} \gamma_{ab} - \nabla^{(0)e} \nabla^{(0)} \gamma \right] = \left( 8\pi T^{(G)}_{ab} - \Lambda g_{ab} \right) \xi^{(0)b},\]  

where \(\gamma_{ab} = \gamma_{ab} - \frac{1}{2} \gamma g^{(0)}_{ab}\).

The gravitational mass \(M_G\) is defined as the integral of the ‘energy current’ \(8\pi T^{(G)}_{tb} - \Lambda g_{tb}\) \(\xi^{(0)b}\) over a spacelike hypersurface \(\Sigma\) orthogonal to the timelike Killing vector field,
\[M_G := \frac{1}{8\pi} \int_{\Sigma} \left( 8\pi T^{(G)}_{tb} - \Lambda g_{tb} \right) \xi^{(0)b} d\Sigma^t,\]  

and ‘\(t\)’ corresponds to the direction of the timelike Killing field. Following [23], we obtain the following expression for the energy current from Eq. (11) after a little algebra,
\[\left[ T^{(G)}_{tb} - \frac{\Lambda}{8\pi} g_{tb} \right] \xi^{(0)b} = \frac{1}{16\pi} \left[ \nabla^{(0)d} \left( \left( \nabla^{(0)c} H_{bced} \right) \xi^{(0)b} - \nabla^{(0)c} \left( H_{bced} \nabla^{(0)d} \xi^{(0)b} \right) \right) \right],\]  

where
\[H_{abcd} = \left( g^{(0)}_{ca} \gamma_{bd} - g^{(0)}_{cb} \gamma_{ad} - g^{(0)}_{ab} \gamma_{cd} + g^{(0)}_{bd} \gamma_{ca} \right).\]  

\(H_{abcd}\) is antisymmetric under the interchange of \((a, d)\) and \((b, c)\). Then since in Eq. (13) the indices are fixed \(a = b\), i.e. timelike, the indices \((d, c)\) must be spacelike. Therefore we can convert the integral in Eq. (12) into a surface integral
\[M_G = \frac{1}{16\pi} \left[ \int \left( \nabla^{(0)c} H_{bced} \right) \xi^{(0)b} dS^d - \int H_{bced} \nabla^{(0)d} \xi^{(0)b} dS^c \right],\]  

where ‘\(dS\)’ denotes the volume element of a closed 2-surface, i.e. the boundary of \(\Sigma\) of the region of interest \(\Sigma\). Since Schwarzschild-de Sitter spacetime is spherically symmetric, the closed surface is a 2-sphere. Also, Eq. (7) gives
\[\gamma = \gamma_{ab} g^{(0)ab} = 0.\]  

We now explicitly evaluate Eq. (15) using Eqs. (7), (8) and (16). The covariant derivative on the background Killing vector field is \(\nabla^{(0)} \xi^{(0)} = \partial_d \xi^{(0)} - \Gamma^{(0)c}_{db} \xi^{(0)}\) where we keep in mind that \(d\) and \(b\)
are summed over as in the equation. Since \( H_{abcd} \) is antisymmetric under the interchange of \( a \) and \( d \), and \( a \) is timelike, \( d \) must be spacelike. Keeping in mind \( b = t \), it is clear that \( \nabla_d \xi^{(0)}_b \) is non-vanishing only when \( d = r \). We also have \( c = r \) in the second integral of Eq. (15). But Eqs. (14), (1) and (10) give \( H_{tt} = 0 \). Thus the second integrand in Eq. (15) is identically vanishing. Now expanding the covariant derivative in the first integral, keeping in mind \( d \) we have to set \( b = t \) and \( d = r \) after making the expansion, and using Eqs. (14), (7), we find that the only non-vanishing term is \(-g^{(0)}_{tt} \Gamma_{tt}^{(0)} f H_{tt} \), where the sum runs over \( \theta \) and \( \phi \) only, and we finally get

\[ M_G = M + \frac{\Lambda r^3}{6}. \quad (17) \]

### 2.2 Region II

Next we consider perturbation over the de Sitter background, i.e. Region II : \( (1 - \frac{\Lambda r^2}{3}) \gg \frac{2M}{r} \) in Eq. (3), which was explicitly done in [23]. The leading perturbation and the background Killing field are the following

\[ g^{(0)}_{tt} = - \left( 1 - \frac{\Lambda r^2}{3} \right) , \quad g^{(0)}_{rr} = \left( 1 - \frac{\Lambda r^2}{3} \right)^{-1}, \quad g^{(0)}_{\theta \theta} = r^2, \quad g^{(0)}_{\phi \phi} = r^2 \sin^2 \theta, \]

\[ \gamma^{(0)}_{tt} = \frac{2M}{r}, \quad \gamma^{(0)}_{rr} = \frac{2M}{r \left( 1 - \frac{\Lambda r^2}{3} \right)^2}, \quad \gamma^{(0)}_{\theta \theta} = 0 = \gamma^{(0)}_{\phi \phi}, \quad (18) \]

and

\[ \xi^{(0)\mu} = \begin{cases} 1, & 0, \ 0, \ 0 \end{cases}, \quad \xi^{(0)\mu} = \begin{cases} - \left( 1 - \frac{\Lambda r^2}{3} \right), & 0, \ 0, \ 0 \end{cases}. \quad (19) \]

