Non-Conservative Minimal Quantum Dynamical Semigroups

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Abstract

Necessary and sufficient conditions for non-conservativity of a class of quantum dynamical semigroups are given. Extensions of well known criteria for conservativity are obtained and interesting connections of the conservativity problem with the von Neumans Theory of the defect indices for symmetric operators are studied.

1 Introduction

The concept of quantum dynamical semigroup (QDS) has become a fundamental notion in the theory of quantum Markov processes. The theory of QDS has been intensively studied in recent years, laying special emphasis to the so called minimal quantum dynamical semigroup as well as to sufficient conditions to ensure its conservativity (markovianity or unitality) [2], [5]. This approach has yields to distinguish a class of minimal conservative QDS. Much less attention has received the class of non-conservative QDS, nevertheless the study of this class is important both from the mathematical point of view as well as for applications in models of quantum physics.

The main aim of this work is to describe a class of non-conservative minimal QDS that naturally arises when a necessary condition for conservativity is not satisfied. In the case when the CP part of the formal generator is zero, a more careful study of the formal generators of this class of QDS permits

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one to observe interesting connections with the von Neumanns theory of the defect indices of symmetric operators.

In section 2 we give the necessary definitions. Section 3 contains several criteria for non-conservativity or explosion of the class of qds introduced in section 2, in particular, Theorems 3.1 and 3.3 extend some criteria of A.M. Chebotarev for conservativity and Corollary 3.5 extends a well known criterion of E.B. Davies. In Section 4 we provide several examples and show the connections of the conservativity problem for the class of minimal qds introduced in Section 2 with the von Neumann Theory of the defect indices of symmetric operator, in the case when the CP part of the formal generator is zero.

2 Preliminaries

Along this work $\mathcal{H}$ will denote a separable complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ $\mathcal{B} = \mathcal{B}(\mathcal{H})$ will denote the von Neumann algebra of all bounded linear operators in $\mathcal{H}$ and $\| \cdot \|_\infty$ will denote the norm in this space.

**Definition 2.1.** A quantum dynamical semigroup on $\mathcal{B}$ is a semigroup $P = (P_t)_{t \geq 0}$ of bounded operators in $\mathcal{B}$ with the following properties

(a) Complete Positivity (CP). $P_t$ is completely positive for every $t \geq 0$, i.e. for every pair of finite sequences $(x_i), (y_j)$ in $\mathcal{B}$

$$\sum_{i,j} y_i^* P_t(x_i^* x_j) y_j \geq 0.$$

(b) (Normality or $\sigma$-weak continuity). For every increasing net $(x_\alpha)$ of positive elements in $\mathcal{B}$ with an upper bound we have

$$P_t(\sup_\alpha x_\alpha) = \sup_\alpha P_t(x_\alpha)$$

for every $t \geq 0$.

(c) (Ultraweak or $*$-weak continuity in $t$). For every trace class operator and every $x \in \mathcal{B}$ we have that

$$\lim_{t \to 0^+} tr(\rho P_t(x)) = tr(\rho x).$$
(d) \( P_t(I) \leq I \) for all \( t \geq 0 \).

A qds \((P_t)_{t \geq 0}\) is conservative (markovian or unital) if \( P_t(I) = I \), for all \( t \geq 0 \).

If a conservative qds is uniformly continuous \( \lim_{t \to 0^+} \sup_{\|x\|_\infty = 1} \|P_t(x) - x\|_\infty = 0 \), then its infinitesimal generator is a bounded linear operator \( \mathcal{L} : \mathcal{B} \to \mathcal{B} \) and there exist a CP normal bounded map \( \phi : \mathcal{B} \to \mathcal{B} \) and a bounded self-adjoint operator \( H \) such that

\[
\mathcal{L}(x) = \phi(x) - G^*x - xG
\]

with \( G = (1/2)\phi(I) - iH \). And conversely any linear operator \( \mathcal{L} \) with the structure (2.1) is the infinitesimal generator of a uniformly continuous conservative qds. This is an important result due to Lindblad and Gorini-Kossakowski-Sudarshan, see [5] and the references therein.

In this work we shall consider unbounded formal generators \( \mathcal{L} \) that associates with every \( x \in \mathcal{B} \) an unbounded sesquilinear form with the structure

\[
\mathcal{L}(x)[u, v] = \phi(x)[u, v] - \langle Gu, xv \rangle - \langle u, xGv \rangle,
\]

\( u, v \in \text{dom}G, \) where

1. \( -G \) is the generator of a \( C_0 \)-semigroup of contractions in \( \mathfrak{h} \), \((W_t)_{t \geq 0}\).

2. \( \phi \) is a linear CP and normal map, i.e., for every \( x \in \mathcal{B} \), \( \phi(x) \) is a sesquilinear form defined on \( \text{dom}G \times \text{dom}G \) such that
   
   \[
   (\text{CP}): \text{for any pair of finite sequences } (u_i) \in \text{dom}G \text{ and } (x_i) \subset \mathcal{B} \text{ we have that }
   \sum_{i,j} \phi(x^*_ix_j)[u_i, u_j] \geq 0.
   \]

3. For every \( u \in \text{dom}G \), \( \phi(\cdot)[u] \) is a normal linear functional on \( \mathcal{B} \), i.e., for any increasing net \((x_\alpha)\) of positive elements of \( \mathcal{B} \) with an upper bound,

   \[
   \phi\left(\sup_\alpha x_\alpha\right)[u] = \sup_\alpha \phi(x_\alpha)[u],
   \]

where \( \phi(\cdot)[u] = \phi(\cdot)[u, u] \) is the quadratic form associated with \( \phi(\cdot) \).
(iii) the estimate
\[ 0 \leq \phi(I)[u] \leq \text{Re}\langle Gu, u \rangle \]
or equivalently
\[ \mathcal{L}(I)[u] \leq 0, \]
holds for every \( u \in \text{dom}G \).

Conditions (i)-(iii) are sufficient to construct a minimal qds \( (P_{t}^{\text{min}})_{t \geq 0} \) that satisfies the master equation
\[
\frac{d}{dt} \langle u, P_{t}^{\text{min}}(x)v \rangle = \mathcal{L}(P_{t}^{\text{min}}(x))[u, v], \quad P_{0}^{\text{min}}(x) = x, \tag{2.2}
\]
u, v \in \text{dom}G, \ x \in \mathcal{B}, which it is shown to be equivalent with the integral equation
\[
\frac{d}{dt} \langle u, P_{t}^{\text{min}}(x)v \rangle = \langle u, W_{t}^{*}xW_{t}v \rangle + \int_{0}^{t} d\tau \phi(P_{\tau}^{\text{min}}(x))[W_{t-\tau}u, W_{t-\tau}v],
\]
u, v \in \text{dom}G, \ x \in \mathcal{B}.

The minimal qds \( (P_{t}^{\text{min}})_{t \geq 0} \) is not necessarily conservative and the problem of finding necessary and sufficient conditions for its conservativity has received the attention of the people working in this topic. A. M. Chebotarev [2] and F. Fagnola [5] have found necessary and sufficient or only sufficient conditions for the conservativity of the class of minimal qds whose formal generator satisfy the additional necessary condition

(iii’) \( \mathcal{L}(I)[u, v] = 0, \ \forall u, v \in \text{dom}G, \)

which is an stronger form of (iii).

