Gravitomagnetism and the significance of the curvature scalar invariants

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The curvature invariants have been subject of interest due to the debate concerning the notions of intrinsic/extrinsic frame-dragging, the use of the electromagnetic analogy in such classification, and the question of whether there is a fundamental difference between the gravitomagnetic fields arising from the translational and rotational motions of the sources (which have been subject of observational and experimental tests, including the dedicated Gravity Probe-B and LARES space missions). In this work we clarify both the algebraic and physical meaning of the curvature invariants and their electromagnetic counterparts. They are seen to yield conditions for the existence of observers measuring vanishing electric/magnetic fields and gravitoelectric/gravitomagnetic tidal tensors, respectively. We determine these observers (in the gravitational sector and in the presence of sources, for the more relevant gravitomagnetic case) obtaining their velocities explicitly in terms of the fields/tidal tensors as measured by an arbitrary observer. The structure of the invariants of the astrophysical setups of interest is studied in detail, and its relationship with the gravitomagnetic effects is dissected. Finally, a new classification for intrinsic/extrinsic gravitomagnetism is proposed.

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References

I. INTRODUCTION

In the last three decades different experiments succeeded in measuring the so-called “gravitomagnetic (GM) field” — which can be described as the inertial force generated by mass/energy currents that is manifest in the precession of gyroscopes, or in the Coriolis-like (apparent) acceleration of a particle in geodesic motion relative to a frame fixed to the distant stars. One can cast (e.g. \[1,2\]) the effects detected in two main types: rotational gravitomagnetism, arising from the rotation of a celestial body, and translational gravitomagnetism, originated by bodies in translational motion with respect to the reference frame.

Translational gravitomagnetism has been detected, to high precision, in a number of ways. The observations of the binary pulsar PSR 1913 +16 (namely the effect of the gravitomagnetic field caused by the motion of each star in the orbit of its companion) \[3\] form one example; other effects that can be cast as translational gravitomagnetism (cf. \[4\]) are the different measurements of the “geodetic” (or de Sitter) precession, namely the precession of the Earth-Moon system along its orbit around the Sun due to the gravitomagnetic field generated by the relative motion of the Sun, detected in the analysis of Lunar Laser Ranging (LLR) data \[10,12\]; the precession of the gyroscopes in the Gravity Probe-B \[13\] due to the translational motion of the Earth relative to the probe; and the precession of the pulsar’s spin vector in the binary systems PSR J0737–3039A/B \[14\] and (with lower precision) PSR B1534+12 \[15\]. It has also been claimed \[16,19\] that the influence on the lunar orbit of the gravitomagnetic field generated by the translational motion of the Earth relative to the Sun has been detected via LLR.\(^1\)

The gravitomagnetic field generated by the rotational motion of celestial bodies has been more elusive, due to its typically smaller magnitude. The only measurements performed to date concern the gravitomagnetic field arising from the Earth’s rotation, detected in the analysis of the LAGEOS Satellites data \[23,24\], by the Gravity Probe-B mission \[9,13\], and by the LARES mission \[25\], with announced accuracies of 10%, 19%, and 2%, respectively. Its detection to a 0.2% accuracy is also the primary goal of the planned LARES 2 mission \[26\].

The curvature invariants have been subject of interest in this context due to the ongoing debate concerning the notions of “intrinsic” vs. “frame-dependent” gravitomagnetism, and the question of whether there is a fundamental difference between the gravitomagnetic fields generated by the rotation and the translational motion of celestial bodies. The use of the curvature invariants in such discussion is motivated by an analogy with the quadratic invariants of the Faraday tensor \(\mathbf{F} \cdot \mathbf{F}\) and \(\mathbf{F} \cdot \mathbf{F} \times \mathbf{F}\). In electromagnetism these invariants give conditions for the vanishing of the electric/magnetic fields for some observers; in particular, when \(\mathbf{F} \cdot \mathbf{F} = 0\) and \(-\mathbf{F} \cdot \mathbf{F} > 0\), there are observers \(\alpha\) measuring zero magnetic field \(\mathbf{B}\). The latter is said to be “frame-dependent”, and an example is the field produced by a uniformly moving (non-spinning) point charge. On the other hand, if \(\mathbf{F} \cdot \mathbf{F} \neq 0\) or \(-\mathbf{F} \cdot \mathbf{F} < 0\), then \(\mathbf{B} \neq 0\) for all observers; \(\mathbf{B}\) is said to be “intrinsic”, and an example is the field produced by a spinning charge. Based on the formal analogy with the quadratic curvature invariants \(\mathbf{R} \cdot \mathbf{R}\) and \(-\mathbf{R} \cdot \mathbf{R}\) (the Kretschmann and Chern-Pontryagin invariants, respectively), a similar classification was (somewhat naively) proposed in \[27\] (and then supported in \[1,28,33\]) for the gravitomagnetic field: the non-vanishing or vanishing of \(-\mathbf{R} \cdot \mathbf{R}\) would signal the presence of intrinsic or frame-dependent gravitomagnetism, respectively. This scheme has then been used to imply a fundamental distinction between the “translational” gravitomagnetic fields mentioned above, and the gravitomagnetic field generated by Earth’s rotational motion implied in \[13,23\].

On the other hand, the scalar invariants of the Weyl tensor (which in vacuum becomes the Riemann tensor), have been studied in the context of a hitherto separate research field \[34,41\]. In particular, the invariant criteria for the classification of a vacuum Riemann tensor as purely electric/magnetic (such classification implying the existence of some observer for which its electric/magnetic part vanishes) are well established since the work by McIntosh et al \[37\] (see also \[41,42\]).

In the present contribution, we start by discussing the stations are based on Earth) directly measure the Earth’s translational gravitomagnetic field (which is zero in the Earth’s rest frame). The only GM field that is being directly tested by LLR is in fact the one generated by the relative translation of the Sun, which is essentially the one involved in the geodetic precession of the Earth-Moon system detected in \[10,12\].
rigorous mathematical implications of the scalar invariants, closing the gap between these two research fields. We show them to yield, in the electromagnetic sector, (sufficient) conditions for the existence of observers for which the electric or magnetic fields vanish; and in the gravitational sector (insufficient) conditions for the vanishing of the gravitoelectric or gravitomagnetic tidal tensors. We fully characterize such observers in the electromagnetic and in the vacuum gravitational sectors; in the presence of gravitational sources, we fully characterize the more relevant case of observers measuring a vanishing gravitomagnetic tidal tensor. We then study in detail and physically interpret the invariant structure of the astrophysical setups of interest, and, finally, dissect their actual implications on the motion of test particles, in particular the relation with the gravitomagnetic effects.

A. Notation and conventions

1. We use the convention \( G = c = 1 \), where \( G \) is the gravitational constant and \( c \) the speed of light, and metric signature \((-+++)\); \( \epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{-g}\epsilon^{\alpha\beta\gamma\delta} \) is the Levi-Civita tensor, with the orientation \([1230] = 1\) (i.e., in flat spacetime, \( \epsilon_{1230} = 1 \)); \( \epsilon_{ijk0} \) denote 4-D spacetime indices, running 0-3; Roman letters \( i,j,k,... \) denote spatial indices, running 1-3. The convention for the Riemann tensor is \( [\alpha\beta\gamma\delta] \equiv \{\alpha\beta\gamma\delta\} - \{\beta\gamma\delta\alpha\} \). The convention for the Riemann-like tensors \( R_{\mu\nu\rho\sigma} \equiv \Gamma^\alpha_{\rho\mu\sigma} - \Gamma^\alpha_{\rho\sigma\mu} \pm \ldots \).

2. Tensors and vectors. To refer to tensors (including 4-vectors) we use either a bold font symbol \( \mathbf{T} \) or abstract index notation \( T^{\alpha\beta\gamma\delta} \). \( \mathbf{S} \cdot \mathbf{T} \) stands for the full contraction \( S^{\alpha\beta\gamma} T_{\alpha\gamma} \ldots \). Round (square) brackets around indices indicate antisymmetrization. \( \star \) denotes the Hodge dual: \( \star F_{\alpha\beta} \equiv \epsilon_{\alpha\beta\mu\nu} F^{\mu\nu}/2 \) for an antisymmetric tensor \( F_{\alpha\beta} = F_{[\alpha\beta]} \); \( \star R_{\alpha\beta\gamma\delta} \equiv \epsilon_{\alpha\beta\mu\nu} R^{\mu\nu\gamma\delta}/2 \) and \( R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\mu\nu} \epsilon^{\mu\nu\gamma\delta}/2 \) are, respectively, the dual in the first and the second pair of indices for Riemann-like tensors \( R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta\gamma\delta]} = R_{\alpha\beta\gamma\delta} \). Arrow notation \( \vec{V} \) denotes the collection of space components of a vector \( V^\alpha \) in a given frame.

3. Observers and reference frames. Following [43–46], an observer (of 4-velocity \( u^\alpha \equiv u^\alpha \)), denoted by \( \mathcal{O} \) or \( \mathcal{O}(u) \), is an entity endowed with a worldline in spacetime (tangent to \( u^\alpha \)), equipped with (besides other possible measurement devices) a clock and a system of axes to perform measurements. By reference frame \( (\mathcal{S}) \), over an extended spacetime region, we understand a 4-D basis (which could be orthonormal, or any coordinate basis) composed of a time-like plus 3 space-like vectors, continuously defined therein; it embodies a congruence of observers (whose worldlines are the integral lines of the time-axis). The projector onto the instantaneous rest space of \( \mathcal{O}(u) \) is

\[
b^\alpha_\beta \equiv \delta^\alpha_\beta + u^\alpha u_\beta.
\]  

4. “Dyadic” notation. Let \( e_i \) be an orthonormal basis in the rest space of \( \mathcal{O}(u) \), \( e_i \cdot u = 0 \). Following [48,49], sometimes we shall denote the collection of space components \( A_{ij} \) of a symmetric tensor \( A \) by \( \vec{A} \); the following notation applies: \( \vec{A} \equiv \epsilon_{ijkl} A_{ij} \). The Faraday tensor and its Hodge dual decompose in terms of the Faraday tensor \( F_{\alpha\beta} \equiv F_{\alpha\beta} \), the electric and magnetic 4-vector fields as measured by an observer \( \mathcal{O}(u) \) of 4-velocity \( u^\alpha \) are given by

\[
E^\alpha \equiv F^\alpha_{\beta} u^\beta, \quad B^\alpha \equiv \star F^\alpha_{\beta} u^\beta.
\]  

Both \( E^\alpha \) and \( B^\alpha \) are spatial with respect to \( u^\alpha \) \((E^\alpha u_\alpha = B^\alpha u_\alpha = 0)\) and thus have 3 independent components each, encoding the 6 independent components of \( F_{\alpha\beta} \) and assembled into associated 3-vectors \( \vec{E} \) and \( \vec{B} \). The Faraday tensor and its Hodge dual decompose in terms of \( E^\alpha \) and \( B^\alpha \) as

\[
F^\alpha_{\beta} = 2u^{[\alpha} E_{\beta]} + \epsilon^{\alpha\beta\gamma\delta} B_{\gamma} u_\delta, \quad \star F^\alpha_{\beta} = 2u^{[\alpha} B_{\beta]} - \epsilon^{\alpha\beta\gamma\delta} E_{\gamma} u_\delta.
\]  

II. ELECTROMAGNETIC SCALAR INVARIANTS

As a preparation for the gravitational case, we start by discussing the electromagnetic invariants and their physical meaning.

In terms of the Faraday tensor \( F \equiv F_{\alpha\beta} \), the electric and magnetic 4-vector fields as measured by an observer \( \mathcal{O}(u) \) of 4-velocity \( u^\alpha \) are given by

\[
E^\alpha \equiv F^\alpha_{\beta} u^\beta, \quad B^\alpha \equiv \star F^\alpha_{\beta} u^\beta.
\]  

Both \( E^\alpha \) and \( B^\alpha \) are spatial with respect to \( u^\alpha \) \((E^\alpha u_\alpha = B^\alpha u_\alpha = 0)\) and thus have 3 independent components each, encoding the 6 independent components of \( F_{\alpha\beta} \) and assembled into associated 3-vectors \( \vec{E} \) and \( \vec{B} \). The Faraday tensor and its Hodge dual decompose in terms of \( E^\alpha \) and \( B^\alpha \) as

\[
F^\alpha_{\beta} = 2u^{[\alpha} E_{\beta]} + \epsilon^{\alpha\beta\gamma\delta} B_{\gamma} u_\delta, \quad \star F^\alpha_{\beta} = 2u^{[\alpha} B_{\beta]} - \epsilon^{\alpha\beta\gamma\delta} E_{\gamma} u_\delta.
\]  

The electromagnetic scalar invariants are the two real, independent relativistic Lorentz scalars that can be constructed from the Faraday tensor:

\[
-\frac{1}{2} \vec{E} \cdot \vec{F} \equiv -\frac{1}{2} F^\alpha_{\beta} F_{\alpha\beta} = E^\alpha E_\alpha - B^\alpha B_\alpha, \quad -\frac{1}{4} \star \vec{E} \cdot \vec{F} \equiv -\frac{1}{4} \star F^\alpha_{\beta} F_{\alpha\beta} = E^\alpha B_\alpha.
\]  

The final expressions in (5), which read \( \vec{E}^2 - \vec{B}^2 \) and \( \vec{E} \cdot \vec{B} \) in 3-vector notation, are thus independent of the observer \( \mathcal{O}(u) \). In particular, if \( \vec{E} \cdot \vec{B} = 0 \) or \( \vec{E}^2 \) is larger, smaller, or equal to \( \vec{B}^2 \) for some observer, then this is true for every observer.

At each point the Faraday tensor can be completely classified in terms of its invariants, and one distinguishes the following cases [50,51]:

(i) \( \vec{E} \cdot \vec{B} \neq 0 \) \( \Leftrightarrow \star \vec{E} \cdot \vec{F} \neq 0 \) \( \Rightarrow \vec{E} \) and \( \vec{B} \) are both non-vanishing for all observers.
(ii) $\vec{E} \cdot \vec{B} = 0$ and $\vec{E}^2 - \vec{B}^2 > 0 (< 0) [\iff \vec{E} \cdot \vec{F} = 0, -\vec{F} \cdot \vec{F} > 0 (< 0)] \Rightarrow$ one can always find observers for which the magnetic field $\vec{B}$ (electric field $\vec{E}$) vanishes. The electromagnetic field is thus classified as purely electric (purely magnetic).

(iii) null case: $\vec{E} \cdot \vec{B} = \vec{E}^2 - \vec{B}^2 = 0 [\iff \vec{E} \cdot \vec{F} = \vec{F} \cdot \vec{F} = 0] \Rightarrow$ either $\vec{E} = \vec{B} = 0$, or $\vec{E}$ and $\vec{B}$ are both non-vanishing for all observers.

The implications in (i) and (iii) are obvious. The proof of statement (ii), as well as the explicit construction of the observers measuring no magnetic or electric field is given in the next subsection; one conclusion is however immediate: the condition $\vec{E}^2 - \vec{B}^2 > 0 (< 0)$ implies the electric (magnetic) field to be non-zero for all observers, such that for a non-zero Faraday tensor there cannot, simultaneously, exist observers for which $\vec{B} = 0$ and observers for which $\vec{E} = 0$.

A. Observers measuring no magnetic/electric fields

Consider two observers $\mathcal{O}(\mathbf{u})$ and $\mathcal{O}'(\mathbf{u}')$; their 4-velocities are related by

$$u^\alpha = \gamma (v^\alpha + u^\alpha); \quad \gamma \equiv -u^\alpha u^\alpha = \frac{1}{\sqrt{1 - v^\alpha v^\alpha}} \tag{7}$$

where $v^\alpha$ is a vector orthogonal to $u^\alpha$, $u^\alpha v^\alpha = 0$, interpreted as the spatial velocity of $\mathcal{O}'(\mathbf{u}')$ relative to $\mathcal{O}(\mathbf{u})$. In a locally inertial frame momentarily comoving with $\mathcal{O}(\mathbf{u})$ (where $u^0 = 0$), $v^i = dx^i/dt$, yielding the ordinary 3-velocity of $\mathcal{O}'(\mathbf{u}')$.

By $\mathbf{[234]}$ the electric and magnetic fields measured by $\mathcal{O}'(\mathbf{u}')$ are related to the ones measured by $\mathcal{O}(\mathbf{u})$ according to

$$E'^\alpha = \left[ 2E^\beta u^\alpha + \epsilon^{\alpha\beta\gamma\delta} B_\gamma u_\delta \right] u'_\beta \tag{8}$$

$$B'^\alpha = \left[ 2B^\beta u^\alpha - \epsilon^{\alpha\beta\gamma\delta} E_\gamma u_\delta \right] u'_\beta \tag{9}$$

To make contact with the textbooks on classical electromagnetism, consider, in flat spacetime, the inertial frames $S$ and $S'$ momentarily comoving with the observers $\mathcal{O}(\mathbf{u})$ and $\mathcal{O}'(\mathbf{u}')$, respectively; in this special case one obtains the well known non-covariant expressions (e.g. Eqs. (11.149) of $\mathbf{[22]}$)

$$\vec{E}' = \gamma \left( \vec{E} + \vec{v} \times \vec{B} \right) - \frac{\gamma^2}{\gamma + 1} \vec{v} \left( \vec{v} \cdot \vec{E} \right), \tag{10}$$

$$\vec{B}' = \gamma \left( \vec{B} - \vec{v} \times \vec{E} \right) - \frac{\gamma^2}{\gamma + 1} \vec{v} \left( \vec{v} \cdot \vec{B} \right), \tag{11}$$

where $\vec{E}'$ and $\vec{B}'$ are space components of the electric and magnetic fields measured by $\mathcal{O}'(\mathbf{u}')$ and expressed in the coordinate system $S'$ (the time components $E'^0$ and $B'^0$ are zero in $S'$).

Let us now prove that observers exist for which $B'^0 = 0$ if, and only if, $\vec{E} \cdot \vec{B} = 0$ and $\vec{E}^2 > B^2$ (i.e., $-\vec{F} \cdot \vec{F} > 0$ and $\vec{F} \cdot \vec{F} = 0$). From (9), the equation $B'^0 = 0$ splits into two components, one parallel to $u^\alpha$: $u^\alpha B'^\alpha u_\beta = 0$, i.e.,

$$B^\beta u'_\beta = B'^0 = 0, \tag{12}$$

plus one orthogonal to $u^\alpha$,

$$B^\alpha = \frac{\epsilon^{\alpha\beta\gamma\delta} u_3 E_\gamma u_\delta}{u^\mu u'_\mu} = \epsilon^{\alpha\beta\gamma\delta} u_3 E_\gamma u_\delta, \tag{13}$$

which implies (12) on its turn. In the rest frame of $\mathcal{O}(\mathbf{u})$, and in 3-vector form, Eq. (13) reads

$$\vec{B} = \vec{v} \times \vec{E}. \tag{14}$$

Hence, it is possible to find an observer for which $\vec{B}'$ vanishes if and only if (14) admits a solution $\vec{v}$. Since $|\vec{v}| < c = 1$, this is the case if and only if $\vec{B}$ lies in the plane orthogonal to $\vec{E}$ and is contained within a circle of radius $|\vec{E}|$, which precisely means $\vec{E} \cdot \vec{B} = 0$ and $\vec{E}^2 > \vec{B}^2$. This concludes the proof.

To obtain the velocities of the observers for which $B'^0 = 0$ (i.e., $\vec{B}' = 0$), it is useful to decompose $v^\alpha$ into its projections parallel and orthogonal to $E'^0$,

$$v'^\alpha_E = \frac{E'_\beta v'^\beta}{E'_\mu E'_\gamma E'^\alpha}; \quad v'^\alpha_{\parallel E} = v'^\alpha - v'^\alpha_E, \tag{16}$$

and to recall the definition of the Poynting vector measured by $\mathcal{O}(\mathbf{u})$,

$$p^\alpha = \frac{1}{4\pi} E_\sigma E^\sigma B'^\alpha, \quad \vec{p} = \frac{1}{4\pi} \vec{E} \times \vec{B}. \tag{15}$$

Since $v^\alpha B^\alpha = 0$, cf. Eq. (12), $v'^\alpha_E$ is also the component of $v'^\alpha$ parallel to $p^\alpha$: $v'^\alpha_E = v'^\alpha_{\parallel E}$. As is clear from (14), $v'^\alpha_{\parallel E}$ is arbitrary. Contracting (13) with $\epsilon^{\alpha\beta\gamma\delta} u^\gamma E^\sigma$ (or, equivalently, taking the cross product of (14) with $\vec{E}$) we obtain

$$v'^\alpha_{\parallel p} = \frac{\epsilon^{\alpha\beta\gamma\delta} E^\beta B'^\delta u^\gamma}{E'_\mu E'_\nu E'^\alpha}, \quad \vec{v}'_{\parallel p} = \frac{\vec{E} \times \vec{B}}{E'^2}, \tag{16}$$

This agrees with results known (in different contexts) in the literature: in Problem §25 of $\mathbf{[30]}$ and Exercise 20.6 of $\mathbf{[52]}$, implicit expressions are obtained for the velocity of the observers measuring aligned fields $\vec{E}'$ and $\vec{B}'$, or, equivalently, for which the Poynting vector vanishes. For a purely electric field, one can transform them into explicit expressions which match (16).
Therefore, the observers \( O' (u') \) for which \( \vec{B}' = 0 \) must move with a velocity that is orthogonal to the magnetic field \( \vec{B} \) as measured by \( O (u) \), cf. Eq. (16), and must have a component along the Poynting vector given by (16), and may have an arbitrary component \( \vec{v}_{\parallel E} \) parallel to \( \vec{E} \), see Fig. 1. In other words, the 4-velocities of these observers are those contained in the timelike plane spanned by \( E^\alpha \) and the timelike vector

\[
T^\alpha = u^\alpha + \frac{\epsilon_{\sigma \tau \beta} E^\beta B^\tau}{E_\nu E^\nu} \tag{17}
\]

(with \( T^\alpha T_\alpha = \vec{B}'^2 / \vec{E}'^2 - 1 < 0 \)), i.e., those of the form

\[
u^\alpha = C \frac{T^\alpha}{\sqrt{-T^\alpha T_\alpha}} + D \frac{E^\alpha}{\sqrt{E^\nu E_\nu}} \quad D^2 - C^2 = -1 . \tag{18}
\]

Replacing \{\( E^\alpha, B^\alpha \)\} \( \rightarrow \) \{\( B^\alpha, -E^\alpha \)\} in the above [compare \( 8 \) to \( 9 \)], one shows that the purely magnetic case (\( \vec{E} \cdot \vec{F} = 0, -\vec{F} \cdot \vec{F} < 0 \)) yields a class of observers measuring no electric field with 4-velocities given by analogues of (17), (18).

To end this section we still mention some additional properties to draw parallels and differences with the gravitational case below. If the Faraday tensor is non-null (\( [\vec{E} \cdot \vec{F}, \vec{F} \cdot \vec{F}] \neq (0, 0) \)) it has exactly two principal null directions (PNDs), spanned by null vectors \( k^\alpha \) that satisfy

\[
k^{[\alpha} F^{\beta]} = 0.
\]

The PNDs generate the timelike principal plane. Observers \( O (u) \) with 4-velocity \( u^\alpha \) lying in this plane are precisely those measuring a vanishing Poynting vector \( p^\alpha \) (see e.g. [10] [20]). For a purely electric (purely magnetic) Faraday tensor one has \( E^\alpha B_\alpha = 0 \), and by (15) the vanishing of \( p^\alpha \) implies the vanishing of \( B^\alpha (E^\alpha) \); hence the observers measuring no magnetic (electric) field are those and only those whose 4-velocity lies in the timelike principal plane. This is illustrated in Fig. 2.

### III. GRAVITATIONAL SCALAR INVARIANTS

Analogously to electromagnetism, the scalar invariants of the curvature tensor are related with the existence of observers for which its electric or magnetic parts vanish; but by contrast with electromagnetism, the invariants do not always yield sufficient conditions for that. We shall first focus on the vacuum case; relevant comments on the non-vacuum case are given at the end.

#### A. Vacuum Riemann tensor

In the vacuum case, characterized by a vanishing Ricci tensor \( (R_{\alpha \beta} \equiv R^\gamma_{\alpha \beta \gamma} = 0) \), the Riemann tensor \( R \equiv R_{\alpha \beta \gamma \delta} \) generically has 10 independent components in any frame, and exhibits the special property that the dual in the first pair of indices equals the dual in the second pair:

\[
^\ast R = R^\ast \quad (\epsilon_{\alpha \beta} \epsilon^\gamma_{\epsilon \delta} = R_{\alpha \beta \gamma \delta} \epsilon^\epsilon_{\gamma \delta}). \tag{19}
\]

Relative to an observer \( O (u) \), the gravitoelectric and gravitomagnetic tidal tensors \( 46 [47] \) (or “electric” and “magnetic” parts of the Riemann tensor, e.g. [57] [58]) are pointwise defined by

\[
E_{\alpha \beta} \equiv R_{\gamma \beta \alpha \delta} u^\gamma u^\delta, \quad H_{\alpha \beta} \equiv ^\ast R_{\alpha \beta \gamma \delta} u^\gamma u^\delta . \tag{20}
\]

\( E_{\alpha \beta} \) and \( H_{\alpha \beta} \) are symmetric, tracefree, and spatial with respect to \( u \) (\( E_{\alpha \beta} = E_{\beta \alpha}, E^\alpha = 0, E_{\alpha \beta} u^\beta = 0 \), and similarly for \( H_{\alpha \beta} \)), thus having 5 independent components.
each. In terms of $E_{\alpha\beta}$ and $H_{\alpha\beta}$ one has decompositions

\[ R_{\alpha\beta\gamma\delta} = 4 \left( 2u_{[\alpha}u_{\gamma]} + \delta^{[\gamma}_{[\alpha}E_{\beta]\delta] \right) + 2\epsilon_{\alpha\beta\gamma\delta}H_{[\epsilon\beta]}u_{\gamma]} , \]

and Chern-Pontryagin scalar properties are independent of the observer, i.e., they are spatial tensors, one real and one purely imaginary, 

\[ \phi_{\alpha\beta} = 4 \left( 2u_{[\alpha}u_{\gamma]} + \delta^{[\gamma}_{[\alpha}H_{\beta]\delta] \right) - 2\epsilon_{\alpha\beta\gamma\delta}E_{[\epsilon\beta]}u_{\gamma]} , \]

which exhibit a certain analogy with (3)-(4).

In contrast to the Faraday tensor, a vacuum Riemann tensor has four real, independent invariants [59]: two quadratic invariants, namely the Kretschmann scalar $R^*R^*$ and Chern-Pontryagin scalar $\Phi = \Phi^* + \Phi^*$, which in terms of the tidal tensors measured by an observer read

\[ \frac{1}{8}R^*R = \frac{1}{8}R_{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \mathbb{E}^\alpha_\beta \mathbb{E}^\beta_\alpha - \mathbb{H}^\alpha_\beta \mathbb{H}^\beta_\alpha ; \]

\[ \frac{1}{16}R^*\Phi = \frac{1}{16}R_{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \mathbb{E}^\alpha_\beta \mathbb{E}^\beta_\alpha - \mathbb{H}^\alpha_\beta \mathbb{H}^\beta_\alpha \]

and are formally analogous to the electromagnetic invariants (3)-(4) but also two cubic invariants (e.g. [55])

\[ A = -\frac{1}{16}R_{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \mathbb{E}^\alpha_\beta \mathbb{E}^\beta_\alpha \mathbb{E}^\gamma_\alpha \mathbb{E}^\delta_\alpha \mathbb{E}^\delta_\alpha = \mathbb{E}^\alpha_\beta \mathbb{E}^\beta_\alpha \mathbb{E}^\gamma_\alpha \mathbb{E}^\delta_\alpha ; \]

\[ B = \frac{1}{16}R_{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \mathbb{H}^\alpha_\beta \mathbb{H}^\beta_\alpha \mathbb{H}^\gamma_\alpha \mathbb{H}^\delta_\alpha \]

which have no electromagnetic counterpart. At any point these four invariants may in principle take any value, independently of each other, and are all needed to determine whether $R_{\alpha\beta\gamma\delta}$ has a purely electric/magnetic character. Analogously to a Faraday tensor, a non-zero vacuum Riemann tensor (or the spacetime) is called purely magnetic (purely magnetic) at a point if there exists an observer $O(u)$ measuring a vanishing gravitomagnetic (gravitoelectric) tidal tensor: $\mathbb{H}^\alpha_\beta = 0$ ($\mathbb{E}^\alpha_\beta = 0$). By (23)-(26) the existence of such an observer clearly requires $\mathbb{E}^\alpha_\beta = 0$ ($\mathbb{H}^\alpha_\beta = 0$), besides $\Phi = 0$ and $\Phi > 0$ ($\Phi < 0$), this is not sufficient.

