Motion of two pairs of point vortices around a circular cylinder

M. N. Moura\textsuperscript{1} and G. L. Vasconcelos\textsuperscript{1,\textsuperscript{a}}

Laboratório de Física Teórica e Computacional, Departamento de Física, Universidade Federal de Pernambuco, 50670-901, Recife, Brazil.

(Dated: August 2, 2012)

The motion of two pairs of counter-rotating point vortices placed in a uniform flow past a circular cylinder is studied analytically. When the motion is restricted to the symmetric subspace—a case that can be realized experimentally by placing a splitter plate in the center plane—, it is found that there is a family of stable equilibria for same-signed vortex pairs. The stability analysis for antisymmetric perturbations is also presented and shows that the equilibria are always unstable in this case. The analysis reported here provides new insights and reveals novel features of this four-vortex system, such as the fact that there is no equilibrium for opposite-signed vortex pairs. It is argued furthermore that opposite-signed equilibria might exist for vortex flows past a cylinder confined in a channel.

PACS numbers: 47.32.C-, 47.15.ki, 47.15.km

\textsuperscript{a}) Corresponding author. Electronic mail: giovani@df.ufpe.br
I. INTRODUCTION

The formation of recirculating eddies in viscous flows past cylindrical structures is a problem of considerable theoretical interest and practical relevance for many applications\textsuperscript{1,2}. In the well-known case of flows past a circular cylinder, a pair of counter-rotating eddies forms behind the cylinder at small Reynolds numbers, which then goes unstable at higher Reynolds numbers and evolves into a von Kármán vortex street. This classical problem was first studied by Föppl\textsuperscript{3} a century ago using a point-vortex model, but only recently it was more fully understood\textsuperscript{4}. The motion of multiple vortex pairs in the presence of a cylinder has also attracted considerable attention; see, e.g., Ref. [5] and references therein. Of particular note is the four-vortex configuration recently observed\textsuperscript{6} in the counterflow of superfluid helium II past a circular cylinder, where stationary eddies formed both downstream and upstream of the cylinder. A possible explanation for this unusual vortex arrangement was given in Ref. [6] in terms of the complex interaction between the normal and superfluid components of He II. It remains an open question whether similar configurations can be observed in classical fluids.

In this paper we investigate the motion of two pairs of point vortices in an inviscid fluid past a circular cylinder. First we treat the case of same-signed vortex pairs and compute the equilibrium configurations. We then study the stability of these equilibria with respect to symmetric perturbations and show that there is a large subset of equilibria that are stable under symmetric perturbations. As for antisymmetric perturbations, it is shown that the equilibria are always unstable in this case. (The locus of same-signed equilibria was first found by Elcrat \textit{et al.}\textsuperscript{7} but the stability analysis has not been carried out before.) Since symmetry can be enforced experimentally by attaching splitter plates to the cylinder in the center plane of the flow\textsuperscript{3}, the family of stable same-signed equilibria in the symmetric subspace may eventually turn out to be of practical relevance, although to the best of our knowledge they have not so far been observed. We also analyze the case of opposite-signed vortex pairs and show that there is no equilibrium configuration in this case, thus correcting an erroneous claim in the literature\textsuperscript{5}. We shall argue furthermore that opposite-signed equilibria are likely to exist for flows past a cylinder placed in a rectangular channel, which might help to explain the unusual vortex configuration seen in superfluid helium mentioned above. A more detailed study of the interesting but difficult problem of vortex flows past a
cylinder in confined geometries is however beyond the scope of the present paper.

The paper is organized as follows. In Sec. II we present the mathematical formulation of the problem. In Sec. III we compute the equilibrium configurations for same-signed vortex pairs and show that there is no equilibrium for opposite-signed vortex pairs. The stability properties of the same-signed symmetric equilibria are studied in Sec. IV and some important implications of our results are discussed in Sec. V. In Sec. VI we summarize our main findings and conclusions.

