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Topological aspects of generalized Harper operators

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Abstract. A generalized version of the TKNN-equations computing Hall conductances for generalized Dirac-like Harper operators is derived. Geometrically these equations relate Chern numbers of suitable (dual) bundles naturally associated to spectral projections of the operators.

Keywords: TKNN-equations, Noncommutative torus, vector bundles, Chern numbers.

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GENERALIZED HARPER OPERATORS

The integer quantum Hall effect (IQHE) reveals a variety of surprising and attractive physical features, and has been the subject of several investigations (see [17, 13] and references therein). In fact, a complete spectral analysis of the Schrödinger operator for a single particle moving in a plane in a periodic potential and subject to an uniform orthogonal magnetic field of strength B (magnetic Bloch electron) is extremely difficult. Thus the need for simpler effective models which hopefully capture (some of) the main physical features in suitable physical regimes.

In the limit of a strong magnetic field, \( B \gg 1 \), the IQHE is well described by an effective Harper operator (cf. [23, 2, 15, 12]). For this model the quantization (in units of \( e^2/h \)) of the Hall conductance has a geometric meaning being related to Chern numbers of suitable naturally bundles associated to spectral projections of the operator. A family of Diophantine equations, the TKNN-equations of [22], provides a recipe for computing such integers. The aim of the present paper is to derive a generalized version of the TKNN-equations yielding the Hall conductances for more general Dirac-like Harper operators. Interest in such generalizations comes also from these Dirac-like operators appearing naturally in important physical models, notably models for the graphene.

With \( \theta = 1/B \), the effective Harper operator is

\[
(H^\theta_{1,0}) (x) = \psi(x - \theta) + \psi(x + \theta) + 2 \cos(2\pi x) \psi(x),
\]

acting on the Hilbert space \( \mathscr{H}_1 = L^2(\mathbb{R}) \). That this operator is the simplest representative of a large family of generalized Harper operators, sharing similar mathematical properties, is our starting point. On the Hilbert space \( \mathscr{H}_1 \) consider the unitary operators

\[
(T_1 \psi)(x) = e^{i2\pi x} \psi(x), \quad (T^\theta_2 \psi)(x) = \psi(x - \theta),
\]
with $\theta \in \mathbb{R}$. They are readily seen to obey the relation

$$T_1 T_2^\theta = e^{i2\pi \theta} T_2^\theta T_1,$$

yielding for the Harper operator the expression $H_{1,0}^\theta = T_1 + (T_1)^\dagger + T_2^\theta + (T_2^\theta)^\dagger$.

For any positive integer $q = 1, 2, \ldots$, on the vector space $\mathbb{C}^q$ consider two unitary $q \times q$ matrices $U_q$ and $V_q$ defined as follows. Let $\{e_0, \ldots, e_{q-1}\}$ be the canonical basis of $\mathbb{C}^q$, then $U_q$ is a diagonal matrix and $V_q$ is a shift matrix acting as

$$U_q : e_j \mapsto e^{i2\pi \frac{j}{q}} e_j, \quad \text{and} \quad V_q : e_j \mapsto e^{i2\pi \frac{j+1}{q}} e_j,$$

where $[\cdot]_q$ stays for modulo $q$. They obey

$$U_q V_q = e^{i2\pi \frac{1}{q}} V_q U_q \quad \text{and} \quad (U_q)^q = I_q = (V_q)^q.$$

Then, on the Hilbert space $\mathcal{H}_q = L^2(\mathbb{R}) \otimes \mathbb{C}^q$ one defines a pair of unitary operators:

$$U_q = T_1 \otimes U_q, \quad V_{q,r}^\theta = T_2^\varepsilon \otimes (V_q)^r,$$

with $\varepsilon(\theta, q, r) = \theta - \frac{r}{q}$ and $T_1$ and $T_2^\varepsilon$ given by (2). The integer $r \in \{0, \pm 1, \ldots, \pm (q-1)\}$ is chosen coprime with respect to $q$. As for the case before (when $q = 1, r = 0$), the operators (4) also obey the relation (3):

$$U_q V_{q,r}^\theta = e^{i2\pi \theta} V_{q,r}^\theta U_q.$$

Following the definition (1) we can introduce the generalized $(q, r)$-Harper operator

$$H_{q,r}^\theta = U_q + (U_q)^\dagger + V_{q,r}^\theta + (V_{q,r}^\theta)^\dagger.$$