The calculation of the mass of the perturbation is essentially the same as before. The only difference we have to remember is that unlike the previous case, the de Sitter background now we are considering now is a \( \Lambda \)-vacuum

\[ R^{(0)}_{ab} - \frac{1}{2} R^{(0)} g^{(0)}_{ab} + \Lambda g^{(0)}_{ab} = 0. \quad (20) \]

The calculation of the mass of the perturbation, using Eqs. (4), (9) and (20) and following the method described above gives

\[ M_G = M. \quad (21) \]

Thus the mass of the perturbation with respect to the background de Sitter spacetime is given by the mass parameter of the Schwarzschild-de Sitter spacetime.

### 2.3 Region III

Finally we consider Region III in Eq. (3), i.e. perturbation of the \( \Lambda = 0 \) Schwarzschild background by a \( \Lambda \) term. The background and the perturbation are

\[ g^{(0)}_{tt} = - \left( 1 - \frac{2M}{r} \right), \quad g^{(0)}_{rr} = \left( 1 - \frac{2M}{r} \right)^{-1}, \quad g^{(0)}_{\theta \theta} = r^2, \quad g^{(0)}_{\phi \phi} = r^2 \sin^2 \theta, \]

\[ \gamma^{(0)}_{tt} = \frac{\Lambda r^2}{3}, \quad \gamma^{(0)}_{rr} = \frac{\Lambda r^2}{3 (1 - \frac{2M}{r})^2}, \quad \gamma^{(0)}_{\theta \theta} = 0 = \gamma^{(0)}_{\phi \phi}, \quad (22) \]

and

\[ \xi^{(0)\mu} = \begin{cases} 1, & 0, \ 0, \ 0 \end{cases}, \quad \xi^{(0)\mu} = \begin{cases} - \left( 1 - \frac{2M}{r} \right), & 0, \ 0, \ 0 \end{cases}. \quad (23) \]
We follow the same procedure described after Eq. (16). The second integral in Eq. (15) can be shown to be vanishing as earlier and evaluating the first integral we get

\[ M_G = \frac{\Lambda r^3}{6}, \]  

(24)

which is the gravitational mass of the perturbation over the background Schwarzschild spacetime.

### 3 Proposal of a new mass function

The three AD mass functions \( M_G \) appearing in Eqs. (17), (21), (24) are positive, but not continuous, since they have been defined with respect to three different backgrounds. But we have discussed that these three regions (Eq. (3)) are merged smoothly with each other. This implies that a satisfactory notion of mass function through these three regions must be continuous as well. In other words, since the three different background Killing fields are smoothly merged in the succession of Regions III, I and II, any satisfactory mass function should also share this crucial notion of continuity. This leads us to propose a new mass function for the Schwarzschild-de Sitter spacetime, in the following reasonable way, by taking into account the mass of the background as well. The phrase ‘background mass’ will be related to the Einstein tensor of the various gravitational backgrounds described in Eq. (3), as we shall see at the end of this section.

First we note that for the Minkowski background, the background curvature is identically vanishing. Therefore we take the background mass to be zero. For the Schwarzschild background we define the background mass to be the Komar mass,

\[ M_B = -\frac{1}{8\pi} \int \nabla_a \xi_t \, dS^a, \]  

(25)

where the integration is performed over a 2-sphere. For Schwarzschild background, i.e. in Region III, \( M_B = M \) anywhere inside that particular perturbation region.

For the de Sitter background we treat the \(-\Lambda g^{(0)}_{ab}\) term appearing in the Einstein equations as the energy-momentum tensor \( (8\pi T^\Lambda_{ab}) \) corresponding to the cosmological constant. Thus the corresponding energy current becomes

\[ T^\Lambda_{ab} \xi^{(0)b} = -\frac{\Lambda}{8\pi} g_{ab} \xi^{(0)b}. \]  

(26)

But we have from the unperturbed \( \Lambda \)-vacuum equation (20),

\[ T^\Lambda_{ab} \xi^{(0)b} = -\frac{1}{8\pi} R^{(0)}_{ab} \xi^{(0)b} = \frac{1}{8\pi} \nabla^{(0)d} \nabla^{(0)}_{d} \xi_{a}^{(0)}, \]  

(27)

using the Killing identity. Thus the gravitational mass \( M_B \) corresponding to the background de Sitter vacuum is

\[ M_B = \int T^\Lambda_{ab} \xi^{(0)b} d\Sigma^t = \frac{1}{8\pi} \int \nabla^{(0)d} \nabla^{(0)}_{d} \xi_{t}^{(0)} d\Sigma^t. \]  

(28)

Since \( \xi^{(0)a} \) is a timelike coordinate Killing field, the index \( d \) is spacelike above, as can be seen by antisymmetrizing the covariant derivative on \( \xi_{t}^{(0)} \). So we can convert the integral in Eq. (28) into a surface integral over a 2-sphere to get

\[ M_B = \frac{1}{8\pi} \int \nabla^{(0)}_{d} \xi_{t}^{(0)} dS^d = \frac{\Lambda r^3}{6}. \]  

(29)
Let us now combine the background ‘gravitational mass’ $M_B$ with the ‘mass’ of the gravitational perturbation $M_G$ in each of the three regions, i.e. Eq.s (17), (21), (24), with Eq.s (25), (29). The result is what we may call the total mass function $U(r, M)$,

$$U(r, M) := M_B + M_G = M + \frac{\Lambda r^3}{6},$$

(30)

a formula valid throughout all three perturbation regions. The total mass function is positive definite, continuous and monotonically increases with $r$.

