SHUFFLE ALGEBRA REALIZATION OF
QUANTUM AFFINE SUPERALGEBRA $U_v(\hat{D}(2, 1; \theta))$

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Abstract. Inspired by [T1], we give shuffle algebra realization of quantum affine superalgebra $U_v(\hat{D}(2, 1; \theta))$ with all simple root systems. We also give shuffle algebra realization of $U_v(\hat{sl}(2|1))$ with odd root system when $v$ is a primitive root of unity of even order, generalizing results in [FJMMT].

1. Introduction

Shuffle algebras are certain (anti)symmetric Laurent polynomials with prescribed poles satisfying the so called wheel conditions, and endowed with an associative algebra structure by shuffle product, which are first studied by the first author and Odesskii in [FO]. They are interesting because they are expected to give realizations of quantum affine and quantum toroidal algebras. The known examples are for type $A$ case. In [SV], Schiffmann and Vasserot constructed an isomorphism between the shuffle algebra associated to $\hat{A}_1$ and the positive part of the elliptic Hall algebra, or equivalently, the positive part of quantum toroidal $\tilde{U}_{v_1,v_2}(\hat{gl}(1))$ algebra, see also [N1] for more details. In [N2], Negut generalize this result to higher rank cases, and proved that the shuffle algebra associated to $\hat{A}_n$ is isomorphic to the positive part of quantum toroidal $\tilde{U}_{v_1,v_2}(\hat{gl}(n))$ algebra for $n \geq 2$. For other types of finite Dynkin diagram, the same problem has been studied in [E1] [E2], and still remains unsolved today.

It is interesting to even further consider the Dynkin diagrams associated to Kac-Moody superalgebras. In [T1], Tsymbaliuk gave the shuffle algebra realization for quantum affine superalgebra $U_v(\hat{sl}(m|n))$ with distinguished simple root system. His results suggest that in the super case, we should consider the antisymmetric rational functions instead of symmetric ones corresponding to the odd simple roots. Note that the Kac-Moody superalgebra admits nonisomorphic simple root systems, and they give different positive parts. Recently in [T2], Tsymbaliuk generalized results in [T1] to all simple root systems associated to $\hat{sl}(m|n)$ and gave shuffle algebra realizations of the corresponding quantum affine superalgebras, making the picture for $A(m|n)$ case complete.

In this paper, we give shuffle algebra realization of positive part of quantum affine superalgebra $U_v(\hat{D}(2, 1; \theta))$ associated to all simple root systems, see the proof of Theorem 3.2 and Theorem 3.3. The problem of giving shuffle algebra realization for quantum toroidal $\tilde{U}_{v_1,v_2}(\hat{D}(2, 1; \theta))$ algebra has been posed in [FJMM] to study the quantization of $\hat{sl}_2$ coset vertex operator algebra, and our motivations start from there.

We give an outline of our proofs and state the meaning of our results. First we define the shuffle algebra $\Omega$ associated to $\hat{D}(2, 1; \theta)$, by finding certain wheel conditions that are used to replace the role of quantum Serre relations in the quantum affine algebra $U_v(\hat{D}(2, 1; \theta))$. Then there is a natural morphism $\varphi$ from $\Omega$ to $U_v(\hat{D}(2, 1; \theta))$ in Drinfeld new realization. To prove the surjectivity of $\varphi$, following ideas in [T1], we construct certain ordered monomials of
quantum affine root vectors as PBW type elements in $U_v(\widehat{\mathfrak{D}}(2,1;\theta))$ and show their images under $\varphi$ constitute a bases for $\Omega$. The difficulty is that the standard specialization map used in [T1], which is one main tool when studying shuffle algebras in type $A$ cases, behaved badly in our case. We overcome this by defining a more complicated specialization map that is compatible with the wheel conditions in our setting. We believe that our results shine a light on solving the similar problem for any finite Dynkin diagrams.

To prove the injectivity of $\varphi$, we choose a different method from Tsymbaliuk’s. Similar to the type $A$ case considered in [HRZ], we note that in our case the ordered monomials of quantum affine root vectors also span the whole algebra, thus the linearly independence of their images in shuffle algebras would give us the injectivity of this morphism. While Tsymbaliuk’s idea is based on the existence of compatible nondegenerate pairings on both sides, see [T1, Proposition 3.4] and [N2] for more details.

As a byproduct, we construct PBW type bases for $U_v(\widehat{\mathfrak{D}}(2,1;\theta))$ in the Drinfeld realization, which shows the benefits of shuffle algebra realizations of quantum affine algebras. Note that the PBW bases for quantum affine algebras has been established a long time ago in the standard Drinfeld-Jimbo presentation, there seems to be missing in literatures a clear proof of PBW property for them in the Drinfeld realization, for more details on this see the introduction in [T1]. This proof of PBW property for quantum affine algebras by giving their shuffle algebra realization is a natural generalization of the usual proof of PBW bases theorem for universal enveloping algebras, which is by comparing it with the symmetric algebra.

The second main result of this paper is to give shuffle algebra realization of $U_v(\widehat{\mathfrak{sl}}(2\mid 1))$ with odd simple root system when $v$ is a primitive root of unity of even order. When $v$ is generic, shuffle algebras are generated by degree one elements. However, when $v$ is a primitive root of unity, the degree one elements only generate a subalgebra, and we need more wheel conditions to describe it. For example, the positive part of $U_v(\widehat{\mathfrak{sl}}(2))$ is isomorphic to the symmetric Laurent polynomials with shuffle product, and under this isomorphism the PBW bases correspond to Hall-Littlewood Laurent polynomials. When $v$ is specialized to a root of unity, $U_v(\widehat{\mathfrak{sl}}(2))$ consists of symmetric Laurent polynomials spanned by “admissible” Hall-Littlewood Laurent polynomials. It is proved that this subspace is governed by certain wheel conditions, see [FJMMT, Proposition 3.5]. For $U_v(\widehat{\mathfrak{sl}}(2\mid 1))$, its positive part is isomorphic to doubly antisymmetric Laurent polynomials with prescribed poles. We show when $v$ is a primitive root of unity of even order, $U_v(\widehat{\mathfrak{sl}}(2\mid 1))$ is also governed by similar wheel conditions, see the proof of Theorem 2.18.

When we initiate this work, the paper [T2] had not came out and the shuffle algebra realization of $U_v(\widehat{\mathfrak{sl}}(n\mid m))$ with non-distinguished simple root system was still unknown, so we gave a detailed proof of shuffle algebra realization for $U_v(\widehat{\mathfrak{sl}}(2\mid 1))$ with odd root system when $v$ is generic. We choose to preserve this part because our arguments differ from Tsymbaliuk’s arguments in some parts and it is also needed for other parts of this paper. Also it can be served as an introduction to shuffle algebras by studying an example with all details.

The paper is organized as follows. In Section 2, we define the shuffle algebra $\Lambda$ corresponding to odd simple root system of $\mathfrak{sl}(2\mid 1)$ and prove the isomorphism $\varphi: \Lambda \overset{\sim}{\to} U_v(\widehat{\mathfrak{sl}}(2\mid 1))$. When $v$ is a primitive root of unity of even order, we consider the subalgebra $\Lambda^w$ of $\Lambda$ generated by degree one elements, and prove that it is isomorphic to the subalgebra $\Lambda^w$ defined by certain wheel conditions. In Section 3, we give shuffle algebra realization of $U_v(\widehat{\mathfrak{D}}(2,1;\theta))$ associated to all simple root systems and prove their PBW property.
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2. SHUFFLE REALIZATION OF $U_v^\subset (\widehat{\mathfrak{sl}(2|1)})$

2.1. $U_v^\subset (\widehat{\mathfrak{sl}(2|1)})$ and a spanning set. Consider the free $\mathbb{Z}$-module $\oplus_{i=1}^{3} \epsilon_i$ with bilinear form $(\epsilon_i, \epsilon_j) = (-1)^{\delta_{i,j}} \delta_{i,j}$. Instead of the distinguished simple root system $\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}$, we choose the simple roots to be $\{\alpha_1 = \epsilon_1 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_2\}$, which both are odd roots. The positive roots are $\Psi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. The Cartan matrix is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Following [Y, Theorem 8.5.1], in the Drinfeld realization, $U_v^\subset (\widehat{\mathfrak{sl}(2|1)})$ is the quantum superalgebra over $\mathbb{C}(v)$ with generators $\{p_i, q_i, i \in \mathbb{Z}\}$ and relations

$$p_ip_j + p_jp_i = 0,$$
$$q_iq_j + q_jq_i = 0,$$
$$p_{i+1}q_j + vp_{j+1}q_{i+1} - vq_{j+1}p_i = -vp_{i+1}q_{j+1} - q_{j+1}p_i,$$

(2.1)

here the parity of generators are given by $p(p_i) = p(q_j) = 1$ and we denote by $[x, y]_v := xy - (-1)^{|x||y|} vxy$ the super bracket. We will simply write $[x, y]$ for $[x, y]_1$. The following formulae can be directly checked from the above defining relations (2.1).

Lemma 2.1. (1) $qSdk = v[p_k, q_s]_{v-1} - vq_kq_s$.
(2) $[p_k, q_s]_{v-1} + v[p_{k+1}, q_{s-1}]_{v-1} = (v - v^{-1})p_{k+1}q_{s-1}$.
(3) $q[p_k, q_s]_{v-1} = v[p_k, q_s]_{v-1}$.
(4) $[p_k, q_s]_{v-1} = vp_k[p_k, q_s]_{v-1}$.

