

\( (\sigma, \tau) \)-DERIVATIONS OF GROUP RINGS

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Abstract. We study \((\sigma, \tau)\)-derivations of a group ring \(RG\) of a finite group \(G\) over an integral domain \(R\) with 1. As an application we extend a well known result on derivation of an integral group ring \(ZG\) to \((\sigma, \tau)\)-derivation on it for a finite group \(G\) with some conditions on \(\sigma\) and \(\tau\). In the process of the extension, a generalization of an application of Skolem-Noether Theorem to derivation on a finite dimensional central simple algebra has also been given for the \((\sigma, \tau)\)-derivation case.

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1. Introduction

Let \(R\) be a commutative ring with 1 and \(A\) be an algebra over \(R\). A derivation on \(A\) is an \(R\)-linear map \(\gamma : A \to A\) satisfying \(\gamma(ab) = \gamma(a)b + a\gamma(b)\) for all \(a, b \in A\). For \(x \in A\), the derivation \(\gamma\) such that \(\gamma(a) = xa - ax\) for all \(a \in A\) is called an inner derivation of \(A\) coming from \(x\). We will denote such a derivation and inner derivation as \(\gamma^\text{usual}\) and \(\gamma^\text{inner}_x\) respectively. Let \(\sigma, \tau\) be two different algebra endomorphisms on \(A\). A \((\sigma, \tau)\)-derivation on \(A\) is an \(R\)-linear map \(\delta\) satisfying \(\delta(ab) = \delta(a)\tau(b) + \sigma(a)\delta(b)\) for \(a, b \in A\). If \(x \in A\), the \((\sigma, \tau)\)-derivation \(\delta_x : A \to A\) such that \(\delta_x(a) = x\tau(a) - \sigma(a)x\), is called a \((\sigma, \tau)\)-inner derivation of \(A\) coming from \(x\). If \(\sigma = \tau = id\), then \(\delta\) and \(\delta_x\) are respectively the usual derivation \(\delta^\text{usual}\) and inner derivation \(\delta^\text{inner}_x\) of \(A\) coming from \(x\). These kinds of derivations were mentioned by Jacobson in [Jac56] (Chapter 7.7). Later on they have been studied extensively in the case of prime and semiprime rings by many authors. A brief history on such cases can be found in [AAH06]. Generalized Witt algebras have been studied with the help of \((\sigma, \tau)\)-derivations in [HLS06], which also contains a study of \((\sigma, \tau)\)-derivations on commutative algebras and unique factorization domains. We however are interested in the \((\sigma, \tau)\)-derivations of group rings. The usual derivation for integral group rings was studied by Spiegel (Theorem 1, [Spi94]). He has shown that for a finite group \(G\), every derivation of \(ZG\) is inner. We extend that result to \((\sigma, \tau)\)-derivations on \(ZG\) over a finite group \(G\) with certain conditions on \(\sigma\) and \(\tau\). The result we have obtained is for a general group ring \(RG\) of a finite group \(G\) over any integral domain \(R\) with 1. The integral group ring case then is obtained as a corollary of the main result. Let \(Z(A)\) denote the center of the algebra \(A\). Our main result can be stated as follows:
Main Theorem 1.1. Let \( G \) be a finite group and \( R \) be an integral domain with 1 with characteristic \( p \geq 0 \) such that \( p \) does not divide the order of \( G \).

(i) If \( R \) is a field and \( \sigma, \tau \) are algebra endomorphisms of \( RG \) such that they fix \( \mathcal{Z}(RG) \) elementwise, then every \((\sigma, \tau)\)-derivation of \( RG \) is \((\sigma, \tau)\)-inner.

(ii) If \( R \) is an integral domain that is not a field and \( \sigma, \tau \) are \( R \)-linear extensions of group homomorphisms of \( G \) such that they fix \( \mathcal{Z}(RG) \) elementwise, then every \((\sigma, \tau)\)-derivation of \( RG \) is \((\sigma, \tau)\)-inner.

Our manuscript has been divided into four sections. The second section contains some properties and interesting results on \((\sigma, \tau)\)-derivations. We have provided a generalized corollary of Skolem-Noether Theorem for the \((\sigma, \tau)\)-derivation case. The third section is devoted to the proof of our main theorem. In the conclusion part we have mentioned how our result can be applied to \( \mathbb{Z}G \).