We have seen earlier that the AD mass functions are related to the gravitational energy density of the perturbation defined over a given background. We shall now see that the gravitational masses $M_B$ appearing in Eq.s (25), (29) can in fact be related to the ‘time-time’ component of the Einstein tensor of the associated backgrounds. To do this, we write $M_B$ as

$$M_B = \frac{1}{8\pi} \int G_{\xi(0)}^{(t \xi(0))} d\Sigma^t = \frac{1}{8\pi} \int \left[ R_{\xi(0)}^{(t \xi(0))} - \frac{1}{2} R^{(0) \xi(0)} g_{\xi(0)} \right] d\Sigma^t,$$

(31)

and use the Killing identity to replace the first term on the right hand side by $-\nabla_{\xi(0)} \nabla_{\xi(0)} \xi_t$. For Regions I and III in Eq. (34), we have $R^{(0)} = 0$, whereas for Region II we have $R^{(0)} = 4\Lambda$. Then for Region II we may replace the second term with $2\Lambda \xi(0) \xi_t$ by using Killing’s identity, and eventually arrive at Eq. (29), whereas for Region III we get Eq. (25). It is clear that calling the left hand side of Eq. (34) the background mass is naturally meaningful due to the appearance of the background Einstein tensor as the integrand on the right hand side.

As before, putting these all together, the total mass or energy function $U(r, M)$ appearing in Eq. (30) for the Schwarzschild-de Sitter spacetime anywhere in Regions I, II, III, can be written in a compact and unified form,

$$U(r, M) := M_B + M_G = \frac{1}{8\pi} \int \left[ R_{\xi(0)}^{(t \xi(0))} - \frac{1}{2} R^{(0) \xi(0)} g_{\xi(0)} \right] \xi(0) d\Sigma^t + M_G.$$

(32)

We note that $U(r, M)$ is positive definite and monotonically increasing with $r$, thereby encompassing the satisfaction of weak energy condition by positive $\Lambda$.

We have thus seen that mathematically it is justified to refer to the function $U(r, M)$ of Eq. (30) as a local and total mass or energy function because of the way it is related to the ‘total’ Einstein tensor (Eq. (32)), corresponding to the background (as discussed above) and as well as to the perturbation (as discussed in the AD formalism in the previous sections). It is interesting to note also that the $\Lambda$ part of the total mass function $U(r, M)$ is formally similar to the Tolman-Oppenheimer-Volkoff mass function (see [1] and references therein) for a spherically symmetric general relativistic star.

Let us now also consider a simple physical example where $U(r, M)$ can naturally be interpreted as a position dependent mass function associated with the spacetime. Specifically, we consider gravitational redshift [1, 2] in Region I of Eq. (3). For two points $r_1$ and $r_2$ in Region I, we find at leading order that

$$\delta \omega \approx \omega \left( \frac{U(r_1, M)}{r_1} - \frac{U(r_2, M)}{r_2} \right),$$

(33)

where $\omega$ is the frequency of the photon emitted at $r = r_1$ and $\delta \omega$ is its frequency shift when detected at $r_2$. We compare this with the result of asymptotically flat (i.e. $\Lambda = 0$) spacetime, in which we get the same formula with $U(r_1, M) = M = U(r_2, M)$, and thus it is manifest that in the above equation $U(r, M)$ acts as a position dependent mass function in the Schwarzschild-de Sitter spacetime. We also note that since \( \left( \frac{4M}{r} + \frac{\Lambda r^2}{3} \right) \leq 1 \) anywhere in the region between the two
horizons, and \((1 - \frac{2U}{r} - \frac{\Lambda r^2}{3}) = 1 - \frac{2U(r,M)}{r}\), analogue of Eq. (33) can be written everywhere in the three perturbation regions,

\[
\delta \omega = \omega \left(1 - \frac{2U(r_1,M)}{r_1}\right)^{-\frac{1}{2}} \left[\left(1 - \frac{2U(r_2,M)}{r_2}\right)^{\frac{1}{2}} - \left(1 - \frac{2U(r_1,M)}{r_1}\right)^{\frac{1}{2}}\right].
\]  

(34)

Thus we can see a physical or observational ground of interpreting \(U(r,M)\) to be a position dependent mass or energy function in the Schwarzschild-de Sitter spacetime. We instead might have considered the Komar integral to construct mass function in region I. But since the Komar mass is related to the derivative of the norm of the timelike Killing vector field (Eq. (25)), it can vanish in region I, as cosmic repulsion and gravitational attraction nearly balance each other there. Consequently, unlike asymptotically flat spacetimes the Komar mass cannot explain or interpret Eq. (33). Eq. (33) also gives an example of the physical scenario where a mass function is required to be constructed in a region where attraction and repulsion are nearly balancing each other, thereby providing a further physical justification to the motivation behind our foregoing calculations.

