HOMOGENEOUS HYPERSURFACES IN COMPLEX HYPERBOLIC SPACES

JÜRGEN BERNDT, JOSÉ CARLOS DÍAZ-RAMOS

Abstract. We study the geometry of homogeneous hypersurfaces and their focal sets in complex hyperbolic spaces. In particular, we provide a characterization of the focal set in terms of its second fundamental form and determine the principal curvatures of the homogeneous hypersurfaces together with their multiplicities.

1. Introduction

An s-representation is the isotropy representation of a semisimple Riemannian symmetric space. A result by Hsiang and Lawson [10] implies that a hypersurface in the Riemannian sphere \( S^m \) is homogeneous if and only if it is a principal orbit of the s-representation of an \((m + 1)\)-dimensional semisimple Riemannian symmetric space \( G/K \) of rank two. The classification of homogeneous hypersurfaces in \( S^m \) can therefore be easily deduced from Cartan’s classification of Riemannian symmetric spaces.

If \( G/K \) is Hermitian symmetric, then \( m \) is odd, say \( m = 2n + 1 \), and the s-representation induces an action on the corresponding complex projective space \( \mathbb{C}P^n \) via the Hopf map \( S^{2n+1} \to \mathbb{C}P^n \). Takagi [12] showed in 1973 that a real hypersurface in \( \mathbb{C}P^n \) is homogeneous if and only if it is a principal orbit of an action that is induced in this way from the s-representation of a semisimple Hermitian symmetric space of rank two. Thus the classification of homogeneous hypersurfaces in \( \mathbb{C}P^n \) can easily be deduced from the classification of Hermitian symmetric spaces.

A remarkable consequence of Takagi’s result is that each homogeneous hypersurface in \( \mathbb{C}P^n \) is a Hopf hypersurface. A real hypersurface \( M \) of an almost Hermitian manifold \( \bar{M} \) is a Hopf hypersurface if the one-dimensional foliation on \( M \) induced by the rank one distribution \( J(\nu M) \) is totally geodesic, where \( \nu M \) is the normal bundle of \( M \) and \( J \) is the complex structure on \( \bar{M} \). In 1989 the first author classified in [1] the homogeneous Hopf hypersurfaces in the complex hyperbolic space \( \mathbb{C}H^n \). Any such Hopf hypersurface is either a horosphere in \( \mathbb{C}H^n \), or a tube around a totally geodesic \( \mathbb{R}H^n \) or \( \mathbb{C}H^k \) for some \( k \in \{0, \ldots, n-1\} \).

For some time it was believed that, as is the case for \( \mathbb{C}P^n \), every homogeneous hypersurface in \( \mathbb{C}H^n \) was a Hopf hypersurface. It came as a kind of surprise when Lohnherr [11] constructed in 1998 a counterexample: the ruled real hypersurface \( W^{2n-1} \) in \( \mathbb{C}H^n \) which is

2000 Mathematics Subject Classification. Primary 53C40; Secondary 53C55.

Key words and phrases. Complex hyperbolic space, real hypersurfaces, constant principal curvatures, minimal ruled submanifolds.
determined by a horocycle in a totally geodesic \( \mathbb{R}H^2 \subset \mathbb{C}H^n \) is a non-Hopf homogeneous real hypersurface. Recently the first author and Tamaru \([7]\) obtained the classification of homogeneous hypersurfaces in \( \mathbb{C}H^n \). A connected real hypersurface in \( \mathbb{C}H^n \), \( n \geq 2 \), is homogeneous if and only if it is holomorphically congruent to

1. (A) a tube around a totally geodesic \( \mathbb{C}H^k \) for some \( k \in \{0, \ldots, n - 1\} \), or
2. (B) a tube around a totally geodesic \( \mathbb{R}H^n \), or
3. (H) a horosphere in \( \mathbb{C}H^n \), or
4. (S) the ruled real hypersurface \( W^{2n-1} \) or one of its equidistant hypersurfaces, or
5. (W) a tube around the minimal ruled submanifold \( W^{2n-k}_\varphi \) for some \( \varphi \in (0, \pi/2] \) and \( k \in \{2, \ldots, n - 1\} \), where \( k \) is even if \( \varphi \neq \pi/2 \).

The construction of the minimal ruled submanifolds \( W^{2n-k}_\varphi \) will be described later in this article. We just mention here that the normal bundle of \( W^{2n-k}_\varphi \) has rank \( k \) and constant Kähler angle \( \varphi \).

The hypersurfaces of type (A), (B) and (H) are Hopf hypersurfaces and their geometry is well understood. The first author gave in [2] a Lie theoretic construction of the homogeneous hypersurfaces of type (S) and investigated their geometry. The aim of this paper is to investigate the geometry of the other homogeneous hypersurfaces and their focal sets \( W^{2n-k}_\varphi \).

The motivation for our investigations originates from the question: Is every real hypersurface with constant principal curvatures in a complex hyperbolic space \( \mathbb{C}H^n \) an open part of a homogeneous hypersurface? Élie Cartan [9] gave an affirmative answer for the corresponding question in real hyperbolic space. Some time ago the first author demonstrated in [1] that for \( \mathbb{C}H^n \) the answer is yes within the class of Hopf hypersurfaces. For arbitrary real hypersurfaces, we obtained recently an affirmative answer in [6] in case of \( \mathbb{C}H^2 \), and for \( n \geq 3 \) we obtained an affirmative answer in [5] provided that the number \( g \) of distinct principal curvatures satisfies \( g \leq 3 \). It is a well-established fact that the classification problem of hypersurfaces with constant principal curvatures is intimately related to the understanding of the geometric structure of their focal sets. Therefore the next step is to understand more thoroughly the geometry of the homogeneous hypersurfaces in \( \mathbb{C}H^n \) and their focal sets. The two main results of this paper are as follows. (1) We determine explicitly the principal curvatures and their multiplicities for all homogeneous hypersurfaces in \( \mathbb{C}H^n \). A consequence is that \( g \in \{2, 3, 4, 5\} \) for any such hypersurface. (2) We give a characterization of the non-totally geodesic focal sets of homogeneous hypersurfaces in \( \mathbb{C}H^n \) in terms of their second fundamental form.

We now describe the contents of this paper. In Section 2 we summarize some basic material about the complex hyperbolic space. In Section 3 we characterize the minimal ruled submanifolds \( W^{2n-k}_\varphi \) in terms of the second fundamental form. More precisely, we prove

**Theorem** (Rigidity of the submanifold \( W^{2n-k}_\varphi \)). Let \( M \) be a \((2n-k)\)-dimensional connected submanifold in \( \mathbb{C}H^n \), \( n \geq 2 \), with normal bundle \( \nu M \subset T\mathbb{C}H^n \) of constant Kähler angle \( \varphi \in (0, \pi/2] \). Assume that there exists a unit vector field \( Z \) tangent to the maximal complex distribution on \( M \) such that the second fundamental form \( II \) of \( M \) is given by the trivial...
symmetric bilinear extension of
\[ 2II(Z, P\xi) = \sin^2(\varphi) \xi \]
for all \( \xi \in \nu M \), where \( P\xi \) is the tangential component of \( J\xi \). Then \( M \) is holomorphically congruent to an open part of the ruled minimal submanifold \( W^{2n-k}_\varphi \). Conversely, the second fundamental form of \( W^{2n-k}_\varphi \) is of this form.

This shows that the minimal ruled submanifold \( W^{2n-k}_\varphi \) has three distinct constant principal curvatures \( \sin(\varphi)/2, -\sin(\varphi)/2 \) and 0 with multiplicities 1, 1 and \( 2n - k - 2 \) with respect to each unit normal vector. In Section 4 we determine the principal curvatures of \( W^{2n-k}_\varphi \) for all \( \varphi \) congruent to an open part of the ruled minimal submanifold \( W^4 \) in \( CH^3 \). In the final part of the paper we derive some geometric information about these particular tubes in terms of some autoparallel distributions of their tangent bundles.

The second author has been supported by project BFM 2003-02949 (Spain) and an EMBARK postdoctoral fellowship from the Irish Research Council for Science, Engineering and Technology.

2. Preliminaries

In this section we summarize some basic facts about the complex hyperbolic space. For details we refer to [1].

Let \( CH^n \), \( n \geq 2 \), be the \( n \)-dimensional complex hyperbolic space equipped with the Fubini-Study metric \( \langle \cdot, \cdot \rangle \) of constant holomorphic sectional curvature \( -1 \). The Riemannian curvature tensor \( \tilde{R} \) on \( CH^n \) is given by
\[ 4\tilde{R}(X, Y)Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X + \langle JX, Z \rangle JY - \langle JY, Z \rangle JX + 2\langle JX, Y \rangle JZ, \]
where \( J \) is the complex structure on \( CH^n \).

We denote by \( CH^n(\infty) \) the ideal boundary of \( CH^n \). Each element \( x \) of \( CH^n(\infty) \) is an equivalence class of asymptotic geodesics in \( CH^n \). We equip \( CH^n \cup CH^n(\infty) \) with the cone topology. Then \( CH^n \cup CH^n(\infty) \) is homeomorphic to a closed ball in the Euclidean space \( R^{2n} \). For each \( o \in CH^n \) and each \( x \in CH^n(\infty) \) there exists a unique geodesic \( \gamma_{ox} : R \to CH^n \) such that \( ||\dot{\gamma}_{ox}|| = 1 \), \( \gamma_{ox}(0) = o \) and \( \lim_{t \to \infty} \gamma_{ox}(t) = x \).

The connected component of the isometry group of \( CH^n \) is the special unitary group \( G = SU(1, n) \). We fix a point \( o \in CH^n \) and denote by \( K \) the isotropy subgroup of \( G \) at \( o \). Then \( K \) is isomorphic to \( S(U(1) \times U(n)) \subset SU(1, n) \) and \( (G, K) \) is a symmetric pair. Let \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) be the corresponding Cartan decomposition of the Lie algebra \( \mathfrak{g} \) of \( G \). As usual we identify \( T_o CH^n \) with \( \mathfrak{p} \).

We now fix a point \( x \in CH^n(\infty) \) and denote by \( \mathfrak{a} \) the one-dimensional linear subspace of \( \mathfrak{p} \) spanned by \( \dot{\gamma}_{ox}(0) \in T_o CH^n \cong \mathfrak{p} \). As the rank of \( CH^n \) is one, \( \mathfrak{a} \) is a maximal abelian subspace of \( \mathfrak{p} \). Let \( \mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 \) be the root space decomposition of \( \mathfrak{g} \).
induced by $a$. Then $n = g_a + g_2a$ is a 2-step nilpotent subalgebra of $g$ which is isomorphic to the $(2n - 1)$-dimensional Heisenberg algebra. The center of $n$ is the one-dimensional subalgebra $g_2a$. Moreover, $g = \mathfrak{k} + a + n$ is an Iwasawa decomposition of $g$. We denote by $A$ and $N$ the connected closed subgroup of $G$ with Lie algebra $a$ and $n$, respectively. The orbit $A \cdot o$ of $A$ through $o$ is just the path $\gamma_o(x) = e^{s\gamma_0}$ of the geodesic $\gamma_o$ in $C H^n$, and the orbits of $N$ are the horospheres in $C H^n$ centered at $x$. The solvable subgroup $AN \subset KAN = G$ acts simply transitively on $C H^n$. Thus we can identify $\alpha + n$ with $T_o C H^n$. The Riemannian metric on $C H^n$ induces an inner product $\langle \cdot, \cdot \rangle$ on $\alpha + n$, and we may identify $CH^n$ with the solvable Lie group $AN$ equipped with the left-invariant Riemannian metric which is induced from $\langle \cdot, \cdot \rangle$.

