Non–Abelian orbifold compactifications of the heterotic string

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Abstract: I consider the construction of heterotic orbifold models based on a toroidal orbifold with non–Abelian point group. I construct an explicit model based on the point group $S_3$ and calculate the spectrum and remnant symmetries. This model provides a simple example of rank reduction of the Yang–Mills gauge group directly in the string theory rather than in the effective field theory.

Keywords: heterotic string theory, compactification, non–Abelian orbifold, massless spectrum

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Contents

1 Introduction 1
2 Strings on orbifold backgrounds 2
3 Specification of the model 6
4 Projecting onto invariant states 10
5 Constraints from one-loop modular invariance 15
6 The $S_3$ orbifolded heterotic string 18
7 Conclusions 27
A Fermionic realization of AKM symmetries and bosonization 29
B Tables and pictures 34

1 Introduction

String theory is the most advanced framework to provide a unified quantum description of all fundamental forces in nature. It turned out that the heterotic string [1–3] has very good properties for string model building [4–6]. To arrive at a phenomenologically appealing model the number of spacetime dimensions has to be reduced from ten to four. One way to do so is to compactify six internal dimensions onto a toroidal orbifold. String kinematics on orbifold backgrounds have been studied extensively in the literature ever since its first treatment in [7, 8]. Nevertheless does the restriction to $E_8 \times E_8$ heterotic orbifold models leave us with a plethora of ways to choose the background geometry and fluxes. All possible six dimensional toroidal orbifolds leading to an $N \geq 1$ supersymmetric spectrum have been recently classified in [9]. Most of the orbifold models constructed up to today use orbifolds with Abelian point groups. By using appropriate Wilson line configurations it has been possible to obtain the exact MSSM spectrum using Abelian orbifolds with point group $Z_2 \times Z_2$ [12, 13] or $Z_6^{II}$ [14].

In most Abelian constructions the rank of the gauge group is not reduced. One possibility to achieve rank reduction in the string model itself is to use a non–diagonal embedding of the space group into the gauge group [15, 16]. In the construction of [16] the space group acts
on the bosonic coordinates $X^I$ describing the Yang–Mills sector by affine transformations, $X^I \rightarrow \theta_I^J X^J + 2\pi \lambda^I$, where $\theta_I^J$ is a lattice automorphism. Hence, by the non–Abelianess of the space group the rank of the gauge group is reduced. However, performing explicit spectrum calculations is quite involved. The main reason is the distinction of the bosonic formulation between Cartan currents $H^I = i \partial X^I$ and root currents $E_\alpha = \exp (i \alpha X)$, $\alpha$ some root of $E_8 \times E_8$. The realization of the Yang–Mills sector by free holomorphic fermions provides a more symmetric treatment. It is a hybrid of the free fermion formulation [5, 17–19] and the usual bosonic formalism. Instead of using a pure free fermion CFT to describe the internal sector of the model, only the Yang–Mills sector is described by a fermion CFT [20].

In this paper, I consider an alternative approach to rank reduction via a non–Abelian point group. I provide an explicit example that shows that rank reduction is possible at the string level. The construction is based on the free fermion realization of the Yang–Mills sector and uses bosonization techniques to analyze the spectrum. Non–Abelian heterotic orbifold models have been previously constructed using the free fermion construction in [19, 21]. For an analysis of discrete torsion in such models see [22].

In section 2, I review the construction of a generic orbifold model and describe the structure of the state space that is essential for understanding the orbifold projection. The material presented here is well known and the purpose of this section is to fix notation. Section 3 parametrizes the family of models under consideration. Apart from the orbifold geometry the only additional parameter is the embedding of the point group in the gauge group. In the section 4, I deal with the projection process itself. Here, I describe an algorithm for the extraction of the transformation properties of string states from the specification. Then, I construct an explicit example in section 6. This model is based on a toroidal orbifold with point group $S_3$. I demonstrate the algorithm by giving intermediate results and show that the remnant gauge and discrete symmetries do not suffer from anomalies in the low energy effective field theory description. Finally, section 7 presents my conclusions.

This paper contains two appendices. In appendix 6.1 properties of the group $S_3$ and its representation theory are reviewed and in appendix A the fermionic realization of affine Kač–Moody (AKM) symmetries is reviewed.

## 2 Strings on orbifold backgrounds

### 2.1 The uncompactified heterotic string

An orbifold state in heterotic string theory is defined by an orbifold CFT. The mother CFT is given by a $(0, 1)$ supersymmetric $\sigma$-model on the worldsheet $\Sigma$ where the fields take values in ten dimensional Minkowski space times a holomorphic $E_8 \times E_8$ torus [1]. The worldsheet CFT is non–interacting and the state space $V$ factorizes as

\[ V = V_{\text{ghosts}} \otimes V_{4d} \otimes V_{\text{bosonic}} \otimes V_{\text{fermionic}} \otimes V_{\text{YM}}. \]  

The non–unitary CFT described by $V_{\text{ghost}}$ accounts for the overcounting due to the invariance under local superdiffeomorphisms. As all orbifold constructions considered here
are compatible with this local symmetry, I do not consider this subsector henceforth. The subsector $V_{4d}$ describes propagation in the uncompactified directions, while $V_{bosonic}$ and $V_{fermionic}$ do so for the six internal dimensions. Moreover, forgetting about worldsheet SUSY, I distinguish between the fermionic and bosonic components of the embedding superfields $X^\mu$ and $\bar{\psi}^{\mu}$, $i = 0, \ldots, 9$. $V_{YM}$ is responsible for the $E_8 \times E_8$ Yang-Mills symmetry from the spacetime point of view. There are several ways to realize this subsector. The only restrictions are that its central charge $c$ be $c = 16$ and this CFT be purely holomorphic. In the bosonic description $V_{YM}$ is given by 16 holomorphic bosons taking values in a torus defined by the root lattice of $E_8 \times E_8$. Alternatively, one can realize it by 16 + 16 holomorphic free Majorana-Weyl fermions $\Xi^I(z)$ with GSO projections being different for the first 16 and second 16 fermions. This way, one reproduces the CFT of the bosonic description [20], cf. appendix A.

The spectrum of the mother CFT enjoys two important symmetries. The first one is Yang-Mills symmetry. It originates from the invariance of the Yang-Mills fermion CFT under orthogonal transformations within both of the groups of 16 fermions $\Xi^I(z)$. The generators $\Omega^{IJ}(z)$ of the $SO(16) \times SO(16)$ symmetry are given by

$$\Omega^{IJ}(z) = i : \Xi^I(z) \Xi^J(z) : .$$  \hspace{1cm} (2.2)

The zero modes of these currents are the generators of the gauge symmetry. Moreover, by adding the zero modes of the spin fields $S^\alpha(z)$ that change boundary conditions from NS to R the gauge group is enhanced to $E_8 \times E_8$ as can be shown by passing to the bosonized description. The second important symmetry is ten dimensional Poincaré symmetry. Under a Poincaré transformation $(\Lambda, a)$ the fields $X^\mu$ and $\bar{\psi}^\mu$ transform as

$$(\Lambda, a) \cdot X^\mu = \Lambda^\mu_\nu X^\nu + a^\mu \hspace{1cm} (2.3a)$$

$$(\Lambda, a) \cdot \bar{\psi}^\mu = \Lambda^\mu_\nu \bar{\psi}^\nu. \hspace{1cm} (2.3b)$$

On the antiholomorphic transverse fermions $\bar{\psi}^i$ the Lorentz generators $J^{rs}$ are given by the zero modes of the currents

$$J^{rs}(z) = i : \bar{\psi}^r(z) \bar{\psi}^s(z) : .$$  \hspace{1cm} (2.4)

Comparing equations (2.2) and (2.4) it becomes apparent that the formal treatment of both subsectors is very similar.

The invariance under local superdiffeomorphisms leads to first class constraints that are implemented in the BRST quantized theory via the cohomology of the BRST operator $Q$. Representatives of the cohomology groups in light cone gauge can be found by solving the mass equations for the matter part $|\Psi\rangle_M$,

$$L_0 - 1 |\Psi\rangle_M = 0 \hspace{1cm} (2.5a)$$

$$\left( L_0 - \frac{1}{2} \right) |\Psi\rangle_M = 0, \hspace{1cm} (2.5b)$$

where $L_0$ and $\bar{L}_0$ are the zero modes of the holomorphic, resp. antiholomorphic Virasoro algebra. The ghost part of these states are only important for string dynamics and are not considered here as I am only concerned with the kinematics.
2.2 Orbifold CFTs

The orbifolded CFT is constructed by keeping only states in the model that are invariant under the action of a finite group $G$ of symmetries of the state space. In addition this requires one to keep only those observables that are invariant under the action of $G$. This construction is described in detail by Dijkgraaf et al. in [25], Ginsparg [24] and Dixon et al. in [7, 8].

There are three important consequences for model building. The first one concerns the remnant symmetries. As the observable algebra $A^G$ of the model is reduced to the fixed points under $G$, it is possible that Poincaré symmetry and Yang–Mills symmetry are reduced by orbifolding, i.e. some generators are projected out. The second one concerns the appearance of additional, twisted representations of $A^G$ that do not arise as subspaces of the original state space. From the field theoretical point of view they correspond to instanton sectors, i.e. one allows the fields to be periodic up to the action of $G$. Equivalently, the worldsheet theory has a discrete gauge symmetry $G$. Since observables are invariant under $G$ all physical correlators are single valued thus leading to a local worldsheet theory. The third and most important consequence is the orbifold projection. Here, the spectrum is truncated in such a way that only states invariant under orbifold group are kept. In the following I will describe the generic orbifold projection.

The inclusion of twisted boundary conditions leads to a decomposition of the total state space into twisted sectors $V_g$ one for each group element $g \in G$, 

$$V = \bigoplus_{g \in G} V_g. \quad (2.6)$$

Let $|\phi, g\rangle \in V_g$ denote a state in which the fields obey boundary conditions twisted by $g$. Let $h \in G$ be another orbifold group element. Then, in the state $U(h)|\phi, g\rangle$ the fields obey boundary conditions given by $hgh^{-1}$. Thus, the operator $U(h)$ changed boundary conditions twisted by $g$ to boundary conditions twisted by $hgh^{-1}$, i.e. $U(h)|\phi, g\rangle \in V_{hgh^{-1}}$. Because $G$ is a finite group, invariant states lie in the image of the projection operator $P_{inv}$,

$$P_{inv} = \frac{1}{|G|} \sum_{g \in G} U(g). \quad (2.7)$$

When analyzing an orbifold model that consists of several subsectors, it is convenient to decompose the individual state spaces into subspaces transforming irreducibly first and then use group theoretical methods to find invariant states by combining irreducible representations. Denote by $\chi^g_\alpha$ the character of the irreducible representation $r_\alpha$ of the centralizer $C_G(g)$. The projector $P^g_\alpha$ is given by [25], $e \in G$ denotes the identity element,

$$P^g_\alpha = \frac{\chi^g_\alpha(e)}{|C_G(g)|} \sum_{h \in C_G(g)} \chi^g_\alpha(h^{-1}) U(h). \quad (2.8)$$
The projectors (2.8) define a decomposition of the twisted state space $V_g$ into subspaces $V_\alpha$ transforming homogeneously under $C_G(g)$,

$$V_g = \bigoplus_\alpha r_\alpha \otimes V_\alpha$$  \hspace{1cm} (2.9a)

$$r_\alpha \otimes V_\alpha = P^g_\alpha(V_g).$$  \hspace{1cm} (2.9b)

States in $r_\alpha \otimes V_\alpha$ are said to transform covariantly under $C_G(g)$.

