Polysymplectic Hamiltonian Field Theory

G. SARDANASHVILY,

Department of Theoretical Physics, Moscow State University, Russia

Abstract

Applied to field theory, the familiar symplectic technique leads to instantaneous Hamiltonian formalism on an infinite-dimensional phase space. A true Hamiltonian partner of first order Lagrangian theory on fibre bundles $Y \to X$ is covariant Hamiltonian formalism in different variants, where momenta correspond to derivatives of fields relative to all coordinates on $X$. We follow polysymplectic (PS) Hamiltonian formalism on a Legendre bundle over $Y$ provided with a polysymplectic $TX$-valued form. If $X = \mathbb{R}$, this is a case of time-dependent non-relativistic mechanics. PS Hamiltonian formalism is equivalent to the Lagrangian one if Lagrangians are hyperregular. A non-regular Lagrangian however leads to constraints and requires a set of associated Hamiltonians. We state comprehensive relations between Lagrangian and PS Hamiltonian theories in a case of semiregular and almost regular Lagrangians. Quadratic Lagrangian and PS Hamiltonian systems, e.g. Yang – Mills gauge theory are studied in detail. Quantum PS Hamiltonian field theory can be developed in the frameworks both of familiar functional integral quantization and quantization of the PS bracket.

Contents

Introduction 2

1 First order Lagrangian formalism on fibre bundles 4

2 Cartan and Hamilton – De Donder equations 7

3 Polysymplectic structure 8

4 PS bracket 10

5 Hamiltonian forms 12

6 Covariant Hamilton equations 14

7 Hamiltonian time-dependent mechanics 16

8 Iso-PS structure 19

9 Associated Hamiltonian and Lagrangian systems 20
Introduction

Applied to field theory, the familiar symplectic Hamiltonian technique takes the form of instantaneous Hamiltonian formalism on an infinite-dimensional phase space, where canonical coordinates are field functions at some instant of time \([19]\). The true Hamiltonian counterpart of classical first order Lagrangian field theory on a fibre bundle \(Y \to X\) is covariant Hamiltonian formalism, where canonical momenta \(p^i_\mu\) correspond to jets \(y^i_\mu\) of field variables \(y^i\) with respect to all coordinates \((x^\mu)\) on a base \(X\). This formalism has been vigorously developed since 1970s in the Hamilton – De Donder, polysymplectic, multisymplectic, \(k\)-symplectic, \(k\)-cosymplectic and other variants (see \([4, 5, 7, 8, 11, 18, 20, 21, 25, 26, 27, 28, 31, 34, 35, 36, 37, 52]\) and references therein).

Here, we are not concerned with higher order Lagrangian theory \([3, 33, 48, 51]\) and poly-Poisson formalism \([22, 32]\).

We follow polysymplectic (PS) Hamiltonian formalism where the Legendre bundle \(\Pi\) \((1.15)\) plays a role of the momentum phase space of field theory on a fibre bundle \(Y \to X\) \([10, 11, 15, 37, 38, 40]\). It is provided with the canonical \(TX\)-valued PS form \((3.1)\) and, if \(Y \to X\) is a vector bundle, with the PS bracket \((4.3)\) (Section 4).

If \(X = \mathbb{R}\), this is the case of Hamiltonian time-dependent (non-autonomous) non-relativistic mechanics on a fibre bundle \(Q \to \mathbb{R}\) \([17, 42, 44]\). Its momentum phase space is the vertical cotangent bundle \(V^*Q\) of \(Q \to \mathbb{R}\) endowed with the vertical (fibrewise) Poisson structure \((7.11)\) (Section 7).

A Hamiltonian in PS theory on the Legendre bundle \(\Pi\) \((1.15)\) is defined as a section \(h\) of the one-dimensional affine bundle \(Z_L \to \Pi\) \((1.25)\) where \(Z_Y\) \((1.22)\) is a homogeneous Legendre bundle endowed with the canonical multisymplectic form \((5.1)\). Its pull-back with respect to a Hamiltonian \(-h\) is a Hamiltonian form \(H\) \((5.2)\) on a Legendre bundle \(\Pi\). It defines the first order covariant Hamilton equation \((6.4) = (6.5)\) on \(\Pi\).

The Legendre bundle \(\Pi\) \((1.15)\) and the homogeneous Legendre bundle \(Z_Y\) \((1.22)\) play a role of the momentum and homogeneous momentum phase spaces in PS Hamiltonian theory, respectively (Sections 5 – 6).
Any first order Lagrangian $L$ on a jet manifold $J^1Y$ yields the Legendre map $\hat{L} : J^1Y \to \Pi$ (1.14). Conversely, any Hamiltonian form $H$ on $\Pi$ defines the momentum map $\hat{H} : \Pi \to J^1Y$ (5.9). With the Legendre and momentum maps, one can relate Lagrangian formalism on $J^1Y$ and PS Hamiltonian formalism on $\Pi$. They are not equivalent in general.

Lagrangian and PS Hamiltonian formalisms are equivalent in a case of a hyperregular Lagrangian $L$ (Theorem 9.1). In Section 9, we state the comprehensive relations between Lagrangian and PS theories in a case of semiregular and almost regular Lagrangians (Definition 1.1).

It should be emphasized that PS Hamiltonian formalism on a Legendre bundle $\Pi$ is equivalent to particular first order Lagrangian theory on $\Pi$. Given the Hamiltonian form $H$ on $\Pi$, the corresponding covariant Hamilton equation (6.4) – (6.5) is the Euler – Lagrange equation (1.8) of the affine first order Lagrangian $L_H$ (Theorem 5.2). Moreover, it follows from the equality (10.15) that a Hamiltonian form $H$ possesses the same classical symmetries as a Lagrangian $L_H$ (Section 10). This fact enable us to describe symmetries of PS Hamiltonian theory similarly to those in Lagrangian formalism.

In Section 11, the vertical extension of Lagrangian and PS Hamiltonian systems are considered. They describe linear deviations of solutions of Euler – Lagrange and covariant Hamilton equations which are Jacobi fields.

Section 12 provides the detailed analysis of quadratic Lagrangian and PS Hamiltonian systems. The most physically relevant example of these systems is Yang – Mills gauge theory of principal connections (Section 13). Its analysis shows that the main ingredients in gauge theory are not directly related with the gauge invariance property, but are common for theories with almost regular quadratic Lagrangians.

Affine Lagrangian and PS Hamiltonian systems also are considered (Section 14). For instance, this is the case of metric-affine gravitation theory [10, 38, 40].

In order to quantize covariant Hamiltonian field theory, one usually attempts to construct different multimomentum generalizations of a Poisson bracket [8, 9, 21, 23, 24]. In Section 16, we discuss some variants of such kind quantization based on the PS bracket (4.3).

At the same time, the above mentioned fact that PS Hamiltonian formalism on a Legendre bundle $\Pi$ is equivalent to particular first order Lagrangian theory on $\Pi$ enables us to quantize PS Hamiltonian field theory in the framework of familiar perturbative quantum field theory (Section 15).

Throughout the work, we follow familiar technique of fibre bundles, jet manifolds and connections [30, 45, 47]. All morphisms are smooth, and manifolds are smooth real and finite-dimensional. Smooth manifolds customarily are assumed to be Hausdorff second-countable and, consequently, locally compact and paracompact. Unless otherwise stated, they are connected.

The standard symbols $\otimes$, $\vee$, and $\wedge$ stand for the tensor, symmetric, and exterior products, respectively. The interior product (contraction) is denoted by $\langle$. By $\partial^A_B$ are meant the partial derivatives with respect to coordinates with indices $^A_B$.

If $Z$ is a manifold, we denote by

$$\pi_Z : TZ \to Z, \quad \pi^*_Z : T^*Z \to Z$$

its tangent and cotangent bundles, respectively. Given manifold coordinates $(z^\alpha)$ on $Z$, they are
equipped with the holonomic coordinates 
\[(z^\lambda, \dot{z}^\lambda), \quad \dot{z}^\lambda = \frac{\partial z^\lambda}{\partial z^\mu} \dot{z}^\mu, \quad (\dot{z}^\lambda, \ddot{z}^\lambda), \quad \ddot{z}^\lambda = \frac{\partial \omega^\mu}{\partial z^\lambda} \dot{z}^\mu,\]

with respect to the holonomic frames \(\{\partial_\lambda\}\) and coframes \(\{dz^\lambda\}\) in the tangent and cotangent spaces to \(Z\), respectively. Any manifold morphism \(f: Z \to Z'\) yields the tangent morphism 
\[Tf: TZ \to TZ', \quad \dot{z}^\lambda \circ Tf = \frac{\partial f^\lambda}{\partial x^\mu} \dot{z}^\mu.\]

We use the compact notation \(\dot{\partial}_\mu = \partial/\partial \dot{z}^\mu\).

Given a fibre bundle \(Y \to X\) endowed with bundle coordinates \((x^\lambda, y^i)\), we denote by \(V_Y\) and \(V^\ast_Y\) its vertical tangent and cotangent bundles provided with holonomic coordinates \((x^\lambda, y^i, \dot{y}^i)\) and \((x^\lambda, y^i, \theta_i)\), respectively.

The symbol \(C^\infty(Z)\) stands for the ring of smooth real functions on a manifold \(Z\).

1 First order Lagrangian formalism on fibre bundles

As was mentioned above, we restrict our consideration to first order Lagrangian formalism on a smooth fibre bundle \(\pi: Y \to X\) over an oriented \((1 < n)\)-dimensional base \(X\) (see Section 7 for a case of \(n = 1\) [10] [15] [45]). Let \(Y\) be provided with an atlas of bundle coordinates \((x^\lambda, y^i)\).

A velocity phase space of first order Lagrangian theory on a fibre bundle \(Y\) is the first order jet manifold \(J^1 Y\) of sections of \(Y \to X\) (or, simply, of \(Y\)). It is endowed with the adapted coordinates \((x^\lambda, y^i, y^i_\lambda)\) possessing transition functions
\[y^i_\lambda = \frac{\partial x^\mu}{\partial x'^\lambda} (\dot{\partial}_\mu + y^i_\mu \partial_i) y^i.\] (1.1)

There are natural fibrations
\[\pi^1: J^1 Y \to X, \quad \pi^1_0: J^1 Y \to Y,\] (1.2)

where the latter is an affine bundle modelled over a vector bundle
\[T^* X \otimes V_Y \to Y.\] (1.3)

There are the canonical imbeddings
\[
\lambda_{(1)}: J^1 Y \to T^* X \otimes TY, \quad \lambda_{(1)} = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i) = dx^\lambda \otimes d_\lambda, \quad \theta_{(1)}: J^1 Y \to T^* Y \otimes V_Y, \quad \theta_{(1)} = (dy^i - y^i_\lambda dx^\lambda) \otimes \partial_i = \theta^i \otimes \partial_i, \quad \] (1.4)

where \(d_\lambda\) denote the total derivatives, and \(\theta^i\) are local contact forms.

A first order Lagrangian of Lagrangian theory on a fibre bundle \(Y \to X\) is defined as a density
\[L = \mathcal{L} \omega: J^1 Y \to \wedge^n T^* X\] (1.6)
on the first order jet manifold \(J^1 Y\) of \(Y\) where, for the sake of simplicity, we denote
\[\wedge^n T^* X = \pi^1_0(\wedge^n T^* X) = J^1 Y \times \wedge^n T^* X = J^1 Y \times \wedge^n T^* Y.\]
The corresponding second-order Euler – Lagrange operator reads

$$\delta L = \mathcal{E}_L : J^2Y \to T^*Y \wedge (\wedge^n T^*X),$$

$$\mathcal{E}_L = (\partial_i \mathcal{L} - d\lambda_i^\lambda) \theta^i \wedge \omega,$$

$$\pi_i^\lambda = \partial_i^\lambda \mathcal{L},$$

$$d\lambda = \partial_\lambda + y_i^\lambda \partial_i + y_{\lambda \mu}^i \partial_\mu,$$ (1.7)  

where $J^2Y$ is the second order jet manifold of $Y \to X$ coordinated by $(x^\lambda, y^i, y_i^\lambda, y_{\lambda \mu}^i = y_{\mu \lambda}^i)$. Its kernel $\text{Ker} \mathcal{E}_L \subset J^2Y$ is the second order Euler – Lagrange equation on $Y$ locally given by equalities

$$\delta_i L (\partial_i - d_\lambda \partial_\lambda) \mathcal{L} = 0.$$ (1.8)  

Remark 1.1. Strictly speaking, the Euler – Lagrange equation (1.8) fails to be a differential equation in general because $\text{Ker} \mathcal{E}_L \subset J^2Y$ need not be a closed subbundle of $J^2Y \to X$ [10, 15].

In a general setting, Lagrangians (1.6) and Euler – Lagrange operators $\delta L$ (1.7) in Lagrangian formalism on a fibre bundle $Y \to X$ are introduced as elements of the variational bicomplex of exterior forms on an infinite order jet manifold $J^\infty Y$ [13, 15, 45]. Its cohomology defines the global variational decomposition

$$dL = \delta L - d_H \Xi_L$$ (1.9)  

er over $J^2Y$ where

$$d_H \phi = dx^\lambda \wedge d\lambda \phi$$ (1.10)  

is the total differential of exterior forms $\phi$ on $J^2Y$, and $\Xi_L$ is some Lepage form which is a Lepage equivalent of a Lagrangian $L$, i.e.,

$$L = h_0(\Xi_L),$$

$$h_0(dx^\lambda) = dx^\lambda, \quad h_0(dy^i) = y_i^\lambda dx^\lambda, \quad h_0(dy_{\mu}^i) = y_{\lambda \mu}^i dx^\lambda.$$  

Defined up to a $d_H$-closed form, a form $\Xi_L$ reads

$$\Xi_L = L + (\pi_i^\lambda - d_\mu \sigma_i^{\mu \lambda}) \theta^i \wedge \omega_\lambda + \sigma_i^{\lambda \mu} \theta^i \wedge \omega_\lambda, \quad \omega_\lambda = \partial_\lambda | \omega,$$ (1.11)  

where $\sigma_i^{\mu \lambda} = -\sigma_i^{\lambda \mu}$ are skew-symmetric local functions on $Y$. Lepage equivalents constitute an affine space modelled over a vector space of $d_H$-exact one-contact Lepage forms

$$\rho = -d_\mu \sigma_i^{\mu \lambda} \theta^i \wedge \omega_\lambda + \sigma_i^{\lambda \mu} \theta^i \wedge \omega_\lambda.$$  

Let us choose the Poincaré – Cartan form

$$H_L = \mathcal{L} \omega + \pi_i^\lambda \theta^i \wedge \omega_\lambda$$ (1.12)  

as the origin of this affine space because it is defined on $J^1Y$.

Given the Lagrangian $L$ (1.6), let us consider the vertical tangent map

$$VL : V_Y J^1Y \to V_Y J^1Y \times \wedge^n Y \times T^*X$$ (1.13)
to \( L \) over \( Y \), where \( V_Y J^1 Y \) denotes the vertical tangent bundle of \( J^1 Y \to Y \). Since \( J^1 Y \to Y \) is an affine bundle modelled over the vector bundle \((1.3)\), we have the canonical vertical splitting

\[
V_Y J^1 Y = J^1 Y \times (T^* X \otimes V Y).
\]

Accordingly, the vertical tangent map \( V_L \) \((1.13)\) yields a linear morphism

\[
J^1 Y \times (T^* X \otimes V Y) \to J^1 Y \times (\wedge^1 T^* X)
\]

over \( J^1 Y \) and the corresponding morphism

\[
\hat{L} : J^1 Y \to \Pi = V^* Y \wedge (\wedge^{n-1} T^* X) \otimes T X \tag{1.14}
\]

over \( Y \). It is called the Legendre map associated to a Lagrangian \( L \).

A fibre bundle

\[
\Pi = V^* Y \wedge (\wedge^{n-1} T^* X) \otimes T X \overset{\pi_{\Pi}}{\to} Y \tag{1.15}
\]

over \( Y \) is called the Legendre bundle. It is provided with the holonomic coordinates \((x^\lambda, y^i, p^\lambda_i)\) possessing transition functions

\[
p^\lambda_i = \det \left( \frac{\partial x^\epsilon}{\partial x'^\mu} \right) \frac{\partial y^j}{\partial y'^i} \frac{\partial x'^\lambda}{\partial x^\mu} p^\mu_j. \tag{1.16}
\]

With respect to these coordinates, the Legendre map \((1.14)\) reads

\[
p^\lambda_i \circ \hat{L} = \pi^\lambda_i. \tag{1.17}
\]

**Remark 1.2.** There is the canonical isomorphism

\[
\Pi = V^* Y \wedge (\wedge^{n-1} T^* X), \quad (p^\lambda_i) \to p^\lambda_i \omega^i \omega^\lambda, \tag{1.18}
\]

where \(\{\omega^i\}\) are fibre bases for the vertical cotangent bundle \(V^* Y\) of \( Y \to X \).  

Certainly, the Legendre map \((1.14)\) need not be a bundle isomorphism. Its range

\[
N_L = \hat{L}(J^1 Y) \subset \Pi \tag{1.19}
\]

is called the Lagrangian constraint space.

**Definition 1.1.** A Lagrangian \( L \) is said to be:

- hyperregular if the Legendre map \( \hat{L} \) is a diffeomorphism;
- regular if \( \hat{L} \) is a local diffeomorphism over \( Y \), i.e., \( \det(\partial_i^\epsilon \partial_j^\nu L) \neq 0 \);
- semiregular if the inverse image \( \hat{L}^{-1}(z) \) of any point \( z \in N_L \) is a connected submanifold of \( J^1 Y \);
- almost regular if the Lagrangian constraint space \( N_L \) \((1.19)\) is a closed imbedded subbundle

\[
i_N : N_L \to \Pi \tag{1.20}
\]

of a Legendre bundle \( \Pi \to Y \) and the Legendre map

\[
\hat{L} : J^1 Y \to N_L \tag{1.21}
\]
is a fibred manifold with connected fibres (i.e., a Lagrangian \( L \) is semiregular).

The Poincaré–Cartan form (1.12) in turn takes its values into a subbundle

\[ J^1Y \times (T^*Y \wedge (\wedge^{n-1} T^*X)) \]

of \( \wedge^n T^*J^1Y \). Hence, it defines a bundle morphism

\[ \tilde{H}_L : J^1Y \to Z_Y = T^*Y \wedge (\wedge^{n-1} T^*X), \quad (1.22) \]

over \( Y \) whose range

\[ Z_L = \tilde{H}_L(J^1Y) \quad (1.23) \]

is an imbedded subbundle \( i_L : Z_L \to Z_Y \) of the fibre bundle \( Z_Y \to Y \). This morphism is called the homogeneous Legendre map. Accordingly, the fibre bundle \( Z_Y \to Y \) (1.22) is said to be the homogeneous Legendre bundle. It is equipped with holonomic coordinates \( (x^\lambda, y^i, p^\lambda_i, p) \) possessing transition functions

\[ p'_i^\lambda = \det \left( \frac{\partial x^\nu}{\partial x'^\mu} \right) \frac{\partial y^i}{\partial y'^i} \frac{\partial x'^\lambda}{\partial x^\mu} P_j^\mu, \quad p' = \det \left( \frac{\partial x^\nu}{\partial x'^\mu} \right) \left( p - \frac{\partial y^i}{\partial y'^i} \frac{\partial x'^\lambda}{\partial x^\mu} P_j^\mu \right). \quad (1.24) \]

With respect to these coordinates, the morphism \( \tilde{H}_L \) (1.22) reads

\[ (p'_i^\mu, p) \circ \tilde{H}_L = (\pi^\mu_i, \mathcal{L} - y^i_{\mu} \pi^\mu_i). \]

A glance at the transition functions (1.24) shows that \( Z_Y \) (1.22) is a one-dimensional affine bundle

\[ \pi_{Z\Pi} : Z_Y \to \Pi \quad (1.25) \]

over the Legendre bundle \( \Pi \) (1.18) modelled over the pull-back vector bundle

\[ \Pi \times X^n \wedge T^*X \to \Pi. \quad (1.26) \]

Moreover, the Legendre map \( \tilde{L} \) (1.14) is exactly the composition of morphisms

\[ \tilde{L} = \pi_{Z\Pi} \circ \tilde{H}_L : J^1Y \to \Pi. \quad (1.27) \]

As was mentioned above, the Legendre bundle \( \Pi \) (1.15) and the homogeneous Legendre bundle \( Z_Y \) (1.22) play a role of the momentum phase space and a homogeneous momentum phase space in PS Hamiltonian theory, respectively (Section 5).

### 2 Cartan and Hamilton–De Donder equations

Note that the Euler–Lagrange equation (1.8) do not exhaust all equations considered in first order Lagrangian theory.

Being a Lepage equivalent of a Lagrangian \( L \), the Poincaré–Cartan form \( H_L \) (1.12) also is a Lepage equivalent of a first order Lagrangian

\[ \mathcal{L} = \tilde{h}_0(H_L) = (\mathcal{L} + (\tilde{g}_\lambda^i - y^i_{\lambda}) \pi^\lambda_i) \omega, \quad \tilde{h}_0(dy^i) = \tilde{g}_\lambda^i dx^\lambda, \quad (2.1) \]
on the repeated jet manifold \( J^1 J^1 Y \), coordinatized by \((x^\lambda, y^i, \hat{y}_\mu^i, y^i_{\mu})\).

The Euler – Lagrange operator for \( L \) (called the Euler – Lagrange – Cartan operator) reads

\[
\delta \mathcal{L} = \mathcal{E}_\mathcal{T} : J^1 J^1 Y \to T^* J^1 Y \wedge (\wedge^n T^* X),
\]

\[
\mathcal{E}_\mathcal{T} = [(\partial_i \mathcal{L} - \delta_\lambda \pi^\lambda_i + \partial_\lambda \pi^\lambda_i (\hat{y}_\mu^i - y^i_{\mu}))dy^i + \partial^\lambda \pi^\mu_i (\hat{y}_\mu^i - y^i_{\mu})dy^i] \wedge \omega, \tag{2.2}
\]

Its kernel \( \text{Ker} \mathcal{E}_\mathcal{T} \subset J^1 J^1 Y \) is the first order Cartan equation on \( J^1 Y \) locally given by equalities

\[
\partial_\lambda \pi^\mu_i (\hat{y}_\mu^i - y^i_{\mu}) = 0, \tag{2.3}
\]

\[
\partial_i \mathcal{L} - \delta_\lambda \pi^\lambda_i + (\hat{y}_\lambda^i - y^i_\lambda) \delta_\lambda \pi^\lambda_i = 0. \tag{2.4}
\]

Since \( \mathcal{E}_{\mathcal{T} J^1 Y} = \mathcal{E}_L \), the Cartan equation (2.3) – (2.4) is equivalent to the Euler – Lagrange on \( J^1 Y \) on integrable sections \( \pi = J^1 s \) of \( J^1 Y \to X \), where \( s \) are sections of \( Y \to X \). These equations are equivalent if a Lagrangian is regular. The Cartan equation (2.3) – (2.4) on sections \( \pi : X \to J^1 Y \) is equivalent to the relation

\[
\pi^*(u|dH_L) = 0, \tag{2.5}
\]

which is assumed to hold for all vertical vector fields \( u \) on \( J^1 Y \to X \).

