Bound states of a system of two bosons with a spherically potential on a lattice

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Abstract. We consider a Hamiltonian of a system of two bosons on a three-dimensional lattice \( \mathbb{Z}^3 \) with a spherically symmetric potential. The corresponding Schrödinger operator \( H(\mathbf{k}) \) this system has four invariant subspaces \( L(123), L(1), L(2), \) and \( L(3) \). The Hamiltonian of this system has a unique bound state over each invariant subspace \( L(1), L(2), \) and \( L(3) \). The corresponding energy values of these bound states are calculated exactly.

1. Introduction. In models of solid state physics [1], [2] and also in lattice quantum field theory [3], discrete lattice operators are considered which are lattice analogs of the two-particle Schrödinger operators in the continuum.

In [4] in order to theoretical explanation of this fact it has been used Hubbard model, namely, the two-particle Hamiltonians on lattice \( \mathbb{Z}^d \). Cold atoms loaded in an optical lattice provide a realization of a quantum lattice gas. The periodicity of the potential gives rise to a band structure for the dynamics of the atoms. The dynamics of the ultracold atoms loaded in the lower or upper band is well described by the Bose-Hubbard hamiltonian [4].

The nature of bound states of two-particle cluster operators for small parameter values was first studied in detail by Minlos and Mamatov [5] and then in a more general setting by Minlos and Mogilner [6]. Studying bound states of a two-particle system Hamiltonian \( H \) on the \( d \)-dimensional lattice \( \mathbb{Z}^d \) reduces to studying the eigenvalues of a family of Schrödinger operators \( H(\mathbf{k}), \mathbf{k} \in \mathbb{T}^d \), where \( \mathbf{k} \) is the total quasimomentum. Moreover, eigenfunctions of \( H(\mathbf{k}) \) are interpreted as bound states of the Hamiltonian \( H \), and eigenvalues, as the bound-state energy. The bound states of \( H \) of a system of two fermions on a one-dimensional lattice were studied in [7], and perturbations of the eigenvalues of a two-particle Schrödinger operator on a one-dimensional lattice were studied in [8], [9].

Bound state of a system of two bosons on a two-dimensional lattice was studied in [10]. Infinity of bound states of a system of two fermions on a two-dimensional lattice was investigated [11], and on a three-dimensional lattice in [12]. The finitness of the number of eigenvalues of the two-particle Schrödinger operator on a \( d \)-dimensional lattice was studied in [13], [14].

In this work we consider bound states of a Hamiltonian \( \hat{H} \) of a system of two bosons on the three-dimensional lattice \( \mathbb{Z}^3 \) with spherically symmetric potential. The corresponding Schrödinger operator \( H(\mathbf{k}) \) this system has four invariant subspaces \( L(1), L(2), \) \( L(3) \) and \( L(123) \). The Hamiltonian of this system has a unique bound state over each invariant subspaces \( L(1) \) (see Theorem 1) and \( L(2), L(3) \) (see Theorem 2). The corresponding energy values of these bound states are calculated exactly. For small \( \beta \) the operator \( H(\mathbf{k}_\beta), \mathbf{k}_\beta = (\pi - 2\beta, \pi, \pi) \) has 10(ten)
eigenfunctions in the invariant subspace \( L(123) \), i.e., operator \( H_{123}(k, \beta) = H(k, \beta)|_{L(123)} \) has 10 (ten) eigenvalues taking into account the multiplicity. One of them is triple eigenvalue \( z_2^{(7)}(\beta) = z_2^{(8)}(\beta) = z_2^{(9)}(\beta) \) (see Theorem 6) is calculated exactly, the remaining seven (three nondegenerate eigenvalues \( z_0(\beta), z_2^{(1)}(\beta), z_2^{(4)}(\beta) \), two double eigenvalues \( z_1^{(2)}(\beta) = z_1^{(3)}(\beta), z_2^{(5)}(\beta) = z_2^{(6)}(\beta) \) (see Theorem 5)) are calculated with accuracy of \( \beta^2 \).

The equation \( \hat{H}(k, \beta)f = zf \) with a small parameter \( \beta \) is solved using three different methods. The first method is the perturbation theory method developed by Rellich and Kato. The second method belongs to Fredholm and is called the Fredholm determinant Friedrichs operator. The third method is Birman-Schwinger principle.

2. Description of the two-particle Hamiltonian. The free Hamiltonian \( \hat{H}_0 \) of a system of two bosons with unit mass on the three-dimensional lattice \( \mathbb{Z}^3 \) usually corresponds to a bounded self-adjoint operator acting in the Hilbert space \( \ell^2_{sym}(\mathbb{Z}^3 \times \mathbb{Z}^3) := \{ f \in \ell_2(\mathbb{Z}^3 \times \mathbb{Z}^3) : f(x, y) = f(y, x) \} \)

by the formula

\[
\hat{H}_0 = -\frac{1}{2} \Delta_1 - \frac{1}{2} \Delta_2.
\]

Here \( \Delta_1 = \Delta \otimes I \) and \( \Delta_2 = I \otimes \Delta \), where \( I \) is the identity operator, the Lattice Laplacian \( \Delta \) is the difference operator describing the transfer of a particle from a site to a neighboring node, i.e.

\[
(\Delta \hat{\psi})(x) = \sum_{j=1}^{3} [\hat{\psi}(x + e_j) + \hat{\psi}(x - e_j) - 2\hat{\psi}(x)], \quad \hat{\psi} \in \ell_2(\mathbb{Z}^3),
\]

where \( e_1 = (1, 0, 0) \), \( e_2 = (0, 1, 0) \), \( e_3 = (0, 0, 1) \) are unit vectors in \( \mathbb{Z}^3 \). The total Hamiltonian \( \hat{H} \) acts in the Hilbert space \( \ell^2_{sym}(\mathbb{Z}^3 \times \mathbb{Z}^3) \) and consists of the difference of the free Hamiltonian \( \hat{H}_0 \) and the interaction potential \( \hat{V} \) two particles ([14], [10])

\[
\hat{H} = \hat{H}_0 - \hat{V},
\]

where

\[
(\hat{V}\hat{\psi})(x, y) = \hat{v}(x - y)\hat{\psi}(x, y), \quad \hat{\psi} \in \ell^2_{sym}(\mathbb{Z}^3 \times \mathbb{Z}^3).
\]

Hereafter, we assume that

\[
\hat{v}(x) = \begin{cases} \pi(|x|), & |x| \leq 2, \\ 0, & |x| \geq 3, \end{cases}
\]

(1)

where \( |x| = |x_1| + |x_2| + |x_3|, \quad x = (x_1, x_2, x_3) \in \mathbb{Z}^3 \). Additionally, we will assume that \( \hat{v}(0) > \hat{v}(1) > \hat{v}(2) > 0 \). Under condition (1), the Hamiltonian \( \hat{H} \) is a bounded self-adjoint operator in \( \ell^2_{sym}(\mathbb{Z}^3 \times \mathbb{Z}^3) \).

