Stability analysis of thin film model

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Abstract. Fluid flow of thin film on an inclined plane can be modeled from theory of lubrication into a partial differential equation of the fluid thickness. The difficulty in solving the equation is strongly non-linear of the model. So that a numerical approach is the alternative way that can be done. The numerical stability of that equation is our main concern in this paper, before we solve numerically the model. We found that the numerical method of FTCS is conditional stable.

1. Introduction
A thin film flow on an inclined channel is considered. The model can be derived from equation of continuity, and equation of momentum called Navier Stokes equation. But since the fluid thickness is much smaller than the fluid length the second equation can be simplified into theory of lubrication, see for example in Acheson [1]. The equations together with the boundary equations were then used to construct a single equation by Wiryanto [2].

Similar problem has been done by King, et. al. [3]. The model is in form of an integral-different equation. The surface boundary condition was applied thin air foil theory, such as in Van Dyke [4, 9]. They worked for steady thin film flow, but involving a fast upward airflow above the film. As the solution, a periodic surface wave was obtained analytically and numerically. They also found that the wave is typically roll wave, such as obtained by Merkin and Needham [5], Rifky Fauzan and Wiryanto [6] occurring in the flow down sluices of canals, as the effect of the frictional resistance of the rough sluice bed.

In this paper, we analyse the numerical stability of the model derived in [2, 8], since the equation is second order partial differential equation and strongly non-linear. A finite difference equation is constructed from the linearized equation to get the stability condition. In this case we use forward time central space, and we obtain that those method is conditional stable.

2. Result and Discussion

2.1. Wave propagation
We consider a thin fluid flowing down on an inclined channel. The sketch of the flow is illustrated in Figure 1. The coordinates are chosen Cartesian with horizontal x—axis along the bottom of the channel and the vertical y—axis perpendicular to the other. Wiryanto and Febrianti [2, 8] derived the fluid flow model from the governing equations based on the lubrication theory into

\[ h_t + \frac{\rho}{3\mu} \left[ -h^3 h_x g \cos \theta + h^3 g \sin \theta \right]_x = 0 \]  \hspace{1cm} (1)
where \( h \) is physically the fluid depth, \( \rho \) is the fluid density, \( \mu \) is viscosity, \( g \) is the acceleration of gravity and \( \theta \) is the inclination of the channel.

![Figure 1. Sketch of coordinates and flow](image)

Equation (1) is the problem that is concerned in this paper. Analytically, it is not easy to be solved. Numerical approach is the alternative way that we suggest. To do so, we like to analyse the character of the solution by considering the linearized equation. For \( h = h_0 \) constant it satisfies (1), so that we can involve a small disturbance to it. The solution is in form of \( h(x,t) = h_0 + \varepsilon \eta(x,t) \) for small \( \varepsilon \). When it is substituted in (1) and we take the first order, we have the linear equation presented as

\[
\eta_t + a \eta_x - b \eta_{xx} = 0
\]

(2)

where \( a = \rho g \sin \theta / \mu \), \( b = \rho g \cos \theta / 3 \mu \) both are constant.

Now, we suppose the solution in \( \eta(x,t) = A e^{i(kx-\omega t)} \). Physically it represents the surface wave travelling with frequency in complex form \( \omega = \omega_\rho + i \omega_i \) and wave number \( k \). \( i \) is defined as \( \sqrt{-1} \). So that the amplitude depends on time \( A e^{\omega dt} \). We expect the amplitude does not increase by increasing \( t \). It can happen for \( \omega_i \leq 0 \). So, we can observe by substituting that solution to the equation (2), giving

\[
(\omega_i - i \omega_r) + ai k + bk^2 = 0
\]

The real and imaginary parts are

\[
\omega_i = -bk^2, \quad \omega_r = ak.
\]

Since \( 0 < \theta < \pi / 2 \) (\( a > 0, b > 0 \)), we obtain that the amplitude decreases by increasing time as \( \omega_i < 0 \) and the wave propagates to the right as \( \omega_r > 0 \).

