SOME CHARACTERIZATIONS OF STRONGLY IRREDUCIBLE SUBMODULES IN ARITHMETICAL AND NOETHERIAN MODULES

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ABSTRACT. The purpose of the present paper is to prove some properties of the strongly irreducible submodules in the arithmetical and Noetherian modules over a commutative ring. The relationship among the families of strongly irreducible submodules, irreducible submodules, prime submodules and primal submodules is proved. Also, several new characterizations of the arithmetical modules are given. In the case when $R$ is Noetherian and $M$ is finitely generated, several characterizations of strongly irreducible submodules are included. Among other things, it is shown that when $N$ is a submodule of $M$ such that $N:_RM$ is not a prime ideal, then $N$ is strongly irreducible if and only if there exist submodule $L$ of $M$ and prime ideal $p$ of $R$ such that $N$ is $p$-primary, $N \subseteq L \subseteq pM$ and for all submodules $K$ of $M$ either $K \subseteq N$ or $L \subseteq p$. In addition, we show that a submodule $N$ of $M$ is strongly irreducible if and only if there is a prime ideal $p$ of $R$ with $pM \not\subseteq N$ and $M_p$ is an arithmetical module.

1. INTRODUCTION

Let $R$ be a commutative ring with non-zero identity and let $M$ be an arbitrary $R$-module. We say that a submodule $N$ of $M$ is a distributive submodule if for all submodules $K, L$ of $M$, the following equivalent conditions are satisfied:

(i) $(K + L) \cap N = (K \cap N) + (L \cap N)$;
(ii) $(K \cap L) + N = (K + N) \cap (L + N)$.

Also, $M$ is said to be distributive module if every submodule of $M$ is a distributive submodule.

We say that $N$ is an irreducible submodule if $N$ is not the intersection of two submodules of $M$ that properly contain it. It is easy to see that if an irreducible submodule $N$ is distributive, then for all submodules $K, L$ of $M$ the condition $K \cap L \subseteq N$ implies that either $K \subseteq N$ or $L \subseteq N$. These considerations motivated us to define a submodule $N$ of an $R$-module $M$ to be strongly irreducible, if for all submodules $K, L$ of $M$, the condition $K \cap L \subseteq N$ implies that either $K \subseteq N$ or $L \subseteq N$.

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The purpose of the present article is to introduce and examine some properties of strongly irreducible submodules. In particular, we relate the notions of strongly irreducible submodules, irreducible submodules, primal submodules and prime submodules of an $R$-module $M$. Also a characterization of strongly irreducible submodules in an arithmetical module is given. Specifically, we show that a submodule $N$ of an arithmetical $R$-module $M$ is strongly irreducible if and only if $N$ is a primal submodule if and only if $N$ is an irreducible submodule. If moreover $R$ is assumed to be Noetherian ring and $M$ finitely generated, then $N$ is strongly irreducible if and only if $N$ is primary, $M_p$ is arithmetical and $N = (pM)^{(n)}$ for some integer $n > 1$, where $p = \text{Rad}(N :_RM)$ with $p \not\in \text{Ass}_R(R/\text{Ann}_R(M))$ and $pM \notin N$.

We recall that an $R$-module $M$ is said to be arithmetical module if $M_m$ is an uniserial module over $R_m$, for each maximal ideal $m$ of $R$, i.e., the submodules of $M_m$ are linearly ordered with respect to inclusion.

A brief summary of the contents of this article will now be given. Let $R$ be a commutative ring and let $M$ be an arbitrary $R$-module. In Section 2, the notion of strongly irreducible submodules are introduced, and some properties of the m are considered. A typical result in this direction, which is a generalization of the main result of Heinzer et al. (see [7, Theorem 2.6]) to strongly irreducible submodules, is the following:

**Theorem 1.1.** Let $(R, m)$ be a quasi-local ring and let $M$ be an $R$-module. Suppose that $N$ is a strongly irreducible submodule of $M$ such that $N \neq N :_m m$. Then the following conditions hold:

(i) the submodule $N :_M m$ of $M$ is cyclic,
(ii) $N = m(N :_M m)$,
(iii) for each submodule $K$ of $M$ either $K \subseteq N$ or $N :_M m \subseteq K$.

The proof of Theorem 1.1 is given in Theorem 2.11. Using this, it is shown that if $N$ is a strongly irreducible $m$-primary submodule of a finitely generated module $M$ over a local Noetherian ring $(R, m)$ with $mM \neq N$, then

$$N = \bigcup \{K | K \text{ is a submodule of } M \text{ and } K \subseteq N :_M m\},$$

and

$$N :_M m = \bigcap \{L | L \text{ is a submodule of } M \text{ and } N \nsubseteq L\}.$$

In Section 3, we give some characterizations of arithmetical and distributive modules. We also establish two characterizations whenever a submodule of an arithmetical module over a commutative ring is strongly irreducible. More precisely, we shall prove:

**Theorem 1.2.** Let $M$ be an arithmetical $R$-module and let $N$ be a submodule of $M$. Then the following statements are equivalent:

(i) $N$ is strongly irreducible.
(ii) $N$ is primal.
(iii) $N$ is irreducible.

Finally, the main results of the Section 4, provide several characterizations for a finitely generated module $M$ over a Noetherian ring $R$ to have a strongly irreducible submodule. In this section, among other things, we shall show that:
Theorem 1.3. Let $M$ be a finitely generated module over a Noetherian ring $R$ and let $N$ be a submodule of $M$. Then, $N$ is strongly irreducible if and only if $N$ is primary, $M_p$ is an arithmetical $R_p$-module and $N = (pM)^{(n)}$ for some integer $n > 1$, where $p = \text{Rad}(N :_R M)$ such that $pM \nsubseteq N$ and $p \notin \text{Ass}_R(R/\text{Ann}_R(M))$.

The proof of Theorem 1.3 is given in Theorem 4.7. One of our tools for proving Theorem 1.3 is the following:

Proposition 1.4. Let $M$ be a non-zero finitely generated module over a local (Noetherian) ring $(R, \mathfrak{m})$ such that $\text{det}_R(R/\text{Ann}_R(M)) > 0$ and $\Gamma_{\mathfrak{m}}(M) \neq 0$. Let $N$ be a strongly irreducible submodule of $M$ such that $\text{Rad}(N :_R M) = \mathfrak{m}$. Then $\Gamma_{\mathfrak{m}}(M) \nsubseteq N$.

Pursuing this point of view further we derive the following consequence of Theorem 1.3.

Corollary 1.5. Let $M$ be a non-zero finitely generated torsion-free module over a Noetherian integral domain $R$. Then there exists a strongly irreducible submodule $N$ of $M$ such that $N :_R M$ is not prime ideal of $R$ if and only if there is a prime ideal $p$ of $R$ with $pM \nsubseteq N$ and $M_p$ is an arithmetical $R_p$-module.

Finally, using Theorem 1.1 we obtain the following proposition which gives us a characterization of a strongly irreducible submodule in a multiplication module over a commutative Noetherian ring.

