Chiral Anomaly on Fuzzy 2-Sphere

Hajime Aoki\textsuperscript{1}, Satoshi Iso\textsuperscript{2} and Keiichi Nagao\textsuperscript{3}

\textsuperscript{1}\textit{Department of Physics, Saga University, Saga 840-8502, Japan}
\textit{and}
\textsuperscript{2,3}\textit{High Energy Accelerator Research Organization (KEK)}
\textit{Tsukuba 305-0801, Japan}

Abstract

We investigate chiral anomaly for fermions in the fundamental representation on noncommutative (fuzzy) 2-sphere. In spite that this system is realized by finite dimensional matrices and no regularization is necessary for either UV or IR, we can reproduce the correct chiral anomaly which is consistent with the calculations done in flat noncommutative space. Like the flat case, there are ambiguities to define chiral currents. We define chiral currents in a gauge-invariant way and a gauge-covariant way, and show that the corresponding anomalous chiral Ward–Takahashi identities take different forms. The Ward–Takahashi identity for the gauge-invariant current contains explicit nonlocality while that for the covariant one is given by a local expression.
1 Introduction

Noncommutative field theory has attracted much interest recently since it was realized that noncommutative geometry appears naturally from string theory in $B_{\mu\nu}$ background\[1\]. Furthermore noncommutative geometry can appear as natural background space-time in matrix models\[2, 3, 4\] that have been proposed as a nonperturbative formulation of superstring theory. Various novel properties\[5\], such as gravity-like induced interactions between objects in noncommutative space, open Wilson lines, UV/IR duality, bi-locality and background independence, demonstrate that Yang-Mills theory in noncommutative space-time may be more appropriately interpreted as a stringy theory than an ordinary local field theory. Since noncommutative field theories can be formulated as matrix models, these novel properties support fundamentality of matrix models as a constructive formulation of superstring\[6\].

Anomalies of noncommutative gauge theories have also been studied extensively \[7]-[20\] and two related features peculiar to noncommutative field theories were found. They are ambiguities to define currents and IR singularity in non-planar diagrams. If the fermions are in the fundamental representation, we can define the following two different types of chiral currents in noncommutative flat space, corresponding to different definitions of local chiral transformations:

$$J_\mu,5(x) = -i\psi_\beta(x) \star \bar{\psi}_\alpha(x)(\gamma_\mu \gamma_5)^{\alpha\beta},$$
$$J'_\mu,5(x) = i\bar{\psi}_\alpha(x) \star \psi_\beta(x)(\gamma_\mu \gamma_5)^{\alpha\beta}.$$  \(1.1\)

Since the fermions transform as $\psi \to U\psi$ under gauge transformation, the first current transforms covariantly $J_\mu \to U J_{\mu,5} U^\dagger$ while the second one invariantly.

The covariant current is shown to satisfy a natural noncommutative generalization of the anomalous Ward–Takahashi(WT) identity (in d=4)

$$\langle D^\mu J_{\mu,5} \rangle \propto \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} \star F_{\lambda\sigma},$$  \(1.2\)
where an ordinary product is replaced by a noncommutative star product, and
\[ D_\mu M = \partial_\mu M - ig[a_\mu, M] \]
is the covariant derivative for adjoint representations. It can be calculated diagrammatically, by point splitting regularization, or by Fujikawa’s method in path integrals. Only planar diagrams contribute in the diagrammatic calculation to the anomaly in the r.h.s. Both sides in the above equation transform as adjoint representations under gauge transformation.

On the contrary, the gauge invariant current \( J'_\mu \) is invariant under gauge transformation and the l.h.s. of the WT identity should be \( \partial^{\mu} J'_\mu \), which is also gauge invariant. Since there are no local gauge invariant quantities constructed from gauge fields in noncommutative field theories, the r.h.s. cannot be expressed locally\(^\text{[10]}\). The authors in ref.\(^\text{[9]}\) obtained a nonlocal expression for the r.h.s. by calculating nonplanar diagrams. It vanishes if we fix the external momenta finite \((k\theta \neq 0)\) and set the regularization cutoff \( \Lambda \) infinity. This result was confirmed in refs.\(^\text{[11, 12]}\). On the other hand, if the external momenta is taken to zero before the cutoff \( \Lambda \) is taken to infinity, a finite anomaly term arises due to IR singularity. The final result can be written by using a generalized star product, \( \ast \)^\text{[9]}. 

As we mentioned at the beginning, a noncommutative field theory can be formulated as a matrix model and if we consider a compact space, the system can be formulated in terms of finite dimensional matrices. One of the simplest examples is a noncommutative 2-sphere (fuzzy 2-sphere). Since wave functions on fuzzy 2-sphere can be expanded in terms of noncommutative analogs of the spherical harmonics, which are constructed from a finite dimensional representation of \( SU(2) \) algebra, field theory on it is formulated as a matrix model of finite size.

In this paper, we consider fermions in the fundamental representation for gauge group on the above mentioned fuzzy 2-sphere and investigate their chiral properties, in particular, calculate chiral anomaly. Chiral anomaly on the fuzzy 2-sphere has been examined in papers \(^\text{[21, 22, 23]}\) from various different approaches. In ref.\(^\text{[21]}\), chiral anomaly was discussed as a Jacobian term of the
fermion measure under a chiral transformation. In ref. [22], it was discussed, along with topologically nontrivial field configurations, based on algebraic K-theory. The model considered in ref. [23] is similar to ours but their treatment is not completely satisfactory. In this paper, we make every step of calculations well-defined without introducing any regularizations or approximations, and clarify the distinction between the WT identities for a covariant and an invariant current. Our results are summarized in eq. (4.44) and eq. (4.60).

In the system we consider, everything is finite dimensional and no regularization is necessary. If both of the action and the measure were invariant under a noncommutative chiral transformation on fuzzy 2-sphere, we could not obtain the correct anomalous WT identity for chiral currents. However, as we show in this paper, a Dirac operator on fuzzy 2-sphere is no longer anti-commutable with a chirality operator at finite $N$ ($N$ is the dimension of matrices) and a careful treatment of the $1/N$ correction will lead to the correct form of anomalous WT identity for chiral currents. In this sense, the chiral anomaly arises in a similar way to the case of the Wilson fermion in the lattice gauge theory [24]. In the case of Wilson fermion, the Wilson term which is introduced in the action to remove doublers violates chiral invariance of the action. On the contrary, a natural Dirac operator on fuzzy 2-sphere does not have the doubling problem and we do not need to add an extra term in the action to remove doublers. Nevertheless, an anti-commutator of the Dirac operator and a natural generalization of the chirality operator acquires an $1/N$ correction and this leads to the correct chiral anomaly. We know the no-go theorem about chiral-invariant Dirac operators in lattice gauge theories [25]. We hope to have some analogous no-go theorem in theories on noncommutative space as well.

Like the flat noncommutative space, there are ambiguities for definitions of chiral currents, depending on definitions of chiral transformations. We define a gauge-invariant current and a gauge-covariant current. The covariant current is shown to satisfy a local anomalous WT identity (4.60) while the invariant current
satisfies nonlocal one (4.44). These results are consistent with the previous results on anomaly in flat space.

The organization of the paper is as follows. In section 2, we explain briefly how to formulate noncommutative gauge theory on fuzzy 2-sphere as a matrix model of finite size. We also show the spectrum of free Dirac operator and its eigenfunctions. In section 3, we introduce two different limits, a commutative limit and a flat limit. Section 4 is the main part of the paper. There we define two different types of chiral transformations and calculate anomalous WT identities for the corresponding chiral currents. We calculate anomaly for the gauge invariant current in subsection 4.1, and then for the gauge covariant current in subsection 4.2. Using various useful identities in appendices, we can calculate anomalies for both cases. Their local forms look very different. We also consider a commutative and flat limit. Section 5 is devoted to discussions. In appendices 6.1 and 6.2, we summarize various useful identities which are used in the calculation of the WT identity. In appendix 6.3, we review the calculation of anomaly for the theory on the commutative 2-sphere in order to compare with the noncommutative result. In appendix 6.4 some detailed calculations are shown.

2 Dirac Operator on Fuzzy 2-Sphere

2.1 Matrix Construction of Fuzzy 2-Sphere

In order to define noncommutative geometry, Connes proposed to generalize ordinary commutative wave functions to noncommutative ones, instead of making the geometry itself noncommutative. In the case of noncommutative (fuzzy) 2-sphere, we first make the coordinates noncommutative by using a \((2L + 1)\)-dimensional representation of the angular momentum operators \(L_i\) which satisfy 
\[
[L_i, L_j] = i\epsilon_{ij} L_k \quad \text{and} \quad (L_i)^2 = L(L + 1).
\]
We then introduce noncommutative
coordinates on fuzzy 2-sphere by

\[ x_i = \alpha L_i. \]  

(2.1)

They are noncommutative as

\[ [x_i, x_j] = i\alpha \epsilon_{ijk} x_k, \]  

(2.2)

and form a sphere with a radius \( \rho \) as,

\[ (x_i)^2 = \alpha^2 L (L + 1) = \rho^2. \]  

(2.3)

Thus, \( \alpha \) gives the noncommutative scale on fuzzy 2-sphere, and the radius is related to \( \alpha \) and \( L \) by the relation \( \rho = \sqrt{\alpha^2 L (L + 1)} \).

Wave functions on fuzzy 2-sphere can be expanded in terms of noncommutative analogs of the spherical harmonics which can be constructed from the above noncommutative coordinates. Similarly to the commutative spherical harmonics, products of \( x_i \) can be decomposed into irreducible representations of \( SO(3) \). They are traceless symmetric products of \( x_i \). Since the noncommutative coordinates \( x_i \) are \( (2L + 1) \)-dimensional matrices, the noncommutative spherical functions are also matrices of the same size. In the fuzzy sphere, there is an upper bound for the angular momentum \( l \) of the noncommutative spherical harmonics \( \hat{Y}_{l,m} \); \( l \leq 2L \). Any hermitian matrix \( M \) can be expanded in terms of \( \hat{Y}_{l,m} \) as

\[ M = \sum_{l=0}^{2L} \sum_{m=-l}^{l} m_{l,m} \hat{Y}_{l,m}. \]  

(2.4)

The total number of the basis wave functions is \( \sum_{l=0}^{2L} (2l + 1) = (2L + 1)^2 \) and gives the number of independent basis of \((2L+1)\)-dimensional hermitian matrices. Some basic properties of \( \hat{Y}_{l,m} \) are summarized in appendix 6.1. In the following, we omit the hat from \( \hat{Y}_{l,m} \) unless there is any confusion with the commutative spherical harmonics.

A noncommutative field theory on the fuzzy sphere can be formulated as a matrix model by expanding fields in terms of the noncommutative spherical
harmonics as above. A hermitian matrix $M$ is mapped to an ordinary function on 2-sphere with the same coefficient $m_{l,m}$ as

$$M \leftrightarrow M(\Omega) = \sum_{l=0}^{2L} \sum_{m=-l}^{l} m_{l,m} Y_{l,m}(\Omega),$$

(2.5)

where $Y_{l,m}(\Omega)$ are ordinary spherical harmonics. Due to the noncommutativity of the coordinates, a product of two matrices is mapped to the so-called star-product of functions on 2-sphere. Derivatives on the fuzzy 2-sphere are expressed by taking a commutator with the $SU(2)$ generator

$$\mathcal{L}_i M = [L_i, M] = (L_i^L - L_i^R) M \leftrightarrow \tilde{\mathcal{L}}_i M(\Omega) = -i\epsilon_{ijk} x_j \partial_k M(\Omega).$$

(2.6)

Here we have introduced a notation $L_i^L$ and $L_i^R$. The superscript $L$ or $R$ means that the operator acts on matrices by left or right multiplication. The superscript $L$ is often omitted in the following when there is no confusion. An integral over 2-sphere is replaced by taking trace over matrices

$$\frac{1}{2L+1} \text{Tr} \leftrightarrow \int \frac{d\Omega}{4\pi} \equiv \int_\Omega.$$

(2.7)

More detailed relation between noncommutative field theories on the fuzzy 2-sphere and matrix models is given in ref.[26].