To explain why, it is useful to define the complex tensor

\[ \mathcal{Q}^\alpha_\beta \equiv \mathbb{E}^\alpha_\beta - i\mathbb{H}^\alpha_\beta . \]

This is a tensor which is spatial relative to the observer, $u_\alpha \mathcal{Q}^\alpha_\beta = 0$, and consists of the sum of two symmetric spatial tensors, one real and one purely imaginary, each of them diagonalizable. Therefore, the existence of observers $O'(u')$ measuring $\mathbb{H}^\alpha_\beta = 0$ ($\mathbb{E}^\alpha_\beta = 0$) implies that the operator $\mathcal{Q}^\alpha_\beta$ has two properties: it is diagonalizable and has real (purely imaginary) eigenvalues. Now, both properties are independent of the observer, i.e., they are shared by the respective tensors $Q^\alpha_\beta$ measured by arbitrary observers $O(u)$. To indicate the origin of this fact, we note that $Q^\alpha_\beta$ can be viewed as a linear operator in (i.e., an endomorphism of) the complexified rest space of $O(u)$. This is a 3-D complex vector space isomorphic to the space of those complex anti-symmetric tensors $X_{\alpha\beta} = X_{[\alpha\beta]}$ satisfying $X_{\alpha\beta} = iX_{\alpha\beta}$ (so-called self-dual bivectors, see e.g. [63]); by virtue of (19) and $R_{\alpha\beta} = 0$ the tensor $-\frac{1}{2}R_{\alpha\beta}$ acts as a trace-free linear operator on this space; moreover, all operators $Q^\alpha_\beta$ are equivalent to this observer-independent operator (see (4.1)-(4.4) of [63], or [55]), meaning that not only they have the same eigenvalues $\lambda_k$ ($k = 1, 2, 3$), i.e., the same characteristic polynomial

\[ c(x) = x^3 - \frac{I}{2}x - \frac{J}{3} = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3), \]

but also the same minimal polynomial $m(x)$, which in 3-D fully determines the algebraic properties of operators.\(^5\) Here $I$ and $J$ are the complex invariants (e.g. [37]-[41])

\[ I = \frac{1}{8}R^*R - \frac{1}{8}R^*\Phi = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \]

\[ = Q^\alpha_\beta Q^\beta_\alpha = \mathbb{E}^\alpha_\beta \mathbb{E}^\beta_\alpha - \mathbb{H}^\alpha_\beta \mathbb{H}^\beta_\alpha - 2\mathbb{E}^\alpha_\beta \mathbb{H}^\beta_\alpha ; \]

\[ J = I + B = -\frac{1}{16}(R^{\alpha\beta\lambda\gamma} - i\mathcal{Q}^{\alpha\beta\lambda\gamma}) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 3\lambda_1\lambda_2\lambda_3 = \mathbb{Q}^\alpha_\beta \mathbb{Q}^\beta_\alpha \]

\[ = \mathbb{E}^\alpha_\beta \mathbb{E}^\beta_\gamma + i\mathbb{H}^\alpha_\beta \mathbb{H}^\beta_\gamma - 3i\mathbb{E}^\alpha_\beta \mathbb{E}^\gamma_\delta - i\mathbb{H}^\alpha_\beta \mathbb{H}^\gamma_\delta \]

One has $\lambda_1 + \lambda_2 + \lambda_3 = Q^\alpha_\alpha = 0$, and the discriminant of the cubic polynomial $c(x)$ equals (up to a factor 2)

\[ \Delta = \lambda_1 - \lambda_2 - \lambda_3 )^2(\lambda_2 - \lambda_3 )^2(\lambda_3 - \lambda_1 )^2 = I^3 - 6J^2 . \]

The eigenvalue problem for $Q^\alpha_\beta$ leads to the Petrov classification of the vacuum Riemann tensor [60]-[63], which can be formulated as follows:

(a) Petrov type I: this is the generic case where all eigenvalues differ ($\Delta \neq 0$), and $m(x) = c(x)$.

(b) Petrov types D and II: both have $\Delta = 0 \neq I$, a double eigenvalue $\lambda = -\frac{J}{I}$ and a single eigenvalue $-2\lambda$, but $m(x) = (x + 2\lambda)(x - \lambda)^2 = c(x)$ for type II while $m(x) = (x + 2\lambda)(x - \lambda)$ for type D; in terms of $Q^\alpha_\beta$ and $h^\alpha_\beta$ defined in (1) this means that

\[ F(Q)^\alpha_\beta = Q^\gamma_\beta Q^\alpha_\gamma + Q^\alpha_\beta Q^\gamma_\beta - 2\lambda^2 h^\alpha_\beta = 0, \]

\[ \lambda = -\frac{J}{I} \]

holds for type D (then implying $m(x) = F(x)$), while $F(Q)^\alpha_\beta \neq 0$ for type II.

\[ ^5 \text{Recall that if a polynomial } p(x) \text{ annihilates an endomorphism } L \text{ of a vector space (i.e., } p(L) = 0 \text{) it has all eigenvalues of } L \text{ (i.e., roots of } c(x) \text{) as roots. The minimal polynomial } m(x) \text{ is the (unique) annihilating polynomial of least degree and leading coefficient } 1, \text{ and only has the eigenvalues as roots, possibly occurring with lower multiplicities than in } c(x); \text{ } L \text{ is diagonalizable precisely if the multiplicities in } m(x) \text{ are 1 for all eigenvalues.} \]

The formal analogy is up to a factor 8 and a minus sign. The sign difference is due to the contraction of one pair of antisymmetric indices within $FF$ and $4FF$ whereas such two pairs are contracted within $R^*R$ and $\Phi^*$. 

\[ ^4 \text{The formal analogy is up to a factor 8 and a minus sign. The sign difference is due to the contraction of one pair of antisymmetric indices within } FF \text{ and } 4FF \text{ whereas such two pairs are contracted within } R^*R \text{ and } \Phi^*. \]
c) Petrov types N, III and O: all have \( I = J = \Delta = 0 \) and a triple eigenvalue 0, but the respective minimal polynomials are \( m(x) = x^3 = c(x), x^2 \) and \( x \) (such that type O corresponds to \( R = 0 \)).

Referring to footnote 5, diagonalizability of the operators \( Q_{\alpha \beta}^\gamma \) precisely means that the Petrov type is I, D or O (where in the last case \( R_{\alpha \beta \gamma \delta} = 0 \Rightarrow Q_{\alpha \beta}^\gamma = 0 \) for all observers, trivially). In particular, the characteristic property \( \{32\} \) for type D ensures diagonalizability of \( Q_{\alpha \beta}^\gamma \) and distinguishes this type from the non-diagonal Petrov type II, a distinction that cannot be made in terms of invariants (which may be equal for both types). From the discussion above it follows that a non-zero vacuum Riemann tensor that is purely electric (purely magnetic) is of Petrov type I or D and the operators \( Q_{\alpha \beta}^\gamma \) have real (purely imaginary) eigenvalues. Now, it turns out that Petrov type is I or D and the operators allows for a basis of orbitors such an observer case that the eigenvalues allows for a basis of orbitors of Petrov types I and D (see e.g. [55, 60, 62, 63] that magnetic (electric) tidal tensor: it is a well known property of Petrov type I or D and the operators \( Q_{\alpha \beta}^\gamma \) are all real (purely imaginary). In the view of (29)-(30) it follows that a non-zero vacuum Riemann tensor that is purely electric (purely magnetic) and of Petrov type D if and only if \( R \neq 0 \) and \( \Delta = 0 \) or \( \Delta > 0 \).

Contrary to the Petrov type I case the invariants are now insufficient to formulate the purely electric or magnetic conditions: we also need condition \( \{32\} \) to discriminate (allowed) Petrov type D from (forbidden) Petrov type II. Note that the first part of \( \{32\} \) on itself implies \( I = 6\lambda^2, J = -\lambda I \) by \( F(Q)_{\alpha \beta}^\gamma = Q_{\alpha \beta}^\gamma \cdot (Q)_{\gamma \beta}^\alpha = 0 \), and thus the last part of \( \{32\} \). It follows that a non-zero vacuum Riemann tensor is purely electric (purely magnetic) and of Petrov type D if and only if \( \lambda = -8A/R \cdot R \) (\( \lambda = -8iB/R \cdot R \)),

\[
Q_{\alpha \beta}^\gamma + \lambda Q_{\alpha \beta}^\gamma - 2\lambda^2 h_{\beta}^\alpha = 0
\]

with \( \lambda = -8A/R \cdot R \) (\( \lambda = -8iB/R \cdot R \)).

Hence we find back the result \[37, 42\] that a non-zero vacuum Riemann tensor is purely electric (purely magnetic), i.e., an observer \( O'u' \) exists for which \( \mathbb{h}_{\alpha \beta}^\prime (E'_{\alpha \beta}) \) vanishes, if and only if the following conditions hold:

\[
\epsilon \cdot R \cdot R = 0 \quad \text{and} \quad R \cdot R > 0 (< 0),
\]

\( J = 0 \) or \( \{J \neq 0 \text{ and } M > 0 \text{ real}\} \) or \( \{32\} \),

where the first two cases of \( \{38\} \) automatically give Petrov type I and \( \{32\} \) fully characterizes Petrov type D. Using (29)-(32) the conditions \( \{34\}-\{38\} \) can be easily tested by calculating \( \Phi\beta\) relative to any observer \( O(u) \).

It follows that one can make formally similar statements to (i)-(iii) of the electromagnetic case in Sec. II replacing \( F \) by \( R \) and adding the condition \( \{38\} \) to (ii):

(i) \( \epsilon \cdot R \cdot R = 0 \) \( \Leftrightarrow \mathbb{E}_{\alpha \beta}^\gamma (\mathbb{H}_{\alpha \beta}^\gamma) \neq 0 \Rightarrow \mathbb{E}_{\alpha \gamma}^\gamma \) and \( \mathbb{H}_{\alpha \gamma}^\gamma \) are both non-vanishing for all observers.

(ii) \( \epsilon \cdot R \cdot R = 0, \mathbb{R} \cdot R > 0 (< 0) \) and \( \{38\} \Rightarrow \) one can always find an observer for which \( \mathbb{H}_{\alpha \gamma}^\gamma (\mathbb{E}_{\alpha \gamma}^\gamma) \) vanishes. At the considered point the vacuum spacetime is classified as purely electric (magnetic).

(iii) All invariants vanish: \( \epsilon \cdot R \cdot R = \mathbb{R} \cdot R = J = 0 \Rightarrow \) either \( R = 0 \) (Petrov type O) or the Petrov type is III or N and \( \mathbb{E}_{\alpha \gamma}^\gamma, \mathbb{H}_{\alpha \gamma}^\gamma \neq 0 \) for all observers.  

\[6\] By \[27\] this means that \( E_{\alpha \beta}^\gamma \) and \( H_{\alpha \beta}^\gamma \) measured by these observers are simultaneously diagonalizable, i.e., they commute (since they are symmetric), which is equivalent to saying that the "super-Poynting vector" \( P^\alpha \), see [18], vanishes for them.

\[7\] That real (purely imaginary) eigenvalues imply \( \{33\} \) follows immediately from \( \{29\}, \{30\} \) and the first expression for \( \Delta \) in \( \{31\} \). Conversely, a cubic polynomial \( p(x) \) with real coefficients and vanishing quadratic term has roots \( -a \) and \( a \pm b \), with \( b \neq 0 \), \( a \neq 0 \) and \( b \neq 0 \), \( a \neq 0 \) and \( \Delta = \pm \sqrt{a^2 - b^2} \) (cf. [37]); hence \( D > 0 \) implies \( b \) real and \( a \) real; all roots are real; given \( \{33\} \) one applies this to \( p(x) = c(x) \), with roots \( \lambda \) and \( D = \Delta (p(x) = ic(x)) \), with roots \( -i\lambda \) and \( D = -\Delta \) to see that the \( \lambda \)'s are all real (purely imaginary). Compare to the more intricate reasoning in [37, 64].

\[8\] In direct analogy with electromagnetism (see footnote 2), the vanishing of all invariants has been proposed as a criterion (Bel's second criterion) for "pure" gravitational radiation, see [65] and also [66, p. 53]. Such criterion is based on "super-energy" [35, 46, 54, 66-68].
The implications in (i) and (iii) are obvious. A vacuum Riemann tensor obeying (i) is either of Petrov type I, II or D, while (ii) implies Petrov type I or D as explained above. Notice that (ii) implies as well that $E_{\alpha\beta}$ ($H_{\alpha\beta}$) is non-zero for all observers; in particular, for a non-zero vacuum Riemann tensor at a point, there cannot, simultaneously, exist observers for which $H_{\alpha\beta} = 0$ and observers for which $E_{\alpha\beta} = 0$.

The possibilities lying outside criteria (i)-(iii) have no counterpart in the formal analogy with electromagnetism, and they all preclude the existence of observers measuring a vanishing $E_{\alpha\beta}$ or $H_{\alpha\beta}$. These are:

(iv) $\mathbf{R} \cdot \mathbf{R} = 0$, $\mathbf{R} \cdot \mathbf{R} \neq 0$, $\mathbf{M} \neq 0$ and either $\mathbb{M} < 0$ real or $\mathbb{M}$ non-real, which is a Petrov type I subcase. Examples of such vacuum solutions are the Lewis metrics for the Lewis class [69, 70] (or, equivalently, the van Stockum exterior solution [62]), describing a special class of the exterior metrics produced by infinite rotating cylinders.

(v) $\mathbf{R} \cdot \mathbf{R} = 0$, $\mathbf{R} \cdot \mathbf{R} \neq 0$, $\mathbb{J} \neq 0$ and $\mathbb{M} = 0$ but without [72] holding, corresponding to Petrov type II [see point (b) above]: an example is the limiting case $aR = 1/2$ of the van Stockum exterior solution [38].

(vi) $\mathbf{R} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{R} = 0$ and $\mathbb{J} \neq 0$ (corresponding to $I = 0 \leftrightarrow \mathbb{M} = -6$), which is a Petrov type I subcase whose only known non-flat vacuum solution is Petrov’s homogeneous metric [60] [71], having a constant $\mathbb{J}$ and a simply-transitive maximal group $G_4$ of motions.

Finally, it is worth mentioning that no vacuum spacetimes are known for which the Riemann tensor is purely magnetic in an open 4-D region (i.e., where $E_{\alpha\beta} = 0$ with respect to some observer congruence). It has therefore been conjectured that no such spacetimes exist (see e.g., [72])9. However, a vacuum Riemann tensor can be purely magnetic in 3-D hypersurfaces (exemplified in Fig. 7) or lower-dimensional sets.

B. Observers measuring no gravitomagnetic/gravitoelectric tidal tensor in vacuum

As explained above, the existence of observers for which the gravitomagnetic (gravitoelectric) tidal tensor vanishes at a point of a non-flat vacuum spacetime requires the Petrov type to be either I or D. Their appearances for both Petrov types are well-known [35, 42, 55, 62]. In the Petrov type I case there is a unique such observer. A vacuum Riemann tensor of type D, on the other hand, shows strong analogy with a Faraday tensor: it has exactly two principal null directions (PNDs) which are spanned by those null vectors $k^\alpha$ that satisfy

$$k^\alpha R_{\gamma \delta \kappa \lambda} k^\gamma k^\delta = 0;$$

these null vectors generate the timelike principal plane, and analogously to the situation in electromagnetism, the observers measuring no $H_{\alpha\beta}$ ($E_{\alpha\beta}$) are precisely those with a 4-velocity in this plane, see Fig. 2.

As proved in general in the companion paper [55], the following algorithm, specified to the purely electric (purely magnetic) case, gives the 4-velocity of the observers $\mathcal{O}'(u')$ for which $H_{\alpha\beta} = 0$ ($E_{\alpha\beta} = 0$) in terms of the tidal tensors measured by an arbitrary observer $\mathcal{O}(u)$:

1. For each non-degenerate eigenvalue $\lambda_k$ of $\mathcal{Q}_{\alpha\beta}^\gamma$ (take $k = 1$ for type D, while $k = 1, 2, 3$ for type I) construct a vector $w^\alpha_k$ in the corresponding eigenspace as follows, where $y^\alpha$ is any spatial vector in the rest space of $\mathcal{O}(u)$ such that $w^\alpha_k \neq 0$.

   - type D: $w^\alpha_k \equiv \mathcal{Q}_{\alpha\beta}^\gamma y^\beta - \lambda_k y^\alpha$, $\lambda = -8\mathbb{J}/(\mathbf{R} \cdot \mathbf{R})$;
   - type I: $w^\alpha_k \equiv (\mathcal{Q}_{\alpha\beta}^\gamma \mathcal{Q}_{\beta\delta}^\gamma + \lambda_k \mathcal{Q}_{\alpha\beta}^\gamma) y^\beta + (\lambda_k^2 - 1/\mathcal{R}^2) \mathbf{R} \cdot \mathbf{R} y^\alpha$, $\lambda_k = \alpha \sqrt{\mathbf{R} \cdot \mathbf{R} / 12} \cos \left(\arccos \Theta_{\alpha\beta} - 2\pi / 3\right)$, $k = 1, 2, 3$, $\Theta_{\alpha\beta} \equiv \text{sgn}(\mathbb{M} / \sqrt{1 + \mathbb{M} / 6})$, $\alpha = \begin{cases} 1, & H_{\alpha\beta} = 0, \\ i, & E_{\alpha\beta} = 0. \end{cases}$

2. The complex vectors $w^\alpha_k$ are orthogonal to $u^\alpha$, non-null, and mutually orthogonal in the type I case; normalize them to unit vectors $w^\alpha_k / \sqrt{w^\alpha_k (w^\alpha_k)} \equiv c^\alpha_k - i b^\alpha_k$, such that $c^\alpha_k (b^\beta)_{\alpha} + b^\alpha_k (c^\gamma)_{\alpha} = 0$ and $c^\alpha_k (c^\beta)_{\alpha} - b^\alpha_k (b^\beta)_{\alpha} = 0$.

3. In the type D case, the observers measuring no $H_{\alpha\beta}$ ($E_{\alpha\beta}$) are those with 4-velocity of the form

$$u^\alpha = C \frac{t^\alpha}{\sqrt{-t^\alpha t_\alpha}} + D \frac{e^\alpha}{\sqrt{e^\alpha e_\alpha}}, \quad D^2 - C^2 = -1,$$

$$t^\alpha \equiv u^\alpha + c^\sigma \gamma \beta c^\beta u^\beta \frac{e_\sigma e^\alpha}{e^\gamma e^\beta},$$

$$e^\alpha \equiv e^\alpha_1 = \Re \left(\frac{w^\alpha_1}{\sqrt{w^\alpha_1 (w^\alpha_1)_{\alpha}}}\right).$$

In the type I case, the unique observer $\mathcal{O}'(u')$ has 4-velocity $u^\alpha = t_1^\alpha / \sqrt{-t_1^\alpha t_1^\alpha}$, where

$$t_1^\alpha = \left(\sum_{k=1}^3 c^\beta_k (c^\gamma_k)_{\beta} - 1\right) u^\alpha + \eta^\alpha \beta \gamma \delta u^\beta \sum_{k=1}^3 c^\delta_k b^\gamma_k.\quad (45)$$

---

9 The conjecture has been proven for Petrov type D [73] and for type I with $\mathbb{J} = 0$ [74]; it has further been shown that, in any vacuum spacetime, $E_{\alpha\beta} \neq 0$ with respect to any observer congruence that is either shear-free [75], non-rotating [76] or geodesic [77]; see [41] for a complete survey.
In the Petrov type D case, the expressions above can also be used to compute the PNDs, generated by the null vectors

\[
k_\pm = \frac{t^\alpha}{\sqrt{-2f_{\alpha}t_{\alpha}}} \pm \frac{c^\alpha}{\sqrt{2c^\alpha_\gamma c_{\gamma}}} .
\]

Notice the formal analogy between \([42, 43]\) and the electromagnetic expressions \([17, 18]\) which yield the 4-velocities of the observers for which the magnetic field vanishes in the purely electric case. An alternative expression for \(t^\alpha\) is \([55]\)

\[
t^\alpha = u^\alpha + \frac{\epsilon^{\alpha\beta\gamma\delta}E_{\beta\mu}H_{\gamma\mu}u_\delta}{3\zeta_5(\xi_5 + 1)} ;
\]

\[
\zeta_5 = \sqrt{\frac{2E_{\alpha\beta}E_{\alpha\beta} + H_{\alpha\beta}H_{\alpha\beta}}{3[E_{\alpha\beta}E_{\alpha\beta} - H_{\alpha\beta}H_{\alpha\beta}]} + \frac{1}{3}} ;
\]

\[
\xi = \frac{1}{4}E_{\alpha\beta}E_{\alpha\beta} - H_{\alpha\beta}H_{\alpha\beta} = \frac{|R \cdot R|}{32} ,
\]

where we recognize the “super-Poynting” vector (see e.g. \([35, 46, 54, 55, 66, 68]\), and compare to \([15]\))

\[
\mathcal{P}^\alpha = \frac{1}{2}e^{\alpha\beta\gamma\delta}E_{\beta\mu}H_{\gamma\mu}u_\delta , \text{ or } \vec{P} = \frac{1}{2}\vec{E} \times \vec{H} ,
\]

the second expression holding in dyadic notation and in the rest frame of \(O(u)\). From Eqs. \((7), (42), \text{ and } 47\) (noticing that \(\gamma = C/\sqrt{-f_{\alpha}}t_{\alpha}\)) it follows that the observers \(O'(u')\) for which \(H_{\alpha\beta}' = 0\) (\(E_{\alpha\beta}' = 0\)) must move, relative to \(O(u)\), with a velocity \(v^\alpha = v^\alpha_P + v^\alpha_P\) that has a component \(v^\alpha_P\) parallel to \(\mathcal{P}^\alpha\) given by

\[
v^\alpha_P = \frac{\epsilon^{\alpha\beta\gamma\delta}E_{\beta\mu}H_{\gamma\mu}u_\delta}{3\zeta_5(\xi_5 + 1)} , \text{ i.e. } \vec{v}_P = \frac{\vec{E} \times \vec{H}}{3\zeta_5(\xi_5 + 1)} ,
\]

and an arbitrary component \(v^\alpha_r\) parallel to \(e^\alpha\), see Fig. 3. Notice the similarity with the electromagnetic counterparts in Eqs. \((16)-\(17\)\) and Fig. 1. These however hold only in the purely electric case (for the observers measuring \(B^\alpha = 0\)); in order to obtain the velocities of the observers measuring \(E^\alpha = 0\) (in purely magnetic fields), one needs to replace \(E^\gamma E_\gamma\) by \(R^\gamma B_\gamma\) in the denominators, and switch \(E\) and \(B\) in Fig. 3. For electromagnetic expressions encompassing both the purely electric and magnetic cases [hence the closest analogues of Eqs. \((17)-\(49\)\)][55], see the companion paper \([55]\).

\[\text{C. General Riemann tensor}\]

In the presence of sources, the Riemann tensor does not obey \([19]\). Generically it has 20 independent components in any frame, and relative to an arbitrary observer \(O(u)\) it may be completely characterized by the three spatial tensors \([78]\)

\[
E_{\alpha\beta} \equiv R_{\alpha\beta\gamma}u^\gamma u^\nu , \quad H_{\alpha\beta} \equiv \star R_{\alpha\beta\gamma}u^\gamma u^\nu ,
\]

\[
F_{\alpha\beta} \equiv \star R \star R_{\alpha\beta\gamma}u^\gamma u^\nu ,
\]

\[
Q_{\alpha\beta} \equiv E_{\alpha\beta} - iH_{\alpha\beta} .
\]

Figure 3: Boosted observers that measure \(H_{\alpha\beta} = 0\) (\(E_{\alpha\beta} = 0\)) in Petrov type D spacetimes. Their velocities \(\vec{v}\) have a component \(\vec{v}_P\) parallel to the super-Poynting vector \(\vec{P}\) given by Eq. \((49)\), and an arbitrary component parallel to the vector \(\vec{r}\) defined in Eq. \((44)\). Notice the analogy with Fig. 1.

\[\text{The tensors } E_{\alpha\beta} \text{ and } F_{\alpha\beta} \text{ are symmetric and spatial relative to } u^\alpha , \text{ thus having 6 independent components each; } H_{\alpha\beta} \text{ is spatial and traceless, possessing 8 independent components; } F_{\alpha\beta} \text{ has no electromagnetic analogue.}\]

\[\text{From the Riemann tensor one can generically construct 14 algebraically independent scalar invariants } [36, 60] . \text{ In particular, the Kretschmann and Chern-Pontryagin scalars are given in terms of the tensors } (50) \text{ by}\]

\[R^\alpha_{\quad \gamma\delta} = 4\epsilon^{\alpha\beta\gamma\delta}u_\delta + \epsilon_{\alpha\beta}^{\gamma\delta}u_\delta u^\nu E^\nu_{\gamma\delta}u^\varphi E^\varphi_{\alpha\beta} + 2\left\{\epsilon_{\gamma\delta}^{\nu\beta}u_\nu u^\chi H^\chi_{\mu\beta}u^\rho H^\rho_{\mu\gamma}\right\} .\]

\[\text{The Riemann tensor may be decomposed as follows:}\]

\[R^\alpha_{\quad \gamma\delta} = C^\alpha_{\quad \beta\gamma\delta} + 2\delta^\alpha_{\gamma}R_{\beta\gamma\delta} - \frac{1}{3}R \epsilon_{\gamma\delta}^\beta .\]

\[\text{Here } R_{\alpha\beta} \equiv R^\gamma_{\quad \gamma\alpha\beta} \text{ is the Ricci tensor, } R \equiv R^\alpha_{\quad \alpha} \text{ the Ricci scalar, and } C_{\alpha\gamma\beta\delta} \equiv C \text{ the Weyl tensor, which has the same symmetries as the Riemann tensor, obeys } (19) \text{ with } R \text{ replaced by } C , \text{ and is moreover trace-free: } C_{\alpha\gamma\beta\delta} = 0 . \text{ One defines the electric and magnetic parts of the Weyl tensor relative to an observer } O(u) \text{ by}\]

\[\varepsilon_{\alpha\beta} \equiv C_{\alpha\gamma\beta\delta}u^\gamma u^\delta ; \quad H_{\alpha\beta} \equiv \star C_{\alpha\gamma\beta\delta}u^\gamma u^\delta .\]

\[\text{These tensors are symmetric, spatial and traceless, and as in } (27) \text{ can be assembled into the complex tensor}\]
By (54) the relation with the tensors (50) is
\[ E_{\alpha\beta} = \mathcal{E}_{\alpha\beta} + \left( \frac{R}{6} + \frac{R_{\gamma} u^\gamma u^\delta}{2} \right) h_{\alpha\beta} - \frac{1}{2} h^{\lambda}_{\alpha} h^{\delta}_{\beta} R_{\gamma\delta}, \] (56)
\[ H_{\alpha\beta} = \mathcal{H}_{\alpha\beta} + \frac{1}{2} \mathcal{C}_{\alpha\beta\gamma} R^{\gamma\mu} u_{\mu} u_{\nu}, \] (57)
\[ F_{\alpha\beta} = -\mathcal{F}_{\alpha\beta} + \left( \frac{R}{3} + \frac{R_{\gamma} u^\gamma u^\delta}{2} \right) h_{\alpha\beta} - \frac{1}{2} h^{\lambda}_{\alpha} h^{\delta}_{\beta} R_{\gamma\delta}, \] (58)
with \( h^{\delta}_{\beta} \) as defined in (1).

In the vacuum case \( R_{\alpha\beta} = 0 \) of the previous subsections the Riemann tensor equals the Weyl tensor. For a non-vacuum Riemann tensor, everything in the previous subsections holds for the Weyl tensor but not (necessarily) for the Riemann tensor itself, i.e., everything holds if one replaces \( R_{\alpha\beta\gamma\delta}, E_{\alpha\beta}, H_{\alpha\beta}, Q_{\alpha\beta} \) by \( C_{\alpha\beta\gamma\delta}, \mathcal{E}_{\alpha\beta}, \mathcal{H}_{\alpha\beta}, \mathcal{Q}_{\alpha\beta} \) in the definitions and results of Secs. IIIA and IIIB referred to as their Weyl generalizations, and the scalars \( \lambda_k \ (k = 1, 2, 3) \) are the common eigenvalues of the operators \(-\frac{1}{2} C_{\alpha\beta\gamma\delta}\) and \( Q^{\alpha\beta} \) for any observer \( \mathcal{O}(u) \). In particular, a non-zero Weyl tensor is called purely electric (purely magnetic) if there exists an observer \( \mathcal{O}(u) \) for which the magnetic (electric) part of the Weyl tensor vanishes; this happens precisely when the Petrov type is I or D and the Weyl generalization of, respectively, (34) or (35) [or, equivalently, of (57)–(58)] holds. The observers \( \mathcal{O}(u') \) for which \( H'_{\alpha\beta} = 0 \) (\( E_{\alpha\beta} = 0 \)) can be obtained from the Weyl generalization of the algorithm in Sec. IIIIB.

Regarding the Riemann tensor itself, one can study the vanishing of the gravitoelectric or gravimagnetic tidal tensor relative to some observer \( \mathcal{O}(u) \); one must however be careful with introducing the terminology ‘purely electric’ or ‘purely magnetic’ here, because for a non-vacuum Riemann tensor the conditions \( E_{\alpha\beta} = 0 \) and \( H_{\alpha\beta} = 0 \) may hold simultaneously, even for the same observer \( \mathcal{O}(u') \) \((75)\).\(^{10}\) We will call the Riemann tensor and the spacetime purely electric (purely magnetic) at a point if an observer \( \mathcal{O}(u') \) exists for which \( E_{\alpha\beta} = 0 \) (\( E_{\alpha\beta} = 0 \)) and moreover \( E_{\alpha\beta} \neq 0 \) (\( E_{\alpha\beta} \neq 0 \)) for all observers \( \mathcal{O}(u') \) (compare to (57) [75], [80]).