The complex structure $J$ on $C H^n$ induces a complex structure on the vector space $\alpha + n \cong T_o C H^n$ which we will also denote by $J$. We define $B := \dot{\gamma}_o(0) \in \alpha$ and $Z := JB \in g_2a$. Note that $g_a$ is $J$-invariant. The Lie algebra structure on $\alpha + n$ is given by the trivial skew-symmetric bilinear extension to $(\alpha + n) \times (\alpha + n)$ of the relations

$$[B, Z] = Z, \quad 2[B, U] = U, \quad [U, V] = \langle JU, V \rangle Z \quad (U, V \in g_a). \quad (1)$$

This shows that $\alpha + n$ is a semidirect sum of the two Lie algebras $\alpha$ and $n$. Let $\text{Exp}_a : n \to N$ be the Lie exponential map. The group structure on the semidirect product $AN$ is given by

$$\left( a, \text{Exp}_n(U + xZ) \right) \cdot \left( b, \text{Exp}_n(V + yZ) \right) = \left( a + b, \text{Exp}_n(U + e^{a/2}V + \left(x + e^{a/2}\langle JU, V \rangle \right)Z) \right) \quad (2)$$

for all $a, b, x, y \in \mathbb{R}$ and $U, V \in g_a$. Here we identify the one-dimensional Lie group $A$ in the canonical way with $\mathbb{R}$ such that $1 \in \mathbb{R}$ corresponds to $\text{Exp}_a(B)$ with the Lie exponential map $\text{Exp}_a : \alpha \to A$. Finally, the Lie exponential map $\text{Exp}_{a+n} : \alpha + n \to AN$ is given by

$$\text{Exp}_{a+n}(aB + U + xZ) = \left( a, \text{Exp}_a\left(\rho(a/2)U + \rho(a)xZ\right) \right) \quad (3)$$

for all $a, x \in \mathbb{R}$ and $U \in g_a$, where the analytic function $\rho : \mathbb{R} \to \mathbb{R}$ is defined by

$$\rho(s) = \begin{cases} e^s - 1, & \text{if } s \neq 0, \\ s, & \text{if } s = 0. \end{cases}$$

The Lie exponential map $\text{Exp}_{a+n}$ is a diffeomorphism. If $V \in g_a$ is a unit vector, then the geodesic $\gamma$ in $C H^n$ with $\gamma(0) = o$ and $\dot{\gamma}(0) = V$ is given by

$$\gamma(t) = (\ln \text{sech}(t/2), \text{Exp}_n(2 \tanh(t/2)V)), \quad (4)$$

and its tangent vector field $\dot{\gamma}$ is given by

$$\dot{\gamma}(t) = -\tanh(t/2)B + \text{sech}(t/2)V. \quad (5)$$

We denote by $\nabla$ the Levi-Civita covariant derivative of $C H^n$. The standard method for calculating the Levi-Civita covariant derivative of a Lie group equipped with a left-invariant
Riemannian metric yields
\[
\nabla_{aB+U+xZ} (bB + V + yZ) = \left( \frac{1}{2} \langle U, V \rangle + x y \right) B - \frac{1}{2} (bU + yJU + xJV) + \left( \frac{1}{2} \langle JU, V \rangle - b x \right) Z,
\]

(6)

where \( a, b, x, y \in \mathbb{R} \) and \( U, V \in g_\alpha \) and all elements in \( a + n \) are considered as left-invariant vector fields on \( AN \cong CH^n \).

3. The ruled submanifolds \( W^{2n-k} \) and \( W^{2n-k}_\varphi \)

The submanifolds \( W^{2n-k} \) and \( W^{2n-k}_\varphi \) were first constructed by the first author and Brück in [3].

Let \( v \) be a linear subspace of \( g_\alpha \). For each \( 0 \neq v \in v \) the Kähler angle of \( v \) with respect to \( v \) is the angle \( \varphi(v) \in [0, \pi/2] \) between \( v \) and the real span of \( Jv \). Thus \( \varphi(v) \in [0, \pi/2] \) is determined by requiring that \( \cos(\varphi(v))\|v\| \) is the length of the orthogonal projection of \( Jv \) onto \( v \). We say that \( v \) has constant Kähler angle \( \varphi \) if \( \varphi(v) = \varphi \) for all nonzero vectors \( v \in v \). The subspaces of \( g_\alpha \) with constant Kähler angle \( \varphi = 0 \) are precisely the complex subspaces of \( g_\alpha \), and the subspaces of \( g_\alpha \) with constant Kähler angle \( \varphi = \pi/2 \) are precisely the real subspaces of \( g_\alpha \).

Let \( w \) be a linear subspace of \( g_\alpha \) such that the orthogonal complement \( w^\perp = g_\alpha \oplus w \) of \( w \) in \( g_\alpha \) has constant Kähler angle \( \varphi \in [0, \pi/2] \). Then \( s = a + w + g_\alpha \) is a subalgebra of \( a + n \). Denote by \( S \) the connected closed subgroup of \( AN \) with Lie algebra \( s \) and by \( N_0^K(S) \) the identity component of the normalizer of \( S \) in \( K \). Then \( N_0^K(S)S \subset KAN \) acts on \( CH^n \) with cohomogeneity one. The orbit \( W^{2n-k}_\varphi = N_0^K(S)S \cdot o = S \cdot o \) of this action containing the point \( o \) is a \((2n-k)\)-dimensional submanifold of \( CH^n \), where \( k = \dim w^\perp \).

If \( \varphi = 0 \), that is, \( w^\perp \) is a complex subspace of \( g_\alpha \), then \( W^{2n-k}_0 \) is a totally geodesic complex hyperbolic subspace \( CH^{n-k'} \), where \( k = 2k' \).

If \( \varphi = \pi/2 \), then \( w^\perp \) is a \( k \)-dimensional real subspace of \( g_\alpha \). If \( k = 1 \), then \( W^{2n-1}_{\pi/2} \) is the ruled real hypersurface \( W^{2n-1} \) determined by a horocycle in a totally geodesic \( \mathbb{R}H^2 \subset CH^n \). The orbits of \( N_0^K(S)S \) form a homogeneous codimension one foliation on \( CH^n \) whose geometry has been investigated by the first author in [2]. If \( k > 1 \), then \( W^{2n-k}_{\pi/2} \) is a \((2n-k)\)-dimensional homogeneous submanifold of \( CH^n \) with totally real normal bundle of rank \( k \). We will sometimes use the notation \( W^{2n-k}_{\varphi} := W^{2n-k}_{\pi/2} \).

If \( 0 < \varphi < \pi/2 \), then \( k \) is even and \( W^{2n-k}_{\varphi} \) is a \((2n-k)\)-dimensional homogeneous submanifold of \( CH^n \) whose normal bundle has constant Kähler angle \( \varphi \) and rank \( k \).

As \( CH^n \) is a two-point homogeneous space, the construction of the submanifolds \( W^{2n-k} \) and \( W^{2n-k}_{\varphi} \) does not depend on the choice of the two points \( o \in CH^n \) and \( x \in CH^n(\infty) \), or equivalently, on the choice of the Iwasawa decomposition of \( G \). Our next aim is to investigate the geometry of the submanifolds \( W^{2n-k}_{\varphi}, 0 < \varphi < \pi/2 \).

Let \( \mathbb{C}w^\perp \) be the complex subspace of \( g_\alpha \) spanned by \( w^\perp \) and \( \mathcal{D} = \mathbb{C}w^\perp \oplus w^\perp \) be the orthogonal complement of \( w^\perp \) in \( \mathbb{C}w^\perp \). As \( \varphi > 0 \), we have \( k = \dim_{\mathbb{C}} \mathbb{C}w^\perp \) and hence \( k = \dim w^\perp = \dim \mathcal{D} \). For each \( \xi \in w^\perp \) we decompose \( J\xi \in \mathbb{C}w^\perp = \mathcal{D} + w^\perp \) into \( J\xi = \mathcal{D} \xi + w^\perp \).
\[ P \xi + F \xi \text{ with } P \xi \in \mathfrak{d} \text{ and } F \xi \in \mathfrak{w} \perp. \] Since \( \mathfrak{w} \perp \) has constant Kähler angle \( \varphi \), we have \( \langle F \xi, F \xi \rangle = \cos^2(\varphi) \langle \xi, \xi \rangle \) and hence \( \langle P \xi, P \xi \rangle = \sin^2(\varphi) \langle \xi, \xi \rangle \). As \( \varphi > 0 \), the homomorphism \( P : \mathfrak{w} \perp \to \mathfrak{d} \) is injective, and as \( \dim \mathfrak{w} \perp = \dim \mathfrak{d} \) we see that \( P : \mathfrak{w} \perp \to \mathfrak{d} \) is an isomorphism.

From \(-\xi = JF \xi = JP \xi + JF \xi = JP \xi + PF \xi + F^2 \xi \) we see that the \( \mathfrak{d} \)-component \((JP \xi)_\mathfrak{d}\) of \( JP \xi \) is equal to \(-PF \xi \), and hence \( \langle (JP \xi)_\mathfrak{d}, (JP \xi)_\mathfrak{d} \rangle = \langle PF \xi, PF \xi \rangle = \sin^2(\varphi) \langle F \xi, F \xi \rangle = \sin^2(\varphi) \cos^2(\varphi) \langle \xi, \xi \rangle = \cos^2(\varphi) \langle P \xi, P \xi \rangle \). Since \( P : \mathfrak{w} \perp \to \mathfrak{d} \) is an isomorphism, this implies that \( \mathfrak{d} \) has constant Kähler angle \( \varphi \) as well.

We denote by \( \mathfrak{c} \) the maximal complex subspace of \( \mathfrak{s} \). Note that \( \mathfrak{a} + \mathfrak{g}_{2\alpha} \subset \mathfrak{c} \), \( \dim C \mathfrak{c} = n - k \) and \( \mathfrak{s} = \mathfrak{c} + \mathfrak{d} \). Then we have the orthogonal decomposition
\[
\mathfrak{a} + \mathfrak{n} = \mathfrak{c} + \mathfrak{d} + \mathfrak{w} \perp.
\]

We denote by \( \mathfrak{A}, \mathfrak{C}, \mathfrak{D} \) and \( \mathfrak{W} \perp \) the left-invariant distributions on \( CH^n \) along \( W_{\varphi}^{2n-k} \) which are induced by \( \mathfrak{a}, \mathfrak{c}, \mathfrak{d} \) and \( \mathfrak{w} \perp \), respectively. By construction, we have \( \mathfrak{C} + \mathfrak{D} = TW_{\varphi}^{2n-k} \) and \( \mathfrak{W} \perp = \nu W_{\varphi}^{2n-k} \).

**Proposition 3.1.** The submanifold \( W_{\varphi}^{2n-k} \), \( 0 < \varphi \leq \pi/2 \), of \( CH^n \) has the following properties:

(i) The maximal holomorphic subbundle \( \mathfrak{C} \) of \( TW_{\varphi}^{2n-k} \) is autoparallel and the leaves of the induced foliation on \( W_{\varphi}^{2n-k} \) are totally geodesic \( CH^{n-k} \subset CH^n \). Hence \( W_{\varphi}^{2n-k} \) is a ruled submanifold of \( CH^n \).

(ii) The following statements are equivalent:

(a) the distribution \( \mathfrak{D} \) on \( W_{\varphi}^{2n-k} \) is integrable;
(b) the distribution \( \mathfrak{A} + \mathfrak{D} \) on \( W_{\varphi}^{2n-k} \) is integrable;
(c) the normal bundle \( \mathfrak{W} \perp \) is flat with respect to the normal connection;
(d) \( \varphi = \pi/2 \).

In this case the leaves of the foliation on \( W_{\varphi}^{2n-k} \) induced by \( \mathfrak{A} + \mathfrak{D} \) are totally geodesic \( RH^{k+1} \subset CH^n \) and the leaves of the foliation on \( W_{\varphi}^{2n-k} \) induced by \( \mathfrak{D} \) are horospheres with center \( x \) in these totally geodesic \( RH^{k+1} \subset CH^n \).

(iii) For each \( 0 \neq \xi \in \mathfrak{w} \perp \) the left-invariant distribution \( \mathfrak{A} + \nu_P \xi \) on \( W_{\varphi}^{2n-k} \) is autoparallel and the leaves of the induced foliation on \( W_{\varphi}^{2n-k} \) are totally geodesic \( RH^2 \subset CH^n \).

(iv) For each \( 0 \neq \xi \in \mathfrak{w} \perp \) the left-invariant distribution \( \nu_P \xi \) on \( W_{\varphi}^{2n-k} \) is integrable and the leaves of the induced foliation on \( W_{\varphi}^{2n-k} \) are horocycles with center \( x \) in the totally geodesic \( RH^2 \subset CH^n \) given by the distribution \( \mathfrak{A} + \nu_P \xi \).

**Proof.** (i) follows immediately from (i) and the fact that the only complex totally geodesic submanifolds of \( CH^n \) are complex hyperbolic spaces.

From (i) we see that \( 2[aB + U, bB + V] = aV - bU + 2(JU, V)Z \) for all \( a, b \in \mathbb{R} \) and \( U, V \in \mathfrak{D} \). This shows that \( \mathfrak{A} + \mathfrak{D} \) is integrable if and only if \( \mathfrak{D} \) is integrable if and only if \( \mathfrak{D} \) is real, that is, \( \varphi = \pi/2 \). On the other hand, (i) implies \( 2\nabla_{aB + U + x\xi} = -xF \xi \) for all \( a, x \in \mathbb{R}, U \in \mathfrak{D} \) and \( \xi \in \mathfrak{W} \perp \). Hence, the normal bundle \( \mathfrak{W} \perp \) is flat if and only if \( F = 0 \), or equivalently, \( \varphi = \pi/2 \). In this case (i) yields
\[
2\nabla_{aB + U} (bB + V) = (U, V)B - bU \in \mathfrak{A} + \mathfrak{D}.
\]
for all \(a, b \in \mathbb{R}\) and \(U, V \in \mathcal{D}\). This shows that \(\mathfrak{A} + \mathcal{D}\) is autoparallel and its leaves are totally geodesic real submanifolds of \(CH^n\). The only real totally geodesic submanifolds of \(CH^n\) are real hyperbolic spaces. Finally, for all \(U, V \in \mathcal{D}\) we have \(2\nabla_U V = (U, V)B\) and \(2\nabla_U B = -U\), which implies that the leaves of \(\mathcal{D}\) are spherical hypersurfaces of the corresponding real hyperbolic subspaces. Since the sectional curvature of a totally geodesic real hyperbolic subspace is \(-1/4\), and the mean curvature vector field of any leaf of \(\mathcal{D}\) is \((1/2)B\), it follows that the leaves of \(\mathcal{D}\) are horospheres centered at \(x\) in the real hyperbolic subspaces. This finishes the proof of (ii).