### 2.2 Dirac Operator

An action for Dirac fermion on fuzzy 2-sphere is given by

$$S_{S_2} = \frac{\alpha}{2g^2} \text{Tr}(\bar{\psi}D\psi),$$

$$D = \sigma_i (\mathcal{L}_i + \rho a_i) + 1,$$

(2.8)\hspace{1cm} (2.9)

where $g$ is a coupling constant. The spinor field $\psi$ and the gauge field $a_i$ are $(2L + 1) \times (2L + 1)$ Hermitian matrices. Dirac fermions on the fuzzy 2-sphere were investigated in refs.[27, 28].
This action (2.8) is invariant under the following gauge transformation:

\begin{align}
\psi & \to U\psi, \\
\bar{\psi} & \to \bar{\psi}U^\dagger, \\
a_i & \to Ua_iU^\dagger + \frac{1}{\rho}(UL_iU^\dagger - L_i).
\end{align}

The last transformation is obtained from a requirement that the combination

\[ A_i = \alpha(L_i + \rho a_i) \]  

transforms covariantly under gauge transformation as \( A_i \to UA_iU^\dagger \). The fermion \( \psi \) transforms as the fundamental representation. The covariant derivative for \( \psi \) is given by

\[ A'_i\psi = \alpha(L_i + \rho a_i)\psi. \]  

It is straightforward to see that \( A'_i\psi \) transforms correctly as

\[ A'_i\psi \to UA'_i\psi. \]  

By noting that \( \sigma_iL_i = J_i^2 - \mathcal{L}_i^2 - \frac{3}{4} \) where \( J_i = \mathcal{L}_i + \frac{\sigma_i}{2} \), eigenvalues for the free Dirac operator can be easily obtained:

\begin{align}
D_0 &= (\sigma_i\mathcal{L}_i + 1) \\
&= j(j+1) - l(l+1) + \frac{1}{4} \\
&= \begin{cases} 
  l + 1 = j + \frac{1}{2} & \text{for } j = l + \frac{1}{2} \\
  -l = -(j + \frac{1}{2}) & \text{for } j = l - \frac{1}{2}
\end{cases}
\end{align}

No doubler modes exist in the spectrum. Various properties of the Dirac operator on the fuzzy 2-sphere are discussed in ref. [29].
The eigenfunctions are given by spinorial-spherical-harmonics:

\[
Y_{l+\frac{1}{2}, m} = |j = l + \frac{1}{2}, j_z = m\rangle = \sqrt{\frac{l + \frac{1}{2} + m}{2l + 1}} Y_{l,m-\frac{1}{2}} \otimes |\uparrow\rangle + \sqrt{\frac{l + \frac{1}{2} - m}{2l + 1}} Y_{l,m+\frac{1}{2}} \otimes |\downarrow\rangle, \quad (2.19)
\]

\[
Y'_{l-\frac{1}{2}, m} = |j = l - \frac{1}{2}, j_z = m\rangle = \sqrt{\frac{l + \frac{1}{2} - m}{2l + 1}} Y_{l,m-\frac{1}{2}} \otimes |\uparrow\rangle - \sqrt{\frac{l + \frac{1}{2} + m}{2l + 1}} Y_{l,m+\frac{1}{2}} \otimes |\downarrow\rangle, \quad (2.20)
\]

which satisfy

\[
D_0 Y_{l+\frac{1}{2}, m} = (l + 1) Y_{l+\frac{1}{2}, m},
\]

\[
D_0 Y'_{l-\frac{1}{2}, m} = -l Y'_{l-\frac{1}{2}, m}. \quad (2.22)
\]

Angular momentum \( l \) takes values \( l = 0, 1, \ldots, 2L \) for \( Y_{l+\frac{1}{2}, m} \) and \( l = 1, \ldots, 2L \) for \( Y'_{l-\frac{1}{2}, m} \). Hence the eigenfunctions of \( D_0 \) are paired between \( Y_{l+\frac{1}{2}, m} \) and \( Y'_{l+\frac{1}{2}, m} \) with positive and negative eigenvalues except for \( Y_{2L+\frac{1}{2}, m} \).

When we calculate anomalous WT identities for chiral currents, we need to evaluate the following type of expectation values in the free Dirac action \( S_0 \):

\[
\langle O \rangle_{S_0} = \frac{1}{Z_{S_0}} \int d\psi d\bar{\psi} O e^{-S_0}, \quad (2.23)
\]

\[
Z_{S_0} = \int d\psi d\bar{\psi} e^{-S_0}, \quad S_0 = \frac{\alpha}{2g^2} Tr(\bar{\psi} D_0 \psi). \quad (2.24)
\]

They can be calculated by expanding the fields \( \psi \) and \( \bar{\psi} \) in terms of \( Y_{l+\frac{1}{2}, m} \) and \( Y'_{l-\frac{1}{2}, m} \), and then using the Wick’s theorem. Expectation values for typical \( O \)'s are given in appendix 6.2.

### 3 Commutative Limit and Flat Limit

In this section, we will consider two different limits of the action (2.8), a commutative limit and a flat limit, and obtain actions for Dirac fermions on a commutative
Then we will review chiral anomalies in these theory. In the next section we will calculate the chiral anomaly on the fuzzy 2-sphere, take these two limits, and then compare with the results of chiral anomalies that will have been reviewed in this section. The calculation of the chiral anomaly in the next section can be done without introducing any kind of regularization and it is a nontrivial check whether they agree or not.

3.1 Commutative 2-Sphere

For the commutative limit, we take the noncommutative parameter $\alpha \to 0$, and the size of matrices $L \to \infty$, with the radius $\rho$ fixed. The action (2.8) can be mapped to a noncommutative field theory on the 2-sphere by using the mapping rules, (2.6) and (2.7). Products are mapped to noncommutative star products. In the commutative limit, this product can be approximated by an ordinary product and we obtain an action for the Dirac fermion on the commutative 2-sphere with the radius $\rho$:

$$S_{S^2} = \frac{\rho}{g^2} \int_{\Omega} \bar{\psi} D \psi, \quad (3.1)$$

$$D = D_0 + \rho \sigma_i a_i, \quad (3.2)$$

$$D_0 = \sigma_i \tilde{L}_i + 1. \quad (3.3)$$

Here, the 2-sphere is embeded in a three-dimensional space, and the Dirac operator can be rewritten as

$$D = \rho[\sigma'_i(i\partial_i + a'_i) + \phi \gamma_3] + 1, \quad (3.4)$$

where we have redefined new $\sigma$ matrices and new gauge fields for later convenience:

$$a'_i = \frac{1}{\rho} \epsilon_{ijk} x_j a_k, \quad (3.5)$$

$$\sigma'_i = \frac{1}{\rho} \epsilon_{ijk} x_j \sigma_k. \quad (3.6)$$
They are tangential components of \( a_i \) and \( \sigma_i \). Similarly the normal components are given by

\[
\phi = \frac{1}{\rho} x_i a_i, \quad \gamma_3 = \frac{1}{\rho} x_i \sigma_i.
\] (3.7) (3.8)

They are a scalar field and a chirality operator on 2-sphere, respectively. Hence the action (3.1) contains a scalar field \( \phi \) as the normal component of \( a_i \).

We can further take a flat limit of (3.1), by considering the vicinity of the north pole on the 2-sphere, taking \( \rho \to \infty \), and mapping \( \rho^2 \int_{\Omega} \to \int \frac{d^2x}{4\pi} \). Then we obtain

\[
S_{R^2} = \frac{1}{4\pi g^2} \int d^2x \bar{\psi} \left[ \sigma''_i (i\partial_i + a''_i) + \phi \sigma_3 \right] \psi,
\] (3.9)

with

\[
\sigma''_i = -\epsilon_{ij} \sigma_j, \quad a''_i = -\epsilon_{ij} a_j,
\] (3.10) (3.11)

where \( i,j = 1,2 \). This is nothing but an action for Dirac fermions on a 2-dimensional commutative flat-space with a Yukawa coupling to a scalar field.

A chiral transformation is defined on the commutative 2-sphere as

\[
\delta \psi = \lambda \gamma_3 \psi, \quad \delta \bar{\psi} = \lambda \bar{\psi} \gamma_3.
\] (3.12) (3.13)

where the chirality operator of (3.8) satisfies

\[
(\gamma_3)^\dagger = \gamma_3, \quad (\gamma_3)^2 = 1, \quad \{ D, \gamma_3 \} = 2\rho \phi.
\] (3.14) (3.15) (3.16)

Anomalous chiral WT identity for the theory on the commutative 2-sphere has been calculated in ref. [31]. Since now we consider the theory with the Yukawa
coupling to the scalar field $\phi$, we show the calculation for the chiral WT identity in the appendix 6.3. The result is

$$\tilde{\mathcal{L}}_i (\bar{\psi} \sigma_i \gamma_3 \psi) - 2 \rho \phi \bar{\psi} \psi = \frac{g^2}{\rho} (-4i\epsilon_{ijk} x_k (\mathcal{L}_i a_j) - 8 \rho \phi). \quad (3.17)$$

This can be rewritten as a more familiar form

$$i \partial_i (\bar{\psi} \sigma_i' \gamma_3 \psi) - 2 \phi \bar{\psi} \psi = \frac{2g^2}{\rho} \epsilon_{ijk} x_i (\partial_j a_k' - \partial_k a_j'), \quad (3.18)$$

by separating $a_i$ and $\sigma_i$ into tangential and normal components as in eqs. (3.5), (3.6), (3.7), (3.8).

Taking the flat limit further, we obtain

$$i \partial_i (\bar{\psi} \sigma_i'' \gamma_3 \psi) - 2 \phi \bar{\psi} \psi = 2g^2 \epsilon_{ij} (\partial_i a_j'' - \partial_j a_i''), \quad (3.19)$$

which agrees with the WT identity in 2-dimensional flat-space. Note that in a usual convention the r.h.s. is divided by $4\pi$. This can be given by multiplying the action (3.1), and then (3.9), by $4\pi$.

### 3.2 Noncommutative Flat Space

Now we take the flat limit from the fuzzy 2-sphere with the noncommutativity fixed. This can be done by considering the vicinity of the north pole on the fuzzy 2-sphere, and taking the radius $\rho \to \infty$, the system size $L \to \infty$, with the noncommutative parameter $\theta = \alpha \rho$ fixed. In this limit, a matrix $M$ is mapped to a function on the flat space and $\text{Tr}$ becomes an integral over this flat space:

$$\frac{1}{2L+1} \text{Tr} \to \frac{1}{4\pi \rho^2} \int dx^2. \quad (3.20)$$

In the vicinity of the north pole, $x_3$ can be replaced by $\rho$ and a commutator between $x_i \ (i = 1, 2)$ becomes $[x_i, x_j] = i \theta \epsilon_{ij}$. Hence

$$[L_i, M] = \frac{1}{\alpha} [x_i, M]$$

$$\to i \theta \epsilon_{ij} \partial_j M(x) = i \rho \epsilon_{ij} \partial_j M(x). \quad (3.21)$$
Then, from the action on the fuzzy 2-sphere (2.8), we obtain an action for the Dirac fermion on the noncommutative 2-dimensional flat-space:

\[
S_{R_{NC}^2} = \frac{1}{4\pi g^2} \int d^2x \left[ \bar{\psi} \left( \sigma''_i (i\partial_i + a''_i) + \sigma_3 \phi \right) \psi \right]_*,
\]  

(3.22)

where \( \sigma''_i \) and \( a''_i \) are defined in eq. (3.10), (3.11), and \([\cdots]_*\) means that any product in the bracket is considered as a star product.

