A CLASS OF WELL-POSED
PARABOLIC FINAL VALUE PROBLEMS

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ABSTRACT. This paper focuses on parabolic final value problems, and well-posedness is proved for a large class of these. The clarification is obtained from Hilbert spaces that characterise data that give existence, uniqueness and stability of the solutions. The data space is the graph normed domain of an unbounded operator that maps final states to the corresponding initial states. It induces a new compatibility condition, depending crucially on the fact that analytic semigroups always are invertible in the class of closed operators. Lax–Milgram operators in vector distribution spaces is the main framework. The final value heat conduction problem on a smooth open set is also proved to be well posed, and non-zero Dirichlet data are shown to require an extended compatibility condition obtained by adding an improper Bochner integral.

1. INTRODUCTION

Well-posedness of final value problems for a large class of parabolic differential equations was recently obtained by the author jointly with A. E. Christensen, and an ample description was given for a broad audience in [5], after the announcement in [4]. The present exposition is more concise and incorporates some improvements that, as indicated, may lead to future developments of the theory.

The below theoretical analysis of the problems shows that they are well posed, i.e., they have existence, uniqueness and stability of solutions \( u \in X \) for given data, say \( (f,u_T) \in Y \), in certain Hilbertable spaces \( X, Y \) that are described explicitly.

This has seemingly closed a gap in the theory, which has remained since the 1950’s, although the well-posedness is decisive for the interpretation and accuracy of numerical schemes that would be used in practice (John [16] made an early contribution in this direction). In rough terms, the results are derived from a useful structure on the reachable set for a general class of parabolic differential equations.

The primary example (addressed in the end of the paper) is the heat conduction problem of characterising the functions \( u(t,x) \) that in a \( C^\infty \)-smooth bounded open set \( \Omega \subset \mathbb{R}^n \) with boundary \( \partial \Omega \) fulfil the equations (whereby \( \Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_n} \)),

\[
\begin{align*}
\partial u(t,x) - \Delta u(t,x) &= f(t,x) & \text{for } t \in [0,T], x \in \Omega, \\
u(t,x) &= g(t,x) & \text{for } t \in [0,T], x \in \partial \Omega, \\
u(T,x) &= u_T(x) & \text{for } x \in \Omega.
\end{align*}
\]

(1)

An area of interest of this could be a nuclear power plant hit by a power failure at time \( t = 0 \): after power is regained at \( t = T > 0 \), and the reactor temperatures \( u_T(x) \) are measured, it would be crucial to calculate backwards to settle whether at an earlier time \( t_0 < T \) the temperatures \( u(t_0,x) \) could cause damage to the fuel rods.

A short plan of the paper is the following: the abstract result on final value problems is given in Section 1.3 below. Its proof then follows in Section 2. The results and proofs for the heat problem in (1) are presented Theorems 5–7 in Section 3.

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1. Background: Phenomena of Instability. It may be recalled that there is a phenomenon of $L_2$-instability in the homogeneous case $f = 0$, $g = 0$ in (1). This classical fact was perhaps first described by Miranker in 1961 [21]; but it was also emphasized more recently by Isakov [15].

The instability results by considering the Dirichlet realization of the Laplace operator, written $-\Delta_D$, and the $L_2(\Omega)$-orthonormal basis $e_1(x), e_2(x), \ldots$ of eigenfunctions associated to the usual ordering of its eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, which via Weyl’s law for the counting function, cf. [6] Ch. 6.4, gives

$$\lambda_j = \Theta(j^{2/n}) \quad \text{for } j \to \infty. \quad (2)$$

The orthonormal basis gives rise to a sequence of final value data,

$$u_{T,j}(x) = e_j(x) \quad \text{for } j \in \mathbb{N}. \quad (3)$$

This sequence clearly lies on the unit sphere in $L_2(\Omega)$ as $\| u_{T,j} \| = \| e_j \| = 1$ for $j \in \mathbb{N}$. But the corresponding solutions $u_j$ to the heat equation $u' - \Delta u = 0$, namely

$$u_j(t,x) = e^{(T-t)\lambda_j} e_j(x), \quad (4)$$

obviously have initial states $u(0,x)$ with $L_2$ norms that because of (2) grow rapidly with the index $j$,

$$\| u_j(0,\cdot) \| = e^{T\lambda_j} \| e_j \| = e^{T\lambda_j} \not\to \infty. \quad (5)$$

Consequently, when a final state $u_T(x)$ is approximated through measurements $\tilde{u}_{T,1}(x)$, $\tilde{u}_{T,2}(x)$, \ldots made with increasing accuracy in $L_2(\Omega)$-norm, it is likely that each difference $\tilde{u}_{T,j+1} - \tilde{u}_{T,j}$ has more non-zero coordinates with respect to the orthonormal basis $(e_n)_{n \in \mathbb{N}}$ than the previous one, which in view of the above rapid blow-up makes it likely that the calculated initial state correction $\tilde{u}_{j+1}(0,x) - \tilde{u}_j(0,x)$ will be of the same size (in $L_2$-norm) as the previous one, $\tilde{u}_{j+1}(0,x) - \tilde{u}_{j}(0,x)$. In this case, it remains entirely unclear whether or not $\tilde{u}_{j+1}(0,x)$ is a more accurate approximation to $u(0,x)$ than $\tilde{u}_j(0,x)$—or if $\tilde{u}_{j+1}(x)$ has been calculated in vein.

The above $L_2$-instability cannot be explained away, of course, but in reality it only indicates that the $L_2$-norm is an insensitive choice for problem (1). It therefore seems reasonable to pose the rhetorical

**Question:** Is the final value heat problem in (1) well posed?

While the answer is “yes”, one could wonder why this has not been proved before. Here it should be mentioned that our description of reachable sets for parabolic problems exploits the previously unavailable structures in the next section.

1.2. Main Tool: Injectivity. The key to the analysis of final value problems lies at a rather different spot, namely, that an analytic semigroup of operators (like $e^{\mathcal{A}t}$) always consists of injections. This enters both at the technical and conceptual level, that is, injectivity enters not just in the proofs, but also in the objects that the theorems are concerned with.

A few aspects of semigroup theory in a complex Banach space $B$ is therefore recalled here. Besides classical references by Davies [7], Pazy [23], Tanabe [27] or Yosida [29], a more recent account is given in [4].

The generator is $A_x = \lim_{\omega \to 0^+} \frac{1}{\omega} (e^{\mathcal{A}\omega} x - x)$, where $x$ belongs to the domain $D(A)$ when the limit exists. $A$ is a densely defined, closed linear operator in $B$ that for some $\omega \geq 0$, $M \geq 1$ satisfies $\| (A - \lambda)^{-n} \|_{\mathcal{B}(B)} \leq M/(\lambda - \omega)^n$ for $\lambda > \omega$, $n \in \mathbb{N}$.

