Holographic Superconductor for a Lifshitz fixed point

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Abstract

We consider black hole duals of Lifshitz-fixed points at finite temperature, which was constructed in a recent work arXiv:0909.0263. We develop holographic techniques from equilibrium, to transport and to superconductivity.
1 Motivation

AdS/CFT correspondence is one of the most important results from string theory [1]. The connection between gauge theory and strings has a long history since the appearance of string models of hadrons in 1960’s. According to Polyakov’s view [2], more concretely, by observation of K.Wilson’s work [3], we can see that in the strong coupling limit of a lattice gauge theory the elementary excitations are represented by strings formed by the colorelectric fluxes. It shows that in a certain limit all the degrees of freedom in the gauge theory should be represented by flux lines (strings) but not fields. It is natural for us to expect an exact duality between gauge fields and strings. This expect has been achieved by the original work [1]. A semi-classic version of this duality has appeared as gauge/gravity duality and such duality has become a powerful tool to understand the strongly coupled gauge theory and it was extended to describe aspects of strongly coupled QCD such as properties of quark gluon plasma in heavy ion collisions at RHIC [4, 5, 6] and hadron physics. To understand strongly coupled gauge theory, the fluid/gravity duality has been developed. To understand gravity itself, like low dimensional quantum gravity, the boundary gauge theory is believed as the complete quantum theory of the bulk gravity. Even in four dimension, one special gravity system like Kerr black hole has been mapped to a CFT$_2$ [7].
More recently, it has been attempted to use this correspondence to describe certain condensed matter systems such as the Quantum Hall effect [8], Nernst effect [9, 10, 11], superconductor [12, 13, 15] and FQHE (fractional quantum hall effect) [16]. All of these phenomena have dual gravitational descriptions. As pointed in [17], there is a large class of interesting strongly correlated electron and atomic systems that can be created and studied in experiments. In some special conditions, these systems exhibit relativistic dispersion relations, so the dynamics near a critical point is well described by a relativistic conformal field theory. It is expected that such field theories which can be studied holographically have dual AdS geometries. To describe more non-relativistic condensed matter systems, this duality has even been extended to non-relativistic conformal field theory which has Schrödinger symmetry [17] or Lifshitz symmetry [18]. With this motivation, 4D black hole solutions supported by the action in [18] with asymptotically Lifshitz spacetimes were investigated [19, 20, 21, 22, 23]. The Lifshitz black hole in any space-time dimensions were found out but supported by a different action from the above [24]. Recently another 4D analytical solution also under a different action was proposed only for z=2 [25]. Beside these, Lifshitz black holes in three-dimensional massive gravity and four-dimensional $R^2$ gravity were also discussed [26, 27]. The problem of embedding those black holes with the action in [18] in string theory was addressed in [28].

In [12], a model of a strongly coupled system which develops superconductivity was constructed based on the holography, which is an Abelian-Higgs model in a warped space time. While the electrons in real materials are non-relativistic, the model in [12] is for relativistic system. Therefore it is natural to ask whether one can develop a similar theory with non-relativistic kinematics [36], especially at Lifshitz-like fixed point. One of purposes of this paper is to answer this question.

This paper is organized as follows. In section 2, we check the thermodynamics of the Lifshitz Black hole and chemical potential background. In section 3, we calculate the scalar correlation function which can be used to calculate several transport coefficients. We find that, the correlation function has a special behavior. In the following section, we study the superconductive phases in this kind of background. We obtain the similar results as ones in the usual AdS black hole background.

