New bounds for a hypergraph Bipartite Turán problem

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Abstract

Let \( t \) be an integer such that \( t \geq 2 \). Let \( K_{2,t}^{(3)} \) denote the triple system consisting of the \( 2t \) triples \( \{a, x_i, y_i\} \), \( \{b, x_i, y_i\} \) for \( 1 \leq i \leq t \), where the elements \( a, b, x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t \) are all distinct. Let \( \text{ex}(n, K_{2,t}^{(3)}) \) denote the maximum size of a triple system on \( n \) elements that does not contain \( K_{2,t}^{(3)} \). This function was studied by Mubayi and Verstraëte [8], where the special case \( t = 2 \) was a problem of Erdős [1] that was studied by various authors [3, 8, 9].

Mubayi and Verstraëte proved that \( \text{ex}(n, K_{2,t}^{(3)}) < t^4 \binom{n}{2} \) and that for infinitely many \( n \), \( \text{ex}(n, K_{2,t}^{(3)}) \geq \frac{2t-1}{3} \binom{n}{2} \). These bounds together with a standard argument show that \( g(t) := \lim_{n \to \infty} \text{ex}(n, K_{2,t}^{(3)})/\binom{n}{2} \) exists and that

\[
\frac{2t - 1}{3} \leq g(t) \leq t^4.
\]

Addressing the question of Mubayi and Verstraëte on the growth rate of \( g(t) \), we prove that as \( t \to \infty \),

\[
g(t) = \Theta(t^{1+o(1)}).
\]

1 Introduction

An \( r \)-graph is an \( r \)-uniform hypergraph. Let \( \mathcal{F} \) be a family of \( r \)-graphs and let \( \text{ex}(n, \mathcal{F}) \) denote the maximum number of edges in an \( r \)-graph on \( n \) vertices containing no member of \( \mathcal{F} \). We call \( \text{ex}(n, \mathcal{F}) \) the Turán number of \( \mathcal{F} \). Determining the asymptotic order of \( \text{ex}(n, \mathcal{F}) \) is generally very difficult. For an excellent survey on the study of hypergraph Turán numbers, see [7]. In this paper, we study a hypergraph Turán problem that is motivated by the study of Turán numbers of complete bipartite graphs as well as by a question of Erdős.

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Definition 1. Let \( r \geq 3 \) be an integer. Let \( G \) be a bipartite graph with an ordered bipartition \((X, Y)\). Suppose that \( Y = \{y_1, \ldots, y_m\} \). Let \( Y_1, \ldots, Y_m \) be disjoint sets of size \( r - 2 \) that are disjoint from \( X \cup Y \). Let \( G^{(r)}_{X,Y} \) denote the \( r \)-graph with vertex set \((X \cup Y) \cup (\bigcup_{i=1}^{m} Y_i)\) and edge set \( \bigcup_{i=1}^{m} \{ e \cup Y_i : e \in E(G), y_i \in e \} \).

Let \( s, t \geq 2 \) be positive integers. If \( G \) is the complete bipartite graph with an ordered bipartition \((X, Y)\) where \(|X| = s, |Y| = t\), then let \( G^{(r)}_{X,Y} \) be denoted by \( K^{(r)}_{s,t} \).

Definition 2. For all \( n \geq r \geq 3 \), let \( f_r(n) \) denote the maximum number of edges in an \( n \)-vertex \( r \)-graph containing no four edges \( A, B, C, D \) with \( A \cup B = C \cup D \) and \( A \cap B = C \cap D = \emptyset \).

Note that \( f_3(n) = \text{ex}(n, K^{(3)}_{2,2}) \), and in general \( f_r(n) \leq \text{ex}(n, K^{(r)}_{2,2}) \). Erdős \([1]\) asked whether \( f_r(n) = O(n^{r-1}) \) when \( r \geq 3 \). Füredi \([3]\) answered Erdős’ question affirmatively. More precisely, he showed that for integers \( n, r \) with \( r \geq 3 \) and \( n \geq 2r \),

\[
\binom{n-1}{r-1} + \left\lfloor \frac{n-1}{r} \right\rfloor \leq f_r(n) < 3.5 \binom{n}{r-1}.
\]

(1)

The lower bound is obtained by taking the family of all \( r \)-element subsets of \([n] := \{1, 2, \ldots, n\}\) containing a fixed element, say 1, and adding to the family any collection of \( \lfloor \frac{n-1}{r} \rfloor \) pairwise disjoint \( r \)-element subsets not containing 1. For \( r = 3 \), Füredi also gave an alternative lower bound construction using Steiner systems. An \((n, r, t)\)-Steiner system \( S(n, r, t) \) is an \( r \)-uniform hypergraph on \([n] \) in which every \( t \)-element subset of \([n] \) is contained in exactly one hyperedge. Füredi observed that if we replace every hyperedge in \( S(n, 5, 2) \) by all its 3-element subsets then the resulting triple system has \( \binom{n}{2} \) triples and contains no copy of \( K^{(3)}_{2,2} \). This slightly improves the lower bound in \([1]\) for \( r = 3 \) to \( \binom{n}{2} \), for those \( n \) for which \( S(n, 5, 2) \) exists. The upper bound in \([1]\) was improved by Mubayi and Verstraëte \([8]\) to \( 3 \binom{n}{r-1} + O(n^{r-2}) \). They obtain this bound by first showing \( f_3(n) = \text{ex}(n, K^{(3)}_{2,2}) < 3 \binom{n}{2} + 6n \), and then combining it with a simple reduction lemma. This was later improved to \( f_3(n) \leq \frac{13}{9} \binom{n}{2} \) by Pikhurko and Verstraëte \([9]\).

Motivated by Füredi’s work, Mubayi and Verstraëte \([8]\) initiated the study of the general problem of determining \( \text{ex}(n, K^{(r)}_{2,t}) \) for any \( t \geq 2 \). They showed that for any \( t \geq 2 \) and \( n \geq 2t \)

\[
\text{ex}(n, K^{(3)}_{2,t}) < t^4 \binom{n}{2},
\]

and that for infinitely many \( n \), \( \text{ex}(n, K^{(3)}_{2,t}) \geq \frac{2t-1}{3} \binom{n}{2} \), where the lower bound is obtained by replacing each hyperedge in \( S(n, 2t+1, 2) \) with all its 3-element subsets.

Mubayi and Verstraëte noted that \( g(t) := \lim_{n \to \infty} \frac{\text{ex}(n, K^{(3)}_{2,t})}{\binom{n}{2}} \) exists and raised the question of determining the growth rate of \( g(t) \). It follows from their results that

\[
\frac{2t-1}{3} \leq g(t) \leq t^4.
\]

(2)
In this paper, we prove that as $t \to \infty$,
\begin{equation}
    g(t) = \Theta(t^{1+o(1)}),
\end{equation}
showing that their lower bound is close to the truth. More precisely, we prove the following.

