Fiber bundle description of number scaling in gauge theory and geometry.

Paul Benioff,
Physics Division, Argonne National Laboratory,
Argonne, IL 60439, USA
e-mail:pbenioff@anl.gov

December 5, 2014

Abstract

This work uses fiber bundles as a framework to describe effects of number scaling on gauge theory and geometry. A brief description of number scaling and fiber bundles over a space time manifold, $M$, is followed by a description of gauge theory. A fiber at point $x$ of $M$ contains a vector space, $\bar{V}_x$, and a set, $\bar{C}_x$, of complex scalars and scaled structures, $\bar{C}_{c,x}, \bar{V}_{c,x}$ for each complex number, $c$. Number scaling induces connections, $c_{x,y} \times c_{x,y}$ between fibers at $x$ and $y$. Connections are given as exponentials of a complex vector field, $\vec{A}(x) + i\vec{B}(x)$. The choice of the gauge group as $GL(1, \mathbb{C})$ for $\bar{V}_x$ and $\bar{C}_x$ gives the result that $\vec{B}$ is massless, and no mass restrictions for $\vec{A}$. In the Mexican hat Higgs mechanism $\vec{B}$ combines with a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. The very speculative possibility that $\vec{A}$ might be the gradient of the Higgs field is noted. The association of $\vec{B}$ to the photon field can be avoided by replacing $\vec{B}$ by two fields, each with different coupling constants. Tangent bundles are used to describe real number scaling in geometry. Each fiber at $x$ contains a tangent space $\bar{T}_x$ and an associated real scalar field, $\bar{R}_x$. Scaled structures, $\bar{T}_{c,x}$ and $\bar{R}_{c,x}$ are also present. Connections between neighbor fields, are defined as in gauge theory with $\vec{B}(x) = 0$ everywhere. Number scaling is shown to affect curve lengths and the geodesic equation. The lack of physical evidence for the $\vec{A}$ field means that either it couples very weakly to matter fields, or that $\vec{A}(x) \sim 0$ for all $x$ in a local region of cosmological space and time. It says nothing about the values outside the local region.

1 Introduction

The use of fiber bundles [1, 2] in the description of gauge theories and other areas of physics and geometry has grown in recent years [3]-[7]. In gauge theories connections between fibers in a bundle are described using Lie algebra representations of gauge groups. For Abelian theories the gauge group is $U(1)$ with $e^{i\alpha(x)}$ as a local gauge transformation. For nonabelian theories [8, 9] the gauge group is $U(n)$ with the exponential of a sum over generators of $su(n)$ as the local gauge transformation. Additional details are given in many texts about field theories [10, 11, 12].

The fiber bundles used to describe gauge theories and geometric quantities are vector bundles over a manifold, $M$. For gauge theories a fiber, $F_x$ at point $x$ of $M$ contains a finite dimensional vector space $\bar{V}_x$. Complex numbers are the scalars for the spaces. For geometric quantities, the vector space at $x$ is a tangent space, $\bar{T}_x$. Real numbers are the scalars for the tangent spaces. In this work $M$ is assumed to be a flat space time manifold. Gauge theories on curved spaces [13] will not be considered.

This work expands the usual treatment of gauge theories and geometric quantities in two ways. In the usual treatments the scalars are treated as if they are outside the vector bundles. There is just one global scalar field for the vector spaces at different locations. Here this is changed by associating local scalar fields with the vector spaces. For gauge theories a fiber at $x$ is expanded to include a complex scalar field, $\bar{C}_x$ with $\bar{V}_x$. For tangent bundles each fiber is expanded to include a real scalar field, $\bar{R}_x$ with $\bar{T}_x$. This expansion is achieved by use of the fiber product [1, 2] of two bundles.

The other expansion is based on a description of mathematical systems of many different types as structures [14, 15]. A structure consists of a base set, and a few basic operations, relations, and constants for which a relevant set of axioms are true. These are referred to as models in mathematical logic [16, 17]. The property of number structures that is relevant here is that they can be scaled. Real or complex numbers,
or numbers of any type, in a number structure can be multiplied (scaled) by a number provided the basic operations and relations can be scaled to compensate for the number scaling. The scaling must be such that the scaled structures satisfy the relevant axioms if and only if the unscaled structures do.

Scaling of fields in general and number structures in particular has been described elsewhere [18]. Other work [19, 20] described the effects of number scaling on gauge theories and geometric quantities. Here this work is extended by the use of fiber bundles. Connections or parallel transfers between fibers include the effects of number scaling. The effects apply to both the scalar and vector space structures in each fiber.

Scaling is by no means new in physics and geometry. It was used almost 100 years ago by Weyl [21] to construct a complete differential geometry. Scaling is an important component of renormalization theory [11], scale invariance of physical quantities, and of conformal field theories [22, 23].

The number structure scaling used here in the connections between fibers is different from these types of scaling. It is based on the fact that parallel transfer of terms in expressions and equations includes scaling of the quantities and of the multiplication operations used in the terms. If $E_y$ is an expression that describes a physical or geometric quantity at $y$, parallel transfer of this expression from $y$ to $x$ along path $p$ corresponds to multiplying the expression by a possibly path dependent scalar.

The scaling used here is different from that in conformal theories in that all quantities are scaled. This includes vector lengths and angles between vectors, and other quantities. It may seem strange and counter-intuitive to scale angles. However, as will be seen later on, the properties of number scaling are such that trigonometric relations are preserved under scaled parallel transfer from one fiber to another.

The plan of the paper is to first describe the salient aspects of scaling number and vector space structures. This is done in Section 2. This is followed, in Section 3 by a brief description of the parts of the theory of fiber bundles and connections that are needed here. Connections are described by exponential integrals over a complex vector field, $\vec{A}(x) + i\vec{B}(x)$.

The usual description of gauge theories, based on $U(1)$ and $U(N)$ transformations of Abelian and non-Abelian theories is expanded, in Section 5, to include the effects of number scaling. The expansion replaces $U(1)$ by $GL(1, C)$ and $U(N)$ by $GL(1, C) \times SU(N)$. Parallel transfer of both the scalars and the vector spaces is described by replacing $GL(1, C)$ by $DGL(1, C)$ consisting of the diagonal elements, $d \times d$, of $GL(1, C) \times GL(1, C)$. The field, $\vec{B}(x)$ is the Lie algebra representation of the $U(1)$ component of the gauge groups.

Section 5 discusses possible physical properties of $\vec{A}$ and $\vec{B}$. The Higgs mechanism for the Mexican hat potential is applied to the complex scalar field as a level section on the scalars in the product fiber bundle. The $\vec{B}$ field combines with a Goldstone boson to give a massive vector boson, a photon, and a Higgs field. The speculative possibility that $\vec{A}$ is the gradient of the Higgs field is noted.

The effect of scaling on curve lengths and the geodesic equation is discussed in section 7. The fiber bundle has the same structure as it does for gauge theories. Complex scalars are replaced by real numbers and the vector spaces are tangent spaces over $M$. The Lie algebra form of the connections is obtained by setting $\vec{B} = 0$ everywhere. $\vec{A}$ is assumed to be the gradient of a scalar field, $\theta(x)$. Both curve lengths and the geodesic equation are affected by the presence of $\vec{A}$.

The fact that all experiments and measurements in physics are limited to a region of space and time occupiable by us, and other observers within communication distances, places restrictions on the $\vec{A}$ field. As noted in Section 8 the lack of evidence of $\vec{A}$ means that $\vec{A}(x) \sim 0$ to within experimental error in the region. $\vec{A}$ can have any value outside the region. The region should be small with respect to the size of the observable universe.

Section 9 concludes the paper. It is noted that number scaling has no effect on comparisons of computations with one another or with results of measurements carried out at different space time points.

It should be noted that the idea of local mathematical structures is not new. It has been described in the context of category theory in which locality refers to structure interpretations in different categories [24]. Here locality refers to locations in space time. Locality in gauge field theories has also been discussed [25], but not with respect to mathematical structures.

---

1 Weyl introduced a nonintegrable real vector field to describe the scaling under parallel transfer of quantities. The problem, as noted by Einstein [21], was that this required that the properties of measuring rods, clocks, and atomic spectra depend on their past history. He also noted that this contradicts the known physical properties of rods, clocks, and atomic spectra. A good description of Weyl’s work, including Einstein’s remarks and subsequent developments of early gauge theory are in a book by O’Raifeartaigh [33].
2 Number scaling

Since number scaling has been described elsewhere, the description here will be relatively brief. It is limited to real and complex numbers. Scaling also applies to rational numbers, and, with slight modifications, to natural numbers and integers.

The description of mathematical systems as structures that satisfy a relevant set of axioms is the basis for number scaling. Examples of structures of use here are the real and complex numbers and vector spaces. Explicit representations of these structures are given by

\[ R = \{ R, +, \times, <, 0, 1 \} \]
\[ C = \{ C, +, \times, 0, 1 \}, \]

and

\[ V = \{ V, +, -, \cdot, |v|, v \}. \]

\( R \) and \( C \) are the base sets, of real and complex number structures, \(+\) and \(\times\) are the basic operations, and \(<\) is an ordering relation. \(0\) and \(1\) are elements of \(R\) and \(C\) that satisfy the respective additive and multiplicative identity axioms. The structures, \(\bar{R}\) and \(\bar{C}\) satisfy the axioms for a complete ordered field and an algebraically complete field of characteristic 0. \(\bar{V}\) satisfies the axioms for a normed vector space. Here \(v\) denotes an arbitrary vector in the base set, \(V\). The operation, \(\cdot\), denotes scalar vector multiplication, and \(|v|\) denotes the real valued norm in \(\bar{V}\).

Here and in the following, structures, such as \(\bar{R}, \bar{C}\), and \(\bar{V}\), are denoted by overlined symbols. This distinguishes them from base sets, \(R, C,\) and \(V\), without overlines.

Number scaling is based on the observation that number structures can be scaled, provided that, where needed, the basic operations and relations are also scaled. The scaling must be such that the truth of the relevant axioms is preserved under the scaling. If this can not be achieved, then scaling is not possible.

It turns out that number systems of different types, and more generally, fields, and systems that include numbers in their description can be scaled. To see this let \(r\) be a positive real number in \(R\). Let \(\bar{R}_r\) denote the structure \(\bar{R}\) scaled by \(r\). The components of \(\bar{R}_r\) are given by

\[ \bar{R}_r = \{ R_r, +_r, \times_r, <_r, 0_r, 1 \}. \]

The subscript, \(r\), on the base operations and relation indicate that they are defined such that 0 is the additive identity, \(a +_r 0 = a\), and \(r\) is the multiplicative identity, \(a \times_r r = a\) for all \(a\) in \(R\). Here \(0\) and \(r\) are numbers in \(R\).

