THE DISTRIBUTION OF RANDOM EVOLUTION IN ERLANG SEMI-MAROV MEDIA

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Abstract. In this paper we study a one-dimensional random motion by having a general Erlang distribution for the sojourn times of the switching process and we obtain solution of the four order hyperbolic PDE for 2-Erlang case.

1. Introduction

In the paper [1] we studied a one-dimensional random motion with the \( m \)-Erlang distribution between consequent epochs of velocity alternations. Let \( f(t, x) \) be the probability density function (pdf) of a particle position at time \( t \), provided that it exists. We obtained the following higher order hyperbolic equations for \( f(t, x) \)

\[
\left( \frac{\partial}{\partial t} - v \frac{\partial}{\partial x} + \lambda \right)^m \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} + \lambda \right)^m f(t, x) - \lambda^{2m} f(t, x) = 0,
\]

where \( v > 0 \) is the velocity of the particle and \( \lambda \) is the parameter of the \( m \)-Erlang distribution. It is assumed that a particle started at \( x = 0 \) and hence, \( f(0, x) = \delta(x) \).

The pdf \( f(t, x) \) can be represented in the following form \( f(t, x) = f_c(t, x) + f_s(t, x) \), where \( f_c(t, x) \) is the absolute continuous part and \( f_s(t, x) \) is the singular part w.r.t. Lebesgue measure on the line.

**Lemma 1.** The singular part \( f_s(t, x) \) of the pdf \( f(t, x) \) is of the following form

\[
f_s(t, x/v) = \delta(t - x/v) e^{-\lambda t} \sum_{i=0}^{m-1} (\lambda t)^i / i!,
\]

**Proof.** It is evident that for \( t = x/v \) the pdf \( f(t, x) \) has the singularity given by Eq.(2). Let us show that for \( t > |x/v| \) the pdf \( f(t, x) \) has no singularity w.r.t. Lebesgue measure on \( \mathbb{R} \). Denote by \( v_k \) the random event "\( k \) velocity alternations occurred". For \( \Delta x = [x, x + \Delta], \Delta > 0 \), consider

\[
P_{\nu_k}(x(t) \in \Delta x) = \sum_{k \geq 1} P(x(t) \in \Delta x, v_k),
\]

which is the probability of the event that at least one alternation occurred and \( x(t) \in \Delta x \). Let us show that for each \( t > 0 \) there exists a constant \( C_t < \infty \) such that

\[
\sup_x \frac{P_{\nu_k}(x(t) \in \Delta x)}{\Delta x} < C_t.
\]

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Denote by \( \theta_k, k \geq 1 \), time between \((k-1)\)th and \(k\)th velocity alternations. Recall that \( \theta_k, k \geq 1 \) are independent \(m\)-Erlang distributed random variables. It is easily verified that

\[
P_{\nu_0}(x(t) \in \Delta x) = \sum_{k \geq 1} P \left( \sum_{i=1}^{k} (-1)^{i+1} \theta_i v + (-1)^k \left( t - \sum_{i=1}^{k} \theta_i v \right) \in \Delta x, \sum_{i=1}^{k} \theta_i < t \right)
\]

\[
= \sum_{k \geq 1} P \left( \sum_{i=1}^{k} (-1)^{i+1} \theta_i - (-1)^k \sum_{i=1}^{k} \theta_i \right) v \in \Delta x - (-1)^k vt, \sum_{i=1}^{k} \theta_i < t \right)
\]

\[
= \sum_{l \geq 0} \left( 2v(\theta_1 + \theta_3 + \ldots + \theta_{2l+1}) \right) \leq \Delta x - vt, \sum_{i=1}^{2l+1} \theta_i < t \right)
\]

\[
= \sum_{l \geq 0} \left( -2v(\theta_2 + \theta_4 + \ldots + \theta_{2l+2}) \right) \leq \Delta x + vt, \sum_{i=1}^{2l+2} \theta_i < t \right)
\]

\[
\leq \sup_x \sum_{l \geq 0} \left( 2v \sum_{i=1}^{l} \theta_{2i-1} \leq \Delta x, 2v \sum_{i=1}^{l} \theta_{2i} < vt - x \right)
\]

\[+ \sup_x \sum_{l \geq 0} \left( -2v \sum_{i=1}^{l} \theta_{2i} \leq \Delta x, 2v \sum_{i=1}^{l} \theta_{2i+1} < vt + x \right).
\]

Since \(|x| \leq vt\) and for every \(m \geq 1\) the pdf \(p_m(x, \lambda)\) of the \(m\)-Erlang distribution with the parameter \(\lambda\) satisfies \(p_m(x, \lambda) \leq \lambda\), we have

\[
\sum_{l \geq 1} P(2v(\theta_1 + \theta_3 + \ldots + \theta_{2l-1}) \leq \Delta x, 2v(\theta_2 + \theta_4 + \ldots + \theta_{2l}) < vt - x)
\]

\[
\leq \frac{\lambda \Delta}{2v} \sum_{l \geq 1} P(\theta_2 + \theta_4 + \ldots + \theta_{2l} < t) \quad (3)
\]

