A generalisation of Mirsky’s singular value inequalities

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Abstract

We prove an $f$-version of Mirsky’s singular value inequalities for differences of matrices. This $f$-version consists in applying a positive concave function $f$, with $f(0) = 0$, to every singular value in the original Mirsky inequalities.

Denote the singular values of a matrix $X$, arranged in non-increasing order, by $\sigma_i(X)$. The main result of this paper is the following singular value inequality:

**Theorem 1.** Let $f$ be a concave function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $f(0) = 0$. Let $X, Y$ be general $n \times n$ complex matrices. Then for any $m \leq n$, and any increasing sequence $(i_1, \ldots, i_m)$ of integers in $\{1, \ldots, n\}$, we have

$$\sum_{k=1}^{m} |f(\sigma_{i_k}(X)) - f(\sigma_{i_k}(Y))| \leq \sum_{k=1}^{m} f(\sigma_k(X - Y)).$$

(1)

Without the application of the function $f$, these inequalities are essentially Mirsky’s singular value inequalities [5] (up to setting $i_k = k$). Their $f$-version is essentially the set of inequalities conjectured by W. Miao that appears in [1] as Conjecture 6 (again with $i_k = k$). We therefore have solved this conjecture. The special case $i_k = k$ and $m = n$ has apparently been proven by Yue and So in their as yet unpublished manuscript [9], where an application is given to low-rank matrix recovery. Our proof technique is completely different from theirs. In [10] Zhang and Qiu also claimed to have proven inequalities (1), but unfortunately their proof is flawed (as pointed out in [9]).

The main ingredient in our work is a set of eigenvalue inequalities for sums of Hermitian matrices, known as the Thompson-Freede (TF) inequalities [7]. These inequalities also come in a version that applies to singular values. Remarkably, this is about the only matrix analytical tool that is needed to prove Theorem 1. Apart from this, the proof is rather elementary and consists in appropriately choosing one of the TF inequalities and combining it with inequalities of the kind $\sigma_i(X) \geq \sigma_j(X)$ for $i < j$, and $\sigma_i(X) \geq 0$. 


In Section 1 we introduce the TF inequalities, for eigenvalues as well as for singular values. We then state their $f$-version, Theorem 2, by which is meant applying a positive, concave function $f$ with $f(0) = 0$ to every singular value in the original TF singular value inequalities. Zhang and Qiu have shown in [10] that all Horn-type singular value inequalities have a valid $f$ version, including the TF inequalities. We give a completely different proof of the $f$-version of the TF singular value inequalities that just like the proof of Theorem 1 is only based on a well-chosen combination of the original TF inequalities. In fact, neither the statement of the theorem nor its proof make any reference to matrix analysis at all. The proof of our main result, Theorem 1, is given in Section 2.

1 The Thompson-Freede inequalities and their $f$-version

Let $A$ and $B$ be $n \times n$ Hermitian matrices. Let $\alpha(i)$, $\beta(i)$ and $\gamma(i)$, for $i = 1, \ldots, n$, be the eigenvalues, sorted in non-ascending order, of $A$, $B$ and $A + B$, respectively.

For a given integer $m \leq n$ let $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_m)$ be two strictly increasing sequences of length $m$ of integers between 1 and $n$ such that $i_m + j_m \leq n + m$. Then the Thompson-Freede (TF) inequalities [7] are

$$\sum_{k=1}^{m} \gamma(i_k + j_k - k) \leq \sum_{k=1}^{m} \alpha(i_k) + \sum_{k=1}^{m} \beta(j_k). \tag{2}$$

These inequalities include as special cases the Lidskii/Wielandt inequalities (take $j_k = k$). They form themselves a subset of Horn’s inequalities, which are of the general form

$$\sum_{k \in K} \gamma(k) \leq \sum_{i \in I} \alpha(i) + \sum_{j \in J} \beta(j), \tag{3}$$

where $I$, $J$ and $K$ are certain subsets of $\{1, \ldots, n\}$ governed by a rather complicated set of recursive constraints (not reproduced here) [3]. We say that $(I, J, K)$ constitutes an admissible triple whenever these constraints are satisfied.