The corresponding $C_0$-semigroup of operators $e^{\mathcal{A}t}$ in $B(B)$ of type $(M, \omega)$: it fulfills that $e^{s\mathcal{A}} e^{t\mathcal{A}} = e^{(s+t)\mathcal{A}}$ for $s,t \geq 0$, $e^{0\mathcal{A}} = I$, $\lim_{t \to 0^+} e^{t\mathcal{A}} x = x$ for $x \in B$, and

$$\| e^{t\mathcal{A}} \|_{\mathcal{B}(B)} \leq M e^{\omega t} \quad \text{for } 0 \leq t < \infty. \quad (6)$$

Indeed, the Laplace transformation $(\lambda I - A)^{-1} = \int_0^\infty e^{-t\lambda} e^{\mathcal{A}t} dt$ gives a bijection of the semigroups of type $(M, \omega)$ onto the resolvents of the stated class of generators.
The well-known result below gives a criterion for $A$ to generate a $C_0$-semigroup $e^{tA}$ that is defined and analytic for $z$ in the open sector

$$S_{\theta} = \{ z \in \mathbb{C} \mid z \neq 0, \ |\arg z| < \theta \}.$$  \hspace{1cm} (7)

It is formulated in terms of the spectral sector

$$\Sigma_{\theta} = \{ \lambda \in \mathbb{C} \mid |\arg \lambda| < \frac{\pi}{2} + \theta \} \cup \{0\}.$$  \hspace{1cm} (8)

**Proposition 1.** When $A$ generates a $C_0$-semigroup of type $(M, \omega)$ and $\omega \in \rho(A)$, then the following properties are equivalent for $\theta \in [0, \frac{\pi}{2})$:

(i) The resolvent set $\rho(A)$ fulfills $\rho(A) \supset \omega + \Sigma_{\theta}$ and

$$\sup \{ \lambda - \omega : ||(\lambda I - A)^{-1}||_{B(\mathbb{C})} \mid \lambda \in \omega + \Sigma_{\theta}, \lambda \neq \omega \} < \infty.$$  \hspace{1cm} (9)

(ii) The semigroup $e^{tA}$ extends to an analytic semigroup $e^{zA}$ defined for $z \in S_{\theta}$ with

$$\sup \{ e^{-\omega t} \|e^{zA}\|_{B(\mathbb{C})} \mid z \in \Sigma_{\theta} \} < \infty \quad \text{whenever} \quad 0 < \theta' < \theta.$$  \hspace{1cm} (10)

In the affirmative case, $e^{zA}$ is differentiable in $B(\mathbb{B})$ for $t > 0$, $(e^{zA})' = Ae^{zA}$ and

$$\sup_{t>0} e^{-\omega t} \|A e^{zA}\|_{B(\mathbb{B})} < \infty.$$  \hspace{1cm} (11)

In case $\omega = 0$, this is just a review of the main parts of Theorem 2.5.2 in [23]. For general $\omega \geq 0$, one can reduce to this case, since $A = \omega I + (A - \omega I)$ yields the operator identity $e^{tA} = e^{\omega t} e^{(A - \omega I)t}$, where $e^{(A - \omega I)t}$ is of type $(M, 0)$. Indeed, the right-hand side is easily seen to be a $C_0$-semigroup, which since $e^{\omega t} = 1 + t\omega + o(t)$ has $A$ as its generator, so the identity results from the bijectiveness of the Laplace transform. In this way, (i) $\iff$ (ii) follows straightforwardly from the case $\omega = 0$, using for both implications that $e^{zA} = e^{\omega t} e^{(A - \omega I)t}$ holds in $S_{\theta}$ by unique analytic extension.

To elucidate the role of injectivity of $e^{zA}$, it is recalled that if $e^{zA}$ is analytic, then $u' = Au$, $u(0) = u_0$ is always uniquely solved by $u(t) = e^{tA}u_0$ for every $u_0 \in B$. Here injectivity clearly is equivalent to the important geometric property that the trajectories of two solutions $e^{zA}v_0$ and $e^{zA}w_0$ of $u' = Au$ have no confluence point in $B$ for $v_0 \neq w_0$. Nevertheless, the literature seems to have focused only on examples of semigroups with non-invertibility of $e^{zA}$, like [23] Ex. 2.2.1.

The reason for stating Proposition 1 for general type $(M, \omega)$ semigroups is that it shows explicitly that cases with $\omega > 0$ only have different estimates in the closed subsectors $\Sigma_{\theta'}$—but the mere analyticity in $S_{\theta}$ is unaffected by the translation by $\omega I$. One therefore has the following improved version of [5] Prop. 1:

**Proposition 2.** If a $C_0$-semigroup $e^{zA}$ of type $(M, \omega)$ on a complex Banach space $B$ has an analytic extension $e^{zA}$ to $S_{\theta}$ for $\theta > 0$, then $e^{zA}$ is injective for every $z \in S_{\theta}$.

**Proof.** Let $e^{n\omega A}u_0 = 0$ hold for some $u_0 \in B$ and $z_0 \in S$. Analyticity of $e^{zA}$ in $S_{\theta}$ carries over by the differential calculus in Banach spaces to the map $f(z) = e^{zA}u_0$. So for $z$ in a suitable open ball $B(z_0, r) \subset S_{\theta}$, a Taylor expansion and the identity $f^{(n)}(z_0) = A^n e^{z_0 A}u_0$ for analytic semigroups (cf. [23] Lem. 2.4.2) give

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (z - z_0)^n f^{(n)}(z_0) = \sum_{n=0}^{\infty} \frac{1}{n!} (z - z_0)^n A^n e^{z_0 A}u_0 = 0.$$  \hspace{1cm} (12)

Hence $f \equiv 0$ on $S_{\theta}$ by unique analytic extension. Now, as $e^{zA}$ is strongly continuous, $u_0 = \lim_{t \to 0^+} e^{zA}u_0 = \lim_{t \to 0^+} f(t) = 0$. Thus the null space of $e^{zA}$ is trivial. \hfill $\Box$

**Remark 1.** The injectivity in Proposition 2 was claimed by Showalter [26] for $z > 0, \theta \leq \pi/4$ and $B$ a Hilbert space (with a flawed proof, as noted in [5] Rem. 1), cf. details on the counter-example in Lemma 3.1 and Remark 3 in [17]). A local version for the Laplacian on $\mathbb{R}^n$ was given by Rauch [24, Cor. 4.3.9].
As a consequence of the above injectivity, for an analytic semigroup \( e^{tA} \) we may consider its inverse that, consistently with the case in which \( e^{tA} \) forms a group in \( \mathcal{B}(B) \), may be denoted for \( t > 0 \) by \( e^{-tA} = (e^{tA})^{-1} \).

Clearly \( e^{-tA} \) maps its domain \( D(e^{-tA}) \), which is the range \( R(e^{tA}) \), bijectively onto \( H \), and it is in general an unbounded, but closed operator in \( B \).

Specialising to a Hilbert space \( B = H \), then also \( (e^{tA})^* = e^{tA^*} \) is analytic, so its null space \( Z(e^{tA}) = \{0\} \) by Proposition 2, whence \( D(e^{-tA}) \) is dense in \( H \).

A partial group phenomenon and commutation properties are restated here:

**Proposition 3.** [5, Prop. 2] The above inverses \( e^{-tA} \) form a semigroup of unbounded operators in \( H \),

\[
e^{-sA}e^{-tA} = e^{-(s+t)A} \quad \text{for } t, s \geq 0.
\]

This extends to \((s,t) \in \mathbb{R} \times ]-\infty,0]\), where \( e^{-(t+s)A} \) may be unbounded for \( t + s > 0 \). Moreover, as unbounded operators the \( e^{-tA} \) commute with \( e^{tA} \) in \( \mathcal{B}(H) \), i.e.,

\[
e^{tA}e^{-tA} \subset e^{-tA}e^{tA} \quad \text{for } t, s \geq 0,
\]

and have a descending chain of domains, \( D(e^{-tA}) \subset D(e^{-sA}) \subset H \) for \( 0 < t < t' \).

The above domains serve as basic structures for the final value problem (1).

