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Some unlimited families of minimal surfaces of general type with the canonical map of degree 8

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Abstract. In this note, we construct nine families of projective complex minimal surfaces of general type having the canonical map of degree 8 and irregularity 0 or 1. For six of these families the canonical system has a non trivial fixed part.

1. Introduction

Let $X$ be a smooth complex surface of general type (see [3] or [1]) and let $\varphi|_{K_X} : X \rightarrow \mathbb{P}^{p_g(X)-1}$ be the canonical map of $X$, where $p_g(X) = \dim \left( H^0(X, K_X) \right)$ is the geometric genus and $K_X$ is the canonical divisor of $X$. A classical result of Beauville [2, Theorem 3.1] says that if the image of $\varphi|_{K_X}$ is a surface, either $p_g \left( \text{im} \left( \varphi|_{K_X} \right) \right) = 0$ or $\text{im} \left( \varphi|_{K_X} \right)$ is a surface of general type. In addition, the degree $d$ of the canonical map of $X$ is less than or equal to 36.

While surfaces with $d = 2$ has been studied thoroughly by Horikawa in his several papers such as [7–10], the case where $d$ bigger than 2 remains to be one of the most interesting open problems in the theory of surfaces. Several surfaces with $d$ bigger than 2 have been constructed, for example with $d = 3, 5, 9$ by Pardini [13] and Tan [18], $d = 6, 8$ by Beauville [2], $d = 4$ by Beauville [2], and Gallego and Purnaprajna [6], $d = 16$ by Persson [14] and Rito [17], $d = 12, 24$ by Rito [15, 16], etc.

In the same paper [2], Beauville also proved that the degree of the canonical map is less than or equal to 9 if $\chi(O_X) \geq 31$. Later, Xiao showed that if the geometric genus of $X$ is bigger than 132, the degree of the canonical map is less than or equal to 8 [19]. In addition, he also proved that if the degree of the canonical map is 8 and geometric genus is bigger than 115, the irregularity $q = h^0 \left( \Omega^1_X \right)$ is less than or equal to 3 (see [20]). Beauville constructed an unlimited family of surfaces with

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d = 8 and arbitrarily high geometric genus [2]. These surfaces have irregularity $q = 3$ and the canonical linear system of these surfaces is base point free.

In this note, we construct nine unlimited families of surfaces with $d = 8$ and $q = 0$ or $q = 1$. Furthermore, for some families the canonical linear systems are not base point free. The following theorem is the main result of this note:

**Theorem 1.** Let $n$ be an integer number such that $n \geq 2$. Then there exist minimal surfaces of general type $X$ with canonical map $\varphi_{|K_X|}$ of degree 8 and the following invariants

| $K^2_X$ | $p_g(X)$ | $q(X)$ | $|K_X|$ is base point free |
|---------|----------|--------|--------------------------|
| $16n - 8$ | $2n + 1$ | 0 | Yes |
| $16n - 16$ | $2n$ | 0 | Yes |
| $16n - 16$ | $2n$ | 1 | Yes |
| $16n - 10$ | $2n$ | 0 | No |
| $16n$ | $2n + 1$ | 0 | No |
| $16n - 8$ | $2n$ | 0 | No |
| $16n - 8$ | $2n$ | 1 | No |
| $16n - 2$ | $2n$ | 0 | No |
| $16n$ | $2n$ | 1 | No |

The approach to construct these surfaces is using $\mathbb{Z}_3^2$—covers with some appropriate branch loci. Note that canonical maps defined by abelian covers of $\mathbb{P}^2$, and in particular the abelian covers with the group $\mathbb{Z}_3^2$, have been studied very explicitly by Du and Gao [5].

2. $\mathbb{Z}_3^2$ coverings

The construction of abelian covers was studied by Pardini [12]. Let $H_{i_1,i_2,i_3}$ denote the nontrivial cyclic subgroup generated by $(i_1, i_2, i_3)$ of $\mathbb{Z}_3^2$ for all $(i_1, i_2, i_3) \in \mathbb{Z}_3^2 \setminus (0, 0, 0)$, and denote by $\chi_{j_1,j_2,j_3}$ the character of $\mathbb{Z}_3^2$ defined by

$$\chi_{j_1,j_2,j_3}(a_1,a_2,a_3):= e^{(\pi a_1 j_1)i}e^{(\pi a_2 j_2)i}e^{(\pi a_3 j_3)i}$$

for all $j_1, j_2, j_3, a_1, a_2, a_3, a_4 \in \mathbb{Z}_2$. For sake of simplicity, from now on we use notations $D_1, D_2, D_3, D_4, D_5, D_6, D_7$ instead of $D(H_{0,0,1,0,0,1}), D(H_{0,1,0,0,1,0}), D(H_{0,1,0,1,0,0}), D(H_{1,0,0,1,0,0}), D(H_{1,0,1,0,0,0}), D(H_{1,1,0,0,0,0}), D(H_{1,1,1,0,0,0})$, respectively. For details about the building data of abelian covers and their notations, we refer the reader to Sections 1 and 2 of R. Pardini’s work ([12]). From [12, Theorem 2.1] we can define $\mathbb{Z}_3^2$—covers as follows:

