BOUNDEDNESS AND COMPACTNESS
OF CAUCHY-TYPE INTEGRAL COMMUTATOR
ON WEIGHTED MORREY SPACES

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Abstract

In this paper we study boundedness and compactness characterizations of the commutators of Cauchy type integrals \( \mathcal{C} \) on bounded strongly pseudoconvex domains \( D \) in \( \mathbb{C}^n \) with boundaries \( bD \) satisfying the minimum regularity condition \( C^2 \), based on the recent results of Lanzani–Stein and Duong et al. We point out that in this setting the Cauchy type integral \( \mathcal{C} \) is the sum of the essential part \( \mathcal{C}^\# \) which is a Calderón–Zygmund operator and a remainder \( R \) which is no longer a Calderón–Zygmund operator. We show that the commutator \( [b, \mathcal{C}] \) is bounded on the weighted Morrey space \( L^{p, \kappa}(bD) \) \((v \in A_p, 1 < p < \infty)\) if and only if \( b \) is in the BMO space on \( bD \). Moreover, the commutator \( [b, \mathcal{C}] \) is compact on the weighted Morrey space \( L^{p, \kappa}(bD) \) \((v \in A_p, 1 < p < \infty)\) if and only if \( b \) is in the VMO space on \( bD \).

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1. Introduction

Recently in the field of the interaction of complex analysis of several variables and harmonic analysis, there has been interest in strongly pseudoconvex domains with boundaries satisfying the minimum regularity condition of class \( C^2 \), with new phenomena emerging in the analysis and geometry of domains of this kind. Lanzani and Stein [13] studied the Cauchy–Szegő projection operator in such domains by introducing a family of Cauchy integrals \( \{ \mathcal{C}_\epsilon \} \), and established the \( L^p(bD) \) \((1 < p < \infty, \) with respect to the Leray–Levi measure) boundedness of \( \mathcal{C}_\epsilon \). Different from the case of smooth strongly pseudoconvex domains, the kernels of these Cauchy integral operators...
do not satisfy the standard size or smoothness conditions for Calderón–Zygmund operators. It is important to study the harmonic analysis of such Cauchy–Szegő operators and Cauchy integral operators in complex analysis, for example, to establish the theory of holomorphic Hardy spaces over such domains. Duong et al. [4] proved the characterization of boundedness and compactness for commutators of such Cauchy type integral operators. In an abuse of notation, we omit the subscript $\epsilon$.

**Theorem A** [4]. Suppose $D \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain whose boundary is of class $C^2$ and is strongly pseudoconvex. Suppose $b \in L^1(bD, d\lambda)$. Then, for $1 < p < \infty$:

1. $b \in \text{BMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}]$ is bounded on $L^p(bD, d\lambda)$;
2. $b \in \text{VMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}]$ is compact on $L^p(bD, d\lambda)$.

Because the characterization of the boundedness and compactness of Calderón–Zygmund operator commutators on certain function spaces has many applications in various areas, such as harmonic analysis, complex analysis, (nonlinear) partial differential equations, researchers have recently been seeking to establish such a characterization for singular integral operators (especially non-Calderón–Zygmund operators) over various function spaces. For example, the characterization of boundedness and compactness was proved for Calderón–Zygmund operator commutators on Morrey spaces over Euclidean space by Di Fazio and Ragusa [3] and Chen et al. [2], and the boundedness of such commutators was proved by Komori and Shirai [9] on weighted Morrey spaces. The characterization for Cauchy integrals and Beurling–Ahlfors transformation commutators on $\mathbb{C}$ over weighted Morrey spaces was established by Tao et al. [16, 17]. In this paper, we extend the characterization of the boundedness and compactness of commutators of Cauchy integral operators in Theorem A to weighted Morrey spaces.

Let $p \in (1, \infty)$. A nonnegative function $v \in L^1(bD)$ is in $A_p(bD)$ if

$$[v]_{A_p(bD)} := \sup_{B \subset bD} \left( \frac{1}{\kappa(B)} \int_B v(z) d\lambda(z) \right)^{1/(1/p-1)} \left( \int_B v(z)^{-1/(p-1)} d\lambda(z) \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B$ in $bD$. A nonnegative function $v \in L^1(bD)$ is in $A_1(bD)$ if there exists a constant $C$ such that for all balls $B \subset bD$,

$$\frac{1}{\kappa(B)} \int_B v(z) d\lambda(z) \leq C \text{essinf}_{z \in B} v(z).$$

For $p = \infty$, we define

$$A_\infty(bD) = \bigcup_{1 \leq p < \infty} A_p(bD).$$

Let $p \in (1, \infty)$, $\kappa \in (0, 1)$ and $v \in A_p(bD)$. The weighted Morrey space $L^{p, \kappa}_v(bD)$ (see [8]) is defined by

$$L^{p, \kappa}_v(bD) := \{ f \in L^p_{\text{loc}}(bD) : \| f \|_{L^{p, \kappa}_v(bD)} < \infty \}$$
with
\[
\|f\|_{L_p^\kappa(bD)} := \sup_B \left\{ \frac{1}{|v(B)|^\kappa} \int_B |f(z)|^p v(z) d\lambda(z) \right\}^{1/p},
\]
where
\[
v(B) = \int_B v(z) d\lambda(z).
\]

For the history of Morrey spaces one can refer to [1].

The main result of our paper is to characterize the boundedness and compactness of the commutator of Cauchy type integral \(C\) on the weighted Morrey space, following the ideas and approach in [4].

**Theorem 1.1.** Suppose \(D \subset \mathbb{C}^n, n \geq 2\), is a bounded domain whose boundary is of class \(C^2\) and is strongly pseudoconvex. Suppose \(b \in L^1(bD), 1 < p < \infty, 0 < \kappa < 1\) and \(v \in A_p\). Then \(b \in \text{BMO}(bD)\) if and only if the commutator \([b, C]\) is bounded on \(L_p^\kappa(bD)\).

**Theorem 1.2.** Suppose \(D \subset \mathbb{C}^n, n \geq 2\), is a bounded domain whose boundary is of class \(C^2\) and is strongly pseudoconvex. Suppose \(b \in L^1(bD), 1 < p < \infty, 0 < \kappa < 1\) and \(v \in A_p\). Then \(b \in \text{VMO}(bD)\) if and only if the commutator \([b, C]\) is compact on \(L_p^\kappa(bD)\).

We also consider the Cauchy–Leray integral on a bounded domain in \(\mathbb{C}^n\), which is strongly \(\mathbb{C}\)-linearly convex and whose boundary \(bD\) satisfies the minimum regularity \(C^{1,1}\) (for the details we refer the reader to Section 3 below); such integral operators are studied by Lanzani and Stein in [12]. They obtained the \(L^p(bD)\) boundedness \((1 < p < \infty)\) of the Cauchy–Leray transform \(C\) by showing that the kernel \(K(w, z)\) of \(C\) satisfies the standard size and smoothness conditions of Calderón–Zygmund operators (for details of these definitions and notation we refer the reader to Section 3), and that \(C\) satisfies a suitable version of the \(T(1)\) theorem. In [4], the authors also obtained the characterization of boundedness and compactness for commutators of such transforms. Following a similar approach to that in the proof of Theorems 1.1 and 1.2, we obtain the following results on the Cauchy–Leray transform and its commutator.

