Meixner polynomials of the second kind
and quantum algebras representing su(1,1)

BY GÁBOR HETYEI*

Department of Mathematics and Statistics, University of North Carolina
at Charlotte, Charlotte, NC 28223, USA

We show how Viennot’s combinatorial theory of orthogonal polynomials may be used to
generalize some recent results of Sukumar and Hodges (Hodges & Sukumar 2007 Proc.
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R. Soc. A 463, 2415–2427 (doi:10.1098/rspa.2007.0003)) on the matrix entries in powers
of certain operators in a representation of su(1, 1). Our results link these calculations to
finding the moments and inverse polynomial coefficients of certain Laguerre polynomials
and Meixner polynomials of the second kind. As an immediate consequence of results by
Koelink, Groenevelt and Van Der Jeugt (Van Der Jeugt 1997 J. Math. Phys. 38, 2728–
2740 (doi:10.1063/1.531984); Koelink & Van Der Jeugt 1998 SIAM J. Math. Anal. 29,
794–822 (doi:10.1137/S003614109630673X); Groenevelt & Koelink 2002 J. Phys. A 35,
65–85 (doi:10.1088/0305-4470/35/1/306)), for the related operators, substitutions into
essentially the same Laguerre polynomials and Meixner polynomials of the second kind
may be used to express their eigenvectors. Our combinatorial approach explains and
generalizes this ‘coincidence’.

Keywords: quantum algebra; su(1, 1); laguerre polynomials;
Meixner polynomials of the second kind

1. Introduction

In two recent papers (Hodges & Sukumar 2007; Sukumar & Hodges 2007),
Sukumar and Hodges introduced a one-parameter family of operator algebras
exhibiting a parity-dependent structure to study a quantum harmonic oscillator. The operators \( \mathcal{R} \) and \( \mathcal{L} \), arising as the half of the square of the creation and
annihilation operators, respectively, generate a subalgebra representing the Lie
algebra su(1, 1). This representation is the direct sum of two discrete series
representations. Sukumar and Hodges observed that calculating the matrix
entries in the powers of \( \mathcal{L} + \mathcal{R} \) leads to some interesting combinatorial questions.
In this paper, we show that these results may be generalized in such a way that the
answer may be expressed in terms of moments and inverse polynomial coefficients
associated to Laguerre polynomials and Meixner polynomials of the second kind.

Almost at the same time, Klimyk (2006) used discrete series representations of
su(1, 1) to study another model of a quantum harmonic oscillator. In Klimyk’s
setting, the operator \( \mathcal{L} + \mathcal{R} \) corresponds to the momentum operator, and

*ghetyei@uncc.edu
Klimyk (2006) noted that the eigenfunctions of this operator may be expressed via substitutions into Meixner–Pollaczek polynomials with the appropriate parameters. This observation is a consequence of the more general results of Van Der Jeugt (1997), Koelink & Van Der Jeugt (1998) and Groenevelt & Koelink (2002), describing the spectra of certain self-adjoint elements in an arbitrary discrete series representation of su(1,1), where besides the Meixner–Pollaczek polynomials, the Laguerre polynomials and the Meixner polynomials of the first kind also appear, depending on the choice of the parameters.

In this paper, we generalize and explain this remarkable ‘coincidence’, using Viennot’s (1983) combinatorial theory of orthogonal polynomials. For that purpose, we will consider tridiagonal operators $T$ defined on a subspace of $\ell^2(\mathbb{Z}_{\geq 0})$ that contains at least the subspace generated by the basis vectors $\{e_0, e_1, e_2, \ldots\}$.

In §3, we express the matrix entries $\langle e_{i+d}, T^m e_i \rangle$ in terms of weighted Motzkin paths. Using Viennot’s (1983) theory, in the case when the matrix of $T$ has non-zero off-diagonal entries wherever this is allowed, we are able to associate an orthogonal polynomial sequence $\{p_n(x)\}_{n \geq 0}$ to $T$ in such a way that the matrix entries $\langle e_0, T^m e_0 \rangle$ and $\langle e_d, T^m e_0 \rangle$ may be expressed in terms of the moments and inverse polynomial coefficients associated to $\{p_n(x)\}_{n \geq 0}$. Applications of these general results to the discrete series representations of su(1,1) follow in §4.

In §5, we see that for those tridiagonal operators $T$ that have non-zero off-diagonal matrix entries wherever this is allowed, the same orthogonal polynomial sequence (OPS) $\{p_n(x)\}_{n \geq 0}$ as in §3 may be associated to $T$ to express its (potential) eigenvectors. For the discrete series representations of su(1,1), we thus recover the necessary conditions stated in the above-mentioned results of Van Der Jeugt (1997), Koelink & Van Der Jeugt (1998) and Groenevelt & Koelink (2002).

The ideas presented in this paper also provide a combinatorial framework to discuss some representations of other Lie (super)algebras over $\ell^2(\mathbb{Z}_{\geq 0})$. In the concluding §6 we outline a sample application. The other example in §6 indicates that we do not have to limit ourself to self-adjoint operators at this level of investigation. It seems harder to replace the Hilbert space $\ell^2(\mathbb{Z}_{\geq 0})$ with another Hilbert space, as in that case one would be facing the dilemma, what notion should replace the notion of the associated orthogonal polynomial sequences. Finding such a generalization seems an interesting challenge for future research.

2. Preliminaries

(a) Viennot’s combinatorial proof of Favard’s theorem

Concerning orthogonal polynomials, in this paper we follow Viennot’s (1983) notation and terminology, but sometimes we complement the facts stated therein with results cited from Chihara’s (1978) classical work. Viennot’s (1983) work is currently out of print but available on the author’s webpage. Some of his most important results were also communicated by Stanton (2003), a recommended source for those readers whose French is not fluent.

The direct way to define an OPS $\{p_n(x)\}_{n \geq 0}$ is to provide a linear form $f : \mathbb{C}[x] \to \mathbb{C}$ and postulate the following three axioms:

- for all $n$, $p_n(x)$ is a polynomial of degree $n$,
- $f(p_m(x)p_n(x)) = 0$ if $m \neq n$, and
- for all $n$, $f(p_n(x)^2) \neq 0$. 

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The map \( f \) is called a *moment functional.* Whenever an OPS exists, each of its elements is determined up to a non-zero constant factor, see the corollary of theorem 2.2 in ch. I of Chihara’s (1978) book.

An equivalent way to define an OPS is by means of Favard’s theorem, see theorems 4.1 and 4.4 in ch. I of Chihara’s (1978) book. This states that a sequence of monic polynomials \( \{p_n(x)\}_{n \geq 0} \) is an orthogonal polynomial sequence if, and only if, it satisfies the initial conditions

\[
p_0(x) = 1, \quad p_1(x) = x - b_0, \tag{2.1}
\]

and a two-term recurrence formula

\[
p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x) \quad \text{for } n \geq 1, \tag{2.2}
\]

where the numbers \( b_n \) and \( \lambda_n \) are constants and \( \lambda_n \neq 0 \) for \( n \geq 1 \). The above form appears in theorem 9 of ch. I in Viennot’s (1983) book; in Chihara’s (1978) work the indices are shifted. The original proof of Favard’s theorem provides only a recursive description of the moment-associated functional \( f \). In proposition 17 of ch. I, Viennot (1983) gave an explicit combinatorial description by expressing the *moments* \( \mu_n := f(x^n) \) as sums of weighted *Motzkin paths*. A Motzkin path of length \( n \), as defined in definition 15 of ch. I by Viennot (1983), is a path \( \omega = (s_0, \ldots, s_n) \) in \( \mathbb{Z} \times \mathbb{Z} \) from \( s_0 \) to \( s_n \) such that the second coordinate of all \( s_i \)'s is non-negative, and each step \( (s_i, s_{i+1}) \) is either a northeast step \((1,1)\) or an east step \((1,0)\) or a southeast step \((-1,1)\). Viennot (1983) introduces the following valuation: the weight \( v(\omega) \) of \( \omega \) is the product of the weights of its steps \((s_{i-1}, s_i)\), where each northeast step has weight \( 1 \), each east step at level \( k \) has weight \( b_k \) and each southeast step starting at level \( k \) has weight \( \lambda_k \). Here the level is the second coordinate of the lattice point. He then defines \( \mu_n := \sum_\omega v(\omega) \) as the total weight of all Motzkin paths \( \omega \) from \((0,0)\) to \((n,0)\).

