Low regularity solutions of two fifth-order KdV type equations

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Abstract: The Kawahara and modified Kawahara equations are fifth-order KdV type equations and have been derived to model many physical phenomena such as gravity-capillary waves and magneto-sound propagation in plasmas. This paper establishes the local well-posedness of the initial-value problem for the Kawahara equation in \( H^s(\mathbb{R}) \) with \( s > -\frac{7}{4} \) and the local well-posedness for the modified Kawahara equation in \( H^s(\mathbb{R}) \) with \( s \geq -\frac{1}{4} \). To prove these results, we derive a fundamental estimate on dyadic blocks for the Kawahara equation through the \([k; Z]\) multiplier norm method of Tao [14] and use this to obtain new bilinear and trilinear estimates in suitable Bourgain spaces.

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1 Introduction

This paper is mainly concerned with the local well-posedness of the initial-value problems (IVP) for the Kawahara equation

\[
\begin{align*}
\left\{\begin{array}{ll}
u_t + uu_x + \alpha u_{xxx} + \beta u_{xxxx} = 0, & x,t \in \mathbb{R}, \\
u(x,0) = u_0(x). &
\end{array}\right.
\end{align*}
\]
and for the modified Kawahara equation

\[
\begin{aligned}
\left\{
\begin{array}{l}
u_t + u^2u_x + \alpha u_{xxx} + \beta u_{xxxx} = 0, \quad x, t \in \mathbb{R}, \\
u(x, 0) = u_0(x),
\end{array}
\right.
\end{aligned}
\]

(1.2)

where \(\alpha\) and \(\beta\) are real constants and \(\beta \neq 0\). Attention will be focused on solutions in Sobolev spaces of negative indices. These fifth-order KdV type equations arise in modeling gravity-capillary waves on a shallow layer and magneto-sound propagation in plasmas (see e.g. [3], [10]).

The well-posedness issue on these fifth-order KdV type equations has previously been studied by several authors. In [11], Ponce considered a general fifth-order KdV equation

\[
u_t + u_x + c_1 uu_x + c_2 u_{xxx} + c_3 u_x u_{xx} + c_4 uu_{xxx} + c_5 u_{xxxx} = 0, \quad x, t \in \mathbb{R}
\]

and established the global well-posedness of the corresponding IVP for any initial data in \(H^4(\mathbb{R})\). In [7] and [8], Kenig, Ponce and Vega studied the local well-posedness of the IVP for the following odd-order equation

\[
u_t + \partial_x^{2j+1}u + P(u, \partial_x u, \cdots, \partial_x^{2j} u) = 0
\]

where \(P\) is a polynomial having no constant or linear terms. They obtained the local well-posedness for

\[u_0 \in H^s(\mathbb{R}) \cap L^2(|x|^m dx),\]

where \(s, m \in \mathbb{Z}^+\). Cui, Deng and Tao in [2] established the local well-posedness in \(H^s\) with \(s > -1\) for the Kawahara equation. Wang, Cui and Deng in a very recent work [15] obtained the local well-posedness in \(H^s\) with \(s \geq -\frac{7}{5}\) for the Kawahara equation by the same method as in [2]. Their method is derived from that of Kenig, Ponce and Vega [9] for the cubic KdV equation. In [13], Tao and Cui studied the low regularity solutions of the modified Kawahara equation (1.2) and proved the local well-posedness of the IVP in any Sobolev space \(H^s(\mathbb{R})\) with \(s \geq \frac{1}{4}\) by employing an approach of Kenig-Ponce-Vega for the generalized KdV equations [5].

Our goal here is to improve the existing low regularity well-posedness results. To this end, we first derive a fundamental estimate on dyadic blocks (see Lemma 3.2 below) for the Kawahara equation by following the idea in the \([k; \mathbb{Z}]\)-multiplier norm method introduced by Tao [14]. We then apply this fundamental estimate to establish new bilinear and trilinear estimates in Bourgain spaces. Combining these estimates with a contraction mapping argument, we are able to prove the following two theorems.

**Theorem 1.1.** Let \(s > -\frac{7}{4}\) and \(u_0 \in H^s(\mathbb{R})\). Then there exist \(b = b(s) \in (\frac{2}{7}, 1)\) and \(T = T(||u_0||_{H^s}) > 0\) such that the IVP (1.1) has a unique solution on \([0, T]\) satisfying

\[u \in C([0, T]; H^s(\mathbb{R})) \quad \text{and} \quad u \in X_{s,b},\]

where \(X_{s,b}\) is a Bourgain type space (defined in the next section). In addition, the dependence of \(u\) on \(u_0\) is Lipschitz.
Theorem 1.2. Let $s \geq -\frac{1}{4}$ and $u_0 \in H^s(\mathbb{R})$. Then there exist $b = b(s) \in (\frac{1}{2}, 1)$ and $T = T(\|u_0\|_{H^s}) > 0$ such that the IVP for the modified Kawahara equation (1.2) has a unique solution on $[0, T]$ satisfying

$$u \in C([0, T]; H^s(\mathbb{R}))$$

and $u \in X_{s,b}$,

and the dependence of $u$ on $u_0$ is Lipschitz.

The proofs of Theorems 1.1 and 1.2 will be provided in the subsequent sections.

2 Linear and bilinear estimates for the Kawahara equation

This section provides the linear and bilinear estimates for the Kawahara equation. We start with a few notation. Denote by $W(t)$ the unitary group generating the solution of the IVP for the linear equation

$$\begin{cases} v_t + \alpha v_{xxx} + \beta v_{xxxx} = 0, & x \in \mathbb{R}, \ t \in \mathbb{R}, \\ v(x, 0) = v_0(x). \end{cases}$$

That is,

$$v(x, t) = W(t)v_0(x) = S_t * v_0(x),$$

where $S_t = e^{itp(\xi)}$ with $p(\xi) = -\beta \xi^5 + \alpha \xi^3$, or

$$S_t(x) = \int e^{i(x\xi + tp(\xi))} d\xi.$$

For $s, b \in \mathbb{R}$, let $X_{s,b}$ denote the completion of the functions in $C^\infty_0$ with respect to the norm

$$\|f\|_{X_{s,b}}^2 = \int \langle \xi \rangle^{2s} (\tau - p(\xi))^{2b} |\hat{f}(\xi, \tau)|^2 d\xi d\tau,$$

where $\langle \xi \rangle = 1 + |\xi|$. It is easy to verify that

$$\|f\|_{X_{s,b}} = \|J^s \Lambda^b W(-t)f\|_{L^2_x}$$

where

$$\hat{\mathcal{J}}g(\xi) = (1 + |\xi|)\hat{g}(\xi), \quad \hat{\mathcal{N}}h(\tau) = (1 + |\tau|)\hat{h}(\tau).$$