The homogeneous Legendre bundle \( Z_Y \) (1.22) admits the canonical multisymplectic Liouville form

\[
\Xi_Y = p\omega + p_\lambda^i dy^i \wedge \omega_\lambda. \tag{2.6}
\]

Accordingly, its imbedded subbundle \( Z_L \) (1.23) is provided with the pull-back De Donder form \( \Xi_L = i_L^* \Xi_Y \). There is the equality

\[
H_L = \hat{H}_L^* \Xi_L = \hat{H}_L^* (i_L^* \Xi_Y). \tag{2.7}
\]

By analogy with the Cartan equation (2.5), the Hamilton – De Donder equation for sections \( \pi \) of \( Z_L \to X \) is written as

\[
\pi^*(u|d\Xi_L) = 0, \tag{2.8}
\]

where \( u \) is an arbitrary vertical vector field on \( Z_L \to X \). Then the following holds [18].

**Theorem 2.1.** Let the homogeneous Legendre map \( \hat{H}_L \) be a submersion. Then a section \( \pi \) of \( J^1 Y \to X \) is a solution of the Cartan equation (2.5) iff \( \hat{H}_L \circ \pi \) is a solution of the Hamilton – De Donder equation (2.8), i.e., the Cartan and Hamilton – De Donder equations are quasi-equivalent.

Note that the Cartan and Hamilton – De Donder equations play a role of the Lagrangian partners of Hamilton equations in PS Hamiltonian theory (Section 9).

3 Polysymplectic structure

Treated as a momentum phase space of fields, the Legendre bundle \( II \) (1.15) is endowed with the following polysymplectic (PS) structure.
There is the canonical bundle monomorphism
\[ \Theta_Y : \Pi \to \left( T^*Y \otimes Y \right)^{n+1} \wedge T^*Y \otimes Y \]
called the tangent-valued Liouville form on \( \Pi \). Strictly speaking, it is \( TX \)-valued, but not a tangent-valued (i.e., \( T\Pi \)-valued) form on \( \Pi \). Therefore, standard technique of tangent-valued forms, as like as that of exterior forms, is not applied to \( \Theta_Y \).

At the same time, there is a unique \( TX \)-valued \((n+2)\)-form \( \Omega_Y = d\lambda \wedge dx_i \wedge \omega \otimes \partial \lambda \) on \( \Pi \) such that the relation \( \Omega_Y \wedge \phi = d(\Theta_Y \wedge \phi) \) holds for an arbitrary exterior one-form \( \phi \) on \( X \). The form \( \Omega_Y \) is called the polysymplectic (PS) form.

**Remark 3.1.** It should be emphasized that, following [20], one often provides \( \Pi \) with an exterior form \( dp^\lambda_i \wedge dx_i \wedge \omega \lambda, \omega \lambda = \partial \lambda \wedge \omega \), which however globally is ill defined because it is not maintained under the transition functions \( 1.16 \) (cf. the form \( 1.13 \)). Our variant of PS formalism on the Legendre bundle \( \Pi \) is based just on the \( TX \)-valued PS form \( \Omega_Y \). \( \bullet \)

Let \( J^1 \Pi \) be the first order jet manifold of a fibre bundle \( \Pi \to X \). It is equipped with the adapted coordinates \((x^\lambda, y^i, p^\lambda_i, y^i_{\mu}, p^\lambda_{\mu i})\). A connection
\[ \gamma = dx^\lambda \otimes (\partial \lambda + \gamma^i_\lambda \partial_i + \gamma^\mu_\lambda \partial^i_\mu) \]
on \( \Pi \to X \) is called the Hamiltonian connection if an exterior form
\[ \gamma \wedge \Omega_Y = (\partial \lambda + \gamma^i_\lambda \partial_i + \gamma^\mu_\lambda \partial^i_\mu) \wedge (dp^\lambda_i \wedge dx_i \wedge \omega) \]
on \( J^1 \Pi \) is closed. Components of a Hamiltonian connection satisfy the conditions
\[ \partial^\lambda_\mu \gamma^i_\lambda - \partial^\lambda_\mu \gamma^i_\mu = 0, \quad \partial \lambda \gamma^\mu_\lambda \partial^i_\mu = 0, \quad \partial \lambda \gamma^i_\lambda + \partial^\lambda_\mu \gamma^i_\mu = 0. \]

If a form \( \gamma \wedge \Omega_Y \) is closed, there is a contractible neighborhood \( U \) of each point of \( \Pi \) where a local form \( \gamma \wedge \Omega_Y \) is exact, i.e.,
\[ \gamma \wedge \Omega_Y = dp^\lambda_i \wedge dx_i \wedge \omega \lambda - (\gamma^i_\lambda dp^\lambda_i - \gamma^\lambda_\mu dy^\mu) \wedge \omega = dH_U \)
on \( U \). It is readily observed that, by virtue of the conditions \( 3.5 \), the second term in the right-hand side of this equality also is closed and, consequently, an exact form on \( U \). In accordance with the relative Poincaré lemma ([10], Remark 4.4.2), this term can be brought into the form \( dH_U \wedge \omega \) where \( H_U \) is a local function on \( U \). Then a form \( H_U \) in the expression \( 3.6 \) reads
\[ H_U = p^\lambda_i dy^i \wedge \omega \lambda - H_U \omega. \]
Example 3.2. Every connection

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)$$  (3.8)

on a fibre bundle $Y \to X$ gives rise to a connection

$$\tilde{\Gamma} = dx^\lambda \otimes [\partial_\lambda + \Gamma^i_\lambda (y) \partial_i + (-\partial_j \Gamma^i_\lambda p^j_\nu + K^\lambda_{\nu \alpha} p^\alpha_\mu - K^\lambda_{\alpha \nu} p^\nu_\mu) \partial_i^j]$$  (3.9)

on a Legendre bundle $\Pi \to X$, where $K$ is a symmetric linear connection

$$K = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)^\nu_\mu \partial^\mu_\nu$$  (3.10)

on the tangent bundle $TX \to X$. Due to the isomorphism (1.18), the connection (3.9) is constructed as follows [10]. It is a tensor product

$$\tilde{\Gamma} = (\Gamma \times K) \otimes \Gamma^* \Gamma$$  (3.11)

over $\Gamma$ of the product connection $\Gamma \times K$ on the pull-back bundle

$$Y \times X^{n-1} \wedge TX \to X$$

and the covertical $\Gamma^* \Gamma$ to $\Gamma$:

$$\Gamma^* \Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)^\nu_\mu \partial^\mu_\nu$$  (3.12)

on the vertical cotangent bundle $V^*Y \to X$. Since the connections $\Gamma \times K$ and $\Gamma^* \Gamma$ are linear connections over $\Gamma$, their tensor product (3.11) is well defined. The connection (3.9) on $\Pi \to X$, by construction, projects onto the connection $\Gamma$ on $Y \to X$. It obeys a relation

$$\tilde{\Gamma}|\Omega_Y = -d(\Gamma|\Theta_Y) = dH$$

where a form

$$H = \Gamma^* \Xi_Y = p^i_\lambda dy^i \wedge \omega - p^i_\lambda \Gamma^i_\lambda \omega$$  (3.13)

globally is well defined. It follows that $\tilde{\Gamma}$ (3.9) is a Hamiltonian connection. $lacklozenge$

Thus, Hamiltonian connections always exist on a Legendre bundle $\Pi \to X$, and every connection $\Gamma$ on $Y \to X$ gives rise to a Hamiltonian connection on $\Pi \to X$.

4 PS bracket

If $Y \to X$ is a vector bundle, the Legendre bundle $\Pi$ (1.15) also is provided with a PS bracket. Note that different generalizations of a Poisson bracket have been suggested in the framework of covariant Hamiltonian field theory [8, 9, 21, 23, 24, 29]. Here, we consider such a bracket in the framework of PS geometry, but it differs from that in our work [29]. In a case of time-dependent mechanics it reduces to the familiar vertical Poisson bracket $\{,\}_V$ (4.11) [17, 44].

Given an exterior bundle

$$\wedge TX = (X \times \mathbb{R}) \oplus TX \oplus \frac{2}{X} TX \oplus \cdots \oplus \frac{n}{X} TX$$

10
let us consider a fibre bundle
\[ \Pi \times_X (\wedge TX \otimes \wedge T^* X) \rightarrow \Pi. \] (4.1)

One can think of its sections
\[ F = \frac{1}{k!} F^\mu_1...\mu_k \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_k} \otimes \omega, \quad |F| = k, \] (4.2)
as being \( TX \)-multivalued densities on \( \Pi \). Let \( \mathcal{T}^*(\Pi) \) denote their real space. It is an exterior algebra with respect to the exterior product
\[ F \wedge G = \frac{1}{(k-1)!q!} \frac{\partial F^\mu_1...\mu_k}{\partial p^\mu_1} \frac{\partial G^\alpha_1...\alpha_q}{\partial y^\alpha_1} \partial_{\mu_2} \wedge \cdots \wedge \partial_{\mu_k} \wedge \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_q} \otimes \omega - \frac{1}{(q-1)!k!} \frac{\partial G^{\alpha_2...\alpha_q}}{\partial p^\alpha_2} \frac{\partial F^\mu_1...\mu_k}{\partial y^\alpha_2} \partial_{\alpha_2} \wedge \cdots \wedge \partial_{\alpha_q} \wedge \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_k} \otimes \omega. \] (4.3)

A manifested PS bracket on elements \( F \) (4.2) of an algebra \( \mathcal{T}^*(\Pi) \) is introduced by the law
\[ \{ F, G \}_{PS} = \frac{1}{k!q!} \frac{\partial F^\mu_1...\mu_k}{\partial p^\mu_1} \frac{\partial G^\alpha_1...\alpha_q}{\partial y^\alpha_1} \partial_{\mu_2} \wedge \cdots \wedge \partial_{\mu_k} \wedge \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_q} \otimes \omega - \frac{1}{(q-1)!k!} \frac{\partial G^{\alpha_2...\alpha_q}}{\partial p^\alpha_2} \frac{\partial F^\mu_1...\mu_k}{\partial y^\alpha_2} \partial_{\alpha_2} \wedge \cdots \wedge \partial_{\alpha_q} \wedge \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_k} \otimes \omega. \] (4.3)

If \( Y \rightarrow X \) is a vector bundle, it is maintained under the transformations (1.16) and, consequently, globally is well defined.

The PS bracket (4.3) obeys the relations
\[ \{ F, G \}_{PS} = -\{ G, F \}_{PS}, \quad |\{ F, G \}_{PS}| = |F| + |G| - 1. \]

The Jacobi identity
\[ \{ S, \{ G, F \} \}_{PS} + \{ G, \{ F, S \} \}_{PS} + \{ F, \{ S, D \} \}_{PS} = 0 \]
also holds if all the products
\[ |S - 1||G - 1|, \quad |G - 1||F - 1|, \quad |F - 1||S - 1| \]
are even. In particular, it follows that a space of \( TX \)-valued densities \( F \in \mathcal{T}^1(\Pi) \) constitute a real Lie algebra with respect to the PS bracket \( \{ \cdot, \cdot \}_{PS} \) (4.3).

If \( \dim X = 1 \), the PS bracket (4.3) always is defined and, as was mentioned above, it is reduced to the Poisson bracket \( \{ \cdot, \cdot \}_V \) (7.11) of the canonical vertical Poisson structure on the vertical cotangent bundle \( V^* Y \) (Section 7).

It should be emphasized that the PS bracket \( \{ F, G \}_{PS} \) (4.3), as like as the Poisson bracket \( \{ \cdot, \cdot \}_V \) (7.11), fails to describe dynamics of Hamiltonian systems \[29\], but in particular it yields the bracket (10.21) of Noether Hamiltonian currents. Similarly, the Poisson bracket \( \{ \cdot, \cdot \}_V \) (7.11) defines a Lie bracket of Noether currents in mechanics \[17\].

The PS bracket (4.3) can be applied to quantization of Hamiltonian field systems (Section 16).
5 Hamiltonian forms

In the framework of PS formalism, dynamics of sections of the Legendre bundle \( \Pi (1.15) \) is described in terms of Hamiltonian forms.

Let us consider the homogeneous Legendre bundle \( Z_Y (1.22) \) and the affine bundle \( Z_Y \rightarrow \Pi (1.25) \) modelled over the pull-back bundle (1.26). The homogeneous Legendre bundle \( Z_Y \) is provided with the canonical multisymplectic Liouville form \( \Xi_Y (2.6) \). Its exterior differential \( d\Xi_Y \) is the canonical multisymplectic form

\[
\Omega_Y = dp \wedge \omega + dp^\lambda_i \wedge dy^i \wedge \omega, \tag{5.1}
\]

Let \( h = -H \omega \) be a section the affine bundle \( Z_Y \rightarrow \Pi (1.25) \). A glance at the transformation law (1.24) shows that it is not a density. By analogy with Hamiltonian time-dependent mechanics (Section 7), \(-h\) is said to be the covariant Hamiltonian of PS Hamiltonian theory. It defines the pull-back

\[
H = h^*\Xi_Y = p^\lambda_i dy^i \wedge \omega - H\omega \tag{5.2}
\]

of the multisimplectic Liouville form \( \Xi_Y \) onto a Legendre bundle \( \Pi \) which is called the Hamiltonian form on \( \Pi \).

The following is a straightforward corollary of this definition.

**Theorem 5.1.** (i) Every connection \( \Gamma (3.8) \) on a fibre bundle \( Y \rightarrow X \) yields a section

\[
\Gamma = \overline{dy}^i \rightarrow dy^i - \Gamma^i_\lambda dx^\lambda
\]

of a fibre bundle \( T^*Y \rightarrow V^*Y \) which gives rise to a section

\[
h_\Gamma : p^\lambda_i \overline{dy}^i \otimes \omega \rightarrow p^\lambda_i dy^i \wedge \omega - p^\lambda_i \Gamma^i_\lambda \omega
\]

of the affine bundle (1.25). Consequently, it defines a Hamiltonian form

\[
h_\Gamma^*\Xi = p^\lambda_i dy^i \wedge \omega - p^\lambda_i \Gamma^i_\lambda \omega \tag{5.3}
\]

on a Legendre bundle \( \Pi \) which coincides with the form \( H_\Gamma (3.13) \).

(ii) Hamiltonian forms constitute a non-empty affine space modelled over the linear space of densities \( \overline{H} = \overline{H}\omega \) on \( \Pi \rightarrow X \) which are sections of the pull-back bundle (1.26).

(iii) Given a connection \( \Gamma \) on \( Y \rightarrow X \), every Hamiltonian form \( H (5.2) \) admits a decomposition

\[
H = H_\Gamma - \overline{H}_\Gamma = p^\lambda_i dy^i \wedge \omega - p^\lambda_i \Gamma^i_\lambda \omega - \overline{H}_\Gamma \omega. \tag{5.4}
\]

Moreover, every Hamiltonian form \( H (5.2) \) admits the canonical decomposition (5.12) as follows.

We agree to call any bundle morphism

\[
\Phi = dx^\lambda \otimes (\partial_\lambda + \Phi^i_\lambda (x^n, y^j, p^m_j) \partial_i) : \Pi \rightarrow J^1Y \tag{5.5}
\]

over \( Y \) the Hamiltonian map.
In particular, let $\Gamma$ be a connection on $Y \to X$. Then the composition
\[ \hat{\Gamma} = \Gamma \circ \pi_{\Pi Y} = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i) : \Pi \to Y \to J^1Y \] (5.6)
is a Hamiltonian map. Conversely, every Hamiltonian map $\Phi : \Pi \to J^1Y$ yields the associated connection
\[ \Gamma_\Phi = \Phi \circ \hat{0} = dx^\lambda \otimes (\partial_\lambda + \Phi^i_\lambda(x^\mu, y^j, 0)\partial_i) \] (5.7)
on $Y \to X$, where $\hat{0}$ is the global zero section of a Legendre bundle $\Pi \to Y$. In particular, we have $\Gamma_{\hat{\Gamma}} = \Gamma$.

**Theorem 5.2.** Every Hamiltonian map $\Phi$ (5.5) defines a Hamiltonian form
\[ H_{\Phi} = -\Phi \vert_{\Theta_{\Pi}} = p_i^\lambda dy^i \wedge \omega_\lambda - p_i^\lambda \Phi^i_\lambda \omega. \] (5.8)

**Proof.** Given an arbitrary connection $\Gamma$ on a fibre bundle $Y \to X$, the corresponding Hamiltonian map (5.6) defines the form $-\hat{\Gamma} \vert_{\Theta_{\Pi}}$ which is exactly the Hamiltonian form $H_{\Gamma}$ (3.13). Since $\Phi - \hat{\Gamma}$ is a $VY$-valued basic one-form on $\Pi \to X$, $H_{\Phi} - H_{\Gamma}$ is a density on $\Pi$. Then the result follows from item (ii) of Theorem 5.1.

The converse also is true.

**Theorem 5.3.** Every Hamiltonian form $H$ (5.2) on a Legendre bundle $\Pi \to Y$ yields the associated Hamiltonian map
\[ \hat{H} : \Pi \to J^1Y, \quad y^i_\lambda \circ \hat{H} = \partial_i^\lambda \mathcal{H}. \] (5.9)

**Proof.** In accordance with Theorem 6.2 below, any Hamiltonian form $H$ admits the associated Hamiltonian connection $\gamma_H$ (6.1) and defines the Hamiltonian map (6.3):
\[ \hat{H} = J^1\pi_{\Pi Y} \circ \gamma_H : \Pi \to J^1\Pi \to J^1Y. \] (5.10)

**Corollary 5.4.** Every Hamiltonian form $H$ (5.2) on a Legendre bundle $\Pi \to Y$ yields the associated connection (5.7):
\[ \Gamma_H = \hat{H} \circ \hat{0} = dx^\lambda \otimes (\partial_\lambda + \partial_i^\lambda \mathcal{H}(x^\mu, y^j, 0)\partial_i) \] (5.11)
on a fibre bundle $Y \to X$.

In particular, we have $\Gamma_{H_{\hat{\Gamma}}} = \Gamma$, where $H_{\Gamma}$ is the Hamiltonian form (3.13) associated to a connection $\Gamma$ on $Y \to X$.

**Corollary 5.5.** Every Hamiltonian form $H$ (5.2) admits the canonical splitting
\[ H = H_{\Gamma_H} - \tilde{H}. \] (5.12)
Remark 5.1. A Hamiltonian form is the main ingredient in PS Hamiltonian formalism because just it defines the covariant Hamilton equation (Section 6). Since Hamiltonian forms are the pull-back of the canonical multisymplectic form \([5.1]\) on the homogeneous Legendre bundle \(Z_Y\), one can treat the latter as a homogeneous momentum phase space of PS formalism by analogy with the cotangent bundle in time-dependent mechanics (Section 7).

6 Covariant Hamilton equations

Let \(\gamma\) \([3.4]\) be a Hamiltonian connection on a Legendre bundle \(\Pi \rightarrow X\).

Given a connection \(\Gamma\) on a fibre bundle \(Y \rightarrow X\), the local form \(H_U\) \([3.7]\) in the expression \([3.6]\) can be written as

\[
H_U = H_{\Gamma} - \tilde{H}_\Gamma \omega,
\]

where \(H_{\Gamma}\) is the Hamiltonian form \([5.3]\) and \(\tilde{H}_\Gamma \omega\) is a local density on \(\Pi\). In accordance with item (ii) of Theorem \([5.1]\) it follows that \(H_U\) is a local Hamiltonian form. Thus, we have proved the following.

**Theorem 6.1.** For every Hamiltonian connection \(\gamma\) \([3.4]\) on a Legendre bundle \(\Pi \rightarrow X\), there exists a local Hamiltonian form \(H_U\) in a neighborhood \(U\) of each point \(q \in \Pi\) such that

\[
\gamma|_{\Omega_Y} = dH_U.
\]

The converse is the following.

**Theorem 6.2.** Every Hamiltonian form \(H\) \([5.2]\) admits a Hamiltonian connection \(\gamma_H\) which obeys the condition

\[
\gamma_H|_{\Omega_Y} = dH, \quad \gamma_H^{\lambda} = \partial^{\lambda}_i \mathcal{H}, \quad \gamma_H^{\lambda}_{\lambda i} = -\partial^i \mathcal{H}.
\]

Proof. It is readily observed that the Hamiltonian form \(H\) \([5.2]\) is the Poincaré–Cartan form \([1.12]\) of an affine first order Lagrangian

\[
L_H = h_0(H) = (p^{\lambda}_i y^i_{\lambda} - \mathcal{H}) \omega
\]

on the jet manifold \(J^1 \Pi\). The Euler–Lagrange operator \([1.7]\) associated to this Lagrangian reads

\[
\delta L_H = \mathcal{E}_H : J^1 \Pi \rightarrow T^* \Pi \wedge (\wedge^{n-1} T^* X),
\]

\[
\mathcal{E}_H = [(y^i_{\lambda} - \partial^i \mathcal{H}) dp^{\lambda}_i - (p^{\lambda}_{\lambda i} + \partial_i \mathcal{H}) dy^i] \wedge \omega.
\]

It is called the Hamilton operator for \(H\). A glance at the expression \([6.3]\) shows that this operator is an affine morphism over \(\Pi\) of constant rank. It follows that its kernel

\[
y^i_{\lambda} = \partial^i \mathcal{H},
\]

\[
p^{\lambda}_{\lambda i} = -\partial_i \mathcal{H}
\]
is an affine closed imbedded subbundle of the jet bundle $J^1\Pi \to \Pi$. Therefore, it admits a global section $\gamma_H$ which thus is a desired Hamiltonian connection obeying the relation (6.1).

**Remark 6.1.** In fact, the Lagrangian (6.2) is the pull-back onto $J^1\Pi$ of a form $L_H$ on the product $\Pi \times J^1Y$.

It should be emphasized that, if $\dim X > 1$, there is a set of Hamiltonian connections associated to the same Hamiltonian form $H$. They differ from each other in soldering forms $\sigma$ on $\Pi \to X$ which fulfill the equation $\sigma[j_\Omega Y] = 0$.

**Example 6.2.** Any connection $\tilde{\Gamma}$ (3.9) for different connections $K$ (3.10) is a Hamiltonian connection for the Hamiltonian form $H$ (3.13).

Being a closed imbedded subbundle of the jet bundle $J^1\Pi \to X$, the kernel (6.4) – (6.5) of the Euler – Lagrange operator $\mathcal{E}_H$ (6.3) defines the Euler – Lagrange equation on $\Pi$. It is a first order dynamic equation called the covariant Hamilton equation.

