A Quantum Mechanical Approach To The Polarization Transport of Photons

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Abstract

Based on quantum mechanical approach the polarization transport of photons which propagate in a medium with slow varying refractive index is studied. The photon polarizations are separated in opposite directions normal to the ray which is called "Spin Hall effect" of photons, and also the rotation of polarization plane, a manifestation of the Berry phase, occurs. This approach can be generalized to other spinning particles in inhomogeneous media as a universal approach.

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I. INTRODUCTION

In recent years progressively increasing attention is given to the phenomena of topological spin transport of quantum particles (see, for instance \[1–9\] and references therein). These effects are closely related to the general notion of Berry phase \[10\] and thus should also be showed up in the electromagnetic wave transport (so-called "Spin Hall effect of photons"). It can lead to the topological splitting of a ray of mixed (non-circular) polarizations into two circularly polarized rays due to the inhomogeneity of medium \[7\]. As it was demonstrated in \[8, 11\], this effect is completely analogous to the spin Hall effect in solids (see \[3, 5\]).

The spin Hall effect of photons has been described based on the wave packet dynamics \[11\] and post geometric optics approximation \[7\] both in inhomogeneous media with slow varying refractive index. In this paper we introduce a quantum mechanical approach to these media. The priority of this approach is that it can be generalized to other spinning particles in inhomogeneous media as a universal approach.

The different polarizations, left and right hand polarizations, are degenerate in a homogeneous isotropic medium \[8, 13\]. In our case this double degeneracy is lifted by an additional term in Hamiltonian which is interpreted as spin-orbit interaction of photons and leads to the Berry phase and Berry curvature. The Berry curvature in momentum space of photons with opposite helicities has the form of opposite-signed 'magnetic monopoles' located at the origin of the momentum space. The Berry phase gives rise to the polarization evolution law \[14\] and the Berry curvature causes an additional effective force which deflects the rays in opposite directions depending on their polarization. The latter phenomenon is a manifestation of the topological spin transport or the Spin Hall effect of photons.

This paper is organized as follow. In section II, the photon Hamiltonian in inhomogeneous media is presented. In order to obtain equations of motion, the Hamiltonian is diagonalized in section III by using a unitary transformation. During this procedure a gauge potential appears which is diagonal in the helicity basis in adiabatic approximation. Section IV is allotted to the derivation of semiclassical equations of motion.
II. HAMILTONIAN IN INHOMOGENEOUS MEDIA

An isotropic medium turns out to be weakly anisotropic due to the presence of a selected direction determined by a gradient of the refractive index. The change in refractive index affects the direction of wave vector $k$ or equivalently the direction of photon’s momentum $p$. Slow variation of refractive index guarantees the adiabatic condition [15] which renders helicity, polarization, as an adiabatic invariant [12].

In the absence of polarization we can construct a wavepacket $|\psi\rangle$, which represents a point particle (photon), by the states $|p\rangle$ in the momentum space. When we consider the polarization degrees of freedom, the states are $|p, \pm\rangle = |p\rangle \otimes |\pm\rangle$ and the constructed wavepacket represent a particle with a spin (polarization). The Hamiltonian in this case will be

$$H(p, x) = H_0(p, x) + H_s,$$

where $H_0$ is the Hamiltonian of the point particle in weakly anisotropic media and $H_s$ is the contribution of spin (polarization) term. As was mentioned, the direction of photon’s momentum varies during the photon’s trajectory, so the Hamiltonian $H_0$ depends on $p$ and $x$ (explicitly through $p = p(x)$).

The spin of the photon always follows the direction of $p$ and the evolution of it is governed by the Hamiltonian $H_s$ as [12]

$$H_s = \kappa \sigma \cdot p,$$

where the constant $\kappa$ is to be determined by experiment. The general Hamiltonian will be

$$H(p, x) = H_0(p, x) I + \kappa \sigma \cdot p, \quad (1)$$

where $I$ is a $2 \times 2$ unit matrix. Double degeneracy of the polarization, left hand and right hand, is lifted by the additional term, second term in the above equation, which is spin-orbit interaction of photons.

