Polynomial bound for the localization length of Lorentz mirror model on the 1D cylinder

Linjun Li

Abstract

We consider the Lorentz mirror model and the Manhattan model on the even-width cylinder $\mathbb{Z} \times (\mathbb{Z}/2n\mathbb{Z}) = \{(x, y) : x, y \in \mathbb{Z}, 1 \leq y \leq 2n\}$. For both models, we show that for large enough $n$, with high probability, any trajectory of light starting from the section $x = 0$ is contained in the region $|x| \leq O(n^{10})$.

1 Introduction

1.1 Main result

Given a positive integer $n$, consider the cylinder

$$C_n = \mathbb{Z} \times (\mathbb{Z}/2n\mathbb{Z}) = \{(x, y) : x, y \in \mathbb{Z}, 1 \leq y \leq 2n\}.$$ 

We think of $C_n$ as a directed graph by defining the set of directed edges as follows. For each $(x, y) \in C_n$, there are four directed edges starting from $(x, y)$ and ending at $(x + 1, y)$, $(x - 1, y)$, $(x, y - 1)$ and $(x, y + 1)$ respectively. Here, for $y \in \mathbb{Z}$, we define $\overline{y} \in \{1, \ldots, 2n\}$ with $y \equiv \overline{y} \pmod{2n}$. We also think of $\mathbb{Z}^2$ as a directed graph with directed edges between vertices $u, v \in \mathbb{Z}^2$ such that $|u - v| = 1$.

We describe the Lorentz mirror model and the Manhattan model on $\mathbb{Z}^2$ as follows. Suppose $p \in (0, 1)$, we designate each vertex of $\mathbb{Z}^2$ a mirror with probability $p$; different vertices receive independent designations. In the Lorentz mirror model, if vertex $u$ is designated a mirror, we position on $u$ a two sided north-west (NW) mirror with probability $\frac{1}{2}$ and a north-east (NE) mirror otherwise; In the Manhattan model, if vertex $u = (x, y)$ is designated a mirror, we position on $u$ a two sided NW mirror if $x - y$ is even and an NE mirror if $x - y$ is odd. The Lorentz mirror model and the Manhattan model on the cylinder $C_n$ is defined in the same way. A ray of light traveling along the edges of $\mathbb{Z}^2$ (or $C_n$) is reflected when it hits a mirror and keeps its direction unchanged at vertices which are not designated a mirror. See Figure 1 for an illustration. For both Lorentz mirror model and Manhattan model, we prove the following

*Department of Mathematics, University of Pennsylvania, Philadelphia, PA. e-mail: linjun@sas.upenn.edu
Figure 1: The pink and blue segments illustrate NW and NE mirrors respectively. The dashed lines illustrates a trajectory of light with starting point $u$, initial edge $e_{in}$, ending point $v$, and terminal edge $e_{ter}$.

**Theorem 1.1.** Given $p \in (0, 1)$, there exists $C_0 > 0$ such that for any $n > C_0$ and $M > C_0 n^{10}$, we have

$$P_p \left( B_{n}^{(M)} \right) \geq 1 - C_0 \exp \left( -\frac{M}{C_0 n^{10}} \right).$$

(1)

Here, $B_{n}^{(M)}$ denotes the event that for any trajectory $L$ on $C_n$, if $L$ contains vertices in $\{ (0, y) : 1 \leq y \leq 2n \}$, then $L$ is contained in $\{ -M, -M+1, \ldots, M \} \times (\mathbb{Z}/2n\mathbb{Z})$.

**Remark 1.2.** The eveness of the width of cylinder is necessary in Theorem 1.1. In the case of the Manhattan model, the definition of the model requires the orientation changes alternatively. As for the Lorentz mirror model, one can show that there always exists an infinite trajectory on the cylinder with odd width (see [KS15]).

For both models, Theorem 1.1 implies that the localization length on $C_n$ is of order $O(n^{10})$. The factor 10 may not be optimal and it is conjectured for Manhattan model that the localization length is less than $O(n)$ (see [Spe12], Page 238). The Manhattan model is related to the random Schrödinger model where one expects full spectrum Anderson localization for arbitrarily small disorder in 2D (see e.g. [Spe12], [Car10], [BOC03], [AALR79]). An exponential upper bound for localization length of random Schrödinger model on 1D cylinder was proved in [Bou13], and one also expects the real localization length to be at most linear in width.

Besides the localization length problem on 1D cylinder, it is a fundamental question to determine the value of $p \in (0, 1)$ such that, almost surely, all trajectories on $\mathbb{Z}^2$ are finite (see e.g. [Gri13], Section 13.3]). In the Lorentz mirror model, the question is open for any $p \in (0, 1)$ while the numerical simulations of
Figure 2: The rectangle illustrates $R_{\{1,\ldots,k_1\},\{1,\ldots,k_2\}}$. The pink and green arrows illustrate the bottom and top edges of $R_{\{1,\ldots,k_1\},\{1,\ldots,k_2\}}$ respectively. The blue points illustrate a trajectory traversing $R_{\{1,\ldots,k_1\},\{1,\ldots,k_2\}}$ from bottom to top (with starting point $u$, ending point $v$, initial edge $e_{in}$ and terminal edge $e_{ter}$) which implies event $E_{k_1,k_2}$.

\[ZKC91\] suggest the finiteness of trajectories for all $p \in (0,1)$. In the Manhattan model, a percolation argument shows that all trajectories are finite when $p > \frac{1}{2}$ (see e.g. [Car10],[Spe12]). Later in [Li20], by an enhancement argument, the same result is proved for $p > \frac{1}{2} - \varepsilon$ with some $\varepsilon > 0$.