**Theorem 1.3.** Let \(D\) be a bounded domain in \(\mathbb{C}^n\) of class \(C^{1,1}\) that is strongly \(\mathbb{C}\)-linearly convex. Let \(1 < p < \infty, v \in A_p\). Then there exists a positive constant \(C\) such that
\[
\|C(f)\|_{L_p^\kappa(bD)} \leq C\|f\|_{L_p^\kappa(bD)}
\]
for every function \(f \in L_p^\kappa(bD)\).

**Theorem 1.4.** Let \(D\) be a bounded domain in \(\mathbb{C}^n\) of class \(C^{1,1}\) that is strongly \(\mathbb{C}\)-linearly convex and let \(b \in L^1(bD), 1 < p < \infty, 0 < \kappa < 1\) and \(v \in A_p\). Let \(C\) be the
Cauchy–Leray transform (as in [12]). Then, for $1 < p < \infty$:

1. $b \in \text{BMO}(bD)$ if and only if the commutator $[b, C]$ is bounded on $L^p_{\kappa}(bD)$;
2. $b \in \text{VMO}(bD)$ if and only if the commutator $[b, C]$ is compact on $L^p_{\kappa}(bD)$.

This paper is organized as follows. In Section 2 we recall the notation and definitions related to a family of Cauchy integrals for bounded strongly pseudoconvex domains in $\mathbb{C}^n$ with minimal smoothness, and then we prove Theorems 1.1 and 1.2. In Section 3 we recall the notation and definitions related to the Cauchy–Leray integral for bounded $C$-linearly convex domains in $\mathbb{C}^n$ with minimal smoothness and give the proofs of Theorems 1.3 and 1.4.

**2. Commutator of Cauchy type integral for bounded strongly pseudoconvex domains with minimal smoothness**

Throughout this section we assume that $D$ is a bounded strongly pseudoconvex domain whose boundary is of class $C^2$.

Let $d(w, z)$ be the quasidistance on the boundary $bD$, which is defined as in [13, Section 2.3] and satisfies the following conditions: there exist constants $A_1 > 0$ and $C_d > 1$ such that for all $w, z, z' \in bD$,

- $d(w, z) = 0$ if and only if $w = z$;
- $d(w, z) \leq C_d(d(w, z') + d(z', z))$;
- $A_1^{-1}d(z, w) \leq d(w, z) \leq A_1d(z, w)$.

Let $d\lambda$ be the Leray–Levi measure on $bD$ (cf. [13, 138]). Then one has

$$d\lambda(w) = \Lambda(w)\,d\sigma(w),$$

where $d\sigma$ is the induced Lebesgue measure on $bD$ and $\Lambda(w)$ is a continuous function such that $c \leq \Lambda(w) \leq \tilde{c}$, $w \in bD$, for some positive constants $c$ and $\tilde{c}$.

We also recall the boundary balls $B_r(w)$ determined via the quasidistance $d$, that is,

$$B_r(w) := \{z \in bD : d(z, w) < r\}, \quad \text{where } w \in bD.$$

According to [13, page 139],

$$c_h^{-1}r^{2n} \leq \lambda(B_r(w)) \leq c_h r^{2n}, \quad 0 < r \leq 1,$$

for some $c_h > 1$. Without loss of generality, by rescaling, we can assume the diameter of the domain to be 1. So we only consider balls of radius less than 1 in this paper.

In [13], the authors defined a family of Cauchy integrals $\{C_\epsilon\}_\epsilon$ and studied their properties when $\epsilon$ is kept fixed. For convenience of notation we henceforth drop explicit reference to $\epsilon$. To study the Cauchy transform $\mathcal{C}$, which is the restriction of such a Cauchy integral on $bD$, one of the key steps in [13] is that they provided a constructive decomposition of $\mathcal{C}$ as follows:

$$\mathcal{C} = \mathcal{C}^\# + \mathcal{R},$$

where $\mathcal{C}^\#$ is the Cauchy transform of the boundary measure and $\mathcal{R}$ is the remainder term.
with essential part
\[
\mathcal{C}^\#(f)(z) := \int_{w \in bD} C^\#(w, z) f(w) \, d\lambda(w), \quad z \in bD.
\]
and remainder
\[
\mathcal{R}(f)(z) := \int_{w \in bD} R(w, z) f(w) \, d\lambda(w), \quad z \in bD.
\]
If we write
\[
\mathcal{C}(f)(z) := \int_{w \in bD} C(w, z) f(w) \, d\lambda(w),
\]
then
\[
C(w, z) = C^\#(w, z) + R(w, z),
\]
where the kernel \( C^\#(w, z) \) satisfies the standard size and smoothness conditions for Calderón–Zygmund operators, that is, there exists a positive constant \( A_2 \) such that for every \( w, z \in bD \) with \( w \neq z \),
\[
|C^\#(w, z)| \leq A_2 \frac{1}{d(w, z)^{2n}};
\]
\[
|C^\#(w, z) - C^\#(w, z')| \leq A_2 \frac{d(z, z')}{d(w, z)^{2n+1}}, \quad \text{if } d(w, z) \geq c d(z, z');
\]
\[
|C^\#(w, z) - C^\#(w', z)| \leq A_2 \frac{d(w, w')}{d(w, z)^{2n+1}}, \quad \text{if } d(w, z) \geq c d(w, w'),
\]
for an appropriate constant \( c > 0 \). However, the kernel \( R(w, z) \) of \( \mathcal{R} \) satisfies a size condition and a smoothness condition for only one of the variables as follows: there exists a positive constant \( C_R \) such that for every \( w, z \in bD \) with \( w \neq z \),
\[
|R(w, z)| \leq C_R \frac{1}{d(w, z)^{2n-1}};
\]
\[
|R(w, z) - R(w, z')| \leq C_R \frac{d(z, z')}{d(w, z)^{2n}}, \quad \text{if } d(w, z) \geq c_R d(z, z'),
\]
for an appropriate large constant \( c_R \).

We also denote by \( BUC(bD) \) the space of all bounded uniformly continuous functions on \( bD \). We first point out that the Leray–Levi measure \( d\lambda \) on \( bD \) is a doubling measure, and satisfies condition (1.1) in [10].

**Lemma 2.1** [4]. The Leray–Levi measure \( d\lambda \) on \( bD \) is doubling, that is, there is a positive constant \( C \) such that for all \( x \in bD \) and \( 0 < r \leq 1 \),
\[
0 < \lambda(B_{2r}(x)) \leq C \lambda(B_r(x)) < \infty.
\]
Moreover, \( \lambda \) satisfies the condition that there exist a constant \( \epsilon_0 \in (0, 1) \) and a positive constant \( C \) such that
\[
\lambda(B_r(x) \setminus B_r(y)) + \lambda(B_r(y) \setminus B_r(x)) \leq C \left( \frac{d(x, y)}{r} \right)^{\epsilon_0}
\]
for all \( x, y \in bD \) and \( d(x, y) \leq r \leq 1 \).

We now recall the bounded mean oscillation (BMO) and vanishing mean oscillation (VMO) spaces on \( bD \). Consider \((bD, d, d\lambda)\) as a space of homogeneous type with \( bD \) compact. Then BMO\((bD)\) is defined as the set of all \( b \in L^1(bD) \) such that
\[
\|b\|_{\text{BMO}(bD)} := \sup_{B \subset bD} \frac{1}{\lambda(B)} \int_B |b(w) - b_B| \, d\lambda(w) < \infty,
\]
where
\[
b_B = \frac{1}{\lambda(B)} \int_B b(z) \, d\lambda(z), \tag{2-3}
\]
and the norm is defined as
\[
\|b\|_{\text{BMO}(bD)} := \|b\|_* + \|b\|_{L^1(bD)}.
\]
The space VMO\((bD)\) is the subspace of BMO\((bD)\) whose elements satisfy the further requirement that
\[
\lim_{a \to 0} \sup_{B \subset bD, r_B = a} \frac{1}{\lambda(B)} \int_B |f(z) - f_B| \, d\lambda(z) = 0,
\]
where \( r_B \) is the radius of \( B \).