Proposition 1.6. Let $M$ be a multiplication module over a Noetherian ring $R$, and suppose that $N$ is a proper submodule of $M$ such that the ideal $N :_R M$ of $R$ is not prime. Then $N$ is strongly irreducible if and only if there exist submodule $L$ of $M$ and prime ideal $p$ of $R$ such that $N$ is $p$-primary, $N \nsubseteq L \subseteq pM$ and for all submodules $K$ of $M$ either $K \subseteq N$ or $L_p \subseteq K_p$.

Throughout this paper, $R$ will always be a commutative ring with non-zero identity and $\mathfrak{a}$ will be an ideal of $R$. For each $R$-module $M$ and for any submodule $N$ of $M$, the submodules $\bigcup_{s \in S}(N :_M s)$ and $\bigcup_{n \geq 0}(0 :_M \mathfrak{a}^n)$ of $M$ are denoted by $S(N)$ and $\Gamma_{\mathfrak{a}}(M)$ respectively, where $S$ is a multiplicatively closed subset of $R$. In the case $S = R \setminus \{p \in \mathfrak{mAss}_R R/MaM\}$, for any integer $m \geq 1$, the submodule $S(\mathfrak{a}^m M)$ is denoted by $(\mathfrak{a} M)^{(m)}$. The radical of $\mathfrak{a}$, denoted by $\text{Rad}(\mathfrak{a})$, is defined to be the set $\{r \in R : r^n \in \mathfrak{a}$ for some $n \in \mathbb{N}\}$. Finally, for any $R$-module $L$, we shall use $Z_R(L)$ (resp. $\mathfrak{mAss}_R L$) to denote the set of zerodivisors on $L$ in $R$ (resp. the set of minimal elements of $\text{Ass}_R L$).

Let $R$ be a Noetherian ring and let $G$ be a non-zero finitely generated $R$-module. For $p \in \text{Supp}(G)$, the $G$-height of $p$, denoted by $ht_Gp$, is defined to be the supremum of lengths of chains of prime ideals of $\text{Supp}(G)$ terminating with $p$. We shall say that an ideal $\mathfrak{a}$ of $R$ is $G$-proper if $G/\mathfrak{a}G \neq 0$, and, when this is the case, we define the $G$-height of $\mathfrak{a}$ (written $ht_Ga$) to be $\inf\{ht_Gp : p \in \text{Supp}(G) \cap V(\mathfrak{a})\}$, where $V(\mathfrak{a})$ denotes the set $\{p \in \text{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$. Also, if $(R, \mathfrak{m})$ is local then we use $\text{depth}_R(G)$ to denote the maximum length of all $G$-sequences contained in $\mathfrak{m}$.

A proper submodule $P$ of an $R$-module $M$ is said to be prime if whenever $rx \in P$ for $r \in R$ and $x \in M$, then $x \in P$ or $r \in (P :_R M)$. (For more information about prime submodules, see [10, 16].)
An $R$-module $M$ is said to be a multiplication module if every submodule of $M$ is of the form $bM$ for some ideal $b$ of $R$. One can easily check that $M$ is a multiplication module if and only if, for all submodules $K$ of $M$, we have $K = (K :_R M)M$.

For any unexplained notation and terminology we refer the reader to [5] or [11].

2. Strongly irreducible submodules

Throughout this section, $R$ will denote a commutative ring (with identity). The purpose of this section is to introduce the concept of strongly irreducible submodules. Several properties of them are considered. The main goals of this section are Theorem 2.11 and Corollary 2.12. We begin with

**Definition 2.1.** Let $M$ be an $R$-module and let $N$ be a submodule of $M$. We say that $N$ is a strongly irreducible submodule of $M$ if for every two submodules $L$ and $K$ of $M$, the inclusion $L \cap K \subseteq N$ implies that either $L \subseteq N$ or $K \subseteq N$.

The first lemma shows that every strongly irreducible submodule in a Noetherian $R$-module $M$ is primary.

**Lemma 2.2.** Let $M$ be an $R$-module and let $N$ be a strongly irreducible submodule of $M$. Then $N$ is irreducible. In particular, if $M$ is Noetherian, then $N$ is a primary submodule of $M$.

**Proof.** Suppose that $N$ is a strongly irreducible submodule of $M$. Let $L$ and $K$ be submodules of $M$ such that $N = L \cap K$. Then $L \cap K \subseteq N$. Since $N$ is strongly submodule, it follows that either $L \subseteq N$ or $K \subseteq N$, and so either $L = N$ or $K = N$. Hence $N$ is irreducible. Now, the second part follows from the proof of [11, Theorem 6.8]. □

In the next lemma we observe that every prime submodule in a multiplication $R$-module $M$ is strongly irreducible.

**Lemma 2.3.** Let $M$ be a multiplication $R$-module and let $N$ be a prime submodule of $M$. Then $N$ is strongly irreducible submodule.

**Proof.** Let $L$ and $K$ be submodules of $M$ such that $L \cap K \subseteq N$. Then $(L \cap K) :_R M \subseteq (N :_R M)$, and so $(L :_R M) \cap (K :_R M) \subseteq (N :_R M)$. As $N :_R M$ is a prime ideal of $R$, it follows that either $L :_R M \subseteq N :_R M$ or $K :_R M \subseteq N :_R M$. Hence, either $(L :_R M)M \subseteq (N :_R M)M$ or $(K :_R M)M \subseteq (N :_R M)M$. Now, since $M$ is a multiplication module, it follows that either $L \subseteq N$ or $K \subseteq N$, as required. □

**Remark 2.4.** Before bringing the next result we fix a notation, which is employed by P. Schenzel in [13] in the case $M = R$. Let $S$ be a multiplicatively closed subset of $R$. For a submodule $K$ of $M$, we use $S(K)$ to denote the submodule $\bigcup_{s \in S}(K :_M s)$.

In particular, for any ideal $a$ of $R$, if $S = R \setminus \bigcup \{p \in \text{mAss}_R M/aM\}$ then for every $n \in \mathbb{N}$; $S(a^nM)$ is denoted by $(aM)^{(n)}$.

**Lemma 2.5.** Let $M$ be an $R$-module and let $N$ be a submodule of $M$. Let $S$ be a multiplicatively closed subset of $R$ such that the submodule $S^{-1}N$ in $S^{-1}M$ is strongly irreducible. Then $S(N)$ is also a strongly irreducible submodule of $M$. 
Proposition 2.6. Let $L$ and $K$ be two submodules of $M$ such that $L \cap K \subseteq S(N)$. Then $S^{-1}(L \cap K) \subseteq S^{-1}(S(N))$, and so it is easy to see that $S^{-1}(L) \cap S^{-1}(K) \subseteq S^{-1}(N)$. Now, the hypothesis on $S^{-1}N$ implies that either $S^{-1}(L) \subseteq S^{-1}(N)$ or $S^{-1}(K) \subseteq S^{-1}(N)$. Hence, either $S^{-1}(L) \cap M \subseteq S^{-1}(N) \cap M$ or $S^{-1}(K) \cap M \subseteq S^{-1}(N) \cap M$; and so either $L \subseteq S(N)$ or $K \subseteq S(N)$, as required.

The following proposition shows that the notion of strongly irreducible submodule behaves well under localization.