**Proposition 1.** Let $Y$ be a smooth projective surface with no 2-torsion. Let $L_\chi$ be divisors of $Y$ such that $L_\chi \neq \mathcal{O}_Y$ for all nontrivial characters $\chi \in (\mathbb{Z}_3^2)^* \setminus \{\chi_{0,0,0}\}$. Let $D_1, D_2, \ldots, D_7$ be effective divisors of $Y$ such that the branch divisor
Surfaces with the canonical map

B := \sum_{i=1}^7 D_i is reduced. Then \{L_x, D_j\}_{j, i} is the building data of a \( \mathbb{Z}_2^3 \)-cover \( f : X \rightarrow Y \) if and only if

\[
2L_{1,0,0} = D_4 + D_5 + D_6 + D_7 \\
2L_{0,1,0} = D_2 + D_3 + D_6 + D_7 \\
2L_{0,0,1} = D_1 + D_3 + D_5 + D_7 \\
2L_{1,1,0} = D_2 + D_3 + D_4 + D_5 \\
2L_{1,0,1} = D_1 + D_3 + D_4 + D_6 \\
2L_{0,1,1} = D_1 + D_2 + D_5 + D_6 \\
2L_{1,1,1} = D_1 + D_2 + D_4 + D_7.
\]

By [12, Theorem 3.1] if each \( D_\sigma \) is smooth and \( B \) is a simple normal crossings divisor, then the surface \( X \) is smooth.

Also from [12, Lemma 4.2, Proposition 4.2] we have:

**Proposition 2.** Let \( f : X \rightarrow Y \) be a smooth \( \mathbb{Z}_3^2 \)-cover with the building data \( D_1, D_2, \ldots, D_7, L_x, \forall x \in (\mathbb{Z}_3^2)^* \setminus \{\chi_{0,0,0}\} \). The invariants of \( X \) are as follows:

\[
2K_X \equiv f^* \left( 2K_Y + \sum_{j=1}^7 D_j \right) \\
K_X^2 = 2 \left( 2K_Y + \sum_{j=1}^7 D_j \right)^2 \\
p_g (X) = p_g (Y) + \sum_{x \in (\mathbb{Z}_3^2)^* \setminus \{\chi_{0,0,0}\}} h^0 (L_x + K_Y) \\
\chi (\mathcal{O}_X) = 8\chi (\mathcal{O}_Y) + \sum_{x \in (\mathbb{Z}_3^2)^* \setminus \{\chi_{0,0,0}\}} \frac{1}{2} L_x (L_x + K_Y).
\]

**Notation 1.** We denote \( P = (k_1, k_2, \ldots, k_7) \) when \( D_1, D_2, \ldots, D_7 \) contain \( P \) with multiplicity \( k_1, k_2, \ldots, k_7 \), respectively.

### 3. Constructions

**3.1. Construction 1**

In this section, we construct the surfaces in the first four rows of Theorem 1.

**3.1.1. Construction and computation of invariants** Let \( F_1 \) denote the Hirzebruch surface with the negative section \( \Delta_0 \) with self-intersection \(-1\) and let \( \Gamma \) denote a fiber of the ruling. Let \( D_2 = 2n \Gamma \) be \( 2n \) fibers in \( F_1 \) and \( D_3, D_6, D_7 \in [2\Delta_0 + 2\Gamma] \) be smooth curves in general position. Let \( f : X \rightarrow F_1 \) be a \( \mathbb{Z}_2^3 \)-cover with the following branch locus

\[
B = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7.
\]
where \( D_1 = D_4 = D_5 = 0 \). By Proposition 1, \( L_{0,1,0} \equiv 3 \Delta_0 + (n + 3) \Gamma \) and \( L_X \) is equivalent to either \( 2 \Delta_0 + 2 \Gamma \) or \( \Delta_0 + (n + 1) \Gamma \) for all \( L_X \neq L_{0,1,0} \). Since each \( D_\sigma \) is smooth and \( B \) is a normal crossings divisor, \( X \) is smooth. Moreover, by Proposition 2, we get

\[
2K_X \equiv f^* (2 \Delta_0 + 2n \Gamma) .
\]