The frame projections of conditions \( E_{\alpha\beta}' \equiv R_{\mu\nu\beta\alpha} u^\mu u^\nu = 0 \) or \( H_{\alpha\beta}' \equiv \ast R_{\mu\nu\beta\alpha} u^\mu u^\nu = 0 \) form an overdetermined system of 6, resp. 8 homogeneous quadratic equations in the components of \( u^\alpha \), which should be augmented with the component form of the inhomogeneous quadratic condition \( q_{\alpha\beta} u^{\alpha} u^{\beta} = -1 \). As for \( E_{\alpha\beta}' = 0 \) no clear-cut general criterion is known for when the resulting system allows for solutions (however, see [75] [80] [81] for some partial results). As

\[ H'_{\alpha\beta} = 0 \Leftrightarrow \begin{cases} H'_{\alpha\beta} = 0, \\ u^{(\alpha} R^{\beta)} \gamma u^{\gamma} = 0. \end{cases} \] (59)

Using the projector orthogonal to \( u^\alpha, h_{\alpha\beta}^{\alpha} = \delta_{\alpha\beta} + u^\alpha u^\beta \), we can still write
\[ u^{(\alpha} R^{\beta)} \gamma u^{\gamma} = 0 \Leftrightarrow h_{\alpha\beta}^{\alpha} R_{\gamma}^{\beta} u^{\gamma} = -8\pi J_{\alpha} = 0. \] (60)

where \( J_{\alpha} \equiv -h_{\alpha\beta} T^{\beta\gamma} u^{\gamma} \) is the spatial mass/energy current density as measured by an observer of 4-velocity \( u^\alpha \), and in the second equality we used Einstein’s field equations \( R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha\beta}) + A_{\mu\nu} \). Equations (59) tell us that an observer \( \mathcal{O}(u) \) measures a zero gravitomagnetic tidal tensor if and only if the magnetic part of the Weyl tensor relative to it vanishes and its 4-velocity \( u^\alpha \) is a Ricci eigenvector [80], [82]. Or, equivalently, if both the magnetic part of the Weyl tensor and the mass-energy currents relative to it vanish.

Define the traceless Ricci tensor as \( S_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{4} R g_{\alpha\beta} \), and its associated invariants [83], [84]:
\[ I_6 \equiv S_{\alpha\beta} S^{\alpha\beta}, \quad I_7 \equiv S_{\alpha\beta} S^{\alpha} S^{\beta}, \quad I_8 \equiv S_{\alpha\beta} S^{\gamma} S^{\beta} S^{\alpha}, \quad I_S \equiv 7 I_6^2 - 12 I_8, \quad J_S \equiv 36 I_6 I_8 - 17 I_6^3 - 12 I_8^2. \]

A timelike eigenvector \( u^\alpha \) of the Ricci tensor \( R_{\alpha\beta} \) (or, equivalently, of \( S_{\alpha\beta} \), both regarded as linear operators in tangent space), exists if and only if \( S_{\alpha\beta} \) is diagonalizable with real eigenvalues\(^{11}\), which, as shown in Appendix A happens precisely in one of the following cases
\[ \begin{align*}
(a) & \quad S_{\alpha\beta} = 0; \\
(b) & \quad I_6 \neq 0 \quad \text{and} \quad S_{\alpha\beta} S^{\gamma}_{\gamma} + 2 \lambda S^{\alpha}_{\gamma} - 3 \lambda^2 \delta^{\alpha}_{\gamma} = 0 \\
& \quad \text{with} \quad \lambda \equiv -I_7/(2 I_6); \\
(c) & \quad I_6 \neq 0 \quad \text{and} \quad S_{\alpha\beta} S^{\delta}_{\gamma} = \frac{1}{4} I_6 \delta^{\alpha}_{\gamma}; \\
(d) & \quad J_S \neq -2 I_6 J_S \quad \text{and} \quad (S^{\alpha}_{\beta} - \lambda \delta^{\alpha}_{\beta})(S^{\beta}_{\gamma} S^{\gamma}_{\delta} + 2 S^{\alpha}_{\delta}) + (3 \lambda^2 - I_6/2) \delta^{\alpha}_{\delta} = 0. \end{align*} \]

\(^{10}\) This happens precisely when \( R_{\alpha\beta\gamma\delta} u^{\delta} = 0 \) and is exemplified by Einstein’s static universe metric, or more generally by 1+3 decomposable spacetimes with \( u^\alpha \) orthogonal to the 3-D spacelike hypersurfaces [79].

\(^{11}\) In general, eigenspaces of \( S_{\alpha\beta} \) corresponding to different eigenvalues are orthogonal [if \( S_{\alpha\beta} u^\beta = \lambda u^\alpha \) and \( S_{\gamma\delta} u^\delta = \mu u^\gamma \) with \( \lambda \neq \mu \) then \( S_{\beta\gamma} u^\beta u^\delta = \lambda \mu u^\gamma + \mu u^\gamma \) and \( v^\alpha u^\alpha = 0 \). If \( S_{\gamma\delta} \) is diagonalizable with real eigenvalues then tangent space is an orthogonal direct sum of the real \( S_{\gamma\delta} \)-eigenspaces, so one of these is timelike and thus contains (all) timelike eigenvectors. Conversely, if a timelike eigenvector exists then its orthogonal complement is a real Euclidean space and the restriction of \( S_{\gamma\delta} \) to this space is a symmetric endomorphism; hence this restriction, and thus \( S_{\gamma\delta} \) itself, is diagonalizable with real eigenvalues.
with \( \lambda = -I_2 I_5/(I_5 + 2 I_6 I_5) \) and \( I_6/2 > 2\lambda^2 \).

Note that this property can be expressed in terms of scalar invariants of the Ricci tensor only in case (e) but not in the more degenerate cases (a)-(d). [This somewhat parallels the situation of condition (34) for Weyl-Petrov type I vs. (55) for type D and \( C_{\alpha\beta\gamma\delta} = 0 \) for type O]. For instance, the invariant relations \( I_5 = J_5 = 0 < I_6 \) identify (b) among the subcases above, but Ricci tensors satisfying these relations and not (62) do exist.

Introducing also the Weyl generalizations of (23)-(26),
\[
\frac{1}{16} \cdot C \cdot C = \frac{1}{16} \alpha^\beta \gamma^\delta C_{\alpha\beta\gamma\delta} = \alpha^\beta \gamma^\delta H_{\alpha\beta} + H_{\alpha\beta},
\]
\( k_C \equiv -\frac{1}{16} \alpha^\beta \gamma^\delta \alpha^\mu \rho \sigma C_{\alpha\beta\gamma\delta} = \alpha^\beta \gamma^\delta \gamma_{\alpha} - 3 \alpha^\beta H_{\alpha\beta}, \]
\( B_C \equiv \frac{1}{16} \alpha^\beta \gamma^\delta \alpha^\mu \rho \sigma C_{\alpha\beta\gamma\delta} = \alpha^\beta H_{\gamma\alpha} - 3 \alpha^\beta \gamma^\delta \gamma_{\alpha}, \]
equations \( (70) \) then imply the following new criterion, providing the necessary and sufficient conditions for the existence of observers measuring \( H_{\alpha\beta} = 0 \).

Criterion: observers \( O'(u') \) measuring a vanishing gravitomagnetic tidal tensor (\( H_{\alpha\beta} = 0 \)) exist in three Petrov types of spacetime, and precisely under the following conditions.

### 1. Petrov type O

In this case \( C_{\alpha\beta\gamma\delta} = 0 \), thus the existence of an observer for which \( H_{\alpha\beta} = 0 \) reduces to the existence of a time-like eigenvector of the Ricci tensor, corresponding to cases (a)-(e) above. The explicit expressions for the 4-velocities of such observers are given, respectively, in items (a)-(e) of Appendix A (to which we refer for further details).

### 2. Petrov type D

First the Weyl tensor needs to be purely electric. A Weyl tensor is purely electric and of type D if and only if the corresponding generalization of (55),
\[
Q_\gamma^\alpha Q_\beta^\gamma + \lambda Q_\beta^\gamma = 2\lambda^2 \epsilon_{\alpha} = 0, \quad \lambda = -\frac{8q_C}{C \cdot C}, \quad (63)
\]
is satisfied. Condition (59) then holds for an observer \( O'(u') \) if and only if \( u^\alpha \) lies in the timelike Weyl principal plane \( \Sigma \), spanned by null vectors \( k^\alpha \) and \( l^\alpha \), with \( k^\alpha l_\alpha = -1 \). These are given by the Weyl generalization of (46) with (49), (51) and (57), identifying \( k^\alpha = k_+^\alpha \), \( l^\alpha = k_-^\alpha \). Such a vector \( u^\alpha \) can be parametrized by
\[
u^\alpha = \frac{1}{\sqrt{q}} (qk^\alpha + l^\alpha), \quad q > 0, \quad (64)
\]
where \( q = (C + D)^2 \) compared to (42) and we need to find when (60) holds for some \( q \). Note that for each \( q > 0 \) the vector \( x^\alpha = qk^\alpha + l^\alpha \) spans the spacelike direction in \( \Sigma \) orthogonal to \( u^\alpha \). Thus condition (60) geometrically means that the vector \( R_{\alpha\beta} u^\beta \) lies in \( \Sigma \) and is orthogonal to \( x^\alpha \), which is respectively expressed by
\[
\epsilon_{\alpha\beta\gamma\delta} k^\beta l^\gamma u^\delta = 0 \quad \text{and} \quad x^\alpha R_{\alpha\beta} u^\beta = 0. \]

With the definitions
\[
K^\alpha \equiv \epsilon_{\alpha\beta\gamma\delta} k^\beta l^\gamma R_{\gamma^\delta} u^\delta; \quad R_{kk} = \epsilon_{\alpha\beta\gamma\delta} k^\alpha k^\beta; \quad L^\alpha \equiv \epsilon_{\alpha\beta\gamma\delta} k^\beta l^\gamma R_{\gamma^\delta} l^\delta; \quad R_{ll} = \epsilon_{\alpha\beta\gamma\delta} l^\alpha l^\beta l^\gamma l^\delta,
\]
these conditions are equivalent to
\[
L^\alpha = -qK^\alpha, \quad R_{ll} = q^2 R_{kk}, \quad q > 0. \quad (65)
\]

Note that the vectors \( K^\alpha \) and \( L^\alpha \) are orthogonal to \( \Sigma \), and are thus spacelike or zero. It follows that an observer exists at \( p \) for which \( H_{\alpha\beta} = 0 \) if and only if either
\[
(2-a) K^\alpha = L^\alpha = 0, \quad R_{kk} = R_{ll} = 0; \quad \text{or}
(2-b) K^\alpha = L^\alpha = 0, \quad R_{kk} R_{ll} > 0; \quad \text{or}
(2-c) K^\alpha L^\alpha = 0, \quad K^\alpha R_{kk} < 0, \quad R_{kk} R_{ll} \left( \frac{K^\alpha K^\alpha}{K^\alpha L^\alpha} \right)^2,
\]
is satisfied on top of (63). The characteristic condition \( K^\alpha = L^\alpha = 0 \) in (2-a) and (2-b) means that the restriction of the Ricci operator \( R_{\alpha\beta} \) to \( \Sigma \) is an endomorphism; if moreover \( R_{kk} = R_{ll} = 0 \) then \( k^\alpha \) and \( l^\alpha \) are both eigenvectors with the same eigenvalue \(-R_{\alpha\beta} k^\beta l^\gamma \) and we have (2-a) [else (2-b)], in which case \( \Sigma \) is a timelike eigenplane. The observers \( O'(u') \) for which (59) holds are those with 4-velocity (64) and the following values of \( q > 0 \):

(2-a) any \( q > 0 \) (all observers with 4-velocity in \( \Sigma \)). Examples: all “doubly aligned” electrovacuum spacetimes which are (at the given point) Weyl purely electric.

(2-b) \( q = \sqrt{R_{kk}/R_{ll}} \) (unique observer). Examples: Gödel universe (and all other “aligned” perfect fluid spacetimes), Som-Raychaudhuri uniform metrics.

(2-c) \( q = -K^\alpha R_{kk} \) (unique observer). Examples: special types of “non-aligned” electrovacuum spacetimes which are (at the given point) Weyl purely electric.

The above mentioned examples are discussed in Sec. VI.

The ‘aligned’ perfect fluids mentioned in case (2-b) consist of Petrov D fluids for which the fluid’s 4-velocity belongs to the time-like principal plane \( \Sigma \). They include all spherically or plane symmetric perfect fluid (including dust) models [60].

### 3. Petrov type I

A Weyl tensor is purely electric and of Petrov type I if and only if the corresponding generalization of (34),
\[
\ast \cdot C \cdot C = B_C = 0, \quad (C \cdot C/8)^3 > 6A_C^2, \quad (66)
\]
\( \Leftrightarrow \{ C \cdot C = 0, C \cdot C > 0 \} \) and \( \{ J_C = 0 \text{ or } M_C > 0 \} \)
is satisfied. In this case there is at each point a unique observer $O'(u')$ for which (59) holds, with 4-velocity $u'^{\alpha} \propto t_1^{\alpha}$ given by the Weyl generalization of (45). Hence, an observer measuring a vanishing gravitomagnetic tidal tensor $H'_{\alpha\beta}$ exists if and only if the condition
\begin{equation}
 t^\alpha R^\beta_{\alpha\gamma} t_1^\gamma = 0
 \end{equation}
is satisfied on top of (66), in which case (59) holds for the unique observer $O'(u')$ of 4-velocity $u'^{\alpha} \propto t_1^{\alpha}$.

Among the spacetimes verifying these conditions are all Petrov type I (locally or globally) static spacetimes: by definition (see e.g. [70]), they admit a hypersurface orthogonal time-like Killing vector field; the congruence of observers tangent to such field is vorticity-free, i.e., they measure no gravitomagnetic field $H^{\alpha} = 2\omega^{\alpha} = 0$, cf. Eq. (137). Hence, by (144), $H_{\alpha\beta} = 0$ for all such observers. Examples of non-vacuum solutions of this kind are the locally static cylindrically symmetric Einstein-Maxwell solutions discussed in Sec. 4 of [87], or Sec. 3 of [88]. Non-static examples are the "gravito-electric" dust models in [89].

An example of the more common situation that one condition but not the other is satisfied is the van Stockum rotating cylinder, discussed in Sec. VI D.

4. What can be said based only on the curvature invariants

It follows from (59) that any invariant condition which ensures that $H_{\alpha\beta} \neq 0$ for all observers (i.e., that the Weyl tensor is non-zero and not purely electric) also ensures that $H'_{\alpha\beta} \neq 0$ for all observers. In line with (66), condition (63) implies $\mathbf{C} \cdot \mathbf{C} = B_C = 0$ and $(\mathbf{C} \cdot \mathbf{C} / 8)^2 = 6A_C^2 > 0$; hence $\mathbf{C} \cdot \mathbf{C} \neq 0$ or $B_C \neq 0$ or $(\mathbf{C} \cdot \mathbf{C} / 8)^2 < 6A_C^2$ (or, less stringently, $\mathbf{C} \cdot \mathbf{C} < 0$) all imply that $H'_{\alpha\beta} \neq 0$ for all observers. In particular, one obtains from (59) and (60) [55], that
\begin{equation}
 \star \mathbf{R} \cdot \mathbf{R} = \mathbf{C} \cdot \mathbf{C} = 16 \mathcal{E}^{\alpha\beta} H_{\alpha\beta},
 \end{equation}
and so a non-zero Chern-Pontryagin scalar, $\star \mathbf{R} \cdot \mathbf{R} \neq 0$, implies $H_{\alpha\beta} \neq 0$ and thus $H'_{\alpha\beta} \neq 0$ for all observers.

Moreover, if there is a non-zero cosmological constant $\Lambda$ but no sources ($T_{\alpha\beta} = 0$, $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$, $R = 4\Lambda \neq 0$) then Eqs. (59)–(58) reduce to
\begin{align}
 \mathcal{E}_{\alpha\beta} &= \mathcal{E}_{\alpha\beta} - \Lambda / 3 (g_{\alpha\beta} + u^\gamma u_\gamma) = -F_{\alpha\beta}, \quad (69) \\
 H_{\alpha\beta} &= H_{\alpha\beta} \quad (\Rightarrow H'_{\alpha\beta} = 0). \quad (70)
\end{align}
From (69) one has $\mathcal{E}_{\alpha\beta}^\ast = -\Lambda$, and $\mathcal{E}_{\alpha\beta}$ is simply obtained as the tracefree part of $\mathcal{E}_{\alpha\beta}$. It follows that the Riemann tensor is never purely magnetic, i.e., $\mathcal{E}_{\alpha\beta} \neq 0$ for all observers, and is purely electric if and only if the Weyl tensor is zero or purely electric, where the last case is equivalent to either (63) (not totally based on curvature invariants) or (66).

IV. INTERPRETATION OF THE INVARIANT STRUCTURE OF THE RELEVANT ELECTROMAGNETIC SETUPS

We are especially interested in understanding physically the invariant $\vec{E} \cdot \vec{B}$, and why the magnetic field vanishes for some observers in certain setups. Consider an arbitrary distribution of charges and currents, and an arbitrary congruence of observers $\mathcal{O}(u)$ of 4-velocity $u^\alpha$. The projections parallel and orthogonal to $u^\alpha$ of the Maxwell field equations $F^{\alpha\beta}_{\gamma} = 4\pi j^\gamma$ and $\star F^{\alpha\beta}_{\gamma} = 0$, respectively, yield the source equations for the magnetic field, which, in an orthonormal frame “adapted” to the observers $\mathcal{O}(u)$, read (see [66], Sec. 3.4.1 for details)
\begin{align}
 \nabla^\perp \times \vec{B} &= \dot{\vec{E}} - \vec{a} \times \vec{B} + 4\pi \vec{j} - \sigma^i \mathcal{E}_{ji} + 2/3 \theta \vec{E} ; \quad (71) \\
 \nabla^\perp \cdot \vec{B} &= -2\mathcal{J} \cdot \vec{E}, \quad (72)
\end{align}
where
\begin{align}
 a^\alpha &= u^\alpha ; \\
 \omega^\alpha &= \frac{1}{2} \mathcal{E}^\alpha_{\gamma\beta \delta} u^\gamma_u u^\beta_u u^\delta_u ; \\
 \sigma^\alpha_{\gamma\beta} &= h^\alpha_{\gamma\beta} u^{(\lambda,\tau)} \frac{1}{3} u^{\tau} ; \\
 \theta &= u^\alpha u^\alpha \quad (73)
\end{align}
are, respectively, the acceleration, vorticity, shear, and expansion scalar of the observer congruence; $h^\alpha_{\gamma\beta}$ is the spatial projector defined in [1]. $\nabla^\perp$ is the spatial projection of the Levi-Civita covariant derivative $\nabla$,
\begin{equation}
 \nabla^\perp X^{\alpha_1 \ldots \alpha_n} = h^\alpha_{\beta_1} \ldots h^\alpha_{\beta_n} \nabla^\gamma X^{\beta_1 \ldots \beta_n}, \quad (74)
\end{equation}
and dot denotes the ordinary time derivative along the observer’s worldline, $X^{\alpha_1 \ldots \alpha_n} \equiv u^\beta \partial^\tau X^{\alpha_1 \ldots \alpha_n}$. Hats in the indices (e.g. $^\ast$) denote tetrad components. In an inertial frame, all the kinematical quantities (73) vanish, $\nabla^\perp = \nabla_i$ and Eqs. (71)–(72) take the well known form
\begin{align}
 \nabla \times \vec{B} &= \dot{\vec{E}} + 4\pi \vec{j} \quad (i) \\
 \nabla \cdot \vec{B} &= 0 \quad (ii) \quad (75)
\end{align}
Based on these equations we make the following observations:

1. If $\dot{\vec{E}} + 4\pi \vec{j} \neq 0$ in an inertial frame, then, according to Eq. (75), $\vec{B}$ cannot vanish in that frame on any spatial 3-D open region (only on 2-surfaces or lower-dimensional sets).

2. If there exists an inertial frame where $\dot{\vec{B}} + 4\pi \vec{j} = 0$ everywhere, and no fields are present other than those arising from the sources, then $\vec{B} = 0$ globally in that frame. This implies $E \cdot \vec{B} = 0$ everywhere. Example, consider a system of $N$ point charges: if an inertial frame exists where they are all at rest, then $\vec{j} = \vec{E} = 0$ and so $\vec{B} = 0 \Rightarrow \vec{E} \cdot \vec{B} = 0$ everywhere (cf. Eq. (80)).
implies (via Green’s theorem) that if, at infinity, \( \mathbf{B} \) point 2, one notes that, when \( \mathbf{B} \) stated as follows: one scales, by some small dimensionless first “post-Coulombian” (1PC) approximation. It can be analogous approximation that we dub, following [90], the approximation (1PN); in electromagnetism we shall use an in order to know \( \mathbf{a} \) to order \( O(4) \), one needs to know \( q\phi/m \) to order \( O(4) \) and \( q\mathbf{A}/m \) to order \( O(3) \). For a system of \( N \) point charges, this amounts to considering a 4-potential\(^{12} \) \( A^0 = (\mathbf{A}, A) \) whose components read, in an inertial frame,

\[
A^0 = \sum_a Q_a \left[ \frac{1}{r_a} \left( 1 + \frac{v_a^2}{2} - \frac{1}{2} \mathbf{a}_a \cdot \mathbf{a}_a \right) - \left( \mathbf{r}_a \cdot \mathbf{v}_a \right)^2 \right];
\]

\[
\mathbf{A} = \sum_a Q_a \mathbf{v}_a \frac{r_a}{r_a},
\]

where \( Q_a \) is the charge of particle “a”, \( \mathbf{r}_a = \mathbf{x}_a - \mathbf{x}_b \), \( \mathbf{x}_a \) is the point of observation, \( \mathbf{v}_a = \partial \mathbf{x}_a / \partial t \) its velocity and \( \mathbf{a}_a = \partial \mathbf{v}_a / \partial t \) its acceleration; for a system of interacting bodies with no external forces, to 1PC accuracy, \( \mathbf{a}_a \) is to be taken in Eq. (77) as the acceleration caused by the Coulomb field produced by the other charges, i.e., \( \mathbf{a}_a = (Q_a/m_a) \sum_{b \neq a} Q_b \mathbf{r}_{ab}/r_{ab}^3 \), with \( \mathbf{r}_{ab} = \mathbf{x}_a - \mathbf{x}_b \). The 1PC electric and magnetic fields, \( \mathbf{E} = -\nabla A^0 - \partial \mathbf{A} / \partial t \) and \( \mathbf{B} = \nabla \times \mathbf{A} \), follow as

\[
\mathbf{E} = \sum_a Q_a (1 + \varphi_a) \frac{\mathbf{r}_a}{r_a^2} - \frac{1}{2} \sum_a Q_a \mathbf{a}_a \frac{r_a}{r_a};
\]

\[
\mathbf{B} = \sum_a \frac{Q_a}{r_a^3} \mathbf{v}_a \times \mathbf{r}_a = \sum_a \mathbf{v}_a \times \left[ \mathbf{E}_a \right]_C,
\]

where

\[
\varphi_a = \frac{\mathbf{v}_a^2}{2} - \frac{1}{2} \left( \mathbf{v}_a \cdot \mathbf{a}_a \right) - \frac{3}{2} \frac{(\mathbf{r}_a \cdot \mathbf{v}_a)^2}{r_a^2} - \frac{1}{2} \frac{\mathbf{a}_a}{r_a^2}
\]

and

\[
\left[ \mathbf{E}_a \right]_C = Q_a \mathbf{r}_a / r_a^3 \] denotes the Coulomb (i.e., 0PC) electric field of particle “a”.

We shall next discuss the electromagnetic invariants and the fields measured by different observers in setups which may be cast as the analogues of the gravitational systems of interest, and where the observations\(^ {[14] } \) above will be exemplified.

\[ \]

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12 These expressions follow from Eqs. (2.73)-(2.74) of [90], for the case of zero gravitational field, setting \( \mu_0 = c_0 = 1 \) therein. They could also be obtained from the exact Léandrew-Wiechert retarded potentials Eqs. (14.8) of [22] (superimposing the potentials of single moving particles given therein, since electromagnetism is linear), expanding them to 1PC order, and noting that to 1PC order the instantaneous relative position \( \mathbf{r}_a \) is related by a quadratic extrapolation to the retarded relative position \( \mathbf{R}_a \), \( \mathbf{r}_a \approx \mathbf{R}_a - \mathbf{v}_R \mathbf{R}_R - \frac{1}{2} \mathbf{a}_R \mathbf{R}_R^2 \).
A. One single point charge

In the inertial rest frame of the charge, one has \( A^α = (\frac{Q}{r^2}, 0, 0, 0) \), \( \vec{E} = (Q/r^2)\hat{e}_r \), \( \vec{B} = 0 \). The two scalar invariants of \( F_{αβ} \) are

\[
\vec{E}^2 - \vec{B}^2 = \frac{Q^2}{r^2} > 0 \ , \quad \vec{E} \cdot \vec{B} = 0 \quad \text{(everywhere)} \ , \quad (81)
\]
telling us that \( F_{αβ} \) is purely electric (everywhere), i.e., everywhere there are observers for which \( B^α = 0 \). Those are the observers at rest in the inertial rest frame of the source ("static" observers), and also observers in purely radial motion, since, as we have seen in Sec. II A, the point \( \vec{v}_{||E} \) along \( \vec{E} \) of the velocity of the observers measuring no magnetic field is arbitrary. In other words, such observers have a 4-velocity of the form\(^{13} \)

\[
u^α = (u^0, u^r, 0, 0).
\]

In order to understand the invariant structure \(^{81} \), let \( S \) and \( S' \) be, respectively, the inertial rest frame of the point charge, and an inertial frame moving relative to it with some velocity \( \vec{v} \) (non-parallel to \( \vec{E} \)). In \( S \), \( \vec{B} = 0 \) globally, which implies \( \vec{E} \cdot \vec{B} = 0 \) everywhere. Observers \( O' \) at rest in \( S' \), in turn, measure a non-zero magnetic field \( \vec{B}' \); but it is such that it is always orthogonal to \( \vec{E}' \), ensuring \( \vec{E}' \cdot \vec{B}' = 0 \), as we shall now explicitly show. By equation \(^{11} \) [or its covariant form \(^{9} \)],

\[
\vec{B}' = -\gamma \vec{v} \times \vec{E} \ , \quad [B'^{αβ} = -\epsilon^{αβγδ}E_γu_δu_β] \ . \quad (82)
\]

Thus, \( \vec{B}' \) is perpendicular to the electric field \( \vec{E} \) measured in the charge’s rest frame and to the velocity \( \vec{v} \); hence it is also perpendicular to \( \vec{E}' \), as is seen from \(^{10} \):

\[
\vec{E}' = \gamma \vec{E} - \frac{\gamma^2}{\gamma + 1} \vec{v} \cdot \vec{E} \ . \quad (83)
\]

B. System of two point charges

We shall now consider two moving charged particles with charges \( Q_1, Q_2 \) of the same sign. If they move with different velocities with respect to some inertial frame, then there is no inertial frame where they are both at rest. From Eq. \(^{75} \) we see that, by contrast with the example in the previous section, in this case the magnetic field cannot vanish globally in an inertial frame. Let us see how this reflects in the invariants. By \(^{79} \) the electric field \( \vec{E} \), at an arbitrary point \( P \) with coordinates \( x \), is

\[
\vec{E} = \frac{Q_1}{r_1^3} (1 + \varphi_1) \vec{r}_1 + \frac{Q_2}{r_2^3} (1 + \varphi_2) \vec{r}_2 - \frac{1}{2} \frac{Q_1}{r_1^3} \vec{a}_1 - \frac{1}{2} \frac{Q_2}{r_2^3} \vec{a}_2 \ , \quad (84)
\]

where \( \vec{r}_1 = \vec{x} - \vec{x}_1 \), \( \vec{r}_2 = \vec{x} - \vec{x}_2 \). By \(^{80} \) the magnetic field is

\[
\vec{B} = \frac{Q_1}{r_1^3} \vec{r}_1 \times \vec{r}_1 + \frac{Q_2}{r_2^3} \vec{r}_2 \times \vec{r}_2 \ . \quad (85)
\]

At an arbitrary point the invariant \( \vec{E} \cdot \vec{B} \) is thus

\[
\vec{E} \cdot \vec{B} = \frac{Q_1}{r_1^3} (\vec{v}_1 \times \vec{r}_1) \cdot \left[ \frac{Q_2}{r_2^3} (1 + \varphi_2) \vec{r}_2 - \frac{1}{2} \frac{Q_1}{r_1^3} \vec{a}_1 - \frac{1}{2} \frac{Q_2}{r_2^3} \vec{a}_2 \right] + \frac{Q_2}{r_2^3} (\vec{v}_2 \times \vec{r}_2) \cdot \left[ \frac{Q_1}{r_1^3} (1 + \varphi_1) \vec{r}_1 - \frac{1}{2} \frac{Q_1}{r_1^3} \vec{a}_1 - \frac{1}{2} \frac{Q_2}{r_2^3} \vec{a}_2 \right] \quad (86)
\]

which is generically non-vanishing. To lowest order,

\[
\vec{E} \cdot \vec{B} \simeq \frac{Q_1 Q_2}{r_1^3 r_2^3} [ (\vec{v}_1 \times \vec{r}_1) \cdot \vec{r}_2 + (\vec{v}_2 \times \vec{r}_2) \cdot \vec{r}_1 ] \ . \quad (87)
\]

As for the invariant \( \vec{E}^2 - \vec{B}^2 \), there is a region between the two charges where \( \vec{B}^2 > \vec{E}^2 \) (magnetic dominance), around the point where \( \vec{E} \simeq \frac{Q_1}{r_1^3} \vec{r}_1 + \frac{Q_2}{r_2^3} \vec{r}_2 = 0 \); elsewhere \( \vec{E}^2 \geq \vec{B}^2 \), and henceforth we restrict attention to this region of electric dominance.

Coplanar motion

Take now the case when \( \vec{v}_1 \), \( \vec{v}_2 \) and the position vectors of the bodies are coplanar (i.e., the two bodies move in the same plane). It follows that \( \vec{E} \cdot \vec{B} \) vanishes in the plane of motion, and is generically non-zero outside that plane. It is easy to see from Eq. \(^{86} \) that in the plane of motion \( \vec{E} \cdot \vec{B} = 0 \): taking the point of observation \( P \) to lie on that plane, then the \( r_3 \), \( u_3 \) and \( a_3 \) all lie on that plane; hence \( (\vec{v}_3 \times \vec{r}_3) \cdot \vec{r}_3 = (\vec{v}_3 \times \vec{r}_3) \cdot \vec{a}_3 = 0 \). This means that, in this plane, there are observers for which the magnetic field vanishes.

