GENERALIZATION OF HYPERBOLIC SMOOTHING APPROACH FOR NON-SMOOTH AND NON-LIPSCHITZ FUNCTIONS

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Abstract. In this study, we concentrate on the hyperbolic smoothing technique for some sub-classes of non-smooth functions and introduce a generalization of hyperbolic smoothing technique for non-Lipschitz functions. We present some useful properties of this generalization of hyperbolic smoothing technique. In order to illustrate the efficiency of the proposed smoothing technique, we consider the regularization problems of image restoration. The regularization problem is recast by considering the generalization of hyperbolic smoothing technique and a new algorithm is developed. Finally, the minimization algorithm is applied to image restoration problems and the numerical results are reported.

1. Introduction. The gradient based methods of optimization have been well studied over the years and they effectively applied to solve many interesting problems. With scientific developments, non-smooth problems have been arisen in many practical problems of engineering, finance, medical and other sciences. They have been transformed into well-known optimization problems such as min-max, min-sum-min and regularization problems [35, 28, 10, 23, 24, 25]. When the objective function \( f \) is not differentiable, it is not possible to use the gradient based optimization methods. Moreover, the optimality conditions may not be used. In order to eliminate this disadvantage, two different approaches come into prominence.

The first approach is the generalization of the notion of differentiation. Indeed, the conditions for differentiability are weakened by the help of the generalizations on differentiation and the well-known gradient based methods are upgraded. These generalization studies started with the rising of the concept of sub-differential [30, 12, 20, 17]. Although the sub-differential based methods have very interesting theoretical background, it is not easy to use for numerical computations.

The second one is smoothing techniques. The idea of these studies is based on the approximation to the non-smooth objective function by smooth functions. After the approximation process, gradient based methods are used to minimize the smoothed function [9]. Therefore the numerical applications of well-known algorithms with smoothing functions have been intensively studied for many years. The first studies

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on smoothing were seen in 1975 by Bertsekas [4] and in 1980 by Zang [47]. One of the cornerstone of the smoothing studies was presented by Nesterov [22]. The comprehensive overview on different smoothing techniques can be found in [13, 32].

The motivation of the smoothing studies emerge from the non-smooth problems. Therefore, many interesting non-smooth problems have been solved by using smoothing functions such as min-max [45], sum-max [36], penalty expressions of constrained optimization problems [19] and regularization problems [37, 14, 21]. Many interesting algorithms are developed and they are effectively applied to the non-smooth optimization problems [44]. On the other hand, not only are the smoothing techniques used for non-differentiable optimization problems but they are also applicable for solving system of equations/inequalities [46] including absolute value equations [7], many different versions of complementarity problems [29] and etc.

One of the important smoothing approach is the hyperbolic smoothing approach that was first studied by Xavier [39, 41, 34, 43]. The idea of hyperbolic smoothing function is rising from the geometric observation of the hyperbolic equation. The hyperbolic smoothing function is applied to solve clustering problems [40, 42, 2], min-max problems [1, 3] and exact penalty function problems [33]. Associated with using a non-Lipschitz regularization term in image restoration problems has remarkable advantages [5, 6], the generalization of hyperbolic smoothing approach is studied for non-Lipschitz type problems in [8] and used for solving regularization problems in [10]. Considering the wide range of application areas of smoothing functions, the hyperbolic smoothing function approach has not been studied in detailed for non-smooth especially non-Lipschitz problems with applications.

Motivating from the above process, our main aim is to disclose the useful properties of the hyperbolic smoothing approach for the non-smooth functions. We especially dwell on the generalizations of hyperbolic smoothing functions for non-Lipschitz functions. We also show the efficiency of the smoothing approach on the regularization problems.

2. Preliminaries. Throughout the paper, we always describe the local minimizers by $x_k^*$ and the global minimizer $x^*$. $\mathbb{R}_+$ denotes the non-negative real numbers and $\|\cdot\|$ denotes the Euclidean norm.

The smoothing function is defined by the following definition:

**Definition 2.1.** [8] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. The function $\bar{f} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is called a smoothing function of $f(x)$, if $\bar{f}(\cdot, \varepsilon)$ is continuously differentiable in $\mathbb{R}^n$ for any fixed $\varepsilon$, and for any $x \in \mathbb{R}^n$,

$$\lim_{z \to x, \varepsilon \to 0} \bar{f}(z, \varepsilon) = f(x).$$

It is well-known that the non-smoothness of non-smooth problems mostly originate from the presence of “$\max\{p, \min\{t, 0\}\}$, $|\cdot|_p$” in the formulation of the $f(x)$ for $p > 0$.