Our aim in this work is to study necessary and sufficient conditions for non-conservativity of the class of minimal qds whose formal generator satisfy only the conditions (i)-(iii). As a corollary we will obtain well known criteria of A.M. Chebotarev and E.B. Davies for conservativity.

3 Criteria for explosion of the minimal qds.

Our analysis is based on the quantity (or observable) \( \mathcal{E}_{t}(I) \), which we call “probability for explosion at time \( t \)”. This quantity is defined as the positive bounded operator given by
\[
\mathcal{E}_{t}(I) := I - P_{t}^{\text{min}}(I).
\]
For the construction of \((P_{t}^{\min})_{t \geq 0}\) it is used the following iterative scheme:

\[
P^{(1)}_{t}(x)[u,v] = \langle u, W_{t}^{*} x W_{t} v \rangle
\]

and

\[
P^{(n)}_{t}(x)[u,v] = \langle u, W_{t}^{*} x W_{t} v \rangle + \int_{0}^{t} d\tau \phi \left( P^{(n-1)}_{\tau}(x) \right) [W_{t-\tau} u, W_{t-\tau} v],
\]

for \(u, v \in \text{dom}G, \ x \in \mathcal{B}\) and \(t \geq 0\) fixed.

It is proved in [2] and [5], that for \(x \geq 0\) the sequence of positive operators \(\left( P^{(n)}_{t}(x) \right)_{n \geq 0}, \ t \geq 0\) fixed, is increasing and bounded, therefore there exists

\[
P^{\min}_{t}(x) = \sup_{n} P^{(n)}_{t}(x),
\]

which is a solution of the master equation (2.2).

Let us consider the sequence \(\left( \mathcal{E}^{(n)}_{t}(I) \right)_{n \geq 0}, \ t \geq 0\) fixed, defined as

\[
\mathcal{E}^{(n)}_{t}(I) := I - P^{(n)}_{t}(I).
\]

We have that

\[
\mathcal{E}^{(1)}_{t}(I)[u,v] = \langle u, (I - W_{t}^{*} W_{t}) v \rangle, \ u, v \in \text{dom}G,
\]

and for \(n \geq 2\) we have

\[
\mathcal{E}^{(n)}_{t}(I)[u] = \langle u, \mathcal{E}^{(1)}_{t}(I) u \rangle + \int_{0}^{t} d\tau \phi \left( \mathcal{E}^{(n-1)}_{\tau}(I) \right) [W_{t-\tau} u] - \int_{0}^{t} d\tau \phi(I) [W_{t-\tau} u],
\]

for \(u \in \text{dom}G\).

By performing a Laplace transform we obtain

\[
\tilde{\mathcal{E}}^{(n)}_{\lambda}(I)[u] = \tilde{\mathcal{E}}^{(1)}_{\lambda}(I)[u] + \int_{0}^{\infty} dt e^{-\lambda t} \phi \left( \tilde{\mathcal{E}}^{(n-1)}_{\lambda}(I) \right) [W_{t} u] - \frac{1}{\lambda} Q_{\lambda}(I)[u]
\]

\[
= \left( \tilde{\mathcal{E}}^{(1)}_{\lambda}(I) - \frac{1}{\lambda} Q_{\lambda}(I) \right) [u] + Q_{\lambda} \left( \tilde{\mathcal{E}}^{(n-1)}_{\lambda}(I) \right) [u],
\]

\(u \in \text{dom}G\), where \(\tilde{\mathcal{E}}^{(n)}_{\lambda}(I)\) is the Laplace transform of the sesquilinear form associated with \(\mathcal{E}^{(n)}_{t}(I)\) and

\[
Q_{\lambda}(x)[u] = \int_{0}^{\infty} dt e^{-\lambda t} \phi(x) [W_{t} u].
\]
It can be shown that $E^{(n)}(I)$ and $Q_n(x)$ are bounded sesquilinear forms in $\mathfrak{h}$ and we shall denote by the same symbols the corresponding bounded operators.

Therefore
\[ E^{(n)}(I)[u] = \ell(I)[u] + Q\left( E^{(n-1)}(I) \right)[u], \quad u \in \text{dom}G, \]
with
\[
\ell(I)[u] = (E^{(1)}(I) - \frac{1}{\lambda}Q\lambda(I))[u] = \int_0^\infty dt e^{-\lambda t} [\langle u, (I - W_t^*W_t)u \rangle
- \int_0^t d\tau \phi(I)[W_{t-\tau}u] = \int_0^\infty dt e^{-\lambda t} \int_0^t d\tau \left( \frac{d}{d\tau} \|W_{t-\tau}u\|^2 - \phi(I)[W_{t-\tau}u] \right)
= \int_0^\infty dt e^{-\lambda t} \int_0^t d\tau \left( \langle GW_{t-\tau}u, W_{t-\tau}u \rangle + \langle W_{t-\tau}u, GW_{t-\tau}u \rangle - \phi(I)[W_{t-\tau}u] \right)
= \int_0^\infty dt e^{-\lambda t} \left( - \int_0^t d\tau \mathcal{L}(I)[W_{t-\tau}u] \right).
\]

Notice that $\ell(I)$ is a positive bounded sesquilinear form since $\mathcal{L}(I)[W_{t-\tau}u] \leq 0, u \in \text{dom} G$.

Consequently we obtain:
\[
E^{(n)}(I)[u] = \ell(I)[u] + Q\lambda \left( \ell(I) + Q\lambda \left( E^{(n-2)}(I) \right) \right)[u]
= \ell(I)[u] + Q\lambda (\ell(I))[u] + Q^2\lambda \left( \ell(I) + Q\lambda \left( E^{(n-3)}(I) \right) \right)[u]
= \ell(I)[u] + Q\lambda (\ell(I))[u] + Q^2\lambda (\ell(I))[u] + \cdots + Q^{n-2}\lambda (\ell(I))[u]
+ Q^{n-1}\lambda \left( E^{(1)}(I) \right) = \sum_{k=0}^{n-2} Q^k\lambda (\ell(I))[u] + Q^{n-1}\lambda \left( \ell(I) + \frac{1}{\lambda}Q\lambda(I) \right)
= \sum_{k=0}^{n-1} Q\lambda (\ell(I))[u] + \frac{1}{\lambda} Q^n\lambda(I), \quad (3.1)
\]
since $E^{(1)}(I) = \ell(I) + \frac{1}{\lambda}Q\lambda(I)$.

It is shown in [2], [5] that the sequence of positive operators $(Q_n^\alpha(I))_{n \geq 1}$ is convergent in $\ast$-weak and strong sense. The sequence $E^{(n)}(I) = I - P_n^{(n)}(I) \geq 0$ is a decreasing sequence of positive elements in $\mathcal{B}$, hence the limit $\lim_n E^{(n)}(I)$
exists in ∗-weak and strong sense. Therefore using the Lebesgue theorem on dominated convergence we obtain

\[ \tilde{E}_\lambda(I)[u] = \int_0^\infty dt e^{-\lambda t} \lim_n \langle u, E_t^n(I)u \rangle = \lim_n \langle u, \tilde{E}_\lambda(I)u \rangle, \]

i.e, \( \tilde{E}_\lambda(I) = \lim_n \tilde{E}_\lambda^{(n)}(I) \) in ∗-weak and strong sense.