III. DIAGONALIZATION OF HAMILTONIAN IN ADIABATIC APPROXIMATION

In order to derive equations of motion, we first diagonalize the Hamiltonian (1). The non-diagonal part of the Hamiltonian is the $H_s$. 

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The photon’s momentum is \( \mathbf{p}(t) = p(t)\mathbf{n} \) where \( \mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \) is the unit vector and \( \theta \) and \( \varphi \) are zenithal and azimuthal angles, thus the Hamiltonian can be diagonalized with the unitary transformation

\[
U(p) = \exp(-i\theta\sigma_2/2)\exp(-i\varphi\sigma_3/2),
\]

in which \( U^\dagger(p)(\sigma \cdot p)U(p) = p\sigma_3 \). Eigenmodes of \( \sigma_3 \) physically describe the helicity quantum number \( \lambda = \hbar^{-1}\sigma \cdot p \). Under this unitary transformation the new Hamiltonian \( \tilde{H} = U^\dagger(p)HU(p) \) becomes

\[
\tilde{H} = H_0(p, xI + A) + \kappa p\sigma_3,
\]

where

\[
A(p) = iU^\dagger(p)\nabla_p U(p).
\]

In deriving the above, we have made use of the identity (see Appendix A)

\[
G^\dagger(x)f(\partial_x)G(x) = f(\partial_x + G^\dagger\partial_x G).
\]

Direct calculation via equation (2) yields the following expression for the components of the vector matrix \( A \)

\[
A_k = 0, \quad A_\theta = \frac{1}{p}\sigma_2, \quad A_\varphi = \frac{1}{p}[\cot\theta\sigma_3 - \sigma_1].
\]

The unitary transformation induces the non-Abelian gauge potential which is a pure gauge one, i.e., the corresponding field strength, \( F_{ij} = \partial_i A_j - \partial_j A_i + i[A_i, A_j] \), is identically zero. The canonical coordinate conjugate to \( p \) is \( x \) which corresponds to the usual derivative \( i\hbar\nabla_p \). In the absence of \( A \), this canonical coordinate is the physical (observable) coordinate. In the presence of \( A \), however, the physical coordinate becomes

\[
r = U^\dagger(i\hbar\nabla_p)U = xI + \hbar A.
\]

So that it now corresponds to \( i\hbar D_p \), where \( D_p \) is the covariant derivative defined by

\[
D_p = I\nabla_p - iA.
\]

As was mentioned, helicity is invariant in the adiabatic approximation in which the diagonal components of \( A \) are retained. Thus \( A \) will be a \( 2 \times 2 \) diagonal matrix in the helicity basis as follow

\[
A = p^{-1}\cot g(0, 0, 1)\sigma_3.
\]
In this representation, the gauge potential (Berry connection) is \( \lambda A \), where \( \lambda = \pm 1 \) and \( A \) is now a vector. The field strength (Berry curvature) becomes \( -\lambda \frac{p_3}{p^2} \) which is the field of a magnetic monopole of charge \( -\lambda \) situated at the origin of the momentum space. Furthermore, the physical coordinate reduces to
\[ r = x + \lambda \hbar A. \] (4)

### IV. BERRY EFFECT IN THE SEMICLASSICAL EQUATIONS OF MOTION

Let’s consider how the presence of the gauge potential \( A \) affects the commutation relations of physical variables and the equations of motion. We have from (4) that
\[ [r_i, r_j] = i\lambda \hbar^2 \varepsilon_{ijk} \frac{p_k}{p^2}; \]
Other commutation relations are \( [p_i, p_j] = 0 \) and \( [r_i, p_j] = i\delta_{ij} \). In the other words the space will be noncommutative [17]. Similar commutation relations are achieved in [16] by parallel transport of photon wave vector in momentum space and also an Aharonov-Bohm effect was predicted. In this paper, by using the quantum mechanical approach, we show that the rotation of polarization plane which is a manifestation of Berry phase, and also the spin Hall effect of photons occur in inhomogeneous media. The Heisenberg equations have the usual canonical form in the generalized variables
\[ \dot{p} = -\frac{i}{\hbar} [p, \tilde{H}], \quad \dot{r} = -\frac{i}{\hbar} [r, \tilde{H}], \]
where the photon Hamiltonian is taken as a function of generalized variables, \( \tilde{H}(p, r) \). Using the commutation relations, the semiclassical equations of motion will be
\[ \dot{p} = -\nabla_r \tilde{H} \quad , \quad \dot{r} = \nabla_p \tilde{H} + \lambda \hbar \frac{p_3}{p^3} \times \dot{p}, \] (5)
up to the first order in \( \hbar \). These relations reduce to the standard ray equations of geometric optics in the classical limit \( \hbar \to 0 \) or in the absence of polarization \( \lambda = 0 \). The second term in the right-hand side of equations (5) constitutes the corrections that describe topological spin transport of the photon. We notice that this term causes an additional displacement of photon of distinct helicity in opposite directions normal to the ray.