**Theorem 2.1 (Viennot 1983).** Let \( \{p_n(x)\}_{n \geq 0} \) be a sequence of monic polynomials, given by equations (2.1) and (2.2) and let \( f : \mathbb{C}[x] \to \mathbb{C} \) be the linear map given by \( f(x^n) := \mu_n \). Then for all \( n, k, \ell \geq 0 \), we have

\[
f(x^n p_k(x) p_\ell(x)) = \lambda_1 \cdots \lambda_\ell \sum_\omega v(\omega),
\]

where the summation is over all Motzkin paths of length \( n \) from level \( k \) to level \( \ell \).

Substituting \( n = 0 \) in theorem (2.1) yields the more difficult implication of Favard’s theorem. A generalization of theorem (2.1) is theorem 1 in ch. III of Viennot’s (1983) work, which allows the computation of the *inverse polynomials* of a system of monic polynomials \( \{p_n(x)\}_{n \geq 0} \) given by equations (2.1) and (2.2), even if some scalars \( \lambda_n \) are zero (and thus \( \{p_n(x)\}_{n \geq 0} \) is not an OPS). The inverse polynomials \( q_n(x) := \sum_{i=0}^{n} q_{n,i} x^i \) are defined by \( x^n = \sum_{i=0}^{n} q_{n,i} p_i(x) \).

**Theorem 2.2 (Viennot 1983).** Let \( \{p_n(x)\}_{n \geq 0} \) be a system of monic polynomials defined by equations (2.1) and (2.2) for some numbers \( \{b_n\}_{n \geq 0} \) and \( \{\lambda_n\}_{n \geq 1} \). The coefficient \( q_{n,k} \) of \( x^k \) in the inverse polynomial \( q_n(x) \) is then the total weight of all weighted Motzkin paths of length \( n \) starting at \((0,0)\) and ending at \((n,k)\).
Motzkin path to the first subsequent southeast step starting at level

The combinatorial model mentioned above is associated to the case in ch. II of Viennot’s (1983) work. The associated moments are given in eqn (31) of his ‘histoires de Laguerre’ of length \( n \) yielding the sequences of numbers \( \{ a_k \}_{k \geq 0}, \{ b_k \}_{k \geq 0} \) and \( \{ c_k \}_{k \geq 1} \), we define the weight \( v_1(\omega) \) of a Motzkin path \( \omega \) as the product of the weights \( v_1(s_{i-1}, s_i) \) of its steps, such that each northeast (resp. east, resp. southeast) step starting at level \( k \) has weight \( a_k \) (resp. \( b_k \), resp. \( c_k \)). Setting

\[
\lambda_k = a_{k-1}c_k \quad \text{for } k \geq 1,
\]

for a Motzkin path \( \omega \) from \((0,0)\) to \((n,0)\), \( v_1(\omega) \) is equal to \( v(\omega) \), as defined in §2a, because each northeast step starting at some level \( k - 1 \) may be matched to the first subsequent southeast step starting at level \( k \). More generally, for a Motzkin path \( \omega \) starting at level \( k \) and ending at level \( l \), we have

\[
v(\omega) = \begin{cases} 
\sum_{\omega} v_1(\omega) & \text{if } k = l, \\
 a_k a_{k+1} \cdots a_{l-1} \sum_{\omega} v_1(\omega) & \text{if } k < l, \\
 c_k c_{l-1} \cdots c_{l+1} \sum_{\omega} v_1(\omega) & \text{if } k > l,
\end{cases}
\]

see lemma 1 in ch. II of Viennot (1983). Algebraically, equation (2.3) corresponds to a switch from the three-term recurrence relation for monic orthogonal polynomials to a general three-term recurrence relation. The combinatorial interest in \( v_1 \) arises when the sequences \( \{ a_k \}_{k \geq 0}, \{ b_k \}_{k \geq 0} \) and \( \{ c_k \}_{k \geq 1} \) consist of non-negative integers, allowing us to think of these weights as making choices from a set of options. In such situations, Viennot (1983) defines a story (‘histoire’) as a pair \((\omega; (p_1, \ldots, p_n))\) of a Motzkin path \( \omega = (s_0, \ldots, s_n) \) and a sequence \((p_1, \ldots, p_n)\) of positive integers satisfying \( 1 \leq p_i \leq v_1(s_{i-1}, s_i) \) for \( 1 \leq i \leq n \). Clearly the moment \( \mu_n \) is the number of stories of length \( n \). Thus, in such situations, we may replace the weighted lattice path model with a model that involves enumerating (non-weighted) combinatorial objects. In particular, permutation enumeration models of the (monic) Laguerre polynomials and of the Meixner polynomials of the second kind (discussed below) may be obtained by specializing Viennot’s (1983) bijection between his ‘histoires de Laguerre’ of length \( n \) associated to the valuation

\[
a_k = k + 1, \quad b_k = 2k + 2 \quad \text{for } k \geq 0 \quad \text{and} \quad c_k = k + 1 \quad \text{for } k \geq 1,
\]

and the permutations of the set \( \{1, \ldots, n+1\} \). For the details, we refer to Viennot’s (1983) work.

The (monic) Laguerre polynomials \( L_n^{(\alpha)}(x) \) are the OPS defined by equations (2.1) and (2.2), where \( b_n = 2n + \alpha + 1 \) and \( \lambda_n = n(n + \alpha) \), see section 5 in ch. II of Viennot’s (1983) work. The associated moments are given in eqn (31’) of ch. II by Viennot’s (1983):

\[
\mu_n = (\alpha + 1)_n = (\alpha + 1)(\alpha + 2) \cdots (\alpha + n).
\]

The combinatorial model mentioned above is associated to the case \( \alpha = 1 \) yielding large Laguerre stories. For general \( \alpha \), we have weighted Laguerre stories, and for \( \alpha = 0 \) we obtain restricted Laguerre stories by limiting \( b_k' \) to \( k \) above. The restricted Laguerre stories are thus a subset of the large Laguerre stories and correspond bijectively to the permutations \( \sigma \in S_{n+1} \) with \( \sigma(1) = n + 1 \).
Another class of particular interest to us are the \textit{Meixner polynomials of the first kind} $m_n(x; \delta, \eta) := (c - 1)^n / c^n \tilde{m}_n(x; \beta, c)$, whose monic variant $\{\tilde{m}_n(x; \beta, c)\}_{n \geq 0}$ is defined by equations (2.1) and (2.2) where

$$b_n = \frac{(1 + c)n + \beta c}{1 - c} \quad \text{and} \quad \lambda_n = \frac{cn(n + \beta - 1)}{(1 - c)^2}. \quad (2.7)$$

Viennot (1983) (see eqn (54’) in ch. 2) has a combinatorial proof of the fact that the moments of the corresponding linear functional are given by

$$\mu_n(\beta, c) = (1 - c)^\beta \sum_{k \geq 0} k^n c^k (\beta)_k / k! \quad (2.8)$$

Here $(\beta)_k = \beta(\beta + 1) \cdots (\beta + k - 1)$.

Finally, we will be interested in the (monic) \textit{Meixner polynomials of the second kind} $M_n(x; \delta, \eta)$, defined by equations (2.1) and (2.2), where

$$b_n = (2n + \eta)\delta \quad \text{and} \quad \lambda_n = (\delta^2 + 1)n(n + \eta - 1). \quad (2.9)$$

A combinatorially interesting special case is $\delta = 0$, implying $b_n = 0$ and $\lambda_n = n(n + \eta - 1)$. The moments associated to $\{M_n(x; \delta, \eta)\}_{n \geq 0}$ may be expressed using the Motzkin paths associated to the Laguerre polynomials $\{L_n^{(\eta-1)}(x)\}_{n \geq 0}$ subject to the restriction that no east step occurs. Motzkin paths with no east steps are also known as Dyck paths. Using the permutation–enumeration model mentioned above, Viennot (1983) (in ch. 2, example 21) shows that $\mu_n(0, \eta)$ is given by

$$\mu_n(0, \eta) = \sum_{\sigma \in Z_n} \eta^{s(\sigma)}. \quad (2.10)$$

Here $Z_n$ is the set of alternating or zig-zag permutations of $\{1, 2, \ldots, n, \}$, starting and ending with a rise, and $s(\sigma)$ is the number of left-to-right minima (‘éléments saillants’) $\sigma(i)$, defined by $\sigma(i) = \min\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$. A generating function for the numbers $\mu_{2n}(0, \eta)$ for positive integer $\eta$ was already computed by Carlitz (1959). He considered the polynomials $f_n^{(k)}(x) = (-i)^n M_n(ix; 0, k)$ for $i = \sqrt{-1}$, but using the substitution $x \mapsto x/i$ we may recover the moments $\mu_{2n}(0, \eta)$. In the next lemma, we state Carlitz’ (1959) result (see his formulae (9.3) and (9.13)) modified for our purposes, together with the outline of a combinatorial proof that extends its validity to all numbers $\eta$.

