WIDE SUBCATEGORIES OF FINITELY GENERATED 
Λ-MODULES.

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Abstract. We explore some properties of wide subcategories of the category mod (Λ) of finitely generated left Λ-modules, for some artin algebra Λ. In particular we look at wide finitely generated subcategories and give a connection with the class of standard modules and standardly stratified algebras. Furthermore, for a wide class $\mathcal{X}$ in mod (Λ), we give necessary and sufficient conditions to see that $\mathcal{X} = \text{pres} (P)$, for some projective Λ-module $P$; and finally, a connection with ring epimorphisms is given.

Introduction.

Wide categories have been extensively studied in the case of modules over commutative rings [6, 11, 17]. One important feature of them is their connection with a classification theorem of the thick subcategories of the derived category of perfect complexes over a commutative noetherian ring in terms of the ring spectrum. For non-commutative rings, as far as we know, there are some interesting results in [7, 8] for the case of wide subcategories of finitely generated left $A$-modules, where $A$ is a finite dimensional, basic and hereditary algebra over an algebraically closed field $k$. On the other hand, in [13] there are stated some nice correspondences between torsion classes and wide classes in mod ($A$), for a finite dimensional $k$-algebra. Furthermore, in the case that $A$ is of finite representation type, explicit bijections are given in [13] between different classes and wide classes in mod ($A$). It seems interesting to us to look at what happens if we consider the more general case of an artin algebra $A$. This paper goes in this direction, we study some elementary properties of wide subcategories of mod ($\Lambda$), for an artin algebra $\Lambda$, where mod ($\Lambda$) is the full subcategory of finitely generated left $\Lambda$-modules.

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Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{C} \subseteq \mathcal{A}$ is wide if $\mathcal{C}$ is closed under extensions, kernels and cokernels of morphisms in $\mathcal{C}$. Thus, a wide subcategory is nothing more than an abelian subcategory which is closed under extensions. Let $\mathcal{X}$ be a class of objects in $\text{mod} \ (\Lambda)$. We denote by $\text{wide} \ (\mathcal{X})$ the smallest wide subcategory of $\text{mod} \ (\Lambda)$ containing $\mathcal{X}$. In this paper we study some special types of wide subcategories of $\text{mod} \ (\Lambda)$.

The paper is organized as follows. In section 2, we give some basic notions and an equivalent description for a subcategory to be wide.

Let $M$ be an indecomposable $\Lambda$-module. In section 3, we prove that $\text{add} \ (M) = \text{wide} \ (M)$ if and only if $\text{Ext}^1_\Lambda(M, M) = 0$ and $\text{End}_\Lambda(M)$ is a division ring. We also give examples where $\text{add} \ (M) \subsetneq \text{wide} \ (M)$, but still $\text{wide} \ (M) = \text{add} \ (N)$ for some other $\Lambda$-module $N$.

In section 4, we deal with the set $\Delta$ of standard $\Lambda$-modules and the class $\mathfrak{g}(\Delta)$ of good $\Lambda$-modules. Here, we prove that $\text{wide} \ (\Delta) = \text{mod} \ (\Lambda)$, for any artin $R$-algebra $\Lambda$. Some consequences, of the above result, are also given in this section. In particular, for a quotient path algebra $\Lambda = kQ/I$, we prove that the quiver $Q$ is directed, in the sense of S. Koenig [11], if and only if $\mathfrak{g}(\Delta)$ is wide.

In section 5, we deal with the category $\text{pres} \ (P)$ of $\Lambda$-modules having a presentation in $\text{add} \ (P)$, for some projective $\Lambda$-module $P$. It is well known that $\text{pres} \ (P)$ is an abelian category, but in general, it is not true (an example is given) that $\text{pres} \ (P)$ is an abelian subcategory of $\text{mod} \ (\Lambda)$. In this section, we characterize wide subcategories $\mathcal{X}$ of the form $\mathcal{X} = \text{pres} \ (P)$ for some $P \in \text{proj} \ (\Lambda)$. We prove that, for a wide subcategory $\mathcal{X}$ of $\text{mod} \ (\Lambda)$, the following statements are equivalent

(a) $\mathcal{X} = \text{pres} \ (P)$ for some $P \in \text{proj} \ (\Lambda)$;
(b) $\mathcal{X}$ has enough projectives and $\text{proj} \ (\mathcal{X}) \subseteq \text{proj} \ (\Lambda)$;
(c) $\mathcal{X}$ is closed under projective covers, that is $\mathcal{P}_0(\mathcal{X}) \subseteq \mathcal{X}$.

If one of the above equivalent conditions hold, then

$$\text{add} \ (P) = \text{proj} \ (\mathcal{X}) = \text{add} \ (\mathcal{P}_0(\mathcal{X})).$$

In particular, $P$ is uniquely determined up to additive closures. Furthermore, the functor $\text{Hom}_\Lambda(P, -) : \mathcal{X} \to \text{mod} \ (\text{End}_\Lambda(P)^{\text{op}})$ is an equivalence of categories.

Finally, in the case of finite dimensional $k$-algebras and assuming that $\mathcal{X}$ is a functorially finite wide subcategory of $\text{mod} \ (A)$, we have that the following statements are equivalent

(a) There is a ring epimorphism $A \to B$ such that $B \in \text{proj} \ (A)$ and $\mathcal{X} = \text{pres} \ (AB)$.
(b) $\mathcal{X}$ has enough projectives and $\text{proj} \ (\mathcal{X}) \subseteq \text{proj} \ (A)$.

If one of the above equivalent conditions hold, there is a basic $A$-module $P$ such that $\text{add} \ (P) = \text{proj} \ (\mathcal{X})$ and $B$ is Morita equivalent to $\text{End}_A(P)^{\text{op}}$. 
Throughout this paper, the term algebra means artin algebra over a commutative artin ring $R$. For an algebra $\Lambda$, the category of finitely generated left $\Lambda$-modules will be denoted by $\text{mod} (\Lambda)$. Unless otherwise specified, we will work with finitely generated $\Lambda$-modules, full subcategories and non-empty classes. We denote by $\text{proj} (\Lambda)$ the full subcategory of $\text{mod} (\Lambda)$ whose objects are the projective $\Lambda$-modules.

Let us denote by $S_1, S_2, \ldots, S_n$ a complete list of non-isomorphic simple $\Lambda$-modules and let $\leq$ denotes this fixed natural ordering of the simple modules. Let $P_i$ be the projective cover of the simple module $S_i$, for each $i = 1, 2, \ldots, n$. We define for each $i$, the standard module $\Delta(i) = P_i/U_i$ where $U_i$ is the trace of $\oplus_{j>i} P_j$ in $P_i$. That is, $U_i = \sum_{f: \oplus_{j>i} P_j \to P_i} \text{Im} f$. Let $\Delta = \{\Delta(1), \Delta(2), \ldots, \Delta(n)\}$ be the set of standard modules. It is well known, that the standard module $\Delta(i)$ is the maximal quotient of $P_i$ which composition factors are only the simple $\Lambda$-modules $S_j$ with $j \leq i$.