It is stated in [8] that there exists the following relations:

$$|t| = \max\{t, 0\} + \max\{-t, 0\}$$

$$\max\{t, y\} = t + \max\{y - t, 0\}$$

$$\min\{t, y\} = t - \max\{t - y, 0\}$$

for \( p = 1 \). Now we recall some ideas about hyperbolic smoothing function. One of the important methods is using density (kernel) function with

\[
\psi(t) \geq 0, \quad \int_{-\infty}^{\infty} \psi(t)dt = 1.
\]

The kernel function for hyperbolic smoothing function is defined as

\[
\psi(t) = \frac{1}{2} \frac{1}{\sqrt{(t^2 + 1)^3}},\tag{1}
\]

for any \( t \in \mathbb{R} \) \([9, 29, 37]\). The smoothing function is obtained as

\[
\theta(t, \varepsilon) = \int_{-\infty}^{\infty} |t-s|\hat{\psi}(s, \varepsilon)ds = \sqrt{t^2 + \varepsilon^2},
\]

where

\[
\hat{\psi}(t, \varepsilon) = \frac{1}{\varepsilon} \psi\left(\frac{t}{\varepsilon}\right).\tag{2}
\]

Another way is obtained from the geometric observation of the following hyperbolic equation

\[
y^2 - t^2 = \varepsilon^2 \Rightarrow y = \pm \sqrt{t^2 + \varepsilon^2}.
\]

Since \( \theta(t, \varepsilon) = \sqrt{t^2 + \varepsilon^2} \) is smooth and approaches to \(|t|\) as \( \varepsilon \to 0 \).

The hyperbolic smoothing function given in \([41, 40]\) for \( \varphi(t) = \max\{t, 0\} \), \( t \in \mathbb{R} \) is defined by

\[
\varphi_1(t, \varepsilon) = \frac{t + \sqrt{t^2 + \varepsilon^2}}{2},
\]

where \( \varepsilon > 0 \) is a parameter.

**Proposition 1.** [1] The function \( \varphi_1(t, \varepsilon) \) has the following properties:

i. \( \varphi_1(t, \varepsilon) \) is convex and \( C^\infty \),

ii. \( 0 \leq \varphi_1(t, \varepsilon) - \varphi(t) \leq \frac{\varepsilon}{2} \).

Another version of hyperbolic smoothing technique is studied in \([33, 15]\) and it is defined as

\[
\varphi_2(t, \varepsilon) = \frac{t + \sqrt{t^q + \varepsilon^q}}{2},
\]

where \( q \in (1, \infty) \) and \( \varepsilon > 0 \).

**Proposition 2.** [33] The function \( \varphi_2(t, \varepsilon) \) has the following properties:

i. \( \varphi_2(t, \varepsilon) \) is continuously differentiable,

ii. \( 0 \leq \varphi_2(t, \varepsilon) - \varphi(t) \leq \frac{\varepsilon}{2} \).

Note that, the properties of the smoothing functions such as convexity, differentiability and etc. are given with respect to the independent variable \( t \) and the \( \varepsilon \) is just a parameter.
The properties of hyperbolic smoothing techniques. We consider generalized hyperbolic smoothing function approach for non-Lipschitz functions. In this section, we handle hyperbolic smoothing functions for φ(t) = |t|^p, 0 < p ≤ 1 and present the useful properties of them.

In [8], the smoothing function
\[
\phi(t, \varepsilon) := \left(\sqrt{t^2 + \varepsilon^2}\right)^p,
\]
for \(\varepsilon > 0\). Now, we define two new smoothing technique for non-Lipschitz functions based on the definition of \(\varphi_2(t, \varepsilon)\). The first one is defined as
\[
\phi_2(t, \varepsilon) := \left(\sqrt{t^q + \varepsilon^q}\right)^p,
\]
where \(q \in (1, \infty)\). The second one is defined by
\[
\phi_3(t, \varepsilon) = \left(|t|^r + \varepsilon^r\right)^{\frac{1}{r}},
\]
where \(r \) is integer such that \(pr > 1\) and \(\varepsilon > 0\). We have the following properties.

Lemma 3.1. Let 0 < p ≤ 1, q > 1 and r > 1 such that pr > 1 then,

(i) for any \(t \in \mathbb{R}\)
\[
0 < \phi_i(t, \varepsilon) - \phi(t) \leq \max\{\varepsilon^p,\varepsilon\},
\]
(ii) \(\phi_i(t, \varepsilon)\) is continuously differentiable,
(iii) the functions \(\phi_i(t, \varepsilon)\) and \(|t|^p\) have the same minimizer,
for all \(\varepsilon > 0\) and \(i = 1, 2, 3\).