The maximal function \( Mf \) is defined as
\[
Mf(z) = \sup_{z \in B \subset bD} \frac{1}{\lambda(B)} \int_B |f(w)| \, d\lambda(w).
\]
The sharp function \( f^\# \) is defined as
\[
f^\#(z) = \sup_{z \in B \subset bD} \frac{1}{\lambda(B)} \int_B |f(w) - f_B| \, d\lambda(w),
\]
where \( f_B \) is defined in (2-3).

Note that from Lemma 2.1, \( d\lambda \) is a doubling measure and hence we have the following results.

**Lemma 2.2 [8].** Let \( v \in A_p(bD), \ p \geq 1 \). Then there exist constants \( C, \sigma > 0 \) such that for every ball \( B \) and measurable subset \( E \subset B \) the inequality
\[
\frac{v(E)}{v(B)} \leq C \left( \frac{\lambda(E)}{\lambda(B)} \right)^{\sigma}
\]
holds.
**Lemma 2.3** [14]. Let $v \in A_p(bD)$, $1 < p < \infty$. There exists a constant $C$ such that for every $f \in L^p_v(bD)$,
\[
\|Mf\|_{L^p_v(bD)} \leq C\|f\|_{L^p_v(bD)},
\]
where $\|f\|_{L^p_v(bD)} = \int_{bD} |f(z)|^p v(z) \, d\lambda(z)$.

**Lemma 2.4** [15]. Let $v \in A_p(bD)$, $1 < p < \infty$. There exists a constant $C$ such that if $\|f\|_{L^p_v(bD)} < \infty$, then
\[
\|f\|_{L^p_v(bD)} \leq C(v(bD))(f_{bD})^p + \|f^\#\|_{L^p_v(bD)}.
\]

**Lemma 2.5** [11]. If $f \in \text{BMO}(bD)$, then there exist positive constants $C_1$ and $C_2$ such that for every ball $B \subset bD$ and every $\alpha > 0$,
\[
\lambda(\{x \in B : |f(x) - f_B| > \alpha\}) \leq C_1 \lambda(B) \exp\left\{-\frac{C_2}{\|f\|_{\text{BMO}(bD)}} \alpha\right\}.
\]

According to [7, Theorem 5.5], we have the following result for BMO functions on $bD$.

**Lemma 2.6.** Let $0 < p < \infty$, $v \in A_\infty(bD)$, $f \in \text{BMO}(bD)$. Then
\[
\|f\|_{\text{BMO}(bD)} \approx \sup_{B \subset bD} \left\{\frac{1}{v(B)} \int_B |f(z) - f_B|^p v(z) \, d\lambda(z)\right\}^{1/p},
\]
where $f_{B,v} = (1/v(B)) \int_B f(z)v(z) \, d\lambda(z)$.

### 2.1. Characterization of BMO($bD$) via the commutator $[b, \mathfrak{C}]$.

**Proof** (Proof of necessity of Theorem 1.1). We first prove necessity, namely that $b \in \text{BMO}(bD)$ implies the boundedness of $[b, \mathfrak{C}]$. We can write
\[
[b, \mathfrak{C}] = [b, \mathfrak{C}^\#] + [b, \mathfrak{R}].
\]

Since the kernel of $\mathfrak{C}^\#$ is a standard kernel on $bD \times bD$, according to [6, Theorem 1.2], we can obtain that $[b, \mathfrak{C}^\#]$ is bounded on $L^{p,\kappa}_v(bD)$ and
\[
\|[b, \mathfrak{C}^\#]\|_{L^{p,\kappa}_v(bD) \to L^{p,\kappa}_v(bD)} \leq \|b\|_{\text{BMO}(bD)}.
\]

Thus, it suffices to show that
\[
\|[b, \mathfrak{R}]\|_{L^{p,\kappa}_v(bD) \to L^{p,\kappa}_v(bD)} \leq \|b\|_{\text{BMO}(bD)}.
\]
Now fix a ball $B = B_r(z_0) \subset bD$ and decompose $f = f \chi_{bD \cap 2B} + f \chi_{bD \setminus 2B} =: f_1 + f_2$. Then

$$\frac{1}{v(B)^\kappa} \int_B |[b, \mathcal{R}] f(z)|^p v(z) \, d\lambda(z)$$

$$\leq \frac{1}{v(B)^\kappa} \int_B |[b, \mathcal{R}] f_1(z)|^p v(z) \, d\lambda(z) + \frac{1}{v(B)^\kappa} \int_B |[b, \mathcal{R}] f_2(z)|^p v(z) \, d\lambda(z)$$

$$=: I + II.$$

For term $I$, by the proof of Theorem 3.2 in [5], we have

$$\frac{1}{v(B)^\kappa} \int_B |[b, \mathcal{R}] f_1(z)|^p v(z) \, d\lambda(z) \leq \frac{1}{v(B)^\kappa} \int_{bD} |[b, \mathcal{R}] f_1(z)|^p v(z) \, d\lambda(z)$$

$$\leq \|b\|_{\text{BMO}(bD)} \frac{1}{v(B)^\kappa} \int_{2B} |f(z)|^p v(z) \, d\lambda(z)$$

$$\leq \|b\|_{\text{BMO}(bD)} \|f\|_{L^p(bD)}.$$

For term $II$, observe that for $z \in B$, by (2-2), we have

$$|[b, \mathcal{R}] f_2(z)|^p \leq \left( \int_{bD \setminus 2B} |b(z) - b(w)||\mathcal{R}(w, z)||f_2(w)| \, d\lambda(w) \right)^p$$

$$\leq \left( \int_{bD \setminus 2B} \frac{|b(z) - b(w)|}{d(w, z)^{2n-1}} |f(w)| \, d\lambda(w) \right)^p$$

$$\leq \left( \int_{bD \setminus 2B} \frac{|f(w)|}{d(w, z_0)^{2n-1}} \{|b(z) - b_B| + |b_B - b(w)|\} \, d\lambda(w) \right)^p$$

$$\leq \left( \int_{bD \setminus 2B} \frac{|f(w)|}{d(w, z_0)^{2n-1}} \, d\lambda(w) \right)^p |b(z) - b_B|^p$$

$$+ \left( \int_{bD \setminus 2B} \frac{|f(w)|}{d(w, z_0)^{2n-1}} |b_B - b(w)| \, d\lambda(w) \right)^p,$$

where $b_B = (1/v(B)) \int_B b(z) v(z) \, d\lambda(z)$. Then

$$II = \frac{1}{v(B)^\kappa} \int_B |[b, \mathcal{R}] f_2(z)|^p v(z) \, d\lambda(z)$$

$$\leq \frac{1}{v(B)^\kappa} \left( \int_{bD \setminus 2B} \frac{|f(w)|}{d(w, z_0)^{2n-1}} \, d\lambda(w) \right)^p \int_B |b(z) - b_B|^p v(z) \, d\lambda(z)$$