**Theorem 1.** For any $t \geq 2$, we have
\begin{equation}
    \text{ex}(n, K_{2,t}^{(3)}) \leq (15t \log t + 40t)n^2.
\end{equation}

**Notation.** Given a hypergraph (or a graph) $H$, throughout the paper, we also denote the set of its edges by $\text{E}(H)$. For example $|H|$ denotes the number of edges of $H$. Given two vertices $x, y$ in a graph $H$, let $N_H(x, y)$ denote the common neighborhood of $x$ and $y$ in $H$. We drop the subscript $H$ when the context is clear.

## 2 Proof of Theorem 1: $K_{2,t}^{(3)}$-free hypergraphs

We will use the a special case of a well-known result of Erdős and Kleitman [2].

**Lemma 1.** Let $H$ be a 3-graph on $3n$ vertices. Then $H$ contains a 3-partite 3-graph, with all parts of size $n$, and with at least $\frac{2}{3} |H|$ hyperedges.

Let us define the sets $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_n\}$ and $C = \{c_1, c_2, \ldots, c_n\}$. Throughout the proof we define various 3-partite 3-graphs whose parts are $A, B$ and $C$.

Suppose $H$ is a $K_{2,t}^{(3)}$-free 3-partite 3-graph on $3n$ vertices with parts $A, B$ and $C$. First let us show that it suffices to prove the following inequality.
\begin{equation}
    |H| \leq (30t \log t + 80t)n^2.
\end{equation}

It is easy to see that inequality (4) and Lemma 1 together imply that any $K_{2,t}^{(3)}$-free 3-graph on $3n$ vertices contains at most $\frac{2}{3}(30t \log t + 80t)n^2$ hyperedges, from which Theorem 1 would follow after replacing $3n$ by $n$.

In the remainder of the section, we will prove (4). Let us introduce the following notion of sparsity.

**Definition 3 (q-sparse and q-dense pairs).** Let $q$ be a positive integer. Let $G$ be a bipartite graph with parts $X, Y$. Let $x, y$ be two different vertices such that $x, y \in X$ or $x, y \in Y$. Then we call $\{x, y\}$ a q-dense pair of $G$ if $|N(x, y)| \geq q$. We call $\{x, y\}$ a q-sparse pair of $G$ if $|N(x, y)| < q$ but $x, y$ are still contained in a copy of $K_{2,q}$ in $G$. Note that it is possible that $\{x, y\}$ is neither q-sparse nor q-dense.
The following Procedure $\mathcal{P}(q)$ about making a bipartite graph $K_{2,q}$-free lies at the heart of the proof. (We think of $q$ as the parameter of the Procedure $\mathcal{P}(q)$, that is changed throughout the proof.)

**Procedure $\mathcal{P}(q)$: Making a graph $K_{2,q}$-free**

**Input:** A bipartite graph $G$ with parts $A$ and $B$.

$G \leftarrow G$, $\psi \leftarrow 1$.

$F(x,y) \leftarrow \emptyset$, $D(x,y) \leftarrow \emptyset$ and $S(x,y) \leftarrow \emptyset$ for every $x,y \in A$ and $x,y \in B$.

**while** $\psi = 1$ **do**

$\psi \leftarrow 0$.

**Step 1:**

For each $q$-sparse pair $\{x,y\}$ of $G$ such that $F(x,y) = \emptyset$, let $S(x,y)$ be the set of vertices spanned by the $q$-dense pairs of $G$ that are contained in $N_G(x,y)$.

Let $F(x,y) \leftarrow \{ab \in G \mid a \in \{x,y\} \text{ and } b \in S(x,y)\}$, and let $D(x,y)$ be a spanning forest of the graph formed by the dense pairs of $G$ that are contained in $S(x,y)$.

If there exists an edge $ab \in G$ such that $ab$ is contained in $F(x,y)$ for at least $q/2$ different pairs $\{x,y\}$, where $x,y \in A$ or $x,y \in B$,

then $G \leftarrow G \setminus \{ab\}$ and $\psi \leftarrow 1$.

**Step 2:**

If there exists a set $M$ of edges in $G$ such that removing all of the edges of $M$ from $G$ decreases the number of $q$-dense pairs by at least $|M|/2$,

then $G \leftarrow G \setminus M$ and $\psi \leftarrow 1$.

**end while**

$G' \leftarrow G$

$F'(x,y) \leftarrow F(x,y)$ for every $x,y \in A$ and $x,y \in B$.

$D'(x,y) \leftarrow D(x,y)$ for every $x,y \in A$ and $x,y \in B$.

$S'(x,y) \leftarrow S(x,y)$ for every $x,y \in A$ and $x,y \in B$.

**Output:** The graph $G'$ and the sets $F'(x,y), D'(x,y), S'(x,y)$ for all $x,y \in A$ and $x,y \in B$.

In the procedure $\mathcal{P}(q)$, initially for all the pairs $\{x,y\}$ (with $x,y \in A$ and $x,y \in B$) the sets $F(x,y), D(x,y), S(x,y)$ are set to be empty. Then as the edges are being deleted during the procedure, possibly, new $q$-sparse pairs $\{x,y\}$ are being created. When this happens, Step 1 redefines the sets $S(x,y), F(x,y), D(x,y)$ and gives them some non-empty values. (They get non-empty values due to the fact that $\{x,y\}$ is $q$-sparse, which implies that $\{x,y\}$ is contained in a copy of $K_{2,q}$, so there is at least one $q$-dense pair in the common neighborhood of $x,y$.) Therefore, these values stay unchanged throughout the rest of the
procedure.

Notice that at the point \( S(x, y) \) was redefined, the pair \( \{x, y\} \) was \( q \)-sparse, so number of common neighbors is less than \( q \). Therefore, as \( S(x, y) \) is a subset of the common neighborhoood of \( x \) and \( y \), we also have \( |S(x, y)| < q \). Moreover, since \( D(x, y) \) is defined as a spanning forest with the vertex set \( S(x, y) \), we have \( |D(x, y)| \leq |S(x, y)| \). Also, it easily follows from the definition of \( F(x, y) \) that \( |F(x, y)| = 2 |S(x, y)| \). Finally, notice that \( D(x, y) \) does not contain any isolated vertices, because its vertex set \( S(x, y) \) spans all of its edges, by definition. Therefore, \( |D(x, y)| \geq |S(x, y)|/2 \). At the end of the procedure, the sets \( F(x, y), D(x, y), S(x, y) \) are renamed as \( F'(x, y), D'(x, y), S'(x, y) \). Note also that if a pair \( \{x, y\} \) never becomes \( q \)-sparse in the process then \( S'(x, y) = D'(x, y) = F'(x, y) = \emptyset \).

**Observation 1.** For every \( x, y \in A \) and \( x, y \in B \), we have

(1) \( |S'(x, y)| < q \).

(2) \( |D'(x, y)| \leq |S'(x, y)| \).

(3) \( |F'(x, y)| = 2 |S'(x, y)| \).

(4) \( |D'(x, y)| \geq |S'(x, y)|/2 \).

