Electrostatics in a Schwarzschild black hole pierced by a cosmic string

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Abstract

We explicitly determine the expression of the electrostatic potential generated by a point charge at rest in the Schwarzschild black hole pierced by a cosmic string. We can then calculate the electrostatic self-energy. From this, we find again the upper entropy bound for a charged object by employing thermodynamics of the black hole.

1 Introduction

It is well known for a long time that a point charge at rest in a static spacetime feels an electrostatic self-force. The calculation is performed by considering the global electrostatic potential determined as the solution of the Maxwell equations in the background metric of the spacetime. However, it would seem that its existence was a curiosity. The situation has recently undergone a change when Bekenstein and Mayo [1] and Hod [2] have derived the upper entropy bound for a charged object by requiring the validity of thermodynamics of the Reissner-Nordström black hole. Their proof takes really into account the expression of the electrostatic self-energy for a point charge at rest in a Schwarzschild black hole which has been previously determined in closed form [3, 4, 5].

The purpose of this work is to extend these results to a new case where it is possible to determine explicitly the electrostatic self-energy. We consider the spacetime, introduced by Aryal et al [6], which describes a Schwarzschild black hole pierced by a cosmic string. It represents a straight cosmic string, infinitely thin, passing through a spherically symmetric black hole. It is obtained by cutting a wedge in the Schwarzschild geometry. So, in the coordinate system \((t, r, \theta, \varphi)\) with \(0 \leq \varphi < 2\pi\), the metric can be written

\[
\begin{align*}
\text{ds}^2 &= - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + B^2 r^2 \sin^2 \theta d\varphi^2
\end{align*}
\]

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where $m$ is a positive parameter and $B$ is related to the linear mass density $\mu$ of the cosmic string by $B = 1 - 4\mu$ with $0 < B < 1$. We only consider the spacetime outside the horizon, i.e. for $r > 2m$.

In section 2, we summarise the Maxwell equations in metric (1) and, in the case $1/2 < B < 1$, we give explicitly the expression of the electrostatic potential generated by a point charge at rest. The proof that this expression obeys the electrostatic equation is fulfilled in section 3. Taking into account the found expression of the electrostatic self-energy, we derive in section 4 the entropy bound for a charged object by employing thermodynamics of the black hole. We add some concluding remarks in section 5.

## 2 Electrostatic potential

The Maxwell equations in metric (1), having as source a point charge $e$ located at the position $(r_0, \theta_0, \varphi_0)$ with $r_0 > 2m$, reduce to

$$\partial_i \left( \sqrt{-g} F^{i0} \right) = -\frac{e}{4\pi} \delta(r - r_0) \delta(\theta - \theta_0) \delta(\varphi - \varphi_0) \quad \text{with} \quad F^{i0} = \partial_i A_0 \quad (2)$$

where $A_0$ is the electrostatic potential. According to (2), the electrostatic equation for $A_0$ can be written

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} A_0 \right) + \frac{1}{r(r-2m) \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} A_0 \right) + \frac{1}{B^2 r(r-2m) \partial^2 \varphi} A_0 = -\frac{e}{4\pi B r^2 \sin \theta} \delta(r - r_0) \delta(\theta - \theta_0) \delta(\varphi - \varphi_0) \quad (3)$$

We point out that the application of the Gauss theorem to equation (2) yields

$$A_0(r, \theta, \varphi) \sim \frac{e}{B r} \quad \text{as} \quad r \to \infty \quad (4)$$

if there is no electric flux through the horizon. Furthermore, we require that the electromagnetic field is regular at the horizon by imposing that $F^{\mu \nu} F_{\mu \nu}$ is finite as $r \to 2m$.

