A Note on a Theorem of Parry

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In [9] Parry shows that a topologically transitive continuous piecewise monotone mapping \( f \) with positive topological entropy \( h(f) \) is conjugate to a uniformly piecewise linear mapping with slope \( \exp(h(f)) \). In this note we generalise Parry’s result somewhat to what we call the class of essentially transitive mappings. This generalisation is of some interest in as much as for mappings with one turning point the converse also holds, i.e., a uniformly piecewise linear mapping \( g \) with one turning point and with slope \( \beta > 1 \) is essentially transitive. (In fact, \( g \) is topologically transitive if and only if the slope \( \beta \) lies in the interval \( (\sqrt{2}, 2) \); if \( \beta \in (1, \sqrt{2}] \) then \( g \) is only essentially transitive.) The proof of our generalisation relies heavily on a result from the kneading theory of Milnor and Thurston [5], [6], which states that a continuous piecewise monotone mapping \( f \) with positive topological entropy \( h(f) \) is semi-conjugate to a uniformly piecewise linear mapping with slope \( \exp(h(f)) \). We show that if \( f \) is essentially transitive then this forces the semi-conjugacy to be a conjugacy.

Let \( a, b \in \mathbb{R} \) with \( a < b \) and put \( I = [a, b] \). The set of continuous mappings which map the interval \( I \) back into itself will be denoted by \( C(I) \). If \( f \in C(I) \) and \( n \geq 0 \) then \( f^n \) will denote the \( n \)th iterate of \( f \), i.e., \( f^n \in C(I) \) is defined inductively by \( f^0(x) = x, f^1(x) = f(x) \) and \( f^n(x) = f(f^{n-1}(x)) \) for each \( x \in I \), or, without arguments, by \( f^0 = \text{id}_I, f^1 = f \) and \( f^n = f \circ f^{n-1} \) for each \( n \geq 2 \).

Mappings \( g, h \in C(I) \) are said to be conjugate if there exists a homeomorphism \( \psi : I \to I \) (which in the present situation just means a continuous and strictly monotone mapping of \( I \) onto itself) such that \( \psi \circ g = h \circ \psi \). Let \( f \in C(I) \); a subset \( B \) of \( I \) is said to be \( f \)-invariant if \( f(B) \subset B \), and the mapping \( f \) is said to be (topologically) transitive if whenever \( F \) is a closed \( f \)-invariant subset of \( I \) then either \( F = I \) or the interior \( \text{int}(F) \) of \( F \) is empty. There are several other standard conditions which are equivalent to that of being transitive; see, for example, Walters [13].

We are here interested in a special class of mappings from \( C(I) \), namely the piecewise monotone mappings. A mapping \( f \in C(I) \) is piecewise monotone if there exists \( p \geq 0 \) and \( a = d_0 < d_1 < \cdots < d_p < d_{p+1} = b \) such that \( f \) is strictly monotone on each of the intervals \( [d_k, d_{k+1}] \), \( k = 0, \ldots, p \). Let \( f \) be piecewise monotone and suppose the minimal choice for the \( d_k \)'s has been made, i.e., so
that \( f \) is not monotone (or, equivalently, is not injective) on any open interval containing \( d_k \) for each \( 1 \leq k \leq p \); then \( d_1, \ldots, d_p \) are called the turning points of \( f \) and the intervals \([d_k, d_{k+1}]\), \( k = 0, \ldots, p \) the laps of \( f \).

The set of piecewise monotone mappings in \( C(I) \) will be denoted by \( M(I) \) and for each \( f \in M(I) \) the set of turning points of \( f \) by \( T(f) \). The mappings in \( M(I) \) are closed under composition: If \( f, g \in M(I) \) then \( g \circ f \in M(I) \) and it is easy to see that \( T(g \circ f) = \{ x \in (a, b) : x \in T(f) \text{ or } f(x) \in T(g) \} \). In particular, \( f^n \in M(I) \) for all \( f \in M(I), n \geq 1 \).

Let \( \ell(f) \) denote the number of laps of \( f \in M(I) \), so \( \ell(f) = \#(T(f)) + 1 \); also let \( h(f) = \inf_{n \geq 1} n^{-1} \log \ell(f^n) \), and thus \( h(f) \geq 0 \).

**Lemma 1** \( h(f) = \lim_{n \to \infty} n^{-1} \log \ell(f^n) \).

*Proof* If \( f, g \in M(I) \) then \( \ell(f \circ g) \leq \ell(f) \ell(g) \), since each lap of \( g \) contains at most \( \ell(f) \) laps of \( f \circ g \), and so in particular \( \ell(f^{m+n}) \leq \ell(f^m) \ell(f^n) \) for all \( m, n \geq 1 \). Put \( a_n = \log \ell(f^n) \); then \( a_{m+n} \leq a_m + a_n \) for all \( m, n \geq 1 \), and hence, as is well-known \( \lim_{n \to \infty} a_n/n = \inf_{n \to \infty} a_n/n \).

Misiurewicz and Szlenk [8] show that \( h(f) \) is the topological entropy of \( f \) (which is why we denote this quantity by \( h(f) \)).
Let $\beta > 0$; a mapping $g \in M(I)$ is said to be uniformly piecewise linear with slope $\beta$ if on each of its laps $g$ is linear with slope $\beta$ or $-\beta$. The result of Parry referred to in the title of this note (Theorem 1 in [9]) states that if $f \in M(I)$ is transitive then $h(f) > 0$ and $f$ is conjugate to a uniformly piecewise linear mapping with slope $\beta = \exp(h(f))$. Parry’s result will be generalised somewhat below to what we call the class of essentially transitive mappings in $M(I)$.

