A Complete Classification Of The Admissible Representations Of Infinite-Dimensional Classical Matrix Groups

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In the present paper a complete description is obtained for the class of admissible (by the terminology of G. I. Ol’shansky) unitary representations of infinite-dimensional analogous classical matrix groups $GL(\infty), Sp(2\infty), O(2\infty)$.

Henceforth these objects we will imagine as matrix of operators that acts in the complex Hilbert space $H$. In a case $GL(\infty)$ suppose, that basis $\{e_1, e_2, \ldots, e_n, \ldots\}$ of $H$ is numerated by the elements from $\mathbb{N}$, and $e_n$ is an infinite row with the unique nonzero $n$-th coordinate that equals to unit.

Let $B(H)$ be a set of bounded operators in $H$, $GL(H) = \{g \in B(H) : \text{there exist } g^{-1} \in B(H)\}, GL(n) = \{g \in GL(H) : ge_i = g^*e_i = e_i \text{ for all } i > n\}$, and $GL(\infty)$ is an inductive limit of the groups $GL(n)$ ($n = 1, 2, 3, ...$).

To define $Sp(2\infty), O(2\infty)$ let’s consider in $H$ orthonormal basis $\{\ldots, e_{-n}, e_{-2}, e_{-1}, e_1, e_2, \ldots, e_n, \ldots\}$ and represent $H$ in the form $H = H_- \oplus H_+$, where $H_-$ and $H_+$ are generated by basis elements with the negative and positive numbers respectively. We put $GL(2n) = \{g \in GL(H) : ge_i = g^*e_i = e_i, \text{if } |i| > n\}$ and define operators $s_+$ and $s_-$ in $H$ by the formulae:

$$s_+e_i = e_{-i}, s_-e_i = (\text{sign } i)e_{-i}.$$ 

Let $GL(2\infty)$ be an inductive limit of $GL(2n), Sp(2\infty)$ ($O(2\infty)$) = $\{g \in GL(2\infty) : s_-g^ts_-^{-1} = g^{-1}(s_+g^ts_+) = g^{-1}\}$, where $g^t$ is a transposed matrix with respect to $g$.

We will denote by $G$ one of the groups : $GL(\infty), Sp(2\infty)$ or $O(2\infty)$. If a more specific setting will be necessary, we’ll specially notice it. We denote by $U(G)$ the unitary subgroup of group $G$ and remind the definition of the admissible representation.

**Definition** (see [2] and [3]). Let $G(n, \infty) = \{g \in G : ge_i = g^*e_i = e_i \text{ for all } i : |i| < n\}$. Factor-representation $\Pi$ of group $G$, that acts in the Hilbert space $H_\Pi$, is called an admissible one, if there exists $n \in \mathbb{N}$ and nonzero vector $\xi \in H_\Pi$ with the property: $\Pi(u)\xi = \xi$ for all $u \in U(G(n, \infty))$.

In some chapters of this paper it’s better to use the equivalent

**Definition** of an admissible representation (see [2]). The representation $\Pi$ is called an admissible one, if the set

$$\bigcup_{n=1}^{\infty} \{\eta \in H_\Pi : \Pi(u)\eta = \eta \text{ for all } u \in U(G(n, \infty))\}$$

is dense in $H_\Pi$. 

Let’s denote by $G^I_n$ the subgroup $GL(\infty)$, which consists of matrices $\begin{pmatrix} I_n & 0 \\ 0 & * \end{pmatrix}$, where $I_n$ is a unit $n \times n$– matrix. When $n = 0$, then $G^I_0 = GL(\infty)$.

Let’s give a construction of the representations of group $G^I_n$, that will be very important in our following reasoning.

Let $\Lambda_m$ be a set of all complex matrices of $m$ rows and infinite number of columns, $\nu_m$—the Gaussian measure on $\Lambda_m$ with the unit covariant operator, $A$ an $m \times m$ selfadjoint matrix, $z$ a matrix of $n$ rows and $m$ columns $(z - n \times m$– matrix).

Let’s define the representation $\Pi_{Az}$ of group $G^I_n$ ($\Pi_A$ of group $GL(\infty) = G^I_0$) in $L^2(\Lambda_m, \nu_m)$ by

$$\left(\Pi_{Az}\begin{pmatrix} I_n & 0 \\ 0 & g \end{pmatrix}\right)\eta(\lambda) = |\det g|^{i\beta} \hat{\alpha}_A(\lambda, g)\eta(\lambda g),$$

where

$$\hat{\alpha}_A(\lambda, g) = \exp \left\{ \frac{-1}{2} \text{Tr}[\lambda(gg^* - 1)\lambda^* - 2iA\lambda(gg^* - 1)\lambda^*] \right\},$$

$I_n$ is $n \times n$ –unit matrix, $\beta$ – real number;

$$\left(\Pi_{Az}\begin{pmatrix} I_n & 0 \\ h & I \end{pmatrix}\right)\eta(\lambda) = \exp[i \ \Re \ \text{Tr}(zh)]\eta(\lambda)$$

$(\eta \in L^2(\Lambda_m, \nu_m))$.

If $M$ is a set of operators in the Hilbert space, and $M'$ is a commutant of $M$, then we can write down

**Theorem 0.1.** Von Neumann algebra $(\Pi_{Az}(G^I_n))'$ is generated by the operators $\tau(u)$, where $u \in \{v \in U(m): \ vA = Av \text{ and } zv = z\} = U(m, A, z)$, that acts in $L^2(\Lambda_m, \nu_m)$ by: $(\tau(u)\eta)(\lambda) = \eta(u^*\lambda)$.

At first this fact was announced in [1] for group $GL(\infty)$.

In [2] and [3] the complete description of spherical representations of $G^I_n$ was obtained. It’s maintained in the following statement:

**Theorem 0.2.** Let $\Pi$ be a factor-representation of $G^I_n$, that acts in a Hilbert space $H_{\Pi}$. If there exists in $H_{\Pi}$ a nonzero vector $\xi$, that fixed by the operators $\Pi(U(G^I_n))$, where $U(G^I_n)$ is the unitary subgroup $G^I_n$, then $\Pi$ is multiple (for some $m, A, z$) by restriction of $\Pi_{Az}$ to subspace $\{\eta \in L^2(\Lambda_m, \nu_m): \tau(u)\eta = \eta \text{ for all } u \in U(m, A, z)\}$.

If $\rho$ is an irreducible representation of $U(m, A, z)$, $\rho_{kl}(1 \leq k, l \leq \dim \rho)$ is its matrix element, then operator

$$P_{k\rho} = \dim \rho \int_{U(m, A, z)} \rho_{kl}(u)du$$

is a minimal orthoproject from $(\Pi_{Az}(G^I_n))'$.

The main result of the paper in a case of group $G^I_n$ is a following

**Theorem 0.3.** Let $\Pi$ be an admissible factor-representation of group $G^I_n$. Then there exist $m, A, z, \rho$ such that $\Pi$ is multiple by restriction of $\Pi_{Az}$ to $P_{k\rho}L^2(\Lambda_m, \nu_m)$.

Classification result (see Theorem 6.15) for the groups $Sp(2\infty)$ and $O(2\infty)$ is obtained by the same way.
Namely, in §1 there was built the set of unitary representations (reducible ones) $\Pi_A$ of these groups, that have almost the same meaning, as representation $\Pi_A^\prime$ for $G_n^\prime$. In Proposition 1.4 the form of their decomposition into irreducible components is obtained.

The main result of the paper in a case of groups $Sp(2\infty)$ and $O(2\infty)$ contains the following theorem, which is proved in §6 (see in greater detail theorem 6.15).

**Theorem 0.4.** Let $\Pi$ be an admissible factor–representation of group $Sp(2\infty)$ or $O(2\infty)$. Then there exists $m \times m$– selfadjoint matrix $A$ such that $\Pi$ is multiple by one of the irreducible components of the representation $\Pi_A^\prime$, that was built in propositions 1.2 — 1.3 (see proposition 1.4).

Give a brief account of the logic of our reasoning. The offered method essentially bases on the following statement, that belongs to G.I. Ol’shansky.

**Theorem 0.5.** If $\Pi$ is an admissible factor–representation of group $G$, that acts in a Hilbert space $H_\Pi$, $\Pi_U$ is its restriction on $U(G)$, then $\Pi_U$ extends by continuity to the representation of semigroup of partial isometrics $U(G)$, which are the limiting points of elements from $U(G)$ concerning to the weak topology in $B(H)$.

All the main classification results (theorems 5.7, 5.10, 6.14, 6.15) follow from the structure of the spherical representations of group $GL(\infty)$ (see [3]) and group of motions $G_n^\prime$ (see[1]), which we’ll identify with the subgroup of $GL(\infty)$, that consists of matrices

$$\begin{pmatrix} I_n & 0 \\ * & * \end{pmatrix},$$

where $I_n$ is a unit $n \times n$– matrix, $*$ is an arbitrary matrix of corresponding size (see theorem 2.1).

For first let’s present the idea of the classification of the admissible representations of $GL(\infty)$ and $G_n^\prime$.

Let $G$ be the one of these groups. If $G = GL(\infty)$, then, as before, $G(p, \infty) = \{g \in G : ge_i = g^*e_i = e_i \text{ for all } i : i \leq n\}$. When $G = G_n^\prime$, then we put $G(p, \infty) = \{g \in G : ge_i = e_i \text{ for all } i : i \leq n + p \text{ and } ge_i = g^*e_i = e_i \text{ for all } i : n < i \leq p + n\}$.

If $\Pi$ is an admissible factor–representation of group $G$, then for sufficiently large $p$ in $H_\Pi$ there exists a unit $\Pi(U(G(p, \infty))$– fixed vector $\xi(p)$.

We assume without loss of generality, that $H_\Pi = [\Pi(G(\xi(p))]$, where $[\Pi(G(\xi(p))]$ is a closure of linear cover of the set $\{\Pi(G(\xi(p) \ (g \in G)\}$.

The groups, we consider, have so-called property of asymptotic abelianness (see definition 3.1), that allows to define for $\Pi$ an important invariant – an asymptotic spherical function (a. s. f.) $\varphi_\Pi$ (see proposition 3.2, definition 3.3 and theorem 4.1). It makes it possible for us to define the rang $r(\Pi)$ of representation $\Pi$.

Henceforth, using the structure of the spherical representation of group $G$ and a fact, that restriction of $\Pi$ to $G(p, \infty)$, that acts in $[\Pi(G(p, \infty))\xi(p)] \subset H_\Pi$, is irreducible, we prove, that for $p > r(\Pi)$ there exists a set of the unit

$$\Pi(U(G(2(p + 1))))$– fixed vectors $\xi_i^U$ such that

\begin{itemize}
  \item[a)] in a case when $G = GL(\infty)$ the subspaces $[\Pi(G(2(p + 1))\xi_i^U] = H_i$ are orthogonal in pairs and
  \end{itemize}

$$\bigoplus_i H_i = [\Pi(G(\xi(p))] = H_\Pi$$
(see lemma 5.3);

b) when $G = G^l_n$, the subspaces $[\Pi(G_{2(p+1)+n}^l) \xi_i^\nu] = H_i$

are orthogonal in pairs and

$$\bigoplus_i H_i = [\Pi(G) \xi(p)] = H_{II}$$

(see lemma 5.8).

That’s why for any vector $\eta \in H_{II}$ in the commutant of $\Pi(G(2(p+1), \infty))$ there exists

an orthoprojection $P_\eta$ such that for some natural $i(\eta)$ $[\Pi(G(2(p+1), \infty)) P_\eta \eta] \subset H_{i(\eta)}$.

Change if we need the orthoprojection $P_\eta$ to the lower one, we suppose, that representation

$(\Pi, G(2(p+1), \infty), [\Pi(G(2(p+1), \infty)) \eta])$ (a restriction of $\Pi$ to $G(2(p+1), \infty)$, that acts in

$[\Pi(G(2(p+1), \infty)) \eta]$) is multiple by $(\Pi, G(2(p+1), \infty), P_\eta H_{II})$.

Therefore $(\Pi, G(2(p+1), \infty), P_\eta H_{II})$ is a restriction of the direct integral of the irreducible

representations of groups $G^l_{2(p+1)+n}$ to the $G(2(p+1), \infty)$ ($n = 0$ corresponds to the case when $G = GL(\infty)$). Besides that, taking to the consideration statements 5.4 – 5.5, 5.9

$(\Pi, G^l_{2(p+1)+n}, H_{i(\eta)})$ we can realize in such form that the restriction of every irreducible

component to $G(2(p+1), \infty)$ is the same representation $\Pi_{\Lambda_\eta}$ (see lemma 5.5, proposition

5.6 and lemma 5.9).

Therefore, the irreducible components of representation

$(\Pi, G(2(p+1), \infty), P_\eta H_{II})$

are unitary equivalent to the irreducible components of representation

$(\Pi_{\Lambda_\eta}, G(2(p+1), \infty), L^2(\Lambda_{r(\Pi)}, \nu_{r(\Pi)}))$.

Now let’s account an isometry $\sigma^{(n)}_q$, that acts in $H$ by

$$\sigma^{(n)}_q(e_i) = e_i \text{ when } i \leq n \text{ and } \sigma^{(n)}_q(e_i) = e_{i+q} \text{ when } i > n.$$

If element–matrix $g = \begin{pmatrix} I_n & 0 \\ h_n & g_n \end{pmatrix}$ belongs to $G$, then $\sigma^{(n)}_q g(\sigma^{(n)}_q)^* = \begin{pmatrix} I_n & 0 & 0 \\ 0 & 0_q & 0 \\ h_n & 0 & g_n \end{pmatrix}$

($0_q$ is $q \times q$ zero–matrix).

Let $g_\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sigma^{(n)}_q g(\sigma^{(n)}_q)^*$. Obviously $g_\sigma \in G(q, \infty)$.

