We continue studying the connection between Jacobi matrices defined on a tree and multiple orthogonal polynomials (MOPs) which was discovered in [7]. In this paper, we consider Angelesco systems formed by two analytic weights and obtain asymptotics of the recurrence coefficients and strong asymptotics of MOPs along all directions (including the marginal ones). These results then are applied to show that the essential spectrum of the related Jacobi matrix is the union of intervals of orthogonality.

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1. Introduction

It is well-known [2] that the spectral theory of one-sided self-adjoint Jacobi matrices can be naturally studied in the context of polynomials orthogonal on the real line and, conversely, many results in the latter topic find operator-theoretic interpretation. In [7], we discovered that a wide class of multiple orthogonal polynomials, e.g., celebrated Angelesco systems, is connected to self-adjoint Jacobi matrices defined on rooted Cayley trees. The present paper makes further step in this direction. We perform a case study of Angelesco systems with two measures of orthogonality given by analytic weights. Our analysis of the related matrix Riemann-Hilbert problem provides the asymptotics of the recurrence coefficients and strong asymptotics of MOPs for all large indices. One application of this precise asymptotic analysis is a characterization of the essential spectrum of the associated Jacobi matrix.

We start this introduction by recalling some definitions and main relations connecting Jacobi matrices on trees and MOPs and then state the main results of the paper. In what follows, we let \( A = \sigma(A) \cup \sigma_{\text{ess}}(A) \) and let asymptotic analysis be a characterization of the essential spectrum of the associated Jacobi matrix. We denote by \( \sigma(A) \) and \( \sigma_{\text{ess}}(A) \) the spectrum and essential spectrum, respectively [33]. In a metric space, \( B_r(X) \) denotes the closed ball with center at \( X \) and radius \( r \). For a complex number \( z \), \( \mathbb{H}z \) and \( 3z \) are its real and imaginary parts, respectively. For a function \( f(z) \), holomorphic in \( \mathbb{C}^+ \), the upper half-plane, its boundary values on \( \mathbb{R} \) are denoted by \( f(x) \).

1.1. Jacobi matrices on trees. Denote by \( \mathcal{T} \) an infinite \((d+1)\)-homogeneous rooted tree (rooted Cayley tree) and by \( \mathcal{V} \) the set of its vertices with \( O \) being the root. On the lattice \( \mathbb{N}^d \), consider an infinite path \((\vec{n}^{(1)}, \vec{n}^{(2)}, \ldots)\) that starts at \( \vec{1} \) (i.e., \( \vec{n}^{(1)} = \vec{1} \)) and satisfies \( \vec{n}^{(j+1)} = \vec{n}^{(j)} + \vec{e}_{k_j}, k_j \in \{1, \ldots, d\} \) for every \( j = 0, 1, \ldots \). Clearly, these are paths for which, as we move from \( \vec{1} \) to infinity, the multi-index of each next vertex is increasing by 1 at exactly one position. Each such path can be mapped to non-selfintersecting path in \( \mathcal{T} \) that starts at \( O \) (see Figure 1 for \( d = 2 \)) and this map is one-to-one. This construction defines a projection \( \Pi : \mathcal{V} \rightarrow \mathbb{N}^d \) as follows: given \( X \in \mathcal{V} \) we consider a path from \( O \) to \( X \), map it to a path on \( \mathbb{N}^d \) and let \( \Pi(X) \) be the endpoint of the mapped path. Every vertex \( Y \in \mathcal{V}, Y \neq O \), has the unique parent, which we denote by \( Y(p) \). This allows us to define the following index function:

\[
i : \mathcal{V} \rightarrow \{1, \ldots, d\}, \quad Y \mapsto i_Y \text{ such that } \Pi(Y) = \Pi(Y(p)) + \vec{e}_{i_Y},
\]

and therefore to distinguish the “children” of each vertex \( Y \in \mathcal{V} \) by denoting \( Z = Y_{(ch),i}x \) when \( Y = Z(p) \), see Figure 1 (for \( d = 2 \)).

![Figure 1. Three generations of \( \mathcal{T} \) (for \( d = 2 \)).](image)

Let \( P := \{a_{\vec{n},i}, b_{\vec{n},i}\}_{\vec{n} \in \mathbb{N}^d, i \in \{1, \ldots, d\}} \) be a collection of real parameters satisfying conditions

\[
\begin{align*}
0 &< a_{\vec{n},i} \quad \text{for all } \vec{n} \in \mathbb{N}^d, \quad i \in \{1, \ldots, d\}, \\
\sup_{\vec{n} \in \mathbb{N}^d, i \in \{1, \ldots, d\}} a_{\vec{n},i} &< \infty, \quad \sup_{\vec{n} \in \mathbb{N}^d, i \in \{1, \ldots, d\}} |b_{\vec{n},i}| < \infty.
\end{align*}
\]
For a function $f$ on $\mathcal{V}$, we denote by $f_Y$ its value at a vertex $Y \in \mathcal{V}$. Given $P$ satisfying (1.2) and $\vec{\kappa} \in \mathbb{R}^d$ with $|\vec{\kappa}| = 1$, we define the corresponding Jacobi operator $J_{\vec{\kappa}}$ by

$$
\begin{align*}
(\mathcal{J}_{\vec{\kappa}} f)_Y & := a_{\vec{\kappa}} f_Y + b_{\vec{\kappa}} f_Y + \sum_{i=1}^d a_{\vec{\kappa},i} f_{Y_i}, & Y \neq O, \\
(\mathcal{J}_{\vec{\kappa}} f)_O & := \sum_{i=1}^d a_{\vec{\kappa},i} f_{O_i} + \sum_{i=1}^d a_{\vec{\kappa},i} f_{O_i}, & Y = O.
\end{align*}
$$

Thus defined operator $J_{\vec{\kappa}}$ is bounded and self-adjoint on $\ell^2(\mathcal{V})$.

The spectral theory of Jacobi matrices on trees enjoyed considerable progress in the last decade, see, e.g., [1, 10, 19, 21, 26, 27]. In this paper, we will study Jacobi matrices on trees that are generated by multiple orthogonality conditions.

1.2. Multiple orthogonal polynomials and recurrence relations. In [7], we investigated properties of the operator $J_{\vec{\kappa}}$ in the case when the coefficients $P$ are the recurrence coefficients for MOPs. We now recall some basic facts about multiple orthogonal polynomials.

Let $\vec{\mu} := (\mu_1, \ldots, \mu_d)$, $d \in \mathbb{N}$, be a vector of positive finite Borel measures defined on $\mathbb{R}$ and $\vec{n}$ be a given a multi-index in $\mathbb{Z}_+^d \setminus \{0\}$, $|\vec{n}| \geq 1$. Type I MOPs $\{A^{(j)}_{\vec{n}} \}_{j=1}^d$ are not identically zero polynomial coefficients of the linear form

$$
Q_{\vec{n}}(x) := \sum_{j=1}^{d} A^{(j)}_{\vec{n}}(x) dx_j, \quad \text{deg} A^{(i)}_{\vec{n}} < n_i, \quad i \in \{1, \ldots, d\},
$$

defined by the conditions

$$
\text{type I} \quad \int x^l Q_{\vec{n}}(x) = 0, \quad l < |\vec{n}| - 1, \quad A^{(i)}_{\vec{n}, -d_i} = 0.
$$

Type II MOPs $P_{\vec{n}}(x)$, deg $P_{\vec{n}}(x) \leq |\vec{n}|$, are not identically zero and defined by

$$
\text{type II} \quad \int P_{\vec{n}}(x) x^l dx_j = 0, \quad l < n_i, \quad i \in \{1, \ldots, d\}.
$$

The polynomials of both types always exist, but their uniqueness is not guaranteed. If deg $P_{\vec{n}}(x) = |\vec{n}|$ for every non-identically zero polynomial $P_{\vec{n}}(x)$ satisfying (1.5), then the multi-index $\vec{n}$ is called normal. In this case $P_{\vec{n}}(x)$ is unique up to a multiplicative factor and we normalize it to be monic, i.e., $P_{\vec{n}}(x) = x^{|\vec{n}|} + \cdots$. It turns out that $\vec{n}$ is normal if and only if the linear form $Q_{\vec{n}}(x)$ is defined uniquely up to multiplication by a constant. In this case, we will normalize it by

$$
\text{n. 2} \quad \int x^{|\vec{n}| - 1} Q_{\vec{n}}(x) = 1.
$$

We will say that vector $\vec{n}$ is called perfect if all the multi-indices $\vec{n} \in \mathbb{Z}_+^d$ are normal.

When $\vec{n}$ is perfect, it is known [35] that the polynomials $P_{\vec{n}}(x)$ and the forms $Q_{\vec{n}}(x)$ satisfy the following nearest-neighbor recurrence relations (NNRRs):

$$
\begin{align*}
& zP_{\vec{n}+e_i}(z) = P_{\vec{n}_i+e_i}(z) + b_{\vec{n},i} P_{\vec{n}-e_i}(z) + \sum_{j=1}^{d} a_{\vec{n},i,j} P_{\vec{n}-e_j}(z), \quad \text{for each } j \in \{1, \ldots, d\}, \\
& zQ_{\vec{n}}(z) = Q_{\vec{n}+e_i}(z) + b_{\vec{n},i} Q_{\vec{n}-e_i}(z) + \sum_{j=1}^{d} a_{\vec{n},i,j} Q_{\vec{n}-e_j}(z),
\end{align*}
$$

For the coefficients $\{a_{\vec{n},i}, b_{\vec{n},i}\}$, we have representations [7]:

$$
\begin{align*}
& a_{\vec{n},j} = \frac{\int P_{\vec{n}}(x) x^{n_j} dx_j}{\int P_{\vec{n}+e_j}(x) x^{n_j-1} dx_j}, \quad b_{\vec{n},e_j} = \int x^{n_j} Q_{\vec{n}}(x) - \int x^{|\vec{n}| - 1} Q_{\vec{n}+e_j}(x).
\end{align*}
$$

If $d > 1$, unlike in one-dimensional case, we can not prescribe $\{a_{\vec{n},i}\}$ and $\{b_{\vec{n},i}\}$ arbitrarily. In fact, these coefficients satisfy the so-called “consistency conditions” which is a system of nonlinear difference equations. This discrete integrable system and the associated Lax pair were studied in [8, 35].

1.3. Angelesco systems and ray limits of NNRR coefficients. We recall that $\vec{\mu}$ is an Angelesco system of measures if

$$
supp \mu_j = \Delta_j := [\alpha_j, \beta_j], \quad \Delta_i \cap \Delta_j = \emptyset, \quad i \neq j, \quad i, j \in \{1, \ldots, d\},
$$
i.e., the supports of measures form a system of $d$ closed segments separated by $d - 1$ nonempty open intervals. We can always assume without loss of generality that $\beta_j < \alpha_{j+1}, j \in \{1, \ldots, d - 1\}$.

Angelesco systems form an important subclass of the perfect systems. They were studied by Angelesco already in 1919, [4]. It is not difficult to see [7] that the corresponding NNRR coefficients satisfy conditions (1.2) and thus define the Jacobi matrix $J_{\vec{\kappa}}$ by (1.3).
The asymptotic behavior of these coefficients \( \{a_{n,j}, b_{n,j}\} \) for the ray sequences regime, namely when

\[
\mathcal{N}_c = \{\vec{n}\} : \quad n_i = c_i|\vec{n}| + o(|\vec{n}|), \quad i \in \{1, \ldots, d\}, \quad |\vec{c}| := \sum_{i=1}^{d} c_i = 1,
\]

was studied in [7] for \( \vec{c} = (c_1, \ldots, c_d) \in (0,1)^d \) (hereafter, \( \lim_{|\vec{n}| \to \infty} \) stands for the limit as \( |\vec{n}| \to \infty \), \( \vec{n} \in \mathcal{N}_c \)). The following theorem was proved.

**Theorem 1.1** ([7]). Let \( \vec{\mu} \) be Angelesco system (1.9) such that for each \( i \in \{1, \ldots, d\} \) the measure \( \mu_i \) is absolutely continuous with respect to the Lebesgue measure on \( \Delta_i \) and the density \( \mu_i'(x) := d\mu_i(x)/dx \) extends to a holomorphic and non-vanishing function in some neighborhood of \( \Delta_i \). Then the ray limits (1.10) of coefficients \( \{a_{n,i}, b_{n,i}\} \) from (1.7) exist for any \( \vec{c} \in (0,1)^d \):

\[
\lim_{\mathcal{N}_c} a_{n,i} = A_{\vec{c},i} \quad \text{and} \quad \lim_{\mathcal{N}_c} b_{n,i} = B_{\vec{c},i}, \quad i \in \{1, \ldots, d\}.
\]

This result and expressions for \( A_{\vec{c},i} \) and \( B_{\vec{c},i} \) were obtained from the strong asymptotics of the MOPs also established in [7]. In Section 2, we will recall the formulas for \( A_{\vec{c},i}, B_{\vec{c},i} \) which show that these numbers depend on \( \Delta_1^{d} \).

1.4. Results and structure of the paper. In this paper, we restrict ourselves to the case \( d = 2 \). Our main technical achievement is an extension of the results in [7] on the strong asymptotics of the Angelesco MOPs to the full range of \( \vec{c} : \vec{c} \in [0,1]^2 \). As a corollary of this extension, we get the following result.

**Theorem 1.2.** Let \( \vec{\mu} \) be as in Theorem 1.1 with \( d = 2 \). Then the ray limits

\[
\lim_{\mathcal{N}_c} a_{n,i} = A_{\vec{c},i} \quad \text{and} \quad \lim_{\mathcal{N}_c} b_{n,i} = B_{\vec{c},i}
\]

exist for any \( c \in [0,1] \) and \( i \in \{1,2\} \), where \( \mathcal{N}_c := \mathcal{N}_{c(1-c)} \) is any sequence satisfying (1.10).

Theorem 1.2 can be used to characterize the essential spectrum of the Jacobi operator \( J_{\vec{c}} \), defined in (1.3), generated by an Angelesco system.

**Definition.** Let \( \mathcal{P} := \{\vec{\alpha}_i, \vec{\beta}_i\}_{i=1,2} \) be a set of real numbers that satisfy (1.2) for \( d = 2 \) and the constants \( A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2} \in [0,1] \) be limits from (1.12) (notice that \( \mathcal{P} \) does not have to be a set of the recurrence coefficients of any Angelesco system, but the limits \( A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2} \in [0,1] \) are generated by some \( \Delta_1 \) and \( \Delta_2 \). We say that \( \mathcal{P} \in \mathcal{P}_{Ang}(\Delta_1, \Delta_2) \) if \( \mathcal{P} \) satisfies

\[
\lim_{\mathcal{N}_c} \vec{\alpha}_i = A_{c,i} \quad \text{and} \quad \lim_{\mathcal{N}_c} \vec{\beta}_i = B_{c,i}
\]

for any \( c \in [0,1] \) and \( i \in \{1,2\} \), where, again, \( \mathcal{N}_c := \mathcal{N}_{c(1-c)} \) is any sequence satisfying (1.10).

By Theorem 1.2, the class \( \mathcal{P}_{Ang}(\Delta_1, \Delta_2) \) is not empty since the recurrence coefficients of any Angelesco system with analytic weights supported on \( \Delta_1 \) and \( \Delta_2 \) belong in \( \mathcal{P}_{Ang}(\Delta_1, \Delta_2) \). Consider Jacobi matrix \( J_{\vec{c}} \) defined in (1.3) with coefficients in \( \mathcal{P}_{Ang}(\Delta_1, \Delta_2) \). The following result gives characterization of its essential spectrum.

**Theorem 1.3.** Let \( J_{\vec{c}} \) be the Jacobi operator defined by (1.3) and corresponding to a collection of parameters \( \mathcal{P} \in \mathcal{P}_{Ang}(\Delta_1, \Delta_2) \), then \( \sigma_{\text{ess}}(J_{\vec{c}}) = \Delta_1 \cup \Delta_2 \). In particular, the essential spectrum of the Jacobi matrix generated by an Angelesco system with analytic weights supported on \( \Delta_1 \) and \( \Delta_2 \) is \( \Delta_1 \cup \Delta_2 \).

We prove this theorem in Section 2. The necessary definitions and statements of main results on strong asymptotics of MOPs are adduced in Section 3. Auxiliary results and their proofs are relegated to Sections 4 and 5. Proofs of the main results can be found in Sections 6 and 7.

2. Expressions for the ray limits and proof of Theorem 1.3

2.1. Expressions for the ray limits. In this subsection we give formulas for the limits in (1.12).

Let \( \Delta_1 = [\alpha_1, \beta_1] \) and \( \Delta_2 = [\alpha_2, \beta_2] \) be two intervals on the real line such that \( \beta_1 < \alpha_2 \). Denote by \( \omega_1 \) and \( \omega_2 \) the arcsine distributions on \( \Delta_1 \) and \( \Delta_2 \), respectively. It is known [32] that

\[
E(\omega_1, \omega_1) \leq E(\nu, \nu), \quad E(\mu, \nu) := -\int \log |x - y|d\mu(x)d\nu(y),
\]
for any probability Borel \( \nu \) measure on \( \Delta_i \). The logarithmic potentials of these measures satisfy
\[
\ell_i - V^{\omega_i} = 0 \quad \text{on} \quad \Delta_i,
\]
for some constants \( \ell_1 \) and \( \ell_2 \), where \( V^{\omega}(z) := -\frac{1}{\nu} \log |z - x|\,d\nu(x) \). Now, given \( c \in (0, 1) \), define
\[
M_c := \{(\nu_1, \nu_2) : \supp(\nu_i) \subseteq \Delta_i, \|\nu_1\| = c, \|\nu_2\| = 1 - c\}.
\]
It is known \([22]\) that there exists the unique pair of measures \((\omega_{c,1}, \omega_{c,2}) \in M_c\) such that
\[
I(\omega_{c,1}, \omega_{c,2}) \leq I(\nu_1, \nu_2), \quad I(\nu_1, \nu_2) := 2E(\nu_1, \nu_1) + 2E(\nu_2, \nu_2) + E(\nu_1, \nu_2) + E(\nu_2, \nu_1),
\]
for all pairs \((\nu_1, \nu_2) \in M_c\). Moreover, \(\supp(\omega_{c,1}) = [\alpha_1, \beta_{c,1}] =: \Delta_{c,1}\) and \(\supp(\omega_{c,2}) = [\alpha_{c,2}, \beta_2] =: \Delta_{c,2}\). Similarly to the case of a single interval, there exist constants \(\ell_{c,i} \in (0, 1)\), \(i \in \{1, 2\}\), such that
\[
\ell_{c,1} - V^{2\omega_{c,1} + \omega_{c,2}} = 0 \quad \text{on} \quad \supp(\omega_{c,1}),
\]
\[
\ell_{c,2} - V^{\omega_{c,1} + 2\omega_{c,2}} = 0 \quad \text{on} \quad \supp(\omega_{c,2}).
\]
The dependence of the intervals \(\Delta_{c,i}\) on the parameter \(c\) is described in greater detail in Section 4.

![Figure 2](image-url)

**Figure 2.** Surface \( \mathcal{R}_c \) when \( \beta_{c,1} = \beta_1 \) and \( \alpha_{c,2} = \alpha_2 \).

Let \( \mathcal{R}_c, c \in (0, 1) \), be a 3-sheeted Riemann surface realized as follows: cut a copy of \( \overline{\mathbb{C}} \) along \( \Delta_{c,1} \cup \Delta_{c,2} \), which henceforth is denoted by \( \mathcal{R}_c^{(0)} \), the second copy of \( \overline{\mathbb{C}} \) is cut along \( \Delta_{c,1} \) and is denoted by \( \mathcal{R}_c^{(1)} \), while the third copy is cut along \( \Delta_{c,2} \) and is denoted by \( \mathcal{R}_c^{(2)} \). These copies are then glued to each other crosswise along the corresponding cuts, see Figure 2. It can be easily verified that thus constructed Riemann surface has genus 0. We denote by \( \pi \) the natural projection from \( \mathcal{R}_c \) to \( \overline{\mathbb{C}} \) and employ the notation \( z \) for a generic point on \( \mathcal{R}_c \) with \( \pi(z) = z \) as well as \( z^{(i)} \) for a point on \( \mathcal{R}_c^{(i)} \) with \( \pi(z^{(i)}) = z \). Since \( \mathcal{R}_c \) has genus zero, one can arbitrarily prescribe zero/pole divisors of rational functions on \( \mathcal{R}_c \) as long as the degree of the divisor is zero. Clearly, a rational function with a given divisor is unique up to multiplication by a constant.

**Proposition 2.1.** Let \( \mathcal{R}_c, c \in (0, 1) \), be as above and \( \chi_c(z) \) be the conformal map of \( \mathcal{R}_c \) onto \( \overline{\mathbb{C}} \) such that
\[
\chi_c(z^{(0)}) = z + O(z^{-1}) \quad \text{as} \quad z \to \infty.
\]
Further, let the numbers \( A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2} \) be defined by
\[
\chi_c(z^{(i)}) =: B_{c,i} + A_{c,i}z^{-1} + O(z^{-2}) \quad \text{as} \quad z \to \infty, \quad i \in \{1, 2\}.
\]
Finally, let \( w_i(z) := \sqrt{(z - \alpha_1)(z - \beta_i)} \) be the branch holomorphic outside of \( \Delta_i \) and normalized so that \( w_i(z)/z \to 1 \) as \( z \to \infty \), in which case
\[
\varphi_i(z) := \frac{1}{2} \left( z - \frac{\beta_i + \alpha_i}{2} + w_i(z) \right)
\]
is the conformal map of \( \overline{\mathbb{C} \setminus \Delta_i} \) onto the complement of the disk \( B_{(\beta_i - \alpha_i)/2}(0) \) satisfying \( \varphi_i(z) = z + O(1) \) as \( z \to \infty \). Then it holds that
\[
\lim_{c \to 0} \begin{cases} 
A_{c,2} = \left( (\beta_2 - \alpha_2)/4 \right)^2 =: A_{0,2}, \\
B_{c,2} = (\beta_2 + \alpha_2)/2 =: B_{0,2}, \\
A_{c,1} = 0 =: A_{0,1}, \\
B_{c,1} = B_{0,2} + \varphi_2(\alpha_1) =: B_{0,1},
\end{cases}
\]
and analogous limits hold when \( c \to 1 \). Moreover, all the constants \( A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2} \) are continuous functions of the parameter \( c \in [0, 1] \).

Even though the expression for \( B_{0,1} \) might seem strange, it has a meaning from the point of view of spectral theory of Jacobi matrices, see (3.5).

We prove Proposition 2.1 in Section 5. It is worth noting that the constants \( A_{c,1} \) and \( A_{c,2} \) are always positive. Indeed, denote by \( \alpha_1, \beta_{c,1}, \alpha_2, \beta_2 \) the ramification points of \( R \), with natural projections \( \alpha_1, \beta_{c,1}, \alpha_2, \beta_2 \), respectively. Then the symmetries of \( R \) and \( \chi_c(z) \) yield that \( \chi_c(z) \) is real and changes from \( -\infty \) to \( \infty \) when \( z \) moves along the cycle

\[
\chi(0) \to \alpha_1 \to \chi(1) \to \beta_{c,1} \to \alpha_{c,2} \to \chi(2) \to \beta_2 \to \chi(0)
\]

whose natural projection is the extended real line. Thus, \( \chi_c(z) \) is increasing when it moves past \( \chi(1) \) and \( \chi(2) \), which yields the claim (this argument also shows that \( B_{c,1} < B_{c,2} \)).

2.2. Proof of Theorem 1.3. Our proof will be based on a characterization of the essential support of a Jacobi matrix on a tree obtained in [11, Theorem 4]. We need some preliminaries to formulate this result. Suppose \( T \) is a 3-homogeneous rooted tree with root at \( O \) (a binary tree), which means that \( O \) has two neighbors and any other vertex has three neighbors. Later in the text, we will use notation \( Z \sim Y \) to indicate that vertices \( Z \) and \( Y \) are neighbors and the symbol \( V \) will denote the set of all vertices of \( T \). Given a real function \( V \) defined on \( V \) and a real positive function \( W \) defined on all edges, we make an assumption

\[
\sup_{Y \in V} |V_Y| < \infty, \quad 0 < W_{Z,Y} < \infty
\]

to introduce \( J \), a bounded self-adjoint Jacobi matrix

\[
(Jf)_Y := V_Y f_Y + \sum_{Z \sim Y} W_{Z,Y} f_Z, \quad Y \in V,
\]

defined on \( \ell^2(V) \). One example one can think of is \( J_c \) introduced in (1.3). Consider a set of distinct vertices (a path) \( \{Y_n\}, n \in N, \) in \( V \) such that \( Y_n \sim Y_{n+1} \) for every \( n \). Clearly, every such path on the tree escapes to infinity, i.e., dist\((O, Y_n) \to \infty, n \to \infty \). We want to define \( R \)-limit (or "right limit") of \( J \) along this path. To do that, suppose \( G \) is a 3-homogeneous tree (without a root), \( O' \) is a fixed vertex on \( G \), and \( J' \) is a bounded self-adjoint operator on \( G \). Recall that \( B_r(Y) \) stands for the ball of radius \( r \) centered at \( Y \) and denote the restriction operator to this ball by \( P_{B_r(Y)} \). Consider two finite matrices:

\[
P_{B_r(Y_n)} J P_{B_r(Y_n)} \quad \text{and} \quad P_{B_r(O')} J' P_{B_r(O')}.
\]

If we identify \( \ell^2(B_r(O')) \) and \( \ell^2(B_r(Y_n)) \) by following the structure of the tree (and there are many ways to do that), then these matrices are defined on the same finite dimensional Euclidean space. If this identification can be done so that all sections of \( J' \) appear as the limits, we call \( J' \) an \( R \)-limit or right limit:

**Definition.** We say that \( J' \) is an \( R \)-limit of \( J \) along \( \{Y_n\} \) if there is a subsequence \( \{n_j\} \) such that

\[
P_{B_r(Y_n)} J P_{B_r(Y_n)} \to P_{B_r(O')} J' P_{B_r(O')} \quad \text{as} \quad j \to \infty
\]

for every fixed \( r \in \mathbb{N} \). Matrix \( J' \) is called simply an \( R \)-limit of \( J \) if there exists a path along which \( J' \) is an \( R \)-limit of \( J \).

**Remark.** For the rigorous definition of \( R \)-limit on more general graphs, see [11].

**Theorem 2.1** (Theorem 4 in [11]). We have

\[
\sigma_{ess}(J) = \bigcup_{J' \text{ is an } R \text{-limit of } J} \sigma(J').
\]

**Remark.** [11, Theorem 4] was stated for the regular trees only, but the proof is valid for rooted trees as well.

**Auxiliary operators \( L_c^{(1)} \) and \( L_c^{(2)} \).** Recall that \( T \) denotes the 3-homogeneous rooted tree with the root denoted by \( O \) and \( V \) stands for the set of all its vertices. There are two edges meeting at the root \( O \). We label one of them type 1 and the other one – type 2. Now, consider two vertices that are at distance 1 from \( O \). Each of them is coincident with exactly three edges. One of the edges for each vertex was labelled already, and we label the remaining two as an edge type 1 and an edge of type 2. We continue inductively by considering all edges that are at distance 2, 3, etc. from \( O \) and calling one
of the unlabelled edges type 1 and the other one type 2. Now that all edges of \( T \) have types assigned to them, we continue by labeling the vertices. If a vertex \( Y \) meets two edges of type 2 and one edge of type 1, we call it a vertex of type 1; otherwise, if it is incident with two edges of type 2 and one edge of type 1, we call it type 2. We do not need to assign any type to the root \( O \). At a vertex \( Y \neq O \) of type \( 1 \), \( Y \), see (1.1), we define both operators \( L^{(1)}_c \) and \( L^{(2)}_c \) by the same formula:

\[
(L^{(l)}_c \psi)_Y = \sum_{j \in \{1,2\}, Y' \sim Y, \text{type of edge } (Y,Y') = j} \sqrt{A_{c,j}} \psi_{Y'} + B_{c,l} \psi_Y, \quad l \in \{1, 2\};
\]

and at the root \( O \) we define the operators \( L^{(1)}_c \) and \( L^{(2)}_c \) differently from each other by

\[
(L^{(l)}_c \psi)_O = \sum_{j \in \{1,2\}, Y' \sim O, \text{type of edge } (O,Y') = j} \sqrt{A_{c,j}} \psi_{Y'} + B_{c,l} \psi_O, \quad l \in \{1, 2\}.
\]

Notice that these operators represent Jacobi matrices on \( T \) when \( c \in (0, 1) \). However, if \( c \in \{0, 1\} \) either \( A_{c,1} \) or \( A_{c,2} \) becomes zero and \( L^{(1)}_c , L^{(2)}_c \) are no longer Jacobi matrices, strictly speaking.

**Remark.** The operators \( L^{(1)}_c \) and \( L^{(2)}_c \) already appeared in [7] as the strong limits of Jacobi matrices on finite trees that correspond to \( \{P_\kappa\} \), the polynomials of the second type (see formula (3.3) and Subsection 4.5 in [7]).

**Lemma 2.1.** If \( J \) has coefficients in \( P_{Ang}(\Delta_1, \Delta_2) \), then the \( R \)-limits of \( J \) and the \( R \)-limits of \( L^{(1)}_c \), \( l \in \{1, 2\} \), are related by the following identity

\[
\bigcup_{c \in [0,1]} \left\{ J' : J' \text{ is an } R \text{ - limit of } L^{(1)}_c \right\} = \left\{ J'' : J'' \text{ is an } R \text{ - limit of } J \right\}.
\]

**Proof.** This follows from the definition of the \( R \)-limit, construction of \( L^{(1)}_c \) and \( L^{(2)}_c \), and from the assumption (1.13). \( \square \)

We further study auxiliary operators \( L^{(1)}_c \) and \( L^{(2)}_c \) in Appendix A.

**Proof of Theorem 1.3.** Assumptions (1.13) characterize the behavior of the coefficients at infinity. Thus, Weyl’s theorem on the essential spectrum [33] implies that any two Jacobi matrices with parameters in \( P_{Ang}(\Delta_1, \Delta_2) \) have the same essential spectra. Moreover, by the same Weyl’s theorem, this essential spectrum is independent of the choice of parameter \( \kappa \) in (1.3). Hence, it is enough to prove the theorem for the Jacobi matrix \( J_\kappa \) generated by some Angelesco system with analytic weights and with \( \kappa = \delta_2 \). In [7, Section 4] we established that \( \Delta_1 \cup \Delta_2 \subseteq \sigma(J_\delta_2) \). Thus, \( \Delta_1 \cup \Delta_2 \subseteq \sigma_{ess}(J_\delta_2) \) as follows from the definition of the essential spectrum.

To prove the opposite inclusion, take any \( J \) for which the coefficients belong to \( P_{Ang}(\Delta_1, \Delta_2) \). The application of Theorem 2.1 and Theorem A.1 to \( L^{(1)}_c \) gives

\[
\bigcup_{c \in [0,1]} \sigma(J') = \sigma_{ess}(L^{(1)}_c) = \Delta_{c,1} \cup \Delta_{c,2},
\]

which yields an inclusion

\[
\bigcup_{c \in [0,1]} \sigma(J') \subseteq \bigcup_{c \in [0,1]} \left( \Delta_{c,1} \cup \Delta_{c,2} \right) = \Delta_1 \cup \Delta_2,
\]

where the last equality follows from the properties of \( \Delta_{c,1} \) and \( \Delta_{c,2} \) (which we also discuss later in Proposition 4.1). Moreover, since

\[
\sigma_{ess}(J) = \bigcup_{c \in [0,1]} \bigcup_{J' \text{ is an } R \text{ - limit of } L^{(1)}_c} \sigma(J')
\]

by Theorem 2.1 and (2.11), we get from (2.12) that \( \sigma_{ess}(J) \subseteq \Delta_1 \cup \Delta_2 \), which proves the theorem. \( \square \)
3. Multiple Orthogonal Polynomials for Angelesco Systems

In this section we state the results on asymptotic behavior of the forms \( Q_n(x) \) and polynomials \( P_n^i(x) \) defined in (1.4) and (1.5), respectively, along ray sequences \( \mathcal{N}_c = \mathcal{N}_{c(1-c)} \) defined in (1.10) under the assumption that the measures of orthogonality are as in Theorem 1.1. Study of strong asymptotics of multiple orthogonal polynomials has long history, see for example [25, 29, 5, 37]. Below, we follow the Riemann-Hilbert approach used in [37].

As before, we assume that the intervals \( \Delta_1 = [\alpha_1, \beta_1] \) and \( \Delta_2 = [\alpha_2, \beta_2] \) are disjoint and \( \beta_1 < \alpha_2 \).

Under the conditions of Theorem 1.2, it holds that

\[ S_{\rho_i}(z) = \exp \left\{ \frac{w_i(z)}{2\pi i} \int_{\Delta_i} \frac{\log(\rho_i w_i(x))}{z-x} \frac{dx}{w_i(x)} \right\}, \quad i \in \{1, 2\}. \]

Then each \( S_{\rho_i}(z) \) is a holomorphic and non-vanishing function in \( \mathbb{C} \setminus \Delta_i \) that is uniquely (up to a sign) characterized by the properties

\[ (S_{\rho_i}+S_{\rho_i-})(\rho_i w_i)(x) \equiv 1, \quad x \in \Delta_i^c, \]

\[ |S_{\rho_i}(z)| \sim |z-x_0|^{-1/4}, \quad z \to x_0 \in \{\alpha_i, \beta_i\}. \]

Then the following theorem holds.

**Theorem 3.1.** Under the conditions of Theorem 1.2, it holds that

\[ P_n^i(z) = (1 + o(1))(S_{\rho_i}(z)/S_{\rho_i}(x))^n(z; \alpha_1)(z-\alpha_1)^n \varphi_2^{n^2}(z) \]

uniformly on bounded subsets of \( \mathbb{C} \setminus (\Delta_0 \cup \Delta_2) \) along any \( \mathcal{N}_0 \) satisfying (3.1), where \( \varphi_2(z) \) was introduced in (2.6) and

\[ S(z; x_0) := \left( \frac{\varphi_2(z) - \varphi_2(x_0)}{\varphi_2(x_0) \varphi_2(z) - A_{0,2}} \frac{\varphi_2(x_0) \varphi_2(z) - z - x_0}{\varphi_2(z) - z - x_0} \right)^{1/2}, \quad z \in \mathbb{C} \setminus \Delta_2, \]

\( x_0 \in (-\infty, \infty) \setminus \Delta_2 \) and the root is chosen so that \( S(z; x_0) = 1 \). An analogous asymptotic formula holds along \( \mathcal{N}_1 \) satisfying (3.1).

Since \( \varphi_2(x) \varphi_2(-x) \equiv A_{0,2} \) for \( x \in \Delta_2 \), and explicit computation shows that \( S(z; x_0)^+S(z; x_0)^-(z-x_0) \equiv 1 \) for \( x \in \Delta_2 \). Since conditions (3.4) uniquely characterize the Szegő function, it holds that

\[ S(x; x_0) = S(-x_0)^+/w_+(z)/S(-x_0)^+/w_+(\infty). \]

We prove Theorem 3.1 in Section 6.
3.2. Szegő Functions on $\mathcal{R}_c$. Let us set $\Delta_{c,i} := \pi^{-1}(\Delta_{c,i})$, $i \in \{1,2\}$, and orient it so that $\mathcal{R}_c^{(0)}$ remains on the left when the cycle is traversed in the positive direction. Put

\begin{equation}
(3.6) \quad w_{c,i}(z) := \sqrt{(z-a_{c,i})(z-b_{c,i})} = z + O(1), \quad z \to \infty,
\end{equation}

to be the branch holomorphic outside of $\Delta_{c,i}$. In what follows, it will be convenient to introduce the following notation

\begin{equation}
F^{(k)}(z) := F(z^{(k)}), \quad k \in \{0,1,2\},
\end{equation}

for a function $F(z)$ defined on $\mathcal{R}_c \setminus (\Delta_{c,1} \cup \Delta_{c,2})$. Then the following proposition holds.