Let $f \in C(I)$; a closed set $C \subset I$ is called an $f$-cycle with period $m \geq 1$ if $C$ is the disjoint union of non-trivial closed intervals $B_0, \ldots, B_{m-1}$ such that $f(B_{k-1}) \subset B_k$ for $k = 1, \ldots, m - 1$ and $f(B_{m-1}) \subset B_0$ (and so in particular $C$ is $f$-invariant). An $f$-cycle $C$ is said to be (topologically) transitive if whenever $F$ is a closed $f$-invariant subset of $C$ then either $F = C$ or $\text{int}(F) = \emptyset$. Thus $f$ being transitive just means that the whole interval $I$ is a transitive $f$-cycle (with period 1). For each subset $B \subset I$ put

$$E(B, f) = \{x \in I : f^n(x) \in B \text{ for some } n \geq 0\},$$

i.e., $E(B, f)$ consists of those points $x$ for which some iterate $f^n(x)$ lies in the set $B$. The complement $I \setminus E(B, f)$ of this set is always $f$-invariant and if $B$ is $f$-invariant then so is $E(B, f)$.

We call a mapping $f \in C(I)$ essentially transitive if there exists a transitive $f$-cycle $C$ such that $I \setminus E(C, f)$ is countable.

**Theorem 1** If the mapping $f \in M(I)$ is essentially transitive then $h(f) > 0$ and $f$ is conjugate to a uniformly piecewise linear mapping with slope $\exp(h(f))$.

**Proof** This will follow directly from Theorem 2, Proposition 2 and Lemma 2. Theorem 2 is a result from the kneading theory of Milnor and Thurston [5], [6] and Lemma 2 is essentially already a part of Parry’s result. Thus the only part which is new here is Proposition 2.

In particular, a transitive mapping is essentially transitive, and so Parry’s result is a special case of Theorem 1. For mappings with one turning point the converse of Theorem 1 holds: In Proposition 3 we show that in this case each uniformly piecewise linear mapping with slope $\beta > 1$ is essentially transitive.

We should point out that an essentially transitive mapping involves a very special situation, as the follows result indicates:

**Proposition 1** Let $f \in M(I)$ and let $C$ be an $f$-cycle such that $I \setminus E(C, f)$ is countable. Then the period of $C$ is of the form $2^p$ for some $p \geq 0$. Moreover, there exists $q \geq 0$ such that each periodic point in $I \setminus E(C, f)$ has a period which divides $2^q$, and each point in $I \setminus E(C, f)$ is eventually periodic.
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Proof This is Proposition 9.2 in Preston [10], the proof of which is based on an idea occurring in Block [1, 2], Misiurewicz [7], and in the proof of Šarkovskii’s theorem (Šarkovskii [11], Stefan [12]) given in Block, Guckenheimer, Misiurewicz and Young [3]. □

Let $V(I) = \{ \psi \in C(I) : \psi \text{ is increasing and surjective} \}$ (where increasing means only that $\psi(x) \geq \psi(y)$ whenever $x \geq y$). A pair $(\psi, g)$ with $\psi \in V(I)$ and $g \in M(I)$ is called a reduction (or semi-conjugacy) of $f \in M(I)$ if $\psi \circ f = g \circ \psi$.

**Theorem 2** Let $f \in M(I)$ with $h(f) > 0$. Then there exists a reduction $(\psi, g)$ of $f$ such that $g$ is uniformly piecewise linear with slope $\exp(h(f))$.

Proof This can be found in Milnor and Thurston [5, 6]. A modification of their proof (not using any complex analysis) is given at the end of this note. □

**Lemma 2** Let $f \in M(I)$; if there exists a transitive $f$-cycle then $h(f) > 0$. In particular, $h(f) > 0$ whenever $f$ is essentially transitive.

Proof Let $C$ be a transitive $f$-cycle; put $m = \text{per}(C)$, let $B$ be one of the $m$ components of $C$, and let $g$ be the restriction of $f^m$ to $B$, which means that $g \in M(B)$. Then $\ell(g^n) \leq \ell(f^{mn})$ for each $n \geq 0$ and thus by Lemma 1

$$h(f) = \lim_{n \to \infty} \frac{1}{mn} \log \ell(f^{mn}) = \lim_{n \to \infty} \frac{1}{mn} \log \ell(g^n) = \frac{1}{m} \log h(g).$$

But $g \in M(B)$ is transitive and so it is enough to show that $h(f) > 0$ for each transitive $f \in M(I)$ (which is already part of Parry’s result). This holds because a transitive mapping $f \in M(I)$ contains some kind of ‘horse-shoe’: There exist $p \geq 1$ and $D, E \subset I$ with $D \cap E = \emptyset$ such that $f^p(D) \cap f^p(E) \supset D \cup E$. Thus $f^p$ is at least 2 to 1 (i.e., $\#((f^p)^{-1}(x)) \geq 2$ for each $x \in I$), and so $f^{pn}$ is at least $2^n$ to 1 for each $n \geq 1$. It follows that $\ell(f^{pn}) \geq 2^n$ for all $n \geq 1$, and hence that

$$h(f) = \lim_{n \to \infty} \frac{1}{pn} \log \ell(f^{pn}) \geq \frac{1}{p} \log 2 > 0.$$

The existence of such a ‘horse-shoe’ follows, for example, from Theorem 2.1 in Preston [10], which states that a transitive mapping in $M(I)$ is either exact or semi-exact. However, the reader can probably can establish the existence directly without too much trouble. □

**Proposition 2** If $(\psi, g)$ is a reduction of an essentially transitive $f \in M(I)$ then $\psi$ must be a homeomorphism, and so in particular $f$ and $g$ are conjugate.
Proof For each \( \psi \in V(I) \) put

\[
\text{supp}(\psi) = \{ x \in I : \psi(J) \text{ is a non-trivial interval for each open interval } J \subset I \text{ containing } x \},
\]

which means of course that

\[
I \setminus \text{supp}(\psi) = \{ x \in I : \text{ there exists an open interval } J \subset I \text{ containing } x \text{ such that } \psi(J) \text{ consists of the single point } \psi(x) \},
\]

and note that \( \psi \) is a homeomorphism if and only if \( \text{supp}(\psi) = I \).

