Quantum mutual information and quantumness vectors for multiqubit systems

Sk Sazim1 · Pankaj Agrawal1

Received: 14 November 2019 / Accepted: 3 June 2020 / Published online: 17 June 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
We introduce a new information theoretic measure of quantum correlations for multiparticle systems. We use a form of multivariate mutual information—the interaction information—and generalize it to multiparticle quantum systems. There are a number of different possible generalizations. We consider two of them. One of them is related to the notion of quantum discord and the other to the concept of quantum dissension. This new measure, called dissension vector, is a set of numbers—quantumness vector. This can be thought of as a fine-grained measure, as opposed to measures that quantify some average quantum properties of a system. These quantities quantify/characterize the correlations present in multiparticle states. We consider some multiqubit states and find that these quantities are responsive to different aspects of quantumness and correlations present in a state. We find that different dissension vectors can track the correlations (both classical and quantum) or quantumness only. As physical applications, we find that these vectors might be useful in several information processing tasks. We consider the role of dissension vectors—(a) in deciding the security of BB84 protocol against an eavesdropper and (b) in determining the possible role of correlations in the performance of Grover search algorithm. Specially, in the Grover search algorithm, we find that dissension vectors can detect the correlations and show the maximum correlations when one expects.

Keywords Quantum correlations · Mutual information · Entanglement

1 Institute of Physics, Sainik School Post, Bhubaneswar, Orissa 751005, India
1 Introduction

In the quantum information science, one of the challenges is to understand the nature of correlations present in a multiparticle system. Because of its complex nature, we still have little success in this respect [1,2]. Correlations, specially quantum correlations, have been very useful for a host of quantum information processing tasks, such as quantum computing [3–5], quantum cryptography [6], and quantum metrology [7]. The quantum correlations also lie at the heart of quantum mysteries and account for many counterintuitive features of the quantum world. Therefore, a understanding of the nature of quantum correlations is very important.

The correlations in a system can be of classical and/or quantum nature. Usually, it is believed that quantum correlations are due to entanglement [2]. However, more recently, it has been suggested that the quantum correlations go beyond the simple idea of entanglement [1]. In particular, it has been argued that quantum discord [8,9] quantifies all types of quantum correlations including entanglement. Discord is the difference between total correlations and classical correlations present in a state. In recent years, it has been one of the main topics of research [1,10–15]. It has been shown that quantum discord also has operational significance [5,16,17]. A number of different measures of quantum correlations similar to quantum discord have been proposed in the literature. These are quantum deficit [18,19], measurement-induced disturbance [20], geometric discord [21], and many more.

It has recently been argued that the quantum discord, and similar other information theoretic measures, actually not only quantifies entanglement, i.e., nonlocal quantumness, but also local quantumness [22,23]. For example, due to the presence of local quantumness, the quantum discord (and other related measures [1]) increases by applying certain kinds of local noise [24]. It is still to be established that a state with zero entanglement and nonzero discord can act as a resource for a nonlocal task (cf., [25,26]). In this sense, the phrase “quantum correlations beyond entanglement” may be a misnomer. However, information theoretic measures like discord do seem to characterize quantum properties of a state beyond entanglement, in particular local quantumness. Such measures appear to characterize the quantum properties of a state more completely. Therefore, it will be useful to generalize the measures like quantum discord to multiparticle systems. There have been several attempts in this direction [1,12,15,27]. We will use multivariate mutual information for our generalization.

There exist a number of different versions of multivariate mutual information [36]. We consider three different versions—interaction information, total information, and binding information. The extension of the definitions of these versions to quantum world present myriad possibilities. However, not all the generalizations seem to have clear physical meaning. We will particularly focus on the generalization of interaction information to the quantum regime. This is because, classically, interaction information corresponds to genuine multivariate correlations.

One important point that we emphasize in this paper is the usefulness of a vector-like quantity to characterize and quantify the quantumness of a state. The correlations in mixed states of a system or even pure states of a multiparticle system are multifaceted. They cannot be characterized by just one number. The set of numbers is called, more generally, a quantumness vector. We can think of this as a fine-grained
measure. A suitable one number can quantify some average quantum properties of a state which may be suitable for some applications. We first illustrate it by considering two-qubit mixed states. We introduce a quantumness vector for characterizing these mixed states. This idea is then extended to multiparticle states. For generalization of quantum discord to $n$-qubit case, we use interaction information, a version of multivariate mutual information [15] that characterizes genuine multivariate correlations in $n$ random variables. It is based on a Venn diagram-type approach. There exist many expressions for this $n$-variable mutual information, all of which are same classically but differ when conditional entropies are generalized to quantum level. For a multiparticle system, one can make measurement on one particle or on more than one particle to probe the different aspects of quantum correlations. This would lead to multiple quantities that can eventually characterize the correlations present in the system. Such physical quantities, quantum dissension, were introduced in our previous work [15]. By emphasizing that we need a set of numbers to characterize correlations, we extend the notion of dissension to dissension vector along two different tracks. In the first track, we proceed in the usual way by which quantum discord was defined as difference of classical information from total amount of information present in the system. Then, we extend the definition to multiparticle case. In the second approach, we consider all possible measurements in the expression of mutual information. In each track, to characterize multiparticle correlations, we will have $n - 1$ quantities based on $(n - 1)$ types of measurements. For example, in the tripartite case, in each track we shall have two quantities that will characterize the correlations. Interestingly, these values can be negative because a measurement on a subsystem can enhance the correlations in the rest of the system. In the case of three-qubit systems, we find that dissension vectors, $\vec{\delta}_1^1$, $\vec{\delta}_2^1$ in track-I and in track-II, which are based on one-particle measurement, quantify correlations, both classical and quantum; more correlations lead to a more negative value. On the other hand, the dissension vector $\vec{\delta}_3^1$ which is based on two-qubit measurement quantifies quantumness of the state—both local and nonlocal. This approach emphasizes the fact that a single quantity alone is not sufficient to characterize the quantum properties of a state. This paves the way for defining quantum correlation as a vector quantity.

We find that these dissension vectors are useful in capturing, local as well as nonlocal quantumness of the multiparticle states. They reveal the complex structures of correlations present in the state. We also find that under non-unital channel, these measures can increase. This suggests that a dissension vector is also characterizing local quantum properties. To put these measures on strong footing, we consider some physical applications. We posit two such applications in quantum informations protocols—(a) Bennett and Brassard quantum key distribution protocol (BB84) [42] and (b) Grover search algorithm [44]. We find that in BB84 protocol, using dissension vector, the respective parties can detect the presence of an eavesdropper. In the case of Grover search algorithm, we find that to achieve success in Grover search algorithm, a substantial amount of dissension should develop during the processes. The dissension vectors can trace the correlations and are maximum where one expects maximum correlations.

The organization of the paper is as follows: In Sect. 2, we discuss classical mutual information and its extension to quantum regime. We discuss correlations and quan-
tumness in Sect. 3. In Sect. 4, we extend the notion of discord along two different tracks and give expressions for dissension vectors for $n$-qubit case. In Sect. 5, we analyze these measures with examples for three- and four-qubit systems. We discuss some features of these measures in Sect. 6. In Sect. 7, we address the possible physical applications of these measures. Finally, we conclude in Sect. 8.

# 2 Mutual information and its generalization to quantum regime

Let us consider two random variables $X$ and $Y$. The common information that they possess is characterized by mutual information

$$ I(X : Y) = H(X) + H(Y) - H(X, Y), $$

where $H(X)$ is Shannon entropy of $X$ and $H(X, Y)$ is the joint entropy. There are many uses of mutual information. Our interest is in its ability to capture correlations between two probability distributions. Using chain rule, one can express mutual information also as:

$$ I(X : Y) = H(X) - H(X|Y), $$

$$ = H(Y) - H(Y|X), $$

$$ = H(X, Y) - (H(X|Y) + H(Y|X)), $$

where $H(X|Y) = H(X, Y) - H(Y)$ is the conditional entropy. In Eq. (2), the last expression of mutual information is symmetric in $X$ and $Y$ unlike the former two. Note that the quantity $H(X|Y) + H(Y|X)$ is metric in its own right and called “variation of information.”

