A VARIABLE COEFFICIENT MULTI-FREQUENCY LEMMA

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Abstract. We show a variable coefficient version of Bourgain’s multi-frequency lemma. It can be used to obtain major arc estimates for a discrete Stein–Wainger type operator considered by Krause and Roos.

1. Introduction

Bourgain’s multi-frequency lemma, first introduced in [Bou89], allows one to estimate expressions of the type
\[ \left\| \sup_{t} |F^{-1}(\sum_{\beta \in \Xi} S(\beta) \sigma_t (\cdot - \beta) \hat{f})| \right\|_2, \]
where \( \Xi \) is a \( \delta \)-separated set of frequencies and \( (\sigma_t)_t \) is a family of multipliers supported in a \( \delta \)-neighborhood of zero. Expressions like (1.1) arise when singular or averaging operators on \( \mathbb{Z}^n \) are treated by the circle method. The coefficients \( S(\beta) \) are usually some type of complete exponential sums.

In this note, we address the problem of extending Bourgain’s lemma to a setting in which the coefficients \( S(\beta) \) in (1.1) also depend on \( t \). This situation recently arose in [Kra18; Roo19]. Contrary to the classical case (1.1), the corresponding operator can no longer be easily represented as the composition of two Fourier multipliers. We defer this application to Section 3 and begin with the statement of our multi-frequency lemma.

Let \( n \geq 1 \) and let \( \chi_0 : \mathbb{R}^n \to [0,1] \) be a smooth bump function that is supported on \( [-1/2, 1/2]^n \) and equals 1 on \( [-1/2, 1/2]^n \). Let \( A \) be a contractive invertible linear map on \( \mathbb{R}^n \) and denote \( \chi(\xi) := \chi_0(A^{-1}\xi) \), so that in particular \( \chi \) equals 1 on \( U := A([-1/2, 1/2]^n) \). Then \( \phi = F^{-1}_\mathbb{R}(\chi) \) is an \( \ell^1 \) normalized bump function, in the sense that \( \|\phi\|_{l^p(\mathbb{Z}^n)} \sim |U|^{-1/p} \). In this article, we write \( A \lesssim B \) if \( A \leq CB \) with a constant \( C \) depending only on the dimension \( n \), unless indicated otherwise by a subscript. We write \( A \sim B \) if \( A \lesssim B \) and \( B \lesssim A \).

For a function \( F \) from a totally ordered set \( T \) to a normed vector space \( H \), we denote the \( r \)-variation seminorm by
\[ \|F(t)\|_{V^r_t(H)} = \sup_{t_0 \leq \cdots \leq t_J} \left( \sum_{j=1}^J \|F(t_j) - F(t_{j-1})\|_H^r \right)^{1/r}, \]
where the supremum is taken over all finite increasing sequences in \( T \). The vector space \( H \) may be omitted if it equals \( \mathbb{C} \).

**Theorem 1.1.** Let \( \Xi \) be a finite set. Let \( g_\beta : \mathbb{Z}^n \to \mathbb{C} \), \( \beta \in \Xi \), be functions such that, for every \( x \in \mathbb{Z}^n \) and every sequence \( (c_\beta)_{\beta \in \Xi} \) of complex numbers, we have
\[ \| \sum_{\beta \in \Xi} \phi(y) g_\beta(x + y) c_\beta \|_{l^2_\mathbb{Z}^n} \leq A_1 |U|^{1/2} \|c_\beta\|_{l_\beta^2}, \]
for some \( A_1 > 0 \). Let \( T \subseteq \mathbb{R} \) be a finite set and let \( (T_t)_{t \in T} \) be a family of translation invariant operators on \( \ell^2(\mathbb{Z}^n) \) such that, for some \( C_1 < \infty \), some \( \eta > 0 \) and every

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Let \( \{f_\beta\}_{\beta \in \Xi} \subset \ell^2(\mathbb{Z}^n) \) be functions with \( \text{supp} \hat{f}_\beta \subset U \) for every \( \beta \). Then, for any \( q \in (2, \infty) \), we have
\[
\left\| \sum_{\beta \in \Xi} g_\beta(x)(T_t f_\beta)(x) \right\|_{\ell^q(\mathbb{Z}^n)} \lesssim \left( \frac{q(\log|\Xi| + 1)}{q - 2} \right)^{q+1} A_1 \left\| f_\beta \right\|_{\ell^q(\mathbb{Z}^n)} \| c_\beta \|_{\ell^q_\beta}. \quad (1.4)
\]

**Remark 1.2.** The classical multi-frequency lemma corresponds to the case \( g_\beta(x) = e^{2\pi i x \cdot \xi_\beta} \) with some \( U \)-separated frequencies \( \xi_\beta \). In this case, (1.2) holds with \( A_1 \sim 1 \).