Every integral section $r$ (i.e., $J^1r = \gamma_H \circ r$) of a Hamiltonian connection $\gamma_H$ associated to a Hamiltonian form $H$ is obviously a solution of the covariant Hamilton equation (6.4) – (6.5). Conversely, if $r : X \to \Pi$ is a global solution of the covariant Hamilton equation (6.4) – (6.5), there exists an extension of the local section

$$J^1r : r(X) \to \text{Ker } \mathcal{E}_H$$

to a Hamiltonian connection $\gamma_H$ which has $r$ as an integral section. Substituting $J^1r$ in (5.10), we obtain the equality

$$J^1(\pi_Y \circ r) = \hat{H} \circ r \tag{6.6}$$

for every solution $r$ of the covariant Hamilton equation (6.4) – (6.5) which thus is equivalent to this equation.

**Remark 6.3.** Similarly to the Cartan equation (2.5), the covariant Hamilton equation (6.4) – (6.5) is equivalent to the condition

$$r^*(u|dH) = 0 \tag{6.7}$$

for any vertical vector field $u$ on $\Pi \to X$.

**Remark 6.4.** The covariant Hamilton equation (6.4) – (6.5) has a standard form

$$S^\lambda_{ab}(x, \phi) \partial_\lambda \phi^b = f_a(x, \phi)$$

for the Cauchy problem or, to be more precise, for the general Cauchy problem since the coefficients $S^\lambda_{ab}$ depend on the variable functions $\phi$ in general [10, 40]. Here $\phi^b$ is a compact notation for variables $r^i$ and $r^\lambda_i$. However, the characteristic form

$$\det(S^\lambda_{ab}c_\lambda), \quad c_\lambda \in \mathbb{R},$$

of this system fails to be different from zero for any $c_\lambda$. One can overcome this difficulty as follows. Let us single out a local coordinate $x^1$ and replace the equation (6.4) with the equation

$$\partial_1 r^i = \partial_1^i \mathcal{H}, \quad d_1 \partial_\lambda^i \mathcal{H} = d_\lambda \partial_1^i \mathcal{H}, \quad \lambda \neq 1, \tag{6.8}$$

$$d_\mu = \partial_\mu + \partial_\mu r^i \partial_i + \partial_\mu r^\lambda_i \partial_\lambda^i.$$
The systems (6.8) and (6.5) have the standard form for the Cauchy problem with the initial conditions
\[ r^i(x') = \phi^i(x'), \quad r^i_\mu(x') = \phi^i_\mu(x'), \quad \partial_\lambda r^i = \partial^i_\lambda H, \quad \lambda \neq 1, \] (6.9)
on a local hypersurface \( S \) of \( X \) transversal to coordinate lines \( x^1 \). If \( r^i \) and \( r^i_\mu \) are solutions of the Cauchy problem (of class \( C^2 \)) for the equations (6.8) and (6.5) with the initial conditions (6.9), they satisfy the equation (6.4). Thus, in order to formulate the Cauchy problem for a covariant Hamilton equation in PS Hamiltonian formalism, one should single a one of the coordinates out and consider the system of equations (6.8) and (6.5).

7 Hamiltonian time-dependent mechanics

As was mentioned above, if \( X = R = \mathbb{R} \), we are in a case of time-dependent Hamiltonian mechanics which admits time reparametrization \( t \to t'(t) \) \cite{17}.

Let \( Q \to R \) be its configuration bundle coordinated by \((t, q^a)\). The corresponding Legendre bundle (1.15) is
\[ \Pi = V^*Q \otimes T^*R \otimes TR. \] (7.1)
It is endowed with holonomic coordinates \((t, q^a, p_a)\) possessing transition functions
\[ p'_a = \frac{\partial q^b}{\partial q'^a} p_b. \]
This transformation law is the same as that of fibre coordinates on the vertical cotangent bundle \( V^*Q \to Q \) of \( Q \to R \). Therefore, we have the canonical isomorphism \( \Pi = V^*Q \) (1.18) of the Legendre bundle \( \Pi \) (7.1) of time-reparametrized mechanics to a momentum phase space \( V^*Q \) of time-dependent mechanics over the time axis \( R = \mathbb{R} \) provided with the canonical Cartesian coordinate \( t \) with transition functions \( t' = t + \text{const.} \) \cite{17, 42, 44}. Accordingly, the homogeneous Legendre bundle \( Z_Q \) (1.22) of time-reparametrized mechanics is isomorphic to the homogeneous momentum phase space \( T^*Q \) of time-dependent mechanics over the time axis \( \mathbb{R} \). It is endowed with the holonomic coordinates \((t, q^a, p_0, p_a)\), possessing transition functions
\[ p'_a = \frac{\partial q^b}{\partial q'^a} p_b, \quad p'_0 = \left( p_0 + \frac{\partial q^b}{\partial t} p_b \right). \] (7.2)

Remark 7.1. Note that, relative to the Cartesian coordinate \( t \), the time axis \( \mathbb{R} \) is provided with the standard vector field \( \partial_t \) and the standard one-form \( dt \) which also is the volume element on \( \mathbb{R} \). As a consequence, there is the one-to-one correspondence between the vector fields \( f \partial_t \), the densities \( f dt \) and the real functions \( f \) on \( \mathbb{R} \). Roughly speaking, one can neglect the contribution of \( TR \) and \( T^*R \) to some expressions. In particular, the canonical imbedding (1.4) of \( J^1Q \) takes a form
\[ \lambda_{(1)} : J^1Q \ni (t, q^a, q'^a) \to (t, q^a, i = 1, q'^a = q'^a) \in TQ, \] (7.3)
\[ \lambda_{(1)} = dt = \partial_t + q'^a \partial_a. \]
In view of the morphism \( \lambda_{(1)} \) (7.3), any connection
\[ \Gamma = dt \otimes (\partial_t + \Gamma^a \partial_a) \] (7.4)
on a fibre bundle $Q \to \mathbb{R}$ can be identified with a nowhere vanishing horizontal vector field
\[
\Gamma = \partial_t + \Gamma^a \partial_a
\]
(7.5) on $Q$ which is the horizontal lift $\Gamma \partial_t$ of the standard vector field $\partial_t$ on $\mathbb{R}$ by means of the connection (7.4). Conversely, any vector field $\Gamma$ on $Q$ such that $dt|\Gamma = 1$ defines a connection on $Q \to \mathbb{R}$. Therefore, the connections (7.4) conventionally are identified with the vector fields (7.5). The integral curves of the vector field (7.5) coincide with the integral sections for the connection (7.4).

A homogeneous momentum phase space $T^*Q$ admits the canonical Liouville form
\[
\Xi_T = p_0 dt + p_a dq^a
\]
(7.6) (cf. (2.6)) and the canonical symplectic form
\[
\Omega_T = d\Xi = dp_0 \wedge dt + dp_a \wedge dq^a
\]
(7.7) (cf. (5.1)) together with the corresponding Poisson bracket
\[
\{f,g\}_T = \partial^0 f \partial_g g - \partial^0 g \partial_f f + \partial^a f \partial_a g - \partial^a g \partial_a f, \quad f,g \in C^\infty(T^*Q).
\]
(7.8)
This bracket yields the coinduced Poisson bracket (7.11) on a momentum phase space $V^*Q$ as follows.

There is the canonical one-dimensional affine bundle
\[
\zeta : T^*Q \to V^*Q
\]
(7.9) (cf. (1.25)). A glance at the transformation law (7.12) shows that it is a trivial affine bundle. Indeed, given a global section $h$ of $\zeta$, one can equip $T^*Q$ with a global fibre coordinate
\[
I_0 = p_0 - h, \quad I_0 \circ h = 0,
\]
possessing the identity transition functions. With respect to the coordinates $(t,q^a,I_0,p_a)$ the fibration (7.9) reads
\[
\zeta : \mathbb{R} \times V^*Q \ni (t,q^a,I_0,p_a) \to (t,q^a,I_0,p_a) \in V^*Q.
\]
(7.10)
Let us consider the subring of $C^\infty(T^*Q)$ which comprises the pull-back $\zeta^* f$ onto $T^*Q$ of functions $f$ on the vertical cotangent bundle $V^*Q$ by the fibration $\zeta$ (7.9). This subring is closed under the Poisson bracket (7.8). Then by virtue of the well-known theorem [17, 50], there exists the degenerate coinduced Poisson structure
\[
\{f,g\}_V = \partial^a f \partial_a g - \partial^a g \partial_a f, \quad f,g \in C^\infty(V^*Q),
\]
(7.11) on the vertical cotangent bundle $V^*Q$ such that
\[
\zeta^*\{f,g\}_V = \{\zeta^* f,\zeta^* g\}_T.
\]
(7.12)
The holonomic coordinates $(t,q^a,p_a)$ on $V^*Q$ are canonical for the Poisson structure (7.11).
With respect to the vertical Poisson bracket (7.11), the Hamiltonian vector fields of functions on $V^*Q$ read

$$\vartheta_f = \partial_i f \partial^i, \quad [\vartheta_f, \vartheta_{f'}] = \vartheta_{\{f, f'\}_V}, \quad f, f' \in C^\infty(V^*Q).$$

(7.13)

They are vertical vector fields on $V^*Q \to \mathbb{R}$. Accordingly, the characteristic distribution of the Poisson structure (7.11) is the vertical tangent bundle $VV^*Q \subset TV^*Q$ of a fibre bundle $V^*Q \to \mathbb{R}$. The corresponding symplectic foliation on the momentum phase space $V^*Q$ coincides with a fibration $V^*Q \to \mathbb{R}$.

One can think of the vertical Poisson bracket (7.11) as being the particular PS bracket (4.3). Indeed, in view of Remark 7.1, the algebra of multivector densities (4.2) comes to the ring $C^\infty(V^*Q)$, whereas the tangent-valued Liouville form (3.1) and the PS form (3.2) are associated to the pull-back forms

$$\Theta = h^*(\Xi_T \wedge dt) = p_i dq^i \wedge dt,$$

$$\Omega = h^*(\Omega_T \wedge dt) = dp_i \wedge dq^i \wedge dt$$

(7.14)

on $V^*Q$, where $h$ is some section of the trivial bundle (7.9). They are independent of the choice of $h$. With $\Omega$ (7.14), the Hamiltonian vector field $\vartheta_f$ (7.13) for a function $f$ on $V^*Q$ is given by the relation

$$\vartheta_f \lrcorner \Omega = -df \wedge dt,$$

while the vertical Poisson bracket (7.11) is written as

$$\{f, g\}_V dt = \vartheta_g \lrcorner \vartheta_f \lrcorner \Omega$$

similarly to the PS bracket (4.3).

In contrast with autonomous Hamiltonian mechanics, the Poisson structure (7.11) fails to provide any dynamic equation on a fibre bundle $V^*Q \to \mathbb{R}$ because Hamiltonian vector fields (7.13) of functions on $V^*Q$ are vertical vector fields, but not connections on $V^*Q \to \mathbb{R}$. Hamiltonian dynamics on $V^*Q$ is described as a particular PS Hamiltonian dynamics in Section 5 [17, 44].

A Hamiltonian on a momentum phase space $V^*Q \to \mathbb{R}$ of non-relativistic mechanics is defined as a global section

$$h : V^*Q \to T^*Q, \quad p_0 \circ h = \mathcal{H}(t, q^j, p_j),$$

(7.15)

of the affine bundle $\zeta$ (7.9). Given the Liouville form $\Xi_T$ (7.6) on $T^*Q$, this section yields the pull-back Hamiltonian form

$$H = (-h)^* \Xi = p_k dq^k - \mathcal{H} dt$$

(7.16)

on $V^*Q$. This is the well-known invariant of Poincaré–Cartan [1].

Given a Hamiltonian form $H$ (7.16), there exists a unique horizontal vector field (7.5):

$$\gamma_H = \partial_t - \gamma^i \partial_i,$$

on $V^*Q$ (i.e., a connection on $V^*Q \to \mathbb{R}$) such that

$$\gamma_H \lrcorner dH = 0.$$  

(7.17)
This vector field, called the Hamilton vector field, reads

$$\gamma_H = \partial_t + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k.$$  \hfill (7.18)

In a different way, a Hamilton vector field $\gamma_H$ is defined by the relation

$$\gamma_H \lfloor \Omega = dH$$

(cf. (6.1)). This vector field yields the first order dynamic Hamilton equation

$$q^k_t = \partial^k \mathcal{H}, \quad p_{tk} = -\partial_k \mathcal{H}$$  \hfill (7.19)

on $V^*Q \to \mathbb{R}$, where $(t, q^k, p_k, q^k_t, p_{tk})$ are the adapted coordinates on the first order jet manifold $J^1V^*Q$ of $V^*Q \to \mathbb{R}$. A solution of the Hamilton equation (7.19) is an integral section for the connection $\gamma_H$ (7.18).

### 8 Iso-PS structure

The canonical PS structure defined by the tangent-valued Liouville form $\Theta_Y$ (3.1) and the PS form $\Omega_Y$ (3.2) is by no means the unique PS structure on the Legendre bundle $\Pi$ (1.15). One can consider its following deformation [40].

Let

$$\psi = \psi^\lambda_\mu(x) dx^\mu \otimes \partial_\lambda$$

be a tangent-valued one-form on $X$ corresponding to some isomorphism of the tangent bundle $TX$ of $X$. Given the the tangent-valued Liouville form $\Theta_Y$ (3.1) and the PS form $\Omega_Y$ (3.2), let us consider their deformations

$$\Theta_\psi = \Theta_Y \lfloor \psi = \psi^\lambda_\mu(x) p^i \omega \wedge dy^i \wedge \partial_\lambda,$$

$$\Omega_\psi = \Omega_Y \lfloor \psi = \psi^\lambda_\mu(x) dp^i \wedge dy^i \wedge \omega \otimes \partial_\lambda,$$  \hfill (8.1) \hfill (8.2)

respectively. In comparison with the canonical forms $\Theta_Y$ and $\Omega_Y$, the forms $\Theta_\psi$ (8.1) and $\Omega_\psi$ (8.2) provide another PS structure on a Legendre bundle $\Pi$. We agree to call it the iso-PS structure. In particular, the relation

$$\Omega_\psi \lfloor \phi = d(\Theta_\psi \lfloor \phi)$$  \hfill (8.3)

(cf. (3.3)) holds for an arbitrary exterior one-form $\phi$ on $X$.

Building on the forms $\Theta_\psi$ and $\Omega_\psi$, one can develop iso-PS Hamiltonian formalism by analogy with the canonical PS one in Sections 3 – 5.

Let us say that that the connection $\gamma$ (3.4) on $\Pi \to X$ is an iso-Hamiltonian connection if an exterior form $\gamma \lfloor \Omega_\psi$ is closed.

An exterior $n$-form $H_\psi$ on $\Pi$ is called the iso-Hamiltonian form if there exists an iso-Hamiltonian connection $\gamma$ such that

$$\gamma \lfloor \Omega_\psi = dH_\phi.$$
The following assertion shows that sets of iso-Hamiltonian connections and iso-Hamiltonian forms on $\Pi$ are not empty.

**Theorem 8.1.** Let $\Gamma$ be the connection (3.8) on $Y \to X$ and $\tilde{\Gamma}$ (3.9) its lift onto $\Pi \to X$.

We have an iso-Hamiltonian form $$H_{\psi} = \psi_\mu^\lambda(p_i^\mu dy^i \wedge \omega_\lambda - \Gamma_i^\lambda p_i^\mu \omega)$$ and the associated iso-Hamiltonian connection $$\gamma = \Gamma + \frac{1}{n}(\psi^{-1})^\mu_\lambda(-\partial_\alpha \psi_\nu^\alpha - K^\alpha_\beta \psi_\nu^\beta + K_\nu^\alpha \psi_\alpha^\beta p_i^\nu dx^\lambda \otimes \partial_\mu^i).$$

Item (ii) of Theorem 5.1 also is extended to iso-Hamiltonian forms. The $n$ as an immediate consequence of Theorem 8.1, we obtain the following corollary.

**Corollary 8.2.** Any iso-Hamiltonian form is given by the expression $$H_{\psi} = H_{\psi} + \tilde{H}_{\psi} \omega = \psi_\mu^\lambda(p_i^\mu dy^i \wedge \omega_\lambda - \mathcal{H}_\psi \omega)$$ where $\Gamma$ is a connection on $Y \to X$.

By analogy with PS Hamiltonian formalism, one can introduce the Hamilton operator and obtain the covariant Hamilton equation associated to the iso-Hamiltonian form (8.4). For sections $r$ of a Legendre bundle $\Pi \to X$, this equation reads $$\psi_\mu^\lambda \partial_\lambda r^i = \partial_\mu H_{\psi}, \quad \partial_\lambda(\psi_\mu^\lambda r^i) = -\partial_i H_{\psi}.$$
where $H_{L^{-1}}$ is the Hamiltonian form (5.8) associated to the Hamiltonian map $\hat{L}^{-1}$. Let $s$ be a solution of the Euler–Lagrange equation (1.8) for a Lagrangian $L$. A direct computation shows that $\hat{L} \circ J^1 s$ is a solution of the covariant Hamilton equation (6.4)–(6.5) for the Hamiltonian form $H$ (9.1). Conversely, if $r$ is a solution of the covariant Hamilton equation (6.4)–(6.5) for the Hamiltonian form $H$ (9.1), then $s = \pi_H \circ r$ is a solution of the Euler–Lagrange equation (1.8) (see the equality (6.6)). Thus, one can state the following.

**Theorem 9.1.** In a case of hyperregular Lagrangians, Lagrangian and PS Hamiltonian formalisms are equivalent. □

Let now $L$ be an arbitrary Lagrangian on a configuration space $J^1Y$.

**Definition 9.2.** A Hamiltonian form $H$ is said to be associated to a Lagrangian $L$ if $H$ satisfies the relations

\[
\hat{L} \circ \hat{H} \circ \hat{L} = \hat{L},
\]

\[
H = H_{\hat{H}} + \hat{H}^* L.
\]

A glance at the relation (9.2) shows that $\hat{L} \circ \hat{H}$ is a projector

\[
p^{\mu}_{\mu}(p) = \partial^i_{\mu} L(x^\mu, y^i, \partial^j_{\lambda} \hat{H}(p)), \quad p \in N_L,
\]

from $\Pi$ onto the Lagrangian constraint space $N_L = \hat{L}(J^1Y)$ (11.19). Accordingly, $\hat{H} \circ \hat{L}$ is a projector from $J^1Y$ onto $\hat{H}(N_L)$.

**Definition 9.3.** A Hamiltonian form is called weakly associated to a Lagrangian $L$ if the condition (9.3) holds on a Lagrangian constraint space $N_L$. □

**Theorem 9.4.** If a Hamiltonian map $\Phi$ (5.5) obeys a relation

\[
\hat{L} \circ \Phi \circ \hat{L} = \hat{L},
\]

then a Hamiltonian form

\[
H = H_{\Phi} + \Phi^* L
\]

is weakly associated to a Lagrangian $L$. If $\Phi = \hat{H}$, then $H$ is associated to $L$. □

**Theorem 9.5.** Any Hamiltonian form $H$ weakly associated to a Lagrangian $L$ fulfills a relation

\[
H|_{N_L} = \hat{H}^* H_L|_{N_L},
\]

where $H_L$ is the Poincaré–Cartan form (1.12). □

**Proof.** The relation (9.3) takes a coordinate form

\[
\mathcal{H}(p) = p^{\mu}_{\mu} \partial^i_{\mu} \mathcal{H} - L(x^\mu, y^i, \partial^j_{\lambda} \hat{H}(p)), \quad p \in N_L.
\]

Substituting (9.4) and (9.8) in (5.2), we obtain the relation (9.7). •
The difference between associated and weakly associated Hamiltonian forms lies in the following. Let $H$ be an associated Hamiltonian form, i.e., the equality \( (9.8) \) holds everywhere on $\Pi$. Acting on this equality by the exterior differential, we obtain the relations

\[
\partial_{\mu} H(p) = -(\partial_{\mu} L) \circ \hat{H}(p), \quad p \in N_L, \\
\partial_{i} H(p) = -(\partial_{i} L) \circ \hat{H}(p), \quad p \in N_L, \\
(p_i^\mu - (\partial^\mu_i L)(x^\mu, y^i, \partial \lambda H))(\partial_{\mu} \partial_{\alpha} \hat{H}) = 0.
\]

The relation \((9.10)\) shows that the associated Hamiltonian form (i.e., the Hamiltonian map $\hat{H}$) is not regular outside a Lagrangian constraint space $N_L$.

**Example 9.1.** Any Hamiltonian form is weakly associated to a zero Lagrangian $L = 0$, while the associated one is only $H \Gamma$ (3.13).

**Example 9.2.** A hyperregular Lagrangian has a unique associated and weakly associated Hamiltonian form \((9.1)\). In a case of a regular Lagrangian $L$, the Lagrangian constraint space $N_L$ is an open subbundle of a vector Legendre bundle $\Pi \to Y$. If $N_L \neq \Pi$, a weakly associated Hamiltonian form fails to be defined everywhere on $\Pi$ in general. At the same time, $N_L$ itself can be provided with the pull-back PS structure with respect to the imbedding $N_L \to \Pi$, so that one may consider Hamiltonian forms on $N_L$.

One can say something more in a case of semiregular Lagrangians (Definition 1.1).

**Lemma 9.6.** The Poincaré–Cartan form $H_L$ for a semiregular Lagrangian $L$ is constant on the connected inverse image $\hat{L}^{-1}(p)$ of any point $p \in N_L$. \(\square\)

**Proof.** Let $u$ be a vertical vector field on an affine jet bundle $J^1Y \to Y$ which takes its values into the kernel of the tangent map $T\hat{L}$ to $\hat{L}$. Then the Lie derivative $L_u H_L$ of $H_L$ along $u$ vanishes. \(\bullet\)

A corollary of Lemma 9.6 is the following.

**Theorem 9.7.** All Hamiltonian forms weakly associated to a semiregular Lagrangian $L$ coincide with each other on a Lagrangian constraint space $N_L$, and the Poincaré–Cartan form $H_L$ (1.12) for $L$ is the pull-back

\[
H_L = \hat{L}^* H, \quad (\pi_i^\lambda y^\lambda_i - \mathcal{L}) \omega = \mathcal{H}(x^\mu, y^i, \pi_j^\mu) \omega,
\]

of any such a Hamiltonian form $H$. \(\square\)

**Proof.** Given a vector $v \in T_p \Pi$, the value $T\hat{H}(v)[H_L(\hat{H}(p))]$ is the same for all Hamiltonian maps $\hat{H}$ satisfying the relation \((9.2)\). Then the result follows from the relation \((9.7)\). \(\bullet\)

Theorem 9.7 enables us to relate the Euler–Lagrange equation for a semiregular Lagrangian $L$ with the covariant Hamilton equations for Hamiltonian forms weakly associated to $L$ [10, 37, 40].

