On Parametric p-Valent Meromorphic Functions

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Abstract: Problem statement: In this research, we studied parametric p-valent meromorphic functions $P_\alpha f$ by considering two classes $M_p^\alpha(\beta)$ and $M_p^\alpha(\lambda, A)$. Approach: With the help of Jack’s Lemma an inclusion relation for the class $M_p^\alpha$ was obtained and it is shown that this class is closed by an integral operator $I_c$. Results: A subordination result for the class $M_p^\alpha(\lambda, A)$ was proved. Consequences of main results with the results for special values of the parameter $\alpha$ were discussed. Conclusion/Recommendations: Our results certainly generalized several results obtained earlier as well as generate new results.

Key words: Meromorphic functions, starlike functions, convolution, generalized hypergeometric functions

INTRODUCTION

Let $M_p$ denotes a class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{-k-p}, p \in \mathbb{N} \{1, 2, 3, \ldots\} \quad (1a)$$

which are analytic and $p$-valent in the punctured unit disk $U^* = \{z: 0 < |z| < 1\} = \mathbb{U}\{0\}$. We say that a function $f(z) \in M_p$ is in the class $M_p^\alpha(\beta)$ if $f(z) \neq 0$ and:

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < -\beta, 0 \leq \beta < p, z \in U$$

Functions in the class $M_p^\alpha(\beta)$ are called $p$-valent meromorphic starlike of order $\beta$.

Let $g_\alpha(z) \in M_p$ be of the form:

$$g_\alpha(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p}(\alpha) z^{-k-p} \quad (1b)$$

whose coefficient $b_{k-p}(\alpha)$ has a parameter $\alpha$ which is either $-p$ or a positive real and it satisfies the relation:

$$b_{k-p}(\alpha+1) = \left(\frac{\alpha+k}{\alpha}\right) b_{k-p}(\alpha) \quad (1c)$$

For $g_\alpha(z)$ given by (1b), a parametric convolution operator $P_\alpha^\beta: M_p \rightarrow M_p$, on the function $f(z)$ of the form (1a) is defined by:

$$P_\alpha^\beta f(z) = \left( f \ast g_\alpha \right)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p}(\alpha) z^{-k-p} \quad (1d)$$

where, "\ast" stands for convolution or Hadamard product.

We have:

$$P_\alpha^\beta f(z) = z^{-p} + \sum_{k=1}^{\infty} \left( \frac{\alpha+k}{\alpha} \right) b_{k-p}(\alpha) z^{-k-p}$$

For $\alpha = -p$, $P_\alpha^\beta f(z) = -\frac{z(P_\alpha^\beta f(z))'}{p} \quad (1e)$

Using (1c) and (1d), we can easily obtain the identity related to parametric $p$-valent meromorphic functions:

$$z(P_\alpha^\beta f(z))' = \alpha P_\alpha^\beta f(z) - (\alpha + p) P_\alpha^\beta f(z) \quad (1e)$$

Several subclasses of $p$-valent meromorphic functions involving various convolution operators have been defined and studied in (Aouf, 2008; Liu and Srivastava, 2001; 2004; Raina and Srivastava, 2006; Srivastava and Patel, 2006; Srivastava et al., 2008; Wang et al., 2009; Yang, 2001). The purpose of this study is to unify the results obtained earlier and to give some new results. Motivated with these earlier works especially the work of Cho (Srivastava and Owa, 1992;
Yang, 2004), in this study, in order to study parametric p-valent meromorphic functions \( P^\alpha f(z) \) of the form (1d), we consider classes \( M^\alpha_\beta(\beta) \) and \( M^\alpha_p(\lambda, A) \) of \( f(z) \in M_p \) satisfying following conditions respectively:

\[
\Re \left[ \frac{\alpha P^{\alpha+1}f(z)}{P^\alpha f(z)} \right] < (\alpha + p - \beta), z \in U \tag{1f}
\]

and for \( \lambda > p, |A| < |\alpha| \):

\[
(1 - \lambda)\alpha z^\alpha P^\alpha f(z) + \lambda \alpha z^\alpha P^{\alpha+1}f(z) < \alpha + \left( 1 + \frac{\lambda}{\alpha} \right) \omega(z), z \in U \tag{1g}
\]

where, ‘\(<\)’ stands for subordination between two analytic functions in \( U \).