By theorem 0.5 $\Pi$ extends by continuity on $\sigma^{(n)}_q$, $(\sigma^{(n)}_q)^*$ and for any nonzero vector

$\eta \in H_{II} (\Pi, G(q, \infty), [\Pi(G(q, \infty)) \Pi(\sigma^{(n)}_q) \eta])$ is multiple by one of the irreducible components

of the representation

$(\Pi_{\Lambda_{\eta}}, G(2(p+1), \infty), L^2(\Lambda_{r(\Pi)}, \nu_{r(\Pi)}))$,

when $q = 2(p+1)$. Therefore in $L^2(\Lambda_{r(\Pi)}, \nu_{r(\Pi)}))$ there exists a vector $f_\eta$, for which

$$(\Pi(g) \eta, \eta) = (\Pi(\sigma^{(n)}_q) \Pi(g) \Pi(\sigma^{(n)}_q)^* \Pi(\sigma^{(n)}_q) \eta, \Pi(\sigma^{(n)}_q) \eta) = \ldots$$
\[(\Pi(g_{\sigma})\Pi(\sigma^{(n)}_q))\eta \cdot \Pi(\sigma^{(n)}_q)\eta) = (\Pi_{A^2}(g_{\sigma}) f_\eta, f_\eta).\]

From this follow the classification statements 5.6–5.7, 5.10.

Description of admissible representations of groups \(Sp(2\infty)\) and \(O(2\infty)\) is essentially based on the structure of the admissible representations of \(GL(\infty)\) and corresponding group of motions. As before from the representation \(\Pi\) of group \(G\), that coincide here with \(Sp(2\infty)\) or \(O(2\infty)\), we pass to the restriction of \(\Pi\) to the subgroup \(G_{K_n} \subset G\), defined in §3 and taking the same part as a group of motions \(G_n^I\) in a case when \(GL(\infty)\). Besides that, this restriction decomposes to the direct integral of spherical representations, for which the explicit realization is obtained (see (30), (31), (32), (33), (34), proposition 6.12 and remark 6.13).

Classification concludes (see theorem 6.15) by using theorems 6.14 and 0.5 as in a case of group \(GL(\infty)\).
§1. Realization Of The Admissible 
Representations Of The Symplectic And 
Orthogonal Groups

In this chapter we denote by $G$ one of the groups $Sp(2\infty)$ or $O(2\infty)$. To define the standard system of generators in $G$ for any matrix $x$ with the elements $x_{jk}$ ($j, k = 1, 2, \ldots$) we introduce the matrices $\gamma^{(0)}_o(x)$, $\gamma^{(0)}_u(x)$ $\in G$ by correlations

\[
(\gamma^{(0)}_o(x) - I)_{jk} = \begin{cases} 
  x_{-jk}, & \text{if } j < 0 \text{ and } k > 0 \\
  0, & \text{otherwise}
\end{cases}
\]

\[
(\gamma^{(0)}_u(x) - I)_{jk} = \begin{cases} 
  x_{j(-k)}, & \text{if } j > 0 \text{ and } k < 0 \\
  0, & \text{otherwise}.
\end{cases}
\]

Matrices of the form $g_0 = \left( \begin{array}{cc} (g^{-1})' & 0 \\ 0 & g \end{array} \right)$, $\gamma^{(0)}_o(x)$, $\gamma^{(0)}_u(x)$, where the block structure corresponds to the decomposition of $H$ into the orthogonal sum of the subspaces $H_-$ and $H_+$, generated by basis elements with the negative and positive numbers respectively, and $(g')_{i,k} = g_{-k,-i}$, are the system of generators for $G$. As has been above, we identify the vectors $f = \sum f_i e_i \in H$ with the rows $(\cdots, f_{-n}, \cdots, f_2, f_{-1}, f_1, f_2, \cdots)$.

If we denote by $t$ an ordinary transpose, then the following correlations are true:

\[
\gamma^{(0)}_o(x) = \gamma^{(0)}_o(x^t), \quad \gamma^{(0)}_u(x) = \gamma^{(0)}_u(x^t),
\]

where \((x^t)_{kj} = x_{(j)(-k)}\) for $Sp(2\infty)$;

\[
\gamma^{(0)}_o(x) = \gamma^{(0)}_o(-x^t), \quad \gamma^{(0)}_u(x) = \gamma^{(0)}_u(-x^t),
\]

when $G = O(2\infty)$.

Let $\Lambda_m$ consists of all the complex matrices $\lambda$ of the form

\[
\begin{pmatrix}
\lambda_{11} & \lambda_{12} & \ldots & \lambda_{1n} \\
\lambda_{21} & \lambda_{22} & \ldots & \lambda_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{m1} & \lambda_{m2} & \ldots & \lambda_{mn}
\end{pmatrix}, \quad \Lambda_m(k) \quad \text{is a set of columns } \vec{\lambda}_k = \begin{pmatrix} \lambda_{1k} \\ \lambda_{2k} \\ \vdots \\ \lambda_{mk} \end{pmatrix}, \quad A \text{ is a selfadjoint operator on } \Lambda_m(k), \rho_k(\vec{\lambda}_k) = \frac{1}{\pi^m} \exp[-(\vec{\lambda}_k)^*\vec{\lambda}_k], \kappa^A_k(\vec{\lambda}_k) = \exp[-i(\vec{\lambda}_k)^*A\vec{\lambda}_k], \nu_m^{(k)} \text{ is a measure on } \Lambda_m(k) \text{ with a density } \rho_k \text{ as regards the Lebesgue measure } d\vec{\lambda}_k, \nu_m = \prod_{k=1}^{\infty} \nu_m^{(k)}.
\]

We denote by $L_m^A$, a Hilbert space, that is an infinite tensor product $\bigotimes_{k=1}^{\infty} L^2(\Lambda_m(k), d\vec{\lambda}_k)$ with the stabilization defined by the sequence $\eta^A_k(\vec{\lambda}_k) = \kappa^A_k(\vec{\lambda}_k) \cdot [\rho_k(\vec{\lambda}_k)]^{1/2}$. Namely, $L_m^A$ is generated by the vectors of the form $f_1 \otimes f_2 \otimes \ldots \otimes f_p \otimes \eta^A_k \otimes \eta^A_k \otimes \ldots$ ($p \in N$).

For $f^{(l)} = f_1^{(l)} \otimes f_2^{(l)} \otimes \ldots \otimes f_p^{(l)} \otimes \eta^A_k \otimes \eta^A_k \otimes \ldots$ ($l = 1, 2$) the scalar product $(f^{(1)} \cdot f^{(2)})$ in $L_m^A$ is evaluated from formula

\[
(f^{(1)} \cdot f^{(2)}) = \prod_{p=1}^{\infty} \int_{\Lambda_m(p)} f_p^{(1)}(\vec{\lambda}_p) f_p^{(2)}(\vec{\lambda}_p) d\vec{\lambda}_p d\vec{\lambda}_p.
\]
In $L^2(\Lambda_m(k), d\tilde{\lambda}_k)$ let’s define the operator of the Fourier transformation $F^A_k$:

$$(F^A_k f_k)(\tilde{\lambda}_k) = \frac{\det(\sqrt{1 + 4A^2})}{(2\pi)^m} \int_{\Lambda_m(k)} \exp\{i \Re [2(\sqrt{1 + 4A^2} \tilde{\lambda}_k)^t \tilde{\lambda}_k]\} \cdot f_k(\tilde{\lambda}_k) d\tilde{\lambda}_k.$$ 

The next statement we can obtain using the ordinary calculations.

**Lemma 1.1** If $A = A^*$ and $A = \pm A^t$, then $F^A_k \eta_k = \eta_k^A$.

From this lemma follows that the operators in the next statement are defined correctly.

**Proposition 1.2** Let in $L^A_m$ the action of operators $\Pi_A(g)$ $(g \in Sp(2\infty))$ is defined by formulas:

$$\begin{align*}
(\Pi_A(g_0)\xi)(\lambda) &= |\det g|^m \xi(\lambda g), \\
(\Pi_A(\gamma_0^{(0)}(x))\xi)(\lambda) &= \\
\exp\{i \Re Tr [\sqrt{1 + 4A^2} \lambda x \lambda^t]\} \xi(\lambda),
\end{align*}$$

where $F^A = \bigotimes_{k=1}^{\infty} F^A_k$.

If $A = A^*$ and $A = -A^t$, then operators $\Pi_A(g)$ define the representation of group $Sp(2\infty)$.

Let’s give a similar realization for group $O(2\infty)$.

**Proposition 1.3.** Let $m = 2k$ and matrix $\lambda = \left(\begin{array}{c} \lambda^{(1)} \\ \lambda^{(2)} \end{array}\right)$, where $\lambda^{(1)}$ consists of the first $k$ rows of $\lambda \in \Lambda_m$, $P = \left(\begin{array}{cc} 0 & -I_k \\ I_k & 0 \end{array}\right)$ ($I_k$ is $k \times k$ unit matrix). If $A^t = PAP^{-1}$, then operators $\Pi_A(g)$ $(g \in O(2\infty))$, that are defined by the correlations:

$$\begin{align*}
(\Pi_A(g_0)\xi)(\lambda) &= |\det g|^m \xi(\lambda g), \\
(\Pi_A(\gamma_0^{(0)}(x))\xi)(\lambda) &= \\
\exp\{i \Re Tr [\sqrt{1 + 4(A^t)^2} \lambda x \lambda^t]\} \xi(\lambda),
\end{align*}$$

$\Pi_A(s^+) = F^{A^t}$, define the representation of group $O(2\infty)$.

For the decomposition of the representation $\Pi_A$ to the irreducible components let us consider two groups:

$$O(A, m) = \{u \in U(m) : u^t = u^* \text{ and } [A, u] = 0\},$$

$$Sp(A, m) = \{u \in U(m) : u^t = Pu^*P^{-1} \text{ and } [A, u] = 0\}.$$ 

Let $(\tau(u)\xi)(\lambda) = \xi(u^{-1}\lambda)$, where $u \in U(m), \xi \in L^A_m$. 

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If \( G(A, m) \) is one of the groups \( O(A, m) \) or \( Sp(A, m) \), \( \rho \) is its irreducible representation, \( a_{kk}^\rho \) is matrix element of \( \rho \), then \( P_k^\rho = \dim \rho \int_{G(A,m)} a_{kk}^\rho (u) \tau(u) \, du \) is an orthogonal projection, that acts in \( L^A_m \).

The following statement was announced at first by G.I. Ol’shansky in [4].

**Proposition 1.4.** Let \( \varphi \) be some set of the operators, that act on \( L^A_m \), \( \varphi' \) is a commutant of \( \varphi \). Then the following statements are true:

i) if \( \Pi_A \) such, as in proposition 1.2, then
\[ [\Pi_A(Sp(2\infty))]' = (\tau(O(A, m)))' = \tau(O(A, m))''; \]

ii) if \( \Pi_A \) such, as in proposition 1.3, then
\[ [\Pi_A(O(2\infty))]' = \tau(Sp(A, m))''; \]

iii) restriction of \( \Pi_A \) to \( P_k^\rho L^A_m \) is irreducible.

§2 Some Properties Of The Representation Of Groups \( GL(\infty) \)

Let \( \Pi \) be an irreducible representation of \( GL(\infty) \), acting in a Hilbert space \( H_\Pi \) with the unit cyclic vector \( \xi \) fixed with respect to the operators \( \Pi(u) \) \((u \in U(GL(\infty)) = U(\infty))\).

Then by the results of the paper [3] the class of the unitary equivalence of \( \Pi \) is defined by a spherical function
\[ \varphi_\Pi = (\Pi(g)\xi, \xi) = |\det(g)|^{i\beta} \det[I_{r(\Pi)} \otimes ch \ln |g| - 2iA \otimes sh \ln |g|], \quad (5) \]
where \( |g| = (g^*g)^{\frac{1}{2}} \), \( r(\Pi) \) is a natural number, \( \beta \) is a real number, \( I_{r(\Pi)} \) is a unit \( r(\Pi) \times r(\Pi) \)– matrix, \( A \) is selfadjoint (s.a.) \( r(\Pi) \times r(\Pi) \)– matrix.

The natural number \( r(\Pi) \) we call the rang of the representation \( \Pi \).

We denote by \( Z_m \) a set \( m \times m \) of the upper triangular matrices with positive elements on the diagonal. Let subgroup \( D_m \subset GL(\infty) \) consists of the matrices of the form \( \begin{pmatrix} z_{mm} & 0 \\ x_m & I \end{pmatrix} \), where \( z_m \in Z_m, \ x_m \) is matrix with the finite number of nonzero elements.

Let us give a classification result for the spherical factor representations of group \( G_n^I \subset GL(\infty) \) that consists of matrices of the form \( \begin{pmatrix} I_n & 0 \\ 0 & * \end{pmatrix} \), where * signifies some matrix of the corresponding size.

**Theorem 2.1.** (see [4]) Let \( \Pi \) be irreducible spherical factor representation \( G_n^I \) that acts in \( H_\Pi \), \( \hat{\alpha}_A(\lambda, g) = \exp\left\{ \frac{i}{2} Tr[\lambda(gg^* - 1)\lambda^* - 2iA\lambda(gg^* - 1)\lambda^*] \right\} \), where \( A \) is s.a. \( m \times m \)– matrix.

Then there exist: s.a. \( m \times m \) – matrix \( A \), \( n \times m \) – matrix \( z \), real number \( \beta \) such, that \( \Pi \) unitary equivalent to the restriction of representation \( \Pi_{Az} \) of group \( G_n^I \) defined in \( L^2(\Lambda_m, \nu_m) \) by formulas:
\[ (\Pi_{Az}\begin{pmatrix} I_n & 0 \\ 0 & g \end{pmatrix}) \eta(\lambda) = |\det g|^{i\beta}\hat{\alpha}_A(\lambda, g)\eta(\lambda g), \]
\[(\Pi_{Az}\left(\begin{pmatrix} I_n & 0 \\ h & I \end{pmatrix}\right))\eta(\lambda) = \exp[i \Re \text{Tr}(z\lambda h)]\eta(\lambda) \quad (\eta \in L^2(\Lambda_m, \nu_m)),\]

to the subspace \([\Pi_{Az}(G^I_n)\xi_0], \) where \(\xi_0 \in L^2(\Lambda_m, \nu_m)\) and corresponds to the function on \(\Lambda_m\) that identically equals to the unit.