2. Useful Results

The following are some interesting properties of the \((\sigma, \tau)\)-derivations which follow directly from the definition. The set of all \((\sigma, \tau)\)-derivations on \( A \) will be denoted by \( \mathcal{D}_{(\sigma, \tau)}(A) \).

Property 2.1. If \( A \) is unital, then for any \((\sigma, \tau)\)-derivation \( \delta \), \( \delta(1) = 0 \).

Property 2.2. \( \mathcal{D}_{(\sigma, \tau)}(A) \) is an \( R \)-module as \( \delta_1 \delta_2 \in \mathcal{D}_{(\sigma, \tau)}(A) \) for \( \delta_1, \delta_2 \in \mathcal{D}_{(\sigma, \tau)}(A) \) and \( r \in R \).

Property 2.3. When \( \sigma(x)a = a\sigma(x) \) (or \( \tau(x)a = a\tau(x) \)) for all \( x, a \in A \), and in particular when \( A \) is commutative, \( \mathcal{D}_{(\sigma, \tau)}(A) \) carries a natural left (or right) \( A \)-module structure by \( (a, \delta) \mapsto a\delta : x \mapsto a\delta(x) \).

Property 2.4. For \( x, y \in A \), the \((\sigma, \tau)\)-inner derivations satisfy: \( \delta_{x+y} = \delta_x + \delta_y \).

Property 2.5. For \((\sigma, \tau)\)-inner derivations \( \delta_x, \delta_y \) for some \( x, y \in A \), \( \delta_x = \delta_y \) if and only if \( (x - y)\tau(a) = \sigma(a)(x - y) \) for all \( a \in A \).

The following lemma will be useful for the proof of the main theorem.

Lemma 2.6. Let \( \sigma \) and \( \tau \) be algebra homomorphisms on \( A \) that fix \( \mathcal{Z}(A) \) elementwise. Then for a \((\sigma, \tau)\)-derivation \( \delta \) on \( A \), we have \( \delta(\alpha^n) = n\alpha^{n-1}\delta(\alpha) \) for all \( \alpha \in \mathcal{Z}(A) \).

Proof. Let \( \alpha \in \mathcal{Z}(A) \). For \( n = 2 \), we have
\[
\delta(\alpha^2) = \delta(\alpha)\tau(\alpha) + \sigma(\alpha)\delta(\alpha) = \delta(\alpha)\alpha + \alpha\delta(\alpha) = 2\alpha\delta(\alpha)
\]
. Let the result be true for some \( n \), that is, \( \delta (\alpha^n) = n\alpha^{n-1}\delta (\alpha) \). Then
\[
\delta (\alpha^{n+1}) = \delta (\alpha^n)\alpha + \alpha^n\delta (\alpha) = (n+1)\alpha^n\delta (\alpha).
\]
Thus the result follows by induction.

Recall Skolem-Noether Theorem (for example, Theorem 4.3.1, [Her68]).

**Theorem 2.7** (Skolem-Noether). Let \( R \) be a simple Artinian ring with center \( F \) and let \( A, B \) be simple subalgebras of \( R \) which contains \( F \) and are finite dimensional over it. If \( \phi \) is an isomorphism of \( A \) onto \( B \) leaving \( F \) elementwise fixed, then there is an invertible \( x \in R \) such that \( \phi(a) = xax^{-1} \) for all \( a \in A \).

We apply the above Skolem-Noether theorem to prove the following result.

**Proposition 2.8.** Let \( A \) be a finite dimensional central simple algebra with 1 over a field \( F \) (that is, \( A \) is a simple algebra finite dimensional over \( F \) such that \( \mathcal{Z}(A) = F \)). Let \( \sigma \) and \( \tau \) be non-zero \( F \)-algebra endomorphisms of \( A \). Then there exist units \( v_1, v_2 \) in \( A \) such that any \( F \)-linear \((\sigma, \tau)\)-derivation \( \delta \) of \( A \) is equal to \( v_1\delta_{\text{inner}}^u v_2 \) for some \( u \in A \).