We say \( f \prec g \), if there exists a Schwartz function \( \omega(z) \) which is analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \), \( z \in U \) such that \( f(z) = g(\omega(z)), z \in U \). Indeed it is known that \( f(z) < g(z), z \in U \Rightarrow f(0) = g(0) \) and \( f(U) \subset g(U) \).

Clearly, if \( \alpha = -p \), the classes \( M^\alpha_p(\beta) \) and \( M^\alpha_\beta(\beta) \) are respectively the classes of p-valent meromorphic starlike and convex functions \( P^\alpha f(z) \) of order \( 0 \leq \beta < p \) if \( P^\alpha f(z), P^{\alpha+1}f(z) \neq 0 \). We denote \( M^\alpha_{-p}(0) = M^\alpha_p \).

**MATERIALS AND METHODS**

To prove our main results, we need following Lemmas:

**Lemma 1:** Let \( q(z) \) be univalent in the unit disk \( U \) and \( \theta \) and \( \varphi \) be analytic in a domain \( E \) containing \( q(U) \) with \( \varphi(w) \neq 0 \) when \( q(U) \) (Eenigenburg et al., 1984). Set:

\[
Q(z) := q(z)\varphi(q(z)), h(z) := \theta(q(z)) + Q(z)
\]

and suppose that either \( Q(z) \) is starlike or \( h(z) \) is convex in \( U \). In addition, assume that:

\[
\Re \frac{zh(z)}{Q(z)} > 0, z \in U
\]

If \( p(z) \) is analytic in \( U \) with \( p(0) = q(0), p(U) \subset E \) and:

\[
\theta(p(z)) + zp'(z)\varphi(p(z)) < \theta(q(z)) + zq'(z)\varphi(q(z))
\]

Then \( p(z) \prec q(z) \) and \( q(z) \) is the best dominant.

**Lemma 2:** Let the (non-constant) function \( \omega(z) \) be analytic in \( U \) with \( \omega(0) = 0 \) Jack (1971) Lemma. If \( |\omega(z)| \) attains its maximum value on the circle \( |z| = r < 1 \) at a point \( z_0 \in U \), then \( z_0 \omega'(z_0) = \gamma \omega(z_0), \gamma \omega(z_0) = e^{i\theta}, \gamma \geq 1 \).

**Theorem 1:** For a parametric convolution operator \( P^\alpha \) defined in (1d), \( M^\alpha_{-p} \subset M^\alpha_p \).

**Proof:** Let \( f \in M^\alpha_{-p} \), we have:

\[
\Re \left[ \frac{\alpha P^{\alpha+1}f(z)}{P^\alpha f(z)} \right] < (\alpha + p) \tag{2a}
\]

Define an analytic function \( \omega(z) \) in \( U \) such that \( \omega(0) = 0 \), by:

\[
\frac{\alpha P^{\alpha+1}f(z)}{P^\alpha f(z)} = (\alpha + p) - \frac{1 - \omega(z)}{1 + \omega(z)} = \frac{\alpha + (\alpha + 2p)\omega(z)}{1 + \omega(z)} \tag{2b}
\]

To show:

\[
\Re \left[ \frac{\alpha P^{\alpha+1}f(z)}{P^\alpha f(z)} \right] < (\alpha + p)
\]

We need to show \( \Re \left[ \frac{1 - \omega(z)}{1 + \omega(z)} \right] > 0 \) or \( |\omega(z)| < 1 \) in \( U \). Differentiating logarithmically (2b), we get:

\[
\frac{z(P^{\alpha+1}f(z)) - z(P^\alpha f(z))}{P^{\alpha+1}f(z)} = \frac{(\alpha + 2p)\omega(z)}{\alpha + (\alpha + 2p)\omega(z)} = \frac{2\omega(z)}{1 + \omega(z)} \tag{2c}
\]

Applying the identity (1e) for \( \alpha \) and \( \alpha + 1 \) in (2c) and using (2b), we get:

\[
\frac{(\alpha + 1)P^{\alpha+1}f(z)}{P^{\alpha+1}f(z)} = \frac{\alpha + 1 + p - p}{1 + \omega(z)}
\]

