On the classification of complex vector bundles of stable rank

CONSTANTIN BĂNICĂ* and MIHAI PUTINAR

1Mathematics Department, University of California, Santa Barbara, CA 93106, USA
E-mail: mputinar@math.ucsb.edu

MS received 12 April 2006

Abstract. One describes, using a detailed analysis of Atiyah–Hirzebruch spectral sequence, the tuples of cohomology classes on a compact, complex manifold, corresponding to the Chern classes of a complex vector bundle of stable rank. This classification becomes more effective on generalized flag manifolds, where the Lie algebra formalism and concrete integrability conditions describe in constructive terms the Chern classes of a vector bundle.

Keywords. Chern class; K-theory; Atiyah–Singer index theorem; Atiyah–Hirzebruch spectral sequence; flag manifold.

1. Introduction

Let $X$ denote a finite CW-complex of dimension $n$. For a natural $r$, one denotes by $\text{Vect}_{\text{top}}^r(X)$ the isomorphism classes of complex vector bundles on $X$, of rank $r$. It is well-known that the map:

$$\text{Vect}_{\text{top}}^r(X) \to \text{Vect}_{\text{top}}^{r+[n/2]}(X), \quad E \mapsto E \oplus 1_{r-[n/2]}$$

is onto whenever $r \geq [n/2]$ (the integer part of $n/2$). Moreover, for $r \geq n/2$, the same map is bijective (see for instance [13]).

On the other hand, Peterson [17] has proved that, if $r \geq n/2$ and $H^{2q}(X, \mathbb{Z})$ has no $(q-1)!$-torsion for any $q \geq 1$, then two vector bundles $E$ and $E'$ of rank $r$ on $X$ are isomorphic if and only if they have the same Chern classes.

An intriguing question is how to determine the range of the total Chern class map:

$$c: \text{Vect}_{\text{top}}^r(X) \to \prod_q H^{2q}(X, \mathbb{Z}).$$

According to previous observations, this would imply (under the above torsion conditions) a classification of all rank $r$ complex vector bundles on $X$, for stable rank $r \geq n/2$.

A few partial answers to this question are known. For instance, a classical result of Wu asserts that any couple of cohomology classes $(c_1, c_2) \in H^2(X, \mathbb{Z}) \times H^4(X, \mathbb{Z})$ coincides with $c(E)$ for a vector bundle $E$, on a basis $X$ of dimension less than or equal to 4. For complex projective spaces the range of $c$ was computed by Schwarzenberger (see Appendix 1, pp. 22 of [12]) and Thomas [21]. A proof of Schwarzenberger–Thomas result is also given in the paper by Switzer [20]. If $n \leq 7$ and $H^7(X, \mathbb{Z})$ has no 2-torsion, then a

*Since deceased.
triple \((c_1, c_2, c_3), c_i \in H^2(X, \mathbb{Z})\) coincides with \(c(E)\) for a rank-3 vector bundle \(E\) on \(X\), if and only if \(c_3 \equiv c_1 c_2 + Sq^2 c_2\) in \(H^8(X, \mathbb{Z}_2)\) (see [3]). Moreover, for a compact complex manifold \(X\) of complex dimension 3, one has the bijection:

\[
c: \text{Vect}_{\text{top}}^3(X) \to \{(c_1, c_2, c_3); c_i \in H^2(X, \mathbb{Z}) \text{ for } i = 1, 2, 3 \text{ and } c_3 \equiv c_1 c_2 + c_1(X)c_2 \text{ (mod 2)}\}.
\]

Here \(c_1(X)\) is the first Chern class of the complex tangent bundle of \(X\).

In the present paper, we study the classification of stable-rank bundles on some compact complex manifolds. First, we extend the classification to manifolds of dimensions 4 and 5 (see Propositions 3.1 and 3.2). We then consider some particular classes of complex manifolds of arbitrary dimensions, including the rational homogeneous manifolds. We list below the results obtained in this case (see Propositions 4.1–4.3 and Theorems 5.1–5.3 for precise statements).

Let \(X\) be a compact complex manifold of complex dimension \(n\).

(a) If \(H^*(X, \mathbb{Z})\) has no torsion, then \(\text{Vect}_{\text{top}}^n(X)\) is a bijection with those \(n\)-tuples of cohomology classes \((c_1, \ldots, c_n)\) satisfying:

\[
\int_X \text{ch}(c_1, \ldots, c_n) \text{ch}(\xi) \tau d(X) \in \mathbb{Z}
\]

for any \(\xi \in K(X)\).

(b) Assume that \(H^*(X, \mathbb{Z})\) has no torsion and that \(H^2(X, \mathbb{Z})\) generates multiplicatively \(H^{\text{even}}(X, \mathbb{Z})\). Then \(\text{Vect}_{\text{top}}^n(X)\) is in bijection with the classes \((c_1, \ldots, c_n)\) satisfying

\[
\int_X \text{ch}(c_1, \ldots, c_n) e^\xi \tau d(X) \in \mathbb{Z},
\]

for any \(\xi \in H^2(X, \mathbb{Z})\).

(c) Assume that \(X = G/B\), where \(G\) is a complex algebraic, semi-simple, simply connected Lie group and \(B\) is a Borel subgroup. Then, \(\text{Vect}_{\text{top}}^n(X)\) is isomorphic to the set of all classes \((c_1, \ldots, c_n)\) with the property

\[
\int_X \text{ch}(c_1, \ldots, c_n) \text{ch}(\mathcal{O}_{X_w}) e^{-\rho} \in \mathbb{Z},
\]

for \(w \in W\) (the Weyl group of \(G\)). Moreover, these integrality conditions are equivalent to

\[
\int_X \text{ch}(c_1, \ldots, c_n) e^\chi \in \mathbb{Z},
\]

for any character \(\chi\) of a fixed maximal torus contained in \(B\).

Here \(\mathcal{O}_{X_w}\) denotes the structure sheaf of the closed Schubert cell \(X_w\) associated with \(w \in W\). Also, \(\rho\) stands for the semisum of all positive roots of \(B\).

(d) If \(X\) is as before and \(G\) is a product of simple groups of type SL or Sp, then the conditions are

\[
\int_X \text{ch}(c_1, \ldots, c_n) e^{\chi - \rho} \in \mathbb{Z},
\]
where \( \chi = \omega_i + \cdots + \omega_k, 1 \leq i_1 \leq \cdots \leq i_k \leq r, 0 \leq k \leq n - 3 \) and \( \omega_1, \ldots, \omega_r \) are the fundamental weights of \( G \) (we have put by convention that \( \chi = 0 \) for \( k = 0 \)).

Actually the classes \( \text{ch}(\xi) \) appearing in assertion (c) can be related to the natural basis \( ([X_w])_{w \in W} \) of \( H^*(X, \mathbb{Z}) \), by using Demazure’s results on the desingularization of Schubert cells [8]. However we ignore whether there exists an accessible transformation which relates the two bases \( \text{ch}(\xi) \) and \( ([X_w])_{w \in W} \) of \( H^*(X, \mathbb{Q}) \). This is discussed in the final part of the paper.

The main tools in the sequel are the Atiyah–Hirzebruch spectral sequence, the integrality condition derived from the Atiyah–Singer index theorem and of course the specific properties of flag manifolds. The present article is motivated by the still unsolved problem concerning the existence of analytic structures on complex vector bundles, and the earlier attempts to find a solution in low dimension [3] via a topological classification of vector bundles at stable rank.

2. The Atiyah–Hirzebruch spectral sequence and a technical result

This preliminary section contains some consequences of the Atiyah–Hirzebruch spectral sequence for topological \( K \)-theory. Among them, the subsequent technical Lemma 2.1 is the basis of all results contained in this paper.

Let \( X \) be a finite CW-complex and let \( X^i \) stand for its \( i \)-dimensional skeleton. Corresponding to the natural filtration,

\[
X = X^n \supset X^{n-1} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset,
\]

there exists the Atiyah–Hirzebruch spectral sequence which converges to \( K(X) \). The second level terms of this sequence are

\[
E_2^{p,q} = H^p(X, K^q(\text{pt.})) = \begin{cases} H^p(X, \mathbb{Z}), & q \text{ even} \\ 0, & q \text{ odd} \end{cases}
\]

The coboundaries of this spectral sequence are denoted by \( d^{p,q}_r \). Due to the concrete form of \( E_2^{p,q} \) one easily finds \( d^{p,q}_r = 0 \) for \( r \) even. Moreover, all coboundaries \( d^{p,q}_r \) have torsion (see [11] or [10]).

It was also proved by Atiyah and Hirzebruch [11] that \( d^{p,q}_r = 0 \) for all \( p, q, r \) \( (r \geq 2) \), whenever \( H^*(X, \mathbb{Z}) \) has no torsion. Indeed, because \( d_3: H(X, \mathbb{Z}) \to H^{r+1}(X, \mathbb{Z}) \) has torsion, it follows that \( d_3 = 0 \), and so on.

For later use, the following detailed analysis of the coboundaries \( d^{p,q}_r \) is needed. First, Atiyah and Hirzebruch [11] noticed that, for a given class \( \alpha \in H^{2p}(X, \mathbb{Z}) \), if \( d_{2k+1} \alpha = 0 \) for all \( k \geq 1 \), then there exists a class \( \xi \in K(X) \) such that

\[
\xi |X^{2p-1} \text{ is trivial and } c_p(\xi) = (p-1)! \alpha
\]

(see also ch. VIII of [10]). Since any \( \xi \in K(X) \) can be represented as \( \xi = [E] - [1_m] \), it follows that there exists a vector bundle \( E \) on \( X \) which is trivial on \( X^{2p-1} \) and its \( p \)-th Chern class is prescribed as \( c_p(E) = (p-1)! \alpha \).

