Conformal Partial Waves:
Further Mathematical Results

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Further results for conformal partial waves for four point functions for conformal primary scalar fields in conformally invariant theories are obtained. They are defined as eigenfunctions of the differential Casimir operators for the conformal group acting on two variable functions subject to appropriate boundary conditions. As well as the scale dimension $\Delta$ and spin $\ell$ the conformal partial waves depend on two parameters $a, b$ related to the dimensions of the operators in the four point function. Expressions for the Mellin transform of conformal partial waves are obtained in terms of polynomials of the Mellin transform variables given in terms of finite sums. Differential operators which change $a, b$ by $\pm 1$, shift the dimension $d$ by $\pm 2$ and also change $\Delta, \ell$ are found. Previous results for $d = 2, 4, 6$ are recovered. The trivial case of $d = 1$ and also $d = 3$ are also discussed. For $d = 3$ formulae for the conformal partial waves in some restricted cases as a single variable integral representation based on the Bateman transform are found.

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Very shortly after the completion of this paper Francis Dolan died, may this be a small contribution to his memory.

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1. Introduction

In conformal field theories the correlation functions of conformal primary operators $\phi_i$ satisfy non trivial identities obtained by expanding various pairs of operators $\phi_i\phi_j$ appearing in the correlation function using the operator product expansion. Assuming that in such correlation functions $\phi_i(x_i)\phi_j(x_j)$ has a convergent operator product expansion in terms of a complete set of of operators with differing spins and scale dimensions then it is natural to formulate bootstrap equations relating the expansions for each pair $i, j$. Individual contributions to each expansion are identified as conformal partial waves if they include the contributions in the operator product expansion of a conformal primary operator $O^{(\ell)}_\Delta$, with scale dimension $\Delta$ and spin $\ell$, and also all its descendants formed by the action of derivatives on $O^{(\ell)}_\Delta$. To find ways of making the bootstrap equations tractable it is important to have succinct mathematical expressions for the conformal partial waves.

For four point correlation functions the conformal partial waves $G^{(\ell)}_\Delta$ are functions of two conformal invariants $u, v$. They can be determined by requiring them to be eigenfunctions of the Casimir operators of the conformal group, subject to appropriate boundary conditions, with eigenvalues given in terms of $\Delta, \ell$. The conformal group is non compact so that $\Delta$ is allowed to vary continuously, although subject to restrictions to ensure unitarity. From the 1970’s onwards, when the conformal bootstrap was first proposed, various formulae for the conformal partial waves $G^{(\ell)}_\Delta$ have been derived although these are often rather involved [1,2,3]. More recently expressions for conformal partial waves as functions of $u, v$ have been obtained in terms of Mellin-Barnes integral representations [4,5,6,7].

Acting on $G^{(\ell)}_\Delta(u,v)$ the quadratic Casimir reduces to a linear second order operator in $u, v$. For $\ell = 0$ solutions may be obtained as a double power series in $u, 1 - v$ although there is no simple generalisation for $\ell > 0$. With a Mellin-Barnes representation where the conformal partial wave is expressed as a $s, t$ integral over $u^s v^t$ the eigenvalue equation reduces to a six term difference equation involving the two independent integration variables $s, t$ [6]. As shown here in the particular case of leading twist, essentially for dimension $d$ when $\Delta = \ell + d - 2$, the conformal partial wave is annihilated by an additional second order differential operator. This renders finding explicit solutions in this case, by various techniques, much more tractable. The leading twist case is particularly relevant in superconformal theories when it corresponds to the contribution of semi-short operators.

In two and four dimensions, with extensions possible for any even dimension, quite simple results have been found which have been found useful in bootstrap applications [8,9,10]. These are described in two previous papers [11,12]. The essential step to achieve a relatively simple form for the conformal partial waves is to express them in terms of symmetric functions of two variables $x, \bar{x}$, which are determined by the conformal invariants.
In terms of \( u, v \) the eigenvalue equation simplifies and the conformal partial waves involve just products of ordinary hypergeometric functions, depending on three parameters, of \( x \) and \( \bar{x} \). The degree of complication is independent of \( \ell \). In the leading twist case the conformal partial wave reduces to a single variable function. In terms of the variables \( x, \bar{x} \) the additional second order operator for leading twist is a reduction of the \( d \)-dimensional wave operator, and the simplifications for even dimensions are a reflection of Huygens principle.

In many respects results for conformal partial waves are a non compact analogue to expressions for two variable harmonic polynomials, for which there is an extensive literature \([13,14,15,16]\) and which also extends to consideration of symmetric functions of \( p \) variables for any \( p \) \([17,18]\). For \( p = 1 \) the harmonic polynomials of interest are just the classical Jacobi polynomial \( P_n^{(\alpha,\beta)}(y) \) which may be defined by requiring them to be orthogonal for \( n = 0,1,2,\ldots \) with respect to the measure \( dy (1−y)^{\alpha}(1+y)^{\beta} \) for \( y \in [−1,1] \). Their symmetric two variable extensions defined in \([13]\) \( P_{nm}^{(\alpha,\beta,\gamma)}(y + \bar{y}, y\bar{y}) \) are orthogonal with respect to the measure \( dy d\bar{y} ((1−y)(1−\bar{y}))^{\alpha}((1+y)(1+\bar{y}))^{\beta} |y−\bar{y}|^{2\gamma+1} \). For \( \gamma = \pm \frac{1}{2} \) there is an explicit construction in terms of the single variable Jacobi polynomials which is identical to the formulae for dimension \( d = 2,4 \) conformal partial waves in terms of hypergeometric functions. In general the correspondence requires \( 2\gamma \to d−3 \). For any \( \gamma \) in \([13,16]\) it was shown that the two variable harmonic polynomials satisfy various recurrence relations for which there may be expected to be equivalent results in the non compact case. Previous results \([12]\) gave partial support for this and we endeavour to derive in this paper expressions for derivative operators relating conformal partial waves.

In section 2 of this paper we define conformal partial waves in terms of eigenfunctions of the quadratic Casimir of the conformal group. Single variable functions \( g_p(a, b; x) \) which play an essential role in the construction of conformal partial waves are introduced. These are hypergeometric functions, up to an additional power \( x^p \), and are the non compact analogs of the Jacobi polynomials \( P_n^{(\alpha,\beta)}(y) \). The equations for the conformal partial waves are simplified by introducing the new variables, \( x, \bar{x} \), so that they become symmetric functions \( F_{\lambda_1,\lambda_2}(a, b; x, \bar{x}) \) with \( \lambda_{1,2} = \frac{1}{2}(\Delta \pm \ell) \) and \( a, b \) determined by the dimensions of of the operators appearing in the four point function. In section 3 we describe a conformally invariant integral which may be expressed in terms of the sum of a conformal partial wave corresponding to the contributions arising from an operator \( \mathcal{O}_\Delta^{(\ell)} \) and also its shadow \( \overline{\mathcal{O}}_{d−\Delta}^{(\ell)} \). This leads to a Mellin-Barnes integral expression which reduces the problem of determining conformal partial waves to solving a multi-term difference equation with polynomial solutions in the two complex variables \( s, t \) of degree \( \ell \). Explicit results are given for \( \ell = 1,2 \). More general solutions have an algebraic complexity which increases rapidly with \( \ell \), although results may be obtained for the leading twist case by solving a
three term difference equation.

In section 4 various recursion relations for conformal partial waves in general dimensions are obtained. These involve the construction two sets of differential operators each of which are related by considering commutators with the differential operator representing the quadratic Casimir. By including the quartic Casimir, which is a fourth order differential operator, these form two closed algebras which ensure that their action in each case generates a finite number of conformal partial waves with \( \Delta, \ell \) differing by \( \pm 1 \) and in the first example also \( a, b \) differing by \( \pm \frac{1}{2} \). We also construct second order differential operators which acting on \( F_{\lambda_1 \lambda_2}(a, b; x, \bar{x}) \) change \( a, b \) by \( \pm 1 \) and the dimension \( d \) by \( \pm 2 \). Applying these shift operators twice leads to eigenvalue equations which are equivalent to those given by the quartic Casimir. In section 5 previous results found in \cite{12} for \( d = 4, 6 \) are obtained from the \( d = 2 \) result using the dimension raising operator. The conformal partial waves are formed from two \( g_p \) functions depending on \( x, \bar{x} \). In section 6 conformal partial waves for \( d = 1 \), when they reduce to just one single variable function \( g_p(x) \) since kinematics require \( x = \bar{x} \) in this case, are found. For \( d = 3 \) solutions for restricted \( \lambda_2 \) are found using an integral representation related to the Bateman transform for solutions of the three dimensional wave equation.

Some supplementary mathematical results are described in five appendices. Appendix A evaluates some relevant two dimensional conformal integrals. In appendix B we show how analogous results are obtained for the two variable extensions of Jacobi polynomials and which may be mostly found in the mathematical literature. In appendix C we show how the apparent singularity in the six dimensional case cancels while in appendix D we obtain some results for conformal partial wave expansions when only leading twist operators are present. An alternative construction for conformal partial waves based on an expansion in terms of Jack polynomials, used previously in \cite{12}, in described briefly in appendix E although the general form is very complicated.

2. Definitions and General Results for Conformal Partial Waves

The four point function for four scalar primary operators \( \phi_i \) with scale dimensions \( \Delta_i \) has the general form

\[
\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = \frac{1}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)} (x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)} (x_{14}^2)^{\frac{1}{2}(\Delta_1+\Delta_4)}} F(u, v),
\]

(2.1)

where

\[
x_{ij} = x_i - x_j, \quad \Delta_{ij} = \Delta_i - \Delta_j,
\]

(2.2)
and $u, v$ are the two independent conformal invariants

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$  \hspace{1cm} (2.3)

The conformal partial waves $G^{(\ell)}_{\Delta}(u, v)$ are so defined so that the conformal partial wave expansion becomes

$$F(u, v) = \sum_{\Delta, \ell} a_{\Delta, \ell} G^{(\ell)}_{\Delta}(u, v).$$  \hspace{1cm} (2.4)

where $G^{(\ell)}_{\Delta}(u, v)$ is the contribution of an operator $O^{(\ell)}_{\Delta}$, with scale dimension $\Delta$ and spin $\ell$, and its descendants to the operator product expansion of $\phi_1 \phi_2$ or $\phi_3 \phi_4$.

The operator $O^{(\ell)}_{\Delta}$ and also all its descendants are eigenvectors of the Casimir operators formed from the generators $M_{AB} = -M_{BA}$, $A, B = 0, 1, \ldots, d + 1$, of the conformal group $SO(d, 2)$ acting on $O^{(\ell)}_{\Delta}$ and which obey the algebra

$$[M_{AB}, M_{CD}] = \eta_{AC} M_{BD} - \eta_{BC} M_{AD} - \eta_{AD} M_{BC} + \eta_{BD} M_{AC},$$  \hspace{1cm} (2.5)

with $\eta_{AB} = \text{diag.}(-1, 1, \ldots, 1, -1)$. For the quadratic and quartic Casimirs constructed from $M_{AB}$ we then have

$$\frac{1}{2} M^{AB} M_{BA} O_{\Delta}^{(\ell)} = C_{\Delta, \ell} O_{\Delta}^{(\ell)},$$

$$\frac{1}{2} M^{AB} M_{BA} M^{CD} M_{DA} O_{\Delta}^{(\ell)} = D_{\Delta, \ell} O_{\Delta}^{(\ell)},$$  \hspace{1cm} (2.6)

where

$$C_{\Delta, \ell} = \Delta(\Delta - d) + \ell(\ell + d - 2),$$

$$D_{\Delta, \ell} = \Delta^2(\Delta - d)^2 + \frac{1}{2} d(d - 1) \Delta(\Delta - d)$$

$$+ \ell^2(\ell + d - 2)^2 + \frac{1}{2}(d - 1)(d - 4) \ell(\ell + d - 2).$$  \hspace{1cm} (2.7)

If $M_{i AB}$ are the generators acting on $\phi_i$ then conformal invariance of the correlation function requires $\sum_i M_{i AB} \langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = 0$. The essential equation determining the conformal partial waves is then

$$\left(\frac{1}{2}(M_1 + M_2)^{AB} (M_1 + M_2)_{BA} - C_{\Delta, \ell}\right) \frac{1}{(x_{12}^2)^{\frac{1}{4}(\Delta_1 + \Delta_2)}} \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}\Delta_{12}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}\Delta_{34}} G^{(\ell)}_{\Delta}(u, v) = 0.$$  \hspace{1cm} (2.8)

This leads directly to the two variable second order equation

$$\mathcal{D} G^{(\ell)}_{\Delta}(u, v) = \frac{1}{2} C_{\Delta, \ell} G^{(\ell)}_{\Delta}(u, v),$$  \hspace{1cm} (2.9)
for
\[ \mathcal{D} = (1 - u - v) \frac{\partial}{\partial v} \left( v \frac{\partial}{\partial v} + a + b \right) + u \frac{\partial}{\partial u} \left( 2u \frac{\partial}{\partial u} - d \right) \]
\[ - (1 + u - v) \left( u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + a \right) \left( u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + b \right), \]
where
\[ a = -\frac{1}{2} \Delta_{12}, \quad b = \frac{1}{2} \Delta_{34}. \]  

(2.10)

The contribution of an operator \( \mathcal{O}_\Delta^{(\ell)} \) to the four point function from the operator product expansion then has the form
\[ \langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle \bigg|_{\phi_1\phi_2,\phi_3\phi_4 \sim \mathcal{O}_\Delta^{(\ell)}} \]
\[ = \frac{1}{(x_{12}^2)^{\frac{1}{2}}(\Delta_1 + \Delta_2)} (x_{34}^2)^{\frac{1}{2}}(\Delta_3 + \Delta_4)} \left( \frac{x_{14}^2}{x_{24}^2} \right)^a \left( \frac{x_{13}^2}{x_{23}^2} \right)^b a_{\Delta,\ell} G_\Delta^{(\ell)}(u, v). \]

(2.12)

It follows directly that the conformal partial waves \( G_\Delta^{(\ell)} \) satisfy symmetry relations under
\[ 1 \leftrightarrow 2 \text{ and } 3 \leftrightarrow 4, \]
\[ G_\Delta^{(\ell)}(u/v, 1/v) = (-1)^\ell v^b G_\Delta^{(\ell)}(u, v) \bigg|_{a \rightarrow -a} = (-1)^\ell v^a G_\Delta^{(\ell)}(u, v) \bigg|_{b \rightarrow -b}, \]
and, by considering \( 1 \leftrightarrow 4, 2 \leftrightarrow 3 \), \( G_\Delta^{(\ell)} \) is also a symmetric function of \( a, b \).

To leading order as \( u \rightarrow 0, v \rightarrow 1, \mathcal{D}(u^p (1-v)^q) \sim (p(2p - d) + q(2p + q - 1))u^p (1-v)^q \)
so that the relevant solutions of (2.9) for conformal partial waves may be determined by imposing the boundary conditions
\[ G_\Delta^{(\ell)}(u, v) = c_\ell u^{\frac{1}{2}}(\Delta - \ell)(1-v)^\ell \left( 1 + \mathcal{O}(u, 1-v) \right). \]

(2.14)

The constant \( c_\ell \) in (2.14) is inessential and may be set to 1 but it is convenient here to allow for more general possibilities, although we require \( c_0 = 1 \).

As previously mentioned for general dimensions \( d \) and \( \ell > 0 \) results for \( G_\Delta^{(\ell)}(u, v) \) do not have any simple form and so are not readily useful. For \( d = 2, 4 \) more explicit expressions have been found [11,12]. These involve using the different variables \( x, \bar{x} \) defined in terms of \( u, v \) by
\[ u = x \bar{x}, \quad v = (1-x)(1-\bar{x}). \]

(2.15)

In the Euclidean domain \( x, \bar{x} \) are conjugates. With (2.15) \( G_\Delta^{(\ell)} \) becomes a symmetric function of \( x, \bar{x} \)
\[ G_\Delta^{(\ell)}(u, v) = F_{\lambda_1,\lambda_2}(x, \bar{x}) = F_{\lambda_1,\lambda_2}(\bar{x}, x), \]

(2.16)
for
\[ \lambda_1 = \frac{1}{2}(\Delta + \ell), \quad \lambda_2 = \frac{1}{2}(\Delta - \ell), \quad \ell = \lambda_1 - \lambda_2 = 0, 1, 2, \ldots. \]

Corresponding to (2.14) we have the boundary conditions

\[ F_{\lambda_1 \lambda_2}(x, \bar{x}) \sim \begin{cases} c_\ell (x\bar{x})^{\lambda_2} x^\ell, & x\bar{x} > 0, \\ c_{\lambda_1 - \lambda_2} x^{\lambda_1} \bar{x}^{\lambda_2}, & x, \bar{x} > 0, \end{cases} \]

where the limit \( \bar{x} \to 0 \) is taken first. The differential operator \( D \) in (2.10) becomes in terms of \( x, \bar{x} \)

\[ \Delta^{(\varepsilon)} \equiv \Delta^{(\varepsilon)}(a, b) = D_x(a, b) + D_{\bar{x}}(a, b) + 2\varepsilon \frac{x\bar{x}}{x - \bar{x}} \left( (1 - x) \frac{\partial}{\partial x} - (1 - \bar{x}) \frac{\partial}{\partial \bar{x}} \right), \]

for \( D \) the single variable differential operator defined by

\[ D_x(a, b) = x^2 (1 - x) \frac{d^2}{dx^2} - (a + b + 1) x^2 \frac{d}{dx} - abx, \]

and for
\[ \varepsilon = \frac{1}{2}(d - 2). \]

Unless necessary the dependence of \( \Delta^{(\varepsilon)} \), and also \( D_x \) in (2.20), on \( a, b \) is suppressed when there is no ambiguity.

The eigenvalue equation (2.10) is then equivalent to

\[ \Delta^{(\varepsilon)} F_{\lambda_1 \lambda_2}(x, \bar{x}) = c_{\lambda_1 \lambda_2} F_{\lambda_1 \lambda_2}(x, \bar{x}), \quad c_{\lambda_1 \lambda_2} = \lambda_1(\lambda_1 - 1) + \lambda_2(\lambda_2 - 1 - 2\varepsilon), \]

for symmetric functions \( F_{\lambda_1 \lambda_2}(x, \bar{x}) \) subject to the boundary conditions (2.18). When it is desirable to exhibit the dependence on \( a, b \) or \( \varepsilon \) the conformal partial waves are written as

\[ F_{\lambda_1 \lambda_2}(x, \bar{x}) \equiv F^{(\varepsilon)}_{\lambda_1 \lambda_2}(a, b; x, \bar{x}). \]

The symmetry relations (2.13) by virtue of (2.16) now become

\[ F_{\lambda_1 \lambda_2}(a, b; x', \bar{x}') = (-1)^\ell v^b F_{\lambda_1 \lambda_2}(-a, b; x, \bar{x}), \]

\[ = (-1)^\ell v^a F_{\lambda_1 \lambda_2}(a, -b; x, \bar{x}), \]

with \( v \) as in (2.15) and

\[ x' = \frac{x}{x - 1}, \quad \bar{x}' = \frac{\bar{x}}{\bar{x} - 1}. \]

The conformal partial waves \( F_{\lambda_1 \lambda_2}(x, \bar{x}) \) may be regarded as functions of the elementary symmetric functions of two variables, \( x + \bar{x}, x\bar{x} \). The limiting result given by (2.18) may be extended for \( x = O(\bar{x}) \) to the form

\[ F_{\lambda_1 \lambda_2}(x, \bar{x}) \sim (x\bar{x})^{\frac{1}{2}\Delta} \left( f(\sigma) + O(x + \bar{x}, \sqrt{x\bar{x}}) \right), \]

where

\[ \Delta = \frac{1}{2}(\Delta + \ell), \quad \Delta = \frac{1}{2}(\Delta - \ell), \quad \ell = \lambda_1 - \lambda_2 = 0, 1, 2, \ldots. \]
\[
\sigma = \frac{x + \bar{x}}{2(x\bar{x})^{\frac{1}{4}}}.
\] (2.26)

Substituting (2.25) into (2.22), with the boundary conditions (2.18), requires
\[
(\sigma^2 - 1)f''(\sigma) + (1 + 2\varepsilon)\sigma f'(\sigma) = \ell(\ell + 2\varepsilon)f(\sigma), \quad f(\sigma) \underset{\sigma \to \infty}{\sim} c_\ell (2\sigma)^\ell.
\] (2.27)

The relevant solutions of (2.27) are Gegenbauer polynomials we may then write
\[
f(\sigma) = \hat{C}_\ell^\varepsilon(\sigma),
\] (2.28)

choosing an alternative normalisation so that
\[
\hat{C}_\ell^\varepsilon(\sigma) = \frac{\ell!}{(2\varepsilon)_\ell} C_\ell^\varepsilon(\sigma), \quad \hat{C}_\ell^\varepsilon(1) = 1.
\] (2.29)

In this case in (2.18)
\[
c_\ell \equiv c^{(\varepsilon)}_\ell = \frac{(\varepsilon)_\ell}{(2\varepsilon)_\ell}.
\] (2.30)

An additional reduction to a single variable function, for any \(\varepsilon\), may be found by considering a limit in which \(\bar{x} \to 0\) since, using \(\Delta^{(\varepsilon)} \sim D_x + \bar{x}^2 \frac{\partial^2}{\partial x^2} - 2\varepsilon \bar{x} \frac{\partial}{\partial x}\),
\[
\Delta^{(\varepsilon)}(\bar{x}^q g(x)) = \bar{x}^q (D_x g(x) + q(q - 1 - 2\varepsilon)g(x)) + O(\bar{x}^{q+1}).
\] (2.31)

Hence the limiting behaviour (2.18) alternatively extends to
\[
F_{\lambda_1 \lambda_2}(x, \bar{x}) \underset{\bar{x} \to 0}{\sim} c_{\lambda_1 - \lambda_2} \bar{x}^{\lambda_2} g_{\lambda_1}(x),
\] (2.32)

so long as \(g_{\lambda_1}\) is a solution of
\[
D_x(a, b)g_p(a, b; x) = p(p - 1)g_p(a, b; x),
\] (2.33)

subject to
\[
g_p(a, b; x) \underset{x \to 0}{\sim} x^p.
\] (2.34)

Since
\[
x^{-p}D_x(a, b)x^p = x^2(1 - x) \frac{d^2}{dx^2} + x(2p - (2p + a + b + 1)x) \frac{d}{dx} - (p + a)(p + b)x + p(p - 1),
\] (2.35)

the functions \(g_p\) defined by (2.33) and (2.34) may be identified as just standard hypergeometric functions,
\[
g_p(a, b; x) = x^p F(p + a, p + b; 2p; x).
\] (2.36)

1 If \(g_p(a; b; x)\) is a solution of (2.33) then so is \(g_{-p+1}(a; b; x)\) but the required solution is determined by the boundary condition (2.34) and we further assume \(p \geq 0\).
The single variable functions $g_p$ defined by (2.36) are essential for constructing general expressions for $F_{\lambda_1,\lambda_2}(x,\bar{x})$ for any $\varepsilon$, they play an analogous role for non compact $SO(1,2)$ to the harmonic Jacobi polynomials $P_n^{(\alpha, \beta)}$ in relation to compact $SO(3)$.

