The renormalization of fluctuating branes, the Galileon and asymptotic safety

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Abstract: We consider the renormalization of \( d \)-dimensional hypersurfaces (branes) embedded in flat \((d+1)\)-dimensional space. We parametrize the truncated effective action in terms of geometric invariants built from the extrinsic and intrinsic curvatures. We study the renormalization-group running of the couplings and explore the fixed-point structure. We find evidence for an ultraviolet fixed point similar to the one underlying the asymptotic-safety scenario of gravity. We also examine whether the structure of the Galileon theory, which can be reproduced in the nonrelativistic limit, is preserved at the quantum level.
1 Introduction

Scalar field theories are ubiquitous in physics, describing a plethora of classical and quantum systems. Because of their relative simplicity, they have often been used as a testing ground for new ideas or techniques. The action of a scalar field is usually assumed to contain a standard kinetic term, especially when quantum or statistical fluctuations of the system are studied. The inclusion of higher-derivative terms may lead to two pathologies: a) The presence of derivatives higher than the second in the equation of motion results in the appearance of modes with negative norm, characterized as ghosts. b) The higher-derivative terms are perturbatively nonrenormalizable and the theory loses predictivity.

We are interested in the systematic study of quantum corrections in scalar field theories with higher-derivative terms. As we have mentioned, such theories are in general nonrenormalizable in the perturbative sense. However, their scale dependence can be studied through the exact renormalization group (ERG). Their renormalizability may result from the presence of a nonperturbative fixed point. The main drawback of the ERG approach is that the integration of the flow equation for the scale-dependent effective action can be achieved only for truncated versions of the action. However, it is still possible to check the reliability of the predictions by expanding the truncation scheme and examining their stability. This procedure has been applied to scalar theories with a general potential and a standard kinetic term, leading to an accurate determination of nontrivial quantities, such as critical exponents [1]. The precision can be improved by going to higher orders of the derivative expansion [2].

The theories we consider in this work describe hypersurfaces, which we term branes, embedded in a higher-dimensional flat spacetime, to which we refer as bulk spacetime. The leading contribution to the action is given by the volume swept by the brane, expressed in terms of the induced metric. It is invariant under arbitrary changes of the brane world-volume coordinates. We can fix this gauge freedom by identifying the brane coordinates
with certain bulk coordinates. This choice is usually characterized as the static gauge. The remaining bulk coordinates become scalar fields of the worldvolume theory. More complicated terms can also be included in the effective action. The crucial property that makes this class of theories interesting is that the effective action can be expressed in terms of geometric quantities, such as the intrinsic and extrinsic curvatures of the hypersurface. In the static gauge these can be written in terms of the scalar fields and their derivatives. In this way we obtain a higher-derivative scalar theory with a particular structure. The ERG flow of the scalar theory describing two-dimensional membranes has been considered in ref. [3]. Here we present the generalization to a $d$-dimensional brane, embedded in a $(d + 1)$-dimensional bulk.

Scalar field theories with derivative interactions have been considered extensively during the last years in the context of particle physics and cosmology under a variety of names, such as: $k$-essence [4], Dirac-Born-Infeld (DBI) inflation [5], the Dvali-Gabadadze-Porrati (DGP) model [6] in the decoupling limit and the Galileon [7], scalar-tensor models with kinetic gravity braiding [8], etc. All these theories are constructed so that the equation of motion does not contain field derivatives higher than the second, even though a large or infinite series of derivative terms can be present in the action. In this way, ghost fields do not appear in the spectrum. The most general scalar-tensor theory with this property was constructed a long time ago [9], and rediscovered recently. It is characterized as the generalized Galileon (see ref. [10] and references therein).

The absence of derivatives higher than the second in the equation of motion is not protected by some underlying symmetry. For example, for the Galileon theory it is known that quantum corrections generate terms that destroy this property. The one-loop corrections computed through dimensional regularization induce a term $\phi \Box^4 \phi$ in the effective action [11]. It is still possible to consider the Galileon as a consistent quantum theory at low energies, for which such a term is subleading. The main motivation for our study stems from the wish to understand the issue of quantum corrections for such derivative theories through the ERG approach.

The connection between the Galileon and the brane picture that we discussed above is provided by ref. [12], which shows that the Galileon theory can be reproduced in the nonrelativistic limit by considering the effective action for the position modulus of a probe brane within a five-dimensional bulk. Derivatives higher than the second can be avoided by employing only Lovelock invariants in the geometric picture. In this work we consider a truncation of the brane effective action that takes into account the lowest-order geometric invariants. Some of these reproduce the structure of the Galileon theory [7]. On the other hand, our truncation includes a contribution involving the extrinsic curvature of the brane that does not have an analogue in the Galileon theory, as it would induce a field derivative higher than the second in the equation of motion. We examine how this term scales under quantum corrections and whether it is consistent to assume that it does not appear in the effective action.

Our study has another very interesting spinoff. If the contributions from the extrinsic curvature are omitted, the ERG flow can be expressed as the evolution of an effective Newton’s constant and the cosmological constant. The picture is similar to that obtained
in ERG studies of gravity, in which the metric is considered as the fundamental field. The \( \beta \)-functions display a fixed-point structure that is analogous to that associated with asymptotic safety \[13\]. Thus we obtain a very useful testing ground for a concept that could provide the UV completion of gravity.

