Discrete scale invariant quantum dynamics and universal quantum beats in Bose gases

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Introduction. – One of the most important tools in physics research is to explore the consequences and implications of symmetries in a physical system, i.e. the invariance of a system under a family of transformations. Any symmetry or invariance can lead to a significant reduction of the complexity of a problem, possibly rendering it solvable. Therefore it is crucial to investigate and understand the features that are invariant due to symmetry transformations. One example of great interest is the symmetry associated with scale invariance. This symmetry has been essential in the study of complicated systems such as the thermodynamics [1–3] and dynamics [4] near critical points, and biological complexes [5]. Without taking into account this symmetry, these problems would be otherwise intractable. Although scale invariance has had resounding success in studying the energetics and critical dynamics of these rather difficult systems, the role of scale invariance on far-from-equilibrium coherent quantum dynamics has yet to be fully understood. In this work we are specifically interested in the scale invariant far-from-equilibrium quantum dynamics, where the governing equations of a system remain unchanged when subject to dilations of the spatial and temporal coordinates:

\[ \vec{x}_i = b \vec{x}_i, \quad i = 1, 2, \ldots, N, \quad t' = b^2 t. \] (1)

Below we use \( \{ \vec{x}_i \} \) to denote the coordinates of the \( N \)-particles with \( i = 1, 2, \ldots, N \), and \( b \) as the scaling factor.

One very promising platform to study the role of scale invariance in far-from-equilibrium dynamics is cold atom systems. Cold atom systems are unique in the tunability present in experiments. It is possible to prepare cold atom systems with a wide range of initial conditions and interaction parameters. This control has led to both experimental and theoretical studies concerning a wide range of dynamical phenomena, such as the dynamics of expansion and collapse [6–9], breathing modes [10,11], solitons [12–14], and quench dynamics [6,15,16]. A few theoretical efforts have also been made to understand far-from-equilibrium coherent quantum dynamics in fermion superfluids [17,18].

One of the simplest cold atom systems that approximately exhibits scale invariance is the two-dimensional Bose gas with interactions of strength \( -g \) (see footnote 1). The semiclassical dynamics of the condensate wave function, \( \phi(\vec{x}, t) \), is described by

\[ i\partial_t \phi(\vec{x}, t) = -\frac{1}{2} \nabla^2 \phi(\vec{x}, t) - g|\phi(\vec{x}, t)|^2 \phi(\vec{x}, t), \] (2)

i.e. the non-linear Schrödinger equation (NLSE). Under the scale transformation, eq. (1), the NLSE is left invariant2. This approximate scale invariant description of the dynamics is valid for large condensates with weak repulsive

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1We can use a contact interaction to effectively describe the short range interactions in cold gases. At the classical level, the interaction strength \( g \) is merely a dimensionless constant, and the system is classically scale invariant. However, the renormalization effect, which depends on the actual range of the interaction, breaks the classical scale invariance. We work in the limit where the renormalization effects are weak. This breaking of the classical scale invariance is discussed later.

2This superficial scale invariance is associated with the scale invariance of the field theory description of a Bose gas with contact interactions at tree level, with no renormalization effects.
\[ \rho(\vec{r},t) = \langle \psi_0 | e^{iHt} \hat{\rho}(\vec{r}) e^{-iHt} | \psi_0 \rangle \]

\[ = \frac{\int D\phi(\vec{x}) D\phi'(\vec{x}) \langle \psi_0 | e^{iHt} | \{ \phi(\vec{x}) \} \rangle \{ \phi'(\vec{x}) | e^{-iHt} \psi_0 \} \phi' (\vec{r}) \{ \phi(\vec{x}) \} \{ \phi'(\vec{x}) \} \}}{\int D\phi(\vec{x}) \{ \phi(\vec{x}) \} | e^{-iHt} \psi_0 \rangle^2}, \]

(3)

interactions [8]. However, this approach does not include quantum fluctuations which can break the scale invariance. We denote this as classical scale invariance.

In this article we study the signatures of the classical scale invariance on the quantum dynamics of an inhomogeneous two-dimensional Bose-Einstein condensate with attractive interactions, beyond the semiclassical approaches used previously [8,19]. Although the scale invariance has been studied in the thermodynamics of this system [20,21], the role of the classical scale invariance in the far-from-equilibrium quantum dynamics has yet to be determined.

The main conclusion here is that the classical scale invariance under the continuous scale transformation, defined in eq. (1), is only present in the semiclassical description of the dynamics and is broken when the dynamics are fully quantized [22,23]. However, there can be an emergent discrete scale invariance induced by the continuous classical scale invariance. The physics we discuss here for two spatial dimensions is reminiscent of the Efimov physics [24] found in three spatial dimensions; where the three-body Hamiltonian has a similar classical scale invariance at resonance. There the classical continuous scale invariance is broken when the motion is quantized, resulting in a discrete scale invariance present in the bound-state spectrum.