**Theorem 9.8.** Let a section $r$ of $\Pi \to X$ be a solution of the covariant Hamilton equation \((6.4) – (6.5)\) for a Hamiltonian form $H$ weakly associated to a semiregular Lagrangian $L$. If $r$ lives in a Lagrangian constraint space $N_L$, a section $s = \pi_{1Y} \circ r$ of $Y \to X$ satisfies the Euler–Lagrange equation \((1.8)\), while its first order jet prolongation

\[
\bar{s} = \hat{H} \circ r = J^1 s
\]
obeys the Cartan equation (2.3) – (2.4).

\[ r = \tilde{\mathcal{L}} \circ \tilde{\pi}, \quad J^1 r = J^1 \tilde{\mathcal{L}} \circ J^1 \tilde{\pi}. \]

If \( r \) is a solution of the covariant Hamilton equation, the exterior form \( \mathcal{E}_H \) vanishes at points of \( J^1 r(X) \). Hence, the pull-back form \( \mathcal{E}_L = (J^1 \tilde{\mathcal{L}})^* \mathcal{E}_H \) vanishes at points \( J^1 \pi(X) \). By virtue of the relation (6.6), we have \( \pi = J^1 s \). Hence, \( s \) is a solution of the Euler – Lagrange equation.

The converse assertion is more intricate ([10], Proposition 4.5.11).

**Theorem 9.9.** Given a semiregular Lagrangian \( L \), let a section \( s \) of a jet bundle \( J^1 Y \to X \) be a solution of the Cartan equation (2.3) – (2.4). Let \( H \) be a Hamiltonian form weakly associated to \( L \) so that the associated Hamiltonian map satisfies a condition

\[ \tilde{\mathcal{H}} \circ \tilde{\mathcal{L}} \circ \pi = J^1 (\pi_0^1 \circ \pi). \]  

(9.12)

Then, a section

\[ r = \tilde{\mathcal{L}} \circ \pi, \quad r_i = \pi_i^1(x^n, \pi^j, \pi^j), \quad r_i = \pi_i, \]

of a Legendre bundle \( \Pi \to X \) is a solution of the Hamilton equation (6.4) – (6.5) for \( H \).

Being restricted to solutions of Euler – Lagrange equations, Theorem 9.9 comes to the following.

**Theorem 9.10.** Given a semiregular Lagrangian \( L \), let a section \( s \) of a fibre bundle \( Y \to X \) be a solution of the Euler – Lagrange equation (1.8) (i.e., \( J^1 s \) is a solution of the Cartan equation (2.3) – (2.4), and \( s = \pi_0^1 \circ J^1 s \)). Let \( H \) be a Hamiltonian form weakly associated to \( L \), and let \( H \) satisfy a relation

\[ \tilde{\mathcal{H}} \circ \tilde{\mathcal{L}} \circ J^1 s = J^1 s. \]

(9.13)

Then a section \( r = \tilde{\mathcal{L}} \circ J^1 s \) of a fibre bundle \( \Pi \to X \) is a solution of the covariant Hamilton equation (6.4) – (6.5) for \( H \).

**Example 9.3.** Let \( L = 0 \). This Lagrangian is semiregular. Its Euler – Lagrange equation comes to the identity \( 0 = 0 \). Every section \( s \) of a fibre bundle \( Y \to X \) is a solution of this equation. Given a section \( s \), let \( \Gamma \) be a connection on \( Y \) such that \( s \) is its integral section. The Hamiltonian form \( H_\Gamma (3.13) \) is associated to \( L = 0 \), and the Hamiltonian map \( \tilde{\mathcal{H}}_\Gamma \) satisfies the relation (9.13). The corresponding Hamilton equation has a solution

\[ r = \tilde{\mathcal{L}} \circ J^1 s, \quad r_i = s^i, \quad r_i^{\lambda} = 0. \]

In view of Theorem 9.10 one may try to consider a set of Hamiltonian forms associated to a semiregular Lagrangian \( L \) in order to exhaust all solutions of the Euler – Lagrange equation for \( L \).

**Definition 9.11.** Let us say that a set of Hamiltonian forms \( H \) weakly associated to a semiregular Lagrangian \( L \) is complete if, for each solution \( s \) of the Euler – Lagrange equation for

\[ s \]
there exists a solution \( r \) of the covariant Hamilton equation for a Hamiltonian form \( H \) from this set such that \( s = \pi_{HV} \circ r \).

A complete family of Hamiltonian forms associated to a given Lagrangian need not exist, or it fails to be defined uniquely. For instance, Example 9.3 shows that the Hamiltonian forms (3.13) constitute a complete family associated to the zero Lagrangian, but this family is not minimal.

By virtue of Theorem 9.10, a set of weakly associated Hamiltonian forms is complete if, for every solution \( s \) of the Euler–Lagrange equation for \( L \), there is a Hamiltonian form \( H \) from this set which fulfills the relation (9.13).

In a case of almost regular Lagrangians (Definition 1.1), one can formulate the following necessary and sufficient conditions of the existence of weakly associated Hamiltonian forms. An immediate consequence of Theorem 9.4 is the following.

**Theorem 9.12.** A Hamiltonian form \( H \) weakly associated to an almost regular Lagrangian \( L \) exists iff the fibred manifold \( J^1Y \to N_L \) (9.21) admits a global section.

**Proof.** A global section of \( J^1Y \to N_L \) can be extended to a Hamiltonian map \( \Phi : \Pi \to J^1Y \) which obeys the relation (9.5).

In particular, on an open neighborhood \( U \subset \Pi \) of each point \( p \in N_L \subset \Pi \), there exists a complete set of local Hamiltonian forms weakly associated to an almost regular Lagrangian \( L \). Moreover, one can always construct a complete set of local Hamiltonian forms associated to \( L \) (10), Proposition 4.5.14). At the same time, a complete set of associated Hamiltonian forms may exists when a Lagrangian is not necessarily semiregular (10, Example 4.5.12).

Given a global section \( \Psi \) of a fibred manifold

\[
\tilde{L} : J^1Y \to N_L,
\]

let us consider the pull-back form

\[
H_N = \Psi^*H_L = i_N^*H
\]

on \( N_L \) called the constrained Hamiltonian form. By virtue of Lemma 9.6, it does not depend on the choice of a section of the fibred manifold (9.14) and, consequently, \( H_L = \tilde{L}^*H_N \). For sections \( r \) of a fibre bundle \( N_L \to X \), one can write the constrained Hamilton equation

\[
r^*(u_N|dH_N) = 0,
\]

where \( u_N \) is an arbitrary vertical vector field on \( N_L \to X \). This equation possesses the following important properties.

**Theorem 9.13.** For any Hamiltonian form \( H \) weakly associated to an almost regular Lagrangian \( L \), every solution \( r \) of the covariant Hamilton equation which lives in a Lagrangian constraint space \( N_L \) is a solution of the constrained Hamilton equation (9.16).

**Proof.** Such a Hamiltonian form \( H \) defines a global section \( \Psi = \tilde{H} \circ i_N \) of the fibred manifold (9.14). Since \( H_N = i_N^*H \) due to the relation (9.11), the constrained Hamilton equation can be written as

\[
r^*(u_N|d\tilde{H}) = r^*(u_N|dH|_{NL}) = 0.
\]
Note that this equation differs from the Hamilton equation (6.7) restricted to $N_L$. This reads

$$r^*(u|dH|_{N_L}) = 0,$$  \hspace{1cm} (9.18)

where $r$ is a section of $N_L \to X$ and $u$ is an arbitrary vertical vector field on $\Pi \to X$. A solution $r$ of the equation (9.18) obviously satisfies the weaker condition (9.17).

**Theorem 9.14.** The constrained Hamilton equation (9.16) is equivalent to the Hamilton – De Donder equation (2.8).

**Proof.** It is readily observed that

$$\hat{L} = \pi_{Z\Pi} \circ \hat{H}_L.$$  

Hence, the projection $\pi_{Z\Pi} (1.25)$ yields a surjection of $Z_L$ onto $N_L$. Given a section $\Psi$ of the fibred manifold (9.14), we have a morphism

$$\hat{H}_L \circ \Psi : N_L \to Z_L.$$  

By virtue of Lemma (9.6), this is a surjection such that

$$\pi_{Z\Pi} \circ \hat{H}_L \circ \Psi = \text{Id}_{N_L}.$$  

Hence, $\hat{H}_L \circ \Psi$ is a bundle isomorphism over $Y$ which is independent of the choice of a global section $\Psi$. Combining (2.7) and (9.15) results in

$$H_N = (\hat{H}_L \circ \Psi)^* \Xi_L$$  

that leads to a desired equivalence.

This proof gives something more. Namely, since $Z_L$ and $N_L$ are isomorphic, the homogeneous Legendre map $\hat{H}_L (1.22)$ fulfils the conditions of Theorem 2.1. Then combining Theorem 2.1 and Theorem 9.14, we obtain the following.

**Theorem 9.15.** Let $L$ be an almost regular Lagrangian such that the fibred manifold (9.14) has a global section. A section $s$ of the jet bundle $J^1 Y \to X$ is a solution of the Cartan equation (2.5) iff $\hat{L} \circ \tilde{s}$ is a solution of the constrained Hamilton equation (9.16).

Theorem 9.15 also is a corollary of Lemma 9.16 below. The constrained Hamiltonian form $H_N$ (9.15) defines the constrained Lagrangian

$$L_N = h_0(H_N) = (J^1 i_N)^*L_H$$  \hspace{1cm} (9.19)

on the first order jet manifold $J^1 N_L$ of a fibre bundle $N_L \to X$.

**Lemma 9.16.** There are the relations

$$\bar{L} = (J^1 \hat{L})^* L_N, \quad L_N = (J^1 \Psi)^* \bar{L},$$  \hspace{1cm} (9.20)

where $\bar{L}$ is the first order Lagrangian (2.11).

The Euler – Lagrange equation for the constrained Lagrangian $L_N$ (9.19) is equivalent to the constrained Hamilton equation (9.16) and, by virtue of Lemma 9.16, is quasi-equivalent to the Cartan equation. At the same time, the Cartan equation of a non-regular Lagrangian system may contain an additional freedom in comparison with the constrained Hamilton equation (Section 12).
10 Lagrangian and Hamiltonian conservation laws

In order to study symmetries of PS Hamiltonian theory, let us use the fact that the Hamiltonian form $H$ is a Poincaré–Cartan form for the Lagrangian $L_H$ and that the covariant Hamilton equation for $H$ is the Euler–Lagrange equation for $L_H$.

We restrict our consideration to classical symmetries defined by projectable vector fields

$$u = u^\lambda \partial_\lambda + u^i \partial_i$$

(10.1)
on a fibre bundle $Y \to X$.

Let us start with Lagrangian conservation laws in first order Lagrangian formalism on a fibre bundle $Y \to X$.

The vector field $u$ admits the canonical decomposition into the horizontal and vertical parts

$$u = u_H + u_V = (u^\lambda \partial_\lambda + y_\lambda^i \partial_i^\lambda) + (u^i \partial_i - y_i^\lambda \partial^\lambda_i)$$

(10.2)
over $J^1Y$ and the first order jet prolongation

$$J^1u = u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_i^\lambda \partial^\lambda_i) \partial^\lambda_i$$

(10.3)
onto $J^1Y$.

Given the first order Lagrangian $L$, the global variational decomposition leads to the corresponding splitting of the Lie derivative $L_{J^1u}L$ of $L$ along $J^1u$:

$$L_{J^1u}L = u_V[E_L + d_H(h_0(u) H_L)],$$

(10.4)
$$\partial_\lambda u^\lambda \mathcal{L} + [u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_i^\lambda \partial^\lambda_i) \partial^\lambda_i] \mathcal{L} = (u^i - y_i^\lambda u^\lambda)E_i - d_\lambda [\pi^\lambda_i (u^\mu y^i_\mu - u^i) - u^\lambda \mathcal{L}],$$

where $\Xi_L = H_L$ is the Poincaré–Cartan form. If $u$ is an exact symmetry of $L$, i.e. $L_{J^1u}L = 0$ we obtain a weak conservation law

$$0 \approx -d_\lambda [\pi^\lambda_i (u^\mu y^i_\mu - u^i) - u^\lambda \mathcal{L}]$$

(10.5)
of a symmetry current

$$\mathcal{J}_u = [\pi^\lambda_i (u^\mu y^i_\mu - u^i) - u^\lambda \mathcal{L}] \omega_\lambda$$

(10.6)
along a vector field $u$ on the shell $E_L = 0$.

The weak conservation law leads to a differential conservation law

$$\partial_\lambda (\mathcal{J}_u^\lambda \circ s) = 0$$
on solutions $s$ of the Euler–Lagrange equation.

It is readily observed that the symmetry current $\mathcal{J}_u$ is linear in a vector field $u$. Therefore, one can consider a superposition of symmetry currents

$$\mathcal{J}_u + \mathcal{J}_{u'} = \mathcal{J}_{u+u'}, \quad \mathcal{J}_{cu} = c\mathcal{J}_u, \quad c \in \mathbb{R},$$

and a superposition of weak conservation laws associated to different symmetries $u$. 

26
For instance, let \( u = u^i \partial_i \) be a vertical vector field on \( Y \rightarrow X \). If it is a symmetry of \( L \), the weak conservation law \((10.5)\) takes a form

\[
0 \approx -d_\lambda (\pi^\lambda_i u^i).
\]

(10.7)

It is called the Noether conservation law of the Noether current

\[
\mathcal{J}_u = -\pi^\lambda_i u^i \omega_{\lambda}.
\]

(10.8)

along a Noether symmetry \( u \).

Given the connection \( \Gamma \) \([3.3]\) on a fibre bundle \( Y \rightarrow X \), a vector field \( \tau \) on \( X \) gives rise to the projectable vector field

\[
\Gamma \tau = \tau^\lambda (\partial_\lambda + \Gamma^i_\lambda \partial_i)
\]

(10.9)

on \( Y \). The corresponding symmetry current \((10.6)\) along \( \Gamma \tau \) reads

\[
\mathcal{J}_{\Gamma} = \tau^\mu \mathcal{J}_{\Gamma}^\lambda = \pi^\lambda (\pi^\mu_i (y^j_{\mu} - \Gamma^j_{\mu i}) - \delta^\lambda_\mu \mathcal{L}) \omega_{\lambda}.
\]

(10.10)

Its coefficients \( \mathcal{J}_{\Gamma}^\lambda \) are components of the tensor field

\[
\mathcal{J}_{\Gamma} = \mathcal{J}_{\Gamma}^\lambda dx^\mu \otimes \omega_\lambda, \quad \mathcal{J}_{\Gamma}^\lambda = \pi^\lambda (y^j_{\mu} - \Gamma^j_{\mu i}) - \delta^\lambda_\mu \mathcal{L},
\]

(10.11)

called the energy-momentum tensor relative to a connection \( \Gamma \) \([15, 41, 45]\). If \( \Gamma \tau \) \((10.9)\) is a symmetry of a Lagrangian \( L \), we have the energy-momentum conservation law

\[
0 \approx -d_\lambda [\pi^\lambda_i \tau^\mu (y^j_{\mu} - \Gamma^j_{\mu i}) - \delta^\lambda_\mu \tau^\mu \mathcal{L}].
\]

(10.12)

Turn now to PS Hamiltonian formalism on a Legendre bundle \( \Pi \). Given the projectable vector field \( u \) \((10.1)\) on \( Y \rightarrow X \), it gives rise to a vector field

\[
\bar{u} = u^\mu \partial_\mu + u^i \partial_i + (-\partial_i u^j \pi_\lambda^i - \partial_\mu u^\mu p^\lambda_i + \partial_\lambda \mu p_i^\mu) \partial_\lambda
\]

(10.13)

on a Legendre bundle \( \Pi \rightarrow Y \) and to a vector field

\[
J\bar{u} = \bar{u} + J^1 u
\]

(10.14)

on \( \Pi \times J^1 Y \). Then we have

\[
\mathbf{L}_{\bar{u}} H = \mathbf{L}_{J\bar{u}} L_H = (-u^i \partial_i \mathcal{H} - \partial_\mu (u^\mu \mathcal{H}) - u^i \partial^\lambda_\lambda \mathcal{H} + p^\lambda_i \partial_\lambda u^i) \omega.
\]

(10.15)

It follows that a Hamiltonian form \( H \) and a Lagrangian \( L_H \) have the same classical symmetries.

Let us apply the first variational formula \((10.4)\) to the Lie derivative \( \mathbf{L}_{J\bar{u}} L_H \) \((6.2)\). It reads

\[
- u^i \partial_i \mathcal{H} - \partial_\mu (u^\mu \mathcal{H}) - u^i \partial^\lambda_\lambda \mathcal{H} + p^\lambda_i \partial_\lambda u^i = -(u^i - y^j_{\mu} u^\mu)(p^\lambda_i + \partial_i \mathcal{H}) + (-\partial_i u^j \pi_\lambda^i - \partial_\mu u^\mu p^\lambda_i + \partial_\lambda \mu p_i^\mu - p_i^\mu u^\mu)(y^j_{\lambda} - \partial_\lambda \mathcal{H}) - d_\lambda [p^\lambda_i (\partial_\mu \mathcal{H} u^\mu - u^i) - u^i (p_i^\mu \partial_\mu \mathcal{H} - \mathcal{H})].
\]

On the shell \((6.4) - (6.5)\), this identity takes a form

\[
- u^i \partial_i \mathcal{H} - \partial_\mu (u^\mu \mathcal{H}) - u^i \partial^\lambda_\lambda \mathcal{H} + p^\lambda_i \partial_\lambda u^i \approx -d_\lambda [p^\lambda_i (\partial_\mu \mathcal{H} u^\mu - u^i) - u^i (p_i^\mu \partial_\mu \mathcal{H} - \mathcal{H})].
\]

(10.16)
If \( L_{\tilde{u}} H = 0 \), we obtain a weak Hamiltonian conservation law
\[
0 \approx -d_\lambda[p_i^\lambda(u^\mu \partial_\mu^i H - u^i) - u^\lambda(p_i^\mu \partial_\mu^i H - H)]
\] (10.17)
of a Hamiltonian symmetry current
\[
\tilde{J}_u = [p_i^\lambda(u^\mu \partial_\mu^i H - u^i) - u^\lambda(p_i^\mu \partial_\mu^i H - H)]\omega_\lambda.
\] (10.18)

In particular, let \( u = u^i \partial_i \) be a vertical vector field on \( Y \rightarrow X \). Then the Lie derivative \( L_\tilde{u} H \) (10.15) takes a form
\[
L_\tilde{u} H = (-u^i \partial_i H + \partial_i u^j p_j^i \partial_\mu^i H + p_i^\mu \partial_\lambda^i u^i)\omega.
\]

The corresponding Noether Hamiltonian current (10.18) reads
\[
\tilde{J}_u = -u^i p_i^\lambda \partial_\lambda \otimes \omega
\] (10.19)
(cf. \( J_u \) (10.8)). This is independent of a Hamiltonian form \( H \), and is defined only by a vertical vector field \( u \). It follows that Noether Hamiltonian currents \( \tilde{J}_u \) (10.19) in PS Hamiltonian theory constitute a real vector space \( \mathcal{J}(\Pi) \) isomorphic to that of vertical vector fields \( u \) on a fibre bundle \( Y \).

Moreover, due to the isomorphism (1.18), Noether Hamiltonian currents (10.19) are represented by \( TX \)-valued densities (4.2):
\[
\tilde{J}_u = -u^i p_i^\lambda \partial_\lambda \otimes \omega.
\] (10.20)

If \( Y \rightarrow X \) is a vector bundle, the PS bracket \( \{,\}_P \) (4.3) provides Noether Hamiltonian currents (10.20) with the Lie bracket
\[
[\tilde{J}_u, \tilde{J}_v] = \{\tilde{J}_u, \tilde{J}_v\}_P = \tilde{J}_{[u,v]}
\] (10.21)
which brings their space \( \mathcal{J}(\Pi) \) into a real Lie algebra, isomorphic to the Lie algebra of vertical vector fields on \( Y \). Similarly, the Poisson bracket \( \{,\}_V \) (7.11) defines a Lie bracket of Noether currents in mechanics [17].

Let now \( \tau = \tau^i \partial_i \) be a vector field on \( X \) and \( \Gamma \tau \) (10.9) its horizontal lift onto \( Y \) by means of a connection \( \Gamma \) on \( Y \rightarrow X \). Given the splitting (5.4) of a Hamiltonian form \( H \), the Lie derivative (10.16) reads
\[
L_{\tilde{\tau}} H = p_i^\lambda([\partial_\lambda + \Gamma_\lambda^i \partial_i, u]^j - [\partial_\lambda + \Gamma_\lambda^i \partial_i, u]^\mu \Gamma_\mu^j \omega - (\partial_\lambda u^\mu \partial_\mu^i H + u^j d\tilde{H}_i)\omega).
\]
where \([,\,]\) is the Lie bracket of vector fields. Then the weak identity (10.16) takes a form
\[
-(\partial_\mu u^\mu + \Gamma_\mu^j \partial_\mu^j) \partial_\lambda^i \omega + p_i^\lambda \Gamma_\lambda^\mu \approx -d_\lambda \tilde{\tau}^\lambda
\] (10.22)
where
\[
R = \frac{1}{2} R_{\lambda \mu} dx^\lambda \wedge dx^\mu \otimes \partial_i, \quad R_{\lambda \mu}^i = \partial_\lambda \Gamma_\mu^i - \partial_\mu \Gamma_\lambda^i + \Gamma_\lambda^j \partial_j^i \Gamma_\mu^\lambda - \Gamma_\mu^j \partial_j^i \Gamma_\lambda^\lambda
\]
is the curvature of a connection \( \Gamma \). The corresponding symmetry current (10.18) reads
\[
\tilde{J}^\lambda = \tau_\mu \tilde{\tau}^\lambda = \tau^\mu (p_i^\lambda \partial_\mu^i \tilde{H}_i - \partial_\mu^i (p_i^\mu \partial_\mu^i \tilde{H}_i - \tilde{H}_i)).
\] (10.22)
The relations (10.23) below show that, on a Lagrangian constraint space \( N_L \), the current (10.22) can be treated as a Hamiltonian energy-momentum current relative to a connection \( \Gamma \).

On solutions \( r \) of the covariant Hamilton equation (6.4) – (6.5), the weak equality (10.17) leads to a differential conservation law

\[
\partial_\lambda (\bar{J}^\lambda_u (r)) = 0.
\]

There is the following relation between differential conservation laws in Lagrangian and PS Hamiltonian formalisms.

**Theorem 10.1.** Let a Hamiltonian form \( H \) be associated to an almost regular Lagrangian \( L \). Let \( r \) be a solution of the covariant Hamilton equation (6.4) – (6.5) for \( H \) which lives in a Lagrangian constraint space \( N_L \). Let \( s = \pi_{\Pi Y} \circ r \) be the corresponding solution of the Euler – Lagrange equation for \( L \) so that the relation (9.13) holds. Then, for any projectable vector field \( u \) on a fibre bundle \( Y \to X \), we have

\[
\bar{J}^\lambda_u (r) = J^\lambda_u (\pi_{\Pi Y} \circ r), \quad \bar{J}^\lambda_u (\bar{L} \circ J^1 s) = J^\lambda_u (s),
\]

where \( J^\lambda u \) is the symmetry current (10.6) on \( J^1 Y \) and \( \bar{J}^\lambda u \) is the symmetry current (10.18) on \( \Pi \).

