1 Introduction

The classical Laguerre Geometry originates, primarily, in the geometry of Speeren (oriented lines) and Zykeln (oriented circles) of a Euclidean plane which, after the Blaschke transformation can be visualized as the structure of conics on a projective cylinder, cf. [4]. This definition can be generalized in various directions: one can investigate structures of oriented hyperplanes (cf. [7]), structures of oriented lines in a higher dimensional Euclidean space (cf. [14]), ovals and ovoidal surfaces replacing the “basic” conic of a cylinder (cf. [1], [3], [10]), structures on degenerate “quadrics” (cf. [9]).

Passing to a, possibly one of the widest generalizations, we can roughly say that a generalized Laguerre space is a geometrical structure based on a projective cone (with its vertex deleted) $S$ defined over a nondegenerate quadric. Several attempts to geometry on $S$ are possible. One can look, primarily, at generators of $S$, considering the underlying point universe as the set of self conjugate points of a degenerate polarity. Following this approach one enters into the world of polar geometry (see e.g. [5]). One can also try to imitate the approach of chain geometry (cf. e.g. [2], [8]) and distinguish as primitives the family of conics on $S$. In this paper we are closer to this second tradition. However, if a chain space contains lines (definable in terms of chains, as it happens e.g. in Benz-Minkowski planes, cf. [2]), it is much more convenient to have these lines distinguished as individuals of some other sort. So, finally, a structure under our consideration has form $L = \langle \text{points}, \text{cycles (chains)}, \text{lines} \rangle$.

With each geometrical structure we can associate the family of its subspaces. In our case we can define the dimension function on the subspaces of $L$ and after that the standard construction of the Grassmann space of $k$-subspaces of $L$ can be applied and pencils of these subspaces can be defined. Then (also a standard one) question in the spirit of Chow appears: can we recover the underlying geometry in terms of a geometry on its $k$-subspaces.

In the paper we answer the above question in a very particular case $k = 1$; moreover, we restrict ourselves to geometry of lines of $L$. Practically, we study in some detail the Grassmannian of (all) 1-dimensional subspaces and structures of pencils of lines. Dealing with (projectively) planar pencils leads to arguments from polar geometry, so we only mention these pencils at the end of the paper and we concentrate upon pencils determined by (projective) cones. The structure
of such pencils seems interesting also on its own right: they do not yield any partial linear space but, instead, they introduce a chain-space-like structure. In both considered cases the answer is affirmative i.e. the structure $L$ can be recovered in terms of respective structures of lines.

Questions concerning structures defined on the set of cycles are addressed in some other papers. In case of Grassmannians of chains one can, perhaps, apply techniques of [12] and [11]. In case of pencils of chains (especially when so called tangent pencils are involved) some troubles appear concerning tangency classes of chains, so that world needs other techniques.

As $k$ increases, $L$ admits $k$-subspaces that carry quite different geometries and investigations on Grassmannians defined on them become much more complex.

1.1 Definitions

Now, let us make the geometry considered in the paper more precise. Let $\mathfrak{P}$ be a finite dimensional Pappian and non-Fanoian projective space. Next, let $T, B$ be two transversal subspaces of $\mathfrak{P}$ and $Q$ be a nondegenerate quadric on $B$. Finally, let $S_0$ be a projective cone with vertex $T$ over a quadric $Q$, contained in $\mathfrak{P}$, and let $S = S_0 \setminus T$. Let $L$ be the set of all (nonempty) sections with $S$ of the lines of $\mathfrak{P}$ which lie on $S_0$. Then $L = A \cup G$, where $A$ consists of the (sections of) lines meeting $T$, and $G$ consists of the lines on $S_0$ that miss $T$; each line in $G$ is contained in a subspace of the form $M + T$, where $M$ is a generator of $Q$. Elements of $G$ are called projective lines, and the elements of $A$ are affine lines. In the family $A$ we have a natural parallelism $\parallel$. Let $\mathcal{C}$ be the family of all nontrivial sections $S \cap A$ which do not contain a line, where $A$ is a plane of $\mathfrak{P}$. Then $\mathcal{C}$ consists of all the conics on $S_0$. Set $\mathcal{I} = \in \subset S \times (L \cup \mathcal{C})$. Finally,

$$L = \langle S, \mathcal{C}, G, A, ||, \mathcal{I} \rangle$$

is a generalized Laguerre space defined over a ruled quadric $Q$ and contained in a projective space $\mathfrak{P}$. If $Q$ is a nonruled quadric then $G = \emptyset$, $L = A$, and most of subsequent results concern “nothing” (structures with the void universe). Thus in the whole paper we assume that $Q$ is ruled.

2 Subspaces

A subspace of $L$ is a subset $X \subset S$ such that the conditions:

- if $|K \cap X| \geq 2$, then $K \subset X$,
- if $a \in X \cap K$ and $K \parallel M \subset X$, then $K \subset X$,
- if $|A \cap X| \geq 3$, then $A \subset X$,
- if $|X \cap A| \geq 2$ and $A \parallel B \subset X$, then $A \subset X$

($\parallel$ is the relation of tangency of cycles in this case)

hold for every $a \in S, K, M \in L, A, B \in \mathcal{C}$.