For any \(aB + xP_\xi, bB + yP_\xi \in \mathfrak{A} + \mathbb{R}P_\xi\) we get from (1) and using \(\langle P_\xi, P_\xi \rangle = \sin^2(\varphi)\langle \xi, \xi \rangle\) that \(2\nabla_{aB + xP_\xi}(bB + yP_\xi) = xy\sin^2(\varphi)B - bxP_\xi \in \mathfrak{A} + \mathbb{R}P_\xi\). From this we easily get the assertion (iii).

Finally, define \(U_\xi = P_\xi/\sin(\varphi)\). Then (6) implies \(2\nabla_{U_\xi} U_\xi = B\) and \(4\nabla_{U_\xi} \nabla_{U_\xi} U_\xi = -U_\xi\). Since the real hyperbolic planes in (iii) have constant sectional curvature \(-1/4\), this shows that the integral curves of \(U_\xi\) are horocycles with center \(x\) in the corresponding real hyperbolic planes. This proves (iv). \(\square\)

The previous proposition implies a nice geometric construction of the ruled submanifolds \(W^{2n-k}_\varphi \subset CH^n\).

**Corollary 3.2.** Let \(k \in \{1, \ldots, n-1\}\), and fix a totally geodesic \(CH^{n-k}\) \(\subset CH^n\) and points \(o \in CH^{n-k}\) and \(x \in CH^{n-k}(\infty)\). Let \(KAN\) be the Iwasawa decomposition of \(SU(1, n)\) with respect to \(o\) and \(x\), and let \(H'\) be the subgroup of \(AN\) which acts simply transitively on \(CH^{n-k}\). Next, let \(W\) be a subspace of \(\nu_oCH^{n-k}\) with constant Kähler angle \(\varphi \in (0, \pi/2)\) such that \(CW = \nu_oCH^{n-k}\). Left translation of \(W\) by \(H'\) to all points in \(CH^{n-k}\) determines a subbundle \(\mathfrak{D}\) of the normal bundle \(\nu CH^{n-k}\). At each point \(p \in CH^{n-k}\) attach the horocycles determined by \(p\) and the linear lines in \(\mathfrak{D}_p\). The resulting subset \(M\) of \(CH^n\) is holomorphically congruent to the ruled submanifold \(W^{2n-k}_\varphi\).

**Proof.** Let \(W^{2n-k}_\varphi\) be the ruled minimal submanifold of \(CH^n\) constructed from the Iwasawa decomposition \(KAN\) associated with \(o\) and \(x\) and from the choice of \(\mathfrak{m}^+ = \nu_oCH^{n-k} \subset W\). We use the above notations. We will show that \(M = W^{2n-k}_\varphi\). Let \(p \in W^{2n-k}_\varphi\). Then there exists an isometry \(s \in S\) with \(p = s(o)\). There is a unique vector \(X\) in the Lie algebra \(s\) of \(S\) such that \(s = \text{Exp}_{a+n}(X)\). We can write \(X = aB + U + V + zZ\) with some \(U \in \mathfrak{c} \oplus \mathfrak{a} + \mathfrak{g}_2\), \(V \in \mathcal{D}\), and \(a, z \in \mathbb{R}\). Note that \([V, U] = 0\) because they are complex orthogonal. We now define \(g = \text{Exp}_{a+n}(\rho(a/2)V)\) and \(h = \text{Exp}_{a+n}(aB + U + zZ)\). Note that \(h \in H'\). Using (2) and (3) we get

\[
g h = \text{Exp}_{a+n}(\rho\left(\frac{a}{2}\right)V) \text{Exp}_{a+n}(aB + U + zZ)
\]

\[
= \left(0, \text{Exp}_n\left(\rho\left(\frac{a}{2}\right)V\right)\right) \cdot \left(a, \text{Exp}_n\left(\rho\left(\frac{a}{2}\right)U + \rho(a)z Z\right)\right)
\]

\[
= \left(a, \text{Exp}_n\left(\rho\left(\frac{a}{2}\right)(U + V) + \rho(a)z Z\right)\right)
\]

\[
= \text{Exp}_{a+n}(aA + U + V + zZ) = \text{Exp}_{a+n}(X) = s.
\]
By construction, \( h(o) \in \mathbb{C}H^{n-k} \), and \( s(o) = g(h(o)) \) is on the horocycle with center \( x \) through \( h(o) \) tangent to \( \mathbb{R}V \). From this we conclude that \( W^{2n-k}_\phi \subset M \). From Proposition 3.1 we already know that \( M \subset W^{2n-k}_\phi \). Altogether this implies \( M = W^{2n-k}_\phi \). 

We now describe the geometry of \( W^{2n-k}_\phi \) in terms of the second fundamental form.

**Proposition 3.3.** The second fundamental form \( II \) of \( W^{2n-k}_\phi \) is given by

\[
II(aB + U + P\xi + xZ, bB + V + P\eta + yZ) = \frac{\sin^2(\phi)}{2}(y\xi + x\eta)
\]

for all \( \xi, \eta \in w^\perp, U, V \in \mathfrak{c} \ominus (a + g_{2\alpha}) \) and \( a, b, x, y \in \mathbb{R} \). Thus \( II \) is given by the trivial symmetric bilinear extension of \( 2II(Z, P\xi) = \sin^2(\phi)\xi \) for all \( \xi \in w^\perp \).

**Proof.** We denote by \((\cdot)^\perp\) the orthogonal projection onto \( \nu W^{2n-k}_\phi \). The Gauß formula and (6) imply

\[
II(aB + U + P\xi + xZ, bB + V + P\eta + yZ) = (\nabla_{aB+U+P\xi+xZ}(bB + V + P\eta + yZ))^\perp
\]

\[
= -\left(\frac{y}{2}JP\xi + \frac{x}{2}JP\eta\right)^\perp
\]

\[
= \frac{\sin^2(\phi)}{2}(y\xi + x\eta),
\]

since \((JP\xi)^\perp = -\sin^2(\phi)\xi\), which follows from the fact that \( w^\perp \) has constant Kähler angle \( \phi \).

As an immediate consequence we get

**Corollary 3.4.** \( W^{2n-k}_\phi \) is a minimal ruled submanifold of \( \mathbb{C}H^n \).

For \( k > 1 \) the previous corollary follows also from the general fact that each singular orbit of a cohomogeneity one action is a minimal submanifold. We will now show that the above equation for the second fundamental form in fact characterizes the minimal ruled submanifolds \( W^{2n-k}_\phi \) in \( \mathbb{C}H^n \).

**Theorem 3.5** (Rigidity of the submanifold \( W^{2n-k}_\phi \)). Let \( M \) be a \((2n - k)\)-dimensional connected submanifold in \( \mathbb{C}H^n, n \geq 2 \), with normal bundle \( \nu M \subset T\mathbb{C}H^n \) of constant Kähler angle \( \phi \in (0, \pi/2] \). Assume that there exists a unit vector field \( Z \) tangent to the maximal complex distribution on \( M \) such that the second fundamental form \( II \) of \( M \) is given by the trivial symmetric bilinear extension of

\[
2II(Z, P\xi) = \sin^2(\phi)\xi
\]

for all \( \xi \in \nu M \), where \( P\xi \) is the tangential component of \( J\xi \). Then \( M \) is holomorphically congruent to an open part of the ruled minimal submanifold \( W^{2n-k}_\phi \). Conversely, the second fundamental form of \( W^{2n-k}_\phi \) is of this form.
Proof. The last statement is a consequence of Proposition 3.3. For the other part we use Corollary 3.2.

We decompose the tangent bundle \( TM \) of \( M \) orthogonally into \( TM = \mathcal{C} + \mathcal{D} \), where \( \mathcal{C} \) is the maximal complex subbundle of \( TM \). For each \( \xi \in \Gamma(\nu M) \) we decompose \( J\xi \) orthogonally into \( J\xi = P\xi + F\xi \) with \( P\xi \in \Gamma(\mathcal{D}) \) and \( F\xi \in \Gamma(\nu M) \). As above one can show that \( \mathcal{D} \) has constant Kähler angle \( \varphi \) as well, and the bundle homomorphisms \( P : \nu M \to \mathcal{D} \) and \( F : \nu M \to \nu M \) are homomorphisms satisfying \( \langle P\xi, P\xi \rangle = \sin^2(\varphi) \langle \xi, \xi \rangle \) and \( \langle F\xi, F\xi \rangle = \cos^2(\varphi) \langle \xi, \xi \rangle \). Note that if \( \varphi = \pi/2 \) then \( F \) is trivial and \( P = J|\nu M \); otherwise, \( P \) and \( F \) are isomorphisms.

For all \( U, V \in \Gamma(\mathcal{C}) \) and \( \xi \in \Gamma(\nu M) \) we have, using (7) and \( \nabla J = 0 \),
\[
\langle \nabla_U V, \xi \rangle = \langle II(U, V), \xi \rangle = 0
\]
and
\[
\langle \nabla_U V, J\xi \rangle = -\langle J\nabla_U V, \xi \rangle = -\langle II(U, J\xi), \xi \rangle = 0.
\]
This shows that \( \mathcal{C} \) is an autoparallel subbundle of \( TM \) and each integral manifold is a totally geodesic submanifold of \( CH^n \). As \( \mathcal{C} \) is a complex subbundle of complex rank \( n - k \), each of these integral manifolds must be an open part of a totally geodesic \( CH^{n-k} \subset CH^n \).

Let \( o \in M \) and \( \mathcal{F}_o \) be the leaf of \( \mathcal{C} \) through \( o \), which is an open part of a totally geodesic \( CH^{n-k} \subset CH^n \). Let \( \gamma : I \to \mathcal{F}_o \) be a curve with \( \gamma(0) = o \). We prove that the normal spaces of \( M \) along \( \gamma \) are uniquely determined by the differential equation
\[
2\tilde{\nabla}_\gamma X + \langle \dot{\gamma}, Z \rangle JX = 0 \tag{8}
\]
along \( \gamma^* \nu \mathcal{F}_o \). Let \( X \in \Gamma(TM) \) and \( \xi \in \Gamma(\nu M) \). Using (7) we get
\[
-\langle \tilde{\nabla}_\gamma \xi, X \rangle = \langle II(\dot{\gamma}, X), \xi \rangle = \langle \dot{\gamma}, Z \rangle \langle X, P\xi \rangle/\langle P\xi, P\xi \rangle = \frac{1}{2} \langle \dot{\gamma}, Z \rangle \langle P\xi, X \rangle,
\]
which implies
\[
\tilde{\nabla}_\gamma \xi = -\frac{1}{2} \langle \dot{\gamma}, Z \rangle P\xi + \nabla^\perp_\gamma \xi, \tag{9}
\]
where \( \nabla^\perp \) is the normal connection of \( M \). Now, let \( X \) be a vector field along \( \gamma \) with \( X_0 \in \nu_o M \) and satisfying (8). We may write \( X = U + J\eta + \xi \) with \( U \in \Gamma(\gamma* \mathcal{C}), \xi, \eta \in \Gamma(\gamma^* \nu M) \) and \( U_0 = \eta_0 = 0 \). Then, using (9) and \( \tilde{\nabla} J = 0 \), we get
\[
0 = 2\tilde{\nabla}_\gamma X + \langle \dot{\gamma}, Z \rangle JX
= 2\tilde{\nabla}_\gamma U + 2J\tilde{\nabla}_\gamma \eta + 2\tilde{\nabla}_\gamma \xi + \langle \dot{\gamma}, Z \rangle JU + \langle \dot{\gamma}, Z \rangle J^2 \eta + \langle \dot{\gamma}, Z \rangle J\xi
= 2\tilde{\nabla}_\gamma U + \langle \dot{\gamma}, Z \rangle JU + P (2\nabla^\perp_\gamma \eta + \langle \dot{\gamma}, Z \rangle F\eta)
+ 2\nabla^\perp_\gamma \xi + \langle \dot{\gamma}, Z \rangle F\xi + F (2\nabla^\perp_\gamma \eta + \langle \dot{\gamma}, Z \rangle F\eta).
\]
We have that \( 2\tilde{\nabla}_\gamma U + \langle \dot{\gamma}, Z \rangle JU \) is tangent to \( \mathcal{C} \) because \( \mathcal{C} \) is a complex autoparallel distribution. Hence, it follows that \( 2\tilde{\nabla}_\gamma U + \langle \dot{\gamma}, Z \rangle JU = 0 \). Since \( U_0 = 0 \), the uniqueness of solutions to ordinary differential equations implies \( U_t = 0 \) for all \( t \) and thus \( X \) is normal to \( \mathcal{F}_o \) along \( \gamma \). Similarly, the component tangent to \( \nu \nu M \) yields \( 2\nabla^\perp_\gamma \eta + \langle \dot{\gamma}, Z \rangle F\eta = 0 \) and since \( \eta_0 = 0 \) we have \( \eta_t = 0 \) for all \( t \). Hence, \( X_t \in \nu_{\gamma(t)} M \) for any \( t \), which proves our previous assertion.
We define $B := -JZ$. The tangent vector $B_o$ determines a point $x \in \mathbb{C}H^n(\infty)$, and thus, $o$ and $x$ give rise to an Iwasawa decomposition of the Lie algebra of the isometry group of $\mathbb{C}H^n$. Let us consider the ruled submanifold $W_{2^{n-k}}$ determined by the above Iwasawa decomposition and the normal space $\nu \perp = \nu_o M$ of constant Kähler angle $\varphi$. The leaf $F_o$ is an open part of the totally geodesic $C^\nu$ and its normal space $\nu$ is obtained by left translation of $M$. Let us consider the ruled submanifold $\nu$, the Codazzi equation, (7) and $\bar{\nu}$ which implies $\nu_p M = \nu_p W_{2^{n-k}}$ for all $p \in F_o$, that is, $\nu_p M$ is obtained by left translation of $\nu_o M$ for all $p \in F_o$. According to Corollary 3.2 it just remains to prove that at any point $p \in F_o$ the horocycles determined by $x$ and the linear lines in $P_{\nu_p} M$ are contained in a neighborhood of $p$ in $M$.

