THE SECOND HOMOTOPY GROUP IN TERMS OF COLORINGS OF
LOCALLY FINITE MODELS AND NEW RESULTS ON ASPHERICITY

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Abstract. We describe the second homotopy group of any CW-complex $K$ by analyzing the universal cover of a locally finite model of $K$ using the notion of $G$-coloring of a partially ordered set. As applications we prove a generalization of the Hurewicz theorem, which relates the homotopy and homology of non-necessarily simply-connected complexes, and derive new results on asphericity for two-dimensional complexes and group presentations.

1. Introduction

Every CW-complex has a locally finite model. This is a classical result of McCord [9, Theorem 3] who considered for any regular CW-complex $K$ the space $\mathcal{X}(K)$ of cells of $K$ with some specific topology, and defined a weak homotopy equivalence $\mu : K \to \mathcal{X}(K)$. The space $\mathcal{X}(K)$ can be viewed as a poset. The interaction between the topological and combinatorial nature of $\mathcal{X}(K)$ allows one to develop new techniques to attack problems of homotopy theory of CW-complexes (see [1]).

In this paper we use locally finite models to describe the second homotopy group of CW-complexes. The notion of $G$-coloring of a poset allows us to classify all the regular coverings of the space $\mathcal{X}(K)$. In particular, we obtain a description of the universal cover $E$ of $\mathcal{X}(K)$ which is used to find an expression for the boundary map of a chain complex whose homology coincides with the singular homology of $E$. By the Hurewicz theorem and McCord’s result, $\pi_2(K) = H_2(E)$. One of the applications of our description is the following result which reduces to the classical Hurewicz theorem when the complex is simply-connected.

Theorem 2.3. Let $K$ be a connected regular CW-complex of dimension 2 and let $K'$ be its barycentric subdivision. Consider the full (one-dimensional) subcomplex $L$ of $K'$ spanned by the barycenters of the 1-cells and 2-cells. If the inclusion of each component of $L$ in $K'$ induces the trivial morphism between the fundamental groups, then $\pi_2(K) = \mathbb{Z}[\pi_1(K)] \otimes H_2(K)$.

We also obtain results on asphericity of 2-complexes and group presentations. Recall that a connected 2-complex $K$ is aspherical if $\pi_2(K) = 0$.

Theorem 3.1. Let $K$ be a 2-dimensional regular CW-complex and let $K'$ be its barycentric subdivision. Consider the full (one-dimensional) subcomplex $L \subseteq K'$ spanned by the barycenters of the 2-cells of $K$ and the barycenters of the 1-cells which are faces of exactly two 2-cells. Suppose that for every connected component $M$ of $L$, $i_*(\pi_1(M)) \leq \pi_1(K')$.
contains an element of infinite order, where \(i_* : \pi_1(M) \to \pi_1(K')\) is the map induced by the inclusion. Then \(K\) is aspherical.

From this result one can deduce for example, the well-known fact that all compact surfaces different from \(S^2\) and \(\mathbb{R}P^2\) are aspherical.

To put our results in perspective, one should recall that it is an open problem, originally posted by Whitehead, whether any subcomplex of an aspherical 2-dimensional CW-complex is itself aspherical. We refer the reader to [5] [7] [13] for more details on Whitehead’s asphericity question.

In Theorem 3.5 we prove a result on asphericity of group presentations which resembles the homological description of \(\pi_2\) by Reidemeister chains (see [5] [10]).

2. COLORINGS AND A DESCRIPTION OF THE SECOND HOMOTOPY GROUP

A poset \(X\) will be identified with a topological space with the same underlying set as \(X\) and topology generated by the basis \(\{U_x\}_{x \in X}\), where \(U_x = \{y \in X \mid y \leq x\}\). If \(X\) and \(Y\) are posets, it is easy to see that a map \(X \to Y\) is continuous if and only if it is order preserving. We denote by \(\mathcal{K}(X)\) the simplicial complex whose simplices are the finite chains of the poset \(X\) (i.e. the classifying space of \(X\)). A result of McCord [9, Theorem 2] shows that there is a natural weak homotopy equivalence \(\mathcal{K}(X) \to X\). Given a regular CW-complex \(K\), its \textit{face poset} \(\mathcal{X}(K)\) is the poset of cells of \(K\) ordered by the face relation. Note that the classifying space of the poset \(\mathcal{X}(K)\) is the barycentric subdivision of \(K\) and therefore, there is a weak homotopy equivalence \(\mu : K \to \mathcal{X}(K)\). In particular the homology groups of the poset \(\mathcal{X}(K)\) coincide with those of \(K\). If \(X = \mathcal{X}(K)\) is the face poset of a regular CW-complex, we can compute its homology by computing the cellular homology of \(K\) in the standard way (see [8, IX, §7]), namely, for each \(n \geq 0\) let \(C_n(X)\) be the free \(\mathbb{Z}\)-module generated by the points \(x \in X\) of height \(h(x) = n\). Recall that \(h(x)\) is one less than the maximum number of points in a chain with maximum \(x\). Choose for each edge \((y, x)\) in the Hasse diagram of \(X\) a number \([x : y] \in \{1, -1\}\) in such a way that for every \(x \in X\) of height 1,