**Proposition 3.1.** Given $c \in (0,1)$ and functions $\rho_1(x), \rho_2(x)$ as in (3.2) and Theorem 1.2, there exists a function $S_c(x)$ non-vanishing and holomorphic in $\mathcal{R}_c \setminus (\Delta_{c,1} \cup \Delta_{c,2})$ such that

\begin{equation}
(3.7) \quad \begin{cases} 
S_{c,1}^{(i)}(x) = S_{c,1}^{(0)}(x)(\rho_{1w_{c,i}})(x), & x \in \Delta_{c,i}, \\
S_{c,2}^{(i)}(z) = S_{c,2}^{(0)}(z)(1 + o(1))^{(i)}, & z \in \mathbb{C}, \\
S_{c,1}^{(0)}(z) \sim |z-x_0|^{-1/4} & \text{as } z \to x_0 \in \{\alpha_1, \beta_{c,1}, \alpha_{c,2}, \beta_2\}. 
\end{cases}
\end{equation}

Properties (3.7) determine $S_c(x)$ uniquely up to a multiplication by a cubic root of unity. Moreover, if $c \to c_0 \in (0,1)$, then

\begin{equation}
S_{c_0}^{(k)}(z) = [1 + o(1)]S_{c}^{(k)}(z),
\end{equation}

locally uniformly in $\mathbb{C} \setminus \Delta_{c,k}$ when $k \in \{1,2\}$, and in $\mathbb{C} \setminus (\Delta_{c,1} \cup \Delta_{c,2})$ when $k = 0$. Furthermore, it holds that

\begin{equation}
(3.9) \quad \frac{S_c^{(k)}(z)}{S_{c_0}^{(k)}(z)} = (1 + o(1)) \begin{cases} 
S \rho_2(z)/S_{\rho_2}(z), & k = 0, \\
1, & k = 1, \\
S \rho_2(z)/S_{\rho_2}(z), & k = 2,
\end{cases}
\end{equation}

as $c \to 0$, where $o(1)$ holds locally uniformly in $\mathbb{C} \setminus \Delta_{c,0}$ when $k \in \{0,1\}$ and uniformly in $\mathbb{C}$ when $k = 2$ (that is, including the traces on $\Delta_2$), while it also holds that

\begin{equation}
(3.10) \quad \lim_{c \to 0} S^{(0)}(x) c^{1/3} = V S \rho_2(x), \quad \lim_{c \to 0} S^{(1)}(x) c^{-2/3} = V^{-2}, \quad \text{and} \quad \lim_{c \to 0} S^{(2)}(x) c^{1/3} = V S \rho_2(x),
\end{equation}

where $V := (2\pi \mu^{(0)}(a_1) |w_2(a_1)| S \rho_2(a_1))^{-1/3}$. Limits analogous to (3.9) and (3.10) also hold as $c \to 1$.

We prove Proposition 3.1 in Section 5.

3.3. Non-Fully Marginal and Non-Marginal Ray Sequences. In this section we assume that sequences $\mathcal{N}_c, c \in [0,1]$, satisfy

\begin{equation}
(3.11) \quad \epsilon_c := 1/\min\{n_1,n_2\} \to 0 \quad \text{as} \quad |\vec{n}| \to \infty, \quad \vec{n} \in \mathcal{N}_c.
\end{equation}

We start by introducing an analog of the functions $\varphi_1(z), \varphi_2(z)$ in the non-fully marginal and non-marginal cases. Given a multi-index $\vec{n}$, let

\begin{equation}
(3.12) \quad c_{\vec{n}} := n_1/|\vec{n}|.
\end{equation}

To alleviate the notation, in what follows we shall use the subindex $\vec{n}$ instead of $c_{\vec{n}}$ for quantities depending on $c_{\vec{n}}$ such that $\mathcal{R}_{\vec{n}} = \mathcal{R}_{c_{\vec{n}}}$, $S_{\vec{n}}(z) = S_{c_{\vec{n}}}(z)$, etc. We shall denote by $\Phi_{\vec{n}}(z)$ a rational function on $\mathcal{R}_{\vec{n}}$ which is non-zero and finite everywhere except at the points on top of infinity, has a pole of order $|\vec{n}|$ at $x^{(i)}$, a zero of multiplicity $n_i$ at $x^{(i)}$ for each $i \in \{1,2\}$, and satisfies

\begin{equation}
(3.13) \quad \frac{\Phi_{\vec{n}}^{(0)} \Phi_{\vec{n}}^{(1)} \Phi_{\vec{n}}^{(2)}(z)}{\Phi_{\vec{n}}^{(0)}(z)} = 1, \quad z \in \mathbb{C}.
\end{equation}

Equality in (3.13) is a simple matter of a normalization since the logarithm of the absolute value of the left-hand side of (3.13) extends to a harmonic function on $\mathbb{C}$ which has a well defined limit at infinity and therefore is a constant.

**Theorem 3.2.** Under the conditions of Theorem 1.2, let $P_{\vec{n}}(z)$ be the polynomials satisfying (1.5). Given $c \in [0,1]$, let $\mathcal{N}_c = \{\vec{n}\}$ be a sequence for which (3.11) holds. Then for all $|\vec{n}|$ large enough, $\vec{n} \in \mathcal{N}_c$, we have that

\begin{equation}
\begin{cases} 
P_{\vec{n}}(z) = (1 + o(1))\gamma_{\vec{n}}(S_{\vec{n}} \Phi_{\vec{n}}^{(0)}(z)), \\
P_{\vec{n}}(x) = (1 + o(1))\gamma_{\vec{n}}(S_{\vec{n}} \Phi_{\vec{n}}^{(0)}(x) + (1 + o(1))\gamma_{\vec{n}}(S_{\vec{n}} \Phi_{\vec{n}}^{(0)}(x),
\end{cases}
\end{equation}
where the relations holds uniformly on closed subsets of \( \mathbb{C}\backslash \Delta_{c,1} \cup \Delta_{c,2} \) and compact subsets \( \Delta_{c,1}^\circ \cup \Delta_{c,2}^\circ \), respectively, and \( \gamma_{\vec{n}} \) is the constant such that
\[
\lim_{z \to \infty} \gamma_{\vec{n}} z^{[\vec{n}]} \left( S_{\vec{n}} \Phi_{\vec{n}} \right)^{(0)}(z) = 1.
\]
When \( c \neq c^*, c^{**} \), see Proposition 4.1 further below, the error rate \( o(1) \) can be replaced by \( O_c(\varepsilon_{\vec{n}}) \), where the dependence of \( O_c(\varepsilon_{\vec{n}}) \) on \( c \) is uniform for \( c \) on compact subsets \( [0,1] \backslash \{c^*, c^{**}\} \).

In the above theorem the functions \( S_{\vec{n}}^{(0)}(z) \) could be replaced by their limits as discussed in Proposition 3.1. However, we can do this only at the expense of the error rate \( O_c(\varepsilon_{\vec{n}}) \).

To describe asymptotic behavior of the forms \( Q_c(x) \), we need to introduce one additional function. Let \( \Pi_{\vec{n}}(z) \) be a rational function on \( \mathfrak{R}_{\vec{n}} \) with the zero/pole divisor and the normalization given by
\[
2\left( x^{(1)} + x^{(2)} \right) - \alpha_1 - \beta_{\vec{n},1} - \alpha_{\vec{n},2} + \beta_2 \quad \text{and} \quad \Pi_{\vec{n}}^{(0)}(x) = 1,
\]
where \( \alpha_1, \beta_{\vec{n},1}, \alpha_{\vec{n},2}, \beta_2 \) are the ramification points of \( \mathfrak{R}_{\vec{n}} \). Then the following theorem holds.

**Theorem 3.3.** Under the conditions of Theorem 1.2, let \( A_{\vec{n}}^{(i)}(z) \) be the polynomials defined in (1.4), \( i \in \{1, 2\} \). Given \( c \in [0,1] \), let \( \mathcal{N}_c = \{\vec{n}\} \) be a sequence for which (3.11) holds. Then for all \( |\vec{n}| \) large enough, \( \vec{n} \in \mathcal{N}_c \), we have that
\[
A_{\vec{n}}^{(i)}(z) = -(1 + o(1)) \frac{\left( \Pi_{\vec{n}}^{(i)}(w_{\vec{n},1}) \right)(z)}{\gamma_{\vec{n}} \left( S_{\vec{n}} \Phi_{\vec{n}} \right)^{(0)}(z)}.
\]
uniformly on closed subsets of \( \mathbb{C}\backslash \Delta_{c,i} \) for \( i \in \{1, 2\} \) when \( c \in (0,1) \), \( i = 2 \) when \( c = 0 \), and \( i = 1 \) when \( c = 1 \), while
\[
A_{\vec{n}}^{(i)}(z) = o(1) \left( \tau_{\vec{n}}(w_{\vec{n},1}) \Phi_{\vec{n}}^{(i)}(z) \right)^{-1},
\]
uniformly on closed subsets of \( \mathbb{C}\backslash \Delta_{c,1} \) for \( i = 1 \) when \( c = 0 \) and of \( \mathbb{C}\backslash \Delta_{c,2} \) for \( i = 2 \) when \( c = 1 \), where \( \tau_{\vec{n}} := \gamma_{\vec{n}} S_{\vec{n}}^{(0)}(x) \), i.e., it is a constant such that \( \lim_{z \to \infty} \tau_{\vec{n}} z [\vec{n}] \Phi_{\vec{n}}^{(0)}(z) = 1 \). Moreover,
\[
A_{\vec{n}}^{(i)}(x) = -(1 + o(1)) \frac{\left( \Pi_{\vec{n}}^{(i)}(w_{\vec{n},1}) \right)_+(x)}{\gamma_{\vec{n}} \left( S_{\vec{n}} \Phi_{\vec{n}} \right)^{(i)}(x)} - (1 + o(1)) \frac{\left( \Pi_{\vec{n}}^{(i)}(w_{\vec{n},1}) \right)_-(x)}{\gamma_{\vec{n}} \left( S_{\vec{n}} \Phi_{\vec{n}} \right)^{(i)}(x)}
\]
uniformly on compact subsets of \( \Delta_{c,i}^\circ \), \( i \in \{1, 2\} \). As in the case of Theorem 3.2, the error rate can be improved to \( O_c(\varepsilon_{\vec{n}}) \) when \( c \in [0,1] \backslash \{c^*, c^{**}\} \) with dependence on \( c \) being locally uniform.

Let \( (\hat{\mu}_1, \hat{\mu}_2) \) be a vector of Markov functions of the measures \( \mu_i \), that is,
\[
\hat{\mu}_i(z) := \int \frac{d\mu_i(x)}{x - z} = \frac{1}{2\pi i} \int_{\Delta_i} \frac{\rho_i(x)}{x - z} \, dx, \quad z \in \mathbb{C}\backslash \Delta_i, \ i \in \{1, 2\}.
\]
Observe also that \( \hat{\mu}_{i+} - \hat{\mu}_{i-}(x) = \rho_i(x), x \in \Delta_i^\circ \), by Sokhotski-Plemelj formulæ. Then one can deduce from orthogonality relations (1.5) that there exist polynomials \( P_{\vec{n}}^{(i)}(z) \) such that
\[
R_{\vec{n}}^{(i)}(z) := (P_{\vec{n}}^{(i)} \hat{\mu}_i - P_{\vec{n}}^{(i)}) \phi_{\vec{n}}(z) = O\left(z^{-n_i+1}\right) \quad \text{as} \quad z \to \infty,
\]
i \in \{1, 2\}. The vector of rational functions \( \left( P_{\vec{n}}^{(1)}/P_{\vec{n}}, P_{\vec{n}}^{(2)}/P_{\vec{n}} \right) \) is called the Hermite-Padé approximant for \( (\hat{\mu}_1, \hat{\mu}_2) \) corresponding to the multi-index \( \vec{n} \). It further can be shown that
\[
R_{\vec{n}}^{(i)}(z) = \frac{1}{2\pi i} \int (P_{\vec{n}}^{(i)})(x)/x - z \, dx, \quad z \in \mathbb{C}\backslash \Delta_i, \ i \in \{1, 2\}.
\]
It also follows from (1.4) that there exists polynomial \( A_{\vec{n}}(x) \) such that
\[
O\left( z^{-[\vec{n}]}\right) = \sum_{i=1}^{2} (A_{\vec{n}}^{(i)} \hat{\mu}_i)(z) - A_{\vec{n}}(z) =: L_{\vec{n}}(z) = \int \frac{Q_z^i(x)}{x - z} \, dz,
\]
where the asymptotic formula is valid for \( z \to \infty \). Then the following result holds.

**Theorem 3.4.** Under the conditions of Theorems 3.2–3.2, it holds for all \( \vec{n} \in \mathcal{N}_c \) with \( |\vec{n}| \) large enough that
\[
R_{\vec{n}}^{(i)}(z) = (1 + o(1)) \gamma_{\vec{n}} \left( S_{\vec{n}} \Phi_{\vec{n}} \right)^{(i)}(z) w_{\vec{n},1}^{(i)}(z),
\]
uniformly on closed subsets of $\mathcal{C}\setminus \Delta_{c,i}$, that is, including the traces on $\Delta_i \setminus \Delta_{c,i}$ for $i \in \{1, 2\}$ when $c \in (0, 1)$, for $i = 2$ when $c = 0$, and for $i = 1$ when $c = 1$, while

$$F^{(i)}_{\beta}(z) = o(1)\tau_{\beta}^{(i)}(z)w_{\gamma_i}^{-1}(z)$$

uniformly on closed subsets of $\mathcal{C}\setminus \Delta_{0,1}$ for $i = 1$ when $c = 0$ and of $\mathcal{C}\setminus \Delta_{1,2}$ for $i = 2$ when $c = 1$. Moreover,

$$L_i(z) = (1 + o(1))\frac{\Pi_i^{(0)}(z)}{\gamma_i(S_i \Phi_i^{(0)}(0))^{(i)}(z)},$$

uniformly on closed subsets of $\mathcal{C}\setminus (\Delta_{c,1} \cup \Delta_{c,2})$. As in Theorems 3.2 and 3.3 the error rate can be improved to $O_c(\varepsilon)$ when $c \in [0, 1]\{c^*, c^{**}\}$ with dependence on $c$ being locally uniform.

Theorems 3.2–3.4 are proven in Chapter 7.

4. On the Supports of the Equilibrium measures

In this section we discuss further properties of the vector equilibrium problem (2.2)–(2.3) as well as prove some auxiliary lemmas needed later.

With the notation introduced after (2.3), the following proposition holds.

**Proposition 4.1.** There exist constants $0 < c^* < c^{**} < 1$ such that

$$\begin{align*}
\beta_{c,1} &< \beta_1, \quad \alpha_{c,2} = \alpha_2, \quad 0 < c < c^1, \\
\beta_{c,1} &< \beta_1, \quad \alpha_{c,2} = \alpha_2, \quad c^* < c \leq c^{**}, \\
\beta_{c,1} &< \beta_1, \quad \alpha_{c,2} > \alpha_2, \quad 1 > c > c^{**}.
\end{align*}$$

Moreover, it holds that

$$\omega_{c,1} \to \omega_{c,1}, \quad \omega_{c,2} \to \omega_{c,2}, \quad \beta_{c,1} \to \beta_{c,1}, \quad \ell_{c,1} \to \ell_{c,1}, \quad V^{\omega_{c,1}} \to V^{\omega_{c,1}} \quad \text{as} \quad c \to c_1 \in (0, 1)$$

for $i \in \{1, 2\}$, where the convergence of potentials is uniform on compact subsets of $\mathcal{C}$. Furthermore,

$$\begin{align*}
\omega_{c,2} &\to \omega_2, \quad \beta_{c,1} \to \alpha_1, \quad \ell_{c,2} \to 2\ell_2, \quad \ell_{c,1} \to V^{\omega_{c,1}}(\alpha_1) \quad \text{as} \quad c \to 0, \\
\omega_{c,1} &\to \omega_1, \quad \alpha_{c,2} \to \beta_2, \quad \ell_{c,1} \to 2\ell_1, \quad \ell_{c,2} \to V^{\omega_{c,1}}(\beta_2) \quad \text{as} \quad c \to 1,
\end{align*}$$

and $V^{\omega_{c,1}} \to V^{\omega_{c,1}}$ uniformly on compact subsets of $\mathcal{C}$ as $c \to 2 - i, \ i \in \{1, 2\}$.

Further, recall the surface $\mathcal{R}_c$, constructed just before Proposition 2.1. Given a rational function $F(z)$ on $\mathcal{R}_c$, we denote its divisor of zeros and poles by $(F)$ and write

$$(F) = m_1 z_1 + \cdots + m_t z_t - k_1 p_1 - \cdots - k_t p_t$$

to mean that $F(z)$ has a zero of order $m_i$ at $z_i$, for each $i \in \{1, \ldots, t\}$, a pole of order $k_i$ at $p_i$, for each $i \in \{1, \ldots, t\}$, and otherwise it is non-vanishing and finite, where necessarily $\sum_{i=1}^t m_i = \sum_{i=1}^t k_i$.

It can be easily checked using Schwarz reflection principle, as it was done in [37, Proposition 2.1] for $c$ rational, that the function

$$H_c(z) := -V^{\omega_{c,1} + \omega_{c,2}}(z) + \frac{\ell_{c,1} + \ell_{c,2}}{3}, \quad z \in \mathcal{R}_c^{(0)},$$

$$V^{\omega_{c,1}}(z) - \ell_{c,1} + \frac{\ell_{c,1} + \ell_{c,2}}{3}, \quad z \in \mathcal{R}_c^{(i)}, \quad i \in \{1, 2\},$$

is harmonic on $\mathcal{R}_c \setminus \{\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)}\}$. Therefore, the function $h_c(z) := 2\partial_x H_c(z)$, where $2\partial_x := \partial_x - i\partial_y$, is rational on $\mathcal{R}_c$. In fact, it holds that

$$\begin{align*}
h^{(0)}_c(z) &= \int \frac{d(\omega_{c,1} + \omega_{c,2})(x)}{z - x}, \quad z \in \mathcal{C}\setminus (\Delta_{c,1} \cup \Delta_{c,2}), \\
h^{(i)}_c(z) &= \int \frac{d\omega_{c,i}(x)}{x - z}, \quad z \in \mathcal{C}\setminus \Delta_{c,i}, \quad i \in \{1, 2\}.
\end{align*}$$

The importance of this function lies in the following: it was shown in [37, Propositions 2.1 and 2.3] that

$$\Phi_i(z) = C_i \exp \left\{ |\beta_i| \int h^{i}_{c\beta}(x)dx \right\} \quad \text{and} \quad \frac{1}{|\beta_i|} \log \left| \Phi_i(z) \right| = H_{c\beta}(z)$$

$^2$Given compactly supported measures $\nu_n, n \in \mathbb{Z}_{\geq 0}, \nu_n \to \nu_0$ as $n \to \infty$ means that $\int f d\nu_n \to \int f d\nu_0$ as $n \to \infty$ for any compactly supported continuous function $f$. 
for $z \in \mathfrak{R}_{\vec{p}}$, where the constant $C_{\vec{p}}$ should be chosen so that (3.13) is satisfied.

**Proposition 4.2.** Let $D_c := \alpha_1 + \beta_{c,1} + \alpha_{c,2} + \beta_2$ be the divisor\(^3\) of the ramification points of $\mathfrak{R}_c$.

It holds that

$$
(h_c) = \infty^{(0)} + \infty^{(1)} + \infty^{(2)} + z_c - D_c
$$

for some $z_c \in \mathfrak{R}_c^{(0)}$ such that $z_c \in [\beta_{c,1}, \alpha_{c,2}]$. Moreover, $z_c$ is a continuous increasing function of $c$ and

$$
\begin{cases}
    z_c = \beta_{c,1}, & c \leq c^*, \\
    z_c = \alpha_{c,2}, & c > c^{**}.
\end{cases}
$$

This proposition has the following implication: point $z_c$ uniquely determines the vector equilibrium measure $(\omega_{c,1}, \omega_{c,2})$. Indeed, choose $z_c \in (\alpha_1, \beta_1)$. Set $\beta_{*,1} = \min\{\beta_1, z_c\}$ and $\alpha_{*,2} = \max\{\alpha_2, z_c\}$.

Construct Riemann surface $\mathfrak{R}_c$ with respect to the cuts $[\alpha_1, \beta_{*,1}]$ and $[\alpha_{*,2}, \beta_2]$ as before. Let $h_*(z)$ be a rational function on $\mathfrak{R}_c$ with the zero/pole divisor

$$(h_*) = \infty^{(0)} + \infty^{(1)} + \infty^{(2)} + z_c - \alpha_1 - \beta_{*,1} - \alpha_{*,2} - \beta_2,$$

where $\alpha_1, \beta_{*,1}, \alpha_{*,2}, \beta_2$ are the ramification points of $\mathfrak{R}_c$ and $z_c \in \mathfrak{R}_c^{(0)}$. Clearly, $h_*(z^{(0)}) + h_*(z^{(1)}) + h_*(z^{(2)}) = 0$ as this sum must be an entire function that vanishes at infinity. Normalize $h_*(z)$ so that $h_*(z^{(0)}) = 1/z + O(1/z^2)$ as $z \to \infty$. Set $c_* := -\lim_{z \to \infty} zh(z^{(1)})$. Then $\mathfrak{R}_c = \mathfrak{R}_{c_*}$, $z_c = z_{c_*}$, and respectively $h_*(z) = h_c(z)$. It further follows from Privalov’s lemma [31, Section III.2] that

$$
d\omega_{c,i}(x) = \left(h^{(i)}_+(x) - h^{(i)}_-(x)\right) \frac{dx}{2\pi i}, \quad i \in \{1, 2\},
$$

and thus, we have recovered the vector equilibrium measure from $z_c$.

**Proof of Propositions 4.1 and 4.2.** Besides relations (2.3), it also holds that the left hand sides of (2.3) are strictly less than zero on $\Delta_1 \setminus \Delta_{c,1}$ and $\Delta_2 \setminus \Delta_{c,2}$, respectively, see [22]. In particular, we can write

$$
V^{\omega_{c,1}}(x) + \frac{1}{2c} V^{\omega_{c,2}}(x) - \frac{\ell_{c,1}}{2c} \left\{ \begin{array}{l}
\equiv 0 \quad \text{on} \quad \text{supp}(\omega_{c,1}), \\
\geq 0 \quad \text{on} \quad \Delta_1 \setminus \text{supp}(\omega_{c,1}),
\end{array} \right.
$$

which, in view of [34, Theorem I.3.3], can be interpreted in the following way: the measure $\frac{1}{c} \omega_{c,1}$ is the weighted logarithmic equilibrium distribution on $\Delta_1$ in the presence of the external field $\frac{1}{2c} V^{\omega_{c,2}}(x)$. Hence, its support maximizes the Mhaskar-Saff functional [34, Chapter IV]:

$$
F_c(K) := \log \text{cap}(K) - \frac{1}{2c} \int V^{\omega_{c,2}} \, d\omega_K,
$$

where $K \subseteq [\alpha_1, \beta_1]$ is compact, $\text{cap}(K)$ is the logarithmic capacity of $K$, and $\omega_K$ is the logarithmic equilibrium distribution on $K$ (when $K$ is an interval, $\omega_K$ is the arc sine distribution on $K$). As mentioned before (2.3), the maximizer of this functional is an interval containing $\alpha_1$ (this was proven in [22]). Therefore, it is enough to consider compact sets $K$ only of the form $[\alpha_1, \beta]$. Thus, the functional $F_c(K)$ reduces to the function

$$
F_*(\beta) := \log \frac{\beta - \alpha_1}{4} - \frac{1}{2c} \int_{\alpha_1}^{\beta} V^{\omega_{c,2}}(x) \, dx\frac{dx}{\pi \sqrt{\beta - x}(x - \alpha_1)},
$$

where we used explicit expressions for the logarithmic capacity and the equilibrium measure of an interval. To find the maximum of $F_*(\beta)$ on $\Delta_1$, let us compute its derivative. To this end, it can be readily checked that

$$
\frac{1}{h} \left( \int_{\alpha_1}^{\beta + h} f(x) \frac{dx}{\pi \sqrt{\beta + h - x}(x - \alpha_1)} - \int_{\alpha_1}^{\beta} f(x) \frac{dx}{\pi \sqrt{\beta - x}(x - \alpha_1)} \right) = \int_{\alpha_1}^{\beta} \frac{1}{h} \left( f\left(x + h \frac{x - \alpha_1}{\beta - \alpha_1}\right) - f(x) \right) \frac{dx}{\pi \sqrt{\beta - x}(x - \alpha_1)}
$$

\(^3\)A divisor is any formal linear combination of points of $\mathfrak{R}_c$ with integer coefficients.
for every differentiable function $f(x)$ on $\Delta_1$. Observe also that $V^{\omega_2}(x)$ is harmonic off $\Delta_2$ and therefore $f_\Delta(x) := V^{\omega_2}(x) = -\frac{1}{2} \log |x-y| d\omega_{\omega_2}(y)$ is a smooth function on $\Delta_1$. Hence, by taking the limit as $h \to 0$ in the above equality, we get

$$F'_c(\beta) = \frac{1}{\beta - \alpha_1} - \frac{1}{4\pi c} \int_{\Delta_1} f'_c(\frac{1}{2} \beta - \alpha_1, x, \frac{1}{2} \beta + \alpha_1, x, \sqrt{1+x}, \sqrt{1-x}) \, dx.$$  

(4.5)

It is also obvious that $f'_c(x) = \left(\frac{y-x}{1-x}\right)^{-1} d\omega_{\omega_2}(y)$, which is an increasing positive function on $\Delta_1$. Thus, $F'_c(\beta)$ is a decreasing function of $\beta$ and therefore has at most one zero. Moreover, it holds that

$$1 - \frac{c}{\beta_2 - \alpha_1} < f'_c(x) < \frac{1 - c}{\alpha_2 - \beta_1}, \quad x \in \Delta_1.$$  

(4.6)

Hence, $F'_c(\beta_1) < 0$ for all $c$ small. As $\lim_{\beta \to \omega_1} F'_c(\beta) = +\infty$, we get that $\beta_{c,1} \in (\alpha_1, \beta_1)$ for all $c$ small. Using $F'_c(\beta_{c,1}) = 0$ and the above estimates, we get from (4.5) that

$$\frac{4c}{1 - c} (\alpha_2 - \beta_1) < \beta_{c,1} - \alpha_1 < \frac{4c}{1 - c} (\beta_2 - \alpha_1)$$  

(4.7)

for all small $c$. This, in particular, implies that $\beta_{c,1} \to \alpha_1$ as $c \to 0$. An analogous argument shows that $\alpha_{c,2}$ approaches $\beta_2$ when $c \to 1$. It further follows from (4.6) that $f'_c(x)$ uniformly converges to zero on $\Delta_1$ as $c \to 1$. Thus, $F'_c(\beta) > 0$ for all $\beta \in \Delta_2$ and all $c$ close to 1. That is, $\Delta_{c,1} = \Delta_1$ in this case. Similarly, we also get that $\Delta_{c,2} = \Delta_2$ for all $c$ small.

Let us now describe what happens to the components of the vector equilibrium measure and their potentials as $c \to 0$. Clearly, $V^{\omega_{c,1}}(x) \to 0$ uniformly on compact subsets of $\mathbb{C} \setminus \Delta_{0,1}$ in this case. To show that $\omega_{c,1} \to \omega_2$ as $c \to 0$, notice that

$$|\sigma|_2 = \int V^{\omega_2} d\sigma = \int V^{\sigma} d\omega_2 \geq \inf_{\Delta_2} V^{\sigma}, \leq \sup_{\Delta_2} V^{\sigma},$$

for any Borel measure $\sigma$ supported on $\Delta_2$ since $\omega_2$ is a probability measure. It follows from (2.3) that $V^{\omega_{c,1}}(x)$ is continuous on $\Delta_2 = \Delta_{c,2}$. Therefore,

$$\begin{cases} 2\ell_2(1-c) \geq \min_{\Delta_2} V^{2\omega_{c,2}} = V^{2\omega_{c,2}}(x_{\min}) = \ell_{c,2} - V^{\omega_{c,1}}(x_{\min}) = \ell_{c,2} + o(1), \\
2\ell_2(1-c) \leq \max_{\Delta_2} V^{2\omega_{c,2}} = V^{2\omega_{c,2}}(x_{\max}) = \ell_{c,2} - V^{\omega_{c,1}}(x_{\max}) = \ell_{c,2} + o(1),
\end{cases}$$

which implies that $\ell_{c,2} = 2\ell_2 + o(1)$ as $c \to 0$. Let $\omega$ be a weak* limit point of $\omega_{c,2}$ as $c \to 0$. Then $\omega$ is a probability measure and

$$V^{\omega}(x) \leq \liminf_{c \to 0} V^{\omega_{c,2}}(x) = \liminf_{c \to 0} (\ell_{c,2} - V^{\omega_{c,1}}(x))/2 = \ell_2, \quad x \in \Delta_2,$$

where the first inequality follows from the Principle of Descent [34, Theorem 1.6.8]. Therefore, $E(\omega, \omega) \leq \ell_2 = E(\omega_2, \omega_2)$, which implies that $\omega = \omega_2$ by the uniqueness of the equilibrium measure. To deduce the behavior of the constants $\ell_{c,1}$ as $c \to 0$, observe that

$$\begin{cases} V^{2\omega_{c,1} + \omega_{c,2}}(x) \leq \ell_{c,1}, \quad x \in (-\infty, \alpha_1], \\
V^{2\omega_{c,1} + \omega_{c,2}}(x) \geq \ell_{c,1}, \quad x \in [\beta_{c,1}, \beta_1],
\end{cases}$$

where the first claim can be easily obtained from (2.3) and the second one was already mentioned at the beginning of the proof. Then

$$V^{2\omega_{c,1} + \omega_{c,2}}(\alpha_1 - \epsilon) \leq \ell_{c,1} \leq V^{2\omega_{c,1} + \omega_{c,2}}(\alpha_1 + \epsilon)$$

for any $\epsilon > 0$ since $\beta_{c,1} < \alpha_1 + \epsilon$ for all $c$ small enough. Hence, we get that

$$V^{\omega_{c,2}}(\alpha_1 - \epsilon) \leq \liminf_{c \to 0} \ell_{c,1} \leq \limsup_{c \to 0} \ell_{c,1} \leq V^{\omega_{c,2}}(\alpha_1 + \epsilon).$$

Since $V^{\omega_{c,2}}(x)$ is continuous on the real line and $\epsilon$ is arbitrary, we get that $\ell_{c,1} \to V^{\omega_{c,2}}(\alpha_1)$ as $c \to 0$. The respective claims for the limits as $c \to 1$ can be shown in a similar fashion.

Let us point out one consequence of the fact that $\omega_{c,2} \to \omega_2$ as $c \to 0$ that will be useful to us later. It holds that

$$f'_c(z) := \int \frac{d\omega_{c,2}(y)}{y-z} \to \int \frac{d\omega_{2}(y)}{y-z} = \frac{1}{\pi} \int_{\alpha_2}^{\beta_2} \frac{1}{y-z} \frac{dy}{\sqrt{(y-\alpha_2)(\beta_2-y)}} = \frac{1}{w_2(z)}.$$
locally uniformly in $\mathbb{C} \setminus \Delta_2$, where, as before, $w_2(z) := \sqrt{(z - \alpha_2)(z - \beta_2)}$. Therefore, we can improve (4.7) to
\[
\frac{4c}{\beta_{c,1} - \alpha_1} = \frac{1}{|w_2(\alpha_1)|} + o(1)
\]
as $c \to 0$, where we again used (4.5).

The facts that $\omega_{c,i} \to \omega_{i,1}$ and $\ell_{c,i} \to \ell_{i,1}$ as $c \to c_\ast \in (0, 1)$, $i \in \{1, 2\}$, were shown in the proof of [37, Proposition 2.1]. Let us now show that $\beta_{c,1} \to \beta_{\ast,1}$ in this case (that is, that $\beta_{c,1}$ is a continuous function of $c$). Weak* convergence of measures necessitates that $\liminf_{c \to c_\ast} \beta_{c,1} \geq \beta_{\ast,1}$. Assume to the contrary that there exists a subsequence $c_n \to c_\ast$ such that $\beta_{c_n,1} < \beta_{\ast,1} := \liminf_{n \to \infty} \beta_{c_n,1}$. Then
\[
\liminf_{n \to \infty} \ell_{c_n,1} = \liminf_{n \to \infty} V_{2\omega_{c_n,1} + \omega_{i,1}^2}(x) \geq V_{2\omega_{\ast,1} + \omega_{i,1}^2}(x) = \ell_{\ast,1}
\]
as $x \in (\beta_{c,1}, \beta_{\ast})$ due to the Principle of Descent [34, Theorem 1.6.8]. However, the above conclusion clearly contradicts the claim $\ell_{c,1} \to \ell_{c,1}$ as $c \to c_\ast$. The convergence $\alpha_{c,2} \to \alpha_{c,2}$ as $c \to c_\ast$ can be shown analogously (unfortunately, this convergence of the endpoints was asserted without justification in the proof [37, Proposition 2.1]). Given the convergence of the endpoint, the uniform convergence of the potentials as $c \to c_\ast \in (0, 1)$ was established in the proof of [37, Proposition 2.1] using harmonicity of $H_c(z)$. The same arguments can be applied to show that $V_{\omega_{c,1}} \to V_{\omega_{1}}$ uniformly on compact subsets of $\mathbb{C}$ as $c \to 2 - i$, $i \in \{1, 2\}$.

Let us now establish the existence of the constants $0 < c^* < c^{**} < 1$ and the monotonicity properties of $\beta_{c,1}$ and $\alpha_{c,2}$. Claim (4.4) was obtained in [37, Proposition 2.3]. There it was further shown that
\[
\beta_{c,1} < \beta_1 \Rightarrow z_c = \beta_{c,1} \quad \text{and} \quad \alpha_{c,2} > \alpha_2 \Rightarrow z_c = \alpha_{c,2}.
\]
Assume now that $\beta_{c,1} = \beta_{c_2,1} < \beta_1$. Then the functions $h_{c_1}(z)$ and $h_{c_2}(z)$ are defined on the same Riemann surface. Their difference has at least four zeros (double zero at $\infty(0)$ and simple zeros at $\infty(1)$ and $\infty(2)$) and at most three poles $\alpha_1, \alpha_2, \beta_2$. This is possible only if the function is identically zero and therefore $c_1 = c_2$ as $h_c(z) = c_2 - 1 + O(z^{-2})$ by (4.2). Since $\beta_{c,1} \to \alpha_1$ as $c \to 0$, this shows the existence of $c^*$ and proves monotonicity of $\beta_{c,1}$ as a function of $c$ (it is a continuous and injective function of $c$). The existence of $c^{**}$ and monotonicity of $\alpha_{c,2}$ are proven analogously. It also follows from (4.9) that $c^* \leq c^{**}$. As it was shown in [37, Proposition 2.3] that $z_c = \beta_{c,1} = \beta_1$ and $z_c^{**} = \alpha_{c,2} = \alpha_2$, we in fact get that $c^* < c^{**}$.