$$+ \left( \int_{bD \setminus 2B} \frac{|f(w)|}{d(w, z_0)^{2n-1}} |b_B - b(w)| \, d\lambda(w) \right)^p v(B)^{1-\kappa}$$

$$=: II_1 + II_2.$$
For $II_1$, recall that $(bD, d, d\lambda)$ is a bounded space of homogeneous type and there exists a positive integer $J_0$ depending on $r$ such that $2^{J_0 + 1} B = bD$ and $2^{J_0} B \subseteq bD$. Then, by the Hölder inequality, Lemmas 2.6 and 2.2,

$$II_1 \lesssim \|f\|_{L^p_{\ast}(bD)}^p \frac{1}{\lambda(2B)} \left( \sum_{j=1}^{J_0} \frac{1}{v(2^{j+1}B)^{(1-\kappa)/p}} \right)^p \int_B |b(z) - b_{B,v}|^p v(z) d\lambda(z)$$

$$\lesssim \|b\|_{\text{BMO}(bD)}^p \|f\|_{L^p_{\ast}(bD)}^p \left( \sum_{j=1}^{J_0} \frac{1}{v(2^{j+1}B)^{(1-\kappa)/p}} \right)^p$$

$$\lesssim \|b\|_{\text{BMO}(bD)}^p \|f\|_{L^p_{\ast}(bD)}^p.$$  

For $II_2$, by the Hölder inequality,

$$II_2 \lesssim v(B)^{1-\kappa} \left( \sum_{j=1}^{J_0} \frac{1}{\lambda(2B)} \int_{2^{j+1}B} |f(w)| |b(w) - b_{B,v}| d\lambda(w) \right)^p$$

$$\lesssim v(B)^{1-\kappa} \left\{ \sum_{j=1}^{J_0} \frac{1}{\lambda(2B)} \left( \int_{2^{j+1}B} |f(w)|^p v(w) d\lambda(w) \right)^{1/p} \right\}^p$$

$$\times \left( \int_{2^{j+1}B} |b(w) - b_{B,v}|^p v(w)^{1-p'} d\lambda(w) \right)^{1/p'}$$

$$\lesssim v(B)^{1-\kappa} \|f\|_{L^p_{\ast}(bD)}^p \times \left\{ \sum_{j=1}^{J_0} \frac{v(2^{j+1}B)^{\kappa/p}}{\lambda(2B)} \left( \int_{2^{j+1}B} |b(w) - b_{B,v}|^p v(w)^{1-p'} d\lambda(w) \right)^{1/p'} \right\}^p$$

$$\lesssim v(B)^{1-\kappa} \|f\|_{L^p_{\ast}(bD)}^p \times \left\{ \sum_{j=1}^{J_0} \frac{v(2^{j+1}B)^{\kappa/p}}{\lambda(2B)} \left( \int_{2^{j+1}B} |b(w) - b_{2^{j+1}B,v_1-p}|^p v(w)^{1-p'} d\lambda(w) \right)^{1/p'} \right\}^p$$

$$+ \left( \int_{2^{j+1}B} |b_{2^{j+1}B,v_1-p} - b_{B,v}|^p v(w)^{1-p'} d\lambda(w) \right)^{1/p'}$$

$$=: v(B)^{1-\kappa} \|f\|_{L^p_{\ast}(bD)}^p \left[ \sum_{j=1}^{J_0} \frac{v(2^{j+1}B)^{\kappa/p}}{\lambda(2B)} (II_{21} + II_{22}) \right]^p.$$  

For $II_{21}$, since $v \in A^p_\rho(bD)$, we have $v^{1-p'} \in A^\rho_\rho(bD)$, where $1/p + 1/p' = 1$. By Lemma 2.6, we can obtain that

$$II_{21} \lesssim \|b\|_{\text{BMO}(bD)} \left[ v^{1-p'}(2^{j+1}B) \right]^{1/p'}.$$
For $I_{22}$, by Lemma 2.6,
\[
|b_{2^{j+1}B,v^{1-\rho'}} - b_{B,v}| \leq |b_{2^{j+1}B,v^{1-\rho'}} - b_{2^{j+1}B}| + |b_{2^{j+1}B} - b_B| + |b_B - b_{B,v}|
\]
\[
\leq \frac{1}{v^{1-\rho'}(2^{j+1}B)} \int_{2^{j+1}B} |b(w) - b_{2^{j+1}B}|v(w)^{1-\rho'} \, d\lambda(w)
\]
\[
+ 2^{2n}(j + 1)\|b\|_{\text{BMO}(bD)} + \frac{1}{v(B)} \int_B |b(w) - b_B|v(w) \, d\lambda(w).
\]

Since $b \in \text{BMO}(bD)$, by Lemmas 2.2 and 2.5, there exist $\tilde{C}_1 > 0$ and $\tilde{C}_2 > 0$ such that for any ball $B$ and $\alpha > 0$,
\[
v(\{z \in B : |b(z) - b_B| > \alpha\}) \leq \tilde{C}_1 v(B)e^{-\tilde{C}_2 \alpha \sigma/\|b\|_{\text{BMO}(bD)}},
\]
for some $\sigma > 0$. Therefore,
\[
\int_B |b(w) - b_B|v(w) \, d\lambda(w) = \int_0^\infty v(\{z \in B : |b(z) - b_B| > \alpha\}) \, d\alpha
\]
\[
\leq v(B) \int_0^\infty e^{-\tilde{C}_2 \alpha \sigma/\|b\|_{\text{BMO}(bD)}} \, d\alpha
\]
\[
\leq v(B)\|b\|_{\text{BMO}(bD)}.
\]

Similarly,
\[
\int_{2^{j+1}B} |b(w) - b_{2^{j+1}B}|v(w)^{1-\rho'} \, d\lambda(w) \leq (j + 1)\|b\|_{\text{BMO}(bD)}v^{1-\rho'}(2^{j+1}B).
\]

Then,
\[
I_{22} \leq (j + 1)\|b\|_{\text{BMO}(bD)}[v^{1-\rho'}(2^{j+1}B)]^{1/\rho'}.
\]

Now together with Lemma 2.2,
\[
H_2 \leq v(B)^{1-k}\|b\|_{\text{BMO}(bD)}^p \|f\|^p_{L^{\rho'}_{\omega^p}(bD)} \left\{ \sum_{j=1}^{j_0} \frac{v(2^{j+1}B)^{\lambda/p} \lambda(2^{j+1}B)}{\lambda(2^{j+1}B)}(j + 1)[v^{1-\rho'}(2^{j+1}B)]^{1/\rho'} \right\}^p
\]
\[
\leq \|b\|^p_{\text{BMO}(bD)} \|f\|^p_{L^{\rho'}(bD)} \left[ \sum_{j=1}^{j_0} \frac{(j + 1)v(B)^{(1-k)/\rho} \gamma^p}{v(2^{j+1}B)^{1-k/p}} \right]^p
\]
\[
\leq \|b\|^p_{\text{BMO}(bD)} \|f\|^p_{L^{\rho'}_{\omega^p}(bD)} \left[ \sum_{j=1}^{\infty} \frac{(j + 1)2^{-(j+1)(1-k)Q\sigma/p}}{} \right]^p
\]
\[
\leq \|b\|^p_{\text{BMO}(bD)} \|f\|^p_{L^{\rho'}(bD)}.
\]
Consequently,

\[ II \leq \|b\|_{\text{BMO}(bD)}^p \|f\|_{L_p^e(bD)}^p. \]

As a result,

\[ \|[b, \mathcal{R}]f\|_{L_p^e(bD)} \leq \|b\|_{\text{BMO}(bD)} \|f\|_{L_p^e(bD)}^p. \]

This completes the proof of the necessity part.