We shall see below that our local mass function \(U(r,M)\) also reproduces the thermodynamics for the Schwarzschild-de Sitter black hole spacetime, but along with a negative pressure term arising due to positive \(\Lambda\).

### 4 Smarr formula and particle creation

Let us compute the variation of this total mass function \(U(r,M)\) (Eq. (30)), subject to the change of the black hole mass parameter \(M\) keeping \(\Lambda\) fixed, and will see that \(U(r,M)\) is compatible with the existing idea of two-temperature de Sitter black hole thermodynamics. For \(\Lambda = 0\) stationary black holes, the area theorem and the constancy of the surface gravity over the event horizon (see [1, 7] and references therein) give rise to the idea of black hole thermodynamics [26, 27, 28, 29, 30] (see also [31] for a review).

The area of the black hole horizon \((r_H)\) is given by

\[
A_H = 4\pi r_H^2,
\]

which, using Eq. (2) we rewrite as

\[
M(A_H) = -\frac{4\Lambda}{3} \left(\frac{A_H}{16\pi}\right)^{\frac{1}{2}} + \left(\frac{A_H}{16\pi}\right)^{\frac{3}{2}}.
\]

(36)

Now we write the mass function \(U(r,M)\) in terms of two new variables: the black hole horizon area \(A_H\) and the volume \(V = \frac{4}{3}\pi r^3\) enclosed by a sphere of radius \(r\) on which we have defined the mass function,

\[
U(A_H, V) = -\frac{4\Lambda}{3} \left(\frac{A_H}{16\pi}\right)^{\frac{1}{2}} + \left(\frac{A_H}{16\pi}\right)^{\frac{3}{2}} + \frac{\Lambda V}{8\pi}.
\]

(37)

the variation of which gives

\[
\delta U(A_H, V) = \left[\frac{-2\Lambda}{(16\pi)^{\frac{3}{2}}} (A_H)^{\frac{1}{2}} + \frac{1}{2(16\pi)^{\frac{3}{2}}} (A_H)^{\frac{3}{2}}\right] \delta A_H + \frac{\Lambda}{8\pi} \delta V.
\]

(38)

The surface gravity \(\kappa_H\) of the black hole horizon is given by

\[
\kappa_H = \left(\frac{M}{r_H^2} - \frac{\Lambda r_H}{3}\right),
\]

(39)
combining which with Eq.s (35), (38) gives

$$\delta U(A_H, V) = \frac{\kappa_H}{8\pi} \delta A_H + \frac{\Lambda}{8\pi} \delta V. \quad (40)$$

A similar calculation with the cosmological horizon yields

$$\delta U(A_C, V) = -\frac{\kappa_C}{8\pi} \delta A_C + \frac{\Lambda}{8\pi} \delta V, \quad (41)$$

where $A_C$ and $\kappa_C = -\left(\frac{M}{r_C} - \frac{\Lambda}{8\pi}\right) > 0$ are the cosmological horizon’s area and the magnitude of surface gravity respectively. The term $\frac{\Lambda}{8\pi}$ appearing in Eq.s (40) and (41) can be interpreted as the negative isotropic pressure due to positive $\Lambda$, and therefore once again justifies the interpretation of $U(r, M)$ as a physical mass or energy function for the spacetime. Eq.s (40), (41) thus connect the variation of our local mass function $U(r, M)$ with the variation of the horizon parameters.

In particular, if we now combine Eq.s (40) and (41) for the same volume $V$, we now get a Smarr formula involving horizon parameters only

$$\kappa_H \delta A_H + \kappa_C \delta A_C = 0 \quad (42)$$

This formula was derived earlier in [22, 32], and we have rederived it using our local mass function. The Smarr formula shows that when the area of the black hole horizon increases, the area of the cosmological horizon decreases and vice versa, which is also expected from Eq.s (2). It is clear from Eq.s (40), (41) that, unlike asymptotically flat or anti-de Sitter spacetimes, one cannot interpret $\delta A_H$ alone as the entropy of the spacetime and instead one defines a ‘total’ entropy $S = \frac{1}{4}(A_H + A_C)$ [32]. We note that [33] also consider this total entropy and the variation of $\Lambda$ as well.