Since \(\theta_i\) is \(m\)-Erlang distributed we have for \(2lm + 1 > t\)

\[
P(\theta_2 + \theta_4 + \ldots + \theta_{2l} < t) \leq \left( e^{\lambda t} - \sum_{i=0}^{2lm} \frac{(\lambda t)^i}{i!} \right) e^{-\lambda t} \leq \frac{(\lambda t)^{2lm+1} e^{-\lambda t}}{2lm!(2lm + 1 - \lambda t)}.
\]

Therefore, taking into account (3), there exists a constant \(A_t\) such that

\[
\sup_x \sum_{l \geq 1} P \left( 2v \sum_{i=1}^{l} \theta_{2i-1} \leq \Delta x, 2v \sum_{i=1}^{l} \theta_{2i} < vt - x \right) \leq A_t \Delta.
\]

In much the same way, we can show that there exists a constant \(B_t\) such that

\[
\sup_x \sum_{l \geq 1} P \left( -2v \sum_{i=1}^{l} \theta_{2i} \leq \Delta x, 2v \sum_{i=1}^{l} \theta_{2i+1} < vt + x \right) \leq B_t \Delta.
\]

Putting \(C_t = A_t + B_t\), we conclude the proof.

**Corollary 1.** The absolute continuous part \(f_c(t, x)\) of the pdf \(f(t, x)\) satisfies Eq.(1) for \(t < \left| \frac{x}{\lambda} \right| \).
Now let us study the behavior of the continuous part \(f_c(t, x)\) close to lines \(t = \pm \frac{\varepsilon}{2}\).

**Lemma 2.** For \(m \geq 2\), we have

\[
\lim_{\varepsilon \to 0} P\{0 < t - x(t) < \varepsilon\} = \frac{\lambda^m e^{m-1} \varepsilon^{m-\lambda t}}{2(m-1)!},
\]

\[
\lim_{\varepsilon \to 0} P\{t + x(t) < \varepsilon\} = 0.
\]

**Proof.** It is easily verified that

\[
P\{0 < t - x(t) \leq \varepsilon\} = P\{t - \frac{\varepsilon}{2} \leq \theta_1 < t\} + \int_0^t P\{\theta_3 \geq t - u, \theta_2 \leq \frac{\varepsilon}{2}, \theta_1 \in du\} + o(\varepsilon),
\]

where \(\theta_i, i = 1, 2, 3\) are independent \(m\)-Erlang distributed random variables with the parameter \(\lambda\). Since \(\int_0^t P(\theta_3 \geq t - u, \theta_2 \leq \frac{\varepsilon}{2}, \theta_1 \in du) = o(\varepsilon)\), passing to the limit, we get

\[
\lim_{\varepsilon \to 0} P\{0 < t - x(t) < \varepsilon\} = \lim_{\varepsilon \to 0} \frac{\lambda^m e^{m-1} \varepsilon^{m-\lambda t}}{2(m-1)!} + e^{\lambda t} \left(\sum_{i=0}^{m-1} \frac{(\lambda t)^i}{i!}\right) = \frac{\lambda^m e^{m-1} \varepsilon^{m-\lambda t}}{2(m-1)!}.
\]

Similarly, \(P\{t + x(t) \leq \varepsilon\} = P\{t \geq \frac{\varepsilon}{2}, \theta_1 \leq \frac{\varepsilon}{2}\} + o(\varepsilon)\) and as it easily seen that

\[
\lim_{\varepsilon \to 0} P\{t + x(t) < \varepsilon\} = 0.
\]

The case where \(m = 1\) will be considered below as an example.

We will seek solutions of Eq.(1) among functions which continuous part \(f_c(t, x)\) satisfies the following conditions

\[
\lim_{x \to t} f_c(t, x) = \lim_{\varepsilon \to 0} P\{0 < t - x(t) < \varepsilon\}, \quad \lim_{x \to t} f_c(t, x) = \lim_{\varepsilon \to 0} P\{t + x(t) < \varepsilon\}.
\]

By applying the transformation \(f(t, x) = \varepsilon^{\lambda t} g(t, x)\) and changing the variable \(y = \frac{t}{\varepsilon}\), we reduce Eq.(1) to

\[
\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right)^m g_c(t, y) - \lambda^{2m} g_c(t, y) = 0,
\]

with the singular part \(g_s(t, y) = \left(\sum_{i=0}^{m-1} \frac{(\lambda t)^i}{i!}\right) \delta(t - y)\).

