We unveil a mechanism enabling a fundamental rogue wave, expressed by a rational function of fourth degree, to reach a peak amplitude as high as a thousand times the background level in a system of coupled nonlinear Schrödinger equations involving both incoherent and coherent coupling terms with suitable coefficients. We obtain the exact explicit vector rational solutions using a Darboux-dressing transformation. We show that both components of such coupled equations can reach extremely high amplitudes. The mechanism is confirmed in direct numerical simulations and its robustness is confirmed upon noisy perturbations. Additionally, we showcase the fact that extremely high peak-amplitude vector fundamental rogue waves (of about 80 times the background level) can be excited even within a chaotic background field.

1. Introduction

Rogue waves or waves of extreme amplitude have been the subject of intense research activity [1–4]. Within the broad framework of the focusing nonlinear Schrödinger (NLS) equation, the analogous physics of light propagation and hydrodynamic waves has led to a significant volume of corresponding considerations in nonlinear optics [4–7]. In 2007, the concept of rogue waves in optics was first used to describe the rare,
extreme fluctuations in the value of an optical field [8]. Since then optical rogue waves have been
generalized to describe many other processes in this area [9–15]. The most common mathematical
description of a rogue wave is the Peregrine soliton, which can be expressed by a rational function
of second degree. This waveform presents a double spatio-temporal algebraic localization on
a finite continuous background [7,16]. The Peregrine soliton is the simplest prototype of a
fundamental rogue wave [5,6]. The dynamics of a fundamental rogue wave was observed in
numerous physical settings, including nonlinear fibres [6], water wave tanks [3] and plasmas [17].

Studies have revealed that, in the presence of higher-order effects and coupling between
components (in multi-component systems), more complex waveforms than the Peregrine
soliton can also arise [18–22]. For example, the mathematical descriptions of fundamental
rogue waves governed by the Sasa–Satsuma (SS) and coupled SS models involve fourth-order
polynomial waveforms [18,21]. The orthogonally polarized fundamental rogue wave governed
by the coupled NLS equations with negative coherent coupling also involves fourth-order
polynomials [19]. Additionally, a model that has been proposed in connection with spinor Bose–
Einstein condensates in [23] also involves fourth-order polynomials [20]. These studies highlight
the relevance of and interest in exploring further the expanded palette of possibilities afforded,
for example, by such multi-component set-ups.

Owing to the energy transfer between different components, the central amplitudes of the
vector fundamental rogue waves are not generally fixed, but rather they can be varied from
zero to triple that of the background [18–22]. It is relevant to also note that the fundamental
rogue waves are not the highest waves that may appear in a chaotic wave field [24]. The highest
waves may appear as a result of superpositions of breathers, solitons and rogue waves [25–27].
In the case of collisions of Peregrine solitons, the amplitude becomes larger than that of a
single Peregrine soliton [27,28]. One study [28] has shown that the Peregrine soliton, still
expressed by a rational function of second degree (second-order polynomials), can reach an
amplitude limit as high as five times the background level because of the energy transfer
between different components and the self-steepening effect. Very recently, Chen et al. [29] have
shown the possibility for one component to grow in an extreme fashion at the expense of the
other component. In particular, in this state-of-the-art case, typical results exhibit an eightfold
peak amplitude increase and the authors state that a maximal enhancement of above 17 can
be achieved.

In this paper, we use this recently emerging platform of vector (i.e. two-component) rogue
waves but bring to bear a drastically different mechanism. We select the rational solutions of
fourth degree as the fundamental rogue-wave solutions. Yet, our two-component NLS variant
features a crucially different, non-sign definite mass (energy in optics) conservation law that
enables each component to grow indefinitely in comparison with the background. As a result of
this, we show that the vector fundamental rogue waves (in both components) can reach a peak
amplitude as high as a thousand times the background level as a result of the coherent coupling
terms. No such case occurs in integrable systems known so far, to the best of our knowledge,
and naturally it significantly eclipses the best-known current result, as per the above discussion.
The physical relevance of the broader class of models within which our system lies naturally then
begs the question of whether a physical realization of such a mechanism may be possible. That
is, while our findings arise in a class of models that have been used in optics and atomic physics,
among other themes, we are not aware of a physical realization of the model for the coefficient
values that our integrable model features. We note that this is often the case for integrable models
such as the so-called Ablowitz–Ladik discretization of the NLS [30], and the integrable spinor
NLS [23] or the integrable non-local NLS [31]. Nevertheless, as is the case with these integrable
models, we expect that the present integrable model and its remarkable rogue-wave properties
will be the source of further studies both on the physical side (to explore the realizability of such
a setting) and on the mathematical side (to explore the model’s properties and, for example, its
usefulness towards perturbative treatments).
2. Mathematical formulation

