BERNOULLI AND EULER NUMBERS FROM DIVERGENT SERIES

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Many special numeric sequences have been studied intensively due to their appearance and applications in Combinatorics, Number Theory, and Analysis. For instance, Fibonacci, Bernoulli, Euler, Eulerian [1], or Stirling numbers [2]. In addition to common techniques to obtain properties and relations among them, summation of divergent series can be used too!

The Bernoulli and Euler numbers are two sequences of rational numbers that play an important role in mathematical analysis. They appear naturally as the coefficients of the Taylor expansions of trigonometric functions and in the computation of sums of series and asymptotic expansions. For instance, in the calculation of \( \zeta(2k) \), where \( k \) is an integer and \( \zeta \) denotes the Riemann zeta function [3], or in the Euler-Maclaurin summation formula [4] Chapter XIII. They also exhibit interesting relations between other numbers, for example with Euler’s constant [5]. Bernoulli numbers have been called an “unifying force” in mathematics due to their presence in several branches as in Analytic Number Theory [6] and Differential Topology [7].

The Bernoulli numbers \( B_n \) (with signs) were introduced by J. Bernoulli in his *Ars Conjectandi* [8, 9], published posthumously in 1713, to give a precise formula for \( 1^k + 2^k + \cdots + n^k \) as a polynomial in \( n \). They are defined as the coefficients in the Taylor expansion at \( z = 0 \) of

\[
\frac{z}{e^z - 1} = -\frac{z}{2} + \frac{z}{2} e^z - 1 = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k, \quad |z| < 2\pi. \tag{1}
\]

From the definition it is deduced that \( B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6} \) and \( \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, n \geq 2. \) This formula allows to find \( B_n \) recursively and it shows that \( B_n \) is always rational. Also, since \( \frac{e^z + 1}{2} \) is an even function, we deduce that \( B_{2k+1} = 0, \) for \( k \geq 1. \)

The Euler numbers are defined as the Taylor expansion at \( z = 0 \) of

\[
\sec(z) = \frac{1}{\cos(z)} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n = 1 + \sum_{n=1}^{\infty} \frac{E_{2n}}{(2n)!} z^{2n}, \quad |z| < \frac{\pi}{2}. \tag{2}
\]

Then it is clear that \( \sum_{k=0}^{n} \binom{2n}{2k} (-1)^k E_{2k} = 0, \) and thus the \( E_{2k} \) are integers. Replacing \( z \) by \( 2iz \) in equation (1) we find the Taylor expansion at \( z = 0 \) of

\[
z \cot(z) = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} (-1)^k 2^{2k} z^{2k}, \quad |z| < \pi, \tag{3}
\]
and from the identity $\tan(z) = \cot(z) - 2 \cot(2z)$, we see that
\[
\tan(z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} z^{2n-1} = -2i \sum_{k=1}^{\infty} \frac{2^{k+1} - 1}{k+1} \frac{(2iz)^k}{k!}, \quad |z| < \frac{\pi}{2}.
\]
Thus the Maclaurin series of $\tan(z)$ and $\sec(z)$ are not trivial but there are elementary recursive methods to obtain them \[10\]. Finally, we obtain from the last equation the power series expansion
\[
-\frac{1}{1+e^z} = \sum_{k=0}^{\infty} \frac{2^{k+1} - 1}{k+1} B_{k+1} \frac{z^k}{k!}, \quad |z| < \pi.
\] (3)

There are many lists of recurrences satisfied by the Bernoulli and Euler numbers, e.g., Nielsen’s classical book \[11\]. We only need one, namely
\[
\frac{2^{k+1} - 1}{k+1} B_{k+1} = \frac{1}{2} - \sum_{l=1}^{k} \binom{k}{l} \frac{2^{l+1} - 1}{l+1} B_{l+1}, \quad k \geq 1.
\] (4)

It can be obtained from the equality $-\frac{1}{1+e^z} e^z = -1 + \frac{1}{1+e^z}$ and the formula \[3\] by equating the corresponding coefficients of $z^k$.