We will investigate such observers in the simple example in Fig. 4 that will prove enlightening for the next section: two particles, with equal charge \( Q \), in circular motion of radius \( d \) and in antipodal positions (e.g., with some rod holding them), so that their velocities are equal in magnitude but opposite in direction: \( \vec{v}_1 = -\vec{v}_2 \).
In this special case \( \vec{a}_1 = a(\vec{r}_1 - \vec{r}_2)/2d \), \( \vec{a}_2 = a(\vec{r}_2 - \vec{r}_1)/2d \) and \( (\vec{v}_2 \times \vec{r}_2) \cdot \vec{r}_1 = (\vec{v}_1 \times \vec{r}_1) \cdot \vec{r}_2 \), such that Eq. (86) simplifies to

\[
\vec{E} \cdot \vec{B} = Q^2 \left[ \frac{2 + \varphi_2 + \varphi_1}{r_1^2 r_2^2} + \frac{a}{4d} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \right]
\]

which has the structure

\[
\begin{cases}
\vec{E} \cdot \vec{B} = 0 \text{ in the plane of motion,} \\
\vec{E} \cdot \vec{B} \neq 0 \text{ elsewhere.}
\end{cases}
\]

That \( \vec{E} \cdot \vec{B} \neq 0 \) at any point outside the plane of motion (denote it by \( \Sigma \)) can be seen as follows. First note that the first line of (88) cannot be zero, since by the 1PC assumptions all terms must be much smaller than the first one \( (2Q^2/r_1^2 r_2^2) \). Then note that \( (\vec{v}_1 \times \vec{r}_1) \cdot \vec{r}_2 = (\vec{v}_1 \times [\vec{r}_1 - \vec{r}_2]) \cdot \vec{r}_2 \). Since \( \vec{r}_1 - \vec{r}_2 \in \Sigma \), the vector \( \vec{v}_1 \times [\vec{r}_1 - \vec{r}_2] \) is orthogonal to \( \Sigma \); hence \( \vec{E} \cdot \vec{B} = 0 \) only if \( \vec{r}_2 \) lies on \( \Sigma \), which is possible only if the point of observation \( P \in \Sigma \); at any point \( P \) outside \( \Sigma \), \( \vec{E} \cdot \vec{B} \neq 0 \). This means that outside \( \Sigma \) the magnetic field is non-vanishing for all observers, and that in \( \Sigma \) there are observers for which \( \vec{B} = 0 \). From Eq. (10), such observers must have a velocity whose component orthogonal to \( \vec{E} \), \( \vec{v}_{\perp E} = \vec{v}_{\parallel P} \), reads

\[
\vec{v}_{\perp E} = \frac{r_2^2 \vec{v}_1 \times \vec{r}_1 r_1 - r_1^2 \vec{v}_1 \times \vec{r}_2 r_2 + r_1 r_2 ([\vec{v}_1 \times \vec{r}_1] \vec{r}_2 - (\vec{r}_2 \times \vec{v}_1) \vec{r}_1)}{r_1^4 + r_2^4 + 2r_1 r_2 r_1^2 \vec{r}_2},
\]

where \( \vec{v}_{1 \times r_1} \) and \( \vec{v}_{1 \times r_2} \) are the components of \( \vec{v}_1 \) orthogonal to \( \vec{r}_1 \) and \( \vec{r}_2 \), respectively. We first notice that \( \vec{v}_{\perp E} \), and, therefore, \( \vec{v} \) (since \( \vec{E} \) at any point of \( \Sigma \) lies on \( \Sigma \), except at the middle point \( \vec{r}_1 = -\vec{r}_2 \) where \( \vec{E} = 0 \) lies on the plane of motion \( \Sigma \). The reason why \( \vec{B} \) vanishes for these observers is especially easy to understand along the axis passing through the two particles, see Fig. 4.

First note that, clearly, the magnetic field at \( P \), as measured by the static observer \( O \), is non-vanishing, because although the magnetic field produced by particle 1 acts in opposite direction to the magnetic field from particle 2, the latter is closer to \( P \) so that the two fields do not cancel out. By choosing an observer \( O' \) moving with 3-velocity \( \vec{v} \) in the same direction as particle 2, one is decreasing particle 2’s velocity, and, at the same time, increasing particle 1’s velocity relative to the observer’s inertial rest frame. That means decreasing the magnetic field \( \vec{B}'_2 \) generated by particle 2 and increasing the magnetic field \( \vec{B}'_1 \) generated by particle 1, so that eventually one can make the (total) magnetic field \( \vec{B}' = \vec{B}'_1 + \vec{B}'_2 \) vanish. Along the axis (with \( r_1 = r + d, r_2 = r - d \), cf. Fig. 4), the observers \( O' \) for which \( \vec{B}' = 0 \) have velocities

\[
\vec{v} = \vec{v}_1 - \frac{2d}{r^2} \Rightarrow v \approx \frac{2v_1 d}{r} = \frac{2J}{M_{\text{tot}} r} (90)
\]

where \( M_{\text{tot}} = M_1 + M_2 = 2M_1 \) is the system’s total mass, and we noted that \( M_{\text{tot}} v_1 d = J \) is the system’s angular momentum as measured in the center of mass frame.

C. Spinning spherical charge

Consider a spinning charged spherical body with mass \( M \), angular momentum \( \vec{J} = J \vec{e}_z \), charge \( Q \) and dipole moment \( \mu_s = (Q/2M) J \vec{e}_z \). The electric and magnetic fields produced are, as measured by the rest observers,

\[
\vec{E} = \frac{Q}{r^2} \vec{e}_r, \quad \vec{B} = \frac{2\mu_s \cos \theta}{r^3} \vec{e}_r + \frac{\mu_s \sin \theta}{r^4} \vec{e}_\theta, \quad (91)
\]

where \( \vec{e}_r \equiv \vec{\partial}_r \) denote coordinate basis vectors. The invariants are given by\(^{14}\):

\[
\begin{align*}
\vec{E}^2 - \vec{B}^2 &= \frac{Q^2}{r^4} - \frac{\mu_s^2 (5 + 3 \cos 2\theta)}{2r^6} > 0, \\
\vec{E} \cdot \vec{B} &= \frac{2\mu_s Q \cos \theta}{r^5} (\vec{E} \cdot \vec{B} = 0 \text{ in the plane } \theta = \pi/2) .
\end{align*}
\]

Since \( \vec{E} \cdot \vec{B} = 0 \) in the equatorial plane (\( \theta = \pi/2 \)), observers \( O' \) exist in this plane for which \( \vec{B}' = 0 \). From Eq. (10), the velocity of those observers is such that its component \( \vec{v}_{\perp E} = \vec{v}_{\parallel P} \) orthogonal to \( \vec{E} \) is

\[
\vec{v}_{\perp E} = \frac{\vec{E} \times \vec{B}}{\vec{E}^2} = \frac{\mu_s}{Q^2 r^2} \vec{e}_\phi ,
\]

\(^{14}\) The first inequality always holds assuming the classical gyromagnetic ratio \( \mu_s/J = Q/2M \), corresponding to a classical source where the charge and mass are identically distributed. In that case

\[
\frac{\mu_s^2}{r^6} \approx \frac{Q^2 J^2}{4M^2} < \frac{1}{4} \frac{Q^2 R^2}{r^6}
\]

where \( R \) is the body’s radius and we have used the fact that, in order for the dominant energy condition to be obeyed, \( R \geq J/M \), see \([22]\). Since \( r > R \) at any point exterior to the particle, we have \( \mu_s^2/r^6 < Q^2/r^4 \Rightarrow \vec{E}^2 - \vec{B}^2 > 0 \).
no restriction being imposed on the (radial) component \( \dot{v}_r = v^r \dot{e}_r \) parallel to \( E \) (apart from the normalization condition \( u^\alpha u_\alpha = -1 \)). That is, observers moving in the equatorial plane \( (\nu^\phi = 0) \) with angular velocity
\[
\frac{d\phi}{dt} = \frac{u^\phi}{u^\nu} = \frac{\mu_s}{Qr^2} = \frac{J}{2Mr^2}
\]
measure a vanishing magnetic field. One might check that these are indeed the only observers for which \( B^\alpha = 0 \) by computing explicitly \( B^\alpha \) for an arbitrary 4-velocity \( u^\alpha = (u^\nu, u^r, u^\theta, u^\phi) \), as done in [91]. If we take the special case \( v^\nu = u^\nu = 0 \), we obtain the velocity field \( \vec{v} = J/(2Mr^2)\vec{e}_\phi \) depicted in Fig. 5.

Figure 5: Observers for which the magnetic field vanishes. (Note that these are not observers “co-rotating” with the same angular velocity of the spinning body). \( \vec{e}_\phi \) is the coordinate basis vector \( \vec{e}_\phi \equiv \vec{\partial}_\phi = r \vec{e}_\phi \). Observer \( O' \) at point \( P \) must have a velocity that decreases the magnetic field generated by the charge elements of the closer hemisphere (e.g., charge element 2), and increases the magnetic field produced by the charge elements of the opposite hemisphere (e.g., charge element 1), such that they eventually cancel out.

The vanishing of \( \vec{B}' \) for such observers can be understood in the same spirit as in the case of the two point charges in coplanar motion of Fig. 4. A rotating charged body may be decomposed in arbitrarily small charge elements in translation; and its electromagnetic field (91) cast as a superposition of the field produced by each such elements. In particular, to 1PC order, we may write for \( \vec{B} \) (cf. Eq. (80))
\[
\vec{B}(\vec{r}) = \int_{\text{body}} \rho_d d^3\vec{x}' \times \frac{(\vec{r} - \vec{x}')}{|\vec{r} - \vec{x}'|^3} d^3\vec{x}',
\]
where \( \vec{u}_d(\vec{x}') \) is the velocity of the charge element \( \rho_d d^3\vec{x}' \) at the point \( \vec{x}' \). Consider the situation in Fig. 5. Relative to an observer at rest at a point \( P \) of the equatorial plane, the charge elements in the closer hemisphere (e.g., charge element 2) move in opposite direction to the ones in the opposite hemisphere (e.g., charge element 1), so their contributions \( \vec{u}_d(\vec{x}') \times (\vec{r} - \vec{x}') \) to the integral (94) have opposite signs. The net field \( \vec{B} \) is different from zero because one hemisphere is closer than the other (leading to a dipole field). The observers \( O' \) for which \( \vec{B}' = 0 \) at \( P \) must move in the same sense as the rotational motion of the body, thereby decreasing the relative velocity of the charge elements in the closer hemisphere (decreasing the magnitude of their magnetic field), and increasing the relative velocity of the elements in the farther hemisphere (increasing the magnitude of their magnetic field), with a suitable velocity \( \vec{v}' \) such that the fields from the two hemispheres cancel out.

Notice the similarity with the result obtained in Sec. IV B the magnitude of the velocity field in Fig. 5 is \( v = J/2Mr \), which, up to a factor of four, matches the asymptotic behavior of the field [90] depicted in Fig. 4.

Using Eq. (11), and noting that \( \vec{B} \cdot \vec{v} = 0 \) to obtain \( \vec{B}' = \gamma \vec{B} - \gamma \dot{\vec{v}} \times \vec{E} \), one can also interpret the vanishing of \( \vec{B}' \) for the observers in Fig. 5 as a cancellation between the magnetic field \( \gamma \vec{B} \) arising from the rotational motion of the source and the magnetic field \( -\gamma \dot{\vec{v}} \times \vec{E} \) arising from the translational motion of the source relative to the observer. The fact that such cancellation may occur only in the equatorial plane is easy to see noting that since the translational magnetic field \( -\gamma \dot{\vec{v}} \times \vec{E} \) is orthogonal to \( \vec{E} \), it can kill the rotational field only if \( \vec{B} \) is also orthogonal to \( \vec{E} \) which, for this setup, happens only in the equatorial plane.

Finally, notice that the observers in Fig. 5 exemplify one case of point 3 of Sec. IV there is no inertial frame where the different charge elements are at rest\(^{15}\), i.e., where \( \vec{J} = 0 \) everywhere; moreover, as in the two-body system of Sec. IV B the magnetic field does not globally vanish in any inertial frame. Yet, in a spatial 2-surface, there are still observers measuring no magnetic field, only they do not form an inertial frame (take e.g. the congruence with \( d\phi/dt \) given by (93), and \( u^\theta = u^\phi = 0 \); such congruence is accelerated and shears\(^{16}\)).

\(^{15}\) In a co-rotating frame the whole spinning body is at rest; but such frame consists of a congruence of observers all having different 4-velocities \( U^\alpha \), thus different inertial rest frames, whilst having the same angular velocity. Actually, no single point in the body is at rest with respect to the inertial frame of a co-rotating observer if the latter lies outside the body. \( \vec{B} \) does not vanish in the co-rotating frame, even though the body is at rest therein (so \( \vec{J} = 0 \)); taking the perspective of such frame, this is justified with the fact that the vorticity of the observer congruence contributes as a source for \( \vec{B} \), cf. Eqs. 17 and 72.

\(^{16}\) For the congruence \( u^\alpha = (0,0,0,d\phi/dt) \), with \( u^\phi = 1/\sqrt{1 - (d\phi/dt)^2}g_{\phi\phi} \), the non-vanishing components of the acceleration and shear are, in the equatorial plane, \( a^\tau = a^\phi = -4a^2/(4\nu^2 - a^2)^{1/2} \), \( \sigma_{\phi\phi} = -4\nu a^2/(4\nu^2 - a^2)^{3/2} \), \( \sigma_{\tau\phi} = \sigma_{\phi\tau} = 2a^2/(4\nu^2 - a^2)^{1/2} \), where \( a \equiv J/M \). The vorticity and expansion vanish in that plane.
D. Further examples — infinite rotating cylinder

Here we consider a simple physical system that exemplifies the remaining cases mentioned in point 4 of Sec. [IV] Consider the electromagnetic field produced by a uniform, rotating, and infinitely long cylinder of radius R and charge density $\rho_c$. The electric and magnetic fields, as measured by static observers, read, in cylindrical coordinates $(r, \phi, z)$,

$$r < R : \quad \vec{E} = 2\pi\rho_c r \vec{e}_r ; \quad \vec{B} = 2\pi\rho_c \Omega (R^2 - r^2) \vec{e}_\phi ;$$
$$r \geq R : \quad \vec{E} = \frac{2\pi\rho_c R^2}{r} \vec{e}_r ; \quad \vec{B} = 0 .$$

It follows that $\vec{E} \cdot \vec{B} = 0$ everywhere, and $\vec{E}^2 - \vec{B}^2 > 0$ ($< 0$) for $r > r_c (< r_c)$, where the critical radius $r_c^2 = R^2 + (1 - \sqrt{1 + 4R^2\Omega^2})/(2\Omega^2)$ defines the boundary between the purely electric/magnetic regions, and lies inside the cylinder ($r_c < R$). The magnetic field $\vec{B}$ vanishes at every point outside the cylinder in the inertial frame of the static observers; this exemplifies one of the situations in point 4 of Sec. [IV] even when there is no inertial frame where the currents are zero everywhere, still $\vec{B}$ can vanish in a 3-D region relative to an inertial frame. Inside the cylinder, for $r > r_c$ (purely electric region), it vanishes for certain observers. Such observers have a velocity whose component orthogonal to $\vec{E}$ is obtained from Eq. (10),

$$\vec{v}_{LE} = \frac{\vec{E} \times \vec{B}}{E^2} = \frac{\Omega (R^2 - r^2)}{r} \vec{e}_r \times \vec{e}_\phi = \Omega \left[ 1 - \frac{R^2}{r^2} \right] \vec{e}_\phi ;$$

i.e., observers with angular velocity $d\phi/dt = \Omega [1 - R^2/r^2]$ (in the sense opposite to the cylinder’s rotation). Such observer congruences are not inertial, as they are shear-free, rotating and accelerating. This exemplifies another situation in point 4 of Sec. [IV] with respect to non-inertial frames, even in a 3-D region where $\vec{j} \neq 0$, one can have $\vec{B} = 0$.

V. INTERPRETATION OF THE INVARIANT STRUCTURE FOR THE RELEVANT ASTROPHYSICAL SETUPS

In the gravitational case we are interested in understanding the curvature invariants of the gravitational fields of current experimental interest, in particular the Chern-Pontryagin invariant $\star^2 \mathbf{R} \cdot \mathbf{R}$ and its vanishing in some setups.

From the differential Bianchi identities $R_{\alpha\beta\gamma\delta} = 0$, written in terms of the electric part $\mathcal{E}_{\alpha\beta} = R_{\alpha\gamma\beta\delta} u^\gamma u^\delta$ and magnetic part $\mathcal{H}_{\alpha\beta} = \ast R_{\alpha\gamma\beta\delta} u^\gamma u^\delta$ of the Weyl tensor with respect to the observers $\mathcal{O}(u)$, see [23], one obtains the source equations for $\mathcal{H}_{\alpha\beta}$ [23], which read in an orthonormal frame “adapted” [40] to the observers,

$$\nabla \mathcal{H}_{ij} = \mathcal{H}_{ij} + \omega^m_{\alpha i} \epsilon_{mik} (\mathcal{E}^k)^j - 3 \pi_{kij} (\mathcal{E}^k) - 2 \pi_{kij} (\mathcal{H}^k) + \mathcal{E}_{ij} \theta + 4 \pi \left[ (p + p) \sigma_{ij} + \nabla_j (\mathcal{J}_j) + 2a_{ij} \mathcal{J}_j \right] + \theta \left( \pi_{ij} + \pi_{ij} / \pi_{kkj} \right) ,$$

$$\nabla_j \mathcal{J}_i = -3 \omega^m_{\alpha i} \epsilon_{mik} \mathcal{E}^k - 4 \pi \left[ (p + p) \omega_i + \epsilon_{ijk} \sigma^m_{\pi m} \pi^k - \pi_{ijk} \omega_j + \nabla_j (\mathcal{J}_j) \right] ,$$

where $\nabla \mathcal{A}_{\alpha\beta} \equiv -\epsilon^{\mu\nu}_{\alpha} A_{\beta \mu \nu} u_\nu$, and the index notation $\langle \mu \nu \rangle$ stands for the spatially projected, symmetric and trace-free part of a rank two tensor:

$$A_{\mu \nu} \equiv h^\alpha_{(\mu} h^\beta_{\nu)} A_{\alpha \beta} - \frac{1}{3} h_{\mu \nu} h_{\alpha \beta} A^{\alpha \beta} ,$$

with $h^\alpha_{\beta}$ defined in Eq. [1]. In these equations $\rho \equiv T^\alpha_{\mu \nu} u^\mu u^\nu$ is the mass/energy density, $\mathcal{J}^\alpha \equiv -h^\alpha \mathcal{J}^\beta u^\nu$ is the spatial mass/energy current density as measured by an observer of 4-velocity $u^\nu$, $p = T^\alpha_{\mu \nu} h_{\alpha \beta} / 3$ is the pressure, and $\pi^{\alpha \beta} \equiv T^{(\alpha \beta)}$ is the traceless spatial projection of $T^{\alpha \beta}$ with respect to $u^\alpha$ (i.e., the traceless stress tensor, cf. [23] Eq. (5.9)). The tensors $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$ are related to the electric and magnetic parts of the Riemann tensor, $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$, by Eqs. (56)-(57). We note in particular that

$$\mathcal{H}_{\alpha\beta} = \mathcal{H}_{\alpha\beta} - 4 \pi \epsilon_{\alpha \beta \gamma \delta} \mathcal{J}^\gamma u^\delta .$$

Equations (95)-(96) exhibit formal similarities with Maxwell’s equations (71)-(72).

The gravitational fields of the astrophysical setups of interest are considered in the literature at post-Newtonian accuracy. Such approximation may be cast as follows (see e.g. [23] [93] [91] [94] [95]). One scales

$$U \sim \epsilon^2 ; \quad v \lesssim \epsilon ; \quad v_s \lesssim \epsilon ,$$

where $U$ is the Newtonian potential and $v_s, v$ are the velocities of the sources and of the test particle. The first post-Newtonian order (1PN) consists of keeping terms up to $O(\epsilon^4) \equiv O(4)$ in the equations of motion (see e.g. [93] Sec. 4.1 (b)). This amounts to considering a metric of the form [19] [94] [95]

$$g_{00} = -1 + 2w - 2w^2 + O(6) ;$$
$$g_{0i} = A_i + O(5) ;$$
$$g_{ij} = \delta_{ij} (1 + 2U) + O(4) ,$$

that the “dot” derivative in [23], Eq. (4.6) therein, denotes $\nabla_u = u^\alpha \nabla_\alpha$ (not an ordinary time derivative, as it does in the present paper); that for a spatial, traceless, and symmetric 2-tensor, $\nabla u \mathcal{E}^{(ij)} = \nabla u \mathcal{E}^{(ij)} = \xi^{(ij)} + 2\xi^{(k)} (\mathcal{E}^{ij} + \xi^{(j)} \mathcal{E}^{ik})$ (see the connection coefficients in [23], with $\Omega = \langle \mathcal{O} \rangle$); and that

\[ \begin{align*}
\omega^\gamma_{\alpha \beta} &= \omega^\gamma_{\alpha \beta} \gamma, \\
\omega^\gamma_{\alpha \beta} &= \omega^\gamma_{\alpha \beta} \gamma.
\end{align*} \]
where \( \vec{A} \) is the "gravitomagnetic vector potential" and the scalar \( w \) consists of the sum of \( U \) plus non-linear terms of order \( \epsilon^4 \), \( w = U + O(4) \). The electric and magnetic parts of the Riemann tensor, as measured by an observer at rest \((u^i = 0)\) in the coordinate system of (98), are, using the 1PN Christoffel symbols in e.g. Eqs. (8.15) of [99],

\[
E_{ij} = -w_{ij} + \hat{\mathcal{A}}_{(i,j)} + 3U_{i,j} + \delta_{ij}(\dot{U} + (\nabla U)^2) + O(6) ;
\]
\[
\mathbb{H}_{ij} = -\frac{1}{2} \epsilon^{i\ell k} \hat{\mathcal{A}}_{k,j} - \epsilon_{ij}^k \dot{U}_{k} + O(5) ;
\]

(99) \( E_{\alpha\beta} = E_{\alpha\beta} = \mathbb{H}_{\alpha\beta} = 0 \) in 1PN order, using the relation (see e.g. [53]) \( \partial \rho / \partial t = -\nabla \cdot \vec{J} + O(\rho^2 \epsilon^3) \), which is an approximation (accurate enough for Eq. (100)) to the conservation equation \( T_{\alpha\beta}^{(\epsilon)} = 0 \). We may also re-write Eqs. (95)-(96) to 1PN order,

\[
\text{curl}H_{ij} = [\mathcal{E}_{ij}]_N + 4\pi \mathcal{J}_{(i,j)} + O(5) ;
\]
\[
H_{i,j} = -4\pi(\nabla \times \vec{J} + O(5)) ;
\]

(102) \( \) and (103)

\[
\text{where} \quad [\mathcal{E}_{ij}]_N = \text{the} \text{traceless} \text{Newtonian (i.e., 0PN) tidal tensor,} \quad \mathcal{E}_{ij} = -U_{i,j} + \delta_{ij}U_{k,k}/3 \quad \text{and we noted that the shear of a 1PN frame vanishes,} \quad \sigma_{ij} = (\Gamma^i_{ij} + \Gamma^k_{ij})/2 - \dot{U}_{ij} = 0 \quad \text{and that} \quad \omega = \nabla \times \vec{A}/2 \sim O(3) ; \quad \theta = 3U \sim O(3) ; \quad \mathcal{J}^i \sim O(\rho^2) \quad \text{and} \quad \pi_{ij} \sim O(\rho^2) \quad \text{and} \quad \rho \sim O(2) \quad \text{via} \quad \nabla^2 U \sim -4\pi\rho .
\]

It is important in this context to notice that the neglect of the terms involving contractions of the tensors \( \mathcal{E}_{ij} \) or \( H_{ij} \) with the kinematical quantities \( \theta, \omega, \sigma_{ij} \) embodies a restriction on the type of reference frame (for e.g., if one chooses an accelerated or rapidly rotating frame, even for weak sources or in the far field regime, one could not neglect the terms involving the vorticity and acceleration); it is reasonable in post-Newtonian frames \( \theta, \omega, \sigma_{ij} \) (such as the one associated to the coordinate system of the metric (98)), because they are as close as possible to inertial frames.

Eqs. (102)-(103), together with (100)-(101), allow one to draw conclusions to some extent analogous to points 1-3 of Sec. [V] using PN frames instead of inertial frames:

1. If, in a PN frame, the right-hand side of Eqs. (102) or (103) is non-zero, then \( H_{ij} \neq 0 \Rightarrow \mathbb{H}_{ij} \neq 0 \) in that frame (it can be zero only on 2-surfaces or lower-dimensional sets).

2. If there exists a PN frame where \( \vec{J} = 0 \) everywhere then \( \mathbb{H}_{ij} = H_{ij} = 0 \) everywhere, and so \( \text{curl} \cdot \text{R} = \mathbb{R} = 0 \) everywhere. Example: system of \( N \) point masses; if there exists a PN frame where they are all at rest then \( \mathbb{H}_{ij} = H_{ij} = 0 \) and \( \text{curl} \cdot \text{R} = \mathbb{R} = 0 \) everywhere, cf. Eq. (105) below.

3. Observation 2 is guaranteed only for PN frames, where (100) holds. For arbitrary frames, the vorticity and shear/expansion of the observer congruence can be arbitrarily large; the terms involving them in (95)-(96) can no longer be neglected, and act as sources for \( H_{ij} \). Example: gravitational field generated by a spinning body; in the frame co-rotating with the body there are no mass-currents \( (\vec{J} = 0) \), but, in spite of that, \( \text{curl}H_{ij} \neq 0 \) and \( \nabla^2 H_{ij} \neq 0 \) generically, implying \( \mathbb{H}_{ij} = H_{ij} = 0 \) and also \( \text{curl} \cdot \text{R} = \mathbb{R} = 0 \) generically.

4. The converse of 2 is not true: when there is no PN frame where \( \vec{J} = 0 \) everywhere, that does not necessarily mean that \( \text{curl} \cdot \text{R} = \mathbb{R} = 0 \) or that \( H_{ij} \) cannot vanish in some region with respect to some frame. Examples: the 2-body system of Sec. [V.B] or the spinning body of Sec. [V.C] although there are no PN frames where both bodies are/the whole body is at rest, still \( \text{curl} \cdot \text{R} = \mathbb{R} = 0 \) in the orbital/equatorial plane, and \( \mathbb{H}_{ij} = H_{ij} = 0 \) with respect to certain observer congruences that do not correspond to PN frames.

Point 1 is the statement that either \( \nabla^2 H_{ij} \neq 0 \) or \( \text{curl}H_{ij} \neq 0 \) imply that \( H_{ij} \) cannot vanish in an open 3-D spatial set. Point 2 follows directly from Eqs. (100)-(101); one may check also that it is consistent with Eqs. (102)-(103) by noting that, when \( \vec{J} = 0 \) everywhere in a PN frame, \( [\mathcal{E}_{ij}]_N = 0 \), cf. Eqs. (101), so that indeed \( \text{curl}H_{ij} = \partial_j H_{i,j} = 0 \).

Regarding point 3 one observes that, for arbitrary frames the different terms involving \( \sigma_{ij}, \theta, \omega, \) and \( \epsilon_{ij} \) in Eqs. (105) cannot in general be neglected. It follows that when \( \vec{J} = 0 \), one can still have \( \text{curl}H_{ij} \neq 0 \) and/or \( \nabla^2 H_{ij} \neq 0 \), implying \( H_{ij} \neq 0 \Rightarrow \mathbb{H}_{ij} \neq 0 \) in any open 3-D spatial set. In general this will also imply \( \text{curl} \cdot \text{R} \neq 0 \). In the case of the stationary gravitational field of a spinning body, from the point of view of the frame rigidly co-rotating \( ^{18} \) with it there are no mass-currents, \( \vec{J} = 0 \); moreover \( \sigma_{ij} = \theta = 0 \) (since the frame is rigid), \( \dot{\mathcal{E}}_{ij} = \pi_{ij} = 0 \) (since the setup is stationary in this frame), and, outside the body, \( \rho = p = \pi_{ij} = 0 \). But still \( \text{curl}H_{ij} = \omega^m \epsilon_{i\ell k} (\mathcal{E}^k_j)^m - 2\omega^k \epsilon_{\ell\ell j} H_{ij}^m \) and \( \nabla^2 H_{ij} = -3\omega^j \mathcal{E}_{ij} \), which are generically non-zero, implying \( \mathbb{H}_{ij} = H_{ij} \neq 0 \) and also \( \text{curl} \cdot \text{R} \neq 0 \) generically, as we shall see explicitly in Sec. [V.C] (Eq. (121) therein).

Regarding point 4 it is also worth mentioning that it is possible, even in a region where \( \vec{J} \neq 0 \), to have \( \text{curl} \cdot \text{C} = 0 \) and \( \mathbb{H}_{ij} = H_{ij} \neq 0 \) with respect to some observers; an example is the Van Stockum interior solution, corresponding to

\[ \omega^m \epsilon_{i\ell k} (\mathcal{E}^k_j)^m - 2\omega^k \epsilon_{\ell\ell j} H_{ij}^m \]
an infinitely long and rigidly rotating cylinder of dust; it is shown in [93] that there is a region in the wall of the cylinder (the inner cylinder) where there are observers for whom \( \mathcal{H}_{\alpha\beta} = 0 \), in analogy with the situation for the magnetic field within a rotating charged cylinder discussed in Sec. IV D. This is consistent with Eqs. (95)–(96), as in a region where \( \mathcal{J} \neq 0 \) we can still have (depending on the kinematical quantites of the chosen frame) \( \text{curl}\mathcal{H}_{ij} = \mathcal{J}_j \), i.e., everywhere there are observers for which \( \mathcal{H}_{\alpha\beta} = 0 \). The same does not apply however to \( \mathcal{H}_{\alpha\beta} \), which is always non-zero when \( \mathcal{J} \neq 0 \) by virtue of Eq. (97).