Lemma 3 For each \( \psi \in V(I) \) the set \( \text{supp}(\psi) \) is non-empty and perfect (i.e., it is closed and contains no isolated points).

Proof Clearly \( \text{supp}(\psi) \) is closed. Moreover, \( \psi \) is constant on each connected component of \( I \setminus \text{supp}(\psi) \). (Let \( J \) be such a component and consider \( c, d \in J \) with \( c < d \). For each \( x \in [c, d] \) there exists an open interval \( J_x \) containing \( x \) on which \( \psi \) is constant. By compactness \([c, d]\) is covered by finitely many of these intervals and hence \( \psi(c) = \psi(d) \).) In particular, this means that \( \text{supp}(\psi) \) is non-empty and perfect (since an isolated point of \( \text{supp}(\psi) \) would be the common end-point of two connected components of \( I \setminus \text{supp}(\psi) \)). □

In fact, each non-empty perfect subset of \( I \) is of the form \( \text{supp}(\psi) \), i.e., if \( D \) is a non-empty perfect subset of \( I \) then there exists a \( \psi \in V(I) \) with \( \text{supp}(\psi) = D \). (This is a classical result in real analysis, and can be found, for example, in Carathéodory [4]. A proof is also given in Preston [10], Proposition 11.1.)

For each \( f \in M(I) \) put \( S(f) = T(f) \cup \{ a, b \} \). A subset \( A \subset I \) will be called \( f \)-almost-invariant if \( f(A \setminus S(f)) \subset A \).

Lemma 4 Let \((\psi, g)\) be a reduction of a mapping \( f \in M(I) \). Then \( \text{supp}(\psi) \) is \( f \)-invariant and \( I \setminus \text{supp}(\psi) \) is \( f \)-almost-invariant.

Proof Let \( x \in I \) with \( f(x) \notin \text{supp}(\psi) \); there thus exists an open interval \( J \) containing \( f(x) \) such that \( \psi(J) \) consists of the single point \( y = \psi(f(x)) \). Hence \( f^{-1}(J) \) is a neighbourhood of \( x \) and so there exists an open interval \( K \) containing \( x \) with \( K \subset f^{-1}(J) \). Then \( g(\psi(K)) = \psi(f(K)) \subset \psi(J) = \{ y \} \), since \( f(K) \subset J \) and \( g \circ \psi = \psi \circ f \). But \( \psi(K) \) is connected and \( g^{-1}(\{ y \}) \) is finite, and therefore \( \psi(K) \) must consist of the single point \( \{ \psi(x) \} \), i.e., \( x \notin \text{supp}(\psi) \). This shows that \( \text{supp}(\psi) \) is \( f \)-invariant.

Now let \( x \in (I \setminus \text{supp}(\psi)) \setminus S(f) \); there thus exists an open interval \( J \) containing \( x \) with \( J \cap S(f) = \emptyset \) such that \( \psi(J) \) consists of the single point \( y = \psi(x) \).
Therefore $f(J)$ is an open interval containing $f(x)$ (since $J \cap S(f) = \emptyset$). But
$$\psi(f(J)) = g(\psi(J)) = g(\{y\})$$
since $g \circ \psi = \psi \circ f$, and so $\psi(f(J))$ consists of the single point $\psi(\{x\})$, i.e., $f(x) \in I \setminus \text{supp}(\psi)$. This shows that $I \setminus \text{supp}(\psi)$ is $f$-almost-invariant.

For each $f \in M(I)$ let $\mathcal{D}(f)$ denote the set of those non-empty perfect subsets of $I$ which are both $f$-invariant and have an $f$-almost-invariant complement. If $(\psi, g)$ is a reduction of $f$ then by Lemmas 9 and 10 supp$(\psi) \in \mathcal{D}(f)$. In fact, the converse also holds: For each $\psi \in V(I)$ with supp$(\psi) \in \mathcal{D}(f)$ there exists a unique $g \in M(I)$ such that $(\psi, g)$ is a reduction of $f$. (This is part of Theorem 5.1 in Preston [10].)

If $C$ is an $f$-cycle then let $C^o$ denote the set obtained by removing the two endpoints from each component of $C$, so if $m$ is the period of $C$ then $\partial C = C \setminus C^o$ consists of exactly $2m$ points. The set $C^o$ is not necessarily $f$-invariant but it is easy to see that it is $f$-almost-invariant, which in turn easily implies that the open set $E(C^o, f)$ is $f$-almost-invariant. Of course, $E(C^o, f) \subset E(C, f)$, since $C^o \subset C$; moreover, $E(C, f) \setminus E(C^o, f)$ is countable, since it is a subset of the countable set $\{x \in I : f^n(x) \in \partial C \text{ for some } n \geq 0\}$. Thus $I \setminus E(C, f)$ is countable if and only if $I \setminus E(C^o, f)$ is.

**Lemma 5** Let $f \in M(I)$, $D \in \mathcal{D}(f)$ and $C$ be a transitive $f$-cycle. Then either $E(C^o, f) \subset D$ or $E(C^o, f) \cap D = \emptyset$.

