Genericness of Big Bounce in isotropic loop quantum gravity cosmology

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The absence of isotropic singularity in loop quantum cosmology can be understood in an effective classical description as the universe exhibiting a Big Bounce. We show that with scalar matter field, the big bounce is generic in the sense that it is independent of quantization ambiguities and details of scalar field dynamics. The volume of the universe at the bounce point is parametrized by a single parameter. It provides a minimum length scale which serves as a cut-off for computations of density perturbations thereby influencing their amplitudes.

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It has been long expected that the existence of singularity in the classical general relativity which has been shown to be quite generic thanks to the singularity theorems, will be removed when classical framework of gravity is extended to a quantum framework of gravity. Despite of tremendous efforts made, unfortunately we still do not have a fully satisfactory theory of quantum gravity. In the last couple of decades two strong contenders have emerged: the string theory approach \[1\] and the loop quantum gravity (LQG) approach \[2\]. The issue of fate of classical cosmological singularities has been addressed head-on within the LQG approach. More precisely, the adaptation of LQG methods to cosmological context, known as loop quantum cosmology (LQC) \[3\], has made impressive progress on the issue of singularities. It has been shown that the isotropic models (flat and closed) \[4\], and more generally the diagonal Bianchi class A models \[5\], \[6\], are free of singularity. It is time now to explore further consequences of LQC corrected cosmologies.

Some of consequences of LQC corrected cosmologies have already been noted. First, there is a natural mechanism for inflation \[7\] \[8\] \[9\] within the context of isotropic cosmologies. Secondly, for the Bianchi IX model there is a suppression of chaotic approach to singularity \[10\]. Thirdly, there is indication of a bounce at the big crunch singularity as well \[11\] \[12\]. All of these have been explored within the framework of an effective Hamiltonian which incorporates the most significant non-perturbative corrections. These modifications stem from the non-trivial definition of the inverse triad operator in LQC \[13\] \[15\] which ensure that the matter density, spin connection components remain bounded as universe approaches zero volume.

The effective Hamiltonian is derived from the admisibility of a continuum approximation \[14\]. This comes about as a requirement on the LQC quantization of the Hamiltonian constraint operator which always leads to a difference equation for the quantum wave function. For large volumes, these wave functions are expected to vary slowly since quantum effects are small. This feature allows one to use an interpolating wave function of a continuous variable which satisfies a differential equation, the ‘Wheeler–DeWitt equation’. Making a further WKB approximation leads one to a Hamilton-Jacobi equation from which the effective Hamiltonian is read-off. This is extrapolated to smaller (but not too small) volumes by replacing the occurrence of inverse triad/volume factors by a function coming from the definition of the inverse triad operator. The validity of the effective Hamiltonian is limited by the validity of WKB approximation and the validity of the continuum approximation.

Recently, we have extended the domain of validity of the continuum approximation by exploiting the non-separable structure of the Kinematical Hilbert space of LQC \[12\] which has infinitely many solutions of the fundamental difference equation. Although each of these may not be slowly varying at smaller volumes, one can choose linear combinations to construct solutions which are slowly varying almost every where. This amounts to an ad-hoc restriction to a sub-class of solution. In the absence of any other criteria to limit the infinity of solutions, such as a physical inner product, this restriction is treated as exploratory. The extraction of effective Hamiltonian then follows the same method as before via a WKB approximation. The validity of the effective Hamiltonian is now limited only by the validity of WKB approximation i.e. to ‘classically accessible regions’. The effective Hamiltonian is derived in \[16\] and is given by,

\[
H_{\text{eff}} = -\frac{1}{\kappa} \left[ \frac{B_+(p)}{4p_0} K^2 + \eta \frac{A(p)}{2p_0} \right] + W_{\text{gg}} + H_m
\]  

where \( \kappa = 16\pi G, p_0 = \frac{\ell}{\kappa} \), \( K\) is the extrinsic curvature (conjugate variable of \( p\)), \( A(p) = |p + p_0|^{\frac{3}{2}} - |p - p_0|^{\frac{3}{2}} \), \( \eta \) takes values 0, 1 for spatially flat, closed models respectively, \( B_+(p) = A(p + 4p_0) + A(p - 4p_0) \), \( \ell_{\text{P}} := \kappa h \) and \( W_{\text{gg}} = \left( \frac{\ell}{\kappa} \right) \left( B_+(p) - 2A(p) \right) \). Apart from the modification of the coefficients of the gravitational kinetic term and the spatial curvature term, the effective Hamiltonian \[14\] differs from the classical Hamiltonian by a non-trivial potential term referred to as quantum geometry potential and denoted as \( W_{\text{gg}} \). It is odd under the reversal of orientation of the triad \( (p \rightarrow -p) \) and for
Given the behaviors of the quantum geometry potential and the matter Hamiltonian during a (super) inflationary region and their opposite signs, it is clear that the extrinsic curvature i.e. existence of classically inaccessible scale factors. The two necessarily cancel each other at a finite, non-zero value of the scale factor. This would be so even after including the contribution of the spatial curvature \( (q) \). But this means that the extrinsic curvature vanishes at that value of the scale factor implying a bounce. Thus we see that a bounce is quite generic and the minimum scale factor defines a new length scale \( L_{\text{bounce}} \). Below this scale, the effective classical picture fails. A graphical illustration of the existence of bounce can be seen in the figure 1.