Let there exists a point \( z_0 \in U \) such that \( |\omega(z_0)| = 1 \), then by lemma 2, \( z_0 \omega'(z_0) = \gamma \omega(z_0), \gamma \omega(z_0) = e^{i\theta}, \gamma \geq 1 \). Therefore:
\begin{align*}
\frac{(\alpha + 1)P^{\alpha+1}f(z_0)}{P^{\alpha+1}f(z_0)} &= \alpha + 1 + p - p \left\{ \frac{1 - \alpha(z_0)}{1 + \alpha(z_0)} \right\} \\
+ \frac{2pz_0 \omega(z_0)}{\alpha + (\alpha + 2p) \alpha(z_0) [1 + \alpha(z_0)]} \\
&= \alpha + 1 + p - p \left\{ \frac{1 - e^\alpha}{1 + e^\alpha} \right\} + \frac{2pe^\alpha}{\alpha + (\alpha + 2p)e^\alpha [1 + e^\alpha]}
\end{align*}

Hence:
\begin{align*}
\text{Re} \left\{ \frac{(\alpha + 1)P^{\alpha+1}f(z)}{P^{\alpha+1}f(z)} \right\} &= (\alpha + 1 + p) \\
+ \text{Re} \left\{ \frac{2pe^\alpha}{\alpha + (\alpha + 2p)e^\alpha [1 + e^\alpha]} \right\} \\
&\geq (\alpha + 1 + p) \cdot \alpha = -p \text{ or } \alpha > 0
\end{align*}

which contradicts (2a), this proves that \( |\omega(z)| < 1 \) in \( U \) and hence Theorem 1 is proved.

Taking \( \alpha = -p \), in Theorem 1, we get following result.

**Corollary 1:** Let \( P^\alpha f(z) \in M_p \) be defined in (1d) with \( \alpha = -p \) and \( P^\alpha f(z) \neq 0 \), if:
\begin{align*}
\text{Re} \left\{ 1 + \frac{z(P^\alpha f(z))}{P^\alpha f(z)} \right\} &< 0
\end{align*}

Then:
\begin{align*}
\text{Re} \left\{ \frac{z(P^\alpha f(z))}{P^\alpha f(z)} \right\} &< 0
\end{align*}

**Theorem 2:** Let an integral operator \( I_c : M_p \rightarrow M_p \) be defined for \( c > 0 \) by:
\begin{equation}
I_c f(z) = \int_0^1 t^{c-1} f(t) \, dt \quad (2d)
\end{equation}

\begin{align*}
&= z^{-p} + \sum_{k \geq 0} \left( \frac{c}{c + k} \right) a_{k+p} z^{k+p}
\end{align*}

The class \( M_p^\alpha \), defined in (1f) is closed under the integral operator \( I_c \).

**Proof:** Let \( f \in M_p^\alpha \), we have:
\begin{align*}
\text{Re} \left\{ \frac{\alpha P^\alpha f(z)}{P^\alpha f(z)} \right\} &= (\alpha + p) \\
\text{Define an analytic function } \omega(z) \text{ in } U \text{ such that } \omega(0) = 0, \text{ by:}
\end{align*}
\begin{equation}
\frac{\alpha P^\alpha f(z)}{P^\alpha f(z)} = (\alpha + p) - p \left\{ \frac{1 - \alpha(z)}{1 + \alpha(z)} \right\}
\end{equation}

\begin{align*}
&= (\alpha + (\alpha + 2p) \alpha(z)) \left\{ 1 + \alpha(z) \right\} \\
&= \alpha + (\alpha + 2p) \alpha(z)
\end{align*}

To show:
\begin{align*}
\text{Re} \left\{ \frac{\alpha P^\alpha I_c f(z)}{P^\alpha I_c f(z)} \right\} &< (\alpha + p)
\end{align*}

We need to show \( \text{Re} \left\{ \frac{1 - \alpha(z)}{1 + \alpha(z)} \right\} > 0 \) or, \( |\omega(z)| < 1 \) in \( U \). Differentiating logarithmically (2h), we get:
\begin{align*}
\frac{z(P^{\alpha+1} f(z))'}{P^{\alpha+1} f(z)} - \frac{z(P^\alpha f(z))'}{P^\alpha f(z)} &= \frac{(\alpha + 2p) \omega'(z)}{\alpha + (\alpha + 2p) \alpha(z)} \\
&= \frac{2pz \omega'(z)}{\alpha + (\alpha + 2p) \alpha(z) [1 + \alpha(z)]}
\end{align*}

Using the series expansion of \( I_c f(z) \) given in (2e), we get the identity:
\begin{equation}
z(P^\alpha I_c f(z))' = cP^\alpha f(z) - (c + p) P^\alpha I_c f(z) \quad (2j)
\end{equation}