1.3. **The Abstract Final Value Problem.** The basic analysis is made for a Lax–Milgram operator \( A \) defined in \( H \) from a \( V \)-elliptic sesquilinear form \( a(\cdot, \cdot) \) in a Gelfand triple, i.e., in a set-up of three Hilbert spaces \( V \hookrightarrow H \hookrightarrow V^* \) having the norms \( \| \cdot \| \), \( | \cdot | \), and \( | | \cdot | | \), respectively. Hereby \( V \) is the form domain of \( a \). Specifically there are constants \( C_j \) such that, for all \( u, v \in V \), one has \( |v| \leq C_1 |u| \leq C_2 |v| \) and \( |a(u,v)| \leq C_3 |u| |v| \) and \( |Ra(u,v)| \geq C_4 |v|^2 \). In fact, \( D(A) \) consists of those \( u \in V \) for which \( a(u,v) = (f,v) \) for some \( f \in H \) holds for all \( v \in V \); then \( Au = f \). It is also recalled that there is a bounded bijective extension \( A^* : V \rightarrow V^* \) given by \( \langle Au,v \rangle = a(u,v) \) for \( u, v \in V \). (The reader may consult [11, Ch. 12], [13] or [5] for more details on the set-up and basic properties of the unbounded, but closed operator \( A^* \) in \( H \).

In this framework, the general final value problem is as follows: for given data \( f \in L^2(0,T;V^*) \) and \( u_T \in H \), determine the \( u \in \mathcal{D}'(0,T;V) \) such that

\[
\begin{align*}
\partial_t u + Au &= f & \text{in } & \mathcal{D}'(0,T;V^*), \\
u(T) &= u_T & \text{in } & H.
\end{align*}
\]

By definition of the vector distribution space \( \mathcal{D}'(0,T;V^*) \), cf. [25], the above equation means that \( \langle u, \varphi' \rangle + \langle Au, \varphi \rangle = \langle f, \varphi \rangle \) holds as an identity in \( V^* \) for every scalar test function \( \varphi \in C_0^\infty([0,T]) \).

A wealth of parabolic Cauchy problems with homogeneous boundary conditions have been treated via triples \((H,V,a)\) and the \( \mathcal{D}'(0,T;V^*) \) set-up in [15]; cf. the work of Lions and Magenes [20], Tanabe [27], Temam [28], Amann [2].

To compare [15] with the Cauchy problem for \( u' + Au = f \) obtained from the initial condition \( u(0) = u_0 \in H \), for some \( f \in L^2(0,T;V^*) \), it is recalled that there is a unique solution \( u \) in the Banach space

\[
X = L^2(0,T;V) \cap C([0,T];H) \cap H^1(0,T;V^*),
\]

\[
\|u\|_X = \left( \int_0^T \|u(t)\|^2 \, dt + \sup_{0 \leq s \leq T} |u(t)|^2 + \int_0^T (\|u(t)\|_2^2 + \|u'(t)\|_2^2) \, dt \right)^{1/2}.
\]

For [15] it would thus be natural to envisage solutions \( u \) in the same space \( X \). This turns out to be true, but only under substantial further conditions on the data \((f,u_T)\).

To formulate these, it is exploited that \( A = -A \) generates an analytic semigroup \( e^{-zA} \) in \( \mathcal{B}(H) \), where \( z \in S_0 \) for \( \theta = \arccot(C_3/C_4) \). This is classical, but crucial for the entire analysis [5, Lem. 4] has a verification of (i), hence of (ii), in Proposition 1. By Proposition 2 it therefore has the inverse \( (e^{-tA})^{-1} = e^{tA} \) for \( t > 0 \).
In control theoretic terms its role is to provide a unique initial state closed operator, and so is Hilbertable space (like that is steered by $f$ must be defined at $(0, t)$ where all terms belong to $X$ as functions of $t$ composite map whence the solution operator $(s, t) \mapsto \Phi(s, t; \cdot)$ defined by the condition $X \in (16)$, if and only if the data $(f, u_T)$ belong to the subspace $Y \subset L_2(0; T; V^*) \oplus H$ defined by the condition

$$u_T - \int_0^T e^{-(T-t)A} f(t) \, dt \in D(e^{TA}). \quad (20)$$

In the affirmative case, the solution $u$ is uniquely determined in $X$ and

$$\|u\|_X \leq c \left( |u_T|^2 + \int_0^T \|f(t)\|^2 \, dt + \left| e^{TA} (u_T - \int_0^T e^{-(T-t)A} f(t) \, dt) \right|^2 \right)^{\frac{1}{2}}$$

whence the solution operator $(f, u_T) \mapsto u$ is continuous $Y \to X$. Moreover,

$$u(t) = e^{-tA} e^{TA} (u(T) - y_f) + \int_0^t e^{-(t-s)A} f(s) \, ds,$$  

where all terms belong to $X$ as functions of $t \in [0, T]$.

The norm on the data space $Y$ in (21) is seen at once to be the graph norm of the composite map

$$L_2(0; T; V^*) \oplus H \xrightarrow{\Phi} H \xrightarrow{e^{TA}} H$$

given by $(f, u_T) \mapsto u_T - y_f \mapsto e^{TA} (u_T - y_f)$. In fact, (20) means that the operator $e^{TA} \Phi$ must be defined at $(f, u_T)$, so the data space $Y$ is its domain. Being an inverse, $e^{TA}$ is a closed operator, and so is $e^{TA} \Phi$; hence $Y = D(e^{TA} \Phi)$ is complete. Consequently $Y$ is a Hilbertable space (like $V^*$).

Thus the unbounded operator $e^{TA} \Phi$ is a key ingredient in the rigorous treatment of (15). In control theoretic terms its role is to provide a unique initial state

$$u(0) = e^{TA} \Phi(f, u_T) \quad (24)$$

that is steered by $f$ to the final state $u(T) = u_T$ at time $T$; cf. (25) below.
Because of \( e^{-(T-t)A} \) and the integral over \([0,T]\), condition (20) clearly involves non-local operators in both space and time as an inconvenient aspect — which is exacerbated by the abstract domain \( D(e^{TA}) \) that for longer lengths \( T \) of the time interval gives increasingly stricter conditions; cf. (17).

Anyhow, (20) is a compatibility condition on the data \((f,u_T)\), and thus the notion of compatibility is generalised. For comparison it is recalled that Grubb and Solonnikov [12] systematically investigated a large class of initial-boundary problems of parabolic (pseudo-)differential equations and worked out compatibility conditions, which are necessary and sufficient for well-posedness in full scales of anisotropic \( L_2 \)-Sobolev spaces. Their conditions are explicit and local at the curved corner \( \partial \Omega \times \{0\} \), except for half-integer values of the smoothness \( s \) that were shown to require so-called coincidence, which is expressed in integrals over the product of the two boundaries \( \{0\} \times \Omega \) and \([0,T] \times \partial \Omega \); hence it also is a non-local condition. However, whilst their conditions are decisive for the solution’s regularity, the above condition (20) is crucial for the existence question; cf. the theorem.

Previously, uniqueness in (15) was shown by Amann [2, Sect. V.2.5.2] in a \( t \)-dependent set-up, but injectivity of the map \( u(0) \leftrightarrow u(T) \) was proved much earlier for problems with \( t \)-dependent sesquilinear forms by Lions and Malgrange [19].

Showalter [26] strived to characterise the possible \( u_T \) via Yosida approximations for \( f = 0 \) and \( A \) having half-angle \( \frac{\pi}{2} \). Invertibility of \( e^{-tA} \) was claimed for this purpose in [26] for such \( A \) (but, as mentioned, not quite obtained).

To make a few more remarks, it is noted that the proof given below exploits that the “height” function \( h \) of the smoothness \( s \) relies crucially on the invertibility of \( e^{-tA} \); cf. (17).