We pass to the momentum representation using the Fourier transform

\[
F : \ell^2_{sym}(\mathbb{Z}^3 \times \mathbb{Z}^3) \to L^2_{sym}(\mathbb{T}^3 \times \mathbb{T}^3).
\]

The Hamiltonian \( H = H_0 - V = F \hat{H}F^{-1} \) in the momentum representation commutes with the unitary operators \( U_s, s \in \mathbb{Z}^3 \), given by

\[
(U_s f)(k_1, k_2) = e^{-i(s, k_1 + k_2)} f(k_1, k_2), f \in L^2_{sym}(\mathbb{T}^3 \times \mathbb{T}^3).
\]

It follows that there exist decompositions of \( L^2_{sym}(\mathbb{T}^3 \times \mathbb{T}^3) \) and the operators \( U_s \) and \( H \) into direct integrals [15]

\[
L^2_{sym}(\mathbb{T}^3 \times \mathbb{T}^3) = \int_{\mathbb{T}^3} \oplus L^2(E_k) d k, \quad U_s = \int_{\mathbb{T}^3} U_s(k) d k, \quad H = \int_{\mathbb{T}^3} \tilde{H}(k) d k.
\]
Here,

$$F_k = \{(k_1, k_2) \in T^3 \times T^3 : k_1 + k_2 = k\}, k \in T^3,$$

and $U^s(k)$ is an operator of multiplication by the function $e^{-i(s, k)}$ in $L_2(F_k)$. The fiber $H(k)$ of $H$ also acts in $L_2(F_k)$ and is unitarily equivalent to $H(k) := H_0(k) - V$, which is called Schrödinger operator. This operator acts in the Hilbert space $L^2_s(T^3) := \{f \in L_2(T^3) : f(-q) = f(q)\}$ by the formula

$$(H(k)f)(q) = \varepsilon_k(q)f(q) - \frac{1}{(2\pi)^2} \int_{T^3} v(q - s)f(s)ds.$$  \hfill (2)

The unperturbed operator $H_0(k)$ is an operator of multiplication by the function

$$\varepsilon_k(q) = \varepsilon\left(\frac{k}{2} + q\right) + \varepsilon\left(\frac{k}{2} - q\right) = 6 - 2\cos \frac{k_1}{2} \cos q_1 - 2\cos \frac{k_2}{2} \cos q_2 - 2\cos \frac{k_3}{2} \cos q_3.$$  \hfill (3)

The perturbation operator $V$ is an integral operator in $L^2_s(T^3)$:

$$(Vf)(q) = \frac{1}{(2\pi)^{3/2}} \int_{T^3} v(q - s)f(s)ds =$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{T^3} \left[\bar{\nu}(0) + 2\bar{\nu}(1) \sum_{j=1}^{3} \cos(q_j - s_j) + 2\bar{\nu}(2) \sum_{j=1}^{3} \cos 2(q_j - s_j) + 4\bar{\nu}(2) \sum_{1 \leq i, j \leq 3} \cos q_i \cos q_j \cos s_i \cos s_j + \sin q_i \sin q_j \sin s_i \sin s_j \right]f(s)ds.$$  \hfill (4)

3. The spectrum and invariant subspaces of the operator $H(k)$. We note that the spectra of the operators $H_0(k)$ and $V$ are known. The operator $H_0(k)$ does not have eigenvalues, its spectrum is continuous and coincides with the range of the function $\varepsilon_k$:

$$\sigma(H_0(k)) = [m(k), M(k)],$$

where

$$m(k) = \min_{q \in T^3} \varepsilon_k(q), \quad M(k) = \max_{q \in T^3} \varepsilon_k(q).$$

The spectrum of $V$ is the set $\{0, \bar{\nu}(0), \bar{\nu}(1), \bar{\nu}(2)\}$ are eigenvalues of $V$. The operator $V$ is a finite range ($\dim \text{Im} V = 13$) and is hence compact. By the Weyl theorem, the essential spectrum of $H(k)$ coincides with the spectrum of $H_0(k)$:

$$\sigma_{ess}(H(k)) = [m(k), M(k)].$$

Let $L^2_s(T)$ be a subspace consists of all odd functions on $T = [-\pi, \pi]$, by $L^2_s(T)$ subspace consists of even functions on $T$.

The system of functions

$$\psi_n^-(p) = \frac{\sin np}{\sqrt{\pi}}, \quad n \in \mathbb{N},$$

forms an orthonormal basis in $L^2_s(T)$, and the system of functions

$$\psi_0^+(p) = \frac{1}{\sqrt{2\pi}}, \quad \psi_n^+(p) = \frac{\cos np}{\sqrt{\pi}}, \quad n \in \mathbb{N},$$

is an orthonormal basis in the space $L^2_s(T)$. 

3
For any self-adjoint operator $B$ acting in a Hilbert space $\mathcal{H}$ and not having an essential spectrum to the right of the point $\mu \in \mathbb{R}$, we denote by $n(\mu, B)$ number of eigenvalues of the operator $B$, lying to the right of $\mu$. We let $N(z, H(k))$ denote the number of eigenvalues of $H(k)$ to the left of $z \leq m(k)$, i.e., $N(z, H(k)) = n(-z, -H(k))$. It follows from the self-adjointness of the operator $H(k) = H_0(k) - V$ and positivity $V$ that
\[
\sigma(H(k)) \cap (M(k), \infty) = \emptyset
\]
and hence $\sigma_{diss}(H(k)) \subset (-\infty, m(k))$. So we are looking for the eigenvalue of $z$ only $z < m(k)$. The number $N(m(k), H(k))$ actually coincides with the number of eigenvalues outside the essential spectrum of the operator $H(k)$.

For any $k \in T^3$ and $z < m(k)$ we define an integral operator
\[
G(k, z) = V^{\frac{1}{2}} r_0(k, z) V^{\frac{1}{2}},
\]
where $r_0(k, z)$ is the resolvent of the unperturbed operator $H_0(k)$. Under condition (1) the operator $V$ is positive, $V^{\frac{1}{2}}$ denotes the positive square root of the positive operator $V$.

The solution $f$ of the Schrödinger equation
\[
H(k)f = zf
\]
and the fixed points $\varphi$ of the operator $G(k, z)$ are connected by the relations
\[
f = r_0(k, z) V^{\frac{1}{2}} \varphi, \quad \varphi = V^{\frac{1}{2}} f.
\]

The Birman-Schwinger principle also holds [10].

**Lemma 1.** The number of eigenvalues of the operator $H(k)$ lying below $z(z < m(k))$ coincides with the number of eigenvalues of $G(k, z)$ greater than unity, i.e. the equality holds
\[
N(z, H(k)) = n(1, G(k, z)).
\]

**Lemma 2.** If the limit operator $\lim_{z \to m(k) - 0} G(k, z) = G(k, m(k))$ exists and is compact, then the equality
\[
N(m(k), H(k)) = n(1, G(k, m(k)))
\]
holds.