For the non-Nariai class spacetimes that we are considering, $r_H \neq r_C$, and thus $\kappa_H \neq \kappa_C$, (in particular, $\kappa_H > \kappa_C$, since $r_H < r_C$) and then Eq.s (10) and (11) show that unlike asymptotically flat or anti-de Sitter spacetimes, there is no unique thermodynamic interpretation in the Schwarzschild-de Sitter spacetime. However, if we could separate the two horizons by a thermally opaque membrane, as considered in e.g. [22, 33], more precisely a barrier through which no radiation can pass in either direction, we could expect two different thermal equilibrium states, of temperatures $\frac{\kappa_H}{2\pi}$, $\frac{\kappa_C}{2\pi}$, corresponding to the black hole and the cosmological horizons respectively. We note that for radiation associated with the black hole alone, both $\delta M, \delta A_H < 0$, whereas for the cosmological horizon alone, $\delta M > 0$ and $\delta A_C < 0$, in Eq.s (10), (11), respectively, which are also indicated by Eq.s (2). On the other hand, the variation of the ‘total’ entropy combined with the variation of the total mass function $U$ gives

$$\frac{\kappa_H \kappa_C}{2\pi (\kappa_H - \kappa_C)} \delta \left(\frac{A_H + A_C}{4}\right) = -\delta U + \frac{\Lambda}{8\pi} \delta V. \quad (43)$$

The above variation is also consistent with the requirement that $(A_H + A_C)$ increases (decreases) as $M$ decreases (increases) (Eq.s (2)). Eq. (43) indicates that with respect to the total entropy one might expect an effective ‘equilibrium temperature’ $T_{\text{eff}} = \frac{\delta S}{2\pi (\kappa_H - \kappa_C)}$. This equilibrium temperature agrees with that derived earlier in [33] with different mass function (and sign convention), provided we set $\delta \Lambda = 0$ throughout in their derivation.

The interpretation of the $\frac{\Lambda}{8\pi}$ term as the pressure-energy can also be found in [36, 37], in the context of anti-de Sitter black holes. The ADM mass parameter was interpreted as enthalpy for the AdS black hole by considering variations of the cosmological constant, which acts as pressure. A similar construction was recently done for de Sitter black holes [38], interprets the term $M + \frac{\Lambda}{8\pi}$ as a total energy. We note that the constructions in these works were based on global Komar integrals defined on the boundaries. Our construction on the other hand, being based on the AD formalism, produces a local function, which the Komar integral does not.
We shall now derive the different thermal equilibrium states using Kruskal patches and canonical quantization for an eternal Schwarzschild-de Sitter spacetime corresponding to each of the horizons, and thereby verify the thermodynamic nature of Eq.s (40), (41). The variation of $U(r, M)$ includes two things – the variation at the boundary, and the local variation (coming from the $\Lambda$ part). Clearly, the variation of energy due to change in boundary is related to the particle creation effects.

It was shown in [39] using canonical quantization and Bogoliubov transformations that a stellar object undergoing gravitational collapse in an asymptotically flat spacetime to form a black hole creates Planckian distribution of particles at late times. For massless quantum fields this distribution can be measured at the future null infinity, and found to have a temperature $\frac{\hbar}{2\kappa_H}$, where $\kappa_H$ is the surface gravity of the black hole future horizon. Later this result was rederived using path integral quantization [40]. This most remarkable result, known as Hawking radiation, was further justified by the renormalization of the quantum energy-momentum tensors (see [41, 42] and references therein). We refer our reader to [43, 44] for excellent pedagogical reviews on this subject.

For an eternal Schwarzschild-de Sitter spacetime, the black hole and the cosmological horizon create thermal particles with temperatures $\frac{\hbar}{2\kappa_U}$ and $\frac{\hbar}{2\kappa_C}$ respectively along an incoming null geodesic, whereas by writing it in terms of $u, v$ coordinates ($r, t, \theta, \phi$) ($\theta, \phi$), the equation of motion for a single mode becomes

$$-\frac{\partial^2 f_{lm}(r, t)}{\partial t^2} + \frac{\partial^2 f_{lm}(r, t)}{\partial r^2} - \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}ight) \left(\frac{l(l + 1) + \frac{M}{r^2} - \frac{\Lambda}{3}\right) f_{lm}(r, t) = 0,$$  

(44)

where we have abbreviated $f(r, t) = e^{-i\omega t} f(r)$ and $r_*$ is the tortoise coordinate defined by,

$$r_* = \int \frac{dr}{(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3})^{\frac{1}{2}}} \ln \left| \frac{r}{r_H} - 1 \right| - \frac{1}{2\kappa_H} \ln \left| \frac{r}{r_C} - 1 \right| + \frac{1}{2\kappa_U} \ln \left| \frac{r}{r_U} + 1 \right|,$$  

(45)

where $\kappa_U = \partial_r \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)$. Thus $r_* \to \mp \infty$ as $r \to r_H, r_C$ respectively.

