Center-Outward Multiple-Output Lorenz Curves and Gini Indices
a measure transportation approach

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Based on measure transportation ideas and the related concepts of center-outward quantile functions, we propose multiple-output center-outward generalizations of the traditional univariate concepts of Lorenz and concentration functions, and the related Gini and Kakwani coefficients. These new concepts have a natural interpretation, either in terms of contributions of central (“middle-class”) regions to the expectation of some variable of interest, or in terms of the physical notions of work and energy, which sheds new light on the nature of economic and social inequalities. Importantly, the proposed concepts pave the way to statistically sound definitions, based on multiple variables, of quantiles and quantile regions, and the concept of “middle class,” of high relevance in various socio-economic contexts.

KEYWORDS. Center-outward quantiles, Measure transportation, Concentration indices, Inequality measurement, Definition of “middle class”.

1. Introduction

1.1 Lorenz curves and the measurement of inequality

Measuring inequalities is a major issue in some areas of economics and the social sciences, and a vast literature has developed on the subject, which even led to the publication of dedicated journals such as The Journal of Economic Inequality. The most popular statistical tools in that context are the various indices of concentration or inequality related with the so-called Lorenz and concentration functions (Lorenz, 1905, Kakwani, 1977), themselves in close relation to integrated quantile functionals.

The concept of quantile is clear and well understood in dimension one, and Lorenz functions originally were developed for univariate quantities (single-output case) only. Due to the lack of a canonical ordering in real spaces of dimension two and higher, however, multivariate extensions (multiple-output case) of the concept are more problematic. The few existing attempts to define Lorenz functions in a multiple-output context
came much later and are, in general, not associated with any sound concept of multivariate quantile; their interpretation, therefore, is not entirely satisfactory, and none of them has met a general consensus among practitioners.

Based on measure transportation ideas, new definitions of multivariate distribution and quantile functions have been proposed recently (Chernozhukov et al., 2017, Hallin et al., 2021) which, contrary to earlier ones, are enjoying the fundamental properties one is expecting from a quantile function; see Hallin (2022) for a survey. In this measure transportation context, convex potentials naturally extend the notion of integrated quantile functions. Building on these concepts of multivariate quantiles and convex potentials, this paper is proposing new multivariate extensions of the classical concepts of Lorenz and concentration functions and the related indices—the best-known of which is the Gini coefficient (Gini, 1912).

1.2 Organisation of the paper

The rest of the paper is organized as follows. We briefly start (Section 2) by reviewing the history of Lorenz curves and inequality measurement. We then propose (Section 3) an overview of center-outward quantile functions. In Section 4, we introduce, in dimension one first, then in arbitrary dimension \( d \), the concept of center-outward Lorenz function. Starting with dimension two, indeed, the space is no longer oriented from \( -\infty \) to \( \infty \), but each direction yields two points at infinity, none of which qualifies as \( -\infty \) nor \( \infty \). This leads to center-outward rather than “left-to-right” orderings. The resulting multivariate indices (Gini, Kakwani) are introduced in Section 5. Section 6 proposes estimators of the newly defined population quantities and establishes their consistency properties. We conclude in Section 7.1 with an application to a dataset relating income, life expectancy, and education. This application nicely illustrates how the proposed concepts yield new quantitative descriptions of multivariate inequalities and concentrations.

2. A brief history of Lorenz curves and Gini coefficients

We start with a brief review of some classical definitions and concepts arising in connection with Lorenz curves and the associated indices. We then propose a general definition that encompasses most previous ones. Due to space considerations, this survey unavoidably is incomplete; we refer to Arnold and Sarabia (2018) for a book-long exposition.

2.1 Traditional single-output concepts

The traditional definition of a Lorenz curve deals with a univariate real-valued positive absolutely continuous random variable \( X \) with Lebesgue density \( f_X(x) > 0 \) for all \( x \geq 0 \). Denoting by \( F_X \) its distribution function, \( F_X(0) = 0 \) and \( F_X \) is strictly increasing, so that the quantile function \( Q_X := F_X^{-1} \) is well defined. Assume that the mean

\[
0 < \mu_X := \mathbb{E}X := \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_0^1 Q_X(t) \, dt
\]
is finite: the classical definition of a Lorenz curve then is as follows (Lorenz, 1905, Gastwirth, 1971).

**Definition 1.** The (relative) Lorenz curve of $X$ is the graph $\{(u, L_X(u)), u \in [0, 1]\}$ of the (relative) Lorenz function

$$
u \mapsto L_X(u) := \frac{\int_0^\nu Q_X(t)dt}{\int_0^1 Q_X(t)dt} = \mu_X^{-1} \mathbb{E}[X \mathbb{1}_{X \leq Q_X(u)}]$$

$$= \mu_X^{-1} (\psi_X(u) - \psi_X(0)), \quad 0 \leq u \leq 1 \quad (2.1)$$

where $\psi_X : (0, 1) \to \mathbb{R}$ is a primitive of $Q_X$, that is, $Q_X(u) = \frac{d\psi_X(u)}{du}$ for $0 < u < 1$, with $\psi_X(1) - \psi_X(0) = \mu_X$.

It immediately follows from the assumptions that $L_X$ is continuous, strictly increasing, and a.e. differentiable, with $0 = L_X(0) < L_X(1) = 1$; $\psi_X$ is a.e. differentiable and convex, with derivative $Q_X$. Without any loss of generality, let us impose $\psi_X(0) = 0$: then,

$$\mu_X = \psi_X(1) \quad \text{and} \quad L_X(u) = \psi_X(u)/\psi_X(1), \quad 0 \leq u \leq 1. \quad (2.2)$$

The numerator $u \mapsto L_X(u) := \int_0^u Q_X(t)dt$ in (2.1) and (2.2)—call it the absolute Lorenz function—represents the contribution to the integral $\int_0^1 Q_X(t)dt = \mu_X$ of the quantile region $(-\infty, u]$, which provides the intuitive interpretation of the concept. This absolute definition still makes sense if the requirement of a positive $X$ is dropped.

In the sequel, the positivity and finite-mean assumptions are implicitly made whenever relative Lorenz functions are mentioned and implicitly relaxed when absolute Lorenz functions are involved.

Associated with the relative Lorenz function are several concentration indices, the most popular of which is the Gini coefficient (Gini, 1912), defined as twice the area comprised between the (relative) Lorenz curve and the main diagonal of the unit square, namely,

$$G_X := 2 \int_0^1 [u - L_X(u)]du. \quad (2.3)$$

Other indices are the Pietra index (Pietra (1915); see Pietra (2014))

$$P_X := \sup_{0 \leq u \leq 1} [u - L_X(u)]$$

and the Amato-Kakwani index, defined as the rescaled length

$$AK_X := \frac{l_X - \sqrt{2}}{2 - \sqrt{2}}$$
of the relative Lorenz curve, where (denoting by $L'_X$ the a.e. derivative of the Lorenz function $u \mapsto L_X(u)$)

$$l_X := \int_0^1 \sqrt{1 + (L'_X(u))^2} \, du.$$  

These three indices all take values in $[0, 1]$. They can be interpreted as a measure of discrepancy between two distribution functions over $[0, 1]$: the uniform, with probability density $f(u) = 1$ and the one with density $f(u) = Q_X(u)/\mu_X$, $u \in [0, 1]$.

Other characterizations of the Gini index are possible. For instance, denoting by $X_1$ and $X_2$ two independent copies of $X$, it can be shown that (Yitzhaki and Olkin, 1991, p. 384)

$$G_X = \frac{\mathbb{E}|X_1 - X_2|}{2\mu_X} = \frac{1}{2\mu_X} \mathbb{E} \left| \begin{array}{cc} X_1 \\ X_2 \end{array} \right| = \frac{2}{\mu_X} \text{Cov}(X, F(X)), \quad (2.4)$$

a definition that does not explicitly involve the quantile function $Q_X$.

Denoting by $g: x \mapsto g(x)$ a continuous and positive real-valued function with finite mean $\mu_g > 0$, the following extension of the Lorenz function was proposed by Kakwani (1977) and is often considered in economics.

DEFINITION 2. The (relative) concentration function of $g(X)$ with respect to $X$ is

$$u \mapsto K^g_X(u) := \frac{\int_0^u g(Q_X(t)) \, dt}{\int_0^1 g(Q_X(t)) \, dt} = \mu_g^{-1} \mathbb{E} \left[ g(X) 1\{X \leq Q_X(u)\} \right] \quad u \in [0, 1]. \quad (2.5)$$

For positive and strictly increasing $g$, $K^g_X(u)$ coincides with $L_{g(X)}(u)$; in particular, $g(x) = x$ yields $K^g_X = L_X$. But $g$ needs not be positive and monotone, nor have finite mean: the numerator $K^g_X(u) := \int_0^u g(Q_X(t)) \, dt$ in (2.5)—call it the absolute concentration function of $g(X)$ then still makes sense.

A case that has attracted particular interest in economics is $g(x) = \mathbb{E}[Y|X = x]$ where $Y$ is some variable of interest: $K^g_X$ then yields

$$u \mapsto K_{Y|X}(u) := \frac{\int_0^u \mathbb{E}[Y|X = Q_X(t)] \, dt}{\int_0^1 \mathbb{E}[Y|X = Q_X(t)] \, dt} = \mu_Y^{-1} \int_0^u \mathbb{E}[Y|X = Q_X(t)] \, dt$$

$$= \mu_Y^{-1} \mathbb{E} \left[ Y 1\{X \leq Q_X(u)\} \right] = \mu_Y^{-1} \left( \psi_{Y/X}(u) - \psi_{Y/X}(0) \right) \quad u \in [0, 1] \quad (2.6)$$

where $\psi_{Y/X}$ denotes an arbitrary primitive of $u \mapsto \mathbb{E}[Y|X = Q_X(u)]$ (the value of which we can assume to be zero at $u = 0$).

For any $g$, there exist infinitely many variables $Y$ such that $g(x) = \mathbb{E}[Y|X = x]$; rather than a particular case of (2.5), (2.6) thus should be considered as an alternative but
equivalent expression involving the joint distribution of a couple of random variables \((X, Y)\). We emphasize this by calling \(K_{Y/X}(u)\) and its numerator

\[
u \mapsto K_{Y/X}(u) := \int_0^u \mathbb{E}[Y \mid X = Q_X(t)]\,dt
\]

the relative and absolute Kakwani functions, respectively, of \(Y\) with respect to \(X\).

Whenever relative Kakwani functions are mentioned or the notation \(\mu_Y\) is used, we tacitly assume that \(0 < \mu_Y < \infty\).

### 2.2 Generalized Lorenz functions

The value at \(u\) of the relative Kakwani function of \(Y\) with respect to \(X\) represents the contribution of the quantile region \((−\infty, Q_X(u)]\) of \(X\) to the mean \(\mu_Y\) of \(Y\), thus splitting the roles: \(Y\) is the variable of interest, but \(X\) determines the quantile regions involved in the definition of \(K_{Y/X}(u)\).