The bounce scale is obviously determined by the conditions \( H^\text{eff} = 0 = K \) with \( H_m = h p^{3/2} \), \( h \) being a constant of proportionality. This is a transcendental equation in \( p \) and the root(s) depend on the constant \( h \). We expect the bounce value to be less than \( 2 j p_0 \) (above which we are in the classical regime). In geometrized units, \( \kappa = 1 = c \), we will refer to all lengths to the scale \( \sqrt{q_0} \). Thus, putting \( p := q p_0 \) the region of interest is \( 0 \ll q \ll 2 j \). This could be further divided into (i) \( 0 \ll q \ll 1 \) and (ii) \( 1 \ll q \ll 2 j \). Since the equation of state variable is a dimensionless function of the scale factor, it is in fact a function of \( q \). The conservation equation can then be solved as: \( \rho^\text{eff}(q) = \bar{\rho} \exp\left(-\frac{1}{2} \int_{2j}^{q}(1 + \omega^\text{eff}(q)) q_0 \right) \). The constant \( \bar{\rho} \) is proportional to \( \bar{\rho} = p^\text{eff}(2j) \). In summary, the equations determining the (non-zero) bounce scale, \( q_{\text{bounce}} \), are,

\[
H_m(q) = 6 \frac{p_0^3}{B_+} q^2 \bar{\rho} \exp\left(-\frac{1}{2} \int_{2j}^{q}(1 + \omega^\text{eff}(q)) \frac{q_0}{q} \right)
\]

\[
\bar{\rho} = \frac{A(q)}{q_0^3/288} \left( B_+(q) - 2 A(q) \right)
\]

In the two regions (i) and (ii), the equations simplify. In the region (i) one has, \( A \to 3 p_0^{3/2} q, B_+ \to 3 (\sqrt{5} - \sqrt{3}) p_0^{3/2} q, \omega^\text{eff}(q) \to -\frac{1}{2} \) and we get,

\[
H_m(q) \to \left[ \frac{\bar{\rho}}{\sqrt{5} - \sqrt{3}} \right] (p_0 \bar{q})^{3/2}
\]

\[
\sqrt{q_{\text{bounce}}} = \frac{1}{2 p_0 h} \left[ 3 q + \frac{(2 - \sqrt{5} + \sqrt{3}) \ell_0^3}{48} \right] .
\]

In region (ii) one has \( A \to 3 p_0^{3/2} (3 q^{1/2} - \frac{1}{8} q^{-3/2}), B_+ \to 3 p_0^{3/2} (6 q^{1/2} - 49 q^{-3/2}), \omega^\text{eff}(q) \to -1 \) and we get,

\[
H_m(q) \to \bar{\rho} (p_0 q) - (p_0 \rho) \left( \frac{3}{2} \right),
\]

and the \( q_{\text{bounce}} \) is determined as a root of the cubic equation,

\[
(2 p_0 \bar{\rho}) q^3 - 3 p_0 q^2 - \frac{1}{12} \left( \frac{\ell_0^3}{p_0^3} \right) = 0
\]
Note that $p_0\bar{\rho}$ is dimensionless. It is easy to see that there is exactly one real root of this equation and in fact, get a close form expression for it.

In the region (i), $h \sim \bar{\rho}/\sqrt{j}$ and the $q_{\text{bounce}} \sim (p_0\bar{\rho})^{-2}$. The solution has explicit $j$ dependence. The inequality for region (i), $q_{\text{bounce}} \ll 1$, implies that the effective density at $p = 2jp_0$ (roughly where the inverse scale factor function attains its maximum) must be larger than $\sqrt{j}$ times the Planck density $\sim \ell_p^{-2}$. Also the bounce scale will be smaller than the Planck scale. This is indicative of the effective continuum model becoming a poor approximation.

In region (ii), $h = \bar{\rho}$. For the flat model $(\eta = 0)$, $q_{\text{bounce}} \sim (p_0\bar{\rho})^{-1/3}$. For the close model, such a simple dependence does not occur. The inequalities, $1 \ll q \ll 2j$, translate into a window for $p_0\bar{\rho}$. Region (ii) bounce scale has no explicit dependence on the ambiguity parameter $j$, the implicit dependence being subsumed in the value of $\bar{\rho}$ which can be treated as a free parameter. The bounce scale is larger than the Planck scale and the density $\bar{\rho}$ is smaller than the Planck density. The relation between $h$ and the bounce scale is displayed in figure 2.