Now we prove that the vector field $\bar{R}$ is a geodesic vector field and all its integral curves are geodesics in $\mathbb{C}H^n$ converging to the point $x \in \mathbb{C}H^n(\infty)$. Since $B$ belongs to the maximal complex distribution we have $\bar{\nabla}_B B \in \Gamma(\mathfrak{c})$. Since $B$ is a unit vector $\langle \bar{\nabla}_B B, B \rangle = 0$. Let $X \in \Gamma(\mathfrak{c} \ominus \mathbb{R}B)$ and $\eta \in \Gamma(\nu M)$ be a local unit normal vector field of $M$. Using the explicit expression for $\bar{R}$, the Codazzi equation, (7) and $\bar{\nabla}_B B \in \Gamma(\mathfrak{c})$ and we get

$$0 = 2\bar{R}_{BP\eta} JX \eta = 2\langle (\nabla^\perp_{B}II)(P\eta, JX) - (\nabla^\perp_{P\eta} II)(B, JX), \eta \rangle$$

$$= -2\langle II(P\eta, \nabla_B JX), \eta \rangle = -2\langle \nabla_B JX, Z \rangle \langle II(P\eta, Z), \eta \rangle$$

$$= -\sin^2(\varphi) \langle \nabla_B JX, Z \rangle = \sin^2(\varphi) \langle \nabla_B B, X \rangle.$$

This implies $\langle \nabla_B B, X \rangle = 0$ and therefore $\nabla_B B = 0$, which means that the integral curves of $B$ are geodesics in $\mathbb{C}H^n$.

Now let $X \in \Gamma(TM \ominus \mathbb{R}B)$ and $\gamma$ an integral curve of $X$. We consider the geodesic variation $F(s, t) = \exp_{\gamma(s)}(tB_{\gamma(s)})$ of $\alpha(t) = F(0, t)$, the geodesic in $\mathbb{C}H^n$ with initial condition $\dot{\alpha}(0) = B_o$. We prove that $d(\alpha(t), F(s, t))$ tends to 0 as $t$ goes to infinity, where $d$ is the Riemannian distance function of $\mathbb{C}H^n$.

The transversal vector field of the geodesic variation $F$, $\zeta(t) = (\partial F/\partial s)(0, t)$, is a Jacobi field along $\alpha$ (thus, $4\zeta'' - \zeta - 3\langle \zeta, Z \rangle Z = 0$) with initial conditions $\zeta(0) = X_{\gamma(0)}$, $\zeta'(0) = \nabla_{X_{\gamma(0)}} B$. Hence, we need to calculate $\nabla_X B$.

Let $\eta \in \Gamma(\nu M)$ be a local unit vector field. Using (7) we have $\langle \nabla_X B, \eta \rangle = \langle II(X, B), \eta \rangle = 0$. We also have $\langle \nabla_X B, B \rangle = 0$. Now, using (7) we get

$$2\langle \nabla_X B, \eta \rangle = -2\langle \nabla_X JZ, J\eta - F\eta \rangle = -2\langle II(X, Z), \eta \rangle = -2\langle II(X, B), F\eta \rangle = -\sin^2(\varphi) \langle X, \eta \rangle / \sin^2(\varphi) = -\langle X, \eta \rangle. \quad (10)$$

Next, let $Y \in \Gamma(\mathfrak{c} \ominus \mathbb{R}B)$ and assume that $X \in \Gamma(\mathfrak{c} \ominus \mathbb{R}B)$. Given $\xi \in \Gamma(\nu M)$ we have $\langle \nabla_{P\eta} JY, P\xi \rangle = \langle II(P\eta, Y), \xi \rangle - \langle II(P\eta, JY), F\xi \rangle = \frac{1}{2} \langle Y, \xi \rangle \langle P\eta, P\xi \rangle$. This, the expression for $\bar{R}$, the Codazzi equation, (7) and $\nabla J = 0$ imply

$$-\sin^2(\varphi) \langle X, Y \rangle = 4\bar{R}_{X_{P\eta} JY \eta} = 4\langle (\nabla^\perp_X II)(P\eta, JY) - (\nabla^\perp_{P\eta} II)(X, JY), \eta \rangle$$

$$= -4\langle II(P\eta, \nabla_X JY), \eta \rangle + 4\langle II(X, \nabla_{P\eta} JY), \eta \rangle$$

$$= -4\langle \nabla_X JY, Z \rangle \langle II(P\eta, Z), \eta \rangle + 4\langle X, Z \rangle \langle II(Z, \nabla_{P\eta} JY), \eta \rangle$$

$$= 2\sin^2(\varphi) \langle \nabla_X B, Y \rangle + \sin^2(\varphi) \langle X, Z \rangle \langle Z, Y \rangle.$$
Hence, if \( X \in \Gamma(\mathfrak{c} \ominus \mathbb{R}B) \) we have, as \( \nabla_X B \in \Gamma(\mathfrak{c}) \), that \( 2\nabla_X B = -X - \langle X, Z \rangle Z \).

Now assume that \( X \in \Gamma(\nu M) \) and write \( X = P\xi \) with \( \xi \in \Gamma(\nu M) \). We have
\[
\langle \nabla_X P_Z, Z \rangle = -\langle \nabla_X Z, J\xi - P\xi \rangle = -\langle II(JY, B), \xi \rangle + \langle II(JY, Z), F\xi \rangle = 0.
\]
This, together with the explicit expression for \( \bar{R} \), the Codazzi equation, (7) and \( \nabla J = 0 \) implies
\[
0 = 2\bar{R}_{P_Z, P_Z} - 2\langle (\nabla^2 P_{\xi} II)(JY, P\xi) - (\nabla^2 J Y)P\xi, P\xi \rangle - 2\langle \nabla Y P\xi, Z \rangle = 0.
\]

Thus we get \( \langle \nabla P_{\xi} B, Y \rangle = 0 \), and as a consequence, we get \( 2\nabla P_{\xi} B = -P\xi \) using (10).

All in all, this implies
\[
\nabla_X B = -\frac{1}{2} X - \frac{1}{2} \langle X, Z \rangle Z \quad \text{for all } X \in \Gamma(TM \ominus \mathbb{R}B).
\]

Therefore, if \( X \in T_{\alpha(0)}M \ominus \mathbb{R}B_{\alpha(0)} \) is a unit vector and \( B_X \) denotes \( \nabla \)-parallel translation of \( X \) along \( \alpha \), we get
\[
\zeta(t) = e^{-t/2}B_X(t) + (e^{-t} - e^{-t/2})\langle X, Z_{\alpha(0)} \rangle Z_{\alpha(t)}.
\]

Note that \( Z_{\alpha(t)} \) is a parallel vector field along \( \alpha \) since \( Z'_{\alpha(t)} = \nabla_{B_{\alpha(t)}} Z = J\nabla_{B_{\alpha(t)}} B = 0 \). We easily see that \( \lim_{t \to \infty} \|\zeta(t)\| = 0 \), which implies \( \lim_{t \to \infty} d(\alpha(t), F(s, t)) = 0 \) as \( d(\alpha(t), F(s, t)) \leq s\|\zeta(t)\| \). Altogether this shows that the integral curves of \( B \) are asymptotic geodesics corresponding to the point \( x \in \mathbb{C}H^n(\infty) \).

Now let \( p \in \mathcal{F}_0 \) and \( \xi_p \in \nu_p M \) be a unit vector. The theorem now follows if we prove that the horocycle determined by \( P\xi_p/\sin(\varphi) \) and the point \( x \in \mathbb{C}H^n(\infty) \) is locally contained in \( M \). To achieve this we will construct a unit local vector field \( \xi \in \Gamma(\nu M) \) such that the previous horocycle is an integral curve of \( P\xi/\sin(\varphi) \).

Let \( \gamma : I \to M \) be a curve in \( M \) satisfying the differential equation
\[
\nabla_{\dot{\gamma}} \dot{\gamma} = \frac{1}{2} \langle \dot{\gamma}, \dot{\gamma} \rangle B, \quad \dot{\gamma}(0) = P\xi_p/\sin(\varphi).
\]

We first prove that \( \gamma \) is parametrized by arc length and that it remains tangent to \( P\nu M \).

Write \( \dot{\gamma} = aB + xZ + X + P\eta \) for some differentiable functions \( a, x : I \to \mathbb{R} \), and vector fields \( X \in \Gamma(\gamma^*(\mathfrak{c} \ominus (\mathbb{R}B + \mathbb{R}Z))) \) and \( \eta \in \Gamma(\gamma^*\nu M) \). Since \( Z = JB \), the definition of \( \gamma \) and (11) show
\[
\frac{dx}{dt} = \frac{d}{dt} \langle \dot{\gamma}, Z \rangle = \langle \nabla_{\dot{\gamma}} \dot{\gamma}, Z \rangle + \langle \nabla_Z \dot{\gamma}, \dot{\gamma} \rangle = \langle xB - \frac{1}{2} JX - \frac{1}{2} JP\eta, \dot{\gamma} \rangle = ax.
\]

Since \( x(0) = 0 \), the uniqueness of solutions to ordinary differential equations implies that \( x(t) = 0 \) for all \( t \).

Let \( Y \in \Gamma(\mathbb{R}B + P\nu M) \) and \( \zeta \in \Gamma(\nu M) \). Then, (11) yields \( \langle \nabla_Y X, \zeta \rangle = \langle II(Y, X), \zeta \rangle = 0 \) and \( \langle \nabla_Y X, J\zeta \rangle = -\langle II(JY, X), \zeta \rangle = 0 \). Moreover, since \( \nabla_Y B \in \Gamma(P\nu M) \) by (11) we have \( \langle \nabla_Y X, B \rangle = -\langle \nabla_Y B, X \rangle = 0 \). Also, \( 2\langle \nabla_X X, B \rangle = -2\langle \nabla_X B, X \rangle = \langle X, X \rangle \) and
\[\langle \tilde{\nabla}_X P \eta, \eta \rangle = -\langle II(X, JX), \eta \rangle - \langle II(X, X), F \eta \rangle = 0.\] Hence,

\[
\frac{d}{dt} \langle X, X \rangle = \frac{d}{dt} \langle \gamma, X \rangle = \langle \nabla_{\dot{\gamma}} \dot{\gamma}, X \rangle + \langle \nabla_{\gamma} X, \dot{\gamma} \rangle = a \langle \nabla_{\dot{\gamma}} X, B \rangle + \langle \nabla_{\gamma} X, X \rangle + \langle \nabla_{\dot{\gamma}} X, P \eta \rangle = \langle \nabla_{\dot{\gamma}} X, X \rangle + a \langle \nabla_{\gamma} X, B \rangle
\]

\[
= \langle \nabla_{\dot{\gamma}} X, X \rangle + \frac{a}{2} \langle X, X \rangle = \frac{1}{2} \frac{d}{dt} \langle X, X \rangle + \frac{a}{2} \langle X, X \rangle.
\]

This yields \( \frac{d}{dt} \langle X, X \rangle = a \langle X, X \rangle \) and since \( \langle X(0), X(0) \rangle = 0 \) we obtain \( \langle X(t), X(t) \rangle = 0 \) for all \( t \) and thus \( X = 0 \).

