On tractable query evaluation for SPARQL

Stefan Mengel¹ and Sebastian Skritek²

¹ CNRS, CRIL UMR 8188, Lens, France; mengel@cril.fr
² Faculty of Informatics, TU Wien; skritek@dbai.tuwien.ac.at

Abstract

Despite much work within the last decade on foundational properties of SPARQL—the standard query language for RDF data—rather little is known about the exact limits of tractability for this language. In particular, this is the case for SPARQL queries that contain the OPTIONAL-operator, even though it is one of the most intensively studied features of SPARQL. The aim of our work is to provide a more thorough picture of tractable classes of SPARQL queries.

In general, SPARQL query evaluation is \textit{PSPACE}-complete in combined complexity, and it remains \textit{PSPACE}-hard already for queries containing only the OPTIONAL-operator. To amend this situation, research has focused on “well-designed SPARQL queries” and their recent generalization “weakly well-designed SPARQL queries”. For these two fragments the evaluation problem is \textit{coNP}-complete in the absence of projection and \textit{Σ²P}-complete otherwise. Moreover, they have been shown to contain most SPARQL queries asked in practical settings.

In this paper, we study tractable classes of weakly well-designed queries in parameterized complexity considering the equivalent formulation as pattern trees. We give a complete characterization of the tractable classes in the case without projection. Moreover, we show a characterization of all tractable classes of \textit{simple} well-designed pattern trees in the presence of projection.

1 Introduction

Driven by the increasing amount of RDF data available on the web, SPARQL—the standard query language for RDF—has received a lot of attention in the last decade. Many different aspects of SPARQL have been studied, for example its expressive power [40, 3, 41, 43], complexity of query evaluation and optimization [37, 42, 38, 12, 11, 33, 31, 39, 28, 10, 30, 13], semantic properties [8, 2, 9], and several more [5, 4, 6, 29, 7, 27, 21, 44]. One of the main features of SPARQL that attracted a lot of interest is the OPTIONAL operator that resembles the left outer join in the Relational Algebra. It allows users to define parts in a query for which an answer is returned if possible. However, in case that providing a complete answer including all optional parts is impossible, only a partial answer covering those parts of the query that can be answered is returned. Thus such partial answers are not lost.

This enables queries to retrieve meaningful answers even over incomplete information or information provided under schemas which users do not have a good understanding of. Given that both these characteristics—being incomplete and not well understood by all users—are part of the nature of web data, the OPTIONAL operator is an essential feature of SPARQL.

Thus research on SPARQL focused on the OPTIONAL operator [37, 31, 39, 26, 33, 12]. Unfortunately early research revealed that in some cases the semantics of the OPTIONAL-operator can be unintuitive and inconsistent with some principles of the semantic web [8]. To deal with this situation, the fragment of \textit{well-designed SPARQL queries} was introduced in [37] and intensively studied later on [31, 39, 26, 33, 12]. The definition of well-designed queries forbids certain patterns of variable distributions over OPTIONALs which turn out to be responsible for the unintuitive semantics. Forbidding them leads to a cleaner semantics for well-designed queries.

Regarding the evaluation of SPARQL queries, it was already shown in [37] that the problem is in general \textit{PSPACE}-complete in combined complexity, where the unintuitive
behavior of the OPTIONAL-operator was identified as the main culprit [12]. Since this behavior is absent in the well-designed fragment, as a side effect, the complexity of the evaluation problem drops to coNP-completeness for queries without projection [37], and \( \Sigma_2P \)-completeness in case projection is allowed [10].

Recently, a generalization of the well-designed queries called weakly well-designed queries [26] was proposed. The main motivation was the observation that only about half of the real-world queries on DBPedia are well-designed. Thus a more relaxed condition on the variables was proposed that covers most of the real-world queries while at the same time not increasing the complexity of query evaluation.

Despite the wealth of research efforts on these restricted classes of queries, only little work was done on actually identifying fragments of SPARQL containing the OPTIONAL-operator for which the evaluation problem is tractable. Some efforts in this direction include [10, 31]. However, all of these results deal only with well-designed queries. Moreover, they rely on the fact that well-designed queries can be seen as CQs extended by optional parts. As a consequence, their approach towards identifying tractable fragments is to investigate to what extend tractable classes of CQs can be applied to these queries. However, the exact limits of tractability have not been explored, yet.

The aim of our work is to close this gap and to provide a more thorough picture of tractable classes of SPARQL queries containing the OPTIONAL-operator. We study the complexity of query evaluation in the model of parameterized complexity where, as usual, we take the size of the query as the parameter. As already argued in [35], this model allows for a more fine-grained analysis than the classical perspectives: on the one hand data complexity which allows impractical algorithms in which the size of the query is considered as a constant and on the other hand combined complexity where the query is assumed to have a size similar to the database which often leads to overly pessimistic results. In parameterized complexity, query answering is considered tractable, formally in \( \text{FPT} \), if, after a preprocessing that only depends on the query, the actual evaluation can be done in polynomial time [22, 23]. This allows for potentially costly preprocessing on the generally small query while the dependency on the generally far bigger database is polynomial for an exponent independent of the query.

Parameterized complexity has found many applications in the complexity of query evaluation problems, see e.g. [25, 24, 34, 14].

We remark that for Boolean Conjunctive Queries (BCQs) of bounded arity, it was shown in seminal work of Grohe, Schwentick and Segoufin [25] and Grohe [24] that the tractable fragment in combined complexity and parameterized complexity coincide. That is, for every class of BCQs of bounded arity the evaluation problem is in \( \text{PTIME} \) if and only if it is in \( \text{FPT} \). In contrast, it is known that for well-designed SPARQL queries with projection this property does not hold. This follows from [30] where it was shown that there are classes of well-designed queries for which the evaluation problem is \( \text{NP} \)-hard, but fixed-parameter tractable. Thus the choice of the tractability notion makes a difference for the results.

To focus on the influence of the OPTIONAL-operator, we restrict ourselves to the \{AND,OPTIONAL\}-fragment of SPARQL, in particular leaving out unions and filters. To infer our results on these queries, we will in fact work in the framework of pattern trees that was originally introduced in [31] for data provided in RDF format and later extended to arbitrary relational vocabulary [10]. Intuitively, pattern trees represent the conjunctive parts of a query at the nodes of the tree while the tree-structure reflects the nesting of the OPTIONALs. Pattern trees constitute a query formalism of their own using the “depth-first approach” semantics suggested in [36]. Our main technical results are characterizations of the tractable classes of pattern trees in the setting without projection and of simple well-designed
pattern trees in the presence of projections.

From these results, one directly gets results for all fragments of \{\textbf{AND, OPTIONAL}\} SPARQL without projection for which the standard semantics and the depth-first semantics of \cite{36} coincide, as is the case for the (weakly) well-designed fragment \cite{31, 26}. This is so because for \{\textbf{AND, OPTIONAL}\} SPARQL queries one can easily compute corresponding pattern trees by essentially just syntactic transformation. These associated pattern trees can then be used to assess the complexity of the queries at hand. Our approach thus has the advantage that, in case further classes of SPARQL queries for which the two possible semantics coincide are discovered in the future, our tractability results immediately carry over.

**Summary of results and organisation of the paper.** We study the following decision problem: Given a pattern tree, a database, and a mapping, is the mapping a solution of the pattern tree over the database? After some preliminaries in Section 2 we will give the following results:

- **Tractable classes for an extension problem.** The semantics of weakly well-designed SPARQL queries is based on the idea of returning \textit{maximal mappings}. Intuitively, first the mandatory part of the query is mapped into the data, and then this partial mapping is extended as much as possible along the optional parts of the query. Thus, testing for extensions of partial solutions to a query is a central task in query evaluation. To formalize this problem, we introduce and analyze a problem called \textit{EXT} in Section 3. We then show that the tractable classes of \textit{EXT} are characterized by the treewidth of an auxiliary structure we call \textit{extension core}. This result will serve as an important building block in later sections and might be of interest in its own right.

- **A complete characterization of tractable classes of pattern trees without projection.** In Section 4 we study the evaluation of pattern trees without projection, i.e., all variables occurring in the query are also part of the result. Using the notions and results developed in Section 3 we provide a complete characterization of all tractable classes of both, pattern trees and weakly well-designed SPARQL queries.

- **A complete characterization of tractable classes of simple well-designed pattern trees with projection.** In Section 5 we study well-designed pattern trees with projection. For technical reasons that we discuss in the conclusion, we will restrict ourselves to simple pattern trees in this section, i.e., pattern trees where no two atoms share the same relation name. This can be seen as analyzing queries by their underlying “graph structure” similar to e.g. \cite{25, 14} while discarding the possibility of taking cores to simplify instances. Again, we provide a complete characterization of the tractable classes.

In Section 6 we discuss our results and potential extensions to conclude the paper.

## 2 Preliminaries

**Graphs.** We consider only undirected, simple graphs \( G = (V, E) \) with standard notations but sometimes write \( t \in G \) to refer to a node \( t \in V(G) \). A graph \( G_2 \) is a subgraph of a graph \( G_1 \) if \( V(G_2) \subseteq V(G_1) \) and \( E(G_2) \subseteq E(G_1) \). A tree is a connected, acyclic graph. A subtree is a connected, acyclic subgraph. A **rooted tree** \( T \) is a tree with one node \( r \in T \) marked as its root. Given two nodes \( t, \hat{t} \in T \), we say that \( \hat{t} \) is an ancestor of \( t \) if \( \hat{t} \) lies on the path from \( r \) to \( t \). Likewise, \( \hat{t} \) is the parent node of \( t \) (\( \hat{t} \) is a child of \( t \)) if \( \hat{t} \) is an ancestor of \( t \) and \( \{t, \hat{t}\} \in E(T) \). For a subtree \( T' \) of \( T \) that contains the root of \( T \), a node \( t \in T \) is a child of \( T' \) if \( t \not\in T' \) and \( \hat{t} \in T' \) for the parent node \( \hat{t} \) of \( t \). We write \( ch(T') \) for the set of all children of \( T' \).
A tree decomposition of a graph \( G = (V, E) \) is a pair \((T, \nu)\), where \( T \) is a tree and \( \nu : V(T) \to 2^V \), that satisfies the following: (1) For each \( u \in V \) the set \( \{ s \in V(T) \mid u \in \nu(s) \} \) is a connected subset of \( V(T) \), and (2) each edge of \( E \) is contained in one of the sets \( \nu(s) \), for \( s \in V(T) \). The width of \((T, \nu)\) is \( \text{max} \{ |\nu(s)| \mid s \in V(T) \} - 1 \). The treewidth of \( G \) is the minimum width of its tree decompositions.

Atoms and Conjunctive queries. We assume familiarity with the relational model, especially with the concept of conjunctive queries (CQs), and refer to [I] for details. In particular, we will heavily use the fact that a conjunctive query can alternatively be seen as a set \( \mathcal{A} \) of atoms on a database \( D \) or as a homomorphism problem between a structure \( A \) associated to these atoms in a canonical way and \( D \). In the following, we will switch between these perspectives whenever convenient. Sets of atoms will be denoted in calligraphic letters \( \mathcal{A}, \mathcal{B}, \ldots \) whereas structures will be denoted as \( A, B, \ldots \).

In the following, we fix some notation. As usual, a \( \sigma \)-structure \( A \) consists of a finite set \( A = \text{dom}(A) \) and a relation \( R^A \subseteq A^r \) for each relation symbol \( R \) of \( \sigma \), for each atom \( r \). For a set \( \mathcal{A} \) of atoms we denote by \( \text{var}(\mathcal{A}) \) the set of variables appearing in \( \mathcal{A} \). Similarly, for a mapping \( \mu \), we denote with \( \text{dom}(\mu) \) the set of elements on which \( \mu \) is defined. For a mapping \( \mu \) and a set of variables \( V \), \( \mu|_V \) is defined. For a mapping \( \mu \) and a set of variables \( V \), we use \( \mu|_V \) to denote the restriction of \( \mu \) to the variables in \( \text{dom}(\mu) \cap V \). We say that a mapping \( \mu \) is an extension of a mapping \( \nu \) if \( \mu|_{\text{dom}(\nu)} = \nu \). By slight abuse of notation, we use operators \( \cup, \cap, \setminus, \) also between sets \( V \), \( \nu \) and tuples \( \bar{v} \) of variables, like in \( V \setminus \nu \).

A homomorphism between two \( \sigma \)-structures \( A \) and \( B \) is a mapping \( \text{dom}(A) \to \text{dom}(B) \) that, for all \( R \in \sigma \), maps all tuples in \( R^A \) to tuples in \( R^B \). We write \( h : A \to B \) to denote a homomorphism \( h \) from \( A \) to \( B \). A minimal substructure \( A' \) of \( A \) such that there is a homomorphism \( A \to A' \) is called a core of \( A \). We recall that all cores of \( A \) are unique up to isomorphism and thus speak of the core of \( A \) which we denote by \( \text{core}(A) \).

