Indecomposable characters on direct limit of symmetric groups with diagonal embeddings

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Abstract

In this paper we obtain the complete description of all indecomposable characters (central positive-definite functions) of inductive limits of the symmetric groups under block diagonal embedding. As a corollary we obtain the full classification of the isomorphism classes of these inductive limits.

Keywords: Infinite symmetric group; Character; Factor representation.

1 Introduction

Consider the space $X = [0, 1)$ with the standard Lebesgue measure $\nu$. Denote by $\text{Aut}_0(X, \nu)$ the group of automorphisms of the space $(X, \nu)$ which preserve the measure $\nu$. In particular, one can consider finite subgroups of this group, which correspond to the so called rational rearrangements of $X$.

Namely, each symmetric group $S_N$ regarded as group of bijections of the set $X_N = \{0, 1, 2, \ldots, N-1\}$ can be embedded into $\text{Aut}_0(X, \mu)$ in the following way. For each $\sigma \in S_N$ define automorphism $T_N(\sigma) \in \text{Aut}_0(X, \mu)$ via the formula

$$T_N(\sigma)(x) = \frac{Nx - \lfloor Nx \rfloor + \sigma(\lfloor Nx \rfloor)}{N}, \ x \in [0, 1).$$

In other words, the map $T_N(\sigma)$ acts on half-closed intervals $[\frac{k}{n_1}, \frac{k+1}{n_1})$, $k \in [0, N-1]$ via the permutation $\sigma$. It is easy to verify that $T_N(\sigma)$ is an injective homomorphism.

In order to understand which automorphism in $T_{n_1,n_2}(S_{n_1,n_2})$ coincides with $T_{n_1}(\sigma)$, let us represent each element $y$ of $X_{n_1,n_2}$ as $y = x_1 + n_1 x_2$, where $x_1 \in X_{n_1}$ and $x_2 \in X_{n_2}$. Then

$$T_{n_1}(\sigma)\left[\frac{k}{n_1} + \frac{i}{n_1n_2}, \frac{k+1}{n_1} + \frac{i+1}{n_1n_2}\right] = \left[\frac{\sigma(k)}{n_1} + \frac{i}{n_1n_2}, \frac{\sigma(k)+i+1}{n_1n_2}\right],$$

where $k \in [0, n_1-1]$, $i \in [0, n_2-1]$. It means that $T_{n_1}(\sigma) = T_{n_1,n_2}(i(\sigma))$ where the permutation $i(\sigma) \in S_{n_1,n_2}$ acts as follows

$$i(\sigma)(x_1 + n_1 x_2) = \sigma(x_1) + n_1 x_2, \ \text{where} \ x_1 \in X_{n_1}, x_2 \in X_{n_2}.$$

In this way we obtain a natural embedding of the symmetric group $S_{n_1}$ into $S_{n_1,n_2}$ which corresponds to the inclusion $T_{n_1}(S_{n_1}) \subset T_{n_1,n_2}(S_{n_1,n_2})$.

1Here we use the notation $[p, q]$ for the set $\{p+1, p+2, \ldots q\} \subset \mathbb{Z}$. 

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If we identify $\mathbb{X}_{n_1n_2}$ with $\mathbb{X}_{n_1} \times \mathbb{X}_{n_2}$ using the correspondence $(x, y) \mapsto x + n_1y$, then $\iota(\sigma)$ acts as follows:

$$\iota(\sigma)((x, y)) = (\sigma(x), y), \ x \in \mathbb{X}_{n_1}, y \in \mathbb{X}_{n_2}.$$ 

Continuing this process and letting $N_k = n_1n_2\ldots n_k$ we obtain the infinite chain of subgroups

$$T_{N_1}(\mathfrak{S}_{N_1}) \subset T_{N_2}(\mathfrak{S}_{N_2}) \subset \ldots \subset T_{N_k}(\mathfrak{S}_{N_k}) \subset \ldots \subset \text{Aut}_0(X, \nu).$$

Their union $\bigcup T_{N_k}(\mathfrak{S}_{N_k})$ is a countable subgroup in $\text{Aut}_0(X, \nu)$. This subgroup of $\text{Aut}_0(X, \nu)$ is naturally isomorphic to the inductive limit $\lim_{\longrightarrow} \mathfrak{S}_{N_k}$ which corresponds to the embedding $\iota(\sigma)$. Note that, in general, different sequences $\{N_k\}$ define non-isomorphic inductive limits. In particular, they might be simple as well as contain a nontrivial normal subgroup. In the present paper we obtain the complete description of pairs of sequences $\{N_k^\prime\}$ and $\{N_k^\prime\}$ for which the corresponding inductive limits are isomorphic (see Theorem 1.3).

For each prime number $p$ denote by $\deg_p(N_k)$ the degree of $p$ in the prime factorization $N_k = \prod_p p^{\deg_p(N_k)}$. In case when for each prime $p$ the sequence $\{\deg_p(N_k)\}_{k=1}^\infty$ is unbounded, i.e. $\lim_{k \to \infty} \deg_p(N_k) = \infty$, the group $\bigcup T_{N_k}(\mathfrak{S}_{N_k})$ is called the group of rational rearrangements of a segment. Denote this group as $\mathfrak{S}_Q$. In particular, $\mathfrak{S}_Q$ is a simple group. In [5] the full description of indecomposable characters on $\mathfrak{S}_Q$ was obtained. Recall that positive definite function $\chi$ on group $G$ is called central or character if it satisfies the following condition

$$\chi(gh) = \chi(hg)$$
for all $g, h \in G$.

In the present paper we consider only normalized characters, i.e. which equal to 1 on the identity element of $G$. A character $\chi$ is called indecomposable if the unitary representation $\Pi_\chi$ of the group $G$, constructed via $\chi$ according to the Gelfand-Naimark-Segal (GNS) construction, is a factor representation. Namely, in this case the operators $\Pi_\chi(G)$ generate a factor of type $\Pi_1[10]$. This definition is equivalent to the following property: indecomposable characters are the extreme points of the simplex of all characters.

An important special case of the group $\lim \mathfrak{S}_{N_k}$, where $N_k = 2^{M_k}$, was studied by A. Dudko [3]. This group is also a simple group and in [3] all indecomposable characters of this group were found. In [4] the full description of indecomposable characters was given for more general symmetric groups which act on the paths of the Bratelli’s diagram. However, the results of papers [3], [4] and [5] did not cover the case of an arbitrary sequence $\{N_k\}$. In the present paper we obtain the description of all characters on groups $\lim \mathfrak{S}_{N_k}$ without any additional conditions on the sequences $\{N_k\}$.

### 1.1 The inductive limit of symmetric groups

In this subsection we define the group $\mathfrak{S}_N$ as an inductive limit of symmetric groups with the diagonal embedding.

We regard the group $\mathfrak{S}_N$ as the group of all bijections (symmetries) of the set $\mathbb{X}_N = \{0, 1, 2, \ldots, N-1\}$. We identify $\mathbb{X}_{NM}$ with $\mathbb{X}_N \times \mathbb{X}_M$ via the isomorphism
Define the group \( S_i \) correspond to the embeddings \( \sigma \), then \( \hat{N_k} \) is a simple group. Denote by \( \imath_{\hat{S}_k} \) the embedding of the group \( S_i \) into \( S_k \), defined as in (1.1).

Remark 1.1. The group \( \hat{S}_k \) is isomorphic to the subgroup \( \bigcup_k \{T_N \in (\hat{S}_k)\} \) of the group \( \text{Auto}(\mathcal{X}, \nu) \).

Suppose that \( \sigma \in S_N \) belongs to the conjugacy class \( C_{\bar{m}} \) in \( S_N \), consisting of permutations of the cycle type \( \bar{m} = (\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_l) \), where \( \bar{m}_i \) is the number of cycles of the length \( i \) in the decomposition of \( \sigma \) into disjoint cycles (see definitions in Subsection 3.1). Then \( \imath_{k,j} (C_{\bar{m}}) \subset C_{\bar{m}}, \) where

\[
\bar{m} = (\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_l) = \left( \frac{k_1 N_j}{N_k}, \frac{k_2 N_j}{N_k}, \ldots, \frac{k_l N_j}{N_k} \right). \tag{1.2}
\]

For each \( \sigma \in S_N \) define \( \text{supp}_{\hat{S}_k} \sigma = \{x \in \mathcal{X}_{\hat{S}_k} : \sigma x \neq x\} \). Note that if \( \sigma \in S_{\hat{S}_k} \), then

\[
\# \left( \text{supp}_{\hat{S}_k} \imath_{k,j}(\sigma) \right) = \# \left( \text{supp}_{\hat{S}_k} \sigma \right) \frac{N_j}{N_k}. \tag{1.3}
\]

Define the multiplicative character \( \text{sgn}_{\hat{S}_k} : \mathcal{X}_{\hat{S}_k} \rightarrow \{-1, 1\} \) via the formula \( \text{sgn}_{\hat{S}_k}(\sigma) = (-1)^{\kappa_{\hat{S}_k}(\sigma)} \). Here by \( \kappa_{\hat{S}_k}(\sigma) \) we denote the minimal number of factors in the decomposition of \( \sigma \) into the product of transpositions. It is known that if \( \sigma \in C_{\bar{m}}, \) then \( \kappa_{\hat{S}_k}(\sigma) = N_k - \sum_{p=1}^{l} \bar{m}_p. \) Note that \( \kappa_{\hat{S}_k}(\sigma) = N_k \kappa_{S_k}(\sigma). \)

Therefore, \( \text{sgn}_{\hat{S}_k}(\sigma) = \left( \text{sgn}_{S_k}(\sigma) \right)^{N_j/N_k}. \) This implies that for each \( s \in \hat{S}_k \) there exists \( M(s) \) such that \( \text{sgn}_{\hat{S}_k}(s) = \text{sgn}_{S_k}(s) \) for all \( i, j > M(s). \)

Thus, there exists a limit \( \text{sgn}_\infty = \lim_{j \rightarrow \infty} \text{sgn}_{\hat{S}_k}(s) \). The function \( \text{sgn}_\infty \) is a multiplicative character on the group \( \hat{S}_k \).

The following statement is immediate.

Proposition 1.1. Denote by \( \mathfrak{A}_k \) the subgroup \( \{g \in \hat{S}_k : \text{sgn}_\infty(g) = 1\} \). Then

(a) \( \mathfrak{A}_k \) is a simple group;

(b) if the sequence \( \hat{n} = \{n_k\}_{k=1}^\infty \) contains infinitely many even numbers, then \( \hat{S}_k \) is a simple group.
1.2 The main result

Let $M_N(\mathbb{C})$ be the algebra of the complex $N \times N$ matrices, and let $I_N$ be the identity $N \times N$ matrix. Denote by $\text{Tr}_N$ an ordinary trace on $M_N(\mathbb{C})$. Set $\text{tr}_N(A) = \frac{1}{N} \text{Tr}_N(A)$, where $A \in M_N(\mathbb{C})$. Define on $M_N(\mathbb{C})$ an inner product $\langle A, B \rangle_N = \text{tr}_N(B^* A)$, $A, B \in M_N(\mathbb{C})$. The elements of $\sigma \in \mathcal{S}_N$ are realized as sequences of positive integers $\Sigma_\sigma = \{ \delta_{\sigma(j)} \}$ in $M_N(\mathbb{C})$, where $\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$ The operators of the left multiplication by $\Sigma_\sigma$ define on $M_N(\mathbb{C})$ a unitary representation $\mathcal{L}_N$ of the group $\mathcal{S}_N$:

$$\mathcal{L}(\sigma) A = \Sigma_\sigma \cdot A, \quad A \in M_N(\mathbb{C}).$$

Put $\varphi_N(\sigma) = \langle \mathcal{L}(\sigma) I_N, I_N \rangle = \text{tr}_N(\Sigma_\sigma)$. Clearly, $\varphi_N$ is a character on $\mathcal{S}_N$ and

$$\varphi_N(\sigma) = 1 - \frac{\# \text{supp}_N \sigma}{N}, \quad \sigma \in \mathcal{S}_N.$$  (1.4)

Note that if $\sigma \in \mathcal{S}_{N_k}$, then for $j > k$ from (1.3) we have

$$\varphi_{N_j}(\{k, j\}(\sigma)) = \varphi_{N_k}(\sigma).$$

Hence, the sequence $\{ \varphi_{N_k} \}$ defines a character $\chi_{\text{nat}}$ on the inductive limit $\mathcal{S}_{\mathbb{N}} = \lim_{\longrightarrow} \mathcal{S}_{N_k}$. In other words, for $\sigma \in \mathcal{S}_{N_k} \subset \mathcal{S}_{\mathbb{N}}$ we have

$$\chi_{\text{nat}}(\sigma) = \varphi_{N_k}(\sigma) = 1 - \frac{\# \text{supp}_{N_k} \sigma}{N_k}. \quad \sigma \in \mathcal{S}_{\mathbb{N}}.$$  (1.5)

Our main result is the following theorem.

**Theorem 1.2.** Let the character $\chi_{\text{nat}}^p$, where $p \in \mathbb{N} \cup \{0, \infty\}$, be defined as

$$\chi_{\text{nat}}^p(\sigma) = (\chi_{\text{nat}}(\sigma))^p \quad \text{when } p \in \mathbb{N} \cup \{0\} \quad \text{and } \chi_{\text{nat}}^\infty(\sigma) = \begin{cases} 1, & \text{if } \sigma = \text{id}, \\ 0, & \text{if } \sigma \neq \text{id.} \end{cases}$$

If $\chi$ is an indecomposable character on $\mathcal{S}_{\mathbb{N}}$, then there exists $p \in \mathbb{N} \cup \{0, \infty\}$ such that $\chi = \chi_{\text{nat}}^p$ or $\chi = \text{sgn}_\infty \cdot \chi_{\text{nat}}^p$, where $(\text{sgn}_\infty \cdot \chi_{\text{nat}}^p)(\sigma) = \text{sgn}_\infty(\sigma) \cdot \chi_{\text{nat}}^p(\sigma), \sigma \in \mathcal{S}_{\mathbb{N}}$.

As a corollary, we also obtain the complete classification of the isomorphism classes of groups $\mathcal{S}_{\mathbb{N}}$.

**Theorem 1.3.** Let $\vec{n} = (n_k')_{k=1}^{\infty}$ and $\vec{n}'' = (n_k'')_{k=1}^{\infty}$, where $n_k', n_k'' > 1$ for all $k$, be the sequences of positive integers. Put $N_k' = \prod_{i=1}^{k} n_i'$ and $N_k'' = \prod_{i=1}^{k} n_i''$. Then, groups $\mathcal{S}_{\mathbb{N}}'$ and $\mathcal{S}_{\mathbb{N}}''$ (see Section 1.1) are isomorphic if for each prime number $p$ the following condition holds:

$$\lim_{k \to \infty} \deg_p(N_k') = \lim_{k \to \infty} \deg_p(N_k'').$$  (1.6)

In other words, groups $\mathcal{S}_{\mathbb{N}}'$ and $\mathcal{S}_{\mathbb{N}}''$ are isomorphic if for each prime $p$ either both sequences $\{\deg_p(N_k')\}_{k=1}^{\infty}$ and $\{\deg_p(N_k'')\}_{k=1}^{\infty}$ are unbounded, or there is a non-negative integer $d_p$ such that $\deg_p(N_k') = \deg_p(N_k'') = d_p$ for all sufficiently large $k$. 

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1.3 \( \mathfrak{H}_1 \)-factor-representations of the group \( \mathfrak{G}_1 \) and spherical representations of \( \mathfrak{G}_1 \times \mathfrak{G}_1 \).