In the following section we establish our notation and summarize the correspondence between the brane and Galileon theories. In section 3 we introduce the effective action that we consider and the flow equation that describes its evolution. In section 4 we derive the \( \beta \)-functions for the couplings of the theory and discuss the effect of quantum corrections on the structure of the brane and Galileon theories. In section 5 we analyze the fixed-point structure for a consistent truncation that preserves only the cosmological-constant and Einstein terms. We discuss the analogy with the asymptotic-safety scenario of gravity. In the final section we present a summary and our conclusions. The ERG formalism has been developed for field theories in Euclidean space. For this reason we assume the analytic continuation to imaginary time throughout the paper.

2 Brane dynamics and the Galileon

Following ref. \[12\], we summarize briefly the connection between the dynamics of fluctuating branes and the Galileon theory. The connection has been established for a four-dimensional brane embedded in five-dimensional flat space. The induced metric in the static gauge is \( g_{\mu \nu} = \eta_{\mu \nu} + \partial_\mu \pi \partial_\nu \pi \), where \( \pi \) denotes the extra coordinate of the bulk space. We preserve the notation \( \eta_{\mu \nu} \) even though we use imaginary time and the bulk metric is Euclidean. The induced extrinsic curvature is \( K_{\mu \nu} = -\partial_\mu \partial_\nu \pi / \sqrt{1 + (\partial \pi)^2} \). We denote its trace by \( K \). The leading terms in the brane effective action are

\[
S_\mu = \mu \int d^4 x \sqrt{g} = \mu \int d^4 x \sqrt{1 + (\partial \pi)^2} \tag{2.1}
\]

\[
S_\nu = \nu \int d^4 x \sqrt{g} K = -\nu \int d^4 x \left( [\Pi] - \gamma^2 [\phi] \right) \tag{2.2}
\]

\[
S_\bar{\kappa} = \frac{\bar{\kappa}}{2} \int d^4 x \sqrt{g} R = \frac{\bar{\kappa}}{2} \int d^4 x \gamma \left( [\Pi]^2 - [\Pi^2] + 2 \gamma^2 ( [\phi^2] - [\Pi] [\phi] ) \right) \tag{2.3}
\]

where \( \gamma = 1/\sqrt{g} = 1/\sqrt{1 + (\partial \pi)^2} \). We have adopted the notation of ref. \[12\], with \( \Pi_{\mu \nu} = \partial_\mu \pi \partial_\nu \pi \) and square brackets representing the trace (with respect to \( \eta_{\mu \nu} \)) of a tensor. Also, we denote \([\phi^n] \equiv \partial \pi \cdot \Pi^n \cdot \partial \pi\), so that \([\phi] = \partial^\mu \pi \partial_\mu \pi \partial^\nu \pi \). The field \( \pi \) has mass dimension \(-1\), as it corresponds to a spatial coordinate. It can be given a more conventional mass dimension through multiplication with the appropriate power of the fundamental energy scale \( M \) of the theory. We implicitly assume that all other scales are expressed in terms of \( M \), which is effectively set equal to 1. The couplings \( \mu, \nu, \bar{\kappa} \) correspond to the effective four-dimensional cosmological constant, the five-dimensional Planck scale \( M_5^3 \) and the four-dimensional Planck scale \( M_4^2 \), respectively.

The effective action of the Galileon theory can be obtained in the nonrelativistic limit \((\partial \pi)^2 \ll 1\). It must be noted, however, that terms with second derivatives of the field, such as \( \pi \Box \pi \), are not assumed to be small (here \( \Box = \eta^{\mu \nu} \partial_\mu \partial_\nu \)). If total derivatives are neglected,
the integrants of the leading terms in the expansion of (2.1)–(2.3) are proportional to \((\partial \pi)^2\). In this way, one obtains three of the terms appearing in the action of the Galileon theory [12]. The term of highest order in this theory can be obtained by including in the brane action the Gibbons-Hawking-York term associated with the Gauss-Bonnet term of five-dimensional gravity. We omit this term in the truncated effective action that we consider, as it complicates significantly the study of the renormalization of the theory. Its effect will be the focus of future work. In the context of asymptotic safety boundary terms have been considered in [14].

The first Gauss-Codazzi equation gives \(R = K^2 - K^\mu\nu K^\mu\nu\). This relation indicates that the truncation of the effective brane action that includes a term \(\sim R\) should also include a term \(\sim K^2\). On the other hand, such a term must be excluded if the equation of motion is assumed not to contain field derivatives higher than the second. Its absence cannot be enforced by some underlying symmetry, and quantum corrections may introduce it even if it is omitted in the tree-level action. In order to study its role in the renormalized theory we include in our truncated action the contribution

\[
S_\kappa = \frac{\kappa}{2} \int d^4x \sqrt{g} K^2 = \frac{\kappa}{2} \int d^4x \gamma \left( [\Pi] - \gamma^2 [\phi] \right)^2.
\]

(2.4)

In the limit \((\partial \pi)^2 \ll 1\), the contribution \([\Pi]^2\) included in this term generates in the integrant a leading contribution \(\sim \pi \Box^2 \pi\). A term \(K^\mu\nu K^\mu\nu\) in the Lagrangian density would produce a contribution \([\Pi]^2\), which would again become \(\sim \pi \Box^2 \pi\) in the nonrelativistic limit. The two leading contributions cancel in \(R = K^2 - K^\mu\nu K^\mu\nu\), so that the structure of the Galileon is generated. On the other hand, if quantum corrections spoil the cancellation, the Galileon theory is not reproduced.