Applying the identity (2j) for \( \alpha \) and \( \alpha + 1 \) in (1e) for \( I_c f(z) \) in (2i) and using (2g), we get:
\begin{align*}
\frac{\alpha P^{\alpha+1} f(z)}{P^\alpha f(z)} &= \frac{\alpha P^{\alpha+1} I_c f(z)}{P^\alpha I_c f(z)} \\
&= \frac{\left\{ \frac{z(P^{\alpha+1} f(z))'}{P^{\alpha+1} f(z)} - \frac{z(P^\alpha f(z))'}{P^\alpha f(z)} \right\}}{1 + \frac{z(P^\alpha f(z))'}{P^\alpha f(z)} + (c + p)} \\
&= (\alpha + p - p)
\end{align*}

Let there exists a point \( z_0 \in U \) such that \( |\omega(z_0)| = 1 \), then by lemma 2, \( z_0 \omega'(z_0) = \gamma \omega(z_0), \omega(z_0) = e^{i\theta}, \gamma \geq 1 \). Therefore:
\[ \frac{\alpha P^{\alpha + i}(z_0)}{P^{\alpha + i}(z_0)} = \alpha + p - p \left[ \frac{1 - e^{\alpha i}}{1 + e^{\alpha i}} \right] \]

\[ = (\alpha + p) + \frac{2pe^{\alpha i}}{c + (c + 2p)e^{\alpha i} + 1 + e^{\alpha i}} \]

Hence:

\[ \text{Re} \left\{ \frac{\alpha P^{\alpha + i}(z)}{P^{\alpha + i}(z)} \right\} = (\alpha + p) + \text{Re} \left\{ \frac{2pe^{\alpha i}}{c + (c + 2p)e^{\alpha i} + 1 + e^{\alpha i}} \right\} \]

\[ \geq (\alpha + p) > -p \text{ or } \alpha > 0 \]

which contradicts (2f), this proves that \(|\omega(z)| < 1\) in \(U\) and hence Theorem 2 is proved.

**Remark:** The results obtained in Theorems 1 and 2, coincide for \(\alpha = n + p > 0\), with the results obtained by Cho in (Srivastava and Owa, 1992).

**Theorem 3:** Let \(f(z) \in M_p\) be in the class \(M_p^\alpha(\lambda, A)\), defined in (1g), then \(\alpha z P^{\alpha + i}(z) \prec \alpha + Az\).

**Proof:** Let \(p(z) = \alpha z P^{\alpha + i}(z)\), we have \(p(z) + \frac{\lambda}{\alpha} z p'(z) = \alpha + \left(1 + \frac{\lambda}{\alpha}\right) Az\). Consider \(q(z) = \alpha + Az\) which is univalent, convex in \(U\). Consider for \(\lambda > p\):

\[ \theta(w) = w, \theta'(w) = \frac{\lambda}{\alpha} (w \in C) \]

which are analytic in \(C\) so that:

\[ \theta(p(z)) + zp'(z)\theta(q(z)) = \frac{\lambda}{\alpha} zp'(z) \]

Set:

\[ Q(z) = zq'(z)\phi(q(z)), h(z) = \theta(q(z)) + Q(z) \]

We obtain that:

\[ Q(z) = \frac{\lambda}{\alpha} zq'(z) \]

which is starlike in \(U\) and:

\[ \text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \text{Re} \left\{ \frac{\lambda}{\alpha} + \frac{2zq'(z)}{q'(z)} \right\} > 0, \alpha = -p \text{ or } \alpha > 0 \]

Hence on applying Lemma 1, we get that \(p(z) \prec q(z)\) or \(\alpha z P^{\alpha + i}(z) \prec \alpha + Az\). \(\alpha + Az\) is the best dominant and the result is sharp with extremal function:

\[ P^{\alpha + i}(z) = \alpha - \omega \left( \int_0^{\alpha} \frac{\alpha}{\alpha + t} dt \right) \]

**Corollary 2:** Let \(f(z) \in M_p\) be in the class \(M_p^\alpha(\lambda, A)\), defined in (1g), then \(\alpha z P^{\alpha + i}(z) \geq \alpha - |A|\).