1.2 Outline

Definition 1.3. For any $(x, y) \in \mathbb{Z}^2$ and positive integer $k$, we denote

$$Q_k(x, y) = \{(x + s, y + t) : s, t \in \mathbb{Z} \text{ and } |s|, |t| \leq k\}.$$ 

For simplicity, we also denote $Q_k = Q_k(0,0)$.

Definition 1.4. Given two intervals $I, J \subset \mathbb{Z}$, we denote the rectangle $R_{I,J} = I \times J$. Suppose $J = \{j_1, j_1 + 1, \ldots, j_2\}$ with integers $j_1 \leq j_2$. We define the bottom edges of $R_{I,J}$ to be $\{(x, j_1 - 1), (x, j_1) : x \in I\}$ and the top edges of $R_{I,J}$ to be $\{(x, j_2), (x, j_2 + 1) : x \in I\}$.

For positive integers $k_1, k_2$, we consider event $E_{k_1,k_2}$ which roughly says that there is a trajectory traversing $R_{\{1,\ldots,k_1\},\{1,\ldots,k_2\}}$ from a bottom edge to a top edge (see Figure 2). The rigorous definition of $E_{k_1,k_2}$ is given in definition 2.1 below. We consider the following statement:

There exists $n > 100$ with $\mathbb{P}_p(E_{100n,n}) \leq c_0$. \hspace{1cm} (*)&

Here, the constant $c_0$ will be chosen in Lemma 2.2 below. We will prove that Theorem 1.1 is true no matter (*) holds or not. More precisely, if (*) holds
for some $n > 100$, then a percolation argument (Lemma 2.2) implies an exponentially decaying probability for a trajectory starting from the origin to reach $\mathbb{Z}^2 \setminus Q_n$ for any $n' > 10^n n$. Since a trajectory on $C_{n'}$ can not tell the difference between being in the cylinder and in $\mathbb{Z}^2$ before getting to distance $n'$, a trajectory on $C_{n'}$ also has an exponentially small probability to reach distance $n'$. On the other hand, if (2) fails, then it is likely to see many up-down closed trajectories in $\{1, 2, \cdots, M\} \times (\mathbb{Z}/2n\mathbb{Z})$ when $M$ is a large polynomial in $n$ (Lemma 2.3). We call any trajectory traveling from section $x = 0$ to section $x = M + 1$ a left-right trajectory. If there exists a left-right trajectory, then the evenness of the width of cylinder implies there are at least two left-right trajectories, say $L_1, L_2$. $L_1, L_2$ must have intersections with any up-down closed trajectories in $\{1, 2, \cdots, M\} \times (\mathbb{Z}/2n\mathbb{Z})$. It turns out that, by modifying the mirror configuration on the intersections, we can transform $L_1, L_2$ to $L_3, L_4$ where $L_3$ starts and ends at section $x = 0$ and $L_4$ starts and ends at section $x = M + 1$; and the number of left-right trajectories decreases. By this observation, in Proposition 2.8 we use a combinatorial enumeration argument to show that it is unlikely to have any left-right trajectory and thus by symmetry, any trajectory starting from $\{(0, y) : 1 \leq y \leq 2n\}$ must localize in $\{-M, -M + 1, \cdots, M\} \times (\mathbb{Z}/2n\mathbb{Z})$. Finally in Section 3 we use a standard bootstrap argument to finish the proof of Theorem 1.4.

1.3 Notations

We now define the notations for trajectories on lattice $\mathbb{Z}^2$ and cylinder $C_n$.

Definition 1.5. Let $L$ be the trajectory of a light which crosses (directed) edges $e_1, e_2, \cdots, e_m$ in order. Suppose $e_i = (u_i, v_i)$ for $1 \leq i \leq m$ and $v_i = u_{i+1}$ for $1 \leq i \leq m - 1$. We call $u_1$ the starting point of $L$ and $v_m$ the ending point of $L$. We also denote $e_{\text{in}}(L) = e_1$ the initial edge of $L$ and $e_{\text{ter}}(L) = e_m$ the terminal edge of $L$. See Figure 1 for an illustration. Define the set of vertices of $L$ as $V(L) = \bigcup_{i=1}^{m} \{u_i, v_i\}$ and the set of inner vertices of $L$ as

$$V_{\text{inn}}(L) = \bigcup_{i=2}^{m-1} \{u_i, v_i\}.$$ 

Moreover, we call $L$ closed if $e_i = e_j$ for some $i \neq j$.

Definition 1.6. In the Manhattan model, we call a trajectory $L$ a Manhattan trajectory if following holds.