Systems of \( N \) point masses. — Systems of point masses are of special interest in this work; for such systems the quantities of the chosen frame) curl \( \mathcal{J} \neq 0 \), as in a region where \( \mathcal{J} \neq 0 \) we can still have (depending on the kinematical quantities of the chosen frame) \( \text{curl}\mathcal{H}_{ij} = \mathcal{J}_j \), i.e., everywhere there are observers for which \( \mathcal{H}_{\alpha\beta} = 0 \). The same does not apply however to \( \mathcal{H}_{\alpha\beta} \), which is always non-zero when \( \mathcal{J} \neq 0 \) by virtue of Eq. (97).

\[ w = \sum_a \frac{M_a}{r_a} \left( 1 + 2\nu^2 - \sum_{b \neq a} \frac{M_b}{r_{ab}} - \frac{1}{2} \vec{a}_a \cdot \vec{v}_a - \left( \frac{\vec{r}_a \cdot \vec{v}_a}{2r_a^2} \right)^2 \right) \]

\[ \mathcal{A} = -4 \sum_a \frac{M_a}{r_a} \vec{v}_a; \quad \mathcal{U} = \sum_a \frac{M_a}{r_a} \]

which reads, in Schwarzschild coordinates,

\[ ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2. \]

The observers at rest in this coordinate system are the Killing or “static” observers \( u \propto \partial/\partial t \), which may be thought of as rigidly attached to the asymptotic inertial rest frame of the source.

This spacetime is of Petrov type D (everywhere); thus the third condition in (98) is satisfied everywhere, and one only has to worry about the quadratic invariants, which have the structure:

\[ \frac{R}{8} = E^\gamma E_{\alpha\gamma} - h^\alpha\gamma h_{\alpha\gamma} = \frac{6M^2}{r^5} > 0 \]

\[ \frac{\mathcal{R}}{16} = \mathcal{E}^\gamma \mathcal{H}_{\alpha\gamma} = 0 \] (everywhere)

Thus this is (everywhere) a purely electric spacetime, i.e., everywhere there are observers for which \( \mathcal{H}_{\alpha\beta} = 0 \), see Sec. III A. Their 4-velocities \( u^{\alpha} \) are obtained from Eqs. (12), (14), (44), (47), and (48). We have, for the auxiliary quantities involved, \( \mathcal{P}^\gamma = 0 \Rightarrow t^\alpha = u^\alpha \), \( \mathcal{J} = \mathcal{A} = -6M^3/r^3 \), \( \lambda = M/r^3 \), \( e^\alpha = -\sqrt{1 - 2M/r}\delta^\alpha_r \); therefore, by (12),

\[ u^{\alpha} = (u^0, u^r, 0, 0) \]

with \( u^r \) arbitrary (under the normalization condition \( u^\alpha u_\alpha = -1 \)). That is, observers which are either static or moving radially, in analogy with the situation in the analogous electromagnetic system of Sec. IV A. One might check these results by computing explicitly \( \mathcal{H}_{\alpha\beta}' \) for an arbitrary \( u^\alpha \), as done in (91). Note that the fact that \( \mathcal{H}_{\alpha\beta} = 0 \) for the static observers means that it globally vanishes in a rigid frame.

One can also get intuition on why \( \mathcal{E}^\gamma \mathcal{H}_{\alpha\gamma} \) remains zero for any observer from arguments analogous to those that explain why \( E^\alpha B_\alpha = 0 \) for the point charge; namely the

A. One single point mass

Drawing a parallel with Sec. IV we will start by studying the invariants of the gravitational field produced by a single point mass. This is the gravitational field effectively involved in the translational form of gravitomagnetism detected in the observations of the binary system\(^{19}\) PSR 1913 +16 (the Hulse-Taylor binary pulsar) [3]. It also describes the relevant contribution to the geodetic precession measured in different systems: the precession of the Earth-Moon system along its orbit around the Sun, detected in the analysis of Lunar Laser Ranging (LLR) data [10,12], the geodetic precession of the gyroscopes in the Gravity Probe-B [13], and the precession of the pulsar’s spin vector in the binary systems PSR J0737−3039 A/B [14] and PSR B1534+12 [15].

The metric is described by the Schwarzschild solution, which reads, in Schwarzschild coordinates,

\[ ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2. \]

The observers at rest in this coordinate system are the Killing or “static” observers \( u \propto \partial/\partial t \), which may be thought of as rigidly attached to the asymptotic inertial rest frame of the source.

This is (everywhere) a purely electric spacetime, i.e., everywhere there are observers for which \( \mathcal{H}_{\alpha\beta} = 0 \), see Sec. III A. Their 4-velocities \( u^{\alpha} \) are obtained from Eqs. (12), (14), (44), (47), and (48). We have, for the auxiliary quantities involved, \( \mathcal{P}^\gamma = 0 \Rightarrow t^\alpha = u^\alpha \), \( \mathcal{J} = \mathcal{A} = -6M^3/r^3 \), \( \lambda = M/r^3 \), \( e^\alpha = -\sqrt{1 - 2M/r}\delta^\alpha_r \); therefore, by (12),

\[ u^{\alpha} = (u^0, u^r, 0, 0) \]

with \( u^r \) arbitrary (under the normalization condition \( u^\alpha u_\alpha = -1 \)). That is, observers which are either static or moving radially, in analogy with the situation in the analogous electromagnetic system of Sec. IV A. One might check these results by computing explicitly \( \mathcal{H}_{\alpha\beta}' \) for an arbitrary \( u^\alpha \), as done in (91). Note that the fact that \( \mathcal{H}_{\alpha\beta} = 0 \) for the static observers means that it globally vanishes in a rigid frame.

One can also get intuition on why \( \mathcal{E}^\gamma \mathcal{H}_{\alpha\gamma} \) remains zero for any observer from arguments analogous to those that explain why \( E^\alpha B_\alpha = 0 \) for the point charge; namely the

\[ 19 \text{ Even though the binary system is a two-body system, the effect being measured is the influence of the translational gravitomagnetic field produced by one body (playing the role of the source) on the motion of the other body. Hence, in what pertains to this effect, the system may effectively be regarded as a one-body (the source) system, the other body being the test particle.} \]
formal similarity between the transformation laws. Let \( O(u) \) and \( O'(u') \) be, respectively, a static observer and an observer moving relative to it with some velocity \( \vec{v} \). For \( O(u) \), \( \mathbb{H}_{\alpha\beta} = 0 \Rightarrow E_{\alpha\gamma}^\prime \mathbb{H}_{\alpha\gamma} = 0 \). The moving observer \( O'(u') \) will in turn measure a non-vanishing gravitomagnetic tidal tensor, \( \mathbb{H}'_{\alpha\beta} \neq 0 \), but it will be such that it is always “orthogonal” to the gravitoelectric tidal tensor \( E_{\alpha\beta} \), in analogy with the situation for the magnetic field [82]. In order to see this, first observe, from the decompositions [21]-[22], that with respect to the congruence of static observers \( (u^a = \phi^a \delta^a_{\phi}) \), the Riemann tensor and its dual are completely described by the electric part:

\[
R_{\alpha\beta} = 4 \left\{ 2u_{[a}u^{[b} + g_{[b]}^{\gamma} E_{b]}^{\delta} \right\};
\]

\[
^{\star}R_{\alpha\beta} = 2\epsilon_{\alpha\beta\gamma} E^{\lambda\gamma} u^{a\gamma} + 2\epsilon^{\gamma\delta\rho\sigma} E_{\lambda[a} u_{\beta]} u^{b\gamma} u^{c\delta} E^{d\rho\sigma}.
\]

Hence, to linear order, the space components of \( E_{\alpha\beta} \) and \( E_{\alpha\beta}^\prime \) yield the tidal tensors \( \xi_{\alpha\beta} = E_{\alpha\beta} \mathbb{H}_{\alpha\beta} = \mathbb{H}_{\alpha\beta}^\prime E_{\alpha\beta} = \mathbb{H}_{\alpha\beta}^\prime E_{\alpha\beta}^\prime \) (using dyadic notation, see point 4 of Sec. IA),

\[
\xi_{\alpha\beta} = \frac{\ddot{\xi}_{\alpha\beta}}{\ddot{\xi}_{\alpha\beta}} \simeq \mathbf{v} \times \ddot{v}; \quad \ddot{\xi}_{\alpha\beta} = \ddot{\xi}_{\alpha\beta} \simeq \mathbf{v} \times \ddot{v} - \ddot{v} \times \ddot{v},
\]

which have formal similarities with Eqs. (82)-83, and, together with \( \mathbb{H}_{\alpha\beta} = \mathbb{H}_{\alpha\beta}^\prime = 0 \), lead immediately to \( E_{\alpha\beta}^\prime \mathbb{H}_{\alpha\beta}^\prime = E_{\alpha\beta} \mathbb{H}_{\alpha\beta} = E_{\alpha\beta} \mathbb{H}_{\alpha\beta}^\prime = 0 \). The verification using the exact expressions for \( E_{\alpha\beta}, \mathbb{H}_{\alpha\beta} \) is also straightforward.

B. Two bodies, coplanar motion — the Earth-Sun system

We consider here the gravitational field generated by two bodies — the Earth and the Sun — orbiting each other, whose (translational) gravitomagnetic effects are implied in [17]-[22]. The metric, accurate to first post-Newtonian order, is, cf. Eqs. (98) and (104).

\[
g_{\alpha\beta} = -1 + 2 \frac{M_\odot}{r_\odot} + 4 \frac{M_\oplus}{r_\oplus} - \frac{M_\odot \vec{v}_\odot \cdot \vec{r}_\odot}{r_\odot^2} - \frac{M_\oplus \vec{v}_\oplus \cdot \vec{r}_\oplus}{r_\oplus^2} - 2 \frac{M_\odot M_\oplus}{r_\odot r_\oplus} \quad \text{or}
\]

\[
g_{\alpha\beta} = \left[ 1 + 2 \frac{M_\odot}{r_\odot} \right] \delta_{\alpha\beta} + \frac{M_\odot}{r_\odot} \vec{a}_\odot \cdot \vec{a}_\odot \quad \text{or}
\]

\[
g_{\alpha\beta} = \left[ 1 + 2 \frac{M_\oplus}{r_\oplus} \right] \delta_{\alpha\beta} + \frac{M_\oplus}{r_\oplus} \vec{a}_\oplus \cdot \vec{a}_\oplus
\]

where \( \odot \equiv \text{Sun}, \oplus \equiv \text{Earth} \). Since \( E_{\alpha\beta} \sim O(2) \) and \( \mathbb{H}_{\alpha\beta} \sim O(3) \), cf. Eqs. (99)-(100), then, generically \( E_{\alpha\gamma}^\prime \mathbb{H}_{\alpha\gamma} \sim O(3) \), i.e., \( \mathbf{R} \cdot \mathbf{R} > 0 \). In this respect we note that the region of magnetic dominance that exists between the two charges in the electromagnetic system of Sec. IVB (around the point where \( \vec{E} = 0 \)), has no counterpart in the present gravitational system, because here we are dealing with tidal tensors, not vector fields, and these add differently. Namely, in the region corresponding to that where, in the electromagnetic system, the electric fields cancel out, the tidal tensors \( E_{\alpha\beta} \) of each body add up instead (in electromagnetism this is analogous instead to the situation with the electric tidal tensor, as defined in [46]-[56]). As for the Chern-Pontryagin invariant \( \mathbb{R} \cdot \mathbf{R} = 16 E_{\alpha\beta}^\prime \mathbb{H}_{\alpha\beta} \), one has, to lowest order, cf. Eqs. (99)-(100), (102)-(105).

\[
E_{\alpha\beta}^\prime \mathbb{H}_{\alpha\beta} = U_{i:j} \left\{ \frac{1}{2} \epsilon_{i\lambda k} A_{k,ij} + \epsilon_{ij} k U_k \right\} + O(7)
\]

\[
eq 6 \left( \sum_{a=\odot,\oplus} \frac{M_a}{r_a} \right) \left[ \sum_{a=\odot,\oplus} \frac{M_a}{r_a^3} (\vec{r}_a \times \vec{v}_a) (i(r_a)_j) \right]
\]

\[
eq 18 M_\odot M_\oplus \left( \vec{r}_\odot \times \vec{v}_\odot \right) \left( \vec{r}_\oplus \times \vec{v}_\oplus \right) \left( \vec{r}_\odot \times \vec{v}_\odot \right) \left( \vec{r}_\oplus \times \vec{v}_\oplus \right)
\]

which agrees with Eq. (7) of [2]. This invariant (namely the expressions between square brackets) exhibits formal similarities with the electromagnetic invariant [87]; it has the structure

\[
E_{\alpha\gamma}^\prime \mathbb{H}_{\alpha\gamma} = 0 \quad \text{in the orbital plane;}
\]

\[
\neq 0 \quad \text{generically.}
\]

That \( E_{\alpha\gamma}^\prime \mathbb{H}_{\alpha\gamma} = 0 \) in the orbital plane \( \Sigma \) can be seen observing that, when the point of observation \( P \) lies on \( \Sigma \), then \( (\vec{r}_\odot, \vec{r}_\oplus) \in \Sigma \); and since also \( \{ \vec{v}_\odot, \vec{v}_\oplus \} \in \Sigma \) (always), it follows that \( (\vec{v}_\odot \times \vec{r}_\odot) \cdot \vec{r}_\oplus = (\vec{v}_\odot \times \vec{r}_\odot) \cdot \vec{r}_\oplus = 0 \), implying \( E_{\alpha\gamma}^\prime \mathbb{H}_{\alpha\gamma} = 0 \). This structure is analogous to that of [87] for coplanar motion, except that here the factor \( (\vec{r}_\odot \times \vec{r}_\oplus) \), which has no electromagnetic counterpart, introduces an additional 1-D region where \( E_{\alpha\gamma}^\prime \mathbb{H}_{\alpha\gamma} = 0 \) (the circle determined by \( \vec{r}_\odot \perp \vec{r}_\oplus \)). The existence of observers for which \( \mathbb{H}_{\alpha\beta} = 0 \) on \( \Sigma \) can also be understood in analogy with the electromagnetic apparatus of Sec. IVB. Let us compute their velocities. The gravitomagnetic tidal tensor measured by an observer \( O' \) moving with velocity \( \vec{v}' \) with respect to the chosen PN frame is, cf. Eq. (105).

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20 To obtain Eq. (7) of [2] from Eq. (112) above, one makes \( \vec{v}_\odot = 0 \) (as in [2] an heliocentric reference frame is used), and notes that \( \vec{r}_\odot \) and \( \vec{r}_\oplus \) read, in the notation therein, respectively, \( \vec{r}_M \) and \( \vec{r}_M^\prime \).
\[
\mathbb{H}'_{ij} = 6 \left[ \frac{M_\odot}{r_\odot^5} (\vec{r} \times [\vec{v} - \vec{a}])_i (\vec{r} \times [\vec{v} - \vec{a}])_j + \odot \leftrightarrow \odot \right] .
\] (113)

Choose, for convenience, the PN frame comoving with the center of mass (CM) of the Earth-Sun system (which is close to the “barycentric” reference frame considered in e.g. [16, 17, 19], or to the heliocentric system in [2]), and take \( z = 0 \) to be the orbital plane \( \Sigma \). Firstly one observes that, in order for \( \mathbb{H}'_{ij} = 0 \), the observer’s velocity \( \vec{v} \) must be parallel to \( \Sigma \), except at some special points on \( \Sigma \), in analogy with the electromagnetic problem in Sec. IV B (coplanar motion). This implies \( \vec{r} \times [\vec{v} - \vec{a}] = (\vec{r} \times [\vec{v} - \vec{a}])^2 \hat{e}_z \), and thus trivially \( \mathbb{H}'_{zz} = 0 \), \( \mathbb{H}'_{z\alpha} = \mathbb{H}'_{\alpha z} = 0 \), whose vanishing amounts to the conditions
\[
\vec{v} \times \vec{a}^{(i)} = \vec{b}^{(i)}, \quad i = x, y, \tag{114}
\]
where the vectors \( \vec{a}^{(i)} \) and \( \vec{b}^{(i)} \) are defined by
\[
\vec{a}^{(i)} = \frac{r^i}{r^5} \frac{M_\odot}{r_\odot} \vec{r}, \quad \vec{b}^{(i)} = \frac{r^i}{r^5} \frac{M_\odot}{r_\odot} \vec{v} \times \vec{r}. \tag{115}
\]
and the solution of (114) splits into two cases:

1. Observer on Earth-Sun axis (\( \vec{r} \times \vec{r}_\odot \neq 0 \)). By (115) and \( \vec{a}^{(x)} \) and \( \vec{a}^{(y)} \) span in this case the orbital plane, therefore one may write \( \vec{v} = \lambda \vec{a}^{(x)} + \mu \vec{a}^{(y)} \); substituting into (114) readily gives the unique solution
\[
\vec{v} = \frac{r^2_\odot}{M_\odot M_\odot} \frac{r^5_\odot}{||\vec{r}^2_\odot \times \vec{r}^5_\odot||^2} \left( (b^{(y)})^\perp \vec{a}^{(x)} - b^{(x)} \vec{a}^{(y)} \right), \tag{116}
\]
where \((b^{(i)})^\perp = \vec{b}^{(i)} \cdot \hat{e}_z \). Hence, at each point of \( \Sigma \) off the Earth-Sun axis, there is a unique observer moving parallel to \( \Sigma \) for which \( \mathbb{H}_{\alpha \beta} = 0 \). This is in contrast with the electromagnetic analogue in Sec. IV B (coplanar motion), where the velocity of the observers measuring \( \vec{B}' = 0 \) had an arbitrary component along the electric field (hence there was an infinite number of such observers at each point). One can say that the situation is similar (in this respect) to purely electric exact solutions of the general Petrov type I.

2. Observer on Earth-Sun axis (\( \vec{r} \times \vec{r}_\odot = 0 \)), depicted in Fig. 6. In this case \( \vec{a}^{(x)} / r^5_\odot = \vec{a}^{(y)} / r^5_\odot \equiv \vec{V} \) and \( \vec{b}^{(x)} / r^5_\odot = \vec{b}^{(y)} / r^5_\odot \equiv \vec{W} \), such that (114) reduces to the single equation \( \vec{v} \times \vec{V} = \vec{W} \). Clearly, the component \( \vec{v}_{||V} \) of \( \vec{v} \) parallel to \( \vec{V} \) is arbitrary; for the orthogonal component one obtains (taking a cross product with \( \vec{V} \))
\[
\vec{v}_{\perp V} = \frac{\vec{V} \times \vec{W}}{V^2} = \frac{M_\odot (r_\odot^3 - r_\odot^3)}{r_\odot^3 M_\odot + M_\odot r_\odot^3} \vec{v}_\odot, \tag{117}
\]
where we noted that \( \vec{V} \) is parallel to the Earth-Sun axis, and the approximation equality follows from the fact that, along the axis, \( \vec{V} \cdot \vec{V} = \vec{V} \cdot \vec{V} = 0 \), and that \( M_\odot \vec{v}_\odot \approx -M_\odot \vec{v}_\odot \) (since the system’s momentum vanishes in the CM frame). We note moreover that, along the axis, the super-Poynting vector as measured by the rest observers, \( \mathcal{P}^i = \epsilon_{ijk} \mathbb{H}^{jk}_{||V} \), reads
\[
\mathcal{P}^i = 9 \left[ \frac{M_\odot^2}{r_\odot^6} \vec{b}^i + \frac{M_\odot^2}{r_\odot^6} \vec{v}^i + \frac{M_\odot^2}{r_\odot^6} (\vec{v}^i + \vec{v}_\odot^i) \right] \vec{v}_\odot = \frac{3M_\odot v_\odot r_\odot^6}{M_\odot^2} \vec{v}_\odot, \tag{118}
\]
which is parallel to \( \vec{v}_{\perp r_\odot} \). This means that \( \vec{v}_{\perp r_\odot} \) is in fact the component of \( \vec{v} \) parallel to \( \vec{P} \); therefore, along the axis, the situation is similar to a purely electric Petrov type D exact solution (and to the electromagnetic case): at each point a class of observers exists for which \( \mathbb{H}_{\alpha \beta} = 0 \); such observers have a velocity consisting of a component \( \vec{v}_{||P} = \vec{v}_{\perp r_\odot} \) along \( \vec{P} \) fixed by Eq. (117), plus an arbitrary component \( \vec{v}_{\perp r_\odot} \) parallel to the Earth-Sun axis.

Using \( M_\odot \gg M_\odot \) we have \( r_\odot \approx 0 \); considering moreover an observation point much farther than the Earth-Sun distance \( r \gg r_\odot \), as depicted in Fig. 6 we obtain, in the special case where \( \vec{v} \) has no component along the axis \( (\vec{v} = \vec{v}_{\perp r_\odot} = \vec{v}_{||P} ) \), the limit
\[
\lim_{r \gg r_\odot} v \geq \frac{3M_\odot v_\odot r_\odot^6}{M_\odot^2} = \frac{3J}{M_\odot r}, \tag{119}
\]
where we noted that, to lowest order (which is the accuracy needed for \( \vec{v} \) in Eq. (113)), \( J = M_\odot v_\odot r_\odot \odot \) is the system’s angular momentum as measured in the center of mass PN frame. This is analogous to the situation in the electromagnetic problem of Sec. IV B and the velocity field (10). The analogy can be made even closer by considering the gravitational counterpart of the system in Fig. 4 i.e., two particles with the same mass.
$M_1 = M_2 = M_{\text{tot}}/2$ and with velocities $\vec{v}_1$ and $-\vec{v}_1$, orbiting each other (no “rod” is necessary in this case) in a circular motion of radius $d$. The velocities of the observers $O'$ (at points $P$ along the axis) for which $\mathbb{H}'_{ij} = 0$ are obtained from (117) setting $M_\odot = M_\odot = M_{\text{tot}}/2$, $|\vec{x}_\odot| = |\vec{x}_\odot| = d$, $v_\odot = v_1$, leading to

$$v = v_1 \frac{(3d^2 + d^3)}{r^3 + 3d^2r} \varepsilon \frac{3v_1d}{r} = \frac{3J}{M_{\text{tot}}r}, \quad (119)$$

where, again, we identified $J = v_1 M_{\text{tot}} d$. This is similar (up to a factor 3/2), for large $r$, to the velocity (90) for which $\vec{B}' = 0$ in the electromagnetic system. The reason why $\mathbb{H}'_{ij} = 0$ for these observers (and not for others) can also be understood by a reasoning analogous to the one we made at the end of Sec. [IV-B]. $\mathbb{H}_{ij}$ is the superposition of the individual gravitomagnetic tidal tensors produced by each body, cf. Eq. (113), which, for the setup analogous to Fig. [IV-B] has non-vanishing components $\mathbb{H}_{zy} = \mathbb{H}_{zy} = (\mathbb{H}_{1})_{zy} + (\mathbb{H}_{2})_{zy}$. The contributions $(\mathbb{H}_{1})_{zy}$ and $(\mathbb{H}_{2})_{zy}$ have opposite signs since $v_1^z = -v_2^z$. Thus, for an observer at rest in the system’s CM frame ($v = 0$), $|\mathbb{H}_{zy}| > |(\mathbb{H}_{1})_{zy}|$, since body 2 is closer to the observer. Increasing the observer’s velocity $v$ (in the sense of the orbital motion) means decreasing $|\mathbb{H}_{zy}|$ whilst increasing $|(\mathbb{H}_{1})_{zy}|$, so that they eventually cancel out, $\mathbb{H}'_{ij} = 0$. These similarities with electromagnetism can be traced back to the facts that, to first post-Newtonian order, $\mathbb{H}_{ij}$, Eq. (105), is linear (so a superposition principle applies just like in electromagnetism), and has a dependence on the velocities of the sources (and transformation laws under a change of PN frame, Eq. (108), that are, to some extent, also analogous to their electromagnetic counterparts.

Finally, we note that this application exemplifies point [4] of Sec. [V] although there is no PN frame where both bodies are at rest, still $\star \mathbf{R} \cdot \mathbf{R} = 0$ in a 2-D region (the orbital plane), where $\mathbb{H}'_{ij} = 0$ for certain families of observers (which do not form PN frames).

### C. The gravitational field of a spinning body

The gravitational field of a compact, spinning body of mass $M$ and angular momentum $J$, whose center of mass is at rest in the given PN frame is, to 1PN order, obtained by substituting $w = U = M/r$, $\vec{A} = 2\vec{r} \times J/r^3$ into Eqs. (98), see e.g. [95, 96]. This coincides with the 1PN limit of the Kerr solution (in isotropic coordinates), which is the field we shall consider here, since this is an exact solution well suited to our methods. Its well-known form in Boyer-Lindquist coordinates is

$$ds^2 = -\frac{\Delta}{\Sigma} (dt - a \sin^2 \theta d\phi)^2 + \frac{\Delta}{\Sigma} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} (adt - (r^2 + a^2) d\phi)^2,$$

where

$$\Delta \equiv r^2 - 2Mr + a^2, \quad \Sigma \equiv r^2 + a^2 \cos^2 \theta, \quad a \equiv \frac{J}{M}.$$

This spacetime is of Petrov type D, so the third condition in (38) is satisfied everywhere; hence it suffices to study the quadratic invariants, which read [90, 99, 100]

$$\mathbf{R} \cdot \mathbf{R} = \frac{48M^2}{\Sigma^6} (r^2 - a^2 \cos^2 \theta)(\Sigma^2 - 16r^2 a^2 \cos^2 \theta);$$

$$\star \mathbf{R} \cdot \mathbf{R} = \frac{96M^2 r a}{\Sigma^6} (3r^2 - a^2 \cos^2 \theta)(r^2 - 3a^2 \cos^2 \theta) \cos \theta.$$

(120)

The structure of these invariants is graphed in Fig. 7. The zeros of $\mathbf{R} \cdot \mathbf{R}$ occur on the shells $r = \pm a \cos \theta$ and $r = \pm (2 \pm \sqrt{3})a \cos \theta$, signaling transitions between regions of electric ($\mathbf{R} \cdot \mathbf{R} > 0$) vs. magnetic ($\mathbf{R} \cdot \mathbf{R} < 0$) dominance. The zeros of $\star \mathbf{R} \cdot \mathbf{R}$ define purely electric/magnetic surfaces and occur for $\theta = \pi/2$ and $r = \pm a \cos \theta/\sqrt{3}$ (purely electric) and $r = \pm \sqrt{3}a \cos \theta$ (purely magnetic). Except for the (purely electric) equatorial plane, all these surfaces lie either inside the event horizon ($r \leq r_+$, $r_+ = M + \sqrt{M^2 - a^2}$) or, in the case of the larger shells (given by $r = \pm (2 + \sqrt{3})a \cos \theta$ when $\mathbf{R} \cdot \mathbf{R} = 0$, and by $r = \pm \sqrt{3}a \cos \theta$ when $\star \mathbf{R} \cdot \mathbf{R} = 0$), they may, for large enough values of $a$, lie partly outside the horizon,22 yet

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22 In order to see this, one observes that, for the largest blue dashed circle in Fig. 7 (the circle $r = \pm (2 + \sqrt{3})a \cos \theta$), the non-extreme condition $a/M < 1$ implies $r_{\text{max}}/r_+ < 2 + \sqrt{3}$, where $r_{\text{max}}$ is the maximum value of the coordinate $r$ along the circle. These regions shall be discussed in detail elsewhere.
still very close to it. Hence, in the astrophysical applications under discussion, which pertain to the “post-

Newtonian zone” \( \mathcal{M} \), where \( r \gg r_{+} \), we have \( \mathbf{R} \cdot \mathbf{R} > 0 \)
everywhere, and the only surface where \( \mathbf{R} \cdot \mathbf{R} = 0 \) is the (purely electric) equatorial plane:

\[
\begin{align*}
\mathop{E^\gamma E_\gamma} - \mathop{H^\gamma H_\gamma} &= \frac{6M^2}{r^6} + O(6) > 0 ; \\
\mathop{E^\gamma H_\gamma} &= \cos \theta \left[ \frac{18JM}{r^7} + O(7) \right] \quad (= 0 \text{ for } \theta = \frac{\pi}{2}), \\
\end{align*}
\]

which is a structure formally analogous to the electromagnetic counterpart \( \mathcal{M} \). The equatorial plane being purely electric means that there are therein observers for which \( \mathbb{H}_{\alpha \beta} = 0 \). Their 4-velocities \( u^\alpha \) are obtained from Eqs. (42), (40), (44), (47), and (48). For the auxiliary quantities involved, we have, in the equatorial plane, \( \mathbf{R} \cdot \mathbf{R} = 48M^2/r^6, \xi = 3M^2/(2r^6) \),

\[
\zeta_{\Xi} = \frac{(\Delta + a^2)}{(\Delta - a^2)^2}, \quad \rho_\alpha = \frac{9aM^2(\Delta + a^2)}{2r^5(\Delta - a^2)^{5/2}} \delta_{\alpha}^{\delta},
\]

\( \mathcal{J} = \Lambda = -6M^3/r^9, \lambda = M/r^3, \) and so

\[
t^\alpha = u^\alpha + \frac{2\rho_\alpha}{\Delta(\Delta - a^2)} = t^0 \left[ \delta_{\alpha}^{0} + \frac{a}{a^2 + r^2} \delta_{\alpha}^{\delta} \right],
\]

\[
e^\alpha = -\frac{\Delta}{r^2(\Delta - a^2)^{5/2}} \delta_{\alpha}^{\delta},
\]

where \( u^\alpha = (-g_{00})^{-1/2} \delta^\alpha_0 \) and \( t^0 = (r - 2M)(a^2 + r^2)\Delta^{-1}(\Delta - a^2)^{-1/2} \). Therefore, by (42),

\[
u^\alpha = (u^0, u^r, 0, 0, a^2 + r^2 u^0), \quad (122)
\]

corresponding to observers with angular velocity

\[
 \frac{d\phi}{dt} = \frac{u^\phi}{u^0} = \frac{a}{a^2 + r^2} \quad (123)
\]

and an arbitrary radial velocity \( dr/dt = u^r/u^0 \) (subject only to the normalization condition \( u^\alpha u^\alpha = -1 \)). One could check these results by computing explicitly \( \mathbb{H}_{\alpha \beta} \) (for an arbitrary \( u^\alpha \)) as done in (111). In the special case \( u^r = 0 \), one obtains the observers depicted in Fig. 8, which coincide with the so-called “Carter canonical observers” (e.g. [101]).