The vanishing conditions \( d_{2k+1} \alpha = 0, k \geq 1 \), are superfluous in the case of a CW-complex \( X \) with torsion-free cohomology.
A deeper analysis of the cohomology operations $d^p,q$ leads to the next result of Buchstaber [7]. Let $p,q$ be positive integers such that $\dim X < p+q$ and let us denote

$$m = \prod_{\nu \geq 2, \nu \text{ prime}} s\left[\frac{\nu - 1}{2}\right].$$

Then for every $\xi \in E_2^{q,0} \cong H^q(X,\mathbb{Z})$, the multiple $m\xi$ is annihilated by all coboundaries $d_{2k+1,k} \geq 1$.

In particular, if $\dim X = 10$, then $12d_{11,1}^4 = 0$ for all $k \geq 1$.

Next we prove that under some natural nontorsion conditions, the range of the Chern class map can be computed, at least in principle. Throughout this paper, $\mathbb{Z}_m$ denotes the finite cyclic group of order $m$ and $\rho$ stands for the class of the integer $p$ in $\mathbb{Z}_m$, or an operation deduced from it.

**Lemma 2.1.** Let $X$ be a topological space having the homotopy type of a finite CW-complex of dimension $n$. Assume that the group $H^{2q}(X,\mathbb{Z})$ has no $(q-1)!$-torsion for $q \leq (n/2) - 1$ and that $d_{2k+1}^{2p} = 0$ for $k \geq 1$ and $p \geq 2$. For every $p \geq 1$ one denotes

$$C_p = \{(c_1,\ldots,c_p);\ c_i \in H^{2i}(X,\mathbb{Z}) \text{ and there exists a vector bundle } E \text{ on } X \text{ with } c_i(E) = c_i, \ 1 \leq i \leq p\}.$$

Then

1. $C_1 = H^2(X,\mathbb{Z}), C_2 = H^2(X,\mathbb{Z}) \times H^4(X,\mathbb{Z})$,
2. There are functions $\varphi_p: C_p \to H^{2p+2}(X,\mathbb{Z}_p!)$ with the property:

$$C_{p+1} = \{(c_1,\ldots,c_{p+1});(c_1,\ldots,c_p) \in C_p, \varphi_{p+1}(c_1,\ldots,c_{p+1}) = \varphi_p(c_1,\ldots,c_p)\}$$

for every $p \geq 1$.

**Proof.**

1. The first equality $C_1 = H^2(X,\mathbb{Z})$ is obvious. In order to prove the second equality, let $(c_1,c_2)$ be an arbitrary element of $H^2(X,\mathbb{Z}) \times H^4(X,\mathbb{Z})$. There exists a complex line bundle $L$ on $X$ with $c_1(L) = c_1$ and, according to the Atiyah–Hirzebruch result mentioned above, there exists a vector bundle $E$ on $X$ such that $c_1(E) = 0$ and $c_2(E) = c_2$. Here we point out that the vanishing assumptions $d_{2k+1}^{2p} = 0$ are essential. Then the vector bundle $L \oplus E$ has the first two Chern classes $c_1$ and $c_2$.
2. The function $\varphi_p: C_p \to H^{2p+2}(X,\mathbb{Z}_p!)$ is defined as follows: Let $(c_1,\ldots,c_p) \in C_p$ and let $E$ be a vector bundle with $c_i(E) = c_i, 1 \leq i \leq p$. Then

$$\varphi_p(c_1,\ldots,c_p) = c_{p+1}(E),$$

that is the image of $c_{p+1}(E)$ through the natural map $H^{2p+2}(X,\mathbb{Z}) \to H^{2p+2}(X,\mathbb{Z}_p!)$. In order to prove that this definition is correct, let $E'$ be another vector bundle with $c_i(E') = c_i$ for $1 \leq i \leq p$. We may assume in addition that $X$ is a CW-complex of dimension $n$ and that $\text{rank}(E') = \text{rank}(E)$. Let $X^k$ denote the $k$-skeleton of $X$. In view of Peterson’s theorem (see the introduction in [12]), $[E|X^{2p+1}] = [E'|X^{2p+1}]$, since $H^p(X,\mathbb{Z}) \cong$
$H^s(X^{2p+1}, \mathbb{Z})$ for $s \leq 2p$ and $H^s(X^{2p+1}, \mathbb{Z}) = 0$ for $s > 2p + 1$. Consequently, the $K$-theory class $\xi = [E|X^{2p+2}] - [E'|X^{2p+2}] \in \tilde{K}(X^{2p+2})$ belongs to the kernel of the restriction map $\rho$:

$$\tilde{K}(X^{2p+2}, X^{2p+1}) \xrightarrow{\rho} \tilde{K}(X^{2p+2}) \xrightarrow{\rho} \tilde{K}(X^{2p+1}).$$

From the above exact sequence of reduced $K$-theory one finds that $\chi = i\eta$ for an element $\eta \in \tilde{K}(X^{2p+2}, X^{2p+1})$.

But $X^{2p+2}/X^{2p+1}$ is a bouquet of $(2p + 2)$-dimensional spheres, whence, by a theorem of Bott (see for instance [10]), $c_i(\eta) = 0$ for $1 \leq i \leq p$ and $c_{p+1}(\eta)$ is a multiple of $p!$. This implies the divisibility of $c_{p+1}(E|X^{2p+2}) - c_{p+1}(E'|X^{2p+2})$ by $p!$. It remains to remark that the restriction map $H^{2p+2}(X, \mathbb{Z}_p) \to H^{2p+2}(X^{2p+2}, \mathbb{Z}_p)$ is one-to-one, and this finishes the proof of the correctness of the definition of $\phi_p$.

Next we prove the equality in assertion (2) for $p \geq 1$. Obviously $\phi_1 = 0$ and this identity agrees with $C_2 = H^2(X, \mathbb{Z}) \times H^4(X, \mathbb{Z})$. For an arbitrary $p$, one remarks that the inclusion ‘$\subset$’ in (2) is a consequence of the definition of $\phi_p$.

In order to prove the converse inclusion, let $(c_1, \ldots, c_{p+1})$ be a $(p+1)$-tuple of cohomology classes which satisfy: $(c_1, \ldots, c_p) \in C_p$ and $c_{p+1} = \phi_p(c_1, \ldots, c_p)$. Let $F$ be a vector bundle on $X$ with the first Chern classes $c_i(F) = c_i, 1 \leq i \leq p$. By the very definition of the function $\phi_p$ one gets $c_{p+1} = c_{p+1}(F) = \phi_p(c_1, \ldots, c_p)$. Hence there exists a cohomology class $\sigma \in H^{2p+2}(X, \mathbb{Z})$, such that

$$c_{p+1} - c_{p+1}(F) = (p!)\sigma.$$ 

The vanishing hypotheses in the statement assures the existence of a vector bundle $G$ on $X$ with $c_i(G) = 0$ for $1 \leq i \leq p$ and $c_{p+1}(G) = (p!)\sigma$.

In conclusion, the vector bundle $E = F \oplus G$ has Chern classes $c_i(E) = c_i, 1 \leq i \leq p + 1$, and the proof of Lemma 2.1 is over.

Let us remark that the sets $C_p$ and the functions $\phi_p$ are obviously compatible with morphisms $f: Y \to X$.

Lemma 2.1 will be used in the sequel in the following form: Let $X$ and $\phi_1, \phi_2, \ldots$ be as before. Then the cohomology classes $(c_1, \ldots, c_{[n/2]})$ are the Chern classes of a complex vector bundle on $X$ if and only if

$$c_{p+1} = \phi_p(c_1, \ldots, c_p) \text{ in } H^{2p+2}(X, \mathbb{Z}_p)$$

for every $p \geq 2$.

Indeed, if $\phi_1 = \phi_2(c_1, c_2)$, then $(c_1, c_2, c_3) \in C_3$, so that $\phi_3(c_1, c_2, c_3)$ makes a perfect sense, and so on. Of course, the above vector bundle can be assume to be exactly of rank equal to $[n/2]$.

The preceding result shows (at least theoretically) what kind of congruences are needed to determine the range of the total Chern class map. The main problem investigated in the following sections is to establish concrete cohomological expressions for the functions $\phi_p$, for particular choices of the base space $X$.

The simple applications of Lemma 2.1 appeared in [3]. For the sake of completeness, we include them in the present section, too.
COROLLARY 2.2.

Let $X$ be a finite CW-complex of dimension less or equal to 7 such that $H^7(X, \mathbb{Z})$ has no 2-torsion. Then $(c_1, c_2, c_3)$ are the Chern classes of a rank 3 vector bundle on $X$ if and only if

$$c_3 \equiv c_1 c_2 + Sq^2 c_2 \quad \text{in} \quad H^6(X, \mathbb{Z}_2).$$

Proof. The torsion conditions in Lemma 2.1 are automatically satisfied. Moreover, the only possible non-zero coboundary in the Atiyah–Hirzebruch spectral sequence is $d_4^3: H^4(X, \mathbb{Z}) \to H^7(X, \mathbb{Z})$.