For our purpose, we shall first construct suitable coordinate systems for the Schwarzschild-de Sitter spacetime. There are two coordinate singularities located at $r_H$ and $r_C$, therefore we require two Kruskal-like patches to remove them. We define the usual outgoing and incoming null coordinates $(u, v)$ as

$$u = t - r_*, \quad v = t + r_*.$$  

(46)

By writing the metric [11] in terms of $u$ and $r$, it is easy to find that $u \to \pm \infty$ as $r \to r_H, r_C$ respectively along an incoming null geodesic, whereas by writing it in terms of $v$ and $r$ gives $v \to \mp \infty$ as $r \to r_H, r_C$ respectively along an outgoing null geodesic. In terms of the null coordinates $(u, v)$ the metric becomes

$$ds^2 = \frac{2M}{r} \left(\frac{r}{r_H} - 1\right) \left(\frac{r}{r_C} - 1\right) \left(\frac{r}{r_U} + 1\right) du dv + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$  

(47)

where $r$ as a function of $(u, v)$ is understood and can be found from Eq. (40). We now define the Kruskal null coordinates for the black hole event horizon as

$$\overline{u} = \frac{1}{\kappa_H} e^{-\kappa_H u}, \quad \overline{v} = \frac{1}{\kappa_H} e^{\kappa_H v},$$  

(48)

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so that $\overline{\pi} \to 0$, $-\infty$ as $r \to r_H, r_C$ respectively, and $\overline{\pi} \to 0$, $+\infty$ as $r \to r_H, r_C$ respectively. The Kruskal null coordinates for the cosmological event horizon can be defined as

$$\overline{\pi} = \frac{1}{\kappa_C} e^{\kappa_C u}, \quad \overline{\pi}' = -\frac{1}{\kappa_C} e^{-\kappa_C v},$$

so that $\overline{\pi}' \to +\infty$, $0$ as $r \to r_H, r_C$ respectively, and $\overline{\pi}' \to -\infty$, $0$ as $r \to r_H, r_C$ respectively. To summarize, the ranges of the various null coordinates are

$$-\infty < u < \infty, \quad -\infty < v < \infty, \quad -\infty < \overline{\pi} \leq 0, \quad 0 \leq \overline{\pi}' < \infty, \quad -\infty < \overline{\pi}' \leq 0.$$

Clearly, there will be both outgoing and incoming mode solutions for the field equation. Since $(\overline{\pi}, \overline{\pi}')$ and $(\pi, \pi')$ are respectively functions of $(u, v)$ only, we have modes in terms of these null coordinates,

$$\psi_{\text{out}} = a_i u_i + a_i^\dagger u_i^\dagger = \overline{\pi}_i \overline{\pi}_i + \overline{\pi}_i^\dagger \overline{\pi}_i^\dagger = \overline{\pi}_i^\dagger \overline{\pi}_i + \overline{\pi}_i^\dagger \overline{\pi}_i^\dagger,$$

$$\psi_{\text{in}} = b_i v_i + b_i^\dagger v_i^\dagger = \overline{\pi}_i \overline{\pi}_i + \overline{\pi}_i^\dagger \overline{\pi}_i^\dagger = \overline{\pi}_i^\dagger \overline{\pi}_i + \overline{\pi}_i^\dagger \overline{\pi}_i^\dagger,$$

$$\psi = \psi_{\text{in}} + \psi_{\text{out}}.$$  \hspace{1cm} (51)

where $(u_i, v_i)$, $(\overline{\pi}_i, \overline{\pi}_i)$ and $(\overline{\pi}_i', \overline{\pi}_i')$ are modes corresponding to the coordinates $(u, v)$, $(\overline{\pi}, \overline{\pi}')$ and $(\overline{\pi}', \overline{\pi}')$ respectively. The index ‘$i$’ corresponds to all discrete and continuous indices. The complex quantities $a_i$, etc. are expansion coefficients and in flat spacetime, they are interpreted as creation and annihilation operators associated with respective modes.

The creation and annihilation operators are defined to satisfy the commutation relations

$$\begin{bmatrix} a_i, a_j^\dagger \end{bmatrix} = \delta_{ij}, \quad \begin{bmatrix} a_i, a_j \end{bmatrix} = 0 = \begin{bmatrix} a_i^\dagger, a_j^\dagger \end{bmatrix}, \quad \begin{bmatrix} a_i, b_j^\dagger \end{bmatrix} = \delta_{ij}, \quad \begin{bmatrix} a_i, b_j \end{bmatrix} = 0 = \begin{bmatrix} b_i^\dagger, b_j^\dagger \end{bmatrix},$$

$$\begin{bmatrix} \overline{\pi}_i, \overline{\pi}_j^\dagger \end{bmatrix} = \delta_{ij}, \quad \begin{bmatrix} \overline{\pi}_i, \overline{\pi}_j \end{bmatrix} = 0 = \begin{bmatrix} \overline{\pi}_i^\dagger, \overline{\pi}_j^\dagger \end{bmatrix}, \quad \begin{bmatrix} \overline{\pi}_i, \overline{\pi}_j^\dagger \end{bmatrix} = \delta_{ij}, \quad \begin{bmatrix} \overline{\pi}_i, \overline{\pi}_j \end{bmatrix} = 0 = \begin{bmatrix} \overline{\pi}_i^\dagger, \overline{\pi}_j^\dagger \end{bmatrix}. \hspace{1cm} (52)$$

The inner product of the modes $(u_i, v_i)$ are defined as

$$(u_i, u_j) = \frac{i}{2} \int \Delta \Sigma (\nabla_a u_j - u_j (\nabla_a u_i^\dagger)) d\Sigma^a = \delta_{ij} \quad (v_i, v_j) = \frac{i}{2} \int \Delta \Sigma (v_j^\dagger (\nabla_a v_j) - v_j (\nabla_a v_j^\dagger)) d\Sigma^a = \delta_{ij}$$

$$(u_i, u_j^\dagger) = 0 = (v_i, v_j^\dagger), \hspace{1cm} (53)$$

where $\Sigma$ is a suitable hypersurface and the direction ‘$a$’ is along its normal. In an asymptotically flat spacetime, one chooses $\Sigma$ to be the past null infinity. But as we discussed earlier, in presence of a de Sitter horizon, infinities are not very meaningful to an observer located within that horizon. So we have to choose $\Sigma$ differently here.