As a consequence, the magnitude of splitting for the rays of left-hand and right-hand polarizations is determined by a contour integral in the momentum space
\[ \delta r = \lambda \int_C \frac{(p \times dp)}{p^3}. \]
where \( C \) is the contour in the \( p \)-space along which the photon moves. Hence the Spin Hall effect of photons is a nonlocal topological effect.

The resulting geometric Berry Phase, \( \sigma \int_C \mathbf{A} \cdot d\mathbf{p} \), has opposite signs for the two polarizations. Therefore, for a linearly polarized wave, this Berry phase leads to the rotation of the polarization plane through the angle

\[
\gamma = \int_C \mathbf{A} \cdot d\mathbf{p} = \int_C \cos \theta d\varphi. \tag{6}
\]

This is the Rytov law for photons, a manifestation of the Berry phase, which is topological in nature as the spin Hall effect is.

The above finding for Berry effect, equations (5) and (6), coincide with the result \([7, 11]\), which were obtained differently.

V. SUMMARY AND CONCLUSION

By using a quantum mechanical approach, it was shown that if we consider the spin degrees of freedom of photon in media with slow varying refractive index, an abelian gauge field will be appeared in photon position operator. This gauge field causes the space to be non-commutative which leads to an additional displacement of photon of distinct helicity in opposite direction normal to the ray (spin Hall effect of photons). Also the rotation of polarization plane occurs which is topological in nature as the spin Hall effect is. The presented quantum mechanical approach can be applied to other spinning particles in inhomogeneous media as a universal approach.

Appendix A: Appearance of pure gauge potential

To prove the identity (3), we expand the operator \( f(\frac{d}{dx}) \) in taylor series, therefore we deal with operator of the form \( (\frac{d}{dx})^n, \, n \in \mathbb{N} \). If we demonstrate that

\[
U^\dagger(x)(\frac{d}{dx})^n U(x) = (\frac{d}{dx} + U^\dagger \frac{dU}{dx})^n, \tag{7}
\]

then the identity will be proved. For this purpose we use priori.

For \( n = 1 \) the expression is true: \( U^\dagger(x) \frac{d}{dx} U(x) = \frac{d}{dx} + U^\dagger \frac{dU}{dx} \). If equation (7) is true for some \( n = i \) then, for \( n = i + 1 \) one has

\[
U^\dagger(\frac{d}{dx})^{i+1} U = U^\dagger(\frac{d}{dx})^i U \frac{d}{dx} U = U^\dagger(\frac{d}{dx})^i U \frac{d}{dx} + U^\dagger(\frac{d}{dx})^i \frac{dU}{dx}
\]
\[ U^\dagger (\frac{d}{dx})^i U \frac{d}{dx} + U^\dagger (\frac{d}{dx})^i U^\dagger \frac{dU}{dx} = U^\dagger (\frac{d}{dx})^i U (\frac{d}{dx} + U^\dagger \frac{dU}{dx}) \]

\[ = (\frac{d}{dx} + U^\dagger \frac{dU}{dx})^i + 1 = (\frac{d}{dx} + U^\dagger \frac{dU}{dx})^i + 1. \]

Thus, by periori, the equality (7) is proven for any power of \((\frac{d}{dx})\) and hence, for any analytic function of it. Three dimensional generalization of equation (3) can be proved in a similar way.

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