Dynamically generated embeddings of spacetime

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Abstract
We discuss how embeddings in connection with the Campbell–Magaard (CM) theorem can have a physical interpretation. We show that any embedding whose local existence is guaranteed by the CM theorem can be viewed as a result of the dynamical evolution of initial data given in a four-dimensional spacelike hypersurface. By using the CM theorem, we establish that for any analytical spacetime, there exist appropriate initial data whose Cauchy development is a five-dimensional vacuum space into which the spacetime is locally embedded. We shall see also that the spacetime embedded is Cauchy stable with respect to the initial data.

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The development of the braneworld scenario [1–4], in which our ordinary spacetime is viewed as a hypersurface of a higher dimensional space, has greatly contributed to increasing recent interest in embedding theorems. The so-called non-compact Kaluza–Klein (NKK) models [5–7] have also motivated the study of embedding problems of the spacetime.

In this context, the Campbell–Magaard (CM) theorem [8–10] and its variants [11–15] are of special interest when the embedding spaces possess only one extra dimension as is the case of the Randall–Sundrum braneworld scenario [3, 4] and the NKK models [5–7].

The CM theorem ensures the existence of local and analytical isometric embedding\(^3\) of any \(n\)-dimensional analytical manifold \(M\) into a Ricci-flat \((n + 1)\)-dimensional space. It can be stated as follow: given an \(n\)-dimensional analytical space \((M, g)\), where \(g\) stands for the metric, for any point \(p \in M\) there exists a vacuum solution of Einstein equations in \((n + 1)\) dimensions into which a neighbourhood of \(p\) in \(M\) can be embedded. The CM theorem is valid for any analytical metric \(g\) irrespective of its signature and for dimension \(n \geq 2\). However, having in mind applications to physics, we shall consider \(n = 4\), throughout the paper, and assume that the metric is Lorentzian. The CM theorem is local, but the extension to the case of global embedding has been studied [16].

Recently some interesting issues concerning the physical interpretation of embeddings in the light of the CM theorem have arisen [17, 18]. It has been argued that, since the CM theorem

\(^3\) Henceforth embedding for us will mean analytical and isometrical embedding.
is based on the Cauchy–Kowalewski theorem it disregards causality and cannot guarantee a continuous dependence between the metric of ambient space and spacetime metric. Therefore, by lacking these important properties, the embeddings obtained by employing the CM theorem would have no physical meaning.

In this paper we intend to show that, as a matter of fact, ambient spaces whose local existence is guaranteed by the CM theorem possess, in a certain domain, those desirable physical properties with respect to appropriate initial data. Indeed, we shall show that each five-dimensional vacuum space into which the spacetime can be locally embedded may be locally viewed as the Cauchy development of some initial data given on a spacelike four-dimensional hypersurface and hence satisfies the stability and causality conditions with respect to these initial data. Further, we shall see that the region of $M$ which is embedded into the Cauchy development of an initial data set possesses also Cauchy stability and domain of dependence property with respect to these initial data.

In order to appropriately address these issues, let us begin our discussion by considering a brief sketch of Magaard’s proof of the CM theorem.

1. The Campbell–Magaard theorem

Consider the Lorentzian metric of the five-dimensional space written in a Gaussian form
\[ ds^2 = g_{ij}(x, \psi) \, dx^i \, dx^j + d\psi^2, \]
where $x = (x^1, \ldots, x^4)$, and Latin indices run from 1 to 4 while the Greek ones go from 1 to 5.

By splitting the vacuum Einstein equations in terms of the extrinsic and intrinsic curvatures of the slices $\psi = \text{const}$, it can be shown that the equations have the following structure:

\[ \frac{\partial^2 g_{ij}}{\partial \psi^2} = F_{ij} \left( \frac{\partial}{\partial \psi}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial^2 g}{\partial x^2}, \frac{\partial^2 g}{\partial x \partial \psi} \right) \]

\[ \nabla_j (\Omega^{ij} - g^{ij} \Omega) = 0 \]

\[ R + \Omega^2 - \Omega_{ij} \Omega^{ij} = 0, \]

where $F_{ij}$ are analytical functions of their arguments, $\nabla_j$ is the covariant derivative with respect to the induced metric $g_{ij} = g_{ij}(x, \psi = \text{const})$; $R$ and $\Omega_{ij}$ denote, respectively, the scalar curvature and the extrinsic curvature of the hypersurface $\psi = \text{const}$; and $\Omega = g^{ij} \Omega_{ij}$.