In the sequel we assume, without restricting the generality, that \(\lambda = 1\). By introducing the function \(f(t, y, z) = \varepsilon^{t} g_c(t, y)\), we reduce Eq.(5) to the following equation

\[
\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right)^m f(t, y, z) - \varepsilon^{2m} \frac{\partial^2}{\partial z^{2m}} f(t, y, z) = 0.
\]

We will seek solutions of this equation by using theory of differentiable functions on commutative algebras [2].
2. MAIN RESULTS

Let \( A_0 \) be an \( 2m \)-dimensional commutative algebra over \( \mathbb{R} \), assume that the set \( e_0, e_1, \ldots, e_{2m-1} \) is a basis of \( A_0 \) with the Cayley table:

\[
e_i e_j = e_{i \oplus j},
\]

where \( i \oplus j = i + j \) (mod \( 2m \)).

Algebra \( A_0 \) has the following matrix representation:

\[
e_k \to P_k = P_1^k,
\]

where \( P_1 = [p_{ij}]_{2m \times 2m} \), \( p_{ii+1} = 1 \) for \( 0 \leq i \leq 2m - 1 \), \( p_{2m0} = 1 \) and \( p_{ij} = 0 \) for the rest of \( i, j \).

Let us put

\[
\begin{align*}
\tau^l_0 &= e_l, \quad l = 0, 1, \ldots, 2m - 1, \\
\tau^l_1 &= e_l \sin s, \quad l = 0, 1, \ldots, 2m - 1, \\
\tau^l_2 &= e_l \cos s, \quad l = 0, 1, \ldots, 2m - 1, \\
\tau^l_{2k} &= e_l \cos ks, \quad \tau^l_{2k+1} = e_l \sin (k+1)s, \quad l = 0, 1, \ldots, 2m - 1, \\
&k = 0, 1, 2, \ldots.
\end{align*}
\]

It is easily that

\[
\begin{align*}
\tau^l_{2n} \tau^l_{2n} &= \frac{1}{2} \left( \tau^l_{2(n-k)} + \tau^l_{2(n+k)} \right), \quad n \geq k, \\
\tau^l_{2n+1} \tau^l_{2n+1} &= \frac{1}{2} \left( \tau^l_{2(n-k)+1} + \tau^l_{2(n+k)+1} \right), \quad n \geq k.
\end{align*}
\]

Let us introduce the following algebra

\[
A = \left\{ \sum_{k=0}^{+\infty} \sum_{l=1}^{2m-1} \left( a_{2k}^l \tau^l_{2k} + a_{2k+1}^l \tau^l_{2k+1} \right) \mid a_j^l \in \mathbb{R} \right\},
\]

where \( \sum_{k=0}^{+\infty} \sum_{l=1}^{2m-1} \left( \left| a_{2k}^l \right|^2 + \left| a_{2k+1}^l \right|^2 \right) < +\infty. \)

It is easily verified that \( A \) is commutative.

We consider the subspace \( B = \{ a_0 \tau^l_1 + a_1 \tau^l_2 + a_2 \tau^l_0 \mid a_i \in \mathbb{R} \} \) of the algebra \( A \).

Let us introduce the function \( f : B \to A \) \((f(t, y, z) = f(e_1 (t \cos s + yi \sin s) + z)) \) as follows

\[
f(t, y, z) = \sum_{k=0}^{+\infty} \sum_{l=0}^{2m-1} \left( v^l_{2k} (t, y, z) \tau^l_{2k} + v^l_{2k+1} (t, y, z) \tau^l_{2k+1} \right).
\]

The function \( f \) is called \( B/A \) differentiable at \( \mathbf{x}_0 \in B \) if there exists \( f' (\mathbf{x}_0) \in A \) such that for any \( \mathbf{h} \in B \)

\[
f' (\mathbf{x}_0) \mathbf{h} = \lim_{\epsilon \to 0} \frac{f(\mathbf{x}_0 + \epsilon \mathbf{h}) - f(\mathbf{x}_0)}{\epsilon}
\]

In [2] proved that if \( f \) is \( B/A \) differentiable, then

\[
\frac{\partial}{\partial t} f = e_1 \cos s \frac{\partial}{\partial z} f
\]

and
\[ \frac{\partial f}{\partial y} = e_1 \text{isin}s \frac{\partial f}{\partial z}. \] (8)

In this case all \( v_{2k}^l (t, y, z) \) are solutions of Eq.(6). Indeed,
\[ \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} \right)^m f - \frac{\partial^{2m}}{\partial z^{2m}} f = e_1^m \left( \cos^2 s - (\text{isin}s)^2 \right)^m - 1 = 0. \]

In the sequel we denote by \( e \) the element \( e_1 \).
We will seek a solution of Eq.(5) in the following form
\[ g_{c} (e (t \cos s + y \text{isin}s)) = e^{e(t \cos s + y \text{isin}s)} \]
Since \( f (e (t \cos s + y \text{isin}s) + z) = g_{c} (e (t \cos s + y \text{isin}s)) e^z \) we have
\[ v_{k}^l (t, y, z) = u_{k}^l (t, y) e^z, l = 0, 1, \ldots, 2m - 1, \quad k = 0, 1, 2, \ldots, \]
where \( g_{c} (t, y) = \sum_{k=0}^{+\infty} \sum_{l=0}^{2m-1} \left( u_{2k}^l (t, y) \tau_{2k}^l + u_{2k+1}^l (t, y) \tau_{2k+1}^l \right) \).