**Remark 1.3.** In (1.4), one can use a different family \( (T_t, \beta)_{t \in \mathbb{T}} \) for each \( \beta \in \Xi \), as long as the bound (1.3) is uniform in \( \Xi \).

## 2. Proof of the Multi-Frequency Estimate

The proof of Theorem 1.1 is based on the arguments introduced in [Bou89] and further developed in [NOT10; Obe13; Kra14; Zor15]. The first point where we deviate from the previous arguments is the following result, which extends [Obe13, Proposition 9.3] and [NOT10, Lemma 3.2].

**Lemma 2.1.** Let \( \Xi \) be a finite set and \( g_\beta, \beta \in \Xi \), complex-valued measurable functions on some measure space \( Y \). Assume that, for some \( A_0 < \infty \), we have
\[
\left\| \sum_{\beta \in \Xi} g_\beta(y) c_\beta \right\|_{L^q_\beta(Y)} \leq A_0 \| c_\beta \|_{\ell^q_\beta}, \quad (2.1)
\]
for every sequence \( (c_\beta)_{\beta \in \Xi} \in \ell^2(\Xi) \). Then, for every \( 2 < r < q \), every countable totally ordered set \( T \), and every collection of sequences \( \{(c_{t, \beta})_{\beta \in \Xi}\}_{t \in T} \subset \ell^2(\Xi) \), we have
\[
\left\| \sum_{\beta \in \Xi} g_\beta(y) c_{t, \beta} \right\|_{\ell^r_\beta(Y)} \lesssim \left( \frac{q}{q - r} + \frac{2}{r - 2} \right) A_0 |\Xi|^\frac{1}{2} \left( \frac{1}{q} - \frac{r}{2} \right) \| c_{t, \beta} \|_{\ell^r_\beta(Y)},
\]
where the implicit constant is absolute.

The proof of Lemma 2.1 relies on the following result.

**Lemma 2.2** ([Zor15, Lemma 2.6]). Let \( B \) be a normed space with norm \( \| \cdot \|_B \) and \( B' \) the dual space of \( B \). Let \( Y \) be a measure space, and \( q \in L^p(Y, B') \), \( p \geq 1 \). Let also \( c = (c_t)_{t \in \mathbb{T}} \subset B \) with a countable totally ordered set \( \mathbb{T} \), and \( q > p \). Then
\[
\| \langle c, g(y) \rangle \|_{V^p_t(Y)} \lesssim \int_0^\infty \min(M(J_\lambda(c))^\frac{1}{p}, \| g \|_{L^p(Y, B')}(J_\lambda(c))^\frac{1}{q})d\lambda,
\]
where \( J_\lambda(c) \) is the greedy jump counting function for the sequence \( c := (c_t)_{t \in \mathbb{T}} \) at the scale \( \lambda \),
\[
M = \sup_{c \in B, \| c \|_B = 1} \| \langle c, g(y) \rangle \|_{L^p_t(Y)},
\]
and the implicit constant is absolute.

We will not need the definition of \( J_\lambda(c) \), only the fact that
\[
J_\lambda(c) \leq \| c_t \|_{V^p_t(B)}/\lambda^r. \quad (2.3)
\]

