RIEMANN–SCHWARZ
REFLECTION PRINCIPLE AND
ASYMPTOTICS OF MODULES
OF RECTANGULAR FRAMES

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Abstract
We investigate asymptotical behavior of the conformal module of a
doubly-connected domain which is the difference of two homothetic rec-
tangles under stretching it along the abscissa axis. Thereby, we give the
answer to a question put by Prof. M. Vuorinen.

1 Introduction
In recent years an interest increases to investigation of conformal modules of
plane quadrilaterals, doubly-connected domains and capacities of condensers
with polygonal boundaries. Special consideration is given to studying of be-
havior of the modules under various deformations of domains, their numerical
calculation, and asymptotics at degeneration. In this regard we can note the
review by R. Kühnau [1] and the papers [2]–[5].

At first, we recall some definitions. Consider a plane doubly-connected do-
main $D$ with nondegenerated boundary components. One of its important
characteristics is the conformal module $m(D)$. There are several equivalent
definitions of $m(D)$; we give some of them.

If $D$ is conformally equivalent to an annulus $\{r_1 < |z| < r_2\}$, then

$$m(D) := \frac{1}{2\pi} \ln \frac{r_2}{r_1}.$$

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Fund of Basic Research.
On the other hand, 
\[ m(D) := \lambda(\Gamma), \]
where \( \lambda(\Gamma) \) is the extremal length of curve-family \( \Gamma \) consisting of all curves joining in \( D \) its boundary components. Besides, 
\[ m(D) := 1/\lambda(\Gamma'), \]
\( \Gamma' \) being the family of all curves in \( D \) separating its boundary components. At last, 
\[ m(D) := 1/\text{Cap}(C), \]
where \( \text{Cap}(C) \) is the conformal capacity of the condenser \( C \) defined by \( D \).

Significant property of module is its invariance under conformal mappings.
It is quasiinvariant under quasiconformal maps (see, e. g., [8]): if \( f \) is an \( H \)-quasiconformal mapping of \( D \) onto \( \tilde{D} \), then
\[ \frac{1}{H} m(D) \leq m(\tilde{D}) \leq H m(D). \]

One of the simplest \( H \)-quasiconformal mappings is the stretching along the abscissa axis \( f_H : x+iy \mapsto Hx+iy \). M. Vuorinen states the following problem: to investigate how the module \( m(D) \) is deformed under \( f_H \) for sufficiently large \( H \). In particular, which is asymptotical behavior of \( m(D) \) if \( D \) is the difference of two homothetic squares?

The main result of the paper is

**Theorem 1.1** If \( D_1 = D_1^\sigma := [-1,1]^2 \setminus [-\sigma,\sigma]^2, \sigma \in (0,1), D_H = D_H^\sigma := f_H(D_1), \) then
\[ m(D_H^\sigma) \sim \frac{1-\sigma}{4\sigma H}, \ H \to \infty. \tag{1} \]

In Section 2 we give a solution to the problem for \( \sigma = 1/2 \), in addition, we deduce an explicit formula for \( m(D_H) \) via elliptic integrals. It should be noted that when \( H = 1 \) an explicit formula for \( m(D_H) \) is well-known, see Remark 1.3 below. In Section 4 the general case is considered. The results of Sections 2 and 4 were announced in [7] and [9]. Preliminarily, in Section 3 we establish continuity of module of quadrilateral under kernel convergence in the sense of Carathéodory.

To prove our results we need to recall some more definitions and facts. A quadrilateral is a simply-connected domain \( D \), conformally equivalent to a disk, with four marked distinct points (prime ends) \( A_k, 1 \leq k \leq 4 \), on the boundary of \( D \); increasing of \( k \) corresponds to the order in which the points occur when we bypass \( \partial D \) in the positive direction. We denote the quadrilateral as \( D(A_1, A_2, A_3, A_4) \) or simply \( D \) if it is clear which points \( A_k \) are fixed. The parts of \( \partial D \) lying between \( A_1 \) and \( A_2, A_3 \) and \( A_4 \) we call horizontal sides of \( D \), the other two parts of the boundary are vertical sides. Let us map conformally
$D(A_1, A_2, A_3, A_4)$ onto a rectangle $[0, a] \times [0, b]$ so that the horizontal sides are mapped onto the horizontal sides of the rectangular. The number

$$m(D) := \frac{a}{b}$$

is called the module of $D$. It is known (see, e. g., [8]) that $m(D)$ is equal to the extremal length $\lambda(\Gamma)$ of the family $\Gamma$ consisting of curves in $D$ joining its vertical sides; therefore, it is invariant under conformal mappings and quasiinvariant under quasiconformal ones. Besides, $m(D) = 1/\lambda(\Gamma')$ where $\Gamma'$ is the family of curves in $D$ joining its horizontal sides.

