An operator approach to BRST invariant transition amplitudes

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Abstract

The transition amplitudes for the free spinless and spinning relativistic particles are obtained by applying an operator method developed long ago by Dirac and Schwinger to the BFV form of the BRST theory for constrained systems.

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The BRST method of quantization \[1, 2, 3\] is now thought as the most effective way of dealing with constrained dynamical systems. In trading the local invariance of gauge systems for the symmetry under global nilpotent transformations generated by the BRST charge, it opened the pathway to the quantum description of many interesting systems, e.g. the first quantized particles, strings and membranes not to mention a better understanding of the Yang-Mills theory.

Recently the BRST invariant transition amplitudes for the spinless and spinning relativistic \[4\] and nonrelativistic \[5\] particles were obtained using the BFV path integral formulation \[2\]. Although the early developments of the BRST theory were done in a path integral environment, the operator method often helps to clarify some aspects unforeseen by the functional approach \[3\]. In this paper we obtain the transition amplitudes for the free spinless and spin one half relativistic particles in the BFV-BRST framework by applying an operator approach developed long ago by Dirac \[7\] in his investigations on the role of the classical action in quantum mechanics and Schwinger \[8\] in his early calculations of effective actions.

Let us start by considering a hamiltonian system with finite degrees of freedom and linearly independent first class constraints $G_a$ ($a = 1, ..., k$), that we choose to be bosonic and real for simplicity and with the corresponding quantum operators generating the algebra (with $U_{ab}^c$ being the structure constants):

$$[G_a, G_b]_- = iU_{ab}^c G_c.$$ (1)

In the BFV-BRST theory \[2\] the original set of coordinate and momentum operators $(q, p)$ is extended as

$$(q, p) \rightarrow (q, p) \oplus (\lambda, \pi) \oplus (\eta, \mathcal{P}) \oplus (\bar{\eta}, \bar{\mathcal{P}}),$$ (2)

where we introduced the bosonic Lagrange multipliers $\lambda^a$ and respective momenta $\pi_a$, a
fermionic pair of conjugate ghost operators \((\eta^a, P_a)\) for each constraint \(G_a\), and a pair of fermionic antighost conjugate operators \((\bar{\eta}^a, \bar{P}_a)\) to take care of each constraint \(\pi_a\) that appear due to the “einbein” character of \(\lambda^a\) that we assume from start. These hermitian operators generate the algebra (the non zero part of it)

\[
[\lambda^a, \pi_b]_-= i\delta^a_b, \quad [\eta^a, P_b]_+ = [\bar{\eta}^a, \bar{P}_b]_+ = \delta^a_b. \tag{3}
\]

Annihilating the physical states, the hermitian nilpotent BRST charge operator in the BFV formulation [2] is given by

\[
Q = G_a \eta^a - \frac{i}{2} U_b^{\ a} P_a \eta^b \eta^c - \frac{i}{2} U_b^{\ ab} \eta^a + \bar{P}_a \pi^a. \tag{4}
\]

Due to the nilpotency of \(Q\) the following extension of the original BRST invariant hamiltonian \(H_0\) is also invariant:

\[
H = H_0 + [\Psi, Q]_+. \tag{5}
\]

Where \(\Psi\) is an arbitrary “gauge fermion”. And also, since \(Q|\text{phys}\rangle = 0\), the above extension of \(H_0\) can be used in the evolution operator without changing the transition amplitudes. In fact for generally covariant systems, like the relativistic particle and string, where the classical \(H_0 \approx 0\), the choice of an appropriate \(\Psi\) will prove essential to ensure integrability.

In the following we use the extended hamiltonian \(H\) and the condition \(Q|\text{phys}\rangle = 0\) to obtain the propagator for the scalar and spinning relativistic particles by the Dirac-Schwinger method.

To illustrate the method we consider the transition amplitude between position eigenstates for a system with one degree of freedom and obeying the Schrödinger equation:

\[
\langle q'', t|q', 0 \rangle = \langle q''| e^{-iHt}|q' \rangle, \tag{6}
\]
where \(|q', t⟩\) are eigenvectors of the position operator \(q(t)\) with eigenvalue \(q'\) (hereafter all operator eigenvalues will be primed) and \(⟨q'', 0|q', 0⟩ = δ(q'' - q')\). We also choose \([q, p]_− = i\), with \(p\) being the conjugate momentum operator, while the following reasoning is also valid for fermionic \(q\) and \(p\) [9].