Proof. (i) It can be observed that \(\phi(t, \varepsilon) \geq \phi(t)\) for \(i = 1, 2, 3\). For \(\varepsilon > 0\), we have
\[
\phi_1(t, \varepsilon) - \phi(t) = \left(\sqrt{t^2 + \varepsilon^2}\right)^p - |t|^p
\]
\[
\leq \left(\sqrt{t^2 + \varepsilon^2 - |t|}\right)^p
\]
\[
\leq \varepsilon^p.
\]
Similarly, we obtain
\[
\phi_2(t, \varepsilon) - \phi(t) \leq \varepsilon^p,
\]
for \(\varepsilon > 0\). Since 0 < \(\frac{1}{r} < 1\), for r > 1 then
\[
\phi_3(t, \varepsilon) - \phi(t) = \left(|t|^r + \varepsilon^r\right)^{\frac{1}{r}} - |t|^p
\]
\[
\leq \left(|t|^r\right)^{\frac{1}{r}} + \varepsilon - |t|^p
\]
\[
\leq \varepsilon.
\]
Therefore, we obtain the proof of Lemma 3.1 (i).
(ii) The differentiability is obvious.
(iii) Since the derivatives of the functions $\phi_i(t, \varepsilon)$ for $i = 1, 2, 3$ are

$$\phi'_1(t, \varepsilon) = \frac{t}{(x^2 + \varepsilon^2)^{\frac{1}{5}}}$$

$$\phi'_1(t, \varepsilon) = \frac{t}{(x^q + \varepsilon^q)^{\frac{1}{5}}}$$

and

$$\phi'_3(t, \varepsilon) = \frac{|t|^{pq-1}{\text{sgn}}(t)}{((|t|^q + \varepsilon^q)^{\frac{1}{q}}},$$

where

$$\text{sgn}(t) = \begin{cases} 
1, & t > 0, \\
0, & t = 0, \\
-1, & t < 0.
\end{cases}$$

Then by considering optimality conditions for smooth optimization the result is obtained.

Lemma 3.2. Let $\varepsilon > 0$. If $p = 1$ then $\phi_i(t, \varepsilon)$ is convex for any $q > 1$ and $i = 1, 2, 3$.

Proof. The proof is straightforward.

Now, let us introduce the effects of changes of parameters on the hyperbolic smoothing functions and compare these three different types of hyperbolic smoothing approaches in terms of numerical point of view.

Example 1. Let $g$ be function from $\mathbb{R}$ to $\mathbb{R}^+$, $g(t) = |t|^p$, $p = \frac{2}{5}$. It is easy to say that the function $g$ is non-smooth and non-Lipschitz. We consider the smoothing functions which are given in (3), (4) and (5).

In the beginning, let us investigate the smoothing function $\phi_1$ and $\phi_2$ in terms of smoothing parameter $\varepsilon$. The smoothing parameter is chosen as $\varepsilon = 0.4$ and $\varepsilon = 0.8$ by fixing $q = 4$ and the graphs of the original and smoothing functions are shown in Fig. 1 (a) and (b), respectively. It can be seen from Fig. 1 that better approximation is obtained when $\varepsilon$ approaches to 0.

![Figure 1](image-url)
If the value of the parameter $q$ is changed for a fixed $\varepsilon = 0.5$ then, the effect can be observed in Fig. 2. The graphs of the smoothing functions are presented in Fig. 2 (a) for $q = \frac{4}{3}$ and Fig. 2 (b) for $q = 4$. It can be deduced that if the parameter $q$ is increased, then better approximation is obtained.

![Figure 2](image2.png)

**Figure 2.** The graphs of smoothing functions $\phi_1(x, \varepsilon)$ and $\phi_2(x, \varepsilon)$ with different $q$ values.

Now, we introduce the effects of parameter changes on $\phi_3(t, \varepsilon)$. The smoothing parameter are chosen as $\varepsilon = 0.4$ (blue and dotted) and $\varepsilon = 0.8$ (green and dashed) by fixing $r = 4$ and the graphs of the original and smoothing functions are shown in Fig. 3 (a). The graphs of the smoothing functions for $r = 3$ (green and dashed) and $r = 6$ (blue and dotted) by fixing $\varepsilon = 0.5$ are presented in 3 (b). It can be concluded that better approximation is obtained if $\varepsilon$ approaches to 0 or/and increasing the parameter $r$.

![Figure 3](image3.png)

**Figure 3.** The graphs of smoothing function $\phi_3(t, \varepsilon)$ with different $\varepsilon$ and $r$ values.

Finally, we compare all the smoothing functions for the similar parameter values. Let us take $\varepsilon = 0.5$ and $q = r = 4$ then, we obtain the graphs of smoothing functions.
at Fig 4. It can be observed that the smoothing function $\phi_3(t, \varepsilon)$ is closer to original function than the other smoothing functions.

**Figure 4.** The graphs of smoothing functions $\phi_1(x, \varepsilon)$, $\phi_2(x, \varepsilon)$ and $\phi_3(x, \varepsilon)$.

4. **The degree of approximation of the hyperbolic smoothing functions.**

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a Lipschitz continuous function and $\psi'(y) > 0$ and $\psi''(y) < 0$ for all $y \in \mathbb{R}_+$ and $\lim_{y \to 0} \psi(y) = 0$. The well known examples of $\psi$ are $y$, $\frac{\alpha y}{1+\alpha y}$ and $\arctan(\alpha y)$ for any $\alpha > 0$.