It is worth pointing out that the term \(S_\nu\) can be omitted if we assume the discrete symmetry \(\pi \rightarrow -\pi\). The same symmetry would eliminate the higher-order contribution related to the Gauss-Bonnet term of the bulk theory. For a probe brane the presence in the action of terms odd in the extrinsic curvature indicates an asymmetry in the fluctuations on either side of the brane. The origin of such terms is not obvious, unless the bulk space is not homogeneous or the brane is viewed as its boundary. These considerations indicate that it seems more natural to include the contributions (2.1), (2.3), (2.4) in a consistent quantum theory than the ones that reproduce the Galileon theory. The terms (2.1), (2.3), (2.4) form the basis for the study of the renormalization of two-dimensional fluid membranes (see ref. [3] and references therein).

3 Flow equation

The focus of our study is the evolution of the scale-dependent effective action

\[
\Gamma_k = \int d^4x \sqrt{g} \left( \mu_k + \nu_k K + \frac{k}{2} K^2 + \frac{\bar{k}}{2} R \right),
\]

(3.1)

with the various invariants expressed through the field \(\pi\). We have included the contributions (2.1)- (2.4) discussed in the previous section, but we now assume that the various couplings depend on the running energy scale \(k\). The action describes the dynamics of a
dimensional brane embedded in \((d + 1)\)-dimensional flat space. We use imaginary time, so that the bulk metric is Euclidean.

The formal treatment of the action (3.1) can be carried out through the ERG. We first introduce the scale \(k\) by adding to the action a term \(R_k(q^2)\) in momentum space, so that fluctuations of the field with characteristic momenta \(q^2 \lesssim k^2\) are cut off \([1, 15]\).

We subsequently introduce sources and define the generating functional for the connected Green functions. Through a Legendre transformation we obtain the generating functional for the 1PI Green functions, from which we subtract the regulating term involving \(R_k(q^2)\). In this way we obtain the scale-dependent effective action \(\Gamma_k[\pi]\). The procedure results in the effective integration of the fluctuations with \(q^2 \gtrsim k^2\). The theory is assumed to possess a fundamental high-energy cutoff \(M\), so that \(\Gamma_k[\pi]\) is identified with the bare action \(S\) for \(k = M\). For \(k = 0\) we obtain the standard effective action. The means for calculating \(\Gamma_k[\pi]\) from \(S\) is provided by the exact flow equation \([15]\)

\[
\partial_t \Gamma_k[\pi] = \frac{1}{2} \frac{\partial_t R_k(-\Box)}{\Gamma_k^{(2)}[\pi] + R_k(-\Box)},
\]

where we have reverted to position space and defined \(t = \ln k\). Here \(\Gamma_k^{(2)}[\pi]\) denotes the second functional derivative of the action with respect to the field. The rhs of the above equation receives contributions only from fluctuations with characteristic momenta \(q^2 \simeq k^2\).

In this sense, the high-energy cutoff \(M\) is only a formal element in the definition of \(\Gamma_k\). It can be replaced by a UV fixed point in the flow of \(\Gamma_k\).

When gauge symmetries, such as the reparametrization invariance of the brane world-volume theory, are present the definition of \(\Gamma_k\) is more involved. We shall not present the details here, and we refer the reader to refs. [13] for the case of gravity, and to ref. [3] for the case of brane reparametrization invariance. In the scale-dependent action the reparametrization invariance is implemented through the use of the background field method. The brane position is determined by the embedding function \(r = (x^\mu, \pi)\) and the induced metric is given by \(g_{\mu\nu} = \partial_\mu r \cdot \partial_\nu r = \eta_{\mu\nu} + \partial_\mu \pi \partial_\nu \pi\). We parametrize the fluctuations around a background configuration \(r\) as \(r + \delta r\). In the static gauge that we have adopted, we have \(\delta r = \delta \pi n\), where \(n\) is the unit vector normal to the brane and \(\delta \pi\) is the fluctuating field. The cutoff \(R_k(\Delta)\) is constructed by means of the operator \(\Delta = -g^{\mu\nu} \nabla_\mu \nabla_\nu\), where the induced metric \(g_{\mu\nu}\) and covariant derivatives \(\nabla_\mu\) compatible with it are expressed in terms of the (background) field \(\pi\). The scale-dependent (background) effective action \(\Gamma_k[\langle \delta \pi \rangle ; \pi]\) depends both on the (background) field \(\pi\) and the expectation value \(\langle \delta \pi \rangle\) of the fluctuating field. It is constructed so as to be invariant under reparametrizations of the background, while the expectation value \(\langle \delta \pi \rangle\) has to transform covariantly on a given background \(\pi\).