From some point of view it is relatively easy to characterize subspaces of $L$; in the projective representation of the Laguerre space $L$ we have started from, a subspace $X$ is a section of $S$ with a projective subspace $\mathfrak{X}$. On the other hand, subspaces may carry quite different geometries. If $\dim(X) = 1$ the situation is clear: $X$ is either a line (affine or projective, then $\dim(\mathfrak{X}) = 1$), or a cycle ($\dim(\mathfrak{X}) = 2$). Roughly speaking, in any case $X$ is a generalized Laguerre space or it is a projective, semiaffine or affine space.
3 Classification of subspaces

In this section we shall give a detailed classification of the subspaces of $\mathcal{L}$ and analyze the arising incidence geometry.

Let us write, generally $\varphi(\mathcal{L})$ for the family of all the subspaces of $\mathcal{L}$ and $\varphi_k(\mathcal{L})$ for the family of $k$-dimensional subspaces of $\mathcal{L}$.

Next, let us write

- $s_{d,w}^{m,m'}$ for the set of all affine cones with $m$-dimensional affine generator defined over a projective cone with $m'$-dimensional projective generator and basis being a $d$-dimensional quadric with index $w$ (note that if $X \in s_{d,w}^{m,m'}$ then the dimension of the vertex of the “projective part” of $X$ is $m' - w - 1$);
- $l_{d,w}^m$ for the set of all affine cones with $m$-dimensional affine generator and the $d$-dimensional quadric with index $w$ as a basis contained in $\mathcal{L}$;
- $s_{d,w}^m$ for the set of all projective cones with $m$-dimensional projective generator (i.e., with $(m - w - 1)$-dimensional vertex) and the $d$-dimensional quadric with index $w$ as a basis contained in $\mathcal{L}$;
- $g_d^m$ for the set of $(m + d)$-dimensional generators of $\mathcal{L}$ which maximal affine subgenerator has dimension $m$.

From the definitions we have

- $l_{d,w}^m = s_{d,w}^m$ and $s_{d,w}^m = s_{d,w}$,
- $g_d^0$ is the set of all $d$-dimensional projective generators of $\mathcal{L}$,
- $g_d^m$ is the set of all $m$-dimensional affine generators of $\mathcal{L}$; in particular
  - $g_1^0 = \mathcal{G}$,
  - $g_1^1 = \mathcal{A}$,
- $l_{d,w}^0$ (0 is the dimension of a point) is the set of all subquadrics contained in $\mathcal{L}$; formally, also $l_{d,w}^0 = s_{d,w}^0$. Thus
  - $l_{d,w}^0 = s_{d,w}^0 = \mathcal{C}$.

Let us write $\nu$ for the dimension of a base of $\mathcal{L}$ i.e. $\nu = \dim(B) + 1$, $t$ for the dimension of a maximal proper subspace of $\mathcal{L}$ so $t$ is the index of $Q$, and $a$ for the dimension of a maximal affine generator of $\mathcal{L}$ i.e. $a = \dim(T) + 1$.

Fundamental properties of the subspaces of $\mathcal{L}$ are given in the subsequent $\S3$.1 $\S3$.5

Fact 3.1. A maximal proper subspace of $\mathcal{L}$ is an element of the following four sets: $l_{\nu-1,t-1}^a$, $l_{\nu-1,t}^a$, $l_{\nu,t}^a$, and $l_{\nu-2,t-1}^a$.

Fact 3.2. If $X \in l_{d,w}^m$ then $\dim(X) = d + m$.

Fact 3.3. If $\varphi(\mathcal{L}) \ni X \subset X' \in l_{d,w}^m$ then $X$ belongs to $s_{d_1,w_1}^{m_1,m_1'}$ or $X$ belongs to $s_{d_2,w_2}^{m_2}$ where $l_{d,w}^m \neq s_{d_1,w_1}^{m_1,m_1'}$, $m_1 = 0, 1, \ldots, m$, $m_1' = 0, 1, \ldots, w$, $d_1 = 0, 1, \ldots, d$, $w_1 = 0, 1, \ldots, w$, and $s_{d_1,w_1}^{m_1,m_1'}$ is well defined.

Theorem 3.4. Let $X \in s_{d,w}^{m,m'}$. Then the geometry of the restriction $\mathcal{L}|X$ of $\mathcal{L}$ to $X$ is the Laguerre space with $\nu_{\mathcal{L}|X} = d + m' - w_1$, $t_{\mathcal{L}|X} = m'$, and $a_{\mathcal{L}|X} = m$.

Theorem 3.5. Let $X \in g_d^m$. Then the geometry of the restriction $\mathcal{L}|X$ of $\mathcal{L}$ to $X$ is a semiaffine linear space (a hole space, c.f. [15], also called a slit space).
4 Pencils, general construction

In accordance with the general approach adopted in incidence geometries for an integer \( k \) and \( X, Z \in \wp(\mathcal{L}) \) with \( \dim(X) = k - 1 \), \( \dim(Z) = k + 1 \) we define the \( k \)-pencil \( p(X, Z) \) to be the set

\[
p(X, Z) = \{ Y \in \wp(\mathcal{L}) : X \subset Y \subset Z, \dim(Y) = k \}.
\]

Actually, the formula (1) defines too wide class of subsets than those which are usually referred to as pencils. For example, usually the set of the cycles through a point on a (projective) sphere is not considered as a pencil. At least two properties of a currently investigated family \( \mathcal{P} \) of pencils should be satisfied:

- \( |p| \geq 3 \) for any \( p \in \mathcal{P} \);
- if \( p_1, p_2 \in \mathcal{P} \) and \( p_1 \subset p_2 \) then \( p_1 = p_2 \).

