Difference Cesàro sequence space defined by a sequence of modulus function

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Abstract

The purpose of this paper is to introduce the difference sequence space $ces(B^p_{\lambda}, F, q)$ using sequence of modulus function $F = (f_i)$. We examine some topological properties of the space and also obtain some inclusion relations.

Keywords: Cesàro sequence space, difference sequence space, paranormed space.

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1. Introduction and preliminaries

Let $w, \ell^0$ denote the spaces of all scalar and real sequences, respectively. For $1 < p < \infty$, the cesàro sequence space $ces_p$ defined by

$$ces_p = \left\{ x \in \ell^0 : \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\},$$

is a Banach space when equipped with the norm

$$||x|| = \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{\frac{1}{p}}.$$

This space was introduced by Shiue [30], which is useful in the theory of matrix operator. Some geometric properties of the cesàro sequence space $ces_p$ were studied by many authors such as Lee [13], Leibowitz [14], Lui et. al [15], Sanhan et. al [25] and Tripathy et. al [33] and references therein. Modulus function has been discussed in [22, 23, 26–29] and references therein.

Ruckle [24] used the idea of a modulus function $f$ to construct a class of FK spaces

$$L(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}.$$
The space $L(f)$ is closely related to the space $\ell_1$ which is an $L(f)$ space with $f(x) = x$ for all real $x \geq 0$.

For any set $E$ of sequences, the space of multipliers of $E$, denoted by $M(E)$, is given by

$$M(E) = \{a \in \mathbb{R} : ax \in E, \text{ for all } x \in E\}.$$  

The notion of the difference sequence space was introduced by Kizmaz [12]. It was further generalized by Et and Çolak [11] as follows

$$Z(\Delta^\mu) = \{x = (x_k) \in \omega : (\Delta^\mu x_k) \in z\},$$

for $z = \ell_\infty$, $c$ and $c_0$, where $\mu$ is a non-negative integer and

$$\Delta^\mu x_k = \Delta^{\mu-1} x_k - \Delta^{\mu-1} x_{k+1}, \quad \Delta^0 x_k = x_k, \quad \forall \ k \in \mathbb{N},$$

or equivalent to the following binomial representation

$$\Delta^\mu x_k = \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} x_{k+v}.$$  

These sequence spaces were generalized by Et and Başarır [10] taking $z = \ell_\infty(p)$, $c(p)$ and $c_0(p)$.

Dutta [3] introduced the following difference sequence spaces using a new difference operator:

$$Z(\Delta_{(n)}^\mu) = \{x = (x_k) \in \omega : (\Delta_{(n)}^\mu x_k) \in z\}, \quad \text{for } z = \ell_\infty, c \text{ and } c_0,$$

where $\Delta_{(n)}^\mu = (\Delta_{(n)}^\mu x_k) = (x_k - x_{k-\eta})$ for all $k, \eta \in \mathbb{N}$.

In [4], Dutta introduced the sequence spaces $\mathcal{C}(\|\cdot\|, \Delta_{(n)}^\mu p)$, $c_0(\|\cdot\|, \Delta_{(n)}^\mu p)$, $\ell_\infty(\|\cdot\|, \Delta_{(n)}^\mu p)$, $m(\|\cdot\|, \Delta_{(n)}^\mu p)$, and $m_0(\|\cdot\|, \Delta_{(n)}^\mu p)$, where $\eta$, $\mu \in \mathbb{N}$ and $\Delta_{(n)}^\mu x_k = (\Delta_{(n)}^\mu x_k) = (\Delta_{(n)}^{\mu-1} x_k - \Delta_{(n)}^{\mu-1} x_{k-\eta})$, and $\Delta_{(n)}^0 x_k = x_k$, for all $k, \eta \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta_{(n)}^\mu x_k = \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} x_{k-\eta v}.$$  

The difference sequence space have been studied by authors [5-9, 18-21, 23, 31, 32, 35] and references therein. Başar and Altay [1] introduced the generalized difference matrix $B = (b_{mk})$ for all $k, m \in \mathbb{N}$, which is a generalization of $\Delta_{(1)}$-difference operator, by

$$b_{mk} = \begin{cases} r, & k = m, \\ s, & k = m - 1, \\ 0, & (k > m) \text{ or } (0 \leq k < m - 1). \end{cases}$$