It only remains to prove that $z_c$ is a continuous increasing function of $c$ on $[c^*, c^{**}]$. To show monotonicity, take $c^* \leq c_1 < c_2 \leq c^{**}$. It follows easily from (4.2) that each $h_c(x^{(0)})$ is a decreasing function of $x \in (\beta_1, \alpha_2)$, which, to prove that $z_c < z_{c_2}$, it is enough to show that $h(x^{(0)}) > 0$ in $(\beta_1, \alpha_2)$, where $h(z) := (h_{c_1} - h_{c_2})(z)$. Notice that $h(x^{(0)}) = -h(x^{(1)}) - h(x^{(2)})$ by (4.2) and therefore it is sufficient to argue that $h(x^{(1)}) < 0$ on $(\beta_1, \infty)$ and $h(x^{(2)}) < 0$ on $(-\infty, \alpha_2)$. These claims are obvious for all $|x|$ large enough since
\[
h(z^{(1)}) = -\frac{c_2 - c_1}{z} + O(z^{-2}) \quad \text{and} \quad h(z^{(2)}) = \frac{c_2 - c_1}{z} + O(z^{-2})
\]
as $z \to \infty$ according to (4.2). As explained after (4.9), $h(z)$ vanishes only at $\infty(0)$, $\infty(1)$, and $\infty(2)$. Therefore, $h(z^{(1)})$ and $h(z^{(2)})$ cannot change sign on $(\beta_1, \infty)$ and $(-\infty, \alpha_2)$, respectively. Hence, these functions are negative everywhere on the considered rays by continuity.

To show continuity of $z_c$ as a function of $c \in [c^*, c^{**}]$, we shall once again use the fact that $h_{c}(x^{(0)})$ is a decreasing function on $(\beta_1, \alpha_2)$. When $c \in (c^*, c^{**})$, $h_{c}(x^{(0)})$ is unbounded on both ends of $(\beta_1, \alpha_2)$ and therefore changes sign from + to − when passing through $z_c$ (recall that $h_c(z)$ has poles at $\beta_1$ and $\alpha_2$ in this case). When $c = c^*$, $h_c(x^{(0)})$ is unbounded only at $\alpha_2$ and, since it is non-vanishing, is negative on $(\beta_1, \alpha_2)$. Similarly, when $c = c^{**}$, it is unbounded at $\beta_1$ only and therefore is positive on $(\beta_1, \alpha_2)$. In any case, $z_c$ is the point where the potential $V_{\infty,1} + \omega_{x,2}(x)$ achieves its minimum on $[\beta_1, \alpha_2]$. Thus, if $z_{c_n} \to z_\ast$ as $c_n \to c_\ast$ when $n \to \infty$, $c_n, c_\ast \in (c^*, c^{**})$, then
\[
V_{\infty,1} + \omega_{x,2}(z_c) \leq \liminf_{n \to \infty} V_{\infty,1} + \omega_{x,2}(z_{c_n}) \leq \liminf_{n \to \infty} V_{\infty,1} + \omega_{x,2}(z_{c_n}) = V_{\infty,1} + \omega_{x,2}(z_\ast),
\]
where the first inequality follows from the weak* convergence of measures and the Principle of Descent [34, Theorem 1.6.8], the second one from the just discussed extremal property of $z_n$, and the last equality holds due to the weak* convergence of measures and the fact that $z_\ast$ does not belong to the supports of the measures in question. Since $V_{\infty,1} + \omega_{x,2}(x)$ is smallest at $z_{c_n}$, we get that $z_\ast = z_{c_\ast}$. 
When $c_\ast = c^\ast$, essentially the same argument works. One just needs to replace $z_{c_\ast} = \beta_1$ with $\beta_1 + \epsilon$ for any $\epsilon > 0$. Since $V^{\omega_\ast + 1}(x)$ is increasing on $[\beta_1, \beta_2]$, this shows that $z_\ast \leq z_{c_\ast} + \epsilon$ for any $\epsilon > 0$ and therefore $z_\ast = z_{c_\ast}$. Clearly, an analogous modification works when $c_\ast = c^{\ast\ast}$.

5. Proof of Propositions 2.1 and 3.1

On several occasions we shall refer to the following consequences of Koebe’s 1/4-theorem, [30, Theorem 1.3]. Given $r > 0$, let

$$a(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad b(z) = \sum_{k=0}^{\infty} b_k z^{-k}, \quad \text{and} \quad d(z) = \sum_{k=-\infty}^{\infty} d_k z^k$$

be univalent in $D_a = \{ |z - z_0| < r \}$, $D_b = \{ |z| > 1/r \}$, and $D_d = \{ |z| > r \}$, respectively. Then,

\begin{equation}
\{ |z - a_0| < r a_1/4 \} \subseteq a(D_a), \quad \{ |z - b_0| < r b_1/4 \} \subseteq b(D_b), \quad \text{and} \quad \{ |z| > 4 r d_1 \} \subseteq d(D_d),
\end{equation}

where $f(D)$ stands for image of a domain $D$ under the function $f(z)$.

5.1. Proof of Proposition 2.1. Recall that $\chi_c(z)$ is univalent on $\mathfrak{H}_c$ and $\chi_c(0)(z) = z + O(z^{-1})$ as $z \to \infty$, see (2.4). Hence, it follows from (5.1) that there exists a finite constant $R$ independent of $c$ such that $\{ |z| > R \} \subset \chi_c(\mathfrak{H}_c)$ for all $c \in (0, 1)$. In particular, it holds that $|\chi_c(x)| \leq R$, $x \in \Delta_{c,1}$, as well as $|B_{c,i}| \leq R$, $i \in \{1, 2\}$, see (2.5), for all $c \in (0, 1)$. For all $c \leq c^{\ast\ast}$ (in which case $\Delta_{c,2} = \Delta_2$), define

$$\phi(z) := \frac{1}{2} \left\{ \begin{array}{ll}
\frac{z - (\beta_2 + \alpha_2)/2 + w_2(z)}{z - (\beta_2 + \alpha_2)/2 - w_2(z)} & z \in \mathfrak{H}_c \setminus \Delta_{c,1}, \\
\frac{w_2(z)}{z} & z \in \mathfrak{H}_c^2.
\end{array} \right.$$  

This is a meromorphic function in $\mathfrak{H}_c \setminus \Delta_{c,1}$ with a simple pole at $\infty_0$, a simple zero at $\infty_2$, and otherwise non-vanishing and finite. It is normalized so that $\phi(z^{\ast\ast}) = z + O(1)$ as $z \to \infty$. Observe that $\phi(z)$ continuously extends to the closed set $\mathfrak{H}_c \setminus \mathfrak{H}_c^2$. It can be readily checked that the image of $\mathfrak{H}_c \setminus \mathfrak{H}_c^2$ under $\phi(z)$ is equal to $\mathbb{T}$ and $\phi(z)$ is one-to-one everywhere except on $\Delta_{c,1}$ that is mapped into an interval

$$\phi(\Delta_{c,1}) := \mathcal{I}_{c,1} = \{ \phi(\alpha_1), \phi(\beta_{c,1}) \} \to \{ \phi(\alpha_1) \} \quad \text{as} \quad c \to 0.$$  

Notice also that $\phi^{\ast\ast}(z) = \phi^{\ast}(z)$ for $z \in \mathbb{T} \setminus \Delta_2$, see (2.6).

Define $f_c(z) := (\chi_c(\phi^{-1}(z)) - B_{c,2})/z$. Then $f_c(z)$ is a holomorphic function in $\mathbb{T} \setminus I_{c,1}$ (there is no pole at the origin as $\phi^{-1}(0) = \infty_2$ and $\chi_c(z) - B_{c,2}$ vanishes there) with bounded traces on $I_{c,1}$ that assumes value 1 at infinity. Hence, it follows from Cauchy’s integral formula that

$$f_c(z) = 1 + \int_{I_{c,1}} \frac{(f_c - f_c^{-\ast})(x)}{x - z} dx, \quad z \in \mathbb{T} \setminus I_{c,1}.$$  

Since the traces $f_c(z)$ are bounded above in absolute value on $I_{c,1}$ independently of $c$ and $|I_{c,1}| \to 0$ as $c \to 0$, we see that $f_c(z) \to 1$ as $c \to 0$ locally uniformly in $\mathbb{T} \setminus \{ \phi(\alpha_1) \}$. Hence, it holds that

$$\chi_c(z) = B_{c,2} + (1 + o(1)) \phi(z)$$  

locally uniformly on $\mathfrak{H}_c \setminus \mathfrak{H}_c^2 \setminus \Delta_{c,1}$. Since the image of $\mathfrak{H}_c \setminus \mathfrak{H}_c^2 \setminus \Delta_{c,1}$ under $\phi(z)$ is $\mathbb{T} \setminus I_{c,1}$ and $|I_{c,1}| \to 0$ as $c \to 0$, for any $\epsilon > 0$ there exists $\delta > 0$ such that the image of $\mathfrak{H}_c \setminus \pi^{-1}(\{ |z - \alpha_1| < \epsilon \}) \setminus \mathfrak{H}_c^1$ under $\chi_c(z)$ contains $\mathbb{T} \setminus \{ |z - B_{c,2} - \phi(\alpha_1)| < \delta \}$. Due to univalency of $\chi_c(z)$ on $\mathfrak{H}_c$, this means that the image of $\mathfrak{H}_c \setminus \pi^{-1}(\{ |z - \alpha_1| < \epsilon \}) \setminus \mathfrak{H}_c^1$ is contained in $\{ |z - B_{c,2} - \phi(\alpha_1)| < \delta \}$. Altogether, we get that

\begin{equation}
\phi^{\ast}(z) = z + \frac{\beta_2 + \alpha_2}{2} + O\left( \frac{1}{z} \right) \quad \text{and} \quad \phi^{\ast\ast}(z) = \frac{(\beta_2 - \alpha_2)^2}{16} \frac{1}{z} + O\left( \frac{1}{z^2} \right),
\end{equation}

the desired limits (2.7) easily follow.

Continuity of $A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}$ as functions of $c$ comes from the continuous dependence of $\alpha_{c,2}$ and $\beta_{c,1}$ on $c$, see Proposition 4.2, and therefore the continuous dependence $\chi_c(z)$ on $c$. 

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\begin{equation}
\chi_c(z) = B_{c,2} + (1 + o(1)) \left\{ \begin{array}{ll}
\phi(z), & z \in \mathfrak{H}_c \setminus \mathfrak{H}_c^2, \\
\phi(\alpha_1), & z \in \mathfrak{H}_c^1,
\end{array} \right.
\end{equation}

where $o(1)$ holds uniformly on the entire surface $\mathfrak{H}_c$. Since

$$\phi^{\ast}(z) = z + \frac{\beta_2 + \alpha_2}{2} + O\left( \frac{1}{z} \right) \quad \text{and} \quad \phi^{\ast\ast}(z) = \frac{(\beta_2 - \alpha_2)^2}{16} \frac{1}{z} + O\left( \frac{1}{z^2} \right),$$

the desired limits (2.7) easily follow.

Continuity of $A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}$ as functions of $c$ comes from the continuous dependence of $\alpha_{c,2}$ and $\beta_{c,1}$ on $c$, see Proposition 4.2, and therefore the continuous dependence $\chi_c(z)$ on $c$. 

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5.2. Auxiliary Estimates, I. In the forthcoming analysis, the following functions will play an important role:

\[
\Upsilon_{c,i}(z) := A_{c,i} \left( \chi_c(z) - B_{c,i} \right)^{-1}, \quad i \in \{1, 2\}.
\]

It follows from the properties of \( \chi_c(z) \), see (2.4) and (2.5), that \( \Upsilon_{c,i}(z) \) is a conformal map of \( \mathcal{R}_c \) onto \( \mathbb{C} \) that maps \( \infty^{(1)} \) into \( \infty \) and \( \infty^{(0)} \) into 0. Moreover, it holds that

\[
\Upsilon_{c,i}(z) = z + O(1) \quad \text{and} \quad \Upsilon_{c,i}^{(0)}(z) = A_{c,i} z^{-1} + O(z^{-2}) \quad \text{as} \quad z \to \infty.
\]

It was explained in [37, Section 7], see [37, Equation (7.2)], that

\[
\Upsilon_{c,i}(z) \to \Upsilon_{c*,i}(z) \quad \text{as} \quad c \to c_* \in (0, 1),
\]

uniformly on \( \mathcal{R}_{c,i} \setminus \mathcal{U} \) for each \( i \in \{1, 2\} \), where \( \mathcal{U} \) is any open set containing ramification points of \( \mathcal{R}_{c,i} \) (if \( \mathcal{U}_c \subset \mathcal{R}_c \) is an open set such that \( \pi(\mathcal{R}^{(k)}_{c} \setminus \mathcal{U}) = \pi(\mathcal{R}_{c}^{(k)} \setminus \mathcal{U}_c) \) for each \( k \in \{0, 1, 2\} \), then the bordered Riemann surfaces \( \mathcal{R}_{c,i} \setminus \mathcal{U} \) and \( \mathcal{R}_{c} \setminus \mathcal{U}_c \) are identical for all \( c \) sufficiently close to \( c_* \) and can be thought of \( \Upsilon_{c,i}(z) \) as a function on \( \mathcal{R}_{c,i} \setminus \mathcal{U} \). On the other hand, when \( c \to 0 \), the following is true.

**Lemma 5.1.** It holds that

\[
\Upsilon_{c,2}(z) = (1 + o(1)) \begin{cases} 
\psi(z), & z \in \mathcal{R}_{c,0} \cup \mathcal{R}_{c}^{(2)}, \\
\psi(\alpha_1), & z \in \mathcal{R}_{c}^{(1)} 
\end{cases}
\]

as \( c \to 0 \), where \( o(1) \) holds uniformly on the entire surface \( \mathcal{R}_c \) and

\[
\psi(z) := \frac{A_{0,2}}{\varphi(z)} = \frac{1}{2} \begin{cases} 
-z - \frac{2}{\alpha_2} + \frac{w(z)}{2}, & z \in \mathcal{R}_{c,0} \setminus \Delta_{c,1}, \\
-z - \frac{2}{\alpha_2} + \frac{w(z)}{2}, & z \in \mathcal{R}_{c}^{(2)},
\end{cases}
\]

that is \( \psi(z) \) maps \( \mathcal{R}_{c}^{(2)} \) conformally onto \( \{ |z| > \frac{2}{\alpha_2} \} \) and \( \psi(0)(z) \psi(2)(z) = A_{0,2} \). Moreover, it holds that \( 4 \)

\[
\Upsilon_{c,1}(z) \sim c |\phi_c(z)|, \quad \Upsilon_{c,1}^{(0)}(z) \sim c |\phi_c(z)|, \quad \text{and} \quad \Upsilon_{c,1}^{(2)}(z) \sim c^2
\]
on \( \mathbb{C} \) (including the traces on \( \Delta_{c,1} \cup \Delta_2, \Delta_{c,1}, \) and \( \Delta_2 \), respectively) as \( c \to 0 \), where

\[
\phi_c(z) := \frac{2}{\beta_{c,1} - \alpha_1} \left( z - \frac{\beta_{c,1} + \alpha_1}{2} + w_{c,1}(z) \right)
\]
is the conformal map of \( \mathbb{C} \setminus \Delta_{c,1} \) onto \( \{ |z| > 1 \} \) that fixes the point at infinity and has positive derivative there. In addition, it holds that \( \Upsilon_{c,1}^{(1)}(z) = z - \alpha_1 + O(c) \) uniformly in \( \mathbb{C} \) as \( c \to 0 \).

**Proof.** Formula (5.6) follows immediately from (5.2), the very definition (5.3), and the first limit in (2.7). It also follows from (5.3) and (5.2) that

\[
\Upsilon_{c,1}^{(2)}(z) = \frac{A_{c,1}}{(1 + o(1)) \varphi(\alpha_1) + (1 + o(1)) \varphi(2)(z)} \sim A_{c,1}
\]
in \( \mathbb{C} \) (including the traces on \( \Delta_2 \) as \( c \to 0 \) since \( |\varphi(2)(z)| < \frac{2}{\alpha_2} < \frac{1}{|\varphi(\alpha_1)|} \), see (2.6). It can be readily verified that the symmetric functions of the branches of a rational function on \( \mathcal{R}_c \) must be rational functions on \( \mathbb{C} \). Since \( \Upsilon_{c,1}^{(1)}(z) \) has a simple pole at infinity, \( \Upsilon_{c,1}^{(0)}(z) \) has a simple zero there, and \( \Upsilon_{c,k}(z), \ k \in \{0, 1, 2\}, \) are otherwise non-vanishing and finite, the product of three branches of \( \Upsilon_{c,1}(z) \) must be a constant. Thus, similarly to (5.9), it holds that

\[
\Upsilon_{c,1}^{(0)}(z) \Upsilon_{c,1}^{(1)}(z) \Upsilon_{c,1}^{(2)}(z) = \frac{A_{c,1}^2}{B_{c,2} - B_{c,1}} = -\frac{A_{c,1}^2}{1 + o(1)} \varphi(\alpha_1) \sim A_{c,1}^2
\]
in \( \mathbb{C} \) as \( c \to 0 \) (recall that \( \varphi(\alpha_1) < 0 \)). For each \( z \notin \Delta_{c,1} \cup \Delta_{c,2} \), let \( \tau \) be the point on the same sheet of \( \mathcal{R}_c \) as \( z \) with \( \pi(\tau) = \tau \) and then extend this definition by continuity to \( \Delta_{c,1} \cup \Delta_{c,2} \). The function \( \Upsilon_{c,1}(\tau) \) is meromorphic on \( \mathcal{R}_c \) and has the same zero/pole divisor and normalization as

\[4\] Given non-negative functions \( A_c(z) \) and \( B_c(z) \), we write \( A_c(z) \leq B_c(z) \) (resp. \( A_c(z) \sim B_c(z) \)) as \( c \to 0 \) on \( K_c \) for some family of closed sets \( \{K_c\} \), if there exists \( \epsilon > 0 \) such that \( A_c(z) \leq C B_c(z) \) (resp. \( C^{-1} A_c(z) \leq B_c(z) \leq C A_c(z) \)) for all \( z \in K_c \) and each \( c \in [0, \epsilon] \), where \( C \) depends only on \( \epsilon \).
Lemma 5.2. For each $0 < \delta \leq (\alpha_2 - \beta_1)/2$ fixed, it holds that

\begin{align*}
\left\{ \begin{array}{l}
c^{-1}|\Upsilon_{c,1}^{(0)}(z)|, \ c^{-1}|\Upsilon_{c,1}^{(1)}(z)|, \ c^{-2}|\Upsilon_{c,1}^{(2)}(z)| \sim 1, \\
(1 - c)^{-2}|\Upsilon_{c,2}^{(0)}(z)|, (1 - c)^{-2}|\Upsilon_{c,2}^{(1)}(z)|, |\Upsilon_{c,2}^{(2)}(z)| \sim 1,
\end{array} \right.
\end{align*}

on $K_{c,0} := \{ z : \text{dist}(z, \Delta_{c,1}) \leq c \delta \}$ for all $c \in (0, 1)$ and that

\begin{align*}
\left\{ \begin{array}{l}
c^{-2}|\Upsilon_{c,1}^{(0)}(z)|, |\Upsilon_{c,1}^{(1)}(z)|, c^{-2}|\Upsilon_{c,1}^{(2)}(z)| \sim 1, \\
(1 - c)^{-1}|\Upsilon_{c,2}^{(0)}(z)|, (1 - c)^{-1}|\Upsilon_{c,2}^{(1)}(z)|, |\Upsilon_{c,2}^{(2)}(z)| \sim 1,
\end{array} \right.
\end{align*}

on $K_{c,2} := \{ z : \text{dist}(z, \Delta_{c,2}) \leq (1 - c)\delta \}$ for all $c \in (0, 1)$, where the constants of proportionality depend only on $\delta$. 

In our analysis, it will be convenient to apply Lemma 5.1 in the following form. As is mentioned above, the sum $\Upsilon_{c,1}^{(0)}(z) + \Upsilon_{c,1}^{(1)}(z) + \Upsilon_{c,1}^{(2)}(z)$ is a rational function on $\mathbb{C}$. Since it has only one pole, which is simple and located at infinity, it is a monic (see (5.4)) polynomial of degree 1. In particular, it holds that

$$\beta_{c,1} - \alpha_1 = 2\Upsilon_{c,1}^{(0)}(\beta_{c,1}) + \Upsilon_{c,1}^{(2)}(\beta_{c,1}) - 2\Upsilon_{c,1}^{(0)}(\alpha_1) - \Upsilon_{c,1}^{(2)}(\alpha_1),$$

where we used the fact that $\Upsilon_{c,1}^{(0)}(\gamma) = \Upsilon_{c,1}^{(1)}(\gamma) = \Upsilon_{c,1}(\gamma)$ for $\gamma \in \{\alpha_1, \beta_{c,1}\}$. Thus, it follows from (4.7) and (5.12) (lower bound) together with (5.9) and (5.11) (upper bound) that

$$c \lesssim 2|\Upsilon_{c,1}^{(0)}(\beta_{c,1})| + |\Upsilon_{c,1}^{(2)}(\beta_{c,1})| + 2|\Upsilon_{c,1}^{(0)}(\alpha_1)| + |\Upsilon_{c,1}^{(2)}(\alpha_1)| \lesssim A_{c,1}^{1/2} + A_{c,1} \lesssim A_{c,1}^{1/2}$$

as $c \to 0$, where we also used the fact that $A_{c,1} \to 0$ as $c \to 0$ for the last inequality. On the other hand, it holds that

$$\Upsilon_{c,1}^{(1)}(z) = -\frac{A_{c,2}}{B_{c,2} - B_{c,1}} + \frac{A_{c,1}A_{c,2}}{B_{c,2} - B_{c,1}} \frac{1}{z} + O\left(\frac{1}{z^2}\right)$$

as $z \to \infty$ by the very definitions (5.3) and (2.4). Therefore, we can deduce from Cauchy's integral formula that

$$\frac{A_{c,1}A_{c,2}}{B_{c,2} - B_{c,1}} = \frac{1}{2\pi i} \int_{\Delta_{c,1}} \left(\Upsilon_{c,2+}^{(1)}(x) - \Upsilon_{c,2-}^{(1)}(x)\right) dx \leq \frac{\beta_{c,1} - \alpha_1}{\pi} \max_{x \in \Delta_{c,1}} |\Upsilon_{c,1}(x)| + Z$$

for any complex number $Z$. Now, if we show that

$$\max_{x \in \Delta_{c,1}} |\Upsilon_{c,1}^{(1)}(x)| + \frac{A_{c,2}}{B_{c,2} - B_{c,1}} \lesssim A_{c,1}^{1/2},$$

as $c \to 0$, inequalities (4.7) and (5.13) together with limits (2.7) will allow us to conclude that $A_{c,1}^{1/2} \lesssim c$ as $c \to 0$, which will finish the proof of (5.7). To prove (5.14), observe that

$$\Upsilon_{c,2}(z) = \frac{A_{c,2}}{\chi_{c,1}(z) - B_{c,2}} = \frac{A_{c,2}}{B_{c,1} - B_{c,2} + A_{c,1} \Upsilon_{c,1}^{-1}(z)} = \frac{A_{c,2}}{B_{c,2} - B_{c,1}} \frac{A_{c,1}}{B_{c,2} - B_{c,1} - \Upsilon_{c,1}(z)}$$

according to their very definition (5.3). Thus,

$$\Upsilon_{c,2}(z) + \frac{A_{c,2}}{B_{c,2} - B_{c,1}} = \frac{A_{c,2}}{B_{c,1} - B_{c,2}} \frac{A_{c,1}}{B_{c,2} - B_{c,1} - (B_{c,2} - B_{c,1}) \Upsilon_{c,1}(z)}$$

The desired estimate (5.14) now follows from (5.11) and (2.7).

To prove the last claim of the lemma, observe that $\Upsilon_{c,1}^{(1)}(z) - (z - \alpha_1)$ is holomorphic in $\mathbb{C} \setminus \Delta_{c,1}$ and

$$|\Upsilon_{c,1}^{(1)}(x) - (x - \alpha_1)| \leq \max_{x \in \Delta_{c,1}} |\Upsilon_{c,1}^{(1)}(x)| + |\Upsilon_{c,1}(x)| + \beta_{c,1} - \alpha_1 \lesssim c, \quad x \in \Delta_{c,1},$$

as $c \to 0$ by (4.7) and (5.11). The desired claim now follows from the maximum modulus principle. □
Proof. We provide the proofs only partially, understanding that the arguments for \( \Upsilon_{c,2}(z) \) are essentially identical. Recall that \( \Upsilon_{c,1}(z) \) is a conformal map of \( \mathbb{R}_c \) onto \( \overline{\mathbb{C}} \) that maps \( \infty^{(0)} \) into 0 and \( \infty^{(1)} \) into \( \infty \). Let \( r := \max\{|\alpha_1|, |\beta_2|\} \). Then it follows from (5.1) and (5.4) that

\[
\{ |z| < A_{c,1}/4(r + \delta) \} \subset \Upsilon_{c,1}^{(0)}(\{ |z| > r + \delta \}) \quad \text{and} \quad \{ |z| > 4(r + \delta) \} \subset \Upsilon_{c,1}^{(1)}(\{ |z| > r + \delta \}).
\]

Thus, it holds that

\[
\frac{A_{c,1}}{4(r + \delta)} \leq |\Upsilon_{c,1}(z)| \leq 4(r + \delta) \quad \text{for all} \quad z \in K_{c,\delta,1} \cup K_{c,\delta,2}.
\]

Since \( A_{c,1} \to ((\beta_1 - \alpha_1)/4)^2 \) by the limit analogous to the one for \( A_{c,2} \) in (2.7), this establishes the desired bounds in (5.15) and (5.16) for all \( c \in [c_1, 1] \) and any \( \epsilon > 0 \) fixed with the constants of proportionality dependent on \( \epsilon \) and \( \delta \). On the other hand, the bounds for \( c \in (0, \epsilon] \) readily follow from (5.7) and (5.8) as

\[
1 \leq |\phi_c(z)| \leq 4 \frac{r \delta + \beta_1 c - \alpha_1}{\beta_2 c - \alpha_1} < 4 + \frac{\delta}{2} \quad \text{and} \quad c|\phi_c(z)| \sim |z - \alpha_1|
\]
on \( K_{c,\delta,1} \) and \( K_{c,\delta,2} \), respectively, as \( c \to 0 \) by elementary estimates and (4.7). The estimates of \( \Upsilon_{c,2}(z) \) can be verified similarly. \( \square \)

Let a function \( \Pi_c(z) \) be defined on \( \mathcal{R}_c \) analogously to the way \( \Pi_{c,1}(z) \) was defined on \( \mathcal{R}_{c,1} \) just before Theorem 3.3. Further, let \( \Pi_{c,i}(z), i \in \{1, 2\} \), be rational functions on \( \mathcal{R}_c \) with the divisors and normalization given by

\[
\Pi_{c,i}(z) = \infty^{(0)} + \infty^{(i)} + 2\infty^{(3-i)} - D_c \quad \text{and} \quad \Pi_{c,i}^{(1)}(z) = \frac{1}{z} + \mathcal{O} \left( \frac{1}{z^2} \right),
\]

where \( D_c \) is the divisor of the ramification points of \( \mathcal{R}_c \), see Proposition 4.2.

**Lemma 5.3.** It holds that

\[
\left( \Pi_{c,i} \right) = \left( \Pi_{c,2} \right)^{2-i}(w_{c,1}w_{c,2})(z)\Pi_{c,3-i}(z) = \begin{cases} \left( \Upsilon_{c,2}^{(1)} - \Upsilon_{c,1}^{(1)} \right)(z), & z \in \mathcal{R}_{c}^{(0)}, \\ \left( \Upsilon_{c,2}^{(0)} - \Upsilon_{c,1}^{(2)} \right)(z), & z \in \mathcal{R}_{c}^{(1)}, \\ \left( \Upsilon_{c,2}^{(1)} - \Upsilon_{c,1}^{(0)} \right)(z), & z \in \mathcal{R}_{c}^{(2)}, \end{cases}
\]

for \( i \in \{1, 2\} \) and

\[
\left( w_{c,1}w_{c,2} \right)(z)\Pi_c(z) = \begin{cases} \left( \Upsilon_{c,2}^{(1)}\Upsilon_{c,1}^{(0)} - \Upsilon_{c,1}^{(1)}\Upsilon_{c,0}^{(2)} \right)(z), & z \in \mathcal{R}_{c}^{(0)}, \\ \left( \Upsilon_{c,2}^{(0)}\Upsilon_{c,1}^{(2)} - \Upsilon_{c,2}^{(2)}\Upsilon_{c,1}^{(0)} \right)(z), & z \in \mathcal{R}_{c}^{(1)}, \\ \left( \Upsilon_{c,2}^{(1)}\Upsilon_{c,1}^{(0)} - \Upsilon_{c,2}^{(0)}\Upsilon_{c,1}^{(1)} \right)(z), & z \in \mathcal{R}_{c}^{(2)}. \end{cases}
\]

Moreover, it holds that

\[
\Pi_c^{(1)}(z) = (1 + o(1)) \frac{\psi^{(1)}(z)}{w_{c,1}(z)} = (1 + o(1)) \frac{\psi^{(2)}(z)}{w_{c,2}(z)}
\]
as \( c \to 0 \), where the first relation holds uniformly in \( \overline{\mathbb{C}} \) (that is, including the traces on \( \Delta_{c,1} \cup \Delta_2 \)) and the second one locally uniformly in \( \overline{\mathbb{C}} \setminus \Delta_{c,1} \).

**Proof.** Representations (5.19) and (5.20) can be easily verified by observing that the right-hand sides are continuous across \( \Delta_{c,1} \) and \( \Delta_{c,2} \) and by comparing the zero/pole divisors and the normalizations of the left-hand and right-hand sides, see (2.5), (5.3), and (5.18). Asymptotic formula (5.21) follows immediately from the first relation in (5.20), asymptotic formulae (5.6) and (5.7), and the last claim of Lemma 5.1. \( \square \)
5.3. Proof of Proposition 3.1. It was shown in [37, Section 6] that the Szegő functions $S_c(z)$ satisfying (3.7) is given by

$$S_c(z) := \exp \left\{ \frac{1}{2\pi i} \sum_{i=1}^{2} \int_{\Delta_c} \log(\rho_i w_{c,i+})(s) C_2(s) \right\},$$

where $C_2(s)$ is the third kind differential on $\mathfrak{R}_c$ with three simple poles at $z, z_1, z_2$ that have the same natural projection $z$ and respective residues $-2, 1, 1$. Limit (3.8) was in fact proven in [37, Section 7]. Thus, it only remains to show the validity of (3.9) and (3.10). In order to do that we shall use an alternative construction of $S_c(z)$ that is more amenable to asymptotic analysis.

Since we are interested in what happens when $c \to 0$, we shall assume that $c \leq \min\{1/2, \epsilon^**\}$ (the choice of $1/2$ is rather arbitrary, but convenient to use in (4.7)). Set

$$D_{c,1}(z) := \left( z - \frac{\beta_{c,1} + \alpha_{c,1}}{4} + \frac{w_{c,1}(z)}{2w_{c,1}(z)} \right)^{1/2}, \quad z \in \overline{\mathbb{C}\setminus\Delta_{c,1}},$$

where we take the branch of the square root such that $D_{c,1}(z)$ is holomorphic and non-vanishing in the domain of the definition and has value 1 at infinity. The traces of $D_{c,1}(z)$ on $\Delta_{c,1}$ satisfy

$$|D_{c,1}(x)|^2 = (D_{c,1+}, D_{c,1-})(x) = \frac{\beta_{c,1} - \alpha_{1}}{4\{w_{c,1}(x)\}}, \quad x \in \Delta_{c,1}.$$

Let $\delta > 0$ be as in Lemma 5.2, that is, $\delta \leq \frac{(\alpha_2 - \beta_1)/2}{2}$. Then it follows from (4.7) that $\delta c \leq |\Delta_{c,1}|/8$.

Using (4.7) once more together with our assumption that $c \leq 1/2$, we get that

$$\delta(\alpha_2 - \beta_1) < \frac{w_{c,1}(x)}{c} < 3\sqrt{\beta_2 - \alpha_1}, \quad |s - \alpha_1| = \delta c, \quad |s - \beta_{c,1}| = \delta c,$$

(5.23) and similar straightforward estimates of $|2\alpha_1 - \beta_{c,1}|$ using (4.7) as well as (5.22) and the maximum modulus principle for holomorphic functions applied to both $D_{c,1}(z)$ and $D_{c,1}^{-1}(z)$ yield that

$$\frac{1}{\delta^{-1/4}} \leq |D_{c,1}(z)| \leq \delta^{-1/4}, \quad 0 < \delta c \leq \text{dist}(z, \{\alpha_1, \beta_{c,1}\}),$$

(5.24) uniformly on the respective sets, where the constants of proportionality do not depend on $c, \delta$. Additionally, since $\beta_{c,1} \to \alpha_1$ as $c \to 0$ and therefore $w_{c,1}(z) = z - \alpha_1 + o(1)$ locally uniformly in $\overline{\mathbb{C}\setminus\Delta_{c,1}}$ as $c \to 0$, it holds locally uniformly in $\overline{\mathbb{C}\setminus\Delta_{c,1}}$ that

$$D_{c,1}(z) = 1 + o(1) \quad \text{as} \quad c \to 0.$$

Now, let $D_{c,\rho_1}(z)$ be the Szegő function of the restriction of $\rho_1(x)$ to $\Delta_{c,1}$ normalized to have value 1 at infinity. That is,

$$D_{c,\rho_1}(z) = \exp \left\{ \frac{w_{c,1}(z)}{2\pi i} \int_{\Delta_{c,1}} \frac{\log \rho_1(x)}{z-x} \, dx \left( w_{c,1}(x) \right) \right\}, \quad z \in \overline{\mathbb{C}\setminus\Delta_{c,1}},$$

(5.26) where we set $\log \rho_1(x) := \log \mu'_1(x) + \log(2\pi) - \pi i/2$, see (3.2) and recall that $\mu'_1(x)$ is positive on $\Delta_1$. Observe that

$$- \int_{\Delta_{c,1}} \frac{1}{w_{c,1}(x)} \, dx = 1 \quad \text{and} \quad \frac{1}{\pi i} \int_{\Delta_{c,1}} \frac{1}{w_{c,1}(x)} \, dx = - \frac{1}{w_{c,1}(x)},$$

(5.27) by Cauchy’s theorem and integral formula. Hence, $D_{c,\rho_1}(z) = D_{c,\mu'_1}(z)$ is a holomorphic and non-vanishing function in $\overline{\mathbb{C}\setminus\Delta_{c,1}}$ with continuous and conjugate-symmetric traces on $\Delta_{c,1}$ that satisfy

$$\rho_1(x)|D_{c,\rho_1 \pm 1} |^2 = (\rho_1 D_{c,\rho_1} + D_{c,\rho_1} - \rho_1) G_{c,\rho_1} := \exp \left\{ - \int_{\Delta_{c,1}} \frac{\log \rho_1(x)}{w_{c,1}(x)} \, dx \right\},$$

(5.28) according to Pleneli-Sokhotski formulæ. Now, analyticity of $\rho_1(x)$ in a neighborhood of $\Delta_1$ implies that $\max_{x \in \Delta_{c,1}} |\rho_1(x)/\rho_1(\alpha_1) - 1| \to 0$ as $c \to 0$. Combining this estimate with (5.27) yields that

$$- \int_{\Delta_{c,1}} \frac{\log \rho_1(x)}{w_{c,1}(x)} \, dx = \log \rho_1(\alpha_1) - \int_{\Delta_{c,1}} \frac{\log(\rho_1(x)/\rho_1(\alpha_1))}{w_{c,1}(x)} \, dx = \log \rho_1(\alpha_1) + o(1)$$
when \( c \to 0 \) as well as that
\[
\frac{w_{c,1}(z)}{\pi i} \int_{\Delta_{c,1}} \frac{\log \rho_1(x)}{z - x} \, dx = \frac{w_{c,1}(z)}{\pi i} \int_{\Delta_{c,1}} \frac{\log (\rho_1(x)/\rho_1(\alpha_1))}{z - x} \, dx - \log \rho_1(\alpha_1) = o(1) - \log \rho_1(\alpha_1)
\]
uniformly on compact subsets of \( \mathbb{C} \setminus \Delta_{0,1} \) when \( c \to 0 \). Thus, it follows from the maximum modulus principle that
\[
(5.29) \quad D_{c,\rho_1}(z) = 1 + o(1) \quad \text{and} \quad G_{c,\rho_1} = (1 + o(1)) \rho_1(\alpha_1)
\]
locally uniformly in \( \mathbb{C} \setminus \Delta_{0,1} \) as \( c \to 0 \). One can also see from its very definition in (5.28) combined with the second formula of (5.29) that \( G_{c,\rho_1} \) extends to a non-vanishing continuous function of \( c \in [0, 1] \) (it is constant for all \( c \geq c^* \)). This observation as well as (5.28) combined with positivity of \( \rho_1(x) \) on \( \Delta_1 \) show that \( |D_{c,\rho_1}(z)| \sim 1 \) uniformly on \( \Delta_{c,1} \) for all \( c \in (0, 1) \). Then the maximum modulus principle for holomorphic functions applied to \( D_{c,\rho_1}(z) \) and \( D_{c,\rho_1}^{-1}(z) \) yields that
\[
(5.30) \quad G_{c,\rho_1}, |D_{c,\rho_1}(z)| \sim 1,
\]
uniformly in \( \mathbb{C} \) for all \( c \in (0, 1) \) (notice that \( |D_{c,\rho_1}(z)| \) is a continuous function on the entire sphere \( \mathbb{C} \) independent of \( c \) when \( c \geq c^* \)).