Therefore, we obtain functions \( u_{0}^l (t, y) \) for \( t \geq |y| \) from the following equation
\[ \sum_{l=0}^{2m-1} u_{0}^l (t, y) \tau_{l}^1 = \sum_{l=0}^{2m-1} u_{0}^l (t, y) e^l \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{e(t \cos s + y \text{isin}s)} ds = J_{0} \left( e^{i \sqrt{y^2 - t^2}} \right) = I_{0} \left( e^{\sqrt{t^2 - y^2}} \right), \]
where \( I_{k} \) (resp. \( J_{k} \)) is the modified Bessel (resp. Bessel) function of the first kind
and \( k \)th order [4].
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and \( k \)th order [4].
It follows from Eqs.(7),(8) the following Cauchy-Riemann type conditions
\[ \frac{\partial}{\partial t} u_{0}^l \stackrel{\oplus 1}{=} \frac{1}{2} u_{2}^l, \]
\[ \frac{\partial}{\partial t} u_{1}^l \stackrel{\oplus 1}{=} \frac{1}{2} u_{3}^l, \]
\[ \frac{\partial}{\partial t} u_{2}^l \stackrel{\oplus 1}{=} u_{0}^l + \frac{1}{2} u_{4}^l, \]
\[ \frac{\partial}{\partial t} u_{2k-1}^l \stackrel{\oplus 1}{=} \frac{1}{2} \left( u_{2k-3}^l + u_{2k+1}^l \right), \]
\[ \frac{\partial}{\partial t} u_{2k}^l \stackrel{\oplus 1}{=} \frac{1}{2} \left( u_{2(k-1)}^l + u_{2(k+1)}^l \right), \] (9)
and
\[ \frac{\partial}{\partial y} u_{0}^l \stackrel{\oplus 1}{=} -\frac{1}{2} u_{1}^l, \]
\[ \frac{\partial}{\partial y} u_{1}^l \stackrel{\oplus 1}{=} u_{0}^l - \frac{1}{2} u_{4}^l, \]
\[ \frac{\partial}{\partial y} u_{2}^l \stackrel{\oplus 1}{=} -\frac{1}{2} u_{3}^l, \]
It is easily seen that
\[ \frac{\partial u_{2k+1}}{\partial y} = \frac{1}{2} \left( u_{2k} - u_{2(k+2)} \right), \]
\[ \frac{\partial u_{2k+2}}{\partial y} = \frac{1}{2} \left( u_{2k-1} - u_{2k+3} \right), \]
(10)

where the basis \( \tau^0 = 1, \tau^1 = \cos ks, \tau^2 = \epsilon^l \cos ks, \tau^3 = \epsilon^l \sin (k+1)s \),
\( l = 0, 1, 2, 3, k = 0, 1, 2, \ldots \)

Therefore, we have \( \tau^0 = 1, \tau^1 = \cos ks, \tau^2 = \epsilon^l \cos ks, \tau^3 = \epsilon^l \sin (k+1)s \),
\( l = 0, 1, 2, 3, k = 0, 1, 2, \ldots \)

Taking into account that \( g_s (e (t \cos s + y \sin s)) = e^{e (t \cos s + y \sin s)} \), we have
\[ u_0^0(t, y) + e u_0^1(t, y) + e^2 u_0^2(t, y) + e^3 u_0^3(t, y) = \frac{1}{2} \int_{-\pi}^{\pi} e^{e (t \cos s + y \sin s)} ds = I_0 \left( e^{\sqrt{t^2 - y^2}} \right). \]

It is easily seen that
\[ I_0 \left( e^{\sqrt{t^2 - y^2}} \right) = \frac{I_0 \left( \sqrt{t^2 - y^2} \right) + \sqrt{t^2 - y^2}}{2} + e^2 \left( \frac{I_0 \left( \sqrt{t^2 - y^2} \right) - \sqrt{t^2 - y^2}}{2} \right) \]
\[ = \frac{I_0 \left( \sqrt{t^2 - y^2} \right) + J_0 \left( \sqrt{t^2 - y^2} \right)}{2} + e^2 \left( \frac{I_0 \left( \sqrt{t^2 - y^2} \right) - J_0 \left( \sqrt{t^2 - y^2} \right)}{2} \right). \]

Therefore, for \( t \geq |y| \), we have \( u_0^1(t, y) = u_0^3(t, y) = 0 \) and
\[ u_0^0(t, y) = \frac{I_0 \left( \sqrt{t^2 - y^2} \right) + J_0 \left( \sqrt{t^2 - y^2} \right)}{2}, \]
\[ u_0^2(t, y) = \frac{I_0 \left( \sqrt{t^2 - y^2} \right) - J_0 \left( \sqrt{t^2 - y^2} \right)}{2}. \]