The same role-splitting can be performed in the definition of Lorenz functions, yielding the relative and absolute Lorenz functions of \(Y\) with respect to \(X\). More precisely, using obvious notation \(F_X, F_Y, Q_X, Q_Y, \psi_X, \text{ and } \psi_Y\) for the couple \((X, Y)\) of random variables, define the relative Lorenz function of \(Y\) with respect to \(X\) as

\[
u \mapsto L_{Y/X}(u) := \frac{\int_0^{F_Y \circ Q_X(u)} Q_Y(t)\,dt}{\int_0^1 Q_Y(t)\,dt} = \frac{\mu_Y^{-1}}{\int_0^1 Q_Y(t)\,dt} \mathbb{E} \left[ Y 1_{\{Y \leq Q_X(u)\}} \right]
\]

\[
= \mu_Y^{-1} \left( \psi_Y(F_Y \circ Q_X(u)) - \psi(0) \right) \quad u \in [0, 1],
\]

reducing to \(L_{Y/X}(u) = \psi_Y(F_Y \circ Q_X(u))/\psi_Y(1)\) if we assume, without loss of generality, that \(\psi_Y(0) = 0\). The numerator \(L_{Y/X}\) of (2.7) is the absolute Lorenz function of \(Y\) with respect to \(X\). To the best of our knowledge, such concept has never been considered in the literature.

Both \(K_{Y/X}\) and \(L_{Y/X}\) extend the traditional Lorenz function \(L_X\), to which they reduce for \(Y = X\) and \(Y \overset{d}{=} X\) (with \(d\) standing for equality in distribution), respectively. While \(K_{Y/X}(u)\) evaluates the contribution to \(\mu_Y\) of the region \((−\infty, Q_X(u)] \times \mathbb{R}\) of the \((x, y)\) space (i.e., the quantile region of order \(u\) of \(X\)), \(L_{Y/X}(u)\) is about the contribution of \(\mathbb{R} \times (−\infty, Q_X(u)]\) (the quantile region of order \(F_Y \circ Q_X(u)\) of \(Y\)). An important difference between them is that \(K_{Y/X}\) depends on the joint distribution \(P_{XY}\) of \(X\) and \(Y\), while \(L_{Y/X}\) only involves their marginals \(P_X\) and \(P_Y\). As a consequence, \(K_{Y/X}\), contrary to \(L_{Y/X}\), cannot be expressed in terms of the marginal integrated quantile functions \(\psi_X\) and \(\psi_Y\). On the other hand, \(L_{Y/X}\) only makes sense if \(X\) and \(Y\) are comparable quantities, which is the case when \(Y = X\), or when \(P_X\) and \(P_Y\) are, for instance, the income distributions within two given socio-economic groups or two countries.
2.3 The multivariate case: classical extensions

Extending the above concepts to multivariate random variables $\mathbf{X}$ and $\mathbf{Y}$ is, of course, highly desirable. Several attempts have been made to extend the definitions of the Lorenz function (2.1) and the Gini index (2.5) to a multivariate setting. The difficulty, of course, lies in the fact that, in dimension 2 and higher, the real space is no longer canonically ordered: as a consequence, the very concept of quantile is problematic. Quoting Arnold and Sarabia (2018, page 149),

Extensions of the Lorenz curve concept to higher dimensions was long frustrated by the fact that the usual definitions of the Lorenz curve involved either order statistics or a quantile function of the corresponding distribution, neither of which has a simple multidimensional analog.

An extension of $L_{Y/X}$ and $K_{Y/X}$ to couples of random vectors $(\mathbf{X}, \mathbf{Y})$ looks even more challenging.

Therefore, rather than generalizing the classical quantile-based definition (2.1), the extensions found in the literature essentially consist in bypassing the role of quantiles in these definitions; as a result, the intuitive interpretation of the univariate definitions sometimes gets lost, and the value of the resulting multivariate concept is disputable. We only briefly review some of them.

Arnold and Sarabia (2018, page 150), for instance, propose, for a positive bivariate random vector $\mathbf{X} = (X_1, X_2)'$ with density $f(x_1, x_2)$ and marginal quantile functions $Q_1$ and $Q_2$, the bivariate relative Lorenz function

\[
(u, v) \in [0, 1]^2 \mapsto L_{X_1, X_2}(u, v) := \frac{1}{E(X_1X_2)} \int_0^{Q_1(u)} \int_0^{Q_2(v)} x_1 x_2 f(x_1, x_2) \, dx_1 \, dx_2 
\]  

(2.8)

and the corresponding Lorenz surface \{ $(u, v, L_{X_1, X_2}(u, v))$, $(u, v) \in [0, 1]^2$ \}. This indeed constitutes a technically correct bivariate extension of Definition 1 and straightforwardly extends to higher dimensions. However, the focus on the product $X_1X_2$ is somewhat ad hoc; the resulting order has not been fully studied.

Starting from Equation (2.4), which characterizes the Gini index without resorting to quantiles, other authors have proposed ingenious multivariate extensions of the Gini index skipping the definition of a Lorenz function. Denoting by $\mathbf{X}, \mathbf{X}_1, \ldots, \mathbf{X}_d$ a $(d + 1)$-tuple of independent copies of a $d$-dimensional variable $\mathbf{X}$ and considering the $(d + 1) \times (d + 1)$ matrix

\[
M_\mathbf{X} := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{X} & \mathbf{X}_1 & \ldots & \mathbf{X}_d \end{pmatrix}',
\]

Koshevoy and Mosler (1997) define the Volume Gini Index of $\mathbf{X}$ as

\[
VG_\mathbf{X} := \frac{1}{(d + 1)!} E |\det M_\mathbf{X}|. 
\]  

(2.9)

This idea of measuring concentration based on volumes of convex bodies actually goes back to Wilks (1960), who proposed a coefficient that coincides with $VG_\mathbf{X}$ up to a scale factor. Oja (1983) pointed out that this coefficient is the expected volume of the simplex
with vertices the random \((d + 1)\)-tuple \(X, X_1, \ldots, X_d\); Koshevoy and Mosler showed that the definition (2.9) corresponds to the volume of a lift zonoid (Mosler, 2002). While reducing, in dimension one, to the traditional concept, the Volume Gini index, however, fails, in dimension two and higher, to enjoy the intuitive interpretation of the traditional Gini concentration index.

Concepts of multivariate inequality/concentration and their measurement still constitute an active area of research; see the recent works by Mosler (2022), Grothe et al. (2021), Andreoli and Zoli (2020), Sarabia and Jorda (2020), Medeiros and de Souza (2015), and the references therein. The latest contribution by Mosler (2022) explores the possibility of aggregating the various attributes of welfare; a compelling approach that differs from but complements ours. The idea of aggregating data is also considered in Sánchez-González and García-Fernández (2020) to construct a notion of middle-class that relies on multiple attributes. By the time we were completing this manuscript, the work by Fan et al. (2022) came to our attention. The multivariate Lorenz curves concepts they are proposing also are based on measure transportation ideas, and are aiming at the same objectives as ours. However, their concepts of multivariate quantiles are not of the center-outward type and do not allow for a clear definition of quantile regions; the interpretation of their Lorenz curves, therefore, is essentially different from ours. Moreover, they do not consider the notions of concentration and Lorenz curves of one variable with respect to another one.

3. CENTER-OUTWARD DISTRIBUTION AND QUANTILE FUNCTIONS

We briefly present here the concept of center-outward quantile underlying the definition our multiple-output Lorenz curves and Gini indices.

3.1 Quantiles and center-outward quantiles

Based on measure transportation ideas, new concepts of multivariate quantile have been proposed recently (Chernozhukov et al., 2017, Hallin et al., 2021, Hallin, 2022) under the name of Monge–Kantorovich and center-outward quantiles, respectively. Contrary to previous proposals, these center-outward quantiles enjoy the essential properties one is expecting from quantiles. In particular, the resulting quantile regions are closed, connected, and nested, with probability contents that do not depend on the underlying distributions (which is not the case, e.g., with depth regions). A distinctive feature of the real space in dimension \(d > 1\) is the nonexistence of a linear, left-to-right ordering from \(+\infty\) to \(-\infty\)): instead, each direction \(u \in S_{d-1}\) defines an arbitrarily remote point, suggesting center-outward partial orderings rather than a unique complete linear ordering from \(-\infty\) to \(+\infty\). Familiar examples of center-outward orderings are the orderings associated with the geometry of elliptical distributions or the various concepts of depth—see, e.g., Serfling and Zuo (2000), Serfling (2002, 2019), Zuo (2021), or Konen and Paindaveine (2022).

The orderings associated with the measure-transportation-based concepts of distribution and quantile functions developed in Hallin et al. (2021) are of a center-outward
nature comparable to measures of outlyingness related to these depth notions, and offer appealing multivariate center-outward generalizations of ranks, signs, and quantile functions. Center-outward quantile functions, in that context, are defined as gradients of a convex potential \( \psi \)—a natural form of integrated quantile function. As we shall see, when substituted for traditional integrated univariate quantiles in (2.1), (2.6), and (2.7), that potential \( \psi \) yields extensions of the univariate concepts of Lorenz and Kakwani functions and the related indices that preserve the intuitive interpretation of the univariate concepts.

### 3.2 Univariate center-outward distribution and quantile functions

Before addressing the general case, let us first examine the consequences, on the familiar univariate concepts, of a center-outward definition of quantile regions. Let the univariate random variable \( X \) have Lebesgue-absolutely continuous distribution \( P_X \) with continuous distribution function \( F_X \), and call \( F_X := 2F_X - 1 \) the center-outward distribution function of \( X \). That center-outward distribution function \( F_X \) is such that

\[
|F_X(x)| = P[x' \leq X \leq x'']
\]

with \( x', x'' \) characterized by

\[
F_X(x') = \min(F_X(x), 1 - F_X(x)) \quad \text{and} \quad F_X(x'') = \max(F_X(x), 1 - F_X(x)).
\]

While the traditional condition \( F_X(x) \leq \tau \) characterizes quantile regions with probability \( \tau \) as nested halflines of the form \( (-\infty, F_X^{-1}(\tau)] \), the condition \( |F_X(x)| \leq \tau \) characterizes quantile regions with probability \( \tau \) as nested central interquantile intervals \([x', x'']\) of the form

\[
C_X(\tau) := [F_X^{-1}_X(-\tau), F_X^{-1}_X(\tau)] = [F_X^{-1}((1 - \tau)/2), F_X^{-1}((1 + \tau)/2)].
\] (3.1)

The intuitive interpretation of these center-outward quantile regions is clear: while the traditional quantile region of order \( \tau \) contains the proportion \( \tau \) of the smallest values of the random variable \( X \sim P_X \), \( C_X(\tau) \) contains the proportion \( \tau \) of its “most central” values.\(^1\)

Let \( P \) be Lebesgue-absolutely continuous. A celebrated theorem by McCann (1995) tells us that there exists a unique potential \( \psi_\pm \) such that \( \psi_\pm(0) = 0 \) with gradient \( \nabla \psi_\pm \)

\textbf{pushing} \( P \) forward\(^2\) to the uniform \( U_1 \) over the unit ball \( S_1 = (-1, 1) \) of \( \mathbb{R} \)—namely, such that \( \nabla \psi_\pm(Z) \sim U_1 \) for any \( Z \sim P \), which we write as \( \nabla \psi_\pm \# P = U_1 \).