Let $\psi \in C^\infty_0$ be a standard bump function and consider the following integral equation

$$u(t) = \psi(\delta^{-1}t)W(t)u_0 - \psi(\delta^{-1}t) \int_0^t W(t-t')u(t')\partial_x u(t') dt'.$$

Denote the right-hand side by $\mathcal{T}(u)$. The goal is to show that $\mathcal{T}(u)$ is contraction on the following complete metric space $Y$, where

$$Y = \{u \in X_{s,b} : \|u\|_{X_{s,b}} \leq 2c_0 \delta^{(1-2b)/2}\|u_0\|_{H^s}\}$$
with metric
\[ d(u, v) = \|u - v\|_{X_{s, b}}, \quad u, v \in Y, \]
where \( c_0 \) is the constant appeared in Proposition 2.1. For this purpose, we need two linear estimates and one bilinear estimate stated in the following propositions.

**Proposition 2.1.** For \( s \in \mathbb{R} \) and \( b > \frac{1}{2} \),
\[
\| \psi(\delta^{-1}t)W(t)u_0 \|_{X_{s, b}} \leq c_0 \delta^{(1-2b)/2} \|u_0\|_{H^s},
\]
\[
\left\| \psi(\delta^{-1}t) \int_0^t W(t - t')f(t') dt' \right\|_{X_{s, b}} \leq c_0 \delta^{(1-2b)/2} \|f\|_{X_{s, b-1}}.
\]

The proof of these estimates follows directly from Kenig, Ponce and Vega [6].

**Proposition 2.2.** For any \( s > -\frac{7}{4} \), there is \( b \) satisfying \( \frac{1}{2} < b < 1 \) such that
\[
\| \partial_x (uv) \|_{X_{s, b-1}} \leq c_1 \|u\|_{X_{s, b}} \|v\|_{X_{s, b}}, \quad (2.1)
\]
where \( c_1 \) is a constant depending on \( s \) and \( b \) only.

Proposition 2.2 will be proved in Section 4 and in the next section we introduce Tao’s \([k; Z]\)-multiplier norm method and prove a fundamental estimate on dyadic blocks for the Kawahara equation from which a variety of bilinear estimates can be derived. Once the estimates in Propositions 2.1 and 2.2 are available, a standard argument then yields that \( T(u) \) is a contraction on \( Y \).

### 3 Fundamental estimate on dyadic blocks for the Kawahara equation

In this section we introduce Tao’s \([k; Z]\)-multiplier norm method and establish the fundamental estimate on dyadic blocks, i.e., Lemma 3.2 for the Kawahara equation from which Proposition 2.2 and other bilinear estimates (see Lemma 5.2 below) could be derived.

Let \( Z \) be any abelian additive group with an invariant measure \( d\xi \). For any integer \( k \geq 2 \), we let \( \Gamma_k(Z) \) denote the hyperplane
\[
\Gamma_k(Z) := \{ (\xi_1, \cdots, \xi_k) \in Z^k : \xi_1 + \cdots + \xi_k = 0 \}
\]
which is endowed with the measure
\[
\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\xi_1, \cdots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1}) d\xi_1 \cdots d\xi_{k-1}.
\]

A \([k; Z]\)-multiplier is defined to be any function \( m : \Gamma_k(Z) \to \mathbb{C} \) which was introduced by Tao in [14]. And the multiplier norm \( \|m\|_{[k; Z]} \) is defined to be the best constant such that the inequality
\[
\left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^k f_j(\xi_j) \right| \leq c \prod_{j=1}^k \|f_j\|_{L^2(Z)}, \quad (3.1)
\]
holds for all test functions \( f_j \) on \( Z \). Tao systematically studied this kind of weighted convolution estimates on \( L^2 \) in [14]. To establish the fundamental estimate on dyadic blocks for the Kawahara equation, we use some notations.

We use \( A \lesssim B \) to denote the statement that \( A \leq CB \) for some large constant \( C \) which may vary from line to line and depend on various parameters, and similarly use \( A \ll B \) to denote the statement \( A \leq C^{-1}B \). We use \( A \sim B \) to denote the statement that \( A \lesssim B \lesssim A \).

Any summations over capitalized variables such as \( N_j, L_j, H \) are presumed to be dyadic, i.e., these variables range over numbers of the form \( 2^k \) for \( k \in \mathbb{Z} \). Let \( N_1, N_2, N_3 > 0 \). It will be convenient to define the quantities \( N_{\max} \geq N_{\text{med}} \geq N_{\min} \) to be the maximum, median, and minimum of \( N_1, N_2, N_3 \) respectively. Similarly define \( L_{\max} \geq L_{\text{med}} \geq L_{\min} \) whenever \( L_1, L_2, L_3 > 0 \). And we also adopt the following summation conventions. Any summation of the form \( L_{\max} \sim \cdots \) is a sum over the three dyadic variables \( L_1, L_2, L_3 \geq 1 \), thus for instance
\[
\sum_{L_{\max} \sim H} := \sum_{L_1, L_2, L_3 \geq 1: L_{\max} \sim H}.
\]
Similarly, any summation of the form \( N_{\max} \sim \cdots \) sum over the three dyadic variables \( N_1, N_2, N_3 > 0 \), thus for instance
\[
\sum_{N_{\max} \sim N_{\text{med}} \sim N} := \sum_{N_1, N_2, N_3 > 0: N_{\max} \sim N_{\text{med}} \sim N}.
\]
If \( \tau, \xi \) and \( p(\xi) \) are given, we define
\[
\lambda := \tau - p(\xi).
\]
Similarly,
\[
\lambda_j := \tau_j - p(\xi_j), \quad j = 1, 2, 3.
\]

In this paper, we do not go further on the general framework of Tao’s weighted convolution estimates. We focus our attention on the \([3; Z]-\)multiplier norm estimate for the Kawahara equation. During the estimate we need the resonance function
\[
h(\xi) = p(\xi_1) + p(\xi_2) + p(\xi_3) = -\lambda_1 - \lambda_2 - \lambda_3,
\]  
(3.2)
which measures to what extent the spatial frequencies \( \xi_1, \xi_2, \xi_3 \) can resonate with each other.