We restrict ourselves to an isotropic single-parameter scaling solution and show how the induced discrete scale invariance leads to distinct features present in the density profile of the gas. First, the spatial profile is dictated by the presence of a logarithmic singularity in the density at short distances. Secondly, this density profile will undergo oscillations, the frequencies of which satisfy a robust discrete scaling relation. Both these effects are universal in the sense that they do not depend on the initial conditions of the condensate and are valid for a wide range of interaction strengths. We then conclude with a discussion on the experimental implications of this work.

**Dynamics and the quantum variational method.**

Consider the expectation value of the density operator \( \hat{\rho}(\vec{r}) = \hat{\phi}^\dagger(\vec{r}) \hat{\phi}(\vec{r}) \) where \( \hat{\phi}^\dagger(\vec{r}) \) is the second quantized annihilation (creation) operator. The unitary evolution of the density is given by

\[ \langle \phi(\vec{x}) | \rangle \]

\[ \approx \delta(\{ \phi'(\vec{x}) \} - \{ \phi(\vec{x}) \}) \]

(4)

The phase of \( \phi_\lambda(\vec{x}) \), is chosen to satisfy the conservation law [28] and is of little importance for the rest of the discussion. The quantity \( \lambda \) represents the width of the condensate and parametrizes the slowly evolving field. The function \( f(x) \) is smooth, normalizable and regular at the origin. This ansatz is motivated by the classical scale invariance of the NLSE and the initial conditions of an inhomogeneous condensate. For more details we refer the reader to the Supplementary Material [Supplementary material.pdf (SM)].

The separation of the slow isotropic and fast degrees of freedom leads to a controllable expansion of the many-body fluctuations valid in the limit of dense condensates. After substituting the split form of \( \phi(\vec{x}) \) into eq. (3) and integrating out \( \delta(\phi(\vec{x})) \), one can show that eq. (3) reduces to (see SM for a full derivation):

\[ \rho(\vec{r},t) = \frac{\int d\lambda \rho_\lambda(\vec{r}) \langle \phi_\lambda(\vec{x}) | e^{-iHt} | \psi_0 \rangle^2}{\int d\lambda | \langle \phi_\lambda(\vec{x}) | e^{-iHt} | \psi_0 \rangle |^2}, \]

(5)

\[ \langle \{ \phi_\lambda(\vec{x}) \} | e^{-iHt} | \psi_0 \rangle = \int D\lambda(t) e^{iS}, \]

where \( | \psi_0 \rangle \) is the initial state of the system, \( H \) is the Hamiltonian for a Bose gas interacting via a short-range attractive interaction of strength \( g \) (see footnote 1).
where $S$ is an effective action for the $\lambda$ degree of freedom:

$$S = \int_0^t \frac{dt'}{2m} \frac{1}{2m} \dot{\lambda}^2 + \frac{V}{2\lambda^2},$$

and $m = C_1 N$ and $V = C_2 g N^2 - C_2 N$. These coefficients depend on the specific shape of the condensate, the function $f(x)$, and are tabulated in ref. [28]. The coefficient $C_1$ parameterizes the effective mass for the $\lambda$ degree of freedom, while the other two coefficients, $C_2$ and $C_3$, represent the kinetic energy of the density and its interaction energy, respectively.

The semiclassical solution to eq. (3) involves minimizing eq. (6) and only consider this contribution to the dynamics. This procedure is equivalent to the NLSE, eq. (2), using the ansatz eq. (4). The motion of $\lambda$ is identical to the classical motion of a particle in an inverse square potential:

$$m \ddot{\lambda}(t) + \frac{V}{\lambda^2}(t) = 0.$$  

(7)

Similar equations for repulsively interacting Bose gases have been studied in both two and three dimensions [7–9].

To quantify the role of fluctuations in $\lambda$, $\delta \lambda$, one can consider the amplitude of the fluctuations, $(\delta \lambda)(\delta \lambda)$, relative to a semiclassical path of size, $\lambda_0$: $(\delta \lambda)(\delta \lambda)/\lambda_0^2$. For attractive potentials, the magnitude of the relative fluctuations, $(\delta \lambda)(\delta \lambda)/\lambda_0^2$, is controlled not by the size of the semiclassical path, $\lambda_0$, but rather by the scale-independent parameter $(mV)^{-1}$. This implies that for small condensates the fluctuations around the semiclassical path are substantial, and full quantization is required. For repulsive potentials $V$ is negative. In this case one finds that the amplitude of the relative fluctuations is controlled by the size of the semiclassical path and quantization is not necessary.