There are various identities for the $g_p$ functions inherited from those for hypergeometric functions such as

$$g_p(a, b; x) = e^{\pm i\pi p} (1 - x)^{-b} g_p(-a, b; x') = e^{\pm i\pi p} (1 - x)^{-a} g_p(a, -b; x')$$

$$= (1 - x)^{-a-b} g_p(-a, -b; x),$$

(2.37)

where $x'$ is defined in (2.24). For $p$ non integer the choice of factors $e^{\pm i\pi p}$ depends on which side of the cut with branch point at $x = 0$ arising from $x^p$ the continuation from $x \in (0, 1)$ to $x' < 0$ is defined. The last equality is a reflection of the identity

$$D_x(a, b) = (1 - x)^{-a-b} D_x(-a, -b) (1 - x)^{a+b}.$$  

(2.38)

The result (2.32) is equivalent to

$$G^\ell_{\Delta}(u, v) \sim c_\ell u^{\frac{1}{2}(\Delta - \ell)} (1 - v)^{\ell} F\left(\frac{1}{2}(\Delta + \ell) + a, \frac{1}{2}(\Delta + \ell) + b; \Delta + \ell; 1 - v\right) \quad \text{as} \quad u \to 0, \quad (2.39)$$

extending (2.14).

The operator $\Delta^{(\varepsilon)}$ also satisfies identities which are of crucial assistance in constructing solutions of the eigenvalue problem. Thus

$$\Delta^{(\varepsilon)} \left(\frac{x\bar{x}}{x - \bar{x}}\right)^{2\varepsilon - 1} = \left(\frac{x\bar{x}}{x - \bar{x}}\right)^{2\varepsilon - 1} \left(\Delta^{(1 - \varepsilon)} + 2(1 - 2\varepsilon)\right),$$

(2.40)

so that letting

$$F^{(\varepsilon)}_{\lambda_1,\lambda_2}(x, \bar{x}) = \left(\frac{x\bar{x}}{x - \bar{x}}\right)^{2\varepsilon - 1} \tilde{F}_{\lambda_1,\lambda_2}(x, \bar{x}),$$

(2.41)

gives equivalently

$$\Delta^{(-\varepsilon + 1)} \tilde{F}_{\lambda_1,\lambda_2}(x, \bar{x}) = \left(\lambda_1(\lambda_1 - 1) + (\lambda_2 - 2\varepsilon + 1)(\lambda_2 - 2)\right) \tilde{F}_{\lambda_1,\lambda_2}(x, \bar{x}).$$

(2.42)

When $\varepsilon = 1$ the conformal partial waves are then reducible to the $\varepsilon = 0$ case for which $\Delta^{(\varepsilon)}$ in (2.19) is just a sum of two independent single variable differential operators.

Furthermore, with $v$ as in (2.15),

$$\Delta^{(\varepsilon)}(a, b) = v^{-a-b} \Delta^{(\varepsilon)}(-a, -b) v^{a+b},$$

(2.43)

which implies

$$F_{\lambda_1,\lambda_2}(a, b; x, \bar{x}) = v^{a+b} F_{\lambda_1,\lambda_2}(-a, -b; x, \bar{x}),$$

(2.44)

corresponding to combining (2.23a, b).
3. Integral Expressions

Results for conformal partial waves can also be obtained in terms of integral expressions for four point correlation functions. To describe these results we first define for any conformal primary operator of scale dimension \( d \), \( \mathcal{O}_{\Delta,\mu_1...\mu_\ell}^{(\ell)} \) an associated dual or shadow operator by

\[
\tilde{\mathcal{O}}_{\Delta,\mu_1...\mu_\ell}^{(\ell)}(x) = k_{\Delta,\ell} \frac{1}{\pi^{2}d} \int d^{d}y \frac{1}{(x - y)^{2d-\Delta}} \mathcal{F}_{\mu_1...\mu_\ell,\nu_1...\nu_\ell}(x - y) \mathcal{O}_{\Delta,\mu_1...\mu_\ell}^{(\ell)}(y),
\]

where \( \mathcal{F}_{\mu_1...\mu_\ell,\nu_1...\nu_\ell}(x) \) is the inversion tensor for symmetric traceless tensors, formed from the symmetrized product of \( \ell \) inversion tensors \( I_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu/x^2 \). The integral in (3.1) is divergent unless \( \Delta < \frac{1}{2}d \) but can be extended to more general \( \Delta \) by analytic continuation so that under conformal transformations (3.1) defines a conformal primary operator of scale dimension \( d - \Delta \). Choosing

\[
k_{\Delta,\ell} = \frac{1}{(\Delta - 1)_{\ell}} \frac{\Gamma(d - \Delta + \ell)}{\Gamma(\Delta - \frac{1}{2}d)},
\]

ensures that

\[
\tilde{\mathcal{O}}_{\Delta,\mu_1...\mu_\ell}^{(\ell)} = \mathcal{O}_{\Delta,\mu_1...\mu_\ell}^{(\ell)}.
\]

In general the shadow operator constructed in (3.1) does not belong to the set of relatively local operators defining a unitary conformal field theory but it plays a useful role in the present discussion.

To construct integral expressions we consider the conformally covariant three point functions [19]

\[
\langle \mathcal{O}_{\Delta,\mu_1...\mu_\ell}^{(\ell)}(x) \varphi_3(x_3) \varphi_4(x_4) \rangle = \frac{1}{(x_3; x_4)_{\frac{1}{2}(\Delta_3+\Delta_4)}} \mathcal{F}_{\Delta,\mu_1...\mu_\ell}^{(\ell)}(x; x_3, x_4),
\]

\[
\langle \varphi_1(x_1) \varphi_2(x_2) \mathcal{O}_{\Delta,\mu_1...\mu_\ell}^{(\ell)}(x) \rangle = \frac{1}{(x_1; x_2)_{\frac{1}{2}(\Delta_1+\Delta_2)}} \mathcal{F}_{\Delta,\mu_1...\mu_\ell}^{(\ell)}(x; x_1, x_2),
\]

where

\[
\mathcal{F}_{\Delta,\mu_1...\mu_\ell}^{(\ell)}(x; x_3, x_4) = \tilde{X}_{\mu_1 \cdots \mu_\ell} \frac{(x_3; x_4)_{\lambda_2}}{(x - x_3)^{\lambda_2+b} (x - x_4)^{\lambda_2-b}},
\]

\[
\mathcal{F}_{\Delta,\mu_1...\mu_\ell}^{(\ell)}(x; x_1, x_2) = X_{\mu_1 \cdots \mu_\ell} \frac{(x_1; x_2)_{\lambda_2}}{(x - x_1)^{\lambda_2-a} (x - x_2)^{\lambda_2+a}},
\]

for

\[
\tilde{X}_\mu = \frac{(x_4 - x_\mu)}{(x_4 - x)^2} - \frac{(x_3 - x_\mu)}{(x_3 - x)^2}, \quad X_\mu = \frac{(x_1 - x_\mu)}{(x_1 - x)^2} - \frac{(x_2 - x_\mu)}{(x_2 - x)^2}.
\]
In (3.5) \( \bar{X}_{\{\mu_1 \ldots \mu_\ell\}} \), \( X_{\{\mu_1 \ldots X_{\mu_\ell}\}} \) denotes symmetrisation and subtraction of traces. With the definition (3.1) and using integrals obtained in [11]

\[
\langle \hat{O} \rangle_{d-\Delta,\mu_1 \ldots \mu_\ell}^{(\ell)}(x) \varphi_3(x_3) \varphi_4(x_4) = \frac{\gamma_{\lambda,b}}{\gamma_{\lambda,a}} \frac{1}{(x_3)^{1/2}(\Delta_3+\Delta_4)} \mathcal{F}_d^{(\ell)}(x_3, x_4) ,
\]

\[
\langle \varphi_1(x_1) \varphi_2(x_2) \hat{O} \rangle_{d-\Delta,\mu_1 \ldots \mu_\ell}^{(\ell)}(x) = \frac{\gamma_{\lambda,a}}{\gamma_{\lambda,a}} \frac{1}{(x_1)^{1/2}(\Delta_1 + \Delta_2)} \mathcal{F}_d^{(\ell)}(x_1, x_2) ,
\]

for

\[
\gamma_{\lambda,a} = \Gamma(\lambda + a) \Gamma(\lambda - a) ,
\]

and the abbreviations

\[
\bar{\lambda}_1 = \frac{1}{2}(d - \Delta + \ell), \quad \bar{\lambda}_2 = \frac{1}{2}(d - \Delta - \ell) .
\]

Assuming (3.5) the expression for a four point function given by

\[
F_\Delta^{(\ell)}(x_1, x_2, x_3, x_4) = \gamma_{\lambda_1,a} \gamma_{\lambda_2,b} \frac{1}{\pi^{d/2}} \int d^d x \mathcal{F}_\Delta^{(\ell)}(x_1, x_2) \mathcal{F}_d^{(\ell)}(x_3, x_4) ,
\]

is then conformally covariant so that

\[
\sum_i M_{iAB} \left( \frac{1}{(x_1)^{1/2}(\Delta_1 + \Delta_2)} \mathcal{F}_d^{(\ell)}(x_1, x_2, x_3, x_4) \right) = 0 .
\]

Since the three point functions in (3.4) have been constructed to be conformally covariant, so that \((M_1 + M_2 + M)_{AB} \langle \varphi_1(x_1) \varphi_2(x_2) \hat{O} \rangle^{(\ell)}_{\Delta,\mu_1 \ldots \mu_\ell} = 0\), we also have

\[
(\frac{1}{2}(M_1 + M_2))_{AB} (M_1 + M_2)_{BA} - C_{\Delta,\ell}) \frac{1}{(x_1)^{1/2}(\Delta_1 + \Delta_2)} \mathcal{F}_d^{(\ell)}(x_1, x_2, x_3, x_4) = 0 .
\]

Using the relations between (3.4) and (3.7) it is straightforward to see that the definition (3.10) ensures that

\[
F_\Delta^{(\ell)}(x_1, x_2, x_3, x_4) = F_{d-\Delta}^{(\ell)}(x_1, x_2, x_3, x_4) .
\]

The expression (3.10) can be simplified by using

\[
X_{\{\mu_1 \ldots X_{\mu_\ell}\}} \bar{X}_{\{\mu_1 \ldots \bar{X}_{\mu_\ell}\}} = \frac{1}{2^\ell c_\ell} (X^2 \bar{X}^2)^{\frac{\ell}{2}} \hat{C}_\ell^\varepsilon(t) , \quad t = \frac{X \cdot \bar{X}}{(X^2 \bar{X}^2)^{\frac{1}{2}}},
\]

for \( \hat{C}_\ell^\varepsilon \) the modified Gegenbauer polynomial defined in (2.29). Hence, as a consequence of conformal invariance,

\[
F_\Delta^{(\ell)}(x_1, x_2, x_3, x_4) = \frac{\gamma_{\lambda_1,a} \gamma_{\lambda_1,b}}{2^\ell c_\ell} \left( \frac{x_{14}^2}{x_{24}^2} \right)^a \left( \frac{x_{14}^2}{x_{13}^2} \right)^b \mathcal{F}_\Delta^{(\ell)}(u, v) ,
\]
where
\[
\frac{1}{\pi^{d/2}} \int d^d x \frac{(X^2 \hat{X}^2)_{\ell} \hat{G}(t)}{((x-x_1)^2)^{\lambda_2 - a} ((x-x_3)^2)^{\lambda_2 + a} ((x-x_2)^2)^{\lambda_2 + b} ((x-x_4)^2)^{\lambda_2 - b}} = \frac{1}{(x_{12}^2)^{\lambda_2} (x_{34}^2)^{\lambda_2}} \left( \frac{x_{14}^2}{x_{24}^2} \right)^a \left( \frac{x_{14}^2}{x_{13}^2} \right)^b \bar{f}_\Delta(u, v). \tag{3.16}
\]

In (3.16) we may write
\[
X^2 = \frac{x_{12}^2}{(x-x_1)^2(x-x_2)^2}, \quad \hat{X}^2 = \frac{x_{34}^2}{(x-x_3)^2(x-x_4)^2}, \tag{3.17}
\]
and
\[
2X \cdot \hat{X} = \frac{x_{13}^2}{(x-x_1)^2(x-x_3)^2} - \frac{x_{24}^2}{(x-x_2)^2(x-x_3)^2} + \frac{x_{24}^2}{(x-x_2)^2(x-x_4)^2} - \frac{x_{14}^2}{(x-x_1)^2(x-x_4)^2}. \tag{3.18}
\]

As a consequence of (3.12)
\[
\mathcal{D} f_\Delta^{(\ell)}(u, v) = \frac{1}{2} C_{\Delta, \ell} f_\Delta^{(\ell)}(u, v), \tag{3.19}
\]
so that, from (2.9), \( f_\Delta^{(\ell)} \) is a linear sum of the conformal partial waves \( G^{(\ell)}_{\Delta} \) and \( G^{(\ell)}_{d-\Delta} \). The coefficients may be determined from the short distance limits \( x_{12}, x_{34} \to 0 \). Assuming \( \Delta < \frac{1}{2} d \), so that \( \lambda_2 < \tilde{\lambda}_2 \), the leading contributions may be found directly using
\[
\mathcal{F}_{\Delta, \mu_1 \ldots \mu_\ell}^{(\ell)}(x; x_1, x_2) \sim_{x_{12} \to 0} (x_{12}^2)^{\lambda_2} x_{12\nu_1} \ldots x_{12\nu_\ell} \frac{1}{((x_1-x)^2)^{\Delta}} \mathcal{T}_{\nu_1 \ldots \nu_\ell, \mu_1 \ldots \mu_\ell}(x_1 - x), \tag{3.20}
\]
from (3.5). Hence
\[
\mathcal{F}^{(\ell)}(x_1, x_2, x_3, x_4) \sim_{x_{12} \to 0} \frac{\gamma_{\lambda_1 a \gamma_{\lambda_1 b}}}{k_{d-\Delta, \ell}} (x_{12}^2)^{\lambda_2} x_{12\mu_1} \ldots x_{12\mu_\ell} \mathcal{F}_{\Delta, \mu_1 \ldots \mu_\ell}^{(\ell)}(x_1; x_3, x_4). \tag{3.21}
\]
while
\[
\mathcal{F}^{(\ell)}_{\Delta, \mu_1 \ldots \mu_\ell}(x_1; x_3, x_4) \sim_{x_{34} \to 0} \frac{1}{(x_{13}^2)^{\Delta}} \mathcal{T}^{(\ell)}_{\mu_1 \ldots \mu_\ell, \nu_1 \ldots \nu_\ell}(x_{13}) x_{43\nu_1} \ldots x_{43\nu_\ell} (x_{43}^2)^{\lambda_2}. \tag{3.22}
\]

Since
\[
1 - v \sim_{x_{12}, x_{34} \to 0} 2 \frac{1}{x_{13}^2} x_{12\mu} I_{\mu \nu}(x_{13}) x_{43\nu}, \tag{3.23}
\]
then combining (3.21) and (3.22) in this limit
\[
\frac{1}{(x_{13}^2)^\ell} x_{12\mu_1} \ldots x_{12\mu_\ell} \mathcal{T}^{(\ell)}_{\mu_1 \ldots \mu_\ell, \nu_1 \ldots \nu_\ell}(x_{13}) x_{43\nu_1} \ldots x_{43\nu_\ell} \sim \left( \frac{1}{2} (1 - v) \right)^\ell. \tag{3.24}
\]
Comparing with the leading behaviour of $G^{(\ell)}_{\Delta}$ in (2.14) and using (3.13) then gives
\[
\begin{align*}
 f^{(\ell)}_{\Delta}(u, v) &= \frac{1}{k_{d-\Delta, \ell}} \gamma_{\lambda, b} \lambda_c \gamma_{\lambda, b} G^{(\ell)}_{\Delta}(u, v) + \frac{1}{k_{\Delta, \ell}} \gamma_{\lambda, a} \gamma_{\lambda, a} G^{(\ell)}_{d-\Delta}(u, v). \quad (3.25)
\end{align*}
\]

Explicit expressions can in principle be obtained by expanding $\hat{C}^{(\varepsilon)}_{\ell}(t)$ as a power series in $t$ and then using (3.18) to expand $t^n$ leads to a sum of elementary conformal integrals [3,4]. Essentially equivalently, we may use the recursion relation for Gegenbauer polynomials, generating all $\hat{C}^{(\varepsilon)}_{\ell}(t)$ starting from $\hat{C}^{(\varepsilon)}_{0}(t) = 1$, in the form
\[
(\ell + 2\varepsilon) \hat{C}^{(\varepsilon)}_{\ell+1}(t) = 2(\ell + \varepsilon) t \hat{C}^{(\varepsilon)}_{\ell}(t) - \ell \hat{C}^{(\varepsilon)}_{\ell-1}(t), \quad (3.26)
\]
which in (3.16) leads to
\[
(\ell + d - 2) f^{(\ell+1)}_{\Delta}(a, b; u, v)
= (\ell + \frac{d}{2} - 1) u^{-\frac{d}{2}} \left( f^{(\ell)}_{\Delta}(a - \frac{1}{2}, b + \frac{1}{2}; u, v) - v f^{(\ell)}_{\Delta}(a + \frac{1}{2}, b + \frac{1}{2}; u, v) \right)
= \left( f^{(\ell)}_{\Delta}(a + \frac{1}{2}, b - \frac{1}{2}; u, v) - f^{(\ell)}_{\Delta}(a - \frac{1}{2}, b - \frac{1}{2}; u, v) \right) - \ell f^{(\ell-1)}_{\Delta}(a, b; u, v). \quad (3.27)
\]

For $d = 2, 4$ this recurrence relation was solved in [11] by using corresponding relations for the single variable $g_p$ functions.

For $\ell = 0$ the conformal integral in (3.16) may be evaluated as a Mellin-Barnes transform [20]
\[
 f^{(0)}_{\Delta}(u, v) = \frac{1}{\gamma_{\lambda, a} \gamma_{\lambda, b}} \int_{C_{s, t}} d\mu_{s, t} \Gamma(-t) \Gamma(-t - a - b) \Gamma(-r + a) \Gamma(-r + b) \times \Gamma(\lambda - s) \Gamma(\bar{\lambda} - s) u^s v^t, \quad (3.28)
\]
for $\lambda = \frac{1}{2} \Delta, \bar{\lambda} = \frac{1}{2}(d - \Delta)$ and
\[
d\mu_{s, t} = \frac{1}{(2\pi i)^2} ds dt, \quad r + t + s = 0. \quad (3.29)
\]

The integration contours $C_{s, t}$ for $s, t$ run from $-i\infty$ to $i\infty$ such that
\[
\text{Re } s < \lambda, \bar{\lambda}, \quad \text{Re } t < 0, -a - b, \quad \text{Re } r < a, b. \quad (3.30)
\]

If the $s$ contour is closed to the right the poles at $s = \lambda + n$ and $s = \bar{\lambda} + n$, for $n = 0, 1, 2, \ldots$, generate the conformal partial wave $G^{(0)}_{\Delta}$ and its dual $G^{(0)}_{d-\Delta}$ in accordance with (3.25). In the remaining $t$-integration it is useful to note that, for $-a - s, -b - s < c < 0, -a - b$,
\[
\int_{-i\infty}^{c+i\infty} dt \Gamma(-t) \Gamma(-t - a - b) \Gamma(s + t + a) \Gamma(s + t + b) v^t = \frac{1}{\Gamma(2s)} \gamma_{s, a} \gamma_{s, b} F(s + a, s + b; 2s; 1 - v). \quad (3.31)
\]
By restricting the contours $C_{s,t}$ so as to pick up only the poles at $s = \lambda + n$ then (3.28) gives for the $\ell = 0$ conformal partial wave a double power series

$$G_{2\lambda}^{(0)}(u, v) = \sum_{n,m \geq 0} \frac{(\lambda - a)_n (\lambda - b)_n}{n!(2\lambda - \varepsilon)_n} \frac{\alpha_{n+m}(\lambda + b)_n}{m!(2\lambda)_{2n+m}} u^{\lambda+n}(1 - v)^m,$$  

(3.32)

where convergence requires for the $m$-sum $v < 1$ and for large $n = O(m)$

$$(1 - v + u)^2 < 4u.$$

Remarkably this coincides with the allowed region of $u, v$ for $x_i$ vectors belonging to $d$-dimensional Euclidean space (letting $x_1 = 0, x_4 \to \infty$ then $1 - v + u = 2 x_2 \cdot x_3 / x_3^2$ and $u = x_2^2 / x_3^2$ so that (3.33) is equivalent to the usual Schwarz inequality). With (2.15) (3.33) becomes $(x - \bar{x})^2 < 0$.