**Lemma 2.3.** For all $\eta \neq 0$, the moments of $\{M_n(x; 0, \eta)\}_{n \geq 0}$ satisfy

$$\mu_{2n}(0, \eta) = E_{2n}^{(\eta)} \quad \text{where} \quad \sec^\eta(t) = \sum_{n \geq 0} E_{2n}^{(\eta)} (2n)! / (2n)^n.$$

**Proof.** Let $E_{2n,k}$ denote the number of those zig-zag permutations on $2n$ elements that start and end with a rise and have $k$ left-to-right minima. Since 1 is the rightmost left-to-right minimum, using the decomposition $\sigma(1) \cdots \sigma(2n) = \sigma(1) \cdots 1 \cdots \sigma(2n)$ we have the recursion formula

$$E_{2n,k} = \sum_{m=0}^{n-1} \binom{2n-1}{2m} E_{2m,k-1} \cdot T_{2n-2m-1}. \quad (2.11)$$

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Here $T_{2n-2m-1}$ counts the number of zig-zag permutations on $2n - 2m - 1$ elements, starting with a descent and ending with a rise. Introducing $\phi_k(t) := \sum_{n \geq 0} E_{2n,k} t^{2n}/(2n)!$, we obtain the recursion
\[
\phi_k(t) = \int_0^t \phi_{k-1}(u) \tan(u) \, du.
\]
Using this, we may show by induction that $\phi_k(t) = (-\ln \cos(t))^k/k!$. Thus, by equation (2.10), $\mu_{2n}(0, \eta)$ is the coefficient of $t^{2n}/(2n)!$ in
\[
\sum_{k \geq 0} \frac{(-1)^k \ln \cos(t)^k \cdot \eta^k}{k!} = \exp(-\eta \ln \cos(t)) = \sec(t)^\eta.
\]
A variant of the Meixner polynomials of the second kind are the Meixner–Pollaczek polynomials $\{P_n^\lambda(x; \phi)\}_{n \geq 0}$, defined by eqn (1.7.1) in the Askey–Wilson scheme by Koekoek & Swarttouw (1998). Setting $\phi = \pi/2$ in equation (2.12) leads to $\delta = 0$. According to eqn (1.7.4) of Koekoek & Swarttouw (1998), the normalized version of $P_n^\lambda(x; \phi)$ is $n!/2 \sin \phi)^n P_n^\lambda(x, \phi)$. As a consequence of equation (2.12), we obtain the following.

**Corollary 2.4.** For $0 < \phi < 1$, the normalized version of the Meixner–Pollaczek polynomials $\{P_n^\lambda(x; \phi)\}_{n \geq 0}$ is $\{2^{-n} M_n(2x, \cot \phi, 2\lambda)\}_{n \geq 0}$.

Finally, we review the inverse polynomials of $\{L_n^\alpha(x)\}_{n \geq 0}$, $\{m_n(x; \beta, c)\}_{n \geq 0}$ and $\{M_n(x; \delta, \eta)\}_{n \geq 0}$. All three OPS are examples of Sheffer orthogonal polynomials defined as polynomials $\{p_n(x)\}_{n \geq 0}$, given by a generating function
\[
\sum_{n \geq 0} p_n(x) \frac{t^n}{n!} = f(t) \exp(xg(t)).
\]
As noted in eqns (11) and (12) of ch. III in Viennot’s (1983) work, the inverse polynomials $\{q_n(x)\}_{n \geq 0}$ of a Sheffer OPS $\{p_n(x)\}_{n \geq 0}$ form a Sheffer OPS, given by
\[
\sum_{n \geq 0} q_n(x) \frac{t^n}{n!} = \frac{1}{f(g(-1)^{-1}(t))} \exp(xg^{-1}(t)).
\]
Here $g^{-1}(t)$ stands for the compositional inverse of $g(t)$.

(c) **Discrete series representations of $\text{su}(1, 1)$ and two applications**

In this subsection, we review the discrete series representations of the Lie algebra $\text{su}(1, 1)$, together with two applications of them in models of a quantum oscillator. Our notation will be close to the one used by Groenevelt & Koelink (2002).
The Lie algebra $\text{su}(1, 1)$ has the generators $H, B, C$ subject to the commutation relations $[H, B] = 2B$, $[H, C] = -2C$ and $[B, C] = H$. According to section 6.4 in the work of Vilenkin & Klimyk (1991), there are four classes of irreducible unitary representations of $\text{su}(1, 1)$: the positive and negative discrete series representations, the principal series representations and the complementary series representations. In this paper, we will focus on the two discrete series representations, defined on the representation space $\ell^2(\mathbb{Z}_{\geq 0})$, whose orthonormal basis vectors we will denote by $\{e_n\}_{n \geq 0}$. The positive discrete series representations $\pi^+_k$ are labelled by $k > 0$. The action is given by

\[
\begin{align*}
\pi^+_k(H)e_n &= 2(k + n)e_n, \\
\pi^+_k(B)e_n &= \sqrt{(n + 1)(2k + n)}e_{n+1}, \\
\pi^+_k(C)e_n &= -\sqrt{n(2k + n - 1)}e_{n-1},
\end{align*}
\]  

(2.15)

and

\[
\begin{align*}
\pi^-_k(H)e_n &= -2(k + n)e_n, \\
\pi^-_k(B)e_n &= -\sqrt{n(2k + n - 1)}e_{n-1}, \\
\pi^-_k(C)e_n &= \sqrt{(n + 1)(2k + n)}e_{n+1},
\end{align*}
\]  

(2.16)

see eqn (2.2) of Groenevelt & Koelink (2002). The negative discrete series representations $\pi^-_k$ are labelled by $k > 0$. The action is given by

where the polynomials $p^{(k)}_n(x; \phi)$ are the Meixner–Pollaczek polynomials.

**Theorem 2.5 (Koelink & Van Der Jeugt).** Introducing $X_\phi = -\cos \phi H + B - C$, the vector

\[
v(x) := \sum_{n \geq 0} p^{(k)}_n(x; \phi)e_n
\]

is a generalized eigenvector for $\pi^+_k(X_\phi)$, with eigenvalue $2x \sin \phi$.

An analogous result for $c = 1$ was worked out by Van Der Jeugt (1997), leading to Laguerre polynomials, who also mentions (in his section VIII) that the case $|c| > 1$ leads to Meixner polynomials of the first kind. The extension of theorem (2.5) to negative discrete series representations is stated in proposition (3.1) of Groenevelt & Koelink (2002). An excellent source treating all cases simultaneously is ch. 3 of Groenevelt’s (2004) thesis.

Klimyk (2006) used the positive series representations of $\text{su}(1, 1)$ to study a model of a quantum oscillator. He defines $\text{su}(1, 1)$ as the Lie algebra generated by $J_0, J_+, J_-$, subject to the commutation relations $[J_0, J_+] = J_+, [J_0, J_-] = -J_-$.
and \([J_-, J_+] = 2J_0\). This definition is equivalent to the above one, via setting \(H := 2J_0\), \(B := J_+\) and \(C := -J_-\). Klimyk (2006) realizes the orthonormal basis vectors \(\{e_n\}_{n \geq 0}\) as polynomials \(e_n^k(y) := ((2k + n - 1)!/n!)^{1/2} y^n\) in the single variable \(y\), and the Hilbert space \(\ell^2(\mathbb{Z}_{\geq 0})\) is realized as the closure of the space of all polynomials in a single variable \(y\), whereas the generators \(J_0, J_+, J_-\) of \(\text{su}(1,1)\) become differential operators. One of Klimyk’s (2006) observations (eqn (17)) is that the eigenfunctions of the momentum operator \(J_1 := (1/2)(J_+ + J_-)\) are of the form

\[
\psi_p(y) = \sum_{n \geq 0} P_n^{(k)} \left( p; \frac{\pi}{2} \right) y^n. \tag{2.17}
\]

with the Meixner–Pollaczek polynomials \(P_n^{(k)}(p; \pi/2)\). Obviously, equation (2.17) may be obtained from theorem (2.5) by substituting \(\phi = \pi/2\). By eqn (1.7.11) of Koekoek & Swarttouw (1998), equation (2.17) is equivalent to \(\psi_p(y) = (1 - iy)^{-k+i:p}(1 + iy)^{-k-i:p}\).