In the sequel in each particular case we shall write down explicitly what types of pencils are currently admitted.

Let us begin with an analysis of possible 1-pencils. We write \( s(a) \) for the set of all lines through a point \( a \) and we call such a set a star. Analogously, one defines stars of cycles; in the paper pencils of cycles are not investigated though, and therefore stars of cycles are not needed here. Then suitable pencils have form

\[
s \cap \{ Y : Y \subset Z \} \text{ where } Z \in \wp(\mathcal{L}), \dim(Z) = 2, \text{ and } s \text{ is a star.}
\]

Two dimensional subspaces of \( \mathcal{L} \) are the elements of the following classes: \( g_0^1 \) (projective planes), \( g_1^1 \) (semiaffine planes), \( g_0^2 \) (affine planes), \( l_1^1,0 \) (affine cones = Laguerre planes), \( G_1^1,0 \) (proper projective pencils), \( G_0^2 \) (proper projective pencils), \( G_1^2 \) (proper semiaffine pencils). Note that the class \( s_{1,1}^1 \) is excluded from \( \wp_2(\mathcal{L}) \); its elements are somehow “strange”, as they are unions of two planes in \( g_1^1 \).

Recall that we restrict ourselves to pencils of lines only, and our pencils should be at least 3-element sets. Consequently, it suffices to consider sets of the form (2) with \( s = s(a) \) and \( a \in Z \) where \( Z \) is one of the following: an affine plane, a projective cone (in this case we assume, additionally, that \( a \) is a vertex of \( Z \); without this assumption the corresponding pencil would consist of one line only), a projective plane, and a semiaffine plane, resp.. The obtained classes of pencils are denoted by \( \mathcal{P}_{A}^{\parallel} \) (proper affine pencils), \( \mathcal{P}_{C}^{\parallel} \) (conic pencils), \( \mathcal{P}_{P}^{\parallel} \) (proper projective pencils), and \( \mathcal{P}_{SP}^{\parallel} \) (proper semiaffine pencils). Note that the elements of a conic pencil and of a proper projective pencil are projective lines, the elements of a proper affine pencil are affine lines, while a proper semiaffine pencil contains one affine line and its remaining lines are projective. In what follows we shall also consider restrictions of pencils in \( \mathcal{P}_{P}^{\parallel} \) to projective lines and such a restriction will be also called a proper semiaffine pencil.

In case of the currently considered geometry we have the notion of a parallelism distinguished; in such a geometry so called “parallel pencils” are frequently considered. We follow this tradition and we consider sets of the form

\[
\{ L \in A : L \parallel L_0, \ L \subset Z \}, \text{ where } Z \in \wp(\mathcal{L}), \dim(Z) = 2.
\]

As above, it suffices to consider the cases when \( Z \) is an affine cone, a semiaffine plane, and an affine plane. The classes of pencils thus obtained are denoted as follows: \( \mathcal{P}_{A}^{\parallel} \) (cylinder pencils), \( \mathcal{P}_{SP}^{\parallel} \) (parallel semiaffine pencils), and \( \mathcal{P}_{A}^{\parallel} \) (parallel affine pencils).

In what follows we shall try to define the underlying Laguerre geometry in terms of the structures of the form

(\text{lines, pencils of lines})

for some, more interesting, systems of pencils.
5 Grassmann spaces and spaces of pencils associated with \( \mathcal{L} \)

One more notion will be used intensively in the sequel: for \( K_1, K_2 \in \mathcal{L} \) we write

\[ K_1 \sim K_2 \iff K_1 \neq K_2 \land \exists a \in S \left[ a \mid K_1, K_2 \right]. \]

5.1 Grassmann space of 1-subspaces

Let us begin with the simplest case when the points of the considered structure are the 1-dimensional subspaces of \( \mathcal{L} \). In symbols,

\[ \varrho_1(\mathcal{L}) = \mathcal{C} \cup \mathcal{L} \quad \text{and} \quad \varrho_2(\mathcal{L}) = g_{0}^{2} \cup g_{1}^{2} \cup g_{0}^{1} \cup l_{1,0}^{1} \cup l_{0,0}^{1} \cup l_{2,1}. \]

In what follows we shall be concerned with the structure

\[ \mathcal{G}_1(\mathcal{L}) := \langle \varrho_1(\mathcal{L}), \varrho_2(\mathcal{L}), \subset \rangle. \]

Let us write, for short, \( \varrho_1 = \varrho_1(\mathcal{L}) \) and \( \varrho_2 = \varrho_2(\mathcal{L}) \).

Through a series of subsequent lemmas we shall distinguish in terms of the geometry of \( \mathcal{G}_1(\mathcal{L}) \) basic types of corresponding subspaces.

The crucial observation consists in the following lemma, which shows when (formally considered) the fundamental axiom of partial linear spaces fails in the structure \( \mathcal{G}_1(\mathcal{L}) \).