**Corollary 3:** Let \(f(z) \in M_p\) be in the class \(M_p^\alpha(\lambda, A)\), defined in (1g), then \(\sum_{i=0}^{\lambda} b_{-p, i} ([\lambda]) \leq \frac{A}{|A|}\). The result is sharp for the extremal function given by:

\[ f_\lambda(z) = z^{-p} + \frac{A}{b_{-p, i} ([\lambda])} z^{-p}, k \geq 1 \]

**RESULTS AND DISCUSSION**

Some consequences of the results are discussed along with some special cases as follows:

For positive real numbers \(a_i, b_i (i = 1, 2, \ldots, q)\) and for positive integers \(A_i, B_i (i = 1, 2, \ldots, q)\) such that \(\sum_{i=0}^{\lambda} (B_i - A_i) \geq 0\), taking:

\[ b_{-p, i} ([\lambda]) = b_{-p, i} ([A_i, A_{i-1}]) \leq \prod_{i=1}^{\lambda} \frac{\Gamma(a_i + A_{i-1}) \Gamma(b_i)}{\Gamma(b_i + B_{i-1}) \Gamma(a_i)} \]

We get for \(f(z) \in M_p\) and for \(\alpha > 0\):

\[ P^{\lambda + i}(z) = \alpha - \omega \left( \int_0^{\lambda} \frac{\alpha}{\alpha + t} dt \right) \]

where, \(-q\Psi_{q+1}(a_1, A_1); ...; (a_q, A_q); (b_1, B_1); ...; (b_q, B_q); z\)

which is a generalized hypergeometric function (Srivastava and Manocha, 1984) and its series representation is given by:

\[ \Psi_{q+1}(a_1, A_1); ...; (a_q, A_q); (b_1, B_1); ...; (b_q, B_q); z \]

\[ = \sum_{i=0}^{\lambda} \prod_{i=1}^{\lambda} \frac{\Gamma(a_i + A_{i-1}) \Gamma(b_i)}{\Gamma(b_i + B_{i-1}) \Gamma(a_i)} z^i \]

and:

\[ b_{-p, i} ([\lambda]) = b_{-p, i} ([A_i + 1, A_{i-1}]) \leq \frac{(a_i + A_{i-1}) b_{-p, i} ([A_i + 1, A_{i-1}])}{a_i} \]

Hence, from:
\[ W([a_i, A_i])f \equiv W((a_i, A_i), (b_i, B_i))f \]

and:
\[ W([a_i + 1, A_i])f \equiv W((a_i + 1, A_i), (b_i, B_i))f \]

with the help of relation (3b), we can easily get the identity:
\[ A_z W([a_i, A_i])f = a_i W([a_i + 1, A_i])f - (a_i + pA_i)W([a_i, A_i])f \]

which can directly be obtained by taking \( \alpha = \frac{a}{A_1} \) in (1e). The operator \( W([a_i, A_i])f \), is itself a generalized operator and is a meromorphic version of the operator considered in (Dziok and Raina, 2004; 2009; Aouf and Dziok, 2008a; 2008b; Sharma, 2010; Dziok et al., 2004). Taking \( A_i = B_i = 1, i = 1, 2, ..., q + 1 \), \( \psi_q(z) \) reduces to the generalized hypergeometric function \( \psi_{q+1}^q F_1(z) \) and we denote:

\[ P^\alpha f = F([a_i])f \equiv z^{-\alpha} q^q F_1(z) * f \tag{3c} \]

Where:
\[ q^q F_1(z) := q^q F(a_i, ..., a_q, 1; b_i, ..., b_q; z) \]

The operator similar to \( F([a_i])f \), has been studied recently by Wang et al. (2009); Raina and Srivastava (2006); Aouf (2008) and Liu and Srivastava (2004). Taking \( q = 2 \) and \( b_1 = 1 \), we get the operator, involving Gauss's hypergeometric function \( \psi_1^1 F_1(z) \):\n
\[ P^\alpha f = G(a_1, a_2, b_1)f := z^{-\alpha} _2 F_1(a_1, a_2; b_1; z) * f \tag{3d} \]

Further, taking \( q = 1 \), we get:
\[ P^\alpha f = L_p(a_1, b_1)f := z^{-\alpha} _2 F_1(a_1; 1; b_1; z) * f \tag{3e} \]

which is studied extensively by Liu and Srivastava (2001); Liu (2000); Aouf and Srivastava (2006) and Srivastava et al. (2008). If \( a_i = n + p, n \in \mathbb{N} \) and \( b_1 = 1 \), we get, Ruscheweyh derivative operator for \( M_p \) class:
\[ P^\alpha f = D^{n+p} f := \frac{z^p}{(1-z)^{n+p}} * f \tag{3f} \]

which is studied by Cho in (Srivastava and Owa, 1992; Uralegaddi and Sommanatha, 1992; Joshi and Srivastava, 1999; Aouf, 1993) and its special case is studied by Aouf and Hossen (1994).