This implies that \( X \) is a minimal surface of general type. Furthermore, by Proposition 2, the invariants of \( X \) are as follows:

\[
K_X^2 = 8 (2n - 1) \quad (1)
\]

\[
p_g (X) = h^0 (\Delta_0 + n \Gamma) = 2n + 1 \quad (2)
\]

\[
\chi (\mathcal{O}_X) = 2n + 2. \quad (3)
\]

From (2) and (3), we get \( q (X) = 0 \).

We show that \(|K_X|\) is not composed with a pencil by considering the following double cover

\[
f_1 : X_1 \longrightarrow \mathbb{P}_1
\]

ramifying on \( D_2 + D_3 + D_6 + D_7 \). We have

\[
K_{X_1} \equiv f_1^* (\Delta_0 + n \Gamma) .
\]

Because \(|\Delta_0 + n \Gamma|\) is not composed with a pencil, \(|K_{X_1}|\) is not composed with a pencil, either. This leads to the fact that \(|K_X|\) is not composed with a pencil and the degree of the canonical map is 8. Moreover, \( \deg (\text{im} \varphi_{|K_X|}) = 2n - 1 \).

3.1.2. Variations

Now by adding a singular point to the above branch locus, we obtain the surfaces described in the second row of Theorem 1. In fact, by Proposition 1, a new branch locus can be formed by adding a point \( P = (0, 1, 1, 0, 0, 1, 1) \) (see Notation 1). And we consider the \( \mathbb{Z}_2^3 \)-cover on \( Y \) instead of \( \mathbb{P}_1 \), where \( Y \) is the blow up of \( \mathbb{P}_1 \) at \( P \). More precisely, let \( P \) be a point in \( \mathbb{P}_1 \) such that \( D_2, D_3, D_6, D_7 \) contain \( P \) with multiplicity 1, 1, 1, 1, respectively. Let \( Y \) be the blow up of \( \mathbb{P}_1 \) at \( P \) and \( E \) be the exceptional divisor. If we abuse notation and denote \( D_2, D_3, D_6, D_7, \Delta_0, \Gamma \) their pullbacks to \( Y \), then \( D_2 = 2n \Gamma + E, D_3 = 2 \Delta_0 + 2 \Gamma - E, D_6 = 2 \Delta_0 + 2 \Gamma - E \) and \( D_7 = 2 \Delta_0 + 2 \Gamma - E \). Let \( f : X \longrightarrow Y \) be a \( \mathbb{Z}_2^3 \)-cover with the following branch locus

\[
B = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7,
\]

where \( D_1 = D_4 = D_5 = 0 \). The building data is as follows:

\[
\begin{align*}
L_{1,0,0} & \equiv 2 \Delta_0 + 2 \Gamma - E \\
L_{0,1,0} & \equiv 3 \Delta_0 + (n + 3) \Gamma - 2E \\
L_{0,0,1} & \equiv 2 \Delta_0 + 2 \Gamma - E \\
L_{1,1,0} & \equiv \Delta_0 + (n + 1) \Gamma - E \\
L_{1,0,1} & \equiv 2 \Delta_0 + 2 \Gamma - E \\
L_{0,1,1} & \equiv \Delta_0 + (n + 1) \Gamma - E \\
L_{1,1,1} & \equiv \Delta_0 + (n + 1) \Gamma - E.
\end{align*}
\]
Similarly to the above, we obtain minimal surfaces of general type with

\[ K^2 = 16n - 16, \ p_g = 2n, q = 0, \ d = 8, \]

and \( \deg(\text{im} \varphi_{|K_X|}) = 2n - 2. \) Moreover, \( \varphi_{|K_X|} \) is a morphism.