**Proof** Let $U \subset I$ be open and $f$-almost-invariant. If $J$ is a (maximal connected) component of $U$ then $f(J \setminus S(f)) \subset U$ and it is easily checked that $f(J \setminus S(f))$ is connected (and in fact an open interval). There thus exists a unique component $K$ of $U$ such that $f(J) \subset \overline{K}$. Iterating this then gives us that for each $n \geq 1$ there exists a unique component $K$ of $U$ such that $f^n(J) \subset \overline{K}$. A component $J$ of $U$ is called periodic if $f^m(J) \subset \overline{J}$ for some $m \geq 1$; the smallest such $m \geq 1$ is called the period of $J$. A component $K$ of $U$ is called eventually periodic if $f^n(K) \subset \overline{J}$ for some periodic component $J$ of $U$ and some $n \geq 0$.

Now suppose $E(C^o, f) \not\subset D$; then $E(C^o, f) \cap (I \setminus D) \neq \emptyset$ and it easily follows that $U = C^o \cap (I \setminus D) \neq \emptyset$, so $U$ is a non-empty $f$-almost-invariant open subset of $I$. Let $J$ be a component of $U$ which is not eventually periodic and for each $n \geq 0$ let $J_n$ be the component of $U$ with $f^n(J) \subset \overline{J_n}$. Then the intervals $\{J_n\}_{n \geq 0}$ are disjoint and $F = \bigcup_{n \geq 1} J_n$ is an $f$-invariant closed subset of $C$ (since $f(J_n) \subset J_{n+1}$ for each $n \geq 1$). But $\text{int}(F) \not\subset \emptyset$ (since $J_1 \subset F$) and $F \neq C$ (since $J \subset C \setminus F$) and this contradicts the fact that $C$ is transitive. Therefore each component of $U$ is eventually periodic and in particular $U$ contains a periodic component. Thus let $J$ be a periodic component of $U$ with period $m$; then $K = \bigcup_{k=0}^{m-1} f^k(J)$ is a closed $f$-invariant subset of $C$ with $\text{int}(K) \neq \emptyset$, and hence $K = C$. But $C^o \cap D \subset K \setminus U$
and $K \setminus U$ is finite. Therefore $C^\circ \cap D = \emptyset$, because $D$ is perfect. This implies that $E(C^\circ, f) \cap D = \emptyset$, since $D$ is $f$-invariant.

We can now complete the proof of Proposition 2. Let $C$ be transitive $f$-cycle such that $I \setminus E(C^\circ, f)$ is countable. By Lemmas 3 and 4 $\text{supp}(\psi) \in D(f)$ and so by Lemma 5 either $E(C^\circ, f) \cap \text{supp}(\psi)$ is empty or $E(C^\circ, f) \subset \text{supp}(\psi)$. But if $E(C^\circ, f) \cap \text{supp}(\psi) = \emptyset$ then $\text{supp}(\psi)$ is countable, which is not possible since by the Baire category theorem a non-empty countable closed subset of $I$ must contain an isolated point. Hence $E(C^\circ, f) \subset \text{supp}(\psi)$ and therefore $\text{supp}(\psi) = I$, since $E(C^\circ, f)$ is dense in $I$. This implies that $\psi$ is a homeomorphism.

Proof of Theorem 1: Let $f \in M(I)$ be essentially transitive. Then by Lemma 2 $h(f) > 0$ and hence by Theorem 1 there exists a reduction $(\psi, g)$ of $f$ such that $g$ is uniformly piecewise linear with slope $\exp(h(f))$. But by Proposition 2 $\psi$ is a homeomorphism and therefore $f$ and $g$ are conjugate.

We next note a result of Misiurewicz and Szlenk [8] which provides us with an alternative method of calculating $h(f)$ for a mapping $f \in M(I)$. This will show in particular that if $g \in M(I)$ is uniformly piecewise linear with slope $\beta > 1$, then $h(g) = \log \beta$. For $f \in C(I)$ let

$$\text{Var}(f) = \sup\left\{ \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| : a = a_0 < x_1 < \cdots < x_n = b \right\}.$$ 

If $f \in M(I)$ and $a = d_0 < d_1 < \cdots < d_N < d_{N+1} = b$, where $d_1, \ldots, d_N$ are the turning points of $f$ then clearly

$$\text{Var}(f) = \sum_{k=0}^{N} |f(d_{k+1}) - f(d_k)|.$$ 

In particular, if $g \in M(I)$ is uniformly piecewise linear with slope $\beta > 0$ then $\text{Var}(g) = (b - a)\beta$.

**Theorem 3** Let $f \in M(I)$; then $h(f) > 0$ holds if and only if

$$\limsup_{n \to \infty} n^{-1} \log \text{Var}(f^n) > 0.$$ 

Moreover, if $h(f) > 0$ then

$$h(f) = \lim_{n \to \infty} n^{-1} \log \text{Var}(f^n).$$
Proof  This is given in Misiurewicz and Szlenk [3]. □

Let \( g \) be uniformly piecewise linear with slope \( \beta > 0 \). Then \( \text{Var}(g^n) = (b-a)\beta^n \) for each \( n \geq 1 \), since \( g^n \) is uniformly piecewise linear with slope \( \beta^n \), and thus \( \lim_{n \to \infty} n^{-1} \log \text{Var}(g^n) = \log \beta \). Hence by Theorem 3 \( h(g) = \log \beta \), provided \( \beta > 1 \).

We now consider the case of a mapping \( f \in M(I) \) having a single turning point, and without loss of generality it can be assumed that \( f \) takes on its maximum there. Moreover, it will be convenient to assume also that \( f(\{a,b\}) \subset \{a,b\} \), i.e., that \( f(a) = f(b) = a \). Note that it is always possible to reduce things to this case by extending the domain of definition of \( f \) to a larger interval. Moreover, this can be done in such a way that for each \( x \in (a',a) \cup (b,b') \) the iterates of \( x \) end up in \([a,b]\) after finitely many steps.

Finally, again without loss of generality, assume that \( I = [0,1] \); let \( S \) denote the set of mappings \( f \in M([0,1]) \) having exactly one turning point and for which \( f(0) = f(1) = 0 \).