\[
\sum_{y < x} [x : y] = 0,
\]

and for each pair \(x, z \in X\) with \(h(x) = h(z) + 2\),

\[
\sum_{z < y < x} [x : y][y : z] = 0.
\]

Here \(y < x\) means that \(y < x\) and there is no \(y < z < x\). The differential \(d : C_n(X) \to C_{n-1}(X)\) is defined by \(d(x) = \sum_{y < x} [x : y]y\) in each basic element \(x\). The homology of this chain complex is then the singular homology of the poset \(X\) (viewed as a topological space). The number \([x : y]\) is the incidence of the cell \(y\) in the cell \(x\) of \(K\) for certain orientations.

Suppose \(p : E \to X = \mathcal{X}(K)\) is a topological covering (\(\mathcal{X}(K)\) considered as a topological space). Using that \(p\) is a local homeomorphism it is easy to prove that \(E\) is also the space associated to a poset. Moreover, for each \(x \in E\), \(p|_{U_x} : U_x \to U_{p(x)}\) is a homeomorphism. In particular \(E\) is the face poset of a regular CW-complex ([4] Proposition 4.7.23). If \(y < x\) in \(E\), \(p(y) < p(x)\) in \(X\). Given a choice of the incidences in \(X\), we define \([x : y] = [p(x) : p(y)]\), which is a coherent choice for the incidences in \(E\). Let \(p : E \to X = \mathcal{X}(K)\) be a regular covering and let \(G\) be its group of deck (covering) transformations. Since \(G\) acts freely
on $E$ and transitively on each fiber, $C_n(E) = \mathbb{Z}G \otimes C_n(X)$ is a free $\mathbb{Z}G$-module with basis \{ $x \in X \mid h(x) = n$ \}. The differential $d : C_n(E) \to C_n(E)$ is a homomorphism of $\mathbb{Z}G$-modules.

In [3] we characterized the regular coverings of locally finite posets (i.e. posets with finite $U_x$, for each $x$) in terms of colorings. We recall this result as it will be required in the description of the universal cover.

Let $X$ be a locally finite poset. We denote by $E(X)$ the set of edges in the Hasse diagram of $X$. An edge-path in $X$ is a sequence $\xi = (x_0, x_1)(x_1, x_2) \ldots (x_{k-1}, x_k)$ of edges, or opposites of edges. The set of closed edge-paths from a point $x_0 \in X$, with certain identifications and the operation given by concatenation, is a group $\mathcal{H}(X, x_0)$ naturally isomorphic to $\pi_1(X, x_0)$ (see [2] and [3]). This construction resembles the definition of the edge-path group of a simplicial complex. Given a group $G$, a $G$-coloring of a locally finite poset $X$ is a map $c$ which assigns a color $c(y, x) \in G$ to each edge $(y, x) \in E(X)$. Given a $G$-coloring, if $y < x$ we define $c(y, x) = c(y, x)^{-1} \in G$. A $G$-coloring of $X$ induces a weight map which maps an edge-path $\xi = (x_0, x_1)(x_1, x_2) \ldots (x_{k-1}, x_k)$ to $w_c(\xi) = c(x_0, x_1)c(x_1, x_2) \ldots c(x_{k-1}, x_k)$. A $G$-coloring $c$ is said to be admissible if for any two chains $x = x_1 < x_2 < \ldots < x_k = y$, $x = x_1' < x_2' < \ldots < x_k' = y$ with same origin and same end, the weights of the edge-paths induced by the chains coincide. An admissible $G$-coloring $c$ induces a homomorphism $W_c : \mathcal{H}(X, x_0) \to G$ which maps the class of a closed edge-path to its weight. The coloring $c$ is connected if $W_c$ is an epimorphism.

Two $G$-colorings $c$ and $c'$ of $X$ are equivalent if there exists an automorphism $\varphi : G \to G$ and an element $g_x \in G$ for each $x \in X$ such that $c'(x, y) = \varphi(g_x c(x, y) g_y^{-1})$ for each $(x, y) \in E(X)$.