By taking \( \theta \to 0 \), we can obtain the commutative limit of (3.22), which again becomes (3.9). In the previous subsection, we first took the commutative limit, \( \alpha \to 0 \) with \( \rho \) fixed, and then took the flat limit \( \rho \to \infty \), while here we first took the flat limit, \( \rho \to \infty \) with \( \theta = \alpha \rho \) fixed, and then the commutative limit, \( \theta \to 0 \). We arrived at the same classical action irrespective of the ordering of taking limits, though there may be some subtle dynamics quantum mechanically.

As we have reviewed in the introduction, chiral anomaly on noncommutative flat-space has been investigated in a number of papers [7]-[20].

4 Chiral Anomaly on Fuzzy 2-Sphere

In this section, we define chiral transformations and calculate the chiral anomaly for fermions in the fundamental representation on the fuzzy 2-sphere (2.8). If we keep the ordering of \( \bar{\psi} \) and \( \psi \) in (3.17), the l.h.s is gauge invariant and it will be difficult to write down its noncommutative analog since we cannot make gauge invariant local operators on noncommutative space from gauge fields. We have to take trace over matrices to make an operator gauge invariant. Hence, the WT identity with a gauge-invariant current cannot be written down unless we introduce explicit nonlocality. On the other hand, local WT identity can be written down if we use a gauge-covariant current, instead. They have been discussed fully in the case of flat noncommutative space as reviewed in the introduction. In the following, we will consider two kinds of chiral transformations and the corresponding chiral currents: in a gauge invariant way and a covariant way.
4.1 Gauge-Invariant Current

We define a local chiral transformation as

\[ \delta \psi = \frac{1}{2L+1}(\sigma_i \psi \{\lambda, L_i\} - \psi \lambda), \quad (4.23) \]

\[ \delta \bar{\psi} = \frac{1}{2L+1}(\{\lambda, L_i\} \bar{\psi} \sigma_i - \lambda \bar{\psi}), \quad (4.24) \]

where the transformation parameter \( \lambda \) is also a \((2L + 1) \times (2L + 1)\) matrix and anti-hermitian. This chiral transformation reduces to the ordinary one (3.12) in the commutative limit. Furthermore, a chiral operator

\[ \Gamma^R = \frac{1}{2L+1}(2\sigma_i L_i^R - 1), \quad (4.25) \]

satisfies

\[ (\Gamma^R)^2 = 1, \quad (4.26) \]

\[ (\Gamma^R)^\dagger = \Gamma^R. \quad (4.27) \]

at finite \( L \). Various properties of this chiral operator are discussed in ref. [29].

The chiral operator (4.25) reduces to the ordinary chiral operator (3.8) in the commutative limit. An anticommutator with the Dirac operator (2.9) becomes

\[ \{\Gamma^R, D\} = \frac{1}{L+1/2}(2(L_i + \rho a_i) L_i^R - D). \quad (4.28) \]

Here \( L_i L_i^R \) and \( D/(L + 1/2) \) vanish in the commutative limit since they are of order \( 1/L \). Hence the anti-commutator becomes proportional to the scalar field as expected:

\[ \{\Gamma^R, D\} \to 2a_i x_i = 2\rho \phi. \quad (4.29) \]

The r.h.s. of the anticommutator (4.28) has correction terms of order \( 1/L \). These terms lead to the correct anomaly as we show in the following.

The chiral transformation (4.23) is compatible with the gauge transformation if \( \lambda \) is a gauge invariant parameter. Namely, \( \delta \psi \) also transforms as the fundamental representation \( \delta \psi \to U \delta \psi \). This is because \( \lambda \) and \( L_i \) are placed in the right
of $\psi$ in (4.23). Since the chiral transformation parameter $\lambda$ is invariant under gauge transformation, the chiral current associated with this transformation is also gauge invariant.

### 4.1.1 Evaluation of the WT Identity

Now we will evaluate the WT identity associated with the chiral transformation (4.23). Under the chiral transformation, the variation of the action (2.8) can be obtained by using the anticommutator (4.28) as

$$\delta S_{\bar{\psi}} = \frac{\alpha}{g^2(2L+1)} \text{Tr} \left[ \frac{1}{2} [L_i, \lambda] \left( \{L_j, \bar{\psi} \sigma_i \sigma_j \psi \} - \bar{\psi} \sigma_i \psi \right) ight]$$

$$+ \{\lambda, L_i\} \left( \bar{\psi} [L_i, \psi] + \rho \bar{\psi} a_i \psi \right) - \lambda \bar{\psi} D \psi.$$

(4.30)

The integration measure also varies as $d\psi' d\bar{\psi}' = J d\psi d\bar{\psi}$ with the Jacobian,

$$J = \left| \begin{array}{ccc} \frac{\partial \psi'}{\partial \psi} & \frac{\partial \psi'}{\partial \bar{\psi}} \\ \frac{\partial \bar{\psi}'}{\partial \psi} & \frac{\partial \bar{\psi}'}{\partial \bar{\psi}} \end{array} \right|^{-1} = 1 + 4 \text{Tr} \lambda + \mathcal{O}(\lambda^2).$$

(4.31)

By combining both variations under the chiral transformation, we have the anomalous chiral WT identity:

$$\langle \delta S_{\bar{\psi}} \rangle_S - 4 \text{Tr} \lambda = 0.$$

(4.32)

The contribution from the Jacobian and the last term in (4.30) are cancelled by the relation

$$\frac{\alpha}{g^2(2L+1)} \langle \text{Tr} \left[ \lambda \bar{\psi} D \psi \right] \rangle_S = -4 \text{Tr} \lambda.$$

(4.33)

This can be easily proved by formally diagonalizing the operator $D$.

---

1If we define chiral transformation for $\psi$ without the second term in (4.23), either of the last term in (4.30) or the contribution from the Jacobian do not appear from the beginning. This chiral transformation also agrees with the commutative one in the commutative limit, and there is no reason to exclude this simpler transformation. But the resulting WT identity becomes the same except for a slight change (of order $1/L$) of the definition of the chiral current. This property is desirable since ambiguities in making noncommutative systems from a commutative one do not affect universal structures of noncommutative systems such as anomaly.
The first term in (4.30) proportional to \([L_i, \lambda]\) gives the divergence of the chiral current in the WT identity. The second term can be rewritten as

\[
\text{Tr} \left( \{\lambda, L_i\} \left( \bar{\psi}[L_i, \psi] + \rho \bar{\psi} a_i \psi \right) \right) = \text{Tr} \left( [\lambda, L_i] \left( \bar{\psi}[L_i, \psi] + \rho \bar{\psi} a_i \psi \right) + 2L_i \lambda \bar{\psi}[L_i, \psi] + 2\rho \lambda \bar{\psi} a_i \psi L_i \right).
\]

(4.34)

The first term in (4.34) is absorbed in the definition of the chiral current. In the following analysis, we consider terms up to the first order in the gauge fields \(a_i\).

We make use of the identity (see eq. (6.24))

\[
\langle \text{Tr} (\lambda^2 \bar{\psi} a_i \psi [\psi, L_i]) \rangle_{S_0} = 0,
\]

(4.35)

and define the scalar field,

\[
\phi = \frac{\alpha}{2\rho} \left[ \{a_i, L_i\} + \rho a_i^2 \right] = \frac{\alpha}{2\rho^2} \left[ (L_i + \rho a_i)^2 - (L_i)^2 \right],
\]

(4.36)

which is a normal component of \(a_i\), and transforms covariantly under the gauge transformation. Then, we can write down the WT identity as

\[
\langle \text{Tr} \left( \frac{\lambda}{2} \left[ L_i, \{L_j, \bar{\psi} \sigma_i \sigma_j \psi \} - \bar{\psi} \sigma_i \psi - 2\bar{\psi}[L_i, \psi] - 2\rho \bar{\psi} a_i \psi \right] \right) - \frac{2\rho^2}{\alpha} \text{Tr} (\lambda \bar{\psi} \phi \psi) \rangle_{S} = -\rho \langle \text{Tr} \bar{\psi}[L_i, a_i] \psi \rangle_{S_0} - \langle \text{Tr} \left( 2L_i \lambda \bar{\psi}[L_i, \psi] \right) \text{Tr} \left( \frac{\alpha \rho}{2g^2} \bar{\psi} \sigma_i a_i \psi \right) \rangle_{S_0} + O((a_i)^2).
\]

(4.37)

The last term in (4.37) came from the first-order perturbative expansion with respect to the gauge field in the action (2.8).

Now we evaluate the last term in eq. (4.37). Here we assume that the background gauge field has only the third component \(a_3\). We can recover the complete
result afterward by using $SO(3)$ invariance. By using the formula (5.28), we have

$$-\langle \text{Tr} \left( 2\vec{\psi} [L_i, \psi] L_i \lambda \right) \text{Tr} \left( \frac{\alpha \rho}{2g^2} \bar{\psi} \sigma_3 a_3 \psi \right) \rangle_{S_0}$$

$$= -\frac{2g^2 \rho}{\alpha (2L+1)^2} 2L \sum_{l=1}^{2L} \sum_{l'=1}^{2L} \sum_{m=-l+\frac{1}{2}}^{l-\frac{1}{2}} \sum_{m'=-l'+\frac{1}{2}}^{l'-\frac{1}{2}} \frac{1}{l(l+1)} \sqrt{(l+\frac{1}{2})^2 - m^2} \sqrt{(l'+\frac{1}{2})^2 - m'^2}
$$

$$\times \left[ \text{Tr} \left( Y_{l,m+\frac{1}{2}}^+ Y_{l',m'\frac{1}{2}}^- \lambda \right) \text{Tr} \left( Y_{l,m-\frac{1}{2}}^+ a_3 Y_{l,m'-\frac{1}{2}}^- \right) - \text{Tr} \left( Y_{l,m-\frac{1}{2}}^+ Y_{l',m'\frac{1}{2}}^- \lambda \right) \right] \text{Tr} \left( Y_{l,m}^+ a_3 \right)$$

$$-\frac{4g^2 \rho}{\alpha (2L+1)^2} \sum_{l=1}^{2L} \sum_{m=-l}^{l} m \text{Tr} (\lambda Y_{lm}) \text{Tr} (Y_{lm}^+ a_3), \quad (4.38)$$

The last term of (4.38) is simplified by using the completeness of the spherical harmonics (6.9) as

$$-\frac{4g^2 \rho}{\alpha (2L+1)^2} \sum_{l=1}^{2L} \sum_{m=-l}^{l} \text{Tr} (\lambda Y_{lm}) \text{Tr} (Y_{lm}^+ a_3) = -\frac{4g^2 \rho}{\alpha (2L+1)} \text{Tr} (\lambda [L_3, a_3]), \quad (4.39)$$

From $SO(3)$ invariance, it gives

$$-\frac{4g^2 \rho}{\alpha (2L+1)} \text{Tr} (\lambda [L_i, a_i]), \quad (4.40)$$

which exactly cancels with the first term of the r.h.s. in eq. (4.37). Hence the first term of (4.38) gives the r.h.s. of the WT identity (4.37).