Başarır and Kayıkcı [2] defined the matrix $B^\mu (b_{mk})$ which reduced the difference matrix $\Delta_{(1)}^\mu$ incase $r = 1$, $s = -1$. The generalized $B^\mu$-difference operator is equivalent to the following binomial representation:

$$B^\mu x = B^\mu (x_k) = \sum_{v=0}^{\mu} \binom{\mu}{v} r^{\mu-v} s^v x_{k-v}.$$  

Let $F = (f_i)$ be a sequence of modulus functions, $q = (q_n)$ be a bounded sequence of strictly positive real numbers, then we define the cesàro sequence space as follows

$$\text{ces}(B^\mu A, F, q) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^\mu A x_i| \right) \right]^{q_n} < \infty \right\}.$$
Taking modulus function \( F' \) instead of \( F \) in the space \( \text{ces}(B^u_A, F, q) \), we can define the composite space \( \text{ces}(B^u_A, F', q) \) as follow

\[
\text{ces}(B^u_A, F', q) = \left\{ x \in w : \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_A x_i| \right) \right]^{q_n} < \infty \right\}.
\]

The following inequality will be used throughout the paper. If \( 0 \leq p_i \leq \sup p_i = H, K = \max(1, 2^{H-1}) \), then

\[
|a_i + b_i|^{p_i} \leq K|a_i|^{p_i} + |b_i|^{p_i},
\]

(1.1)

for all \( i \) and \( a_i, b_i \in C \). Also \( |a|^{p_i} \leq \max(1, |a|^H) \) for all \( a \in C \).

We examine some topological properties of the space \( \text{ces}(B^u_A, F, q) \) and also obtain some inclusion relations.

2. Topological properties

**Theorem 2.1.** Let \( F = (f_i) \) be a sequence of modulus function and \( q = (q_n) \) be a bounded sequence of positive real numbers. Then \( \text{ces}(B^u_A, F, q) \) is a linear space over the field of complex number \( C \).

**Proof.** Let \( x, y \in \text{ces}(B^u_A, F, q) \) and \( \alpha, \beta \in C \). Then there exist positive number \( M_\alpha \) and \( N_\beta \) such that \( |\alpha| \leq M_\alpha \) and \( |\beta| \leq N_\beta \). From condition (ii) and (iii) of definition of modulus function and by using inequality (1.1), we have

\[
\sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_A (\alpha x_i + \beta y_i)| \right) \right]^{q_n} \leq \max(1, 2^{H-1}) \left( \max(1, M^{H_H}_\alpha) \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_A x_i| \right) \right]^{q_n} + \max(1, N^{H_H}_\beta) \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_A y_i| \right) \right]^{q_n} \right).
\]

This implies that \( \alpha x + \beta y \in \text{ces}(B^u_A, F, q) \). This proves that \( \text{ces}(B^u_A, F, q) \) is a linear space. This completes the proof of the theorem. \( \square \)

**Theorem 2.2.** Let \( F = (f_i) \) be a sequence of modulus function and \( q = (q_n) \) be a bounded sequence of positive real numbers, \( \text{ces}(B^u_A, F, q) \) is a topological linear space, paranormed by

\[
g(x) = \left( \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_A x_i| \right) \right]^{q_n} \right)^{\frac{1}{q}},
\]

where \( H = \sup q_n < \infty \) and \( K = \max(1, H) \).

**Proof.** Clearly \( g(x) = g(-x) \). It is trivial \( B^u_A x_i = 0 \) for \( x = 0 \). Since \( f_i(0) = 0 \), we get \( g(x) = 0 \) for \( x = 0 \). Since \( \frac{p_i}{H} \leq 1 \), Using the Minkowski’s inequality, we have

\[
\left( \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_A (x_i + y_i)| \right) \right]^{q_n} \right)^{\frac{1}{q}} \leq \left( \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_A x_i| + |B^u_A y_i| \right) \right]^{q_n} \right)^{\frac{1}{q}}
\]

\[
\leq \left( \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_A x_i| \right) \right]^{q_n} \right)^{\frac{1}{q}}
\]