Let \( \chi \) be an arbitrary character on \( \mathfrak{G}_1 \). Consider GNS-representation \( (\pi_\chi, \mathcal{H}_\chi, \xi_\chi) \) of \( \mathfrak{G}_1 \) corresponding to \( \chi \), where \( \xi_\chi \) is the unit cyclic vector for \( \pi_\chi (\mathfrak{G}_1) \) in the Hilbert space \( \mathcal{H}_\chi \) such that \( \chi(\sigma) = \langle \pi_\chi(\sigma) \xi_\chi, \xi_\chi \rangle \) for all \( \sigma \in \mathfrak{G}_1 \). Denote by \( M \) the \( \mathfrak{w}^\ast \)-algebra generated by a set of operators \( \pi_\chi (\mathfrak{G}_1) \). Character \( \chi \) defines a finite, normal faithful trace \( \text{tr} \) on \( M \). Namely,

\[
\text{tr}(\pi_\chi(\sigma)) = \chi(\sigma), \sigma \in \mathfrak{G}_1.
\]

Taking into account the definition of GNS-construction, we assume that \( \mathcal{H}_\chi = L^2(M, \text{tr}) \), where inner product \( \langle \cdot, \cdot \rangle \) is defined as follows

\[
\langle m_1, m_2 \rangle = \text{tr}(m_2^* m_1), \quad m_1, m_2 \in M,
\]

and \( \xi_\chi \) is an identity operator from \( M \). Finally, we recall that the operators \( \pi_\chi(\sigma), \sigma \in \mathfrak{G}_1 \) act on \( L^2(M, \text{tr}) \) by left multiplication

\[
L^2(M, \text{tr}) \ni v \xrightarrow{\pi_\chi(\sigma)} \pi_\chi(\sigma) \cdot v \in L^2(M, \text{tr}). \tag{1.7}
\]

Denote by \( \mathcal{B}(\mathcal{H}_\chi) \) the set of all bounded linear operators on \( \mathcal{H}_\chi \), and put

\[
M' = \{ A \in \mathcal{B}(\mathcal{H}_\chi) : AB = BA \text{ for all } B \in M \}.
\]

Since \( \text{tr} \) is a central state on \( M \); i.e. \( \text{tr}(m_1 m_2) = \text{tr}(m_2 m_1) \) for all \( m_1, m_2 \in M \), the mapping

\[
L^2(M, \text{tr}) \ni v \xrightarrow{\pi_\chi(\sigma)} v \cdot \pi_\chi(\sigma^{-1}) \in L^2(M, \text{tr}) \tag{1.8}
\]

define an unitary operator \( \pi_\chi^*(\sigma) \in M' \) on \( \mathcal{H}_\chi \). Denote by \( J \) the antilinear isometry which acts as follows: \( L^2(M, \text{tr}) \ni m \xrightarrow{J} m^\ast \). It is clear that \( J^2 = I \).

It follows from the above that

\[
JMJ \subset M' \text{ and } M \subset JM'J. \tag{1.9}
\]

Let us prove that

\[
M = JM'J. \tag{1.10}
\]

Let operator \( A \) belongs to \( JM'J \). Since \( \xi_\chi \) is cyclic for \( M \); i.e. a set \( M\xi_\chi \) is norm dense in \( L^2(M, \text{tr}) \), there exists a sequence \( \{A_n\}_{n \in \mathbb{N}} \subset M \) such that

\[
\lim_{n \to \infty} \|A_n \xi_\chi - A_n \xi_\chi\|_{L^2(M, \text{tr})} = 0. \tag{1.11}
\]

Hence, applying centrality of \( \chi \), we have

\[
\lim_{m, n \to \infty} \|A_m^\ast \xi_\chi - A_n^\ast \xi_\chi\|_{L^2(M, \text{tr})} = 0.
\]

It follows from this that sequence \( \{A_n^\ast \xi_\chi\} \) converges in norm to \( \eta \in L^2(M, \text{tr}) \). Hence for each \( U' \in JMJ \) we obtain next chain of equalities:

\[
\langle (U')^\ast \xi_\chi, \eta \rangle = \lim_{n \to \infty} \langle (U')^\ast \xi_\chi, A_n^\ast \xi_\chi \rangle \tag{1.11}
\]
\[
= \lim_{n \to \infty} \langle A_n \xi_\chi, U' \xi_\chi \rangle = \langle A \xi_\chi, U' \xi_\chi \rangle = \langle (U')^\ast \xi_\chi, A^\ast \xi_\chi \rangle.
\]
Therefore, using the cyclicity of $\xi_\chi$ for $JMJ$, we have
\[ A^*\xi_\chi = \eta; \quad \text{i.e.} \quad \lim_{n \to \infty} \|A_n^*\xi_\chi - A^*\xi_\chi\|_{L^2(M, \mu)} = 0. \] (1.12)

**Lemma 1.4.** Put $\omega(x) = \langle x\xi_\chi, \xi_\chi \rangle$, where $x \in B(H_\eta)$. Then

- **1.** $\omega$ is a central state on $M'$ and on $JM'$; i.e.
  \[ \omega(A'B') = \omega(B'A') \text{ for all } A', B' \in M', \]
  \[ \omega(AB) = \omega(BA) \text{ for all } A, B \in JM'; \]
  \[ \omega(A) = \omega(B) \text{ for all } A, B \in JM'; \]
  \[ \omega(\lambda) = \lambda \omega(1) \text{ for all } \lambda \in \mathbb{C}. \]

**Proof.** **Property 1.** Since $\xi_\chi$ is cyclic vector for $M$, there exists the sequences $\{A_n\}, \{B_n\}$ in $M$ such that
\[ \lim_{n \to \infty} \|A_n\xi_\chi - A_n\xi_\chi\|_{L^2(M, \mu)} = \lim_{n \to \infty} \|B_n\xi_\chi - B_n\xi_\chi\|_{L^2(M, \mu)} = 0. \]

Hence, using (1.12), we obtain
\[ \lim_{n \to \infty} \|B_n\xi_\chi - B_n\xi_\chi\|_{L^2(M, \mu)} = 0. \]

Therefore, $\omega(AB) = \lim_{n \to \infty} \omega(A_nB_n) = \lim_{n \to \infty} \operatorname{tr}(A_nB_n) = \lim_{n \to \infty} \operatorname{tr}(B_nA_n) = \lim_{n \to \infty} \omega(B_nA_n) = \omega(BA)$. We leave it to the reader to verify that $\omega(A'B') = \omega(B'A')$ for all $A', B' \in M'$.

**Property 2.** Since $J^2 = I$, then, by cyclicity of the vector $\xi_\chi$ for $M$, there exists the sequences $\{A_n\}, \{B_n\}$ in $M$ such that
\[ \lim_{n \to \infty} \|A_n\xi_\chi - A_n\xi_\chi\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|B_n^*J\xi_\chi - B_n\xi_\chi\| = 0. \]

Hence, using the equality $\|B_n^*\xi_\chi - JB_nJ\xi_\chi\| = \|(B')^*\xi_\chi - (JB_nJ)^*\xi_\chi\|$, which is due to the fact that $\omega$ is central state on $M'$, we obtain for any $U' \in JM$ and $V \in M$
\[ \langle AB'U'\xi_\chi, V\xi_\chi \rangle = \lim_{n \to \infty} \langle A_nJU_nB_n^*JU_n'\xi_\chi, V\xi_\chi \rangle = \lim_{n \to \infty} \langle JB_nJU_n^*\xi_\chi, V\xi_\chi \rangle = \lim_{n \to \infty} \langle JB_nJ^*\xi_\chi, V\xi_\chi \rangle = \langle B'U'\xi_\chi, V\xi_\chi \rangle. \]

Consequently, $AB' = B'A$. In particular, this establishes equality (1.10).

Now we define the representation $\pi_\chi^{(2)}$ of the group $\mathfrak{S}_\bar{\mathfrak{n}} \times \mathfrak{S}_\bar{\mathfrak{n}}$ as follows
\[ \pi_\chi^{(2)}((\sigma_1, \sigma_2)) = \pi_\chi(\sigma_1)\pi_\chi^*(\sigma_2), \quad (\sigma_1, \sigma_2) \in \mathfrak{S}_\bar{\mathfrak{n}} \times \mathfrak{S}_\bar{\mathfrak{n}}. \] (1.14)

Hence, applying (1.9) and (1.8), we obtain
\[ \pi_\chi^{(2)}((\sigma, \sigma))\xi_\eta = \xi_\eta \text{ for all } \sigma \in \mathfrak{S}_\bar{\mathfrak{n}}. \] (1.15)

Denote by $C(M)$ center of $w^*$-algebra $M$.

**Proposition 1.5.** $\left( \pi_\chi^{(2)}(\mathfrak{S}_\bar{\mathfrak{n}} \times \mathfrak{S}_\bar{\mathfrak{n}}) \right)' = C(M).$
Proof. It follows from (1.8) and (1.10) that 
\[ \pi(2)(\mathfrak{S}_n \times \mathfrak{S}_n)' = M \cap M' = C(M). \]

Define an orthogonal projection \( E_k \) by
\[ E_k = \frac{1}{N_k!} \sum_{\sigma \in \mathfrak{S}_N} \pi^{(2)}(\sigma, \sigma). \] (1.16)

It is clear that \( E_k \geq E_{k+1} \). Therefore, there exists the limit \( E = \lim_{k \to \infty} E_k \). It follows from (1.16) that
\[ E \mathcal{H}_\chi = \{ \eta \in \mathcal{H}_\chi : \pi(2)((\sigma, \sigma))\eta = \eta \text{ for all } \sigma \in \mathfrak{S}_n \}. \] (1.17)

**Proposition 1.6.** The following three conditions are equivalent:

- (i) \( \pi_\chi \) is a factor-representation;
- (ii) representation \( \pi^{(2)}_\chi \) is irreducible;
- (iii) \( \dim E \mathcal{H}_\chi = 1 \).

**Proof.** The equivalence (i) and (ii) follows from proposition 1.5.

Let us prove that (i) \( \Rightarrow \) (iii). On the contrary, suppose that \( \dim E \mathcal{H}_\chi \geq 2 \).

Then there exists unit vector \( v \in E \mathcal{H}_\chi \) such that
\[ \langle v, \xi_\chi \rangle = 0. \] (1.18)

Since \( \xi_\chi \) is cyclic vector for \( M \), then
\[ \|v - m\xi_\chi\|_{\mathcal{H}_\chi} < 1 \text{ for some } m \in M. \] (1.19)

Hence, using (1.15) and applying an equality
\[ \pi^{(2)}_\chi((\sigma, \sigma)) m \pi^{(2)}_\chi((\sigma^{-1}, \sigma^{-1})) = \pi_\chi(\sigma) m \pi_\chi(\sigma^{-1}), \]
we have
\[ \|v - \pi_\chi(\sigma) m \pi_\chi(\sigma^{-1}) \xi_\eta\|_{\mathcal{H}_\chi} < 1 \text{ for all } \sigma \in \mathfrak{S}_n. \] (1.20)

Consequently,
\[ \left\| v - \frac{1}{N_k!} \sum_{\sigma \in \mathfrak{S}_N} \pi_\chi(\sigma) m \pi_\chi(\sigma^{-1}) \xi_\eta \right\|_{\mathcal{H}_\chi} < 1. \] (1.21)

It easy to check that sequence \( \left\{ m_k = \frac{1}{N_k!} \sum_{\sigma \in \mathfrak{S}_N} \pi_\chi(\sigma) m \pi_\chi(\sigma^{-1}) \right\} \subset M \) converges in strong operator topology to an operator \( Em \in C(M) \). Emphasize that we identify here \( Em \in \mathcal{H}_\chi = L^2(M, \text{tr}) \) with the corresponding left multiplication operator from \( M \). By (1.21),
\[ \|v - Em\xi_\chi\|_{\mathcal{H}_\chi} < 1. \]
The following fact is immediate.

Remark 2.1. Here an element \( \sigma \in \mathfrak{S}_N \) acts on \( \mathfrak{X}_N \) as follows: \( \sigma \) maps an element \( (x,y) \in \mathfrak{X}_{N_k} \times \prod_{j=k+1}^{\infty} \mathfrak{X}_{n_j} \) to \( (\sigma(x),y) \) (see also (1.1)).

Define the action of the automorphism \( O \in \text{Aut}_0(\mathfrak{X}_N, \nu_N) \) on \( x = (x_1, x_2, \ldots) \in \mathfrak{X}_{N_k} \setminus \{n_1 - 1, n_2 - 1, \ldots, n_k - 1, \ldots\} \) in the following way: \( O x = (y_1, y_2, \ldots) \), where

\[
y_p = \begin{cases} x_p + 1 \pmod{n_p}, & \text{if } p \leq \min \{i : x_i < n_i - 1\}, \\ x_p, & \text{if } p > \min \{i : x_i < n_i - 1\}. \end{cases}
\]

Also, define \( O \) at \( (n_1 - 1, n_2 - 1, \ldots, n_k - 1, \ldots) \) as

\[
O(n_1 - 1, n_2 - 1, \ldots, n_k - 1, \ldots) = (0, 0, \ldots, 0, \ldots).
\]

The following fact is immediate.

\[
\|v - a \xi_\chi\|_H = \sqrt{1 + |a|^2 - 2\Re(a < v, \xi_\chi)} \geq \sqrt{1 + |a|^2} < 1, \text{ where } a \in \mathbb{C}.
\]

To prove that (iii) implies (i) suppose contrary, that there exist an orthogonal projection \( Q \in \mathcal{C}(M) \) with the properties:

\[
v = Q \xi_\chi \neq 0, \quad w = (I - Q) \xi_\chi \neq 0.
\]

Since \( v, w \) are mutually orthogonal vectors from \( EH_\chi \), then \( \dim EH_\chi \geq 2. \)

2 The realizations of \( \Pi_1 \)-representations

In this section we give the explicit construction of a type \( \Pi_1 \) factor representation of the group \( \mathfrak{S}_N \) and the corresponding irreducible representation of the group \( \mathfrak{S}_N \times \mathfrak{S}_N \).

2.1 Preliminaries

Denote by \( \nu_n \) the uniform probability measure on the set \( \mathfrak{X}_m = \{0, 1, \ldots, m - 1\} \), i.e. \( \nu_n(j) = \frac{1}{m} \) for all \( j \in \mathfrak{X}_m \). Let \( \mathfrak{X}_N = \prod_{k=1}^{\infty} \mathfrak{X}_{n_k} \). For \( x = (x_1, x_2, \ldots) \in \mathfrak{X}_N \)
we set \( \hat{x} = (x_1, x_2, \ldots, x_j) \in \prod_{k=1}^{j} \mathfrak{X}_{n_k} \). Each element \( y \in \prod_{k=1}^{j} \mathfrak{X}_{n_k} \) defines a cylindric set

\[
\hat{\mathfrak{X}}_y = \{ x \in \mathfrak{X}_N : \hat{x} = y \} \subset \mathfrak{X}_N.
\]

Now introduce the probability measure \( \nu_N = \prod_{k=1}^{\infty} \nu_{n_k} \) on \( \mathfrak{X}_N \) by the formula

\[
\nu_N(\hat{\mathfrak{X}}_y) = \frac{1}{n_1 n_2 \cdots n_j} = \frac{1}{N_j}.
\]

Let \( \text{Aut}_0(\mathfrak{X}_N, \nu_N) \) be the group of automorphisms of the Lebesgue space \( (\mathfrak{X}_N, \nu_N) \) which preserve the measure \( \nu_N \). It follows from the definition of \( \mathfrak{S}_N \) that

\[
\mathfrak{S}_N \subset \text{Aut}_0(\mathfrak{X}_N, \nu_N).
\]
Lemma 2.1. Let \( x = (x_1, x_2, \ldots) \in X_{\mathbb{R}} \) and \( O^m x = ((O^m x)_1, (O^m x)_2, \ldots) \).

Then

a) the following equalities hold: \( i x = i (O^N x) \), \( O^m \left( \mathfrak{H}(i z) \right) = \mathfrak{H}(i (O^m x)) \) and 
\[
\bigcup_{m=0}^{N_j-1} \mathfrak{H}(i (O^m x)) = X_{\mathbb{R}};
\]

b) for the map \( j O \), defined as follows
\[
\begin{align*}
  j O z &= \begin{cases} 
    O z, & \text{if } z \in \bigcup_{m=0}^{N_j-2} O^m \left( \mathfrak{H}(i z) \right); \\
    O^{-(N_j-1)} z, & \text{if } z \in O^{N_j-1} \left( \mathfrak{H}(i z) \right),
  \end{cases}
\end{align*}
\]

where \( z = (z_1, z_2, \ldots) \in X_{\mathbb{R}} \), the period of each \( z \in X_{\mathbb{R}} \) equals \( N_j \);

c) for the element \( 0 = (0, 0, \ldots) \in X_{\mathbb{R}} \) we have 
\[
\begin{align*}
  j O z &= \left( \left( j O z \right)_1, \left( j O z \right)_2, \ldots, \left( j O z \right)_j, z_{j+1}, z_{j+2}, \ldots \right),
\end{align*}
\]

where
\[
\begin{align*}
  \left( j O z \right)_p &= \begin{cases} 
    z_p + 1 \mod n_p, & \text{if } p \leq \min \{ i : z_i < n_i - 1 \} \leq j, \\
    z_p, & \text{if } \min \{ i : z_i < n_i - 1 \} < p \leq j, \\
    0, & \text{if } p < j \leq \min \{ i : z_i < n_i - 1 \}. \quad (2.2)
  \end{cases}
\end{align*}
\]

Define an invariant metric \( \rho \) on the group \( \text{Aut}_0 (X_{\mathbb{R}}, \nu_{\mathbb{R}}) \) as follows
\[
\rho(\alpha, \beta) = \nu_{\mathbb{R}} (x \in X_{\mathbb{R}} : \alpha(x) \neq \beta(x)), \ \alpha, \beta \in \text{Aut}_0 (X_{\mathbb{R}}, \nu_{\mathbb{R}}). \quad (2.3)
\]

For an automorphism \( \alpha \in \text{Aut}_0 (X_{\mathbb{R}}, \nu_{\mathbb{R}}) \) denote by \([\alpha]\) the subgroup in \( \text{Aut}_0 (X_{\mathbb{R}}, \nu_{\mathbb{R}}) \) defined as follows: \( \beta \in [\alpha] \), if for almost all \( z \in X_{\mathbb{R}} \) the equality
\[
\beta(z) = \alpha^{d(\beta, z)}(z) \quad (2.4)
\]
holds, where \( d(\beta, \cdot) \) is a measurable function on \( (X_{\mathbb{R}}, \nu_{\mathbb{R}}) \) with values in \( \mathbb{Z} \). Denote by \( \Sigma_j \) a \( \sigma \)-algebra on \( X_{\mathbb{R}} \) generated by collection of the cylindric subsets \( \{ \mathfrak{H}_y \}, y \in \bigcup_{k=1}^{\ell} X_{n_k} \).