Recall that in the coordinates adopted the extrinsic curvature assumes the simple form:
\[ \Omega_{ij} = -\frac{1}{2} \frac{\partial g_{ij}}{\partial \psi}. \]

It is well known that, owing to the Bianchi identities, the second and third equations need to be imposed only on the hypersurface, since they are propagated by the first one. In this sense, it is said that the Einstein equations consist of the dynamical equation (2) plus constraint equations (3) and (4) for $\Omega_{ij}$ and $g_{ij}$.

Let us now consider the hypersurface $\psi = 0$. According to the Cauchy–Kowalewski theorem, for any point in this hypersurface, say the origin, there is an open set in five dimensions containing that point, where equation (2) always has a unique analytical solution $g_{ij}(x, \psi)$ provided that the following analytical initial conditions are specified:

\[ g_{ij}(x, 0) = g_{ij}(x) \]

\[ \frac{\partial g_{ij}}{\partial \psi} \bigg|_{\psi=0} = -2 \Omega_{ij}(x). \]
From the perspective of the embedding problem these initial conditions represent, respectively, the metric and the extrinsic curvature of the hypersurface $\Psi = 0$, whereas the solution of equation (2) gives the metric of the $(n + 1)$-dimensional space. Thus, if there is a solution for the constraint equations for any given metric $g_{ik}$, then the theorem is proved, since the solution found $\bar{g}_{ij}(x, \psi)$ substituted in (1) will give rise to a metric that satisfies the vacuum Einstein equation $R_{\mu\nu} = 0$. Clearly, the embedding map is then given by the equation $(x, \psi = 0)$.

It turns out, as Magaard has proved [9, 12], that the constraint equations always have a solution. Indeed, by simple counting operation we can see that there are $n(n + 1)/2$ unknown functions (the independent elements of extrinsic curvature) and $n + 1$ constraint equations. The metric $g_{ij}(x)$ must be considered as a given datum. For $n \geq 2$, the number of variables is equal to or greater than the number of equations. Magaard has shown that after the elimination of equation (4), the first-order differential equation (3) can be written in a canonical form (similar to (2)) with respect to $n$ components of $\Omega_{ij}$ conveniently chosen. Taking initial conditions for these components in such a way that the right-hand side of the mentioned equation is analytical at the origin, the Cauchy–Kowalewski theorem can be applied once more to guarantee the existence of a solution for the constraint equations.

2. Dynamically generated embedding

From the above we see that Magaard’s proof of the CM theorem is formulated in terms of an initial value problem. Roughly speaking we can say that a $(3 + 1)$ spacetime is taken as part of the initial data and it is ‘propagated’ along a spacelike extra dimension by the dynamical part of the Einstein vacuum equations to generate the higher dimensional space. Nevertheless, it is clear enough that, despite some similarities, the CM theorem is not concerned with real dynamical propagation since the initial data ‘evolve’ along a spacelike direction. Therefore there is no reason why we should expect a causality relation between different slices of the higher dimensional space.

However, we can look at this picture from a different perspective. Indeed, supported by the CM theorem, we know that given any point $p \in M$ there is a five-dimensional vacuum space $(\tilde{M}, \tilde{g})$ into which a neighbourhood of $p$ in $M$ is embedded. Now we can determine an open subset of $\tilde{M}$, say, $\tilde{O}$, containing the point $p$ at which there exists a four-dimensional hypersurface $\Sigma$, which is spacelike everywhere, acausal and that contains the point $p$ (see appendix A). The embedding of $\Sigma$ into $(\tilde{O}, \tilde{g})$ induces a positive definite metric $h$ in the hypersurface. Let $K$ be the extrinsic curvature of $\Sigma$ in $(\tilde{O}, \tilde{g})$. The metric and the extrinsic curvature are analytical fields in the hypersurface $\Sigma$; thus they belong to the local Sobolev space $W^m(\Sigma)$ for any $m$. The set $(h, K, \Sigma)$ constitutes appropriate initial data for the Einstein vacuum equations, since $h$ and $K$ satisfy the vacuum constraint equations in the hypersurface $\Sigma$ and fulfil the required condition of regularity (see [19], pp 248–9, and [20], for instance).