We limit ourselves to the case where $1/2 < B < 1$ which is physically justified for a cosmic string since $\mu \ll 1$. Without loss of generality, we put $\varphi_0 = \pi$ to simplify. The expression of the electrostatic potential $A_0$ satisfying equation (3) with the desired boundary conditions can be expressed as the following sum

$$A_0(r, \theta, \varphi) = V^*(r, \theta, \varphi) + V_B(r, \theta, \varphi) + \frac{em}{B r r_0} \quad (5)$$

where the expressions of $V^*$ and $V_B$ are given below.

To express the potential $V^*$, we must consider the regions of the spacetime delimited by the hypersurfaces $\varphi = \text{constant}$ as shown on Figure 1. We have

$$V^*(r, \theta, \varphi) = \begin{cases} V_C(r, \sigma_0(\theta, \varphi)) + V_C[r, \sigma_1(\theta, \varphi)] & 0 < \varphi < \pi/B - \pi \\ V_C[r, \sigma_0(\theta, \varphi)] & \pi/B - \pi < \varphi < 3\pi - \pi/B \\ V_C[r, \sigma_0(\theta, \varphi)] + V_C[r, \sigma_{-1}(\theta, \varphi)] & 3\pi - \pi/B < \varphi < 2\pi \end{cases} \quad (6)$$
where $V_C$ is the Copson potential [7] which is a solution to electrostatic equation (3) with $B = 1$, i.e. for the Schwarzschild black hole. Its expression is

$$V_C[r, \sigma] = \frac{e}{rr_0} \frac{(r - m)(r_0 - m) - m^2 \sigma}{[(r - m)^2 + (r_0 - m)^2 - m^2 - 2(r - m)(r_0 - m)\sigma + m^2\sigma^2]^{1/2}}$$

The variables $\sigma_n$ in formula (6) are the following functions of $\theta$ and $\varphi$

$$\begin{align*}
\sigma_0(\theta, \varphi) &= \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos B(\varphi - \pi), \\
\sigma_1(\theta, \varphi) &= \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos B(\varphi + \pi), \\
\sigma_{-1}(\theta, \varphi) &= \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos B(\varphi - 3\pi).
\end{align*}$$

The potential $V_B$ is given by the integral expression

$$V_B(r, \theta, \varphi) = \frac{1}{2\pi B} \int_0^\infty V_C[r, k(\theta, x)] F_B(\varphi, x) \, dx$$

where the function $k$ is given by

$$k(\theta, x) = \cos \theta \cos \theta_0 - \sin \theta \sin \theta_0 \cosh x$$

and the function $F_B$ by

$$F_B(\varphi, x) = -\frac{\sin(\varphi - \pi/B)}{\cosh x/B + \cos(\varphi - \pi/B)} + \frac{\sin(\varphi + \pi/B)}{\cosh x/B + \cos(\varphi + \pi/B)}.$$  

In the Schwarzschild black hole, sum (5) with $V_1 = 0$ and $V^* = V_C$ yields the electrostatic potential that we have already obtained [8]. On the other hand for the cosmic string, i.e. $m = 0$, we find our previous result [9], already known in the case of a wedge in flat space [10, 11].

The electrostatic self-potential in a neighbourhood of the point charge is $A_0(r, \theta, \varphi) - V_C[r, \sigma_0(\theta, \varphi)]$. In consequence, the electrostatic self-energy $W_{self}$ is

$$W_{self}(r_0, \theta_0) = \frac{e^2 m}{2B^2 r_0^2} - \frac{e \sin \pi/B}{2\pi B} \int_0^\infty V_C[r_0, k(\theta_0, x)] \frac{dx}{\cosh x/B - \cos \pi/B}. $$

From (12), we can deduce the electrostatique self-force which has been already obtained in the Schwarzschild black hole [3, 4, 5] and in the cosmic string [12, 13].
3 Checking of the electrostatic solution

We must firstly verify that sum (3) is a solution to equation (3). The potential \( V^* \) is obviously a local solution since the Copson potential \( V_C \) expressed in variables \( r, \theta \) and \( \phi \) with \( \phi = B\varphi \) obeys the electrostatic equation for the Schwarzschild black hole. As a consequence, the function \( V_C[r, k(\theta, x)] \) satisfies