Proposition 2.6. Let $M$ be an $R$-module and let $N$ be a strongly irreducible primary submodule of $M$. Let $S$ be a multiplicatively closed subset of $R$ such that $\text{Rad}(N :_R M) \cap S = \emptyset$. Then $S^{-1}N$ is a strongly irreducible submodule of $S^{-1}M$.

Proof. Suppose that $\Omega_1$ and $\Omega_2$ are two submodules of $S^{-1}M$ such that $\Omega_1 \cap \Omega_2 \subseteq S^{-1}N$. Then, in view of [14, Ex. 9.11], there exist submodules $K$ and $L$ of $M$ such that $\Omega_1 = S^{-1}K$ and $\Omega_2 = S^{-1}L$. Hence $S^{-1}K \cap S^{-1}L \subseteq S^{-1}N$, and so $S(K) \cap S(L) \subseteq S(N)$. Now, as $N$ is strongly irreducible, it follows that either $S(K) \subseteq N$ or $S(L) \subseteq N$. Therefore, either $S^{-1}(S(K)) \subseteq S^{-1}N$ or $S^{-1}(S(L)) \subseteq S^{-1}N$, and so either $S^{-1}K \subseteq S^{-1}N$ or $S^{-1}L \subseteq S^{-1}N$, as required.

Lemma 2.7. Let $\mathfrak{p}$ be a prime ideal of $R$ and let $M$ be an $R$-module. Suppose that $N$ is a $\mathfrak{p}$-primary submodule of $M$ such that $N_\mathfrak{p}$ is a strongly irreducible submodule of $M_\mathfrak{p}$. Then $N$ is a strongly irreducible submodule of $M$.

Proof. Let $L$ and $K$ be two submodules of $M$ such that $L \cap K \subseteq N$. Then $L_\mathfrak{p} \cap K_\mathfrak{p} \subseteq N_\mathfrak{p}$. Since $N_\mathfrak{p}$ is strongly irreducible submodule, it follows that either $L_\mathfrak{p} \subseteq N_\mathfrak{p}$ or $K_\mathfrak{p} \subseteq N_\mathfrak{p}$. Now, as $N$ is $\mathfrak{p}$-primary submodule, it readily follows that either $L \subseteq N$ or $K \subseteq N$, as required.

The next lemma investigates how the strongly irreducible property behaves under the faithfully flat extensions.

Lemma 2.8. Assume that $T$ is a commutative ring which is a faithfully flat $R$-algebra. Let $M$ be an $R$-module and assume that $N$ is a submodule of $M$ such that $N \otimes_R T$ is a strongly irreducible submodule of $M \otimes_R T$. Then $N$ is a strongly irreducible submodule of $M$.

Proof. Suppose that $L$ and $K$ are two submodules of $M$ such that $L \cap K \subseteq N$. Then, in view of [11, Theorem 7.4], $(L \otimes_R T) \cap (K \otimes_R T) \subseteq N \otimes_R T$. Now, as $N \otimes_R T$ is a strongly irreducible submodule of $M \otimes_R T$, it follows that either $L \otimes_R T \subseteq N \otimes_R T$ or $K \otimes_R T \subseteq N \otimes_R T$. Now, if $L \otimes_R T \subseteq N \otimes_R T$, then we have

$$(L + N)/N \otimes_R T = ((L \otimes_R T) + (N \otimes_R T))/(N \otimes_R T) = 0,$$

and so by the faithfully flatness of $T$ over $R$ we have $L \subseteq N$. A similar argument also shows that if $K \otimes_R T \subseteq N \otimes_R T$, then $K \subseteq N$. This completes the proof.
Lemma 2.9. Let $M$ be an $R$-module and let $U$ be a submodule of $M$. Assume that $N$ is a strongly irreducible submodule of $M$ containing $U$. Then $N/U$ is a strongly irreducible submodules of $M/U$.

Proof. Suppose that $L$ and $K$ are two submodules of $M$ such that $U \subseteq L \cap K$ and $K/U \cap L/U \subseteq N/U$. Then $K \cap L \subseteq N$, and so as $N$ is strongly irreducible it follows that either $K \subseteq N$ or $L \subseteq N$. Hence either $K/U \subseteq N/U$ or $L/U \subseteq N/U$, as required. $\square$

Proposition 2.10. Let $M$ be an $R$-module and let $N$ be a submodule of $M$. Then $N$ is strongly irreducible submodule if and only if for all cyclic submodules $L$ and $K$ of $M$ the condition $K \cap L \subseteq N$ implies that either $K \subseteq N$ or $L \subseteq N$.

Proof. One direction is clear. To prove the converse, suppose that $T$ and $S$ are two submodules of $M$ such that $T \cap S \subseteq N$ and $T \nsubseteq N$. Then there exists $y \in T$ such that $y \notin N$. Now, for all $x \in S$, we have $Rx \cap Ry \subseteq N$. According to hypothesis $Rx \subseteq N$, and so $S \subseteq N$, as required. $\square$

We are now ready to state and prove the main theorem of this section which provides some properties of a strongly irreducible submodule $N$ of an arbitrary module $M$ over a quasi-local ring $(R, m)$ which is properly contained in $N :_M m$.

It is easy to see that if $R$ is Noetherian and $\text{Rad}(N :_R M) = m$, then $N$ is properly contained in $N :_M m$. This result plays an important role in the Section 4 where we restrict attention to the case $R$ is a Noetherian ring.

Theorem 2.11. Let $(R, m)$ be a quasi-local ring and let $M$ be an $R$-module. Let $N$ be a strongly irreducible submodule of $M$ such that $N \neq N :_M m$. Then

(i) the submodule $N :_M m$ of $M$ is cyclic,

(ii) $N = m(N :_M m)$,

(iii) for each submodule $K$ of $M$ either $K \subseteq N$ or $N :_M m \subseteq K$.

Proof. In order to show (i), in view of the hypothesis $N \neq N :_M m$, there exists an element $x \in N :_M m$ such that $x \notin N$. It is enough for us to show that $N :_M m = Rx$. To do this, let $y$ be an arbitrary element of $(N :_M m)\setminus Rx$. We claim that $Rx \cap Ry \subseteq N$. To this end, set $z \in Rx \cap Ry$. Then there exist elements $a, b \in R$ such that $z = ax = by$. Now, if $b$ is a unit in $R$, then $y \in Rx$, which is a contradiction. Thus we may assume that $b$ is not unit. Then $b \in m$, and so it follows from $y \in N :_M m$ that $by \in N$. That is $z \in N$, and hence $Rx \cap Ry \subseteq N$. Now, since $N$ is strongly irreducible it follows from $x \notin N$ that $Rx \subseteq N$, i.e., $y \in N$. Therefore it follows that $N :_M m = N \cup Rx$, and so $N :_M m = N$ or $N :_M m = Rx$. Consequently, as $N :_M m \neq N$, it yields that $N :_M m = Rx$, and so the submodule $N :_M m$ of $M$ is cyclic.