## 2 Equilibrium

In this part, we begin with the equilibrium properties of the strongly coupled thermal field at Lifshitz-like fixed point, by analyzing the Lifshitz black hole solutions.
2.1 Lifshitz black hole solutions

We start from the Lifishitz scaling

\[ t \to \lambda^z t, \; x \to \lambda x, \]

where \( z \) is the so-called dynamical exponent. In the gravitational dual, the spacetime metric which geometrizes these symmetries was first found in [18]:

\[ ds^2 = L^2 \left( -\frac{dt^2}{r^{2z}} + \frac{dx^2 + dy^2}{r^2} + \frac{dr^2}{r^2} \right), \]

where \( 0 < r < \infty \) and \( L \) sets the scale for the radius of curvature of the geometry. For \( z = 1 \), this is just the anti-deSitter spacetime supported by the usual Einstein-Hilbert action plus a negative cosmological constant. For \( z > 1 \) these spacetimes are candidate duals to nonrelativistic field theories and the corresponding action must contain some matter fields besides the pure gravity term.

\[ S = \int d^4x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu} \right), \]

where \( F_{(p),(q)} \) are the field strength of the gauge fields with \( p=1,2 \), \( F_{(2)} = dA_{(1)} \), \( F_{(3)} = dB_{(2)} \). The 4D cosmological constant \( \Lambda = -\frac{z^2 + z + 4}{2L^2} \) and the topological coupling \( c \) is related to \( z \) by \( 2z = (cL)^2 \). The tidal forces diverge on the “horizon” at \( r \to \infty \) unless \( z = 1 \) and this implies that the metric (2) has no global extension [29]. To describe the physics of the dual field theory at finite temperature, black hole solutions with asymptotical Lifshitz metric [2] were proposed in [19, 20, 21, 22, 23].

The action that differs from (3) can also support a Lifshitz black hole solution [21]. It includes a massless scalar coupled appropriately to a U(1) gauge field:

\[ S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu} \right), \]

with the cosmological constant \( \Lambda = -\frac{(z+1)(z+2)}{2L^2} \). The solution to this system is [4]

\[ ds^2 = L^2 \left( -f(r) \frac{dt^2}{r^{2z}} + \frac{dx^2 + dy^2}{r^2} + \frac{dr^2}{r^2 f(r)} \right), \]

\[ F_{rt} = -\frac{\sqrt{2(z-1)(z+2)L}}{r^{z+3}}, \; e^{\lambda\phi} = r^4; \]

\[ f(r) = 1 - \frac{r^{z+2}}{r^2 H}, \; (z-1)\Lambda^2 = 4, \]

\[ ^4 \text{Here we choose a simple form in [30] with } d=2 \text{ compared to the original one given in [24]. This solution can be extended to any spacetime dimension.} \]
and the relative thermodynamic quantities can be calculated.

Recently, another 4D black hole solution which asymptotes to the Lifishiz spacetime \(^{(2)}\) with only \(z = 2\) was constructed by a strongly-coupled scalar (without kinetic terms) and a massive vector field \([25]\). The supporting action is

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} (R - 2\Lambda) - \int d^4x \sqrt{-g} \left( \frac{e^{-2\phi}}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu + (e^{-2\phi} - 1) \right),
\]

with \(\Lambda = -\frac{z^2 + z + 4}{2}\), \(m^2 = 2z\) and the field strength \(F = dA\). The gravitational constant and curvature radius are set to \(8\pi G_4 = 1\) and \(L = 1\). With this convention the equations of motion of each field are

\[
F^2 = -4, \quad \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} e^{-2\phi} F^{\mu\nu}) = m^2 A^\nu,
\]

\[
R_{\mu\nu} = e^{-2\phi} F_{\mu\lambda} F^{\lambda}_\nu + m^2 A_\mu A^\nu + \Lambda g_{\mu\nu} + (2e^{-2\phi} - 1) g_{\mu\nu},
\]

and a black hole solution of this system is \(^3\)

\[
ds^2 = -f(r) \frac{dt^2}{r^2} + \frac{dx^2 + dy^2}{r^2} + \frac{dr^2}{r^2 f(r)},
\]

\[
f(r) = 1 - \frac{\ell^2}{r^2}, \quad e^{-2\phi} = 1 + \frac{\ell^2}{r^2}, \quad A = \frac{f(r)}{\sqrt{2r^2}} dt.
\]

In the rest parts of this paper, we will use this solution to discuss the transport and superconductivity.