Using the form of the operators \(\Pi_{Az}(g) \ (g \in G^I_n)\) we can prove the following statement.

**Theorem 2.2.** Let subgroup \(D_{mn} \subset G^I_n(G^I_0 = GL(\infty), D_m = D_m)\) consists of matrices of the form \(\begin{pmatrix} I_n & 0 \\ z_m & 0 \\ * & * \\ * & I \end{pmatrix}\) (\(z_m \in \mathbb{Z}_m\)), \(\Pi\) is the same as in theorem 2.1. Then \([\Pi(D_{mn})\xi] = [\Pi(G^I_n)\xi]\), where \(\xi\) is an arbitrary nonzero \(\Pi(U(G^I_n))\) – fixed vector.

§3. Asymptotic Properties

Of The Admissible Representations

Let \(G = Sp(2\infty)\) or \(O(2\infty)\). The subgroups \(G_d, \ G_o, \ G_u \subset G\) consist of matrices of the form \(g_0 = \begin{pmatrix} (g^{-1})^t & 0 \\ 0 & g \end{pmatrix}\), \(\gamma_o(0)(x), \gamma_u(0)(x)\) respectively, where \(x\) is a matrix with the elements \(x_{jk}\) \((j, k = 1, 2, \ldots)\), and the block structure is defined by the decomposition of \(H\) into the orthogonal sum of the subspaces \(H_-\) and \(H_+\) (see §1).

If \((x^t)_{jk} = x_{jk}\), then the correlations are true

\[
\gamma_o(0)(x) = \gamma_o(0)(x^t), \quad \gamma_u(0)(x) = \gamma_u(0)(x^t)\]  for \(Sp(2\infty)\);

\[
\gamma_o(0)(x) = \gamma_o(0)(-x^t), \quad \gamma_u(0)(x) = \gamma_u(0)(-x^t)\]  for \(G = O(2\infty)\).  

(7)

Let \(b\) be a matrix with the elements \(b_{jk}\), where \(j, k = 1, 2, \ldots\). As in §1 let’s define the matrices \(\gamma_o(n)(b)\) and \(\gamma_u(n)(b) \in G\) by the correlations

\[
(\gamma_o(n)(b) - I)_{jk} = \begin{cases} b_{(j-n)(k-n)}, & \text{if } j < -n \text{ and } k > n \\ 0, & \text{otherwise} \end{cases}
\]

\[
(\gamma_u(n)(b) - I)_{jk} = \begin{cases} b_{(j-n)(k-n)}, & \text{if } j > n \text{ and } k < -n \\ 0, & \text{otherwise}. \end{cases}
\]

Let \(Y(\infty, n)\) be a set of all the local nonzero \((\infty \times n)\)-matrices with the infinite number of rows and \(n\) columns. If \(x = [x_{jk}], y = [y_{jk}]\) \((1 \leq j < \infty, 1 \leq k \leq n)\), then we define matrix \(\theta(n)(x, y) \in G\) by the correlation

\[
(\theta(n)(x, y) - I)_{im} = \begin{cases} x_{(i-n)m}, & \text{if } i > n \text{ and } 1 \leq m \leq n \\ -(x^t)_{(-i)(-m-n)}, & \text{if } -n \leq i \leq -1 \text{ and } m < n \\ y_{(-i-n)m}, & \text{if } i < -n \text{ and } 1 \leq m \leq n \\ y^t_{(-i)(m-n)}, & \text{if } -n \leq i \leq -1 \text{ and } m > n \\ 0, & \text{otherwise}, \end{cases}
\]

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where \( y^\dagger = y^t \) for \( G = Sp(2\infty) \) and \( y^\dagger = -y^t \) for \( G = O(2\infty) \).

At last we denote by \( \delta^{(n)}(a) \) an element from \( G \), that defined by \( n \times n \) matrix \( a = [a_{jk}] \) \((1 \leq j, k \leq n)\) with respect to the correlation

\[
(\delta^{(n)}(a) - I)_{im} = \begin{cases} a_{(i-m)n}, & \text{if } -n \leq i \leq n \text{ and } 1 \leq m \leq n \\ 0, & \text{otherwise.} \end{cases}
\]

Let

\[
g_n = \begin{pmatrix} (g^{-1})^t & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \\ 0 & 0 & g \end{pmatrix},\]

where \( (g')_{ik} = g(-k)(-i) \) \((k, i < 0)\).

Let \( G_d(n, \infty) \) \((G_d(0, \infty) = G_d), \Gamma^{(n)}_O \) \((n \geq 1)\), \( \Gamma^{(n)}_U \) \((n \geq 1)\), \( \Delta^{(n)} \) \((n \geq 1)\) are the subgroups of \( G \), which consist of the matrices of the form \( g_n, \gamma^{(n)}_o(b), \gamma^{(n)}(b), \delta^{(n)}(a) \) respectively. Moreover, when \( G = Sp(2\infty) \), \( a \) is symmetric matrix, and when \( O(2\infty) \) is antisymmetric one. A set of all the elements from \( G \) of the form \( \theta^{(n)}(x, y) \) we denote by \( \Theta^{(n)} \) \((n \geq 1)\). Observe that \( \Theta^{(n)} \) is not a group.

The following correlations are true:

\[
\begin{align*}
\gamma^{(n)}_o(b)\theta^{(n)}(x, y) & = \gamma^{(n)}_o(-b) \\
\delta^{(n)}(-(bx)^2x)\theta^{(n)}(0, bx)\theta^{(n)}(x, y) & = \gamma^{(n)}(b)\theta^{(n)}(x, y) \gamma^{(n)}_o(-b) \\
\delta^{(n)}((by)^tx)\theta^{(n)}(by, 0)\theta^{(n)}(x, y) & = \gamma^{(n)}(x, y) \\
\theta^{(n)}(x_1, y_1)\theta^{(n)}(x_2, y_2) & = \delta^{(n)}(x_2^t y_1 - y_2^t x_1 - x_2^t y_2 + y_2^t x_2)\theta^{(n)}(x_2, y_2)\theta^{(n)}(x_1, y_1), \\
g_n \theta^{(n)}(x, y) & = \gamma^{(n)}(g x, (g^{-1})^t y).
\end{align*}
\]

Let subgroup \( K_n \) be generated by \( \Theta^{(n)} \) and \( \Delta^{(n)} \). From the correlations (8) it follows, that the elements of the group \( G(n, \infty) \), that is formed by \( G_d(n, \infty), \Gamma^{(n)}_O, \Gamma^{(n)}_U \) naturally act by the automorphisms on \( K_n \). We denote by \( GK_n \) a subgroup of \( G \) generated by \( G(n, \infty) \) and \( K_n \).

Let’s introduce a useful property of the infinite-dimensional groups.

Suppose, that group \( G \) is an inductive limit of the finite-dimensional matrix groups \( G(n) \) \((G(n) \subset G(n + 1) \) \((n = 1, 2, 3, \ldots)\)).

**Definition 3.1.** Group \( G \) is called an asymptotic Abelian one \( (a.a.) \), if there exists a sequence \( u_n \) \((n \in N)\) of the elements of subgroup \( U(G) \) that coincide with a set of the unitary elements of the group \( G \) with the properties:

i) for any \( l, n \in N \) and \( g \in G(l) \) there exists \( k(g, n) \in N \) such that for \( k \geq k(g, n) \) \( u_k g u_k^* \in \{h \in G : hp = ph \text{ for all } p \in G(n)\} = G'(n) \);

ii) for any \( n \in N \) there exists \( l(n) \in N \), for which \( u_k u_l^* \in G'(n) \) when \( k > l \).
space $H_{nl}$ with a cyclic vector $\xi$. Moreover, set

$$\bigcup_{n=1}^{\infty} \{ \eta \in H_{nl} : \Pi(u)\eta = \eta \text{ for all } u \in U(G) \cap \{ h \in G : h g = g h \text{ for all } g \in G(n) \} \} \text{ dense in } H_{nl} \text{ (This property, as was shown by G.I. Ol’shansky, is one of the equivalent definitions of the admissible representation in a case of the large class of the infinite-dimensional groups, that includes } GL(\infty), \text{ Sp}(2\infty), \text{ O}(2\infty) \).

For any unit vectors $\eta_1, \eta_2 \in H_{nl}$

$$\varphi_{\eta_1}(g) = \lim_{l \to \infty} (\Pi(u_l g u_l^*) \eta_1, \eta_1) = \lim_{l \to \infty} (\Pi(u_l g u_l^*) \eta_2, \eta_2) = \varphi_{\eta_2}(g).$$

Moreover, the limits do not depend on a choice of the sequence $u_l$ ($l \in N$) from the definition 3.1, and $\varphi_{\eta_1} = \varphi_{\eta_2}$ is an indecomposable spherical function on $G$.

**Proof.** For any $\epsilon > 0$ there exists $n \in N$ and unit vectors $\xi(\epsilon) \in H_{nl}$ ($i = 1, 2$) with the properties: $\Pi(u)\xi(\epsilon) = \xi(\epsilon), \Pi(u)\eta_i(\epsilon) = \eta_i(\epsilon)$ ($i = 1, 2$) for all $u \in U(G) \cap G_n'$;

$$||\xi - \xi(\epsilon)|| < \epsilon; \quad ||\eta_i - \eta_i(\epsilon)|| < \epsilon. \quad (9)$$

By the definition 3.1 we choose $l \in N$ such, that $u_k u_l^* \in G_n'$ for all $k > l$. Then, using (9) we get:

$$2\epsilon > |(\Pi(u_k g u_k^*) \eta_i, \eta_i) - (\Pi(u_k g u_k^*) \eta_i(\epsilon), \eta_i(\epsilon))| = |(\Pi(u_k g u_k^*) \eta_i, \eta_i) - (\Pi(u_l g u_l^*) \eta_i(\epsilon), \eta_i(\epsilon))|.$$

Hence and from (9) it follows that

$$|(\Pi(u_k g u_k^*) \eta_i, \eta_i) - (\Pi(u_l g u_l^*) \eta_i, \eta_i)| < 4\epsilon \text{ for all } k > l.$$

Since $\epsilon$ is an arbitrary, then $(\Pi(u_k g u_k^*) \eta_i, \eta_i)$ is a fundamental sequence. Therefore the limits of sequences $(\Pi(u_k g u_k^*) \eta_i, \eta_i) (k \in N)$ exist for every $i = 1, 2$.

Moreover, from the correlation

$$\lim_{l \to \infty} \Pi(u_l u u_l^*) \eta_1 = \eta_1 \quad \forall u \in U(G)$$

we get, that

$$\varphi_{\eta_1}(ugv) = \varphi_{\eta_2}(g) \quad \forall u, v \in U(G).$$

Therefore $\varphi_{\eta_1}$ and $\varphi_{\eta_2}$ are the spherical functions on $G$.

Let’s prove the coincidence of $\varphi_{\eta_1}$ and $\varphi_{\eta_2}$.

As $\xi$ is a cyclic vector, then for any $\delta > 0$ there exist the collections $\{g_{ik_i} \}_{k_i=1}^{N_i} (i = 1, 2)$ of the elements from $G(n)$ and numbers $\{c_{ik_i} \}_{k_i=1}^{N_i} (i = 1, 2)$ from $C$ with the properties

$$\sum_{k_i=1}^{N_i} c_{ik_i} \Pi(g_{ik_i})\xi - \eta_i \quad < \delta \quad (i = 1, 2) \quad (10)$$

Since $\Pi$ is factor-representation, then, using (10), we get

$$2\delta > |\lim_{l \to \infty} (\Pi(u_l g u_l^*) \eta_i, \eta_i) -$$

$$\lim_{l \to \infty} (\Pi(u_l g u_l^*) \sum_{k_i=1}^{N_i} c_{ik_i} \Pi(g_{ik_i})\xi \sum_{k_i=1}^{N_i} c_{ik_i} \Pi(g_{ik_i})\xi )| =$$
where $p$ asymptotic Abelian ones, the set vectors from (a.s.f.) and it is denoted by group $GL$ group $\Pi$. 

O proposition 3.2 by the admissible representation $\Pi$ dimensional groups $G$ $\Pi$ vector for guarantees a.a. of $GK$. To prove this theorem it suffices to notice, that the groups from the condition are the Let Theorem 4.1. We define the Proof The proof is completely similar to the proof of the Multiplicativity Theorem from.

Definition 3.3. A spherical function on a.a. group $G$ defined in accordance with the proposition 3.2 by the admissible representation $\Pi$ is called an asymptotic spherical function (a.s.f.) and it is denoted by $\varphi_{\Pi}^{(a)}$.

Theorem 3.4. Let $G$ be an a.a. matrix group, that is an inductive limit of finite-dimensional groups $G(n)(n \in N), \Pi$ is the same, as in the proposition 3.2, $\xi$ is a unit cyclic vector for $\Pi$ fixed with respect to $\Pi(u_l)$ ($l \in N$), where $u_l$ is a sequence from $U(G)$ that guarantees a.a. of $G$.

hen subspace $H_{\eta} = \{\eta \in H_{\Pi} : \Pi(u_l)\eta = \eta \text{ for all } l \in N\}$ is one-dimensional.

Proof The proof is completely similar to the proof of the Multiplicativity Theorem from §4.

Definition 3.5. We define the rang $r(\Pi)$ of the admissible factor-representation $\Pi$ of group $G$, that coincides to the one of the following groups: $GL(\infty)$, $G_{n}^{I}$, $Sp(2\infty)$, $O(2\infty)$, $GK_n$ as a rang of the representation $\Pi^{(a)}$ that defined by the indecomposable s.f. $\varphi_{\Pi}^{(a)}$ of group $GL(\infty)$ ($G = GL(\infty)$); $GL(n, \infty)$, where $G = G_{n}^{I}$; $G_d (G = Sp(2\infty)$ or $O(2\infty) )$; $G_d \cap G(n, \infty) (G = GK_n)$ (see §2).