For convenience, throughout the paper we (informally) say that the sets \( F'(x, y), D'(x, y), S'(x, y) \) are defined by applying Procedure \( \mathcal{P}(q) \) to a graph \( G \) to obtain the graph \( G' \), instead of saying that the input to Procedure \( \mathcal{P}(q) \) is \( G \) and the output is the graph \( G' \) and the sets \( F'(x, y), D'(x, y), S'(x, y) \).

**Claim 1.** Let the sets \( F'(x, y), D'(x, y), S'(x, y) \) (for \( x, y \in A \) and \( x, y \in B \)) be defined by applying Procedure \( \mathcal{P}(q) \) to a bipartite graph \( G \) to obtain \( G' \). Let \( N(x, y) \) denote the number of common neighbors of vertices \( x, y \) in the graph \( G \). Then

\[
\frac{|F'(x, y)|}{4} \leq |D'(x, y)| < q.
\]

Moreover \( |F'(x, y)| \leq 2 |N(x, y)| \).

**Proof.** Combining the parts (3) and (4) of Observation 1 we have \( |F'(x, y)|/4 \leq |D'(x, y)| \). Combining the parts (1) and (2) of Observation 1 we obtain \( |D'(x, y)| < q \), proving the first part of the claim.

To prove the second part, notice that \( S'(x, y) \) is a common neighborhood of \( x, y \) in some subgraph \( G \) of \( G \), we have \( |S'(x, y)| \leq |N(x, y)| \). Combining this with part (3) of Observation 1 we obtain \( |F'(x, y)| \leq 2 |N(x, y)| \), as required.

Finally, let us note the following properties of the graph obtained after applying the procedure.
Observation 2. Let the sets $F'(x, y), D'(x, y), S'(x, y)$ (for $x, y \in A$ and $x, y \in B$) be defined by applying Procedure $P(q)$ to a bipartite graph $G$ to obtain $G'$. Then

1. Every edge $ab$ in $G'$ is contained in at most $q/2$ members of $\{F'(x, y) : x, y \in A\}$ and in at most $q/2$ members of $\{F'(x, y) : x, y \in B\}$.

2. For any set $M$ of edges in $G'$, removing the edges of $M$ from $G'$ decreases the number of $q$-dense pairs by less than $|M|/2$.

Definition 4. Let $H$ be a 3-partite 3-graph with parts $A, B$ and $C$.

For each $1 \leq i \leq n$, let $G_i[H](A, B)$ be the bipartite graph with parts $A$ and $B$, whose edge set is $\{ab \mid a \in A, b \in B, abc_i \in E(H)\}$. The graphs $G_i[H](B, C)$ and $G_i[H](A, C)$ are defined similarly.

Definition 5 (Applying Procedure $P(q)$ to a hypergraph). Let $H$ be a 3-partite 3-graph with parts $A, B$ and $C$. We define the hypergraph $H'$ as follows:

For each $1 \leq i \leq n$, let $G_i'[H](A, B)$, $G_i'[H](B, C)$, $G_i'[H](A, C)$ be the graphs obtained by applying the procedure $P(q)$ to the graphs $G_i[H](A, B)$, $G_i[H](B, C)$, $G_i[H](A, C)$ respectively.

For each edge $ab$ which was removed from $G_i[H](A, B)$ by the procedure $P(q)$ (i.e. $ab \in G_i[H](A, B) \setminus G_i'[H](A, B)$) we remove the hyperedge $abc_i$ from $H$ (it may have been removed already). Similarly for each edge $bc$ (resp. $ac$) which was removed from $G_i[H](B, C)$ (resp. $G_i[H](A, C)$) by the procedure $P(q)$ we remove the hyperedge $abc$ (resp. $abc_i$) from $H$. Let the resulting hypergraph be $H'$. More precisely, $H' = \{a_i b_j c_k \in H \mid a_i b_j \in G_i'[H](A, B), b_j c_k \in G_i'[H](B, C), a_i c_k \in G_i'[H](A, C)\}$.

We say $H'$ is obtained from $H$ by applying the Procedure $P(q)$.

Remark 1. Let $H'$ be obtained by applying the Procedure $P(q)$ to the hypergraph $H$. Then,

$$|H| - |H'| \leq \sum_{1 \leq i \leq n} (|G_i[H](A, B)| - |G_i'[H](A, B)|) + \sum_{1 \leq i \leq n} (|G_i[H](B, C)| - |G_i'[H](B, C)|)$$

$$+ \sum_{1 \leq i \leq n} (|G_i[H](A, C)| - |G_i'[H](A, C)|).$$

Indeed, if $a_i b_j c_k \in H \setminus H'$ then it is easy to see that $a_i b_j \in G_k[H](A, B) \setminus G_k'[H](A, B)$ or $b_j c_k \in G_i[H](B, C) \setminus G_i'[H](B, C)$ or $a_i c_k \in G_j[H](A, C) \setminus G_j'[H](A, C)$.

Lemma 2. Let $q \geq 2$ be an even integer and $G$ be a bipartite graph with parts $A$ and $B$. Suppose $G'$ is the graph obtained by applying Procedure $P(q)$ to $G$. Then $G'$ is $K_{2,q}$-free.

Proof. Let us define a $q$-broom of size $k$ to be a set of $q$-sparse pairs $\{x_0, x_j\}$ (with $1 \leq j \leq k$), and a $q$-dense pair $\{y, z\}$ such that $\{y, z\}$ is contained in the common neighborhood of $x_0, x_j$ for every $1 \leq j \leq k$. Note that either $\{x_0, x_1, \ldots, x_k\} \subseteq A$ and $\{y, z\} \subseteq B$ or $\{x_0, x_1, \ldots, x_k\} \subseteq B$ and $\{y, z\} \subseteq A$. 


Claim 2. There is no \( q \)-broom of size \( q/2 \) in \( G' \).

Proof. Suppose by contradiction that there is a set of \( q \)-sparse pairs \( \{x_0, x_j\} \) (with \( 1 \leq j \leq q/2 \)), and a \( q \)-dense pair \( \{y, z\} \) such that \( \{y, z\} \) is contained in the common neighborhood of \( x_0 \) and \( x_j \) for every \( 1 \leq j \leq q/2 \). Then the edge \( x_0 y \) is contained in the sets \( F'(x_0, x_j) \) for every \( 1 \leq j \leq q/2 \), which contradicts Observation 2.

Let us suppose for a contradiction (to Lemma 2) that \( G' \) contains a copy of \( K_{2,q} \). Then \( G' \) contains at least one \( q \)-dense pair. Without loss of generality we may assume there is a \( q \)-dense pair \( \{a, a_1\} \) in \( A \). Suppose \( \{a, a_j\} \) (for \( 1 \leq j \leq p \)) are all the \( q \)-dense pairs of \( G' \) containing the vertex \( a \). For each \( 1 \leq j \leq p \), let \( B_j \subseteq B \) be the common neighborhood of \( a \) and \( a_j \) in \( G' \). By definition, \( |B_j| \geq q \) for \( 1 \leq j \leq p \).

Claim 3. For any \( J \subseteq \{1, 2, \ldots, p\} \), we have \( |\bigcup_{j \in J} B_j| > 2 |J| \).