By dyadic decomposition of the variables \( \xi_j, \lambda_j \), as well as the function \( h(\xi) \), one is led to consider
\[
\| X_{N_1, N_2, N_3; H; L_1, L_2, L_3} \|_{[3; \mathbb{R} \times \mathbb{R}]},
\]  
(3.3)
where \( X_{N_1, N_2, N_3; H; L_1, L_2, L_3} \) is the multiplier
\[
X_{N_1, N_2, N_3; H; L_1, L_2, L_3}(\xi, \tau) := \chi_{[h(\xi)\sim H]} \prod_{j=1}^{3} \chi_{[|\xi_j|\sim N_j]} \chi_{[|\lambda_j|\sim L_j]}.
\]  
(3.4)
From the identities
\[ \xi_1 + \xi_2 + \xi_3 = 0 \]
and
\[ \lambda_1 + \lambda_2 + \lambda_3 + h(\xi) = 0 \]
on the support of the multiplier, we see that \( X_{N_1, N_2, N_3; H, L_1, L_2, L_3} \) vanishes unless
\[ N_{\text{max}} \sim N_{\text{med}}, \] (3.5)
and
\[ L_{\text{max}} \sim \max(H, L_{\text{med}}). \] (3.6)

From the definition of the resonance function, i.e., (3.2), we obtain the following algebraic smoothing relation

**Lemma 3.1.** If \( N_{\text{max}} \sim N_{\text{med}} \gtrsim 1 \), then
\[ \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} \gtrsim N_{\text{max}}^4 N_{\text{min}}. \] (3.7)

**Proof.** Noticing that \( p(\xi_j) = -\beta \xi_j^5 + \alpha \xi_j^3, j = 1, 2, 3 \), we have
\[ h(\xi) = -\lambda_1 - \lambda_2 - \lambda_3 = p(\xi_1) + p(\xi_2) + p(\xi_3) \]
\[ = \xi_1 \xi_2 \xi_3 \left( 3\alpha - 5\beta (\xi_1^2 + \xi_1 \xi_2 + \xi_2^2) \right). \]
Since \( \xi_1^2 + \xi_1 \xi_2 + \xi_2^2 \sim \max\{\xi_1^2, \xi_2^2\} \), and if \( N_{\text{max}} \sim N_{\text{med}} \gtrsim 1 \) and \( \beta \neq 0 \), we obtain that
\[ \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} \geq \frac{1}{3} (|\lambda_1 + \lambda_2 + \lambda_3|) \gtrsim N_{\text{max}}^4 N_{\text{min}}. \]

Under the condition of Lemma 3.1, we see that we may assume that
\[ H \sim N_{\text{max}}^4 N_{\text{min}}, \] (3.8)
since the multiplier in (3.4) vanishes otherwise.

Now we are in the position to state the fundamental estimate on dyadic blocks for the Kawahara equation.

**Lemma 3.2.** Let \( H, N_1, N_2, N_3, L_1, L_2, L_3 > 0 \) obey (3.5), (3.6), (3.8).
\( \circ ((++)) \text{Coherence} \) If \( N_{\text{max}} \sim N_{\text{min}} \) and \( L_{\text{max}} \sim H \), then we have
\[ (3.3) \lesssim L_{\text{min}}^{1/2} N_{\text{max}}^{-2} L_{\text{med}}^{1/2}. \] (3.9)
\( \circ ((+-)) \text{Coherence} \) If \( N_2 \sim N_3 \gg N_1 \) and \( H \sim L_1 \gtrsim L_2, L_3 \), then
\[ (3.3) \lesssim L_{\text{min}}^{1/2} N_{\text{max}}^{-2} \min(H, \frac{N_{\text{max}}}{N_{\text{min}}} L_{\text{med}})^{1/2}. \] (3.10)

Similarly for permutations.
\( \circ \) In all other cases, we have
\[ (3.3) \lesssim L_{\text{min}}^{1/2} N_{\text{max}}^{-2} \min(H, L_{\text{med}})^{1/2}. \] (3.11)
Proof. The fundamental estimate on dyadic blocks for the Kawahara equation is new. We prove it by using the tools Tao developed in [14].

In the high modulation case \( L_{\text{max}} \sim L_{\text{med}} \gg H \) we have by an elementary estimate employed by Tao (see (37) p.861 in [14])

\[
(3.3) \lesssim L^{1/2} N_{min}^{1/2} \lesssim L^{1/2} N_{max}^{-2} N_{min}^{1/2} N_{max}^{2} \lesssim L^{1/2} N_{max}^{-2} H^{1/2}.
\]

For the low modulation case: \( L_{\text{max}} \sim H \), by symmetry we may assume that \( L_1 \geq L_2 \geq L_3 \).

By Corollary 4.2 in Tao’s paper [14], we have

\[
(3.3) \lesssim L_3^{1/2} \left\{ \xi_2: |\xi_2 - \xi_2^0| \ll N_{min}; |\xi - \xi_2 - \xi_3^0| \ll N_{min}; \right. \\
p(\xi_2) + p(\xi - \xi_2) = \tau + O(L_2) \mid^{1/2}
\]

for some \( \tau \in \mathbb{R}, \xi, \xi_1^0, \xi_2^0, \xi_3^0 \) satisfying

\[
|\xi_j^0| \sim N_j (j = 1, 2, 3); |\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_{min}; |\xi + \xi_1^0| \ll N_{min}.
\]

To estimate the right-hand side of the expression (3.12) we shall use the identity

\[
p(\xi_2) + p(\xi - \xi_2) = p(\xi) + q(\xi, \xi_2)
\]

where

\[
q(\xi, \eta) = 5\beta\xi\eta(\xi - \eta)(\xi^2 - \xi\eta + \eta^2) - 3\alpha\xi\eta(\xi - \eta).
\]

We need to consider three cases: \( N_1 \sim N_2 \sim N_3, N_1 \sim N_2 \gg N_3 \) and \( N_2 \sim N_3 \gg N_1 \).