§4 . Properties Of The Admissible Representations Of Group $GL(\infty)$ And The Group Of Motions

We may take the group of motions to be the subgroup of $G_{n}^{I} \subseteq GL(\infty)$ (see §2)

Theorem 4.1. Let $\Pi$ be an admissible representation of $G$, where $G$ coincides with the one of the groups $GL(\infty)$, $G_{n}^{I}$, $Sp(2\infty)$, $O(2\infty)$, $GK_n$; $\xi_1$, $\xi_2$ are $\Pi(U(G(n, \infty))$ -fixed vectors from $H_{\Pi}$. hen $(\Pi(g)\xi_1 \xi_1) = (\Pi(g)\xi_2 \xi_2)$ for all $g \in G(n, \infty)$.

To prove this theorem it suffices to notice, that the groups from the condition are the asymptotic Abelian ones, the set $\bigcup_{n} \{\eta \in H_{\Pi} : \Pi(U(G_{n}^{I}))\eta = \eta\}$ is dense in $H_{\Pi}$ and to use the proposition 3.2.

In $GL(\infty)$ we consider a subset $Y_p$ that consists of matrices of the form

$$
\begin{pmatrix}
  y_{11} & y_{12} & \cdots & y_{1p} & \delta_1 & 0 & 0 & 0 & \cdots & \cdots \\
  \ast & \ast & \cdots & \ast & \ast & \delta_2 & 0 & 0 & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \ast & \ast & \cdots & \ast & \ast & \ast & \ast & \delta_s & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix},
$$

where $p \geq 1$, $\delta_s > 0 \ \forall \ s \in N$. 

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Lemma 4.2. Let $\Pi$ be an admissible factor–representation of group $GL(\infty)$, $\xi$ is its cyclic vector fixed with respect to the operators $\Pi(U(G(p, \infty)))$. Then $[\Pi(Y_p)\xi] = [\Pi(GL(\infty))\xi]$.

To prove this, it suffices to notice, that $GL(\infty) = Y_p U(G(p, \infty))$.

Lemma 4.3. Let $\Pi$, $p$, $\xi$ are the same as in lemma 4.2, $K_{2p}$ is a subgroup of $GL(\infty)$ that consists of matrices of the form $\begin{pmatrix} g_{11} & 0 \\ * & I \end{pmatrix}$, where $g_{11}$ is an arbitrary $(2p) \times (2p)$ matrix, $p \geq r(\Pi)$. Then $H_n = [\Pi(K_{2p})\xi]$.

Proof. It is easy to check it, that the closure of the set $K_{2p}GL(p, \infty)$ includes $Y_p$. Next, by the statements 3.2 and 4.1 a restriction of $\Pi$ to $GL(p, \infty)$, that acts in a Hilbert space $[\Pi(GL(p, \infty))\xi]$, is a factor–representation. According to the theorem 2.2 (for $n = 0$) $[\Pi(GL(p, \infty))\xi] = [\Pi(GL(p, \infty) \cap K_{2p})\xi]$. From this and from lemma 3.2 we get $[\Pi(GL(\infty))\xi] = [\Pi(Y_p)\xi] = [\Pi(K_{2p}GL(p, \infty))\xi] = [\Pi(K_{2p})\xi]$. Lemma 4.3 is proved.

The following analog of the statement above might be proved by the same way for group $G^I_n \subset GL(\infty)$.

Lemma 4.4. Let $\Pi$ be an admissible factor–representation of group $G^I_n$, $\xi$ is a cyclic vector fixed with respect to operators $\Pi(u)(u \in U(G(p + n, \infty)))$, $K^{(n)}_{2p}$ is a subgroup of $G^I_n$, that consists of the matrices of the form $\begin{pmatrix} I_n & 0 \\ * & g_{22} \\ * & * \end{pmatrix}$, where $g_{22}$ is an arbitrary $(2p) \times (2p)$ matrix.

If $p \geq r(\Pi)$ (see definition 3.5), then $[\Pi(K^{(n)}_{2p})\xi] = [\Pi(G^I_n)\xi]$.

§5. A Description Of The Admissible Representations Of The Groups $GL(\infty)$ And $G^I_n$

Let $\Pi$ be an admissible factor–representation of $G$, that in first two statements of this chapter may be one of the groups: $GL(\infty)$, $G^I_n$, $Sp(2\infty)$, $O(2\infty)$.

Isometry $\sigma_q^{(n)}$, that acts in $H$ by the formula:

$$\sigma_q^{(n)}(e_i) = e_i \text{ for } i \leq n \text{ and } \sigma_q^{(n)}(e_i) = e_{i+q} \text{ for } i > n$$

($n = 0$ corresponds to $GL(\infty)$), is a strong limit of the elements of unitary subgroup $U(G^I_n)$. In a case of groups $Sp(2\infty)$, $O(2\infty)$ this property belongs to isometry $\sigma_q$ defined by the correlation $\sigma_q(e_i) = e_{i+(\text{sign } i)}q$.

We denote by $\Pi_U$ a restriction of $\Pi$ to $U(G)$. By the results of paper $[3]$ $\Pi_U$ is a continuous, if $U(G) \subset B(H)$ and $U(H_n) \subset B(H_q)$ are the topological groups with respect to the strong operator topology. Therefore, $\Pi_U$ extends by the continuity to the representation of semigroup $U(G)$, that includes all the isometries, which are the limiting points with respect to the weak operator topology on $U(G)$. Obviously, $\sigma_q^{(n)}$, $\sigma_q$ belong to $U(G)$.

Theorem 5.1. If $E_l$ is an orthoprojection to the subspace $\Pi(\sigma) H_n$, where $\sigma$ equals to $\sigma_q^{(n)}$ or $\sigma_q$, then $\Pi(g) E_l - E_l \Pi(g) = [\Pi(g), E_l] = 0$ for all $g \in G(l, \infty)$. 

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Proof. Since \( \Pi \) is extended by the continuity to \( \sigma \), then for arbitrary \( g \in G(l, \infty) \) \( \sigma^* g \sigma \in G \) and \( \Pi(\sigma^*) \Pi(g) \Pi(\sigma) = \Pi(\sigma^* g \sigma) \).

Therefore, \( \Pi(\sigma^*) E_i \Pi(g) E_i \Pi(\sigma) \) is a unitary operator in \( H_i \). Hence, accounting that \( \Pi(\sigma) \) is an isometry, we get that \( E_i \Pi(g) E_i \) is a unitary operator on \( E_i H_i \).

Because of the same reason \( E_i \Pi(g) (I - E_i) = 0 \) or \( E_i \Pi(g) = E_i \Pi(g) E_i \).

If we use the similar reasoning for \( g^{-1} \), we’ll get : \( E_i \Pi(g^{-1}) = E_i \Pi(g^{-1}) E_i \). Therefore, \( \Pi(g) E_i = E_i \Pi(g) \). The proof of theorem 5.1 is complete.

Let \( \xi \) be a cyclic vector for the representation \( \Pi \), \( \Pi(u)\xi = \xi \) for all \( u \in U(G(p, \infty)), p > r(\Pi) \) and \( q \in N \).

Proposition 5.2. Put \( \xi_q = \Pi(\sigma)\xi \), where \( \sigma \) equals to \( \sigma_q^{(n)} \) or to \( \sigma_q \), \( H_q = [\Pi(G(q, \infty))\xi_q] \). Then

- a) \( H_q = \Pi(\sigma) H_i \);
- b) representation \( (\Pi, G, H_i) \) is unitary,
equivalent to the representation defined by the chain of mappings:

\[
G \xrightarrow{i_q} G(q, \infty) \xrightarrow{\Pi} (\Pi(G(q, \infty)), H_q, \xi_q),
\]

where \( i_q(g) \) for \( g \in G \) is defined by the correlations

\[
\sigma^* i_q(g) \sigma = g, i_q(g)e_l = e_l \quad \text{for} \quad l = 1, 2, 3, \ldots, q.
\]

Proof. We note, that \( \Pi(\sigma^*) \Pi(G(q, \infty)) \Pi(\sigma) \xi = [\Pi(G) \xi] = H_i \). Therefore, \( \Pi(\sigma^*) [\Pi(G(q, \infty)) \xi_q] = H_i \) or \( E_q [\Pi(G(q, \infty)) \xi_q] = \Pi(\sigma) H_i \), where \( E_q = \Pi(\sigma) \Pi(\sigma^*) \).

But by the theorem 5.1 \( E_q \in \Pi(G(q, \infty))' \). Therefore \( \Pi(\sigma) H_i = E_q [\Pi(G(q, \infty)) \xi_q] = [\Pi(G(q, \infty)) \xi_q] = H_i \). Thus correlation (a) is proved.

Statement (b) follows from this chain of equalities:

\[
(\Pi(g)\xi, \xi) = (\Pi(\sigma^* i_q(g) \sigma) \xi, \xi) = (\Pi(\sigma^*) \Pi(i_q(g)) \Pi(\sigma) \xi, \xi) = (\Pi(i_q(g)) \xi_q, \xi_q).
\]

The following statements 5.3–5.7 refer only to group \( GL(\infty) \). Let \( q = 2(p+1) \), where \( p \geq r(\Pi) \).

Lemma 5.3. In \( H_i \) there exists a set of vectors \( \{\xi_i\} \subseteq N \), \( \xi_1 = \xi \) with the properties:

- a) the subspaces \( H_q(i) = [\Pi(G_i^q) \xi] \) (\( q = 2(p+1) \)) are orthogonal in pairs for the different values of index \( i \) and \( \oplus_i H_q(i) = H_i \);
- b) \( \Pi(\xi_i) = \xi_i \) for all \( i \in N \) and \( u \in U(GL(q, \infty)) \).

Proof. Let \( \{g_k\} \) \( (k \in N) \) be a dense subset in \( GL(q) \); \( \xi_1, \xi_2, \ldots, \xi_n \) are the unit vectors from \( H_i \), for which the subspaces \( H_q(i) = [\Pi(G_i^q) \xi] \) \( (1 \leq i \leq n) \) are orthogonal in pairs. If \( k(n) = \min\{k : \Pi(g_k) \xi \notin \oplus_{i=1}^n H_q(i) \} \) and \( H(n) = \{H_i - \oplus_{i=1}^n H_q(i) \} \neq 0 \), then put \( \xi_{n+1} = \frac{P_n \Pi(g(n)) \xi}{||P_n \Pi(g(n)) \xi||} \), where \( P_n \) is an orthoprojection on \( H(n) \).

If we continue this process, we will get the system of vectors \( \{\xi_i\} \) \( (i \in N) \) (possibly the finite one) such that \( \Pi(g_k) \xi \in \oplus_{i=1}^{\infty} H_q(i) \) for all \( k \in N \) and \( \Pi(u)\xi_i = \xi_i \) \( \forall u \in U(q, \infty) \).
Now this statement follows from 4.3.

Let \( E_q(i) \) be the orthoprojection on \( H_q(i) \) \((H_q(i) = E_q(i) H_\parallel)\). From lemma 5.3 it follows that exists \( i \), for which \( E_q(i) e_q \neq 0 \). If \( E_q(i) E_q = V_q(i) \{E_q E_q(i) E_q\}^{\frac{1}{2}} \) is a polar decomposition of \( E_q(i) E_q \), \( e_q(i) = V_q^{*}(i) V_q(i) \leq E_q \), \( f_q(i) = V_q(i) V_q^{*}(i) \leq E_q(i) \), then from theorem 5.1 and lemma 5.3 we get, that \( V_q(i) \in (\Pi(\text{GL}(q, \infty)))' \).

Since \((\Pi(\text{GL}(\infty)))'' = ((\Pi(\text{GL}(\infty)))')'\) is a factor, then by the proposition 5.2 \( w^* \)-algebra \((\Pi(\text{GL}(q, \infty)))''\) of operators, that act in \( H_q \), is also factor. Because of this, accounting theorem 5.1, we get, that the next chain of the mappings

\[
a \in (\Pi(\text{GL}(q, \infty)))'' E_q \rightarrow e_q(i) a \in (\Pi(\text{GL}(q, \infty)))'' e_q(i) \rightarrow
\]

\[
\rightarrow V_q(i) e_q(i) a V_q(i)^* = f_q(i) a \in (\Pi(\text{GL}(q, \infty)))'' f_q(i)
\]

is an isomorphism \((\Pi(\text{GL}(q, \infty)))'' E_q = E_q(\Pi(\text{GL}(q, \infty)))'' E_q\) on \( f_q(i) (\Pi(\text{GL}(q, \infty)))'' f_q(i) = f_q(i) (\Pi(\text{GL}(q, \infty)))'' \subset (\Pi(\text{GL}(q, \infty)))'' E_q(i) \).

Therefore, by the proposition 5.2 the class of unitary equivalence of the representation \( \Pi \) of group \( \text{GL}(\infty) \) is determined up to the multiplies by the restriction of \( \Pi \) to \( \text{GL}(q, \infty) \), that acts in some invariant subspace \( f_q(i) H_\Pi \subset H_q(i) \) with a cyclic vector \( \xi_q(i) = V_q(i) \xi_q \).

Let

\[
\Pi(G_q^l_i, H_q(i), \xi_i) = \int_S (\Pi_s(G_q^l_i), H_q(i, s), \xi_i(s)) d\mu(s) \tag{11}
\]

be a decomposition of the restriction of representation \( \Pi \) to group \( G_q^l_i \), that acts in \( H_q(i) \), into a direct integral of the irreducible spherical representations, that corresponds to center \( C_i \) of algebra \((\Pi(G_q^l_i))''\), where \( S \) is a spectrum of \( C_i \), \( \mu \) is a probability measure on \( S \).

Using the classification of the spherical representations of group \( \text{GL}(\infty) \), the proposition 3.2, theorem 4.1 and the fact that \( q > 2(r(\Pi) + 1) \), we can prove the following statement.