More generally one considers the collection $\mathcal{A}_{q,r}^\theta$ of bounded operators on the Hilbert space $\mathcal{H}_q$ generated by the unitaries $U_q$ and $V_{q,r}^\theta$. Technically $\mathcal{A}_{q,r}^\theta$ is a $C^*$-algebra, i.e. an involutive algebra closed with respect to the operator norm, and it is named the (ir)rational rotation algebra or the noncommutative torus algebra [19, 8].

By writing the operator in (1) as $H_{1,0}^\theta = D_\theta + C$ with

$$(D_\theta \psi)(x) = \psi(x - \theta) + \psi(x + \theta), \quad (C \psi)(x) = 2 \cos(2\pi x) \psi(x),$$

in particular, the generalized $(2, 1)$-Harper operator $H_{2,1}^\theta$ is just

$$H_{2,1}^\theta = \begin{pmatrix} C & D_{\theta - \frac{1}{2}} \\ D_{-\frac{1}{2}} & -C \end{pmatrix},$$

acting on $\mathcal{H}_2 = L^2(\mathbb{R}) \otimes \mathbb{C}^2$. This operator provides an interesting effective model for the IQHE on graphene [3, 14, 21]. Moreover, Dirac-like operators like $H_{2,1}^\theta$ can be used to describe effective models for electrons interacting with the periodic structure of a crystal through a periodic (internal) magnetic field and subjected to the action of an external strong magnetic field [9, 12].
**BLOCH-FLOQUET TRANSFORM**

For rational deformation parameter, $\theta = M/N$ ($M$ and $N$ taken to be coprime here and after), any family of operators $\mathcal{A}^\theta_{q,r}$ can be decomposed in a continuous way according to a generalized version of the Bloch theorem. More explicitly we have the following.

**Proposition A.** Let $\theta = M/N$. For any (admissible) pair $(q,r)$ the bounded operator algebra $\mathcal{A}^\theta_{q,r}$ on $\mathcal{H}_q$ admits a bundle representation $\Pi_{q,r}$ over the ordinary two-torus $\mathbb{T}^2$. That is to say, there is a Hermitian vector bundle $E_{N,q} \to \mathbb{T}^2$ together with a unitary transform $\mathcal{F}_{q,r} : \mathcal{H}_q \to L^2(E_{N,q})$ such that

$$\Pi_{q,r}(\mathcal{A}^\theta_{q,r}) = \mathcal{F}_{q,r} \mathcal{A}^\theta_{q,r} \mathcal{F}_{q,r}^{-1} \subset \Gamma(\text{End}(E_{N,q})).$$

The vector bundle $E_{N,q}$ has rank $N$ and (first) Chern number $C_1(E_{N,q}) = q$.

Here $L^2(E_{N,q})$ denotes the Hilbert space of square integrable sections of the vector bundle $E_{N,q}$ and $\Gamma(\text{End}(E_{N,q}))$ denotes the collection of continuous sections of the endomorphism bundle $\text{End}(E_{N,q}) \to \mathbb{T}^2$, i.e. the vector bundle with fibers $\text{End}(\mathbb{C}^q)$ associated with the vector bundle $E_{N,q}$. The unitary map $\mathcal{F}_{q,r}$ implementing the bundle representation of $\mathcal{A}^\theta_{q,r}$ is called (generalized) Bloch-Floquet transform [9, 11]. For the details of the proof of Prop. A, that we briefly sketch, we refer to [10] (see also [20]).

