ON THE NONEXISTENCE OF POSITIVE SOLUTION OF SOME SINGULAR NONLINEAR INTEGRAL EQUATIONS

NGUYEN THANH LONG

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We consider the singular nonlinear integral equation

\[ u(x) = \int_{\mathbb{R}^N} g(x, y, u(y)) \frac{dy}{|y-x|^\sigma} \]

for all \( x \in \mathbb{R}^N \), where \( \sigma \) is a given positive constant and the given function \( g(x, y, u) \) is continuous and \( g(x, y, u) \geq M |x|^{\beta_1} |y|^{\beta} (1 + |x|)^{-\gamma_1} (1 + |y|)^{-\gamma} u^\alpha \) for all \( x, y \in \mathbb{R}^N, u \geq 0 \), with some constants \( \alpha, \beta, \beta_1, \gamma, \gamma_1 \geq 0 \) and \( M > 0 \). We prove in an elementary way that if

\[ 0 \leq \alpha \leq (N + \beta - \gamma)/(\sigma + \gamma_1 - \beta_1), \quad (1/2)(N + \beta + \beta_1 - \gamma - \gamma_1) < \sigma < \min\{N, N + \beta + \beta_1 - \gamma - \gamma_1\}, \quad \sigma + \gamma_1 - \beta_1 > 0, \quad N \geq 2, \]

the above nonlinear integral equation has no positive solution.

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1. Introduction

We consider the nonexistence of positive solutions of the following singular nonlinear integral equation

\[ u(x) = b_N \int_{\mathbb{R}^N} \frac{g(x, y, u(y)) dy}{|y-x|^\sigma} \quad \forall x \in \mathbb{R}^N, \quad (1.1) \]

where \( b_N = 2((N-1)\omega_{N+1})^{-1} \) with \( \omega_{N+1} \) being the area of unit sphere in \( \mathbb{R}^{N+1} \), \( N \geq 2 \), \( \sigma \) is a given positive constant with \( 0 < \sigma < N \), and \( g: \mathbb{R}^{2N} \times \mathbb{R} \rightarrow \mathbb{R} \) is given continuous function satisfying the following.

There exist the constants \( \alpha, \beta, \beta_1, \gamma, \gamma_1 \geq 0 \) and \( M > 0 \) such that

\[ g(x, y, u) \geq M |x|^{\beta_1} |y|^{\beta} (1 + |x|)^{-\gamma_1} (1 + |y|)^{-\gamma} u^\alpha \quad \forall x, y \in \mathbb{R}^N, u \geq 0, \quad (1.2) \]

and some auxiliary conditions below.

In the case of \( \sigma = N - 1 \), \( g(x, y, u(y)) = g(y, u(y)) \), the integral equation (1.1) is a consequence of the following nonlinear Neumann problem
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\[ \Delta v = \sum_{i=1}^{N+1} v_{x_ix_i} = 0, \quad x \in \mathbb{R}^N, \ x_{N+1} > 0, \quad (1.3) \]

\[ -v_{x_{N+1}}(x,0) = g(x,v(x,0)) = 0, \quad x \in \mathbb{R}^N, \quad (1.4) \]

of which the boundary value \( u(x) = v(x,0) \) together with some auxiliary conditions will be a solution of the equation

\[ u(x) = b_N \int_{\mathbb{R}^N} \frac{g(y,u(y))dy}{|y-x|^\sigma} \quad \forall x \in \mathbb{R}^N. \quad (1.5) \]

In [3] the authors have studied a problem (1.3), (1.4) for \( N = 2 \) with the Laplace equation (1.3) having the axial symmetry

\[ u_{rr} + \frac{1}{r} u_r + u_{zz} = 0 \quad \forall r > 0, \ \forall z > 0, \quad (1.6) \]

and with the nonlinear boundary condition of the form

\[ -u_z(r,0) = I_0 \exp(-r^2/r_0^2) + u^\alpha(r,0) \quad \forall r > 0, \quad (1.7) \]

where \( I_0, r_0, \alpha \) are given positive constants. The problem (1.6), (1.7) is the stationary case of the problem associated with ignition by radiation. In the case of \( 0 < \alpha \leq 2 \) the authors in [3] have proved that the following nonlinear integral equation

\[ u(r,0) = \frac{1}{2\pi} \int_0^{+\infty} \left[ I_0 \exp(-s^2/r_0^2) + u^\alpha(s,0) \right] s ds \int_0^{2\pi} \frac{d\theta}{\sqrt{r^2 + s^2 - 2rs \cos \theta}} \quad \forall r > 0, \quad (1.8) \]

associated to the problem (1.6), (1.7) has no positive solution. Afterwards, this result has been extended in [8] to the general nonlinear boundary condition

\[ -u_z(r,0) = g(r,u(r,0)) \quad \forall r > 0. \quad (1.9) \]