1. For every horizontal edge $e = (u, v)$ in $L$, $v = u + (1, 0)$ if the $y$-coordinate of $u$ is odd; and $v = u + (-1, 0)$ if the $y$-coordinate of $u$ is even.

2. For every vertical edge $e = (u, v)$ in $L$, $v = u + (0, -1)$ if the $x$-coordinate of $u$ is odd; and $v = u + (0, 1)$ if the $x$-coordinate of $u$ is even.

Note that for any trajectory $L$ in the Manhattan model, either $L$ is a Manhattan trajectory or the inverse of $L$ is a Manhattan trajectory.
2 Dichotomy from a crossing event

Definition 2.1. Given two intervals $I, J \subset \mathbb{Z}$, denote by $E_{I,J}$ the event that there is a trajectory $L$ such that following holds:

1. $e_{in}(L)$ is a bottom edge of $R_{I,J}$ and $e_{ter}(L)$ is a top edge of $R_{I,J}$,
2. $V_{inn}(L) \subset R_{I,J}$.

See Figure 2 for an illustration. We also denote by $E'_{I,J}$ the event that there exists a trajectory $L$ satisfying the two properties above with the starting and ending points having the same $x$-coordinate.

For $I = \{1, 2, \cdots , k_1\} \text{ and } J = \{1, 2, \cdots, k_2\}$ with positive integers $k_1, k_2$, we also use $E_{k_1,k_2} (E'_{k_1,k_2})$ to denote $E_{I,J} (E'_{I,J})$ for simplicity, respectively.

Lemma 2.2. There exists a constant $0 < c_0 < 1$ such that following holds. For any $p \in (0, 1)$, if there exists $n > 100$ with $\mathbb{P}_p(E_{100n,n}) \leq c_0$, then for any $m$ with $m > c_0^{-1}n$, we have

$$\mathbb{P}_p(A_m) \geq 1 - \exp \left(-c_0 \frac{m}{n}\right).$$

(2)

Here, $A_m$ denotes the event that for any trajectory $L$ in $\mathbb{Z}^2$, if the starting point of $L$ is in $Q_1$, then $V(L) \subset Q_m$.

Remark 2.3. In the Lorentz mirror model, it was proved in [KS15] that $\mathbb{P}_p((A_m)^c) \geq \frac{1}{2m+1}$ for any $p \in (0,1)$. Thus Lemma 2.2 implies that there exists $c_0 > 0$ with $\mathbb{P}_p(E_{100n,n}) \geq c_0$ for any $n > 100$. The situation is different in Manhattan model where $\mathbb{P}_p((A_m)^c)$ decays exponentially for $p$ larger than or near $\frac{1}{2}$ (see [Li20]).

Proof of Lemma 2.2. For each positive integer $k$ and $(x,y) \in \mathbb{Z}^2$, denote by $E_{k}^{inn}(x,y)$ the event that there is a trajectory with starting point in $Q_{\left\lfloor \frac{2}{3}k \right\rfloor}(x,y)$ and ending point in $\mathbb{Z}^2 \setminus Q_k(x,y)$. Suppose $\mathbb{P}_p(E_{100n,n}) \leq c_0$ for some $n > 100$. Observe that there are four translational and $90^\circ$-rotational images of $R_{\{1,\ldots,100n\},\{1,\ldots,n\}}$ splitting the annulus $Q_{40n} \setminus Q_{36n}$, namely

$$R_1 = R_{\{-50n,\cdots,50n-1\},\{38n,\cdots,39n-1\}},$$
$$R_2 = R_{\{-50n,\cdots,50n-1\},\{-38n,\cdots,-37n-1\}},$$
$$R_3 = R_{\{-38n,\cdots,-37n-1\},\{-50n,\cdots,50n-1\}} \text{ and }$$
$$R_4 = R_{\{38n,\cdots,39n-1\},\{-50n,\cdots,50n-1\}}.$$

Any trajectory with starting point in $Q_{36n}$ and ending point in $\mathbb{Z}^2 \setminus Q_{40n}$ must traverse one of $R_i$’s in the short direction (see Figure 3 and its caption). Thus by translational and $90^\circ$-rotational invariance, we have

$$\mathbb{P}_p(E_{40n}^{inn}(0,0)) \leq 4\mathbb{P}_p(E_{100n,n}) \leq 4c_0.$$

(3)
Figure 3: The largest box is $Q_{40n}$, the green box is $Q_{36n}$ and the four red rectangles are $R_1, R_2, R_3$ and $R_4$. Any trajectory starting from $Q_{36n}$ ending at $Z^2 \setminus Q_{40n}$ must traverse one of the red rectangles in the short direction.

By translational invariance again, we have $P_p(\mathcal{E}_{40n}^{\text{ann}}(x, y)) \leq 4c_0$ for any $(x, y) \in \mathbb{Z}^2$ such that $x - y$ is even (here, the evenness is only required in Manhattan model). Define a graph $\mathcal{G}$ with vertex set

$$V = \{Q_{40n}(50nx, 50ny) : x, y \in \mathbb{Z}\}$$

and there is an edge connecting $v_1, v_2 \in V$ if and only if $v_1 \cap v_2 \neq \emptyset$. Given $v' \in V$ with $v' = Q_{40n}(x', y')$, the box $v'$ is called \textit{tamed} if $\mathcal{E}_{40n}^{\text{ann}}(x', y')$ holds.