Let \( \Gamma_{c,2} := \chi_c(\Delta_{c,2}) \), which are clockwise oriented analytic Jordan curves (recall that \( \Delta_{c,2} \) is oriented so that \( \mathfrak{R}(0) \) remains on the left when \( \Delta_{c,2} \) is traversed in the positive direction and that \( \chi_c(z) \) is conformal on \( \mathfrak{R}c \) and maps \( \mathfrak{R}(0) \) into \( \infty(0) \). The function
\[
(5.31) \quad S_{c,2}(z) := \exp \left\{ \frac{\pi}{2\pi i} \int_{\Gamma_{c,2}} \frac{\log(D_{c,1}D_{c,\rho_1})(\pi(\chi^{-1}(s)))}{\pi \chi_c(z)} ds \right\}
\]
is holomorphic and bounded in \( \mathfrak{R}_c \setminus \Delta_{c,2} \) and has value 1 at \( \infty(0) \). It follows from Plemelj-Sokhotski formulae that
\[
(5.32) \quad S_{c,2,-}(x) = S_{c,2+}(x)(D_{c,1}D_{c,\rho_1})(x), \quad x \in \Delta_{c,2}.
\]
Observe also that \( (D_{c,1}D_{c,\rho_1})(\pi(z)) \) is holomorphic in a neighborhood of \( \Delta_{c,2} \). Therefore, \( S_{c,2}(z) \) can be continued analytically across each side of \( \Delta_{c,2} \). In fact, this continuation has an integral representation similar to (5.31), where one simply needs to homologically deform \( \Gamma_{c,2} \) within the domain of holomorphy of \( (D_{c,1}D_{c,\rho_1})(\pi(\chi_c^{-1}(s))) \). Moreover, it holds that
\[
(5.33) \quad S_{c,2}(z) = 1 + o(1) \quad \text{as} \quad c \to 0 \quad \text{and} \quad |S_{c,2}(z)| \sim 1, \quad c \in (0, c^*],
\]
uniformly on \( \mathfrak{R}_c \) (again, this means including the traces on \( \Delta_{c,2} \)). Indeed, observe that the analytic curves \( \Gamma_{c,2} \) approach the circle \( \{|z - B_{0,2}| = (\beta_2 - \alpha_2)/4\} \) by (2.7) and (5.2). Let \( \delta > 0 \) be small enough so that the integrand in (5.31) is analytic in a neighborhood of the closure of the annular domain bounded by \( \Gamma_{c,2} \) and \( C_{\delta} := \{|z - B_{0,2}| = 2\delta + (\beta_2 - \alpha_2)/4\} \). Assuming that \( C_{\delta} \) is clockwise oriented, it follows from Cauchy’s theorem that \( \Gamma_{c,2} \) can be replaced by \( C_{\delta} \) whenever \( z \in \mathfrak{R}(2) \), i.e., whenever \( \chi_c(z) \) is interior or on \( \Gamma_{c,2} \). Then it trivially holds that
\[
|S_{c,2}(z)| \leq \exp \left\{ \frac{|C_{\delta}|}{2\pi \delta} \max_{z \in C_{\delta}} \log(D_{c,1}D_{c,\rho_1})(\pi(\chi_c^{-1}(s))) \right\},
\]
for \( z \in \mathfrak{R}(2) \), where \( |C_{\delta}| \) is the arclength of \( C_{\delta} \). The desired limit in \( \mathfrak{R}(2) \) now follows from (5.25) and (5.29) while the uniform boundedness follows from (5.24) and (5.30). Clearly, the estimates in the remaining part of \( \mathfrak{R}_c \) can be obtained analogously by deforming \( \Gamma_{c,2} \) into the circles \( \{|z - B_{0,2}| = -2\delta + (\beta_2 - \alpha_2)/4\} \).

As a part of the final piece of our construction, let \( \Gamma_{c,1} := \chi_c(\Delta_{c,1}) \). Similarly to \( \Gamma_{c,2} \), these are clockwise oriented analytic Jordan curves that collapse into a point \( B_{0,1} \) by (2.7) and (5.2). Let
\[
(5.34) \quad S_{c,1}(z) := \exp \left\{ \frac{1}{2\pi i} \int_{\Gamma_{c,1}} \frac{\log \left[S_{\rho_2}(\pi(\chi_c^{-1}(s)))S_{\rho_1}(z)\right]}{s - \chi_c(z)} ds \right\},
\]
which is a holomorphic and bounded function on \( \mathfrak{R}_c \) that has value 1 at \( \infty(1) \) and whose traces on \( \Delta_{c,1} \) are continuous and satisfy
\[
(5.35) \quad S_{c,1-}(x) = S_{c,1+}(x)S_{\rho_2}(x)/S_{\rho_1}(x), \quad x \in \Delta_{c,1},
\]
by Plemelj-Sokhotski formulae. Notice that all the observation about analytic continuations (contour deformation) made for $S_{c,2}(z)$ apply to $S_{c,1}(z)$ as well. Since the Cauchy kernel is integrated against the pullback of a fixed function $S_{p_2}(z)/S_{p_2}(\infty)$ from $\Delta_{c,1}$ while the curves $\Gamma_{c,1}$ collapse into a point, straightforward estimates of Cauchy integrals as well as analytic continuation (deformation of a contour) technique yield that

$$S_{c,1}(z) = 1 + o(1) \quad \text{as} \quad c \to 0 \quad \text{and} \quad |S_{c,1}(z)| \sim 1, \quad c \in (0, e^{**}] ,$$

locally uniformly on $(\mathcal{R}_c^{(0)} \cup \mathcal{R}_c^{(2)}) \setminus \Delta_{c,1}$ and uniformly on $\mathcal{R}_c$, respectively. To examine what happens to $S_{c,1}(z)$ on $\mathcal{R}_c^{(1)}$, given $\epsilon > 0$, let $C_\epsilon := \{ |z - B_{c,1}| = \epsilon \}$ be clockwise oriented circle. It follows from (5.32) that the Jordan curve $\chi_c^{-1}(C_\epsilon)$ belongs to $\mathcal{R}_c^{(0)}$ and is homologous to $\Delta_{c,1}$ for all $c$ sufficiently small. A straightforward computation shows that

$$\int_{C_\epsilon} \log \left[ \frac{S_{p_2} (\pi (\chi_c^{-1}(s)))}{s - B_{c,1}} \right] \frac{ds}{2\pi i} = \log \left[ \frac{S_{p_2} (\alpha_1)}{S_{p_2}(\infty)} \right] + O \left( \max_{z \in \pi (\chi_c^{-1}(C_\epsilon))} \left| \log \frac{S_{p_2}(z)}{S_{p_2}(\alpha_1)} \right| \right).$$

It further follows from (2.7) and (5.2) that Jordan curves $\pi (\chi_c^{-1}(C_\epsilon))$ converge to the analytic Jordan curve $\pi (\chi_0^{-1}(C_\epsilon))$ (recall that $\varphi_2(z) = \varphi_0(z)$) and the latter curves collapse into a point $\alpha_1$ as $c \to 0$. Hence, by taking the limit as $c \to 0$ and then the limit as $c \to 0$ of the $O(\cdot)$ in (5.37) gives 0. Therefore, analytic continuation (deformation of a contour) technique and (5.34) imply that

$$\lim_{c \to 0} S_{c,1}(\infty^{(1)}) = \lim_{c \to 0} \exp \left\{ \frac{1}{2\pi i} \int_{C_\epsilon} \log \left[ \frac{S_{p_2} (\pi (\chi_c^{-1}(s)))}{s - B_{c,1}} \right] \frac{ds}{2\pi i} \right\} = \frac{S_{p_2} (\alpha_1)}{S_{p_2}(\infty)}. $$

Finally, we are ready to state an alternative formula for the functions $S_c(z)$ when $c \leq e^{**}$. Since relations (3.7) characterize $S_c(z)$ up to multiplication by a cubic root of unity, it follows from the normalization of $D_{c,1}(z)$ and $D_{c,0}(z)$ at infinity, the normalization of $S_{c,1}(z)$ and $S_{c,2}(z)$ at $\infty^{(0)}$, and relations (3.4), (5.22), (5.28), (5.32), and (5.35) that

$$\lim_{c \to 0} \frac{S_c(z)}{S_c(\infty^{(0)})} = \left\{ \begin{array}{ll}
\frac{S_{p_2}^{-1}(\infty)(D_{c,1}D_{c,0}, S_{p_2}(z)),}{1 - \frac{\pi^{(0)}}{\varphi_2 + B_{c,1}} G_{c,1}(D_{c,1}D_{c,0})^{-1}(z),}
\frac{(S_{p_2}(\infty)S_{p_2}(z))^{-1},}{z \in \mathcal{R}_c^{(0)} \setminus (\Delta_{c,1} \cup \Delta_{c,2}),}
\end{array} \right.$$

Now, it follows from (5.33) and (5.36) that

$$S_{c}^{(2)}(z)/S_{c}^{(0)}(\infty) = (1 + o(1))(S_{p_2}(\infty)S_{p_2}(z))^{-1}$$

uniformly in $\overline{\mathcal{C}}$ (that is, including the traces on $\Delta_2$) as $c \to 0$. Similarly, it follows from (5.25), (5.29), (5.33), and (5.36) that

$$S_{c}^{(0)}(z)/S_{c}^{(0)}(\infty) = (1 + o(1))S_{p_2}(z)/S_{p_2}(\infty)$$

locally uniformly in $\overline{\mathcal{C}} \setminus \Delta_{0,1}$ as $c \to 0$. Further, it follows from the middle relation in (3.7) and the last two asymptotic formulae that

$$\frac{S_{c}^{(1)}(z)}{S_{c}^{(0)}(\infty)} = \frac{1}{S_{c}^{(0)}(\infty)^2} \frac{S_{c}^{(0)}(\infty)}{S_{c}^{(2)}(z)} = (1 + o(1)) \frac{S_{p_2}(\infty)^2}{S_{p_2}(\infty)^2}$$

locally uniformly in $\overline{\mathcal{C}} \setminus \Delta_{0,1}$ as $c \to 0$. Since relations (5.40)–(5.42) also provide asymptotics for the ratios of $S_{c}^{(b)}(\infty)/S_{c}^{(0)}(\infty)$, the limits in (3.9) easily follow. In fact, we deduce from (5.40) and (5.42) that

$$S_{c}^{(2)}(\infty) = (1 + o(1))S_{p_2}(\infty)$$

and

$$S_{c}^{(1)}(\infty) = (1 + o(1)) \left( \frac{S_{p_2}(\infty)}{S_{p_2}(\infty)} \right)^2 .$$

On the other hand, it follows from the normalization $D_{c,1}(z)$ and $D_{c,0}(z)$ at infinity, (3.2), (4.8), (5.29), (5.33), and (5.38) that

$$\lim_{c \to 0} \frac{1}{c} \frac{S_{c}^{(1)}(\infty)}{S_{c}^{(0)}(\infty)} = \frac{2\pi \mu_1(\alpha_1)|w_2(\alpha_1)|S_{p_2}(\alpha_1)}{S_{p_2}(\infty)} .$$

Plugging in the second asymptotic formula of (5.43) into (5.44) yields the first limit in (3.10). The other two now follow from (5.43).
5.4. Auxiliary Estimates, II. The sole purpose of this subsection is to state the following lemma that follows from (5.24), (5.30), (5.33), (5.36), (5.39), as well as the analogous results for \( c \in [c^*, 1) \) and \( c \to 1 \).

**Lemma 5.4.** It holds uniformly on \( \mathfrak{R}_c \) for all \( c \in (0, 1) \) that
\[
\left| \frac{S_c^{(0)}(\infty)}{S_c^{(2)}(\infty)} \right| \left( 1 - c \right) \left| \frac{S_c^{(0)}(\infty)}{S_c^{(2)}(\infty)} \right| \sim 1.
\]
Moreover, let \( \delta > 0 \) be such that \( 0 < \delta c \leq |\Delta_c, 1|/8 \) and \( 0 < \delta(1 - c) \leq |\Delta_c, 2|/8 \) for all \( c \in (0, 1) \). Then it holds for all \( c \in (0, 1) \) that
\[
\left| \frac{S_c^{(0)}(z)}{S_c^{(0)}(\infty)} \right| \sim \delta^{-1/4}
\]
uniformly on each circle \( \{ |z - \alpha_1| = \delta c \}, \{ |z - \beta_{c, 1}| = \delta c \}, \{ |z - \alpha_{c, 2}| = \delta(1 - c) \}, \) and \( \{ |z - \beta_2| = \delta(1 - c) \} \); and
\[
1 \leq \left| \frac{S_c^{(0)}(z)}{S_c^{(0)}(\infty)} \right| \leq \delta^{-1/4}
\]
uniformly on \( \{ \delta c \leq \text{dist}(z, \{ \alpha_1, \beta_{c, 1} \}) \} \) and \( \{ \delta(1 - c) \leq \text{dist}(z, \{ \alpha_{c, 2}, \beta_2 \}) \} \). In addition, it holds for all \( c \in (0, 1) \) and each \( i \in \{ 1, 2 \} \) that
\[
\left| \frac{S_c^{(i)}(z)}{S_c^{(i)}(\infty)} \right| \sim \delta^{1/4}
\]
uniformly on circles \( \{ |z - \alpha_{c, i}| = \delta(i - 1 - (-1)^i c) \} \) and \( \{ |z - \beta_{c, i}| = \delta(i - 1 - (-1)^i c) \} \); and
\[
\delta^{1/4} \leq \left| \frac{S_c^{(i)}(z)}{S_c^{(i)}(\infty)} \right| \leq 1
\]
uniformly on \( \{ \delta(i - 1 - (-1)^i c) \leq \text{dist}(z, \{ \alpha_{c, i}, \beta_{c, i} \}) \} \).

6. Proof of Theorem 3.1

Let \( \alpha_1 \leq x_{\bar{r}, 1} < x_{\bar{r}, 2} < \ldots < x_{\bar{r}, n_1} \leq \beta_1 \) be the zeros of \( P_{\bar{r}}(x) \) on \( \Delta_1 \). Then we can write
\[
P_{\bar{r}}(x) = P_{\bar{r}, 1}(x)P_{\bar{r}, 2}(x), \quad P_{\bar{r}, 1}(x) := \prod_{i=1}^{n_1} (x - x_{\bar{r}, i}).
\]
Observe that the polynomials \( \{ P_{\bar{r}, 1}(x) \}_{\bar{r} \in \mathcal{N}_0} \) form a normal family in a neighborhood of \( \Delta_2 \). As \( \text{deg}(P_{\bar{r}, 2}) = n_2 \) and it holds that
\[
\int x^l P_{\bar{r}, 2}(x)P_{\bar{r}, 1}(x) \, d\mu_2(x) = 0, \quad l \in \{ 0, \ldots, n_2 - 1 \},
\]
by (1.5), the asymptotics of \( P_{\bar{r}, 2}(z) \) follows from [9, Theorem 2.7]. Namely, it holds that
\[
P_{\bar{r}, 2}(z) = (1 + o(1)) \left( S_{\rho_2}(z)/S_{\rho_2}(\infty) \right) \left( \prod_{i=1}^{n_1} S(z; x_{\bar{r}, i}) \right) \varphi_2^{n_2}(z)
\]
uniformly on compact subsets of \( \mathbb{C} \setminus \Delta_2 \). Thus, to obtain the asymptotic formula for \( P_{\bar{r}}(z) \), we only need to show that all the zeros \( \{ x_{\bar{r}, i} \}_{i=1}^{n_1} \) approach \( \alpha_1 \). We shall do it in a slightly more general setting.

**Lemma 6.1.** Suppose that \( \mu_2 \) is an absolutely continuous Szegő measure, i.e., \( \int_{\Delta_2} \log \mu_2(x) \, dx > -\infty \), and that \( \mathcal{N}_0 \) is any marginal sequence, that is, \( n_1/n_2 \to 0 \) as \( \bar{m} \to \infty \) for \( \bar{n} \in \mathcal{N}_0 \). Assuming formula (6.1) remains valid, it holds that \( x_{\bar{r}, n_1} \to \alpha_1 \) as \( \bar{m} \to \infty \) for \( \bar{n} \in \mathcal{N}_0 \). Moreover,
\[
\lim_{|\bar{m}| \to \infty} \lim_{\bar{n} \in \mathcal{N}_0} \left( \frac{P_{\bar{r}, 2}(z)}{P_{\bar{r}}(z)} - z \right) = -B_{0,i}, \quad i \in \{ 1, 2 \}.
\]

**Proof.** Assume to the contrary that there exists \( \epsilon > 0 \) such that \( \alpha_1 + \epsilon \leq x_{\bar{r}, n_1} \) along some subsequence \( \mathcal{N}' \subset \mathcal{N}_0 \). Let \( \rho_{\bar{r}, 1}(x) := P_{\bar{r}, 1}(x)/(x - x_{\bar{r}, n_1}) \). Then it follows from (1.5) that
\[
\int_{x_{\bar{r}, n_1}}^{x_{\bar{r}, n_1}^2} \rho_{\bar{r}, 1}^2(x)P_{\bar{r}, 2}(x)/(x - x_{\bar{r}, n_1}) \, d\mu_1(x) = \int_{x_{\bar{r}, n_1}}^{x_{\bar{r}, n_1}^2} \rho_{\bar{r}, 2}^2(x)P_{\bar{r}, 2}(x)/(x - x_{\bar{r}, n_1}) \, d\mu_1(x),
\]
and
\[
\lim_{|\bar{m}| \to \infty} \lim_{\bar{n} \in \mathcal{N}_0} \left( \frac{P_{\bar{r}, 2}(z)}{P_{\bar{r}}(z)} - z \right) = -B_{0,i}, \quad i \in \{ 1, 2 \}.
\]
(since all the zeros of $P_{n,2}(x)$ belong to $\Delta_2$, it has a constant sign on $\Delta_1$). As the zeros of the monic polynomial $P_{n,1}(z)$ belong to $\Delta_1$, we have that $|P_{n,1}(x)| \leq |\beta_1 - \alpha_1|^n, x \in \Delta_1$. Moreover, since each $S(z;x_0)$ is a non-vanishing function in $\mathcal{U}(\Delta_2)$, compactness of $\Delta_1$ implies that there exists a constant $C_1 > 1$ such that $C_1^{-1} \leq |S(x;x_0)| \leq C_1$ for any $x, x_0 \in \Delta_1$. Therefore, we can deduce from (6.1) that

$$\int_{x_{\alpha_1}}^{\beta_1} \rho_{\alpha_1}^2(x) |P_{n,2}(x)|(x - x_{\alpha_1}) \, d\mu_1(x) \leq C_2^3 |\varphi_2(\alpha_1 + \epsilon)|^{n_2}$$

for some absolute constant $C_2 > 0$. On the other hand, by restricting the interval of integration from $[\alpha_1, x_{\alpha_1}]$ to $[\alpha_1, \alpha_1 + \epsilon/2]$ and then using (6.1), the lower estimate of the Szegő functions $S(z;x_0)$, the facts that $\mu_1'(x)$ is non-vanishing and $|\varphi_2(x)|$ is decreasing for $x < \alpha_2$ we get that

$$\int_{\alpha_1}^{x_{\alpha_1}} \rho_{\alpha_1}^2(x) |P_{n,2}(x)|(x - x_{\alpha_1}) \, d\mu_1(x) \geq C_3^3 \min_{\alpha_1 \neq \alpha_2} \left( \int_{\alpha_1}^{\alpha_1 + \epsilon/2} L_{n_1-1}^2(x) \, dx \right) |\varphi_2(\alpha_1 + \epsilon/2)|^{n_2} \geq C_4^3 |\varphi_2(\alpha_1 + \epsilon/2)|^{n_2}$$

for some constants $C_3, C_4 > 0$ that might depend on $\epsilon$, but are independent of $\vec{n}$, where $L_n(x)$ is the $n$-th monic orthogonal polynomial with respect to $dx$ on $[\alpha_1, \alpha_1 + \epsilon/2]$ (rescaled Legendre polynomial) and the last estimate follows from [15, Table 18.3.1]. Since $n_1/n_2 \to 0$ and $|\varphi_2(x)|$ is decreasing on $(-\infty, \alpha_2)$, we have that

$$C_4^{n_1/n_2} |\varphi_2(\alpha_1 + \epsilon/2)| > C_2^{n_1/n_2} |\varphi_2(\alpha_1 + \epsilon)|$$

for all $|\vec{n}|$ large, $\vec{n} \in N_0$. Hence, the above estimate shows that (6.4)–(6.5) is incompatible with (6.3). Thus, it indeed holds that $x_{\vec{n}, \alpha_1} \to \alpha_1$ as $|\vec{n}| \to \infty$, $\vec{n} \in N_0$. Further, it holds that

$$\lim_{z \to \infty} \left( \frac{P_{n+\vec{e}_1,1}(z)}{P_{n,1}(z)} - z \right) = -\sum_{i=1}^{n_1+1} x_{\vec{n}+\vec{e}_1,i} + \sum_{i=1}^{n_1} x_{\vec{n},i} = -\alpha_1 + o(1) - \sum_{i=1}^{n_1} (x_{\vec{n}+\vec{e}_1,i+1} - x_{\vec{n},i}).$$

It is known that the zeros of $P_{n}(z)$ and $P_{n+\vec{e}_1}(z)$ interlace, see for example [7, Lemma A.2]. Therefore,

$$0 \leq \sum_{i=1}^{n_1} (x_{\vec{n}+\vec{e}_1,i+1} - x_{\vec{n},i}) \leq x_{\vec{n}+\vec{e}_1,n_1} - x_{\vec{n},1} = o(1),$$

where the last conclusion follows from the fact that $x_{\vec{n},1}, x_{\vec{n}+\vec{e}_1,n_1} \to \alpha_1$ (observe that $\{\vec{n} + \vec{e}_1 : \vec{n} \in N_0\}$ is also a marginal sequence). Thus,

$$\lim_{|\vec{n}| \to \infty} \lim_{z \to \infty} \left( \frac{P_{n+\vec{e}_1,1}(z)}{P_{n,1}(z)} - z \right) = -\alpha_1.$$

Furthermore, it follows from the explicit definition (3.5) that

$$S^2(z;x_0) = \frac{1 - B_{0,2} + O(z^{-2})}{1 - B_{0,2} + O(z^{-2})} \frac{1 - B_{0,2} + O(z^{-2})}{1 - B_{0,2} + O(z^{-2})},$$

where we used (2.6) to get that $\varphi_2(z) = z - B_{0,2} + O(z^{-1})$ as $z \to \infty$. Since

$$B_{0,2} + \varphi_2(x_0) - x_0 - A_{0,2} \varphi_2^{-1}(x_0) = 2(B_{0,2} + \varphi_2(x_0) - x_0),$$

we have that $S(z;x_0) = 1 - (B_{0,2} + \varphi_2(x_0) - x_0)z^{-1} + O(z^{-2})$ as $z \to \infty$. Now, interlacing of the zeros $\{x_{\vec{n}+\vec{e}_1,i}\}_{i=1}^{n_1+1}$ and $\{x_{\vec{n},i}\}_{i=1}^{n_1}$, their convergence to $\alpha_1$, and monotonicity of $\varphi_2(z)$ yield similarly to (6.6) that

$$\lim_{|\vec{n}| \to \infty} \lim_{z \to \infty} \left( \frac{P_{n+\vec{e}_1,1}(z)}{P_{n,1}(z)} \prod_{i=1}^{n_1+1} S(z;x_{\vec{n}+\vec{e}_1,i}) - 1 \right) = -(B_{0,2} + \varphi_2(\alpha_1) - \alpha_1).$$

Hence, it follows from (6.1), (6.6), (6.8), and (2.7) that the limit in (6.2) when $i = 1$ is equal to

$$\lim_{|\vec{n}| \to \infty} \lim_{z \to \infty} \left( \frac{P_{n+\vec{e}_1,1}(z)}{P_{n,1}(z)} \prod_{i=1}^{n_1+1} S(z;x_{\vec{n}+\vec{e}_1,i}) - z \right) = -B_{0,1}.$$
To prove Theorems 3.2–3.4 we use the extension to multiple orthogonal polynomials [20] of by now classical approach of Fokas, Its, and Kitaev [16, 17] connecting orthogonal polynomials to matrix Riemann-Hilbert problems. The RH problem is then analyzed via the non-linear steepest descent method of Deift and Zhou [14].

As was agreed in Section 3.3, we label quantities dependent on \( n \) only by the subindex \( \bar{n} \) as in \( \beta_{\bar{n},1} := \beta_{e,1} \), \( \Delta_{\bar{n},i} := \Delta_{e,1} \), etc. If \( \Delta \) is a closed interval, we denote by \( \Delta^c \) the open interval with the same endpoints. Moreover, when convenient, we write \( \alpha_{\bar{n},1} = \alpha_1 \) and \( \beta_{\bar{n},2} = \beta_2 \) even though they do not depend on the index \( \bar{n} \).

Throughout this section, the reader must keep in mind the definition of constants \( c^* \) and \( c^{**} \) in Proposition 4.1. Moreover, we would like to use the symbol \( c \) as a free parameter from the interval \([0,1]\), as was done in the previous sections. Thus, we slightly modify the notation from the statement of Theorems 3.2–3.4 and assume that we deal with a sequence of multi-indices \( \mathcal{N}_e \) such that

\[
c_R = n_1/|\bar{n}| \to c_0 \in [0,1] \quad \text{and} \quad n_1, n_2 \to \infty \quad \text{as} \quad |\bar{n}| \to \infty, \quad \bar{n} \in \mathcal{N}_e.
\]

We let \( [A]_{i,j} \) to stand for \((i,j)\)-th entry of a matrix \( A \) and \( E_{i,j} \) to be the matrix whose entries are all zero except for \( [E_{i,j}]_{i,j} = 1 \). We set \( I \) to be the identity matrix, \( \sigma_3 := \text{diag}(1,-1) \) to be the third Pauli matrix, and \( \sigma(\bar{n}) := \text{diag}((\bar{n}), -n_1, -n_2) \). Finally, for compactness of notation, we introduce transformations \( T_i, i \in \{1,2\} \), that act on \( 2 \times 2 \) matrices in the following way:

\[
T_1 \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \quad \text{and} \quad T_2 \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} e_{11} & 0 & e_{12} \\ 0 & 1 & 0 \\ e_{21} & 0 & e_{22} \end{pmatrix}.
\]

### 7.1. Initial RH Problem

Let the measures \( \mu_1, \mu_2 \) be as in Theorem 1.2 and the functions \( \rho_1(x), \rho_2(x) \) be given by (3.2). Consider the following Riemann-Hilbert problem (RHP-\( Y \)): find a \( 2 \times 2 \) matrix function \( Y(z) \) such that

\begin{enumerate}
\item \( Y(z) \) is analytic in \( \mathbb{C}\backslash(\Delta_1 \cup \Delta_2) \) and \( \lim_{z \to \infty} Y(z)z^{-\sigma(\bar{n})} = I \);
\item \( Y(z) \) has continuous traces on \( \Delta^c \) that satisfy \( Y_+(x) = Y_-(x)(I + \rho_i(x)E_{1,i+1}) \), \( i \in \{1,2\} \);
\item the entries of the \((i+1,1)\)-st column of \( Y(z) \) behave like \( O(\log|z - \xi|) \) as \( z \to \xi \in [\alpha_i, \beta_i] \), while the remaining entries stay bounded, \( i \in \{1,2\} \).
\end{enumerate}

### 7.2. Opening of the Lenses

Given \( c \in (0,1) \) and \( \delta > 0 \), denote by \( U_{c,\delta,c} \) an open square with vertices \( c \pm \delta e, c \pm i\delta e \) when \( e \in [\alpha_1, \beta_{c,1}] \) and \( e \pm (1-c)\delta, e \pm i(1-c)\delta \) when \( e \in [\alpha_{c,2}, \beta_{2}] \). Define \( \delta_1(c), i \in \{1,2\}, \) via

\[
\delta_1(c) := \begin{cases} \min\{\beta_{c,1} - \alpha_1, \beta_1 - \beta_{c,1}\}, & c < c^*, \\ \min\{\beta_1 - \alpha_1, \alpha_2 - \beta_1\}, & c^* \leq c, \end{cases}
\]

\[
\delta_2(c) := \begin{cases} \min\{\beta_2 - \alpha_2, \alpha_2 - \beta_1\}, & c \leq c^{**}, \\ \min\{\beta_2 - \alpha_{c,2}, \alpha_{c,2} - \alpha_2\}, & c^{**} < c. \end{cases}
\]

Of course, it holds that \( \delta_1(c) \) (resp. \( (1-c)\delta_2(c) \)) is constant for \( c \geq c^* \) (resp. \( c \leq c^{**} \)). Moreover, \( \delta_1(c) \) (resp. \( \delta_2(c) \)) approaches a non-zero constant as \( c \to 0^+ \) (resp. \( c \to 1^- \)) by (4.8) and it approaches 0 as \( c \to c^{**}^- \) (resp. \( c \to c^{*+} \)). Set \( \delta(c) := \min\{\delta_1(c), \delta_2(c)\} \). For brevity, we write

\[
U_c := [U_{c,\delta,c}], \quad \bar{n} \in \mathcal{N}_e, \quad c \in E_{\bar{n}} := E_{c\bar{n}}, \quad \mathcal{E}_c := [\alpha_1, \beta_{c,1}, \alpha_{c,2}, \beta_2],
\]

assuming that \( \delta \in (0, \delta(c_c)) \). In particular, all the domains \( U_c \) are disjoint and \( \beta_1 \notin \overline{U}_{\beta_{c,1}} \) when \( c < c^* \) while \( \alpha_2 \notin \overline{U}_{\alpha_{c,2}} \) when \( c > c^{**} \), again, for all \( |\bar{n}| \) large enough, \( \bar{n} \in \mathcal{N}_e \).
Lemma 7.2. \( \zeta_{\beta_{1,1}} \left( \Gamma_{\beta_{1,1}}^+ \cap U_{\beta_{1,1}} \right) \subset I_\pm := \{ z : \arg(z) = \pm 2\pi/3 \} \), \( \zeta_{\alpha_{1,1}} \left( \Gamma_{\alpha_{1,1}}^+ \cap U_{\alpha_{1,1}} \right) \subset I_\pm \), and \( \Gamma_{\beta_{1,1}}^\pm \) consist of straight line segments outside of \( U_{\alpha_{1,1}} \) and \( U_{\beta_{1,1}} \), see Figure 3. When \( c_* = c^* \), we slightly modify (7.2) and require that

\[
\tilde{\zeta}_{\beta_{1,1}} \left( \Gamma_{\beta_{1,1}}^+ \cap U_{\beta_{1,1}} \right) \subset I_\pm \quad \tilde{\zeta}_{\beta_{1,1}}(z) := \zeta_{\beta_{1,1}}(z) - \zeta_{\beta_{1,1}}(\beta_1),
\]

with an analogous modification holding for \( c_* = c^{**} \) at \( \alpha_{1,2} \). We denote by \( \Omega_{\alpha_{i}}^\pm \) the domains delimited by \( \Gamma_{\beta_{1,1}}^\pm \) and \( \Delta_{\alpha_{i}} \), see Figure 3.

Given \( Y(z) \), the solution of RHP-Y, set

\[
X(z) := Y(z) \begin{pmatrix} T_i & 1/ho_i(z) & 0 \\ -1/ho_i(z) & 0 & 1 \\ \end{pmatrix}, \quad z \in \Omega_{\alpha_{i}}^\pm, \quad i \in \{1, 2\},
\]

otherwise.

It can be readily verified that \( X(z) \) solves the following Riemann-Hilbert problem (RHP-X):

(a) \( X(z) \) is analytic in \( \mathbb{C} \setminus \bigcup_{i=1}^2 \left( \Delta_i \cup \Gamma_{\beta_{1,1}}^+ \cup \Gamma_{\alpha_{i}}^- \right) \) and \( \lim_{z \to \infty} X(z)z^{-\sigma(i)} = I \);

(b) \( X(z) \) has continuous traces on \( \bigcup_{i=1}^2 \left( \Delta_i^+ \cup \Gamma_{\beta_{1,1}}^+ \cup \Gamma_{\alpha_{i}}^- \right) \) that satisfy

\[
X_+(s) = X_-(s) \begin{pmatrix} T_i & -1/ho_i(s) & 0 \\ 1/ho_i(s) & 0 & 1 \\ 0 & 1 & 1 \\ \end{pmatrix}, \quad s \in \Delta_{\beta_{1,1}},
\]

\[
T_i \begin{pmatrix} 0 & \rho_i(s) \\ -1/ho_i(s) & 0 \\ \end{pmatrix}, \quad s \in \Gamma_{\alpha_{i}}^+ \cup \Gamma_{\alpha_{i}}^-.
\]

for each \( i \in \{1, 2\} \);

(c) the entries of the first and \((i + 1)\)-st columns of \( X(z) \) behave like \( \mathcal{O}(\log|z - \xi|) \) as \( z \to \xi \in \{ \alpha_i, \beta_i \} \), while the remaining entries stay bounded, \( i \in \{1, 2\} \).