To prove (ii), in view of (i), we have $N :_M m = Rx$, where $x \in (N :_M m)\setminus N$. Hence $mx \subseteq N$, and so $m \subseteq N :_R x$. As $x \notin N$, it yields that $m = N :_R x$. Thus $m(N :_M m) = (N :_R x)x$. So it is enough for us to show that $N \subseteq (N :_R x)$. To do this, let $y \in N$. Then, as $N \nsubseteq (N :_M m) = Rx$, it follows that $y = rx$, for some $r \in R$. Since $x \notin N$, it yields that $r$ is not unit and so $r \in m$. Hence $y \in mx$, and thus $y \in (N :_R x)x$, as required.
Finally, in order to prove (iii) suppose that $K$ is an arbitrary submodule of $M$ such that $(N :_M m) \not\subseteq K$. We have to show that $K \subseteq N$. To do this, let $y \in K$. Since by part (i) $N :_M m = Rx$, for some $x \in (N :_M m) \setminus N$, and $(N :_M m) \not\subseteq K$, it follows that $x \notin K$. Moreover, $Rx \cap Ry \subseteq N$. Because, if $w \in Rx \cap Ry$, then there are elements $r, s \in R$ such that $w = rx = sy$. Since $y \in K$ and $x \notin K$, it follows that $r$ is not unit, and so $r \in m$. Hence $w \in mx$, i.e., $w \in N$. Now, as $N$ is strongly irreducible we deduce that either $Rx \subseteq N$ or $Ry \subseteq N$. As $x \notin N$, it yields that $y \in N$, and hence $K \subseteq N$, as required. \hfill \Box

Before bringing the final result of this section we recall that a proper submodule $N$ of an $R$-module $M$ is said to be sheltered if the set of submodules of $M$ strictly containing $N$ has a smallest member $S$, called the shelter of $N$ (see [3, Exercise 18, p. 238]).

Corollary 2.12. Let $(R, m)$ be a local (Noetherian) ring and let $M$ be a finitely generated $R$-module. Suppose $N$ is a strongly irreducible submodule of $M$ such that $\text{Rad}(N :_R M) = m$. Then $N$ is sheltered and its shelter is $N :_M m$.

Proof. Since $N$ is a strongly irreducible and $\text{Rad}(N :_R M) = m$, it follows from Lemma 2.2 that $N$ is an $m$-primary submodule of $M$. Hence it is easy to see that $N \neq N :_M m$. Now, the assertion follows from Theorem 2.11. \hfill \Box

3. STRONGLY IRREDUCIBLE SUBMODULES IN ARITHMETICAL MODULES

The first main result of this section, gives us several characterizations of an arithmetical module over a commutative ring. Before stating that theorem, let us recall that an $R$-module $M$ is called an arithmetical module if $M_m$ is uniserial $R_m$-module for each maximal ideal $m$ of $R$, i.e., the submodules of $M_m$ are linearly ordered with respect to inclusion. It is clear that every primary submodule of an arithmetical module is strongly irreducible.

Theorem 3.1. Let $M$ be an $R$-module. Then the following statements are equivalent:

(i) $M$ is an arithmetical module.
(ii) $M$ is a distributive module.
(iii) $(K + L) :_R N = (K :_R N) + (L :_R N)$ for all submodules $K, L, N$ of $M$ with $N$ is finitely generated.
(iv) $K :_R (L \cap N) = (K :_R N) + (L :_R N)$ for all submodules $K, L, N$ of $M$ with $L$ and $N$ are finitely generated.
(v) Every finitely generated submodule of $M$ is a multiplication module.

Proof. (i) $\Rightarrow$ (ii): Let $M$ be an arithmetical $R$-module and we show that the lattice of the submodules of $M$ is distributive, i.e., for all submodules $K, L, N$ of $M$, we have

$$(K + L) \cap N = (K + N) \cap (L + N).$$

To do this, in view of [1, Corollary 3.4 and Proposition 3.8], it is enough for us to show that for all maximal ideals $m$ of $R$, we have

$$(K_m + L_m) \cap N_m = (K_m + N_m) \cap (L_m + N_m).$$

Since the submodules of $M_m$ are linearly ordered with respect to inclusion, it follows that either $K_m \subseteq N_m$ or $N_m \subseteq K_m$. Now, the desired result easily follows from this and the modular law (see [15, Proposition 1.2]).
(ii) $\Rightarrow$ (i): Let $M$ be a distributive $R$-module. Then, it easily follows from [14, Ex. 9.11] that, for all maximal ideals $m$ of $R$, the $R_m$-module $M_m$ is also distributive. So without loss of generality we may assume that $R$ is a quasi-local ring with the unique maximal ideal $m$, and we must show that the the submodules of $M$ are linearly ordered with respect to inclusion. To do this, suppose that the contrary is true, i.e., there exist two submodules $K$ and $L$ of $M$ such that $K \not\subseteq L$ and $L \not\subseteq K$. Then there exist elements $x, y \in M$ such that $x \in K \setminus L$ and $y \in L \setminus K$. Now, since $M$ is a distributive module, it follows that
\[
R(x + y) \cap Rx + R(x + y) \cap Ry = R(x + y) \cap (Rx + Ry) = R(x + y),
\]
and so there are elements $w_1, w_2 \in M$ such that
\[
x + y = w_1 + w_2, \quad \text{where } w_1 \in R(x + y) \cap Rx \text{ and } w_2 \in R(x + y) \cap Ry.
\]
Therefore, there exist elements $r, s, t \in R$ such that $w_1 = rx = s(x + y)$ and $w_2 = ty$. Hence $(r - s)x = sy$, and so the elements $r - s$ and $s$ are not units. This means that $r, s \in m$. On the other hand, since $x + y = rx + ty$, it follows that $(1 - r)x = (t - 1)y$, and so $1 - r \in m$, which is a contradiction. Consequently one of $K \subseteq L$ and $L \subseteq K$ must hold.

(i) $\Rightarrow$ (iii): Let $M$ be an arithmetical $R$-module and let $K, L, N$ be submodules of $M$ such that $N$ is finitely generated. We show that
\[
(K + L) :_R N = (K :_R N) + (L :_R N).
\]
To do this, in view of [11, Corollary 3.4 and Proposition 3.8], we may assume that $R$ is a quasi-local ring. Then the submodules of $M$ are linearly ordered with respect to inclusion. Hence, without loss of generality we may assume that $K \subseteq L$. Then
\[
(K + L) :_R N = L :_R N \text{ and } K :_R N \subseteq L :_R N.
\]
Now, the assertion follows.

(iii) $\Rightarrow$ (i): In view of [11, Corollary 3.4 and Proposition 3.8], we may assume that $R$ is a quasi-local ring with the unique maximal ideal $m$. To establish (i), suppose, on the contrary, that $M$ is not arithmetical, and seek a contradiction. Then, there exist two submodules $K$ and $L$ of $M$ such that $K \not\subseteq L$ and $L \not\subseteq K$. Thus, there exist elements $x, y \in M$ such that $x \in K \setminus L$ and $y \in L \setminus K$. By the hypothesis we have
\[
(Rx + Ry) :_R R(x + y) = (Rx :_R R(x + y)) + (Ry :_R R(x + y)).
\]
hence $R = (Rx :_R R(x+y)) + (Ry :_R R(x+y))$, and so $1 = a + b$, where $a \in (Rx :_R R(x+y))$ and $b \in (Ry :_R R(x+y))$. Therefore, there exist elements $r, s \in R$ such that $ax + ay = rx$ and $bx + by = sy$. Hence $(a - r)x = ay$ and $bx = (s - b)y$. Since $x \notin Ry$ and $y \notin Rx$, it follows that $a, b \in m$, so that $1 \in m$, which is a contradiction.