For each \( \beta \in (0,2] \) there is exactly one mapping in \( S \) which is uniformly piecewise linear with slope \( \beta \). This is the mapping \( u_\beta \) defined by

\[
u_\beta(x) = \begin{cases} \beta x & \text{if } 0 \leq x \leq \frac{1}{2}, \\
\beta - \beta x & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}
\]
By Theorem \( h(u_\beta) = \log \beta \) for each \( \beta \in (1, 2] \). In particular, if \( \alpha, \beta \in (1, 2] \) with \( \alpha \neq \beta \) then \( u_\alpha \) and \( u_\beta \) are not conjugate. (If \( f, g \in M(I) \) are conjugate then \( \ell(f^n) = \ell(g^n) \) for all \( n \geq 1 \), and so \( h(f) = h(g) \).

Note if \((\psi, g)\) is a reduction of \( f \in S \) (with \( g \in M([0, 1]) \)) then in fact \( g \in S \). Thus if \( f \in S \) with \( h(f) > 0 \) then by Theorem \([\text{III}]\) there exists \( \psi \in V([0,1]) \) such that \( \psi \circ f = u_\beta \circ \psi \), where \( \beta = \log h(f) \). Moreover, if \( f \) is essentially transitive then by Theorem \([\text{I}]\) \( f \) and \( u_\beta \) are conjugate. In fact, the following result shows that the converse is true (since if \( f \) and \( g \) are conjugate and \( f \) is essentially transitive then so is \( g \)).

**Proposition 3** The mapping \( u_\beta \) is essentially transitive for each \( \beta \in (1, 2] \).

**Proof** Suppose first that \( \beta \in (\sqrt{2}, 2] \). Let \( C \) be a \( u_\beta \)-cycle with period \( m \), let \( B \) be one of the \( m \) components of \( C \) and let \( g \) be the restriction of \( f^m \) to \( B \). Then \( g \) is uniformly piecewise linear with slope \( \beta^m \) and, since \( g \) has only one turning point, \( \beta^m \leq 2 \). This is only possible if \( m = 1 \), i.e. there are no \( u_\beta \)-cycles with period \( m > 1 \). Now let \( J = [u_\beta^2(\frac{1}{2}), u_\beta(\frac{1}{2})] \); then \( u_\beta(J) = J \), and so \( J \) is a \( u_\beta \)-cycle with period 1. Moreover, if \( K \) is any \( u_\beta \)-cycle with period 1 then \( J \subset K \), since \( \frac{1}{2} \in K \). Thus if \( K \) is any \( u_\beta \)-cycle with \( K \subset J \) then \( K = J \), and from this it is straightforward to show that \( J \) is transitive. (If \( J \) is not transitive then there exists a closed \( u_\beta \)-invariant subset \( F \) of \( J \) with \( \text{int}(F) \neq \emptyset \) and \( F \neq J \). Then \( U = \text{int}(F) \) is a non-empty \( u_\beta \)-almost-invariant subset of \( J \). An argument similar to that employed in the proof of Lemma \([\text{V}]\) shows that \( J \) contains a periodic
component which can be used to define a $u_\beta$-cycle $K \subset F$, and this contradicts
the fact that if $K$ is any $u_\beta$-cycle with $K \subset J$ then $K = J$.) But it is clear
that $[0, 1] \setminus E(J^0, u_\beta) \subset \{0, 1\}$, which shows that $u_\beta$ is essentially transitive. Now
suppose that $\beta \in (1, \sqrt{2}]$; then $d = (1 + \beta)^{-1} \beta$ is the unique fixed point of $u_\beta$ in
$\left(\frac{1}{2}, 1\right)$. Let $c = 1 - d$; thus $u_\beta(c) = d$. Then $u_\beta^2([c, d]) \subset [c, d]$, and it is easy to see
that the restriction of $u_\beta^2$ to $[c, d]$ is conjugate to $u_\beta$.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure.png}
\caption{A geometric representation of the system.
}
\end{figure}

But $u_\beta^m(x) \in [c, d]$ for some $m \geq 0$ for each $x \in (0, 1)$, and for each $x \in [0, 1]$
the set $\{y \in [0, 1] : u_\beta^n(y) = x \text{ for some } n \neq 0\}$ is countable; it thus follows that
if $u_{\beta^2}$ is essentially transitive then so is $u_\beta$. Therefore $u_\beta$ is essentially transitive
for each $\beta \in (1, 2]$.

If $\beta \in (0, 1]$ then $u_\beta$ is certainly not essentially transitive: In this case it is easy
to see that $Z_s(u_\beta) = [0, 1]$, and so there is no transitive $u_\beta$-cycle.

Let $\beta \in (1, 2]$; then $u_\beta$ is essentially transitive and thus there exists a transitive
$u_\beta$-cycle $C$ such that $[0, 1] \setminus E(C^0, u_\beta)$ is countable. The proof of Proposition \[8\]
in fact shows that $C$ has period $2^p$, where $p \geq 0$ is the smallest integer such that
$2^{p+1} \log \beta > \log 2$. Thus $C$ has period 1 if $\beta \in (\sqrt{2}, 2]$, period 2 if $\beta \in (\sqrt{2}, \sqrt{2}]$, period 4 if $\beta \in (\sqrt{2}, \sqrt{2}]$ and so on. The same then holds true for an essentially
transitive mapping $f \in S$.