Let $E := \{|z| < 1\}, U := \{\Im z > 0\}, E^+ := E \cap U$, $S_{\gamma\delta} := \{e^{i\varphi} | \gamma < \varphi < \delta\}$, $0 < \delta - \gamma < 2\pi$. We denote by $[a, b]$ the segment with endpoints $a, b \in \mathbb{C}$.

The elliptic integral of the first kind

$$K(r) := \int_0^1 \frac{d\xi}{\sqrt{(1 - \xi^2)(1 - r^2\xi^2)}}.$$  

It is known (see, e. g., [10], [11]) that

$$\lim_{r \to 0} \left( K(r') - \ln \frac{4}{r} \right) = 0, \quad (2)$$

where as usual $r' = \sqrt{1 - r^2}$. From (2) it follows that $K(r) \sim \ln \frac{1}{r}$ as $r \to 1$.

Therefore,

$$K(r) \sim \frac{1}{2} \ln \frac{1}{1 - r}, \quad \frac{K(r)}{K(r')} \sim \frac{1}{\pi} \ln \frac{1}{1 - r}, \quad r \to 1. \quad (3)$$

From (2) we also obtain that

$$\frac{K(r')}{K(r)} \sim \frac{2}{\pi} \ln \frac{1}{r}, \quad r \to 0. \quad (4)$$

**Remark 1.2** The ring domain consisting of the unit disk minus a radial slit from 0 to $r$, $0 < r < 1$, is usually called the Grötzsch ring and its modulus is often denoted

$$\mu(r) = \frac{\pi K'(r)}{2 K(r)},$$

see [11]. The asymptotic formula (4) can be refined by use of the results from [10], Theorem 5.13.

**Remark 1.3** When $H = 1$ an explicit formula for $m(D_H^\sigma)$ is well-known (see, e. g., [6]):

$$m(D_H^\sigma) = \mu \left( \left( \frac{l - l'}{l + l'} \right)^2 \right), \quad l = \mu^{-1} \left( \frac{2}{\pi} \frac{1 - \sigma}{1 + \sigma} \right), \quad l' = \sqrt{1 - l^2}.$$
2 The case $\sigma = 1/2$

Consider the part $Q_H$ of $D_H$ lying in the first quarter of the plane. It is the union of three rectangles of the same size. Let us map conformally one of the rectangles with vertices at the points $(H+i)/2$, $H+i/2$, $H/2 + i$, and $H + i$ onto the quarter of the unit disk $U_1 := \{ z \mid |z| < 1, \Re z > 0, \Im z > 0 \}$ by the mapping $f$ so that $f((H+i)/2) = 0$, $f(H+i/2) = 1$, and $f(H/2 + i) = i$. Let $e^{i\kappa} = f(H + i)$.

By the Riemann-Schwarz reflection principle $f$ could be extended up to the conformal mapping of the rectangle $[0, H] \times [0, 1]$ onto the unit disk $E$; we will designate the extension also through $f$. Then $f$ maps conformally $Q_H$ onto the domain which is three quarters of the unit disk, besides, $f(H/2) = -i$, $f(H) = e^{-i\kappa}$, $f(i/2) = -1$, and $f(i) = -e^{-i\kappa}$.

![Diagram](image)

The function $g(\zeta) := (if(\zeta))^{2/3}$ maps conformally $Q_H$ onto $E^+$ at an appropriate choice of a branch of the power function. We have $g(H/2) = 1$, $g(H) = e^{i\alpha}$, $f(i/2) = -1$, $f(i) = -e^{-i\beta}$ where

$$\alpha = (\pi - 2\kappa)/3, \quad \beta = \pi - 2\kappa/3. \quad (5)$$

The module of $D_H$, by the symmetry principle for quasiconformal mappings (see, e. g., [8]) is equal

$$m(D_H) = 1/(4\lambda(\Gamma)) \quad (6)$$

where $\Gamma$ is the family of all curves in $Q_H$ which join $[H/2, H]$ and $[i/2, i]$. Because of conformal invariance of the module we obtain

$$\lambda(\Gamma) = \lambda(\Gamma'), \quad (7)$$

where $\Gamma'$ is a family of all curves in $E^+$ connecting $S_{0\alpha}$ to $S_{\beta\pi}$. By the symmetry principle,

$$\lambda(\Gamma') = 2\lambda(\bar{\Gamma}) \quad (8)$$
where $\tilde{\Gamma}$ is the family of all curves which join $S_{-\alpha, \alpha}$ and $S_{\beta, 2\pi - \beta}$ in $E^+.$