For a given class $\Theta$ of $\Lambda$-modules, we denote by $\mathfrak{F}(\Theta)$ the full subcategory of $\text{mod} (\Lambda)$, whose objects are the $\Lambda$-modules $M$ which have a $\Theta$-filtration. That is, $M \in \mathfrak{F}(\Theta)$ if there is a finite chain

$$0 = M_0 \leq M_1 \leq M_2 < \cdots < M_t = M$$

of submodules of $M$ such that each quotient $M_i/M_{i-1}$ is isomorphic to a module in $\Theta$. In case of the class $\Theta := \Delta$, the modules in $\mathfrak{F}(\Delta)$ are called good modules.

If $\Lambda \in \mathfrak{F}(\Delta)$ then the pair $(\Lambda, \leq)$ is said to be a (left) standardly stratified algebra. A standardly stratified algebra is called quasi-hereditary if the endomorphism ring of each standard module is a division ring. Quasi-hereditary algebras were introduced by L.L. Scott in [16].

Associated with the category $\mathfrak{F}(\Delta)$, there is the class of the $\Delta$-injective $\Lambda$-modules

$$\mathcal{I}(\Delta) := \{Y \in \text{mod} (\Lambda) : \text{Ext}^1_\Lambda (-, Y)|_{\mathfrak{F}(\Delta)} = 0\}.$$ 

In the theory of standardly stratified algebras, the subcategory $\mathfrak{F}(\Delta) \cap \mathcal{I}(\Delta)$ is of special interest. In fact, this subcategory was characterized by C. M. Ringel in [15] and by I. Agoston, D. Happel, E. Lukacs and L. Unger in [1], for quasi-hereditary algebras and for standardly stratified algebras, respectively. Their characterization is the following theorem.

**Theorem 1.1** ([15] [1]). Let $(\Lambda, \leq)$ be a standardly stratified algebra. Then, there is a (generalized) tilting basic module $T$ such that $\text{add}(T) = \mathfrak{F}(\Delta) \cap \mathcal{I}(\Delta)$.

Finally, we recall that for any $M \in \text{mod} (\Lambda)$, the class $\text{add}(M)$ consists of all $\Lambda$-modules which are direct summands of finite coproducts of copies of $M$. 
2. Wide subcategories

We start this section introducing some basic definitions, which will be used in the sequel.

Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ be a class in $\mathcal{A}$. We say that $\mathcal{C}$ is **closed under extensions** if, for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, the fact that $X, Z \in \mathcal{C}$ implies that $Y \in \mathcal{C}$. We say that $\mathcal{C}$ is **thick** if $\mathcal{C}$ is closed under direct summands and for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, the fact that two of the terms of the exact sequence belong to $\mathcal{C}$ implies that the third one belongs to $\mathcal{C}$. We say that $\mathcal{C} \subseteq \mathcal{A}$ satisfies the **intersection property** if for any sub-objects $X$ and $Y$ of $\mathcal{C}$, which are in $\mathcal{C}$, we have that $X \cap Y \in \mathcal{C}$.

A subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is called **admissible** if it is thick and any $\mathcal{C} \subseteq \mathcal{C}$ satisfies the intersection property (see [14]). Note that, the fact that $\mathcal{C}$ is admissible in $\mathcal{A}$ does not mean that the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{A}$ admits a right or a left adjoint functor. Therefore, the usual definition of an admissible subcategory of a triangulated category is not related to the above one.

Following M. Hovey, in [6], we say that the subcategory $\mathcal{C} \subseteq \mathcal{A}$ is **wide** if it is closed under extensions, kernels and cokernels of morphisms in $\mathcal{C}$. Wide categories have been extensively studied in the case of modules over commutative rings [6, 11, 17]. For the case of finitely generated modules over a hereditary algebra, we recommend the reader to see [7, 8]. An easy observation is that any wide category $\mathcal{C}$ is closed under isomorphisms and furthermore it is an additive subcategory of $\mathcal{A}$.

Given a class $\mathcal{X} \subseteq \mathcal{A}$, we denote by wide $\mathcal{X}$ the smallest wide subcategory of $\mathcal{A}$ containing $\mathcal{X}$. Note that the class $\mathcal{F}(\mathcal{X})$ of all the $\mathcal{X}$-filtered objects is a subclass of wide $\mathcal{X}$, since $\mathcal{F}(\mathcal{X})$ is the smallest subclass in $\mathcal{A}$, which is closed under extensions and contains $\mathcal{X}$.

Let $\mathcal{C}$ be an abelian category, which is an additive subcategory of an abelian category $\mathcal{A}$. We recall that $\mathcal{C}$ is an abelian subcategory of $\mathcal{A}$ if the inclusion functor $i: \mathcal{C} \rightarrow \mathcal{A}$ is exact. The following is a nice criteria to check that an additive subcategory is an abelian subcategory.

**Remark 2.1.** Let $\mathcal{C}$ be an additive subcategory of an abelian category $\mathcal{A}$. Then, $\mathcal{C}$ is an abelian subcategory of $\mathcal{A}$ if and only if $\mathcal{C}$ is closed under kernels and cokernels (in $\mathcal{A}$) of morphisms in $\mathcal{C}$.

**Corollary 2.2.** Let $\mathcal{C}$ be a wide subcategory of an abelian category $\mathcal{A}$. Then, $\mathcal{C}$ is abelian and the inclusion functor $i: \mathcal{C} \rightarrow \mathcal{A}$ is exact. Moreover, $\text{Ext}^1_{\mathcal{C}}(X, Y) = \text{Ext}^1_{\mathcal{A}}(X, Y)$ for any $X, Y \in \mathcal{C}$.

**Proof.** It follows from the definition of wide category and Remark 2.1. $\square$

**Proposition 2.3.** Let $\mathcal{C}$ be a subcategory of an abelian category $\mathcal{A}$. Then, $\mathcal{C}$ is admissible if and only if $\mathcal{C}$ is wide.
Proof. $(\Rightarrow)$ Assume that $C$ is admissible. Let us show that $C$ is closed under kernels and cokernels. Let $f : A \to B$ be a morphism in $C$. Consider the sub-objects $M$ and $N$ of $A \oplus B \in C$, where

$$M := \{(a, f(a)) : a \in A\}$$

and

$$N := \{(a, 0) : a \in A\}.$$

Since $g : A \to M$, given by $a \mapsto (a, f(a))$, is an isomorphism and $N \simeq A$, we get that $M$ and $N \in C$. Therefore, using that $A \oplus B \in C$, it follows that $\text{Ker}(f) \simeq M \cap N \in C$. Thus, the exact sequence

$$0 \to \text{Ker}(f) \to A \to \text{Im}(f) \to 0$$

gives us that $\text{Im}(f) \in C$. Finally, by considering the exact sequence

$$0 \to \text{Im}(f) \to A \to \text{Coker}(f) \to 0,$$

we get that $\text{Coker}(f) \in C$.

$(\Leftarrow)$ Suppose that $C$ is wide. We assert that $C$ is closed under direct summands. Indeed, let $C \in C$ and consider a decomposition $C = A \oplus B$ in $\text{mod}(\Lambda)$ and let $\alpha := \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} : C \to C$. Thus $B \simeq \text{Ker}(\alpha) \in C$, proving that $C$ is closed under direct summands.