If the potential $\hat{v}$ has form (1) and $k = k_{\pi} = (\pi, \pi, \pi)$ then the spectrum of $H(k_{\pi}) = 6I - V$ consists only of eigenvalues $6 - \hat{v}(0)$, $6 - \hat{v}(1), 6 - \hat{v}(2)$, and the essential spectrum [6]. Moreover, $6 - \hat{v}(0)$ is the nondegenerate eigenvalue of the operator $H(k_{\pi})$ with the normalized eigenfunction $\psi^+_0(p) = \psi^+_0(p_1)\psi^+_0(p_2)\psi^+_0(p_3)$. The number $6 - \hat{v}(1)$ is the triple eigenvalue with the corresponding normalized eigenfunctions
\[
\psi^{(1)}_1(p) = \psi^+_1(p_1)\psi^+_1(p_2)\psi^+_1(p_3), \quad \psi^{(2)}_1(p) = \psi^+_1(p_1)\psi^+_1(p_2)\psi^-_1(p_3), \quad \psi^{(3)}_1(p) = \psi^+_1(p_1)\psi^-_1(p_2)\psi^+_1(p_3).
\]

$6 - \hat{v}(2)$ is the ninefold eigenvalue with the corresponding normalized eigenfunctions
\[
\psi^{(1)}_2(p) = \psi^+_2(p_1)\psi^+_0(p_2)\psi^+_1(p_3), \quad \psi^{(2)}_2(p) = \psi^+_2(p_1)\psi^+_2(p_2)\psi^+_1(p_3), \quad \psi^{(3)}_2(p) = \psi^+_2(p_1)\psi^+_0(p_2)\psi^+_1(p_3), \quad \psi^{(4)}_2(p) = \psi^+_2(p_1)\psi^-_0(p_2)\psi^+_1(p_3), \quad \psi^{(5)}_2(p) = \psi^+_2(p_1)\psi^+_1(p_2)\psi^-_1(p_3), \quad \psi^{(6)}_2(p) = \psi^-_2(p_1)\psi^+_0(p_2)\psi^-_1(p_3),
\]

$\psi^{(7)}_2(p) = \psi^-_2(p_1)\psi^-_1(p_2)\psi^+_1(p_3), \quad \psi^{(8)}_2(p) = \psi^+_2(p_1)\psi^-_1(p_2)\psi^-_1(p_3), \quad \psi^{(9)}_2(p) = \psi^-_2(p_1)\psi^-_0(p_2)\psi^-_1(p_3).$
The potential \( \tilde{v} \) is spherically symmetric, because its Fourier transform

\[
v(p_1, p_2, p_3) = (F \tilde{v})(p_1, p_2, p_3)
\]

are even for all the arguments \( p_1, p_2, p_3 \in [-\pi, \pi] \). The function \( \varepsilon_k \) also has this property (see (3)). Because the subspace

\[
L^{++}(T^3) = L^0_2(T) \otimes L^0_2(T) \otimes L^0_2(T)
\]

is invariant under the operator \( H(k) \).

**Lemma 3.** The subspace \( L^{++}(T^3) = L(123) \) is invariant under action \( H(k) \).

The space of even functions \( L^0_2(T^3) \) can be represented as a direct sum

\[
L^0_2(T^3) = L(123) \oplus L(1) \oplus L(2) \oplus L(3).
\]

The subspaces \( L(1), L(2), L(3) \) can be represented as a tensor product

\[
L(1) := L^0_2(T) \otimes L^0_2(T) \otimes L^0_2(T), \quad L(2) := L^0_2(T) \otimes L^0_2(T) \otimes L^0_2(T),
\]

\[
L(3) := L^0_2(T) \otimes L^0_2(T) \otimes L^0_2(T).
\]

**Lemma 4.** The subspaces \( L(1), L(2) \) and \( L(3) \) are invariant under the operator \( H(k) \).

**Proof.** Here we show that the subspace \( L(1) \) is invariant under the operator \( H(k) \). The invariance of the subspaces \( L(2) \) and \( L(3) \) under the operator \( H(k) \) is proved similarly.

We prove that the subspace \( L(1) \) is invariant first with respect to \( H_0(k) \) and then with respect to \( V \). It follows from representation (3) that the function \( \varepsilon_k \) belongs to the subspace \( L(123) \), and it follows from the inclusion \( f \in L(1) \) that \( \varepsilon_k f \in L(1) \).

Simple calculations show that the function \( (V f)(p) \) is equal to

\[
(V f)(p) = \frac{\bar{v}^2(2)}{2\pi^3} \sin p_2 \sin p_3 \sqrt{\pi} \sin q_2 \sin q_3 f(q) dq = C \sin p_2 \sin p_3
\]

for \( f \in L(1) \) which belongs to the subspace \( L(1) = L^0_2(T) \otimes L^0_2(T) \otimes L^0_2(T) \). Hence, we prove the invariance of \( L(1) \) with respect to \( V \), and it follows that \( L(1) \) is invariant with respect to \( H(k) = H_0(k) + V \). The lemma is proved.

We denote by \( H_{\alpha}(k) = H_0(k) - \bar{v}(2) V_\alpha \), \( \alpha = 1, 2, 3 \) the restriction of the operator \( H(k) \) in the invariant subspace \( L(\alpha) \), that is \( H_{\alpha}(k) := H(k)|_{L(\alpha)}, \alpha = 1, 2, 3 \).

The restriction \( H_0(\alpha)(k) := H_0(k) \) of the unperturbed operator \( H_0(k) \) is the multiplication by the function \( \varepsilon_k \). Now let us see how the restriction \( V_\alpha \) of the operator \( V \) acts in the invariant subspace \( L(\alpha) \), \( \alpha = 1, 2, 3 \):

\[
(V_{\alpha} f)(p) = \frac{1}{2\pi^3} \sqrt{\pi} \sin p_\beta \sin s_\beta \sin p_\gamma \sin s_\gamma f(s) ds, \{\alpha, \beta, \gamma\} = \{1, 2, 3\}.
\]

Next we study eigenvalues and eigenfunctions of the operator \( H(k_\beta) \), \( k_\beta = (\pi - 2\beta, \pi, \pi) \).

**Theorem 1.** For each \( \beta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) the operator \( H_1(k_\beta) \) has a unique nondegenerate eigenvalue \( z^{(1)}_2(\beta) = 6 - \sqrt{\tilde{v}^2(2) + 4\sin^2 \beta} \) corresponding to the eigenfunction

\[
f^{(1)}_2(p) = \frac{C \sin p_2 \sin p_3}{6 - 2\sin \beta \cos p_1 - z^{(1)}_2(\beta)} \in L(1) = L^0_2(T) \otimes L^0_2(T) \otimes L^0_2(T).
\]
**Proof.** We consider the equation \( H_1(k_\beta)f = zf \) for the eigenvalues. This equation is equivalent to

\[
(6 - 2 \sin \beta \cos p_1 - z)f(p) = \frac{\bar{v}(2)}{2\pi^3} \int_{T^3} \sin p_2 \sin p_3 \sin q_2 \sin q_3 f(q) d\mathbf{q}.
\]

Using the notation

\[
C_f = \frac{1}{2\pi^3} \int_{T^3} \sin q_2 \sin q_3 f(q) d\mathbf{q},
\]

we obtain the expression

\[
f(p) = \frac{\bar{v}(2)}{6 - 2 \sin \beta \cos p_1 - z} C_f \sin q_2 \sin q_3 d\mathbf{q},
\]

for the eigenfunction \( f(p) \). Substituting it in (9), we derive the homogeneous equation for the coefficient \( C_f \):

\[
C_f = C_f \frac{\bar{v}(2)}{2\pi^3} \int_{T^3} \frac{\sin^2 q_2 \sin^2 q_3 dq_1 dq_2 dq_3}{6 - 2 \sin \beta \cos q_1 - z}
\]

or

\[
C_f \left(1 - \frac{\bar{v}(2)}{2\pi} \int_{T} \frac{dq_1}{6 - 2 \sin \beta \cos q_1 - z} \cdot \frac{1}{\pi} \int_{T} \sin^2 q_2 dq_2 \cdot \frac{1}{\pi} \int_{T} \sin^2 q_3 dq_3 \right) = 0.
\]