In [7] the problem (1.3), (1.4) is considered for \( N = 2 \) and for a function \( g \) continuous, nondecreasing and bounded below by the power function of order \( \alpha \) with respect to the third variable and it is proved that for \( 0 < \alpha \leq 2 \) such a problem has no positive solution.

In [1, 2] we have considered the problem (1.3), (1.4) for \( N \geq 3 \). The function \( g : \mathbb{R}^N \times [0, +\infty) \rightarrow [0, +\infty) \) is continuous, nondecreasing with respect to variable \( u \), satisfies the condition (1.2) with \( y = 0 \) and some auxiliary conditions. In the case of \( 0 \leq \alpha \leq N/(N - 1) \), \( N \geq 2 \) we have proved that the problem (1.3), (1.4) has no positive solution [1, 2].

In [5, 6] the authors have proved the nonexistence of a positive solution of the problem (1.3), (1.4) with

\[ g(x,u) = u^\alpha. \quad (1.10) \]

In [6] it is proved with \( 1 \leq \alpha < N/(N - 1), N \geq 2 \), and in [5] with \( 1 < \alpha < (N + 1)/(N - 1), N \geq 2 \). We also note that the function \( g(x,u) = u^\alpha \) does not satisfy the conditions in the papers [1, 7, 8].
In this paper, we consider the nonlinear integral equation (1.1) for \( (1/2)(N + \beta + \beta_1 - y - y_1) < \sigma < \min\{N, N + \beta + \beta_1 - y - y_1\}, \sigma + y_1 - \beta_1 > 0, N \geq 2 \). The function \( g(x, y, u) \) is continuous, satisfies the condition (1.2) of which (1.10) is a special case. By proving elementarily we generalize the results from [1–10] that for \( 0 \leq \alpha \leq (N + \beta - y)/(\sigma + y_1 - \beta_1) \) (1.1) has no continuous positive solution.

2. The theorem of nonexistence of positive solution

Without loss of generality, we can suppose that \( b_N = 1 \) with a change of the constant \( M \) in the assumption (1.2) of \( g \). We write in the integral equation (1.1):

\[
u(x) = Tu(x) = \int_{\mathbb{R}^N} g(x, y, u(y)) \frac{dy}{|y-x|^{\sigma}} \quad \forall x \in \mathbb{R}^N. \tag{2.1}
\]

Then we have the main result as follows.

**Theorem 2.1.** Let \( g: \mathbb{R}^{2N} \times [0, +\infty) \to \mathbb{R} \) be a continuous function satisfying the following hypothesis. There exist constants \( M > 0, \alpha, \beta, \beta_1, \gamma, \gamma_1 \geq 0 \) with

\[
\frac{1}{2}(N + \beta + \beta_1 - y - y_1) < \sigma < \min\{N, N + \beta + \beta_1 - y - y_1\}, \quad \sigma + y_1 - \beta_1 > 0, \quad N \geq 2,
\]

such that

\[
g(x, y, u) \geq M|x|^{\beta_1} |y|^{\beta} (1 + |x|)^{-y_1} (1 + |y|)^{-\gamma} u^\alpha \quad \forall x, y \in \mathbb{R}^N, u \geq 0. \tag{2.3}
\]

If \( 0 \leq \alpha \leq (N + \beta - y)/(\sigma + y_1 - \beta_1) \) then, the integral equation (2.1) has no continuous positive solution.

**Remark 2.2.** The result of theorem is stronger than that in [1, 7]. Indeed, corresponding to the same equation (1.5), the following assumptions which were made in [1, 7] are not needed here.

\( (G_1) g(y, u) \) is nondecreasing with respect to variable \( u \), that is,

\[
(g(y, u) - g(y, v))(u - v) \geq 0 \quad \forall u, v \geq 0, \ y \in \mathbb{R}^N. \tag{2.4}
\]

\( (G_2) \) The integral \( \int_{\mathbb{R}^N} (g(y, 0) dy)/(1 + |y|)^{N-1} \) exists and is positive.