**Lemma 5.1.** Let \( Y, Y' \in \varrho_1 \) and \( Z, Z' \in \varrho_2 \). If \( Y, Y' \subset Z, Z' \) and \( Y \neq Z' \), \( Z \neq Z' \) then \( Y, Y' \in \mathcal{L} \) and \( Z, Z' \in l_{1,0}^{1} \cup l_{0,0}^{1} \cup l_{2,1}. \)

**Proof.** Suppose that \( Y \in \mathcal{C} \); then \( Y, Y' \) uniquely determines the subspace in \( \varrho_2 \) which contains \( Y, Y' \). This proves that \( Y, Y' \notin \mathcal{C} \). Then possible \( Z, Z' \) are those elements of \( \varrho_2 \) that contain a line. On the other hand the planes of \( \mathcal{L} \), i.e. the elements of \( g_{0}^{1}, g_{0}^{2} \), and \( g_{0}^{2} \), if distinct may have at most a line in common. Moreover, no plane can cross a 2-dimensional subspace that is not a plane in two lines. This finally yields our claim. \( \square \)

Let us see that the lines \( Y, Y' \) from 5.1 are such that \( Y \parallel Y' \) or \( Y \sim Y' \), because otherwise \( Y, Y' \) uniquely determines the subspace in \( \varrho_2 \) which contains \( Y, Y' \).

As a direct consequence of 5.1 we obtain

**Lemma 5.2.** Let \( Y \in \varrho_1 \). Then

\[ Y \in \mathcal{L} \iff (\exists Y' \in \varrho_1, \exists Z, Z' \in \varrho_2)[Y \neq Y' \land Z \neq Z' \land Y, Y' \subset Z, Z']. \] (4)

Consequently, the class \( \mathcal{L} \) is definable in terms of \( \mathcal{G}_1(\mathcal{L}) \).

**Proof.** The right-to-left implication of 4 follows from 5.1. Let \( Y \in \mathcal{L} \) be arbitrary. If \( Y \notin \mathcal{C} \) we take arbitrary point \( a \in Y \), a cycle \( C_1 \) through \( a \), the cone \( Z \) with base \( C_1 \) and generator \( Y \), \( b \in C_1 \) with \( b \neq a \), \( Y' \in \mathcal{A} \) with \( b \in Y' \parallel Y \), \( C_2 \in \mathcal{L} \) with \( a, b \in C_2, C_2 \notin Z \), and the cone \( Z' \) with base \( C_2 \) and generator \( Y \). If \( Y \notin \mathcal{G} \), analogously, we take arbitrary point \( a \in Y \), a cycle \( C_1 \) through \( a \). Next we consider three-dimensional subspace \( V \) of \( \mathcal{G} \) such that \( C_1, Y \in V \) and we take the intersection \( Z \) of \( V \) and \( \mathcal{G} \). Let \( b \in C_1, b \neq a, Y' \in \mathcal{G} \) with \( b \in Y' \parallel Y, C_2 \in \mathcal{C} \) with \( a, b \in C_2, C_2 \notin Z \), and \( Y \sim Y' \). Let \( C_2 \in \mathcal{C} \) with \( a, b \in C_2, C_2 \notin Z \). On the end we consider subspace \( V' \) of \( \mathcal{G} \) with \( C_2, Y, Y' \subset V' \) and we take the intersection \( Z' \) of \( V' \) and \( \mathcal{L} \). \( \square \)
Lemma 5.3. Let $Z \in \wp_2$. The following two equivalences hold:

$$Z \in l_1^0 \cup d_1^0 \iff (\exists Y, Y' \in L)[Y, Y' \subset Z \land Y \neq Y'] \land (\forall Y, Y' \in L)[Y, Y' \subset Z \land Y \neq Y' \implies (\exists Z' \in \wp_2)[Y, Y' \subset Z' \neq Z]]$$  \hspace{1cm} (5)

$$Z \in l_2^1 \iff (\exists Z' \in \wp_2)(\exists Y, Y' \in L)[Y \neq Y' \land Z \neq Z' \land Y, Y' \subset Z, Z'] \land Z \notin (l_1^0 \cup d_1^0)$$  \hspace{1cm} (6)

Consequently, the class of cones and the class of Minkowski planes contained in $L$ both are definable in $G_1(L)$.

Proof. Let $Z \in l_1^0 \cup d_1^0$. By elementary geometry of planes from $l_1^0$ and $d_1^0$, there exist $Y, Y' \in L$ such that $Y, Y' \subset Z$ and $Y \neq Y'$.

Let $Y, Y' \in L$ and let $Y, Y' \subset Z$ and $Y \neq Y'$. Thus there exists a cycle $C_1$ such that $Z$ is the cone with base $C_1$ and generator $Y$. Let $a = Y \cap C_1$ and $b = Y' \cap C_1$. Thus $a \neq b$. Consider a cycle $C_2$ with $a, b \in C_2$ and $C_2 \notin Z$. Let $Z'$ be the cone with base $C_2$ and generator $Y$. Then $Z' \notin \wp_2$, $Y, Y' \subset Z'$ and $Z' \neq Z$.

Assume the right-hand-side of (5). If $Z \in l_1^0$ then there does not exist $Y \in L$ with $Y \subset Z$ and we have a contradiction. If $Z \in l_2^1$ then there exist $Y, Y' \in L$ such that $Y \neq Y'$, $Y', Y'' \subset Z$ and $Y \neq Y'$. Thus for every $Z' \in \wp_2$ such that $Y, Y' \subset Z'$ we get $Z = Z'$. This contradicts our assumption. If $Z \in g_1 \cup g_2$ then $Y, Y', Y'' \in L$ with $Y', Y'' \subset Z$ and $Y \neq Y'$ uniquely determine $Z$. Thus we have a contradiction again. Hence $Z \in l_1^0 \cup d_1^0$, which completes the proof of (5).