From Theorem 1, we directly get following results.

**Corollary 4:** Let for \( f(z) \in M_p \) the operator \( W(a_i, A_i) \) be defined in (3a). If:

\[ \text{Re} \left( \frac{W([a_i, A_i])f}{W([a_i + 1, A_i])f} \right) < \frac{(a_i + 1 + pA_i)}{(a_i + 1)} \]

Then:

\[ \text{Re} \left( \frac{W([a_i + 1, A_i])f}{W([a_i, A_i])f} \right) \]

**Corollary 5:** Let for \( f(z) \in M_p \) the operator \( W([a_i]) \) be defined in (3c). If:

\[ \text{Re} \left( \frac{F([a_i + 2])f}{F([a_i + 1])f} \right) \]

Then:

\[ \text{Re} \left( \frac{F([a_i + 1])f}{F([a_i])f} \right) \]

**Corollary 6:** Let for \( f(z) \in M_p \) the operator \( G(a_1, a_2, b_1) \) be defined in (3d). If:

\[ \text{Re} \left( \frac{G(a_1 + 2, a_2, b_1)f}{G(a_1 + 1, a_2, b_1)f} \right) \]

Then:

\[ \text{Re} \left( \frac{G(a_1, a_2, b_1)f}{G(a_1, a_2, b_1)f} \right) \]

**Corollary 7:** Let for \( f(z) \in M_p \) the operator \( L_p(a_1, b_1) \) be defined in (3e). If:

\[ \text{Re} \left( \frac{L_p(a_1 + 2, b_1)f}{L_p(a_1 + 1, b_1)f} \right) \]

Then:
Corollary 8: Let for \( f(z) \in M_p \) the operator \( D^{n+p-1}f \) be defined in (3f) (Srivastava and Owa, 1992). If:

\[
\text{Re} \left\{ \frac{D^{n+p}f}{D^{n+p-1}f} \right\} < \frac{(n+1+2p)}{(n+1+p)}
\]

Then:

\[
\text{Re} \left\{ \frac{D^{n+p}f}{D^{n+p-1}f} \right\} < \frac{(n+2p)}{(n+p)}
\]

Corollary 9: Let the operator \( I_c \) be defined by (2d) and the parametric class \( M^\alpha \) be defined in (1f), if \( I_c f \in M^\alpha \), then \( I_c f \in M^\alpha \).

Proof: By Theorem 2, if \( I_c f \in M^\alpha \), then \( I_c f \in M^\alpha \) and further, by Theorem 1, we get the result.

Corollary 10: Let the parameter \( \alpha \) be positive in the definition of parametric class \( M^\alpha \) defined in (1f):

\[
f \in M^\alpha \Leftrightarrow I_c f \in M^\alpha
\]

Proof: Taking \( c = \alpha > 0 \), we see from (1d) and (2d), that:

\[
P^{\alpha-1}I_c f(z) = P^\alpha f(z)
\]

This directly proves the result.

From Theorem 2, we get following special case:

Corollary 11: Let for \( f(z) \in M_p \) the operators \( W([a_i, A])f \) and \( I_c f \) be defined by (3a) and (2d) respectively. If:

\[
\text{Re} \left\{ \frac{W([a_i, A])f}{W([a_i, A])I_c f} \right\} < \frac{(a_i + pA_i)}{a_i}
\]

Then:

\[
\text{Re} \left\{ \frac{W([a_i, A])f}{W([a_i, A])I_c f} \right\} < \frac{(a_i + pA_i)}{a_i}
\]

Similar to Corollaries 5-8, we can further find special results of the Corollary 11.

Convolution condition follows for the class \( M^\alpha(\lambda, A) \).

Corollary 12: Let for \( \lambda > p, |A| < |\alpha|, f(z) \in M^\alpha(\lambda, A) \) if and only if:

\[
\alpha + \left(1 + \frac{\lambda}{\alpha}\right)Ae^{\alpha} - z^p f \ast \left[ \alpha + \sum_{k=1}^\infty (\alpha + \lambda k) a_{k-1} b_k f(\alpha) A^k \right]
\]

\[
eq 0, z \in U, 0 \leq \theta \leq 2\pi
\]

From Theorem 3, we get following results.