Proof. Let us assume for contradiction that there exists a \( J \subseteq \{1, 2, \ldots, p\} \) such that \( |\bigcup_{j \in J} B_j| \leq 2 |J| \). Let \( G^* \) be obtained from \( G' \) by deleting all the edges from \( a \) to \( \bigcup_{j \in J} B_j \). For each \( j \in J \), the pair \( \{a, a_j\} \) has no common neighbor in \( G^* \) since we have removed all the edges from \( a \) to \( B_j \). Thus the pair \( \{a, a_j\} \) is not \( q \)-dense in \( G^* \). So in forming \( G^* \) from \( G' \) the number of \( q \)-dense pairs decreases by at least \( |J| \), while the number of edges decreases by \( |\bigcup_{j \in J} B_j| \leq 2 |J| \) edges, contradicting Observation 2.

Let \( B' = \bigcup_{1 \leq j \leq p} B_j \). For each vertex \( v \in B' \) and let

\[
J(v) := \{ j \mid v \in B_j \},
\]

\[
D(v) := \{ \{v, u\} \mid \{v, u\} \text{ is } q\text{-dense in } G' \text{ and } \{v, u\} \subseteq B_j \text{ for some } j \in J(v) \}.
\]

In the next two claims, we will prove two useful inequalities concerning \( |J(v)| \) and \( |D(v)| \).

Claim 4. For each \( v \in B' \), \( |J(v)| > 2 |D(v)| \).

Proof. Suppose for contradiction that there is a vertex \( v \in B' \) such that \( |J(v)| \leq 2 |D(v)| \). Let us delete all the edges of the form \( va_j, j \in J(v) \), from \( G' \) and let the resulting graph be \( G^* \). Since we deleted \( |J(v)| \) edges, by Observation 2 the number of \( q \)-dense pairs decreases by less than \( |J(v)|/2 \leq |D(v)| \). So there exists \( \{v, u\} \in D(v) \) such that \( \{v, u\} \) is (still) \( q \)-dense in \( G^* \). That is, \( |N^*(v, u)| \geq q \), where \( N^*(v, u) \) denotes the common neighborhood of \( v \) and \( u \) in \( G^* \). Clearly each pair of vertices in \( N^*(v, u) \) is contained in a copy of \( K_{2,q} \) in \( G^* \) (and hence in \( G' \)).

For each pair of vertices in \( N^*(v, u) \), since it is contained in a copy of \( K_{2,q} \) in \( G' \), it is either \( q \)-sparse or \( q \)-dense in \( G' \). Note that \( a \in N^*(v, u) \). If all the pairs \( \{a, x\} \) with \( x \in N^*(v, u) \setminus \{a\} \) are \( q \)-sparse in \( G' \) then the set of these pairs together with \( \{v, u\} \) is a \( q \)-broom of size at least \( q - 1 \geq q/2 \) in \( G' \), which contradicts Claim 2. So there exists a vertex \( x \in N^*(v, u) \setminus \{a\} \) such that \( \{a, x\} \) is \( q \)-dense in \( G' \). Since \( v \) is adjacent to both \( a \) and \( x \), by the definition of \( J(v) \), \( x = a_j \) for some \( j \in J(v) \). However, by definition, in forming \( G^* \) we have removed \( vx \) from \( G' \). This contradicts \( x \in N^*(v, u) \) and completes the proof.
Claim 5.

\[ \sum_{v \in B'} |D(v)| \geq \frac{1}{2} \sum_{1 \leq j \leq p} |B_j|. \]

Proof. Fix any \( j \) with \( 1 \leq j \leq p \). Since \( \{a, a_j\} \) is \( q \)-dense in \( G' \), every pair \( \{x, y\} \subseteq B_j \) is contained in some copy of \( K_{2,q} \) and hence is either \( q \)-dense or \( q \)-sparse in \( G' \). Let \( v \) be any vertex in \( B_j \) and let \( S(v) = \{ y \in B_j \mid \{v, y\} \) is \( q \)-sparse in \( G' \} \). By definition, the set \( \{v, y\} \mid y \in S(v) \) together with \( \{a, a_j\} \) is a \( q \)-broom of size \( |S(v)| \). By Claim 2, \( |S(v)| \leq q/2 - 1 \leq |B_j|/2 - 1 \). Since \( |D(v)| + |S(v)| \geq |B_j| - 1 \), we have

\[ |D(v)| \geq \frac{1}{2} |B_j|. \]  

(5)

Note that (5) holds for every \( j = 1, \ldots, p \) and every \( v \in B_j \).

Let us define an auxiliary bipartite graph \( G_{aux} \) with a bipartition \((\{1, 2, \ldots, p\}, B')\) in which a vertex \( j \in \{1, 2, \ldots, p\} \) is joined to a vertex \( y \in B' \) if and only if \( y \in B_j \). Let \( J \) be an arbitrary subset of \( \{1, 2, \ldots, p\} \). The neighborhood of \( J \) in \( G_{aux} \) is precisely \( \bigcup_{j \in J} B_j \). By Claim 3, \( |\bigcup_{j \in J} B_j| > 2 |J| \geq |J| \). Since this holds for every \( J \subseteq \{1, 2, \ldots, p\} \), by Hall’s theorem [5] there exist distinct vertices \( w_j \in B_j \), for \( j = 1, \ldots, p \). By (5), for every \( j \in \{1, 2, \ldots, p\} \), \( |D(w_j)| \geq \frac{1}{2} |B_j| \). Hence

\[ \sum_{v \in B'} |D(v)| \geq \sum_{1 \leq j \leq p} |D(w_j)| \geq \frac{1}{2} \sum_{1 \leq j \leq p} |B_j|. \]

Using Claim 4 and Claim 5 we have

\[ \sum_{v \in B'} |J(v)| > \sum_{v \in B'} 2 |D(v)| \geq 2 \sum_{1 \leq j \leq p} \frac{1}{2} |B_j| = \sum_{1 \leq j \leq p} |B_j|, \]

which contradicts (6). This completes proof of Lemma 2. \( \square \)

In the next subsection we will prove a general lemma about making an arbitrary hypergraph \( K_{1.2,q} \)-free (for any given value of \( q \)). This lemma is used several times in the following subsections.
2.1 Applying Procedure \( \mathcal{P}(q) \) to an arbitrary hypergraph \( H \)

Let \( q \) be an even integer and let \( q \geq t \). Let \( H \) be an arbitrary \( K_{2,q}^{(3)} \)-free 3-partite 3-graph with parts \( A, B \) and \( C \). In this subsection we will prove the following lemma that estimates the number of edges removed from the graphs \( G_i = G_i[H](A,B) \) for \( 1 \leq i \leq n \), when the Procedure \( \mathcal{P}(q) \) is applied to them. This lemma together with Remark \( \square \) will allow us to estimate the number of edges removed from \( H \) when the Procedure \( \mathcal{P}(q) \) is applied to it.

Throughout this subsection, \( N_i(x,y) \) denotes the set of common neighbors of the vertices \( x,y \) in the graph \( G_i \).