From (3.1) we obtain the following explicit formula for \( \tilde{E}_\lambda(I) \):

\[ \tilde{E}_\lambda(I) = \frac{1}{\lambda} \lim_n Q_n^\lambda(I) + \sum_{n \geq 0} Q_n^\lambda(\ell_\lambda(I)), \quad (3.2) \]

the limits taken in ∗-weak or strong sense.

By \( R_{\lambda}^{\min} \) we denote the resolvent map associated with the qds \( (P_t^{\min})_{t \geq 0} \), i.e, for every \( x \in \mathcal{B} \), \( R_{\lambda}^{\min}(x) \) is the operator in \( \mathcal{B} \) defined by means of the sesquilinear form

\[ R_{\lambda}^{\min}(x)[u, v] = \int_0^\infty dt e^{-\lambda t} \langle u, P_t^{\min}(x)v \rangle, \quad u, v \in \mathfrak{h}. \]

Therefore one has the following criterion for explosion or non-conservativity of a minimal qds.

**Theorem 3.1.** If \( \mathcal{L} \) is a formal generator satisfying (i)-(iii), then the following are equivalent

(i) \( (P_t^{\min})_{t \geq 0} \) is non-conservative (or explosive)

(ii) \( \ell_\lambda(I) \neq 0 \) or \( \lim_n Q_n^\lambda(I) \neq 0 \).

(iii) \( R_{\lambda}^{\min}(I) < \frac{1}{\lambda} I \)

**Proof.** The equivalence of (i) and (ii) follows directly from (3.2).

Notice that for \( \lambda > 0 \)

\[ R_{\lambda}^{\min}(I)[u] = \int_0^\infty dt e^{-\lambda t} \langle u, P_t^{\min}(I)u \rangle = \int_0^\infty dt e^{-\lambda t} \|u\|^2 - \int_0^\infty dt e^{-\lambda t} \langle u, E_t(I)u \rangle = \frac{1}{\lambda} \|u\|^2 - \tilde{E}_\lambda(I)[u], \]

therefore \( R_{\lambda}^{\min}(I) < \frac{1}{\lambda} I \) if and only if \( \tilde{E}_\lambda(I) > 0 \). This proves that (i) and (iii) are equivalent. \( \square \)

As a simple corollary we obtain Chebotarev’s criterion for conservativity.
Corollary 3.2. If in addition $L$ satisfies the condition (iii)' then $\ell_{\lambda}(I) = 0$ and $\tilde{E}_{\lambda}(I) = \frac{1}{\lambda} \lim_{n} Q_{\lambda}^{n}(I)$. Hence $(P_{t}^{\text{min}})_{t \geq 0}$ is conservative if and only if $\lim_{n} Q_{\lambda}^{n}(I) = 0$.

**Proof.** Condition (iii)' implies that $L(I)[W_{t}u] = 0$, $\forall t \geq 0$ and $u \in \text{dom} \ G$. Therefore

$$\ell_{\lambda}(I)[u] = \int_{0}^{\infty} dt e^{-\lambda t} \left( - \int_{0}^{t} d\tau L(I)[W_{t-\tau}u] \right) = 0, \ \forall u \in \text{dom} \ G.$$  

This implies that $\ell_{\lambda}(I) = 0$ as an element of $B$ since $\text{dom} \ G$ is dense in $h$. Hence, it follows from (3.2) that

$$\tilde{E}_{\lambda}(I) = \frac{1}{\lambda} \lim_{n} Q_{\lambda}^{n}(I)$$

in weak, $\ast$-weak and strong sense. \hfill $\square$

If $E_{t}(x) := x - P_{t}^{\text{min}}(x)$, $x \in B$, then we have from the master equation (2.2) that

$$E_{t}(x)[u, v] = - \int_{0}^{t} d\tau L(P_{\tau}^{\text{min}}(x))[u, v] =$$

$$= \int_{0}^{t} d\tau L(E_{\tau}(x))[u, v] - tL(x)[u, v], \ u, v \in \text{dom} \ G.$$

Hence with $x = I$ we obtain that

$$E_{t}(I)[u, v] = -tL(I)[u, v] + \int_{0}^{t} d\tau L(E_{\tau}(I))[u, v],$$

$u, v \in \text{dom} \ G$.

Assuming that (iv) $L(I)[u, v] = 0$, $\forall u, v \in D \subset \text{dom} \ G$, we obtain $D$ a dense subspace of $h$, after performing a Laplace transform, that

$$\tilde{E}_{\lambda}(I)[u, v] = \frac{1}{\lambda} \int_{0}^{\infty} dt e^{-\lambda t} L(E_{t}(I))[u, v] = \frac{1}{\lambda} L(\tilde{E}_{\lambda}(I))[u, v],$$

$u, v \in D$. 

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Therefore $\tilde{E}_\lambda(I)$ is a positive solution of the equation.

$$\mathcal{L}(x)[u, v] = \lambda\langle u, xv \rangle, \; u, v \in D, \; \lambda > 0, \; x \in \mathcal{B}. \quad (3.3)$$

Notice that (iv) is a weaker form of Chevotarev’s condition (iii)’. The dense subspace $D$ is not necessarily a core for $G$.

**Theorem 3.3.** Assume that $\mathcal{L}$ is a formal generator satisfying (i)-(iii) and (iv). Then the following are equivalent

(i) $(P_{t_{\min}}^\lambda)_{t \geq 0}$ is non-conservative,

(ii) There exists a positive, bounded solution $x$ of (3.3) for some $\lambda > 0$.

**Proof.** If $(P_{t_{\min}}^\lambda)_{t \geq 0}$ is non-conservative, equation (3.3) has the nontrivial solution $0 < x = \frac{\tilde{E}_\lambda(I)}{\|\tilde{E}_\lambda(I)\|_\infty}$ for any $\lambda > 0$.

Conversely, if $0 < x \leq I$ is a positive bounded solution of (3.3) for some $\lambda > 0$, then $\mathcal{L}(x)$ has a bounded extension to the whole $\mathfrak{h}$ and $\mathcal{L}(x)[u, v] = \lambda\langle u, xv \rangle$ holds for every $u, v \in \text{dom} G$, therefore for any $u \in \text{dom} G$

$$e^{-\lambda t}\phi(x)[W_t u] = e^{-\lambda t}(\langle GW_t u, xW_t u \rangle + \langle W_t u, xGW_t u \rangle + \lambda\langle W_t u, xW_t u \rangle)$$

$$= -\frac{d}{dt} e^{-\lambda t}\langle W_t u, xW_t u \rangle. \quad (3.4)$$

hence

$$Q_\lambda(x)[u] = \int_0^\infty dt e^{-\lambda t}\phi(x)[W_t u] = -\int_0^\infty dt \frac{d}{dt} e^{-\lambda t}\langle W_t u, xW_t u \rangle = \langle u, xu \rangle,$$