Since \( F_X \) is monotone increasing (hence the derivative of a convex function) and such that \( F_X(X) \sim U_1 \) (hence pushes the distribution \( P_X \) of \( X \) forward to \( U_1 \)), McCann's theorem entails \( F_X \) is \( \nabla \psi_X \) Lebesgue-a.e. Assuming that \( X \) has an everywhere

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1In dimension \( d = 1 \), it is easy to see that the notions of “most central” and “Tukey-deepest” coincide, so that these center-outward quantile regions also coincide with the Tukey depth regions—an equivalence that no longer holds in dimensions \( d \geq 2 \), though.

2We adopt this convenient terminology and the corresponding notation from measure transportation, where a mapping \( G \) is \textbf{pushing} a distribution \( P_1 \) \textbf{forward} to a distribution \( P_2 \) (notation: \( G \# P_1 = P_2 \)) if \( Z \sim P_1 \) entails \( G(Z) \sim P_2 \).
positive density, the inverse \( Q_{X \pm} \) of \( F_{X \pm} \) is well defined and is pushing \( U_1 \) forward to \( P_X \): call it the center-outward quantile function of \( X \). McCann’s theorem again implies that, for some unique convex potential \( \psi_{X \pm} \) such that \( \psi_{X \pm}(0) = 0 \), the center-outward quantile function \( Q_{X \pm}(u) := \nabla \psi_{X \pm}(u) = d\psi_{X \pm}(u)/du \); elementary algebra yields that \( \psi_{X \pm}(u) \) equals \( 2\psi_X((u+1)/2) \) where \( \psi_X \), defined in Section 2, is a primitive of \( Q_X \). Up to this point, \( \psi_{X \pm} \) is defined on \((-1,1)\) only; this domain is classically extended to the real line by letting \( \psi_{X \pm}(1) := \lim_{u \uparrow 1} \psi_{X \pm}(u) \), \( \psi_{X \pm}(-1) := \lim_{u \downarrow -1} \psi_{X \pm}(u) \), and \( \psi_{X \pm}(u) := \infty \) for \( |u| > 1 \).

3.3 Multivariate center-outward quantiles and transportation to the unit ball

Since a one-to-one correspondence between the traditional distribution function \( F_X \) and the center-outward \( F_{X \pm} \), hence between \( Q_X \) and \( Q_{X \pm} \), exists, the concepts \( F_{X \pm} \) and \( Q_{X \pm} \) of center-outward distribution and quantile functions are of limited interest in dimension \( d = 1 \); their major advantage is that their definitions and their interpretation, unlike those of \( F_X \) and \( Q_X \), readily extend to dimensions \( d \geq 2 \), for which no sound counterparts to \( F_X \) and \( Q_X \) are available.

Let \( X \) be a \( d \)-dimensional random vector with distribution \( P_X \) in the class \( \mathcal{P}_d \) of Lebesgue-absolutely continuous distributions over \( \mathbb{R}^d \). In view of the aforementioned theorem by McCann, a center-outward distribution function \( F_{X \pm} \) extending the univariate one \( F_{X \pm} \) can be defined as follows. Denote by \( S_d, S_{d-1} \) the open unit ball in \( \mathbb{R}^d \) and the unit hypersphere centered at the origin, respectively, by \( U_d \) the spherical uniform over the ball \( S_d \), and by \( V_{d-1} \) the uniform over \( S_{d-1} \). For \( d = 1 \), the spherical uniform \( U_d = U_1 \) reduces to the Lebesgue uniform over the unit ball \( S_1 = (-1,1) \). For \( d \geq 2 \), however, \( U_d \) is the product of the uniform \( U_{[0,1]} \) over \([0,1]\) (for the distances to the origin) and the uniform \( V_{d-1} \) over \( S_{d-1} \) (for the directions), which no longer coincides with the Lebesgue uniform.

**Definition 3.** Call center-outward distribution function of \( X \) the a.e. unique gradient of convex function \( \nabla \psi_{X \pm}^* := F_{X \pm} \) pushing \( P_X \) to \( U_d \), that is, such that \( F_{X \pm} \# P_X = U_d \).

This is the definition proposed in Hallin et al. (2021), where \( F_{X \pm} \) is shown to enjoy all the properties expected from a distribution function. In particular, for a measure \( P_X \) in the class \( \mathcal{P}_d^\pm \subset \mathcal{P}_d \) of all absolutely continuous distributions with density \( f \) such that for all \( c > 0 \) there exists \( 0 \leq \lambda_c \leq \Lambda_c < \infty \) for which

\[
\lambda_c \leq \inf_{x \in cS_d} f(x) \leq \sup_{x \in cS_d} f(x) \leq \Lambda_c,
\]

the center-outward distribution function \( F_{X \pm} \) is a homeomorphism between the punctured unit ball \( S_d \setminus \{0\} \) and \( \mathbb{R}^d \setminus F_{X \pm}^{-1}(0) \) (Figalli, 2018); the condition \( P \in \mathcal{P}_d^\pm \) is partially relaxed in del Barrio et al. (2020) and Hallin et al. (2021), where convex supports are allowed. Then, the center-outward distribution function \( F_{X \pm} \) is continuously invertible except, possibly, at the origin: denote by \( Q_{X \pm} \) its inverse. This inverse is the \( U_d \)-a.s. unique gradient \( \nabla \psi_{X \pm} \) of a convex potential function from \( S_d \) to \( \mathbb{R}^d \) pushing \( U_d \) forward
to $P_X$, and naturally qualifies as the center-outward quantile function of $X$ (see Hallin et al. (2021) for details).

The function $\psi_X$, which so far is defined on $S_d$ only, is extended to the closed ball $\overline{S}_d$ by lower semi-continuity: $\psi_X(u) := \liminf_{x \to u, |\psi_X(x)| < 1} \psi_X(u)$ for $|u| = 1$ and to $\mathbb{R}^d$ by setting $\psi_X(u) := +\infty$ for $u \notin \overline{S}_d$ (see, e.g. (A.18) in Figalli (2018)). With this extension, the potentials $\psi_X$ and $\psi^*_X$ are Fenchel–Legendre conjugates, i.e., satisfy

$$\psi^*_X(x) := \sup_{u \in \overline{S}_d} \{ u \cdot x - \psi_X(u) \},$$

(throughout, we use the dot product notation $u \cdot y$ for the scalar product of $u$ and $y$).

Finally, the potentials $\psi_X$ and $\psi^*_X$ are uniquely determined if, without any loss of generality, we impose $\psi_X(0) = 0$. Figalli (2018) also shows that for spherically symmetric distributions, i.e., for distributions with densities $f_X$ of the form $f_{\text{rad}}(\|x - \mu_X\|)$ for some $\mu_X$, the center-outward quantile function has the particular form

$$Q_X(u) = \mu_X + q(\|u\|) \frac{u}{\|u\|}, \quad u \in S_d$$

where $q$ is the traditional univariate quantile function of $\|X - \mu\|$. The quantile region of order $\tau$ of $U_d$, thus, is the closed ball with radius $\tau$ centered at the origin.

Associated with $F_X$ and $Q_X$ are the quantile regions $C_X(\tau) := Q_X(\tau S_d)$ and contours $C_X(\tau) := Q_X(\tau S_{d-1})$, $\tau \in (0, 1)$; the set $C_X(0) := \cap_{\tau \in (0, 1)} C_X(\tau)$, which (Figalli, 2018) is convex, compact, and has Lebesgue measure zero, can be considered as a multivariate median.

The intuition behind this definition of the center-outward quantile regions is the same as in the univariate case (3.1): $C_X(\tau)$ contains the proportion $\tau$ of the “most central” values of $X \sim P_X$. That intuition is illustrated, for $d = 2$, in Figure 1.

### 3.4 Vector quantiles and transportation to the unit cube

The definition of center-outward quantiles in dimension $d \geq 2$ is based on a transport pushing $P_X$ forward to the spherical uniform $U_d$ over the unit ball. Several authors (including Carlier et al. (2016), Chernozhukov et al. (2017), Deb and Sen (2019), Ghosal and Sen (2019)) rather privilege a transport to the Lebesgue uniform over the unit cube $[0, 1]^d$. That choice produces, under the names of vector ranks and vector quantile functions, alternative $d$-variate distribution and quantile functions $F_{\square}$ and $Q_{\square} := F_{\square}^{-1}$, say.

As long as the objective is a characterization (in the empirical case, a consistent estimation) of $P_X$ or (in the empirical case) the construction of distribution-free tests, that choice of a cubic uniform reference is perfectly fine. However, when it comes to defining quantile regions and contours, $Q_{\square}$ is running into the same conceptual difficulties as the inverse $Q := F^{-1}$ of the traditional multivariate distribution function

$$F : x = (x_1, \ldots, x_d) \mapsto F(x) := P_X[(-\infty, x_1] \times \ldots \times (-\infty, x_d)].$$

Just as $Q$, $Q_{\square}$ is based on a $d$-tuple of marginal linear orderings rather than a unique center-outward one, and depends on a $d$-tuple $(\tau_1, \ldots, \tau_d)$ of marginal orders rather than
Center-outward multiple-output Lorenz curves and Gini indices

Figure 1. Left panel: the two-dimensional unit ball, with its quantile regions of order $\tau = 0.146$, 0.268, 0.39, 0.634, and 0.878 (the balls $\tau S_d$ with radius $\tau$ centered at the origin). Right-hand panel: a numerical approximation, based on a sample of points generated from a banana-shaped mixture $P$ of three normals, of the quantile regions $Q_{\pm}(\tau S_d)$ (same $\tau$ values) of $P$.

A unique $\tau$. This does not yield (see, e.g., Genest and Rivest (2001)) a satisfactory concept of quantile regions and contours. An inconvenient feature of unit cubes, moreover, is that there are many of them: besides $[0,1]^d$, constructed over the canonical coordinate system, all orthogonal transformations of $[0,1]^d$, yielding an infinite number of plausible quantile functions $Q_{\Box}$, are equally natural. All these $Q_{\Box}$’s carry the same information; as quantiles, however, their interpretations may be extremely different. For instance, in dimension $d = 2$, the same point $x$ can be of the form $x = Q^{(1)}_{\Box}(\tau_1 \approx 1, \tau_2 \approx 0)$ for some given unit square, of the form $x = Q^{(2)}_{\Box}(\tau_1 \approx 0, \tau_2 \approx 1)$ for some other one, and finally of the form $x = Q^{(3)}_{\Box}(\tau_1 \approx 1/\sqrt{2}, \tau_2 \approx 1/\sqrt{2})$ for a third one: in the first two cases, it is an extreme but in the third case it is not. Finally, $Q_{\Box}$, just as $Q$, is highly non-equivariant under orthogonal transformations of $X$ while, thanks to the rotational invariance of $U_d$ and the unit ball (see Proposition 2.2 in Hallin et al. (2022)), $Q_{\pm}$ is.