**Theorem 2.1.** ([3] Corollary 3.5) Let $X$ be a connected locally finite poset and let $G$ be a group. There exists a correspondence between the set of equivalence classes of regular coverings $p : E \to X$ of $X$ with $\text{Deck}(p)$ isomorphic to $G$ and the set of equivalence classes of admissible connected $G$-colorings of $X$.

Here $\text{Deck}(p)$ denotes the group of deck transformations of $p$. The covering associated to an admissible connected $G$-coloring $c$ is the covering that corresponds to the subgroup $\ker(W_c)$ of $\mathcal{H}(X, x_0) \cong \pi_1(X, x_0)$. Theorem 3.6 of [3] tells us explicitly how to construct the covering $E(c)$ corresponding to $c$. It is the poset $E(c) = \{(x, g) \mid x \in X, g \in G\}$ with the relations $(x, g) \prec (y, g'c(x, y))$ whenever $x \prec y$ in $X$. The covering map being the projection onto the first coordinate. The group $G$ acts on $E(c)$ by left multiplication in the second coordinate.

Now, let $K$ be a regular CW-complex and suppose $c$ is any $G$-coloring of $X = \mathcal{X}(K)$ which corresponds to the universal cover, that is, $c$ is an admissible and connected $G$-coloring such that $E = E(c)$ is simply-connected or, equivalently, $W_c : \mathcal{H}(X, x_0) \to G$ is an isomorphism. The second homotopy group of $K$ is $\pi_2(K) = \pi_2(\mathcal{X}(K)) = H_2(E)$. The homology of $E$ can be computed using the chain complex described above. In the case that $K$ is two-dimensional, this computation is easier. In this case $E$ is a poset of height two and $C_3(E) = 0$. A chain $\alpha \in C_2(E) = \mathbb{Z}G \otimes C_2(X)$ is a finite sum of the form

$$\alpha = \sum_{h(x) = 2} \sum_{g \in G} n_x^x g_x$$
where \( n_e^x \in \mathbb{Z} \). The isomorphism between \( C_2(E) \) and \( \mathbb{Z}G \otimes C_2(X) \) identifies \((x, 1) \in E \) with \( x \). Thus, \( d : \mathbb{Z}G \otimes C_2(X) \rightarrow \mathbb{Z}G \otimes C_2(X) \) maps \( x \) to
\[
\sum_{y < x} [x : y](y, c(y, x)^{-1}) = \sum_{y < x} [x : y]c(y, x)^{-1}y \in C_1(E) = \mathbb{Z}G \otimes C_1(X)
\]
and then \( d(\alpha) = \sum_{h(x) = 2 \in G} \sum_{y < x} [x : y]n^x_{h, c(y, x)}y \).

Therefore \( \pi_2(K) = \ker(d) \) has the following description
\[
\pi_2(K) = \{ \sum_{h(x) = 2 \in G} n^x_{h, c(y, x)}y \mid \sum_{x > y} [x : y]n^x_{h, c(y, x)} = 0 \forall y \in X \text{ with } h(y) = 1 \text{ and } \forall h \in G \}.
\]

On the other hand, Theorem 4.4 and Remark 4.6 in [3] provide a concrete way to describe a coloring \( \hat{c} \) which corresponds to the universal cover. Let \( X \) be a locally finite poset and let \( D \) be a subdiagram (=subgraph) of the Hasse diagram of \( X \). Suppose that the poset which corresponds to \( D \) is simply-connected and that \( D \) contains all the points of \( X \) (for instance, a spanning tree). Let \( G \) be the group generated by the edges \( e \in E(X) \) which are not in \( D \) with the following relations. For each pair of chains
\[
x = x_1 \prec x_2 \prec \ldots \prec x_k = y,
\]
\[
x = x'_1 \prec x'_2 \prec \ldots \prec x'_l = y
\]
with same origin and same end, we put a relation
\[
\prod_{(x_i, x_{i+1}) \notin D} (x_i, x_{i+1}) = \prod_{(x'_i, x'_{i+1}) \notin D} (x'_i, x'_{i+1}).
\]

According to Theorem 4.4 in [3] \( G \) is isomorphic to \( \pi_1(X) \). Moreover, let \( \hat{c} \) be the \( G \)-coloring defined by \( \hat{c}(e) = \overline{e} \), the class of \( e \) in \( G \), for each \( e \in E(X) \). If \( e \in D, \overline{e} = 1 \in G \). Then \( W_{\hat{c}} : H(X, x_0) \rightarrow G \) is an isomorphism, so \( \hat{c} \) corresponds to the universal cover of \( X \). This coloring can be used in the formula above to compute \( \pi_2(K) \).