All integrals in parentheses are explicit:

\[
\frac{1}{2\pi} \int_{T} \frac{dq_1}{6 - 2 \sin \beta \cos q_1 - z} = \frac{1}{\sqrt{(z - 6)^2 - 4 \sin^2 \beta}},
\]

\[
\frac{1}{\pi} \int_{T} \sin^2 q_2 dq_2 = \frac{1}{\pi} \int_{T} \sin^2 q_3 dq_3 = 1.
\]

Using these equalities, (11) looks like this:

\[
C_f \left(1 - \frac{\bar{v}(2)}{\sqrt{(z - 6)^2 - 4 \sin^2 \beta}} \right) = 0.
\]

From the zero of product \( C_f = 0 \) or

\[
\Delta_1(\beta; z) = 1 - \frac{\bar{v}(2)}{\sqrt{(z - 6)^2 - 4 \sin^2 \beta}} = 0
\]

we come to terms. Condition \( C_f = 0 \) contradicts that \( f \) is an eigenfunction, so we come to \( \Delta_1(\beta; z) = 0 \). We solve equation (13) with respect to \( z \) \( z < 6 - 2 \sin \beta \) get the following result

\[
z_2^{(1)}(\beta) = 6 - \sqrt{v^2(2) + 4 \sin^2 \beta}.\]

Thus the number \( z_2^{(1)}(\beta) \) found will be the eigenvalue of the operator \( H_3(k_\beta) \). Putting this eigenvalue in (10) takes the following (8) for the eigenfunction.

We introduce unitary operator \( U_{ij} : L(j) \rightarrow L(i) \), that changes the arguments \( p_i \) and \( p_j \). For example, \( U_{23} : L(3) \rightarrow L(2) \). \((U_{23}f)(p_1, p_2, p_3) = f(p_1, p_3, p_2)\).

The operators \( H_2(k_\beta) \) and \( H_3(k_\beta) \) are unitary equivalent, i.e.,

\[
H_2(k_\beta) = U_{23} H_3(k_\beta) U_{23}^{-1}.
\]
Therefore we provide all confirmations for the operator $H_2(k_\beta)$.

**Theorem 2.** a) If $\bar{v}(2) > \sin \beta$, then the operator $H_2(k_\beta)$ has unique simple eigenvalue

$$z_2^{(2)}(\beta) = 6 - \bar{v}(2) - \frac{1}{\bar{v}(2)} \sin^2 \beta,$$

and corresponding eigenfunction is

$$f_2^{(2)}(p) = \frac{C \sin p_1 \sin p_3}{6 - 2 \sin \beta \cos p_1 - z_2^{(2)}(\beta)} \in L(2) = L_2^0(T) \otimes L_2^0(T) \otimes L_2^0(T).$$

b) If $\bar{v}(2) < \sin \beta$, then the operator $H_2(k_\beta)$ has no eigenvalues outside the essential spectrum.

c) If $\bar{v}(2) = \sin \beta$, then the left edge $m(k_\beta) := 6 - 2 \sin \beta$ the essential spectrum of the operator $H_2(k_\beta)$ is resonance.

**Proof.** Equation $H_2(k_\beta) f = z f$ for the eigenfunctions have a non-zero solution if and only if the number $z < 6 - 2 \sin \beta$ be a solution of the equation

$$\Delta_2(\beta, z) := 1 - \frac{v(2)}{2} \int_0^{\pi} \frac{\sin^2 q_1 dq_1}{6 - 2 \sin \beta \cos q_1 - z} = 0. \quad (14)$$

We use the following identity $\sin^2 q = -(\cos^2 q - 1)$ and the formula for division with a remainder

$$\frac{\cos^2 q_1 - 1}{6 - 2 \sin \beta \cos q_1 - z} = -\frac{\cos q_1}{2 \sin \beta} - \frac{6 - z}{4 \sin^2 \beta} + \left[\frac{(6 - z)^2}{4 \sin^2 \beta} - 1\right] \frac{1}{6 - 2 \sin \beta \cos q_1 - z}$$

we write the formula (14) in the following

$$1 - \frac{\bar{v}(2)(6 - z)}{2 \sin^2 \beta} + \frac{(6 - z)^2}{4 \sin^2 \beta - 1} \left[\frac{2\bar{v}(2)}{6 - z - 4 \sin^2 \beta - 1}\right] \int_{-\pi}^{\pi} \frac{dq_1}{6 - 2 \sin \beta \cos q_1 - z} = 0. \quad (15)$$

The following expression is known to be appropriate in all $z < m(k_\beta)$. Using equality (12) we write equation (15) in the form

$$1 - \frac{\bar{v}(2)(6 - z)}{2 \sin^2 \beta} + \frac{(6 - z)^2}{4 \sin^2 \beta - 1} \left[\frac{2\bar{v}(2)}{\sqrt{(6 - z)^2 - 4 \sin^2 \beta}}\right] = 0. \quad (16)$$

If $\bar{v}(2) > \sin \beta$ the equation (16) has unique solution

$$z = 6 - \bar{v}(2) - \frac{1}{\bar{v}(2)} \sin^2 \beta.$$

The in turn proves the a) confirmation of the theorem.

Let $\bar{v}(2) < \sin \beta$. We show that the operator $H(k_\beta)$ has no eigenvalues outside of essential spectrum. The operator $H(k_\beta)$ does not have eigenvalues lying to the right on the essential spectrum (see (7)). Therefore we show that $H(k_\beta)$ has no eigenvalues lying to the left of $m(k_\beta)$. We show that the Birman-Schwinger operator $G(k_\beta, z)$ corresponding to the operator $H(k_\beta)$ has no eigenvalues greater than 1 at $z = m(k_\beta)$. As the operator $V_2$ is projector, the Birman-Schwinger $G(k_\beta, z)$ operator take the form:

$$G(k_\beta, z) = \bar{v}(2) V_2 r_0 (k_\beta, z) V_2, \quad z < m(k_\beta).$$
Since the rank of the operator $V_2$ is 1, the rank of the operator $G(k_\beta, m(k_\beta))$ is also 1. We see its effect on element $f$:

$$
(G(k_\beta, m(k_\beta))f)(p) = \frac{\bar{v}(2)}{\sin \beta}(V_2f)(p). 
$$

(17)

From the representation (17) we find that generator $G(k_\beta, m(k_\beta))$ has a unique non-zero simple eigenvalue $\frac{\bar{v}(2)}{\sin \beta}$.

Since $\bar{v}(2) < \sin \beta$ the operator $G(k_\beta, m(k_\beta))$ has no eigenvalues greater than 1 and according to Lemma 2, the operator $H_2(k_\beta)$ has no eigenvalues below $m(k_\beta)$. Part b) of the theorem was proved.

Let $\bar{v}(2) = \sin \beta$, then the Schrödinger equation $H_2(k_\beta)f = m(k_\beta)f$ has a non-zero solution

$$
f(p) = \frac{\sin p_1 \sin p_3}{1 - \cos p_1},
$$

but this solution does not belong in the space $L(2)$. This proves part c) of the theorem.

We studied the eigenvalues and eigenvectors of the operator $H(k_\beta)$ in the subspaces $L(\alpha), \alpha = 1, 2, 3$. Now we study the eigenvalues and eigenvectors of the operator $H(k_\beta)$ in $L(123)$.