Denote with $\alpha \in \mathbb{Z}$, $|\alpha| < q$ the unique solution of $\beta q - \alpha r = 1$ (due to $q$ and $r$ being coprime) and be $M_0 = qM - rN$. Then, a simple check shows that the unitary operators

$$A_{q,r}^\theta = (T_1)^{\frac{1}{2q}} \otimes (U_q)^\alpha, \quad B_{q,r}^\theta = T_2^\frac{M_0}{q} \otimes (V_q)^r$$

commute, $[A_{q,r}^\theta, B_{q,r}^\theta] = 0$, while commuting with any element in $\mathcal{A}^\theta_{q,r}$. They generate a (indeed maximally) commutative sub-algebra of the commutant of $\mathcal{A}^\theta_{q,r}$, and in particular, of symmetries for the operator (5). Were $N$ a multiple of $q$ this commutative sub-algebra would reduce to a direct sum of $q$ copies of a commutative algebra on $L^2(\mathbb{R})$. Thus to avoid this degeneracy, we take $q$ and $N$ to be coprime as well. This entails there exist two integers $d_r$ and $n_r$ such that $qd_r + Nn_r = 1$, a fact we shall exploit momentarily. Moreover, the commutative algebra generated by $A_{q,r}^\theta$ and $B_{q,r}^\theta$ is isomorphic to the algebra of continuous functions over the ordinary 2-torus $\mathbb{T}^2$.

A generalized simultaneous eigenvectors of $A_{q,r}^\theta$ and $B_{q,r}^\theta$ is a $\Xi_k \in S'(\mathbb{R}) \otimes \mathbb{C}^q$ ($S'(\mathbb{R})$ is the space of tempered distributions) such that

$$A_{q,r}^\theta \Xi_k = e^{i2\pi k_1} \Xi_k, \quad B_{q,r}^\theta \Xi_k = e^{i2\pi k_2} \Xi_k.$$

For any $k = (k_1,k_2) \in [0,1]^2 \simeq \mathbb{T}^2$, the generalized eigenvectors make up a $N$-dimensional space, a basis of which being given by a fundamental family of distribution $\Upsilon^{(j)}(k) = (\varphi_0^{(j)}(k), \ldots, \varphi_{q-1}^{(j)}(k)) \in S'(\mathbb{R}) \otimes \mathbb{C}^q$, for indices $j = 0, \ldots, N-1$, with elements $\zeta_\ell^{(j)}(k)$, $\ell = 0, \ldots, q-1$, defined by

$$\zeta_\ell^{(j)}(k) = \sqrt{\frac{|M_0|}{N}} \sum_{m \in \mathbb{Z}} e^{-i2\pi k_1 (\tau + mq)} \delta \left[ - \frac{M_0}{N}(k_2 + j) - mM_0 - \tau \frac{M_0}{q} \right].$$


Here the permutation $\tau : \ell \mapsto \tau_\ell$ of the set $\{0, \ldots, q-1\}$ is defined by $\ell = [\tau_\ell N]_q$ and, as usual, the Dirac delta function $\delta(-x_0)$ acts on functions $f : \mathbb{R} \to \mathbb{C}$ as the evaluation at the point $x_0$, i.e. $\langle \delta(-x_0); f \rangle = f(x_0)$.

We let $\mathcal{H}_{q,r}(k) \subset S'(\mathbb{R}) \otimes \mathbb{C}^q$ denote the $N$-dimensional vector space spanned by the distributions $\Upsilon^{(0)}(k), \ldots, \Upsilon^{(N-1)}(k)$. The total space of the vector bundle $E_{N,q}$ is just the disjoint union of the spaces $\mathcal{H}_{q,r}(k)$ glued together with transition functions coming from pseudo-periodic conditions satisfied by the $\Upsilon^{(j)}$'s. Indeed, from (9) one deduces that $\Upsilon^{(j)}(k_1 + 1, k_2) = \Upsilon^{(j)}(k_1, k_2)$ while $\Upsilon^{(j)}(k_1, k_2 + 1) = \Upsilon^{(j+1)}(k_1, k_2)$ for $j = 0, \ldots, N-2$ and $\Upsilon^{(N-1)}(k_1, k_2 + 1) = e^{i2\pi q} \Upsilon^{(0)}(k_1, k_2)$. Also, there is an identification