Let $\mathcal{E}_m^{\text{(per)}}$ denote the event that there exists a path $v_1, v_2, \ldots, v_t$ in $\mathcal{G}$ consisted of tamed boxes such that $v_1 = Q_{40n}$ and $v_t \cap Q_{m-40n} = \emptyset$. For constant $c_0$ small enough (not depending on $n$), by [Gri13, Page 16], we have

$$P_p(\mathcal{E}_m^{\text{(per)}}) \leq \exp\left(-c_0\frac{m}{n}\right)$$

for $m > c_0^{-1}n$. Thus it suffices to prove $(A_m)^c \subset \mathcal{E}_m^{\text{(per)}}$ when $m > 10^9n$. To see this, assume $(A_m)^c$ and let $L$ be a trajectory with starting point in $Q_1$ and ending point outside $Q_m$. Suppose the vertices of $L$ are $u_1, u_2, \ldots, u_t$ in order. For each $1 \leq i \leq t$, pick $(x_i, y_i) \in \mathbb{Z}^2$ with $u_i \in Q_{36n}(50nx_i, 50ny_i)$ and let $v_i = Q_{40n}(50nx_i, 50ny_i)$; we can also assume $v_1 = Q_{40n}$. Note that $v_t \cap Q_{m-40n} = \emptyset$ since $u_t \not\in Q_m$. Then $\{v_1, v_2, \ldots, v_t\}$ provides us a path of tamed boxes which implies $\mathcal{E}_m^{\text{(per)}}$. Thus $(A_m)^c \subset \mathcal{E}_m^{\text{(per)}}$ and our lemma follows.

\textbf{Lemma 2.4.} For positive integers $n, m > 100$, if $P_p(\mathcal{E}_{m,n}) \geq c_0$, then

$$P_p(\mathcal{E}_{m,2n}) \geq \frac{(1-p)^2c_0^2}{m^2}.$$  

(5)
Proof. Let $I = \{1, 2, \ldots, m\}$, $J_{\text{down}} = \{1, 2, \ldots, n\}$ and $J_{\text{up}} = \{n + 1, n + 2, \ldots, 2n\}$. For $1 \leq i, j \leq m$, let $A_{i,j}$ be the event that there exists a trajectory $L$ with $e_{\text{in}}(L) = ((i, 0), (i, 1))$, $e_{\text{ter}}(L) = ((j, n), (j, n + 1))$ and $V_{\text{inn}}(L) \subset R_{I, J_{\text{down}}}$. 

Claim 2.5. For Lorentz mirror model, we have $\mathbb{P}_p(A_{i,j}) = \mathbb{P}_p(A_{j,i})$ for any $1 \leq i, j \leq m$. For Manhattan model, the same is true when $n$ is even.

Proof. Let $\tilde{\Omega}$ be the configuration space restricted on $R_{I,J_{\text{down}}}$. We define a bijection $\phi : \tilde{\Omega} \rightarrow \tilde{\Omega}$ as follows. Suppose $\omega \in \tilde{\Omega}$. For each $(x, y) \in R_{I, J_{\text{down}}}$, designate a mirror on $(x, y)$ for $\phi(\omega)$ if and only if there is a mirror on $(x, n+1-y)$ in $\omega$. Then position an NE (NW) mirror on $(x, y)$ if $\omega$ has an NW (NE) mirror on $(x, n+1-y)$ respectively. Note that $\phi(\omega)$ is the mirror image of $\omega$ along the horizontal line $y = \frac{n+1}{2}$. We have $\omega \in A_{i,j}$ if and only if $\phi(\omega) \in A_{j,i}$. Here, in the case of Manhattan model, we used the evenness of $n$ to guarantee $\phi(\omega) \in \tilde{\Omega}$. Since $\mathbb{P}_p(\omega) = \mathbb{P}_p(\phi(\omega))$ for every $\omega \in \tilde{\Omega}$, we have $\mathbb{P}_p(A_{i,j}) = \mathbb{P}_p(A_{j,i})$. \qed

With the claim above, we now prove the lemma for two models.

1. Lorentz mirror model: We have

$$\bigcup_{1 \leq i,j \leq m} A_{i,j} = E_{m,n},$$

and thus there exist $i', j' \in \{1, 2, \ldots, m\}$ with $\mathbb{P}_p(A_{i',j'}) \geq \frac{c_0}{m^2}$. Let us define $\tilde{A}_{i',j'}$ to be the event that exists a trajectory $L$ with $e_{\text{in}}(L) = ((i', n), (j', n + 1))$, $e_{\text{ter}}(L) = ((i', 2n), (j', 2n + 1))$ and $V_{\text{inn}}(L) \subset R_{I, J_{\text{up}}}$. Since $R_{I, J_{\text{down}}}$ is a translation of $R_{I, J_{\text{down}}}$, we have

$$\mathbb{P}_p(\tilde{A}_{i',j'}) = \mathbb{P}_p(A_{i',j'}) = \mathbb{P}_p(A_{j',i'}).$$

by Claim 2.5. Moreover,

$$\mathbb{P}_p(A_{i',j'} \cap A_{j',i'}) = \mathbb{P}_p(A_{i',j'})\mathbb{P}_p(A_{j',i'}) \geq \frac{c_0}{m^2}.$$