We will show how to obtain \[4\] also by summing divergent series. The use of this method is not new. Garabedian \[12\], for instance, showed that
\[
B_{n+1} = \frac{(-1)^n(n+1)}{2^{n+1}-1} \sum_{k=1}^{n+1} \frac{1}{2^k} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j (j+1)^n,
\]
from summing
\[
\sigma_k := 1^k - 2^k + 3^k - 4^k + \cdots, \quad k \in \mathbb{N},
\]
using Cesàro and Abel summability. Radkowski \[13\] also provided a proof using calculus of finite differences. Similarly, Namias \[14\] deduced some other recurrences using Stirling’s asymptotic series and the duplication formula for the Gamma function, although the results can be also obtained in an elementary way \[15\]. We will use the sum of $\sigma_k$ and the linearity of a summation method that can sum it to obtain \[4\] and some other simple recurrences for Bernoulli and Euler numbers.

Let us recall that a series $\sigma = \sum_{n=0}^{\infty} a_n$ is said to be *Abel summable* with sum $A(\sigma)$ if for all $x \in \mathbb{R}$ with $0 \leq x < 1$, the associated generating series $\sum_{n=0}^{\infty} a_n x^n$ is convergent and $A(\sigma) := \lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n$ exists. In particular, the series $\sigma_k$ is Abel summable \[10\] \[17\]: if we replace $x = e^{-y}$ in the series
\[
1^k x - 2^k x^2 + 3^k x^3 - \cdots = \sum_{n=0}^{\infty} (-1)^n (n+1)^k e^{-(n+1)y} = (-1)^{k} \frac{d^k}{dy^k} \left( \frac{1}{1 + e^y} \right),
\]
we can use the expansion \[4\] and take $y \to 0^+$ to find the value $A(\sigma_k) = \frac{2^{k+1} - 1}{k+1} B_{k+1}$, $k \geq 1$. It is also clear that $A(\sigma_0) = \frac{1}{2}$. In the same way
\[
A\left( 1^k - 3^k + 5^k - 7^k + \cdots \right) = (-1)^{\lfloor k/2 \rfloor} E_k/2,
\] (5)
where $\lfloor , \rfloor$ denotes the floor function. Indeed, setting $x = e^{-2y}$ in the generating series $\sum_{n=0}^{\infty} (-1)^n (2n+1)^k x^n = (-1)^k e^y \frac{d^k}{dy^k} \left( \frac{1}{e^{-2y} + 1} \right)$, we can let $y \to 0^+$ and then the formula follows from equation \[2\].
Is it possible to attribute a sum to a divergent series in a way compatible with the usual rules of calculus? For Euler the answer was positive! This is evidenced in his work *De seriebus divergentibus* [13] on the Wallis series $\sum_{n=0}^{\infty}(-1)^n n!$, where he found the sum $\int_{0}^{1} \frac{\sin t}{1 + t} dt \approx 0.5963473625$ [10]. In the same spirit, this was the belief of Hardy as he exhibited in his book *Divergent Series* [4]. Nowadays, the theory of summability attempts to answer this question.

We denote by $D$ the $\mathbb{C}$-vector space of complex sequences $(a_n)_{n \geq 0}$ and by $\mathcal{C}$ subspace of sequences such that $\lim_{n \to +\infty} a_0 + \cdots + a_n$ exists. We can think of elements of $D$ as formal numerical series $\sigma = \sum_{n=0}^{\infty} a_n$. The space $\mathcal{C}$ is the domain of the *sum homomorphism* $S: \mathcal{C} \to \mathcal{C}$, which associates a series $\sigma$ to its sum $S(\sigma)$, i.e., the limit of its partial sums. From this point of view, a summability method is a map $S^*: \mathcal{C}^* \to \mathbb{C}$ on some linear subspace $\mathcal{C} \subseteq \mathcal{C}^* \subseteq D$ such that the following rules are satisfied:

1. **Regularity rule:** If $\sigma \in \mathcal{C}$, then $S^*(\sigma) = S(\sigma)$.
2. **Translation rule:** $S^* \left( \sum_{n=0}^{\infty} a_n \right) = a_0 + S^* \left( \sum_{n=1}^{\infty} a_n \right)$.
3. **Linearity rule:** $S^*$ is a $\mathbb{C}$-linear map.

Cesàro and Abel summability are examples of summability methods satisfying such rules. It is worth noting that such axioms were implicitly used by Euler.