In quantum regime, mutual information is written in terms of von Neumann entropy of density matrices. Intuitively, this quantity solely should characterize the correlations between two subsystems of a bipartite system. But in reality, it does not. It is sometimes suggested that the mutual information quantifies the total correlations of a bipartite system [28]. However, in general what it characterizes about the state is somewhat elusive [29,30]. Also, the generalization of this quantity to quantum regime leads to many new features and complexities. One way of generalization is that of replacing the probability distributions with density matrices, and another is using relative entropy, i.e., for a bipartite state $\rho_{xy},$

$$ I_q^q(X : Y) = S(X) + S(Y) - S(X, Y), $$

$$ = S(\rho_{XY} \parallel \rho_X \otimes \rho_Y), $$

where $S(X) = -\text{Tr}(\rho_X \log_2 \rho_X)$ represents von Neumann entropy and $S(\rho \parallel \sigma) = \text{Tr}\rho(\log_2 \rho - \log_2 \sigma)$ is relative entropy.
2.1 Quantum conditional entropy and mutual information

Possible generalizations of Eq. (2) for the bipartite quantum state $\rho_{XY}$ are:

\[
I_Y(X : Y) = S(X) - S(X|Y),
\]
\[
I_X(X : Y) = S(Y) - S(Y|X),
\]
\[
I_a(X : Y) = S(X, Y) - (S(X|Y) + S(Y|X)),
\]
where $S(X|Y)$ is the quantum conditional entropy. If we directly extend the classical conditional entropy expression to quantum domain, then $S(X|Y) = S(X, Y) - S(Y)$, which is negative for pure entangled states. This negativity of conditional entropy was explained in references [31–35]. However, there is an alternate view which says that to know a state we have to make a measurement [9]. This is then the meaning of “conditional.” So, conditional entropy can also be expressed as:

\[
S(X|Y) = \sum_i p_i S(\rho_{X|\Pi^Y_i}),
\]

where $\rho_{X|\Pi^Y_i} = \frac{1}{p_i} \text{Tr}_Y (\mathbb{I}_2 \otimes \Pi^Y_i) \rho_{XY} (\mathbb{I}_2 \otimes \Pi^Y_i)$ with $p_i = \text{Tr}(\mathbb{I}_2 \otimes \Pi^Y_i) \rho_{XY} (\mathbb{I}_2 \otimes \Pi^Y_i)$. $\mathbb{I}_p$ is the identity matrix of order $p$ and $\{\Pi^Y_i; i = 1, 2\}$ are, in general, the rank-one positive operator valued measure (POVM) on part $Y$. The definition in Eq. (5) is always positive.

2.2 Multiparticle mutual information

Our goal in this paper is to examine multiparticle systems. So, we need a generalization of the bipartite mutual information to a multipartite situation. We will use the usual generalization based on Venn diagram approach. In this approach, the mutual information for three variables $X$, $Y$ and $Z$ is defined as

\[
I_0(X : Y : Z) = I(X : Y) - I(X : Y|Z),
\]

where $I(X : Y|Z) = H(X|Z) + H(Y|Z) - H(X, Y|Z)$ is conditional mutual information [36]. This can be immediately generalized to $n$-variate mutual information. Using chain rules, this generalization will lead to the multivariate mutual information as:

\[
I_0(X_1 : \ldots : X_n) = \sum_{p=1}^{n} (-1)^{p-1} \sum_{\{l_p\}} H(X_{l_1}, X_{l_2}, \ldots, X_{l_p}),
\]

where $\{l_p\}$ in the sum denotes that if $p = k$, then indices $l_1, l_2, \ldots, l_k (k \leq n)$ will survive with each $l_i$ varying from 1 to $n$ and $l_i \neq l_j$. In the literature, this quantity is also known as the “interaction information.” By analogy, one can write the multiparticle
mutual information of the state $\rho_{x_1,x_2,...,x_n}$ as:

$$I_q^0 (x_1 : x_2 : \ldots : x_n) = \sum_{p=1}^{n} (-1)^{p-1} \sum_{|I_p|} S(x_{l_1}, x_{l_2}, \ldots, x_{l_p}),$$  \hspace{1cm} (8)

where $x_i$ here stands for $x_i^{th}$ subsystem. This generalization has not been explored much. In this paper, we will use this generalization and define a vector-type correlation measure to characterize and quantify multiparticle correlations.

However, there exist at least two more mutual information like quantities in the literature. First one is known as “total correlation.” The total correlation for three variables is

$$I_t(X : Y : Z) = I(X : Y) + I(XY : Z),$$  \hspace{1cm} (9)

where $I(XY : Z) = I(X : Z) + I(Y : Z|X)$. This quantity can also be generalized for multi-variables, i.e.,

$$I_t(X_1 : \ldots : X_n) = \sum_{i=1}^{n} H(X_i) - H(X_1, \ldots, X_n).$$  \hspace{1cm} (10)

It can be generalized to quantum regime for the state $\rho_{x_1,x_2,...,x_n}$

$$I_q^0 (x_1 : x_2 : \ldots : x_n) = \sum_{p=1}^{n} S(x_i) - S(x_1, x_2, \ldots, x_n)
= S(\rho_{x_1,x_2,...,x_n} \parallel \otimes_{i=1}^{n} \rho_{x_i}).$$  \hspace{1cm} (11)

The second line of Eq. (11) shows that it is a distance between the state and tensor products of its marginals. This generalization has been used in the literature [28] to capture total correlations in a multiparticle quantum state. Note that the above generalization is always positive [37].

Another quantity is the “dual total correlation,” or “binding information,” or sometime known as “secrecy monotone” [38]. For three random variables, it is expressed as

$$I_b(X : Y : Z) = I(X : YZ) + I(Y : Z|X),$$  \hspace{1cm} (12)

where $I(X : YZ) = I(X : Y) + I(X : Z|Y)$. The above quantity can be generalized for multi-variables, i.e.,

$$I_b(X_1 : \ldots : X_n) = \sum_{i=1}^{n} H(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) - (n - 1)H(X_1, \ldots, X_n).$$  \hspace{1cm} (13)
Fig. 1 (Color online) Venn diagram: The information theoretic quantities for three random variables: X, Y, and Z.

The total correlation, 
\[ I_t(X : Y : Z) = I_b(X : Y : Z) + I_0(X : Y : Z), \]
and the binding information, 
\[ I_b(X : Y : Z) = I(X : Y | Z) + I(X : Z | Y) + I(Y : Z | X) + I_0(X : Y : Z), \]
where \( I_0(X : Y : Z) \) is the interaction information.

The quantity \( I_b(x_1 : x_2 : \ldots : x_n) \) can easily be extended for the multiparticle quantum state, \( \rho_{x_1x_2...x_n} \)

\[ I_b^q(x_1 : x_2 : \ldots : x_n) = \sum_{i=1}^{n} S(x_1, \ldots , x_{i-1}, x_{i+1}, \ldots , x_n) - (n - 1) S(x_1, \ldots , x_n). \]  

(14)

Note that the above quantity is also always positive \([39]\) and for pure states \( I_b^q(x_1 : x_2 : \ldots : x_n) = I_b^q(x_1 : x_2 : \ldots : x_n) \). This quantity has been used in the literature for capturing correlations in a quantum state \([39]\) and to detect the shared secret correlations between the parties \([38]\). The total correlation \( I_t^q(x_1 : x_2 : \ldots : x_n) \) and the binding information \( I_b^q(x_1 : x_2 : \ldots : x_n) \) are monotones under complete positive trace preserving (CPTP) map \([38,39]\). Moreover, for two particle quantum systems, Eqs. (8, 11 and 14) reduce to \( I_b^q(x_1 : x_2) \). Figure 1 depicts the relations between the possible generalizations of multivariate mutual information. These relations may not hold for quantum case. From the diagram, it is clear that only \( I_0 \) characterizes genuine multiparticle correlations. Other two generalizations \( I_t \) and \( I_b \) also contain bipartite correlations.

An important feature of interaction information is: Negative of tripartite quantum interaction information (\( I_0^q \)) is useful in determining the achievable rate for a particular secret sharing task, the information scrambling \([40,41]\). If interaction information is negative, then one can conclude that achievable rate will be nonzero for such a secret sharing task. If we carefully inspect Table 1, it shows that \( I_0^q \) may not capture total or genuine correlations in the quantum state, but it may be useful in characterizing certain highly entangled states. Figure 2 indicates that \( I_0^q \) can be negative for three-qubit mixed states. We are using generalization of \( I_0 \) to quantum domain.
0 0.2 0.4 0.6 0.8 1
-0.8 -0.6 -0.4 -0.2 0

Fig. 2 (Color online) Figure shows the plot of interaction information $I_0^q$ with the mixing parameter $p$ for the three-qubit mixed states $\rho_{GW} = (1 - p)|W_3\rangle\langle W_3| + p|G_3\rangle\langle G_3|$ (black dashed line) and $\rho_{We} = (1 - p)|G_3\rangle\langle G_3| + p|G_3\rangle\langle G_3|$ (solid red line). It shows interaction information can be negative for three-qubit states.

### Table 1 Comparison between three types of mutual information

| State     | $I_0^q$ | $I_b^q$ | $I_t^q$ |
|-----------|---------|---------|---------|
| $|G_4\rangle$ | 2       | 4       | 4       |
| $|C\rangle$       | -2      | 4       | 4       |
| $|HS\rangle$       | -2.755  | 4       | 4       |
| $|W_2^2\rangle$ | 0.49    | 4       | 4       |
| $|G_2\rangle^\otimes 2$ | 0       | 4       | 4       |

For four-qubit case, $I_t^q$ may be used to characterize different highly entangled states which may not be possible using the other two. The states are $|C\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$, $|HS\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |1100\rangle + \omega(|1010\rangle + |0101\rangle) + \omega^2(|1001\rangle + |0110\rangle)$, $|W_n^r\rangle = 1/\sqrt{r} \sum_{i=0}^{r-1} |P\rangle^r_0 \langle i| \langle i| \langle i|$. Here, $P$ denotes all possible combinations, $\omega = e^{-i\pi/3}$.