In order to prove the sufficiency of Theorem 1.1, we need the following lemma, which can be obtained by the proof of (3.10) in [5, Theorem 3.2] and of (2.18) in [4, Theorem 1.1].

**Lemma 2.7.** Denote by \( C_1(w, z) \) and \( C_2(w, z) \) the real and imaginary parts of \( C(w, z) \), respectively. Then at least one of \( C_1 \) and \( C_2 \) satisfies the following argument. There exist positive constants \( \gamma_0, A \) such that for every ball \( B = B_r(z_0) \subset bD \) with \( r < \gamma_0 \), there exists another ball \( \tilde{B} = B_r(w_0) \subset bD \) with \( Ar \leq d(w_0, z_0) \leq (A + 1)r \) such that for every \( z \in B \) and \( w \in \tilde{B} \), \( C_1(w, z) \) does not change sign and

\[ |C_1(w, z)| \geq \frac{c}{d(w, z)^{2n}}. \]

**Proof (Proof of sufficiency of Theorem 1.1).** We turn to the proof of the sufficiency condition, namely that if \([b, \mathcal{C}]\) is bounded on \( L^p_{e, \kappa}(bD) \), then \( b \in \text{BMO}(bD) \). We mainly follow the method and technique in [4].

Assume that \( b \) is in \( L^1(bD) \) and that \( \|[b, \mathcal{C}]\|_{L_p^e(bD) \to L_p^e(bD)} < \infty \). Let \( \gamma_0 \) be the constant in Lemma 2.7. We test the \( \text{BMO}(bD, d\lambda) \) condition on the case of balls with big radius and small radius.

**Case 1:** In this case we work with balls with large radius, \( r \geq \gamma_0 \).

By (2-1) and by the fact that \( \lambda(B) \geq \lambda(B_{\gamma_0}(z_0)) \approx \gamma_0^{2n} \),

\[ \frac{1}{\lambda(B)} \int_B |b(w) - b_B| d\lambda(w) \leq \gamma_0^{-2n} \|b\|_{L^1(bD, d\lambda)}. \]

**Case 2:** In this case we work with balls with small radius, \( r < \gamma_0 \).

We aim to prove that for every fixed ball \( B = B_r(z_0) \subset bD \) with radius \( r < \gamma_0 \),

\[ \frac{1}{\lambda(B)} \int_B |b(w) - b_B| d\lambda(w) \leq \|[b, \mathcal{C}]\|_{L_p^e(bD) \to L_p^e(bD)}. \]

Now let \( \tilde{B} = B_r(w_0) \) be the ball chosen as in Lemma 2.7. Without loss of generality we may assume that \( C_1(w, z) \) does not change sign and

\[ |C_1(w, z)| \geq \frac{c}{d(w, z)^{2n}}. \quad (2-4) \]

Let \( m_b(\tilde{B}) \) be the median value of \( b \) on the ball \( \tilde{B} \) with respect to the measure \( d\lambda \). Following the sufficiency proof of Theorem 1.1(1) in [4], we define sets \( F_1 = \{ w \in \tilde{B} : b(w) \leq m_b(\tilde{B}) \}, F_2 = \{ w \in \tilde{B} : b(w) \geq m_b(\tilde{B}) \}, E_1 = \{ z \in B : b(z) \geq m_b(\tilde{B}) \} \) and
$E_2 = \{z \in B : b(z) < m_b(\tilde{B})\}$. Then it is immediate that $\tilde{B} = F_1 \cup F_2$, $B = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, and from definition of the median value we can see that

$$\lambda(F_i) \geq \frac{1}{2} \lambda(\tilde{B}), \quad i = 1, 2. \quad (2-5)$$

It is clear that

$$\begin{cases} b(z) - b(w) \geq 0, & (z, w) \in E_1 \times F_1, \\ b(z) - b(w) < 0, & (z, w) \in E_2 \times F_2. \end{cases}$$

Then, for $(z, w) \in (E_1 \times F_1) \cup (E_2 \times F_2)$,

$$|b(z) - b(w)| = |b(z) - m_b(\tilde{B}) + m_b(\tilde{B}) - b(w)| \geq |b(z) - m_b(\tilde{B})|. \quad (2-6)$$

Together with (2-5), (2-6) and (2-4), we have

$$\frac{1}{\lambda(B)} \int_{E_1} |b(z) - m_b(\tilde{B})| \, d\lambda(z)$$

$$\leq \frac{1}{\lambda(B)} \frac{\lambda(F_1)}{\lambda(B)} \int_{E_1} |b(z) - m_b(\tilde{B})| \, d\lambda(z)$$

$$\leq \frac{1}{\lambda(B)} \int_{E_1} \int_{F_1} \frac{1}{\lambda(B)} |b(z) - b(w)| \, d\lambda(w) \, d\lambda(z)$$

$$\leq \frac{1}{\lambda(B)} \int_{E_1} \int_{F_1} |C_1(w, z)(b(z) - b(w))| \, d\lambda(w) \, d\lambda(z)$$

$$\leq \frac{1}{\lambda(B)} \int_{E_1} ||[b, c](\chi_{F_1})(z)|| \, d\lambda(z).$$

Then by Hölder’s inequality and the fact that $\|[b, c]||_{L^p_s(x, dy)} \rightarrow L^{s_1}_p(dy) < \infty$,

$$\frac{1}{\lambda(B)} \int_{E_1} |b(z) - m_b(\tilde{B})| \, d\lambda(z)$$

$$\leq \frac{1}{\lambda(B)} \left( \int_{E_1} v^{-(p'/p)} \, d\lambda(z) \right)^{1/p'} \left( \int_{E_1} ||[b, c](\chi_{F_1})(z)||^p \, d\lambda(z) \right)^{1/p}$$

$$\leq \frac{1}{\lambda(B)} \left( \int_B v^{-(p'/p)} \, d\lambda(z) \right)^{1/p'} (v(B))^{s/p} ||[b, c]||_{L^p_s(x, dy)}$$

$$\leq (v(B))^{(s-1)/p} ||[b, c]||_{L^p_s(dx)} \|\chi_{F_1}||_{L^{s_1}_p(dy)}$$

$$\leq (v(B))^{(s-1)/p} (v(\tilde{B}))^{(1-s)/p} ||[b, c]||_{L^p_s(dx)} \|\chi_{F_1}||_{L^{s_1}_p(dy)}$$

$$\leq ||[b, c]||_{L^p_s(dx)} \rightarrow L^{s_1}_p(dy).$$

Similarly,

$$\frac{1}{\lambda(B)} \int_{E_2} |b(z) - m_b(\tilde{B})| \, d\lambda(z) \leq ||[b, c]||_{L^p_s(dx)} \rightarrow L^{s_1}_p(dy).$$
and hence,
\[
\frac{1}{\lambda(B)} \int_B |b(z) - m_b(\tilde{B})| \, d\lambda(z) \leq \|[b, \mathcal{C}]\|_{L^p_\kappa(bD) \to L^p_\kappa(bD)}.
\]

Consequently,
\[
\frac{1}{\lambda(B)} \int_B |b(z) - b| \, d\lambda(z) \leq \frac{2}{\lambda(B)} \int_B |b(z) - m_b(\tilde{B})| \, d\lambda(z)
\]
\[
\lesssim \|[b, \mathcal{C}]\|_{L^p_\kappa(bD) \to L^p_\kappa(bD)}.
\]

This finishes the proof of sufficiency of Theorem 1.1. \(\square\)

2.2. Characterization of VMO\((bD, d\lambda)\) via the Commutator \([b, \mathcal{C}]\). Based on Lemma 2.1, we have the following fundamental lemma from [10, Lemma 1.2].