We end this note by giving a proof of Theorem \[2\]. The proof is essentially that to
be found in Milnor and Thurston \[5\], \[6\] but without using any complex analysis. Fix a mapping $f \in M(I)$ with $h(f) > 0$ and put $r = \exp(-h(f))$; thus $r = 1/\beta$
and $0 < r < 1$. By Lemma 6 $\beta = \lim_{n \to \infty} \ell(f^n)^{1/n}$, and hence $r$ is the radius of convergence of the power series $\sum_{n \geq 0} \ell(f^n)t^n$; in particular this means that the series $L(t) = \sum_{n \geq 0} \ell(f^n)t^n$ converges for all $t \in (0, r)$.

**Lemma 6** $\lim_{t \downarrow r} L(t) = \infty$.

**Proof** By definition $\ell(f^n) \geq (\exp(h(f)))^n = \beta^n$ for each $n \geq 0$ and it therefore follows that $L(t) \geq \sum_{n \geq 0} (\beta t)^n = r(r - t)^{-1}$ for all $t \in (0, r)$. □

Let $J$ denote the set of non-trivial closed intervals $J \subset I$; for $J \in J$ and $n \geq 0$ denote by $\ell(f^n|J)$ the number of laps of $f^n$ which intersect the interior of $J$ (and so in fact $\ell(f^n|J) = \#(T(f^n) \cap \text{int}(J)) + 1$). Then $\ell(f^n|J) \leq \ell(f^n|I) = \ell(f^n)$, and thus in particular the series $L(J, t) = \sum_{n \geq 0} \ell(f^n|J)t^n$ converges for all $t \in (0, r)$. Now, noting that $L(I, t) = L(t) \neq 0$, put $\Lambda(J, t) = L(J, t)/L(I, t)$ for each $t \in (0, r)$, thus $0 \leq \Lambda(J, t) \leq 1$, because $L(J, t) \leq L(I, t)$.

**Lemma 7** Let $J, K \in J$ intersect in a single point. Then

$$\lim_{t \downarrow r} |\Lambda(J \cup K, t) - \Lambda(J, t) - \Lambda(K, t)| = 0 .$$

**Proof** For each $n \geq 0$

$$\ell(f^n|J) + \ell(f^n|K) - 1 \leq \ell(f^n|J \cup K) \leq \ell(f^n|J) + \ell(f^n|K) ,$$

and thus

$$|\Lambda(J \cup K, t) - \Lambda(J, t) - \Lambda(K, t)| = L(I, t)^{-1}|L(J \cup K, t) - L(J, t) - L(K, t)|$$

$$\leq L(I, t)^{-1} \sum_{n \geq 0} t^n = (L(t)(1 - t))^{-1} .$$

But by Lemma 6 $\lim_{t \downarrow r} (L(t)(1 - t))^{-1} = 0$ (since $r < 1$). □

**Lemma 8** Let $J \in J$ be such that $f$ is monotone on $J$. Then

$$\lim_{t \downarrow r} |r\Lambda(f(J), t) - \Lambda(J, t)| = 0 .$$

**Proof** Since $f$ is monotone on $f$ it follows that $\ell(f^{n+1}|J) = \ell(f^n|f(J))$ for each $n \geq 0$, and thus

$$L(J, f) = \sum_{n \geq 0} \ell(f^n|J)t^n = 1 + \sum_{n \geq 0} \ell(f^{n+1}|J)t^{n+1}$$

$$= 1 + t \sum_{n \geq 0} \ell(f^n|f(J))t^n = 1 + tL(f(J), t) .$$
Hence
\[
|r\Lambda(f(J), t) - \Lambda(J, t)| = L(I, t)^{-1}|rL(f(J), t) - L(J, t)|
\]
\[
\leq L(I, t)^{-1}(|rL(f(J), t) - tL(f(J), t)| + |tL(f(J), t) - L(J, t)|)
\]
\[
\leq |r - t| + L(I, t)^{-1} = |r - t| + L(t)^{-1},
\]
and by Lemma 6 \(\lim_{t \uparrow r}(|r - t| + L(t)^{-1}) = 0\). \(\square\)

**Lemma 9** Let \(J \in \mathcal{J}\). If \(f^m\) is monotone on \(J\) then \(\limsup_{t \uparrow r} \Lambda(J, t) \leq r^m\).

**Proof** Since \(f\) is monotone on \(f^k(J)\) for each \(k = 0, \ldots, m - 1\) it follows from Lemma 8 that
\[
\lim_{t \uparrow r} |r^{k+1}\Lambda(f^{k+1}(J), t) - r^k\Lambda(f^k(J), t)| = \lim_{t \uparrow r} r^k|r\Lambda(f^k(J), t) - \Lambda(f^k(J), t)| = 0.
\]
Hence \(\lim_{t \uparrow r} |r^m\Lambda(f^m(J), t) - \Lambda(J, t)| = 0\), and so
\[
\limsup_{t \uparrow r} \Lambda(J, t) = \limsup_{t \uparrow r} r^m\Lambda(f^m(J), t) \leq r^m,
\]
since \(\Lambda(f^m(J), t) \leq 1\) for all \(t \in (0, r)\). \(\square\)

**Lemma 10** There exists a sequence \(\{t_n\}_{n \geq 1}\) from \((0, r)\) with \(\lim_{n \to \infty} t_n = r\) such that \(\{\Lambda(J, t_n)\}_{n \geq 1}\) converges for all \(J \in \mathcal{J}\).