For general $\ell$ (3.28) may be extended to

$$f_{\Delta}^{(\ell)}(u, v) = \frac{1}{\gamma_{1,a} \gamma_{1,b}} \int_{C_{s,t}} d\mu_{s,t} \Gamma(-t) \Gamma(-t - a - b) \Gamma(-r + a) \Gamma(-r + b) \times \Gamma(\lambda_2 - s) \Gamma(\lambda_2 - s) \alpha_{\ell}(a, b; r, t, s) u^a v^t,$$

(3.34)

so that from (3.28) $\alpha_0 = 1$. Since the symmetry (2.13) also applies to $f_{\Delta}^{(\ell)}(u, v)$ we must have

$$\alpha_{\ell}(a, b; r, t, s) = (-1)^{\ell} \alpha_{\ell}(-a, b; t + b, r - b, s) = (-1)^{\ell} \alpha_{\ell}(a, -b; t + a, r - a, s).$$

(3.35)

Applying the eigenvalue equation (3.19), using the result (2.10) for the operator $\mathcal{D}$ and suppressing the $a, b$ arguments in $\alpha_{\ell}$, gives

$$t(t + a + b)(\alpha_{\ell}(r + 1, t - 1, s) - \alpha_{\ell}(r, t, s))$$

$$+ (r - a)(r - b)(\alpha_{\ell}(r - 1, t + 1, s) - \alpha_{\ell}(r, t, s))$$

$$- (s - \lambda_2)(s - \bar{\lambda}_2)(\alpha_{\ell}(r + 1, t, s - 1) + \alpha_{\ell}(r, t + 1, s - 1) - 2 \alpha_{\ell}(r, t, s))$$

$$= (2s(\frac{1}{2}d - \lambda_2 - \bar{\lambda}_2) + 2\lambda_2 \bar{\lambda}_2 + \frac{1}{2} C_{\Delta, \ell} \alpha_{\ell}(r, t, s) = \ell(2s + \ell - 1) \alpha_{\ell}(r, t, s).$$

(3.36)

using (2.7) for $C_{\Delta, \ell}$ and (2.17),(3.9) for $\lambda_2, \bar{\lambda}_2$. Equivalent difference equations have been given in [5] in a more general context. In order that the analytic structure of the conformal partial waves remain valid for any $\ell$ we require that $\alpha_{\ell}$ is a polynomial in $r, t, s$. For any $\ell = 0,1,2 \ldots$ there are solutions $\alpha_{\ell}(r, t, s) = \sum_{n+m \leq \ell} c_{nm} w^n s^m$ where $w = t - r + a + b$ which depend on the $\frac{1}{2}(\ell + 1)(\ell + 2)$ coefficients $c_{nm}$. With $w$ as defined above (3.35) requires that these are constrained by $c_{nm}(a, b) = (-1)^{\ell-n} c_{nm}(-a, b) = (-1)^{\ell-n} c_{nm}(a, -b)$. For the assumed expression for $\alpha_{nm}$ the recurrence relation (3.36) generates terms $w^n s^m$ with
\[ n + m \leq \ell + 1 \] but the equation is an identity when \( n + m = \ell + 1 \) and also \( n = \ell, m = 0 \) so that there remain \( \frac{1}{2} \ell(\ell + 3) \) linearly independent equations determining \( c_{nm} \) uniquely up to an overall constant factor. The existence of polynomial solutions is crucially dependent on the factor \( \ell(2s + \ell - 1) \) appearing on the right hand side of (3.36). This in turn is dependent on the choice of \( \lambda_2, \bar{\lambda}_2 \) in (3.34). As will be demonstrated below these polynomial solutions ensure that the required boundary conditions (2.14) are satisfied.

More directly from (3.27)

\[
(\ell + d - 3) \alpha_\ell(a, b; r, t, s) = (\ell + \frac{1}{2}d - 2)((\lambda_1 + a - 1)(\bar{\lambda}_1 - b - 1)(b - r)\alpha_{\ell-1}(a - \frac{1}{2}, b + \frac{1}{2}; r + \frac{1}{2}, t, s + \frac{1}{2})
+ (\lambda_1 - a - 1)(\bar{\lambda}_1 - b - 1) t \alpha_{\ell-1}(a + \frac{1}{2}, b + \frac{1}{2}; r + \frac{1}{2}, t - 1, s + \frac{1}{2})
+ (\lambda_1 - a - 1)(\bar{\lambda}_1 + b - 1)(a - r)\alpha_{\ell-1}(a + \frac{1}{2}, b - \frac{1}{2}; r - \frac{1}{2}, t, s + \frac{1}{2})
+ (\lambda_1 + a - 1)(\bar{\lambda}_1 + b - 1)(t + a + b)\alpha_{\ell-1}(a - \frac{1}{2}, b - \frac{1}{2}; r - \frac{1}{2}, t, s + \frac{1}{2}))
- (\ell - 1)(\lambda_1 + a - 1)(\lambda_1 - a - 1)(\bar{\lambda}_1 + b - 1)(\bar{\lambda}_1 - b - 1)(\lambda_2 - s)(\bar{\lambda}_2 - s)
\times \alpha_{\ell-2}(a, b; r, t, s),
\] (3.37)

where the \( \lambda_1, \bar{\lambda}_1 \) factors are a consequence of the normalisation of the integral in (3.34). Starting \( \alpha_0 = 1 \) this gives \((d - 1)\ell - 1 \alpha_\ell \) iteratively, for any \( \ell = 1, 2, \ldots \), as a polynomial in all variables.

For \( \ell = 1 \)

\[ \alpha_1(r, t, s) = \lambda_2 \bar{\lambda}_2 w - ab(s - \frac{1}{2}d + 1), \] (3.38)

and for \( \ell = 2, a = b = 0, \)

\[
\frac{2(d - 1)}{\lambda_2 + 1}(\lambda_2 + 1) \alpha_2(r, t, s) = (2\lambda_2 + 1)(2\bar{\lambda}_2 + 1)(\lambda_2 + \bar{\lambda}_2 + 2) w^2
- (\lambda_2 + \bar{\lambda}_2 + 2\lambda_2\bar{\lambda}_2) s(s - 2\lambda_2 - 2\bar{\lambda}_2 - 2)
\] (3.39)

For any \( \ell \) in order to satisfy the boundary condition (2.39) it is sufficient to consider only the contribution of the pole at \( s = \lambda_2 \) and then require

\[
\alpha_\ell(r, t, \lambda_2) = (d - \Delta - 1)\ell c_\ell \beta_\ell(r, t),
\] (3.40)

where

\[
\beta_\ell(r, t) = \sum_{n=0}^{\ell} \binom{\ell}{n} (-1)^n (-t)_n (-t - a - b)_n (-r + a)_{\ell - n} (-r + b)_{\ell - n}
\] (3.41)

\[ = (a - r)_{\ell} (b - r)_{\ell} {}_3F_2\left( -\ell, -t, -t - a - b; r - a - \ell + 1, r - b - \ell + 1; 1 \right), \]
since then, using (3.31), the $t$-integral gives $\sum_{n=0}^{\epsilon} \binom{n}{\ell} (-1)^{n+\alpha} F(\lambda_1 + a, \lambda_1 + b; 2\lambda_1; 1 - v)$, so that there is the necessary $(1 - v)^{\ell}$ factor required by (2.39). In writing (3.41) in terms of a $\mathcal{F}_2$-function it is useful to note the Pochhammer symbol identity \[ (x)_n = (-1)^n \frac{(x)_N}{(1 - x - N)_n}. \] (3.42)

Defining also
\[
\begin{align*}
\gamma_{\ell}(r, t) &= (b - r)_{\ell}(a + \alpha + 1)_{3\mathcal{F}_2}\left(-\ell, -t, -\lambda_1 + a + 1; r - b - \ell + 1, \lambda_2 + a; 1\right), \\
\delta_{\ell}(r, t) &= (-1)^{\ell}(-t - a - b)_{\ell}(\lambda_2 + a)_{3\mathcal{F}_2}\left(-\ell, -r + a, -\lambda_1 + a + 1; t + a + b - \ell + 1, \lambda_2 + a; 1\right),
\end{align*}
\] (3.43)

then, with the aid of Sheppard identity [21]^2,
\[
\beta_{\ell}(r, t) = \gamma_{\ell}(r, t) = \delta_{\ell}(r, t) \quad \text{for} \quad -r - t = \lambda_2,
\] (3.44)

and
\[
\begin{align*}
\gamma_{\ell}(a, b; r, t) &= (\lambda_1 + a - 1)(b - r)_{\ell}\left(a - \frac{1}{2}, b + \frac{1}{2}; r - \frac{1}{2}, t\right) \\
&\quad + (\lambda_1 - a - 1)_{\ell}\left(a + b - \frac{1}{2}; r - \frac{1}{2}, t - 1\right), \\
\delta_{\ell}(a, b; r, t) &= (\lambda_1 + a - 1)(t + a + b)_{\ell}\left(a - \frac{1}{2}, b + \frac{1}{2}; r - \frac{1}{2}, t\right) \\
&\quad + (\lambda_1 - a - 1)_{\ell}\left(a - r - 1; b + \frac{1}{2}; r - \frac{1}{2}, t\right),
\end{align*}
\] (3.45)

which ensures that (3.40) satisfies (3.37) for $s = \lambda_2$. In the above $\beta_{\ell}$ satisfies both symmetries in (3.35) while $\gamma_{\ell}(a, b; r, t) = (-1)^{\ell}\gamma_{\ell}(-a, b; t + b, r - b)$, and similarly for $\delta_{\ell}$, but $(-1)^{\ell}\delta_{\ell}(a, -b; t + a, r - a) = \delta_{\ell}(a, b; r, t)$.

If $\lambda_0 = 1 \mp b$, or $\lambda_2 = \varepsilon \pm b$, then (3.37) simplifies to three term relations which have the solutions
\[
\alpha_{\ell}(r, s)|_{\lambda_0 = 1 - b} = (-1)^{\ell}(2b)_{\ell}\gamma_{\ell}(r, t), \quad \alpha_{\ell}(r, s)|_{\lambda_0 = 1 + b} = (2b)_{\ell}\delta_{\ell}(r, t),
\] (3.46)

There are analogous solutions of (3.37) for $\lambda_1 = 1 \pm a$ but, given (3.35) and (3.34), these are irrelevant for the conformal partial wave $G^\ell_\Delta$. The solutions for $\alpha_{\ell}$ in (3.46) in terms of $\mathcal{F}_2$ functions correspond to the leading twist conformal partial waves.

\[
(\gamma)_{\ell}(\delta)_{\ell}\mathcal{F}_2(-\ell, \alpha, \beta; \gamma, \delta; 1) = (\gamma)_{\ell}(\delta - \alpha)_{\ell}\mathcal{F}_2\left(-\ell, \alpha, \delta - \beta; \gamma, \alpha - \delta + 1 - \ell; 1\right)
\]
\[= (\gamma - \alpha)_{\ell}(\delta - \alpha)_{\ell}\mathcal{F}_2\left(-\ell, \alpha + \beta - \gamma - 1 - \ell, \alpha - \delta + 1 - \ell; 1\right).\]
In general $\alpha_\ell$ has no very compact form but it may be expressed as a quadnomial sum

$$(d-2)\ell \alpha_\ell = \sum_{m+n+p+q=\ell} \frac{\ell!}{m!n!p!q!} (-1)^{p+n}(2\lambda_2 + \ell - 1)\ell_q(2\lambda_2 + \ell - 1)_n$$

\[
\times (\lambda_1 + a - q)q(\lambda_1 + b - q)(\lambda_1 + a - m)_m(\lambda_1 + b - m)_m
\]
\[
\times (d-2 + \ell + n - q)q(\frac{1}{2}d - 1)\ell_q(\frac{1}{2}d - 1 + a + b + n)_p
\]
\[
\times (\lambda_2 - s)_p q(-\ell)_n,
\]

where the various Pochhammer symbols may be rewritten using (3.42) and

$$(x - n)_n = (-1)^n (1 - x)_n.$$  \quad (3.48)

The result (3.47) is symmetric under $a \leftrightarrow b$. It can be reexpressed in a variety of similar forms, such as those obtained by imposing (3.35) or letting $\lambda_2, \lambda_1 \leftrightarrow \bar{\lambda}_2, \bar{\lambda}_1$, albeit with no significant simplification. Such identities may be obtained from (3.47) by repeatedly writing some of the summations in terms of a terminating $3F_2(1)$ function and using the Sheppard identity.

In two dimensions there are significant simplifications using complex, or light cone, coordinates. If $x^2 = z\bar{z}$, $x'^2 = z'\bar{z}'$ then $(x^2x'^2)^{\frac{1}{2}\ell} \hat{C}_\ell^0(x \cdot x'/(x^2x'^2)^{\frac{1}{2}}) = \frac{1}{2}((z\bar{z}')^\ell + (\bar{z}z')^\ell)$ and hence

$$\frac{(X^2\bar{X}^2)^{\frac{1}{2}\ell} \hat{C}_\ell^0(t)}{(x - x_1)^2} \frac{\lambda_2 - a}{(x - x_2)^2} \frac{\lambda_2 + a}{(x - x_3)^2} \frac{\lambda_2 + b}{(x - x_4)^2}$$

\[
\times (3\lambda_2 - 2\lambda_1 - 2\lambda_3 - 2) = \frac{1}{2} (z_{12} \bar{z}_{43})^\ell f_{\lambda_1}(z; z_i) f_{\lambda_2}(\bar{z}; \bar{z}_i) + \frac{1}{2} (\bar{z}_{12} z_{43})^\ell f_{\lambda_2}(z; z_i) f_{\lambda_1}(\bar{z}; \bar{z}_i),
\]

where

$$f_{\lambda}(z; z_i) = \frac{1}{(z - z_1)^{\lambda - a} (z - z_2)^{\lambda + a} (z - z_3)^{1 - \lambda + b} (z - z_4)^{1 - \lambda - b}}.$$  \quad (3.50)

The essential integral, which is evaluated in appendix A, has the form

$$\frac{1}{\pi} \int d^2x \ (z_{12} \bar{z}_{43})^\ell f_{\lambda_1}(z; z_i) f_{\lambda_2}(\bar{z}; \bar{z}_i)$$

\[
= \frac{1}{(x_{12})^{\lambda_2} (x_{34})^{1 - \lambda_1}} \frac{(x_{14}^{2})^a}{(x_{24}^{2})^b}
\]
\[
\times \left( \frac{\gamma_{1, b}}{\gamma_{1 - \lambda_2, b}} \frac{\Gamma(1 - 2\lambda_1)}{\Gamma(2\lambda_2)} g_{\lambda_1}(x) g_{\lambda_2}(\bar{x}) + \frac{\gamma_{1 - \lambda_1, a}}{\gamma_{\lambda_2, a}} \frac{\Gamma(2\lambda_1 - 1)}{\Gamma(2 - 2\lambda_2)} g_{1 - \lambda_1}(x) g_{1 - \lambda_2}(\bar{x}) \right),
\]

for

$$x = \frac{z_{12} z_{34}}{z_{13} z_{24}}, \quad \bar{x} = \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{13} \bar{z}_{24}}.$$  \quad (3.52)
Since \(1 - x = z_{14}/z_{13} z_{24}\) it is easy to see that this is in accord with the definition (2.3) of \(x, \bar{x}\). Using repeatedly \(\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x\), from which follow the identities \(\Gamma(1-2\lambda_1)/\Gamma(2\lambda_2) = \Gamma(1-2\lambda_2)/\Gamma(2\lambda_1)\) and \(\gamma_{1-\lambda_1,a}/\gamma_{\lambda_2,a} = \gamma_{1-\lambda_2,a}/\gamma_{\lambda_1,a}\), then from (3.16), (3.49) and (3.25) in two dimensions we get

\[
G^{(\ell)}_{\Delta}(u, v) = \frac{1}{2}(g_{\lambda_1}(x) g_{\lambda_2}(\bar{x}) + g_{\lambda_1}(\bar{x}) g_{\lambda_2}(x)).
\]  

(3.53)

4. Recurrence Relations

In the discussion of orthogonal polynomials recurrence relations involving two or more terms play a significant role. Here we derive analogous results for conformal partial waves, extending and simplifying the results given in [12]. They play a role in combining conformal partial waves into superconformal blocks.

We first consider recurrence relations for the single variable functions \(g_p\) satisfying (2.33) and whose derivation is a model for obtaining recurrence relations for the two variable conformal partial waves. With the differential operator \(D_x(a, b)\) defined by (2.36) it is straightforward to obtain

\[
D_x(a + \frac{1}{2}, b + \frac{1}{2}) x^{-\frac{\lambda}{2}} = x^{-\frac{\lambda}{2}} D_x(a, b) - x^\frac{\lambda}{2} \frac{d}{dx} + \frac{3}{4} x^{-\frac{\lambda}{2}},
\]

\[
D_x(a + \frac{1}{2}, b + \frac{1}{2}) x^\frac{\lambda}{2} \frac{d}{dx} = \left( x^\frac{\lambda}{2} \frac{d}{dx} - x^{-\frac{\lambda}{2}} \right) D_x(a, b) - \frac{1}{4} x^\frac{\lambda}{2} \frac{d}{dx}.
\]

(4.1)

Since \(g_p\) is defined by being a solution of (2.33), (4.1) ensures that \(x^{-\frac{\lambda}{2}} g_p(a, b; x)\) and \(x^\frac{\lambda}{2} \frac{d}{dx} g_p(a, b; x)\) can be expressed in terms of \(g_{p'}(a + \frac{1}{2}, b + \frac{1}{2}; x)\) for just \(p' = p \pm \frac{1}{2}\) in the form

\[
x^{-\frac{\lambda}{2}} g_p(a, b; x) = g_{p-\frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}; x) - \sigma_p g_{p+\frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}; x),
\]

\[
x^\frac{\lambda}{2} \frac{d}{dx} g_p(a, b; x) = p g_{p-\frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}; x) + (p - 1) \sigma_p g_{p+\frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}; x).
\]

(4.2)

In the result for \(x^{-\frac{\lambda}{2}} g_p(a, b; x)\) the coefficient of \(g_{p-\frac{1}{2}}(a - \frac{1}{2}, b + \frac{1}{2}; x)\) is determined by requiring the normalisation condition (2.34) for the leading \(x^{p-\frac{\lambda}{2}}\) term. By applying (2.37) and writing in (4.2) \(\sigma_p = \sigma_{p}(a, b)\) then (4.2) may be extended to

\[
x^{-\frac{\lambda}{2}} g_p(a, b; x) = g_{p-\frac{1}{2}}(a \pm \frac{1}{2}, b \mp \frac{1}{2}; x) + \sigma_p(\pm a, \mp b) g_{p+\frac{1}{2}}(a \pm \frac{1}{2}, b \mp \frac{1}{2}; x),
\]

\[
x^{-\frac{\lambda}{2}} (1 - x) g_p(a, b; x) = g_{p-\frac{1}{2}}(a - \frac{1}{2}, b - \frac{1}{2}; x) - \sigma_p(-a, -b) g_{p+\frac{1}{2}}(a - \frac{1}{2}, b - \frac{1}{2}; x).
\]

(4.3)

Assuming the normalisation condition (2.34) for \(g_{p+\frac{1}{2}}\), \(\sigma_p\) may be determined to be

\[
\sigma_p(a, b) = \frac{(p + a)(p + b)}{2p(2p - 1)}.
\]

(4.4)
There are also relations between \( g_p \) for differing \( p \) but the same \( a, b \). Although they follow from standard identities a derivation which may be extended later to conformal partial waves is obtained by first defining
\[
    f_0 = \frac{1}{x} - \frac{1}{2}, \quad f_1 = (1 - x) \frac{d}{dx} - \frac{1}{2}(a + b),
\]
which satisfy the commutation relations
\[
    [D_x, f_0] = -2f_1 + 2f_0, \quad [D_x, f_1] = -2f_0 D_x - ab.
\]
These relations ensure that \( f_0 g_p, f_1 g_p \) may be expanded in terms of only \( g_{p \pm 1}, g_p \) giving the four term relations
\[
    f_0 g_p = g_{p-1} + \alpha_p g_p + \beta_p g_{p+1}, \quad (4.7a)
\]
\[
    f_1 g_p = p g_{p-1} + \alpha_p g_p - (p - 1) \beta_p g_{p+1}. \quad (4.7b)
\]
where the coefficient of \( g_{p-1} \) is determined by imposing (2.34) and (4.6) determines
\[
    \alpha_p = - \frac{ab}{2p(p-1)}. \quad (4.8)
\]
To determine \( \beta_p \) it is necessary to take account of the normalisation of \( g_{p+1} \). This may be achieved by using \([f_1, f_0] = -f_0^2 + \frac{1}{4}\) from which we may obtain
\[
    \beta_p = \frac{(p + a)(p + b)(p - a)(p - b)}{4p^2(2p - 1)(2p + 1)}. \quad (4.9)
\]
By iterating (4.3) \( \beta_p = \sigma_p(-a, b) \sigma_{p+a} (a - \frac{1}{2}, -b - \frac{1}{2}) \).