\( \square \)

**Proof.** The proof follows from the relations (9.4), (9.8) and (9.11). \( \bullet \)

By virtue of Theorems (9.8 – 9.10) it follows that:

- if \( J^\lambda u \) in Theorem (10.1) is a conserved symmetry current, then the symmetry current \( \bar{J}^\lambda u \) (10.23) is conserved on solutions of the Hamilton equation which live in a Lagrangian constraint space,

- if \( \bar{J}^\lambda u \) in Theorem (10.1) is a conserved symmetry current, then the symmetry current \( J^\lambda u \) (10.23) is conserved on solutions \( s \) of the Euler – Lagrange equation which obey the condition (9.13).

However, Theorem (10.1) fails to provide straightforward relations between symmetries of Lagrangians and associated Hamiltonian forms. In Section 12, we can obtain such relations between symmetries of constrained Lagrangians \( L_N \) (9.19) and original quadratic Lagrangians \( L \) (12.21) (Theorem 12.6).

### 11 Lagrangian and Hamiltonian Jacobi fields

The vertical extension of Lagrangian theory on a fibre bundle \( Y \to X \) onto the vertical tangent bundle \( VY \) of \( Y \to X \) describes the linear deviations of solutions of the Euler – Lagrange equation which are Jacobi fields [15, 46]. Accordingly, the vertical extension of PS Hamiltonian formalism on the Legendre bundle \( \Pi \) (1.15) onto the vertical Legendre bundle \( \Pi_{VY} \) (11.6) describes Jacobi fields of solutions of the covariant Hamilton equations [15, 30].

Let \( VY \) be the vertical tangent bundle of \( Y \to X \) endowed with holonomic coordinates \((x^\lambda, y^i, \dot{y}^i)\). The configuration space of first order Lagrangian theory on \( VY \to X \) is the jet manifold \( J^1 VY \). There is the canonical isomorphism

\[
J^1 VY_{J^1 Y} = V J^1 Y, \quad \dot{y}^i_{\lambda} = (\dot{y}^i)_{\lambda},
\]
where, in comparison with $V_Y J^1 Y$ in the expression (1.13), $V J^1 Y$ is the vertical tangent bundle of $J^1 Y \to X$ which is provided with holonomic coordinates $(x^\lambda, y^i, \dot{y}^i, \dot{y}_\lambda^i)$. Due to this isomorphism, first order Lagrangian formalism on $V Y$ can be developed as the vertical extension of Lagrangian theory on $Y$.

**Lemma 11.1.** Similar to the canonical isomorphism between fibre bundles $TT^* Z$ and $T^* T Z$ [25], the isomorphism

$$VV^* Y = V^* V Y, \quad p_i \leftrightarrow \dot{v}_i, \quad \dot{p}_i \leftrightarrow \dot{y}_i,$$

(11.1)

can be established by inspection of the transformation laws of holonomic coordinates $(x^\lambda, y^i, p_i = \dot{y}_i)$ on $V^* Y$ and $(x^\lambda, y^i, v^i = \dot{y}^i)$ on $V Y$.

It follows that any exterior form $\phi$ on a fibre bundle $Y$ gives rise to an exterior form

$$\phi_V = \partial_V (\phi) = \dot{y}^i \partial_i (\phi)$$

(11.2)
on $V Y$ so that $d\phi_V = (d\phi)_V$. For instance,

$$\partial_V f = \dot{y}^i \partial_i f, \quad \partial_V (dy^i) = d\dot{y}^i, \quad f \in C^\infty(Y).$$

The form $\phi_V$ (11.2) is called the vertical extension of $\phi$ on $Y$.

Let $L$ be the Lagrangian (1.6) on $J^1 Y$. Its vertical extension $L_V$ (11.2) onto $V J^1 Y$ (but not $V L$ (1.13) onto $V Y J^1 Y$) reads

$$L_V = \partial_V L = (\dot{y}^i \partial_i + \dot{y}_\lambda^i \partial_\lambda^i) L \omega.$$  

(11.3)

The corresponding Euler – Lagrange equation (1.8) takes a form

$$\dot{\delta}_i L_V = \delta_i L = 0,$$

(11.4)

$$\dot{\delta}_i L_V = \partial_V \delta_i L = 0,$$

(11.5)

$$\partial_V = \dot{y}^i \partial_i + \dot{y}_\lambda^i \partial_\lambda^i + \dot{y}_\mu^i \partial_\mu^i.$$

The equation (11.4) is exactly the Euler – Lagrange equation (1.8) for an original Lagrangian $L$. In order to clarify the meaning of the equation (11.3), let us suppose that $Y \to X$ is a vector bundle. Given a solution $s$ of the Euler – Lagrange equation (11.4), let $\delta s$ be a Jacobi field, i.e., $s + \varepsilon \delta s$ also is a solution of the Euler – Lagrange equation (11.4) modulo the terms of order $> 1$ in a small parameter $\varepsilon$. Then it is readily observed that a Jacobi field $\delta s$ satisfies the Euler – Lagrange equation (11.5), which therefore is called the variation equation of the equation (11.4) [6, 30, 46].

The Lagrangian $L_V$ (11.3) yields a Legendre map

$$\hat{L}_V : V J^1 Y \longrightarrow \Pi_{V Y} = V^* V Y \wedge V^* (^{n-1} T^* X),$$

(11.6)

where $\Pi_{V Y}$ is called the vertical Legendre bundle.

**Lemma 11.2.** Due to the isomorphism (11.1) there exists a bundle isomorphism

$$\Pi_{V Y} = V^* V Y, \quad p^\lambda_i \longleftrightarrow \dot{p}_i^\lambda, \quad q^\lambda_i \longleftrightarrow p^\lambda_i,$$

(11.7)
written with respect to the holonomic coordinates \((x^\lambda, y^i, \dot{y}^i, p_i^\lambda, q_i^\lambda)\) on \(\Pi_{VY}\) and \((x^\lambda, y^i, \dot{y}^i, p_i^\lambda, \dot{p}^\lambda_i)\) on \(V\Pi\).

In view of the isomorphism (11.7), the Legendre map (11.6) takes a form

\[
\begin{align*}
\hat{L}_V = V\hat{L} : VJ^1Y &\to \Pi_{VY} = V\Pi, \\
p_i^\lambda = \partial^*_i \mathcal{L}_V = \pi_i^\lambda, \quad \dot{p}^\lambda_i = \partial^*_i \mathcal{L} = \partial_V \pi_i^\lambda.
\end{align*}
\]

It is called the vertical Legendre map.

Let \(Z_{VY}\) be the homogeneous Legendre bundle (11.22) over \(VY\) endowed with the corresponding coordinates \((x^\lambda, y^i, \dot{y}^i, p_i^\lambda, q_i^\lambda, p)\). There is a fibre bundle

\[
\zeta : Z_Y \to Z_{VY}, \quad (x^\lambda, y^i, \dot{y}^i, p_i^\lambda, q_i^\lambda, p) \circ \zeta = (x^\lambda, y^i, \dot{y}^i, p_i^\lambda, q_i^\lambda, \dot{p}).
\]  

(11.9)

Then the vertical tangent morphism \(V\pi_{ZH} : Z_{VY} \to Z_Y \to \Pi_{VY} = V\Pi\).

Owing to this fact, one can develop PS Hamiltonian formalism on a momentum phase space \(\Pi_{VY}\) as the vertical extension of PS Hamiltonian theory on \(\Pi\). The corresponding canonical conjugate pairs are \((y^i, \dot{p}^\lambda_i)\) and \((\dot{y}^i, p_i^\lambda)\). In particular, due to the isomorphism (11.7), \(V\Pi\) is endowed with the canonical PS form (11.12) which reads

\[
\Omega_{VY} = [dp_i^\lambda \wedge dy^i + dp_i^\lambda \wedge d\dot{y}^i] \wedge \omega \otimes \partial_\lambda.
\]

Let \(Z_{VY}\) be the homogeneous Legendre bundle (11.22) over \(VY\) with the corresponding coordinates \((x^\lambda, y^i, \dot{y}^i, p_i^\lambda, q_i^\lambda, p)\). It can be endowed with the multisymplectic Liouville form \(\Xi_{VY}\). Sections of the affine bundle

\[
Z_{VY} \to V\Pi,
\]

(11.11)

by definition, provide Hamiltonian forms on \(V\Pi\).

Let us consider the following particular case of these forms which are related to those on a Legendre bundle \(\Pi\). Due to the fibre bundle (11.9):

\[
\zeta : Z_Y \to Z_{VY},
\]

the vertical tangent bundle \(VZ_Y\) of \(Z_Y \to X\) is provided with an exterior form

\[
\Xi_V = \zeta^* \Xi_{VY} = \dot{p}^\lambda_i dy^i + (p_i^\lambda d\dot{y}^i + p_i^\lambda \dot{y}^i) \wedge \omega_\lambda,
\]

which is exactly the vertical extension (11.12) of the canonical multisymplectic Liouville form \(\Xi\) on \(Z_Y\). Given the affine bundle \(Z_Y \to \Pi\) (1.25), we have the fibre bundle \(VZ_Y \to V\Pi\) (11.10) where \(V\pi_{ZH}\) is the vertical tangent map to \(\pi_{ZH}\). Let \(h\) be a section of an affine bundle \(Z_Y \to \Pi\) and \(H = h^* \Xi\) the corresponding Hamiltonian form (11.2) on \(\Pi\). Then a section \(Vh\) of the fibre bundle (11.10) and the corresponding section \(\zeta \circ Vh\) of the affine bundle (11.11) defines the Hamiltonian form

\[
\begin{align*}
H_V &= (Vh)^* \Xi_V = (p_i^\lambda d\dot{y}^i + p_i^\lambda d\dot{y}^i) \wedge \omega_\lambda - \mathcal{H}_V \omega, \\
\mathcal{H}_V &= \partial_V \mathcal{H}, \quad \partial_V = \dot{y}^i \partial_i + \dot{p}^\lambda_i \partial_{\lambda}^i.
\end{align*}
\]

(11.12)
on VII. It is called the vertical extension of \( H \) (or, simply, the vertical Hamiltonian form). In particular, given the splitting \( 11.1 \) of \( H \) with respect to a connection \( \Gamma \) on \( Y \to X \), we have the corresponding splitting

\[
\mathcal{H}_V = p^\lambda \Gamma^i_\lambda + \dot{y}^i \partial_i + \partial \tilde{\mathcal{H}}
\]

of \( H \) with respect to the canonical vertical prolongation

\[
VT : VY \to J^1 VY,
\]

\[
VT = dx^\lambda \otimes (\partial \lambda + \Gamma^i_\lambda \partial_i + \partial \tilde{\mathcal{H}})
\]

of \( \Gamma \) onto \( VY \to X \).

**Theorem 11.3.** Let \( \gamma \) \( 3.4 \) be a Hamiltonian connection on \( \Pi \) associated to a Hamiltonian form \( H \). Then its vertical prolongation \( V\gamma \) \( 11.13 \) on \( V\Pi \to X \) is a Hamiltonian connection associated to the vertical Hamiltonian form \( H_V \) \( 11.12 \).

**Proof.** The proof follows from a direct computation. We have

\[
V\gamma = \gamma + dx^\mu \otimes [\partial \gamma^i_\mu \dot{\partial}_i + \partial \gamma^\lambda_\mu \dot{\partial}_\lambda].
\]

Components of this connection obey the equation

\[
\dot{\gamma}^i_\mu = \partial \gamma^i_\mu \partial \mathcal{H} = \partial \gamma^\lambda_\mu \partial \mathcal{H}, \quad \dot{\gamma}^\lambda_\mu = -\partial \gamma^\lambda_\mu \partial \mathcal{H} = -\partial \gamma^\lambda_\mu \partial \mathcal{H}
\]

and the equation \( 11.4 \).

In order to clarify the meaning of the equation \( 11.14 \), let us suppose that \( Y \to X \) is a vector bundle. Given a solution \( r \) of the Hamilton equation \( 6.4 \) \(- 6.5 \) for \( H \), let \( \varphi \) be a Jacobi field, i.e., \( r + \varepsilon \varphi \) also is a solution of the same Hamilton equation modulo terms of order \( > 1 \) in \( \varepsilon \). Then it is readily observed that a Jacobi field \( \varphi \) satisfies the equation \( 11.14 \). At the same time, the Lagrangian \( L_{H_V} \) \( 6.2 \) on \( J^1 V \Pi \), defined by the Hamiltonian form \( H_V \) \( 11.12 \), takes a form

\[
L_{H_V} = h_0 (H_V) = p^\lambda_\mu (y^\lambda_\mu - \partial \gamma^\lambda_\mu \partial \mathcal{H}) - \dot{y}^i (p^\lambda_\mu \partial \gamma^\lambda_\mu \partial \mathcal{H}) + d\lambda (p^\lambda_i \dot{y}^i),
\]

where \( p^\lambda_\mu, \dot{y}^i \) play a role of the Lagrange multipliers.

In conclusion, let us study the relationship between the vertical extensions of Lagrangian and PS Hamiltonian formalisms. The Hamiltonian form \( H_V \) \( 11.12 \) on \( \Pi \) yields the vertical Hamiltonian map

\[
\hat{H}_V = V \hat{H} : V\Pi \to VJ^1 VY
\]

\[
y^\lambda_\mu = \partial \gamma^\lambda_\mu \partial \mathcal{H} = \partial \gamma^\lambda_\mu \partial \mathcal{H}, \quad \dot{y}^i = \partial \gamma^\lambda_\mu \partial \mathcal{H}.
\]

**Theorem 11.4.** Let a Hamiltonian form \( H \) on \( \Pi \) be associated to a Lagrangian \( L \) on \( J^1 Y \). Then the vertical Hamiltonian form \( H_V \) \( 11.12 \) is weakly associated to the Lagrangian \( L_V \) \( 11.3 \).

**Proof.** If the morphisms \( \hat{H} \) and \( \hat{L} \) obey the relation \( 9.2 \), then the corresponding vertical tangent morphisms satisfy the relation

\[
V \hat{L} \circ V \hat{H} \circ V_i N = V_i N.
\]

The condition \( 9.3 \) for \( H_V \) reduces to the equality \( 9.4 \) which is fulfilled if \( H \) is associated to \( L \).

\[
\bullet
\]
12 Quadratic Lagrangian and Hamiltonian systems

Field theories with almost regular quadratic Lagrangians admit comprehensive PS Hamiltonian formulation \[10, 11, 15\].

Given a fibre bundle \(Y \rightarrow X\), let us consider the quadratic Lagrangian \(L\) \[1.6\]:

\[
L = \frac{1}{2} \alpha^{\lambda \mu \nu \lambda} y^i \partial_{\lambda} y^j \partial_{\mu} y^i + b^i_\lambda y^i \partial_{\lambda} y^i + c(y^i, y^\nu), \tag{12.1}
\]

where \(a, b\) and \(c\) are local functions on \(Y\). This property is coordinate-independent due to the affine transformation law \[1.1\] of coordinates \(y^i_\lambda\). The associated Legendre map \(\hat{L}\) \[1.14\] is given by the coordinate expression

\[
p^i_\lambda \circ \hat{L} = a^{\lambda \mu \nu \lambda} y^j \partial_{\lambda} y^i + b^{i}_\lambda, \tag{12.2}
\]

and is an affine morphism over \(Y\). It yields the corresponding linear morphism

\[
\hat{a}: T^*X \otimes Y \rightarrow N_L \subset \Pi, \quad p^i_\lambda \circ \hat{a} = a^{\lambda \mu \nu \lambda} y^i \partial_{\lambda}, \tag{12.3}
\]

where \(\partial_{\lambda}\) are fibred coordinates on the vector bundle \[1.3\].

Let the Lagrangian \(L\) \[12.1\] be almost regular, i.e., the morphism \(\hat{a}\) \[12.3\] is of constant rank. Then the Lagrangian constraint space \(N_L\) \[12.2\] is an affine subbundle of the Legendre bundle \(\Pi \rightarrow Y\), modelled over the vector subbundle \(N_L\) \[12.3\] of \(\Pi \rightarrow Y\). Hence, \(N_L \rightarrow Y\) has a global section \(s\). For the sake of simplicity, let us assume that \(s = \hat{0}\) is the canonical zero section of \(\Pi \rightarrow Y\). Then \(N_L = N_L\). Accordingly, the kernel of the Legendre map \[12.2\] is an affine subbundle of the affine jet bundle \(J^1Y \rightarrow Y\), modelled over the kernel of the linear morphism \(\hat{a}\) \[12.3\]. Then there exists a connection \(\Gamma : Y \rightarrow \text{Ker} \hat{L} \subset J^1Y\),

\[
a^{\lambda \mu \nu \lambda} \Gamma^j_i + b^{j}_\lambda = 0, \tag{12.4}
\]

on \(Y \rightarrow X\). Connections \[12.4\] constitute an affine space modelled over a vector space of soldering forms

\[
\phi = \phi^i_\lambda dx^\lambda \otimes \partial_i
\]

on \(Y \rightarrow X\), satisfying the conditions

\[
\phi^{i}_\lambda \phi^j_\mu = 0 \tag{12.5}
\]

and, as a consequence, the conditions \(\phi^i_\lambda b^{j}_\lambda = 0\). If the Lagrangian \[12.1\] is regular, the connection \[12.4\] is unique.

**Remark 12.1.** If \(s \neq \hat{0}\), one can consider connections \(\Gamma\) taking their values into \(\text{Ker} \hat{L}\).

A matrix \(a\) in the Lagrangian \(L\) \[12.1\] can be seen as a global section of constant rank of a tensor bundle

\[
\Lambda^\lambda T^*X \otimes [\tilde{V}(TX \otimes V^*Y)] \rightarrow Y.
\]

Then it satisfies the following corollary of the well-known theorem on a splitting of a short exact sequence of vector bundles \[15\].

**Corollary 12.1.** Given a \(k\)-dimensional vector bundle \(E \rightarrow Z\), let \(a\) be a fibre metric of rank \(r\) in \(E\). There is a splitting

\[
E = \text{Ker} a \oplus E', \tag{12.6}
\]
where $E' = E/\text{Ker } a$ is the quotient bundle, and $a$ is a non-degenerate fibre metric in $E'$.

**Theorem 12.2.** There exists a linear bundle map

$$
\sigma : \Pi \to T^* X \otimes V Y, \quad f^\lambda \circ \sigma = \sigma^{ij}_{\lambda} \mu^\mu,
$$

(12.7)
such that $\mu \circ \sigma \circ i_N = i_N$.

*Proof.* The map (12.7) is a solution of algebraic equations

$$
a^\lambda_{ij} \sigma^{jk}_{\mu} \sigma^\alpha_{kb} = a^\lambda_{b}. 
$$

(12.8)

By virtue of Corollary 12.1 there exists the bundle splitting

$$
T X^* \otimes V Y = \text{Ker } a \oplus E',
$$

(12.9)

and an atlas of this bundle such that transition functions of Ker $a$ and $E'$ are mutually independent. Since $a$ is a non-degenerate section of

$$
\wedge^n T^* X \otimes (T E'^*) \to Y,
$$

there exist fibre coordinates $(\overline{y}^A)$ on $E'$ such that $a$ is brought into a diagonal matrix with non-vanishing components $a_{AA}$. Due to the splitting (12.9), we have the corresponding bundle splitting

$$
T X \otimes V^* Y = (\text{Ker } a)^* \oplus E'^*.
$$

Then a desired map $\sigma$ is represented by a direct sum $\sigma_1 \oplus \sigma_0$ of an arbitrary section $\sigma_1$ of a fibre bundle

$$
\wedge^n T X \otimes (\wedge^2 \text{Ker } a) \to Y
$$

and the section $\sigma_0$ of a fibre bundle

$$
\wedge^n T X \otimes (\wedge^2 E') \to Y
$$

which has non-vanishing components $\sigma^{AB} = (a_{AA})^{-1}$ with respect to the fibre coordinates $(\overline{y}^A)$ on $E'$. We have relations

$$
\sigma_0 = \sigma_0 \circ a \circ \sigma_0, \quad a \circ \sigma_1 = 0, \quad \sigma_1 \circ a = 0. \quad (12.10)
$$

•

**Remark 12.2.** Using the relations (12.10), one can write the above assumption, that the Lagrangian constraint space $N_L \to Y$ admits a global zero section, in the form

$$
b^\mu_i = a^\mu_{ij} \sigma^{jk}_{\mu} b^\nu_k. \quad (12.11)
$$

•
With the relations (12.4), (12.8) and (12.10), we obtain a splitting
\[ J^1Y = S(J^1Y) \oplus F(J^1Y) = \text{Ker} \tilde{L} \oplus \text{Im}(\sigma \circ \tilde{L}), \]  
(12.12)
where in fact, \( \sigma = \sigma_0 \) owing to the relations (12.10) and (12.11). Then with respect to the coordinates \( S_{\lambda} \) and \( F_{\lambda} \), the Lagrangian constraint space equations (12.4). Similarly to the splitting (12.12) of a configuration space \( J^1Y \), the Lagrangian brought into the form (12.14), factorizes through the covariant differential relative to any such connection.