$u \in \text{dom} G$, i.e.,

$$Q_\lambda(x) = x. \quad (3.5)$$

Since $\phi(x)$ is positive and

$$\phi(x)[W_t u] \leq \|x\|_\infty \phi(I)[W_t u] \geq -\frac{d}{dt}\|W_t u\|^2,$$

we obtain from (3.4) that

$$-\frac{d}{dt} e^{-\lambda t}\langle W_t u, xW_t u \rangle \leq -e^{-\lambda t} \frac{d}{dt}\|W_t u\|^2, \; u \in \text{dom} G.$$
Therefore
\[ \langle u, xu \rangle = -\int_0^\infty dt \frac{d}{dt} e^{-\lambda t} \langle W_t u, xW_t u \rangle \leq -\int_0^\infty dt e^{-\lambda t} \frac{d}{dt} \|W_t u\| \]
\[ = \|u\|^2 - \lambda \int_0^\infty dt e^{-\lambda t} \|W_t u\| = \lambda \int_0^\infty dt e^{-\lambda t} \langle u, (I - W_t^* W_t) \rangle \]
\[ = \lambda \langle u, \tilde{E}_\lambda^{(1)}(I) u \rangle, \]
consequently
\[ 0 < x \leq \lambda \tilde{E}_\lambda^{(1)}(I) = \lambda \ell(I) + Q(I), \]
If \( \ell(I) \neq 0 \) the proof is finished. In the case \( \ell(I) = 0 \), from the above estimate we obtain
\[ 0 < x \leq Q(I), \]
hence using (3.5) one gets for every \( n \geq 1 \),
\[ 0 < x = Q^n(I) \leq Q^n(I). \]
Therefore
\[ 0 < x \leq \lim_n Q^n(I), \]
and this proves that \( (P^{\text{min}}_t)_{t \geq 0} \) is non-conservative.

The predual semigroup \( (P_t^\dagger)_{t \geq 0} \) of a qds \( (P_t)_{t \geq 0} \) is the family of bounded operators on the Banach space \( (\mathcal{T}(h), \| \cdot \|_1) \) of trace-class operators with the trace norm \( \| \rho \|_1 = \text{tr}|\rho| \), defined by means of the relation
\[ \text{tr}(P_t(x) \rho) = \text{tr}(x P_t^\dagger(\rho)), \]
for \( x \in \mathcal{B} \) and \( \rho \in \mathcal{T}(h) \).

If \( \rho = |v\rangle\langle u| \) is the projector \( (|v\rangle\langle u|) \omega := \langle u, \omega \rangle v \), for \( u, v, \omega \in \mathfrak{h} \), in particular we have that
\[ \langle u, P_t(x)v \rangle = \text{tr}(P_t(x)|v\rangle\langle u|) = \text{tr}(x P_t^\dagger(|v\rangle\langle u|)), \]
\( x \in \mathcal{B} \).

Since \( (P_t)_{t \geq 0} \) is \( \omega^* \)-continuous, therefore \( P_t^\dagger \) is continuous with respect to the weak topology on \( \mathcal{T}(h) \). Hence, by a well known result (see [1], Corollary 3.1.8, p. 168), \( (P_t^\dagger)_{t \geq 0} \) is strongly continuous and therefore a \( C_0 \)-semigroup in \( \mathcal{T}(h) \) and the weak and strong generators coincide.

We need the following assumption on the CP coefficient \( \phi \) of the formal generator \( \mathcal{L} \).
(v) There exists a Hilbert space $\mathfrak{k}$ with the inner product $\langle \langle \cdot, \cdot \rangle \rangle$, densely and continuously included in $\mathfrak{h}$, and $\phi(I)$ is a bounded sesquilinear form on $\mathfrak{k} \times \mathfrak{k}$. Moreover we assume that $D \subset \mathfrak{k}$.

From (v) and the Lax-Milgram Theorem it follows that there exists a positive and self-adjoint operator $\Lambda$ on $\mathfrak{h}$, with $\text{dom}\Lambda^{1/2} = \mathfrak{k}$ and

$$\langle \langle u, v \rangle \rangle = \langle \Lambda^{1/2}u, \Lambda^{1/2}v \rangle,$$

for any $u, v \in \mathfrak{h}$. Furthermore we can assume $\Lambda \geq I$.

On $\mathcal{T}(\mathfrak{h})$ let us consider the injective, contractive and completely positive linear map $\beta : \mathcal{T}(\mathfrak{h}) \to \mathcal{T}(\mathfrak{h})$ defined by

$$\beta(\rho) = \Lambda^{-1/2}\rho\Lambda^{-1/2}.$$

A linear map $\beta : \mathcal{T}(\mathfrak{h}) \to \mathcal{T}(\mathfrak{h})$ is completely positive if

$$\sum_{i,j} \text{tr}\left[ x_i x_j \beta(\sigma_i^* \sigma_j) \right] \geq 0$$

for any pair of sequences $(x_j) \subset \mathcal{B}$, $(\sigma_j) \subset \mathcal{T}_2(\mathfrak{h})$, where $\mathcal{T}_2$ is the space of Hilbert-Schmidt operators in $\mathfrak{h}$.

$\widetilde{T} = \beta(\mathcal{T})$ will denote the range of $\beta$, $\widetilde{T}$ has a natural structure of Banach space with the norm

$$\|\rho\|_{\widetilde{T}} := \|\beta^{-1}(\rho)\|_1, \quad \rho \in \widetilde{T}.$$  

The map $\beta$ results to be an isometric isomorphism from $\mathcal{T}$ onto $\widetilde{T}$.

We shall denote by $\mathcal{V}$ the subspace of $\mathcal{T}(h)$ of rank-one operators $|u\rangle\langle u|$, $u, v \in D$. Since $D \subset k$, it follows that $\mathcal{V} \subset \mathcal{T}$.

Assumption (v) was introduced in [3] in a different context. Some important consequences of this assumption where studied in [3] and [6]. In particular it was proved there that for any map $\phi$ satisfying (ii) and (v) there exists a map $\phi^\dagger : \mathcal{T} \to \mathcal{T}$ contractive and completely positive satisfying the relation

$$\text{tr}\left[ x\phi^\dagger(|v\rangle\langle u|) \right] = \phi(x)[u, v],$$

for any $x \in \mathcal{B}$ and $u, v \in D$.

An element $\rho \in \mathcal{T}(h)$ belongs to the domain $\text{dom}\mathcal{L}^\dagger$ of the generator $\mathcal{L}^\dagger$ of $(\mathcal{P}^\dagger_t)_{t \geq 0}$ if and only if there exists the limit

$$\lim_{t \to 0^+} \frac{1}{t} \|\mathcal{P}^\dagger_t(\rho) - \rho\|_1.$$
But, since the weak and strong generators coincide, $\rho \in \text{dom}L^\dagger$ if and only if for every $x \in \mathcal{B}$ the limit
\[
\lim_{t \to 0^+} \frac{1}{t} \text{tr} \left( x(P_t^\dagger(\rho) - \rho) \right)
\]
exists.

The following is another criterion for explosion of a minimal qds.