For a structure \( A \) and a set \( A' \subseteq \text{dom}(A) \), we write \( \text{dom}(A) \setminus A' \) to denote the restriction of \( A \) to \( \text{dom}(A) \setminus A' \) (we provide a formal definition of this concepts in the appendix). For two structures \( A, B \), the structure \( A \setminus B \) contains the relations in \( A \) but not in \( B \). The treewidth of a set of atoms or a structure is the treewidth of the respective Gaifman graph.

We sometimes write CQs \( q \) as \( \text{Ans}(\bar{x}) \leftarrow B \), where the body \( B = \{ R_1(\bar{v}_1), \ldots, R_m(\bar{v}_m) \} \) is a set of atoms and \( \bar{x} \) are the free variables. A Boolean CQ (BCQ) is a CQ with no free variables. We define \( \text{var}(q) = \text{var}(B) \). The existential variables are implicitly given by \( \text{var}(B) \setminus \bar{x} \). The result \( q.D \) of \( q \) over a database \( D \) is the set of tuples \( \{ \mu(\bar{x}) \mid \mu : B \to D \} \).

Pattern trees (PTs). A pattern tree (short: PT) \( p \) over a relational schema \( \sigma \) is a tuple \((T, \lambda, \chi)\) where \( T \) is a rooted tree and \( \lambda \) maps each node \( t \in T \) to a set of relational atoms over \( \sigma \). The set \( \chi \) of variables denotes the free variables of the PT. We may write \((\sigma, T, \lambda, \chi)\) to emphasize that \( \sigma \) is the root node of \( T \).

For a PT \((T, \lambda, \chi)\) and a subtree \( T' \subseteq T \), we write \( \lambda(T') \) to denote the set \( \bigcup_{t \in V(T')} \lambda(t) \). We may write \( \text{var}(t) \) instead of \( \text{var}(\lambda(t)) \), and \( \text{var}(T') \) instead of \( \text{var}(\lambda(T')) \). We further define \( \text{fvar}(t) = \text{var}(t) \cap \chi \) and \( \text{evar}(t) = \text{var}(t) \setminus \chi \) as the free and existential variables in \( T' \), respectively. These definitions extend naturally to subtrees \( T' \) of \( T \). We call a PT \((T, \lambda, \chi)\) projection free if \( \chi = \text{var}(T) \). We may write \((T, \lambda)\) to emphasize a PT to be projection free.

We define the order \( t \prec t' \) among nodes \( t \in T \) as \( t_1 \prec t_2 \) if \( t_1 \) is visited before \( t_2 \) in a depth-first, left-to-right traversal of \( T \). Also, for \( t \in T \), and a (not necessarily proper) subtree \( T' \) of \( T \), let \( T'_{\prec t} \) be the subtree of \( T \) that contains all nodes \( t' \in T' \) with \( t' \prec t \).

Semantics of PTs. Evaluating a PT \( p \) with free variables \( \chi \) over a database \( D \) returns a set \( p.D \) of mappings \( \mu : V \to \text{dom}(D) \) with \( V \subseteq \chi \). Intuitively, the idea of the evaluation is to evaluate the root node first, resulting in a set of mappings. Then, in a top-down left-to-right traversal of the tree these mappings are extended as far as possible by the solutions.
at the different nodes. The semantics of \( p(D) \) is, however, usually defined by providing a characterization of the mappings generated by this idea of a “top-down evaluation”. We follow this approach and use the characterization of solutions for weakly well-designed pattern trees from \[24\] which also works as a definition for the semantics of arbitrary PTs studied here.

Definition 1 (\([26]\) pp-solution). For a PT \( p = ((T, r), \lambda) \) and a database \( D \), a mapping \( \mu: V \to \text{dom}(D) \) (with \( V \subseteq \text{var}(T) \)) is a potential partial solution (pp-solution) to \( p \) over \( D \) if there is a subtree \( T' \) of \( T \) containing \( r \) such that \( \mu: \lambda(T') \to D \).

Observe that if \( \mu \) is a pp-solution, then although this might be witnessed by different subtrees \( T' \), there exists a unique maximal such subtree \( T' \). We will denote it by \( T_\mu \). Also, for a mapping \( \mu \) and some node \( t \in T \), let \( \mu_{\approx t} \) denote the restriction \( \mu|_{\text{var}(T_{\approx t})} \).

Definition 2 (\( p(D) \)). Let \( p = (T, \lambda, \kappa) \) be a PT and \( p' = (T, \lambda) \) be the same but projection-free PT. A mapping \( \mu \) is in \( p'(D) \) if (1) \( \mu \) is a pp-solution to \( p' \) over \( D \), and (2) there exists no child node \( t' \) of \( T_\mu \) and homomorphism \( \mu': \lambda(t') \to D \) extending \( \mu_{\approx t'} \).

For PTs with projection, we have \( p(D) = \{ \mu|_X \mid \mu \in p'(D) \} \).

Example 3. Consider the PT \( p_1 \) in Figure 1. Intuitively it asks for tickets \( t \) and tries to assign seats to each ticket. In doing so, it first tries to find a seat in the ticket class (left child). If this is not possible, it tries to return any seat (right child). This reflects the intuitive semantics of pattern trees that nodes earlier in the order \( \prec \) are evaluated first.

Assume the database \( D = \{ \text{ticket}(1), \text{class}(1, E), \text{seatclass}(1, E), \text{seatclass}(2, F), \text{empty}(1), \text{empty}(2) \} \). The mapping \( \mu = \{ (x, 1), (s, 2), (c, F) \} \) is a pp-solution, as it maps the root and the second child node into \( D \), and these two nodes contain exactly the variables in \( \text{dom}(\mu_1) \). But \( \mu \) is not “maximal” according to Definition 2. When testing for an extension to the first child, we may not test \( \mu_1 \), since \( t \) is the only variable in \( \text{dom}(\mu) \) occurring in a node that precedes the first child in the order \( \prec \). Thus \( p_1(D) = \{ \{ (x, 1), (s, 1), (c, E) \} \} \).

Parameterized complexity. We only give a bare-bones introduction to parameterized complexity and refer the reader to \[20\] for more details. Let \( \Sigma \) be a finite alphabet. A parameterization of \( \Sigma^* \) is a polynomial time computable mapping \( \kappa: \Sigma^* \to N \). A parameterized problem over \( \Sigma \) is a pair \( (L, \kappa) \) where \( L \subseteq \Sigma^* \) and \( \kappa \) is a parameterization of \( \Sigma^* \). We refer to \( x \in \Sigma^* \) as the instances of a problem, and to the numbers \( \kappa(x) \) as the parameters.

A parameterized problem \( E = (L, \kappa) \) belongs to the class \( \text{FPT} \) of fixed-parameter tractable problems if there is an algorithm \( A \) deciding \( L \), a polynomial \( p \), and a computable function \( f: N \to N \) such that the running time of \( A \) is at most \( f(\kappa(x)) \cdot p(|x|) \).

In this paper, for classes \( P \) of PTs, we study the problem \( p\text{-EVAL}(P) \) defined below. We will also use the evaluation problem \( p\text{-CQ-EVAL}(Q) \) for classes \( Q \) of CQs.
Let $E = (L, \kappa)$ and $E' = (L', \kappa')$ be parameterized problems. An \emph{FPT-Turing reduction} from $E$ to $E'$ is an algorithm deciding $E$ with runtime at most $f(\kappa(x)) \cdot p(|x|)$ for a computable function $f$ and a polynomial $p$ where the algorithm can make calls to an oracle for $E'$ such that every oracle query $y$ satisfies $\kappa'(y) \leq f(\kappa(x))$. An FPT-Turing reduction is called a \emph{many-one reduction}, if it makes only one oracle call and accepts if and only if the answer to oracle call is positive. Unless stated otherwise, all our FPT-reductions will be many-one.

Of the rich parameterized hardness theory, we will only use the class $W[1]$ of parameterized problems. To keep this introduction short, we only define a parameterized problem $(L, \kappa)$ to be $W[1]$-hard if there is a problem $(L', \kappa')$ that reduces to $(L, \kappa)$. It is generally conjectured that FPT $\neq W[1]$ and thus in particular $W[1]$-hard problems are not in FPT. We will take the hardness results on the problems $(L', \kappa')$ from the literature, often from the following result:

\begin{itemize}
\item \textbf{Theorem 4} (\cite{24}). Let $Q$ be a decidable class of Boolean CQs. If there is a constant $c$ such that the treewidth of the core of each CQ in $Q$ is bounded by $c$, then $p$-CQ-EVAL$(Q)$ is $W[1]$-hard.
\end{itemize}

We will also study HOM($\mathcal{C}$) and p-Hom($\mathcal{C}$) for a class $\mathcal{C}$ of structures. The input of both problems consists of a structure $A \in \mathcal{C}$ and a structure $B$, and the question is whether there exists a homomorphism $h: A \rightarrow B$. For p-Hom($\mathcal{C}$), the parameter is the size of $A$.

## 3 The Extension Problem

Definition\cite{2} shows that there are two potential sources of complexity for the evaluation problem: First, to determine whether some mapping is a pp-solution, and second to check if a pp-solution is “maximal”. As we will discuss in the next section, for projection free PTs, the test for a pp-solution is easy. Hardness is thus exclusively due to testing maximality.

This problem is closely related to the homomorphism problem, and can be easily reduced to it. However, done naively, this reduction loses the information about the parts of the pattern tree already mapped into the database, which might result in the reduction of an easy instance to a hard instance of the homomorphism problem.

Thus, for a class $\mathcal{C}$ of pairs of structure, in this section we study the following problem.

\begin{itemize}
\item \textbf{EXT($\mathcal{C}$)}
\end{itemize}

\begin{itemize}
\item \textbf{INPUT:} A pair $(A, B) \in \mathcal{P}$ of structures, a structure $C$, and a homomorphism $h: A \rightarrow C$.
\item \textbf{QUESTION:} Exists a homomorphism $h': B \rightarrow C$ compatible with $h$?
\end{itemize}

The problem p-EXT($\mathcal{C}$) is the problem EXT($\mathcal{C}$) parameterized by the size of $(A, B)$.

As mentioned earlier, the main difference between HOM and EXT is that in addition to a structure, the input of EXT gets another structure and a homomorphism that is already guaranteed to map this additional structure into the target. When looking for tractable classes, this additional input has to be taken into account.

To do so, we introduce the idea of the \emph{extension core}. Towards its definition, for a set of elements $A$, let $S_A$ be the $\{R_a \mid a \in A\}$-structure (where each $R_a$ is a unique relation symbol) with $\text{dom}(S_A) = A$ and $R^S_A = \{(a)\}$.
Definition 5 (Extension Core). Let \((A, B)\) be a pair of structures. The extension core \(\text{extcore}(A, B)\) is the structure 
\[
\text{extcore}(A, B) = (\text{core}(A \cup B \cup S_{\text{dom}(A)}) \setminus S_{\text{dom}(A)}) \setminus \text{dom}(A).
\]

Said differently, the extension core is constructed by introducing a new relation for every domain element in \(A\) and then computing the core, the extension core accounts on the one hand for the possibility that parts of \(B\) might be folded into \(A\) (and thus the extension of the homomorphism to these parts is guaranteed), and on the other hand for the fact that the mapping on \(\text{dom}(A)\) is fixed. Removing \(\text{dom}(A)\) is then possible because the mapping is already provided for these values.

The notion of the extension core allows us to formulate an exact characterization of the tractable classes \(C\) of the extension problem \(\text{EXT}(C)\). To this end, we define the treewidth of \(\text{extcore}(C)\) to be the maximum of the treewidth of \(\text{extcore}(A, B)\) for \((A, B) \in C\) if this maximum exists and \(\infty\) otherwise.

Theorem 6. Assume that \(\text{FPT} \neq \text{W}[1]\) and let \(C\) be a decidable class of pairs of structures. Then the following statements are equivalent:
1. The treewidth of \(\text{extcore}(C)\) is bounded by a constant.
2. The problem \(\text{EXT}(C)\) is in \(\text{PTIME}\).
3. The problem \(\text{p-EXT}(C)\) is in \(\text{FPT}\).

The theorem is shown using a sequence of lemmas. However, before the first lemma, we state an easy but important observation that we use tacitly throughout this section.

Observation 7. Extension cores are unique up to isomorphism.

For any two structures \(A, B\), we have \(\text{core}(\text{extcore}(A, B)) = \text{extcore}(A, B)\)

The first result describes a crucial property of extension cores that will be used several times throughout the remainder of this section.

Lemma 8. An instance \((A, B), D, h\) of \(\text{EXT}\) is a positive instance of \(\text{EXT}\) if and only if there exists a homomorphism \(h'\): \((A \cup S) \rightarrow D\) that extends \(h\), where \(S\) is the structure 
\[
\text{core}(A \cup B \cup S_{\text{dom}(A)}) \setminus S_{\text{dom}(A)}
\]
from the definition of extension cores.