Consider now a non-negative, concave (hence non-decreasing) function $f$ on $[0, +\infty)$ such that $f(0) = 0$. When $A$ and $B$ are positive semidefinite, the eigenvalues of $A$, $B$ and $A + B$ also satisfy what one could call the $f$-version of Horn’s inequalities:

$$\sum_{k \in K} f(\gamma(k)) \leq \sum_{i \in I} f(\alpha(i)) + \sum_{j \in J} f(\beta(j)). \tag{4}$$

These inequalities are also satisfied for general matrices $A$ and $B$ when $\alpha$, $\beta$ and $\gamma$ are the singular values of $A$, $B$ and $A + B$, respectively. Note that the non-negativity of singular values is essential here; although the eigenvalues of Hermitian $A$, $B$ and $A + B$ satisfy all Horn inequalities, they do not in general satisfy their $f$-versions.

Zhang and Qiu [10] have recently proven this $f$-version by exploiting a theorem by Bourin and Uchiyama (Corollary 2.6 in [4]) which states that for all $A$ and $B$ and any positive concave function $f$ there exist unitary matrices $U$ and $V$ such that

$$f(|A + B|) \leq UF(|A|)U^* + VF(|B|)V^*. \tag{5}$$
Thus, in particular, the singular values of $A$, $B$ and $A + B$ satisfy an $f$-version of the TF inequalities.

Below we give an alternative proof of the latter statement based uniquely on the fact that these singular values satisfy the original TF inequalities.

**Theorem 2.** Let $\alpha(i)$, $\beta(i)$ and $\gamma(i)$, for $i = 1, \ldots, n$, be sequences of non-negative numbers, sorted in non-ascending order, and satisfying all TF inequalities \cite{2}. Let $f$ be a non-negative, concave function on $[0, +\infty)$ such that $f(0) = 0$. Then $\alpha(i)$, $\beta(i)$ and $\gamma(i)$ satisfy the $f$-version of the TF inequalities. To wit, for a given integer $m \leq n$ let $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_m)$ be two strictly increasing sequences of length $m$ of integers between 1 and $n$ such that $i_m + j_m \leq n + m$. Then

$$\sum_{k=1}^{m} f(\gamma(i_k + j_k - k)) \leq \sum_{k=1}^{m} f(\alpha(i_k)) + \sum_{k=1}^{m} f(\beta(j_k)). \quad (6)$$

**Proof.** Any function $f$ satisfying the assumptions of Theorem \cite{2} can be uniformly approximated as a finite or infinite positive linear combination of ‘hook’ functions $h_t(x) := \min(x, t)$ with $t > 0$; that is, for any such $f$ there exists a positive measure $d\mu(t)$ on $(0, \infty)$ such that $f(x) = \int_0^\infty h_t(x) \, d\mu(t)$. By linearity of LHS and RHS of \cite{2} in $f$ it therefore suffices to prove \cite{3} for hook functions only. Furthermore, by a scaling argument it is clear that we can restrict to $f(x) = h(x) := \min(x, 1)$.

Let $a$ and $b$ be index values, $1 \leq a, b \leq m$, such that the following hold:

$$\alpha(i_a) < 1 \leq \alpha(i_{a-1}), \quad \beta(j_b) < 1 \leq \beta(j_{b-1}).$$

Then $h(\alpha(i_k)) = 1$ for $k < a$ and $h(\alpha(i_k)) = \alpha(i_k)$ for $k \geq a$, and similar identities hold for $\beta$. Inequality \cite{4} then reduces to

$$\sum_{k=1}^{m} h(\gamma(i_k + j_k - k)) \leq (a - 1) + \sum_{k=a}^{m} \alpha(i_k) + (b - 1) + \sum_{k=b}^{m} \beta(j_k). \quad (7)$$

We will first consider the case that $a + b - 1 \leq m$. Making the replacements $m \to m' := m - (a - 1) - (b - 1)$, $i_k \to i_{k+a-1}$ and $j_k \to j_{k+b-1}$ in \cite{2} yields

$$\sum_{k=1}^{m'} \gamma(i_{k+a-1} + j_{k+b-1} - k) \leq \sum_{k=1}^{m'} \alpha(i_{k+a-1}) + \sum_{k=1}^{m'} \beta(j_{k+b-1}) = \sum_{k=a}^{m-2b+1} \alpha(i_k) + \sum_{k=b}^{m-a+1} \beta(j_k). \quad (8)$$

Since $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_m)$ are strictly increasing sequences, they satisfy $i_{k+a-1} \leq i_{k+a+b-2} - (b - 1)$ and $j_{k+b-1} \leq j_{k+a+b-2} - (a - 1)$. Furthermore, the sequence $(\gamma(1), \ldots, \gamma(n))$ is non-increasing. Therefore, the LHS of \cite{8} is bounded below as

$$\sum_{k=1}^{m'} \gamma(i_{k+a-1} + j_{k+b-1} - k) \geq \sum_{k=1}^{m'} \gamma(i_{k+a+b-2} + j_{k+a+b-2} - (k + a + b - 2)) = \sum_{k=a+b-1}^{m} \gamma(i_k + j_k - k),$$

where
so that
\[ \sum_{k=a+b}^{m} \gamma(i_k + j_k - k) \leq \sum_{k=a}^{m-b+1} \alpha(i_k) + \sum_{k=b}^{m-a+1} \beta(j_k). \]

For the other case, \( a + b - 1 > m \), the same inequality holds trivially.