\[
+ \left( \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_A y_i| \right) \right]^{q_n} \right)^{\frac{1}{q}}.
\]
Hence \( g(x) \) is subadditive. For the continuity of multiplication, let us take any complex number \( \alpha \). By definition, we have
\[
g(\alpha x) = \left( \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{\infty} |B^u_{\Lambda}(\alpha x_i)| \right) \right]^q \right)^{\frac{1}{q}} \leq C_{\alpha}^F g(x),
\]
where \( C_{\alpha} \) is a positive integer such that \( |\alpha| \leq C_{\alpha} \). Now, let \( \alpha \to 0 \) for any fixed \( x \) with \( g(x) \neq 0 \). By definition for \( |\alpha| < 1 \), we have
\[
\sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{\infty} |\alpha B^u_{\Lambda} x_i| \right) \right]^q < \epsilon, \quad \text{for } n > n_0(\epsilon).
\]
(2.1)

Also, for \( 1 < n < n_0 \), taking \( \alpha \) small enough, since \( F = (f_i) \) is continuous, we have
\[
\sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{\infty} |\alpha B^u_{\Lambda} x_i| \right) \right]^q < \epsilon.
\]
(2.2)

Now, (2.1) and (2.2) together imply that \( g(\alpha x) \to 0 \) as \( \alpha \to 0 \). This completes the proof of the theorem. \( \square \)

**Theorem 2.3.** Let \( F = (f_i) \) be a sequence of modulus function and \( q = (q_n) \) be a bounded sequence of positive real numbers, \( \text{ces}(B^u_{\Lambda}, F, q) \) is a complete paranormed space with paranorm defined by
\[
g(x) = \left( \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_{\Lambda} x_i| \right) \right]^q \right)^{\frac{1}{q}},
\]
where \( H = \sup q_n < \infty \) and \( K = \max(1, H) \).

**Proof.** In view of Theorem 2.2 it suffices to prove the completeness of \( \text{ces}(B^u_{\Lambda}, F, q) \). Let \( (x^{(s)}) \) be a Cauchy sequence in \( \text{ces}(B^u_{\Lambda}, F, q) \). Then \( g(x^{(s)} - x^{(t)}) \to 0 \) as \( t \to \infty \), that is
\[
\sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_{\Lambda} (x^{(s)}_i - x^{(t)}_i)| \right) \right]^q \to 0, \quad \text{as } s, t \to \infty,
\]
(2.3)

which implies that for each \( i \), \( |x^{(s)}_i - x^{(t)}_i| \to 0 \) as \( s, t \to \infty \) and so \( (x^{(s)}) \) is a Cauchy sequence in \( C \) for each fixed \( i \). Since \( C \) is complete, as \( s \to \infty \), \( x^{(s)}_i \to x_i \), for each \( i \). Now from (2.3), we have that for \( \epsilon > 0 \), there exists a natural number \( N \) such that
\[
\sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_{\Lambda} (x^{(s)}_i - x^{(t)}_i)| \right) \right]^q < \epsilon^K, \quad \text{for } s, t > N.
\]
(2.4)

Since for any fixed natural number \( M \), we have from (2.4)
\[
\sum_{n=1}^{M} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_{\Lambda} (x^{(s)}_i - x^{(t)}_i)| \right) \right]^q < \epsilon^K, \quad \text{for } s, t > N,
\]
by taking \( t \to \infty \) in the above expression we obtain
\[
\sum_{n=1}^{M} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^u_{\Lambda} (x^{(s)}_i - x^{(t)}_i)| \right) \right]^q < \epsilon^K, \quad \text{for } s > N.
\]
Since \( M \) is arbitrary, by taking \( M \to \infty \), we obtain
\[
\sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{\infty} |B^{\mu}_{A}(x_i)| \right) \right] q_n < e^k, \quad \text{for } s > N,
\]
i.e., \( g(x^s - x) < e \) for \( s > N \). To show that \( x \in \text{ces}(B^{\mu}_{A}, F, q) \), let \( t > M \) and fix \( n_0 \). Since \( \frac{p_n}{q_n} \leq 1 \) and \( K \geq 1 \), using Minkowski’s inequality and the definition of modulus function, we have
\[
\left( \sum_{n=1}^{n_0} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{\infty} |B^{\mu}_{A}(x_i)| \right) \right] q_n \right)^{\frac{1}{q}} = \left( \sum_{n=1}^{n_0} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{\infty} |B^{\mu}_{A}(x_i - x^{(t)}_i)| \right) + f_i \left( \frac{1}{n} \sum_{i=1}^{\infty} |B^{\mu}_{A}(x^{(t)}_i)| \right) \right] q_n \right)^{\frac{1}{q}}
\leq \left( \sum_{n=1}^{n_0} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{\infty} |B^{\mu}_{A}(x_i - x^{(t)}_i)| \right) \right] q_n \right)^{\frac{1}{q}}
+ \left( \sum_{n=1}^{n_0} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{\infty} |B^{\mu}_{A}(x^{(t)}_i)| \right) \right] q_n \right)^{\frac{1}{q}}
< e + g(x^t).
\]