### 2.2 Thermodynamics

We first review the thermodynamics of this black hole proposed in \([25]\). Our calculation procedure follows \([29]\). According to the AdS/CFT dictionary, within a semiclassical regime the partition function of the bulk theory is to be equivalent to the partition function of the dual field theory as a path integral over metrics. Given the dominant saddle \(g_*\), the partition is

\[
Z = e^{-S_E[g_*]},
\]

where \(S_E[g_*]\) is the Euclidean action evaluated on the saddle. This action must contain extrinsic boundary terms and intrinsic boundary terms in order to render the finiteness of the on-shell action, the internal energy and pressure of the boundary theory and to hold

\(^5\)There is a factor 1/\(\sqrt{2}\) missing in the expression (2.5) of the massive vector field in \([25]\).
the conformal Ward identity. This was already given in [25]:

\[ S_E = -\frac{1}{2} \int d^4x \sqrt{g} (R - 2\Lambda) + \int d^4x \sqrt{g} \left( \frac{e^{-2\phi}}{4} F^2 + \frac{m^2}{2} A^2 + (e^{-2\phi} - 1) \right) \]

\[ + \int_{r \to 0} d^3x \sqrt{\gamma} K - \frac{1}{2} \int_{r \to 0} d^3x \sqrt{\gamma} \left( -\frac{27}{8} + \frac{7}{2}\phi^2 + \frac{7}{2}\phi \right) \]

\[ -\frac{1}{2} \int_{r \to 0} d^3x \sqrt{\gamma} \left( \left( \frac{17}{2} + 7\phi \right) A^2 + \frac{13}{2} A^4 \right), \]  

where \( \gamma \) is the induced metric on the boundary \( r \to 0 \) and \( K \) is the trace of the extrinsic curvature. The Dirichlet boundary condition is imposed on the massive vector field in the above action.

One saddle is obtained after making the Wick rotating \( \tau = it \) to (8):

\[ ds_*^2 = f(r) \frac{d\tau^2}{r^2} + \frac{dx^2 + dy^2}{r^2} + \frac{dr^2}{r^2 f(r)}, \quad A = -i \frac{f(r)}{\sqrt{2r^2}} d\tau \]

It’s well known that the temperature of the system is the inverse of the periodicity of the Euclidean time

\[ T = \frac{1}{\beta} = \frac{1}{2\pi r_H^2}, \]  

which is required by the absence of the conical singularity at \( r = r_H \).

Given the temperature, then we can evaluate the action (10) on the dominant saddle (11):

\[ S_E[g_*] = -\beta \frac{L_x L_y}{2r_H^4} = -2\pi^2 L_x L_y T, \]

and the free energy

\[ F = -T \log Z = TS_E[g_*] = -\frac{L_x L_y}{2r_H^4} = -2\pi^2 L_x L_y T^2, \]

as given in [25].

As a check, the entropy

\[ S = -\frac{\partial F}{\partial T} = 4\pi^2 L_x L_y T, \]

is coincide with the Hawking entropy \( S = 2\pi A \) with the area of the event horizon \( A = L_x L_y / \pi r_H^2 \) and the unit convention \( 8\pi G = 1. \)

\[^6\text{That is } c_N = 0 \text{ in the expression (3.3) in [25]. We rewrite that expression into Euclidean space and substitute the specific values of } c_0-c_5. \text{ There is a minus sign difference of } c_1-c_5 \text{ here from those given in Appendix A of [25].} \]
The boundary stress tensor resulting from (10) is

\[ T_{\mu\nu} \equiv -\frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{\mu\nu}} = K_{\mu\nu} - \left( \frac{17}{2} + 7\phi + 13A^2 \right) A_{\mu}A_{\nu} - K\gamma_{\mu\nu} \]