Sukumar & Hodges (2007) considered another model of a quantum oscillator, using (besides others) the operators \(L, R, S\), acting on \(\ell^2(\mathbb{Z}_{\geq 0})\) as follows:

\[
\begin{align*}
L e_{2n} &= \frac{\sqrt{(2n + 1 + \alpha)(2n + 2)}}{2} \cdot e_{2n+2} \\
L e_{2n-1} &= \frac{\sqrt{(2n + 1 + \alpha)(2n)}}{2} \cdot e_{2n+1} \\
R e_{2n+2} &= \frac{\sqrt{(2n + 1 + \alpha)(2n + 2)}}{2} \cdot e_{2n} \\
R e_{2n+1} &= \frac{\sqrt{(2n + 1 + \alpha)(2n)}}{2} \cdot e_{2n-1} \\
S e_n &= \frac{2n + 1 + \alpha}{2} \cdot e_n
\end{align*} \tag{2.18}
\]

The parameter \(\alpha \in \mathbb{R}\) is assumed to satisfy \(|\alpha| \leq 1\). As noted by Sukumar & Hodges (2007) in equation (2.1), the operators \(L, R, S\) obey the same commutation rules as the generators \(B, -C, H\) above, thus we obtain a representation \(\pi\) of \(\text{su}(1,1)\) by setting

\[
\pi(B) := L, \quad \pi(C) := -R \quad \text{and} \quad \pi(H) := S. \tag{2.19}
\]

This representation may be written as a direct sum of two representations by considering the restrictions of the operators \(L, R, S\) to the closure of the subspaces generated by \(\{e_{2n}\}_{n \geq 0}\) and \(\{e_{2n+1}\}_{n \geq 0}\), respectively. Introducing \(\pi_\beta\) to denote the representation of \(\text{su}(1,1)\) given by

\[
\begin{align*}
\pi_\beta(B)e_n &= \sqrt{(n + 1)(n + \beta)} e_{n+1}, \\
\pi_\beta(C)e_n &= -\sqrt{n(n - 1 + \beta)} e_{n-1} \\
\pi_\beta(H)e_n &= 2 \left( n + \frac{\beta}{2} \right) e_n
\end{align*} \tag{2.20}
\]

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it is not difficult to see that the first restriction is equivalent to $\pi_{(\alpha+1)/2}$, whereas
the second is equivalent to $\pi_{(\alpha+3)/2}$. Comparing equation (2.20) with (2.15) we
obtain $\pi_\beta = \pi_{\beta/2}$ for $\beta > 0$.

**Corollary 2.6.** For $\alpha \in (-1,1)$, the representation $\pi$ of $su(1,1)$ given by
equation (2.19) is isomorphic to the direct sum of the positive discrete series
representations $\pi^+_{(\alpha+1)/4}$ and $\pi^+_{(\alpha+3)/4}$.

**Remark 2.7.** For $\beta = 0$, we get $\pi_0(B)e_0 = \pi_0(C)e_0 = \pi_0(H)e_0 = 0$, thus we may
decompose $\pi_0$ as a direct sum of zero acting on the linear span of $e_0$ and of the
restriction of $\pi_0$ acting on the orthogonal complement $e_0^\perp$ of $e_0$. By shifting basis
vector indices in $e_0^\perp$, we may show that the restriction of $\pi_0$ to $e_0^\perp$ is equivalent
to $\pi_2 = \pi_1^+$.

It should be emphasized that corollary (2.6) is applicable only to the
representation of $su(1,1)$, induced by the operators $L,R,S$. In the work of
Sukumar & Hodges (2007), these operators are expressed in terms of the
annihilation operator $A$ and its adjoint $A^\dagger$. The algebra generated by $A$ and
$A^\dagger$ is a representation of a Lie superalgebra properly containing $su(1,1)$. In
this paper, we focus on the combinatorial statements related to the operators
$L,R,S$ and generalizations of these. In particular, Hodges & Sukumar (2007)
show (equation (5.2)) that for $\alpha = 1$ we have

$$
\langle e_0, (L + R)^{2m} e_0 \rangle = E_{2m} \quad \text{and} \quad \langle e_1, (L + R)^{2m} e_1 \rangle = T_{2m+1} \quad \text{for all } m \geq 0. \tag{2.21}
$$

Here the numbers $\{E_{2m}\}_{m \geq 0}$ resp. $\{T_{2m+1}\}_{m \geq 0}$ are the secant resp. tangent
numbers, given by

$$
\sec(x) = \sum_{m \geq 0} \frac{E_{2m}}{(2m)!} x^{2m} \quad \text{and} \quad \tan(x) = \sum_{m \geq 0} \frac{T_{2m+1}}{(2m + 1)!} x^{2m+1}.
$$

In the light of corollary (2.6), equation (2.21) is equivalent to

$$
\langle e_0, \pi_{1/2}^+ (B - C)^m e_0 \rangle = E_{2m} \quad \text{and} \quad \langle e_0, \pi_1^+ (B - C)^m e_0 \rangle = T_{2m+1}. \tag{2.22}
$$

### 3. Matrix entries of tridiagonal operators on $\ell^2(\mathbb{Z}_{\geq 0})$

**Definition 3.1.** Consider the complex Hilbert space $\ell^2(\mathbb{Z}_{\geq 0})$ with orthonormal
basis vectors $e_0, e_1, e_2, \ldots$. We call a linear operator $T : D(T) \subset \ell^2(\mathbb{Z}_{\geq 0}) \rightarrow \ell^2(\mathbb{Z}_{\geq 0})$, defined on at least the linear span of $\{e_0, e_1, e_2, \ldots\}$, a
**tridiagonal operator**, if there exist real numbers $\{l_n\}_{n \geq 0}, \{d_n\}_{n \geq 0}$ and $\{u_n\}_{n \geq 0}$ such that $Te_n = d_n e_0 + l_n e_1$ and, for each $n \geq 1$, we have $Te_n = u_n e_{n-1} + d_n e_n + l_n e_{n+1}$.

Equivalently, the restriction of $T$ to the linear span of $\{e_0, e_1, e_2, \ldots\}$ may be
represented by a tridiagonal matrix

$$
\begin{pmatrix}
  d_0 & u_0 & 0 & 0 & 0 & \cdots \\
  l_0 & d_1 & u_1 & 0 & 0 & \cdots \\
  0 & l_1 & d_2 & u_2 & 0 & \cdots \\
  0 & 0 & l_2 & d_3 & u_3 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
$$

such that all entries are real numbers.
The letters \( \{l_n\}_{n \geq 0}, \{u_n\}_{n \geq 0} \) and \( \{d_n\}_{n \geq 0} \) should remind the reader of the words ‘lower’, ‘upper’ and ‘diagonal’, as it is customary in numerical analysis. We define the LDU decomposition of the tridiagonal operator \( T \) as the sum of operators \( T = L + D + U \), where the operators \( L, D, U \) are given by

\[
Le_n = l_ne_{n+1}, \quad Ue_{n+1} = u_ne_n \quad \text{and} \quad De_n = d_ne_n \quad \text{for} \quad n \geq 0.
\]

**Remark 3.2.** As it is a fortunate coincidence that the operator \( L \) used by Sukumar & Hodges (2007) corresponds to the operator \( L \) in this setting, modulo corollary (2.6). The choice of \( D \) will depend on what we want to compute: to recover the formulas regarding the matrix entries of \( L + R \) and \( L + R + S \) in the work of Sukumar & Hodges (2007) and Hodges & Sukumar (2007), we would need to equate \( D \) to zero or to the appropriate restriction of \( S \).

The following lemma may be considered as a generalization of equations (4.2) and (4.3) in Hodges & Sukumar (2007). A reader making this comparison should note that, for technical reasons, we read the words \( X \) consisting of the letters \( L, D, U \) right to left.

**Lemma 3.3.** Let \( X \) be a word of length \( m \) consisting of the letters \( L, D, U \), where \( T = L + D + U \) is the LDU decomposition of a tridiagonal operator \( T : D(T) \subset \ell^2(\mathbb{Z}_{\geq 0}) \rightarrow \ell^2(\mathbb{Z}_{\geq 0}) \) and let \( i \geq 0 \) and \( d \geq -i \) be integers. Associate to the pair \((X, i)\) a lattice path starting at \((0, i)\) such that each \( L \) is a southeast step, each \( U \) is a northeast step, each \( D \) is an east step and we read the letters in \( X \) right to left. Then, \( \langle e_{i+d}, Xe_i \rangle \) is not zero only if the lattice path associated to \((X, i)\) is a Motzkin path of length \( m \) from \((0, i)\) to \((m, i+d)\).

**Proof.** Applying \( X \) to \( e_i \) involves applying the operators \( L, D, U \) in it, reading \( X \) from right to left. While applying these operators, at every instance of the calculation our partial result is a scalar multiple of a single basis vector. Each \( L \) increases the index of this basis vector by 1, each \( U \) decreases it by 1 and each \( D \) leaves the index unchanged. Thus, the end result is either zero or a scalar multiple of \( e_{i+d} \), where \( d \) is the difference between the number of \( L \)s and \( U \)s in \( X \) (and may be negative). If at any instance the number of \( U \)s read exceeds the number of \( L \)s by more than \( i \), then we get zero by \( Ue_0 = 0 \). Therefore, if \( \langle e_{i+d}, Xe_i \rangle \neq 0 \), then \((X, i)\) must represent a Motzkin path starting at \((0, i)\) and necessarily ending at \((m, i+d)\). \[\blacksquare\]

The next lemma generalizes equations (4.6) and (4.7) of Hodges & Sukumar (2007). Our formulas look somewhat different mainly because of our convention of reading \( X \) right to left.