**Example 2.2.** Consider the regular CW-complex \( K \) in Figure [1]. It has three 0-cells, \( a, b, c \), six 1-cells, \( q, r, s, t, u, v \) and four 2-cells, \( w, x, y, z \). The Hasse diagram of \( X(K) \) appears in Figure [2]. Let \( D \) be the subdiagram of the Hasse diagram given by the solid edges. It is easy to see that the space corresponding to \( D \) is simply connected because it is a contractible finite space (=dismantlable poset) [11, Section 4]. The group \( G \) generated by the dotted edges \( e_1, e_2, e_3, e_4, e_5 \) with relations \( e_1e_1 = 1, e_1 = e_5, e_2 = e_3, e_2 = e_5, e_3 = e_4 \) is then isomorphic to the fundamental group of \( K \). Hence \( \pi_1(K) = \mathbb{Z}_2 \).

For each \( h \in G \) and each point of \( X(K) \) of height 1 we have one equation. These twelve equations describe \( \pi_2(K) \). Denote \( \gamma \) the generator of \( G \). Then \( \overline{e_1} = \overline{e_2} = \gamma \). We can choose the incidences \([p_0 : p_1]\) according to the orientations of the cells in Figure [1]. For instance, the equation corresponding to \( u \) and \( h \in G \) is 0 = \([w : u]n^w_h + [y : u]n^y_h \overline{\gamma} = n^w_h + n^y_h \gamma \).

For each \( h \in G \) the equations are
- Equations for \( q \) : \( n^w_h + n^z_h = 0 \)
- Equations for \( r \) : \( n^x_h + n^y_h = 0 \)
- Equations for \( s \) : \( -n^w_h - n^z_h = 0 \)
- Equations for \( t \) : \( -n^y_h - n^z_h = 0 \)
- Equations for \( u \) : \( n^w_h + n^y_h \gamma = 0 \)
- Equations for \( v \) : \( n^x_h + n^y_h \gamma = 0 \)
Therefore $\pi_2(K) = \{ n(w - \gamma w - x + \gamma x + y - \gamma y - z + \gamma z) \mid n \in \mathbb{Z} \}$ is isomorphic to $\mathbb{Z}$. In fact $K$ is just the real projective plane, so the results are not surprising. However, the example shows how to carry on the computation of $\pi_2$ for arbitrary regular CW-complexes.

**Theorem 2.3.** Let $K$ be a connected regular CW-complex of dimension 2 and let $K'$ be its barycentric subdivision. Consider the full (one-dimensional) subcomplex $L$ of $K'$ spanned by the barycenters of the 1-cells and 2-cells. If the inclusion of each component of $L$ in $K'$ induces the trivial morphism between the fundamental groups, then $\pi_2(K) = \mathbb{Z}[\pi_1(K)] \otimes H_2(K)$.

**Proof.** Let $D$ be the subgraph of the Hasse diagram of $\mathcal{X}(K)$ induced by the points of height 1 and 2. Then $L$ is the classifying space $K(D)$ of the poset associated to $D$ and the weak homotopy equivalence $\mu : K' \to \mathcal{X}(K)$ restricts to a weak equivalence $L \to D$. Moreover, for each component $L_i$ of $L$, $\mu|_{L_i}$ is a weak equivalence between $L_i$ and a component $D_i$ of $D$. If $L_i \hookrightarrow K'$ induces the trivial map in $\pi_1$, then so does the inclusion $D_i \hookrightarrow \mathcal{X}(K)$. By [3, Remark 4.2] each admissible $G$-coloring of $\mathcal{X}(K)$ is equivalent to another which is trivial in the edges of $D$. In particular, if $c$ is a $\pi_1(K)$-coloring which corresponds to the universal cover, then it is equivalent to a coloring $c'$ such that $c'(x, y) = 1$ for each $(x, y) \in D$. Then $\tilde{X} = E(c')$ is also the universal cover of $\mathcal{X}(K)$ and

$$d = 1 \otimes \delta : C_2(\tilde{X}) = \mathbb{Z}[\pi_1(K)] \otimes C_2(\mathcal{X}(K)) \to C_1(\tilde{X}) = \mathbb{Z}[\pi_1(K)] \otimes C_1(\mathcal{X}(K)),$$

where $\delta : C_2(\mathcal{X}(K)) \to C_1(\mathcal{X}(K))$ is the boundary map of the chain complex associated to $\mathcal{X}(K)$. Since $\mathbb{Z}[\pi_1(K)]$ is a free $\mathbb{Z}$-module, $\pi_2(K) = H_2(\tilde{X}) = \mathbb{Z}[\pi_1(K)] \otimes H_2(K)$ by the Künneth formula. □
When $K$ is simply-connected, the previous result reduces to the Hurewicz Theorem for dimension 2.