We will also frequently using the following formulae for super bracket, see [Y, 6.9].

Lemma 2.2 (6.9, [Y]). Let $U$ be a superalgebra over $\mathbb{C}(v)$. For any $X, Y, Z \in U$ and $a, b, c \in \mathbb{C}(v)$, we have

$$[[X, Y]_a, Z]_b = [X, [Y, Z]_a]_b + (1)^{p(X)p(Y)}[X, Z]_{b^{-1}}Y^1,$$
$$[X, [Y, Z]_a]_b = [X, [Y, Z]_a]_b + (1)^{p(X)p(Y)}[Y, [X, Z]_{b^{-1}}]_{a^{-1}}.$$

(2.2)

Following [T1, Subsection 2.2], let $r_i = [p_i, q_i]_{v-1}$, see also [HRZ, Definition 3.9], [Z, Definition 3.11]. Then $\{p_i, q_j, r_k\}_{i, j, k \in \mathbb{Z}}$ are quantum affine root vectors corresponding to positive roots. Let $H$ be the set of functions $h: \Psi^+ \times \mathbb{Z} \rightarrow \mathbb{N}$ with finite support and such that $h(\alpha_i, k) \leq 1$. For each $h \in H$ we have the ordered monomial $E_h = \prod_{i \in \mathbb{Z}} P_i^{h(\alpha_i, i)} \prod_{j \in \mathbb{Z}} r_j^{h(\alpha_1 + \alpha_2, j)} \prod_{k \in \mathbb{Z}} q_k^{h(\alpha_2, k)}$. Let $U' \subset U_v^\subset (\widehat{\mathfrak{sl}(2|1)})$ be the spanning set of these $E_h$ over $\mathbb{C}(v)$.

Proposition 2.3. For any $i, j, k, s \in \mathbb{Z}$, the elements $[p_i, q_j]_{v-1}, [p_i, r_k]_{v-1}, [q_j, r_k]_{v-1}, [r_k, r_s]_{v^2}$ are all belonging to $U'$.

Proof. We can assume $i, j \geq k \geq s \geq 0$, other cases are similar. First by Lemma 2.1 (2) we have $[p_i, q_j]_{v-1} = (-v)^i r_{i+j} + (v - v^{-1}) \sum_{k=1}^{j} (-v)^{k-1} p_{i-k}q_{j+k}$. Hence $[p_i, q_j]_{v-1} \in U'$. Next by Lemma 2.1 (2),(4) we have $[p_i, r_k]_{v-1} = (v^{-1} - v) \sum_{l=1}^{i-k} (-v)^{l-1} p_{k+l}q_{i-l}$. Hence $[p_i, r_k]_{v-1} \in U'$, and similarly by Lemma 2.1 (2),(3) we get $[q_j, r_k]_{v-1} \in U'$. Finally, we deal
with \([r_k, r_s]_{v^2}\). Furthermore, we can assume \(s = 0\). By Lemma 2.1 and Lemma 2.2 we have
\[
[r_k, r_0]_{v^2} = [p_k, q_0]_{v-1}, [p_0, q_0]_{v-1}]_{v^2}
\]
\[
= [p_k, q_0, [p_0, q_0]_{v-1} v] + v[[p_k, [p_0, q_0]_{v-1}] v, q_0]_{v-2}
\]
\[
= -[[p_k, [p_1, q-1] v] v, q_0]_{v-2}
\]
\[
= v[[p_1, r_{k-1}] v, q_0]_{v-2}
\]
\[
= [r_1, r_{k-1}]_{v^2} + v[p_1, [r_{k-1}, q_0]_{v-1}]
\]
\[
= [r_1, r_{k-1}]_{v^2}.
\]
Hence by induction on \(k\) we know \([r_k, r_s]_{v^2}\) all belong to \(U'\).

Now we have our main theorem of this subsection, that \(U'\) actually equals to \(U_{v'}^{\tilde{\mathfrak{sl}}(2|1)}\).

**Theorem 2.4.** The ordered monomials \(E_h = \prod_{i \in \mathbb{Z}} P_i \in \mathfrak{h}(\tilde{\mathfrak{sl}}(2|1))\) is a spanning set of \(U_{v'}^{\tilde{\mathfrak{sl}}(2|1)}\).

**Proof.** By the commutation relations given in Proposition 2.3, it is clear that any element of \(U_{v'}^{\tilde{\mathfrak{sl}}(2|1)}\) can be written by a linear combination of \(E_h\). \(\square\)

### 2.2. Shuffle algebra \(\Lambda\)

Let \(\Lambda = \bigoplus_{n,m \in \mathbb{N}} \Lambda_{n,m}\) be graded vector spaces over \(\mathbb{C}(v)\), where \(\Lambda_{n,m}\) consists of rational functions \(F\) in the variables \(\{x_1, \ldots, x_n, y_1, \ldots, y_m\}\) and satisfies the following conditions:

1. \(F\) is skew symmetric with respect to \(\{x_i\}_{i=1}^n\) and \(\{y_j\}_{j=1}^m\).
2. \(F = \frac{f}{\prod_{i,j}(x_i - y_j)}\), where \(f \in \mathbb{C}(v)[x_i^{\pm 1}, y_j^{\pm 1}]\) is a Laurent polynomial.

Denote by \(\mathfrak{S}_n\) the symmetric group of order \(n\). For \(F \in \Lambda_{k_1,l_1}, G \in \Lambda_{k_2,l_2}\), we define the shuffle product \(F \star G \in \Lambda_{k_1+k_2,l_1+l_2}\) as
\[
F \star G = \text{ASym}_{\mathfrak{S}_{k_1+k_2} \times \mathfrak{S}_{l_1+l_2}}\left(F\left(\{x_i, y_j\}_{1 \leq i \leq k_1}\right) \cdot G\left(\{x_i, y_j\}_{k_1 \leq i \leq k_1+k_2}\right)\right)
\]
\[
= \prod_{1 \leq i \leq k_1} \frac{x_i + v^{-1}y_j}{x_i - y_j} \prod_{k_1 + 1 \leq i \leq k_1 + k_2} \frac{y_j + v^{-1}x_i}{y_j - x_i},
\]
where \(\text{ASym}\) means anti-symmetrization with respect to \(\{x_i\}_{1 \leq i \leq k_1+k_2}\) and \(\{y_j\}_{1 \leq j \leq l_1+l_2}\). Standardly, we have

**Proposition 2.5.** Under the shuffle product, \(\Lambda\) is an associative \(\mathbb{C}(v)\)-algebra.

**Proof.** See the proof of [FHHSY, Lemma 2.3]. \(\square\)

### 2.3. Isomorphism between \(U_{v'}^{\tilde{\mathfrak{sl}}(2|1)}\) and \(\Lambda\)

There is a natural \(\mathbb{C}(v)\)-algebra morphism \(\varphi\) from \(U_{v'}^{\tilde{\mathfrak{sl}}(2|1)}\) to \(\Lambda\). Our aim is to prove \(\varphi\) is actually an isomorphism.

**Proposition 2.6.** \(p_i \mapsto x^i, q_j \mapsto y^j\) induces a \(\mathbb{C}(v)\)-algebra morphism \(\varphi: U_{v'}^{\tilde{\mathfrak{sl}}(2|1)} \to \Lambda\).

**Proof.** This is straightforward to check. In particular, by checking the third relation in (2.1), under \(\varphi\) the quantum affine root vector \(r_k\) has the following explicit from \(\varphi(r_k) = \frac{1-v^{-2}}{x-y} x^{k+1}\). \(\square\)
Lemma 2.7. We have
\[
\frac{1}{x-y} \cdots \frac{1}{x-y} \frac{y^0 \ast y^1 \ast \cdots \ast y^{k-1}}{n} = c(v) \cdot \prod_{1 \leq i < j \leq n} (x_i - x_j) \prod_{1 \leq t < \ell < n+k} (y_t - y_\ell) \prod_{1 \leq i \leq n} (x_i - y_j),
\]
where \( c(v) \neq 0 \). Similar result holds also for \( x^0 \ast x^1 \ast \cdots \ast x^{k-1} \frac{1}{x-y} \cdots \frac{1}{x-y} \frac{y^0 \ast y^1 \ast \cdots \ast y^{k-1}}{n} \).

Proof. By definition, under anti-symmetrization, \( \prod_{i < j} (x_i - x_j) \prod_{k < l} (y_k - y_l) \) is a factor. And by comparing degrees between the two sides, we know it is the only factor. So we only need to prove \( c(v) \neq 0 \), i.e., \( \text{Asym}(\prod_{i < j} (x_i + v^{-1} y_j)) \prod_{i < j} (y_i + v^{-1} x_j)) y_{n+2} \cdots y_{n+k+1} \neq 0 \).

First, when \( k = 0 \), let \( \Gamma_n = \prod_{1 \leq i < j \leq n} (x_i + v^{-1} y_j)(y_i + v^{-1} x_j) \), then the coefficient of \( x_1^{n-1} y_1^{n-1} \) in \( \text{Asym}(\Gamma_n) = (1 + v^{-2} + \cdots + v^{-2n+2}) \text{Asym}(\Gamma_{n-1}(x_2, \ldots, x_n, y_2, \ldots, y_n)) \), now by induction on \( n \) we are done. When \( k > 0 \), let \( \Delta_{n,k} = \prod_{1 \leq i < j < n} (x_i - x_j) \prod_{1 \leq i < j \leq n+k} (y_i - y_j) \), then the coefficient of \( y_1^{n+k-1} \) in \( \text{Asym}(\Delta_{n,k-1}(x_1, \ldots, x_n, y_1, \ldots, y_{n+k-1})) \prod_{1 \leq i \leq n} (x_i + v^{-1} y_{n+k}) y_{n+k+1} \) is \( (-1)^n v^{-n} \Delta_{n,k-1} \), hence by induction on \( k \) we are done. \( \square \)

Proposition 2.8. \( \Lambda \) is generated by \( \{x, y\}_{i,j \in \mathbb{Z}} \), i.e., \( \varphi \) is surjective.