There is a representation of the operations and relation in \(\bar{R}_r\), based on those in \(\bar{R}\), that satisfies the axiom validity preservation requirement. This representation is given by

\[ \bar{R}_r = \{ R, +, \frac{\times}{r}, <, 0, r \}. \]

References to \(\bar{R}_r\) in the following refer to this representation unless otherwise indicated.

This structure shows that addition and the relation, \(<\), are the same in the scaled structure as in the unscaled one. The only operation that needs to be changed is multiplication. This is shown by the observation that multiplication in \(\bar{R}_r\) corresponds to multiplication in \(\bar{R}\) scaled by \(1/r\).

Much use is made here and in the following of the fact that \(R\) is the same base set for both \(\bar{R}\) and \(\bar{R}_r\), scaled real number structures. This is a consequence of the fact for any \(r \neq 0\), \(rR = R.\) Multiplication by \(r\) is a bijection from \(R\) to \(\bar{R}\).

The fact that real number axiom validity is invariant under scaling of number structures can be extended to properties in general. If \(P\) is a property of numbers, then for each \(a\) in \(R\), \(P(a)\) is true for \(\bar{R}\) if and only if \(P_r(ra)\) is true for \(\bar{R}_r\). \(P_r\) is obtained from \(P\) by replacing all basic operations used to describe \(P\) by the corresponding ones in \(\bar{R}_r\).

A important consequence of scaling is that the elements of the base set, \(R\), must be separated from the values they have in different structures. Here and in the following the elements of \(\bar{R}\) will be referred to as numbers. This is to distinguish them from the different number values they may have in different scaled structures.

\[ \text{Variation of operations in abstract algebras has been noted in passing in [24]. Here scaling of basic operations in structures is a special example of this.} \]
This distinction is not needed if one works with just one structure, $\tilde{R}$. In this case there is no difference between number and number value. This is the usual setup in mathematical analysis. The distinction can also be ignored when the scaling factor is the identity.

Here this is not the case. Each real number structure provides a background or reference frame for giving values to the elements of $R$. The values depend on the scaling factor. For example, the value of the number, 1, in the base set, $R$, is $1_v$ in $\tilde{R}$. In a scaled structure, $\tilde{R}_r$, the numbers, 1, $\sqrt{2}$, and $r$ in $R$ have values, $(1/r)_v$, $(\sqrt{2}/r)_v$, and $1_v$ in $\tilde{R}_r$. These are different from the values, $1_v$, $\sqrt{2}_v$, $r_v$ they have in $\tilde{R}$. The elements of $R$ are called numbers to distinguish them from the values they have in different structures. The subscript, $v$, is used to distinguish number values from from numbers.

The distinction between numbers and number values is reminiscent of the distinction in mathematical logic between syntactic structures, as meaningless strings of symbols, and semantic structures, as meaningful models of the syntactic structures. However these two distinctions are not the same. This can be seen by noting that, in the absence of scaling, the concepts of numbers and number values coincide. Yet the distinction between syntactic and semantic structures remains.

The description of number scaling shows that, in the presence of scaling, there is just one number where the distinction between elements of $\tilde{R}$ and their values in a structure is not needed. This is the number 0. This number has the same value, 0, in all scaled structures. It is like a number vacuum in that the value, 0, is invariant under all scalings.

One needs to be careful in the use of scaling factors $\neq 1$. For example, $(ra)^{2r}$ in $\tilde{R}_r$ corresponds to $r(a)^{2r}$ in $\tilde{R}$. It does not correspond to $(ra)^{2r}$. The subscript, $r$, on the power, as in $2_r$, or on any power, indicates that the implied multiplication(s) is $\times/r$.

The same holds for negative powers. Here $(ra)^{-1r}$ in $\tilde{R}_r$ corresponds to the number, $ra^{-1}$, in $\tilde{R}$. It does not correspond to $(ra)^{-1r}$. More generally a term, $(ra)^m(rb)^n(rc)^{-pr}$ in $\tilde{R}_r$ corresponds to the term, $r(a^n)b^n(c^{-p})$ in $\tilde{R}$.

This can be extended to analytic functions with $R$ the domain and range. Let $f_r$ and $f$ denote corresponding analytic functions in $\tilde{R}_r$, and $\tilde{R}$. The term correspondences give

$$f_r(ra) = rb \leftrightarrow rf(a) = rb \leftrightarrow f(a) = b.$$  

This shows that the relation between $f_r$ and $f$ is given by $f_r(ra) = rf(a)$.

A nice property of number scaling is that equations are preserved under an isomorphic map from $\tilde{R}_r$ to $\tilde{R}$. This follows from the equivalences of equations for general mathematical terms. If $T_r(ra, rb, \cdots)$ and $U_r(ra, rb, \cdots)$ are terms in $\tilde{R}_r$, then

$$T_r(ra, rb, \cdots) = U_r(ra, rb, \cdots) \leftrightarrow rT(a, b, \cdots) = rU(a, b, \cdots)$$

$$\leftrightarrow T(a, b, \cdots) = U(a, b, \cdots).$$  

The left-hand equation is in $\tilde{R}_r$ and the two right-hand ones are in $\tilde{R}$. The terms can be quite general in that they can include integrals and derivatives. This preservation property extends to isomorphic maps from $\tilde{R}_r$ to $\tilde{R}_s$.

One can define a structure group acting on the set, $\{\tilde{R}_r : r \in R\}$, of scaled real number structures. For each $s$ in $R$ define $W_s$ by

$$W_s\tilde{R}_r = \tilde{R}_{sr}.$$  

The $W_s$ form a commutative group under the composition,

$$W_sW_t = W_{st} = W_{ts}.$$  

The implied multiplication of subscripts $sr, st$, and $ts$ is that in $\tilde{R}$. Also $W_{s^{-1}}$ is the inverse of $W_s$.

The description for scaling of complex numbers is quite similar to that for the real numbers. The structure, $C$, scaled by the complex number, $c$, is given by

$$\tilde{C}_c = \{C, +, \times_{\tilde{C}}, 0, c\}.$$  

$\tilde{C}_c$ is defined so that it satisfies the complex number axioms if $\tilde{C}$ does. As was the case for real numbers, the complex number $d$ in $C$ has the same properties in $\tilde{C}$ as $cd$ does in $\tilde{C}_c$.

It may seem strange to contemplate a structure, $\tilde{C}_c$, in which a number, which is complex or even imaginary in $\tilde{C}$, can serve as the identity in another structure. However one can show that $c$ is a real number.
in $\tilde{C}_c$. To achieve this it is sufficient to show that $c = c^*$. That is, $c$ must be equal to its complex conjugate in $\tilde{C}_c$. This follows from the observation that any number, $d$, has the same properties in $\tilde{C}_c$ as $cd$ does in $\tilde{C}_c$. Thus
\begin{equation}
    d = d^* \iff cd = cd^* \iff cd = (cd)^*.
\end{equation}
The first two equations are in $\tilde{C}_c$ and the last is in $\tilde{C}_c$. Here $*_{c}$ is the complex conjugation operation in $\tilde{C}_c$.

The distinction between the elements or numbers in the base set and the values they have in different scaled real number structures, holds for complex numbers in $C$ and the values they have in different structures, $\tilde{C}_c$. The value of a complex number, $d$, in $C$ depends on the structure containing it. The value of $d$ in $\tilde{C}_c$ is $d_{c}$, in $\tilde{C}_c$ the value of $d$ is $(d/c)_c$.

Vector spaces are affected by scaling of the fields on which they are based. If the vector space $V_c$ of Eq. 2 is based on $\tilde{C}_c$, then the structure for the vector space based on $\tilde{C}_c$ of Eq. 9 is given by
\begin{equation}
    V_c = \{ V, \pm, c | \psi|_c, cv \}. \tag{11}
\end{equation}
Here $v$ is an arbitrary vector in $V$.

The definition of scaling for the vector space is defined to satisfy the requirement that $\tilde{V}_c$, $\tilde{C}_c$ satisfy the vector space and complex number axioms if and only if $\tilde{V}_c, \tilde{C}_c$ do. Eq. 11 shows that $cv$ and $\cdot/c$ are the same vector and scalar vector multiplication operation in $\tilde{V}_c$ as $v$ and $\cdot$ are in $\tilde{V}$. Also $| - |_c$ is the same normalization operation in $\tilde{V}_c$ as $| - |$ is in $\tilde{V}$.

There is another vector space structure that satisfies this requirement. It is given by
\begin{equation}
    V'_c = \{ V, \pm, c |v|, v \}. \tag{12}
\end{equation}
This is simpler than is $V_c$ in that vectors are not scaled between the structures. This structure is not used here because it fails the equivalence between $n$ dimensional vector spaces and their representations based on complex numbers. Thus one has $V \equiv \tilde{C}^n$ and $V_c \equiv (\tilde{C}_c)^n$, but not $V'_c \equiv (\tilde{C}_c)^n$.

## 3 Fiber bundles
Fiber bundles have been much used to describe gauge theories and other areas of physics and geometry [3-7]. Here they will be used to describe the effects of number scaling on gauge theories and on some geometric objects. A fiber bundle [1 2] can be described by $\mathcal{B} = \{ E, \pi, M \}$. Here $E$ and $M$ are the total and base spaces, and $\pi$ is a projection of $E$ onto $M$. For each point, $x$, in $M$, $\pi^{-1}(x)$ is a fiber, at $x$, in $E$.

A fiber bundle is trivial if the total space is a product as in $E = M \times F$. Here $F$ is a fiber, and
\begin{equation}
    \pi^{-1}(x) = (x, F) \text{ for all } x \in M. \tag{13}
\end{equation}
Fiber bundles are locally trivial if one has a collection $U_\alpha$ of open sets on $M$ such that the restriction, $\mathcal{B}|U_\alpha$, of $\mathcal{B}$ to $U_\alpha$, is trivial as in
\begin{equation}
    \mathcal{B}|U_\alpha = \{ U_\alpha \times F, \pi_\alpha, U_\alpha \}. \tag{14}
\end{equation}

Different local regions must be consistent with one another. This means that, for any two regions, $U_\alpha$ and $U_\beta$ such that $U_\alpha \cap U_\beta$ is not empty, $\pi_\beta \pi^{-1}_\alpha(x) = x$ for all $x$ in $U_\alpha \cap U_\beta$.

Vector bundles are special cases of fiber bundles where the fiber is a vector space, $\tilde{V}$. The vector bundles of interest here are trivial because $M$ is restricted to be Minkowski space. In this case a vector bundle is $\mathcal{V} = \{ M \times \tilde{V}, \pi, M \}$. For each $x$ in $M$ $\pi^{-1}(x) = (x, \tilde{V})$.