**Lemma 5.4.** There exist selfadjoint matrix \( A \) with a size \( r(\Pi) \times r(\Pi) \) and a real number \( \beta \) such that

\[
(\Pi_s(g)\xi_i(s), \xi_i(s)) ||\xi_i(s)||^{-1} = \det(|g|)^{i\beta} \det[I_r(\Pi) \otimes \cosh(\ln|g|) - 2i A \otimes \sinh(\ln|g|)]
\]

\( \forall g \in \text{GL}(q, \infty) \) and \( \mu - \) almost all (a.a.) \( s \in S \). Moreover, for \( \mu - \) a.a. \( s \in S \)

\[
[\Pi_s(G_q^l_i)\xi_i(s)] = [\Pi_s(N_q)\xi_i(s)], \text{ where } N_q \text{ consists of matrices of the form } \begin{pmatrix}
I_q & 0 \\
* & I
\end{pmatrix}.
\]

From this and from the complete classification of the spherical representations of groups of motions \( G_n^i \) (see [\text{II}] ) it follows

**Lemma 5.5.** There exist \( \text{isometry } V \), that maps \( H_q(i) \) to \( L^2(S, \mu) \otimes L^2(\Lambda_{s(\Pi)}, \nu_{s(\Pi)}) \), and \( \mu - \) measurable mapping \( z \) from \( S \) to the set of all the complex \( q \times r(\Pi) \) matrices such that the action of the operators \( \Pi_A(g) = V \Pi(g) E_q(i) V^{-1} \) \((g \in G_q^l_i)\) is determined by correlations:

\[
(\Pi_A(g)\eta)(s) = \Pi_{A z(s)} \eta(s) \tag{12}
\]
(see (6)), where \( \eta(s) \in L^2(\Lambda_r(\Pi), \nu_{r(\Pi)}) \) for all \( s \in S \).

Moreover, there exists a \( \mu \)-measurable mapping \( f_q(i, \cdot) \) from \( S \) to the set of the orthoprojections of \( w^* - \) algebra \( (\Pi_{A_z(s)}(GL(q, \infty)))' \subset B(L^2(\Lambda_r(\Pi), \nu_{r(\Pi)})) \), for which \( (V f_q(i) V^{-1} \eta)(s) = f_q(i, s) \eta(s) \).

For \( u \in U(\Pi, A) = \{ u \in U(\Pi) : [u, A] = 0 \} \) we define operator \( \tau(u) \) in \( L^2(\Lambda_r(\Pi), \nu_{r(\Pi)}) \) by
\[
(\tau(u) \xi)(\lambda) = \xi(u^* \lambda) .
\]

Denote by \( \tilde{\tau}(u) \) a natural extension of \( \tau \) to \( L^2(S, \mu) \otimes L^2(\Lambda_r(\Pi), \nu_{r(\Pi)}). \) Namely, \( \tilde{\tau}(u) = I \otimes \tau(u) \).

If \( \kappa \) is irreducible representation of group \( U(\Pi, A), \kappa_{i, k} \) is a matrix element of operator \( \kappa(g) \), then
\[
P^\kappa_i = \dim(\kappa) \int_{U(\Pi, A)} \kappa_{i, k}(v) \tau(v) \, dv \quad \text{is an orthoprojection from } w^* - \text{algebra}
\]
\[
(\Pi_{A_z}(GL(q, \infty)))' \quad \text{(see (6)), that acts in } L^2(\Lambda_r(\Pi), \nu_{r(\Pi)}).
\]

Obviously operators \( \tilde{P}^\kappa_i = I \otimes P^\kappa_i \) and \( \tilde{\tau}(u) \forall u \in U(\Pi, A) \) belong to \( \tilde{\Pi}_A(GL(q, \infty))' \).

Without loss of generality suppose, that for \( \kappa \) and \( i \) introduced above \( \tilde{P}^\kappa_i V f_q(i) V^{-1} \neq 0 \).

Let \( w[V f_q(i) V^{-1} \tilde{P}^\kappa_i V f_q(i) V^{-1}]^{1/2} \) be a polar decomposition of operator \( \tilde{P}^\kappa_i V f_q(i) V^{-1} \). Since \( (\tilde{\Pi}_A(GL(q, \infty)))'' V f_q(i) V^{-1} \) is a factor, orthoprojection \( w^* w \leq V f_q(i) V^{-1}, w \), \( V f_q(i) V^{-1} \in (\tilde{\Pi}_A(GL(q, \infty))' \), then, using the statement of theorem 0.1 and the fact that orthoprojection \( P^\kappa_i \) is a minimal in \( (\Pi_{A_z(s)}(GL(q, \infty)))' \subset B(L^2(\Lambda_r(\Pi), \nu_{r(\Pi)})) \), we get:

1) the restriction of \( \tilde{\Pi}_A \) to \( GL(q, \infty) \), that acts in \( V f_q(i) V^{-1}(L^2(S, \mu) \otimes L^2(\Lambda_r(\Pi), \nu_{r(\Pi)})) \), is multiple by the representation of subgroup \( GL(q, \infty) \) in \( w w^* (L^2(S, \mu) \otimes L^2(\Lambda_r(\Pi), \nu_{r(\Pi)})) \) defined by correlation \( w \tilde{\Pi}_A(g) w^* = \tilde{\Pi}_A(g) w w^* \);
2) if \( f(s) \) is \( \mu \)-measurable field of orthoprojections, that determines orthoprojection \( w w^* \), then there exists subset \( S' \subset S \) of the positive measure such that
\[
f(s) = \begin{cases} P^\kappa_i, & \text{if } s \in S' \\ 0, & \text{if } s \notin S' \end{cases}
\]

The facts above are summed by the following statement.

**Proposition 5.6.** In the commutant of representation \( (\Pi, GL(q, \infty), H_q) \) (see proposition 5.2) there exists an orthoprojection \( f \) such, that \( (\Pi, GL(q, \infty), f H_q) \) is irreducible and unitary equivalent to the restriction of \( \Pi_{A_z} \) to group \( GL(q, \infty) \), that acts in \( P^\kappa_i L^2(\Lambda_r(\Pi), \nu_{r(\Pi)})) \).

From this and from proposition 5.2 (b) it follows the main classification

**Theorem 5.7.** An arbitrary admissible factor–representation \( \Pi \) of group \( GL(\infty) \) has a type \( I \) and it is multiple by representation \( \Pi_{A_z} \) of group \( G_n^I \) for \( n = 0, z = 0 \) (\( G_o^I = \)
GL(\infty)\), that acts in \(P^\kappa L^2(\Lambda_{r(\Pi)}, \nu_{r(\Pi)})\), for some selfadjoint matrix \(A\) of size \(r(\Pi) \times r(\Pi)\) and orthoprojection \(P^\kappa_i\), where \(\kappa\) is irreducible representation of \(U(r(\Pi), A)\), (see (6))

A statements analogous to statements (5.3) – (5.6), from which follows theorem 5.7, might be modified for group \(G^l_n\).

Let \(\Pi\) be an admissible factor–representation of group \(G^l_n\) \((n \geq 1)\), \(\xi\) is a cyclic vector for \(\Pi, r(\Pi)\) is a rang of \(\Pi\) (see the definition 3.5), \(\Pi(u)\xi = \xi\) for all \(u \in U(G(p + n, \infty))\), \(q = n + 2(p + 1), p \geq r(\Pi)\).

The following statement is analogous to lemma 5.3 for group \(G^l_n\).

**Lemma 5.8.** In \(H_\Pi\) there exists a set of unit vectors \(\xi_i (i \in N) (\xi_1 = 1)\) with the properties:

a) subspaces \(H_q(i) = [\Pi(G^l_q)^i] (q = n + 2(p + 1))\) are orthogonal in pairs for different \(i\) and \(\otimes \in H_q(i) = H_\Pi\);

b) \(\Pi(u)\xi_i = \xi_i\) for all \(i\) and \(u \in U(G(q, \infty))\).

**Proof** might be done by using the statement of lemma 4.4 and analogous to the ground of lemma 5.3.

If \(E_q, \xi_q\) are the same as in proposition 5.2, then repeating the reasoning we’ve done for \(GL(\infty)\), let’s find the orthoprojection \(f_q(i) \in (\Pi(G(q, \infty)))^i\) such that \(f_q(i)H_\Pi \subset H_q(i)\) and \((\Pi, G(q, \infty), H_q)\) is multiple by \((\Pi, G(q, \infty), f_q(i)H_q(i)\).

Hence accounting theorem 2.1, statements 3.2, 4.1 and the decomposition
\(\Pi(G^l_q), H_q(i), \xi_i = \int(I_{\Pi}(G^l_q), H_q(i,s), \xi_i(s))d\mu(s)\), where the objects we’ve met are described in (11), we get the statement analogous to lemma 5.5 for \(G^l_n\).

**Lemma 5.9.** There exist: s.a. \(r(\Pi) \times r(\Pi) – \text{matrix} A, \mu – \text{measurable mapping} z\) from \(S\) to the set of matrices of \(q = n + 2(p + 1)\) rows and \(r(\Pi)\) columns of a form \(z(s) = (z_n^s)\), where \(z_n\) is independent of \(s\) matrix of height \(n\); isometry \(V\), that maps \(H_q(i)\) to \(L^2(S,\mu) \otimes L^2(\Lambda_{r(\Pi)}, \nu_{r(\Pi)})\) such that operators \(\Pi_{Az}(g) = \Pi_{Az}(g)V^{-1} (g \in G^l_I)\), where \(\Pi_i\) is a restriction of \(\Pi\) to \(H_q(i)\), are defined in \(L^2(S,\mu) \otimes L^2(\Lambda_{r(\Pi)}, \nu_{r(\Pi)})\) by the correlations (6) and (12).

Let \(U(r(\Pi), A, z_n) = \{u \in U(r(\Pi), A) : z_n u = z_n\}\). As before we define by the irreducible representation \(\kappa\) of group \(U(r(\Pi), A, z_n)\) the orthoprojection \(P^\kappa_i\).

The reasoning for group \(GL(\infty)\), introducing in this chapter with a slight change might be carried over to the case of group \(G^l_n\) and they’re leading to the following statement.

**Theorem 5.10.** For the admissible factor–representation \(\Pi\) of group \(G^l_n\) there exist: s.a. \(r(\Pi) \times r(\Pi) – \text{matrix} A, n \times r(\Pi) – \text{matrix} z\) and orthoprojection \(P^\kappa_i\) such that \(\Pi\) is multiple by restriction of representation \(\Pi_{Az}\) (see (6)) to subspace \(P^\kappa_i L^2(\Lambda_{r(\Pi)}, \nu_{r(\Pi)})\).

§6 . A Description Of The Admissible Representations Of Groups \(Sp(2\infty)\) And \(O(2\infty)\)
The main result of this chapter is theorem 6.15, that establishes a completeness of the collection of the admissible factor–representations, that was built in §1, for groups \( Sp(2\infty) \) and \( O(2\infty) \). We suppose that \( G \) coincides with the one of the groups \( Sp(2\infty) \) or \( O(2\infty) \). Otherwise we’ll specially notice it.

Let \( \Theta^{(n)}_1 \) be a subgroup of \( \Theta^{(n)} \), that consists of matrices of the form \( \theta^{(n)}(x,0) \), \( GN_n = G_d(n, \infty) \Theta^{(n)}_1 \). Notice, that subgroup \( GN_n \) is naturally isomorphic to the group of motions \( G^I_n \).

**Proposition 6.1.** If \( \Pi \) is an admissible factor–representation of group \( G \), of the rang \( r(\Pi) \) that acts in a Hilbert space \( H_\Pi \), \( \eta \) is a nonzero vector from \( H_\Pi \) and \( n \geq r(\Pi) \), then the following properties are true :

i) in a subspace \( H_\eta = [\Pi(GN_n)\eta] \) there exists a nonzero vector \( \eta_U \), for which \( \Pi(u)\eta_U = \eta_U \) \( \forall \ u \in U(GN_n) \);

ii) if \( (\Pi(GN_n), H_\eta, \eta) = \int (\Pi_s(GN_n), H_\eta(s), \eta(s))d\mu(s) \) is a decomposition of \( (\Pi, GN_n, H_\eta) \) into a direct integral of the factor–representations corresponding to the center of \( (\Pi(GN_n))^\prime \), then for \( \mu – \text{a.a.} \ s \in S \) \( (\Pi_s, GN_n, H_\eta(s)) \) is multiple by \( (\Pi_{Ax(s)}', GN_n, P_s^0L^2(\Lambda_{r(\Pi)}), \nu_{r(\Pi)}) \)

(see theorem 5.10 and (6)). The rang of matrix \( z(s) \) being equals to \( r(\Pi) \) for \( \mu – \text{a.a.} \ s \in S \).

**Proof** is based on the theorem 4.1, the classification statements 5.7, 5.10 and the analyses of the concrete realization of representations of group \( G_d \), that is isomorphic to \( GL(\infty) \).

**Remark 6.2** Let \( \Pi \) be the same as in a proposition 6.1, \( \xi \) is a cyclic vector for \( \Pi \) \( (H_\Pi = [\Pi(G)\xi]) \). hen for \( \Pi \) and \( G \) the theorem 5.1 and the proposition 5.2 are true. Namely, a class of the unitary equivalence of representation \( \Pi \) of group \( G \) is uniquely determined by restriction of \( \Pi \) to \( G(q, \infty) \), that acts in \( H_q = [\Pi(q, \infty)\xi_q] \), where \( \xi_q = \Pi(\sigma_q)\xi \ (\sigma_qe_i = e_{i+(\text{sign})q}) \) (see proposition 5.2).

We denote by \( E_q \) the projection of \( H_\Pi \) to \( H_q \).

From the proposition 6.1 there follows an important

**Lemma 6.3.** Let \( \Pi \) be the same as in conditions of statements 6.1 – 6.2, \( q = r(\Pi) \). There exists \( \Pi(U(GK_q)) \) – fixed unit vector \( \eta^{(u)}_q \in H_\Pi \) with a property: if \( E^{(u)}_q \) is an orthoprojection of \( H_\Pi \) to \( [\Pi(GK_q)\eta^{(u)}_q] \), then \( E^{(u)}_q \xi_q \neq 0 \).