Notice the similarity with the velocity field in Fig. 5 which makes the magnetic field vanish in the analogous electromagnetic problem: both velocities depend only on \( r \) and on the ratio \( a = J/M \), and asymptotically they match up to a factor of 2. Note also the similarities with the velocity fields (118) or (119) for which \( \mathbb{H}_{\alpha \beta} = 0 \) in systems of two bodies orbiting each other: in the post-

Newtonian regime, \( r \gg r_{+} \Rightarrow r \gg a \) and, from Eq. (123), \( v \approx a/r \equiv J/(Mr) \); hence, for large \( r \), the velocities match up to a factor of three. The vanishing of \( \mathbb{H}_{\alpha \beta} \) for such observers can also be understood (in the PN regime) by the same reasoning we made in Sec. V B by thinking about the rotating body as a set of translating elements and adding up their individual gravitomagnetic tidal tensors. This parallels what happens in electromagnetism, where the vanishing of \( \mathbf{B} \) for some observers in the equatorial plane of a spinning charge, Fig. 5, can be explained by the same reasoning that explains its vanishing in the motion plane of the system of two charges in Fig. 4. One thus concludes that, although very different from a system of one single point source of Sec. V A a spinning body is not, in the PN regime, substantially different from the two-body systems of Sec. V B in what pertains to the structure of the curvature invariants (and the existence of observers for which \( \mathbb{H}_{\alpha \beta} = 0 \)).

Finally, we note that the velocity field in Fig. 8 provides another example of point 4 of Sec. V although there is no PN frame where all the mass currents are zero, still \( \mathbf{R} \cdot \mathbf{R} = 0 \) in a 2-D spatial surface (the equatorial plane), where \( \mathbb{H}_{\alpha \beta} = 0 \) for certain observer congruences that are not PN frames.

VI. NON-VACUUM EXAMPLES

A. Doubly aligned electrovacuum spacetimes — the Kerr-Newman solution

In electrovacuum gravitational fields \( T_{\alpha \beta} \) reduces to the stress-energy tensor of the electromagnetic field,
where. In the case where the charge, mass, and electromagnetic field energy densities are uniform, the metric reads (cf. Eq. (28) of [119])

\[ A_i dx^i = \omega r^2 d\phi, \quad h_{ij} dx^i dx^j = dr^2 + r^2 d\phi^2 + dz^2. \]  

Again the Weyl tensor is purely electric and of Petrov type D, with (63) being satisfied for \( \lambda = 2\omega^2/3 \). The analysis is similar to that for the Gödel universe, yielding the same PNDs \( k^{\alpha} \) and \( l^{\alpha} \), and (126) and (127) holding for, respectively, the observers measuring a vanishing \( H^{\alpha}_\beta \) and \( H^{\alpha}_\beta \).

D. Van Stockum cylinder

The van Stockum interior solution (e.g. [70, 98]), describes the gravitational field of an infinite rigidly rotating cylinder of dust. Its line element is of the form (124)
with
\[ A_i dx^i = \omega r^2 d\phi , \quad h_{ij} dx^i dx^j = r^2 d\phi^2 + e^{-\omega^2 r^2} (dr^2 + dz^2) . \]
\( \text{(129)} \)

Therein \( A \cdot C = 0 \), \( M_{\mathcal{C}} = 162e^{4r^4} f(r)/(2 - 4e^{2r^2}) \), where \( f(r) = 1 - 4e^{-2r^2} \); hence \( \textbf{(60)} \) is satisfied for \( \omega > 1/2 \), and so, within this regime, the Weyl tensor is purely electric and of Petrov type I, as is well known \( \text{[58]} \).

This means that, at each point, there is a unique observer for which \( H_{\alpha \beta} = 0 \); the question is whether \( \text{(59ii)} \) is also satisfied for such observer. The Ricci tensor of the perfect fluid type, criterion (b) of Sec. \( \text{[41]} \) and (b-2) of Appendix \( \text{[A.1]} \), there is a unique observer for which \( \text{(59ii)} \) is satisfied, having 4-velocity parallel to \( u^\alpha \); \( u^\alpha \equiv U_{\alpha} = \delta_0^\alpha \).

The physical interpretation is that this is the observer at rest with respect to the dust; all other observers moving with respect to \( u^\alpha \) measure a non-vanishing spatial mass-energy current \( \mathcal{J}^\alpha \). The magnetic part of the Weyl tensor \( H_{\alpha \beta} \equiv \mathcal{C}_{\alpha \beta \mu} U^\mu U^\nu \) as measured by this observer, however, is not zero, having non-vanishing components \( H_{\alpha \beta} = H_{\beta \alpha} = -u^\rho r \). Therefore \( \text{(59ii)} \) is not obeyed, and so observers for which \( \mathcal{E}_{\alpha \beta} = 0 \) do not exist.

VII. DYNAMICAL IMPLICATIONS OF THE INVARIANTS. GRAVITOMAGNETISM.

In the previous sections we made use of the insight that the analogy \( \{ - \mathbf{F} \cdot \mathbf{F}, - e \mathbf{F} \cdot \mathbf{F} \} \leftrightarrow \{ \mathbf{R} \cdot \mathbf{R}, - \mathbf{R} \cdot \mathbf{R} \} \) (see footnote 4) between electromagnetic invariants and gravitational invariants in vacuum gives us to interpret the structure of the latter. It is crucial, however, to realize that this is a purely formal analogy. For in one case one is dealing with quantities built on electromagnetic fields \( E^\alpha, B^\alpha \); in the other case with gravitational tidal tensors \( \mathcal{E}_{\alpha \beta}, \mathcal{H}_{\alpha \beta} \); and these objects do not play analogous dynamical roles. The fields \( E^\alpha \) and \( B^\alpha \) govern effects like the Lorentz force and the precession of a magnetic dipole, and have as closest gravitational counterpart the so-called gravitoelectric \( (G^\alpha) \) and gravitomagnetic \( (H^\alpha) \) inertial fields, governing effects like the (fictitious) inertial force that drives a particle in geodesic motion, or the “precession” of a gyroscope. The tensors \( \mathcal{E}_{\alpha \beta}, \mathcal{H}_{\alpha \beta} \), by contrast, govern gravitational tidal effects, such as the geodesic deviation, the spin-curvature force on a spinning particle, or the differential precession of spinning particles (and their electromagnetic analogues, from a physical point of view, are the electromagnetic tidal tensors \( \mathcal{E}_{\alpha \beta}, B_{\alpha \beta} \), as argued in \( \text{[40, 55]} \)). This means that the use, in some literature \( \text{[27, 100]} \), of the formal analogy between the invariants to infer about effects like the (inertial) gravitomagnetic force on a test particle or gyroscope precession, is not a good physical guiding principle. The effects involved on both sides are different, and may actually be opposite, as we shall exemplify next.

It is likewise crucial to distinguish and understand the relation between the “gravitoelectromagnetic” inertial fields \( G^\alpha \) and \( H^\alpha \) (the ones involved in the frame-dragging effects under debate in the literature) and the electric and magnetic parts of the curvature \( \mathcal{E}_{\alpha \beta}, \mathcal{H}_{\alpha \beta} \), as well as the invariants they form, which we shall also discuss next.

A. A magnetic dipole in the field of a spinning charge vs. a gyroscope in the Kerr spacetime.

The equations of motion for a spinning particle with magnetic moment \( \mu^\alpha \) (and no charge nor electric dipole moment) in an electromagnetic field in flat spacetime are, under the Mathisson-Pirani spin condition (e.g. \( \text{[91]} \)),
\[ \frac{DP^\alpha}{d\tau} = B^\beta \mu_\beta \ ; \quad (a) \quad \frac{D F S^\alpha}{d\tau} = \epsilon^\alpha_{\beta \gamma \delta} U^\beta \mu^\gamma B^\delta \ , \quad \text{(b)} \]
\( \text{(130)} \)

where \( U^\alpha, P^\alpha \) and \( S^\alpha \) are, respectively, the particle’s 4-velocity, 4-momentum, and spin angular momentum 4-vector; \( B^\alpha = *F^\alpha \beta U_\beta \) and \( B_{\alpha \beta} = *F_{\alpha \gamma \beta} U^\gamma \) are, respectively, the magnetic field and “magnetic tidal tensor” \( \mathcal{B}_{\alpha \beta} \) as measured by the particle; \( D/d\tau = U^\alpha \nabla_\alpha \) is the usual (Levi-Civita) covariant derivative, and \( D_F = d/d\tau \) is the Fermi-Walker covariant derivative, which reads, for a spatial vector \( X^\alpha \) \( (X^\alpha U_\alpha = 0) \),
\[ \frac{D F X^\alpha}{d\tau} = \frac{DX^\alpha}{d\tau} - X^\beta B_{\alpha \beta} U^\alpha \ . \]
\( \text{(131a)} \)

The equations of motion for a spinning pole-dipole particle in a gravitational field are (under the same spin condition), e.g. \( \text{[91]} \).
\[ \frac{DP^\alpha}{d\tau} = -\mathcal{H}^\beta S_\beta \ ; \quad (a) \quad \frac{d S^i}{d\tau} = (S \times \Omega)^i \ . \quad \text{(b)} \]
\( \text{(131b)} \)

Equation \( \text{(131b)} \) is the spin-curvature force, which causes the particle to deviate from geodesic motion; it consists of a coupling between \( S^\alpha \) and the gravitomagnetic tidal tensor as measured by the particle, \( \mathcal{H}_{\alpha \beta} = 4 \alpha \beta \rho U^\mu U^\nu \). Equation \( \text{(131b)} \) is the space part of equation \( D_F S^\alpha/d\tau = 0 \) as measured in the particle’s center of mass frame (stating that \( S^\alpha \) is Fermi-Walker transported). The use of a simple derivative in \( \text{(131b)} \) manifests the fact that, by contrast with the Larmor precession in \( \text{(130)} \), the so-called “precession” of a gyroscope is not a covariant, locally measurable effect. Indeed, \( S^\alpha \) is fixed with respect to a comoving, locally non-rotating system of axes (mathematically defined, precisely, as a Fermi-Walker transported frame; for this reason one says that gyroscopes define the local “compass of inertia”, see e.g. \( \text{[27, 112]} \)). The quantity \( \Omega \) in Eq. \( \text{(131b)} \) is thus just the angular velocity of rotation of the spatial axes \( e_i \) of the chosen frame relative to a locally non-rotating one. In the context of the measurement of frame-dragging, the triad \( e_i \) is chosen to be rotationally locked to the “distant stars” (how such frame is constructed is discussed in Sec.
Table I: Opposite effects: magnetic dipoles in the equatorial plane of a spinning charge (where \( \mathbf{\cdot F} \cdot \mathbf{F} = 0, \ -\mathbf{F} \cdot \mathbf{F} > 0 \)) vs gyroscopes in the equatorial plane of a spinning celestial body (where \( \mathbf{\cdot R} \cdot \mathbf{R} = 0, \ \mathbf{R} \cdot \mathbf{R} > 0 \)).

| Magnetic dipole moving with angular velocity | Gyroscope moving with angular velocity |
|---------------------------------------------|---------------------------------------|
| \( \frac{d\phi}{dt} = \frac{a}{2r^2} \) (Fig. 3) | \( \frac{d\phi}{dt} = \frac{a}{a^2 + r^2} \) (Fig. 8) |

| No Larmor precession | Gyroscope precesses: |
|----------------------|----------------------|
| \( \vec{B} = 0 \Rightarrow D\vec{S} = 0 \) | \( d\vec{S} \neq \frac{a}{dt} \neq 0 \) |

| A force acts on it | No force: |
|-------------------|----------|
| \( B_{\alpha\beta} \neq 0 \Rightarrow \frac{DP^\alpha}{d\tau} \neq 0 \) | \( H_{\alpha\beta} = 0 \Rightarrow \frac{DP^\alpha}{d\tau} = 0 \) |

Table VII C below; in such case \( \vec{D} \) yields minus the precession rate of the gyroscope with respect to the distant stars.

If the invariant conditions

\[ \mathbf{\cdot F} \cdot \mathbf{F} = 0, \quad -\mathbf{F} \cdot \mathbf{F} > 0 \]  \hspace{1cm} (132)

are satisfied in some region, then there are observers for which \( B^\alpha = 0 \) everywhere, then by Eq. (130b) means that magnetic dipoles carried by such observers do not undergo Larmor precession. But it tells us nothing, a priori, about the force on the particle. By contrast, what the conditions

\[ \mathbf{\cdot R} \cdot \mathbf{R} = 0, \quad \mathbf{R} \cdot \mathbf{R} > 0 \]  \hspace{1cm} (133)

together with (38) tell us (in vacuum) is that there are observers for which \( H_{\alpha\beta} = 0 \), which by Eq. (131a) means that gyroscopes comoving with them feel no gravitational force. It does not tell us (in general) about gyroscope precession. Hence the effects at stake are different; for seemingly analogous setups they may even be opposite.

A realization of this contrast is summarized in Table I we have seen in Sec. VII C that, in the equatorial plane of the spinning charge, conditions (132) are satisfied, implying that observers with angular velocity (93) measure no magnetic field. Hence magnetic dipoles comoving with them do not undergo Larmor precession; they feel however a force, Eq. (130a), since \( B_{\alpha\beta} \neq 0 \) always for an observer moving in a non-uniform field (due to the laws of electromagnetic induction) as discussed in detail in [91]. We have also seen in Sec. VII C that, in the equatorial plane of the spacetime around a spinning body, conditions (133) and (38) are satisfied, implying that \( H_{\alpha\beta} = 0 \) for observers with angular velocity (123). This velocity field has some similarities with (93); namely their asymptotic limits match up to a factor of two. However, for gyroscopes moving with these velocities, the situation is precisely the opposite: by Eq. (131a), no force is exerted on them, but they precess (with respect to the distant stars). This last point deserves to be discussed in detail. To first post-Newtonian order, in terms of the metric potentials in (98), the precession frequency (let us denote it by \( -\vec{\Omega}_* \)) of the spin vector of a gyroscope with respect to a frame anchored to the distant stars reads

\[ -\vec{\Omega}_* = -\frac{1}{2} \vec{v} \times \vec{a} + \frac{3}{2} \vec{v} \times \nabla U - \frac{1}{2} \nabla \times \vec{A} \quad (134) \]

(cf. e.g. Eqs. (40.33) of [23], Eqs. (3.4.38) of [27]), where \( \vec{v} \) is the gyroscope’s velocity with respect to the PN frame and \( \vec{a} = \nabla U \vec{U} \) are the spatial components of its covariant acceleration. The first term, \( \vec{a} \times \vec{v}/2 = \vec{\Omega}_{\text{Thomas}} \), is the Thomas precession; because of it, \( \vec{\Omega}_* \) depends on the gyroscope’s acceleration. Hence, to determine \( \vec{\Omega}_* \) for gyroscopes moving with the velocities depicted in Fig. 8 we must say how they accelerate. It is natural to consider two cases: i) gyroscopes in circular motion with angular velocity \( d\phi/dt \) given by (123), and ii) gyroscopes at rest in boosted PN frames momentarily moving with \( \vec{v} = (d\phi/dt)\vec{e}_\phi \). To 1PN order, the Thomas precession is the same in both cases: in case i), the exact acceleration of the gyroscope is \( \vec{a} = [M/r^2 - J^2/(M^2r^3)]\vec{e}_z \); hence, using \( v \approx J/(Mr) \),

\[ \vec{\Omega}_{\text{Thomas}} = -\frac{1}{2} \vec{v} \times \vec{a} = \frac{J}{2r^3} \left[ 1 - \frac{M}{r} \left( \frac{J}{M^2} \right)^2 \right] \vec{e}_z \]

\[ = \frac{J}{2r^3} \left[ 1 + O(\epsilon^5) \right] \vec{e}_z \approx \frac{J}{2r^3} \vec{e}_z \]

where \( \vec{e}_z = -\vec{e}_\theta \). In case ii), \( \vec{v} \times \vec{a} = -\vec{v} \times \nabla U + O(\epsilon^5/L) \); hence, to the accuracy at hand, \( \vec{\Omega}_{\text{Thomas}} \) is the same. Observing that, in the equatorial plane, \( \nabla \times \vec{A} = 2J/r^3 \) and \( \vec{v} \times \nabla U = \vec{e}_z vM/r^2 \), the overall precession with respect to the distant stars is (for both cases)

\[ -\vec{\Omega}_* = \frac{3J}{2r^3} \neq 0 \]

A question that naturally arises is whether there are, in the equatorial plane, velocity fields for which gyroscopes do not precess with respect to the distant stars. The answer is affirmative, but, again, acceleration dependent. If one considers gyroscopes comoving with boosted PN frames (i.e., gyroscopes moving with constant coordinate velocity, \( dv/dt = 0 \)), then \( -\vec{\Omega}_* = 2\vec{v} \times \nabla U - \frac{1}{2} \nabla \times \vec{A} \), and the condition \( \vec{\Omega}_* = 0 \) yields \( v = J/(2Mr) \). This is half the asymptotic limit of the velocity (123) for which \( H_{\alpha\beta} = 0 \), but is precisely the same as the velocity (93) for which \( \vec{B} = 0 \) in the equatorial plane of a spinning charge (Fig. 3). Indeed, this is physically the analogue of the latter: \( H' = -4\vec{v} \times \nabla U + \nabla \times \vec{A} = 2\vec{\Omega}_* \), is the gravitomagnetic field (see below) as measured in the PN rest frame of the gyroscope; so solving for \( \vec{\Omega}_* = 0 \) amounts to finding (at each point) a boosted PN frame where the gravitomagnetic field \( H' \) vanishes. Its velocity is given by Eq. (142) below, analogous to Eq. (16). Analogously to the electromagnetic case, this can be cast as a cancellation between the gravitomagnetic fields generated by the rotation and relative translation of the source. One must
note, however, that this has nothing to do with the curvature invariants; it comes from the analogy (discussed in Sec. VII B below) between the transformation laws for the GEM fields in the PN regime and the electromagnetic fields.

B. “Gravitoelectromagnetic fields” (GEM Fields)

The inertial GEM fields have been defined in different ways in the literature, from the linearized theory approaches in e.g. [27] [103] [106], to the exact formulations in e.g. [44] [46] [50] [107] [113]. Here we will follow the exact approach in [46], which we believe to be physically motivated, and which leads, in the corresponding limit, to the GEM fields usually defined in post-Newtonian approximations, e.g. [19] [94] [96].

Consider a congruence of observers of 4-velocity $u^\alpha$, and a test particle of worldline $z^\alpha(\tau)$ and 4-velocity $dz^\alpha/d\tau = U^\alpha$. Take it, for simplicity, to be a point-like monopole particle, and assume that there are no external forces, so that $z^\alpha(\tau)$ is geodesic. Let $U^{(\alpha)} \equiv h^\alpha_\beta U^\beta$ be the spatial projection of the particle’s velocity with respect to the observers, cf. Eq. [1]; it can be interpreted as the relative velocity of the particle with respect to the observers. It is the variation of $U^{(\alpha)}$ along $z^\alpha(\tau)$ that one casts as inertial forces; the precise definition of such variation involves some subtleties however. For that we need a connection (i.e., a covariant derivative) for spatial vectors; however the space projection of the spacetime (Levi-Civita) covariant derivative, $h^\alpha_\beta \nabla_U U^{(\beta)}$, which might seem the most obvious, is not the one we seek, as it yields the Fermi-Walker derivative of $U^{(\alpha)}$ (i.e., its variation with respect to a system of Fermi-Walker transported axes). We seek a connection that yields the variation of $U^{(\alpha)}$ with respect to a system of spatial axes undergoing a transport law specific to the reference frame one chooses. Given a congruence of observers, the most natural choice would be spatial triads co-rotating with the observers. That is, for an orthonormal basis $e_\alpha$, whose general transport law along the observer congruence can be written as (e.g. [53])

$$\nabla_U e_\alpha = \Omega^\alpha_\beta e_\beta; \quad \Omega^\alpha_\beta = 2u^{[\alpha} a^{\beta]} + \epsilon^{\alpha\mu\nu} \Omega^\mu u^\nu,$$

that amounts to choosing $\Omega^\alpha$ (the angular velocity of rotation of the spatial axes relative to Fermi-Walker transport) equal to the observer’s vorticity: $\Omega^\alpha = \omega^\alpha$. If the congruence is rigid, this ensures that the $e_\alpha$ point to fixed neighboring observers (cf. Eq. [109] below). One might argue [102] [114] that this is the closest generalization of the Newtonian concept of reference frame; we dub it the congruence adapted frame. For more details we refer to Sec. 3 of [46]. The connection that yields the variation of a spatial vector $X^\alpha$ with respect to such frame is $\nabla_U X^\alpha = h^\alpha_\gamma \nabla_O X^\gamma + u^\alpha \epsilon^{\alpha\beta\lambda} \lambda^\beta X^\gamma \omega^\lambda$, cf. Eq. (51) of [46]; and the inertial or “gravitoelectromagnetic” force on a test particle is the variation of $U^{(\alpha)}$ along $z^\alpha(\tau)$ with respect to $\tilde{\nabla}$, that is $\tilde{\nabla}_U U^{(\alpha)} \equiv \tilde{D} U^{(\alpha)}/d\tau$. Since, for geodesic motion, $\nabla_U U^\alpha = 0$, it follows, using [1], that $\tilde{\nabla}_U U^{(\alpha)} = -\gamma(\nabla_U u^\alpha + \epsilon^{\alpha\beta\lambda} \lambda^\beta X^\gamma \omega^\lambda)$, where $\gamma = -U_\alpha u^\alpha$. Finally, from the decomposition (cf. Eq. (73))

$$\nabla_\beta \omega^\alpha \equiv u_{\alpha\beta} \nabla_\beta u^\alpha = -u_\beta \nabla_\beta u^\alpha - \epsilon_\alpha_\beta_\gamma \omega^\gamma u_\delta + \sigma_\alpha_\beta + \frac{\theta}{3} h^3_{\alpha\beta} \theta,$$

we have [46] (noting that $\nabla_\beta U^\alpha = U^{\beta} \nabla_\beta u^\alpha$)

$$\tilde{D} U^{(\alpha)}/d\tau = \gamma \left[ \gamma G^\alpha + \epsilon^{\alpha\beta\gamma\delta} u^\beta H^\gamma - \sigma_\alpha_\beta U^\beta - \frac{\theta}{3} h^3_{\alpha\beta} U^\beta \right],$$

where

$$G^\alpha = -\nabla_\beta u^\alpha; \quad H^\alpha = 2\omega^\alpha$$

are, respectively, the “gravitoelectric” and “gravitomagnetic” fields. These are exact GEM fields, herein defined in terms of the kinematical quantities of the observers’ congruence; they play in [136] a role analogous to the electric and magnetic field in the Lorentz force. One should keep in mind that $G^\alpha$ is minus the observers’ acceleration, and $H^\alpha$ twice their vorticity. For the observers at rest ($u^\alpha = 0$) in a given coordinate system, $G^i = G_{0i}/g_{00}$. $H^i = -\epsilon^{ij}_{k} F_{0j}/g_{00}$; hence, to first post-Newtonian (1PN) order,\cite{24}

$$\tilde{G} = \nabla w - \frac{\partial \tilde{A}}{\partial t} + O(6); \quad \tilde{H} = \nabla \times \tilde{A} + O(5), \quad (138)$$

which match the GEM fields in Eqs. (3.21) of [94], or Eqs. (2.5) of [94]. The 1PN limit of [136] takes the form\cite{25}

$$d^2 \bar{x}/dt^2 = (1 + u^2 - 2U) \bar{G} + \bar{v} \times \bar{H} - \gamma \frac{\partial U}{\partial t} \sigma - 4(\bar{G} \cdot \bar{v}) \bar{v}. \quad (139)$$

which matches\cite{26} Eq. (7.17) of [94]. Linearizing Eqs. [138]-[139] one obtains (up to some factors depending on the conventions) the GEM fields and geodesic equation of the linearized theory approaches [27] [103] [106] [113].

1. post-Newtonian frames where $\tilde{H} = 0$

The transformation laws for the GEM fields in a change of reference frame exhibit some similarities to their electromagnetic counterparts. The exact forms are given in

\begin{footnotesize}

\begin{itemize}
\item [24] Using the 1PN Christoffel symbols, e.g. Eqs. (8.15) of [96], identifying $w \rightarrow U + \Psi, A_i \rightarrow -U_i$ in the notation therein.
\item [25] Noting that, to 1PN, $\tilde{D} U^{(\alpha)}/d\tau = (U^{(\alpha)})^2 d^2 x^\alpha/dt^2 + 2u^\alpha \partial_t U - v^2 G^{\alpha} + 4u^\alpha \tilde{G} \cdot \tilde{v}, \sigma_{\alpha\beta} = 0, \theta = 3d_t U$ and $\gamma^2/(U^{(2)}) = 1 - 2U + O(4)$.
\item [26] Noting that by $\tilde{G} \equiv G_{\alpha}^\beta \partial_j$, we denote the spatial components of $G^\alpha$ in the PN coordinate basis, and that, to 1PN order, $G_i = G^i(1 + 2U) \delta_{ij} = G^i + 2UG^i$.
\end{itemize}
\end{footnotesize}
Eqs. (8.3) of [44]. To 1PN order, the GEM fields of a boosted PN frame can be obtained applying a post-Galilean coordinate transformation (e.g. Eqs. (13) of [97]) to the metric, and then computing expressions [138] for the boosted potentials. In the case of $\mathcal{H}$ we have

$$
\mathcal{H}' = \mathcal{H} - 4\bar{v} \times \mathcal{G}
$$

(cf. Eqs. (5) of [9], Eq. (4.20b) of [94]), formally similar to the post-Coulombian limit of Eq. (11), apart from the factor of 4 in the second term. It is clear from (140) that when $\mathcal{G} \cdot \mathcal{H} = 0$ and $\mathcal{G}^2 > \mathcal{H}^2$ one can always find a boost velocity $\bar{v}$ such that

$$
\mathcal{H}' = 0 \Rightarrow 4\bar{v} \times \mathcal{G} = \mathcal{H}.
$$

This is in analogy with the situation in electromagnetism in Sec. [II A] for the vanishing of $\mathcal{H}$. Here $\bar{v}$ is such that its component $\bar{v}_LG$ orthogonal to $\mathcal{G}$ reads (taking the cross product of (141) with $\mathcal{G}$)

$$
\bar{v}_LG = \frac{\mathcal{G} \times \mathcal{H}}{4\mathcal{G}^2},
$$

in analogy with Eq. (16); and likewise no condition is imposed on $\bar{v}_LG$. An example is the case of the equatorial plane of the field produced by a spinning body, where $\mathcal{G} \perp \mathcal{H}$ (and $\mathcal{G}^2 > \mathcal{H}^2$), and indeed, as we have seen in Sec. VII A at each point one can find PN frames where $\mathcal{H}' = 0$ at that point.

However (contrary to what has been suggested in some literature [1] [2] [27] [32]), this has nothing to do with field invariants: firstly, $\mathcal{G} \cdot \mathcal{H}$ and $\mathcal{G}^2 - \mathcal{H}^2$ are not frame invariant (as $\mathcal{G}$ and $\mathcal{H}$ are actually mere artifacts of the reference frame, which vanish in a locally inertial one); secondly, they do not have any obvious relation with the curvature invariants. Indeed, as one may check computing $\star \mathbf{R} \cdot \mathbf{R} = 16\epsilon_{\alpha\beta\gamma} h^{\alpha\beta} h^{\gamma\delta}$ using Eqs. (145)-(146) below, one can have e.g. $\mathcal{G} \cdot \mathcal{H} \neq 0$ whilst $\star \mathbf{R} \cdot \mathbf{R} = 0$, or $\star \mathbf{R} \cdot \mathbf{R} \neq 0$ whilst $\mathcal{G} \cdot \mathcal{H} = 0$.

2. Relation between GEM fields and tidal tensors

It is of crucial importance to distinguish between the gravitational tidal tensors $\mathbb{E}_{\alpha\beta}$, $\mathbb{H}_{\alpha\beta}$ and the inertial fields $\mathcal{G}$, $\mathcal{H}$. A first obvious difference is that whereas $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ are physical fields, governing physical forces such as the spin-curvature force exerted on a gyroroscope, Eq. (31b) (which is the covariant derivative of the 4-momentum), $\mathcal{G}$ and $\mathcal{H}$ are artifacts of the reference frame, governing fictitious forces and torques, such as the inertial force in Eq. (136), or the gyroscope “precession” in Eq. (131a) (an ordinary derivative of $\mathcal{S}$). The exact relation between the two types of objects is complicated in general; it is given by Eqs. (109)-(110) of [46]. In this work we are interested in two special cases where it becomes simpler: exact stationary fields, and arbitrary fields to first post-Newtonian order.