The exact flow equation has the form

\[
\partial_t \Gamma_k[\langle \delta \pi \rangle ; \pi] = \frac{1}{2} \frac{\partial_t R_k(\Delta)}{\Gamma_k^{(2,0)}[\langle \delta \pi \rangle ; \pi] + R_k(\Delta)},
\]

In the limit \(\langle \delta \pi \rangle \to 0\) the background invariance is promoted to full reparametrization invariance. The effective action can be identified with \(\Gamma_k[\pi] \equiv \Gamma_k[0; \pi]\). The difficulty we
have to face is that the flow equation (3.3) is not a closed relation for \( \Gamma_k[0; \pi] \). It becomes closed if we make the ansatz \( \Gamma_k^{(2;0)}[0; \pi] = \Gamma_k^{(0;2)}[0; \pi] \equiv \Gamma_k^{(2)}[\pi] \). We obtain eq. (3.2), where now the d’Alembertian \(-\Box\) is replaced by the d’Alembertian \(\Delta\) constructed with the full induced metric. As truncation ansatz for the effective action \( \Gamma_k[\pi] \) we choose eq. (3.1), which is reparametrization invariant by construction. In this way, the invariance is preserved by the evolution, even though the full dependence of the functional \( \Gamma_k[\langle \delta\pi \rangle; \pi] \) on the two fields \( \pi \) and \( \langle \delta\pi \rangle \) is not taken into account. (For example, this functional in principle includes a separate wavefunction renormalization for the fluctuation field \( \langle \delta\pi \rangle \).)

There is a more intuitive, albeit less rigorous, way to generate the flow equation. The one-loop correction to a tree-level action of the form (3.1) is proportional to the logarithm of the fluctuation determinant around a given background. In order to compute it, we employ the static gauge and expand the field as \( r + \delta\pi n \), keeping only the terms quadratic in \( \delta\pi \). The resulting expression depends on the various couplings appearing in (3.1). These are now the bare ones and have no \( k \)-dependence. The contribution of fluctuations with characteristic momenta below a given scale \( k \) can be excluded if we add to the Lagrangian density a term \( \sim \delta\pi R_k(\Delta)\delta\pi \). It must be kept in mind that the theory (3.1) has a geometric origin, which must be preserved even when we employ the static gauge and express the action in terms of the field \( \pi \). The cutoff must be constructed in a way consistent with this property. This can be achieved if we construct the d’Alembertian employing the full induced metric, expressed in terms of \( \pi \). A “renormalization-group improvement” of the effective action can be achieved by taking its logarithmic derivative with respect to \( k \) and substituting the running couplings, which are \( k \)-dependent, for the bare ones. The resulting expression is the flow equation we discussed above.

4 \( \beta \)-functions

Extracting information from the flow equation requires an appropriate parametrization and truncation of the scale-dependent effective action. For this purpose we employ the truncation (3.1). In order to calculate the trace in the rhs of the flow equation we need the second functional derivative of (3.1) on an arbitrary background. We find

\[
\Gamma_k^{(2)}[\pi] = \kappa_k \Delta^2 + \mu_k \Delta + V^{\mu\nu} \nabla_\mu \nabla_\nu + U + \mathcal{O}(K^4, \nabla K),
\]

(4.1)

where

\[
V^{\mu\nu} = 2\nu_k (K^{\mu\nu} - K g^{\mu\nu}) + \kappa_k \left[ -\frac{1}{2} (3K^2 - 4K^{\rho\sigma} K_{\rho\sigma}) g^{\mu\nu} + 2K K^{\mu\nu} \right] + \tilde{\kappa}_k \left\{ \frac{1}{2} R^{\mu\nu} - R g^{\mu\nu} \right\} - (K^2 - K^{\rho\sigma} K_{\rho\sigma}) g^{\mu\nu} + 2K K^{\mu\nu} + 2K^{\mu\sigma} K^{\nu}_{\sigma} \]

(4.2)

\[
U = \mu_k (K^2 - K^{\rho\sigma} K_{\rho\sigma})
\]

(4.3)

and the covariant derivatives are evaluated with the full induced metric. The first Gauss-Codazzi equation allows us to express \( K^2 - K^{\rho\sigma} K_{\rho\sigma} \) in terms of \( R \) in the above expressions. A similar simplification can be carried for \( K^{\mu\sigma} K^{\nu}_{\sigma} \). However, we have preserved the expression in a form similar to that given in ref. [3] for the two-dimensional brane.
We substitute the above expressions in the rhs of the flow equation and expand the denominator in powers of the curvatures. The trace of the resulting terms can be computed through the heat kernel expansion, as described in [16]. The details of this procedure have been presented in ref. [3] for the case \( d = 2 \) and we do not repeat them here. We insert the truncation (3.1) in the lhs of the flow equation and match the contributions that involve the same curvature invariants on both sides of the equation. In this way we obtain the \( \beta \)-functions for the various couplings. They are

\[
\partial_t \mu_k = \frac{1}{(4\pi)^{d/2}} \frac{1}{2} Q_{\frac{d}{2}} [G_k \partial_t R_k]
\]

(4.4)

\[
\partial_t \nu_k = -\frac{1}{(4\pi)^{d/2}} \frac{d-1}{2} Q_{\frac{d}{2}+1} [G_k^2 \partial_t R_k] \nu_k
\]

(4.5)

\[
\partial_t \kappa_k = \frac{1}{(4\pi)^{d/2}} \left\{ \frac{d+4}{4} Q_{\frac{d}{2}+1} [G_k^2 \partial_t R_k] \kappa_k + (d^2 - 1) Q_{\frac{d}{2}+2} [G_k^2 \partial_t R_k] \nu_k^2 \right\}
\]

(4.6)

\[
\partial_t \bar{\kappa}_k = \frac{1}{(4\pi)^{d/2}} \left\{ \frac{1}{6} Q_{\frac{d}{2}-1} [G_k \partial_t R_k] - Q_{\frac{d}{2}} [G_k^2 \partial_t R_k] \mu_k - 2 Q_{\frac{d}{2}+2} [G_k^2 \partial_t R_k] \nu_k^2 \right\}
\]

\[-d Q_{\frac{d}{2}+1} [G_k^2 \partial_t R_k] \kappa_k - \frac{3(d-2)}{4} Q_{\frac{d}{2}+1} [G_k^2 \partial_t R_k] \bar{\kappa}_k \right\},
\]