II. FORMULATION OF THE PROBLEM

We consider the two-dimensional motion of a pair of vortex pairs around a circular cylinder of radius $a$, in the presence of a uniform stream of velocity $U$, as illustrated in Fig. 1. The vortices are considered to be point-like and the fluid is treated as incompressible, inviscid, and irrotational, except at the vortex positions where the vorticity is singular (i.e., a delta function). Under such conditions one has a potential flow: the fluid velocity field is given by $\vec{v} = \vec{\nabla} \phi$, where $\phi(x, y)$ is the velocity potential which satisfies Laplace equation, $\nabla^2 \phi = 0$. It is convenient to work in the complex $z$-plane, where $z = x + iy$, with the origin placed at the center of the cylinder. The upper and lower vortices of the vortex pair downstream of the cylinder are located at positions $z_1 = x_1 + iy_1$ and $z_3 = x_3 + iy_3$, and have circulations $\pm \Gamma_1$, respectively, whereas the positions of the upper and lower vortices of the vortex pair upstream of the cylinder are denoted by $z_2 = x_2 + iy_2$ and $z_4 = x_4 + iy_4$, with respective circulations denoted by $\pm \Gamma_2$; see Fig. 1.

The complex potential $w(z) = \phi(x, y) + i\psi(x, y)$ for the flow, where $\psi$ is the stream func-
tion, is obtained by a direct application of the circle theorem, yielding
\[
w(z) = U \left( z + \frac{a^2}{z} \right) + \frac{\Gamma_1}{2\pi i} \log \left( \frac{z - z_1}{z - a^2/\bar{z}_2} \right) + \frac{\Gamma_2}{2\pi i} \log \left( \frac{z - z_3}{z - a^2/\bar{z}_4} \right),
\]
where the bar denotes complex conjugation. In the right-hand side of Eq. (1), the first two terms account for the uniform stream and its image by the cylinder (a dipole at the origin), whereas the other two terms represent the contributions from the vortices at \( z_j, j = 1, 2, 3, 4 \), and their respective images which are located (inside the cylinder) at \( a^2/\bar{z}_j \).

Introducing dimensionless variables
\[
z' = \frac{z}{a}, \quad t' = \frac{U}{a} t, \quad w' = \frac{w}{Ua}, \quad \kappa_i = \frac{\Gamma_i}{2\pi U a},
\]
Eq. (1) can be rewritten as
\[
w(z) = z + \frac{1}{z} + i\kappa_1 \log \left( \frac{z - z_1}{1 - z_1 \bar{z}_3} \right) + i\kappa_2 \log \left( \frac{z - z_2}{1 - z_2 \bar{z}_4} \right),
\]
where the prime notation has been dropped. To calculate the velocity, \( \vec{\nu}_j = (u_j, v_j) \), of a given vortex located at position \( z_j \), one must subtract from the complex potential \( \Psi(z) \) the contribution of the vortex itself and then evaluate the derivative of the resulting “effective potential” at the vortex position \( z = z_j \). For example, for the upper vortex located at \( z_1 \) one has
\[
u_1 - iv_1 = \frac{d}{dz} \left[ w(z) - i\kappa_1 \log(z - z_1) \right] \bigg|_{z=z_1},
\]
which yields
\[
u_1 - iv_1 = 1 - \frac{1}{z_1} + i\kappa_1 \left( -\frac{1}{z_1 - z_3} + \frac{\bar{z}_3}{1 - z_1 \bar{z}_3} + \frac{\bar{z}_1}{1 - z_1 \bar{z}_1} \right)
+ i\kappa_2 \left( -\frac{1}{z_1 - z_4} + \frac{1}{z_1 - z_2} + \frac{\bar{z}_2}{1 - z_1 \bar{z}_2} - \frac{\bar{z}_4}{1 - z_1 \bar{z}_4} \right).
\]
Similar procedure gives the velocity \( \vec{\nu}_2 = (u_2, v_2) \) for the upper vortex at \( z_2 \):
\[
u_2 - iv_2 = 1 - \frac{1}{z_2} + i\kappa_2 \left( -\frac{1}{z_2 - z_4} + \frac{\bar{z}_4}{1 - z_2 \bar{z}_4} + \frac{\bar{z}_2}{1 - z_2 \bar{z}_2} \right)
+ i\kappa_1 \left( -\frac{1}{z_2 - z_3} + \frac{1}{z_2 - z_1} + \frac{\bar{z}_1}{1 - z_2 \bar{z}_1} - \frac{\bar{z}_3}{1 - z_2 \bar{z}_3} \right).
\]
The velocity of the lower vortices can be obtained from Eq. (5) by a proper interchange of the indexes: for the vortex located at \( z_3 \) one makes \( 1 \leftrightarrow 3 \) and \( 2 \leftrightarrow 4 \), whereas for the vortex at \( z_4 \) one takes \( 1 \leftrightarrow 4 \) and \( 2 \leftrightarrow 3 \), together with the change \( \kappa_j \rightarrow -\kappa_j \).
Here we are interested in finding the equilibrium positions for the vortex system described above, which corresponds to solving the equations \( u_j = v_j = 0 \), for \( j = 1, \ldots, 4 \). For \( \kappa_1 \neq \kappa_2 \) this amounts to finding the zeros of polynomials of very high order—a difficult task that can be accomplished only numerically. The problem is relatively easier when the two vortex pairs have the same strength, i.e., \( |\kappa_1| = |\kappa_2| \), as we shall now discuss.