It follows from the first two equations of (10) that
\[ u_1^1 = -2 \frac{\partial}{\partial y} u_0^0 = - \frac{\partial}{\partial y} \left[ I_0 \left( \sqrt{t^2 - y^2} \right) - J_0 \left( \sqrt{t^2 - y^2} \right) \right] = \frac{y}{\sqrt{t^2 - y^2}} \left( I_1 \left( \sqrt{t^2 - y^2} \right) + J_1 \left( \sqrt{t^2 - y^2} \right) \right), \]

\[ u_3^1 = -2 \frac{\partial}{\partial y} u_0^0 = - \frac{\partial}{\partial y} \left[ I_0 \left( \sqrt{t^2 - y^2} \right) + J_0 \left( \sqrt{t^2 - y^2} \right) \right] = \frac{y}{\sqrt{t^2 - y^2}} \left( I_1 \left( \sqrt{t^2 - y^2} \right) - J_1 \left( \sqrt{t^2 - y^2} \right) \right), \]

\[ u_0^0 = -2 \frac{\partial}{\partial y} u_0^0 = 0, \]

\[ u_2^0 = -2 \frac{\partial}{\partial y} u_3^0 = 0. \]

Then it follows from the Cauchy-Riemann type conditions (9) that

\[ u_2^0 (t, y) = 2 \frac{\partial u_0^1 (t, y)}{\partial t} = 0; \]

\[ u_2^1 (t, y) = 2 \frac{\partial u_3^0 (t, y)}{\partial t} = \frac{\partial}{\partial t} \left[ I_0 \left( \sqrt{t^2 - y^2} \right) - J_0 \left( \sqrt{t^2 - y^2} \right) \right] = \frac{t}{\sqrt{t^2 - y^2}} \left( I_1 \left( \sqrt{t^2 - y^2} \right) + J_1 \left( \sqrt{t^2 - y^2} \right) \right); \]

\[ u_2^2 (t, y) = 2 \frac{\partial u_0^3 (t, y)}{\partial y} = 0; \]

\[ u_2^3 (t, y) = 2 \frac{\partial u_0^3 (t, y)}{\partial t} = \frac{\partial}{\partial t} \left[ I_0 \left( \sqrt{t^2 - y^2} \right) + J_0 \left( \sqrt{t^2 - y^2} \right) \right] = \frac{t}{\sqrt{t^2 - y^2}} \left( I_1 \left( \sqrt{t^2 - y^2} \right) - J_1 \left( \sqrt{t^2 - y^2} \right) \right). \]

Similarly, for \( u_3^0 \) we have

\[ u_3^0 = 2 \frac{\partial}{\partial t} u_1^0 = 2 \frac{\partial}{\partial t} \left[ \frac{y}{\sqrt{t^2 - y^2}} \left( I_1 \left( \sqrt{t^2 - y^2} \right) + J_1 \left( \sqrt{t^2 - y^2} \right) \right) \right] = - \frac{2ty}{(t^2 - y^2)^{3/2}} \left( I_1 \left( \sqrt{t^2 - y^2} \right) + J_1 \left( \sqrt{t^2 - y^2} \right) \right) \]

\[ + \frac{ty}{t^2 - y^2} \left( I_0 \left( \sqrt{t^2 - y^2} \right) + I_2 \left( \sqrt{t^2 - y^2} \right) + J_0 \left( \sqrt{t^2 - y^2} \right) - J_2 \left( \sqrt{t^2 - y^2} \right) \right); \]
Next, it follows from (9) that

\[ u_3^2 = 2 \frac{\partial}{\partial t} u_1^3 = 2 \frac{\partial}{\partial t} \left[ \frac{y}{\sqrt{t^2 - y^2}} \left( I_1 \left( \sqrt{t^2 - y^2} \right) - J_1 \left( \sqrt{t^2 - y^2} \right) \right) \right] \]

\[ = - \frac{2ty}{\sqrt{(t^2 - y^2)^3}} \left( I_1 \left( \sqrt{t^2 - y^2} \right) - J_1 \left( \sqrt{t^2 - y^2} \right) \right) \]

\[ + \frac{2ty}{t^2 - y^2} \left( I_0 \left( \sqrt{t^2 - y^2} \right) + I_2 \left( \sqrt{t^2 - y^2} \right) - J_0 \left( \sqrt{t^2 - y^2} \right) + J_2 \left( \sqrt{t^2 - y^2} \right) \right) \]

It is easily seen that \( u_1^3 = u_3^2 = 0 \).