**Lemma 3.4.** For all \( p, i \geq 0 \) and \( d \geq -i \), we have

\[
\langle e_{i+d}, XDL^p e_i \rangle = d_{p+i} \cdot \langle e_{i+d}, XL^p e_i \rangle \tag{3.2}
\]

and

\[
\langle e_{i+d}, XUL^{p+1} e_i \rangle = l_{p+i} u_{p+i} \cdot \langle e_{i+d}, XL^p e_i \rangle. \tag{3.3}
\]

The proof is immediate. Using equation (2.4) and lemmas (3.3) and (3.4), we may express each \( \langle e_{i+d}, T^m e_i \rangle \) in terms of a total weight of weighted Motzkin paths.
Theorem 3.5. Let \( i \geq 0 \) and \( d \geq -i \) be integers, \( T : D(T) \subset \ell^2(\mathbb{Z}_{\geq 0}) \to \ell^2(\mathbb{Z}_{\geq 0}) \) a tridiagonal operator and \( T = L + D + U \) its LDU decomposition. Then, we have

\[
\langle e_{i+d}, T^m e_i \rangle = \begin{cases} 
\sum_\omega v(\omega) & \text{if } d = 0, \\
\sum_\omega v(\omega) l_{i+1} \cdots l_{i+d-1} & \text{if } d > 0, \\
\sum_\omega v(\omega) u_{i-2} \cdots u_d & \text{if } d < 0.
\end{cases}
\]

Here \( \sum_\omega v(\omega) \) is the total weight of all weighted Motzkin paths of length \( m \) starting at \((0, i)\) and ending at \((m, i+d)\) such that each northeast step has weight 1, each southeast step starting at level \( n \) has weight \( \lambda_n = l_{n-1} u_{n-1} \) and each east step at level \( n \) has weight \( d_n \). In particular, if \( D \) is identically zero, then \( \sum_\omega v(\omega) \) is the total weight of all weighted Dyck paths of length \( m \) starting at \((0, i)\) and ending at \((m, i+d)\) such that each northeast step has weight 1 and each southeast step starting at level \( n \) has weight \( \lambda_n = l_{n-1} u_{n-1} \).

As a consequence of theorem (2.1), setting \( d = 0 \) and \( i = 0 \) in theorem (3.5) yields the following algebraic statement.

Corollary 3.6. If \( l_n u_n \neq 0 \) for all \( n \geq 0 \), then for all \( m \), \( \langle e_0, T^m e_0 \rangle \) is the \( m \)th moment of the functional associated to the OPS given by equations (2.1) and (2.2) where \( b_n = d_n \) and \( \lambda_n = l_{n-1} u_{n-1} \).

Remark 3.7. As noted in the preliminaries, an OPS and the associated moments determine each other only up to a constant factor. In corollary (3.6), we understand that the moments were defined as total weights of weighted Motzkin paths as in theorem (2.1). If we obtain the moments by some other means, we always need to check whether an adjustment by a constant factor is necessary. It suffices to check whether \( \mu_0 = 1 \) is satisfied.

When we apply theorem (3.5) to \( d \neq 0 \), theorem (2.2) becomes useful, at least for the case \( i = 0 \).

Theorem 3.8. For all \( m, d \geq 0 \) we have

\[
\langle e_d, T^m e_0 \rangle = l_0 l_1 \cdots l_{d-1} \mu_{m,d},
\]

where \( \mu_{m,d} \) is the coefficient of \( x^d \) in the degree \( m \) inverse polynomial for system of monic polynomials given by equations (2.1) and (2.2) where \( b_n = d_n \) and \( \lambda_n = l_{n-1} u_{n-1} \).

4. Matrix entries in discrete series representations of su(1,1)
It is easy to see that \( l_n u_n \neq 0 \) holds in all cases, thus we may apply corollary (3.6) to compute \( \langle e_0, (\pi (-C) + c \cdot \pi (H) + \pi (B))^m e_0 \rangle \) for every discrete series representation \( \pi \), all \( c \in \mathbb{R} \) and all \( m \in \mathbb{N} \). In particular, using equations (2.15) and (2.16), we obtain the following corollary.

**Corollary 4.1.** For any \( k > 0 \), \( m \in \mathbb{N} \) and \( c \in \mathbb{R} \), the matrix entry \( \langle e_0, (\pi_k^+ (-C) + c \cdot \pi_k^+ (H) + \pi_k^+ (B))^m e_0 \rangle \) is the \( m \)th moment of the functional associated to the OPS \( \{p[k, c]_n (x)\}_{n \geq 0} \) given by equations (2.1) and (2.2) where

\[
\begin{align*}
    b_n &= 2c(k + n) \quad \text{and} \quad \lambda_n = n(n + 2k - 1). \\
    \end{align*}
\]

Similarly, the matrix entry \( \langle e_0, (\pi_k^- (-C) + c \cdot \pi_k^- (H) + \pi_k^- (B))^m e_0 \rangle \) is the \( m \)th moment of the functional associated to the OPS \( \{p[k, -c]_n (x)\}_{n \geq 0} \).

The appropriate parts of the next proposition (restricted to the cases \( |c| = 1 \), \( |c| > 1 \) and \( |c| < 1 \), respectively) are either implied or explicitly stated in some equivalent form in the works of Van Der Jeugt (1997), Koelink & Van Der Jeugt (1998) and Groenevelt & Koelink (2002). A detailed and deep simultaneous discussion of all three cases may be found in ch. 3 of Groenevelt’s (2004) thesis. Here we make a summary statement with respect to our bases and outline the proof that ensues from applying Meixner’s classical method.

**Proposition 4.2 (Groenevelt–Koelink–Van Der Jeugt).** The OPS \( \{p[k, c]_n (x)\}_{n \geq 0} \) defined in corollary (4.1) is given by

\[
p[k, c]_n (x) = \begin{cases} 
    \left( \sqrt{c^2 - 1} - c \right)^n m_n \left( \frac{x}{2\sqrt{c^2 - 1}} - k; 2k, \frac{c - \sqrt{c^2 - 1}}{c + \sqrt{c^2 - 1}} \right) & \text{if } |c| > 1, \\
    L_n^{(2k-1)}(x) & \text{if } c = 1, \\
    (-1)^n L_n^{(2k-1)}(-x) & \text{if } c = -1, \\
    \left( \sqrt{1 - c^2} \right)^n M_n \left( \frac{x}{\sqrt{1 - c^2}}; c \sqrt{1 - c^2}; 2k \right) & \text{if } |c| < 1.
\end{cases}
\]

Here the polynomials \( \{L_n^{(2k-1)}(x)\}_{n \geq 0} \) are Laguerre polynomials, and the polynomials \( \{m_n(x; 2k, (c - \sqrt{c^2 - 1})/(c + \sqrt{c^2 - 1}))\}_{n \geq 0} \) and \( \{M_n(x; c/\sqrt{1 - c^2}; 2k)\}_{n \geq 0} \), respectively, are Meixner polynomials of the first and second kind, respectively.