**Theorem 4.1.** Let $\varepsilon > 0$, $0 < p \leq 1$, $q > 1$ and $r > 1$ such that $pr > 1$. Then, for $i = 1, 2, 3$ we have

$$|\psi(\phi_i(t, \varepsilon)) - \psi(|t|^p)| \leq L \max\{\varepsilon, \varepsilon^p\},$$

where $L > 0$ is a constant.

**Proof.** Since the function $\psi$ is non-negative, increasing and Lipschitz continuous, we have

$$\psi(\phi_i(t, \varepsilon)) - \psi(|t|^p) \leq L (\phi_i(t, \varepsilon) - |t|^p),$$

with $L > 0$ for any $\varepsilon > 0$. By the Lemma 3.1

$$\psi(\phi_i(t, \varepsilon)) - \psi(|t|^p) \leq L \max\{\varepsilon, \varepsilon^p\}.$$
Theorem 4.3. Let $t^*$ be a local minimizer of $\psi(|t|^p)$ and $\bar{t}$ is a local minimizer of $\psi(\phi_i(t, \varepsilon))$ for $i = 1, 2, 3$. Then, we have $\psi(\phi_i(\bar{t}, \varepsilon)) \to \psi(|t^*|^p)$ for $\varepsilon \to 0$.

Proof. Since the function $\psi$ is increasing,

\[
\psi(\phi_i(\bar{t}, \varepsilon)) - \psi(\bar{t})^p \leq \psi(\phi_i(\bar{t}, \varepsilon)) - \psi(|t^*|^p).
\]

\[
\leq \psi(\phi_i(t^*, \varepsilon)) - \psi(|t^*|^p),
\]

\[
\leq L \max\{\varepsilon, \varepsilon^p\}.
\]

for $L > 0$.

Let us define the function $H$ from $\mathbb{R}^n$ to $\mathbb{R}$ by $H(x) = \sum_{k=1}^{m} \psi(|h_k(x)|^p)$, where $h_k(x) : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable functions and $0 < p \leq 1$. The function $H(x)$ is used generally for $\ell_p$ norm regularization problems. The function $H$ is non-Lipschitz if $h_k(x) = 0$ for any $k = 1, 2, \ldots, m$ and $x \in \mathbb{R}^n$. The smoothing function is defined by $\bar{H}(x, \varepsilon) = \sum_{k=1}^{m} \psi(\phi_i(h_k(x), \varepsilon))$. Now, we present the following results.

Theorem 4.4. Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$

\[
0 \leq \bar{H}(x, \varepsilon) - H(x) \leq K \max\{\varepsilon, \varepsilon^p\},
\]

(7)

Where $K > 0$ is constant.

Proof. From Lemma 3.1 and Theorem 4.1 we obtain

\[
\bar{H}(x, \varepsilon) - H(x) = \sum_{k=1}^{m} \psi(\phi_i(h_k(x), \varepsilon)) - \sum_{i=1}^{m} \psi(|h_k(x)|^p)
\]

\[
= \sum_{k=1}^{m} (\psi(\phi_i(h_k(x), \varepsilon)) - \psi(|h_k(x)|^p))
\]

\[
\leq L \sum_{k=1}^{m} (\phi_i(h_k(x), \varepsilon) - |h_k(x)|^p)
\]

\[
\leq L m \max\{\varepsilon, \varepsilon^p\},
\]

for $i = 1, 2, 3$. Hence the inequality (7) is obtained.

Corollary 2. Let $\varepsilon \to 0$ then, the function $\bar{H}(x, \varepsilon)$ approaches to $H(x)$.

Proof. It can be easily obtained from Theorem 4.4.

Theorem 4.5. Let $\{\varepsilon_j\} \to 0$ and $x^j$ be a minimizer of $\bar{H}(x, \varepsilon_j)$. Suppose that $\bar{x}$ is an accumulation point of $\{x^j\}$. Then $\bar{x}$ is a minimizer of $H(x)$.

Proof. By considering Theorem 4.4 and Corollary 2, the proof is obtained.

Theorem 4.6. Suppose that $x^*$ is a local minimizer of $H(x)$ and $\bar{x}$ is a local minimizer of $\bar{H}(x, \varepsilon)$. Then, we have

\[
|\bar{H}(\bar{x}, \varepsilon) - H(x^*)| \leq K \varepsilon,
\]

(8)

where $K > 0$ is constant. Moreover, we have $\bar{H}(\bar{x}, \varepsilon) \to H(x^*)$ for $\varepsilon \to 0$.

Proof. From Lemma 3.1 and Theorem 4.4, we obtain

\[
|\bar{H}(\bar{x}, \varepsilon) - H(x)| \leq |\bar{H}(\bar{x}, \varepsilon) - H(x^*)|
\]

\[
\leq |\bar{H}(x^*, \varepsilon) - H(x^*)|
\]

\[
\leq K \max\{\varepsilon, \varepsilon^p\}.
\]
Thus, the inequality (8) is obtained and the proof is completed.