A vector NLS system with coherent and incoherent nonlinear couplings governing the dynamics of two orthogonally polarized modes in a nonlinear optical fibre is [32]

\[ iQ_{1z} + Q_{1tt} + 2(A|Q_1|^2 + B|Q_2|^2)Q_1 + CQ_1^*Q_2^2 = 0 \]  

(2.1a)

and

\[ iQ_{2z} + Q_{2tt} + 2(D|Q_1|^2 + E|Q_2|^2)Q_2 + FQ_1^2Q_2^2 = 0, \]

(2.1b)

where \( Q_1 \) and \( Q_2 \) are the complex envelopes of the two field components, with \( z \) and \( t \) the propagation distance and retarded time, respectively. The potential applications of four-wave mixing (FWM) in coupled NLS equations have been discussed in numerous references (e.g. [33–36]). Typical experimentally relevant values, as discussed in [32], are, for example, \( A = B = D = E = 1 = 2C = 2F \). However, as discussed in, for example, [34] other combinations are physically possible, such as \( A = E = 1, B = D = \mu \) and \( C = F = 2(1 - \mu) \), where \( \mu \) is a real parameter satisfying \( 0 < \mu < 1 \) (with \( \mu = 1/3 \) being relevant, for example, for dielectric materials with purely electronic response).

Here, we will use the combination of relevant coefficients discussed in [37], which, nevertheless, amounts to an integrable system, namely \( 2A = -B = -C = D = -2E = F = 2 \). As indicated above, and similarly to a number of other integrable variants, we are not aware of a direct physical application of this setting, yet the physical relevance of the model for different parameters renders it a ripe testbed for mathematical and computational studies. For this choice and if \( Q_2 \equiv kQ_1 \) with \( k \) being pure imaginary, the model can be reduced to the scalar NLS equation. Furthermore, the appearance of case examples where the inter-component interaction may feature a negative sign in the cubic cross-coupling term (for a recent example from atomic physics, see [38]), and given the rather remarkable properties of the rogue waves of this system (including the unprecedented mechanism discussed below), suggests, in our view, that the interest in considering this system as a prototype is justified.

Using the Darboux-dressing transformation [39–42], we obtain the fundamental rogue-wave solutions

\[ Q_1 = \lim_{\lambda \to i/\sqrt{a_1^2 - a_2^2}} a_1 e^{i(2a_1^2 - 2a_2^2)z} - 2iS_{13} \]

(2.2a)

and

\[ Q_2 = \lim_{\lambda \to i/\sqrt{a_1^2 - a_2^2}} i a_2 e^{i(2a_1^2 - 2a_2^2)z} - 2iS_{14}, \]

(2.2b)

where the matrix elements are obtained from the matrices

\[
S = A \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^* & 0 \\ 0 & 0 & 0 & \lambda^* \end{pmatrix} A^{-1}, \quad A = \begin{pmatrix} \psi_1 & -\psi_2 & \psi_3 & -\psi_4 \\ \bar{\psi}_2 & \psi_1 & \bar{\psi}_4 & \bar{\psi}_3 \\ -\psi_4 & \bar{\psi}_3 & \psi_1 & -\psi_2 \\ \bar{\psi}_4 & \psi_3 & -\psi_1 & \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = e^{B(\lambda)z + C(\lambda)t} \mathcal{F} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

(2.3)

with

\[
\mathcal{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-iz(2a_1^2 - 2a_2^2)} & 0 \\ 0 & 0 & 0 & e^{-iz(2a_1^2 - 2a_2^2)} \end{pmatrix}.
\]

(2.4)
Here, $F_0 = (z_1, z_2, z_3, z_4)^T$ is a complex constant vector, $a_1$ and $a_2$ are real constants corresponding to the background heights and $S_{ij}$ ($j = 3, 4$) represent the entry of matrix $S$ in the first row and $j$th column. We have provided the analytical forms of the rogue-wave solutions used in appendix A. In general, such fundamental rogue-wave solutions are expressed by rational functions of fourth degree, but not by those of second degree. When $Q_2 \equiv (a_2/a_1)iQ_1$ ($|a_2/a_1| < 1$), such fundamental rogue-wave solutions revert to the rational functions of second degree, which are the same as the Peregrine soliton of the focusing NLS equation. For the higher-order rogue waves, we note that they can be obtained via the Darboux-dressing transformation; see [43] for a similar analysis. However, we do not focus on them in this paper, but rather on the different mechanism of fundamental rogue waves of equations (2.1).