**Remark 2.3.** In the case of \( N \geq 2 \), we have also obtained some results concerning in the papers [2, 7, 9] in the cases as follows:

(a) \( \beta = \beta_1 = \gamma = \beta = 0, \sigma = N - 1, 0 \leq \alpha \leq N/(N - 1) \) (see [2]).

(b) \( \beta = \beta_1 = \gamma = \beta = 0, 0 \leq \alpha \leq N/\sigma \) (see [7]).

(c) \( \beta_1 = \gamma = 0, 0 < \sigma < \min\{N, N + \beta - y_1\}, 0 \leq \alpha \leq (N + \beta)/(\sigma + y_1) \) (see [9]).

First, we need the following lemma.
Lemma 2.4. For every \( p \geq 0, q \geq 0, 0 < \sigma < N, x \in \mathbb{R}^N \). Put

\[
A[p, q](x) = \int_{\mathbb{R}^N} \frac{|y|^p (1 + |y|)^{-q} dy}{|y - x|^{\sigma}},
\]

we have

\[
A[p, q](x) = +\infty, \quad \text{if } q - p \leq N - \sigma,
\]

\[
A[p, q](x) \text{ convergent and } A[p, q](x) \geq \left( \frac{1}{N + p} + \frac{1}{q} \right) \frac{\omega_N}{2^\sigma} |x|^{p + N - \sigma} (1 + |x|)^{-q}, \quad \text{if } q - p > N - \sigma,
\]

where \( \omega_N \) is the area of unit sphere in \( \mathbb{R}^N \).

The proof of lemma can be found in [9].

Proof of Theorem 2.1. We prove by contradiction. Suppose that there exists a continuous positive solution \( u(x) \) of the integral equation (2.1). We suppose that there exists \( x_0 \in \mathbb{R}^N \), such that \( u(x_0) > 0 \). Since \( u \) is continuous, then there exists \( r_0 > 0 \) such that

\[
u(x) > \frac{1}{2} u(x_0) \equiv L \quad \forall x \in \mathbb{R}^N, \ |x - x_0| \leq r_0.
\]

It follows from (2.1), (2.3), (2.8) and the monotonicity of the integral operator

\[
u(x) = Tu(x) \geq M |x|^\beta_1 (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} |y|^\beta (1 + |y|)^{-\gamma} \frac{u^a(y) dy}{|y - x|^{\sigma}} \geq M |x|^\beta_1 (1 + |x|)^{-\gamma_1} L^a \int_{|y - x_0| \leq r_0} |y|^\beta (1 + |y|)^{-\gamma} \frac{dy}{|y - x|^{\sigma}} \geq ML^a (1 + |x_0| + r_0)^{-\sigma} |x|^\beta_1 (1 + |x|)^{-\gamma - \gamma_1} \int_{|y - x_0| \leq r_0} |y|^\beta (1 + |y|)^{-\gamma} dy,
\]

for all \( x \in \mathbb{R}^N \).

Using the inequality

\[
|y - x| \leq |y| + |x| \leq (1 + |x_0| + r_0) (1 + |x|) \quad \forall x, y \in \mathbb{R}^N, \ |y - x_0| \leq r_0,
\]

we obtain from (2.9), (2.10) that

\[
u(x) \geq u_1(x) = m_1 |x|^\beta_1 (1 + |x|)^{-\gamma_1} \quad \forall x \in \mathbb{R}^N,
\]
where
\begin{equation}
\begin{aligned}
p_1 = \beta_1, \\
q_1 = \sigma + \gamma_1,
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
m_1 = ML^a (1 + |x_0| + r_0)^{-\sigma} \int_{|y-x_0| \leq r_0} |y|^{-\beta (1 + |y|)^{-\gamma} dy}.
\end{aligned}
\end{equation}

Using again the equality (2.1), it follows from (2.3), (2.11) that
\begin{equation}
\begin{aligned}
u(x) = Tu(x) & \geq M|x|^\beta (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} |y|^{-\beta (1 + |y|)^{-\gamma} u_1^\sigma(y) dy} \\
& \geq M|x|^\beta (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} |y|^{-\beta (1 + |y|)^{-\gamma} (m_1 |y|^p_1 (1 + |y|)^{-\gamma q_1})^a dy} \\
& = Mm_1^a |x|^\beta (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} |y|^{-\beta + \alpha p_1} (1 + |y|)^{-\gamma - \alpha q_1} dy \\
& = Mm_1^a |x|^\beta (1 + |x|)^{-\gamma_1} A(\beta + \alpha p_1, \gamma + \alpha q_1)(x) \quad \forall x \in \mathbb{R}^N.
\end{aligned}
\end{equation}

Now, we consider separately the cases of different values of \( \alpha \).