Finally, let $M_1$ and $M_2$ be sub-objects of $M$ with $M_1, M_2$ and $M \in C$. Consider the morphism

$$f : M_1 \oplus M_2 \to M, \quad (m_1, m_2) \mapsto m_1 + m_2.$$

Then, since $C$ is closed under kernels of morphisms in $C$, we get

$$M_1 \cap M_2 \simeq \{(m, -m) : m \in M_1 \cap M_2\} = \text{Ker}(f) \in C;$$

proving that $C$ is admissible. □

Corollary 2.4. Let $C$ be a wide subcategory of $\text{mod}(\Lambda)$. If $\Lambda \Lambda \in C$ then $C = \text{mod}(\Lambda)$.

Proof. For every $M \in \text{mod}(\Lambda)$ there exists an exact sequence

$$\Lambda \Lambda \xrightarrow{f} \Lambda \Lambda \to M \to 0,$$

and so we have that $M \simeq \text{Coker}(f) \in C$. □

3. Wide finitely generated subcategories

Let $\Lambda$ be an artin $R$-algebra. For any $M \in \text{mod}(\Lambda)$, we sometimes use $(M, -)$ to denote the functor $\text{Hom}_{\Lambda}(M, -)$.

Consider a wide subcategory $\mathcal{X}$ of $\text{mod}(\Lambda)$. We say that $\mathcal{X}$ is wide finitely generated if there is some $X \in \mathcal{X}$ such that $\mathcal{X} = \text{wide}(X)$. We also say that $\mathcal{X}$ is wide additively generated if there is some $X \in \mathcal{X}$ such that $\mathcal{X} = \text{add}(X)$.
In what follows, for some indecomposable \( M \in \text{mod}(\Lambda) \), we give necessary and sufficient conditions for the category \( \text{add}(M) \) to be wide. For doing so, we need the following result.

**Proposition 3.1.** Let \( M \in \text{mod}(\Lambda) \) be such that \( \text{End}_\Lambda(M) \) is a semi-simple ring. Then, the category \( \text{add}(M) \) is closed under kernels, images and cokernels of morphisms in \( \text{add}(M) \). In particular, \( \text{add}(M) \) is an abelian subcategory of \( \text{mod}(\Lambda) \) and \( (M, -) : \text{add}(M) \to \text{mod}(\Gamma) \) is an equivalence of categories, where \( \Gamma := \text{End}_\Lambda(M)^{\text{op}} \).

**Proof.** It is well known (see [2, 3]) that the evaluation functor \( (M, -) : \text{mod}(\Lambda) \to \text{mod}(\Gamma) \) induces an equivalence of \( \Lambda \)-categories between \( \text{add}(M) \) and \( \text{proj}(\Gamma) \).

By hypothesis we know that \( \Gamma \) is a semi-simple ring. Thus, we have that \( (M, -) : \text{add}(M) \to \text{mod}(\Gamma) \) is an equivalence of categories.

Firstly we assert that \( \text{add}(M) \) has the following property: For all \( X, E \in \text{add}(M) \) any monomorphism (epimorphism) \( \alpha : X \to E \) is a split-mono (split-epi). In particular, the class \( \text{add}(M) \) is closed under cokernels (kernels) of monomorphisms (epimorphisms). Indeed, let \( \alpha : X \to E \) be a monomorphism with \( X, E \in \text{add}(M) \). Therefore \( (M, \alpha) : (M, X) \to (M, E) \) is a monomorphism in \( \text{mod}(\Gamma) \). Thus \( (M, \alpha) \) is a split-mono, since \( \Gamma \) is a semi-simple ring.

But now, using the fact that \( (M, -) : \text{add}(M) \to \text{mod}(\Gamma) \) is full and faithful, we get that \( \alpha \) is a split-mono.

Let \( f : A \to B \) be a morphism in \( \text{add}(M) \). We prove that \( \text{Im}(f) \) and \( \text{Coker}(f) \) belong to \( \text{add}(M) \). Note that, from the exact sequence \( 0 \to \text{Im}(f) \to B \to \text{Coker}(f) \to 0 \) and the fact that \( \text{add}(M) \) is closed under cokernels of monomorphisms, we reduce the problem to prove only that \( \text{Im}(f) \in \text{add}(M) \).

Indeed, consider the morphism \( (M, f) : (M, A) \to (M, B) \) of \( \Gamma \)-modules and its factorization through its image \( \text{Im}(M, f) \). Since \( (M, -) : \text{add}(M) \to \text{mod}(\Gamma) \) is an equivalence of categories and \( \Gamma \) is a semi-simple ring, we have that \( \text{Im}(M, f) = (M, M') \) and \( (M, f) : (M, A) \to (M, B) \) is the composition of a split-epi \( (M, \alpha) : (M, A) \to (M, M') \) and a split-mono \( (M, \beta) : (M, M') \to (M, B) \). In particular, we get that \( f = \beta \alpha \), where \( \alpha \) is a split-epi and \( \beta \) is a split-mono. Thus \( \text{Im}(f) \cong M' \in \text{add}(M) \).

Finally, to prove that \( \text{Ker}(f) \in \text{add}(M) \), we use the exact sequence \( 0 \to \text{Ker}(f) \to A \to \text{Im}(f) \to 0 \).

Given an artin algebra \( \Lambda \) and \( M \in \text{mod}(\Lambda) \), we denote by \( \ell_\Lambda(M) \) the length of a Jordan-Hölder composition series of \( M \).

**Theorem 3.2.** Let \( M \) be an indecomposable \( \Lambda \)-module. Then, \( \text{add}(M) = \text{wide}(M) \) if and only if \( \text{Ext}^1_\Lambda(M, M) = 0 \) and \( \text{End}_\Lambda(M) \) is a division ring.

**Proof.** \( \Rightarrow \) Assume that \( \text{add}(M) = \text{wide}(M) \). Let us see that \( \text{Ext}^1_\Lambda(M, M) = 0 \). Indeed, consider an exact sequence

\[
\eta : 0 \to M \xrightarrow{\alpha} L \xrightarrow{\beta} M \to 0
\]
in \(\text{mod}(\Lambda)\). Since \(\text{add}(M)\) is closed under extensions and \(M\) is indecomposable, we get that \(L \simeq M^t\). Thus \(t \ell_\Lambda(M) = \ell_\Lambda(L) = 2 \ell_\Lambda(M)\) and so \(L \simeq M^2\).

Consider the evaluation functor \((M, -) : \text{mod}(\Lambda) \to \text{mod}(\Gamma)\), where \(\Gamma := \text{End}_\Lambda(M)^{op}\). By applying the functor \((M, -)\) to \(\eta\), we get the exact sequence in \(\text{mod}(\Gamma)\)

\[
(M, \eta) : 0 \to (M, M) \xrightarrow{(M, \alpha)} (M, L) \xrightarrow{(M, \beta)} (M, M).
\]

We claim that \(\text{Im}((M, \beta)) = (M, M)\).