The single variable recurrence relations for \( g_p \) can be extended to \( F_{\lambda_1 \lambda_2} \) in a similar fashion. The relations satisfy the symmetry
\[
    F^{(\varepsilon)}_{\lambda_2 - \varepsilon \lambda_1 + \varepsilon}(x, \bar{x}) = F^{(\varepsilon)}_{\lambda_1 \lambda_2}(x, \bar{x}). \quad (4.10)
\]
This is in part a consequence of \( C_{\Delta, \ell} = C_{\Delta, \ell - d + 2}, D_{\Delta, \ell} = D_{\Delta, \ell - d + 2} \). The symmetry relation (4.10) depends on the choice of normalisation in (2.30) (with the normalisation in (2.29) we may identify \( \tilde{C}_\ell^{\varepsilon}(t) = \tilde{C}_{\ell - 2k}^{\varepsilon}(t)\)).

We first consider the corresponding operators acting on two variable conformal partial waves to those considered in (4.1). Defining
\[
    H_1 = (x \bar{x})^{-\frac{1}{2}} \left( x \frac{\partial}{\partial x} + \bar{x} \frac{\partial}{\partial \bar{x}} \right), \quad H_2 = (x \bar{x})^\frac{1}{2} \left( \frac{\partial^2}{\partial x \partial \bar{x}} - \varepsilon \frac{1}{x - \bar{x}} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial \bar{x}} \right) \right),
\]
\[
    H_3 = (x \bar{x})^{-\frac{1}{2}} \left( x \frac{\partial}{\partial x} - \bar{x} \frac{\partial}{\partial \bar{x}} + 2\varepsilon \frac{x + \bar{x}}{x - \bar{x}} \right) \left( D_x(a, b) - D_{\bar{x}}(a, b) \right),
\]
(4.11)
then these satisfy the algebra
\[
\Delta^{(e)}(a + \frac{1}{2}, b + \frac{1}{2}) (\bar{x})^{-\frac{1}{2}} = (\bar{x})^{-\frac{1}{2}} \Delta^{(e)}(a, b) + (\frac{1}{2} + \varepsilon) (\bar{x})^{-\frac{1}{2}} - \mathcal{H}_1,
\]
\[
\Delta^{(e)}(a + \frac{1}{2}, b + \frac{1}{2}) \mathcal{H}_1 = (\mathcal{H}_1 - (\bar{x})^{-\frac{1}{2}}) \Delta^{(e)}(a, b) + (\frac{1}{2} + \varepsilon) \mathcal{H}_1 - 2 \mathcal{H}_2,
\]
\[
\Delta^{(e)}(a + \frac{1}{2}, b + \frac{1}{2}) \mathcal{H}_2 = (\mathcal{H}_2 - \frac{1}{2} \mathcal{H}_1 - \varepsilon (\bar{x})^{-\frac{1}{2}}) \Delta^{(e)}(a, b) - (\frac{1}{2} + \varepsilon) \mathcal{H}_2 + \frac{1}{2} \mathcal{H}_3,
\]
and
\[
\Delta^{(e)}(a + \frac{1}{2}, b + \frac{1}{2}) \mathcal{H}_3 = \mathcal{H}_3 \Delta^{(e)}(a, b) + (\frac{1}{2} - \varepsilon) \mathcal{H}_3 - (\bar{x})^{-\frac{1}{2}} \Delta^{(e)}(a, b).
\] (4.12)

Here $\Delta_4$ is the fourth order Casimir operator
\[
\Delta^{(e)}_4(a, b) = \left(\frac{x \bar{x}}{x - \bar{x}}\right)^{2\varepsilon} (D_x(a, b) - D_{\bar{x}}(a, b)) \left(\frac{x \bar{x}}{x - \bar{x}}\right)^{-2\varepsilon} (D_x(a, b) - D_{\bar{x}}(a, b)),
\] (4.14)

which satisfies
\[
[\Delta^{(e)}(a, b), \Delta^{(e)}_4(a, b)] = 0.
\] (4.15)

Acting on conformal partial waves
\[
\Delta^{(e)}_4 F_{\lambda_1 \lambda_2} = c_{4, \lambda_1 \lambda_2} F_{\lambda_1 \lambda_2},
\] (4.16)

where
\[
c_{4, \lambda_1 \lambda_2} = (\lambda_1 - \lambda_2) (\lambda_1 - \lambda_2 + 2\varepsilon) (\lambda_1 + \lambda_2 - 1) (\lambda_1 + \lambda_2 - 1 - 2\varepsilon).
\] (4.17)

The formula for $c_{4, \lambda_1 \lambda_2}$ may be obtained by considering the limit $\bar{x} \to 0$ in (4.16) when $\Delta^{(e)}_4(a, b) \sim D_x(a, b) - \bar{x}^2 \partial^2 / \partial x^2 + 4\varepsilon \bar{x} \partial / \partial x - 2\varepsilon (1 + 2\varepsilon)$ and using (2.32). The expression (4.17) is related to the quartic Casimir given in (2.6) since, with the relations (2.17), $D_{\Delta, \varepsilon} = -2c_{4, \lambda_1 \lambda_2} + 4c_{\lambda_1 \lambda_2}^2 + d(d - 1) c_{\lambda_1 \lambda_2}$.

In general $(x \bar{x})^{-\frac{1}{2}} F_{\lambda_1 \lambda_2}(a, b; x, \bar{x})$ has an expansion in terms of the linearly independent basis $F_{\lambda_1 - \frac{1}{2} + n \lambda_2 - \frac{1}{2} + m}(a + \frac{1}{2}, b + \frac{1}{2}; x, \bar{x})$ for $n, m = 0, 1, 2, \ldots$. However consistency conditions arising from (4.12) and (4.13) restrict this to only four terms of the form
\[
(x \bar{x})^{-\frac{1}{2}} F_{\lambda_1 \lambda_2}(a, b) = r F_{\lambda_1 - \frac{1}{2} \lambda_2 - \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) + s F_{\lambda_1 + \frac{1}{2} \lambda_2 - \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) + t F_{\lambda_1 - \frac{1}{2} \lambda_2 + \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) + u F_{\lambda_1 + \frac{1}{2} \lambda_2 + \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}).
\] (4.18)

Using (4.12) we then get from (4.18) for $i = 1, 2, 3$
\[
\mathcal{H}_i F_{\lambda_1 \lambda_2}(a, b) = r_i F_{\lambda_1 - \frac{1}{2} \lambda_2 - \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) + s_i F_{\lambda_1 + \frac{1}{2} \lambda_2 - \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) + t_i F_{\lambda_1 - \frac{1}{2} \lambda_2 + \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) + u_i F_{\lambda_1 + \frac{1}{2} \lambda_2 + \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}),
\] (4.19)
where
\[ r_1 = (\lambda_1 + \lambda_2)r, \quad s_1 = -(\lambda_1 - \lambda_2 - 1)s, \quad t_1 = (\lambda_1 - \lambda_2 + 1 + 2\varepsilon)t, \]
\[ u_1 = -(\lambda_1 + \lambda_2 - 2 - 2\varepsilon)u, \]
\[ r_2 = (\lambda_1 + \varepsilon)\lambda_2 r, \quad s_2 = -(\lambda_1 - 1 - \varepsilon)\lambda_2 s, \quad t_2 = -(\lambda_1 + \varepsilon)(\lambda_2 - 1 - 2\varepsilon)t, \]
\[ u_2 = (\lambda_1 - 1 - \varepsilon)(\lambda_2 - 1 - 2\varepsilon)u, \]
\[ r_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2 + 2\varepsilon)(\lambda_1 + \lambda_2 - 1)r, \]
\[ s_3 = -(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - 1)(\lambda_1 + \lambda_2 - 1 - 2\varepsilon)s, \]
\[ t_3 = (\lambda_1 - \lambda_2 + 2\varepsilon)(\lambda_1 + \lambda_2 - 1)(\lambda_1 + \lambda_2 - 1 - 2\varepsilon)s, \]
\[ u_3 = -(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2 + 2\varepsilon)(\lambda_1 + \lambda_2 - 1 - 2\varepsilon)u. \]

(4.20)

The justification of (4.18) follows by showing that (4.13) is then satisfied for \( \Delta^{(\varepsilon)} \) having the eigenvalue (4.17).

The coefficients \( s, t, u \) in (4.18) can be determined by analysing the limit \( x, \bar{x} \to 0, \)
\[ r = 1, \quad s = -\frac{\lambda_1 - \lambda_2 + 2\varepsilon}{\lambda_1 - \lambda_2 + \varepsilon}\sigma_{\lambda_1}(a, b), \quad t = -\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2 + \varepsilon}\sigma_{\lambda_2 - \varepsilon}(a, b), \]
\[ u = \frac{(\lambda_1 + \lambda_2 - 1)(\lambda_1 + \lambda_2 - 2\varepsilon)}{(\lambda_1 + \lambda_2 - 1 - \varepsilon)(\lambda_1 + \lambda_2 - \varepsilon)}\sigma_{\lambda_1}(a, b)\sigma_{\lambda_2 - \varepsilon}(a, b). \]

(4.21)

By combining (4.18) with (2.23a, b) we may then obtain
\[ u^{-\frac{1}{2}}F_{\lambda_1\lambda_2}(a, b) = F_{\lambda_1 - \frac{1}{2}\lambda_2 - \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) - \frac{\ell + 2\varepsilon}{\ell + \varepsilon}\sigma_{\lambda_1}(a, b)F_{\lambda_1 + \frac{1}{2}\lambda_2 - \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) \]
\[ - \frac{\ell}{\ell + \varepsilon}\sigma_{\lambda_2 - \varepsilon}(a, b)F_{\lambda_1 - \frac{1}{2}\lambda_2 + \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) \]
\[ + f\Delta\sigma_{\lambda_1}(a, b)\sigma_{\lambda_2 - \varepsilon}(a, b)F_{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}), \]
\[ u^{-\frac{1}{2}}vF_{\lambda_1\lambda_2}(a, b) = F_{\lambda_1 - \frac{1}{2}\lambda_2 - \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) + \frac{\ell + 2\varepsilon}{\ell + \varepsilon}\sigma_{\lambda_1}(a, b)F_{\lambda_1 + \frac{1}{2}\lambda_2 - \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) \]
\[ + \frac{\ell}{\ell + \varepsilon}\sigma_{\lambda_2 - \varepsilon}(a, b)F_{\lambda_1 - \frac{1}{2}\lambda_2 + \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}) \]
\[ + f\Delta\sigma_{\lambda_1}(a, b)\sigma_{\lambda_2 - \varepsilon}(a, b)F_{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}), \]
\[ u^{-\frac{1}{2}}vF_{\lambda_1\lambda_2}(a, b) = F_{\lambda_1 - \frac{1}{2}\lambda_2 - \frac{1}{2}}(a - \frac{1}{2}, b - \frac{1}{2}) - \frac{\ell + 2\varepsilon}{\ell + \varepsilon}\sigma_{\lambda_1}(a, b)F_{\lambda_1 + \frac{1}{2}\lambda_2 - \frac{1}{2}}(a - \frac{1}{2}, b - \frac{1}{2}) \]
\[ - \frac{\ell}{\ell + \varepsilon}\sigma_{\lambda_2 - \varepsilon}(a, b)F_{\lambda_1 - \frac{1}{2}\lambda_2 + \frac{1}{2}}(a - \frac{1}{2}, b - \frac{1}{2}) \]
\[ + f\Delta\sigma_{\lambda_1}(a, b)\sigma_{\lambda_2 - \varepsilon}(a, b)F_{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{2}}(a - \frac{1}{2}, b - \frac{1}{2}), \]

(4.22)

for
\[ f\Delta = \frac{(\Delta - 1)(\Delta + 2 - d)}{(\Delta - \frac{1}{2}d)(\Delta - \frac{1}{2}d + 1)}. \]

(4.23)
By iterating (4.22) we may rederive the result for \( u^{-1}F_{\lambda_1,\lambda_2}(a, b) \) in terms of \( F_{\lambda_1',\lambda_2'}(a, b) \) for \( (\lambda_1', \lambda_2') = (\lambda_1 \pm 1, \lambda_2 \pm 1), (\lambda_1 \pm 1, \lambda_2), (\lambda_1, \lambda_2 \pm 1), (\lambda_1, \lambda_2) \) given in [12]. Furthermore, eliminating \( F_{\lambda_1 - \frac{1}{2}\lambda_2 - \frac{1}{2}} \) and \( F_{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{2}} \) from (4.22) leads to

\[
(\ell + \frac{1}{2}d - 1)u^{-\frac{1}{2}}(\ell_1 + b - \frac{1}{2})(\lambda_2 + b - \frac{1}{2} - \varepsilon)(F_{\lambda_1,\lambda_2}(a - \frac{1}{2}, b + \frac{1}{2}) - vF_{\lambda_1,\lambda_2}(a + \frac{1}{2}, b + \frac{1}{2}))
+ (\ell_1 - b - \frac{1}{2})(\lambda_2 - b - \frac{1}{2} - \varepsilon)(F_{\lambda_1,\lambda_2}(a + \frac{1}{2}, b - \frac{1}{2}) - F_{\lambda_1,\lambda_2}(a - \frac{1}{2}, b - \frac{1}{2}))
\]

\[
= (\ell + d - 2)\frac{1}{\lambda_1}(\lambda_2 - \frac{1}{2} - \varepsilon)(\lambda_1 - b - \frac{1}{2})(\lambda_1 + b - \frac{1}{2})F_{\lambda_1,\lambda_2}(a, b)
+ \ell\frac{1}{\lambda_2 - \varepsilon}(\lambda_2 - b - \frac{1}{2} - \varepsilon)(\lambda_2 + b - \frac{1}{2} - \varepsilon)F_{\lambda_1,\lambda_2}(a, b),
\]

(4.24)

which is equivalent to (3.27).

Using (4.18) and (4.19) with (4.20) and (4.21) gives

\[
(H_2 - \lambda_2 H_1 + \lambda_2(\lambda_2 - \varepsilon)(x\bar{x})^{-\frac{1}{2}})F_{\lambda_1,\lambda_2}(a, b) = u^{\lambda_2}H_2 u^{-\lambda_2}F_{\lambda_1,\lambda_2}(a, b)
= \sigma_{\lambda_2 - \varepsilon}(a, b)(2\lambda_2 - 1 - 2\varepsilon)(\ell F_{\lambda_1,\lambda_2}(a + \frac{1}{2}, b + \frac{1}{2})
+ (\Delta - 1)(\Delta - 2\varepsilon)^{-1}\sigma_{\lambda_1}(a, b)F_{\lambda_1 + \frac{1}{2},\lambda_2 + \frac{1}{2}}(a + \frac{1}{2}, b + \frac{1}{2}))
\]

\[
= 0 \text{ for } \lambda_2 = \varepsilon - b \text{ or } \lambda_2 = \varepsilon - a.
\]

(4.25)

There are further multi-term recurrence relations for conformal partial waves, for fixed \( \varepsilon \) and \( a, b \), which extend the single variable results (4.7a, b). To obtain these we define

\[
\mathcal{F}_0 = \frac{1}{x} + \frac{1}{\bar{x}} - 1, \quad \mathcal{F}_1 = (1 - x)\frac{\partial}{\partial x} + (1 - \bar{x})\frac{\partial}{\partial \bar{x}} - a - b,
\]

\[
\mathcal{F}_2 = \frac{x - \bar{x}}{x\bar{x}}(D_x - D_{\bar{x}}),
\]

\[
\mathcal{F}_3 = \left[(1 - x)\frac{\partial}{\partial x} - (1 - \bar{x})\frac{\partial}{\partial \bar{x}} - 2\varepsilon \frac{x + \bar{x} - 2}{x - \bar{x}}\right](D_x - D_{\bar{x}}),
\]

which satisfy the commutation relations with \( \Delta^{(\varepsilon)} \), generalising (4.6),

\[
[\Delta^{(\varepsilon)}, \mathcal{F}_0] = -2\mathcal{F}_1 + 2(1 + \varepsilon)\mathcal{F}_0, \quad [\Delta^{(\varepsilon)}, \mathcal{F}_1] = \mathcal{F}_2 - \mathcal{F}_0 \Delta^{(\varepsilon)} - 2ab,
\]

\[
[\Delta^{(\varepsilon)}, \mathcal{F}_2] = 2\mathcal{F}_3 + 2(1 + \varepsilon)\mathcal{F}_2,
\]

\[
[\Delta^{(\varepsilon)}, \mathcal{F}_3] = -\mathcal{F}_0 \Delta_4^{(\varepsilon)} - 4\varepsilon \mathcal{F}_3 + \mathcal{F}_2(\Delta^{(\varepsilon)} - 2\varepsilon(1 + 2\varepsilon)).
\]

(4.27)

As a consequence of (4.27) it is sufficient to assume

\[
\mathcal{F}_1 F_{\lambda_1,\lambda_2} = r_i F_{\lambda_1,\lambda_2 - 1} + s_i F_{\lambda_1 - 1,\lambda_2} + t_i F_{\lambda_1 + 1,\lambda_2} + u_i F_{\lambda_1,\lambda_2 + 1} + w_i F_{\lambda_1,\lambda_2},
\]

(4.28)
Imposing (4.27) then leads to

\[ r_1 = \lambda_2 r_0, \quad s_1 = (\lambda_1 + \varepsilon)s_0, \quad t_1 = -(\lambda_1 - 1 - \varepsilon)t_0, \quad u_1 = -(\lambda_2 - 1 - 2\varepsilon)u_0, \]
\[ r_2 = (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - 1)r_0, \quad s_2 = -(\lambda_1 - \lambda_2 + \varepsilon)(\lambda_1 + \lambda_2 - 1)s_0, \]
\[ t_2 = - (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - 1 - 2\varepsilon)t_0, \quad u_2 = (\lambda_1 - \lambda_2 + 2\varepsilon)(\lambda_1 + \lambda_2 - 1 - 2\varepsilon)u_0, \]
\[ r_3 = -\lambda_2(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - 1)r_0, \]
\[ s_3 = (\lambda_1 + \varepsilon)(\lambda_1 - \lambda_2 + \varepsilon)(\lambda_1 + \lambda_2 - 1)s_0, \]
\[ t_3 = -(\lambda_1 - 1 - \varepsilon)(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - 1 - 2\varepsilon)t_0, \]
\[ u_3 = (\lambda_2 - 1 - 2\varepsilon)(\lambda_1 - \lambda_2 + 2\varepsilon)(\lambda_1 + \lambda_2 - 1 - 2\varepsilon)u_0, \quad (4.29) \]

and also \( w_i \) is determined giving

\[ w_1 = (1 + \varepsilon)w_0, \quad w_3 = -(1 + \varepsilon)w_2, \]
\[ w_0 = - (c_{\lambda_1 \lambda_2} + 2\varepsilon) \frac{ab}{2d_{\lambda_1 \lambda_2}}, \quad w_2 = -c_{4, \lambda_1 \lambda_2} \frac{ab}{2d_{\lambda_1 \lambda_2}}, \quad (4.30) \]

where

\[ d_{\lambda_1 \lambda_2} = -\frac{1}{4}(c_{4, \lambda_1 \lambda_2} - c_{\lambda_1 \lambda_2}^2 - 2\varepsilon c_{\lambda_1 \lambda_2}) = \lambda_1(\lambda_1 - 1)(\lambda_2 - \varepsilon)(\lambda_2 - 1 - \varepsilon). \quad (4.31) \]

The coefficients are the determined by taking in (4.29)

\[ r_0 = \frac{\lambda_1 - \lambda_2 + 2\varepsilon}{\lambda_1 - \lambda_2 + \varepsilon}, \quad s_0 = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2 + \varepsilon}, \]
\[ t_0 = \frac{(\lambda_1 + \lambda_2 - 1)(\lambda_1 + \lambda_2 - 2\varepsilon)}{(\lambda_1 + \lambda_2 - 1 - \varepsilon)(\lambda_1 + \lambda_2 - \varepsilon)} \frac{\lambda_1 - \lambda_2 + 2\varepsilon}{\lambda_1 - \lambda_2 + \varepsilon} \beta_{\lambda_1}, \quad (4.32) \]
\[ u_0 = \frac{(\lambda_1 + \lambda_2 - 1)(\lambda_1 + \lambda_2 - 2\varepsilon)}{(\lambda_1 + \lambda_2 - 1 - \varepsilon)(\lambda_1 + \lambda_2 - \varepsilon)} \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2 + \varepsilon} \beta_{\lambda_2 - \varepsilon}. \]

Explicitly the result for \( F_0 F_{\lambda_1 \lambda_2} \) becomes

\[ F_0 F_{\lambda_1 \lambda_2} = \frac{\ell + 2\varepsilon}{\ell + \varepsilon} F_{\lambda_1 \lambda_2 - 1} + \frac{\ell}{\ell + \varepsilon} F_{\lambda_1 - 1 \lambda_2} - (c_{\lambda_1 \lambda_2} + 2\varepsilon) \frac{ab}{2d_{\lambda_1 \lambda_2}} F_{\lambda_1 \lambda_2}, \]
\[ + f_{\Delta} \left( \frac{\ell + 2\varepsilon}{\ell + \varepsilon} \beta_{\lambda_1} F_{\lambda_1 + 1 \lambda_2}(x, \bar{x}) + \frac{\ell}{\ell + \varepsilon} \beta_{\lambda_2 - \varepsilon} F_{\lambda_1 \lambda_2 + 1} \right) \],

with \( f_{\Delta} \) given by (4.23). The result for \( F_2 F_{\lambda_1 \lambda_2} \) vanishes for \( \ell = 0 \) so that

\[ (D_x - D_{\bar{x}}) F_{\lambda \lambda}(x, \bar{x}) = 0. \quad (4.34) \]
There are also second order differential operators relating conformal partial waves for dimensions differing by 2. Defining, in terms of $F_2$ in (4.26),

$$\mathcal{E}_+ = \left( \frac{x\bar{x}}{x - \bar{x}} \right)^2 F_2 = \frac{x\bar{x}}{x - \bar{x}} \left( D_x - D_{\bar{x}} \right),$$  

(4.35)

then we may verify that

$$\mathcal{E}_+ \Delta^{(\varepsilon)} = (\Delta^{(\varepsilon+1)} + 2 + 2\varepsilon) \mathcal{E}_+. \quad (4.36)$$

For $\bar{x} \to 0$ followed by $x \to 0$, $\mathcal{E}_+ \sim (x^2 \frac{\partial^2}{\partial x^2} - \bar{x}^2 \frac{\partial^2}{\partial \bar{x}^2})$ so that, assuming this limit, $\mathcal{E}_+ x^{\lambda_1} \bar{x}^{\lambda_2} \sim (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - 1) x^{\lambda_1} \bar{x}^{\lambda_2+1}$. Hence, with the normalisation of $F^{(\varepsilon)}_{\lambda_1\lambda_2}$, determined by (2.30),

$$\mathcal{E}_+ F^{(\varepsilon)}_{\lambda_1\lambda_2}(x, \bar{x}) = \frac{\lambda_1 + \lambda_2 - 1}{2(2\varepsilon + 1)} (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2 + 2\varepsilon) F^{(\varepsilon+1)}_{\lambda_1\lambda_2+1}(x, \bar{x}). \quad (4.37)$$

As a special case $\mathcal{E}_+ F^{(\varepsilon)}_{\lambda\lambda} = 0$ which is equivalent to (4.34).