Using the definition of \( \gamma \) we obtain

\[
\frac{d}{dt} \langle \dot{\gamma}, \dot{\gamma} \rangle = 2 \langle \nabla_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle = a \langle \dot{\gamma}, \dot{\gamma} \rangle.
\]

The definition of \( \gamma \), the fact that \( B \) is geodesic and (11) yield

\[
\frac{da}{dt} = \frac{d}{dt} \langle \dot{\gamma}, B \rangle = \langle \nabla_{\dot{\gamma}} \dot{\gamma}, B \rangle + \langle \nabla_{\gamma} B, \dot{\gamma} \rangle = \frac{1}{2} \langle \dot{\gamma}, \dot{\gamma} \rangle - \frac{1}{2} \langle P \eta, \dot{\gamma} \rangle + \frac{1}{2} \langle \dot{\gamma}, \dot{\gamma} \rangle - \frac{1}{2} \langle P \eta, P \eta \rangle.
\]

Now we calculate \( \frac{d}{dt} \langle P \eta, P \eta \rangle \). Let \( \xi, \zeta \in \Gamma(\nu M) \) and \( Y \in \Gamma(\mathfrak{c}) \). Since \( \mathfrak{c} \) is autoparallel, we have \( \langle \nabla_B P \xi, Y \rangle = 0 \). Using (7) we obtain \( \langle \nabla_B P \xi, \zeta \rangle = \langle II(B, P \xi), \zeta \rangle = 0 \). On the other hand we have \( JP \xi = -\xi - PF \xi - F^2 \xi = -PF \xi - \sin^2(\varphi) \xi \), which gives, using (7),

\[
\langle \nabla_B P \xi, P \zeta \rangle = \langle \nabla_B P \xi, J \zeta - F \zeta \rangle = -\langle \nabla_B JP \xi, \zeta \rangle - \langle II(B, P \xi), F \zeta \rangle = \langle II(B, PF \xi), \zeta \rangle + \sin^2(\varphi) \langle \nabla_B \xi, \zeta \rangle = \langle P \nabla_B \xi, P \zeta \rangle.
\]

This readily implies,

\[
\nabla_B P \xi = P \nabla_B^\perp \xi \quad \text{for all } \xi \in \Gamma(\nu M).
\]

Using again (7) we get

\[
2 \langle \nabla_{P \xi} P \xi, Y \rangle = -2 \langle \nabla_{P \xi} Y, J \xi - F \xi \rangle = 2 \langle JY, Z \rangle \langle II(P \xi, Z), \xi \rangle + 2 \langle Y, Z \rangle \langle II(P \xi, X), F \xi \rangle = -\sin^2(\varphi) \langle JZ, Y \rangle \langle \xi, \xi \rangle + \sin^2(\varphi) \langle Y, Z \rangle \langle \xi, F \xi \rangle = \langle P \xi, P \xi \rangle \langle B, Y \rangle.
\]

Clearly, equation (7) implies \( \langle \nabla_{P \xi} P \xi, \zeta \rangle = \langle II(P \xi, P \xi), \zeta \rangle = 0 \). Using (7) and the fact that \( JP \xi = -PF \xi - \sin^2(\varphi) \xi \) we obtain

\[
\langle \nabla_{P \xi} P \xi, P \zeta \rangle = \langle \nabla_{P \xi} P \xi, J \zeta - F \zeta \rangle = -\langle \nabla_{P \xi} JP \xi, \zeta \rangle - \langle II(P \xi, P \xi), F \zeta \rangle = \langle II(P \xi, PF \xi), \zeta \rangle + \sin^2(\varphi) \langle \nabla_{P \xi} \xi, \zeta \rangle = \langle P \nabla_{P \xi} \xi, P \zeta \rangle.
\]

Altogether this implies,

\[
\nabla_{P \xi} P \xi = \frac{1}{2} \langle P \xi, P \xi \rangle B + P \nabla_{P \xi}^\perp \xi \quad \text{for all } \xi \in \Gamma(\nu M).
\]
Finally, equations (13) and (14) yield  
\[
\frac{d}{dt}(P\eta, P\eta) = \frac{d}{dt}(\gamma, P\eta) = (\nabla_\gamma \dot{\gamma}, P\eta) + (\nabla_{\dot{\gamma}} P\eta, \dot{\gamma}) \\
= \frac{a}{2}(P\eta, P\eta) + \langle P\nabla^\perp_B \eta, P\eta \rangle + \langle P\nabla^\perp_{P\eta} \eta, P\eta \rangle \\
= \frac{a}{2}(P\eta, P\eta) + \langle P\nabla^\perp_\xi \eta, P\eta \rangle = \frac{a}{2}(P\eta, P\eta) + \frac{1}{2} \frac{d}{dt}(P\eta, P\eta),
\]
and hence  
\[
\frac{d}{dt}(P\eta, P\eta) = a(P\eta, P\eta).
\]

Putting \( b = \langle \dot{\gamma}, \dot{\gamma} \rangle \) and \( c = \langle P\eta, P\eta \rangle \) we then have the initial value problem:

\[
a' = \frac{1}{2}(b - c), \quad b' = ab, \quad c' = ac, \quad a(0) = 0, \quad b(0) = c(0) = 1.
\]

Again, the uniqueness of solutions to differential equations yields \( a(t) = 0, b(t) = c(t) = 1 \) for all \( t \). Therefore, \( \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 1 \) and \( \dot{\gamma}(t) \in \nu M \) for all \( t \) as desired.

Let us assume then that \( \gamma : I \to M \) is a curve satisfying equation (12). There exists a unit normal vector field \( \eta \) of \( M \) in a neighborhood of \( p \) such that \( \dot{\gamma}(t) = P\eta_{\eta(t)}/\sin(\varphi) \) for all sufficiently small \( t \). Since \( B \) is nonsingular and \( \gamma \) is normal to \( B \), there exists a hypersurface \( N \) in \( M \) containing \( \gamma \) and transversal to \( B \) in a neighborhood of \( p \). The restriction of \( \eta \) to \( N \) is a smooth unit normal vector field along \( N \). We define \( \xi \) as the unit normal vector field on a neighborhood of \( p \) satisfying \( \xi = \eta \) on \( N \) and such that \( \xi \) is obtained by \( \nabla^\perp \)-parallel translation along the integral curves of \( B \). The smooth dependance on initial conditions of ordinary differential equations implies that \( \xi \) is smooth. Also, note that \( \nabla^\perp_B \xi = 0 \) and that \( \xi \) is a local unit vector field extending \( \xi_p \in \nu_p M \).

The definition of \( \xi \) and equations (11) and (13) yield \( [B, P\xi] = \nabla_B P\xi - \nabla_{P\xi} B = \frac{1}{2} P\xi \), and hence the distribution generated by \( B \) and \( P\xi \) is integrable. Let \( U \) denote the integral submanifold through \( p \). We prove that \( U \) is an open part of a totally geodesic \( \mathbb{RH}^2 \subset CH^n \).

Since \( B \) is geodesic we have \( \nabla_B B = 0 \). Equation (11) implies \( 2\nabla_{P\xi} B = -P\xi \), and by definition of \( \xi \) we have using (13) that \( \nabla_B P\xi = P\nabla^\perp_B \xi = 0 \). Now we prove that \( 2\nabla_{P\xi} P\xi = (P\xi, P\xi) B \).

Let \( \eta \in \nu M \) and denote by \( S_\eta \) the shape operator of \( M \) with respect to the normal vector \( \eta \). Equation (17) implies that \( S_\eta B = 0 \) for all \( \eta \), and thus, for any \( \eta, \xi \in \nu M \) the Ricci equation of \( M \) reads

\[
\langle R^\perp_{BP\xi}\eta, \xi \rangle = \langle \hat{R}(B, P\xi)\eta, \xi \rangle + \langle [S_\eta, S_\xi] B, P\xi \rangle = 0,
\]
where \( R^\perp \) denotes the curvature tensor of the normal connection \( \nabla^\perp \). The previous equation, \( 2[B, P\xi] = P\xi \), and the definition of \( \xi \) imply

\[
0 = R^\perp_{BP\xi}\xi = \nabla^\perp_B \nabla^\perp_{P\xi} \xi - \nabla^\perp_{P\xi} \nabla^\perp_B \xi - \nabla^\perp_{[B,P\xi]} \xi = \nabla^\perp_B \nabla^\perp_{P\xi} \xi - \frac{1}{2} \nabla^\perp_{P\xi} \xi,
\]
that is,

\[
2\nabla^\perp_B \nabla^\perp_{P\xi} \xi = \nabla^\perp_{P\xi} \xi. \tag{15}
\]
By definition of $\xi$, we have along $\gamma$ that $2\nabla P_\xi P\xi = 2\sin^2(\varphi)\nabla_\varphi \gamma = 2\sin^2(\varphi)\nabla_\varphi \gamma = \langle P\xi, P\xi \rangle B$. On the other hand, $(14)$ yields $2\nabla P_\xi P\xi = \langle P\xi, P\xi \rangle B + 2P\nabla P_\xi \xi$, and hence $\nabla P_\xi \xi = 0$ along $\gamma$. Now, let us take $\alpha$ an integral curve of $B$ through $\alpha(0) = \gamma(s)$. We have just seen that $\nabla P_\xi \xi|_{\alpha(0)} = \nabla P_\xi \xi|_{\alpha(s)} = 0$. Moreover, using $(15)$ and the fact that $S_\eta B = 0$ for any $\eta \in \nu M$, we obtain

$$2\nabla_\alpha \nabla P_\xi \xi|_t = 2\nabla_B \nabla P_\xi \xi|_{\alpha(t)} - 2S\nabla P_\xi \xi|_{\alpha(t)} = \nabla P_\xi \xi|_{\alpha(t)}.$$ 

Therefore, by the uniqueness of solutions to differential equations we get $\nabla P_\xi \xi|_{\alpha(t)} = 0$ for all $t$, and as a consequence $2\nabla P_\xi P\xi = \langle P\xi, P\xi \rangle B$ along the integral submanifold $U$. Hence, $U$ is an open part of a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$.

We define $\tilde{P}\xi = P\xi/\|P\xi\| = P\xi/\sin(\varphi)$ along $U$. From $(14)$ we obtain $2\nabla P_\xi \tilde{P}\xi = B$ since $\xi$ is unit normal. Using this and $(11)$ we get

$$\tilde{\nabla} P_\xi \tilde{\nabla} P_\xi \tilde{P}\xi + \langle \tilde{\nabla} P_\xi \tilde{P}\xi, \tilde{\nabla} P_\xi \tilde{P}\xi \rangle \tilde{P}\xi = \frac{1}{2} \tilde{\nabla} P_\xi B + \frac{1}{4} \langle B, B \rangle \tilde{P}\xi = 0.$$ 

From this we see that the integral curves of $\tilde{P}\xi$ are horocycles with center $x$ at infinity contained in an open part of a totally geodesic real hyperbolic plane contained in $\mathbb{C}H^n$. Corollary 3.2 the rigidity of totally geodesic submanifolds of Riemannian manifolds (see e.g. [4], p. 230), and of horocycles in real hyperbolic planes (see e.g. [4], pp. 24-26), then imply the assertion.

Remark 3.6. The proof shows that the differential equation $(8)$ characterizes left translation of the normal spaces by $S_\xi$.

4. THE TUBES AROUND $W^{2n-k}$ AND $W^\varphi_{2n-k}$

To accomplish the task of investigating the geometry of orbits of the cohomogeneity one actions on $\mathbb{C}H^n$ we will deal with two different possibilities depending on the constant Kähler angle $\varphi \in (0, \pi/2)$ or $\varphi = \pi/2$ of $\mathfrak{m}$. For this we first we recall a few properties of the solvable foliation already studied in [2].

4.1. The solvable foliation. The solvable foliation is the foliation on $\mathbb{C}H^n$ arising from $k = 1$. In this case $\varphi = \pi/2$ and the orbit $S \cdot o = W^{2n-1}$ is a minimal homogeneous ruled real hypersurface. Its principal curvatures are $1/2, -1/2$ and $0$ with multiplicities $1, 1$ and $2n - 3$. The following theorem shows that this eigenvalue structure is characteristic of this orbit.

Theorem 4.1 (Rigidity of the submanifold $W^{2n-1}$). Let $M$ be a connected real hypersurface in $\mathbb{C}H^n$, $n \geq 2$, with three distinct principal curvatures $1/2, -1/2$ and $0$ and multiplicities $1, 1$ and $2n - 3$, respectively. Then $M$ is holomorphically congruent to an open part of the minimal homogeneous ruled real hypersurface $W^{2n-1}$.