Proof. Solving the instance \((A, B), D, h\) of \(\text{EXT}\) is equivalent to solving the instance \(((A \cup B \cup S_{\text{dom}(A)}), (D \cup h(S_{\text{dom}(A)})))\) of \(\text{HOM}\) (where \(h(S_{\text{dom}(A)})\) denotes the structure \(S_{\text{dom}(A)}\) where all elements \(a \in \text{dom}(h)\) are replaced by \(h(a)\)) which in turn is equivalent to deciding the existence of \(h'\): \((A \cup S) \rightarrow D\) extending \(h\).

Next, we show the positive result, i.e. that the problem \(\text{EXT}(C)\) can be solved efficiently if the treewidth of the extension cores in \(C\) is bound.

Lemma 9. Let \(C\) be a class of pairs of structures such that the treewidth of \(\text{extcore}(A, B)\) for all \((A, B) \in C\) is bounded by some constant \(c\). Then \(\text{EXT}(C)\) is in \(\text{PTIME}\).

Proof (sketch). Given an instance \((A, B), D, h\) of \(\text{EXT}(C)\), we know from Lemma 8 that we can equivalently solve the problem whether there is an extension \(h'\): \((A \cup S) \rightarrow D\) of \(h\). This in turn can be shown to be equivalent to deciding an instance \((L, T)\) of \(\text{HOM}\), that we can derive from \(((A \cup S), D)\) (intuitively by replacing “variables” in \((A \cup S)\) according to \(h\) by “constants” from \(\text{dom}(D)\) and then reducing the resulting structure). It can then be shown that when applying the same transformation (replacing elements according to \(h\) and reducing the structure) to the pair \((\text{extcore}(A, B), D)\), the resulting structure is identical to \((L, T)\). Since \(\text{extcore}(A, B)\) has bounded treewidth and this transformation does not increase
the treewidth, it follows that the treewidth of $L$ is bounded, and thus the instance $(L, T)$ can be solved in polynomial time \[^{13}\]. Finally, since all transformations can be computed in polynomial time as well, we get the desired result.

The next result shows that the above lemma is optimal by using the characterization of tractable classes (for both, PTIME and FPT) of p-Hom($C$) provided in \[^{24}\].

**Lemma 10.** Let $C$ be a decidable class of pairs of structures and let $\text{extcore}(C)$ be the class of extension cores of the pairs in $C$. Then $p\text{-Hom}(\text{extcore}(C)) \subseteq_{\text{FPT}} p\text{-EXT}(C)$.

**Proof (idea).** Given an instance $(L, R)$ of $p\text{-Hom}(\text{extcore}(C))$, first compute a pair $(A, B) \in C$ such that $\text{extcore}(A, B) = L$. Next, define a structure $D$ and a homomorphism $h: A \rightarrow D$ such that for $S = \text{core}(A \cup B \cup S_{\text{dom}(A)}) \setminus S_{\text{dom}(A)}$ the following holds: There exists a homomorphism $h': (A \cup S) \rightarrow D$ extending $h$ if and only if there exists a homomorphism $h: L \rightarrow R$. By Lemma \[^{8}\] this implies that $(A, B), D, h$ is a positive instance of $\text{EXT}(C)$ if and only if $(L, R)$ is a positive instance of $p\text{-Hom}(\text{extcore}(C))$ and thus proves the case. \[\blacktriangle\]

Theorem \[^{6}\] now follows immediately. $(1) \Rightarrow (2)$ follows from Lemma \[^{9}\]. The implication $(2) \Rightarrow (3)$ follows immediately. Finally, if the treewidth of $\text{extcore}(P)$ is not bounded, then $p\text{-Hom}(\text{extcore}(C))$ is not in FPT by \[^{24}\]. Thus, by Lemma \[^{10}\] the problem $p\text{-EXT}(P)$ is not in FPT, which shows $(3) \Rightarrow (1)$.

We will make use of $\text{EXT}$ and extension cores throughout the paper, but usually for sets of atoms. In this case, we implicitly assume their common representation as structures.

## 4 Projection Free Pattern Trees

We start with our investigation of PTs by looking at projection-free PTs. As already mentioned at the beginning of the previous section, given a PT $p = ((T, r), \lambda)$, a database $D$, and a mapping $\mu$, deciding whether $\mu$ is a pp-solution is feasible in polynomial time. The algorithm could proceed as follows: First it identifies the set $N = \{ t \in T \mid \text{var}(t) \subseteq \text{dom}(\mu) \text{ and } \mu(\tau) \in D \text{ for all } \tau \in \lambda(t) \}$. Now if $r \notin N$, then clearly $\mu \notin p(D)$. Otherwise let $T'$ be the maximal subtree of $T$ that contains $r$ and is built exclusively from nodes in $N$. Then $\mu$ is a pp-solution if and only if $\text{var}(T') = \text{dom}(\mu)$.

Consequently, the coNP-hardness of deciding $\mu \in p(D)$ originates exclusively from testing whether $\mu$ can be extended. Essentially, the reason for this test being hard is that it can be the same as any homomorphism test. However, testing the possibility of extending a mapping to a single node being hard does not necessarily make the complete problem of testing the existence of any extension hard as well, as illustrated by the following example.

**Example 11.** Consider the PT $p_2$ from Figure \[^{1}\]. Let $D$ be some database containing at least $a(1)$ and assume the mapping $\mu = \{(x, 1)\}$. Then $\mu$ clearly is a pp-solution. In this case testing whether $\mu \in p(D)$ boils down to deciding whether there exists either a mapping $\mu_1: \lambda(t_1) \rightarrow D$ or a mapping $\mu_2: \lambda(t_2) \rightarrow D$. Deciding the existence of $\mu_1$ is as hard deciding whether $D$ contains a clique of size $n$. However, observe the homomorphism $h: \lambda(t_2) \rightarrow \lambda(t_1)$. Thus, whenever $\mu_1$ exists, $\mu_2$ exists as well. As a result, testing for the existence of a $\mu_1$ is not necessary. Instead, the easy test for $\mu_2$ is sufficient.

We formalize the observation of Example \[^{11}\] by identifying for each subtree of a PT exactly those child nodes that potentially have to be tested for an extension. Let $p = (T, \lambda)$ be a PT and $T'$ a subtree of $T$. To start with, consider a set $C(T')$ of pairs of sets
of atoms \( C(T') = \{ (\lambda(T'_\text{ext}), \lambda(t_i)) \mid t_i \in ch(T') \} \). From this set of initial candidates, we eliminate redundant pairs as illustrated in Example 11. That is, if there are two pairs \((T_i, \lambda(t_i)), (T_j, \lambda(t_j)) \in C(T')\) with \(t_i \neq t_j\) such that there exists a homomorphism
\[
h: T_i \cup \lambda(t_i) \to T_j \cup \lambda(t_j)
\]
with \(h|_{\text{var}(T)}(x) = x\) for all \(x \in \text{var}(T)\), remove \((T_j, \lambda(t_j))\) from \(C(T')\). Once there are no more such pairs, we denote the resulting set \(C(T')\) as \(\text{csts}(T', T)\), and refer to its elements as \textit{critical subtrees}.

Note that the construction of \(\text{csts}(T')\) is not deterministic because in each elimination step, there might be several choices for elements to remove. Most of these choices lead to the same result, since the composition of two homomorphisms is again a homomorphism. However, when there are mutual homomorphisms between two pairs in \(C(T')\), then different choices may lead to different results. Considering all possible elimination sequences, we thus get a set \(\text{CST}(T') = \{ \text{csts}_1(T'), \ldots, \text{csts}_i(T') \} \) of sets of critical subtrees. As we will see, for our purposes all sets of critical subtrees in \(\text{CST}(T')\) are equivalent.

The first observation is a direct consequence of the definition of the sets \(\text{csts}_i(T')\).

\begin{itemize}
  \item \textbf{Observation 12.} For every child node \(t \in ch(T')\) and every \(\text{csts}_i(T')\), either there is a pair \((T, \lambda(t)) \in \text{csts}_i(T')\), or there is a node \(t' \in ch(T')\) such that \((T', \lambda(t')) \in \text{csts}_i(T')\) and there exists a homomorphism \(h: T \cup \lambda(t) \to T' \cup \lambda(t')\) such that \(h|_{\text{var}(T)}\) is the identity mapping.
\end{itemize}

The next property holds because the only reason for \(\text{CST}(T')\) containing more than one element are mutual homomorphisms between two potential candidates. This implies that both extension cores have the same treewidth. Recall that the treewidth of \(\text{extcore}(\text{csts}_i(T'))\) is the maximal treewidth of \(\text{extcore}(\lambda(T'_\text{ext}), \lambda(t))\) over \((\lambda(T'_\text{ext}), \lambda(t)) \in \text{csts}_i(T')\).

\begin{itemize}
  \item \textbf{Proposition 13.} Let \((T, \lambda, \mathcal{X})\) be a PT, \(T'\) a subtree of \(T\), and \(\text{csts}_i(T'), \text{csts}_j(T') \in \text{CST}(T')\). Then, for any \(c \geq 1\), the treewidth of \(\text{extcore}(\text{csts}_i(T'))\) is less or equal than \(c\) if and only if the treewidth of \(\text{extcore}(\text{csts}_j(T'))\) is less or equal than \(c\).
\end{itemize}

Thus in the following we ignore the ambiguities in the construction of the set of critical subtrees and let \(\text{csts}(T')\) be the result of an arbitrary run of the construction algorithm. All our results are invariant under this choice. Finally, for a PT \(p = (T, \lambda)\), we define \(\text{csts}(p) = \bigcup_{T' \text{ subtree of } T} \text{csts}(T')\), and for a class \(\mathcal{P}\) of pattern trees \(\text{csts}(\mathcal{P}) = \bigcup_{p \in \mathcal{P}} \text{csts}(p)\).

\begin{itemize}
  \item \textbf{Theorem 14.} For every decidable class \(\mathcal{P}\) of projection free PTs, the problems \(\text{EVAL}(\mathcal{P})\) and \(\text{EXT}(\text{csts}(\mathcal{P}))\) are equivalent under \(\text{FPT-Turing}\) reductions.
\end{itemize}

\begin{itemize}
  \item \textbf{Proof.} The reduction of \(\text{EVAL}(\mathcal{P})\) to \(\text{EXT}(\text{csts}(\mathcal{P}))\) is given by the following algorithm to decide \(\mu \in p(D)\) for a mapping \(\mu\), a database \(D\), and a PT \(p = (T, \lambda)\): First, decide whether \(\mu\) is a pp-solution. If this is the case, compute \(T'_\mu\) and \(\text{csts}(T'_\mu)\), otherwise return “no”. Second, for every \((T, \lambda(t)) \in \text{csts}(T'_\mu)\), call an oracle for \(\text{EXT}\) on instance the \((T, \lambda(t)), D, \mu\). If any of these calls returns “yes” \(\mu\) is not maximal and thus we answer “no”, otherwise we return “yes”. The correctness of this algorithm follows from Definition 2 and the discussion on critical subtrees and Observation 12.
\end{itemize}

Next we show that \(\text{EXT}(\text{csts}(\mathcal{P}))\) reduces to \(\text{EVAL}(\mathcal{P})\). Let \(\mathcal{P}\) be a class of PTs, and let \((A, B) \in \text{csts}(\mathcal{P})\). Moreover, let \(C\) be a structure over the same vocabulary \(\sigma\) as \(A \cup B\), and let \(h: A \to C\) be a homomorphism. Thus, \((A, B), C, h\) is an instance of \(\text{EXT}(\text{csts}(\mathcal{P}))\). We will show how to check whether \(h\) can be extended to a homomorphism \(A \cup B \to C\) with the help of an oracle for \(\text{EVAL}(\mathcal{P})\). Towards this goal, we first find a projection free PT \(p = (T, \lambda)\) with a subtree \(T'\) of \(T\) and a child node \(t \in ch(T')\) such that \(\lambda(T'_t) \simeq A\), \(\lambda(t) \simeq B\), and such that \((\lambda(T'_t), \lambda(t)) \in \text{csts}(T')\) (recall that we assume the implicit translation between sets of atoms and structures, indicated by \(\simeq\)). By definition, such a combination exists, and because \(\mathcal{P}\) is computable there is an algorithm to construct it.
Next, we construct a new \( \sigma \)-structure \( D \) with \( \text{dom}(D) = \text{dom}(C) \times (\text{dom}(A) \cup \text{dom}(B)) \): for every relation symbol \( R \in \sigma \), the structure \( D \) contains the interpretation

\[
R^D = \{(c_1, b_1), \ldots, (c_\ell, b_\ell) \mid (c_1, \ldots, c_\ell) \in R^C \text{ and } (b_1, \ldots, b_\ell) \in R^{A \cup B}\}.
\]

Note that two homomorphisms \( h_C \colon (A \cup B) \to C \) and \( h_A \colon (A \cup B) \to (A \cup B) \) can be combined to a homomorphism \( h_{C \times A} \colon (A \cup B) \to D \) and that every homomorphism \( h_{i} \colon (A \cup B) \to D \) has such a representation as a product. Clearly, constructing \( D \) is in \( \text{FPT} \).