$$\tilde{T}_{q,r} : \mathcal{H}_q \to L^2(E_{N,q}) \simeq \int_{\mathbb{T}^2} \mathcal{H}_{q,r}(k) \, dz(k)$$

which is very reminiscent of the usual direct integral decomposition of the Bloch theory. We stress that Prop. A does not only states that any $H \in \mathcal{A}_{q,r}^\theta$ can be decomposed as a direct integral operator $H = \int_{\mathbb{T}^2} h(k) \, dz(k)$ with $h(k)$ an $N \times N$ matrix acting on $\mathcal{H}_{q,r}(k)$, but also that such a decomposition is continuous with respect to the topology of the vector bundle $E_{N,q}$, thus amounting to a bundle representation. For $H \in \mathcal{A}_{q,r}^\theta$, we denote $\tilde{H} = \Pi_{q,r}(H)$. For the generators, when acting on the basis $\{\Upsilon^{(j)}(k)\}$ one finds

$$\tilde{U}_q(k_1, k_2) = e^{i2\pi \frac{M_0}{N} k_2} (U_N)^{qM}, \quad \tilde{V}^\theta_{q,r}(k_1, k_2) = e^{i2\pi n k_1} (V_{N;k_1})^{d_r}. \quad (10)$$

Here $U_N$ is the diagonal matrix $U_N : e_j \mapsto e^{i2\pi \frac{M_j}{N}} e_j$; $V_{N;k_1}$ is the twisted shift matrix sending $e_j$ to $e_{j+1}$ for $j = 0, \ldots, N-2$ while $e_{N-1}$ to $e^{i2\pi q_{k_1}} e_0$. The matrices in (10) commute up to $e^{i2\pi \frac{M_j}{N} q_{d_r}} = e^{i2\pi \frac{M_j}{N}}$ being $q_{d_r} = 1 - n_r N$ as before, $\tilde{U}_q \tilde{V}^\theta_{q,r} = e^{i2\pi \frac{M_j}{N}} \tilde{V}^\theta_{q,r} \tilde{U}_q$, thus providing a representation of $\mathcal{A}_{q,r}^\theta$. Moreover, their pseudo-periodic conditions

$$\tilde{U}_q(k_1 + 1, k_2 + 1) = e^{i2\pi \frac{M_j}{N}} \tilde{U}_q(k_1, k_2), \quad \tilde{V}^\theta_{q,r}(k_1 + 1, k_2 + 1) = \tilde{V}^\theta_{q,r}(k_1, k_2),$$

match those of the basis $\{\Upsilon^{(j)}(k)\}$ thus making $\Pi_{q,r}$ a representation of $\mathcal{A}_{q,r}^\theta$ as bundle endomorphisms, as expressed in (8).

The bundle $E_{N,q}$ comes equipped with the Berry connection

$$\omega_{i,j}(k) = \langle \Upsilon^{(i)}(k) ; d\Upsilon^{(j)}(k) \rangle, \quad i, j = 0, \ldots, N-1, \quad (11)$$

Its curvature $K = d\omega$ is constant, $K(k) = \left( \frac{2\pi q}{T_N} \, I_N \right) \, dk_1 \wedge dk_2$ (up to an exact form) and when integrated it results in the first Chern number of the bundle being

$$C_1(E_{N,q}) = \frac{1}{2\pi} \int_{\mathbb{T}^2} \text{Tr}_N(K) = q.$$
GENERALIZED TKNN-EQUATIONS

For a rational $\theta = M/N$, the spectrum of $H_{q,r}^\theta$ in (5) has $N + 1$ energy bands if $N$ is odd or $N$ energy bands if $N$ is even [16, 6, 10]. These include the inf-gap (from $-\infty$ to the minimum of the spectrum) and the sup-gap (from the maximum of the spectrum to $+\infty$).

