Nonlinear algorithms of residual stresses reconstruction for axisymmetric problem

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Abstract. A non-destructive testing method for three-dimensional stress of a glass cylinder based on photoelasticity is presented. Based on simulated stresses in the glass cylinder, intensity images output from a circular polariscope are calculated by Jones calculus. Therefore, the isoclinic angle and optical retardation can be simulated. Then the residual stresses in the glass cylinder can be reconstructed in cylindrical coordinates. By comparing the reconstructed stresses with numerical simulated stresses, there is good agreement but with some mismatches.

1. Introduction
With a large demand of the compression molded glass products, especially precision glass lenses, nondestructive method for stress measurement inside the glass components are needed. However, due to the nonlinear problem of the stress. Stress reconstruction in photoelastic material is a problem[1]. In order to study the reconstruction error of the stress reconstruction method, a residual stress reconstruction process of a glass cylinder is analyzed. Firstly, residual stresses in the glass cylinder are simulated by finite element method (FEM). Based on the simulated stresses, intensity images emerging from a circular polarizer can be calculated by Jones calculus. With six step phase-shifting technique (six step PST) [2], the isoclinic angle and optical retardation of the glass cylinder was then obtained. Therefore, the residual stresses were reconstructed through the isoclinic angle and optical retardation by integrated photoelasticity. The error of reconstruction method is presented by comparing the reconstructed stresses with the stresses through FEM.

2. Numerical simulation of a circular polariscope
Figure 1 shows a schematic of a circular polarizer. In Fig. 1, the polarizer and analyzer are identical. S represents the slow axis, F represents the fast axis. ξ and γ represents the fast axis of two quarter wave plates; φ is isoclinic angle of the specimen. The orientation of the polarizer is fixed at π/2, β is the orientation of the analyzer.
According to Jones Calculus, the light emitting from the analyzer can be described as [3]:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \cos^2 \beta & \cos \beta \sin \beta \\ -\cos \beta \sin \beta & \sin^2 \beta \end{pmatrix} \times \begin{pmatrix} i \cos^2 \gamma + \sin^2 \gamma (i-1) \sin \gamma \cos \gamma \\ (i-1) \sin \gamma \cos \gamma i \sin^2 \gamma + \cos^2 \gamma \end{pmatrix}$$

$$\times J(\Delta, \phi) \times \begin{pmatrix} i \cos^2 \varepsilon + \sin^2 \varepsilon (i-1) \sin \varepsilon \cos \varepsilon \\ (i-1) \sin \varepsilon \cos \varepsilon i \sin^2 \varepsilon + \cos^2 \varepsilon \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(1)

$J(\Delta, \phi)$ is the Jones matrix of the specimen, and $\Delta$ and $\phi$ are optical retardation and isoclinic angle of the specimen, respectively. The $J(\Delta, \phi)$ can also be expressed as:

$$J(\Delta, \phi) = \begin{pmatrix} e^{i\Delta} \cos^2 \phi + \sin^2 \phi & (e^{i\Delta} - 1) \sin \phi \cos \phi \\ (e^{i\Delta} - 1) \sin \phi \cos \phi & e^{i\Delta} \sin^2 \phi + \cos^2 \phi \end{pmatrix}$$

(2)

Along the light path, the object can be divided into $n$ layers with a thickness of $dz$. The relation between the outgoing light $E_{out}$ and the incident light $E_{in}$ is expressed as [3, 4]:

$$E_{out} = \sum_{k=1}^{n} (1 + A_k dz) E_{in}$$

(3)

where, $E = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$, $A = -\frac{1}{2} i C_0 \begin{pmatrix} \sigma_{xx} - \sigma_{yy} & 2\tau_{xy} \\ 2\tau_{xy} & -(\sigma_{xx} - \sigma_{yy}) \end{pmatrix}$. $\sigma_{xx}$, $\sigma_{yy}$ and $\tau_{xy}$ are stress components in XY plane, $i$ is imaginary. Then, the Jones matrix of the photoelastic object is obtained as:

$$J_{stress} = \sum_{k=1}^{n} (1 + A_k dz) = J(\Delta, \phi)$$

(4)

2.1. Experiment simulation

The residual stresses in a thermal treated glass cylinder are simulated by MSC/MARC[5] based on a two-dimensional axisymmetric model. Figure 2 shows the residual stresses in cylindrical coordinates. The stress distribution displays half of the normal cross section of the cylinder. In the stress distribution figures, compressive stress is represented by negative value and tensile stress is represented by positive value.