By combining (4.36) with (2.40) we may obtain a related operator which has the reverse effect

$$\mathcal{E}^{(\varepsilon)}_- = \left( \frac{x\bar{x}}{x - \bar{x}} \right)^{2\varepsilon-2} \left( D_x - D_{\bar{x}} \right) \left( \frac{x\bar{x}}{x - \bar{x}} \right)^{-2\varepsilon+1},$$  

(4.38)

such that

$$\mathcal{E}^{(\varepsilon)}_- \Delta^{(\varepsilon)} = (\Delta^{(\varepsilon-1)} - 2\varepsilon) \mathcal{E}^{(\varepsilon)}_-.$$

(4.39)

Similarly to (4.37) we then have

$$\mathcal{E}^{(\varepsilon)}_- F^{(\varepsilon)}_{\lambda_1\lambda_2}(x, \bar{x}) = 2(2\varepsilon - 1) (\lambda_1 + \lambda_2 - 2\varepsilon) F^{(\varepsilon-1)}_{\lambda_1\lambda_2+1}(x, \bar{x}). \quad (4.40)$$

It is crucial that both $\mathcal{E}_+$ and $\mathcal{E}^{(\varepsilon)}_-$, defined in (4.35) and (4.38), are symmetric under $x \leftrightarrow \bar{x}$. It is easy to see from (4.14)

$$\mathcal{E}^{(\varepsilon+1)}_- \mathcal{E}_+ = \Delta_4^{(\varepsilon)}. \quad (4.41)$$

---

3 To solve this equation it is more convenient to revert to the $u, v$ variables when

$$\mathcal{E}_+ = u \left( (1 - v) \frac{\partial^2}{\partial v^2} - 2uv \frac{\partial^2}{\partial u \partial v} - u^2 \frac{\partial^2}{\partial u^2} + (a + b + 1) \left( (1 - v) \frac{\partial}{\partial v} - u \frac{\partial}{\partial u} \right) - ab \right).$$

Then $\mathcal{E}_+ G(u, v) = 0$ has solutions, analytic for $1 - v \approx 0$, $u^\lambda F(\lambda + a, \lambda + b; 2\lambda; 1 - v)$ for any $\lambda$. This is in accord with the conformal partial wave when $\ell = 0$ given by (3.32).
From the result from $\mathcal{F}_2 F_{\lambda_1, \lambda_2}$,
\[
\left( \frac{x - \bar{x}}{x \bar{x}} \right)^2 \frac{\ell + 1 + \varepsilon}{2(2\varepsilon + 1)} F^{(\varepsilon+1)_{\lambda_1, \lambda_2}}(x, \bar{x})
\]
\[=
F^{(\varepsilon)_{\lambda_1, \lambda_2-2}}(x, \bar{x}) - F^{(\varepsilon)_{\lambda_1-1, \lambda_2-1}}(x, \bar{x})
\]
\[- (\lambda_1 + \lambda_2 - 2 - 2\varepsilon)(\ell + 1 + \varepsilon) \frac{ab}{2d_{\lambda_1, \lambda_2-1}} F^{(\varepsilon)_{\lambda_1, \lambda_2-1}}(x, \bar{x})
\]
\[- \frac{(\lambda_1 + \lambda_2 - 1 - 2\varepsilon)(\lambda_1 + \lambda_2 - 2 - 2\varepsilon)}{(\lambda_1 + \lambda_2 - 1 - \varepsilon)(\lambda_1 + \lambda_2 - 2 - \varepsilon)} \left( \beta_{\lambda_1} F^{(\varepsilon)_{\lambda_1+1, \lambda_2-1}}(x, \bar{x}) - \beta_{\lambda_2-1-\varepsilon} F^{(\varepsilon)_{\lambda_1, \lambda_2}}(x, \bar{x}) \right). \tag{4.42}
\]

An equivalent result was obtained in [12], which, along with (4.33), was derived using manipulations based on a series expansion in terms of Jack polynomials.\(^4\)

Further recurrence relations for conformal partial waves can be derived by using the second order operator
\[
\mathcal{D}^{(\varepsilon)} = (x \bar{x})^{-\frac{1}{2}} \mathcal{H}_2 = \frac{1}{(x - \bar{x})^\varepsilon} \frac{\partial^2}{\partial x \partial \bar{x}} (x - \bar{x})^\varepsilon, \tag{4.43}
\]
with $\mathcal{H}_2$ given in (4.11), or equivalently
\[
4 \mathcal{D}^{(\varepsilon)} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} - \frac{2\varepsilon}{r} \frac{\partial}{\partial r} \quad \text{for} \quad t = \frac{1}{2}(x + \bar{x}), \ r = \frac{1}{2}(x - \bar{x}). \tag{4.44}
\]

It is then clear that $\mathcal{D}^{(\varepsilon)} \hat{F} = 0$ is just the spherically symmetric, or $S$-wave, projection of the wave equation in $2(\varepsilon + 1)$ dimensions, although here we require only solutions even in $r$.

From its definition (4.43) $\mathcal{D}^{(\varepsilon)}$ is invariant under reflection $x \rightarrow -x, \bar{x} \rightarrow -\bar{x}$. $\mathcal{D}^{(\varepsilon)}$ also satisfies, analogous to (2.40),
\[
\mathcal{D}^{(\varepsilon)} \frac{1}{(x - \bar{x})^{2\varepsilon-1}} = \frac{1}{(x - \bar{x})^{2\varepsilon-1}} \mathcal{D}^{(-\varepsilon+1)}. \tag{4.45}
\]

The wave equation is invariant under conformal transformations which preserve the spherical symmetry. In this case for
\[
x' = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \bar{x}' = \frac{\alpha \bar{x} + \beta}{\gamma \bar{x} + \delta}, \quad \alpha \delta - \beta \gamma = 1, \tag{4.46}
\]
so that $\left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \text{SL}(2, \mathbb{R})$, then
\[
\mathcal{D}^{(\varepsilon)} \rightarrow \mathcal{D}'^{(\varepsilon)} = \Omega^{2+\varepsilon} \mathcal{D}^{(\varepsilon)} \Omega^{-\varepsilon}, \tag{4.47}
\]

\(^4\) The analogous formula in the compact case was given in [17] and is rederived in appendix B.
where
\[
\Omega = (\gamma x + \delta)(\gamma \bar{x} + \delta) = \frac{1}{(\alpha - \gamma x')(\alpha - \gamma \bar{x}')}. \tag{4.48}
\]

With \(u, v\) defined in (2.15), we may obtain
\[
\Delta^{(\varepsilon)}(a, b + 1) u^{-b+1+\varepsilon} \mathcal{D}^{(\varepsilon)} u^{-b-\varepsilon} = u^{-b+1+\varepsilon} \mathcal{D}^{(\varepsilon)} u^{-b-\varepsilon} \Delta^{(\varepsilon)}(a, b), \tag{4.49}
\]
Hence we may obtain
\[
u \mathcal{D}^{(\varepsilon)} u^{-b-\varepsilon} F_{\lambda_1 \lambda_2}(a, b; x, \bar{x}) = (\lambda_1 + b)(\lambda_2 + b - \varepsilon) u^{-b-\varepsilon} F_{\lambda_1 \lambda_2}(a, b + 1, x, \bar{x}). \tag{4.50}
\]
Conversely using (2.44)
\[
u v \mathcal{D}^{(\varepsilon)} u^{-b-\varepsilon} v^{a+b} F_{\lambda_1 \lambda_2}(a, b; x, \bar{x}) = (\lambda_1 - b)(\lambda_2 - b - \varepsilon) u^{-b-\varepsilon} v^{a+b} F_{\lambda_1 \lambda_2}(a, b - 1; x, \bar{x}). \tag{4.51}
\]
Of course there are similar results for \(a \leftrightarrow b\). Consistency (4.50) and (4.51) requires the fourth order eigenvalue equation,
\[
u^2 \mathcal{D}^{(\varepsilon)} v \mathcal{D}^{(\varepsilon)} u^{-\varepsilon} F_{\lambda_1 \lambda_2}(0, 0; x, \bar{x}) = d_{\lambda_1 \lambda_2} F_{\lambda_1 \lambda_2}(0, 0; x, \bar{x}). \tag{4.52}
\]
for \(d_{\lambda_1 \lambda_2}\) as in (4.31).

The right hand side of (4.50) vanishes when \(\lambda_2 = \varepsilon - b\). Using \(\mathcal{D}^{(\varepsilon)} u^{N+1} \mathcal{D}^{(\varepsilon)} u = u^N \mathcal{D}^{(\varepsilon)} u^{N+1} u\) and applying (4.50) repeatedly this may be extended to
\[
\mathcal{D}^{(\varepsilon)} u^{N} ((x, \bar{x})^{b+N-1-\varepsilon} F_{\lambda_1 \lambda_2}(x, \bar{x}))|_{\lambda_2 = \varepsilon - b + 1 - N} = 0, \quad N = 1, 2, \ldots . \tag{4.53}
\]
This result leads to additional constraints on conformal partial waves for particular values of the twist \(\Delta - \ell\). For \(N = 1\) (4.53) coincides with the vanishing condition in (4.25).

Corresponding to (4.50) and (4.51) there are analogous single variable relations
\[
x \frac{d}{dx} x^b g_p(a, b; x) = (p + b) x^b g_p(a, b + 1; x), \tag{4.54}
\]
\[
x(1 - x) \frac{d}{dx} x^{-b}(1 - x)^{a+b} g_p(a, b; x) = (p - b) x^{-b}(1 - x)^{a+b} g_p(a, b - 1; x).
\]

5. Even Dimensional Results

For even \(d\) the solutions are all expressible in terms of the symmetric, antisymmetric combinations of \(g_p, g_q\), as defined in (2.36), for suitable \(p, q\),
\[
\mathcal{F}_{pq}(x, \bar{x}) = g_p(x) g_q(\bar{x}) \pm g_p(\bar{x}) g_q(x) \underset{x \to 0, x \to 0}{\sim} (x\bar{x})^q x^{p+q-2}, \quad \text{for} \quad p - q = 1, 2, \ldots . \tag{5.1}
\]
Assuming \( g_p(\bar{x}) \) transforms in the conjugate fashion to \( g_p(x) \) in (2.37) then from (5.1)

\[
F^\pm_{pq}(x, \bar{x}) = (-1)^{p-q}v^{-b}F^\pm_{pq}(x', \bar{x}') \bigg|_{a \to -a} = (-1)^{p-q}v^{-a}F^\pm_{pq}(x', \bar{x}') \bigg|_{b \to -b},
\]

(5.2)

which is sufficient to verify (2.23a, b) in each case. The identity (4.10) follows in each case from \( F^\pm_{pq} = \pm F^\pm_{qp} \).

For \( d = 2 \), or \( \varepsilon = 0 \), \( \Delta^{(0)} \) becomes a sum of single variable operators so that it is trivial to separate variables and the conformal partial waves are just, since from (2.30) \( c_\ell^{(0)} = \frac{1}{2} \) for \( \ell \neq 0 \),

\[
2 F^{(0)}_{\lambda_1 \lambda_2}(x, \bar{x}) = F^+_{\lambda_1 \lambda_2}(x, \bar{x}),
\]

(5.3)

agreeing of course with (3.53).

For \( \varepsilon = 1 \)

\[
(\lambda_1 - \lambda_2 + 1) F^{(1)}_{\lambda_1 \lambda_2}(x, \bar{x}) = \frac{x\bar{x}}{x - \bar{x}} F^-_{\lambda_1 \lambda_2 - 1}(x, \bar{x}).
\]

(5.4)

With the operator \( \mathcal{E}_+ \) defined by (4.35), and using the eigenvalue equation (2.33), it is very easy to verify that (5.4) and (5.3) are in accord with (4.37) raising \( \varepsilon \) by one.

To obtain results for \( \varepsilon = 2 \) using (4.37) directly, which becomes

\[
\mathcal{E}_+ F^{(1)}_{\lambda_1 \lambda_2}(x, \bar{x}) = \frac{1}{6}(\lambda_1 + \lambda_2 - 1) (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2 + 2) F^{(2)}_{\lambda_1 \lambda_2 + 1}(x, \bar{x}),
\]

(5.5)

it is necessary to calculate the action of \( \mathcal{E}_+ \), as given by (4.35), on \( F^{(2)}_{\lambda_1 \lambda_2} \). Using (2.33) again we get

\[
(\lambda_1 - \lambda_2 + 1) \mathcal{E}_+ F^{(1)}_{\lambda_1 \lambda_2}(x, \bar{x})
= - \left( \frac{x\bar{x}}{x - \bar{x}} \right)^3 \left\{ 2 \left( (1-x) \frac{\partial}{\partial x} + (1-\bar{x}) \frac{\partial}{\partial \bar{x}} - a - b \right) F^-_{\lambda_1 \lambda_2 - 1}(x, \bar{x}) + (\lambda_1 - \lambda_2 + 1)(\lambda_1 + \lambda_2 - 2) \left( \frac{1}{x} - \frac{1}{\bar{x}} \right) F^+_{\lambda_1 \lambda_2 - 1}(x, \bar{x}) \right\}.
\]

(5.6)

As a consequence of (4.7a, b) we have

\[
\left( \frac{1}{x} - \frac{1}{\bar{x}} \right) F^+_{pq}(x, \bar{x}) = F^-_{p-1,q}(x, \bar{x}) - F^-_{p,q-1}(x, \bar{x}) + (\alpha_p - \alpha_q) F^-_{pq}(x, \bar{x}) + \beta_p F^-_{p+1,q}(x, \bar{x}) - \beta_q F^-_{p,q+1}(x, \bar{x}),
\]

(5.7a)

\[
(1-x) \frac{\partial}{\partial x} + (1-\bar{x}) \frac{\partial}{\partial \bar{x}} - a - b \right) F^-_{pq}(x, \bar{x})
= p F^-_{p-1,q}(x, \bar{x}) + q F^-_{p,q-1}(x, \bar{x}) + (\alpha_p + \alpha_q) F^-_{pq}(x, \bar{x}) - (p-1)\beta_p F^-_{p+1,q}(x, \bar{x}) - (q-1)\beta_q F^-_{p,q+1}(x, \bar{x}),
\]

(5.7b)
for $\alpha_p, \beta_p$ is given by (4.8).

Using (5.7a, b) in (5.6) then gives from (5.5)

$$
\frac{1}{6}(\lambda_1 - \lambda_2 + 3)(\lambda_1 - \lambda_2 + 2)(\lambda_1 - \lambda_2 + 1) F^{(2)}_{\lambda_1 \lambda_2}(x, \bar{x})
= \left(\frac{x \bar{x}}{x - \bar{x}}\right)^3 \left\{ (\lambda_1 - \lambda_2 + 1) F^-_{\lambda_1 \lambda_2 - 3}(x, \bar{x}) - (\lambda_1 - \lambda_2 + 3) F^-_{\lambda_1 - 1 \lambda_2 - 2}(x, \bar{x})
- \frac{\lambda_1 + \lambda_2 - 4}{\lambda_1 + \lambda_2 - 2} \left( (\lambda_1 - \lambda_2 + 1) F^-_{\lambda_1 + 1 \lambda_2 - 2}(x, \bar{x}) - (\lambda_1 - \lambda_2 + 3) F^-_{\lambda_1 \lambda_2 - 3}(x, \bar{x}) \right) 
- (\lambda_1 + \lambda_2 - 4)(\lambda_1 - \lambda_2 + 1)(\lambda_1 - \lambda_2 + 3) \frac{ab}{2d_{\lambda_1 \lambda_2}} F^-_{\lambda_1 \lambda_2 - 2}(x, \bar{x}) \right\},
$$

(5.8)

with $d_{\lambda_1 \lambda_2} = \lambda_1(\lambda_1 - 1)(\lambda_2 - 2)(\lambda_2 - 3)$. The right hand side of (5.8) vanishes when $\lambda_1 - \lambda_2 = -1, -2, -3$. That the expression for $F^{(2)}_{\lambda_1 \lambda_2}(x, \bar{x})$ given by (5.8) is finite when $x = \bar{x}$ is verified in appendix C.

For particular twist the solutions simplify as expected from (4.53). For the case when $N = 1$ we require solutions of

$$
D^{(\varepsilon)} \hat{F}(x, \bar{x}) = 0,
$$

(5.9)

for $\hat{F}(x, \bar{x})$ symmetric in $x, \bar{x}$. The general solutions of (5.9) involve an unconstrained single variable function $f$. For even $\varepsilon$ the solutions, which may be found using (4.45), are quite simple. In the first three cases

$$
\hat{F}(x, \bar{x}) = \begin{cases} 
  f(x) + f(\bar{x}), & \varepsilon = 0, \\
  f(x) - f(\bar{x}), & \varepsilon = 1, \\
  f(x) - f(\bar{x}) - \frac{1}{(x - \bar{x})^2} (f(x) - f(\bar{x})) + f'(x) + f'(\bar{x}), & \varepsilon = 2.
\end{cases}
$$

(5.10)

It is easy to verify that $\hat{F}(x, \bar{x})$ is regular as $x \to \bar{x}$ so long as $f$ is a smoothly differentiable function, so that from (5.10)

$$
\hat{F}(x, x) = 2f(x), f'(x), -\frac{1}{12} f'''(x) \quad \text{for} \quad \varepsilon = 0, 1, 2.
$$

(5.11)

It is straightforward to verify how the solutions obtained above for $\varepsilon = 0, 1, 2$ connect with the special cases given by (5.10). For $\varepsilon = 0$ we have from (5.3)

$$
F^{(0)}_{\lambda_1 \lambda_2}(x, \bar{x}) |_{\lambda_2 = -b} = (x \bar{x})^{-b} \left( f(x) + f(\bar{x}) \right), \quad f(x) = x^b g_{\lambda_1}(x),
$$

(5.12)

and for $\varepsilon = 1$ from (5.4)

$$
(\lambda_1 - \lambda_2 + 1) F^{(1)}_{\lambda_1 \lambda_2}(x, \bar{x}) |_{\lambda_2 = 1 - b} = \frac{(x \bar{x})^{1-b}}{x - \bar{x}} \left( f(x) - f(\bar{x}) \right), \quad f(x) = x^b g_{\lambda_1}(x).
$$

(5.13)
For \( \varepsilon = 2 \) we require

\[
\frac{1}{6} (\lambda_1 - \lambda_2 + 3)(\lambda_1 - \lambda_2 + 2)(\lambda_1 - \lambda_2 + 1) F^{(2)\lambda_1\lambda_2}(x, \bar{x}) \bigg|_{\lambda_2 = 2 - b} = \frac{(x\bar{x})^{2-b}}{(x - \bar{x})^3} \left( f(x) - f(\bar{x}) - \frac{1}{2} (x - \bar{x}) (f'(x) + f'(\bar{x})) \right). 
\]  
(5.14)

From (5.8), using the expansion of \( F(a - b - 1, -1; -2b - 2; x) \), this requires

\[
f(x) - \frac{1}{2} x f'(x) = (\lambda_1 - 1 + b) x^{b+1} g_{\lambda_1}(x),
\]  
(5.15)

and

\[
x^{-b - \frac{1}{2}} f'(x) = - (\lambda_1 + 1 + b) g_{\lambda_1 - 1}(x) + \frac{1}{2} (\lambda_1 - 1 + b) \left( -1 + \frac{a(b + 2)}{\lambda_1(\lambda_1 - 1)} \right) g_{\lambda_1}(x) \\
- (\lambda_1 - 1 + b) \frac{\lambda_1 - 2 - b}{\lambda_1 - b} g_{\lambda_1 + 1}(x). 
\]  
(5.16)

(5.15) is satisfied by taking

\[
f(x) = -2x^{b+1} g_{\lambda_1}(a, b - 1; x). 
\]  
(5.17)

This also satisfies (5.16), as may be verified by using (5.15) to eliminate \( f'(x) \) and then using (4.7a) for \( g_{\lambda_1}(x)/x \) and also

\[
\frac{1}{x} g_p(a, b - 1; x) = g_{p-1}(a, b; x) + \hat{\alpha}_p(a, b) g_p(a, b; x) + \hat{\beta}_p(a, b) g_{p+1}(a, b; x), 
\]  
(5.18)

where

\[
\hat{\alpha}_p(a, b) = - \frac{a(p + b - 1)}{2p(p - 1)}, \\
\hat{\beta}_p(a, b) = - \frac{(p + a)(p + b)(p - a)(p + b - 1)}{4p^2(2p + 1)(2p - 1)}. 
\]  
(5.19)

(5.18) may be derived from (4.3) so that \( \hat{\alpha}_p(a, b) = -\sigma_p(a, b - 1) + \sigma_{p-\frac{1}{2}}(-a - \frac{1}{2}, b - \frac{1}{2}) \), \( \hat{\beta}_p(a, b) = -\sigma_p(a, b - 1) \sigma_{p+\frac{1}{2}}(-a - \frac{1}{2}, b - \frac{1}{2}) \).