**Lemma 2.8.** Let \(b \in \operatorname{VMO}(bD, d\lambda)\). Then, for any \(\xi > 0\), there is a function \(b_\xi \in \operatorname{BUC}(bD)\) such that
\[
\|b_\xi - b\|_* < \xi.
\]
Moreover, \(b_\xi\) satisfies the following conditions: there is an \(\epsilon \in (0, 1)\) such that
\[
|b_\xi(w) - b_\xi(z)| < C_\xi d(w, z)^\epsilon, \quad \text{for all } w, z \in bD. \tag{2-7}
\]

For each \(0 < \eta \ll 1\), we let \(R^\eta(w, z)\) be a continuous extension of the kernel \(R(w, z)\) of \(\mathcal{R}\) from \(bD \times bD\) to \(bD \times bD\) such that
\[
R^\eta(w, z) = R(w, z), \quad \text{if } d(w, z) \geq \eta;
\]
\[
|R^\eta(w, z)| \lesssim \frac{1}{d(w, z)^{2n-1}}, \quad \text{if } d(w, z) < \eta;
\]
\[
R^\eta(w, z) = 0, \quad \text{if } d(w, z) < \eta/c, \quad \text{for some } c > 1.
\]

Let \(\mathcal{R}^\eta\) be the integral operator associated to the kernel \(R^\eta(w, z)\). Then we have the following approximation result.

**Lemma 2.9.** Let \(b \in \operatorname{BUC}(bD)\) satisfy
\[
|b(w) - b(z)| < c \, d(w, z)^\epsilon, \quad \text{for some } c \geq 1, \, \epsilon \in (0, 1), \, \text{for all } w, z \in bD. \tag{2-8}
\]

Then, for \(1 < p < \infty, \, 0 < \kappa < 1\) and \(v \in A_p\),
\[
\|[b, \mathcal{R}] - [b, \mathcal{R}^\eta]\|_{L^p_\kappa(bD) \to L^p_\kappa(bD)} \to 0
\]
as \(\eta \to 0\).
PROOF. Let \( f \in L_v^{p,k}(bD) \). For any \( z \in bD \),

\[
[b, \Re] f(z) - [b, \Re^\eta] f(z) = \left| \int_{\frac{1}{2} \leq d(w,z) < \eta} (b(z) - b(w)) R^\eta(w, z) f(w) \, d\lambda(w) 
- \int_{d(w,z) < \eta} (b(z) - b(w)) R(w, z) f(w) \, d\lambda(w) \right|
\leq \int_{d(w,z) < \eta} \frac{d(w,z)^n}{d(w,z)^{2n-1}} |f(w)| \, d\lambda(w).
\]

Then

\[
\| [b, \Re] f - [b, \Re^\eta] f \|_{L_v^{p,k}(bD)} \leq \eta^{\ell+1} \| f \|_{L_v^{p,k}(bD)},
\]

which implies that

\[
\lim_{\eta \to 0} \| [b, \Re] - [b, \Re^\eta] \|_{L_v^{p,k}(bD) \to L_v^{p,k}(bD)} = 0.
\]

This concludes the proof.

We can now give the proof of Theorem 1.2.

PROOF (Proof of Theorem 1.2). We begin with sufficiency. Assume that \( v \in A_p, 1 < p < \infty \) and that \( [b, \mathcal{C}] \) is compact on \( L_v^{p,k}(bD) \). Then \( [b, \mathcal{C}] \) is bounded on \( L_v^{p,k}(bD) \). By Theorem 1.1, we have \( b \in \text{BMO}(bD) \). Without loss of generality, we may assume that \( \|b\|_{\text{BMO}(bD)} = 1 \).

To show \( b \in \text{VMO}(bD) \), we seek a contradiction. In its simplest form, the contradiction is that there is no bounded operator \( T : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N}) \) with \( T e_j = T e_k \neq 0 \) for all \( j, k \in \mathbb{N} \). Here, \( e_j \) is the standard basis for \( \ell^p(\mathbb{N}) \).

The main step is to construct the approximation to a standard basis in \( \ell^p \), namely a sequence of functions \( \{g_j\} \) such that \( \|g_j\|_{L_v^{p,k}(bD)} \approx 1 \), and such that for a nonzero \( \phi \), we have \( \|\phi - [b, \mathcal{C}] g_j\|_{L_v^{p,k}(bD)} < 2^{-j} \).
Suppose that $b \notin \text{VMO}(bD)$. Then there exist $\delta_0 > 0$ and a sequence \{\(B_j\)\}_{j=1}^\infty := \{B_r(z_j)\}_{j=1}^\infty$ of balls such that
\[
\frac{1}{\lambda(B_j)} \int_{B_j} |b(z) - b_{B_j}| \, d\lambda(z) \geq \delta_0.
\]
Without loss of generality, we assume that $r_j < \gamma_0$ for all $j$, where $\gamma_0$ is as in Lemma 2.7.

Now choose a subsequence \{\(B_{j_i}\)\} of \{\(B_j\)\} such that
\[
\frac{1}{4c_\lambda} r_{j_i} \leq \frac{1}{4c_\lambda} r_{j_i},
\]
where $c_\lambda$ is the constant as in (2-1).

For the sake of simplicity we drop the subscript $i$, that is, we denote \{\(B_{j_i}\)\} by \{\(B_j\)\}.

Following the sufficiency proof of Theorem 1.1(2) in [4], for each such $B_j$, we can choose a corresponding ball $\tilde{B}_j$ and the corresponding disjoint subsets $E_{j,1}, E_{j,2} \subset B_j$ and $\tilde{F}_{j,1}, \tilde{F}_{j,2} \subset \tilde{B}_j$, such that $B_j = E_{j,1} \cup E_{j,2}$, and we have
\[
\lambda(\tilde{F}_{j,i}) \geq \frac{\lambda(\tilde{B}_j)}{4}, \quad i = 1, 2,
\]
and
\[
\frac{1}{\lambda(B_j)} \int_{E_{j,1}} |b(z) - m_{\tilde{B}_j}| \, d\lambda(z) \leq \frac{1}{\lambda(B_j)} \int_{E_{j,1}} \| [b, \mathcal{C}] (\chi_{\tilde{F}_{j,1}})(z) \| \, d\lambda(z).
\]

Next, by using Hölder’s inequality and $v \in A_p$,
\[
\frac{1}{\lambda(B_j)} \int_{E_{j,1}} |b(z) - m_{\tilde{B}_j}| \, d\lambda(z)
\leq \frac{1}{\lambda(B_j)} \left( \int_{E_{j,1}} v^{-(p'/p)} \, d\lambda(z) \right)^{1/p'} \left( \int_{E_{j,1}} \| [b, \mathcal{C}] (\chi_{\tilde{F}_{j,1}})(z) \|^p v(z) \, d\lambda(z) \right)^{1/p}
\leq \frac{1}{\lambda(B_j)} \lambda(B_j) v(B_j)^{-1/p} \| [b, \mathcal{C}] (\chi_{\tilde{F}_{j,1}}) \|_{L^{p'}_{\mathcal{C}}(bD)}
\leq \| [b, \mathcal{C}](f_j) \|_{L^{p'}_{\mathcal{C}}(bD)},
\]
where in the above inequalities we denote
\[
f_j := v(B_j)^{\kappa-1/p} \chi_{\tilde{F}_{j,1}}.
\]