(i) $\Rightarrow$ (iv): Let $M$ be an arithmetical $R$-module and suppose that $K, L, N$ are submodules of $M$ such that $L$ and $N$ are finitely generated. It is clear that $K :_R L \subseteq K :_R (L \cap N)$ and $K :_R N \subseteq K :_R (L \cap N)$, and so $(K :_R L) + (K :_R N) \subseteq K :_R (L \cap N)$. Now, in order to show the opposite inclusion, in view of [11, Proposition 3.8], it is enough for us to show that, for all maximal ideals $m$ of $R$,
\[
(K :_R (L \cap N))/(K :_R L) + (K :_R N)_m = 0.
\]
To do this, we have
\[(K:R (L \cap N))_m \subseteq K_m : R_m (L \cap N)_m = K_m : R_m (L_m \cap N_m),\]
and in view of \[1, \text{Corollary 3.4 and Proposition 3.8},\]
\[((K : R L) + (K : R N))_m = (K_m : R_m L_m) + (K_m : R_m N_m).\]
Now, since the submodules of \(M_m\) are linearly ordered (with respect to inclusion), we may assume that \(L_m \subseteq N_m\). Then
\[(K : R (L \cap N))_m \subseteq (K_m : R_m L_m)\]
and \(((K : R L) + (K : R N))_m = (K_m : R_m L_m)\)
Therefore
\[(K : R (L \cap N))_m \subseteq ((K : R L) + (K : R N))_m.\]
Now the assertion follows easily from \[1, \text{Corollary 3.4}.\]

(iv) \(\implies\) (i): According to the definition we need to show that for every maximal ideal \(m\) of \(R\), the submodules of the \(R_m\)-module \(M_m\) are linearly ordered (with respect to inclusion). To this end, suppose that the contrary is true and look for a contradiction.

Then, there exist two submodules \(K_m\) and \(L_m\) of \(M_m\) (see \[14, \text{Ex. 9.11}\]) such that \(K_m \not\subseteq L_m\) and \(L_m \not\subseteq K_m\), where \(K\) and \(L\) are submodules of \(M\). Thus there are elements \(x, y \in M\) such that \(x/1 \in K_m \setminus L_m\) and \(y/1 \in L_m \setminus K_m\). Now, by hypothesis (iv), we have
\[R = (Rx \cap Ry : R Rx \cap Ry) = (Rx \cap Ry : R Rx) + (Rx \cap Ry : R Rx) = (Ry : R Rx) + (Ry : R Rx).\]
Hence \(R_m = (R_m y : R_m R_m x) + (R_m x : R_m R_m y)\). But \(R_m y : R_m R_m x\) and \(R_m x : R_m R_m y\) are proper ideals in \(R_m\), we achieve a contradiction.

Finally, the equivalence between (ii) and (v) follows from \[2, \text{Proposition 7}].\]

The next result, which is the second main theorem of this section, gives us two characterizations of strongly irreducible submodules of the arithmetical modules in terms of primal and irreducible submodules. To this end, recall that a proper submodule \(N\) of an \(R\)-module \(M\) is called primal submodule of \(M\) if \(Z_R(M/N)\), the set of zero-divisors of the \(R\)-module \(M/N\), is an ideal of \(R\) (see \[4, \text{Section 2, P. 193}\]). Then, it is easy to see that \(\mathfrak{p} := Z_R(M/N)\) is a prime ideal of \(R\), called the adjoint prime ideal of \(N\). Also, in this case we say that \(N\) is a \(\mathfrak{p}\)-primal submodule of \(M\).

**Theorem 3.2.** Let \(M\) be an arithmetical \(R\)-module and let \(N\) be a submodule of \(M\). Then the following statements are equivalent:

(i) \(N\) is irreducible.
(ii) \(N\) is strongly irreducible.
(iii) \(N\) is primal.

**Proof.** (i) \(\implies\) (ii): Let \(N\) be an irreducible submodule of \(M\) and suppose that \(K\) and \(L\) are two submodules of \(M\) such that \(K \cap L \subseteq N\). Then, as \(M\) is arithmetical, it follows from Theorem 3.1 that
\[N = N + (K \cap L) = (N + K) \cap (N + L).\]
Since \(N\) is irreducible, it yields that either \(N = N + K\) or \(N = N + L\), and so either \(K \subseteq N\) or \(L \subseteq N\), as required.
The implication (ii) $\implies$ (i) follows from Lemma 2.2. In order to show (ii) $\implies$ (iii), suppose that $N$ is a strongly irreducible and let $a, b \in Z_R(M/N)$. Then there exist elements $x, y \in M\setminus N$ such that $ax, by \in N$, and so $Rx \cap Ry \not\subseteq N$. Hence, there exists $z \in Rx \cap Ry$ such that $z \not\in N$. Thus, $z = rx = sy$ for some elements $r, s \in R$, and so $(a - b)z = rax - sby \in N$.

Therefore $(a - b)(z + N) = N$, i.e., $a - b \in Z_R(M/N)$. Moreover, for every $c \in R$, we have $ac(x + N) = N$, and so $ac \in Z_R(M/N)$. This shows that $Z_R(M/N)$ is an ideal of $R$, and hence $N$ is a primal submodule.

(iii) $\implies$ (ii): Let $N$ be a primal submodule of $M$. Then $Z_R(M/N)$ is a prime ideal of $R$; say $p := Z_R(M/N)$. Suppose that $S$ is the multiplication closed subset $R \setminus p$ of $R$. It is then easily seen that $S(N) = N$. Now, since the submodules of the $R_p$-module $M_p$ are linearly ordered with respect to inclusion, it follows that $N_p$ is a strongly irreducible submodule of $M_p$. Hence, in view of Lemma 2.5, $N$ is also a strongly irreducible submodule of $M$, as required. 

\[ \square \]

4. **Strongly irreducible submodules in Noetherian modules**

The purpose of this section is to give a characterization for a finitely generated module $M$ over a Noetherian ring $R$ to have a strongly irreducible submodule. The main goal is Theorem 4.7. To this end, as an application of Theorem 2.11, we first prove the following proposition which is needed in the proof of that theorem.

**Proposition 4.1.** Let $R$ be a Noetherian ring and let $M$ be a finitely generated $R$-module. Suppose that $N$ is a strongly irreducible submodule and assume that $pM_p \neq N_p$, where $p = \text{Rad}(N :_R M)$. Then the following conditions are hold:

(i) The submodule $(N :_M p)_p$ of $M_p$ is cyclic.

(ii) $N_p = p(N_p :_{M_p} pR_p)$.

(iii) For each submodule $K$ of $M$ either $K \subseteq N$ or $(N_p :_{M_p} pR_p) \subseteq K_p$.