6. Results for One and Three Dimensions

For odd \( d \) no such explicit formulae as for even \( d \) described above are known. When \( d = 1 \) or \( \varepsilon = -\frac{3}{2} \), corresponding to conformal quantum mechanics, conformal partial waves are rather trivial. The conformal group \( SO(1, 2) \) acting on four points leaves just
one invariant instead of the two given by \( u, v \). The necessary constraint is obtained by imposing \( x = \bar{x} \). In this case from (2.19)
\[
\Delta^{(-\frac{1}{2})}(a, b)F(t) = \frac{1}{2} D_t(2a, 2b)F(t) + O((x - \bar{x})^2), \quad t = \frac{1}{2}(x + \bar{x}),
\]
and
\[
\Delta^{(-\frac{1}{2})}(a, b)(x - \bar{x})^2 = 2(2 - (a + 2)(b + 2))(x - \bar{x})^2.
\]
Hence (2.22) may be restricted to \( x = \bar{x} \) for \( \varepsilon = -\frac{1}{2} \) and the conformal partial waves are given in (2.16) for this case with \( \ell = 0 \) by
\[
F^{(-\frac{1}{2})}_{\lambda\lambda}(x, x) = g_{2\lambda}(2a, 2b; x) = x^\Delta F(\Delta + 2a, \Delta + 2b; 2\Delta; x).
\]

The requirement that \( \ell = \lambda_1 - \lambda_2 = 0 \) follows since for the conformal group \( SO(1, 2) \) the representations are determined just by the scale dimension \( \Delta \).

For \( d = 3 \) or \( \varepsilon = \frac{1}{2} \) solutions of (5.9) of a similar form to (5.10) are not possible, reflecting the difference between the properties of the wave equation in even and odd dimensions.\(^5\) However when \( d = 3 \) solutions of the wave equation can be found in terms of the Bateman transform [22], which is related to twistor transforms. Hence we may obtain a solution of \( D^{(\frac{1}{2})}\hat{F} = 0 \) in terms of the equivalent integral expressions
\[
\hat{F}(x, \bar{x}) = \frac{1}{\pi} \int_0^\pi ds \ f(X) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} f(X),
\]
for
\[
X = \cos^2 s \ x + \sin^2 s \ \bar{x} = t + r \frac{1}{2} (z + z^{-1}).
\]
Clearly from (6.4)
\[
\hat{F}(x, x) = f(x).
\]
To verify (6.4) provides a solution of (4.53), with \( N = 0 \), it is sufficient to use
\[
D^{(\frac{1}{2})} f(X) = -\frac{1}{4} \frac{\partial}{\partial s} \left( \frac{\sin 2s}{x - \bar{x}} f'(X) \right) \sim 0,
\]
where \( \sim \) denotes equality up to terms which vanish under integration in (6.4). For this to be valid \( f(X) \) should have no branch point at \( X = 0 \). Subject to this condition we suppose that (6.4) is in fact a general solution but this depends on global issues not considered here.

\(^5\) For \( d = 2, 4 \) the solutions of the spherically symmetric wave equation are \( f(t + r) + g(t - r), \frac{1}{\sqrt{2}}(f(t + r) + g(t - r)) \) where the dependence on \( t \pm r \) reflects Huygens principle. The solutions in (5.10) are obtained by requiring them to be even in \( r \), so that \( g = \pm f \). Huygens principle does not hold in odd dimensions.
The integral representation (6.4) satisfies various identities. As a consequence of (4.47) for the transformation (4.46), assuming (6.6) and \( \hat{F}(x, \bar{x}) \) given by (6.4), we must have

\[
\Omega^{-\frac{1}{2}} \hat{F}(x', \bar{x}') = \frac{1}{\pi} \int_0^\pi \frac{1}{\gamma X + \delta} f\left(\frac{\alpha X + \beta}{\gamma X + \delta}\right).
\]  

(6.8)

To verify this we may note that, with \( x', \bar{x}' \) given by (4.46) and \( \Omega \) by (4.48),

\[
\frac{\alpha X + \beta}{\gamma X + \delta} = \cos^2 s' x' + \sin^2 s' \bar{x}', \quad \frac{1}{\gamma X + \delta} = \frac{1}{\Omega^2} \frac{ds'}{ds},
\]

(6.9)

for

\[
\cos^2 s' = \frac{k \cos^2 s}{k \cos^2 s + k^{-1} \sin^2 s}, \quad \frac{ds'}{ds} = \frac{1}{k \cos^2 s + k^{-1} \sin^2 s}, \quad k = \left(\frac{\alpha - \gamma \bar{x}'}{\alpha - \gamma x'}\right)^{\frac{1}{2}}.
\]

(6.10)

A corollary of (6.8) for the transformation \( x, \bar{x} \to 1/x, 1/\bar{x} \) gives

\[
\frac{1}{\pi} \int_0^\pi ds \ f(X) = \frac{u^{\frac{1}{2}}}{\pi} \frac{1}{\pi} \int_0^\pi ds \ \frac{1}{X} f\left(\frac{u}{X}\right).
\]

(6.11)

Also a special case ensures that, assuming (6.4),

\[
\hat{F}(x', \bar{x}') = \frac{v^{\frac{1}{2}}}{\pi} \frac{1}{\pi} \int_0^\pi ds \ \frac{1}{1 - X} f\left(\frac{X}{X - 1}\right), \quad x' = \frac{x}{x - 1}, \quad \bar{x}' = \frac{\bar{x}}{\bar{x} - 1}
\]

(6.12)

For functions \( f(X) \) expressible as a Laurent series then \( \hat{F}(x, \bar{x}) \) given by (6.4) is a corresponding sum of Legendre polynomials using

\[
\frac{1}{\pi} \int_0^\pi X^n ds = u^{\frac{n}{2}} P_n(\sigma),
\]

(6.13)

where \( \sigma \) is defined in (2.26) so that (6.5) becomes \( X = (x\bar{x})^{\frac{1}{2}}(\sigma + \sqrt{\sigma^2 - 1} \cos 2s) \). The relation (6.11) is automatic if we identify

\[
P_n(\sigma) = P_{-n-1}(\sigma).
\]

(6.14)

To construct conformal partial waves we use, for \( X \) defined by (6.5),

\[
\Delta^{(\frac{1}{2})}(a, b) f(X) = D_X (a + b - \frac{1}{2}, 1) f(X) + \frac{1}{4} \frac{\partial}{\partial s} \left( \sin 2s \left( \frac{x\bar{x}}{x - \bar{x}} (2 - x - \bar{x}) f'(X) + (x - \bar{x}) (X f'(X) + (a + b - \frac{1}{2}) f(X)) \right) \right)
\]

\[
- (a - \frac{1}{2}) (b - \frac{1}{2}) (x + \bar{x}) f(X),
\]

(6.15)
and hence
\[
\Delta^{(\frac{1}{2})}(a, b)(u/X)^{q}f(X) \sim (u/X)^{q}(D_{X}(q + a + b - \frac{1}{2}, 1 - q)f(X) + q(q - 2)f(X) \nonumber
\]
\[
- (q + a - \frac{1}{2})(q + b - \frac{1}{2})(x + \bar{x})f(X)).
\]

Writing then
\[
F^{(\frac{1}{2})}_{\lambda_{1}, \lambda_{2}}(x, \bar{x}) = \frac{1}{\pi} \int_{0}^{\pi} ds \left( \frac{u}{X} \right)^{\lambda_{2}}f(X) \quad \text{for} \quad \lambda_{2} = \frac{1}{2} - b, \quad (6.17)
\]
the eigenvalue equation (2.22) becomes in this case using (6.16)
\[
D_{X}(a, b + \frac{1}{2})f(X) = \lambda_{1}(\lambda_{1} - 1)f(X).
\]

This is exactly of the form in (2.33) so that (6.18) may be solved by taking
\[
f(X) = g_{\lambda_{1}}(a, b + \frac{1}{2}; X),
\]
with \(g_{\lambda_{1}}\) given explicitly by (2.36). With this result we then have
\[
F^{(\frac{1}{2})}_{\lambda_{1}, \lambda_{2}}(a, b; x, \bar{x}) \big|_{\lambda_{2} = \frac{1}{2} - b} = \frac{1}{\pi} \int_{0}^{\pi} ds \left( \frac{u}{X} \right)^{\frac{1}{2} - b}g_{\lambda_{1}}(a, b + \frac{1}{2}; X)
\]
\[
= u^{\frac{1}{2} - b} \frac{1}{\pi} \int_{0}^{\pi} ds X^{\ell}F(\lambda_{1} + a, \ell + 1; 2\lambda_{1}; X).
\]
Since \(F(\lambda_{1} + a, \ell + 1; 2\lambda_{1}; X) = (1 - X)^{-\ell - 1}F(\lambda_{1} - a, \ell + 1; 2\lambda_{1}; X/(X - 1))\) then (6.12) shows that (6.20) satisfies (2.23a). Using the integral
\[
c_{n} = \frac{1}{\pi} \int_{0}^{\pi} \cos^{2n}s \, ds = \left( \frac{\frac{1}{2}}{n!} \right), \quad (6.21)
\]
and the power series expansion of the hypergeometric function in (6.20) when \(X \rightarrow x \cos^{2} s\) gives directly
\[
F^{(\frac{1}{2})}_{\lambda_{1}, \lambda_{2}}(x, \bar{x}) \big|_{\lambda_{2} = \frac{1}{2} - b} \sim c_{\ell}(x\bar{x})^{\lambda_{2}}x^{\ell}F(\lambda_{1} + a, \ell + \frac{1}{2}; 2\lambda_{1}; x),
\]
in accord with the limiting form (2.32) for the choice of \(c_{\ell}\) given by (6.21), which is identical to (2.30) for \(\varepsilon = \frac{1}{2}\).

In an attempt to understand the form of expressions for more general solutions we consider using (4.51)
\[
u^{b+\frac{3}{2}} - a - b + 1 D^{(\frac{1}{2})}u^{b-\frac{1}{2}}v^{a+b} F^{(\frac{1}{2})}_{\lambda_{1}, \lambda_{2}}(a, b) = -2b(\lambda_{1} - b)F^{(\frac{1}{2})}_{\lambda_{1}, \lambda_{2}}(a, b - 1) \quad \text{for} \quad \lambda_{2} = \frac{1}{2} - b, \quad (6.23)
\]
starting from (6.17). To apply this it is sufficient to note that
\[
\begin{align*}
&u^{1-p}v^{1-q} \mathcal{D}^{(\frac{1}{2})}(u^p v^q F(X)) \\
&\sim - \left( qu \left( (1 - X) \frac{\partial}{\partial X} - p - q - \frac{1}{2} \right) - pv \left( X \frac{\partial}{\partial X} + p + q + \frac{1}{2} \right) - pq \right) F(X) \\
&\sim - \left( q \left( (1 - X) \frac{\partial}{\partial X} - X + p + \frac{1}{2} \right) - (p + q)v \left( X \frac{\partial}{\partial X} + p + \frac{1}{2} \right) - (p + q)qu \right) F(X),
\end{align*}
\]
using (6.7) and
\[
\begin{align*}
&\left( u (1 - X) \frac{\partial}{\partial X} + v X \frac{\partial}{\partial X} - X (1 - X) \frac{\partial}{\partial X} + X \right) F(X) \\
&= \frac{1}{2} (1 + u - v) F(X) + \frac{\partial}{\partial s} \left( \frac{1}{4} \sin 2s (x - \bar{x}) F(X) \right).
\end{align*}
\]
Applying (6.23) to (6.17) then gives
\[
2b(\lambda_1 - b) F^{(\frac{1}{2})}(a, b - 1)|_{\lambda_2 = \frac{1}{2} - b} = \frac{1}{\pi} \int_0^\pi ds \left( \frac{u}{X} \right)^{\frac{1}{2} - b} \tilde{f}(X),
\]
for
\[
\tilde{f}(X) = \left( (a + b) \left( X (1 - X) \frac{\partial}{\partial X} - (b + \frac{1}{2})X - b \right) - (a - b)v \left( X \frac{\partial}{\partial X} - b \right) - (a^2 - b^2)u \right) f(X).
\]
With \( f(X) \) given by (6.19) and using (4.54) we may then obtain directly from (6.27)
\[
\tilde{f}(X) = (a + b)(\lambda_1 - a) g_{\lambda_1}(a - 1, b + \frac{1}{2}; X) \\
- (a - b)(\lambda_1 + a) v g_{\lambda_1}(a + 1, b + \frac{1}{2}; X)
+ (a^2 - b^2)(1 + v - u) g_{\lambda_1}(a, b + \frac{1}{2}; X).
\]
For \( a = \pm b \) this simplifies to just one term. More generally by repeated application of \( \mathcal{D}_- \) we may express \( F^{(\frac{1}{2})}(a, b - n) \) solely in terms of \( g_{\lambda_1}(b - n, b + \frac{1}{2}; X) \) but this is just equivalent to (6.20) for \( a = b - n \).

7. Discussion

Explicit expressions for conformal partial waves have proved quite useful in recent years in attempts to revive the bootstrap approach to conformal field theories. It is clear that identifying the sum over conformal blocks in the \( s \) and \( t \) channels of particular four
point functions can lead to non trivial constraints \[8\], which may be of phenomenological interest. On a theoretical level it would be very nice to apply such ideas in three dimensions, where there is a plethora of conformal field theories. This paper was begun in an attempt to find an tractable expression for three dimensional conformal waves which could be used for similar bootstrap calculations.

In four dimensions there is a very close connection between the solutions of superconformal Ward identities and the explicit expressions for conformal partial waves \[23\] which may be expected to generalise to other dimensions \[24\]. In particular the conformal partial waves for leading twist have a special form and are related to the contributions of semi-short operators in the operator product expansion. In this paper we have shown how leading twist conformal partial waves are simplified by being directly related to solutions of \( D^{(\varepsilon)} F = 0 \) which is just a restriction of massless wave equation. The operator \( D^{(\varepsilon)} \) also appeared in the discussion in \[24\]. This observation demonstrates that the form for conformal partial waves must be very different in even and odd dimensions. Although we have found and expression for leading twist in three dimensions there is no straightforward extension to the arbitrary twist. In even dimensions it is clear, at least in retrospect, that the construction is very simple starting in two dimensions where there is, using \( x, \bar{x} \), separation of variables and then using the dimension raising operator \( \mathcal{E}_+ \). Although the various recurrence relations obtained in section 3 relate different conformal partial waves they do not allow expressions for the general case to be obtained from leading twist. A natural guess is to suppose that the three dimensional conformal partial waves can be given by a similar integral expression as in the leading twist case but involving products of functions \( g_p, g_q \), with different arguments, but a precise form has eluded us so far.

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Appendix A. Two dimensional conformal integrals

Although the Symanzik formula \[20\] provides a general form for a certain class of conformal integrals in general dimensions in two dimensions using complex variables there are further extensions. We consider here integrals of the form

\[ I_n = \frac{1}{\pi} \int d^2x \, f_n(z) \bar{f}_n(\bar{z}), \quad f_n(z) = \prod_{i=1}^{n} \frac{1}{(z - z_i)^{h_i}}, \quad \bar{f}_n(\bar{z}) = \prod_{i=1}^{n} \frac{1}{(\bar{z} - \bar{z}_i)^{\bar{h}_i}}, \quad (A.1) \]

where

\[ \sum_{i=1}^{n} h_i = \sum_{i=1}^{n} \bar{h}_i = 2, \quad h_i - \bar{h}_i \in \mathbb{Z}. \quad (A.2) \]

The first condition is necessary for \( I_n \) to transform covariantly under conformal transformations, \( z \to (az + b)/(cz + d) \), \( \bar{z} \to (\bar{a}\bar{z} + \bar{b})/(\bar{c}\bar{z} + \bar{d}) \), and the second for \( I_n \) to be single valued. The integral in (A.1) reduces to the form discussed by Symanzik when \( h_i = \bar{h}_i \). Convergence of the integral requires \( h_i + \bar{h}_i < 2 \) for all \( i \) although \( I_n \) may be extended by analytic continuation.

When \( n = 2 \)

\[ I_2 = K_{12} (-1)^{h_1 - \bar{h}_1} \pi \delta^2(x_1 - x_2), \quad (A.3) \]

for

\[ K_{12} = \frac{\Gamma(1 - h_1) \Gamma(1 - h_2)}{\Gamma(h_1) \Gamma(h_2)} = \frac{\Gamma(1 - \bar{h}_1) \Gamma(1 - \bar{h}_2)}{\Gamma(h_1) \Gamma(h_2)}, \quad (A.4) \]

with equality of the two forms following from (A.2) noting that \( \Gamma(h)\Gamma(1 - h) = \pi / \sin \pi h \).

Here we show how evaluation of \( I_n \) for \( n > 2 \) can be reduced to solving differential equations. To this end we note that

\[ \left( \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z} \sum_{i=2}^{n} \frac{h_i}{z - z_i} \right) f_n(z) = \left( \frac{\partial^2}{\partial z_1^2} + \sum_{i=2}^{n} \frac{1}{z_{1i}} \left( h_i \frac{\partial}{\partial z_1} - h_1 \frac{\partial}{\partial z_i} \right) \right) f_n(z), \quad (A.5) \]

so that \( I_n \) satisfies the second order equation

\[ \left( \frac{\partial^2}{\partial z_1^2} + \sum_{i=2}^{n} \frac{1}{z_{1i}} \left( h_i \frac{\partial}{\partial z_1} - h_1 \frac{\partial}{\partial z_i} \right) \right) I_n = 0. \quad (A.6) \]

There is of course a corresponding equation for \( z_i \to \bar{z}_i, \ h_i \to \bar{h}_i \).

For \( n = 3 \) the form of \( I_n \) for non coincident \( x_i \) is dictated just by conformal invariance

\[ I_3 = K_{123} z_{12}^{h_3 - 1} z_{23}^{h_1 - 1} z_{31}^{h_2 - 1} \bar{z}_{12}^{\bar{h}_3 - 1} \bar{z}_{23}^{\bar{h}_1 - 1} \bar{z}_{31}^{\bar{h}_2 - 1}, \quad (A.7) \]
where the overall constant can be obtained by extending the result for \( h_i = \bar{h}_i \)

\[
K_{123} = \frac{\Gamma(1 - h_1) \Gamma(1 - h_2) \Gamma(1 - h_3)}{\Gamma(h_1) \Gamma(h_2) \Gamma(h_3)} = \frac{\Gamma(1 - \bar{h}_1) \Gamma(1 - \bar{h}_2) \Gamma(1 - \bar{h}_3)}{\Gamma(h_1) \Gamma(h_2) \Gamma(h_3)}.
\]  

(A.8)

It is easy to verify that (A.8) satisfies (A.6) subject to (A.2), which is also necessary to ensure the required symmetries under permutations of \( \{z_i\} \) or \( \{\bar{z}_i\} \).