Next, it follows from (9) that

\[ u_4^0 - 2 \frac{\partial u_3^2}{\partial t} - 2u_0^0 = 2 \frac{\partial}{\partial t} \frac{t}{\sqrt{t^2 - y^2}} \left( I_1 \left( \sqrt{t^2 - y^2} \right) + J_1 \left( \sqrt{t^2 - y^2} \right) \right) - 2u_0^0 \]

\[ = \frac{-2y^2}{\sqrt{(t^2 - y^2)^3}} \left( I_1 \left( \sqrt{t^2 - y^2} \right) + J_1 \left( \sqrt{t^2 - y^2} \right) \right) \]

\[ + \frac{t^2}{t^2 - y^2} \left( I_0 \left( \sqrt{t^2 - y^2} \right) + I_2 \left( \sqrt{t^2 - y^2} \right) + J_0 \left( \sqrt{t^2 - y^2} \right) - J_2 \left( \sqrt{t^2 - y^2} \right) \right) \]

\[ - I_0 \left( \sqrt{t^2 - y^2} \right) - J_0 \left( \sqrt{t^2 - y^2} \right) ; \]

Also it is easily verified that \( u_1^3 = u_3^2 = 0 \).

By using well-known integrals for Bessel functions [3-5], we have

\[ \int_{-t}^{t} u_0^0 dy = \sinh t + \sin t , \int_{-t}^{t} u_2^0 dy = \sinh t - \sin t , \int_{-t}^{t} u_1^1 dy = \int_{-t}^{t} u_1^3 dy = 0, \]

\[ \int_{-t}^{t} u_2^2 dy = 2 \int_{-t}^{t} \frac{\partial u_3^2}{\partial t} dy = 2 \left( \frac{\partial}{\partial t} \int_{-t}^{t} \left( u_0^0(t) - u_0^0(t,-t) \right) dt \right) \]

\[ = 2 \cosh t - 2 \cos t , \]

\[ \int_{-t}^{t} u_3^3 dy = 2 \int_{-t}^{t} \frac{\partial u_3^2}{\partial t} dy = 2 \left( \frac{\partial}{\partial t} \int_{-t}^{t} u_0^0(t) - u_0^0(t,-t) dt \right) \]

\[ = 2 \cosh t + 2 \cos t - 4. \]

As an example, we obtain the pdf for the case, where \( m = 1 \). For this case \( e_1 = 1 \) and hence, we can consider functions \( \sum_{l=0}^{4} u_k^l(t,y) \), \( k = 0,1,2,\ldots \) as solutions of Eq.(5) for \( m = 1 \).
For $t \leq |y|$ consider the function $g_c(t, y) = g_\epsilon(t, y) + g_s(t, y)$ of the following form:

\[
g_c(t, y) = \frac{1}{2} (u_0^0(t, y) + u_0^2(t, y)) + \frac{1}{4} (u_1^0(t, y) + u_1^2(t, y) + u_2^0(t, y) + u_2^2(t, y))
\]

\[
= I_0 \left( \frac{\sqrt{t^2 - y^2}}{2} \right) + \frac{t + y}{2\sqrt{t^2 - y^2}} t_1 \left( \sqrt{t^2 - y^2} \right)
\]

and $g_s(t, y) = \delta(t - y)$.

It is easily seen that function $g_c(t, y)$ is a solution of the equation for $t < y$

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} \right) g(t, y) - g(t, y) = 0,
\]

(11)

In addition, we have $\lim_{y \uparrow t} g_c(t, y) = \frac{1}{2} (1 + t)$ and $\lim_{y \downarrow t} g_c(t, y) = \frac{1}{2}$.

To avoid cumbersome calculations we put $v = 1$.

Therefore, $f(t, x) = e^{-t} g(t, x)$ is a solution of the equation:

\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} + 1 \right) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + 1 \right) f_c(t, x) - f_c(t, x) = 0,
\]

(12)

\[
f_s(t, x) = \delta(t - x) e^{-t}.
\]

In addition, $f_c(t, x)$ satisfies the following conditions:

\[
\lim_{x \uparrow t} f_c(t, x) = \frac{1}{2} \left( e^{-t} + te^{-t} \right), \quad \lim_{x \downarrow t} f_c(t, x) = \frac{1}{2} e^{-t},
\]

and for all $t > 0$ we have $\int_{-t}^t f(t, x) dx = 1$.

For a small $\varepsilon > 0$ consider the probability $P \{0 < t - x(t) < \varepsilon \}$.

Let us verify that $\lim_{\varepsilon \downarrow 0} P \{0 < t - x(t) < \varepsilon \}$, i.e.,

\[
\lim_{\varepsilon \downarrow 0} \frac{P \{0 < t - x(t) < \varepsilon \}}{\varepsilon} = \left( \frac{e^{-t} + te^{-t}}{2} \right).
\]

Indeed, it is easily seen that

\[
P \{0 < t - x(t) \leq \varepsilon \} = P \left\{ t - \frac{\varepsilon}{2} \leq \theta_1 < t \right\} + \int_0^t P \left\{ \theta_3 \geq t - u, \theta_2 \leq \frac{\varepsilon}{2}, \theta_1 \in du \right\}
\]

+ $o(\varepsilon)$,

where $\theta_i$, $i = 1, 2, 3$ are independent exponentially distributed random variables.