Let $Z \in l_1^0$. Thus $Z \notin (l_1^0 \cup d_1^0)$. By elementary geometry of Minkowski plane, there exist $Y, Y' \in L$ with $Y \neq Y'$, $Y \sim Y'$ and $Y, Y' \subset Z$. Let $c = Y \cap Y'$ and let $C_1$ be a cycle such that $c \notin C_1 \subset Z$. Thus there exist points $a = Y \cap C_1$, $b = Y' \cap C_1$. Of course $a \neq b$. Let $C_2$ be a cycle such that $a, b \in C_2$ and $C_2 \notin Z$. Consider the intersection $Z'$ of $L$ and the 3-subspace $V$ of $\wp$ with $C_2, Y, Y' \subset V$.

Let $Z \in \wp_2$: assume that the right-hand-side of (6) holds. If $Z \in l_1^1$ then there does not exist $Y \in L$ with $Y \subset Z$ and we have a contradiction. If $Z \in g_1 \cup g_2$ then a pair $Y, Y' \in L$ with $Y, Y' \subset Z$ and $Y \neq Y'$ uniquely determines $Z$. Thus we have a contradiction again. Hence $Z \in l_1^1$, and we have completed the proof of (6).

By elementary geometry of planes from $l_1^1$, $d_1^0$ and $l_1^0$ we get

Lemma 5.4. Let $Y \in \wp_1$. We have

$$Y \in G \iff Y \in L \land (\exists Z \in \wp_2)[Y \subset Z];$$  \hspace{1cm} (7)

$$Y \in A \iff Y \in L \land Y \notin G.$$  \hspace{1cm} (8)

Further, let $Z \in \wp_2$. Then

$$Z \in d_1^0 \iff Z \in (d_1^0 \cup l_1^0) \land (\exists Y \in G)[Y \subset Z];$$  \hspace{1cm} (9)

$$Z \in l_1^1 \iff Z \in (d_1^0 \cup l_1^0) \land Z \notin d_1^0.$$  \hspace{1cm} (10)

Proof. The right-to-left implication of (7) is evident. Let $Y \in G$. (Of course $Y \in L$.) Thus there exists a base $Q$ of $L$ with $Y \subset Q$. If $Q \notin l_1^1$ we take $Z := Q$.

1. $Q \notin l_1^1$. Let $V_1$ be the subspace of $\wp$ spanned by $Q$ and let $W_1$ be a non-tangent to $Q$ hyperplane of $V_1$ such that $Y \subset W_1$. Then $Q_1 = W_1 \cap Q$ is a ruled quadric with $\dim(Q_1) = \dim(Q) - 1$ and $Y \subset Q_1$. If $Q_1 \in l_1^1$ then we take $Z := Q_1$. 

6
2. \( Q_1 \notin \mathcal{P}_{1,1} \). Let \( V_2 \) be the subspace of \( \mathcal{P} \) spanned by \( Q_1 \) and let \( W_2 \) be a non tangent to \( Q_1 \) hyperplane of \( V_2 \) such that \( Y \subset W_2 \). Then \( Q_2 = W_2 \cap Q_1 \) is a ruled quadric with \( \dim(Q_2) = \dim(Q_1) - 1 \) and \( Y \subset Q_2 \). If \( Q_2 \notin \mathcal{P}_{1,1} \) then after a finite number of steps analogous to the above we find \( Q_i \notin \mathcal{P}_{1,1} \) with \( Y \subset Q_i \) and we set \( Z := Q_i \).

Now equivalences of (8), (9) and (10) are evident. \( \square \)

**Corollary 5.5.** The structure of “conic” pencils i.e. the structure
\[
(G, \mathcal{P}_{1,0}, \subset)
\]
is definable in \( G_1(\mathcal{L}) \).

### 5.2 “Conic” pencils

Now, we assume that \( \nu \geq 3 \). Then for every point \( a \) of \( \mathcal{L} \) there exists \( S' \in \mathcal{P}_{1,0} \) with vertex \( a \). Let us pay attention to the structure
\[
\mathcal{E} := (G, \mathcal{P}_{1,0}, \subset) \cong (G, \mathcal{P}_2^0).
\]

Let us begin with the following characterization of the adjacency relation of projective lines:

**Lemma 5.6.** Let \( L_1, L_2 \in G \). Then
\[
L_1 \sim L_2 \iff (\exists S' \in \mathcal{P}_{1,0})[L_1, L_2 \subset S'] \lor (\exists M_1, M_2 \in G)(\exists S_1, S_2 \in \mathcal{P}_{1,0})
\]
\[
[ L_1, M_1, M_2 \subset S_1 \wedge L_2, M_1, M_2 \subset S_2 \wedge M_1 \neq M_2].
\]

**Proof.** The right-to-left implication of (11) is evident; the projective lines which are contained in a cone in \( \mathcal{P}_{1,0} \) all pass through its vertex.

The point is to prove the converse implication. Let \( a \parallel L_1, L_2 \) and let \( Z \) be a 2-dimensional subspace that contains \( L_1, L_2 \).