Analogously, by Proposition 1, a point \((0, 0, 0, 0, 0, 2, 2)\) can be added to the original branch locus. In fact, let \( P \) be a point in \( \mathbb{P}^1 \) such that \( D_6, D_7 \) contain \( P \) with multiplicity 2, 2, respectively. Let \( Y \) be the blow up of \( \mathbb{P}^1 \) at \( P \) and \( E \) be the exceptional divisor. If we abused notation and denote \( D_2, D_3, D_6, D_7, \Delta_0, \Gamma \) their pullbacks to \( Y \), then \( D_2 = 2n\Gamma, D_3 = 2\Delta_0 + 2\Gamma, D_6 = 2\Delta_0 + 2\Gamma - 2E \) and \( D_7 = 2\Delta_0 + 2\Gamma - 2E. \) Let \( f : X \rightarrow Y \) be a \( \mathbb{Z}_2^3 \)-cover with the following branch locus

\[ B = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7, \]

where \( D_1 = D_4 = D_5 = 0. \) The building data is as follows:

\[
\begin{align*}
L_{1,0,0} &\equiv 2\Delta_0 + 2\Gamma - 2E \\
L_{0,1,0} &\equiv 3\Delta_0 + (n + 3)\Gamma - 2E \\
L_{0,0,1} &\equiv 2\Delta_0 + 2\Gamma - E \\
L_{1,1,0} &\equiv \Delta_0 + v(n + 1)\Gamma \\
L_{1,0,1} &\equiv 2\Delta_0 + 2\Gamma - E \\
L_{0,1,1} &\equiv \Delta_0 + (n + 1)\Gamma - E \\
L_{1,1,1} &\equiv \Delta_0 + (n + 1)\Gamma - E.
\end{align*}
\]

We get minimal surfaces of general type with

\[ K^2 = 16n - 16, \ p_g = 2n, q = 1, \ d = 8, \]

and \( \deg(\text{im} \varphi_{|K_X|}) = 2n - 2. \) Furthermore, \( \varphi_{|K_X|} \) is a morphism. Therefore we obtain the surfaces described in the third row of Theorem 1. The Albanese pencil of these surfaces \( X \rightarrow \text{Alb}(X) \) is the pullback of the Albanese pencil of the intermediate surface \( Z, \) where \( Z \) is obtained by the \( \mathbb{Z}_2 \)-cover ramifying on \( 2L_{1,0,0}. \) For details about the surfaces with \( q > 0, \) we refer the reader to the work of Mendes Lopes and Pardini [11].

Remark 1. These surfaces in the first three rows of Theorem 1 can be obtained by taking three iterated \( \mathbb{Z}_2 \)-covers. First, we ramify on \( D_2, D_3, D_6, \) and \( D_7 \) (i.e. \( B = 2L_{0,1,0} \)) and we get Horikawa’s surfaces with \( K^2 = 2p_g - 4 \) [7]. The second cover ramifies only on nodes (i.e. \( B = 2L_{1,0,0} \)). These nodes come from the intersection points between \( D_2 + D_3 \) and \( D_6 + D_7. \) The last cover ramifies on nodes coming from the intersection points between \( D_2 \) and \( D_3, \) and \( D_6 \) and \( D_7 \) (i.e. \( B = 2L_{0,0,1} \)) (see [4, Prop. 3.1]). Moreover, the following diagram commutes
Now, by Proposition 1, a point \((0, 0, 1, 0, -1, 1, 2)\) can be imposed on the original branch locus, where \(-1\) in the fifth component means the exceptional divisor is added to \(D_5\). In fact, let \(P\) be a point in \(\mathbb{P}^1\) such that \(D_3, D_6, D_7\) contain \(P\) with multiplicity 1, 1, 2, respectively. Let \(Y\) be the blow up of \(\mathbb{P}^1\) at \(P\) and \(E\) be the exceptional divisor. If we abuse notation and denote \(\phi: X \to Y\) be a \(\mathbb{Z}_2^3\)-cover with the following branch locus

\[
B = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7,
\]

where \(D_1 = D_4 = 0\) and \(D_5 = E\). The building data is as follows:

\[
\begin{align*}
L_{1,0,0} &\equiv 2\Delta_0 + 2\Gamma - E \\
L_{0,1,0} &\equiv 3\Delta_0 + (n+3)\Gamma - 2E \\
L_{0,0,1} &\equiv 2\Delta_0 + 2\Gamma - E \\
L_{1,1,0} &\equiv \Delta_0 + (n+1)\Gamma \\
L_{1,0,1} &\equiv 2\Delta_0 + 2\Gamma - E \\
L_{0,1,1} &\equiv \Delta_0 + (n+1)\Gamma \\
L_{1,1,1} &\equiv \Delta_0 + (n+1)\Gamma - E.
\end{align*}
\]

We get minimal surfaces of general type with

\[
K^2 = 16n - 10, \quad pg = 2n, \quad q = 0,
\]

and \(\text{deg}(\text{im} \phi|_{K_X}) = 2n - 2\). Moreover, \(|K_X|\) is not base point free (we will prove this in the next Sect. 3.1.3). Therefore, we obtain the surfaces described in the fourth row of Theorem 1.