**Proof.** We apply Meixner’s method to classify all OPS \( \{P_n(x)\}_{n \geq 0} \) given by equations (2.1) and (2.2) where

\[
\begin{align*}
    b_n &= dn + f \quad \text{and} \quad \lambda_n = n(gn + h), \\
\end{align*}
\]

see section 3 of ch. 6 in Chihara (1978). By equation (4.2) we have

\[
\begin{align*}
    d &= 2c, \quad f = 2ck, \quad g = 1 \quad \text{and} \quad h = 2k - 1.
\end{align*}
\]

The case \( d^2 - 4g = 4(c^2 - 1) > 0 \) is thus equivalent to \( |c| > 1 \). Following section 3 of ch. 6 in Chihara (1978), we set \( \rho := \sqrt{d^2 - 4g} = 2\sqrt{c^2 - 1} \). Note next that by
replacing \( p[k, c]_n(x) \) with

\[
q[k, c]_n(x) := p[k, c]_n(x - \gamma_{k,c})
\]
yields an OPS \( \{q_n[k, c](x)\}_{n \geq 0} \) satisfying equation (4.3) with

\[
d = 2c, \quad f = 2ck + \gamma_{k,c}, \quad g = 1 \quad \text{and} \quad h = 2k - 1.
\]

We want to choose \( \gamma_{k,c} \) in such a way that

\[
f = \frac{2(g + h)}{d + \rho}, \quad \text{i.e.} \quad 2ck + \gamma_{k,c} = \frac{4k}{2c + 2\sqrt{c^2 - 1}}
\]
is satisfied. This yields

\[
\gamma_{k,c} = \frac{2k}{c + \sqrt{c^2 - 1}} - 2kc = 2k \left( c - \sqrt{c^2 - 1} \right) - 2kc = -2k\sqrt{c^2 - 1},
\]

and we may apply the results stated in section 3 of ch. 6 in Chihara (1978) to \( \{q_n[k, c](x)\}_{n \geq 0} \). The parameter ‘c’ appearing in the definition of the corresponding Meixner polynomials of the first kind becomes

\[
\frac{d - \rho}{d + \rho} = \frac{2c - 2\sqrt{c^2 - 1}}{2c + 2\sqrt{c^2 - 1}} = \frac{c - \sqrt{c^2 - 1}}{c + \sqrt{c^2 - 1}},
\]

and the parameter \( \beta \) becomes \( 1 + (2k - 1)/1 = 2k \). We obtain

\[
\left( \frac{1}{\sqrt{c^2 - 1} - c} \right)^n q[k, c]_n \left( 2\sqrt{c^2 - 1}x \right) = m_n \left( x; 2k, \frac{c - \sqrt{c^2 - 1}}{c + \sqrt{c^2 - 1}} \right)
\]
or, equivalently,

\[
q[k, c]_n(x) = \left( \sqrt{c^2 - 1} - c \right)^n m_n \left( \frac{x}{2\sqrt{c^2 - 1}}; 2k, \frac{c - \sqrt{c^2 - 1}}{c + \sqrt{c^2 - 1}} \right).
\]

The formula for \( p[k, c]_n(x) \) now follows from \( p[k, c]_n(x) = q[k, c]_n(x + \gamma_{k,c}) \).

The case \( d^2 - 4g = 4(c^2 - 1) < 0 \) is equivalent to \( |c| < 1 \). The parameter \( \delta \) of the corresponding Meixner polynomials of the second kind becomes \( 2c/\sqrt{4 - 4c^2} = c/\sqrt{1 - c^2} \) and the parameter \( \eta \) becomes \( 1 + (2k - 1)/1 = 2k \). We obtain

\[
\left( \frac{1}{\sqrt{1 - c^2}} \right)^n p[k, c]_n \left( x\sqrt{1 - c^2} \right) = M_n \left( x; \frac{c}{\sqrt{1 - c^2}}, 2k \right).
\]

The case \( d^2 - 4g = 4(c^2 - 1) = 0 \) is equivalent to \( |c| = 1 \). For \( c = 1 \), equation (4.2) gives \( b_n = 2n + 2k \) and \( \lambda_n = n(n + 2k - 1) \), which define the monic Laguerre polynomials \( p[k, 1]_n(x) = L_n^{(2k-1)}(x) \). Similarly, we also have \( p[k, -1]_n(x) = (-1)^n L_n^{(2k-1)}(-x) \). □

**Remark 4.3.** Inspired by theorem (2.5) and corollary (2.4), for \( |c| < 1 \) we may introduce a \( \phi \) such that \( 0 < \phi < \pi \) and \( c = \cos(\phi) \) hold. Then, we obtain

\[
p[k, \cos(\phi)]_n(x) = \sin(\phi)^n M_n \left( \frac{x}{\sin(\phi)}; \cot(\phi), 2k \right).
\]
As indicated in the preliminaries, a combinatorial interpretation of the ‘Laguerre case’ $|c| = 1$ may be obtained using Viennot’s (1983) ‘Laguerre stories’. The ‘best’ combinatorial interpretations (linked to permutation enumeration with no special weights) are associated to the case $k \in \{0, 1\}$. In the light of corollary (2.6), this means that a combinatorial study of the matrix entries of the operator $R + L + S$ appearing in the work of Sukumar & Hodges (2007) is easiest when $(\alpha - 1)/2$ or $(\alpha + 1)/2$ belongs to $\{0, 1\}$. As a consequence of equation (2.6), we obtain
\[
\langle e_0, (L + S + R)^n e_0 \rangle = \left(\frac{\alpha + 1}{2}\right)_n \text{ for } \alpha \in (-1, 1].
\] (4.5)

The case $\alpha = -1$ may be handled using remark (1.7). Similarly, for $e_1$, we have
\[
\langle e_1, (L + S + R)^n e_1 \rangle = \left(\frac{\alpha + 3}{2}\right)_n \text{ for all } \alpha \in [-1, 1].
\] (4.6)

Another combinatorially interesting case is $c = 0$ where proposition (4.2) gives $p[k, 0]_n(x) = M_n(x; 0, 2k)$. As a consequence, the operator $L + R$ appearing in the work of Sukumar & Hodges (2007) satisfies
\[
\langle e_0, (L + R)^{2n} e_0 \rangle = \mu_{2n} \left(0, \frac{(\alpha + 1)}{2}\right) \text{ for } \alpha \in (-1, 1].
\] (4.7)

The case $\alpha = -1$ may again be handled using remark (2.7). Similarly, for $e_1$, we have
\[
\langle e_1, (L + R)^{2n} e_1 \rangle = \mu_{2n} \left(0, \frac{(\alpha + 3)}{2}\right) \text{ for all } \alpha \in [-1, 1].
\] (4.8)

Here $\mu_{2n}(0, \eta)$ stands for the $(2n)$th moment associated to the Meixner polynomials of the second kind with parameters $(0, \eta)$. For $\eta = 1$, equation (2.10) takes a very simple form, as is noted in example 21 of ch. II by Viennot (1983). Thus, by applying corollary (4.1) to $c = 0$ and $k = 1/2$, we obtain a combinatorial explanation for the first half of equation (2.22).

**Remark 4.4.** Viennot’s (1983) work offers a nice model for the moments $\mu_{2n}(0, 1)$ by further restricting the appropriate ‘Laguerre stories’. This corresponds to enumerating certain zig-zag permutations. For other values of $\eta$, we could not avoid considering ‘weighted stories’ in Viennot’s (1983) model. The case $\alpha = 0$ in the work of Sukumar & Hodges (2007) does not correspond to $\eta = 1$, yet Sukumar & Hodges (2007) find a simple, non-weighted model by considering the enumeration of transposition zig-zag classes. It is an interesting question for future research whether this enumeration question may be related to Viennot’s (1983) theory by constructing a highly non-trivial bijection, similar to Viennot’s (1983) bijection between all permutations and his ‘Laguerre stories’.

For other values of $\eta$ we may use lemma (2.3), which gives
\[
\langle e_0, \pi^+_k (B - C)^m e_0 \rangle = E^{(2k)}_m \text{ where } sec^{2k}(t) = \sum_{n \geq 0} E^{(2k)}_n \frac{t^n}{n!}. \tag{4.9}
\]

In particular, substituting $k = 1$ into equation (4.9) yields the second half of equation (2.22). Indeed, as noted in eqn (3.2) of Sukumar & Hodges (2007), we have $E^{(2)}_n = T_{2n+1}$. Using equation (4.9) for an arbitrary parameter $k$ allows
Meixner polynomials and \( \text{su}(1,1) \)

us to rewrite equations (4.7) and (4.8) as

\[
\langle e_0, (L + R)^{2n} e_0 \rangle = E_{2n}^{(\alpha+1)/2} \quad \text{for} \ \alpha \in (-1, 1] \quad (4.10)
\]

and

\[
\langle e_1, (L + R)^{2n} e_1 \rangle = E_{2n}^{(\alpha+3)/2} \quad \text{for all} \ \alpha \in [-1, 1]. \quad (4.11)
\]

Finally, in the case \( |c| > 1 \), leading to Meixner polynomials of the first kind, we may use equation (2.8) to find

\[
\langle e_0, (\pi + k(-C) + c \cdot \pi + (H) + \pi + (B))^m e_0 \rangle.
\]

We omit the details for this case and mention only one, combinatorially interesting, subcase. For \( k = 1/2 \) and \( c = 3/\sqrt{8} \), the parameter \( \beta \) of the associated Meixner polynomial becomes 1, whereas the parameter \( 'c' \) becomes 1/2. As it is explained in ch. II, §8 of Viennot’s (1983) work, the unitary Meixner polynomials of the first kind \( \{\hat{m}_n(x;1,1/2)\}_{n \geq 0} \) are the Kreweras polynomials whose moments \( \mu_n \) are the numbers of bicoloured permutations of \( \{1,2,\ldots,n\} \) where the two colours are used to colour the descents of each permutation.