We denote by $H_{123}(k)$ the restriction of the operator $H(k)$ in the invariant subspace $L(123)$. The restriction $H_{0123}(k) := H_0(k)$ of the unperturbed operator $H_0(k)$ unchanged and coincides with $H_0(k)$. The restriction $V_{123} = V|_{L(123)}$ of the operator $V$ acts on the element $f \in L(123)$ according to the formula

$$
(V_{123}f)(p) = \frac{1}{(2\pi)^3} \int_\mathbb{T}^3 \left[ \bar{v}(0) + 2\bar{v}(1) \sum_{j=1}^{3} \cos p_j \cos q_j + 2\bar{v}(2) \sum_{j=1}^{3} \cos 2p_j \cos 2q_j + 
\right.
$$

$$
+ 4\bar{v}(2) \sum_{1 \leq i < j \leq 3} \cos p_i \cos p_j \cos q_i \cos q_j \right] f(q)dq.
$$

We use the orthonormal basis in the space $L_2^+(T)$. Let $L^+(n)$ be a one-dimensional subspace spanned by the vector $\psi_n^+, n \in \mathbb{Z}_+: = \{0\} \cup \mathbb{N}$. The space $L_2^+(T)$ can be decomposed into the direct sum

$$
L_2^+(T) = \sum_{n=0}^{\infty} \oplus L^+(n)
$$

This decomposition produce another decomposition

$$
L_2^+(T) \otimes L_2^+(T) \otimes L_2^+(T) = L_2^+(T) \otimes \sum_{n=0}^{\infty} \oplus L^+(n) \otimes \sum_{m=0}^{\infty} \oplus L^+(m) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \oplus R_{123}^+(n, m),
$$

where $R_{123}^+(n, m) := L_2^+(T) \otimes L^+(n) \otimes L^+(m)$.

**Lemma 5.** For any $n, m \in \mathbb{Z}_+$ the subspace $R_{123}^+(n, m)$ is invariant under action $H(k_1, \pi, \pi)$.

Proof. We show the proof of this lemma for the space $R_{123}^+(0, 0)$. This lemma is proved similar for the remaining spaces $R_{123}^+(n, m)$.

Let

$$
(f\psi_0^+(p_1, p_2, p_3), p_1, p_2, p_3) := \psi_0^+(p_2)\psi_0^+(p_3), \quad f \in L_2^+(T), \quad \psi_0^+ \in L^+(0)
$$

is an arbitrary element of $R_{123}^+(0, 0)$. We consider the action of $H(k_1, \pi, \pi) = H_0(k_1, \pi, \pi) - V_{123}$ on $f\psi_0^+(p_1, p_2, p_3)$:

$$
(H_0(k_1, \pi, \pi)f\psi_0^+(p_1, p_2, p_3))(p_1, p_2, p_3) = \left[(6 - 2\cos \frac{k_1}{2} \cos p_1)f(p_1)\right]\psi_0^+(p_2)\psi_0^+(p_3),
$$

8
(V_{123} f \psi_n^+ \psi_0^+ (p_1, p_2, p_3) = \left[ \frac{1}{2\pi} \int_{\mathbb{T}} \{ \bar{\psi}(0) + 2 \bar{\psi}(1) \cos p_1 \cos q_1 + + 2 \bar{\psi}(2) \cos 2 p_1 \cos 2 q_1 \} f(q_1) dq_1 \right] \psi_0^+(p_2) \psi_0^+(p_3).

(18)

In obtaining equality (18), we used the orthogonality of the system \{ \psi_n^+ \}_{n=0}^\infty. Thus we have

\( \langle H_{123}(k_1, \pi, \pi) f \psi_n^+ \psi_0^+ \rangle = \left[ 6 - 2 \cos k_1 \right] \cos p_1 f(p_1) - \frac{1}{2\pi} \int_{\mathbb{T}} \{ \bar{\psi}(0) + 2 \bar{\psi}(1) \cos p_1 \cos q_1 + 2 \bar{\psi}(2) \cos 2 p_1 \cos 2 q_1 \} f(q_1) dq_1 \psi_0^+(p_2) \psi_0^+(p_3) \in \mathcal{R}_{123}^+(0,0), \)

The proof of the lemma is complete.

It follows from (19)

\[ H_{123}(k_1, \pi, \pi)|_{\mathcal{R}_{123}^+(0,0)} = H_{123}^{(0,0)}(k_1) \otimes I_0 \otimes I_0, \]

where \( I_0 \) is identity operator in \( L^+(0), H_{123}^{(0,0)}(k_1) \) := \( H_0(k_1) - V_{123}^{(0,0)} \) is the two-particle Schrödinger operator acts in the Hilbert space \( L_2^+(\mathbb{T}) \):

\[ (H_0(k_1) f)(p) = (6 - 2 \cos k_1) \cos p) f(p), \]

\[ (V_{123}^{(0,0)} f)(p) = \frac{1}{2\pi} \int_{\mathbb{T}} \{ \bar{\psi}(0) + 2 \bar{\psi}(1) \cos p \cos q + 2 \bar{\psi}(2) \cos 2 p \cos 2 q \} f(q) dq. \]

(21)

A similar form can be obtained for the other \( n \) and \( m \). The restriction \( H_{123}^{(n,m)}(k_1, \pi, \pi) = H_{123}^{(n,m)}(k_1, \pi, \pi)|_{\mathcal{R}_{123}^+(n,m)} \) has the form

\[ H_{123}^{(n,m)}(k_1, \pi, \pi) = H_{123}^{(n,m)}(k_1) \otimes I_n \otimes I_m, \]

where \( I_n \) is the identity operator in \( L^+(n), H_{123}^{(n,m)}(k_1) := H_0(k_1) - V_{123}^{(n,m)} \) is the two-particle Schrödinger operator acts in the Hilbert space \( L_2^+(\mathbb{T}) \). \( H_0(k_1) \) is the operator of multiplication by the function \( 6 - \cos k_1 \cos p \), where \( (V_{123} f)(p) \equiv 0 \) for \( f \in \mathcal{R}_{123}^+(m,n), n + m \geq 3 \). Therefore we only list the restrictions of the operator \( V_{123} \) in \( \mathcal{R}_{123}^+(0,1), \mathcal{R}_{123}^+(1,0), \mathcal{R}_{123}^+(2,0), \mathcal{R}_{123}^+(0,2) \) and \( \mathcal{R}_{123}^+(1,1) \).

Simple calculations show that

\[ (V_{123}^{(1,0)} f)(p) = (V_{123}^{(0,1)} f)(p) = \frac{1}{2\pi} \int_{\mathbb{T}} \{ \bar{\psi}(1) + 2 \bar{\psi}(2) \cos p \cos q \} f(q) dq, \]

\[ (V_{123}^{(1,1)} f)(p) = (V_{123}^{(2,0)} f)(p) = (V_{123}^{(0,2)} f)(p) = \frac{1}{2\pi} \int_{\mathbb{T}} f(q) dq, \]

These imply the following relations

\[ H_{123}^{(1,0)}(k_1) = H_{123}^{(0,1)}(k_1), \quad H_{123}^{(2,0)}(k_1) = H_{123}^{(0,2)}(k_1) = H_{123}^{(1,1)}(k_1). \]

Note that the operators \( H_{123}^{(0,0)}(\pi - 2 \beta), H_{123}^{(1,0)}(\pi - 2 \beta), H_{123}^{(0,1)}(\pi - 2 \beta), H_{123}^{(1,1)}(\pi - 2 \beta), H_{123}^{(2,0)}(\pi - 2 \beta), H_{123}^{(0,2)}(\pi - 2 \beta) \) have only nondegenerate eigenvalues for small \( \beta > 0 \).