Replacement of $\tilde{V}$ in the vector bundle by its definition as a structure as in Eq. 2 means that for each point, $x$, in $M$
\begin{equation}
    \pi^{-1}(x) = (x, \tilde{V}) = \{(x, V), (x, \cdot), (x, | - |, (x, v)\}
    = \{ V_x, \cdot_x, - |x, v_x \} = \tilde{V}_x. \tag{13}
\end{equation}
The second line is just a rewrite of the first line with more efficient representation of the location association with the vector space. It also suggests that one can regard $(x, \tilde{V}) = \tilde{V}_x$ as a local mathematical structure, as a vector space at $x$. This representation of the values of $\pi^{-1}$ will be used throughout this work.

The scalar field, $S$, associated with $\tilde{V}_x$, is a number structure that satisfies the field axioms. Here $S = \tilde{C}$ (complex) or $S = \tilde{R}$ (real) numbers. For gauge theories $S = \tilde{C}$, in geometry, $S = \tilde{R}$.

So far the scalar field, $\tilde{S}$ is not part of the vector bundle. In a sense it is a global structure as it is not associated with points of $M$. This can be remedied by defining another fiber bundle, a scalar bundle,
The projection, $\mathcal{S} = \{M \times \tilde{S}, \pi_S, M\}$ where for each $x$ in $M$, $\pi_S^{-1}(x) = (x, \tilde{S})$. Representation of $\tilde{S}$ as a structure satisfying the relevant axioms and carrying out the same replacements as in Eq. 13 gives $\pi_S^{-1}(x) = \tilde{S}_x$.

The fiber product [1] of $\mathcal{S}$ and $\mathcal{V}$ is represented by

$$\mathcal{S} \oplus \mathcal{V} = \{(M \times \tilde{S}) \oplus (M \times \tilde{V}), p, M\} = \{M \times (\tilde{S} \times \tilde{V}), p, M\} = \mathcal{SV}.$$  \hspace{1cm} (14)

The projection, $p$, is related to $\pi$ and $\pi_S$ by

$$p^{-1}(x) = \pi_S^{-1}(x) \times \pi^{-1}(x).$$ \hspace{1cm} (15)

Here

$$(x, \tilde{S} \times \tilde{V}) = (x, \tilde{S}) \times (x, \tilde{V})$$ \hspace{1cm} (16)

is used.

This use of the fiber product achieves the desired aim of replacing a global scalar structure not associated with any point of $M$ by a local structure at the same point of $M$ as is the local vector space. For each $V_x$, $\tilde{S}_x$ is the associated scalar field with $| - |_x$ a map from $V_x$ to $S_x$ and $\cdot_x$ a map from $S_x \times V_x$ to $V_x$. The maps are defined on the base sets of the structures.

A main point of this work is to use fiber bundles to describe the effect of number scaling on gauge theories and on some geometric objects. This can be done by expanding the fibers to include all scaled pairs of scalar fields and vector spaces. For gauge theories with complex scalars the pairs are $\tilde{C}_c \times \tilde{V}_c$ for all complex numbers. $\tilde{C}_c$ and $\tilde{V}_c$ are given by Eqs. 9 and 11. For geometric objects the pairs are $\tilde{R}_c \times \tilde{T}_c$. $\tilde{R}_c$ is a scaled real number structure and $\tilde{T}_c$ is a scaled tangent space.

Inclusion of number scaling in a fiber bundle description can be achieved by first defining fiber bundles,

$$\mathcal{CV}_c = \{M \times (\tilde{C}_c, \tilde{V}_c), p_c, M\}$$ \hspace{1cm} (17)

for all complex numbers $c$. Here $\mathcal{CV}_c$ is a fiber product as defined by Eq. 13.

One then defines a sum bundle by

$$\sum_c \mathcal{CV}_c = \{M \times \bigcup_c (\tilde{C}_c, \tilde{V}_c), Q, M\} = \mathcal{CV}.$$ \hspace{1cm} (18)

Here $\bigcup_c (\tilde{C}_c, \tilde{V}_c)$ is the collection of all pairs, $(\tilde{C}_c, \tilde{V}_c)$. $Q$ projects the total space onto $M$. The inverse map $Q^{-1}$ defines the fibers by

$$Q^{-1}(x) = (x, \bigcup_c (\tilde{C}_c, \tilde{V}_c)) = \bigcup_c (\tilde{C}_{c,x}, \tilde{V}_{c,x})$$ \hspace{1cm} (19)

Here $\tilde{C}_{c,x}$ and $\tilde{V}_{c,x}$ are defined from $(x, \tilde{C}_c)$ and $(x, \tilde{V}_c)$ as was done for $\tilde{V}$ in Eq. 13. The projection $Q$ is related to the individual projections, $p_c$ by

$$Q(x, (\tilde{C}_c, \tilde{V}_c)) = p_c(x, (\tilde{C}_c, \tilde{V}_c)).$$ \hspace{1cm} (20)

The structure pair, $\tilde{C}_x, \tilde{V}_x$, for $c = 1$ will be referred to as the reference pair.

An important aspect of the contents of each fiber is that all complex number structures, $\tilde{C}_{c,x}$, in the fiber at $x$ have the same base set, $C_x$. The number values of the numbers in $C_x$ depend on the scaling factor for the structure containing $C_x$. Similarly, the vector space structures for all $c$ at $x$ have the same base set $V_x$. These results follow from Eqs. 9, 11 and 13.

A structure group, $W_{CV}$ with elements, $W_{CV,d}$, that are structure isomorphisms for any complex number $d$, can be defined on the fibers. The action of $W_{CV,d}$ on $\bigcup_c (\tilde{C}_c, \tilde{V}_c)$ is given by

$$W_{CV,d} \bigcup_c (\tilde{C}_c, \tilde{V}_c) = \bigcup_c (\tilde{C}_{c,d}, \tilde{V}_{c,d}).$$ \hspace{1cm} (21)

The group $W_{CV}$ is a continuous commutative group with $W_{CV,d^{-1}}$ the inverse of $W_{CV,d}$.

The group $W_{CV}$ can be particularized to each point $x$ as $W_{CV,x}$. Here the elements $W_{CV,d,x}$ of $W_{CV}$ act on the fiber at $x$.

Associated with the continuous structure group, $W_{CV}$ is the group, $DGL(1, C)$. This group consists of the diagonal elements, $d \times d$, of $GL(1, C) \times GL(1, C)$. That is

$$DGL(1, C) = \{d \times d : d \in GL(1, C)\}. \hspace{1cm} (22)$$

$0$ is excluded as a value of $d$ because the action of $W_{CV,0}$ on a structure annihilates it.
There is an obvious isomorphism between \( W_{CV} \) and \( DGL(1, C) \) given by \( d \times d \to W_{CV,d} \) for all \( d \) in \( GL(1, C) \).

The bundle, \( \mathcal{C} \mathcal{W} \) and a similar tangent bundle are not principal \( G \) bundles. The reason is that each fiber contains a pair of distinguished reference structures, \( \bar{C}_x, \bar{V}_c \) or \( R_x, T_x \). These are the structure that are used in the absence of scaling.

The definition of fiber bundles is quite general in that the fibers can contain much material. For example a fiber can include a field structure, \( \bar{S} \) and some or much of the mathematical analysis based on \( \bar{S} \). A fiber can include integrals and derivatives of functions, charts of \( M \) and other systems. In all cases the fiber bundle is represented here by \( \{ M \times F, \pi, M \} \). The bundle is trivial on \( M \) as Minkowski or Euclidean space.

### 4 Connections

Connections or parallel transports [25] play important roles in gauge theories and geometry. They account for the fact that, for vector spaces at different points of \( M \), a choice of a basis or coordinate system at point \( y \) does not determine the choice at point \( x \) of \( M \). This is the "Naheinformationsprinzip", 'no information at a distance' principle [25]. Applied to number scaling it says that the choice of a number scaling factor at point \( y \) does not determine the choice at point \( x \).

The purpose of connections is to determine the relations between bases or coordinate systems in vector spaces at different \( M \) locations. For number scaling connections determine the relations between scaled number systems at different \( M \) points. As maps these connect, or parallel transport, number structures between points of \( M \).

For neighboring points, \( x, x + \Delta x \) on \( M \) one defines a connection \( W_{CV}(x, x + \Delta x) \) that connects the elements in the fiber \( Q^{-1}(x + \Delta x) \) to those in \( Q^{-1}(x) \). Here \( Q \) is the projection in the fiber bundle given by Eq. [25] If \( \bigcup_c(\bar{C}_{c,x+\Delta x}, \bar{V}_{c,x+\Delta x}) \) is the fiber at \( x + \Delta x \) then

\[
W_{CV}(x, x + \Delta x) \bigcup_c(\bar{C}_{c,x+\Delta x}, \bar{V}_{c,x+\Delta x}) = W_{CV,c,x+\Delta x} \bigcup_c(\bar{C}_{c,x+\Delta x}, \bar{V}_{c,x+\Delta x})
\]

The second line is the fiber at \( x \). It shows that the connection multiplies scaling factors by \( c_{x,x+\Delta x} \).

For points \( y \) distant from \( x \) the \( W_{CV}(x, y) \) may depend not only on \( y \) and \( x \) but on the curves from \( y \) to \( x \). If \( p \) is a path from \( y \) to \( x \), then the action of \( W_{CV}(x, y) \) on the fiber, \( Q^{-1}(y) \) gives the fiber \( Q^{-1}(x) \) where

\[
W_{CV}^p(x, y) \bigcup_c(\bar{C}_{c,y}, \bar{V}_{c,y}) = \bigcup_c(\bar{C}_{c,x}, \bar{V}_{c,x})
\]

If \( q \) is a path from \( z \) to \( y \) on \( M \) and \( p \) is a path from \( y \) to \( x \) then \( W_{CV}^{pq}(x, z) \) connects fiber \( Q^{-1}(z) \) to \( Q^{-1}(x) \). The associated scaling factor, for each \( C_c, \bar{V}_c \) is \( e^{pq}(x, y) \times e^{pq}(y, z) \) where

\[
e^{pq}(x, z) = e^p(x, y)e^q(y, z).
\]

Here \( pq \) is the concatenation of \( p \) with \( q \). If \( pq \) is a closed path from \( x \) to \( x \), then, in general, \( e^{pq}(x, x) \neq 1 \).

The relationships between fiber contents and the connections between two representative fibers, \( F_x \) and \( F_y \), are shown in Figure [1]. Connections, \( W_{CV}(x, y) \), with their associated scale factors, \( c_{x,y} \) are shown as are the reference and scaled contents of each fiber. The structure group elements are indicated by \( W_{CV,c} \) and \( W_{CV,cx,y} \). The isomorphisms, \( I_{x,y} \) map the structure pairs \( C_c, \bar{V}_c \) in \( F_y \) to the same scaled structure pairs in \( F_x \).

To avoid index clutter, the figure represents the special case in which the connections are path independent. If \( y = x + \Delta x \), then the figure is valid for all connections, path dependent or independent.