**Lemma 3.** Let \( q \geq t \) be an even integer. Let \( H \) be an arbitrary \( K_{2,q}^{(3)} \)-free 3-partite 3-graph with parts \( A, B \) and \( C \). Let \( G_i = G_i[H](A,B) \) for \( 1 \leq i \leq n \). For each \( 1 \leq i \leq n \) and any \( x,y \in A \) or \( x,y \in B \), let \( F_i(x,y) \) be defined by applying the procedure \( \mathcal{P}(q) \) to \( G_i \), and let the resulting graph be \( G_i' \). Then,

\[
\sum_{1 \leq i \leq n} |G_i \setminus G_i'| < \frac{2}{q} \left( \sum_{u,v \in A} \sum_{1 \leq i \leq n} |F_i(u,v)| + \sum_{u,v \in B} \sum_{1 \leq i \leq n} |F_i(u,v)| \right) + 2tn^2.
\]

**Proof of Lemma 3.** First let us prove the following claim.

**Claim 6.** Let \( u,v \in A \) or \( u,v \in B \). Then \( \{u,v\} \) is \( q \)-dense in less than \( t \) of the graphs \( G_i \), \( 1 \leq i \leq n \).

**Proof.** Without loss of generality, suppose that \( u,v \in A \). Suppose for contradiction that \( \{u,v\} \) is \( q \)-dense in \( t \) of the graphs \( G_i \), \( 1 \leq i \leq n \). Without loss of generality suppose \( \{u,v\} \) is \( q \)-dense in \( G_1, \ldots, G_t \). Then \( |N_i(u,v)| \geq q \geq t \) for \( i = 1, \ldots, t \). Therefore, we can greedily choose \( t \) distinct vertices \( y_1, \ldots, y_t \) such that for each \( i \in [t] \), \( y_i \in N_i(u,v) \). For each \( i \in [t] \), since \( y_i \in N_i(u,v) \) we have \( uy_i c_i, vy_i c_i \in E(H) \). However, the set of hyperedges \( \{uy_i c_i, vy_i c_i \in E(H) \mid 1 \leq i \leq t \} \) forms a copy of \( K_{2,t}^{(3)} \) in \( H \), a contradiction. \( \square \)

Note that when procedure \( \mathcal{P}(q) \) is applied to \( G_i \) (to obtain \( G'_i \)), Step 1 and Step 2 may be applied several times (and each time one of these steps is applied it may delete an edge of \( G_i \)).

For each \( i \in [n] \), let \( m_i \) denote the number of \( q \)-dense pairs of \( G_i \). By Claim \( \square \) we know that each pair \( \{u,v\} \) with \( u,v \in A \) or \( u,v \in B \), is \( q \)-dense in less than \( t \) different graphs \( G_i \) (for \( 1 \leq i \leq n \)). Therefore,

\[
\sum_{1 \leq i \leq n} m_i \leq \sum_{u,v \in A} (t - 1) + \sum_{u,v \in B} (t - 1) = 2 \binom{n}{2} (t - 1).
\]  

(7)

For each \( i \in [n] \), let \( \alpha_i \) denote the total number of edges that were removed by Step 1 when procedure \( \mathcal{P}(q) \) is applied to \( G_i \) and \( \beta_i \) be the number of edges removed by Step 2
when procedure \( \mathcal{P}(q) \) is applied to \( G_i \). Then \( \alpha_i + \beta_i = |G_i \setminus G'_i| \), so \( \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} |G_i \setminus G'_i| \).

First, we bound \( \sum_{i=1}^{n} \beta_i \). Let \( i \in [n] \). Observe that whenever a set \( M \) of edges were removed by Step 2 of Procedure \( \mathcal{P}(q) \) applied to \( G_i \), the number of \( q \)-dense pairs decreased by at least \( |M|/2 \). Hence \( \beta_i \leq 2m_i \). So summing up over all \( 1 \leq i \leq n \), and using (7), we get

\[
\sum_{1 \leq i \leq n} \beta_i \leq 2 \sum_{1 \leq i \leq n} m_i \leq 2n(n-1)(t-1) < 2tn^2. \tag{8}
\]

Next, we bound \( \sum_{i=1}^{n} \alpha_i \). Let \( i \in [n] \). If an edge \( xy \) was removed from \( G_i \) by Step 1 of the procedure \( \mathcal{P}(q) \) then there are vertices \( z_1, z_2, \ldots, z_{q/2} \) such that \( xy \in F'_i(x, z_j) \) for every \( j \in \{1, 2, \ldots, q/2 \} \) or \( xy \in F'_i(y, z_j) \) for every \( j \in \{1, 2, \ldots, q/2 \} \). So

\[
\alpha_i \leq \frac{1}{q/2} \left( \sum_{u,v \in A} |F'_i(u, v)| + \sum_{u,v \in B} |F'_i(u, v)| \right).
\]

Therefore,

\[
\sum_{1 \leq i \leq n} \alpha_i \leq \frac{2}{q} \left( \sum_{1 \leq i \leq n} \sum_{u,v \in A} |F'_i(u, v)| + \sum_{1 \leq i \leq n} \sum_{u,v \in B} |F'_i(u, v)| \right).
\]

This is equivalent to the following.

\[
\sum_{1 \leq i \leq n} \alpha_i \leq \frac{2}{q} \left( \sum_{u,v \in A} \sum_{1 \leq i \leq n} |F'_i(u, v)| + \sum_{u,v \in B} \sum_{1 \leq i \leq n} |F'_i(u, v)| \right). \tag{9}
\]

Combining this inequality with (8) completes the proof of Lemma 3.

\[ \square \]

### 2.2 The overall plan

Let us define the sequence \( q_0, q_1, \ldots, q_k \) as follows. Let \( q_0 = 2^l \) where \( l \) is an integer such that \( q_0 = 2^l \leq t^2 < 2^{l+1} = 2q_0 \). For each \( 1 \leq j \leq k \), let \( q_j = \frac{q_{j-1}}{2} \), and \( q_k = \frac{q_0}{2} \). Clearly \( \frac{q_0}{q_k} = 2^k \), moreover

\[
2^k = \frac{q_0}{q_k} \leq \frac{t^2}{l^2} = t.
\]

So we have

\[
k \leq \log t. \tag{10}
\]

Now we apply the procedure \( \mathcal{P}(q_0) \) to the hypergraph \( H \) (recall Definition 5) to obtain a \( \overline{K}_{1,2,q_0} \)-free hypergraph \( H_0 \). For each \( 0 \leq j < k \) we obtain \( \overline{K}_{1,2,q_{j+1}} \)-free hypergraph \( H_{j+1} \) by applying the procedure \( \mathcal{P}(q_{j+1}) \) to the hypergraph \( H_j \).

This way, in the end we will get a \( \overline{K}_{1,2,q_k} \)-free hypergraph \( H_k \). In the following section, we will upper bound \( |H| - |H_0| \). Then in the next section, using the information that \( H_j \) is
$K_{1,2,q_0}$-free, we will upper bound $|H_{j+1}| - |H_j|$ for each $0 \leq j < k$. Then we sum up these bounds to upper bound the total number of deleted edges (i.e., $|H| - |H_k|$) from $H$ to obtain $H_k$. Finally, we bound the size of $H_k$, which will provide us the desired bound on the size of $H$.