The first term of (4.38), which we call $H_3$, can be evaluated as

$$H_3 = \frac{2g^2 \rho}{\alpha (2L+1)^2} \sum_{l=1}^{2L} \sum_{l'=1}^{2L} \sum_{m=-l+\frac{1}{2}}^{l-\frac{1}{2}} \sum_{m'=-l'+\frac{1}{2}}^{l'-\frac{1}{2}} \frac{1}{l(l+1)} \sqrt{(l+\frac{1}{2})^2 - m^2} \sqrt{(l'+\frac{1}{2})^2 - m'^2}
$$

$$\times \left[ \text{Tr} \left( Y_{l,m+\frac{1}{2}}^+ Y_{l',m'\frac{1}{2}}^- \lambda \right) \text{Tr} \left( [L_+, Y_{l,m+\frac{1}{2}}^+] a_3 [L_-, Y_{l,m+\frac{1}{2}}^-] \right) \right]
$$

$$-\text{Tr} \left( Y_{l,m-\frac{1}{2}}^+ Y_{l',m'-\frac{1}{2}}^- \lambda \right) \text{Tr} \left( [L_-, Y_{l,m-\frac{1}{2}}^+] a_3 [L_+, Y_{l,m-\frac{1}{2}}^-] \right)$$

$$= \frac{2g^2 \rho}{\alpha (2L+1)^2} \sum_{l=1}^{2L} \sum_{m=-l}^{l} \frac{1}{l(l+1)} \text{Tr} \left( [L_+, \lambda Y_{l,m}^+] a_3 [L_-, Y_{l,m}] \right) - \text{Tr} \left( [L_-, \lambda Y_{l,m}^+] a_3 [L_+, Y_{l,m}] \right). \quad (4.41)$$
Here we have used (6.10), (6.17), and then (6.9). Using the identity (6.18), and then (6.11), this term becomes

\[ H_3 = \frac{2g^2\rho}{\alpha(2L + 1)^2L(L + 1)} \sum_{l=1}^{2L} \sum_{m=-l}^{l} \sum_{n=0}^{\infty} \frac{1}{\{L(L + 1)\}^n} \times \left[ \text{Tr} \left( [L_+, \lambda Y^+_l]^a_3 \left[ L_-, (L^L_i L^R_i)^n Y^+_l \right] \right) - \text{Tr} \left( [L_-, \lambda Y^+_l]^a_3 \left[ L_+, (L^L_i L^R_i)^n Y^+_l \right] \right) \right] \]

\[ = \frac{g^2\rho}{\alpha} \sum_{n=0}^{\infty} \frac{1}{\{L(L + 1)\}^{n+1}} \left[ \text{Tr} (L_{i_n} \cdots L_{i_1} L_+ \lambda) \text{Tr} (a_3 L_- L_{i_1} \cdots L_{i_n}) - \text{Tr} (L_{i_n} \cdots L_{i_1} L_- \lambda) \text{Tr} (a_3 L_- L_{i_1} \cdots L_{i_n}) \right. \]

\[ + \text{Tr} (L_{i_n} \cdots L_{i_1} L_+ \lambda) \text{Tr} (L_+ a_3 L_{i_1} \cdots L_{i_n}) - [L_+ \leftrightarrow L_-]. \] (4.42)

By SO(3) invariance, the term in general background gauge configurations can be determined as

\[ H = \frac{-2i\epsilon_{ijk}g^2\rho}{\alpha} \sum_{n=0}^{\infty} \frac{1}{\{L(L + 1)\}^{n+1}} \times \left[ \text{Tr} (L_{i_n} \cdots L_{i_1} L_k \lambda) \text{Tr} (a_k L_j L_{i_1} \cdots L_{i_n}) - \text{Tr} (L_{i_n} \cdots L_{i_1} L_j L_k \lambda) \text{Tr} (a_k L_{i_1} \cdots L_{i_n}) \right. \]

\[ - \text{Tr} (L_{i_n} \cdots L_{i_1} \lambda) \text{Tr} (L_i a_k L_j L_{i_1} \cdots L_{i_n}) + \text{Tr} (L_{i_n} \cdots L_{i_1} \lambda) \text{Tr} (L_i a_k L_{i_1} \cdots L_{i_n}) \right] \]

\[ = \frac{-2ig^2\rho}{\alpha} \sum_{n=0}^{\infty} \frac{1}{\{L(L + 1)\}^{n+1}} \times \left[ \text{Tr} (L_{i_n} \cdots L_{i_1} \lambda) [\epsilon_{ijk} \text{Tr} (L_i [L_j, a_k] L_{i_1} \cdots L_{i_n}) - i\text{Tr} (a_i a_L L_{i_1} \cdots L_{i_n})] \right. \]

\[ - \text{Tr} (L_{i_n} \cdots L_{i_1} \lambda) [\epsilon_{ijk} \text{Tr} ([L_j, a_k] L_{i_1} \cdots L_{i_n}) - i\text{Tr} (a_i L_{i_1} \cdots L_{i_n})]. \] (4.43)

Finally, the WT identity (4.37) for the gauge invariant chiral current becomes

\[ \langle \text{Tr} \left( \frac{\lambda}{2} [L_i, J^5_i] \right) - \frac{2\rho^2}{\alpha} \text{Tr} (\lambda \bar{\psi} \phi \psi) \rangle_s = H, \] (4.44)

where the chiral current is given by

\[ J^5_i = \{ L_j, \bar{\psi} \sigma_j \psi \} - \bar{\psi} \sigma_i \psi - 2\bar{\psi} [L_i, \psi] - 2\rho \bar{\psi} a_i \psi. \] (4.45)

The r.h.s. of the WT identity (4.44), \( H \), contains, in addition to an anomaly term, an extra term which makes the scalar term in l.h.s. of (4.44) normal ordered.
We will see this in the next subsection. Note that we have considered terms up to the first order in the gauge fields, and there are higher order terms which are neglected in the above WT identity.

Generic property of the WT identity for the gauge invariant current is that the chiral transformation parameter \( \lambda \) and the gauge field \( a_i \) are inserted in different traces in (4.43). This means that we cannot write down explicit local expressions for the WT identity because the gauge field \( a_i \) is always in a trace, namely, in an integral over the sphere. This is consistent with a general argument that there is no local gauge invariant quantity constructed from gauge fields on noncommutative space.

### 4.1.2 \( \lambda = 1 \) Case

In order to see that \( H \) contains an extra term (vev of the scalar term) mentioned above, we evaluate \( H \) for a special case \( \lambda = 1 \). We again set the background gauge configuration \( a_i = a_3 \delta_i^3 \). Then \( H_3 \) becomes

\[
H_3|_{\lambda=1} = -\frac{2g^2\rho}{\alpha(2L+1)^2} \sum_{l=1}^{2L} \sum_{m=-l}^{l+\frac{1}{2}} \frac{1}{l(l+1)} \left\{ \left( l + \frac{1}{2} \right)^2 - m^2 \right\} \times \left[ \text{Tr}(Y_{l,m-\frac{1}{2}}^\dagger a_3 Y_{l,m-\frac{1}{2}}) - \text{Tr}(Y_{l,m+\frac{1}{2}}^\dagger a_3 Y_{l,m+\frac{1}{2}}) \right]
\]

\[
= \frac{4g^2\rho}{\alpha(2L+1)} \sum_{l=1}^{2L} \sum_{m=-l}^{l} \frac{m}{l(l+1)} \text{Tr}(Y_{lm}^\dagger a_3 Y_{lm}). \tag{4.46}
\]

The summation over \( l, m \) in this equation can be evaluated by using the identity (6.18) and the completeness of the spherical harmonics (6.10). We refer the detailed calculation to appendix 6.4. We finally obtain

\[
H_3|_{\lambda=1} = \frac{8g^2\rho}{\alpha(2L+1)} \text{Tr}(a_3 L_3). \tag{4.47}
\]

From \( SO(3) \) invariance, a general form is given by

\[
H|_{\lambda=1} = \frac{8g^2\rho}{\alpha(2L+1)} \text{Tr}(a_i L_i). \tag{4.48}
\]
Because of the relation
\[-\frac{2\rho^2}{\alpha} \langle \text{Tr} \bar{\psi} \phi \psi \rangle_{S_0} = \frac{8g^2\rho}{\alpha(2L + 1)} \text{Tr}(a_3 L_3), \tag{4.49}\]
(4.48) becomes a vev of the scalar term in the l.h.s. of WT identity (4.37), and can be interpreted as a normal ordering constant for it.

4.1.3 Flat Limit

The flat noncommutative limit corresponds to considering a vicinity of the north pole. More precisely, this means that the support of the background gauge field $a_i$ and the chiral transformation parameter $\lambda$ are localizes around the north pole, and $L_i$ in the traces including these fields can be replaced by $L_3$ in the leading order. Hence the flat limit of the anomaly term (4.43) vanishes because the second line and the third line of (4.43) are cancelled each other. On the contrary, in an almost constant chiral transformation where $\lambda$ is close to 1, $L_i$ in the trace in which $\lambda$ is inserted cannot be replaced with $L_3$ and the cancellation does not occur. These results are consistent with the calculations in the flat noncommutative space[9, 11].

4.2 Gauge-Covariant Current

We define another type of a chiral transformation as
\[
\delta \psi = \frac{1}{L + \frac{1}{2}} \sigma_i \lambda \psi L_i, \tag{4.50}\n\]
\[
\delta \bar{\psi} = \frac{1}{L + \frac{1}{2}} L_i \bar{\psi} \sigma_i \lambda. \tag{4.51}\n\]
This chiral transformation reduces to (3.12) in the commutative limit. The algebra of the chiral transformation closes up to the gauge transformation since
\[
\delta_1 \delta_2 \psi = \frac{1}{(L + \frac{1}{2})^2} \sigma_i \lambda_1 \lambda_2 \psi L_i + \frac{L(L + 1)}{(L + \frac{1}{2})^2} \lambda_1 \lambda_2 \psi. \tag{4.52}\n\]
The chiral transformation (4.50) is compatible with the gauge transformation, (2.10), (2.11) if \( \lambda \) transforms covariantly as
\[
\lambda \rightarrow U\lambda U^\dagger.
\] (4.53)

Since the chiral transformation parameter \( \lambda \) transforms gauge covariantly as (4.53), the associated current is also gauge covariant. Thus, the local WT identity is expected to be written down.

### 4.2.1 Evaluation of the WT Identity

Now, we will evaluate the chiral WT identity for the covariant current. Under the chiral transformation (4.50), the action (2.8) varies as
\[
\begin{align*}
\delta S &= \frac{\alpha}{g^2(2L + 1)} \text{Tr}\left( \bar{\psi}\sigma_i\sigma_j [L_i + \rho a_i, \lambda] \psi L_j + 2\bar{\psi}\lambda [L_i, \psi] L_i \right. \\
& \quad + \left. \frac{2\rho^2}{\alpha} \bar{\psi}\lambda\phi\psi - \rho^2\bar{\psi}\lambda a_i^2\psi - 2\rho\bar{\psi}\lambda a_i [L_i, \psi] - \rho\bar{\psi}\lambda [L_i, a_i] \psi \right),
\end{align*}
\] (4.54)

where \( \phi \) is the gauge covariant scalar field defined in (4.36). The integration measure is invariant. Therefore, the WT identity becomes
\[
\begin{align*}
\langle \text{Tr} \left( \lambda \left[ L_i + \rho a_i, -\psi_\beta L_j\psi_\alpha (\sigma_j, \sigma_j)^{\alpha\beta} \right] \right) \rangle - \frac{2\rho^2}{\alpha} \text{Tr} \left( \bar{\psi}\lambda\phi\psi \right) s_0 \\
= -\rho\langle \text{Tr} \left( \bar{\psi}\lambda [L_i, a_i] \psi \right) \rangle s_0 \\
-\langle \text{Tr} \left( 2\bar{\psi}\lambda [L_i, \psi] L_i \right) \text{Tr} \left( \frac{\alpha\rho}{2g^2} \bar{\psi}\sigma_j a_j \psi \right) \rangle s_0 + \mathcal{O}(a_i^2)
\end{align*}
\] (4.55)

up to the first order in the gauge field \( a_i \). The last term came from the first-order perturbative expansion in the gauge fields in (2.8).