Proof. We need to prove for each monomial \( m(a_1, \ldots, a_n, b_1, \ldots, b_m) = x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_m^{b_m}, a_i, b_j \in \mathbb{Z}, \prod_{i < j} (x_i - x_j) \prod_{k < l} (y_k - y_l) \text{Sym}(m(a_1, \ldots, a_n, b_1, \ldots, b_m)) \in \Lambda_{n,m} \), where \( \text{Sym} \) denotes symmetrization with respect to \( \mathfrak{S}_n \times \mathfrak{S}_m \). We can assume \( n \leq m \) and \( m = n + k \). For each \( \sigma = \pi \times \tau \in \mathfrak{S}_n \times \mathfrak{S}_m \), let \( F_\sigma \in \Lambda_{n,m} \) be
\[
F_\sigma = \frac{x^{a_{\sigma(1)}} \cdots x^{a_{\sigma(n)}} y^{b_{\sigma(1)}} \cdots y^{b_{\sigma(n)}}}{x-y} \cdots \frac{x^{a_{\sigma(n)}} \cdots x^{a_{\sigma(n)}} y^{b_{\sigma(n)}}}{x-y} \cdots \frac{y^{b_{\sigma(n+1)+1}}}{y^{b_{\sigma(n+1)+1}}} \cdots \frac{y^{b_{\sigma(n+k)+1}}}{y^{b_{\sigma(n+k)+1}}}.
\]
Then
\[
\sum_{\sigma \in \mathfrak{S}_n \times \mathfrak{S}_m} F_\sigma = \text{Sym}(m) \cdot \text{ASym}(\prod_{i < j} (x_i + v^{-1} y_j)) \prod_{i < j} (y_i + v^{-1} x_j) y_{n+2} \cdots y_{n+k+1}) \prod_{i \leq n} (x_i - y_j).
\]

By Lemma 2.7 we get the desired element. \( \square \)

Now by Theorem 2.4 and the surjectivity of \( \varphi \), we know \( \{\varphi(E_h)\}_{h \in H} \) span the shuffle algebra \( \Lambda \). Thus proving their linear independence would gives us the linear independence of \( \{E_h\}_{h \in H} \) and that \( \varphi \) is actually an isomorphism. The following proposition follows from the technique of specialization introduced in [T1], we will treat this particular case as an example to explain it. Note that this technique of specialization has been frequently used in the studies of shuffle algebras, see [FHHSY], [N2].

Proposition 2.9. \( \{\varphi(E_h)\}_{h \in H} \) are linearly independent in \( \Lambda \).

Proof. For any \( h \in H \), denote its degree by \( \deg(h) = d = (d_1, d_2, d_3) \in \mathbb{N}^3 \) such that \( d_1 = \sum_{k \in \mathbb{Z}} h(a_1, k), d_3 = \sum_{k \in \mathbb{Z}} h(a_2, k), d_2 = \sum_{k \in \mathbb{Z}} h(a_1 + a_2, k) \). If \( \deg(h) = (d_1, d_2, d_3) \), we call \( \text{gr}(h) = (d_1 + d_2, d_2 + d_3) \in \mathbb{N}^2 \) its grading and we have \( \varphi(E_h) \in \Lambda_{\text{gr}(h)} \). We hope to prove in each graded part \( \Lambda_{n,m} \), the elements \( \{\varphi(E_h), \text{gr}(h) = (n,m)\} \) are linearly independent. For any \( h, h' \in H \) such that \( \text{gr}(h) = \text{gr}(h') = (n,m) \), we say \( \deg h' < \deg h \) if \( d_1' < d_1 \), and it induces a complete order on the set of degree of functions that has grading \( (n,m) \) and we list
them as $D_{n,m} = \{d_1 < \cdots < d_j\}$. Now for $f \in \Lambda_{n,m}$ and $d \in D_{n,m}$, define the specialization 
\[ \phi_d(f) \in \mathbb{C}(v)(z_{1,1}, \ldots, z_{1,d_1}, z_{2,1}, \ldots, z_{2,d_2}, z_{3,1}, \ldots, z_{3,d_3}) \]
by specializing:
\[ x_i \mapsto z_{1,i}, 1 \leq i \leq d_1 \]
\[ x_i \mapsto z_{2,i-d_1}, d_1 + 1 \leq i \leq n \]
\[ y_j \mapsto -v z_{2,j}, 1 \leq j \leq d_2 \]
\[ y_j \mapsto -v z_{3,j-d_2}, d_2 + 1 \leq j \leq m \]
(2.5)

Note that $\phi_{d_1}$ is just identity, and $\phi_{d_{d-1}}$ can be seen as first specialize by $\phi_{d}$ and then specialize $z_{1,d_1}$ and $y_{d_2+1}$ both to $z_{2,d_2+1}$. In addition we let $\phi_{d_0}$ be the zero map. Hence we have a filtration on $\Lambda_{n,m}$:
\[ \Lambda_{n,m} = \text{Ker}(\phi_{h_0}) \supset \cdots \supset \text{Ker}(\phi_{h_{d-1}}) \supset \text{Ker}(\phi_{h_d}) = 0. \]

Now for any $a_1 < \cdots < a_{d_1}, b_1 \leq \cdots \leq b_{d_2}, c_1 < \cdots < c_{d_3}$, $\varphi(E_h)$ equals to
\[
\text{Asym}_{\mathfrak{S}_n \times \mathfrak{S}_m}(x_1^{a_1} \cdots x_{d_1}^{b_1+1} \cdots x_n^{b_3+1}, y_{d_2+1}^{c_1} \cdots y_m^{c_3}) \prod_{i \leq d_1} (x_i + v^{-1} y_j) \prod_{i > d_1} (x_i + y_j) \prod_{1 \leq i \leq j \leq d_2} (x_{i+d_1} + y_{i+j}) / \prod_{1 \leq i < j \leq d_2} (x_i - y_j),
\]
(2.6)
it is a sum of terms corresponding to elements of $\mathfrak{S}_n \times \mathfrak{S}_m$. Without loss of generality, we will focus on the numerator part. First let us compute $\phi_{d}(\varphi(E_h))$. In this case the terms which do not specialize to zero are corresponding to those $\sigma \times \tau \in \mathfrak{S}_n \times \mathfrak{S}_m$ such that $\sigma = \sigma_1 \times \pi, \tau = \pi \times \tau_1$, where $\sigma_1 \in \mathfrak{S}_{d_1}, \pi \in \mathfrak{S}_{d_2}, \tau_1 \in \mathfrak{S}_{d_3}$. Hence we have $\phi_{d}(\varphi(E_h)) = Z \cdot \text{Asym}_{\mathfrak{S}_{d_1}}(a_{d_1}^{a_1} \cdots z_{1,d_1}^{a_{d_1}}) \cdot \text{Asym}_{\mathfrak{S}_{d_3}}(b_{d_3}^{c_1} \cdots z_{3,d_3}^{c_{d_3}}) \prod_{1 \leq i < j \leq d_2} (z_{2,i} - v z_{2,j})$.

(2.7)

where $Z = c \cdot \prod_{1 \leq i < j \leq d_2} (z_{2,i} - z_{2,j})(z_{2,i} - z_{3,k})(z_{2,j} - z_{3,k}) \prod_{1 \leq i < j \leq d_2} (z_{2,i} - z_{3,j})^2$ is a common factor for all $h$, and $c \in \mathbb{C}[v, v^{-1}]$ is some nonzero constant. From the above formula we know if we take all $h \in H$ such that $\deg h = d$, then under $\phi_d$ the images of $\{\varphi(E_h)\}$ are linearly independent, hence $\{\varphi(E_h)\}_{\deg h = d}$ is linearly independent set. It is also clear from the above explicit formula for $\varphi_d(E_h)$ that for $d' < \deg h$, we have $\phi_{d'}(\varphi(E_h)) = 0$. Hence for any $\deg h = d_k$ we have $\varphi(E_h) \subset \text{Ker}(\phi_{d_{k-1}}) - \text{Ker}(\phi_{d_k})$ and thus collect them all the set $\{\varphi(E_h)\}_{h \in D_{n,m}}$ is linearly independent.

\[ \square \]