Consider now $D(\Sigma)$, the domain of dependence of $\Sigma$ relative to $(\tilde{O}, \tilde{g})$, i.e., the region which is causally determined by the data in $\Sigma$. Since $\Sigma$ is an acausal hypersurface of $\tilde{O}$, then $D(\Sigma)$ is open in $\tilde{O}$ (see [21], p 425). Of course $D(\Sigma)$ is a non-empty set, since $\Sigma \in D(\Sigma)$. By construction, the five-dimensional manifold $(D(\Sigma), \tilde{g})$ is a solution of the Einstein vacuum equations; hence $(D(\Sigma), \tilde{g})$ is a Cauchy development for the Einstein vacuum equations of the initial data $(h, K, \Sigma)$.

By writing $h \in W^m(\Sigma)$ we mean that the norm of $h$ together with its covariant derivatives of order equal or less than $m$ are square integrable in any open set $U$ of $\Sigma$ with compact closure. For the sake of simplicity, we shall assume that the norm and derivatives are calculated with respect to an Euclidean metric. Here we are adopting the notation of Sobolev spaces used in [19].
As we have mentioned before $D(\Sigma)$ is open, and thus the non-empty set $M \cap D(\Sigma)$ is a neighbourhood of $p$ in $M$ contained in $D(\Sigma)$. Therefore $M \cap D(\Sigma)$ is embedded into $D(\Sigma)$, i.e., in a Cauchy development of $(h, K, \Sigma)$ (see figure 1). In other words the dynamical evolution of the initial data $(h, K, \Sigma)$ generates a five-dimensional vacuum space into which the spacetime is locally embedded. In this sense, we can say that this local embedding is dynamically generated by the physical propagation of those initial data.

More precisely this result can be stated as follows: Consider an analytical spacetime $(M, g)$. For any $p \in M$ there are initial data $(h, K, \Sigma)$ whose Cauchy development for the Einstein vacuum equations is a five-dimensional vacuum space into which a neighbourhood of $p$ in $M$ is analytically and isometrically embedded.

Furthermore, the Einstein vacuum equations admit a well-posed initial value formulation with respect to the data $(h, K, \Sigma)$ (see, for example, [19, 20, 22, 23]). Therefore, the general properties of solutions of the vacuum Einstein equations, related to the hyperbolic character of the differential equations, are applicable to our solution $(D(\Sigma), \tilde{g})$. This ensures that the dependence of the solution $(D(\Sigma), \tilde{g})$ on the initial data $(h, K, \Sigma)$ is continuous (Cauchy stability). As a consequence the spacetime embedded into $(D(\Sigma), \tilde{g})$ is stable in a similar sense too, as we describe in the next section.

Another important property is that any change of data outside $\Sigma$ does not affect the solution in the future domain of dependence (causality). Thus it follows that any perturbation outside $\Sigma$ will not disturb the embedding of spacetime in $D(\Sigma)$.

### 3. Cauchy stability

Consider an analytical spacetime $(M, g)$ and let $(h, K, \Sigma)$ be a set of analytical initial data with a Cauchy development $(D(\Sigma), \tilde{g})$ in which the spacetime is locally embedded, around $p \in M$. Additionally let us admit that this initial data set satisfies the following property: the image of $p$ through the embedding lies in $\Sigma$. In other words, corresponding to the set $(h, K, \Sigma)$ there are some neighbourhoods $O$ of $p$ in $M$ and a map $\phi : O \subset M \rightarrow D(\Sigma)$ which is an embedding, with $\phi(p) \in \Sigma$.

Now we denote by $(h', K')$ a new set of initial data which satisfies the vacuum Einstein constraint equation in $\Sigma$. For the sake of simplicity let us assume that the fields $h'$ and $K'$ are $C^\infty$ in $\Sigma$. In this case, the new generated metric $g'$ is a $C^\infty$ field.