But it is known that $d_3^2$ is a cohomological operation, which coincides with Streenrod’s square $Sq^3$ (see [10]). Therefore $2 \cdot d_3^3 = 0$, which, in conjunction with the lack of 2-torsion in $H^7(X, \mathbb{Z})$ shows that $d_3^3 = 0$.

At this moment we are able to apply Lemma 2.1 and we are seeking a concrete form of the function $\varphi_2$. This is obtained by Wu’s formula:

$$w_6 = w_2 w_4 + Sq^2 w_4, \quad (2.1)$$

referring to the universal Stiefel–Whitney classes (see for instance, ch. III.1.16 of [13]).

Accordingly, the Chern classes $(c_1, c_2, c_3)$ of a rank 3 vector bundle on $X$ satisfy

$$c_3 \equiv c_1 c_2 + Sq^2 c_2 \quad \text{in} \quad H^6(X, \mathbb{Z}_2). \quad (2.2)$$

In conclusion, the function $\varphi_2: H^2(X, \mathbb{Z}) \times H^4(X, \mathbb{Z}) \to H^6(X, \mathbb{Z}_2)$ is precisely

$$\varphi_2(c_1, c_2) = (c_1, c_2 + Sq^2 c_2)^*. \quad \square$$

COROLLARY 2.3.

Let $X$ be a connected, compact complex manifold of dimension 3. Then

$$\text{Vect}_{\text{top}}^3(X) \cong \{(c_1, c_2, c_3); c_i \in H^{2i}(X, \mathbb{Z}), c_1 c_2 + c_1(X)c_2 \equiv c_3 \text{ (mod 2)}, \quad i = 1, 2, 3\}. \quad (\star)$$

Indeed, in this case $Sq^2 c_2 = c_1(X)c_2$ (mod 2), where $c_1(X)$ stands for the first Chern class of the tangent complex bundle of $X$.

The congruence relation $c_3 \equiv c_1 c_2 + c_1(X)c_2$ (mod 2) can alternatively be obtained in Corollary 2.3 from the topological Riemann–Roch theorem.

For higher dimensional CW-complexes (i.e., of dimension greater than 7) we ignore whether a finite result like Corollary 2.2 holds. It would be interesting for instance to identify the functions $\varphi_3, \varphi_4$ among the identities satisfied by the universal Chern classes and the secondary and tertiary cohomological operations of order 6 and respectively 24.

For the rest of the paper we confine ourselves to only the simpler case of compact complex manifolds.

3. The stable classification on low dimensional compact complex manifolds

The Atiyah–Singer index theorem [2] provides some integrality conditions from which one can read the functions $\varphi_3$ and $\varphi_4$, on compact complex manifolds of dimensions 4 and 5 satisfy certain torsion-free cohomological conditions.
PROPOSITION 3.1.

Let $X$ be a compact, connected complex manifold of complex dimension 4 such that $H^0(X, \mathbb{Z})$ and $H^1(X, \mathbb{Z})$ have no 2-torsion. Then $(c_1, c_2, c_3, c_4)$ are the Chern classes of a (unique) rank 4 vector bundle on $X$, if and only if:

$$
\begin{align*}
    c_3 & \equiv c_1c_2 + \text{Sq}^2c_2 \quad \text{in} \quad H^6(X, \mathbb{Z}_2), \\
    c_4 & \equiv (c_1 + c_1(X))(c_3 - c_1c_2) + \frac{1}{2}(c_2(c_2 - c_2(X))) \\
    & \quad + c_1(X)(c_3 - (c_1 + c_1(X))c_2)(\text{mod} \ 6).
\end{align*}
$$

(3.1)

(3.2)

Proof. We will see below that (3.1) implies that the right-hand term of (3.2) is an integer cohomology class in spite of the denominator 2.

The general torsion and vanishing assumptions in Lemma 1 are fulfilled because $2d_3 = 0$ and $d_3^1 = 0$.

Let $E$ be a rank-4 complex vector bundle on $X$ with Chern classes $c_i = c_i(E), i = 1, 2, 3, 4$. As before, formula (2.1) of Wu implies the congruence (3.1), whence

$$
\varphi_2(c_1, c_2) = (c_1c_2 + \text{Sq}^2c_2) \quad \text{in} \quad H^6(X, \mathbb{Z}_2).
$$

(3.3)

The next function $\varphi_3$ is obtained from the Atiyah–Singer index theorem.

First, recall the notation $\int_X \xi$ for the coefficient of the component of degree 8 in the cohomology class $\xi \in H^8(X, \mathbb{Q})$, via the isomorphism $H^8(X, \mathbb{Q}) \cong \mathbb{Q}$ given by the volume element.

The Atiyah–Singer index theorem [2] implies that

$$
\int_X \text{ch}(E) \tau d(X) \in \mathbb{Z},
$$

(3.4)

for any vector bundle $E$ on $X$.

By denoting $\tau_i = c_i(T_X)$, where $T_X$ is the complex tangent bundle of $X$, a straightforward computation shows that relation (3.3) becomes

$$
\begin{align*}
    & - \frac{1}{180}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - \tau_1\tau_3 + \tau_4) \\
    & + \frac{1}{24}c_1\tau_1\tau_2 + \frac{1}{24}(c_1^2 - 2c_2)(\tau_1^2 + \tau_2) \\
    & + \frac{1}{12}(c_1^3 - 3c_1c_3 + 3c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) \in \mathbb{Z}.
\end{align*}
$$

(3.5)

The last formula can be simplified by replacing $E$ with only a few particular vector bundles. For instance, if $E = 1_4$ (the trivial bundle), then one finds that the first bracket is divisible by 180. Then for $E = 1_3 \oplus L$, where $L$ is a line bundle with $c_1(L) = c_1$, one gets

$$
\begin{align*}
    & \frac{1}{24}c_1\tau_1\tau_2 + \frac{1}{24}c_2^2(\tau_1^2 + \tau_2) + \frac{1}{12}c_1^3 + \frac{1}{24}c_4 \in \mathbb{Z}.
\end{align*}
$$

(3.6)

By simplifying (3.4) correspondingly one obtains exactly the relation (3.2).

Moreover, if $(c_1, c_2, c_3)$ satisfies (3.1), then by Lemma 2.1, there exists a vector bundle $F$ on $X$ with $c_i(F) = c_i, i = 1, 2, 3$. In view of the preceding computation one finds the
identity

\[ c_4(F) - (c_1 + c_1(X))(c_3 - c_1c_2) + 6\sigma \]

\[ = \frac{1}{2}(c_2(c_2 - c_2(X)) + c_1(X)(c_3 - (c_1 + c_1(X))c_2)), \]

for some \( \sigma \in H^3(X, \mathbb{Z}) \). This shows that the expression on the right-hand term always represents an integer class of cohomology and that one can identify the function \( \varphi_3 : C_3 \rightarrow H^8(X, \mathbb{Z}_2) \) with

\[ \varphi_3(c_1, c_2, c_3) = \left[ (c_1 + c_1(X))(c_3 - c_1c_2) \right. \]

\[ + \frac{1}{2}(c_2(c_2 - c_2(X)) + c_1(X)(c_3 - (c_1 + c_1(X))c_2)) \right]. \]

Thus the proof of Proposition 3.1 is over. \( \square \)

By taking into account the properties of the Steenrod operation \( Sq \),

\[ Sq^2 \eta = c_1(X)\eta, \quad \eta \in H^6(X, \mathbb{Z}_2), \]

\[ Sq^4 \xi = 0, \quad \xi \in H^2(X, \mathbb{Z}_2), \]

\[ Sq^2(\xi c_2) = \xi Sq^2 c_2 + Sq^4 \xi \cdot Sq^1 c_2 + \xi^2 c_2, \]

one deduces from (3.1) the relation

\[ \xi (c_1c_2 + c_3 - (c_1(X) + \xi) c_2) \equiv 0 \pmod{2} \]  \hspace{1cm} (3.5)

for every \( \xi \in H^2(X, \mathbb{Z}) \).

But \( H^6(X, \mathbb{Z}) \) was supposed to be without 2-torsion. Hence \( H^3(X, \mathbb{Z}) \) has no 2-torsion, too, and the restriction of scalars map \( H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z}_2) \) is onto.

That means that relation (3.5) is equivalent to (3.1), because one does not lose information by evaluating (3.1) on the classes of \( H^2(X, \mathbb{Z}_2) \). \( \square \)

In conclusion, relations (3.1) and (3.5) may be interchanged in Proposition 3.1. Moreover, the above proof shows that relation (3.5) is sufficient to be satisfied by a subset of \( H^2(X, \mathbb{Z}) \) whose image in \( H^2(X, \mathbb{Z}_2) \) is a basis.

Before going on to complex manifolds of dimension 5 we recall that the Chern character in \( K \)-theory has a formula:

\[ \text{ch}(\xi) = \text{rank} \xi + \sum_{k=1}^\infty P_k(c_1(\xi), \ldots, c_k(\xi)), \quad \xi \in K(X), \]

where \( P_k \) are some universal weighted polynomials with rational coefficients (see for instance [12]).