The case \( N_1 \sim N_3 \gg N_2 \) follows by symmetry. By (3.13) and (3.12), we have

\[
p(\xi) + (\xi - \xi_2) \left( 5\beta\xi_2(\xi^2 - \xi\xi_2 + \xi_2^2) - 3\alpha\xi_2 \right) = \tau + O(L_2).
\]

(i) If \( N_1 \sim N_2 \sim N_3 \), we see from (3.14) that \( \xi_2 \) variable is contained in one interval of length \( O(L_2 N_{max}^{-4}) \), and then

\[
(3.3) \lesssim L_3^{1/2} N_{max}^{1/2} = L_{min}^{1/2} L_{med}^{1/2} N_{max}^{-2},
\]

so (3.9) follows.

(ii) If \( N_1 \sim N_2 \gg N_3 \), the same computation as in the case (i) gives that

\[
(3.3) \lesssim L_{min}^{1/2} L_{med}^{1/2} N_{max}^{-2}.
\]

(iii) If \( N_2 \sim N_3 \gg N_1 \), we see from (3.14) that \( \xi_2 \) variable is contained in one interval of length \( O(L_2 N_{max}^{3} N_{min}^{-1}) \), and then

\[
(3.3) \lesssim L_3^{1/2} N_{min}^{1/2} N_{max}^{3/2} = L_{min}^{1/2} L_{med}^{1/2} N_{min}^{-1/2} N_{max}^{-3/2}.
\]

But \( \xi_2 \) is also contained in an interval of length \( \ll N_{min} \). The claim (3.10) follows. \( \Box \)
4 Proof of Proposition 2.2

This section is devoted to the proof of Proposition 2.2 with the fundamental estimate on dyadic blocks in Lemma 3.2.

Proof. By Plancherel it suffices to show that
\[
\left\| \frac{\xi_1 + \xi_2}{\xi_1 > -s < \xi_2 > -s < \xi_3 > s < \tau_1 - p(\xi_1) > b < \tau_2 - p(\xi_2) > b < \tau_3 - p(\xi_3) > 1-b} \right\|_{[3; \mathbb{R} \times \mathbb{R}]} \lesssim 1.
\]
(4.1)

By dyadic decomposition of the variables \( \xi_j, \lambda_j (j = 1, 2, 3), h(\xi) \), we may assume that \( |\xi_j| \sim N_j, |\lambda_j| \sim L_j (j = 1, 2, 3), |h(\xi)| \sim H \). By the translation invariance of the \([k; Z]-multiplier norm, we can always restrict our estimate on \( L_j \gtrsim 1 (j = 1, 2, 3) \) and \( \max(N_1, N_2, N_3) \gtrsim 1 \). The comparison principle and orthogonality (see Schur’s test in [14], p851) reduce the multiplier norm estimate (4.1) to showing that
\[
\sum_{N_{\text{max}} \sim N_{\text{med}}} \sum_{N \gtrsim 1} \left\| X_{N_1, N_2, N_3; L_{\text{max}}; L_1, L_2, L_3} \right\|_{[3; \mathbb{R} \times \mathbb{R}]} \lesssim 1
\]
(4.2)

and
\[
\sum_{N_{\text{max}} \sim N_{\text{med}}} \sum_{N_{\text{max}} \sim N_{\text{med}}} \sum_{L_{\text{med}} \leq L_{\text{max}}} \left\| X_{N_1, N_2, N_3; H; L_1, L_2, L_3} \right\|_{[3; \mathbb{R} \times \mathbb{R}]} \lesssim 1
\]
(4.3)

for all \( N \gtrsim 1 \). Estimates (4.2) and (4.3) will be accomplished by the fundamental estimate Lemma 3.2 and some delicate summation.

Fix \( N \gtrsim 1 \). This implies (3.8). We first prove (4.3). By (3.11) we reduce to
\[
\sum_{N_{\text{max}} \sim N_{\text{med}}} \sum_{N_{\text{max}} \sim N_{\text{med}}} \sum_{L_{\text{med}} \gtrsim N_{\text{med}}} \left\| X_{N_1, N_2, N_3; L_{\text{max}}; L_1, L_2, L_3} \right\|_{[3; \mathbb{R} \times \mathbb{R}]} \lesssim 1.
\]
(4.4)

By symmetry we only need to consider two cases: \( N_1 \sim N_2 \sim N, N_3 = N_{\text{min}} \) and \( N_1 \sim N_3 \sim N, N_2 = N_{\text{min}} \).

(i) In the first case \( N_1 \sim N_2 \sim N, N_3 = N_{\text{min}} \), the estimate (4.4) can be further reduced to
\[
\sum_{N_{\text{max}} \sim N_{\text{med}}} \sum_{L_{\text{med}} \gtrsim N_{\text{med}}} \left\| X_{N_1, N_2, N_3; L_{\text{max}}; L_1, L_2, L_3} \right\|_{[3; \mathbb{R} \times \mathbb{R}]} \lesssim 1.
\]
then performing the \( L \) summations, we reduce to
\[
\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \left\| X_{N_1, N_2, N_3; L_{\text{max}}; L_1, L_2, L_3} \right\|_{[3; \mathbb{R} \times \mathbb{R}]} \lesssim 1,
\]

which is true if \( 4 + 2s > 0 \). So, (4.4) is true if \( s > -2 \).

(ii) In the second case \( N_1 \sim N_3 \sim N \), \( N_2 = N_{\min} \), the estimate (4.4) can be reduced to

\[
\sum_{N_{\max} \sim N_{\text{med}} \sim N} \sum_{L_{\max} \sim L_{\text{med}} \geq N^4 N_{\min}} \frac{N}{< N_{\min} >^s L_{\min}^b L_{\text{med}}^b L_{\max}^{1-b} N_{\min}^{1/2} L_{\min}^{1/2}} \lesssim 1.
\]

Before performing the \( L \) summations, we need pay a little more attention to the summation of \( N_{\min} \). So we reduce to

\[
\sum_{N_{\max} \sim N_{\text{med}} \sim N, N_{\min} \leq 1} \sum_{L_{\max} \sim L_{\text{med}} \geq N^4 N_{\min}} \frac{N N_{\min}^{1/2}}{L_{\min}^b L_{\text{med}}^b L_{\max}^{1-b}} + \sum_{N_{\max} \sim N_{\text{med}} \sim N, 1 \leq N_{\min} \leq N} \sum_{L_{\max} \sim L_{\text{med}} \geq N^4 N_{\min}} \frac{N N_{\min}^{1/2-s}}{L_{\min}^b L_{\text{med}}^b L_{\max}^{1-b}} \lesssim 1,
\]

which is obviously true if \( s > -\frac{7}{2} \). So, (4.4) is true if \( s > -\frac{7}{2} \).