\[ + \frac{1}{2} \left( -\frac{27}{8} + \frac{7}{2}\phi + \frac{7}{2}\phi^2 \right) \gamma_{\mu\nu} + \frac{1}{2} \left( \left( \frac{17}{2} + 7\phi \right) A^2 + \frac{13}{2} A^4 \right) \gamma_{\mu\nu}, \]

then the internal energy and pressure of boundary theory following [25] are respectively

\[ E = -L_xL_y\sqrt{-\gamma}T^t_t = \frac{L_xL_y}{2r_H^4}, \]

\[ P = \frac{1}{2}L_xL_y\sqrt{-\gamma}T^i_i = L_xL_y\sqrt{-\gamma}T^x_x = \frac{L_xL_y}{2r_H^4}. \]

Thus

\[ E = P = -F = \frac{1}{2}TS. \]

The first law of thermodynamics \( E + P = TS \) is satisfied as given in [25].

### 2.3 Finite chemical potential

Consider a massless vector fluctuation \( A_{\mu} \) in the Lifshitz black hole background. That is to add a Maxwell term to the original action (6),

\[ S = -\frac{1}{4} \int d^4x \sqrt{-g}F_{\mu\nu}F^{\mu\nu}. \]

For simplicity, we ignore the back reaction. This means \( A_{\mu} \) is a small perturbation and the metric is still as the same as (8). This vector field is expected to support a charge current operator \( J_{\mu} \) in the dual field. The equation of motion of \( A_{\mu} \) is

\[ \frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}F^{\mu\nu}) = 0 \]

If we only consider the zero component of \( A_{\mu} \), \( A = \phi(r)dt \), then we have

\[ \phi'' + \frac{z-1}{r}\phi' = 0. \]

Near the boundary,

\[ \phi = \phi_{(0)} + \phi_{(1)}r^{2-z}, \]

where \( \phi_{(0)} \) and \( \phi_{(1)} \) are chemical potential and charge density respectively in the dual field theory. In the special case \( z = 2 \),

\[ A_0 = \mu + Q \log r. \]

Due to the singularity in the boundary, charge can not be allowed in our system.

\[ \text{There are also some minus sign differences between the expression we give and the one in (3.4) of [25].} \]
3 Transport

In this part, we would like to study the transport coefficients of the hydrodynamics in the thermal field theory which is expected to dual of the Lifshitz black hole.

3.1 Scalar correlation function

Now, we focus on the solution found in [25]. Remember that this solution contains a non-trivial dilaton, and a massive vector field which turns the source from the boundary. We rewrite the metric

\[ ds^2 = -\frac{f dt^2}{r^{2s}} + \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{fr^2}, \quad f = 1 - \frac{r^2}{r_H^2}. \] (24)

Consider a massive real scalar field theory in this curve background, with the action

\[ S = -\frac{1}{2} \int d^4x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2). \] (25)

After inserting the metric into the scalar action, we obtain

\[ S = -\frac{1}{2} \int d^4x [-\frac{1}{rf}(\partial_t \phi)^2 + \frac{f}{r^3}(\partial_r \phi)^2 + \frac{1}{r^3}((\partial_x \phi)^2 + (\partial_y \phi)^2) + \frac{m^2}{r^3} \phi^2] \] (26)

So the equation of motion is

\[ f \partial_r^2 \phi - \frac{2f}{r^3} \partial_r \phi - \frac{r^2}{f} \partial_t^2 \phi - \partial_x^2 \phi - \partial_y^2 \phi - \frac{m^2}{r^2} \phi = 0. \] (27)

Assume the scalar field satisfies

\[ \phi(r, x) = \int d^3x' \phi(0, x') G(r, x; 0, x'), \] (28)

where \( G(u, x; 0, x') \) is the propagator from the boundary to the bulk spacetime. Due to the translational invariance in the \( x, y, t \) directions, the equation (28) in Fourier space becomes

\[ \tilde{\phi}_k(r) = \tilde{\phi}(u, k) = \tilde{G}(u, k) \tilde{\phi}(0, k), \] (29)

where \( k = (\omega, \vec{k}) \). Then the equation of motion becomes

\[ f \partial_r^2 \tilde{G} - \frac{2f}{r^3} \partial_r \tilde{G} - \frac{r^2 \omega^2}{f} \tilde{G} + k^2 \tilde{G} - \frac{m^2}{r^2} \tilde{G} = 0. \] (30)