Lemma 2.2. Let \( \overline{\mathfrak{S}}_{\mathbb{R}} \) be the closure of the group \( \mathfrak{S}_{\mathbb{R}} \) with respect to the metric \( \rho \).

Then

a) \( \mathfrak{S}_{N_j} = \{ \beta \in [j O] : \beta \Sigma_j = \Sigma_j \} \);

b) \( O \in \overline{\mathfrak{S}}_{\mathbb{R}} \);

c) the action of automorphism \( O \) on \( (X_{\mathbb{R}}, \nu_{\mathbb{R}}) \) is ergodic;

d) for every \( l \in \mathbb{Z} \setminus \{0\} \) the automorphism \( O^l \) acts freely on \( X_{\mathbb{R}} \); i.e. if there is an \( x \in X_{\mathbb{R}} \) such that \( O^l x = x \), then \( l = 0 \).
Proof. The property a) is a consequence of Lemma 2.1 c). From the parts b) and c) of Lemma 2.1 we have

\[ \nu_\hat{n}(x \in \mathcal{X}_\hat{n} : Ox \neq jOx) \leq \frac{1}{N_j}, \]

Taking this and part a) into account, we obtain the part b) of Lemma 2.2. Therefore, the ergodicity of the automorphism \( O \) is equivalent to the ergodicity of the action of \( \hat{S}_n \). And finally, the property d) follows from the definition of the automorphism \( O \).

Lemma 2.1 (b,c) and Lemma 2.2 (a) imply that action of each automorphism \( \sigma \in \hat{S}_n \) on \( x \in \mathcal{X}_\hat{n} \) can be expressed in the following way

\[ \sigma(x) = O^{d(\sigma,x)}(x), \quad (2.5) \]

where \( d(\sigma, x) \in \mathbb{Z} \). The uniqueness of the function \( d(\sigma, \cdot) \) in (2.5) is a consequence of Lemma 2.2 (d). Note that if \( \gamma, \sigma \in \hat{S}_n \), then

\[ d(\gamma \sigma, x) = d(\gamma, \sigma(x)) + d(\sigma, x). \quad (2.6) \]

Remark 2.2. It follows from (2.4) and (2.5) that \( [O] = \mathcal{S}_n \).

2.2 Construction of a II₁ factor representation of the group \( \mathcal{S}_n \)

In the Hilbert space \( \mathcal{H} = L^2(\mathcal{X}_\hat{n}, \nu_\hat{n}) \otimes l^2(\mathbb{Z}) \) define the unitary operator \( F(\sigma) \), where \( \sigma \in \mathcal{S}_n \), as follows:

\[ (F(\sigma)\eta)(x, m) = \eta(\sigma^{-1}(x), m - d(\sigma^{-1}, x)) \quad \text{for all } \eta \in \mathcal{H}. \quad (2.7) \]

Equality (2.7) implies that the map \( \sigma \mapsto F(\sigma) \) is a unitary representation of the group \( \mathcal{S}_n \), which can be extended by continuity with respect to the metric \( \rho \) (see (2.3)) to the representation of the group \( \mathcal{S}_n \). Thus (2.7) define the representation of the group \( \mathcal{S}_n \).

Denote by \( \mathcal{B}(\mathcal{H}) \) the set of all bounded linear operators acting on \( \mathcal{H} \). Put

\[ F(\mathcal{S}_n)' = \{ A \in \mathcal{B}(\mathcal{H}) : AB = BA \text{ for all } B \in F(\mathcal{S}_n) \}. \]

Denote by \( F(\mathcal{S}_n)'' \) the dual *-algebra generated by operators \( F(\mathcal{S}_n) \). Let \( \mathcal{I} \) be the function on \( \mathcal{X}_\hat{n} \) that is identically one on \( \mathcal{X}_\hat{n} \). Define function \( \delta_i \) on \( \mathbb{Z} \), where \( i \in \mathbb{Z} \) as \( \delta_i(m) = \begin{cases} 1, & \text{if } m = i \\ 0, & \text{if } m \neq i. \end{cases} \) Put \( \xi_0 = \mathcal{I} \odot \delta_0 \). It is easy to check that

\[ (F(s)F(\sigma)\xi_0, \xi_0) = (F(\sigma)F(s)\xi_0, \xi_0) \quad \text{for all } \sigma, s \in \mathcal{S}_n. \]

Thus, the vector state \( \text{tr} \) on \( F(\mathcal{S}_n)'' \) defined as

\[ \text{tr}(A) = (A\xi_0, \xi_0), \quad A \in F(\mathcal{S}_n)'', \quad (2.8) \]

is central. Namely, the equality \( \text{tr}(AB) = \text{tr}(BA) \) holds for all \( A, B \in F(\mathcal{S}_n)'' \). In particular, it follows from (1.5) that for any \( s \in \mathcal{S}_n \) we have

\[ \chi_{\text{nat}}(s) = \text{tr} (F(s)) \quad (2.9) \]
Indeed, Lemma 2.2 (d) and formulas (2.7) and (1.5) imply that for $\sigma \in \mathcal{G}_N, \subset \mathcal{G}_N$ we have
\[
\text{tr} (\mathcal{F}(\sigma)) = \nu_{\mathcal{N}} (x \in \mathbb{X}_{\mathcal{N}} : \sigma x = x) = \frac{\# \{ x \in \mathbb{X}_{\mathcal{N}} : \sigma x = x \}}{N_k} = \chi_{\text{nat}}(\sigma). \quad (2.10)
\]

Now consider two families of operators $\{\mathcal{M}(f)\}_{f \in L^\infty(\mathbb{X}_{\mathcal{N}}, \nu_k)}$ and $\{\mathcal{F}(O^k)\}_{k \in \mathbb{Z}}$ which belong to $\mathcal{F}(\mathcal{G}_N)'$ and whose action on an element $\eta \in \mathcal{H}$ is defined in the following way:
\[
(\mathcal{M}(f)\eta) (x, m) = f (O^m x) \eta(x, m), \quad (\mathcal{F}(O^k)\eta) (x, m) = \eta(x, m - k). \quad (2.11)
\]
Using (2.7), one can check that $\mathcal{M}(f)$ and $\mathcal{F}(O^k)$ belong to $\mathcal{F}(\mathcal{G}_N)'$. Thus, we obtain the following statement:

**Lemma 2.3.** Let $N'$ be the $w^*$-subalgebra of $\mathcal{F}(\mathcal{G}_N)'$, which is generated by the operators $\{\mathcal{M}(f)\}_{f \in L^\infty(\mathbb{X}_{\mathcal{N}}, \nu_k)}$ and $\mathcal{F}(O^k)$. Then, vector $\xi_0$ is cyclic for $N'$, i.e. the closure of the set $N'\xi_0$ coincides with $\mathcal{H}$.

**Lemma 2.4.** Let $y \in \prod_{k=1}^{\infty} \mathbb{X}_{\mathcal{N}_k}$, $J_{\mathcal{A}_y}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{A}_y, \\ 0, & \text{if } x \notin \mathbb{A}_y \end{cases}$ (see (8.1)) and operator $\mathcal{M}(J_{\mathcal{A}_y})$ acts on $\eta \in \mathcal{H}$ as follows
\[
(\mathcal{M}(J_{\mathcal{A}_y})\eta) (x, m) = J_{\mathcal{A}_y}(x) \eta(x, m). \quad (2.12)
\]
Then, operator $\mathcal{M}(J_{\mathcal{A}_y})$ belongs to algebra $\mathcal{F}(\mathcal{G}_N)'$.

**Proof.** Applying lemma 2.1 (a) we obtain that
\[
O^{N_k} \mathbb{A}_y = \mathbb{A}_y \text{ for all } k \geq j. \quad (2.13)
\]
For $k \geq j$ define an automorphism $^kD_y$ as
\[
^kD_y x = \begin{cases} x, & \text{if } x \in \mathbb{A}_y, \\ O^{N_k} x, & \text{if } x \notin \mathbb{A}_y. \end{cases} \quad (2.14)
\]
In view of Lemma 2.2 (b) the automorphism $^kD_y$ belongs to the group $\overline{\mathcal{G}_N}$. Since the representation $\mathcal{F}$ of $\mathcal{G}_N$ can be extended to a representation of $\overline{\mathcal{G}_N}$ (see (2.7)) it suffices to prove that
\[
\lim_{k \to \infty} \mathcal{F}(^kD_y) = \mathcal{M}(J_{\mathcal{A}_y}). \quad (2.15)
\]
Here “$w - \lim$” stands for the limit in the weak operator topology.

Using (2.7) and (2.14) we obtain
\[
\mathcal{F}(^kD_y) \xi_0 = J_{\mathcal{A}_y} \otimes \delta_0 + (J - J_{\mathcal{A}_y}) \otimes \delta_{N_k}.
\]
From here, taking into account the weak convergence of the sequence $\delta_{N_k}$ to zero-vector in $l^2(\mathbb{Z})$, we get that the sequence $\mathcal{F}(^kD_y) \xi_0$ converges weakly in $\mathcal{H}$ to the vector $J_{\mathcal{A}_y} \otimes \delta_0 = \mathcal{M}(J_{\mathcal{A}_y}) \xi_0$.

Now (2.15) is a consequence of Lemma 2.3.

The following statement is a direct corollary of Lemma 2.4. \qed
Corollary 2.5. The algebra $\mathcal{F}(\mathfrak{S}_n)''$ contains the family of operators $\{\mathfrak{M}(f)\}_{f \in L^\infty(\mathbb{X}_n,\nu_n)}$, acting on an element $\eta \in \mathcal{H}$ as follows

$$\mathfrak{M}(f)\eta(x,m) = f(x)\eta(x,m). \quad (2.16)$$

In particular, vector $\xi_0$ is a cyclic vector for the algebra $\mathcal{F}(\mathfrak{S}_n)''$ (see Lemma 2.3) and due to (2.7) the following relations hold

$$\mathcal{F}(\sigma)\mathfrak{M}(f)(\mathcal{F}(\sigma))^{-1} = \mathfrak{M}(f), \quad \text{where} \quad f(x) = f(\sigma^{-1}(x)), \quad \sigma \in \overline{\mathfrak{S}_n}. \quad (2.17)$$

Proposition 2.6. The algebra $\mathcal{F}(\mathfrak{S}_n)''$ is a $\text{II}_1$-factor.

Proof. Suppose that operator $A$ belongs to $\mathcal{F}(\mathfrak{S}_n)'' \cap \mathcal{F}(\mathfrak{S}_n)'$. Since $\mathcal{H} = L^2(\mathbb{X}_n,\nu_n) \otimes l^2(\mathbb{Z})$, we have

$$A\xi_0 = \sum_{i \in \mathbb{Z}} f_i \otimes \delta_i \quad \text{where} \quad f_i \in L^2(\mathbb{X}_n,\nu_n), \quad \delta_i(m) = \begin{cases} 1, & \text{if} \ m = i, \\ 0, & \text{if} \ m \neq i. \end{cases}$$

Recall that $O \in \overline{\mathfrak{S}_n}$ and representation $\mathcal{F}$ of $\mathfrak{S}_n$ can be extended by continuity with respect to the metric $\rho$ to a representation of $\overline{\mathfrak{S}_n}$. Therefore, the following equality holds

$$A\xi_0 = \sum_{i \in \mathbb{Z}} \mathcal{F}(O^{-i}) \left( (O^i) f_i \otimes \delta_0 \right). \quad (2.18)$$

Hence, for any $f \in L^\infty(\mathbb{X}_n,\nu_n)$ we have

$$A\mathfrak{M}(f)\xi_0 \overset{(2.16)}{=} \mathfrak{M}(f) A\xi_0 = \sum_{i \in \mathbb{Z}} f f_i \otimes \delta_i. \quad (2.19)$$

We thus get

$$\|A\|^2 \int_{\mathbb{X}_n} |f(x)|^2 \, d\nu_n = \|A\|^2 \|\mathfrak{M}(f)\xi_0\|^2 \geq \int_{\mathbb{X}_n} |f(x)|^2 \left( \sum_{i \in \mathbb{Z}} |f_i(x)|^2 \right) \, d\nu_n.$$  

It follows from this that

$$\left\| \sum_{i \in \mathbb{Z}} |f_i|^2 \right\|_{L^\infty(\mathbb{X}_n,\nu_n)} \leq \|A\|^2.$$

In particular, $f_i \in L^\infty(\mathbb{X}_n,\nu_n)$ and $\mathfrak{M}(f_i) \in \mathcal{F}(\mathfrak{S}_n)''$ for all $i$ (see Corollary 2.5). Hence, using (2.18), we obtain

$$A\xi_0 = \sum_{i \in \mathbb{Z}} \mathcal{F}(O^{-i}) \mathfrak{M} \left( (O^i) f_i \right) \xi_0 \overset{(2.19)}{=} \sum_{i \in \mathbb{Z}} \mathfrak{M}(f_i) \mathcal{F}(O^{-i}) \xi_0. \quad (2.20)$$

The last equality, Lemma 2.3 and Corollary 2.6 imply that

$$A = \sum_{i \in \mathbb{Z}} \mathfrak{M}(f_i) \mathcal{F}(O^{-i}). \quad (2.21)$$
Therefore, the equality $F(O)A\xi_0 = A F(O^i)\xi_0$ is equivalent to relations $^0f_i = f_i,$ where $i \in \mathbb{Z}$. It follows from Lemma 2.2 that function $f_i$ should be constant almost everywhere. In other words, there are constants $c_i, i \in \mathbb{Z}$ such that $f_i \equiv c_i$ almost everywhere and $A\xi_0 = \sum_{i \in \mathbb{Z}} c_i F(O^{-i})\xi_0$.

Finally, note that equality $M(f) A\xi_0 = A M(f^i)\xi_0,$ $f \in L^\infty (\mathbb{X}_n, \nu_n)$ is equivalent to

$$c_i \cdot f = c_i \cdot f^i, \quad i \in \mathbb{Z}.$$ 

Since $f$ is arbitrary, Lemma 2.2 implies that $c_i = 0$ for all $i \neq 0$. Therefore, $A\xi_0 = c_0\xi_0$. By Lemma 2.3, $A$ is a scalar operator.

### 2.3 Construction of an irreducible representation of the group $\hat{S}_n \times \hat{S}_n$

For the $\Pi_1$ factor representation $F$ there is a corresponding irreducible representation $F^{(2)}$ of the group $\hat{S}_n \times \hat{S}_n$ acting in the Hilbert space $\mathcal{H} = L^2 (\mathbb{X}_n, \nu_n) \otimes l^2 (\mathbb{Z})$ such that

$$F^{(2)} (g, id) = F(g),$$

$$F^{(2)} (g, g)\xi_0 = \xi_0$$

for all $g \in \hat{S}_n$ and $F^{(2)} (id, \hat{S}_n) \subset F (\hat{S}_n)'$; (2.22)

In order to define $F^{(2)}$ let us introduce the antiunitary operator $J$ acting on $\mathcal{H}^\ast$ as follows:

$$(J \eta) (x, m) = \overline{\eta (O^m x, -m)}, \quad \eta \in \mathcal{H} = L^2 (\mathbb{X}_n, \nu_n) \otimes l^2 (\mathbb{Z}).$$ (2.23)

Then direct calculations show that

$$J F(g)\xi_0 = F (g^{-1})^\ast \xi_0, \quad g \in \hat{S}_n;$$

$$(J \eta, J \zeta) = (\zeta, \eta)$$

for all $\zeta, \eta \in \mathcal{H}$.