We will use the following fact:

**Assume that $S^*$ satisfies the above rules. Then if it sums a series, the value we find through the rules is the value $S^*$ assigns to the series.**

As a first example, we consider $F = 1 + 1 + 2 + 3 + 5 + \cdots = \sum_{n=0}^{\infty} F_n$, the series of Fibonacci numbers; if $S^*$ sums $F$, then $S^*(F) = 2 + S^* \left( \sum_{n=2}^{\infty} F_n \right) = 2 + S^* \left( \sum_{n=0}^{\infty} F_{n+2} - F_n \right) = 2 + (S^*(F) - 1) + S^*(F)$ and thus $S^*(F) = -1$. As second example, we take the geometric series $s_z = 1 + z + z^2 + z^3 + \cdots$. We have $S^*(s_z) = \frac{1}{1-z}$, $z \neq 1$, since $S^*(s_z) = 1 + z S^*(s_z)$. In particular, for $z = -1$ we recover the usual value $S^*(s_0) = S^*(1 - 1 + 1 - 1 + \cdots) = \frac{1}{2}$. The same conclusion is true for any positive integer $S^*(s_0) = S^*(1 - 1 - 1 + 1 + \cdots) = \frac{1}{2}$. The principle of induction allows us to conclude the proof.

**Proposition 1.** Let $S^*$ be a summability method satisfying rules 2 and 3. If $S^*$ sums the series $\sigma_k$ for all integers $k \geq 0$, then $S^*(\sigma_k) = \frac{2^{k+1}-1}{k+1} B_{k+1}$ for all $k \geq 1$.

**Proof.** For $k = 1$ we note that

$$(1 - 2 + 3 - 4 + 5 - 6 + \cdots) + (0 + 1 - 2 + 3 - 4 + 5 - \cdots) = 1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

Then by rules 2 and 3, $S^*(\sigma_0) = S^*(\sigma_1) + S^*(\sigma_1) = 2S^*(\sigma_1)$ and $S^*(\sigma_1) = \frac{1}{2} S^*(s_0) = \frac{1}{4}$. Now we assume the formula holds for $S^*(\sigma_1), \ldots, S^*(\sigma_{k-1})$, $k \geq 2$.

By the binomial theorem, we see that

$$S^*(\sigma_k) = S^* \left( 1 - \sum_{n=2}^{\infty} (-1)^n (1 + (n-1)) \right) = \frac{1}{2} - \sum_{j=1}^{k} \binom{k}{j} S^*(\sigma_j).$$

Using the induction hypothesis and equation [4], we conclude that the formula is valid for $k$. The principle of induction allows us to conclude the proof.

The previous reasoning provides another way to prove recursion [4]: Take $S^* = A$ as Abel summability. Since $A$ satisfies rules 2 and 3, we can replace the value
\[
A(\sigma_k) = \frac{a^{k+1}-1}{k+1} B_{k+1}
\]
in the recursion obtained in the previous proof. In fact, we can easily generalize equation (5) using the same type of argument.

**Proposition 2.** Let \( a \) and \( k \) be positive integers. Then the Bernoulli numbers satisfy the recursion formula

\[
\frac{2^{k+1} - 1}{k+1} B_{k+1} = \sum_{n=0}^{a-1} (-1)^n (n+1)^k + \frac{(-1)^a}{2} a^k - (-1)^a \sum_{j=1}^{k} \binom{k}{j} a^{k-j} 2^{j+1} - \frac{1}{j+1} B_{j+1}.
\]

**Proof.** The divergent series \( a^{-k} \sigma_k = \sum_{n=0}^{\infty} (-1)^n \left( \frac{n+1}{a} \right)^k \) is Abel summable and \( A(a^{-k} \sigma_k) = \frac{a^{k+1}-1}{k+1} B_{k+1} \). The formula follows from the binomial theorem and rules 2 and 3 since

\[
A(a^{-k} \sigma_k) = \frac{1}{a^k} \sum_{n=0}^{a-1} (-1)^n (n+1)^k + A \left( \sum_{n=0}^{\infty} (-1)^n \left( \frac{n-a+1}{a} \right)^k \right)
\]

\[
= \frac{1}{a^k} \sum_{n=0}^{a-1} (-1)^n (n+1)^k + (-1)^a A \left( \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{k} \binom{k}{j} \left( \frac{n+1}{a} \right)^j \right).
\]

\( \square \)

Formula (6) corresponds to the case \( a = 1 \) of Proposition 2. The formula above is simple in the sense that it can be deduced directly from (3) by equating corresponding coefficients of \( z^k \) in the identity \( \sum_{n=0}^{\infty} (-1)^n e^{(n+1)z} = (1-e^{az})^{-1} = 1 - \frac{1}{1+e^z} \).