**Proof of Lemma 2.1.** We apply Lemma 2.2 with \( B = B' = \ell^2(\Xi) \), and \( g(y) = (g_\beta(y))_{\beta \in \Xi} \). By the hypothesis (2.1), we have \( M \leq A_0 \), where \( M \) was defined in (2.2). Moreover,
\[
\| g \|_{L^2(Y, B')} = \left( \sum_{\beta \in \Xi} \| g_\beta \|_{L^2}^2 \right)^{1/2} \leq |\Xi|^{1/2} A_0,
\]
where we used (2.1) with \((c_\beta)\) being indicator functions of points. Hence, we obtain
\[
\left\| \sum_{\beta \in \Xi} g_\beta(y) c_{t, \beta} \right\|_{V_\phi^a} \lesssim A_0 \int_0^\infty \min(J_{\lambda/2}^{1/2}, |\Xi|^{1/2} J_{\lambda/2}^{1/2}) d\lambda.
\]
Using (2.3) with \(a := \|c_{t, \beta}\|_{V_\phi^a(\ell_2^n)}\) and splitting the integral at \(\lambda_0 = a |\Xi|^{-1/(2r(1/2-1/q))}\), we obtain
\[
\begin{align*}
\int_0^{\lambda_0} |\Xi|^{1/2} (a^r/\lambda^r)^{1/4} d\lambda + \int_{\lambda_0}^\infty (\lambda')^{1/2} d\lambda \\
= |\Xi|^{1/2} a^r/q(-r/q + 1)^{-1/2} \lambda_0^{-r/q + 1} - a^r/(r/2 + 1)^{-1} \lambda_0^{-r/2 + 1} \\
= a |\Xi|^{1/2} (1 - r/q)^{-1} + (r/2 - 1)^{-1}.
\end{align*}
\]

Proof of Theorem 1.1. From the hypothesis (1.3) and Minkowski’s inequality, it follows that
\[
\left\| \sum_{\beta \in \Xi} \sum_{y \in \mathbb{Z}^n} g_\beta(x) g_\beta(\tilde{\phi}_\beta(x)) \phi(y) g_\beta(y) \right\|_{V_\phi^a(\ell_2^n)} \leq C_1 \left( \frac{r}{r - 2} \right) \|f_\beta\|_{\ell_2^n(\ell_2^n)} \|f_\beta\|_{\ell_2^n(\ell_2^n)},
\]
initially for \(r \in (2, 3)\), but by monotonicity of the variation norms also for \(r \in (2, \infty)\).

We use the Fourier uncertainty principle. Let \(R_y f(x) = f(x - y)\). By the frequency support assumption on \(f_\beta\), we have
\[
f_\beta = f_\beta \ast (\phi \tilde{\phi}),
\]
where \(\phi\) is an \(\ell^\infty\) normalized bump function with \(\text{supp} \phi \subseteq 4U\) such that \(\phi \ast \phi \equiv 1\) on \(U\). It follows that
\[
\text{LHS of (1.4)} = \left\| \sum_{\beta} g_\beta(x) T_x f_\beta(x) \right\|_{V_\phi^a} \lesssim \left\| \sum_{\beta} \phi(y) g_\beta(x) (T_x f_\beta(x)) \right\|_{V_\phi^a}
\]
\[
\leq \left\| \phi(y) \right\|_{\ell_2^n} \left\| \sum_{\beta \in \Xi} g_\beta(x) (T_x f_\beta(x)) \right\|_{V_\phi^a}
\]
\[
\leq \left\| \phi \right\|_{\ell_2^n} \left\| \sum_{\beta \in \Xi} g_\beta(x) (T_x f_\beta(x)) \right\|_{V_\phi^a} \lesssim |U|^{-1/2} \left\| \sum_{\beta \in \Xi} \phi(y) g_\beta(x) (T_x f_\beta(x)) \right\|_{V_\phi^a} \lesssim |U|^{-1/2} \left\| \sum_{\beta \in \Xi} \phi(y) g_\beta(x) (T_x f_\beta(x)) \right\|_{V_\phi^a}.
\]
For each fixed \(x\), we will apply Lemma 2.1 with the functions
\[
g_\beta(y) = \phi(y) g_\beta(x + y).
\]
By the hypothesis (1.2), the estimate (2.1) holds with
\[
A_0 \leq A_1 |U|^{1/2}.
\]
By Lemma 2.1, for any \(2 < r < q\), we obtain
\[
\text{LHS of (1.4)} \lesssim A_1 \left( \frac{q}{q - r} + \frac{2}{r - 2} \right) \left| \Xi \right|^{(1 - \frac{1}{r}) q - \frac{2}{q - r}} \|f_\beta\|_{\ell_2^n(\ell_2^n)} \|f_\beta\|_{\ell_2^n(\ell_2^n)}.
\]
By (2.4), we obtain
\[
\text{LHS of (1.4)} \lesssim C_1 A_1 \left( \frac{q}{q - r} + \frac{2}{r - 2} \right) \left| \Xi \right|^{(1 - \frac{1}{r}) q - \frac{2}{q - r}} \left( \frac{r}{r - 2} \right)^{\eta} \|f_\beta\|_{\ell_2^n(\ell_2^n)} \|f_\beta\|_{\ell_2^n(\ell_2^n)}.
\]
Choosing $r$ such that $r - 2 = (q - 2)(\log|\Xi| + 1)^{-1}$, this implies (1.4). \qed