Case 1. \( 0 \leq \alpha \leq (N - \sigma + \beta - \gamma)/(\sigma + \gamma_1 - \beta_1) \). We obtain from (2.6), (2.13) with \( p = \beta + \alpha p_1, q = \gamma + \alpha q_1, q - p = \gamma - \beta + \alpha (q_1 - p_1) = \gamma - \beta + \alpha (\sigma + \gamma_1 - \beta_1) \leq N - \sigma \), that
\begin{equation}
u(x) = +\infty \quad \forall x \in \mathbb{R}^N.
\end{equation}

It is a contradiction.

Case 2. \( (N - \sigma + \beta - \gamma)/(\sigma + \gamma_1 - \beta_1) < \alpha < (N + \beta - \gamma)/(\sigma + \gamma_1 - \beta_1) \). Using (2.7) with \( p = \beta + \alpha p_1, q = \gamma + \alpha q_1, q - p = \gamma - \beta + \alpha (q_1 - p_1) = \gamma - \beta + \alpha (\sigma + \gamma_1 - \beta_1) > N - \sigma \), we deduce from (2.13) that
\begin{equation}
u(x) \geq u_2(x) = m_2 |x|^{p_2} (1 + |x|)^{-q_2} \quad \forall x \in \mathbb{R}^N,
\end{equation}
where
\begin{equation}
\begin{aligned}
p_2 &= \alpha p_1 + \beta + \beta_1 + N - \sigma, \\
q_2 &= \alpha q_1 + \gamma + \gamma_1,
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
m_2 &= M m_1^a \left( \frac{1}{N + \beta + \alpha p_1} + \frac{1}{\gamma + \alpha q_1} \right) \frac{\omega_N}{2^\sigma}.
\end{aligned}
\end{equation}
Suppose that
\begin{equation}
\begin{aligned}
u(x) \geq u_{k-1}(x) = m_{k-1} |x|^{p_{k-1}} (1 + |x|)^{-q_{k-1}} \quad \forall x \in \mathbb{R}^N,
\end{aligned}
\end{equation}
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If \( y + \alpha q_{k-1} - \beta - \alpha \sigma < N - \sigma \), then, using (2.1), (2.3), (2.7), and (2.17), we obtain

\[
\begin{align*}
 u(x) &= Tu(x) \geq M|x|^\beta (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} |y|^\beta (1 + |y|)^{-\gamma} \frac{u(y)dy}{|y - x|^{\sigma}} \\
&\geq M|x|^\beta (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} |y|^\beta (1 + |y|)^{-\gamma} \frac{u_k(y)dy}{|y - x|^{\sigma}} \\
&\geq Mm^a_{k-1} |x|^\beta (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} |y|^\beta (1 + |y|)^{-\gamma} \frac{|y|^\sigma_{\alpha p_{k-1}} (1 + |y|)^{-\alpha q_{k-1}} (y)dy}{|y - x|^{\sigma}} \\
&= Mm^a_{k-1} |x|^\beta (1 + |x|)^{-\gamma_1} A[\beta + \alpha p_{k-1}, y + \alpha q_{k-1}](x) \\
&\geq Mm^a_{k-1} \left( \frac{1}{N + \beta + \alpha p_{k-1}} + \frac{1}{y + \alpha q_{k-1}} \right) \frac{\omega_N}{2^\sigma} |x|^\beta \frac{M_{\alpha p_{k-1} + N - \sigma (1 + |x|)^{-\gamma_1} - \alpha q_{k-1} - \frac{\gamma}{2}}. \tag{2.18}
\end{align*}
\]