**Proof.** As \( \eta^{(u)}_q \) we should take the unit \( \Pi(U(GN_q)) \) – fixed vector from subspace \( H_{\xi_q} = [\Pi(GN_q)\xi_q] \), that exists by proposition 6.1 (i). It satisfied to all the conditions of the lemma, but it may be not \( \Pi(U(GK_q)) \) – fixed. The last property follows from the complete description of the structure of an admissible representations of group \( U(GK_q) \), that was found by .. Kirillov in \( \square \), and from the fact that vector \( \eta^{(u)}_q \) is \( \Pi(U(GN_q)) \) – fixed.

Let \( v_q[E_qE^{(u)}_qE_q]^{1} = E^{(u)}_qE_q \) be a polar decomposition of operator \( E^{(u)}_qE_q \), \( v_q^*v_q = e_q \leq E_q \), \( v_qv_q^* = e^{(u)}_q \leq E^{(u)}_q \).

Diminishing, if we need, orthoprojection \( e_q \) and accounting the fact that \( v_q \in \Pi(G(q, \infty))' \), nd \( \Pi(G(q, \infty))'' \) is a factor in \( H_q \), and using the statements 6.1 – 6.2 we get
Proposition 6.4. Let Π be the same as in statements 6.1 – 6.3. The representation $(Π, G(q, ∞), H_q)$, that is a restriction of Π to the group $G(q, ∞)$ and that acts in $H_q = \Pi(G(q, ∞))ξ_q$ is multiple by $(Π, G(q, ∞), e_q(u)H_q)$, where $e_q(u)H_q \subset [Π(GK_q)v_qξ_q] ⊂ [Π(GK_q)η_q(u)] = H_q(u)$, $η_q(u)$ is a unit $Π(U(GK_q))$ – fixed vector.

Proof follows from the correlation $v_qΠ(g)v_q^* = v_qe_qΠ(g)v_q^*Π(g) = e_qΠ(g) ∀g \in G(q, ∞)$.

Let’s begin the analysis of restriction of Π to $GK_q$, that acts in $H_q(u)$.

Proposition 6.5. Let $C(M)$ be a center of $w^*$- algebra $M$, Π, $η_q(u)$ are the same as in the condition of proposition 6.4, $q = r(Π)$. hen $\Pi(GN_q)'' ⊂ C(Π(GK_q)''')$.

Proof Let $Θ^{(n,q)} = Θ^{(n)} ∩ Θ^{(q)} (n ≥ q)$, $Θ_1^{(n,q)} = Θ_1^{(n)} ∩ Θ^{(n,q)}$, $G_{K_q,q}$ ($G_{N_q,q}$) is generated by $G(n, ∞)$, $Δ_q$ and $Θ^{(n,q)}$ ($G_d(n, ∞)$ and $Θ_1^{(n,q)}$). Obviously $G_{K_q,q} = GK_q$, $G_{N_q,q} = GK_q$.

If we prove the correlations

$$C(Π(GK_q)''')E_q(u) = \bigcap_{n≥q} C(Π(GK_{n,q})''')E_q(u), \quad (14)$$

$$C(Π(GN_q)''')E_q(u) = \bigcap_{n≥q} C(Π(GN_{n,q})''')E_q(u), \quad (15)$$

then from the fact that $GN_q ⊂ GK_q$, our statement will follow.

Let $c ∈ C(Π(GK_q)''')E_q(u)$. There exist the collections of the elements $\left\{ g_{ik,i} \right\}_{ik,i}^{N_i} \subset GK_q \cap G(i)$ and the complex numbers $\left\{ c_{ik,i} \right\}_{ik,i}^{N_i} (i \in N)$ with the properties:

$$\left\| \sum_{k_i = 1}^{N_i} c_{ik,i}Π(g_{ik,i}) \right\| < ||c||,$$

$$\lim_{i → ∞} \left\| \sum_{k_i = 1}^{N_i} c_{ik,i}Π(g_{ik,i})f - cf \right\| = 0 \quad ∀ \; f \in H_q(u). \quad (16)$$

Next we notice, that in $U(G_{d}(q, ∞))$ there exist a sequence of elements $u_i (i \in N)$ with the property

$$u_i g_{ik,i} u_i^* = g'_{ik,i} ∈ GK_{i,q} \; ∀ \; i ≥ q. \quad (17)$$

Hence accounting (16) and the fact that vector $η_q(u)$ is $Π(U(G_{d}(q, ∞)))$ – fixed we get

$$\lim_{i → ∞} \left\| \sum_{k_i = 1}^{N_i} c_{ik,i}Π(g'_{ik,i})η_q(u) - cη_q(u) \right\| = 0.$$
This correlation and (17) lead to the following chain of equalities:

\[
0 = \lim_{i \to \infty} \left\| \Pi(g) \left( \sum_{k_i=1}^{N_i} c_{ik_i} \Pi(g'_{ik_i}) \eta_q^{(u)} - c \eta_q \right) \right\|
\]

\[
= \lim_{i \to \infty} \left\| \sum_{k_i=1}^{N_i} c_{ik_i} \Pi(g'_{ik_i}) \Pi(g) \eta_q^{(u)} - c \Pi(g) \eta_q \right\| = 0 ,
\]

that are true for all \( g \in GK_q \).

Hence accounting (16), we get the property (14). Correlation (15) are proved by the same way. The proposition 6.5 is proved.

**Proposition 6.6.** Let \( \Pi, q, \eta_q^{(u)}, H_q^{(u)} \) are the same as in proposition 6.5, \((\Pi, GK_q, H_q^{(u)}) = \int_S (\Pi_s, GK_q, H_q^{(u)}(s))d\mu(s) \) is a decomposition of \((\Pi, GK_q, H_q^{(u)}) \) into a direct integral of the factor–representations, that corresponds to the center of \( \Pi(GK_q)^w E_q^{(u)} \), \( \eta_q^{(u)} = \int_S \eta_q^{(u)}(s)d\mu(s) \), then

i) for \( \mu \)–a. a. \( s \in S \) subspace \( \{ \eta(s) \in H_q^{(u)}(s) \} : \Pi_s(u)\eta(s) = \eta(s) \forall u \in U(GK_q) \} \) is one-dimensional;

ii) \( \Pi(GN_q)^w \) is a factor in \( H_q^{(u)}(s) \);

iii) \( H_q^{(u)}(s) = [\Pi_s(GK_q)\eta_q^{(u)}(s)] = [\Pi_s(GN_q)\eta_q^{(u)}(s)] \) for \( \mu \)–a.a. \( s \in S \).

**Proof.** Since \( GK_q \) is an .. group (see definition 3.1), and \((\Pi_s, GK_q, H_q^{(u)}(s)) \) is a cyclic factor–representation, then the property (i) is a corollary of general theorem 3.4.

Statement (ii) follows from the proposition 6.5.

To prove (iii) we notice, that subspace \( H^{(u)}(s) = H_q^{(u)}(s) \cap [\Pi_s(GN_q)\eta_q^{(u)}(s)] \) is \( \Pi_s(GN_q) \)–invariant. If \( H^{(u)}(s) \neq 0, \) \( P^{(u)}(s) \) is an orthoprojection from \( H_q^{(u)}(s) \) to \( H^{(u)}(s) \), then by the property (ii) in \( \Pi_s(GN_q)^w \) there exists a partial isometry \( w_s \), for which \( w_s w^*_s \leq P^{(u)}(s) \) and \( w_s \eta_q^{(u)}(s) \neq 0 \). By the construction the vector \( w_s \eta_q^{(u)}(s) \) is \( \Pi_s(U(GN_q)) \)–fixed. Since \( \Pi_s \) is an admissible representation for \( \mu \)–a.a. \( s \in S \), then from the structure of the admissible representations of groups \( U(Sp(2\infty)) \) and \( U(O(2\infty)) \), that was completely described by ..Kirillov in [1], we get that vector \( w_s \eta_q^{(u)}(s) \) is \( \Pi_s(U(GK_q)) \)–fixed. But the last fact contradicts the property (i). Proposition 6.6 is proved.

The next statement follows from statements 3.2, 6.1, 6.6 and classification theorem 5.10.

**Proposition 6.7.** Let \( \Pi \) be the same as in proposition 6.6. hen there exists \( \mu \)–measurable field of the isometries \( V_s (s \in S) \), that maps \( H_q^{(u)}(s) \) to \( L_2(\Lambda_q, \nu_q) \) such that operators \( V(s)\Pi_s(g)V_s^{-1} = \Pi_s(g) \) act in \( L_2(\Lambda_q, \nu_q) \) for \( g \in GN_q \) by:

\[
(\Pi_s(gq)\eta)(\lambda) = \hat{\alpha}_A(\lambda, g)\eta(\lambda g) ,
\]

\[
(\Pi_s(\theta(q)(x, 0))\eta)(\lambda) = \exp[i\Re Tr(z(s)\lambda h)]\eta(\lambda) ,
\]

where \( \hat{\alpha}_A \) is defined in the condition of theorem 2.1, \( z(s) \) is \( q \times q \)–matrix, \( A \) is s.a. \( q \times q \)–matrix and \( q = r(\Pi) \).
Moreover $A$ does not depend on $s$, the mapping $s \rightarrow z(s)$ is $\mu$–measurable and the range of $z(s)$ equals to $q$ for $\mu$ – a.a. $s \in S$.

Now we should get (side by side with (18)) explicitly the form of the actions of operators $	ilde{\Pi}_{s}(\theta^{(q)}(x, y))$, $	ilde{\Pi}_{s}(\gamma_{u}^{(q)}(b))$ and $\tilde{\Pi}_{s}(\gamma_{o}^{(q)}(b))$ (see § 3).

The next statement follows from the commutation relations (8) and from propositions 6.6 (iii)– 6.7.

**Proposition 6.8.** Let $\Pi$ satisfies to all the requests of proposition 6.1, and $\tilde{\Pi}_{s}$ ($s \in S$) are built as in proposition 6.7. There exists $\mu$–measurable mapping $h$ from $S$ to the set of symmetric (antisymmetric ) for $G = Sp(2\infty)$ $(G = O(2\infty))$ $q \times q$–matrices such that the operators $\tilde{\Pi}_{s}(g)$ ($g \in GK_{q}$) act in $L^{2}(\Lambda_{q}, \nu_{q})$ by:

\[
\tilde{\Pi}_{s}(\theta^{(q)}(a))\eta)(\lambda) = \exp\{i \Re Tr(ah(s))\} \eta(\lambda),
\]

\[
\tilde{\Pi}_{s}(g_{q})\eta)(\lambda) = \delta_{A}(\lambda, g)\eta(\lambda g),
\]

\[
\tilde{\Pi}_{s}(\theta^{(q)}(x, 0))\eta)(\lambda) = \exp[i \Re Tr(z(s)\lambda h)]\eta(\lambda),
\]

\[
\tilde{\Pi}_{s}(\theta^{(q)}(0, y))\eta)(\lambda) = U(y, \lambda, s).
\]

Moreover $U$ and $M$ are the unitary scalar functions.

In the following statement we’ll find an explicit form of cocycle $U$ and function $M$, that are introduced in proposition 6.8.

**Lemma 6.9.** If $h(s)$, $z(s)$ ($s \in S$) are the same as in conditions of propositions 6.7 – 6.8, then the following correlations are true:

i) $U(y, \lambda, s) = \exp\{2i \Re Tr[\lambda^{*}Az^{-1}(s)(yh(s))^{t}] + (\lambda^{*}Az^{-1}(s)(yh(s))^{t})^{*} + 2(z^{-1}(s)(yh(s))^{t})^{*}Az^{-1}(s)(yh(s))^{t}]\}$;

ii) $h(s)$ for $\mu$–a.a. $s \in S$ is an invertible;

iii) $M(b, \lambda, s) = \exp\{-i \Re Tr[R(s)\lambda b\lambda^{t}]\}$,

where $R(s) = \frac{1}{4}z^{t}(s)h^{-1}(s)z(s)$. 

**Proof.** In the following reasoning we omit some standard details of metric character.

Let $Y(\infty, q)$ be a set of all local nonzero $(\infty \times q)$–matrices of infinite number of rows and $q$ columns. If $y \in Y(\infty, q)$, then we denote by $y(k)$ a matrix from $Y(\infty, q)$, that has only one nonzero $k$–th row, which coincides with the $k$–th row of matrix $y$. Since operator $\tilde{\Pi}_{s}(\theta^{(q)}(0, y(k)))$ commutes with all the $\tilde{\Pi}_{s}(u)$, where $u \in U(G_{d}(q, \infty))$ and satisfies to the correlation $ue_{m} = e_{m}$ for $m = \pm k$, then it’s easy to show, that

a) $U(y(k), \lambda, s)$ depends only on the variable $k$–th column of matrix $\lambda \in \Lambda_{q}$;

b) $U(y, \lambda, s) = \prod_{k} U(y(k), \lambda, s)$. 

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We denote by $Y_q(\infty, q)$ a subset in $Y(\infty, q)$, that consists of matrices, which has the elements of first $q$ rows equal to zero. Let $L_q$ be a subgroup in $G_d(q, \infty) \in G$, that consists of matrices of the form $l_q(x) = \begin{pmatrix} (l(x))^{-1} & 0 \\ 0 & l(x) \end{pmatrix}$, where $l(x) = \begin{pmatrix} I_q & 0 \\ 0 & I_q x \end{pmatrix}$, $x$ is an arbitrary local nonzero $q \times \infty -$ matrix.

Element $\lambda \in \Lambda_q$ we will write as $\lambda = (\lambda_q, \lambda(q, \infty))$, where $\lambda_q$ is $q \times q$ – matrix, that consists of first $q$ columns of matrix $\lambda$.