**Proposition 3.4.** If $\mathcal{L}$ is a formal generator satisfying (i)-(iii) and (iv) with $\phi$ satisfying condition (v). Then the subspace $\mathcal{V}$ generated by the rank-one operators $\langle v | u \rangle$, $u, v \in D$ is contained in the domain $\text{dom}L^\dagger$ of the generator of the predual semigroup $(P_{t_{\min}}^\dagger)$ and
\[
\mathcal{L}(|v\rangle\langle u|) = \phi^\dagger(|v\rangle\langle u|) - |v\rangle\langle Gu| - |Gv\rangle\langle u|.
\]

Moreover the following conditions are equivalent:
(i) $(P_{t_{\min}}^\dagger)_{t \geq 0}$ is non-conservative.
(ii) The orthogonal complement (or annihilator) in $\mathcal{B}$ of $(\lambda - \mathcal{L}^\dagger)(\mathcal{V})$ is non-trivial for any $\lambda > 0$.

**Proof.** For $u, v \in D$ and every $x \in \mathcal{B}$ the master equation (2.2) can be written in the form
\[
\text{tr}(P_{t_{\min}}^\dagger(x)|v\rangle\langle u|) = \text{tr}(x|v\rangle\langle u|) + \int_0^t d\tau \mathcal{L}(P_{t_{\min}}^\dagger(x))[u, v].
\]

Therefore we have
\[
\frac{1}{t} \text{tr} \left( x \left( P_{t_{\min}}^\dagger(|v\rangle\langle u|) - |v\rangle\langle u| \right) \right) = \frac{1}{t} \int_0^t d\tau \left( \phi(P_{t_{\min}}^\dagger(x))[u, v] - \langle Gu, P_{t_{\min}}^\dagger(x)v \rangle - \langle u, P_{t_{\min}}^\dagger(x)Gv \rangle \right) =
\]
\[
= \frac{1}{t} \int_0^t d\tau \text{tr} \left( P_{t_{\min}}^\dagger(x) \left[ \phi^\dagger(|v\rangle\langle u|) - |v\rangle\langle Gu| - |Gv\rangle\langle u| \right] \right) =
\]
\[
= \frac{1}{t} \int_0^t d\tau \text{tr} \left( xP_{t_{\min}}^\dagger(\phi^\dagger(|v\rangle\langle u|) - |v\rangle\langle Gu| - |Gv\rangle\langle u|) \right).
\]

From the weak continuity of $(P_{t_{\min}}^\dagger)_{t \geq 0}$ we obtain
\[
\lim_{t \to 0^+} \frac{1}{t} \text{tr} \left( x \left( P_{t_{\min}}^\dagger(|v\rangle\langle u|) - |v\rangle\langle u| \right) \right) =
\]
\[
= \text{tr} \left( x \left( \phi^\dagger(|v\rangle\langle u|) - |v\rangle\langle Gu| - |Gv\rangle\langle u| \right) \right).
\]
This proves that $\mathcal{V} \subset \text{dom}\mathcal{L}^\dagger$ and

$$\mathcal{L}^\dagger(|v\rangle\langle u|) = \phi^\dagger(|v\rangle\langle u|) - |u\rangle\langle Gu| - |Gv\rangle\langle u|$$

To prove the equivalence of conditions (i) and (ii) observe that $x$ is an element in the orthogonal complement in $\mathcal{B}$ of $(\lambda - \mathcal{L}^\dagger)(\mathcal{V})$ for some $\lambda > 0$, if and only if

$$0 = \text{tr} \left( x (\lambda - \mathcal{L}^\dagger) (|v\rangle\langle u|) \right) = \text{tr} \left( x (\lambda|v\rangle\langle u| - \mathcal{L}^\dagger (|v\rangle\langle u|)) \right)$$

$$= \text{tr} \left( x (\lambda|v\rangle\langle u| - \phi^\dagger (|v\rangle\langle u|) + |v\rangle\langle Gu| + |Gv\rangle\langle u|) \right)$$

$$= (\lambda - \mathcal{L})(x)[u, v],$$

for any $u, v \in D$. The result follows from the equivalence of conditions (i) and (ii) in Theorem 3.3.

The following Corollary is an extension of a criterion for conservativity due to E. B. Davis (see [5], Prop. 3.3.2).

**Corollary 3.5.** Assume that $\mathcal{L}$ is a formal generator satisfying (i)-(iii) and (iv) with $\phi$ satisfying condition (v). Then the following are equivalent

(i) $(P^\text{min}_t)_{t \geq 0}$ is conservative.

(ii) The subspace $\mathcal{V}$ of rank-one operators $|v\rangle\langle u|$, $u, v \in D$ is a core for $\mathcal{L}^\dagger$.

**Proof.** The subspace $\mathcal{V}$ is dense in $\mathcal{T}(\mathfrak{h})$. By Proposition 3.1 in [4], $\mathcal{V}$ is a core for $\mathcal{L}^\dagger$ if and only if $R(\lambda - \mathcal{L}^\dagger) = (\lambda - \mathcal{L}^\dagger)(\mathcal{V})$ is dense in $\mathcal{T}(\mathfrak{h})$ for some $\lambda > 0$. This condition holds if and only if the orthogonal complement (or annihilator) in $\mathcal{B}$ of $(\lambda - \mathcal{L}^\dagger)(\mathcal{V})$ is trivial for some $\lambda > 0$. The result follows from Proposition 3.4.

4 Examples.

**Example 4.1.** On $\mathfrak{h} = L_2(0, \infty)$ we shall consider operators induced by the differential form

$$\tau_f u = \frac{1}{2i} ((fu)' + fu'),$$

where $f \in C^\infty(0, \infty)$, $f > 0$, $f'$ is bounded and $\int_0^\infty dx f(x)^{-1} = \infty$. Notice that the function $f(x) = (1 + x)^\alpha$, $0 \leq \alpha \leq 1$ satisfies these conditions.
We denote by $H_{1,0}$ the minimal operator induced by $\tau_f$, it is defined by

$$\text{dom} H_{1,0} = C_0^\infty(0, \infty) \quad \text{and} \quad H_{1,0} u = \tau_f u, \quad u \in \text{dom} H_{1,0}. $$

The maximal operator $H_1$ induced by $\tau_f$ is defined by

$$\text{dom} H_1 = \{ u \in L_2(0, \infty) : u \text{ is absolutely continuous and } \tau_f u \in L_2(0, \infty) \}$$

and

$$H_1 u = \tau_f u, \quad u \in \text{dom} H_1. $$

One can show that $H_0$ is a symmetric operator and that $H_{1,0}$ and $H_1$ are formal adjoints of each other. Moreover if $u \in \text{dom} H_{1,0}$ following [7], Theorem 6.29, pg. 160, one can see that $u(x) = \omega(x) + cf^{1/2}(x)$, a.e. in $(0, \infty)$, where $\omega$ is absolutely continuous and

$$\tau_f \omega = H_{1,0}^* u.$$

Therefore $u$ is absolutely continuous, and $\tau_f u = \tau_f \omega = H_1^* u \in L_2(0, \infty)$; since $u \in L_2(0, \infty)$, we can conclude that $u \in \text{dom} H_1$ and this proves that $H_{1,0}^* = H_1$.