**Remark 2.10.** Using the above specialization map $\phi_{d}$, we can also give a proof of surjectivity of $\varphi$, see [T1, Lemma 3.19]. For any $F \in \Lambda_{n,m}$, let $\tilde{h} \in H$ be such that $\deg(\tilde{h}) = (d_1, d_2, d_3)$ and $\text{gr}(h) = (n, m)$. Without loss of generality, we assume $n \leq m$. If $\phi_{d'}(F) = 0$ for any $d' < d$, then we see $\phi_{d}(F)$ has $\prod_{1 \leq i < j < d_2} (z_{2,i} - z_{2,j})(z_{2,i} - z_{3,k})$ as a factor. Since $\phi_{d}$ can be seen as first specialize by $\phi_{d_1}$ and then specialize $z_{1,d_1}$ and $z_{3,d_2+1}$ both to $z_{2,d_2+1}$, hence $\phi_{d}(F) = 0$ gives the factor $z_{1,d_1} - z_{3,d_2+1}$, and taking symmetrization gives us the desired factor. Moreover since $F$ is antisymmetric, we see $\phi_{d}(F)$ is exactly some linear combination of elements $\phi_{d}(\varphi(E_h))$. Note that if we choose $d = \tilde{d_1}$, that is if $d_1 = 0$, then $\phi_{d}(F)$ automatically has the factor $\prod_{1 \leq i < j < d_2} (z_{2,i} - z_{2,j})^2 \prod_{1 \leq i < j < d_2} (z_{2,j} - z_{3,k})$. Thus there is some $G_1$ which is some linear
combinations of $\varphi(E_h)$ such that $\deg h = d_1$ and $\phi_{d_1}(F) = \phi_{d_1}(G_1)$, that is $\phi_{d_1}(F - G_1) = 0$. Hence we have $G_2$ that is some linear combination of $\varphi(E_h)$ such that $\deg h = d_2$ and $\phi_{d_2}(F - G_1) = \phi_{d_2}(G_2)$. Repeat this procedure, we have $G_1, \ldots, G_l$ which are all linear combinations of $\varphi(E_h)$ such that $\phi_{d_l}(F) = \phi_{d_l}(G_1 + \cdots + G_l)$. Hence $F$ is some linear combination of $\varphi(E_h)$ and $\varphi$ is surjective. Similar arguments can also prove the surjectivity for $U^>_{\hat{v}}(\mathfrak{S}(2, 1; \theta))$, once we define the appropriate specialization map, see the proof of Theorem 3.2 and Theorem 3.3.

**Corollary 2.11.** $\{E_h\}_{h \in H}$ are PBW type bases for $U^>_{\hat{v}}(\mathfrak{sl}(2|1))$.

**Proof.** By Theorem 2.4 we know they span the whole $U^>_{\hat{v}}(\mathfrak{sl}(2|1))$, and by Proposition 2.9 we know they are linearly independent. \hfill \Box

**Theorem 2.12.** $\varphi: U^>_{\hat{v}}(\mathfrak{sl}(2|1)) \rightarrow \Lambda$ is an isomorphism.

**Proof.** Since $\{E_h\}_{h \in H}$ are basis and $\{\varphi(E_h)\}$ are linearly independent, $\varphi$ is injective. \hfill \Box

### 2.4. When $v$ is a primitive root of unity

When $v$ is generic, we see in subsection 2.3 that $\Lambda$ is generated by $\Lambda_1 = \Lambda_{0,1} \oplus \Lambda_{1,0}$. Never the less, when $v$ is a primitive root of unity, the algebra $\Lambda^\ast$ generated by $\Lambda_1$ is only a subalgebra of $\Lambda$.

This subsection generalizes results from [FJMMT]. Denote by $S = \oplus_k S_k$ the vector space of symmetric Laurent polynomials over $\mathbb{C}(v)$. For $F \in S_k, G \in S_l$, we can define the shuffle product $F \star G \in S_{k+l}$ as

$$F \star G = \text{Sym}_{\mathfrak{S}_{k+l}} \left( F(\{x_i\}_{1 \leq i \leq k}) \cdot G(\{x_j\}_{k \leq j \leq k+l}) \cdot \prod_{i \leq k}^{j > k} \frac{x_i - vx_j}{x_i - x_j} \right).$$

Using this shuffle product $S$ becomes an associative algebra. For any partition $\lambda$ of length $n$, the Hall-Littlewood polynomial with $n$ variables is defined as

$$P_{\lambda}(x_1, \ldots, x_n; v) = \text{Sym}_{\mathfrak{S}_n} \left( x_{\lambda_1} \cdots x_{\lambda_n} \prod_{1 \leq i < j \leq n} \frac{x_i - vx_j}{x_i - x_j} \right),$$

and it is well known that the Hall-Littlewood polynomials form a $\mathbb{C}(v)$ basis of $S$. By definition of this shuffle product, we have $P_{\lambda}(x_1, \ldots, x_n; v) = x_{\lambda_1} \ast \cdots \ast x_{\lambda_n}$, hence $S$ is generated by degree one elements $\{x^i, i \in \mathbb{Z}\}$. When $v$ is specialized to a primitive root of unity of order $t$, $F \in S$ is said to satisfy the wheel condition if $F(x, vx, \ldots, v^{t-1}x, x_{t+1}, \ldots) = 0$ for any $x \in \mathbb{C}$. Denote by $S^w \subset S$ the subspace consisting of elements satisfying the wheel condition. For a partition $\lambda$ of length $n$, denote by $m_i(\lambda)$ the number of $i$ appearing in $\lambda$, we say $\lambda$ is admissible if $m_i(\lambda) \leq t - 1$ for any $i \in \mathbb{Z}$. One main result of [FJMMT] is

**Theorem 2.13** (Proposition 3.5, [FJMMT]). When $v$ is a primitive root of unity of order $t$, the Hall-Littlewood polynomials $P_{\lambda}$ in which $\lambda$ is admissible form a basis of $S^w$ over $\mathbb{C}$.

Now let $v$ be a primitive root of unity of order $2t$, by the proof of Lemma 2.7, we know for any $k \in \mathbb{Z}$

$$\underbrace{x^k \ast \cdots \ast x^k}_{m} = 0,$$

if and only if $m \geq t$. Hence
Proposition 2.14. When \( v \) is a primitive root of unity of order \( 2t \), \( \{ \varphi(E_h), h \in H, h(\alpha_1 + \alpha_2, l) \leq t - 1 \} \) form a \( C \) basis of \( \Lambda^\xi \).

Similar to [FJMMT], we prove that \( \Lambda^\xi \) is also governed by certain wheel conditions.

Definition 2.15. When \( v \) is a primitive root of unity of order \( 2t \), \( F \in \Lambda \) is said to satisfying the wheel condition if \( F(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0 \) once \( \frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_t}{y_t} = -v^{-1} \). We denote this set by \( \Lambda^w \).

Proposition 2.16. \( \Lambda^w \) is a subalgebra of \( \Lambda \) under shuffle product. Hence \( \Lambda^\xi \) is a subalgebra of \( \Lambda^w \).

Proof. Let \( F, G \in \Lambda^w \), we shall prove that each term of (2.4) is zero under the specialization \( \frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_t}{y_t} = -v^{-1} \). Note that each term is corresponding to a permutation \( \sigma \times \tau \). For \( 1 \leq i \leq k_1 + k_2 \), we denote \( \text{sgn}(i) = 1 \) if \( 1 \leq i \leq k_1 \) and \( \text{sgn}(i) = 2 \) otherwise. Define similarly for \( 1 \leq j \leq l_1 + l_2 \). Now let \( \sigma \in \mathfrak{S}_{k_1+k_2}, \tau \in \mathfrak{S}_{l_1+l_2} \), we see if there is some \( 1 \leq i \leq t \) such that \( \text{sgn}(\sigma^{-1}(i)) = 1, \text{sgn}(\tau^{-1}(i)) = 2 \) or \( \text{sgn}(\sigma^{-1}(i+1)) = 2, \text{sgn}(\tau^{-1}(i)) = 1 \), then this term is specialized to zero. Otherwise it must happen that \( \text{sgn}(\sigma^{-1}(i)) = \text{sgn}(\tau^{-1}(i)) \) for all \( 1 \leq i \leq t \), then since \( F, G \in \Lambda^w \) this term is also specialized to zero. Here \( \sigma^{-1}(t+1) = \sigma^{-1}(1) \).

Certainly, if \( F \in \Lambda^w_{n,m} \) and \( \min\{n,m\} < t \), then \( F \in \Lambda^\xi \). Slightly further, we have

Proposition 2.17. If \( F \in \Lambda^w_{t,t} \), then \( F \in \Lambda^\xi \).

Proof. If \( t = 1 \) it is trivial, hence we assume \( t \geq 2 \). Since each \( F \in \Lambda^w \) is of the form \( \prod_{j \neq i} (x_i - x_j) \prod_j (y_k - y_l) \cdot g \) and \( g \) is a symmetric polynomial with respect to \( \{x_i\} \) and \( \{y_j\} \), we know \( F \) satisfies the wheel condition if and only if \( g \) satisfies the wheel condition. Let \( \{\chi_1, \ldots, \chi_t\}, \{\psi_1, \ldots, \psi_t\} \) be separately the elementary symmetric polynomials of \( \{x_1, \ldots, x_t\} \) and \( \{y_1, \ldots, y_t\} \), then \( g = G(\chi_1, \ldots, \chi_t, \psi_1, \ldots, \psi_t) \) for some polynomial \( G \). The wheel condition says \( g(x, \ldots, v^{2t-2}x, -vx, \ldots, -v^{2t-1}x) = 0 \), it is equivalent to \( G(0, \ldots, 0, (-1)^{t-1}x^t, 0, \ldots, 0, x^t) = 0 \) for any \( x \in \mathbb{C} \). Hence \( g \) satisfies the wheel condition if and only if \( g \) belongs to the ideal generated by \( \{\chi_1, \ldots, \chi_{t-1}, \psi_1, \ldots, \psi_{t-1}, \chi_t + (-1)^{t} \psi_t\} \).