The dynamical effects of these fluctuations in $\lambda$ can be more effectively addressed by solving the quantum mechanical problem:

$$\rho(\vec{r}, t) = \langle \rho_\lambda(\vec{r}) \rangle(t) = \frac{\int d\lambda \rho_\lambda(\vec{r}) |\psi(\lambda, t)|^2}{\int d\lambda |\psi(\lambda, t)|^2},$$

(8)

which is equivalent to eqs. (5) and (6). A full derivation of eq. (8) is provided in the SM. The last equality in eq. (8) indicates that the density at any given time, $t$, can be obtained by simply averaging $\rho_\lambda(\vec{r})$ in eq. (4) over $|\psi(\lambda, t)|^2$, or $\langle \rho_\lambda(\vec{r}) \rangle(t)$.

The quantity $\psi(\lambda, t)$ is the transition amplitude for the slow degrees of freedom, $\langle \lambda | e^{-iH_\lambda t} | \psi_0 \rangle$. The Hamiltonian governing this transition amplitude is given by

$$H_\lambda = \hat{P}^2 + \frac{V}{2\lambda^2} + \delta H_\lambda,$$

(9)

where $\delta H_\lambda$ is a small correction due to the many-body fluctuations [28], the effect of which will be discussed towards the end. In this effective description, $\lambda$ is now an operator measuring the width of the condensate with eigenstates $|\lambda \rangle$: $\hat{P}(\lambda) = \lambda |\lambda \rangle$, representing a condensate with wave function $\phi_\lambda(x)$. The operator $\hat{P}_\lambda$ is the momentum conjugate to $\lambda$. Finally, it is important to note that eq. (9) with $\delta H_\lambda = 0$ is classically scale invariant, i.e. $H' = b^{-2}H$, under eq. (1). This scaling is due to the classical scale invariance in the semiclassical model.

The spectrum of eq. (9) with $\delta H_\lambda = 0$ consists of a continuous set of scattering states, $\psi^{(1)}_s$ and $\psi^{(2)}_s$, with energies $E = \frac{k^2}{2m}$, and a discrete set of bound states, $\psi_b$, with energies $E_n = -\frac{k^2}{2m}$, where (up to normalization factors):

$$\psi^{(1)}_s = \text{Re} \sqrt{k \lambda} J_n(k \lambda), \quad \psi^{(2)}_s = \text{Re} \sqrt{k \lambda} Y_n(k \lambda),$$

$$\psi_b = \sqrt{k \lambda} K_n(k_n \lambda), \quad k_n = k_0 \exp \left(-\frac{n \pi}{\sqrt{mV}} \right).$$

(10)

The functions $J_n(x)$, $Y_n(x)$, and $K_n(x)$ are the Bessel $J$, Bessel $Y$ and modified Bessel $K$ functions of order $n$, and $n = 1, 2, 3, \ldots$. Here we focus on condensates with $mV > 1/4$ and $g \ll 1$, or from the discussion after eq. (9), $C_2/C_3N < g \ll 1$.

It is important to notice that eq. (9) is singular near $\lambda = 0$, and is strictly speaking ill defined. As a result, when the motion of $\lambda$ is quantized, it is necessary to introduce a UV scale, $k_0$, which regularizes the singular potential. In practice, this scale is associated with the confinement along the $z$-direction for a quasi-two-dimensional systems [29]. This UV scale breaks the continuous classical scale invariance discussed in the introduction. However, a discrete scale invariance can be induced as a result of the classical scale invariance for $mV > 1/4$. Discrete scale invariance occurs when the system will rescale according to eq. (1) for certain values of the scaling parameter. This is evident in the bound-state spectrum as $k_n/k_{n-1} = \exp(\pi/\sqrt{mV})$. This piece of physics is reminiscent of Efimov physics, where a similar phenomenon occurs [24]. The discrete scale invariance is explicit in the bound-state spectrum which is equally spaced on a logarithmic scale. In addition to the presence of the discrete scale invariance on the bound-state spectrum, both the scattering and bound-state eigenfunctions have an envelope that depletes as $(k \lambda)^{1/2}$ as $\lambda \to 0$. This feature is robust and depends only on the classical scale invariance of eq. (9).