Indeed, suppose that this is not the case, and so \(\ell_\Gamma(\text{Im}((M, \beta))) < \ell_\Gamma(M, M)\). Furthermore, the exact sequence of \(\Gamma\)-modules

\[
0 \to (M, M) \to (M, L) \to \text{Im}((M, \beta)) \to 0,
\]

gives us the following

\[
\ell_\Gamma(M, L) = \ell_\Gamma(M, M) + \ell_\Gamma(\text{Im}((M, \beta))) < 2 \ell_\Gamma(M, M).
\]

Using now that \(L \simeq M^2\), we get \(\ell_\Gamma(M, L) = 2 \ell_\Gamma(M, M)\); which is a contradiction, proving that \(\text{Im}((M, \beta)) = (M, M)\) implies that the exact sequence \(\eta\) splits and hence \(\text{Ext}^1_\Lambda(M, M) = 0\).

We prove now that \(\text{End}_\Lambda(M)\) is a division ring. Suppose that this is not the case, and so the radical of the ring \(\text{End}_\Lambda(M)\) is non-zero. Then, there exists a non-zero nilpotent endomorphism \(f : M \to M\). Note that \(\text{Ker}(f) \neq 0\), since \(M\) is non-zero.

Since \(\text{add}(M)\) is wide, we get that \(\text{Ker}(f) \in \text{add}(M)\). By using Krull-Schmidt theorem and the fact that \(M\) is indecomposable, we conclude that \(\text{Ker}(f) \simeq M^t\), for some \(t \geq 1\). Therefore \(\text{Ker}(f) = M\) and then \(f = 0\), which is a contradiction; proving that \(\text{End}_\Lambda(M)\) is a division ring.

\((\Leftarrow)\) It follows from Proposition 3.1 and using the fact that \(\text{Ext}^1_\Lambda(M, M) = 0\) implies that \(\text{add}(M)\) is closed under extensions. \(\square\)

**Remark 3.3.** Let \(\Lambda\) be a finitely dimensional \(k\)-algebra, over an algebraically closed field \(k\).

1. Any indecomposable \(\Lambda\)-module \(M\), which is post-projective or pre-injective, satisfies that \(\text{End}_\Lambda(M) = k\) and \(\text{Ext}^1_\Lambda(M, M) = 0\) \([3, \text{Lemma VIII.2.7}]\).
2. Let \(\Lambda\) be a hereditary algebra of finite representation type. Then, any indecomposable \(\Lambda\)-module \(M\) satisfies that \(\text{End}_\Lambda(M) = k\) and \(\text{Ext}^1_\Lambda(M, M) = 0\) \([3, \text{Corollary VII.5.14}]\).

Therefore, by Theorem 3.2 each such module produces an example of a wide additively generated subcategory of \(\text{mod}(\Lambda)\).

**Remark 3.4.** In general we could have that \(\text{add}(M) \subsetneq \text{wide}(M)\), but still \(\text{wide}(M) = \text{add}(N)\) for some other \(\Lambda\)-module \(N\). To see this, take an algebra which is of finite representation type and an indecomposable module which has
a self-extension. Then it is clear that \( \text{wide}(M) \) is of the form \( \text{add}(N) \), since for finite representation type algebras \( \Lambda \) every full subcategory which is closed by direct summands is of the type \( \text{add}(N) \), just take \( N \) the direct summand of the indecomposable in \( N \). It is also clear that if \( M \) has self-extension, then \( \text{wide}(M) = \text{add}(N) \) properly contains \( \text{add}(M) \).

Even if we take \( M \) indecomposable without self-extensions it can happen that \( \text{wide}(M) = \text{add}(N) \) which properly contains \( \text{add}(M) \), for instances of that, see the following examples.

**Example 3.5.** Let \( n > 2 \) and \( \Lambda = k\tilde{A}_{n-1}/J^{n+1} \) be a symmetric Nakayama algebra where \( J \) denotes the arrow ideal of the path \( k \)-algebra \( \tilde{A}_{n-1} \) [3 Proposition V.3.8]. Note that \( P_1/\text{rad}P_1 \cong P_1/\text{soc}(P_1) \). Moreover, we have that

\[
\text{wide}(P_1) = \text{add}[(P_1/\text{rad}P_1) \oplus (P_1/\text{rad}P_1) \oplus P_1 \oplus (P_2/\text{rad}^2P_2) \oplus (P_2/\text{rad}^2P_2)]
\]

Indeed, the reader can check that

\[
\text{add}[(P_1/\text{rad}P_1) \oplus (P_1/\text{rad}P_1) \oplus P_1 \oplus (P_2/\text{rad}^2P_2) \oplus (P_2/\text{rad}^2P_2)]
\]

is closed under extensions, kernels and cokernels of morphisms.

**Example 3.6.** Let \( \Lambda = k\tilde{A}_2/I \) where \( \tilde{A}_2 \) is the quiver

\[
\begin{array}{ccc}
1 & \overset{\gamma}{\rightarrow} & 3 \\
\downarrow & & \\
2 & \overset{\alpha}{\rightarrow} & \beta \\
\end{array}
\]

and \( I = \langle \beta\alpha \rangle \). Then

\[
\text{wide}(P_2) = \text{add}[(P_2/\text{rad}P_2) \oplus (P_2/\text{rad}P_2) \oplus P_2 \oplus P_3 \oplus (P_2/\text{rad}^3P_2)].
\]

In what follows, we give some discussion and a generalization of Theorem 3.2.

**Remark 3.7.** Let \( M = \oplus_{i=1}^t M_i \) with \( M_i \) indecomposable and \( M_i \not\cong M_j \) for \( i \neq j \).

1. If \( \text{Ext}_\Lambda^1(M,M) = 0 \) and \( \text{End}_\Lambda(M) \) is a product of division rings, then \( \text{add}(M) = \text{wide}(M) \).

   This follows directly from Proposition 3.1 since \( \text{Ext}_\Lambda^1(M,M) = 0 \) implies that \( \text{add}(M) \) is closed under extensions.

2. The converse of (1) does not hold in general.

To see this, consider an artin algebra \( \Lambda \) of finite representation type, which is not semi-simple and such that the set of indecomposable \( \Lambda \)-modules (up to isomorphism) is given by the set \( \{M_1, M_2, \ldots, M_n\} \) with \( n \geq 2 \). Take \( M := \oplus_{i=1}^n M_i \). It is clear that \( \text{add}(M) = \text{mod}(\Lambda) \)
and therefore \( \text{add}(M) \) is wide, but \( \text{End}_\Lambda(M) \) is not semi-simple neither \( \text{Ext}_\Lambda^1(M, M) = 0 \).

The next result states that the converse of item (1) in Remark 3.7 holds whenever \( M \) is a projective \( \Lambda \)-module.

**Proposition 3.8.** Let \( P = \bigoplus_{i=1}^t P_i \) with each \( P_i \) indecomposable projective \( \Lambda \)-module such that \( P_i \neq P_j \) for \( i \neq j \). Then, the following statements are equivalent.

(a) \( \text{add}(P) = \text{wide}(P) \).
(b) \( \text{End}_\Lambda(P) = \times_{i=1}^t \text{End}_\Lambda(P_i) \) and every \( \text{End}_\Lambda(P_i) \) is a division ring.
(c) \( \text{End}_\Lambda(P) \) is a product of division rings.