Hence

\[
\begin{align*}
 u(x) \geq u_k(x) &= m_k |x|^{p_k} (1 + |x|)^{-q_k} \quad \forall x \in \mathbb{R}^N, \tag{2.19}
\end{align*}
\]

where the sequences \( \{p_{k-1}\}, \{q_{k-1}\} \) and \( \{m_{k-1}\} \) are defined by the recurrence formulas

\[
\begin{align*}
p_k &= \alpha p_{k-1} + \beta + \beta_1 + N - \sigma, \\
q_k &= \alpha q_{k-1} + y + y_1, \\
m_k &= Mm^a_{k-1} \left( \frac{1}{N + \beta + \alpha p_{k-1}} + \frac{1}{y + \alpha q_{k-1}} \right) \frac{\omega_N}{2^\sigma}, \quad k \geq 2. \tag{2.20}
\end{align*}
\]

Note that \( (N - \sigma + \beta - y)/(\sigma + y_1 - \beta_1) < 1 < (N + \beta - y)/(\sigma + y_1 - \beta_1) \), hence we obtain from (2.16), (2.20) that

\[
\begin{align*}
p_k &= \begin{cases} 
(\beta + \beta_1 + N - \sigma)(k - 1) + \beta_1, & \text{if } \alpha = 1, \\
(\beta + \beta_1 + N - \sigma) + \beta_1 \alpha^{k-1}, & \text{if } N - \sigma + \beta - y < \alpha < \frac{N + \beta - y}{\sigma + y_1 - \beta_1}, \alpha \neq 1,
\end{cases} \tag{2.21}
\end{align*}
\]

\[
\begin{align*}
q_k &= \begin{cases} 
(k - 1)(y + y_1) + \sigma + y_1, & \text{if } \alpha = 1, \\
(y + y_1) + (\sigma + y_1) \alpha^{k-1}, & \text{if } N - \sigma + \beta - y < \alpha < \frac{N + \beta - y}{\sigma + y_1 - \beta_1}, \alpha \neq 1.
\end{cases} \tag{2.22}
\end{align*}
\]

It follows from (2.1), (2.3), and (2.18) that

\[
\begin{align*}
u(x) \geq Mm^a_k |x|^\beta (1 + |x|)^{-\gamma_1} A[\beta + \alpha p_k, y + \alpha q_k](x) \quad \forall x \in \mathbb{R}^N. \tag{2.23}
\end{align*}
\]
So, from (2.22), (2.23), we only need to choose the natural number $k \geq 2$ such that

$$y + aq_k - \beta - \alpha p_k \leq N - \sigma < y + aq_{k-1} - \beta - \alpha p_{k-1},$$  

(2.24)

since $A[\beta + \alpha p_k, y + aq_k](x) = +\infty$.

On the other hand, by (2.21), (2.22) the inequalities (2.24) equivalent to

$$k - 1 < \frac{\sigma}{N - \sigma + \beta + \beta_1 - \gamma - \gamma_1} \leq k, \quad \text{if } \alpha = 1,$$  

(2.25)

or

$$k - 1 < \frac{1}{\ln \alpha} \ln \left( \frac{\alpha(y_1 - \beta_1) - (N - \sigma + \beta - \gamma)}{\alpha(\sigma + y_1 - \beta_1) - (N + \beta - \gamma)} \right) \leq k,$$  

(2.26)

if

$$\frac{N - \sigma + \beta - \gamma}{\sigma + y_1 - \beta_1} < \alpha < \frac{N + \beta - \gamma}{\sigma + y_1 - \beta_1}, \quad \alpha \neq 1.$$  

(2.27)

By (2.23)–(2.26) we choose $k$ as follows.

(i) If $\alpha = 1$, we choose $k$ satisfying $\sigma/(N - \sigma + \beta + \beta_1 - \gamma - \gamma_1) \leq k < 1 + \sigma/(N - \sigma + \beta + \beta_1 - \gamma - \gamma_1)$.

(ii) If $(N - \sigma + \beta - \gamma)/(\sigma + y_1 - \beta_1) < \alpha < (N + \beta - \gamma)/(\sigma + y_1 - \beta_1)$ and $\alpha \neq 1$, we choose $k$ satisfying $k_0 \leq k < k_0 + 1$, where

$$k_0 = \frac{1}{\ln \alpha} \ln \left( \frac{(y_1 - \beta_1)\alpha - (N - \sigma + \beta - \gamma)}{(\sigma + y_1 - \beta_1)\alpha - (N + \beta - \gamma)} \right).$$  

(2.28)