Quantum dynamics of the condensate. – At this stage one can consider the dynamics of a condensate which is initially prepared with a size $\lambda_0$. For specificity we will assume the initial density profile to be a Gaussian: $f(x) = \pi^{-1} \exp(-x^2)$. The initial probability amplitude can be chosen as:

$$\psi(\lambda, t = 0) = \langle \lambda |\psi_0 \rangle = \frac{1}{(\pi)^{1/4}} \frac{e^{-\lambda^2}}{\sqrt{\sigma}},$$

(11)

where the spreading, $\sigma$, is fixed by requiring that the energy of the effective model is identical to the microscopic

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The density profile is a Gaussian of width \( \lambda_0 \). Initially the density profile is the result one would obtain by employing the NLSE, \( \rho \approx |\psi(\lambda, t)|^2 \). However, our calculations show that \( \langle \lambda^{-2} \rangle \) is actually dominated by contributions at small \( \lambda \) far away from the most probable value in \( |\psi(\lambda, t)|^2 \) (see footnote 4).

These anomalous contributions from small \( \lambda \) alter the density profile at length scales \( \lambda_0 \ll \rho \ll \lambda_{sc}^{(1)}(t) \). The behaviour of the density at these length scales will be governed by the depletion of the scattering eigenstates, which is explicitly shown in fig. 1(a). In this limit \( |\psi(\lambda, t)|^2 \) depletes linearly and following eq. (8), it results in a logarithmic singularity in the density profile:

\[
\lim_{r/\rho \to 0} \rho(\bar{r}, t) = \frac{1}{\pi} \frac{\lambda_0^2}{\nu^2} \log^\alpha \left( \frac{\sqrt{\bar{r}}}{\rho} \right),
\]

as shown in fig. 1(c). The power of the logarithm \( \alpha = 1 \) for the discrete scale invariant system under consideration. Note that eq. (12) obeys: \( \rho(b\bar{r}, b^2t) \sim b^{-2+\nu}\rho(\bar{r}, t) \). The exponent \( \eta \) is known as the anomalous dimension, and in this system \( \eta = -2 \).

At length scales \( \lambda \ll \lambda_0 \), the dynamics will be governed by the bound-state contribution to the transition amplitude as shown in fig. 1(b). There will still be a logarithmic singularity now regulated by \( \lambda_0 \) due to the depletion of the transition amplitude. However, the pre-factor of \( \lambda_0^2/\nu^2 \) will be replaced by an oscillatory function. These oscillations are due to the interference, or beating, of different bound states and is shown in fig. 2(a). In figs. 2(b), (c), the frequency spectrum of these quantum beats is shown for two interaction strengths. The frequencies of the quantum beats are \( \omega_{n,\nu} = E_{n+\nu} - E_n \), with \( E_n \) given by eq. (10), and \( n, \nu = 1, 2, 3, \ldots \). The exact location of these beat frequencies and their spectral weight will specifically depend on the UV parameter and initial conditions, \( \lambda_0 \). However, the effect of the induced discrete scale invariance of the system is manifest in the organization of these frequencies. From eq. (10), the beat frequencies are

\[
\log(\omega_{n,\nu}m\nu^2) = \log \left( \frac{k_0^2\nu^2}{2} \right) - \frac{2\pi}{\sqrt{mV}} n - \log \left( 1 - e^{-\pi mV} \right).
\]

Equation (13) indicates that one can organize the frequency spectrum into a series of families specified by the fixed parameter \( \nu \). In this logarithmic scale, the frequencies in each family with given \( \nu \) will be equally spaced from one another by an amount \( 2\pi/\sqrt{mV} \). The spacing between family members of different \( n \) is a universal quantity and is independent of the UV parameter and initial conditions \( \lambda_0 \). In practice, there will be many families present, each shifted with respect to one another but with the same intra-family spacing. The overall shift between adjacent families, \( \nu \) and \( \nu + 1 \), is also universal, as seen by the third term in eq. (13). Our numerical solutions

\[\text{The most probable value of } \lambda \text{ in } |\psi(\lambda, t)|^2 \text{ depends where the probability is concentrated. When the scattering states dominate, the most probable value is } \lambda_{sc}^{(1)}(t). \text{ When } gN \text{ is appreciable, the most probable size is } \lambda_0 \text{ the size of the bound states. In both cases the dominant contributions to the density will come from } \lambda \text{ much smaller than the most probable value.} \]

\[\text{These anomalous contributions originate from the short distance, } k\lambda \ll mV, \text{ structure of the effective wave function. The attractive case is discussed in the main text. For repulsive interactions, one can show that } \langle \lambda^{-2} \rangle \sim \langle \lambda_{sc}^{(1)}(t) \rangle. \text{ This implies that the density profile can be approximated by the semiclassical solution, which was observed in ref. [8].} \]
The Hamiltonian has an imaginary contribution is easily un-
covered with the semiclassical solution for a Bose gas with
an initial size \( \tau/\sigma \) anan.