We claim that \( h \) can be extended to a homomorphism \( h' \colon (A \cup B) \to C \) if and only if the mapping \( \mu \) defined as \( \mu = h \times \text{id} \) (where \( \text{id} \) is the identity mapping on \( \text{dom}(A \cup B) \) is not an answer to \( p \) on \( D = D \), i.e. if \( \mu \notin p(D) \).

To prove the claim, first assume that \( \mu \notin p(D) \). Observe that \( \mu = \mu_{\text{ext}} \). Then by Definition 2 the mapping \( \mu \) cannot be extended to a mapping \( \mu' \) such that \( \mu'(\tau) \in D \) for every atom \( \tau \in \lambda(t) \). But since \( \text{id} \colon (A \cup B) \to (A \cup B) \) is a homomorphism, the second component of \( \mu \) could be extended to \( \lambda(t) = B \). Hence the only reason why there is no such extension of \( \mu \) is because of the first component. It thus follows that there cannot be a homomorphism \( h' \colon (A \cup B) \to C \) that extends \( h \). This completes this direction of the proof.

Next assume that \( \mu \notin p(D) \). Clearly, \( \mu(\tau) \in D \) holds for all atoms \( \tau \in \lambda(T') \subseteq A \). Consequently, there must be a node \( t' \in \text{ch}(T') \) such that there exists an extension \( \mu_{\text{ext}} \) of \( \mu_{\text{ext}} \) with \( \mu_{\text{ext}}(\tau) \in D \) for all atoms \( \tau \in \lambda(T') \). We claim that \( t' \) must in fact be \( t \). To see this, towards a contradiction, assume that \( t' \neq t \), and that there exists such an extension \( \mu_{\text{ext}} \). Then \( \mu'_{\text{ext}} \) decomposes into homomorphisms \( \mu'_{\text{ext}} = h_C \times h_A \). Now \( h_A \) is a homomorphism \( h_A \colon (T'_{\text{ext}}) \cup \lambda(t') \to (T') \cup \lambda(t) = A \cup B \) that is the identity on \( \text{var}(T'_{\text{ext}}) \). This gives the desired contradiction, since the existence of \( h_A \) would have lead to the elimination of \( (\lambda(T'), \lambda(t)) \) from \( \text{csts}(T') \). Thus \( t = t' \) and there exists an extension \( \mu' \) of \( \mu \) with \( \mu'(\tau) \in D \) for each atom \( \tau \in \lambda(t) \). Again, \( \mu' \) decomposes into homomorphisms \( \mu' = h_C \times h_A \), and we have that \( h_C \colon (A \cup B) \to C \) is a homomorphism that extends \( h \).

Combining the above results, we thus get the following dichotomy for projection-free PTs.

**Theorem 15.** Let \( \mathcal{P} \) be a decidable class of without projections and assume \( \text{FPT} \neq \text{W}[1] \). Then \( \text{p-Eval}(\mathcal{P}) \) is in \( \text{FPT} \) if and only if the treewidth of \( \text{extcore}(\lambda(T'), \lambda(t)) \) for all \( (\lambda(T'), \lambda(t)) \in \text{csts}(\mathcal{P}) \) is bounded by a constant.

### 5 Pattern Trees with Projection

We now turn towards well-designed pattern trees with projection. As already mentioned in the Introduction, for technical reasons to be discussed in Section 6, in addition to the restriction to well-designed pattern trees, we also restrict our study to simple pattern trees. However, most of the tractability results presented in this section also hold for non-simple well-designed pattern trees. Even more, all tractability results can be extended to arbitrary PTs. The restriction to well-designed pattern trees is because we cannot characterize the tractable classes of simple non-well-designed pattern trees yet.

We start by defining simple pattern trees, and then introduce well-designed pattern trees.

**Definition 16 (Simple PTs).** A PT \( p = (T, \lambda, \mathcal{X}) \) is a simple pattern tree if at the one hand no atom \( R(\vec{v}) \) occurs in \( \lambda(t) \) and \( \lambda(t') \) for \( t \neq t' \in T \), and on the other hand there are no two atoms \( R_i(\vec{v}_i), R_j(\vec{v}_j) \in \lambda(T) \) with \( R_i = R_j \).

Well-designed pattern trees (wdPTs) restrict the distribution of variables among the nodes of a pattern tree.
Definition 17 (Well-Designed Pattern Tree (wdPT)). A PT \( (T, \lambda, \mathcal{X}) \) is well-designed if for every variable \( y \in \text{var}(T) \), the set of nodes of \( T \) where \( y \) appears is connected.

An immediate consequence of this definition is that for every variable \( y \in \text{var}(T) \), there exists a unique node \( t \in T \) containing \( y \) such that all nodes \( t' \in T \) with \( y \in \text{var}(t') \) are descendants of \( t \). Because of this, the semantics of wdPTs can be described by reformulating Definition 2 in a much simpler way. That is, given a wdPT \( p = (T, \lambda, \mathcal{X}) \) and a database \( D \), a pp-solution \( \mu \) is in \( p(D) \) if and only if there exists no child node \( t' \) of \( T_\mu \) and homomorphism \( \mu' : \lambda(t') \rightarrow D \) extending \( \mu \).

The main result of this section will be a complete characterization of the tractable classes of simple wdPTs. However, before we are able to do so, we first need to introduce yet another property of (nodes of) PTs. Pattern trees may contain nodes that are irrelevant for query answering. Instead of resorting to normal forms to exclude pattern trees containing such nodes like in [31], here we may assume wdPTs to contain such nodes.

In order to correctly characterize tractability, we must thus be able to identify such nodes to exclude them from our tractability considerations. We formalize this concept of nodes potentially influencing the result of a simple wdPT after we introduced one more piece of notation necessary for the definition. Let \( (T, r) \) be a rooted tree and \( t \in T \). Then \( \text{branch}(t) \) denotes the set of nodes on the path from \( r \) to the parent node of \( t \). Moreover, \( \text{cbranch}(t) = \text{branch}(t) \cup \{t\} \).

Definition 18 (Relevant Nodes). Let \( p = ((T, r), \lambda, \mathcal{X}) \) be a simple wdPT and \( t \in T \). Then node \( t \in T \) is relevant if and only if \( \text{var}(T') \setminus \text{var}(\text{branch}(t)) \neq \emptyset \) where \( T' \) is the subtree of \( T \) rooted in \( t \). We use \( \text{relv}(T) \) to denote the set of relevant nodes in \( T \).

It follows immediately from [31] that this efficiently decidable notion indeed captures the intended meaning.

Proposition 19. Let \( p = (T, \lambda, \mathcal{X}) \) be a simple wdPT. A node \( t \in T \) is relevant if and only if there exists a database \( D \) such that \( p(D) \neq p'(D) \) where \( p' \) is retrieved by removing the subtree of \( p \) rooted in \( t \).

With this notion settled, we can now aim towards our main result. The overall idea of our algorithm for \( p\text{-Eval}(P) \) can be described as follows. Given a wdPT \( p \), a database \( D \), and a mapping \( \mu \), we construct a set of CQs \( q \) and associated databases \( D' \) such that \( \mu \in p(D) \) if and only if for at least one of these CQs \( q \) the tuple \( \mu(\vec{x}) \) (where \( \vec{x} \subseteq \text{dom}(\mu) \)) are the free variables of \( q \) is in \( q(D') \). We will define three tractability criteria that ensure this algorithm to be in \( \text{FPT} \). Intuitively, the third tractability condition guarantees that deciding \( \mu(\vec{x}) \in q(D') \) is in \( \text{PTIME} \). The second condition guarantees the size of each relation in \( D' \) to be polynomially bounded in the size of the input, and the first condition guarantees that \( D' \) can be computed efficiently. Below, we will first introduce these three tractability criteria and show that each of them is indeed necessary. Afterwards we show that they are also sufficient by describing the \( \text{FPT} \) algorithm.

Note that some of the following notions and definitions are based on ideas and similar notions in [10] and [30]. However, many of them are refined carefully to provide a far more fine-grained complexity analysis. Note further that the tractability results also work on wdPTs that are not simple, and only the lower bounds hold for simple queries only.

One important property of PTs that influences all three tractability criteria are the (number and distribution of) variables that occur in more than one node.

Definition 20 ([10] Interface \( I(t, t') \)). Let \( (T, \lambda, \mathcal{X}) \) be a wdPT, \( t, t' \in T \), and \( T' \) a subtree of \( T \). The interface \( I(t, t') \) of \( t \) and \( t' \) is the set \( I(t, t') = \text{var}(t) \cap \text{var}(t') \).
While the size of the interface $I(t, t')$ (i.e., the number of variables in each interface) for all pairs of nodes $t, t'$ can be used for the definition of tractable classes (cf. [10]), this restriction is quite coarse. To provide a more fine grained tractability criteria, we first recall the notion of an $S$-component from [19]: Let $G = (V, E)$ be a graph, and $S \subseteq V$. Then let $C$ be the set of connected components of $G[V \setminus S]$, and for each $C \in C$, let $S_C \subseteq S$ be the set of nodes in $S$ that have (in $G$) an edge to some node in $C$. I.e., $S_C = \{ v \mid (v, v') \in E \text{ for some } v' \in C \}$. The set $S$ of $S$-components of $G$ now is the set $\{G[C \cup S_C] \mid C \in C\}$.

For a set $S$ of variables, the notion of $S$-components extends to sets of atoms in the obvious way via the Gaifman graph. We will thus talk about $S$-components of sets of atoms.

**Definition 21 (Interface Components).** Let $p = ((T, r), \lambda, \mathcal{X})$ be a wdPT, $t \in T$ a node of $T$ (but not the root node), and $\hat{t}$ the parent node of $t$. The set of **interface components** $I_C(t)$ of $t$ is a set of set of atoms, defined as the union of:

1. The set $\{\{\tau\} \mid \tau \in \lambda(t) \text{ and } \text{var}(\tau) \subseteq I(t, \hat{t})\}$ consisting of singleton sets for every atom $\tau \in \lambda(t)$ which contains only “interface variables”, i.e. variables from $I(t, \hat{t})$.
2. The set of all $I(t, \hat{t})$-components of $\lambda(t)$.

Hence, interface components of “type (1)” are sets of single atoms, while interface components of “type (2)” are sets of possibly several atoms.

To understand why interface components are essential for our results, recall that solutions to wdPTs must be “maximal” (it must not be possible to extended the mapping to some node). Now a mapping cannot be extended to some node, if and only if it cannot be extended to any one of its interface components. Thus instead of testing the complete node at once (which might be intractable), we test the maximality of a mapping component by component (which might be tractable).

For each interface component, in general we are especially interested in the existential interface variables occurring in it. For a wdPT $(T, \lambda, \mathcal{X})$ and a node $t \in T$ with parent node $\hat{t}$, we therefore define the **inherited variables of an interface component** $S \in I_C(t)$ as the set $\text{var}_I(S) = (I(t, \hat{t}) \cap \text{var}(S)) \setminus \mathcal{X}$.

However, just for the first tractability condition the free variables are actually of interest. Thus, for $((T, r), \lambda, \mathcal{X}), t, \hat{t}$, and $S$ as before, let $\text{var}_I^+(S) = \text{var}_I(S) \cup (\text{var}(\hat{t}) \cap \text{var}(S))$. Also, for a set $\mathcal{V}$ of variables, recall the definition of the structure $S_{\mathcal{V}}$ from Section 3.

**Tractability condition (a):** There is a constant $c$, such that for each $p = ((T, r), \lambda, \mathcal{X}) \in \mathcal{P}$, the treewidth of $\text{extcore}(S_{\mathcal{V}_I^+(S), S})$ is bounded by $c$ for all relevant nodes $t \in T$ (with $t \neq r$) and all $S \in I_C(t)$.

Intuitively, condition (a) guarantees that for each such interface component, given some mapping $\mu$ on (a subset of) the free variables plus a mapping on the inherited variables of this interface component, deciding whether the mapping can be extended in such a way that all atoms of the interface component are mapped into some database is in polynomial time.

Next, we show that tractability condition (a) is indeed necessary.

**Lemma 22.** Let $\mathcal{P}$ be a decidable class of simple wdPTs such that tractability condition (a) is not satisfied. Then $p\text{-Eval}(\mathcal{P})$ is coW[1]-hard.