In general, for a \( r \)-tuple of cohomology classes \( (c_1, \ldots, c_r), c_i \in H^{2i}(X, \mathbb{Z}), 1 \leq i < r \), one defines

\[ \text{ch}(c_1, \ldots, c_r) = r + \sum_{k=1}^\infty P_k(c_1, \ldots, c_k), \]

by putting \( c_{r+1} = 0, c_{r+2} = 0 \) and so on. Of course, when \( (c_1, \ldots, c_r) \) are the Chern classes of a vector bundle \( E \) of rank \( r \), the above expression represents the Chern character of \( E \).
PROPOSITION 3.2.

Let $X$ be a connected, compact complex manifold of complex dimension 5. One assumes that the groups $H^i(X, \mathbb{Z})$ and $H^j(X, \mathbb{Z})$ have no 2-torsion, that $H^8(X, \mathbb{Z})$ has no 6-torsion and that $H^9(X, \mathbb{Z})$ has no 12-torsion.

Then $(c_1, c_2, c_3, c_4, c_5)$ are the Chern classes of a (unique) vector bundle of rank 5 on $X$ if and only if

$$c_3 \equiv c_1 c_2 + \text{Sq}^2 c_2 \quad \text{in} \quad H^8(X, \mathbb{Z}).$$

and

$$\int_X \text{ch}(c_1, \ldots, c_5) e^\xi \text{td}(X) \in \mathbb{Z},$$

for every $\xi \in H^2(X, \mathbb{Z}).$

Proof. The cohomology of the manifold $X$ satisfies the assumptions in Lemma 2.1. Moreover, the coboundaries $d_{2k+1}^2$ of Atiyah–Hirzebruch spectral sequence vanish for $k > 1$ and $p \geq 2$. Indeed, $d_2^4$ vanishes because $2d_3^4 = 0$ and on the other hand its target, $H^7(X, \mathbb{Z})$, has no 2-torsion. Similarly $d_3^5 = 0$ and also $d_5^3 = 0$ by dimension reasons. Consequently $d_5^4, E_5^2 = H^4 / \text{Im} d_3 \rightarrow E_5^{3,0} = H^3$. But the group $H^3(X, \mathbb{Z})$ has no 12-torsion and on the other hand, $12 - d_3^{1,2} = 0$ by \[1\]. Hence $d_3^4 = 0$ as desired.

Let $E$ be a vector bundle on $X$ of rank 5 and let $(c_1, \ldots, c_5)$ denote its Chern classes. The congruence (3.6) follows from Wu’s formula as before. By applying the index theorem to $E \otimes L$, where $L$ is a line bundle with $c_1(L) = \xi$ one gets (3.7).

Conversely, assume that the cohomology classes $(c_1, \ldots, c_5)$ satisfy (3.6) and (3.7). In view of Lemma 2.1 we have to prove that $c_3 = \varphi_2(c_1, c_2), c_4 = \varphi_3(c_1, c_2, c_3)$ and $c_5 = \varphi_4(c_1, c_2, c_3, c_4)$.

We already know the form of the function $\varphi_2$ derived from Wu’s formula: $\varphi_2(c_1, c_2) = c_1 c_2 + \text{Sq}^2(c_2)$ (mod 2). Consequently, there exists a vector bundle $E$ on $X$ with $c_i(E) = c_i, i = 1, 2, 3$. Then by definition, $\varphi_3(c_1, c_2, c_3)$ is the image of $c_4(E)$ in $H^8(X, \mathbb{Z})$, and thus we are led to prove that $c_4(E) = c_4$.

The hypotheses on $H^*(X, \mathbb{Z})$ imply

$$H^8(X, \mathbb{Z}) \cong \text{Hom}_\mathbb{Z}(H^2(X, \mathbb{Z}), \mathbb{Z}).$$

Therefore it suffices to prove that $\xi c_4 - \xi c_4(E)$ is a multiple of 6 for every $\xi \in H^2(X, \mathbb{Z})$.

The assumption (3.7) and the index theorem yield

$$\int_X \text{ch}(c_1, \ldots, c_5)(1 - e^\xi) \text{td}(X) \in \mathbb{Z}$$

and

$$\int_X \text{ch}(E)(1 - e^\xi) \text{td}(X) \in \mathbb{Z}$$

respectively.
It is worth mentioning that $c_5$ and $c_5(E)$ do not matter in the integrality conditions (3.8) and (3.9). In fact, these conditions can be reduced modulo $\mathbb{Z}$ to expressions such as
\[
\frac{1}{6} \xi \cdot c_4 + P(\xi, c_1, c_2, c_3, c(X)),
\]
\[
\frac{1}{6} \xi \cdot c_4(E) + P(\xi, c_1, c_2, c_3, c(X)),
\]
where $P$ is a polynomial with rational coefficients.

In conclusion, $\xi(c_4 - c_4(E))$ is divisible by 6 for any $\xi \in H^2(X, \mathbb{Z})$, whence $\epsilon_4 = \phi_3(c_1, c_2, c_3)$. Therefore, there exists a vector bundle $F$ on $X$, with $c_i(F) = c_i, i = 1, 2, 3, 4.$

By simply writing the index theorem for $F$,
\[
\int_X \text{ch}(F) t d(X) \in \mathbb{Z}
\]
and comparing this relation to the assumption (3.7) one gets that $c_4 - c_4(F)$ is a multiple of 4!

This finishes the proof of Proposition 3.2. \hfill $\square$

As in the preceding proof, it is sufficient to ask in the statement of Proposition 3.2 if the class $\xi$ belongs to a system of generators of $H^2(X, \mathbb{Z})$ and $\xi = 0$.

We do not know whether condition (3.6) is equivalent, like in the case of $\dim X = 4$, to a congruence involving only the cap product operation. Also, the stable classification of complex vector bundles on 6-dimensional manifolds (more exactly a concrete form of the function $\phi_3$) is unknown to us.

4. The stable-rank classification on compact complex manifolds with torsion-free cohomology

In this section we prove that Atiyah–Singer integrality conditions (3.3) are the only restrictions imposed to an $n$-tuple of cohomology classes $(c_1, \ldots, c_n)$ in order to be the Chern classes of a vector bundle on a compact complex manifold $X$, whose cohomology is torsion free, $n = \dim \mathbb{C}X.$

PROPOSITION 4.1.

Let $X$ be a connected, compact complex manifold of dimension $n$, with torsion-free cohomology. Then the cohomology classes $(c_1, \ldots, c_n)$ are the Chern classes of a (unique) vector bundle of rank $n$ on $X$, if and only if
\[
\int_X \text{ch}(c_1, \ldots, c_n) \text{ch}(\xi) t d(X) \in \mathbb{Z}, \quad \xi \in K(X).
\]

Proof. Conditions (4.1) are necessary by the Atiyah–Singer index theorem. Conversely, let $(c_1, \ldots, c_n)$ be an $n$-tuple of cohomology classes which satisfies (4.1). In order to construct a vector bundle $E$ with $c_i(E) = c_i, 1 \leq i \leq n$, it is sufficient to prove that
\[
\epsilon_3 = \phi_2(c_1, c_2), \ldots, \quad \epsilon_n = \phi_{n-1}(c_1, \ldots, c_{n-1})
\]
and apply them to Lemma 2.1.
However, in this case Lemma 2.1 can be avoided and a direct argument is possible. So we prove inductively that there are vector bundles $E^p$ on $X$, with the properties: $c_i(E^p) = c_i, 1 \leq i \leq p$ and $1 \leq p \leq n$.

The case $p = 1$ is obvious. Assume that there exists a vector bundle $E^{p-1}$ with $c_i(E^{p-1}) = c_i$ for $1 \leq i \leq p - 1$. We have to prove that $c_p - c_p(E^{p-1})$ is a multiple of $(p - 1)!$. Indeed, in that case the result of Atiyah and Hirzebruch mentioned in the introduction provides a vector bundle $F$ with $c_i(F) = 0$ for $1 \leq i \leq p - 1$ and $c_p(F) = c_p - c_p(E^{p-1})$. Hence the vector bundle $E^p = E^{p-1} \oplus F$ has the first Chern classes $c_i(E^p) = c_i, 1 \leq i \leq p$.

Since

$$H^{2p}(X, \mathbb{Z}_{(p-1)!}) \cong \text{Hom}_\mathbb{Z}(H^{2n-2p}(X, \mathbb{Z}), \mathbb{Z}_{(p-1)!}),$$

it is sufficient to verify that $\eta(c_p - c_p(E^{p-1}))$ is a multiple of $(p - 1)!$ for any $\eta \in H^{2n-2p}(X, \mathbb{Z})$. For that aim, consider a class $\xi \in K(X)$ with $c_i(\xi) = 0$ for $0 \leq i \leq n - p$ and $c_{n-p}(\xi) = (-1)^{n-p-1}(n - p - 1)!\eta$, so that

$$\text{ch}(\xi) = \eta + \text{higher order terms}.$$

By hypothesis

$$\int_X \text{ch}(c_1, \ldots, c_n)\text{ch}(\xi)td(X) \in \mathbb{Z},$$

and by the index theorem,

$$\int_X \text{ch}(E^{p-1})\text{ch}(\xi)td(X) \in \mathbb{Z}.$$

In both integrals the classes $c_{p+1}, \ldots, c_n$ and $c_{p+1}(E^{p-1}), \ldots, c_n(E^{p-1})$ do not matter. Consequently, they reduce to

$$\frac{1}{(p - 1)!} \eta c_p + \mathbb{Q}(c_1, \ldots, c_{p-1}, c(\xi), c(X)) \in \mathbb{Z}$$

and

$$\frac{1}{(p - 1)!} \eta c_p(E^{p-1}) + \mathbb{Q}(c_1, \ldots, c_{p-1}, c(\xi), c(X)) \in \mathbb{Z}$$

respectively, where $\mathbb{Q}$ is a polynomial with rational coefficients.