For \( n = 4 \) conformal invariance shows that the result must have the form

\[
I_4 = z_{12}^{h_3 + h_4 - 1} z_{23}^{h_1 + h_4 - 1} z_{31}^{h_2 - 1} z_{24}^{h_3 + \bar{h}_4 - 1} \bar{z}_{23}^{\bar{h}_1 + \bar{h}_4 - 1} \bar{z}_{31}^{\bar{h}_2 - 1} \bar{z}_{24}^{\bar{h}_3 - \bar{h}_4} \mathcal{I}(x, \bar{x}),
\]

(A.9)

for \( \mathcal{I}(x, \bar{x}) \) an undetermined functions of the invariants \( x, \bar{x} \) defined here by (3.52). However imposing (A.6) leads to the differential equation

\[
\left( x(1 - x) \frac{\partial^2}{\partial x^2} + (h_3 + h_4 + (h_2 - h_4 - 2)x) \frac{\partial}{\partial x} + h_4(h_2 - 1) \right) \mathcal{I}(x, \bar{x}) = 0,
\]

which has a simple hypergeometric form. The relevant solutions of (A.10) and its conjugate can be expressed as

\[
\mathcal{I}(x, \bar{x}) = K_4 F(1 - h_2, h_4; h_3 + h_4; x) F(1 - \bar{h}_2, \bar{h}_4; \bar{h}_3 + \bar{h}_4; \bar{x})
\]

\[
+ \bar{K}_4 (-1)^{h_1 + h_4 - h_2 - h_4} x^{h_1 + h_2 - 1} F(1 - h_3, h_1; h_1 + h_2; x)
\]

\[
\times \bar{x}^{\bar{h}_1 + \bar{h}_2 - 1} F(1 - \bar{h}_3, \bar{h}_1; \bar{h}_1 + \bar{h}_2; \bar{x}).
\]  

(A.11)

Although solving the differential equation allows a more general form than (A.11) the result given is required by the symmetry relations

\[
\mathcal{I}(x, \bar{x}) = (1 - x)^{-h_4} (1 - \bar{x})^{-\bar{h}_4} \mathcal{I}(x', \bar{x}') \bigg|_{h_1 \leftrightarrow h_2, \bar{h}_1 \leftrightarrow \bar{h}_2}, \quad x' = \frac{x}{x - 1}, \quad \bar{x}' = \frac{\bar{x}}{\bar{x} - 1},
\]

\[
\mathcal{I}(x, \bar{x}) = (1 - x)^{h_2 - 1} (1 - \bar{x})^{\bar{h}_2 - 1} \mathcal{I}(x', \bar{x}') \bigg|_{h_3 \leftrightarrow h_4, \bar{h}_3 \leftrightarrow \bar{h}_4},
\]

(A.12)

and

\[
\mathcal{I}(x, \bar{x}) = (-1)^{h_4 - \bar{h}_4} \mathcal{I}(1 - x, 1 - \bar{x}) \bigg|_{h_1 \leftrightarrow h_3, \bar{h}_1 \leftrightarrow \bar{h}_3}.
\]

(A.13)

All other symmetry relations can be obtained by combining (A.12) and (A.13). The relations in (A.12) follow easily from standard hypergeometric identities which give \( F(a, b; c; x) = (1 - x)^{-a} F(c - a, b; c; x') = (1 - x)^{-a} F(a, c - b; c; x') \) if both \( K_4, \bar{K}_4 \) are invariant under \( 1 \leftrightarrow 2 \) and also \( 3 \leftrightarrow 4 \). To satisfy (A.13) we need to use the further
hypergeometric identities

\[ F(1 - h_2, h_4; h_1 + h_4; 1 - x) \]
\[ = \frac{\Gamma(h_1 + h_4) \Gamma(h_1 + h_2 - 1)}{\Gamma(1 - h_3) \Gamma(h_1)} F(1 - h_2, h_4; h_3 + h_4; x) \]
\[ + \frac{\Gamma(h_1 + h_4) \Gamma(h_3 + h_4 - 1)}{\Gamma(1 - h_2) \Gamma(h_4)} x^{h_1 + h_2 - 1} F(1 - h_3, h_1; h_1 + h_2; x), \]
\[ (1 - x)^{h_2 + h_3 - 1} F(1 - h_1, h_3; h_2 + h_3; 1 - x) \]
\[ = \frac{\Gamma(h_2 + h_3) \Gamma(h_1 + h_2 - 1)}{\Gamma(1 - h_4) \Gamma(h_2)} F(1 - h_2, h_4; h_3 + h_4; x) \]
\[ + \frac{\Gamma(h_2 + h_3) \Gamma(h_3 + h_4 - 1)}{\Gamma(1 - h_1) \Gamma(h_3)} x^{h_1 + h_2 - 1} F(1 - h_3, h_1; h_1 + h_2; x). \]

(A.14)

With the aid of (A.14) and the associated conjugates we may determine

\[ K_4 = \frac{\Gamma(1 - h_1) \Gamma(1 - h_2) \Gamma(h_1 + h_2 - 1)}{\Gamma(h_1) \Gamma(h_2) \Gamma(2 - h_1 - h_2)}, \]
\[ \bar{K}_4 = \frac{\Gamma(1 - h_3) \Gamma(1 - h_4) \Gamma(h_3 + h_4 - 1)}{\Gamma(h_3) \Gamma(h_4) \Gamma(2 - h_3 - h_4)}. \]

(A.15)

As for \( K_{12}, K_{13} \) these are symmetric under \( h_i \leftrightarrow \bar{h}_i \). The overall scale is fixed since we require \( K_4 |_{h_4 = \bar{h}_4 = 0} = K_{123}, \bar{K}_4 |_{h_4 = \bar{h}_4 = 0} = 0 \). To verify that (A.13) is satisfied it is necessary to repeatedly use \( \Gamma(h) \Gamma(1 - h) = \pi / \sin \pi h \) and identities such as

\[ \sin \pi h_1 \sin \pi h_3 \]
\[ \sin \pi (h_3 + h_4) \sin \pi (h_1 + h_4) \]
\[ \quad + (-1)^{h_3 + h_4 - h_1 - h_4} \sin \pi h_2 \sin \pi \bar{h}_4 \]
\[ \quad \sin \pi (h_3 + h_4) \sin \pi (h_2 + h_3) = (-1)^{h_4 - \bar{h}_4}, \]

(A.16)

which depend on (A.2).

An alternative more direct evaluation may be obtained by introducing the conformal covariant differential operator

\[ \mathcal{D}^{(h, h')}_{z, z'} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(h')_n}{(h + h')_n} (z - z')^n \frac{\partial^n}{\partial z^n}, \]

(A.17)

which satisfies

\[ \mathcal{D}^{(h, h')}_{z, z'} \frac{1}{(w - z)^{h + h'}} = \frac{1}{(w - z)^h (w - z')^{h'}}. \]

(A.18)

Using this operator we may write

\[ \frac{1}{(z - z_1)^{h_1} (z - z_2)^{h_2}} \frac{1}{(\bar{z} - \bar{z}_1)^{\bar{h}_1} (\bar{z} - \bar{z}_2)^{\bar{h}_2}} \]
\[ = \mathcal{D}^{(h_1, h_2)}_{z_1, z_2} \mathcal{D}^{(\bar{h}_1, \bar{h}_2)}_{\bar{z}_1, \bar{z}_2} \frac{1}{(z - z_1)^{h_1 + h_2} (\bar{z} - \bar{z}_1)^{\bar{h}_1 + \bar{h}_2}} \]
\[ + K_4 (-1)^{h_2 - \bar{h}_2} z_{12}^{1 - h_1 - h_2} \bar{z}_{12}^{1 - \bar{h}_1 - \bar{h}_2} \mathcal{D}^{(1 - h_2, 1 - h_1)}_{\bar{z}_1, \bar{z}_2} \mathcal{D}^{(1 - \bar{h}_2, 1 - \bar{h}_1)}_{\bar{z}_1, \bar{z}_2} \pi \delta^2(x - x_1), \]

(A.19)
with $K_4$ as in (A.15). The coefficient is dictated by the requirement that using (A.19) in
the integral expression for $I_3$ gives the result (A.7).

Using (A.19) in $I_4$ gives

$$I_4 = K_4(-1)^{h_2-h_3} z_{12}^{1-h_1-h_2} \tilde{z}_{12}^{1-h_1-h_2} \times
\mathcal{D}(1-h_2,1-h_1)_{z_{13}}^{h_1-h_3} \mathcal{D}(1-h_2,1-h_1)_{\tilde{z}_{13}}^{1-h_3-h_4}
+ K_4(-1)^{h_3-h_4} z_{34}^{1-h_3-h_4} \tilde{z}_{34}^{1-h_3-h_4}
\times \mathcal{D}(h_1,h_2)_{z_{13}}^{h_4-1} \mathcal{D}(\tilde{h}_1,\tilde{h}_2)_{\tilde{z}_{13}}^{\bar{h}_4-1} \tilde{z}_{14}^{1-h_4-1},$$

(A.20)

where agreement with (A.9) and (A.11) follows from

$$\mathcal{D}(h_1,h_2)_{z_{13}}^{h_3-1} z_{14}^{h_3-1} = z_{13}^{h_1} z_{24}^{h_3-1} z_{23}^{h_1+h_4-1} F(1-h_3,h_1;h_1+h_2;x),$$
$$\mathcal{D}(1-h_2,1-h_1)_{z_{13}}^{h_3} z_{14}^{h_4} = z_{13}^{h_2-1} z_{24}^{h_4} z_{23}^{h_1+h_4-1} F(1-h_2,h_4;h_3+h_4;x),$$

(A.21)

together with their conjugates.

The verification of (A.21) follows from the integral representations

$$\mathcal{D}(h_1,h_2)_{z_{13}}^{h_3-1} z_{14}^{h_3-1} = \frac{\Gamma(h_1+h_2)}{\Gamma(1-h_3)\Gamma(1-h_4)} \int_0^1 d\alpha \frac{\alpha^{-h_4}(1-\alpha)^{-h_3}}{(\alpha z_{32} + (1-\alpha)z_{42})^{h_2}(\alpha z_{31} + (1-\alpha)z_{41})^{h_1}} \times
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \Gamma(-s)\Gamma(h_1+s)\Gamma(1-h_3+s)\Gamma(1-h_1-h_4-s) \left(\frac{\tilde{z}_{14} \tilde{z}_{23}}{z_{13} z_{24}}\right)^s.$$  

(A.22)

Appendix B. Compact case

In this appendix we demonstrate how the recurrence relations for conformal partial
waves are closely related to those for symmetric two variable Jacobi polynomials associated
with the root system $BC_2$ described in [16] which contains related results.

For the elementary single variable polynomials we first consider the differential operator

$$\hat{D}_x = x(1-x) \frac{d^2}{dx^2} + (1+\beta-(2+\gamma)x) \frac{d}{dx},$$

(B.1)

and then Jacobi polynomials of degree $n$ may be defined by the eigenvalue equation

$$-\hat{D}_x P_n^{(\alpha,\beta)}(y) = n(n+\gamma+1) P_n^{(\alpha,\beta)}(y), \quad y = 2x - 1,$$

(B.2)
where we require
\[ \gamma = \alpha + \beta . \] (B.3)

It is convenient here to adopt a non standard normalisation for Jacobi polynomials \( R_n^{(\alpha,\beta)}(y) \) such that \( R_n^{(\alpha,\beta)}(1) = 1 \). Recursion relations may be obtained by following the same procedure as in section 3. Defining \( \tilde{f}_0 = x - \frac{1}{2}, \tilde{f}_1 = x(1-x)\frac{d}{dx} \) then the commutators
\[ [\tilde{D}_x, \tilde{f}_0] = 2\tilde{f}_1 - (2 + \gamma)\tilde{f}_0 + \frac{1}{2}(\beta - \alpha), \quad [\tilde{D}_x, \tilde{f}_1] = -2\tilde{f}_0\tilde{D}_x + \gamma\tilde{f}_1 , \] (B.4)
require
\[
\tilde{f}_0 R_n^{(\alpha,\beta)} = a R_n^{(\alpha,\beta)} + b R_{n+1}^{(\alpha,\beta)} + \frac{1}{2} \gamma (\beta - \alpha) R_n^{(\alpha,\beta)} , \\
\tilde{f}_1 R_n^{(\alpha,\beta)} = (n + \gamma + 1) a R_{n-1}^{(\alpha,\beta)} - nb R_n^{(\alpha,\beta)} - n(n + \gamma + 1)(\beta - \alpha) R_n^{(\alpha,\beta)} .
\] (B.5)

Imposing the normalisation conditions, so that \( \tilde{f}_0 R_n^{(\alpha,\beta)} |_{x=1} = \frac{1}{2}, \tilde{f}_1 R_n^{(\alpha,\beta)} |_{x=1} = 0 \), determines
\[
a = \frac{n(n + \beta)}{(2n + \gamma)(2n + \gamma + 1)}, \quad b = \frac{(n + \gamma + 1)(n + \alpha + 1)}{(2n + \gamma)(2n + \gamma + 1)} .
\] (B.6)

The corresponding two variable operator is then constructed using \( \tilde{D} \) defined in (B.1)
\[ \tilde{\Delta}^{(\varepsilon)} = -\tilde{D}_x - \tilde{D}_{\bar{x}} - 2\varepsilon \frac{1}{x - \bar{x}} \left( x(1-x) \frac{\partial}{\partial x} - \bar{x}(1-\bar{x}) \frac{\partial}{\partial \bar{x}} \right) , \] (B.7)
and the eigenvalue equations becomes
\[ \tilde{\Delta}^{(\varepsilon)} R_{nm}^{(\alpha,\beta,\varepsilon)} = \tilde{c}_{nm} R_{nm}^{(\alpha,\beta,\varepsilon)} , \] (B.8)

where \( R_{nm}^{(\alpha,\beta,\varepsilon)}(y, \bar{y}) \) are symmetric polynomials, \( y = 2x - 1, \bar{y} = 2\bar{x} - 1 \), with the convenient normalisation
\[ R_{nm}^{(\alpha,\beta,\varepsilon)}(1,1) = 1 . \] (B.9)

For polynomial solutions such that
\[ R_{nm}^{(\alpha,\beta,\varepsilon)}(y, \bar{y}) \xrightarrow{\bar{z} \to \infty} k_{nm} \bar{x}^n P_m^{(\alpha,\beta)}(y) , \quad n \geq m , \] (B.10)
then (B.8) determines
\[ \tilde{c}_{nm} = n(n + 1 + \gamma + 2\varepsilon) + m(m + 1 + \gamma) , \quad m, n = 0, 1, \ldots , m \leq n . \] (B.11)
In a similar fashion to (2.40)
\[ \tilde{\Delta}^{(\varepsilon)} (x - \bar{x})^{1-2\varepsilon} = (x - \bar{x})^{1-2\varepsilon} (\tilde{\Delta}^{(1-\varepsilon)} + (2 + \gamma)(1 - 2\varepsilon)) . \] 

(B.12)

Also for
\[ \tilde{\Delta} = -\frac{1}{x - \bar{x}} (D_x - \bar{D}_\bar{x}) , \]

(B.13)

then analogous to (4.36)
\[ \tilde{\Delta}^{(\varepsilon+1)} = (\Delta^{(\varepsilon+1)} + 2 + \gamma + 2\varepsilon) \tilde{\Delta}^{(\varepsilon)} . \]

(B.14)

Using (B.12) if
\[ \tilde{\Delta}^{(\varepsilon+1)} = (x - \bar{x})^{1-2\varepsilon} \tilde{\Delta}^{(\varepsilon+1)} = (x - \bar{x})^{-2\varepsilon} (D_x - \bar{D}_\bar{x})(x - \bar{x})^{1+2\varepsilon} , \]

(B.15)

then
\[ \tilde{\Delta}^{(\varepsilon+1)} \Delta^{(\varepsilon+1)} = (\Delta^{(\varepsilon)} + 2 + \gamma + 2\varepsilon) \tilde{\Delta}^{(\varepsilon+1)} . \]

(B.16)

As a consequence of (B.14) and (B.16)
\[ \tilde{\Delta}^{(\varepsilon)} R^{(\alpha,\beta,\varepsilon)}_{n,m} = \alpha^{(\varepsilon)} R^{(\alpha,\beta,\varepsilon+1)}_{n-1,m} , \quad \tilde{\Delta}^{(\varepsilon+1)} R^{(\alpha,\beta,\varepsilon+1)}_{n,m} = \beta^{(\varepsilon)} R^{(\alpha,\beta,\varepsilon)}_{n+1,m} . \]

(B.17)

Requiring (B.9) the coefficient \( \beta^{(\varepsilon)} \) may be directly determined by
\[ \beta^{(\varepsilon)} = (x - \bar{x})^{-2\varepsilon} (D_x - \bar{D}_\bar{x})(x - \bar{x})^{1+2\varepsilon} \bigg|_{x = \bar{x} = 1} = 2(1 + 2\varepsilon)(\alpha + \varepsilon + 1) . \]

(B.18)

To calculate \( \alpha^{(\varepsilon)} \) it is sufficient to note that
\[ \tilde{\Delta}^{(\varepsilon)} \Delta^{(\varepsilon)} = \tilde{\Delta}^{(\varepsilon+1)} \tilde{\Delta}^{(\varepsilon)} = (x - \bar{x})^{-2\varepsilon} (D_x - \bar{D}_\bar{x})(x - \bar{x})^{1+2\varepsilon} \]

(B.19)

is a fourth order Casimir operator such that
\[ \tilde{\Delta}^{(\varepsilon)} R^{(\alpha,\beta,\varepsilon)}_{nm} = \tilde{\alpha}_{4,nm} R^{(\alpha,\beta,\varepsilon)}_{nm} , \]

(B.20)

and hence
\[ \alpha^{(\varepsilon)} \beta^{(\varepsilon)} = \tilde{\alpha}_{4,nm} \]

(B.21)

Applying (B.10) with (B.19) dictates
\[ \tilde{\alpha}_{4,nm} = (n - m)(n - m + 2\varepsilon)(n + m + 1 + \gamma)(n + m + 1 + \gamma + 2\varepsilon) . \]

(B.22)

Just as \( F_{\lambda_1,\lambda_2}(x, \bar{x}) \) determine the conformal partial waves \( G^{(\ell)}_{\Delta} \), there are harmonic polynomials associated with \( R^{(\alpha,\beta,\varepsilon)}_{nm} \) given by
\[ R^{(\alpha,\beta,\varepsilon)}_{nm}(y, \bar{y}) = Y^{(\alpha,\beta)}_{nm}(\sigma, \tau) , \quad \sigma = x\bar{x} , \quad \tau = (1 - x)(1 - \bar{x}) , \]

(B.23)
which play a role [23] in the decomposition of correlation functions. The differential operator acting on \( Y_{nm}(\sigma, \tau) \) becomes

\[
\tilde{D} = - (1 - \sigma - \tau) \left( \frac{\partial}{\partial \sigma} \sigma \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau} \tau \frac{\partial}{\partial \tau} + \beta \frac{\partial}{\partial \sigma} + \alpha \frac{\partial}{\partial \tau} \right) + 4 \sigma \tau \frac{\partial^2}{\partial \sigma \partial \tau} + (d + 2\alpha) \sigma \frac{\partial}{\partial \sigma} + (d + 2\beta) \tau \frac{\partial}{\partial \tau},
\]

(B.24)

so that

\[
\tilde{D} Y_{nm}^{(\alpha, \beta)}(\sigma, \tau) = \tilde{c}_{nm} Y_{nm}^{(\alpha, \beta)}(\sigma, \tau).
\]

(B.25)

The relevant boundary conditions are then

\[
Y_{nm}^{(\alpha, \beta)}(\sigma, \tau) \propto \sigma^n \left( 1 - \frac{\tau}{\sigma} \right)^{n-m}, \quad \sigma \to \infty, \quad \tau = O(\sigma).
\]

(B.26)

It is easy to see that

\[
\begin{align*}
R_{nm}^{(\alpha, \beta, 0)}(y, \bar{y}) &= \frac{1}{2} \left( R_n^{(\alpha, \beta)}(y) R_m^{(\alpha, \beta)}(\bar{y}) + R_n^{(\alpha, \beta)}(\bar{y}) R_m^{(\alpha, \beta)}(y) \right), \\
R_{nm}^{(\alpha, \beta, 0)}(y, \bar{y}) &= 2(\alpha + 1) \frac{R_{n+1}^{(\alpha, \beta)}(y) R_m^{(\alpha, \beta)}(\bar{y}) - R_{n+1}^{(\alpha, \beta)}(\bar{y}) R_m^{(\alpha, \beta)}(y)}{y - \bar{y}}.
\end{align*}
\]

(B.27)

Starting from (B.27) and using (B.17) with \( a^{(0)} \) given by (B.21) gives

\[
(n - m + 1)(n + m + 2 + \gamma) R_{nm}^{(\alpha, \beta, 1)}(y, \bar{y}) = 2(\alpha + 1) \frac{R_{n+1}^{(\alpha, \beta)}(y) R_m^{(\alpha, \beta)}(\bar{y}) - R_{n+1}^{(\alpha, \beta)}(\bar{y}) R_m^{(\alpha, \beta)}(y)}{y - \bar{y}}.
\]

(B.28)

The results (B.27) and (B.28) correspond to (5.3) and (5.4) for conformal partial waves.