**Proof.** For a well-designed pattern tree $p$, let the **relevant component set** $\text{rcs}(p)$ contain all the pairs $(S_{\mathcal{V}_I^+(S), S})$ as defined in tractability condition (a). Moreover, for a class $\mathcal{C}$ of pattern trees, let $\text{rcs}(\mathcal{C}) = \bigcup_{p \in \mathcal{C}} \text{rcs}(p)$. We will — by an FPT-reduction — reduce $\text{Hom}(\text{extcore}(\text{rcs}(\mathcal{P})))$ to the co-problem of $\text{Eval}(\mathcal{P})$. The result thus follows from [24], since $\text{extcore}(\text{rcs}(\mathcal{P})) = \text{core}(\text{extcore}(\text{rcs}(\mathcal{P})))$ does not have bounded treewidth by assumption.
Thus, assume an instance \((E,F)\) of \(\text{HOM}(\text{extcore}(\text{rCs}(\mathcal{P})))\). First of all, find a wdPT \(p = ((T,\tau),\lambda,\mathcal{X}) \in \mathcal{P}\) with a relevant node \(t \in T\) with \(t \neq \tau\) and an interface component \(S \in \mathcal{I}\mathcal{C}_t\) such that \(E = \text{extcore}(S_{\mathcal{V}_t(S)})\). They exist by definition and since \(\mathcal{P}\) is decidable, finding them is in FPT.

Since \(t\) is a relevant node, there exists at least one node \(t' \in T\) with \(\text{fvar}(t') \setminus \text{fvar}(\text{branch}(t')) \neq \emptyset\) such that either \(t = t'\) or \(t'\) is a descendant of \(t\). Among all nodes satisfying this property, pick \(t'\) to be the node with the minimal distance to \(t\).

We define the set \(N\) of nodes as follows. If \(t = t'\), then \(N = \emptyset\), otherwise set \(N = c\text{branch}(t') \setminus c\text{branch}(t)\).

We define a database \(D\) over the set of relational symbols in \(p\) as follows. First, \(\text{dom}(D) = \text{dom}(F) \cup \{d\}\), where \(d\) is a fresh value not occurring in \(\text{dom}(F)\). The relations in \(D\) contain the following tuples:

- For each relation symbol \(R\) mentioned outside of \(\lambda(c\text{branch}(t'))\) set \(R^D = \emptyset\).
- For each relation symbol \(R\) mentioned in \(\lambda(\text{branch}(t))\), let \(R^D\) contain the single tuple \((d,\ldots,d)\).
- For each relation symbol \(R\) mentioned in \(\lambda(c\text{branch}(t') \setminus \text{branch}(t)) \setminus S\), let \(R^D\) contain all possible tuples \((a_1,\ldots,a_k) \in \text{dom}(F) \cup \{d\}\).
- For each relation symbol \(R\) mentioned in \(S\), observe that there exists a relation symbol \(R'\) in the vocabulary of \(E\) that was derived from \(R\) during the computation of the extension core. That is, the arity of \(R'\) is less or equal than the arity of \(R\). Let \(k\) be the arity of \(R\), let \(\{i_1,\ldots,i_k\} \subseteq \{1,\ldots,k\}\) be those positions of \(R\) containing values from \(\mathcal{V}_t(S)\), and \(\{a_1,\ldots,a_m\} = \{1,\ldots,k\} \setminus \{i_1,\ldots,i_k\}\) those positions of \(R\) that contain values from \(\text{var}(S) \setminus \mathcal{V}_t(S)\). Then, for every \((a_1,\ldots,a_m) \in (R')^P\), let \(R^D\) contain all tuples \((a_1,\ldots,a_k)\) with \((a_0,\ldots,a_{m+1}) = (d,\ldots,d)\), i.e. we extend \((a_1,\ldots,a_m) \in (R')^T\) by assigning \(d\) to the missing positions.

Finally, we define the last part of the instance of \(\text{EVAL}(\mathcal{P})\), the mapping \(\mu\) as \(\mu(x) = d\) for all \(x \in \text{fvar}(\text{branch}(t))\).

It is now the case that \(\mu \in \text{p}(D)\) if and only if there is no homomorphism from \(E\) to \(F\). We prove this property in two steps. First, we show that \(\mu \in \text{p}(D)\) only depends on whether \(\mu\) can be extended to \(t\) or not. After this we show that such an extension of \(\mu\) exists if and only if there is a homomorphism \(h : E \rightarrow F\).

First, observe that the only possible extension \(\mu'\) of \(\mu\) such that \(\mu'(\tau) \in D\) for every \(\tau \in \lambda(\text{branch}(t))\) is obviously \(\mu'\) mapping every variable in \(\text{var}(\text{branch}(t))\) to \(d\). It follows immediately, that for all nodes \(t'' \neq t\) with their parent node in \(\text{branch}(t)\) the mapping \(\mu'\) cannot be extended to \(\lambda(t'')\), since for all relation symbols \(R\) mentioned in \(\lambda(t'')\) the relation \(R^D\) is empty.

Next — in order to conclude \(\mu \in \text{p}(D)\) if and only if there exists no extension \(\mu''\) of \(\mu'\) with \(\mu''(\tau) \in D\) for all \(\tau \in \lambda(t)\) — assume \(\mu''\) exists. Then \(\mu''\) can be obviously extended to \(\mu'''\) with \(\mu'''(\tau) \in D\) for all \(\tau \in c\text{branch}(t')\) since for all atoms on \(N \cup \{t'\}\), every possible atom over \(\text{dom}(D)\) is contained in \(D\). Since the other direction — \(\mu''\) exists, therefore there is an extension of \(\mu'\) to \(t\) is trivial, it remains to show that the extension \(\mu''\) exists if and only if there is a homomorphism \(h : E \rightarrow F\).

To see that this is the case, observe that every such homomorphism \(h\) in combination with \(\mu'\) gives a mapping from \(S\) into \(D\), and vice versa, every mapping \(\mu : S \rightarrow D\) restricted to \(\text{dom}(F)\) gives the desired homomorphism. For the remaining atoms in \(\lambda(t) \setminus S\), observe that every possible mapping sends them into \(D\), since \(D\) again contains every possible atom for these relations.

\(\blacksquare\)
The second source of hardness are the existential variables shared between a component and its predecessors, and the second condition restricts the number of such shared variables.

**Definition 23 (Interface Component Width).** Let \( p = (T, \lambda, \mathcal{X}) \) be a wdPT, \( t \in T \), and \( S \in \mathcal{I}_T \). The width of the interface component \( S \) is \( |V_t(S)| \). For a node \( t \in T \), the interface component width of \( t \) is the maximum width of any interface component \( S \) of \( t \). The interface component width of \( p \) is the maximum interface component width of all \( t \in \text{rel}(T) \).

**Tractability condition (b):** There is a constant \( c \) such that for all \( p \in \mathcal{P} \) the interface component width of \( p \) is at most \( c \).

**Lemma 24.** Let \( \mathcal{P} \) be a decidable class of simple wdPTs such that tractability condition (b) is not satisfied. Then \( p\text{-}\text{Eval}(\mathcal{P}) \) is \( \text{coW}[1] \)-hard.

**Proof.** We show the result by an FPT-reduction of the problem of model checking FO sentences \( \phi_k \) of the following form:

\[
\phi_k = \forall x_1 \ldots \forall x_k \exists y \bigwedge_{i \in k} E_i(x_i, y)
\]

By [15], model checking for this class of sentences is \( W[1] \)-hard. Therefore, let \( \phi_k \) and a database \( E \) with relations \( E^{\mathcal{I}_k} \) over \( \text{dom}(E) \) be given.

First, compute a wdPT \( p = (T, \lambda, \mathcal{X}) \in \mathcal{P} \) with an interface component width of at least \( k \). W.l.o.g. we assume \( p \) to contain only binary atoms: Since we assume a bounded arity, binary atoms can be easily simulated with atoms of higher arity. Consider the relevant node \( t \in T \) and an interface component \( S \in \mathcal{I}_T \) such that the interface component width of \( S \) is at least \( k \). Since we assume relations to be of some bounded arity, \( S \) cannot be of type (1) (Definition 21). W.l.o.g., we thus assume that \( S \) is of type (2).

Since \( t \) is relevant, there exists some \( t' \) which is either \( t \) itself or some descendant of \( t \) such that \( \text{fvar}(t') \setminus \text{fvar}(\text{branch}(t')) \neq \emptyset \). Among all possible candidates, choose some \( t' \) at a minimal distance to \( t \).

For the definition of the database \( D \), recall that we assume each relation symbol to occur at most once in \( p \). We define \( D \) as follows. First, \( \text{dom}(D) = \text{dom}(E) \). Based on this, the database contains the following relations:

- For each relation symbol \( R \) mentioned outside of \( \lambda(\text{branch}(t')) \) set \( R^{D} = \emptyset \).
- For each relation symbol \( R \) mentioned in \( \lambda(\text{branch}(t')) \setminus S \), set \( R^{D} \) contain all possible tuples \((a_1, \ldots, a_k) \in \text{dom}(E)\)
- For the relation symbols \( R \) mentioned in \( S \), proceed as follows. Choose \( k \) interface variables \( v_1, \ldots, v_k \in \mathcal{I}_i(S) \). Let \( L = \text{var}(S) \setminus \text{var}(\text{branch}(t)) \) be the “local variables” of \( S \). Observe that \( L \neq \emptyset \), since otherwise \( S \) could not be an interface component (it requires at least one variable to connect the variables from \( \mathcal{I}_i(S) \)). By the same reasoning, for each of the variables \( v_i \), there must exist at least one atom \( R_i(v_i, z_i) \) or \( R_i(z_i, v_i) \) for some \( z_i \in L \). We will assume \( R_i(v_i, z_i) \) in the following, the other case is analogous. Now for each \( v_i \), fix one such atom. Based on this, we define the following relations to be contained in \( D \):
  - For each of the selected atoms \( R_i(v_i, z_i) \), let \( R_i^{D} = E_i^{\mathcal{I}_k} \). i.e., we assume \( R_i \) to take the place of \( E_i \).
  - For every atom \( R_i(z_1, z_2) \in S \) such that \( z_1, z_2 \in L \), define \( R_i^{D} \) to contain all tuples \( \{(d, d) \mid d \in \text{dom}(E)\} \).
  - For the remaining atoms \( R_i(z_1, z_2) \in S \), define \( R_i^{D} = (\text{dom}(E))^2 \).
Then \( \mu \) is an arbitrary mapping \( \text{fvar}(\text{branch}(t)) \to \text{dom}(E) \).

It now follows by the same arguments as in the proof of Lemma 24 that we have \( \mu \notin p(D) \)
if and only if for every extension \( \mu' \) of \( \mu \) to \( \text{var}(\text{branch}(t)) \), there exists an extension \( \nu \) of \( \mu' \)
such that \( \nu(\tau) \in D \) for all \( \tau \in S \).

To close this proof, we thus only need to show that such an extension exists if and only if \( \phi_k \) is satisfied.

First, assume that \( \phi_k \) is satisfied. Then, for all \( z \in L \), define \( \nu(z) \) to be the value of \( y \) in \( \phi_k \). This clearly maps \( S \) into \( D \).

Next, assume that \( \phi_k \) is not satisfied. Thus there exists some assignment to \( x_1, \ldots, x_k \)
such that no suitable \( y \) value exists. Then for the mapping \( \mu' \) assigning exactly those values
to the selected interface variables \( v_1, \ldots, v_k \), there exists no extension of \( \mu' \) to \( S \). This concludes the proof.

It remains to define the tractability condition ensuring that the evaluation problem for
the resulting CQs will be tractable. For this, we first need to introduce the notion of an

**Lemma 25.** Let \( \mathcal{P} \) be a decidable class of simple \( \text{wdPT}s \) such that tractability condition
\( (c) \) is not satisfied. Then \( \text{p-Eval}(\mathcal{P}) \) is \( \mathcal{W}[1] \)-hard.

**Proof.** First, assume that the interface component width of the instances is bounded.

Otherwise, the result follows from Lemma 24. In particular, we may thus assume that all instances\n\( \text{extcore}(\mathcal{S}_{\text{var}(T')}, \lambda(T') \cup \bigcup_{i=1}^{n}(\text{cia}(S_i))) \) for all \( p \in \mathcal{P} \) are of bounded arity.

Let \( \text{solcheck}(\mathcal{P}) \) be the class of all structures \( \text{extcore}(\mathcal{S}_{\text{var}(T')}, \lambda(T') \cup \bigcup_{i=1}^{n}(\text{cia}(S_i))) \) for \( \mathcal{P} \) as defined in tractability condition \( (c) \). We reduce \( \text{EXT}(\text{solcheck}(\mathcal{P})) \) to \( \text{p-Eval}(\mathcal{P}) \) via
an FPT reduction. The result then follows directly from Theorem 4 since by assumption there does not exist a constant \( c \) such that the treewidth of the extension core of each pair \( (A, B) \in \text{solcheck}(\mathcal{P}) \) is less or equal than \( c \).

Thus, let \( (A, B), D, h \) be an instance of \( \text{EXT}(\text{solcheck}(\mathcal{P})) \). We show how to
construct a database \( D' \) and a mapping \( \mu \) such that \( \mu \in p(D') \) if and only if there exists an
extension \( h' \) of \( h \) that is a homomorphism from \( B \) to \( D \).