In conclusion, $(p - 1)!$ divides $\eta(c_p - c_p(E^{p-1}))$ and the proof of Proposition 4.1 is complete. \qed

Before continuing with an application of the proposition we recall some needed terminology. A nonsingular projective (complex) variety $X$ of dimension $n$ is said to possess an algebraic cell decomposition

$$X = X^n \supset X^{n-1} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset,$$

if $X^i$ are closed algebraic subsets of $X$ and each difference $X^i \setminus X^{i-1}$ is a disjoint union of locally closed submanifolds $U^i$ which are isomorphic with the affine space $C^i$. The sets $U_{ij}$ are the cells of this decomposition and their closures are called the closed cells.
If the projective manifold $X$ admits an algebraic cell decomposition, then $H^{\text{odd}}(X, \mathbb{Z}) = 0, H^{\text{even}}(X, \mathbb{Z})$ is torsion free and moreover $H^{2p}(X, \mathbb{Z})$ is freely generated by the fundamental classes of all closed cells on codimension $p$ (see [9]). Concerning the $K$-theory, it is known that $K_{\text{alg}}(X)$ is a free group with base $[\mathcal{O}_Y]$, where $\mathcal{O}_Y$ are the structure sheaves of the closed cells (see [11]). Moreover, the natural map

$$\varepsilon: K_{\text{alg}}(X) \to K(X)$$

(4.2)
is in that case isomorphic.

The surjectivity of $\varepsilon$ follows easily from the following statement: if $\xi \in K(X)$ has the property $c_i(\xi) = 0$ for $1 \leq i \leq p - 1$ then there exists an algebraic class $\xi' \in K_{\text{alg}}(X)$ with $c_i(\xi') = c_i(\varepsilon(\xi'))$ for $1 \leq i \leq p$.

If $\xi \in K(X)$ is as before, then $c_p(\xi) = (-1)^{p-1}(p-1)!\eta$, with $\eta \in H^{2p}(X, \mathbb{Z})$. Since $\eta - \sum n_i[Y_i]$ is the closed cells of dimension $p$, we define $\xi' = \sum n_i[\mathcal{O}_{Y_i}]$. But for an irreducible subvariety $Y$ of codimension $p$ one has $c_i(\mathcal{O}_Y) = 0$ for $0 < i < p$ and $c_p(\mathcal{O}_Y) = (-1)^{p-1}(p-1)!Y$ (see p. 297 of [2] for an argument valid when $Y$ is nonsingular) (in general, by simply removing the singular locus of $Y$). This concludes the surjectivity. The injectivity of $\varepsilon$ follows similarly.

**PROPOSITION 4.2.**

Let $X$ be a projective, nonsingular variety of dimension $n$, which has an algebraic cell decomposition. The cohomology classes $(c_1, \ldots, c_n)$ are the Chern classes of a (unique) vector bundle of rank $n$ on $X$ if and only if

$$\int_X \text{ch}(c_1, \ldots, c_n) \text{ch}(\mathcal{O}_Y) td(X) \in \mathbb{Z},$$

(4.3)

for every closed cell $Y$ of $X$, of dimension greater than or equal to 3.

**Proof.** The necessity follows from the index theorem or from the Riemann–Roch–Hirzebruch formula.

If we drop the condition $\dim Y \geq 3$, the statement is equivalent to Proposition 4.1. So the only thing to be proved is that the closed cells $Y$ of dimension less than 3 do not matter in the integrality formula (4.3). This is obvious when we compare with the proof of Proposition 4.1, because any vector bundle $E$ on $X$ with $c_i(E) = c_i, 1 \leq i < p$ satisfies $c_p = c_p(E)$ in $H^{2p}(X, \mathbb{Z}[p-1])$ for $p \leq 2$. \hfill $\square$

Otherwise, one repeats the proof of Proposition 4.1 by evaluating $c_p - c_p(E)$ on $\eta = [Y]$ and by taking $\xi = \varepsilon[\mathcal{O}_Y]$, correspondingly.

**Remark.** Let $Y$ be a connected, nonsingular, closed algebraic subset of dimension $d$ of a projective nonsingular variety $X$ of dimension $n$.

The Riemann–Roch–Grothendieck theorem applied to the inclusion $i: Y \hookrightarrow X$ yields

$$i_* (td(Y)) = ch(\mathcal{O}_Y) td(X).$$
This identity and the projection formula prove that for any cohomology classes \((c_1, \ldots, c_n)\) one has
\[
\int_X \text{ch}(c_1, \ldots, c_n) \text{ch}(\mathcal{E}_Y) td(X) = \int_Y \text{ch}(i^* c_1, \ldots, i^* c_n) td(Y)
\]
\[
= \int_Y \text{ch}(i^* c_1, \ldots, i^* c_d) td(Y)
\]
\[
+ (n-d) \chi(\mathcal{E}_Y)
\]
because \(i^*(c_{d+1}) = \cdots = i^*(c_n) = 0\).

This remark shows that condition (4.3) in Proposition 4.2 can be replaced by the Riemann–Roch formula, written on every closed cell, provided that all closed cells are non-singular. A familiar example of that kind is \(X = \mathbb{P}^n(\mathbb{C})\), the complex projective space of dimension \(n\). Actually Le Potier [16] had a similar point of view on \(\mathbb{P}^n(\mathbb{C})\), in what concerns the integrability conditions satisfied by the Chern classes of a vector bundle.

Quite specifically, in that case \((c_1, \ldots, c_n)\) are the Chern classes of a vector bundle on \(\mathbb{P}^n(\mathbb{C})\), if and only if

\[
\int_{\mathbb{P}^3} \text{ch}(c_1, c_2, c_3) td(\mathbb{P}^3) \in \mathbb{Z}, \ldots, \int_{\mathbb{P}^n} \text{ch}(c_1, \ldots, c_n) td(\mathbb{P}^n) \in \mathbb{Z}.
\]

Recall that \(td(\mathbb{P}^n) = [h/(1 - e^{-h})]^{n+1}\), where \(h\) is the class of a hyperplane.

An equivalent set of integrality conditions, such as

\[
\int_{\mathbb{P}^n} \text{ch}(c_1, \ldots, c_n) e^{kh}[h/(1 - e^{-h})]^{n+1} \in \mathbb{Z}, \quad 0 \leq k \leq n - 3,
\]

is provided by the next result.

**PROPOSITION 4.3.**

Let \(X\) be a complex compact manifold of dimension \(n\). Assume that \(H^*(X, \mathbb{Z})\) is torsion-free and that \(H^{\text{even}}(X, \mathbb{Z})\) is generated multiplicatively by \(H^2(X, \mathbb{Z})\). Then \((c_1, \ldots, c_n)\) are the Chern classes of a (unique) vector bundle of rank \(n\) on \(X\) if and only if

\[
\int_X \text{ch}(c_1, \ldots, c_n) e^{\xi} td(X) \in \mathbb{Z}, \quad \xi \in H^2(X, \mathbb{Z}). \quad (4.4)
\]

Moreover, the class \(\xi\) can be chosen of the form \(\xi = \xi_{j_1} + \cdots + \xi_{j_k}\), where \(0 \leq k \leq n - 3, 1 \leq j_1 \leq \cdots \leq j_k \leq m\) and \(\xi_1, \ldots, \xi_m\) is a basis of \(H^2(X, \mathbb{Z})\) (we put \(\xi = 0\) for \(k = 0\)).

**Proof.** Conditions (4.4) are necessary by the Atiyah–Singer index theorem.

Conversely, let \((c_1, \ldots, c_n)\) be the cohomology classes which satisfy (4.4). For proving the existence of a vector bundle on \(X\) with the prescribed Chern classes \((c_1, \ldots, c_n)\) one may use Lemma 2.1 or one may argue as in the proof of Proposition 4.1. As a matter of fact, we have to prove that for any \(p \geq 3\) and any vector bundle \(E\) with \(c_j(E) = c_j\) for \(1 \leq i < p\), the classes \(c_p\) and \(c_p(E)\) have the same image in \(H^{2p}(X, \mathbb{Z}/(p-1)\mathbb{Z})\) or equivalently, \(\eta(c_p - c_p(E))\) is divisible by \((p-1)!\) for a set of generators \(\eta\) of \(H^{2n-2p}(X, \mathbb{Z})\).