### III. EQUILIBRIUM CONFIGURATIONS

#### A. Same-Signed Equilibria

Here we consider the case where the two vortex pairs have the same sign and strength, so that \( \Gamma_1 = \Gamma_2 = \Gamma \), which implies \( \kappa_1 = \kappa_2 = \kappa \). From symmetry considerations, it is clear that the equilibrium configuration must be such that vortices are located at the vertices of a rectangle centered at the origin:

\[
\begin{align*}
z_1 &= z_0, & z_2 &= -\bar{z}_0, & z_3 &= \bar{z}_0, & z_4 &= -z_0,
\end{align*}
\]

as indicated in Fig. 2. As can be inferred from Fig. 2, a necessary condition for a steady configuration to exist is that the upper (lower) vortices be of negative (positive) circulation, hence only the case \( \Gamma < 0 \) is of interest here, which implies \( \kappa > 0 \); see Eq. (3). In this case, it is easy to convince oneself that if we happen to find a configuration in which the velocity \( \vec{v}_1 \) of the first vortex vanishes, then the velocities of the other three vortices will also vanish. Thus the problem reduces to solving Eq. (5) for \( u_1 = v_1 = 0 \), with \( \kappa_1 = \kappa_2 = \kappa \), subjected to the condition (7).

![FIG. 2. Symmetric configuration for two same-signed vortex pairs.](image-url)
After some algebraic manipulation, one finds that the locus of possible equilibrium positions for the first vortex is obtained by solving the equation $P(x, y) = 0$, where $P(x, y)$ is a polynomial of order 14 given in Eq. (A1) of the Appendix. Solving this equation in the first quadrant yields a curve $C_0$, shown in Fig. 3 with the equilibrium positions for the three other vortices being obtained by successive reflections of $C_0$ about the $x$ and $y$ axes. For each point $(x_0, y_0)$ on the curve $C_0$, the corresponding vortex intensity $\kappa$ is given by

$$\kappa = \frac{Q(x_0, y_0)}{R(x_0, y_0)},$$

where $Q(x, y)$ and $R(x, y)$ are polynomials given in Eqs. (A3) and (A4), respectively. Fig. 4 shows a plot of the vortex intensity $\kappa$ for points on the curve $C_0$. 

FIG. 3. Loci of equilibrium points for same-signed vortex pairs.

FIG. 4. Vortex intensity $\kappa$ along the equilibrium curve for the vortex in the first quadrant.
FIG. 5. Loci of equilibrium points (solid curves) for same-signed vortex pairs with \( \kappa_1 = 0.5 \) and \( 0 < \kappa_2 < 6 \); the dashed line indicates the loci of the symmetric equilibria.

It is interesting to note that, differently from the case of a single vortex pair behind a cylinder\(^4\), the stationary positions in the four-vortex case lie in a bounded region close to the cylinder. In other words, equilibrium configurations exist only up to a certain maximum vortex strength, beyond which the vortex-vortex interactions cannot be cancelled by the oncoming stream. The equilibria shown in Fig. 3 were first found numerically by Elcrat et al.\(^7\). They were also obtained by Shashikanth\(^5\) who considered the problem of two symmetric pairs of point vortices interacting with a neutrally buoyant cylinder, but there the stability properties of the equilibria are quite different from the case of a fixed cylinder studied here; see below.