Fan et al. (2022) are building their approach to multiple-output Lorenz functions on such a $Q_{\Box}$, which does not really allow for any interpretation based on the contribution of the proportion $\tau$ of “most central,” “smallest,” or “less extreme” values of $X \sim P_X$.

4. Center-outward Lorenz and Kakwani functions

When generalizing the classical univariate definitions (2.1) and (2.6) of Lorenz and Kakwani functions, two distinct points of view can be adopted:
(a) either these classical functions are seen as expectations of a family of truncated random variables, e.g., of the form $X 1_{\{X \in (-\infty, Q_X(u)]\}}$, or
(b) as integrals of quantile functions, with the physical interpretation of variations of potential or work.
Each of these two interpretations yields center-outward versions which we develop in Sections 4.1 and 4.2, respectively.

### 4.1 Center-outward Lorenz, Kakwani, and income share functions

Emphasis, in the classical univariate definition (2.1) of the Lorenz function, is on quantile regions of the form $(-\infty, Q(u)]$: the numerator, in (2.1), is the contribution of the quantile region $(-\infty, Q(u)]$ to the mean $\mu_X := \int_0^1 Q(t)dt$ (assumed to be finite). In the center-outward approach, emphasis is on center-outward quantile regions of the form (see (3.1))

$$C_{\pm}(\tau) := [Q_{\pm}(-\tau), Q_{\pm}(\tau)] = [Q((1-\tau)/2), Q((1+\tau)/2)]$$

and their contribution to $\mu_X$. This change of paradigm suggests the following definitions.

Let the random vectors $X$ and $Y$ take values in $\mathbb{R}^d_X$ and $\mathbb{R}^d_Y$, respectively. The notation $P_X$, $P_Y$, and, whenever $X$ and $Y$ are defined on the same probability space, $P_{XY}$ (all assumed to be Lebesgue-absolutely continuous) is used in an obvious way, as well as $F_{X_{\pm}}$, $Q_{X_{\pm}}$, $F_{Y_{\pm}}$, and $Q_{Y_{\pm}}$.

**Definition 4.** (Center-outward Lorenz and Kakwani functions)

(i) Call (absolute) center-outward Lorenz function of $X$ the mapping $u \mapsto L_{X_{\pm}}(u)$, where

$$L_{X_{\pm}}(u) := \mathbb{E}\left[ X \mathbb{1}_{\{X \in C_{X_{\pm}}(u)\}} \right], \quad 0 \leq u \leq 1; \tag{4.1}$$

(ii) assuming $d_X = d = d_Y$, call (absolute) center-outward Lorenz function of $Y$ with respect to $X$ the mapping $u \mapsto L_{Y/X_{\pm}}(u)$, where

$$L_{Y/X_{\pm}}(u) := \mathbb{E}\left[ Y \mathbb{1}_{\{Y \in C_{X_{\pm}}(u)\}} \right], \quad 0 \leq u \leq 1; \tag{4.2}$$

(iii) call (absolute) center-outward Kakwani function of $Y$ with respect to $X$ the mapping $u \mapsto K_{Y/X_{\pm}}(u)$, where

$$K_{Y/X_{\pm}}(u) := \mathbb{E}\left[ Y \mathbb{1}_{\{X \in C_{X_{\pm}}(u)\}} \right], \quad 0 \leq u \leq 1. \tag{4.3}$$

Note that (iii), unlike (ii), involves a joint distribution $P_{XY}$ for $X$ and $Y$.

Instead of trimmed expectations, as in (4.1)–(4.3), one also may like to consider conditional expectations and replace, e.g., $\mathbb{E}\left[ X \mathbb{1}_{\{X \in C_{X_{\pm}}(u)\}} \right]$ with

$$\mathbb{E}\left[ X \middle| X \in C_{X_{\pm}}(u) \right] = \mathbb{E}\left[ X \mathbb{1}_{\{X \in C_{X_{\pm}}(u)\}} \right] / u,$$

yielding conditional center-outward Lorenz and Kakwani functions which sometimes are easier to interpret.
DEFINITION 5. (Conditional center-outward Lorenz and Kakwani functions)

Call (absolute) conditional center-outward Lorenz function of $X$, (absolute) conditional center-outward Lorenz function of $Y$ with respect to $X$, and (absolute) conditional center-outward Kakwani function of $Y$ with respect to $X$ the functions

$$L_{X \pm}^{\text{cond}} (u) := L_{X \pm} (u)/u, \quad L_{Y / X \pm}^{\text{cond}} (u) := L_{Y / X \pm} (u)/u, \quad \text{and} \quad K_{Y / X \pm}^{\text{cond}} (u) := K_{Y / X \pm} (u)/u,$$

respectively.

Conditional Lorenz and Kakwani functions convey the same information as their unconditional counterparts. In the absence of concentration, these conditional functions take value one for all $u$.

Just as $X$ and $Y$ themselves, these center-outward functions are vector-valued ($L_{X \pm}$ and $L_{X \pm}^{\text{cond}}$ are $d_X$-dimensional, $L_{Y / X \pm}$, $L_{Y / X \pm}^{\text{cond}}$, $K_{Y / X \pm}$, and $K_{Y / X \pm}^{\text{cond}}$ are $d_Y$-dimensional). Relative versions $L_{X \pm}^{\text{cond}}$, $L_{X \pm}^{\text{cond}}$, $L_{Y / X \pm}$, $L_{Y / X \pm}^{\text{cond}}$, $K_{Y / X \pm}$, and $K_{Y / X \pm}^{\text{cond}}$ can be obtained by dividing each component of the corresponding Lorenz and Kakwani absolute functions by its value at $u = 1$ (provided that the latter are finite). These relative functions then define curves running from $(0, \ldots, 0)$ to $(1, \ldots, 1)$ for $u = 1$ which, if the components of $X$ (respectively, the components of $Y$) all are non-negative, lie in the unit cube. These curves offer much information on the contributions of the central quantile regions to the mean of $X$ or the mean of $Y$, with a large variety of possible behaviors. While a curve that coincides with the diagonal (for instance, $L_{X \pm}^{\text{cond}} (u) = (u, \ldots, u)'$ for $0 \leq u \leq 1$) indicates the absence of concentration\(^3\), deviations the diagonal values may yield very diverse interpretations: for instance, an $i$th component of $L_{X \pm}^{\text{cond}} (u) - (u, \ldots, u)'$ positive for $0 \leq u < u_0$, then negative for $u_0 < u \leq 1$ indicates a high contribution of $X$’s central quantile regions (the “middle class”) to the mean of $X$’s $i$th components, with the more extremal quantiles contributing less. Mutatis mutandis, similar conclusions can be made for $L_{Y / X \pm}$ and $K_{Y / X \pm}$, allowing for a very detailed analysis of concentrations. Importantly, this analysis incorporates the dependencies between the components of the random vector, enabling a genuine notion of joint multivariate centrality.

REMARK 1. The word “concentration” is not to be understood in the sense of concentration of probability measures. To comply with the traditional terminology of Lorenz curves where concentration refers to the concentration of income in the hands of a few rich people, we also describe the situation in which all individual incomes are the same —i.e., $X = \mu_X$ almost surely—as the absence of concentration.

Center-outward versions of relative Lorenz functions actually have been considered before, under a slightly different form, for univariate non-negative variables. For instance, Davidson (2018), in a study of household income in Canada, proposes an analysis of the income share function ($IS_X$) of the “middle class” based on the income share

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\(^3\)The center-outward quantile function $Q_{X \pm}$ of a degenerate random variable $X$ taking value $\mu_X$ with probability one ($X \sim \delta_{\mu_X}$, the Dirac distribution at $\mu_X$) can be defined as mapping any $s \in S_d$ to $\mu_X$. Indeed, there is only one way to push $U_d$ forward to $\delta_{\mu_X}$. Then, $L_{X \pm}^{\text{cond}} (u) = \int_0^u \int_{S_{d-1}} Q_{X \pm} (ts) dtdV_{d-1} (s)$ is well defined and equal to $u\mu_X$ (hence $L_{X \pm}^{\text{cond}} (u) = \mu_X$).
function

\[ IS_X(a, b) := \mu_X^{-1} \int_{a X(1/2)}^{b X(1/2)} x dF_X(x), \]  

(4.5)

where \( X \) denotes the (nonnegative) revenue of a Canadian household chosen at random, \( X(1/2) := Q_X(1/2) \) is the median of \( X \), and \( 0 \leq a < b \). For the sake of consistency, the version we are providing here is the relative one which, of course, requires \( 0 < \mu_X < \infty \). The relative center-outward Lorenz function for a univariate random variable \( X \)

\[ L_X \pm (u) = \frac{1}{2 \mu_X} \left[ \int_{[0, u]} Q_X \pm (t) dU_{[0, 1]}(t) + \int_{[0, u]} Q_X \pm (-t) dU_{[0, 1]}(t) \right] \]

\[ = \mu_X^{-1} \int_{[1-u, 1+u]} Q_X(t) dU_{[0, 1]}(t) = IS_X \left( \frac{Q_X \left( \frac{1-u}{2} \right)}{X(1/2)}, \frac{Q_X \left( \frac{1+u}{2} \right)}{X(1/2)} \right), \quad u \in [0, 1) \]

(4.6)

offers a natural quantile-based slight modification of Davidson’s concept (4.5); the equal-tails choice of the integration domain, moreover, provides a sound justification of the terminology middle-class.

We finally stress that the concepts proposed in Definition 4 fill a gap by providing a new tool that has been long awaited. Indeed, Davidson (2018) concludes that

An ideal definition [of middle class] would have to be based on all sorts of socio-economic characteristics of individuals and households ...

Due to the lack of a multivariate version of the central intervals (of a form similar to \([a X(1/2), b X(1/2)]\) defining the “middle class,” he regretfully concludes that, desirable as it is, defining a generalized income share function based on such a multivariate characterization of the middle class is “well beyond the scope of this work.” The same concern is expressed by Atkinson and Brandolini (2013) in their study of the evolution of the concept of Middle Class. The definitions, for univariate \( Y \) and multivariate \( X \) (the socio-economic variables characterizing the “middle class”), of the center-outward Kakwani functions \( K_Y/X \pm \) and \( K_Y^{\text{cond}}/X \pm \) of \( Y \) with respect to \( X \) and their relative counterparts are providing the perfect solution to that need. Figure 2 provides, for three classical distributions, the classical Lorenz functions along with the newly proposed center-outward ones.