2.3 Toroidal orbifolds

A $\sigma$-model with target space being a toroidal orbifold provides a simple example of an orbifold CFT and the geometry of the target space is important to understand the spacetime theory. Some general information on $\mathbb{Z}_2$ or $\mathbb{Z}_3$ orbifolds can be found in [8, 9, 26]. A geometric orbifold $\mathcal{O} = M/G$ can be defined as a quotient space of a smooth manifold $M$ by the action of a finite group $G$. Two points $x_1$ and $x_2$ are identified if

$$x_1 \sim x_2 \leftrightarrow x_1 = g \cdot x_2,$$  \hspace{1cm} (2.10)

for some group element $g \in G$. The group $G$ is called the orbifold group. The group action may not be free, i.e. there may exist fixed points. Geometric orbifolds fail to be manifolds at the fixed points. Toroidal orbifolds are special cases of geometric orbifolds. They are obtained by taking the manifold $M$ to be a torus. An $n$–dimensional torus $T^n$ is defined as the quotient of $n$–dimensional Euclidean space $\mathbb{R}^n$ by a lattice $\Lambda$, i.e.

$$T^n = \mathbb{R}^n/\Lambda, \quad \Lambda = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n.$$  \hspace{1cm} (2.11)

The lattice $\Lambda$ consists of all integer linear combinations of the generators $e_i \in \mathbb{R}^n$. The number of linearly independent generators is called the rank of the lattice $\Lambda$. A map $\sigma : \Lambda \to \Lambda$, which maps the lattice to itself as a subset of $\mathbb{R}^n$ is called a lattice automorphism. In this case, $\sigma$ induces a well-defined map from the torus $T^n$ to itself. The map $\sigma : T^n \to T^n$ is called a torus automorphism. The set of all torus automorphisms constitutes a group under composition. A subgroup $P$ of torus automorphisms is called a point group. Every point group is finite. The quotient space

$$\mathcal{O} = T^n/P$$  \hspace{1cm} (2.12)

is called a toroidal orbifold. To distinguish between inequivalent classes of geometric strings, it is necessary to introduce the notion of a space group. Instead of constructing the orbifold $\mathcal{O}$ by dividing out first the action of the lattice on $\mathbb{R}^n$ and then the action of the point group on the torus $T^n$, it is possible to construct the orbifold directly by dividing out the action of the space group $S$ on $\mathbb{R}^n$,

$$\mathcal{O} \cong T^n/P \cong \mathbb{R}^n/S.$$  \hspace{1cm} (2.13)

If there are no rototranslations the space group $S = P \rtimes \Lambda$ is a semidirect product of groups [9]. The group $S$ acts on Euclidean space by translations and rotations as ($x \in \mathbb{R}^n$)

$$(g, \lambda) \cdot x = g \cdot x + 2\pi \lambda.$$  \hspace{1cm} (2.14)
3 Specification of the model

3.1 Definition of the model

From the geometric point of view, the orbifold model is completely specified by the space group $S$. The space group $S$ is a subgroup of the Poincaré group which is realized by the generators $P^\mu$ and $J^{\mu\nu}$. A lattice element $\lambda \in \Lambda$ acts on Euclidean space by $(e, \lambda) \cdot x = x + 2\pi \lambda$ and corresponds to a translation. The quantum mechanical realization is provided by the unitary operator

$$U(e, \lambda) = \exp \left( 2\pi i \lambda_i P^i \right).$$

(3.1)

I assume for simplicity that there are no rototranslations, i.e. a point group element $g \in P$ embeds into the space group $S$ as $(g, 0)$. In the heterotic string only the connected part of the Poincaré group has an equivalent as a quantum mechanical operator. Therefore, the rotation matrices $g$ must be orientation preserving, i.e. $g \in \text{SO}(8)$. Every rotation matrix $g$ of finite order can be written in the form

$$g = \exp \left( 2\pi i \xi^i_g D^i_g \right),$$

(3.2)

where the matrices $D^i_g$ generate rotations in mutually orthogonal planes, hence, constitute a basis of the Lie algebra $\mathfrak{so}(8)$. The four component vector $\xi^g$ is called the twist vector for the element $g$. Denote by $J^i_g$ the quantum mechanical operators corresponding to $D^i_g$. Then, the geometric action of $g$ is realized on the state space by the unitary operator

$$U(g, 0) = \exp \left( 2\pi i \xi^i_g J^i_g \right).$$

(3.3)

There is a restriction on the choice of the twist vector $\xi^g$. The quantum mechanical operators $U(g, 0)$ constitute a representation of the point group. Thus, the order of $U(g, 0)$ and $g$ must be identical. The spectrum of the heterotic string contains spinors and the order of $U(g, 0)$ can be doubled. In order to obtain a well–defined action on spinors and $N \geq 1$ SUSY of spacetime, it is sufficient to require [8]

$$\sum_1^8 \xi^i_g \equiv 0 \mod 2.$$

(3.4)

The twist vector is not uniquely defined by the rotation matrix $g$, because one can always add integers to each entry $\xi^i_g$ without changing $g$. Because the orbifold CFT can be different for difference choices of the twist vector, it is part of the defining data of the model. This phenomenon is also known from Abelian orbifold model building and gives rise to models related by discrete torsion [32]. Although the choice of Cartan basis depends on the sector $g$ the precise relation between two Cartan basis for different elements is not important, because the basis is only used to construct the appropriate twisted sector and the spectrum does not depend on the other sectors. To avoid a clash in notation, define a new (fermionic) twist vector $\xi^{(F)}_g = \xi^g$. In addition to the geometric degrees of freedom the heterotic orbifold has Yang–Mills degrees of freedom. These degrees of freedom are realized by 32 real holomorphic Majorana fermions
Ξ^I(z). By equation (2.2), there is a representation of the group SO(16) × SO(16) on the fermionic state space that is infinitesimally generated by operators Ω^{IJ}. An action of the point group is specified by giving a homomorphism φ: SO(8) → SO(16) × SO(16). Giving the homomorphism φ is equivalent to selecting an so(8)-subalgebra g of so(16) ⊕ so(16). To the generators J^g_i there corresponds an element K^g_i ∈ g by the homomorphism φ. The unitary maps on the state space are then given by

$$U(g, 0) = \exp \left( 2\pi i (\xi^{(YM)}_g)^i K^g_i \right).$$

### 3.2 Twisted closed string boundary conditions

Throughout this section the space group element (g, λ) is fixed. Denote by n = ord(g) the order of the rotation part.

#### 3.2.1 Free bosons with values in toroidal orbifolds

The equations of motion, $\partial \bar{\partial} X^\mu(z, \bar{z}) = 0$, have the same form as in the untwisted sector. But the boundary conditions now read as

$$X(z e^{2\pi i}, \bar{z} e^{-2\pi i}) = g X(z, \bar{z}) + 2\pi \lambda,$$  

(3.6)

where index contractions are implicit. The general solution to the equations of motion subject to the orbifold boundary conditions (3.6) read

$$X(z, \bar{z}) = x_0 - i a_0 \log z - i \bar{a}_0 \log \bar{z} + i \sum_{k \in \mathbb{Z} \setminus \{0\}} \left( a_{k \frac{z}{n}} + \bar{a}_{k \frac{\bar{z}}{n}} \right).$$  

(3.7)

The orbifold boundary conditions lead to restrictions on the parameters $a_k, \bar{a}_k$ and $x_0$,

$$gx_0 + 2\pi \lambda = x_0 + 2\pi (a_0 - \bar{a}_0),$$  

(3.8a)

$$ga_{k \frac{z}{n}} = e^{-\frac{2\pi i k}{n}} a_{k \frac{z}{n}}$$ and

$$g\bar{a}_{k \frac{z}{n}} = e^{\frac{2\pi i k}{n}} \bar{a}_{k \frac{z}{n}}.$$  

(3.8b)

The constructing elements $(e, \lambda)$ correspond to winding modes and are already present on the torus. This suggests to introduce the quantities

$$w := \frac{1}{2} (a_0 - \bar{a}_0) \text{ and}$$

$$p := \frac{1}{2} (a_0 + \bar{a}_0).$$  

(3.9a)

$w$ is called the winding vector of the string and $p$ the momentum of the string. The latter name is motivated by the fact that in the quantum theory the corresponding operator generates translations in spacetime. Equations (3.8) require invariance of both the winding vector $w$ and the momentum $p$ under the action of the point group element $g$, just set $k = 0$ there and use the definitions (3.9).
Equation (3.2) represents the point group element $g$ as rotations in mutually orthogonal planes defined by the matrices $D_g^{\alpha}$. An orthogonal change of basis leaves the bosonic operator algebra invariant, so that without loss of generality it is assumed that $g$ is blockdiagonal,

$$g = \text{diag} \left( D(2\pi \xi_1^g), \ldots, D(2\pi \xi_4^g) \right),$$

(3.10)

where $D(\alpha)$ is a $2 \times 2$ rotation matrix about $\alpha$ clockwise. The representation (3.10) allows to analyze the constraints (3.8) separately for each plane $D^i_g$. If $\xi^i_g \in \mathbb{Z}$, then the modes $a_k^i$ and $\bar{a}_k^i$ are integer moded and there the momentum $p$ and center of mass coordinate $x_0$ are unconstrained. The winding $w$ must be equal to the translation $\lambda$, i.e. $2w = \lambda$. The solutions are the same as for the torus. These planes are called fixed tori.

If the point group acts non-trivially on the plane, i.e. $\xi^i_g \notin \mathbb{Z}$, the constraints (3.8) become non-trivial. Denote by $X^1$ and $X^2$ the coordinate directions of the plane $D^i_g$ and introduce complex coordinates by virtue of

$$Z(z, \bar{z}) = X^1(z, \bar{z}) + iX^2(z, \bar{z})$$

(3.11a)

$$Z^*(z, \bar{z}) = X^1(z, \bar{z}) - iX^2(z, \bar{z}).$$

(3.11b)

The corresponding mode operators are denoted by $A_k^i$ and $\bar{A}_k^i$, the center of mass coordinate by $Z_0$. In complex coordinates, the action of $g$ is diagonal and reads as

$$gZ = e^{-2\pi i \xi^i_g} Z.$$

(3.12)

The constraint equations (3.8) can be solved and read as

$$Z_0 = \frac{2\pi}{1 - e^{-2\pi i \xi^i_g}} \lambda$$

(3.13a)

$$\xi^i_g - \frac{k}{n} \notin \mathbb{Z} \rightarrow A_k^i = 0$$

(3.13b)

$$\xi^i_g + \frac{k}{n} \notin \mathbb{Z} \rightarrow \bar{A}_k^i = 0.$$  

(3.13c)

There is no momentum $p = 0$ and winding $w = 0$ and the center of mass coordinate is fixed to a point, the string is localized at $Z_0$. The points $Z_0$ are called fixed points. Twisted strings wind themselves around the singularities\footnote{In geometric orbifolds there are always singularities at the fixed points. This follows from the fact that parallel transport of tangent vectors around them has non trivial holonomy, but the geometry is flat near the fixed points.} of the metric. Moreover, the oscillators in the direction of this plane are fractionally moded. Notably, there is a shift $\delta c$ in the zero–point energy of this twisted sector or equivalently, the ground state in the twisted sector has non–zero conformal weight. This shift reads as [8]

$$\delta c = \frac{1}{2} \sum_{i=1}^4 \eta_i \left( 1 - \eta_i \right),$$

(3.14)

where $\eta_i \in \xi^i_g + \mathbb{Z}$ with $0 \leq \eta_i < 1$. 

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3.2.2 Free fermions with values in orbifolds

The worldsheet SUSY current is kept invariant under the action of the orbifold group if the action on the antiholomorphic fermions is defined via equation (2.3b). This leads to twisted sectors, where the fields $\psi(\bar{z})$ obey

$$\psi(\bar{z}e^{-2\pi i}) = g\psi(\bar{z}).$$

(3.15)

In the fermionic realization of Yang–Mills symmetry the point group acts on the fermions $\Xi^I(z)$ by (3.5) and defines an additional fermionic orbifold sector. The antiholomorphic fermion sector and the Yang–Mills sector are structurally identical and only the Yang–Mills sector is described in the following. For simplicity it is assumed that the point group acts only on the first 16 fermions. The orbifold is specified by giving a finite subgroup of $\text{SO}(16)$ transformations $g$ in the form (3.5),

$$g = \exp\left(2\pi i(\xi^{(YM)}_g)^i K^g_i\right),$$

(3.16)

where $\xi^{(YM)}_g$ is the twist vector for $g$ and $K^g_i$ form a basis for a Cartan subalgebra of $\text{so}(16)$ and are normalized, such that

$$\frac{1}{2}\text{Tr}\left(K^g_i K^g_j\right) = \delta_{ij}.$$

(3.17)

Consider the sector with boundary conditions twisted by $g$. The fermionic mode algebra is left invariant by an orthogonal transformation. Thus, it is no loss of generality to assume that the matrix $g$ is block diagonal,

$$g = \text{diag}\left(D(2\pi(\xi^{(YM)}_g)^1), \ldots, D(2\pi(\xi^{(YM)}_g)^8)\right).$$

(3.18)

The fermions $\Xi^I(z)$ can be grouped into complex fermions $\Psi^I(z)$ subject to the boundary conditions, cf. section A.2,

$$\Psi^I(ze^{2\pi i}) = \pm e^{2\pi i(\xi^{(YM)}_g)^i}\Psi^I(z),$$

(3.19)

where $\pm$ distinguishes between the NS or R sector. The real fermions $\Xi^I(z)$ transform in the vector representation of $\text{SO}(16)$, so that $\Omega^{IJ}(z)$ transforms in the adjoint representation. Thus, under the orthogonal transformation $g$ the currents $\Omega^{IJ}(z)$ behave like

$$\Omega^{IJ}(z) \xrightarrow{g} g_R^T g_I^T \Omega^{RS}(z).$$

(3.20)

As for the free boson, twisted boundary conditions imply the existence of fractionally numbered modes. Rewrite the currents $\Omega^{IJ}(z)$ is a Cartan–Weyl basis, $H^I(z)$ and $E_\alpha(z)$, w.r.t. the Cartan subalgebra defined by $K^g_i$. Then, the twisted boundary conditions imply that

$$H^I(ze^{2\pi i}) = H^I(z)$$

(3.21a)

$$E_\alpha(ze^{2\pi i}) = e^{2\pi i(\xi^{(YM)}_g)\cdot \alpha} E_\alpha(z).$$

(3.21b)

Only currents with $\xi^{(YM)}_g \cdot \alpha \in \mathbb{Z}$ for all roots $\alpha$ are periodic and, therefore, have zero–modes. Denote the Lie algebra generated by the zero–modes by $\mathfrak{g}$. The twisted state space carries only representations of $\mathfrak{g}$ instead of $\mathfrak{so}(16)$. 