Because \( h(x) \leq x \) and \( \alpha, \beta \geq 0 \) (here is where the argument would break down when considering eigenvalues instead of singular values) this implies
\[ \sum_{k=a+b}^{m} h(\gamma(i_k + j_k - k)) \leq \sum_{k=a}^{m-b+1} \gamma(i_k + j_k - k) \leq \sum_{k=a}^{m} \alpha(i_k) + \sum_{k=b}^{m} \beta(j_k). \] (9)

For the remaining terms in the LHS of (7) we have
\[ \sum_{k=1}^{a+b-2} h(\gamma(i_k + j_k - k)) \leq a + b - 2 = (a - 1) + (b - 1), \] (10)

and taking the sum of (9) and (10), inequality (7) follows.

\[ \square \]

2 Proof of Theorem 1

Let us replace the matrices \( X, Y \) and \( X - Y \) in the statement of Theorem 1 by matrices \( C, A \) and \( B \), respectively, with \( A + B + C = 0 \), and let us denote their singular values by \( \gamma(i), \alpha(i) \) and \( \beta(i) \), respectively. The proof starts with a number of simple reductions.

As in the proof of Theorem 2, it is enough to prove Theorem 1 for the function \( f(x) := h(x) = \min(1, x) \), as all other functions under consideration can be written as positive linear combinations of \( t h(x/t) \). Whereas in the proof of Theorem 2 we merely exploited linearity of the LHS and RHS in \( f \), here we must also use the triangle inequality for the absolute value in the LHS.

It therefore suffices to prove the following inequality
\[ \sum_{k=1}^{m} |\min(1, \gamma(i_k)) - \min(1, \alpha(i_k))| \leq \sum_{k=1}^{m} \min(1, \beta(k)). \] (11)

Let us define the index set \( I = \{i_1, \ldots, i_m\} \), and the indices \( a, b \) and \( c \) for which the following holds:
\[ \gamma(i_c) < 1 \leq \gamma(i_{c-1}) \]
\[ \alpha(i_a) < 1 \leq \alpha(i_{a-1}) \]
\[ \beta(b) < 1 \leq \beta(b - 1). \]

We can assume that \( a \leq c \); otherwise we just swap the roles of \( A \) and \( C \).

As the contribution to the LHS of (11) of the terms with \( k < a \) is exactly zero, removing the indices \( i_1, \ldots, i_{a-1} \) from \( I \) and removing the \( a - 1 \) smallest \( \beta \)'s from the RHS turns one
instance of (11) into another. Thus, henceforth we only need to consider the case \( a = 1 \), which is:

\[
\sum_{k=1}^{c-1} (1 - \alpha(i_k)) + \sum_{k=c}^{m} |\gamma(i_k) - \alpha(i_k)| \leq (b - 1) + \sum_{k=b}^{m} \beta(k). \tag{12}
\]

Let us partition \( I = \{i_1, \ldots, i_m\} \) into two subsets \( I_C \) and \( I_A \), where \( I_C \) is the set of indices \( i \in I \) for which \( \gamma(i) \geq \alpha(i) \) and \( I_A \) is the set of remaining indices. Clearly, the indices \( i_1, \ldots, i_{c-1} \) are always in \( I_C \), and never in \( I_A \). Because \( |x - y| = \max(x - y, y - x) \), to prove (12) it suffices to prove the following inequality for all such partitions \( I_C \) and \( I_A \) of \( I \) (keeping the requirement that \( i_1, \ldots, i_{c-1} \in I_C \)), regardless for which of the \( i \) the inequality \( \gamma(i) \geq \alpha(i) \) holds:

\[
\sum_{k \in I_C} (h(\gamma(k)) - \alpha(k)) - \sum_{k \in I_A} (\gamma(k) - \alpha(k)) \leq (b - 1) + \sum_{k=b}^{m} \beta(k),
\]

or, equivalently

\[
\sum_{k \in I_C} h(\gamma(k)) + \sum_{k \in I_A} \alpha(k) \leq \sum_{k \in I_C} \alpha(k) + \sum_{k \in I_A} \gamma(k) + (b - 1) + \sum_{k=b}^{m} \beta(k). \tag{13}
\]