The random variable $\theta_1$ is time of the first velocity alternation, $\theta_2$ is time between the first and the second velocity alternations and $\theta_3$ is time between the second and the third velocity alternations.

We have that $P \{ t - \frac{\varepsilon}{2} \leq \theta_1 < t \} = e^{-t + \frac{\varepsilon}{2}} - e^{-t}$ and as it easy to calculate

\[
\int_0^t P \left\{ \theta_3 \geq t - u, \theta_2 \leq \frac{\varepsilon}{2}, \theta_1 \in du \right\} = (1 - e^{-\frac{\varepsilon}{2}}) \int_0^t e^{-t + u} e^{-u} du = (1 - e^{-\frac{\varepsilon}{2}}) t e^{-t}.
\]

Whence, it is easily verified that $\lim_{\varepsilon \downarrow 0} \frac{P \{0 < t - x(t) \leq \varepsilon \}}{\varepsilon} = \left( \frac{e^{-t} + te^{-t}}{2} \right)$.  

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Similarly, \( P \{ t + x ( t ) \leq \varepsilon \} = P \{ \theta_2 \geq t - \frac{\varepsilon}{2}, \theta_1 \leq \frac{\varepsilon}{2} \} + o ( \varepsilon ) \). This implies that
\[
\lim_{\varepsilon \downarrow 0} \frac{P \{ t + x ( t ) < \varepsilon \}}{\varepsilon} = \frac{1}{2} e^{-t} = \lim_{x \downarrow t} f_c ( t, x ) .
\]

Therefore, \( f_c ( t, x ) \) is a solution of the Goursat problem for the linear second order hyperbolic equation that ensures the uniqueness of the solution of Eq.(12) with conditions (4). It means that \( f ( t, x ) \) is the pdf of the particle’s position for \( m = 1 \).

It is relevant to remark that function \( f ( t, x ) \) coincides with the result obtained in [5].

Now, we turn to the case \( m = 2 \) and continue to calculate integrals of \( u_k \).

It follows from \( u_4^0 = 2 \frac{\partial u_0^3}{\partial t} - 2u_0^3 \) that
\[
\int_{-t}^{t} u_4^0 dy = 2 \left( \frac{\partial}{\partial t} \int_{-t}^{t} u_2^3 dy - u_2^1 ( t, t ) - u_2^1 ( t, -t ) \right) - 2\sinh t - 2\sin t = 4 ( \sinh t + \sin t - t ) - 2\sinh t - 2\sin t = 2\sinh t + 2\sin t - 4t .
\]

Next, it follows from \( u_3^2 = 2 \frac{\partial u_0^3}{\partial t} - 2u_0^3 \) that
\[
\int_{-t}^{t} u_3^2 dy = 2 \left( \frac{\partial}{\partial t} \int_{-t}^{t} u_2^3 dy - u_2^3 ( t, t ) - u_2^3 ( t, -t ) \right) - 2\sinh t + 2\sin t = 4\sinh t - 4\sin t - 2\sinh t + 2\sin t = 2\sinh t - 2\sin t .
\]

For \( t \leq |y| \) we introduce the function \( g_c ( t, y ) = g_c ( t, y ) + g_s ( t, y ) \), where
\[
g_c ( t, y ) = \frac{1}{2} u_0^2 ( t, y ) + \frac{1}{4} \left( u_1^1 ( t, y ) + u_1^1 ( t, y ) + u_2^1 ( t, y ) + u_2^1 ( t, y ) + u_3^1 ( t, y ) + u_3^1 ( t, y ) \right) ,
\]
\[
g_s ( t, y ) = \delta ( t - y ) + t\delta ( t - y ) .
\]

By construction, the function \( g_c ( t, y ) \) is a solution of the following equation
\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} \right) g ( t, y ) - g ( t, y ) = 0 .
\]

Therefore, the function \( f_c ( t, x ) = e^{-t} g_c ( t, x ) \) is a solution of Eq.(1) for \( m = 2 \) \((\lambda = v = 1)\).

We put \( f ( t, x ) = f_c ( t, x ) + e^{-t} g_s ( t, x ) \). Taking into account the values of integrals of functions, which are involved in the expression for \( g_c ( t, y ) \), we have that \( f_{-t} f ( t, x ) dx = 1 \), for all \( t \geq 0 \).

Let us prove that \( \lim_{x \uparrow t} f_c ( t, x ) = \lim_{x \downarrow 0} \frac{P \{ 0 < t - x ( t ) < \varepsilon \}}{\varepsilon} \) and \( \lim_{x \downarrow -t} f_c ( t, x ) = \lim_{x \downarrow 0} \frac{P \{ t + x ( t ) < \varepsilon \}}{\varepsilon} \).