The two triangle diagrams in the figure are represented by the equation,

\[
W_{CV}(x, y) = W_{CV,cx,y} I_{x,y}.
\]

Here \( W_{CV,cx,y} \) is an element of the structure group in \( F_x \).

---

4 It is interesting to speculate on whether this condition can be dropped and that any scaled structure pair can serve as a reference. This would correspond to some type of invariance under number scaling. This is a question for future work.
Figure 1: Relations between connections and fiber contents for two points, \( y \) and \( x \) of manifold, \( M \). If the connection is nonintegrable, then \( y \) is a neighbor point of \( x \). The reference pair, \( \bar{C}, \bar{V} \) is shown in each fiber as well as the scaled structures, \( \bar{C}_c, \bar{V}_c \) for any complex number, \( c \). The structure groups are indicated by \( W_{CV,c} \) in \( F_y \). To avoid clutter, indices, \( y \) and \( x \), are not shown on the contents of the two fibers. The effect of the connection, \( W_{CV}(x,y) \), relating scaled structures in \( F_y \) to those in \( F_x \) is shown by the two triangle diagrams, one based on \( \bar{C}, \bar{V} \) and the other on \( \bar{C}_c, \bar{V}_c \). The number, \( c_{x,y} \) within \( F_x \), is a number in \( \bar{C}_x \). The maps, \( I_{x,y} \), are isomorphic maps from pairs of scaled structures in \( F_y \) to the same scaled structures in \( F_x \).

The Lie algebra representation of the connection scaling factors for neighboring points is by means of a pair of vector fields as in

\[
\begin{align*}
c_{x,x+\bar{dx}} = e^{-\langle \bar{A}(x)+i\bar{B}(x) \rangle \cdot \bar{dx}}, \\
\end{align*}
\]  

(27)

The order of the location pair, \( x, x+\bar{dx} \), in the subscript specifies the direction of the connection, from \( x+\bar{dx} \) to \( x \). The connection in the opposite direction is related to \( c_{x,x+\bar{dx}} \) by

\[
\begin{align*}
c_{x+\bar{dx},x} = (c_{x,x+\bar{dx}})^{-1}. \\
\end{align*}
\]  

(28)

For points \( y \) that are distant from \( x \), the connection factor, \( c_{x,y} \), is expressed by

\[
\begin{align*}
c_{x,y}^P = e^{\int_y^x \langle \bar{A}(p(z))+i\bar{B}(p(z)) \rangle \cdot dp(z)} = e^{\int_y^x \bar{A}(p(z)) \cdot dp(z)} e^{i \int_y^x \bar{B}(p(z)) \cdot dp(z)}. \\
\end{align*}
\]  

(29)

The integrals are expressed as line integrals along a path \( p \) from \( y \) to \( x \). This result shows that the complex scale factor, \( c_{x,y}^P \), is the product of a real scale factor and a phase factor.

Eq. (29) applies if both \( \bar{A} \) and \( \bar{B} \) are nonintegrable. If either field is integrable, then the integral in the exponent for the integrable field is path independent. If both vector fields are integrable, then there are scalar fields, \( \theta \) and \( \phi \) where \( \bar{A} \) and \( \bar{B} \) are the gradients of \( \theta \) and \( \phi \). The expression for \( c_{x,y}^P \) simplifies to

\[
c_{x,y} = e^{\theta(x)-\theta(y) + i(\phi(x)-\phi(y))}. \\
\]  

(30)

5 Gauge theories

5.1 Abelian theories

The usual description of Abelian, and nonabelian, gauge theories is a special case of gauge theories based on the fiber bundle \( CV \). The vector field \( \psi \) is a level section on the reference vector spaces, \( V_x \) in the bundle. The connection vector fields, \( \bar{A} \) and \( \bar{B} \) are 0 everywhere.
For Abelian gauge theories the $U(1)$ connection, $U(x, x + \vec{dx})$ parallel transports the vector $\psi(x + \vec{dx})$ in $\vec{V}_{x+\vec{dx}}$ to a vector in $\vec{V}_x$. The action of this connection is expressed by

$$U(x, x + \vec{dx})\psi(x + \vec{dx}) = e^{i\vec{H}(x)\cdot\vec{dx}}\psi(x + \vec{dx}).$$  \hspace{1cm} (31)

The exponential is the Lie algebra representation of the connection, $\vec{H}$ is a vector field, and $\psi(x + \vec{dx})_x$ is the same state in $\vec{V}_x$ as $\psi(x + \vec{dx})$. This is shown by the map, $I_{x,x+\vec{dx}}$, in Fig. 1.

Eq. (31) is used to replace ordinary derivatives in Lagrangians by covariant derivatives. These are given by

$$D_{\mu,x} = \lim_{dx^\mu \to 0} \frac{e^{-iH_\mu(x)(x)dx^\mu}\psi(x + dx^\mu)_x - \psi(x)}{dx^\mu}. \hspace{1cm} (32)$$

The presence of number scaling opens up another option. This is based on the fact that the Lie algebra representation of the connection $c_{x,x+\vec{dx}}$, shown in Eq. (27), already includes a $U(1)$ gauge component. This is a consequence of the fact that the general complex linear group, $GL(1, C) = GL(1, R^+) \times U(1)$. This factorization is shown explicitly by writing Eq. (27) in the form,

$$c_{x,x+\vec{dx}} = e^{-\vec{A}(x)\cdot\vec{dx}}e^{-i\vec{B}(x)\cdot\vec{dx}}.$$  \hspace{1cm} (33)

Expansion of the gauge group from $U(1)$ to $GL(1, C)$ results in the replacement of the covariant derivative of Eq. (32) by

$$D_{\mu,x}\psi = \lim_{dx^\mu \to 0} \frac{e^{-(A_\mu(x)+iB_\mu(x))dx^\mu}\psi(x + dx^\mu)_x - \psi(x)}{dx^\mu}. \hspace{1cm} (34)$$

Expansion of the exponential to first order in small quantities and inclusion of coupling constants, $-g_r$ and $-g_i$, gives,

$$D_{\mu,x}\psi = (\partial_{\mu,x} + g_rA_\mu(x) + ig_iB_\mu(x))\psi. \hspace{1cm} (35)$$

Use of this in Lagrangian densities with the requirement that retained terms must be invariant under local $U(1) = e^{i\alpha(x)}$ gauge transformations gives the well known \[10, 11\] result that the $B$ field is replaced by

$$B'_\mu(x) = B_\mu(x) - g_i^{-1}(\partial_{\mu}\alpha(x)). \hspace{1cm} (36)$$

The $\vec{A}$ field is unaffected in that

$$A'_\mu(x) = A_\mu(x). \hspace{1cm} (37)$$

Also the $\vec{B}$ field is massless. There are no restrictions on the mass, if any, for the $\vec{A}$ field.

Expansion of the covariant derivative in the Lagrangian density,

$$L_x = |D_{\mu,x}\psi|^2 - m^2\psi^*\psi,$$  \hspace{1cm} (38)

for Klein Gordon fields gives

$$L_x = \partial_{\mu}\psi\partial^{\mu}\psi^* + g_rA_\mu\partial^{\mu} (\psi\psi^*) + ig_iB_\mu(\psi\partial^{\mu}\psi^* - \psi^*\partial^{\mu}\psi)$$

$$+ (g_r^2A^2_\mu - m^2)\psi^*\psi. \hspace{1cm} (39)$$

The second and third terms describe interactions of the $\vec{A}$ and $\vec{B}$ fields with the matter field, $\psi$. The presence of $g_r^2A^2_\mu$ in the mass term shifts the square of the mass from $m^2$ to $m^2 - g_r^2A_\muA^\mu$. The term, $g_i^2B_\muB^\mu\psi^*\psi$ is not present as it is not invariant under local $U(1)$ gauge transformations.

For Dirac fields the fibers include a spinor space with the complex numbers. Each fiber, $F_x$, contains all pairs, $\vec{C}_{c,x}, \vec{S}_{c,x}$ for $c$ in $\vec{C}_x$. Here $\vec{S}_x$ is a spinor space just as $\vec{V}_x$ is a vector space. $\vec{S}_{c,x}$ is a scaled spinor space just as $\vec{V}_{c,x}$ is a scaled vector space.

The connections that relate the contents of the fiber $F_{x+\vec{dx}}$ to those of $F_x$ are elements of $DGL(1, C)$. The covariant derivative, $D_{\mu,x}\psi$, appearing in the Dirac Lagrangian density is given by Eq. 35.

Expansion of the covariant derivative in the Dirac Lagrangian density,

$$L_x = \bar{\psi}\gamma^{\mu}D_{\mu,x}\psi - m\bar{\psi}\psi,$$  \hspace{1cm} (40)

The associated parallel transfer, $\vec{C}_{x+\vec{dx}} \rightarrow \vec{C}_x$ can be implemented by a connection with value 1.
and addition of a Yang Mills term $G_{\mu,\nu}$ where
\[ G_{\mu,\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \]
gives
\[ \mathcal{L}_x = \bar{\psi} i\gamma^\mu (\partial_\mu x + g_r A_\mu(x) + ig_i B_\mu(x)) \psi - m \bar{\psi} \psi - \frac{1}{4} G_{\mu,\nu} G^{\mu,\nu}. \]

This is the QED Lagrangian with an extra term accounting for the presence of the $\vec{A}$ field. Because the $\vec{A}$ field is unaffected by the requirement of invariance under local $U(1)$ gauge transformations, a mass term, $\beta A_\mu A^\mu$ and possibly a dynamical term to treat $\vec{A}$ as a dynamical variable can be added. Whether $\beta = 0$ or $\neq 0$ is unknown at this point.

The Lagrangian in Eq. 42 shows the $\vec{B}$ field as the electromagnetic or photon field. In this case it must have spin 1 and be nonintegrable [29]. The nonintegrability requirement follows from the Aharonov Bohm effect [32] [33]. Additional support for this interpretation is given in the description of the Higg’s mechanism [34] in the next section.

The great accuracy of the QED Lagrangian in describing experimental results [30, 31] without the $\vec{A}$ field present means that the coupling constant, $g_r$, must be very small compared to the fine structure constant. Another possibility is that the $\vec{A}$ field is close to zero in all regions of space and time in which experiments that test or depend on QED accuracy have been carried out.

5.2 Nonabelian gauge theories

For nonabelian gauge theories invariance of terms in Lagrange densities is extended to include local $SU(n)$ gauge transformations. The overall gauge group is extended from $GL(1, C)$ to $GL(1, C) \times SU(n)$.

The covariant derivative is obtained in the same way as for the Abelian gauge theories. The resulting covariant derivative differs from that of Eq. 34 by the addition of a term that is the sum over the $n^2 - 1$ generators of the Lie algebra, $su(n)$. One obtains,
\[ D_{\mu,\nu} \psi = (\partial_{\mu,\nu} + g_r A_\mu(x) + ig_i B_\mu(x) + ig w^a_{\mu}(x) T^a) \psi. \]

Here $\vec{w}$ is an $n^2 - 1$ component real vector field and the $T^a$ are the generators of $su(n)$. For the simplest case with $n = 2$, the $T^a$ are the three Pauli operators. As was the case for Abelian theories the only difference between Lagrangians using this covariant derivative and that without the $\vec{A}$ field present are terms describing the interaction of $\vec{A}$ with the matter field.