It is relevant to highlight that the conserved ‘energy’ (in the context of optics) for the model of equation (2.1a,b) reads

$$E = \int_{-\infty}^{\infty} |Q_1|^2 - |Q_2|^2 \, dt,$$

which differs significantly from the customary conservation of the sum of the two-component squared $L^2$ norms in FWM models [32] (or their individual conservation, e.g. in Manakov-type models [30,32]). This is particularly crucial because it enables the dynamical evolution of the mass of the two components in such a way that both may grow indefinitely with respect to the background, but retain their relative size with respect to each other so that the conservation law (2.6) is preserved. This appears to be the principal mechanism at work, enabling the dramatic intensity enhancements that will arise parametrically in the examples that will follow in the numerical illustration below. For completeness, additional conservation laws of the integrable model at hand, such as its Hamiltonian ($\mathcal{H}$) and momentum ($\mathcal{M}$), are given as [37]

$$\mathcal{M} = i \int_{-\infty}^{+\infty} \left[ (Q_1 Q_{1\text{t}}^* - Q_{1\text{t}}^* Q_1 - Q_2 Q_{2\text{t}}^* + Q_{2\text{t}}^* Q_2) \right] dt,$$

$$\mathcal{H} = \int_{-\infty}^{+\infty} \left[ \left( |Q_{1\text{t}}|^2 + |Q_{2\text{t}}|^2 \right) - \left( |Q_1|^4 + |Q_2|^4 \right) + 4 |Q_1|^2 |Q_2|^2 + \left( Q_1^2 Q_2^2 + Q_1^2 Q_2^2 \right) \right] dt.$$

3. Numerical verification

To illustrate the fundamental rogue-wave dynamics, we use the parameters $a_1 = 2, \ a_2 = 1, \ z_1 = 3 + i, \ z_3 = 0, \ z_4 = 3$, but with varying structural parameter $z_2$ in figure 1. In figure 1a, we show the standard fundamental rogue-wave dynamics of our solutions (2.2), obtained with structural parameter $z_2 = 4$. It can be seen that the vector fundamental rogue waves reach amplitudes as high as $0 \sim 3$ times the background level. Such a realization can be viewed as the standard case, which can also be observed in the SS equation [18,21], as well as in the coupled NLS equations of [19,20]. However, in stark contrast with the previous studies [18–21,40], we will show that the fundamental rogue waves of equations (2.1) could reach an extremely high peak amplitude.

In figure 1b–d, depending on the relative values of the structural parameter $z_2$, the fundamental rogue-wave solutions, still expressed by rational functions of the fourth degree, can remarkably reach amplitudes as high as a thousand times the background level. When $z_2 = 1/3, \ Q_1$ and $Q_2$ reach an amplitude limit at least as high as 26 times the background level. When $z_2 = 1/6$, both components are at least 100 times the background level. When $z_2 = 12/100, \ Q_1$ and
Figure 1. (a) The fundamental rogue-wave dynamics of our solutions (2.2), obtained with the parameters $a_1 = 2$, $a_2 = 1$, $z_1 = 3 + i, z_2 = 2$ and $z_2 = 4$. The same is shown for different parameters $z_2 = 1/3$ (b), $z_2 = 1/6$ (c) and $z_2 = 12/100$ (d). (Online version in colour.)