### 2.3 Can mutual information be negative?

One feature of the multivariate mutual information, as given by Venn diagram approach, is that it can be negative. Sometimes, it is considered a negative aspect of this approach. However, as we will see, the negative value characterizes a very special type of correlations [36]. For this, we consider mutual information of three variables $X$, $Y$, and $Z$, as given in Eq. (6). In this definition, both $I(X : Y)$ and $I(X : Y \mid Z)$ are nonnegative, but $I_0(X : Y : Z)$ can be negative, when $I(X : Y) < I(X : Y \mid Z)$. This situation will occur when knowing $Z$ enhances the correlation between $X$ and $Y$. Let us take a well-known example of “modulo 2 addition ($\oplus$) of two binary random variables (XOR gate).” Suppose $X \oplus Y = Z$. If $X$ and $Y$ are independent, then $I(X : Y) = 0$. However, once we know the value of $Z$, knowing the value of $X$ uniquely determines the value of $Y$. Hence, the knowledge of $Z$ enhances the correlation between $X$ and $Y$.
i.e., $I(X : Y | Z)$ is nonzero. This implies when $I_0(X : Y : Z)$ is negative, it captures certain aspect of the correlations among the variables $X$, $Y$, and $Z$.

The generalization of Eq. (6) in the quantum regime, for the state $\rho_{XYZ}$, is

$$I^q_0(X : Y : Z) = I^q(X : Y) - I^q(X : Y | Z),$$

where $I^q(X : Y | Z) = S(X | Z) + S(Y | Z) - S(XY | Z)$ is conditional mutual information. Let us consider the case of a three-qubit GHZ state $|G_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. If we trace out any one qubit from the state, then the reduced density matrix is a mixture of product states, i.e., $\rho_r = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. For this state, the mutual information is $I^q(X : Y) = 1$. Now, the conditional mutual information $I^q(X : Y | Z)$ for $|G_3\rangle$ will depend on the measurement basis. We know $S(XY | Z) = 0$ in any measurement basis, but it is not the case for other two terms $S(X | Z)$ and $S(Y | Z)$. The measurement on qubit $Z$ in computational basis ($|0\rangle$, $|1\rangle$) will give $S(X | Z) = S(Y | Z) = 0$, i.e., $I^q(X : Y | Z) = 0$. So, the total mutual information is $I^q_0(X : Y : Z) = 1$, i.e., positive. It is not surprising because the state of remaining two qubits, after measurement on one qubit, does not have enhanced entanglement. But for the measurement on qubit $Z$ in Hadamard basis ($|+\rangle$, $|−\rangle$), the mutual information $I^q(X : Y | Z) = 2$ means total mutual information $I^q_0(X : Y : Z) = −1$, i.e., negative. This is expected, since now the state of two qubits $XY$ is a Bell state; so measurement on $Z$ qubit has enhanced the entanglement in $XY$ subsystem. The essence of this discussion is that in both classical and quantum regime multivariate mutual information can be negative, characterizing a special type of correlations.

Let us re-express Eqs. (6 & 15) such that we find the following compact expression:

$$I_0(X : Y : Z) = I(X : Y) + I(X : Z) - I(X : Y | Z).$$

The above expression is positive if $I(X : Y) + I(X : Z) \geq I(X : Y | Z)$, i.e., when mutual information is monogamous. Otherwise, it is polygamous. Hence, a negative $I_0(X : Y : Z)$ means that the correlation between $X$ and the joint system $YZ$ is more than the sum of the individual ones. It is well understood that the entanglement among other correlations is always monogamous in nature, while classical correlations are not. However, the presence of strong entanglement as well as classical correlations between $X$ and joint system $YZ$ makes the situation complicated.

### 3 Correlations and quantumness

Whether a quantum state (of more than one particle) has correlations or not, it is often far from obvious. This is because the meaning of the word “correlation,” as often used in the literature, is quite fluid. We know the meaning of correlation in classical world, but in the case of a quantum state, there are classicality and quantumness. This makes the nature of correlations very complex. If we take the intuitive meaning of correlations [2], then quantum correlations are nonlocal in nature and can be taken as due to entanglement of the state only. They exist due to the nonlocal quantumness of a state. A state can also have classical correlations [9] and local quantumness [22]. When we
speak of quantumness of a state, it can be local or nonlocal in character. Information theoretic measures like quantum discord, and its generalization like dissension, characterize and quantify both types of quantumness. Next, we emphasize the need of a vector measure to characterize the quantumness of a state. We then expand on local and nonlocal quantumness.

3.1 Quantum discord: is one number sufficient?

In reference [8,9], authors have given a way of quantifying quantum correlations present in bipartite two-qubit states through quantum discord. To do so, they used different generalizations of the mutual information to quantum regime. Let us consider the bipartite state $\rho_{xy}$. Then using Eqs. (3) and (4), the discord is defined in the following way:

$$\delta_j(\rho_{xy}) = \inf_{\Pi_j} \{ I_0^q(x : y) - I_j(i : j) \},$$

(17)

where $I_j(i : j) = S(i) - S(i|j)$ with $i, j; i \neq j = x, y$. Here, the measurement bases $\Pi_j$ are $\{ \cos \theta |0\rangle + e^{i\phi} \sin \theta |1\rangle, - \sin \theta |0\rangle + e^{i\phi} \cos \theta |1\rangle \}$ with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. (Note that throughout the manuscript, we have considered this basis as our single particle measurement basis.) Obviously, the above definition is not symmetric in the parties. When $j = y$, it is usual discord and for $j = x$ it is $\delta_x(\rho_{xy})$. Sometimes, one of the discords is zero, even when other is nonzero. If we take that the quantum discord captures ‘quantumness’ present in a state, it is quite clear that we need both the discords to know the exact quantumness of the state.

One can define “another discord” as:

$$\delta_a(\rho_{xy}) = \inf_{\{ \Pi_x, \Pi_y \}} \{ I_0^q(x : y) - I_a(x : y) \}. $$

(18)

This definition is symmetric in the parties.

**Lemma 1** The quantity, $\delta_a(\rho_{xy})$, is nothing but the sum of the two discords $\delta_x(\rho_{xy})$ and $\delta_y(\rho_{xy})$.

**Proof** Using Eq.(4), we have

$$\delta_a(\rho_{xy}) = \inf_{\{ \Pi_x, \Pi_y \}} \{ I_0^q(x : y) - I_a(x : y) \}
= \inf_{\{ \Pi_x, \Pi_y \}} \{ S(x) + S(x|y) - S(xy) + S(y|x) - S(xy) \}
= \inf_{\Pi_y} \{ I_0^q(x : y) - I_y(x : y) \} + \inf_{\Pi_x} \{ I_0^q(x : y) - I_x(x : y) \}
= \delta_x(\rho_{xy}) + \delta_y(\rho_{xy}) .$$

Hence, proved.

Let us compute the above quantities for the following examples. For this purpose, we introduce a vector-type quantity $\{ \delta_x, \delta_y \}$ instead of using $\delta_x$ and $\delta_y$ separately. It is a quantumness vector—discord vector. For example, consider the following two-qubit states. Both for product states $\rho_p = |\phi\rangle \otimes |\chi\rangle$ and classically correlated states
\[ \rho_c = p|\phi\rangle\langle\phi| + p|\phi_\perp\rangle\langle\phi_\perp|, \] where the discords are \( \{\delta_x, \delta_y\} = \{0, 0\} \), \( \delta_a = 0 \), and the classical-quantum state \( \rho_{cq} = \frac{1}{2}(|++\rangle\langle++| + |--\rangle\langle--|) \) and the quantum-classical states \( \rho_{qc} = \frac{1}{2}(|++\rangle\langle++| + |00\rangle\langle00|) \) have the discords \( \{\delta_x, \delta_y\} = \{0.02\} \) and \( \{\delta_x, \delta_y\} = \{0, 0.2\} \), respectively, but the \( \delta_a = 0.2 \) for both the states, whereas a quantum–quantum separable state \( \rho_{qq} = \frac{1}{2}(|00\rangle\langle00| + |++\rangle\langle++|) \) has \( \{\delta_x, \delta_y\} = \{0.15, 0.15\} \), \( \delta_a = 0.3 \). As expected, the maximally entangled state \( |G_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \) will have discords \( \{\delta_x, \delta_y\} = \{1, 1\} \), \( \delta_a = 2 \). Hence, the vector-type quantification of correlation reveals more information about the correlation of a state than \( \delta_x \) or \( \delta_y \) alone.

### 3.2 Local and nonlocal quantumness

As has been argued in [22], the quantities like quantum discord not only characterize nonlocal quantumness (i.e., entanglement), but also local quantumness. Same will continue to hold for the generalizations of the discord that we will discuss. Let us recall these aspects of quantumness.