The following lemma is contained in [37, Lemma 8.1].

**Lemma 7.2.** RHP-X is solvable if and only if RHP-Y is solvable. When solutions of RHP-X and RHP-Y exist, they are unique and connected by (7.4).

7.3. Auxiliary Parametrices. The following Riemann-Hilbert problem (RHP-N) is essentially obtained by discarding the jumps of \( X(z) \) outside of \( \Delta_{\beta_{1,1}} \cup \Delta_{\alpha_{2}} \):

(a) \( N(z) \) is analytic in \( \mathbb{C} \setminus \left( \Delta_{\beta_{1,1}} \cup \Delta_{\alpha_{2}} \right) \) and \( \lim_{z \to \infty} N(z)z^{-\sigma(i)} = I \);

(b) \( N(z) \) has continuous traces on \( \Delta_{\alpha_{i}} \) that satisfy \( N_+(s) = N_-(s)T_i \begin{pmatrix} 0 & \rho_i(s) \\ -1/ho_i(s) & 0 \\ \end{pmatrix} \);

(c) it holds that \( N(z) = \mathcal{O}(|z - e|^{-1/4}) \) as \( z \to e \in \mathbb{R} \).

---

**Figure 3.** The squares \( U_{\alpha_{1,1}}, U_{\beta_{1,1}}, \) and \( U_{\beta_{1}} \), arcs \( \Gamma_{\beta_{1,1}}^\pm \), domains \( \Omega_{\alpha_{i}}^\pm \) (shaded), and the extension domains \( O_{\alpha_{i}} \) (darker shaded regions).
Let \( S_R(z) := S_{c_R}(z) \) be the one granted by Proposition 3.1. Put
\[
S(z) := \text{diag}(S_{R_{1,1}}^{(0)}(z), S_{R_{1,1}}^{(1)}(z), S_{R_{1,1}}^{(2)}(z))
\]
for \( z \in \mathbb{C}(\Delta_{\bar{R}_{1,1}} \cup \Delta_{\bar{R}_{2,1}}) \). Further, let \( \Phi_{R_{1,1}}(z) \), \( w_{R_{1,1}}(z) := w_{c_R,1}(z) \), and \( \Upsilon_{R_{1,1}}(z) := \Upsilon_{c_R,1}(z) \) be the functions given by (3.13), (3.6), and (5.3), respectively. Define
\[
M(z) := S^{-1}(\infty) \begin{pmatrix}
1 & 1/w_{R_{1,1}}(z) & 1/w_{R_{1,1}}(z) \\
\Upsilon_{R_{1,1}}^{(0)}(z) & \Upsilon_{R_{1,1}}^{(1)}(z)/w_{R_{1,1}}(z) & \Upsilon_{R_{1,1}}^{(2)}(z)/w_{R_{2,2}}(z) \\
\Upsilon_{R_{2,2}}^{(0)}(z) & \Upsilon_{R_{2,2}}^{(1)}(z)/w_{R_{1,1}}(z) & \Upsilon_{R_{2,2}}^{(2)}(z)/w_{R_{2,2}}(z)
\end{pmatrix} S(z).
\]

Then RHP-\( N \) is solved by \( N(z) := C(MD)(z) \), see [37, Section 8.2], where \( C \) is a diagonal matrix of constants such that
\[
\lim_{z \to \infty} CD(z)z^{-\sigma(\bar{c}_R)} = I \quad \text{and} \quad D(z) := \text{diag} \left( \Phi_{R_{1,1}}^{(0)}(z), \Phi_{R_{1,1}}^{(1)}(z), \Phi_{R_{1,1}}^{(2)}(z) \right).
\]

Since the jump matrix in RHP-\( N \) (b) has determinant 1, it follows from RHP-\( N \) (a,b) that \( \det(N)(z) \) is holomorphic in \( \mathbb{C} \setminus E_{\bar{c}} \) with at most square root singularities at the points of \( E_{\bar{c}} \). Thus, \( \det(N)(z) \) is a constant and \( \det(N)(z) = 1 \) by RHP-\( N \) (a). Therefore, it holds that \( \det(M)(z) = \det(D)(z) = \det(C) = 1 \) due to the second relation in (3.7) and (3.13). Moreover, it follows from (5.19) and (5.20) that
\[
M^{-1}(z) = S^{-1}(\infty) \begin{pmatrix}
\Pi_{R_{1,1}}^{(0)}(z) & \Pi_{R_{1,1}}^{(0)}(z) & \Pi_{R_{1,1}}^{(2)}(z) \\
\Pi_{R_{1,1}}^{(0)}(z) & \Pi_{R_{1,1}}^{(1)}(z) & \Pi_{R_{1,1}}^{(2)}(z) \\
\Pi_{R_{2,2}}^{(0)}(z) & \Pi_{R_{2,2}}^{(1)}(z) & \Pi_{R_{2,2}}^{(2)}(z)
\end{pmatrix} S(\infty).
\]

We use the following convention: \( |A(z)| \lesssim |B(z)| \) (resp. \( |A(z)| \sim |B(z)| \)) if all the individual entries satisfy \( ||A_{i,j}(z)|| \lesssim ||B_{i,j}(z)|| \) (resp. \( ||A_{i,j}(z)|| \sim ||B_{i,j}(z)|| \)). Moreover, if the constants appearing in inequalities \( \lesssim \) and \( \sim \) do depend on a certain parameter, say \( \delta \), we write \( \lesssim_{\delta} \) and \( \sim_{\delta} \). Furthermore, we shall write \( \tilde{A}(z) = \mathcal{O}(1) \) if all the individual entries satisfy \( ||A_{i,j}(z)|| \lesssim_{\delta} 1 \).

**Lemma 7.3.** It holds that \( M^{\pm 1}(z) = \mathcal{O}_{\delta}(1) \) uniformly for \( z \) such that \( 0 < \delta c_R \leq \text{dist}(z, \{\alpha_1, \beta_{\bar{R},1}\}) \) and \( 0 < \delta(1-c_R) \leq \text{dist}(z, \{\alpha_{\bar{R},2}, \beta_2\}) \), where the estimate is independent of the parameter \( c_R \). Moreover, it holds that \( M^{-1}(z) \) is
\[
\sim \begin{pmatrix}
\delta^{-1/4} & \delta^{-1/4} & 1 - c_R \\
\delta^{-1/4} & \delta^{-1/4} & \delta^{-1/4} \\
(1 - c_R)\delta^{-1/4} & (1 - c_R)\delta^{-1/4} & 1
\end{pmatrix}
\]
uniformly on \( |z - \alpha_1| = \delta c_R, |z - \beta_{\bar{R},1}| = \delta c_R \) and on \( |z - \alpha_{\bar{R},2}| = \delta(1 - c_R), |z - \beta_2| = \delta(1 - c_R) \), respectively, where the constants of proportionality are independent of \( c_R \) and \( \delta \). Finally, it holds that \( M^{-1}(z) \) is equal to
\[
\mathcal{O} \left( \begin{pmatrix}
1 & \delta^{-1/4} & 1 - c_R \\
\delta^{-1/4} & 1 & \delta^{-1/4} \\
(1 - c_R)\delta^{-1/4} & (1 - c_R)\delta^{-1/4} & 1
\end{pmatrix} \right)
\]
uniformly on \( |z - \alpha_1| = \delta c_R, |z - \beta_{\bar{R},1}| = \delta c_R \) and on \( |z - \alpha_{\bar{R},2}| = \delta(1 - c_R), |z - \beta_2| = \delta(1 - c_R) \), respectively, with \( \mathcal{O}(\cdot) \) holding independently of \( c_R \) and \( \delta \).

**Proof.** Consider first \( z \) on one of the circles from the statement of the lemma. It follows from (3.10) and Lemma 5.4 that
\[
S(\infty) \sim \text{diag} \left( c_R^{-1/3}(1 - c_R)^{-1/3}, c_\bar{R}^{2/3}(1 - c_\bar{R})^{-1/3}, c_\bar{R}^{-1/3}(1 - c_\bar{R})^{2/3} \right),
\]
where the constants of proportionality are independent of \( c_R \). It further follows from Lemma 5.4 that
\[
|S(z)| \sim S(\infty)\text{diag} \left( \delta^{-1/4}, \delta^{-1/4}, 1 \right)
\]
and
\[
|S(z)| \sim S(\infty)\text{diag} \left( \delta^{-1/4}, 1, \delta^{-1/4} \right)
\]
uniformly on \( |z - \alpha_1| = \delta c_R, |z - \beta_{\bar{R},1}| = \delta c_R \) and on \( |z - \alpha_{\bar{R},2}| = \delta(1 - c_R), |z - \beta_2| = \delta(1 - c_R) \), respectively, where the constants of proportionality are independent of \( c_R \) and \( \delta \). Moreover, we deduce
from Lemma 5.2 and (5.23) that $S(x)(MS^{-1})(z)$ is
\[
\sim \begin{pmatrix}
1 & c^{-1} \delta^{-1/2} \\
\delta^{-1/2} & c^2 \\
(1-\delta^{-1/2}) & c^{-1} \delta^{-1/2} & 1
\end{pmatrix}
\quad \text{and} \quad
\sim \begin{pmatrix}
1 & 1 & (1-\delta^{-1/2})^{-1} \\
\delta^{-1/2} & c^{-1} \delta^{-1/2} & 1 \\
1-\delta^{-1/2} & (1-\delta^{-1/2})^2 \\
\end{pmatrix}
\]
uniformly on $|z-\alpha_1| = \delta c_\text{R}, |z-\alpha_2| = \delta c_\text{R}$ and on $|z-\alpha_2| = \delta(1-\delta), |z-\beta_2| = \delta(1-\delta)$, respectively, where the constants of proportionality are independent of $c_\text{R}$ and $\delta$. The combination of the above three estimates yields the desired asymptotics of $M(z)$ on the circles around $\alpha_1, \beta_1, \alpha_2, \beta_2$.

It further follows from Lemma 5.4 that
\[
|S_{\pm}(x)| \lesssim S(x) \text{diag}(\delta^{-1/4}, 1, 1)
\]
uniformly for $x \in (\alpha_1 + \delta c_\text{R}, \beta_1 - \delta c_\text{R}) \cup (\alpha_2 + \delta(1-\delta), \beta_2 - \delta(1-\delta))$ where the constants of proportionality are independent of $c_\text{R}$ and $\delta$. Analogously, it follows from Lemma 5.2 and (5.23) that the above estimate of $S(x)(MS^{-1})(z)$ on the circles remains valid as an upper estimate on $(\alpha_1 + \delta c_\text{R}, \beta_1 - \delta c_\text{R}) \cup (\alpha_2 + \delta(1-\delta), \beta_2 - \delta(1-\delta))$. The last two observations and the maximum modulus principle for holomorphic functions show that $M(z) = \mathcal{O}(1)$ uniformly for $z$ such that $0 < \delta c_\text{R} \leq \text{dist}(z, \{\alpha_1, \beta_1\})$ and $0 < \delta(1-\delta) \leq \text{dist}(z, \{\alpha_2, \beta_2\})$, where the estimate is independent of the parameter $c_\text{R}$.

Finally, as $\det(M)(z) \equiv 1$, the estimates of $M^{-1}(z)$ follow in a straightforward fashion from the ones for $M(z)$. 

Besides $N(x)$, we shall also need matrix functions that solve RHP-X within the domains $U_e$, introduced at the beginning of Section 7.2, with an additional matching condition on the boundary. More precisely, let $\tilde{\epsilon_0}$ be given by (3.11). For each $e \in \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ we are seeking a solution of the following RHP-$P_e$:

(a,b,c) $P_e(z)$ satisfies RHP-X(a,b,c) within $U_e$;

(d) $P_e(s) = M(s)(I + o(1))D(s)$ uniformly on $\partial U_e \backslash \bigcup_{i=\bar{1}}^2 (\Delta_i \cup \Gamma_{\bar{1},i} \cup \Gamma_{\bar{1},i})$, where
\[
\|o(1)\|_{[j,k]} \leq C_{\tilde{e}_0} \begin{cases}
\delta^{-1/2}, & e = \alpha_1, \\
\delta^{-3/2}, & e = \beta_1 \text{ when } c_* < c^*, \\
(\delta(z_* - \beta_1))^{-1/2}, & e = \beta_1 \text{ when } c_* > c^*,
\end{cases}
\]
for some constant $C > 0$ independent of $n$ and $\delta$, and analogous estimates hold around $\alpha_2, \beta_2$ (in the cases $c_* = c^*$ and $c_* = c^*$ we cannot specify the exact rate of the error term), where the point $z_*$, or more precisely $z_\epsilon_0$ was defined in Proposition 4.2.

We will solve RHP-$P_e$ only for $e \in \{\alpha_1, \beta_1\}$ understanding that the solutions for $e \in \{\alpha_2, \beta_2\}$ can be constructed similarly. Solution of each RHP-$P_e$ will require a construction, carried out in the next subsection, of a local conformal map around $\alpha_1$ and $\beta_1$. Recall that these maps were already used in (7.2).

7.4. Conformal Maps. In this subsection we construct local conformal maps needed to solve problems RHP-$P_e$. To this end, recall the definition, given right after (4.1), and properties, described in Proposition 4.2, of a function $h_\epsilon(z)$ that is rational on the surface $\mathcal{R}_\epsilon$.

4.1. Local maps around $\alpha_1$. Given $e \in \{0, 1\}$, define
\[
\zeta_{c,\alpha_1}(z) := \left(\frac{1}{4} \int_{\alpha_1}^{x} \left(h_\epsilon^{(0)}(s) - h_\epsilon^{(1)}(s)\right) ds\right)^2, \quad \Re z < \beta_{c,1}.
\]
Since $h_\epsilon^{(0)}(x) = h_\epsilon^{(1)}(x)$ on $\Delta_{c,1}$, the function $\zeta_{c,\alpha_1}(z)$ is holomorphic in the region of definition. When $\omega$ is a real measure on the real line, it trivially holds that
\[
\int_{\alpha_1}^{x} \frac{d\omega(x)}{x - (x_0 \pm iy)} = \int_{\alpha_1}^{x} \frac{d\omega(x)}{(x - x_0)^2 + y^2} = \pm iy \int_{\alpha_1}^{x} \frac{d\omega(x)}{(x - x_0)^2 + y^2}.
\]
Therefore, if the traces of $(x - z)^{-1}d\omega(x)$ exist at $x_0$, they are necessarily conjugate-symmetric. In particular, it follows from (4.2) that the integrand in (7.8) is purely imaginary on $\Delta_{c,1}$ and therefore $\zeta_{c,\alpha_1}(x) < 0$ for $x \in \Delta_{c,1}$. It also clearly follows from (4.2) that $\zeta_{c,\alpha_1}(x) > 0$ for $x < \alpha_1$. Moreover, since $h_\epsilon(z)$ has a pole at $\alpha_1$, a ramification point of $\mathcal{R}_\epsilon$ of order 2, $\zeta_{c,\alpha_1}(z)$ has a simple zero at $\alpha_1$. 
Lemma 7.4. There exist $\delta_{\alpha_1} > 0$, $A_{\alpha_1} > 0$, and $D_{\alpha_1} > 0$, independent of $c$, such that each $\zeta_{c,\alpha_1}(z)$ is conformal in $\{|z-\alpha_1| < \delta_{\alpha_1}, c\}$, $4A_{\alpha_1}c \leq |\zeta'_{c,\alpha_1}(\alpha_1)|$, and $|\zeta'_{c,\alpha_1}(z)| \leq D_{\alpha_1}c$ when $|z-\alpha_1| < \delta_{\alpha_1}c$ for all $c \in (0,1)$.

Proof. We start by proving the estimate on the size of $|\zeta'_{c,\alpha_1}(\alpha_1)|$. Assume first that $c \leq c^*$. Since $\alpha_1$ is a simple pole of $h_c(z)$ and $h_c^{(0)}(x) < 0$ for $x < \alpha_1$ by (4.2), it holds that $h_c^{(0)}(x) = u_c(\alpha_1-x)^{-1/2} + O(1)$ for $x < \alpha_1$ and sufficiently close to $\alpha_1$, where the branch of the square root is principal and $u_c < 0$. Since $h_c^{(1)}(x) = -u_c(\alpha_1-x)^{-1/2} + O(1)$ around $\alpha_1$, it can be readily checked that $\zeta'_{c,\alpha_1}(\alpha_1) = -u_c^2$. It was shown in [6, Equation 2.7] that $h_c(z)$ solves

\begin{equation}
(7.9) \quad h^3 - (1-c + c^2) \frac{z - d_c}{\Pi(z)} h - \frac{c - c^2}{\Pi(z)} = 0,
\end{equation}

where $\Pi(z) := (z - \alpha_1)(z - \alpha_2)(z - \beta_2)$ and $d_c$ is the point such that the discriminant of (7.9), whose numerator is a cubic polynomial, vanishes at $\beta_{c,1}$ and has an additional double zero. By plugging the identity $h_c^{(0)}(x) = u_c(\alpha_1-x)^{-1/2} + O(1)$ into (7.9), it is easy to verify that

\begin{equation}
(7.10) \quad u_c^2 = (1-c + c^2) \frac{d_c - \alpha_1}{(\alpha_2 - \alpha_1)(\beta_2 - \alpha_1)}.
\end{equation}

The numerator of the discriminant of (7.9) is equal to

\begin{equation}
(7.11) \quad 4(1-c + c^2)^3 (z - d_c)^3 - 27(c - c^2)^2 (z - \alpha_1)(z - \alpha_2)(z - \beta_2)
\end{equation}

and must have a single sign change, which happens at $\beta_{c,1}$. If $d_c \leq \alpha_1$ were true, then the discriminant would have been positive at $\alpha_2, \beta_2$ and non-negative at $\alpha_1$, that is, it would have been positive on $(\alpha_1, \beta_2)$, which contradicts vanishing at $\beta_{c,1}$. On the other hand if, $d_c \geq \beta_{c,1}$ were to be true, then the discriminant would have been strictly negative at $\beta_{c,1}$, which, again, leads to a contradiction. Thus, $\alpha_1 < d_c < \beta_{c,1}$. Now, (7.9) yields that

\begin{equation}
(7.12) \quad \frac{d_c - \alpha_1}{c} = \frac{1-c}{(\alpha_2 - d_c)(\beta_2 - d_c)} h_c^2(d_c) \geq \frac{(1-c^*)(\alpha_2 - \beta_1)^3}{(\alpha_2 - \alpha_1)(\beta_2 - \alpha_1)},
\end{equation}

where we used (4.2) to observe that $h_c^{(2)}(d_c) \leq 1/(\alpha_2 - \beta_1)$. The above inequality and (7.10) clearly yield the desired estimate for $|\zeta'_{c,\alpha_1}(\alpha_1)| = u_c^2$ when $c \leq c^*$. In fact, when $c \to 0$, it actually follows from the first equality in (7.12) that

\begin{equation}
(7.13) \quad \frac{c}{d_c - \alpha_1} = \frac{(\alpha_2 - d_c)(\beta_2 - d_c)}{1-c} h_c^{(2)}(d_c) (h_c^{(2)}(d_c))^3 \to (\alpha_2 - \alpha_1)(\beta_2 - \alpha_1) \left( \int \frac{d\omega_2(x)}{x - \alpha_1} \right)^3 = \frac{1}{|w_2(\alpha_1)|}
\end{equation}

due to (4.2), the last conclusion of Proposition 4.1, and the formula before (4.8). In this case, (7.10) and (7.13) yield that

\begin{equation}
(7.14) \quad \zeta'_{c,\alpha_1}(\alpha_1) = -u_c^2 = -c + o(1) \quad \text{as} \quad c \to 0.
\end{equation}

When $c \in [c^*, c^{**}]$ the surface $\mathfrak{X}_c$ is always the same. Hence, one can argue using local coordinates that the pull-backs of $h_c(z)$ from a fixed circular neighborhood of $\alpha_1$ to a fixed neighborhood in $\mathbb{C}$ continuously depend on $c$. Since each $|\zeta'_{c,\alpha_1}(\alpha_1)| > 0$ for $c \in (0,1)$, the desired estimate follows from compactness of $[c^*, c^{**}]$. When $c^{**} \approx c$, $h_c(z)$ satisfies an equation similar to (7.9). Using this equation, we again can argue that the estimate holds as $c \to 1$, thus, proving that it holds uniformly for all $c \in (0,1)$.

It remains to study conformality of $\zeta_{c,\alpha_1}(z)$. Denote by $\delta_{\alpha_1}(c)$ the supremum of $\delta$ such that $\zeta_{c,\alpha_1}(z)$ is conformal in $\{|z-\alpha_1| < 2\delta c\}$. We take $\delta_{\alpha_1} := \inf_{c \in (0,1)} \delta_{\alpha_1}(c)$. Since $\delta_{\alpha_1}(c) > 0$ for $c \in (0,1)$ and continuously depends on $c$, we only need to study what happens as $c \to 0$ and $c \to 1$ to prove that $\delta_{\alpha_1} > 0$. Assume first that $c \to 0$. Set $\tilde{\zeta}_{c,\alpha_1}(s) := c^{-2} \zeta_{c,\alpha_1}(z(s))$, where $z(s) := \alpha_1 + |\Delta_{c,1}|(1-s)/2$. Then it follows from (7.8) that

\begin{equation}
\tilde{\zeta}_{c,\alpha_1}(s) = \left( \frac{|\Delta_{c,1}|}{8c} \int_1^s (h_c^{(0)} - h_c^{(1)}) (t) dt \right)^2,
\end{equation}

where $h_c^{(0)} - h_c^{(1)}$ has two consecutive roots in $[1,s]$; these roots are $\alpha_1 + |\Delta_{c,1}|(1-s)/2$ and $\alpha_1 + |\Delta_{c,1}|(1-s)/2 + \delta_{\alpha_1}(c).$
4.2.2 Proof. 

\hat{\phi} \text{ is real in the gap between } J \text{ and } \frac{1}{2} J + \alpha \text{, so do the constants } \hat{\phi} \text{. Thus, we only need to check their limits as } c \to 0 \text{ and } c \to 1. \text{ The finiteness of } D_{\phi} \text{ easily follows from (}4.8\text{), (}7.16\text{), and (}7.17\text{).}

4.2.1 \text{ Let } D_{\phi}(c) := c^{-1} \max_{|z - \alpha| < \delta_{\phi}} |\zeta_{\phi}(z)|. \text{ These constants are finite for each } c \in (0,1) \text{, and the function } D_{\phi}(c) \text{ is analytic in } \{ |z - \alpha| < 2 \delta_{\phi} \}. \text{ Moreover, } \zeta_{\phi}(z) \text{ continuously depends on } c \text{, so do the constants } D_{\phi}(c). \text{ Thus, we only need to check their limits as } c \to 0 \text{ and } c \to 1. \text{ The finiteness of } D_{\phi} := \sup_{c \in (0,1)} D_{\phi}(c) \text{ now easily follows from (}4.8\text{), (}7.16\text{), and (}7.17\text{).} \square

4.4.2 Local maps around } \beta_{c,1} \text{ when } c \in (0,c^\ast). \text{ Given } c \in (0,c^\ast], \text{ define}

\zeta_{\beta_{c,1}}(z) := \left( \frac{3}{4} \int_{0}^{z} \left( h^{(0)}(x) - h^{(1)}(x) \right) dx \right)^{2/3}, \quad \alpha_1 < \Re z < \alpha_2,

where the choice of the root function can be made such that } \zeta_{\beta_{c,1}}(z) \text{ is holomorphic with a simple zero at } \beta_{c,1} \text{ and is positive for } x > \beta_{c,1}. \text{ Indeed, since } h_c(z) \text{ is bounded at } \beta_{c,1} \text{, which is a ramification point of order 2, we can write}

\begin{equation}
\frac{d}{dx} \left( \frac{1}{2} \sqrt{x - \alpha_1} - \frac{1}{2} \sqrt{x - \alpha_2} \right)^2 = \frac{1}{4} \left( \log \left( \frac{\beta_1 + \alpha_1}{2} - z - \sqrt{(z - \alpha_1)(z - \beta_1)} \right) - \log \left( \frac{\beta_1 - \alpha_1}{2} \right) \right)^2,
\end{equation}

which allows us to conclude that } \lim_{c \to 1} D_{\phi}(c) > 0 \text{ as desired.}

Finally, let } D_{\phi}(c) := c^{-1} \max_{|z - \alpha| < \delta_{\phi}} |\zeta_{\phi}(z)|. \text{ These constants are finite for each } c \in (0,1) \text{, and the function } D_{\phi}(c) \text{ is analytic in } \{ |z - \alpha| < 2 \delta_{\phi} \}. \text{ Moreover, } \zeta_{\phi}(z) \text{ continuously depends on } c \text{, so do the constants } D_{\phi}(c). \text{ Thus, we only need to check their limits as } c \to 0 \text{ and } c \to 1. \text{ The finiteness of } D_{\phi} := \sup_{c \in (0,1)} D_{\phi}(c) \text{ now easily follows from (}4.8\text{), (}7.16\text{), and (}7.17\text{).} \square

4.4.2 Local maps around } \beta_{c,1} \text{ when } c \in (0,c^\ast). \text{ Given } c \in (0,c^\ast], \text{ define}

\begin{equation}
\zeta_{\beta_{c,1}}(z) := \left( \frac{3}{4} \int_{0}^{z} \left( h^{(0)}(x) - h^{(1)}(x) \right) dx \right)^{2/3}, \quad \alpha_1 < \Re z < \alpha_2,
\end{equation}

where the choice of the root function can be made such that } \zeta_{\beta_{c,1}}(z) \text{ is holomorphic with a simple zero at } \beta_{c,1} \text{ and is positive for } x > \beta_{c,1}. \text{ Indeed, since } h_c(z) \text{ is bounded at } \beta_{c,1} \text{, which is a ramification point of order 2, we can write}

\begin{equation}
\frac{d}{dx} \left( \frac{1}{2} \sqrt{x - \alpha_1} - \frac{1}{2} \sqrt{x - \alpha_2} \right)^2 = \frac{1}{4} \left( \log \left( \frac{\beta_1 + \alpha_1}{2} - z - \sqrt{(z - \alpha_1)(z - \beta_1)} \right) - \log \left( \frac{\beta_1 - \alpha_1}{2} \right) \right)^2,
\end{equation}

which allows us to conclude that } \lim_{c \to 1} D_{\phi}(c) > 0 \text{ as desired.}

Finally, let } D_{\phi}(c) := c^{-1} \max_{|z - \alpha| < \delta_{\phi}} |\zeta_{\phi}(z)|. \text{ These constants are finite for each } c \in (0,1) \text{, and the function } D_{\phi}(c) \text{ is analytic in } \{ |z - \alpha| < 2 \delta_{\phi} \}. \text{ Moreover, } \zeta_{\phi}(z) \text{ continuously depends on } c \text{, so do the constants } D_{\phi}(c). \text{ Thus, we only need to check their limits as } c \to 0 \text{ and } c \to 1. \text{ The finiteness of } D_{\phi} := \sup_{c \in (0,1)} D_{\phi}(c) \text{ now easily follows from (}4.8\text{), (}7.16\text{), and (}7.17\text{).} \square

4.4.2 Local maps around } \beta_{c,1} \text{ when } c \in (0,c^\ast). \text{ Given } c \in (0,c^\ast], \text{ define}

\begin{equation}
\zeta_{\beta_{c,1}}(z) := \left( \frac{3}{4} \int_{0}^{z} \left( h^{(0)}(x) - h^{(1)}(x) \right) dx \right)^{2/3}, \quad \alpha_1 < \Re z < \alpha_2,
\end{equation}

where the choice of the root function can be made such that } \zeta_{\beta_{c,1}}(z) \text{ is holomorphic with a simple zero at } \beta_{c,1} \text{ and is positive for } x > \beta_{c,1}. \text{ Indeed, since } h_c(z) \text{ is bounded at } \beta_{c,1} \text{, which is a ramification point of order 2, we can write}

\begin{equation}
\frac{d}{dx} \left( \frac{1}{2} \sqrt{x - \alpha_1} - \frac{1}{2} \sqrt{x - \alpha_2} \right)^2 = \frac{1}{4} \left( \log \left( \frac{\beta_1 + \alpha_1}{2} - z - \sqrt{(z - \alpha_1)(z - \beta_1)} \right) - \log \left( \frac{\beta_1 - \alpha_1}{2} \right) \right)^2,
\end{equation}

which allows us to conclude that } \lim_{c \to 1} D_{\phi}(c) > 0 \text{ as desired.}

Finally, let } D_{\phi}(c) := c^{-1} \max_{|z - \alpha| < \delta_{\phi}} |\zeta_{\phi}(z)|. \text{ These constants are finite for each } c \in (0,1) \text{, and the function } D_{\phi}(c) \text{ is analytic in } \{ |z - \alpha| < 2 \delta_{\phi} \}. \text{ Moreover, } \zeta_{\phi}(z) \text{ continuously depends on } c \text{, so do the constants } D_{\phi}(c). \text{ Thus, we only need to check their limits as } c \to 0 \text{ and } c \to 1. \text{ The finiteness of } D_{\phi} := \sup_{c \in (0,1)} D_{\phi}(c) \text{ now easily follows from (}4.8\text{), (}7.16\text{), and (}7.17\text{).} \square

\begin{lem}
There exist } \delta_{\beta_1} > 0 \text{ and } A_{\beta_1} > 0 \text{, independent of } c \in (0,c^\ast), \text{ such that each } \zeta_{\beta_{c,1}}(z) \text{ is conformal in } \{ |z - \beta_{c,1}| < \delta_{\beta_1} \} \text{ and } A_{\beta_1} c^{-1/3} \leq \zeta'_{\beta_{c,1}}(\beta_{c,1}) \text{ for all } c \in (0,c^\ast].
\end{lem}

\textbf{Proof.} \text{ Since } \zeta'_{\beta_{c,1}}(\beta_{c,1}) \neq 0 \text{ for } c \in (0,c^\ast], \text{ to prove the second claim, we only need to consider what happens as } c \to 0. \text{ Similarly to considerations preceding (}7.15\text{), let } h_c(s) := h_c(\beta_{c,1} + |\Delta_{c,1}|(s - 1/2)\text{.}
Lemma 7.6. To prove the first one, it is enough to observe that Proposition 4.1, we can write the ray sequences as
c
Observe that

There exist \( \delta, \epsilon \)
formally given by

to powers of

Proof. (we can adjust the constant

Since the functions

This construction will be used only for
the ray sequences \( \mathcal{N}_{c,\epsilon} \) with infinitely many indices \( \bar{u} \) such that \( c_{\bar{u}} > c^* \). By Proposition 4.2, \( h_{c^*}(z) \) is bounded at \( \beta_1 \) while \( h_c(z) \) has a simple pole at \( \beta_1 \) for all \( c > c^* \) and a simple zero \( x_c \) that approaches \( \beta_1 \) as \( c \to c^* \). Since the functions \( h_c(z) \) converge around \( \beta_1 \) to \( h_{c^*}(z) \) as \( c \to c^* \) by (4.2) and Proposition 4.1, we can write

4.3.1

for some \( \epsilon_c > 0 \) such that \( \epsilon_c \to 0^+ \) as \( c \to c^* \), where \( f_c(z) \) is a holomorphic function that is real on \((\alpha_1, \alpha_2)\) (observe that the Puiseux expansion of \((h_c(0) - h_c(1))(x) \) around \( \beta_{c,1} \) does not have the integral powers of \( x - \beta_1 \)). Similarly, it holds that \( \zeta_{c,c_{\beta_1}^*}^{3/2}(z) = (z - \beta_1)^{3/2} f_{c^*}(z) \) for some holomorphic function \( f_{c^*}(z) \) that is real on \((\alpha_1, \alpha_2)\) and is positive at \( \beta_1 \). Since the right-hand side of (7.22) converges to \( \zeta_{c,c_{\beta_1}^*}^{3/2}(z) \) as \( c \to c^* \), the functions \( f_c(z) \) converge to \( f_{c^*}(z) \) (in particular, \( f_c(\beta_1) > 0 \) for all \( c \) sufficiently close to \( c^* \)).

|lem:4.3 | Lemma 7.6. There exist \( c' > c^* \) and a fixed neighborhood of \( \beta_1 \) such that for every \( c \in (c^*, c'] \) there exists a function \( \zeta_{c,c_{\beta_1}}(z) \) conformal in this neighborhood, such that

4.3.4

(we can adjust the constant \( \delta_c > 0 \) from Lemma 7.5 so that the neighborhood of conformality is given by \( \{ |z - \beta_1| < \delta_{c,c'} \} \). Moreover, \( \zeta_{c,c_{\beta_1}}(z) \) is positive for \( x > \beta_1 \) and converges to \( \zeta_{c^* , c_{\beta_1}^*} \) as \( c \to c^* \).