If $y \in Y_q(\infty, q)$, then from the correlation $l_q(x)\theta^{(q)}(0, y) = \theta^{(q)}(0, y)l_q(x)$, setting $\alpha_{\Lambda}(\lambda, g) = \exp\{Tr[iA\lambda(gh^*-1)\lambda^*] \}$ and accounting (21) – (22), we get:

$$\alpha_{\Lambda}((\lambda_q, \lambda(q, \infty)), l(x))U(y, (\lambda_q, \lambda(q, \infty) + \lambda_q x), s) =$$

$$\alpha_{\Lambda}((\lambda_q, \lambda(q, \infty) + 2z^{-1}(yh(s))^t), l(x))U(y, \lambda, s). \tag{24}$$

Let

$$\begin{align*}
a(\lambda, x, y) &= \exp\{-2iTr[\lambda^*(q, \infty)A^{-1}(yh(s))^t + \\
&+ (\lambda^*(q, \infty)A^{-1}(yh(s))^t)^* + x^*\lambda^*qAz^{-1}(s)(yh(s))^t + \\
&+ (x^*\lambda^*qAz^{-1}(s)(yh(s))^t)^* + 2(z^{-1}(s)(yh(s))^t)^*Az^{-1}(s)(yh(s))^t] \}
\end{align*}$$

Since $\alpha_{\Lambda}(\lambda, g) = \exp\{Tr[i\lambda(gg^*-1)\lambda^*] \}$, then using (24) we get:

$$a(\lambda, x, y)U(y, (\lambda_q, \lambda(q, \infty) + \lambda_q x), s) =$$

$$a(\lambda, 0, y)U(y, \lambda, s). \tag{25}$$

From the properties (a) – (b), accounting a form of the action of operators $\hat{\Pi}_s(g_q)$, $\hat{\Pi}_s(\theta^{(q)}(0, y))$ (see (20) and (22)) and the inclusion $y \in Y_q(\infty, q)$, it’s easy to get that $U(y, \lambda, s)$ does not depend on $\lambda_q$.

Hence, accounting (25), we get that $a(\lambda, 0, y)U(y, \lambda, s) = c(y, s)$ does not depend on $\lambda$.

Therefore,

$$U(y, \lambda, s) = c(y, s)\exp\{-2iTr[\lambda^*(q, \infty)A^{-1}(yh(s))^t + \\
&+ (\lambda^*(q, \infty)A^{-1}(yh(s))^t)^* + 2(z^{-1}(s)(yh(s))^t)^*Az^{-1}(s)(yh(s))^t] \}$$

for all $y \in Y_q(\infty, q)$ and $\lambda \in \Lambda_q$.

Taking into account this correlation it’s easy to check, that $U(y, \lambda, s) = U(y, \lambda, s)c^{-1}(y, s)$ is 1–cocycle of action of group $\theta^{(q)}(0, Y_q(\infty, q))$ to $\Lambda_q$. Since $U$ is also 1–cocycle, and $c(y, s)$ does not depend on $\lambda$, then $c(\cdot, s)$ is a multiplicative character of group $Y_q(\infty, q)$).

If $g = \begin{pmatrix} I_q & 0 \\ 0 & g_1 \end{pmatrix}$, $g_q = \begin{pmatrix} (g^{-1})^t & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & g \end{pmatrix}$, then from the correlation

$$\hat{\Pi}_s(g_q)\hat{\Pi}_s(\theta^{(q)}(0, y))(\hat{\Pi}_s(g_q))^* = \hat{\Pi}_s(\theta^{(q)}(0, (g_1^{-1})^t y)), \text{ accounting (20), (22) and using the simple calculations, we get } c(y, s) = c((g_1^{-1})^t y, s) \forall g_1. \text{ Therefore, } c \equiv 1. \text{ Correlation (i) is proved.}$$

To prove (ii) and (iii) we get the correlation, that connects $z(s)$ and $h(s)$ ($s \in S$) (see (19)–(23)).
Obviously the isometry \( V_s \) (\( s \in S \)) from the proposition 6.7 maps \( \Pi_s(U(GK_q)) \) – fixed vector \( \eta_q(s) \) into the vector \( \xi_0 \in L^2(\Lambda_q, \nu_q) \) determined by the function, that equals to unit on \( \Lambda_q \).

Let
\[
\begin{cases}
    s_q^-(e_i) (s_q^+(e_i)) = \begin{cases} s_q^-(e_i) (s_q^+(e_i)), & \text{if } |i| > q \\
e_i, & \text{if } |i| \le q.
\end{cases}
\end{cases}
\]

Since \( \Pi \) is an admissible representation, then it extends by the continuity to \( s_q^- (s_q^+) \). Moreover, operators \( \Pi(s_q^-)(\Pi(s_q^+)) \) we can approximate by the elements from \( \Pi(U(GK_q)) \). Hence \( \Pi(s_q^+)\eta_q(s) = \eta_q(s) \) (see lemma 6.3 and proposition 6.6). Therefore \( \hat{\Pi}_s(s_q^+)\xi_0 = \xi_0 \) This condition and the correlation \( s_q^- \theta(q)(x,0)s_q^+ = \theta(q)(0, \pm x) \) lead to the equality:

\[
(\hat{\Pi}_s(\theta(q)(x,0))\xi_0 , \xi_0 ) = (\hat{\Pi}_s(\theta(q)(0, \pm x))\xi_0 , \xi_0 ).
\]

We calculate the both sides of this equality using ((ii)) and the correlations (21) – (22), and we get

\[
\exp \left\{ -\frac{1}{4} Tr(x^*xz(s)z^*(s)) \right\} = \exp \{ -Tr[x^*x(4h(s)((z^{-1}A^2z^{-1})^t h^*(s) + h(s)((z^{-1}z^{-1})^t h^*(s))]) \}
\]

Therefore,

\[
z(s)z^*(s) = 4h(s)(z^{-1}(s))^t (1 + 4A^2)^t(z^{-1}(s))^t h^*(s).
\]

Since \( rang \ z(s) = q \) for \( \mu - a.a. \ s \in S \) (see proposition 6.1), then from (26) we get the property (ii).

Let’s pass to the proof of (iii).

At first we notice that from (iii) it follows the correctness of the definition of matrix \( R(s) = \frac{1}{2}z^t(s)h^{-1}(s)z(s) \).

Put

\[
T(b, \lambda, s) = \exp\{-i \Re Tr[R(s)\lambda b] \}.
\]

Correlation \( \theta(q)(0, y)\gamma_u(q)(-b)\theta(q)(0, -y) = \gamma_u(q)(-b)\delta(q)((by)^t y)\theta(q)(by, 0) \) (see (8)) leads to the equality:

\[
M(-b, \lambda, s) = M(-b, \lambda, s) \exp\{i \Re Tr[[(by)^t yh(s) + z(s)\lambda by]] \}.
\]

Hence, accounting the definition of \( T(b, \lambda, s) \), we get

\[
M(-b, \lambda, s)T(b, \lambda, s) = M(-b, \lambda, s)T(b, \lambda, s).
\]

Since \( y \) is an arbitrary, and \( rang \ h(s) = rang \ z(s) = q \), then \( M(-b, \lambda, s)T(b, \lambda, s) = c(b, s) \) does not depend on \( \lambda \). Hence, accounting the definition of \( T(b, \lambda, s) \) and correlation \( g g^q\gamma_u(q)(b)(g^{-1} = \gamma_u(q)(gbg^t) \) (see (8)), we get \( c(b, s) = c(gbg^t, s) \) \( \forall g \). Therefore, \( c(b, s) = 1 \) for \( \mu - a.a. \ s \in S \). Lemma 6.9 is proved.
Lemma 6.10 Let \( R(s) = \frac{1}{2}z^t(s)h^{-1}(s)z(s) \), \[ R(s)R^*(s)|^\frac{1}{2}u(s) = R(s) \] is a polar decomposition of \( R(s) \). hen for \( \mu-a.a. \ s \in S \) the following correlations are true: 

i) \( I + 4(A^t)^2 = 4R(s)R^*(s) \); 

ii) \( -A = u^*(s)A^t u(s) \).

Proof Let \( g_q = \begin{pmatrix} (g^{-1})^t & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & g \end{pmatrix} \), \( \xi_0 \) is a vector from \( L^2(\Lambda_q, \nu_q) \) defined by

the function, that equals to unit. Suppose, that matrix \( g \) has a form \( \begin{pmatrix} g(n) & 0 \\ 0 & I \end{pmatrix} \), where \( g(n) \) is \( n \times n \) matrix, for which \( (g(n)g^*(n) - I_n) \) is invertible. Identify \( C_q^n \) with a set of all complex \( n \times q \)-matrices.

By (20)

\[ (\hat{\Pi}_s(g_q)\xi_0)(\lambda) = \exp \left[ -\frac{1}{2} Tr((1 - 2iA)\lambda (gg^* - 1)\lambda^* \right] . \]

Hence and from (21) using the ordinary calculations we get

\[ (\hat{\Pi}_s(g_q)\xi_0)(\lambda) = c(g, s) \int_{C_q^n} \exp\left[ -\frac{1}{2} Tr(z(s)1 - 2iA)^{-1} \right] . \]

\[ \cdot z^*(s) x_n^*(g(n)g^*(n) - I_n)^{-1} x_n \] \[ (\hat{\Pi}_s(\theta(q)(x, 0))\xi_0)(\lambda) dx_n , \]

where \( x_n \in C_q^n, x = \begin{pmatrix} x_n \\ 0 \end{pmatrix} \), \( c(g, s) \) depends only on \( g \) and \( s \).

From the definition of \( s_{\pm}^{(q)} \) and from (27) the next chain of equalities follows:

\[ ((\hat{\Pi}_s(s_{\pm}^{(q)}))^{-1} \hat{\Pi}_s(g_q)\hat{\Pi}_s(s_{\pm}^{(q)})\xi_0)(\lambda) \]

\[ = \left( \hat{\Pi}_s((g^{-1})^t)\xi_0 \right)(\lambda) = \exp \left\{ -\frac{1}{2} Tr \left[ (1 - 2i\lambda)((g^{-1})^t g^{-1} - 1)\lambda \right] \right\} \]

\[ \cdot \int_{C_q^n} \exp \left[ -\frac{1}{2} Tr(z(s)1 - 2iA)^{-1} z^*(s) x_n^*(g(n)g^*(n) - I_n)^{-1} x_n \right] . \]

Next, accounting (22) and a statement of lemma 6.9(i), we get:

\[ \exp \left\{ -\frac{1}{2} Tr \left[ (1 - 2i\lambda)((g^{-1})^t g^{-1} - 1)\lambda \right] \right\} = c(g, s) \cdot \int_{C_q^n} \exp \left[ -\frac{1}{2} Tr \left[ z(s)(1 - 2iA)^{-1} z^*(s) x_n^*(g(n)g^*(n) - I_n)^{-1} x_n \right] - 

\cdot 2(1 - 2iA)z^{-1}(s)h(s)x^t \lambda^* - 2h^*(s)(z^*(s))^{-1}(1 - 2iA)\lambda \bar{x} + 

+ 4\bar{x}h^*(s)(z^*(s))^{-1}(1 - 2iA)z^{-1}(s)h(s)x^t \right\} dx . \]
The calculation of integral in the right side leads to the correlation

\[
\exp \left\{ -\frac{1}{2} \text{Tr} \left[ (1 - 2iA)\lambda((g^{-1})^t \bar{g}^{-1} - 1)\lambda^* \right] \right\} = \\
\exp \left\{ 2 \text{Tr} \left[ (1 - 2iA) \left( z^*(s)(h^*(s))^{-1} \bar{z}(s)(1 - 2iA^t)^{-1} z^*(s)h^{-1}(s)z(s) \otimes \right. \right. \right. \\
\left. \left. \left. (g(n)g^*(n))^t \right) - 1 \right) + 4(1 - 2iA) \right\}^{-1} \{(1 - 2iA)\lambda(n)\lambda^*(n)\}. \\
\] (28)

Here \( \lambda(n) \) consists of first \( n \) columns of matrix \( \lambda \), the action of the operator \( a \otimes b \) on \( \lambda \in \Lambda_q \) is defined by: \( (a \otimes b)\{\lambda\} = a\lambda b \).

If we use the ordinary calculation, we’ll see that (28) is true if and only if \( 4(1 - 2iA) = z^*(s)(h^*(s))^{-1} \bar{z}(s)(1 - 2iA^t)^{-1} z^*(s)h^{-1}(s)z(s) \). Therefore,

\[
1 + 4(A^t)^2 = 4R(s)R^*(s) \\
-4A = R^*(s)(1 + 4(A^t)^2)^{-1} A^t R(s). \\
\] (29)

If \( |R(s)R^*(s)|^\frac{1}{2} u(s) = R(s) \) is a polar decomposition of \( R(s) \), then from (29) it follows that \( -A = u^*(s)A^t u(s) \). Lemma 6.10 is proved.

For \( \mu \)-a.a. \( s \in S \) representation \( \hat{\Pi}_s \) of group \( GK_q \) is defined by the set of parameters \( \{A, z(s), h(s)\} \), that was introduced in propositions 6.7 – 6.8. If \( \varepsilon \) is a non negative number, then we denote by \( \vartheta(\varepsilon), \varsigma(\varepsilon) \) the \( 2 \times 2 \)-matrices of the form \( \varsigma(\varepsilon) = \begin{pmatrix} 0 & i\varepsilon \\ -i\varepsilon & 0 \end{pmatrix}, \vartheta(\varepsilon) = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix} \).

From the propositions 6.9 – 6.10 using the standard methods of elementary theory of matrices we can get the following statement.