Theorem 2.3 can be restated as follows: If every closed edge-path of $K'$ containing no vertex of $K$ is equivalent to the trivial edge-path, then $\pi_2(K) = \mathbb{Z}[\pi_1(K)] \otimes H_2(K)$.

There is an obvious generalization of Theorem 2.3 to connected regular CW-complexes with no restriction on the dimension.

Corollary 2.4. Let $K$ be a connected regular CW-complex. If every closed edge-path of $K'$ containing only vertices which are barycenters of 1, 2 or 3-dimensional simplices is equivalent to the trivial edge-path, then $\pi_2(K) = \mathbb{Z}[\pi_1(K)] \otimes H_2(K)$.

The following is another application of our methods (compare with [12]).

Theorem 2.5. Let $X$ and $Y$ be two connected CW-complexes. If $Y$ is simply-connected, then $\pi_2(X \vee Y) = \pi_2(X) \oplus (\mathbb{Z}[\pi_1(X)] \otimes \pi_2(Y))$.

Proof. Since each CW-complex is homotopy equivalent to a simplicial complex, it suffices to prove the result for face posets $X$ and $Y$ of regular CW-complexes. Here, $X \vee Y$ denotes the space whose Hasse diagram is obtained from the diagrams of $X$ and of $Y$ by identifying a minimal element of each. Let $c$ be a coloring of $X \vee Y$ corresponding to the universal cover. Then $c$ is a $G$-coloring with $G \simeq \pi_1(X \vee Y) \simeq \pi_1(X)$. Since $Y$ is simply-connected, there is an equivalent $G$-coloring $c'$ which is trivial in $Y$ (once again by Lemma 4.1 or Remark 4.2 in [3]). The restriction of $c'$ to $X$ is an admissible connected $G$-coloring. Moreover, if a closed edge-path in $X$ is in $\ker(W_\nu|_X)$, then it is in $\ker(W_\nu) = 0$. Thus, it is trivial in $\mathcal{H}(X \vee Y)$ and then in $\mathcal{H}(X)$, since the inclusion $X \hookrightarrow X \vee Y$ induces an isomorphism between the fundamental groups. Therefore, $c'|_X$ corresponds to the universal cover of $X$.

Let $\widetilde{X} \vee \widetilde{Y} = E(c')$ be the universal cover of $X \vee Y$. Note that

$$C_n(\widetilde{X} \vee \widetilde{Y}) = \mathbb{Z}[G] \otimes C_n(X \vee Y) = (\mathbb{Z}[G] \otimes C_n(X)) \oplus (\mathbb{Z}[G] \otimes C_n(Y))$$

for $n = 1, 2$. Since $c'|_X$ corresponds to the universal cover of $X$ and $c'|_Y$ is trivial, the differential

$$d : (\mathbb{Z}[G] \otimes C_2(X)) \oplus (\mathbb{Z}[G] \otimes C_2(Y)) \to (\mathbb{Z}[G] \otimes C_1(X)) \oplus (\mathbb{Z}[G] \otimes C_1(Y))$$

has the form $d = d_{\widetilde{X}} \oplus (1_{\mathbb{Z}[G]} \otimes d_Y)$, where $d_{\widetilde{X}} : C_2(\widetilde{X}) \to C_1(\widetilde{X})$ is the differential in the chain complex associated to the universal cover of $X$ and $d_Y : C_2(Y) \to C_1(Y)$ is the differential in the complex associated to $Y$. By the Künneth formula, $\pi_2(X \vee Y) = \ker(d) = H_2(\widetilde{X}) \oplus (\mathbb{Z}[G] \otimes H_2(Y)) = \pi_2(X) \oplus (\mathbb{Z}[G] \otimes \pi_2(Y))$. □

3. RESULTS ON ASPHERICITY

We use the methods developed above to study asphericity of two-dimensional complexes and group presentations.