**Proof** Let \(I_o\) be a countable dense subset of \(I\) with \(\{a, b\} \subset I_o\) and \(T(f^n) \subset I_o\) for each \(n \geq 1\); let \(\mathcal{J}_o\) be the set of intervals \(J = [c, d] \in \mathcal{J}\) such that \(c, d \in I_o\), thus \(\mathcal{J}_o\) is countable. Now if \(J \in \mathcal{J}\) and \(\{s_n\}_{n \geq 1}\) is any sequence from \((0, r)\) then the sequence \(\{\Lambda(J, s_n)\}_{n \geq 1}\) is bounded, and so there exists a subsequence \(\{n_k\}_{k \geq 1}\) such that \(\{\Lambda(J, s_n)\}_{k \geq 1}\) converges. Therefore, since \(\mathcal{J}_o\) is countable, a sequence \(\{t_n\}_{n \geq 1}\) from \((0, r)\) with \(\lim_{n \to \infty} t_n = r\) can be found (using the standard diagonal argument) such that \(\{\Lambda(J, t_n)\}_{n \geq 1}\) converges for every \(J \in \mathcal{J}_o\). In fact this sequence then converges for all \(J \in \mathcal{J}\): First consider \(J = [c, d] \in \mathcal{J}\) with \(c, d \in I_o\) and \(d \notin I_o\). Let \(\varepsilon > 0\) and choose \(m \geq 1\) so that \(r^m < \varepsilon\); since \(d \notin T(f^m) \cup \{a, b\}\) there exist \(u, v \in I_o\) such that \(c < u < d < v\) and \(f^m\) is monotone on \([u, v]\). Then \([c, u]\) and \([c, v]\) are both in \(\mathcal{J}_o\), and \(\Lambda([c, u], t) \leq \Lambda(J, t) \leq \Lambda([c, v], t)\) for all \(t \in (0, r)\); hence by Lemmas 7 and 9
\[
\limsup_{n \to \infty} \Lambda(J, t_n) - \liminf_{n \to \infty} \Lambda(J, t_n) \leq \lim_{n \to \infty} \Lambda([c, v], t_n) - \lim_{n \to \infty} \Lambda([c, u], t_n)
\]
\[
= \lim_{n \to \infty} \Lambda([u, v], t_n) \leq r^m < \varepsilon.
\]
Therefore, since $\varepsilon > 0$ was arbitrary, the sequence $\{\Lambda(J, t_n)\}_{n \geq 1}$ converges. The same argument also gives that this sequence converges when $J = [c, d]$ with $c \notin I_o$ and $d \in I_o$. Finally, if $J = [c, d]$ and $c \notin I_o$, $d \notin I_o$ then choose $u \in I_o$ with $c < u < d$; then $\{\Lambda([c, u], t_n)\}_{n \geq 1}$ and $\{\Lambda([u, d], t_n)\}_{n \geq 1}$ both converge, and so by Lemma 7 $\{\Lambda(J, t_n)\}_{n \geq 1}$ converges. \(\Box\)

Now fix a sequence $\{t_n\}_{n \geq 1}$ as in Lemma 10 and put $\Lambda(J) = \lim_{n \to \infty} \Lambda(J, t_n)$ for each $J \in \mathcal{J}$. Then by Lemmas 7, 8 and 9

(1) If $J, K \in \mathcal{J}$ intersect in a single point then $\Lambda(J \cup K) = \Lambda(J) + \Lambda(K)$.
(2) If $J \in \mathcal{J}$ and $f$ is monotone on $J$ then $r\Lambda(f(J)) = \Lambda(J)$.
(3) If $J \in \mathcal{J}$, $m \geq 1$ and $f^m$ is monotone on $J$ then $\Lambda(J) \leq r^m$.

Also define a mapping $\pi : I \to [0, 1]$ by letting $\pi(a) = 0$ and $\pi(x) = \Lambda([a, x])$ for all $x \in (a, b]$.

**Lemma 11** The mapping $\pi : I \to [0, 1]$ is continuous, increasing and surjective.

*Proof* It is clear that $\pi$ is increasing, and if it is continuous then it is surjective, because $\pi(a) = 0$ and $\pi(b) = \Lambda(I) = 1$. Let $x \in I$ and $\varepsilon > 0$; choose $m \geq 1$ so that $r^m < \varepsilon$. Then there exists $\delta > 0$ such that $\{w \in T(f^m) : \{\delta - x\} \in \{x\}\}$. Now if $U = I \cap (x - \delta, x + \delta)$ then $U$ is a neighbourhood of $x$ in $I$, and it follows from (1) and (3) that $|\pi(y) - \pi(x)| < \varepsilon$ for all $y \in U$, since if $y > x$ (resp. $y < x$) then $f^m$ is monotone on $[x, y]$ (resp. on $[y, x]$). This shows $\pi$ is continuous. \(\Box\)

**Lemma 12** There exists a unique mapping $\alpha : [0, 1] \to [0, 1]$ with $\pi \circ f = \alpha \circ \pi$.

*Proof* If $\alpha : [0, 1] \to [0, 1]$ is a mapping with $\pi \circ f = \alpha \circ \pi$ then $\alpha(z) = \pi(f(x))$ whenever $x \in I$ is such that $\pi(x) = z$. Conversely, this relation can be used to define a mapping $\alpha$ with $\pi \circ f = \alpha \circ \pi$, provided $\pi(f(x)) = \pi(f(y))$ whenever $x, y \in I$ are such that $\pi(x) = \pi(y)$. Let $x, y \in I$ with $x < y$ and $\pi(x) = \pi(y)$, and consider $u, v$ with $x \leq u < v \leq y$ so that $f$ is monotone on $[u, v]$. Then $\pi(u) = \pi(v)$ and hence by (1) and (2)

$$r\Lambda(f([u, v])) = \Lambda([u, v]) = \Lambda([a, v]) - \Lambda([a, u]) = \pi(v) - \pi(u) = 0,$$

i.e., $\Lambda(f([u, v])) = 0$; thus $\pi(f(u)) = \pi(f(v))$, because $f(u)$ and $f(v)$ are the end-points of the interval $f([u, v])$. But $[u, v]$ can be written as $[x, y] = [u_0, u_n] = [u_0, u_1] \cup \cdots \cup [u_{n-1}, u_n]$ with $f$ monotone on each of the intervals $[u_j, u_{j+1}], 0 \leq j \leq n - 1$, and therefore

$$\pi(f(x)) = \pi(f(u_0)) = \pi(f(u_1)) = \cdots = \pi(f(u_n)) = \pi(f(y)).$$
This shows that $\alpha$ exists. The uniqueness of $\alpha$ follows immediately from the fact that $\pi$ is surjective.