**Proposition 6.11.** There exists \( \mu \)-measurable mapping \( w(\cdot) \) from \( S \) into \( U(q) \) with the following properties:

i) if \( G = Sp(2\infty) \), then

\[
w(s)Aw^*(s) = d_{sp}(A) = \begin{pmatrix} \varsigma(\lambda_1) & 0 & \ldots & 0_2 & 0 \\ 0 & \varsigma(\lambda_2) & \ldots & 0_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_2 & 0 & \ldots & \varsigma(\lambda_k) & 0 \\ 0 & 0 & 0 & 0 & 0_p \end{pmatrix},
\]

where \( 0_p \) is zero \( p \times p \)-matrix, \( \lambda_j > 0 \) for \( j = 1, 2, \ldots, k \) (\( k \) may be equals to zero)

and \( 2k + p = q \);

\( \bar{w}(s)u(s)w^*(s) = I_q \) (see lemma 6.10);

ii) if \( G = O(2\infty) \), then

\[
w(s)Aw^*(s) = d_o(A) = \begin{pmatrix} \vartheta(\lambda_1) & 0 & \ldots & 0_2 & 0 \\ 0 & \vartheta(\lambda_2) & \ldots & 0_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_2 & 0 & \ldots & \vartheta(\lambda_k) & 0 \\ 0 & 0 & 0 & 0 & 0_{2p} \end{pmatrix},
\]

where \( 2p + 2k = q \), and the rest properties of the parameters...
of matrix $d_o(A)$ are the same as in (i);

$$u_a = \tilde{w}(s)u(s)w^*(s) = \begin{pmatrix} \varsigma(-i) & 0_2 & \cdots & 0_2 \\ 0_2 & \varsigma(-i) & \cdots & 0_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0_2 & 0_2 & \cdots & \varsigma(-i) \end{pmatrix};$$

iii) $R_c = \tilde{w}(s)R(s)w^*(s) = \frac{1}{2}\sqrt{1 + 4w(s)A^2w^*(s)}\tilde{w}(s)a(s)w^*(s)$ does not depend on $s$.

Consider the unitary operator $W$, that acts in $L^2(S, \mu) \otimes L^2(\Lambda_q, \nu_q)$ by

$$(W\eta)(s, \lambda) = \eta(s, w^*(s))\lambda, \quad \text{where} \quad \eta \in L^2(S, \mu) \otimes L^2(\Lambda_q, \nu_q).$$

The next statement follows from the propositions 6.6 – 6.8, 6.11 and has the same meaning for description of the representations of group $G$, as the lemma 5.5 in a case of $GL(\infty)$.

**Proposition 6.12.** We denote by $V$ the isometry of a Hilbert space

$$H_q^{(V)} = \underset{S}{\bigoplus} H_q^{(V)}(s) d\mu(s) \rightarrow L^2(S, \mu) \otimes L^2(\Lambda_q, \nu_q),$$

that defined by $(V\eta)(s) = V_s\eta(s)$, where $\eta(s) \in H_q^{(V)}(s)$, $V_s\eta(s) \in L^2(\Lambda_q, \nu_q)$ for $\mu$ – a.a. $s \in S$ (see propositions 6.6 – 6.7). For $g \in GK_q$ we put

$$\tilde{\Pi}(g) = WV\Pi(g)V^*W^*, \quad \varsigma(s) = z(s)w^*(s).$$

Then the action of operators $\tilde{\Pi}(g)$ in $L^2(S, \mu) \otimes L^2(\Lambda_q, \nu_q)$ is defined by correlations:

$$\tilde{\Pi}(\theta^q(x, 0))\eta|s, \lambda\rangle = \exp[i\mathfrak{R}Tr(\varsigma(s)\lambda h)]\eta|s, \lambda\rangle; \quad (30)$$

$$\tilde{\Pi}(g)\eta|s, \lambda\rangle =$$

$$\begin{cases} \hat{\alpha}_{d_o(A)}(\lambda, g)\eta|s, \lambda g\rangle, \text{if} \ g_q \in Sp(2\infty) \\ \hat{\alpha}_{d_o(A)}(\lambda, g)\eta|s, \lambda g\rangle, \text{if} \ g_q \in O(2\infty) \end{cases}; \quad (31)$$

$$\tilde{\Pi}(\gamma^q_u(b))\eta|s, \lambda\rangle =$$

$$\begin{cases} \exp\left\{-\frac{i}{2}Tr[\sqrt{1 + 4(d_{sp}(A))^2}\lambda b\lambda^t}\right\}, \text{if} \ b = b^t \quad (32) \\ \exp\left\{-\frac{i}{2}Tr[\sqrt{1 + 4(d_o(A))^2} u_o\lambda b\lambda^t]\right\}, \text{if} \ b = -b^t \end{cases}$$

(see proposition 6.11(ii) and lemma 6.9(iii));

$$\tilde{\Pi}(\theta^q(0, y))\eta|s, \lambda\rangle = U(y, \lambda, s) \cdot$$

$$\cdot \left[\frac{dv_q(\lambda + 2\varsigma^{-1}(s)(yh(s))^t)}{dv_q(\lambda)}\right]^{\frac{1}{2}}\eta|s, \lambda + 2\varsigma^{-1}(s)(yh(s))^t\rangle. \quad (33)$$
The form of the unitary cocycle from the last correlation is given in the condition of lemma 6.9 (see (22)). Moreover in the corresponding expression we should change \( z(s) \) into \( \zeta(s) \), and \( A \) – into \( w(s)Aw^*(s) \) (see proposition 6.11).

The advantage of the realization of \( \tilde{\Pi} \) by correlations \((20) – (23)\) is that the form of the action of operators \( \tilde{\Pi}(g_q) \) and \( \tilde{\Pi}(\gamma_u^{(q)}(b) \) does not depend on \( s \).

Since \( \tilde{\Pi} \) is an admissible representation, then it extends by continuity to the group, generated by \( GK_q \) and \( s_\pm^{(q)} \) for \( G = Sp(2\infty) \) or \( GK_q \) and \( s_\pm^{(q)} \) for \( G = O(2\infty) \).

Change if we need \( \Pi(U(GK_q)) \) – fixed vector \( \eta_q \) (see lemma 6.3 and proposition 6.6 ) to \( c\eta_q \), where \( c \) is an operator from the center of \( w^* \)– algebra \( (\Pi(GK_q))' \), we may take \( WV\eta_q = \xi_0 \) (see proposition 6.12). Vector \( \xi_0 \) is determined by the function on \( S \times \Lambda_q \), that equals to the unit. Therefore, \( \tilde{\Pi}(s_\pm^{(q)})\xi_0 = \xi_0 \). Hence and from the correlations \((20), (22)\) we get
\[
(\tilde{\Pi}(s_\pm^{(q)}))^{-1}\tilde{\Pi}(\theta^{(q)}(x, 0)\xi_0 = \tilde{\Pi}(\theta^{(q)}(0, \pm x)\xi_0 .
\]

Thus
\[
\begin{align*}
\exp\{i\Re Tr(\zeta(s)\lambda h)\} & \eta(s, \lambda) \\
& \longrightarrow \exp\{2i\Re Tr[\pm \lambda^* d_q(A)\zeta^{-1}(s) h^t(s)(\zeta^t(s))^{-1}y^t \pm \\
& \pm \bar{y}\zeta(s)^{-1}h(s)\zeta^{-1}(s)d_q(A)\zeta^{-1}(s)(\zeta^{-1})^ty^t] · \\
& \left[ \frac{dv_q(\lambda + 2\zeta^{-1}(s) h^t(s) (\zeta^{-1}(s))^ty^t)}{dv_q(\lambda)} \right] \quad (34)
\end{align*}
\]

where \( \n \) means one of the indexes \( o \) or \( sp \).

From this correlations we get an important

**Remark 6.13.** Since \( \zeta(s) = z(s)w^*(s) \), then by the statements 6.10 – 6.12 the next correlation is true
\[
R = \tilde{w}(s)R(s)w^*(s) = \frac{1}{4}\zeta^t(s)h^{-1}(s)\zeta(s) ,
\]
and that’s why the form of the action of the operator \( \tilde{\Pi}(s_\pm^{(q)}) \) does not depend on \( s \in S \).

We denote by \( \tilde{G}(q, \infty) \) a group generated by all the elements \( g_q, \gamma_u^{(q)}(b), s_\pm^{(q)} \). Obviously \( G(q, \infty) \subset \tilde{G}(q, \infty) \). Therefore, for \( g \in G(q, \infty) \) \( \tilde{\Pi}(g) = I_{L^2(S, \nu)} \otimes \Pi(g) \), where \( \Pi(g) \) is a unitary operator in \( L^2(\Lambda_q, \nu_q) \) (see statements 6.12 – 6.13). We denote by \( \ell_q \) a natural isomorphism of groups \( G(q, \infty) \) and \( G \) \( (\ell_q : g \in G(q, \infty) \longrightarrow \sigma_q^{-1}gg_q \in G) \).

The total of our reasoning is the following statement.

**Theorem 6.14.** Let \( \Pi \) be the same as in propositions 6.1 – 6.4, \( \xi \) is a cyclic vector for \( \Pi \), \( \xi_q = \Pi(\sigma_q) \) and \( \Pi_A \) is a representation of group \( G \) built in \( \S 1 \) (see propositions 1.2 – 1.3), then the restriction of \( \Pi \) to \( G(q, \infty) \), that acts in \( [\Pi(G(q, \infty))\xi_q] \), is multiple by the irreducible component of the representation, that defined by the chain of mappings:
\[
g \in G(q, \infty) \longrightarrow \ell_q(g) \in G \longrightarrow \Pi_A(\ell_q(g)) .
\]
To prove this theorem it suffices to notice that \( \Pi_A \circ \ell_q \) is naturally unitary equivalent to \( \tilde{\Pi} \).

To formulate the main classification result let us notice the important moments of our reasoning.

We have supposed, that the admissible factor–representation \( \Pi \) of group \( G \) is cyclic with the respect to the unit vector \( \xi \in H_\Pi \) (see remark 6.2). Then we’ve considered a restriction of \( \Pi \) to the subgroup \( G(q, \infty) \), that acts in \( [\Pi(G(q, \infty))\Pi(\sigma_q)\xi] \). There was shown that, it is a subrepresentation of the spherical representations of larger group \( GK_q \) (see proposition 6.4), for which in the proposition 6.12 explicit realization was given. Moreover the form of the operators \( \tilde{\Pi}(g) \) (see (31), (32) and (34)) for \( g \in G(q, \infty) \) does not depend on \( s \in S \) (see also remark 6.13). At last in theorem 6.14 we’ve got the form of representation \( (\Pi, \sigma_q \xi) \) (see theorem 1.4). Theorem 0.5 found by G.I. Ol’shansky let us turn to the representation \( \Pi \) of group \( G \).

Because of this fact \( \Pi \) extends by continuity to \( \sigma_q \). Therefore the next correlation is true:

\[
(\Pi(g)\xi, \xi)_{H_\Pi} = (\Pi(\sigma_q g \sigma_q^* + I_q(0))\Pi(\sigma_q)\xi, \Pi(\sigma_q)\xi)_{H_\Pi},
\]

where \( I_q(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \), nd \( \sigma_q g \sigma_q^* + I_q(0) \) belongs to \( G(q, \infty) \).

Next, by theorem 6.14 in \( L_q^A \) (see §1), there exists vector \( f_\xi \), for which

\[
(\Pi(\sigma_q g \sigma_q^* + I_q(0))\Pi(\sigma_q)\xi, \Pi(\sigma_q)\xi)_{H_\Pi} = (\Pi_A \circ \ell_q(\sigma_q g \sigma_q^* + I_q(0))f_\xi, f_\xi)_{L_q^A}.
\]

Hence and from (35) we get

\[
(\Pi(g)\xi, \xi)_{H_\Pi} = (\Pi_A(g)f_\xi, f_\xi)_{L_q^A}.
\]

This correlation and theorem 1.4 lead us to the main result of this chapter.

**Theorem 6.15.** Let \( \Pi \) be an admissible factor–representation of group \( G \) (>G = Sp(2\infty) or \( O(2\infty) \)) . hen there exist: a natural number \( q \), \( q \times q \) – self-adjoint matrix \( A \), that satisfies to the corresponding conditions of propositions 1.2–1.3, an admissible representation \( \rho \) of group \( O(A,q) \subset U(q) \) for \( G = Sp(2\infty) \) or \( Sp(A,m) \subset U(q) \) for \( G = O(2\infty) \) (see §1) such that \( \Pi \) is multiple by restriction of \( \Pi_A \) to \( P_k L_q^A \) (see theorem 1.4).
References

[1] A. Kirillov: *Representations of an infinite-dimensional unitary group*, Dokl. Akad. Nauk SSSR 212 (1973), 288–290; English transl. in Soviet Math. Dokl. 14 (1973).

[2] G. I. Ol’shansky: *Unitary representations of infinite-dimensional classical groups* $U(p, \infty), SO_0(p, \infty), Sp(p, \infty)$ and of the corresponding motion groups, Functional Anal. Appl., 12 (1978), No 3, 32 – 44.

[3] G. I. Ol’shansky: *Unitary representation of infinite-dimensional pairs* $(G, K)$ and the formalism of R. Howe, Dokl. Akad. Nauk SSSR 269 (1983), 33 – 36.

[4] G. I. Ol’shansky: *Construction of unitary representations of infinite-dimensional classical groups*, Dokl. Akad. Nauk SSSR 250 (1980), 284 – 288.

[5] G. I. Ol’shansky: *Method of holomorphic extensions in theory of unitary representations of infinite-dimensional classical groups*, Functional Anal. Appl., 22 (1988), 4, 23–37.

[6] A. Vershik and S. V. rv: *Characters and factor representations of the infinite unitary group*, Dokl. Akad. Nauk SSSR 267 (1982), No 2, 272 – 276.

[7] R. S. Ismagilov: *Infinite groups and their representations*, Proceedings of the international mathematic congress, Warsaw. (1983), 861 – 875.

[8] N. I. Nessonov: *Description of representations of the group invertible operators in a Hilbert space, containing the identity representation of the unitary subgroup*, Functional Anal. Appl. 17, (1983), No 1, 79 – 80.

[9] N. I. Nessonov: *A complete classification of the representations of GL($\infty$), containing the identity representation of the unitary subgroup*, th. USSR Sbornik, 58 (1987), No 1, 127 – 147.

[10] N. I. Nessonov: *A complete description of indecomposable spherical functions on the infinite-dimensional group of motions*, Dokl. Akad. Nauk USSR (1987), No 6, 7 – 9.

[11] N. I. Nessonov: *Description of an admissible representations of infinite-dimensional matrix groups with the coefficient in finite-dimensional algebra*, Functional Anal. Appl. 26, (1992), No 2.

[12] N. I. Nessonov: *Representations of infinite-dimensional matrix groups and associated dynamical systems*, Operator algebras and operator theory, Proc. OATE 2 Conf., Romania. (1989), Longman Group UK Limited 1992.