Secondly, \( e^{-tA}u(0) \) solves \( Au + f \), and for \( u(0) \neq 0 \) there is the precise property in non-selfadjoint dynamics that the “height” function \( h(t) = \|e^{-tA}u(0)\| \) is strictly positive \( (h > 0) \), strictly decreasing \( (h' < 0) \) and strictly convex. (26)

Whilst this holds if \( A \) is self-adjoint or normal, it was emphasized in [1] that it suffices that \( A \) is just hyponormal. Recently this was followed up by the author in [17], where the stronger logarithmic convexity of \( h(t) \) was proved equivalent to the weaker property of \( A \) that \( 2(R(Ax|x)|^2 \leq \Re(A^2x|x|^2 + |Ax|^2|x|^2 \text{ for } x \in D(A^2)) \).

The stiffness inherent in strict convexity reflects that \( u(T) = e^{-TA}u(0) \) is confined to a dense, but very small space, as by the analyticity

\[
u(T) \in \bigcap_{n \in \mathbb{N}} D(A^n).\]

For \( u' + Au = f \), the possible final data \( u_T \) will hence be a sum of an arbitrary vector \( y_f \) in \( H \) and a term \( e^{-TA}u(0) \) of great stiffness, cf. (27). Thus \( u_T \) can be prescribed in the affine space \( y_f + D(e^{TA}) \). As any \( y_f \neq 0 \) will shift \( D(e^{TA}) \subset H \) in some arbitrary direction, \( u(T) \) can be expected anywhere in \( H \) (unless \( y_f \in D(e^{TA}) \) is known). So neither \( u(T) \in D(e^{TA}) \) nor (27) can be expected to hold for \( y_f \neq 0 \)—not even if \( |y_f| \) is much smaller than \( \|e^{-TA}u(0)\| \). In view of this conclusion, it seems best for final value problems to consider inhomogeneous problems from the very beginning.
2. Proof of Theorem 2

The point of departure is the following well-known result, which is formulated as a theorem here only to indicate that it is a cornerstone in the proof. It is also emphasized that the Lax–Milgram operator $A$ need not be selfadjoint.

**Theorem 2.** Let $V$ be a separable Hilbert space with $V \subseteq H$ algebraically, topologically and densely, and let $A$ denote the Lax–Milgram operator induced by a $V$-elliptic sesquilinear form, as well as its extension $A \in \mathbb{B}(V, V^*)$, cf. Section [7.3] When $u_0 \in H$ and $f \in L_2(0, T; V^*)$ are given, then the Cauchy problem

$$\begin{cases} \partial_t u + Au = f & \text{in } \mathcal{D}'(0, T; V^*), \\ u(0) = u_0 & \text{in } H, \end{cases}$$

(28)

has a uniquely determined solution $u(t)$ belonging to the space $X$ in [16].

This is a special case of a classical result of Lions and Magenes [20, Sect. 3.4.4] on $t$-dependent forms $a(t; u, v)$. The conjunction of the properties $u \in L_2(0, T; V)$ and $u' \in L_2(0, T; V^*)$, which appears in [20], is clearly equivalent to the property in (16) that $u$ belongs to the intersection of $L_2(0, T; V)$ and $H^1(0, T; V^*)$.

To clarify a redundancy, it is first noted that in Theorem 2 the solution space $X$ is a Banach space, which can have its norm in [16] written in the form

$$\|u\|_X = \left( \|u\|_{L_2(0, T; V)}^2 + \sup_{0 \leq t \leq T} |u(t)|^2 + \|u\|_{H^1(0, T; V^*)}^2 \right)^{1/2}. \quad (29)$$

Here there is a well-known inclusion $L_2(0, T; V) \cap H^1(0, T; V^*) \subset C([0, T]; H)$ and an associated Sobolev inequality for vector functions (5) has an elementary proof

$$\sup_{0 \leq t \leq T} |u(t)|^2 \leq (1 + \frac{C_2}{C_1^2}) \int_0^T \|u\|^2 dt + \int_0^T \|u'\|^2 dt. \quad (30)$$

Hence one can safely omit the space $C([0, T]; H)$ in (16). Likewise $\sup |u|$ can be removed from $\|u\|_X$, as one obtains an equivalent norm (similarly $\int_0^T \|u(t)\|^2 dt$ is redundant in (16)). Thus $X$ is more precisely a Hilbert space; but (16) is kept as stated in order to emphasize the properties of the solutions.

However, two refinements of the above theory is needed. For one thing, the next result yields well-posedness of (28), which is a well-known corollary to the proofs in [20]. But a short explicit argument is also possible:

**Proposition 4.** The unique solution $u$ of (28), given by Theorem 2 depends as an element of $X$ continuously on the data $(f, u_0) \in L_2(0, T; V^*) \oplus H$, i.e.

$$\|u\|_X^2 \leq c(\|u_0\|^2 + \|f\|_{L_2(0, T; V^*)}^2). \quad (31)$$

Here and in the sequel, $c$ denotes as usual a constant in $[0, \infty]$ of unimportant value, which moreover may change from place to place.

**Proof.** As $u \in L_2(0, T; V)$ while $u'$, $f$ and $Au$ belong to the dual space $L_2(0, T; V^*)$, one has the identity $\mathfrak{R}(\partial u, u) + \mathfrak{R}(Au, u) = \mathfrak{R}(f, u)$ in $L_1$. Now, a classical regularisation yields $\partial_t |u|^2 = 2\mathfrak{R}(\partial u, u)$, so by Young’s inequality and the $V$-ellipticity,

$$\partial_t |u|^2 + 2C_4 |u|^2 \leq 2|\langle f, u \rangle| \leq C_4^{-1} \|f\|^2 + C_4 |u|^2. \quad (32)$$

Using again that $|u(t)|^2$ and $\partial_t |u(t)|^2$ are in $L_1(0, T)$, integration yields

$$|u(t)|^2 + C_4 \int_0^t \|u(s)\|^2 ds \leq |u_0|^2 + C_4^{-1} \|f\|^2_{L_2(0, T; V^*)}. \quad (33)$$
This gives \( \|u\|_{L^2(0,T;V')}^2 \leq C_4^{-1} \|u_0\|^2 + C_4^{-2} \|f\|_{L^2(0,T;V')}^2 \) for the first term in \( \|u\|_X \). As \( u \) solves (28), it is clear that \( \|\partial_t u(t)\|_2 \leq (\|f(t)\|_2 + \|Au\|_2) \), hence
\[
\int_0^T \|\partial_t u(t)\|^2 dt \leq 2 \int_0^T \|f(t)\|^2 dt + 2 \|A\|_B(V;V')^2 \int_0^T \|u\|^2 dt,
\] (34)
where \( \int_0^T \|u\|^2 dt \) is estimated above. Finally \( \sup \|u\| \) can be covered via (30).

Secondly, as \( e^{-TA} \) extends to an analytic semigroup in \( V^* \), cf. [5 Lem. 5], so that \( e^{-TA} f(t) \) is defined. Theorem 2 can be supplemented by the explicit expression:
\[
u(t) = e^{-TA} u_0 + \int_0^t e^{-(T-t)A} f(s) \, ds \quad \text{for } 0 \leq t \leq T.
\] (35)

This is of course the Duhamel formula, but even for analytic semigroups the proof that it does give a solution requires \( f(t) \) to be Hölder continuous ([23 Cor. 4.3.3] is slightly more general, though), whereas above \( f \in L_2(0,T;V^*) \). As the present space \( X \) even contains non-classical solutions, (35) requires a new proof here—but it suffices to reinforce the usual argument by the injectivity of \( e^{-TA} \) in Proposition 2.