So far, the formulas are both right for solution (8) and (12). According to [25], this equation has an exact solution through replacing \( u \equiv \frac{r^2}{r_H^2} \). We can define the correlation function by the coefficients of the asymptotic solution near \( u = 0 \).
3.2 Real time correlation

Since we have the Lifshitz black hole solution \((12)\), we can consider the hydrodynamic behavior of their dual field theories from the gravity side. The transport coefficients come from the real time correlation function. To obtain such a real time correlation, we first substitute the equation of motion \((27)\) and the formula \((28)\) into the action \((26)\) to get the on-shell action of scalar field

\[
S = -\frac{1}{2} \int d^3x \int^\infty_\epsilon du \left( [m^2 \phi - \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \phi)] \phi + \partial_\mu (\sqrt{-g} g^{\mu \nu} \phi \partial_\nu \phi) \right). \tag{31}
\]

Then integrated out the coordinate \(u\), and set \(m = 0\), the boundary action is

\[
S = -\frac{1}{2} \int d^3x \left[ (\sqrt{-g} g^{rr} \partial_r \phi) \right]^{\infty}_\epsilon = \int d^3k d\omega \phi(0, -\vec{k}, -\omega) F(\vec{k}, \omega) \phi(0, \vec{k}, \omega), \tag{32}
\]

with

\[
F = -\left[ \frac{1}{8\pi^3} \frac{f}{r^3} \tilde{G}(r, -\vec{k}, -\omega) \partial_t \tilde{G}(r, \vec{k}, \omega) \right]^{\infty}_\epsilon. \tag{33}
\]

Following the method in \([31]\), we obtain the Minkowski retarded Green’s function

\[
G_R(\vec{k}, \omega) = \lim_{r \to \epsilon} \frac{2}{8\pi^3} \frac{f}{r^3} \tilde{G}(r, -\vec{k}, -\omega) \partial_t \tilde{G}(r, \vec{k}, \omega). \tag{34}
\]

If we use the exact solution below \((35)\), this correlation function will be divergent, we should search for the coefficient of the \(r^4\) term. In the massless case, the behavior of the solution near \(u = 0\) is

\[
\phi(u, \vec{k}, \omega) = 1 - \frac{u}{4}(\vec{k}^2 + 2i\omega) - \frac{u^2}{64} \left( (\vec{k}^2)^2 + 4\omega^2 \right) \left[ -3 + 2\psi \left( \frac{1}{2}(-1 + i\omega - \sqrt{1 - \vec{k}^2 - \omega^2}) \right) + 2\psi \left( \frac{1}{2}(-1 + i\omega + \sqrt{1 - \vec{k}^2 - \omega^2}) \right) + 2\gamma_E + 2\ln u \right] + O(u^3). \tag{35}
\]

Then, from \((34)\) we get the real time retarded Green’s function

\[
G_R(\vec{k}, \omega) = \frac{2}{8\pi^3} \frac{4}{64} \left( (\vec{k}^2)^2 + 4\omega^2 \right) \left[ -3 + 2\psi \left( \frac{1}{2}(-1 + i\omega - \sqrt{1 - \vec{k}^2 - \omega^2}) \right) + 2\psi \left( \frac{1}{2}(-1 + i\omega + \sqrt{1 - \vec{k}^2 - \omega^2}) \right) + 2\gamma_E \right].
\]

Remember \(u = \frac{x^2}{r_H^2}\), \(\gamma_E\) is Euler’s constant, and \(\psi\) is the digamma function. To calculate the transport coefficients which determined by the real-time scalar correlation function, we should note that the Kubo-like formula should be modified. We leave this to the future work.
4 Superconductivity

In this section, we shall build an Abelian-Higgs model \[32, 12\] in the Lifshitz black hole background and study the superconductive phases. We introduce a new $U(1)$ gauge field $A_\mu$ which is different from that in the action (6) and also introduce a complex scalar $\psi$. We assume that the background response is negligible for simplicity.