4. **Perturbations of a nondegenerate eigenvalue.** Our main goal is to study the behavior of the nondegenerate (simple) eigenvalues of \( H_{123}(k_\beta) \).
Theorem 3 (the Kato-Rellich theorem)[15]. Let \( z_0 \) be an isolated nondegenerate eigenvalue of \( A \), and \( B \) be a bounded operator. Then for small \( \beta \in \mathbb{R} \), the operator \( T(\beta) = A - \beta B \) has unique nondegenerate eigenvalue \( z_0(\beta) \) in some neighborhood \( \cup_\delta(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \delta \} \) of \( z_0 \). Moreover, \( z_0(\beta) \) is analytic in \( \beta \) in a neighborhood of \( \beta = 0 \).

In addition, we have the representation for \( z_0(\beta) \)

\[
z_0(\beta) = z_0 - \sum_{n=1}^{\infty} a_n \beta^n = z_0 - \sum_{n=1}^{\infty} c_n \beta^n. \tag{22}
\]

If \( A \) and \( B \) are self-adjoint, then the Rayleigh-Schrödinger coefficients \( a_n \) and \( b_n \) are defined by

\[
a_n = -\frac{1}{2\pi i} \int_{|z-z_0| = \varepsilon} (\psi_0^+, [B(A - zI)^{-1}]^n \psi_0^+) dz, \tag{23}
\]

\[
b_n = -\frac{1}{2\pi i} \int_{|z-z_0| = \varepsilon} (\psi_0^+, (A - zI)^{-1} [B(A - zI)^{-1}]^n \psi_0^+) dz. \tag{24}
\]

Here, \( \varepsilon > 0 \) is chosen such that there is exactly one simple eigenvalue of \( T(\beta) \) inside the circle \( |z - z_0| < \varepsilon \), and corresponding to the eigenvalue \( z_0 \) [15]. The series \( \sum_{n=1}^{\infty} a_n \beta^n \) in (22) is the Rayleigh-Schrödinger series for \( z_0(\beta) \), and its convergence radius is greater than zero.

Denote by

\[
(W_1 f)(p) = 2 \cos pf(p), \quad f \in L^2_2(\mathbf{T}), \tag{25}
\]

and using equality (21), we write \( H_{123}^{(0,0)}(k_1) \) in the form

\[
H_{123}^{(0,0)}(k_1) = H_{123}^{(0,0)}(\pi) - \cos \frac{k_1}{2} W_1. \tag{26}
\]

In particular, using the identity \( \cos\left(\frac{\pi}{2} - \beta\right) = \sin \beta \), we obtain

\[
H_{123}^{(0,0)}(\pi - 2\beta) = H_{123}^{(0,0)}(\pi) - \sin \beta W_1. \tag{26}
\]

One can easily see that the operator \( H_{123}^{(0,0)}(\pi) \) has three nondegenerate eigenvalues \( z_0 = 6 - \upsilon(0), z_1 = 6 - \upsilon(1) \) and \( z_2 = 6 - \upsilon(2) \) with the corresponding eigenfunctions \( \psi_0^+, \psi_1^+ \) and \( \psi_2^+ \). Therefore, we can apply the Kato-Rellich theorem to the operator \( H_{123}^{(0,0)}(\pi - 2\beta) \).

Theorem 4. There exists \( \delta > 0 \) such that for each \( \beta \in (0, \delta) \), the operator \( H_{123}^{(0,0)}(\pi - 2\beta) \) has three nondegenerate eigenvalues \( z_0(\beta), z_1^{(1)}(\beta) \) and \( z_2^{(4)}(\beta) \) in a small neighborhood of \( z_0 = 6 - \upsilon(0), z_1 = 6 - \upsilon(1) \) and \( z_2 = 6 - \upsilon(2) \) respectively. Moreover, we have the asymptotic formulas

\[
z_0(\beta) = 6 - \upsilon(0) - \frac{2}{\upsilon(0) - \upsilon(1)} \beta^2 + O(\beta^4), \quad \beta \to 0,
\]

\[
z_1^{(1)}(\beta) = 6 - \upsilon(1) - \frac{\upsilon(0) + 2\upsilon(2) - 3\upsilon(1)}{(\upsilon(0) - \upsilon(1))(\upsilon(1) - \upsilon(2))} \beta^2 + O(\beta^4), \quad \beta \to 0,
\]

\[
z_2^{(4)}(\beta) = 6 - \upsilon(2) - \frac{\upsilon(1) - 2\upsilon(2)}{\upsilon(2)(\upsilon(1) - \upsilon(2))} \beta^2 + O(\beta^4), \quad \beta \to 0.
\]

To prove this theorem, we need the following lemma.

Lemma 6. For \( \psi_0^+ \) and \( \psi_m^+, m \in \mathbb{N} \), we have the equalities

\[
(H_{123}^{(0,0)}(\pi) - zI)^{-1} \psi_m^+ = \begin{cases} \frac{\psi_m^+}{6 - \upsilon(m) - z}, & m = 0, 1, \\ \frac{\psi_m^+}{6 - z}, & m \geq 2, \end{cases} \tag{27}
\]
To calculate the eigenvalue \( z \) obtain the representation

\[
W_1 \psi_m^+ = \begin{cases} 
\sqrt{2} \psi_1^+, & m = 0, \\
2 \psi_0^+ + \psi_2^+, & m = 1, \\
\psi_{m-1}^+ + \psi_{m+1}^+, & m \geq 2. 
\end{cases}
\]  

(28)

**Proof.** From the equality

\[
(H_{123}^{(0,0)}(\pi) - zI) \psi_m^+ = \begin{cases} 
(6 - \bar{v}(m) - z)\psi_m^+, & m = 0,1, \\
(6 - z)\psi_m^+, & m \geq 2,
\end{cases}
\]

we obtain relation (27). Formula (28) is a direct corollary of the definition of \( W_1 \) and the identity

\[
2 \cos p_1 \cos mp_1 = \cos(m - 1)p_1 + \cos(m + 1)p_1.
\]

**Proof of Theorem 4.** We prove theorem for the eigenvalue \( z_0(\beta) \). The theorem is similar proved for the eigenvalues \( z_1^{(1)}(\beta) \) and \( z_2^{(4)}(\beta) \). Using the Kato-Rellich theorem for \( z_0(\beta) \), we obtain the representation

\[
\begin{align*}
\psi_0 + \frac{1}{z_0 - z} \sum_{n=1}^{\infty} c_n \beta^n \\
&= d_1(z)\psi_1^+ + d_3(z)\psi_3^+ + \ldots + d_{2n-1}(z)\psi_{2n-1}^+.
\end{align*}
\]