To each gap $g$ one associates a spectral projection $P_g$ with the convention that $P_0 = 0$ for the inf-gap $g = 0$ and $P_{\text{max}} = I$ for the sup-gap $g = N_{\text{max}}$ with $N_{\text{max}} = N - 1$ or $N_{\text{max}} = N$ according to whether $N$ is odd or even. As usual, the projection $P_g$ is defined via the Riesz formula for the operator $H_{q,r}^\theta$,

$$P_g = \frac{1}{12\pi i} \int_\Lambda (\lambda I - H_{q,r}^\theta)^{-1} \, d\lambda. \quad (12)$$

The closed rectifiable path $\Lambda \subset \mathbb{C}$ encloses the spectral subset $I_g = [\epsilon_0, \epsilon_g] \cap \sigma(H_{q,r}^\theta)$ (intersecting the real axis in $\epsilon_0$ and $\epsilon_g$) with the real numbers $\epsilon_0, \epsilon_g \in \mathbb{R} \setminus \sigma(H_{q,r}^\theta)$ being such that $-\infty < \epsilon_0 < \min \sigma(H_{q,r}^\theta)$ and $\epsilon_g$ in the gap $g$.

The Hall conductance associated with the energy spectrum up to the gap $g$ is related to the projection $P_g$ via the Kubo formula (linear response theory). Its value is an integer number $t_g$; it is by now well known that $t_g$ is to be thought of as the Chern number of a bundle determined by the projection $P_g$ [22, 4, 1].

Any such a spectral projection $P_g$ yields a projection $\Pi_{q,r}(P_g) \in \Gamma(\text{End}(E_{N,q}))$, via the representation $\Pi_{q,r}$ in (8), and thus a vector subbundle $L_{q,r}(P_g) \subset E_{N,q}$. The related (first) Chern number $C_1(L_{q,r}(P_g))$ measures the degree of non triviality of the bundle $L_{q,r}(P_g)$. The geometric interpretation of the Hall conductance is none other than the equality $t_g = C_1(L_{q,r}(P_g))$. On the other hand, the Chern number $C_1(L_{q,r}(P_g))$ obeys a Diophantine equation which then provides a TKNN-type equation for the conductance $t_g$. We have the following.

**Proposition B.** For any projection $P$ in the algebra $\mathcal{A}_{q,r}^\theta$ there exists a “dual” vector bundle $L_{\text{ref}}(P) \to T^2$ s.t. the following duality between Chern numbers holds:

$$C_1(L_{q,r}(P)) = q \left[ \frac{1}{N} \text{Rk}(L_{\text{ref}}(P)) + \left( \frac{M}{N} - \frac{r}{q} \right) C_1(L_{\text{ref}}(P)) \right]. \quad (13)$$

Before we sketch the proof of this result we turn to its interpretation in terms of conductances of the generalized Harper operators in (5). As mentioned, if $P_g$ is its spectral projection up to the gap $g$, the associated Hall conductance $t_g$ is the number $C_1(L_{q,r}(P_g))$. For the dual number we have $C_1(L_{\text{ref}}(P_g)) = -s_g$, with $s_g$ identified with the Hall conductance of the energy spectrum up to the gap $g$ but in the opposite limit of a weak magnetic field ($B \ll 1$) [22, 1, 9]. Writing $d_g = \text{Rk}(L_{\text{ref}}(P_g))$, relation (13) translates to the generalized TKNN-equations

$$N t_g + (qM - rN) s_g = q d_g, \quad g = 0, \ldots, N_{\text{max}}. \quad (14)$$

When $q = 1$ and $r = 0$, the above reduces to

$$N t_g + M s_g = d_g, \quad g = 0, \ldots, N_{\max}, \quad (15)$$
which is the original TKNN-equation derived in [22] for the Harper operator (1). In its
spirit then, the integer \(d_g\) in the right-hand side coincides with the labeling of the gap
when \(N\) is odd, i.e. \(d_g = g\) for \(N\) odd. When \(N\) is even \(d_g = g\) if \(0 \leq g \leq N/2 - 1\) and
\(d_g = g + 1\) if \(N/2 \leq g \leq N_{\text{max}} = N - 1\). We remark that the bound
\[
2|s_g| < N
\] (16)
(already present in [22]) still holds, owing to the bound \(2|C_1(L_{\text{ref}}(P_g))| < N\) for the
spectral projections into the gaps of the Hofstadter operator [6].