**Proof.** Let \( \delta \) be an \( F \)-linear \((\sigma, \tau)\)-derivation of \( A \). As \( \sigma, \tau \) are non-zero endomorphisms on \( A \), by Schur’s Lemma, they are actually isomorphisms on \( A \). Let \( A_2 \) be the ring of \( 2 \times 2 \) matrices over \( A \). Then \( A_2 \) is simple and finite dimensional over \( F \) with dimension \( 4 \times [A : F] \). Let
\[
B = \left\{ \left( \begin{array}{cc} \sigma(a) & \delta(a) \\ 0 & \tau(a) \end{array} \right) \mid a \in A \right\} \quad \text{and} \quad C = \left\{ \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right) \mid a \in A \right\}.
\]
Then \( B \) and \( C \) are simple subalgebras of \( A_2 \). Define a mapping \( \Psi : C \to B \) such that
\[
\Psi \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right) = \left( \begin{array}{cc} \sigma(a) & \delta(a) \\ 0 & \tau(a) \end{array} \right).
\]
As \( \sigma, \tau \) are \( F \)-algebra homomorphisms and \( \delta \) is additive and \( F \)-linear, we get \( \Psi \) is additive and \( F \)-linear as well. Also since \( \delta(ab) = \delta(a)\tau(b) + \sigma(a)\delta(b) \), we have
\[
\Psi \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right) \Psi \left( \begin{array}{cc} b & 0 \\ 0 & b \end{array} \right) = \left( \begin{array}{cc} \sigma(a) & \delta(a) \\ 0 & \tau(a) \end{array} \right) \left( \begin{array}{cc} \sigma(b) & \delta(b) \\ 0 & \tau(b) \end{array} \right) = \left( \begin{array}{cc} \sigma(ab) & \delta(ab) \\ 0 & \tau(ab) \end{array} \right) = \Psi \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right) \Psi \left( \begin{array}{cc} b & 0 \\ 0 & b \end{array} \right).
\]
Thus, \( \Psi \) is multiplicative as well. Now, since \( \sigma, \tau \) are \( F \)-algebra endomorphisms and so \( \sigma(1) = \alpha \sigma(1) = \alpha \tau(1) = \tau(1) = \alpha \) and \( \delta(1) = \alpha \delta(1) = 0 \) for \( \alpha \in F \), we get \( \Psi \) fixes \( F \) elementwise. Also, \( \Psi \) is one-one and onto as \( \sigma \) and \( \tau \) are bijections. Thus \( \Psi \) is an isomorphism of \( C \) onto \( B \) leaving \( F \) elementwise fixed. Also, \( C \) is isomorphic to \( A \), which is simple. So we have \( A_2 \) is a simple Artinian ring (as it is finite dimensional over its center) with \( \mathcal{Z}(A_2) = F \) and \( A(\cong C) \), \( B \) are simple subalgebras of \( A_2 \) that are finite dimensional.
over $F$. Also $\Psi$ is an isomorphism of $A(\cong C)$ onto $B$ leaving $F$ elementwise fixed. Therefore by Theorem 2.7 there exists an invertible matrix $(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}) \in A_2$ such that

$$\Psi \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}^{-1}.$$ 

That is,

$$\begin{pmatrix} \sigma(a) & \delta(a) \\ 0 & \tau(a) \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Solving, we get the following set of equations:

$$\sigma(a)x + \delta(a)z = xa \quad (2.1)$$
$$\sigma(a)y + \delta(a)w = ya \quad (2.2)$$
$$\tau(a)z = za \quad (2.3)$$
$$\tau(a)w = wa \quad (2.4)$$

for all $a \in A$. Note that the above equations are symmetric in $w$ and $z$. Since we cannot have $z = w = 0$ as then $(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix})$ will not be invertible, we assume one of them is non-zero, say, $z$. As $\sigma, \tau$ are automorphisms on the central simple algebra $A$, by Skolem-Noether Theorem 2.7, there exist units $s, t$ in $A$ such that $\sigma(a) = s^{-1}as$ and $\tau(a) = t^{-1}at$ for all $a \in A$. From equation 2.3 we get $t^{-1}atz = za$, that is, $atz = tza$ for all $a \in A$. This implies $tz \in \mathbb{Z}(A) = F$.

Let us write $z = t^{-1}\alpha$ for some $\alpha \in F$. Hence, $z^{-1}$ exists. Then from the equation 2.1 we get for all $a \in A$,

$$\delta(a) = xaz^{-1} - \sigma(a)xz^{-1} \quad (2.5)$$
$$= xaa^{-1}t - s^{-1}asx\alpha^{-1}t$$
$$= \alpha^{-1}(xat - s^{-1}asxt)$$
$$= \alpha^{-1}s^{-1}(sxa - asx)t$$
$$= v_1(ua - au)v_2$$

where $v_1 = \alpha^{-1}s^{-1}$, $v_2 = t$ and $u = sx$. So there exist units $v_1, v_2$ in $A$ such that $\delta(a) = v_1\delta_u^{inner}(a)v_2$ for all $a \in A$, where $\delta_u^{inner}$ is the inner derivation of $A$ coming from $u$. Hence proved.