If \( Z \in \mathcal{P}_{1,0} \) we are done.

Let \( Z \notin \mathcal{P}_{1,0} \). Let \( Q \) be a base of \( \mathcal{L} \) such that \( L_1, L_2 \subset Q \). Consider a hyperplane \( V \) of projective space spanned by \( Q \), tangent to \( Q \) at the point \( a \). Thus \( V \cap Q \) contains a cone \( S' \in \mathcal{P}_{1,0} \) with vertex \( a \) such that \( L_1, L_2 \subset S' \).

Let \( Z \in \mathcal{P}_{0,1} \cup \mathcal{P}_{1,1} \). Let \( b \parallel L_1 \) and \( b \neq a \). Thus there exists a cycle \( C \) with \( b \parallel C \). Consider \( Z_1 \in \mathcal{P}_2 \) spanned by \( L_1 \) and \( C \). Thus \( Z_1 \subset \mathcal{P}_{0,1} \cup \mathcal{P}_{1,1} \). By above, there exists a cone \( S_1 \in \mathcal{P}_{1,0} \) with vertex \( a \) such that \( Z_1 \subset S_1 \). Let a cycle \( C_1 \) be a base of the cone \( S_1 \) with \( b \parallel C_1 \). Let \( c, d \parallel C_1 \) with \( \neq (c, d, b) \). Thus there exist \( M_1, M_2 \in G \) such that \( a \parallel M_1, a \parallel M_2 \). Of course \( M_1, M_2 \subset S_1 \). Consider the lines \( M_1, M_2 \), and \( L_2 \). Note that no plane of \( \mathcal{P} \) contains \( M_1 \cup M_2 \cup L_2 \). Hence \( M_1, M_2 \), and \( L_2 \) together span a 3-space \( V_1 \) of \( \mathcal{P} \). Thus \( S \cap V_1 \in \mathcal{P}_2 \). Now it is easily seen that \( S \cap V_1 \in \mathcal{P}_{1,0} \). Let \( S_2 = S \cap V_1 \). Hence we proved the right-hand-side of (11).

From 5.6 we learn that the adjacency of lines is definable in terms of the geometry of \( \mathcal{E} \). To prove that the whole geometry of \( \mathcal{L} \) can be interpreted in the geometry of \( \mathcal{E} \) it suffices to prove that the concurrency of lines in \( G \) is definable in terms of their adjacency.

**Lemma 5.7.** Let \( L_1, L_2 \) be two distinct projective lines contained in a cone \( S' \in \mathcal{P}_{1,0} \) with vertex \( a \). If \( L \in \mathcal{L} \) and \( L \sim L_1, L_2 \) then \( a \parallel L \).

**Proof.** Let \( L \in \mathcal{L} \) and \( L \sim L_1, L_2 \). Suppose that \( a \parallel L \). Then there exist two points \( b, c \) such that \( b \neq c \), \( b \parallel L \), \( b \parallel L_1 \), and \( c \parallel L_2 \). But, by elementary geometry of planes from \( \mathcal{P}_{1,0} \), there exists a cycle \( C \) with \( b, c \parallel C \). Hence we have a contradiction. \( \square \)
Lemma 5.8. Let \( S' \in \mathfrak{d}^1_{1,0} \), \( a \) be the vertex of \( S' \), and \( L \in \mathcal{G} \). Then
\[
a \mid L \iff (\forall L' \in \mathcal{G})[L' \subset S' \implies L \sim L' \lor L = L'].
\] (12)

Proof. Let \( a \mid L \). Assume that \( L' \in \mathcal{G} \) and \( L' \subset S' \). Thus \( a \mid L' \), so \( L \sim L' \lor L = L' \).
Assume the right-hand-side of (12). Consider \( L'_1, L'_2 \in \mathcal{G} \) with \( L'_1, L'_2 \subset S' \) and \( L'_1 \neq L'_2 \). From assumption we get \( L \sim L'_1, L'_2 \lor L = L'_1 \lor L = L'_2 \). By 5.7 \( a \mid L \).

In view of 5.8 the family
\[
\{ \{ L \in \mathcal{G} \mid (\forall L' \in \mathcal{G})[L' \subset S' \implies L \sim L' \lor L = L'] \} : S' \in \mathfrak{d}^1_{1,0} \}
\]
coinsides with the family \( \{ s(a) \cap \mathcal{G} : a \in S \} \) and thus the latter is definable in \( \mathcal{C} \). Since, clearly, \( \langle S, \mathcal{G}, \epsilon \rangle \cong \langle \{ s(a) \cap \mathcal{G} : a \in S \}, \mathcal{G}, \exists \rangle \), we conclude with

Corollary 5.9. The structure \( \langle S, \mathcal{G} \rangle \) is definable in \( \mathcal{C} \).

Note also that in terms of the adjacency of projective lines we can distinguish two cases: adjacent lines are on a cone and adjacent lines are on a (affine, semi-affine, or projective) plane. Note that the case when \( L_1, L_2 \) (\( L_1 \sim L_2 \)) are on a cone may cover also the case when they are on a Minkowski plane contained in \( \Sigma \).

Proposition 5.10. Let \( L_1, L_2 \in \mathcal{G}, L_1 \sim L_2 \). The following conditions are equivalent
- There is \( S' \in \mathfrak{d}^1_{1,0} \) such that \( L_1, L_2 \subset S' \).
- The formula
\[
(\forall K_1, K_2 \in \mathcal{G})[L_1, L_2 \sim K_1, K_2 \implies K_1 \sim K_2 \lor K_1 = K_2]
\] (13)
holds.