(4.7)

The regularized propagators are

\[
G_k(z) = \frac{1}{\kappa_k z^2 + \mu_k z + R_k(z)},
\]

(4.8)

while the \( Q \)-functionals are defined as

\[
Q_n[f] = \frac{1}{\Gamma(n)} \int_0^\infty dz \: z^{n-1} f(z) \quad n > 0
\]

\[
Q_n[f] = (-1)^n f^{(n)}(0) \quad n \leq 0.
\]

(4.9)

Some qualitative properties of the evolution are immediately apparent. The \( \beta \)-function of \( \nu_k \) vanishes for \( \nu_k = 0 \). This is an expected result, as setting \( \nu_k = 0 \) in the tree-level action induces the discrete symmetry \( \pi \to -\pi \), which protects this value at the quantum level as well. For \( \nu_k \neq 0 \), which is a necessary assumption in order to reproduce the Galileon theory in the nonrelativistic limit, the \( \beta \)-function of \( \kappa_k \) does not vanish. It is apparent from eq. (4.6) that a contribution \( \sim \nu_k^2 \mathcal{K}^2 \) is induced through quantum fluctuations. In the nonrelativistic limit a term \( \sim \nu_k^2 \pi \Box \pi \) will be generated, which is not present in the Galileon theory. A similar phenomenon occurs for a scalar field coupled to gravity [17]. On the other hand, the analysis of the one-loop corrections to the Galileon through the use of dimensional regularization shows that the lowest-order induced term is \( \sim \nu_k^2 \pi \Box \pi \) [11].

In order to understand this point we need to make contact with perturbation theory. With the appropriate approximations, the \( \beta \)-functions (4.4)-(4.7) can reproduce standard perturbative results. For \( \nu_k = \kappa_k = 0 \) the scale-dependent effective action (3.1) has the same structure as Einstein gravity with a cosmological constant [13]. The one-loop contribution to the cosmological constant can be obtained if we set \( \mu_k = 1 \) in the rhs of eq. (4.4).
This is the bare value of this parameter that leads to a canonically normalized kinetic term when $\sqrt{g}$ is expanded in powers of $(\partial \pi)^2$ and the leading term is retained. Independently of the choice of cutoff function $R_k(q^2)$, we obtain (with $z = q^2$)

$$\partial_t \mu_k = \partial_t \left[ \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \ln(q^2 + R_k(q^2)) \right].$$

(4.10)

The trivial integration of this equation for $k$ in the range $[0, M]$ reproduces the one-loop contribution to the vacuum energy arising from the quantum fluctuations of a single massless mode in a theory with a fundamental high-energy cutoff $\sim M$. It must be noted that in the brane theory the renormalization of the cosmological constant is the same as that of the leading kinetic term at low energies. This is obvious from the form of the propagator (4.8), in which $z = q^2$ is multiplied by $\mu_k$. As a result the field $\pi$ has a large anomalous dimension.

We can obtain the one-loop correction to $\kappa_k$ in a similar fashion, by substituting the bare couplings for the running ones in the rhs of Eq. (4.6). We assume that the bare theory does not contain a term $\sim K^2$ and the kinetic term is canonically normalized. With these assumptions we can set $\mu_k = 1$ and $\kappa_k = 0$ in the rhs of (4.6) and replace $\nu_k$ by a constant value $\nu_M$. We obtain

$$\partial_t \kappa_k = - \frac{2(d^2 - 1)}{d(d + 2)} \nu_M^2 \partial_t \left[ \int \frac{d^d q}{(2\pi)^d} \frac{q^4}{(q^2 + R_k(q^2))^2} \right].$$

(4.11)

The integration of this equation for $k$ in the range $[0, M]$ results in a momentum integral with a quartic divergence for $d = 4$, which is cut off by a high-energy scale $\sim M$. This quantum correction would not be visible if dimensional regularization was used. On the other hand, the regularization with an explicit cutoff, such as the one employed in the context of the ERG, picks up corrections with possible quadratic or quartic divergences.

The correction of Eq. (4.11) induces a term $\sim \nu_M^2 K^2$, which in the nonrelativistic limit becomes $\sim \nu_M^2 \pi \Box^2 \pi$. This term is not present in the Galileon theory, and is of a lower order than the term $\sim \nu_M^2 \pi \Box^4 \pi$ expected from an analysis based on dimensional regularization.

We emphasize that this conclusion does not require a specific choice of the cutoff function $R_k(z)$, and thus is ERG-scheme independent.

A cross-check of the $\beta$-functions (4.4)-(4.7) can be obtained if we set $\mu_k = \nu_k = 0$. For $d = 2$ the resulting theory can describe two-dimensional fluid membranes in three-dimensional space. The couplings $\kappa_k$ and $\bar{\kappa}_k$ correspond to the bending and Gaussian rigidities. The $\beta$-functions of these couplings were computed in Ref. [3]. They agree with those derived through perturbation theory [18] if the anomalous dimension of the fluctuating field is set to zero.