It follows that \( \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{\infty} |B^{\mu}_{A}(x_i)| \right) \right] q_n \) converges, so that \( x = (x_i) \in \text{ces}(B^{\mu}_{A}, F, q) \) and the space is complete. This completes the proof of the theorem. \( \Box \)

3. Inclusion relations

**Theorem 3.1.** If \( q = (q_n) \) and \( p = (p_n) \) are bounded sequences of positive real numbers with \( 0 < q_n \leq p_n < \infty \), for each \( n \) and \( F = (f_i) \) be a sequence of modulus function, then \( \text{ces}(B^{\mu}_{A}, F, q) \subseteq \text{ces}(B^{\mu}_{A}, F, p) \).

**Proof.** Let \( x \in \text{ces}(B^{\mu}_{A}, F, q) \). Then
\[
\sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^{\mu}_{A}x_i| \right) \right] q_n < \infty.
\]
This implies that \( f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^{\mu}_{A}x_i| \right) \leq 1 \) for sufficiently large values of \( n \), say \( n \geq n_0 \) for some fixed \( n_0 \in \mathbb{N} \). Since \( F = (f_i) \) is increasing and \( q_n \leq p_n \), we have
\[
\sum_{n \geq n_0} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^{\mu}_{A}x_i| \right) \right] p_n \leq \sum_{n \geq n_0} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^{\mu}_{A}x_i| \right) \right] q_n < \infty,
\]
which implies that \( x \in \text{ces}(B^{\mu}_{A}, F, p) \) and this completes the proof of the theorem. \( \Box \)

**Theorem 3.2.** If \( u = (u_n) \) and \( v = (v_n) \) are bounded sequences of positive real numbers with \( 0 < u_n, v_n < \infty \), and \( q_n = \min(u_n, v_n) \), then
\[
\text{ces}(B^{\mu}_{A}, F, q) = \text{ces}(B^{\mu}_{A}, F, u) \cap \text{ces}(B^{\mu}_{A}, F, v).
\]

**Proof.** It follows from Theorem 3.1 that
\[
\text{ces}(B^{\mu}_{A}, F, q) \subseteq \text{ces}(B^{\mu}_{A}, F, u) \cap \text{ces}(B^{\mu}_{A}, F, v).
\]
For any complex number \( \lambda, |\lambda|^{q_n} \leq \max(|\lambda|^{u_n}, |\lambda|^{v_n}) \), thus
\[
\text{ces}(B^{\mu}_{A}, F, u) \cap \text{ces}(B^{\mu}_{A}, F, v) \subseteq \text{ces}(B^{\mu}_{A}, F, q),
\]
and the proof of the theorem is complete. \( \Box \)
Theorem 3.3. If $H = \sup p_k < \infty$ and $F = (f_i)$ be a sequence of modulus function, then $\ell_\infty \subset M(\ces(B^H_{\Lambda}, F, q))$.