Being symmetric the operator $H_{1,0}$ is closable and its closure $\overline{H}_{1,0}$ is symmetric, moreover $\overline{H}_{1,0} = H_{1,0}^* = H_1$. Notice that

$$\text{dom} \overline{H}_{1,0} = \{ u \in \text{dom} H_1 : u(0) = 0 \}.$$ 

Now let us consider the equations

$$H_{1,0}^* u = \pm i u, \quad u \in \text{dom} H_{1,0}^*.$$ 

The solutions of these equations are respectively

$$u_+(x) = c_1 f(x)^{1/2} e^{-\int_0^x \frac{d\tau}{f(\tau)}},$$

and

$$u_-(x) = c_2 f(x)^{-1/2} e^{\int_0^x \frac{d\tau}{f(\tau)}}, \quad c_1, c_2 \text{ nonzero.}$$

Notice that

$$\|u_+\|^2 = c_1^2 \int_0^\infty dx f(x)^{-1} e^{-2\int_0^x \frac{d\tau}{f(\tau)}} = \frac{1}{2} c_1^2 < \infty,$$

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since \( \int_0^\infty \frac{d\tau}{f(\tau)} = \infty \), therefore \( u_+ \in L_2(0, \infty) \). Similarly one can show that 
\( u_- \not\in L_2(0, \infty) \). This proves that the defect indices of the symmetric operator 
\( \tilde{H}_{1,0} \) are 
\( n_+ (\tilde{H}_{1,0}) = 1 \) and 
\( n_- (\tilde{H}_{1,0}) = 0 \).

By the von Neumann Theorem we have that 
\[ \text{dom}\, H_1 = \text{dom}\, \tilde{H}_{1,0} \oplus N_+ \oplus N_- \]
and
\[ H_1 (\omega + v_+ + v_-) = \tilde{H}_{1,0} \omega + iv_+ - iv_- , \]
\( \omega \in \text{dom}\, \tilde{H}_{1,0}, \ v_+ \in N_+, \ v_- \in N_- \), where
\[ N_+ = \mathcal{N}(i-H_1) = \mathcal{R}(-i-\tilde{H}_{1,0})^\perp \]
and 
\[ N_- = \mathcal{N}(-i-H_1) = i = \mathcal{R}(i-\tilde{H}_{1,0})^\perp \]
are the defect subspaces of \( \tilde{H}_{1,0} \) and \( \oplus \) denotes direct sum.

But we have shown that \( N_- = \{0\} \), therefore 
\[ \text{dom}\, H_1 = \text{dom}\, \tilde{H}_{1,0} \oplus N_+ \]
and
\[ H_1 (\omega + v_+) = \tilde{H}_{1,0} \omega + iv_+, \]
\( \omega \in \text{dom}\, \tilde{H}_{1,0}, \ v_+ \in N_+ \).

Then for \( u \in \text{dom}\, H_1, \ u = \omega + v_+, \ \omega \in \text{dom}\, \tilde{H}_{1,0}, \ v_+ \in N_+ \) we have that
\[ \langle iH_1 u, u \rangle = -i\langle \tilde{H}_{1,0} \omega, \omega \rangle + 2i \text{Im} \langle \omega, v_+ \rangle - \|v_+\|^2 , \]
hence
\[ Re \langle iH_1 u, u \rangle = -\|v_+\|^2 \leq 0. \]
This proves that \( iH_1 \) is dissipative.

If \( \Theta(iH_1) = \{ \langle iH_1 u, u \rangle : u \in \text{dom}\, H_1, \ \|u\| = 1 \} \) is the numerical range of 
\( iH_1 \), then we have 0 for \( \lambda_0 > 0 \)
\[ \delta = \text{dist.} (\lambda_0, \overline{\Theta(iH_1)}) > 0 . \]

Therefore
\[ \delta \leq \| (iH_1 u, u) - \lambda_0 \| = \| (iH_1 - \lambda_0 I) u, u \| \leq \| (iH_1 - \lambda_0 I) u \| , \]
for any \( u \in \text{dom}H_1, \|u\| = 1 \). Hence the operator \((iH_1 - \lambda_0 I)^{-1}\) there exists, it is bounded and closed on \( R(iH_1 - \lambda_0 I) \). Then \( R(iH_1 - \lambda_0) \) is closed and therefore

\[ R(iH_1 - \lambda_0 I) = \mathfrak{h}, \]

since \( R(iH_1 - \lambda_0(I)^\perp = \{0\} \) for any \( \lambda_0 > 0 \).

By the Lumer-Phillips Theorem we conclude that \(-G = iH_1\) is the generator of a \( C_0\)-semigroup of contractions \((W_t)_{t \geq 0}\) in \( \mathfrak{h} \).

Let us consider the formal generator \( \mathcal{L} \) that associates with every element \( x \in \mathcal{B} = \mathcal{B}(L_2(0, \infty)) \) the sesquilinear form

\[ \mathcal{L}(x)[u, v] = -\langle Gu, xv \rangle - \langle u, xGv \rangle, \quad (4.1) \]

\( u, v \in \text{dom}G \). In this case the CP part of \( \mathcal{L} \) is zero, \( \phi(x) = 0, x \in \mathcal{B} \).

\( \mathcal{L} \) satisfies conditions (i)-(iii) in section 3 and for \( u, v \in D = \text{dom}\bar{H}_{1,0} \subset \text{dom}H_1 = \text{dom}G \) we have that

\[ \mathcal{L}(I)[u, v] = \langle i\bar{H}_{1,0}u, v \rangle + \langle u, i\bar{H}_{1,0}v \rangle = i(-\langle \bar{H}_{1,0}u, v \rangle + \langle u, \bar{H}_{1,0}v \rangle) = 0, \]

since \( H_{1,0} \) is symmetric. Hence \( \mathcal{L} \) satisfies also condition (iv) in the previous section with \( D = \text{dom}\bar{H}_{1,0} \).

Since \( \phi = 0 \) we have that \( \lim_n Q^n_\lambda(I) = 0 \). If \( \ell_\lambda(I) = 0 \), we have from (4.1) that for every \( u \in \text{dom}G, \int_0^t \tau \mathcal{L}(I)(W_{t-}\tau) = 0 \) a.e. \( t \geq 0 \) and taking derivative we obtain \( \mathcal{L}(I)[u] = 0 \) for all \( u \in \text{dom}G \). This implies that \( H_1 \) is symmetric, but we know that \( \bar{H}_{1,0} \) is maximal symmetric and \( \bar{H}_{1,0} \subset \subset H_1 \). Therefore \( \ell_\lambda(I) \neq 0 \) and the minimal semigroup constructed from the formal generator \( \mathcal{L} \) is non-conservative.

The minimal qds constructed from the formal generator (4.1) is

\[ P_t^{\text{min}}(x) = W_t^*xW_t, \quad x \in \mathcal{B}. \]

Observe that \((P_t^{\text{min}})_{t \geq 0}\) is conservative \((P_t^{\text{min}}(I) = I, \ t \geq 0)\), if and only if the \( C_0\)-semigroup \((W_t)_{t \geq 0}\) is a semigroup of isometries: \( \|W_t u\| = \|u\|, u \in \mathfrak{h} \).
Example 4.2. The adjoint semigroup \((U_t)_{t \geq 0}\) defined by \(U_t = W_t^*\), \(t \geq 0\) is a strongly continuous semigroup of contractions with the infinitesimal generator \((-G)^* = -iH_1^* = -i\bar{H}_{1,0}\). The associated minimal qds
\[
P_t^{\min}(x) = U_t^* x U_t, \quad t \geq 0, \quad x \in \mathcal{B},
\]
is conservative, because its formal generator is defined by
\[
\mathcal{L}(x)[u, v] = -\langle i\bar{H}_{1,0}u, xv \rangle - \langle u, x\bar{H}_{1,0}v \rangle.
\]
for \(x \in \mathcal{B}, \ u, v \in \text{dom} \bar{H}_{1,0}\). Therefore we have \(\phi(x) = 0, \ x \in \mathcal{B}\) and hence \(\lim_n Q^n_s(I) = 0\); moreover for \(u \in \text{dom} \bar{H}_{1,0}\)
\[
\ell_s(I)[u] = \int_0^\infty dte^{-\lambda t} \int_0^t d\tau \mathcal{L}(I)[W_{t-\tau}u] = 0,
\]
since \(\bar{H}_{1,0}\) is symmetric.