2.3 Making $H$ $K_{1,2,q_0}$-free

First, we are going to prove an auxiliary lemma that is similar to Lemma A.4 of [8]. In an edge-colored multigraph $G$, an $s$-frame is a collection of $s$ edges all of different colors such that it is possible to pick one endpoint from each edge with all the selected endpoints being distinct.

**Lemma 4.** Let $G$ be an edge-colored multigraph with $e$ edges such that each edge has multiplicity at most $p$ and each color class has size at most $q$. If $G$ contains no $t$-frame then $|G| \leq \binom{t-1}{2}p + tq$.

**Proof.** Consider a maximum frame $S$, say with edges $e_1, \ldots, e_s$ such that for every $i \in \{1,2,\ldots,s\}$, $e_i$ has color $i$ and that there exist $x_1 \in e_1, x_2 \in e_2, \ldots, x_s \in e_s$ with $x_1, \ldots, x_s$ being distinct. By our assumption, $s \leq t-1$. Let $f$ be any edge with a color not in $[s]$. Then both vertices of $f$ must be in $\{x_1, \ldots, x_s\}$, otherwise $e_1, \ldots, e_s, f$ give a larger frame, a contradiction. On the other hand, each edge with both of its vertices in $\{x_1, \ldots, x_s\}$ has multiplicity at most $p$. Hence there are at most $\binom{s}{2}p$ edges with colors not in $\{1,2,\ldots,s\}$. The number of edges with color in $\{1,2,\ldots,s\}$ is at most $sq$ by our assumption. So $|G| \leq \binom{s}{2}p + sq \leq \binom{t-1}{2}p + tq$. □

Let us recall that $H$ is 3 partite $K_{2,3}^{(3)}$-free hypergraph with $A, B, C$. For convenience we denote $G_i = G_i[H](A, B)$ where $1 \leq i \leq n$. For each $1 \leq i \leq n$ and any $x, y \in A$ or $x, y \in B$, let $F'_i(x,y)$, $D'_i(x,y)$ and $S'_i(x,y)$ be defined by applying the procedure $\mathcal{P}(q_0)$ on $G_i$ and let the obtained graph be $G'_i$.

First, observe that $t^2/2 < q_0 \leq t^2$ according to our definition.

**Claim 7.** Let $u, v \in A$ or $u, v \in B$. Then $\sum_{1 \leq i \leq n} |F'_i(u,v)| \leq 6t^3$.

**Proof.** Let $D^*$ be an edge-colored multigraph in which a pair of vertices $e$ is an edge of color $i \in [n]$ whenever $e$ is an edge of $D'_i(u,v)$. The number of edges of color $i$ in $D^*$ is $|D'_i(u,v)|$. By Claim 1 we have $|D'_i(u,v)| < q_0$. Hence the number of edges in each color class of $D^*$ is less than $q_0$.

Let $xy$ be an arbitrary edge of $D^*$ and let $I = \{ i \in [n] \mid xy \in D'_i(u,v) \}$. For each $i \in I$, the pair $\{x, y\}$ is $q_0$-dense in $G_i$ by the definition of $D'_i(u,v)$. Therefore, by Claim 3 we have $|I| < t$. So $xy$ has multiplicity less than $t$ in $D^*$. Since $xy$ is arbitrary, the multiplicity of each edge of $D^*$ is less than $t$. 

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Next, observe that $D^*$ contains no $t$-frame. Indeed, otherwise without loss of generality we may assume that $D^*$ contains $t$ edges $x_1y_1, \ldots, x_ty_t$, where $x_iz_j$ has color $i$ for each $i \in [t]$ and $y_1, \ldots, y_t$ are distinct. For each $i \in [t]$ since $x_iz_i \in D^*_i(u,v)$, in particular $y_i \in N_i(u,v)$ (where $N_i(u,v)$ denotes the common neighborhood of $u$ and $v$ in $G_i$), which means that $uy_ic_i, vy_ic_i \in H$. But now, \{uy_i, vy_i | i \in [t]\} forms a copy of $K_{2,t}^{(3)}$, contradicting $H$ being $K_{2,t}^{(3)}$-free.

Therefore, applying Lemma 4, we have $|D^*| \leq (t^2 - 1) t + tq_0$. By Claim 1, we have

$$\sum_{1 \leq i \leq n} \frac{|F'_i(u,v)|}{4} \leq |D'_i(u,v)|.$$ 

So

$$\sum_{1 \leq i \leq n} \frac{|F'_i(u,v)|}{4} \leq \sum_{1 \leq i \leq n} |D'_i(u,v)| = |D^*| \leq \left(\frac{t - 1}{2}\right) t + tq_0 < \frac{3}{2} t^2,$$

which proves the claim. □

By Lemma 3, we have

$$\sum_{1 \leq i \leq n} |G_i \setminus G'_i| < \frac{2}{q_0} \left( \sum_{u,v \in A} \sum_{1 \leq i \leq n} |F'_i(u,v)| + \sum_{u,v \in B} \sum_{1 \leq i \leq n} |F'_i(u,v)| \right) + 2tn^2.$$

Combining it with Claim 7 we get

$$\sum_{1 \leq i \leq n} |G_i \setminus G'_i| < \frac{2}{q_0} \left( \sum_{u,v \in A} 6t^3 + \sum_{u,v \in B} 6t^3 \right) + 2tn^2.$$

Therefore, as $q_0 > t^2/2$, we have

$$\sum_{1 \leq i \leq n} |G_i \setminus G'_i| < \frac{4}{t^2} \left( 12t^3 \left(\frac{n}{2}\right) \right) + 2tn^2 < 26tn^2.$$

So,

$$\sum_{1 \leq i \leq n} |G_i \setminus G'_i| = \sum_{1 \leq i \leq n} |G_i[H](A,B) \setminus G'_i[H](A,B)| < 26tn^2.$$

By symmetry, using the same arguments, we have

$$\sum_{1 \leq i \leq n} |G_i[H](B,C) \setminus G'_i[H](B,C)| < 26tn^2,$$

and

$$\sum_{1 \leq i \leq n} |G_i[H](A,C) \setminus G'_i[H](A,C)| < 26tn^2.$$

Therefore, by Remark 1, we have

$$|H| - |H_0| < 78tn^2. \quad (11)$$
2.4 Making a $K_{1,2,q_j}$-free hypergraph $K_{1,2,q_j+1}$-free

In this subsection, we fix a $j$ with $0 \leq j < k$. Recall that $H_j$ is $K_{1,2,q_j}$-free, and $H_{j+1}$ is obtained by applying the $P(q_{j+1})$ to $H_j$. Our goal in this subsection is to estimate $|H_j| - |H_{j+1}|$. The key difference between arguments in this subsection and in the previous subsection is that now in addition to $H_j$ being $K^{(3)}_{2,t}$-free we can also utilize the fact that $H_j$ is $K_{1,2,q_j}$-free. In particular, this extra condition leads to Claim 8, which improves upon Claim 7. For convenience of notation, in this subsection, let $G_i = G[H_j](A,B)$ for each $1 \leq i \leq n$. For every $1 \leq i \leq n$ and every $u, v \in A$ or $u, v \in B$ let the sets $F'_i(u, v)$ and $D'_i(u, v)$ be defined by applying the procedure $P(q_{j+1})$ to the graph $G_i$, to obtain the graph $G_i'$. **Claim 8.** Let $u, v \in A$ or $u, v \in B$. Then $\sum_{1 \leq i \leq n} |F'_i(u, v)| < 2q_j t$.