Proof. $a \in \ell_\infty$ implies $|a_i| < 1 + [i]$ for some $i > 0$ and all $i$. Hence, $x \in \ces(B^H_{\Lambda}, F, q)$ implies

$$
\sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |a_i x_i| \right) \right]^{q_n} < (1 + [i])^H \sum_{n=1}^{\infty} \left[ f_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^H_{\Lambda} x_i| \right) \right]^{q_n},
$$

which gives $\ell_\infty \subset M(\ces(B^H_{\Lambda}, F, q))$. This completes the proof of the theorem.

\[ \square \]

Theorem 3.4. For any sequence of modulus function $F = (f_i)$ and $v \in \mathbb{N}$,

(i) $\ces(B^H_{\Lambda}, F^v, q) \subset \ces(B^H_{\Lambda}, q)$, if $\lim_{t \to \infty} \frac{f(t)}{t} = \beta > 0$.

(ii) $\ces(B^H_{\Lambda}, q) \subset \ces(B^H_{\Lambda}, F^v, q)$, if there exists a positive constants $\alpha$ such that $f(t) \leq \alpha t$, for all $t \geq 0$.

Proof. (i) By Maddox [12, Proposition 1], we have

$$
\beta = \lim_{t \to \infty} \frac{f(t)}{t} = \inf \left\{ \frac{f(t)}{t} : t > 0 \right\},
$$

so that $0 \leq \beta \leq f(1)$. Let $\beta > 0$, by definition of $\beta$, we have $\beta t \leq f(t), \forall t \geq 0$. Since $F = (f_i)$ is increasing we have $\beta^2 t \leq f^2(t)$. So by induction we have $\beta^v t \leq f^v(t)$. Let $x \in \ces(F^v, q, B^H_{\Lambda})$, Using inequality $|\lambda|^{q_i} \leq \max(1, |\lambda|^H)$, we have

$$
\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |B^H_{\Lambda} x_i| \right)^{q_n} \leq \sum_{n=1}^{\infty} \left[ \beta^{-v} f^n \left( \frac{1}{n} \sum_{i=1}^{n} |B^H_{\Lambda} x_i| \right) \right]^{q_n}
$$

\[ \leq \max(1, \beta^{-vH}) \sum_{n=1}^{\infty} \left[ f^n \left( \frac{1}{n} \sum_{i=1}^{n} |B^H_{\Lambda} x_i| \right) \right]^{q_n}, \]

and hence $x \in \ces(B^H_{\Lambda}, q)$.

(ii) Since $f_i(t) \leq \alpha t$, for all $t \geq 0$ and $F = (f_i)$ is an increasing function, we have $f_i^v(t) \leq \alpha^v t$ for each $v \in \mathbb{N}$. Let $x \in \ces(B^H_{\Lambda}, q)$. Using inequality $|\lambda|^{q_i} \leq \max(1, |\lambda|^H)$, we have

$$
\sum_{n=1}^{\infty} \left[ f^n \left( \frac{1}{n} \sum_{i=1}^{n} |B^H_{\Lambda} x_i| \right) \right]^{q_n} \leq \max(1, \alpha^{vH}) \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |B^H_{\Lambda} x_i| \right)^{q_n},
$$

and hence $x \in \ces(B^H_{\Lambda}, F^v, q)$.

\[ \square \]

Theorem 3.5. Let $m, v \in \mathbb{N}$ be such that $m < v$. If there exists a positive constant $\alpha$ such that $f(t) \leq \alpha t$ for all $t \geq 0$, then

$$
\ces(B^H_{\Lambda}, q) \subset \ces(B^H_{\Lambda}, F^m, q) \subset \ces(B^H_{\Lambda}, F^v, q).
$$

Proof. Let $r = v - m$. Since $f_i(t) \leq \alpha t$, we have $f^r(t) \leq M r f^m_i(t) < M^v t$, where $M = 1 + [\alpha]$. Let $x \in \ces(B^H_{\Lambda}, q)$, we have

$$
\sum_{n=1}^{\infty} \left[ f^n \left( \frac{1}{n} \sum_{i=1}^{n} |B^H_{\Lambda} x_i| \right) \right]^{q_n} < M^r H \sum_{n=1}^{\infty} \left[ f^m_i \left( \frac{1}{n} \sum_{i=1}^{n} |B^H_{\Lambda} x_i| \right) \right]^{q_n}
$$

\[ < M^{vH} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |B^H_{\Lambda} x_i| \right)^{q_n}, \]

and the required inclusion follows. This completes the proof.

\[ \square \]

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