**Theorem 3.** The unique solution \( u \) in \( X \) provided by Theorem 2 is given by (35), where each of the three terms belongs to \( X \).

**Proof.** Once (35) has been shown, Theorem 2 yields for \( f = 0 \) that \( u(t) \in X \), hence \( e^{-TA} u_0 \in X \). For general \( (f, u_0) \) the last term containing \( f \) is then also in \( X \).

To obtain (35) in the above context, note that all terms in \( \partial_t u + Au = f \) are in \( L_2(0,T;V^*) \).

Therefore \( e^{-(T-t)A} \) applies to both sides as an integration factor, so
\[
\partial_t (e^{-(T-t)A} u(t)) = e^{-(T-t)A} \partial_t u(t) + e^{-(T-t)A} Au(t) = e^{-(T-t)A} f(t).
\] (36)

Indeed, \( e^{-(T-t)A} u(t) \) is in \( L_1(0,T;V^*) \) and its derivative in \( \mathcal{G}'(0,T;V^*) \) follows a Leibniz rule, as one can prove by regularisation since \( u(t) \in V = D(A) \) for \( t \) a.e.

The right-hand side above is in \( L_2(0,T;V^*) \), hence in \( L_1(0,T;V^*) \), so when the Fundamental Theorem for vector functions (cf. [28 Lem. III.1.1]) is applied and followed by commutation of \( e^{-(T-t)A} \) with the integral (via Bochner’s identity),
\[
e^{-(T-t)A} u(t) = e^{-TA} u_0 + \int_0^t e^{-(T-t)A} f(s) \, ds
\] (37)
\[
= e^{-(T-t)A} e^{-TA} u_0 + e^{-(T-t)A} \int_0^t e^{-(t-s)A} f(s) \, ds.
\]

Since \( e^{-(T-t)A} \) is injective, cf. Proposition 2 (35) now results at once.

As all terms in (35) are in \( C([0,T];H) \), we may safely evaluate at \( t = T \), which in view of (19) gives that \( u(T) = e^{-TA} u(0) + y_f \); this is the flow map \( u(0) \mapsto u(T) \). Owing to the invertibility of \( e^{-TA} \) once again, this flow is inverted by
\[
u(0) = e^{TA}(u(T) - y_f).
\] (38)

In other words, the solutions in \( X \) to \( u_t' + Au = f \) are for fixed \( f \) parametrised by the initial states \( u(0) \) in \( H \). Departing from this observation, one may give an intuitive

**Proof of Theorem 1.** When (15) has a solution \( u \in X \), then \( u(T) = u_T \) is reached from the initial state \( u(0) \) determined from the bijection in (38). This gives that \( u_T - y_f = e^{-TA} u(0) \in D(e^{TA}) \), so (20) is necessary.

In case \( u_T \), \( f \) do fulfill (30), then \( u_0 = e^{TA}(u_T - y_f) \) is a well-defined vector in \( H \), so Theorem 2 yields a function \( u \in X \) solving \( u_t' + Au = f \) and \( u(0) = u_0 \). By (38) this \( u \) has final state \( u(T) = e^{-TA} e^{TA}(u_T - y_f) + y_f = u_T \), hence solves (15).

In the affirmative case, one obtains (22) by insertion of formula (38) for \( u(0) \) into (35), that each term in (22) is a function belonging to \( X \) was seen in Theorem 3.
Uniqueness of \( u \) in \( X \) is obvious from the right-hand side of \((22)\). The solution can hence be estimated in \( X \) by insertion of \((33)\) into the inequality in Proposition\(^3\) which gives \( \| u \| _X^2 \leq \text{c} (|e^{\text{F}}(ur - yj)|^2 + \| f \|_{L^2(0,T;V')}^2) \). Here one may add \( |u|^2 \) on the right-hand side to arrive at the expression for \( \| (f,u) \|_Y \) given in Theorem 1.

**Remark 2.** The above arguments seem to extend to Lax–Milgram operators \( A \) that are only \( V \)-coercive, i.e. fulfil \( \| Au(u,u) \|_X \geq C \| u \|^2 - k \| u \|^2 \) for \( u \in V \). In fact, it was observed already in \((20)\) that Theorem\(^2\) holds verbatim for such \( A \), for since \( A + k \) is \( V \)-elliptic, the unique solvability in \( X \) of \( \nu' + (A + k)\nu = e^{-kt}f \), \( \nu(0) = u_0 \) yields a unique solution \( u = e^{kt}\nu \) of \( u' + Au = f \), \( u(0) = u_0 \). Since the same translation trick gave the improved version of Proposition\(^2\) also \( V \)-coercive \( A \) generate analytic semigroups of injections; so the proofs of the Duhamel formula and of Theorem\(^1\) seem applicable in their present form. However, the estimates in Proposition\(^4\) need to be modified using Grönwall’s lemma. The details of this are left for future work.

**Remark 3.** Recently Almog, Grebenkov, Helffer, Henry \((1,9,8)\) studied variants of the complex Airy operator via triples \((H,V,a)\), and Theorem\(^1\) is expected to extend to final value problems for those of their realisations that have non-empty spectrum.

### 3. THE HEAT PROBLEM WITH FINAL TIME DATA

To follow up on Theorem\(^1\) it is now applied to the heat equation and its final value problem. In the sequel \( \Omega \) stands for a \( C^\infty \) smooth, open bounded set in \( \mathbb{R}^n \), \( n \geq 2 \) as described in \((11)\) App. C. In particular \( \Omega \) is locally on one side of its boundary \( \Gamma = \partial \Omega \). For such sets we consider the problem of finding the \( u \) satisfying

\[
\begin{align*}
\partial_t u(t,x) - \Delta u(t,x) &= f(t,x) \quad \text{in } \Omega = ]0,T[ \times \Omega, \\
\gamma_0 u(t,x) &= g(t,x) \quad \text{on } S = ]0,T[ \times \Gamma, \\
r_T u(x) &= u_T(x) \quad \text{at } \{T\} \times \Omega.
\end{align*}
\]

(39)

Hereby the trace of functions on \( \Gamma \) is written in the operator notation \( \gamma_0 u = u|_\Gamma \). Similarly \( \gamma_1 \) is also used for traces on \( S \), while \( r_T \) denotes the trace at \( t = T \).

Moreover, \( H^1_0(\Omega) \) is the subspace obtained by closing \( C^\infty_0(\Omega) \) in the Sobolev space \( H^1(\Omega) \). Dual to this one has \( H^{-1}(\Omega) \), which identifies with the set of restrictions to \( \Omega \) from \( H^{-1}(\mathbb{R}^n) \), endowed with the infimum norm; Chapter 6 and Remark 9.4 in \((11)\) could be references for this and basic facts on boundary value problems.

#### 3.1. The Boundary Homogeneous Case

In case \( g \equiv 0 \) in \((35)\), the main result in Theorem\(^1\) applies for

\[
V = H^1_0(\Omega), \quad H = L_2(\Omega), \quad V^* = H^{-1}(\Omega).
\]

(40)

Indeed, the boundary condition \( \gamma_0 u = 0 \) is then imposed via the condition that \( u(t) \in V \) for all \( t \), or rather through use of the Dirichlet realization of the Laplacian \( -\Delta_\partial \) (denoted by \( -\Delta_\partial \) in the introduction), which is the Lax–Milgram operator \( A \) induced by the triple \((L_2(\Omega),H^1_0(\Omega),s)\) for

\[
s(u,v) = \sum_{j=1}^n (\partial_j u | \partial_j v)_{L_2(\Omega)}.
\]

(41)

In fact, Poincaré’s inequality yields that \( s(u,v) \) is \( H^1_0(\Omega) \)-elliptic and symmetric, so \( A = -\Delta_\partial \) is a selfadjoint unbounded operator in \( L_2(\Omega) \), with \( D(-\Delta_\partial) \subset H^1_0(\Omega) \).