Example 4.3. Take \(\mathfrak{h}\) and \(H_1\) as in Example 4.1 and consider the CP map that associates with every element \(x \in \mathcal{B}\) the sesquilinear form defined for \(u, v \in \text{dom} H_1\) by
\[
\phi(x)[u, v] = \langle Lu, xLv \rangle,
\]
where \(L\) is the operator of multiplication by a complex-valued function \(\ell(s), \ s \in (0, \infty)\). Then we have that
\[
\phi(I)[u, v] = \langle u, |\ell|^2 v \rangle,
\]
\(u, v \in \text{dom} H_1\), i.e., \(\phi(I)\) coincides with the operator of multiplication by the positive function \(|\ell(s)|^2, \ s \in (0, \infty)\).

Assume that \(-G = -\frac{1}{2}\phi(I) + iH_1\), with \(\text{dom} G = \text{dom} H_1\), is the generator of a strongly continuous semigroup of contractions in \(\mathfrak{h}\), \((W_t)_{t \geq 0}\), and let us consider the formal generator \(\mathcal{L}\) that associates with every element \(x \in \mathcal{B}\) the sesquilinear form
\[
\mathcal{L}(x)[u, v] = \phi(x)[u, v] - \langle Gu, xu \rangle - \langle u, xGv \rangle,
\]
\(u, v \in \text{dom} G\).
\( \mathcal{L} \) satisfies conditions (i) and (ii) in Section 2, moreover

\[
\mathcal{L}(I)[u] = \phi(I)[u] - \left( \frac{1}{2} \phi(I) - iH_1 \right) u, u \right) \\
- \left( u, \left( \frac{1}{2} \phi(I) - iH_1 \right) u \right) = \langle u, |\ell|^2 u \rangle - \frac{1}{2} \langle |\ell|^2 u, u \rangle \\
+ \langle iH_1 u, u \rangle - \frac{1}{2} \langle |\ell|^2 u \rangle + \langle u, iH_1 u \rangle \\
= 2Re\langle iH_1 u, u \rangle \leq 0,
\]

since \( iH_1 \) is dissipative. Hence \( \mathcal{L} \) satisfies also condition (iii) in Section 2.

Notice that for \( u \in \text{dom} \bar{H}_{1,0} \) we have that

\[
\mathcal{L}(I)[u] = 2Re\langle iH_1 u, u \rangle = 2Re\langle i\bar{H}_{1,0} u, u \rangle = 0,
\]

since \( \bar{H}_{1,0} \) is symmetric. Therefore \( \mathcal{L} \) satisfies our condition (iv) in the previous section with \( D = \text{dom} \bar{H}_{1,0} \).

The minimal qds constructed from this formal generator \( \mathcal{L} \) is non-conservative because as in Example 4.1, \( \ell_\lambda(I) = 0 \) implies that \( H_1 \) is symmetric but \( \bar{H}_{1,0} \not\subseteq H_1 \) and \( \bar{H}_{1,0} \) is maximal symmetric.

To observe the connection of the conservativity problem for formal generators (4.1) with the von Neumann theory of the defect indices of a symmetric operator, we prove the following.

**Proposition 4.4.** Let \( \mathcal{L} \) be the formal generator given by equation (4.1). Then the following conditions are equivalent

(i) The defect index \( n_+(\bar{H}_{1,0}) \) of the closed symmetric operator

\( \bar{H}_{1,0} = iG|_D \) is positive, \( n_+(\bar{H}_{1,0}) > 0 \).

(ii) The equation

\[
\mathcal{L}(x)[u, v] = \lambda \langle u, xv \rangle, \quad u, v \in \text{dom} \bar{H}_{1,0},
\]

has a positive, bounded solution \( x \in \mathcal{B} \) for some \( \lambda > 0 \).
Proof. Assume that $N_+(\bar{H}_{1,0}) = \mathcal{N}(i - H_{1,0}^*) \neq \{0\}$ and take $u \in N_+(\bar{H}_{1,0})$, $u \neq 0$. Let $x \in \mathcal{B}$ be the projector $x = |u\rangle\langle u|$, then we have for every $v \in \text{dom}\bar{H}_{1,0}$ that

$$\mathcal{L}(x)[v] = \langle i\bar{H}_{1,0}v, |u\rangle\langle u|v\rangle + \langle v, |u\rangle\langle u|i\bar{H}_{1,0}v\rangle$$

$$= -i\langle v, H_{1,0}^*u \rangle\langle u, v\rangle + i\langle v, u \rangle\langle H_{1,0}^*u, v\rangle$$

$$= -i\langle u, i\bar{u} \rangle\langle u, v\rangle + \langle v, u \rangle\langle i\bar{u}, v\rangle = 2\langle u, v \rangle\langle v, u\rangle$$

$$= 2\langle v, xv\rangle.$$

Using the polarization identity we obtain that

$$\mathcal{L}(x)[u, v] = 2\langle u, xv\rangle, \quad u, v \in \text{dom}\bar{H}_{1,0}.$$ 

Therefore (ii) holds with $\lambda = 2$ if $n_+(\bar{H}_{1,0}) > 0$.

Conversely, assume that (ii) holds and $n_+(\bar{H}_{1,0}) = 0$, i.e., $N_+(\bar{H}_{1,0}) = \mathcal{N}(i - H_{1,0}^*) = \{0\}$. Therefore

$$\mathcal{R}(I - i\bar{H}_{1,0}) \perp = \mathcal{R}(-i - H_{1,0}^*) \perp = \mathcal{N}(i - H_{1,0}^*) = \{0\}.$$ 

Take $u \in \text{dom}G = \mathcal{R}((I + G^{-1}))$ and let $v = (I + G)u$. Since $(I - i\bar{H}_{1,0})\text{dom}\bar{H}_{1,0} = \mathcal{R}(I - iH_{1,0})$ is dense in $\mathfrak{h}$, for any $\epsilon > 0$ there exists $v_\epsilon = (I - i\bar{H}_{1,0})u_\epsilon = (I + G)u_\epsilon$, $u_\epsilon \in \text{dom}\bar{H}_{1,0}$, such that $\|v - v_\epsilon\| < \epsilon$.

Therefore we have that

$$\|u - u_\epsilon\|^2 = \|(I + G)^{-1}v - (I + G)^{-1}v_\epsilon\| \leq \|v - v_\epsilon\| < \epsilon,$$

by the Hille-Yosida Theorem.