- 9 -
4 Projecting onto invariant states

Constructing an orbifold model requires the removal of states not invariant under the orbifold group. Geometric orbifold models are defined by a space group. The methods presented in section 2.2 require that the acting group is finite. However, the space group is infinite. By the space group composition law every space group element $(g, \lambda)$ can be decomposed as follows:

$$(g, \lambda) = (g, 0) \circ (e, g^{-1} \lambda).$$

(4.1)

It is possible to arrive at a state space that is invariant under the whole space group, by projecting onto states that are invariant under the translations subgroup. The translation subgroup does not act on the bosonic oscillators. Therefore, it is sufficient to require its action on highest weight states with momentum $p$ and winding $w$ to be trivial, i.e.

$$U(e, \lambda) |p, w\rangle = e^{2\pi i p \cdot \lambda} |p, w\rangle,$$

(4.2)

where $\lambda \in \Lambda$ is an arbitrary lattice vector. The state is invariant under translations if the phase factor is equal to 1 for every lattice translation. Equivalently, the momentum $p$ must lie in the dual lattice $\Lambda^*$. This quantization of momentum is the only restriction that arises from invariance under the translations. The problem of constructing a CFT invariant under an infinite group has been reduced to constructing an orbifold CFT. For constructing invariant states in the sector twisted by $g \in P$ it is enough to project onto states that are invariant under the action of the centralizer $\mathcal{C}_P(g)$. A general state $|\Psi\rangle$ in the full state space is a tensor product of states from every subsector, i.e.

$$|\Psi\rangle = |\psi, b\rangle \otimes |\psi, f\rangle \otimes |\psi, YM\rangle,$$

(4.3)

where the state $|\psi, b\rangle$ is from the bosonic sector, the state $|\psi, f\rangle$ from the antiholomorphic fermion sector and $|\psi, YM\rangle$ from the Yang–Mills sector.

If the centralizer is Abelian, then it is always possible to find a basis in the state space, on which the centralizer elements $U(g) = U(g, 0)$ act diagonal. The reason is their representation theory: abelian groups only have one-dimensional irreducible representations, i.e. the group acts via multiplication by phase factors there. However, if the centralizer is non-Abelian, then there are multidimensional irreducible representations. The state space $V_i$ of each subsector can be decomposed into parts transforming irreducibly, i.e.

$$V_i = \bigoplus r_\alpha \otimes V_{i,\alpha},$$

(4.4)

where $r_\alpha$ denotes an irreducible representation of the centralizer and $V_{i,\alpha}$ is a subspace that is invariant under the centralizer. Invariant states (4.3) are then constructed, by taking states from $V_{i,\alpha}$ for each subsector and combine them to an invariant state.
4.1 Remnant symmetries

The orbifold observable algebra $A^G$ contains only observables of the original theory that are invariant under the whole orbifold group. States represent this observable algebra. In the original heterotic string, there are two important symmetries:

1. **Poincaré symmetry** generated by the momentum operators $P^\mu$ and the rotation generators $J^{\mu\nu}$. The space group acts non-trivially on the rotation generators and it is possible to project out some of them out by restricting to the algebra of invariant observables. Lorentz symmetry is reduced by orbifolding to a subgroup $L \subset \text{SO}(8)$,

$$\text{SO}(8) \xrightarrow{\text{orbifolding}} L \subset \text{SO}(8).$$

(4.5)

For phenomenological reasons the point group acts only on six space-like coordinate directions. Notably, the point group leaves the four dimensional *helicity operator* $J_{12}$ invariant. Massless particles in four dimensions are classified according to irreducible representations of the stabilizer group of the standard momentum $(k,k,0,0)$. The helicity operator $J_{12}$ generates rotations in transverse four-dimensional Minkowski space. Its eigenvalues determine the helicity of the particle. While $J_{12}$ is always fixed by the point group, there can be additional unbroken generators $J_{ij}$ from the compact directions. At first, $J_{ij}$ seems to generate a continuous symmetry. But the models have to be invariant under lattice translations. Thus, only rotations generated by $J_{ij}$ that leave invariant the torus lattice, i.e. are torus automorphisms, are symmetries of the model. Because the torus automorphism group is always finite, the unbroken symmetry is always a discrete group. These symmetries need not to commute with the four-dimensional supercharge and are then called *R symmetries* \[.\] Let $R$ be a rotation that commutes with all point group elements and leaves the lattice invariant. $U(R,0)$ can map between different space group conjugacy classes and, hence, changes the localization of the string. Consider a string with constructing element $(g,\lambda)$, then

$$U(R,0)U(g,\lambda)U(R,0)^{-1} = U(g,R\lambda).$$

(4.6)

Because $R$ is a lattice automorphism, $R\lambda$ is again a lattice vector. But $(g,R\lambda)$ and $(g,\lambda)$ need not to be conjugated in $S$. Compatibility with the orbifold geometry requires in addition that $U(R,0)$ does not change the conjugacy classes.

2. **Yang–Mills symmetry** is generated by the zero-modes of the currents $\Omega^{rs}(z)$ and the spin fields $S^\alpha(z)$. The orbifold projection projects some of them out. The invariant zero–modes generate a subalgebra $\mathfrak{g}_0 \subset \mathfrak{e}_8 \otimes \mathfrak{e}_8$. The Yang–Mills gauge group of the effective, four–dimensional theory is broken to a subgroup $G_0$,

$$\mathfrak{e}_8 \times \mathfrak{e}_8 \rightarrow G_0.$$

(4.7)

Physical states in the orbifold model transform in representations of $\mathfrak{g}_0$. These representations correspond to their charges under the remaining Yang–Mills symmetry.
4.2 Analysis of the boson sector

Sectors with non-trivial winding number $w$ are generically massive. Because I am only interested in the effective massless theory, I only consider the generic case henceforth. By equation (3.8) the momentum $p$ of the string has to vanish in directions that do not belong to fixed tori. Along fixed tori, the string is free to move, but the momentum can only have discrete values. If the point group element $g$ has a fixed point on the orbifold, the string is localized there.

Naively, there is one conjugacy class for each fixed point on the torus. Naive conjugacy classes correspond to orbits $[(g, \lambda)]_{\Lambda}$ of the space group under conjugation by the translation subgroup, i.e.

$$[(g, \lambda)]_{\Lambda} = \{(e, \mu) \circ (g, \lambda) \circ (e, -\mu), \mu \in \Lambda\}.$$  \hfill (4.8)

Let’s call those orbits $\Lambda$–orbits. The naive picture is true if the representative $(g, \lambda)$ is mapped into its own $\Lambda$–orbit under conjugation by an element of the point group. But, it can also happen that $(g, \lambda)$ is mapped into a different $\Lambda$–orbit. In this case, the two orbits are identical under the whole space group and the two fixed points have to be identified. Identification of fixed points are realized, e.g. in a $\mathbb{Z}_4$ orbifold [15]. To each point group element there can correspond several space group conjugacy classes. Each of them describes strings localized at different fixed points. From the dynamical point of view they are identical. However, by introducing Wilson lines, it is possible to change the spectrum independently. For each conjugacy class $f$ there is a highest weight state $|0, 0; f\rangle$, where the momentum and the winding have been set to zero. The highest weight state is invariant under rotations and, therefore, also invariant under the centralizer.

Consider the action of an element $h \in P$, commuting with $g$, on the oscillators $a^i_{\frac{k}{n}}$. The element $h$ acts on them by conjugation,

$$U(h)^{a^i_{\frac{k}{n}}}U(h)^{-1} = h^ia^j_{\frac{k}{n}}.$$  \hfill (4.9)

Because $h$ and $g$ commute, coordinate directions in which $g$ acts differently are not mixed. Notably, fixed tori are mapped to fixed tori by $h$. Lorentz symmetry in the transverse spacetime directions is reduced by the compactification to a subgroup $L \subset SO(8)$. Similar to the general case described in (4.4), the oscillators $a^i_{\frac{k}{n}}$ can be decomposed into irreducible representations of both the residual Lorentz group $L$ and the centralizer $C_P(g)$.

The conformal weight of the fractionally moded oscillators is found by the fact that $i \partial X^\mu(z, \bar{z})$ is a primary field. The commutator with $L_0$ evaluate to

$$[L_0, a^i_{\frac{k}{n}}] = -\frac{k}{n}a^i_{\frac{k}{n}}.$$  \hfill (4.10)

Applying $a^i_{\frac{k}{n}}$ to a state raises its conformal weight by $\frac{k}{n}$. In the right–moving sector similar considerations can be made. This time the oscillators $\bar{a}^i_{\frac{k}{n}}$ transform in the complex conjugate representation, by equation (3.8).
4.3 Analysis of the Yang–Mills sector

Consider a sector with boundary conditions twisted by \( g \). In the bosonized description the fermions \( \Psi_i(z) \) are realized by bosons \( \phi_i^\ell(z) \). The momentum of the bosons corresponds to the eigenvalues of the operators \( H_0^i \) and is constrained to values in a shifted \( E_8 \) root lattice,

\[
(p^i) \in \xi_0 + \Lambda_{E_8}.
\]  

(4.11)

States have to be invariant under the action of the centralizer \( C_P(g) \). In the definition (3.5) of the action of the point group there is an ambiguity. In general, the centralizer is only represented projectively [25].

4.4 Finding representations of \( g \) at a given conformal weight.

The unprojected state space is independent of the orbifold projection and of the definition of the operators \( U(h,0) \). I use the bosonized description to find the states and representations of \( g_0 \) at a given conformal weight. Recall that \( g \) is the Lie algebra generated by the zero-modes of the currents \( \Omega_{rs}^i(z) \).

An algorithm to find all irreducible representations of the algebra \( g \) at a conformal weight \( L_0 = A \) is given now. A state is characterized by its momentum \( (p^i) \) and the oscillator numbers \( N_{osc,i}^i \). By equations (A.16) and (4.11) one has to find \( E_8 \) lattice vectors \( \alpha \) and non-negative integers \( n_i \), such that

\[
A = \frac{1}{2}(\alpha + \xi_{g}^{(YM)})^2 + \sum_{i=1}^{8} n_i.
\]  

(4.12)

The Diophantine problem (4.12) is equivalent to finding lattice points inside a sphere of radius \( \sqrt{2A} \) in \( \mathbb{R}^8 \) shifted by the vector \( \xi \). For small \( A \) it can be solved numerically, e.g. by using GAP [28]. Denote by \( W_A \) the collection of momenta \( (p^i) = \alpha + \xi_{g}^{(YM)} \) that solve (4.12) including degeneracies. Since the state space represents the zero-mode algebra \( g \) and the momenta are the eigenvalues of the operators \( H_0^i \in g \), the collection \( W_A \) must split into collections of weights of irreducible representations of \( g \). Equivalently, the eigenspace \( V_A \subset V \) of \( L_0 = A \) decomposes into irreducible representations of \( g \) as

\[
V_A = \bigoplus_i L(\lambda_i, q_i),
\]  

(4.13)

where \( \lambda_i \in W_A \) are highest weight vectors of \( g_s \) and \( q_i \) is a tuple of charges \( Q_i \). \( g_s \) is the semisimple part of \( g \). The charges \( Q_i \) form a basis of the \( u(1) \) parts and are linear combinations of the generators \( H_0^i \).

4.5 Projecting onto \( C_P(g) \) covariant states.

In this section I decompose twisted state space into irreducible representations of the centralizer and the invariant algebra \( g_0 \). Let be \( h \in C_P(g) \). If \( X \in g \) is arbitrary, it is left invariant by \( g \), i.e. \( U(g)XU(g)^{-1} = X \), by definition of \( g \). But the element \( U(h)XU(h)^{-1} \)
is also left invariant by $g$, because $g$ and $h$ commute. Hence, the operator $U(h)$ induces a Lie algebra automorphism $g \to g$. Notably, it maps irreducible representation of $g$ into themselves and it suffices to restrict to a representation $L(\lambda_i, q_i)$ from the decomposition (4.13).