Proof. Let \( F(z;\epsilon) \) be a family of holomorphic and non-vanishing functions in \( \{|z| < r_0\} \) that are positive at the origin and continuously depend on the parameter \( \epsilon \in [0, \epsilon_0] \). Consider the equation

4.3.2

where \( p_c > 0 \) is a parameter that we shall fix in a moment. The solution of this cubic equation is formally given by

\[
\begin{align*}
\left\{ \begin{array}{l}
u(z;\epsilon) = 2p_c + p_c^3 \epsilon + \epsilon^2 v^{-1/3}(z;\epsilon), \\
u(z;\epsilon) = g(z;\epsilon) - p_c^3 + \sqrt{g(z;\epsilon)}(g(z;\epsilon) - 2p_c^3). \\
\end{array} \right.
\end{align*}
\]

Observe that \( g'(x;\epsilon) = (x - \epsilon)[(3x - \epsilon) F(x;\epsilon) + x(x - \epsilon) F'(x;\epsilon)] \). The expression in the square brackets is negative at 0 and positive at \( \epsilon \). Since \( F(0;\epsilon) > \delta > 0 \), independently of \( \epsilon \in [0, \epsilon_0] \) for some \( \delta, \epsilon_0 > 0 \) sufficiently small, the derivative of the expression in the square brackets, that is, \( 3F(x;\epsilon) + (5x - \epsilon) F'(x;\epsilon) + x(x - \epsilon) F''(x;\epsilon) \), is positive on \([0, \epsilon] \) for all \( \epsilon \in [0, \epsilon_0] \), where we might...
Lemma 7.7. Let $\delta > 0$ be as in (4.1) and $\delta_{\beta_1}$ as in Lemma 7.5. There exists $\delta_{\beta_1} \in (0, \delta_{\beta_1})$ such that given $c \in (0, c^*)$ and $\delta \in (0, \delta_{\beta_1})$, it holds that
\begin{align*}
\left| \frac{H_c^{(0)}(x) - H_c^{(1)}(x)}{(x + iy)} \right| &\leq -B_{\delta_1} \delta^{3/2} c, \\
&\quad \times \in [\beta_{c,1} + \delta c, \alpha_2 - \delta c], \quad y \in [-\delta c/2, \delta c/2],
\end{align*}
where $B_{\delta_1} > 0$ is a constant independent of $c$ and $\delta$. Moreover, for any fixed $\delta > 0$ small enough there exists $c_3 > 0$ and $\epsilon > 0$ such that
\begin{align*}
\left| \frac{H_c^{(0)}(x) - H_c^{(1)}(x)}{(x + iy)} \right| &\leq -\epsilon.
\end{align*}
for all $c \in (0, c_5)$, $x \in [\alpha_1 + \delta, \alpha_2 - \delta]$, and $y \in [-\delta/2, \delta/2]$. Finally, for any $c \in (0, 1)$, it holds that

$$(7.29) \quad (H^{(0)}_c - H^{(1)}_c)(x + i\delta c) \geq B_{\beta_1} \delta^{3/2} c, \quad x \in [\alpha_1, \beta_{c,1}].$$

**Proof.** Since $h_c(z) = 2\partial_z h_c(z)$ and $\beta_{c,1}$ is a ramification point of $\mathfrak{R}_c$ belonging to both $\mathfrak{R}^{(0)}_c$ and $\mathfrak{R}^{(1)}_c$, it holds that

$$(4.5.3a) \quad (H^{(0)}_c - H^{(1)}_c)(z) = \Re \left( \int_{\beta_{c,1}}^z (h^{(0)}_c - h^{(1)}_c)(s) ds \right), \quad \alpha_1 < \Re z < \alpha_2.$$  

It further follows from (4.2) that

$$\partial_z \Re \left( h^{(0)}_c - h^{(1)}_c \right)(x + iy) = \int \frac{y^2 - (t - x)^2}{((t - x)^2 + y^2)^2} d(2\omega_{c,1} + \omega_{c,2})(t) < 0$$
when $|y| < \delta c \leq \text{dist}(x, \Delta_{c,1} \cup \Delta_{c,2})$. Therefore, it holds that

$$\left( H^{(0)}_c - H^{(1)}_c \right)(x + iy) \leq \left( H^{(0)}_c - H^{(1)}_c \right) (\beta_{c,1} + \delta c + iy)$$
for all $x \in [\beta_{c,1} + \delta c, \alpha_2 - \delta c]$ and $y \in [-\delta c/2, \delta c/2]$. Now, by combining (7.18) and (7.30) we get that

$$(3.1) \quad (H^{(0)}_c - H^{(1)}_c)(z) = -\frac{4}{3} \Re \left( \zeta^{3/2}_{\beta_{c,1}}(z) \right), \quad \alpha_1 < \Re z < \alpha_2,$$
for all $c \in (0, c^*)$. Take $\delta_{\beta_1} \leq \sin(\pi/6)\delta_{\beta_1}$. Since each map $\zeta_{\beta_{c,1}}(z)$ is conformal in $|z - \beta_{c,1}| < \delta_{\beta_1} c$ and $\delta < \sin(\pi/6)\delta_{\beta_1}$, every point $\beta_{c,1} + \delta c + iy$ lies within a disk of conformality of $\zeta_{\beta_{c,1}}(z)$ when $|y| < \delta c/2$. Since $\text{Arg}(\delta c + iy) \in [-\pi/6, \pi/6]$ when $|y| < \delta c/2$ and $\zeta_{\beta_{c,1}}(x)$ is positive for $x > \beta_{c,1}$, it is negative for $x < \beta_{c,1}$ and has a positive derivative at $\beta_{c,1}$, there exists $\delta_c > 0$ such that

$$\Re \left( \zeta^{3/2}_{\beta_{c,1}}(\beta_{c,1} + \delta c + iy) \right) \geq \frac{1}{2} \left| \zeta^{3/2}_{\beta_{c,1}}(\beta_{c,1} + \delta c + iy) \right|$$
for all $|y| < \delta c/2$ and $\delta < \delta_c$. Since the maps $\zeta_{\beta_{c,1}}(z)$ continuously depend on $c$ and have a rescaled conformal limit as $c \to 0$, see (7.21), the constants $\delta_c$ can be chosen so that $\delta_c \geq \delta_{\beta_1} > 0$ for all $c \in (0, c^*)$ and some $\delta_{\beta_1} > 0$. Thus,

$$\left( H^{(0)}_c - H^{(1)}_c \right)(x + iy) \leq \frac{2}{3} \left| \zeta^{3/2}_{\beta_{c,1}}(\beta_{c,1} + \delta c + iy) \right| \leq -B_{\beta_1} \delta^{3/2} c$$
for $x \in [\beta_{c,1} + \delta c, \alpha_2 - \delta c]$, $y \in [-\delta c/2, \delta c/2]$, and a constant $B_{\beta_1} > 0$ independent of $c$ by Lemma 7.5 and (5.1), which finishes the proof of (7.27).

Estimate (7.28) follows in straightforward fashion from the observation that the left-hand side of (7.28) converges to $V^{\omega}(\alpha_1) - V^{\omega}(x + iy)$ as $c \to 0$ uniformly on the considered set by Proposition 4.1 and (4.1), where $\omega_2$ is the arcwise distribution on $\Delta_2$.

To prove (7.29), observe that for each $x \in \Delta_{c,1}$ fixed, the functions $(H^{(0)}_c - H^{(1)}_c)(x + iy)$ are increasing for $y \in [0, \infty)$ and vanish at $y = 0$ by (4.1) and (2.3). Moreover, since these functions have the same value at conjugate-symmetric points, it is enough to consider only the upper half-plane. As the right-hand side of (7.29) is positive whenever $c, \delta > 0$, we can assume without loss of generality that $\delta < \min\{\delta_{\alpha_1}, \delta_{\beta_1}, \min_{c \in [c', 1]} \delta_{\beta_1}(c)\}$, where $\delta_{\alpha_1}, \delta_{\beta_1}, c'$, and $\delta_{\beta_1}(c)$ were introduced in Lemmas 7.4, 7.5, 7.6, and 7.7, respectively.

Suppose that $|x + i\delta c - \alpha_1| < \delta_{c,1} c$. Then it follows from Lemma 7.4 together with (5.1) that

$$(7.32) \quad \left| \zeta_{c,\alpha_1}(x + i\delta c) \right| \geq \left( A_{\alpha_1}/4 \right)^{1/2} \delta^{1/2} c.$$ 
It clearly holds that $\text{Arg}(x + i\delta c) \in \left[ \arctan(\delta/\delta_{c,1}), \pi/2 \right]$. Since $\zeta_{c,\alpha_1}(z)$ is conformal, negative for $z > \alpha_1$, and positive for $z < \alpha_1$, there exists $\delta_c > 0$ such that

$$(7.33) \quad \text{Arg} \left( \zeta^{1/2}_{c,\alpha_1}(x + i\delta c) \right) \in (0, \pi - \arctan(\delta/\delta_{c,1}))/2$$
for all $\delta \in (0, \delta_c)$. Since the maps $\zeta_{c,\alpha_1}(z)$ continuously depend on $c$ and have a rescaled conformal limit as $c \to 0$, see (7.16), and a conformal limit as $c \to 1$, see (7.17), the constants $\delta_c$ can be chosen so that $\delta_c \geq \delta_e > 0$ for all $c \in (0, 1)$. However, as mentioned before, without loss of generality we can consider only $\delta \in (0, \delta_e)$. Furthermore, similarly to (7.31), it holds that

$$\left( H^{(0)}_c - H^{(1)}_c \right)(z) = 4\Re \left( \zeta^{1/2}_{c,\alpha_1}(z) \right), \quad \Re z < \beta_{c,1},$$
by (7.8). Thus, combining the above expression with (7.32) and (7.33) gives us

$$\left( H_c^{(0)} - H_c^{(1)} \right)(x + i\delta c) \geq \sin(\arctan(\delta/\delta_{\alpha_1}))/2 \left| \zeta_{\beta_1}^{1/2}(x + i\delta c) \right| \geq B' \delta^{3/2} c$$

for some $B' > 0$, independent of $c$ and $\delta$.

Now, we shall examine what happens when $x$ lies in the vicinity of $\beta_{c,1}$. Unfortunately, there are three different constructions of the conformal maps in this case. Thus, we first assume that $c \in (0, c^*)$ and $|x + i\delta - \beta_{c,1}| < \delta_{\beta_1}c$, see Lemma 7.5. Then it follows from Lemma 7.5 and (5.1) that

$$\left| \zeta_{\beta_1}^{1/2}(x + i\delta c) \right| \geq (A_{\beta_1}/4)^{3/2} \delta^{3/2} c.$$ 

In the considered case $\text{Arg}(x + i\delta c) \in [\pi/2, \pi - \arctan(\delta/\delta_{\beta_1})]$. Since the conformal maps $\zeta_{\beta_1}^{1/2}(z)$ continuously depend on $c$, have a rescaled limit when $c \to 0$, see (7.21), are positive for $z > \beta_{c,1}$ and negative for $x < \beta_{c,1}$, (7.33) gets now replaced by

$$\text{Arg} \left( \zeta_{\beta_1}^{1/2}(x + i\delta c) \right) \in (5\pi/8, (3\pi - \arctan(\delta/\delta_{\beta_1}))/2)$$

for all $\delta \in (0, \delta_a)$ and a possibly adjusted constant $\delta_a > 0$. Thus, combining the above observations with (7.31) gives us that

$$\left( H_c^{(0)} - H_c^{(1)} \right)(x + \xi) \geq (4/3) \sin(\arctan(\delta/\delta_{\beta_1}))/2 \left| \zeta_{\beta_1}^{3/2}(x + i\delta c) \right| \geq B'' \delta^{5/2} c$$

for some $B'' > 0$, independent of $\delta$ and $c$. Let now $c'$ be the same as in Lemma 7.6 and $|x + i\delta - \beta_1 c| < \delta_{\beta_1} c$ for any $c \in (c^*, c']$, again, see Lemma 7.6. Then it follows from (7.23) that

$$\left( H_c^{(0)} - H_c^{(1)} \right)(z) = -\frac{4}{3} \Re \left( \zeta_{c,\beta_1}^{3/2}(z) - \hat{\zeta}_{c,\beta_1}(\beta_1 + \epsilon_c) \zeta_{c,\beta_1}^{1/2}(z) \right).$$

Since $\hat{\zeta}_{c,\beta_1}(x)$ is positive for $x > \beta_1$ and negative for $x < \beta_1$, it holds that

$$-\frac{4}{3} \Re \left( \zeta_{c,\beta_1}^{3/2}(z) - \hat{\zeta}_{c,\beta_1}(\beta_1 + \epsilon_c) \zeta_{c,\beta_1}^{1/2}(z) \right) > -\frac{4}{3} \Re \left( \zeta_{c,\beta_1}^{3/2}(z) \right)$$

for $z$ with $\text{Arg}(z) \in (0, \pi)$. Since the maps $\hat{\zeta}_{c,\beta_1}(z)$ continuously depend on $c \in [c^*, c']$, where we set $\hat{\zeta}_{c^*,\beta_1}(z) := \zeta_{c^*,\beta_1}(z)$, see Lemma 7.6, the constant $\delta_a$ can be adjusted so that (7.35) remains valid with $\zeta_{c,\beta_1}(z)$ replaced by $\hat{\zeta}_{c,\beta_1}(z)$ for $|\delta| < \delta_a$ and $c \in [c^*, c']$. Hence, we can proceed exactly as in the case $c \in (0, c^*)$, perhaps, at the expense of possibly adjusting the constant $B''$ in (7.36). Further, when $c \in [c', 1)$, it follows from (7.26) that

$$\left( H_c^{(0)} - H_c^{(1)} \right)(z) = 4 \Re \left( \zeta_{1/2,\beta_1}(z) \right), \quad \alpha_1 < \Re z < \alpha_2.$$ 

It also follows from Proposition 4.2 and Lemma 7.7 that $|\zeta_{c,\beta_1}^{1/2}(\beta_1)|$ is bounded away from 0 independently of $c \in [c', 1)$ (the bound does depend on $c'$). Notice also that in this case (7.33) remains valid with $\delta_{\alpha_1}$ replaced by $\min_{c \in [c', 1]} \delta_{\beta_1}(c)$. Therefore, (7.34) remains valid as well, where we need to replace $\zeta_{c,\alpha_1}(z)$ by $\hat{\zeta}_{c,\beta_1}(z)$ and, perhaps, adjust $B'$. It only remains to examine what happens when $\alpha_1 + \delta' c \leq x \leq \beta_{c,1} - \delta' c$ for some $\delta' > 0$. To this end, let us denote by $\bar{h}_c(x)$ the following function:

$$\bar{h}_c(x) := 2i\Im \left( h_c^{(0)}(x) \right) = h_c^{(0)}(x) - h_c^{(1)}(x) = h_c^{(0)}(x) - h_c^{(1)}(x) = 2i(3 \Im \left( h_c^{(1)}(x) \right)) = -2i(3 \Im \left( h_c^{(1)}(x) \right)), \quad x \in \Delta_{c,1}^\circ.$$ 

Let us show that $\bar{h}_c(x) \neq 0$ for $x \in \Delta_{c,1}^\circ$. Indeed, if $\bar{h}_c(x') = 0$ for some $x' \in \Delta_{c,1}^\circ$, then $h_c^{(0)}(x') = h_c^{(0)}(x') = h_c^{(1)}(x') = h_c^{(1)}(x')$ and this value is real. That is, there exist $x', x'' \in \Delta_{c,1}^\circ(\pi(x') = \pi(x'')) = x'$ at which $h_c(z)$ assumes the same non-zero real value. On the other hand, when $c \in (c^*, c^*)$, $h_c(z)$ has simple poles at $\alpha_1, \beta_1, \alpha_2, \beta_2$. Therefore, it can be clearly seen from (4.2) that $h_c^{(0)}(x)$ assumes every non-zero real value twice, once on $(-\infty, \alpha_1) \cup (\beta_2, \infty)$ and once on $(\beta_1, \alpha_2)$. Furthermore, (4.2) also shows that $h_c^{(1)}(x)$ and $h_c^{(2)}(x)$ assume every non-zero real value once on $(-\infty, \alpha_1) \cup (\beta_1, \infty)$ and $(-\infty, \alpha_2) \cup (\beta_2, \infty)$, respectively. As $h_c(z)$ has four zeros/poles, it assumes every value exactly four times. Thus, if $\bar{h}_c(x')$ were zero, $\bar{h}_c(z)$ would assume a given real value six times, which is impossible. Since the proof for the case $c \in (0, c^*) \cup [c^*, 1)$ is quite similar, the claim follows.
For the next step, we would like to argue that
\[
\hat{h}_{\text{min}} := \inf_{c \in (0,1)} \min_{\alpha_1 + \delta' c \leq s \leq \beta_e, 1 - \delta' c} \left| \hat{h}_c(x) \right| > 0.
\]
For that, it will be convenient to consider the rescaled function \( \hat{h}_c(s) := \hat{h}_c(\beta_e, 1 + |\Delta_{\gamma,1}(s - 1)/2). \) These functions are purely imaginary and non-vanishing on \((-1,1).\) It follows from (4.8) that there exists \( \delta'' > 0 \) such that
\[
\hat{h}_{\text{min}} \geq \inf_{c \in (0,1) - 1 + \delta'' c \leq s \leq 1 - \delta'' c} \left| \hat{h}_c(s) \right|.
\]
For each \( c \) fixed, the minimum over \( s \) is clearly non-zero and continuously depends on \( c. \) On the other hand, exactly as in Lemma 7.5, it holds that
\[
\tag{7.37}
\hat{h}_c(s) \to - \frac{i}{\omega_2(\alpha_1)} \sqrt{\frac{1 - s}{1 + s}}
\]
as \( c \to 0 \) uniformly on \([-1 + \delta'', 1 - \delta''], \) which again, has a non-zero minimum of the absolute value. Moreover, a computation similar to the one leading to (7.17) gives us that
\[
\tag{7.38}
\hat{h}_c(s) \to - \frac{4i}{\sqrt{\beta_1 - \alpha_1} \sqrt{1 - s^2}}
\]
as \( c \to 1 \) uniformly on \([-1 + \delta'', 1 - \delta''], \) which also has a non-zero minimum of the absolute value. Hence, it indeed holds that \( \hat{h}_{\text{min}} > 0. \)

Now, observe that \( h_c(x) \) is a trace of a function analytic across \( \Delta_{e,1}, \) namely, of
\[
\hat{h}_c(z) := \begin{cases} h_c^{(0)}(z) - h_c^{(1)}(z), & \Im z > 0, \\ h_c^{(1)}(z) - h_c^{(0)}(z), & \Im z < 0. \end{cases}
\]
Therefore, for each \( x' \in [\alpha_1 + \delta' c, \beta_e, 1 - \delta' c] \) fixed, there exists \( \delta(c; x') > 0 \) such that
\[
\tag{7.39}
|\tilde{H}_c(z; x')| \geq (\hat{h}_{\text{min}}/4)|z - x'|, \quad \tilde{H}_c(z; x') := \int_{x'}^{z} \hat{h}_c(s)ds,
\]
for all \( |z - x'| < \delta(c; x')c \) by (5.1). Notice that \( \delta(c; x')c \) can be taken to be the radius of the largest disk of conformality of \( \tilde{H}_c(z; x'). \) Observe also that \( \delta(c; x') \) continuously depends on \( x' \) and therefore there exists \( \delta(c) > 0 \) such that \( \delta(c; x') \geq \delta(c) \) for all \( x' \in [\alpha_1 + \delta' c, \beta_e, 1 - \delta' c]. \) Since \( \delta(c) \) can be made to continuously depend on \( c \) and the limits (7.37) and (7.38) hold not only on \((-1,1), \) but in some neighborhood of \((-1,1)\) as well, the constant \( \delta \) can be adjusted so that \( \delta(c) > \delta \) for all \( c \in (0,1). \)

Since the functions \( \tilde{H}_c(z; x') \) are conformal in \( |z - x'| < \delta c \) for each \( x' \in [\alpha_1 + \delta' c, \beta_e, 1 - \delta' c] \) and are purely imaginary on the real axis, the same continuity and compactness arguments we have been employing throughout the lemma imply that
\[
\tag{7.40}
\Re \left( \tilde{H}_c(x' + iy; x') \right) \geq C|\tilde{H}_c(x' + iy; x')|
\]
for all \( y \in (0, \delta c) \) and \( x' \in [\alpha_1 + \delta' c, \beta_e, 1 - \delta' c] \), where \( C > 0 \) is constant independent of \( c. \) Since \( h_c(z) = 2\partial_c \tilde{H}_c(z), \) it follows from (7.39) and (7.40) that
\[
\tag{7.41}
\left( H_c^{(0)} - H_c^{(1)} \right)(x + i\delta c) = \Re \left( \tilde{H}_c(x + i\delta c; x) \right) \geq (C\hat{h}_{\text{min}}/4)\delta c.
\]
The estimate in (7.29) now follows from (7.34), (7.36), and (7.41).

**5. Local Parametrices.** Below, we construct solutions of RHP- \( P_c \) for \( c \in \{\alpha_1, \beta_{\mathfrak{N}}, \} \) \( n \in \mathcal{N}_c. \) Recall that the squares \( U_c \) have diagonals of length \( 2\delta c, \) where \( \delta \leq \delta(c) \) see Section 7.2. Additionally, we assume that \( \delta \leq \min(\delta_{\alpha_1}, \delta_{\beta_{\mathfrak{N}}}, \delta_c, c \}) \) and \( \delta \leq \min(\delta_{\alpha_1}, \delta_{\beta_{\mathfrak{N}}, c^*}) \), depending on \( c^*, \) see Lemmas 7.4–7.7. Then the maps constructed in Section 7.4 are conformal in the corresponding squares \( U_c. \)

**5.1. Matrix \( P_{\alpha_1}. \)** Let \( \Psi(\zeta) \) be a matrix-valued function such that
\begin{align*}
(\text{a}) \quad \Psi(\zeta) & \text{ is holomorphic in } C\setminus(I_+ \cup I_- \cup (-\infty, 0)], \text{ see (7.2)}; \\
(\text{b}) \quad \Psi(\zeta) & \text{ has continuous traces on } I_+ \cup I_- \cup (-\infty, 0) \text{ that satisfy }
\end{align*}
\[
\Psi_+(\zeta) = \Psi_-(\zeta) \begin{cases} 0 & \zeta \in (-\infty, 0), \\ -1 & \zeta \in I_\pm, \end{cases}
\]}
where \( I_{\pm} \) are oriented towards the origin;
(c) \( \Psi(\zeta) = \mathcal{O}(\log |\zeta|) \) as \( \zeta \to 0 \);
(d) \( \Psi(\zeta) \) has the following behavior near \( \infty \):
\[
\Psi(\zeta) = \frac{\zeta^{1/2} e^{\sigma_3/4}}{\sqrt{2}} \left( \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right) \left( I + \mathcal{O} \left( \zeta^{-1/2} \right) \right) \exp \left\{ 2\zeta^{1/2} \sigma_3 \right\}
\]
uniformly in \( C \setminus (I_+ \cup I_- \cup (-\infty, 0]) \).

Solution of RHP-\( \Psi \) was constructed explicitly in [28] with the help of modified Bessel and Hankel functions. Observe that the jump matrices in RHP-\( \Psi \) have determinant one. Therefore, it follows from RHP-\( \Psi \) (d) that \( \det(\Psi(\zeta)) = \sqrt{2} \).

Let \( \zeta_{\alpha,\alpha_1}(z) := \zeta_{\alpha,\alpha_1}(z) \), see (7.8), which is conformal in \( U_{\alpha_1} \). It holds due to Lemma 7.4 and (5.1) that
\[
\det(s) \text{ is holomorphic in } E_{\alpha_1} \text{ of RHP-} \Psi \text{ from RHP-} \Psi \text{ and Lemma 7.3, see (7.8), which is conformal in } U_{\alpha_1}.
\]

Let \( D(z) \) be given by (7.6). Note also that the matrix \( \sigma_3 \Psi(\zeta) \) also satisfies RHP-\( \Psi \), but with the orientation of all the rays in RHP-\( \Psi \) reversed and \( i \) replaced by \( -i \) in the asymptotic formula of RHP-\( \Psi \) (d). Relation (7.43) and RHP-\( \Psi \) (a,b,c) imply that the matrix
\[
P_{\alpha_1}(z) := E_{\alpha_1}(z) T_1 \left( \sigma_3 \Psi(\zeta) \right) \left( \frac{1}{|n|^{2}} \zeta_{\alpha,\alpha_1}(z) \right) \rho_1^{-\sigma_3/2}(z) \left( \Phi_n^0(\zeta) / \Phi_n^1(\zeta) \right)^{-\sigma_3/2}(z) D(z),
\]
satisfies RHP-\( \Psi \) (a,b,c) for any holomorphic prefactor \( E_{\alpha_1}(z) \). As \( \zeta_+^{1/4} = i \zeta_-^{1/4} \) on \( (-\infty, 0) \), where we take the principal branch, it can be easily checked that
\[
\frac{\zeta_+^{1/4}}{\sqrt{2}} \left( \begin{array}{cc} 1 & -i \\ -i & 1 \end{array} \right) = \frac{\zeta_-^{1/4}}{\sqrt{2}} \left( \begin{array}{cc} 1 & -i \\ -i & 1 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)
\]
there. Then RHP-\( \Psi \) (d) implies that
\[
E_{\alpha_1}(z) := M(z) T_1 \left( \frac{1}{|n|^{2}} \zeta_{\alpha,\alpha_1}(z) \right) \rho_1^{-\sigma_3/2}(z) \left( \Phi_n^0(\zeta) / \Phi_n^1(\zeta) \right)^{-\sigma_3/2}(z) D(z),
\]
is holomorphic in \( U_{\alpha_1} \setminus \{ \alpha_1 \} \). Since the first and second columns of \( M(z) \) has at most quarter root singularities at \( \alpha_1 \) and the third one is bounded, see Lemma 7.3, \( E_{\alpha_1}(z) \) is in fact holomorphic in \( U_{\alpha_1} \) as desired. Finally, RHP-\( \Psi \) (d) follows from RHP-\( \Psi \) (d) and (7.42).

Recall that \( \det(M(z)) = \det(D(z)) = 1 \) as explained between (7.5) and (7.6). Hence, it holds that \( \det(E_{\alpha_1}(z)) = 1 / \sqrt{2} \) and respectively \( \det(P_{\alpha_1}(z)) = 1 \).

7.5.2. Matrix \( P_{\beta,\alpha_1}(z) \) when \( c_+ \leq c^* \) and \( c_- \leq c^* \). Below, given \( N_{\beta} \), with \( c_+ \leq c^* \), we solve RHP-\( \Psi \) along the subsequence \( N_{\beta}^\ast := \{ \bar{n} \in N_{\beta} : c_+ \leq c^* \} \), when such a subsequence is infinite. Clearly, \( N_{\beta}^\ast \) only omits finitely many terms from \( N_{\beta} \) when \( c_+ < c^* \).

Given \( \sigma \in C \setminus (-\infty, 0) \) and \( s \in (-\infty, \infty) \), let \( \Phi(\zeta; s) \) be a matrix-valued function such that
(a) \( \Phi(\zeta; s) \) is holomorphic in \( C \setminus (I_+ \cup I_- \cup (-\infty, \infty)) \);
(b) \( \Phi(\zeta; s) \) has continuous traces on \( I_+ \cup I_- \cup (-\infty, 0) \cup (0, \infty) \) that satisfy
\[
\Phi_{\sigma}(\zeta; s) = \Phi_{\sigma^{-}}(\zeta; s),
\]
where
\[
\begin{align*}
&\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta \in (-\infty, 0), \\
&\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \zeta \in I_+, \\
&\begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}, \quad \zeta \in (0, \infty);
\end{align*}
\]
(c) \( \Phi(\zeta; s) = \mathcal{O}(1) \) and \( \Phi(\zeta; s) = \mathcal{O}(\log |\zeta|) \) when \( \sigma \neq 1 \) as \( \zeta \to 0 \);
(d) $\Phi(\zeta; s)$ has the following behavior near $\infty$:

$$\Phi(\zeta; s) = \frac{\zeta^{-s/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left( I + O\left( \zeta^{-1/2} \right) \right) \exp \left\{ -\frac{2}{3} (\zeta + s)^{3/2} \sigma_3 \right\}$$

uniformly in $\mathbb{C} \setminus (I_+ \cup I_- \cup (-\infty, \infty))$.

As in the previous subsection, notice that $\det(\Phi(\zeta; s)) = \sqrt{2}$.

Besides RHP-$\Phi_s$, we shall also need RHP-$\Phi$ obtained from RHP-$\Phi_0$ by replacing RHP-$\Phi_0(d)$ with

(d) $\Phi(\zeta; s)$ has the following behavior near $\infty$:

$$\Phi(\zeta; s) = \frac{\zeta^{-s/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left( I + O\left( \zeta^{-1/2} \right) \right) \exp \left\{ -\frac{2}{3} (\zeta + s)^{3/2} \sigma_3 \right\}$$

When $\sigma = 1$ and $s = 0$, the Riemann-Hilbert problem RHP-$\Phi$ is well known [13] and is solved using Airy functions. In fact, in this case RHP-$\Phi_1(d)$ can be improved to

(7.46) \[ \Phi_1(\zeta; 0) = \frac{\zeta^{-s/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left( I + O\left( \zeta^{-1/2} \right) \right) \exp \left\{ -\frac{2}{3} (\zeta + s)^{3/2} \sigma_3 \right\} \]

uniformly in $\mathbb{C} \setminus (I_+ \cup I_- \cup (-\infty, \infty))$. More generally, when $\sigma = 1$, the solvability of these two problems for all $s \in (-\infty, \infty)$ was shown in [23] with further properties investigated in [24]. The solvability of the general case $\sigma \in \mathbb{C} \setminus (-\infty, 0)$ was obtained in [36]. In [37, Theorem 4.1] it was shown that RHP-$\Phi_0(d)$ can be replaced by

(7.47) \[ \Phi_0(\zeta; s) = \frac{\zeta^{-s/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left( I + O\left( \left| \frac{s}{|k| + 1} \right| \right) \right) \exp \left\{ -\frac{2}{3} (\zeta + s)^{3/2} \sigma_3 \right\} \]

which holds uniformly for $\zeta \in \mathbb{C} \setminus (I_+ \cup I_- \cup (-\infty, \infty))$ and $s \in (-\infty, \infty)$ when $\sigma \neq 0$, and uniformly for $s \in [0, \infty)$ when $\sigma = 0$, and that RHP-$\Phi(d)$ can be replaced by

(7.48) \[ \Phi(\zeta; s) = \frac{\zeta^{-s/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left( I + O\left( \left| \frac{s}{|k| + 1} \right| \right) \right) \exp \left\{ -\frac{2}{3} (\zeta + s)^{3/2} \sigma_3 \right\} \]

uniformly for $\zeta \in \mathbb{C} \setminus (I_+ \cup I_- \cup (-\infty, 0])$ and $s \in (-\infty, 0]$.

Let $\zeta_{\beta, 1}(z) := \zeta_{\beta, 1}(z)$ be the functions defined in (7.18) that are conformal in $U_{\beta, 1}$, see Lemma 7.5. It follows from (4.3) and (7.18) that

(4.70) \[ \zeta_{\beta, 1}(z) = \left( -\frac{3}{4|\vec{n}|} \log \left( \frac{\Phi_0^{(0)}(\Phi_0^{(1)})}{\Phi_0^{(1)}} \right) \right)^{2/3}, \quad z \in U_{\beta, 1}. \]

According to Lemma 7.5 and (5.1), it holds that

(7.50) \[ \left\{ |z| < A_{\beta, 1} \delta_{\beta, 1}^{3/2} \right\} \subset |\vec{n}|^{2/3} \zeta_{\beta, 1}(U_{\beta, 1}), \]

where $A_{\beta, 1}$ is independent of $\vec{n}$ with $\vec{n} \in N_{c_{\beta, 1}}^\infty$.

Assume now that $c_{\beta, 1} < c^*$. Recall that this is case $\beta_1 \in U_{\beta, 1}$ for all $|\vec{n}|$ large enough. Relation (7.49) and RHP-$\Phi_1(a,b,c)$ imply that the matrix

(7.51) \[ P_{\beta, 1}(z) := E_{\beta, 1}(z)T_1 \left( \Phi_1 \left( |\vec{n}|^{2/3} \zeta_{\beta, 1}(z); 0 \right) \rho_1^{-\sigma_3/2}(z) \left( \Phi_0^{(0)}/\Phi_0^{(1)} \right)^{-\sigma_3/2}(z) \right) J(z), \]

satisfies RHP-$P_{\beta, 1}(a,b,c)$ for any holomorphic prefactor $E_{\beta, 1}(z)$. As in the previous subsection, RHP-$N(b)$ implies that

(7.52) \[ E_{\beta, 1}(z) := M(z)T_1 \left( \frac{|\vec{n}|^{2/3} \zeta_{\beta, 1}(z)^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right)^{-1} \rho_1^{-\sigma_3/2}(z) \left( \Phi_0^{(0)}/\Phi_0^{(1)} \right)^{-\sigma_3/2}(z) \right) D(z), \]

is holomorphic in $U_{\beta, 1}$. Requirement RHP-$P_{\beta, 1}(d)$ now follows from (7.46) and (7.50).

Assume now that $c_{\beta, 1} = c^*$. Recall (7.3). Observe also that $\beta_{\beta, 1} \leq \beta_3$ for $\vec{n} \in N_{c_{\beta, 1}}^\infty$ and therefore $s_{\vec{n}} := |\vec{n}|^{2/3} \zeta_{\beta, 1}(\beta_3) \geq 0$. Then, similarly to (7.51), we get from (7.49) and RHP-$\Phi_0(a,b,c)$ that

(7.53) \[ P_{\beta, 1}(z) := E_{\beta, 1}(z)T_1 \left( \Phi_0 \left( |\vec{n}|^{2/3} \zeta_{\beta, 1}(z); s_{\vec{n}} \right) \rho_1^{-\sigma_3/2}(z) \left( \Phi_0^{(0)}/\Phi_0^{(1)} \right)^{-\sigma_3/2}(z) \right) J(z), \]
Again, we stress that the map defined in (7.26), whose properties were described in Lemma 7.7. It follows from (4.3) and (7.26) that

\[ s \in \partial U_{\beta_1}. \]

Since \( \zeta_{\beta_1}(\beta_1) \to 0 \) as \( |\hat{n}| \to \infty \), \( \hat{n} \in N_{\infty}^{\infty} \), and \( \hat{\zeta}_{\beta_1}(z) \) is bounded below in modulus on \( \partial U_{\beta_1}, \) RHP-\( P_{\beta_1} \) (d) follows. As in the previous subsection, we point out that \( \det(P_{\beta_1}(z)) = 1. \)

7.5.3. Matrix \( P_{\beta_1}(z) \) when \( c_{\infty} = c^* \) and \( c_{\infty} > c^* \). Below, we solve RHP-\( P_{\beta_1} \) along the subsequence \( N_{\infty}^{\infty} := \{ \hat{n} \in N_{\infty} : c_{\infty} > c^* \} \), when such a subsequence is infinite. Let \( \hat{\zeta}_{\beta_1}(z) := \hat{\zeta}_{\infty,\beta_1}(z) \) be the conformal map in \( U_{\beta_1} \) constructed in Lemma 7.6. As before, it follows from (4.3) that

\[ \frac{3}{2} \hat{\zeta}_{\beta_1}(z) - \hat{\zeta}_{\beta_1}(\beta_1 + \epsilon_{\beta_1}) \hat{s}_{\beta_1}(z) = - \frac{3}{4|\hat{n}|} \log \left( \frac{\Phi^{(0)}_{\hat{n}} / \Phi^{(1)}_{\hat{n}}}{\Phi^{(0)}_{\hat{n}} / \Phi^{(1)}_{\hat{n}} - \sigma_{s_1} / 2} \right), \quad z \in U_{\beta_1}. \]

Let \( s_{\beta_1} := -|\hat{n}|^{2/3} \hat{\zeta}_{\beta_1}(\beta_1 + \epsilon_{\beta_1}). \) As above, it follows from (7.54) and RHP-\( \hat{\Phi} \) that

\[ P_{\beta_1}(z) := E_{\beta_1}(z) T_1 \left( \Phi \left( |\hat{n}|^{2/3} \hat{\zeta}_{\beta_1}(z); s_{\beta_1} \right) \rho_1^{-\sigma_{s_1} / 2} \left( \Phi^{(0)}_{\hat{n}} / \Phi^{(1)}_{\hat{n}} \right)^{-\sigma_{s_1} / 2} \right) D(z), \]

satisfies RHP-\( P_{\beta_1} \), where \( E_{\beta_1}(z) \) is given by (7.52) with \( \zeta_{\beta_1}(z) \) replaced by \( \hat{\zeta}_{\beta_1}(z) \), and it follows from (7.48) that RHP-\( P_{\beta_1} \) (d) is satisfied with

\[ a(1) \geq \max \left\{ \left| s_{\beta_1}(\beta_1 + \epsilon_{\beta_1}, |\hat{n}|^{-1/3}) \right| \right\}. \]

Again, we stress that \( \det(P_{\beta_1}(z)) = 1. \)

7.5.4. Matrix \( P_{\beta_1}(z) \) when \( c_{\infty} > c^* \). The construction of \( P_{\beta_1}(z) \) in the considered case is absolutely identical to the one of \( P_{\alpha_1}(z) \) in Section 7.5.1.