To calculate the eigenvalue \( z_0(\beta) \) with accuracy of \( \beta^2 \), it suffices to find the coefficients \( a_1, a_2, \) and \( b_1 \) of Rayleigh-Schrödinger series (23) and (24). We consider the scalar product \( \langle \psi_0^+, [W_1(H_{123}^{(0,0)}(\pi) - zI)^{-1}n\psi_0^+ \rangle \), which we need for calculating the coefficients \( a_n \) and \( b_n \). Using formulas (27) and (28), we obtain

\[
\left[ W_1(H_{123}^{(0,0)}(\pi) - zI)^{-1} \right]^{2n-1} \psi_0^+ = d_1(z)\psi_1^+ + d_3(z)\psi_3^+ + \ldots + d_{2n-1}(z)\psi_{2n-1}^+.
\]

where the coefficients \( d_1(z), d_3(z), \ldots, d_{2n-1}(z) \) depend only on \( z \). It follows from this and the orthogonality of the system \( \{\psi_m^+\}_{m \in \mathbb{Z}^+} \) that

\[
(\psi_0^+, [W_1(H_{123}^{(0,0)}(\pi) - zI)^{-1}]^{2n-1}\psi_0^+) = 0 \quad \text{for any } n \in \mathbb{N}.
\]  

(29)

Taking Rayleigh-Schrödinger coefficients into account, we hence find that \( a_{2n-1} = 0 \) for all \( n \in \mathbb{N} \). It follows from the relation

\[
(\psi_0^+, (H_{123}^{(0,0)}(\pi) - zI)^{-1}[W_1(H_{123}^{(0,0)}(\pi) - zI)^{-1}2n-1\psi_0^+]) =
\]

\[
= (\langle [H_{123}^{(0,0)}(\pi) - zI)^{-1}\psi_0^+, [W_1(H_{123}^{(0,0)}(\pi) - zI)^{-1}2n-1\psi_0^+] \rangle =
\]

\[
= \frac{1}{z_0 - z} \sum_{n=1}^{\infty} c_n \beta^n [W_1(H_{123}^{(0,0)}(\pi) - zI)^{-1}]^{2n-1}\psi_0^+ = 0
\]

and expression (24) that \( b_{2n-1} = 0 \) for all \( n \in \mathbb{N} \).

Simple calculations show that

\[
[W_1(H_{123}^{(0,0)}(\pi) - zI)^{-1}]^{2} \psi_0^+ = \frac{2\psi_0^+ + \sqrt{2}\psi_2^+}{(6 - \bar{v}(0) - z)(6 - \bar{v}(1) - z)}.
\]

It follows that

\[
(\psi_0^+, [W_1(H_{123}^{(0,0)}(\pi) - zI)^{-1}]^{2} \psi_0^+) = \frac{2}{(z_0 - z)z_1 - z_0}.
\]

Multiplying this equality by \( (2\pi i)^{-1} \) and integrating along the circle \( |z - z_0(\pi)| = \varepsilon \), we obtain

\[
a_2 = \frac{2}{z_1 - z_0} = \frac{2}{\bar{v}(0) - \bar{v}(1)}.
\]
Because $c_1 = a_1$ and $c_2 = a_2 - a_1 b_1$, we conclude that $c_1 = 0$ and $c_2 = a_2$. Taking into account that $\sin \beta \sim \beta$ for $\beta \to 0$, we complete the proof.

Now we give the following assertion for the operator $H_{123}^{(1,0)}(\pi - 2\beta)$ since the operators $H_{123}^{(1,0)}(\pi - 2\beta)$ and $H_{123}^{(0,1)}(\pi - 2\beta)$ are equal.

**Theorem 5.** There exists $\delta > 0$ such that for any $\beta \in (0, \delta)$ the operator $H_{123}^{(1,0)}(\pi - 2\beta)$ has two different nondegenerate eigenvalues $z_1^{(2)}(\beta)$ and $z_2^{(5)}(\beta)$ with the asymptotic forms as $\beta \to 0$

\[
z_1^{(2)}(\beta) = 6 - \bar{v}(1) - \frac{2}{\bar{v}(1) - \bar{v}(2)} \beta^2 + O(\beta^4), \]

\[
z_2^{(5)}(\beta) = 6 - \bar{v}(2) - \frac{\bar{v}(1) - 3\bar{v}(2)}{\bar{v}(2)(\bar{v}(1) - \bar{v}(2))} \beta^2 + O(\beta^4). \]

**Proof.** We consider the equation $H_{123}^{(1,0)}(\pi - 2\beta)f = zf$ for the eigenvalues. This equation is equivalent to

\[
(6 - 2 \sin \beta \cos p - z)f(p) = \frac{1}{2\pi} \int_T [\bar{v}(1) + 2\bar{v}(2) \cos p \cos q]f(q)dq. \quad (30)
\]

Using notation

\[
C_1 = \frac{1}{2\pi} \int_T f(q)dq, \quad C_2 = \frac{1}{2\pi} \int_T \cos qf(q)dq, \quad (30)
\]

we obtain the expression

\[
f(p) = \frac{\bar{v}(1)C_1 + 2\bar{v}(2)C_2 \cos p}{6 - 2 \sin \beta \cos p - z} \quad (31)
\]

for the eigenfunction $f(p)$. Substituting (31) in (30), we derive the homogeneous system of equations for the coefficients $C_1$ and $C_2$

\[
C_1 = C_1 \frac{\bar{v}(1)}{2\pi} \int_T \frac{dq}{6 - 2 \sin \beta \cos q - z} + C_2 \frac{\bar{v}(2)}{\pi} \int_T \frac{\cos qdq}{6 - 2 \sin \beta \cos p - z},
\]

\[
C_2 = C_1 \frac{\bar{v}(1)}{2\pi} \int_T \frac{\cos qdq}{6 - 2 \sin \beta \cos q - z} + C_2 \frac{\bar{v}(2)}{\pi} \int_T \frac{\cos^2 qdq}{6 - 2 \sin \beta \cos q - z}.
\]

Taking into account that $\sin \beta \sim \beta$ for $\beta \to 0$, we obtain the sought assertion of theorem. This system of equation has a non-zero solution if and only if the determinant

\[
\Delta_3(\beta, z) = \left| \begin{array}{cc}
1 - \bar{v}(1)\Delta_{11}(\beta, z) & -2\bar{v}(2)\Delta_{12}(\beta, z) \\
-\bar{v}(1)\Delta_{12}(\beta, z) & 1 - 2\bar{v}(2)\Delta_{22}(\beta, z)
\end{array} \right|
\]

is equal to zero. Here,

\[
\Delta_{11}(\beta, z) = \frac{1}{2\pi} \int_T \frac{dq}{6 - 2 \sin \beta \cos q - z}, \quad \Delta_{12}(\beta, z) = \frac{1}{2\pi} \int_T \frac{\cos qdq}{6 - 2 \sin \beta \cos q - z},
\]

\[
\Delta_{22}(\beta, z) = \frac{1}{2\pi} \int_T \frac{\cos^2 qdq}{6 - 2 \sin \beta \cos q - z}. \quad (32)
\]

Now we determine the eigenvalues of the operator $H_{123}^{(1,0)}(\pi - 2\beta)$ using the indeterminate coefficient method as follows:

\[
z_1^{(2)}(\beta) = 6 - \bar{v}(1) + a \beta^2.
\]
Putting $z = z_1^{(2)}(\beta)$ in $\Delta_{11}, \Delta_{12}$ and $\Delta_{22}$ in (32) we get

$$\Delta_3(\beta, z_1^{(2)}(\beta)) = \frac{1}{\pi^2(1)} \begin{vmatrix} -(a\bar{v}(1) + 2)\beta^2 & -2\bar{v}(2)\beta \\ -\bar{v}(1)\beta & \bar{v}(1)(\bar{v}(1) - \bar{v}(2)) \end{vmatrix} + O(\beta^4) = 0.$$ 

Calculating the determinant, we find a value of

$$a = \frac{2}{\bar{v}(1) - \bar{v}(2)}.$$

So the eigenvalue $z_1^{(2)}(\beta)$ looks like this:

$$z_1^{(2)}(\beta) = 6 - \bar{v}(1) - \frac{2}{\bar{v}(1) - \bar{v}(2)}\beta^2 + O(\beta^4).$$

In the same way the following equation can be obtained for the eigenvalue $z_2^{(5)}(\beta)$:

$$z_2^{(5)}(\beta) = 6 - \bar{v}(2) - \frac{\bar{v}(1) - 3\bar{v}(2)}{\bar{v}(2)(\bar{v}(1) - \bar{v}(2))}\beta^2 + O(\beta^4).$$

The operators $H_{123}^{(1,1)}(\pi - 2\beta), H_{123}^{(2,0)}(\pi - 2\beta)$ and $H_{123}^{(0,2)}(\pi - 2\beta)$ are equal. Therefore we will formulate following statement for the operator $H_{123}^{(1,1)}(\pi - 2\beta)$.