**Proof.** First of all, we note that as $N$ is a strongly irreducible submodule of $M$, it follows from Lemma 2.2 that $N$ is a $p$-primary submodule of $M$. Moreover, in view of Proposition 2.6, $N_p$ is a strongly irreducible submodule of $M_p$. Also, since $p = \text{Rad}(N :_R M)$ and $R$ is Noetherian, it follows that there exists an integer $n \geq 1$ such that $p^n M \subseteq N$. Then, it is easy to see that $N_p \subseteq N_p :_{M_p} pR_p$. Hence, it follows from Theorem 2.11 that $N_p = p(N_p :_{M_p} pR_p)$ and the submodule $(N_p :_{M_p} pR_p)$ of $M_p$ is cyclic.

Now, in order to show (iii), let $K$ be an arbitrary submodule of $M$ such that $(N_p :_{M_p} pR_p) \not\subseteq K_p$. Then, it follows from Theorem 2.11 that $K_p \subseteq N_p$, and hence as $N$ is $p$-primary it is easy to see that $K \subseteq N$. 

\[ \square \]

The next result of this section investigates whenever $N$ is a strongly irreducible submodule of a module $M$ over a local (Noetherian) ring $(R, m)$, then $N$ and $N :_M m$ are comparable (under containment) to all submodules of $M$.

**Theorem 4.2.** Let $(R, m)$ be a local (Noetherian) ring and let $M$ be a finitely generated $R$-module. Suppose that $N$ is a strongly irreducible submodule of $M$ such that $mM \neq N$
and \( \text{Rad}(N: R M) = \mathfrak{m} \). Then \( N \) and \( N: \mathfrak{m} \) are comparable by set inclusion to all submodules of \( M \). In fact,

\[
N = \bigcup \{ K \mid K \text{ is a submodule of } M \text{ and } K \subseteq N: \mathfrak{m} \},
\]

and

\[
N: \mathfrak{m} = \bigcap \{ L \mid L \text{ is a submodule of } M \text{ and } N \subsetneq L \}.
\]

**Proof.** Let \( K \) be an arbitrary submodule of \( M \). We must show that either \( N \subseteq K \) or \( K \subseteq N \). To do this, in view of Theorem 2.11 either \( K \subseteq N \) or \( N: \mathfrak{m} \subseteq K \). Now, if \( K \nsubseteq N \), then it follows that \( N: \mathfrak{m} \subseteq K \), and so \( N \subseteq K \). That is \( N \) is comparable to all submodules of \( M \). Also, if \( N: \mathfrak{m} \nsubseteq K \), it follows that \( K \subseteq N \), and therefore \( K \subseteq N: \mathfrak{m} \). Thus \( N: \mathfrak{m} \) is also comparable to all submodules of \( M \). Now, we show that \( N = \bigcup \{ K \mid K \text{ is a submodule of } M \text{ and } K \nsubseteq N: \mathfrak{m} \} \). To do this, if \( K \) is an arbitrary submodule of \( M \) such that \( K \nsubseteq N: \mathfrak{m} \), then \( N: \mathfrak{m} \nsubseteq K \), and so it follows from Theorem 2.11 that \( K \subseteq N \). Therefore

\[
\bigcup \{ K \mid K \text{ is a submodule of } M \text{ and } K \nsubseteq N: \mathfrak{m} \} \subseteq N.
\]

On the other hand, in view of Lemma 2.2 and the hypothesis \( \text{Rad}(N: R M) = \mathfrak{m} \), we deduce that \( N \) is an \( \mathfrak{m} \)-primary submodule of \( M \), so that there exists an integer \( k \geq 1 \) such that \( \mathfrak{m}^k M \subseteq N \). Hence we obtain that \( N \subsetneq N: \mathfrak{m} \), and so

\[
N \subseteq \bigcup \{ K \mid K \text{ is a submodule of } M \text{ and } K \nsubseteq N: \mathfrak{m} \}.
\]

Finally, in order to show

\[
N: \mathfrak{m} = \bigcap \{ L \mid L \text{ is a submodule of } M \text{ and } N \subsetneq L \},
\]

if \( L \) is an arbitrary submodule of \( M \) such that \( N \subsetneq L \), then in view of Theorem 2.11 we have \( N: \mathfrak{m} \subseteq L \). Hence

\[
N: \mathfrak{m} \subseteq \bigcap \{ L \mid L \text{ is a submodule of } M \text{ and } N \subsetneq L \},
\]

Moreover, since \( N \subsetneq N: \mathfrak{m} \) it follows that

\[
\bigcap \{ L \mid L \text{ is a submodule of } M \text{ and } N \subsetneq L \} \subseteq N: \mathfrak{m},
\]

and this completes the proof. \( \square \)

As an application, we derive the following consequence of Theorem 4.2, which shows that every strongly irreducible submodule \( N \) of finitely generated module \( M \) over a Noetherian ring \( R \) of dimension one, is a distributive submodule, whenever the ideal \( N: R M \) of \( R \) contains a regular element on \( M \). Recall that a submodule \( N \) of an \( R \)-module \( M \) is called a **distributive submodule** if for all submodules \( L \) and \( K \) of \( M \),

\[
(K \cap L) + N = (K + N) \cap (L + N).
\]

**Corollary 4.3.** Let \( R \) be a Noetherian ring and let \( M \) be a finitely generated \( R \)-module such that \( \dim M = 1 \). Suppose that \( N \) is a strongly irreducible submodule of \( M \) such that the ideal \( N: R M \) contains a regular element on \( M \). Then \( N \) is a distributive submodule.

**Proof.** According to the definition it is enough to show that for all submodules \( L \) and \( K \) of \( M \), we have

\[
(K \cap L) + N = (K + N) \cap (L + N).
\]
To do this, it suffices to check the equation locally at each prime ideal \( p \) in \( \text{Supp}(M) \). Now, if \( N :_R M \nsubseteq p \), then \( p \nsubseteq \text{Supp}(M/N) \), and so \( N_p = M_p \). Hence the equation clearly holds in this case. We therefore may assume that \( N = M \). Then, it follows from \( N :_R M \nsubseteq Z_R(M) \) that \( \text{ht}_M(N :_R M) = \text{ht}_M p = 1 \). Hence \( \text{Rad}(N :_R M) = p \), and therefore \( \text{Rad}(N_p :_{R_p} M_p) = pR_p \). On the other hand, in view of Proposition 2.6 the submodule \( N_p \) is strongly irreducible in \( M_p \). Consequently, without loss of generality we may assume that \( (R, p) \) is local. Now, in view of Theorem 4.2, either \( K \subseteq N \) or \( N \subseteq K \). If \( K \subseteq N \), then

\[(K \cap L) + N = N \quad \text{and} \quad (K + N) \cap (L + N) = N \cap (L + N) = N,
\]

and so the equation holds in this case. Also, if \( N \subseteq K \) then by the modular law (see [15, Proposition 1.2]), we have

\[(K \cap L) + N = K \cap (L + N) = (K + N) \cap (L + N),
\]

as required. \(\square\)

The following proposition gives us a characterization of strongly irreducible submodule in a multiplication module over a commutative Noetherian ring.