Let us map conformally the unit disk $E$ onto the upper half-plane $U$ so that the points $e^{i\beta}, e^{-i\beta}, e^{-i\alpha},$ and $e^{i\alpha}$ are mapped on $-1/l, -1, 1,$ and $1/l,$ $l > 1.$ Now we express $l$ through $\alpha$ and $\beta.$ Equating the anharmonic ratios

$$\frac{-1/l + 1}{-1/l - 1} \cdot \frac{1/l - 1}{1/l + 1} = \frac{e^{i\beta} - e^{-i\beta}}{e^{i\beta} - e^{-i\alpha}} \cdot \frac{e^{i\alpha} - e^{-i\alpha}}{e^{i\alpha} - e^{-i\beta}}$$

we obtain

$$l = \sqrt{\frac{1 - \cos(\alpha + \beta) - \sqrt{2} \sin \beta \sin \alpha}{1 - \cos(\alpha + \beta) + \sqrt{2} \sin \beta \sin \alpha}}.$$ (9)

Besides,

$$\lambda(\tilde{\Gamma}) = \frac{2K(l)}{K(l')}.$$ (10)

From (6), (7), (8), and (10) we have

$$m(D_H) = \frac{K(l')}{16K(l)}.$$ (11)

Now we find the relation between $\kappa$ and $H.$ For this purpose we map conformally $E$ onto the upper half-plane $U$ by a function $\varphi$ so that $-e^{-i\kappa}, -e^{i\kappa}, e^{-i\kappa},$ and $e^{i\kappa}$ are mapped on $-1/k, -1, 1,$ and $1/k.$ We note that $k$ satisfies the condition

$$\frac{2K(k)}{K(k')} = H$$ (12)

because the quadrilateral, which is the upper half-plane $U$ with fixed points $-1/k, -1, 1,$ and $1/k,$ is conformally equivalent to the rectangle of length $H$ and height 1 under the mapping $\varphi \circ f.$ From the equality of cross-ratios

$$\frac{-1/k + 1}{-1/k - 1} \cdot \frac{1/k - 1}{1/k + 1} = \frac{-e^{-i\kappa} + e^{i\kappa}}{-e^{-i\kappa} - e^{i\kappa}} \cdot \frac{e^{i\kappa} - e^{-i\kappa}}{e^{i\kappa} - e^{-i\kappa}}$$

we have

$$\kappa = \arcsin \frac{1 - k}{1 + k}.$$ (13)

Therefore, we prove

**Theorem 2.1** For $\sigma = 1/2$ the module of $D_H = D_H^\sigma$ is defined by (11) where $l$ is found from (9) taking into account (5), (12), and (13).

**Corollary 2.2** We have

$$m(D_H) \sim \frac{1}{4H}, \quad H \to \infty.$$
Actually, from (13) it follows that \( \kappa \sim (1 - k)/2 \) as \( H \to \infty \). Now, taking into account (9) and (5), we obtain
\[
1 - l \simeq \sqrt{\sin \beta} = \sqrt{\sin \frac{2\kappa}{3}} \simeq \sqrt{1 - k}.
\]
Thus, using (11), (12), and (3), we have
\[
m(D_H) = K'(l) \sim \frac{\pi}{16 K(1) / \ln[(1 - l)^{-1}]} \sim \frac{\pi}{8 \ln[(1 - k)^{-1}]} \sim \frac{1}{4H}, \quad H \to \infty.
\]

3 Convergence of domains and their modules

At first, we recall some results of the theory of prime ends of a sequence of domains converging to a kernel [12].