## 5 Fixed points and asymptotic safety

Explicit expressions for the $\beta$-functions can be obtained for specific forms of the cutoff function $R_k(z)$. The results are particularly simple for the choice

$$R_k(z) = \left[ \kappa_k(k^4 - z^2) + \mu_k(k^2 - z) \right] \theta(k^2 - z).$$

(5.1)
Despite its unconventional form, the cutoff function generates the required behavior for the effective propagator $1/G_k(q^2)$: For $z = q^2 > k^2$ the effective propagator is the perturbative one $(1/G_k(z) = \kappa_k z^2 + \mu_k z)$, so that the corresponding fluctuations remain unaffected by the presence of the cutoff. For $z < k^2$, $1/G_k(z)$ is finite and constant $(1/G_k(z) = \kappa_k k^4 + \mu_k k^2)$ and the low-energy fluctuations are suppressed. It has been verified through several studies that, when these criteria are fulfilled, the predictions obtained in the limit $k \to 0$ are independent of the specific form of $R_k(z)$ [1].

For $R_k(z)$ given by eq. (5.1) the function $f(z)$ in eqs. (4.4)-(4.7) has the general form $f(z) = [G_k(z)]^m \partial_t R_k(z)$, with $m$ a positive integer. Our cutoff choice leads to the appearance of terms $\sim \partial_t \kappa_k, \partial_t \mu_k$ in $\partial_t R_k(z)$. For $\kappa_k = 0$, the term $\partial_t \mu_k$ would correspond to the anomalous dimension of the field. We shall neglect these contributions in our analysis, as they are not expected to affect the qualitative features of the evolution. They must be included, however, if quantitative precision is required.

For $n \geq 0$, the $Q$-functionals become

$$Q_n[f] = \frac{k^{2n}}{\Gamma(n+1)} f(0),$$

with

$$f(0) = \left( \frac{1}{\kappa_k k^4 + \mu_k k^2} \right)^m (4\kappa_k k^4 + 2\mu_k k^2).$$

The evolution equations (4.4)-(4.7) take the form

$$\partial_t \mu_k = - \frac{k^d}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} + 1 \right)} \frac{2\kappa_k k^2 + \mu_k}{\kappa_k k^4 + \mu_k},$$

$$\partial_t \nu_k = - \frac{k^d}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} + 2 \right)} (d-1) \frac{(2\kappa_k k^2 + \mu_k)\nu_k}{(\kappa_k k^4 + \mu_k)^2},$$

$$\partial_t \kappa_k = \frac{2k^d}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} + 2 \right)} \left\{ \frac{d+4}{4} \frac{2\kappa_k k^2 + \mu_k}{\kappa_k k^4 + \mu_k} + \frac{4}{d+4} \frac{(2\kappa_k k^2 + \mu_k)\nu_k^2}{(\kappa_k k^4 + \mu_k)^3} \right\},$$

$$\partial_t \bar{k}_k = \frac{k^d}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} + 2 \right)} \left\{ \frac{d(d+2)}{12} \frac{2\kappa_k k^2 + \mu_k}{\kappa_k k^4 + \mu_k} - 8 \frac{(2\kappa_k k^2 + \mu_k)\nu_k^2}{(\kappa_k k^4 + \mu_k)^3} \right\} - \left[ \frac{3(d-2)}{2} \frac{\kappa_k k^2 + \mu_k}{(\kappa_k k^4 + \mu_k)^2} \right].$$

The structure of the above equations is typical of the ERG, with terms involving various powers of the effective propagator appearing in the $\beta$-functions. The class of theories that we are considering involves only generalized kinetic terms. For this reason couplings such as $\kappa_k, \mu_k$ that multiply the leading terms appear often in the denominator in the $\beta$-functions.

As the theory has a geometric origin, the fundamental field $\pi$ has mass dimension $-1$ because it corresponds to a spatial coordinate. As we have already mentioned, it can be given a more conventional mass dimension through multiplication with the appropriate power of the fundamental energy scale $M$ of the theory. Throughout the paper we assume that all scales are expressed in terms of $M$. The scaling dimensions of the various couplings
can be deduced from eqs. (5.4)-(5.7) if we remove the explicit factors of $k$ through the appropriate redefinitions. If we define

$$\mu_k = k^d \tilde{\mu}_k, \quad \nu_k = k^{d-1} \tilde{\nu}_k, \quad \kappa_k = k^{d-2} \tilde{\kappa}_k, \quad \bar{\kappa}_k = k^{d-2} \bar{\kappa}_k,$$

eqs. (5.4)-(5.7) become

$$\partial_t \tilde{\mu}_k = -d \tilde{\mu}_k + \frac{1}{(4\pi)^{d/2} \Gamma \left(\frac{d}{2} + 1\right)} \frac{2 \tilde{\kappa}_k + \tilde{\mu}_k}{\tilde{\kappa}_k + \tilde{\mu}_k} (5.9)$$

$$\partial_t \tilde{\nu}_k = -(d-1) \tilde{\nu}_k - \frac{1}{(4\pi)^{d/2} \Gamma \left(\frac{d}{2} + 2\right)} (d-1) \frac{(2 \tilde{\kappa}_k + \tilde{\mu}_k) \tilde{\nu}_k}{(\tilde{\kappa}_k + \tilde{\mu}_k)^2} (5.10)$$