We can go further and recover the usual formulas to determine the Bernoulli numbers in terms of Euler numbers and vice versa.

**Proposition 3.** The Bernoulli and Euler numbers are related by the formulas

\[
\sum_{j=1}^{k} \binom{k}{j} 2^{j+1} - \frac{1}{j+1} B_{j+1} = \frac{1}{2} - \frac{(-1)^{k/2}}{2} E_k,
\]

\( (6) \)

\[
2^{k+1} \frac{2^{k+1} - 1}{k+1} B_{k+1} = \sum_{j=0}^{k} \binom{k}{j} (-1)^{j/2} E_j,
\]

\( (7) \)

valid for all integers \( k \geq 1 \).

**Proof.** To prove (6), we first calculate the Abel sum of \( 1^k - 3^k + 5^k - 7^k + \cdots \) using equation (4). Then we use the binomial theorem and rules 2 and 3 to obtain

\[
A \left( \sum_{n=0}^{\infty} (-1)^n (2n+1)^k \right) = \frac{1}{2} - \sum_{j=1}^{k} \binom{k}{j} 2^{j+1} - \frac{1}{j+1} B_{j+1}.
\]

Equating both results we get (6). Similarly, for (7) it is enough to consider the series \( 2^k \sigma_k = 2^k - 4^k + 6^k - 8^k + \cdots \) and the relation

\[
A(2^k \sigma_k) = A \left( \sum_{n=1}^{\infty} (-1)^{n+1} (2n-1+1)^k \right) = \sum_{j=0}^{k} \binom{k}{j} A \left( \sum_{n=1}^{\infty} (-1)^{n+1} (2n-1)^j \right).
\]

\( \square \)
The formulas (6) and (7) are of course elementary. They can be deduced by equating the coefficients of \( z^k \) in \( e^z \frac{1}{1+ze} = e^z \frac{1}{e^z+e^{-z}} = \frac{1}{1+ze} \), respectively.

We invite the reader to calculate the Abel sum of 
\[
\sum_{n=0}^{\infty} (-1)^n (an+q)k, \quad k \geq 1,
\]
\( a, q \in \mathbb{R} \) and \( a > 0 \) as we did here (setting \( x = e^{-ag} \) in the generating series and using rules 2 and 3) to conclude that Bernoulli and Euler numbers also satisfy
\[
\frac{q^k}{2} \sum_{j=1}^{k} \binom{k}{j} q^{k-j} a^j 2^{j+1} - 1 \frac{B_{j+1}}{j+1} = \frac{(-1)^k}{2} \sum_{j=0}^{k} \binom{k}{j} \left( \frac{a}{2} - q \right)^{k-j} \left( \frac{a}{2} \right)^j (-1)^{\lfloor j/2 \rfloor} E_j.
\]

Unfortunately, this expression does not provide new information since it can be deduced directly from (7) by equating the corresponding coefficient of \( q^{k-j} a^j \).