Consider a sequence of simply-connected domains \( G_n \) on the Riemann sphere \( \mathbb{C} \) converging to a kernel \( G \) with respect to a fixed point \( S_0 \in \mathbb{C} \). We assume that the boundaries of \( G_n \) and \( G \) are nondegenerate, i.e., each of them contains more than one point. Further we provide \( G_n \) and \( G \) with metrics induced from the sphere; in the case when all \( G_n \) are contained in a fixed Euclidean disk it is possible to change the spherical metrics by the Euclidean one.

Consider a section \( \gamma \) of \( G \), i.e., a Jordan arc in \( G \) with endpoints on \( \partial G \).

Let \( \gamma_n \) be a section of \( G_n \). We say that the sequence \( (\gamma_n) \) is a section of the sequence \( \tilde{G} := (G_n) \) lying over \( \gamma \) if the following conditions are fulfilled:

1. For any neighborhood \( U \) of \( \gamma \) there exists \( n_0 \) such that \( \gamma_n \) lies in \( U \) for any \( n \geq n_0 \);
2. if points \( p_1, p_2 \in G \) are separated by \( \gamma \) in \( D \), then there exists \( n_1 \) such that \( p_1 \) and \( p_2 \) are separated by \( \gamma_n \) in \( G_n \) for any \( n \geq n_1 \).

In [12] it is proved

**Lemma 3.1** For any section \( \gamma \) of \( G \) there exists a section \( \tilde{\gamma} := (\gamma_n) \) of \( \tilde{G} \) lying over \( \gamma \).

Let \( (\gamma_m) \) be a chain of sections of \( G \) which defines a prime end \( P \) of \( G \), and \( \tilde{\gamma}_m := (\gamma_{mm}) \) is a section of \( \tilde{G} \) lying over \( \gamma_m \). It is possible to introduce a natural relation of equivalence on the set of all such sequences \( (\gamma_m) \) (in more detail see [12]); the classes of equivalence \( \tilde{P} \) of such sequences are called prime ends of \( \tilde{G} \).

If a sequence of sections \( (\gamma_m) \) defines a prime end \( P \) of \( G \) and prime end \( \tilde{P} \) of \( \tilde{G} \) contains \( (\gamma_m) \) where \( \tilde{\gamma}_m \) is a section of \( \tilde{G} \) lying over \( \gamma_m \), then we will say that \( \tilde{P} \) is the prime end of \( \tilde{G} \) corresponding to prime end \( P \) of \( G \). The described correspondence \( \Phi : P \mapsto \tilde{P} \) is a bijection between the set of prime ends of \( G \) and the set of prime ends of \( \tilde{G} \).

Now consider a sequence \( (P_n) \) where \( P_n \) is a point or a boundary prime end of \( G_n \). Let \( \tilde{P} = \Phi(P) \) be the prime end of \( \tilde{G} \) corresponding to a prime end \( P \).
of $G$ and $P$ is defined by a sequence $(\gamma_m)$. Let $\tilde{P}$ be defined by $(\gamma_m)$, where $\gamma_m$ lies over $\gamma_m$ for any $m$, and $\tilde{\gamma}_m := (\gamma_m)$.

We will say that the sequence $(P_n)$ converges to $\tilde{P}$ if for any $m$ there exists $n_0$ such that $\gamma_{mn}$ separates $P_n$ from $S_0$ in $G_m$ for any $n \geq n_0$.

Let $f_n$ and $f$ be conformal mappings of $E$ onto $G_n$ and $G$, extended to $\overline{E}$ up to homeomorphisms of prime ends. Let $P_n$ be a sequence consisting of points or boundary prime ends of $G_n$, and let $P$ be a boundary prime end of $G$. Denote $\zeta_n = f_n^{-1}(P_n)$, $\zeta_0 = f^{-1}(P)$.

Theorem 6, p. 75, from [12] implies

Theorem 3.2 A sequence $(P_n)$ converges to the prime end $\tilde{P}$ of $\tilde{G}$ corresponding to $P$ if and only if $\zeta_n \to \zeta_0$.

From Theorem 3.2 we deduce the following result on continuity of the module of quadrilateral under kernel convergence of domains.

Theorem 3.3 Let $A_n$, $B_n$, $C_n$, and $D_n$ be distinct boundary prime ends of $G_n$. Let the sequences $(A_n)$, $(B_n)$, $(C_n)$, and $(D_n)$ converge to the prime ends $A$, $B$, $C$ and $D$ of $G$ corresponding to distinct prime ends $A$, $B$, $C$, and $D$ of $G$. Then $m(G_n(A_n,B_n,C_n,D_n)) \to m(G(A,B,C,D))$.