Proof. Assume that there is \( S' \in \mathfrak{d}^1_{1,0} \) such that \( L_1, L_2 \subset S' \). Let \( K_1, K_2 \in \mathcal{G} \) and \( L_1, L_2 \sim K_1, K_2 \).
By 5.7 \( a \mid K_1, K_2 \) where \( a \) is the vertex of \( S' \). Hence \( K_1 \sim K_2 \lor K_1 = K_2 \).

Assume (13). Let \( Z \in \mathfrak{d}^2 \) with \( L_1, L_2 \subset Z \). Thus either \( Z \in \mathfrak{d}^1_{1,0} \) or \( Z \in \mathfrak{g}^2 \) or \( Z \in \mathfrak{g}^1 \). Evidently, if \( Z \in \mathfrak{d}^1_{1,0} \) then there exists \( S' \in \mathfrak{d}^1_{1,0} \) with \( L_1, L_2 \subset S' \). However if \( Z \in \mathfrak{g}^2 \) or \( Z \in \mathfrak{g}^1 \) then \( Z \) contains \( K_1, K_2 \) which contradicts (13). Finally, there is \( S' \in \mathfrak{d}^1_{1,0} \) such that \( L_1, L_2 \subset S' \).

5.3 Planar pencils

Usually, when one deals with structures with lines then he considers planar pencils. Consequently, one should primarily consider the structure
a) \( \mathfrak{S} = \langle \mathcal{G}, \mathcal{P}_{\mathcal{G}}^p \rangle \),
b) \( \mathfrak{S} = \langle \mathcal{G}, \mathcal{P}_{\mathcal{G}}^p \cup \mathcal{P}_{\mathcal{G}}^{ps} \rangle \), and
c) \( \mathfrak{S} = \langle \mathcal{L}, \mathcal{P}_{\mathcal{L}} \rangle \), where \( \mathcal{P}_{\mathcal{L}} = \{ \{ L \in \mathcal{L} : a \in L \subset Z \} : a \in Z \in \mathfrak{g}^1 \cup \mathfrak{g}^2 \cup \mathfrak{g}^3 \} \).

The obtained structures are partial linear spaces. Here some standard methods used in Grassmann geometries of polar and projective spaces can be used (cf. [6], [13]; in any case we begin with determining maximal cliques of the collinearity of the considered structure and maximal strong subspaces.)
5.3.1 Case \( \text{a} \)

Now the maximal cliques of \( \mathfrak{S} \) fall into two classes:

- (a1) \( \{ T(Z) : Z \in \mathfrak{g}_0^1 \} \), where \( T(Z) = \{ L \in \mathfrak{G} : L \subset Z \} \),
- (a2) \( \{ [a, Y] : a \in Y \in \mathfrak{g}_1^2 \} \), where \( [a, Y] = \{ L \in \mathfrak{G} : a \in L \subset Y \} \).

Simultaneously, they are maximal strong subspaces of \( \mathfrak{S} \); actually, they are projective spaces. Within the partial linear space \( \mathfrak{S} \) we have \( \dim(T(Z)) = 2 \) and \( \dim([a, Y]) = t - 1 \). If \( t \neq 3 \) then the above two types of maximal cliques can be distinguished in terms of the geometry of \( \mathfrak{S} \).

5.3.2 Case \( \text{b} \)

The maximal cliques of \( \mathfrak{S} \) (and, at the same time, maximal strong subspaces of \( \mathfrak{S} \)) fall into two classes:

- (b1) \( \{ T(Z) : Z \in \mathfrak{g}_0^0 \cup \mathfrak{g}_0^1 \} \), where \( T(Z) \) is as in (a1),
- (b2) \( \{ [a, Y] : a \in Y \in \mathfrak{g}_1^2 \} \), \( [a, Y] \) is as in (a2).

Note that the subspace \( T(Z) \) carries the geometry of projective plane (when \( Z \in \mathfrak{g}_0^0 \)) or the geometry of affine plane (when \( Z \in \mathfrak{g}_0^1 \)). In any case, from the point of view of \( \mathfrak{S} \), \( \dim(T(Z)) = 2 \). Thus these two types of subspaces in the class \( \text{b} \) are distinguishable within \( \mathfrak{S} \).

In \( \mathfrak{S} \) we have \( \dim([a, Y]) = \dim(Y) - 1 = t + a - 1 \); thus maximal strong subspaces of the form \([a, Y]\) have dimension 2 only in the case when \( t + a = 3 \). If \( t = 2, a = 1 \) then \( Y \in \mathfrak{g}_1^2 \) and \([a, Y]\) is a semiaffine plane (a projective plane with one point deleted) and thus a subspace of the form \([a, Y]\) is distinguishable from subspaces of the form \( T(Z) \).