Let us consider the example states from the previous subsection. We found that both the \( X \)-discord and \( Y \)-discord are zero for \( \rho_p \) and \( \rho_c \). But \( X \)-discord is zero and \( Y \)-discord is nonzero for \( \rho_{cq} \) and both discords are nonzero for \( \rho_{qq} \). Now, consider the state \( \rho_p \). In Hadamard basis, there is no superposition in it, but in computational basis, there is. So, one can mask the local superposition of the state \( \rho_p \) and hence discord is zero. State \( \rho_c \) is the mixture of orthogonal states, and there is no local quantumness in it. But the states \( \rho_{cq} \) and \( \rho_{qq} \) are the mixture of non-orthogonal states. State \( \rho_{cq} \) has local superposition in one part and that’s why one of the discords is nonzero for this case, but for \( \rho_{qq} \), both parts have local superposition. Formally, one can define local quantumness as follows:

**Definition** We say that a mixture of non-orthogonal separable states has local quantumness (i.e., local superposition), if it cannot be masked by writing down the state in another decomposition.

To illustrate the various features of a state, let us consider the generalized Werner state [22]:

\[ \rho_{W,g} = \frac{(1 - p)}{4} \mathbb{I}_4 + p \rho_k, \tag{19} \]

where \( \rho_k = \langle\psi\rangle_k |\psi\rangle_k \) with \( |\psi\rangle_k = \frac{1}{\sqrt{1+k^2}} (|00\rangle + k|11\rangle) \). Here, \( p \) is classical mixing parameter, whereas \( k \) is the nonlocal parameter due to its role in nonlocal superposition. The state [Eq. (19)] is separable if \( p \leq \frac{1+k^2}{1+4k+4k^2} \), but discord is nonzero. Nonetheless, one can show that the above state [Eq. (19)] has both local and nonlocal quantumness (Fig. 3). For \( p \leq \frac{1+k^2}{1+4k+4k^2} \), one rewrite the state as valid mixture of non-orthogonal states, and the state has only local quantumness.

For example, if we consider \( k = 1 \), the state coincides with the Werner state, \( \rho_{Wer} = \frac{(1-p)}{4} \mathbb{I}_4 + p \rho_{G_2} \). Figure 4 depicts the behavior of both the discords with mixing parameter \( p \) for the two-qubit Werner state. The Werner state is separable for \( p \leq \frac{1}{3} \). This is because one can always rewrite Werner state in such a way that the
state is a valid mixture of non-orthogonal states whenever $p < \frac{1}{3}$ [20]. Rewriting the Werner state in that form, we have

$$\rho_{\text{Wer}} = (1 - 3p) \frac{I}{4} + \frac{p}{2}(|++\rangle\langle++| + |+-\rangle\langle-+| + |--\rangle\langle-\rangle + |00\rangle\langle00| + |11\rangle\langle11|), \quad (20)$$

where $|\tilde{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$. This decomposition is valid only for $p \leq \frac{1}{3}$. This is precisely the region of $p$, where Werner state is not entangled. Since $\langle+|0\rangle \neq 0$ and $\langle++|\tilde{+}\rangle \neq 0$, this state is a mixture of separable non-orthogonal states; so it is expected to have nonzero discord due to local quantumness.

### 4 Dissension vectors

From the discussion of the last section, it is clear that a vector-type correlation measure is better in describing the quantum properties of a state. Using multivariate mutual information, we will now generalize the quantum discord to $n$-qubit system, calling it dissension. We will introduce two types of quantumness vectors—called dissension vectors.

Let us consider a state $\rho_{x_1 x_2 \ldots x_n}$ in Hilbert space $H_2 \otimes H_2 \otimes \cdots \otimes H_2$ where $x_i$ qubit is with $i^{th}$ party. The mutual information for this state is (Eq. 8)

$$I_0^q(x_1 : x_2 : \ldots : x_n) = \sum_{p=1}^{n} (-1)^{p-1} \sum_{[i_p]} S(x_{i_1}, x_{i_2}, \ldots, x_{i_p}). \quad (21)$$
Using chain rule, we can now introduce conditional entropies. These conditional entropies are to be understood in terms of measurements. In this way, one can introduce one party, two party, ..., \((n - 1)\)-party measurements in the above expression of mutual information [Eq.(21)] and each leads one to an expression for new mutual information. When more than one party is involved, joint measurement is to be implemented. Following reference [15], one can have mutual information with all possible conditionals which we called track-II-type definition, but following [8,9] one can have mutual information with smaller number of conditionals which we call track-I (or discord track) definition of mutual information.

**Notation.** Before defining the dissension vectors, we want to clarify what we mean by the notation \(\vec{\delta}_m^t\). The index \(m\) defines the number of particles on which the joint measurement has been performed and \(t\) indicates the track.

### 4.1 Track-I

Let us consider the most general situation where we have state \(\rho_{x_1x_2...x_n}\) with \(n\) number of qubits. On the basis of \(m\)-party joint measurement (one can employ local measurements simultaneously), we will have \((n - 1)\) expressions for mutual information \(\{I_m^1(x_1 : x_2 : \ldots : x_n); m = 1, 2, \ldots, (n - 1)\}:

\[
I_m^1(x_1 : x_2 : \ldots : x_n) = \sum_{k=1}^{m-1} (-1)^{k-1} \sum_{\{l_k\}}^{n} S(x_{l_1}, x_{l_2}, \ldots, x_{l_k}) \\
+ (-1)^{m-1} \sum_{\{k_{m-1}\}; k_1=2} S(x_1, x_{k_1}, \ldots, x_{k_{m-1}})
\]
In Eq. (22), if we put $\delta_{I_0}^1 I^1 = (\rho_{x_1} I_0 - I^1_m)$, and then the dissension,

$$\delta_{I_0}^1 = \inf_{\Pi_m} [(\rho_{x_1} I_0 - I^1_m)],$$  \hspace{1cm} (23)

where minimization is done over $m$-party measurement. The expressions of mutual information in Eq. (22) are not symmetric under interchange of parties. For example, if we take $\delta_{I_0}^1$, we can have $n$ number of different expressions which are very different from one another. In Eq. (22), if we put $m = 1$, we can have one type of $I^1_1$; let us name it $I_{x_1}^1$. Now exchanging $x_n$ with $x_1, x_2, \ldots, x_{n-1}$, respectively, one can have others. So, we have $n$ number of $\delta_{I_0}^1$. We label them as $\delta_{I_0}^1 I^1_p = (\rho_{x_1} I_0 - I^1_{x_p}); p = 1, 2, \ldots, n$. This leads us to define dissension vector

$$\tilde{\delta}_{I_0}^1 = \{\delta_{I_0}^1 I^1_p; p = 1, 2, \ldots, n\}. \hspace{1cm} (24)$$

In this way with some particular choice of entries, one can have $n - 1$ vectors, i.e., $\tilde{\delta}_{I_0}^1; i = 1, 2, \ldots, (n - 1)$.

### 4.2 Track-II

Next, we extend the definitions of mutual information in this track to all possible $m$ party conditionals and we have the expression for mutual information, i.e., $I^2_m, m = 1, 2, \ldots, (n - 1)$:

$$I^2_m = \sum_{k=1}^{m-1} (-1)^{k+1} \sum_{\{l_k\}} S(x_{l_1}, x_{l_2}, \ldots, x_{l_k}) + (-1)^{m-1} \sum_{\{k_{m-1}; k_1=2\}} \sum_{\{l_{k-1}; k_1=2\}} S(x_{l_1}, x_{l_2}, x_{l_{k-1}}, x_{l_{k-1}+1}) - S(x_{l_1}, x_{l_2}, x_{l_{k-1}}, x_{l_{k-1}+1}) + \cdots \hspace{1cm} (25)$$

The dissension function in this track is defined as $D^2_m = (\rho_{x_1} I_0 - I^2_m)$). Therefore, the dissensions are

$$\delta_{I_0}^2 = \inf_{\Pi_m} [(\rho_{x_1} I_0 - I^2_m)]. \hspace{1cm} (26)$$
If we interchange parties, the mutual information in the Eq. (25) will not remain same except for \( m = (n - 1) \). For example if we consider \( m = 1 \) in Eq. (25), we will get one \( I^2_q \); let us call it as \( I^2_{x_n} \). Now interchanging \( x_n \) with \( x_1, x_2, \ldots, x_{n-1} \), respectively, we will get others. In this way, we will have \( n \) numbers of \( \delta^2_{x_p} \). If we label them as \( \delta^2_{x_p} = (-1)^p (I^0_q - I^2_{x_p}) \), \( p = 1, 2, \ldots, n \), we have dissension vector

\[
\vec{\delta}^2_1 = \{ \delta^2_{x_p}; p = 1, 2, \ldots, n \}.
\]

(27)

With some particular choice of entries, one can have \( n - 2 \) vectors, i.e., \( \vec{\delta}^2_i; i = 1, 2, \ldots, (n - 2) \) and one symmetric quantity \( \delta^2_{n-1} \). We call these quantities dissension vectors in track-II.

5 Simple illustrations

In this section, we will present our numerical results for a set of three-qubit and four-qubit states. It will illustrate the usefulness of the dissension vectors. We will consider track-I and track-II dissension vectors, as defined in the last section. We will see that both tracks are most of the time useful.