![FIG. 2: The plot shows how the dimensionless parameter $h p_0$ varies as a function of the bounce scale $q_{\text{bounce}}$ as determined by the equations (1 3). $p_0 = \frac{1}{\sqrt{3}}\ell_p^2$ has been chosen for the plot.](image)

Clearly, as $h \to 0$, the scale $p_0 \to 0$ and so does the bounce scale. It also vanishes as $\bar{\rho} \to \infty$. However in the non-singular evolution implied by isotropic LQC (inverse scale factor having a bounded spectrum) implies that there is a maximum energy density attainable and the density $\bar{\rho}$ can be thought of as this maximum energy density. Correspondingly there is a minimum scale factor or minimum proper volume since in LQC the fiducial coordinate volume is absorbed in the definition of the triad. For volumes smaller than the minimum volume, the WKB approximation fails and the effective classical picture cannot be trusted. The quantum geometry potential plays a crucial role in this result.

A remark about the physical justification for the approximations used is in order. The results use the effective Hamiltonian picture which is based on a continuum approximation followed by the WKB approximation. The physical justification thus hinges on the physical justification for these two approximations. The continuum approximation for the geometry is physically expected to be a good approximation for length scales larger than the discreteness scale set by $4p_0 \simeq 4\mu_0$, the step size in the fundamental difference equation. The WKB approximation is valid in a sub-domain of the continuum approximation, determined by slow variation of amplitude and phase. Mathematically, the amplitude variation begins to get stronger around $2p_0$ while the phase variation is stronger at the turning point which determines the bounce scale. Thus, physically, the effective Hamiltonian (including the quantum geometry potential) is trustworthy for the bounce scale larger than $\sim p_0$. As shown by the figure 2, there is a range of $\bar{\rho} < l_p^{-2}$ such that the bounce scale is consistent with the physical domain of validity of the effective Hamiltonian. Note also that the behavior of the matter Hamiltonian as $\bar{\rho} \to 0$ is dependent on $p \ll 2jp_0$ and hence the bounce scale is also smaller than $2jp_0$.

We will discuss now a possible implication of the minimum proper length on the inflationary cosmology. The standard inflationary scenario is considered a successful paradigm not only because it can effectively solve the traditional problems of standard classical cosmology, but also because it provides a natural mechanism of generating classical seed perturbations from quantum fluctuations. These seed perturbations are essential in a theory of large scale structure formation but there is no mechanism of generating the initial perturbation within the classical setup. A quantum field living on an inflating background quite generically produces scale-invariant power spectrum of primordial density perturbations which is consistent with the current observations.

However, one major problem that plagues almost all potential driven inflationary models is that these models generically predict too much amplitude for density perturbation. Considering the fluctuations of quantum scalar field on an inflating classical background, one can show that these models naturally predict density perturbations at horizon re-entry to be $\frac{\delta \rho}{\rho} \sim 1 - 10^3$. But CMB anisotropy measurements indicates $\frac{\delta \rho}{\rho} \sim 10^{-5}$. Thus it is very difficult to get desired amplitude for density perturbation from the standard inflationary scenarios unless one introduces some fine tuning in inflaton potential.

An interesting suggestion to get an acceptable amount of density perturbation from inflationary scenario was made by Padmanabhan. The basic idea of the
suggestion is that any proper theory of quantum gravity should incorporate a zero-point proper length. This in turn damps the propagation of modes with proper wavelength smaller than the zero-point proper length. This mechanism reduces the amplitude of density perturbation by an exponential damping factor. The computations of [2] show that with the energy density \( \rho_0 \) attainable during inflation to be order of the Planck energy density and the introduced cutoff \( (L) \) of the order of \( (G\hbar/c^3)^{1/2} \), one can indeed get the necessary amount of damping. In the picture discussed above, we already have a correlation between \( \rho \) and the bounce scale.

As discussed above, the effective model derived from semi-classical LQC already shows the existence of a minimum proper length which can play the role of the zero-point length. Furthermore, this scale is not put in by hand but arises generically and naturally from the non-singularity of the effective model [1] and is correlated self-consistently with the maximum attainable energy density whose existence is also guaranteed in a non-singular evolution. It has no explicit dependence on the quantization ambiguity parameter \( j \). With the genericness of (exponential) inflation shown in [4], one can expect that the effective LQC model has the potential to produce an acceptable primordial power spectrum as well as an acceptable amplitude for density perturbation. A detailed analysis of primordial density fluctuations incorporating semi-classical LQC modifications is being carried out and will be reported elsewhere [23].

Apart from the possible phenomenological implications of the existence of a bounce, there are some theoretical implications as well. Within the WKB approximation used in deriving the effective Hamiltonian, existence of bounce corresponds to existence of classically in-accessible regions (volumes). This can also be interpreted as limiting the domain of validity of continuum geometry or the kinematical framework of general relativity. Since the exact quantum wave functions do connect the two regions of the triad variable, there is also the possibility of tunneling to and from the oppositely oriented universe \( (p < 0) \) through these regions. Because of this, the bounce can be expected to be ‘fuzzy’. If and how the tunneling possibility between oppositely oriented universe affects ‘discrete symmetries’ needs to be explored.

Finally, the bounce result has been derived using genericness of inflationary regime \( (p \ll 2jp_0) \). It is reasonable to assume that the maximum energy density would be comparable or less than the Planck density. In such a case the bounce scale will be greater than \( p_0 \). Thus, both the results regarding genericness of bounce and genericness of inflation would follow even if the underlying assumption of slowly varying wave functions is valid only down to the bounce scale.

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