3. An application

For a function $f : \mathbb{Z}^n \to \mathbb{C}$, consider the operator

$$Cf(x) = \sup_{\lambda \in \mathbb{R}} \left| \sum_{y \in \mathbb{Z}^n \setminus \{0\}} f(x - y)e(\lambda |y|^{2d})K(y) \right|, \quad (x \in \mathbb{Z}^n),$$

(3.1)

where $K$ is a Calderon-Zygmund kernel that satisfies the conditions as in [Roo19] and $e(\lambda) = e^{2\pi i \lambda}$. For instance, one can take $K$ to be the Riesz kernel.

Here we use Theorem 1.1 to estimate the major arc operators arising in the proof of the $\ell_2^2$ bounds of $C$ in [Roo19]. We begin by recalling the approach, notation, and some results from [Roo19].

First, we apply a dyadic decomposition to $K$ and write

$$K = \sum_{j \geq 1} K_j,$$

(3.2)

where $K_j := K \cdot \psi_j$ and $\psi_j(\cdot) = \psi(2^{-j} \cdot)$ for some appropriately chosen non-negative smooth bump function $\psi$ which is compactly supported. Define the multiplier

$$m_{j,\lambda}(\xi) := \sum_{y \in \mathbb{Z}^n} e(\lambda |y|^{2d} + \xi \cdot y)K_j(y).$$

(3.3)

For a multiplier $m(\xi)$ defined on $T^n$, we define

$$m(D)f(x) := \int_{T^n} m(\xi)\hat{f}(\xi)e(x \cdot \xi)d\xi, \quad x \in \mathbb{Z}^n.$$

(3.4)

We also define the continuous version of the multiplier $m_{j,\lambda}$ by

$$\Phi_{j,\lambda}(\xi) = \int_{\mathbb{R}^n} e(\lambda |y|^{2d} + \xi \cdot y)K_j(y)dy.$$

(3.5)

The goal is to prove

$$\left\| \sup_{\lambda \in \mathbb{R}} \left[ \sum_{j \geq 1} m_{j,\lambda}(D)f \right] \right\|_{\ell^2} \lesssim \|f\|_{\ell^2}.$$

(3.6)

Define the major arcs (in the variable $\lambda$)

$$X_j = \bigcup_{a/q \in \mathbb{Q}, (a,q) = 1} \{ \lambda \in \mathbb{R} : |\lambda - a/q| \leq 2^{-2dj} + \epsilon_1 j \},$$

(3.7)

where $\epsilon_1 > 0$ is a small fixed number that depends only on $d$. The complement $\mathbb{R} \setminus X_j$ will be called a minor arc.

The contribution of the minor arcs was estimated in [Kra18; Roo19] using a $TT^*$ argument in the spirit of [SW01], the result being that there exists $\gamma > 0$ such that

$$\| \sup_{\lambda \notin X_j} |m_{j,\lambda}(D)f|\|_{\ell^2(\mathbb{Z})} \lesssim 2^{-j\gamma} \|f\|_{\ell^2},$$

(3.8)

holds for every $j \geq 1$.

On the major arcs in the variable $\lambda$, we have a good approximation of the discrete multiplier $m_{j,\lambda}$ by the continuous multiplier $\Phi_{j,\lambda}$. For convenience, define

$$\Phi_{j,\lambda'} = \Phi_{j,\lambda'} \cdot 1_{|\lambda'| \leq 2^{-2d} + \epsilon_1 j}.$$

(3.9)

For an integer $1 \leq s \leq \epsilon_1 j$, define

$$\mathcal{R}_s = \{ (a/q, b/q) \in \mathbb{Q} \times \mathbb{Q}^n \mid (a, b, q) = 1, q \in \mathbb{Z} \cap [2^{s-1}, 2^s) \}.$$

(3.10)
For \((\alpha, \beta) \in \mathcal{R}_s\), define a complete Gauss sum
\[
S(\alpha, \beta) = q^{-n} \sum_{r=(r_1, \ldots, r_n) \atop 0 \leq r_1, \ldots, r_n < q} e(\alpha| r |^{2d} + \beta \cdot r).
\] (3.11)