Therefore, we obtain two interesting variant of hyperbolic smoothing functions for non-Lipschitz optimization. We can conclude that the above smoothing process is applicable for $\ell_p$ minimization for $0 < p \leq 1$.

5. Application to regularization problems. In this section we present the application of the smoothing techniques in solving image restoration problems. The image restoration problem can be described as reconstructing an image from the observed image. Our aim is to computationally estimate the original image. This is one of the common problems especially in medical imaging, biological engineering, astronomical imaging and other areas [11, 18, 26, 38]. The most common mathematical model is represented as follows:

$$y = Hx + e,$$

(9)

where $e \in \mathbb{R}^n$ represents the noise, $H$ is an $m \times n$ blurring matrix, $x \in \mathbb{R}^n$ is the underlying image and $y \in \mathbb{R}^m$ is an observed image [8, 11]. We consider the regularized least squares optimization problem in the process of the restoration of the image. The regularization problem is defined by the following:

$$\min_{x \in \mathbb{R}^n} f(x) = F(x) + \rho \sum_{i=1}^{n} \psi(d_i^T x),$$

(10)

where $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth function, $\rho$ is a regularization parameter, $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is potential function and $d_i^T \in \mathbb{R}^n$ is the $l$-th row of the matrix $D$ which is defined as

$$D = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & -1
\end{pmatrix}.$$

We define the function $F(x)$ as

$$F(x) = ||Hx - y||,$$

(11)

where $H \in \mathbb{R}^{m \times n}$ is a matrix of type block-Toeplitz-Toeplitz-block type and $y \in \mathbb{R}^m$ is observed image. Moreover, the function $\psi$ is called as potential function. There are many different types of potential functions such as convex, non-convex, smooth and non-smooth. We consider the potential defined as $\psi_1(t) = |t|^p$ for $p \in (0, 1)$, $\psi_2(t) = \frac{\alpha |t|}{1 + \alpha |t|}$, and $\psi_3(t) = \ln(1 + \alpha |t|)$, $\alpha > 0$. By considering the smoothing process introduced in Section 2, the smooth approximation to the function $\psi$ is illustrated shortly as

$$\tilde{\psi}_k(t, \varepsilon) = \psi_k(\phi_i(t, \varepsilon)),$$

(12)

where $q > 1$ such that $pq > 1$, for $i, k = 1, 2, 3$.

If the function $F$ is defined as (11) and $\tilde{\psi}$ is defined as (12), then the problem in (10) transforms into the following smoothing $\ell_2 - \ell_p$ regularization problem:

$$\min_{x \in \mathbb{R}^n, \varepsilon} \tilde{f}(x, \varepsilon) = F(x) + \rho \sum_{l=1}^{n} \tilde{\psi}_k (d_l^T x_l, \varepsilon),$$

(13)

where $\varepsilon > 0$. Since finding exact solution of regularization problem in (13) is not easy task, the approximate solution is useful for these types of problems. The
τ−approximate solution for inequality constrained optimization problems is defined as follows:

**Definition 5.1.** Assume τ > 0, x^j and x^{j+1} are the consecutive solutions of problem (13) depending on the different ε values, a point x^{j+1} is called τ−approximate solution of problem (10), if

\[ |\tilde{f}(x^{j+1}, \varepsilon_{j+1}) - \tilde{f}(x^j, \varepsilon_j)| \leq \tau. \]

There are many interesting variants of problem (13) that has been studied in many paper [14, 10, 16] and there exist many algorithms in order to solve it. We construct the following algorithm to solve the problem given in (13).

**Algorithm 1**

Step 1 Determine ε_0 > 0, q_0 > 1, ρ_0 > 0, η < 1, τ > 0, N > 1 and choose the initial point x^0. Let j = 0 and go to Step 2.

Step 2 Use x^j as the starting point and apply local solver to problem in (10). Let x^{j+1} be the solution. Go to Step 3.

Step 3 If x^{j+1} is τ−approximate solution for problem then stop and x^{j+1} is the optimal solution. Otherwise, determine ε_{j+1} = ηε_j, q_{j+1} = Nq_j, ρ_{j+1} = Nρ_j and j = j + 1, then go to Step 2.

**Theorem 5.2.** Assume that for ρ ∈ [ρ_0, ∞), τ ∈ (0, τ_0] and r ∈ (1, r_0] the set

\[ \arg\min_{x \in \mathbb{R}^n} \tilde{f}(x, \varepsilon) \neq \emptyset. \]

Let x^j is generated by Algorithm 1 when ηN < 1. If \{x^j\} has a limit point, then the limit point of x^j is the solution for (P).