We view $R_{I, J_{\text{down}} \cup J_{\text{up}}}$ as a concatenation of $R_{I, J_{\text{up}}}$ and $R_{I, J_{\text{down}}}$. Then the event $A_{i',j'} \cap A_{j',i'}$ implies that there exists a trajectory $L$ with $e_{\text{in}}(L) = ((i', 0), (i', 1))$, $e_{\text{ter}}(L) = ((i', 2n), (i', 2n + 1))$ and $V_{\text{inn}}(L) \subset R_{I, J_{\text{down}} \cup J_{\text{up}}}$. Thus $A'_{i',j'} \cap \tilde{A}_{j',i'} \subset E'_{m,2n}$ and the lemma follows.

2. Manhattan model: If $n$ is even, the same argument for Lorentz mirror model implies the lemma. Otherwise, if $n$ is odd, we apply the argument above for $n - 1$. Note that $E_{m,n} \subset E_{m,n-1}$ and we imply the following. There exists $1 \leq i' \leq m$ such that $\mathbb{P}_p(E_{i'}^{(\text{cro})}) \geq \frac{c_0}{m}$, where $E_{i'}^{(\text{cro})}$ denotes the event that there is a trajectory $L$ with $e_{\text{in}}(L) = ((i', 0), (i', 1))$, $e_{\text{ter}}(L) = ((i', 2n - 2), (i', 2n - 1))$ and $V_{\text{inn}}(L) \subset R_{I, \{1, \ldots, 2n-2\}}$. Let $E_{i'}^{*}$ be the event that there is no mirror on $(i', 2n - 1)$ and $(i', 2n)$. Then by viewing $R_{I, J_{\text{down}} \cup J_{\text{up}}}$ as a concatenation of $R_{I, \{2n-1, 2n\}}$ and $R_{I, \{1, \ldots, 2n-2\}}$, we imply $E_{i'}^{*} \cap E_{i'}^{(\text{cro})} \subset E_{m,2n}$. Our lemma follows from

$$\mathbb{P}_p(E_{i'}^{*} \cap E_{i'}^{(\text{cro})}) = \mathbb{P}_p(E_{i'}^{*})\mathbb{P}_p(E_{i'}^{(\text{cro})}) \geq (1 - p)^{c_0} \frac{c_0}{m^2}.$$
Definition 2.6. Given positive integers \( n, k_1, k_2 \) with \( k_1 < k_2 \), define \( T_{k_1,k_2}^{(n)} = \{ k_1 + 1, k_1 + 2, \ldots, k_2 \} \times (\mathbb{Z}/2n\mathbb{Z}) \).

Definition 2.7. Given positive integers \( n, N \) and a subset \( S \subset T_{0,N}^{(n)} \), let graph \( G \) be obtained from \( \mathbb{Z} \times (\mathbb{Z}/2n\mathbb{Z}) \) by removing the vertices in \( S \) and the edges joined to \( S \). We say \( G \) is \( N \)-good if \((0,n)\) and \((N+1,n)\) live in different connected components of \( G \).

Proposition 2.8. Given \( p \in (0,1) \), there exists \( C_1 > 0 \) such that following holds. For any integer \( n > C_1 \), if \( \mathbb{P}_p(\mathcal{E}_{n100},n) \geq c_0 \), then

\[
\mathbb{P}_p(\mathcal{E}_{n}^{(loc)}) \geq \frac{1}{2} \tag{6}
\]

Here, \( \mathcal{E}_{n}^{(loc)} \) denotes the event that for any trajectory \( L \) on \( \mathcal{C}_n \), if \( L \) contains vertices in \( \{(0,y) : 1 \leq y \leq 2n\} \), then \( V(L) \subset T_{[C_1,n^{10}]}^{(n)} \).

Proof. Denote \( N = |C_1,n^{10}| \). We let \( E_r^+ = \{((0,y),(1,y)) : 1 \leq y \leq 2n\} \), \( E_l^{-} = \{((1,y),(0,y)) : 1 \leq y \leq 2n\} \), \( E_r^{-} = \{((N,y),(N+1,y)) : 1 \leq y \leq 2n\} \) and \( E_l^{+} = \{((N+1,y),(N,y)) : 1 \leq y \leq 2n\} \). Given a trajectory \( L \) with \( V_{inn}(L) \subset T_{0,N}^{(n)} \), we call \( L \) is

- a left-right trajectory if \((e_{in}(L),e_{ter}(L)) \in E_r^{+} \times E_l^{+} \);
- a right-left trajectory if \((e_{in}(L),e_{ter}(L)) \in E_l^{-} \times E_r^{-} \);
- a left-left trajectory if \((e_{in}(L),e_{ter}(L)) \in E_l^{+} \times E_l^{-} \);
- a right-right trajectory if \((e_{in}(L),e_{ter}(L)) \in E_r^{-} \times E_r^{+} \).