To obtain the above amplitude Dirac [7] wrote it in the following way:

\[
⟨q'', t|q', 0⟩ = e^{iW(q'', q'; t)},
\]

where \(W(q'', q'; t)\) is a complex function of the end point coordinates and time. It is easy to verify that this function is determined by the following relations:

\[
-\frac{∂W(q'', q'; t)}{∂t} = \frac{⟨q'', t|H(q(t), p(t))|q', 0⟩}{⟨q''|t|q', 0⟩},
\]

(8)

\[
\frac{∂W(q'', q'; t)}{∂q''} = \frac{⟨q'', t|p(t)|q', 0⟩}{⟨q''|t|q', 0⟩},
\]

(9)

\[
-\frac{∂W(q'', q'; t)}{∂q'} = \frac{⟨q'', t|p(0)|q', 0⟩}{⟨q''|t|q', 0⟩},
\]

(10)

\[
W(q'', q'; 0) = -i \ln δ(q'' - q').
\]

(11)

To solve this problem Schwinger [8] noticed that the above equations relate the transition amplitude to the solution of the Heisenberg equations for \(q(t)\) and \(p(t)\). If we solve for \(p(t)\) in terms of \(q(t)\) and \(q(0)\) and insert this, in a time ordered fashion, on (8)-(10) we are left with a set of first order equations to integrate. In the following we apply the above Dirac-Schwinger method to the BFV-BRST models of the scalar and spinning particles.

The classical mechanics of a spinless relativistic particle with parametrized world line \(X^\mu(s)\) (\(\mu = 0, ..., 3\), \(s ∈ [0, 1]\) and momentum \(P^\mu(s)\), is described by the vanishing hamiltonian \(H_0 = P^2 - m^2 ≈ 0\) [10]. In the canonical quantization of this system \(X^\mu\) and \(P^\mu\) become operators that generate the algebra (the nonzero part)
\[ [X^\mu, P^\nu] = i\eta^{\mu\nu}, \quad (\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)) \] and \( H_0 \) becomes the gauge generator \( G = P^2 - m^2 \). The BFV-BRST charge for this system with a single abelian generator is simply
\[
Q = \eta G + \bar{P}\pi. \tag{13}
\]

In order to construct an evolution operator in the parameter time \( s \), we rely on the BRST extended hamiltonian (13), a popular choice for \( \Psi \) that ensures the integrability of this system, is given by
\[
\Psi = \mathcal{P}\lambda, \tag{14}
\]
with this choice the extended hamiltonian reads
\[
H = [\Psi, Q]_+ = \lambda(P^2 - m^2) + i\mathcal{P}\bar{P}. \tag{15}
\]

From the above \( H \) we have the Heisenberg equations (the dot means derivative with respect to the parameter \( s \)):
\[
\begin{align*}
\dot{X}^\mu &= 2\lambda P^\mu, \\
\dot{\lambda} &= 0, \\
\dot{\pi} &= -(P^2 - m^2), \\
\dot{\mathcal{P}} &= \mathcal{P}, \\
\dot{\bar{\mathcal{P}}} &= -\mathcal{P}.
\end{align*} \tag{16-19}
\]

With the solutions
\[
\begin{align*}
X^\mu(s) &= X^\mu(0) + 2\lambda P^\mu s, \\
\pi(s) &= \pi(0) - (P^2 - m^2)s.
\end{align*} \tag{20-21}
\]
\[ \eta(s) = \eta(0) + \bar{P}s, \]
\[ \bar{\eta}(s) = \bar{\eta}(0) - P s. \]

Note that in (20) there is an ambiguity, since we can change \( \lambda \rightarrow -\lambda \) and \( P^\mu \rightarrow -P^\mu \) and end up with the same \( X^\lambda(s) \) [10]. A remedy for this situation is to restrain the eigenvalues of \( \lambda \) to \((-\infty, 0]\) or \([0, \infty)\).