4.1 Superconducting phases

Consider the Lagrangian density

$$
\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |\nabla \psi - i A \psi|^2 - V(|\psi|),
$$

then the equations of motion for $A$ and $\psi$ are respectively

$$
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) = iq [\psi^* (\partial^\nu - iq A^\nu) \psi - \psi (\partial^\nu + iq A^\nu) \psi^*],
$$

$$
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} (\partial^\mu \psi - iq A^\mu \psi)) - iq A^\mu (\partial_\mu \psi - iq A_\mu \psi) - \frac{\psi}{2|\psi|} V'(|\psi|) = 0.
$$

We will work in the probe limit, in which $A_\mu$ and $\psi$ is taken small so that their backreactions on the spacetime metric can be ignored. The metric is still a 4D Lifshitz black hole with $z=2$ in (8). Taking the ansatz $A = \phi(r) dt, \psi = \psi(r)$, the equations of motion (37) and (38) reduce to

$$
\phi'' + \frac{z - 1}{r} \phi' - \frac{2\psi^2}{r^2 f(r)} \phi = 0,
$$

$$
\psi'' + \left[ \frac{f'(r)}{f(r)} - \frac{z + 1}{r} \right] \psi' + \frac{r^{2z-2} \phi^2}{f^2(r)} \psi - \frac{V'(\psi)}{2r^2 f(r)} = 0.
$$

where $\psi$ can be taken real which is allowed by the r-component of (37). For simplicity we will specialize to simple potential $V(\psi) = m^2 \psi^2$ with $m^2 < 0$ but above the Breitenlohner-Freedman bound. Then the bulk fields will behave near the boundary $r \to 0$ as

$$
\phi = \mu + r^{\nu_2 - z} \ldots
$$

$$
\psi = \psi(0) r^{\nu_2} + \psi(1) r^{\nu_1} + \ldots
$$

with $\nu_\pm = \frac{z+2}{2} \pm \sqrt{m^2 + \left( \frac{z+2}{2} \right)^2}$. At the horizon, for $A$ to have a finite form, $\phi(r_H) = 0$ and (40) then implies $\psi'(r_H) = -\frac{m^2}{2r_H} \psi(r_H)$. 

10
For \( z = 2 \), there is only chemical potential \( \mu \) in the dual field, which will set the scale of the critical temperature of the superconductor, \( T_c \propto \mu^z \). Here we also set \( m^2 = -3 \), \( \nu_\pm \) in (42) is simplified to \( \nu_- = 1 \), \( \nu_+ = 3 \). The Condensate of the scalar operator \( \mathcal{O} \) in the dual field to \( \psi \) is
\[
\langle \mathcal{O} \rangle = \psi(1) \tag{43}
\]
with the boundary condition \( \psi(0) = 0 \). \( \mathcal{O}/T_c^2 \) is a dimensionless quantity. We can solve the equations (39) and (40) numerically and finally get a condensation curve shown in figure 1. The curve is similar to that obtained in BCS theory and that in \( z=1 \) holographic superconductor [12]. The condensate goes to a constant as the temperature goes to zero.