**Proof.** For each $i \in [n]$ we denote the set of common neighbors of $u, v$ in $G_i$ as $N_i(x, y)$. For each $i \in [n]$, since $H_j$ is $K_{1,2,q_j}$-free, $G_i$ is $K_{2,q_j}$-free and so $|N_i(u, v)| < q_j$. Without loss of generality let us assume $u, v \in A$. For each vertex $w$ of $B$, let $I_w = \{ i \in \{1, 2, \ldots , n\} \mid w \in N_i(u, v)\}$. We claim that $|I_w| < q_j$. Indeed, for each $i \in I_w$, we have $uw_{i}, vw_{i} \in H_j$. So the set of edges $\{uw_{i}, vw_{i} \mid i \in I_w\}$ form a copy of $K_{1,2,q_j}$ in $H_j$. Thus if $|I_w| \geq q_j$, then $H_j$ contains a copy of $K_{1,2,q_j}$, a contradiction. Therefore, $|I_w| < q_j$, as desired.

Consider an auxiliary bipartite graph $G_{AUX}$ with parts $B$ and $[n]$ where the vertex $i \in [n]$ is adjacent to $b \in B$ in $G_{AUX}$ if and only if $b \in N_i(u, v)$. Then by the discussion in the previous paragraph, each vertex $w \in B$ has degree $|I_w| < q_j$, and each vertex $i \in [n]$ has degree $|N_i(u, v)| < q_j$. In other words, the maximum degree in $G_{AUX}$ is less than $q_j$.

We claim that $G_{AUX}$ does not contain a matching of size $t$. Indeed, suppose for a contradiction that the edges $i_1b_{i_1}, i_2b_{i_2}, \ldots , i_{t}b_{i_{t}}$ (i.e., $b_{i_l} \in N_i(u, v)$ for $1 \leq l \leq t$) form a matching of size $t$ in $G_{AUX}$. Then the set of hyperedges $ub_{i_l}c_{i_l}, vb_{i_l}c_{i_l}, 1 \leq l \leq t$, form a copy of $K^{(3)}_{2,t}$ in $H_j$, a contradiction, as desired.

Since $G_{AUX}$ does not contain a matching of size $t$, by the König-Egerváry theorem it has a vertex cover of size less than $t$. This fact combined with the fact that the maximum degree of $G_{AUX}$ is less than $q_j$, implies that the number of edges of $G_{AUX}$ is less than $q_j t$. On the other hand, the number of edges in $G_{AUX}$ is $\sum_{i \in [n]} |N_i(u, v)|$. Therefore, $\sum_{i \in [n]} |N_i(u, v)| < q_j t$. This, combined with the fact that for each $i \in [n]$, $|N_i(u, v)| \geq |F'_i(u, v)| / 2$ (see Claim 7), completes the proof of the lemma.

By Lemma 3, we have

$$\sum_{1 \leq i \leq n} |G_i \setminus G'_i| \leq \frac{2}{q_{j+1}} \left( \sum_{u, v \in A} \sum_{1 \leq i \leq n} |F'_i(u, v)| + \sum_{u, v \in B} \sum_{1 \leq i \leq n} |F'_i(u, v)| \right) + 2tn^2.$$

Now using Claim 8, we have
Since $q_{j+1} = q_j / 2$, we have
\[
\sum_{1 \leq i \leq n} |G_i \setminus G'_i| < 8tn^2 + 2t^2 = 10tn^2.
\]

So,
\[
\sum_{1 \leq i \leq n} |G_i \setminus G'_i| = \sum_{1 \leq i \leq n} |G_i[H_j](A, B) \setminus G'_i[H_j](A, B)| < 10tn^2.
\]

By symmetry, using the same arguments, we have
\[
\sum_{1 \leq i \leq n} |G_i[H_j](B, C) \setminus G'_i[H_j](B, C)| < 10tn^2,
\]
and
\[
\sum_{1 \leq i \leq n} |G_i[H_j](A, C) \setminus G'_i[H_j](A, C)| < 10tn^2.
\]

Therefore, by Remark 1 we have
\[
|H_j| - |H_{j+1}| < 30tn^2. \tag{12}
\]

### 2.5 Putting it all together

By (11) and (12) we have
\[
|H| - |H_k| = |H| - |H_0| + \sum_{0 \leq j < k} (|H_j| - |H_{j+1}|) < 78tn^2 + k(30tn^2).
\]

By (10) we have $k \leq \log t$, so we obtain,
\[
|H| - |H_k| < 78tn^2 + 30t \log tn^2. \tag{13}
\]

Notice that $H_k$ is $K_{1,2,q_k}$-free and $q_k < 2t$. Therefore $H_k$ is $K_{1,2,2t}$-free. Moreover, we know that the hypergraph $H_k$ is 3-partite and $K_{2,t}^{(3)}$-free with parts $A, B, C$ (as it is a subhypergraph of $H$). Now we bound the size of $H_k$.

**Claim 9.** We have $|H_k| \leq 2tn^2$.

**Proof.** Suppose for a contradiction that $|H_k| > 2tn^2$. For any pair $\{a, b\}$ of vertices with $a \in A$ and $b \in B$, let $\text{codeg}(a, b)$ denote the number of hyperedges of $H_k$ containing the pair
\[
\sum_{1 \leq i \leq n} |G_i \setminus G'_i| \leq \frac{8q_jt}{q_{j+1}} \binom{n}{2} + 2tn^2 < \frac{4tq_j}{q_{j+1}} n^2 + 2tn^2.
\]
\{a, b\}. Then the number of copies of \(K_{2,1,1}\) in \(H_k\) of the form \(\{abc, a'bc\}\) where \(a, a' \in A, b \in B, c \in C\) is

\[
\sum_{b,c \in C} \binom{\text{codeg}(b, c)}{2}.
\]

As the average codegree (over all the pairs \(b \in B, c \in C\)) is more than \(2t\), by convexity, this expression is more than

\[
\left(\frac{2t}{2}\right) n^2 > (2t - 1)^2 \binom{n}{2}.
\]

This means there exist a pair \(a, a' \in A\) and a set of \((2t - 1)^2 + 1 > (t - 1)(2t - 1) + 1\) pairs \(S := \{bc \mid b \in B, c \in C\}\) such that \(abc, a'bc \in E(H_k)\) whenever \(bc \in S\). Let \(G_{\text{Aux}}\) be a bipartite graph whose edges are elements of \(S\). Since \(G_{\text{Aux}}\) has \(|S| \geq (t - 1)(2t - 1) + 1\) edges, it either contains a matching \(M\) with \(t\) edges or a vertex \(v\) of degree \(2t\) (see Lemma A.3 in [8] or the last paragraph of our proof of Claim 8 for a proof). In the former case, the set of all hyperedges of the form \(abc, a'bc\) with \(bc \in M\), form a copy of \(K_{2,t}^3\) in \(H_k\), a contradiction. In the latter case, let \(u_1, u_2, \ldots, u_{2t}\) be the neighbors of \(v\) in \(G_{\text{Aux}}\). Then the set of hyperedges \(\{avu_i, a'vu_i \mid 1 \leq i \leq 2t\}\) form a copy of \(K_{1,2t}\) in \(H_k\), a contradiction again. This completes the proof of the claim. \(\square\)

Combining (13) with Claim 9, we have \(|H| \leq 80tn^2 + 30t \log tn^2\), thus proving (14), which implies Theorem 11 as desired.