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} V_C \right) + \frac{1}{r(r - 2m)} \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} V_C \right) - \frac{1}{r(r - 2m)} \frac{\partial^2}{\partial x^2} V_C = 0.
\]

Then, according to its expression (9), potential \( V_B \) obeys electrostatic equation (3) without second member if we have

\[
\int_0^\infty \frac{\partial^2}{\partial x^2} V_C[r, k(\theta, x)] F_B(\varphi, x) dx + \frac{1}{B^2} \int_0^\infty V_C[r, k(\theta, x)] \frac{\partial^2}{\partial \varphi^2} F_B(\varphi, x) dx = 0.
\]

Now from expression (11), we verify that

\[
\frac{\partial^2}{\partial x^2} F_B(\varphi, x) + \frac{1}{B^2} \frac{\partial^2}{\partial \varphi^2} F_B(\varphi, x) = 0
\]

which ensures the condition mentioned above after two successive integrations by part. We notice that the potential \( e m / B r r_0 \) is a homogeneous solution to electrostatic equation which is regular at the horizon.

We secondly check that sum (3) is continuous. At \( \varphi = 0 \), it is clear because \( V^*(r, \theta, 0) = V^*(r, \theta, 2\pi) \) since \( \sigma_1(r, \theta, 0) = \sigma_{-1}(r, \theta, 2\pi) \). At \( \varphi = \pi / B - \pi \), we introduce \( \epsilon \) by setting \( \varphi = \pi / B - \pi + \epsilon \) and then the potential \( V_B \) becomes

\[
V_B(r, \theta, \pi / B - \pi + \epsilon) = \frac{\sin \epsilon}{2\pi B} \int_0^\infty V_C[r, k(\theta, x)] \frac{dx}{\cosh x / B - \cos \epsilon}
- \frac{\sin(2\pi / B + \epsilon)}{2\pi B} \int_0^\infty V_C[r, k(\theta, x)] \frac{dx}{\cosh x / B - \cos(2\pi / B + \epsilon)}.
\]

We write down the following integral

\[
\sin \epsilon \int_0^x \frac{dy}{\cosh y - \cos \epsilon} = 2 \arctan \left( \tan \frac{x}{2} \cot \frac{\epsilon}{2} \right) \quad \text{with} \quad \epsilon \neq 0.
\]

By integrating by part the first term of expression (14), we get

\[
\frac{1}{2} V_C[r, k(\theta, \infty)] - \frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial y} V_C[r, k(\theta, By)] \arctan \left( \tan \frac{y}{2} \cot \frac{\epsilon}{2} \right) dy.
\]

But the function \( \arctan \) is bounded by \( \pi / 2 \) and consequently we may take the limit \( \epsilon \to 0 \) inside the integral. We obtain thereby

\[
\frac{1}{2} V_C[r, k(\theta, \infty)] - \frac{1}{2} \{V_C[r, k(\theta, \infty)] - V_C[r, k(\theta, 0)]\} \quad \text{as} \quad \epsilon \to 0 \quad \epsilon > 0.
\]
We have thus prove that integral expression (14) verifies
\[
\lim_{\epsilon \to 0, \epsilon > 0} V_B(r, \theta, \pi/B - \pi + \epsilon) = \frac{1}{2} V_C[r, k(\theta, 0)] - \frac{\sin 2\pi/B}{2\pi B} \int_0^\infty V_C[r, k(By)] \frac{dy}{\cosh y - \cos 2\pi/B}
\]
and another with \(\epsilon < 0\) yielding \(-V_C/2\) in formula (15). On the other hand, the potential \(V^*\) verifies
\[
\lim_{\epsilon \to 0, \epsilon > 0} (V^*(r, \theta, \pi/B - \pi + \epsilon) - V^*(r, \theta, \pi/B - \pi - \epsilon)) = -V_C[r, k(\theta, 0)].
\]
By combining results (15) and (16), we thus obtain that the potentials \(V_B\) and \(V^*\) are both discontinuous at \(\varphi = \pi/B - \pi\) whereas their sum is regular. Of course, the potential \(V^* + V_B\) is also continuous at \(\varphi = 3\pi - \pi/B\) by symmetry. In conclusion, the electrostatic potential \(A_0\) is a smooth function only singular at the position of the point charge. Furthermore, it is easy to show that the derivative of the electrostatic potential \(A_0\) with respect to \(\varphi\) is everywhere continuous.