Now, the subbundle \(L_{q,r}(P) \subset E_{N,q}\) determined by the projection valued section
\(\Pi_{q,r}(P) = P(\cdot)\), for a projection \(P \in \mathcal{A}_{q,r}^\theta\), will have as fiber over \(k \in T^2\) the space
\[
L_{q,r}(P)|_k = \text{Range}(P(k)) \subset \mathcal{H}_{q,r}(k).
\] (17)
We need a dual bundle representation, \(\Pi_{q,r}^\text{ref}\) of \(\mathcal{A}_{q,r}\), s.t.
\[
P(k_1,Nk_2) = P_{\text{ref}}(k_1,M_0k_2),
\] (18)
and \(P_{\text{ref}}(\cdot) = \Pi_{q,r}^\text{ref}(P)\). We are lead to the representation
\[
U_q^\text{ref}(k) = e^{i2\pi k_2 (\mathbb{U}_N)q^M}, \quad V_q^\text{ref}(k) = V_{q,r}^\theta(k) = e^{i2\pi n_{k_1} (\mathbb{V}_{N;k_1})^M r}.
\] (19)
which obey \(U_q^\text{ref}(\cdot)V_{q,r}^\text{ref}(\cdot) = e^{i2\pi \frac{d}{N} M} V_{q,r}^\text{ref}(\cdot)U_q^\text{ref}(\cdot)\). As elements in \(C(T^2) \otimes \text{Mat}_N(\mathbb{C}) \simeq C(T^2;\text{Mat}_N(\mathbb{C}))\) they yield a representation of \(\mathcal{A}_{q,r}^\theta\) as endomorphisms of the trivial
bundle \(T^2 \times \mathbb{C}^N \rightarrow T^2\). Then, any projection \(P\) in \(\mathcal{A}_{q,r}^\theta\) is mapped to a projection-valued
section \(P_{\text{ref}}(\cdot) = \Pi_{q,r}^\text{ref}(P)\) which defines a vector subbundle \(L_{\text{ref}}(P) \rightarrow T^2\) of the trivial
vector bundle \(T^2 \times \mathbb{C}^N\). It will have as fiber over \(k \in T^2\) the space
\[
L_{\text{ref}}(P)|_k = \text{Range}(P_{\text{ref}}(k)) \subset \mathbb{C}^N.
\] (20)
Then, equation (18) say that the vector bundle \(L_{q,r}(P)\) “winded” around \(N\) times in the
second direction is (locally) isomorphic to the vector bundle \(L_{\text{ref}}(P)\) “winded” around
\(M_0\) times in the same direction. There is however an extra twist, due to the bundle \(E_{N,q}\),
of which \(L_{q,r}(P)\) is a subbundle, being not trivial. Indeed, an analysis of the transition
functions lead to the bundle isomorphism
\[
\phi_{(1,N)}^* L_{q,r}(P) \simeq \phi_{(1,M_0)}^* L_{\text{ref}}(P) \otimes \det(E_{N,q}).
\] (21)
Here \(\det(E_{N,q}) \rightarrow T^2\) is the determinant line bundle and the extra operation \(\phi_{(1,N)}^*\)
(the pullback) stays for the extra winding by \(N\) (for the bundle \(L_{q,r}(P)\)) and the same
for \(\phi_{(1,M_0)}^*\) (for the bundle \(L_{\text{ref}}(P)\)). Formula (13) is the relation among corresponding
first Chern numbers. Using the fact that \(C_1(\phi_{(1,N)}^* L_{q,r}(P)) = NC_1(L_{q,r}(P))\) and
\(C_1(\phi_{(1,M_0)}^* L_{\text{ref}}(P)) = M_0C_1(L_{\text{ref}}(P))\), as well as the identity \(C_1(\det(E_{N,q})) = C_1(E_{N,q}) = q\), the relation (13) follows from (21) by standard arguments.
THE IRRATIONAL CASE