We now write \( P \) and \( \bar{P} \) in terms of \( \eta(s), \eta(0), \bar{\eta}(s) \) and \( \bar{\eta}(0) \), so that we get the time ordered hamiltonian:

\[ H_{ord} = \lambda(P^2 - m^2) - \frac{i}{s^2} \left( \bar{\eta}(s) \eta(s) - \bar{\eta}(s) \eta(0) \right) + \bar{\eta}(0) \eta(0) + \eta(s) \bar{\eta}(0) - [\bar{\eta}(0), \eta(s)]_+ , \]

with the anticommutator being

\[ [\bar{\eta}(0), \eta(s)]_+ = s. \]

We are now in position to integrate (3), using the basis \( |P', \lambda', \eta', \bar{\eta}', s\rangle \) (with the ghost eigenvalues being Grassmann numbers):

\[ W = -s\lambda'(P'^2 - m^2) - \frac{i}{s} (\bar{\eta}' - \bar{\eta}'')(\eta'' - \eta') - i \ln s + \Phi , \]

where \( \Phi \) is a \( s \) independent function of the dynamical variables. Using (3), (10) and the solutions (20)-(23) we have

\[ \left( \frac{\partial}{\partial P''^\mu} + \frac{\partial}{\partial P'\mu} \right) \Phi = \left( \frac{\partial}{\partial \lambda''} + \frac{\partial}{\partial \lambda'} \right) \Phi = 0 , \]
\[ \frac{\partial}{\partial \eta'} \Phi = \frac{\partial}{\partial \eta''} \Phi = \frac{\partial}{\partial \bar{\eta}'} \Phi = \frac{\partial}{\partial \bar{\eta}''} \Phi = 0 , \]

so that with aid of (11) and using the Berezin definition for a Grassmann \( \delta \)–function [11]:

\[ \Phi = \Phi(P''^\mu - P'^\mu; \lambda'' - \lambda') = -i \ln i \delta^4(P''^\mu - P'^\mu) \delta(\lambda'' - \lambda') . \]
Now that we have found $W$ we must impose the invariance under $Q$:

$$Q|P', \pi', \eta', \bar{\eta}' , 0\rangle = 0, \quad Q|P'', \pi'', \eta'', \bar{\eta}'', s\rangle = 0,$$

(29)

where we changed our basis from $\lambda$ to $\pi$ eigenvectors for later convenience, and used the fact that $Q^\dagger = Q$. Among the several ways for this to be true [3], we choose the boundary conditions:

$$\pi'' = \pi' = \eta'' = \eta' = \bar{\eta}'' = \bar{\eta}' = 0.$$  

(30)

The condition on $\bar{\eta}$ is a consistency one for $\pi = [Q, \bar{\eta}]_+$ and $Q$ to annihilate $|\text{phys}\rangle$.

We now Fourier transform $e^{iW}$ to the above basis choosing $\lambda \geq 0$,

$$K(P'', P') = i \int_0^\infty d\lambda' \; e^{i\lambda'(\pi'' - \pi')} e^{-i[s\lambda'(P''^2 - m^2) + \frac{1}{2}(\eta'' - \eta')(\eta'' - \eta')]} \delta^4(P'' - P').$$

(31)

With the BRST invariant boundary conditions and defining $s\lambda' \equiv T$:

$$K(P'', P') = i \int_0^\infty dT \; e^{-iT(P''^2 - m^2)} \delta^4(P'' - P') = \frac{\delta^4(P'' - P')}{P''^2 - m^2 - i\epsilon}.$$  

(32)

The choice of positive $\lambda$ led to the momentum space Feynman propagator, had we chosen negative values for $\lambda$ we would end up with the complex conjugate of it. We now turn to the more involved model of a spin one half relativistic particle.