Now it is easy to check that for \( 1 \leq r < t \),
\[
\frac{x}{x-y} \cdots \frac{x}{x-y} \frac{1}{x-y} \cdots \frac{1}{x-y} = \frac{c(v) \cdot \chi_r \cdot \Delta_{t,0}}{\prod (x_i - y_j)},
\]
\[
\frac{1}{x-y} \cdots \frac{1}{x-y} \frac{y}{x-y} \cdots \frac{y}{x-y} = \frac{c(v) \cdot \psi_r \cdot \Delta_{t,0}}{\prod (x_i - y_j)},
\]
\[
\frac{x+y}{x-y} \cdots \frac{x+y}{x-y} \frac{x}{x-y} \cdots \frac{x}{x-y} = \frac{c(v) \cdot (\chi_t + (-1)^{t} \psi_t + L) \cdot \Delta_{t,0}}{\prod (x_i - y_j)},
\]
where \( L \) belongs to the ideal generated by \( \{\chi_1, \ldots, \chi_{t-1}, \psi_1, \ldots, \psi_{t-1}\} \). Now by the proof of Proposition 2.8, we know if \( F \in \Lambda^\xi_{n,m} \), then for any symmetric polynomial \( G \in S_{n,m} \) we have \( G \cdot F \in \Lambda^\xi \). Hence the above elements also generate an ideal and it equals to \( \Lambda^w_{t,t} \).

Viewing Proposition 2.17 as a toy model and starting point of induction, we can now prove the general case.
Theorem 2.18. When specializing \( v \) to the primitive root of unity of order \( 2t \), for any \( f \in \Lambda \), \( f \) belongs to \( \Lambda^c \) if and only if \( f \) satisfies the wheel condition.

Proof. We will focus on the symmetric factor. For \( k,l \geq 0 \), let \( F \in \Lambda_{t+k,t+l} \), then the corresponding symmetric factor \( g \) satisfies the wheel condition if and only if \( g \) belongs to the ideal generated by \( \{ \chi_{k+1}, \ldots, \chi_{t+k-1}, \psi_{t+1}, \ldots, \psi_{t+l-1}, \chi_{t+k} \psi_k + (-1)^l \chi_{l+1} \} \), here \( \chi_0 = \psi_0 = 1 \). Now it is easy to check that \( \chi_{t+i} \) is generated by shuffle product of \( x \) and \( \psi_{t+j} \) is generated by shuffle product of \( \psi_{t+j-1} \) and \( y \). By Proposition 2.17 and induction on \( k,l \), we get \( \Lambda^c = \Lambda^w \).

3. SHUFFLE REALIZATION OF \( U_v^>(\mathfrak{G}(2, 1; \theta)) \)

3.1. Drinfeld realization and spanning set. The exceptional Lie superalgebras \( \mathfrak{G}(2,1; \theta) \) with \( \theta \in \mathbb{C} \) and \( \theta \neq 0, -1 \) form a one-parameter family of superalgebras of rank 3 and dimension 17. There are four different simple root systems and corresponding Dynkin diagrams, first we choose the completely fermionic one. Namely the simple roots are \( \{ \alpha_1, \alpha_2, \alpha_3 \} \) with parities \( p(\alpha_i) = 1 \) for \( i = 1, 2, 3 \) and Cartan matrix \( A = (a_{ij})_{1 \leq i,j \leq 3} \) where

\[
A = \begin{pmatrix}
0 & 1 & \theta \\
1 & -\theta & -1 \\
\theta & -\theta -1 & 0
\end{pmatrix}.
\]

The positive roots are \( \Psi^+ = \{ \alpha_1 < \alpha_1 + \alpha_3 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \alpha_2 < \alpha_2 + \alpha_3 < \alpha_3 \} \) with a fixed ordering. The odd positive roots are \( \Psi^+' = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3 \} \), the even positive roots are \( \Psi^0 = \{ \alpha_2 = \alpha_1 + \alpha_2, \alpha_{23} = \alpha_2 + \alpha_3, \alpha_{13} = \alpha_1 + \alpha_3 \} \). The quantum affine superalgebra \( U_v^>(\mathfrak{G}(2, 1; \theta)) \) has been studied in [HSTY], where the Drinfeld realization is obtained. We consider its positive part. We assume that \( v \in \mathbb{C} \) is generic, that is \( v^{ku} \neq 1 \) for all \( u \in \{ 1, \theta, \theta + 1 \} \) and \( k \in \mathbb{N} \). \( U_v^>(\mathfrak{G}(2, 1; \theta)) \) is the \( \mathbb{C} \)-superalgebra with generators \( \{ e_{i,k} \}_{1 \leq i \leq 3} \), in which the parities are \( p(e_{i,k}) = 1 \) for any \( i = 1, 2, 3 \) and \( k \in \mathbb{N} \), and the following relations

\[
e_{i,k} e_{r,l} + v^{a_{i,j}} e_{j,l} e_{i,k+1} = v^{a_{i,j}} e_{i,k} e_{j,l+1} + e_{j,l+1} e_{i,k}, \quad a_{ij} \neq 0, k,l \in \mathbb{Z}
\]

where \( [u]^v = \frac{u^v - v^u}{u - v} \) for \( u \in \mathbb{C} \).

We define the quantum affine root vectors by \( E_{\alpha_i}(k) = e_{i,k}, E_{\alpha_{ij}}(k) = [e_{i,k}, e_{j,0}]_v^{-a_{ij}}, E_{\alpha_{123}}(k) = [e_{1,k}, e_{2,0}]_v^{-a_{123}} e_{3,0} \) for any \( k \in \mathbb{Z} \). Let \( H \) be the set of functions \( h: \Psi^+ \times \mathbb{Z} \rightarrow \mathbb{N} \) with finite support and such that \( h(\beta, k) \leq 1 \) if \( \beta \in \Psi^+_1 \). For each \( h \in H \) we have the ordered monomial \( E_h := \prod_{(\beta,k) \in \Psi^+ \times \mathbb{Z}} E_\beta(k)^{h(\beta,k)} \). Let \( U' \subset U_v^>(\mathfrak{G}(2, 1; \theta)) \) be the spanning set of these \( E_h \) over \( \mathbb{C} \).

Theorem 3.1. Any element of \( U_v^>(\mathfrak{G}(2, 1; \theta)) \) is a linear combination of some \( E_h \).

Proof. Same to the proof of Proposition 2.3, by repeatedly using Lemma 2.2, we get the commutation relations between these quantum affine root vectors. Specifically, for any \( \beta, \beta' \in \Psi^+ \), \( k,l \in \mathbb{Z} \), we have \( [E_\beta(k), E_{\beta'}(l)]_{v^{-\beta, \beta'}} \in U' \). In the following we will use the symbol \( \approx \) to denote an equation without considering the coefficients, for example if \( A = vB + [\theta]_vC \), then we have \( A \approx B + C \). For \( \beta = \alpha_i, \beta' = \alpha_j \) or \( \beta = \alpha_i, \beta' = \alpha_{ij} \) or \( \beta = \alpha_{ij} \), it is the same as Proposition 2.3. For \( \beta = \alpha_{ij}, \beta' = \alpha_k, k \neq i,j \), we have \( [e_{1,r}, e_{2,0}]_{v^{-1}} \approx [e_{1,r}, e_{3,0}]_{v^{-1}} e_{2,0} \approx E_{\alpha_{123}}(r + k) + \sum E_{\alpha_{12}}(l)E_{\alpha_3}(l') + \sum E_{\alpha_1}(l)E_{\alpha_{23}}(l') \in U' \). For \( \beta = \alpha_{ijk}, \beta' = \alpha_i \), we have
\begin{equation}
\left[\left[ e_{1,r}^1, e_{2,0}^v \right], e_{3,0}^v \right], e_{2,k} \right]_v^v \approx \left[ \left[ E_{a_{13}(r-k)}, e_{2,k} \right], e_{2,k} \right]_v^v + \sum \left[ \left[ e_{1,r} w, e_{3,0}^v \right], e_{2,k} \right]_v^v \in U'.
\end{equation}

The remaining cases are similar.

3.2. Shuffle algebra $\Omega$. Consider $\Omega = \bigoplus_{k_1, k_2, k_3 \in \mathbb{N}_0} \Omega_{k}$, where $\Omega_{k}$ consists of rational functions $F$ in the variables $\{ x_{i,r} \}_{1 \leq r \leq k_i}$ which satisfies:

1. $F$ is antisymmetric with respect to $\{ x_{i,r} \}_{1 \leq r \leq k_i}$ for any $1 \leq i \leq 3$.
2. $F = \left( \prod_{1 \leq i < j \leq 3, 1 \leq r \leq k_i, 1 \leq s \leq k_j} [x_{i,r} - x_{j,s}] \right)$, where $f \in \mathbb{C}[x_{i,r}^{-1}]_{1 \leq r \leq k_i}$ is a Laurent polynomial.
3. $F$ satisfies the wheel condition, that is $F(\{ x_{i,r} \}_{1 \leq r \leq k_i}) = 0$ once $x_{1,r} = v^{-1} x_{2,s} = v^\theta x_{3,w}$ or $x_{1,r} = v x_{2,s} = v^{-\theta} x_{3,w}$ for some $1 \leq r \leq k_1, 1 \leq s \leq k_2, 1 \leq w \leq k_3$.