Let $\alpha : [0, 1] \to [0, 1]$ be the unique mapping with $\pi \circ f = \alpha \circ \pi$.

**Lemma 13** The mapping $\alpha$ is uniformly piecewise linear with slope $\beta$.

**Proof** Let $[c, d]$ be a lap of $f$ on which $f$ is increasing, and let $z \in (\pi(c), \pi(d)]$. Then there exists $x \in (c, d]$ with $\pi(x) = z$, and hence by (1) and (2)

$$\alpha(z) - \alpha(\pi(c)) = \alpha(\pi(x)) - \alpha(\pi(c)) = \pi(f(x)) - \pi(f(c))$$
$$= \Lambda([a, f(x)]) - \Lambda([a, f(c)]) = \Lambda([f(c), f(x)]) = \Lambda(f([c, x]))$$
$$= \beta\Lambda([c, x]) = \beta(\Lambda([a, x]) - \Lambda([a, c]))$$
$$= \beta(\pi(x) - \pi(c)) = \beta(z - \pi(c)).$$

This shows that $\alpha$ is linear with slope $\beta$ on $[\pi(c), \pi(d)]$. If $[u, v]$ is a lap of $f$ on which $f$ is decreasing then a similar calculation shows that $\alpha$ is linear on $[\pi(u), \pi(v)]$ with slope $-\beta$. Therefore $\alpha$ is uniformly piecewise linear with slope $\beta$.

Now let $\gamma : [0, 1] \to I$ be the linear rescaling given by $\gamma(t) = a + (b - a)t$, and define $\psi, g : I \to I$ by $\psi = \gamma \circ \pi$ and $g = \gamma \circ \alpha \circ \gamma^{-1}$. Then $\psi \in V(I)$ and $g \in M(I)$ is uniformly piecewise linear with slope $\beta$ (because $\alpha$ is); moreover,

$$\psi \circ f = \gamma \circ \pi \circ f = \gamma \circ \alpha \circ \pi = \gamma \circ \alpha \circ \gamma^{-1} \circ \gamma \circ \pi = g \circ \psi,$$

i.e., $(\psi, g)$ is a reduction of $f$. This completes the proof of Theorem 2.

**Remark:** In Milnor and Thurston [5, 6] it is shown that for each $J \in \mathcal{J}$ there exists a meromorphic function $L_1(J, \cdot) : D = \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C} \cup \{\infty\}$ with $L_1(J, t) = L(J, t)$ for all $t \in (0, r)$. There thus also exists a meromorphic function $\Lambda_1(J, \cdot) : D \to \mathbb{C} \cup \{\infty\}$ with $\Lambda_1(J, t) = \Lambda(J, t)$ for all $t \in (0, r)$ (with of course $\Lambda_1(J, \cdot) = L_1(J, \cdot)/L_1(I, \cdot)$). Now since $0 \leq \Lambda(J, t) \leq 1$ for all $t \in (0, r)$, it follows that $0 \leq \Lambda_1(J, r) \leq 1$ and $\lim_{t \uparrow r} \Lambda(J, t) = \Lambda_1(J, r)$. In particular, $\Lambda(J) = \Lambda_1(J, r)$. The construction in Lemma 10 is therefore not really necessary.

**References**

[1] Block, L. (1977): Mappings of the interval with finitely many periodic points have zero entropy. Proc. of the A.M.S., 67, 357-360.
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[2] Block, L. (1979): Simple periodic orbits of mappings of the interval. Trans. of the A.M.S., 254, 391-398.

[3] Block, L., J. Guckenheimer, M. Misiurewicz and L.-S. Young (1980): Periodic points and topological entropy of one dimensional maps. Global Theory of Dynamical Systems. Proceedings 1979. Z. Nitecki and C. Robinson, eds., Springer Lecture Notes in Math., 819, 18-34.

[4] Carathéodory, C. (1918): Vorlesungen über reelle Funktionen. Reprinted by Chelsea Publishing Company (1968), New York.

[5] Milnor, J., Thurston, W. (1977): On iterated maps of the interval. I. The kneading matrix, and II. Periodic points. Preprint, Princeton University

[6] Milnor, J., Thurston, W. (1988): On iterated maps of the interval. in: Springer Lecture Notes in Mathematics, Vol. 1342.

[7] Misiurewicz, M. (1980): Invariant measures for continuous transformations of [0, 1] with zero topological entropy. Ergodic Theory Proceedings, 1978. M. Denker and K. Jacobs, eds., Springer Lecture Notes in Math., 729, 144-152.

[8] Misiurewicz, M., Szlenk, W. (1980): Entropy of piecewise monotone maps. Studia Math., 67, 45-63.

[9] Parry, W. (1966): Symbolic dynamics and transformations of the unit interval. Trans. of the A.M.S., 122, 368-378.

[10] Preston, C. (1988): Iterates of piecewise monotone mappings on an interval. Springer Lecture Notes in Mathematics, Vol. 1347.

[11] Šarkovskii, A. (1964): Coexistence of cycles of a continuous map of the line into itself. Ukr. Mat. Z., 16, 61-71.

[12] Štefan, P. (1977): A theorem of karkovskii on the coexistence of periodic orbits of continuous endomorphisms of the real line. Comm. Math. Phys., 54, 237-248.

[13] Walters, P. (1982): An Introduction to Ergodic Theory. Graduate Texts in Math., Vol. 79, Springer-Verlag, New York Heidelberg Berlin