Case 3. $\alpha = (N + \beta - \gamma)/(\sigma + y_1 - \beta_1)$. Note that by $\beta + \alpha p_1 = \beta + \alpha \beta_1$ and $y + aq_1 = N + \beta + \alpha \beta_1$, we rewrite (2.13) as follows

$$u(x) \geq Mm_1^0 |x|^\beta (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} \frac{|y|^{\beta + a\beta_1} (1 + |y|)^{-\gamma - a\gamma_1}}{|y - x|^{\sigma}} dy$$

$$= Mm_1^0 |x|^\beta (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} \frac{|y|^{\beta + a\beta_1} (1 + |y|)^{-N - \beta - a\beta_1}}{|y - x|^{\sigma}} dy$$

$$= Mm_1^0 |x|^\beta (1 + |x|)^{-\gamma_1} A[\beta + \alpha \beta_1, N + \beta + \alpha \beta_1](x)$$

(2.29)

for all $x \in \mathbb{R}^N$.

On the other hand, for every $x \in \mathbb{R}^N$, $|x| \geq 1$, we have

$$A[\beta + \alpha \beta_1, N + \beta + \alpha \beta_1](x) \geq \int_{\mathbb{R}^N} \frac{|y|^{\beta + a\beta_1} (1 + |y|)^{-N - \beta - a\beta_1}}{(|y| + |x|)^{\sigma}} dy$$

$$= \omega_N \int_0^{+\infty} \frac{r^{\beta + a\beta_1 + N - 1}}{(1 + r)^{N + \beta + a\beta_1}} dr$$

$$\geq \omega_N \int_1^{+\infty} \frac{r^{\beta + a\beta_1 + N - 1}}{(1 + r)^{N + \beta + a\beta_1}} dr = \omega_N B(x).$$  

(2.30)
Notice that for every $r$ such that $1 \leq r \leq |x|$ we have

$$
\left( \frac{r}{1+r} \right)^{\beta + a\beta_1} \geq \frac{1}{2^{\beta + a\beta_1 + N}}, \quad \frac{1}{(r + |x|)^{\sigma-1}} \geq \min \left\{ 1, 2^{1-\sigma} \right\}. \quad (2.31)
$$

Then

$$
B(x) = \int_{1}^{|x|} \left( \frac{r}{1+r} \right)^{\beta + a\beta_1 + N} \frac{1}{(r + |x|)^{\sigma-1}} \frac{dr}{r(r + |x|)} \geq \frac{1}{2^{\beta + a\beta_1 + N}} \int_{1}^{|x|} \min \left\{ 1, 2^{1-\sigma} \right\} \frac{dr}{r(r + |x|)} \geq \frac{1}{2^{\beta + a\beta_1 + N}} \min \left\{ 1, 2^{1-\sigma} \right\} \ln \left( \frac{1 + |x|}{2} \right). \quad (2.32)
$$

It follows from (2.29), (2.30), (2.32) that

$$
u_2(x) = \begin{cases} 0, & \text{if } |x| \leq 1, \\ C_2 |x|^\beta (1 + |x|)^{-\gamma_1} \left( \ln \left( \frac{1 + |x|}{2} \right) \right)^{s_2}, & \text{if } |x| \geq 1, \end{cases} \quad (2.33)
$$

with

$$s_2 = 1, \quad C_2 = Mm^2 \omega_N \frac{1}{2^{\beta + a\beta_1 + N}} \min \left\{ 1, 2^{1-\sigma} \right\}. \quad (2.34)
$$

Suppose that

$$u(x) \geq v_{k-1}(x) = \begin{cases} 0, & \text{if } |x| \leq 1, \\ C_{k-1} |x|^\beta (1 + |x|)^{-\gamma_1} \left( \ln \left( \frac{1 + |x|}{2} \right) \right)^{s_{k-1}}, & \text{if } |x| \geq 1, \end{cases} \quad (2.35)
$$

and $C_{k-1}, s_{k-1}$, are positive constants.