### 3.1.3. The fixed part of the canonical system

In this section, we show that the canonical linear system \(|K_X|\) of the surfaces in the fourth row of Theorem 1 has a nontrivial fixed part. Indeed, the \(\mathbb{Z}_2^3\)-cover \(\phi: X \to Y\) factors through \(X_2\), where \(X_2\) is obtained by the \(\mathbb{Z}_2\)-cover ramifying on \(2L_{1,1,1}, 2L_{1,0,1}\). The linear system \(|K_{X_2}|\) is base point free. The surface \(X\) is obtained by the \(\mathbb{Z}_2\)-cover ramifying on the pullback of \(D_5 = E\) and some \(A_1\) points. So the moving part of \(|K_X|\) is
the pullback of $|K_X|$. Therefore, the fixed part of $|K_X|$ is $\frac{1}{2} f^* (E)$. More precisely, we consider the $\mathbb{Z}_2^3$—cover as the composition of the following $\mathbb{Z}_2$—covers

![Diagram](image)

The first cover ramifies on $D_2 + D_7$ (i.e. $B = 2L_{1,1,1}$) and we get a surface $X_1$ with $K_{X_1} = f_1^* (-\Delta_0 + (n - 2) \Gamma)$. Moreover, $f_1^* (E) = E_1$ with $E_1^2 = -2$, $g (E_1) = 0$. The second cover ramifies on $D_3 + D_6$ (i.e. $B = 2L_{1,0,1}$). We have

$$K_{X_2} = f_2^* f_1^* (\Delta_0 + n \Gamma - E).$$

So $|K_{X_2}|$ is base point free. Moreover, $f_2^* (E_1) = E_2$ with $E_2^2 = -4$, $g (E_2) = 1$. The last cover ramifies on $f_2^* f_1^* (E)$ and $8n + 6$ nodes (i.e. $B = 2L_{1,0,0}$). These nodes come from the intersection points between $D_2$ and $D_7$, and $D_3$ and $D_6$. And we obtain $f_3^* (E_2) = 2E_3$ with $E_3^2 = -2$, $g (E_3) = 1$. In addition, by the projection formula (see [5, Corollary 2.3]), we get

$$h^0 (K_X) = h^0 (f_3^* (K_{X_2})) = 2n. \quad (4)$$

On the other hand, $K_X = f_3^* (K_{X_2}) + R$, where $R$ is the ramification of $f_3$. Hence,

$$K_X = f_3^* (K_{X_2}) + E_3. \quad (5)$$

From (4) and (5), the elliptic curve $E_3$ is the fixed part of $|K_X|$.

### 3.2. Construction 2

In this section, we construct the surfaces in the last five rows of Theorem 1.

#### 3.2.1. Construction and computation of invariants

Let $D_3 = \Gamma$, $D_4 \in |\Delta_0 + \Gamma| + \Delta_0$, $D_7 = (2n + 1) \Gamma$ be in $\mathbb{F}_1$ and $D_5, D_6 \in |2\Delta_0 + 2\Gamma|$ be smooth curves in general position in $\mathbb{F}_1$. Let $f : X \to \mathbb{F}_1$ be a $\mathbb{Z}_2^3$—cover with the following branch locus

$$B = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7,$$

where $D_1 = D_2 = 0$. By Proposition 1, $L_{1,0,0} \equiv 3\Delta_0 + (n + 3) \Gamma$ and $L_X$ is equivalent to either $2\Delta_0 + 2\Gamma$, $\Delta_0 + (n + 2) \Gamma$ or $\Delta_0 + (n + 1) \Gamma$ for all $L_X \neq$
Since each $D_\sigma$ is smooth and $B$ is a normal crossings divisor, $X$ is smooth. Furthermore, by Proposition 2, we get
\[ 2K_X \equiv f^* (2\Delta_0 + (2n + 1) \Gamma) . \]
This implies that $X$ is a minimal surface of general type. Moreover, by Proposition 2, the invariants of $X$ are as follows:
\[ K_X^2 = 16n \]  
\[ p_g (X) = h^0 (\Delta_0 + n \Gamma) = 2n + 1 \]  
\[ \chi (\mathcal{O}_X) = 2n + 2. \]
From (7) and (8), we get $g (X) = 0$.