We may assume that \(\eta = \zeta_1, \ldots, \zeta_{n-p}\), where \(\zeta_i = \xi_{j_l}\) for \(1 \leq i \leq n-p\) and certain indices \(j_1, \ldots, j_{n-p}\). Let us denote

\[
K(\zeta_1, \ldots, \zeta_{n-p}) = 1 - \sum_{i} e^{\xi_i} + \sum_{j \neq k} e^{\xi_j + \xi_k} + \cdots + (-1)^{n-p} e^{\xi_{j_1} + \cdots + \xi_{j_{n-p}}},
\]
so that assumption (4.4) becomes
\[ \int_X \text{ch}(c_1, \ldots, c_n) K(\zeta_1, \ldots, \zeta_{n-p}) td(X) \in \mathbb{Z}. \]  

(4.5)

On the other hand, the index theorem applied to bundles of the form \( E \otimes L, \text{rk} L = 1 \), yields
\[ \int_X \text{ch}(E) K(\zeta_1, \ldots, \zeta_{n-p}) td(X) \in \mathbb{Z}. \]  

(4.6)

Next it is sufficient to remark that
\[ K(\zeta_1, \ldots, \zeta_{n-p}) = (1 - e^{\zeta_1}) \cdots (1 - e^{\zeta_{n-p}}) \]
\[ = (-1)^{n-p} \eta + \text{higher order terms} \]

This shows that the classes \( c_{p+1}, \ldots, c_n \) and \( c_{p+1}(E), \ldots, c_n(E) \) do not matter in relations (4.5) and (4.6), and moreover by subtracting (4.6) from (4.5) one finds that \( (p-1)! \) divides \( \eta(c_p - c_p(E)) \).

This completes the proof of Proposition 4.3.

\[ \square \]

5. The stable-rank classification on flag manifolds

A notable class of algebraic manifolds which fit into the conditions of Proposition 4.2 are the flag manifolds, or generalized flag manifolds, following the terminology adopted by some authors.

Let \( G \) denote a semi-simple, connected and simply-connected complex algebraic linear group. One denotes by \( T \) a maximal torus of \( G \), by \( N(T) \) the normalizer of \( T \) in \( G \) and by \( W = N(T)/T \) the corresponding Weyl group. Let \( B \) be a Borel subgroup of \( G \) which contains \( T \), \( \Phi \) the root system with respect to \( T \) and \( \Phi^+ \subset \Phi \) the positive roots corresponding to the choice of \( B \). The system of simple (positive) roots is denoted by \( \Delta \subset \Phi^+ \).

The parabolic subgroup \( P_I = BW_I B \) associated to a subset \( I \subset \Delta \) is obtained from the subgroup \( W_I \subset W \) generated by the reflections corresponding to \( I \). Any parabolic subgroup of \( G \) can be obtained by such a standard construction (see for instance [5]).

Let \( P = P_I \) be a parabolic subgroup of \( B \). We simply denote \( W_P \) instead of \( W_I \). Let \( w \in W \) and let \( n(w) \) denote a representative of \( w \) in \( N(T) \). One easily remarks that the double \( (B, P) \) coset \( Bn(w)P \) depends only on \( wW_P \) and hence it may be denoted by \( BwP \).

This is the so-called open Bruhat cell associated to \( wW_P \). The images of the Bruhat cells through the natural projection \( \pi: G \to G/P \) give an algebraic cell decomposition of the flag manifold \( G/P \). The corresponding closed cells, called the Schubert cells of \( G/P \), are denoted by \( X_p(w) \), and they are labelled as \( w \in W/W_P \). In fact, \( B \) acts (from the left) on \( G/P \) and the Schubert cells are exactly the closures of the corresponding orbits.

If \( I \) is empty, then \( P = B \) and the corresponding homogeneous space \( G/B \) is called the full flag manifold of \( G \). In that case the Schubert cells \( X_w \) are labelled by the Weyl group \( W \).

The flag manifolds are projective rational homogeneous varieties, and conversely any such abstract variety is isomorphic to a flag manifold (see [5]).
Recall now the Atiyah–Hirzebruch morphism for the full flag manifold $G/B$. Let $\hat{T}$ be the group of characters of $T$ and let $R(T) = Z[\hat{T}]$ be the representation ring of $T$. Any character $\chi \in \hat{T}$ extends trivially over the unipotent radical of $B$ and it gives a complex representation $\chi : B \to C^\ast$. Thus one obtains a line bundle $L(\chi) = G \times_B C^\ast$ on $G/B$. The association $\chi \mapsto L(\chi)$ is a morphism of groups $\hat{T} \to \text{Pic}(G/B)$ which extends to a morphism of rings $R(T) \to K(X)$, known as the Atiyah–Hirzebruch morphism.

Since the group $G$ was supposed to be simply connected, the map $\hat{T} \to \text{Pic}(G/B)$ is actually an isomorphism. The first Chern class gives an isomorphism $\text{Pic}(G/B) \cong H^2(G/B, \mathbb{Z})$. For simplicity, we identify the integer weights of $G$ with the characters of $T$ and with elements of $H^2(G/B, \mathbb{Z})$.

Consider a flag manifold $X = G/P$ of dimension $n$, as above. Proposition 4.2 tells us that a system of cohomology classes $(c_1, \ldots, c_n)$ of $X$ is the total Chern class of a vector bundle on $X$ if and only if

$$\int_X \text{ch}(c_1, \ldots, c_n) \text{ch}(\mathcal{O}_{X_{\mathbb{C}}(w)}) \text{td}(X) \in \mathbb{Z}, \quad w \in W/W_P.$$ 

Our next aim is to lift these conditions on the full flag manifold $G/B$. Let $\pi : G/B \to G/P$ be the canonical projection. This is a fibration with fibre the rational manifold $P/B$. The induced morphism $\pi^* : H^\ast(G/P, \mathbb{Z}) \to H^\ast(G/B, \mathbb{Z})$ is one-to-one and identifies the cohomology of $G/P$ with the ring of $W_P$-invariants in $H^\ast(G/B, \mathbb{Z})$.

Let $w \in W$. One has

$$\pi_\ast(\mathcal{O}_{X_w}) = \mathcal{O}_{X_{\mathbb{C}}(w)}; \quad R^q \pi_\ast(\mathcal{O}_{X_w}) = 0, \quad q \geq 1. \tag{5.1}$$

This is a subtle result essentially proved by Ramanan and Ramanathan (see also Proposition 2 of §5 in [8]). Consequently $\pi_\ast(\mathcal{O}_{X_w}) = (\mathcal{O}_{X_{\mathbb{C}}(w)})$ at the level of algebraic $K$-theories.

Riemann–Roch–Grothendieck theorem and the projection formula yield

$$\begin{align*}
\pi_\ast[\text{ch}(c_1, \ldots, c_n) \text{ch}(\mathcal{O}_{X_w}) \text{td}(G/B)] &= \text{ch}(c_1, \ldots, c_n) \pi_\ast[\text{ch}(\mathcal{O}_{X_w}) \text{td}(G/B)] \\
&= \text{ch}(c_1, \ldots, c_n) \text{ch}(\mathcal{O}_{X_{\mathbb{C}}(w)}) \text{td}(G/P). \tag{5.2}
\end{align*}$$

Recall that $\text{td}(G/B) = e^{-\rho}$, where $\rho$ is the semisum of all positive roots of $G$ (see [6]). By evaluating formula (5.2) on $G/P$ one obtains

$$\begin{align*}
\int_{G/B} \text{ch}(c_1, \ldots, c_n) \text{ch}(\mathcal{O}_{X_w}) e^{-\rho} &= \int_{G/P} \text{ch}(c_1, \ldots, c_n) \text{ch}(\mathcal{O}_{X_{\mathbb{C}}(w)}) \text{td}(G/P).
\end{align*}$$

In particular, this shows that the left-hand term does not depend on the representative $w$ of the class $wW_P \in W/W_P$ and explains the meaning of (5.3).

On the other hand, it is known that the Atiyah–Hirzebruch map is onto (see [13]). Therefore any class $\text{ch}(\xi)$ with $\xi \in K(G/B)$ is an entire combination of elements of the form $e^\tau$ with $\tau \in \hat{T} = H^2(G/B, \mathbb{Z})$. Hence we have proved the following.

**Theorem 5.1.** Let $X = G/P$ be a flag manifold of dimension $n$. The $n$-tuple of cohomology classes $(c_1, \ldots, c_n)$ of $X$ represents the Chern classes of a (unique) vector bundle of rank
n on \( X \) if and only if

\[
\int_{G/B} \text{ch}(c_1, \ldots, c_n) \text{ch}(\mathcal{O}_{X_w}) e^{-\rho} \in \mathbb{Z} \quad \text{for } w \in W/W_P, \tag{5.3}
\]

or equivalently, if and only if

\[
\int_{G/B} \text{ch}(c_1, \ldots, c_n) e^\chi \in \mathbb{Z} \quad \text{for } \chi \in \hat{T}. \tag{5.4}
\]

Remarks.

(1) According to Proposition 4.2, it is sufficient to consider in relation (5.3) only cosets \( wW_P \) with the property that \( \dim X_P(w) \geq 3 \).

Let \( Y \) be a Schubert cell in \( X \). Then \( \pi^{-1}(Y) \) is an irreducible \( B \)-invariant subset of \( G/B \), so it coincides with a Schubert cell of \( G/B \). Thus for any coset \( wW_P \) one may find a representative \( w \in W \), such that \( \pi^{-1}(X_P(w)) = X_w \) (\( w \) is the element of maximal length in \( wW_P \)). Thus \( w \in W \) is called \( P \)-saturated. In that case the vanishing conditions (5.1) easily follow from the fact that \( \pi/X_w: X_w \to X_P(w) \) is a fibration of fiber \( P/B \), and \( H^q(P/B, O) = 0 \) for \( q \geq 1 \). Thus one can avoid ref. [19] and the corresponding integrality conditions (which replace (5.3)) are

\[
\int_{G/B} \text{ch}(c_1, \ldots, c_n) \text{ch}(\mathcal{O}_{X_w}) e^{-\rho} \in \mathbb{Z},
\]

for any \( P \)-saturated word \( w \in W \) of length \( l(w) \geq \dim(P/B) + 3 \).