We also fix an $3 \times 3$ matrix of rational functions $\omega_{i,j}(z)_{1 \leq i,j \leq 3} \in \text{Mat}_{3 \times 3}(\mathbb{C}(z))$ by setting

\begin{align}
\omega_{i,j}(z) &= -\omega_{j,i}(z) = \frac{z - v^{-\alpha_{ij}}}{z - 1}, & 1 \leq i < j \leq 3 \\
\omega_{i,i}(z) &= 1, & 1 \leq i \leq 3.
\end{align}

Denote by $\hat{\mathfrak{S}}_k = \mathfrak{S}_{k_1} \times \mathfrak{S}_{k_2} \times \mathfrak{S}_{k_3}$. For any $F \in \Omega_{k_1}, G \in \Omega_{k_2},$ define their shuffle product $F \ast G \in \Omega_{k_1+1}$ by $F \ast G =
\begin{equation}
\text{ASym}_{\hat{\mathfrak{S}}_k} \left( F(\{ x_{i,r} \}_{1 \leq r \leq k_i}), G(\{ x_{j,s} \}_{1 \leq j \leq m}) \right) \prod_{l \leq j \leq m} \omega_{i,j}(x_{i,r} / x_{j,s}).
\end{equation}

We know $\Omega$ is $\ast$-closed, and $\Omega$ becomes an associative $\mathbb{C}$-algebra under $\ast$.

For an ordered monomial $E_h$, define its degree $\deg(E_h) = \deg(h) = d \in \mathbb{N}^7$ as a collection of $d_{\beta} := \sum_{r \in \mathbb{Z}} h(\beta, r) \in \mathbb{N}$ ($\beta \in \Psi^+$) ordered with respect to the ordering on $\Psi^+$. We consider the lexicographical ordering on $\mathbb{N}^7$:

\[ \{ d_{\beta} \}_{\beta \in \Psi^+} > \{ d'_{\beta} \}_{\beta \in \Psi^+} \text{ iff there is } \gamma \in \Psi^+ \text{ such that } d_{\gamma} > d'_{\gamma} \text{ and } d_{\beta} = d'_{\beta} \text{ for all } \beta < \gamma. \]

Identifying simple roots as a basis for $\mathbb{N}^3$, for any $d \in \mathbb{N}^7$ we define its grading $\text{gr}(d) = \sum_{\beta \in \Psi^+} d_{\beta} \beta \in \mathbb{N}^3$. Let us now define for any degree $d$ a specialization map

\begin{equation}
\phi_d: \Omega_{\text{gr}(d)} \rightarrow \mathbb{C}(v)[\{ w_{\beta, s}^1 \mid 1 \leq s \leq d_{\beta} \}].
\end{equation}

Denote $\text{gr}(d) = k$. For $1 \leq i \leq 3$, we split the variables $\{ x_{i,r} \}_{1 \leq r \leq k_i}$ into groups $\{ x_{i,s}^\beta \}_{\beta \in \Psi^+}$ corresponding to each $\beta \in \Psi^+$ and $1 \leq s \leq d_{\beta}$. Now for any $F \in \Omega_{k}$ define the specialization map $\phi_d(F)$ by specializing:

\begin{align}
x_{1,s}^{\beta} &\mapsto w_{\beta, s}, \\
x_{2,s}^{\beta} &\mapsto v w_{\beta, s}, \\
x_{3,s}^{\beta} &\mapsto v^\theta w_{\beta, s}, \\
x_{3,s}^{\beta} &\mapsto v^{-\theta} w_{\beta, s}.
\end{align}

Since $F \in \Omega_{k}$ is antisymmetric with respect to $\hat{\mathfrak{S}}_k$, different choices of our splitting of the variables only occur different signs in the specialization $\phi_d(F)$ and we can ignore them.

**Theorem 3.2.** $e_{i,k} \mapsto x_i^k$ induces a $\mathbb{C}$-algebra isomorphism $\varphi: U^>(\mathfrak{D}(2, 1; \theta)) \tilde{\rightarrow} \Omega$.

**Proof.** It is straightforward to check that the assignment $e_{i,k} \mapsto x_i^k$ induces an algebra morphism $\varphi$ from $U^>(\mathfrak{D}(2, 1; \theta))$ to $\Omega$. In particular, we have $\varphi(E_{a_{ij}}(k)) = (1 - v^{-2a_{ij}} v^1) x_{i,1} / x_{j,1}$ for $1 \leq i < j \leq 3$.
where \( \{r_{\beta,1}, \ldots, r_{\beta,d_3}\} \) is the support of \( h \) restricted on \( \beta \) and the shuffle element \( u_{\beta,1}^{r_{\beta,1}} \cdots u_{\beta,d_3}^{r_{\beta,d_3}} \) is defined as monomial basis of antisymmetric polynomials for odd root \( \beta \) or as Hall-Littlewood basis of symmetric polynomials for even root \( \beta \). And the function \( y_{i,j}^{\beta,\beta'}(a,b) \) is defined as \( y_{1,2}^{\beta,\beta'} = a - b, y_{2,1}^{\beta,\beta'} = a - v^{-2}b \) for any \( \beta < \beta' \); \( y_{1,3}^{\beta,\beta'} = a - b, y_{2,3}^{\beta,\beta'} = a - v^{2}b \) if \( \beta' = \alpha_{13}, \alpha_{123} \); \( y_{1,3}^{\beta,\beta'} = a - v^{-2}b, y_{2,3}^{\beta,\beta'} = a - b \) if \( \beta' = \alpha_{3}, \alpha_{23} \); \( y_{3,1}^{\beta,\beta'} = a - v^{-2}b, y_{3,2}^{\beta,\beta'} = a - v^{2}b \) if \( \beta = \alpha_{13}, \alpha_{123} \); \( y_{i,i}^{\beta,\beta'} = 1 \) for any \( \beta < \beta' \) and \( 1 \leq i \leq 3 \). Same to [T1] we have to prove that \( \phi_{d'}(\varphi(E_h)) = 0 \) for any \( d' < \deg(h) \). Recall that each term of \( \varphi(E_h) \) is corresponding to some permutation \( \sigma \times \tau \times \mu \), and we will prove that each term is zero under specialization \( \phi_{d'} \). Let \( \deg(h) = (d_3), \deg(h') = (d'_3) \), then there are the following cases.

- If \( d'_1 < d_{a_1} \), then for some \( 1 \leq i \leq d_{a_1} \), we have \( \sigma(i) > d_{a_1} \), and then this term is zero under \( \phi_{d'_3} \) because the \( x_{1,i}^{\sigma(i)} \) is not mapping to some \( w_3^{\beta,s} \) and some \( x_2r \) or \( x_3r \). And for any \( 1 \leq i \leq d_{a_1} \), we have \( 1 \leq \sigma(i) \leq d_{a_1} \), then since \( d'_1 < d_{a_1} \) and \( \phi_{d'_3} \) is zero under \( \phi_{d'_3} \), there will be some \( x_{1,i} \) for \( 1 \leq i \leq d_{a_1} \) and \( x_2r \) (or \( x_3r \)) that is mapping to some \( w_3^{\beta,s} \) or \( v w_3^{\beta,s} \) (or \( v^a w_3^{\beta,s} \)), and for any such \( \sigma \) the term contains the factor \( x_{1,i} - v^{-1} x_2r \) (or \( x_{1,i} - v^{-\theta} x_3r \)), and for such arguments in the last case, the term is zero under \( \phi_{d'_3} \); otherwise for any \( d_{a_1} + 1 \leq i \leq d_{a_1} + d_{a_13} \), we have \( \sigma(i) > d_{a_1} \), then there will be some \( x_{1,s}^{\beta,s} \) and \( x_2r \) corresponding to each term that will be mapping to some \( w_3^{\beta,t} \) (or \( v w_3^{\beta,t} \)).

- If \( d_{a_1} = d_{a_1}, d'_{a_{13}} < d_{a_{13}} \), then if there is some \( 1 \leq i \leq d_{a_1} \) such that \( \sigma(1) > d_{a_1} \), same for the arguments in the last case, the term is zero under \( \phi_{d'_3} \); otherwise for any \( d_{a_1} + 1 \leq i \leq d_{a_1} + d_{a_{13}} \), we have \( \sigma(i) > d_{a_1} \), then there will be some \( x_{1,s}^{\beta,s} \) and \( x_2r \) corresponding to each term that will be mapping to some \( w_3^{\beta,t} \) (or \( v w_3^{\beta,t} \)).

- If \( d_{a_1} = d_{a_1}, d'_{a_{13}} = d_{a_{13}}, d'_{a_{12}} < d_{a_{12}} \), then if for some \( d_{a_1} + d_{a_{13}} + 1 \leq i \leq d_{a_1} + d_{a_{13}} + d_{a_{12}} \) we have \( \sigma(i) \leq d_{a_1} + d_{a_{13}} + d_{a_{12}} \), same for the arguments in the above cases we have the term is zero under \( \phi_{d'_3} \); otherwise there will be some \( x_{1,s}^{\beta,s} \) and \( x_3r \) corresponding to each term that will be mapping to some \( w_3^{\beta,t} \) (or \( v w_3^{\beta,t} \)).

- If \( d'_{a_2} < d_{a_2} \) and \( d'_3 = d_3 \) for any \( \beta < a_2 \), then if there is some \( \beta = \alpha_1, \alpha_{13}, \alpha_{12}, \alpha_{123} \) and such that \( \sigma(x_{1,s}^{\beta}) = x_{1,s}^{\beta'} \) for some \( \beta' \neq \beta \), same for arguments in the above cases the term is zero.
under $\phi_d$; otherwise there will be some $x_{2,s}$ and $x_{3,r}$ corresponding to each term that will be mapping to some $vw_{\alpha_{23},t}$ and $v^{-\theta}w_{\alpha_{23},t}$ and the term contains the factor $x_{2,s} - \theta^{\beta+1}x_{3,r}$.