Combining (2.7), (2.11), (2.16) and (2.23) we obtain

$$J M(f) J = M(g^i), \quad J F(O^i) J = F^{(i)} (O^i).$$ (2.24)

These equalities, Lemma 2.3 and Corollary 2.5 imply that

$$N'' = J F (\mathcal{S}_n)^\ast J \subset F (\mathcal{S}_n)'.$$ (2.25)

Therefore, the operators $\{ F^{(2)} (g, h) = F(g) J F(h) J \}_{g, h \in \mathcal{S}_n}$ define a representation of the group $\hat{S}_n \times \hat{S}_n$. It is easy to check that $F^{(2)}$ satisfies conditions (2.22).

**Proposition 2.7.** The representation $F^{(2)}$ is irreducible.

**Proof.** In view of Proposition 2.6 it suffices to show that

$$J F (\mathcal{S}_n)^\ast J = F (\mathcal{S}_n)'.$$ (2.26)
Denote for convenience the factor $\mathcal{F}(\mathbb{S}_n)$ as $M$. Consider arbitrary bounded operators $A'$ and $B$ such that $A' \in M'$ and $B \in \mathcal{J}M'\mathcal{J}$. In order to prove (2.26) it is enough to check that

$$A'B = BA'.$$  \hfill (2.27)

According to Corollary 2.5 vector $\xi_0$ is a cyclic vector for $M$. Hence, there is a sequence $\{B_n\}_{n=1}^{\infty} \subset M$ such that

$$\|B\xi_0 - B_n\xi_0\|_\mathcal{H} = 0. \hfill (2.28)$$

In particular,

$$\lim_{l,m \rightarrow \infty} \|B_l\xi_0 - B_m\xi_0\|_\mathcal{H} = 0. \hfill (2.29)$$

Since $(UV\xi_0, \xi_0) = (VU\xi_0, \xi_0)$ for all $U, V \in M$ we have

$$\|B^*_l\xi_0 - B^*_m\xi_0\|_\mathcal{H} = \|B_l\xi_0 - B_m\xi_0\|_\mathcal{H}. \hfill (2.30)$$

Relations (2.29), (2.31) and equalities $V'B_n = B_nV'$ imply that the sequence $B_n\xi_0$ is a Cauchy sequence for any $V' \in M'$. Namely,

$$\lim_{l,m \rightarrow \infty} \|B_lV'\xi_0 - B_mV'\xi_0\|_\mathcal{H} = \lim_{l,m \rightarrow \infty} \|V'(B_l\xi_0 - B_m\xi_0)\| = 0. \hfill (2.32)$$

Therefore, we can define the linear operators $\hat{B}$ and $B^\sharp$ as follows

$$\hat{B}V'\xi_0 = \lim_{n \rightarrow \infty} B_nV'\xi_0, \quad B^\sharp V'\xi_0 = \lim_{n \rightarrow \infty} B^*_nV'\xi_0, \quad V' \in M'. \hfill (2.33)$$

Denote by $\mathcal{D}(\hat{B})$ and $\mathcal{D}(B^\sharp)$ the domains of the operators $\mathcal{D}(\hat{B})$ and $\mathcal{D}(B^\sharp)$. It is clear that $M'\xi_0 \subset \mathcal{D}(\hat{B})$ and $M'\xi_0 \subset \mathcal{D}(B^\sharp)$. For any $W' \in \mathcal{J}M\mathcal{J} \subset M'$ and $V' \in M'$, we have $W'\hat{B} = BW'$. Now, it follows from (2.28) that

$$(BW'\xi_0, V'\xi_0) = (W'B\xi_0, V'\xi_0) \overset{2.28}{=} \lim_{n \rightarrow \infty} (W'B_n\xi_0, V'\xi_0) = \lim_{n \rightarrow \infty} (B_nW'\xi_0, V'\xi_0) = \lim_{n \rightarrow \infty} (W'\xi_0, B^\sharp V'\xi_0) = (W'\xi_0, B^\sharp V'\xi_0).$$

Thus, $(W'\xi_0, B^\sharp V'\xi_0) = (W'\xi_0, B^\sharp V'\xi_0)$ for all $W' \in \mathcal{J}M\mathcal{J} = N'$ and $V' \in M'$. (see Lemma 2.3). By Lemma 2.3 the set $\{W'\xi_0\}_{W' \in \mathcal{J}M\mathcal{J}}$ is norm dense in $\mathcal{H}$. Therefore,

$$(\eta, B^\sharp V'\xi_0) = (\eta, B^\sharp V'\xi_0) \hfill (2.31)$$

for all $\eta \in \mathcal{H}$ and $V' \in M'$. Hence, for arbitrary $U', V' \in \mathcal{J}M\mathcal{J} = N'$ we have

$$(A'BU'\xi_0, V'\xi_0) = (A'U'B\xi_0, V'\xi_0) \overset{2.31}{=} (A'U'B_n\xi_0, V'\xi_0) = (A'U'B_n\xi_0, V'\xi_0) \overset{2.31}{=} (A'U'B_n\xi_0, V'\xi_0) = (A'U'\xi_0, B^\sharp V'\xi_0) = (A'U'\xi_0, B^\sharp V'\xi_0) = (BA'U'\xi_0, V'\xi_0).$$

This proves the equality (2.27) and concludes the proof. \hfill □
3 Preliminaries about the representation theory of the symmetric groups

In this section we remind some notions and facts from the representation theory of the symmetric groups $\mathfrak{S}_N$ which are used in the proof of the main theorem.

3.1 The minimal element of the conjugacy class

Consider the symmetric group $\mathfrak{S}_N$. Denote by $C_m$ the conjugacy class of the group $\mathfrak{S}_N$, which consists of permutations of the cycle type $m = (m_1, m_2, \ldots, m_N)$, where $m_i$ is the number of independent cycles of length $i$.

Definition 3.1. The support of a permutation $s \in \mathfrak{S}_N$ is the subset $\text{supp}_s = \{x \in \mathcal{X}_N \mid sx \neq x\}$.

Remark 3.1. Further, we also use the notation $m = (m_1, m_2, \ldots, m_l)$ for a cycle type $m$ instead of $m = (m_1, m_2, \ldots, m_N)$ if $m_{l+1} = \ldots = m_N = 0$ (here $l < N$).

Clearly, for $\sigma \in C_m \subset \mathfrak{S}_N$ we have $\sum_{i=1}^{N} i \cdot m_i = N$ and $\# \text{supp}_\sigma = \sum_{i=2}^{N} i \cdot m_i$.

For any distinct elements $n_0, n_1, \ldots, n_{j-1} \in \mathcal{X}_N$ denote by $(n_0 n_1 \ldots n_{j-1})$ the cyclic permutation $c \in \mathfrak{S}_N$ such that $c(n_i) = n_{i+1 \pmod{j}}$. Denote by $s_i$ the transposition $(i \ i+1)$. The elements $\{s_1, s_2, \ldots, s_{N-1}\}$ are also known as the Coxeter generators of the symmetric group $\mathfrak{S}_N$.

Definition 3.2. Consider a conjugacy class $C_m$ of the symmetric group $\mathfrak{S}_N$. Let $\{i_1, i_2, \ldots, i_p\}$, where $1 < i_1 < i_2 < \ldots < i_p$, be the set of those $i \in \{2, 3, \ldots, N\}$ for which $m_i \geq 1$. The minimal element of the conjugacy class $C_m$ is the permutation $C_m$ defined as follows:

$$
C_m = (1 \ 2 \ \ldots \ i_1) \cdots ((m_{i_1} - 1) \ i_1 + 1 \ (m_{i_1} - 1) \ i_1 + 2 \ \ldots \ m_{i_1} \ i_1) \cdots $$

$$
(m_{i_1} \ i_1 + 1 \ m_{i_1} \ i_1 + 2 \ \ldots \ m_{i_1} \ i_1 + i_2) \cdots $$

$$
(m_{i_2} \ i_2 + 1 \ m_{i_2} \ i_2 + 1 \ m_{i_2} \ i_2 + 2 \ \ldots \ m_{i_2} \ i_2 + i_3) \cdots $$

It is clear that $\sum_{i=1}^{p} i \cdot m_{i_q} = N - m_1$ and $\text{supp}_c \sigma_m = \{1, 2, \ldots, N - m_1\} \subset \mathcal{X}_N$. The crucial property of $C_m$ is the following decomposition into the product of Coxeter generators:

$$
C_m = s_{j_1} s_{j_2} \cdots s_{j_r},
$$

where $j_1 < j_2 < \ldots < j_r$ are elements of $\{1, 2, \ldots, N\}$ and $r = N - m_1 - \sum_{i=1}^{p} m_{i_q}$.

The existence of such decomposition follows from the equality

$$
(i \ i+1 \ \ldots \ i+j) = s_i s_{i+1} \cdots s_{i+j-1}.
$$

For any permutation $\sigma \in \mathcal{S}_N$ denote by $\kappa_N(\sigma)$ the minimal number of factors in the decomposition of $\sigma$ into the product of transpositions. Define the sign (or signature) of permutation $\sigma$ as $\text{sgn}_N(\sigma) = (-1)^{\kappa_N(\sigma)}$. 

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3.2 Young tableaux and diagrams

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j) \), where \( \lambda_1 \geq \ldots \geq \lambda_j \geq 1 \), be a partition of a positive integer \( N = |\lambda| \) into positive integer summands \( \lambda_1, \lambda_2, \ldots, \lambda_j \) \((\lambda \vdash N)\); i.e. \( \sum_{i=1}^{j} \lambda_i = N \). Denote by \( \lambda \) the corresponding Young diagram consisting of \( j \) rows of the length \( \lambda_1, \lambda_2, \ldots, \lambda_j \) (the \( i \)-th row consists of \( \lambda_i \) boxes). The conjugate or transposed Young diagram \( \lambda' \) is obtained from \( \lambda \) by replacement of its rows by its columns.

A standard Young tableau \( T \) of the shape \( \lambda \) is the diagram \( \lambda \), whose boxes are filled with positive integers from 1 to \( N \) such that each number occurs exactly once and \( n_{pq} \) is obtained for \( n_{pq} = \min \{n_{p+1,q}, n_{p,q+1}\} \) for all \( p, q \) (see Figure 1). In this paper we consider only standard Young tableaux.

\[
\begin{array}{cccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_j \\
\lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \ldots & \lambda_{1\lambda_1} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} & \ldots & \lambda_{2\lambda_2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_{n_1} & \lambda_{n_2} & \lambda_{n_3} & \lambda_{n_4} & \ldots & \lambda_{n_{\lambda_j}} \\
\end{array}
\]

Figure 1: A standard Young tableau of the shape \( \lambda \).

Denote the set of all standard Young tableaux of the shape \( \lambda \) by \( \text{Tab}(\lambda) \). The number \( c(p, q) = q - p \) is called the content of the box with coordinates \((p, q)\) of the diagram \( \lambda \) (see [1]) or the content of the element \( n_{pq} \) of tableau \( T \in \text{Tab}(\lambda) \) (see [2]). We set \( a_i = c(p, q) \) for \( n_{pq} = i \). For each tableau \( T \) define two functions \( \mathfrak{R}_T \) and \( \mathcal{C}_T \) (row and column numbers) on the set \( \{1, 2, \ldots, N = |\lambda|\} \) as follows

\[
\mathfrak{R}_T(n_{pq}) = p, \quad \mathcal{C}_T(n_{pq}) = q. \tag{3.2}
\]

3.3 Realization of the irreducible representations of \( \mathfrak{S}_N \)

In this subsection we remind the explicit constructions of the irreducible representations of the finite group \( \mathfrak{S}_N \) [3, 7]. It is known that irreducible representations of \( \mathfrak{S}_N \) are parameterized by the Young diagrams \( \lambda \) with \( N \) boxes. Let \( \mathcal{R}_\lambda \) be one of those unitary irreducible representations which acts in the vector space \( V_\lambda \). There exists an orthonormal basis \( \{v_T\}_{T \in \text{Tab}(\lambda)} \) in \( V_\lambda \) such that the Coxeter generators \( s_i = (i, i + 1) \) acts on elements of this basis as follows:

- if \( \mathfrak{R}_T(i) = \mathfrak{R}_T(i + 1) \), then \( \mathcal{R}_\lambda(s_i)v_T = v_T \);
- if \( \mathcal{C}_T(i) = \mathcal{C}_T(i + 1) \), then \( \mathcal{R}_\lambda(s_i)v_T = -v_T \);
- if \( \mathfrak{R}_T(i) \neq \mathfrak{R}_T(i + 1) \) and \( \mathcal{C}_T(i) \neq \mathcal{C}_T(i + 1) \), then after the permutation of only two elements \( i \) and \( i + 1 \) in \( T \) we obtain a tableau \( T' \) again. Then the matrix of the operator \( \mathcal{R}_\lambda(s_i) \) in basis \( \{v_T, v_{T'}\} \) in the two-dimensional space span\(\{v_T, v_{T'}\} \) is

\[
\begin{pmatrix}
\frac{1}{d} & \sqrt{1 - \frac{1}{d^2}} \\
\sqrt{1 - \frac{1}{d^2}} & -\frac{1}{d}
\end{pmatrix}, \tag{3.3}
\]

where \( d = a_{i+1} - a_i \) and \( a_i \) is the content of the box of \( T \) that contain \( i \).
Hence, we have the following statement.

Lemma 3.3. Let $\sigma \in S_N$. Put $\hat{R}_\lambda(\sigma) = \text{sgn}(\sigma)R_\lambda(\sigma)$. Then the representations $\hat{R}_\lambda$ and $R_\lambda'$ of the group $S_N$ are unitary equivalent.

Define a normalized character of the irreducible representation $R_\lambda$ as follows

$$\chi_\lambda(\sigma) = \frac{\text{Tr}(R_\lambda(\sigma))}{\text{Tr}(R_\lambda(\text{id}))},$$

(3.4)

where $\text{Tr}$ is the standard trace of a finite-dimensional operator, $\sigma \in S_N$ and $\text{id}$ is the identity element of $S_N$. Note that Lemma 3.3 implies that

$$\chi_\lambda'(\sigma) = \text{sgn}(\sigma)\chi_\lambda(\sigma)$$

(3.5)

for all $\sigma \in S_N$ and $\lambda \vdash N$.

Lemma 3.4. Suppose that permutation $\sigma \in S_N$ is expressed as a product of distinct Coxeter generators:

$$\sigma = s_{i_1} \ldots s_{i_r} \text{ where } i_1 < i_2 < \ldots < i_r.$$

Then for any tableau $T$ the following equality holds

$$|(R_\lambda(\sigma)v_T, v_T)| = \prod_j \frac{1}{|a_{i_j+1} - a_{i_j}|},$$

(3.6)

where the product is taken by all indices $j$ for which numbers $i_j$ and $i_j + 1$ are in different rows and different columns of tableau $T$; i.e. $\text{R}_T(i_j) \neq \text{R}_T(i_j + 1)$ and $C_T(i_j) \neq C_T(i_j + 1)$.

Proof. Consider an arbitrary Coxeter generator $s_i$ and a basis vector $v_T \in V_\lambda$, where $T \in \text{Tab}(\lambda)$. If numbers $i$ and $i + 1$ are in the same row or in the same column of $T$ we have $R_\lambda(s_i)v_T = \pm v_T$. Thus, in this case for any permutation $\sigma \in S_N$ we have

$$(R_\lambda(\sigma s_i)v_T, v_T) = \pm (R_\lambda(\sigma)v_T, v_T).$$

(3.7)

Otherwise, according to (3.5), we have

$$R_\lambda(s_i)v_T = \pm \frac{1}{a_{i+1} - a_i}v_T + \sqrt{1 - \frac{1}{(a_{i+1} - a_i)^2}}v_{T'},$$

(3.8)

where $T' = s_i(T)$ is the tableau obtained from $T$ by permutation of numbers $i$ and $i + 1$ (recall that $a_i$ and $a_{i+1}$ are the contents of the boxes of $T$ which contain $i$ and $i + 1$ respectively).