On the algebra $\mathcal{A}_{q,r}^\theta$ there is a faithful trace defined by

$$\tau \left( (U_q)^n(V_{q,r}^\theta)^m \right) = \delta_{n,0} \delta_{m,0}$$
on monomials, and extended by linearity. Derivations $\partial_j : \mathcal{A}_{q,r}^\theta \to \mathcal{A}_{q,r}^\theta$, for $j = 1, 2$, defined on monomials by

$$\partial_1((U_q)^n(V_{q,r}^\theta)^m) = i2\pi n (U_q)^n(V_{q,r}^\theta)^m, \quad \partial_2((U_q)^n(V_{q,r}^\theta)^m) = i2\pi m (U_q)^n(V_{q,r}^\theta)^m,$$
on are extended by linearity and Leibniz rule. Lastly, we need the first Connes-Chern number which, for a projection $P \in \mathcal{A}_{q,r}^\theta$ (in the domain of the derivations) computes the integer (an index of a Fredholm operator)

$$C_1(P) = \frac{1}{i2\pi} \tau(P(\partial_1(P)\partial_2(P) - \partial_2(P)\partial_1(P))).$$

Let $H_{q,r}^\theta \in \mathcal{A}_{q,r}^\theta$ be the Hofstadter operator (5) with associated spectral projection $P_{g}^\theta$ for the gap $g$ as in (12). For $\theta \in I \subset \mathbb{R}$, the functional expression of $H_{q,r}^\theta \in \mathcal{A}_{q,r}^\theta$ is fixed and $H_{q,r}^\theta$ depends on the parameter $\theta$ only through the fundamental commutation relation which defines $\mathcal{A}_{q,r}^\theta$. Now, if the gap $g$ is open for all $\theta \in I$ (with $I$ sufficiently small), the functions $\theta \mapsto C_1(P_{g}^\theta)$ is constant in the interval $I$ [5]. On the other hand, from the structure of the group $K_0(\mathcal{A}_{q,r}^\theta)$, one deduces [18, 7] that

$$\tau(P_{g}^\theta) = m P_{g}^\theta - \theta C_1(P_{g}^\theta), \quad (22)$$

with the integer $m(\cdot) \in \mathbb{Z}$ uniquely determined by the condition $0 \leq \tau(\cdot) \leq 1$. From (22), the integer $m(\cdot)$ is constant for $\theta \in I$. Hence, formula

$$C_{q,r}(P_g) = q \left[ m(P_g) - \frac{\tau}{q} C_1(P_g) \right] = q \left[ \tau(P_g) + \left( \theta - \frac{\tau}{q} \right) C_1(P_g) \right] \in \mathbb{Z} \quad (23)$$

is well defined and extends (13) for irrational values $\theta \in I$ (for which the gap $g$ remains open). Indeed, for a rational $\theta = M/N$ one has natural identifications $\text{Rk}(L_{ref}(P)) = \tau(P)$ and $C_1(L_{ref}(P)) = C_1(p)$ [10] and for the rational torus, formula (23) is the same as (13). We think of (23) as relating conductances for the Harper operator $H_{q,r}^\theta$ in (5), thus generalizing (14) to

$$t_g + (q\theta - r)s_g = qd_g, \quad (24)$$

with $P_g$ once again the spectral projections of the Harper operator $H_{q,r}^\theta$, and now identifying $t_g = C_{q,r}(P_g)$ and $s_g = -C_1(P_g)$ as before, whereas $d_g = \tau(P_g)$.
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