Turn now to PS Hamiltonian formalism. Let \( L \) be an almost regular quadratic Lagrangian brought into the form (12.14), \( \sigma = \sigma_0 + \sigma_1 \) the linear map (12.7) and \( \Gamma \) the connection (12.4). Similarly to the splitting (12.12) of a configuration space \( J^1Y \), we have the following decomposition of a momentum phase space:
\[ \Pi = R(\Pi) \oplus P(\Pi) = \text{Ker} \sigma_0 \oplus N_L, \] 
(12.16)
where
\[ p_\lambda^i = R_\lambda^i + P_\lambda^i = [p_\lambda^i - a^{ij}_\lambda \sigma^{j \mu}_{\mu k' k}] + [a^{ij}_\lambda \sigma^{j \mu}_{k' k}]. \] 
(12.17)

The relations (12.10) lead to the equalities
\[ \sigma_{0 \mu a} R_\lambda^a = 0, \quad \sigma_{1 \mu a} P_\lambda^a = 0, \quad R_\lambda^i F_\mu^j = 0. \] 
(12.18)
Relative to the coordinates (12.17), the Lagrangian constraint space \( N_L \) is given by the equations
\[ R_\lambda^i = p_\lambda^i - a^{ij}_\lambda \sigma^{j \mu}_{\mu k' k} p_\lambda^a = 0. \] 
(12.19)

Let the splitting (12.9) be provided with adapted fibre coordinates \((\overline{y}^i, \overline{F}^A)\) such that the matrix function \( a \) is brought into a diagonal matrix with non-vanishing components \( a_{AA} \). Then the Legendre bundle \( \Pi \) is endowed with the dual (non-holonomic) fibre coordinates \((p_a, p_A)\) where \( p_A \) are coordinates on a Lagrangian constraint space \( N_L \), given by the equalities \( p_a = 0 \). Relative to these coordinates, \( \sigma_0 \) becomes the diagonal matrix
\[ \sigma_0^{AA} = (a_{AA})^{-1}, \quad \sigma_0^{aa} = 0, \] 
(12.20)
while \( \sigma_1^{aA} = \sigma_1^{AB} = 0 \). Let us write
\[ p_a = M_a^i p_\lambda^i, \quad p_A = M_A^i p_\lambda^i, \] 
(12.21)
where \( M \) are the matrix functions on \( Y \) which obeys the relations
\[ M_a^i a^{ij}_\lambda = 0, \quad (M^{-1})_{ij}^\lambda \sigma_0_{\mu i} = 0, \] 
\[ M_A^i (a \circ \sigma_0)_{\mu i} = M_A^i, \quad (M^{-1})_{j}^\mu M_A^i = a_{jk}^{\mu \nu} \sigma_0_{\nu i}. \] 
(12.22)
Let us consider the affine Hamiltonian map
\[ \Phi = \hat{\Gamma} + \sigma : \Pi \to J^1Y, \quad \Phi^i = \Gamma^i_\lambda + \sigma^i_\lambda \mu \rho^\mu_j, \] (12.23)
and the Hamiltonian form
\[ H(\Gamma, \sigma_1) = H_\Phi + \Phi^*L = p^i_\lambda dy^i \wedge \omega_\lambda - [\Gamma^i_\lambda p^\lambda_\mu + \frac{1}{2} \sigma^i_\lambda \rho^\mu_j - c^i_\omega](\omega_\lambda - (\mathcal{R}^i_\lambda + P^i_\lambda)dy^i) \]
(12.24)

\[ = (R^i_\lambda + P^i_\lambda)dy^i \wedge \omega_\lambda - [(R^i_\lambda + P^i_\lambda)\Gamma^i_\lambda + \frac{1}{2} \sigma^i_\lambda \rho^\mu_j + \sigma^i_\lambda \rho^\mu_j - c^i_\omega]. \]

**Theorem 12.3.** The Hamiltonian forms \( H(\Gamma, \sigma_1) \) (12.24) parameterized by connections \( \Gamma \) (12.4) are weakly associated to the Lagrangian (12.1), and they constitute a complete set. \( \square \)

**Proof.** By the very definitions of \( \Gamma \) and \( \sigma \), the Hamiltonian map (12.23) satisfies the condition (9.2). Then \( H(\Gamma, \sigma_1) \) is weakly associated to \( L \) (12.1) in accordance with Theorem 9.4. Let us write the corresponding Hamilton equation (6.4) for a section \( r \) of a Legendre bundle \( \Pi \to X \). It reads
\[ J^1s = (\hat{\Gamma} + \sigma) \circ r, \quad s = \pi_{HY} \circ r. \] (12.25)
Due to the surjections \( S \) and \( F \) (12.12), the Hamilton equation (12.26) is brought into the two parts
\[ S \circ J^1s = \Gamma \circ s, \quad \partial \lambda r^i - \sigma^i_\lambda (\omega_\mu \sigma^\mu_j \partial \mu r^j + b^\alpha_k) = \Gamma^i_\lambda \circ s, \] (12.26)
\[ F \circ J^1s = \sigma \circ r, \quad \sigma^i_\lambda (\omega_\mu \sigma^\mu_j \partial \mu r^j + b^\alpha_k) = \sigma^i_\lambda \rho^\alpha_k. \] (12.27)
Let \( s \) be an arbitrary section of \( Y \to X \), e.g., a solution of the Euler – Lagrange equation. There exists the connection \( \Gamma \) (12.24) such that the relation (12.26) holds, namely, \( \Gamma = S \circ \Gamma' \) where \( \Gamma' \) is a connection on \( Y \to X \) which has \( s \) as an integral section. It is easily seen that, in this case, the Hamiltonian map (12.23) satisfies the relation (9.13) for \( s \). Hence, the Hamiltonian forms (12.24) constitute a complete set. \( \bullet \)

It is readily observed that, if \( \sigma_1 = 0 \), then \( \Phi = \hat{H}(\Gamma) \), and the Hamiltonian forms \( H(\Gamma, \sigma_1 = 0) \) (12.24) are associated to the Lagrangian (12.1). For different \( \sigma_1 \), we have different complete sets of Hamiltonian forms (12.24). Hamiltonian forms \( H(\Gamma, \sigma_1) \) and \( H(\Gamma', \sigma_1) \) (12.24) of such a complete set differ from each other in the term \( \phi^i_\lambda \mathcal{R}^i_\lambda \), where \( \phi \) are the soldering forms (12.23). This term vanishes on the Lagrangian constraint space (12.19). Accordingly, the covariant Hamilton equations for different Hamiltonian forms \( H(\Gamma, \sigma_1) \) and \( H(\Gamma', \sigma_1) \) (12.24) differ from each other in the equations (12.26).

Since the Lagrangian constraint space \( N_L \) (12.19) is an imbedded subbundle of \( \Pi \to Y \), all Hamiltonian forms \( H(\Gamma, \sigma_1) \) (12.24) define a unique constrained Hamiltonian form \( H_N \) (9.13) on \( N_L \) which reads
\[ H_N = i^*_N H(\Gamma, \sigma_1) = P^\lambda_i dy^i \wedge \omega_\lambda - [P^\lambda_i \Gamma^i_\lambda + \frac{1}{2} \sigma^i_\lambda \rho^\mu_j - c^i_\omega](\omega_\lambda - (\mathcal{R}^i_\lambda + P^i_\lambda)dy^i). \] (12.28)
In view of the relations (12.18), the corresponding constrained Lagrangian \( L_N \) (9.19) on \( J^1N_L \) takes a form
\[ L_N = h_0(H_N) = (P^\lambda_i \mathcal{F}^i_\lambda - \frac{1}{2} \sigma^i_\lambda \rho^\mu_j + c^i_\omega) \omega. \] (12.29)
It is the pull-back onto $J^1N_L$ of a Lagrangian

$$L_{H(\Gamma,\sigma_1)} = R^\lambda_i(S^i_\lambda - \Gamma^i_\lambda) + P^\lambda_iF^i_\lambda - \frac{1}{2}\sigma^i_0\lambda\mu P^\lambda_iP^\mu_j - \frac{1}{2}\sigma^i_\lambda\lambda\mu R^\lambda_iR^\mu_j + c'$$  \hspace{1cm} (12.30)

on $J^1\Pi$ for any Hamiltonian form $H(\Gamma,\sigma_1)$ \hspace{1cm} (12.24).

In fact, the Lagrangian $L_N$ \hspace{1cm} (12.29) is defined on the product $N_L \times \Pi$ for any Hamiltonian form $H(\Gamma,\sigma_1)$ \hspace{1cm} (12.24). Since the momentum phase space $\Pi$ \hspace{1cm} (12.16) is a trivial bundle $\text{pr}_2: \Pi \to N_L$ over the Lagrangian constraint space $N_L$, one can consider the pull-back

$$L_\Pi = (P^\lambda_iF^i_\lambda - \frac{1}{2}\sigma^i_0\lambda\mu P^\lambda_iP^\mu_j + c')\omega$$  \hspace{1cm} (12.31)

of the constrained Lagrangian $L_N$ \hspace{1cm} (12.29) onto $\Pi \times J^1Y$.

In a case of quadratic Lagrangians, we can improve Theorem 9.13 as follows.

**Theorem 12.4.** For every Hamiltonian form $H$ \hspace{1cm} (12.24), the Hamilton equations \hspace{1cm} (6.5) and \hspace{1cm} (12.27) restricted to a Lagrangian constraint space $N_L$ are equivalent to the constrained Hamilton equation \hspace{1cm} (9.16).

**Proof.** Due to the splitting \hspace{1cm} (12.16), we have the corresponding splitting of the vertical tangent bundle $V\Pi$ of a Legendre bundle $\Pi \to X$. In particular, any vertical vector field $u$ on $\Pi \to X$ admits the decomposition

$$u = [u - u_{TN}] + u_{TN}, \quad u_{TN} = u^i\partial_i + a^i_\lambda u^{\mu}_\lambda u^\mu_\lambda \partial^i_\lambda,$$

such that $u_{TN} |_{N_L}$ is a vertical vector field on a Lagrangian constraint space $N_L \to X$. Let us consider the equations

$$r^*(u_{TN}|dH) = 0$$  \hspace{1cm} (12.32)

where $r$ is a section of $\Pi \to X$ and $u$ is an arbitrary vertical vector field on $\Pi \to X$. They are equivalent to the pair of equations

$$r^*(a^i_\lambda u^{\mu}_\lambda \partial^i_\lambda|dH) = 0,$$  \hspace{1cm} (12.33)

$$r^*(\partial_i|dH) = 0.$$  \hspace{1cm} (12.34)

The equation \hspace{1cm} (12.34) obviously is the Hamilton equation \hspace{1cm} (6.5) for $H$. Bearing in mind the relations \hspace{1cm} (12.21) and \hspace{1cm} (12.10), one can easily show that the equation \hspace{1cm} (12.33) coincides with the Hamilton equation \hspace{1cm} (12.27). The proof is completed by observing that, restricted to a Lagrangian constraint space $N_L$, the equation \hspace{1cm} (12.32) is exactly the constrained Hamilton equation \hspace{1cm} (9.17).

Theorem 12.4 shows that, restricted to a Lagrangian constraint space, the Hamilton equation for different Hamiltonian forms \hspace{1cm} (12.24) associated to the same quadratic Lagrangian \hspace{1cm} (12.1) differ from each other in the equations \hspace{1cm} (12.26). These equations are independent of momenta and play a role of the gauge-type conditions as follows.

By virtue of Theorem 9.13, the constrained Hamilton equation is quasi-equivalent to the Cartan equation. A section $\pi$ of $J^1Y \to X$ is a solution of the Cartan equation for an almost regular quadratic Lagrangian \hspace{1cm} (12.1) if $r = \hat{L} \circ \pi$ is a solution of the Hamilton equations \hspace{1cm} (6.5) and \hspace{1cm} (12.27). In particular, let $\pi$ be such a solution of the Cartan equation and $\pi_0$ a section of a
fibre bundle $T^*X \otimes \mathcal{V} \to X$ which takes its values into $\text{Ker} \mathcal{L}$ (see (12.3)) and projects onto a section $s = \pi^1_0 \circ \mathfrak{s}$ of $Y \to X$. Then the affine sum $\mathfrak{s} + \mathfrak{s}_0$ over $s(X) \subset Y$ is also a solution of the Cartan equation. Thus, we come to the notion of a gauge-type freedom of the Cartan equation for an almost regular quadratic Lagrangian $L$. One can speak of the gauge classes of solutions of the Cartan equation whose elements differ from each other in the above-mentioned sections $\mathfrak{s}_0$. Let $z$ be such a gauge class whose elements project onto a section $s$ of $Y \to X$. For different connections $\Gamma$ (12.4), we consider a condition $S \circ s = \Gamma \circ s$, $s \in z$. (12.35)

Lemma 12.5. (i) If two elements $\mathfrak{s}$ and $\mathfrak{s}'$ of the same gauge class $z$ obey the same condition (12.35), then $\mathfrak{s} = \mathfrak{s}'$. (ii) For any solution $\mathfrak{s}$ of the Cartan equation, there exists a connection (12.4) which fulfills the relation (12.26). Let us consider the affine sum $\mathfrak{s} + \mathfrak{s}'$ over $s(X)$ $\subset$ $Y$. This is a local section of the vector bundle $\text{Ker} \mathcal{L} \to Y$ over $s(X)$.

Proof. (i) Let us consider the affine difference $\mathfrak{s} - \mathfrak{s}'$ over $s(X) \subset Y$. We have $S(\mathfrak{s} - \mathfrak{s}') = 0$ iff $\mathfrak{s} = \mathfrak{s}'$. (ii) In the proof of Theorem 12.3, we have shown that, given $s = \pi^0_0 \circ \mathfrak{s}$, there exists the connection $\Gamma$ (12.4) which fulfills the relation (12.26). Let us consider the affine difference $\mathfrak{s}(s - J^1 s)$ over $s(X) \subset Y$. This is a local section of the vector bundle $\text{Ker} \mathcal{L} \to Y$ over $s(X)$. Let $\phi$ be its prolongation onto $Y$. It is easy to see that $\Gamma + \phi$ is a desired connection.

Due to the properties in Lemma 12.5, one can treat (12.35) as a gauge-type condition on solutions of the Cartan equation. The Hamilton equation (12.26) exemplifies this gauge-type condition when $s = J^1 s$ is a solution of the Euler – Lagrange equation. At the same time, the above-mentioned freedom characterizes solutions of the Cartan equation, but not of the Euler – Lagrange one. First of all, this freedom reflects the degeneracy of the Cartan equation (2.3). Therefore, e.g., in Hamiltonian gauge theory (Section 13), the above mentioned freedom is not related directly to the familiar gauge invariance. Nevertheless, the Hamilton equation (12.26) are not gauge invariant, and thus can play a role of gauge conditions in gauge theory.

Now let us study symmetries of the Lagrangians $L_N$ (12.29) and $L_{\Pi}$ (12.31) [2, 15]. We aim to show that, under certain conditions, they inherit Noether (i.e. vertical classical) symmetries of an original Lagrangian $L$ (12.14) (Theorems 12.6 – 12.7).

Let a vertical vector field $u = u^i \partial_i$ on $Y \to X$ be a classical (Noether) symmetry of the Lagrangian $L$ (12.14), i.e.,

$$L_{\mathcal{J}^1 u} L = (u^i \partial_i + d_\lambda u^i \partial^i_\lambda) \mathcal{L} \omega = 0.$$ (12.36)

Since

$$J^1 u(y^i_\lambda - \Gamma^i_\lambda) = \partial_k u^i(y^k_\lambda - \Gamma^k_\lambda),$$ (12.37)

one easily obtains from the equality (12.36) that

$$u^k \partial_k a^\mu_{ij} + \partial_i u^k a^\mu_{kj} + a^\lambda_{ik} \partial_j u^k = 0.$$ (12.38)

It follows that the summands of the Lagrangian (12.14) are invariant separately, i.e.,

$$J^1 u(a^\mu_{ij} \mathcal{F}^j_\mu) = 0, \quad J^1 u(c') = u^k \partial_k c' = 0.$$ (12.39)

The equalities (12.14), (12.37) and (12.38) give the transformation law

$$J^1 u(a^\mu_{ij} \mathcal{F}^j_\mu) = - \partial_i u^k a^\lambda_{kj} \mathcal{F}^j_\mu.$$ (12.40)
The relations (12.10) and (12.37) lead to the equality
\[ a_{ij}^\mu \left[ u^k \partial_k \sigma_{0j}^{\mu} - \partial_k u^j \sigma_{0i}^{\mu} - \sigma_{0i}^{\mu} \partial_k u^j \right] a_{\alpha\nu}^n = 0. \] (12.41)

Let us compare symmetries of the Lagrangian \( L \) (12.14) and the Lagrangian \( L_N \) (12.29). Given the Legendre map \( \tilde{L} \) (12.2) and the tangent morphism
\[ T\tilde{L} : TJ^1Y \to TN_L, \quad \dot{p}_A = (\dot{y}^i \partial_i + \dot{y}^k \partial_k)(M_{\lambda\mu}^{\lambda\mu} F^\mu_I), \]
let us consider the map
\[ T\tilde{L} \circ J^1u : J^1Y \ni (x^\lambda, y^i, \lambda^i) \rightarrow \]
\[ u^i \partial_i + [u^k \partial_k (M_{\lambda i}^{\lambda i} \lambda^i + M_{\lambda i}^{\lambda i} J^1u (a^\mu_{ij} F^\mu_I)) \partial A = \]
\[ u^i \partial_i + [u^k \partial_k (M_{\lambda i}^{\lambda i} \lambda^i + M_{\lambda i}^{\lambda i} J^1u (a^\mu_{ij} F^\mu_I))] \partial A = \]
\[ u^i \partial_i + [u^k \partial_k (a \circ \sigma_0)]_{j}^{\mu} \partial_i - (a \circ \sigma_0)]_{j}^{\mu} \partial_i + u^k \partial_k \lambda^i \partial_\mu \in TN_L, \]
where the relations (12.22) and (12.40) have been used. Let us assign to a point \((x^\lambda, y^i, \lambda^i) \in N_L\) some point
\[ (x^\lambda, y^i, \lambda^i) \in \tilde{L}^{-1}(x^\lambda, y^i, \lambda^i) \] (12.43)
and then the image of the point (12.43) under the morphism (12.42). We obtain the map
\[ v_N : (x^\lambda, y^i, \lambda^i) \rightarrow u^i \partial_i + [u^k \partial_k (a \circ \sigma_0)]_{j}^{\mu} \lambda^i \partial_\mu - (a \circ \sigma_0)]_{j}^{\mu} \partial_i k \partial_\mu \] (12.44)
which is independent of a choice of the point (12.43). Therefore, it is a vector field on a Lagrangian constraint space \( N_L \). This vector field gives rise to a vector field
\[ Jv_N = u^i \partial_i + [u^k \partial_k (a \circ \sigma_0)]_{j}^{\mu} \lambda^i \partial_\mu - (a \circ \sigma_0)]_{j}^{\mu} \partial_i k \partial_\mu + d_\lambda u^i \partial_\mu \] (12.45)
on \( N_L \times J^1Y \).

**Theorem 12.6.** The Lie derivative \( L_{Jv_N} L_N \) of the Lagrangian \( L_N \) (12.29) along the vector field \( Jv_N \) (12.45) vanishes, i.e., any Noether symmetry \( u \) of the Lagrangian \( L \) (12.14) yields the symmetry \( v_N \) (12.44) of the Lagrangian \( L_N \) (12.29).

**Proof.** One can show that
\[ v_N (P^\lambda_I) = -\partial_i u^k P^\lambda_k \] (12.46)
on the constraint space \( \mathcal{R}_i^\lambda = 0 \). Then the invariance condition \( Jv_N (\mathcal{L}_N) = 0 \) falls into the three equalities
\[ Jv_N (\sigma_{0ij}^{\mu} P^\lambda_I P^\mu_j) = 0, \quad Jv_N (P^\lambda F^\mu_I) = 0, \quad Jv_N (c') = 0. \] (12.47)
The latter is exactly the second equality (12.39). The first equality (12.47) is satisfied due to the relations (12.41) and (12.46). The second one takes a form
\[ Jv_N (P^\lambda(y_\lambda^i - \Gamma_\lambda^i)) = 0. \] (12.48)
It holds owing to the relations (12.37) and (12.46). 

39
Turn now to symmetries of the Lagrangian $L_\Pi$ (12.31). Since $L_\Pi$ is the pull-back of $L_N$ onto $\Pi \times J^1Y$, its symmetry must be an appropriate lift of the vector field $v_N$ (12.44) onto $\Pi$.

Given a vertical vector field $u$ on $Y \to X$, let us consider its canonical lift (10.13):

$$\tilde{u} = u^i \partial_i - \partial_i u^j p^\lambda_j \partial^\lambda,$$  \hspace{1cm} (12.49)

onto the Legendre bundle $\Pi$. It readily observed that the vector field $\tilde{u}$ is projected onto the vector field $v_N$ (12.44).

Let us additionally suppose that the one-parameter group of automorphisms of $Y$ generated by $u$ preserves the splitting (12.16), i.e., $u$ obeys the condition

$$u^k \partial_k (\sigma_0 l^m_\lambda e^\nu_m) + \sigma_0 l^m_\lambda a^\nu_m \partial_j u^k - \partial_k u^i l^k_{\lambda} e^\nu_m = 0.$$  \hspace{1cm} (12.50)

The relations (12.37) and (12.50) lead to the transformation law

$$J^1u(F^i_\mu) = \partial_j u^i F^j_\mu.$$  \hspace{1cm} (12.51)

**Theorem 12.7.** If the condition (12.50) holds, the vector field $\tilde{u}$ (12.49) is a symmetry of the Lagrangian $L_\Pi$ (12.31) iff $u$ is a Noether symmetry of the original Lagrangian $L$ (12.14). \hfill $\Box$

**Proof.** Due to the condition (12.50), the vector field $\tilde{u}$ (12.49) preserves the splitting (12.16), i.e.,

$$\tilde{u}(P^\lambda_i) = -\partial_i u^k p^\lambda_k, \hspace{1cm} \tilde{u}(R^\lambda_k) = -\partial_k u^i R^\lambda_i.$$  \hspace{1cm} (12.52)

The vector field $\tilde{u}$ gives rise to the vector field (10.14):

$$\tilde{J} \tilde{u} = u^i \partial_i - \partial_i u^j p^\lambda_j \partial^\lambda + d_\lambda u^i \partial^\lambda,$$  \hspace{1cm} (12.53)

on $\Pi \times J^1Y$, and we obtain the Lagrangian symmetry condition

$$(u^i \partial_i - \partial_j u^j p^\lambda_j \partial^\lambda + d_\lambda u^i \partial^\lambda)\mathcal{L}_\Pi = 0.$$  \hspace{1cm} (12.54)

It is readily observed that the first and third terms of a Lagrangian $L_\Pi$ are separately invariant due to the relations (12.39) and (12.51). Its second term is invariant owing to the equality (12.41).

Conversely, let the invariance condition (12.51) hold. It falls into the independent equalities

$$J \tilde{u}(\sigma_0 l^i_\mu p^\lambda_i) = 0, \hspace{1cm} J \tilde{u}(p^\lambda_i F^j_\lambda) = 0, \hspace{1cm} u^i \partial_i c' = 0,$$  \hspace{1cm} (12.55)

i.e., the Lagrangian $L_\Pi$ is invariant iff its three summands are separately invariant. One obtains at once from the second condition (12.55) that the quantity $F$ is transformed as the dual of momenta $p$. Then the first condition (12.55) shows that the quantity $\sigma_0 p$ is transformed by the same law as $F$. It follows that the term $aFF$ in the Lagrangian $L$ (12.14) is transformed as $a(\sigma_0 p)(\sigma_0 p) = \sigma_0 pp$, i.e., it is invariant. Then this Lagrangian is invariant due to the third equality (12.55). \hfill $\bullet$

**Remark 12.3.** At the same time, a Lagrangian $L_\Pi$ may possess additional non-classical symmetries which do not come from symmetries of an original Lagrangian $L$. For instance, let
us assume that \( Y \to X \) is an affine bundle modelled over a vector bundle \( Y \to X \). In this case, the Legendre bundle \( \Pi \) is isomorphic to the product

\[
\Pi = Y \times_X \left( \overline{\text{Ker}} \sigma_0 \oplus \overline{N}_L \right),
\]

such that transition functions of coordinates \( p^\lambda_i \) are independent of \( y^i \). Then the splitting (12.16) takes a form

\[
\Pi = Y \times_X \left( \overline{\text{Ker}} \sigma_0 \oplus \overline{N}_L \right),
\]

where \( \overline{\text{Ker}} \sigma_0 \) and \( \overline{N}_L \) are fibre bundles over \( X \) such that

\[
\overline{\text{Ker}} \sigma_0 = \pi^* \text{Ker} \sigma_0,
\]

and \( N_L = \pi^* \overline{N}_L \) are their pull-backs onto \( Y \). The splitting (12.56) keeps the coordinate form (12.17). The splittings (12.12) and (12.56) lead to the decomposition

\[
\Pi \times J^1 Y = \left( \overline{\text{Ker}} \sigma_0 \oplus \overline{N}_L \right) \times \left( \overline{\text{Ker}} \hat{L} \oplus \text{Im}(\sigma_0 \circ \hat{L}) \right).
\]

In view of this decomposition, let us associate to any section \( \xi \) of \( \overline{\text{Ker}} \sigma_0 \to X \) the vector field

\[
u_\Pi = \xi_a (M^{-1})^a_i \partial^i_\lambda,
\]

on \( \Pi \). Its lift (10.14) onto \( \Pi \times J^1 Y \) keeps the coordinate form

\[
J u_\Pi = \xi_a (M^{-1})^a_i \partial^i_\lambda.
\]

It is readily observed that the Lie derivative of the Lagrangian \( L_{\Pi} \) along the vector field (12.59) vanishes, i.e., \( u_\Pi \) is a symmetry of \( L_{\Pi} \). Moreover, the vector fields (12.58), parameterized by sections \( \xi \) of \( \overline{\text{Ker}} \sigma_0 \to X \), is a gauge symmetry of \( L_{\Pi} \). However, it does not come from symmetries of an original Lagrangian \( L \).