Let us now define the Bogoliubov transformation coefficients (see e.g. [42] for details) and consider the outgoing $u$ and $\overline{\pi}$ modes first,

$$(\overline{\pi}_i, u_j) = \alpha_{ij}, \quad \begin{bmatrix} \overline{\pi}_i, u_j^\dagger \end{bmatrix} = \beta_{ij}. \hspace{1cm} (54)$$

Let us now consider the equality between the first two mode expansions in the first of Eq.s (71). We use Eq.s (83), (84) and the commutation relations (52) to get

$$\alpha_{ik} \beta_{kj}^\dagger - \beta_{ik} \alpha_{kj}^\dagger = \delta_{ij}, \quad \alpha_{ik} \beta_{kj}^\dagger - \beta_{kj} \alpha_{ik} = 0. \hspace{1cm} (55)$$

Subject to these relations and the commutations, one can then take the inverse transformations

$$a_i = \alpha_{ij} \overline{\pi}_j - \beta_{ij}^\dagger \overline{\pi}_j. \hspace{1cm} (56)$$
If $|0\rangle_K$ denotes the vacuum associated with the respective Kruskal modes, then the $(u, v)$ observer will ‘see’ particles in $|0\rangle_K$ in the $i$-th mode as the following,

$$\langle 0|_K a^\dagger_i a_i |0\rangle_K = \Sigma_j |\beta_{ij}\rangle^2 \quad \text{(no sum on } i) .$$  \hspace{1cm} (57)$$

Thus, all we have to do now is to determine the Bogoliubov coefficient $\beta_{ij}$.

In order to do that we note first that Eq. (44) admits plane wave solutions infinitesimally close to the horizons in $(u, v)$ null coordinates,

$$u(\omega, l, m) \sim \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega u} Y_{lm}(\theta, \phi), \quad v(\omega, l, m) \sim \frac{1}{\sqrt{4\pi \omega}} r Y_{lm}(\theta, \phi),$$  \hspace{1cm} (58)$$

whereas near the black hole horizon the mode with respect to $(\overline{\omega}, \overline{\omega})$ becomes

$$\overline{u}(\omega, l, m) \sim \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega \overline{u}} Y_{lm}(\theta, \phi), \quad \overline{v}(\omega, l, m) \sim \frac{1}{\sqrt{4\pi \omega}} r Y_{lm}(\theta, \phi),$$  \hspace{1cm} (59)$$

and near the cosmological horizon the mode with respect to $(\overline{\omega}, \overline{\omega})$ becomes

$$\overline{u}'(\omega, l, m) \sim \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega \overline{u}} Y_{lm}(\theta, \phi), \quad \overline{v}'(\omega, l, m) \sim \frac{1}{\sqrt{4\pi \omega}} r Y_{lm}(\theta, \phi),$$  \hspace{1cm} (60)$$

along with their negative frequency counterparts.

We shall consider a mode outgoing at the future cosmological horizon and trace in back to the past black hole horizon, where we shall determine the Bogoliubov coefficients by integrating over the entire past black hole horizon between this traced back and the outgoing Kruskal mode in Eq. 44. Consequently, the surface $\Sigma$ in Eq. 53 is a closed null hypersurface on the past black hole horizon. As in the case of asymptotically flat spacetime, during this backtracing, there will be some part of the wave which will be backscattered due to the effective potential barrier in Eq. 44 to the future cosmological horizon and hence will be disconnected from the wave outgoing at the past black hole horizon. This will be the usual greybody effect associated with the black hole horizon.

With all the above equipments, our task is now thus to compute $\beta_{ij}$. We note that on any $r =$ constant hypersurface, $dt = d(t - r(r)) = du = e^{Huis} d\overline{\omega}$, using Eq.s 45. There will be a $\partial_r$, coming from the normal direction of the hypersurface volume element. But $\partial_r e^{-i\omega u} = i\omega e^{-i\omega u} = -\partial_r e^{-i\omega u} = -e^{-\kappa H u} (\partial_{\overline{\omega}} e^{-i\omega u})$, which means for these modes $d\overline{\omega} = du \partial_r$. Putting these in all together it is straightforward to calculate

$$\alpha_{\omega', \omega} = \frac{ik}{4\pi \sqrt{\omega' \omega}} \int_{\Sigma, r=\rho_H} \left[ \left( \omega' - \frac{\omega}{\kappa H} \right) e^{i\omega' \overline{\omega}} e^{-\frac{i\omega}{H} \ln(-\kappa H)} \right] d\overline{\omega},$$  \hspace{1cm} (61)$$