4.2 Center-outward Lorenz and Kakwani potential functions

An alternative point of view on (2.1) consists of emphasizing the interpretation of the classical univariate Lorenz and Kakwani functions as variations of a potential function \( \psi \). That potential describes how the mass of the (uniform) reference measure is optimally (monotonically, in the sense of cyclical monotonicity) reorganized into the probability distribution \( P_X \) of \( X \). The potential variation \( \psi(u_1) - \psi(u_2) \) between two points \( u_1 \)
and $u_2$ can be understood as the work to be exerted on an object of unit mass that has to travel along any arbitrary smooth\footnote{A path between two points $u_1$ and $u_2$ is said to be \textit{smooth} whenever it admits a parametrisation $r : [0, 1] \to \mathbb{R}^d$ such that $r(0) = u_1$, $r(1) = u_2$, and $t \mapsto r(t)$ is continuously differentiable.} path from $u_1$ to $u_2$.

This interpretation is underlying the definitions we now propose. Recall that $V_d\rightarrow V_0$ denotes the uniform distribution over the unit hypersphere $S_{d\rightarrow 1}$ which, for $d = 1$, reduces to the uniform discrete distribution $V_0$ over $S_0 = \{-1, 1\}$.

**Definition 6.** (Lorenz potential functions)
Assuming that they are well-defined,

(i) call (absolute) \textit{center-outward Lorenz potential function of $X$} the map

$$u \mapsto \Lambda_X^\pm(u) := \int_{S_{d\rightarrow 1}} \psi_X^\pm(us) dV_{d\rightarrow 1}(s) = \mathbb{E}[\psi_X^\pm(uS)], \quad u \in [0, 1]; \quad (4.7)$$
assuming \( d_X = d = d_Y \), call (absolute) center-outward Lorenz potential function of \( Y \) with respect to \( X \) the map

\[
u \mapsto \Lambda_{Y/X}(\nu) := \int_{S_{d-1}} \psi_{Y}(\nu \cdot (\mathbf{F}_{Y}(\nu) \circ \mathbf{Q}_{X}(\nu))) \, dV_{d-1}(s), \quad \nu \in [0, 1).
\]

Contrary to the Lorenz functions in Definition 4, which take values in \( \mathbb{R}^{d_X} \) or \( \mathbb{R}^{d_Y} \), Lorenz potential functions are real-valued.

From a physical perspective, \( \Lambda_{X}(\nu) \) in (4.7) can be interpreted as the expected work it takes, in the field with potential \( \psi \), to move a particle with unit mass from zero to a random point \( x \) uniformly distributed over the quantile contour of order \( \nu \).

The concept of Lorenz potential function nicely simplifies in the case of (nontrivial) spherical distributions. In that case, we have

\[
\frac{\Lambda_{X}(\nu)}{\Lambda_{X}(1)} = \mathcal{L}_{\|X\|}(\nu), \quad \nu \in [0, 1)
\]

The relative Lorenz potential function of a spherical random variable \( X \) thus reduces to the classical univariate Lorenz function for \( \|X\| \), since \( \psi_{X}(\nu) \) then only depends on \( \nu \).

5. Multivariate center-outward concentration indices

5.1 Center-outward Gini and Pietra concentration indices

Let us assume that all components of \( X \) (for the center-outward Lorenz function of \( X \)) or \( Y \) (for the center-outward Lorenz function of \( Y \) with respect to \( X \)) are positive-valued with finite means, so that the corresponding relative functions (see Section 4.1) make sense. The following indices then are natural generalizations of the traditional univariate ones.

**Definition 7.** (Center-Outward Gini and Pietra indices)

(i) Call

\[
G_X := \frac{2}{\sqrt{d}} \int_0^1 \| (u, \ldots, u)' - \mathcal{L}_{X}(u) \| \, du
\]

and

\[
P_X := \frac{1}{\sqrt{d}} \sup_{0 \leq u \leq 1} \| (u, \ldots, u)' - \mathcal{L}_{X}(u) \| \quad (5.2)
\]

the Gini and Pietra center-outward G-indices of \( X \) respectively;

(ii) call

\[
G_{Y/X} := \frac{2}{\sqrt{d}} \int_{u=0}^1 \| (u, \ldots, u)' - \mathcal{L}_{Y/X}(u) \| \, du
\]

and

\[
P_{Y/X} := \frac{1}{\sqrt{d}} \sup_{0 \leq u \leq 1} \| (u, \ldots, u)' - \mathcal{L}_{Y/X}(u) \| \, du
\]

(5.4)
the Gini and Pietra center-outward G-indices of Y with respect to X, respectively;

(iii) call

\[ G_{Y/X}^A := 2 \int_0^1 \frac{u - \Lambda_{Y/X}^A(u)}{\Lambda_{Y/X}^A(1)} \, du \quad (5.7) \]

and

\[ P_{Y/X}^A := \sup_{0 \leq u \leq 1} \frac{u - \Lambda_{Y/X}^A(u)}{\Lambda_{Y/X}^A(1)} \quad (5.8) \]

the center-outward Gini and Pietra potential concentration indices of Y with respect to X, respectively.

5.2 Center-outward Gini and Pietra potential indices

Definition 6 and the interpretation of Lorenz functions as potentials naturally suggest different indices.

DEFINITION 8. (Center-outward Gini and Pietra potential concentration indices)

(i) Call

\[ G_{X}^A := 2 \int_0^1 \frac{u - \Lambda_{X}^A(u)}{\Lambda_{X}^A(1)} \, du \quad (5.7) \]

and

\[ P_{X}^A := \sup_{0 \leq u \leq 1} \frac{u - \Lambda_{X}^A(u)}{\Lambda_{X}^A(1)} \quad (5.8) \]

the center-outward Gini and Pietra potential concentration indices of X, respectively;

(ii) assuming \( d_X = d = d_Y \), call

\[ G_{Y/X}^A := 2 \int_0^1 \frac{u - \Lambda_{Y/X}^A(u)}{\Lambda_{Y/X}^A(1)} \, du \quad (5.9) \]

and

\[ P_{Y/X}^A := \sup_{0 \leq u \leq 1} \frac{u - \Lambda_{Y/X}^A(u)}{\Lambda_{Y/X}^A(1)}, \quad (5.10) \]

the center-outward Gini and Pietra potential concentration indices of Y with respect to X, respectively.
These indices take value zero when $\Lambda_{Y/X}(u)/\Lambda_{Y/X}(1) = u$ for almost all $0 \leq u \leq 1$, that is, when $X$ is an almost sure constant. This, however, is not necessary: the same indices also vanish if, for instance, the expected slope of the line joining $0$ and $\psi(uS)$, where $S \sim V_{d-1}$, does not depend on $u$.

5.3 Center-outward quantile function and Gini mean difference

A global multivariate Gini index also can be constructed as an extension, based on center-outward quantile functions, of the characterization (2.4) of the traditional concept.

**Definition 9.** Call

$$G_{X}^{KM} := \frac{1}{2\kappa} \int \int \| Q_{X}\pm(x) - Q_{X}\pm(y) \| dU_{d}(x)dU_{d}(y),$$

(5.11)

where $\kappa := \int \| Q_{\pm}(x) \| dU_{d}(x)$, the Koshevoy–Mosler multivariate center-outward Gini index of $X$.

It is easy to see that $G_{X}^{KM}$ is similar to the “Gini mean difference” proposed and studied by Koshevoy and Mosler (1997). The introduction of the center-outward quantile functions provides a new interpretation of that coefficient. Indeed, the integral in (5.11) is a measure of the variability of the transport plan $Q_{X}\pm$ in an $L^1(U_d)$ sense. From a geometric point of view, the value of this index is an intrinsic property of elements of the Wasserstein tangent space at $U_d$. Furthermore, that concept coincides, in dimension $d=1$, with the classical Gini coefficient $G_X$.

**Proposition 1.** For a nonnegative real-valued random variable $X$ with $\mathbb{E}X \neq 0$,

$$G_{X}^{KM} = G_X.$$

**Proof.** This is a consequence of the change of variable formula and the relationships from Section 2.1. Indeed,

$$G_X = \frac{1}{2\mu} \int \int |Q(t) - Q(s)| \mathbb{1}_{\{0 \leq s \leq 1\}} \mathbb{1}_{\{0 \leq t \leq 1\}} dsdt$$

$$= \frac{1}{2\mu} \int \int |Q_{\pm}(t) - Q_{\pm}(s)| \mathbb{1}_{\{-1 \leq s \leq 1\}} \mathbb{1}_{\{-1 \leq t \leq 1\}} 2 dsdt = G_{X}^{KM}. \quad \Box$$

6. Estimation

So far, only population concepts have been considered. In practice, one is dealing with observations

$$(X, Y)^{(n)} := \left( (X_1^{(n)}, Y_1^{(n)}), \ldots, (X_n^{(n)}, Y_n^{(n)}) \right), \quad n \in \mathbb{N}. \quad (6.1)$$
Whenever it can be assumed that these observations are an i.i.d. sample with distribution $P_{X,Y}$, $X_i^{(n)} \sim P_X \in \mathcal{P}^{d_X}$, and $Y_i^{(n)} \sim P_Y \in \mathcal{P}^{d_Y}$, the empirical counterparts (described in Section 6.2 below) constitute estimators of the population Lorenz and Kakwani functions $L_{X \pm}$, $L_{Y/X \pm}$, $K_{Y/X \pm}$, etc. of Definition 4; the consistency properties of these estimators are investigated in Section 6.2. Depending on the context, this assumption of an i.i.d. sample, however, may be unrealistic; the same estimators, then, should be considered from a purely descriptive point of view and consistency is meaningless.

In the remainder of this section, we thus make the assumption that (6.1) is an i.i.d. sample and consider the problem of estimating the corresponding population Lorenz and Kakwani functions. Whenever necessary (i.e., when dealing with a.s. convergence), we tacitly assume that the sequence $\{(X, Y)^{(n)} \}, n \in \mathbb{N}$ is defined on a single probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For simplicity, the superscripts $(n)$ are omitted when no confusion is possible.

### 6.1 Empirical distribution and quantile functions

The empirical counterparts $F_{X \pm}^{(n)}$ of $F_{X \pm}$ and $Q_{X \pm}^{(n)}$ of $Q_{X \pm}$ are obtained as the solution of an optimal matching problem between the sample values $X_1^{(n)}, \ldots, X_n^{(n)}$ and a “regular” grid $\pi^{(n)}$ with gridpoints $\pi_1^{(n)}, \ldots, \pi_n^{(n)}$ such that the empirical distribution over $\pi^{(n)}$ converges weakly to $U_d$. More precisely, $\left( F_{X \pm}^{(n)}(X_1^{(n)}), \ldots, F_{X \pm}^{(n)}(X_n^{(n)}) \right)$ is defined as the minimizer $\left( \pi_1^{(n)}, \ldots, \pi_n^{(n)} \right)$, over the $n!$ possible permutations $\pi \in \Pi_n$ of the integers $\{1, \ldots, n\}$, of the sum of squared Euclidean distances $\sum_{i=1}^{n} \| X_i^{(n)} - \pi_i^{(n)} \|^2$, while $Q_{X \pm}^{(n)}$ is characterized by $Q_{X \pm}^{(n)}(\pi_i^{(n)}) = X_i^{(n)}$, $i = 1, \ldots, n$.