Corollary 13: For \( p < \lambda_1 < \lambda_2, |A| < |\alpha|, M^\alpha(\lambda_2, A) \subset M^\alpha(\lambda_1, A) \).

Proof: Let \( f(z) \in M^\alpha(\lambda_2, A) \). Consider:

\[
(1 - \lambda_1)\alpha z^p f(z) + \lambda_1 \alpha z^p f(z) = \left(1 - \frac{\lambda_1}{\lambda_2}\right)\alpha z^p f(z)
\]

\[
+ \frac{\lambda_1}{\lambda_2} \left[(1 - \lambda_2)\alpha z^p f(z) + \lambda_2 \alpha z^p f(z)\right] - \alpha + \left(1 + \frac{\lambda_1}{\alpha}\right)Az
\]

On using the given hypothesis and the result of Theorem 3. This proves the result.

Corollary 14: For \( p < \lambda < 2p^2, |A| < |\alpha|, M^\alpha(\lambda, A) \subset M^\alpha(\beta) \).

Where:

\[
\beta = p - \frac{1}{\lambda}\left(2|\alpha| + \frac{\lambda|A|}{\alpha}\right)
\]

Proof: With the given hypothesis and from Corollary 2 of Theorem 3, we get:

\[
\left[ \alpha z^p f(z) - \alpha z^p f(z) \right] = \left[ 1 + \frac{(1 - \lambda)\alpha z^p f(z) + \lambda \alpha z^p f(z) - \alpha z^p f(z)}{\lambda} \right]
\]

\[
\leq \frac{1}{\lambda} \left( \frac{2|\alpha| + \lambda|A|}{1 - |\alpha|} \right) |f| z^p f(z)
\]
Hence:

\[
\left| \frac{\alpha p^{\alpha+1} f(z)}{P^\alpha f(z)} - \alpha \right| < \frac{1}{\lambda} \left( \frac{2|\alpha| + \frac{\lambda}{\alpha} |A|}{1 - \frac{\lambda}{\alpha}} \right)
\]

Or:

\[
\text{Re} \left\{ \frac{\alpha p^{\alpha+1} f(z)}{P^\alpha f(z)} \right\} < \alpha + p - \beta
\]

Where:

\[
\beta = p - \frac{1}{\lambda} \left( \frac{2|\alpha| + \frac{\lambda}{\alpha} |A|}{1 - \frac{\lambda}{\alpha}} \right)
\]

which proves the result.

**Corollary 15:** Let \( f(z) \in M_p^\alpha(\lambda,A) \) defined in (1g), then \( I_1 f(z) \in M_p^\alpha(p-1) \) , where:

\[
I_1 f(z) = \frac{1}{z^{\alpha+1}} \int_0^z t^\alpha f(t)dt
\]

**Proof:** From Corollary 3 of Theorem 3, we get:

\[
\sum_{k=0}^{\infty} a_k b_{k+\alpha}([\alpha]) \leq \frac{|A|}{\alpha} < 1
\]

Therefore:

\[
\left| \frac{\alpha p^{\alpha+1} f(z)}{P^\alpha f(z)} - \alpha \right| \leq \left\{ \sum_{k=1}^{\infty} k \frac{1}{k+1} \sum_{\alpha} \left| a_k b_{k+\alpha}([\alpha]) \right| \right\} < 1
\]

which proves the result.

Results for suitable values of parameter \( \alpha \), based on Corollaries 12-15 can also be obtained.

**CONCLUSION**

An inclusion relation for the class \( M_p^\alpha \) is obtained and it is shown that this class is preserved under the operator \( I_1 \) \((c>0)\). The subordinate condition for the class \( M_p^\alpha(\lambda,A) \) provides a convolution condition and a subordination which gives an inclusion relation, a sharp coefficient inequality. It is verified that \( M_p^\alpha(\lambda,A) \subset M_p^\beta(\beta) \) for some \( \beta \) and the operator \( I_1 : M_p^\alpha(\lambda,A) \to M_p^\alpha(p-1) \). The results verify the inclusions related to starlike, convex and close-to-convex classes. Necessary and sufficient coefficient conditions for the classes \( M_p^\alpha \) and \( M_p^\alpha(\lambda,A) \) can also be obtained for \( a_{k+\alpha} \geq 0, k \geq 1 \). With the help of coefficient conditions several more results can be derived.

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