First of all, find a \( p = (T, \lambda, \bar{f}) \in \mathcal{P} \) and a subtree \( T' \) of \( T \) containing the root of \( T \)
such that \( (A, B) = (\mathcal{S}_{\text{var}(T')}, \lambda(T') \cup \bigcup_{i=1}^{n}(\text{cia}(S_i))) \) for some combination \( (S_1, \ldots, S_n) \in \mathcal{I}_T \times \cdots \times \mathcal{I}_T \)
where \( \{t_1, \ldots, t_n\} = \text{ch}(T') \cap \text{relv}(T) \).

For the definition of a database \( D' \) and a mapping \( \mu \) such that \( \mu \in p(D') \) if and only if \( h' \) exists, we need to define the following
sets of nodes first. Let \( \text{ch}(T') \cap \text{relv}(T) = \{t_1, \ldots, t_n\} \). For every \( t_i \in \text{ch}(T') \), we define the set \( N_i \) of nodes as follows:

- If \( \text{fvar}(t_i) \setminus \text{fvar}(\text{branch}(t_i)) \neq \emptyset \) (i.e., \( t_i \) contains some “new” free variable):
  Then \( N_i = \emptyset \).
On tractable query evaluation for SPARQL

Otherwise, let \( s_i \in T \) be a descendant of \( t_i \) such that \( \text{fvar}(s_i) \setminus \text{fvar}(\text{branch}(t_i)) \neq \emptyset \) and such that this property holds for no other node \( s'_i \in T \) on the path from \( t_i \) to \( s_i \). Then \( N_i = \text{cbbranch}(s_i) \setminus \text{cbbranch}(t_i) \).

Finally, let \( N = \bigcup_{i=1}^{n} N_i \). Now all notions are in place to describe the database \( D' \).

For all atoms \( R(\vec{g}) \in \lambda(T) \setminus (\lambda(T') \cup \lambda(ch(T')) \cup \lambda(N)) \), we set \( \text{R}^{D'} = \emptyset \). I.e., for all atoms neither in \( T' \) nor in any of the child nodes of \( T' \) (or their extensions to some node with a “new” free variable), no matching values exist in the database.

For all atoms \( R(\vec{g}) \in \lambda(T') \), we do the following. The relations for the atoms in \( \lambda(T') \) are designed in such a way that all free variables \( x \in \text{fvar}(T') \) in these atoms can only be mapped to \( h(x) \). Since this way the values for all free variables are fixed, in the remainder, for atoms in \( \lambda(T') \) we will only describe the values for those positions containing existentially quantified variables (recall that we only consider simple queries, thus these positions are uniquely defined). By slight abuse of notation (i.e., just ignoring the free variables), for every atom \( R(\vec{g}) \in \lambda(T') \), we define \( \text{R}^{D'} = \text{R}^{D} \).

For all atoms \( R(\vec{g}) \in \lambda(N) \), set \( \text{R}^{D'} = \text{dom}(D')^{k} \), where \( k \) is the arity of \( R \) (where \( \text{dom}(D') \) is implicitly defined as the union over all elements mentioned in the definition of \( D' \)).

Finally, we define the relations for the atoms in \( ch(T') \). Thus consider \( t_i \in ch(T') \). Let \( \vec{v} \) be the set of the inherited variables of the interface component \( S_i \in \mathcal{I} \mathcal{C}_{t_i} \) selected for the construction of \( B \). We use \( R(\vec{v}) \) to denote the atom \( \text{cia}(S_i) \).

For all atoms \( R(\vec{g}) \in \lambda(t_i) \setminus S_i \), set \( \text{R}^{D'} = \text{dom}(D)^{k} \) where \( k \) is the arity of \( R \). For the atoms in \( S_i \), we distinguish between \( S_i \) being of type (1), or of type (2).

If \( S_i \) consists of a single atom of type (1), i.e. \( S_i \) is of the form \( R(\vec{v}) \in \lambda(t) \), define \( \text{R}^{D'} = \text{dom}(D)^{k} \setminus \text{R}^{D} \).

If \( S_i \) is of type (2), we distinguish between two types of variables: Those that occur in \( I \mathcal{L}_{t_i}(S_i) \), and those that are “new” in \( S_i \), i.e. those that do not appear in some node \( t' \in \text{branch}(t_i) \). For these “new” variables, we will use as domain the set \( \text{dom}(D)^{[\vec{v}]} \), i.e. the set of all possible assignments of values from \( D \) to the vector \( \vec{v} \). Moreover, we assume some ordering of the values in \( \vec{v} \) to be in such a way that for a tuple \( \vec{a} \in \text{dom}(D)^{[\vec{v}]} \), the value at position \( i \) corresponds to the variable \( v_i \in \vec{v} \). For the remaining variables (i.e. the variables in \( \vec{v} \)), we will use values from \( \text{dom}(D) \). Then for each atom \( R(\vec{g}) \in S_i \), the values in \( \text{R}^{D'} \) are defined as follows.

- If \( \vec{g} \subseteq \vec{v} \), then \( \text{R}^{D'} = \text{dom}(D)^{k} \), where \( k \) is the arity of \( R \).
- If \( \vec{g} \) contains “new” variables, i.e. \( \vec{g} = \vec{g} \cap (\text{var}(S_i) \setminus \vec{v}) \neq \emptyset \), then \( \text{R}^{D'} \) contains all tuples satisfying the following three properties.

1. all the “new” variables \( \vec{g} \) get assigned the same value (say \( \vec{a} \in \text{dom}(D)^{[\vec{v}]} \)),
2. for all variables \( v_i \in \vec{v} \cap \vec{g} \), the value of \( v_i \) is consistent with \( \vec{a} \) (say \( \vec{a} \cap \vec{g} = \{ v_{i_1}, \ldots, v_{i_m} \} \) and let \( \vec{b} \) be the values assigned to \( \{ v_{i_1}, \ldots, v_{i_m} \} \)), and
3. there exists no tuple \( \vec{r} \notin \text{R}^{D} \) such that \( \vec{r} \) projected onto \( \{ v_{i_1}, \ldots, v_{i_m} \} \) is \( \vec{b} \) (i.e., \( \vec{b} \notin \pi_{i_{1},\ldots,i_{m}}(\text{R}^{D}) \)).

Note that all of these relations can be constructed in polynomial time because the arity of all relations is assumed to be bound.

We claim that, with the definition above, for an assignment \( \mu' \) to \( \vec{v} \), we have \( (\mu', D) \models R(\vec{v}) \) (i.e., \( R(\mu'(\vec{v})) \in D \)) if and only if all extensions \( \mu'' \) of \( \mu' \) to \( \text{var}(\lambda(t_i)) \) do not map all atoms in \( S_i \) into \( D' \), and thus also not \( \lambda(t_i) \) (since \( S_i \subseteq \lambda(t_i) \)). To see this, first assume that \( (\mu', D) \models R(\vec{v}) \). Let \( \mu'' \) be an extension of \( \mu' \) to \( \text{var}(S_i) \) that satisfies conditions 1. and 2.
Then all variables in \( \text{var}(S_i) \setminus \vec{v} \) take the same value under \( \mu'' \), and this value is exactly the tuple \( \mu'(\vec{v}) \). But then \( \mu'' \) does not satisfy condition 3., because \( (\mu', D) \models R(\vec{v}) \) (This implies \( \mu'(\vec{v}) \in R^D \), and thus \( \mu'(\vec{v}) \) projected onto the variables in \( \vec{v} \cap \vec{y} \) gives exactly \( \mu''_{|\vec{v}-\vec{y}}(\vec{v} \cap \vec{y}) \), contradicting 3.). So \( \mu'' \) does not map all atoms in \( S_i \) into \( D' \). For the other direction, assume that no extension \( \mu'' \) of \( \mu' \) maps all atoms in \( S_i \) into \( D' \). Then this is in particular true for those assignments satisfying 1. and 2. Consequently, \( \mu'_{|\vec{v}} = \mu' \) does not map all atoms in \( S_i \) on \( D \) due to 3. But this means that \( (\mu', D) \models R(\vec{v}) \), proving the claim.

We now claim that the assignment \( \mu \) setting all free variables \( x \) in \( T' \) to \( h(x) \) is an answer to \( p \) on \( D' \) if and only if the required extension \( h' \) of \( h \) exists. First, observe that \( \mu \in p(D') \) if and only if

- there is an extension \( \mu' \) of \( \mu \) to \( \text{var}(T') \) that maps all atoms in \( \lambda(T') \) into \( D' \) (of course, in general every subtree \( T'' \) with \( \text{fvar}(T'') = \text{dom}(\mu) \) is a potential candidate, but given the construction of \( D' \), the subtree \( T'' \) is the only possible candidate) and

- for all \( t_i \in \text{ch}(T') \), we have that there does not exists an extension of \( \mu'_{t_i} \) onto \( \lambda(t_i) \cup \lambda(N_i) \).

(In fact, extending the mapping to any descendant of \( t_i \) that contains some additional free variable would work. However, the only nodes with non-empty relations in \( D' \) are those mentioned in \( N \).)

By the construction of \( D' \) for atoms in \( \lambda(N) \), for every \( t_i \in \text{ch}(T') \) it follows immediately that there exists an extension of \( \mu'_{t_i} \) onto \( \lambda(t_i) \cup \lambda(N_i) \) if and only if there exists an extension to \( \lambda(t_i) \): Since for the atoms in \( \lambda(N) \) the database \( D' \) contains all possible tuples, every extension \( \mu'' \) of \( \mu'_{t_i} \) onto \( \lambda(t_i) \) can be further extended to all atoms in \( \lambda(N_i) \).

Note that the existence of an extension of \( \mu'_{t_i} \) onto \( \lambda(t_i) \) is, as we have seen before, equivalent to \( \mu' \) satisfying \( R(\vec{v}) \), the atom introduced for the node \( t_i \) in \( q \). So \( \mu \in p(D') \) if and only if there is an extension \( h' \) of \( h \) that is a homomorphism from \( B \) into \( D \). This completes the proof. \( \blacksquare \)

Having defined the three tractability conditions, we next show how they may be used to design an FPT-algorithm for EVAL(\( P \)). The algorithm is outlined in Algorithm 1.

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**Algorithm 1** EvalFPT(\( p, D, \mu \))

1. \( T = T[\text{rel}(T)] \quad \triangleright \text{Remove all nodes from } T \text{ that are not relevant} \)
2. \( \text{for all subtrees } T' \text{ of } T \text{ with } \text{fvar}(T') = \text{dom}(\mu) \text{ do} \)
3. \( \text{Let } \{t_1, \ldots, t_n\} = \text{ch}(T') \cap \text{rel}(T') \)
4. \( \text{for all } (S_1, \ldots, S_n) \in \mathcal{IC}_{t_1} \times \cdots \times \mathcal{IC}_{t_n} \text{ do} \)
5. \( q = \text{"Ans}(\vec{x}) \leftarrow \lambda(T') \cup \{\text{cia}(S_1), \ldots, \text{cia}(S_n)\}" \quad \triangleright \text{Let } \vec{x} \text{ contain all } x \in \text{fvar}(T') \)
6. \( D' = D \cup \bigcup_{v=1}^{n} \{R_i(\nu(\vec{v})) \mid \nu \in \text{stop}(S_i, D)\} \quad \triangleright \text{Assume } \text{cia}(T') S_i = R_i(\vec{v}) \)
7. \( \text{if } \mu(\vec{v}) \in q(D') \text{ then } \text{exit(YES)} \)
8. \text{exit(NO)}

---

First of all, we discuss the only notion in Algorithm 1 not yet introduced in this section: \( \text{stop}(S, D) \) for an interface component \( S \) and a database \( D \). Recall that we said earlier that the intention of the interface components is to ensure a mapping to be maximal not by testing for extensions to the complete node, but to do these tests component wise.

The idea how to realize this is to store in \( D' \) for each interface component \( S \) those variable assignments \( \nu \) to its inherited variables such that there exists no extension \( \nu' : S \rightarrow D \) of \( \nu \cup \mu \). These are exactly the values stored in \( \text{stop}(S, D) \).

Formally, for a wdPT \( (T, \lambda, X) \), a subtree \( T' \) of \( T \), a child node \( t \in \text{ch}(T') \), an interface component \( S \in \mathcal{IC}_t \), a database \( D \), and a mapping \( \mu : X' \rightarrow \text{dom}(D) \) (for \( X' \subseteq X \)), consider
the mappings \( \text{extend}(\mathcal{S}, \mathcal{D}) = \{ \eta | \nu(\mathcal{S}) \rightarrow \text{dom}(\mathcal{D}), \eta \text{ extends } \mu|_{\text{var}(\mathcal{S})} \text{ and } \eta(\tau) \in \mathcal{D} \text{ for all } \tau \in \mathcal{S} \} \). I.e., \( \text{extend} \) contains exactly those mappings on \( \nu(\mathcal{S}) \) that can be extended in a way that maps \( \mathcal{S} \) into \( \mathcal{D} \). We thus get \( \text{stop}(\mathcal{S}, \mathcal{D}) = \{ \nu: \nu(\mathcal{S}) \rightarrow \text{dom}(\mathcal{D}) | \nu \notin \text{extend}(\mathcal{S}, \mathcal{D}) \} \).