Clearly, we can assume that \( \hat{n} \in N_{\infty} \) is such that \( c_{\infty} > c^* \). Let \( \zeta_{\infty,\beta_1}(z) := \zeta_{\infty,\beta_1}(z) \) be the conformal map defined in (7.26), whose properties were described in Lemma 7.7. As before from (4.3) and (7.26) that

\[ \zeta_{\infty,\beta_1}(z) = \left( \frac{1}{4|\hat{n}|} \log \left( \frac{\Phi^{(0)}_{\hat{n}} / \Phi^{(1)}_{\hat{n}}}{\Phi^{(0)}_{\hat{n}} / \Phi^{(1)}_{\hat{n}} - \sigma_{s_1} / 2} \right) \right)^{2}, \quad z \in U_{\beta_1}. \]

According to Lemma 7.7 and (5.1) theorem and since \( n_{1}^{2} \leq |\hat{n}|^{2} \), it holds that

\[ \{ z : < A_{\beta_1}\delta(z_{\infty} - \beta_1) n_{1}^{2} \} \subset |\hat{n}|^{2} \zeta_{\infty,\beta_1}(U_{\beta_1}), \]

where \( \delta_{\beta_1}(c) \) is continuous and non-vanishing on \( (c^*, 1) \). Similarly to (7.44), a solution of RHP-\( P_{\beta_1} \) is given by

\[ P_{\beta_1}(z) := E_{\beta_1}(z) T_1 \left( \Phi \left( |\hat{n}|^{2} \zeta_{\infty,\beta_1}(z) \right) \rho_1^{-\sigma_{s_1} / 2} \left( \Phi^{(0)}_{\hat{n}} / \Phi^{(1)}_{\hat{n}} \right)^{-\sigma_{s_1} / 2} \right) D(z), \]

where

\[ E_{\beta_1}(z) := M(z) T_1 \left( \left[ \left| \frac{|\hat{n}|^{2} \zeta_{\infty,\beta_1}(z)}{\sqrt{2}} \right| \right] \rho_1^{-\sigma_{s_1} / 2} \right)^{-1}. \]

It again holds that \( \det(P_{\beta_1}(z)) = 1. \)

7.6. Solution of RHP-\( X \). Set \( \partial U_{\tilde{\xi}} := U_{\alpha_1} \cup U_{\beta_1} \cup U_{\alpha_2} \cup U_{\beta_2} \) and \( \Gamma_{\tilde{\xi}} := \Gamma_{\hat{\alpha}_1}^{+} \cup \Gamma_{\hat{\beta}_1}^{+} \cup \Gamma_{\hat{\alpha}_2}^{+} \cup \Gamma_{\hat{\beta}_2}^{+}. \) Put

\[ \Sigma_{\hat{\alpha},\delta} := \partial U_{\hat{\alpha}} \cup \left( \Gamma_{\hat{\alpha}} \cup \{ \hat{\beta}_1 \} \cup [\alpha_2, \alpha_{\tilde{\xi}}] \right) \backslash \cup_{\hat{\beta}} \],

see Figure 4. For definitiveness, we agree that all the segments in \( \Sigma_{\hat{\alpha},\delta} \) are oriented from left to right and all the polygons are oriented counter-clockwise. We shall further denote by \( \Sigma_{\hat{\alpha},\delta,1} \) and \( \Sigma_{\hat{\alpha},\delta,2} \) the left and right, respectively, connected components of \( \Sigma_{\hat{\alpha},\delta} \).

For what is to come, we shall need uniform boundedness of the Cauchy operators on \( \Sigma_{\hat{\alpha},\delta} \). For convenience, we formulate this claim as a lemma.
Lemma 7.9. Given $r > 1$, there exists a constant $C_r > 0$ such that for all $\delta > 0$ it holds that

$$
\|C_{\pm} f\|_{L^r(\Sigma_{\delta, \delta})} \leq C_r \|f\|_{L^r(\Sigma_{\delta, \delta})},
$$

where $C f(z) = \frac{1}{2\pi i} \oint_{\Sigma_{\delta, \delta}} \frac{f(t) dt}{t-z}$ and $C_{\pm} f(s)$ are the traces of $C f(z)$ on the left ($-$) and right ($+$) handsides of $\Sigma_{\delta, \delta}$.

Proof. Recall the following known fact, see [12, Equation (7.11)], if $R_1, R_2$ are two semi-infinite rays with a common endpoint, then

(7.55) $$
\|C_{R_1} f\|_{L^r(R_2)} \leq C_r \|f\|_{L^r(R_1)},
$$

for some constant $C_r > 0$ (we can take $C_2 = 1$), where $C_{R_1}$ is the Cauchy operator defined on $R_1$. Moreover, the same estimate holds when $R_2 = R_1$ and $C_{R_1}$ is replaced by the trace operators $C_{R_1 \pm}$, see [12, Equations (7.5)–(7.7)]. Trivially, the same estimate holds when $R_2$ is replaced by an interval disjoint from $R_1$ (may be for an adjusted constant $C_r$). Since we can embed any two segments with a common endpoint into semi-infinite rays with a common endpoint and embed a function from $L^r$ space of the corresponding ray by extending it by zero, the desired estimate then follows from (7.55) (again, with an adjusted constant $C_r$).

Given the global parametrix $N(z) = C(MD)(z)$ solving RHP-$N$, see (7.5) and (7.6), and local parametrixes $P_e(z)$ solving RHP-$P_e$ and constructed in the previous section, consider the following Riemann-Hilbert Problem (RHP-$Z$):

(a) $Z(z)$ is a holomorphic matrix function in $\mathbb{C} \setminus \Sigma_{\delta, \delta}$ and $Z(x) = I$;
(b) $Z(z)$ has continuous traces on $\Sigma_{\delta, \delta}$ that satisfy

(7.56) $$
Z_+(s) = Z_-(s) \begin{cases} 
(MD)(s) T_i \begin{pmatrix} 1 \\ \rho_i(s) \\ 1 \end{pmatrix} (MD)^{-1}(s), & s \in \Gamma_{\delta} \setminus \mathcal{U}_{\delta}, \\
(MD)(s) T_i \begin{pmatrix} 1 \\ 0 \\ \rho_i(s) \end{pmatrix} (MD)^{-1}(s), & s \in \Delta \setminus (\Delta_{\delta, 1} \cup \mathcal{U}_{\delta}), \\
P_e(s) (MD)^{-1}(s), & s \in \partial U_e, e \in \{\alpha_1, \beta_1, \alpha_2, \beta_2\};
\end{cases}
$$

(c) around the points of $\Sigma_{\delta, \delta} \setminus (\Sigma_{\delta, \delta} \cup \{\beta_1, \alpha_2\})$ the function $Z(z)$ is bounded and around $\beta_1$ (resp. $\alpha_2$) its entries are bounded except for those in the second (resp. third) column that behave like $O(\log |z - \beta_1|)$ (resp. $O(\log |z - \alpha_2|)$).

To show existence and prove size estimates of the matrix function $Z(z)$, let us first estimate the size of its jump:

(7.57) $$
\|V\|_{L^\infty(\Sigma_{\delta, \delta})} \lesssim \frac{\epsilon_{\bar{u}}}{\delta^4} \min \{\min \{z_{\epsilon}, -\beta_1 - \alpha_2 - z_{\epsilon}\}^{-1/2}, \frac{\epsilon_{\bar{u}}}{\|\epsilon^*\|}, (\epsilon^*, \epsilon^*), (\epsilon^*, c^*)\},
$$

with the constant in $\lesssim$ being independent of $\delta$ and $\bar{u}$. Moreover, it also holds that $\|V\|_{L^\infty(\Sigma_{\delta, \delta})} = o(1)$ when $c_\epsilon \in \{\epsilon^*, \epsilon^*\}$. 

Lemma 7.10. Let $V(z)$ be given by (7.56) and RHP-$Z(b)$. Then it holds that

(7.58) $$
\|V\|_{L^\infty(\Sigma_{\delta, \delta})} \lesssim \frac{\epsilon_{\bar{u}}}{\delta^4} \min \{\min \{z_{\epsilon}, -\beta_1 - \alpha_2 - z_{\epsilon}\}^{-1/2}, \frac{\epsilon_{\bar{u}}}{\|\epsilon^*\|}, (\epsilon^*, \epsilon^*), (\epsilon^*, c^*)\},
$$

with the constant in $\lesssim$ being independent of $\delta$ and $\bar{u}$. Moreover, it also holds that $\|V\|_{L^\infty(\Sigma_{\delta, \delta})} = o(1)$ when $c_\epsilon \in \{\epsilon^*, \epsilon^*\}$. 

Proof. We shall prove (7.57) separately for different parts of \( \Sigma_{\vec{n},\delta} \). In fact, we shall do it only on \( \Sigma_{\vec{n},\delta} \) understanding that the estimates on \( \Sigma_{\vec{n},\delta} \) can be carried out in the same fashion. For \( s \in \partial U_{\vec{n}} \), \( e \in \{ \alpha_1, \beta_1 \} \), it holds that \( V(s) = P_e(s)(MD)^{-1}(s) - I \). Therefore, the desired estimate (7.57) follows from Lemma 7.3 and \( \text{RHP-} P_e(d) \). Let now \( s = x \in \Delta \setminus (\Delta_{\vec{n},1} \cup \overline{U}_{\vec{n}}) \), which is non-empty when \( c_* < c_* \). In this case, it holds that
\[
V(x) = (MD)(x)T_I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (MD)^{-1}(x) - I = \rho_1(x) \frac{\Phi_0^{(0)}(s)}{\Phi_0^{(1)}(s)} M(x) E_{1,2} M^{-1}(x).
\]

Estimate (7.57) now follows from Lemma 7.3 and the estimate

\[
(7.58) \quad \left| \Phi_0^{(0)}(s)/\Phi_0^{(1)}(s) \right| = \exp \left\{ \left[ \rho_1(s) \left( H_0^{(0)}(s) - H_1^{(1)}(s) \right) \right] \right\} \leq \exp \left\{ -B_\delta \delta^{3/2} \right\} \leq \frac{\varepsilon_{\vec{n}}}{B_\delta \delta^{3/2}},
\]

see (4.3) and (7.27). Lastly, let \( s \in \Gamma_{\vec{n},1}^+ \setminus U_{\vec{n}} \). Then it holds that
\[
V(s) = (MD)(s)T_I \begin{pmatrix} 1 & 0 \\ 1/\rho_1(s) & 0 \end{pmatrix} (MD)^{-1}(s) - I = \frac{1}{\rho_1(s)} \frac{\Phi_0^{(0)}(s)}{\Phi_0^{(1)}(s)} M(s) E_{2,1} M^{-1}(s).
\]

The desired estimate (7.57) can be deduced exactly as in the second step of the proof with (7.29) used instead of (7.27).
\[ \square \]

It is essentially a standard argument in the theory of orthogonal polynomials to deduce existence of \( Z(z) \) from Lemma 7.10, see [12, Chapter 7].

**Lemma 7.11.** Given \( N_{\vec{n}} \), \( c_* \in [0,1] \), there exists a constant \( M(N_{\vec{n}}) \) such that a solution of \( \text{RHP-} Z \) exists for all \( |\vec{n}| \geq M(N_{\vec{n}}) \) and it satisfies

\[
(7.59) \quad \max_{i,j} \left| \left( Z(z) - I \right)_{i,j} \right| \lesssim \delta^{-1} \| V \|_{L^\infty(\Sigma_{\vec{n},\delta})}
\]

for all \( z \in \mathbb{C} \) when \( c_* \in [c^*, c^{**}] \), \( |z - \beta_1| \geq \delta/5 \) when \( c_* \in (0, c^*) \), \( \text{dist}(z, \{ \alpha_1, \beta_1 \}) \geq \delta/5 \) when \( c_* = 0 \), \( |z - \alpha_2| \geq \delta/5 \) when \( c_* \in (c^{**}, 1) \), and \( \text{dist}(z, \{ \alpha_2, \beta_2 \}) \geq \delta/5 \) when \( c_* = 1 \), where the constant in \( \lesssim \) is independent of \( \vec{n} \) and \( \delta \).

**Proof.** Let \( C \) and \( C_0 \) be the operators defined in Lemma 7.9 and \( C_V : L^\infty(\Sigma_{\vec{n},\delta}) \to L^\infty(\Sigma_{\vec{n},\delta}), r > 1 \), be an operator defined by \( C_VM := C_0(FV) \) for any \( 2 \times 2 \) matrix function \( F(s) \) in \( L^\infty(\Sigma_{\vec{n},\delta}) \). Then it follows from Lemmas 7.9 and 7.10 that

\[
(7.60) \quad \| C_V \|_r \leq C_r \| V \|_{L^\infty(\Sigma_{\vec{n},\delta})} = o(1).
\]

Let \( M(N_{\vec{n}}) \) be such that the above norm is less than \( 1/2 \) for all \( \vec{n} \in N_{\vec{n}}, |\vec{n}| \geq M(N_{\vec{n}}) \). Then the operator \( I - C_V \) is invertible in \( L^\infty(\Sigma_{\vec{n},\delta}) \) for all such \( \vec{n} \). Hence, one can readily verify that
\[
Z(z) = I + C(UV)(z), \quad U(s) := (I - C_V)^{-1}(I)(s).
\]

The above formula and Hölder inequality immediately yield that

\[
(7.61) \quad \max_{i,j} \left| \left( Z(z) - I \right)_{i,j} \right| \lesssim \frac{\| UV \|_{L^\infty(\Sigma_{\vec{n},\delta})}}{\text{dist}(z, \Sigma_{\vec{n},\delta})} \lesssim \delta^{-1} \| V \|_{L^\infty(\Sigma_{\vec{n},\delta})}
\]

for \( \text{dist}(z, \Sigma_{\vec{n},\delta}) \geq \delta/5 \), where the constant in \( \lesssim \) is independent of \( \vec{n} \) and \( \delta \) (it involves the arelengths of \( \Sigma_{\vec{n},\delta} \), but the latter are uniformly bounded above and below).

It can be readily seen from \( \text{RHP-} Z \) that \( V(s) \) can be analytically continued off each connected component of \( \Sigma_{\vec{n},\delta} \). Hence, solutions of \( \text{RHP-} Z \) for the same value of \( \vec{n} \) and different values of \( \delta \) are, in fact, analytic continuations of each other. Thus, using (7.61) together with (7.61) where \( \delta \) is replaced by \( \delta/2 \), we get that (7.61) in fact holds for \( \text{dist}(z, \{[\beta_{c_*}, \beta_1] \cup [\alpha_2, \alpha_{c_*}, \beta_2]) \cup \overline{U}_{\vec{n}} \}) \geq \delta/5 \). The set \( \{[\beta_{c_*}, \beta_1] \cup [\alpha_2, \alpha_{c_*}, \beta_2]) \cup \overline{U}_{\vec{n}} \} \) is not empty only when \( c_* \in [0, c^*) \cup (c^{**}, 1) \). In particular, we have obtained the proof of the lemma for \( c_* \in [c^*, c^{**}] \). When \( c_* \in (0, c^*) \), set \( I_{\vec{n},\delta} := [\beta_1 + i\delta c_*/3, \beta_1 + i\delta c_*/3), \overline{U}_{\vec{n}} \). Observe that \( V(s) \) extends as an analytic matrix function into \( O_{\vec{n},\delta} \), and still satisfies (7.61) there by (7.27). Thus, we can analytically continue \( Z(s) \) into \( O_{\vec{n},\delta} \) by multiplying it by \( I + V(z) \) there. This continuation will still have a jump matrix satisfying (7.57) and therefore itself will satisfy (7.61) away from its jump contour. This finishes the proof of the lemma when \( c_* \in (0, c^*) \cup (c^{**}, 1) \) (the proof for the case \( c_* \in (c^{**}, 1) \) is identical). The proof in the case \( c_* = 0 \) (and therefore in the case \( c_* = 1 \)) is similar and uses (7.28) instead of (7.27).
Lemma 7.12. A solution of RHP-$X$ is given by

\[ X(z) := CZ(z) \begin{cases} (MD)(z), & z \in \mathbb{C} \setminus \overline{U}_\delta, \\ P_c(z), & z \in U_c, \end{cases} \]

where $Z(z)$ solves RHP-$Z$, $N(z) := C(MD)(z)$ solves RHP-$N$, see (7.5)–(7.6), and $P_c(z)$ solve RHP-$P_c$, see section 7.5.

7.7. Proof of Theorems 3.2–3.4. We are now ready to prove the main results of Section 3. We stop using the notation $c_*$ and resume writing $c$ as in the statements of Theorems 3.2–3.4.

7.7.1. Proof of Theorem 3.2. Let $K$ be a closed subset of $\mathbb{C} \setminus (\Delta_{c,1} \cup \Delta_{c,2})$. It follows from Proposition 4.1 that the constant $\delta$ in the definition of the contour $\Sigma_{\delta, \delta}$ can be adjusted so that $K$ lies outside of each $\overline{U}_\delta$ as well as $\overline{U}_\delta$ for all $|\vec{n}|$ large enough. Then it holds that

\[ Y(z) = CZM(z), \quad z \in K, \]

by (7.4) and Lemma 7.12, where we need to write $Y_{\pm}(z)$ and $Z_{\pm}(z)$ for $z \in \Delta_{\delta, i}$, $i \in \{1, 2\}$. Set

\[ B_k(z) := [Z(z)]_{1,k+1} - \delta_{0k} = o(1), \quad k \in \{0, 1, 2\}, \]

where $\delta_{ij}$ is the usual Kronecker symbol. Observe that $B_k(\infty) = 0$ and $B_k(0) = 0$.

\[ \left| B_k(z) \right| = \begin{cases} O_{\delta,c}(\varepsilon), & c \notin [c^*, c^{**}], \\ O_{\delta,c}(1), & c \in [c^*, c^{**}], \end{cases} \]

uniformly in $\mathbb{C} \setminus (\alpha_1, \beta_1)$ when $c = 0$, in $\mathbb{C} \setminus (\beta_1)$ when $c \in (0, c^*)$, in $\mathbb{C}$ when $c \in [c^*, c^{**}]$, in $\mathbb{C} \setminus (\alpha_2)$ when $c \in (c^{**}, 1)$, and in $\mathbb{C} \setminus (\alpha_2, \beta_2)$ when $c = 1$ by (7.57) and (7.59), where the dependence on $c$ of $O_{\delta,c}(\varepsilon)$ is uniform on compact subsets of $[0, c^*) \cup (c^{**}, 1]$. Then it follows from (7.1), (7.63), the definition of $M(z)$ in (7.5), and of $C$, $D(z)$ in (7.6) that

\[ P_\delta(z) = [Y(z)]_{1,1} = [C][ZM(z)]_{1,1} = (D(z)]_{1,1} = 0, \]

where $s_{\delta, i} := s_{\delta}^{(0)}(\alpha)/s_{\delta}^{(0)}(\infty), i \in \{1, 2\}$. The first asymptotic formula of the theorem now follows from (7.65), (5.5)–(5.8), and (3.10).

Let now $K$ be a closed subset of $\Delta_{c,1} \cup \Delta_{c,2}$. Again, we can adjust $\delta$ so that $K$ does not intersect $\overline{U}_\delta$ for all $|\vec{n}|$ large enough. Hence,

\[ Y_{\pm}(x) = C[ZM_{\pm}D_{\pm}](x)(I \pm \rho_{\pm}^{-1}(x)E_{i+1,1}), \quad x \in K \cap \Delta_{c,i}, \]

for $i \in \{1, 2\}$, again by (7.4) and Lemma 7.12. Thus, we get for $x \in K \cap \Delta_{c,i}$ that

\[ P_\delta(z) = \gamma_{\delta}(S_{\delta} \Phi_{\delta})^{(0)}(x) \begin{pmatrix} 1 + B_0(x) + B_1(x)Y_{n,1,1}^{(0)}(x) + B_2(x)Y_{n,1,2}^{(0)}(x) \\ \pm \gamma_{\delta}(\rho_{\pm} w_{n,1,1}^{-1}(x)) (S_{\delta} \Phi_{\delta})^{(0)}(x) \begin{pmatrix} 1 + B_0(x) + B_1(x)Y_{n,1,1}^{(0)}(x) + B_2(x)Y_{n,1,2}^{(0)}(x) \end{pmatrix} \end{pmatrix}. \]

Since $F_{\pm}^{(0)}(x) = F_{\pm}^{(1)}(x)$ on $\Delta_{c,i}$ for any rational function $F(z)$ on $\mathcal{R}_{\delta}$, the second asymptotic formula of the theorem now follows from (3.7), (7.65), and (5.5)–(5.8).

7.7.2. Proof of Theorem 3.3. Similarly to the matrix $Y(z)$ defined in (7.1), set

\[ \hat{Y}(z) := \begin{pmatrix} L_{\delta}(z) & -A_{\delta}^{(1)}(z) & -A_{\delta}^{(2)}(z) \\ -d_{\delta,1} L_{\delta}(z) & d_{\delta,1} A_{\delta,1}(z) & d_{\delta,1} A_{\delta,2}(z) \\ -d_{\delta,2} L_{\delta}(z) & d_{\delta,2} A_{\delta,1}(z) & d_{\delta,2} A_{\delta,2}(z) \end{pmatrix}, \]

\[ \hat{Y}(z) := \begin{pmatrix} L_{\delta}(z) & -A_{\delta}^{(1)}(z) & -A_{\delta}^{(2)}(z) \\ -d_{\delta,1} L_{\delta}(z) & d_{\delta,1} A_{\delta,1}(z) & d_{\delta,1} A_{\delta,2}(z) \\ -d_{\delta,2} L_{\delta}(z) & d_{\delta,2} A_{\delta,1}(z) & d_{\delta,2} A_{\delta,2}(z) \end{pmatrix} \]
where the constants $d_{\hat{n},i}$ are chosen so that the polynomials $d_{\hat{n},i}A^{(i)}_{\hat{n},i}(z)$ are monic. It was shown in [20, Theorem 4.1] that

$$\hat{Y}(z) = (Y^T(z))^{-1}.\tag{7.68}$$

Hence, it follows from (7.63) that on closed subsets of $\mathbb{C}\setminus(\Delta_{1,1} \cup \Delta_{c,2})$ it holds that

$$\hat{Y}(z) = C^{-1} (Z^{-1})^T(z) (M^{-1})^T(z)D^{-1}(z)\tag{7.69}$$

(as before, the contour $\Sigma_{\hat{n},d}$ can be adjusted to accommodate any such closed set, moreover, one needs to write $\hat{Y}_\pm(z)$ for $z \in \Delta_{1} \setminus \Delta_{c,1}$). The above equation and (7.67) yield that

$$A^{(i)}_{\hat{n}}(z) = -\left(C^{-1} (Z^{-1})^T(z) (M^{-1})^T(z)D^{-1}(z)\right)_{1,i+1}, \quad z \in K.\tag{7.70}$$

Let us rewrite (7.7) as

$$M^{-1}(z) = \text{diag} \left( \frac{1}{S^{(0)}_{\hat{n}}(z)}, \frac{w_{\hat{n},1}(z)}{S^{(1)}_{\hat{n}}(z)}, \frac{w_{\hat{n},2}(z)}{S^{(2)}_{\hat{n}}(z)} \right) \Pi(z)S(\infty),$$

which serves as a definition of the matrix $\Pi(z)$. Notice that $\tau_{\hat{n},l}$, defined in the statement of the theorem, is equal to $|C|_{1,1}$. Thus, it follows from (7.69) that

$$A^{(i)}_{\hat{n}}(z) = -\left[(Z^{-1})^T(z)S(\infty)\Pi^T(z)\right]_{1,i+1} \frac{w_{\hat{n},i}(z)}{\tau_{\hat{n},l}(S^{(l)}_{\hat{n}}(\tau_{\hat{n},l}))}, \quad z \in K.\tag{7.71}$$

Similarly to (7.64), set

$$\hat{R}_k(z) := \left[(Z^{-1})^T(z)\right]_{1,k+1} - \delta_{0k} = o(1), \quad k \in \{0, 1, 2\}.$$

Observe that all the jump matrices in RHP-$Z(b)$ have determinant one. Since $Z(\infty) = I$, we therefore get that $\det(Z(z)) = 1$. Hence, the functions $\hat{R}_k(z)$ do obey the estimate of (7.65) as well. Again, it holds that $\hat{R}_k(\infty) = 0$. Thus,

$$\left[(Z^{-1})^T(z)S(\infty)\Pi^T(z)\right]_{1,i+1} = S^{(0)}_{\hat{n},l}(\infty) \left( \Pi^{(i)}_{\hat{n},1}(z) + \hat{R}_{0}(z)\Pi^{(i)}_{\hat{n},2}(z) \right)$$

$$+ s_{\hat{n},1}^{-1}\hat{R}_{1}(z)\Pi^{(i)}_{\hat{n},1}(z) + s_{\hat{n},2}^{-1}\hat{R}_{2}(z)\Pi^{(i)}_{\hat{n},2}(z), \quad z \in K,$$

where, as before, $s_{\hat{n},l} = S^{(0)}_{\hat{n},l}/S^{(l)}_{\hat{n},l}(\infty)$. Now, observe that

$$\Pi_{\hat{n},l}(z)/\Pi_{\hat{n},l}(z) = A^{(i)}_{\hat{n},l}(z), \quad l \in \{1, 2\},$$

which follows from comparing zero/pole divisors and the normalizations at $\infty^{(l)}$ of the left- and right-hand sides of the above equality (recall that $\Pi^{(0)}_{\hat{n},l}(\infty) = 1$ and $\Pi^{(0)}_{\hat{n},l}(\infty) = -z^{-1} + O(z^{-2})$, which can be seen from (5.19)). Therefore, it follows from (7.70) that

$$A^{(i)}_{\hat{n}}(z) = -\left(1 + \hat{R}_{0}(z) - \frac{\gamma_{\hat{n}}(z)}{S^{(0)}_{\hat{n}}(z)}\hat{R}_{1}(z) - \frac{\gamma_{\hat{n}}(z)}{S^{(1)}_{\hat{n}}(z)}\hat{R}_{2}(z) \right) \frac{\Pi^{(i)}_{\hat{n},l}(z)}{\tau_{\hat{n},l}(S_{\hat{n}}(\tau_{\hat{n},l}))^{(i)}(z)},$$

Hence, the first asymptotic formula of the theorem follows from (7.65), (5.5)–(5.8) (here, one needs to recall that $\hat{R}_{1}(\infty) = 0$ and therefore the estimate for $(\gamma_{\hat{n}}(z)\hat{R}_{1}(z))$ around infinity follows from the maximum principle), (3.10), and the fact that $A^{(i)}_{\hat{n}}c_{\hat{n}}^{-1} = c_{\hat{n}}^{2}$, shown in the proof of Lemma 5.1. When $c = 0$ and $i = 1$, we also deduce from (7.71) and the maximum modulus principle that

$$A^{(1)}_{\hat{n}}(z) = c_{\hat{n}}^{-1} \frac{S^{(1)}_{\hat{n}}(\gamma_{\hat{n}}(z))}{S^{(1)}_{\hat{n}}(z)} \frac{\Pi^{(1)}_{\hat{n},l}(z)}{\tau_{\hat{n},l}(S_{\hat{n}}(\tau_{\hat{n},l}))^{(i)}(z)},$$

where we also used (3.9) and $o(1)$ behaves like the right-hand side of (7.65). Recall that $\Pi^{(1)}_{\hat{n}}(z)$ has a double zero at infinity. Therefore,

$$|\Pi^{(1)}_{\hat{n},l}(\gamma_{\hat{n}}(z))| = \left|\left(\gamma_{\hat{n}}(z) - \gamma_{\hat{n}}(z)\right)\gamma_{\hat{n}}(z)\right| \frac{w_{\hat{n},1}(z)}{w_{\hat{n},2}(z)} = O\left(c_{\hat{n}}^{2}\right)$$

uniformly on closed subsets $\mathbb{C}\setminus\Delta_{0,1}$ by (5.20), (5.6)–(5.8), and the maximum modulus principle. Clearly, the last two estimates prove the second asymptotic formula of the theorem (the case $c = 1$ and $i = 2$ can be treated similarly).
Finally, (7.66) and (7.68) give us

\[ \hat{Y}_\pm(x) = C^{-1}(Z^{-1})^T(x)(M_{\pm}^{-1})^T(x)D_{\pm}^{-1}(x)(I + \rho_i^{-1}(x)E_{1,i+1}) \]

on any compact subset of \( \Delta^c_{i,i} \), \( i \in \{1, 2\} \). Analogously to (7.71), the above formula yields that

\[ A_n^{(i)}(x) = -\left(1 + \hat{B}_0(x) - \frac{\gamma^{(i)}_{n,1}z}{s_{n,1}A_{n,1}} \hat{B}_1(x) - \frac{\gamma^{(i)}_{n,2}z}{s_{n,2}A_{n,2}} \hat{B}_2(x) \right) \frac{(\Pi^{(i)}_{n,1}w_{n,i})_{\pm}(x)}{\gamma_n(S_{n}\Phi_{n})_{\pm}^{(i)}(x)} + \rho_i^{-1}(x) \left(1 + \hat{B}_0(x) - \frac{\gamma^{(i)}_{n,1}z}{s_{n,1}A_{n,1}} \hat{B}_1(x) - \frac{\gamma^{(i)}_{n,2}z}{s_{n,2}A_{n,2}} \hat{B}_2(x) \right) \frac{(\Pi^{(i)}_{n,2}w_{n,i})_{\pm}(x)}{\gamma_n(S_{n}\Phi_{n})_{\pm}^{(i)}(x)}. \]

Once again, (7.65) and (5.5)–(5.8) imply that

\[ A_n^{(i)}(x) = -(1 + o(1)) \frac{(\Pi^{(i)}_{n,1}w_{n,i})_{\pm}(x)}{\gamma_n(S_{n}\Phi_{n})_{\pm}^{(i)}(x)} + (1 + o(1)) \rho_i^{-1}(x) \frac{(\Pi^{(i)}_{n,2}w_{n,i})_{\pm}(x)}{\gamma_n(S_{n}\Phi_{n})_{\pm}^{(i)}(x)} \]

uniformly on compact subsets of \( \Delta^c_{i,i} \). Since

\[ \mp \rho_i^{-1}(x) \Pi_{n,\pm}^{(0)}(0)(S_{n}\Phi_{n})_{\pm}^{(i)}(x) = (\Pi_{n,\pm}^{(i)}w_{n,i})_{\pm}(x)/(S_{n}\Phi_{n})_{\pm}^{(i)}(x), \quad x \in \Delta_{n,i}, \]

by (3.7), the last asymptotic formula of the theorem follows.

7.7.3. Proof of Theorem 3.4. As in the previous two subsections, given a closed set \( K \) in \( \overline{\mathbb{C}} \,(\Delta_1 \cup \Delta_2) \), we can adjust the contour \( \Sigma_{n,i} \) so that \( K \) lies in the unbounded component of its complement. Hence, using the notation of the previous two subsections, we get from (7.1), (7.5), (7.6), (7.63), and (7.65) that

\[ R_n^{(i)}(z) = \gamma_n S_n^{(i)}(z)w_{n,1}^{-1}\left(1 + B_0(z) + s_{n,1}B_1(z)\right) \]

for \( z \in K, \ i \in \{1, 2\} \). The first asymptotic formula of the theorem now follows from (7.65), (5.5)–(5.8), (3.10), and the maximum modulus principle applied to \( (\gamma_n^{(i)}B_1(z)) \) to extend the desired estimates to the neighborhood of infinity. As in the proof of Theorem 3.3, it holds when \( c = 0 \) and \( i = 1 \) that

\[ R_n^{(1)}(z) = o(1)\gamma_n\Phi_n^{(1)}(z)w_n^{-1}(z) \]

uniformly on closed subsets of \( \overline{\mathbb{C}} \,(\Delta_{1,1} \cup \Delta_{c,2}) \), from which the last asymptotic formula of the theorem follows, as usual, by (7.65) (holding for \( B_0(z) \) as well), (5.5)–(5.8), (3.10), and since \( A_{n,1} \sim n^c \) as shown in Lemma 5.1.

8. Proof of Theorem 1.2

While proving Theorem 1.2 we first consider the case of fully marginal sequences and then consider separately the asymptotic behavior of \( a_{n,1}, a_{n,2} \) and \( b_{n,1}, b_{n,2} \).

8.1. Fully Marginal Ray Sequences. In this section we only consider sequences \( N_0 \) and \( N_1 \) satisfying (3.1). Again, we present the proof only in the case of \( c = 0 \). Recurrence formula (1.7) for \( P_n(x) \) can be rewritten as

\[ z - b_{n,i} = \frac{P_{n+\epsilon_i}(z)}{P_n(z)} + a_{n,1} \frac{P_{n+\epsilon_i}(z)}{P_n(z)} + a_{n,2} \frac{P_{n+\epsilon_i}(z)}{P_n(z)}, \quad i \in \{1, 2\}. \]

One can easily see from (8.1) that

\[ b_{n,i} = \lim_{z \to \infty} \left( \frac{P_{n+\epsilon_i}(z)}{P_n(z)} - z \right). \]
Thus, the limiting behavior of $b_{\vec{n},1}, b_{\vec{n},2}$ follows from Theorem 3.1 and (6.2) in Lemma 6.1. Moreover, since the rays $\{\vec{n} \pm \vec{e}_i : \vec{n} \in \mathbb{N}_0\}$ are also fully marginal, we can use Theorem 3.1 to rewrite (8.1) for $i = 2$ as

$$(8.3) \quad z - b_{\vec{n},2} = (1 + o(1))\varphi_2(z) + (1 + o(1)) \frac{a_{\vec{n},1}}{S(z; \alpha_1)(z - \alpha_1)} + (1 + o(1)) \frac{a_{\vec{n},2}}{\varphi_2(z)}.$$  

Recall that $S(z; \alpha_1) = 1 - (B_{0,1} - \alpha_1)/z + O(z^{-2})$ by (6.7) and (2.7). Hence, if we use (2.6) to obtain the first four terms of the power series expansion of $\varphi_2(z)$ at infinity, we then can rewrite (8.3) as

$$(8.4) \quad z - b_{\vec{n},2} = (1 + o(1)) \left( z - B_{0,2} - \frac{A_{0,2}}{z} + \frac{A_{0,2}B_{0,2}}{z^2} + O\left(\frac{1}{z^3}\right) \right) + \frac{a_{\vec{n},1}}{z} \left( 1 + B_{0,1} + O\left(\frac{1}{z^2}\right) \right) + \frac{a_{\vec{n},2}}{z} \left( 1 + B_{0,2} + O\left(\frac{1}{z^2}\right) \right).$$

It follows immediately from (8.4) that

$$a_{\vec{n},1} + a_{\vec{n},2} = (1 + o(1))A_{0,2} \quad \text{and} \quad B_{0,1}a_{\vec{n},1} + B_{0,2}a_{\vec{n},2} = (1 + o(1))B_{0,2}A_{0,2},$$

from which the limits of $a_{\vec{n},1}, a_{\vec{n},2}$ easily follow (recall that $\varphi_2(z)$ is non-vanishing).