Now we show the low modulation case (4.2). In this case \( L_{\max} \sim N^4 N_{\min} \). We first deal with the contribution where (3.9) holds. In this case we have \( N_1, N_2, N_3 \sim N \geq 1 \), so we reduce to

\[
\sum_{L_{\max} \sim N^5} \frac{N^{-s} N}{L_{\min}^b L_{\text{med}}^b L_{\max}^{1-b}} L_{\min}^{1/2} N_{\min}^{-2} L_{\text{med}}^{1/2} \lesssim 1. \tag{4.5}
\]

Performing the \( L \) summations, we reduce to

\[
\frac{1}{N^{1+s} N^{5(1-b)}} \lesssim 1,
\]

which is true if \( 1 + s + 5(1 - b) > 0 \). So, (4.5) is true if \( s > -\frac{7}{2} \) and \( \frac{1}{2} < b < \frac{6+s}{5} \).

Now we deal with the cases where (3.10) holds. By symmetry we only need to consider two cases

\[
N \sim N_1 \sim N_2 \gg N_3; \quad H \sim L_3 \gtrsim L_1, L_2 \\
N \sim N_2 \sim N_3 \gg N_1; \quad H \sim L_1 \gtrsim L_2, L_3.
\]

In the first case we reduce by (3.10) to

\[
\sum_{N_3 \ll N} \sum_{N_1 \leq L_1, N_2 \leq N_3} \frac{N_3 < N_3 >^s}{N_2^s L_{1}^b L_{2}^b L_{3}^{1-b}} L_{\min}^{1/2} N_{\min}^{-2} \min \left( N^4 N_3, \frac{N}{N_3 L_{\text{med}}} \right)^{1/2} \lesssim 1. \tag{4.6}
\]

Decompose the left-hand side of (4.6) into the following two terms:

\[
\sum_{N_3 \ll N} \sum_{N_1 \leq L_1, N_2 \leq N_3} \frac{N_3 < N_3 >^s}{N_2^s L_{1}^b L_{2}^b L_{3}^{1-b}} L_{\min}^{1/2} N_{\min}^{-2} \min \left( N^4 N_3, \frac{N}{N_3 L_{\text{med}}} \right)^{1/2} + \sum_{N_3 \ll N} \sum_{N_1 \leq L_1, N_2 \leq N_3} \frac{N_3 < N_3 >^s}{N_2^s L_{1}^b L_{2}^b L_{3}^{1-b}} L_{\min}^{1/2} N_{\min}^{-2} \min \left( N^4 N_3, \frac{N}{N_3 L_{\text{med}}} \right)^{1/2} =: I_1 + I_2.
\]
We estimate the above two terms separately.

We first consider the estimate of $I_1$. If $N^4 I_3 = \frac{N}{N^3} L_{med}$, then $N_3 = \left( \frac{L_{med}}{N^4} \right)^{1/2}$. We divide two cases: $\left( \frac{L_{med}}{N^4} \right)^{1/2} \geq 1$ and $\left( \frac{L_{med}}{N^4} \right)^{1/2} < 1$ to estimate $I_1$. When $\left( \frac{L_{med}}{N^4} \right)^{1/2} \geq 1$,

$$I_1 \leq \sum_{N_3 \leq 1} \sum_{1 \leq L_1, L_2 \leq N^4 N_3} \frac{N_3}{N^2 s L_{min}^{-b/2} L_{med} (N^4 N_3)^{1-b}} N^{-2} N^2 N_3^{3/2}.$$  

(4.7)

Performing the $N_3$ summation in (4.7), we have

$$I_1 \lesssim \sum_{1 \leq L_1, L_2 \leq N^4} \frac{1}{N^2 s L_{min}^{-b/2} L_{med} N^4 (1-b)} \lesssim 1$$

if $2s + 4 - b > 0$. That means that $I_1 \lesssim 1$ if $s > -\frac{7}{4}$ and $\frac{1}{2} < b < 4 + 2s$.

When $\left( \frac{L_{med}}{N^4} \right)^{1/2} < 1$,

$$I_1 \leq \sum_{N_3 < \left( \frac{L_{med}}{N^4} \right)^{1/2}} \sum_{1 \leq L_1, L_2 \leq N^4 N_3} \frac{N_3}{N^2 s L_{min}^{-b/2} L_{med} (N^4 N_3)^{1-b}} N^{-2} N^2 N_3^{3/2}$$

$$+ \sum_{\left( \frac{L_{med}}{N^4} \right)^{1/2} < N_3 < 1} \sum_{1 \leq L_1, L_2 \leq N^4 N_3} \frac{N_3}{N^2 s L_{min}^{-b/2} L_{med} (N^4 N_3)^{1-b}} N^{-2} N^2 N_3^{3/2}.$$  

Performing the $N_3$ summation, we have

$$I_1 \lesssim \sum_{1 \leq L_1, L_2 \leq N^4} \frac{\left( \frac{L_{med}}{N^4} \right)^{-b/2}}{N^2 s L_{min}^{-b/2} L_{med} N^4 (1-b)} + \sum_{1 \leq L_1, L_2 \leq N^4} \frac{N^{-b/2}}{N^2 s L_{min}^{-b/2} L_{med} N^4 (1-b)}$$

$$\lesssim \sum_{1 \leq L_1, L_2 \leq N^4} \frac{1}{N^2 s L_{min}^{-b/2} L_{med} N^4 (1-b)} N^{\frac{3}{2} (1+b)} + \sum_{1 \leq L_1, L_2 \leq N^4} \frac{N^{-b/2}}{N^2 s L_{min}^{-b/2} L_{med} N^4 (1-b)}$$

$$=: I_{11} + I_{12}.$$  

Performing the $L$ summations, we see that $I_{11} \lesssim 1$ if $2s + 4(1-b) + \frac{3}{2} (\frac{1}{2} + b) > 0$ and $I_{12} \lesssim 1$ if $2s + 4(1-b) + \frac{3}{2} > 0$. This implies that $I_1 \lesssim 1$ if $s > -\frac{7}{4}$ and $\frac{1}{2} < b < \frac{4s + 11}{8}$.