### 4.2 Conductivity

In order to compute the electric conductivity, we follow the standard procedure in [29]. Firstly, let \( A_x \) and the metric \( g_{tx} \) have a zero spatial momentum fluctuations\(^\text{8}\)

\[
A = \phi(r) dt + A_x(r) e^{-i\omega t} dx,
\]
\[
ds^2 = -f(r) \frac{dt^2}{r^2} + \frac{dx^2 + dy^2}{r^2} + \frac{dr^2}{r^2 f(r)} + 2 g_{tx}(r) e^{-i\omega t} dt dx.
\]

Then the Maxwell and Einstein equations
\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu \nu}) = 2 \psi^2 A^\nu, \quad R_{\mu \nu} = F_{\rho \mu} F^\rho_{\nu} \tag{46}
\]

\(^8\)Although in this paper we work in the probe limit and do not discuss the thermal conductivity, the fluctuation of the metric \( g_{tx} \) can be ignored.
give the equations of the fluctuation of $A_x$ and $g_{tx}$:

$$A''_x + \left[ \frac{f'(r)}{f(r)} - \frac{z - 1}{r} \right] A'_x + \left[ \frac{\omega^2 r^{2z-2} - 2\psi^2}{f^2(r)} - \frac{2\phi'}{2z} \right] A_x = -\frac{r^{2z} \phi'}{f(r)} \left[ g'_{tx} + \frac{2}{r} g_{tx} \right], \quad (47)$$

$$\frac{g'_{tx}}{2} + \frac{g_{tx}}{r} + \phi' A_x = 0. \quad (48)$$

Finally, we find

$$A''_x + \left[ \frac{f'(r)}{f(r)} - \frac{z - 1}{r} \right] A'_x + \left[ \frac{\omega^2 r^{2z-2} - 2\psi^2}{f^2(r)} - \frac{2\phi'^2}{2z} \right] A_x = 0. \quad (49)$$

The last term of $\phi'^2$ coming from the fluctuation of $g_{tx}$ can be ignored since we work in the probe limit.

On the horizon, we choose the ingoing wave boundary conditions,

$$A_x \propto f(r)^{-i\omega r H/2}. \quad (50)$$

Near the boundary, the field behaves as

$$A_x = A_x(0) + A_x(1) r^z + \ldots, \quad (51)$$

where $A_x(0)$ gives the background electric field in the dual field theory, $E_x = i\omega A_x(0)$ and $A_x(1)$ is related to the expectation of electric current $J_x$.

For the gauge field (44) and the metric (45), the Maxwell action reduces to

$$S = -\frac{2}{4} \int d^4x \sqrt{-g} \left[ g^{rr}(g^{tx} A'_x + 2g^{tx} \phi' A'_x + g^{tt} \phi'^2) - \omega^2 A_x^2 (g^{tt} g^{tx} - g^{tx} g^{tx}) \right], \quad (52)$$

then the expectation of the electric current can be obtained from this action,

$$\langle J^x \rangle = \frac{\delta S_{\text{on-shell}}}{\delta A_x(0)} = -\lim_{r \to 0} \frac{\delta S}{\delta \partial_r A_x(0)}, \quad (53)$$

with the notation $\partial_r A_x(0) = A'_x(r)$. From the above, we get

$$\langle J^x \rangle = \lim_{r \to 0} \sqrt{-g} g^{rr} (g^{tx} A'_x + g^{tx} \phi') = z A_{x(1)} + (2 - z) \phi_{(1)} g_{tx(0)}, \quad (54)$$

which gives the electric conductivity

$$\sigma(\omega) = \frac{\langle J^x \rangle}{E_x} = -\frac{i}{\omega} \frac{z A_{x(1)}}{A_x(0)}. \quad (55)$$

All left is to solve equation (49) to obtain the electric conductivity in (55). We plot the real and imaginary part of conductivity at $T > T_c$ in Fig.2 and Fig.3. Near $\omega = 0$, we observe the delta function both for the real and imaginary, as shown in Fig.5. There also appears a gap with small fluctuations for the real part at some finite $\omega$. 

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Figure 2: Real part of the conductivity at $T > T_c$.

Figure 3: Imaginary part of the conductivity at $T > T_c$.

Figure 4: Real part of the conductivity at $T < T_c$. 
Figure 5: Imaginary part of the conductivity at $T < T_c$.

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