Hence \( -\Delta = \Delta_\partial \) generates an analytic semigroup of operators \( e^{s_\partial} \) in \( B(L_2(\Omega)) \), and the bounded bijective extension \( \Delta : H^1_0(\Omega) \rightarrow H^{-1}(\Omega) \) induces the analytic semigroup \( e^{s_\partial} \) on \( V^* = H^{-1}(\Omega) \). As done previously, we set \( (e^{s_\partial})^{-1} = e^{-s_\partial} \).
Moreover, when $g = 0$ in (39), then the solution and data spaces amount to

$$X_0 = L_2(0, T; H^1_0(\Omega)) \bigcap C([0, T]; L_2(\Omega)) \bigcap H^1(0, T; H^{-1}(\Omega)), \quad (42)$$

$$Y_0 = \left\{ (f, u_T) \in L_2(0, T; H^{-1}(\Omega)) \oplus L_2(\Omega) \bigg| u_T - y_f \in D(e^{-T\Delta_0}) \right\}. \quad (43)$$

Here, with $y_f$ given in (18), the data norm from Theorem 1 specialises to

$$\| (f, u_T) \|^2_{Y_0} = \int_0^T \| f(t) \|^2_{H^{-1}(\Omega)} \, dt + \int_\Omega \| u_T \|^2 + | e^{-T\Delta_0}(u_T - y_f) |^2 \, dx. \quad (44)$$

Now Theorem 1 straightforwardly gives the following result, which first appeared in [5] even though the problem is entirely classical:

**Theorem 4.** Let $A = -\Delta_0$ be the Dirichlet realization of the Laplacian in $\Omega$ and $\Delta$ its extension, as introduced above. When $g = 0$ in the final value problem (39) and $f \in L_2(0, T; H^{-1}(\Omega))$, $u_T \in L_2(\Omega)$, then there exists a solution $u$ in $X_0$ of (39) if and only if the data $(f, u_T)$ are given in $Y_0$, i.e. if and only if

$$u_T - \int_0^T e^{(T-s)\Delta} f(s) \, ds \quad \text{belongs to} \quad D(e^{-T\Delta_0}). \quad (45)$$

In the affirmative case, $u$ is uniquely determined in $X_0$ and $\| u \|_{X_0} \leq c \| (f, u_T) \|_{Y_0}$. Furthermore the difference in (45) equals $e^{T\Delta_0} u(0)$ in $L_2(\Omega)$.

3.2. The Inhomogeneous Case. When $g \neq 0$ on the surface $\Sigma$, cf. (39), then the solution $u(t, \cdot)$ belongs to the full Sobolev space $H^1(\Omega)$ for each $t > 0$, so here the solution space is denoted by $X_1$,

$$X_1 = L_2(0, T; H^1(\Omega)) \bigcap C([0, T]; L_2(\Omega)) \bigcap H^1(0, T; H^{-1}(\Omega)). \quad (46)$$

Clearly $X_1$ is a Banach space when normed analogously to (29),

$$\| u \|_{X_1} = (\| u \|^2_{L_2(0, T; H^1(\Omega))} + \sup_{0 \leq t \leq T} \| u(t) \|^2_{L_2(\Omega)} + \| u \|^2_{H^1(0, T; H^{-1}(\Omega))})^{1/2}. \quad (47)$$

Here $H^1, H^{-1}$ are not dual on $\Omega$, so the previously mentioned redundancy does not extend to the term $\sup_{0 \leq t \leq T} \| u \|_{L_2}$ above.

For $g \neq 0$ the standard strategy is, of course, to try to reduce to a semi-homogeneous problem by choosing $w$ so that $\gamma_0 w = g$ on $\Sigma$, using the classical

**Lemma 1.** $\gamma_0 : H^1(\Omega) \rightarrow H^{1/2}(\Sigma)$ is a continuous surjection having a bounded right inverse $\tilde{K}_0$, that is, $\gamma_0 \tilde{K}_0 g = g$ for every $g \in H^{1/2}(\Sigma)$ and

$$\| w \|_{H^1(\Omega)} = \| \tilde{K}_0 g \|_{H^{1/2}(\Sigma)} \leq C \| g \|_{H^{1/2}(\Sigma)}. \quad (48)$$

In lack of a reference with details, the reader is referred to [5] for a sketch of how this lemma follows using standard techniques.

Another preparation is based on the fine theory of the elliptic problem

$$-\Delta u = f, \quad \gamma_0 u = g. \quad (49)$$

Indeed, it exploited below that $\mathcal{P} = I - \Delta_0^{-1} \Delta$ is a well-known projection in $H^1(\Omega)$, along $H^1_0(\Omega)$ and onto the closed subspace of harmonic $H^1$-functions,

$$Z(-\Delta) = \{ u \in H^1(\Omega) \mid -\Delta u = 0 \}. \quad (50)$$

To recall this, (49) may conveniently be treated via the matrix operator $(-\Delta_0^{-1})$, with an inverse in row form $(-\Delta_0^{-1}_0 \tilde{K}_0)$ that applies to the data $(\gamma_0 f)$, for then the basic composition
formulae appear in two identities on $H^1(\Omega)$ and $H^{-1}(\Omega) \oplus H^{1/2}(\Gamma)$,
\[
I = (-\Delta_{\Omega}^{-1} K_0) (-\Delta_{\Omega}^{-1} ) = \Delta_{\Omega}^{-1} \Delta + K_0 \gamma_0,
\]
(51)
\[
\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = (-\Delta_{\Gamma}^{-1} ) (-\Delta_{\Omega}^{-1} K_0) = (\Delta_{\Omega}^{-1} \gamma_0 \Delta_{\Omega}^{-1} \gamma_0 K_0).
\]
(52)
In particular the first formula yields that $\mathcal{Q} = I - \Delta_{\Omega}^{-1} \Delta = K_0 \gamma_0$ on $H^1(\Omega)$. However, it should be emphasized that the simplicity of the formulas (51) and (52) relies on a specific choice of $K_0$, which is recalled here:

As $\Delta_{\Omega} = \Delta_{H^0_0}$ holds in the distribution sense, $\mathcal{P} = \Delta_{\Omega}^{-1} \Delta$ clearly fulfills $\mathcal{P}^2 = \mathcal{P}$, it is bounded $H^1 \rightarrow H_0^1$ and equals $I$ on $H_0^1$, so $\mathcal{P}$ is the projection onto $H_0^1(\Omega)$ along its null space, which clearly is the space in (50). Therefore $H^1$ is a direct sum,
\[
H^1(\Omega) = H_0^1(\Omega) + Z(-\Delta),
\]
(53)
so that $\mathcal{Q} = I - \mathcal{P} = I - \Delta_{\Omega}^{-1} \Delta$ is the projection onto $Z(-\Delta)$ along $H_0^1(\Omega)$.

Since $\gamma_0 : H^1(\Omega) \rightarrow H^{1/2}(\Gamma)$ is surjective with null-space $H_0^1$, it has an inverse $K_0$ on the complement $Z(-\Delta)$, which by the open mapping principle is bounded $K_0 : H^{1/2}(\Gamma) \rightarrow Z(-\Delta) \rightarrow H^1(\Omega)$. Since in this construction $K_0 = (\gamma_0 | Z(-\Delta))^{-1}$, it is also a right-inverse, i.e. $\gamma_0 K_0 = I_{H^{1/2}(\Gamma)}$. The rest of (52) now follows at once.