Another version of the above result is as follows:

**Corollary 2.9.** Let $A$ be a finite dimensional central simple algebra with 1 over a field $F$. Let $\sigma$ and $\tau$ be non-zero $F$-algebra endomorphisms of $A$. Then any $F$-linear $(\sigma, \tau)$-derivation of $A$ is a $(\sigma, \tau)$-inner derivation of $A$.

**Proof.** Following the same steps as Proposition 2.8 and taking $\tau(a) = t^{-1}at$ for some invertible $t$, we get finally as in equation 2.5

$$\delta(a) = xaz^{-1} - \sigma(a)xz^{-1} \quad \text{for all } a \in A \text{ and where } z = t^{-1}\alpha, \alpha \in \mathbb{Z}(A) = F.$$ We can write for all $a \in A$:
\[
\delta(a) = xtt^{-1}att^{-1}z^{-1} - \sigma(a)xz^{-1} \\
= xtt(a)t^{-1}z^{-1} - \sigma(a)xz^{-1} \\
= xtt(a)t^{-1}z^{-1} - \sigma(a)xz^{-1} \\
= xz^{-1}t(a) - \sigma(a)xz^{-1} \\
= u\tau(a) - \sigma(a)u \\
= \delta_u(a)
\]
where \( u = xz^{-1} \). Thus a \((\sigma, \tau)\)-derivation \( \delta \) of \( A \) is a \((\sigma, \tau)\)-inner derivation of \( A \). Hence proved. \( \Box \)

Let us prove some results regarding \((\sigma, \tau)\)-derivations on group rings.

**Lemma 2.10.** Let \( R \) be a commutative ring with 1. For algebra homomorphisms \( \sigma, \tau : RG \to RG \), if a map \( \delta : G \to RG \) is such that \( \delta(gh) = \delta(g)\tau(h) + \sigma(g)\delta(h) \) for all \( g, h \in G \), then \( \delta \) extends linearly to a \((\sigma, \tau)\)-derivation of \( RG \).

**Proof.** Define \( \Delta : RG \to RG \) such that
\[
\Delta \left( \sum_{g \in G} r_g g \right) = \sum_{g \in G} r_g \delta(g),
\]
where \( r_g \in R \). Then \( \Delta \) is \( R \)-linear. For \( \sum_{g \in G} r_g g, \sum_{h \in G} s_h h \in RG \),
\[
\Delta \left( \sum_{g \in G} r_g g \sum_{h \in G} s_h h \right) = \Delta \left( \sum_{g, h \in G} r_g s_h gh \right) \\
= \sum_{g, h \in G} r_g s_h \delta(gh) = \sum_{g, h \in G} r_g s_h (\delta(g)\tau(h) + \sigma(g)\delta(h)) \\
= \sum_{g \in G} r_g \delta(g) \sum_{h \in G} s_h \tau(h) + \sum_{g \in G} r_g \sigma(g) \sum_{h \in G} s_h \delta(h) \\
= \Delta \left( \sum_{g \in G} r_g g \right) \tau \left( \sum_{h \in G} s_h h \right) + \sigma \left( \sum_{g \in G} r_g g \right) \Delta \left( \sum_{h \in G} s_h h \right).
\]
Hence, \( \Delta \) is a \((\sigma, \tau)\)-derivation of \( RG \). \( \Box \)

The following remark is an analogue of the necessary and sufficient conditions for an element being central in a group ring for the \((\sigma, \tau)\)-case.

**Remark 2.11.** Let \( R \) be a commutative ring with 1 and \( \sum_{g \in G} a_g g \in RG \), where \( a_g \in R \). Let the algebra homomorphisms \( \sigma, \tau : RG \to RG \) be such that they are \( R \)-linear extensions of group homomorphisms of \( G \) and they fix \( Z(RG) \) elementwise. We obtain the necessary and
sufficient condition for an element \( a = \sum_{g \in G} a_g g \in RG \) to satisfy the condition \( a \sigma(b) = \sigma(b) a \) for every \( b \in RG \). Note that \( \sigma(h)^{-1} g \tau(h) \) is a group element for \( g, h \in G \). Thus for any \( h \in G \), we get:

\[
\left( \sum_{g \in G} a_g g \right) \tau(h) = \sigma(h) \left( \sum_{g \in G} a_g g \right)
\]

\[\iff\]

\[
\sum_{g \in G} a_g \sigma(h)^{-1} g \tau(h) = \sum_{g \in G} a_g g
\]

\[\iff\]

\[a_g = a_{\sigma(h)^{-1} \tau(h)}\].