Next, we show that for any $\sigma \in S_N$ whose support $\text{supp}_\lambda \sigma$ does not contain $i + 1$ we have

$$(R_\lambda(\sigma s_i)v_T, v_T) = \pm \frac{1}{a_{i+1} - a_i}(R_\lambda(\sigma)v_T, v_T).$$

(3.9)

Indeed, it suffices to check that the vector $R_\lambda(\sigma)v_T'$ is orthogonal to $v_T$. It follows from the fact that $R_\lambda(\sigma)v_T'$ is a linear combination of vectors $v_{T''}$ which correspond to those tableau $T''$ in which number $i + 1$ is written in the same
box as in $T'$. In particular, for all $T''$ we have $T'' \neq T$ and hence, $(v_{T''}, v_T) = 0$. Then $(\mathcal{R}_\lambda(\sigma)v_T, v_T) = 0$, so the required orthogonality is proved.

Therefore, for any permutation $\sigma \in S_N$ whose support does not contain $i+1$ we have

$$(\mathcal{R}_\lambda(\sigma)\sigma_{i_1}v_T, v_T) = \pm k_i(T) \cdot (\mathcal{R}_\lambda(\sigma)v_T, v_T),$$

where

$$k_i(T) = \begin{cases} \frac{1}{a_{i+1} - a_i}, & \text{if } \mathcal{R}_T(i) \neq \mathcal{R}_T(i+1) \text{ and } \mathcal{C}_T(i) \neq \mathcal{C}_T(i+1), \\ 1, & \text{otherwise}. \end{cases}$$

Now, let us apply this observation to $(\mathcal{R}_\lambda(\sigma)v_T, v_T)$. Note that the sequence $\{i_j\}$ is strictly increasing, so for all $j$ the support of permutation $s_{i_1} \ldots s_{i_{j-1}}$ does not contain $i_j + 1$. Therefore,

$$(\mathcal{R}_\lambda(\sigma)v_T, v_T) = (\mathcal{R}_\lambda(s_{i_1} \ldots s_{i_r})v_T, v_T) =$$

$$= \pm k_{i_r}(T)(\mathcal{R}_\lambda(s_{i_1} \ldots s_{i_{r-1}})v_T, v_T) = \ldots = \pm \prod_{j=1}^{r} k_{i_j}(T)(v_T, v_T).$$

Thus,

$$|(\mathcal{R}_\lambda(\sigma)v_T, v_T)| = \prod_{j=1}^{r} |k_{i_j}(T)|$$

and we obtain the formula (3.6). Lemma 3.4 is proved.

3.4 Upper bound for the characters of the symmetric group

In the proof of the main theorem we use the following important bound for the characters of the symmetric group (see [8] and also [9]).

Proposition 3.5 (Roichman, 1996). There exist absolute constants $a \in (0, 1)$ and $b > 0$ such that for any Young diagram $\lambda$ with $N = |\lambda| \geq 4$ boxes and for any $\sigma \in S_N$ the following inequality holds:

$$|\chi_\lambda(\sigma)| \leq \left( \max\left\{ \frac{\lambda_1}{N}, \frac{\lambda'_1}{N}, a \right\} \right)^{b \#(\text{supp}_N(\sigma))}. \quad (3.10)$$

Here $\lambda_1$ ($\lambda'_1$) is the number of boxes in the first row (column) of the diagram $\lambda$.

4 The approximation theorem for the characters on $\hat{S}_n$

In this section we state and prove the approximation theorem for characters on the group $\hat{S}_n$ which is a crucial part of the proof of Theorem 1.2.

Firstly, let us recall the definition of a character.

Definition 4.1. A function $\chi : G \to \mathbb{C}$ on a group $G$ is called a character if the following conditions hold

(a) $\chi$ is central, i.e. $\chi(gh) = \chi(hg)$ for all $g, h \in G$;
(b) \( \chi \) is a positive-definite function, i.e. for any elements \( g_1, \ldots, g_k \in G \) the matrix \( [\chi(g_i g_j^{-1})] \) is a Hermitian and positive-semidefinite matrix;

(c) \( \chi \) is normalized, i.e. \( \chi(\text{id}) = 1 \), where \( \text{id} \) is the identity of the group \( G \).

If additionally

(d) \( \chi \) is indecomposable, i.e. \( \chi \) cannot be represented as a sum of two linear independent functions that satisfy (a) and (b),

then \( \chi \) is called an indecomposable character.

**Remark 4.1.** Let \( \pi_\chi \) be the representation of \( G \), obtained from \( \chi \) via the Gelfand-Naimark-Segal (shortly GNS) construction. Then the property (d) is equivalent to \( \pi_\chi \) being a factor representation.

The following fact is an analogue of the approximation theorem from [6] for the characters of the standard infinite symmetric group \( S_\infty \).

**Proposition 4.2.** Each indecomposable character \( \chi \) on \( S_\infty \) is a weak limit of some sequence of normalized irreducible characters of the groups \( S_N \). Namely, there is an increasing sequence \( \{k(l)\}_{l=1}^\infty \) of positive integers and there exist the partitions \( k(l) \mapsto N_{k(l)} \) such that

\[
\lim_{l \to \infty} \chi_{k(l)}(g) = \chi(g) \text{ for all } g \in S_\infty.
\]

**Proof.** Consider the GNS-representation \( (\pi_\chi, \mathcal{H}_\chi, \xi_\chi) \) of the group \( S_\infty \) acting in the Hilbert space \( \mathcal{H}_\chi \) with the cyclic and separating vector \( \xi_\chi \), such that \((\pi_\chi(g)\xi_\chi, \xi_\chi) = \chi(g)\) for all \( g \in S_\infty \). Denote by \( B(\mathcal{H}_\chi) \) the set of all bounded linear operators on \( \mathcal{H}_\chi \). For any conjugation-invariant subset \( S \subset B(\mathcal{H}_\chi) \) define its commutant by

\[
S' = \{ A \in B(\mathcal{H}_\chi) : AB = BA \text{ for all } B \in S \}, \quad (S')' = S''.
\]

We denote by \( [S\mathcal{B}] \) the smallest closed subspace containing \( S\mathcal{B} \), where \( \mathcal{B} \) is a subset of \( \mathcal{H}_\chi \).

Since \( \chi \) is a character, then according to the GNS-construction we have

\[
[\pi_\chi(S_\infty)\xi_\chi] = [\pi_\chi(S_\infty)'\xi_\chi] = \mathcal{H}_\chi. \tag{4.1}
\]

For convenience we denote the \( \ast \)-algebra \((\pi_\chi(S_\infty))''\), which is a factor of type \( II_1 \), as \( M \). The faithful normal normalized trace \( \text{tr} \) on \( M \) is a vector state defined by \( \xi_\chi \), i.e. \( \text{tr}(a) = (a\xi_\chi, \xi_\chi) \) for \( a \in M \). Denote by \( L^2(M, \text{tr}) \) the Hilbert space which is the completion of \( M \) with respect to the norm which correspond to the inner product \( \langle a, b \rangle = \text{tr}(b^*a) \), where \( a, b \in M \).

From now on we suppose that \( \mathcal{H}_\chi = L^2(M, \text{tr}) \). Then we can assume that the operators \( a \in M \) acts by left multiplication; i.e.

\[
L^2(M, \text{tr}) \ni \eta \mapsto a\eta \in L^2(M, \text{tr}).
\]

In this case the operators \( a' \in M' \) are being realized by operators of the right multiplication by the elements of \( a \in M \)

\[
L^2(M, \text{tr}) \ni \eta \mapsto a' \eta \cdot a \in L^2(M, \text{tr}).
\]
Each element \( g \in \mathcal{S}_N \) defines an inner automorphism \( \text{Ad}_g \) of the factor \( M \):

\[
\text{Ad}_g(a) = \pi \chi(g)a\pi\chi(g^{-1}), \quad a \in M.
\]

The map

\[
M \ni a \mapsto \frac{1}{N_k!} \sum_{g \in \mathcal{S}_N} \text{Ad}_g(a) \in M
\]

is the conditional expectation \( \text{E} \) which projects \( M \) onto the subalgebra \( N_k \mathcal{M} = \{ a \in M : \text{Ad}_g(a) = a \text{ for all } g \in \mathcal{S}_N \} \).

If \( \lambda \vdash N_k \) and \( P_{\lambda} = \frac{\dim \lambda}{N_k!} \sum_{g \in \mathcal{S}_N} \chi_\lambda(g) P_{\lambda} \) for all \( g \in \mathcal{S}_N \), then \( P_{\lambda} \) belongs to the center of the finite-dimensional algebra \( \pi \chi(\mathcal{S}_N)_\lambda \) and

\[
N_k \mathcal{E}(\pi \chi(g)) = \sum_{\lambda \vdash N_k} \chi_\lambda(g) P_{\lambda} \text{ for all } g \in \mathcal{S}_N.
\]

The conditional expectations \( N_k \mathcal{E} \) are orthogonal projections in \( L^2(M, \text{tr}) \). Since

\[
\lim_{k \to \infty} N_k \mathcal{E}(a) = \infty \mathcal{E}(a) \text{ for any } a \in M,
\]

where \( \infty \mathcal{E} = \lim_{k \to \infty} N_k \mathcal{E} \). Since \( \mathcal{M} = \pi \chi(\mathcal{S}_N)_\lambda \) is a factor, formula \( (4.2) \) implies that

\[
\infty \mathcal{E}(a) = \text{tr}(a)I \text{ for any } a \in M.
\]

Thus, combining \( (4.3) \) and \( (4.4) \), we obtain

\[
\lim_{k \to \infty} \sum_{\lambda \vdash N_k} (\chi_\lambda(g) - \chi(g))^2 \text{tr}(P_{\lambda}) = 0 \text{ for any } g \in \mathcal{S}_N.
\]

Consider two sequences of positive reals \( \{\epsilon_l\} \) and \( \{\delta_l\} \) which satisfy the following conditions:

\[
\lim_{l \to \infty} \max \{\epsilon_l, \delta_l\} = 0 \quad \text{and} \quad \lim_{l \to \infty} \frac{\epsilon_l N_l!}{\delta_l} = 0.
\]

Now, using \( (4.5) \) we find for each \( l \) a positive integer \( k(l) \) such that

\[
\sum_{\lambda \vdash N_m} (\chi_\lambda(g) - \chi(g))^2 \text{tr}(P_{\lambda}) < \epsilon_l \text{ for all } g \in \mathcal{S}_N, \text{ and } m \geq k(l).
\]

Put \( \Lambda(m, g) = \{ \lambda \vdash N_m : (\chi_\lambda(g) - \chi(g))^2 > \delta_l \} \), where \( g \in \mathcal{S}_N \), and \( k\Lambda(m) = \bigcup_{g \in \mathcal{S}_N} \Lambda(m, g) \). Applying \( (4.7) \), we obtain

\[
\sum_{\lambda \vdash N_m \& (\lambda \not\in \Lambda(m))} \text{tr}(P_{\lambda}) > 1 - \frac{\epsilon_l N_l!}{\delta_l}. \quad (4.8)
\]
This inequality and (4.6) imply that for each \( l \) there exists a partition \( k(l) \lambda \vdash N_k(l) \) such that
\[
|\chi_{k(l) \lambda}(g) - \chi(g)| \leq \sqrt{\delta_l} \quad \text{for all } g \in \mathcal{G}_N.
\]
Since \( \lim_{l \to \infty} \delta_l = 0 \) we have \( \lim_{l \to \infty} \chi_{k(l) \lambda}(g) = \chi(g) \) for all \( g \in \mathcal{G}_\infty \). Thus, the sequences \( \{k(l)\}_l \) and \( \{k(l) \lambda \vdash N_k(l)\}_l \) satisfy the required conditions. Proposition 4.2 is proved. \( \square \)

5 Technical lemmas

In this section we establish several technical facts which are used in the proof of Theorem 1.2.

First, let us introduce some notations. Let \( ^n \Upsilon \) be the set of all partitions of a positive integer \( n \). Denote by \( (n) \) the partition which consists of only one part and which corresponds to the Young diagram with only one row. Put \( (1^n) = (n) \), i.e. \( (1^n) \) is the partition which consists of \( n \) parts equal 1 and which corresponds to the Young diagram with only one column. For an arbitrary partition \( \mu \), denote \( ^n \Upsilon_\mu = \{ \lambda \in ^n \Upsilon : \lambda \setminus (1^i) = \mu \} \) and \( ^n \Upsilon'_\mu = \{ \lambda \in ^n \Upsilon : \lambda \setminus (1^i) = \mu' \} \), where \( \lambda = (\lambda_1, \ldots, \lambda_i) \). It is clear that \( ^n \Upsilon_\mu = \{ \lambda \in ^n \Upsilon : \lambda' \setminus (1^i) = \mu' \} \) and \( ^n \Upsilon'_\mu = \{ \lambda \in ^n \Upsilon' : \lambda' \setminus (1^i) = \mu' \} \).

Define in the set of all sequences \( \{k \lambda \vdash N_k\}_k \) of partitions the subsets \( C_\infty \), \( \widehat{C}_\infty \), \( C_\mu \) and \( \widehat{C}_\mu \), where \( \mu \) is a partition and \( |\mu| < \infty \), as follows:

- (\( C_\infty \)) there is a subsequence \( \{k_i\} \) such that
  \[
  \lim_{i \to \infty} |k_i \lambda \setminus (k_i \lambda_1)| = \infty;
  \]

- (\( \widehat{C}_\infty \)) there is a subsequence \( \{k_i\} \) such that
  \[
  \lim_{i \to \infty} |k_i \lambda \setminus (1^{k_i \lambda_1'})| = \infty;
  \]

- (\( C_\mu \)) there is a subsequence \( \{k_i\} \) such that \( k_i \lambda \in \langle n_k \rangle_\mu \), where \( \mu \) is independent of \( i \);

- (\( \widehat{C}_\mu \)) there is a subsequence \( \{k_i\} \) such that \( k_i \lambda \in \langle n_k \rangle'_{\mu} \), where \( \mu \) is independent of \( i \).

Remark 5.1. Clearly, \( \widehat{C}_\mu \) (\( \widehat{C}_\infty \)) is the set of all sequences which can be obtained by transposing the sequences \( \{k \lambda \vdash N_k\}_k \) in \( C_\mu \) (\( C_\infty \)).

The following statement is trivial.

Lemma 5.1. The union \( C_\infty \cup \widehat{C}_\infty \cup \bigcup \mu C_\mu \cup \bigcup \mu \widehat{C}_\mu \) coincides with the set of all sequences of partitions \( \{k \lambda \vdash N_k\}_k \).
Lemma 5.2. \( S \) is defined as \( \text{sgn} \). Therefore, \( (\text{sgn}) \) type.

Remark 5.2. Suppose that \( \sigma \) belongs to the conjugacy class \( C_m \) of the group \( S_m \), which consists of the permutations of the cycle type \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell}) \) (see Subsection 3.3). Recall that \( i_{k,j} (j > k) \) is the embedding of \( S_m \) into \( S_{m_j} \) (see Section 1.3 and also (1.1)). Then \( i_{k,j} (C_m) \subset C_m \), where

\[
\lambda = \left( \lambda_1, \lambda_2, \ldots, \lambda_{\ell} \right) = \left( \frac{N_{\ell}}{N_k}, \frac{N_{\ell-1}}{N_k}, \ldots, \frac{N_{1}}{N_k} \right), \quad \frac{N_{\ell}}{N_k} = \prod_{i=k+1}^{\ell} \frac{n_i}{n_i}.
\]

In particular, if \( \sigma \in S_{m_1} \), then \( \# (\text{supp}_{m_j} i_{k,j}(\sigma)) = \# (\text{supp}_{m_j} \sigma) \cdot \frac{N_j}{N_k} \) and \( \kappa_{m_j}(\sigma) = \frac{N_j}{N_k} \kappa_{m_k}(\sigma) \) (see Proposition 5.3). From now on we identify \( g \in S_{m_j} \) with its image \( i_{k,j}(g) \in S_{m_j} \) while taking into account the changes of the cycle type.

Recall that \( \text{sgn}_{m_j} \) is a one-dimensional representation of the group \( S_{m_j} \) defined as

\[
\text{sgn}_{m_j}(s) = \begin{cases} 
-1, & \text{if } s \in S_{m_j} \text{ is an odd permutation}, \\
1, & \text{if } s \in S_{m_j} \text{ is an even permutation}.
\end{cases}
\]

The following statement is immediate.

Lemma 5.2. For any \( s \in S_N \) there exists a positive integer \( N(s) \) such that \( \text{sgn}_{m_j}(s) = \text{sgn}_{m_k}(s) \) for all \( j, k > N(s) \).