**Proof.** We only show that the first and the second statements are equivalent. Clearly, the second statement implies the third. We leave for the reader to show that the third implies the second.

Let us show the equivalence between the first the second statement. We first assume that \( \text{add}(P) \) is a wide subcategory of \( \text{mod}(\Lambda) \).

In order to get the decomposition \( \text{End}_\Lambda(P) = \times_{i=1}^t \text{End}_\Lambda(P_i) \) as rings, it is enough to see that \( \text{Hom}_\Lambda(P_i, P_j) = 0 \), for any \( i \neq j \). Indeed, suppose there exists a non-zero morphism \( f : P_i \to P_j \) for some \( i \neq j \). Since \( \text{add}(P) \) is wide, we have that \( 0 \to \text{Ker}(f) \to P_i \to \text{Im}(f) \to 0 \) and \( 0 \to \text{Im}(f) \to P_j \to \text{Coker}(f) \to 0 \) are exact sequences in \( \text{add}(P) \), and thus both of them split. So, we get the decompositions \( P_i = \text{Ker}(f) \oplus \text{Im}(f) \) and \( P_j = \text{Im}(f) \oplus \text{Coker}(f) \).

Using now that \( P_i \) and \( P_j \) are indecomposable and the fact that \( \text{Im}(f) \neq 0 \), it follows that \( P_i \simeq \text{Im}(f) = P_j \); contradicting that \( P_i \neq P_j \).

We prove now that \( \text{End}_\Lambda(P_i) \) is a division ring. Indeed, let \( g : P_i \to P_i \) be a non-zero morphism. Since \( \text{add}(P) \) is wide, we have that \( 0 \to \text{Ker}(g) \to P_i \to \text{Im}(g) \to 0 \) is an exact sequence in \( \text{add}(P) \), and so it splits giving us \( P_i = \text{Ker}(g) \oplus \text{Im}(g) \). Since \( P_i \) is indecomposable and \( g \neq 0 \), it follows that \( g \) is an isomorphism.

The other implication follows from item (1) of the former remark. □

4. **Standard modules and wide subcategories**

Let \( \Lambda \) be an artin \( R \)-algebra and \( D := \text{Hom}_\Lambda(-, I) : \text{mod}(\Lambda) \to \text{mod}(\Lambda^{op}) \) be the usual duality functor, where \( I \) is the injective envelope of \( R/\text{rad}(R) \).

In this section we will prove that the category \( \text{wide}(\Delta) \) coincides with \( \text{mod}(\Lambda) \), where \( \Delta \) is the set of standard modules. Furthermore, we analyse the case when the category \( \mathfrak{F}(\Delta) \), of \( \Delta \)-filtered modules, is wide. In order to do that, we will need some preliminary results. Recall that \( \text{top}(M) \) denotes the module \( M/\text{rad}(M) \), for any \( M \in \text{mod}(\Lambda) \).

**Lemma 4.1.** Let \( \mathcal{X} \subseteq \text{mod}(\Lambda) \) be a wide subcategory and \( M \in \mathcal{X} \). If \( \text{top}(M) \) is a direct summand of \( \text{soc}(M) \) then \( \text{top}(M) \in \mathcal{X} \).
Proof. Let \( \text{top}(M) \) be a direct summand of \( \text{soc}(M) \). Then there is a monomorphism \( \mu : \text{top}(M) \to \text{soc}(M) \). Consider now the natural projection \( \pi : M \to \text{top}(M) \) and the inclusion \( i : \text{soc}(M) \to M \). Therefore the morphism \( f := i \mu \pi : M \to M \) satisfies that \( \text{Im}(f) = \text{top}(M) \); and since \( \mathcal{X} \) is wide we get that \( \text{top}(M) = \text{Im}(f) \in \mathcal{X} \). \( \square \)

Proposition 4.2. Let \( \mathcal{X} \) be a wide subcategory of \( \text{mod}(\Lambda) \) and let
\[
\{M_1, M_2, \cdots, M_i\} \subseteq \mathcal{X}
\]
be such that \( \text{top}(M_i) \) is equal to the simple \( S_i \) for all \( i \). If each module \( M_i \) has only composition factors among the simple \( \Lambda \)-modules \( S_j \) with \( j \leq i \), then \( \{S_1, S_2, \cdots, S_1\} \subseteq \mathcal{X} \).

Proof. We will proceed by induction on \( t \). By Lemma 4.1 the result is clear for \( t = 1 \). Let us assume that \( \{S_1, S_2, \cdots, S_{j-1}\} \subseteq \mathcal{X} \). To prove that \( S_j \in \mathcal{X} \) we will use induction on \( \ell_A(M_j) \). If \( \ell_A(M_j) = 1 \) then \( S_j = M_j \in \mathcal{X} \).

Let \( \ell_A(M_j) > 1 \). Consider the exact sequence
\[
\varepsilon : 0 \to \text{Tr}_{\mathcal{X}}(\text{soc} M_j) \to M_j \to M_j' \to 0.
\]
If \( \text{Tr}_{\mathcal{X}}(\text{soc} M_j) = 0 \) then \( \text{soc}(M_j) = S_j^{\text{soc}} \) and by Lemma 4.1 we get \( S_j \in \mathcal{X} \).

Assume that \( \text{Tr}_{\mathcal{X}}(\text{soc} M_j) \neq 0 \). Then \( \ell_A(M_j') < \ell_A(M_j) \) with \( M_j' \) having composition factors among the simple \( \Lambda \)-modules \( S_1, S_2, \cdots, S_j \). Furthermore, since \( \text{Tr}_{\mathcal{X}}(\text{soc} M_j) \in \mathfrak{I}(\{S_1, S_2, \cdots, S_{j-1}\}) \subseteq \mathcal{X} \), \( M_j \in \mathcal{X} \) and \( \mathcal{X} \) is wide, from the exact sequence \( \varepsilon \) we get that \( M_j' \in \mathcal{X} \). Thus, by induction, since \( \ell_A(M_j') < \ell_A(M_j) \), we conclude that \( S_j \in \mathcal{X} \). \( \square \)

Theorem 4.3. For the set \( \Delta \) of standard \( \Lambda \)-modules, we have that
\[
\text{wide}(\Delta) = \text{mod}(\Lambda).
\]

Proof. As we know \( \Delta = \{\Delta(1), \Delta(2), \cdots, \Delta(n)\} \) is the set of standard \( \Lambda \)-modules with respect to the natural order \( \leq \) on the set \( \{1, 2, \cdots, n\} \), where \( n \) is the number of simple \( \Lambda \)-modules (up to isomorphism). It is also well known, that \( \text{top}(\Delta(i)) = S_i \) and \( \Delta(i) \) has composition factors among the simple \( \Lambda \)-modules \( S_j \) with \( j \leq i \). Then, by Proposition 4.2 we get that \( \text{wide}(\Delta) \) contains all the simple \( \Lambda \)-modules; proving the result. \( \square \)

Corollary 4.4. Let \( \mathfrak{I}(\Delta) \) be an abelian subcategory of \( \text{mod}(\Lambda) \). Then \( \Lambda \) is a quasi-hereditary algebra and \( \text{gl.dim}(\Lambda) \leq 1 + \text{pd} \, D(\Lambda) \), where \( D \) is the usual duality functor.