The solution to eq. (7) for a collapsing Bose gas is

\[
\rho(t) = N \lambda(t) \frac{r}{\lambda(t)},
\]

\[
\lambda(t) = \sqrt{1 - (2t - nT)/T^2},
\]  

for \( t \in [nT - T/2, nT + T/2] \) with \( n = 0, 1, 2, \ldots \) and period, \( T \):

\[
T = 2\sqrt{\frac{m}{\lambda}}.
\]  

We note that there is only one fundamental frequency which depends on the initial conditions of the problem, a consequence of the classical scale invariance of the system. Since this solution oscillates with a period \( T \), the frequency spectrum only contains frequencies \( \omega_n = 2\pi n/T \) and \( n = 1, 2, 3, \ldots \). These points are shown alongside the quantum frequency spectrum in figs. 2(b), (c).

Effects of anisotropic fluctuations. – The results found in eqs. (12) and (13) neglect fluctuations of wave lengths smaller than \( \lambda \), that is when \( \delta H \lambda \) is set to zero. These fluctuations correspond to corrections which are of higher order in \( g \) and can be expanded perturbatively in the limit \( C_2/C_2 N < g \ll 1 \). These corrections have two main effects which we will now discuss.

The first effect of the fluctuations is to generate an imaginary correction to the Hamiltonian. The fact that our Hamiltonian has an imaginary contribution is easily understood as the result of the coupling between the long wave length isotropic degrees of freedom and the short

wave length fluctuations — phonons. This effect introduces a term \( i \Im \delta H \) to \( H \), where

\[
\Im \delta H = \frac{g^2 C_4 N^2}{2\lambda^2}.
\]  

This correction is suppressed by an additional factor of \( g \). All the eigenstates of energy \( E_n \) now acquire a finite lifetime of order \( 1/gE_n \). This implies that the beat frequencies in the spectrum, eq. (13), will have non-zero widths. The widths associated with the beats with large \( \nu, \nu \gg \sqrt{mV/(2\pi)} \), will be of \( O(g) \) and can be neglected.

The second effect of the fluctuations is to renormalize the parameters \( C_1, C_2, \) and \( C_3 \), and the coupling constant \( g \). The renormalization of the coefficients \( C_1, C_2 \) and \( C_3 \), do not qualitatively alter the physics present in the semiclassical model. However, the fluctuations have a more dramatic effect on \( g \); the fluctuations replace the bare interaction strength with a function that depends on the UV scale, \( k_0 \) [30–34]6. This effect explicitly breaks the scale invariance. The effects of the renormalization are of order \( O(g \log(k_0\lambda)) \), which is small in the limit under consideration. As a result the scale invariant Hamiltonian found in eq. (9) is a good approximation for the dynamics.

Discussion. – Practically, to observe the log singularity and the universal quantum beats discussed in figs. 1 and 2, it is most convenient to work in the limit where \( N \gg g \) is not too large, \( g \). The parameter controls the quantum nature of the isotropic motion. If \( N \gg 1 \), the spectrum shown in fig. 2 will approach the semiclassical solution, eq. (15), obscuring the quantum beats. As an example of this system, we consider an experimental set-up similar to ref. [21], where \( ^{133}\text{Cs} \) atoms were placed in a two-dimensional trap with \( g \approx 0.01 \). The beats typically occur for \( t > \sqrt{V/m\lambda^3} \) (see discussion after eq. (13)). A contour plot showing the time scale for a single quantum beat for various \( N \) and \( \lambda_0 \) is provided in fig. 3.

In this article we have discussed the role of classical scale invariance in the quantum dynamics of two-dimensional Bose gases. Although the scale invariance is only present in the semiclassical dynamics, nevertheless scale invariance is not completely broken when quantized. The resulting discrete scale invariance has important consequences on the far-from-equilibrium dynamics, see eqs. (12) and (13).

6When \( \log(k_0\lambda) \) becomes appreciable, it is necessary to replace the coupling constant with \( g = 2\nu/(\log(a/\lambda)) \), where \( a \) is the size of the two-body bound state in two dimensions: \( a = k_0^{-1} e^{-2\nu}/g \). This specifically breaks the scale invariance and is known as a quantum anomaly. For more details see ref. [26].
in the limit of dense condensates in low dimensions. This method is quite general, and can be applied to a wide number of systems. This method is akin to a mean field for dynamics and is a great tool for providing a basic understanding for far-from-equilibrium quantum dynamics. In the future we plan to extend this work to study the dynamics of Fermi gases in low dimensions.

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