Combining the above estimates, we have that
\[
0 < \delta_0 \leq \| [b, \mathcal{C}](f_j) \|_{L^{p'}_{\mathcal{C}}(bD)}.
\]
Note that
\[ \|f_j\|_{L^p(B_D)} = v(B_j)^{(k-1)/p} \sup_B \left\{ \frac{1}{|v(B)|^k} \int_B |\chi_{\overline{F}_{j,1}}(z)|^p v(z) \, d\lambda(z) \right\}^{1/p} \]

\[ = v(B_j)^{(k-1)/p} \sup_B \left\{ \frac{v(B \cap \overline{F}_{j,1})}{|v(B)|^k} \right\}^{1/p} \]

Since \( v \in A_p \), it follows that there exist positive constants \( C_1, C_2 \) and \( \sigma \in (0, 1) \) such that, for any measurable set \( E \subset B \),
\[ \left( \frac{\lambda(E)}{\lambda(B)} \right)^{p} \leq C_1 \frac{v(E)}{v(B)} \leq C_2 \left( \frac{\lambda(E)}{\lambda(B)} \right)^{\sigma} \]

Combining this and (2-9),
\[ \sup_B \left\{ \frac{v(B \cap \overline{F}_{j,1})}{|v(B)|^k} \right\}^{1/p} \leq \sup_B \left\{ \frac{v(B \cap \overline{F}_{j,1})^{(1-k)}}{|v(B)|^k} \right\}^{1/p} \leq v(\overline{F}_{j,1})^{(1-k)/p} \leq v(B_j)^{(1-k)/p} \approx v(B_j)^{(1-k)/p} \]

and
\[ \sup_B \left\{ \frac{v(B \cap \overline{F}_{j,1})}{|v(B)|^k} \right\}^{1/p} \geq \left\{ \frac{v(\overline{F}_{j,1})}{|v(B_j)|^k} \right\}^{1/p} \geq v(\overline{F}_{j,1})^{(1-k)/p} \approx v(B_j)^{(1-k)/p} \]

This implies that \( \|f_j\|_{L^p(B_D)} \approx 1 \). Consequently, \( \{f_j\} \) is a bounded sequence in \( L^{p,k}_v(bD) \) with a uniform \( L^{p,k}_v(bD) \) lower bound away from zero.

Since \([b, \mathcal{C}]\) is compact, we obtain that the sequence \([\{b, \mathcal{C}\}(f_j)\}]\ has a convergent subsequence, denoted by \([\{b, \mathcal{C}\}(f_j)\}]\). We denote the limit function by \( g_0 \), that is,
\[ [b, \mathcal{C}](f_j) \to g_0 \text{ in } L^{p,k}_v(B_D), \quad \text{as } i \to \infty. \]

Moreover, \( g_0 \neq 0 \).

After taking a further subsequence, labelled \( g_j \), we may assume that
- \( \|g_j\|_{L^{p,k}_v(B_D)} \approx 1 \);
- \( g_j \) are disjointly supported; and
- \( \|g_0 - [b, \mathcal{C}]g_j\|_{L^{p,k}_v(B_D)} < 2^{-j} \).

Take \( a_j = j^{-(p+1)/2} \), so that \( \{a_j\} \in \ell^p \setminus \ell^1 \). It is immediate that \( \gamma = \sum_j a_jg_j \in L^{p,k}_v(B_D) \), and hence, \([b, \mathcal{C}]\gamma \in L^{p,k}_v(B_D) \). But \( g_0 \sum_j a_j \equiv \infty \), and yet
\[ \left\| g_0 \sum_j a_j \right\|_{L^p(B_D)} \leq \|([b, \mathcal{C}]\gamma)\|_{L^{p,k}_v(B_D)} + \sum_j a_j\|g_0 - [b, \mathcal{C}]g_j\|_{L^{p,k}_v(B_D)} < \infty. \]

This contradiction shows that \( b \in \text{VMO}(bD) \).

We now turn to necessity. Recall that \( \mathcal{C} = \mathcal{C}^d + \mathcal{R} \). Since the kernel \( \mathcal{C}^d \) is a standard kernel, \([b, \mathcal{C}^d] \) is compact on \( L^{p,k}_v(bD) \). Therefore, we only need to show that \([b, \mathcal{R}] \) is also compact on \( L^{p,k}_v(bD) \).
From Lemma 2.8, for any \( \xi > 0 \), there exists \( b_\xi \in BUC(bD) \) such that \( \|b - b_\xi\|_* < \xi \). Then, by Theorem 1.1,

\[
\|[b, R]f - [b_\xi, R]f\|_{L^{p_\star}_*(bD)} \leq C_p \|f\|_{L^{p_\star}_*(bD)} \|b - b_\xi\|_* < \xi C_p \|f\|_{L^{p_\star}_*(bD)}.
\]

Thus, to prove that \([b, R]\) is compact on \( L^{p_\star}_*(bD)\), it suffices to prove that \([b_\xi, R]\) is compact on \( L^{p_\star}_*(bD)\). By Lemma 2.8 and (2-7), without loss of generality, we may assume that \( b \in BUC(bD)\) and (2-8) holds. By Lemma 2.9, it suffices to prove that, for any fixed \( \eta \) satisfying \( 0 < \eta \ll 1 \), \([b, \mathcal{R}^\eta]\) is compact on \( L^{p_\star}_*(bD)\).

Since \( R(w, z) \) is continuous on \( bD \times bD \setminus \{(z, z) : z \in bD\} \), for any \( f \in L^{p_\star}_*(bD) \), we see that \([b, \mathcal{R}^\eta]\) is continuous on \( bD \). To conclude the proof, we now argue that the image of the unit ball of \( L^{p_\star}_*(bD)\) under the commutator \([b, \mathcal{R}^\eta]\) is an equicontinuous family. Compactness follows from the Arzelà–Ascoli theorem.

It remains to prove equicontinuity. For any \( z, w \in bD \) with \( d(w, z) < 1 \),

\[
[b, \mathcal{R}^\eta]f(z) - [b, \mathcal{R}^\eta]f(w)
\]

\[
= b(z) \int_{bD} R^\eta(u, z)f(u) \, d\lambda(u) - b(w) \int_{bD} R^\eta(u, w)f(u) \, d\lambda(u)
\]

\[
+ b(w) \int_{bD} R^\eta(u, w)f(u) \, d\lambda(u) + \int_{bD} R^\eta(u, w)b(u)f(u) \, d\lambda(u)
\]

\[
= (b(z) - b(w)) \int_{bD} R^\eta(u, z)f(u) \, d\lambda(u)
\]

\[
+ b(w) \int_{bD} (R^\eta(u, z) - R^\eta(u, w))f(u) \, d\lambda(u)
\]

\[
+ \int_{bD} (R^\eta(u, w) - R^\eta(u, z))b(u)f(u) \, d\lambda(u)
\]

\[
= (b(z) - b(w)) \int_{bD} R^\eta(u, z)f(u) \, d\lambda(u)
\]

\[
+ \int_{bD} (R^\eta(u, w) - R^\eta(u, z))(b(u) - b(w))f(u) \, d\lambda(u)
\]

\[
=: I(z, w) + II(z, w).
\]

For \( I(z, w) \), by Hölder’s inequality,

\[
|I(z, w)|
\]

\[
= |(b(z) - b(w))| \int_{bD} R^\eta(u, z)f(u) \, d\lambda(u)
\]

\[
\leq c|(b(z) - b(w))|v(bD)^{1/p'} \left( \int_{bD} |R^\eta(u, z)|^{p' / p} v^{(p' / p)^{-}} \, d\lambda(u) \right)^{1/p'}
\]
where the last inequality is due to the fact that \( v \in A_p, R^n(u, z) \in C(bD \times bD) \) and \( bD \) is bounded.