5.1 Three-qubit states

For the three-qubit states, the dissensions have been extensively calculated in the work [15]. For the sake of completeness, we have analyzed the dissension vectors for three-qubit states and discuss some of the results. For three qubits, the dissension vectors are \( (\vec{\delta}^1_1, \vec{\delta}^1_2) \) in track-I and in track-II, \( \vec{\delta}^2_1 \) along with the symmetric discord \( \delta^2_{n-1} \). Here in this work, we consider two-qubit joint measurement basis as \( \{ \cos \theta|00\rangle + \sin \theta|11\rangle, - \sin \theta|00\rangle + \cos \theta|11\rangle, \cos \eta|01\rangle + \sin \eta|10\rangle, - \sin \eta|01\rangle + \cos \eta|10\rangle \} \), where \( \theta \in [0, \pi] \) and \( \eta \in [0, \pi] \).

We will consider the following three-qubit states to show the usefulness of the dissension vectors. Let us consider a product state \( \rho_{pro} = |000\rangle \langle 000| \) and a classical state \( \rho_{ccc} = \frac{1}{2}(|000\rangle \langle 000| + |111\rangle \langle 111|) \). The later has only classical correlations, but the former has none. Examples of three-qubit separable states which may have only local quantumness are \( \rho_{cqc} = \frac{1}{2}(|000\rangle \langle 000| + |110\rangle \langle 110|) \), \( \rho_{qqc} = \frac{1}{2}|1\rangle \langle 1| + |0\rangle \langle 0| \), and \( \rho_{qqq} = \frac{1}{2}|++\rangle \langle ++| + |000\rangle \langle 000| \). We further consider two classes of pure highly entangled states—GHZ state \( |G_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \) and W-state \( |W\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |010\rangle + |001\rangle) \). From Table 2, it is clear that the dissension vector \( \vec{\delta}^1_1 \simeq \kappa \vec{\delta}^2_1 \), where \( \kappa \) is just a scale factor. It seems that only one of them is sufficient for our characterization and following the similar reasoning, we find that the dissension vector \( \vec{\delta}^1_1 \) captures more information than the symmetric quantity, \( \delta^2_{n-1} \). Then, we may not need the track-II dissensions at all. However, this might not be the case always as one will find in the following subsection.

It is evident from Table 2 that the dissension vectors characterize the correlations in the above three-qubit states. For the state, \( \rho_{pro} \), the dissension vectors are null,
depicting it does not have any local as well as nonlocal quantumness. The state $\rho_{ccc}$ has only classical correlations, no local and nonlocal quantumness, whereas the states $\rho_{ccq}$, $\rho_{qqc}$, and $\rho_{qqq}$ have only local quantumness. From Table 2, we observe that the dissension vectors based on the measurement on one qubit, $\vec{\delta}_1$ and $\vec{\delta}_2$, quantify correlations, both classical and quantum, or more accurately mixedness of qubits. In the case of separable states, classical state qubits are maximally mixed, and the dissension vector is most negative for this state. In the case of classical state, $\rho_{ccc}$, and GHZ state, $\rho_{G3}$, the system qubits are maximally mixed, and the dissension vectors are identical. On the other hand, the dissension vector based on two-qubit measurements, $\vec{\delta}_1$, is positive and increases with the quantumness of the state. In the case of separable states, from $\rho_{ccc}$ to $\rho_{qqq}$, as quantumness is increasing, this vector is becoming larger. It takes larger values for entangled states and is largest for GHZ state, as one would expect.

To illustrate the above facts, we also consider the following mixed entangled states, $\rho_{WG} = (1 - p)\rho_W + p\rho_{G3}$ and $\rho_{Wer} = (1 - p)^2 I + p\rho_{G3}$. Out of these states, $\rho_{wer}$ can be thought as the Werner state in three-qubit scenario. In Fig. 5, we have plotted the behavior of the dissension vectors as a function of mixing parameter $p$. We note from Fig. 5 that for both mixed states, $\vec{\delta}_1$ and $\vec{\delta}_2$ are more negative for larger correlations, while $\vec{\delta}_1$ is more positive with more quantumness in the states.

5.2 Four-qubit states

In case of four-qubit system, there are three dissension vectors ($\vec{\delta}_1^1$, $\vec{\delta}_1^2$, $\vec{\delta}_1^3$) in track-I and two vectors ($\vec{\delta}_2^1$, $\vec{\delta}_2^2$) and one symmetric discord $\delta_2^3$ in track-II. The single-qubit and two-qubit measurement strategies are discussed above. For three-qubit measurements, we have considered eight orthogonal non-maximally entangled three-qubit GHZ class states with at least four parameters on which we will perform optimization.

In the previous subsection, we find that the dissension vectors from one track might suffice to describe the quantumness of a multiparticle state. Below, we discuss that for four-qubit states, some dissension vectors from track-II are necessary to distinguish

Table 2 Track-I and track-II dissension vectors for a few three-qubit states

| State  | $\vec{\delta}_1^1$ | $\vec{\delta}_1^2$ | $\vec{\delta}_1^3$ |
|--------|---------------------|--------------------|---------------------|
| $\rho_{pro}$ | $[0, 0, 0]$ | $[0, 0, 0]$ | $[0, 0, 0]$ |
| $\rho_{ccc}$ | $[-2, -2, -2]$ | $[0, 0, 0]$ | $[-3, -3, -3]$ |
| $\rho_{ccq}$ | $[-1.2, -1.2, -1.6]$ | $[0, 0, 0]$ | $[-1.8, -1.8, -2.6]$ |
| $\rho_{qqc}$ | $[-0.99, -0.99, -0.78]$ | $[0, 0, 0.22]$ | $[-1.6, -1.6, -1.17]$ |
| $\rho_{qqq}$ | $[-0.67, -0.67, -0.67]$ | $[0.15, 0.15, 0.15]$ | $[-1.06, -1.06, -1.06]$ |
| $\rho_{G3}$ | $[-2, -2, -2]$ | $[1, 1, 1]$ | $[-3, -3, -3]$ |
| $\rho_{W}$ | $[-1.08, -1.08, -1.08]$ | $[0.92, 0.92, 0.92]$ | $[-1.75, -1.75, -1.75]$ |

Note that we have not considered $\delta_2^3$ because it is simply equal to the sum of elements in $\delta_1^2$. 

 Springer
them. Hence, we may stick with either track-II dissension vectors or consider more relevant dissension vectors from both the tracks. The second choice seems more useful as we know the dissension vector which contains bipartite discords is better in characterizing states than the symmetric discord one. For four-qubit case, we are considering them. Hence, we may stick with either track-II dissension vectors or consider more relevant dissension vectors from both the tracks. The second choice seems more useful as we know the dissension vector which contains bipartite discords is better in characterizing states than the symmetric discord one. For four-qubit case, we are considering them. Hence, we may stick with either track-II dissension vectors or consider more relevant dissension vectors from both the tracks. The second choice seems more useful as we know the dissension vector which contains bipartite discords is better in characterizing states than the symmetric discord one. For four-qubit case, we are considering

the W-state, \(|W\rangle = \frac{1}{\sqrt{2}}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)\), and the \(\Omega\)-state, \(|\Omega\rangle = \frac{1}{\sqrt{2}}(|0\psi^+\rangle + |1\psi^-\rangle)\), where \(|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)\).

From Table 3, it is clear that the state \(\rho_{pro}\) has no quantumness. The state \(\rho_{cl}\) has only classical correlations. Local quantumness is increasing in other listed separable states (\(\rho_{3qC}, \rho_{2qC}, \rho_{c3q}, \) and \(\rho_{4q}\)). The behavior of \(\vec{\delta}_1\) is same as that of \(\vec{\delta}_2\) for three-qubit states. It captures the quantumness of a state. It becomes larger for a more quantum state. The behavior of \(\vec{\delta}_2\) is similar to \(\vec{\delta}_1\) in the three-qubit case. It is more negative for states with more correlations, whether classical or quantum. This suggests that for a \(n\)-qubit state, dissension vectors based on \((n - 1)\)-qubit measurement quantify quantumness of a state and become more positive with increasing quantumness. On the other hand, dissension vectors based on \((n - 2)\)-qubit measurement quantify

![Fig. 5](Color online) The figure shows how the dissension vectors behave as a function of mixing parameter \(p\) for the three-qubit mixed states \(\rho_{Gw}\) and \(\rho_{uer}\). The subfigures [(i)–(iii)] depict the behavior of the dissensions for the state \(\rho_{Gw}\) and [(iv)–(vi)] for \(\rho_{uer}\) where subfigures (i) and (iv) depict \(\delta_1^1\); (ii) and (v) depict \(\delta_1^2\) and (iii) and (vi) depict \(\delta_2^2\). (Note that we have plotted one of the elements from each dissension vectors. This is because within a vector each element is same as the states are symmetric.)
correlation, both classical and quantum, and become more negative with increasing correlations.

The four-qubit mixed state, Werner-like state, $\rho_{Wer} = \frac{(1-p)}{16} I + p \rho_{G_4}$, shows the similar features. The dissension vectors are plotted in Fig. 6. The plots indicate that as $p$ approaches unity (i.e., state becoming more entangled), the dissension vectors, $\delta_1$, and $\delta_2^2$, are becoming more negative, while $\delta_3^1$ is becoming more positive.

6 Some features of dissension vectors

Here in this section, we look into some attributes of dissension vectors.