We note in passing that the symmetry condition (7) can be relaxed if one allows for different vortex strengths, i.e., \( \kappa_1 \neq \kappa_2 \). In this case, one can still find equilibrium configurations that are symmetric with respect to the middle plane but without fore-and-aft symmetry. These equilibria can be computed by fixing the strength of one of the vortex pairs, say, \( \kappa_1 \), and then varying the strength of the other, so that for each pair of values \( (\kappa_1, \kappa_2) \) one needs to solve the equilibrium equations numerically. One example is shown in Fig. 5 for the case where the intensity of the vortex downstream was kept fixed at \( \kappa_1 = 0.5 \), while the intensity of the vortex upstream was varied in the interval \( 0 < \kappa_2 < 6 \).

B. Opposite-Signed Vortex Pairs: No Equilibrium

As already mentioned, the question about the (possible) existence of equilibrium configurations for two opposite-signed vortex pairs is of particular interest. In what follows we will answer this question in the negative: no equilibrium is possible for two opposite-
signed vortex pairs in a flow past a cylinder in an otherwise unbounded domain. (In Ref. [5] it was erroneously claimed that such configurations exist.) To see this, first note that for opposite-signed vortex pairs the fact that the velocity of the first pair vanishes does not automatically ensures that the velocity of the second pair also vanishes. Indeed, setting $\kappa_1 = -\kappa_2 = \kappa$ in Eq. (5) and solving for $u_1 = v_1 = 0$, under conditions (7), yields a polynomial curve of the form $P_1(x_0, y_0) = 0$ for the putative equilibrium (this is the curve shown in Fig. 15 of Ref. [5]), with a vortex strength given by a rational function: $\kappa = Q_1(x_0, y_0)/R_1(x_0, y_0) > 0$. [The polynomials $P_1(x, y)$, $Q_1(x, y)$, and $R_1(x, y)$ are not given here for brevity.] However, when solving Eq. (6) for $u_2 = v_2 = 0$ one finds a vortex strength of the form $\kappa = -Q_1(x_0, y_0)/R_1(x_0, y_0) < 0$, in contradiction with the previous result. Hence no symmetric equilibrium is possible for opposite-signed vortex pairs.

This result can also be easily proved by the following geometrical argument. Consider the upper vortex of the first vortex pair, draw the arrows corresponding to the velocities induced by the other three vortices and all the images inside the cylinder, and suppose that the resulting velocity precisely cancels the velocity $U$ of the uniform stream, so that $\vec{v}_1 = 0$. Now do the same for the upper vortex of the second vortex pair. It is easy to convince oneself that the velocities induced at this vortex by the other vortices (including the images) correspond to a mere reflection about the $y$ axis of the velocities drawn at the first vortex. It then follows that the resulting vortex-induced velocity at the second vortex cannot be canceled by the velocity $U$ of the uniform stream, hence $\vec{v}_2 \neq 0$. This argument can be generalized for the case where the pair of opposite-signed vortex pairs have different strength, i.e., $|\kappa_1| \neq |\kappa_2|$, and the symmetry condition is relaxed. Here again one cannot satisfy both equations $\vec{v}_1 = 0$ and $\vec{v}_2 = 0$ simultaneously. We conjecture, however, that opposite-signed equilibria might exist for vortex flows past a cylinder confined in a rectangular channel; see Sec. V for further discussion about this case.

IV. STABILITY ANALYSIS

To perform the stability analysis of the equilibria found in Sec. III A, one needs in principle to linearize the equations of motion for the four vortices around the corresponding equilibrium positions, which will result in a dynamical system described by an $8 \times 8$ matrix, whose eigenvalues define the nature of the equilibria. It is more convenient, however, to perform
the stability analysis of our vortex system with respect to symmetric and anti-symmetric perturbations separately, as discussed next.