In a rigid frame in a stationary spacetime we have (Eqs. (111)-(112) of [46]),

$$
\mathbb{E}_{ij} = -\nabla^j G_i + G_i G_j + \frac{1}{4} \left( H^2 h_{ij} - H_j H_i \right),
$$

$$
\mathbb{H}_{ij} = -\frac{1}{2} \left[ \nabla^j H_i + (\mathcal{G} \cdot \mathcal{H}) h_{ij} - 2G_j H_i \right],
$$

where $h_{\alpha\beta}$ is the spatial metric, cf. Eq. (1), and $\nabla^a$ the connection defined by Eq. (7), whose restriction to the spatial directions (which equals that of $\nabla$) yields the Levi-Civita connection of $h_{\alpha\beta}$.

In an arbitrary spacetime, to 1PN order, we have, from Eqs. (99)-(100) and (138),

$$
\mathbb{E}_{ij} = -\nabla^j G_i + G_i G_j + \frac{1}{6} \varepsilon_{ijk} \dot{\mathcal{H}}^k - \mathcal{U} \delta_{ij} + O(6),
$$

$$
\mathbb{H}_{ij} = -\frac{1}{2} \nabla^j H_i - \varepsilon_{ijk} \mathcal{G}^k + O(5),
$$

in agreement with Eqs. (3.38) and (3.41) of [94].

3. Uniform gravitomagnetic fields

A pedagogical example that illustrates how crucial it is to distinguish between the GEM inertial fields $\mathcal{G}$, $\mathcal{H}$ and the GEM tidal tensors $\mathbb{E}_{\alpha\beta}$, $\mathbb{H}_{\alpha\beta}$ (showing that there is no direct relation between $\star \mathbf{R} \cdot \mathbf{R}$ and $\mathcal{H}$) is to consider spacetimes with uniform gravitomagnetic fields. Examples of such spacetimes are the Gödel universe and the particular class of the Som-Raychaudhuri solutions considered in Sec. [VIC]. These metrics, described by the line elements [124], [125] and [128], have special properties. Observers at rest in their coordinate systems, $u^\alpha = \delta^\alpha_0$, form a rigid congruence ($\sigma_{\alpha\beta} = 0$, $\theta = 0$) with zero acceleration [as in all metrics of the form (124)], and uniform vorticity $\omega = \omega \epsilon_z$. That is, in a frame adapted to such observers, the gravitoelectric field vanishes, and there is a non-zero uniform gravitomagnetic field [133], cf. Eq. (137):

$$
\mathcal{G} = 0, \quad \mathcal{H} = 2\omega \epsilon_z.
$$

---

27 Restricting ourselves to post-Newtonian frames, we can still say that $\mathcal{G} \cdot \mathcal{H}$ is invariant, to 1PN order, under changes of PN frame, since, as follows from Eqs. (5) of [9], $\mathcal{G} \cdot \mathcal{H} = \mathcal{G}^2 - \mathcal{H}^2 + O(7)$; however, $\mathcal{G}^2 - \mathcal{H}^2$, to that accuracy, is not.

28 To obtain Eq. (3.38) of [94] from (145), one notes that $\nabla_j G_i \approx G_{ij} - \mathcal{G}_{ij} - \epsilon_{ijk} \mathcal{G}^k = G_{ij} - 2G_i G_j + \delta_{ij} \mathcal{G}^2$, and $\nabla \times \mathcal{G} = -\partial \mathcal{H}/\partial t$, cf. Eq. (94) of [39].
In terms of the curvature, the situation is precisely the opposite. From Eqs. (143)-(144) one obtains

$$E_{ij} = \frac{1}{4} \left( \vec{H}^2 h_{ij} - H_i H_j \right) ; \quad H_{\alpha\beta} = 0 ,$$

(148)
i.e., these observers measure a non-zero gravitoelectric tidal tensor and a vanishing gravitomagnetic tidal tensor everywhere. Actually, one may check that the only solutions to the system $R_{\alpha\beta\gamma\delta} x^\gamma x^\delta = 0$ are $x^\alpha \propto \delta^\alpha_z$ and thus spacelike, such that for all observers the gravitoelectric tidal tensor is non-zero, and so the Riemann tensor is purely electric according to the definition in Sec. III C. Hence, then follows from Eq. (53) that $\star \mathbf{R} \cdot \mathbf{R} = 0$ everywhere.

Thus, from the point of view of the curvature, the metrics (124)-(128) represent purely electric spacetimes, with $E_{\alpha\beta} \neq 0$ and $H_{\alpha\beta} = 0$ globally with respect to a rigid congruence of observers, namely the rest observers in the coordinates of (124); from the point of view of the GEM inertial fields, by contrast, one would say that they are purely magnetic, since $\vec{G} = 0$ and $\vec{H} \neq 0$ for the same observers. It is actually impossible to make $\vec{H}$ vanish in any rigid frame, which can be seen as follows. From Eq. (91) of [16], we have, for a rigid frame,

$$\nabla \times \vec{H} = -16\pi \vec{J} ,$$

where $\vec{J}^\alpha$ is the spatial mass/energy current as defined in Sec. III C. Hence, $\vec{H} = 0$ requires $\vec{J} = 0$ in that frame. The Ricci tensor of the Gödel universe is of perfect fluid type, conditions (b) of Sec. III C and (b-2) of Appendix A holding with $\lambda = \omega^2/2$, $w^\alpha \propto \delta^\alpha_0$. Then, as shown in Appendix A (see also A 1), there is a unique congruence of observers for which $\vec{J}^\alpha = 0$, which are the observers at rest in the coordinates of (124), $w^\alpha(\tau = 0) = \delta^\alpha_0 \propto w^\alpha$. That is, the observers comoving with the fluid. Since $\omega^\alpha \neq 0$ for those observers, no rigid congruences with vanishing vorticity exist in this spacetime, that is, $\vec{H} \neq 0$ for frames adapted to any rigid congruence of observers. A similar proof can be made for the metric (124) with (128), only now the Ricci tensor is of Segre type [(11)1,1], conditions (d) of Sec. III C and (D) and (d2) of Appendix A holding with $\lambda = 3\omega^2/2$, $c = \omega^2$, thereby admitting the single time-like eigenvector $\delta^\alpha_0$. More strongly, it was actually shown in Sec. II.C of [120] that, for the Gödel universe, $\omega^\alpha \neq 0$ and thus $\vec{H} \neq 0$ for frames adapted to any shear-free observer congruence $(\sigma_{\alpha\beta} = 0)$. The same proof applies to the special Som-Raychaudhuri metrics.29

These metrics thus possess a globally intrinsic gravitomagnetic field $\vec{H}$, according to the classification scheme proposed in Sec. VII A.

To see the consequences in terms of motion of test particles, consider gyroscopes at rest ($U^\alpha = \delta^\alpha_0$) in the coordinate system of (124). These feel no spin-curvature force, since $H_{\alpha\beta} = 0$; cf. Eq. (131)); they will actually remain at rest in that coordinate system since, moreover, they feel no gravitoelectric field (as the frame is freely falling), cf. Eqs. (147). However they precess relative to the frame adapted to these observers (i.e., to the basis vectors of the coordinate system of $h_{ij}$) with angular velocity $-\vec{\omega} = -\omega^z z = -\vec{H}/2$, like a magnetic dipole under a uniform magnetic field, cf. Eq. (131)). Moreover, there is no rigid (and even not a shear-free) frame relative to which the gyroscopes do not precess. In fact, the only precession effect that vanishes due to the vanishing of $H_{\alpha\beta}$ is the so-called “differential precession”, that is, the precession of a gyroscope relative to a system of axes anchored to a set of neighboring, infinitesimally close gyroscopes, as this is a tidal effect governed precisely by $H_{\alpha\beta}$, see Eq. (3.11) of [122] (cf. also Sec. 2.3 of [17]). We believe this to be enough to convince the reader about the importance of distinguishing between GEM inertial and tidal fields, and that indeed the invariant $\star \mathbf{R} \cdot \mathbf{R}$ is not a good test for “intrinsic gravitomagnetic field”.

C. What the invariants say about the gravitomagnetic field

As is explicit from Eqs. (23), (20/52)-(53), it is the electric and magnetic parts of the curvature (and their possible vanishing for some observers), not the GEM fields, that are directly related with the curvature invariants. However, still there are special cases where indeed from the curvature invariants one can infer information about the gravitomagnetic field itself.

First let us discuss what can be understood as a physically meaningful gravitomagnetic field, that can be identified with the effects that have been under experimental and observational scrutiny. The gravitomagnetic field $\vec{H}$ is an inertial field, i.e., a reference frame artifact, that can always be gauged away by choosing a locally inertial frame. Thus, locally, it has no physical meaning; yet it may reflect global physical properties of a given spacetime. For instance, the “precession” of a gyroscope (at a finite $r$) in the Kerr spacetime with respect to a frame anchored to the distant stars, discussed in Sec. VII A reflects an effect — frame-dragging — which is physical, and intrinsic in the sense that it distinguishes the Kerr metric from a static solution (e.g. the Schwarzschild spacetime). Its non-local nature is manifest in the fact that in order to measure it one needs to lock the frame to the distant stars by means of telescopes [128]. Thus, one can say that frame-dragging is manifest when at some point a system of locally non-rotating axes (defined mathematically by the Fermi-Walker transport law, or physically by guiding gyroscopes, see Sec. VII A) rotates rel-

29 Both metrics (124)-(128) are of Petrov type D with the two Weyl PNDs spanned by the null vectors $\delta^\alpha_0 \pm \delta^\alpha_0$; in an adapted Newman-Penrose (complex null) frame, the relevant Newman-Penrose Weyl scalars and spin coefficients are $\Psi_2 = \Psi_2 \propto \omega^2$, $\Psi_k = 0$ for $k \neq 2$, and $\kappa = \sigma = \tau = 0$, $\nu = \lambda = \pi = 0$, $\rho = \mu = i\omega/\sqrt{2}$; thus none of the criteria (1)-(5) in proposition B.1 of [120] is fulfilled, and so the spacetimes do not admit any shear- and vorticity-free observer congruence.
ative to an inertial frame at infinity. In other words, when \( \vec{H} \) is non-vanishing in a reference frame with axes rotationally locked to an inertial frame at infinity (star-fixed axes). This is however a concept that makes sense only in a special class of spacetimes. In general, one has no way of determining the rotation of a system of axes at one point relative to another system of axes at a different point (since in a curved spacetime there is a priori no natural way of comparing vectors in different tangent spaces). This is possible only if (at least within some approximation) the spacetime admits shear-free observer congruences. In order to see this, consider an orthonormal tetrad frame \( \mathbf{e}_a \), whose time axis \( \mathbf{e}_0 = \mathbf{u} \) is the 4-velocity of some congruence of observers. Let \( \xi^\alpha \) be a connecting vector between the worldlines of the observers, \( \mathcal{L}_a \xi^\alpha = 0 \), and \( Y^\alpha = (h^u)^\alpha_{\beta} \xi^\beta \) its space projection; \( Y^\alpha \) evolves in the tetrad as (Eq. (41) of [46])

\[
\dot{Y}_i = \left( \sigma_{ij} + \frac{1}{3} \theta \delta_{ij} + \omega_{ij} - \Omega_{ij} \right) Y^j .
\]

(149)

If the congruence is rigid (\( \sigma_{ij} = \theta = 0 \)), and one chooses spatial triads \( \mathbf{e}_i \) co-rotating with the observers, \( \omega_{ij} = \Omega_{ij} \) (see Sec. VIIB), \( Y^\alpha \) is constant in the tetrad, \( \dot{Y}_i = 0 \). Hence the triads \( \mathbf{e}_i \) point to fixed neighboring observers. If the congruence is inertial at infinity, this means (since it is rigid) that the \( \mathbf{e}_i \) are locked to an inertial frame at infinity. Hence, by measuring the precession of a gyroscope with respect to the local axes \( \mathbf{e}_i \), one is in fact measuring it with respect to the distant stars, and it has thus a clear meaning in terms of frame-dragging. If the congruence is not rigid but only expands (i.e., no traceless shear, \( \sigma_{ij} = 0 \)), then \( \dot{Y}_i = \theta Y_i/3 \); i.e., \( Y^\alpha \), albeit not constant, has a fixed direction on the tetrad, so similar arguments still apply. When the congruence shears (\( \sigma_{ij} \neq 0 \)), however, one has no way of locking the frame to an inertial frame at infinity, and therefore the gravitomagnetic field measured in a frame adapted to such congruence generically has no relevant physical meaning.

We thus conclude that if an asymptotically flat spacetime admits a shear-free observer congruence which is inertial at infinity, the frame adapted to it has axes fixed with respect to the distant stars, and the gravitomagnetic field \( \vec{H} \) measured therein has a meaning in terms of precession of gyroscopes and deflection of test particles with respect to the distant stars [this is the case of any post-Newtonian frame to 1PN order, as \( \sigma_{ij} = 0 \) for the rest observers in the 1PN metric [98]]. Now, the connection with the curvature invariants and with \( \mathbb{H}_{\alpha\beta} \) is the following. Since \( \vec{H} \) is twice the vorticity of the observers, cf. Eq. (137), the vanishing of \( \vec{H} \) requires the congruence to be vorticity-free (also known as a “normal” congruence). If the spacetime is conformally flat (i.e., \( C_{\alpha\beta\gamma\delta} = 0 \) everywhere) then there are as many shear-free normal congruences as in flat spacetime, as follows from Eq. (6.15) in [60]. Assume now the generic case that \( C_{\alpha\beta\gamma\delta} \neq 0 \). From Eq. (110) of [46], we have, in the tetrad frame above,

\[
\mathcal{H}_{ij} = -\nabla_j \omega_i + \delta_{ij} \nabla^\gamma \omega^\delta + 2G\omega_i + \nabla^i \sigma_{m(j} t^{m)j} .
\]

(150)

Hence, relative to shear- and vorticity-free observer congruences (\( \sigma_{\alpha\beta} = \omega^{\alpha} = 0 \)), the magnetic part of the Weyl tensor vanishes, \( \mathcal{H}_{\alpha\beta} = 0 \) (cf. also [82], Theorem 3). Hence the Weyl tensor is necessarily purely electric, which comes down to the conditions

\[
\mathbf{C} \cdot \mathbf{C} = 0, \quad \mathbf{C} \cdot \mathbf{C} > 0,
\]

(151)

plus the Weyl generalization of [58] (see Sec. IIIIC), or equivalently to one of the conditions [63] or [66]. These are not however, in general, sufficient conditions for the existence of shear- and vorticity-free congruences (they only ensure that \( \mathcal{H}_{\alpha\beta} = 0 \) for some \( u^\alpha \)). For instance, as we have seen in Sec. VIIB3, the metrics [124]-[128] are Riemann and thus Weyl purely electric [see 148 and 59] but do not admit a shear- and vorticity-free observer congruence. Only in the special case of vacuum (or Einstein, \( R_{\alpha\beta} = \Lambda g_{\alpha\beta} \)) Petrov type D solutions, it is known (see [121], Theorem 2.1, Appendix B and Proposition B.1 therein) that the invariant conditions [151], when they hold in some open 4-D spacetime region, are indeed sufficient to ensure the existence of shear- and vorticity-free congruences. And since, in vacuum, \( \mathbf{C} = \mathbf{R} \), one can say that when \( \mathbf{R} \mathbf{R} = 0 \), \( \mathbf{R} \mathbf{R} > 0 \) in some open 4-D region, a shear-free normal congruence exists therein. If such congruence is inertial at infinity, then this means that there is a frame rotationally locked to the distant stars (and where frame dragging is a well-defined notion) where \( \vec{H} \) globally vanishes.

This is all one can say about \( \vec{H} \) based on the curvature invariants. It is of limited applicability for the astrophysical systems under discussion. Among the systems studied in the present paper, only the fields of a single non-spinning/spinning body can be seen as Petrov type D vacua, as they are approximately described by the Schwarzschild/Kerr solutions (as for the two-body metric in Sec. VIIB although the exact solution is not known, its post-Newtonian limit is already incompatible with the type D at any point off the Earth-Sun axis, as we have seen therein). Schwarzschild’s solution is purely electric everywhere, so there are indeed shear-free normal congruences everywhere, one of them the static observers \( u \propto \partial / \partial t \). That is, \( \vec{H} = 0 \) everywhere relative to the static observers. In the case of Kerr spacetime, the only purely electric region outside the horizon is the equatorial plane; this is a 3-D hypersurface, not an open 4-D

---

30 This is not the only gravitomagnetic field that has a meaning in terms of frame-dragging. For instance, the gravitomagnetic field \( \mathcal{H}_{\alpha\beta\gamma\delta} \) measured in the so-called “locally non-rotating frames” considered in [32][110][122] associated to a shearing congruence (the zero angular momentum observers), and where \( \Omega_{ij} \neq \omega_{ij} \) (the triads \( \mathbf{e}_i \) are tied to the background symmetries), signals frame-dragging and vanishes in a static spacetime; however, this frame is not tied to the distant stars, thus it does not correspond to the gravitomagnetic field under experimental scrutiny.
spacetime region. Hence, in spite of \( \mathbf{r} \cdot \mathbf{R} = 0 \) at the equatorial plane, there is no congruence which is shear- and vorticity-free therein; and the fact that \( \mathbf{r} \cdot \mathbf{R} \neq 0 \) elsewhere implies that such congruences do not exist at all in this spacetime. This means that \( \mathbf{H} \neq 0 \) in a frame adapted to any non-shearing congruence of observers in the Kerr spacetime.

D. New criteria for intrinsic/extrinsic gravitomagnetism

Given the interest on these notions in the literature, and the unsatisfactory character of the existing ones, in this section we propose new criteria for extrinsic/intrinsic gravitomagnetism. Similarly to previous approaches in the literature \cite{2, 27}, we start from the observation of the situation for electromagnetic fields in flat spacetime to get insight, but devise criteria that are more physically motivated and that make sense in view of knowledge gathered in the previous sections.

For electromagnetism in flat spacetime, the following classification seems reasonable:

a) globally extrinsic (intrinsic) magnetic field: there is (there is not) a globally inertial frame where \( \mathbf{B} = 0 \) everywhere in the region of interest. Example of globally extrinsic \( \mathbf{B} \): Coulomb field of a point charge.

b) Locally extrinsic (intrinsic) magnetic field: there are (there are not), at the given point, observers measuring \( \mathbf{B} = 0 \). Amounts to the notion of “purely electric” field, given by the invariant conditions ii) of Sec. \[1\] Examples of globally intrinsic but locally extrinsic magnetic field: equatorial plane of a spinning charge; motion plane of two charged bodies in co-planar motion. Example of (globally, and at every point locally) intrinsic magnetic field: field of spinning charge outside the equatorial plane.

Note that a) implies b), but not the other way around. The distinction between globally/locally extrinsic, and casting globally inertial frames as preferred in this context, seems to make sense from the analysis in Sec. \[IV\] as indeed there is a substantial difference between e.g. the Coulomb field of a point charge and the field in the equatorial plane of a spinning charge. In the former, \( \mathbf{B} = 0 \) everywhere in the inertial rest frame of the charge; this may be cast as the vanishing of \( \mathbf{B} \) everywhere for a family of observers all with the “same” 4-velocity (the observers “at rest with respect to the charge”), since in flat spacetime we have a well-defined notion of parallelism,\(^{31}\) and can thus talk about the relative velocity of distant observers. In the case of a spinning charge, as we have seen in Sec. \[IV\] \( \mathbf{B} \) can be made to vanish everywhere in the equatorial plane, but \textit{not in an inertial frame}; only with respect to shearing observer congruences (of angular velocity \( \omega \)). With respect to an inertial frame, \( \mathbf{B} \) vanishes only at a point (different in general for different inertial frames). That is, observers exist for which \( \mathbf{B} = 0 \), but their 4-velocity differs from point to point.

To generalize this to curved spacetime, the obvious difficulty is that there are no globally inertial frames, and the parallelism of vectors (thus the relative velocity of observers) at different points is not a well-defined notion. There is a local notion of difference (with respect to the Levi-Civita connection) between the 4-velocities of (infinitesimally close) neighboring observers in a congruence, which is given by, cf. Eqs. \( (73), (135) \),

\[
\nabla_{\alpha} u^\beta = -\epsilon_{\alpha\beta\gamma\delta} u^\gamma \omega^\delta + \sigma_{\alpha\beta} X^\gamma + \frac{\theta}{3} X^\alpha ,
\]

for any spatial vector \( X^\alpha \) orthogonal to \( u^\alpha \) \( (X^\alpha u_\alpha = 0) \). That tells us that the observer’s 4-velocity differs from that of its neighbors when the congruence has shear, expansion or vorticity. However, congruences where they all vanish do not exist in general, as is well known (in vacuum, in particular, this would require the spacetime to be locally static, see Theorem 4 in \[82\] and Theorem 2 in \[125\] for Petrov type I, and Theorem 2.2 in \[120\] for Petrov type D). We propose generalizing criteria a)-b) to general relativity by replacing “globally inertial frames” by “shear-free frames” (i.e., allowing the preferred frame to have vorticity and expansion, but no traceless shear). The justification is that such replacement, in flat spacetime, leaves the above classification unchanged for all the examples studied (which would not be the case for vorticity-free or expansion-free frames\(^{32}\)). Moreover, shearfree (not vorticity-free or expansion-free) frames are the case of post-Newtonian frames (to 1PN order), which may be regarded as the closest entity in a curved spacetime to the globally inertial frame of flat spacetime.

In this generalized form, the criteria can be closely mirrored for the gravitational field.

Starting with the curvature tensor,

\[
\text{c) globally extrinsic (intrinsic) gravitomagnetic curvature: there is (there is not) a non-shearing congruence of observers measuring } \mathbf{H}_{\alpha\beta} = 0 \text{ everywhere within the region of interest. Examples of globally extrinsic: all (locally or globally } \text{[20]} \text{) static spacetimes, e.g. Schwarzschild; FLRW metrics; uniform gravitomagnetic fields (e.g. Gödel universe).}
\]

\[31\] Namely parallelism with respect to the Levi-Civita connection. It amounts to saying that two observers have the same 4-velocity if \( u^\alpha = u'^\alpha \) in a rectangular coordinate system.

\[32\] For instance the velocity field \[93\] for which \( \mathbf{B} = 0 \) in the equatorial plane of a spinning charge is vorticity and expansion free, see Footnote 16; however \( \mathbf{B} \neq 0 \) (except at a point) in this plane with respect to any inertial frame.
### Electromagnetism (flat spacetime)

|                | Examples                                                                 |
|----------------|--------------------------------------------------------------------------|
| Globally extrinsic $\vec{B}$  
(inertial frames exist where $\vec{B} = 0$ globally) | • EM field of any static charge distribution  
(e.g. Coulomb field) |
| Globally intrinsic $\vec{B}$  
(no inertial frames where $\vec{B} = 0$ globally) | Locally extrinsic  
(Observers measuring $\vec{B} = 0$)  
 Locally intrinsic  
($\vec{B} \neq 0$ for all observers) |
|                | • equatorial plane of spinning charge  
• motion plane of 2-body systems  
• spinning charge outside equatorial plane  
• 2-body systems outside motion plane |

### Gravity

|                | Examples                                                                 |
|----------------|--------------------------------------------------------------------------|
| Globally extrinsic $H_{\alpha\beta}$  
(shearfree frames exist where $H_{\alpha\beta} = 0$ globally) | • any locally static spacetime  
(e.g. Schwarzschild)  
• vacuum spacetimes with globally extrinsic $\vec{H}$  
• spacetimes with uniform $\vec{H}$ (e.g. Gödel)  
• FLRW metrics |
| Globally intrinsic $H_{\alpha\beta}$  
(no shearfree frames where $H_{\alpha\beta} = 0$ globally) | Locally extrinsic  
(Observers measuring $H_{\alpha\beta} = 0$)  
 Locally intrinsic  
($H_{\alpha\beta} \neq 0$ for all observers) |
|                | • equatorial plane of spinning body  
• orbital plane of 2-body systems  
• spinning body outside equatorial plane  
• 2-body systems outside orbital plane |

|                | Examples                                                                 |
|----------------|--------------------------------------------------------------------------|
| Globally extrinsic $\vec{H}$  
(shearfree frames exist where $\vec{H} = 0$ globally) | • any locally static or spherically symmetric spacetime  
(e.g. Schwarzschild)  
• vacuum spacetimes with globally extrinsic $H_{\alpha\beta}$  
• all conformally flat spacetimes (e.g. FLRW) |
| Globally intrinsic $\vec{H}$  
(no shearfree frames where $\vec{H} = 0$ globally) | Exact theory  
PN theory: locally extrinsic  
(PN frames where $\vec{H} = 0$ at a point)  
 PN theory: locally “intrinsic”  
($\vec{H} \neq 0$ in all PN frames) |
|                | • Kerr spacetime everywhere  
• any vacuum open spacetime region with locally intrinsic $H_{\alpha\beta}$  
• Gödel universe  
• equatorial plane of spinning body  
• orbital plane of 2-body systems  
• spinning body outside equatorial plane  
• 2-body systems outside orbital plane |

### Table II: Proposed classification scheme for magnetic field $\vec{B}$, gravitomagnetic tidal tensor $H_{\alpha\beta}$, and gravitomagnetic field $\vec{H}$.

Note that “globally extrinsic” implies “locally extrinsic” everywhere in the region of interest, but not the other way around. Locally intrinsic implies globally intrinsic, but not the other way around. (The examples given pertain to the systems studied in this paper, thus are not exhaustive.)

d) Locally extrinsic (intrinsic) gravitomagnetic curvature – there are (there are not), at the given point, observers measuring $H_{\alpha\beta} = 0$. In (non-flat) vacuum amounts to the notion of “purely electric curvature”, given by condition ii) of Sec. III A in general, it coincides with one of the possibilities [III C 1], [III C 3] in the criterion of Sec. III C. Examples of globally intrinsic but locally extrinsic magnetic curvature: equatorial plane of spinning body; orbital plane of 2-body systems. Example of (globally, and at every point locally) intrinsic magnetic curvature: Kerr spacetime outside the equatorial plane.

Note that c) implies d), but not the other way around. Both globally and locally extrinsic gravitomagnetic curvature require the Weyl tensor to be zero or purely electric and are thus of Weyl-Petrov type O, D or I, see the criterion of Sec. III C for the local case. In Appendix A we have determined the observers measuring $H_{\alpha\beta} = 0$ in case criterion d) is satisfied; if it holds over a 4-D region then, in those subcases where a unique observer congruence measuring $H_{\alpha\beta} = 0$ exists [namely (b2), (d2), and (e) of Appendix A (2-b), (2-c) of Sec. III C 2, and Sec. III C 3], it is easy to test criterion c), i.e., whether this congruence is shear-free; if this holds true then the spacetime exhibits globally extrinsic (else globally intrinsic) gravitomagnetic curvature; in the other subcases criterion c) may be more difficult to test.

As for the gravitomagnetic field $\vec{H}$:

e) globally extrinsic (intrinsic) gravitomagnetic field: there is (there is not) a non-shearing congruence
f) Locally extrinsic (intrinsic) gravitomagnetic field (only for PN approximation): there are (there are not) PN frames where $\vec{\mathcal{H}} = 0$ at the given point. It amounts to the conditions $\hat{G} \cdot \hat{H} = 0$, $\hat{G}^2 > \hat{H}^2$ (see Sec. VII B 1). Examples of globally intrinsic but locally extrinsic $\vec{\mathcal{H}}$: equatorial plane of spinning body; orbital plane of 2-body systems. Example of (globally, and at every point locally) intrinsic $\vec{\mathcal{H}}$: metric of spinning body outside the equatorial plane.

Criterion e) does not translate into a condition on the invariants, although it has a relation with the invariants of the Weyl tensor, in the sense that shear- and vorticity-free observer congruences exist only when the Weyl tensor is purely electric (and thus of Petrov type D or I); but not the other way around, cf. Sec. VII C. An easy way to test criterion e) was given in Proposition B.1 of [120] for the Petrov type D case, while for Petrov type I one simply needs to check whether the Weyl generalization of $\mathcal{H}_{\alpha\beta}$ is satisfied, and if so whether the unique observer measuring vanishing $\mathcal{H}_{\alpha\beta}$ (with 4-velocity proportional to $\hat{t}^\alpha$ given by the Weyl generalization of (45)) is shear- and vorticity-free.

Extrinsic gravitomagnetic curvature $\mathcal{H}_{\alpha\beta}$ vs. extrinsic gravitomagnetic field $\vec{\mathcal{H}}$.—In the presence of sources, a spacetime can have globally extrinsic $\mathcal{H}_{\alpha\beta}$ whilst not globally extrinsic $\vec{\mathcal{H}}$: examples are the Gödel universe or the Som-Raychaudhuri metrics studied in Sec. VII B 3. And the other way around: examples are conformally flat spacetimes having a Ricci tensor not obeying criteria (a)-e) of Sec VII C [so that condition (50) is not obeyed for any observer, whilst there being as many shear- and vorticity-free observer congruences as in flat spacetime], e.g. the pure radiation metrics in [152]. However, in a vacuum or Einstein spacetime ($R_{\alpha\beta} = \Lambda g_{\alpha\beta}$), these notions are equivalent. Indeed, by (70), a globally extrinsic $\mathcal{H}_{\alpha\beta}$ ($\vec{\mathcal{H}}$) then comes down to the existence of a shear-free observer congruence for which $\mathcal{H}_{\alpha\beta} = 0$ ($\vec{\mathcal{H}} = \vec{2}\mathcal{H} = 0$). Taking an orthonormal frame “adapted” to the observers and substituting $\mathcal{H}_{ij} = \sigma_{ij}$ together with the Einstein space conditions $\rho + p = \mathcal{J}_t = 0 = \mathcal{J}_i$ into the differential Bianchi identity [96] one obtains $\mathcal{E}_{ij}\omega^j = 0$; if $\omega^j \neq 0$ then $\mathcal{Q}^\alpha_\beta = \mathcal{C}^\alpha_\beta$, seen as an operator in the rest space of the observer, would have an eigenvalue 0, in contradiction with a result of Brans [33]; hence $\omega^j = \mathcal{H}^i/2 = 0$. Conversely, by (150), $\omega^i = 0 = \sigma_{ij}$ implies $\mathcal{H}_{ij} = 0$, and the equivalence is established. This generalizes the proof made in [133] considering the special case of rigid congruences in vacuum axistationary spacetimes. In general, when $\vec{\mathcal{H}}$ is globally extrinsic, it means that there is a non-shearing frame (the frame adapted to the shear- and vorticity-free congruence) relative to which all gyroscopes whose center of mass is at rest do not precess. If such frame is inertial at infinity, then it means that no gyroscope at rest in such frame precesses with respect to the distant stars.