6 Possible physical properties of the $\vec{A}$ and $\vec{B}$ fields

The presence of the $\vec{A}$ and $\vec{B}$ vector fields in the connections raises the question regarding the relevance of these fields to physics. One way to investigate this question begins with noting that each fiber bundle, $CV_c$, in the sum used to define $\mathfrak{W}$ in Eq. 18 is a fiber product of the bundles, $\{M \times C_c, \pi_c, M\}$ and $\{M \times V_c, V, M\}$. This structure allows one to define a product field, $\psi_{CV} = \psi_C \times \psi_V$ as a level section over the bundle $CV_c$ for $c = 1$. Here $\psi_C$ is a complex scalar field where $\psi_C(x)$ is a complex number in $C_x$ and $\psi_V$ is the vector field in Eqs. [38] [39] [42].

A complex scalar field, such as $\psi_C$, is a common field for vector fields $\psi_V$ based on $n$ dimensional vector spaces for all $n$. If $n = 1$ then the two fields can be identified. The fact that $\psi_C$ is present for vector spaces of all finite dimensions suggests that it may have possible relevance to physics.

The presence of $\psi_C$ allows one to construct a gauge theory based on $\psi_C$. The theory would be closely associated with all gauge theories based on $\psi_V$. One would have a scalar gauge theory that is associated with all vector space gauge theories.

A good property of $\psi_C$ is that it fits easily into a ”Mexican hat” description of the Higg’s mechanism [34]. The Lagrangian density is [10]
\[ \mathcal{L}_x = (D_\mu \psi_C^\dagger D^\mu \psi_C) + \mu^2 \psi_C^\dagger \psi_C - \lambda (\psi_C^\dagger \psi_C)^2 - \frac{1}{4} G_{\mu,\nu} G^{\mu,\nu}. \]

For the Abelian case, $G_{\mu,\nu}$ is the $\vec{B}$ field strength given by Eq. 41. The field, $\psi_C$ can be written as
\[ \psi_C = \frac{1}{\sqrt{2}} (\psi_{C,1} + i \psi_{C,2}). \]
Here $\psi_{C,1}$ and $\psi_{C,2}$ are both real. From now on, to save on notation, the subscript, $C$, will be dropped from the fields.

The application of the Higgs mechanism follows the description in [10]. From the nonzero value of the potential minimum, given by $|\langle 0 | \bar{\psi} \psi | 0 \rangle| = v^2/2 = \mu^2/2\lambda$, the field, $\psi$, has a nonzero vacuum expectation value,

$$|\langle 0 | \bar{\psi} \psi | 0 \rangle| = v/\sqrt{2} = \mu/\sqrt{2\lambda}.$$ 

The rotational symmetry around the Mexican hat potential is broken by a specific choice, $\langle 0 | \bar{\psi} \psi | 0 \rangle = v$ and $\langle 0 | \bar{\psi} \psi' | 0 \rangle = 0$. Replacement of $\psi_1$ by $\psi_1' + v$, where $\psi_1'$ is a shifted field, gives objectionable mixing terms in the Lagrangian. These can be removed by expressing $\psi$ as

$$\psi(x) = \frac{1}{\sqrt{2}}(v + \alpha(x))e^{i\beta(x)/v} = \psi'(x)e^{i\beta(x)/v},$$

and defining a new field $B'_\mu(x)$, by

$$B'_\mu(x) = B_\mu(x) + \frac{1}{g_i v} \partial_\mu \beta(x).$$

In the unitary gauge the covariant derivative becomes,

$$D_\mu \psi = (\partial_\mu x + g_i A_\mu(x) + ig_i B_\mu(x))(\frac{1}{\sqrt{2}}(v + \alpha(x))e^{i\beta(x)/v})$$

$$= \frac{1}{\sqrt{2}}[\partial_\mu \alpha(x) + (g_i A_\mu(x) + ig_i B'_\mu(x))(v + \alpha(x))]e^{i\beta(x)/v}.\tag{48}$$

Use of this in the Mexican hat Lagrangian gives mass terms $(1/2)(g_i v)^2 A_\mu A^\mu$ and $(1/2)(g_i v)^2 B'_\mu B'^\mu$ for the $A$ and $\vec{B}$ fields. The massless $\vec{B}$ combines with $\beta(x)$ to becomes a massive vector boson, $\vec{B}$. The mass, $\mu^2/2$ of the $\alpha(x)$ field arises from the mass term, $\mu^2(\alpha(x))^2$ in the potential.

The $A$ field is also massive. However this is not new in that nothing prevents addition of a mass term to the Lagrangian at the outset. Also there are mixing terms, $g_i A_{\mu}(x)\partial^\mu \alpha(x) + g_i A^\mu \partial_\mu \alpha(x)$, in the Lagrangian. Similar terms for the $\vec{B}$ field were removed by use of the unitary gauge. However this option does not seem to be available for the $A$ field.

These results show that the Higgs mechanism can be used to combine the massless phase part, $\vec{B}$, of the connection with $\beta(x)$ to give a massive vector boson, $\vec{B}$. The remaining real field, $\alpha(x)$, with mass term, $\mu^2\alpha(x)^2$, is the Higgs field [33].

The possible relationship of $A$ to the Higgs field is intriguing. If $A$ is integrable, then $\vec{A}(x) = \nabla_x \theta(x)$ where $\theta(x)$ is a real scalar field, presumably of spin 0. This suggests the possibility that $\theta(x)$ is the Higgs's field. In this case it would replace $\alpha(x)$ in the above derivation. Whether this possibility is true or not is open at this point. More work is needed.

So far the description of gauge theories and the Higgs mechanism is based on the gauge group $DGL(1, C)$ (Eq. 22) for Abelian theories and $DGL(1, C) \times SU(n)$ for nonabelian theories. The $U(1)$ factor of $U(1) \times SU(n)$ in the usual description of gauge theories is taken to be the phase part of the connection that accounts for number scaling. One result of this is that the $\vec{B}$ field appears as the photon field in the Lagrangians.

This association of $\vec{B}$ with the photon can be avoided by adding an extra $U(1)$ factor to the gauge group where $U(1)$ becomes part of the connection for the vector spaces, but not for the complex scalars. In this case the connection group for the vector spaces is expanded to become

$$GL(1, C) \times U(n) = GL(1, \vec{R}) \times U(1)_1 \times U(1)_2 \times SU(n).\tag{49}$$

Here $U(1)_1$ and $U(1)_2$ are the respective phase components in the connection arising from number scaling and from the usual $U(1)$ gauge transformations on vector spaces.

In this case there are two $\vec{B}$ fields, $\vec{B}_1$ and $\vec{B}_2$ with $\vec{B}_1$ and $\vec{B}_2$ appearing in the respective definitions of $U(1)_1$ and $U(1)_2$ as $U_j(x, x + \vec{d} x) = \exp(ig_j \vec{B}_j(x) \cdot \vec{d} x)$ for $j = 1, 2$. Here $g_1$ and $g_2$ are the respective coupling constants for the number scaling and the usual gauge groups. The term, $ig_i B'_\mu(x)$ in the covariant derivative in Eqs. 33 and 34 becomes a sum of two field components, $ig_1 \vec{B}_{1,\mu}(x) + ig_2 \vec{B}_{2,\mu}(x)$, each with their own coupling constant.

In this case Eq. 36 becomes

$$g_1 \vec{B}_{1,\mu}(x) + g_2 \vec{B}_{2,\mu}(x) = g_1 B_{1,\mu}(x) + g_2 B_{2,\mu}(x) - \partial_\mu \alpha(x).\tag{50}$$
This equation gives two options: either $\vec{B}_1$ is the massless electromagnetic field and any mass is possible for $\vec{B}_2$, or $\vec{B}_2$ is the massless electromagnetic field with any mass possible for $\vec{B}_1$. These options follow from setting $\vec{B}_2 = \vec{B}_2$ with $\vec{B}_1$ satisfying Eq. \[30\] or from setting $\vec{B}_1 = \vec{B}_1$ with $\vec{B}_2$ satisfying Eq. \[30\] if $g_2 \neq g_2$, then the physical properties of the field whose primed value equals its unprimed value are not known. As is the case for the $A$ field, any mass is possible for the field. It is not possible from the appearance of these fields in the covariant derivatives and Lagrangian to tell which field is the photon field. The one distinction is that, unlike the case for $\vec{B}_1$, $\vec{B}_2$ is not part of the connection for the complex scalars. However, if $g_1 = g_2$, then the sum field $\vec{B}_1 + \vec{B}_2$ with $g_1$ as a coupling constant, is the same as one field in covariant derivatives and Lagrangians. In this case, and only in this case, $\vec{B}_1$ and $\vec{B}_2$ correspond to an arbitrary separation of $B$ into the sum of two parts. This separation has no physical significance.

7 Number scaling and geometry

7.1 Fiber bundles

The fiber bundle structure for geometric quantities is quite similar to that for gauge theories. The vector spaces in geometry are tangent spaces over $M$ and complex numbers as scalars are replaced by real numbers. The fiber product \( \prod \) of the real number and tangent space bundle $s$, in covariant derivatives and Lagrangians. In this case, and only in this case, the definition of scaled structures for real numbers and tangent spaces follows that for complex numbers.

The fiber bundle, $\mathcal{R}$, is defined from Eq. \[14\] as

$$\mathcal{R} = \{M \times \tilde{R}, \pi_r, M\}$$

(51)

is defined from Eq. \[14\] as

$$\mathcal{R}T = \{M \times (\tilde{R} \times \tilde{T}), \pi_{rt}, M\}.$$  

(52)

As was the case for gauge theories, this bundle replaces a single global real number structure as the scalars for all the tangent spaces with a local structure $\tilde{R}_x$ for $\tilde{T}_x$ at each $x$ in $M$.