6.1 Why track-II is necessary?

Let us consider the following states: $|\psi\rangle_{1234} = |0\rangle|G_3\rangle$, $\rho^0_{1234} = \frac{1}{2}(|0G_3\rangle\langle 0G_3| + |1G_3\rangle\langle 1G_3|)$ and $\rho^G_{1234} = \frac{1}{2}(|0G_3\rangle\langle 0G_3| + |+G_3\rangle\langle +G_3|)$. These states are different in the sense that the first state is product in $1|234$ cut, but other two are not. The last one has local quantumness in $1|234$ cut. From Table 4, it is clear that only using track-I dissension vectors, we cannot distinguish the above states, but it will be possible to
Table 4 Track-I and track-II dissension vectors for a few specific four-qubit states

| State | $\bar{\delta}_1^1$ | $\bar{\delta}_1^2$ | $\bar{\delta}_3^1$ | $\bar{\delta}_2^2$ |
|-------|-------------------|-------------------|-------------------|-------------------|
| $\rho_{1,234}$ | $[-1, 0, 0, 0]$ | $[0, -3, -3, -2]$ | $[0, 1, 1, 1]$ | $[-3, -6, -6, -5]$ |
| $\rho^c_{1,234}$ | $[-1, 0, 0, 0]$ | $[0, -3, -3, -2]$ | $[0, 1, 1, 1]$ | $[-6, -7, -7, -6]$ |
| $\rho^q_{1,234}$ | $[-1, 0, 0, 0]$ | $[0, -3, -3, -2]$ | $[0, 1, 1, 1]$ | $[-4.8, -6.6, -6.6, -5.6]$ |

Table 5 Track-I average dissensions for few three-qubit states: $|\psi\rangle_{i,j,k} = |0\rangle_i |G_2\rangle_{jk}$, $\rho_{ccq}$ and $\rho_{qqc}$

| State | $\langle \delta_1^1 \rangle$ | $\langle \delta_1^2 \rangle$ | $\langle \delta_2^2 \rangle$ |
|-------|-------------------|-------------------|-------------------|
| $|\psi\rangle_{i,j,k}$ | $-\frac{1}{3}$ | $\frac{2}{3}$ | $-\frac{2}{3}$ |
| $\rho_{ccq}$ | $-\frac{4}{3}$ | $0$ | $-\frac{31}{15}$ |
| $\rho_{qqc}$ | $-0.92$ | $0.07$ | $-1.46$ |

The table shows that one will not be able to distinguish the states from the track-I dissension vectors, whereas one will be if he/she considers track-II dissension vectors. (Note that for this particular case, $\bar{\delta}_1^1 \equiv \bar{\delta}_1^2$.)

Table 5 introduces the concept of average dissension that is calculated as an average of the dissension vectors. Let us define the average dissension quantities:

$$
\langle \delta_1^\ell \rangle = \frac{1}{n} \sum_{k=1}^{n} \delta_1^{\ell,k},
$$

where $\ell = 1, 2$ denotes the track in which we are calculating them. Similarly, we can have different quantities like $\{\langle \delta_1^\ell \rangle; i = 1, 2, \ldots, n - 1\}$, except the quantity, $\delta_2^n$, which is a symmetric quantity and sum of all bipartite discord. Here, we will illustrate these measures particularly for some three-qubit states.

Table 5 shows the average dissension quantities for different states. As expected, once we look at the average properties, some states cannot be distinguished. For example, $\rho_{ccq}$, $\rho_{qcq}$, and $\rho_{qcc}$ have same average quantumness but have different entries in the dissension vector $\delta_1^2$, i.e., $\{-1.8, -1.8, -2.6\}$, $\{-1.8, -2.6, -1.8\}$ and $\{-2.6, -1.8, -1.8\}$, respectively. (Same problem arises for the vector $\delta_1^1$.) Similar analysis goes for the states ($|\psi\rangle_{1,23}$, distinguish them if we consider the dissension vector $\delta_2^2$ from track-II. So, sometimes the track-II dissension vectors are needed.

### 6.2 Average quantumness of multiqubit states

A vector measure characterizes a state in a fine-grained manner. Sometimes, one may be interested in average correlation properties. For some quantum tasks, average properties may be relevant. For such tasks, two states with different vector measures, but same “average” properties may both be suitable. Therefore, in this section, we consider average of the dissension vectors. We will investigate whether our measures are good in characterizing the states if we take average in a particular dissension quantity. Let us define the average dissension quantities:

$$
\langle \delta_1^\ell \rangle = \frac{1}{n} \sum_{k=1}^{n} \delta_1^{\ell,k},
$$

where $\ell = 1, 2$ denotes the track in which we are calculating them. Similarly, we can have different quantities like $\{\langle \delta_1^\ell \rangle; i = 1, 2, \ldots, n - 1\}$, except the quantity, $\delta_2^n$, which is a symmetric quantity and sum of all bipartite discord. Here, we will illustrate these measures particularly for some three-qubit states.

Results are presented in Table 5. As expected, once we look at the average properties, some states cannot be distinguished. For example, $\rho_{ccq}$, $\rho_{qcq}$, and $\rho_{qcc}$ have same average quantumness but have different entries in the dissension vector $\delta_1^2$, i.e., $\{-1.8, -1.8, -2.6\}$, $\{-1.8, -2.6, -1.8\}$ and $\{-2.6, -1.8, -1.8\}$, respectively. (Same problem arises for the vector $\delta_1^1$.) Similar analysis goes for the states ($|\psi\rangle_{1,23}$,
Fig. 7 (Color online) The figure depicts the behavior of the “dissension vectors” of ρ_{cl} under non-unital channel parameter n. Here, the non-unital channel (Λ_{nu}) is applied on the first qubit. The dissension vectors \( \delta_1^1 \) (blue dashed line), \( \delta_1^2 \) (red solid line), and \( \delta_2^1 \) (black solid line) are nonzero for finite value of n. (Note that we have plotted one of the elements from each dissension vectors. This is because within a vector each elements is same.)

Therefore, it will be always advantageous if we consider dissension vectors instead of their averages.

### 6.3 Behavior of quantumness under local noise

For almost all quantum processing devices, effect of noise is inevitable. This leads us to examine the behavior of our dissension vector under local noise. From a property of a measure of quantum correlations, e.g., \( Q \), for the bipartite state \( \rho_{12} \),

\[
Q(\rho_{12}) \geq Q(\Lambda_{12}[\rho_{12}]),
\]

where \( \Lambda_{12} = \Lambda_1 \otimes \Lambda_2 \) are local channels. Under global operations, the situation may be different. One can create or increase entanglement under such operations.

It is evident that our measures are also affected by the local noise. In this respect, we can define two important classes of channels—a unital/semiclassical channel \( \Lambda_{u/sc} \) is defined as \( \Lambda_{u/sc}(I_2) = I_2 \), while for a non-unital channel \( \Lambda_{nu}, \Lambda_{nu}(\frac{1}{2}) \neq \frac{1}{2} \). Streltsov et al. [24] have shown that a local quantum channel acting on a single qubit can create “quantumness” in a multiqubit system iff it is neither semiclassical nor unital. This result holds for the dissension vector also. In our vector-type measure, at least one of the elements will be affected. For example, let us consider a classical state \( \rho_{cl} = \frac{1}{2}(|0000\rangle\langle0000| + |1111\rangle\langle1111|) \). Now, application of non-unital channel \{ \( E_1 = |0\rangle\langle0|, E_2 = |n\rangle\langle1| \) with \( |n\rangle = \frac{1}{1+n^2}(|0\rangle + n|1\rangle) \) \( n \in \mathbb{R} \) on any subsystem will make the state non-classical and will have nonzero element in the vector (Fig. 7).

### 7 Physical applications of dissension vectors

The state of a n-bit classical computer can always be expressed by n-qubit classically correlated states. This is because a classically correlated states are described by joint probability distribution and have specific form of dissension vectors. Also, there

\[ \text{Springer} \]
exist states other than classically correlated states. (Here, we are excluding product states and maximally mixed states.) Will the use of such states yield some quantum enhancement in the computation? Below, we will show some examples where dissension vectors may have a role to play.

### 7.1 BB84 protocol

In 1984, Bennett and Brassard developed a cryptographic protocol [42] where they utilized the quantum no-cloning theorem [43]. In this protocol, Alice prepares a qubit in one of the four states \{\mid 0 \rangle, \mid 1 \rangle, \mid + \rangle, \mid - \rangle\} and sends it to Bob. Bob measures the state in either \{\mid 0 \rangle, \mid 1 \rangle\} basis or in \{\mid + \rangle, \mid - \rangle\} randomly. This process goes for many rounds, and at the end, both publicly declare the basis of preparation and measurement, respectively. There are two possible outcomes—1. basis match, i.e., perfect correlations, and 2. basis do not match. If in second case they find any correlation, then they may conclude that someone (Eve) was eavesdropping in between the state transmission. Otherwise, they will use the correlated bit string (of case 1) as one-time pad, the fundamental object of cryptography.