Consider a Cartan–Weyl basis $I^i = H^i, F_\alpha$ of $g$ and fix a choice of positive roots. If there are $u(1)$ factors, appended them to the basis elements $I^i$ of the Cartan subalgebra. States of the form $F_{-\alpha_1} \cdots F_{-\alpha_n} |\lambda_i, q_i\rangle$ ($\alpha_i$ positive roots) span the subspace $L(\lambda_i, q_i)$ for a highest weight state $|\lambda_i, q_i\rangle$.

The operator $U(g)U(h)U(gh)^{-1}$ acts as the identity on $g$. Therefore, its action on the highest weight representation $L(\lambda_i, q_i)$ is completely determined by its action on the highest weight state $|\lambda_i, q_i\rangle$. However, in a highest weight module the weight space for $\lambda_i$ is always one–dimensional [29]. Thus, using Schur’s lemma and unitarity of $U(g)$ the action has to be multiplication by a phase factor,

$$U(g)U(h)U(gh)^{-1} = e^{ia(g,h)}.$$ (4.14)

If the phase factor $e^{ia(g,h)}$ is non–trivial, the centralizer is projectively represented. By the remark below it can be redefined to a linear representation by multiplying with an appropriate phase function $f(g)$,

$$U(g) \to f(g)^{-1}U(g).$$ (4.15)

The projection onto irreducible representations of $C_P(g)$ is accomplished by using the projection formulas (2.8). According to (2.9), the representation $L(\lambda_i, q_i)$ decomposes as

$$L(\lambda_i, q_i) = \bigoplus_\alpha r_\alpha \otimes V_\alpha,$$ (4.16)

where $r_\alpha$ are irreducible representations of the centralizer. Because $L(\lambda_i, q_i)$ are irreducible, the subspaces $V_\alpha$ are not representations of $g$ anymore, in general. In the orbifold model only the action of the modes invariant under the whole orbifold group $P$ is well–defined in all sectors. Hence, the subspaces $V_\alpha$ carry representations of $g_0$ and can be decomposed into irreducible representations of it.

### 4.6 Analysis of the fermion sector

Structurally, the antiholomorphic fermion sector is the same as the Yang–Mills sector. The only formal difference is that the number of complex fermions this time is 4 instead of 8 (or 16 if both $E_8$ factors are involved) and there is no enhanced AKM symmetry. In the bosonized description the momenta $(q^i)$ are given this time by (A.19), i.e.

- $(q^i) \in - \xi_g^{(F)} + \Lambda_{SO(8)}$ for the NS–sector and
- $(q^i) \in - \xi_g^{(F)} + \lambda_s + \Lambda_{SO(8)}$ for the R–sector,

where $\lambda_s$ is a highest weight vector for the spinor representation of $SO(8)$ and $\xi_g^{(F)}$ denotes the twist vector for the $g$–twisted sector. The minus signs in front of the shift vectors in
(4.18) take into account that the representations in the antiholomorphic sector are complex conjugated. Finding representations at a given conformal weight and the projection on $C_P(g)$ covariant states goes along the same lines as for the Yang–Mills sector.

4.7 Analysis of the full state space

The state space of the full orbifold CFT is a subspace of the tensor product of the bosonic, fermionic and the Yang–Mills sector that is invariant under the whole space group $S$. Not all states in the orbifold state space are physical. They have to satisfy the mass equations

$$
(L_0 - 1)|\Psi\rangle_M = 0 \\
\left( L_0 - \frac{1}{2} \right)|\Psi\rangle_M = 0
$$

(4.19a)

(4.19b)

on the matter parts of physical states. The matter part consists of the bosonic, fermion and the Yang–Mills sector only. Since the total matter stress energy tensor is the sum of the stress energy tensors of the individual sectors, the mass equations in the $g$–twisted sector can be rewritten as

$$
(L_0^{(YM)} + \frac{1}{2}(p + w)^2 + N_{osc} + \delta c - \frac{1}{2} m^2 - 1)|\Psi\rangle_M = 0 \\
\left( \bar{L}_0^{(fermion)} + \frac{1}{2}(p - w)^2 + \bar{N}_{osc} + \delta c - \frac{1}{2} m^2 - \frac{1}{2} \right)|\Psi\rangle_M = 0,
$$

(4.20a)

(4.20b)

where $m^2$ is the mass in four dimensions, $p \in \Lambda^*$ the internal momentum and $2w \in \Lambda$ is the winding vector. $N_{osc}$ and $\bar{N}_{osc}$ are the contribution of the bosonic oscillators from both compactified and uncompactified directions in the holomorphic, resp. antiholomorphic sector of the model. $\delta c$ is the shift of zero–point energy (3.14) in the twisted sectors. Because of the symmetric action of the orbifold group, it is the same in both chiral parts. The operators $L_0^{(YM)}$ and $\bar{L}_0^{(fermion)}$ evaluate to the conformal weight of the Yang–Mills part and the fermionic part, respectively, and can be calculated by the methods presented above.

5 Constraints from one-loop modular invariance

In perturbative string theory a $g$ loop amplitude is obtained by first calculating the CFT correlation function of appropriate vertex operators on a genus $g$ Riemann surface and, then, by integrating the result over the moduli space of punctured Riemann surfaces. In order to yield well-defined amplitudes, the correlation functions have to be single-valued on the moduli space. This condition is called modular invariance. In this section I find the one-loop contribution to the string free energy. This contribution is given by calculating the zero-point function on a torus with modular parameter $\tau$. I then calculate their behaviour under modular transformations and give sufficient conditions for modular invariance. These conditions turn out to be very similar to the one known from Abelian orbifold model building and reduce to them upon choosing an Abelian point group.
The state space of the CFT decomposes into sectors labelled by the conjugacy classes \([g]\) of the point group \(P\). The partition function then takes the form

\[
Z(\tau, \bar{\tau}) = \sum_{[g]} \frac{1}{|C_P(g)|} \sum_{h \in C_P(g)} \text{Tr}_{V_g} \left( U(h) q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \right) \]

\[
= \sum_{[g]} \frac{1}{|C_P(g)|} \sum_{h \in C_P(g)} Z_{g,h}(\tau, \bar{\tau}),
\]

(5.1)

where \(q = e^{2\pi i \tau}\) is the nome and \(U(\cdot)\) denotes the (projective) representation of \(h \in C_P(g)\) on the subspace \(V_g\). The partition function can be rewritten as [25],

\[
Z(\tau, \bar{\tau}) = \frac{1}{|P|} \sum_{[g,h]} Z_{g,h}(\tau, \bar{\tau}).
\]

(5.2)

Note that compared to [25] our definition of the objects \(Z_{g,h}(\tau, \bar{\tau})\) contains already possible discrete torsion phases. They are defined as the partition function of the CFT on the torus with boundary conditions \(g\) in the \(a\)-direction and \(h\) in the \(b\)-direction, where \(a\) and \(b\) denote a canonical basis of the first homology group of the torus. I call \(Z_{g,h}(\tau, \bar{\tau})\) the (non-chiral) orbifold characters.

Since the total states space is a tensor product of four different CFTs, the orbifold characters decompose into a product of four characters,

\[
Z_{g,h}(\tau, \bar{\tau}) = Z_{g,h}^B(\tau, \bar{\tau}) Z_{g,h}^{YM}(\tau) Z_{g,h}^{\text{fermion}}(\bar{\tau}) Z_{g,h}^{\text{rest}}(\tau, \bar{\tau}).
\]

(5.3)

The character \(Z_{g,h}^{\text{rest}}(\tau)\) stems from the ghosts and the uncompactified directions. It is invariant under \(S\)-modular transformations and cancels some \(e^{-\frac{2\pi i}{24}}\) factors arising from \(T\)-modular transformations. \(Z_{g,h}^B(\tau, \bar{\tau})\) describes the contribution from the bosonic part and \(Z_{g,h}^{\text{fermion}}(\bar{\tau})\) and \(Z_{g,h}^{YM}(\tau)\) describe the contributions from the fermionic and the Yang–Mills sector, respectively. The latter characters are chiral, because the associated CFT is chiral.

It has been shown in [31] that \(Z^B(\tau, \bar{\tau})\) is modular invariant at one-loop in arbitrary orbifold compactifications of the heterotic string. Moreover, there it is also shown that the difference between the holomorphic and antiholomorphic fermionic ground state energies have to be equal modulo 1. This is the same condition that follows by directly calculating the fermionic contributions to the partition function.

### 5.1 The chiral orbifold characters

The calculation of chiral orbifold characters for the fermionic sector and the Yang–Mills sector is similar. Henceforth I only describe the calculation for the Yang–Mills sector and just state the results for the fermionic part.

Let \(h \in P\) commute with \(g\) and consider the action \(U(h)\) in the twisted sector \(V_g\). \(U(h)\) is possibly a projective representation, so that its action on the state space is only defined up to some phase factor. So, a priori, the chiral orbifold characters are only defined up to a phase factor. However, the action on the fields \(\Xi^I(z)\) is still well-defined. In the sector
\( V_g \) the real fermions \( \Xi^I \) are grouped into complex fermions \( \Psi^i \) subject to the boundary conditions, c.f. appendix A,

\[
\Psi^i(ze^{2\pi i}) = \pm e^{2\pi i(\xi_g^{(YM)})^i} \Psi^i(z), \tag{5.4}
\]

where \( \pm \) distinguishes between the NS and R sector. Thus, the twisted state space \( V_g \) is a tensor product of 8 complex, twisted fermions as described in appendix A. The choice of basis for the Cartan subalgebra enters in the calculation through the grouping of the real fermions into complex fermions and through the twist vector \( \xi_g^{(YM)} \). Denote by \( H_0^I \) the zero modes of the Cartan currents. Because \( g \) and \( h \) commute, \( h \) induces an automorphism of the invariant subalgebra \( g = so(16)^g \subset so(16) \) and also of any representation of \( g \). Denote the exponential group of \( g \) by \( G \). Because \( h \) has finite order, it is conjugated to an element in the maximal torus of exponentiated Cartan elements, \( R \in G \),

\[
RU(h)R^{-1} = \exp \left( 2\pi i c^i (H_0^i - \xi_g^{(YM)}) \right) = G_\zeta. \tag{5.5}
\]

For later convenience I added the constant factor \( e^{-2\pi i \zeta \cdot \xi_g} \) to \( G_\zeta \). This factor does not change the adjoint action of \( h \) on the fermionic modes. The operators \( R \) have conformal weight 0 and, hence, commute with \( L_0 \). Thus, the chiral orbifold character for the Yang–Mills sector can be written as

\[
Z_{g,h}^{(YM)}(\tau) = \text{Tr}_{V_g} \left( G_\zeta q^{L_0-c/24} \right). \tag{5.6}
\]

Since the fields \( \Psi^i \) transform in the vector representation of \( so(16) \), the operator \( G_\zeta \) induces a gradation of type \( \zeta \) on the fermionic Fock space and the chiral orbifold characters can be evaluated explicitly in terms of Jacobi theta functions, see appendix A for more details. Using the general characters \( Z_{\xi,\zeta} \) introduced in appendix A the chiral orbifold characters evaluate to

\[
Z_{g,h}^{(YM)}(\tau) = Z_{\xi_g^{(YM)}}(\zeta, \zeta_g^h)(\tau). \tag{5.7}
\]

The vector \( \xi_g^h \) corresponds to the chosen \( G_\zeta \). Note that the choice of \( \zeta \) is not unique. If we replace \( \zeta \to \zeta + \alpha \), where \( \alpha \in \Lambda^*_{so(16)} \) is a weight of \( so(16) \), it still induces the same automorphism of \( g \) but might act differently on states \( |p\rangle \in V_g \). However, a simple calculation shows that

\[
G_{\zeta+\alpha}|p\rangle = G_\zeta |p\rangle. \tag{5.8}
\]

This means that the operator \( G_\zeta \) is uniquely defined as are the chiral orbifold characters. The additional factor in \( G_\zeta \) ensures that the twisted Fock vacuum transforms trivially under \( G_\zeta \). Additionally, one finds that, \( \alpha \in \Lambda^*_{so(16)} \),

\[
Z_{\xi_g^{(YM)},\xi_g^{(YM)}}(\tau) = Z_{\xi_g^{(YM)},\xi_g^{(YM)}}(\zeta, \zeta_g^h)(\tau), \tag{5.9a}
\]

\[
Z_{\xi_g^{(YM)},\xi_g^{(YM)}}(\tau) = e^{-2\pi i \alpha \cdot \xi_g^h} Z_{\xi_g^{(YM)},\xi_g^{(YM)}}(\zeta, \zeta_g^h)(\tau). \tag{5.9b}
\]
If this condition is also fulfilled for $N_{T}lows$, that and antiholomorphic fermions are equal modulo $N$.