**Proposition 4.4.** Let \( R \) be a Noetherian ring and let \( M \) be a finitely generated multiplication \( R \)-module. Suppose that \( N \) is a proper submodule of \( M \) such that the ideal \( N :_R M \) of \( R \) is not prime. Then \( N \) is strongly irreducible if and only if there exists a submodule \( L \) of \( M \) and a prime ideal \( p \) of \( R \) such that \( N \subseteq L \subseteq pM \) and that \( N \) is \( p \)-primary and for all submodules \( K \) of \( M \) either \( K \subseteq N \) or \( L_p \subseteq K_p \).

**Proof.** First, let \( N \) be a strongly irreducible submodule of \( M \). Then it follows from Lemma 2.2 that \( N \) is primary, and so \( \text{Rad}(N :_R M) \) is a prime ideal of \( R \). Let \( p = \text{Rad}(N :_R M) \), and put \( L = N :_M p \). Now, in view of Proposition 4.1, it is enough for us to show that \( N \nsubseteq L \subseteq pM \). To do this, as \( R \) is Noetherian it follows that there exists an integer \( n \geq 1 \) such that \( p^nM \subseteq L \). Thus \( p^tM \subseteq N \) and \( p^{t-1}M \nsubseteq N \) (even if \( t = 1 \), simply because \( N \neq M \)), and so it yields that \( N_p \subseteq N_p :_{R_p} pR_p \). Hence \( N \nsubseteq L \). In order to show \( L \subseteq pM \), since \( M \) is a multiplication \( R \)-module, there exists an ideal \( a \) of \( R \) such that \( L = N :_M p = aM \). Hence \( aM \subseteq N \). Now, as the ideal \( N :_R M \) is not prime, it follows that \( pM \nsubseteq N \), and so \( a \subseteq p \), note that \( N \) is \( p \)-primary. Therefore \( aM \subseteq pM \), i.e., \( L \subseteq pM \).

In order to show the converse, let \( L \) be a submodule of \( M \) and suppose that \( p \) is a prime ideal of \( R \) such that \( N \nsubseteq L \subseteq pM \) and assume that \( N \) is \( p \)-primary. We show that \( N \) is a strongly irreducible submodule. To do this, let \( T \) and \( S \) be two submodules of \( M \) such that \( T \cap S \subseteq N \). Now, suppose contrary is true, i.e., \( T \nsubseteq N \) and \( S \nsubseteq N \). Then, it follows from hypothesis that \( L_p \subseteq T_p \cap S_p \). Whence \( L_p \subseteq N_p \), and so as \( N \) is \( p \)-primary it easily follows that \( L \subseteq N \), which is a contradiction. \(\square\)

The following two propositions will serve to shorten the proof of the main theorem of this section.
Proposition 4.5. Let \((R, \mathfrak{m})\) be a local (Noetherian) ring and let \(M\) be a non-zero finitely generated \(R\)-module such that \(\operatorname{deth}_R(R/\operatorname{Ann}_R(M)) > 0\) and that \(\Gamma_\mathfrak{m}(M) \neq 0\). Suppose \(N\) is a strongly irreducible submodule of \(M\) such that \(\operatorname{Rad}(N :_R M) = \mathfrak{m}\). Then \(\Gamma_\mathfrak{m}(M) \not\subseteq N\).

Proof. In view of Proposition 4.1 the submodule \(N :_M \mathfrak{m}\) of \(M\) is cyclic, and so there exists \(w \in M\) such that \(N :_M \mathfrak{m} = Rw\). Now, in order to prove the claim, suppose the contrary is true, that is \(\Gamma_\mathfrak{m}(M) \subseteq N\). Then \(\Gamma_\mathfrak{m}(M) \subseteq Rw\), and so there exists an ideal \(\mathfrak{a}\) of \(R\) such that \(\Gamma_\mathfrak{m}(M) = \mathfrak{a}w\). On the other hand, since \(\Gamma_\mathfrak{m}(M)\) is finitely generated it follows that there exists \(t \in \mathbb{N}\) such that \(\mathfrak{m}^t \mathfrak{a}w = 0\). Now, since \(N \subseteq Rw\) and the \(R\)-module \(M/N\) has finite length, it follows that the \(R\)-module \(M/Rw\) also has finite length. Thus there exists \(l \in \mathbb{N}\) such that \(\mathfrak{m}^lM \subseteq Rw\). Therefore \(\mathfrak{m}^{l+t} \mathfrak{a}M \subseteq \mathfrak{m}^t \mathfrak{a}w = 0\), and so \(\mathfrak{m}^t \mathfrak{a} \subseteq \operatorname{Ann}_R(M)\).

Hence

\[(\mathfrak{m}/\operatorname{Ann}_R(M))^{l+t}(\mathfrak{a} + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)) = 0,\]

and so

\[(\mathfrak{a} + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)) \subseteq \Gamma_\mathfrak{m}(R/\operatorname{Ann}_R(M)).\]

Now, since by hypotheses \(\operatorname{deth}_R(R/\operatorname{Ann}_R(M)) > 0\), it follows from [4, Lemma 2.1.1] that \(\Gamma_\mathfrak{m}(R/\operatorname{Ann}_R(M)) = 0\), and so \(\mathfrak{a} \subseteq \operatorname{Ann}_R(M)\). Hence \(\mathfrak{a}w = 0\), and so \(\Gamma_\mathfrak{m}(M) = 0\), which is a contradiction. \(\square\)

Proposition 4.6. Let \(R\) be a Noetherian ring and let \(M\) be a finitely generated \(R\)-module. Suppose \(N\) is a strongly irreducible submodule of \(M\) such that \(\mathfrak{p}M \not\subseteq N\) and that \(\mathfrak{p} \not\in \operatorname{Ass}_R(R/\operatorname{Ann}_R(M))\), where \(\mathfrak{p} = \operatorname{Rad}(N :_R M)\). Then, the ideal \(N_\mathfrak{p} := (N :_M \mathfrak{p})\) of \(M_\mathfrak{p}\) contains a regular element on \(M_\mathfrak{p}\).

Proof. In view of Lemma 2.2 the submodule \(N\) of \(M\) is \(\mathfrak{p}\)-primary and hence it follows from \(\mathfrak{p}M \not\subseteq N\) that \(\mathfrak{p}M_\mathfrak{p} \not\subseteq N_\mathfrak{p}\). So according to Proposition 2.6 it may be assumed that \((R, \mathfrak{p})\) is a local ring, and we must show that the ideal \(\mathfrak{b} := (N :_R M)\) contains a regular element on \(M\). To this end, in view of [4, Lemma 2.1.1], it is enough for us to show that \(\Gamma_{\mathfrak{b}}(M) = 0\). Since \(\operatorname{Rad}(\mathfrak{b}) = \mathfrak{p}\), it is sufficient to establish that \(\Gamma_{\mathfrak{b}}(M) = 0\).

In view of Proposition 4.1 either \(\Gamma_{\mathfrak{b}}(M) \subseteq N\) or \((N :_M \mathfrak{p}) \subseteq \Gamma_{\mathfrak{b}}(M)\). If \(\Gamma_{\mathfrak{b}}(M) \subseteq N\), then the assertion follows from the proof of Proposition 4.5.