3. Proof of Theorem 1.1

Finally, we are now in a position to prove our main theorem. Let \( RG \) be the group ring of a finite group \( G \) over an integral domain \( R \) with 1 of characteristic \( p \) such that \( p \) does not divide the order of \( G \).

3.1. Proof of part (i). Let \( R \) be a field and \( \sigma \) and \( \tau \) be algebra endomorphisms on \( RG \) such that they fix the center of \( RG \) elementwise. Let \( \delta \) be a \((\sigma, \tau)\)-derivation of the group ring \( RG \). We need to show that \( \delta \) is a \((\sigma, \tau)\)-inner derivation of \( RG \).

As \( G \) is finite and the characteristic of \( R \) does not divide the order of \( G \), we have \( RG \) is semisimple. So by Wedderburn Structure Theorem, \( RG \) can be written as a sum of simple algebras in the following way:

\[RG \cong RG e_1 \oplus RG e_2 \oplus \cdots \oplus RG e_k\]

where

(i) \( e_1, e_2, \ldots, e_k \) is a unique collection of idempotents in \( RG \) such that they form a central primitive decomposition of 1.

(ii) \( RG e_i \) is simple and \( RG e_i \cong M_{n_i}(D_i) \) where \( D_i \) is a finite dimensional division algebra over \( D \) for all \( 1 \leq i \leq k \), and

(iii) \( Z(RG e_i) \cong Z(M_{n_i}(D_i)) \cong Z(D_i) = F_i \), say, such that \( F_i \) is a finite dimensional field extension over \( R \) for all \( 1 \leq i \leq k \).

Now, let \( i, j \) be distinct integers with \( 1 \leq i, j \leq k \). As \( e_i e_j = 0 \), we have

\[
0 = \delta(e_i e_j) = \delta(e_i) \tau(e_j) + \sigma(e_i) \delta(e_j)
\]

\[= \delta(e_i) e_j + e_i \delta(e_j)\]
as $\sigma, \tau$ fix all central elements. Multiplying with $e_j$ from the left, we get $\delta(e_i)e_j = 0$ as $e_j$ is a central idempotent and $e_j e_i = 0$. Thus for each $j$ such that $1 \leq j \leq k$ and $i \neq j$, we get $\delta(e_i)e_j = 0$, that is, $\delta(e_i)$ is orthogonal to $e_j$. Therefore, we must have $\delta(e_i) \in RGe_i$. Now, if $\alpha \in RG$, then as $\tau(e_i) = e_i$, we have

$$\delta(\alpha e_i) = \delta(\alpha) e_i + \sigma(\alpha) \delta(e_i) \in RGe_i.$$ 

So we get $\delta(RGe_i) \subseteq RGe_i$. Let us denote $\delta, \sigma$ and $\tau$ restricted to $KGe_i$ as $\delta_i, \sigma_i$ and $\tau_i$ respectively. Then $\delta_i : RGe_i \rightarrow RGe_i$ is a $(\sigma_i, \tau_i)$-derivation on $RGe_i$.

Now, let $\alpha \in F_i = \mathcal{Z}(D_i)$. Since $\alpha$ is algebraic over $R$, $\alpha$ satisfies a unique monic irreducible polynomial of minimal degree in $R[x]$. Let the polynomial be $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Then $p(\alpha) = \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$. Recall that $\delta(1) = 0$. Applying $\delta_i$ to this, we get using Lemma 2.6

$$n\alpha^{n-1}\delta_i(\alpha) + a_{n-1}(n-1)\alpha^{n-2}\delta_i(\alpha) + \cdots + a_1\delta_i(\alpha) = 0.$$ 

That is, $[D_x(p(x))]_{x=\alpha} \delta_i(\alpha) = 0$, where $D_x(p(x))$ denotes the derivative of the polynomial $p(x)$. As $p(x)$ is a minimal polynomial that $\alpha$ satisfies, $[D_x(p(x))]_{x=\alpha}$ can never be zero. Also it belongs to $F_i$ and so is invertible. This gives us $\delta_i(\alpha) = 0$.