To establish further recurrence relations for \( R_{nm}^{(\alpha, \beta, \varepsilon)} \) following the same method as in the non compact case we define

\[
\begin{align*}
\tilde{F}_0 &= x + \bar{x} - 1, \\
\tilde{F}_1 &= x(1 - x) \frac{\partial}{\partial x} + \bar{x}(1 - \bar{x}) \frac{\partial}{\partial \bar{x}}, \\
\tilde{F}_2 &= (x - \bar{x}) (\tilde{D}_x - \tilde{D}_{\bar{x}}), \\
\tilde{F}_3 &= \left( x(1 - x) \frac{\partial}{\partial x} - \bar{x}(1 - \bar{x}) \frac{\partial}{\partial \bar{x}} + 2\varepsilon \frac{x(1 - x) + \bar{x}(1 - \bar{x})}{x - \bar{x}} \right) (\tilde{D}_x - \tilde{D}_{\bar{x}}),
\end{align*}
\]

(B.29)

and then, similarly to (4.27),

\[
\begin{align*}
[\tilde{\Delta}^{(\varepsilon)}, \tilde{F}_0] &= - 2 \tilde{F}_1 + (2 + \gamma + 2\varepsilon) \tilde{F}_0 + \alpha - \beta, \\
[\tilde{\Delta}^{(\varepsilon)}, \tilde{F}_1] &= \tilde{F}_2 - \gamma \tilde{F}_1 - \tilde{F}_0 \tilde{\Delta}^{(\varepsilon)}, \\
[\tilde{\Delta}^{(\varepsilon)}, \tilde{F}_2] &= - 2 \tilde{F}_3 + (2 + \gamma - 2\varepsilon) \tilde{F}_2, \\
[\tilde{\Delta}^{(\varepsilon)}, \tilde{F}_3] &= \tilde{F}_0 \tilde{\Delta}^{(\varepsilon)}_4 + \gamma \tilde{F}_3 - \tilde{F}_2 (\tilde{\Delta}^{(\varepsilon)} + 2\varepsilon(1 + \gamma)).
\end{align*}
\]

(B.30)
Just as in the non compact case in section 4 the algebra (B.30) dictates that

\[ \tilde{F}_i R^{(\alpha,\beta,\varepsilon)}_{nm} = a_i R^{(\alpha,\beta,\varepsilon)}_{n-1,m} + b_i R^{(\alpha,\beta,\varepsilon)}_{n+1,m} + c_i R^{(\alpha,\beta,\varepsilon)}_{m-1,m} + d_i R^{(\alpha,\beta,\varepsilon)}_{n,m+1} + u_i R^{(\alpha,\beta,\varepsilon)}_{n,m} , \]  

(B.31)

where

\[
\begin{align*}
a_1 &= (n + 1 + \gamma + 2\varepsilon)a_0 , & b_1 &= -n b_0 , & c_1 &= (m + 1 + \gamma + \varepsilon)c_0 , & d_0 &= -(m - \varepsilon)d_0 , \\
a_2 &= -(n - m + 2\varepsilon)(n + m + 1 + \gamma + 2\varepsilon)a_0 , & b_2 &= -(n - m)(n + m + 1 + \gamma)b_0 , \\
c_2 &= (n - m)(n + m + 1 + \gamma + 2\varepsilon)c_0 , & d_2 &= (n - m + 2\varepsilon)(n + m + 1 + \gamma)d_0 , \\
a_3 &= -(n + 1 + \gamma)(n - m + 2\varepsilon)(n + m + 1 + \gamma + 2\varepsilon)a_0 , \\
b_3 &= (n + 2\varepsilon)(n - m)(n + m + 1 + \gamma)b_0 , \\
c_3 &= (m + 1 + \gamma - \varepsilon)(n - m)(n + m + 1 + \gamma + 2\varepsilon)c_0 , \\
d_3 &= -(m + \varepsilon)(n - m + 2\varepsilon)(n + m + 1 + \gamma)d_0 ,
\end{align*}
\]  

as well as

\[
\begin{align*}
u_0 &= - \frac{(\alpha - \beta)\gamma (2\tilde{c}_{n,m} + (\gamma + 2)(\gamma + 2\varepsilon))}{(2n + 2 + \gamma + 2\varepsilon)(2n + \gamma + 2\varepsilon)(2m + 2 + \gamma)(2m + \gamma)} , \\
u_2 &= - \frac{2(\alpha - \beta)\gamma \tilde{c}_{4,nn}}{(2n + 2 + \gamma + 2\varepsilon)(2n + \gamma + 2\varepsilon)(2m + 2 + \gamma)(2m + \gamma)} , \\
u_1 &= \frac{1}{2}(2 + \gamma + 2\varepsilon)u_0 + \frac{1}{2}(\alpha - \beta) , & u_3 &= \frac{1}{2}(2 + \gamma - 2\varepsilon)u_2 .
\end{align*}
\]  

(B.33)

The coefficients \( a_0, b_0, c_0, d_0 \) are determined by applying the normalisation conditions at \( x = \bar{x} = 1 \) in (B.31) which imply \( a_i + b_i + c_i + d_i + u_i = 1, \ i = 0, 0, i = 1, 2, 3 \). Hence

\[
\begin{align*}
a_0 &= \frac{(n + \varepsilon)(n + \beta + \varepsilon)(n - m)(n + m + 1 + \gamma)}{(2n + \gamma + 2\varepsilon)(2n + 1 + \gamma + 2\varepsilon)(n - m + \varepsilon)(n + m + 1 + \gamma + \varepsilon)} , \\
b_0 &= \frac{(n + 1 + \gamma + \varepsilon)(n + 1 + \alpha + \varepsilon)(n - m + 2\varepsilon)(n + m + 1 + \gamma + 2\varepsilon)}{(2n + 1 + \gamma + 2\varepsilon)(2n + 2 + \gamma + 2\varepsilon)(n - m + \varepsilon)(n + m + 1 + \gamma + \varepsilon)} , \\
c_0 &= \frac{m(m + \beta)(n - m + 2\varepsilon)(n + m + 1 + \gamma)}{(2m + \gamma)(2m + 1 + \gamma)(n - m + \varepsilon)(n + m + 1 + \gamma + \varepsilon)} , \\
d_0 &= \frac{(m + 1 + \gamma)(m + 1 + \alpha)(n - m)(n + m + 1 + \gamma + 2\varepsilon)}{(2m + 1 + \gamma)(2m + 2 + \gamma)(n - m + \varepsilon)(n + m + 1 + \gamma + \varepsilon)} .
\end{align*}
\]  

(B.34)

The results for \( a_i, b_i, c_i, d_i \) are related for each \( i \) as a consequence of the symmetry relations

\[ R^{(\alpha,\beta,\varepsilon)}_{nm} = R^{(\alpha,\beta,\varepsilon)}_{-n-1-\gamma-2\varepsilon-m-1-\gamma} = R^{(\alpha,\beta,\varepsilon)}_{m-\varepsilon n+\varepsilon} , \]  

(B.35)
From (B.13) and (B.29) \( \tilde{E}_+ = -(x - \bar{x})^{-2} \tilde{F}_2 \) so that (B.17) and (B.31) for \( i = 2 \) with

\[
\begin{align*}
    a_2 &= - \frac{(n + \varepsilon)(n + \beta + \varepsilon) \tilde{c}_{4, nm}}{(2n + \gamma + 2\varepsilon)(2n + 1 + \gamma + 2\varepsilon)(n - m + \varepsilon)(n + m + 1 + \gamma + \varepsilon)}, \\
    b_2 &= - \frac{(n + 1 + \gamma + \varepsilon)(n + 1 + \alpha + \varepsilon) \tilde{c}_{4, nm}}{(2n + 1 + \gamma + 2\varepsilon)(2n + 2 + \gamma + 2\varepsilon)(n - m + \varepsilon)(n + m + 1 + \gamma + \varepsilon)}, \\
    c_2 &= - \frac{m(m + \beta) \tilde{c}_{4, nm}}{(2m + \gamma)(2m + 1 + \gamma)(n - m + \varepsilon)(n + m + 1 + \gamma + \varepsilon)}, \\
    d_2 &= \frac{(m + 1 + \gamma)(m + 1 + \alpha) \tilde{c}_{4, nm}}{(2m + 1 + \gamma)(2m + 2 + \gamma)(n - m + \varepsilon)(n + m + 1 + \gamma + \varepsilon)},
\end{align*}
\]

(B.36)

determine \( R_{nm}^{(\alpha, \beta, \varepsilon + 1)} \) in terms of \( R_{n', m'}^{(\alpha, \beta, \varepsilon)} \) for \( n' = n, n + 2, m' = m \) and \( n' = n + 1, m' = m, m + 1, \) in agreement with the result in [17].

**Appendix C. Finiteness for \( d = 6 \)**

The linear combination of \( \mathcal{F}_{pq} \)'s appearing in (5.8) is determined in order to ensure the result is finite as \( \bar{x} \to x \). To verify this we consider the Wronskian

\[
W_{pq}(x) = g_p'(x) g_q(x) - g_q'(x) g_p(x). \tag{C.1}
\]

The defining equation (2.33) then gives

\[
\frac{d}{dx} \left( (1 - x)^{a+b-1} W_{pq}(x) \right) = (p - q)(p + q - 1) x^{-1} (1 - x)^{a+b-1} g_p(x) g_q(x). \tag{C.2}
\]

Hence

\[
\frac{d}{dx} \left( (1 - x)^{a+b-1} \left( (p - q) W_{p+1,q-1} - (p - q + 2) W_{pq} \right) \right) = (p - q)(p - q + 2)(p + q - 1) x^{-1} (1 - x)^{a+b-1} \left( g_{p+1} g_{q-1} - g_p g_q \right). \tag{C.3}
\]

Using the identity (4.7) in \( (g_{p+1}(x)/x) g_q(x) = g_{p+1}(x) (g_q(x)/x) \) gives

\[
g_{p+1} g_{q-1} - g_p g_q = (\alpha_{p+1} - \alpha_q) g_{p+1} g_q + \beta_{p+1} g_{p+2} g_q - \beta_q g_{p+1} g_{q+1}. \tag{C.4}
\]

As a consequence of (C.3) this leads to the relation

\[
(p - q) W_{p+1,q-1} - (p - q + 2) W_{pq} = \frac{p + q - 1}{p} \frac{(p - q)(p - q + 2)}{p - q + 1} (\alpha_{p+1} - \alpha_q) W_{p+1,q} \tag{C.5}
\]

\[
+ \frac{p + q - 1}{p + q + 1} ((p - q) \beta_{p+1} W_{p+2,q} - (p - q + 2) \beta_q W_{p+1,q+1}).
\]

Since \( \mathcal{F}_{pq}(x, \bar{x}) = (x - \bar{x}) W_{pq}(t) + O((x - \bar{x})^3), t = \frac{1}{2}(x + \bar{x}) \), the coefficients in (5.8) are exactly of the form necessary according to (C.5) to cancel the singular terms as \( x \to \bar{x} \).
Appendix D. Free Theory and Leading Twist

In four dimensions the conformal partial wave expansion of the four point function obtained from elementary scalar fields, with \( \Delta = 1 \), has only contributions from twist two operators. Here we extend this to other cases. With \( \mathcal{D}(\varepsilon) \) given by (4.43) it is easy to see that \( \mathcal{D}(\varepsilon) u^\alpha v^\alpha = 0 \) has solutions only for \( (\alpha, \beta) = (0, 0), (-\varepsilon, 0), (0, -\varepsilon) \). Assuming all operators in the four point function have the same scale dimension, \( \Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = \varepsilon \) so that \( a = b = 0 \), then we identify the four point function for elementary scalars as satisfying \( \mathcal{D}(\varepsilon) u^\varepsilon F(u, v) = 0 \) so that it has the general form

\[
F(u, v) = 1 + C u^\varepsilon + C' \left( \frac{u}{v} \right)^\varepsilon,
\]

where 1 corresponds to the identity in the operator product expansion and the two point function has been normalised to 1. Crossing symmetry requires also \( C = C' = 1 \) but we relax that here for generality.

As a consequence of (4.53), with \( b = 0, N = 1 \), the conformal partial wave expansion is restricted to be

\[
F(u, v) = \sum_{\lambda_1} a_{\lambda_1} F_{\lambda_1 \varepsilon}(x, \bar{x}).
\]

To determine the coefficients \( a_{\lambda_1} \) it is sufficient to consider only \( x = \bar{x} \) as the solutions of \( \mathcal{D}(\varepsilon) f(x, \bar{x}) = 0 \) are determined in terms of \( f(x, x) \). From (D.1) we have

\[
F(u, v)\big|_{x=\bar{x}} = 1 + C x^{2\varepsilon} + C' \left( \frac{x}{1-x} \right)^{2\varepsilon},
\]

For \( \varepsilon = \frac{1}{2} \) then from (6.4) and (6.6) with (6.17) and (6.19) the partial wave expansion (D.2) requires

\[
x^{-\frac{1}{2}} + C x^{\frac{3}{2}} + C' \frac{x^{\frac{3}{2}}}{1-x} = \sum_{\lambda_1} a_{\lambda_1} g_{\lambda_1}(0, \frac{1}{2}; x).
\]

This has a solution

\[
a_{-\frac{1}{2}} = 1, \quad a_{\frac{1}{2}} = C + C' \quad a_{p+\frac{1}{2}} = (C(-1)^p + C') 2^{-(2p-1)} \quad \text{for} \quad p = 1, 2, \ldots.
\]

Note that in this case \( F_{-\frac{1}{2}, \frac{1}{2}} = F_{0, 0} = 1 \).

When \( \varepsilon = 1 \) we may use (5.13), with \( b = 0, \) so that (D.2) with (D.3) become

\[
F(u, v)\big|_{x=\bar{x}} = x^{2} \sum_{\lambda_1} \frac{1}{\lambda_1} a_{\lambda_1} g_{\lambda_1}'(x) \quad \Rightarrow \quad -\frac{1}{x} + C x + C' \frac{1}{1-x} + a = \sum_{\lambda_1} \frac{1}{\lambda_1} a_{\lambda_1} g_{\lambda_1}(x),
\]

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for some arbitrary constant $a$. Matching the power expansions on both sides gives

$$a_{-1} = 1, \quad a_{p+1} = (C(-1)^p + C') \frac{p!(p+1)!}{(2p)!}, \quad p = 0, 1, \ldots,$$

(D.7)

agreeing with the result in [11] since for $\epsilon = 1$ $F_{-11} = F_{00} = 1$.

In the six dimensional case, $\epsilon = 2$, we may use from (5.14) and (5.17), with $a, b = 0$, $(\lambda_1 + 1)\lambda_1(\lambda_1 - 1)F_{\lambda_12}(x, x) = x^4(\lambda g_{\lambda_1}(0, -1; x))''$. Hence (D.2), with (D.3), reduces to

$$\frac{1}{x^4} + C + C' \frac{1}{(1-x)^4} = \sum_{\lambda_1} \frac{1}{(\lambda_1+1)\lambda_1(\lambda_1-1)} a_{\lambda_1} (x g_{\lambda_1}(0, -1; x))'',$$

(D.8)

or

$$-\frac{1}{x^2} + Cx^2 + C' \frac{1}{x(1-x)} + \frac{a}{x} + b + c x = \sum_{\lambda_1} \frac{6}{(\lambda_1+1)\lambda_1(\lambda_1-1)} a_{\lambda_1} g_{\lambda_1}(0, -1; x),$$

(D.9)

with $a, b, c$ arbitrary. The solution is now

$$a_{-2} = 1, \quad a_{p+2} = (C(-1)^p + C') \frac{(p+2)!(p+3)!}{12(2p+1)!}, \quad p = 0, 1, \ldots,$$

(D.10)

and now $F_{-22} = F_{00} = 1.$

### Appendix E. Expressions in terms of Jack Polynomials

Jack polynomials are a class of symmetric polynomials depending on a parameter [25]. For two variables, which are sufficient for our purpose, we consider $P^{(\epsilon)}_{\lambda_1\lambda_2}(x, \bar{x})$, $\lambda_1 - \lambda_2 \in \mathbb{N}$, forming a basis for homogeneous symmetric functions in $x, \bar{x}$ of degree $\lambda_1 + \lambda_2$. Moreover

$$P^{(\epsilon)}_{\lambda_1\lambda_2}(x, \bar{x}) = (x\bar{x})^{\lambda_2} P^{(\epsilon)}_{\lambda_1\lambda_2}(x, \bar{x}),$$

(E.1)

where $P^{(\epsilon)}_{\ell_0}(x, \bar{x})$ is a polynomial of degree $\ell$. Here the normalisation is chosen so that

$$P^{(\epsilon)}_{\lambda_1\lambda_2}(x, x) = x^{\lambda_1 + \lambda_2}, \quad P^{(\epsilon)}_{\lambda_1\lambda_2}(x, \bar{x}) \sim c^{(\epsilon)}_{\lambda_1\lambda_2} x^{\lambda_1} \bar{x}^{\lambda_2},$$

(E.2)

with $c^{(\epsilon)}_{\lambda_1\lambda_2}$ given by (2.30). For two variables Jack polynomials are expressible in terms of single variable Gegenbauer polynomials where with the conventions in (2.29)

$$P^{(\epsilon)}_{\lambda_1\lambda_2}(x, \bar{x}) = (x\bar{x})^{\frac{1}{2}(\lambda_1 + \lambda_2)} \hat{C}^{(\epsilon)}_{\lambda_1 - \lambda_2}(\sigma),$$

(E.3)

for $\sigma$ given by (2.26).
A formal solution for $F_{\lambda_1 \lambda_2}(x, \bar{x})$ was obtained previously [12], following [14], as a double series in terms of Jack polynomials $P^{(\varepsilon)}_{\lambda_1 \lambda_2}(x, \bar{x})$ where

$$F^{(\varepsilon)}_{\lambda_1 \lambda_2}(x, \bar{x}) = \sum_{m,n \geq 0} r_{mn} P^{(\varepsilon)}_{\lambda_1 + m \lambda_2 + n}(x, \bar{x}). \quad (E.4)$$

For $r_{00} = 1$ (E.4) satisfies (2.18) and indeed the first term in the expansion in (E.4) is identical with the leading short distance contribution given by (2.25) and (2.28).

The coefficients $r_{mn}$ in (E.4) are determined, starting from $r_{00}$, using

$$\Delta^{(\varepsilon)}(a, b) P^{(\varepsilon)}_{\lambda_1 \lambda_2} = (\lambda_1 (\lambda_1 - 1) + \lambda_2 (\lambda_2 - 1 - 2\varepsilon)) P^{(\varepsilon)}_{\lambda_1 \lambda_2}$$

$$- \frac{\lambda_1 - \lambda_2 + 2\varepsilon}{\lambda_1 - \lambda_2 + \varepsilon} (\lambda_1 + a)(\lambda_1 + b) P^{(\varepsilon)}_{\lambda_1+1 \lambda_2}$$

$$- \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2 + \varepsilon} (\lambda_2 + a - \varepsilon)(\lambda_2 + b - \varepsilon) P^{(\varepsilon)}_{\lambda_1 \lambda_2+1}, \quad (E.5)$$

which leads to a recursion relation for $r_{mn}$. Requiring $r_{mn} = 0$ for $m - n + \lambda_1 - \lambda_2 = -1$ there is a general solution for any $\varepsilon$ but it is very unwieldy, although the result was used in [12] to derive various recursion relations for $F^{(\varepsilon)}_{\lambda_1 \lambda_2}(x, \bar{x})$ for arbitrary $\varepsilon$.

If $\lambda_1 = \lambda_2 = \lambda$ the result simplifies to

$$r_{mn} = \frac{(\lambda + a)_m (\lambda + b)_m (\lambda + a - \varepsilon)_n (\lambda + b - \varepsilon)_n}{(2\lambda)_m (2\lambda - \varepsilon)_n} \frac{(2\varepsilon)_{m-n} (m-n+\varepsilon)}{n! (m-n)! (\varepsilon)_{m+1}}, \quad (E.6)$$

and if $\varepsilon = -\frac{1}{2}$ then

$$r_{mn} = \begin{cases} 
\frac{(2\lambda+2a)_{2n}(2\lambda+2b)_{2n}}{(2n)!((4\lambda)_{2n}),} & m = n, \\
\frac{(2\lambda+2a)_{2n+1}(2\lambda+2b)_{2n+1}}{(2n+1)!((4\lambda)_{2n+1}),} & m = n + 1, \\
0, & m > n + 1. 
\end{cases} \quad (E.7)$$

It is easy to see that the summation reproduces (6.3), using (E.2).

When $\lambda_2 = \varepsilon - b - N$, or similarly when $\lambda_2 = \varepsilon - a - N$, it is easy to see that in the solution (E.4) the summation over $n$ truncates to $n = 0, 1, \ldots, N$. Acting on Jack polynomials $P^{(\varepsilon)}_{\lambda_1 \lambda_2}(x, \bar{x})$ with the differential operator $\mathcal{D}^{(\varepsilon)}$, defined by (4.43), gives

$$\mathcal{D}^{(\varepsilon)} P^{(\varepsilon)}_{\lambda_1 \lambda_2} = (\lambda_1 + \varepsilon)\lambda_2 P^{(\varepsilon)}_{\lambda_1-1 \lambda_2-1} \Rightarrow \mathcal{D}^{(\varepsilon)} P^{(\varepsilon)}_{\ell_0} = 0. \quad (E.8)$$

Assuming (E.3) this is equivalent to the standard differential equation for $C^{(\varepsilon)}_{\ell}$. Applying (E.8) to the solution (E.4) for $(x\bar{x})^{-\lambda_2} F^{(\varepsilon)}_{\lambda_1 \lambda_2}(x, \bar{x})$, expressed in terms of $P^{(\varepsilon)}_{\ell+m,n}$, $n =
0, 1, \ldots, N, is an alternative justification for (4.53). For \( N = 0 \) there is then just a single series so that

\[
F^{(\varepsilon)}_{\lambda_1 \varepsilon-b}(x, \bar{x}) = \sum_{m=0}^{\infty} \frac{(\lambda_1 + a)_m (\lambda_1 + b + \varepsilon)_m}{m! (2\lambda_1)_m} P^{(\varepsilon)}_{\lambda_1 + m \varepsilon-b}(x, \bar{x}). \tag{E.9}
\]

For the case of primary interest here, \( \varepsilon = \frac{1}{2} \), Gegenbauer polynomials become ordinary Legendre polynomials \( P_\ell \) and Jack polynomials are expressible just by \( P^{(\frac{1}{2})}_\ell(x, \bar{x}) = (x\bar{x})^{\frac{\ell}{2}} P_\ell(\sigma) \) for \( \sigma \) is defined in (2.26). Using the representation provided by (6.13) it is easy to see, using (E.1), that (E.9) is identical with (6.4) and (6.19) since \( c^{(1)}_\ell = c_\ell \) in (6.21). In general for \( \varepsilon = \frac{1}{2} \) the Jack polynomial expansion (E.4) can be recast as

\[
F^{(\frac{1}{2})}_{\lambda_1 \lambda_2}(x, \bar{x}) = u^{\lambda_2} \frac{1}{\pi} \int_0^\pi d\sigma X^{\ell} \sum_{m,n \geq 0, m-n+\ell \geq 0} r_{mn} \left( \frac{u}{X} \right)^n X^m, \tag{E.10}
\]

where we may note that, as a consequence of (6.25) and (6.11), for any integer \( p \)

\[
(p + 1) X^{p+1} + p uX^{p-1} - (p + \frac{1}{2}) (x + \bar{x})X^p \sim 0, \quad X^p \sim \frac{u^{p+\frac{1}{2}}}{X^{p+1}}. \tag{E.11}
\]

(E.11) is equivalent to the standard recurrence relation for Legendre polynomials.

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