With this in place, we describe the idea of Algorithm \[\text{Algorithm 1}\] Recall that given \( \mu \), we have to find a mapping \( \mu' \) extending \( \mu \) that on the one hand (1) is a pp-solution, and on the other hand (2) is maximal. Unlike the case without projection, where \( T_\mu \) is easy to find, because of the presence of existential variables, there might be up to an exponential number of candidates for \( T_\mu \); all subtrees \( T' \) of \( T \) with \( \text{fvar}(T') = \text{dom}(\mu) \). After removing all irrelevant nodes (they might make evaluation unnecessarily hard), we thus check all of these candidates.

If the required mapping \( \mu' \) exists then, as discussed earlier, for each child node of \( T' \) there exists at least one interface component to which \( \mu' \) cannot be extended. Not knowing which interface components these are, the algorithm iterates over all possible combinations (line 4). In lines 5–7, the algorithm now checks whether there exists an extension of \( \mu \) that maps all of \( \lambda(T') \) into \( \mathcal{D} \) (ensured by adding \( \lambda(T') \) to \( q \), but none of the interface components \( \mathcal{S}_1, \ldots, \mathcal{S}_n \)). The latter property is equivalent to asking that for each \( \mathcal{S}_i \), \( \mu' \) must assign a value to its inherited interface variables that cannot be extended. This is guaranteed by adding the atoms \( \text{cia}(\mathcal{S}_i) \) to \( q \) and providing in \( \mathcal{D}' \) exactly the values from \( \text{stop}(\mathcal{S}_i, \mathcal{D}) \).

In order to see that this indeed gives an FPT algorithm in case tractability conditions (a), (b), and (c) are satisfied, note that condition (b) ensures that the arity of each of the new relations for the atoms \( \text{cia}(\mathcal{S}) \) is at most \( c \). Thus the size of these relations (and thus the number of possible mappings in \( \text{stop}(\mathcal{S}, \mathcal{D}) \)) is at most \( |\text{dom}(\mathcal{D})|^c \). Next, condition (a) ensures that for each mapping \( \nu: \nu(\mathcal{S}) \rightarrow \text{dom}(\mathcal{D}) \) deciding membership in \( \text{stop}(\mathcal{S}, \mathcal{D}) \) is in PTIME. Finally, condition (c) ensures that the test in line 7 is feasible in polynomial time.

We note that the algorithm is an extension and refinement of the FPT algorithm presented in [30]. An inspection of [30] reveals that the conditions provided there imply our tractability conditions (a), (b), and (c), but not vice-versa. In fact, our conditions explicitly describe the crucial properties of their restrictions that allow the problem to be in FPT.

From Algorithm \[\text{Algorithm 1}\] we thus derive the following result.

\begin{theorem}
Let \( \mathcal{P} \) be a decidable class of simple wdPTs. If the tractability conditions (a), (b), and (c) hold for \( \mathcal{P} \), then \( p\text{-}\text{EVAL}(\mathcal{P}) \) can be solved in FPT.
\end{theorem}

The correctness of the algorithm follows immediately from the above discussion. For the runtime, observe that in addition to what we already discussed, the number of loop-iterations in lines 2 and 6 are bounded in the size of \( p \), which is the parameter for the problem.

Combining Theorem 26 with Lemmas 22, 24 and 25 we get the following characterization.

\begin{theorem}
Assume that \( \text{FPT} \not\equiv \text{W}[1] \), and let \( \mathcal{P} \) be a decidable class of simple wdPTs. Then the following statements are equivalent.

1. The tractability conditions (a), (b), and (c) hold for \( \mathcal{P} \).
2. \( \text{EVAL}(\mathcal{P}) \) is in FPT.
\end{theorem}

We mentioned at the beginning of this section that most tractability results also hold for arbitrary wdPTs instead of just simple ones. Observe that we make use of simple wdPTs only in Lemmas 22, 24 and 25 as well as in the definition of relevant nodes. In fact, whenever a arbitrary well-designed pattern tree satisfies the tractability conditions for all nodes (instead of just the relevant ones), Algorithm \[\text{Algorithm 1}\] also provides an FPT algorithm for the evaluation problem.
6 Relationship with SPARQL and Conclusion

The results of Sections 4 and 5 give a fine understanding of the tractable classes of PTs without projection and wdPTs in the presence of projection. In particular they show the different sources of hardness. As laid out in the introduction, there is a strong relationship between (weakly) well-designed SPARQL queries and classes of PTs, namely the weakly well-designed (wwdPT) and well-designed (wdPTs) pattern trees, respectively: For every (weakly) well-designed SPARQL query, an equivalent (weakly) well-designed pattern tree can be computed in polynomial time, and vice versa, in a completely syntactic way.

Since our results for projection-free PTs apply to all classes of PTs, they therefore immediately apply to (weakly) well-designed \{AND, OPTIONAL\}-SPARQL queries as well. Note that the correspondence is unfortunately less tight for the case with projections. Not only because we study only well-designed pattern trees instead of arbitrary ones, but recall that our characterization only applies for classes of \textit{simple} well-designed pattern trees. However, RDF triples and SPARQL triple patterns, in the relational model, are usually represented with a single (ternary) relation. Thus, there is no direct translation to and from simple (well-designed) pattern trees. As a consequence, in the presence of projection, our characterization of tractable classes of simple well-designed pattern trees does not imply an immediate characterization of the tractable classes of well-designed \{AND, OPTIONAL\}-SPARQL queries.

Nevertheless, our results also give interesting insights to SPARQL with projections. First of all, Algorithm 1 can directly be applied without any changes also for queries in which relation symbols can appear several times and thus in particular for well-designed pattern trees that result from the translation of well-designed SPARQL queries. Moreover, our result determines completely the tractable classes that can be characterized by analyzing only the underlying graph structure of the queries, i.e., the Gaifman graph. Indeed, since simple queries can simulate all other queries sharing the same Gaifman graph by duplicating relations, Gaifman graph based techniques have exactly the same limits as simple queries. Thus, our work gives significant information on limits of tractability for SPARQL queries in the same way as e.g. \[25, 14\] did in similar contexts.

Let us mention the major stumbling block towards a characterization of non-simple well-designed pattern trees with projections: In the proof of Lemma 24, we have used a reduction from quantified conjunctive queries. Unfortunately, the tractable classes for the non-simple fragment and the correct notion of cores for that problem are not well understood which limits our result to simple queries since we are using the respective results from \[15\]. Note that we might have been able to give a more fine-grained result in sorted logics by using \[16\], but since this would, in our opinion, not have been very natural in our setting, we did not pursue this direction. Thus getting an even better understanding of non-simple pattern trees would either need progress on quantified conjunctive queries or a reduction from another problem that is better understood.

For future work, there are further SPARQL features that we did not include in the framework studied in this paper. The most prominent among them are of course FILTER expressions. Let us remark that pattern trees with FILTER expressions are easily seen to subsume conjunctive queries with inequalities—just consider pattern trees consisting of only one node—and thus in particular also graph embedding problems. The tractable fragments of the latter are a notorious problem that has resisted the efforts of the parameterized complexity community for a long time now, even though there has recently been progress in the area, see e.g. \[32, 17\]. Thus showing a complete characterization of the tractable classes
for SPARQL queries with FILTER is probably very hard. Still, it would be interesting to give algorithms extending our results to that fragment and maybe giving lower bounds based on the conjectured dichotomy for embedding problems.
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A  Proofs for Section 3 (The Extension Problem)

A.1 Additional Definitions

The proofs in this section require a few concepts that have already been introduced informally in the main part of this paper. In order to use these concepts in the proof we first provide a formal definition for them.

We start by recalling two well-known operators from the Relational Algebra: the projection $\pi$ and the selection $\sigma$. For a relation $R$ of arity $k$, we denote each of the $k$ positions of a tuple $(a_1, \ldots, a_k) \in R$ by their index $i$. Then the projection $\pi_{i_1, \ldots, i_q}(a_1, \ldots, a_k)$ returns the tuple $(a_{i_1}, \ldots, a_{i_q})$, and for a relation $R$ the projection $\pi_{i_1, \ldots, i_q}(R) = \{\pi_{i_1, \ldots, i_q}(a_1, \ldots, a_k) \mid (a_1, \ldots, a_k) \in R\}$. For the selection $\sigma_{i_1=v_1, \ldots, i_q=v_q}$ we say that a tuple $(a_1, \ldots, a_k)$ satisfies the selection condition $(i_1 = v_1, \ldots, i_q = v_q)$ if $a_{i_j} = v_j$ for all $1 \leq j \leq q$. For a relation $R$ we get $\sigma_{i_1=v_1, \ldots, i_q=v_q}(R) = \{(a_1, \ldots, a_k) \in R \mid (a_1, \ldots, a_k) \text{ satisfies } (i_1 = v_1, \ldots, i_q = v_q)\}$.

**Projection of a Structure.** Let $S$ be a $\sigma$-structure, and consider $V \subseteq \text{dom}(S)$. Then the *projection of $S$ under $V$*, denoted by $S \setminus V$, returns the following $\sigma'$-structure (the set of relational symbols in $\sigma'$ is defined implicitly by the newly introduced relations described below): For every relation symbol $R \in \sigma$ and each $(a_1, \ldots, a_k) \in R^S$, let $a_1, \ldots, a_k$ be those positions of $(a_1, \ldots, a_k)$ that do not contain values from $V$, i.e. $a_{i_j} \notin V$ for $q \leq j \leq \ell$, and let $\{i_1, \ldots, i_m\} = \{1, \ldots, k\} \setminus \{a_1, \ldots, a_k\}$ be those positions such that $a_{i_j} \in V$ for $1 \leq j \leq m$.

Then in structure $S \setminus V$ the relation

$$\{R_{i_1=a_{i_1}, \ldots, i_m=a_{i_m}}\setminus V \text{ contains the tuple } (a_{i_1}, \ldots, a_{i_m})\}.$$

These are all the tuples in any relation in $S \setminus V$.

**Projection of a Pair of Structures under a homomorphism.** Next, let $(Q, D)$ be a pair of $\sigma$-structures and let $h$ be a mapping on (a subset of) $\text{dom}(Q)$. We define the *projection of $(Q, D)$ under $h$* as the $\sigma'$-structures $(Q', D')$ as follows (the vocabulary $\sigma'$ is again defined implicitly):

We start by defining $Q'$. For every relation symbol $R \in \sigma$ and every tuple $(a_1, \ldots, a_k) \in R^Q$, we distinguish two cases:

- If $\{a_1, \ldots, a_k\} \subseteq \text{dom}(h)$ and $(h(a_1), \ldots, h(a_k)) \in D^Q$, then ignore $(a_1, \ldots, a_k)$ (i.e., no tuple derived from $(a_1, \ldots, a_k)$ occurs in $Q'$).
- Otherwise, let $\{i_1, \ldots, i_\ell\} \subseteq \{a_1, \ldots, a_k\}$ be those positions of $(a_1, \ldots, a_k)$ such that $a_{i_j} \in \text{dom}(h)$ for $1 \leq j \leq \ell$, and $\{a_1, \ldots, a_m\} = \{1, \ldots, k\} \setminus \{i_1, \ldots, i_\ell\}$ those positions such that $a_{i_j} \notin \text{dom}(h)$.

Then $\{R_{i_1=h(a_{i_1}), \ldots, i_m=h(a_{i_m})}^{Q'} \text{ contains the tuple } (a_{i_1}, \ldots, a_{i_m})\}$.

Observe that the second case includes the possibility that $\ell = k$, i.e. that $a_i \in \text{dom}(h)$ for all $1 \leq i \leq k$, but that $(h(a_1), \ldots, h(a_k)) \notin D^Q$. In this case, the result is a $0$-ary relation symbol $R_{i_1=h(a_{i_1}), \ldots, k=h(a_{i_k})}^{Q'} \subseteq D^Q$ contains the empty tuple.

Next we define $D'$. For every relation symbol $R_{i_1=b_{i_1}, \ldots, i_\ell=b_{i_\ell}} \in \sigma'$ introduced by the definition of $Q'$ and every tuple $(a_1, \ldots, a_m) \in (R_{i_1=b_{i_1}, \ldots, i_\ell=b_{i_\ell}}^{Q'})$, let $b_{i_j}$ be the arity of the original relation symbol $R$ from $\sigma$. Then the positions $1, \ldots, m$ in $R_{i_1=b_{i_1}, \ldots, i_\ell=b_{i_\ell}}$ correspond to positions $j_1, \ldots, j_m$ in $R$. In fact, $\{j_1, \ldots, j_m\} = \{1, \ldots, k\} \setminus \{i_1, \ldots, i_\ell\}$. Then

$$\{R_{i_1=b_{i_1}, \ldots, i_\ell=b_{i_\ell}}^{D'} \text{ contains the tuples } \pi_{j_1, \ldots, j_m}(\sigma_{i_1=b_{i_1}, \ldots, i_\ell=b_{i_\ell}}(R^D))\}.$$
Observation 28. Let two $\sigma$-structures $Q, D$ and a mapping $h: Q \to \text{dom}(D)$ be given where $Q \subseteq \text{dom}(Q)$, and let $Q', D'$ the projection of $(Q, D)$ under $h$. Then there exists some extension $h': Q \to D$ of $h$ if and only if there exists a homomorphism $h: Q' \to D'$ (i.e., if $(Q', D')$ is a positive instance of $\text{HOM}$).