This result was proved in [3] for $n \geq 3$. The analogous statement for $n = 2$ is more involved and follows from the classification of real hypersurfaces with constant principal curvatures in the complex hyperbolic plane [6]. Any other orbit of the action of $S$ is an equidistant hypersurface to this minimal one. Any two such orbits are congruent to each
other if and only if their distance to $S \cdot o = W^{2n-1}$ is the same. None of them is ruled by a totally geodesic $\mathbb{C}H^{n-1}$. Let $M(r)$ denote an orbit of $S$ at a distance $r > 0$ from $S \cdot o$. The shape operator of $M(r)$ has exactly three eigenvalues

$$\lambda_{1/2} = \frac{3}{4} \tanh \left( \frac{r}{2} \right) \pm \frac{1}{2} \sqrt{1 - \frac{3}{4} \tanh^2 \left( \frac{r}{2} \right)}, \quad \lambda_3 = \frac{1}{2} \tanh \left( \frac{r}{2} \right).$$

with corresponding multiplicities $m_1 = 1$, $m_2 = 1$ and $m_3 = 2n - 3$. The Hopf vector field $J\xi$ has nontrivial projection onto the principal curvature spaces of $\lambda_1$ and $\lambda_2$.

The subspace $a + w^\perp + Jw^\perp + g_{2a}$ of $a + n$ is a subalgebra of $a + n$, and the orbit through $o$ of the corresponding connected closed subgroup of $AN$ is a totally geodesic $\mathbb{C}H^2$. The action of the connected closed subgroup of $S$ with Lie algebra $a + Jw^\perp + g_{2a}$ induces the solvable foliation on this totally geodesic $\mathbb{C}H^2$. The relevant geometric information of the solvable foliation on $\mathbb{C}H^n$ is contained in the “slice” $\mathbb{C}H^2$. We describe in more detail the geometry of the leaves of the solvable foliation on $\mathbb{C}H^2$ in what follows.

Let $\gamma$ be the geodesic in $\mathbb{C}H^2$ determined by $\gamma(0) = o$ and $\dot{\gamma}(0) = \xi$, where $\xi \in w^\perp$ is a unit vector. Let $r \in \mathbb{R}$ and denote by $M(r)$ the leaf of the solvable foliation containing $\gamma(r)$. According to (1) the tangent vector field $\dot{\gamma}$ of the geodesic $\gamma$ can be written with respect to left-invariant vector fields as

$$\dot{\gamma}(t) = -\tanh(t/2)B + \text{sech}(t/2)\xi.$$

Then $\{Z, J\xi, \text{sech}(r/2)B + \tanh(r/2)\xi\}$ is an orthonormal basis of $T_{\gamma(r)}M(r)$. The distribution on $M(r)$ generated by $Z$ and $J\xi$ is integrable by (11). Moreover, using (6) we get

$$\nabla_Z Z = B, \quad 2\nabla_{J\xi} J\xi = B, \quad 2\nabla_Z J\xi = \xi, \quad 2\nabla_{J\xi} Z = \xi. \quad (16)$$

Thus the shape operator of the leaf of this distribution containing $\gamma(r)$ with respect to the unit normal vector $\text{sech}(r/2)B_{\gamma(r)} + \tanh(r/2)\xi_{\gamma(r)} \in T_{\gamma(r)}M(r)$ is given by the matrix

$$\frac{1}{2} \begin{pmatrix} 2 \text{sech} \left( \frac{r}{2} \right) & \tanh \left( \frac{r}{2} \right) \\ \tanh \left( \frac{r}{2} \right) & \text{sech} \left( \frac{r}{2} \right) \end{pmatrix}$$

with respect to the basis $\{Z_{\gamma(r)}, J\xi_{\gamma(r)}\}$. Using the Gauss equation and (110) we get that the Gaussian curvature of the leaf through $\gamma(r)$ is equal to zero. For topological reasons it is clear that the leaf is a Euclidean plane $\mathbb{R}^2$.

On the other hand, using Lemma (6) we get

$$\nabla_{\text{sech}(\frac{r}{2})B + \tanh(\frac{r}{2})\xi} \left( \text{sech} \left( \frac{r}{2} \right)B + \tanh \left( \frac{r}{2} \right)\xi \right) = -\frac{1}{2} \tanh \left( \frac{r}{2} \right)\dot{\gamma}(r).$$

Hence, every integral curve of $\text{sech}(r/2)B + \tanh(r/2)\xi$ is a geodesic in $M(r)$.

All in all, this means

**Theorem 4.2.** The leaves of the solvable foliation on $\mathbb{C}H^2$ are diffeomorphic to $\mathbb{R}^3$ and are foliated orthogonally by a one-dimensional totally geodesic foliation and a two-dimensional foliation whose leaves are Euclidean planes.
4.2. Constant Kähler angle $\varphi = \pi/2$. In this case $w^\perp$ has constant Kähler angle $\varphi = \pi/2$, that is, $w^\perp$ is real. This means that the normal bundle $\nu W^{2n-k}$ of $W^{2n-k}$ is totally real. We recall that the second fundamental form of $W_{2n-k}$ is given by the trivial symmetric bilinear extension of $\Pi(Z, J\xi) = (1/2)\xi$ for all $\xi \in \nu W^{2n-k}$. Thus the eigenvalues of the shape operator of $W^{2n-k}$ with respect to any unit vector $\xi \in \nu W^{2n-k}$ are $1/2, -1/2$ and $0$ with multiplicities $1, 1$ and $2n - 2 - k$ respectively. The corresponding principal curvature spaces are $\mathbb{R}(Z + J\xi)$, $\mathbb{R}(Z - J\xi)$ and $\mathbb{T} W^{2n-k} \ominus (\mathbb{R}Z + \mathbb{R}J\xi)$ respectively.

The above information allows us to calculate the shape operator of the principal orbits using Jacobi field theory. Every principal orbit of this action is a tube around $W^{2n-k}$. We denote by $M(r)$ the tube at distance $r > 0$ and fix $o \in W^{2n-k}$ and a unit vector $\xi \in \nu_o W^{2n-k}$. Let $\gamma$ be the geodesic in $\mathbb{C}H^n$ given by the initial conditions $\gamma(0) = o$ and $\dot{\gamma}(0) = \xi$. We recall that the Jacobi equation in the complex hyperbolic space of constant holomorphic sectional curvature $-1$ along $\gamma$ reads $4\zeta'' - \zeta - 3\langle J\dot{\gamma}, \zeta \rangle J\dot{\gamma} = 0$.

For any $X \in T_o \mathbb{C}H^n$ we denote by $B_X$ the parallel displacement of the vector $X$ along $\gamma$. If $X \in T_o W^{2n-k}$ we denote by $\zeta_X$ the Jacobi field along $\gamma$ defined by the initial conditions $\zeta_X(0) = X$ and $\zeta_X'(0) = -\mathcal{S}_\xi(X)$. If $X$ is a principal curvature vector, that is, $\mathcal{S}_\xi X = \lambda X$ for some $\lambda \in \mathbb{R}$, then the Jacobi equation can be solved explicitly to get

$$\zeta_X(t) = f_\lambda(t) B_X(t) + \langle X, J\xi \rangle g_\lambda(t) J\dot{\gamma}(t)$$

with

$$f_\lambda(t) = \cosh\left(\frac{t}{2}\right) - 2\lambda\sinh\left(\frac{t}{2}\right), \quad g_\lambda(t) = \left(\cosh\left(\frac{t}{2}\right) - 1\right)\left(1 + 2\cosh\left(\frac{t}{2}\right) - 2\lambda\sinh\left(\frac{t}{2}\right)\right).$$

If $X \in \nu_o W^{2n-k} \ominus \mathbb{R}\xi$ we define the Jacobi field $\zeta_X$ along $\gamma$ by the initial conditions $\zeta_X(0) = 0$ and $\zeta_X'(0) = X$. In this case we have

$$\zeta_X(t) = p(t) B_X(t) + \langle X, J\xi \rangle q(t) J\dot{\gamma}(t)$$

with

$$p(t) = 2\sinh\left(\frac{t}{2}\right), \quad q(t) = 2\sinh\left(\frac{t}{2}\right)\left(\cosh\left(\frac{t}{2}\right) - 1\right).$$

Using the above formulas one easily gets

$$\zeta_X(t) = \begin{cases} 
\cosh\left(\frac{t}{2}\right) B_Z(t) - \frac{1}{2}\sinh(t) B_{J\xi}(t), & \text{if } X = Z, \\
-\sinh\left(\frac{t}{2}\right) B_Z(t) + \cosh(t) B_{J\xi}(t), & \text{if } X = J\xi, \\
\cosh\left(\frac{t}{2}\right) B_X(t), & \text{if } X \in T W^{2n-k} \ominus (\mathbb{R}Z + \mathbb{R}J\xi), \\
2\sinh\left(\frac{t}{2}\right) B_X(t), & \text{if } X \in \nu W^{2n-k} \ominus \mathbb{R}\xi.
\end{cases}$$

We define the endomorphism $D(r)$ of $T_{\gamma(r)} M(r) \ominus \mathbb{R}\gamma(r)$ by $D(r)B_X(r) = \zeta_X(r)$ for all $X \in T_o \mathbb{C}H^n \ominus \mathbb{R}\xi$. Jacobi field theory shows that the shape operator of $M(r)$ at $\gamma(r)$
with respect to $-\gamma(r)$ is given by $S(r) = D'(r)D(r)^{-1}$. In our case $S(r)$ is represented by the matrix
\[
S(r) = \frac{1}{2} \begin{pmatrix}
\tanh^2 \left( \frac{r}{2} \right) & -\text{sech}^2 \left( \frac{r}{2} \right) \\
-\text{sech}^2 \left( \frac{r}{2} \right) & 2 \left( 1 + \frac{1}{2} \text{sech}^2 \left( \frac{r}{2} \right) \right) \tanh \left( \frac{r}{2} \right) \\
\tanh \left( \frac{r}{2} \right) \text{Id}_{2n-2-k} & \coth \left( \frac{r}{2} \right) \text{Id}_{k-1}
\end{pmatrix}.
\]
with respect to the orthogonal sum decomposition
\[
T_{\gamma(r)}M(r) = B_{RZ+RJ\xi}(r) + B_{\overline{\nu},W^{2n-k}\oplus(RZ+RJ\xi)}(r) + B_{\overline{\nu},W^{2n-k}\oplus\mathbb{R}\xi}(r),
\]
where $B_V$ denotes the parallel translation of any vector subspace $V \subset T_0CH^n$ along $\gamma_\xi$.

A straightforward calculation shows that $M(r)$ has four principal curvatures
\[
\lambda_1 = \frac{3}{4} \tanh \left( \frac{r}{2} \right), \quad \lambda_2 = \frac{3}{4} \tanh \left( \frac{3}{4} \tanh \left( \frac{r}{2} \right) \right), \quad \lambda_3 = \frac{1}{2} \tanh \left( \frac{r}{2} \right), \quad \lambda_4 = \frac{1}{2} \coth \left( \frac{r}{2} \right),
\]
with corresponding multiplicities $m_1 = m_2 = 1$, $m_3 = 2n - 2 - k$ and $m_4 = k - 1$. The Hopf vector field on $M$ has nontrivial orthogonal projection onto the principal curvature spaces of $\lambda_1$ and $\lambda_2$. A special situation occurs when $r = \ln(2 + \sqrt{3})$. In this case we have $\lambda_2 = \lambda_4$ and the principal curvatures are $\lambda_1 = 0$, $\lambda_2 = \lambda_4 = \sqrt{3}/2$ and $\lambda_3 = \sqrt{3}/6$ with multiplicities 1, $k$ and $2n - k - 2$ respectively.

The previous calculations show that the interesting part of the shape operator of both the singular orbit $W^{2n-k}$ and the principal orbit $M(r)$ concerns the vectors $Z$ and $J\xi$. More precisely, let $\xi \in \nu_\nu W^{2n-k}$ be a unit vector. Consider the subalgebra $\tilde{g} = a + R\xi + R J\xi + g_{2\alpha}$ of $a + g_{\alpha} + g_{2\alpha}$, and let $\tilde{G}$ be the connected closed subgroup of $AN$ with Lie algebra $\tilde{g}$. The orbit $\tilde{G} \cdot o$ is a totally geodesic $CH^2$ in $CH^n$. This $CH^2$ defines a “slice” of the action of $N^R(S)S$ through $o$. Next, $\tilde{h} = s \cap \tilde{g}$ is a subalgebra of $\tilde{g}$ of codimension one. Let $\tilde{H}$ be the connected closed subgroup of $\tilde{G}$ with Lie algebra $\tilde{h}$. Then $\tilde{H}$ acts on $CH^2 = \tilde{G} \cdot o$ with cohomogeneity one and gives exactly the solvable foliation of $CH^2$ described in Section 4.1. The orbits of the action of $\tilde{H}$ on $CH^2$ are the equidistant hypersurfaces to the orbit $\tilde{H} \cdot o$. On the other hand, the intersection of the orbits of the cohomogeneity one action of $N^R(S)S$ on $CH^n$ with the slice $CH^2$ also gives tubes around $\tilde{H} \cdot o$ because $CH^2 = \tilde{G} \cdot o$ is totally geodesic in $CH^n$. Thus, the geometry of the orbits of the action of $G$ on $CH^n$ in the slice $CH^2$ is exactly the geometry of the orbits of the action of $\tilde{H}$ on $CH^2$. This study was accomplished in the previous subsection.