We show that $|K_X|$ is not composed with a pencil by considering the following double cover
\[ g_1 : Y_1 \longrightarrow \mathbb{P}_1 \]
ramifying on $D_4 + D_5 + D_6 + D_7$. We have
\[ K_{Y_1} \equiv g_1^* (\Delta_0 + n \Gamma) . \]
Because $|\Delta_0 + n \Gamma|$ is not composed with a pencil, $|K_{Y_1}|$ is not composed with a pencil, either. This yields that $|K_X|$ is not composed with a pencil and the degree of the canonical map is 8.

### 3.2.2. The fixed part of the canonical system
In this section, we show that the canonical linear system $|K_X|$ has a nontrivial fixed part. In fact, the $\mathbb{Z}_2^3$–cover $f: X \longrightarrow Y$ factors through $X_2$, where $X_2$ is obtained by the $\mathbb{Z}_2^2$–cover ramifying on $2L_{1,1,1}, 2L_{0,1,1}$. The linear system $|K_{X_2}|$ is base point free. The surface $X$ is obtained by the $\mathbb{Z}_2$–cover ramifying on the pullback of $D_3 = \Gamma$ and some $A_1$ points. So the moving part of $|K_X|$ is the pullback of $|K_{X_2}|$. Therefore, the fixed part of $|K_X|$ is $\frac{1}{2} f^* (\Gamma)$. More precisely, we consider the $\mathbb{Z}_2^3$–cover as the compositions of the following $\mathbb{Z}_2$–covers

The first cover ramifies on $D_4 + D_7$ (i.e. $B = 2L_{1,1,1}$). We get a surface $X_1$ with $K_{X_1} \equiv f_1^* (\Delta_0 + (n - 2) \Gamma)$. Furthermore, $f_1^* (D_3) = \Gamma_1$ with $g (\Gamma_1) = 0$. The
second cover ramifies on \( D_5 + D_6 \) (i.e. \( B = 2L_{0,1,1} \)). We get surface of general type \( X_2 \) with

\[
K_{X_2} \equiv f_2^* f_1^* (\Delta_0 + n\Gamma).
\]

Hence, \( |K_{X_2}| \) is base point free and \( \deg \left( \text{im} \varphi_{|K_{X_2}|} \right) = 2n - 1 \). Furthermore, 
\( f_2^* (\Gamma_1) = \Gamma_2 \) with \( g (\Gamma_2) = 3 \). The last cover ramifies on \( f_2^* f_1^* (D_3) \) and \( 8n + 12 \) nodes (i.e. \( B = 2L_{0,1,0} \)). These nodes come from the intersection points between \( D_4 \) and \( D_7 \), and \( D_5 \) and \( D_6 \). And we get 
\( f_3^* (\Gamma_2) = 2\Gamma_3 \) with \( g (\Gamma_3) = 3 \). In addition, by the projection formula, we get

\[
h^0 (K_X) = h^0 (f_3^* (K_{X_2})) = 2n + 1. \tag{9}
\]

On the other hand, \( K_X \equiv f_3^* (K_{X_2}) + R \), where \( R \) is the ramification of \( f_3 \). Hence,

\[
K_X \equiv f_3^* (K_{X_2}) + \Gamma_3. \tag{10}
\]

Therefore, from (9) and (10), the curve \( \Gamma_3 \) is the fixed part of \( |K_X| \).

### 3.2.3. Variations

By Proposition 1, the branch locus can be imposed a point \((0, 0, 0, 1, 1, 1, 1)\). In fact, let \( P \) be a point in \( \mathbb{P}_1 \) such that \( D_4, D_5, D_6, D_7 \) contain \( P \) with multiplicity 1, 1, 1, 1, respectively. Let \( Y \) be the blow up of \( \mathbb{P}_1 \) at \( P \) and \( E \) be the exceptional divisor. If we abuse notation and denote \( D_3, D_4, D_5, D_6, D_7, \Delta_0, \Gamma \) their pullbacks to \( Y \), then \( D_3 = \Gamma, D_4 = 2\Delta_0 + \Gamma - E, D_5 = 2\Delta_0 + 2\Gamma - E, D_6 = 2\Delta_0 + 2\Gamma - E \) and \( D_7 = (2n + 1)\Gamma - E \). Let \( f : X \longrightarrow Y \) be a \( \mathbb{Z}_2^3 \)-cover with the following branch locus

\[
B = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7,
\]

where \( D_1 = D_2 = 0 \). The building data is as follows:

\[
\begin{align*}
L_{1,0,0} & = 3\Delta_0 + (n + 3)\Gamma - 2E \\
L_{0,1,0} & = \Delta_0 + (n + 2)\Gamma - E \\
L_{0,0,1} & = \Delta_0 + (n + 2)\Gamma - E \\
L_{1,1,0} & = 2\Delta_0 + 2\Gamma - E \\
L_{1,0,1} & = 2\Delta_0 + 2\Gamma - E \\
L_{0,1,1} & = 2\Delta_0 + 2\Gamma - E \\
L_{1,1,1} & = \Delta_0 + (n + 1)\Gamma - E.
\end{align*}
\]