Thus if $\beta \in RGe_i$ and $\alpha \in \mathcal{Z}(RGe_i) = F_i$, then

$$\delta_i(\alpha \beta) = \delta_i(\alpha) \tau_i(\beta) + \sigma_i(\alpha) \delta_i(\beta) = \sigma_i(\alpha) \delta_i(\beta) = \alpha \delta_i(\beta).$$

Thus, $\delta_i$ is a $(\sigma_i, \tau_i)$-derivation of the finite dimensional simple algebra $RGe_i$ and is linear over $\mathcal{Z}(RGe_i) = F_i$. Hence by Corollary 2.9, $\delta_i$ is a $(\sigma_i, \tau_i)$-inner derivation of $RGe_i$.

Hence by the definition of a $(\sigma_i, \tau_i)$-inner derivation, there exists $x_i \in RGe_i$ such that $\delta_i(a) = \delta_{x_i}(a) = x_i a - ax_i$ for all $a \in RGe_i$. That is, $\delta_i = \delta_{x_i}$. For each $i, 1 \leq i \leq k$, there exists some $x_i \in RGe_i$ such that $\delta_i = \delta_{e_i}$. Then $x = x_1 + x_2 + \cdots + x_k \in RG$. Then by Property 2.4, $\delta = \delta_1 + \delta_2 + \cdots + \delta_k = \delta_{x_1} + \delta_{x_2} + \cdots + \delta_{x_k} = \delta_x$. Thus, $\delta$ is a $(\sigma, \tau)$-inner derivation on $RG$.

3.2. Proof of Part (ii). Let $R$ be an integral domain with 1 that is not a field and $\sigma, \tau$ are $R$-linear extensions of group homomorphisms of $G$ such that they fix $\mathcal{Z}(RG)$ elementwise. Let $K$ be the field of fractions of $R$, that is, $K$ is the smallest field containing $R$ such that every nonzero element of $R$ is a unit in $K$.

Define $\sigma', \tau' : KG \rightarrow KG$ such that

$$\sigma' \left( \sum_{g \in G} r_g g \right) = \sum_{g \in G} r_g \sigma(g)$$
and
\[ \tau' \left( \sum_{g \in G} r_g g \right) = \sum_{g \in G} r_g \tau(g) \]
where \( r_g \in K \). Then \( \sigma', \tau' \) are \( K \)-algebra endomorphisms of \( KG \). It is known that (see, for example, Lemma 3.6.4, [Sch78]), the center \( Z(RG) \) of a group ring \( RG \) of a group \( G \) over a commutative ring \( R \) is the \( R \)-linear span of finite class sums of \( G \), that is,
\[ Z(RG) = \left\{ \sum_{x \in G} a_x C_x : C_x = \sum_{y \in G, y \sim x} y, \ a_x \in R \right\}, \]
where \( y \sim x \) for \( x, y \in G \) means \( y \) is conjugate to \( x \). Then, as \( \sigma, \tau \) fix \( Z(RG) \) elementwise, we have
\[ \sigma' \left( \sum_{x \in G} a_x C_x \right) = \sum_{x \in G} a_x \sigma(C_x) = \sum_{x \in G} a_x C_x \]
and
\[ \tau' \left( \sum_{x \in G} a_x C_x \right) = \sum_{x \in G} a_x \tau(C_x) = \sum_{x \in G} a_x C_x. \]
So \( \sigma' \) and \( \tau' \) fix \( Z(KG) \) elementwise too. Now define \( \Delta : KG \to KG \) such that
\[ \Delta \left( \sum_{g \in G} r_g g \right) = \sum_{g \in G} r_g \delta(g), \]
where \( r_g \in K \). Then by Lemma 2.10, \( \Delta \) is a \( (\sigma', \tau') \)-derivation of \( KG \). By part (i) of Theorem 1.1, \( \Delta \) is \( (\sigma', \tau') \)-inner derivation of \( KG \). Let \( x \in KG \) be such that \( \Delta = \Delta_x \), that is, \( \Delta(a) = x \tau'(a) - \sigma'(a)x \) for all \( a \in KG \).

Now, every finite integral domain is a field and the case has already been proved in part (i). So let \( R \) be an infinite integral domain that is not a field.