Then we have proved that $u_\epsilon \in \text{dom}\bar{H}_{1,0}$, $u_\epsilon \rightarrow u$ and $(I - i\bar{H}_{1,0})u_\epsilon \rightarrow (I + G)u$, hence $-i\bar{H}_{1,0}u_\epsilon \rightarrow Gu_\epsilon$, as $\epsilon \rightarrow 0$. This implies that $u \in \text{dom}\bar{H}_{1,0}$ and hence $iH_1 = -G = i\bar{H}_{1,0}$, i.e., $H_1 = \bar{H}_{1,0}$.

The relation $\mathcal{L}(x)[u, v] = \lambda\langle u, xv\rangle$ holds for $u, v \in \text{dom}\bar{H}_{1,0}$ and some $\lambda > 0$, therefore for every $t \geq 0$ and $u, v \in \text{dom}G$, we have that

$$-\lambda\langle W_tu, xW_tv\rangle - \langle GW_tu, xW_tv\rangle - \langle W_tu, xGW_tv\rangle = 0.$$ 

Equivalently we have that

$$\frac{d}{dt}e^{-\lambda t}\langle W_tu, xW_tv\rangle = 0,$$

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and integrating we obtain

\[ 0 = \int_0^\infty \frac{d}{dt} e^{-\lambda t} \langle W_t u, x W_t v \rangle = \langle u, x v \rangle, \]

for all \( u, v \in \text{dom}G \). Then \( x = 0 \) and this finishes the proof.

Since \( n_+(\bar{H}_{1,0}) > 0 \), Proposition 4.2 and Theorem 3.3 give another proof that the minimal qds of Example 4.1 is non-conservative or explosive.

The above proposition holds in the case when \( -G = iH^* \), with \( H \) any maximal symmetric closed operator in a Hilbert space \( \mathfrak{h} \). It says that in the case when \( iH \) is the restriction of a generator of a strongly continuous semigroup of contractions \( (W_t)_{t \geq 0} \) in \( \mathfrak{h} \), then \( (W_t)_{t \geq 0} \) is a semigroup of isometries if and only if \( n_+(H) = 0 \).

Given a closed symmetric operator \( H \) it naturally arises the question of whether or not is \( iH \) the restriction of a generator of a strongly continuous semigroup isometries. The following proposition give an answer.

**Proposition 4.5.** Let \( H \) be a closed symmetric operator in a Hilbert space \( \mathfrak{h} \) with finite defect indices \((n_+, n_-)\), then

(i) if \( n_+ \leq n_- \) the operator \( iH \) is the restriction of a generator of a strongly continuous semigroup isometries in \( \mathfrak{h} \).

(ii) if \( n_+ > n_- \) then \( iH \) is not the restriction of a generator of a strongly continuous semigroup of isometries in \( \mathfrak{h} \).

**Proof.** (i) If \( 0 = n_+ \leq n_- \) then \( H \) is maximal symmetric (or selfadjoint if \( n_- = 0 \)). Therefore the arguments in Example 4.1 help to prove that \( -iH^* \) generates a strongly continuous semigroup of contractions in \( \mathfrak{h} \). The adjoint semigroup \( (U_t = W_t^*)_{t \geq 0} \) is generated by \( iH \), since \( n_+ = 0 \), this semigroup \( (U_t)_{t \geq 0} \) is of isometries by Proposition 4.4. If \( H \) is selfadjoint, \( iH \) generates a unitary group.

If \( 0 < n_+ \leq n_- \), then the defect subspace \( N_+ \) of \( H \) is isometrically isomorphic with a subspace \( F_- \) of \( N_- \), let us denote by \( V \) the isometry \( V : N_+ \to F_- \). By the von Neumann Theorem, associated with \( V \) there exists a closed symmetric extension \( H_V \) of \( H \) defined as

\[ \text{dom}H_V = \text{dom}H + \{ v + Vv : v \in N_+ \} \]
and
\[ H_V(u + v + Vv) = H_u + iv - iVv = H^*(u + v + Vv) \]
for \( u \in \text{dom}H \) and \( v \in N_+ \).

Since \( \mathcal{R}(-i - H_V) = \mathcal{R}(-i - H) + N_+ = \mathcal{R}(-i - H) + \mathcal{R}(-i - H)^\perp = \mathfrak{h} \)
we have that \( n_+(H_V) = 0 \), hence we are in the case \( 0 = n_+(H_V) \leq n_-(H_V) \).

So we can proceed as above to prove that \( iH_V \) generates a \( C_0 \)-semigroup of isometries in \( \mathfrak{h} \), and hence \( iH \) is the restriction of a generator of a strongly continuous semigroup of isometries in \( \mathfrak{h} \).

(ii) If \( n_+ > n_- \) then there exists an isometry \( V' \) from the defect subspace \( N_- \) of \( H \) onto a proper subspace of \( N_+ \), and associated with \( V' \) exists a maximal symmetric extension \( H_{V'} \) of \( H \). The semigroup of contractions generated by the dissipative operator \( iH_{V'}^* \) is not a semigroup of isometries since \( n_+(H_{V'}) > 0 \).

\[ \square \]

**Example 4.6.** In \( \mathfrak{h} = \ell_2(\mathbb{C}) \), with the complete orthonormal system \( (e_n)_{n \geq 0} \), let \( V \) be the isometry defined by
\[ Ve_n = e_{n+1}, \quad n \geq 0. \]

So we have that \( D(V) = \mathfrak{h} \) and \( \mathcal{R}(V) = \text{span}\{e_n, n \geq 1\} \).

Therefore from the von Neumann Theorem, there exists a symmetric operator \( H \) given by the Cayley transform
\[ H = i(I + V)(I - V)^{-1}, \]
if and only if \( \mathcal{R}(I - V) \) is dense in \( \mathfrak{h} \). But \( v \in \mathcal{R}(I - V)^\perp \) implies that
\[ \langle v, e_n - e_{n+1} \rangle = 0, \quad n \geq 0, \]

hence
\[ 0 = \sum_{k=0}^{n-1} \langle v, e_k - e_{k+1} \rangle = \langle v, e_0 \rangle - \langle v, e_n \rangle, \quad n \geq 1, \]
or
\[ \langle v, e_0 \rangle = \langle v, e_n \rangle, \quad n \geq 1. \]

This implies that \( v = 0 \) and hence \( \mathcal{R}(I - V) \) is dense in \( \mathfrak{h} \).
The isometry $V$ is closed, therefore $H$ is closed and $\text{dom} V = \mathcal{R}(i + H)$, $\mathcal{R}(V) = \mathcal{R}(I - H)$. Hence we obtain

$$N_+(H) = \mathcal{R}(-i - H)^\perp = \{0\} \quad \text{and} \quad N_-(H) = \mathcal{R}(i - H)^\perp = \text{span}\{e_0\}. $$

Then $n_+(H) = 0$ and $n_-(H) = 1$. By Proposition 4.4 $iH$ is not the restriction of a generator of a $C_0$-semigroup of isometries.

A similar result is obtained when $V_m$ is the isometry defined by

$$V_m e_n = e_{n+m}, \quad n \geq 0 \quad \text{and} \quad m > 1 \text{ fixed}. $$

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