Similarly, $\bar{2262}$

In this section I demonstrate the methods developed in the previous sections by explicitly constructing a heterotic orbifold model with point group $S_{3}$. This geometry occurs in the 2262nd $Q$–class in the classification of [9] and is known to allow for $N = 1$ SUSY. I find the remnant symmetries and the transformation properties of the massless single string states.

Finally, I discuss its untwisted geometric moduli.
Table 1: Character table and composition rules for the group $S_3$. The irreducible representation with character $\chi_{\text{inv}}$, $\chi_{\text{alt}}$ or $\chi_{\text{def}}$ are called the trivial $r_{\text{inv}}$, alternating $r_{\text{alt}}$ or defining $r_{\text{def}}$ representation, respectively [28, 29].

|       | $r_{\text{inv}}$ | $r_{\text{alt}}$ | $r_{\text{def}}$ |
|-------|------------------|------------------|------------------|
| $\chi_{\text{inv}}$ | 1 1 1            |                  |                  |
| $\chi_{\text{alt}}$ | 1 -1 1           |                  |                  |
| $\chi_{\text{def}}$ | 2 0 -1           |                  |                  |

6.1 Properties and representations of the group $S_3$

The symmetric group $S_3 = \{e, \tau, \sigma, \tau\sigma, \tau\sigma^2\}$ has six elements and is presented by

$$S_3 = \langle \tau, \sigma | \tau^2 = \sigma^3 = (\tau\sigma)^2 = e \rangle.$$  \hspace{1cm} (6.1)

In this notation $S_3$ is the free group in two generators, $\tau$ and $\sigma$, with the only relations given on the right–hand side. $\tau$ has order 2 and $\sigma$ has order 3. By the identification $\tau \equiv (1,2)$ and $\sigma \equiv (1,2,3)$ the familiar realization of the symmetric group as permutations of three objects, 1, 2 and 3, is recovered. The presentation notation emphasizes the relations among the group elements rather than some specific realization. The group $S_3$ splits into three conjugacy classes,

$$[e] = \{e\}, \ [\tau] = \{\tau, \tau\sigma, \tau\sigma^2\}, \ [\sigma] = \{\sigma, \sigma^2\}.$$  \hspace{1cm} (6.2)

The number of irreducible representations of a finite group is equal to the number of conjugacy classes: one two dimensional irreducible representation and two one dimensional representations. The latter satisfy $r(\tau) = \pm 1$ and $r(\sigma) = 1$. The first choice of sign denotes the invariant or trivial representation $r_{\text{inv}}$, the second is called the alternating representation $r_{\text{alt}}$. The third irreducible representation $r_{\text{def}}$ is called the defining representation. It can be realized by $GL(2, \mathbb{Z})$ matrices as

$$r_{\text{def}}(\tau) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad r_{\text{def}}(\sigma) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$  \hspace{1cm} (6.3)

The character table together with the composition rules of these irreducible representations can be found in table 1. The projectors onto irreducible components in terms of elements of the group ring $\mathbb{C}S_3$ are given, according to equation (2.8):

$$P_{\text{inv}} = \frac{1}{6} (e + \tau)(e + \sigma + \sigma^2), \quad P_{\text{alt}} = \frac{1}{6} (e - \tau)(e + \sigma + \sigma^2), \quad P_{\text{def}} = \frac{1}{3} (2e - \sigma - \sigma^2).$$  \hspace{1cm} (6.4)

The centralizer of $\tau$ in $S_3$ is $C_{S_3}(\tau) = \langle \tau \rangle \cong \mathbb{Z}_2$ and for $\sigma$ it is given by $C_{S_3}(\sigma) = \langle \sigma \rangle \cong \mathbb{Z}_3$. Both groups are cyclic and, hence, Abelian. Their character tables and fusion rules can be found in table 2.
\[ \chi^+ | \begin{array}{l} e \\ \tau \end{array} \right| \begin{array}{l} 1 \\ 1 \end{array} \quad \otimes \quad \begin{array}{l} r_+ \\ r_- \end{array} \right| \begin{array}{l} r_+ \\ r_- \end{array} \]

\[ \chi^- | \begin{array}{l} e \\ \tau \end{array} \right| \begin{array}{l} 1 \\ -1 \end{array} \quad \otimes \quad \begin{array}{l} r_+ \\ r_- \end{array} \right| \begin{array}{l} r_+ \\ r_- \end{array} \]

\[ \chi^0 | e \right| \begin{array}{l} 1 \\ 1 \end{array} \quad \otimes \quad r_0 \quad r_+ \quad r_- \\
\chi^+ | e \right| \begin{array}{l} 1 \\ \zeta^2 \end{array} \quad \otimes \quad r_0 \quad r_+ \quad r_- \\
\chi^- | e \right| \begin{array}{l} 1 \\ \zeta \end{array} \quad \otimes \quad r_0 \quad r_+ \quad r_- \\

\begin{array}{l|ll}
| e | [\sigma] | [\sigma^2] \hline
\chi^0 & 1 & 1 & 1 \\
\chi^+ & 1 & \zeta & \zeta^2 \\
\chi^- & 1 & \zeta^2 & \zeta \\
\end{array}

\begin{array}{l|ll}
| r_0 | r_+ | r_- \hline
r_0 & r_0 & r_+ & r_- \\
r_+ & r_- & r_0 & r_+ \\
r_- & r_+ & r_- & r_+ \\
\end{array}

Table 2: Character table and composition rules for the groups \( \mathbb{Z}_2 \) (left) and \( \mathbb{Z}_3 \) (right).

The irreducible representation of \( \mathbb{Z}_2 \) with character \( \chi^+ \) or \( \chi^- \) will be called the trivial \( r_+ \) or alternating \( r_- \) representation, respectively. For \( \mathbb{Z}_3 \), the irreducible representation with character \( \chi_0, \chi^+ \) or \( \chi^- \) will be called the trivial \( r_0 \), defining \( r_+ \) representation or complex conjugated defining \( r_- \) representation, respectively. The symbol \( \zeta = \exp(2\pi i/3) \) is a third root of unity.

### 6.2 The geometric orbifold

A toroidal orbifold is completely specified by its lattice \( \Lambda \) of translations and the point group. \( \Lambda \) is an \( SU(3) \times SU(3) \times SU(3) \) lattice. The coordinate directions \( X^0, X^1, X^2, X^9 \) are not compactified and are, henceforth, not considered. \( \Lambda \) is generated by the vectors \( e_i \in \mathbb{R}^6 \),

\[
\begin{align*}
e_1 &= \alpha (1,0,0,0,0,0), \\
e_2 &= \alpha \left(-\frac{1}{2}, \frac{1}{2} \sqrt{3}, 0,0,0,0\right), \quad (6.5a) \\
e_3 &= \alpha (0,0,1,0,0,0), \\
e_4 &= \alpha \left(0,0,-\frac{1}{2}, \frac{1}{2} \sqrt{3}, 0,0\right), \quad (6.5b) \\
e_5 &= \alpha (0,0,0,0,1,0), \\
e_6 &= \alpha \left(0,0,0,0,-\frac{1}{2}, \frac{1}{2} \sqrt{3}\right). \quad (6.5c)
\end{align*}
\]

The parameter \( \alpha > 0 \) is an arbitrary scale factor and is adjusted, such that the winding modes are massive. Geometrically, \( 2\pi \alpha \) is the circumference of the compactification torus. The lattice factors into three, mutually orthogonal rank 2 lattices each being isomorphic to an \( SU(3) \) root lattice. One could also assign different scale factors to each of the factors. In the following the particular scale factors are not important, as long as they do not give rise to massless winding modes, so that I take them simply as equal.

The point group \( P \) is generated by the elements \( \tau \) and \( \sigma \), which act on \( \Lambda \) in the lattice basis (6.5) by

\[
\sigma = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad \tau = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.
\]
The generators $\tau$ and $\sigma$ have order 2 and 3, respectively, and satisfy $(\tau\sigma)^2 = e$. According to the presentation (6.1), the point group $P$ is isomorphic to $S_3$. In the third torus $T^2$ the element $\sigma$ acts trivially and the quotient $T^2/P$ is a $\mathbb{Z}_2$ orbifold. In the Euclidean basis, $\tau$ is a reflection in the first and second torus,

$$x^4 \mapsto -x^4 \text{ and } x^6 \mapsto -x^6. \quad (6.7a)$$

Two reflections at orthogonal planes are equivalent to a rotation about $\pi$ in the plane spanned by their normal vectors. The orthogonal matrices $\sigma$ and $\tau$ can be rewritten in the form (3.3),

$$\tau \equiv \exp \left( 2\pi i \frac{1}{2} (J_{46} - J_{78}) \right) \quad (6.8a)$$

$$\sigma \equiv \exp \left( 2\pi i \frac{1}{3} (J_{34} - J_{56}) \right). \quad (6.8b)$$

The twist vectors $\xi_{e}^{(F)}$ and basis elements $D_{i}^{\sigma}$ defining the action in the antiholomorphic fermion sector for the representatives $e$, $\tau$ and $\sigma$ of the point group conjugacy classes are given by

$$\xi_{e}^{(F)} = (0,0,0,0), \quad (D_{i}^{\tau}) = (J_{34}, J_{56}, J_{78}, J_{12}) \quad (6.9a)$$

$$\xi_{\tau}^{(F)} = \left( \frac{1}{2}, -\frac{1}{2}, 0, 0 \right), \quad (D_{i}^{\tau}) = (J_{46}, J_{78}, J_{35}, J_{12}) \quad (6.9b)$$

$$\xi_{\sigma}^{(F)} = \left( \frac{1}{3}, -\frac{1}{3}, 0, 0 \right), \quad (D_{i}^{\sigma}) = (J_{34}, J_{56}, J_{78}, J_{12}). \quad (6.9c)$$

In this model the point group acts only on the first 16 fermions by a standard embedding. The point group is embedded into SO(16) as a subgroup acting only on the first six fermions. Denote a basis for a Cartan subalgebra of $so(16)$ by the matrices $K_{i,j}$ generating counterclockwise rotations in the $\Xi^i - \Xi^j$–plane, i.e. it is an antisymmetric $16 \times 16$ matrix whose non-zero entries are 1 in the $i$th row and $j$th column and $-1$ in the $j$th row and $i$th column. The twist vectors $\xi_{g}^{(YM)}$ and the basis matrices $K_{i}^{\sigma}$ are given by

$$\xi_{e}^{(YM)} = (0,0,0,0,0,0,0), \quad (K_{i}^{\tau}) = (K_{1,2}, K_{3,4}, K_{5,6}, K_{7,8}, K_{9,10}, K_{11,12}, K_{13,14}, K_{15,16}) \quad (6.10a)$$

$$\xi_{\tau}^{(YM)} = \left( \frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0 \right), \quad (K_{i}^{\tau}) = (K_{2,4}, K_{5,6}, K_{1,3}, K_{7,8}, K_{9,10}, K_{11,12}, K_{13,14}, K_{15,16}) \quad (6.10b)$$

$$\xi_{\sigma}^{(YM)} = \left( \frac{1}{3}, -\frac{1}{3}, 0, 0, 0, 0, 0 \right), \quad (K_{i}^{\sigma}) = (K_{1,2}, K_{3,4}, K_{5,6}, K_{7,8}, K_{9,10}, K_{11,12}, K_{13,14}, K_{15,16}). \quad (6.10c)$$
Figure 1: Visualization of the actions of rotations about $2\pi/3$ and a reflection at the $e_1$-axis on different regions $R_i$ and $S_i$ of the fundamental domain of the torus. The rotation maps $R_1 \rightarrow R_2 \rightarrow R_3 \rightarrow R_1$ and $S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_1$ and the reflection maps $S_1 \leftrightarrow R_2$, $S_2 \leftrightarrow R_1$ and $S_3 \leftrightarrow R_3$.

(a) A possible choice of fundamental domain (gray) for the $S_3$ action on $T^6$. This is also the region where untwisted strings are localized.

(b) In the $\tau$-twisted sector there are 4 fixed points. The string is localized on a torus $F_1 \times F_2$. $F_i$ are fixed circles and shown as bold dashed lines.

(c) In the $\sigma$-twisted sector there are 9 fixed points $(A_1, A_2)$, etc.. In the third torus the string is localized in the bulk.