where all the constants including those arising from summation of the discrete indices and angular integral have been dumped into the constant $k$. We are yet to choose the limit of the above integration. The above integration is done on the entire past black hole horizon, therefore we choose the limit of $\overline{\omega}$ in Eq. 61 to be $-\infty$ to $0$ (Eq. 56). With this, the integral in Eq. 61 looks exactly the same as in asymptotically flat spacetime. Analytically continuing this to the complex plane, and treating $\overline{\omega} = 0$ as a branch cut one obtains

$$|\alpha_{\omega', \omega}|^2 = e^{\frac{i\omega}{H}} |\beta_{\omega', \omega}|^2 .$$  \hspace{1cm} (62)$$

Then from Eq. 65 we get

$$\int d\omega' \Gamma(\omega, \omega') \delta(\omega - \omega') = -k' \int d\omega' \left( 1 - e^{\frac{i\omega}{H}} \right) |\beta_{\omega', \omega}|^2 ,$$  \hspace{1cm} (63)$$

13
where \( \Gamma \) stands for possible greybody effect as in the asymptotically flat spacetimes \([39, 43]\) and \( k' \) is some positive constant arising from summation of the discrete indices and angular integrations. We note that for \( \kappa_H \to 0 \), \( \beta \to 0 \) and there is no particle creation. We also note that if instead one computes the Bogoliubov transformation between two different Kruskal modes, one might have non-thermal spectra \([43]\).

Thus the \((u, v)\) observer will ‘see’ the Kruskal vacuum corresponding to the black hole horizon is filled with thermal distribution of outgoing particles

\[
I(\omega) \sim \frac{\Gamma(\omega)}{e^{\frac{\omega}{\kappa_H}} - 1},
\]

with temperature \( T_H = \frac{\kappa_H}{2\pi} \).

A similar analysis can be done for the cosmological event horizon as the following. We consider ingoing \( v \) modes at the future black hole horizon and incoming \( \overline{v'} \) modes at the past cosmological event horizon. We trace the incoming mode back to the past cosmological horizon. There will be some part backscattered to the black hole and thus generating the greybody effect. We compute the Bogoliubov transformation coefficient at the past cosmological event horizon and find thermal spectrum with temperature \( T_C = \frac{\kappa_C}{2\pi} \). If we set \( \kappa_C = 0 \), there will be no particle creation for the cosmological horizon.

For fermionic field the commutations are replaced with anticommutations and Eqs. \((55)\) is modified with a ‘+’ in place of ‘−’. This will give Fermi-Dirac distribution with the same respective temperatures.

5 Discussions

Let us summarize the results now. We have followed \([23]\) to construct a mass function in each of the three different perturbation regions (Eq. \((3)\)) of the Schwarzschild-de Sitter spacetime. The main motivation behind this comes from the lack of asymptotic region in between the black hole and the cosmological horizon. The continuity of the mass functions in different perturbation regions led us to define a new, ‘total’ mass function by adding the AD mass with the mass of the background. We have also shown that such addition is justified since the mass function thus obtained is related to the total Einstein tensor as described earlier. The resulting final mass function is positive definite, continuous and its \( \Lambda \)-part monotonically increases with the radial distance from the black hole. Thus it takes care of the weak energy condition satisfied by a positive \( \Lambda \). We have also related our local mass function with the gravitational redshift effect to give it a natural physical interpretation.

We note that the perturbation scheme described in Eq. \((3)\) was the only crucial ingredient for our calculations. Such scheme is clearly not valid for Schwarzschild-de Sitter spacetimes with comparable sizes of horizons, but as we have argued earlier, to do physics in the universe we live in, such construction is reasonable and should be sufficient.

The most useful feature of this mass function is manifest in particular when we consider Region I in Eq. \((3)\) as the following. Since the norm of the timelike Killing vector field vanishes at the two horizons, it reaches a maximum in between, where a geodesic in nearly undeflected. Clearly, this region corresponds to Region I. Hence if there is a geodesic observer, he/she will ‘feel’ that there is no mass within of the spacetime at all. This of course cannot be acceptable.

We have rederived the two-temperature thermodynamic relations by varying this mass function for the Schwarzschild-de Sitter spacetime. Apart from the surface gravity terms, we have obtained a term due to negative isotropic pressure exerted by a positive \( \Lambda \). This once again justifies the physical validity of our mass function.

Finally we have computed particle creation in this spacetime for both the horizons using canonical quantization. Thus the thermodynamic nature of Eqs. \((40), (41)\) are verified. The \( \frac{\Lambda}{4\pi} \) term should be regarded as change in the (negative) pressure-energy measured by an observer due to the infinitesimal displacement of the observer from his/her initial position, which is purely local, in
contrast to the asymptotically flat spacetimes. Thus the mass function we constructed takes care of the local variation of energy of the ambient de Sitter spacetime as well, which in fact can be comparable with the energy of the created particles.

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