A. Symmetric Perturbations

1. Linear Stability Analysis

Let us consider arbitrary displacements from the equilibrium positions of the two vortices located in the upper half-plane:

\[ z_1 = z_0 + \xi_1 + i\eta_1, \quad z_2 = -\bar{z}_0 + \xi_2 + i\eta_2, \]  

(9)

where \( \xi_i \) and \( \eta_i \) are (infinitesimally small) real numbers. For symmetric perturbations, the positions of the two lower vortices are given by

\[ z_3 = -\bar{z}_0 + \xi_1 - i\eta_1, \quad z_4 = -z_0 + \xi_2 - i\eta_2. \]  

(10)

Because this symmetry is preserved by the dynamics, we need to focus only on the motion of the upper vortices. After inserting Eqs. (9) and (10) into Eqs. (5) and (6) and linearizing the resulting equations of motion for \( \xi_i \) and \( \eta_i \), one obtains the following dynamical system:

\[
\begin{pmatrix}
\dot{\xi}_1 \\
\dot{\eta}_1 \\
\dot{\xi}_2 \\
\dot{\eta}_2
\end{pmatrix} = A
\begin{pmatrix}
\xi_1 \\
\eta_1 \\
\xi_2 \\
\eta_2
\end{pmatrix},
\]  

(11)

where \( A \) is a 4 \times 4 matrix given by

\[
A = \begin{pmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial y_2} \\
\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial y_2} \\
\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial y_2} \\
\frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial y_2}
\end{pmatrix},
\]  

(12)

with the derivatives evaluated at the equilibrium position. Out of the 16 elements of \( A \) only seven of them are independent of one another. These elements can be written explicitly in
terms of rational functions of the equilibrium coordinates $x_0$ and $y_0$, but the corresponding expressions are rather long and will not be given here.

From the analysis of the eigenvalues of the matrix $A$ one finds that the stability nature of the equilibrium point varies with its location along the curve $C_0$. It is therefore convenient to separate this curve into five different regions, according to the stability properties of the equilibrium, as indicated in Fig. 6. In regions I and II the eigenvalues of $A$ are of the form $\lambda_1^2 = \alpha < 0$ and $\lambda_2^2 = \beta < 0$, with $\alpha \neq \beta$. Thus, in these two regions the equilibrium is a center-center in the four-dimensional symmetric subspace. (The reason to divide this region into two subregions has to do with the distinct nature of the equilibrium point with respect to antisymmetric perturbations; see below.) In the transition point from region II to region III, a bifurcation occurs such that one of the two centers turns into a saddle. Thus, the equilibrium point in region III becomes a saddle-center with $\lambda_1^2 > 0$ and $\lambda_2^2 < 0$. The nature of the equilibrium point under symmetric perturbations in region IV is the same as in region III. (Here again the distinction between these two regions is because of the different behavior under antisymmetric perturbations.) In going from region IV to region V, another bifurcation occurs with the remaining center turning into a saddle, so that the equilibrium point in region V becomes a saddle-saddle, with eigenvalues $\lambda_1^2 > 0$ and $\lambda_2^2 > 0$. In regions III–V the equilibria are therefore always unstable since there is at least one real positive eigenvalue.
FIG. 7. Trajectories of the two upper vortices for $\kappa = 1.23$ and initial conditions corresponding to the perturbations $\Delta z_1 = 0.226 + i0.003$ and $\Delta z_2 = 0.195 + i0.028$ of the respective equilibrium positions. For sufficiently long time the trajectories will fill a compact neighborhood of the equilibrium points. Here the equilibrium was chosen from region I of Fig. 6.

2. Nonlinear Dynamics

As is well known, the equations of motion for point vortices in a two-dimensional inviscid flow can be formulated as a Hamiltonian system. The dynamics of point vortices in the presence of rigid boundaries was shown by Lin to be also Hamiltonian with the same canonical symplectic structure as in the absence of boundaries. For two pairs of vortices placed in a uniform stream past a circular cylinder, the eight-dimensional phase space has a four-dimensional invariant subspace corresponding to symmetric orbits, as already noted. A detailed discussion of the nonlinear dynamics of this four-dimensional Hamiltonian system is, of course, beyond the scope of the present paper. However, a few interesting observations are in order.