**Theorem 6.** For each $\beta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$ the operator $H_{123}^{(1,1)}(\pi - 2\beta)$ has a unique nondegenerate eigenvalue $z_2^{(7)}(\beta) = 6 - \sqrt{\bar{v}^2(2) + 4\sin^2\beta}$ corresponding to the eigenfunction

$$f_2^{(7)}(p) = \frac{C}{6 - 2\sin\beta \cos p_1 - z_2^{(7)}(\beta)} \in L_2^e(\mathcal{T}).$$

The proof is similar as the proof of Theorem 1.

**Conclusion.** Operator $H(k_\beta)$ has a unique nondegenerate eigenvalue $z_0(\beta)$ lying in the neighborhood of $z_0 = 6 - \bar{v}(0)$ for small $\beta$. In neighborhood of $z_1 = 6 - \bar{v}(1)$ the operator $H(k_\beta)$ has three eigenvalues $z_1^{(1)}(\beta)$ (see Theorem 4.) and $z_1^{(2)}(\beta) = z_1^{(3)}(\beta)$ (see Theorem 5.), $z_1^{(1)}(\beta)$ is nondegenerate, and $z_1^{(2)}(\beta) = z_1^{(3)}(\beta)$ is the double eigenvalue. The ninefold eigenvalue of the operator $H(k_\beta)$ at small perturbations breaks down into four different eigenvalues lying in a small neighborhood of $z_2 = 6 - \bar{v}(2)$. Moreover, $z_2^{(4)}(\beta)$ (see Theorem 4) is nondegenerate, $z_2^{(5)}(\beta) = z_2^{(6)}(\beta)$ (see Theorem 5) and $z_2^{(2)}(\beta) = z_2^{(3)}(\beta)$ (see Theorem 2) are double, $z_2^{(1)}(\beta) = z_2^{(7)}(\beta) = z_2^{(8)}(\beta) = z_2^{(9)}(\beta)$ (see Theorem 1 and Theorem 6) is four fold eigenvalue of the operator $H(k_\beta)$. For any perturbation $\beta > 0$, the essential spectrum $\{6\}$ of $H(k_\beta)$, i.e., the infinite-multiplicity eigenvalue $z_{\infty}(k_\beta) = 6$, becomes the essential spectrum $\{6 - 2\sin\beta, 6 + 2\sin\beta\}$.

The eigenvalue $z_0(\beta)$ for all small $\beta$ lies to the left of the eigenvalue $z_0 = 6 - \bar{v}(0)$ of the unperturbed operator $H(k_\beta)$. Threecfold eigenvalue $z_1 = 6 - \bar{v}(1)$ of the operator $H(k_\beta)$ for small perturbations splits into two different eigenvalues $z_1^{(1)}(\beta)$ (simple) and $z_1^{(2)}(\beta) = z_1^{(3)}(\beta)$, (double). The double eigenvalue $z_1^{(2)}(\beta) = z_1^{(3)}(\beta)$ always lies to the left of the unperturbed eigenvalue $z_1$, the other $z_1^{(1)}(\beta)$ either lies to the right or lies to the left of $z_1$, depending on the sign of $\bar{v}(0) + 2\bar{v}(2) - 3\bar{v}(1)$.
Under the condition \( \bar{v}(1) < 2\bar{v}(2) \) and small \( \beta \) the relation

\[
\begin{align*}
\bar{z}_2^{(1)}(\beta) < z_2^{(2)}(\beta) < 6 - \bar{v}(2) < z_2^{(4)}(\beta) < z_2^{(5)}(\beta)
\end{align*}
\]

hold. This means that when the ninefold eigenvalue \( 6 - \bar{v}(2) \) splits into four, two of them \( z_2^{(5)}(\beta) \), \( z_2^{(4)}(\beta) \) lies to the right, the other two \( z_2^{(2)}(\beta), z_2^{(1)}(\beta) \) are to the left.

As \( \beta \) increases, some of the eigenvalues of the operator \( H(k\beta) \) disappear by absorbing the continuous spectrum, for example, if \( \bar{v}(2) < 1 \) the eigenvalues \( z_2^{(2)}(\beta) = z_2^{(3)}(\beta) \) (see Theorem 2) exist only at \( \beta \in [0, \arcsin \bar{v}(2)] \). The eigenvalues \( z_0(\beta), z_2^{(2)}(\beta) = z_2^{(3)}(\beta) \) and \( z_2^{(1)}(\beta) = z_2^{(7)}(\beta) = z_2^{(8)}(\beta) = z_2^{(9)}(\beta) \) exist for all \( \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \).

Now we give some information about the eigenfunctions of the operator \( H(k\beta) \). General form of the eigenfunctions of the corresponding fourfold eigenvalue \( z_2^{(1)}(\beta) = z_2^{(7)}(\beta) = z_2^{(8)}(\beta) = z_2^{(9)}(\beta) \) has the form:

\[
\begin{align*}
C_1 f_2^{(1)}(p) + C_7 f_2^{(7)}(p) + C_8 f_2^{(8)}(p) + C_9 f_2^{(9)}(p) &= \\
&= C_1 \sin p_2 \sin p_3 + C_7 \cos p_2 \cos p_3 + C_8 \cos 2p_2 + C_9 \cos 2p_3, \\
&\quad 6 - 2 \sin \beta \cos p_1 - z_2^{(1)}(\beta),
\end{align*}
\]

where \( C_1, C_7, C_8, C_9 \) are arbitrary constants, \( f_2^{(1)} \in \mathcal{L}(1), f_2^{(7)} \in \mathcal{R}^{(1,1)}_{123}, f_2^{(8)} \in \mathcal{R}^{(2,0)}_{123} \) and \( f_2^{(9)} \in \mathcal{R}^{(0,2)}_{123} \). The general form of the eigenfunction of the corresponding double eigenvalue \( z_2^{(2)}(\beta) = z_2^{(3)}(\beta) \) is:

\[
\begin{align*}
C_2 f_2^{(2)}(p) + C_3 f_2^{(3)}(p) &= C_2 \sin p_1 \sin p_3 + C_3 \sin p_1 \sin p_2, \\
&\quad 6 - 2 \sin \beta \cos p_1 - z_2^{(2)}(\beta) \in \mathcal{L}(2) \oplus \mathcal{L}(3),
\end{align*}
\]

where \( C_2, C_3 \) are arbitrary constants.
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