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5 The existence of $(h, K, \Sigma)$ was shown in the previous section. Possibly the initial data set is not unique. The results obtained in this section are applicable separately to each one of all possible initial data sets.
Let $V$ be an open set of $J^+(\Sigma)$, the causal future of $\Sigma$ in $(D(\Sigma), \tilde{g})$, with compact closure and $\mathcal{U} \subset \Sigma$ be some neighbourhood of $J^+(\Sigma) \cap \Sigma$, the causal past of $\overline{V}$ (the closure of $V$) in $\Sigma$, with compact closure in $\Sigma$. According to the Cauchy stability theorem (see [19], p 253, and [20]), for any $\varepsilon > 0$ there is some $\delta > 0$ such that any initial data $(h', K')$ on $\Sigma$ close to $(h, K)$ in $\mathcal{U}$ with respect to the local Sobolev norm, i.e., $\|h' - h, \mathcal{U}\|_m < \delta$ and $\|K' - K, \mathcal{U}\|_m < \delta (m > 4)$, give rise to a new metric $g^*$ which is near the old one $\tilde{g}$ in $V$, i.e., $\|g^* - \tilde{g}, V\|_m < \varepsilon$.

Now let $V$ be such that $V \cap (M \cap D(\Sigma)) = N$ is a non-empty set, where $M \cap D(\Sigma)$ means $\varphi(O)$, the image of $O$ through the embedding. And let $g'$ be the induced metric on $N$ by the embedding of $N$ in $(V, g^*)$. We shall see that if $\delta$ is sufficiently small then $g'$ will be a Lorentzian metric and it will be close to the spacetime metric $g$ in $N$.

For the sake of simplicity, let us make some assumptions. First, we assume that $D(\Sigma)$ is covered by Gaussian coordinates (1) adapted to the embedding, in which the embedding map is $(x, \psi) = (0)$. If this was not the case, we could find a neighbourhood $S$ of $\Sigma$ and a neighbourhood $\tilde{O}$ of $p$ in $D(\Sigma)$ such that the domain of dependence of $S$ relative to $(\tilde{O}, \tilde{g})$, $D(S, \tilde{O})$ is covered by (1). We would proceed in the following manner. Since the embedding exists, we know that $D(\Sigma)$ is a timelike hypersurface of $(D(\Sigma), \tilde{g})$. By the usual procedure we construct, from the geodesics that cross $M \cap D(\Sigma)$ orthogonally, Gaussian normal coordinates in a neighbourhood $\tilde{O}$ of $p$ in $D(\Sigma)$. Now make $\tilde{S} = \Sigma \cap \tilde{O}$. Then we concentrate our analysis in the region $D(S, \tilde{O})$.

Second, let us assume that the Sobolev norm is evaluated with respect to an Euclidean metric defined on $D(\Sigma)$ and that in coordinates (1) the Euclidean metric has the canonical form, i.e., diag$(+1, +1, +1, +1)$. Then, for example, $\|g^* - \tilde{g}, V\|_{m=0} = \left[ \int_V |g^* - \tilde{g}|^2 d^4 x d\psi \right]^2$, where

$$|g^* - \tilde{g}| = \left[ 4 \sum_{i,j=1}^4 (g^*_{ij} - \tilde{g}_{ij})^2 + 2 \sum_{i=1}^4 (g^*_{55} - 1)^2 \right]^{1/2}.$$ 

As we have mentioned, in the given coordinates, the embedding map is $(x, \psi) = (0)$. Thus the induced metric in $N$ by the new solution $g^*$ is given by $g^*_{ij}(x) = \tilde{g}_{ij}(x, \psi = 0)$. Let us show that if $\varepsilon$ is sufficiently small the induced metric $g'$ in $N$ is Lorentzian.

Metrics which are $C^\infty$ in the whole manifold belong to local Sobolev spaces $W^m$ for any $m$. Thus they obey some important inequalities which hold on Sobolev spaces. For example, according to lemma 7.4.1 in [19] (p 235), we have that for $m \geq 3, |g^* - \tilde{g}| \leq P\|g^* - \tilde{g}, V\|_m$ on $V$, where $P$ is a positive constant (depending on $V$). Thus if $\|g^* - \tilde{g}, V\|_m \leq \varepsilon$, it follows that all components satisfy the inequality $|g^*_{\mu\nu} - \tilde{g}_{\mu\nu}| \leq P\varepsilon$ on $V$.