### 8.2. Asymptotics of $a_{\vec{n},1}, a_{\vec{n},2}$ along Non-fullly Marginal Sequences.

From now on we are assuming that ray sequences $\mathcal{N}_c$ satisfy (3.11). It can be deduced from orthogonality relations (1.5) and definition (3.14) that

$$R_{\vec{n}}^{(i)}(z) = -\frac{h_{\vec{n},i}}{2\pi i} \frac{1}{z^{n_{i+1}} + O(z^{-n_{i+2}})}, \quad h_{\vec{n},i} := \int P_{\vec{n}}(x) x^{n_{i+1}} d\mu_i(x),$$

$i \in \{1, 2\}$. In particular, we have that $m_{\vec{n},i} = -2\pi i/h_{\vec{n} - \vec{e}_i,i}$ in (7.1). Then it follows from the first and second asymptotic formulæ of Theorem 3.4, the definition of constants $\gamma_{\vec{n}}$ and $\tau_{\vec{n}}$ in Theorems 3.2 and 3.3, respectively, and the definition of the matrix $C$ in (7.6) that

$$-\frac{h_{\vec{n},i}}{2\pi i} = \frac{1 + o(1)}{s_{\vec{n},i}} \frac{[C]_{1,1}}{[C]_{1+1,i+1}} \quad \text{or} \quad -\frac{h_{\vec{n},i}}{2\pi i} = o(1) \frac{[C]_{1,1}}{[C]_{1+1,i+1}},$$

where, as before, $s_{\vec{n},i} = S_{\vec{n}}^{(0)}(x)/S_{\vec{n}}^{(1)}(x)$, $i \in \{1, 2\}$, the first formula holds for $i \in \{1, 2\}$ when $c \in (0, 1)$, $i = 2$ when $c = 0$, and $i = 1$ when $c = 1$, and the second formula holds for the remaining cases. Furthermore, we get from (7.1) that

$$-\frac{2\pi i}{h_{\vec{n} - \vec{e}_i,i}} = m_{\vec{n},i} = \lim_{z \to x^*} z^{1-|\vec{n}|} [Y(z)]_{i+1,i+1}.$$  

Analogously to the computation after (7.63)–(7.65) we get that $[Y(z)]_{i+1,i+1}$ is equal to

$$[C]_{1+1,i+1} + \frac{S_{\vec{n}}^{(0)}(z)}{S_{\vec{n}}^{(1)}(\infty)} \left( s_{\vec{n},i} \Upsilon_{\vec{n},i}(z) + B_{0,i}(z) + s_{\vec{n},i} B_{1,i}(z) \Upsilon_{\vec{n},i}(z) + s_{\vec{n},2} B_{2,i}(z) \Upsilon_{\vec{n},i}(z) \right) \Phi_{n_i}^{(0)}(z)$$

in a neighborhood of infinity, where $B_{k,i}(z) := [Z(z)]_{i+1,k+1} - \delta_{ik}$, $k \in \{0, 1, 2\}$, satisfy (7.65). Since $B_{k,i}(x) = 0$ and $\Upsilon_{\vec{n},i}(z) = A_{\vec{n},i}z^{-1} + O(z^{-2})$ as $z \to x^*$, see (5.3), we get that

$$-\frac{2\pi i}{h_{\vec{n} - \vec{e}_i,i}} = (s_{\vec{n},i} A_{\vec{n},i} + o(1)) \frac{[C]_{1+1,i+1}}{[C]_{1,1}}.$$  

Now, it is well known, see for example [7, Lemma A.1], that $a_{\vec{n},i} = h_{\vec{n},i}/h_{\vec{n} - \vec{e}_i,i}$. Therefore, it follows from (8.5) and (8.8) that

$$a_{\vec{n},i} = (1 + o(1))(A_{\vec{n},i} + s_{\vec{n},i}^{-1} o(1)) \quad \text{or} \quad a_{\vec{n},i} = o(1)(s_{\vec{n},i} A_{\vec{n},i} + o(1))$$

$i \in \{1, 2\}$, where the first formula holds for $i \in \{1, 2\}$ when $c \in (0, 1)$, $i = 2$ when $c = 0$, and $i = 1$ when $c = 1$, and the second formula holds for the remaining cases. The desired limits of $a_{\vec{n},i}$ therefore follow from continuity of the constants $A_{\vec{n},i}$ with respect to the parameter $c$, see Proposition 2.1, asymptotic formulæ (3.10), and the estimates $A_{c,1} \sim c^3$ as $c \to 0$ ($A_{c,2} \sim (1-c)^2$ as $c \to 1$), see (5.11) and after.
8.3. **Asymptotics of** $b_{\tilde{n},1}, b_{\tilde{n},2}$ **along Non-fully Marginal Sequences.** Excluding the cases $i = 1$ when $c = 0$ and $i = 2$ when $c = 1$, we get from (8.6)–(8.8) and (5.6)–(5.8) that

\[
P_{\tilde{n}+\epsilon_{i}}(z) = (1 + o(1))A_{\tilde{n}+\epsilon_{i}}(\text{S}_{\epsilon_{i}}\Phi_{\epsilon_{i}})^{(0)}(z)
\]

in some neighborhood of the point at infinity. Replacing the sequence $\mathcal{N}_c$ with $\{\tilde{n} + \epsilon_{i} : \tilde{n} \in \mathcal{N}_c\}$, we get from (8.2), Theorem 3.2, and (8.9) that

\[
b_{\tilde{n},i} = -(1 + o(1)) \lim_{z \to \infty} \left( \frac{A_{\tilde{n}+\epsilon_{i}}}{Y_{\tilde{n}+\epsilon_{i}}(z)} - z \right) = (1 + o(1))B_{\tilde{n}+\epsilon_{i}},
\]

where we also used (2.5) and (5.3). The desired claim now follows from Proposition 2.1. Out of the two exceptional cases, we shall only consider the case $i = 1$ when $c = 0$ understanding that the other one can be treated similarly. Assume for the moment that the measure $\mu_2$ is, in fact, the arcsine distribution on $\Delta_2$, that is,

\[
d\mu_2(x) = \frac{dx}{2\pi \sqrt{(x-\alpha_2)(\beta_2-x)}} = \frac{dx}{2\pi \text{Im}w_2(x)},
\]

Recall the notation of Section 6 where we wrote $P_\epsilon(z) = P_{\tilde{n},1}(z)P_{\tilde{n},2}(z)$ with polynomial $P_{\tilde{n},i}(z)$ having all its zeros on $\Delta_i$. We would like to show that when $\mu_2$ is of the form (8.10), formula (6.1) still holds along any marginal ray sequence $\mathcal{N}_0$. To this end, we shall use $2 \times 2$ Riemann-Hilbert analysis of orthogonal polynomials. Since this method has been described in detail in Section 7, we shall only outline the main steps.

It follows from (1.5) and (8.10) that the Riemann-Hilbert problem

(a) $Y(z)$ is analytic in $C \setminus \Delta_2$ and $\lim_{z \to \infty} Y(z)z^{-n_2} = I$;

(b) $Y(z)$ has continuous traces on each $\Delta_2^+$ that satisfy $Y_+(x) = Y_-(x)\begin{pmatrix} 1 & (P_{\tilde{n},1}/w_{2+})(x) \\ 0 & 1 \end{pmatrix}$;

(c) the entries of the first column of $Y(z)$ are bounded and the entries of the second column behave like $O(|z-\xi|^{-1/2})$ as $z \to \xi \in \{\alpha_2, \beta_2\}$;

is solved by

$Y(z) := \begin{pmatrix} P_{\tilde{n},2}(z) & R_{\tilde{n}}^{(2)}(z) \\ m^*_{\tilde{n},2}P_{\tilde{n},2}^{*}(z) & m^*_{\tilde{n},2}R_{\tilde{n}}^{(2)}(z) \end{pmatrix}$,

where $P_{\tilde{n},2}^{*}(z)$ is the monic polynomial of degree $n_2 - 1$ orthogonal to lower degree polynomials with respect to the weight $P_{\tilde{n},1}(x)d\mu_2(x)$ and

$R_{\tilde{n}}^{(2)}(z) = \frac{1}{2\pi i} \int P_{\tilde{n},2}(x)P_{\tilde{n},1}(x)d\mu_2(x) \frac{1}{x-z} = \frac{1}{m^*_{\tilde{n},2}z^{-n_2}} + O(z^{-n_2-1})$.

Let $\Gamma_2$ be a Jordan curve encircling $\Delta_2$ counter-clockwise and containing $\Delta_1$ in its exterior. Set

$X(z) := Y(z) \begin{pmatrix} 1 & 0 \\ (-w_2/P_{\tilde{n},1})(z) & 1 \end{pmatrix} z \in \Omega_2,$

otherwise,

where $\Omega_2$ is the interior domain of $\Gamma_2$. Then $X(z)$ solves the following Riemann-Hilbert problem:

(a) $X(z)$ is analytic in $C \setminus (\Delta_2 \cup \Gamma_2)$ and $\lim_{z \to \infty} X(z)z^{-n_2} = I$;

(b) $X(z)$ has continuous traces on $\Delta_2^+ \cup \Gamma_2$ that satisfy

$X_+(s) = X_-(s) \begin{pmatrix} 0 & (P_{\tilde{n},1}/w_{2+})(s) \\ -(w_{2+}/P_{\tilde{n},1})(s) & 0 \end{pmatrix}, \quad s \in \Delta_2$;

(c) the entries of the first column of $Y(z)$ are bounded and the entries of the second column behave like $O(|z-\xi|^{-1/2})$ as $z \to \xi \in \{\alpha_2, \beta_2\}$.

The solution of the above Riemann-Hilbert problem is given by $X(z) = C(ZL)(z)$, where

$L(z) := \begin{pmatrix} 1 & 1/w_2(z) \\ 1/\tilde{w}_2(z) & \tilde{w}_2(z)/w_2(z) \end{pmatrix} (S_{\tilde{n},2})_{\mu_2}^{(0)}(z)$.
Indeed, as in Section 7, we only need to verify that the jump of 
mentioned in Section 2.2, these operators appear in [7, Formula (4.20)] used with
in a neighborhood of the point at infinity. It follows from (8.11), (8.12), and (3.9) that
\[
P_{\vec{e}_1}(s) = \left( \begin{array}{c} \tilde{\varphi}_2(s) \\ \tilde{\varphi}_2(s) \end{array} \right) - \tilde{\varphi}_2(s) - \tilde{\varphi}_2(s),
\]
Observe that
\[
(\varphi_\tilde{e}_1(s))^{\frac{1}{2}} = \varphi_\tilde{e}_1(s) \prod_{i=1}^{n_1} b(s;x_{i}), \quad b(z;x_0) := \tilde{\varphi}_2(z) - \tilde{\varphi}(2,0) - \tilde{\varphi}_2(x_0) - 1.
\]
Notice that \( \inf_{\Gamma_1 \geq 2} |\tilde{\varphi}_2(s)| > 1 \) and \( \inf_{\Gamma_1 \geq 2, x_0 \in \Gamma_2} |b(s;x_0)| > 0 \) by the compactness of \( \Delta_1 \) and \( \Gamma_2 \).
Therefore, there exist positive constants \( C_1 > 1 \) and \( C_2 < 1 \) such that
\[
\sup_{\Gamma_1 \geq 2} |(w_2 P_{\tilde{e}_1}(s))^{\frac{1}{2}}(s)^{-1} \leq C_1 C_2^{n_1+n_2} = (C_1^{n_1/(2n_2+n_2)} C_2)^{2n_2+n_2} = o(1)
\]
as \( n_1/n_2 \to 0 \). This finishes the proof of the identity \( X(z) = C(\mathbf{ZL})(z) \) from which (6.1) easily
follows. Observe that \( \mu_2 \) as in (8.10) is a Szegö weight. Hence, Lemma 6.1 is applicable. Therefore,
\[
\lim_{|\vec{n}| \to \infty, \vec{n} \in N_0} \lim_{z \to i} \frac{P_{\vec{n}+\vec{e}_1}(z)}{P_{\vec{n}}(z)} = -B_{0,1}
\]
by (6.2). On the other hand, it should be clear from the above argument that the proof in Section 7
will work if \( \mu_2 \) is as in (8.10). Therefore, Theorem 3.2 for such a choice of \( \mu_2 \) gives us that
\[
\lim_{|\vec{n}| \to \infty, \vec{n} \in N_0} \lim_{z \to i} \frac{\tau_{\vec{n}+\vec{e}_1}(z)}{\tau_{\vec{n}}(z)} = -B_{0,1}
\]
in a neighborhood of the point at infinity. It follows from (8.11), (8.12), and (3.9) that
\[
\lim_{|\vec{n}| \to \infty, \vec{n} \in N_0} \lim_{z \to i} \frac{\tau_{\vec{n}+\vec{e}_1}(z)}{\tau_{\vec{n}}(z)} = -B_{0,1},
\]
where \( \tau_{\vec{n}} \) was defined in Theorem 3.3. Observe that (8.13) is a statement about Riemann surfaces
\( \mathcal{R}_{\vec{n}} \) for \( \vec{n} \in N_0 \) and is independent of the original measures \( \mu_1, \mu_2 \). By Theorem 3.2, (8.12) holds
for measures \( \mu_1, \mu_2 \) as in Theorem 1.2, which we are currently proving. Hence, polynomials \( P_{\tilde{e}_1}(z) \),
\( \vec{n} \in N_0 \), satisfy (8.11) by (8.13) and (3.9). The final claim of the theorem now follows from (8.2).

APPENDIX A.

In this Appendix, we will study the operators \( \mathcal{L}_c^{(1)} \) and \( \mathcal{L}_c^{(2)} \) defined in (2.10). As we have already
mentioned in Section 2.2, these operators appear in [7, Formula (4.20)] used with \( \vec{K} = \vec{e}_1 \) and \( \vec{K} = \vec{e}_2 \), respectively. In what follows, we denote by \( \delta^{(Y)} \) the delta function (Kronecker symbol) of the vertex \( Y \).
Consider two functions
\[
m_I(z) := \langle (\mathcal{L}_c^{(1)} - z)^{-1} \delta^{(O)}, \delta^{(O)} \rangle, \quad m_{II}(z) := \langle (\mathcal{L}_c^{(2)} - z)^{-1} \delta^{(O)}, \delta^{(O)} \rangle.
\]
Given the function \( \chi_c(z) \) from Proposition 2.1 and \( c \in (0,1) \), [7, Equation (4.22)] yields that
\[
m_I(z) = \frac{-1}{\chi_c^{(0)}(z) - B_{c,1}}, \quad m_{II}(z) = \frac{-1}{\chi_c^{(0)}(z) - B_{c,2}},
\]
where, as usual, \( \chi_c^{(0)}(z) \) are the values taken from the zero-th sheet \( \mathcal{R}_{c}^{(0)} \). By the Spectral Theorem
[3], they can also be written in the form
\[
m_I(z) = \int_{\mathbb{R}} \frac{d\sigma_c^{(1)}(x)}{x - z}, \quad m_{II}(z) = \int_{\mathbb{R}} \frac{d\sigma_c^{(2)}(x)}{x - z},
\]
where \( \sigma^{(l)}_\beta \) is the spectral measure of \( \delta^{(l)} \) with respect to \( L^{(l)}_c \), \( l \in \{1, 2\} \). The properties of the conformal map \( \chi_c(z) \) imply that the functions \( m_1(z) \) and \( m_{11}(z) \) satisfy:

(A) \( m_1(z) \) and \( m_{11}(z) \) have no poles since \( \chi_c^{(l)}(z) \neq B_{c,l} \) for \( z \in \mathbb{R}^{(0)}_c \) by conformality;

(B) both \( m_1(z) \) and \( m_{11}(z) \) are Herglotz-Nevanlinna functions in \( \mathbb{C}^+ \), i.e., they are analytic, have positive imaginary part, and are continuous up to the boundary. Moreover, \( 3m_1(x) = 3m_{11}(x) = 0 \) for \( x \in \mathbb{R} \{ \Delta_{c,1} \cup \Delta_{c,2} \} \) and \( 3m_1^{(l)}(x) = 0 \) for \( x \in \Delta_{c,1} \cup \Delta_{c,2}^{(l)} \).

We will use the following notation. If \( Y, Z \in \mathcal{V} \) and \( Y \sim Z \), then deleting the edge \((Y, Z)\) that connects them leaves us with two subtrees. The one containing \( Y \) will be called \( T_{[Y,Z]} \), the other one will be called \( T^c \). The restriction of any Jacobi matrix \( J \) to a subtree \( T^c \) will be denoted by \( J^c \).

We learned from (A) and (B) above that \( \sigma^{(1)}_O \) and \( \sigma^{(2)}_O \) are absolutely continuous measures with supports equal to \( \Delta_{c,1} \cup \Delta_{c,2} \). We need this for the following lemma.

**Lemma A.1.** If \( c \in (0,1) \), then \( L^{(1)}_0 \) and \( L^{(2)}_0 \) have no eigenvalues.

**Proof.** Suppose that \( L^{(l)}_c \), \( l \in \{1, 2\} \), has an eigenvector \( \Psi \). Since \( \sigma^{(l)}_O \) is purely absolutely continuous as just explained, the restriction of \( L^{(l)}_c \) to the cyclic subspace generated by \( \delta^{(l)} \) has no eigenvalues by the spectral theorem. Therefore, we must have \( \Psi_0 = 0 \). Now, consider the restrictions of \( \Psi \) to \( T_{[O,O(c)(1,1)]} \) and \( T_{[O,O(c)(1,2)]} \). One of these functions is not identically equal to zero and the one that is not must be an eigenvector of the corresponding operator: either \( J_{[O,O(c)(1,1)]} \) or \( J_{[O,O(c)(1,2)]} \). By construction, these operators are identical to either \( L^{(1)}_c \) or \( L^{(2)}_c \) and, as we established earlier, this implies that \( \Psi_{O(c)(1,1)} = \Psi_{O(c)(1,2)} = 0 \). Repeating the argument, we can now show that \( \Psi = 0 \) identically on the whole tree which gives a contradiction. \( \square \)

The following observation holds for a general Jacobi matrix \((2.8) \text{ and } (2.9) \). Let \( \sigma_Y \) denote the spectral measure of \( \delta^{(Y)} \) with respect to \( J^c \), i.e.,

\[
m_Y(z) := \left\langle (J - z)^{-1} \delta^{(Y)}, \delta^{(Y)} \right\rangle = \int_{\mathbb{R}} \frac{d\sigma_Y(x)}{x-z}, \quad z \in \mathbb{C}^+.\]

If we delete all edges connecting \( Y \) to its neighbors, say \( l \) of them, we will be left with the vertex \( Y \) and \( l \) subtrees \( T_{[Y,Y_j]} \). The restrictions of \( J \) to these subtrees are also Jacobi matrices and we previously denoted them by \( J_{[Y,Y_j]} \). Let

\[
m_{[Y,Y_j]}(z) := \left\langle (J_{[Y,Y_j]} - z)^{-1} \delta^{(Y_j)}, \delta^{(Y_j)} \right\rangle = \int_{\mathbb{R}} \frac{d\sigma_{[Y,Y_j]}(x)}{x-z}, \quad z \in \mathbb{C}^+.\]

Then the following lemma holds.

**Lemma A.2.** For every \( z \in \mathbb{C}^+ \), we have

\[
m_Y(z) = \frac{1}{V_Y - \sum_{j=1}^l W_{Y,Y_j} m_{[Y,Y_j]}(z) - z}.\]

**Proof.** Let \( f := (J - z)^{-1} \delta^{(Y)} \). Clearly, \( J f = zf + \delta^{(Y)} \), that is,

\[
(J f)_X = \begin{cases} V_X f_X + \sum_{Z \sim X} W^{1/2}_{X,Z,XY} f_X = zf_X, & X \neq Y, \\
V_Y f_Y + \sum_{j=1}^l W^{1/2}_{Y,Y_j} f_{Y_j} = zf_Y + 1, & X = Y.\end{cases}
\]

Set \( f^{(j)} := - (W^{1/2}_{Y_j,Y_j} f_{Y_j})^{-1} f_{[Y,Y_j]} \), which is a renormalized restriction of \( f \) to the set of vertices \( V_{[Y,Y_j]} \) of \( T_{[Y,Y_j]} \). Observe that

\[
(J f^{(j)})_X = \begin{cases} (J f^{(j)})_X = zf^{(j)}_X, & X \neq Y, \\
V_{Y_j} f^{(j)}_{Y_j} + \sum_{Z \sim Y_j, Z \neq Y} W^{1/2}_{Z,Y_j} f^{(j)}_Z = zf^{(j)}_{Y_j} + 1, & X = Y.\end{cases}
\]

where both relations follow from the first line of \((A.5) \) (for the second relation we need to separate the summand corresponding to \( Z = Y \), bring it to the other side of the equation, and then divide by it). It follows immediately from \((A.6) \) that

\[
J f^{(j)}_Y = zf^{(j)}_Y + \delta^{(Y)} \Rightarrow f^{(j)}_Y = (J f^{(j)}_Y - z)^{-1} \delta^{(Y)}.\]
The claim of the lemma follows from the second equality in (A.5) since \( f_Y = \langle (\mathcal{J} - z)^{-1}\delta(Y), \delta(Y) \rangle = m_Y(z) \) and similarly \( f_{Y_i} = -(W_{Y,Y_i}^2 f_Y) f_{Y_i} = -W_{Y,Y_i}^2 m_Y(z)m_{Y(Y_i)} \).

Let us now return to the operators \( \mathcal{J} = \mathcal{L}_c^{(l)}, l \in \{1, 2\} \). Take any vertex \( Y \neq O \). Deleting the edge \((Y, Y_{(p)})\) leaves us with two subtrees. As before, we denote by \( T_{[Y, Y_{(p)}]} \) the one containing \( Y_{(p)} \), and let \( m_{Y}^{(l)}(z) \) and \( m_{Y,Y_{(p)}}^{(l)}(z) \) to be given by (A.2) and (A.3), respectively (with \( \mathcal{J} = \mathcal{L}_c^{(l)} \)).

**Lemma A.3.** For every \( Y \neq O \), the function \( m_{Y}^{(l)}(z) \) is meromorphic in \( \mathbb{C} \setminus (\Delta_{c,1} \cup \Delta_{c,2}) \) and the function \( m_{Y}^{(l)}(z) \) is analytic there.

**Proof.** Recall that the functions \( m_l(z) \) and \( m_{11}(z) \) are in fact analytic in \( \mathbb{C} \setminus (\Delta_{c,1} \cup \Delta_{c,2}) \). We shall prove the desired claims inductively on \( n \), the distance from \( Y \) to the root \( O \). Assume first that \( n = 1 \). Let \( \nu \) be the type of \( Y \). Formula (A.4) applied at the vertex \( O \) to the operator \( \mathcal{L}_c^{(l)} \) restricted to the subtree \( T_{[O,Y]} \) gives

\[
m_{Y}^{(l)}(z) = \frac{1}{B_{c,l} - A_{c,3-l} m_{O,Z}^{(l)}(z) - z},
\]

where \( Z \) is the other “child” of \( O \) and we used an obvious fact that the restriction of \( \mathcal{L}_c^{(l)} \) from \( T_{[Y,O]} \) to the subtree \( T_{[O,Z]} \) is the same as the restriction of \( \mathcal{L}_c^{(l)} \) from \( T \) to \( T_{[O,Z]} \). Since the restriction of \( \mathcal{L}_c^{(l)} \) to \( T_{[O,Z]} \) is \( \mathcal{L}_c^{(l-1)} \), \( m_{O,Z}^{(l)}(z) \) is equal to either \( m_l(z) \) when \( \nu = 2 \) or \( m_{11}(z) \) when \( \nu = 1 \). In any case, \( m_{Y}^{(l)}(z) \) is meromorphic outside \( \Delta_{c,1} \cup \Delta_{c,2} \).

Suppose now that the claims are true for all vertices up to the distance \( n \). Consider any \( Y \) such that its distance from the root is \( n + 1 \). Let \( \nu \) be the type of \( Y \). As in the first part of the proof, apply (A.4) at the vertex \( Y_{(p)} \) of the subtree \( T_{[Y,Y_{(p)}]} \) to get

\[
m_{Y}^{(l)}(z) = \frac{1}{B_{c,\nu} - A_{c,\nu} m_{Y_{(p)},Y_{(p)}}(z) - A_{c,3-l} m_{Y_{(p)},Z}(z) - z},
\]

where \( \nu_{(p)} \) is the type of \( Y_{(p)} \) and \( Z \) is the “sibling” of \( Y \). The first function in the denominator is meromorphic outside \( \Delta_{c,1} \cup \Delta_{c,2} \) by the inductive assumption and the other one is either \( m_l(z) \) or \( m_{11}(z) \). Thus, \( m_{Y}^{(l)}(z) \) is also meromorphic outside \( \Delta_{c,1} \cup \Delta_{c,2} \). This way we get the claim for \( n + 1 \) and so we proved the first statement of the lemma.

Now, apply (A.4) to \( m_{Y}^{(l)}(z) \). The functions involved are \( m_{Y,Y_{(p)}}(z), \ z \in \{1, 2\}, \) and \( m_{Y,Y_{(p)}}(z) \). The first two are \( m_l(z) \) and \( m_{11}(z) \) and they are analytic in the considered domain. The third one is meromorphic there by the first statement of the lemma. Notice that \( m_{Y}^{(l)}(z) \) can not have poles by Lemma A.1 thus it is analytic outside \( \Delta_{c,1} \cup \Delta_{c,2} \).

**Lemma A.4.** Let \( Y \in \mathcal{V} \) and \( c \in (0,1) \). If \( \sigma_{Y}^{(l)} \) is the spectral measure of \( Y \) with respect to \( \mathcal{L}_c^{(l)} \), \( l \in \{1, 2\} \), then it is absolutely continuous and its support is equal to \( \Delta_{c,1} \cup \Delta_{c,2} \).

**Proof.** The measure \( \sigma_{Y}^{(l)} \) is purely absolutely continuous and is supported on \( \Delta_{c,1} \cup \Delta_{c,2} \) as explained before Lemma A.1. Fix \( Y \neq O \) and let \( \nu_Y \) be the type of \( Y \). Further, let \( m_{Y}^{(l)}(z) \) and \( m_{Y,Y_{(p)}}^{(l)}(z) \) be given by (A.2) and (A.3), respectively, with \( \mathcal{J} = \mathcal{L}_c^{(l)} \). Then is follows from (2.10) and (A.4) that

\[
\Im m_{Y}^{(l)}(E + i\epsilon) = \frac{A_{c,\nu} \Im m_{Y_{(p)},Y_{(p)}}^{(l)}(E + i\epsilon) + \sum_{i=1}^{2} A_{c,\nu} \Im m_{Y,Y_{(p)}}^{(l)}(E + i\epsilon) + \epsilon}{B_{c,\nu} - A_{c,\nu} m_{Y_{(p)},Y_{(p)}}^{(l)}(E + i\epsilon) - \sum_{i=1}^{2} A_{c,\nu} m_{Y,Y_{(p)}}^{(l)}(E + i\epsilon) - (E + i\epsilon)^2} \leq \frac{1}{A_{c,\nu} \Im m_{Y,Y_{(p)}}^{(l)}(E + i\epsilon) + \sum_{i=1}^{2} A_{c,\nu} \Im m_{Y,Y_{(p)}}^{(l)}(E + i\epsilon) + \epsilon} \leq \frac{1}{A_{c,\nu} \Im m_{Y,Y_{(p)}}^{(l)}(E + i\epsilon)},
\]

because the imaginary parts of all \( m \)-functions are positive in \( \mathbb{C}^+ \). Notice now that the restriction of \( \mathcal{L}_c^{(l)} \) to any subtree of the type \( T_{[Z_{(p)},Z]} \) is in fact equal to either \( \mathcal{L}_c^{(1)} \) or \( \mathcal{L}_c^{(2)} \). Therefore, \( m_{Y,Y_{(p)}}^{(l)}(z) \) is either \( m_l \) or \( m_{11} \). The properties (A) and (B) of \( m_l \) and \( m_{11} \) listed above can be now applied to get

\[
\sup_{E \in I, 0 < \epsilon < 1} \Im m_{Y}^{(l)}(E + i\epsilon) < \infty
\]
Theorem A.1. We have that \( \sigma^l(\mathcal{L}_c^{(l)}) = \sigma_{\text{ess}}(\mathcal{L}_c^{(l)}) = \Delta_{c,1} \cup \Delta_{c,2} \), \( l \in \{1, 2\} \), where, as before, we understand that \( \Delta_{0,1} := \{ \alpha_1 \} \) and \( \Delta_{1,2} := \{ \beta_2 \} \).

Proof. If \( c \in (0, 1) \), Lemma A.4 shows that \( \delta^{(Y)} \) belongs to the absolutely continuous subspace of \( \mathcal{L}_c^{(l)} \) for all \( Y \). Since all linear combinations of \( \delta^{(Y)} \) must belong to this subspace and are dense in \( \ell^2(Y) \), this subspace is in fact the whole space \( \ell^2(Y) \). Thus, \( \sigma(\mathcal{L}_c^{(l)}) = \sigma_{\text{ess}}(\mathcal{L}_c^{(l)}) \) and it is equal to \( \Delta_{c,1} \cup \Delta_{c,2} \) by Lemma A.4 and the Spectral Theorem.

Let \( c \in \{0, 1\} \). We shall consider \( \mathcal{L}_c^{(2)} \) only, other cases can be handled similarly. By (2.7), we have \( A_{0,1} = 0 \) and \( A_{0,2} > 0 \). Thus, the operator \( \mathcal{L}_c^{(2)} \) decouples into the following direct sum

\[
\mathcal{L}_c^{(2)} = A_1 \oplus \left( \bigoplus_{n=1}^\infty A_2 \right)
\]

where \( A_1 \) is one-sided Jacobi matrix

\[
A_1 := \begin{pmatrix}
B_{0,2} & \sqrt{A_{0,2}} & 0 & 0 \\
0 & B_{0,2} & \sqrt{A_{0,2}} & 0 \\
0 & 0 & \ddots & \ddots \\
0 & 0 & \ddots & B_{0,2} & \sqrt{A_{0,2}} \\
\end{pmatrix}
\]

and \( A_2 \) is one-sided Jacobi matrix given by

\[
A_2 := \begin{pmatrix}
B_{0,1} & \sqrt{A_{0,2}} & 0 & 0 \\
0 & B_{0,2} & \sqrt{A_{0,2}} & 0 \\
0 & 0 & \ddots & \ddots \\
\end{pmatrix}
\]

This direct sum decomposition implies that \( \sigma(\mathcal{L}_c^{(2)}) = \sigma(A_1) \cup \sigma(A_2) \). It is well known that \( \sigma(A_1) = [B_{0,2} - 2\sqrt{A_{0,2}}, B_{0,2} + 2\sqrt{A_{0,2}}] = [\alpha_2, \beta_2] \), see (2.7) for the second equality, and that

\[
\hat{m}_1(z) := \langle (A_1 - z)^{-1} \delta^{(0)}, \delta^{(0)} \rangle = \frac{B_{0,2} - z + \sqrt{(z - B_{0,2})^2 - 4A_{0,2}}}{2A_{0,2}} = \frac{B_{0,2} - z + w_2(z)}{2A_{0,2}}.
\]

Furthermore, since the restriction of \( A_2 \) from \( \ell^2(\mathbb{Z}_{\geq 0}) \) to \( \ell^2(\mathbb{N}) \) is equal to \( A_1 \) and therefore \( m_{[0,1]}(z) = \hat{m}_1(z) \) in the notation of (A.3), we get from (A.4) that

\[
\hat{m}_2(z) := \langle (A_2 - z)^{-1} \delta^{(0)}, \delta^{(0)} \rangle = \frac{-1}{A_{0,2}\hat{m}_1(z) + z - B_{0,1}},
\]

where \( w_2(z) \) was introduced in the Proposition 2.1. One can readily check that \( \Im \hat{m}_2(x) > 0 \) for \( x \in (\alpha_2, \beta_2) \), \( \Im \hat{m}_2(x) = 0 \) for \( x \notin [\alpha_2, \beta_2] \), and that \( \hat{m}_2(z) \) has the unique pole at a point \( \bar{x} \in \mathbb{R} \) given by

\[
A_{0,2}\hat{m}_1(\bar{x}) + \bar{x} - B_{0,1} = 0
\]

which implies that \( \bar{x} = \alpha_1 \) thanks to (2.7). In other words, \( \sigma(A_2) = \alpha_1 \cup [\alpha_2, \beta_2] \). Now, the statement about the spectrum and essential spectrum follows from direct sum decomposition (A.7). \( \square \)

References

[1] C. Allard and R. Froese. A Mourre estimate for a Schrödinger operator on a binary tree. Rev. Math. Phys. 12, no. 12, 1655–1667, 2000.
[2] N.I. Akhiezer. The classical moment problem and some related questions in analysis, Hafner Publishing Co., New York, 1965.
[3] N.I. Akhiezer, I.M. Glazman. Theory of linear operators in Hilbert space. Dover Publications, Inc., New York, 1993.
[4] A. Angelesco. Sur deux extensions des fractions continues algébriques. Comptes Rendus de l’Académie des Sciences, Paris, 168:262–265, 1919.
[5] A. I. Aptekarev. Asymptotics of polynomials of simultaneous orthogonality in the Angelesco case. Mat. Sb. (N.S.), 136(178)(1):56–84, 1988.
[6] A.I. Aptekarev, A.I. Bogolubsky, and M. Yattselev. Convergence of ray sequences of Frobenious-Padé approximants. Math. Sb., 208(3):4–27, 2017. https://doi.org/10.4213/sm8632.
A.I. Aptekarev, S.A. Denisov, and M.L. Yattselev. Self-adjoint Jacobi matrices on trees and multiple orthogonal polynomials. *Trans. Amer. Math. Soc.*, 373(2), 875–917, 2020. [https://doi.org/10.1090/tran/7959](https://doi.org/10.1090/tran/7959). 1, 2, 3, 4, 7, 23, 43, 45

A. I. Aptekarev, M. Derevyagin, W. Van Assche. Discrete integrable systems generated by Hermite–Padé approximants. *Nonlinearity*, 29(5):1487–1506, 2016. 3

L. Baratchart and M. Yattselev. Convergent interpolation to Cauchy integrals over analytic arcs with Jacobi-type weights. *Int. Math. Res. Not.*, 2010(22):4211–4275, 2010. [https://doi.org/10.1093/imrn/rnq026](https://doi.org/10.1093/imrn/rnq026). 22

J. Breuer, R. Frank. Singular spectrum for radial trees. *Rev. Math. Phys.*, 21, no. 7, 929–945, 2009. 3

J. Breuer, S. Denisov, and L. Eliaz. On the essential spectrum of Schrödinger operators on trees. *Math. Phys. Anal. Geom.*, 21(4), 2018. Art. 33, 25 pp. 6

P. Deift. *Orthogonal Polynomials and Random Matrices: a Riemann-Hilbert Approach*, volume 3 of *Courant Lectures in Mathematics*. Amer. Math. Soc., Providence, RI, 2000. 38, 39

P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, and X. Zhou. Strong asymptotics for polynomials orthogonal with respect to varying exponential weights. *Comm. Pure Appl. Math.*, 52(12):1491–1552, 1999. 36

P. Deift and X. Zhou. A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the mKdV equation. *Ann. of Math.*, 137:295–370, 1993. 24

W.J.W. Olver et al. editors. NIST digital library of mathematical functions. [http://dlmf.nist.gov](http://dlmf.nist.gov). gov. 23

A.S. Fokas, A.R. Its, and A.V. Kitaev. The isomonodromy approach to matrix models in 2D quantum gravitation. *Ann. Henri Poincaré*, 5, no. 6, 1097–1115, 2004. 3

A.A. Gonchar and E.A. Rakhmanov. On convergence of simultaneous Padé approximants for systems of functions with different branch orders. *J. Approx. Theory*, 68(5):1159–1200, 2016. [http://dx.doi.org/10.4153/CJM-2015-043-3](http://dx.doi.org/10.4153/CJM-2015-043-3). 8, 11, 14, 16, 19, 24, 25, 26, 36, 40

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