The empirical distribution function $F_{X \pm}^{(n)}$ arising from this optimization procedure is defined at the observations only, and the empirical quantile function $Q_{X \pm}^{(n)}$ at the gridpoints. Both, however, can be extended into $F_{X \pm}^{(n)}$ with domain $\mathbb{R}^d$ and $Q_{X \pm}^{(n)}$ with domain $\mathbb{S}_d$ by means of an interpolation that preserves the properties (cyclical monotonicity) of the transport map (i.e., being the gradient of a convex potential). Indeed, as shown in Hallin et al. (2021), there exist smooth and Fenchel–Legendre conjugate empirical potentials $\psi_{X \pm}^{(n)}$ and $\psi_{X \pm}^{(n)}$ such that

$$F_{X \pm}^{(n)}(X_i) := \nabla \psi_{X \pm}^{(n)}(X_i) = F_{X \pm}^{(n)}(X_i) \quad \text{and} \quad Q_{X \pm}^{(n)}(X_i) := \nabla \psi_{X \pm}^{(n)}(X_i) = Q_{X \pm}^{(n)}(X_i)$$

for all $i = 1, \ldots, n$; these potentials can be constructed from $F_{X \pm}^{(n)}$ (or from $Q_{X \pm}^{(n)}$ in a similar fashion) via the following Yosida–Moreau regularization.

For any realization $x^{(n)}$ of $X^{(n)}$, owing to the cyclical monotonicity property of optimal transport, there exists $\{ \psi_{i, \pm}^{*}, i = 1, \ldots, n \}$ such that

$$x_i^{(n)} \cdot \psi_i^{(n)} - \psi_{i, \pm}^{*} \geq \max_{1 \leq j \leq n} (x_i^{(n)} \cdot \psi_j^{(n)} - \psi_{j, \pm}^{*}), \quad i = 1, \ldots, n$$
and the equality is strict when the maximum is restricted to $j \neq i$. Defining the convex map

$$x \mapsto \phi^*(x) = \max_{1 \leq j \leq n} (x \cdot \psi_j^{(n)} - \psi_i^{*}, i = 1, \ldots, n,$$

and its regularized version

$$\tilde{\psi}^*(x) := \phi_\epsilon(x) := \inf_{y \in \Theta^{(n)}} \left[ \phi^*(y) + \frac{1}{2\epsilon} |y - x|^2 \right]$$

(6.2)

where

$$\epsilon := \frac{1}{2} \min_i \left( (x_i \cdot \psi_i^{(n)} - \psi_i^{*}) - \max_{j \neq i} (x_j \cdot \psi_j^{(n)} - \psi_j^{*}) \right),$$

the desired regularized empirical distribution function is $\tilde{F}_{X \pm}^{(n)} := \nabla \tilde{\psi}_{X \pm}^{(n)}$, which moreover satisfies the Glivenko-Cantelli property

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \| \tilde{F}_{X \pm}^{(n)}(x) - F_{X \pm}(x) \| = 0 \quad \text{P}_X \text{-a.s.}$$

(6.3)

As above, one can also estimate the potential corresponding to the center-outward quantile function. For any realisation $x^{(n)}$ of $X^{(n)}$ there exists an $n$-tuple $\{\psi_i, i = 1, \ldots, n\}$ of real numbers such that

$$x_i^{(n)} \cdot \psi_i^{(n)} \geq \max_{1 \leq j \leq n} (x_j^{(n)} \cdot \psi_j^{(n)} - \psi_i^{*}), \quad i = 1, \ldots, n,$$

where the inequality is strict for $j \neq i$; define $\tilde{\psi}_X^{(n)}$ as

$$u \mapsto \tilde{\psi}_X^{(n)}(u) := \max_{1 \leq j \leq n} (x_j^{(n)} \cdot u - \psi_j^{*}), \quad i = 1, \ldots, n.$$ (6.4)

Similar interpolations $F_{Y \pm}^{(n)}, \tilde{F}_{Y \pm}^{(n)}, Q_{Y \pm}^{(n)}$, and $\tilde{Q}_{Y \pm}^{(n)}$ can be constructed for $F_{Y \pm}$ and $Q_{Y \pm}$; we refer to Hallin et al. (2021) for details.

### 6.2 Estimation of center-outward Lorenz and Kakwani functions

As estimators of the Lorenz and Kakwani functions $L_{X \pm}, L_Y / X \pm$, and $K_Y / X \pm$ of Definition 4, we propose the statistics

(a) $u \mapsto \tilde{L}_{X \pm}^{(n)}(u) := \frac{1}{n} \sum_{i=1}^{n} X_i 1\{\|\tilde{F}_{X \pm}^{(n)}(X_i)\| \leq u\}$,

(b) $u \mapsto \tilde{L}_{Y / X \pm}^{(n)}(u) := \frac{1}{n} \sum_{i=1}^{n} Y_i 1\{\|\tilde{F}_{Y \pm}^{(n)}(Y_i)\| \leq u\}$, and

(c) $u \mapsto \tilde{K}_{Y / X \pm}^{(n)}(u) := \frac{1}{n} \sum_{i=1}^{n} Y_i 1\{\|\tilde{F}_{X \pm}^{(n)}(X_i)\| \leq u\}$, respectively $(0 \leq u \leq 1)$.

Remark that, replacing $F_{X \pm}^{(n)}$ by $\tilde{F}_{X \pm}^{(n)}$ in (a) and (c) yields exactly the same estimators: computing $\tilde{F}_{X \pm}^{(n)}$, thus, is not necessary. For (b), quite on the contrary, computing $\tilde{F}_{X \pm}^{(n)}$ is required as $F_{X \pm}^{(n)}$ is not defined at the $Y_i$'s.
As estimators of the conditional counterparts of these curves (Definition 5), we propose
(a) \( u \mapsto \hat{L}^{(n)}_{X \pm}(u) := \frac{n}{n_0 + n_S \lfloor u(n_R + 1) \rfloor} \hat{L}^{(n)}_{X \pm}(u) \),
(b) \( u \mapsto \hat{L}^{(n)}_{Y/X \pm}(u) := \frac{n}{n_0 + n_S \lfloor u(n_R + 1) \rfloor} \hat{L}^{(n)}_{Y/X \pm}(u) \), and
(c) \( u \mapsto \hat{K}^{(n)}_{Y/X \pm}(u) := \frac{n}{n_0 + n_S \lfloor u(n_R + 1) \rfloor} \hat{K}^{(n)}_{Y/X \pm}(u) \), respectively \((0 \leq u \leq 1)\).

We then have the following consistency results.

**Proposition 2.** Let \((X, Y)^{(n)}\) be as in (6.1). Assuming that \(E\|X\|\) and \(E\|Y\|\) exist and are finite,

(i) \( \hat{L}^{(n)}_{X \pm}, \hat{L}^{(n)}_{Y/X \pm}, \) and \( \hat{K}^{(n)}_{Y/X \pm} \) are uniformly \((in u \in [0, 1])\) strongly consistent estimators of \( L_{X \pm}, L_{Y/X \pm}, \) and \( K_{Y/X \pm}, \) respectively;

(ii) for any fixed \( \varepsilon > 0, \hat{L}^{(n)}_{X \pm}(u), \hat{L}^{(n)}_{Y/X \pm}(u), \) and \( \hat{K}^{(n)}_{Y/X \pm}(u) \) are uniformly \((in u \in [\varepsilon, 1])\) strongly consistent estimators of \( L^{\text{cond}}_{X \pm}, L^{\text{cond}}_{Y/X \pm}, \) and \( K^{\text{cond}}_{Y/X \pm}, \) respectively.

**Proof.** (i) It follows from the Glivenko–Cantelli result (6.3) that, for all \( i \),

\[
\mathbb{I}\left\{ \|F_{X \pm}^{(n)}(X_i)\| \leq u \right\} = \mathbb{I}\left\{ \|F_{X \pm}(X_i)\| \leq u \right\} + o_{a.s.}(1)
\]

and

\[
\mathbb{I}\left\{ \|F_{Y \pm}^{(n)}(Y_i)\| \leq u \right\} = \mathbb{I}\left\{ \|F_{Y \pm}(Y_i)\| \leq u \right\} + o_{a.s.}(1)
\]

as \( n \to \infty \), uniformly in \( u \). Since \( E\|Y\| < \infty \) entails that \( o_{a.s.}(n^{-1}) \sum_{i=1}^{n} Y_i \) is \( o_{a.s.}(1), \)

\[
\frac{1}{n} \sum_{i=1}^{n} \left[ Y_i \mathbb{I}\{ \|F_{X \pm}^{(n)}(X_i)\| \leq r \} \right] = \frac{1}{n} \sum_{i=1}^{n} \left[ Y_i \mathbb{I}\{ \|F_{X \pm}(X_i)\| \leq r \} \right] + o_{a.s.}(1).
\]

Further, the class of functions \( \{(x, y) \mapsto y \mathbb{I}\{F_{X \pm}(x) \leq r \}\}_{r \in [0, 1]} \) is pointwise measurable and has a finite bracketing number. Hence, it is Glivenko–Cantelli, and

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Y_i \mathbb{I}\{ \|F_{X \pm}^{(n)}(X_i)\| \leq r \} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Y_i \mathbb{I}\{ \|F_{X \pm}(X_i)\| \leq r \} = E \mathbb{I}\{ \|F_{X \pm}(X)\| \leq r \},
\]

uniformly in \( r \). The claim for \( \hat{K}^{(n)}_{Y/X \pm} \) follows. The proof for \( \hat{L}^{(n)}_{X \pm} \) and \( \hat{L}^{(n)}_{Y/X \pm} \) follows along the same lines.

Turning to (ii), note that \( \tilde{L}^{(n)}_{X \pm}, \tilde{L}^{(n)}_{Y/X \pm}, \) and \( \tilde{K}^{(n)}_{Y/X \pm} \) all are obtained as a rescaling by \( n/(n_S \lfloor (n_R + 1)u \rfloor) \) of the estimators of Proposition 2. The claim follows from the fact that these estimators and the rescaling function are bounded and strongly converge, uniformly in \( u \in [\varepsilon, 1] \) for all \( \varepsilon > 0 \). \( \square \)
REMARK 2. Further asymptotic results such as asymptotic distributions for the estimators in Proposition 2, of course, are highly desirable. Such results, however, are much beyond the scope of the present work. Although the study of the asymptotic behavior of (functions of) empirical optimal transport plans and potentials is a very active area of research, many challenging questions remain unsolved; we refer to Gunsilius and Schennach (2019), Hütter and Rigollet (2021) and the references therein for an overview of some of the results currently available.

For the estimation of the Lorenz potential functions, we will require the \( n \)-point grid \( G^{(n)} \) to exhibit a specific nature. Factorizing \( n \) into \( n = n_Rn_S + n_0 \) with remainder term \( n_0 \leq \min(n_R, n_S) \), construct \( G^{(n)} \) as the outer product of the radial grid

\[
\{1/(n_R + 1), 2/(n_R + 1), \ldots, n_R/(n_R + 1)\}
\]

over \([0,1]\) and a uniform grid over \( S_{d-1} \) such that the discrete uniform measure over these gridpoints converges weakly as \( n_S \to \infty \) to the uniform measure \( V_{d-1} \) on \( S_{d-1} \), then add \( n_0 \) copies of the origin to this grid of \( n_{SN_R} \) points. Call this a regular grid.