The gradient $\nabla^* \psi$ ($\nabla^* \psi$ is the covariant derivative compatible with $g^*$) determines the normal direction of the hypersurface $\psi = 0$ with respect to the new solution $g^*$. The induced metric will be Lorentzian if $\nabla^* \psi$ is spacelike with respect to the metric $g^*$. In coordinates (1), we have

$$g^*(\nabla^* \psi, \nabla^* \psi) = (g^*)^{55}.$$ 

Now, this metric component can be written as

$$(g^*)^{55} = \frac{\det g^*_{ij}}{\det g^*_{\mu\nu}}.$$ 

Since $\det g^*_{\mu\nu} < 0$ on $V$, $(g^*)^{55}$ will be positive if $\det(g^*_{ij}) < 0$. We know that the determinant is a continuous function with respect to the matrix components in the usual norm. So there is

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6 By the symbol $\sim$ we mean that derivatives are taken only in tangent directions of $\Sigma$. 

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5009
some $\xi > 0$ which depends only on $\tilde{g}$ and $V$ (an estimate of $\xi$ is given in appendix B) such that for $\varepsilon < \xi$ we have

$$|\text{det} g^*_i j - \text{det} \tilde{g}_i j| < \inf_{\tilde{V}} |\text{det} \tilde{g}_i j|.$$ 

This means that $\text{det} g^*_i j < 0$ for all points in $V$, since $\text{det}(\tilde{g}_i j)$ is negative in $V$. This shows that $g^*$ is Lorentzian for $\varepsilon < \xi$.

Now let us compare the induced metric $g^*$ with the original spacetime metric $g$. It is known that for a field in a Sobolev space the norm of the restriction of that field to a hypersurface is related to its norm in the manifold (see [19], lemma 7.4.3, p 235). According to the mentioned lemma 7.4.3, there is a positive constant $Q$ (depending on $N$ and $V$) such that

$$\|g^* - \tilde{g}, N\|_m \leq Q \|g^* - \tilde{g}, V\|_{m+1},$$

where $g^* - \tilde{g}, N\|_m$ is defined on $N$ with the induced Euclidean metric. Thus, for example,

$$\|g^* - \tilde{g}, N\|_0 = \left(\int_N (|g^* - \tilde{g}|^2 d^4x)\right)^{1/2}.$$ 

On the other hand $\|g^* - \tilde{g}, N\|_m \leq \|g^* - \tilde{g}, V\|_{m+1}$ (by $\sim$ we mean that derivatives are taken only in directions tangent to $N$). Now consider $g^*$, the induced metric on $N$. In these special coordinates it is easy to see $\|g^* - g, N\|_m \leq \|g^* - \tilde{g}, N\|_m$, since the induced metric has fewer components than the higher dimensional metric. Therefore $\|g^* - g, N\|_m \leq Q \|g^* - \tilde{g}, V\|_{m+1}$.

From the results obtained above we can now show that if the initial data are sufficiently close to $(h, K)$ on a neighbourhood of the causal past of $N$ in $\Sigma$, then the induced metric by the new solution is Lorentzian and close to $g$ in $N$.

More precisely, let $N$ be an open set in $D^+(\Sigma) \cap M$ with compact closure in $M$ and $\eta > 0$, there exist some neighbourhood $U$ of $J^-(\tilde{N}) \cap \Sigma$ of compact closure in $\Sigma$ and some $\delta > 0$ such that $C^\infty$ initial data $(h^*, K^*)$ close to $(h, K)$ in $U$, i.e., $\|h^* - h, U\|_m < \delta$ and $\|K^* - K, U\|_{m-1} < \delta$, give rise to a metric $g^*$ which induces a Lorentzian metric in $N$ which is near $g$, i.e., $\|g^* - g, N\|_{m-1} < \eta$.

In order to see this it suffices to take $m > 4$ ([20]) and make appropriate choices for $\varepsilon$ and $V$ in the Cauchy stability theorem. Indeed, let $V$ be any neighbourhood of $N$ in $D^+(\Sigma)$ with compact closure. Since $\tilde{N} \subset \tilde{V}$ then $J^-(\tilde{N}) \subset J^-(\tilde{V})$. Thus, taking $U$ to be some neighbourhood of $J^-(\tilde{V}) \cap \Sigma$ of compact closure in $\Sigma$, the Cauchy stability theorem ensures that there exists $\delta$ such that the new metric generated by $g^*$ satisfies $\|g^* - \tilde{g}, V\|_m < \varepsilon$ for any $\varepsilon > 0$. Now if we take $\varepsilon < \min(\xi, \frac{\eta}{Q})$, we guarantee that the induced metric $g^*$ is Lorentzian in $N$ and that $\|g^* - g, N\|_{m-1} < \eta$.