Proof. It is well known that \( \mathfrak{I}(\Delta) \) is closed under extensions, and hence, by Remark 2.1 we get that \( \mathfrak{I}(\Delta) \) is a wide subcategory of \( \text{mod}(\Lambda) \). Then, by Theorem 4.3 we conclude that \( \mathfrak{I}(\Delta) = \text{mod}(\Lambda) \). Therefore \( \Lambda \in \mathfrak{I}(\Delta) \) and \( \mathfrak{I}(\Delta) \) is closed under submodules. Thus, from [12, Proposition 3.21] we get that \( \Lambda \) is quasi-hereditary and \( \text{gl.dim}(\Lambda) \leq 1 + \text{pd} \, T \), where \( T \) is the
characteristic tilting $\Lambda$-module associated to $\Lambda$. Furthermore [15, Corollary 4] gives us that $T = \oplus_{i=1}^n I_i$, where $I_i$ is the injective envelope of the simple $S_i$. In particular, we have that $\text{pd} (T) = \text{pd} D(\Lambda_\Lambda)$. □

It is well known that $\mathcal{F}(\Delta)$ is closed under extensions, direct summands and kernels of epimorphisms. A natural question is to give necessary and sufficient conditions to get that $\mathcal{F}(\Delta)$ is a thick subcategory of $\text{mod} (\Lambda)$. The following result characterizes, in the case of standardly stratified algebras, when the category $\mathcal{F}(\Delta)$ is closed under cokernels of monomorphisms. We denote by $\mathcal{P}^{<\infty}(\Lambda)$ the class of all $\Lambda$-modules of finite projective dimension.

**Proposition 4.5.** Let $\Lambda$ be a standardly stratified algebra. Then, $\mathcal{F}(\Delta)$ is a thick subcategory of $\text{mod} (\Lambda)$ if and only if $\mathcal{F}(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$.

**Proof.** ($\Leftarrow$) If $\mathcal{F}(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$ then the conclusion is trivial, since $\mathcal{P}^{<\infty}(\Lambda)$ is a thick subcategory of $\text{mod} (\Lambda)$.

($\Rightarrow$) Assume that $\mathcal{F}(\Delta)$ is closed under cokernels of monomorphisms. In order to obtain that $\mathcal{F}(\Delta) = \mathcal{P}^{<\infty}(\Lambda)$, by using [12, Proposition 3.18], it is enough to prove that $(\text{add} T)^\wedge \subseteq \mathcal{F}(\Delta)$, where $T$ is the characteristic tilting $\Lambda$-module. For the notation of $(\text{add} T)^\wedge$, see [12, p. 397].

Let $M \in (\text{add} T)^\wedge$. Then we have an exact sequence

$$0 \to T_k \to T_{k-1} \to \cdots \to T_0 \to M \to 0$$

with $T_i \in \text{add} T$ for $i = 0, 1, \cdots, k$. Since $\text{add} T \subseteq \mathcal{F}(\Delta)$ and $\mathcal{F}(\Delta)$ is closed under cokernels of monomorphisms, we get that $M \in \mathcal{F}(\Delta)$. Therefore $(\text{add} T)^\wedge \subseteq \mathcal{F}(\Delta)$, proving the result. □

It is well known that if $\Lambda$ is a quasi-hereditary algebra then its global dimension is finite. So we get the following corollary.

**Corollary 4.6.** Let $\Lambda$ be a quasi-hereditary algebra. If $\mathcal{F}(\Delta)$ is a thick subcategory of $\text{mod} (\Lambda)$ then $\mathcal{F}(\Delta) = \text{mod} (\Lambda)$.

**Proof.** It follows from Proposition 4.5. □

Next, we consider a quotient path $k$-algebra $\Lambda := kQ/I$ and characterize, in this case, when the category $\mathcal{F}(\Delta)$ is wide. For doing so, we will make use of the following well known definition.

**Definition 4.7.** Let $\Lambda = kQ/I$ be a quotient path $k$-algebra and $\leq$ be a linear order on the set of vertices $Q_0 = \{1, 2, \cdots, n\}$. We say that the algebra $\Lambda$ is triangular, with respect to the partially ordered set $(Q_0, \leq)$, if $Q$ does not have oriented cycles and $\text{Hom}_\Lambda(\Lambda e_i, \Lambda e_j) = 0$ for $i < j$.

We remark that the above definition is equivalent to say that the ordinary quiver of $\Lambda$ is directed. The definition of directed quiver was introduced by S. Koenig in [10].
Theorem 4.8. Let $\Lambda = kQ/I$ be a quotient path $k$-algebra, $\leq$ be a linear order on $Q_0$ and $\Delta$ be the set of standard $\Lambda$-modules (with respect to the linear order $\leq$ on $Q_0$). Then, the following statements are equivalent.

(a) $\mathcal{F}(\Delta) = \text{mod}(\Lambda)$.
(b) $\mathcal{F}(\Delta)$ is wide.
(c) $\Lambda$ is triangular with respect to the partially ordered set $(Q_0, \leq)$.
(d) $\mathcal{F}(\Delta)$ is an abelian subcategory of mod($\Lambda$).

Proof. Let $Q_0 = \{v_1, v_2, \cdots, v_n\}$ where $v_1 < v_2 < \cdots < v_n$. For simplicity, we consider $\Delta(i) := \Delta(v_i)$, $S_i := S_{v_i}$ and $P_i := P_{v_i}$.

(a) $\Leftrightarrow$ (b) It follows from Theorem 4.3.

(a) $\Rightarrow$ (c) Since $\mathcal{F}(\Delta) = \text{mod}(\Lambda)$, we get that $\Delta(i) = S_i$ for any $i$; otherwise $S_i$ could not be in $\mathcal{F}(\Delta)$ and so $P_n = \Delta(n) = S_n$. Therefore $v_n$ is a sink and then $\text{Hom}_\Lambda(\Lambda e_j, \Lambda e_n) \cong e_j \Lambda e_n = 0$ for any $j < n$. We claim that $v_{n-1}$ is a sink in $Q - \{v_n\}$. Indeed, we know that $S_{n-1} = \Delta(n-1) = P_{n-1}/\text{Tr}P_n(P_{n-1})$. Suppose that $v_{n-1}$ is not a sink in $Q - \{v_n\}$. Then there exists $t < n - 1$ and an arrow $v_{n-1} \to v_t$ in $Q$. This implies that the simple $S_t$ is a composition factor of $\Delta(n-1) = S_{n-1}$, which is a contradiction. Hence $v_{n-1}$ is a sink in $Q - \{v_n\}$. Inductively, we get that $v_{n-i}$ is a sink in $Q - \{v_n, v_{n-1}, \cdots, v_{n-i+1}\}$. Therefore $Q$ does not have oriented cycles and $\text{Hom}_\Lambda(\Lambda e_i, \Lambda e_j) \cong e_i \Lambda e_j = 0$ for all $i < j$.