Since \( b \) is bounded, if we let \( d(w, z) < \eta/c \cdot c_R \), by a discussion similar to [10, page 645], we can obtain that

\[
|II(z, w)| = \left| \int_{bD} (R^n(u, w) - R^n(u, z))(b(u) - b(w))f(u) d\lambda(u) \right|
\]

\[
\leq c\|b\|_{L^\infty(bD)} \int_{bD \setminus B_R(z)} \frac{d(w, z)}{d(u, z)^{2n}} |f(u)| d\lambda(u)
\]

\[
\leq c\|b\|_{L^\infty(bD)} v(bD)^{(k-1)/p} |f|_{L^{p,x}(bD)} d(w, z)
\]

\[
\times \left( \int_{bD \setminus B_R(z)} \frac{1}{d(u, z)^{2n}} \right)^{-1/p'} \lambda(u)
\]

\[
\leq c\|b\|_{L^\infty(bD)} v(bD)^{(k-1)/p} |f|_{L^{p,x}(bD)} d(w, z)
\]

As a consequence, \( \{[b, \mathcal{R}^n](U)\} \) is an equicontinuous family, where \( \mathcal{U} \) is the unit ball in \( L^{p,x}_b(bD) \). This finishes the proof of Theorem 1.2. \( \square \)

3. Commutator of Cauchy–Leray integral for strongly \( \mathbb{C} \)-linearly convex domains with minimal smoothness

In this section we focus on the bounded domain \( D \subset \mathbb{C}^n \) which is strongly \( \mathbb{C} \)-linearly convex and whose boundary satisfies the minimal regularity condition of class \( C^{1,1} \) [12].

Suppose \( D \) is a bounded domain in \( \mathbb{C}^n \) with defining function \( \rho \) satisfying the following two conditions.

1. \( D \) is of class \( C^{1,1}_b \), that is, the first derivatives of its defining function \( \rho \) are Lipschitz, and \( |\nabla \rho(w)| > 0 \) whenever \( w \in \{w : \rho(w) = 0\} = bD \).
2. \( D \) is strongly \( \mathbb{C} \)-linearly convex, that is, \( D \) is a bounded domain of \( C^1 \), and at any boundary point it satisfies either of the two equivalent conditions

\[
|\Delta(w, z)| \geq c|w - z|^2,
\]

\[
d_F(z, w + T^C_w) \geq \tilde{c}|w - z|^2,
\]
for some $c, \tilde{c} > 0$, where

$$\Delta(w, z) = \langle \partial \rho(w), w - z \rangle$$

and $d_E(z, w + T_w^C)$ denotes the Euclidean distance from $z$ to the affine subspace $w + T_w^C$. Note that $T_w^C := \{ v : \langle \partial \rho(w), v \rangle = 0 \}$ is the complex tangent space referred to the origin, and $w + T_w^C$ is its geometric realization as an affine space tangent to $bD$ at $w$.

On $bD$ there is a quasidistance $d$, which is defined as

$$d(w, z) = |\Delta(w, z)|^{1/2} = |\langle \partial \rho, w - z \rangle|^{1/2}, \quad w, z \in bD.$$ 

Let $d\lambda$ be the Leray–Levi measure $d\lambda$ on $bD$ (see [12]). According to [12, Proposition 3.4], $d\lambda$ is also equivalent to the induced Lebesgue measure $d\sigma$ on $bD$ in the following sense:

$$d\lambda(w) = \tilde{\Lambda}(w) d\sigma(w) \quad \text{for } \sigma\text{-almost every } w \in bD,$$

and there are two strictly positive constants $c_1$ and $c_2$ so that

$$c_1 \leq \tilde{\Lambda}(w) \leq c_2 \quad \text{for } \sigma\text{-almost every } w \in bD.$$ 

We also denote by $B_r(w) = \{ z \in bD : d(w, z) < r \}$ the boundary balls determined via the quasidistance $d$. By [12, Proposition 3.5], we also have

$$\lambda(B_r(w)) \approx r^{2n}, \quad 0 < r \leq 1. \quad (3-1)$$

The Cauchy–Leray integral of a suitable function $f$ on $bD$, denoted $C(f)$, is formally defined by

$$C(f)(z) = \int_{bD} f(w) \frac{\Delta(w, z)^n}{d\lambda(w)}, \quad z \in D.$$ 

When restricting $z$ to the boundary $bD$, we have the Cauchy–Leray transform $f \mapsto C(f)$, defined as

$$C(f)(z) = \int_{bD} f(w) \frac{\Delta(w, z)^n}{d\lambda(w)}, \quad z \in bD,$$

where the function $f$ satisfies the Hölder-like condition

$$|f(w_1) - f(w_2)| \leq d(w_1, w_2)^\alpha, \quad w_1, w_2 \in bD,$$

for some $0 < \alpha \leq 1$.

Take $K(w, z)$ to be the function defined for $w, z \in bD$, with $w \neq z$, by

$$K(w, z) = \frac{1}{\Delta(w, z)^n}.$$

This function is the ‘kernel’ of the operator $C$, in the sense that

$$C(f)(z) = \int_{bD} K(w, z) f(w) d\lambda(w),$$
whenever \( z \) lies outside of the support of \( f \) and \( f \) satisfies the Hölder-like condition for some \( \alpha \). The size and regularity estimates that are relevant for us are:

\[
|K(w, z)| \leq \frac{1}{d(w, z)^{2n}},
\]

\[
|K(w, z) - K(w', z)| \leq \frac{d(w, w')}{d(w, z)^{2n+1}}, \quad \text{if } d(w, z) \geq c_K d(w, w'), \tag{3-2}
\]

\[
|K(w, z) - K(w, z')| \leq \frac{d(z, z')}{d(w, z)^{2n+1}}, \quad \text{if } d(w, z) \geq c_K d(z, z'),
\]

for an appropriate constant \( c_K > 0 \). Moreover, for the size estimates we actually have

\[
|K(w, z)| = \frac{1}{d(w, z)^{2n}}. \tag{3-3}
\]

We need the \( L^p(bD) \) boundedness of the Cauchy–Leray transform in [12, Theorem 5.1].

**Lemma 3.1.** The Cauchy–Leray transform \( f \mapsto C(f) \), initially defined for functions \( f \) that satisfy the Hölder-like condition for some \( \alpha \), extends to a bounded linear operator on \( L^p(bD) \) for \( 1 < p < \infty \).

**Proof (Proof of Theorems 1.3 and 1.4).** We point out that the proof of Theorem 1.3 follows from the proof of Theorem 1.5 in [5], and the proof of Theorem 1.4 follows from the proofs of Theorems 1.1 and 1.2. In fact, these are simpler, since the operator \( C \) is a Calderón–Zygmund operator. \( \square \)

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