Similarly to the above, we get minimal surfaces of general type with

\[
K^2 = 16n - 8, \quad p_g = 2n, \quad q = 0, \quad d = 8,
\]

and \( \deg \left( \text{im} \varphi_{|K_X|} \right) = 2n - 2 \). Moreover, \( \frac{1}{2} f^* (\Gamma) \) is the fixed part of \( |K_X| \) and the following diagram commutes
So we obtain the surfaces in the sixth row of Theorem 1.

Analogously, by Proposition 1, we can put a point \((0, 0, 0, 2, 2, 0)\) into the original branch locus. In fact, let \(P\) be a point in \(\mathbb{P}^1\) such that \(D_5, D_6\) contain \(P\) with multiplicity 2, 2, respectively. Let \(Y\) be the blow up of \(\mathbb{P}^1\) at \(P\) and \(E\) be the exceptional divisor. If we abuse notation and denote \(D_3, D_4, D_5, D_6, D_7, \Delta_0, \Gamma\) their pullbacks to \(Y\), then \(D_3 = \Gamma, D_4 = 2\Delta_0 + \Gamma, D_5 = 2\Delta_0 + 2\Gamma - 2E, D_6 = 2\Delta_0 + 2\Gamma - 2E\) and \(D_7 = (2n + 1)\Gamma\). Let \(f: X \rightarrow Y\) be a \(\mathbb{Z}_2^3\)-cover with the following branch locus

\[
B = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7,
\]

where \(D_1 = D_2 = 0\). The building data is as follows:

\[
\begin{align*}
L_{1,0,0} &\equiv 3\Delta_0 + (n + 3)\Gamma - 2E \\
L_{0,1,0} &\equiv \Delta_0 + (n + 2)\Gamma - E \\
L_{0,0,1} &\equiv \Delta_0 + (n + 2)\Gamma - E \\
L_{1,1,0} &\equiv 2\Delta_0 + 2\Gamma - E \\
L_{1,0,1} &\equiv 2\Delta_0 + 2\Gamma - E \\
L_{0,1,1} &\equiv 2\Delta_0 + 2\Gamma - 2E \\
L_{1,1,1} &\equiv \Delta_0 + (n + 1)\Gamma.
\end{align*}
\]

Similarly to the above, we get minimal surfaces of general type with

\[
K^2 = 16n - 8, \ p_g = 2n, \ q = 1, \ d = 8,
\]

and \(\deg(\text{im} \varphi_{|K_X|}) = 2n - 2\). Furthermore, \(\frac{1}{2}f^* (\Gamma)\) is the fixed part of \(|K_X|\) and the following diagram commutes
Thus, we obtain the surfaces in the seventh row of Theorem 1. The Albanese pencil of these surfaces \( X \longrightarrow Alb \, (X) \) is the pullback of the Albanese pencil of the intermediate surface \( Z \), where \( Z \) is obtained by the \( \mathbb{Z}_2 \)-cover ramifying on \( 2L_{0,1,1} \).

Similarly, by Proposition 1, a new branch locus can be formed by adding a point \((0,0,−1,1,2,0,1)\), where \(-1\) in the third component means the exceptional divisor \( E \) is added to \( D_3 \). In fact, let \( P \) be a point in \( \mathbb{F}_1 \) such that \( D_4, D_5, D_7 \) contain \( P \) with multiplicity \( 1,2,1 \), respectively. Let \( Y \) be the blow up of \( \mathbb{F}_1 \) at \( P \) and \( E \) be the exceptional divisor. If we abuse notation and denote \( D_4, D_5, D_6, D_7, \Delta_0, \Gamma \) their pullbacks to \( Y \), then \( D_4 = 2\Delta_0 + \Gamma - E, D_5 = 2\Delta_0 + 2\Gamma - 2E, D_6 = 2\Delta_0 + 2\Gamma \) and \( D_7 = (2n + 1) \Gamma - E \). Let \( f : X \longrightarrow Y \) be a \( \mathbb{Z}_2^3 \)-cover with the following branch locus

\[
B = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7,
\]

where \( D_1 = D_2 = 0 \) and \( D_3 = \Gamma + E \). The building data is as follows:

\[
\begin{align*}
L_{1,0,0} &\equiv 3\Delta_0 + (n + 3) \Gamma - 2E \\
L_{0,1,0} &\equiv \Delta_0 + (n + 2) \Gamma \\
L_{0,0,1} &\equiv \Delta_0 + (n + 2) \Gamma - E \\
L_{1,1,0} &\equiv 2\Delta_0 + 2\Gamma - E \\
L_{1,0,1} &\equiv 2\Delta_0 + 2\Gamma \\
L_{0,1,1} &\equiv 2\Delta_0 + 2\Gamma - E \\
L_{1,1,1} &\equiv \Delta_0 + (n + 1) \Gamma - E.
\end{align*}
\]