### 3 Concluding remarks

Recall that given a bipartite graph \(G\) with an ordered bipartition \((X,Y)\), where \(Y = \{y_1, \ldots, y_m\}\), \(G_{X,Y}^{(r)}\) is the \(r\)-graph with vertex set \((X \cup Y) \cup (\bigcup_{i=1}^m Y_i)\) and edge set \(\bigcup_{i=1}^m \{e \cup Y_i : e \in E(G), y_i \in e\}\), where \(Y_1, \ldots, Y_m\) are disjoint \((r - 2)\)-sets that are disjoint from \(X \cup Y\). A standard reduction argument such as the one used in the proof of Theorem 1.4 in [8] can be used to show the following.

**Proposition 1.** Let \(n, r \geq 3\) be integers and \(G\) a bipartite graph with an ordered bipartition \((X,Y)\). There exists a constant \(c_r\) depending only on \(r\) such that

\[
\text{ex}(n, G_{X,Y}^{(r)}) \leq c_r n^{r-3} \cdot \text{ex}(n, G_{X,Y}^{(3)}).
\]

Thus, by Theorem 11 and Proposition 11 for all \(r \geq 4\), we have \(\text{ex}(n, K_{2,t}^{(r)}) \leq c_r t \log t \binom{n}{r-1}\) for some constant \(c_r\), depending only on \(r\). On the other hand, taking the family of all \(r\)-element subsets of \([n]\) containing a fixed element shows that \(\text{ex}(n, K_{2,t}^{(r)}) \geq \binom{n-1}{r-1}\). Recall that in the \(r = 3\) case, a better lower bound of \(\Omega(t^{n/2})\) was shown by Mubayi and Verstraëte [8]. For \(r = 4\), we are able to improve the lower bound to \(\Omega(t^{3/2})\) as follows.
Proposition 2. We have

\[ \text{ex}(n, K_{2,t}^{(4)}) \geq (1 + o(1)) \frac{t - 1}{8} n^3. \]

Proof. (Sketch.) Consider a \( K_{2,t} \)-free graph \( G \) with \( (1 + o(1))^{\frac{t-1}{2}} n^{3/2} \) edges where each vertex has degree \( (1 + o(1)) \sqrt{(t - 1)} \sqrt{n} \). (Such a graph exists by a construction of Füredi [8].) Let us define a 4-graph \( H = \{abcd | ab, cd \in G \text{ and } ac, ad, bc, bd \notin G \} \). In other words, let the edges of \( H \) be the vertex sets of induced 2-matchings in \( G \). Via standard counting, it is easy to show that \( |H| = (1 + o(1))^{\frac{t-1}{2}} n^3 \). It remains to show \( H \) is \( K_{2,t}^{(4)} \)-free.

Claim 10. If \( axyz, bxyz \in H \), then there is a vertex \( c \in \{x, y, z\} \) such that \( ac, bc \in G \).

Proof. By our assumption, \( \{a, x, y, z\} \) and \( \{b, x, y, z\} \) both induce a 2-matching in \( G \). Without loss of generality, suppose \( ax, yz \in G \). If \( bx \in G \) then we are done. Otherwise, we have \( by, xz \in G \) or \( bz, xy \in G \), both contradicting \( \{ax, yz\} \) being an induced matching in \( G \).

Suppose for contradiction that \( H \) has a copy of \( K_{2,t}^{(4)} \) with edge set \( \{ax_iy_iz_i, bx_iz_i | 1 \leq i \leq t\} \). By Claim [10], for each \( 1 \leq i \leq t \), there exists a vertex \( w_i \in \{x_i, y_i, z_i\} \) such that \( aw_i, bw_i \in G \). This yields a copy of \( K_{2,t} \) in \( G \), a contradiction.

For \( r \geq 5 \), we do not yet have a lower bound that is asymptotically larger than \((\frac{n-1}{r-1})^2\). It would be interesting to narrow the gap between the lower and upper bounds on \( \text{ex}(n, K_{2,t}^{(r)}) \).

It will be interesting to have a systematic study of the function \( \text{ex}(n, G_{X,Y}^{(r)}) \). Mubayi and Verstraëte [8] showed that \( \text{ex}(n, K_{s,t}^{(3)}) = O(n^{3-1/s}) \) and that if \( t > (s-1)! > 0 \) then \( \text{ex}(n, K_{s,t}^{(3)}) = \Omega(n^{3-2/s}) \) and speculated that \( n^{3-2/s} \) is the correct order of magnitude. The case when \( G \) is a tree is studied in [4], where the problem considered there is slightly more general. The case when \( G \) is an even cycle has also been studied. Let \( C_{2t}^{(r)} \) denote \( G_{X,Y}^{(r)} \) where \( G \) is the even cycle \( C_{2t} \) of length \( 2t \). It was shown by Jiang and Liu [3] that \( c_1 t^\left(\frac{n}{r-1}\right) \leq \text{ex}(n, C_{2t}^{(r)}) \leq c_2 t^5 \left(\frac{n}{r-1}\right) \), for some positive constants depending \( c_1, c_2 \) on \( r \). Using results in this paper and new ideas, we are able to narrow the gap to \( c_1 t^{\left(\frac{n}{r-1}\right)} \leq \text{ex}(n, C_{2m}^{(r)}) \leq c_2 t^2 \log t^{\left(\frac{n}{r-1}\right)} \), for some positive constants \( c_1, c_2 \) depending on \( r \). We would like to postpone this and other results on the topic for a future paper.

Finally, motivated by results on \( K_{2,t}^{(r)} \) and \( C_{2t}^{(r)} \), we pose the following question.

Question 1. Let \( r \geq 3 \). Let \( G \) be the family of bipartite graphs \( G \) with an ordered bipartition \((X, Y)\) in which every vertex in \( Y \) has degree at most 2 in \( G \). Is it true that \( \forall G \in \mathbb{G} \) there is a constant \( c \) depending on \( G \) such that \( \text{ex}(n, G_{X,Y}^{(r)}) \leq c \left(\frac{n}{r-1}\right)^2 \)?
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