Now we evaluate the last term in eq. (4.55) for a special background gauge field, where \( \sigma_j a_j \) is replaced by \( \sigma_3 a_3 \). By using the formula (6.28), it can be
rewritten as

\[-\langle \text{Tr} \left( 2 \bar{\psi} \lambda [L_i, \psi] L_i \right) \text{Tr} \left( \frac{\alpha \rho}{2g_2} \bar{\psi} \sigma_3 \psi \right) \rangle \rangle_{s_0} = -\frac{2g^2 \rho}{\alpha (2L+1)^2} \sum_{l=1}^{2L} \sum_{l'=1}^{2L} \sum_{m=-l+\frac{1}{2}}^{l-\frac{1}{2}} \sum_{m'=-l'+\frac{1}{2}}^{l'-\frac{1}{2}} \frac{1}{l(l+1)} \sqrt{(l+1)^2 - m^2} \sqrt{\left( l' + \frac{1}{2} \right)^2 - m'^2}
\times \left[ \text{Tr} \left( Y_{l,m+\frac{1}{2}}^{\dagger} \lambda Y_{l',m'+\frac{1}{2}} \right) \text{Tr} \left( Y_{l,m'-\frac{1}{2}} \sigma_3 \alpha \right) - \text{Tr} \left( Y_{l,m-\frac{1}{2}}^{\dagger} \lambda Y_{l',m'-\frac{1}{2}} \right) \text{Tr} \left( Y_{l',m'+\frac{1}{2}} \sigma_3 \alpha \right) \right]
- \frac{4g^2 \rho}{\alpha (2L+1)^2} \sum_{l} \sum_{m=-l}^{l} m \text{Tr} (\lambda Y_{lm}) \text{Tr} (Y_{lm}^\dagger a_3).
\]

As in (4.38), the last term of (4.56) can be calculated by using the completeness of the spherical harmonics (6.9), and exactly cancels with the first term of the r.h.s. in eq.(4.55). Hence the first term of (4.56) gives the r.h.s. of the WT identity (4.55).

The first term of (4.56), which we call $I_3$, when $\lambda = 1$, turns out to be exactly equal to $H_3|_{\lambda=1}$ (see eq.(4.46)). Following the same steps in $H_3$ case, we obtain

\[ I_{\lambda=1} = H_{\lambda=1} = \frac{8g^2 \rho}{\alpha (2L+1)^2} \text{Tr}(a_i L_i). \]

This term corresponds to the normal-ordering constant for the scalar term in the l.h.s. of the WT identity (4.55).

We then evaluate the first term of (4.56) for $\lambda \neq 1$. Using (6.16) and (6.17),

\[ I_3 = \frac{2g^2 \rho}{\alpha (2L+1)} \sum_{l=1}^{2L} \sum_{l'=1}^{2L} \sum_{m=-l+\frac{1}{2}}^{l-\frac{1}{2}} \sum_{m'=-l'+\frac{1}{2}}^{l'-\frac{1}{2}} \frac{1}{l(l+1)} \times \left[ \text{Tr} \left( Y_{l,m+\frac{1}{2}}^{\dagger} \lambda Y_{l',m'+\frac{1}{2}} \right) \text{Tr} \left( [L_+, Y_{l',m'+\frac{1}{2}}^{\dagger}] a_3 [L_-, Y_{l,m+\frac{1}{2}}] \right) - \text{Tr} \left( Y_{l,m-\frac{1}{2}}^{\dagger} \lambda Y_{l',m'-\frac{1}{2}} \right) \text{Tr} \left( [L_-, Y_{l',m'-\frac{1}{2}}^{\dagger}] a_3 [L_+, Y_{l,m-\frac{1}{2}}] \right) \right]. \]

Comparing with eq.(4.31), the only difference is the position of $\lambda$: $\lambda$ is placed before the $Y_{lm}$ in (4.58), while it was after $Y_{lm}$ in (4.31). Then we can follow the
same steps in $H_3$, and obtain

$$I = \frac{-2ig^2\rho}{\alpha} \sum_{n=0}^{\infty} \frac{1}{\{L(L+1)\}^{n+1}} \times \left[ \text{Tr} (L_{i_{n}} \cdots L_{i_{1}}) \left[ \varepsilon_{ijk} \text{Tr} (\lambda[L_{i_{1}}, a_{j}]L_{k}L_{i_{1}} \cdots L_{i_{n}}) - i\text{Tr} (\lambda a_{i}L_{i_{1}} \cdots L_{i_{n}}) \right] - \text{Tr} (L_{i_{n}} \cdots L_{i_{1}}) \left[ \varepsilon_{ijk} \text{Tr} (\lambda[L_{j}, a_{k}]L_{i_{1}} \cdots L_{i_{n}}) - i\text{Tr} (\lambda a_{i}L_{i_{1}} \cdots L_{i_{n}}) \right] \right].$$

(4.59)

Due to the position of $\lambda$ in (4.58), $\lambda$ and $a_i$ are placed in the same trace, namely in the same integral when we express traces by integrals. Therefore the WT identity can be written locally for an arbitrary chiral parameter $\lambda$. The WT identity (4.58) for the covariant current becomes

$$\langle \text{Tr} (\lambda [L_i + \rho a_i, -\psi\gamma L_j \bar{\psi}_{\alpha}(\sigma_i \sigma_j)^{\alpha\beta}]) - \frac{2\rho^2}{\alpha} \text{Tr} (\bar{\psi} \lambda \phi \psi) \rangle_S = I. \quad (4.60)$$

We will further evaluate the anomaly in two limiting cases below.

### 4.2.2 Commutative Limit

We now take the commutative limit. In this limit, we can replace $L_i$ by the classical coordinate $x_i/\alpha$, and $L_i$ by $\hat{L}_i$. The second line of eq.(4.59) becomes

$$= \frac{-2ig^2\rho}{\alpha} \sum_{n=0}^{\infty} \frac{1}{\{L(L+1)\}^{2n+1}} (2L + 1)^2 \frac{1}{\alpha^{4n+1}} \rho^{4n+3} \int_{\Omega} (\varepsilon_{ijk} \lambda[L_{i_{1}, a_{j}]x_{k}x_{l}^{2n} - i\lambda(a \cdot x)x_{l}^{2n}] \int (x_{l}^{'})^{2n}$$

$$= \frac{-2ig^2\rho}{\alpha} \sum_{n=0}^{\infty} \frac{1}{\{L(L+1)\}^{2n+1}} (2L + 1)^2 \frac{1}{\alpha^{4n+1}} \rho^{4n+3} \int_{\Omega} (\varepsilon_{ijk} \lambda[L_{i_{1}, a_{j}]x_{k} - i\lambda \rho \phi})$$

$$= \frac{-2ig^2\rho}{\alpha} \sum_{n=0}^{\infty} \frac{1}{\{L(L+1)\}^{2n+2}} (2L + 1)^2 \frac{1}{\alpha^{4n+3}} \rho^{4n+3} \int_{\Omega} (\varepsilon_{ijk} \lambda[L_{i_{1}, a_{j}]x_{k} - i\lambda \rho \phi}) (4.61)$$

Similarly, the third line becomes

$$= \frac{-2ig^2\rho}{\alpha} \sum_{n=0}^{\infty} \frac{1}{\{L(L+1)\}^{2n+2}} (2L + 1)^2 \frac{1}{\alpha^{4n+3}} \rho^{4n+2} \int_{\Omega} (\varepsilon_{ijk} \lambda[L_{i_{1}, a_{j}]x_{k} - i\lambda \rho \phi})$$

$$= \frac{2ig^2\rho}{\rho} (2L + 1)^2 \sum_{n=0}^{\infty} \frac{1}{(2n+3)} \int_{\Omega} (\varepsilon_{ijk} \lambda[L_{i_{1}, a_{j}]x_{k} - i\lambda \rho \phi}). \quad (4.62)$$

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Here we have used the following formula:

$$
\int_{\Omega} x_{i_1} \cdots x_{i_{2n}} = \frac{(\rho^2)^n}{(2n + 1)!!} \left[ \delta_{i_1 i_2} \delta_{i_3 i_4} \cdots \delta_{i_{2n-1} i_{2n}} + \cdots \right],
$$

(4.63)

where there are \( (2n - 1)!! \) ways of contraction in the r.h.s.. We thus obtain the commutative limit of \( I \):

$$
I \longrightarrow -\frac{2ig^2}{\rho} \int_{\Omega} (\epsilon_{ijk} \lambda L_i a_j x_k - i\lambda \rho \phi).
$$

(4.64)

The commutative limit of the WT identity (4.60) becomes

$$
\mathcal{L}_i(\bar{\psi} \sigma_i \gamma_3 \psi) - 2\rho \phi \bar{\psi} \psi - 4g^2 \phi = \frac{4g^2}{\rho} (-i\epsilon_{ijk} x_i \mathcal{L}_j a_k - 2\rho \phi),
$$

(4.65)

which completely agrees with (3.17). The last term in the l.h.s. is the normal-ordering constant for the second term, as we have already seen in the noncommutative case at \( \lambda = 1 \).

### 4.2.3 Flat Limit

We then take another limit of (4.59), the flat limit. Since we consider the vicinity of the north pole, the background gauge field \( a_i \) is assumed to have supports only around the north pole and we can replace \( L_i \) by \( L_3 \) if they are in the trace in which \( a_i \) and \( \lambda \) are inserted:

$$
I \rightarrow \frac{-2ig^2\rho}{\alpha} \sum_{n=0}^{\infty} \frac{1}{[L(L + 1)]^{n+1}}
\times \left[ \text{Tr}((L_3)^n)\epsilon_{ij} \text{Tr}(\lambda[L_i, a_j](L_3)^{n+1}) - i\text{Tr}(\lambda a_3 (L_3)^{n+1}) \right]
$$

$$
- \text{Tr}((L_3)^n)\epsilon_{ij} \text{Tr}(\lambda[L_i, a_j](L_3)^{n+1}) - i\text{Tr}(\lambda a_3 (L_3)^{n}) \right].
$$

(4.66)

Here \( i, j \) take values 1 or 2. By the same reasoning, \( L_3 \) in the trace in which \( a_i \) is inserted can be replaced with a c-number \( \sqrt{L(L + 1)} \). \( Tr(L_3^n) \) can be evaluated exactly, and it vanishes for odd \( n \). We then obtain

$$
I \rightarrow \frac{-4ig^2\rho}{\alpha} [\epsilon_{ij} \text{Tr}(\lambda[L_i, a_j]) - i\text{Tr}(\lambda a_3)],
$$

(4.67)
and the WT identity (4.60) becomes

\[ \langle \text{Tr} \left( \lambda \left[ L_i + \rho a_i, -\bar{\psi} L_3 \bar{\psi} \sigma_3 \right] \right) \rangle_S = \frac{-4ig^2\rho}{\alpha} \left[ \epsilon_{ij} \text{Tr}(\lambda[L_i, a_j]) - i\text{Tr}(\lambda a_3) \right]. \] (4.68)

By using (3.20), (3.21), (3.10), (3.11), we obtain

\[ \langle \int d^2x \left[ \lambda(i\partial_i + \tilde{a}'')(-\sigma''_{i\alpha\beta}(\gamma''_{3\beta\gamma}\bar{\psi}\gamma)\bar{\psi}\gamma) + \lambda[a_3, -\bar{\psi}\bar{\psi}] - 2\bar{\psi}\lambda\phi\psi \right] \rangle_S = \int d^2x \left[ 4g^2\lambda\epsilon_{ij}\partial_j a'_j \right]_S, \] (4.69)

where

\[ \gamma''_3 = \frac{\alpha}{\rho} L_3^R \sigma_3, \] (4.70)

\( \tilde{a}'' \) are adjoint operators of \( a'' \), and \([ \cdots ]_* \) means that the products in the bracket are replaced by the star products. Note that the last term of the r.h.s. in (4.69) is a sub-leading contribution in \( 1/\rho \) and can be ignored. The second term in the l.h.s. of (4.69) vanishes up to the first order in the gauge field \( a_i \), since \( \langle \lambda[a_3, -\bar{\psi}\bar{\psi}] \rangle_{S_0} = 0 \) due to (6.24). Therefore the WT identity becomes

\[ \langle \int d^2x \left[ \lambda(i\partial_i + \tilde{a}'')(\sigma''_{i\alpha\beta}(\gamma''_{3\beta\gamma}\bar{\psi}\gamma)\bar{\psi}\gamma) - 2\bar{\psi}\lambda\phi\psi \right] \rangle_S = \int d^2x \left[ 4g^2\lambda\epsilon_{ij}\partial_j a'_j \right]_S \] (4.71)
in the flat limit.