Criterion f) has no relation with any field invariants (see Sec. VII B 1) and is a notion that makes sense only in the framework of the post-Newtonian approximation. One might argue that no inertial fields should ever be dubbed “locally intrinsic”, as they can always be made to vanish by switching to a locally inertial frame. Still this notion (as long as limited to the PN framework), seems useful to distinguish the situation in static spacetimes from e.g. the equatorial plane of the field of a spinning body, or the orbital plane of a 2-body system. The gravitomagnetic field in this regime is formally very similar to the magnetic field, and what is said in point b) above applies to $\vec{\mathcal{H}}$ and to the analogous gravitational systems, replacing inertial frames by PN frames. Hence a formally analogous criterion seems to make sense. Moreover, $\vec{\mathcal{H}}$ in this framework always has a meaning in terms of precession of gyroscopes with respect to the distant stars, since the basis vectors of PN coordinate systems are locked to inertial frames at infinity.

Regarding the astrophysical setups of interest, these criteria clearly distinguish between the gravitational field of a single translating non-spinning body, which has globally extrinsic gravitomagnetic curvature and field, and the fields of a spinning body or of a system of two bodies orbiting each other; but not between these last two fields, as they both have gravitomagnetic curvature and field which is globally intrinsic, locally extrinsic in the equatorial/orbital planes, and locally intrinsic, generically, elsewhere. The proposed scheme is summarized in Table II.

VIII. CONCLUSION

Motivated by the recent interest in the curvature invariants and their formal analogies with the invariants of the Maxwell tensor, in the context of the debate on the notions of “intrinsic”/“extrinsic” gravitomagnetism and their detection in solar system based experiments and astronomical observations, we thoroughly discussed in this work the invariants, their mathematical meaning

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33 Brans stated his result for pure vacuum, but the proof is unaltered when adding a cosmological constant.
and physical interpretation, and what they actually tell us about the motion of test particles.

We started with a rigorous discussion of the algebraic meaning of the invariants. The quadratic invariants of the Maxwell tensor give conditions for the existence of observers for which one of the fields (magnetic or electric) vanishes; an explicit expression [Eq. (10)] for their velocities was derived. The invariants of the Riemann tensor in vacuum, and of the Weyl tensor in general, are analogously related with conditions for the existence of observers for which the corresponding magnetic or electric parts vanish. The explicit expressions for the velocities of such observers were also obtained, which, for a special class of spacetimes [Petrov type D vacua, Eq. (19)], exhibit a strong formal analogy with the electromagnetic counterpart. In the gravitational case, however, the quadratic invariants are not sufficient. For the vacuum Riemann tensor (or the Weyl tensor in general) such conditions involve also the cubic invariants, and the invariants are even insufficient if a certain relation between them holds [see (32) and (37)-(38) in Sec. III A]. For the Riemann tensor in the presence of sources, we have derived a set of conditions for the vanishing of \( \mathbb{H}_{\alpha\beta} \), which involve moreover the scalar invariants of the Ricci tensor (whose algebraic classification is reformulated), and fully determined the corresponding observers. These are seen to consist of the intersection between the set of observers measuring a vanishing magnetic part of the Weyl tensor and the time-like eigenspace of the Ricci tensor. But here the curvature invariants do not suffice either, a fortiori.

A consequence of this, considering the proposal in the literature \([1, 2, 27]\) of using the Chern-Pontryagin invariant \( \star \mathbf{R} \cdot \mathbf{R} \) as a probe for intrinsic gravitomagnetism, is that even though its non-vanishing has a clear meaning, implying that \( \mathbb{H}_{\alpha\beta} \neq 0 \) for all observers, the converse is not true, i.e., the condition \( \star \mathbf{R} \cdot \mathbf{R} = 0 \) alone does not have a special significance (it does not ensure that \( \mathbb{H}_{\alpha\beta} = 0 \) for some observer, even in vacuum). Thus, even prior to physical considerations, one notes that such criteria are based on incomplete conditions.

Then we investigated the physical principles behind the behavior of the invariants in different systems, with emphasis on \( \star \mathbf{F} \cdot \mathbf{F} \) and \( \star \mathbf{R} \cdot \mathbf{R} \), and the question of why \( B^0 \) and \( \mathbb{H}_{\alpha\beta} \) vanish for certain observers in some systems and not in others. An explanation based on a loose notion of relative motion has been suggested in \([27]\) (p. 358): “spacetime geometry and the corresponding curvature invariants are affected and determined, not only by mass-energy, but also by mass-energy currents relative to other mass, that is, mass-energy currents not generable nor eliminable by any Lorentz transformation” (with a similar explanation for the electromagnetic case). This is not, however, satisfactory, since in a curved spacetime the relative motion of distant objects is not possible to define unambiguously (as there is no global notion of parallelism). Different definitions of relative velocity have been proposed in the literature (see \([128, 129]\)); however, a direct relation of any of these with the curvature invariants seems to be ruled out by simple arguments. \(^{34}\)

We looked instead into the field equations — the Maxwell equations for \( B^\alpha \), in its general form for arbitrary frames in arbitrary spacetimes, and, on the gravitational side, the so-called “higher order field equations” for \( \mathcal{H}_{\alpha\beta} \) and \( \mathbb{H}_{\alpha\beta} \) — since they are always valid, and checked what one can say about the invariants based on them (points \([14]\) of Secs. \([17, 1]\)). Concerning the explanation above, our results show that it is partially correct, but as a feature of the weak field slow motion approximation: if the system is such that a post-Newtonian frame exists where mass currents \( \mathcal{J} \) vanish everywhere, then (to 1PN accuracy) \( \star \mathbf{R} \cdot \mathbf{R} = 0 \); the converse, however, is not true (i.e., when there is no PN frame where \( \mathcal{J} = 0 \) everywhere, that does not ensure \( \star \mathbf{R} \cdot \mathbf{R} \neq 0 \)). This is in close analogy with the electromagnetic invariants in flat spacetime: if the setup is such that an inertial frame exists where all currents (charge and displacement) are zero, then \( \star \mathbf{F} \cdot \mathbf{F} = 0 \); but the converse is not true. In the more general cases of electromagnetic fields in a curved spacetime, or gravity outside the PN regime, things are more complicated because one has no inertial or PN frames, and in generic frames the observer’s vorticity, shear and expansion act as sources of the fields (in addition to the currents).

We studied and physically interpreted the structure of the invariants in the astrophysical setups of interest — the field of a single non-spinning body (the one effectively involved in the measured gravitomagnetic effects in binary pulsars, and in the geodetic precession of the Earth-Moon system in the Sun’s field), described by the Schwarzschild solution, the field of a system of two bodies to first post-Newtonian order (involved in the LLR measurements of the Moon’s orbit), and the exact Kerr field (which approximately describes the field of a spinning body, namely the Earth) — and their electromagnetic analogues: the exact fields of single non-spinning and spinning charged bodies, and two-body systems to first “post-Coulombian” approximation. The electromagnetic analogy proved illuminating to explain the structure of the gravitational invariants; to post-Newtonian order, in particular, a similar reasoning can be employed. We found that the invariant structure of the field of a single non-spinning body (where \( \star \mathbf{R} \cdot \mathbf{R} = 0 \) everywhere) is clearly different from that of a spinning body, in agreement with the analysis in \([2, 27]\), but that the latter (contrary to the claim in \([2]\)) is not substantially different from that of a system of two bodies orbiting each other: \( \star \mathbf{R} \cdot \mathbf{R} = 0 \) in the equatorial/orbital plane, \( \star \mathbf{R} \cdot \mathbf{R} \neq 0 \) generically elsewhere. In the post-Newtonian framework, this structure can actually be physically explained in

\(^{34}\) From the intrinsic relative velocities proposed in \([128]\), only one yields a symmetric notion of rest (i.e., \( A \) being comoving with \( B \) implies \( B \) to be comoving with \( A \)), and is not transitive (i.e. \( A \) being comoving with \( B \), and \( B \) being comoving with \( C \), does not mean that \( A \) comoves with \( C \) ).
both cases using the same reasoning. This closely mirrors
the situation in electromagnetism and can be traced back
to the fact that $\mathbb{H}_{\alpha\beta}$ is linear to 1PN order, such that a
superposition principle applies (like in electromagnetism)
and one can treat for these matters a spinning body as
an assembly (in the spirit of [18]) of translating mass el-
ments. We hope this may shed some light on this issue.

However, in spite of the insight it gives into the invari-
ant structures, it is crucial to realize that the analogy be-
tween the invariants of $F_{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$ is purely formal,
as it relates objects that do not play analogous physical
roles in the two theories. The effects involved are differ-
ent, and may actually be opposite (Sec. VII A). Focusing
on magnetism/gravitomagnetism (and taking, as probes,
magnetic dipoles/gyroscopes), the invariants of $F_{\alpha\beta}$ give
conditions for the existence of velocity fields for which
the magnetic field $B^\alpha$ vanishes, i.e., for which magnetic
dipoles do not undergo Larmor precession (but they feel
a force in general, as the magnetic tidal tensor $B_{\alpha\beta}$ is
non-vanishing for a particle moving in an inhomogeneous field); the invariants of $R_{\alpha\beta\gamma\delta}$ give (in vacuum) conditions
for the existence of velocity fields for which the gravit-
omegnetic tidal tensor $\mathbb{H}_{\alpha\beta}$ vanishes, i.e., a gyroscope feels no force (not that it does not precess relative to the
“distant stars”). Hence the use of the invariants and the
electromagnetic analogy in the discussion [1, 27] of gy-
roscopic precession and the gravitomagnetic deflection of
test particles (that have been under experimental and ob-
servational scrutiny) is essentially misguided, as these are
effects governed by the gravitomagnetic field $H^\alpha$ (the dy-
namical analogue of $B^\alpha$), not the tidal tensor $\mathbb{H}_{\alpha\beta}$. One
should not confuse GEM inertial fields with tidal tensors;
a pedagogical example are spacetimes with uniform $H^\alpha$,
e.g. the Gödel universe (Sec. VII B 3), where one has, in
a rigid frame, $E_{\alpha\beta} = 0$ and $\mathbb{H}_{\alpha\beta} = 0$ everywhere, whilst $G^\alpha = 0$ and $H^\alpha \neq 0$ (being purely electric from the
point of view of the curvature, and exactly the opposite in
terms of the GEM fields).

The curvature invariants are locally measurable quan-
tities that are built on GEM tidal tensors, not on inertial
fields, which are reference frame artifacts (that vanish in
locally inertial frames), and as such cannot be directly re-
ected in invariants. Yet still there are special cases
where one can infer about the gravitomagnetic field $\vec{H}$
from the invariants. $\vec{H}$ has a clear meaning in terms of
gyroscopic precession and test particle deflection relative
to the distant stars if it is measured in a shear-free frame
which is inertial at infinity. As discussed in Sec. VII C
in a vacuum Petrov type D spacetime, shear-free frames
where $\vec{H}$ is globally zero (i.e., shear- and vorticity-free
observer congruences) exist if and only if $\mathbf{R} \cdot \mathbf{R} = 0$ and
$\mathbf{R} \cdot \partial \mathbf{R} > 0$ hold in an open 4-D region. Concerning
the astrophysical setups of interest, this tells us that, in
the Kerr spacetime (describing approximately the field
of a spinning body), $\vec{H} \neq 0$ with respect to any non-
shearing frame, in any open 4-D region; and that in the
Schwarzschild spacetime there are shear-free frames (e.g.
the one adapted to the static observers, $u \propto \partial / \partial t$,
which

is star fixed), where $\vec{H} = 0$ everywhere. Hence, in these
two special cases, one can indeed imply, based on the cur-
vature invariants, a distinction between them in terms of
gravitomagnetic field $\vec{H}$/frame dragging, which, to some
extent, supports the claim in [27]. But since, amongst
the systems under discussion, these are the only ones of
Petrov type D (it does not apply to the 2-body system,
or others in general), this is all one can tell, from the
invariants (with the present knowledge), about $\vec{H}$.

We note (Sec. VII B 1), on the other hand, that in the
post-Newtonian regime (which pertains to all the grav-
itomagnetic effects detected to date, and those that one
hopes to detect in the near future), criteria for the van-
isishing or not of $\vec{H}$ in PN frames, based on the “scalars”
$G^2 - \vec{H}^2$ and $\vec{G} \cdot \vec{H}$, formally analogous to the elec-
 tromagnetic invariants, can be devised. These quantities,
however, are not field invariants, nor do they have a
straightforward relation with the curvature invariants.
Such analogy originates instead from the similarity be-
tween the transformation laws for the post-Newtonian
GEM fields and those for the electromagnetic fields.

Concluding, curvature invariants tell us about the
gravitomagnetic tidal field (magnetic curvature); they do
not tell us directly (or at all, in general), about the grav-
itomagnetic field $H^\alpha$ and the frame-dragging effects that
have been under experimental scrutiny, which are based
on spin precession and orbital effects (including preces-
sion) caused by the gravitomagnetic “force” $\vec{v} \times \vec{H}$
on test particles in (approximately) geodesic motion. Approp-
ate probes to measure magnetic curvature would be the
force on a gyroscope, or gravity gradiometers, as proposed in e.g. [123, 131].

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Appendix A: Algebraic classification of the Ricci tensor

We revise here the algebraic classification of the Ricci
tensor. We take the eigenvalue degeneracy of the trace-
less Ricci operator $S^\alpha_\beta \equiv R^\alpha_\beta - \frac{1}{2} R \delta^\alpha_\beta$ on (complexified)
tangent space at $p$, using the invariants $\mathbb{R}$. The char-

\[ \mathbb{R} = \mathbf{R} \cdot \mathbf{R} \]
acteristic polynomial of $S_{\alpha\beta}^\alpha$ is
\[ k(x) = x^4 - \frac{1}{2}I_6x^2 - \frac{1}{3}I_7x + \frac{1}{8}I_6^2 - \frac{1}{4}I_8. \]
Its discriminant equals $(I_S^3 - J_S^2)/432$, so there is a degenerate eigenvalue if and only if \[ I_S^3 = J_S^2. \]

Suppose $\lambda$ is such eigenvalue. Because of $S_{\alpha\beta}^\alpha = 0$ the four eigenvalues can then be written as $\lambda, \lambda, -\lambda + c$ and $-\lambda - c$, and one has
\[ I_6 = 4\lambda^2 + 2c^2, \quad I_7 = -6\lambda c^2, \quad I_8 = 4\lambda^4 + 12c^2\lambda^2 + 2c^4, \quad I_S = 4(c^2 - 4\lambda^2), \quad J_S = 8(c^2 - 4\lambda^2)^3, \]
which explicitly verifies (A1) and implies $J_S + 2I_6I_S = 24c^2(c^2 - 4\lambda^2)^2$. This leads to four main cases, fully characterized by the invariant conditions indicated in square brackets, and with several subcases based on the real or non-real character of the eigenvalues and the minimal polynomial $m(x)$ of $S_{\alpha\beta}^\alpha$.\(^{36}\)

(A) $\lambda = c = 0$ [$I_S = J_S = 0 = I_6$]. There is one quadruple eigenvalue $\lambda = 0$. The allowed minimal polynomials are $m(x) = x^p$ with $p \in \{1, 2, 3\}$.

(B) $c^2 = 4\lambda^2 \neq 0$ [$I_S = J_S = 0 \neq I_6$, implying $I_6 > 0$]. There is one triple, real eigenvalue $\lambda = -I_7/(2I_6) \neq 0$ and one simple, real eigenvalue $-3\lambda$. The allowed minimal polynomials are $m(x) = (x - \lambda)^p(x + 3\lambda)$ with $p \in \{1, 2, 3\}$.

(C) $c = 0 \neq \lambda$ [$I_S + 2I_6I_S = I_7 = 0 \neq I_6$, implying $I_6 > 0$]. There are two double, real eigenvalues $\lambda = \pm \sqrt{I_6}/2 \neq 0$ and $-\lambda$. The allowed minimal polynomials are $m(x) = (x - \lambda)^p(x + \lambda)$ with $p \in \{1, 2\}$.

(D) $(c^2 - 4\lambda^2)c \neq 0$ [$I_S^3 = J_S^2$, $J_S + 2I_6I_S \neq 0$]. There is one double, real eigenvalue $\lambda = -I_7I_S/(I_S + 2I_6I_S)$ and two simple eigenvalues $-\lambda \pm c$, with $c^2 = I_6/2 - 2\lambda^2$. The simple eigenvalues are real (non-real and complex conjugate) when $c^2 > 0$ $(c^2 < 0)$, and the allowed polynomials $m(x)$ are $(x - \lambda)^p((x + \lambda)^2 - c^2)$ with $p \in \{1, 2\}$ $(p = 1)$.

Finally one has the non-degenerate case
\[ (E) I_S^3 \neq J_S^2. \] All eigenvalues are simple, such that $m(x) = k(x)$. If $I_S^3 > J_S^2$ all four eigenvalues are real, while if $I_S^3 < J_S^2$ two are real and two non-real and complex conjugate.

Considering the further split of the subcases b) with $p = 1$ and d) with $p = 1$, $c^2 > 0$ into two branches each (see below) we retrieve the fifteen algebraic Ricci types listed in Table 5.1 of \(^{60}\).

The condition that $S_{\alpha\beta}^\alpha$ is diagonalizable corresponds to $p = 1$ in (A)-(D) and is automatic in (E)\(^{37}\); adding the condition that all eigenvalues are real [which only gives a further restriction in cases (D) and (E)] one arrives at the conditions (a)-(e) in Sec. III C. In the extended Segre notation of \(^{60}\) these cases precisely cover Ricci type [111, 1] and its degenerations, as indicated below. Here symbols 1 enclosed by round brackets refer to coinciding eigenvalues, and the 1 after the comma refers to the eigenvalue corresponding to the unique timelike eigenspace $T$ of $S_{\alpha\beta}^\alpha$, which is also indicated and contains the 4-velocities of precisely those observers $O'(u')$ for which $\epsilon_{(59)i)-(60)i}$ holds.

(a) coincides with type [(111, 1)] (Einstein space type, $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$), where $T$ is the full 4-D tangent space.

(b) splits into types [(11, 1)] and [(111), 1]. $\lambda$ is the triple eigenvalue. Take any $y^\alpha$ such that $u^\alpha = S_{\alpha\beta}^\beta y^\beta - \lambda y^\alpha \neq 0$; then $w^\alpha$ spans the eigendirection of the simple eigenvalue $-\lambda$, and the type is
\[ (b1) [1(11, 1)] \] (tachyonic fluid type) if $w^\alpha w_\alpha > 0$, where $T$ is the $\lambda$-eigenspace, i.e., the 3-D orthogonal complement of $w^\alpha$;

(b2) [(111), 1] (perfect fluid type) if $w^\alpha w_\alpha < 0$, where $T$ is 1-D and spanned by $w^\alpha$.

(c) is type [(11)(1, 1)] (non-null Einstein-Maxwell type), where $\lambda_\pm = \pm \sqrt{I_6}/2$ are the double eigenvalues and $T$ is 2-D. Compute $w^\alpha = S_{\alpha\beta}^\alpha u^\beta - \lambda_\pm u^\alpha$ for any timelike $u^\alpha$; if $w^\alpha w_\alpha < 0$ then $\lambda_-$ corresponds to $T$, else $\lambda_+$.\(^{38}\)

(d) splits into types [(11), 1, 1] and [(111), 1, 1]. $\lambda$ is the double eigenvalue and $-\lambda \pm c$ are the simple ones. Take any $y^\alpha$ such that $u^\alpha = (S_{\alpha\beta}^\beta + (\lambda_+ \pm \lambda_-)) y^\beta$.

\(^{37}\) Note that in the respective cases (b)-(d) of Sec. III C the polynomials $m(x) = (x - \lambda)(x + 3\lambda)$, $x^2 - I_6/4$ and $(x - \lambda)(x + \lambda)^2 - c^2$ are annihilating and of degree less than 4, such that there is at least one degenerate eigenvalue by footnote 5. The conditions $I_6 \neq 0$ in (b), (c) and $J_S + 2I_6I_S \neq 0$ in (d) of Sec. III C now ensure that there are at least two, resp. three different eigenvalues, such that $m(x)$ must be the minimal polynomial, having the distinct eigenvalues as roots (see again footnote 5). The complete list of eigenvalues then follows from $S_{\alpha\beta}^\alpha = 0$; e.g., when $m(x) = (x - \lambda)(x + 3\lambda)$ the complete list must be $\{\lambda, \lambda, \lambda, -3\lambda\}$ and we are in case (b) of the above Ricci classification.

\(^{38}\) Including also case (E), some subcases cannot occur due to the Lorentzian signature of the metric, namely $m(x) = k(x)$ in (A), (C) and (D) with $c^2 < 0$, and one pair resp. two pairs of non-real complex conjugate eigenvalues in (C) and (E). If the weak energy condition is assumed then those cases with two non-real eigenvalues, i.e., (D) with $c^2 < 0$ and (E) with $I_S^3 < J_S^2$, cannot occur either (see Ch. 5 of \(^{60}\)).
c) $\delta^{\alpha}_{\beta}(S_{\alpha}^{\beta}, y_{\beta}^i - \nu_{y_{\beta}^i}) \neq 0$; then $w_{\alpha}^I$ and $w_{\alpha}^N$ span the eigendirections of the simple eigenvalues, and the type is

(d1) $[11(1,1)]$ if $w_{\alpha}^I(w_+)^\alpha w_{\beta}^I(w_-)_\beta > 0$, where both $w_{\alpha}^I$ and $w_{\alpha}^N$ are spacelike and $\mathcal{T}$ is the 2-D orthogonal complement of $w_{\alpha}^I$ and $w_{\alpha}^N$;

(d2) $[11], 1]$ if $w_{\alpha}^N(w_+)^\alpha w_{\beta}^I(w_-)_\beta < 0$, where one of $w_{\alpha}^I$, $w_{\alpha}^N$ is timelike (the other one being spacelike) and spans the 1-D space $\mathcal{T}$.

(e) is type $[111,1]$, where all eigenvalues $\lambda_i$, $i = 1,4$ are simple. For each $i$ take $y_{\alpha i}^I$ such that $w_{\alpha i}^I = \prod_{j \neq i} S_{\alpha j}^I y_{\alpha j}^I - \lambda_j y_{\alpha j}^I \neq 0$; then one of the vectors $w_{\alpha i}^I$ is timelike (the other ones being spacelike) and spans the 1-D space $\mathcal{T}$.

1. Locally extrinsic gravitomagnetic curvature — alternative route

The existence of observers $w_{\alpha}^I$ for $\Xi_{\alpha \beta} = 0$ at a point $p$ (locally extrinsic gravitomagnetic curvature) requires one of the conditions (a)-(e) of Sec. III C to hold in any case, i.e., the Ricci type must be $[111,1]$ or one of its degenerations regardless of the Weyl-Petrov type. This can be easily tested and it is useful to do this a priori, along with verifying the necessary Weyl condition $C_{\alpha \beta \gamma \delta} = 0$ or $[63]$ or $[66]$.

Alternatively, for each allowed Ricci type, we can look when there are observers with 4-velocity satisfying ($[59]$) (i.e., those in the timelike Ricci eigenspace $\mathcal{T}$) that also satisfy ($[59]$); this is especially useful when the Weyl tensor is of Petrov type I, where the use of $l_{\alpha i}^I$ may be cumbersome due to the possibly intricate expressions for the Weyl eigenvalues ($[11]$). If the Ricci type is $[111,1]$ (vacuum type) the criterion reduces to the validity of either ($[34]$) or ($[35]$) or $R_{\alpha \beta \gamma \delta} = 0$. If the Ricci type is $[111,1]$ (perfect fluid type), $[11], 1]$ or $[111,1]$ then there is only one 4-velocity $u_{\alpha}^I$ that verifies ($[60]$), namely the one proportional to $w_{\alpha}^I$ constructed in (b2), the timelike $w_{\alpha}^I$ or $w_{\alpha}^N$ in (d2), or the timelike $w_{\alpha}^I$ in (e), respectively; hence if the spacetime is of one of these Ricci types at $p$ then it has locally extrinsic gravitomagnetic curvature if and only if ($[59]$) or equivalently *$C_{\alpha \beta \gamma \delta} w_{\gamma}^I w_{\delta}^I = 0$ holds*, where $w_{\alpha}^I$ is joint notation for the timelike vector mentioned in each case; this alternative criterion is especially useful for Ricci types $[111], 1]$ or $[1111], 1$, where the construction of $w_{\alpha}^I$ is straightforward, whereas type $[111,1]$ may be more difficult to treat if Descartes’ method for solving quartics is to be used to find the distinct Ricci eigenvalues $\lambda_i$. For Ricci types $[1111,1]$, (Einstein-Maxwell type) or $[11[1,1]$ the space $\mathcal{T}$ is 2-D and the vectors $u_{\alpha}^I$ satisfying ($[60]$) are given by

$$u_{\alpha}^I = \frac{(qk_{\alpha} + \bar{l}_{\alpha})}{\sqrt{2q}}, \quad q > 0,$$  (A2)

where the null vectors $k_{\alpha}$ and $\bar{l}_{\alpha}$ are chosen along the null directions of $\mathcal{T}$ and normalized by $k_{\alpha}l_{\alpha} = -1$. In terms of the symmetric tensors

$$A_{\alpha \beta} = \star C_{\alpha \gamma \beta \delta} k_{\gamma}k_{\delta}, \quad B_{\alpha \beta} = \star C_{\alpha \gamma \delta \beta} (k_{\gamma}l_{\delta} + \bar{l}_{\gamma}k_{\delta}),$$

$$C_{\alpha \beta} = \star C_{\alpha \gamma \beta \delta} l_{\gamma}l_{\delta}, \quad A^\alpha = A^{\alpha \beta} l_{\beta}, \quad C^\alpha = C_{\alpha \beta} k_{\beta}$$

the condition ($[59]$) for such $u_{\alpha}^I$ becomes

$$A_{\alpha \beta}q^2 + B_{\alpha \beta}q + C_{\alpha \beta} = 0.$$  (A3)

By contraction with $\bar{k}_{\alpha}$ (or $l_{\alpha}$) and use of $A_{\alpha} = -B_{\alpha \beta}k_{\beta}$ (or $C_{\alpha} = -B_{\alpha \beta}l_{\beta}$) this implies

$$qA_{\alpha} = C_{\alpha}.$$  

Note that the vectors $A_{\alpha}^I$ and $C_{\alpha}^I$ are orthogonal to $\mathcal{T}$, and are thus spacelike or zero. Moreover, if $A_{\alpha}^I = C_{\alpha}^I = 0$ then $B_{\alpha \beta} = 0$ (because of *$C_{\alpha \gamma \delta \beta} = 0$* and $A_{\alpha \beta}$ and $C_{\alpha \beta}$ are orthogonal to $\mathcal{T}$ (i.e., $A_{\alpha \beta}k_{\beta} = A_{\alpha \beta}l_{\beta} = 0$ and thus $A_{\alpha \beta} = 0$ or $A_{\alpha \beta}A_{\alpha \beta} = 0$, and analogously for $C_{\alpha \beta}$). It follows that a spacetime that is of Ricci type $[[11(1,1)]$ or $[[111,1)]$ at $p$ has locally extrinsic gravitomagnetic curvature if and only if one of the following holds:

(i) $A_{\alpha \beta} = B_{\alpha \beta} = C_{\alpha \beta} = 0$;

(ii) $A_{\alpha} = C_{\alpha} = 0$ (i.e., $B_{\alpha \beta} = 0$, $A_{\alpha \beta}C_{\alpha \beta} < 0$),  (A3)

with $q = -A^\alpha C_{\alpha}^I$ (i.e., $A^\gamma C_{\alpha \gamma \beta \delta} A_{\alpha \beta} = A^\gamma l_{\beta} A_{\alpha \beta}C_{\alpha \beta}$);

(iii) $A^\alpha C_{\alpha}^I = 0$, $A_{\alpha}C_{\alpha} > 0$,  (A3) with $q = -A^\alpha C_{\alpha}^I$.

In case (i) the Weyl-Petrov type is necessarily O or D, where the Weyl principal plane $\Sigma$ equals $\mathcal{T}$ for type D; $[59]$ is satisfied by all observers $\mathcal{O}^I(u^I)$ with 4-velocity $u_{\alpha}^I$ in this case, and by the unique observer given by the respective indicated values of $q$ in cases (ii) and (iii). Finally, for Ricci type $[[11(1,1)]$ (tachyonic fluid type) one needs to suitably parameterize the 2-D variety of the unit timelike vectors $u_{\alpha}^I$ within the 3-D space $\mathcal{T}$ (i.e., the orthogonal complement of the unique, spacelike simple eigenvector $w_{\alpha}^I$) and check when at least one of these vectors solves ($[59]$); for purely electric Petrov type D (i.e., when ($[63]$) is verified to hold) this is the case for all (respectively exactly one) $u_{\alpha}^I \in \Sigma$ when $w_{\alpha}^I$ is orthogonal to $\Sigma$ (respectively the projection of $v_{\alpha}^I$ onto $\Sigma$ is spacelike, the unique observer lying in $\Sigma$ and orthogonal to it). In the Petrov type I case this is more involved and left to be treated on a case-by-case basis.

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