Let \( \mathcal{E}_{(tra)} \) denote the event that there exists a left-right trajectory. By symmetry, in order to prove the proposition, it suffices to prove

\[
\mathbb{P}_p(\mathcal{E}_{(tra)}) \leq \frac{1}{4} \tag{7}
\]

For any \( \ell \in \{1, \ldots, \lceil \frac{C_1}{1000n^{9}} \rceil \} \), denote by \( \mathcal{E}_{\ell}^{(wind)} \) the event that there is a closed trajectory \( L \) such that \( V(L) \) is \( N \)-good and \( V(L) \subset T_{2000\ell n, 2000\ell n + 100n}^{(n)} \).

Claim 2.9. We have \( \mathbb{P}_p(\mathcal{E}_{\ell}^{(wind)}) \geq \frac{c_1}{n^2} \) for each \( \ell \in \{1, \ldots, \lceil \frac{C_1}{1000n^{9}} \rceil \} \) where

\[
c_1 = \frac{1-p^2}{1000}.
\]

Proof. We identify the bottom edges and top edges of \( R_{\{1,\ldots,100n\},\{1,\ldots,2n\}} \) and the resulting graph is isomorphic to \( T_{2000\ell n, 2000\ell n + 100n}^{(n)} \). By definition 2.4, a trajectory satisfying conditions of \( \mathcal{E}_{100n,2n}^{(n)} \) becomes a closed trajectory in \( T_{2000\ell n, 2000\ell n + 100n}^{(n)} \) and after removing vertices on this closed trajectory, \((0,n)\) and \((N+1,n)\) live in different connected components. Thus the claim follows from Lemma 2.4 (with \( m = 100n \)). □
Let $\Omega$ be the configuration space restricted on $\{0, 1, \cdots, N + 1\} \times (\mathbb{Z}/2n\mathbb{Z})$ and we view $E_{\text{tra}}$ and $E_{\ell}^{\text{wind}}$ as subsets of $\Omega$. Denote

$$E_{\ell}^{\text{wind}} = \left\{ \omega \in \Omega : \left| 1 \leq \ell \leq \frac{C_1}{1000}n^9 : \omega \in E_{\ell}^{\text{wind}} \right| \geq \frac{c_1 C_1}{2000}n^5 \right\}.$$ 

Since $\{E_{\ell}^{\text{wind}} : 1 \leq \ell \leq \frac{C_1}{1000}n^9\}$ is a family of independent events, by Claim 2.9 and Hoeffding’s inequality (see e.g. [Hoe94]), we have

$$P_p(E_{\text{wind}}) \geq 1 - \exp \left( -\frac{c_1^2 C_1}{2000}n \right). \quad (8)$$

**Claim 2.10.** There exists a subset $S \subset (E_{\text{wind}} \cap E_{\text{tra}}) \times \Omega$ such that following holds:

1. for each $\omega \in E_{\text{wind}} \cap E_{\text{tra}}$, $|\{\omega' : (\omega, \omega') \in S\}| \geq \frac{c_1 C_1}{2000}n^5$,
2. for each $(\omega, \omega') \in S$, $P_p(\{\omega\}) \leq C_p P_p(\{\omega'\})$ for a constant $C_p$ depending on $p$,
3. for each $\omega' \in \Omega$, $|\{\omega \in E_{\text{wind}} \cap E_{\text{tra}} : (\omega, \omega') \in S\}| \leq 10^6n^5$.

**Proof of the claim.** For each $\omega \in E_{\text{wind}} \cap E_{\text{tra}}$, we construct $\omega' \in \Omega$ from $\omega$ by the procedure described below. Then we declare $(\omega, \omega') \in S$ if and only if $\omega'$ is one of the resulting configurations from the procedure.

Fix an $\omega \in E_{\text{wind}} \cap E_{\text{tra}}$. Let

$$W = \left\{ 1 \leq \ell \leq \frac{C_1}{1000}n^9 : \omega \in E_{\ell}^{\text{wind}} \right\}$$

and for each $\ell \in W$, choose a closed trajectory $L^{(\ell)}$ such that $V(L^{(\ell)})$ is $N$-good and $V(L^{(\ell)}) \subset T^{(n)}_{2000n, 200\ell n + 100n}$. In the Manhattan model, we also assume $L^{(\ell)}$ to be a Manhattan trajectory; otherwise we replace $L^{(\ell)}$ by its inverse. We now construct $\omega'$ for each of the two models:

1. Lorentz mirror model: By the eveness of width of the cylinder and a pairing consideration, the number of left-right trajectories is even. Thus we can select two different left-right trajectories $L_1, L_2$. Then we pick an arbitrary $\ell \in W$. For $i = 1, 2$, let $u_i$ be the vertex where $L_i$ touches $L^{(\ell)}$ for the first time. Let $u_3 \in V(L_2)$ be the vertex right before $L_2$ reaches $u_2$. We modify $\omega$ first on $u_1$ and next on $u_2$ as follows: if there is a mirror on $u_1$, we remove it; otherwise, we add an NE mirror on $u_1$. Denote the resulting configuration by $\tilde{\omega}$. In the configuration $\tilde{\omega}$, shine a light from the initial edge $e_{in}(L_1)$. The light will first travel along $L_1$ until reaching $u_1$, after that it continues along $L^{(\ell)}$ (or its inverse) and then reaches $u_2$. We modify $\tilde{\omega}$ on $u_2$ such that the light will continue on the edge $(u_2, u_3)$ (and then travel along the inverse of $L_2$ until reaching $E_{\ell}^{-1}$). Let $\omega'$ be the resulting configuration (see Figure 4 and its caption).
Figure 4: In the top cylinder, we have two left-right trajectories \( L_1 \) (the pink curve) and \( L_2 \) (the yellow curve), and a closed trajectory \( L^{(\ell)} \) (the blue curve). After modifying the configurations on \( u_1, u_2 \) (the pink and yellow vertices), in the bottom cylinder, we have a new left-left trajectory (the red curve) and a new right-right trajectory (the green curve).

2. Manhattan model: By a pairing consideration, there exists a left-right Manhattan trajectory \( L_1 \) and a right-left Manhattan trajectory \( L_2 \). We pick an arbitrary \( \ell \in W \). Let \( u_1 \) be the vertex where \( L_1 \) touches \( L^{(\ell)} \) for the first time; and let \( u_2 \) be the vertex where \( L_2 \) touches \( L^{(\ell)} \) for the last time. Let \( \omega' \) be the resulting configuration after modifying \( \omega \) as follows: for each \( v \in \{ u_1, u_2 \} \), if \( \omega \) has no mirror on \( v \), then we add an mirror on \( v \); otherwise we remove the mirror of \( \omega \) on \( v \). See Figure 5 and its caption.

In both models, we declare \( (\omega, \omega') \in S \) if and only if \( \omega' \) is constructed from \( \omega \) by the above procedure. We show that the three items in the claim hold. To see item 1, note that by definition of \( E^{(wind)} \), there are at least \( C_1 \frac{n^3}{200} \) many choices for \( \ell \). As for item 2, note that \( \omega' \) is obtained from \( \omega \) by modifying the configuration on two vertices. Finally we show item 3. Suppose \( (\omega, \omega') \in S \). Then \( \omega' \) has exactly two more left-left trajectories \( L', L'' \) than \( \omega \) and \( L'' \) is the inverse of \( L' \). Suppose \( \omega' \) differs from \( \omega \) on two vertices \( u_1, u_2 \) and let 

\[
\ell_0 = \max \left\{ 1 \leq \ell \leq \left[ \frac{C_1}{1000} n^3 \right] : V(L') \cap T^{(n)}_{200\ell n,200\ell n+100n} \neq \emptyset \right\}.
\]

Then by our construction, we have \( u_i \in T^{(n)}_{200\ell_i n,200\ell_i n+100n} \) for \( i = 1, 2 \). For each \( \omega' \in \Omega \), there are at most \( 2n \) choices for the left-left trajectory \( L' \); and after choosing \( L' \), there are at most \( |T^{(n)}_{200\ell_1 n,200\ell_1 n+100n}|^2 \) possible choices for \( u_1, u_2 \); and finally there are at most 4 possible choices for configurations on \( u_1, u_2 \). Thus for each \( \omega' \in \Omega \),

\[
|\{ \omega \in E^{(wind)} \cap E^{(tra)} : (\omega, \omega') \in S \}| \\
\leq 2n \times (200n^2)^2 \times 4 \\
\leq 10^6 n^5
\]

10
Figure 5: In the top cylinder, we have a left-right Manhattan trajectory $L_1$ (the pink curve) and a right-left Manhattan trajectory $L_2$ (the yellow curve). The blue curve illustrates the closed Manhattan trajectory $L^{(t)}$. After modifying the configurations on $u_1, u_2$ (the pink and yellow vertices), in the bottom cylinder, we have a new left-left Manhattan trajectory (the red curve) and a new right-right Manhattan trajectory (the green curve).

and our claim follows. □

**Claim 2.11.** We have

$$P_p(E^{(wind)} \cap E^{(tra)}) \leq \frac{10^{10}}{c_1C_1}C_p.$$  \hspace{1cm} (9)

**Proof.** Let us consider the following quantity,

$$\sum_{\omega \in E^{(wind)} \cap E^{(tra)}} \int_{\Omega} \mathbb{1}_{(\omega, \omega') \in S} dP_p(\omega').$$  \hspace{1cm} (10)

By item 1 and item 2 in Claim 2.10, (10) is lower bounded by

$$\geq \sum_{\omega \in E^{(wind)} \cap E^{(tra)}} \int_{\Omega} \mathbb{1}_{(\omega, \omega') \in S} dP_p(\omega') \geq \frac{c_1C_1}{2000}n^5C_p^{-1}P_p(E^{(wind)} \cap E^{(tra)}).$$  \hspace{1cm} (11)
On the other hand, we can upper bound (10) by item 3 in Claim 2.1 as follows,

\[
\sum_{\omega \in \mathcal{E}(\text{wind}) \cap \mathcal{E}(\text{tra})} \int_{\Omega} \mathbb{1}_{(\omega, \omega') \in S} d\mathbb{P}_p(\omega')
\]

\[
= \int_{\Omega} \left( \sum_{\omega \in \mathcal{E}(\text{wind}) \cap \mathcal{E}(\text{tra})} \mathbb{1}_{(\omega, \omega') \in S} \right) d\mathbb{P}_p(\omega')
\]

\[
\leq \int_{\Omega} 10^6 n^5 d\mathbb{P}_p(\omega')
\]

\[
= 10^6 n^5.
\]

Combining (12) and (11), our claim follows.