### 4. Final remarks

From this analysis we can conclude that for each ambient space whose existence is guaranteed by the CM theorem there corresponds an initial data set $(h, K, \Sigma)$ with respect to which it possesses, in a certain domain, the desirable physical properties of causality and stability. This cannot be ensured by the CM theorem itself, but by an indirect reasoning as discussed above. As a direct consequence of this result we have found that for any analytical spacetime there exist initial data in whose Cauchy development for the vacuum Einstein equations it can be locally embedded. The embedding is Cauchy stable and obeys the domain of dependence property with respect to the initial data set $(h, K, \Sigma)$.

The extension of the above result to the case where a cosmological constant is included in the field equations can easily be done. On the other hand the same analysis cannot be applied to the case of a brane in a straightforward way. The problem is that in this case we cannot guarantee the existence of a smooth spacelike hypersurface in an open set of $\tilde{M}$, at least by
the method employed here, because, as is known, the brane must be embedded in a space whose metric has a discontinuity in the derivative along the normal direction with respect to the brane. Thus this metric is not analytical in any open set containing the point \( p \) and for this reason we cannot use the Cauchy–Kowalewski theorem to guarantee the existence of a function whose gradient is everywhere timelike as we have done in appendix A. Nevertheless, this question deserves further investigation.

The analyticity of the initial data in \( \Sigma \) is a restriction imposed by the CM theorem. However, it is not to be considered as an unphysical condition. Indeed, it must be realized that a great part of the physical solutions, even in the relativistic regime, are analytical in a certain domain. The crucial point here is the domain of convergence. What seems to be unrealistic is a field which is analytical in the whole manifold. Thus, we can say that if we could handle the initial data in the spacelike four-dimensional hypersurface, it would be physically feasible to prepare initial data which are analytical in the interior of a compact domain \( S \subset \Sigma \) containing the point \( p \) in order to generate the desired embedding in the interior of \( D(S) \).

Moreover we have seen that \( C^\infty \) initial data sufficiently close to an analytical initial data set give rise to embeddings of \( C^\infty \) spacetimes which are near the original analytical spacetime. This may suggest the possibility of extending the CM theorem to a less restrictive differential class of embedddings. We are currently investigating this possibility.

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Appendix A

Consider the five-dimensional vacuum space \( (\tilde{M}, \tilde{g}) \) into which the spacetime \( (M, g) \) is locally embedded around the point \( p \in M \). Let us take the following equation for the unknown function \( \phi \):

\[
\tilde{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi = -1.
\]  
(A.1)

In Gaussian coordinates this equation can be written in the following form:

\[
\frac{\partial \phi}{\partial \psi} = \pm \sqrt{-1 + \tilde{g}^{ij} \partial_i \phi \partial_j \phi}.
\]  
(A.2)

Let us consider the equation with the positive sign. According to the Cauchy–Kowalewski theorem, there is an open set \( \tilde{O} \) in \( \tilde{M} \) containing the point \( p \) where the equation has a solution provided the initial condition

\[
\phi|_{\psi=0} = f(x)
\]  
(A.3)

ensures that the right-hand side of that equation be an analytical function of its arguments \( (\partial_i \phi) \) at the point \( p \). This can be achieved by choosing \( f(x) \) in such a way that the following inequality is satisfied at the point \( p \):

\[
(g^{ij} \partial_i f \partial_j f)|_p > 1.
\]  
(A.4)

This can always be done. For example, take \( f(x) = \lambda V_i x^i \), where \( \lambda > 1 \) and \( V_i \) is the component of a unit spacelike vector with respect to the spacetime metric at the point \( p \).