**Proof.** Assume \( \overline{x} \) is a limit point of \{x^j\}. Then there exists set \( J \subset \mathbb{N} \), such that \( x^j \to \overline{x} \) for \( j \in J \). We have to show that \( \overline{x} \) is the optimal solution for problem (10). Thus, it is sufficient to show \( f(\overline{x}) \leq \inf_{x \in \mathbb{R}^n} f(x) \). By considering the Step 2 in Algorithm 1 and for any \( x \in \mathbb{R}^n \),

\[ \tilde{f}(x^j, \varepsilon_j) \leq \tilde{f}(x^j, \varepsilon_j) = f(x) + K\varepsilon_j. \]

When \( j \to \infty \), we have \( f(\overline{x}) \leq f(x) \). \( \square \)

5.1. **The numerical results.** We perform the numerical experiments on the four different test images in restoration problem by considering the above 3 different smoothing functions. The function \( \phi_1 \) proposed in [8] and it is applied to image restoration problems in [10]. Numerical results are obtained by using the program Matlab R2015A on PC with configuration of Intel Core i3, 8GB RAM. The test images are House image of size 256 × 256 and, Barbara, Cameraman and Peppers images of size 512 × 512. The original images are shown in Fig. 2.

There are many interesting ways of adding noise to original image [27], we consider the well-known distributions such as Gaussian noise with parameter value as 0.01 and Salt & Pepper noise to the original image with parameter value as 0.05. We construct the optimization problems based on the depicted way at (10) for each images. We apply Algorithm 1 to constructed problems for each images. The stopping tolerance value is determined as τ = 10^{-4} and 500 iterations are allowed as
maximum number of iterations. The $\psi_1$ is considered as potential function. We compare our results with the well-known total-variation (TV) algorithm in [31].

The quality of restoration is provided by the Algorithm 1 with different smoothing functions and TV algorithm has been measured by using Peak Signal to Noise Ratio (PSNR) values. All of the numerical results for the images of “Barbara”, “Cameraman”, “House” and “Peppers” are reported. The different types of potential functions and different values of $p$ are considered. The total iteration number (Iter), the PSNR value and total running time (Time) are reported for each of the smoothing functions and TV algorithm. The detailed numerical results is presented in Table 1.
Table 1. The computational results.

| Noisy Image | Denoised by $\Phi_1$ | Denoised by $\Phi_2$ | Denoised by $\Phi_3$ | Denoised by TV |
|-------------|----------------------|----------------------|----------------------|----------------|
|             | PSNR | Iter | Time | PSNR | Iter | Time | PSNR | Iter | Time |
| Barbara     | 20.156 | 98  | 23.272 | 10.545 | 116 | 23.378 | 11.194 | 112 | 24.684 | 13.757 | 124 | 24.661 | 14.278 |
| Cameraman   | 18.161 | 96  | 23.155 | 11.121 | 120 | 23.152 | 13.440 | 89  | 23.340 | 9.0128 | 99  | 23.147 | 11.554 |
| House       | 18.149 | 96  | 25.126 | 12.285 | 119 | 24.976 | 13.705 | 93  | 25.201 | 11.0368 | 102 | 24.943 | 10.386 |
| Peppers     | 20.075 | 98  | 26.591 | 8.9565 | 117 | 26.430 | 11.302 | 86  | 27.496 | 10.9960 | 140 | 27.111 | 14.11  |

In the first experiment, we consider Barbara image Fig. 5 (a). The Algorithm 1 is applied to the image combining with the different smoothing functions. The denoised images are shown in Fig. 6. The numerical results are reported in the first part of the Table 1. We see that combining Algorithm 1 with $\phi_3$ presents the best quality image and fast results for both Gaussian and Salt & Pepper noised image among the other smoothing functions and TV algorithm. As the second experiment, the Algorithm 1 is applied to the Cameraman image (Fig. 5 (b)) combining with the different smoothing functions. The denoised images are shown in Fig. 7. The numerical results are reported in the second part of the Table 1. The Algorithm 1 with $\phi_3$ provide best quality restoration with low computational effort. At the third experiment, the Algorithm 1 is applied to the House image (Fig. 5 (c)) combining with the different smoothing functions. The denoised images are shown in Fig. 8. The numerical results are reported in the third part of the Table 1. The Algorithm 1 with $\phi_3$ provide best quality restoration with low computational time. At the last experiment, the Algorithm 1 is applied to the Peppers image (Fig. 5 (d)) combining with the different smoothing functions. The denoised images are shown in Fig 9. The numerical results are reported in the last part of the Table 1. The Algorithm 1 with $\phi_3$ provide best quality restorations among the other methods.

According to all numerical results, the smoothing function $\phi_3$ is the best fitted smoothing technique for these types of problems among the other smoothing techniques and standard TV algorithm. The Algorithm 1 with smoothing functions

![Image](image_url)
restore the Salt & Pepper noised images than the Gaussian noised images. As a consequence, it is seen that CPU time cost is increasing when the dimension is increased in application of Algorithm 1 with different smoothing functions and TV algorithm. Moreover, the changes of the total number of iterations parallel to CPU time while the quality values of the images are independent from CPU time and total number of iterations. On the whole performance order depending on the quality of the image can be summarized as below:

$$\phi_2 \leq TV \leq \phi_1 \leq \phi_3.$$  \hspace{1cm} (14)