5 Conclusions and Discussions

In this paper, we have calculated chiral anomaly for fermions in the fundamental representation on the fuzzy 2-sphere. This system can be formulated as a matrix model of finite size and no regularization is necessary. In spite of this, we can reproduce the anomalous chiral WT identity. Our final results for the WT identities are written in eq.(4.44) and eq.(4.60). We have obtained WT identities for two types of chiral currents, a gauge invariant current (4.44) and a covariant current (4.60).
The anomaly term is contained in $H$ of eq. (4.43) and $I$ of eq. (4.59) respectively. $H$ and $I$ have the same expression except the location of the chiral transformation parameter $\lambda$. In $H$, $\lambda$ and the background gauge field $a_i$ are inserted in different traces, while in $I$, they are in the same trace. $\text{Tr}$ becomes an integral over 2-sphere. Hence, for the covariant case, if we take $\text{Tr}\lambda$ out of the WT identity (4.60) and the corresponding anomaly term $I$ in eq. (4.59), we obtain a local expression for the anomaly term in the WT identity. On the contrary, for the invariant case, even after taking $\text{Tr}\lambda$ out of $H$, the gauge field $a_i$ is still in the other trace, namely in an integral, and the WT identity has a nonlocal form. If we put $\lambda = 1$, $H$ and $I$ become the same and we can obtain the same global form of the chiral anomaly.

When we take a flat limit, the small difference between $H$ and $I$ causes a big difference to the final results. By assuming that both of the chiral transformation parameter $\lambda$ and the gauge field $a_i$ are localized around the north pole of the sphere, the anomaly for the covariant case becomes a star generalization of the anomaly in the commutative theory as we saw in section 4.2.3. The anomaly for the invariant case, however, vanishes as in section 4.1.3. These results are consistent with the previous results [7]-[20]. A merit of our calculation is that our system is finite and we could obtain the results (4.43) and (4.59), which interpolate between local $\lambda$ and global $\lambda$.

In this paper, we have only evaluated the anomaly in the leading order of the gauge field. Though the calculation of higher orders is very complicated, they can be guessed by the gauge covariance. For the gauge invariant and covariant currents respectively, $H$ of eq. (4.43) and $I$ of eq. (4.59) can have the following
simple gauge invariant completions:

\[
H_G = -\frac{2ig^2\rho^2}{\alpha} \sum_{n=0}^{\infty} \frac{1}{\{\alpha L(L+1)\}^{n+1}} \times [\text{Tr}(L_{i_n} \cdots L_{i_1}\lambda)\text{Tr}(A_{i_1}B_{i_1}A_{i_1} \cdots A_{i_n}) - \alpha \text{Tr}(L_{i_n} \cdots L_{i_1}L_{i_1}\lambda)\text{Tr}(B_{i_1}A_{i_1} \cdots A_{i_n})],
\]

\[(5.1)\]

\[
I_G = -\frac{2ig^2\rho^2}{\alpha} \sum_{n=0}^{\infty} \frac{1}{\{\alpha L(L+1)\}^{n+1}} \times [\text{Tr}(L_{i_n} \cdots L_{i_1})\text{Tr}(\lambda A_{i_1}B_{i_1}A_{i_1} \cdots A_{i_n}) - \alpha \text{Tr}(L_{i_n} \cdots L_{i_1}L_{i_1})\text{Tr}(\lambda B_{i_1}A_{i_1} \cdots A_{i_n})],
\]

\[(5.2)\]

where \(B_i\) is the magnetic field defined by

\[
B_i = \frac{1}{2} \epsilon_{ijk}F_{jk} = \frac{1}{\alpha^2\rho^2}(\epsilon_{ijk}A_jA_k - i\alpha A_i),
\]

\[(5.3)\]

\[
F_{ij} = \frac{1}{\alpha^2\rho^2} ([A_i, A_j] - i\alpha \epsilon_{ijk}A_k).
\]

\[(5.4)\]

They are gauge covariant fields and vanish when \(a_i = 0\). The above forms of anomalies are invariant under gauge transformations and become \(H\) of \((4.43)\) and \(I\) of \((4.59)\) in the first order of the gauge field. In the commutative limit, the anomaly for the covariant current becomes the same as the r.h.s. of \((4.64)\) except that \([L_i, a_j]\) is replaced by a gauge covariant form \((\{L_i, a_j\} - [L_j, a_i] + \rho [a_i, a_j]) / 2\). It is also similar in the flat limit, the final expression becomes the same as the r.h.s. of eq.\((4.68)\) except \([L_i, a_j]\) is replaced by the same covariant form. For the case of the invariant current, it is interesting to see whether the above gauge invariant completion is consistent with the perturbative form of the anomaly discussed in [10].

A motivation to consider chiral anomaly on fuzzy 2-sphere is to define topological invariants on finite noncommutative space. In commutative space, we have fully understood the topological structures of the gauge configuration space and its relation to the index of Dirac operators or anomalies. They have been extensively utilized in many situations, for example, in constructing the chiral fermions in the Kalza-Klein compactification.
We have been struggling to build a constructive formulation of superstrings and, at present, matrix models or noncommutative field theories are considered to be promising candidates. From this perspective, it is necessary to investigate various possibilities to make chiral fermions in finite dimensional matrix models. One possibility was investigated in ref. [32] where orbifold matrix models were proposed. Another interesting possibility to make chiral fermions will be to define index of some Dirac operators in a compactified noncommutative space and then make use of the Dirac operator with a nontrivial index. In infinite dimensional noncommutative space, solitons have been constructed [33] in terms of the so-called shift operators. But the shift operator is formally written as

$$S = \sum_{n=0}^{\infty} |n+1\rangle \langle n|,$$

and the construction essentially makes use of the infinite dimensionality. In finite noncommutative geometries, topologically nontrivial field configurations have been constructed based on algebraic K-theory and projective modules [34, 22]. Though they are mathematically beautiful, it seems difficult to apply this idea to matrix models for square matrices, such as [6].

To overcome the above difficulty, it is interesting to apply the ideas related to Ginsparg–Wilson(GW) relation [35] in lattice gauge theory to matrix models or noncommutative field theories. GW relation represents the remnant chiral symmetry of chiral continuum theories. Another important idea originates from the observation that in the presence of a mass defect, a chiral fermion appears at the boundary. So far a domain wall fermion [36] and a vortex fermion [37] are constructed on the lattice and from the former model, a practical solution of GW relation is derived [38]. More abstractly, GW relation plays a crucial role in discussing the chiral symmetry [39, 40] or the index theorem at a finite lattice spacing [41, 39]. In a forthcoming paper [42] we show that by making use of the ideas related to GW relation, it is possible to define topological invariants or indices of Dirac operators in the finite dimensional fuzzy 2-sphere with general
6 Appendix

6.1 Useful Identities

$L_i$’s are $(2L+1)$-dimensional representation matrices of the angular momentum operators and satisfy

\[
[L_i, L_j] = i\epsilon_{ijk}L_k, \tag{6.1}
\]

\[
(L_i)^2 = L(L+1), \tag{6.2}
\]

\[
L_\pm = L_1 \pm iL_2, \tag{6.3}
\]

\[
[L_+, L_-] = 2L_3, \tag{6.4}
\]

\[
[L_3, L_\pm] = \pm L_\pm. \tag{6.5}
\]

$L_i M = [L_i, M]$ are adjoint operators and satisfy

\[
[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{6.6}
\]

Noncommutative spherical harmonics $Y_{lm}$ satisfy an orthonormality and a completeness relation:

\[
\frac{1}{2L+1} \text{Tr}(Y_{lm}^\dagger Y_{l'm'}) = \delta_{ll'}\delta_{mm'}, \tag{6.7}
\]

\[
\frac{1}{2L+1} \sum_{l=0}^{2L} \sum_{m=-l}^l (Y_{lm}^\dagger)_{ij} (Y_{lm})_{kp} = \delta_{ip}\delta_{jk}. \tag{6.8}
\]

Total number of basis wave functions becomes $\sum_{l=0}^{2L}(2l+1) = (2L+1)^2$ and agrees with the number of independent $(2L+1)$-dimensional hermitian matrices. Thus for any matrices $A$ and $B$, we have

\[
\frac{1}{2L+1} \sum_{lm} \text{Tr}(Y_{lm}^\dagger A)\text{Tr}(Y_{lm}B) = \text{Tr}(AB), \tag{6.9}
\]

\[
\frac{1}{2L+1} \sum_{lm} \text{Tr}(Y_{lm}^\dagger AY_{lm}B) = \text{Tr}(A)\text{Tr}(B). \tag{6.10}
\]
There are various useful identities when $L_i$’s act on $Y_{lm}$:

\[
[L_i, [L_i, Y_{lm}]] = l(l + 1)Y_{lm}, \quad (6.11)
\]
\[
[L_3, Y_{lm}] = mY_{lm}, \quad (6.12)
\]
\[
L_i Y_{lm} L_i = \left[ L(L + 1) - \frac{1}{2} l(l + 1) \right] Y_{lm}, \quad (6.13)
\]
\[
L_i [L_i, Y_{lm}] = \frac{l(l + 1)}{2} Y_{lm}, \quad (6.14)
\]
\[
[L_i, Y_{lm}] L_i = -\frac{l(l + 1)}{2} Y_{lm}, \quad (6.15)
\]
\[
[L_+, Y_{lm}] = \sqrt{l(l + 1) - m(m + 1)} Y_{l,m+1}, \quad (6.16)
\]
\[
[L-, Y_{lm}] = \sqrt{l(l + 1) - m(m - 1)} Y_{l,m-1}. \quad (6.17)
\]

Using the above equations, we can prove the following identity:

\[
\frac{1}{l(l + 1) + 2\epsilon} Y_{lm} = \sum_{n=0}^{\infty} \frac{1}{2L(L + 1)} \left[ 1 - \frac{l(l + 1) + 2\epsilon}{2L(L + 1)} \right] Y_{lm},
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{2L(L + 1)} \left[ \frac{L_i^L L_i^R - \epsilon}{L(L + 1)} \right] Y_{lm}. \quad (6.18)
\]