Numerical integration of the equations of motion in the symmetric subspace indicate that the neutrally stable equilibria in regions I-II (see Fig. 6) are indeed nonlinearly stable, in the sense that for small (but finite) perturbations the vortex trajectories remain bounded. An example of this case is shown in Fig. 7, where the orbits of the vortices tend to fill densely a compact neighborhood of the equilibrium points as time progresses. In regions III–V, where there is at least one real positive eigenvalue, numerical integration of the equations of motion indicates that for generic (symmetric) perturbations the vortices will eventually move off to infinity. There are however certain initial conditions for which surprising trajectories occur: one vortex pair goes to infinity downstream while the other pair approaches a periodic orbit.
FIG. 8. Trajectories of the two upper vortices for $\kappa = 0.92$ and initial conditions corresponding to the perturbations $\Delta z_1 = -0.5 + i 0.5$ and $\Delta z_2 = 0.5 + i 0.5$. Here the equilibrium point was chosen from region V of Fig. 6.

B. Antisymmetric Perturbations

Assuming general displacements of the two upper vortices as shown in Eq. (9), the antisymmetric perturbations of the two lower vortices are given by

$$ z_3 = \bar{z}_0 - \xi_1 + i \eta_1, \quad z_4 = -z_0 - \xi_2 + i \eta_2. \quad (13) $$

Since the four-dimensional antisymmetric subspace is invariant under the linear dynamics, we can once again focus only on the two upper vortices when carrying out the stability analysis for this case.

Linearizing Eqs. (5) and (6) for the perturbations given in (9) and (13), one obtains

$$
\begin{pmatrix}
\dot{\xi}_1 \\
\dot{\eta}_1 \\
\dot{\xi}_2 \\
\dot{\eta}_2
\end{pmatrix} = B
\begin{pmatrix}
\xi_1 \\
\eta_1 \\
\xi_2 \\
\eta_2
\end{pmatrix}, \quad (14)
$$
where the matrix $B$ is given by

$$
B = \begin{pmatrix}
\frac{\partial u_1}{\partial x_1} - \frac{\partial u_1}{\partial x_3} & \frac{\partial u_1}{\partial y_1} + \frac{\partial u_1}{\partial y_3} & \frac{\partial u_1}{\partial x_2} - \frac{\partial u_1}{\partial x_4} & \frac{\partial u_1}{\partial y_2} + \frac{\partial u_1}{\partial y_4} \\
\frac{\partial v_1}{\partial x_1} - \frac{\partial v_1}{\partial x_3} & \frac{\partial v_1}{\partial y_1} + \frac{\partial v_1}{\partial y_3} & \frac{\partial v_1}{\partial x_2} - \frac{\partial v_1}{\partial x_4} & \frac{\partial v_1}{\partial y_2} + \frac{\partial v_1}{\partial y_4} \\
\frac{\partial u_2}{\partial x_1} - \frac{\partial u_2}{\partial x_3} & \frac{\partial u_2}{\partial y_1} + \frac{\partial u_2}{\partial y_3} & \frac{\partial u_2}{\partial x_2} - \frac{\partial u_2}{\partial x_4} & \frac{\partial u_2}{\partial y_2} + \frac{\partial u_2}{\partial y_4} \\
\frac{\partial v_2}{\partial x_1} - \frac{\partial v_2}{\partial x_3} & \frac{\partial v_2}{\partial y_1} + \frac{\partial v_2}{\partial y_3} & \frac{\partial v_2}{\partial x_2} - \frac{\partial v_2}{\partial x_4} & \frac{\partial v_2}{\partial y_2} + \frac{\partial v_2}{\partial y_4}
\end{pmatrix},
$$

with the derivatives evaluated at the equilibrium positions.

Upon computing the eigenvalues of $B$ along the curve $C_0$ of equilibria one finds the following behavior. In region I (see Fig. 6), the eigenvalues of $B$ are two pairs of real numbers with opposite sign, i.e., $\lambda_3^2 > 0$ and $\lambda_4^2 > 0$, thus characterizing a saddle-saddle equilibrium point in the antisymmetric subspace. In the transition point from region I to region II, a bifurcation occurs with one of the saddles becoming a center, so that in region II the eigenvalues of $B$ are of the form $\lambda_3^2 > 0$ and $\lambda_4^2 < 0$, corresponding to a saddle-center. This situation does not change in region III. But in going from region III to region IV, a reversed bifurcation occurs with the center reverting to a saddle, so that in region IV the equilibrium in the anti-symmetric subspace becomes a saddle-saddle again, i.e., $\lambda_3^2 > 0$ and $\lambda_4^2 > 0$. In region V the equilibrium remains a saddle-saddle.