What we have seen thus far is that the application of corollary (3.6) to positive discrete series representations of \( \text{su}(1,1) \) is linked to the use of Laguerre polynomials and of Meixner polynomials of both kinds. The same orthogonal polynomial sequences need to be used to apply theorem (2.2) to the same representations. The only difference is that, in each case, instead of the moments we would need to use the inverse polynomials of the same OPS. We may find these inverse polynomials using equations (2.13) and (2.14). For brevity’s sake, in the rest of the section we only outline how to find the inverse polynomials of Laguerre and Meixner polynomials, and leave the adaptation of these data to theorem (2.2) to the reader.

The Laguerre polynomials \( \{L_n^{(\alpha)}(x)\}_{n \geq 0} \) satisfy eqn (29) in ch. II of Viennot’s (1983) work:

\[
\sum_{n \geq 0} L_n^{(\alpha)}(x) \frac{t^n}{n!} = (1 + t)^{-\alpha-1} \exp \left( x \frac{t}{1+t} \right), \quad (4.12)
\]

thus we must set \( f(t) = (1 + t)^{-\alpha-1} \) and \( g(t) = t/(1 + t) \) in equation (2.13). As indicated in table 4 of ch. III by Viennot (1983), this implies \( g^{(-1)}(t) = t/(1-t) \). Thus, the inverse polynomials of \( \{L_n^{(\alpha)}(x)\}_{n \geq 0} \) are given by

\[
\sum_{n \geq 0} Q_n(x) \frac{t^n}{n!} = \left( \frac{1}{1-t} \right)^{\alpha+1} \exp \left( x \frac{t}{1-t} \right). \quad (4.13)
\]

Hence, we have

\[
\mu_{n,d} = [x^d]Q_n(x) = \frac{n!}{d!} \sum_{j=0}^{n-d} (-1)^j \binom{-\alpha - 1}{j} (-d)^{n-d-j} \binom{n - d - j}{n - d} (-1)^{n-d-j} \quad = \frac{(-1)^n}{d!} \binom{-d - \alpha - 1}{n - d}.
\]

implying

\[
\mu_{n,d} = \binom{n}{d} (\alpha + 1 + d)_{n-d}. \quad (4.14)
\]
The generating function for the unitary Meixner polynomials \( \{ \hat{m}_n(x; \beta, c) \}_{n \geq 0} \) may be found in table 4 of ch. III by Viennot (1983):

\[
\sum_{n \geq 0} \hat{m}_n(x; \beta, c) \frac{t^n}{n!} = \left( 1 + \frac{tc}{1 - c} \right)^{-\beta} \left( \frac{1 - c + t}{1 - c + ct} \right)^x,
\]

thus we must set \( f(t) = (1 + tc/(1 - c))^{-\beta} \) and \( g(t) = \ln(1 - t + c) - \ln(1 - c + ct) \) in equation (2.13). As indicated in table 4 of ch. III by Viennot (1983), this implies \( g^{(-1)}(t) = (1 - c)(\exp(t) - 1)/(1 - c \exp(t)) \), and the inverse polynomials of \( \{ \hat{m}_n(x; \beta, c) \}_{n \geq 0} \) are given by

\[
\sum_{n \geq 0} Q_n(x) \frac{t^n}{n!} = \left( \frac{1 - c}{1 - c \exp(t)} \right)^\beta \exp \left( \frac{x(1 - c)(\exp(t) - 1)}{1 - c \exp(t)} \right). \tag{4.15}
\]

The Meixner polynomials of the second kind \( \{ M_n(x; \delta, \eta) \}_{n \geq 0} \) satisfy eqn (69) in ch. II of Viennot’s (1983) work:

\[
\sum_{n \geq 0} M_n(x; \delta, \eta) \frac{t^n}{n!} = ((1 + \delta t)^2 + t^2)^{-\eta/2} \exp \left( x \arctan \left( \frac{t}{1 + \delta t} \right) \right),
\]

thus we must set \( f(t) = ((1 + \delta t)^2 + t^2)^{-\eta/2} \) and \( g(t) = \arctan(t/(1 + \delta t)) \) in equation (2.13). As indicated in table 4 of ch. III by Viennot (1983), this implies \( g^{(-1)}(t) = \tan(t)/(1 - \delta \tan t) \), and the inverse polynomials of \( \{ M_n(x; \delta, \eta) \}_{n \geq 0} \) are given by

\[
\sum_{n \geq 0} Q_n(x) \frac{t^n}{n!} = \left( \frac{\sec(t)^2}{(1 - \delta \tan(t))^2} \right)^{\eta/2} \exp \left( x \frac{\tan(t)}{1 - \delta \tan(t)} \right). \tag{4.16}
\]

In particular, we have

\[
\mu_{n,d} = \frac{n!}{d!} [t^n] \sec(t)^\eta \tan(t)^d \quad \text{for} \ \delta = 0. \tag{4.17}
\]

5. Eigenvectors of closable tridiagonal operators

In this section, we return to general tridiagonal operators on the complex Hilbert space \( \ell^2(\mathbb{Z}_{\geq 0}) \), but it will be convenient to assume that they are closable. This assumption was not necessary in §3, where we were only interested in calculating the matrix entries in the first column for the powers of tridiagonal operators. Formally, we are still able to state theorem (5.1) below without assuming closability. However, the theorem implies that the eigenvectors of tridiagonal operators of the stated form must arise as infinite linear combinations of the basis vectors. In all applications, it seems easier to show that such vectors belong to the domain of the operator when the operator is closable.

As noted at the end of section 2.1 by Groenevelt & Koelink (2002), for any (positive or negative) discrete series representation \( \pi \) of \( \text{su}(1, 1) \), the operators \( \pi(B) \), \( \pi(C) \) and \( \pi(H) \) are closable. As a consequence, the operators \( T_c \) defined in equation (4.1) are closable. They are even self-adjoint, i.e. Jacobi operators. We refer the reader to Koelink (2004) for the spectral theory of Jacobi operators.

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This theory was used by Van Der Jeugt (1997), Koelink & Van Der Jeugt (1998) and Groenevelt & Koelink (2002) to obtain a complete description of the spectra of the operators $T_c$. The main theorem of this section generalizes only the necessary conditions stated in their results. Even this generalization suffices to highlight the fact: the (potential) eigenvectors of any (closable) tridiagonal operator $T$ may be expressed using the same OPS, whose moments and inverse polynomial coefficients may be used to express the first columns in the matrices of powers of $T$.

**Theorem 5.1.** Let $T$ be a tridiagonal operator with LDU decomposition $T = L + D + U$ such that $l_n u_n \neq 0$ holds for all $n \geq 0$ and let $z \in \mathbb{C}$ be any complex number. If $z$ is an eigenvalue of $T$, then the associated eigenspace is one dimensional, generated by

$$v_z := \sum_{n \geq 0} \frac{p_n(z)}{\prod_{i=0}^{n-1} u_i} e_n.$$

Here \( \{p_n(x)\}_{n \geq 0} \) is the sequence of monic polynomials defined by equations (2.1) and (2.2), satisfying $b_n = d_n$ and $\lambda_n = u_{n-1} l_{n-1}$.

**Proof.** A vector of the form $\sum_{n \geq 0} h_n \cdot e_n$ is an eigenvector of $T$ associated to the eigenvalue $z$ only if

$$d_0 h_0 + u_0 h_1 = zh_0 \quad (5.1)$$

and

$$l_{n-1} h_{n-1} + d_n h_n + u_n h_{n+1} = zh_n \quad \text{holds for } n \geq 1. \quad (5.2)$$

Introducing $p_0 := h_0$ and

$$p_n := h_n \cdot u_0 u_1 \cdots u_{n-1} \quad \text{for } n \geq 1,$$

we may rewrite equation (5.1) as

$$p_1 = (z - d_0) p_0, \quad (5.3)$$

while multiplying both sides of equation (5.2) by $u_0 u_1 \cdots u_{n-1}$ yields the equivalent equation

$$p_{n+1} = (z - d_n) p_n - u_{n-1} l_{n-1} p_{n-1} \quad \text{for } n \geq 1. \quad (5.4)$$

If $p_0 = 0$, then equations (5.3) and (5.4) imply $p_n = 0$ for all $n$. We obtain a non-zero eigenvector only if $p_0 \neq 0$ and then, without loss of generality, we may assume that $p_0 = 1$. Furthermore, equation (5.4) implies that $p_n$ must be obtained by substituting $z$ into a monic polynomial sequence $\{p_n(x)\}_{n \geq 0}$ satisfying Favard’s recursion formula with $b_n = d_n$ and $\lambda_n = u_{n-1} l_{n-1}$ for $n \geq 1$. Therefore, any eigenvector associated to $z$ is of the form stated in the theorem. 