Figure 2: Fundamental domain and localization of strings in the $T^6/S_3$ orbifold.
The point group has three conjugacy classes and so there are three twisted sectors, labeled by the elements $e$, $\tau$ and $\sigma$. The geometry of this orbifold might be unfamiliar to the reader. In order to find a fundamental domain for the action of $S_3$ on the torus $T^6$ it is helpful to first get used to the action of $\tau$ and $\sigma$ acting on a two dimensional torus. In figure 1 the fundamental region of the torus has been divided into six subdomains. The action of a counterclockwise rotation about $2\pi/3$ and a reflection at the $e_1$-axis are described in its caption. Divide now the torus $T^6$ into subdomains of the form $D \times D' \times D''$ where $D$, $D'$ and $D''$ are one of the subdomains $R_i$ or $S_i$ from figure 1. One obtains this way $6^3 = 216$ subdomains of $T^6$. Using the definition of $\sigma$ and $\tau$, one can find exactly $6^2 = 36$ of those subdomains that are not identified under the action of $S_3$. The union of those subdomains constitutes a fundamental domain. A possible choice of fundamental domain is shown in figure 2. Figure 2 also shows the regions of $T^6/S_3$ in which the twisted strings are localized. From there it can be seen that the effective geometry for the $\sigma$ and $\tau$ twisted sectors is six dimensional, while it is ten dimensional for the untwisted string.

In the $\sigma$ twisted sector there are nine fixed-tori $T^6_{(P,Q)}$. Each of the tori can be labelled by a pair $(P,Q)$ where $P \in \{A_1,B_1,C_1\}$ is a fixed point in the first torus and $Q \in \{A_2,B_2,C_2\}$ is one in the second torus. Note that $\tau$ interchanges $C_1 \leftrightarrow B_1$ and $C_2 \leftrightarrow B_2$. Nevertheless this does not identify some of the fixed-tori, because $\tau$ does not commute with $\sigma$ and so changes boundary conditions to $\sigma^2$. There are no restrictions on the momentum in the third torus. In the $\tau$ twisted sector four fixed-tori can be found. Each of these tori intersect the torus $T^6_{(A_1,A_2)}$ in exactly one point. All other fixed-tori are mutually disjoint.

6.3 The physical spectrum

This $S_3$ model possesses four dimensional $N = 1$ SUSY and an $E_6 \times U(1)$ Yang–Mills symmetry. Additionally, there is an $R$ symmetry that contains $\mathbb{Z}_4^R$ and a $\mathbb{Z}_2$ parity. The spectrum can be found in table 3 in the appendix.

6.3.1 Orbifolded fermion sector

The fermionic sector consists of eight antiholomorphic fermions $\psi^\beta(z)$ and can be analyzed by the method described in section 4 with the shift vectors and Cartan subalgebras given in equation (6.9). States are classified according to irreducible representations of the subalgebra $\mathfrak{g}_0$ of $\mathfrak{so}(8)$ that is invariant under the action of the point group. In the case at hand, the breaking scheme is

$$\mathfrak{so}(8) \rightarrow \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) = \mathfrak{g}_0.$$  \hspace{1cm} (6.11)

The generators of the various invariant $\mathfrak{u}(1)$ algebras are given by

$$R_0 = J_{12},$$  \hspace{1cm} (6.12a)

$$R_1 = J_{35} - J_{46}$$  \hspace{1cm} (6.12b)

$$R_2 = J_{78}.$$  \hspace{1cm} (6.12c)
(a) The rotation $\beta$ exchanges the fixed points on the third torus: $B \rightarrow C \rightarrow D \rightarrow B$

\[
(A_1, A_2) \rightarrow (A_1, A_2) \quad (B_1, A_2) \rightarrow (A_1, C_2) \quad (C_1, A_2) \rightarrow (A_1, B_2)
\]

\[
(A_1, B_2) \rightarrow (B_1, A_2) \quad (B_1, B_2) \rightarrow (B_1, C_2) \quad (C_1, B_2) \rightarrow (B_1, B_2)
\]

\[
(A_1, C_2) \rightarrow (C_1, A_2) \quad (B_1, C_2) \rightarrow (C_1, C_2) \quad (C_1, C_2) \rightarrow (C_1, B_2)
\]

(b) The rotation $\alpha$ permutes the 9 fixed points of the $\sigma$-twisted sector.

**Figure 3:** The action of the rotations $\alpha$ and $\beta$ on the localization of the states. Only $\beta^3$ and $\alpha^4$ are compatible with the orbifold geometry. For $\alpha$ the different localization points of winding modes in the $\tau$-sector have to be taken into account. The labels for the fixed-points are the same as in figure 2.

The charge $R_0$ generates rotations in the uncompactified directions $x^1$ and $x^2$ and corresponds to the helicity operator. The charges $R_1$ and $R_2$ generate rotations in the compactified directions and must respect the lattice structure of $\Lambda$. As lattice automorphism groups are always finite, $R_1$ and $R_2$ generate a discrete symmetry. Call the eigenvalues of $2R_1$ and $2R_2$ $D$ charges ($D$ for discrete). Table 4 gives the states in the fermion sector with conformal weight $\leq 1/2$.

**Discrete symmetries and $R$ symmetry** The charge $R_2$ generates rotations in the $x^7 - x^8$-plane. In order to be compatible with the orbifold geometry, these rotations have to be lattice automorphisms, cf. section 4. In the $x^7 - x^8$-plane the torus is defined by an SU(3) root lattice. It is a hexagonal lattice and allows only for rotations about $2\pi/6$. Hence, there is a $\mathbb{Z}_{12}$ symmetry generated by

\[
\beta = \exp \left( 2\pi i \frac{1}{12} \cdot 2J_{78} \right).
\]  

(6.13)

The order of the generator is larger than 6, because $J_{78}$ can have half-integer eigenvalues. In the orbifold model, the operators $U(\beta)$ map states localized at one fixed point to another, see figure 3. Compatibility with the orbifold geometry requires that only $\beta^3$ is a symmetry of the full model and the discrete symmetry is reduced to $\mathbb{Z}_4$. Similarly, $R_1$ generates rotations in the $x^3 - x^5$-plane together with an opposite rotation in the $x^4 - x^6$-plane by the same angle. The lattice $\Lambda$ defining the torus $T^6$ in these planes is a square lattice.
Therefore, rotation angles have to be restricted to multiples of $2\pi/4$. Consequently, there exists a $\mathbb{Z}_8$ symmetry generated by

$$\alpha = \exp \left( 2\pi i \frac{1}{8} \cdot 2(J_{35} - J_{46}) \right).$$  \hspace{1cm} (6.14)

$U(\alpha)$ permutes the fixed points of the $\sigma$-twisted sector, cf. figure 3. Thus, only $\alpha^4$ is compatible with the orbifold geometry and the symmetry reduces to a $\mathbb{Z}_2$ parity.

The charges under $\mathbb{Z}_8 \times \mathbb{Z}_{12}$ are shown in table 4 as “$D$ charges”. From the spectrum it can be deduced that the supercharge in four dimensions must be charged under $\mathbb{Z}_2 \times \mathbb{Z}_4$ with charge $(0, 1)$ and $\beta^3$ generates an $R$ symmetry. The generator $\alpha^4$ commutes with the supercharge and is a discrete non–$R$ symmetry. The $R$ and $\mathbb{Z}_2$ charges of the particles are listed in table 3.

### 6.3.2 Orbifolded Yang–Mills Sector

In the Yang–Mills sector the invariant subalgebra $g_0$ of $\mathfrak{so}(16)$ is obtained from the symmetry breaking

$$\mathfrak{so}(16) \rightarrow \mathfrak{so}(10) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) = g_0.$$  \hspace{1cm} (6.15a)

In the fermionic realization the AKM symmetry is enhanced. The enhanced symmetry breaking pattern is

$$\mathfrak{e}_8 \rightarrow \mathfrak{e}_6 \oplus \mathfrak{u}(1).$$  \hspace{1cm} (6.15b)

The rank of the symmetry algebra has been reduced. This is one of the features that distinguishes Abelian orbifold models from non–Abelian orbifold models. Table 5 shows the states in this sector with conformal weight $\leq 1$. The U(1) charge $Q$ of $E_6 \times U(1)$ is related to the two U(1) charges, $A_1$ and $A_2$, of $SO(10) \times U(1) \times U(1)$ by

$$Q = A_1 + 2A_2.$$  \hspace{1cm} (6.16)

### 6.3.3 Massless states

The full string state is a vector in the ghost extended state space. Because both the world–sheet super current and the stress–energy tensor are invariant under orbifolding, the conformal and superconformal ghost sectors need not to be changed. An on–shell string state has to satisfy the equations of motion (4.20) and has to be invariant under the action of the point group $P$. Using intermediate results from tables 4 and 5 it is possible to recover the full massless spectrum from table 3.

### 6.4 Discussion

The $T^6/S_3$ orbifold model described in this section exhibits a non-chiral spectrum in four dimensions. Therefore, its gauge symmetries are automatically free of chiral anomalies. This is an important requirement for consistency of the model [27, 33–36].
From figure 2 it can be seen that the $\tau$- and $\sigma$-twisted sectors are indeed six dimensional, because both have fixed-tori. From table 4 one sees that in the $\tau$-twisted sectors the states comprise a 4D, $\mathcal{N} = 1$ chiral supermultiplet. The effective six-dimensional theory on the fixed-tori is $\mathcal{N} = (1,0)$ supersymmetric and is localized in the $X^1, X^2, X^3, X^5$-transversal directions. Two chiral multiplets then organize into a hypermultiplet. The intermediate gauge group is $E_7 \times SU(2)$ and the hypermultiplet transforms as $(56,1)$. Note that on every fixed-torus one finds only 28 hypermultiplets. But, because $56$ is a pseudoreal representation of $E_7$, 28 hypermultiplets are sufficient to realize one half-hypermultiplet transforming as $(56,1)$ \cite{37,38}. Similarly, one finds at each fixed-torus 4 half-hypermultiplets transforming as $(1,2)$ built up from one half-hypermultiplet each, as $2$ is also pseudoreal. As these states involve holomorphic oscillators, they can be identified with $\tau$-twisted moduli of this orbifold.

A similar analysis can be applied to the $\sigma$-twisted sector. As before one finds that the effective six-dimensional theory at each of the 9 fixed-tori is $\mathcal{N} = (1,0)$ supersymmetric and that the fields are localized in the $X^1, X^2, X^7, X^8$-transversal direction. The intermediate gauge group is $E_7 \times U(1)$ in this case. Matter organizes into one half-hypermultiplet transforming as $56_0$ and 7 singlet half-hypermultiplets. The latter involve again oscillators and are therefore identified with $\sigma$-twisted moduli.

Note that the above $g$-twisted states are only invariant under the centralizer of $g$ in the point group. This means that the twisted sectors can be almost considered as the twisted sectors of a $\mathbb{Z}_2$- or $\mathbb{Z}_3$-orbifold. The spectrum agrees almost with the one given in table 2 in \cite{39}. One has to take into account that in the $S_3$-orbifold the number of fixed-tori and twisted sectors is different from the Abelian orbifolds considered there. For example, the $\mathbb{Z}_3 = \langle \sigma \rangle$ orbifold model has two twisted sectors corresponding to the elements $\sigma$ and $\sigma^2$.

In the $S_3$ orbifold model both elements are related by conjugation by $\tau$ and are therefore identified. Since in the $\sigma$-twisted sectors the number of fixed-tori is the same in both models, the only difference that has to be taken into account is that instead of having matter particles transforming as half-hypermultiplets, the matter particles transform as half-hypermultiplets. Conversely, in the $\tau$-twisted sector the number of fixed-tori is reduced from 16 in \cite{39} to 4 in the $S_3$-orbifold. Hence, one ends up with only one quarter of the matter contents of \cite{39}.

Since the twisted matter is organized into half-hypermultiplets, dimensional reduction to four dimensions automatically yields a non-chiral spectrum. This means that this particular orbifold model is not phenomenologically viable. It is not possible to obtain a chiral theory from this orbifold without introducing non-trivial background fields \cite{40}.