Actually, taking into account conformal invariance of the module of quadrilateral, by Theorem 3.2 we have

$$m(G_n(A_n,B_n,C_n,D_n)) = m(E(a_n,b_n,c_n,d_n)) \to E(a,b,c,d) = m(G(A,B,C,D))$$

where $a_n$, $b_n$, $c_n$, and $d_n$ are preimages of $A_n$, $B_n$, $C_n$, and $D_n$ under the map $f_n$, and $a$, $b$, $c$, and $d$ are preimages of $A$, $B$, $C$, and $D$ under the map $f$.

Remark 3.4 The statement of Theorem 3.3 is valid not only for univalent domains, it is true for $p$-valent domains (Riemann surfaces) as well (see, e. g., [13]).

4 General case

As in the case $\sigma = 1/2$, consider the part $Q_H$ of $D_H$ lying in the first quarter of the plane. Let us shift $Q_H$ to the left on the value $\sigma H$; as a result we receive the domain $\tilde{Q}_H$. By the symmetry principle, the module of $D_H$ is equal to

$$m(D_H) = \frac{1}{4\lambda(\Gamma)}$$

where $\Gamma$ is the family of curves joining $[0, (1 - \sigma)H]$ and $[-\sigma H + i\sigma, -\sigma H + i]$ in the quadrilateral $\tilde{Q}_H$. 

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Now consider the domain
\[ \tilde{Q} := \bigcup_{H > 0} \tilde{Q}_H = (\mathbb{R} \times (0, 1)) \setminus ((-\infty, 0] \times (0, \sigma]). \]

Let us map conformally \( \tilde{Q} \) onto the horizontal strip \( \mathbb{R} \times (0, 1) \) with keeping the infinite prime ends and the origin. For this purpose we map conformally the upper half-plane \( U \) onto \( \tilde{Q} \) and \( G = \{ 0 < \Im \omega < 1 \} \) by the functions
\[ z = C \int_1^\zeta \sqrt{\frac{\zeta - s}{\zeta - 1}} \, d\zeta, \quad \omega = \frac{\ln \zeta}{\pi}, \]
where \( C > 0, \ 0 < s < 1. \)

Near \( \zeta = 0 \) we have
\[ z = C \sqrt{s} \ln \zeta + \sum_{k=0}^\infty \sigma_k \zeta^k. \]

Since the intersection of the domain \( \tilde{Q} \) and the left half-plane is a half-strip of width \((1 - \sigma)/\pi\), we have \( C \sqrt{s} = (1 - \sigma)/\pi \). Near \( \zeta = \infty \)
\[ z = C \ln \zeta + \sum_{k=0}^\infty \beta_k \zeta^k, \]
therefore, similarly we obtain \( C = 1/\pi \). Then \( s = (1 - \sigma)^2 \) and
\[ z = \frac{1}{\pi} \int_1^\zeta \sqrt{\frac{\zeta - (1 - \sigma)^2}{\zeta - 1}}. \]

Considering it we conclude that for sufficiently large \( M > 0 \) we have
\[ z = (1 - \sigma)\omega + \sum_{k=0}^\infty \sigma_k e^{k\pi \omega}. \quad (15) \]
in the half-plane \( \Pi^-_\mu := \{ \Re \omega < -M, \ 0 < \Im \omega < 1 \} \), and
\[ z = \omega + \sum_{k=0}^\infty \beta_k e^{-k\pi \omega}. \quad (16) \]
in the half-plane \( \Pi^+_\mu := \{ \Re \omega > M, \ 0 < \Im \omega < 1 \} \).

From rectilinearity of the boundary arcs of the domains and the Riemann-Schwarz reflection principle it follows that convergence of the series (15) and (16) is uniform in the closed half-planes \( \Pi^-_\mu \) and \( \Pi^+_\mu \).