**Remark 5.2.** In the special case when the tridiagonal operator $T$ is a Jacobi operator, $T$ is self-adjoint and thus has only real eigenvalues.

The connecting coefficients $\{\lambda_n\}_{n \geq 1}$ resp. $\{b_n\}_{n \geq 0}$ for the polynomials $\{p_n(x)\}_{n \geq 0}$ used in theorem (5.1) above are the same as the ones used in corollary (3.6) and theorem (3.8). Thus, the same OPS may be used to calculate the first column of matrix entries of the powers of $T$ as in the calculation of its eigenvectors. In particular, for any operator $T_c$ defined in equation (4.1) we must use the appropriate OPS $\{p[k, c]_n(x)\}_{n \geq 0}$ from proposition (4.2). This leads to the use of
Laguerre polynomials, and Meixner polynomials of both kinds, as in the above cited results of Van Der Jeugt (1997), Koelink & Van Der Jeugt (1998) and Groenevelt & Koelink (2002).

We conclude this section with an observation regarding the case $c = 0$ and $k = 1/2$, leading to $p[1/2, 0](x) = M_n(x; 0, 1)$, by proposition (4.2). A combinatorial model for the coefficients of the powers of $x$ in $M_n(x; 0, 1)$ is mentioned in example 21 of ch. II by Viennot (1983). Up to sign, the coefficient of $x^j$ in $M_n(x; 0, 1)$ is the number of permutations of $\{1, 2, \ldots, n\}$ having $j$ odd cycles. Thus, for this specific choice of parameters, not only the selected matrix entries of the powers of $T_0$, but also the description of the eigenvectors has a combinatorial interpretation. Recall that the same selection of parameters leads to the first part of equation (2.22), which is equivalent to the first part of equation (2.21) that also caught the attention of Hodges & Sukumar (2007). Therefore, if someone wanted to build a ‘purely combinatorial model’ of a quantum oscillator in the future, starting with $\alpha = 1$ in the model proposed Hodges & Sukumar (2007) or with $k = 1/2$ in the model proposed by Klimyk (2006), and focusing on understanding the action of the moment operator seems a very reasonable first step.

6. Beyond self-adjoint operators and beyond $su(1,1)$

Up to this point, our applications of theorems (3.5), (3.8) and (5.1) involved self-adjoint operators from discrete series representations of $su(1,1)$. In this final section, we outline two examples indicating that these theorems may also be useful in other applications where the operator is not self-adjoint or where it belongs to the representation of an algebra that is not $su(1,1)$.

Our first example is $\pi^+_k(B + C)$, where $\pi^+_k$ is a positive discrete series representation of $su(1,1)$ for some $k > 0$. In analogy to corollary (4.1), the matrix entry $\langle e_0, \pi^+_k(B + C)^m e_0 \rangle$ is the $m$th moment of the functional associated to the OPS $\{f^{(2k)}_n(x)\}_{n \geq 0}$ given by equations (2.1) and (2.2), where

$$b_n = 0 \quad \text{and} \quad \lambda_n = -n(n + 2k - 1).$$

It is easy to verify that we have

$$f^{(2k)}_n(x) = (-i)^n M_n(ix; 0, 2k),$$

where the polynomials $M_n(x; 0, 2k)$ are Meixner polynomials of the second kind. Thus, for $k \in (1/2)\mathbb{Z}_{>0}$, the polynomials $f^{(2k)}_n(x)$ are identical to the polynomials studied by Carlitz (1959). The applications of theorems (3.5) and (3.8) are no different from the self-adjoint setting. By theorem (5.1), a complex number $z \in \mathbb{C}$ is an eigenvalue $\pi^+_k(B + C)$ only if the expression

$$u^{(k)}_z := \sum_{n \geq 0} \frac{f^{(2k)}_n(z)}{\sqrt{(n-1)!(2k)_n}} e_n$$

represents a vector belonging to $l^2(\mathbb{Z}_{\geq 0})$. Then the discussion of the convergence issues is analogous to the case of $\pi^+_k(B - C)$ that was worked out by Koelink & Van Der Jeugt (1998): here we obtain that all eigenvalues belong to $i\mathbb{R}$. This is
not surprising, since \(i\pi \frac{k}{j}(B + C)\) is self-adjoint. However, the way we were able to handle the example indicates that the validity of our main results is not restricted to self-adjoint operators.

Our second example is the sum \(A + A^\dagger\), where \(A\) and \(A^\dagger\) are the operators defined in equation (2.1) of Sukumar \& Hodges (2007):

\[
A e_{2n} = \sqrt{2n} \cdot e_{2n-1}, \quad A e_{2n+1} = \sqrt{2n + 1 + \alpha} \cdot e_{2n}
\]

and

\[
A^\dagger e_{2n} = \sqrt{2n + 1 + \alpha} \cdot e_{2n+1}, \quad A^\dagger e_{2n-1} = \sqrt{2n} \cdot e_{2n}.
\]

The operator \(A + A^\dagger\) is self-adjoint, but it belongs to a representation of a Lie superalgebra properly containing \(su(1,1)\). Theorems (3.5), (3.8) and (5.1) are obviously applicable, and the OPS \(\{p_n(x)\}_{n \geq 0}\) appearing in all of them is given by equations (2.1) and (2.2) where

\[
b_n = 0, \quad \lambda_{2n+1} = 2n + 1 + \alpha \quad \text{and} \quad \lambda_{2n} = 2n.
\]

We obtain the generalized Hermite polynomials, see eqn (2.46) of ch. V in Chihara (1978). Choosing \(\alpha = 0\) yields the usual Hermite polynomials. Sukumar \& Hodges (2007) note that this choice corresponds to the standard harmonic oscillator. For non-zero \(\alpha\), the parity-dependent nature of the quantum algebra defined by Sukumar \& Hodges (2007) is reflected in the parity-dependent nature of the recursion defining the generalized Hermite polynomials.

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References

Carlitz, L. 1959 Bernoulli and Euler numbers and orthogonal polynomials. Duke Math. J. 26, 1–16. (doi:10.1215/S0012-7094-59-02601-8)

Chihara, T. S. 1978 An introduction to orthogonal polynomials. New York, NY: Gordon and Breach Science Publishers.

Groenevelt, W. 2004 Tensor product representations and special functions. Doctoral thesis, Delft University Press, Delft.

Groenevelt, W. \& Koelink, E. 2002 Meixner functions and polynomials related to Lie algebra representations. J. Phys. A 35, 65–85. (doi:10.1088/0305-4470/35/1/306)

Hodges, A. \& Sukumar, C. V. 2007 Bernoulli, Euler, permutations and quantum algebras. Proc. R. Soc. A 463, 2401–2414. (doi:10.1098/rspa.2007.0001)

Klimyk, A. U. 2006 The \(su(1,1)\)-models of quantum oscillator. Ukr. J. Phys. 51, 1019–1027.

Koekoek, R. \& Swarttouw, R. F. 1998 The Askey-scheme of hypergeometric orthogonal polynomials and its \(q\)-analogue. See http://fa.its.tudelft.nl/koekoek/askey/.

Koelink, E. 2004 Spectral theory and special functions, Laredo lectures on orthogonal polynomials and special functions. In Advances in the theory of special functions and orthogonal polynomials (eds R. Álvarez-Nodarse \et al.), pp 45–84. Hauppauge, NY: Nova Science Publishers.

Koelink, E. \& Van Der Jeugt, J. 1998 Convolutions for orthogonal polynomials from Lie and quantum algebra representations. SIAM J. Math. Anal. 29, 794–822. (doi:10.1137/S003614109630673X)

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Stanton, D. 2003 Enumeration and special functions. In *Orthogonal polynomials and special functions* (Leuven 2002). Lecture Notes in Mathematics, no 1817 (eds E. Koelink & W. Van Assche), pp 137–166. Berlin, Germany: Springer.

Sukumar, C. V. & Hodges, A. 2007 Quantum algebras and parity-dependent spectra. *Proc. R. Soc. A* **463**, 2415–2427. (doi:10.1098/rspa.2007.0003)

Van Der Jeugt, J. 1997 Coupling coefficients for Lie algebra representations and addition formulas for special functions. *J. Math. Phys.* **38**, 2728–2740. (doi:10.1063/1.531984)

Viennot, X. 1983 Une théorie combinatoire des polynômes orthogonaux généraux. Lecture Notes LACIM, UQAM, 219p. Montreal, Australia: LACIM, Université du Québec à Montréal (réed. 1991). See http://web.mac.com/xgviennot.

Vilenkin, N. Ja. & Klimyk, A. U. 1991 *Representation of Lie groups and special functions*, vol. 1. Dordrecht, The Netherlands: Kluwer Academic Publishers.