Hence we get $\{E_h\}_{h \in H}$ are PBW type bases for $U_\nu^+(\tilde{\Theta}(2,1;\theta))$ and $\varphi$ is injective.

For surjectivity of $\varphi$, by Remark 2.10 we only need to prove that given $h \in H$ such that $\text{gr}(h) = k$ and $\deg(h) = (d_\beta)_{\beta \in \Psi^+}$, if for any $\varphi(h') = \varphi(h)$ and $\deg(h') < \deg(h)$ we have $\phi_d(\varphi(F)) = 0$, then $\phi_d(F)$ is a linear combination of some $\phi_d(\varphi(E_h))$ for any $F \in \Omega_k$. Actually, we only need to consider the case where there are only two positive roots $\beta, \tilde{\beta}$ such that $d_\beta, d_{\tilde{\beta}} \neq 0$, and this can be done by case by case study. We give details of proof for some cases, other cases are similar.

- For cases such as $(\beta, \beta') = (\alpha_i, \alpha_j)$, $(\alpha_i, \alpha_{ij})$, $(\alpha_{ij}, \alpha_j)$, where $1 \leq i < j \leq 3$, it is the same as Remark 2.10.
- For $(\beta, \beta') = (\alpha_1, \alpha_{23})$, $(\alpha_{13}, \alpha_2)$, $(\alpha_{12}, \alpha_3)$, we consider the case $(\beta, \beta') = (\alpha_{13}, \alpha_2)$. We have $\phi_d(\varphi(E_h)) = \prod_{1 \leq s < r \leq d_\beta}(w_{\beta,s} - w_{\beta,r})^2 \prod_{1 \leq s < r \leq d_{\tilde{\beta}}}(w_{\tilde{\beta},s} - w_{\tilde{\beta},r}) \prod_{1 \leq s, \tilde{s} \leq d_\beta}(w_{\beta,s} - v^2 w_{\beta,r}) \cdot f$, in which $f \in \mathbb{C}[w_{\beta,s}, w_{\beta,r}]_{1 \leq s, \tilde{s} \leq d_\beta, 1 \leq r, \tilde{r} \leq d_{\tilde{\beta}}}$.

Now for any $F \in \Omega_{(d_\beta, d_{\tilde{\beta}}, d_\beta)}$, $F$ is antisymmetric, hence $\phi_d(F)$ has the factor $\prod_{1 \leq s < r \leq d_\beta}(w_{\beta,s} - w_{\beta,r})^2 \prod_{1 \leq s < r \leq d_{\tilde{\beta}}}(w_{\tilde{\beta},s} - w_{\tilde{\beta},r})$. Under specialization $\phi_d$ the wheel condition becomes $\phi_d(F) = 0$ once $w_{\beta,s} = v^2 w_{\beta,r}$, hence giving us the factor $\prod_{1 \leq s < r \leq d_\beta}(w_{\beta,s} - v^2 w_{\beta,r})$. Finally, let $\deg(h') = (d_\beta - 1, d_{\alpha_{123}} = 1, d_{\alpha_{123}} - 1)$, then $\deg(h') < \deg(h)$, hence $\phi_d(F) = 0$, and gives us the last factor $\prod_{1 \leq s \neq r \leq d_{\beta}}(w_{\beta,s} - w_{\beta,r})$.

- For $(\beta, \beta') = (\alpha_{13}, \alpha_{12})$, $(\alpha_{12}, \alpha_{23})$, $(\alpha_{13}, \alpha_{23})$, we consider the case $\beta = \alpha_{12}, \beta' = \alpha_{23}$. We have $\phi_d(\varphi(E_h)) = \prod_{1 \leq s < r \leq d_\beta}(w_{\beta,s} - w_{\beta,r})^2 \prod_{1 \leq s < r \leq d_{\tilde{\beta}}}(w_{\tilde{\beta},s} - w_{\tilde{\beta},r})^2 \prod_{1 \leq s \neq r \leq d_\beta}(w_{\beta,s} - w_{\beta,r}) \cdot f$. The anti-symmetrization gives the factor $\prod_{1 \leq s < r \leq d_\beta}(w_{\beta,s} - w_{\beta,r})^2 \prod_{1 \leq s \neq r \leq d_\beta}(w_{\beta,s} - w_{\beta,r}) \cdot f$. Under specialization $\phi_d$ the wheel condition becomes $\phi_d(F) = 0$ once $w_{\beta,s} = w_{\beta,r}$, hence giving us the factor $\prod_{1 \leq s \neq r \leq d_{\beta}}(w_{\beta,s} - v^{2g} w_{\beta,r})$. Finally, let $\deg(h') = (d_\beta - 1, d_{\alpha_{123}} = 1, d_{\alpha_{123}} - 1)$, then $\deg(h') < \deg(h)$, hence $\phi_d(F) = 0$, and gives us the last factor $\prod_{1 \leq s \neq r \leq d_{\beta}}(w_{\beta,s} - v^{2g} w_{\beta,r})$.

- For $(\beta, \beta') = (\alpha_{1}, \alpha_{123})$, $(\alpha_{12}, \alpha_2)$, $(\alpha_{12}, \alpha_{3})$, we consider the case $\beta = \alpha_1, \beta' = \alpha_{123}$. We have $\phi_d(\varphi(E_h)) = \prod_{1 \leq s < r \leq d_\beta}(w_{\beta,s} - w_{\beta,r})^2 \prod_{1 \leq s < r \leq d_{\tilde{\beta}}}(w_{\tilde{\beta},s} - w_{\tilde{\beta},r})^3 \prod_{1 \leq s \neq r \leq d_\beta}(w_{\beta,s} - v^{-2} w_{\beta,r}) \cdot f$. The anti-symmetrization gives the factor $\prod_{1 \leq s < r \leq d_\beta}(w_{\beta,s} - w_{\beta,r}) \prod_{1 \leq s < r \leq d_{\tilde{\beta}}}(w_{\tilde{\beta},s} - w_{\tilde{\beta},r})^3 \prod_{1 \leq s \neq r \leq d_\beta}(w_{\beta,s} - w_{\beta,r}) \cdot f$. The wheel condition becomes $\phi_d(F) = 0$ once $w_{\beta,s} = v^{-2} w_{\beta,r}$ or $w_{\beta,s} = v^{-2} w_{\beta,r}$. Let $\deg(h') = (d_\beta - 1, d_{\alpha_{123}} = 1, d_{\alpha_{123}} - 1)$, then $\phi_d(F) = 0$ gives the factor $\prod_{1 \leq s, \tilde{s} \leq d_\beta}(w_{\beta,s} - w_{\beta,r})$.

- For $(\beta, \beta') = (\alpha_{12}, \alpha_{123})$, $(\alpha_{13}, \alpha_{123})$, $(\alpha_{123}, \alpha_{23})$, we consider the case $\beta = \alpha_{12}, \beta' = \alpha_{123}$. We have $\phi_d(\varphi(E_h)) = \prod_{1 \leq s < r \leq d_\beta}(w_{\beta,s} - w_{\beta,r})^2 \prod_{1 \leq s < r \leq d_{\tilde{\beta}}}(w_{\tilde{\beta},s} - w_{\tilde{\beta},r})^3 \prod_{1 \leq s \neq r \leq d_\beta}(w_{\beta,s} - v^{-2} w_{\beta,r}) \cdot f$. The anti-symmetrization gives the factor $\prod_{1 \leq s < r \leq d_\beta}(w_{\beta,s} - w_{\beta,r}) \prod_{1 \leq s < r \leq d_{\tilde{\beta}}}(w_{\tilde{\beta},s} - w_{\tilde{\beta},r})^2 \prod_{1 \leq s \neq r \leq d_\beta}(w_{\beta,s} - v^{-2} w_{\beta,r}) \cdot f$. The wheel condition becomes $\phi_d(F) = 0$ once $w_{\beta,s} = v^{-2} w_{\beta,r}$ or $w_{\beta,s} = v^{-2} w_{\beta,r}$ or $w_{\beta,s} = v^{-2} w_{\beta,r}$ or $w_{\beta,s} = v^{-2} w_{\beta,r}$, hence giving us the factor $\prod_{1 \leq s, \tilde{s} \leq d_\beta}(w_{\beta,s} - v^{-2} w_{\beta,r})$.
\[ v^{-2\theta}w_{\beta,r} \prod_{1 \leq s \leq d_{\beta}} (w_{\beta,s} - v^{-2}w_{\beta,r})(w_{\beta,s} - v^{2\theta}w_{\beta,r}) \] The remaining factors come from the anti-symmetrization.

This completes our proof. \( \square \)

3.3. Generalization to all Dynkin diagrams associated to \( \mathfrak{D}(2,1;\theta) \). * In this subsection, we give shuffle algebra realization of quantum affine algebras corresponding to all Dynkin diagrams associated to \( \mathfrak{D}(2,1;\theta) \), making the picture for this exceptional Lie superalgebra complete.