### 6.5 Untwisted moduli of the $S_3$ orbifold

Connected to given a state in string theory there is associated a family of infinitesimally deformations. Introducing complex coordinates allows one to distinguish between complex structure moduli and Kähler moduli. A possible choice of complex coordinates with this
property is given by
\[ z^1 = x^3 + ix^5, \quad z^2 = x^6 + ix^4, \quad z^3 = x^7 + ix^8. \] (6.17)

In the coordinates \( z^i \) the maps \( \tau \) and \( \sigma \) act holomorphically on the torus \( T^6 \). Their action on a tuple \((z_1, z_2, z_3)\) can be deduced from equation (6.6) and reads as
\[ (z_1, z_2, z_3) \rightarrow \tau (z_1, -z_2, -z_3) \] (6.18a)
\[ (z_1, z_2, z_3) \rightarrow \sigma (z_1 \cos \frac{2\pi}{3} - iz_2 \sin \frac{2\pi}{3}, -iz_1 \sin \frac{2\pi}{3} + z_2 \cos \frac{2\pi}{3}, z_3). \] (6.18b)

In complex coordinates \( \delta H_{ab} \) and \( \delta H_{\bar{a}b} \) are the deformations of the complex structure and Kähler structure, respectively. Massless deformations are given by
\[ \delta H_{1\bar{1}} + \delta H_{2\bar{2}}, \quad \delta H_{11} - \delta H_{22}, \quad \delta H_{2\bar{3}}, \quad \delta H_{33} \] (6.19)
together with their complex conjugates. Hence, there are two complex structure moduli and two Kähler moduli. These numbers agree with the numbers given in table 3 as well as with the number of global \((2,1)\) and \((1,1)\) forms on the orbifold as can be checked explicitly using the holomorphic action (6.18).

7 Conclusions

In this paper I constructed a heterotic orbifold model with non Abelian point group \( S_3 \). However, the methods presented here apply to more general toroidal symmetric orbifolds as well. The only restriction is that the left moving Yang–Mills sector and right moving fermion sector are not mixed by orbifolding. The Yang–Mills sectors is realized by a free fermion CFT, so that the analysis of the state space is formally the same as for the right moving fermion sector. In this model the YM gauge group is broken from \( E_8 \times E_8 \) down to \( E_6 \times U(1) \times E_8 \), i.e. it reduces the rank by one. Rank reduction on the orbifold itself cannot be achieved by using Abelian point groups. Hitherto, it could only be achieved by using a non–diagonal embedding of the space group or in the low energy effective action via some Higgs mechanism. Compared to Abelian orbifold model building the only new technical points are the appearance of multidimensional representations of the point group. Additionally, the method used here to analyze the spectrum requires one to change the Cartan subalgebra for each twisted sector. The first point is handled by using the representation theory of finite groups, while the second point is more subtle. The fermionic realization treats all Yang–Mills degrees of freedom symmetrically and allows so for freely regrouping of the real fermions into complex fermions. The drawback is here that the space group action has to be embedded into \( SO(16) \times SO(16) \). All in all the methods presented here complement the well-known bosonic framework and could be used to extend the search for phenomenologically realistic string models to include non–Abelian point groups as well. However, most different point groups have to be used.

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### A Fermionic realization of AKM symmetries and bosonization

The fermionic realization of the $E_8 \times E_8$ affine symmetry and its relation to the usual bosonized description play an important role in my analysis of the orbifolding process. For the reader’s convenience I provide a short overview on this subject. It is pretty standard and I refer the reader to any good textbook on conformal field theory for more details.

#### A.1 Twisted complex fermions and their bosonization

The CFT of $n$ real Majorana–Weyl fermions $\psi^i(z)$, $i = 1, \ldots, n$ is defined by the OPE and stress tensor $T(z)$ given by

$$\psi^i(z)\psi^j(w) \sim \frac{\delta^{ij}}{z-w},$$

$$T(z) = -\frac{1}{2} \sum_{i=1}^n :\psi^i(z)\partial\psi^i(z):.$$  \hspace{1cm} (A.1a, b)

For two fermions one can define complex fermions $\Psi(z) = \frac{1}{\sqrt{2}} (\psi^1(z) + i\psi^2(z))$ and the fermion number current $j(z) = i :\Psi(z)\Psi(z):$. $\text{W.r.t.}$ the $U(1)$ current $j(z)$ the field $\Psi(z)$ has charge $+1$ and $\bar{\Psi}(z)$ has charge $-1$. It turns out that the fermion field operators can be expressed as composite operators of a single free boson $\phi(z)$ on the same world sheet as

$$\Psi(z) = :\exp(i\phi(z)):, \quad \bar{\Psi}(z) = :\exp(-i\phi(z)):, \quad j(z) = i\partial\phi(z).$$  \hspace{1cm} (A.2)

These identifications are called bosonization rules. The irreducible representations of the bosonic field algebra are characterized by momenta $p$. For bosonization to work, the boson has to be compactified on a circle of radius 1, i.e. the momenta $p$ in the NS sector have to be integral $p \in \mathbb{Z}$, while the momenta in the R sector fulfill $p \in 1/2 + \mathbb{Z}$ to account for the antiperiodic boundary conditions. Moreover, the spin field $\sigma(z)$ is given by $:\exp\left(\frac{i}{2}\phi(z)\right):$; i.e. it changes the boundary conditions from periodic to antiperiodic and vice versa. It is important to notice that the fermion number, i.e. the zero mode of $j(z)$, is identical to the momentum. It is clear that this construction can be generalized to arbitrary boundary conditions obeyed by the complex fermions. The most generic boundary conditions compatible with the form of the stress tensor are characterized by a twist parameter $0 \leq s < 1$ and take the form

$$\Psi(z e^{2\pi i}) = -e^{-2\pi i s}\Psi(z).$$  \hspace{1cm} (A.3)

The choice $s = 0$ corresponds to the R sector and $s = 1/2$ to the NS sector. In the bosonized description the momenta take values in a shifted lattice $p \in s - 1/2 + \mathbb{Z}$.

A generic state space is either specified by a certain number of fermionic oscillators acting on a ground state or, equivalently, by giving the ground state momentum $p$ and the number $N_{\text{osc}}$ of bosonic oscillators acting on $|p\rangle$. The conformal weight $L_0$ of such a state is then given by

$$L_0 = \frac{1}{2}p^2 + N_{\text{osc}} + \delta c.$$  \hspace{1cm} (A.4)
The number $\delta c$ is equal to the conformal weight of the ground state, i.e.

$$\delta c = \frac{1}{2} \left( s - \frac{1}{2} \right)^2. \quad \text{(A.5)}$$

### A.2 Bosonized description and enhanced Yang–Mills symmetry

The bosonization representation from section A.1 can be used to describe the fermionic realization of $\text{SO}(8)$ and $\text{SO}(16)$ symmetry in the heterotic string. For definiteness only the holomorphic realization of $\text{SO}(16)$ in terms of 16 Majorana–Weyl fermions $\Xi^I(z)$, $I = 1, \ldots, 16$, is discussed. The antiholomorphic realization of $\text{SO}(8)$ goes along the same lines. In view of orbifolds, generalized twisted boundary conditions are included in the discussion.

The 16 Majorana–Weyl fermions $\Xi^I(z)$ can be grouped into 8 complex fermions $\Psi^i(z)$,

$$\Psi^i(z) = \frac{1}{\sqrt{2}} \left( \Xi^{2i-1}(z) + i\Xi^{2i}(z) \right) \quad \text{(A.6a)}$$

$$\bar{\Psi}^i(z) = \frac{1}{\sqrt{2}} \left( \Xi^{2i-1}(z) - i\Xi^{2i}(z) \right). \quad \text{(A.6b)}$$

The complex fermions (A.6) should obey twisted boundary conditions. In the NS–sector the twist parameters $s^i$ are therefore given by

$$s^i \equiv \frac{1}{2} + \xi^i \text{ mod } 1, \quad \text{(A.7a)}$$

while in the R–sector they read as

$$s^i \equiv \xi^i \text{ mod } 1. \quad \text{(A.7b)}$$

By section A.1, each complex fermion $\Psi^i(z)$ can be represented by a free boson $\phi^i(z)$, i.e.

$$\Psi^i(z) = : \exp \left( i\phi^i(z) \right) :, \quad \bar{\Psi}^i(z) = : \exp \left( -i\phi^i(z) \right) :. \quad \text{(A.8)}$$

The momentum $p^i$ of the boson $\phi^i$ takes values in a one–dimensional lattice,

$$p^i \in s^i - \frac{1}{2} + \mathbb{Z}. \quad \text{(A.9)}$$

A Cartan subalgebra of $\mathfrak{so}(16)$ is spanned by the zero–modes of the currents $H^i(z) \equiv K^{2i-1,2i}(z)$. In terms of the complex fermions $\Psi^i(z)$, they read as

$$H^i(z) = K^{2i-1,2i}(z) = i : \Xi^{2i-1}(z)\Xi^{2i}(z) := j^i(z), \quad \text{(A.10)}$$

where $j^i(z)$ is the fermion number current. Therefore, the Cartan generators $H^i_0$ of $\text{SO}(16)$ correspond to fermion number operator $F^i$ for the complex fermions $\Psi^i(z)$. Now, by equation (A.2) the fermion number current $j^i(z)$ can be expressed in terms of the free bosons $\phi^i(z)$ as

$$i\partial \phi^i(z) = j^i(z) = H^i(z), \quad \text{(A.11)}$$
which identifies the eigenvalues of the Cartan generators $H_i^0$ with the momentum (A.9).

The GSO projection selects states according to their $G$ parity, which is determined by their fermion number, i.e. their momentum in the bosonized description. The $G$ parity operator for a system of 8 complex fermions is given by

$$G = \prod_{i=1}^{8} (-1)^{F^i + \frac{1}{2} - s^i}.$$  \hspace{1cm} (A.12)

The shift of the fermion number by $\frac{1}{2} - s^i$ takes into account that the ground states are assigned positive $G$ parity. The $G$ parity operator intertwines between the different fermion sectors. A momentum $(p_1^i, \ldots, p_8^i)$ is not projected out by the GSO projection if

$$\sum_{i=1}^{8} p^i + \frac{1}{2} - s^i \in 2\mathbb{Z}.$$  \hspace{1cm} (A.13)

Therefore, in the GSO projected state space of the sector twisted by $s$ the momenta are given by

$$p^i = s^i - \frac{1}{2} + n^i, \quad \sum_{i=1}^{8} n^i \in 2\mathbb{Z}.$$  \hspace{1cm} (A.14)

Consider the untwisted case, i.e. $\xi = 0$. In the NS–sector it is $s^i = \frac{1}{2}$ and the momenta $(p^i)$ are restricted to values whose sum is even. This lattice coincides with the root lattice $\Lambda_{\text{SO}(16)}$ of the Lie group SO(16). In the R–sector all momenta are shifted by $\lambda_s = (\frac{1}{2}, \ldots, \frac{1}{2})$, which corresponds to the highest weight vector of a spinor representation of SO(16). In the twisted case, i.e. $\xi \neq 0$, the momenta are shifted by the vector $(\xi^i)$,

$$\begin{align*}
(p^i) &\in \xi + \Lambda_{\text{SO}(16)} & \text{in the NS–sector} \\
(p^i) &\in \xi + \lambda_s + \Lambda_{\text{SO}(16)} & \text{in the R–sector}. \hspace{1cm} (A.15a, b)
\end{align*}$$

In the bosonized description, the fermionic and bosonic stress energy tensors coincide, by section A.1. In particular, the operator $L_0$ can be expressed in terms of the bosons $\phi^i(z)$,

$$L_0 = \frac{1}{2} \sum_{i=1}^{8} (F^i)^2 + \sum_{i=1}^{8} N^i_{\text{osc}},$$  \hspace{1cm} (A.16)

where $N^i_{\text{osc}}$ is the number operator for the boson $\phi^i(z)$. Thus, the weights of states including their multiplicities at a given conformal weight can be obtained by solving (A.16). In the untwisted sector, the zero–modes of the currents $\Omega^I(z)$ are the generators of SO(16)–transformations. They have conformal weight 0 and consequently the weights obtained by solving (A.16) have to form complete sets of weights of representations of SO(16).

The momenta $(p^i)$ are constraint to take values in $\Lambda_{\text{SO}(16)}$ and its spinor coset $\lambda_s + \Lambda_{\text{SO}(16)}$. The spinor weight $\lambda_s$ satisfies $2\lambda_s \in \Lambda_{\text{SO}(16)}$. Hence, the union of $\Lambda_{\text{SO}(16)}$ and the spinor coset is a lattice,

$$\Lambda_{E_s} = \Lambda_{\text{SO}(16)} \cup (\lambda_s + \Lambda_{\text{SO}(16)}),$$  \hspace{1cm} (A.17)
which turns out to be isomorphic to the root lattice of the group $E_8$. Moreover, all roots $\alpha \in \Lambda_{E_8}$ have length $\alpha^2 = 2$. This implies that the vertex operator $V_\alpha(z)$ is a current for every root of $E_8$. By considering the bosonic realization of Yang–Mills symmetry, it is possible to show that the currents $E_\alpha(z) = c_\alpha V_\alpha(z)$ and $H^i(z)$ satisfy an AKM algebra for $E_8$ [41]. $c_\alpha$ is a Klein cocycle ensuring the correct commutation relations. Thus, the AKM symmetry is enhanced from $SO(16)$ to $E_8$. The additional currents correspond to the spin fields $S_\alpha(z)$ creating the ground states in the R–sector from the NS-vacuum. They have conformal weight 1 by equation (A.5).