### 6.2 Expectation Values in the Free Theory

The free part of the action $S_0$ (2.8) becomes

\[
S_0 = \frac{\alpha(2L + 1)}{2g^2} \left[ \sum_{l=0}^{2L} b_{l+\frac{1}{2},m}^\dagger b_{l+\frac{1}{2},m}(l + 1) - \sum_{l=1}^{2L} b_{l-\frac{1}{2},m}^\dagger b_{l-\frac{1}{2},m} \right] \quad (6.19)
\]

by expanding the fields $\psi$ and $\bar{\psi}$ in terms of spinorial-spherical harmonics $\mathcal{V}_{l+\frac{1}{2},m}$ and $\mathcal{V}_{l-\frac{1}{2},m}$ introduced in eq. (2.20) as

\[
\psi = \sum_{l=0}^{2L} \sum_{m=-\frac{1}{2}}^{l+\frac{1}{2}} b_{l+\frac{1}{2},m}^\dagger \mathcal{V}_{l+\frac{1}{2},m} + \sum_{l=1}^{2L} \sum_{m=-l+\frac{1}{2}}^{l-\frac{1}{2}} b_{l-\frac{1}{2},m}^\dagger \mathcal{V}_{l-\frac{1}{2},m}, \quad (6.20)
\]
\[
\bar{\psi} = \sum_{l=0}^{2L} \sum_{m=-\frac{1}{2}}^{l+\frac{1}{2}} b_{l+\frac{1}{2},m} \mathcal{V}_{l+\frac{1}{2},m}^\dagger + \sum_{l=1}^{2L} \sum_{m=-l+\frac{1}{2}}^{l-\frac{1}{2}} b_{l-\frac{1}{2},m} \mathcal{V}_{l-\frac{1}{2},m}^\dagger. \quad (6.21)
\]
Then we can calculate the following expectation values,

\[
\langle O \rangle_{S_0} = \frac{1}{Z_{S_0}} \int d\psi d\bar{\psi} O e^{-S_0},
\]

\[
Z_{S_0} = \int d\psi d\bar{\psi} e^{-S_0},
\]

by using the Wick’s theorem. Thus, we obtain the following formulae:

- \( \langle \text{Tr}(\bar{\psi}A\psi B) \rangle_{S_0} = -\frac{4g^2}{\alpha(2L + 1)} \text{Tr}(AB). \) (6.24)

- \( \langle \text{Tr}(\bar{\psi}A\sigma_3\psi B) \rangle_{S_0} = -\frac{4g^2}{\alpha(2L + 1)} \sum_{l=1}^{2L} \sum_{m=-l}^{l} \frac{m}{l(l+1)} \text{Tr}(Y_{lm}^\dagger AY_{lm} B). \) (6.25)

- \( \langle \text{Tr}(\bar{\psi}A\sigma_{\pm}\psi B) \rangle_{S_0} = -\frac{2g^2}{\alpha(2L + 1)} \sum_{l=1}^{2L} \sum_{m=-l+\frac{1}{2}}^{l-\frac{1}{2}} \sqrt{(l + \frac{1}{2})^2 - m^2} \text{Tr}\left(Y_{l,m+\frac{1}{2}}^\dagger AY_{l,m-\frac{1}{2}} B\right). \) (6.26)

- \( \langle \text{Tr}(\bar{\psi}A\psi B)\text{Tr}(\bar{\psi}C\psi D) \rangle_{S_0} \)

\[
= \frac{8g^4}{\alpha^2(2L + 1)^2} \text{Tr}(AB)\text{Tr}(CD)
- \frac{8g^4}{\alpha^2(2L + 1)^2} \sum_{l=1}^{2L} \sum_{l'=1}^{2L} \sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} \frac{mm'}{l(l+1)l'(l'+1)} \text{Tr}(Y_{lm}^\dagger AY_{l'm'} B)\text{Tr}(Y_{l'm'}^\dagger CY_{lm} D)
- \frac{4g^4}{\alpha^2(2L + 1)^2} \sum_{l=1}^{2L} \sum_{l'=1}^{2L} \sum_{m=-l+\frac{1}{2}}^{l-\frac{1}{2}} \sum_{m'=-l'+\frac{1}{2}}^{l'-\frac{1}{2}} \frac{1}{l(l+1)l'(l'+1)} \sqrt{(l + \frac{1}{2})^2 - m^2} \sqrt{(l' + \frac{1}{2})^2 - m'^2}
\times \left[ \text{Tr}(Y_{l,m-\frac{1}{2}}^\dagger AY_{l',m'-\frac{1}{2}} B)\text{Tr}(Y_{l',m'+\frac{1}{2}}^\dagger CY_{l,m+\frac{1}{2}} D)
+ \text{Tr}(Y_{l,m+\frac{1}{2}}^\dagger AY_{l',m'-\frac{1}{2}} B)\text{Tr}(Y_{l',m'+\frac{1}{2}}^\dagger CY_{l,m-\frac{1}{2}} D) \right].
\] (6.27)
\[ \langle \text{Tr}(\bar{\psi}A\psi)\text{Tr}(\bar{\psi}C\sigma_{3}\psi D)) \rangle_{S_0} \]
\[ = \frac{16g^4}{\alpha^2(2L + 1)^2} \sum_{l=1}^{2L} \sum_{m=-l}^{l} \frac{m}{l(l+1)} \text{Tr}(AB)\text{Tr}(Y_{lm}^{+}CY_{lm}D) \]
\[ - \frac{4g^4}{\alpha^2(2L + 1)^2} \sum_{l=1}^{2L} \sum_{m=-l}^{l} \sum_{l'=-l}^{l} \frac{m}{l(l+1)} \sqrt{(l + 1)^2 - m^2} \text{Tr}(Y_{l,m}^{+}A)\text{Tr}(Y_{l',m'}^{+}B)\text{Tr}(Y_{l',m'}^{+}CY_{lm}^{-1}CY_{lm}D) \]
\[ \times \left[ \text{Tr}(Y_{l,m+\frac{1}{2}}^{+}AY_{l',m'+\frac{1}{2}}^{+}B)\text{Tr}(Y_{l',m'+\frac{1}{2}}^{+} CY_{l,m-\frac{1}{2}}^{+}D) \right] \]
\[ - \frac{8g^4}{\alpha^2(2L + 1)^2} \sum_{l=1}^{2L} \sum_{m=-l}^{l} \frac{m}{l(l+1)} \left[ \text{Tr}(Y_{lm}^{+}AB)\text{Tr}(CY_{lm}D) + \text{Tr}(AY_{lm}B)\text{Tr}(Y_{lm}^{+}CD) \right] . \] (6.28)

\[ \langle \text{Tr}(\bar{\psi}A\psi)\text{Tr}(\bar{\psi}C\sigma_{\pm}\psi D)) \rangle_{S_0} \]
\[ = \frac{8g^4}{\alpha^2(2L + 1)^2} \sum_{l=1}^{2L} \sum_{m=-l}^{l} \frac{m}{l(l+1)} \sqrt{(l + 1)^2 - m^2} \text{Tr}(AB)\text{Tr}(Y_{l,m+\frac{1}{2}}^{+}CY_{l,m+\frac{1}{2}}D) \]
\[ + \frac{4g^4}{\alpha^2(2L + 1)^2} \sum_{l=1}^{2L} \sum_{m=-l}^{l} \sum_{l'=-l}^{l} \frac{m}{l(l+1)} \sqrt{(l + 1)^2 - m^2} \text{Tr}(Y_{l,m}^{+}A)\text{Tr}(Y_{l',m'}^{+}B)\text{Tr}(Y_{l',m'}^{+}CY_{lm}^{-1}CY_{lm}D) \]
\[ \times \left[ \text{Tr}(Y_{l,m+\frac{1}{2}}^{+}AY_{l',m'+\frac{1}{2}}^{+}B)\text{Tr}(Y_{l',m'+\frac{1}{2}}^{+} CY_{l,m-\frac{1}{2}}^{+}D) \right] \]
\[ - \frac{4g^4}{\alpha^2(2L + 1)^2} \times \left[ \sum_{l=1}^{2L} \sum_{m=-l}^{l} \frac{m}{l(l+1)} \sqrt{(l + 1)^2 - m^2} \text{Tr}(Y_{l,m+\frac{1}{2}}^{+}A)\text{Tr}(CY_{lm,\frac{1}{2}}^{+}D) \right] \]
\[ + \sum_{l'=1}^{2L} \sum_{m'=-l'+\frac{1}{2}}^{\prime} \frac{(l' + 1)^2 - m'^2}{l'(l'+1)} \text{Tr}(AY_{l',m'+\frac{1}{2}}B)\text{Tr}(Y_{l',m'+\frac{1}{2}}^{+}CD) \right] . \] (6.29)

Here \( A, B, C, D \) are arbitrary \((2L + 1)\)-dimensional matrices and \( \sigma_{\pm} = \frac{\sigma_{1} \pm i\sigma_{2}}{2} \).
6.3 Chiral Anomaly on $S^2$

In this appendix, we will derive anomalous WT identity in commutative 2-sphere, (3.17), (3.18). Under the chiral transformation (3.12), the action (3.1) varies as

$$\delta S_{S^2} = \frac{\rho}{g^2} \int_{\Omega} \left[ (\tilde{L}_i \lambda) \bar{\psi} \sigma_i \gamma_3 \psi + 2 \rho \lambda \bar{\psi} \phi \psi \right], \quad (6.30)$$

and the measure changes $d\psi' d\bar{\psi}' = J d\psi d\bar{\psi}$. The Jacobian is calculated as

$$J = \exp \left[ -2 \int_{\Omega} \lambda(x) \sum_n \phi_n^\dagger(x) \gamma_3 \phi_n(x) \right] \quad (6.31)$$

$$= 1 - 2 \int_{\Omega} \lambda(x) \sum_n \phi_n^\dagger(x) \gamma_3 \phi_n(x), \quad (6.32)$$

where $\phi_n$ is a complete set of eigenfunctions for the hermitian Dirac operator $D$ and satisfies $D \phi_n = \lambda_n \phi_n$ and $\int_{\Omega} \phi_n^\dagger(x) \phi_n(x) = \delta_{nm}$.

Here we introduce spherical harmonics $Y_{lm}$, which satisfy

$$\int_{\Omega} Y_{lm}^\ast(x) Y_{l'm'}(x) = \delta_{ll'} \delta_{mm'}, \quad (6.33)$$

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} Y_{lm}(x) Y_{lm}^\ast(x') = 4\pi \delta(\Omega - \Omega'), \quad (6.34)$$

$$\sum_{m=-l}^{m=l} Y_{lm}^\ast(x') Y_{lm}(x) = (2l + 1) P_l(\cos \alpha), \quad (6.35)$$

$$\sum_{m=-l}^{m=l} Y_{lm}^\ast(x) Y_{lm}(x) = 2l + 1, \quad (6.36)$$

$$\sum_{m=-l}^{m=l} Y_{lm}^\ast(x) \tilde{\mathcal{L}}_i Y_{lm}(x) = 0, \quad (6.37)$$

$$\sum_{m=-l}^{m=l} Y_{lm}^\ast(x) \tilde{\mathcal{L}}_i \tilde{\mathcal{L}}_j Y_{lm}(x) = \frac{(2l + 1)(l + 1)}{2} \left[ \delta_{ij} - \frac{x_i x_j}{\rho^2} \right], \quad (6.38)$$

where $\alpha$ in the third line is the angle between $\Omega$ and $\Omega'$.  