Then, using (2.1), (2.3), (2.35), we have

$$u(x) \geq M|x|^\beta (1 + |x|)^{-\gamma_1} \int_{\mathbb{R}^N} \frac{|y|^\beta (1 + |y|)^{-\gamma_1} v_{k-1}^a(y) dy}{|y - x|^\sigma} \geq M|x|^\beta (1 + |x|)^{-\gamma_1} \int_{|y| \geq 1} \frac{|y|^\beta (1 + |y|)^{-\gamma_1} v_{k-1}^a(y) dy}{(|y| + |x|)^\sigma} = M|x|^\beta (1 + |x|)^{-\gamma_1} C_{k-1}^a \times \int_{|y| \geq 1} \frac{|y|^\beta (1 + |y|)^{-\gamma_1} |y|^\alpha(\beta_1 - \sigma) (1 + |y|)^{-\sigma y_1} (\ln ((1 + |y|)/2))^{\alpha y_1} dy}{(|y| + |x|)^\sigma} \geq MC_{k-1}^a |x|^\beta (1 + |x|)^{-\gamma_1} \int_{|y| \geq 1} \frac{|y|^\beta + \alpha(\beta_1 - \sigma) (1 + |y|)^{-\sigma y_1} (\ln ((1 + |y|)/2))^{\alpha y_1} dy}{(1 + |y|)^{\gamma_1} (|y| + |x|)^\sigma} \geq MC_{k-1}^a |x|^\beta (1 + |x|)^{-\gamma_1} \int_{1}^{+\infty} \frac{1}{(1 + r)^{\gamma_1} (r + |x|)^\sigma} dr. \quad (2.36)$$
Considering $|x| \geq 1$, we have
\[
\int_{1}^{+\infty} r^{\beta+\alpha(\beta_1-\sigma)+N-1} \left( \frac{\ln \left( (1+r)/2 \right)}{\ln \left( |x|/2 \right)} \right)^{as_{k-1}} dr \\
\geq \left( \ln \left( \frac{1+|x|}{2} \right) \right)^{as_{k-1}} \int_{|x|}^{+\infty} r^{\beta+\alpha(\beta_1-\sigma)+N-1} dr \\
= \frac{1}{2^{\gamma+\alpha \gamma_1+\sigma}} \left( \ln \left( \frac{1+|x|}{2} \right) \right)^{as_{k-1}} + \frac{\alpha \gamma_1}{\sigma 2^{\gamma+\alpha \gamma_1+\sigma}} \times \frac{1}{|x|^{\gamma}} \times \left( \ln \left( \frac{1+|x|}{2} \right) \right)^{as_{k-1}}.
\]

We deduce from (2.36), (2.37) that
\[
u_k(x) \geq v_k(x) = \begin{cases} 
0, & \text{if } |x| \leq 1, \\
C_k |x|^\beta (1+|x|)^{-\gamma_1} \left( \ln \left( \frac{1+|x|}{2} \right) \right)^{sk}, & \text{if } |x| \geq 1,
\end{cases}
\]
where
\[
s_k = as_{k-1}, \quad C_{k-1} = \frac{1}{\sigma 2^{\gamma+\alpha \gamma_1+\sigma}} Mw_N C_{k-1}^{\alpha}, \quad k \geq 3.
\]

From (2.34), (2.39) we obtain
\[
s_k = s_2 \alpha^{k-2} = \alpha^{k-2} = \left( \frac{N+\beta-\gamma}{\sigma+\gamma_1-\beta_1} \right)^{k-2}, \quad C_k = \frac{1}{d} \left( dC_2 \right)^{\alpha^{k-2}},
\]
where
\[
d = \frac{1}{\sigma 2^{\gamma+\alpha \gamma_1+\sigma}} Mw_N \left( \frac{\alpha^{k-1}}{\sigma^{k-1}} \right), \quad \alpha = \frac{(N+\beta-\gamma)}{(\sigma+\gamma_1-\beta_1)}, \quad \alpha > 1.
\]

Then, with $|x| \geq 1$, we rewrite (2.38) in the form
\[
u_k(x) \geq v_k(x) = \frac{1}{d} |x|^\beta (1+|x|)^{-\gamma_1} \left( dC_2 \ln \left( \frac{1+|x|}{2} \right) \right)^{\alpha^{k-2}}.
\]

Choosing $x_1$ such that $dC_2 \ln((1+|x_1|)/2) > 1$. By (2.42), we deduce that $u(x_1) = +\infty$. It is a contradiction.