Theorem 3.1. Let $K$ be a 2-dimensional regular CW-complex and let $K'$ be its barycentric subdivision. Consider the full (one-dimensional) subcomplex $L \subseteq K'$ spanned by the barycenters $b(\tau)$ of the 2-cells $\tau$ of $K$ and the barycenters of the 1-cells which are faces of exactly two 2-cells. Suppose that for every connected component $M$ of $L$, $i_M(\pi_1(M)) \leq \pi_1(K')$ contains an element of infinite order, where $i_M : \pi_1(M) \to \pi_1(K')$ is the map induced by the inclusion. Then $K$ is aspherical.
Proof. Let $c$ be a $G$-coloring of $\mathcal{X}(K)$ which corresponds to the universal cover. We will use the equations describing $\pi_2(K)$ to show that if $\alpha = \sum \sum n_g^x g x \in \pi_2(K)$, then $n_g^x = 0$ for every $g \in G$ and every $x$ with $h(x) = 2$. Let $x = \tau$ be a maximal element of $\mathcal{X}(K)$. Then $W = W_c : \mathcal{H}(\mathcal{X}(K), x) \to G$ is an isomorphism.

Let $Y$ be the subspace of $\mathcal{X}(K)$ consisting of the 2-cells and the 1-cells which are faces of exactly two 2-cells. Note that $L = K(Y)$, so there is a weak homotopy equivalence $L \to Y$.

Since $i_*(\pi_1(L, b(\tau)))$ contains an element of infinite order and $W$ is an isomorphism, there is a closed edge-path $\xi$ at $x$ in $Y$ of weight $w(\xi) \in G$ of infinite order. We may assume that $\xi$ is an edge-path of minimum length satisfying this property. Suppose $\xi$ is the edge-path $x = x_0 \searrow w_0 \nearrow x_1 \searrow \ldots \searrow w_{k-1} \nearrow x_k = x$. By the minimality of $\xi$, $x_{i+1} \neq x_i$ for every $0 \leq i < k$. Since $x_i$ and $x_{i+1}$ are the unique two elements covering $w_i$, the equation corresponding to $w_i$ and an element $g \in G$ is

$$[x_i : w_i] n_g^{w_i} + [x_{i+1} : w_i] n_g^{w_{i+1}} = 0.$$  

In particular, given $g \in G$, if $n_g^{w_i} \neq 0$, then $n_g^{w_{i+1}} \neq 0$.

Let $h \in G$. Suppose that $n_h^{w_i} \neq 0$. Applying the previous assertion $k$ times we obtain that $n_h^{w(\xi)} \neq 0$. Repeating this reasoning we deduce that $n_g^{w_l(\xi)} \neq 0$ for every $l \geq 0$. However, $w(\xi) \in G$ has infinite order and this contradicts the fact that only finitely many $n_g^x$ can be non-zero.

Note that from the previous result one deduces the well-known fact that all compact surfaces different from $S^2$ and $\mathbb{R}P^2$ are aspherical. Any triangulation $K$ of such surfaces satisfies the hypotheses of the theorem since every edge of $K$ is face of exactly two 2-simplices and the links of the vertices are connected.

**Example 3.2.** The pinched two-handled torus and the wedge of two torii (Figure 3) are aspherical by Theorem 3.1.

**Figure 3.** Aspherical two-complexes.

**Remark 3.3.** It is well-known that the fundamental group of any 2-dimensional aspherical complex is torsion-free (see [3, Proposition 2.45]). Theorem 3.1 says that if the 2-complex $K$ has a torsion-free fundamental group and the maps $i_* : \pi_1(M) \to \pi_1(K')$ are non-trivial then $K$ is aspherical.

We derive from Theorem 3.1 a result on asphericity of group presentations. This result resembles in some sense the homological description of $\pi_2$ using Reidemeister chains [10, Thm 3.8] (See also [5]). Given a group presentation $P$, let $K_P$ be the usual two-dimensional complex...
CW-complex associated to the presentation, which has one 0-cell, one 1-cell for each generator and one 2-cell for each relator. The presentation \( P \) is called aspherical. In order to study asphericity of \( P \), we will construct a digraph \( D_P \) associated to \( P \) together with a \( G \)-coloring. First note that the notion of a \( G \)-coloring naturally extends to directed graphs. A \( G \)-coloring of a digraph \( D \) is a labeling of the edges of \( D \) by elements in \( G \). We allow loops and parallel edges which could have different colors. The color of the inverse of an edge \( e \) is the inverse \( c(e)^{-1} \) of the color of \( e \). A \( G \)-coloring \( c \) induces a weight map \( w_c \). If \( \alpha = e_0 e_1 \ldots e_n \) is a cycle in the underlying undirected graph of \( D \) (for each \( i, e_i \) is an edge of \( D \) or \( e_i^{-1} \) is an edge of \( D \)), then \( w_c(\alpha) = c(e_0) c(e_1) \ldots c(e_n) \).