Moreover, since $\gamma_0 \mathcal{P} = \gamma_0 \Delta_{\Omega}^{-1} \Delta = 0$, the definition of $\mathcal{Q}$ gives (51) thus:
\[
K_0 \gamma_0 = K_0 \gamma_0 (\mathcal{P} + \mathcal{Q}) = K_0 \gamma_0 \mathcal{Q} = I_{Z(-\Delta)} \mathcal{Q} = \mathcal{Q} = I - \Delta_{\Omega}^{-1} \Delta.
\]
(54)

**Remark 4.** $K_0$ is an example of a Poisson operator; these are amply discussed within the pseudo-differential boundary operator calculus, for example in [10].

**Remark 5.** The $H^r(S)$-norm can be chosen so that this is a Hilbert space; cf. the use of local coordinates in [11] (8.10). The vast subject of Sobolev spaces on $C^\infty$ surfaces generally requires distribution densities as explained in [13] Sect. 6.3] (cf. [11] Sect. 8.2) for a concise review. But the surface measure on $S$ induces a well-known identification of densities with distributions on $S$, and within this framework, a systematic exposition can be found in [18] Sect. 4, albeit in an $L_p$-setting with anisotropic mixed-norm Triebel–Lizorkin spaces $F^s_{p,q}(S)$, which are the correct boundary data spaces for parabolic problems having different integrability properties in $x$ and $t$. Cf. also the application to the heat problem [39] in [18] Sect. 6.5, and the more detailed discussion in [22] Ch. 7.

Now, when splitting the solution of (39) as $u = v + w$ for $w = \tilde{K}_0 g$, cf. Lemma [11] then $v$ should satisfy $v' - \Delta v = \tilde{f}$, $\gamma_0 v = 0$ and $rT v = \tilde{u}_T$ for data
\[
\tilde{f} = f - (\partial_t w - \Delta w), \quad \tilde{u}_T = u_T - rT w.
\]
(55)
For this problem to be solved by $v$, (45) stipulates that $D(e^{-T\Delta_0})$ should contain
\[
\tilde{u}_T - y_j = u_T - y_j - (rT w - y_j).
\]
(56)
But the presence of the term $rT w - y_j$ makes it impossible just to transfer condition (45) of being a member of $D(e^{-T\Delta_0})$ from $\tilde{u}_T - y_j$ to $u_T - y_j$. Thus the compatibility condition (45) destroys the trick of reducing to homogeneous boundary conditions, despite the linearity of the problem.

To find the correct compatibility conditions on $(f,g,u_T)$, the strategy in [5] was (with hindsight) to use Lemma [11] to get a solution formula for the corresponding linear initial value problem instead. This is motivated by the fact that, for the present space $X_1$ of low regularity, no compatibility condition is needed for this:
\[
\partial_t u - \Delta u = f \quad \text{in } Q, \quad \gamma_0 u = g \quad \text{on } S, \quad r_0 u = u_0 \quad \text{at } \{0\} \times \Omega.
\]
(57)
Proof. Here the last term is only integrable on $\Omega$, cf. Proposition 1. As a remedy, one can use the improper Bochner integral

$$
\int_{0}^{T} e^{{(t-s)}\Delta} \hat{f}(\partial_{t} - \Delta) w(s) ds
$$

while this is well known per se, an explanation is given to account below for the crucial existence of an improper integral showing up when $g \neq 0$ in (59).

**Proposition 5.** The heat initial value problem (57) has a unique solution $u \in X_1$ for given $f \in L_2(0,T;H^{-1}(\Omega))$, $g \in H^{1/2}(\Omega)$, $u_0 \in L_2(\Omega)$, and there is an estimate

$$
\|u\|_{X_1}^2 \leq c(\|u_0\|_{L_2(\Omega)}^2 + \|f\|_{L_2(0,T;H^{-1}(\Omega))}^2 + \|g\|_{H^{1/2}(\Omega)}^2).
$$

**Proof.** Let $I = [0,T]$. Setting $w = \tilde{K}_0 g$ by means of Lemma 1 we tentatively write $u = v + w$ for some $v \in X_0$ solving (57) for data

$$
\hat{f} = f - (\partial_t - \Delta)w, \quad \tilde{g} = 0, \quad \tilde{u}_0 = u_0 - w(0).
$$

Here $w(0)$ is well defined, as $w \in H^1(\Omega) \subset H^1([0,T];L_2(\Omega)) \subset C([0,T];L_2(\Omega))$ by a Sobolev embedding, which (e.g. as in [5, Rem. 4]) also gives the estimate

$$
\sup_{0 \leq t \leq T} \|w(t)\|_{L_2(\Omega)} \leq c(\|w\|_{L_2(L_2(\Omega))} + \|\partial_t w\|_{L_2(L_2(\Omega))}) \leq c\|w\|_{H^1(\Omega)}^2.
$$

To show that $w \in X_1$ and $\|w\|_{X_1}^2 \leq c\|w\|_{H^1(\Omega)}^2$, it is noted that estimates of the two remaining terms in $\|w\|_{X_1}^2$, cf. (47), can be read off from the obvious inequalities

$$
\|w\|_{L_2(L_2(\Omega))}^2 + \|\partial_t w\|_{L_2(L_2(\Omega))}^2 \leq c\|w\|_{H^1(\Omega)}^2 + c'\|w\|_{H^1(\Omega)}^2 \leq c\|w\|_{H^1(\Omega)}^2.
$$

This also entails $\hat{f} \in L_2(0,T;H^{-1}(\Omega))$, and $\tilde{u}_0 \in L_2(\Omega)$, so by Theorem 2 the boundary homogeneous problem for $v$ has a solution in $X_0$; cf. (42). Hence (57) has the solution $u = v + w \in X_1$; its uniqueness is easily carried over from Theorem 2.

Finally the estimate in Theorem 3 (that is a consequence of Proposition 4) gives

$$
\|u\|_{X_1}^2 \leq 2(\|\tilde{u}_0\|_{L_2(\Omega)}^2 + \|w\|_{X_1}^2) \leq c(\|\tilde{u}_0\|_{L_2(\Omega)}^2 + \|\hat{f}\|_{L_2(L_2(\Omega))}^2 + \|w\|_{X_1}^2),
$$

which via (61) and (48) entails the stated estimate (58). □

Obviously the formula in Theorem 3 now applies directly to the function $v = u - w$ in the above proof, which as a crucial addendum to Proposition 5 yields

$$
u(t) - w(t) = e^{\Delta t_0} (u_0 - w(0)) + \int_{0}^{t} e^{(t-s)\Delta} (f - (\partial_t - \Delta)w) ds.
$$

The next step is to rewrite the contributions from $w$ so that $g = \gamma_0 w$ can be reintroduced. First a regularisation of $w$ in $H^1(0,t;L_2(\Omega))$ leads to the Leibniz rule

$$
\partial_s (e^{(t-s)\Delta} \gamma_0 w(s)) = e^{(t-s)\Delta} \gamma_0 \partial_t w(s) - \Delta \gamma_0 e^{(t-s)\Delta} \gamma_0 w(s).
$$

Here the last term is only integrable on $[0,t-e]$ for $e > 0$, as it has a singularity at $s = t$; cf. Proposition 1. As a remedy, one can use the improper Bochner integral

$$
\int_{0}^{t} \Delta \gamma_0 e^{(t-s)\Delta} \gamma_0 w(s) ds = \lim_{e \to 0} \int_{0}^{t-e} \Delta \gamma_0 e^{(t-s)\Delta} \gamma_0 w(s) ds.
$$

**Lemma 2.** For every $w \in H^1(\Omega)$ the limit (65) exists in $L_2(\Omega)$ and

$$

w(t) = e^{\Delta t_0} w(0) + \int_{0}^{t} e^{(t-s)\Delta} \gamma_0 \partial_t w(s) ds - \int_{0}^{t} \Delta \gamma_0 e^{(t-s)\Delta} \gamma_0 w(s) ds.
$$

(For general background material on $\tilde{K}_0$ the reader could consult Section III.6 in [3]; and [12] for the fine theory including compatibility conditions.)