We thirdly determine the asymptotic form of the electrostatic potential \(A_0\). From expression (3) of \(V^*\), we have immediately
\[
V^*(r, \theta, \varphi) \sim \begin{cases} 
2e(r_0 - m)/rr_0 & 0 < \varphi < \pi/B - \pi \\
e(r_0 - m)/rr_0 & \pi/B - \pi < \varphi < 3\pi - \pi/B \quad \text{as} \quad r \to \infty \\
2e(r_0 - m)/rr_0 & 3\pi - \pi/B < \varphi < 2\pi
\end{cases}
\]
On the other hand, from expression (3) of \(V_B\) we get
\[
V_B(r, \theta, \varphi) \sim \frac{e(r_0 - m)}{rr_0} g(\varphi) \quad \text{as} \quad r \to \infty
\]
with
\[
g(\varphi) = \frac{1}{2\pi} \int_0^\infty \left[ \frac{\sin(\varphi + \pi/B)}{\cosh x + \cos(\varphi + \pi/B)} - \frac{\sin(\varphi - \pi/B)}{\cosh x + \cos(\varphi - \pi/B)} \right] dx
\]
which can be integrated by elementary methods
\[
g(\varphi) = \begin{cases} 
1/B - 2 & 0 < \varphi < \pi/B - \pi \\
1/B - 1 & \pi/B - \pi < \varphi < 3\pi - \pi/B \\
1/B - 2 & 3\pi - \pi/B < \varphi < 2\pi
\end{cases}
\]
By using (17) and (18) with (19), we obtain that the electrostatic potential (3) has the desired asymptotic form (4).

At last, we must verify that the electromagnetic field derived from the electrostatic potential (3) is regular at the horizon. This point results of the fact that \(V_C\) tends to \(e/rr_0\) when \(r \to 2m\).
4 Entropy bound for a charged object

We now consider the spacetime which describes a Reissner-Nordström black hole pierced by a cosmic string. It is obtained by cutting a wedge in the Reissner-Nordström geometry. In the coordinate \((t, r, \theta, \varphi)\) with \(0 \leq \varphi < 2\pi\), the metric can be written

\[
d s^2 = -\left(1 - \frac{2E}{Br} + \frac{q^2}{B^2r^2}\right)dt^2 + \left(1 - \frac{2E}{Br} + \frac{q^2}{B^2r^2}\right)^{-1}dr^2 + r^2d\theta^2 + \frac{B^2r^2}{sin^2 \theta}d\varphi^2
\]

where \(E\) and \(q\) are two parameters. We only consider the spacetime outside the outer horizon, i.e. \(r > (E + \sqrt{E^2 - q^2})/B\) by assuming that \(E^2 > q^2\). Following [6, 14], we interpret \(E\) as the energy of the black hole. Clearly, \(q\) is the electric charge of the black hole. For \(q = 0\), metric (20) reduces to metric (1) by setting \(m = E/B\). The horizon area \(\mathcal{A}\) of the black hole defined by metric (20) has the expression

\[
\mathcal{A}(E, q) = \frac{4\pi}{B} \left(E + \sqrt{E^2 - q^2}\right)^2
\]

and the entropy \(S_{BH}\) of the black hole is given by

\[
S_{BH}(E, q) = \frac{1}{4} \mathcal{A}(E, q).
\]  