For the Lorenz potential functions of Definition 6, we propose the estimators

(a) \( u \mapsto \hat{\Lambda}^{(n)}_{X^{\pm}}(u) := \frac{1}{n_S} \sum_{i \in S_u^{(n)}} \hat{\psi}^{(n)}_{X^{\pm}}(\Theta_i^{(n)}) \)

(b) \( u \mapsto \hat{\Lambda}^{(n)}_{Y/X^{\pm}}(u) := \frac{1}{n_S} \sum_{i \in S_u^{(n)}} \hat{\psi}^{(n)}_{Y^{\pm}/X^{\pm}}(F^{(n)}_{Y^{\pm}/X^{\pm}}(Q^{(n)}_{X^{\pm}}(\Theta_i^{(n)}))), \)

with \( \hat{\psi}^{(n)}_{X^{\pm}}, \hat{\psi}^{(n)}_{Y^{\pm}}, F^{(n)}_{Y^{\pm}/X^{\pm}}, \) and \( Q^{(n)}_{X^{\pm}} \) as in Section 6.1 and \( S_u^{(n)} := \{ i : \| \Theta_i^{(n)} \| = \frac{(n_R+1)u}{n_R+1} \} \).

PROPOSITION 3. Consider a regular grid \( G^{(n)} \) constructed as above. Then, for any fixed \( u \in [0,1) \), \( \hat{\Lambda}^{(n)}_{Y/X^{\pm}} \) and \( \hat{\Lambda}^{(n)}_{Y/X^{\pm}} \) are strongly consistent estimators of \( \Lambda_{X^{\pm}} \) and \( \Lambda_{Y/X^{\pm}} \), respectively, uniformly over \([0, 1 - \epsilon]\) for any \( \epsilon > 0 \).

PROOF. We actually only need to prove the second part of the claim since, for any fixed \( u < 1 \), there exists an \( \epsilon > 0 \) such that \( u \in [0, 1 - \epsilon] \).

We start by proving that the empirical potentials a.s. converge to their population counterparts uniformly on the closed ball \((1 - \epsilon)S^d \). The claim then follows from the weak convergence to \( U_d \) of the empirical distribution over the regular grid \( G^{(n)} \).

By the almost sure convergence of empirical measures (see, e.g., Varadarajan (1958)), the sequences \( P^{(n)}_{X^{(n)}}(\omega) \) of empirical measures over \( X_1, \ldots, X_n \) weakly converges to \( P_X \) for any \( \omega \in \Omega_0 \subseteq \Omega \) where \( \Omega_0 \) has probability one. Denoting by \( P_{G^{(n)}} \) the empirical distribution over \( G^{(n)} \), \( P^{(n)}_{G^{(n)}} \) similarly weakly converges to \( U_d \). Using Villani (2009, Theorem 5.20) and the uniqueness of the optimal transport plan for \( P_X \in P_d \), it follows from a subsequence argument that the sequence of optimal transport plans from \( P^{(n)}_{X^{(n)}}(\omega) \) to \( P^{(n)}_{G^{(n)}} \) converges weakly to the optimal transport plan from \( P_X \) to \( U_d \). As explained in

\[ \text{This amounts to considering the grid consisting of the union of the points of the } n_R \text{ spherical grids multiplied by } k/(n_R + 1), 1 \leq k \leq n_R. \]
Hallin et al. (2021, proof of Proposition 3.3), one can deduce from that weak convergence and Del Barrio and Loubes (2019, Theorem 2.8) the pointwise convergence, for each \( \omega \in \Omega_0 \), of the corresponding empirical potentials. Now—see the proof of Theorem 1.52 in Santambrogio (2015)—the empirical optimal potentials have the same modulus of continuity as the cost function, here the squared Euclidean distance. Since a sequence of real-valued equicontinuous functions converging pointwise on a compact set converges uniformly over that set, the above convergence of potentials is uniform in \( u \in (1 - \epsilon)S^d \).

Since the grids \( \mathbf{G}^{(n)} \) are constructed as the product of a radial grid and a uniform grid over \( S_{d-1} \), both converging in distribution to uniform measures, over \([0,1]\) and \( S_{d-1} \), respectively, the measures \( \frac{1}{nS} \sum_{i \in S_u} \delta_{\mathbf{G}^{(n)}_i} \) converge weakly to the uniform measure \( V_{d-1}^u \) on \( uS_{d-1} \) uniformly in \( u \). To see this, note that \( |[(n_R + 1)u] - (n_R + 1)u|/(n_R + 1) \) converges to zero uniformly in \( u \). This uniform weak convergence, combined with the uniform convergence of the potential over the closed ball \((1 - \epsilon)S^d\), ensures that the result holds for \( \hat{\lambda}_{X \pm}^{(n)} \).

The result for \( \hat{\lambda}_{Y \pm}^{(n)} / X \pm \) follows essentially from the same argument, exploiting further the continuity of \( \omega_{Y \pm} / X_{Y \pm} \) as well as the Glivenko–Cantelli result from Equation (6.3). Further, one also requires the uniform convergence of \( Q_{X \pm}^{(n)} \) over \((1 - \epsilon)S^d\); this follows from Segers (2022, Theorem 1.1).

Interestingly, this result would not necessarily hold for arbitrary grids converging weakly to \( U_d \), and the special “product structure” form of the grid is required here as one needs the empirical measure underlying the empirical contour of level \( u \) to converge weakly to the uniform on \( uS_{d-1} \).

### 6.3 Estimation of multivariate Gini and Pietra indices

Define

\[
\begin{align*}
\hat{L}_{X \pm}(u) := & \hat{L}_{X \pm}^{(n)}(u) \odot \frac{1}{n} \sum_{i=1}^{n} X_i, \\
\hat{L}_{Y \pm}(u) := & \hat{L}_{Y \pm}^{(n)}(u) \odot \frac{1}{n} \sum_{i=1}^{n} Y_i, \\
\hat{K}_{Y \pm}(u) := & \hat{K}_{Y \pm}^{(n)}(u) \odot \frac{1}{n} \sum_{i=1}^{n} Y_i, \\
\hat{K}_{Y \pm}(u) := & \hat{K}_{Y \pm}^{(n)}(u) \odot \frac{1}{n} \sum_{i=1}^{n} Y_i, \\
\hat{L}_{Y \pm}/X_{Y \pm}(u) := & \hat{L}_{Y \pm}/X_{Y \pm}^{(n)}(u) \odot \frac{1}{n} \sum_{i=1}^{n} Y_i, \\
\hat{K}_{Y \pm}/X_{Y \pm}(u) := & \hat{K}_{Y \pm}/X_{Y \pm}^{(n)}(u) \odot \frac{1}{n} \sum_{i=1}^{n} Y_i.
\end{align*}
\]

(6.6)

where \( \odot \) denotes Kronecker/componentwise division. The Gini and Pietra indices of Definition 7 easily can be estimated by replacing, in equations (5.1)–(5.6), \( L_{X \pm}(u), L_{Y \pm}(u), \) and \( K_{Y \pm}(u) \) with \( \hat{L}_{X \pm}(u), \hat{L}_{Y \pm}(u), \) and \( \hat{K}_{Y \pm}(u) \).

**Proposition 4.** For any \( u \in (0,1) \), the estimators of the indices defined in (5.1)–(5.6) obtained by substituting, in these definitions, \( L_{Y \pm}(u), L_{Y \pm}(u), \) and \( K_{Y \pm}(u) \) with their empirical counterparts are strongly consistent.
The proof follows directly from Proposition 2, the strong law of large numbers, Slutsky’s theorem, and the continuous mapping theorem.

Now, let us turn to the estimation of the concepts of Gini and Pietra potential concentration indices proposed in Definition 8.

**Proposition 5.** Denote by \( \hat{G}_{X_\pm}^\Lambda, \hat{P}_{X_\pm}^\Lambda, \hat{G}_{Y/X_\pm}^\Lambda, \) and \( \hat{P}_{Y/X_\pm}^\Lambda \) the empirical counterparts of the potential concentration indices of Definition 8, obtained by plugging-in \( \hat{\Lambda}_{X_\pm} \) and \( \hat{\Lambda}_{Y/X_\pm} \) instead of \( \Lambda_{X_\pm} \) and \( \Lambda_{Y/X_\pm} \) respectively, in (5.7)–(5.10). Then, assuming that \( \text{supp}(P_X) \) is compact, \( \hat{G}_{X_\pm}^\Lambda \) and \( \hat{P}_{X_\pm}^\Lambda \) are strongly consistent estimators of \( G_{X_\pm}^\Lambda \) and \( P_{X_\pm}^\Lambda \), respectively. If, furthermore, \( \text{supp}(P_Y) \) is compact, \( \hat{G}_{Y/X_\pm}^\Lambda \) and \( \hat{P}_{Y/X_\pm}^\Lambda \) are strongly consistent estimators of \( G_{Y/X_\pm}^\Lambda \) and \( P_{Y/X_\pm}^\Lambda \), respectively.

**Remark 3.** The crucial point, when proving the consistency of Gini and Pietra potential concentration indices, is the uniform convergence of the empirical potentials to the actual ones, which in turn guarantees that \( \hat{\Lambda}_{X_\pm}(u) \) and \( \hat{\Lambda}_{Y/X_\pm}(u) \) converge to \( \Lambda_{X_\pm}(u) \) and \( \Lambda_{Y/X_\pm}(u) \) in a sufficiently strong sense for \( \hat{G}_{X_\pm}^\Lambda \) and \( \hat{P}_{X_\pm}^\Lambda \) (respectively, \( \hat{G}_{Y/X_\pm}^\Lambda \) and \( \hat{P}_{Y/X_\pm}^\Lambda \)) to be consistent. The assumption of a compactly supported \( X \) (respectively, compactly supported \( X \) and \( Y \)) entails such a convergence. Compactness here is sufficient but certainly not necessary, and we expect the results of Proposition 5 to hold in more general situations. Relaxing compactness, however, would require a better understanding of the behavior of empirical potentials in the neighborhood of the unit sphere. This behavior is related to the boundary behavior of solutions of the Monge–Ampère equation—a highly nontrivial problem in the noncompact case, see Segers (2022).

**Proof of Proposition 5.** Owing to compactness, almost sure uniform convergence of the potentials \( \psi_{X_\pm} \) and \( \psi_{Y_\pm} \) follows from Theorem 1.52 in Santambrogio (2015). The uniform convergence of the empirical quantile function follows again from from Segers (2022, Theorem 1.1). Therefore, because of the more stringent compactness assumption, the convergence of the estimators of Proposition 3 holds uniformly in \( u \in [0,1] \). The claim then follows from a trivial application of Slutsky’s lemma and the continuous mapping theorem.