Hence, there exists a one-dimensional representation \( \text{sgn}_\infty \) of the group \( S_\infty \) defined as follows

\[
\text{sgn}_\infty(s) = \lim_{k \to \infty} \text{sgn}_{m_k}(s), \quad s \in S_\infty.
\]

Remark 5.3. If the sequence \( \hat{n} = \{n_k\}_{k=1}^\infty \) contains infinitely many even numbers, then \( \text{sgn}_\infty(s) = 1 \) for all \( s \in S_\infty \).

Remark 5.4. Suppose that there are only finitely many even numbers in the sequence \( \hat{n} = \{n_k\}_{k=1}^\infty \). Then one can choose two sequences \( \{s_n\} \) and \( \{\sigma_n\} \) in \( S_\infty \) such that \( \text{sgn}_\infty(s_n) = 1 \) and \( \text{sgn}_\infty(\sigma_n) = -1 \) for all \( n \) but which converge to the same automorphism \( s \in \overline{S_\infty} \) with respect to the metric \( \rho \) (see (2.3))

\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} \sigma_n = s.
\]

Therefore, \( \text{sgn}_\infty \) cannot be extended by continuity to the closure \( \overline{S_\infty} \) of the group \( S_\infty \) with respect to the metric \( \rho \).
Lemma 5.3. Let $\chi$ be a character of the group $\mathfrak{S}_n$ and let $\{k \lambda \vdash N_k\}_{k=1}^{\infty}$ be a sequence of partitions such that $\lim_{k \to \infty} \chi_{k \lambda_i}(g) = \chi(g)$ for all $g \in \mathfrak{S}_n$. If $\{k \lambda\}_{k=1}^{\infty} \in C_{\infty} \cup \hat{C}_{\infty}$, then $\chi(g) = \begin{cases} 1, & \text{if } g = \text{id}; \\ 0, & \text{if } g \neq \text{id}. \end{cases}$

Proof. Clearly, $\chi(\text{id}) = 1$. Take any element $g \neq \text{id}$ of the group $\mathfrak{S}_n$. Due to Lemma 3.3, it is sufficient to consider the case when $\{k \lambda\}_{k=1}^{\infty}$ be a subsequence which satisfies the condition

$$\lim_{i \to \infty} |k \lambda/ (k \lambda_1)| = \infty. \quad (5.3)$$

Suppose that $g \in \mathfrak{S}_{N_k} \subset \mathfrak{S}_n$.

It is sufficient to consider the case when $k_i = i$ for all $i$, i.e. when the sequence $\{k_i\}_{i=1}^{\infty}$ coincides with the sequence $\{k\}_{k=1}^{\infty}$ (the proof of the general case is analogous). Then, Proposition 3.5 implies that

$$|\chi(k \lambda)(g)| \leq \left( \max \left\{ \frac{k \lambda_1}{N_k}, a \right\} \right)^{b \cdot \#(\text{supp}_N(g))}, \text{ where } a \in (0, 1), b > 0.$$

Hence, after passing to the limit $k \to \infty$ we obtain $k \to \infty$

$$|\chi(g)| \leq \limsup_{k \to \infty} \left( \frac{k \lambda_1}{N_k} \right)^{b \cdot \#(\text{supp}_N(g))} = \limsup_{k \to \infty} \left( \frac{k \lambda_1}{N_k} \right)^{b \cdot \#(\text{supp}_N(g))}.$$ 

Therefore,

$$|\chi(g)| \leq \limsup_{k \to \infty} \left( 1 - \frac{N_k - k \lambda_1}{N_k} \right)^{b \cdot \#(\text{supp}_N(g))} =$$

$$= \limsup_{k \to \infty} \exp \left( - \frac{|k \lambda/ (k \lambda_1)|}{N_1} \cdot \frac{b}{N_1} \cdot \#(\text{supp}_N(g)) \right).$$

Since $\text{supp}_N(g) > 1$, then, using (5.3), we obtain that $\chi(g) = 0$. This finishes the proof. \hfill \Box

Lemma 5.4. Let $\chi$ be a character of the group $\mathfrak{S}_n$ and let $\{k \lambda \vdash N_k\}_{k=1}^{\infty}$ be a sequence of partitions such that $\lim_{j \to \infty} \chi_{j \lambda_j}(\sigma) = \chi(\sigma)$ for all $\sigma \in \mathfrak{S}_n$. If $\{k \lambda \vdash N_k\}_{k=1}^{\infty} \in C_{\mu}$ for some partition $\mu$, then $\chi(\sigma) = \chi_{\text{nat}}(\sigma)^{|\mu|}$.

Proof. As in the proof of Lemma 5.3 it suffices to consider the case when $\lambda \setminus (\lambda_1) = \mu$ for all $k$. If $\mu$ is the empty partition, then $\chi_{(\lambda)}(\sigma) = 1$ for each $\sigma \in \mathfrak{S}_{N_k}$ and the statement is trivial. Hence, we can now suppose that

$$|\mu| \geq 1. \quad (5.4)$$

Recall that Tab $(\lambda)$ is the set of all (standard) Young tableaux of the shape $(\lambda)$ filled with numbers $1, 2, \ldots, N_k$. Let $R_{\mu}(\lambda)$ be the irreducible representation of the group $\mathfrak{S}_{N_k}$ defined in Subsection 3.3 and let $\chi_{(\lambda)}$ be its normalized character (see (3.4)).
The hook length formula implies that
\[
\# \left( \text{Tab}(\lambda) \right) = \dim \lambda = \frac{\dim \mu}{|\mu|!} \cdot \frac{N_k!}{(N_k - |\mu| - m)! \prod_{i=1}^{m} (N_k - |\mu| - i + 1 + \mu'_i)}
\]
(5.5)
where \(\mu' = (\mu'_1, \mu'_2, \ldots, \mu'_m)\) and \(\dim \lambda\) is the dimension of representation \(R(\lambda)\).

Let \(S^{N_k}_{|\mu|}\) be the family of all \(|\mu|\)-element subsets \(\{i_1 < i_2 < \ldots < i_{|\mu|}\}\) of the set \(\{1, 2, \ldots, N_k\}\). Clearly, \(#S^{N_k}_{|\mu|} = \binom{N_k}{|\mu|}\). Denote by \(S(T, \mu)\) the subset of those elements of \(\{1, 2, \ldots, N_k\}\) which are located in the boxes of the diagram \(\lambda\) \(\setminus (\lambda_{\mu}) = \mu\) in a tableau \(T \in \text{Tab}(\lambda)\). Since for each \(T \in \text{Tab}(\lambda)\) we have \(#S(T, \mu) = |\mu|\) we can regard \(S(T, \mu)\) as an element of \(S^{N_k}_{|\mu|}\). Note that a tableau \(T\) is defined uniquely by the filling of diagram \(\lambda\) \(\setminus (\lambda_{\mu}) = \mu\).

Denote by \(\text{Tab}(\lambda)\) the set of all tableaux \(T \in \text{Tab}(\lambda)\) such that \(S(T, \mu)\) consists only of subsets \(\{i_1 < i_2 < \ldots < i_{|\mu|}\}\) which satisfy \(i_{l + 1} - i_l > 1\) for all \(l = 1, 2, \ldots, |\mu| - 1\).

Consider an arbitrary element \(\sigma \in \mathfrak{S}_{N_k}\). Suppose that \(\sigma\) is an element of \(\mathfrak{S}_{N_k}\) and belongs to the conjugacy class \(\mathcal{C}_m\) of the group \(\mathfrak{S}_{N_k}\) that contains the permutations of the cycle type \(m = (k_{m_1}, k_{m_2}, \ldots, k_{m_l})\) (see Subsection 3.1). For any \(j > k\) denote by \(\sigma_{(m)}\) the minimal element of the conjugacy class \(\mathcal{C}_m\) that contains \(i_{k+1}(\sigma)\). In view of Remark 5.2 there is a rational number \(\alpha\) independent of \(j\) such that
\[
\text{supp}_{\alpha} \sigma_{(m)} = \{1, 2, \ldots, \alpha N_j\} \text{ for all } j > k.
\]
(5.6)

Now take an arbitrary positive integer parameter \(Q > |\mu|\) which will tend to infinity. Denote by \(\text{Tab}_Q(\lambda)\) the subset of those tableaux in \(\text{Tab}(\lambda)\) whose first \(Q\) boxes of the first row are filled with numbers \(1, 2, \ldots, Q\). It is clear that
\[
\# \left( \text{Tab}(\lambda) \cap \text{Tab}_Q(\lambda) \right) \geq \dim \mu \cdot \frac{(N_j - Q)(N_j - Q - 3) \cdots (N_j - Q - 3(|\mu| - 1))}{|\mu|!}.
\]
(5.7)

According to (5.1), we have
\[
\chi^{(\lambda)}(\sigma) = \frac{\sum_{T \in \text{Tab}(\lambda)} \left( R^{(\lambda)}(\sigma_{(m)}) \right)_{VT, VT}}{\#(\text{Tab}(\lambda))}
\]
It follows from (5.5) and (5.7) that
\[
\lim_{j \to \infty} \frac{\# \left( \text{Tab}(\lambda) \cap \text{Tab}_Q(\lambda) \right)}{\#(\text{Tab}(\lambda))} = 1.
\]
Thus, in order to compute the limit \(\lim_{j \to \infty} \chi^{(\lambda)}(\sigma)\) we need to estimate the matrix elements \(R^{(\lambda)}(\sigma_{(m)})_{VT, VT}\) for \(T \in \text{Tab}(\lambda) \cap \text{Tab}_Q(\lambda)\).
Take a tableau $T \in \widehat{\mathsf{Tab}}(\lambda) \cap \mathsf{Tab}_Q(\lambda)$ such that at least one element $e$ from $\text{supp}_{\lambda_j}(\sigma_{(m)})$ (see (5.6)) there is in the diagram $\mu$. Then either $e$ or $e - 1$ belongs to $\{ j_1 \}_{i=1}^l$ from (5.1).

Let us first consider the case when the transposition $s_e = (e \ e + 1)$ is contained in the decomposition $s_j = s_{j_1} s_{j_2} \cdots s_{j_r}$, where $j_1 < j_1 + 1$ (see (5.1)). Then, according to the definitions of sets $\mathsf{Tab}(\lambda)$ and $\mathsf{Tab}_Q(\lambda)$, we have

$$e + 1 \in \langle 2 \lambda_1 \rangle, \quad a_e + 1 \geq Q.$$  (5.8)

In other words, the number $e + 1$ is contained in the first row of the tableau $T$. Since $\sigma_{(m)} = s_{j_1} s_{j_2} \cdots s_{j_r}$ satisfies the conditions of Lemma 3.4 we have

$$| (R(\lambda)(\sigma_{(m)}) v_T, v_T) | = \prod_{i} \frac{1}{|a_i + 1 - a_i|},$$

where the product is taken over all indices $i$ such that elements $i_1$ and $i_1 + 1$ are contained in different rows and different columns of the tableau $T$ (recall that $a_i$ is the content of the box of $T$ that contains $i$). Since $e$ and $e + 1$ are in different rows and columns of $T$, we have

$$| (R(\lambda)(\sigma_{(m)}) v_T, v_T) | \leq \frac{1}{|a_e + 1 - a_e|}.$$  (5.9)

According to our assumption, $e$ is contained in the diagram $\mu$. Therefore,

$$-|\mu| \leq a_e \leq |\mu| - 2.$$  

Thus, using (5.8) and (5.9) we obtain

$$| (R(\lambda)(\sigma_{(m)}) v_T, v_T) | \leq \frac{1}{Q - |\mu| + 2}.  \quad (5.10)$$

If the transposition $s_{e-1} = (e - 1 \ e)$ is contained in the decomposition $\sigma_{(m)} = s_{j_1} s_{j_2} \cdots s_{j_r}$, where $j_1 < j_1 + 1$ (see (5.1)) then we can obtain the estimate (5.10) in a similar way.

Now let us estimate the number of tableaux $T \in \widehat{\mathsf{Tab}}(\lambda) \cap \mathsf{Tab}_Q(\lambda)$, whose rows, starting from the second (i.e. rows of the diagram $\mu$), contain only elements of the set

$$\{1, 2, \ldots, N_j\} \setminus (\text{supp}_{\lambda_j}(\sigma_{(m)}) = \{1 + \alpha N_j, 2 + \alpha N_j, \ldots, N_j\} \quad (5.6).$$

In other words, it means that $\text{supp}_{\lambda_j}(\sigma_{(m)})$ is contained in the first row of $T$. Denote the set of all tableaux (not necessarily from $\widehat{\mathsf{Tab}}(\lambda) \cap \mathsf{Tab}_Q(\lambda)$) that satisfy this property by $\mathsf{Tab}_Q^\alpha(\lambda)$. Note that for all sufficiently large $j$ the inequality $|\mu| < Q \leq \alpha N_j$ holds. Therefore, applying (5.6), we obtain

$$\mathsf{Tab}_Q^\alpha(\lambda) = \{ T : S(T, \mu) \subset \{1 + \alpha N_j, 2 + \alpha N_j, \ldots, N_j\} \},$$

$$\# \mathsf{Tab}_Q^\alpha(\lambda) = \dim \mu \cdot \left( N_j - \alpha N_j \right)^{|\mu|}. \quad (5.11)$$

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Applying the bound (5.10) to matrix elements in the first sum, we obtain

\[
\mathcal{T}_0 = \sum_{T \in \text{Tab}^0(\lambda)} (R_\lambda(\sigma^{(m)}) v_T, v_T),
\]

\[
\mathcal{T}_1 = \sum_{T \in \text{Tab}(\lambda) \setminus \text{Tab}^0(\lambda)} (R_\lambda(\sigma^{(m)}) v_T, v_T).
\]

(5.12)

From the definition of \( R_\lambda(\sigma^{(m)}) \) (see Subsection 5.3) we have

\[
R_\lambda(\sigma^{(m)}) v_T = v_T \quad \text{for all } T \in \text{Tab}^0(\lambda).
\]

Hence, using (5.11) we obtain

\[
\mathcal{T}_0 = \dim \mu \cdot \left( N_j - \alpha N_j \right).
\]

(5.13)

In order to estimate \( \mathcal{T}_1 \) consider two subsets

\[
\text{Tab}^{10} = (\text{Tab}(\lambda) \setminus \text{Tab}^0(\lambda)) \cap \left( \text{Tab}(\lambda) \cap \text{Tab}^0(\lambda) \right),
\]

\[
\text{Tab}^{11} = (\text{Tab}(\lambda) \setminus \text{Tab}^0(\lambda)) \setminus \left( \text{Tab}(\lambda) \cap \text{Tab}^0(\lambda) \right).
\]

Then \( \text{Tab}(\lambda) \setminus \text{Tab}^0(\lambda) = \text{Tab}^{10} \cup \text{Tab}^{11} \) and hence

\[
\mathcal{T}_1 = \sum_{T \in \text{Tab}^{10}} (R_\lambda(\sigma^{(m)}) v_T, v_T) + \sum_{T \in \text{Tab}^{11}} (R_\lambda(\sigma^{(m)}) v_T, v_T).
\]

(5.14)

Applying the bound (5.10) to matrix elements in the first sum we obtain

\[
\mathcal{T}_1 \leq \frac{\# \text{Tab}^{10}}{Q - |\mu| + 2} + \# \text{Tab}^{11}.
\]

(5.15)

Next, combining (5.3) and (5.7) gives

\[
\# \text{Tab}^{11} \leq \frac{\dim \mu}{|\mu|!} \left( \prod_{i=1}^{\mu} \left( N_j - |\mu| - m + i \right) \prod_{i=1}^{\mu} \left( N_j - Q - 3(|\mu| - i) \right) \right).
\]

Therefore, there exists a positive constant \( C \) independent of \( j \) such that

\[
\# \text{Tab}^{11} \leq C N_j^{\mu - 1} \quad \text{for all } j.
\]

Hence, applying formula (5.3) and bound (5.15) we obtain the inequality

\[
\frac{\# \text{Tab}^{11}}{\# \text{Tab}(\lambda)} \leq \frac{1}{Q - |\mu| + 2} + \frac{C_1}{N_j},
\]

(5.16)

where \( C_1 \) is a positive constant independent of \( j \).