(2) As \( \int_{G/B} \text{ch}(c_1, \ldots, c_n) e^\tau = \int_{G/B} \text{ch}(c_1, \ldots, c_n, 0, \ldots, 0) e^\tau + \) and integer, the integrality conditions (5.3) shows that the classification of stable-rank vector bundles on \( G/P \) can be deduced from the same classification on \( G/B \).

Namely \( \pi^*: \text{Vect}_{\text{top}}^t(G/P) \to \text{Vect}_{\text{top}}^t(G/B) \) is a one-to-one map and a vector bundle \( E \) on \( G/B \) (of stable-rank) is in the range of \( \pi^* \) if and only if the Chern classes \( c_i(E) \) are \( W_P \)-invariant when \( i \geq 1 \). Of course, the two stable ranks on \( G/P \) and \( G/B \) do not coincide and the precise meaning of \( \pi^*(E) \) is up to a trivial direct summand.

By keeping the notation introduced in the first part of this section we denote by \( w_0 \) in the sequel the element of maximal length in \( W \). Also \( w_1, \ldots, w_r \) are the fundamental weights of \( G \) with respect to the Borel subgroup \( B \).

**Theorem 5.2.** Let \( X = G/P \) be a flag manifold and assume that \( H^2(X, \mathbb{Z}) \) generates \( H^\text{even}(X, \mathbb{Z}), n = \dim X \). Then \( \text{Vect}_{\text{top}}^t(X) \) is in bijection, via Chern classes, with the n-tuples \( (c_1, \ldots, c_n) \) satisfying

\[
\int_{G/B} \text{ch}(c_1, \ldots, c_n) e^{X - \rho} \in \mathbb{Z}, \tag{5.5}
\]

for every character \( \chi \in \hat{T} \) of the form \( \chi = \omega_{i_1} + \cdots + \omega_{i_k} \), where \( 0 \leq k \leq n - 3 \), \( 1 \leq i_1 \leq \cdots \leq i_k \leq r \) and \( \omega_1, \ldots, \omega_r \) are the fundamental weights corresponding to simple roots \( \alpha \) for which \( w_0 \sigma_\alpha \) is \( P \)-saturated.

**Proof.** One applies Proposition 4.3. Let \( \alpha \) be a root with \( w_0 \sigma_\alpha \) \( P \)-saturated (\( \sigma_\alpha \) is the associated reflection). Then \( X_P(w_0 \sigma_\alpha) \) is a hypersurface in \( X \), whose corresponding line bundle \( L_P(\alpha) = \mathcal{O}_X(X_P(w_0 \sigma_\alpha)) \) has the first Chern class \( [X_P(w_0 \sigma_\alpha)] \).
By varying \( \alpha \) one gets a basis for \( \text{Pic}(X) \cong H^2(X, \mathbb{Z}) \). On the other hand,

\[
\pi^*L_P(\alpha) = \mathcal{O}_{G/B}(X_{w_0\alpha}) = L(\alpha),
\]

where \( \omega_\alpha \) is the fundamental weight associated to \( \alpha \). Since for any line bundle \( L \) on \( X \) one has

\[
\int_X \text{ch}(c_1, \ldots, c_n) \text{ch}(L)td(X) = \int_{G/B} \text{ch}(c_1, \ldots, c_n) \text{ch}(\pi^*L)e^{-\rho},
\]

the proof of Theorem 5.2 is over. \( \square \)

Let \( S(\hat{T}) \) denote the symmetric algebra of \( \hat{T} \), i.e. the polynomials with integer coefficients acting on the Lie algebra of \( T \). The isomorphism

\[
\hat{T} \cong H^2(G/B, \mathbb{Z}), \quad \chi \mapsto c_1(L(\chi)),
\]

extends to a morphism of rings

\[
c: S(\hat{T}) \to H^*(G/B, \mathbb{Z}).
\]

Demazure \(^8\) proved that \( c \) is onto precisely when \( G \) is a product of groups of type \( \text{SL} \) and \( \text{Sp} \). In particular in that case \( H^2(G/B, \mathbb{Z}) \) generates \( H^{\text{even}}(G/B, \mathbb{Z}) \), and so this fits into the conditions of Theorem 5.2. However, for a parabolic subgroup \( P \subset G, H^2(G/P, \mathbb{Z}) \) may not generate the even cohomology of \( G/P \). In spite of that, the next result still holds.

**Theorem 5.3.** Let \( X = G/P \) be an \( n \)-dimensional flag manifold with \( G \) a product of simple Lie groups of types \( \text{SL} \) and \( \text{Sp} \).

Then \( \text{Vect}^n_{\text{top}}(X) \) is in bijection, via Chern classes, with the set of \( n \)-tuples \( (c_1, \ldots, c_n) \), \( c_i \in H^2(G/P, \mathbb{Z}), 1 \leq i \leq n \), satisfying

\[
\int_{G/B} \text{ch}(c_1, \ldots, c_n)e^{-\rho} \in \mathbb{Z},
\]

where \( \chi = \omega_{i_1} + \cdots + \omega_{i_k}, 0 \leq k \leq \dim(G/B) - 3, 1 \leq i_1 \leq \cdots \leq i_k \leq r \).

**Proof.** If \( P = B \), then the statement reduces to Theorem 5.2. Otherwise one uses the equality \( \text{ch}(c_1, \ldots, c_n) = \text{ch}(c_1, \ldots, c_n, 0, \ldots, 0) + n - \dim(G/B) \) and Remark 2.

We close this section by a few comments on the Lie algebra interpretation of the integrality conditions. The morphism \( c \) induces a ring morphism

\[
c_Q: S_Q(\hat{T}) \to H^*(G/B, \mathbb{Q}),
\]

where \( S_Q(\hat{T}) = \mathbb{Q} \otimes \mathbb{Z} S(\hat{T}) \). The Weyl group \( W \) acts on \( S_Q(\hat{T}) \) and a classical result asserts that \( c_Q \) is onto and \( \ker c_Q \) is the ideal generated by the homogeneous \( W \)-invariant polynomials of positive degree \(^5\).

For any \( w \in W \) one defines the operators

\[
A_w: S_Q(\hat{T}) \to S_Q(\hat{T}), \quad D_w: S_Q(\hat{T}) \to \mathbb{Q}
\]

by \( A_{\alpha}f = (f - \sigma_\alpha f)/\alpha, f \in S_Q(\hat{T}) \) for a simple root \( \alpha \), \( A_w = A_{\alpha_1} \cdots A_{\alpha_l}, D_w f = (A_w f)(0) \) for any irreducible decomposition \( w = \sigma_{\alpha_1} \cdots \alpha_{\alpha_l} \) (see \(^4\)).
The operators $A_w$ and $D_w$ descend to the rational cohomology of $G/B$ (we use, for convenience, the same letters)

$$A_w : H^*(G/B, \mathbb{Q}) \to H^*(G/B, \mathbb{Q}), \quad D_w : H^*(G/B, \mathbb{Q}) \to \mathbb{Q}.$$ 

According to Bernstein–Gelfand–Gelfand \[4\], the integrality condition (5.4) can be expressed as

$$\int_{G/B} \text{ch}(c_1, \ldots, c_n)e^\tau = D_{w_0}(\text{ch}(c_1, \ldots, c_n)e^\tau),$$

for any $\tau \in \hat{T}$. Recall that $w_0$ stands for the maximal length element in $W$.

Concerning the integrality conditions (5.3), their translation into Lie algebra terms faces an additional difficulty. More exactly, $H^*(G/B, \mathbb{Q})$ has two natural bases as $\mathbb{Q}$-vector space

$$(\{X_w\})_{w \in W} \quad \text{and} \quad (\text{ch}(\mathcal{O}_{X_w}))_{w \in W},$$

related by some linear formulae:

$$\text{ch}(\mathcal{O}_{X_w}) = \sum_{w' \in W} q_{w,w'}[X_w], \quad w \in W,$$

where $q_{w,w'} \in \mathbb{Q}$, $q_{w,w'} = 0$ unless $w' \leq w$ and $q_{w,w} = 1$. Therefore, by \[4\], the expression in (5.4) becomes

$$\int_{G/B} \text{ch}(c_1, \ldots, c_n)\text{ch}(\mathcal{O}_{X_w})e^{-\rho} = \sum_{w' \in W} q_{w,w'}D_{w'}(\text{ch}(c_1, \ldots, c_n)e^{-\rho}).$$

The next natural question is how to compute the matrix $(q_{w,w'})_{w,w' \in W}$. The results of Demazure \[8\] on the desingularization of the Schubert cells of the flag manifolds give an algorithm for computing this matrix. Unfortunately we were not able to put this algorithm into a simple, manageable form (see the subsequent Appendix).

Another solution to this problem would be to find for every $w \in W$ a concrete polynom in $\mathcal{S}_{\mathbb{Q}}(\hat{T})$ whose image through $c_{\mathbb{Q}}$ is $\text{ch}(\mathcal{O}_{X_w})$.