Finally, by (8) and Claim 2.11, we have

\[
\mathbb{P}_p(\mathcal{E}(\text{tra})) \leq \mathbb{P}_p(\mathcal{E}(\text{wind})) + \mathbb{P}_p(\mathcal{E}(\text{wind}) \cap \mathcal{E}(\text{tra}))
\]

\[
\leq \exp(-c_0^2 C_1 n) + \frac{10^{10}}{c_1 C_p}
\]

\[
\leq \frac{1}{4}
\]

by letting \( C_1 > 10^{11} c_1^{-1} C_p + 10^4 c_1^{-2} \). Thus (7) holds and our proposition follows.

\[\square\]

3 Proof of the main theorem

Proof of Theorem 1.1. We claim that there exists \( C_2 > 0 \) such that for any \( n > C_2 \), we have \( \mathbb{P}_p(\mathcal{E}(\text{loc})) \geq \frac{1}{2} \). To see this, by Proposition 2.8 if \( \mathbb{P}_p(\mathcal{E}_{100n,n}) \geq c_0 \) for each \( n > \max\{100, C_1\} \), then our claim follows by letting \( C_2 = \max\{100, C_1\} \). Otherwise, there is \( n_0 > 100 \) with \( \mathbb{P}_p(\mathcal{E}_{100n_0,n_0}) \leq c_0 \). Then by Lemma 2.2 for any \( n > c_0^{-1} n_0 \), we have \( \mathbb{P}_p(A_n) \geq 1 - \exp(-c_0 \frac{n}{n_0}) \). By a union bound of probability for vertices in \( \{(0, 2y) : 1 \leq y \leq n\} \), we imply that with probability at least \( 1 - n \exp(-c_0 \frac{n}{n_0}) \), any trajectory on \( C_n \) containing vertices in \( \{(0, y) : 1 \leq y \leq 2n\} \) is contained in \( \{-n, -n + 1, \ldots, n\} \times (\mathbb{Z}/2n\mathbb{Z}) \). In particular, we have

\[
\mathbb{P}_p(\mathcal{E}(\text{loc})) \geq 1 - n \exp\left(-c_0 \frac{n}{n_0}\right) \geq \frac{1}{2}
\]

for \( n > 100(c_0^{-1} n_0)^2 \). This proves our claim with \( C_2 = 100(c_0^{-1} n_0)^2 \).

Suppose \( n > C_2 \) and let \( N = \lfloor C_1 n^{10} \rfloor \). For each \( k \in \mathbb{Z} \), let

\[
\mathcal{C}_n^{(k)} = T_{(4k+1)N-1,(4k+3)N}^{(n)}
\]

and denote \( \mathcal{E}_n^{(obs)} \) to be the event that any trajectory containing vertices in \( \{(4k + 2)N, y) : 1 \leq y \leq 2n\} \) is contained in \( \mathcal{C}_n^{(k)} \). Then by translational
invariance and the claim above, we have \( P_p(\mathcal{E}_k^{(obs)}) \geq \frac{1}{2} \) for each \( k \in \mathbb{Z} \). For any \( M > 100N \), we have
\[
\left( \bigcup_{k=1}^{\lfloor M - 4N \rfloor} \mathcal{E}_k^{(obs)} \right) \cap \left( \bigcup_{k=1}^{\lfloor M - 4N \rfloor} \mathcal{E}_{-k}^{(obs)} \right) \subset B_n^{(M)}.
\]
Since \( \mathcal{C}_n^{(k_1)} \cap \mathcal{C}_n^{(k_2)} = \emptyset \) when \( k_1 \neq k_2 \), \( \{\mathcal{E}_k^{(obs)} : k \in \mathbb{Z}\} \) is a family of independent events. Thus our theorem follows from
\[
P_p \left( \left( \bigcup_{k=1}^{\lfloor M - 4N \rfloor} \mathcal{E}_k^{(obs)} \right) \cap \left( \bigcup_{k=1}^{\lfloor M - 4N \rfloor} \mathcal{E}_{-k}^{(obs)} \right) \right)
\geq 1 - 2 \left( \frac{1}{2} \right)^{\lfloor M - 4N \rfloor}
\geq 1 - C_0 \exp \left( - \frac{M}{C_0 n^{10}} \right)
\]
by letting \( C_0 > \max \left\{ \frac{4C_1}{\log(2)} + 8, C_2 \right\} \).

\[\square\]

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