All of these preliminary results promise the further studies.
6. **Conclusion.** In this study, we present the generalized hyperbolic smoothing approach for some sub-class of non-Lipschitz functions. We propose two new smoothing functions based on the hyperbolic smoothing technique for non-Lipschitz problems and we introduce the useful properties of the smoothing functions. The new variants provide better approximation than the previous smoothing functions. We apply this smoothing technique to solve image restoration problems. The smoothing functions $\phi_2$ and $\phi_3$ are considered for the first time in image restoration problems. The obtained results are satisfactory and promising. Moreover, it is easy to apply the problem at hand and easily adapt to new problems. Therefore, the algorithm with hyperbolic smoothing functions is user friendly. As a result, it can be observed from the processed images and numerical results that our method is useful for annihilating the noise on test images. These results showed that our method is faithful for working such problems.

For future work, it can be interesting to investigate the efficiency of different smoothing techniques acting on regularization problems as in [38]. We consider the hyperbolic smoothing techniques with one of the well-known algorithm and we obtain remarkable improvements on PSNR values (see Table 1). Therefore, the numerical efficiency of hyperbolic smoothing techniques can be re-evaluated with different minimization algorithms. The hyperbolic smoothing technique can be reconsidered for other image processing problems such as deblurring, segmentation, detection and etc.

**REFERENCES**

[1] A. M. Bagirov, A. Al Nuaimat and N. Sultanova, Hyperbolic smoothing function method for minimax problems, *Optimization*, 62 (2013), 759–782.

[2] A. M. Bagirov, B. Ordin, G. Ozturk and A. E. Xavier, An incremental clustering algorithm based on hyperbolic smoothing, *Comput. Optim. Appl.*, 61 (2015), 219–241.

[3] A. M. Bagirov, N. Sultanova, A. Al Nuaimat and S. Taheri, Solving minimax problems: Local smoothing versus global smoothing, *Numerical Analysis and Optimization*, Springer Proceedings in Mathematics and Statistics, 235 (2018), 23–43.

[4] D. P. Bertsekas, Nondifferentiable optimization via approximation, *Math. Programming Stud.*, 3 (1975), 1–25.
[5] W. Bian and X. Chen, Smoothing neural network for constrained non-Lipschitz optimization with applications, IEEE Trans. Neural Netw. Learn. Syst., 23 (2012), 399–411.
[6] W. Bian and X. Chen, Linearly constrained non-Lipschitz optimization for image restoration, SIAM J. Imaging Sci., 8 (2015), 2294–2322.
[7] L. Caccetta, B. Qu and G. Zhou, A globally and quadratically convergent method for absolute value equations, Comput. Optim. Appl., 48 (2011), 45–58.
[8] X. Chen, Smoothing methods for nonsmooth, nonconvex minimization, Math. Prog. Ser. B., 134 (2012), 71–99.
[9] C. Chen and O. L. Mangasarian, A class of smoothing functions for nonlinear and mixed complementarity problems, Comput. Optim. Appl., 5 (1996), 97–138.
[10] X. Chen, M. K. Ng and C. Zhang, Non-Lipschitz ℓp-regularization and box constrained model for image restoration, IEEE Trans. Image Process., 21 (2012), 4709–4721.
[11] X. Chen and W. Zhou, Smoothing nonlinear conjugate gradient method for image restoration using nonsmooth nonconvex minimization, SIAM J. Imaging Sciences, 3 (2010), 765–790.
[12] F. H. Clarke, Optimization and Nonsmooth Analysis, John Wiley & Sons, New York, 1983.
[13] C. Grossmann, Smoothing techniques for exact penalty methods, Contemporary Mathematics, In book Panorama of Mathematics: Pure and Applied, 658 (2016), 249–265.
[14] Y. Huang, H. Liu and W. Cong, A note on the smoothing quadratic regularization method for non-Lipschitz optimization, Numer. Algor., 69 (2015), 863–874.
[15] X. Jiang and Y. Zhang, A smoothing-type algorithm for absolute value equations, J. Ind. Manag. Optim., 9 (2013), 789–798.
[16] M. Kang and M. Jung, Simultaneous image enhancement and restoration with non-convex total variation, J. Sci. Comput., 87 (2021), Paper No. 83, 46 pp.
[17] K. C. Kiwiel, Methods of Descent for Nondifferentiable Optimization, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1985.
[18] G. Landi, A modified Newton projection method for ℓ1-regularized least squares image de-blurring, J. Math. Imaging Vis., 51 (2015), 195–208.
[19] S.-J. Lian, Smoothing approximation to ℓ1 exact penalty function for inequality constrained optimization, Appl. Math. Comput., 219 (2012), 3113–3121.
[20] M. M. Mäkelä and P. Neittaanmäki, Nonsmooth Optimization, World Scientific, Singapore, 1992.
[21] N. Mau Nam, L. T. H. An, D. Giles and N. Thai An, Smoothing techniques and difference of convex functions algorithms for image reconstructions, Optimization, 69 (2020), 1601–1633.
[22] Y. Nesterov, Smooth minimization of non-smooth functions, Math. Program., 103 (2005), 127–152.
[23] M. Nikolova, Minimizers of cost functions involving nonsmooth data-fidelity terms. Application to the processing outliers, SIAM J. Numer. Anal., 40 (2002), 965–994.
[24] M. Nikolova, M. K. Ng, S. Zheng and W. K. Ching, Efficient reconstruction of piecewise constant images using nonsmooth nonconvex minimization, SIAM J. Imaging Sci., 1 (2008), 2–25.
[25] O. N. Onak, Y. Serinagaoglu-Dogrusoz and G.-W. Weber, Effects of a priori parameter selection in minimum relative entropy method on inverse electrocardiography problem, Inverse Probl. Sci. Eng., 26 (2018), 877–897.
[26] C. T. Pham, G. Gamard, A. Kopylov and T. T. T. Tran, An algorithm for image restoration with mixed noise using total variation regularization, Turk. J. Elec. Eng. Comp. Sci., 26 (2018), 2831–2845.
[27] C. T. Pham, T. T. T. Tran and G. Gamard, An efficient total variation minimization method for image restoration, Informatica, 31 (2020), 539–560.
[28] R. A. Polyak, Smooth optimization methods for minimax problems, SIAM J. Control Optim., 26 (1988), 1274–1286.
[29] L. Qi and D. Sun, Smoothing functions and smoothing Newton method for complementarity and variational inequality problems, J. Optim. Theory Appl., 113 (2002), 121–147.
[30] R. T. Rockafellar and R. J.-B. Wets, Variational Analysis, Springer, Berlin, 1998.
[31] L. I. Rudin, S. Osher and E. Fatemi, Nonlinear total variation based noise removal algorithms, Physica D, 60 (1992), 259–268.
[32] B. Saheya, C.-H. Yu and J.-S. Chen, Numerical comparisons based on four smoothing functions for absolute value equations, J. Appl. Math. Comput., 56 (2018), 131–149.
[33] A. Sahiner, G. Kapusuz and N. Yilmaz, A new smoothing approach to exact penalty functions for inequality constrained optimization problems, Numer. Algebra Contol Optim., 6 (2016), 161–173.