In the whole process, the average state of Alice and Bob is

\[
\rho_{BB84} = \frac{1}{4} (\mid 00 \rangle \langle 00 \mid + \mid 11 \rangle \langle 11 \mid + \mid 0+ \rangle \langle 0+ \mid + \mid 1- \rangle \langle 1- \mid).
\] (30)

The state \(\rho_{BB84}\) is a mixture of two maximally classically correlated states in two different bases. The state has no entanglement. The state has nonzero local quantumness in Bob’s part. This means local quantumness on Bob’s side is providing the security in the protocol. Let us examine it.

Assume that to read the key, Eve will perform a universal state independent Buzek–Hillary (B–H) cloning operations\(^1\) [56,57] on the state, while it is transmission from Alice to Bob. Now, before applying the cloning operation the average initial state of Alice, Bob, and Eve is

\[
\rho_i = \rho_{BB84} \otimes \mid 0 \rangle \langle 0 \mid_{eve}.
\]

After the Eve’s action, the state becomes

\[
\rho_f = \frac{1}{4} \left[ \mid 000 \rangle \langle 000 \mid + \mid 111 \rangle \langle 111 \mid + \frac{2}{3} \mathbb{I}_2 \otimes \mid \phi^+ \rangle \langle \phi^+ \mid \\
+ \frac{1}{3} (\mid 011 \rangle \langle 011 \mid + \mid 100 \rangle \langle 100 \mid + \sigma_z \otimes (\mid \psi^+ \rangle \langle \phi^+ \mid + \mid \phi^+ \rangle \langle \psi^+ \mid)) \right],
\] (31)

where \(\mid \psi^+ \rangle = \frac{1}{\sqrt{2}} (\mid 00 \rangle + \mid 11 \rangle), \mid \phi^+ \rangle = \frac{1}{\sqrt{2}} (\mid 01 \rangle + \mid 10 \rangle), \sigma_z = \mid 0 \rangle \langle 0 \mid - \mid 1 \rangle \langle 1 \mid,\) and \(\mathbb{I}_2\) is a identity matrix of order 2. The dissension vectors for the states \(\rho_i\) and \(\rho_f\) are given in Table 6.

---

\(^1\) The universal state independent B–H cloning operations are defined as

00 → \(\sqrt{2}(00 + \phi^+), 10 → \sqrt{2}(11 + \phi^+)\), where the first ket in the left-hand side is the target and the second one is the ancilla where the target ket will be copied.
Table 6 Dissensions for the states in the BB84 protocol

| State | $\vec{\delta^1}$ | $\vec{\delta^2}$ | $\vec{\delta^3}$ |
|-------|-----------------|-----------------|-----------------|
| $\rho_i$ | $\{0, 0, -0.40\}$ | $\{0, 0, 0\}$ | $\{0, 0, -0.79\}$ |
| $\rho_f$ | $\{-0.40, -0.24, -0.24\}$ | $\{0, 0.29, 0.29\}$ | $\{-0.73, -0.41, -0.41\}$ |

From dissension vectors, it is evident that the correlations in the state $\rho_f$ have increased due to Eve’s action. And hence, the loss of security of BB84 protocol may be decided based on dissension vectors.

7.2 Dissension in Grover search algorithm

Grover search algorithm was introduced for accelerating the data search from an “unstructured database” \cite{44,45}. It was believed that entanglement may be necessary to achieve such speedup \cite{46}. Later, it was shown that the entanglement is not directly related to the probability of success in the search \cite{47}. Also, it is not clear whether there is a relationship between the probability of success and the correlations that go beyond entanglement (particularly captured by well-known measure, quantum discord) \cite{48}. Recently, it was suggested that the success probability relies on the depletion of quantum coherence \cite{49,50}.

Here, we investigate possible role of the dissension vectors in Grover search algorithm. In this algorithm, the initial $n$-qubit database can be expressed by

\[
|\psi_0\rangle = \sqrt{\frac{j}{2^n}} |\chi\rangle + \sqrt{1 - \frac{j}{2^n}} |\chi^\perp\rangle,
\]

where $j$ are the number of solutions in Grover search algorithm and $|\chi\rangle = \frac{1}{\sqrt{j}} \sum_x |x\rangle$ ($\{|\chi\rangle, |\chi^\perp\rangle\}$ form a basis). In the next step, a Grover operation (called iteration) is applied repeatedly to improve the proportion of solutions. The Grover operation, $G = A\mathcal{O}$, consists of Oracle $\mathcal{O} = \mathbb{I} - 2\langle \chi | \chi \rangle$ and an inversion operation $A = 2\langle \psi_0 | \psi_0 \rangle - \mathbb{I}$. After $r$ iterations, the global state takes the form

\[
|\psi_r\rangle = G^r |\psi_0\rangle = \sin \theta_r |\chi\rangle + \cos \theta_r |\chi^\perp\rangle,
\]

where $\theta_r = (r + \frac{1}{2})\beta$ and $\beta = 2 \arctan \sqrt{\frac{j}{2^n-j}}$. The final step is to obtain $|\chi\rangle$ with high probability by performing measurement on $|\psi_r\rangle$. The probability of the success is given by $P_{\text{succ}} = \sin^2 \theta_r$. The optimal time to stop the iteration is at $r_{\text{opt}} = C_I[\frac{\pi - \beta}{2\beta}]$ times, where $C_I[m]$ denotes the closest integer to $m$.

We will only consider the simplest situation of single solution (i.e., $j = 1$) and will assume that the solution is located at $|0\rangle$. Then, the final density matrix generated by
Fig. 8 (Color online) The figure depicts the behavior of the dissension vectors of $\rho_r$ with the Grover iteration $r$. The inset figure shows the plot of success probability with $r$. It indicates that while the dissension vectors $\vec{\delta}^1_1$ and $\vec{\delta}^2_1$ are decreasing with $r$ and reaches its minimum at $r = 4$, the $\vec{\delta}^1_6$ is increasing up to $r = 4$ but remains always positive. It shows that the correlations during the Grover search increase with $r$ and reach its maximum and then start decreasing.

Grover search has the following form:

$$\rho_r = \begin{pmatrix} a^2 & ab & ab & ab & \cdots \\ ab & b^2 & b^2 & b^2 & \cdots \\ ab & b^2 & b^2 & b^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{2^n \times 2^n}, \tag{34}$$

where $a = \sin \theta_r$ and $b = \frac{1}{\sqrt{2^n-j}} \cos \theta_r$. And the reduced density matrix of any $k$-qubits is defined as

$$\rho^k_r = \begin{pmatrix} a^2 + (2^{n-k} - 1)b^2 & ab + (2^{n-k} - 1)b^2 & ab + (2^{n-k} - 1)b^2 & ab + (2^{n-k} - 1)b^2 & \cdots \\ ab + (2^{n-k} - 1)b^2 & 2^{n-k}b^2 & 2^{n-k}b^2 & 2^{n-k}b^2 & \cdots \\ ab + (2^{n-k} - 1)b^2 & 2^{n-k}b^2 & 2^{n-k}b^2 & 2^{n-k}b^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{2^k \times 2^k}. \tag{35}$$

We investigate behavior of the dissension vectors of the state $\rho_r$ for $n = 7$. The quantum correlations in the state increase with the Grover iterations and reach its maximum at some $r$ value. (Here, it is 4.) Then, it starts to decrease and becomes zero when the final projective measurement is performed to obtain the solution (Fig. 8).

Recently, in Ref. [51], it was shown that the some modified form of dissensions may be useful in characterizing the average state merging cost. Apart from that the
dissensions may be useful in multiparticle entanglement distributions [52,53], quantum cryptography [55], and quantum interferometry [54].

8 Conclusion

By considering the dissension vector as a measure of the quantumness of a multiqubit state, we have argued that a vector quantity, as a fine-grained measure, does a better job in characterizing and quantifying the quantum properties of a state. We considered two tracks of these measures for $n$-qubit states. In particular, for three-qubit and four-qubit systems, we showed how various classes of states can be distinguished and characterized using these measures. In particular, we saw that in the case of $n$-qubit states, for $(n−2)$-qubit measurements, the dissension vectors $\vec{\delta}_{n-2}^1$ and $\vec{\delta}_{n-2}^2$ quantify correlations, both classical and quantum. More correlated states have more negative values for these vectors. On the other hand, for $(n−1)$-qubit measurements, the dissension vector $\vec{\delta}_{n-1}^1$ quantifies quantumness of a state and is always positive. We have also considered the effect of local noise and how to quantify average quantumness. We also discussed applications of these measures in the context of BB84 protocol and Grover search algorithm.

Acknowledgements Author SS would like to thank Mr. Abhishek Deshpande, Dr. Indranil Chakraborty, and Prof. V. Ravishankar for having useful discussions. We thank anonymous referee for his valuable comments. PA acknowledges the support from the Department of Science and Technology, India, through the project DST/ICPS/QuST/Theme-1/2019.