Now we consider the estimate of the second term $I_2$. The estimate is a little simple compared to the estimate of $I_1$. We do not need distinguish the cases $\left( \frac{L_{med}}{N^4} \right)^{1/2} \geq 1$ and $\left( \frac{L_{med}}{N^4} \right)^{1/2} < 1$. We could get the following estimate for $I_2$ in a unified way

$$I_2 \leq \sum_{1 < N_3 \ll N} \sum_{1 \leq L_1, L_2 \leq N^4 N_3} \frac{N_3 N_3^s}{N^2 s L_{min}^{-b/2} L_{med} (N^4 N_3)^{1-b}} N^{-2} N^{1/2} L_{med}^{-1/2} N_3^{-1/2}.$$  

(4.8)
Performing the $N_3$ summation in (4.8) and noticing that if $s - \frac{1}{2} + b < 0$, we have

$$I_2 \lesssim \sum_{1 \leq L_1, L_2 \leq N^5} \frac{1}{N^{2s} L_{\min}^{b-1/2} L_{\text{med}}^{b-1/2} N^{4(1-b)} N^{3/2}} \lesssim 1$$

under condition that $2s + 4(1 - b) + \frac{3}{2} > 0$. If $s - \frac{1}{2} + b \geq 0$, we have

$$I_2 \lesssim \sum_{1 \leq L_1, L_2 \leq N^5} \frac{N^{s-1/2+b}}{N^{2s} L_{\min}^{b-1/2} L_{\text{med}}^{b-1/2} N^{4(1-b)} N^{3/2}} \lesssim 1$$

under condition that $2s + 4(1 - b) + \frac{3}{2} > \frac{1}{2} + b$. That means that $I_2 \lesssim 1$ if $s > -\frac{7}{4}$ and $\frac{1}{2} < b \leq \min \left(\frac{4s+11}{8}, \frac{s+6}{5}\right)$. Combining the estimates for $I_1$ and $I_2$, we obtain the desired estimate (4.9).

Now we deal with the case $N \sim N_2 \sim N_3 \gg N_1$; $H \sim L_1 \gtrsim L_2, L_3$. In this case we reduce by (3.10) to

$$\sum_{1 \leq L_1, L_2 \leq N^5} \sum_{N_1 \leq \mathcal{N}} \frac{N^{1+s} L_{\min}^{1/2}}{N^{s} \mathcal{N}^{1/2} L_2 L_3 (N_4 N_1)^b \min (H, N^{1/2} L_{\text{med}})} \lesssim 1. (4.9)$$

Decompose the left-hand side of (4.9) into the following two terms:

$$\sum_{N_1 \leq \mathcal{N}} \sum_{1 \leq L_1, L_2, L_3 \leq N^4 N_1} \frac{N^{1+s}}{N L_2 L_3 (N_4 N_1)^b} L_{\min}^{1/2} N_1^{1/2} +$$

$$\sum_{1 \leq N_1 < \mathcal{N}} \sum_{1 \leq L_1, L_2, L_3 \leq N^4 N_1} \frac{N^{1+s}}{N_1 N_4 L_2 L_3 (N_4 N_1)^b} L_{\min}^{1/2} N^{-2} N^{5/4} L_{\text{med}}^{1/4} =: J_1 + J_2.$$

In $J_1$, we assume $N_1 \gtrsim N^{-4}$, otherwise the summation of $L$ vanishes. Performing the summation of $L$, we get

$$J_1 \lesssim \sum_{N^{-4} \leq N_1 \leq 1} \frac{N N_1^{s-b}}{N_4 b} \lesssim \frac{N N^{(-4)(s-b)}}{N_4 b} \lesssim 1.$$

If we take $\frac{1}{2} < b < \frac{3}{4}$ in $J_2$, then performing the summation of $L$ implies that

$$J_2 \lesssim \sum_{1 \leq N_1 < N} \frac{N^{1/4} N_1^{s-b}}{N_4 b} \lesssim \frac{N^{1/4} N^{-s-b}}{N_4 b} \lesssim 1,$$

if $4b + s + b - \frac{1}{4} > 0$. The condition $4b + s + b - \frac{1}{4} > 0$ is always true if $s > -\frac{7}{4}$ and $\frac{1}{2} < b \leq 1$. So, $J_2 \lesssim 1$ if $s > -\frac{7}{4}$ and $\frac{1}{2} < b < \frac{3}{4}$. Combining the estimates for $J_1$ and $J_2$, we get the needed estimate (4.9).
To finish the proof of (4.2) it remains to deal with the cases where (3.11) holds. This reduces to

\[
\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{L_{\text{max}} \sim N^4 N_{\text{min}}} \frac{N_3 < N_3 >^s}{N_1 >^s N_2 >^s L_1^b L_2^b L_3^b} L_{\text{min}}^{1/2} N^{-2 \min (H, L_{\text{med}})^{1/2}} \lesssim 1. \tag{4.10}
\]

To estimate (4.10), by symmetry we need to consider two cases: \( N_1 \sim N_2 \sim N, N_3 = N_{\text{min}} \) and \( N_1 \sim N_3 \sim N, N_2 = N_{\text{min}} \).

(i) When \( N_1 \sim N_2 \sim N, N_3 = N_{\text{min}} \), the estimate (4.10) further reduces to

\[
\sum_{N_1 \sim N_2 \sim N} \frac{N_3 < N_3 >^s}{N^2 s L_1^b L_2^b L_{\text{med}} (N^4 N_3)^{1/2} - b L_{\text{min}}^{1/2} N^{-2} L_{\text{med}}^{1/2}} \lesssim 1,
\]

then performing the \( L \) summations, we reduce to

\[
\sum_{N_1 \sim N_2 \sim N} \frac{N_3 < N_3 >^s}{N^2 + 2 s N^4 (1 - b) N_{\text{min}}^{1/2}} \lesssim 1,
\]

which is true if \( 2 + 2 s + 4(1 - b) > 0 \). So, (4.10) is true if \( s > -2 \) and \( \frac{1}{2} < b < \frac{s + 3}{2} \).