Let the element \( x \) as obtained above be of the form \( x = \sum_{g \in G} x_g g \in KG \), where \( x_g \in K \). Let \( h \in G \). Then \( \Delta(h) = 1 \delta(h) \in RG \). But \( \Delta(h) = \Delta_x(h) \). This means
\[ \Delta(h) = x \tau'(h) - \sigma'(h)x = x \tau(h) - 1 \sigma(h)x \sum_{g \in G} x_g g \tau(h) - \sum_{g \in G} x_g \sigma(h)g \in RG. \]

Recall that \( \sigma \) and \( \tau \) are \( R \)-linear extensions of group homomorphisms of \( G \). Now, let \( g_i \tau(h) = \sigma(h)g_{k_i} \), for some \( g_i, g_{k_i} \in G \), say. Then \( \sigma(h)^{-1}g_i \tau(h) = g_{k_i} \). Hence in the above equation, we have \( \sum_{g \in G} \left( x_g - x_{\sigma(h)^{-1}g \tau(h)} \right) g \tau(h) \in RG \). That is, \( x_g - x_{\sigma(h)^{-1}g \tau(h)} \in R \), where \( x_g, x_{\sigma(h)^{-1}g \tau(h)} \in K \). Note that \( x_{\sigma(h)^{-1}g \tau(h)} \) is the coefficient of the group element \( \sigma(h)^{-1}g \tau(h) \).

Now, we want to find an element \( u \in KG \) such that \( x - u \in RG \) and \( \Delta_x = \Delta_{x-u} \). Then we can conclude that \( \Delta \) restricted to \( RG \) which is \( \delta \), is of the form \( \delta_{x-u} \) and hence is a
Let us denote the support of $x$ by $\text{Supp}(x) = \{g \mid x_g \neq 0\}$. Let $F = \{x_g \mid g \in \text{Supp}(x)\}$, that is $F$ is the collection of all those elements $x_g \in K$ which appear in the expression of $x$. As $G$ is finite, we have $\sum_{\pi} F$ finite, we have $F = \{x_{\pi(h)^{-1}g\pi(h)} \in F \mid x_{\pi(h)^{-1}g\pi(h)} \in R, \ h \in G\}$. That is, $F_g$ is the collection of all those elements of $F$ that appear as coefficients of $\pi(h)^{-1}g\pi(h)$ for $h \in G$ and such that the difference of those elements with $x_g$ belongs to $R$. Note that $x_g$ itself belongs to $F_g$ as for $h = 1$, $x_{g} - x_{g} = 0 \in R$. For every $g \in \text{Supp}(x)$, we can form the corresponding $F_g$ in this manner. Now, if $F_{g_i} \cap F_{g_j} \neq \emptyset$ for some $g_i, g_j \in \text{Supp}(x)$, $g_i \neq g_j$, it follows that $F_{g_i} = F_{g_j}$. This is because, if $x_{g_0} \in F_{g_i} \cap F_{g_j}$, then $x_{g_0} - x_{g_0} \in R$ and $x_{g_0} - x_{g_j} \in R$ and this implies $x_{g_i} - x_{g_j} \in R$. Let $g_1, g_2, \ldots, g_t \in \text{Supp}(x)$ be the representatives of the distinct $F_g$'s. That is, $F_{g_1}, F_{g_2}, \ldots, F_{g_t}$ are distinct subsets of $F$ which means $F_{g_i} \cap F_{g_j} = \emptyset$ for every $1 \leq i, j \leq t, i \neq j$. Note that $F = \bigcup_{i=1}^{t} F_{g_i}$. As $R$ is infinite, so is $K$. Thus for each $F_{g_i}$, $1 \leq i \leq t$, which is a finite subset of $K$ we can find an element $x'_{g_i} \in K$ such that $x'_{g_i}, \notin F_{g_i}$ and $x'_{g_i} + R = y + R$ in the abelian group $(K/R, +)$ for every $y \in F_{g_i}$. That is, $x'_{g_i} - y \in R$ for every $y \in F_{g_i}$. Fix this $x'_{g_i}$ for the corresponding $F_{g_i}$.

Define the map:

$$((\cdot)) : F = \bigcup_{i=1}^{t} F_{g_i} \rightarrow K$$

where $((F_{g_i})) = x'_{g_i}$, that is, $((x_g)) = x'_{g_i}$ for every $x_g \in F_{g_i}$, where $x'_{g_i}$ is the element we fixed for $F_{g_i}$ such that $x'_{g_i}, \notin F_{g_i}$ and $x_g - x'_{g_i} \in R$ for every $x_g \in F_{g_i}$.