2.2. Numerical stability

In solving (2), we use a finite difference method by forward time central space (FTCS). The finite differenceequation of (2) is

\[
\eta_j^{n+1} = \eta_j^n - \frac{\Delta t}{\Delta x} \left[ \frac{a}{2} (\eta_{j+1}^n - \eta_{j-1}^n) - b \frac{\eta_{j+1}^n - 2 \eta_j^n + \eta_{j-1}^n}{\Delta x^2} \right]
\]

(3)

where \( \eta_j^n \approx \eta(x_j,t_n) \) for \( x_j = j \Delta x, t_n = n \Delta t \), \( j = 0, 1, \ldots, J \); \( n = 0, 1, \ldots \). We discretise the space into \( J \) subspace with length \( \Delta x \) and discrete time \( \Delta t \).

The stability condition can be obtained by applying von Neumann stability, expressing \( \eta_j^n = U^n e^{i\varphi} \) to (3) giving

\[
U = 1 - \frac{\Delta t}{\Delta x} \left[ \frac{a}{2} (e^{i\varphi} - e^{-i\varphi}) - b e^{i\varphi} - 2 + e^{-i\varphi} \right]
\]

After some algebraic operations, it becomes

\[
U = 1 - \frac{\Delta t}{\Delta x} \left[ \frac{a}{2} (2 - 2e^{i\varphi}) - b e^{i\varphi} - 2 + e^{-i\varphi} \right]
\]

\[
U = 1 - \frac{\Delta t}{\Delta x} \left[ \frac{a}{2} (2 - 2\cos \varphi) - b \cos \varphi - 2 + 2 \cos \varphi \right]
\]

\[
U = 1 - \frac{\Delta t}{\Delta x} \left[ \frac{a}{2} (2 - 2\cos \varphi) - b \cos \varphi \right]
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\[
U = 1 - \frac{\Delta t}{\Delta x} \left[ \frac{a}{2} (2 - 2\cos \varphi) - b \right]
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U = 1 - \frac{\Delta t}{\Delta x} \left[ \frac{a}{2} (2 - 2\cos \varphi) - b \right]
\]
\[ U = (1 - \alpha + \alpha \cos \phi) - \beta i \sin \phi \]

where \( \alpha = 2b \Delta t / \Delta x^2 \), \( \beta = a \Delta t / (2 \Delta x) \). The numerical method is stable if \( |U| \leq 1 \), by increasing time it is not followed by increasing \( \eta^n \). Meanwhile, we write \( U \) in complex form \( U = U_r + i U_i \) and the relation between the real and imaginary parts is

\[ \frac{(U_r - (1 - \alpha))^2}{\alpha^2} + \frac{U_i^2}{\beta^2} = 1. \]

(4)

Therefore, the stability condition is obtained from elliptic (4) that is inside in the unity circle \( |U| \leq 1 \). This is satisfied for \( 0 < \alpha < 1 \) and \( \beta \) is not too large. For simplifying we can choose \( \beta \leq \alpha \). For larger \( \beta \), it is possible that part of the elliptic (4) is outside of \( |U| \leq 1 \). In Figure 2, we illustrate (4) inside \( |U| \leq 1 \) for \( \beta < \alpha \), shown by elliptic of solid line. For \( \beta = \sqrt{1 - (1 - \alpha)^2} \), part of the elliptical is outside of the circle, shown by dash line. From that analysis we can express the stability condition for FTCS method is \( \Delta t < \frac{\Delta x^2}{2b} \). The time step \( \Delta t \) used in the FTCS method must be chosen relatively small depending on the space discretization \( \Delta x \). This condition will help us in solving (2) numerically so that the iteration is stable (Figure 2).

**Figure 2.** The stability area shown inside of the elliptic for small \( \beta \) so that it is in the circle

3. **Conclusion**

A model of thin film flow in form of partial differential equation was analysed to observe the character of the solution. We found that the solution represents the surface wave propagating with decreasing
the amplitude. Forward time central space is a numerical method that can be applied to solve the model with conditionally stable, the time step should be chosen of order square of the space discretization.

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