Appendix: The Chern character of Schubert cells

Let $T \subset B \subset G, \Delta \subset \Phi^+ \subset \Phi$ be as in §5 and let $n = \dim(G/B)$. Throughout this appendix, $\hat{B}$ stands for the opposite Borel subgroup of $B$ and $w_0 = s_{\beta_1} \cdots s_{\beta_n}$ is the element of $W$ of maximal length, written in a reduced form. One denotes

$$\alpha_1 = \beta_1, \alpha_2 = s_{\beta_1}(\beta_2), \ldots, \alpha_n = s_{\beta_1} \cdots s_{\beta_{n-1}}(\beta_n).$$

Then $\Phi^+ = \{\alpha_1, \ldots, \alpha_n\}$ and $s_{\alpha_1} \cdots s_{\alpha_i} = s_{\beta_i} \cdots s_{\beta_1}$ for $1 \leq i \leq n$. 
Let $\psi$: $Z \to X = G/B$ be the desingularization associated to the decomposition of $w_0$ into $s_{i_0}, \ldots, s_{i_k}$ [3]. The space $Z$ is constructed as a sequence of $\mathbb{P}^1$-fibre bundles $f_i: Z^i \to Z^{i-1}$, each of them being endowed with a canonical cross-section $\sigma_i$: $Z^{i-1} \to Z^i$.

\[
Z = Z^n \xrightarrow{f_n/\sigma_n} Z^{n-1} \xrightarrow{\cdots} Z^1 \xrightarrow{f_1/\sigma_1} Z^0 = \text{point}.
\]

For any $i = 1, 2, \ldots, n$ one denotes $Z_i = f_n^{-1} \cdots f_{i+1}^{-1}(\text{Im } \sigma_i)$ and one remarks that $Z_i$ are smooth hypersurfaces in $Z$. Following [3], each intersection $Z_K = \bigcap_{i \in K} Z_i$, $K \subset [1, n]$ is still smooth and of codimension $|K|$. Let $\xi_i$ denote the fundamental class $[Z_i] \in H^2(Z, \mathbb{Z}), i = 1, \ldots, n$. Consequently $[Z_K] =: \xi_K = \prod_{i \in K} \xi_i$.

Demazure proves that the cohomology ring $H^*(Z, \mathbb{Z})$ is $\mathbb{Z}$-free, with generators $(\xi_K)_{K \subset [1, n]}$ and relations

\[
\xi^2_1 = 0, \xi^2_2 + \langle \alpha_1^\vee, \alpha_2 \rangle \xi_1 \xi_2 = 0, \ldots,
\]

\[
\xi^n_1 + \langle \alpha_1^\vee, \alpha_n \rangle \xi_1 \xi_n + \cdots + \langle \alpha_n^\vee, \alpha_1 \rangle \xi_n \cdots \xi_1 = 0.
\]

Also, the following relations hold [3]:

1. $\psi_*(\xi_K) = 0$ for length $(w_K) \neq |K|$,
2. $\psi_*(\xi_K) = [X_{w_Kw_0}]$ if length $(w_K) = |K|$,

where

\[
w_K = \prod_{i \in K} s_{i}, \quad K \subset [1, n].
\]

In view of p. 58 of [3], the class of the tangent bundle of $Z$ in $K(Z)$ can be explicitly described as

\[
[T_Z] = \sum_{r=1}^n \left[ \bigotimes_{i=1}^n (\mathcal{O}(Z_i))^{(\alpha_i^\vee, \alpha_i)} \right].
\]

Accordingly, the Todd character of $T_Z$ is

\[
td(Z) = \prod_{r=1}^n \frac{\sum_{i=1}^r (\alpha_i^\vee, \alpha_i) \xi_i}{1 - \exp(-\sum_{i=1}^r (\alpha_i^\vee, \alpha_i) \xi_i)}.
\]  \hspace{1cm} (A1)

From now on $K = [1, \ldots, k], k \leq n$ is fixed. From the exact sequence,

\[
0 \to T_{Z_K} \to T_Z|Z_K \to N_{Z_K/Z} \to 0,
\]

the isomorphism $N_{Z_K/Z} \cong \mathcal{O}(Z_k)|Z_K \oplus \cdots \oplus \mathcal{O}(Z_k)|Z_K$ and from the Riemann–Roch–Grothendieck formula applied to the inclusion $i_K: Z_K \to Z$ one obtains

\[
\text{ch}(\mathcal{O}(Z_K)) = (-1)^k \prod_{i=1}^k \frac{\xi_i^2}{1 - e^{\beta_i}}.
\]  \hspace{1cm} (A2)

Notice that for every element $w \in W$ there exists a decomposition of $w_0$ into simple reflections, such that $w|_{[1, \ldots, k]} = w_0$, where $k = \text{length}(w)$ (see [3]).
On the other hand, for \( K = [1, \ldots, k] \), one knows that

\[
\psi^*(\mathcal{O}_Z^K) = \mathcal{O}_{X_w} \quad \text{and} \quad R^q \psi^*(\mathcal{O}_Z^K) = 0, \quad q \geq 1.
\]

Finally, Riemann–Roch–Grothendieck formula yields

\[
\text{ch}(\mathcal{O}_{X_w}) e^{-\rho} = \psi^*(\text{ch}(\mathcal{O}_Z^K) \text{td}(Z)). \tag{A3}
\]

In conclusion, by substituting (A1) and (A2) into (A3) and by knowing the action of \( \psi^* \) on the cohomology, one gets an algorithm of computing \( \text{ch}(\mathcal{O}_{X_w}) \) in terms of the Cartan integers \( \langle \alpha_i^w, \alpha_j \rangle \), a reduced decomposition of \( w \) and the generators \( [X_{\alpha_i^w}], w' \in W \) of \( H^*(G/B, \mathbb{Q}) \). However, as mentioned before in \( \S 5 \), some further simplifications of this algorithm are worth having.

**Remark.** It is likely that the recent paper by Kostant and Kumar \cite{Kostant-Kumar} can lead to a better understanding of the integrality conditions in \( \S 5 \).

**Acknowledgements**

The present article was completed in 1990, a few months before the sudden and tragic death of the first author. The second author was partially supported by a grant from the National Science Foundation. The main results were announced in C Bănică and M Putinar, Fibrés vectoriels complexes de rang stable sur les variétés complexes compactes, *C. R. Acad. Sci. Paris. t. Serie I* 314 (1992) 829–832. The second author thanks his colleagues at the Indian Statistical Institute, Bangalore and the Tata Institute, Mumbai for their interest in resurrecting this work.

**References**

1. Atiyah M F and Hirzebruch F, Vector bundles on homogeneous spaces, *Proc. Symp. Pure Math.* (1961) (RI: Am. Math. Soc. Providence) vol. III, pp. 7–38
2. Atiyah M F and Singer I M, The index of elliptic operators on compact manifolds, *Bull. Am. Math. Soc.* 69 (1963) 422–433
3. Bănică C and Putinar M, On complex vector bundles on projective threefolds, *Invent. Math.* 88 (1987) 427–438
4. Berstein I N, Gelfand I M and Gelfand S I, Schubert cells and cohomology of the spaces \( G/P \), in: Representation theory (eds) Gelfand et al (1982) (Cambridge Univ. Press: London Math. Soc. Lecture Notes No. 69) pp. 115–140
5. Borel A, Linear algebraic groups (1969) (New York: W. A. Benjamin)
6. Borel A and Hirzebruch F, Characteristic classes and homogeneous spaces II, *Am. J. Math.* 81 (1959) 315–382
7. Buhstaber V M, Modules of differentials of the Atiyah–Hirzebruch spectral sequence (in Russian), *Math. Sb.* 78 (1969) 307–320
8. Demazure M, Désingularisation des variétés de Schubert généralisées, *Ann. Ec. Norm. Sup.* 7 (1974) 53–88
9. Fulton W, Intersection theory (1984) (Berlin: Springer-Verlag)
10. Griffiths P and Adams J, Topics in algebraic and analytic geometry (1974) (Princeton: Princeton Univ. Press)
11. Grothendieck A, Sur quelques propriétés fondamentales en théorie des intersections, in: *Sem. Chevalley* (1958) exposé 4
Complex vector bundles of stable rank

[12] Hirzebruch F, Topological methods in algebraic geometry (1966) (Berlin: Springer-Verlag)
[13] Husemoller D H, Fibre bundles (1966) (New York: McGraw-Hill)
[14] Kostant B, Lie algebra cohomology and generalized Schubert cells, *Ann. Math.* **77** (1963) 72–144
[15] Kostant B and Kumar S, \(\Gamma\)-equivariant \(K\)-theory of generalized flag varieties, *J. Diff. Geom.* **32**(2) (1990) 549–603
[16] Le Potier J, Fibrés vectoriels sur \(\mathbb{P}^4\) (exposé fait à Paris VII)
[17] Peterson F P, Some remarks on Chern classes, *Ann. Math.* **69** (1959) 414–420
[18] Pittie H V, Homogeneous vector bundles on homogeneous spaces, *Topology* **11** (1972) 199–203
[19] Ramanan S and Ramanathan A, Projective normality of the flag varieties and Schubert varieties, *Invent. Math.* **79** (1985) 217–224
[20] Switzer R M, Rank 2 bundles over \(\mathbb{P}^3\) and the e-invariant, *Indiana Univ. Math. J.* **28**(6) (1979) 961–974
[21] Thomas A, Almost complex structures on complex projective spaces, *Trans. Am. Math. Soc.* **193** (1974) 123–132