The pseudoclassical behavior of a spinning relativistic particle is described by an action where the local diffeomorphism invariance is generalized to a local world line supersymmetry [12]. From this model we have five second class and four first class constraints, that after applying the Dirac algorithm reduce to the two first class constraints [4]:

$$G_1 = P^2 - m^2, \quad G_2 = \zeta \cdot P - \zeta_5 m,$$

(33)

where the $\zeta\mu(s)$ and $\zeta_5(s)$ are Grassmann odd variables describing the spin degrees of freedom (spin one half). In the quantum version of the above model the $\zeta(s)$’s obey

$$[\zeta_\mu, \zeta_\nu]_+ = -2\eta_{\mu\nu}, \quad [\zeta_5, \zeta_5]_+ = 2$$

(34)
and the gauge algebra is

\[ [G_2, G_2]_+ = -2(P^2 - m^2) = -2G_1. \] (35)

As in the spinless case the original phase space will be extended with the same variables for the gauge generator \( G_1 \) and the additional ones for \( G_2 \), with reversed statistics, obeying

\[ [\lambda_2, \pi_2]_+ = 1, \quad [\eta_2, \mathcal{P}_2]_- = [\bar{\eta}_2, \bar{\mathcal{P}}_2]_- = i, \] (36)

From the gauge algebra we have that the single surviving structure constant is \( U_{22}^1 = 2i \), so that the BRST charge for this model reads

\[ Q = \eta_1(P^2 - m^2) + \eta_2(\zeta \cdot P - \zeta_5 m) + \mathcal{P}_1 \eta_2^2 + \bar{\mathcal{P}}_1 \bar{\eta}_1 + \bar{\mathcal{P}}_2 \pi_2. \] (37)

The generalization of the last section gauge fermion is

\[ \Psi = \mathcal{P}_a \lambda^a, \] (38)

leading to the extended Hamiltonian

\[ H = [Q, \Psi]_+ = \lambda_1(P^2 - m^2) - i\lambda_2(\zeta \cdot P - \zeta_5 m) - 2i\eta_2 \mathcal{P}_1 + i\mathcal{P}_1 \bar{\eta}_1 + \mathcal{P}_2 \bar{\mathcal{P}}_2. \] (39)

The Heisenberg equations of motion are:

\[ \dot{X}^\mu = 2\lambda_1 P^\mu - i\lambda_2 \zeta^\mu, \quad \dot{\zeta}^\mu = -2\lambda_2 P^\mu, \] \[ \dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = 0, \] \[ \dot{\eta}_1 = \bar{\mathcal{P}}_1 + 2\eta_2 \lambda_2, \quad \dot{\bar{\eta}}_1 = -\mathcal{P}_1, \] \[ \dot{\eta}_2 = \bar{\mathcal{P}}_2, \quad \dot{\bar{\eta}}_2 = \mathcal{P}_2, \] \[ \dot{\pi}_1 = -2\lambda_2 m, \quad \dot{\bar{\pi}}_1 = -(P^2 - m^2), \] \[ \dot{\pi}_2 = -(\zeta \cdot P - \zeta_5 m) - 2\eta_2 \mathcal{P}_1, \] \[ \dot{\bar{\pi}}_2 = 2i\lambda_2 \mathcal{P}_1. \] (40-47)
with the solutions

\[ X^\mu(s) = X^\mu(0) + (2\lambda_1 P^\mu - i\lambda_2 \zeta^\mu(0))s, \quad (48) \]

\[ \zeta^\mu(s) = \zeta^\mu(0) - 2\lambda_2 P^\mu s, \quad (49) \]

\[ \zeta_5(s) = \zeta_5(0) - 2\lambda_2 ms, \quad (50) \]

\[ \pi_1(s) = \pi_1(0) - (P^2 - m^2)s, \quad (51) \]

\[ \pi_2(s) = \pi_2(0) - (2\eta_2(0)P_1 + \zeta(0) \cdot P - \zeta_5(0)m)s + [\lambda_2(P^2 - m^2) - \bar{P}_2 P_1]s^2, \quad (52) \]

\[ \eta_1(s) = \eta_1(0) + (\bar{P}_1 - 2\lambda_2\eta_2(0))s - \lambda_2\bar{P}_2 s^2, \quad (53) \]

\[ \bar{\eta}_1(s) = \bar{\eta}_1(0) - P_1 s, \quad (54) \]

\[ \eta_2(s) = \eta_2(0) + \bar{P}_2 s, \quad (55) \]

\[ \bar{\eta}_2(s) = \bar{\eta}_2(0) + P_2(s) - i\lambda_2 P_1 s^2. \quad (56) \]