Define \(\chi_s(\cdot) := \chi_0(2^{10s} \cdot)\). Define
\[
L^s_{j, \lambda}(\xi) = \sum_{(\alpha, \beta) \in \mathcal{R}_s} S(\alpha, \beta) \Phi^*_{j, \lambda - \alpha}(\xi - \beta) \chi_s(\xi - \beta).
\] (3.12)

Define the error term
\[
E_{j, \lambda}(\xi) := m_{j, \lambda}(\xi) \cdot 1_{X_j}(\lambda) - \left( \sum_{1 \leq s \leq \epsilon_{1,j}} L^s_{j, \lambda}(\xi) \right).
\] (3.13)

By a Sobolev embedding argument in the spirit of Krause and Lacey [KL17] applied to the sup over \(\lambda\), it was proved in [Roo19, Proposition 3.2] that there exists \(\gamma > 0\) such that
\[
\| \sup_{\lambda \in X_j} |E_{j, \lambda}(D)f| \|_{\ell^2} \lesssim 2^{-\gamma j} \|f\|_{\ell^2}.
\] (3.14)

It remains to bound the contribution from the multiplier
\[
\sum_{j \geq 1} \sum_{1 \leq s \leq \epsilon_{1,j}} L^s_{j, \lambda}(\xi) = \sum_{s \geq 1} \sum_{j \geq \epsilon_{1}^{-1}s} L^s_{j, \lambda}(\xi).
\] (3.15)

To simplify notation, we introduce
\[
L^s_{\lambda} = \sum_{j \geq \epsilon_{1}^{-1}s} L^s_{j, \lambda} \quad \text{and} \quad \Phi^s_{\lambda}(\xi) = \sum_{j \geq \epsilon_{1}^{-1}s} \Phi^*_{j, \lambda}(\xi) \chi_s(\xi).
\] (3.16)

By the triangle inequality applied to the sum over \(s \geq 1\), it suffices to prove that there exists \(\gamma > 0\) such that
\[
\| \sup_{\lambda \in \mathbb{R}} |L^s_{\lambda}(D)f| \|_{\ell^2} \lesssim 2^{-\gamma s} \|f\|_{\ell^2},
\] (3.17)
for every \(s \geq 1\). This estimate is where our variable coefficient multi-frequency lemma, Theorem 1.1, will be useful. The next two lemmas verify its assumptions (1.2) and (1.3), respectively. Let
\[
\mathcal{A}_s = \{ \alpha \in \mathbb{Q} : (\alpha, \beta) \in \mathcal{R}_s \text{ for some } \beta \},
\]
\[
\mathcal{B}_s(\alpha) = \{ \beta \in \mathbb{Q}^n : (\alpha, \beta) \in \mathcal{R}_s \}.
\] (3.18)

Moreover, define
\[
L^s_{\alpha, 2}(\xi) := \sum_{\beta \in \mathcal{B}_s(\alpha)} S(\alpha, \beta) \chi_s(\xi - \beta).
\] (3.19)

We have

**Lemma 3.1** ([Roo19, Proposition 3.3]). *There exists \(\gamma > 0\) depending on \(d\) and \(n\) such that*
\[
\| \sup_{\alpha \in \mathcal{A}_s} |L^s_{\alpha, 2}(D)f| \|_{\ell^2} \lesssim 2^{-\gamma s} \|f\|_{\ell^2},
\] (3.20)
*for every \(s \geq 1\).*

**Lemma 3.2.** *For every \(r \in (2, 3)\), we have*
\[
\|\|\Phi^s_{\lambda}(D)f\|_{V^{r}_{\lambda \in [0,1]}}\|_{\ell^2} \lesssim d, n (r - 2)^{-1} \|f\|_{\ell^2}.
\] (3.21)
Proof of Lemma 3.2. By the transference principle of Magyar, Stein, and Wainger in [MSW02, Proposition 2.1], it suffices to prove that
\[
\|\Phi^s_t(D)f\|_{L^2(\mathbb{R}^n)} \lesssim d,n \ (r - 2)^{-1}\|f\|_{L^2(\mathbb{R}^n)},
\]
with constants independent of \(s\). This was essentially established in Guo, Roos and Yung [GRY17], with minor changes detailed in Roos [Roo19, Section 7]. \(\square\)