(c) $\Rightarrow$ (a) Since $Q$ does not have oriented cycles and $\text{Hom}_\Lambda(\Lambda e_i, \Lambda e_j) = 0$ for $i < j$, we get that $\text{Tr}_{\oplus j\neq i}P_j(P_i) = \text{Tr}_{\oplus j\neq i}P_j(P_i)$. Therefore

$$\Delta(i) = P_i/\text{Tr}_{\oplus j\neq i}P_j(P_i) = P_i/\text{Tr}_{\oplus j\neq i}P_j(P_i) \cong S_i.$$

Then, we get that

$$\mathcal{F}(\Delta(1), \Delta(2), \cdots, \Delta(n)) = \mathcal{F}(S_1, S_2, \cdots, S_n) = \text{mod}(\Lambda).$$

Finally, from Theorem 4.3 and the using the fact that $\mathcal{F}(\Delta)$ is closed under extensions, we get that the items (b) and (d) are equivalent. \hfill $\Box$

Let $\Lambda$ be an artin $R$-algebra. We finish this section with the following proposition.

Proposition 4.9. Let $\mathcal{X}$ be a wide subcategory of mod($\Lambda$) and $I$ a two sided ideal of $\Lambda$ such that $\Lambda/I \in \mathcal{X}$. Then $\text{soc}(\Lambda/I) \in \mathcal{X}$.

Proof. Let us see first that

$$(*) \quad \forall t \in \Lambda/I \quad \text{ann}_{\Lambda/I}(t) \in \mathcal{X}.$$

Indeed, for any $t \in \Lambda$, the morphism $\phi_t : \Lambda/I \to \Lambda/I$, given by $\phi_t(x) := xt$ satisfies $\text{ann}_{\Lambda/I}(t) = \text{Ker}(\phi_t) \in \mathcal{X}$, proving $(*)$.

Since $\text{soc}(\Lambda/I)$ is a finitely generated $R$-module, we have that $\text{soc}(\Lambda/I) = \langle x_1, x_2, \cdots, x_m \rangle_R$. Thus, from $(*)$ and Proposition 2.3, we get that

$$\text{ann}(\text{soc}(\Lambda/I)) = \cap_{i=1}^m \text{ann}_{\Lambda/I}(x_i) \in \mathcal{X}.$$
and therefore \( \frac{\Lambda/I}{\text{ann}(\text{soc}(\Lambda/I))} \in \mathcal{X} \). Finally, since
\[
\text{add} (\text{soc}(\Lambda/I)) = \text{add} \left[ \frac{\Lambda/I}{\text{ann}(\text{soc}(\Lambda/I))} \right],
\]
we get that \( \text{soc}(\Lambda/I) \in \mathcal{X} \). □

5. Wide categories of type \( \text{pres}(P) \) with \( P \) projective

Let \( \Lambda \) be an artin \( R \)-algebra. In this section, we characterise when a wide subcategory \( \mathcal{X} \) of \( \text{mod}(\Lambda) \) is of the form \( \text{pres}(P) \) for some \( P \in \text{proj}(\Lambda) \).

Let \( P \in \text{proj}(\Lambda) \). We recall, for details see [2], that \( \text{pres}(P) \) is the full subcategory of \( \text{mod}(\Lambda) \), whose objects are all the \( \Lambda \)-modules \( M \) admitting a presentation in \( \text{add}(P) \), that is, an exact sequence \( P_1 \to P_0 \to M \to 0 \) with \( P_0, P_1 \in \text{add}(P) \). Using the Horseshoe Lemma, we see that \( \text{pres}(P) \) is closed under extensions.

Let \( \Gamma := \text{End}_{\Lambda}(P)^{op} \). In this case, it is well known, see [2, Proposition 2.5], that the evaluation functor \((P, -) : \text{pres}(P) \to \text{mod}(\Gamma)\) is an equivalence of categories. So, by using this equivalence, we can translate the abelian structure from \( \text{mod}(\Gamma) \) to \( \text{pres}(P) \). In general, it can happen that \( \text{pres}(P) \) is not an abelian subcategory of \( \text{mod}(\Lambda) \), although \( \text{pres}(P) \) is an abelian category. In what follows, we give an example illustrating this situation.

Example 5.1. Let \( \Lambda = k\tilde{A}_2/I \) where \( \tilde{A}_2 \) is the quiver

\[\begin{array}{ccc}
2 & \alpha & 3 \\
\gamma & \beta & \\
1
\end{array}\]

and \( I = J^5 \). Recall that \( J \) denotes de arrow ideal in \( k\tilde{A}_2 \). Let \( P := P_1 \) be the projective \( \Lambda \)-module attached to the vertex 1. We assert that \( \text{pres}(P) \) is not an abelian subcategory of \( \text{mod}(\Lambda) \). Observe that in this case \( \text{pres}(P) \) is equivalent to \( \text{mod}(k[x]/(x^2)) \).

Indeed, consider the natural projection \( \pi : P \to P/\text{rad}^2(P) \). Note that \( P/\text{rad}^2(P) \simeq \text{rad}^1(P) \) and hence we have a monomorphism \( \mu : P/\text{rad}^2(P) \to P \). Let \( g := \mu \pi : P \to P \). Note that \( \ker(g) \simeq P_3/\text{rad}^3(P_3) \notin \text{add}(P) \). Therefore \( \text{Im}(g) \notin \text{pres}(P) \); proving that \( \text{pres}(P) \) is not an abelian subcategory of \( \text{mod}(\Lambda) \).

For a given subcategory \( \mathcal{X} \subseteq \text{mod}(\Lambda) \), we consider the class \( \text{P}_0(\mathcal{X}) := \{ P_0(X) : X \in \mathcal{X} \} \), where \( P_0(X) \) is the projective cover of \( X \). We recall that, an exact sequence in \( \mathcal{X} \) is just an exact sequence in \( \text{mod}(\Lambda) \), whose terms are all in \( \mathcal{X} \).
Recall that \( \text{proj} (\mathcal{X}) \) denotes the full subcategory of \( \mathcal{X} \) whose objects are relatively projective, that is, \( P \in \text{proj} (\mathcal{X}) \) if and only if for any exact sequence \( 0 \to A \to B \xrightarrow{f} C \to 0 \) in \( \mathcal{X} \), the map \( \text{Hom}_\Lambda (P, f) : \text{Hom}_\Lambda (P, B) \to \text{Hom}_\Lambda (P, C) \) is surjective. We say that \( \mathcal{X} \) has enough projectives if any \( M \in \mathcal{X} \) admits an exact sequence \( 0 \to K \to P \to M \to 0 \) in \( \mathcal{X} \) with \( P \in \text{proj} (\mathcal{X}) \).

**Theorem 5.2.** Let \( \mathcal{X} \) be a wide subcategory of \( \text{mod} (\Lambda) \). Then, the following statements are equivalent.

1. \( \mathcal{X} = \text{pres} (P) \) for some \( P \in \text{proj} (\Lambda) \).
2. \( \mathcal{X} \) has enough projectives and \( \text{proj} (\mathcal{X}) \subseteq \text{proj} (\Lambda) \).
3. \( \mathcal{X} \) is closed under projective covers, that is \( \text{proj} (\mathcal{X}) \subseteq \mathcal{X} \).