Finally, let us estimate

\[
\left| \chi_{\lambda}(\sigma^{(m)}) - \frac{\mathcal{T}_0}{\# \text{Tab}(\lambda)} \right| = \left| \chi_{\lambda}(\sigma^{(m)}) - \frac{\dim \mu}{\# \text{Tab}(\lambda)} \left( N_j - \alpha N_j \right) \right|.
\]
Combining (5.1), (5.12) and (5.16) we obtain
\[
\left| \chi(\lambda) (\sigma_{(m)}) - \frac{\dim \mu}{\# \text{Tab}(\lambda)} \cdot \left( N_j - \alpha N_j \right) \right| \leq \frac{|\Sigma_1|}{\# \text{Tab}(\lambda)} \leq \frac{1}{Q - |\mu| + 2} + C_1.
\]
Passing to the limit \( j \to \infty \) gives us the inequality
\[
\left| \chi(\sigma) - (1 - \alpha)^{|\mu|} \right| \leq \frac{1}{Q - |\mu| + 2}.
\]

Since \( Q \) can be chosen arbitrarily large we have \( \chi(\sigma) = (1 - \alpha)^{|\mu|} \). The statement of lemma 5.4 now follows from (2.10) and (5.6).

**Corollary 5.5.** Let \( \chi \) be a character of the group \( \mathbb{S}_N \), and let \( \{ k \lambda \vdash N_k \}_{k=1}^\infty \) be a sequence of partitions such that \( \lim_{j \to \infty} \chi_{\lambda}^k(\sigma) = \chi(\sigma) \) for all \( \sigma \in \mathbb{S}_N \). If \( \{ k \lambda \vdash N_k \}_{k=1}^\infty \in \tilde{C}_\mu \) for some partition \( \mu \), then \( \chi(\sigma) = \text{sgn}_\infty(\sigma) \cdot \chi_{\text{nat}}(\sigma)^{|\mu|} \).

**Proof.** The statement follows directly from Lemma 5.4, Remark 5.1 and formula (5.17).

Denote by \( \text{ex Char} \) the set of functions on \( \mathbb{S}_N \) which are claimed in Theorem 1.2 to be indecomposable. Namely, we put
\[
\text{ex Char}^+(\mathbb{S}_N) = \{ \chi_{\text{nat}}^p : p \in \mathbb{N} \cup \{0\} \}, \\
\text{ex Char}^-(\mathbb{S}_N) = \{ \text{sgn}_\infty \cdot \chi_{\text{nat}}^p : p \in \mathbb{N} \cup \{0\} \}, \\
\text{ex Char} = \text{ex Char}^+(\mathbb{S}_N) \cup \text{ex Char}^-(\mathbb{S}_N) \cup \{ \chi_{\text{nat}} \}.
\]

**Lemma 5.6.** \( \text{ex Char}(\mathbb{S}_N) \) is a subset of the set of all characters on \( \mathbb{S}_N \), and the indecomposable characters belongs to \( \text{ex Char} \).

**Proof.** It is clear that \( \chi_{\text{nat}}^\infty = \text{sgn}_\infty \cdot \chi_{\text{nat}}^\infty \) is a character on \( \mathbb{S}_N \). It remains to check that \( \chi_{\text{nat}}^p \) and \( \text{sgn}_\infty \cdot \chi_{\text{nat}}^p \) are characters on the group \( \mathbb{S}_N \).

Recall that \( \mathcal{F} \) is a unitary representation of the group \( \mathbb{S}_N \) which acts in the Hilbert space \( \mathcal{H} \) of all \( \lambda \in \mathbb{C}^N \) (see Subsection 2.2.1). Moreover, this representation satisfies the following property: for \( \xi_0 = 1 \otimes \xi_0 \in \mathcal{H} \) the equality
\[
\chi_{\text{nat}}(\sigma) = (\mathcal{F}(\sigma)\xi_0, \xi_0)_{\mathcal{H}} \quad (see \quad (2.3))
\]
holds for all \( \sigma \in \mathbb{S}_N \). Thus, \( \chi_{\text{nat}} \) is a character on \( \mathbb{S}_N \).

Now for any \( p \in \mathbb{N} \cup \{0\} \) consider the unitary representation \( \mathcal{F}^op \) acting on \( \mathcal{H}^{op} \). It is clear that
\[
\chi_{\text{nat}}^p(\sigma) = (\mathcal{F}^op(\sigma)\xi_0^{op}, \xi_0^{op})_{\mathcal{H}^{op}} \quad \text{for all} \quad \sigma \in \mathbb{S}_N.
\]

Since \( \tau(B) = (B\xi_0^{op}, \xi_0^{op})_{\mathcal{H}^{op}} \) is a vector state, \( \chi_{\text{nat}}^p \) is a character on \( \mathbb{S}_N \). Similarly, \( \text{sgn}_\infty \cdot \mathcal{F}^op \) is also a unitary representation acting on \( \mathcal{H}^{op} \) and the same argument implies that \( \text{sgn}_\infty \cdot \chi_{\text{nat}}^p \) is also a character on \( \mathbb{S}_N \).

If \( \chi \) is indecomposable characters then, applying Proposition 4.2, Lemmas 5.1, 5.3, 5.4 and Corollary 5.3 we obtain that \( \chi \) belongs to \( \text{ex Char}(\mathbb{S}_N) \).

**Lemma 5.7.** Each character from \( \text{ex Char}(\mathbb{S}_N) \) is indecomposable.
Proof. It is clear that character \( \chi^+_0 = \text{sgn}_\infty \chi^+_n \) is indecomposable if and only if character \( \chi^+_p = \chi^+_n \) is indecomposable.

First we recall that, by Proposition 2.0, character \( \chi_{\text{nat}} \) is indecomposable.

**First proof.** We suppose the opposite; i.e. some character \( \chi^+_m \) is not extreme point in the set of all normalized characters. Then there exist the numbers \( \alpha^+_p, \alpha^-_p \in [0, 1] \) with the property

\[
\chi^+_m(\sigma) = \alpha^+_0 + \alpha^-_0 \text{sgn}_\infty(\sigma) + \sum_{j=1}^{m-1} \alpha^+_j \chi^+_j(\sigma) + \sum_{j=1}^{m-1} \alpha^-_j \chi^-_j(\sigma)
\]

\[
+ \sum_{j=m+1}^\infty \alpha^+_j \chi^+_j(\sigma) + \sum_{j=m+1}^\infty \alpha^-_j \chi^-_j(\sigma) + \alpha_\infty \chi_\infty(\sigma) \quad \text{for all } \sigma \in \mathcal{S}_n.
\]

(5.18)

Take the sequence \( \{\sigma_n\} \subset \mathcal{A}_n \), satisfying the following conditions:

\( \sigma_n \neq \text{id} \) for all \( n \) and \( \lim_{n \to \infty} \chi_{\text{nat}}(\sigma_n) = 0. \)

Substituting \( \sigma_n \) instead \( \sigma \) into (5.18) and passing to the limit \( n \to \infty \), we obtain

\[
\alpha^+_0 + \alpha^-_0 = 0 \Rightarrow \alpha^+_0 = 0 \text{ and } \alpha^-_0 = 0.
\]

(5.19)

Put \( \mathcal{A}_n = \{\sigma \in \mathcal{S}_n : \text{sgn}_\infty(\sigma) = 1\} \). Since a set \( \{\chi_{\text{nat}}(\sigma)\}_{\sigma \in \mathcal{A}_n} \) is dense in \([0, 1]\), it follows from (5.18) that

\[
\gamma^m = \sum_{j=1}^{m-1} (\alpha^+_j + \alpha^-_j) \gamma^j + \sum_{j=m+1}^\infty (\alpha^+_j + \alpha^-_j) \gamma^j \quad \text{for all } \gamma \in [0, 1).
\]

An easy computation shows that

\[
\alpha^+_j = \alpha^-_j = 0 \quad \text{for all naturale } j \neq m.
\]

Hence, using (5.18) and (5.19), we obtain that

\[
\chi^+_m(\sigma) = \alpha_\infty \chi_\infty(\sigma) \quad \text{for all } \sigma \in \mathcal{S}_n.
\]

Therefore, \( \chi^+_m = \chi_\infty \). Since character \( \chi_\infty \) is indecomposable as a regular character of the ICC group (Proposition 7.9 [10]), this contradicts the assumption that \( \chi^+_m \) is decomposable character.

**Second proof.** By above we can to suppose that \( m < \infty \). If a character \( \chi^+_m \) is not indecomposable then the corresponding GNS-representation \( (\pi, \xi, \mathcal{H}) \), where \( \xi \) is unit cyclic vector such that \( \chi^+_m(\pi(g)\xi, \xi)_{\mathcal{H}} \) for all \( g \in \mathcal{S}_n \), is not factor-representation. Therefore, there are nonzero orthogonal projection \( E \) in the center \( C(M) \) of \( \text{w}^*\)-algebra \( M \), generated by operators \( \{\pi(s)\}_{s \in \mathcal{A}_n} \), and \( \delta \in (0, 1) \) such that

\[
\|E\xi\|^2 = \delta > 0.
\]

(5.20)

Since \( \xi \) is cyclic vector, there exist finite subset \( \{s_i\}_{i=1}^{K_\rho} \subset \mathcal{S}_n \) and a collection \( \{\theta_i\} \) of the complex numbers with the property

\[
\left\|E\xi - \sum_{i=1}^{K_\rho} \theta_i \pi(s_i)\xi\right\| < \epsilon.
\]

(5.21)
Let \( q_n, r_n, \Theta_n, \mathcal{J}_n \in \mathfrak{S}_n \) be the objects are the same as in Corollary 8.2. It follows from (5.21) that
\[
\chi^+_m(s_i, \mathcal{J}_n(s_j)) = \chi^+_m(s_i) \cdot \chi^+_m(\mathcal{J}_n(s_j)) \quad \text{for all } i, j.
\] (5.22)
Since \( \chi^+_m \) is continuous under the topology, defined on \( \mathfrak{S}_n \) by metric \( \rho \), we have from (5.21)
\[
\| (I - E)\mathbf{\xi} - \left( I - \sum_{i=1}^{K_p} \partial_i \pi(\mathcal{J}_n(s_i)) \right) \mathbf{\xi} \| < \epsilon \quad \text{for all } n > K.
\] (5.23)
For simplicity of the notations we put \( A = \sum_{i=1}^{K_p} \partial_i \pi(s_i) \) and \( A_n = I - \sum_{i=1}^{K_p} \partial_i \pi(\mathcal{J}_n(s_i)) \).
Without loss of generality we suppose that \( A = A^* \) and \( A_n = A^*_n \).
Now we obtain the following chain of inequalities
\[
\langle E\mathbf{\xi}, (I - E)\mathbf{\xi} \rangle \geq \langle A\xi, (I - E)\mathbf{\xi} \rangle - \epsilon \| (I - E)\mathbf{\xi} \|
\] (5.21)
\[
\geq \langle A\xi, A_n\mathbf{\xi} \rangle - \epsilon \| A\mathbf{\xi} \| - \epsilon \| (I - E)\mathbf{\xi} \|
\] (5.22)
\[
\geq \langle A\xi, A_n\mathbf{\xi} \rangle - \epsilon \| E\mathbf{\xi} \| + \epsilon - \epsilon \| (I - E)\mathbf{\xi} \| \geq \langle A\xi, A_n\mathbf{\xi} \rangle - 2\epsilon - \epsilon^2.
\]
Hence, applying (5.22), we have
\[
0 = \langle E\mathbf{\xi}, (I - E)\mathbf{\xi} \rangle \geq \langle A\xi, \xi \rangle \langle A_n\mathbf{\xi}, \mathbf{\xi} \rangle - 2\epsilon - \epsilon^2
\]
\[
\geq \langle E\mathbf{\xi}, \xi \rangle - \epsilon ((I - E)\mathbf{\xi}, \mathbf{\xi} - \epsilon) - 2\epsilon - \epsilon^2 \geq \delta(I - \delta) - 3\epsilon.
\]
This inequality is false for \( \epsilon < \frac{\delta - \delta^2}{3} \).

6 The proof of Theorem 1.2

In this section we prove the main result of the present paper.

Proof of Theorem 1.2 Suppose that \( \chi \) is an indecomposable character on \( \mathfrak{S}_n \).
According to Proposition 1.2 there exist a subsequence \( \{k(l)\}_{l=1}^\infty \) and a sequence \( \{k(l)\lambda \vdash N_{k(l)}\}_{l=1}^\infty \) of partitions such that
\[
\chi(g) = \lim_{l \to \infty} \chi_{k(l)\lambda}(g) \quad \text{for any } g \in \mathfrak{S}_n.
\] (6.1)
Lemma 5.1 implies that three cases are possible:
- The sequence \( \{k(l)\lambda \vdash N_{k(l)}\}_{l=1}^\infty \) belongs to the union \( C_\infty \cup \hat{C}_\infty \). In this case Lemma 5.3 implies that \( \chi = \chi^0_{\text{nat}} \).
- The sequence \( \{k(l)\lambda \vdash N_{k(l)}\}_{l=1}^\infty \) belongs to \( C_\mu \) for some partition \( \mu \). In this case Lemma 5.4 implies that \( \chi = \chi^0_{\text{nat}} \).
- The sequence \( \{k(l)\lambda \vdash N_{k(l)}\}_{l=1}^\infty \) belongs to \( \hat{C}_\mu \) for some partition \( \mu \). In this case Corollary 5.7 implies that \( \chi = \text{sgn}_\infty \chi^0_{\text{nat}} \).
Thus, the character \( \chi \) equals either \( \chi^0_{\text{nat}} \), or \( \text{sgn}_\infty \chi^0_{\text{nat}} \) for some \( p \in \mathbb{N} \cup \{0, \infty\} \), i.e. \( \chi \in \text{ex Char} \) (see 5.17). In other words, we proved that the set of all indecomposable characters on \( \mathfrak{S}_n \) is a subset of \( \text{ex Char} \). Finally, Lemma 5.7 implies that all these functions are indeed indecomposable characters on \( \mathfrak{S}_n \). \( \square \)
7 The proof of Theorem 1.3

In this section we prove Theorem 1.3 about the isomorphism classes of groups \( \mathfrak{S}_n \).

We need the following simple lemmas in the proof of Theorem 1.3.

Lemma 7.1. Let \( \tilde{n}^i = \{n_k^i\}_{k=1}^{\infty} \) and \( \tilde{n}'' = \{n_k''\}_{k=1}^{\infty} \), where \( n_k^i, n_k'' > 1 \) for all \( k \), be the sequences of positive integers. Put \( N_k^i = \prod_{i=1}^{k} n_k^i \) and \( N_k'' = \prod_{i=1}^{k} n_k'' \). Denote

\[
\text{Div}(\tilde{n}^i) = \{N \in \mathbb{N} : N \text{ divides } N_k^i \text{ for some } k\}, \quad (7.1)
\]

\[
\text{Div}(\tilde{n}'') = \{N \in \mathbb{N} : N \text{ divides } N_k'' \text{ for some } k\} \quad (7.2)
\]

Then, the following conditions are equivalent:

(a) For each prime number \( p \) the following condition holds:

\[
\lim_{k \to \infty} \text{deg}_p(N_k^i) = \lim_{k \to \infty} \text{deg}_p(N_k'') \quad (7.3)
\]

(b) For any element \( N_j^i \) of the sequence \( \{N_k^i\}_{k=1}^{\infty} \) there is an element \( N_j'' \) of the sequence \( \{N_k''\}_{k=1}^{\infty} \) such that \( N_j^i \text{ divides } N_j'' \) and vice versa.

(c) \( \text{Div}(\tilde{n}^i) = \text{Div}(\tilde{n}'') \).

Proof. The implications \( (a) \Rightarrow (b) \) and \( (b) \Rightarrow (c) \) are trivial. It remains to prove the implications \( (b) \Rightarrow (a) \) and \( (c) \Rightarrow (b) \).

The case \( (b) \Rightarrow (a) \).

Let the condition \( (a) \) does not holds for some prime \( p \).

First we consider the case when \( \lim_{k \to \infty} \text{deg}_p(N_k^i) < \infty \) and \( \lim_{k \to \infty} \text{deg}_p(N_k'') < \infty \). For the sake of definiteness, we will assume that there exist a positive integers \( s \) and \( D_p \) with the properties:

1. \( \frac{N_k^i}{p^s} = pL_k^i + \text{rem}_k^i \) for all \( k > D_p \), where \( \text{rem}_k^i \in \mathbb{N} \), \( \text{rem}_k^i < p \) and \( L_k^i \) is nonnegative integer; i.e. \( \lim_{k \to \infty} \text{deg}_p(N_k^i) = s \);
2. \( p^{s+1} | N_k'' \) for all \( k > D_p \) or \( N_k'' = L_k'' p^s + \text{rem}_k'' \) for all \( k \), where \( 1 \leq \text{rem}_k'' < p^s \) and \( L_k'' \) is nonnegative integer; i.e. \( \lim_{k \to \infty} \text{deg}_p(N_k'') \neq s \).