Therefore it may be assumed that \((N :_M \mathfrak{p}) \subseteq \Gamma_{\mathfrak{b}}(M)\). Then, there exists an integer \(n \geq 1\) such that \(N \subseteq (0 :_M \mathfrak{p}^n)\), and so \(\mathfrak{p}^nN = 0\). On the other hand, since \(\mathfrak{p} = \operatorname{Rad}(N :_R M)\), there is an integer \(s \geq 1\) such that \(\mathfrak{p}^sM \subseteq N\). Hence \(\mathfrak{p}^{s+t}M \subseteq \mathfrak{p}^tN = 0\), and so \(\mathfrak{p}^{s+t}M = 0\). Therefore \(M\) has finite length. Now, it is easy to see that the \(R\)-module \(R/\operatorname{Ann}_R(M)\) is Artinian, and so \(\operatorname{Ass}_R(R/\operatorname{Ann}_R(M)) = \{\mathfrak{p}\}\), which is a contradiction. \(\square\)

Now we are prepared to prove the main theorem of this section, which gives a characterization for a finitely generated module \(M\) over a Noetherian ring \(R\) to have a strongly irreducible submodule.

Theorem 4.7. Let \(R\) be a Noetherian ring and let \(M\) be a finitely generated \(R\)-module. Let \(N\) be a submodule of \(M\). Then, \(N\) is strongly irreducible if and only if \(N\) is primary,
Lemma 2.2 that \( N \) do this end, in view of Proposition 2.6 the submodule \( N \text{Rad}(M) \). Moreover, it is easy to see that \( N = (pM)^{(n)} \) if and only if \( N_p = p^nM_p \). Therefore, without loss of generality we may assume that \((R, p)\) is local, and that \((pM)^{(n)} = p^nM\). Then, Proposition 4.6 shows that the ideal \((N :_RM)\) contains a regular element on \( M \), and according to Proposition 4.1 the submodule \( N :_RM \) of \( M \) is cyclic. Hence, there exists an element \( w \in M \) such that \( N :_M p = Rw \), and thus by Proposition 4.1 we have \( N = pw \). Now, Since by the Krull intersection theorem, \( \bigcap_{k \geq 1} p^kM = 0 \), and since \( w \neq 0 \), it follows that \( w \in p^nM \), where \( t \) is the greatest integer \( i \) such that \( w \in p^iM \). Then \( N = pw \subseteq p^{i+1}M \). Now, we show that \( p^nM \) is a cyclic submodule. To achieve this, suppose the contrary is true. Then \( p^{i+1}M \not\subseteq p^iM \cup Rw \), and there exists an element \( y \in p^iM \) such that \( y \not\in p^{i+1}M \cup Rw \). Hence, it follows from \( N \subseteq p^{i+1}M \) that \( y \not\in N \). Also, one easily sees from \( y \not\in p^{i+1}M \cup Rw \) that \( w \not\in Rw \). Next, we show that \( Rw \subseteq N \). To do this, let \( x \in Rw \cap Ry \). Then there exist elements \( a, b \in R \) such that \( x = aw = by \). As \( w \not\in Ry \), it follows that \( a \in p \), and so \( aw \in pw \). This shows that \( x \in N \). Since \( N \) is strongly irreducible, it yields that either \( Rw \subseteq N \) or \( Ry \subseteq N \), which is a contradiction. Therefore \( p^nM \) is a cyclic submodule of \( M \). Hence, as \( p \) contains a regular element on \( M \) (note that \( N :_RM \) contains a regular element on \( M \) and \( p = \text{Rad}(N :_RM) \)), it follows from [12, Theorem 2.3] that \( R/\text{Ann}_R(M) \) is a PID and that \( M \cong R/\text{Ann}_R(M) \). Consequently, every proper submodule of \( M \) is of the form \( p^nM \) for some integer \( v \geq 1 \); and this shows that the submodules of \( M \) are linearly ordered with respect to inclusion, i.e., \( M \) is an arithmetical \( R \)-module and \( N = p^nM \) for some integer \( n > 1 \), as required.

In order to prove the converse, since \( M_p \) is an arithmetical \( R \)-module, it follows that every submodule of \( M_p \) is strongly irreducible. In particular \( N_p \) is strongly irreducible submodule of \( M_p \). Now, as \( N \) is a primary submodule of \( M \), it follows from Lemma 2.5 that \( N \) is strongly irreducible.

\[ \square \]

**Corollary 4.8.** Let \( R \) be a Noetherian integral domain and let \( M \) be a non-zero finitely generated torsion-free \( R \)-module. Then there exists a strongly irreducible submodule \( N \) of \( M \) with \( N :_RM \) is not prime ideal of \( R \) if and only if there is a prime ideal \( p \) of \( R \) with \( pM \not\subseteq N \) and \( M_p \) is an arithmetical \( R_p \)-module.

**Proof.** Let \( N \) be a strongly irreducible submodule of \( M \) such that \( N :_RM \) is not prime ideal of \( R \). Then, \( N :_RM \neq 0 \) and in view of Lemma 2.2, \( N \) is primary. Hence \( \text{Rad}(N :_RM) \) is a prime ideal of \( R \). Say \( p = \text{Rad}(N :_RM) \). Then, since \( M \) is torsion-free, it is easy to see that \( p \not\in \text{Ass}_R(R/\text{Ann}_R(M)) \), and so in view of Theorem 4.7 the \( R_p \)-module \( M_p \) is arithmetical. Moreover, as the ideal \( N :_RM \) is not prime, one can easily check that \( pM \not\subseteq N \).
Conversely, let \( p \) be a prime ideal of \( R \) such that \( M_p \) is an arithmetical \( R_p \)-module. Then \( N := (pM)^{(2)} \) is a \( p \)-primary submodule of \( M \). Whence in view of Theorem 4.7, \( N \) is a strongly irreducible submodule of \( M \). Also, the ideal \( N :_R M \) is not a prime ideal in \( R \), because if \( N :_R M \) is a prime ideal, then
\[
N :_R M = \text{Rad}(N :_R M) = \text{Rad}((pM)^{(2)} :_R M) = p,
\]
and so \( pM \subseteq N \), which is a contradiction. \( \square \)

Before we state the final result of this paper, recall that the radical of a submodule \( N \) of an \( R \)-module \( M \), denoted by \( \text{rad}_M(N) \), is defined as the intersection of all prime submodules containing \( N \).

**Proposition 4.9.** Let \( R \) be a Noetherian ring and let \( M \) be a finitely generated \( R \)-module. Let \( N \) be a strongly irreducible submodule of \( M \) such that \( \text{rad}_M(N) = N \). Then \( N \) is a prime submodule of \( M \).

**Proof.** In view of Lemma 2.2 the submodule \( N \) of \( M \) is primary and hence \( \text{Rad}(N :_R M) \) is a prime ideal of \( R \). Hence by applying [9, Theorem 5] we deduce that \( (N :_R M) \) is a prime ideal of \( R \), say \( p := (N :_R M) \), and so in view of [11, Theorem 6.6] we have \( \text{Ass}_R(M/N) = \{p\} \). Hence it follows from [14, Corollary 9.36] that \( Z_R(M/N) = p \). Now, it is easy to see that \( N \) is a prime submodule of \( M \). \( \square \)

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