The definition of scaled structures for real numbers and tangent spaces follows that for complex numbers and vector spaces in gauge theory. The scaled tangent space, $\tilde{T}_r$ is given by

$$\tilde{T}_r = \{T, \pm_r, \odot_r, \cdot_r, \vec{a}_r\} = \{T, \pm, \odot_r, \cdot_r, r \vec{a}\}.$$  

(53)

The associated scaled real number structure, $\tilde{R}_r$ is given by Eq. \[4\]. The righthand term is the representation of the components of $\tilde{T}_r$ in terms of those in $\tilde{T}$. Vectors on $\tilde{T}_r$ are scaled tuples in $\tilde{R}_r^4$. The operators, $\odot$ and \(\cdot\) denote scalar vector multiplication and vector dot product in $\tilde{T}$. The scaled vectors are denoted by $r\vec{a} = r \odot \vec{a}$ where $\vec{a}$ is an arbitrary vector in $\tilde{T}$. The dot product of two vectors, $\vec{w}_r$ and $\vec{v}_r$ in $\tilde{T}_r$ is related to that in $\tilde{T}$ by

$$\phantom{\vec{w}_r \cdot_r \vec{v}_r} = (\eta_{\mu,\nu})_r \times_r r \mu \times_r r \nu = r \eta_{\mu,\nu} \times_r r \mu \times_r r \nu = r (\vec{w} \cdot v).$$

(54)

sum over repeated indices implied. Here $\eta_{\mu,\nu} = (-1, 1, 1, 1)$.

The fiber bundle, $\mathcal{R}\mathcal{T}$ is the sum of all the scaled bundles. It is similar to that for complex vector spaces in Eq. \[14\] in that

$$\mathcal{R}\mathcal{T} = \sum_r \mathcal{RT}_r = \{M \times \bigcup_r (\tilde{T}_r \times \tilde{T}), \pi_{rt}, M\}.$$  

(55)

Since $M$ is a Minkowski space, it is flat. As a result, tangent spaces $\tilde{T}_x$ at all points, $x$, of $M$ are equivalent. Each tangent space is a local representation of all of $M$. That is, for each $x$ in $M$, $\tilde{T}_x = \tilde{R}_x^4$ (sometimes represented as $\mathbb{R}^4 \times \mathbb{C}$).

Coordinate systems on $\tilde{R}_x^4$ are defined by use of global coordinate charts, $\phi_x : M \rightarrow \tilde{T}_x$. Here $\phi_x$ is a homeomorphic map of $M$ onto $\tilde{T}_x$. Two charts $\phi_y$ and $\phi_x$ are the same if for all points, $p$ on $M$, $\phi_y(p)$ is the same number tuple in $\tilde{R}_y^4$ as $\phi_x(p)$ is in $\tilde{R}_x^4$. An isomorphism, $I_{x,y} : \tilde{R}_y^4 \rightarrow \tilde{R}_x^4$, expresses the sameness of number tuples in the structures. The relation between $\phi_x, \phi_y$ and $I_{x,y}$ is given by

$$I_{x,y} \phi_y(p) = \phi_x(p)$$

(56)

for all $p$ in $M$.  

12
The structure group for each fiber, $F_x$, in $\mathfrak{F}$ is taken to be $W_{RT,x}$. The elements of $W_{RT,x}$ are represented by $W_{RT,s,x}$, where $s$ is a positive real number in $R_x$. For each $r$ in $R_x$,

$$W_{RT,s}R_rT_r = R_{sr}, T_{sr}.$$  

(57)

The subscript, $x$, is deleted here and from now on if the fiber location of the vector space and number structures are clear.

### 7.2 Connections

Connections between structure contents of neighboring fibers in the tangent bundle are elements of a gauge group, $W_{RT}$. For each point pair, $x, x + d\vec{d}x$ on $M$, the map, $W_{RT}(x, x + d\vec{d}x)$ connects the contents of $F_{x+d\vec{d}x}$ to those of $F_x$.

The action of $W_{RT}(x, x + d\vec{d}x)$ is represented by positive real number pairs in the group $GL(1, R^+)$ of pairs of the same elements in $GL(1, R^+) \times GL(1, R^+)$. One has

$$W_{RT}(x, x + d\vec{d}x) = \bar{R}_{r_x+dx} \bar{T}_{r_x+dx} = R_{r_x+dx} R_{x+dx}, T_{x+dx}.$$  

(58)

This map connects the pair $\bar{R}_r, \bar{T}_r$ in the fiber, $F_{x+dx}$, to the pair, $\bar{R}_{r_x+dx}, \bar{T}_{r_x+dx}$ in $F_x$. The effect of the connection is to change the scaling by a positive real factor, $r_{x+dx}$.

The scaling factor, $r_{x+dx}$ is obtained from Eq. 59 by setting $\vec{B} = 0$. One has

$$r_{x+dx} = e^{-\bar{A}(x)} = (r_{x+dx})^{-1}. $$  

(59)

For $y$ distant from $x$, Eq. 59 gives

$$r^p_{x,y} = e^{\int_{s(x)}^{s(y)} \bar{A}(p(s))}. $$  

(60)

The dependence on a path $p(s)$ from $p(1) = y$ to $p(0) = x$ is indicated in the integrand.

To help understand the relations between the quantities in different fibers, it is useful to have at hand a figure for tangent spaces that is similar to Fig. 1. This is supplied by Figure 2. Eq. 58 is represented in the figure by the triangle diagrams. These show that for any vector $\vec{v}_{x+dx}$ in $T_{r_x+dx}$,

$$W_{RT}(x, x + dx)\vec{v}_{x+dx} = W_{RT,x+dx}I_{x+dx} \vec{v}_{x+dx}$$  

(61)

The fact that $\vec{v}_{x}$ and $\vec{v}_{x+dx}$ are the same vectors in their respective spaces is shown in the figure by the map, $I_{x+dx}$ where

$$I_{x+dx} \vec{v}_{x+dx} = \vec{v}_{x}. $$  

(62)

### 7.3 Scaling of geometric quantities

As noted in the introduction, one effect of number scaling is that both lengths of vectors and angles between vectors are scaled. The scaling of angles seems strange and counterintuitive. However, because of the scaling of operations for number structures and vector spaces, Eq. 55 the scaling of angles does not seem to cause problems. For instance, trigonometric relations are preserved under scaling.

As an example, consider the relation, $\sin^2 \phi_y + y \cos^2 \phi_y = 1_y$. This equation applies to angles in the tangent space $T_y$. Since it also uses numbers in $R_y$, it is an equation in the fiber, $F_y$.

The corresponding equation in $F_x$ is obtained by the connection, $W_{RT}(x, y)$. One has

$$W_{RT}(x, y) [\sin^2 \phi_y + y \cos^2 \phi_y] = W_{RT}(x, y) 1_y$$  

(63)

$$\equiv W_{RT}(x, y) [\sin^2 \phi_y] + x W_{RT}(x, y) [\cos^2 \phi_y] = W_{RT}(x, y) 1_y$$  

$$\equiv r_{x,y} \sin^2 \phi_x + x r_{x,y} \cos^2 \phi_x = r_{x,y} 1_x$$  

$$\equiv \sin^2 \phi_x + x \cos^2 \phi_x = 1_x.$$
Figure 2: Relations between connections and fiber contents for two points, y, and x of manifold, M. If the connection is nonintegrable, then y is a neighbor point of x. The reference pair, $\bar{R}, \bar{T}$ is shown in each fiber as well as the scaled structures, $\bar{R}_r, \bar{T}_r$ for any positive real number, $r$. The subscript locations on the real number and tangent space structures are not shown as their location in the fibers is clear. The structure groups are indicated by $W_{RT,r}$ in $F_y$. The effect of the connection, $W_{RT}(x,y)$, in relating structures in $F_y$ to those in $F_x$ is shown by the two triangle diagrams, one based on $\bar{R}, \bar{T}$ and the other on $\bar{R}_r, \bar{T}_r$, both in $F_y$. The number, $r_{x,y}$ in $F_x$, is the same number in $\bar{R}_x$ as $r_{x,y}$ is in the connection group, $GL(1, R^+)$. The maps, $I_{x,y}$, are isomorphic maps from pairs of scaled structures in $F_y$ to the same scaled structures in $F_x$.

Here

$$W_{RT}(x,y) \sin^2 \phi_y = r_{x,y} \sin \phi_x \times r_{x,y} \sin \phi_x = r_{x,y} \sin^2 \phi_x$$

is used. The same equivalences hold for $\cos \phi$.

The reason trigonometric relations like the one just described are unaffected by scaling is that they are local relations. The relation based on $\phi_y$ is at point $y$ of $M$. The relation using $\phi_x$ is at point $x$ on $M$. The scaling factors cancel in the equations.

Scaling factors do not cancel in the description of nonlocal geometric quantities. These include integrals and derivatives of functionals over $M$. Two examples are considered here. The integral expressing the length of a path is one and the geodesic equation is the other.

### 7.3.1 Path lengths

Path lengths are simple one dimensional examples of nonlocal geometric properties. As such they show the effects of number scaling. The usual expression for the length of a timelike path, $p$, from $x$ to $y$, parameterized by a real variable, $\gamma$, is given by

$$L(p) = \int \sqrt{-\eta_{\mu,\nu} \frac{dp^\mu}{d\gamma} \frac{dp^\nu}{d\gamma}} \, d\gamma.$$  \hspace{1cm} (64)

Here $\eta_{\mu,\nu} = -1, 1, 1, 1$ is the metric tensor for Minkowski space.

The tangent bundle viewpoint taken here results in two descriptions of the path length in Eq. 64. One is local and the other is nonlocal. The local description regards the integral as a limit of a sum of integrands, all of which are quantities in a fiber, $F_x$, in a tangent bundle over $M$. A coordinate chart, $\phi_x : M \to \bar{R}_x^4$, is used to lift the path $p$ on $M$ to a path on $\bar{R}_x^4$ in $F_x$. The path length integral is well defined in $F_x$ with $\gamma$ a real parameter in $\bar{R}_x$.

The nonlocal description of the path length puts the integrand, for each value of $\gamma$, in the fiber, $F_{p(\gamma)}$. As noted before, integrals over quantities in different fibers are not defined. This is fixed by use of connections to map the integrands from $p(\gamma)$ to a common location, $x$. The integral is well defined as the mapped
integrands are all quantities in the base set \( \mathbb{R} \). This base set is the same for all scaled real structures, \( \tilde{R}_{r,x} \) in \( F_x \).

These nonlocal descriptions are the ones of interest here. They extend the nonlocal description of gauge theories to integrals over space time regions of \( M \).

The resulting expression for the length of \( p \) in \( F_x \) is

\[
L(p)_x = \int W_{RT}(x, p(\gamma)) \sqrt{-\eta_{\mu,\nu} \frac{dp^\mu}{d\gamma} \frac{dp^\nu}{d\gamma}} d\gamma
\]

Here \( r^p_{x,p(\gamma)} \) is the real Lie algebra representation of the operator, \( W_{RT}(x, p(\gamma)) \) that connects \( \tilde{R}_{p(\gamma)} \) in \( F_{p(\gamma)} \), to \( W_{RT}(x, p(\gamma)) \tilde{R}_x = r^p_{x,p(\gamma)} x \) in \( F_x \). The scale factor \( r^p_{x,p(\gamma)} \) is obtained from Eq. 65 by replacing the limit 1 with \( \gamma \). This gives

\[
r^p_{x,p(\gamma)} = e^{\int_0^{\gamma} \tilde{\Lambda}(p(s)) \cdot \tilde{p}(ds)}.
\]

In the absence of scaling, \( W_{RT}(x, p(\gamma)) = I_{x,p(\gamma)} \). This is obtained from Eq. 59 by setting \( r^p_{x,p(\gamma)} = 1 \). This shows that the usual setup is a special case of scaling where the scaling factor and the resulting connections are the identity everywhere.