Now we are ready to prove (3.17). We linearize the supremum and aim to prove
\[
\|L^s_{\lambda(x)}(D)f(x)\|_{L^2} \lesssim 2^{-\gamma s}\|f\|_{L^2},
\]
where \(\lambda : \mathbb{Z}^n \to (0, 1]\) is an arbitrary function. For each \(x \in \mathbb{Z}^n\), \(\alpha(x)\) is defined as the unique \(\alpha \in \mathcal{A}_s\) such that \(|\lambda(x) - \alpha| \leq 2^{-3s}\) (say), or as an arbitrary value from the complement of \(\mathcal{A}_s\) if no such \(\alpha\) exists (in this case, \(L^s_{\lambda(x)}(\xi) = 0\)). By definition, the term we need to bound in (3.23) can be written as
\[
\sum_{\beta \in \mathcal{B}_s(\alpha(x))} \int S(\alpha(x), \beta)\Phi^s_{\lambda(x) - \alpha(x)}(\xi - \beta)\hat{F}_\beta(\xi)e(\xi)d\xi,
\]
where
\[
\hat{F}_\beta(\xi) = \hat{f}(\xi)\check{\chi}_s(\xi - \beta),
\]
and \(\check{\chi}_s(\cdot) = \check{\chi}_0(2^{10s}\cdot)\) for some appropriately chosen compactly supported smooth bump function \(\check{\chi}_0\) with \(\check{\chi}_0\check{\chi}_0 = \chi_0\). We apply Theorem 1.1 with
\[
\Xi = \{b/q : b \in \mathbb{Z}^n, q \in \mathbb{Z} \cap [2^{s-1}, 2^s]\},
\]
\(t = \lambda, \phi = \mathcal{F}^{-1}_{\mathbb{Z}^n}(\chi_s), U\) the support of \(\chi_s\), and \(T_1 = \Phi^s_t(D) = \Phi^s_t(D)\), and any fixed \(q\), say, \(q = 3\). The hypothesis (1.3) with \(\eta = 1\) is then given by Lemma 3.2. In (1.4), we take
\[
f_\beta(y) = F_\beta(y)e(-\beta y),
\]
and
\[
g_\beta(x) = 1_{\beta \in \mathcal{B}_s(\alpha(x))} \cdot S(\alpha(x), \beta)e(\beta x).
\]
Since \(\Phi^s_1 = 0\) for all \(s\), the \(V^q\) norm on the left-hand side of (1.4) controls the supremum over \(\lambda\). We apply Theorem 1.1 and bound term (3.24) by
\[
s^2A_1\|f\|_{L^2},
\]
where \(A_1\) is the constant in (1.2) under the above choice of \(g_\beta\). It remains to prove that
\[
A_1 \lesssim 2^{-\gamma s}\text{ for some } \gamma > 0.
\]
To do so, we will apply Lemma 3.1.

Regarding the left hand side of (1.2), we apply a change of variable and write it as
\[
\left\|\sum_{\beta \in \Xi} \phi(y - x)g_\beta(y)c_\beta \right\|_{L^2_x} = \left\|\sum_{\beta \in \mathcal{B}_s(\alpha(y))} \phi(y - x)S(\alpha(y), \beta)e(\beta y)c_\beta \right\|_{L^2_x}.
\]
We write a linearization of the left hand side of (3.20) as
\[
\sum_{\beta \in \mathcal{B}_s(\alpha(y))} \int S(\alpha(y), \beta)\hat{F}_\beta(\xi)e(\xi \gamma)d\xi = \sum_{\beta \in \mathcal{B}_s(\alpha(y))} S(\alpha(y), \beta)\check{F}_\beta(y),
\]
where
\[
\hat{F}_\beta(\xi) = \check{\chi}_s(\xi - \beta)\hat{f}(\xi).
\]
In the end, we just need to pick
\[
\hat{f}(\xi) = \sum_{\beta \in \mathcal{B}_s^c} c_\beta \cdot \check{\chi}_s(\xi - \beta)e(x(\beta - \xi)),
\]

\[\]
The desired estimate (3.30) follows as
\[
\|f\|_{L^2} \sim |U|^{1/2} \|c_\beta\|_{H^2}.
\] (3.35)

This finishes the proof of (3.30), thus the proof of the desired estimate (3.23).

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