We can see that due to the nilpotency of \( \lambda_2 \) we can replace \( \zeta^\mu(s) \) and \( \zeta_5(s) \) by \( \zeta^\mu(0) \) and \( \zeta_5(0) \) in \( H \) without changing anything. Choosing the operator initial conditions \( \zeta^\mu(0) = \gamma_5 \gamma^\mu \) and \( \zeta_5(0) = \gamma_5 \), with \( \gamma^\mu \) and \( \gamma_5 \) being the Dirac matrices, we get closer to the usual description of field theoretical fermions. As in the spinless case we use the above solutions to obtain the time ordered hamiltonian:

\[
H_{ord} = \lambda_1(P^2 - m^2) - i\lambda_2\gamma_5(\gamma \cdot P - m) - \frac{1}{s^2} \left[ i\left( \bar{\eta}_1(s)\eta_1(s) - \bar{\eta}_1(0)\eta_1(0) \right) \right. \\
+ \left. \bar{\eta}_1(0)\eta_1(0) + \eta_1(s)\bar{\eta}_1(0) - [\bar{\eta}_1(0), \eta_1(s)]_+ - \bar{\eta}_2(s)\eta_2(s) \right. \\
+ \left. \bar{\eta}_2(s)\eta_2(0) - \bar{\eta}_2(0)\eta_2(0) + \eta_2(s)\bar{\eta}_2(0) + [\bar{\eta}_2(0), \eta_2(s)]_- \right], \quad (57)
\]

where the (anti)commutators are

\[
[\bar{\eta}_1(0), \eta_1(s)]_+ = s, \quad [\bar{\eta}_2(0), \eta_2(s)]_- = is. \quad (58)
\]

Unlike the case of a single bosonic constraint the contribution of the fermionic ghost anticommutator exactly cancels the commutator of the bosonic ghost in the hamiltonian. This was expected due to the supersymmetry of the model.
Integrating for $W$ in the basis $|P', \lambda', \eta', \bar{\eta}', s\rangle$ we have:

$$W = -s \lambda'_1 (P'^2 - m^2) + i s \lambda'_2 \gamma_5 (\gamma \cdot P' - m) + \frac{i}{2} [-i(\bar{\eta}'_1'' - \bar{\eta}'_1')(\eta'_1'' - \eta'_1) + (\bar{\eta}'_2'' - \bar{\eta}'_2')(\eta'_2'' - \eta'_2)] + \Phi,$$

repeating the same analysis as in the previous case we have for the $s$ independent $\Phi$:

$$\Phi = -i \ln i \delta (\lambda'_1'' - \lambda'_1') \delta (\lambda'_2'' - \lambda'_2') \delta^4 (P'' - P').$$

It is easy to see that the same invariant boundary conditions of the spinless case hold:

$$\pi''_a = \pi'_a = \eta''_a = \eta'_a = \bar{\eta}''_a = \bar{\eta}'_a = 0.$$

To impose the above condition on $\pi_a$ we fourier transform $e^{iW}$ in both $\lambda'_1$ ($\lambda'_1 \geq 0$) and $\lambda'_2$, in the Berezin sense for $\lambda'_2$ [11]. Defining $s \lambda'_1 \equiv T$ and $s \lambda'_2 \equiv \theta$ we get

$$K(P'', P') = i \int_0^\infty dT e^{-iT(P'^2 - m^2)} \int d\theta e^{-\theta \gamma_5 (\gamma \cdot P' - m)} \delta^4 (P'' - P')$$

$$= -\gamma_5 \frac{\gamma \cdot P' - m}{P'^2 - m^2 - i\epsilon} \delta^4 (P'' - P').$$

That is the momentum space Feynman propagator for the Dirac equation, modulo a $-\gamma_5$ factor that is unavoidable in these supersymmetric models of spinning particles [4, 10].

**Discussion**

In this article we obtained the BRST invariant transition amplitudes for relativistic particles in the operator framework provided by the Dirac-Schwinger method. It remains a challenge to apply this procedure to more involved first quantized problems like the higher spin point particles and the spinning string.

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