References

1. Modi, K., Brodutch, A., Cable, H., Paterek, T., Vedral, V.: The classical-quantum boundary for correlations: discord and related measures. Rev. Mod. Phys. 84, 1655–1707 (2012)
2. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. 81, 865–942 (2009)
3. Jozsa, R.: An introduction to measurement based quantum computation. arXiv:quant-ph/0508124 (2005)
4. Chaves, R., de Melo, F.: Noisy one-way quantum computations: the role of correlations. Phys. Rev. A 84, 022324 (2011)
5. Datta, A., Shaji, A., Caves, C.M.: Quantum discord and the power of one qubit. Phys. Rev. Lett. 100, 050502 (2008)
6. Gisin, N., Ribordy, G., Tittel, W., Zbinden, H.: Quantum cryptography. Rev. Mod. Phys. 74, 145–195 (2002)
7. Giovannetti, V., Lloyd, S., Maccone, L.: Quantum metrology. Phys. Rev. Lett. 96, 010401 (2006)
8. Ollivier, H., Zurek, W.H.: Quantum discord: a measure of the quantumness of correlations. Phys. Rev. Lett. 88, 017901 (2001)
9. Henderson, L., Vedral, V.: Classical, quantum and total correlations. J. Phys. A Math. Gen. 34(35), 6899 (2001)
10. Horodecki, M., Horodecki, P., Horodecki, R., Oppenheim, J., De Sen, A., Sen, U., Synak-Radtke, B.: Local versus nonlocal information in quantum-information theory: formalism and phenomena. Phys. Rev. A 71, 062307 (2005)
11. Usha Devi, A.R., Rajagopal, A.K.: Generalized information theoretic measure to discern the quantumness of correlations. Phys. Rev. Lett. 100, 140502 (2008)
12. Modi, K., Paterek, T., Son, W., Vedral, V., Williamson, M.: Unified view of quantum and classical correlations. Phys. Rev. Lett. 104, 080501 (2010)
13. Okrasa, M., Walczak, Z.: Quantum discord and multipartite correlations. Europhys. Lett. 96(6), 60003 (2011)
14. Rulli, C.C., Sarandy, M.S.: Global quantum discord in multipartite systems. Phys. Rev. A 84, 042109 (2011)
15. Chakrabarty, I., Agrawal, P., Pati, A.K.: Quantum dissension: generalizing quantum discord for three-
quibit states. Eur. Phys. J. D 65(3), 605–612 (2011)
16. Laflamme, R., Cory, D., Negrevergne, C., Viola, L.: NMR quantum information processing and entan-
glement. Quantum Inf. Comput. 2, 166–176 (2002)
17. Madhok, V., Datta, A.: Interpreting quantum discord through quantum state merging. Phys. Rev. A 83, 032323 (2011)
18. Horodecki, M., Horodecki, K., Horodecki, P., Horodecki, R., Oppenheim, J., De Sen, A., Sen, U.: Local information as a resource in distributed quantum systems. Phys. Rev. Lett. 90, 100402 (2003)
19. Oppenheim, J., Horodecki, M., Horodecki, P., Horodecki, R.: Thermodynamical approach to quanti-
fying quantum correlations. Phys. Rev. Lett. 89, 180402 (2002)
20. Rajagopal, A.K., Rendell, R.W.: Separability and correlations in composite states based on entropy methods. Phys. Rev. A 66, 022104 (2002)
21. Dakić, B., Vedral, V., Brukner, Č.: Necessary and sufficient condition for nonzero quantum discord. Phys. Rev. Lett. 105, 190502 (2010)
22. Agrawal, P., Chakrabarty, I., Sazim, S., Pati, A.K.: Local, nonlocal quantumness and information theoretic measures. Int. J. Quantum Inform. 14, 1640034 (2016). arXiv:1502.00857
23. Bellomo, G., Plastino, A., Plastino, A.R.: Quantumness and the role of locality on quantum correlations. Phys. Rev. A 93, 062322 (2016)
24. Streltsov, A., Kampermann, H., Bruß, D.: Behavior of quantum correlations under local noise. Phys. Rev. Lett. 107, 170502 (2011)
25. Gian Luca Giorgi: Quantum discord and remote state preparation. Phys. Rev. A 88, 022315 (2013)
26. Horodecki, P., Tuziemski, J., Mazurek, P., Horodecki, R.: Can communication power of separable correlations exceed that of entanglement resource? Phys. Rev. Lett. 112, 140507 (2014)
27. Giorgi, G.L., Bellomo, B., Galve, F., Zambrini, R.: Genuine quantum and classical correlations in multipartite systems. Phys. Rev. Lett. 107, 190501 (2011)
28. Groisman, B., Popescu, S., Winter, A.: Quantum, classical, and total amount of correlations in a quantum state. Phys. Rev. A 72, 032317 (2005)
29. Walczak, Z.: Total correlations and mutual information. Phys. Lett. A 373, 1818–1822 (2009). arXiv:0806.4861
30. Li, N., Luo, S.: Total versus quantum correlations in quantum states. Phys. Rev. A 76, 032327 (2007)
31. Cerf, N.J., Adami, C.: Negative entropy and information in quantum mechanics. Phys. Rev. Lett. 79, 5194–5197 (1997)
32. Cerf, N.J., Adami, C.: Quantum extension of conditional probability. Phys. Rev. A 60, 893–897 (1999)
33. Horodecki, M., Oppenheim, J., Winter, A.: Partial quantum information. Nature 436, 673–676 (2005). arXiv:quant-ph/0505062
34. del Rio, L., Aberg, J., Renner, R., Dahlsten, O., Vedral, V.: The thermodynamic meaning of negative entropy. Nature 471, 61–63 (2011)
35. Horodecki, M., Oppenheim, J., Winter, A.: Quantum state merging and negative information. Commun. Math. Phys. 269(1), 107–136 (2007)
36. Cover, T.M., Thomas, J.A.: Elements of Information Theory. Wiley, New York (1991)
37. Herbut, F.: On mutual information in multipartite quantum states and equality in strong subadditivity of entropy. J. Phys. A Math. Gen. 37(10), 3535 (2004)
38. Cerf, N.J., Massar, S., Schneider, S.: Multiparticle classical and quantum secrecy monotones. Phys. Rev. A 66, 042309 (2002)
39. Kumar, A.: Multiparty quantum mutual information: an alternative definition. Phys. Rev. A 96, 012332 (2017)
40. Ding, D., Hayden, P., Walter, M.: Conditional mutual information of bipartite unitaries and scrambling. J. High Energy Phys. 12, 145 (2016). arXiv:1608.04750
41. Sharma, K., Wakakuwa, E., Wilde, M.M.: Conditional quantum one-time pad. arXiv e-prints arXiv:1703.02903 (2017)
42. Bennett, C.H., Brassard, G.: Quantum cryptography: Public key distribution and coin tossing. In: Proceedings of IEEE International Conference on Computers, Systems and Signal Processing, vol. 175, p. 8. New York (1984)
43. Wootters, W.K., Zurek, W.H.: A single quantum cannot be cloned. Nature 299, 802–803 (1982)
44. Grover, L.K.: Quantum mechanics helps in searching for a needle in a haystack. Phys. Rev. Lett. 79, 325–328 (1997)
45. Childs, A.M., van Dam, W.: Quantum algorithms for algebraic problems. Rev. Mod. Phys. 82, 1–52 (2010)
46. Jozsa, R., Linden, N.: On the role of entanglement in quantum-computational speed-up. Proc. R. Soc. Lond. A 459, 2011 (2003)
47. Braunstein, S.L., Pati, A.K.: Speed-up and entanglement in quantum searching. Quantum Inform. Comput. 2(5), 399–409 (2002)
48. Cui, J., Fan, H.: Correlations in the Grover search. J. Phys. A Math. Theor. 43, 045305 (2010)
49. Shi, H.-L., Liu, S.-Y., Wang, X.-H., Yang, W.-L., Yang, Z.-Y., Fan, H.: Coherence depletion in the Grover quantum search algorithm. arXiv e-prints arXiv:1610.08656 (2016)
50. Anand, N., Pati, A.K.: Coherence and entanglement monogamy in the discrete analogue of analog Grover search. arXiv e-prints arXiv:1611.04542 (2016)
51. Chakrabarty, I., Deshpande, A., Chatterjee, S.: Quantum residual correlation: interpreting through state merging. arXiv e-prints arXiv:1410.7067 (2014)
52. Streltsov, A., Kampermann, H., Bruß, D.: Quantum cost for sending entanglement. Phys. Rev. Lett. 108, 250501 (2012)
53. Chuan, T.K., Maillard, J., Modi, K., Paterek, T., Paternostro, M., Piani, M.: Quantum discord bounds the amount of distributed entanglement. Phys. Rev. Lett. 109, 070501 (2012)
54. Girolami, D., Souza, A.M., Giovannetti, V., Tufarelli, T., Filgueiras, J.G., Sarthour, R.S., Soares-Pinto, D.O., Oliveira, I.S., Adesso, G.: Quantum discord determines the interferometric power of quantum states. Phys. Rev. Lett. 112, 210401 (2014)
55. Stefano, P.: Quantum discord as a resource for quantum cryptography. Sci. Rep. 4, 6956 (2014)
56. Bužek, V., Hillery, M.: Quantum copying: beyond the no-cloning theorem. Phys. Rev. A 54, 1844–1852 (1996)
57. Bužek, V., Hillery, M.: Universal optimal cloning of arbitrary quantum states: from qubits to quantum registers. Phys. Rev. Lett. 81, 5003–5006 (1998)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.