Similarly to Theorem 2 and Proposition 4, one may depart from well-posedness of (57).
Theorem 5. For given data \((f, g, u_T)\) in \(L_2(0,T;H^{-1}(\Omega)) \oplus H^{1/2}(S) \oplus L_2(\Omega)\), the final value problem (39) is solved by a function \(u \in X_1\), whereby

\[
X_1 = L_2(0,T;H^1(\Omega)) \bigcap C([0,T];L_2(\Omega)) \bigcap H^1(0,T;H^{-1}(\Omega)),
\]
if and only if the data in terms of (153) and (71) satisfy the compatibility condition
\[ u_T - y_f + z_g \in D(e^{-T \Delta_0}). \] (74)
In the affirmative case, \( u \) is uniquely determined in \( X_1 \) and has the representation
\[ u(t) = e^{\Delta_0} e^{-T \Delta_0} (u_T - y_f + z_g) + \int_0^t e^{(t-s)\Delta} f \, ds - \int_0^t \Delta_0 e^{(t-s)\Delta} K_0 \, ds, \] (75)
where the three terms all belong to \( X_1 \) as functions of \( t \in [0, T] \).

**Proof.** Given a solution \( u \in X_1 \), the bijection (2) yields \( u_T = e^{T \Delta_0} u(0) + y_f - z_g \), so that (74) necessarily holds. Inserting its inversion \( u(0) = e^{-T \Delta_0} (u_T - y_f + z_g) \) into the solution formula from Proposition 6 yields (75); thence uniqueness of \( u \).

If (74) does hold, \( u_0 = e^{-T \Delta_0} (u_T - y_f + z_g) \) is a vector in \( L_2(\Omega) \), so the initial value problem with data \((f, g, u_0)\) can be solved using Proposition 5. This yields a \( u \in X_1 \) that also solves (153), since \( u(T) = u_T \) holds by (2) and the choice of \( u_0 \).

The final regularity statement follows from the fact that \( X_1 \) also is the solution space for the Cauchy problem in Proposition 5, the improper integral in (75) is a solution in \( X_1 \) to (57) for data \((f, g, u_0) = (0, g, 0)\), according to Proposition 6; the integral containing \( f \) solves (57) for data \((f, 0, 0)\), hence is in \( X_1 \); the first term in (75) solves (57) for data \((0, 0, e^{-T \Delta_0} v)\) for the vector \( v = u_T - y_f + z_g \).

Exploiting the above theorem, we let \( Y_1 \) stand for the set of admissible data corresponding to \( X_1 \). Within the broader space \( L_2(0, T; H^{-1}(\Omega)) \oplus H^{1/2}(\Gamma) \oplus L_2(\Omega) \), the data space \( Y_1 \) is the subspace given via the map \( \Phi_1(f, g, u_T) = u_T - y_f + z_g \) as
\[ Y_1 = \{(f, g, u_T) \mid u_T - y_f + z_g \in D(e^{-T \Delta_0})\}, \] (76)
where \( Y_1 \) is endowed with the graph norm of \( e^{-T \Delta_0} \Phi_1 \), that is, of the composite map \( (f, g, u_T) \mapsto e^{-T \Delta_0} (u_T - y_f + z_g) \). Taking \( t = 0 \) in (75), it is clear that \( e^{-T \Delta_0} \Phi_1(f, g, u_T) \) is the initial state \( u(0) \) steered by \( f, g \) to the final one \( u(T) = u_T \).

In some details, the above-mentioned graph norm is given by
\[ \| (f, g, u_T) \|_{Y_1}^2 = \| u_T \|_{L_2(\Omega)}^2 + \| f \|^2_{H^{1/2}(\Gamma)} + \| f \|_{L_2(0, T; H^{-1}(\Omega))}^2 \]
\[ + \int_0^T \| e^{-T \Delta_0} \left( u_T - \int_0^T e^{(T-s)\Delta} f \, ds + \int_0^T \Delta_0 e^{(T-s)\Delta} K_0 g \, ds \right) \|^2 \, dx. \] (77)
Hereby \( \| e^{-T \Delta_0} (u_T - y_f + z_g) \|_{L_2(\Omega)}^2 \) is written with explicit integrals to emphasize the complexity of the fully inhomogeneous boundary and final value problem (153).

Completeness of \( Y_1 \) follows from continuity of \( \Phi_1 \), cf. Lemma 5 concerning \( z_g \). Indeed, its composition to the left with the closed operator \( e^{-T \Delta_0} \) in \( L_2(\Omega) \) (cf. Proposition 5) is also closed. Therefore its domain \( D(e^{-T \Delta_0} \Phi_1) = Y_1 \) is complete with respect to the graph norm in (77). This is induced by an inner product when \( H^{-1}(\Omega) \) is given the equivalent norm \( \| f \|_1 = s(\Delta_0^{-1} f, \Delta_0^{-1} f)^{1/2} \), and when \( H^{1/2}(\Gamma) \) is normed as in Remark 5. Hence \( Y_1 \) is a Hilbert space.

These preparations concerning the data space \( Y_1 \) lead to the following stability result:

**Theorem 6.** The unique solution \( u \) of problem (153) lying in the Banach space \( X_1 \) depends continuously on the data \((f, g, u_T)\) in the Hilbert space \( Y_1 \), when these are given the norms in (77) and (75), respectively.

**Proof.** Boundedness of the solution operator \((f, g, u_T) \mapsto u\) is seen by inserting the expression \( u_0 = e^{-T \Delta_0} (u_T - y_f + z_g) \) from (2) into the estimate in Proposition 5
\[ \| u \|^2_{X_1} \leq c\left( \| e^{-T \Delta_0} (u_T - y_f + z_g) \|_{L_2(\Omega)}^2 + \| f \|_{L_2(0, T; H^{-1}(\Omega))}^2 + \| g \|_{H^{1/2}(\Gamma)}^2 \right). \] (78)
Adding \( \| u_T \|^2_{L_2(\Omega)} \) on the right-hand side, one arrives at \( \| (f, g, u_T) \|_{Y_1}^2 \). \( \square \)
Taken together, Theorem 5 and Theorem 6 yield the final result:

**Theorem 7.** When $\Omega$ is a $C^\omega$ smooth, open bounded set in $\mathbb{R}^n$ for $n \geq 2$, and $T > 0$, then the fully inhomogeneous final value heat conduction problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= f \quad \text{in } [0,T] \times \Omega, \\
\gamma_0 u &= g \quad \text{on } [0,T] \times \Gamma, \\
u(T) &= u_T \quad \text{in } \Omega,
\end{aligned}
$$

(79)

is well posed with solutions $u$ and data $(f,g,u_T)$ belonging to the spaces $X_1$ and $Y_1$ defined in (45), (76), and normed as in (47), (77), respectively.

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