Based on Fig. 2, light intensity images of six step PST can be calculated, shown in Fig. 3. In the calculation, refraction of light passing through the glass cylinder is not considered. And the light wavelength is 683 nm, the photoelastic constant of BK7 is $C_0 = 2.77 e-6$/Mpa.
3. Residual stress reconstruction

As shown in Fig. 4, along the light ray $s$, in the 3D region LMN formed by plane $z=Z_0+h$ and plane $z=Z_0$, the shear forces of plane $z=Z_0+h$ and plane $z=Z_0$ can be expressed as:

$$ T_u = \int_{d}^{R} (\int_{d}^{\zeta} \tau_{xz} dy) dx = \frac{1}{2C_0} \int_{d}^{R} V_z dx $$  \hspace{1cm} (5)  

$$ T_l = \int_{d}^{R} (\int_{d}^{\zeta} \tau_{xz} dy) dx = \frac{1}{2C_0} \int_{d}^{R} V_z dx $$  \hspace{1cm} (6)
where, $R$ is the outer diameter of this area; $d$ is the distance to the axis of symmetry of light $s$. In the auxiliary plane $z=zh$, $V_1$ and $V_2$ are denoted as $V_1'$ and $V_2'$, respectively.

Based on the equilibrium condition of the 3D region LMN:

$$
C_0 \int_L^M \sigma_z dy = \frac{1}{2h} \left( \int_0^N V_{2}^2 dx - \int_0^N V_{1}^2 dx \right) - V_i
$$

(7)

The shear stress $\tau_c$ and axial stress $\sigma_c$ can be calculated, through Eqs. (6) and (7), respectively [6, 7].

3.1. Circumferential stress and radial stress

The circumferential stress $\sigma$ and radial stress $\sigma_r$, can be calculated through the equation of equilibrium:

$$
\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_c}{\partial z} = 0
$$

(8)

and the generalized sum rule:

$$
\sigma_\theta + \sigma_r = \sigma_c - 2 \int_0^r \frac{\partial \tau_c}{\partial z} dt + C_1
$$

(9)

where, $C_1$ is an constant and depends on boundary conditions. Therefore, the circumferential stress $\sigma$ and radial stress $\sigma_r$ can be obtained as:

$$
\sigma_c(r) = -\int_0^r \frac{\partial \tau_c}{\partial z} dt + \frac{1}{r^2} \int_0^r r \sigma_z dt + \frac{1}{2} C_2 + C_4
$$

(10)

$$
\sigma_r(r) = \sigma_c - \int_0^r \frac{\partial \tau_c}{\partial z} dt - \frac{1}{r^2} \int_0^r r \sigma_z dt - \frac{1}{2} C_2 + C_4
$$

(11)

where, $C_2$ and $C_4$ are constants depending on $C_1$. In order to acquire $C_2$ and $C_4$, a cost function $PF$ is introduced [4].

$$
PF = \frac{1}{n} \sum_{m=1}^n \left[ (V_{1c}(i) - V_{1m}(i))^2 \right]
$$

(12)

Subscript $m$ denotes experiment values and subscript $c$ denotes calculated values. The relation between $V_{1c}(i)$ and $V_{1m}(i)$ can be expressed as:

$$
C_2 \left( C_0 \int_L^M \left( \frac{x^2}{r^4} - \frac{1}{2r^2} \right) dy \right) + C_4 \left( C_0 \int_L^M dy \right) = V_{1m} - V_{1c}'
$$

(13)

Therefore, the $C_2$ and $C_4$ on each layer can be obtained and optimized by using the least square method.

4. Results and discussion

Based on the method in section 3, the reconstructed stresses are shown in Fig. 5. Compressive stress is represented by negative value and tensile stress is represented by positive value.

![Fig. 5 Residual stresses calculation of the normal cross section in cylindrical coordinates: (a) $\sigma_z$, (b) $\sigma_r$, (c) $\sigma_\theta$, (d) $\tau_c$](image-url)
Comparing Fig. 2 and Fig. 5, the reconstructed stresses shows the same magnitude and distribution with the FEM simulation but the axial stress, radial stress and circumferential stress have certain errors.

Figure 6 shows the $C_2$ and $C_4$ along the layer. It can be seen that the $C_2$ is a very small value, and can be ignored. $C_4$ is the mainly factor.

5. Conclusion
In this paper, the residual stress distribution in a thermal treated glass cylinder is calculated by FEM simulation. Based on the stress tensor, intensity images from the circular polariscope were simulated and the stresses reconstruction was carried out. By comparing the reconstructed stresses with the stresses by numerical simulation, it shows that the method can be used to reconstruct the 3D stresses but with some errors. In the calculation of circumferential stress and radial stress the constant, which depends on boundary conditions, in generalized sum rule plays an import factor.

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