After these reductions, we come to the core of the argument. Let us define the additional index sets

\[
I_R = \{i_1, \ldots, i_{m-b+1}\}, \quad I_R^\complement = \{i_{m-b+2}, \ldots, i_m\},
\]

\[
I_L = \{i_b, \ldots, i_m\}, \quad I_L^\complement = \{i_1, \ldots, i_{b-1}\},
\]

\[
J = \{b, \ldots, m\}. \tag{14}
\]

Note that \( I_R, I_L \) and \( J \) have size \( m-b+1 \) and \( I_R^\complement \) and \( I_L^\complement \) have size \( b-1 \). To simplify notations we adopt the notations \( \gamma(K) := \sum_{k \in K} \gamma(k) \), etc., and \( I_{CL} := I_C \cap I_L \), \( I_{CR} := I_C \cap I_R \), \( I_{CR}^\complement := I_C \cap I_R^\complement \), etc.

Inequality (13), and hence the inequality of Theorem 1, is a straightforward consequence of the following theorem, which will be proven below:

**Theorem 3.** For all \( n \times n \) matrices \( A, B \) and \( C \) such that \( A + B + C = 0 \), for any partitioning of \( I = \{i_1, \ldots, i_m\} \) (with \( m \leq n \)) into \( I = I_C \cup I_A \), and with the notations just introduced,

\[
\gamma(I_{CL}) + \alpha(I_{AL}) \leq \alpha(I_{CR}) + \gamma(I_{AR}) + \beta(J). \tag{15}
\]

Note we do not restrict \( I_C \) to contain \( i_1, \ldots, i_{c-1} \) here.

The simplest non-trivial examples of inequality (15) are

\[
\gamma(I_L) \leq \alpha(I_R) + \beta(J), \tag{16}
\]

which are obtained by setting \( I_C = I \) and \( I_A = \emptyset \). One can easily verify that these are just instances of the TF singular value inequalities. What Theorem 3 is actually saying is that in (16) one can freely replace any \( \alpha(i) \) with the corresponding \( \gamma(i) \) and vice-versa, and still have a valid inequality.
To see how (13) follows from this, note that all terms in the LHS of (13) are bounded above by 1. Therefore, and because \( I_{CL} \cup I_{AL} = I_L \),

\[
\gamma(I_{CL}) + \alpha(I_{AL}) \leq b - 1.
\]

Furthermore, as singular values are non-negative, we also have

\[
0 \leq \alpha(I_{CR}) + \gamma(I_{AR}).
\]

Adding these two inequalities to inequality (15) of Theorem 3, we get

\[
\gamma(I_C) + \alpha(I_A) \leq \alpha(I_C) + \gamma(I_A) + \beta(J).
\]

Since \( h(x) = \min(1, x) \leq x \) this yields (13). \( \square \)

**Proof of Theorem 3**

The essential idea is to consider the following pairing of elements \( t_k \) of \( I_R \) with elements \( s_k \) of \( I_L \):

\[
(t_k, s_k) := (i_k, i_{k+b-1}), \quad k = 1, \ldots, m - b + 1.
\]

Clearly, we have \( s_k - t_k = i_{k+b-1} - i_k \geq b - 1 \). For every such pair, exactly one out of four possibilities arises concerning membership of the sets \( I_{AL}, I_{CL}, I_{AR}, I_{CR} \). We can partition the set \( K := \{1, \ldots, m - b + 1\} \) accordingly as \( K = K_1 \cup K_2 \cup K_3 \cup K_4 \), with

\[
\begin{align*}
K_1 &= \{k : s_k \in I_{CL}, t_k \in I_{AR}\} \\
K_2 &= \{k : s_k \in I_{AL}, t_k \in I_{CR}\} \\
K_3 &= \{k : s_k \in I_{CL}, t_k \in I_{CR}\} \\
K_4 &= \{k : s_k \in I_{AL}, t_k \in I_{AR}\}.
\end{align*}
\]

These sets have the following unions:

\[
\begin{align*}
K_1 \cup K_3 &= \{k : s_k \in I_{CL}\} \\
K_2 \cup K_4 &= \{k : s_k \in I_{AL}\} \\
K_1 \cup K_4 &= \{k : t_k \in I_{AR}\} \\
K_2 \cup K_3 &= \{k : t_k \in I_{CR}\}.
\end{align*}
\]

With the four subsets \( K_i \) as a starting point, we will write down a number of valid inequalities, the sum of which is exactly (15).