Similarly to the above, we get minimal surfaces of general type with

\[
K^2 = 16n - 2, \quad p_g = 2n, \quad q = 0, \quad d = 8,
\]

and \( \deg(\text{im} \varphi|_{K_X}) = 2n - 2 \). Moreover, \( \frac{1}{2} f^* (\Gamma + E) \) is the fixed part of \( |K_X| \) and the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi|_{K_X}} & X_2 \\
\downarrow & & \downarrow 4:1 \\
\mathbb{P}^{2n-1} & \xrightarrow{\varphi|_{K_{X_2}}} & X_1 \\
\downarrow & & \downarrow \varphi^*|_{K_{X_1}} \\
Y & \xrightarrow{f_2} & X_1 \\
\downarrow & & \downarrow 2L_{0,1,1} \\
2L_{0,1,1} \quad & \xrightarrow{f_1} & \quad 2L_{0,1,1} \\
\downarrow & & \downarrow 2L_{0,1,0} \\
\mathbb{F}_1 & \xrightarrow{\varphi} & \quad \mathbb{F}_1 \\
\end{array}
\]

Therefore, we obtain the surfaces in the eighth row of Theorem 1.

Finally, for \( n \geq 3 \) by Proposition 1, a point \( P = (0,0,−1,1,2,2,1) \) can be added to the original branch locus, where \(-1\) in the third component means the exceptional divisor is added to \( D_3 \). In fact, let \( P \) be a point in \( \mathbb{F}_1 \) such that \( D_4, D_5, D_6, D_7 \) contain \( P \) with multiplicity \( 1,2,2,1 \), respectively. Let \( Y \) be the blow up of \( \mathbb{F}_1 \) at \( P \) and \( E \) be the exceptional divisor. If we abuse notation and denote
$D_4, D_5, D_6, D_7, \Delta_0, \Gamma$ their pullbacks to $Y$, then $D_4 = 2\Delta_0 + \Gamma - E$, $D_5 = 2\Delta_0 + 2\Gamma - 2E$, $D_6 = 2\Delta_0 + 2\Gamma - 2E$ and $D_7 = (2n + 1) \Gamma - E$. Let $f : X \to Y$ be a $\mathbb{Z}_2^3$–cover with the following branch locus

$$B = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7,$$

where $D_1 = D_2 = 0$ and $D_3 = \Gamma + E$. The building data is as follows:

$$
\begin{align*}
L_{1,0,0} &\equiv 3\Delta_0 + (n + 3) \Gamma - 3E \\
L_{0,1,0} &\equiv \Delta_0 + (n + 2) \Gamma - E \\
L_{0,0,1} &\equiv \Delta_0 + (n + 2) \Gamma - E \\
L_{1,1,0} &\equiv 2\Delta_0 + 2\Gamma - E \\
L_{1,0,1} &\equiv 2\Delta_0 + 2\Gamma - E \\
L_{0,1,1} &\equiv 2\Delta_0 + 2\Gamma - 2E \\
L_{1,1,1} &\equiv \Delta_0 + (n + 1) \Gamma - E.
\end{align*}
$$

After contracting the $-1$ curve arising from the fiber passing through $P$, we get minimal surfaces of general type with

$$K^2 = 16n - 16, \ p_g = 2n - 2, \ q = 1, \ d = 8,$$

and $\deg(\text{im } \varphi|_{K_X}) = 2n - 4$.

Furthermore, $\frac{1}{2} f^* (\Gamma + E)$ is the fixed part of $|K_X|$ and the following diagram commutes

Thus, taking $m = n - 1, m \geq 2$, we obtain the surfaces in the last row of Theorem 1. The Albanese pencil of these surfaces $X \to Alb (X)$ is the pullback of the Albanese pencil of the intermediate surface $Z$, where $Z$ is obtained by the $\mathbb{Z}_2$–cover ramifying on $2L_{0,1,1}$.

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