$$\partial_t \tilde{\kappa}_k = -(d-2) \tilde{\kappa}_k + \frac{2}{(4\pi)^{d/2} \Gamma \left(\frac{d}{2} + 2\right)} \left\{ \frac{d+4}{4} \frac{(2 \tilde{\kappa}_k + \tilde{\mu}_k) \tilde{\kappa}_k}{(\tilde{\kappa}_k + \tilde{\mu}_k)^2} + \frac{4(d^2-1)}{d+4} \frac{(2 \tilde{\kappa}_k + \tilde{\mu}_k) \tilde{\nu}_k^2}{(\tilde{\kappa}_k + \tilde{\mu}_k)^3} \right\}$$

$$\partial_t \tilde{\bar{\kappa}}_k = -(d-2) \tilde{\bar{\kappa}}_k + \frac{1}{(4\pi)^{d/2} \Gamma \left(\frac{d}{2} + 2\right)} \left\{ \frac{d(d+2)}{12} \frac{2 \tilde{\kappa}_k + \tilde{\mu}_k}{\tilde{\kappa}_k + \tilde{\mu}_k} - \frac{8}{d+4} \frac{(2 \tilde{\kappa}_k + \tilde{\mu}_k) \tilde{\nu}_k^2}{(\tilde{\kappa}_k + \tilde{\mu}_k)^3} \right\} - \left[ \frac{(d+2)}{2} \tilde{\mu}_k + 2d \tilde{\kappa}_k + \frac{3(d-2)}{2} \tilde{\bar{\kappa}}_k \right] \frac{2 \tilde{\kappa}_k + \tilde{\mu}_k}{(\tilde{\kappa}_k + \tilde{\mu}_k)^2}. (5.12)$$

This is the most convenient form of the evolution equations for the determination of their fixed points.

As a first check we can compute the $\beta$-functions of $\kappa_k$, $\bar{\kappa}_k$ for two-dimensional fluid membranes. In the membrane theory the volume (now area) term is considered subleading. This means that we can get the relevant equations by setting $d = 2$, $\mu_k = \nu_k = 0$ in eqs. (5.6), (5.7). We obtain

$$\partial_t \kappa_k = \frac{3}{4\pi}, \quad \partial_t \bar{\kappa}_k = -\frac{5}{6\pi}. (5.13)$$

These expressions reproduce the results of refs. [3, 18] for the renormalization of the bending and Gaussian rigidities of fluctuating membranes in a three-dimensional bulk space. It must be pointed out, however, that the relation $\mu_k = 0$ is not consistent with eq. (5.4), which becomes

$$\partial_t \mu_k = \frac{k^2}{2\pi} (5.14)$$

for $d = 2$, $\mu_k = 0$. Neglecting the area term can be viewed only as a low-energy approximation. Setting $\mu_k = \nu_k = 0$ in eqs. (5.6), (5.7) provides a generalization of the evolution for branes of arbitrary dimensionality.

An important point, which we have already discussed in the previous section, is the stability of the conditions $\nu_k = 0$ and $\kappa_k = 0$ under quantum corrections. The first one is expected to be stable, as it is protected by the symmetry $\pi \rightarrow -\pi$. The evolution equation (5.5) explicitly demonstrates that $\partial_t \nu_k$ vanishes for $\nu_k = 0$. On the other hand the condition $\kappa_k = 0$ does not enhance the symmetry of the action and is not expected to survive at the quantum level. Eq. (5.6) indicates that corrections $\sim \nu_k^2$ are generated for $\kappa_k$ under renormalization. It is noteworthy that, if we set $\nu_k = 0$, we have $\partial_t \kappa_k = 0$ for
\( \kappa_k = 0 \) and \( \mu_k \neq 0 \). We believe that this is an accidental feature. Notice also that the \( \beta \)-function does not vanish if we first set \( \mu_k = 0 \) and then take the limit \( \kappa_k \to 0 \).

The analysis of the fixed points of the system of equations (5.9)-(5.12) and their stability for various dimensionality goes beyond the scope of this work. We shall analyze the flow in a reduced parameter space which is relevant for the issue of asymptotic safety in gravity. For \( \mu_k, \tilde{\kappa}_k \neq 0 \) we can consistently assume that \( \nu_k = \kappa_k = 0 \), as then the associated \( \beta \)-functions vanish. The reduced action (3.1) contains only the Einstein and cosmological-constant terms. It must be emphasized that the theory we are considering is not dynamical gravity. The action (3.1) involves only one fluctuating scalar degree of freedom that has geometric origin. Despite the different nature of the theory, we find that the flows display striking similarity with what has been observed in the analysis of gravitational theories.

The evolution of the couplings is described by eqs. (5.9), (5.12) with \( \tilde{\kappa}_k = 0 \). We concentrate on the case \( d = 4 \) which is closest to four-dimensional gravity. In order to make the analogy with gravity more apparent we define the dimensionless cosmological and Newton’s constants through the relations

\[
\tilde{\mu}_k = \frac{\Lambda_k}{8\pi G_k}, \quad \tilde{\kappa}_k = -\frac{1}{8\pi G_k}
\]

(5.15)

Their evolution is given by

\[
\partial_t \Lambda_k = -2\Lambda_k + \frac{1}{6\pi} G_k (3 - 2\Lambda_k)
\]

(5.16)

\[
\partial_t G_k = 2G_k + \frac{1}{12\pi \Lambda_k} G_k^2 (3 - 4\Lambda_k).
\]

(5.17)

This system of equations has two fixed points at which the \( \beta \)-functions vanish: a) the Gaussian one, at \( \Lambda_k = G_k = 0 \), and b) a nontrivial one, at \( \Lambda_k = 9/8, G_k = 18\pi \).