13 PS Hamiltonian gauge theory

Yang–Mills gauge theory of principal connections provides the most physically relevant example of a quadratic Lagrangian and PS Hamiltonian systems [10, 15]. The peculiarity of gauge theory lies in the fact that the splittings (12.12) and (12.16) of its configuration and momentum phase spaces are canonical.

Let \( P \to X \) be a principal bundle with a structure Lie group \( G \). Being \( G \)-equivariant, principal connections on \( P \to X \) are represented by sections of the affine bundle

\[
C = J^1 P/G \to X,
\]

called the bundle of principal connections. It is modelled over a vector bundle \( T^* X \otimes V_G P \), where \( V_G P = VP/G \to X \) is the fibre bundle in Lie algebras \( g \) of the group \( G \). Given a basis \( \{ \varepsilon_r \} \) for \( g \), we obtain local fibre bases \( \{ e_r \} \) for \( V_G P \). The connection bundle \( C \) is coordinated by \( (x^\mu, a^\nu_\mu) \) such that, written relative to these coordinates, sections \( A = A^\mu_\mu dx^\mu \otimes e_r \) of \( C \to X \) are familiar local connection one-forms, regarded as gauge potentials.
There is one-to-one correspondence between the sections \( \xi = \xi^r e_r \) of \( V_G P \to X \) and the vector fields on \( P \) which are infinitesimal generators of one-parameter groups of vertical automorphisms (gauge transformations) of \( P \). Any section \( \xi \) of \( V_G P \to X \) yields the vector field

\[
u_\xi = u^r_\mu \partial^\mu_r = (\partial_\mu \xi^r + c^r_{pq} a^p_\mu a^q_\lambda) \partial_\mu^r
\] (13.2)
on \( C \), where \( c^r_{pq} \) are the structure constants of a Lie algebra \( g \).

The configuration space of gauge theory is the first order jet manifold \( J^1 C \) equipped with the adapted coordinates \((x^\lambda, a^m_\lambda, a^m_\mu \lambda)\). This configuration space admits the canonical splitting

\[
J^1 C = C_+ \oplus C_- = C_+ \oplus (C \times \frac{2}{X} T^* X \otimes V_G P),
\] (13.3)
\[
a^r_\mu \lambda = \frac{1}{2}(a^r_\mu + a^r_\lambda - c^r_{pq} a^p_\mu a^q_\lambda) + \frac{1}{2}(a^r_\mu - a^r_\lambda + c^r_{pq} a^p_\mu a^q_\lambda),
\] with the corresponding projections

\[
S : J^1 C \to C_+ , \quad S^r_\mu \lambda = a^r_\mu \lambda + a^r_\lambda - c^r_{pq} a^p_\mu a^q_\lambda,
\] (13.4)
\[
F : J^1 C \to C_- , \quad F^r_\mu \lambda = a^r_\mu \lambda - a^r_\lambda + c^r_{pq} a^p_\mu a^q_\lambda,
\] (13.5)
where \( F \) is the strength of gauge fields.

Gauge theory of principal connections on \( P \to X \) is characterized by almost regular first order Yang–Mills Lagrangian

\[
L_{YM} = \frac{1}{4} a^G_{pq} g^{\lambda \mu} g^{\delta \nu} F^p_\lambda F^q_\mu \sqrt{|g|} \omega , \quad g = \det(g_{\mu \nu}),
\] (13.6)
on \( J^1 C \), where \( a^G \) is a non-degenerate \( G \)-invariant metric on the Lie algebra \( g_r \) and \( g \) is a non-degenerate world metric on \( X \). It possesses the gauge symmetries \( u_\xi \) (13.2). Their jet prolongation onto \( J^1 C \) read

\[
 J^1 u_\xi = (\partial_\mu \xi^r + c^r_{pq} a^p_\mu a^q_\lambda) \partial^\mu_r + (c^r_{pq} (a^p_\lambda \xi^q + a^p_\mu \partial_\lambda \xi^q) + \partial_\lambda \partial_\mu \xi^r) \partial^\mu_\lambda,
\] (13.7)
and we have transformation laws

\[
 J^1 u_\xi (F^r_\lambda) = c^r_{pq} F^p_\lambda \xi^q ,
 J^1 u_\xi (S^r_\mu \lambda) = c^r_{pq} a^p_\lambda \xi^q + c^r_{pq} a^p_\mu \partial_\lambda \xi^q + \partial_\lambda \partial_\mu \xi^r .
\] (13.8)

The Euler–Lagrange operator of the Yang–Mills Lagrangian \( L_{YM} \) (13.6) is

\[
\delta L_{YM} = \mathcal{E}_\nu \theta^\nu \wedge \omega = (\delta^\nu_\lambda d_\lambda + c^\nu_\lambda a^p_\lambda)(a^G_g g^{\mu \alpha} g^{\lambda \beta} F^q_{\alpha \beta} \sqrt{|g|}) \theta^\mu \wedge \omega .
\]

Its kernel defines the Yang–Mills equation

\[
\mathcal{E}^\nu_r = (\delta^\nu_\lambda d_\lambda + c^\nu_\lambda a^p_\lambda)(a^G_g g^{\mu \alpha} g^{\lambda \beta} F^q_{\alpha \beta} \sqrt{|g|}) = 0.
\] (13.9)

A momentum phase space of gauge theory is the Legendre bundle

\[
\pi_{HC} : \Pi \to C , \quad \Pi = \frac{\mathbb{A}}{C} T^* X \otimes TX \otimes [C \times C]^*,
\] (13.10)
endowed with holonomic coordinates \((x^\lambda, a^\rho_\lambda, p^\mu_\mu)\). The Legendre bundle \(\Pi\) (13.10) admits the canonical decomposition (12.16):

\[
\Pi = \Pi^+ \oplus C \Pi^-,
\]

(13.11)

\[
p^\mu_\mu = \mathcal{R}^{(\mu)} + \mathcal{P}^{(\mu)} = p^\mu_\mu + p^{[\mu]}_\mu = \frac{1}{2}(p^\mu_\mu + p^{\lambda}_\mu\gamma + \frac{1}{2}(p^\mu_\mu - p^{\lambda}_\mu).
\]

The Legendre map associated to the Lagrangian (13.6) takes a form

\[
p^{(\mu)}_\mu \circ \tilde{L}_{YM} = 0,
\]

(13.12)

\[
p^{[\mu]}_\mu \circ \tilde{L}_{YM} = a^G_{mn}g^\mu_\alpha g^\lambda_\beta f^n_{\alpha\beta}\sqrt{|g|}.
\]

(13.13)

A glance at this morphism shows that \(\text{Ker} \tilde{L}_{YM} = C^+\), and the Lagrangian constraint space is

\[
N_L = \tilde{L}_{YM}(J^1C) = \Pi^-.
\]

(13.14)

It is defined by the equation \(p^{(\mu)}_\mu = 0\) (13.12). Obviously, \(N_L\) is an imbedded submanifold of \(\Pi\), and the Lagrangian \(L_{YM}\) is almost regular.

Let us consider connections \(\Gamma\) on a fibre bundle \(C \to X\) which take their values into \(\text{Ker} \tilde{L}_{YM}\), i.e.,

\[
\Gamma : C \to C^+, \quad \Gamma_{\lambda\mu} - \Gamma^{\nu}_{\lambda\nu} + c^r_{pq}a^p_\lambda a^q_\mu = 0.
\]

(13.15)

Given a symmetric linear connection \(K\) (3.10) on \(T^*X\), every principal connection \(B\) on a principal bundle \(P \to X\) gives rise to a connection \(\Gamma_B : C \to C^+\) such that

\[
\Gamma_B \circ B = S \circ J^1B.
\]

It reads

\[
\Gamma_B^{\lambda\mu} = \frac{1}{2}[(\partial^\mu B^\lambda_\mu + \partial^\lambda B^\mu_\mu - c^r_{pq}a^p_\lambda a^q_\mu + c^r_{pq}(a^p_\lambda B^q_\mu + a^p_\mu B^q_\lambda))] - K^{\lambda\beta}_\mu(a^r_\beta - B^r_\beta).
\]

(13.16)

Given the connection (13.16), the corresponding Hamiltonian form (12.24):

\[
H_B = p^{\lambda\mu} da^r_\mu \wedge \omega_\lambda - p^{\lambda\mu} \Gamma_{\lambda\mu} \omega - \tilde{H}_{YM}\omega,
\]

(13.17)

\[
\tilde{H}_{YM} = \frac{1}{4}a^G_{mn}g^\mu_\alpha g^\lambda_\beta p^{[\mu]}_n p^{[\nu]}_m \sqrt{|g|},
\]

is associated to the Lagrangian \(L_{YM}\) (13.6). It is the Poincaré–Cartan form of a Lagrangian

\[
L_H = [p^{\mu\lambda}_r(a^r_\mu - \Gamma_{\lambda\mu}) - \tilde{H}_{YM}\omega]
\]

(13.18)

on \(\Pi \times J^1C\). The pull-back of any Hamiltonian form \(H_B\) (13.17) onto the Lagrangian constraint space \(N_L\) (13.14) is the constrained Hamiltonian form (9.15):

\[
H_N = i_N H_B = p^{[\lambda\mu]}(da^r_\mu \wedge \omega_\lambda + \frac{1}{2}c^r_{pq}a^p_\lambda a^q_\mu) - \tilde{H}_{YM}\omega.
\]

(13.19)

The corresponding constrained Lagrangian \(L_N\) on

\[
N_L \times J^1C = \Pi^- \otimes J^1C
\]

(13.20)

reads

\[
L_N = (p^{[\lambda\mu]}_r f^r_{\lambda\mu} - \tilde{H}_{YM}\omega).
\]

(13.21)
Its pull-back $L_\Pi$ onto $\Pi \times J^1C$ is

$$L_\Pi = (p^\lambda_\mu F^\mu_\lambda - \tilde{H}_{YM})\omega.$$  \hfill (13.22)

Note that, in contrast with the Lagrangian \textcolor{red}{[13.15]}, the constrained Lagrangian $L_N$ \textcolor{red}{[13.21]} possesses gauge symmetries as follows. Gauge symmetries $u_\zeta$ \textcolor{red}{(13.2)} of the Yang – Mills Lagrangian give rise to vector fields \textcolor{red}{(12.49)}:

$$\tilde{u}_\zeta = (\partial_\mu \zeta^\mu + c^\mu_{qp} p^\mu_\lambda \partial^q_{\lambda} - c^\mu_{qr} p^\mu_\lambda \partial^r_{\lambda}).$$  \hfill (13.23)

on $\Pi$. Vector fields $J^1u_\zeta$ \textcolor{red}{(13.7)} and $\tilde{u}_\zeta$ \textcolor{red}{(13.23)} provide gauge symmetries

$$J\tilde{u}_\zeta = J^1u + \tilde{u}$$  \hfill (13.24)

of the Lagrangians $L_N$ \textcolor{red}{(13.21)} and $L_\Pi$ \textcolor{red}{(13.22)} in accordance with Theorems \textcolor{red}{[12.6]} – \textcolor{red}{[12.7]}.

The Hamiltonian form $H_B$ \textcolor{red}{(13.17)} yields the covariant Hamilton equation which consist of the equation \textcolor{red}{(13.13)} and the equations

$$a_\mu^m = a_\mu^m = 2\Gamma^m_{\mu\phi}(\lambda\nu),$$  \hfill (13.25)

$$p^\lambda_\mu = c^q_{\mu} r^p_{\lambda} [\lambda\nu] - c^q_{\nu} B^p_\lambda (\lambda\mu) + K^\nu_{\lambda\mu} \mu p^p_{\lambda\nu}.$$  \hfill (13.26)

The Hamilton equations \textcolor{red}{(13.25)} and \textcolor{red}{(13.13)} are similar to the equations \textcolor{red}{(12.25)} and \textcolor{red}{(12.27)}, respectively. The Hamilton equations \textcolor{red}{(13.13)} and \textcolor{red}{(13.26)} restricted to the Lagrangian constraint space \textcolor{red}{(13.12)} are precisely the constrained Hamilton equation \textcolor{red}{(9.16)} for the constrained Hamiltonian form $H_N$ \textcolor{red}{(13.19)}, and they are equivalent to the Yang – Mills equation \textcolor{red}{(13.9)} for gauge potentials $A = \pi_{IC} \circ r$.

Different Hamiltonian forms $H_B$ lead to different equations \textcolor{red}{(13.25)}. This equation is independent of momenta and, thus, it exemplifies the gauge-type condition \textcolor{red}{(12.26)}:

$$\Gamma_B \circ A = S \circ J^1A.$$  \hfill (12.26)

A glance at this condition shows that, given a solution $A$ of the Yang – Mills equation, there always exists a Hamiltonian form $H_B$ (e.g., $H_B = A$) which obeys the condition \textcolor{red}{(9.13)}, i.e.,

$$\hat{H_B} \circ \hat{L}_{YM} \circ J^1A = J^1A.$$  \hfill (9.13)

Consequently, the Hamiltonian forms $H_B$ \textcolor{red}{(13.17)} parameterized by principal connections $B$ constitute a complete set.

It should be emphasized that the gauge-type condition \textcolor{red}{(13.25)} differs from the familiar gauge conditions in gauge theory which single out a representative of each gauge coset (with the accuracy to Gribov’s ambiguity). Namely, if a gauge potential $A$ is a solution of the Yang – Mills equation, there exists a gauge conjugate potential $A'$ which also is a solution of the Yang – Mills equation and satisfies a given gauge condition. At the same time, not every solution of the Yang – Mills equation is a solution of the system of the Yang – Mills equation and a certain gauge condition. In other words, there are solutions of the Yang – Mills equation which are not singled out by the gauge conditions known in gauge theory. In this sense, this set of gauge conditions is not complete. In gauge theory, this lack is not essential since one can think of all gauge conjugate
potentials as being physically equivalent, but not in the case of other non-regular Lagrangian systems, e.g., that of Proca fields \([10]\), Example 4.6.5).

In the framework of the PS Hamiltonian description of quadratic Lagrangian systems, there is a complete set of gauge-type conditions in the sense that, for any solution of the Euler–Lagrange equation, there exist constrained Hamilton equation equivalent to this Euler–Lagrange equation and a supplementary gauge-type condition which this solution satisfies.

In gauge theory where gauge conjugate solutions are treated physically equivalent, one may replace the equation (13.25) with a condition on the quantity

\[
(S \circ J^1 A)^\lambda_{\mu} = \frac{1}{2} (\partial_\lambda A^\mu_\nu + \partial_\nu A^\mu_\lambda - c_{\rho \sigma} A^\rho_\lambda A^\sigma_\mu),
\]

which supplements the Yang–Mills equation and plays a role of a gauge condition due to the gauge transformation law (13.8). In particular,

\[
g^\lambda_\mu (S \circ J^1 A)^\gamma_\mu = \alpha^\gamma(x) \tag{13.27}
\]

reovers the familiar generalized Lorentz gauge condition.

14 Affine Lagrangian and Hamiltonian systems

Let us turn now to a case of an affine Lagrangian system on a fibre bundle \(Y \to X\) whose Lagrangian is given by the coordinate expression

\[
L = L_\omega, \quad L = b^i_\lambda y^i_\lambda + c, \tag{14.1}
\]

where \(b\) and \(c\) are local functions on \(Y\). The corresponding Legendre map \(\hat{L} (1.14)\) takes a form

\[
p^\lambda_\iota \circ \hat{L} = b^\lambda_\iota. \tag{14.2}
\]

We have the commutative diagram

\[
J^1 Y \xrightarrow{\hat{L}} Q \subset \Pi
\]

\[
\begin{array}{c}
Y \\
\gamma \searrow b
\end{array}
\]

\[
b = b^\lambda_\iota \omega_\lambda \otimes dy^i,
\]

where \(Q = b(Y)\) is the image of a section \(b\) of a Legendre bundle \(\Pi \to Y\). Clearly, the Lagrangian (14.1) is almost regular without fail.

Let \(\Gamma\) be an arbitrary connection (3.8) on a fibre bundle \(Y \to X\), and let \(\hat{\Gamma}\) the associated Hamiltonian map (5.6). This Hamiltonian map satisfies the condition (9.5), where \(\hat{L}\) is the Legendre morphism (14.2). Let us consider the Hamiltonian form (9.6) corresponding to \(\hat{\Gamma}\). It reads

\[
H = H_\Gamma + L \circ \Gamma = p^\lambda_\iota dy^i \wedge \omega_\lambda - (p^\lambda_\iota - b^\lambda_\iota) \Gamma^i_\lambda \omega + c\omega, \tag{14.3}
\]

and is weakly associated to the affine Lagrangian (14.1). The corresponding Hamiltonian map

\[
y^\lambda_\iota \circ \hat{H} = \Gamma^i_\lambda \tag{14.4}
\]

45
coincides with \( \hat{\Gamma} \), i.e., \( H (14.3) \) is associated to \( L \).

The Hamiltonian form \( H (14.3) \) is affine in canonical momenta. It follows that the Hamilton equation (6.4) for \( H \) reduces to the gauge-type condition

\[
\partial_\lambda r^i = \Gamma^i_\lambda,
\]

whose solutions are integral sections of the connection \( \Gamma \).

Conversely, for each section \( s \) of a fibre bundle \( Y \to X \), there exists a connection \( \Gamma \) on \( Y \) whose integral section is \( s \). Then, the corresponding Hamiltonian map (14.4) obeys the condition (9.13). It follows that the Hamiltonian forms (14.3) parameterized by connections \( \Gamma \) on a fibre bundle \( Y \to X \) constitute a complete family.

The most physically relevant examples of affine Lagrangian and PS Hamiltonian systems are Dirac spinor fields and metric affine-gravitation theory [10, 38, 40].

15 Functional integral quantization

The fact that PS Hamiltonian system with the Hamiltonian form \( H (5.2) \) on a Legendre bundle \( \Pi \) is equivalent to a first-order Lagrangian system on \( \Pi \) with the Lagrangian \( L_H (6.2) \) enables us to quantize this PS Hamiltonian system in the framework of familiar perturbative quantum field theory [2, 39].

If there is no constraints and the matrix \( \partial^2 \mathcal{H}/\partial p^i_\mu \partial p^j_\nu \) is non-degenerate and positive-definite, this quantization is given by the generating functional

\[
Z = N^{-1} \exp \left\{ \int (L_H + \Lambda + iJ_iy^i + iJ_i^p p^i_\mu)dx \right\} \prod_x [dp(x)][dy(x)]
\]

of Euclidean Green functions, where \( \Lambda \) comes from the normalization condition

\[
\int \exp \left\{ \left( -\frac{1}{2} \partial_\mu \partial^\mu p^i_\nu \partial_\nu p^i_\mu + \Lambda \right)dx \right\} \prod_x [dp(x)] = 1.
\]

If a Hamiltonian \( \mathcal{H} \) is degenerate, the Lagrangian \( L_H (6.2) \) may admit gauge symmetries. In this case, integration of a generating functional along gauge group orbits must be finite. If there are constraints, the Lagrangian system with the Lagrangian \( L_H (6.2) \) restricted to a constraint space is quantized.

In order to verify this functional integral quantization scheme, we apply it to PS Hamiltonian systems associated to Lagrangian systems with quadratic Lagrangians (12.1). Note that, in the framework of perturbative quantum field theory, any Lagrangian is split into a sum of some quadratic Lagrangian (12.1) and an interaction term quantized as a perturbation.

For instance, let the Lagrangian (12.1) be hyperregular, i.e., the matrix function \( a \) is non-degenerate. Then there exists a unique associated Hamiltonian system whose associated Hamiltonian form \( H (9.1) \) is quadratic in momenta \( p^i_\mu \), and so is the corresponding Lagrangian \( L_H (6.2) \). If the matrix function \( a \) is positive-definite on an Euclidean space-time, the generating functional (15.1) is a Gaussian integral of momenta \( p^i_\mu (x) \). Integrating \( Z \) with respect to \( p^i_\mu (x) \), one restarts the generating functional of quantum field theory with the original Lagrangian (12.1). We extend this result to theories with almost regular Lagrangians \( L (12.1) \), e.g., Yang–Mills gauge theory. The key point is that, though such a Lagrangian \( L \) yields Lagrangian constraints \( N_L \),
and admits different associated Hamiltonian forms $H$, all the Lagrangians $L_H$ coincide on a constraint space $J^1Y \times_N N_L$, and we have a unique constrained Lagrangian system with a Lagrangian $L_N$, which is equivalent to the original one.

Let us quantize a Lagrangian system with the Lagrangian $L_N = \mathcal{L}_N \omega$ \eqref{eq:12.29} on a product $J^1Y \times_N N_L$. In the framework of a perturbative quantum field theory, we should assume that $X = \mathbb{R}^n$ and $Y \to X$ is a trivial affine bundle. It follows that both the original coordinates $(x^i, y^i, p_i^i)$ and the adapted coordinates $(x^i, y^i, p_a, p_A)$ on the Legendre bundle $\Pi$ are global. Passing to field theory on an Euclidean space $\mathbb{R}^n$, we also assume that the matrix $a$ in the Lagrangian $L$ \eqref{eq:12.14} is positive-definite, i.e., $a_{AA} > 0$.

Let us start with the Lagrangian \eqref{eq:12.29} without gauge symmetries. Since a Lagrangian of Euclidean Green functions of a Lagrangian system in question reads

$$Z = N^{-1} \int \exp \{ \int (\mathcal{L}_N + \frac{1}{2} \text{tr} \ln \sigma_0 + iJ_iy^i + iJ^A p_A) \omega \} \prod_x [dp_a(x)][dy(x)],$$

where $\mathcal{L}_N$ with respect to the adapted coordinates is given by an expression

$$\mathcal{L}_N = M^{-1} \sum_A (a_{AA})^{-1} (p_A)^2 + c',$$

and $\sigma_0$ is a square matrix

$$\sigma_0^{AB} = M^{-1} i^A M^{-1} \sigma_0^{ij} \delta^{AB} (a_{AA})^{-1}.$$

The generating functional \eqref{eq:15.2} is a Gaussian integral of functional variables $p_A(x)$. Its integration with respect to $p_A(x)$ under the condition $J^A = 0$ restarts a generating functional

$$Z = N^{-1} \int \exp \{ \int (\mathcal{L} + iJ_iy^i) \omega \} \prod_x [dy(x)],$$

of the original Lagrangian field system on $Y$ with the Lagrangian \eqref{eq:12.14}. However, the generating functional \eqref{eq:15.2} can not be rewritten with respect to the original variables $p_i^a$, unless $a$ is a non-degenerate matrix function.