It follows from Lemma 2 that for \( m = 2 \) we have
\[
\lim_{x \downarrow 0} \frac{P \{ t - x ( t ) < \varepsilon \}}{\varepsilon} = \frac{1}{2} t e^{-t} .
\]

and
\[
\lim_{x \downarrow 0} \frac{P \{ t + x ( t ) < \varepsilon \}}{\varepsilon} = 0.
\]

It easily verified that \( \lim_{y \uparrow t} u_4^0 ( t, y ) = 0 \), \( \lim_{y \downarrow t} u_3^0 ( t, y ) = 0 \) and consequently

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It is easily seen that each solution of Eq.(11) is a solution of Eq.(14). By changing the solution $f$ of Eq.(11)

$$
\lim_{y \to t} g_c(t, y) = \lim_{y \to t} \frac{t + y}{2 \sqrt{t^2 - y^2}} I_1 \left( \sqrt{t^2 - y^2} \right) = \frac{t}{2},
$$

$$
\lim_{y \to -t} g_c(t, y) = \lim_{y \to -t} \frac{t + y}{2 \sqrt{t^2 - y^2}} I_1 \left( \sqrt{t^2 - y^2} \right) = 0. \quad (15)
$$

Thus,

$$
\lim_{x \to t} f_c(t, x) = \frac{1}{2} t e^{-t} = \lim_{\varepsilon \downarrow 0} \frac{P \{ t - x (t) < \varepsilon \} - P \{ t - x (t) > \varepsilon \}}{\varepsilon},
$$

$$
\lim_{x \to -t} f_c(t, x) = 0 = \lim_{\varepsilon \downarrow 0} \frac{P \{ t + x (t) < \varepsilon \} - P \{ t + x (t) > \varepsilon \}}{\varepsilon}. \quad (16)
$$

Let us show that conditions (15) with the condition $\int_{-t}^{t} g(t, y) e^{-t} dx = 1$ insure the uniqueness of the solution $g_c(t, y)$ for Eq.(14) and consequently, the uniqueness of the solution $f_c(t, x)$ of Eq.(12).

It is easily seen that each solution of Eq.(11) is a solution of Eq.(14). By changing the variables $s = t + y, \, p = t - y$, we reduce Eq.(14) to

$$
\frac{\partial^4}{\partial s^2 \partial p^2} G(s, p) - G(s, p) = 0. \quad (17)
$$

Passing to the Fourier transform $\hat{G}(s, \alpha) = \int_{0}^{\infty} G(s, p) e^{i \alpha p} dp$ in Eq.(17), we get the ordinary differential equation of order 4. Taking into account that $\lim_{y \to t} g_c(t, y) = 0$, we have

$$
\hat{G}(0, \alpha) = 0. \quad (18)
$$

Hence, at most four independent solutions of the ordinary differential equation satisfy the initial condition (18) for each $\alpha$. Passing to the inverse Fourier transform, we have four independent solutions of Eq.(14) with the condition $\lim_{x \to t} f_c(t, x) = 0$ and just two of them satisfy Eq.(14) but not Eq.(11). By construction, one of these solutions $g_c(t, y)$ is given by Eq.(13). As another solution we can take

$$
g_2(t, y) = u_0^2(t, y) + u_4^0(t, y).
$$

It is easily verified that no linear combination $c(t, y)$ of functions $g_c(t, y)$ and $g_2(t, x)$ satisfies conditions (16) and $\int_{-t}^{t} (c(t, x) + g_c(t, y)) e^{-t} dx = 1$ for all $t > 0$, but solution $g_c(t, y)$.

Therefore, the function $f(t, x)$ is the pdf of the particle position at time $t$ for $m = 2, \, v = \lambda = 1$ and has the following form

$$
f(t, x) = -J_0 \left( \frac{\sqrt{t^2 - x^2}}{2} \right) e^{-t} + \frac{(t + x) e^{-t}}{2 \sqrt{t^2 - x^2}} I_1 \left( \sqrt{t^2 - x^2} \right) \left( \frac{x^2 e^{-t}}{2 \sqrt{(t^2 - x^2)^3}} \right) \left( I_1 \left( \sqrt{t^2 - x^2} \right) + J_1 \left( \sqrt{t^2 - x^2} \right) \right) + \frac{t^2 e^{-t}}{4 (t^2 - x^2)} \left( I_0 \left( \sqrt{t^2 - x^2} \right) + I_2 \left( \sqrt{t^2 - x^2} \right) + J_0 \left( \sqrt{t^2 - x^2} \right) - J_2 \left( \sqrt{t^2 - x^2} \right) \right) + \delta(t - x) e^{-t} + t \delta(t - x) e^{-t}.
$$
In much the same way as the pdf $f(t,x)$ of the particle position for $m = 2$ was obtained we can also get solutions of Eq.(1) with conditions (2) and (4) for each $m > 2$. 
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