The evolution of the couplings is depicted in fig. 1 for increasing \( k \). For \( \Lambda_k > 0 \) the Gaussian fixed point is UV unstable, while all flows converge to the nontrivial one, which is UV stable. This indicates that the nontrivial fixed point can provide a UV completion of the theory by allowing the limit \( k \to \infty \) to be taken. The flows in the region \( \Lambda_k < 0 \) converge towards the Gaussian fixed point, which is now UV stable. The two regions, of positive or negative \( \Lambda_k \), are disconnected, as the \( \beta \)-function of \( G_k \) diverges on the line \( \Lambda_k = 0 \), while the flows are in opposite directions on either side of this line.

The presence of the nontrivial fixed point and the form of the flows around it display a strong similarity with the corresponding flows for gravity in the Einstein-Hilbert truncation, in which only the cosmological and Newton’s constants are retained [13]. In gravity the flows for \( \Lambda_k < 0 \) or for large positive \( \Lambda_k \) can display a strong sensitivity to the choice of the cutoff function. However, their qualitative form in the vicinity of the fixed points is stable and provides support for the asymptotic safety scenario, which assumes a UV completion of gravity through a nontrivial fixed point. A nice feature of our flows is that they display stream lines connecting the region near the UV fixed point with the physical IR region in the limit \( k \to 0 \).

An important conclusion of our study is that the asymptotic safety scenario can be realized even within scalar theories, which are in general much simpler to analyze. In
this sense these results are similar to those obtained by considering the gravitational flows induced by matter fields in the large \( N \) limit \cite{19}. It must be emphasized, however, that the theory we are considering has an underlying gauge symmetry, the reparametrization invariance of the worldvolume, which must be preserved in the cutoff theory. In this sense it poses difficulties analogous to those encountered when trying to preserve the general covariance of gravity.

6 Conclusions

The focus of this work has been on understanding the effect of quantum corrections on the structure of higher-derivative theories. Such theories are in general nonrenormalizable in the perturbative sense. For this reason we employed the ERG, which has the potential to reveal nonperturbative features, such as fixed points not easily accessible to perturbative methods. On the other hand, it must be kept in mind that the ERG approach relies heavily on the use of truncated versions of the effective action, which may not capture all the physics.
The analysis of a general higher-derivative theory would involve too many parameters. For this reason we limited our discussion to the class of theories that describe $d$-dimensional fluctuating branes within a bulk space of $d + 1$ dimensions. The physical degree of freedom is the position modulus $\pi$ of the brane, which can be viewed as a scalar field of the worldvolume theory. The structure of the Lagrangian density is constrained by the reparametrization invariance of the brane worldvolume. The various terms correspond to geometric invariants, involving the extrinsic and intrinsic curvatures of the brane expressed in terms of $\pi$.

In the nonrelativistic limit the classical brane theory can reproduce the structure of the Galileon theory \cite{12}. An important question is whether this feature remains valid at the quantum level as well. We found evidence that quantum corrections spoil the correspondence. They generate a geometric term in the brane theory $\sim K^2$, where $K$ denotes the trace of the extrinsic curvature. Even if the term is absent at the classical level, it will appear upon renormalization. In the nonrelativistic limit this term becomes $\sim \pi \Box^2 \pi$, a contribution not present in the Galileon theory. On the other hand, the analysis of the quantum corrections to the Galileon theory through the use of dimensional regularization indicates that the lowest-order correction is $\sim \pi \Box^4 \pi$ \cite{11}. The discrepancy can be resolved by noting that the ERG analysis employs an explicit cutoff as a regulator of momentum integrals. For this reason it is sensitive to corrections with quadratic or quartic divergences. The term $\sim \pi \Box^2 \pi$ is induced by a correction with a quartic divergence, which is not visible through dimensional regularization. It must be noted that our conclusion does not depend on the specific choice of the infrared cutoff that we employ in the context of the ERG, and is, therefore, ERG-scheme independent.

We considered the action of eq. (3.1), written in terms of geometric invariants. These can be expressed through the position modulus $\pi$ according to eqs. (2.1)-(2.4). The $\beta$-functions for the couplings of the theory are given by eqs. (4.4)-(4.7). They form the main result of this work. For the particular choice (5.1) for the cutoff function, the $\beta$-functions can be written in the form (5.9)-(5.12), without an explicit reference to the running scale $k$. In an approximation consistent with these equations, we considered a truncation of the action that preserves only the cosmological-constant and Einstein terms. Despite the similarity with dynamical gravity, the theory has only one fluctuating scalar degree of freedom. It is remarkable, therefore, that the most prominent feature of the flow diagram is qualitatively similar to the one in the asymptotic-safety scenario for gravity. There is an attractive UV fixed point, which can be employed in order to obtain a UV completion of the theory.

The fixed points and the related flows predicted by eqs. (4.4)-(4.7) for various values of $d$ will be the focus of future research. The coupling $\nu_k$ can be consistently set to zero if we assume a symmetry in the fluctuations on either side of the brane. The reduced system involves three couplings ($\mu_k$, $\kappa_k$, $\bar{\kappa}_k$) and possesses novel fixed points. It forms a consistent framework in which to study the renormalization-group evolution of a higher-derivative theory with nontrivial features. The analogy with the evolution of $d$-dimensional gravity is a very interesting issue.
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