V. DISCUSSIONS

In Table I we summarize the linear stability analysis of the same-signed equilibria with respect to both symmetric and antisymmetric perturbations. We recall that $\lambda_1^2$ and $\lambda_2^2$ refer to the two pairs of eigenvalues for the symmetric modes, whereas $\lambda_3^2$ and $\lambda_4^2$ denote the eigenvalues for the antisymmetric modes. In Table I we have indicated with the letter C (from “center”) the regions where $\lambda_i^2 < 0$, implying a pair of purely imaginary eigenvalues, and with the label S (from “saddle”) the regions for which $\lambda_i^2 > 0$, giving a pair of real eigenvalues of opposite signs.

From a practical standpoint, the most relevant information in Table I is perhaps the fact that the equilibrium configurations located in regions I and II are neutrally stable under symmetric perturbations (in fact they are nonlinearly stable as shown in Sec. IV A 2).
TABLE I. Regions of stability along the locus of equilibria shown in Fig. 6, where C denotes “center” ($\lambda_j^2 < 0$) and S denotes “saddle” ($\lambda_j^2 > 0$).

| Region | I | II | III | IV | V |
|--------|---|----|-----|----|---|
| $\lambda_1^2$ | C | C | S | S | S |
| $\lambda_2^2$ | C | C | C | C | S |
| $\lambda_3^2$ | S | S | S | S | S |
| $\lambda_4^2$ | S | C | C | S | S |

Since symmetry can be enforced by placing a splitter plate in the middle plane behind the cylinder, this equilibrium could in principle be observed in experiments. Of course, the difficulty here is to generate the vortices in front of the cylinder. This may however be possible by placing a sufficiently long splitter plate in front of the cylinder, which would have the tendency of generating vortices in front of the cylinder with the same sign of the vortices behind it. (To the best of our knowledge such vortex configurations have not so far been observed in experiments.)

It is also important to recall here that there exists no equilibrium point for two opposite-signed vortex pairs; see Sec. III B. This means, in particular, that the type of vortex configuration observed in the counterflow of superfluid helium past a cylinder, where the vortex pair in front of the cylinder has the opposite sign of the vortex pair behind it, cannot be realized in the flow of a classical fluid past a cylinder in an otherwise unbounded domain. It might be the case, however, that such configurations may exist for flows past a cylinder confined in a rectangular channel, which was the geometry used in the experiments with superfluid helium mentioned before. In this case, the channel walls tend to generate vortices in front of the cylinder with the opposite sign of the vortices behind it. Indeed, recirculating eddies in front of a circular cylinder placed near a plane boundary are observed when the gap between the cylinder and the plane is sufficiently small. It is thus likely that confining the cylinder between two plane walls may induce the formation of an opposite-signed vortex pair in front of the cylinder.

From a theoretical perspective, the treatment of point-vortex dynamics in the presence of a cylinder placed between two plane walls is a much more complicated problem because of the infinitely many vortex images that one has to consider. It may nevertheless be possible
to tackle this difficult problem using a recent methodology developed by Crowdy\cite{Crowdy} to deal with vortex dynamics in multiply connected domains. We are currently investigating this possibility. The existence of stationary configurations for opposite-signed vortex pairs (if found) would be of considerable interest because it would show that one needs not to invoke the two-fluid model of superfluid helium to explain the experimental observations reported in Ref. [6].

VI. CONCLUSIONS

We have presented a detailed study of the stationary configurations for two pairs of point vortices placed in a uniform flow past a circular cylinder. We have shown that in the case of same-signed vortex pairs there exists a family of symmetric equilibria that although unstable under generic perturbations are nonlinearly stable with respect to symmetric perturbations. Furthermore, we have proven that there is no opposite-signed equilibrium for vortex flows past a cylinder in an unbounded domain, although such equilibria may exist in confined geometries, which could explain the unusual vortex configurations recently observed in the counterflow of superfluid helium past a cylinder in a rectangular channel.