(ii) When \( N_1 \sim N_3 \sim N, N_2 = N_{\text{min}} \), the estimate (4.10) can be reduced to

\[
\sum_{N_1 \sim N_3 \sim N} \sum_{N_2 \sim N} \frac{N^{1 + s} L_{\text{min}}^{1/2} N^{-2}}{N^s L_2^b L_1^b L_{\text{med}}^{1/2} \min (H, L_{\text{med}})^{1/2}} \lesssim 1. \tag{4.11}
\]

Before performing the \( L \) summations, as before we need pay a little more attention to the summation of \( N_2 \). Decompose the left-hand side of (4.11) into the following two terms:

\[
\sum_{N_2 \leq 1} \sum_{L_{\text{max}} \sim N^4 N_2} \frac{N}{L_{\text{min}}^{1/2} L_{\text{med}}^{1/2} L_{\text{max}}^{1- b}} N^{-2} L_{\text{med}}^{1/4} (N^4 N_2)^{1/4} + \sum_{1 \leq N_2 \leq N_1} \sum_{L_{\text{max}} \sim N^4 N_2} \frac{N}{N_2 L_{\text{min}}^{b-1/2} L_{\text{med}}^{1- b} L_{\text{max}}^{1- b}} N^{-2} L_{\text{med}}^{1/2} =: J_3 + J_4.
\]

It is easily seen that \( J_3 \lesssim 1 \) for any \( \frac{1}{2} < b < 1 \). For \( J_4 \), if \( s + 1 - b \geq 0 \), we always have \( J_4 \lesssim 1 \) for any \( \frac{1}{2} < b < 1 \). If \( s + 1 - b < 0 \), we have \( J_4 \lesssim 1 \) under condition that \( 4(1 - b) + s + 1 + (1 - b) > 0 \). So, (4.11) is true if \( s > -\frac{7}{2} \) and \( \frac{1}{2} < b < \frac{s + 6}{5} \). This completes the proof of Proposition (2.2). \( \square \)
5 A trilinear estimate and local well-posedness of the modified Kawahara equation

In this section we will prove a trilinear estimate in Bourgain spaces, from which and the linear estimates presented in Section 2, the local well-posedness of the initial-value problem for the modified Kawahara equation Theorem 1.2 could be derived.

Lemma 5.1. Let \( s \geq -\frac{1}{4} \). For all \( u_1, u_2, u_3 \) on \( \mathbb{R} \times \mathbb{R} \) and \( \frac{1}{2} < b \leq 1 \), we have
\[
\| \partial_x (u_1 u_2 u_3) \|_{X_{s,b}} \lesssim \| u_1 \|_{X_{s,b}} \| u_2 \|_{X_{s,b}} \| u_3 \|_{X_{s,b}}.
\] (5.1)

This is the first trilinear estimate in Bourgain spaces associated to the class of Kawahara equations. It seems difficult to obtain this kind of trilinear estimates by the method firstly presented by Bourgain, Kenig-Ponce-Vega for KdV. We reduce the trilinear estimate by the \( TT^* \) identity Tao developed in [14] to a bilinear estimate, then prove the bilinear estimate by the fundamental estimate on dyadic blocks in Lemma 3.2.

Proof. By duality and Plancherel it suffices to show that
\[
\left\| \frac{(\xi_1 + \xi_2 + \xi_3) < \xi_4 >^s}{< \tau_4 - p(\xi_4) >^{1-b} \prod_{j=1}^3 < \xi_j >^s < \tau_j - p(\xi_j) >^b} \right\|_{[4, \mathbb{R} \times \mathbb{R}]} \lesssim 1.
\]

We estimate \(|\xi_1 + \xi_2 + \xi_3| \) by \(< \xi_4 >\). Applying the fractional Leibnitz rule, we have
\[
< \xi_4 >^{s+1} \lesssim < \xi_4 >^{1/2} \sum_{j=1}^3 < \xi_j >^{s+1/2}
\]
where we assume \( s > -1/2 \), and symmetry to reduce to
\[
\left\| \frac{< \xi_1 >^{-s} < \xi_3 >^{-s} < \xi_2 >^{1/2} < \xi_4 >^{1/2}}{< \tau_4 - p(\xi_4) >^{1-b} \prod_{j=1}^3 < \tau_j - p(\xi_j) >^b} \right\|_{[4, \mathbb{R} \times \mathbb{R}]} \lesssim 1.
\]

We may replace \(< \tau_2 - p(\xi_2) >^b \) by \(< \tau_2 - p(\xi_2) >^{1-b} \) (this is true for any \( b \geq \frac{1}{2} \)). By the \( TT^* \) identity (see Lemma 3.7 in [14], p847), the estimate is reduced to the following bilinear estimate. \( \square \)

Lemma 5.2. (Bilinear estimate). Let \( s \geq -\frac{1}{4} \). For all \( u, v \) on \( \mathbb{R} \times \mathbb{R} \) and \( 0 < \epsilon \ll 1 \), we have
\[
\| uv \|_{L^2(\mathbb{R} \times \mathbb{R})} \lesssim \| u \|_{X_{s+1/2,1/2}(\mathbb{R} \times \mathbb{R})} \| v \|_{X_{s+1/2,1+\epsilon}(\mathbb{R} \times \mathbb{R})}.
\] (5.2)

This lemma can be proved in the same way as Proposition 2.2 by using the fundamental estimate on dyadic blocks in Lemma 3.2. But we should point out that there is some differences between this lemma and Proposition 2.2. Lemma 5.2 is an asymmetric bilinear estimate while Proposition 2.2 is a symmetric bilinear estimate. This leads to the lack of some symmetry in the proof of Lemma 5.2. On the other hand, since there is no derivative in the left-hand side of (5.2), the proof of Lemma 5.2 is rather simpler than that of Proposition 2.2.
Proof. By Plancherel it suffices to prove that
\[
\left\| \frac{\langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{1/2}}{<\tau_1 - p(\xi_1) >^{1/2+\epsilon} <\tau_2 - p(\xi_2) >^{1/2-\epsilon}} \right\|_{[3; R \times R]} \lesssim 1. \quad (5.3)
\]

By dyadic decomposition and orthogonality as in the proof of Proposition 2.2, we reduce the multiplier norm estimate (5.3) to showing that
\[
\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{L_1, L_2, L_3 \geq 1} \frac{<N_1 >^{-s} <N_2 >^{1/2}}{L_1^{1/2+\epsilon} L_2^{1/2-\epsilon}} \lesssim 1 \quad (5.4)
\]
and
\[
\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{L_{\text{max}} \sim L_{\text{med}}} \sum_{H \leq L_{\text{max}}} \frac{<N_1 >^{-s} <N_2 >^{1/2}}{L_1^{1/2+\epsilon} L_2^{1/2-\epsilon}} \lesssim 1 \quad (5.5)
\]
for all $N \gtrsim 1$.