Theorem is proved completely. □

Remark 2.5. In the case of $g(x,u)$ we have not a conclusion about $\alpha > N/(N-1)$ and $N \geq 2$, yet. However, when $g(x,u) = u^\alpha, N/(N-1) \leq \alpha < (N+1)/(N-1), N \geq 2$, Hu in [5] have proved that the problem (1.3), (1.4) has no positive solution. In the limiting case $\alpha = (N+1)/(N-1)$, positive solutions do exist (see [4–6]). In particular, for this
value of $\alpha$, the authors of [4] gave explicit forms for all nontrivial nonnegative solutions $u \in C^2(\mathbb{R}^{N+1}_+ \cap C^1(\mathbb{R}^{N+1}_+)$ of the problem
\[ -\Delta u = au^{\alpha+2/(N-1)} \quad \text{in} \ x' \in \mathbb{R}^N, \ x_{N+1} > 0, \]
\[ -u_{x_{N+1}}(x',0) = bu^\alpha(x',0) \quad \text{on} \ x_{N+1} = 0. \] (2.43)

They proved the following results:
(i) if $a > 0$ or $a \leq 0, b > B = \sqrt{a(1-N)/(N+1)}$, then $u(x) = C(|x - x_0|^2 + \beta)^{(1-N)/2}$ for some $C > 0, \beta \in \mathbb{R}$ and $x_0 = (x_0^1, \ldots, x_0^{N+1}) \in \mathbb{R}^{N+1}$, where $x_0^i = (b/(N-1))C^2/(N-1)$ and $\beta = (a/(N+1)(N-1))C^4/(N-1)$;
(ii) if $a = 0$ and $b = 0$, then $u(x) = C$ for some $C > 0$;
(iii) if $a = 0$ and $b < 0$, then $u(x) = Cx_1 + (-C/b)^{-(N-1)/(N+1)}$ for some $C > 0$;
(iv) if $a < 0$ and $b = B$, then $u(x) = ((2B/N - 1)x_1 + C)^{(1-N)/2}$ for some $C > 0$;
(v) if $a < 0$ and $b < B$, then there is no nontrivial nonnegative solution of the problem.

References
[1] D. T. T. Binh, T. N. Diem, D. V. Ruy, and N. T. Long, On nonexistence of positive solution of a nonlinear Neumann problem in half-space $\mathbb{R}^n_+$, Demonstratio Mathematica 31 (1998), no. 4, 773–782.
[2] D. T. T. Binh and N. T. Long, On the nonexistence of positive solution of Laplace equation in half-space $\mathbb{R}^N_+$ with a nonlinear Neumann boundary condition, Demonstratio Mathematica 33 (2000), no. 2, 365–372.
[3] F. V. Bunkin, V. A. Galaktionov, N. A. Kirichenko, S. P. Kurdyumov, and A. A. Samarskii, A nonlinear boundary value problem of ignition by radiation, Akademiya Nauk SSSR. Zhurnal Vychislitel’no˘ı Matematiki i Matematichesko˘ı Fiziki 28 (1988), no. 4, 549–559, 623 (Russian), translated in U.S.S.R. Comput. Math. and Math. Phys. 28 (1988), no. 2, 157–164 (1989).
[4] M. Chipot, I. Shafrir, and M. Fila, On the solutions to some elliptic equations with nonlinear Neumann boundary conditions, Advances in Differential Equations 1 (1996), no. 1, 91–110.
[5] B. Hu, Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition, Differential Integral Equations 7 (1994), no. 2, 301–313.
[6] B. Hu and H.-M. Yin, The profile near blowup time for solution of the heat equation with a nonlinear boundary condition, Transactions of the American Mathematical Society 346 (1994), no. 1, 117–135.
[7] N. T. Long and D. T. T. Binh, On the nonexistence of positive solution of a nonlinear integral equation, Demonstratio Mathematica 34 (2001), no. 4, 837–845.
[8] N. T. Long and D. V. Ruy, On a nonexistence of positive solution of Laplace equation in upper half-space with Cauchy data, Demonstratio Mathematica 28 (1995), no. 4, 921–927.
[9] ______, On the nonexistence of positive solution of some nonlinear integral equation, Demonstratio Mathematica 36 (2003), no. 2, 393–404.
[10] D. V. Ruy, N. T. Long, and D. T. T. Binh, On a nonexistence of positive solution of Laplace equation in upper half-space, Demonstratio Mathematica 30 (1997), no. 1, 7–14.

Nguyen Thanh Long: Department of Mathematics and Computer Science, University of Natural Science, Vietnam National University-Ho Chi Minh City, 227 Nguyen Van Cu Street, District 5, Ho Chi Minh City, Vietnam
E-mail address: longnt@hcmc.netnam.vn