4.3. **Constant Kähler angle** $\varphi \in (0, \pi/2)$. Again, we assume the notation above, and consider the singular orbit $W^{2n-k}_\varphi$ of the cohomogeneity one action determined by the Lie group $N^R(S)S$, where $S$ is the connected, simply connected Lie group whose Lie algebra is $s = a + w + g_{2\alpha}$, where $m^+$ has constant Kähler angle $\varphi \in (0, \pi/2)$. In this case we have that $k$ is an even number. The second fundamental form of $W^{2n-k}_\varphi$ is given by the
trivial bilinear extension of \( II(Z, P\xi) = (\sin^2(\varphi)/2)\xi \), for each unit \( \xi \in \nu W^{2n-k}_\varphi \). Thus, the eigenvalues of the shape operator with respect to \( \xi \) are \( \sin(\varphi)/2, -\sin(\varphi)/2 \) and 0, with multiplicities 1, 1 and \( 2n - k - 2 \).

It is convenient to introduce the notation

\[
P\xi = P\xi/\|P\xi\| = P\xi/\sin(\varphi) \quad \text{and} \quad F\xi = F\xi/\|F\xi\| = F\xi/\cos(\varphi)
\]

for each unit vector \( \xi \in \nu W^{2n-k}_\varphi \). Then, the eigenspaces of \( \sin(\varphi)/2, -\sin(\varphi)/2 \) and 0 of the shape operator of \( W^{2n-k}_\varphi \) with respect to \( \xi \) are \( \mathbb{R}(Z + P\xi), \mathbb{R}(-Z + P\xi) \) and \( TW^{2(n-k)}_\varphi \ominus (\mathbb{R}Z + \mathbb{R}P\xi) \), respectively.

The shape operator of the principal orbits can be calculated using Jacobi field theory as in the previous section. We delete the calculations, which are straightforward (although long) and directly give the matrix representation of the shape operator \( \mathcal{S}(r) \) in direction \(-\hat{\gamma}_\xi(r)\) of the orbit at a distance \( r > 0 \) from \( W^{2n-k}_\varphi \)

\[
\mathcal{S}(r) = \begin{pmatrix}
s(r) & \frac{1}{2}\tanh\left(\frac{r}{2}\right) \text{Id}_{2n-k-2} \\
\frac{1}{2}\tanh\left(\frac{r}{2}\right) \text{Id}_{2n-k-2} & \frac{1}{2}\coth\left(\frac{r}{2}\right) \text{Id}_{k-2}
\end{pmatrix},
\]

with respect to the direct sum decomposition

\[
T_{\gamma_\xi(r)}M(r) = \mathcal{B}_{\mathbb{R}Z + \mathbb{R}P\xi + \mathbb{R}F\xi}(r) + \mathcal{B}_{TW^{2n-k}_\varphi \ominus (\mathbb{R}Z + \mathbb{R}P\xi)}(r) + \mathcal{B}_{\nu W^{2n-k}_\varphi \ominus (\mathbb{R}Z + \mathbb{R}P\xi)}(r).
\]

Here, \( s(r) \) is a symmetric \( 3 \times 3 \) real matrix whose explicit entries we do not provide (they can be obtained after some elementary but long calculations). The characteristic polynomial of \( s(r) \) is

\[
p_{r,\varphi}(x) = -x^3 + \frac{1}{2}\left\{\text{csch}\left(\frac{r}{2}\right)\text{sech}\left(\frac{r}{2}\right) + 4\tanh\left(\frac{r}{2}\right)\right\}x^2 - \frac{1}{4}\left\{2\text{sech}^2\left(\frac{r}{2}\right) + 5\tanh^2\left(\frac{r}{2}\right)\right\}x
\]

\[-\frac{1}{8}\text{csch}^3\left(\frac{r}{2}\right)\left\{\sin^2(\varphi) - \sin^2\left(\frac{r}{2}\right) - 2\sinh^4\left(\frac{r}{2}\right)\right\}.
\]

If we introduce the variable \( 6x = \coth(r/2)z - \text{csch}(r/2)\text{sech}(r/2) - 4\tanh(r/2) \), then the polynomial equation \( p_{r,\varphi}(x) = 0 \) transforms into \( z^3 - 3z + \beta_{r,\varphi} = 0 \), where \( \beta_{r,\varphi} = 27\sin^2(\varphi)\tanh^2(r/2)\text{sech}^4(r/2) - 2 \). The discriminant of this cubic equation is \( \Delta_{r,\varphi} = 27(\beta_{r,\varphi}^2 - 4) \). It is easy to prove that \( \Delta_{r,\varphi} < 0 \) for all \( r > 0 \), which means that the above cubic equation has exactly three distinct real roots for any \( r \). They can be calculated explicitly as follows. Let \( u_{r,\varphi}^i, i \in \{1, 2, 3\} \), denote each cubic root of the unit complex number \( (\beta_{r,\varphi} + \sqrt{\beta_{r,\varphi}^2 - 4})/2 \). Then, \(-u_{r,\varphi}^i - 1/u_{r,\varphi}^i\) is a solution to \( z^3 - 3z + \beta_{r,\varphi} = 0 \) and hence, the eigenvalues of \( s(r) \) are given by

\[
\lambda_i(r) = -\frac{1}{6}\left(\coth\left(\frac{r}{2}\right)\left(u_{r,\varphi}^i + 1/u_{r,\varphi}^i\right) + \text{csch}\left(\frac{r}{2}\right)\text{sech}\left(\frac{r}{2}\right) + 4\tanh\left(\frac{r}{2}\right)\right), \quad i \in \{1, 2, 3\}.
\]

On the other hand, \( p_{r,\varphi}(1/2\tanh(r/2)) \neq 0 \) and \( p_{r,\varphi}(1/2\coth(r/2)) \neq 0 \). Thus, neither \((1/2)\tanh(r/2)\) nor \((1/2)\coth(r/2)\) are eigenvalues of \( s(r) \). This implies that \( M(r) \) has five
dependence on initial conditions of the solution to ordinary differential equations implies

B

then

R

a totally geodesic real hyperbolic plane

H

geodesic perpendicular to

The following two statements hold.

Theorem 4.3. The following two statements hold.

(i) Let \( D \) be the rank one distribution on \( M(r) \) defined by \( B_B^p(r) \), \( p \in M(r) \), and denote by \( D^\perp \) the orthogonal complement of \( D \) in \( TM(r) \). Then both \( D \) and \( D^\perp \) are integrable. Moreover, \( D \) is autoparallel, that is, each of its leaves is totally geodesic in \( M(r) \). If \( p \in M(r) \) and \( R \mathbb{H}^2 \) is the totally geodesic real hyperbolic plane in \( \mathbb{C}H^3 \) which is determined by \( \xi(p) \) and \( B_o \), where \( o \in W_\varphi^4 \) is the footpoint of \( \xi(p) \), then the leaf of \( D \) through \( p \) is parametrized by the parallel curve through \( p \) in \( R \mathbb{H}^2 \) of the geodesic in \( R \mathbb{H}^2 \) through \( o \) and in direction \( B_o \).

(ii) Let \( E \) be the rank two distribution on \( M(r) \) defined by \( B_B^p(r) + B_{PF\xi(p)}^p(r) \). Then \( E \) is autoparallel and each integral manifold has constant sectional curvature \(-(1/4)\text{sech}(r/2)\). If \( p \in M(r) \) and \( R \mathbb{H}^3 \) is the totally geodesic real hyperbolic 3-space in \( \mathbb{C}H^3 \) which is determined by \( \xi(p) \), \( PF\xi(p) \) and \( B_o \), where \( o \in W_\varphi^4 \) is the footpoint of \( \xi(p) \), then the leaf of \( E \) through \( p \) is the parallel surface through \( p \) in \( R \mathbb{H}^3 \) of the totally geodesic \( R \mathbb{H}^2 \) in \( R \mathbb{H}^3 \) through \( o \) determined by \( PF\xi(p) \) and \( B_o \).

Proof. Let \( p \in M(r) \) and \( o \in W_\varphi^4 \) the footpoint of \( \xi(p) \). The vectors \( B_o \) and \( \xi(p) \) determine a totally geodesic real hyperbolic plane \( R \mathbb{H}^2 \subset \mathbb{C}H^3 \) through \( o \). Let \( \tilde{p} \in M(r) \cap R \mathbb{H}^2 \). Since the normal exponential map of \( W_\varphi^4 \subset \mathbb{C}H^3 \) is a diffeomorphism and \( W_\varphi^4 \cap R \mathbb{H}^2 \) is the
path of the geodesic determined by $B_o$ we have that $\xi(\bar{p}) \in T_o R^2$, where $\bar{o} \in W^4_\varphi \cap R^2$ is the footpoint of $\xi(\bar{p})$. Since $R^2$ is totally geodesic, $B^\varphi_B(r)$ is tangent to $M(r) \cap R^2$. This proves that $M(r) \cap R^2$ is an integral manifold of $D$ through $p$. Moreover, if $X \in \Gamma(M(r) \cap R^2)$ and $\nabla M(r)$ denotes the Levi-Civita connection of $M(r)$, it is clear that $\nabla M(r) X \in \Gamma(TM(r))$. On the other hand, since $R^2$ is totally geodesic, $\nabla X X \in \Gamma(T R^2)$ and hence $\nabla X X \in \Gamma(D)$, which proves that $D$ is autoparallel and the first part of (i) follows.

Similarly, let $R^3 \subset CH^3$ be the totally geodesic real hyperbolic space determined by $B_o$, $\xi(p)$ and $PF \xi(p)$. Then the integral submanifold of $\mathcal{E}$ through $p$ is $M(r) \cap R^3$, and since $R^3$ is totally geodesic and intersects $M(r)$ perpendicularly, we see that $\mathcal{E}$ is autoparallel. The curvature of the integral submanifolds of $\mathcal{E}$ is $-(1/4) \text{sech}(r/2)$ as they are equidistant to a totally geodesic $R^2 \subset R^3$ obtained as the intersection of $W^4_\varphi$ and $R^3$. This proves (ii).

Now we prove that $D^\perp$ is integrable. We define the vector field $\bar{\xi}$ along $CH^3 \setminus W^4_\varphi$ by

$$\bar{\xi}_{\exp(r, \eta)} = L_{\exp(r, \eta)} \eta \quad \text{for all unit vectors } \eta \in \nu W^4_\varphi \text{ and } r > 0.$$ 

Let $\eta \in \nu W^4_\varphi$ be a unit vector and denote also by $\eta$ the unit vector field on $CH^3$ obtained by left translation to all points of $CH^3$. Let $\gamma_\eta$ be the geodesic in $CH^3$ such that $\dot{\gamma}_\eta(0) = \eta$. According to (5) we have $\dot{\gamma}_\eta(r) = - \tanh(r/2) B_{\gamma_\eta(r)} + \text{sech}(r/2) \eta_{\gamma_\eta(r)}$ where $B$ and $\eta$ are considered as left-invariant vector fields. Using (5) we get

$$\nabla \dot{\gamma}_\eta \left( \text{sech} \left( \frac{r}{2} \right) B + \tanh \left( \frac{r}{2} \right) \eta \right) = - \frac{1}{2} \text{sech} \left( \frac{r}{2} \right) \tanh \left( \frac{r}{2} \right) B + \text{sech} \left( \frac{r}{2} \right) \nabla \dot{\gamma}_\eta B + \frac{1}{2} \text{sech}^2 \left( \frac{r}{2} \right) \eta + \tanh \left( \frac{r}{2} \right) \nabla \dot{\gamma}_\eta \eta = 0.$$ 

This proves that

$$B^\varphi_B(r) = \text{sech} \left( \frac{r}{2} \right) B_p + \tanh \left( \frac{r}{2} \right) \bar{\xi}_p.$$  

(17)