Fix $N \gtrsim 1$. We first prove (5.5). By (3.11) we reduce to
\[
\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{L_{\text{max}} \sim L_{\text{med}}} \sum_{H \leq L_{\text{max}}} \frac{<N_1 >^{-s} <N_2 >^{1/2}}{L_1^{1/2+\epsilon} L_2^{1/2-\epsilon}} \lesssim 1 \quad (5.6)
\]

We consider two cases: $s \geq 0$ and $s < 0$.

(i) In the first case $s \geq 0$, the estimate (5.6) can be further reduced to
\[
\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{L_{\text{max}} \sim L_{\text{med}}} \frac{N^{1/2} <N_{\min}>^{-s} <N_{\min}>^{1/2}}{L_{\text{min}}^{1/2+\epsilon} L_{\text{med}}^{1/2-\epsilon}} L_{\text{min}}^{1/2} N_{\min}^{1/2} \lesssim 1,
\]
then performing the $L$ summations, we reduce to
\[
\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \frac{N^{1/2} N_{\min}^{1/2} <N_{\min}>^{-s}}{(N^4 N_{\min})^{1/2-\epsilon}} \lesssim 1,
\]
which is always true for $s \geq 0$.

(ii) In the second case $s < 0$, the estimate (5.6) can be reduced to
\[
\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{L_{\text{max}} \sim L_{\text{med}}} \frac{N^{1/2-s}}{L_{\text{min}}^{1/2+\epsilon} (N^4 N_{\min})^{1/2-\epsilon}} L_{\text{min}}^{1/2} N_{\min}^{1/2} \lesssim 1.
\]
Performing the $L$ summations, we reduce to
\[
\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \frac{N^{1/2-s} N_{\min}^{\epsilon}}{N^{2-4\epsilon}} \lesssim 1,
\]
which is true if \( s > -\frac{3}{2} \). So, (5.6) is true if \( s > -\frac{3}{2} \).

Now we show the low modulation case (5.4). In this case we may assume \( L_{\text{max}} \sim N_{\text{max}}^4 N_{\text{min}} \). We first deal with the contribution where (3.9) holds. In this case we have \( N_1, N_2, N_3 \sim N \gtrsim 1 \), so we reduce to
\[
\sum_{L_{\text{max}} \sim N^5} \frac{N^{-s} N^{1/2}}{L_{\text{min}}^{1/2+\epsilon} L_{\text{med}}^{1/2-\epsilon}} L_{\text{min}}^{1/2} N^{-2} L_{\text{med}}^{1/2} \lesssim 1. \tag{5.7}
\]
Performing the \( L \) summations, we reduce to
\[
\frac{N^{5\epsilon}}{N^{3/2+s}} \lesssim 1,
\]
which is true if \( s > -\frac{3}{2} \).

Now we deal with the cases where (3.10) holds. Since the lack of symmetry, we need to consider three cases

\[
\begin{align*}
N &\sim N_1 \sim N_2 \gg N_3; \quad H \sim L_3 \gtrsim L_1, L_2 \\
N &\sim N_2 \sim N_3 \gg N_1; \quad H \sim L_1 \gtrsim L_2, L_3 \\
N &\sim N_1 \sim N_3 \gg N_2; \quad H \sim L_2 \gtrsim L_1, L_3.
\end{align*}
\]

In the first case we reduce by (3.10) to
\[
\sum_{N_3 \ll N} \sum_{1 \ll L_1, L_2 \ll N^4 N_3} \frac{N^{1/2-s}}{L_{\text{min}}^{1/2+\epsilon} L_{\text{med}}^{1/2-\epsilon}} L_{\text{min}}^{1/2} N^{-2} \min\left( N^4 N_3, \frac{N}{N_3} L_{\text{med}} \right)^{1/2} \lesssim 1. \tag{5.8}
\]
Performing the \( N_3 \) summation we reduce to
\[
\sum_{1 \ll L_1, L_2 \ll N^4 N_3} \frac{N^{1/2-2-s+5/4}}{L_{\text{min}}^{1/2+\epsilon} L_{\text{med}}^{1/2-\epsilon}} L_{\text{min}}^{1/2} L_{\text{med}}^{1/4} \lesssim 1
\]
which is true if \( s \geq -\frac{1}{2} \).

Now we deal with the second case \( N \sim N_2 \sim N_3 \gg N_1; \quad H \sim L_1 \gtrsim L_2, L_3 \). In this case we make use of the first half of (3.10) and reduce to
\[
\sum_{N_1 \ll N} \sum_{1 \ll L_1, L_2 \ll N^4 N_1} \frac{N^{1/2}}{L_{\text{min}}^{1/2+\epsilon} L_{\text{med}}^{1/2-\epsilon}} L_{\text{min}}^{1/2} N_1^{1/2} \lesssim 1 \tag{5.9}
\]
which is true if \( s > -1 \).

In the third case \( N \sim N_1 \sim N_3 \gg N_2; \quad H \sim L_2 \gtrsim L_1, L_3 \), we similarly reduce by using the first half of (3.10) to
\[
\sum_{N_2 \ll N} \sum_{1 \ll L_1, L_3 \ll N^4 N_2} \frac{< N_2 >^{1/2}}{N^s L_1^{1/2+\epsilon} L_{\text{med}}^{1/2-\epsilon}} L_{\text{min}}^{1/2} N_2^{1/2} \lesssim 1 \tag{5.10}
\]
which is true if \( s > -\frac{3}{2} \).
To finish the proof of (5.4) it remains to deal with the cases where (3.11) holds. This reduces to
\[
\sum_{N_{\text{max}} \sim N_{\text{med}}} \sum_{L_{\text{max}} \sim N^4 N_{\text{min}}} <N_2>^{1/2} <N_1>^s L_1^{1/2+\epsilon} L_2^{1/2-\epsilon} N^{-2} L_{\text{med}}^{1/2} \lesssim 1
\] (5.11)
which is true if \( s > -\frac{3}{2} \). This finishes the proof of Lemma 5.2.

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