As was the case for the \( B \) field, if \( \tilde{A} \) is nonintegrable, then the integral in the exponent of Eq. 65 depends on the path between \( x \) and \( y \). If \( \tilde{A} \) is integrable, then it can be expressed as the gradient of a real field, \( \theta(x) \). This is the same real field that appears in Eq. 33 for gauge theories. Eq. 66 then simplifies to

\[
r^p_{x,p(\gamma)} = e^{\theta(x) - \theta(p(\gamma))}.
\]

It is of interest to note that Einstein’s comments (footnote 1) on Weyl’s paper [21], applied here, suggest that \( \tilde{A} \) should be integrable. The reason is that, if \( \tilde{A} \) is nonintegrable, then the effect of scaling on some physical quantities will depend on their past histories.

Use of Eq. 67 for \( r^p_{x,p(\gamma)} \) in Eq. 65 gives

\[
L(p)_x = \int e^{\theta(x) - \theta(p(\gamma))} \sqrt{-\eta_{\mu,\nu} \frac{dp^\mu}{d\gamma} \frac{dp^\nu}{d\gamma}} d\gamma.
\]

The scaling factor depends on the path endpoints, \( x \) and \( p(\gamma) \), and not on the path from one point to the other.

Eq. 68 holds for time like paths. For space like paths, \( -\eta_{\mu,\nu} \) is replaced by \( \eta_{\mu,\nu} \). For null paths, such as those travelled by light rays, the path length is 0. This is the only case for which the path length is independent of scaling. It is a consequence of the fact that 0 is the only number whose value is unaffected by scaling.

Eqs. 65 and 68 show that the length of space and time like paths are affected by scaling. One consequence of this is that the proper time, measured by clocks carried along the path, is affected by scaling. This follows from replacement of the arbitrary parameter \( \gamma \) by the proper time, \( \tau \), and the use of 36

\[
d\tau = (-\eta_{\mu,\nu} dp^\mu dp^\nu / d\gamma d\gamma)^{1/2} d\gamma.
\]

At reference point, \( x \), of \( M \), the elapsed proper time along \( p \) from 0 to \( T \) is given by

\[
\tau_0 = e^{\theta(x)} \int_0^T e^{-\theta(p(\tau))} d\tau.
\]

Here \( \tau_0 \) is the elapsed proper time in the presence of scaling. In the absence of scaling \( \theta(p(\tau)) = \theta(x) \) for all \( \tau \).

The choice of reference locations for the path lengths in Eqs. 68 and 70 is not limited to \( x \) or any point on the path. Any location, \( z \), in \( M \) can be chosen. The resulting path length, as a numerical value in the fiber, \( F_z \), is obtained by replacing \( x \) by \( z \) in Eqs. 68 and 70.

\[\text{Gauge theories are nonlocal because the derivatives involve comparison of field values in neighboring fibers.}\]
7.3.2 Geodesics

Geodesics are also affected by number scaling. The minimum path length between $x$ and $y$ is found by varying $L(p)_x^y$ with respect to $p$. Addition of a small path change, $\delta p$, to $p$ gives

$$L(p + \delta p)_x = \int e^{\theta(x) - \theta(p(\gamma)) + \delta\theta} \sqrt{-\eta_{\alpha,\beta} \frac{d(p^\mu + \delta p^\mu)}{d\gamma} \frac{d(p^\nu + \delta p^\nu)}{d\gamma}} d\gamma.$$  \hspace{1cm} (71)

The Euler Lagrange equations for $L(p)$ are obtained by expansion of Eq. (71) to obtain

$$L(p)_x + L(\delta p)_x = e^{\theta(x)} \int e^{-\theta(p(\gamma))}[1 - \frac{d\theta(p)}{d\rho} \delta\rho^\mu] \times (-\eta_{\mu,\nu} \frac{dp^\mu}{d\gamma} \frac{dp^\nu}{d\gamma})^{1/2} [1 + \eta_{\mu,\nu} \frac{dp^\mu}{d\gamma} \frac{dp^\nu}{d\gamma})^{-1} (-\eta_{\alpha,\beta} \frac{d\delta\rho^\alpha}{d\gamma} \frac{d\delta\rho^\beta}{d\gamma})] d\gamma.$$ \hspace{1cm} (72)

Here $e^{\theta(x)}$ has been moved outside the integral as a constant factor, and a Taylor expansion of the exponent has been used. The factor in the second line was obtained by factoring $(-\eta_{\mu,\nu} \frac{dp^\mu}{d\gamma} \frac{dp^\nu}{d\gamma})^{1/2}$ from $\delta[(-\eta_{\mu,\nu} \frac{dp^\mu}{d\gamma} \frac{dp^\nu}{d\gamma})^{1/2}]$ and expanding the resulting square root \[36\]. Removal of the $L(p)$ component and changing the arbitrary variable, $\gamma$, to the proper time, $\tau$ as in Eq. (69) gives

$$\delta L(p) = \delta\tau(p) = e^{\theta(x)} \int e^{-\theta(p(\tau))} (-\eta_{\alpha,\beta} \frac{dp^\alpha}{d\tau} \frac{d\delta\rho^\beta}{d\tau}) - \frac{d\theta(p)}{d\rho} \delta\rho^\mu d\tau = 0.$$  \hspace{1cm} (73)

Integration of the first term by parts and assuming $\delta\rho^\beta = 0$ at the integral endpoints gives, after index relabeling,

$$\delta\tau(p) = e^{\theta(x)} \int e^{-\theta(p(\tau))} [\eta_{\alpha,\beta} \frac{dp^\alpha}{d\tau} - \eta_{\alpha,\beta} \frac{d\theta(p)}{d\rho} \frac{dp^\mu}{d\tau} \frac{dp^\alpha}{d\tau} - \frac{d\theta(p)}{d\rho} \delta\rho^\mu] d\tau = 0.$$ \hspace{1cm} (74)

This equation is satisfied if the terms in the square brackets sum to 0. Multiplying by $\eta^{\alpha,\beta}$ and using the fact that $\eta_{\alpha,\beta} = 0$ if and only if $\alpha \neq \beta$ gives

$$\frac{d}{d\tau} \frac{dp^\alpha}{d\tau} - \frac{d\theta(p)}{d\rho} \frac{dp^\mu}{d\tau} \frac{dp^\alpha}{d\tau} - \eta^{\alpha,\beta} \frac{d\theta(p)}{d\rho} = 0.$$ \hspace{1cm} (75)

This is the geodesic equation in the presence of number scaling. Comparison of this with the equation from general relativity \[36\]

$$\frac{d^2 p^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu,\nu} \frac{dp^\mu}{d\tau} \frac{dp^\nu}{d\tau} = 0,$$ \hspace{1cm} (76)

shows that the Christoffel symbol is 0 unless $\alpha = \nu$. In this case $\Gamma^\nu_{\mu,\nu} = -d\theta/dp^\mu$, independent of $\nu$.

One can use the chain rule of differentiation to write Eq. (75) in the revealing form,

$$\left[\frac{d}{d\tau} - \vec{A}(p(\tau)) \cdot \nabla_{p(\tau)}\right] \frac{dp^\alpha}{d\tau} - \eta^{\alpha,\beta} A_\alpha(p(\tau)) = 0.$$ \hspace{1cm} (77)

The term in the square brackets is similar to a covariant derivative. It includes some of the effects of number scaling. The remaining effects are shown by the presence of the term, $\eta^{\alpha,\beta} A_\alpha(p(\tau))$. This arises from the effect of path variation on the field $\theta$ in the exponential, $e^{-\theta(p + \delta p)}$.

It is tempting to follow the argument in general relativity \[36\] by saying that the first two terms of either Eq. (77) or (75) represent the motion of an unaccelerated particle in the presence of the field, $\theta$. In this case the third term acts like a force on a particle, also generated by $\theta$. However, one must keep in mind that the effect of $\theta$, or, more accurately, of the gradient vector field, $\vec{A}(x) = \nabla_{p(\tau)}\theta$ on particle motion, arises from the connection between the fibers of the tangent bundle. This in turn arises from the observation that numbers can be scaled and the scaling factor can depend on space and time. Note that if $\vec{A}(x) = 0$ everywhere, then there is no scaling and Eqs (75) or (77) describe a straight line path.

8 Experimental restrictions on $\vec{A}$

In sections 5 and 6 it was seen that the $\vec{B}$ field is the photon field. This was based on the assumption that the overall gauge group was $DGL(1, C)$ for Abelian theories and $DGL(1, C) \times SU(N)$ for nonabelian
Theories. If another $U(1)$ factor that acts just on the vector spaces is added, then the associated Lie algebra vector field is the photon. This would conflict with the use of the scalar field over the complex scalars in the description of the Higgs mechanism. This leaves the $\vec{A}$ and possibly the $\vec{B}$ fields looking for relevance in gauge theory physics.

The results in the last section show that the presence of the $\vec{A}$ field affects geometric quantities. This must be reconciled with the lack of definite physical evidence for the effects of number scaling.

One way to do this is to use a scenario in which $M$ represents cosmological space and time. Time varies from the big bang, about $14 \times 10^9$ years ago, to the present and beyond. Space positions vary over the observable universe. General relativity is neglected here.

This description of $M$ means that the tangent bundle, described in the previous section, is a bundle of fibers over all of space and time, from the big bang to the present, and over all locations of the universe. It follows that numerical values of geometric and physical quantities are local in that they are expressed as real numerical quantities in a fiber $F_x$ at some location $x$. Examples of this are curve lengths in Eq. 68 and the geodesic equation, Eq. 77. Both are numerical values in $F_x$.

A direct comparison of these scaled quantities with experimental results must take account of the fact that all experiments and measurements are necessarily local. They must occur in a region of space and time in which there are or can be occupied by observers. The reason is that observers are needed to implement the relevant measurement procedures. This is the case whether one is measuring properties of locally produced systems or of incoming radiation emitted by systems at cosmological distances.

Consider all geometric or physical quantities whose description is limited to a region of space and time that is or can be occupied by observers. This includes such things as lengths of paths in the region or wave packets whose extent is limited to the region. The lack of evidence for number scaling effects in this region means that $A(x) \approx 0$ to the limits of experimental accuracy in the region. It says nothing about the values of $A(x)$ outside the region. They can be quite large and vary rapidly in very far away regions in space and time, including those near and after the big bang.