Hence, if \( p^{s+1} | N_k'' \) then, by property i), \( \frac{N_k^i}{N_k''} = \frac{A_j'}{pA_k} \), where \( A_j' = \frac{N_j^i}{p^s} \in \mathbb{N} \), \( A_k = \frac{N_k''}{p^s} \in \mathbb{N} \), \( p \nmid A_j' \) and \( k > D_p \). Here the notation \( l \nmid j \) means that \( \text{‘} 1 \text{’}\).

Therefore, \( N_k'' \) not divides \( N_j'' \) for all \( j \).

Now we take \( N_j'' \) with \( j > D_p \) and assume that holds second part from property ii); i.e. \( N_k'' = L_k'' p^s + \text{rem}_k'' \) for all \( k \). Then \( \frac{N_j''}{\text{rem}_k''} = \frac{L_j'' + \text{rem}_k''}{\text{rem}_k''} \). Hence, since \( 1 \leq \text{rem}_k'' < p^s \), then \( N_j'' \) not divides \( N_k'' \) for all \( k \).

Now we assume that \( \lim_{k \to \infty} \text{deg}_p(N_k^i) = \infty \) and \( \lim_{k \to \infty} \text{deg}_p(N_k'') = s < \infty \).

Then there exists natural \( D_p' \) such that for all \( k, j > D_p' \)

\[
N_j'' = p^{s+1}A_j', \quad \frac{N_k''}{p^s} = L_k'' p + \text{rem}_k'', \quad \text{where } A_j', L_k'', \text{rem}_k'' \in \mathbb{N} \text{ and } \text{rem}_k'' < p. \quad (7.4)
\]
Therefore, if \( j > D'_p \) then \( N'_j \) not divides \( N''_k \) for all \( k \). This proves the implication (b) \( \Rightarrow \) (a).

At last we will prove that (c) \( \Rightarrow \) (b). Suppose that there exists \( j \) with the property: \( N'_j \nmid N''_k \) for all \( k \); i.e. \( N'_j \not\in \text{Div}(\hat{n}'') \). Since \( N'_j \in \text{Div}(\hat{n}') \), this contradicts the condition (c). \( \square \)

The following statement is the direct consequence of Theorem 1.2.

**Lemma 7.2.** Let \( \chi^0 \) be a trivial character on \( S_n \). Put \( CH(S_n) = \text{ex Char}(S_n) \setminus \{\chi^0, \text{sgn}\} \). For any element \( \sigma \in S_n \setminus \{\text{id}\} \) we have \( \chi_{\text{nat}}(\sigma) = \max_{\chi \in CH(S_n)} |\chi(\sigma)| \).

**Proof of Proposition 1.3.** Let \( n' = \{n'_k\}_{k=1}^{\infty} \) and \( n'' = \{n''_k\}_{k=1}^{\infty} \), where \( n'_k, n''_k > 1 \) for all \( k \). Suppose that there exists an isomorphism \( \alpha : S_n' \rightarrow S_n'' \). Then the map

\[
\chi \ni \text{ex Char}(S_n') \mathcal{P} \chi \circ \alpha \in \text{ex Char}(S_n'')
\]

is a bijection of a set \( \text{ex Char}(S_n') \) onto \( \text{ex Char}(S_n'') \). By definition \( \alpha^* \), we have

\[
\alpha^* (\text{ex Char}^+ (S_n'')) = \text{ex Char}^+ (S_n'').
\]

Let \( \chi^\prime_{\text{nat}} \) and \( \chi_{\text{nat}}^{''} \) be the natural characters on \( S_n' \) and \( S_n'' \), respectively (see (7.5), (2.9)). Using the characterization of the natural character from lemma 7.2 we obtain

\[
\alpha^* (\chi_{\text{nat}}^{''}) = \chi_{\text{nat}}^{''} \circ \alpha = \chi_{\text{nat}}^{'}.
\]

Thus, from (1.5) it follows that

\[
\chi_{\text{nat}}^{''} (S_n') = \chi_{\text{nat}}^{''} (\alpha (S_n')) = \chi_{\text{nat}}^{'} (S_n') = \left\{ \frac{L}{N'_k} : k \in \mathbb{N}, 0 \leq L \leq N'_k \right\}
\]

\[
= \left\{ \frac{p}{q} : q \in \text{Div}(n'), 0 \leq p \leq q, \gcd(p, q) = 1 \right\}.
\]

Hence

\[
\left\{ \frac{p}{q} : q \in \text{Div}(n'), 0 \leq p \leq q, \gcd(p, q) = 1 \right\}
\]

\[
= \left\{ \frac{p}{q} : q \in \text{Div}(n''), 0 \leq p \leq q, \gcd(p, q) = 1 \right\}.
\]

Therefore, \( \text{Div}(n') = \text{Div}(n'') \). Finally, Lemma 7.1 implies that the condition (1.6) holds. \( \square \)

**8 Appendix**

Take the natural numbers \( p, q, r \) such that \( 1 < p < q < r \). Let \( \mathcal{X}_n = \prod_{k=1}^{\infty} \mathcal{X}_{n_k} \).

For \( x = (x_1, x_2, \ldots) \in \mathcal{X}_n \) we set \( \varphi_{x} = (x_{q+1}, x_{q+2}, \ldots, x_r) \in \prod_{k=q+1}^{r} \mathcal{X}_{n_k} \). Each
element \( y \in \prod_{k=q+1}^{r} X_{n_k} \) defines a cylindric set

\[
\mathcal{A}_y = \{ x \in X_{\hat{n}} : \mathcal{A}_y^q x = y \} \subset X_{\hat{n}}. \tag{8.1}
\]

Put \( N_r = \frac{N_r}{N_p} \).

Let us introduce subgroup \( \mathcal{S}_n \subset S_{N_r} \) by

\[
\mathcal{S}_n = \{ \sigma \in S_{\hat{n}} : (\sigma(x))_i = x_i \text{ for all } x \in X_{\hat{n}} \text{ and } i \notin \{q + 1, q + 2, \ldots, r\} \}.
\]

It is clear that \( \mathcal{S}_n \) is isomorphic to \( S_{N_r} \). For large enough \( r \) we find the nonnegative integer numbers \( m \) and \( \text{rem} \) such that

\[
N_r = N_p m + \text{rem}, \quad \text{where } N_r > N_p \text{ and } \text{rem} < N_p. \tag{8.2}
\]

Further we denote by \( T \) automorphism \( O^{N_r} \), where \( O \) has been defined in chapter 2.1.

Set \( 0 = (0, 0, \ldots, 0, \ldots) \in X_{\hat{n}} \). Then the subsets in the collection \( \{ \mathcal{A}_y \}_{i=0}^{N_r-1} \), where \( y_i = T^i(0) \), are pairwise disjoint and \( \bigcup_{i=0}^{N_r-1} \mathcal{A}_y = X_{\hat{n}} \). For simplicity of notation, we denote \( \mathcal{A}_y \) by \( \mathcal{B}_i \) and define periodic automorphism \( q^T \in \mathcal{S}_n \) as follows

\[
q^T(x) = \begin{cases} 
T(x) = O^{N_r}(x), & \text{if } x \in \bigcup_{j=0}^{N_r-2} \mathcal{B}_j; \\
T^{(1 - N_r)}(x) = O^{N_r(1 - N_r)} x, & \text{if } x \in \mathcal{B}_{N_r-1}. 
\end{cases} \tag{8.3}
\]

Put

\[
E_k = \bigcup_{i=km}^{(k+1)m-1} \mathcal{B}_i. \tag{8.4}
\]

Hence we can conclude that

\[
\nu_n(E_k) = \frac{m}{N_r} = \frac{m}{N_p m + \text{rem}} = \frac{1}{N_p} - \frac{\text{rem}}{N_p (mN_p + \text{rem})} \tag{8.5}
\]

for all \( k \in 0, N_p - 1 \).

We now define automorphism \( \nu^T \) as follows

\[
\nu^T(x) = \begin{cases} 
q^{m} x, & \text{if } x \in \bigcup_{j=0}^{N_r-2} \mathcal{E}_j; \\
q^{T^{m(1 - N_r)}} x, & \text{if } x \in \mathcal{E}_{N_r-1}; \\
x & \text{if } x \in X_{\hat{n}} \setminus \bigcup_{j=0}^{N_r-1} \mathcal{E}_j.
\end{cases} \tag{8.6}
\]

Hence we obtain that \( \nu^T \mathcal{N}_r(x) = x \) for all \( x \in X_{\hat{n}} \) and

\[
\nu^T(E_k) = \mathcal{E}_{k+1} \mod N_p \tag{8.7}
\]
We recall that
\[ (\frac{q}{q})^T \mathcal{P}_{\mathcal{A}_y} = \mathcal{P}_{\mathcal{A}_y} \text{ for all } y \in \prod_{k=1}^{p} \mathcal{X}_{n_k}. \tag{8.8} \]
If \( \mathcal{O}_p = (0, \ldots, 0) \), then the cylindric sets from the collection \( \left\{ (\frac{q}{q})^T \mathcal{P}_{\mathcal{A}_0} = \mathcal{P}_{\mathcal{A}_0} \right\}_{i=0}^{N_p-1} \),
where \( \mathcal{Y}_i = \frac{q}{q}(T^i(\mathcal{O})) \), are pairwise disjunct and \( \bigcup_{i=0}^{N_p-1} \mathcal{P}_{\mathcal{A}_y} = \mathcal{X}_{\mathcal{N}} \). Therefore, the following expression
\[ \Theta(x) = \begin{cases} (\frac{q}{q})^{i-j} (\frac{q}{q})^{j-i} x, & \text{if } x \in E_i \cap \mathcal{P}_{\mathcal{A}_y}; \\ x, & \text{if } x \in \mathcal{X}_{\mathcal{N}} \setminus \left( \bigcup_{j=0}^{N_p-1} E_j \right) \end{cases} \tag{8.9} \]
define automorphism from \( \mathcal{S}_{\mathcal{N}} \). An ordinary verification shows that \( \Theta^2 x = x \) for all \( x \in \mathcal{X}_{\mathcal{N}} \).
Denote by \( \mathcal{E} = \frac{q}{q} \mathcal{N} \) the subgroup, consisting of the automorphisms \( \alpha_r \in \mathcal{S}_{\mathcal{N}} \), acting as follows
\[ \alpha_r(x) = \begin{cases} (\frac{q}{q})^{k_1} (x), & \text{if } x \in E_i; \\ x, & \text{if } x \in \mathcal{X}_{\mathcal{N}} \setminus \left( \bigcup_{j=0}^{N_p-1} E_j \right) \end{cases} \tag{8.10} \]
where \( \{k_0, k_1, \ldots, k_{N_p-1}\} \) is the collection of the integer numbers, belonging to the set \( \{0, 1, \ldots, N_p - 1\} \). Hence, using (8.9), we have
\[ \sigma_r(x) = \Theta^{-1} \alpha_r \Theta (x) = \begin{cases} (\frac{q}{q})^{k_1} (x), & \text{if } x \in \mathcal{P}_{\mathcal{A}_y} \cap \left( \bigcup_{j=0}^{N_p-1} E_j \right); \\ x, & \text{if } x \in \mathcal{X}_{\mathcal{N}} \setminus \left( \bigcup_{j=0}^{N_p-1} E_j \right) \end{cases} \tag{8.11} \]
Now we take an arbitrary automorphism \( \sigma \in \mathcal{S}_{\mathcal{N}} \). By lemma 2.22, for every \( \sigma \in \mathcal{S}_{\mathcal{N}} \), exists a collection of the nonnegative integers \( \{k_1(\sigma)\}_{i=0}^{N_p-1} \), where \( k_i(\sigma) \in \{0, 1, \ldots, N_p - 1\} \), such that \( \sigma(x) = (\frac{q}{q})^{k_1(\sigma)} (x) \) if \( x \in \mathcal{P}_{\mathcal{A}_y} \). Since \( q > p \), the cylindric sets \( \mathcal{P}_{\mathcal{A}_y} \) are invariant under the group \( \mathcal{S}_{\mathcal{N}} \); i. e. \( (\frac{q}{q})^{k_1(\sigma)} \mathcal{P}_{\mathcal{A}_y} = \mathcal{P}_{\mathcal{A}_y} \) for all \( x \in \mathcal{S}_{\mathcal{N}} \), and \( i = 0, 1, \ldots, N_p - 1 \). Therefore, automorphism \( \sigma_r \in \mathcal{S}_{\mathcal{N}} \), from (8.11), where we will write \( k_i(\sigma) \) instead of \( k_i \), is well defined. If \( \rho \) is the metric introduced in (2.3), then, using (8.2), (8.3) and (8.11), we have
\[ \rho(\sigma, \sigma_r) \leq \frac{\text{rem}}{mN_p + \text{rem}} \leq \frac{1}{m}, \text{ where } mN_p = N_p - \text{rem}, \text{rem} < N_p. \tag{8.12} \]
From the above, we obtain the following statement.

**Proposition 8.1.** Let the natural numbers \( p, q, r \) satisfy the inequality \( 1 < p < q < r \) and the condition (8.2). There exist injective homomorphism \( \mathcal{J} : \mathcal{S}_{\mathcal{N}} \to \mathcal{S}_{\mathcal{N}} \) and automorphism \( \Theta \in \mathcal{S}_{\mathcal{N}} : \mathcal{S}_{\mathcal{N}} \) such that
\[ \rho(\Theta \Theta^{-1}, \mathcal{J}(\sigma)) \leq \frac{1}{m} \text{ for all } \sigma \in \mathcal{S}_{\mathcal{N}}. \tag{8.13} \]
Corollary 8.2. For a fixed $p > 1$ take two sequences of the positive integer $q_n$, $r_n$ such that $p \leq q_n < r_n$ and $\lim_{n \to \infty} q_n = \lim_{n \to \infty} (r_n - q_n) = \infty$. Let $N_n = N_p m_n + (\text{rem})_n$, where $m_n > 1$ and $(\text{rem})_n < N_p$. Then for each $n$ there exist injective homomorphism $I_n : \mathfrak{S}_{N_p} \to \mathfrak{S}_{N_n}$ and automorphism $\Theta_n \in \mathfrak{S}_{N_p} \cdot \mathfrak{S}_{N_n}$ such that

$$\rho(\Theta_n \sigma \Theta_n, I_n(\sigma)) \leq \frac{1}{m_n} \text{ for all } \sigma \in \mathfrak{S}_{N_p}.$$  

(8.14)

Since $\lim_{n \to \infty} (r_n - q_n) = \infty$, we have $\lim_{n \to \infty} \rho(\Theta_n \sigma \Theta_n, I_n(\sigma)) = 0$.

References

[1] I. G. Macdonald. Symmetric Functions and Hall Polynomials, Second edition, 1998, Bookcraft(Bath), 475 pages.

[2] T. Ceccherini, F. Scarabotti, F. Tolli. Representation Theory of the Symmetric Groups. (The Okounkov-Vershik Approach, Character Formula, and Partition Algebras), Cambridge university press, 2010, 412 pages.

[3] A. Dudko. Characters on the full group of an ergodic hyperfinite equivalence relation, J. Funct. Anal., 261 (2011), 1401–1414.

[4] A. Dudko and K. Medynets, On characters of inductive limits of symmetric groups, J. Funct. Anal., 264 (2013), 1565–1598.

[5] E. E. Goryachko and F. V. Petrov, Indecomposable characters of the group of rational rearrangements of a segment, J. Math. Sci. (N. Y.), 174 (2011), 7–14.

[6] A. M. Vershik, S. V. Kerov. Asymptotic theory of characters of the symmetric group, Functional Analysis and Its Applications volume 15, pages 246–255 (1981).

[7] A. Okounkov, A.M. Vershik. A new approach to the representation theory of symmetric groups. Selecta Math. (N.S.), 4:581–605, 1996.

[8] Y. Roichman. Upper bound on the characters of the symmetric groups. Invent. Math., 125(3):451–485, 1996.

[9] V. Féray, P. Śniady. Asymptotics of Characters of Symmetric Groups Related to Stanley Character Formula. Annals of Mathematics 173, no. 2 (2011): 887–906.

[10] M. Takesaki. Theory of operator algebras I, Springer, Berlin, 2002, volume 1, 415 pages.