**Remark.** Not all series can be summed with methods satisfying rules 1 to 3. For instance, the sum \( s \) of \( 1 + 1 + 1 + \cdots \) must satisfy \( s = 1 + s \) which is impossible for a finite value. Another example is the series \( S = \sum_{n=1}^{\infty} n \): if it would be summable for some \( S^* \), then
\[
S^*(\sigma) - S^*(\sigma) = S^*(1 + 2 + 3 + \cdots) - S^*(0 + 1 + 2 + \cdots) = S^*(1 + 1 + \cdots),
\]
and \( 1 + 1 + 1 + \cdots \) would be summable with sum equals to 0. However, there are methods that assign the controversial value \( -\frac{1}{12} \) to \( \sigma \). For instance, interpreting \( \sigma \) as the value of the analytic continuation of \( \zeta(z) \) at \( z = -1 \). Another example is the constant of a series method of Ramanujan [19, 20, 4, p. 327, 346]. Naturally, such methods can not satisfy the rules 1 to 3. It is curious that Ramanujan wrote [19] p. 135]
\[
\sigma = 1 + 2 + 3 + 4 + \cdots,
\]
\[
4\sigma = 4 + 8 + 12 + \cdots.
\]
Subtracting both equations he found \( -3\sigma = 1 - 2 + 3 - 4 + 5 - 6 + \cdots = \frac{1}{4} \), so again \( \sigma = -\frac{1}{12} \), although this reasoning is not compatible with our approach.

**Remark.** All formulas we have obtained here are well-known, elementary and they admit direct proofs by using power series. Thus it is natural to wonder whether the method we used is widely applicable to more complicated recurrences or to general sequences of numbers. This might not be the case since we have used only linear recursions and the binomial theorem. However, this point of view gives a natural interpretation of formulas (6), (7) and the one in Proposition 2 in terms of the divergent series involved.

**References**
[1] T. K. Petersen, *Eulerian Numbers*, Birkhäuser (2015).
[2] R. P. Stanley, *Enumerative Combinatorics: Vol. 1* (2nd edn.), 49, Cambridge University Press (2012).
[3] T. J. Osler and J. Zeng, Finding \( \zeta(2n) \) from a recursion relation for Bernoulli numbers, *Math. Gaz.* 91 (March 2007) pp. 123–126.
[4] G. H. Hardy, *Divergent series* (2nd edn.), NY, AMS Chelsea Publishing (1992).
[5] H. Chien-Lih, Relations between Euler’s Constant, Riemann’s Zeta Function and Bernoulli Numbers, *Math. Gaz.* 89 (March 2005) pp. 57–59.
[6] T. Arakawa, T. Ibukiyama and M. Kaneko, *Bernoulli Numbers and Zeta Functions*, Springer Monographs in Mathematics, Japan Springer (2014).
[7] B. Mazur, Bernoulli Numbers and the Unity of Mathematics, accessed September 2018 at: [www.math.harvard.edu/~mazur/papers/slides.Bartlett.pdf](http://www.math.harvard.edu/~mazur/papers/slides.Bartlett.pdf)
[8] J. Bernoulli, *Ars conjectandi*, Basel, Thurneysen Brothers (1713).
[9] J. Bernoulli, J. Bernoulli, and Sylla, E.D., *The Art of Conjecturing, Together with Letter to a Friend on Sets in Court Tennis*, Johns Hopkins University Press (2006).
[10] D. Lawson, Super-Pascal, *Math. Gaz.* 64 (October 1980) pp. 177–180.
[11] N. Nielsen, *Traité Élémentaire des nombres de Bernoulli*, Gauthier-Villars, Paris (1923).
[12] H. L. Garabedian, A new formula for the Bernoulli numbers, *Bull. AMS.* 46 (1940) pp. 531–533.
[13] G. Rądkowski, A short proof of the explicit formula for Bernoulli numbers, *Amer. Math. Monthly* 111 (2004) pp. 432–434.
[14] V. Namias, A simple derivation of Stirling’s asymptotic series, *Amer. Math. Monthly* 93 (1986) pp. 25–29.
[15] E. Y. Deeba and D. M. Rodriguez, Stirling’s Series and Bernoulli Numbers, *Amer. Math. Monthly* 98 (1991) pp. 423–426.
[16] V. S. Varadarajan, Euler and his work in infinite series. *Bull. AMS.* 44 (2007) pp. 515–539.
[17] K. Knopp, *Theory and Application of Infinite Series*, NY, Dover (1990).
[18] L. Euler, De seriebus divergentibus, *Opera Omnia* I (14) (1760) pp. 585–617.
[19] B. Berndt, *Ramanujan’s Notebooks: Part 1*, NY, Springer-Verlag (1985).
[20] B. Candelpergher, *Ramanujan Summation of Divergent Series*, Vol. 2185. LNM, Springer International Publishing (2017).

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