Hence for each $x_g \in F$ we have found an element $((x_g)) \in K$ such that whenever $x_g - x_{\sigma(h)^{-1}g\pi(h)} \in R$ for $h \in G$, we will have $((x_g)) = ((x_{\sigma(h)^{-1}g\pi(h)}))$.

Now consider the element $u = \sum_{g \in G} ((x_g))g$ in $KG$. Note that by construction $x - u \in RG$ and for every $h \in G$, $((x_g)) = ((x_{\sigma(h)^{-1}g\pi(h)}))$. Then by Remark 2.11, for every $h \in G$ we get

$$\left(\sum_{g \in G} ((x_g))g\right) \tau(h) = \sigma(h) \left(\sum_{g \in G} ((x_g))g\right).$$

We can write it as:

$$\left(\sum_{g \in G} x_gg - \left(\sum_{g \in G} x_gg - \sum_{g \in G} ((x_g))g\right)\right) \tau(h) = \sigma(h) \left(\sum_{g \in G} x_gg - \left(\sum_{g \in G} x_gg - \sum_{g \in G} ((x_g))g\right)\right).$$

The above equation can be written as $(x - (x - u)) \tau(a) = \sigma(a) (x - (x - u))$ for every $a \in KG$. Hence, by Property 2.3, we get $\Delta_x = \Delta_{x-u}$. Now, $x - u \in RG$. Thus $\Delta, \sigma', \tau'$
restricted to $RG$ gives $\delta, \sigma, \tau$ respectively and we get

$$\delta(a) = \delta_{x-u}(a) = (x-u)\tau(a) - \sigma(a)(x-u),$$

for all $a \in RG$. Thus, $\delta$ is a $(\sigma,\tau)$-inner derivation of $RG$. Hence Theorem 1.1 is proved.

4. Conclusion

Part (ii) of our main theorem can be applied to the integral group ring $\mathbb{Z}G$ of a finite group $G$, with the same conditions on $\sigma$ and $\tau$. Thus we get the following result regarding $(\sigma, \tau)$-derivations of integral group rings which extends Theorem 1 of [Spi94] to the $(\sigma, \tau)$-case.

**Corollary 4.1.** Let $G$ be a finite group and $\sigma, \tau$ be $\mathbb{Z}$-linear extensions of group homomorphisms of $G$ such that they fix $\mathbb{Z}(\mathbb{Z}G)$ elementwise. Then every $(\sigma, \tau)$-derivation of $\mathbb{Z}G$ is $(\sigma, \tau)$-inner.

**Proof.** The result follows directly from our main theorem. Independently also one can prove this corollary by following the same steps as in the proof of our main theorem with a slightly different approach towards the end. Once we get the $(\sigma, \tau)$-derivation $\delta$ of $\mathbb{Z}G$ extended to $(\sigma', \tau')$-derivation $\Delta$ of $\mathbb{Q}G$ is a $(\sigma', \tau')$-inner derivation of $\mathbb{Q}G$ and is of the form $\Delta_x(a) = x\tau(a) - \sigma(a)x$ for all $a \in \mathbb{Q}G$, where $x = \sum_{g \in G} x_g g \in \mathbb{Q}G$, we can construct an element $u$ in $\mathbb{Q}G$ such that $\Delta_x = \Delta_{x-u}$ with $x-u \in ZG$ by considering the ‘fractional part’ $\{x_g\}$ of each $x_g$. By fractional part $\{x_g\}$ of $x_g$ we mean, $0 \leq \{x_g\} < 1$ and $\{x_g\} + \mathbb{Z} = x_g + \mathbb{Z}$ in the abelian group $(\mathbb{Q}/\mathbb{Z}, +)$. For every $h \in G$ we will get $\sum_{g \in G} (x_g - x_{\sigma(h)^{-1}g\tau(h)}) g \tau(h) \in ZG$, that is, $x_g - x_{\sigma(h)^{-1}g\tau(h)} \in \mathbb{Z}$ with $x_g, x_{\sigma(h)^{-1}g\tau(h)} \in \mathbb{Q}$. So we must have $\{x_g\} = \{x_{\sigma(h)^{-1}g\tau(h)}\}$. Then $u = \sum_{g \in G} \{x_g\} g$ will serve our purpose. Thus the $(\sigma, \tau)$-derivation $\delta$ of $\mathbb{Z}G$ with the given conditions on $\sigma$ and $\tau$ is a $(\sigma, \tau)$-inner derivation $\delta_{x-u}$ of $\mathbb{Z}G$. $\square$

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