Now, let $p \in M(r)$. Let us assume without loss of generality that $\gamma_{\xi(p)}(0) = 0$ and write $\eta = \xi(p) \in \nu W^4_\varphi$. The formulas for $\dot{\gamma}_\eta(r)$ and $B^\varphi_B(r)$ show that $D^\perp_p$ is spanned by $Z_p$, $P \eta_p$, $PF \eta_p$ and $F \eta_p$. Let $X, Y \in \Gamma(D^\perp)$. Using (17) we get

$$\langle \nabla_{X_p} Y, B^\varphi_B(r) \rangle = - \langle Y_p, \nabla_{X_p} B_B(r) \rangle = - \text{sech} \left( \frac{r}{2} \right) \langle Y_p, \nabla_{X_p} B \rangle - \tanh \left( \frac{r}{2} \right) \langle Y_p, \nabla_{X_p} \bar{\xi} \rangle.$$ 

The first term on the right-hand side of this equation may be calculated using (6) so we turn our attention to $\nabla_{X_p} \bar{\xi}$. Let $\chi_X$ be a curve such that $\chi_X(0) = X_p$. The curve $\chi_X$ can be written as $\chi_X(t) = \exp_{g_X(t) \cdot 0} (s_X(t)(j_X(t) \bar{F}) \eta + h_X(t) \eta))$ for certain smooth functions $g_X : I \rightarrow \bar{H}$ and $s_X, j_X, h_X : I \rightarrow \mathbb{R}$ satisfying $s_X(0) = r, j_X(0) = 0, h_X(0) = 1$ and $j_X^2 + h_X^2 = 1$. Taking derivatives on the last equality we get $h'_X(0) = 0$. Since $\bar{\xi}_{\chi_X(t)} = j_X(t) \bar{F} \eta + h_X(t) \eta_{\chi_X(t)}$, (6) yields

$$\nabla_{\chi_X(0)} \bar{\xi} = \nabla_{\chi_X(0)} \left( j_X \bar{F} \eta + h_X \eta \right) = j'_X(0) \bar{F} \eta_p + \nabla_{\chi_X(0)} \eta.$$
Again, the second term can be calculated using (3). All in all this means (interchanging the roles of \(X\) and \(Y\)) that
\[
\langle [X,Y], p, \mathcal{E}_B^0(r) \rangle = \langle \nabla_X Y - \nabla_Y X, \mathcal{E}_B^0(r) \rangle = -\tanh \left( \frac{r}{2} \right) \langle j'(0) Y_p - j_Y(0) X_p, \bar{F}_p \rangle.
\]
Note that the vector fields \(Z, P \xi, PF \xi, F \xi\) restricted to \(M(r)\) form a global frame field of \(\mathcal{D}^\perp\), and at the point \(p\) we have \(\bar{\xi}_p = \eta_p\). Therefore it is clear that the result follows if we prove \(j_X'(0) = 0\) for \(X_p \in \{Z_p, P\eta_p, PF\eta_p\}\).

The curve \(\alpha(t) = \exp_{a+n} (tX_0)\) is tangent to \(X_0\) for \(t = 0\). Then, \(\chi_X(t) = \gamma(t) \alpha(t)\) is tangent to \(X_p\) at \(t = 0\). We define
\[
U_Z = \text{sech}^2 \left( \frac{r}{2} \right) Z, \quad U_{\eta} = \text{sech} \left( \frac{r}{2} \right) \eta + \text{sech} \left( \frac{r}{2} \right) \sin^2(\varphi) Z, \quad U_{PF\eta} = \text{sech} \left( \frac{r}{2} \right) PF\eta.
\]
Using (2), (3) and (4) we get
\[
\dot{j}_X(0) = 0 \quad \text{for } \alpha(t) = \exp_{a+n} (tX_0), \quad \text{at the point } p, \quad \text{and}\]
\[
\bar{F}_p = \eta_p, \quad \text{therefore it is clear that the result follows if we prove } \dot{j}_X(0) = 0 \quad \text{for } X_p \in \{Z_p, P\eta_p, PF\eta_p\}.
\]

The curve \(\alpha(t) = \exp_{a+n} (tX_0)\) is tangent to \(X_0\) for \(t = 0\). Then, \(\chi_X(t) = \gamma(t) \alpha(t)\) is tangent to \(X_p\) at \(t = 0\). We define
\[
U_Z = \text{sech}^2 \left( \frac{r}{2} \right) Z, \quad U_{\eta} = \text{sech} \left( \frac{r}{2} \right) \eta + \text{sech} \left( \frac{r}{2} \right) \sin^2(\varphi) Z, \quad U_{PF\eta} = \text{sech} \left( \frac{r}{2} \right) PF\eta.
\]
Using (2), (3) and (4) we get
\[
\chi_X(t) = \left( \ln \text{sech}^2 \left( \frac{r}{2} \right) \eta, \exp \left( 2 \tanh \left( \frac{r}{2} \right) \eta \right) \cdot \left( 0, \exp \left( tX \right) \right) \right).
\]
On the other hand, we have \(\chi(t) := \chi_X(t) = \exp_{g(t),o} (s(t)(j(t)\bar{F}\eta + h(t)\eta))\). We may write \(g(t) = (b(t), \exp_n(x(t)Z + V(t)))\) for certain functions \(b, x : I \to \mathbb{R}\) and \(V : I \to \mathfrak{v}_0\). Taking into account that \(g(t)\) is an isometry, we get, using again (2) and (4),
\[
\chi(t) = \exp_{g(t),o} (s(t)(j(t)\bar{F}\eta + h(t)\eta)) = g(t) \exp_o (s(t)(j(t)\bar{F}\eta + h(t)\eta))
\]
\[
= (b(t), \exp_n(x(t)Z + V(t))) \cdot \left( \ln \text{sech}^2 \frac{s(t)}{2}, \exp_n \left( 2 \tanh \frac{s(t)}{2} (j(t)\bar{F}\eta + h(t)\eta) \right) \right)
\]
\[
= \left( b(t) + \ln \text{sech}^2 \frac{s(t)}{2}, \exp_n \left( V(t) + 2e^{b(t)/2} \tanh \frac{s(t)}{2} (j(t)\bar{F}\eta + h(t)\eta) \right) \right)
\]
\[
\left( x(t) + e^{b(t)/2} \tanh \frac{s(t)}{2} (jV(t), j(t)\bar{F}\eta + h(t)\eta) \right) Z \right).
\]
As \(\exp_n : \mathfrak{n} \to N\) is a diffeomorphism, we easily get \(j(t) \tanh (s(t)/2) = 0\) for all \(t\) from the previous two equations by comparing the \(\bar{F}\eta\)-component. This eventually implies \(j'(0) = 0\), and finishes the proof for the second part of (i).

\[\square\]

\section*{References}

[1] J. Berndt: Real hypersurfaces with constant principal curvatures in complex hyperbolic space, \textit{J. Reine Angew. Math.} \textbf{395} (1989), 132–141.

[2] J. Berndt: Homogeneous hypersurfaces in hyperbolic spaces, \textit{Math. Z.} \textbf{229} (1998), 589–600.

[3] J. Berndt, M. Brück: Cohomogeneity one actions on hyperbolic spaces, \textit{J. Reine Angew. Math.} \textbf{541} (2001), 209–235.

[4] J. Berndt, S. Console, C. Olmos: \textit{Submanifolds and holonomy}, Chapman & Hall/CRC Research Notes in Mathematics \textbf{434}, Chapman & Hall/CRC, Boca Raton, FL, 2003.

[5] J. Berndt, J.C. Díaz-Ramos: Real hypersurfaces with constant principal curvatures in complex hyperbolic spaces, to appear in \textit{J. London Math. Soc.}, preprint \url{arXiv:math.DG/0507361}.
[6] J. Berndt, J.C. Díaz-Ramos: Real hypersurfaces with constant principal curvatures in the complex hyperbolic plane, to appear in Proc. Amer. Math. Soc., preprint arXiv:math.DG/0605118.

[7] J. Berndt, H. Tamaru: Cohomogeneity one actions on noncompact symmetric spaces of rank one, to appear in Trans. Amer. Math. Soc., preprint arXiv:math.DG/0505490.

[8] J. Berndt, F. Tricerri, L. Vanhecke: Generalized Heisenberg groups and Damek–Ricci harmonic spaces, Lecture Notes in Mathematics 1598, Springer–Verlag, Berlin, 1995.

[9] É. Cartan: Familles de surfaces isoparamétriques dans les espaces à courbure constante, Ann. Mat. Pura Appl., IV. Ser. 17 (1938), 177–191.

[10] W.-Y. Hsiang and H.B. Lawson Jr.: Minimal submanifolds of low cohomogeneity, J. Differential Geom. 5 (1971), 1–38.

[11] M. Lohnherr: On ruled real hypersurfaces of complex space forms, PhD Thesis, University of Cologne, 1998.

[12] R. Takagi: On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math. 10 (1973), 495–506.

Department of Mathematics, University College, Cork, Ireland
Email: j.berndt@ucc.ie, jc.diazramos@ucc.ie
Homogeneous hypersurfaces in the complex hyperbolic space

| Type | Group acting | Principal curvatures | Multiplicities | Comments |
|------|--------------|----------------------|----------------|----------|
| (A)  | $S(U(1,k)U(n-k))$ | $\frac{1}{2} \tanh \frac{r}{2}$, $\frac{1}{2} \coth \frac{r}{2}$, $\coth r$ | $2(n-k-1)$, $2k$, $1$ | Tubes around totally geodesic $\mathbb{CH}^k$, $0 \leq k \leq n-1$. Two principal curvatures if $k \in \{0, n-1\}$. |
| (B)  | $SO^0(1,n)$ | $\lambda_1 = \frac{1}{2} \tanh \frac{r}{2}$, $\lambda_2 = \frac{1}{2} \coth \frac{r}{2}$, $\lambda_3 = \tanh r$ | $n-1$, $n-1$, $1$ | Tubes around totally geodesic $\mathbb{R}H^n$. If $r = \ln(2 + \sqrt{3})$ then $\lambda_2 = \lambda_3$. |
| (H)  | $N$ (nilpotent part of Iwasawa decomposition) | $1/2$ | $2(n-1)$ | Horosphere foliation. |
| (S)  | $S$ (Lie algebra of $S$: $s = a + w + g_{2\alpha}$ with $w$ hyperplane in $g_{\alpha}$) | $\frac{3}{2} \tanh \frac{r}{2} + \frac{1}{2} \sqrt{1 - \frac{3}{2} \tanh^2 \frac{r}{2}}$, $\frac{3}{2} \tanh \frac{r}{2} - \frac{1}{2} \sqrt{1 - \frac{3}{2} \tanh^2 \frac{r}{2}}$, $\frac{1}{2} \tanh \frac{r}{2}$ | $1$, $1$, $2n-3$ | Solvable foliation. |
| (W)$_{\pi/2}$ | $N_{K,n}^0(S)S$ (Lie algebra of $S$: $s = a + w + g_{2\alpha}$ with $w$ such that $g_{\alpha} \ominus w$ is real) | $\lambda_1 = \frac{3}{4} \tanh \frac{r}{2} + \frac{1}{4} \sqrt{1 - \frac{3}{2} \tanh^2 \frac{r}{2}}$, $\lambda_2 = \frac{3}{4} \tanh \frac{r}{2} - \frac{1}{4} \sqrt{1 - \frac{3}{2} \tanh^2 \frac{r}{2}}$, $\lambda_3 = \frac{1}{2} \tanh \frac{r}{2}$ | $1$, $1$, $2n-k-2$ | Tubes around $W_{n/2}^{2n-k}$, $2 \leq k \leq n-1$. If $r = \ln(2 + \sqrt{3})$ then $\lambda_2 = \lambda_4$. |
| (W)$_{\varphi}$ | $N_{K,n}^0(S)S$ (Lie algebra of $S$: $s = a + w + g_{2\alpha}$ with $w$ such that $g_{\alpha} \ominus w$ has constant Kähler angle $0 < \varphi < \pi/2$) | $-\frac{1}{6} \left( \coth \frac{r}{2} \left( u_{r,\varphi}^2 + \frac{1}{\tan^2 \frac{r}{2}} \right) + \csch \frac{r}{2} \sech \frac{r}{2} + 4 \tanh \frac{r}{2} \right)$, $-\frac{1}{6} \left( \coth \frac{r}{2} \left( u_{r,\varphi}^2 + \frac{1}{\tan^2 \frac{r}{2}} \right) + \csch \frac{r}{2} \sech \frac{r}{2} + 4 \tanh \frac{r}{2} \right)$, $\frac{1}{2} \tanh \frac{r}{2}$ | $1$, $1$, $2n-k-2$ | Tubes around $W_{\varphi}^{2n-k}$, $2 \leq k \leq n-1$, $k$ even. The number $u_{r,\varphi}^2$ is the $i$th cubic root of $(\beta_{r,\varphi} + \sqrt{\beta_{r,\varphi}^2 - 4})/2$, where $\beta_{r,\varphi} = 27 \sin^2(\varphi) \tan^2(r/2) \sech^2(r/2) - 2$. |

$\lambda_4 = \frac{3}{4} \coth \frac{r}{2}$, $k-1$. Four principal curvatures if $k = 2$. |