Finally consider the problem of estimating the Koshevoy–Mosler multivariate center-outward Gini index \( G_{X_\pm}^{KM} \) introduced in Definition 9. Denote by \( \Delta \) the double integral appearing in the numerator of (5.11). A natural estimator of \( \Delta \) is

\[
\Delta^{(n)} := \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \| Q_{X_\pm}^{(n)}(\theta_i^{(n)}) - Q_{X_\pm}^{(n)}(\theta_j^{(n)}) \| = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \| X_i - X_j \|.
\]

This quantity is a U-statistic with kernel \( h(x,y) = \| x - y \| \). The following result thus readily follows from, e.g., Theorem 12.3 in van der Vaart (1998).
PROPOSITION 6. Let $X_1, X_2,$ and $X'_2$ be i.i.d. random variables with common distribution $P_X$. If $\mathbb{E}\|X\|^2 < \infty$, then
\[ \sqrt{n}(\hat{\Delta}(n) - \mathbb{E}\|X_1 - X_2\|) \rightsquigarrow \mathcal{N}(0, \sigma^2) \quad \text{as } n \to \infty, \]
where $\sigma^2 = 4 \text{Cov}(\|X_1 - X_2\|, \|X_1 - X'_2\|)$. 

Based on this, for any consistent estimator $\hat{\kappa}$ of $\kappa := \int \|Q_\pm(u)\|dU_d(u)$, it follows from Slutsky’s Lemma that
\[ \sqrt{n}\left(\frac{\hat{\Delta}(n)}{\hat{\kappa}} - G^{KM}_{X}\right) \rightsquigarrow \mathcal{N}\left(0, \frac{\sigma^2}{\kappa^2}\right) \quad \text{as } n \to \infty. \]

A natural choice for $\hat{\kappa}$ is $\sum_{i=1}^{n}\|X_i\|/n$.

7. Empirical Examples

7.1 Socio-economic inequalities and well-being

In this entire section, the curves we are depicting are piecewise constant over the interval $u \in [0, n_R/(n_R + 1)]$. For the sake of readability, however, we are showing linear interpolations of their points of discontinuity. For the same reason, $u$ is rescaled by $(n_R + 1)/n_R$, so that the curves reach the top corner of the unit square or cube.

7.1.1 The dataset  The relation between socio-economic inequalities and well-being is a fundamental topic in economics and social sciences. Sarabia and Jorda (2013) recently proposed a study of well-being across 132 countries using the generalized Lorenz curve introduced in (Arnold and Sarabia, 2018), see (2.8).

Well-being, of course, is the resultant of a number of socio-economic variables. Sarabia and Jorda are basing their analysis on education, income, and health, as measured by the indicators used by the United Nations in the construction of the Human Development Index (see, for instance, the latest report by the United Nations Development Programme, 2020): Education ($X_1$) is based on the average between the Mean Years of Schooling and the Expected Years of Schooling indices, Income ($X_2$) on transformations of Gross National Income per capita$^6$, and Health ($Y$) on life expectancy at birth. Each of these variables takes values between zero and one by construction.

In their application, Sarabia and Jorda (2013) consider, in dimension two, a parametric version of their Lorenz curve concept (2.8), assuming bivariate Sarmanov–Lee distributions (a form of multivariate beta distribution, with beta marginals). This requires, in dimension two, the estimation of five parameters—two per marginal plus the dependence parameter. They repeat their analysis for each five-year period between 1980 and 2000. As an illustration of our concepts, we apply our methods (which do not require

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$^6$Gross National Income per capita is measured in Purchasing Power Parity international dollars. The Index we consider is further slightly different from that proposed by the UN. We did not apply the logarithmic transformation as it modifies the distribution of interest and, in particular, changes the tails behavior.
the somewhat arbitrary choice of a specific distribution, nor the estimation of its parameters) to the most recent versions of the same three variables, collected from the United Nations website. From that dataset, we only retained the 144 countries for which data were available since 1990 and (in line with the quantisation suggested in Table 1 of Hallin and Mordant (2021)) factorized $n = 144$ into $n = n_R n_S + n_0$ with $n_R = 8$, $n_S = 18$, and $n_0 = 0$.

The assumption of an i.i.d. sample seems hard to justify for this dataset, though, and we feel that the analysis below, as well as the one by Sarabia and Jorda (despite their Sarmanov–Lee distributional assumption), should be taken from a purely descriptive perspective.

7.1.2 Center-outward quantiles. As a first step, let us show how the proposed center-outward approach determines empirical quantiles. Figure 3 displays the quantile obtained from the empirical optimal transport map between the observations and the spherical grid. In the upper rightmost corner are Singapore, Switzerland, and Luxembourg, while the Central African Republic is the lower leftmost point. The most central contours consist of Egypt, Brazil or Mexico, to mention but three. A positive interrelation between the two indices is quite visible. An interactive version of the plot enabling to select certain empirical center-outward quantile contours and providing additional information when hovering over the points is provided as digital supplemental material.

![Figure 3. Center-outward quantiles for the Health (Life Expectancy) and Income indices (n = 144 countries in 2015). Each dot represents a country. The dots are coloured according to their center-outward quantile orders (their center-outward ranks, rescaled between $1/(n_R + 1)$ and $n_R/(n_R + 1)$).](image)

7.1.3 Center-outward Lorenz and Kakwani functions, Gini and Pietra coefficients. In this section, we are dealing with the empirical versions $\hat{L}_X^{(n)}$, $\hat{L}_Y^{(n)}/X$, and $\hat{K}_Y^{(n)}/X$ of the concepts introduced in Definitions 4 and 5.

The results are shown in Figure 4, where we provide plots of $\hat{j}_X^{(n)}(X = (X_1, X_2))$ with $X_1 =$Education and $X_2 =$Income) for the years 1990 and 2019. The impact of Income clearly is higher than that of Education. These plots also indicate that the curves
\( \hat{L}_{X \pm} \), between 1990 and 2020, have evolved towards the main diagonal of the unit cube, meaning that the empirical mean of \( X \) conditionally on being in a central quantile region of level \( u \) is getting closer to the overall empirical mean—less concentration, thus, in 2020 than in 1990.

Figure 5 yields, for the period 1990 - 2020, the graphs of the empirical conditional relative Kakwani functions \( \hat{K}_{Y/X \pm}^{\text{cond}} \) of \( Y = \text{Health conditional on } X = (\text{Education, Income}) \), which represents, as a function of \( u \in [0, 1] \), the relative “expected health” for a “middle-class of dimension \( u \)” characterized in terms of national Education and Income indices. The results show, for instance, that, conditionally on the fact that a country belonged to the 20% center-outward quantile region for Education and Income, its expected health
score in 1990 was much higher than for the countries in the 70% center-outward quantile region. This was much less the case in 2020, though, and the evolution between 1990 and 2020, again, clearly indicates that the discrepancies between “middle-class” and “outlying” (in terms of Education and Income) countries significantly reduced over the past thirty-year period.

| Year | $G_{X\pm}$ | $P_{X\pm}$ | $GK_{Y/X\pm}$ | $PK_{Y/X\pm}$ |
|------|-------------|-------------|---------------|---------------|
| 1990 | 0.1140      | 0.0834      | 0.0568        | 0.0134        |
| 2000 | 0.1274      | 0.0977      | 0.0567        | 0.0155        |
| 2010 | 0.1093      | 0.0757      | 0.0558        | 0.0084        |
| 2020 | 0.1000      | 0.0640      | 0.0559        | 0.0065        |

Table 1. Empirical Gini and Pietra coefficients associated with the empirical Lorenz and Kakwani functions shown in Figures 4 and 5.

Table 1 further provides the empirical Gini and Pietra coefficients $G_{X\pm}$, $P_{X\pm}$, $GK_{Y/X\pm}$, and $PK_{Y/X\pm}$ associated with the Lorenz and Kakwani functions shown in Figures 4 and 5; their variations between 1990 and 2020 are in line with the evolution of the Lorenz and Kakwani functions they are summarizing.

7.1.4 Center-outward Lorenz potential functions  
Next, we propose graphs of the Lorenz potentials defined in Equation (4.7) and estimated as in Section 6.2. As in Figure 3, these potentials are for the variable $X$ consisting of the Income and Education Indices of 144 countries in the year 2015. The potential increase was steeper in 1990 than it is in 2020, indicating that the spread of $X$ has decreased between 1990 and 2020. This complements the information provided by the Lorenz curves of Figure 4.

Figure 6. The (estimated) Lorenz potential functions associated with the joint distributions of the Education and Income Indices in 1990 and 2020, respectively.
7.2 The 2014 Spanish Health Survey

7.2.1 The dataset  We further applied some of our concepts to the results of the European Health Survey for Spain provided by the Instituto Nacional de Estadística (2014). This dataset contains information about individuals’ state of health in Spain in 2014. The data is gathered from a survey that took place in a framework set by the European Statistical System. We refer to the documentation on the website of the Instituto Nacional de Estadística for more information about the data collection process and the European Health survey context.

Motivated by socio-economic applications, we consider work absenteeism (measured in days over a one-year period) and its relation to other characteristics of the individuals. Our aim with this dataset is to show that it is possible to go beyond the bivariate examples presented in the previous section and exhibit various curve patterns.

We start by exhibiting the Lorenz center-outward curve for $X = (X_1, X_2)$ where $X_1$ is rescaled \(^7\) age and $X_2$ rescaled \(^7\) work absenteeism. The result are shown in Figure 7. The Lorenz curve (left panel) indicates a significant concentration; that concentration, however, is due to absenteeism (as shown by the center panel) rather than age (right panel).

More information is provided by incorporating more variables and computing, e.g., (estimated) conditional relative Kakwani functions (as defined in Definition 5). Figure 8 provides plots of these conditional Kakwani functions for $Y = \text{Work Absenteeism}$ and $X = (X_1, X_2, X_3)$ with $X_1 = \text{rescaled Age}$, $X_2 = \text{rescaled Weight}$, and $X_3 = \text{rescaled Alcohol Consumption}$. Figure 9 adds to $X$ a fourth component $X_4 = \text{Sports Practice}$, the rescaled time spent on practicing sports. The estimators are those proposed in Section 6.2. Figure 8 thus describes, as a function of $u \in [0, 1]$, the expected Absenteeism

\[^7\]Rescaling is performed by dividing each observation by its largest observed value, thus yielding a [0-1] range and ensuring some balance between the components involved in the matching problem that leads to the center-outward ranks and signs.
for individuals belonging to middle-class regions of order $u$, where the construction of these middle classes is based on Age, Weight, and Alcohol Consumption. For the sake of readability, the value at $u$ of the Kakwani function has been multiplied by the number $n/(n_0 + n_S[(n_R + 1)u]) ≈ u^{-1}$ of observations in the quantile region of order $u$. Figure 9 has the same interpretation, now with $X$ defining a four-dimensional “middle class.”

Both Figures 8 and 9 indicate a concentration of Absenteeism on “extreme” (in terms of Age, Weight, Alcohol Consumption, and Sports Practice) observations, with a relatively small contribution of the corresponding central values.
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