Therefore, the solution \( \phi \) is a function whose gradient is everywhere timelike in \( (\tilde{O}, \tilde{g}) \), and this means that the stable causality condition holds on \( (\tilde{O}, \tilde{g}) \). Now let us assume without loss of generality that \( \phi(p) = 0 \). Considering that the gradient of \( \phi \) does not vanish in \( \tilde{O} \), we
know that the inverse image $\phi^{-1}(0)$ is a hypersurface $\Sigma$ of $\bar{O}$. Since the gradient of $\phi$ which is orthogonal to $\Sigma$ is everywhere timelike, we can conclude that $\Sigma$ is a spacelike hypersurface. Moreover, $\Sigma$ is acronhral. Indeed, every future directed timelike curve which leaves $\Sigma$ cannot return to $\Sigma$ since $\phi$ does not change sign along these curves. It happens that an acronhral spacelike hypersurface is acausal [21]. Therefore, $\Sigma$ is an acausal spacelike hypersurface in $(\bar{O}, \tilde{g})$.

Appendix B

In this section we want to determine an estimate of how near $\tilde{g}$ the new metric $g^*$ must be in order to induce a Lorentzian metric in $N$. As described above, this is achieved if the condition

$$|\det g^*_ij - \det \tilde{g}_{ij}| < \inf \{ \det \tilde{g}_{ij} \}$$

holds on $V$. As we have seen, for \( m \geq 3 \), \( \| g^* - \tilde{g} \|_m < \epsilon \) implies that \( |g^*_{\mu\nu} - \tilde{g}_{\mu\nu}| < P\epsilon \) on $V$.

Then, there exists a continuous field $\gamma_{\mu\nu}$, with $|\gamma_{\mu\nu}| < 1$ on $V$, such that $g^*_{\mu\nu} = \tilde{g}_{\mu\nu} + \theta \gamma_{\mu\nu}$, where $\theta = P\epsilon$. We want to evaluate $\det g^*_ij$. First let us define $\omega^{ijklm}$ as a totally anti-symmetric four index ‘tensor’ with $\omega^{1234} = 1$. Thus, we can write

$$\det g^*_ij = \omega^{ijklm}(\tilde{g}_{ik} + \theta \gamma_{ik}) (\tilde{g}_{lj} + \theta \gamma_{lj}) (\tilde{g}_{mk} + \theta \gamma_{mk}) (\tilde{g}_{ln} + \theta \gamma_{ln}).$$

This expression can be rewritten in the following form:

$$\det g^*_ij = \det \tilde{g}_{ij} + \theta I_1 + \theta^2 I_2 + \theta^3 I_3 + \theta^4 |\det \gamma_{ij}|,$$

where

$$I_1 = \omega^{ijklm}(\tilde{g}_{ik} + \theta \gamma_{ik}) (\tilde{g}_{lj} + \theta \gamma_{lj}) (\tilde{g}_{mk} + \theta \gamma_{mk}) (\tilde{g}_{ln} + \theta \gamma_{ln})$$

$$I_2 = \omega^{ijklm}(\tilde{g}_{ik} + \theta \gamma_{ik}) (\tilde{g}_{lj} + \theta \gamma_{lj}) (\tilde{g}_{mk} + \theta \gamma_{mk}) (\tilde{g}_{ln} + \theta \gamma_{ln})$$

$$I_3 = \omega^{ijklm}(\tilde{g}_{ik} + \theta \gamma_{ik}) (\tilde{g}_{lj} + \theta \gamma_{lj}) (\tilde{g}_{mk} + \theta \gamma_{mk}) (\tilde{g}_{ln} + \theta \gamma_{ln}).$$

Therefore, we have

$$|\det g^*_ij - \det \tilde{g}_{ij}| \leq \theta |I_1| + \theta^2 |I_2| + \theta^3 |I_3| + \theta^4 |\det \gamma_{ij}|.$$

In order to make estimates of $I_1$, $I_2$, $I_3$ it is necessary to eliminate $\gamma_{ij}$. To this end, let us introduce $\chi_{ij}$, a $4 \times 4$ matrix with all elements equal to unity. Using this matrix, we get the following estimate:

$$|\omega^{ijklm}(\tilde{g}_{ik} + \theta \gamma_{ik}) (\tilde{g}_{lj} + \theta \gamma_{lj}) (\tilde{g}_{mk} + \theta \gamma_{mk}) (\tilde{g}_{ln} + \theta \gamma_{ln})| < |\omega^{ijklm}(\tilde{g}_{ik} + \theta \gamma_{ik}) (\tilde{g}_{lj} + \theta \gamma_{lj}) (\tilde{g}_{mk} + \theta \gamma_{mk}) (\tilde{g}_{ln} + \theta \gamma_{ln})|.$$