[34] M. Souza, A. E. Xavier, C. Lavor and N. Maculan, Hyperbolic smoothing and penalty techniques applied to molecular structure determination, Oper. Res. Lett., 39 (2011), 461–465.

[35] P. Taylan, G.-W. Weber and F. Yerlikaya-Ozkurt, A new approach to multivariate adaptive regression splines by using Tikhonov regularization and continuous optimization, TOP, 18 (2010), 377–395.

[36] A. H. Tor, Hyperbolic smoothing method for sum-max problems, Neural, Parallel Sci. Comput., 24 (2016), 381–391.

[37] S. Voronin, G. Ozkaya and D. Yoshida, Convolution based smooth approximations to the absolute value function with application to non-smooth regularization, Preprint, (2015). arXiv:1408.6795.

[38] C. Wu, J. Zhan, Y. Lu and J.-S. Chen, Signal reconstruction by conjugate gradient algorithm based on smoothing $l_1$-norm, Calcolo, 56 (2019), Paper No. 42, 26 pp.

[39] A. E. Xavier, Penalizacao Hiperbolica, I Congresso Latino-Americano de Pesquisa Operacional e Engenharia de Sistemas, 8 a 11 de Novembro, Rio de Janeiro, Brasil, 1982.

[40] A. E. Xavier, The hyperbolic smoothing clustering method, Patt. Recog., 43 (2010), 731–737.

[41] A. E. Xavier and A. A. F. de Oliveira, Optimal covering of plane domains by circles via hyperbolic smoothing, J. Global Optim., 31 (2005), 493–504.

[42] A. E. Xavier and V. L. Xavier, Solving the minimum sum-of-squares clustering problem by hyperbolic smoothing and partition into boundary and gravitational regions, Patt. Recog., 44 (2011), 70–77.

[43] V. L. Xavier and A. E. Xavier, Accelerated hyperbolic smoothing method for solving the multisource Fermat-Weber and k-Median problems, Knowl. Based Syst., 191 (2020), 105226.

[44] N. Yilmaz and A. Sahiner, On a new smoothing technique for non-smooth, non-convex optimization, Numer. Algebra Contol Optim., 10 (2020), 317–330.

[45] H. Yin, An adaptive smoothing method for continuous minimax problems, Asia-Pac. J. Oper. Res., 32 (2015), 1540001, 19 pp.

[46] L. Yuan, C. Fei, Z. Wan, W. Li and W. Wang, A nonmonotone smoothing Newton method for system of nonlinear inequalities based on a new smoothing function, Comput. Appl. Math., 38 (2019), Paper No. 91, 11 pp.

[47] I. Zang, A smoothing out technique for min-max optimization, Math. Programm., 19 (1980), 61–77.

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