The same analysis can be repeated for the antiholomorphic fermions, which represent an $SO(8)$–AKM symmetry. The fermions can be bosonized by four bosons $\phi^i(z)$. In this case the GSO projection in the NS–sector is changed. The NS–ground state is assigned negative $G$ parity. It turns out, that the momenta $q^i$ of the bosons $\phi^i$ in the NS–sector is given by,

$$q^i = \xi^i + n^i, \quad \sum_{i=1}^4 n^i \in 2\mathbb{Z} + 1. \quad (A.18)$$

The vacuum is again assigned $G_t$-charge 0. Define the GSO projected characters $Z_{\xi,t}(\tau)$ by

$$Z_{\xi,t}(\tau) = \text{Tr} \frac{1 + G}{2} G_t q^{L_0-8/24}, \quad (A.21)$$

where $G$ is the $G$-parity operator. The character $Z_{\xi,t}(\tau)$ counts the number of states weighted by their $G_t$-charge in the GSO-projected state space. It receives contributions from both the NS-sector and the R-sector.

**A.3 Characters and modular properties**

In the fermionic realization of Yang–Mills symmetry there are eight independent complex fermions. It is possible to define a different gradation to every complex fermion. The operator $G_t$ generating the gradation has therefore eight parameter $t^i$ and acts on the fermionic modes $b^i_n$ and $\bar{b}^i_n$ by

$$G_t b^i_n G_t^{-1} = \exp \left(-2\pi i t^i\right) b^i_n \quad (A.20a)$$

$$G_t \bar{b}^i_n G_t^{-1} = \exp \left(2\pi i t^i\right) \bar{b}^i_n. \quad (A.20b)$$

The vacuum is again assigned $G_t$-charge 0. Define the GSO projected characters $Z_{\xi,t}(\tau)$ by

$$Z_{\xi,t}(\tau) = \text{Tr} \frac{1 + G}{2} G_t q^{L_0-8/24}, \quad (A.21)$$

where $G$ is the $G$-parity operator. The character $Z_{\xi,t}(\tau)$ counts the number of states weighted by their $G_t$-charge in the GSO-projected state space. It receives contributions from both the NS-sector and the R-sector.
The complex fermions in the NS-sector are described by the twist parameter \( s^i = \frac{1}{2} + \xi^i \) according to equation (A.7a). The state space is a tensor product of the eight complex fermion state spaces. The action of the operators \( G_t \) on the \( i \)-th state space is exactly the one considered in section (A.1). The effect of the \( G \)-parity operator can be included to the operator \( G_t \) by defining \( G'_t \) with \( t'^i = t^i + \frac{1}{2} \). Then it holds true that \( G'_t = GG_t \). Using the tensor product structure of the state space, one can show that the contribution of the NS-sector to \( Z_{\xi,t}(\tau) \) is,

\[
Z_{\xi,t}^{NS}(\tau) = \text{Tr}_{NS} \left( \frac{1 + G}{2} G_t q^{L_0 - 8/24} \right) = \frac{1}{2} \left( \prod_{i=1}^{8} \chi^{\xi^i + \frac{1}{2}, t^i}(\tau) + \prod_{i=1}^{8} \chi^{\xi^i, t^i + \frac{1}{2} + \frac{1}{4}}(\tau) \right),
\]

where \( \chi_{s,t}(\tau) \) is the contribution from one twisted complex fermion. The functions \( \chi_{s,t}(\tau) \) can be expressed in terms of Jacobi theta functions and read, using the conventions of [42],

\[
\chi_{s,t}(\tau) = e^{-2\pi i (s - \frac{1}{2})} \frac{\vartheta_{s - \frac{1}{2}, t}(0, \tau)}{\eta(\tau)}.
\]

In the R-sector the twist parameters are \( s^i = \xi^i \) according to equation (A.7b). By a similar calculation as in the NS-sector, the contribution \( Z_{\xi,t}^{R}(\tau) \) of the R-sector can be obtained and reads as

\[
Z_{\xi,t}^{R}(\tau) = \text{Tr}_{R} \left( \frac{1 + G}{2} G_t q^{L_0 - 8/24} \right) = \frac{1}{2} \left( \prod_{i=1}^{8} \chi^{\xi^i, t^i}(\tau) + \prod_{i=1}^{8} \chi^{\xi^i + \frac{1}{2}, t^i + \frac{1}{4}}(\tau) \right).
\]

The full character \( Z_{\xi,t}(\tau) \) is the given by

\[
Z_{\xi,t}(\tau) = Z_{\xi,t}^{NS}(\tau) + Z_{\xi,t}^{R}(\tau).
\]

Consider now the behaviour of the contributions \( Z_{\xi,t}^{NS/R}(\tau) \) under \( T \)-modular transformations. Using the modular properties of the Jacobi theta function one can show that

\[
Z_{\xi,t}^{NS}(\tau) \xrightarrow{T} e^{-\frac{\pi i}{12}} e^{\pi i \xi^2} Z_{\xi,t+\xi}(\tau), \quad \text{and} \quad Z_{\xi,t}^{R}(\tau) \xrightarrow{T} e^{-\frac{\pi i}{12}} e^{\pi i (\xi^2 - \sum_i \xi^i)} Z_{\xi,t+\xi}(\tau).
\]

The contributions do not transform homogeneously under \( T \)-modular transformations and the characters \( Z_{\xi,t}(\tau) \) are not mapped to themselves, unless

\[
\sum_i \xi^i \equiv 0 \mod 2.
\]

Similar considerations for \( S \)-modular transformations require in addition that the gradation parameters \( t^i \) have to obey

\[
\sum_i t^i \equiv 0 \mod 2.
\]
If the twist vector $\xi$ and the gradation parameter $t$ both satisfy their restrictions, the set of characters is closed under modular transformations and it can be shown that

$$Z_{\xi,t}(\tau) \xrightarrow{T} e^{-\frac{8\pi}{12} e^{\pi i \xi^2}} Z_{\xi,t+\xi}(\tau)$$  \hspace{1cm} (A.29a)

$$Z_{\xi,t}(\tau) \xrightarrow{S} e^{-2\pi i \xi t} Z_{t,-\xi}(\tau).$$  \hspace{1cm} (A.29b)

The calculations are the same for the anti-holomorphic fermions representing SO(8)-Lorentz symmetry. The central charge of this CFT is $c = 4$ and the phase factor under $T$-modular transformations is changed. For completeness, their modular transformation properties are given,

$$Z_{\xi,t}(\tau) \xrightarrow{T} e^{-\frac{4\pi}{12} e^{\pi i \xi^2}} Z_{\xi,t+\xi}(\tau)$$  \hspace{1cm} (A.30a)

$$Z_{\xi,t}(\tau) \xrightarrow{S} e^{-2\pi i \xi t} Z_{t,-\xi}(\tau).$$  \hspace{1cm} (A.30b)

B  Tables and pictures

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– 34 –
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| $N = 1$ SUSY multiplet | $E_6 \times U(1)$ charges | Degeneracy | Interpretation                          |
|------------------------|--------------------------|------------|----------------------------------------|
| Gravity $(0, 0)$        | $1_0$                    | 1          | SUGRA multiplet                         |
| Chiral $(1, 0)$         | $1_0$                    | 1          | Dilaton/Dilatino                        |
| Chiral $(1, 0)$         | $1_0$                    | 2          | Kähler moduli                           |
| Chiral $(1, 0)$         | $1_0$                    | 2          | Complex structure moduli                |
| Vector $(0, 0)$         | $78_0 \oplus 1_0$        | 1          | YM gauge bosons                         |
| Vector $(0, 0)$         | $248 \times 1_0$        | 1          | YM gauge bosons                         |
| Chiral $(-1, 0)$        | $27_2 \oplus 27_{-2} \oplus 1_0$ | 1         | Charged matter                          |
| Chiral $(1, 0)$         | $27_{-1} \oplus 27_1$   | 1          | Charged matter                          |
| Chiral $(1, 0)$         | $1_0 \oplus 1_3 \oplus 1_{-3}$ | 1         | Charged matter                          |
| Chiral $(2, 1)$         | $1_{3/2} \oplus 1_{-3/2}$ | 4          | Charged matter                          |
| Chiral $(0, 1)$         | $1_{3/2} \oplus 1_{-3/2} \oplus 27_{1/2} \oplus 27_{-1/2}$ | 4         | Charged matter                          |
| Chiral $(0, 1)$         | $7 \times 1_0$          | 9          | Charged matter                          |
| Chiral $(1, 0)$         | $1_3 \oplus 1_{-3} \oplus 27_{-1} \oplus 27_1$ | 9         | Charged matter                          |

Table 3: The full physical, massless spectrum of the $S_3$ model. It shows the different types of massless particles together with a possible interpretation and quantum numbers, like geometric transformation properties and representations under the $E_6 \times U(1) \times E_8$ low–energy Yang–Mills gauge group. However, all particles are uncharged under $E_8$ and $E_8$ charges are not shown explicitly. The given degeneracy is due to the geometry (fixed tori, etc.). The additional quantum numbers $(r, c)$ are the charges under the $R$ symmetry $\mathbb{Z}_4^R$ and the $\mathbb{Z}_2$ parity introduced in section 6.3.1. Only the $R$ charge of the state with highest helicity is given.
Table 4: String states with conformal weight $\leq 1/2$ in the fermion sector of the $S_3$ model. It is shown the conformal weight, the transformation properties in Minkowski space, the degeneracy and the representation under the centralizer group and the $D$ charges of discrete symmetry $\mathbb{Z}_8 \times \mathbb{Z}_{12}$. Only the $D$ charges of the highest helicity states are shown.
| Sector | Conformal weight | $E_6 \times U(1)$ quantum numbers | Centralizer irrep. | $SO(10) \times U(1) \times U(1)$ quantum numbers |
|--------|-----------------|----------------------------------|-------------------|-----------------------------------------------|
|        |                 |                                  |                   | $NS$-sector                                   | $R$-sector                                   |
| $e$    | 0               | $1_0$                            | $r_{\text{inv}}$  | $1_{0,0}$                                     | $16_{1/2} \oplus \overline{10}_{1/2}$       |
|        | 1               | $78_0$                           | $r_{\text{inv}}$  | $45_{0,0} \oplus 1_{0,0}$                    | $1_0,0$                                     |
|        |                 | $1_0$                            |                   |                                               | $16_{1/2} \oplus \text{c.c.}$               |
|        | $27_2 \oplus 27_{-2}$ | $1_0$                            | $r_{\text{alt}}$ | $10_{0,1} \oplus 1_{2,0} \oplus \text{c.c.}$ | $1_{0,0}$                                    |
|        |                 | $1_0$                            |                   |                                               | $16_{1/2} \oplus \text{c.c.}$               |
|        | $27_{-1} \oplus 27_{1}$ | $1_3 \oplus 1_{-3}$              | $r_{\text{def}}$ | $10_{-1,0} \oplus 1_{1,-1} \oplus \text{c.c.}$ | $1_{1,0}$                                    |
|        |                 | $1_0$                            |                   |                                               | $16_{0,-1/2} \oplus \text{c.c.}$           |
| $\tau$ | $1/4$           | $1_{3/2} \oplus 1_{-3/2}$        | $r_{\text{r}}$    | $1_{1/2,1/2} \oplus \text{c.c.}$             | $1_{0,0}$                                    |
|        | $3/4$           | $27_{1/2} \oplus 27_{-1/2}$      | $r_{\text{r}}$    | $10_{-1/2,1/2} \oplus 1_{3/2,-1/2} \oplus \text{c.c.}$ | $1_{1/2,1/2} \oplus \text{c.c.}$          |
|        |                 | $1_{3/2} \oplus 1_{-3/2}$        |                   |                                               | $16_{1/2,0} \oplus \text{c.c.}$           |
| $\sigma$ | $1/9$          | $1_0$                            | $r_{\text{r}}$    | $1_{0,0}$                                    |                                             |
|        | $4/9$           | $1_0$                            | $r_{\text{r}}$    | $1_{0,0}$                                    |                                             |
|        | $7/9$           | $27_{-1} \oplus 27_{1}$          | $r_{\text{r}}$    | $10_{-1,0} \oplus 1_{1,-1} \oplus \text{c.c.}$ | $1_{1,0} \oplus \text{c.c.}$              |
|        |                 | $1_3 \oplus 1_{-3}$              |                   |                                               | $16_{0,-1/2} \oplus \text{c.c.}$          |

Table 5: String states with conformal weight $\leq 1$ in the Yang–Mills sector of the $S_3$-model. The conformal weight, the representation under the Yang–Mills gauge group as well as the representation under the centralizer group are shown. On the right hand side the decomposition into irreducible representations of $g_0$ is shown.