For every \( k \in K_1 \) we consider the inequality \( \gamma(s_k) \leq \gamma(t_k) \), which is valid since \( s_k \geq t_k \). Summing over all \( k \in K_1 \), we get

\[
\sum_{k \in K_1} \gamma(s_k) \leq \sum_{k \in K_1} \gamma(t_k).
\]

Analogously, we have

\[
\sum_{k \in K_2} \alpha(s_k) \leq \sum_{k \in K_2} \alpha(t_k).
\]

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For the remaining pairs, corresponding to \( k \in K_3 \cup K_4 \), we will write down a single, but more complicated inequality. Letting \( r \) denote the number of these remaining pairs, \( r = |K_3| + |K_4| \), we write

\[
\sum_{k \in K_3} \gamma(s_k) + \sum_{k \in K_4} \alpha(s_k) \leq \sum_{k \in K_3} \alpha(t_k) + \sum_{k \in K_4} \gamma(t_k) + \sum_{k=b}^{b+r-1} \beta(k). \tag{21}
\]

Since the overall number of pairs is exactly \( m - b + 1 \), we have \( r \leq m - b + 1 \), so that \( b + r - 1 \leq m \). The index \( k \) in the final summation therefore does not exceed the bound \( m \).

Assuming (21) is correct, the sum of (19), (20) and (21) yields, after adding some more \( \beta \)-terms to the RHS,

\[
\sum_{k \in K_1 \cup K_3} \gamma(s_k) + \sum_{k \in K_2 \cup K_4} \alpha(s_k) \leq \sum_{k \in K_1 \cup K_4} \gamma(t_k) + \sum_{k \in K_2 \cup K_3} \alpha(t_k) + \sum_{k=b}^{m} \beta(k).
\]

By the identities (18), this is exactly inequality (15).

It remains to prove inequality (21). We will do so by showing that it is essentially one of the TF inequalities. Rearranging (21) gives

\[
\sum_{k \in K_3} \gamma(s_k) - \sum_{k \in K_4} \gamma(t_k) \leq \sum_{k \in K_3} \alpha(t_k) - \sum_{k \in K_4} \alpha(s_k) + \sum_{k=b}^{b+r-1} \beta(k). \tag{22}
\]

The TF inequality that we need is the eigenvalue inequality

\[
\sum_{l=1}^{r} \gamma(i_l + j_l - l) \leq \sum_{l=1}^{r} \alpha(i_l) + \beta(j_l). \tag{23}
\]

for eigenvalues \( \hat{\alpha}(k) \), \( \hat{\beta}(k) \) and \( \gamma(k) \) of Hermitian matrices \( \hat{A} \), \( \hat{B} \) and \( \hat{A} + \hat{B} \), respectively. In particular, we take \( j_l = l + b - 1 \) (so that \( j_l - l = b - 1 \)), and let the indices \( i_l \) be the elements of the set

\[
\{t_k : k \in K_3\} \cup \{2n + 1 - s_k : k \in K_4\}
\]

sorted in decreasing order. Then (23) becomes

\[
\sum_{k \in K_3} \gamma(t_k+b-1) + \sum_{k \in K_4} \gamma(2n+1-s_k+b-1) \leq \sum_{k \in K_3} \hat{\alpha}(t_k) + \sum_{k \in K_4} \hat{\alpha}(2n+1-s_k) + \sum_{l=1}^{r} \beta(l + b - 1).
\]

Because \( s_k \geq t_k + b - 1 \), this implies the weaker inequality

\[
\sum_{k \in K_3} \gamma(s_k) + \sum_{k \in K_4} \gamma(2n+1-t_k) \leq \sum_{k \in K_3} \hat{\alpha}(t_k) + \sum_{k \in K_4} \hat{\alpha}(2n+1-s_k) + \sum_{l=b}^{b+r-1} \beta(l). \tag{24}
\]
If we let \( \hat{A} \) and \( \hat{B} \) be the Wielandt matrices

\[
\hat{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix},
\]

then, for all \( k = 1, \ldots, n \), we have \( \hat{\alpha}(k) = \alpha(k) \) and \( \hat{\alpha}(2n + 1 - k) = -\alpha(k) \), and similar identities for \( \hat{\beta}(k) \) and \( \hat{\gamma}(k) \). Using these identities, \((24)\) reduces to the singular value inequality \((22)\), which ends the proof.

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