Let \( P = \langle a_1, a_2, \ldots, a_k \mid r_1, r_2, \ldots, r_s \rangle \) be a presentation of a group \( G \). The vertices of the directed graph \( D_P \) are the letters \( a_i \) which appear in total exactly twice in the words \( r_1, r_2, \ldots, r_s \). So, \( a_i \) appears either with exponent 2 or \(-2 \) in one of the relators and does not appear in any other relator, or it appears twice (in the same relator or in two different relators) with exponent 1 or \(-1 \) each time. Each vertex of \( D_P \) will be the source of exactly two oriented edges and the target of two directed edges. Let \( r = r_j = a_i^{\epsilon_{i_0}} a_j^{\epsilon_{i_1}} \ldots a_k^{\epsilon_{i_{t-1}}} \) be one of the relators of \( P \), \( \epsilon_i = \pm 1 \) for every \( l \in \mathbb{Z} \). We consider \( r \) as a cyclic word, so for example \( a_i \) comes after \( a_{i_0} \) and \( a_{i_0} \) comes after \( a_{i_{t-1}} \). Suppose \( a_{i_l} \) is a vertex of \( D_P \). We consider the first letter \( a_{i_{l+m}} \) coming after \( a_{i_1} \), which is a vertex of \( D_P \) (i.e. the minimum \( m > 0 \) such that \( a_{i_{l+m}} \in D_P \)). It could be a letter different from \( a_{i_1} \) or the same letter if \( a_{i_l} \) appears twice in \( r \) or if it appears once and no other \( a_{i_s} \) is a vertex of \( D_P \). Then \((a_{i_j}, a_{i_{l+m}}) \) is a directed edge of \( D_P \) and the color corresponding to that edge is the subword \( g^{a_{i_{l+1}} a_{i_{l+2}} \ldots a_{i_{l+m-1}}} h \in G \) where \( g = 1 \) if \( \epsilon_l = 1 \) and \( g = a_{i_l} \) if \( \epsilon_l = -1 \), \( h = 1 \) if \( \epsilon_{l+m} = -1 \) and \( h = a_{i_{l+m}} \) if \( \epsilon_{l+m} = 1 \).

The next example illustrates the situation.

**Example 3.4.** Figure 4 shows the digraph \( D_P \) corresponding to the presentation \( P = \langle a, b, c, d, e \mid b^3ca^{-1}b^{-1}dba, c^{-1}debe \rangle \). Its vertices are \( a, c, d \) and \( e \).

![Figure 4. The digraph \( D_P \) associated to \( P \).](image)

**Theorem 3.5.** Let \( P \) be a presentation of a group \( G \). Suppose that every relator in \( P \) contains a letter which is a vertex of \( D_P \). If each component of \( D_P \) contains a cycle whose weight has infinite order in \( G \), then \( P \) is aspherical.
Proof. We subdivide $K_P$ barycentrically to obtain a regular CW-complex $K$ as usual. Each 1-cell corresponding to a generator $a$ in $P$ is subdivided in two 1-cells $e_{a_0}$ and $e_{a_1}$ sharing the unique vertex $v$ of $K_P$ and a new vertex $v_a$. The 2-cell $f_r$ corresponding to a relator $r$ of $P$ is subdivided in $2m$ 2-cells where $m$ is the number of letters in $r$, adding a new 0-cell $v_r$ in the interior of the original 2-cell. Let $L$ be the 1-dimensional subcomplex of $K'$ defined as in the statement of Theorem 3.1. The vertices of $L$ are the barycenters of the 2-cells of $K$ and the barycenters of the 1-cells which are faces of exactly two 2-cells. In the interior of the cell $f_r$ there are exactly $4m$ vertices of $L$ (the barycenters of the $2m$ 2-cells and the barycenters of the $2m$ edges from $v_r$ to $v$ and to each $v_a$). This 1-dimensional complex of $4m$ vertices is a cycle that we denote $C_r$. The remaining vertices of $L$ are the barycenters $b(e_{a_0})$ and $b(e_{a_1})$ for each letter $a$ which is a vertex of $D_P$. We show that the hypotheses of the theorem ensure that the hypotheses of Theorem 3.1 are fulfilled.

![Figure 5](image.png)

**Figure 5.** The 2-cell $f_r$. Here $r$ is the first relator of $P$ in Example 3.4. The simplices of the subcomplex $L$ are drawn with thick lines and the edge-path $\gamma_i$ homotopic to $b^2c \in G$ appears with dotted segments.