From (15) and (16) we deduce that on the vertical segments in \( \tilde{Q} \), lying on the lines \( \Re \omega = -\bar{\sigma} H \), where
\[ \bar{\sigma} = \frac{\sigma}{1 - \sigma}, \]
we have
\[ \Re z(\omega) = -\bar{\sigma} H + O(1), \quad H \to \infty. \]

In the same way, on the segments, lying on the lines \( \Re \omega = (1 - \sigma)H, \)
\[ \Re z = (1 - \sigma)H + O(1), \quad H \to \infty, \]
Therefore,
\[ \lambda(\Gamma) \sim m(\tilde{P}) \quad (17) \]
where \( \tilde{P} \) is the quadrilateral which is the rectangle
\[ P := [-\bar{\sigma} H, (1 - \sigma)H] \times [0, 1] \]
with the segments \([-\bar{\sigma} H, -\bar{\sigma} H + i]\) and \([0, (1 - \sigma)H]\) as vertical sides.

Let us map \( U \) onto \( P \) by the function
\[ z = C \int_0^\zeta \frac{d\xi}{\sqrt{(1 - \xi^2)(1 - k^2\xi^2)}} + C_1 \]
where
\[ C_1 = \frac{(1 - \sigma)^2 - \sigma}{1 - \sigma} \quad H, \quad C = \frac{(1 - \sigma)^2 + \sigma}{2K(k)(1 - \sigma)}. \]
and \( k \in (0, 1) \) is defined from the relation
\[ \frac{2K(k)}{K(k')} = \frac{(1 - \sigma)^2 + \sigma}{1 - \sigma} H. \quad (18) \]
The mapping takes the points \(-\bar{\sigma} H + i, \bar{\sigma} H, 0, \) and \((1 - \sigma)H, \) i. e., the vertices of quadrilateral \( P, \) into \((-1/k), (-1), a, \) and \( 1, \)
\[ a = \text{sn} \left[ \frac{\sigma - (1 - \sigma)^2}{\sigma + (1 - \sigma)^2} K(k), k \right]. \quad (19) \]
Here \( \text{sn}[\cdot, k] \) is the Jacobi elliptic sine corresponding to the parameter \( k \) (see, e. g., [11]).

Now we map the upper half-plane \( U \) onto itself conformally so that the points \(-1/k, -1, a, \) and \( 1 \) move into \(-1/\nu, -1, 1, \) and \( 1/\nu \) (\( 0 < \nu < 1 \)). Then the module of the quadrilateral \( \tilde{P} \) is equal to
\[ m(\tilde{P}) = \frac{2K(\nu)}{K(\nu')} \quad (20) \]
where \( \nu \) is defined from the equality of the cross-ratios:
\[ \frac{1}{1 - \nu} - \frac{1}{\nu - 1} = \frac{1}{1 + \nu} \quad \frac{1}{1 + \nu} \]
\[ \frac{1}{1 - \nu} - \frac{1}{\nu - 1} = \frac{1}{1 + \nu} \quad \frac{1}{1 + \nu} \]
\[ \frac{1}{1 - \nu} - \frac{1}{\nu - 1} = \frac{1}{1 + \nu} \quad \frac{1}{1 + \nu} \]
\[ or \]
\[ \frac{1}{1 + \nu} = \sqrt{\frac{1 - a}{1 + ka}} \cdot \sqrt{\frac{1 - k}{2}}. \quad (21) \]
Therefore, for finding the asymptotics of \( m(P) \) we need to know the asymptotic behavior of \( a \) as \( H \to \infty \). It is possible to do using (19), but we prefer to apply geometric considerations which are based on rectilinearity of the boundary arcs and the reflection principle. Let us prove the following auxiliary statement.

**Lemma 4.1** Let \( Q \) be a quadrilateral which is the square \([0,1]^2\) with vertices at the points \( c, 1, 1+i, \) and \( i \), where \( c \in (0,1) \). Denote \( Q_H = f_H(Q) \). Then

\[
m(Q_H) \sim (1-c)H, \quad H \to \infty.
\]

Proof. Let \( \bar{Q}_H = (1/H)Q_H \). Since \( m(\bar{Q}_H) \) is a monotonic function of \( H \), it is sufficient to consider the sequence \( H_n = 2^n \) and to prove that

\[
m(Q_{H_n}) \sim (1-c)H_n, \quad n \to \infty.
\]

For short we will write \( Q_n \) instead of \( Q_{H_n} \).

Consider the domains \( Q_n^1, Q_n^2, \ldots, Q_n^{2n} \), where

\[
Q_n^k = [0,1] \times [\frac{k-1}{2n}, \frac{k}{2n}].
\]

Let us glue \( Q_n^k \) and \( Q_n^{k+1} \) along \( \{(x,y) \mid 0 \leq x \leq 1, y = j/(2n)\} \) for odd \( k \), and along \( \{(x,y) \mid c \leq x \leq 1, y = j/(2n)\} \) for even \( k \). As a result, we obtain the domain \( G_n \) which is the unit square with \((n-1)\) horizontal slits (Fig. 2).