Similar inequalities hold for every term that appears in $I_1$, $I_2$, $I_3$. Then we have $|I_1| < |J_1|$, for $s = 1, 2, 3$, where $|J_s|$ is obtained from $I_s$ by substituting each one of its terms for the corresponding estimate as above. It happens that $|J_s|$ depends only on $\tilde{g}_{ij}$. Indeed, it is a function constituted of sum and multiplication of the components of $\tilde{g}_{ij}$. Now for the sake of simplicity let us consider that $\theta < 1$, (i.e., $\epsilon < \frac{1}{\theta}$), then we have

$$|\det g^*_ij - \det \tilde{g}_{ij}| < P\epsilon (|J_1| + |J_2| + |J_3| + 4!)$$

recalling that $|\det \gamma_{ij}| < 4!$. Since $\tilde{g}_{ij}$ is continuous in $\tilde{M}$, the right-hand side is bounded in any compact set such as $\tilde{V}$. The same is valid for $\det \tilde{g}_{ij}$. Now define $\xi_1$ as the positive quantity

$$\xi_1 = \frac{\inf \{ \det \tilde{g}_{ij} \}}{P[\sup_{V}(|J_1| + |J_2| + |J_3| + 4!)].}$$
The value of $\xi_1$ depends only on $\tilde{g}$ and $V$ and the chart (possibly one can give a definition independent of coordinates, but this definition is sufficient for our purpose). Now define $\xi = \min \left\{ \frac{1}{\sqrt{r}}, \xi_1 \right\}$. Taking $\varepsilon < \xi$, we get

$$|\det g^\ast_{ij} - \det \tilde{g}_{ij}| < \inf_V |\det \tilde{g}_{ij}|.$$ 

Considering that $\det \tilde{g}_{ij} < 0$ on $V$, we conclude that $\det g^\ast_{ij} < 0$ on $V$ and hence the induced metric in $N$ is Lorentzian.

References

[1] Arkani-Hamed N, Dimopoulos S and Dvali G 1998 Phys. Lett. B 429 263
[2] Antoniadis I, Arkani-Hamed N, Dimopoulos S and Dvali G 1998 Phys. Lett. B 436 257
[3] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 3370
[4] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 4690
[5] Overduin J M and Wesson P S 1997 Phys. Rep. 283 303
[6] Wesson P and Ponce de Leon J 1992 Math. Phys. 33 3883
[7] Wesson P S 1999 Space-Time-Matter (Singapore: World Scientific)
[8] Campbell J 1926 A Course of Differential Geometry (Oxford: Claredon)
[9] Magoard L 1963 Zur einbettung riemannscher Raume in Einstein-Raume und konformeueclidische Raume PhD Thesis Kiel
[10] Romero C, Tavakol R and Zalaletdinov R 1996 Gen. Rel. Grav. 28 365
[11] Anderson E and Lidsey J 2001 Class. Quantum Grav. 18 4831
[12] Dahia F and Romero C 2002 J. Math. Phys. 44 5804
[13] Dahia F and Romero C 2002 J. Math. Phys. 43 3097
[14] Anderson E, Dahia F, Lidsey J and Romero C 2003 J. Math. Phys. 44 5108
[15] Seahra S and Wesson P 2003 Class. Quantum Grav. 20 1321
[16] Katzourakis N 2004 Preprint math-ph/0407067
[17] Anderson E 2004 Preprint gr-qc/0409122
[18] Anderson E and Tavakol R 2003 Preprint gr-qc/0309063
[19] Hawking S and Ellis G 1973 The Large Scale Structure of Space-Time (Cambridge: Cambridge University Press)
[20] Choquet-Bruhat Y and York J W 1980 The Cauchy problem General Relativity and Gravitation vol 1 ed A Held (New York: Plenum)
[21] O’Neill B 1983 Semi-Riemannian Geometry (New York: Academic)
[22] Wald R 1984 General Relativity (Chicago, IL: University of Chicago Press)
[23] Friedrich H and Rendall A 2000 Preprint gr-qc/0002074