These restrictions on values of $A$ apply to a region that is occupiable by observers. The exact size of this region is not known. However it must include, at least, regions already occupied by observers. This includes the surface of the earth. Since we, as observers, are in principle capable of interplanetary travel, the region should include the solar system.

The region should also include other planets and areas around other nearby stars which are in reasonable communication distances from us. A generous estimate of the size of this region is that of a sphere about 2,500 light years in diameter roughly centered on the solar system.

The exact size of this region is not important. What is relevant is that the region is a small fraction of the observable universe. The size is predicated on the idea that observers in locations outside the region are too far away for us to know that they exist or for us to communicate with them, now or in the future.

9 Conclusion

This work represents some advances over earlier work on the topic of number scaling [12]. These include advances in the explanation of number scaling and in the use of fiber bundles. The advances in number scaling consist of a clearer separation of numbers, as elements of the base set of a type (real, complex, etc.) of number structure, from the values they have in a particular scaled structure.

Values of numbers in a base set are associated with satisfaction of properties in different scaled number structures. If a real number, $a$, satisfies property $P$ in $\tilde{\mathbb{R}}$, then the number, $ra$, satisfies property $P_r$ in $\tilde{\mathbb{R}}_r$. The property $P_r$ is obtained from $P$ by replacing all multiplication operations, $\times$, in the description of $P$ by $\times/r$.

An advantage of the use of fiber bundles is that the bundles make a clear distinction between local mathematical structures, as elements of $M$ based fibers, and global mathematical structures that do not belong to any fiber. These exist outside of space and time and are outside of any fiber bundle whose base space is $M$. Elements of some global structures serve as connections between the fibers.

Yet there is much more work to be done. This includes extension to include the effects of scaling on many other physical and geometric quantities, besides the few described here, and expansion to include general relativity. Also more work is needed to determine the physical nature, if any, of the $\vec{A}$ and $\vec{B}$ fields. One may hope that these fields correspond to the Higgs’s field and the photon field, but much more work is needed to see if this is true or not.

If one used complex geometry, as in [3], then $\vec{B}$ would also affect geometric quantities.
Another area that needs additional investigation is the nature of the base set elements of number structures and of other types of structures. In particular the relationship between these elements and physical systems needs investigation. Quantum mechanical representations of numbers may be useful here. This is especially the case when one considers that expressions of numbers and number properties, as alphabet symbol strings, are usually described classically. This follows from the observation that symbols in a string are assumed to have particular intrinsic values irrespective of whether or not they are being examined.

One point about the effect of number scaling mentioned in earlier work [19] deserves to be emphasized again. This is that scaling plays no role in comparisons between outputs of experiments with one another or with outputs of computations as theoretical predictions of experimental results.

This might seem surprising in that comparison of a computational output as a number at $x$ with a numerical experimental result at $y$ can only be done locally at some point of $M$. In the presence of scaling, transference of the number at $y$ to $x$ requires multiplication of the experimental result by a scaling factor. The scaled number is then compared locally with the computational result.

The problem is that physics gives no hint of any such scaling. This is not a problem when one realizes that outputs of experiments and computations, as physical processes, are never numbers. They are physical systems in physical states that are interpreted as numbers. These interpretations are local and refer to the output locations.

Comparison of the outputs requires physical transportation of the information in the output states to a common location. At this location, the two output states, interpreted as numbers at the same point, are compared locally. No number scaling is involved.

In conclusion one hopes that this work is a contribution to the more general topic of the basic relationship between mathematics and physics. This is a topic that has interested this author [39], and many others [38, 40, 41, 42]. Clearly there is much work to be done.

Acknowledgement

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics, under contract number DE-AC02-06CH11357.

References

[1] D. Husemöller, *Fibre Bundles*, Second edition, Graduate texts in Mathematics, v. 20, Springer Verlag, New York, (1975).

[2] D. Husemöller, M. Joachim, B. Jurco, and M. Schottenloher, *Basic Bundle Theory and K-Cohomology Invariants*, Lecture Notes in Physics, 726, Springer, Berlin, Heidelberg, (2008), DOI 10.1007/ 978-3-540-74956-1, e-book.

[3] Y. Manin, *Gauge field theory and complex geometry*, second edition, Translated from Russian by N. Koblitz and J. King, Grundlehren der mathematischen wissenschaften, 289, Springer Verlag, Berlin, (1997).

[4] Y. Guttmann and H. Lyre, ”Fiber Bundle Gauge Theories and Fields Dilemma”, (2000) arXiv:physics/0005051.

[5] M. Daniel and G. Vialet, ”The geometrical setting of gauge theories of the Yang Mills type”, Reviews of Modern Phys., 52, pp 175-197, (1980).

[6] W. Drechsler and M. Mayer, *Fiber bundle techniques in gauge theories*, Springer Lecture notes in Physics #67, Springer Verlag, Berlin, (1977).

[7] C. Pfeifer, ”Tangent bundle exponential map and locally autoparallel coordinates for general connections with applications to Finslerian geometries”, arXiv:1406.5413.

[8] C. N. Yang and R. L. Mills, ”Conservation of Isotopic Spin and Isotopic Gauge Invariance,” Phys. Rev., 96, 191-195, (1954).

[9] R. Utiyama, ”Invariant theoretical interpretation of Interaction,” Phys. Rev. 101, 1597, (1956).
10. T. P. Cheng and L. F. Li, *Gauge Theory of Elementary Particle Physics*, Oxford University Press, Oxford, UK, (1984), Chapter 8.

11. M. Peskin and D. Schroeder, *An introduction to quantum field theory*, Addison Weseley Publishing Co. Reading, MA, (19950, Chapters 4 & 15.

12. I. Montvay and G. Münster, *Quantum fields on a lattice*, Cambridge University Press, UK,(1994), Chapter 3.

13. R. Wald, *Quantum field theory in curved space time and black hole thermodynamics*, University of Chicago Press, Chicago, IL. USA, 1994.

14. *Proof and other dilemmas*, B. Gold and R. Siomons, Editors, Mathematics and philosophy, Spectrum Series, Mathematical Association of America, Washington DC, 2008, Chapter III.

15. *Meaning in Mathematics*, J. Polkinghorne, Ed., Oxford University Press, Oxford, UK, 2011.

16. J. Barwise, "An Introduction to First Order Logic," in *Handbook of Mathematical Logic*, J. Barwise, Ed. North-Holland Publishing Co. New York, 1977. pp 5-46.

17. H. J. Keisler, "Fundamentals of Model Theory”, in *Handbook of Mathematical Logic*, J. Barwise, Ed. North-Holland Publishing Co. New York, (1977). pp 47-104.

18. P. Benioff, "Representations of each number type that differ by scale factors," Advances in Pure Mathematics, 3, 394-404, (2013); arXiv:1102.3658.

19. P. Benioff, "Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry,” in *Recent Developments in Bosons Research*, I. Tremblay, Ed., Nova publishing Co., (2013), Chapter 3. arXiv:1211.3381.

20. P. Benioff, "Effects of mathematical locality and number scaling on coordinate chart use”, in *Quantum Information and Computation XII*, E. Donkor, A. Pirich, H. Brandt, M. Frey, and S. Lomonaco, , Eds., Proceedings of SPIE, Vol. 9123: SPIE: Bellingham, WA, 2014, 98227, arXiv:1405.3217.

21. H. Weyl, "Gravitation and Electricity”, Sitzungsberichte der Königlich Preussichen Akademie der Wissenschaften, Jan.-June, pp 465-480, 1918.

22. P. Ginsparg, "Applied Conformal Field Theory" in *Fields, Strings and Critical Phenomena*, (Les Houches, Session XLIX, 1988) Ed. by E. Brézin and J. Zinn Justin, 1989

23. M. Gaberdiel, "An introduction to conformal field theory”, Rept. Prog. Phys., 63, 607-667, (2000), arXiv:hep-th/9910156.

24. J. L. Bell, "From absolute to local mathematics”, Synthese, 69 pp. 409-426, (1986).

25. G. Mack, "Physical principles, geometrical aspects, and locality properties of gauge field theories,” Fortshritte der Physik, 29, 135, (1981).

26. J. Randolph, *Basic Real and Abstract Analysis*, Academic Press, Inc. New York, NY, (1968), P. 26.

27. J. Shoenfield, *Mathematical Logic*, Addison Weseley Publishing Co. Inc. Reading Ma, (1967), p. 86; Wikipedia: Complex Numbers.

28. R. Kadison and J. Ringrose, *Fundamentals of the Theory of Operator Algebras: Elementary theory*, Academic Press, New York, (1983), Chap 1.

29. T. Wu and C. Yang, "Concept of nonintegrable phase factors and global formulation of gauge fields”, Phys, Rev. D, 12, 3845-3857, (1975).

30. B. Odom, D. Hanneke, B. D’Urso, and G. Gabrielse, "New Measurement of the Electron Magnetic Moment Using a One-Electron Quantum Cyclotron”, Phys. Rev. Lett. 97, 030801 (2006).

31. G. Gabrielse, D. Hanneke, T. Kinoshita, M. Nio, and B. Odom, "New Determination of the Fine Structure Constant from the Electron g Value and QED”, Phys. Rev. Lett. 97, 030802 (2006), Erratum, Phys. Rev. Lett. 99, 039902 (2007).
[32] Y. Aharonov and D. Bohm, Phys. Rev. 115, 485, (1959).
[33] L. O’Raifeartaigh, The Dawning of Gauge Theory, Princeton series in Physics, Princeton University press, Princeton, NJ, 1997.
[34] P. W. Higgs, "Broken Symmetries and the Masses of Gauge Bosons". Phys. Rev. Lett. 13 (16): 508, 1964.
[35] D. Bailin and A. Love, Introduction to Gauge theory, Graduate student series in physics, Adam Hilger, Bristol, UK, Accord, MA, USA, (1986).
[36] S. Carroll, "Lecture notes on general relativity", December, 1997, NSF-ITP/97-147; arXiv:gr-qc/971209.
[37] H. A. Smith, "Alone in the Universe", American Scientist, 99, No. 4, p. 320, (2011).
[38] M. Tegmark, "The mathematical univerese", Found. Phys., 38, 101-150, 2008.
[39] P. Benioff, "Towards a coherent theory of physics and mathematics: the theory-experiment connection", Found. Phys., 35, 1825-1856, (2005).
[40] E. Wigner, "The unreasonable effectiveness of mathematics in the natural sciences", Commun. Pure Appl. Math. 13, No. 1, (1960). Reprinted in E. Wigner, Symmetries and Reflections, (Indiana Univ. Press, Bloomingtong, IN 1966), pp222-237.
[41] R. Omnes, "Wigner’s "Unreasonable Effectiveness of Mathematics”, Revisited,” Foundations of Physics, 41, 1729-1739, (2011).
[42] A. Plotnitsky, "On the reasonable and unreasonable effectiveness of mathematics in classical and quantum physics" Found. Phys. 41, pp. 466-491, (2011).