VII. ACKNOWLEDGEMENTS

This work was supported financially in part by the Brazilian agencies CNPq and FACEPE.

Appendix A: The Locus of Same-Signed Equilibrium Points

After setting $\kappa_1 = \kappa_2 = \kappa$ in Eq. (5), taking the real and imaginary parts, and solving for $u_1 = 0$ and $v_1 = 0$, one finds after some manipulation that the equilibrium positions for the first vortex lie on the curve given by $P(x, y) = 0$, where

$$P(x, y) = (2 + y^2) r^{12} + (2 - 16y^2) r^{10} + (-4 - 19y^2 + 16y^4) r^8$$
$$+ 4 \left(-1 - 8y^2 + 8y^4\right) r^6 + \left(2 + 19y^2 + 96y^4 - 16y^6\right) r^4$$
$$- 2 \left(-1 + 8y^2 + 16y^4 + 32y^6\right) r^2 + y^2 \left(-1 + 16y^2 + 16y^4\right). \quad (A1)$$
with $r^2 = x^2 + y^2$. The corresponding vortex intensity is given by

$$\kappa = \frac{Q(x, y)}{R(x, y)},$$

(A2)

where the polynomials $Q(x, y)$ and $R(x, y)$ take the form

$$Q(x, y) = 2y (-1 + x^4 + 2x^2y^2 + y^4) \times \left( -x^6 + (-1 + y^2) (1 + y^2)^2 + x^4 (1 + 15y^2) + x^2 (1 - 6y^2 - 15y^4) \right),$$

(A3)

and

$$R(x, y) = x^{10} - 12x^8y^2 + 4x^4y^2 (3 + 10y^2) - 2x^6(1 + 4y^2 + 11y^4) + 4y^4 (2 - y^2 - 2y^4 + y^6) + x^2 (1 - 8y^2 - 6y^4 + 40y^6 + 13y^8).$$

(A4)

REFERENCES

1. M. M. Zdravkovich, *Flow around circular cylinders*, Vol. 1: Fundamentals; Vol. 2: Applications (Oxford University Press, Oxford, 1997).
2. B. M. Sumer and J. Fredse, *Hydrodynamics around cylindrical structures* (World Scientific, Singapore, 2006).
3. L. Föppl, “Wirbelbewegung hinter einem Kreiszylinder,” Sitzb. Bayer. Akad. Wiss. 1, 1 (1913).
4. G. L. Vasconcelos, M. N. Moura, and A. M. J. Schakel, “Vortex motion around a circular cylinder,” Phys. Fluids 23, 123601 (2011).
5. B. N. Shashikanth, “Symmetric pairs of point vortices interacting with a neutrally buoyant two-dimensional circular cylinder,” Phys. Fluids 18, 127103 (2006).
6. T. Zhang and S. W. Van Sciver, “Large-scale turbulent flow around a cylinder in counterflow superfluid $^4$He (He (II))”, Nature Phys. 1, 36 (2005).
7. A. Elcrat, B. Fornberg, M. Horn, and K. Miller, “Some steady vortex flows past a circular cylinder,” J. Fluid Mech. 409, 13 (2000).
8. L. M. Milne-Thomson, *Theoretical Hydrodynamics*, 5th ed. (Dover, New York, 1996).
9. L. M. Lerman and Y. L. Umanskiy, *Four-dimensional Integrable Hamiltonian Systems with Simple Singular Points* (American Mathematical Society, Providence, 1998).
10. P. G. Saffman, *Vortex Dynamics* (Cambridge University Press, Cambridge, 1992).
11C. C. Lin, “On the motion of vortices in two dimensions–I and II,” Proc. Natl. Acad. Sci. U.S.A. 27, 570 (1941).

12W-J. Lin, C. Lin, S-C. Hsieh, and S. Dey, “Flow characterization around a circular cylinder placed horizontally above a plane boundary,” J. Eng. Mech. 135, 697 (2009).

13D. G. Crowdy, “A new calculus for two-dimensional vortex dynamics,” Theor. Comput. Fluid Dyn. 24, 9 (2010).