Since each relator contains a letter which is a vertex of $D_P$, the components of $D_P$ are in bijection with the components of $L$. Suppose $a$ and $c$ are vertices of $D_P$ and that there is an edge $(a, c) \in D_P$ (or $(c, a)$). Then, there is a relator $r$ of $P$ such that $a$ and $c$ are letters of $r$. Since $a, c \in D_P$, $b(e_{a_1})$ and $b(e_{c_1})$ are vertices of $L$ and they lie in the 2-cell $f_r$ of $K_P$ corresponding to $r$. Moreover, there is an edge in $L$ from $b(e_{a_1})$ to the cycle $C_r$ and an edge from $b(e_{c_1})$ to $C_r$. Therefore there is an edge-path in $L$ from $b(e_{a_1})$ to $b(e_{c_1})$ entirely contained in $f_r$ (see Figure 5). A cycle $\alpha$ in $D_P$ with base point $a$, has associated then a closed edge-path $\xi$ in $L$ at $b(e_{a_1})$. We will show that the order of $\xi$ in the edge-path group $E(K', b(e_{a_1}))$ is infinite or, equivalently, that the order of $\hat{\xi} = (v, b(e_{a_1}))\xi(b(e_{a_1}), v) \in E(K', v)$ is infinite. The edge-path $\hat{\xi}'$ obtained from $\hat{\xi}$ by inserting the edge-paths $(b(e_l), v)(v, b(e_l))$ at each vertex $b(e_l)$ ($l$ a letter in $\alpha$) is
equivalent to $\hat{\xi}$. Suppose $a = l_0, l_1, \ldots, l_k = a$ are the vertices of $\alpha$. The edge-path $\hat{\xi}'$ is a composition of closed edge-paths $\gamma_i$ in $K'$ at $v$, each of them contained in a 2-cell $f_{r_i}$. The edge-path $\gamma_i$, as an element of $\pi_1(K, v)$, is homotopic to a loop contained in the boundary of $f_{r_i}$, which is, as an element of $G$, the color of the edge $(l_i, l_{i+1})$ in $\alpha$. Thus, $\hat{\xi}' \in \pi_1(K, v) \simeq G$ coincides with the weight of $\alpha$ and the first one has infinite order provided the second one does.

In Example 3.4 there is an edge from $c$ to $d$ with color $c^{-1}d$, an edge from $a$ to $d$ with color $a^{-1}b^{-1}d$ and an edge from $a$ to $c$ with color $b^2c$. Therefore, there is a cycle with base point $e$ whose weight is $c^{-1}d(a^{-1}b^{-1}d)^{-1}b^2c = c^{-1}bab^2c \in G$. It is easy to verify that this element has infinite order, since $a + 3b$ clearly has infinite order in the abelianization $G/[G : G]$. Since $D_P$ has a unique component and both relators of $P$ have at least one letter in $D_P$, Theorem 3.5 applies. This shows that $P$ is aspherical.

References

[1] J.A. Barmak. Algebraic topology of finite topological spaces and applications. Lecture Notes in Mathematics Vol. 2032. Springer (2011) xviii+170 pp.

[2] J.A. Barmak and E.G. Minian. Minimal finite models. J. Homotopy Relat. Struct. 2(2007), No. 1, 127-140.

[3] J.A. Barmak and E.G. Minian. G-colorings of posets, coverings and presentations of the fundamental group. Available in arXiv:1212.6442v2

[4] A. Björner, M. Las Vergnas, B. Sturmfels, N.White and G. Ziegler. Oriented matroids. Encyclopedia of Mathematics and its Applications. Cambridge University Press (1999).

[5] W. Bogley. J.H.C. Whitehead’s asphericity question, in Two-dimensional homotopy and combinatorial group theory. London Mathematical Society Lecture Note Series 197, Cambridge University Press (1993).

[6] A. Hatcher. Algebraic Topology. Cambridge University Press (2002) xii+544 pp.

[7] J. Howie, Some remarks on a problem of J. H. C. Whitehead. Topology 22 (1983), 475-485.

[8] W.S. Massey. A basic course in algebraic topology. Graduate Texts in Mathematics, 127. Springer-Verlag, New York, 1991. xvi+428 pp

[9] M.C. McCord. Singular homology groups and homotopy groups of finite topological spaces. Duke Math. J. 33 (1966), 465-474.

[10] A. Sieradski. Algebraic topology for two dimensional complexes, in Two-dimensional homotopy and combinatorial group theory. London Mathematical Society Lecture Note Series 197, Cambridge University Press (1993).

[11] R.E. Stong. Finite topological spaces. Trans. Amer. Math. Soc. 123 (1966), 325-340.

[12] G. Whitehead. Homotopy groups of joins and unions. Trans. Amer. Math. Soc. 83 (1956), 55-69.

[13] J. H. C. Whitehead, On adding relations to homotopy groups. Ann. of Math. (2), 42 (1941), no. 2. 409-428.

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