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We then evaluate the Jacobian using Fujikawa’s method\cite{43} as

\[ \mathcal{A}(x) = \sum_n \phi_n^\dagger(x) \gamma_3 \phi_n(x) \]

\[ = \lim_{M \to \infty} \sum_n \phi_n^\dagger(x) \gamma_3 \exp \left( -\frac{\lambda_n^2}{M^2} \right) \phi_n(x) \]

\[ = \lim_{M \to \infty} \sum_{l=0}^\infty \sum _{m=-l}^{l} \sum _{s=\pm 1} \chi^+_s Y^*_l m(x) \gamma_3 \exp \left( -\frac{D^2}{M^2} \right) Y_{l m}(x) \chi_s \]

\[ = \lim_{M \to \infty} \sum_{l=0}^\infty \sum _{m=-l}^{l} e^{\frac{-i(l+1)^2}{M^2}} Y^*_l m(x) \text{tr} \left[ \gamma_3 \exp \left( -\frac{D'}{M^2} \right) \right] Y_{l m}(x), \quad (6.39) \]

where \( \chi_s \) in the third line are complete basis for the spinor space. In the last line, \( \text{tr} \) is a trace over spinor space, and

\[ D' = D^2 - (\tilde{\mathcal{L}}_i)^2 \]

\[ = \sigma_i \tilde{\mathcal{L}}_i + 1 + \rho [(\tilde{\mathcal{L}}_i a_i) + i \varepsilon_{ijk} (\tilde{\mathcal{L}}_i a_j + 2a_i a_j)] + \rho^2 a_i a_i. \quad (6.40) \]

Only the first term \( \sigma_i \tilde{\mathcal{L}}_i \) and the sixth term \( 2\rho a_i \tilde{\mathcal{L}}_i \) in eq.\((6.40)\) can act as angular momentum operators on the factors in the right in eq.\((6.39)\), and can take the value of the order of \( l \), and thus \( M \), when acting on \( Y_{lm} \). The other \( \tilde{\mathcal{L}}_i \)'s in eq.\((6.40)\) act only on \( a_i \) in the round bracket. Thus, the terms except for the first and the sixth terms in eq.\((6.40)\) act just as c-number, and take the value of the order of 1 in eq.\((6.39)\). Therefore, when we Taylor-expand \( \exp \left( -\frac{D'}{M^2} \right) \), only the following terms can survive in the large \( M \) limit in eq.\((6.39)\):

\[ \text{tr} \left[ \gamma_3 \exp \left( -\frac{D'}{M^2} \right) \right] \to \text{tr} \left[ \gamma_3 \left( 1 - \frac{D'}{M^2} + \frac{1}{2 M^4} (\sigma_i \tilde{\mathcal{L}}_i + 2 \rho a_j \tilde{\mathcal{L}}_j)^2 \right) \right]. \quad (6.41) \]

After taking trace over the spinor space, this becomes

\[ \frac{-2}{M^2} [x_i \tilde{\mathcal{L}}_i / \rho + i \varepsilon_{ijk} x_k (\tilde{\mathcal{L}}_i a_j) + 2x_i a_i] + \frac{4}{M^4} x_i a_j \tilde{\mathcal{L}}_i \tilde{\mathcal{L}}_j. \quad (6.42) \]

By using \((6.36),(6.37),(6.38)\),

\[ \mathcal{A}(x) = \lim_{M \to \infty} \sum_{l=0}^\infty e^{\frac{-i(l+1)^2}{M^2}} (2l + 1) \left( \frac{-2}{M^2} \right) [i \varepsilon_{ijk} x_k (\tilde{\mathcal{L}}_i a_j) + 2x_i a_i]. \quad (6.43) \]
The summation over the variable \( l \) can be transferred to the integral of the continuous variable \( l \) as,

\[
A(x) = M \int_0^\infty dl e^{-l^2} (2MI) \left( -\frac{2}{M^2} \right) i\epsilon_{ijk} x_k (\tilde{L}_i a_j + 2x_i a_i) \quad (6.44)
\]

\[
= 2(-i\epsilon_{ijk} x_k (\tilde{L}_i a_j) - 2\phi \rho) \quad (6.45)
\]

\[
= 2\rho \epsilon_{ijk} x_i \partial_j a_k, \quad (6.46)
\]

where \( a_k' \) and \( \phi \) are defined in eqs. (3.5), (3.7).

Using the above results, the WT identity, \( \langle \delta S_{S^2} + 2 \int_\Omega \lambda A \rangle = 0 \), is written as

\[
\frac{\rho}{g^2} \left( \int_\Omega \lambda(x) (\tilde{L}_i (\bar{\psi} \sigma_i \gamma_3 \psi) - 2\rho \bar{\psi} \phi \psi) \right)_S
\]

\[
= \int_\Omega \lambda(x) \left[ -4i\epsilon_{ijk} x_k (\tilde{L}_i a_j) - 8\phi \rho \right] \quad (6.47)
\]

\[
= \int_\Omega \lambda(x) \left[ 4\rho \epsilon_{ijk} x_i \partial_j a_k' \right], \quad (6.48)
\]

which gives (3.17) or (3.18).

### 6.4 Evaluation for \( \lambda = 1 \) Case

In this appendix, we evaluate \( H_3 |_{\lambda=1} \) in (4.46). By making use of the identities (6.12) and (6.18), we have

\[
H_3 |_{\lambda=1} = \frac{4g^2 \rho}{\alpha (2L+1)} \sum_{l=0}^{2L} \sum_{m=-l}^{l} \sum_{n=0}^{\infty} \frac{1}{2L(L+1)} \left\{ \frac{1}{L(L+1)} \right\}^n
\]

\[
\times \text{Tr} \left( Y_{lm}^{\dagger} a_3 (L_i^L L_i^R)^n [L_3, Y_{lm}] \right)
\]

\[
\equiv \frac{4g^2 \rho}{\alpha} \frac{1}{2L(L+1)} \sum_{n=0}^{\infty} 1 \left\{ \frac{1}{L(L+1)} \right\}^n
\]

\[
\times [\text{Tr} (a_3 L_{i_n} \cdots L_{i_1} L_3) \text{Tr} (L_{i_1} \cdots L_{i_n}) - \text{Tr} (a_3 L_{i_n} \cdots L_{i_1}) \text{Tr} (L_3 L_{i_1} \cdots L_{i_n})]
\]

\[
= \frac{4g^2 \rho}{\alpha} \frac{1}{2L(L+1)} \sum_{n=0}^{\infty} f_n. \quad (6.49)
\]

In the first line, \( l = 0 \) term is added in the sum since \([L_3, Y_{00}] = 0\). In the end of this appendix, we will justify it more carefully. Several \( f_n \) with small \( n \) can be
evaluated as

\begin{align*}
  f_0 & = (2L + 1) \text{Tr}(a_3 L_3), \\
  f_1 & = -\frac{1}{3} (2L + 1) \text{Tr}(a_3 L_3), \\
  f_2 & = \frac{1}{3} (2L + 1) \text{Tr}(a_3 L_3) - \frac{1}{6L(L+1)} (2L + 1) \text{Tr}(a_3 L_3), \\
  f_3 & = \frac{1}{6L(L+1)} (2L + 1) \text{Tr}(a_3 L_3) \\
  & \quad - \frac{2L + 1}{15 \left\{ L(L+1) \right\}^2} \left[ L(L+1) (2L^2 + 2L + 1) + (L^2 + L - 1) (L^2 + L - 2) \right] \\
  & \quad \times \text{Tr}(a_3 L_3).
\end{align*}

Thus, the sum \( \sum f_n \) is proportional to \( \text{Tr}(a_3 L_3) \):

\[ \sum_{n=0}^{\infty} f_n = C \text{Tr}(a_3 L_3). \quad (6.54) \]

In order to evaluate the value of \( C \), we replace the index 3 by a general index \( i \) and sum over \( i \). We then set \( a_i = L_i \). Then the r.h.s. of eq.(6.54) becomes

\[ C \text{Tr}(a_3 L_3)|_{3 \to i, a_3 \to L_i} = CL(L+1)(2L+1). \quad (6.55) \]

On the other hand, from eq.(6.49) we obtain

\[
\sum_{n=0}^{\infty} f_n|_{3 \to i, a_3 \to L_i} =
= \frac{1}{L(L+1)} \left[ L(L+1) \text{Tr}(L_i \cdots L_i) \text{Tr}(L_i \cdots L_i) \\
- \text{Tr}(L_{i_{n+1}} L_i \cdots L_i) \text{Tr}(L_i \cdots L_i) \right] \\
= L(L+1) \{\text{Tr}1\}^2 - \lim_{n \to \infty} \frac{1}{\{L(L+1)\}^n} \{\text{Tr}(L^{n+1})\}^2 \\
= 4 \{L(L+1)\}^2 ,
\]

(6.56)
where we have used the following identity

\[
\lim_{n \to \infty} \frac{1}{\{L(L + 1)\}^n} \text{Tr} \left(L_{i_{n+1}} \cdots L_{i_1}\right) \text{Tr} \left(L_{i_1} \cdots L_{i_{n+1}}\right)
\]

\[
= \lim_{n \to \infty} \frac{1}{\{L(L + 1)\}^n} \frac{1}{2L + 1} \sum_{lm} \text{Tr}[Y^\dagger_{lm} L_{i_{n+1}} \cdots L_{i_1} Y_{lm} L_{i_1} \cdots L_{i_{n+1}}]
\]

\[
= \lim_{n \to \infty} \frac{1}{\{L(L + 1)\}^{n+1}} \sum_{lm} [L(L + 1) - \frac{1}{2}(l + 1)]^n
\]

\[
= L(L + 1).
\]  

(6.57)

From eqs. (6.54), (6.55), (6.56), we find

\[
\sum_{n=0}^{\infty} f_n = \frac{4L(L + 1)}{2L + 1} \text{Tr}(a_3 L_3).
\]  

(6.58)

Thus, from eq. (6.49), we obtain

\[
H_3|_{\lambda=1} = \frac{8g^2 \rho}{\alpha(2L + 1)} \text{Tr}(a_3 L_3).
\]  

(6.59)

In the remainder of this appendix, we treat \( l = 0 \) part more carefully by introducing a regulator \( \epsilon \) and justify the above calculation. Eq. (6.49) becomes

\[
\frac{4g^2 \rho}{\alpha(2L + 1)} \sum_{l=0}^{2L} \sum_{m=-l}^{l} \sum_{n=0}^{\infty} \frac{1}{2L(L + 1) \{L(L + 1)\}^n} \times \text{Tr} \left(Y^\dagger_{lm} a_3 \left[L_3, (L_e^R L_i^L - \epsilon) Y_{lm}\right]\right).
\]  

(6.60)
Then, after replacing 3 by \(i\) (\(i\) is summed) and setting \(a_i = L_i\), we have

\[
\sum_{n=0}^{\infty} f_n|_{i=3, a_i \rightarrow L_i} = \frac{1}{(2L + 1)} \sum_{lmn} \frac{1}{\{L(L + 1)\}^n} \left[ L(L + 1) \text{Tr} \left( Y_{tm}^{\dagger} (L_i^L L_i^R - \epsilon)^{n} Y_{tm} \right) \right.
\]

\[
- \text{Tr} \left( Y_{tm}^{\dagger} L_i^L L_i^R (L_i^L L_i^R - \epsilon)^n Y_{tm} \right) \right]
\]

\[
= \frac{1}{(2L + 1)} \left[ L(L + 1)(2L + 1)^3 \right.
\]

\[
- \lim_{n \rightarrow \infty} \frac{1}{\{L(L + 1)\}^n} \sum_{lm} \text{Tr} \left( Y_{tm}^{\dagger} (L_i^L L_i^R - \epsilon)^{n+1} Y_{tm} \right)
\]

\[
- \sum_{lmn} \frac{\epsilon}{\{L(L + 1)\}^n} \text{Tr} \left( Y_{tm}^{\dagger} (L_i^L L_i^R - \epsilon)^n Y_{tm} \right) \right]
\]

\[
= \frac{1}{(2L + 1)} \left[ L(L + 1)(2L + 1)^3 - 0 - 2L(L + 1)(2L + 1) \right]
\]

\[= 4L(L + 1)^2, \quad (6.61) \]

which exactly agrees with (6.56).

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