If one of the above equivalent conditions holds, then \( \text{add} (P) = \text{proj} (\mathcal{X}) = \text{add} (\text{proj} (\mathcal{X})) \).

In particular, \( P \) is uniquely determined up to additive closures, and the functor \( \text{Hom}_\Lambda (P, -) : \mathcal{X} \to \text{mod} (\text{End}_\Lambda (P)^{\text{op}}) \) is an equivalence of categories.

**Proof.** (a) \( \Rightarrow \) (b) It is clear that \( \text{add} (P) \subseteq \text{proj} (\mathcal{X}) \). We assert that, for any \( M \in \text{pres} (P) \), there is an exact sequence \( 0 \to K \to P_0 \to M \to 0 \) in \( \mathcal{X} \), with \( P_0 \in \text{add} (P) \).

Indeed, let \( M \in \text{pres} (P) \). Then there is an exact sequence \( P_1 \xrightarrow{f} P_0 \to M \to 0 \), with \( P_0, P_1 \in \text{add} (P) \). Using the fact that \( \mathcal{X} \) is wide, we get that \( \text{Im} (f) \in \text{pres} (P) \); and hence \( 0 \to \text{Im} (f) \to P_0 \to M \to 0 \) is the desired exact sequence. This, in particular, implies that \( \mathcal{X} \) has enough projectives, since \( \text{add} (P) \subseteq \text{proj} (\mathcal{X}) \).

Let us prove that \( \text{proj} (\mathcal{X}) \subseteq \text{add} (P) \). Consider \( Q \in \text{proj} (\mathcal{X}) \). Then, there is an exact sequence \( \eta : 0 \to K \to P_0 \to Q \to 0 \) in \( \mathcal{X} \), with \( P_0 \in \text{add} (P) \).

Thus, \( \eta \) splits and then \( Q \in \text{add} (P) \).

(b) \( \Rightarrow \) (a) Let \( P := \bigoplus_{i = 1}^t P_i \), where \( \{ P_1, P_2, \cdots, P_t \} \) is a complete list of indecomposable pairwise non-isomorphic \( \Lambda \)-modules in \( \text{proj} (\mathcal{X}) \subseteq \text{proj} (\Lambda) \).

Furthermore, \( \text{add} (P) \subseteq \mathcal{X} \), since \( P \in \mathcal{X} \). Let \( M \in \text{pres} (P) \). Then there is an exact sequence \( P_1 \xrightarrow{f} P_0 \to M \to 0 \), with \( P_0, P_1 \in \text{add} (P) \). Since \( \text{add} (P) \subseteq \mathcal{X} \), it follows that \( M \cong \text{Coker} (f) \in \mathcal{X} \); proving that \( \text{pres} (P) \subseteq \mathcal{X} \).

Let \( M \in \mathcal{X} \). Using that \( \mathcal{X} \) has enough projectives and \( \text{proj} (\mathcal{X}) = \text{add} (P) \), we get an exact sequence \( 0 \to K \to Q_0 \to M \to 0 \) in \( \mathcal{X} \), with \( Q_0 \in \text{add} (P) \).

Doing the same with \( K \), we get an exact sequence \( Q_1 \to Q_0 \to M \to 0 \), with \( Q_0, Q_1 \in \text{add} (P) \). Therefore \( M \in \text{pres} (P) \) and thus \( \mathcal{X} \subseteq \text{pres} (P) \).

(b) \( \Rightarrow \) (c) Let \( M \in \mathcal{X} \). Then there is an epimorphism \( Q \to M \), with \( Q \in \text{proj} (\mathcal{X}) \subseteq \text{proj} (\Lambda) \). Therefore, the projective cover \( P_0 (M) \) is a direct summand of \( Q \) and thus \( P_0 (M) \in \mathcal{X} \).

(c) \( \Rightarrow \) (b) Since \( \mathcal{P}_0 (\mathcal{X}) \subseteq \mathcal{X} \) and \( \mathcal{X} \) is closed under kernels of epimorphisms, we have an exact sequence \( 0 \to K \xrightarrow{M} P_0 (M) \to M \to 0 \) in \( \mathcal{X} \), for any \( M \in \mathcal{X} \). By taking \( M := Q \in \text{proj} (\mathcal{X}) \), we get that \( Q \) is a direct summand of
$P_0(M)$ and thus $Q \in \proj(A)$. Finally, it is clear that $P_0(M) \in \proj(X)$ and then $X$ has enough projectives. □

Let $\varphi : A \to B$ be a ring epimorphism of finite dimensional $k$-algebras, and let $F_\varphi : \mod(B) \to \mod(A)$ be its associated restriction functor. It is well known that $\mod(B)$ can be identified, as a full subcategory of $\mod(A)$, with the essential image of the functor $F_\varphi$. Let $X$ be a class of finitely generated $A$-modules. We say that $X$ is precovering (or contravariantly finite) if for any $M \in \mod(\Lambda)$, there is a morphism $f : X \to M$, with $X \in X$, such that $(Z, f) : (Z, X) \to (Z, M)$ is bijective for any $Z \in X$. Dually, there is the notion of preenveloping (or covariantly finite) class. The class $X$ is functorially finite if it is precovering and preenveloping.

The following proposition appears in [13, Proposition 4.1] and it is essentially done in [5] (see also in [9, Theorem 1.6.1]). We recall that two ring epimorphisms $f : A \to B$ and $g : A \to C$ are equivalent if there is a (necessarily unique) isomorphism of rings $h : B \to C$ such that $g = hf$.

**Proposition 5.3.** Let $A$ be a finite dimensional $k$-algebra. By assigning to a given ring epimorphism, the essential image of its associated restriction functor, one gets a bijection between

(a) equivalence classes of ring epimorphisms $A \to B$ with $\dim_k(B)$ finite and $\Tor_A^1(B, B) = 0$;

(b) functorially finite wide subcategories of $\mod(A)$.

In case of wide functorially finite subcategories, we have that Theorem 5.2 can be connected, with the proposition above, as can be seen in the following corollary.

**Corollary 5.4.** Let $A$ be a finite dimensional $k$-algebra. Then, for a functorially finite wide subcategory $X$ of $\mod(A)$ the following statements are equivalent.

(a) There is a ring epimorphism $A \to B$ such that $B \in \proj(A)$ and $X = \pres_A(B)$.

(b) $X$ has enough projectives and $\proj(X) \subseteq \proj(A)$.

If one of the above equivalent conditions hold, there is a basic $A$-module $P$ such that $\add(P) = \proj(X)$ and $B$ is Morita equivalent to $\End_A(P)^{\text{op}}$.

**Proof.** (a) ⇒ (b) Suppose (a) holds. Then by Proposition 5.3 we get $X = \mod(B)$ and so $X$ has enough projectives. Furthermore, any indecomposable projective $B$-module is a direct summand of $g_B$, and hence projective as an $A$-module as $B$ is so.

(b) ⇒ (a) Assume that (b) holds. by Proposition 5.3 there is a ring epimorphism $A \to B$ with $\dim_k(B)$ finite and $X = \mod(B)$. But then $B \in \proj(X) \subseteq \proj(A)$.

Finally, the last assertion follows from Theorem 5.2.
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