The Reissner-Nordström black hole pierced by a cosmic string linearised with respect to its electric charge \(q\) is described by metric (1) plus an electromagnetic test field having the electrostatic potential

\[
A_0^{\text{ext}}(r, \theta, \varphi) = \frac{q}{Br}.
\]

Moreover, the black hole entropy (22) reduces to

\[
S_{BH}(E, q) \approx \frac{2\pi}{B} \left(2E^2 - q^2\right).
\]

The original method of Bekenstein [15] for finding the entropy bound for a neutral object in the Schwarzschild black hole has been recently extented for charged object in the Reissner-Nordström black hole [1, 2]. Referring to [1, 2], we recall that the energy \(\mathcal{E}\) of a charged object with a mass \(\mu\), an electric charge \(e\) and a radius \(R\) located at the position \((r_0, \theta_0)\) in metric (1), in presence of the exterior electrostatic potential (23), has the expression

\[
\mathcal{E} = \sqrt{1 - \frac{2E}{Br_0} + \frac{eq}{Br_0}} + W_{\text{self}}(r_0, \theta_0)
\]

where \(W_{\text{self}}\) is the electrostatic self-energy [12]. When the charged object is just outside the horizon, its energy (25), for a very small proper length \(R\), is

\[
\mathcal{E}_{\text{last}} \sim \frac{\mu RB}{4E} + \frac{eq}{2E} + W_{\text{self}}(2E/B, \theta_0) \quad \text{as} \quad R \to 0.
\]
In this state, the system formed by the black hole and the charged object has an entropy $S_{BH}(E, q) + S$ where $S$ is the entropy of the charged object. When the charged object falls in the horizon, the final state is a Reissner-Nordström black hole with the new parameters

$$E_f = E + \mathcal{E}_{last} \quad \text{and} \quad q_f = q + \varepsilon.$$  \hfill (27)

But in this final state, the entropy is $S_{BH}(E_f, q_f)$. We now write down the generalised second law of thermodynamics

$$S_{BH}(E_f, q_f) \geq S_{BH}(E, q) + S.$$ \hfill (28)

We can calculate $\Delta S_{BH} = S_{BH}(E_f, q_f) - S_{BH}(E, q)$ from expression (24). We keep only linear terms in $\mathcal{E}_{last}$. By this way, we thus exclude a possible gravitational self-force which should be quadratic in $\mu$ as in a cosmic string [13]. We find

$$\Delta S_{BH} = \frac{4\pi}{B} \left[ 2E\mathcal{E}_{last} - eq - \frac{e^2}{2} \right].$$ \hfill (29)

By inserting (26) into (29), we get

$$\Delta S_{BH} = \frac{4\pi}{B} \left[ \frac{\mu RB}{2} + 2E \left( \frac{e^2}{8E} + \frac{e^2B}{8E g(\pi)} \right) - \frac{e^2}{2} \right].$$ \hfill (30)

where $g(\pi) = 1/B - 1$ by formula (19). According to inequality (28), we obtain then from (30) the desired entropy bound

$$S \leq 2\pi \left[ \mu R - \frac{e^2}{2} \right],$$ \hfill (31)

initially derived by Zaslavskii [16] in another context.

5 Conclusion

We have determined the explicit expressions of the electrostatic potential and self-energy in the Schwarzschild black hole pierced by a cosmic string. We can extend our method to the static, spherically symmetric spacetimes pierced by a cosmic string when the electrostatic potential is known in absence of a cosmic string: Brans-Dicke [17] and Reissner-Nordström [18].

We have found again the upper entropy bound for a charged object by employing thermodynamics of the Reissner-Nordström black hole pierced by a cosmic string. To prove this, we have used the value of the electrostatic self-energy at the horizon of the Schwarzschild black hole pierced by a cosmic string. This result confirms the physical importance of the electrostatic self-force.
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