We will consider \( G_n \) as a quadrilateral with vertices \( c, 1, c+i, \) and \( i \). By the symmetry principle, \( m(G_n) = m(Q_n)/(2n) \). The domains \( G_n \) converge to the rectangle \( G := [c,1] \times [0,1] \) as \( n \to \infty \), and the sequences of their vertices converge to four distinct prime ends of \( \bar{G} = (G_n) \) corresponding to the vertices of \( G \). To show that we take \((1/4)\) of concentric circles with radius \( r_m \to 0 \) as
sections $\gamma_m$ which define prime end being a vertex of $G$. Let $\gamma_{mn}$ be a section of $G_n$ which is the union of $\gamma_m$ and, if it is necessary, a segment connecting one of its endpoint to one of the nearest points of $\partial G_n$. By Theorem 3.3 $m(G_n) \to m(G) = 1 - c$, and Lemma 4.1 is proved.

**Remark 4.2** The quadrilateral $Q_H$ could be considered as a generalized long quadrilateral. Asymptotics of the modules of long quadrilaterals was investigated in [16], [17], [15], [18], [19], and other papers where various methods for computing the modules were suggested.

Consider the quadrilateral $P^*$ which is the rectangle $P$ with the segments $[0, (1 - \sigma)H]$ and $[-\bar{\sigma}H + i, (1 - \sigma)H + i]$ as horizontal sides. Taking into account conformal invariance of the module, by Lemma 4.1 we have

$$m(P^*) \sim (1 - \sigma)H, \quad H \to \infty. \quad (22)$$

Now we can describe the behavior of $a$ as $H \to \infty$. Let us map $P^*$ conformally onto $U$ such that the points $(1 - \sigma)H + i, -\bar{\sigma}H + i, 0$, and $(1 - \sigma)H$ are mapped into $-1/\mu, -1, 1$, and $1/\mu, 0 < \mu < 1$. Then

$$m(P^*) = \frac{K(\mu')}{2K(\mu)}, \quad (23)$$

We should note that because of $m(P^*) \to \infty$ as $H \to \infty$, by (23) and (14), we have $\mu \to 0, H \to \infty$. Comparing the anharmonic ratios of the points in $\partial U$, corresponding to each other under conformal automorphism, we obtain

$$\frac{1}{1 + \mu} \cdot \frac{1}{-1/\mu - 1/\mu} = \frac{a - 1}{a + 1/k} \cdot \frac{1/k + 1/k}{1/k - 1}$$

or

$$\frac{1}{1 + ak} \cdot \frac{2k}{1 - k} = \frac{(\mu - 1)^2}{4\mu}.$$ 

Therefore, taking into account that $a, k \to 1$ as $H \to \infty$ we have

$$1 - a \sim \frac{1 - k}{\mu}. \quad (24)$$

By (22), (23), and (3),

$$\frac{1}{(1 - \sigma)H} \sim 2\frac{K(\mu)}{K(\mu')} \sim \frac{4}{\pi} \ln \frac{1}{\mu}. \quad (25)$$

Now we use the asymptotic behavior 3 of the elliptic integrals. With use of that and by (18)

$$\ln \frac{1}{1 - k} \sim \frac{\pi}{K(\mu')} \sim \frac{\pi}{2} \cdot \frac{(1 - \sigma)^2 + \sigma}{1 - \sigma} H.$$
Now from (3), (20), (21), (24), and (25) we obtain

\[ m(\tilde{P}) = \frac{2K(\nu)}{K(\nu)} \sim \frac{2}{\pi} \ln \frac{1}{1 - \nu} \sim \frac{2}{\pi} \ln \frac{1 + ak}{1 - a} + \frac{1}{\pi} \ln \frac{1}{1 - k} \]

\[ \sim \frac{2}{\pi} \ln \frac{1}{1 - k} \quad - \quad \frac{1}{\pi} \ln \frac{1}{\mu} \sim (1 - \sigma + \tilde{\sigma})H - (1 - \sigma)H = \tilde{\sigma}H. \]

Because of (14) and (17) it completes the proof of (1).

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