If \( t = 1, a = 2 \) then \([a, Y]\) is an affine plane. At the same time, \( \mathfrak{g}_0^0 = \emptyset \) and thus every maximal subspace of \( \mathfrak{S} \) is an affine plane and there is no way to distinguish the two types (b1), (b2) following the above way. In any other case these two types are distinguishable. Finally, for \( Y_1, Y_2 \in \mathfrak{g}_1^2 \) and points \( a \in Y \) it suffices to characterize the relation \( a_1 = a_2 \) in terms of the subspaces \( X_1 = [a_1, Y_1], X_2 = [a_2, Y_2] \) and the geometry of \( \mathfrak{S} \) to get that the structure \( (S, \mathfrak{G}) \) can be defined in \( \mathfrak{S} \).

5.3.3 Case \( \text{c} \)

In this case the maximal cliques of \( \mathfrak{S} \) fall into two classes:

- (c1) subsets of \( T^+(Z) = \{ L \in \mathfrak{L} : L \subset Z \} \), where \( Z \in \mathfrak{g}_0^2 \cup \mathfrak{g}_1^1 \cup \mathfrak{g}_0^3 \), and
- (c2) \( \{ [a, Y]^* : a \in Y \in \mathfrak{g}_1^2 \} \), where \( [a, Y]^* = \{ L \in \mathfrak{L} : a \in L \subset Y \} \).

If \( Z \in \mathfrak{g}_0^2 \) then \( T^+(Z) \) is a clique. If \( Z \in \mathfrak{g}_0^3 \) then \( T^+(Z) \) is not any clique as it contains a pair of parallel lines; a clique \( \mathcal{K} \) contained in \( T^+(Z) \) is a selector of the horizon of \( Z \): it contains exactly one line in each of the directions on \( Z \). If \( Z \in \mathfrak{g}_1^1 \) then a clique contained in \( T^+(Z) \) has form \( T(Z) \cup \{ L \} \), where \( L \) is an affine line on \( Z \) and \( T(Z) \) is defined in (a1). Note that cliques of type \( \text{b} \) are subspaces of \( \mathfrak{S} \) only when \( Z \in \mathfrak{g}_0^2 \) or \( Z \in \mathfrak{g}_0^3 \) and a clique \( \mathcal{K} \) contained in \( T^+(Z) \) is a selector of the horizon of \( Z \) which is a pencil, and cliques of the form \( [a, Y]^* \) are subspaces of \( \mathfrak{S} \). If \( \mathcal{K} \) is a clique contained in \( T^+(Z) \) then \( \mathcal{K} \) spans in \( \mathfrak{S} \) the subspace \( T^+(Z) \). As previously, if \( \dim(Y) \neq 3 \) then the class of cliques \( [a, Y]^* \) is distinguishable and then we can reconstruct \( (S, \mathfrak{L}) \) in \( \mathfrak{S} \).
5.3.4 Pencils of affine lines

Note that the structure $\mathcal{G} = \langle A, P_A \rangle$ with $P_A = \{ L \in A : a \in L \subset Z \} : a \in Z \in g_3^2 \}$ is useless to characterize the geometry of $\mathcal{L}$. One can even expect that under some additional assumptions the structure $\langle S, A \rangle$ can be defined in $\mathcal{G}$; clearly, $\mathcal{G}$ is definable in $\langle S, A \rangle$, but $\mathcal{L}$ is not definable in the latter.

References

[1] A. Barlotti, K. Strambach, Collineation groups of ovals and of ovoidal Laguerre planes, J. Geom. 57 (1996), no.1–2, 36–57
[2] W. Benz, Vorlesungen über Geometrie der Algebren, Springer Verlag 1973 Springer V. 1973
[3] W. Benz, H. Mäurer, Über die Grundlagen der Laguerre-Geometrie. Ein Bericht Jber. Deutsch. Math.-Verein. 67 (1964/65) Abt. I, 14–42
[4] W. Blaschke, Über die Laguerrasche Geometrie orientierter Geraden in der Ebene I, Arch. Math. Phys. 18 (1911), 132–140
[5] P. J. Cameron, Projective and polar spaces, QMW Maths Notes 13, 1991.
[6] A. M. Cohen, Point-Line Spaces Related to Buildings, in: Handbook of Incidence Geometries, ed. by F. Buekenhout, Elsevier 1995, 647–737
[7] H. Guściora, On the geometry of oriented equiaxial hyperquadrics, Bull Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 17 (1969), 29–35
[8] A. Herzer, Chain Geometries, in: Handbook of Incidence Geometries, ed. by F. Buekenhout, Elsevier 1995, 781–842
[9] Ch. Lefèvre-Percsy, Espaces polaires dégénérés des espaces projectifs, Simon Stevin 55 (1981), no. 4, 237–246
[10] H. Mäurer, Möbius- und Laguerre-Geometrien über konvexen Semiflächen, Math. Z. 98 (1967), 355–386
[11] E. Michalak, K. Prażmowski, Grassmannians of spheres in Möbius and in Euclidean spaces, mimeographed.
[12] V. Pambuccian, Sphere tangency as single primitive notion for hyperbolic and Euclidean geometry, Forum Math. 15(2003), 943–947
[13] M. Pankov, K. Prażmowski, M. Żynel, Geometry of polar Grassmann spaces, Demonstratio Math. 39(2006) no. 3, 625–637.
[14] K. Prażmowski, Multidimensional Euclidean geometry of cycles and axes, Algebra, Geom. Appl. Semin. Proc., Erevan, vol. 3–4 (2004), 5–18
[15] K. Radziszewski, Subspaces and parallelity in semi affine partial linear spaces, Abh. Math. Sem. Univ. Hamburg 73(2003), 131–144

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