Besides the simple root system with complete fermionic roots, there are three other simple root systems associated to \( \mathfrak{D}(2,1;\theta) \), which all contains one fermionic root and two bosonic roots. The only difference in these three cases is the position of fermionic root, hence we only need to consider the case corresponding to the following Cartan matrix

\[
A = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 0 & -\theta \\
0 & -1 & 2
\end{pmatrix},
\]

where \( \theta \neq 0, -1 \). We denote the corresponding Lie superalgebra by \( \mathfrak{D}_2(2,1;\theta) \). Let \( d_1 = d_2 = 1, d_3 = \theta \), so that \( (d_{ij}a_{ij})_{1 \leq i,j \leq 3} \) is symmetric. The positive roots are \( \Psi^+ = \{ \alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \alpha_1 + 2\alpha_2 + \alpha_3 < \alpha_2 + \alpha_3 < \alpha_2 + \alpha_3 + \alpha_3 \} \) with a fixed ordering. We denote the longest positive root by \( \gamma \) and denote the other positive roots by \( \alpha_{ij} \) as before. Still we assume that \( v \in \mathbb{C} \) is generic, that is \( v^{ku} \neq 1 \) for all \( u \in \{1, \theta, \theta + 1\} \) and \( k \in \mathbb{N} \). The positive part of quantum affine superalgebra \( \mathcal{U}_v^+(\mathfrak{D}_2(2,1;\theta)) \) is the \( \mathbb{C} \)-superalgebra with generators \( \{e_{i,k}\}_{1 \leq i \leq 3} \) in which the parities are \( p(e_{i,k}) = i - 1 \) for any \( k \in \mathbb{N} \), and the following relations:

\[
[e_{i,k}, e_{j,l}] = 0, \quad a_{ij} = 0, k, l \in \mathbb{Z} \\
[e_{i,k}, e_{j,l+1}v^{-d_{a_{ij}}}] = -[e_{j,l}, e_{i,k+1}]v^{-d_{a_{ij}}}, \quad a_{ij} \neq 0, k, l \in \mathbb{Z}
\]

\[
\text{Sym}_{k,l}[e_{i,k}, [e_{i,l}, e_{2,s}v^{-d_{a_{12}}}v^{-d_{a_{12}a_{2}}}]] = 0, \quad i = 1, 3, k, l, s \in \mathbb{Z}
\]

(3.8)

The quantum affine root vectors \( E_\gamma(k) \) and the ordered monomials \( E_h \) are also defined similarly as before. Especially, we have \( E_\gamma(k) = [E_{a_{13}}(k), E_{a_{2}}(0)]_{1^{i=\theta}} \). Standard arguments show that these ordered monomials span the whole positive part. Note that the difference between this case and the case for type \( A(2|2) \) with distinguished simple root system is that there is no commutation relations between quantum affine root vectors \( E_{a_{13}} \) and \( E_{a_{2}} \), and there is one more quantum affine root vector \( E_h \) in the ordered monomials \( E_h \).

Consider \( \Omega' = \bigoplus_{h \in \{0,1,2,3\}} \mathcal{O}_{\mathfrak{h}}' \), where \( \mathcal{O}_{\mathfrak{h}}' \) consists of rational functions \( F \) in the variables \( \{x_{i,r}\}_{1 \leq r \leq k_i} \) which satisfies:

(1) \( F \) is symmetric with respect to \( \{x_{i,r}\}_{1 \leq r \leq k_i} \) for \( i = 1, 3 \) and antisymmetric with respect to \( \{x_{2,r}\}_{1 \leq r \leq k_2} \).

(2) \( F = \prod_{1 \leq i \leq 2, 1 \leq r \leq k_i, 1 \leq s \leq k_i+1} f(x_{i,r}-x_{i,s}) \), where \( f \in \mathbb{C}[x_{i,r}]_{1 \leq r \leq k_i} \) is a Laurent polynomial.

(3) \( F \) satisfies the wheel condition, that is \( F(\{x_{i,r}\}_{1 \leq r \leq k_i}) = 0 \) once \( x_{1,r_1} = v^2x_{1,r_2} = vx_{2,s} \) or \( x_{3,t_1} = v^{2\theta}x_{3,t_2} = v^\theta x_{2,s} \) for some \( 1 \leq r_1, r_2 \leq k_1, 1 \leq s \leq k_2, 1 \leq t_1, t_2 \leq k_3 \).

Let \( \omega_{ij}(z) = z^{-d_{a_{ij}}} \), then \( \Omega' \) becomes an associative algebra under the shuffle product similar to (3.3) except that we take symmetrization instead of anti-symmetrization with respect to \( \{x_{1,r}\} \) and \( \{x_{3,s}\} \). Now we have

*The results in this subsection have been previously worked out by Tsymbaliuk (private communication).
Theorem 3.3. \( e_{i,k} \mapsto x_i^k \) induces a \( \mathbb{C} \)-algebra isomorphism \( \varphi : U_\kappa^\gamma(\widehat{\mathfrak{S}}_2(2,1;\theta)) \rightarrow \Omega' \).

Proof. The only difficulty is that we need to define the specialization map corresponding to \( \Omega' \). Now for any \( E_h \), we label the variables in \( \varphi(E_h) \) by \( \{x_{i,s}^\beta\}_{i \in [\beta],1 \leq s \leq d_\beta} \) for \( \beta \neq \gamma \) and by \( \{x_{1,s}^\beta,x_{2,1,s}^\beta,x_{2,2,s}^\beta,x_{3,s}^\beta\}_{1 \leq s \leq d_\beta} \) for \( \beta = \gamma \). Now define the specialization \( \phi_d(\varphi(E_h)) \in \mathbb{C}[w_{\beta,s}^\pm] \) by specializing:

\[
x_{1,s}^\beta \mapsto w_{\beta,s}, \quad x_{2,s}^\beta \mapsto v^{-1}w_{\beta,s}, \quad x_{3,s}^\beta \mapsto v^{-1-\theta}w_{\beta,s}, \quad \beta \neq \gamma
\]

\[
x_{1,s}^\gamma \mapsto w_{\beta,s}, \quad x_{2,1,s}^\gamma \mapsto v^{-1}w_{\beta,s}, \quad x_{2,2,s}^\gamma \mapsto v^{-1-2\theta}w_{\beta,s}, \quad x_{3,s}^\gamma \mapsto v^{-1-\theta}w_{\beta,s}
\]

(3.9)

Explicitly we have \( c_d(\varphi(E_h)) = c \cdot \prod_{\beta \neq \gamma} G_{\beta,\gamma} \prod_{\beta \in \Psi} G_{\beta} \prod_{\beta \in \Psi} w_{\beta,1}^{r_{\beta,1}} \cdots w_{\beta,d_\beta}^{r_{\beta,d_\beta}} \) where \( c \) is some non-zero constant and we have

- \( G_{\beta} = 1, \beta = \alpha_1, \alpha_2, \alpha_3 \).
- \( G_{\beta} = \prod_{1 \leq s \neq r \leq d_\beta} (w_{\beta,s} - v^2w_{\beta,r}), \beta = \alpha_2 \).
- \( G_{\beta} = \prod_{1 \leq s \neq r \leq d_\beta} (w_{\beta,s} - v^2w_{\beta,r}), \beta = \alpha_3 \).
- \( G_{\beta} = \prod_{1 \leq s \neq r \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r}), \beta = \alpha_1 \).
- \( G_{\beta} = \prod_{1 \leq s \neq \beta \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r})(w_{\beta,s} - v^{-2}w_{\beta,r}), \beta = \alpha_3 \).
- \( G_{\beta} = \prod_{1 \leq s \neq \beta \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r})(w_{\beta,s} - v^{-2}w_{\beta,r}), \beta = \gamma \).
- \( G_{\beta,\gamma} = \prod_{1 \leq s \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r}), (\beta, \beta') = (\alpha_1, \alpha_2), (\alpha_2, \alpha_3), (\alpha_1, \alpha_2, \alpha_3), (\alpha_2, \alpha_3) \).
- \( G_{\beta,\gamma} = \prod_{1 \leq s \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r}), (\beta, \beta') = (\alpha_1, \alpha_3) \).
- \( G_{\beta,\gamma} = \prod_{1 \leq s \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r}), (\beta, \beta') = (\alpha_2, \alpha_3) \).
- \( G_{\beta,\gamma} = \prod_{1 \leq s \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r}), (\beta, \beta') = (\alpha_1, \gamma) \).
- \( G_{\beta,\gamma} = \prod_{1 \leq s \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r}), (\beta, \beta') = (\alpha_2, \gamma) \).
- \( G_{\beta,\gamma} = \prod_{1 \leq s \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r}), (\beta, \beta') = (\alpha_3, \gamma) \).
- \( G_{\beta,\gamma} = \prod_{1 \leq s \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r}), (\beta, \beta') = (\alpha_2, \beta_3) \).
- \( G_{\beta,\gamma} = \prod_{1 \leq s \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r}), (\beta, \beta') = (\alpha_1, \beta_3) \).
- \( G_{\beta,\gamma} = \prod_{1 \leq s \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r}), (\beta, \beta') = (\alpha_2, \beta_3) \).
- \( G_{\beta,\gamma} = \prod_{1 \leq s \leq d_\beta} (w_{\beta,s} - v^{-2}w_{\beta,r}), (\beta, \beta') = (\alpha_3, \beta_3) \).

Now same to the proof of Theorem 3.2, we have \( \phi_d(\varphi(E_h)) = 0 \) for any \( d' < \deg(h) \) and by looking at each pair of positive roots the wheel conditions give us the vanishing factors as above, thus completing our proof.

\[\square\]

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