In order to overcome this difficulty, let us consider a Lagrangian system on the whole Legendre manifold $\Pi$ with the Lagrangian $L_\Pi$ \eqref{eq:12.31}. Since this Lagrangian is constant along the fibres of a vector bundle $\Pi \to N_L$, the integration of the generating functional of this field model with respect to variables $p_a(x)$ should be finite. One can choose the generating functional in a form

$$Z = N^{-1} \int \exp \{ \int (\mathcal{L}_{\Pi} - \frac{1}{2} \sigma_1^{ij} p_i^a p_j^a + \frac{1}{2} \text{tr} \ln \sigma + iJ_iy^i + iJ^A p_A) \omega \} \prod_x [dp_a(x)][dy(x)].$$

Its integration with respect to momenta $p^A(x)$ restarts the generating functional \eqref{eq:15.3} of the original Lagrangian system on $Y$.

**Remark 15.1.** Strictly speaking, since a Lagrangian $L_\Pi$ may possess gauge symmetries (Remark \ref{eq:12.3}), in order to obtain the generating functional \eqref{eq:15.3}, one can follow a procedure of quantization of gauge-invariant Lagrangian systems. In a case of the Lagrangian $L_\Pi$ \eqref{eq:12.31}, this procedure is rather trivial because the space of momenta variables $p_a(x)$ coincides with the translation subgroup of the gauge group $\text{Aut Ker} \sigma_0$. 

47
Now let us suppose that the Lagrangian $L_\Pi$ (Theorem 12.7) is invariant under some gauge group $G_X$ of vertical automorphisms of a fibre bundle $Y \to X$ (and the induced automorphisms of $\Pi \to X$) which acts freely on a space of sections of $Y \to X$. Its infinitesimal generators are represented by vertical vector fields $u = u^i(x^\mu, y^j)\partial_i$ on $Y \to X$ which give rise to the vector fields $\pi = J\tilde{u}$ (12.53):

$$\pi = u^i\partial_i - \partial_j u^i p^i_{\mu}\partial^\mu + d_\lambda u^\lambda \partial_i, \quad d_\lambda = \partial_\lambda + y^i\partial_i,$$

(15.5)
on $\Pi \times J^1Y$. Let us also assume that $G_X$ is indexed by $m$ parameter functions $\xi^\tau(x)$ such that $u = u^i(x^\lambda, y^j, \xi^\tau)\partial_i$, where

$$u^i(x^\lambda, y^j, \xi^\tau) = u^i(x^\lambda, y^j)\xi^\tau + u^i_\mu(x^\lambda, y^j)\partial_\mu \xi^\tau$$

(15.6)
are linear first order differential operators on a space of parameters $\xi^\tau(x)$ 15.16. The vector fields $u(\xi^\tau)$ must satisfy the commutation relations

$$[u(\xi^q), u(\xi^p)] = u(c^r_{pq}\xi^p, \xi^q),$$

where $c^r_{pq}$ are structure constants. The Lagrangian $L_\Pi$ (12.31) is invariant under the above mentioned gauge transformations iff its Lie derivative $L_\pi L_\Pi$ along vector fields (15.5) vanishes, i.e.,

$$(u^i \partial_i - \partial_j u^i p^i_{\mu}\partial^\mu + d_\lambda u^\lambda \partial_i)L_\Pi = 0.$$ (15.7)

Since the operator $L_\pi$ is linear in momenta $p^i_\mu$, the condition (15.7) falls into the independent conditions (12.55). It follows that a Lagrangian $L_\Pi$ is gauge-invariant iff its three summands are separately gauge-invariant.

Since $S^i_\lambda = y^i_\lambda - F^i_\lambda$, one can easily derive from the formula (12.51) the transformation law

$$\pi(S^i_\mu) = d_\mu u^i - \partial_j u^i F^i_\mu = d_\mu u^i - \partial_j u^i(y^j_\mu - S^j_\mu) = \partial_\mu u^i + \partial_j u^i S^j_\mu$$

(15.8)
of $S$. A glance at this expression shows that the gauge group $G_X$ acts freely on a space of sections $S(x)$ of the fibre bundle Ker $\hat{L} \to Y$ in the splitting (12.11). Let the number $m$ of parameters of a gauge group $G_X$ do not exceed the fibre dimension of Ker $\hat{L} \to Y$. Then some combinations $b^r_\mu S^i_\mu$ of $S^i_\mu$ can be used as the gauge condition

$$b^r_\mu S^i_\mu(x) - \alpha^r(x) = 0,$$

similar to the generalized Lorentz gauge (13.27) in Yang–Mills gauge theory.

Turn now to quantization of a Lagrangian system with the gauge-invariant Lagrangian $L_\Pi$ (12.31). In accordance with the well-known quantization procedure, let us modify the generating functional (15.4) as follows

$$Z = \mathcal{N}^{-1} \int \exp\left\{\int (L_\Pi - \frac{1}{2}\sigma_1^{ij}\sigma_{\mu
u}^j p^i_\nu + \frac{1}{2}\tr \ln \sigma - \frac{1}{2}h_{rs}\alpha^r\alpha^s + iJ_i y^i + iJ^i_\mu p^i_\mu)\omega\right\}$$

$$\Delta \prod_x \delta(b^r_\mu S^i_\mu(x) - \alpha^r(x))[da(x)][dp(x)][dy(x)] =$$

$$\mathcal{N}^{-1} \int \exp\left\{\int (L_\Pi - \frac{1}{2}\sigma_1^{ij}\sigma_{\mu
u}^j p^i_\nu + \frac{1}{2}\tr \ln \sigma - \frac{1}{2}h_{rs}b^r_\mu b^s_\lambda S^i_\mu S^j_\lambda + iJ_i y^i + iJ^i_\mu p^i_\mu)\omega\right\}$$

$$\Delta \prod_x [dp(x)][dy(x)],$$

(15.9)
where
\[
\int \exp \{ \int \left( -\frac{1}{2} h_{rs} \alpha^r \alpha^s \right) \omega \} \prod_x [d\alpha(x)]
\]
is a Gaussian integral, and the factor $\Delta$ is defined by the condition
\[
\Delta \int \prod_x \delta(u(\xi)(b^{\mu}_i S^i_{\mu})_x)] [dx] = 1.
\]
We have the linear second order differential operator
\[
M^\mu_\nu \xi^\nu = u(\xi)(b^{\mu}_i S^i_{\mu}(x)) = b^{\mu}_i (\partial_\mu u^i(\xi) + \partial_j u^i(\xi) S^i_j)
\]  \hspace{1cm} (15.10)
on the parameter functions $\xi(x)$, and obtain $\Delta = \det M$. Then the generating functional \hspace{1cm} (15.9)
takes a form
\[
Z = N^{n+1} \int \exp \{ \int (\mathcal{L} - \frac{1}{2} h_{rs} b^{\mu}_i b^{\nu}_j S^i_{\mu} \bar{S}^j_{\nu} - \tau_r M^\mu_\nu c^\nu + iJ_i y^i + iJ^i_\mu p^i_\mu) \omega \} \prod_x [d\tau][dc][d\rho(x)][dy(x)],
\]  \hspace{1cm} (15.11)
where $\tau_r$, $c^\nu$ are odd ghost fields. Integrating $Z$ \hspace{1cm} (15.11) with respect to momenta under the condition $J^i_\mu = 0$, we come to the generating functional
\[
Z = N^{n+1} \int \exp \{ \int (\mathcal{L} - \frac{1}{2} h_{rs} b^{\mu}_i b^{\nu}_j S^i_{\mu} \bar{S}^j_{\nu} - \tau_r M^\mu_\nu c^\nu + iJ_i y^i) \omega \} \prod_x [d\tau][dc][dy(x)]
\]  \hspace{1cm} (15.12)
of the original Lagrangian system on $Y$ with the gauge-invariant Lagrangian $L$ \hspace{1cm} (12.14).

Note that the Lagrangian
\[
L' = L - \frac{1}{2} h_{rs} b^{\mu}_i b^{\nu}_j S^i_{\mu} \bar{S}^j_{\nu} - \tau_r M^\mu_\nu c^\nu
\]  \hspace{1cm} (15.13)
fails to be gauge-invariant, but it admits the BRST symmetry whose odd operator reads
\[
\vartheta = u'(x^\mu, y^i, c^\nu) \partial_i + d_\lambda u'(x^\mu, y^i, c^\nu) \partial_\lambda + \tau_r (x^\mu, y^i, c^\nu) \partial_{\tau_r} + v'(x^\mu, y^i, c^\nu) \partial_{\partial c^\nu} + d_\lambda v'(x^\mu, y^i, c^\nu) \partial_{\partial c^\nu} + d_\mu d_\lambda v'(x^\mu, y^i, c^\nu) \partial_{\partial c^\mu} \partial_{\partial c^\nu},
\]  \hspace{1cm} (15.14)
Its components $u'(x^\mu, y^i, c^\nu)$ are given by the expression \hspace{1cm} (15.6) where parameter functions $\xi^r(x)$ are replaced with the ghosts $c^r$. The components $\tau_r$ and $v'$ of the BRST operator $\vartheta$ can be derived from the condition
\[
\vartheta(L') = -h_{rs} M^\mu_\nu b^{\nu}_j S^j_{\lambda} c^\lambda - \tau_r M^\mu_\nu c^\nu + \tau_r (\vartheta(b^{\nu}_j S^j_{\lambda})) = 0
\]
of the BRST invariance of $L'$. This condition falls into the two independent relations
\[
h_{rs} M^\mu_\nu b^{\nu}_j S^j_{\lambda} + \tau_r M^\mu_\nu = 0,
\]
\[
\vartheta(c^\mu)(\vartheta(c^\nu)(b^{\nu}_j S^j_{\lambda})) = u(c^\nu)(u(c^\nu)(b^{\nu}_j S^j_{\lambda})) + u(v')(b^{\nu}_j S^j_{\lambda}) = u(\frac{1}{2} c^\nu c^\nu c^\lambda + v')(b^{\nu}_j S^j_{\lambda}) = 0.
\]
Hence, we obtain
\[ \mathcal{F}_r = -h_{r,s} b^{s \lambda} \mathcal{S}^j_{\lambda}, \quad \nu^r = -\frac{1}{2} c^r_{pq} c^p c^q. \]

In particular, let us turn to Yang–Mills gauge theory of principal connections in Section 13. Its constrained Lagrangian \( L_\Pi \) \(^{[13,22]}\) is invariant under the gauge transformations \(^{[13,24]}\). In view of the transformation law \(^{[15,10]}\), one can chose the gauge condition \(^{[15,18]}\):
\[ g^{\lambda \mu} S_{\lambda \mu}^r (x) - \alpha^r (x) = \frac{1}{2} g^{\lambda \mu} (\partial_\lambda a_\mu^r (x) + \partial_\mu a_\lambda^r (x)) - \alpha^r (x) = 0,\]
which is the familiar generalized Lorentz gauge. The corresponding second-order differential operator \(^{[15,10]}\) reads
\[ M_\xi^s \xi^s = g^{\lambda \mu} \left( \frac{1}{2} c^r_{pq} (\partial_\lambda a_\mu^r + \partial_\mu a_\lambda^r) \xi^q + c^r_{pq} a^p_\mu \partial_\lambda \xi^q + \partial_\lambda \partial_\mu \xi^r \right). \]

Passing to the Euclidean space and repeating the above quantization procedure, we come to the generating functional
\[ Z = N^{-1} \int \exp \left\{ \left( g^{\lambda \mu} \mathcal{F}_{\lambda \mu}^r - a^{\mu \nu}_{\lambda} g_{\lambda \beta} p^\rho_{\mu \nu} \sqrt{|g|} - \frac{1}{8} a^G_{\mu \nu} g^{\mu \nu} g^{\lambda \mu} (\partial_\lambda a_\mu^r + \partial_\mu a_\lambda^r) - g^{\lambda \mu} \tilde{\alpha}_{\rho}^r \left( \frac{1}{2} c^r_{pq} (\partial_\lambda a_\mu^r + \partial_\mu a_\lambda^r) \xi^q + c^r_{pq} a^p_\mu \partial_\lambda \xi^q + \partial_\lambda \partial_\mu \xi^r \right) + i \mathcal{J}^r_{\mu \nu} \right\} \prod_x \left[ d\bar{c}(x) [dp(x)] [da(x)] \right]. \]

Its integration with respect to momenta restarts the familiar generating functional of gauge theory.

16 Algebraic quantization. Quantum PS bracket

Canonical quantization of time-dependent non-relativistic mechanics on a fibre bundle \( Q \rightarrow \mathbb{R} \) in Section 7 is adequately formulated is geometric quantization of the vertical Poisson bracket \( \{, \}_V \) \(^{[11]}\) \(^{[12]}\) \(^{[13]}\) \(^{[14]}\). This fact motivates us to investigate quantization of the PS bracket \( \{, \}_{PS} \) \(^{[15]}\).

Let us note that one can quantize only linear spaces of variables, and therefore the condition of \( Y \rightarrow X \) to be a vector bundle is not a loss of generality.

In particular, in order to quantize the PS bracket \( \{, \}_{PS} \), one can be based on the fact that this bracket defines the Lie bracket \(^{[10,21]}\) of Noether Hamiltonian currents which brings a vector space \( \mathcal{J}(\Pi) \) of these currents into a Lie algebra. Then a representation of this algebra by operators acting in some space can be treated as a variant of quantization of the PS bracket \( \{, \}_{PS} \). However, such kind quantization fails to be a quantization of fields, but that of currents in the spirit of the well known current algebra approach \(^{[19]}\).

In a different way, we can restrict our consideration to a subspace of linear functions in \( y^i \) and \( p^\lambda_i \) represented as \( (n - 1) \)-forms \( F^\lambda \omega^\lambda \) due to the isomorphism \(^{[11,18]}\) and can modify the PS bracket \( \{, \}_{PS} \) \(^{[13]}\) as
\[ \{ F, G \}_r = \int_W \{ F, G \}_{PS} \]
where \( W \) is some compact \((n-1)\)-dimensional submanifold of \( X \). This bracket leads us to a nuclear algebra of canonical commutation relations whose representations can be investigated in a standard way [14, 43].

References

[1] V. Arnold (Ed.), *Dynamical Systems III, IV* (Springer, Berlin, 1990).

[2] D. Bashkirov and G. Sardanashvily, Covariant Hamiltonian field theory. Path integral quantization, *Int. J. Theor. Phys.* 43 (2004) 1317; arXiv: hep-th/0402057.

[3] C. Campos, M. de León, D. Martín de Diego and J. Vankerschaver, Unambiguous formalism for higher order Lagrangian field theories, *J. Phys. A* 42 (2009) 475207.

[4] F. Cantrijn, A. Ibort and M. De León, On the geometry of multisymplectic manifolds, *J. Austral. Math. Soc. Ser. A* 66 (1999) 303.

[5] J. Cariñena, M. Crampin and I. Ibort, On the multisymplectic formalism for first order field theories, *Diff. Geom. Appl.* 1 (1991) 345.

[6] W. Dittrich and M. Reuter, *Classical and Quantum Dynamics* (Springer-Verlag, Berlin, 1994).

[7] A. Echeverria-Enriquez, M. Munos-Lecanda and N. Roman-Roy, Geometry of multisymplectic Hamiltonian first-order field theories, *J. Math. Phys.* 41 (2002) 7402.

[8] M. Forger and S. Romero, Covariant Poisson Brackets in Geometric Field Theory, *Commun. Math. Phys.* 256 (2005) 375.

[9] M. Forger and M. Salles, On covariant Poisson brackets in classical field theory, arXiv: 1501.03780.

[10] G. Giachetta, L. Mangiarotti and G. Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory* (World Scientific, Singapore, 1997).

[11] G. Giachetta, L. Mangiarotti and G. Sardanashvily, Covariant Hamiltonian equations for field theory, *J. Phys. A* 32 (1999) 6629; arXiv: hep-th/9904062.

[12] G. Giachetta, L. Mangiarotti and G. Sardanashvily, Covariant geometric quantization of nonrelativistic time-dependent mechanics, *J. Math. Phys.* 43 (2002) 56; arXiv: quant-ph/0012036.

[13] G. Giachetta, L. Mangiarotti and G. Sardanashvily, Lagrangian supersymmetries depending on derivatives. Global analysis and cohomology, *Commun. Math. Phys.*, 259 (2005) 103.

[14] G. Giachetta, L. Mangiarotti and G. Sardanashvily, *Geometric and Topological Algebraic Methods in Quantum Mechanics* (World Scientific, Singapore, 2005).
[15] G. Giachetta, L. Mangiarotti and G. Sardanashvily, *Advanced Classical Field Theory* (World Scientific, Singapore, 2009).

[16] G. Giachetta, L. Mangiarotti and G. Sardanashvily, On the notion of gauge symmetries of generic Lagrangian field theory, *50* (2009) 012903; arXiv: 0807.0303.

[17] G. Giachetta, L. Mangiarotti and G. Sardanashvily, *Geometric Formulation of Classical and Quantum Mechanics* (World Scientific, Singapore, 2010).

[18] Gotay, M. (1991). A multisymplectic framework for classical field theory and the calculus of variations, in *Mechanics, Analysis and Geometry: 200 Years after Lagrange* (North Holland, Amsterdam) p. 203.

[19] M. Gotay, A multisymplectic framework for classical field theory and the calculus of variations. II. Space + time decomposition, *Diff. Geom. Appl.* 1 (1991) 375.

[20] C. Günther, The polysymplectic Hamiltonian formalism in field theory and calculus of variations, I: The local case, *J. Diff. Geom.* 25 (1987) 23.

[21] F. Helein and J. Kouneiher, Covariant Hamiltonian formalism for the calculus of variations with several variables, *Adv. Theor. Math. Phys.* 8 (2002) 565.

[22] D. Iglesias, J. Marrero and M. Vaquero, Poly-Poisson structures, *Lett. Math. Phys.* 103 (2013) 1103.

[23] I. Kanatchikov, On field theoretic generalizations of a Poisson algebra, *Rep. Math. Phys.* 40 (1997) 225; arXiv: hep-th/9710069.

[24] I. Kanatchikov, Ehrenfest theorem in orecanonical quantization, *J. Geom. Symm. Phys.* 37 (2015) 43; arXiv: 1501.00480.

[25] J. Kijowski and W. Tulczyjew, *A Symplectic Framework for Field Theories* (Springer-Verlag, Berlin, 1979).

[26] O. Krupkova, Hamiltonian field theory, *J. Geom. Phys.* 43 (2002) 93.

[27] M. de León, D. Martín de Diego and A. Santamaría-Merini, Symmetries in classical field theory, *Int. J. Geom. Methods Mod. Phys.* 1 (2004) 651.

[28] M. de Leon, M. Saldago and S. Vilarino, Methods of differential geometry in classical field theories: $k$-symplectic and $k$-cosymplectic approaches, arXiv: 1409.5604.

[29] L. Mangiarotti and G. Sardanashvily, On the bracket problem in covariant Hamiltonian field theory, arXiv: hep-th/9903220.

[30] L. Mangiarotti and G. Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).

[31] J. Marsden, G. Patrick and S. Shkoller, Multisymplectic geometry, variational integrators and nonlinear PDEs, *Commun. Math. Phys.* 199 (1998) 351.

[32] N. Martinez, Poly-symplectic groupoids and poly-Poisson structures, arXiv: 1409.0695.
[33] P. Prieto-Martínez and N. Roman-Roy, A new multisymplectic unified formalism for second order classical field theories, *arXiv*: 1402.4087.

[34] A. Rey, N. Roman-Roy and M. Saldago, M. Gunther’s formalism (k-symplectic formalism) in classical field theory: Skinner-Rusk approach and the evolution operator, *J. Math. Phys.* 46 (2005) 052901.

[35] N. Roman-Roy, Multisymplectic Lagrangian and Hamiltonian formalisms of classical field theories, *SIGMA* 5 (2009) 100.

[36] O. Rossi and D. Saunders, Dual jet bundles, Hamiltonian systems and connections, *J. Geom. Phys.* 35 (2014) 178.

[37] G. Sardanashvily and O. Zakharov, On application of the Hamilton formalism in fibred manifolds to field theory, *Diff. Geom. Appl.* 3 (1993) 245.

[38] G. Sardanashvily, Multimomentum Hamiltonian formalism in field theory, *arXiv*: hep-th/9403172.

[39] G. Sardanashvily, Multimomentum Hamiltonian formalism in quantum field theory, *Int. J. Theor. Phys.* 33 (1994) 2373; *arXiv*: hep-th/9404001.

[40] G. Sardanashvily, *Generalized Hamiltonian Formalism for Field Theory* (World Scientific, Singapore, 1995).

[41] G. Sardanashvily, Stress-energy-momentum tensors in constraint field theories, *J. Math. Phys.* 38 (1997) 847.

[42] G. Sardanashvily, Hamiltonian time-dependent mechanics, *J. Math. Phys.* 39 (1998) 2714.

[43] G. Sardanashvily, Nonequivalent representations of nuclear algebras of canonical commutation relations. Quantum fields, *Int. J. Theor. Phys.* 41 (2002) 1541; *arXiv*: hep-th/0202038.

[44] G. Sardanashvily, Geometric formulation of non-autonomous mechanics, *Int. J. Geom. Methods Mod. Phys.* 10 (2013) 1350061; *arXiv*: 1303.1735.

[45] G. Sardanashvily, *Advanced Differential Geometry for Theoreticians. Fiber bundles, jet manifolds and Lagrangian theory* (Lambert Academic Publishing, Saarbrucken, 2013); *arXiv*: 0908.1886.

[46] G. Sardanashvily, Deviation differential equations. Jacobi fields, *arXiv*: 1304.0706.

[47] D. Saunders, *The Geometry of Jet Bundles* (Cambridge Univ. Press, Cambridge, 1989).

[48] D. Saunders and M. Crampin, On the Legendre map in higher-order field theories, *J. Phys. A* 23 (1990) 3169.

[49] S. Treiman, R. Jackiw and D. Gross, *Lectures on current algebra and its applications*. Princeton Series in Physics (Princeton Univ. Press, Princeton, 1972).

[50] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds* (Birkhäuser, Basel, 1994).
[51] L. Vitagliano, The Lagrangian-Hamiltonian formalism for higher order field theories, *J. Geom. Phys.* 60 (2010) 857873.

[52] O. Zakharov, Hamiltonian formalism for nonregular Lagrangian theories in fibered manifolds, *J. Math. Phys.* 33 (1992) 607.