We need to take care of one special case: Assume that, for a pair $(Q, D)$ and a homomorphism $h$ such that $Q$ is already a projection of some structure $L$ under a set $V \subseteq \text{dom}(L)$, we want to get the projection of $(Q, D)$ under $h$. I.e. the vocabulary of $Q$ already contains relation symbols of the form $R_{i_1=b_1, \ldots, i_\ell=b_\ell}$. If for any of the $i_j$ ($1 \leq j \leq \ell$) it is the case that $b_j \in \text{dom}(h)$, then in the resulting structure we replace $b_j$ in the name of the resulting atom by $h(b_j)$. In certain situations, this renaming of the relational symbols will ensure the resulting structures to be over the same relational schema, which is a prerequisite for finding homomorphisms.

A.2 Full Proofs of Section 3

Lemma 9. Let $C$ be a class of pairs of structures such that the treewidth of $\text{extcore}(A, B)$ for all $(A, B) \in C$ is bounded by some constant $c$. Then $\text{EXT}(C)$ is in $\text{PTIME}$.

Proof. Let $(A, B) \in C, D$, and $h$ be an instance of $\text{EXT}(P)$. By Lemma 8, this problem is equivalent to asking for the existence of a homomorphism $h': (A \cup S) \to D$ that extends $h$, where $S = \text{core}(A \cup B \cup S_A) \setminus S_A$ and $A = \text{dom}(A)$. By Observation 28, this is equivalent to the instance $((A \cup S)', D')$ of $\text{HOM}$, where $((A \cup S)', D')$ is the projection of $((A \cup S), D)$ under $h$.

For $E = \text{extcore}(A, B)$ and $F = D$, we next show that $((A \cup S)', D') = (E', F')$ where $(E', F')$ is the projection of $(E, F)$ under $h$. First of all, observe that $D' = F'$ does not necessarily holds, since it depends on the result of the projections of the left hand side. However, if the left hand side coincide, the equality $D' = F'$ obviously holds.

We thus show that $(A \cup S)' = E'$. First of all, we have that $A \cup S = A \cup \langle \text{core}(A \cup B \cup S_A) \setminus S_A \rangle \subseteq (A \cup B)$. Moreover, since $h: A \to D$, by the first case in the case distinction if the definition of the projection of a pair of structures under a homomorphism, $(A \cup S)'$ does not contain any relation derived from $A$. It is thus safe to conclude that $(A \cup S)' = S'$, where $(S', D')$ is the projection of $(S, D)$ under $h$.

Next, recall that $S = \text{core}(A \cup B \cup S_A) \setminus S_A$, and that $\text{extcore}(A, B) = S \setminus A$. Thus the only difference between $E' = (S \setminus A)'$ and $(A \cup S)' = S'$ is that in the first case, the projection of $S$ under $A$ is computed, before the projection under $h$. However, since $\text{dom}(h) = \text{dom}(A) = A$, it can be easily checked that this results in the same structures, and thus $E' = (A \cup S)'$.

Now the treewidth of $E$ is bounded by $c$, and therefore also the treewidth of $E'$ (taking subgraphs does not increase the treewidth). As a result, the existence of a homomorphism $E' \to F'$ can be decided in polynomial time [24], which proves the lemma.

Lemma 29. Let $A$ and $B$ be structures and let $S$ be the structure $\text{core}(A \cup B \cup S_A)$ from the definition of the extension core. If $h$ is a homomorphism from $S$ to itself, then $h$ is bijective.

Proof. Since $S$ is a core, any homomorphism from $S$ to itself is an isomorphism.

Lemma 10. Let $C$ be a decidable class of pairs of structures and let $\text{extcore}(C)$ be the class of extension cores of the pairs in $C$. Then $\text{p-Hom}(\text{extcore}(C)) \leq_{\text{FPT}} \text{p-EXT}(C)$.
Proof. Let \( \mathcal{C} \) be a decidable class of pairs of structures, and let \((L, T)\) be an instance of \( p\text{-Hom}(\text{extcore}(\mathcal{C})) \). We reduce this problem to an instance of \( p\text{-EXT}(\mathcal{P}) \).

In a first step, we compute a pair \((A, B)\) \( \in \mathcal{C} \) such that \( \text{extcore}(A, B) = L \). We will define a structure \( D \) and homomorphism \( h: A \to D \) such that there exists a homomorphism \( h': (A \cup B) \to D \) that is an extension of \( h \) if and only if there exists a homomorphism from \( L \) to \( T \). However, we will define \( D \) and homomorphism \( h \) in a slightly different setting.

Let \( S \) be the structure \( \text{core}(A \cup B \cup S_A) \setminus S_A \) from the definition of extension cores where \( A = \text{dom}(A) \). By Lemma 29, the desired extension \( h' \) of \( h \) exists if and only if there exists a homomorphism \( h'' : (A \cup S) \to D \) that extends \( h \). Observe that the structure \( D \) and homomorphism \( h \) are still the same as above. We will thus work in the latter setting with \( S \) instead of \( B \) as this turns out to be easier.

We define the structure \( D \) over the same vocabulary as \( S \) as follows:

- The domain \( \text{dom}(D) = \text{dom}(T) \times \text{dom}(S) \), i.e. the elements represent pairs of elements from \( T \) and \( S \), respectively.
- For each relation symbol \( R \) of arity \( k \), and every tuple \((a_1, \ldots, a_k) \in R^S\), the relation \( R^D \) contains the following tuples:

  Let \( \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, k\} \) be all those positions of \((a_1, \ldots, a_k)\) such that \( a_{i_j} \in \text{dom}(A) \), and let \( \{o_1, \ldots, o_m\} = \{1, \ldots, k\} \setminus \{i_1, \ldots, i_k\} \) be all those positions such that \( a_{o_j} \notin \text{dom}(A) \), i.e. \( a_{o_j} \in \text{dom}(B) \setminus \text{dom}(A) \). Let furthermore \( \ell \) be the relation symbol derived for \((a_1, \ldots, a_k) \in R^S \) when computing the projection \( S \setminus \text{dom}(A) = \text{extcore}(A, B) \). Now, for every pair \((d_1, d_2)\) of tuples \( d_1 = (d_{o_1}, \ldots, d_{o_m}) \in (R')^T \) and \( d_2 = (d_{i_1}, \ldots, d_{i_k}) \in \text{dom}(T)^\ell \), we add the tuple \( (d_1, a_1, \ldots, d_k, a_k) \) to \( R^D \). (Observe that by slight abuse of notation, in order to simplify the description we denote the positions in \( d_1 \) and \( d_2 \) according to the position in \( R \) they originate from.) Thus, intuitively, we replace all domain elements from \( \text{dom}(A) \) with all possible combinations of elements from \( \text{dom}(T) \).

These are all the tuples in \( D \).

It is worth pointing out that in case \( R' \) is not part of the vocabulary of \( T \) or \( (R')^T \) is empty, then by this definition \( R^D \) is the empty relation. The resulting instance will therefore be a simple “no” instance, because \( R^S \) is non-empty. However, in this case we also have that \((R')^T \) is non-empty, and therefore also \((L, T)\) is a trivial “no” instance.

Finally, we define the mapping \( h : \text{dom}(A) \to \text{dom}(D) \) as \( h(a) = (d, a) \) for some arbitrary but fixed element \( d \in \text{dom}(T) \).

It remains to prove that there indeed exists a homomorphism \( g : L \to T \) if and only if \( h \) can be extended to a homomorphism \( h' : S \to D \).

First assume that \( g \) exists. Then define an extension \( h' \) of \( h \) to \( \text{dom}(S) \) as \( h'(a) = (g(a), a) \) for all \( a \in \text{dom}(S) \setminus \text{dom}(A) \). The mapping \( g \) is indeed defined on all these elements, since \( \text{dom}(S) \setminus \text{dom}(A) = \text{dom}(\text{extcore}(A, B)) = \text{dom}(L) \) because of \( \text{extcore}(A, B) = L \). For \( a \in \text{dom}(A) \) we need not define \( h' \) since \( h \) is already defined on these elements, and \( h' \) extends \( h \). It now follows immediately from the construction of \( D \) that \( h' \) is indeed the required homomorphism.

For the other direction, assume that \( h' \) exists. First, observe that \( D \) projected onto the second component of its domain elements gives \( S \). Thus \( h' \) is a bijection in this second coordinate by Lemma 29. Let \( \pi_2 \) be the projection to the second coordinate. Then \( \pi_2 \circ h \) is an automorphism of \( S \), and thus there is a \( n \in \mathbb{N} \) such that \( (\pi_2 \circ h)^n = \text{id} \) (where \( \text{id} \) denotes the identity mapping). Consequently, w.l.o.g. we assume that \( \pi_2 \circ h = \text{id} \). For every \( a \in \text{dom}(L) = \text{dom}(\text{extcore}(A, B)) \) define \( g(a) \) to be the value \( d \) such that \( h'(a) = (d, a) \). Then again by definition of \( D \) it follows immediately that for all relation symbols \( R \) and tuples \( \bar{a} \in R^L \) we have \( g(\bar{a}) \in R^T \).
Observing that all constructions can be done efficiently completes the proof.

A.3 Relationship to CQs

Since it will turn out to be a useful tool in Section 5, and to substantiate our claim that EXT is an interesting problem on its own right, we observe the following relationship between EXT and CQ-EVAL which is reminiscent of the relationship between the problems HOM and the evaluation problem for Boolean CQs. For a CQ $q = \text{Ans}(x) \leftarrow B$, let $\text{extcq}(q) = (\{\text{Ans}(x)\}, B)$. Furthermore, for a class $Q$ of CQs let $\text{extcq}(Q) = \bigcup_{q \in Q} \text{extcq}(q)$.

Then the following immediate corollary of Theorem 6 provides a complete characterization via the treewidth of the extension core of all tractable classes of CQ-EVAL over schemas with bounded arity but an unbounded number of free variables.

**Corollary 30.** For every recursively enumerable class $Q$ of CQs, the problems $\text{CQ-EVAL}(Q)$ and $\text{EXT}(\text{extcq}(Q))$ are equivalent under many-one reductions.

B Proofs for Section 4 (Projection Free Pattern Trees)

**Proposition 13.** Let $(T, \lambda, X)$ be a PT, $T'$ a subtree of $T$, and $\text{csts}_i(T'), \text{csts}_j(T') \in \text{CST}(T')$. Then, for any $c \geq 1$, the treewidth of $\text{extcore}(\text{csts}_i(T'))$ is less or equal than $c$ if and only if the treewidth of $\text{extcore}(\text{csts}_j(T'))$ is less or equal than $c$.

**Proof.** As already mentioned in the main part of the paper, note that the composition of two homomorphisms is again a homomorphism. Next, given the initial set $C(T')$, consider two sequences $\delta_1, \delta_2$ of deletions of pairs $(T_i, \lambda(t_i))$ such that they result in different critical subtrees $\text{csts}_1(T')$ and $\text{csts}_2(T')$. W.l.o.g. assume that there is a pair $(T_i, \lambda(t_i)) \in \text{csts}_1(T') \setminus \text{csts}_2(T')$. Thus in $C(T')$, there was some $(T_j, \lambda(t_j))$ that witnessed the deletion of $(T_i, \lambda(t_i))$ in the sequence $\delta_2$, i.e. there exists a homomorphism $h: T_j \cup \lambda(t_j) \rightarrow T_i \cup \lambda(t_i)$. However, since $(T_i, \lambda(t_i))$ cannot be removed from $\text{csts}_1(T')$, we have $(T_j, \lambda(t_j)) \notin \text{csts}_1(T')$. However, the only pair that can witness the deletion of $(T_j, \lambda(t_j))$ without still witnessing (because of the composition of homomorphisms) a possible deletion of $(T_i, \lambda(t_i))$ is the pair $(T_j, \lambda(t_j))$ itself. We thus have both, a homomorphism from $(T_j, \lambda(t_j))$ to $(T_i, \lambda(t_i))$ and vice versa. Since in addition these homomorphisms are the identity on $\text{dom}(T_i)$ and $\text{dom}(T_j)$, respectively, they also give homomorphisms between the two extension cores. Thus, by [24] both extension cores are isomorphic, and therefore have the same treewidth.