$Q_2$ reach an amplitude limit as high as at least 1200 times the background level. In addition, the $Q_1$ component has one peak, while the $Q_2$ component has two peaks. This is in line with the feature that such equations also admit the vector single peak–double peak solitons on top of a vanishing background [37]. For completeness, in figure 2, we illustrate the profile of the modulus of both spatial fields (divided by their respective backgrounds) near and at the instance of peak formation. We show how the fundamental rogue waves reach the ultra-high peak amplitude in both fields within a short time. It is worth noticing how the second component forms a peak first on one side of the first component maximum, subsequently symmetrizes and then the peak appears on the other side of the first component maximum. It is also relevant to note that, when $|\lambda| < \sqrt{a_1^2 - a_2^2}$, Akhmediev breathers arise in the model, which means that the solutions exhibit localization in $z$ but periodicity along $t$. On the other hand, when $|\lambda| > \sqrt{a_1^2 - a_2^2}$, Kuznetsov–Ma solitons appear, which means that the solutions exhibit localization in $z$ but periodicity along $t$. Akhmediev breathers and Kuznetsov–Ma solitons with ultra-high peak amplitude are shown in figure 3.

Now we show that such vector rogue waves could be generated in the modulation instability (MI) regime. We take the background solutions as $Q_{10} = a_1 e^{i(kz + \omega t)}$ and $Q_{20} = ia_2 e^{i(kz + \omega t)}$ with $k = 2a_1^2 - 2a_2^2 - \omega^2$. A perturbed nonlinear background can be written as $Q_1 = (a_1 + p_1) e^{i(kz + \omega t)}$ and $Q_1 = i(a_2 + p_2) e^{i(kz + \omega t)}$, where $p_1$ and $p_2$ are small perturbations. The $p_1$ and $p_2$ are $t$-periodic with frequency $\Omega$. Using linear stability analysis, we obtain the eigenvalues of the linear system as $\pm \sqrt{-4a_1^2 \Omega^2 + 4a_2^2 \Omega^2 + \Omega^4 - 2\omega \Omega}$. When the eigenvalue has a negative imaginary part, MI arises, which means that $a_1^2 > a_2^2$. Therefore, when $|a_1| > |a_2|$, a baseband MI [13], which includes frequencies that are arbitrarily close to zero, is present, i.e. $0 < \Omega^2 < 4a_1^2 - 4a_2^2$. In figure 4, we
show the logarithmic gain plot ln $G(\Omega)$ versus $\Omega$, where $G(\Omega) = \sqrt{+4a_1^2\Omega^2 - 4a_2^2\Omega^2 - \Omega^4}$. The gain maximum is found to be 1.09861 at $\Omega = 1.73205$.

We also use numerical simulations to investigate the robustness of these fundamental rogue waves in figure 5. Here, the split-step Fourier spectral method is used to deal with equations (2.1). We can observe that both with (figure 5b) and without (figure 5a) imposing a 5% white noise perturbation [21] on top of the solutions with $z_2 = 1/3$ (figure 1b), the waveforms are found to robustly persist in the evolution dynamics. In the presence of a perturbation, owing to the spontaneous MI of the homogeneous state, both fields develop an unstable background after the fundamental rogue-wave propagation. Similar case scenarios have been explored for other values of the parameters (e.g. for the case of $z_2 = 1/6$ in figure 1c) and the results of figure 5 have been found to be representative of the dynamical robustness of the states at hand.

Importantly, we even find that such unusual extremely high peak-amplitude fundamental rogue waves can be excited in a chaotic background field. To do this, we use the plane-wave solutions as initial conditions at $z = 0$, perturbed by random noise of a strength of 2%. Specifically, we multiply the plane waves in $Q_1$ and $Q_2$ by the factors $(1 + 0.02f_j(t))(j = 1, 2)$, respectively, where $f_j(t)$ are real random functions whose mean value is 0 and variance is $1/3$. In addition, $f_j(t)$ are taken as Gaussian-distributed and Gaussian-correlated functions with a correlation length of $1/2$ [44].

It is relevant to note here that the context of MI evolution upon random perturbations has been revisited in optical and hydrodynamic systems [24] (including in experiments [45]). It is found in these works that, in addition to fundamental rogue waves, more general structures in the form of solitons on finite backgrounds (as well as breathers) clearly arise. The statistical analysis of such occurrences in the context of the present model would constitute an interesting possibility for future work.

Before that, to highlight the differences between chaotic background fields under different conditions, we firstly consider two special cases for: (i) the integrable parameter set $2A = -B = -C = D = -2E = F = 2$ and $f_1(t) = f_2(t)$ and (ii) the non-integrable parameter set $2A = -B = D =
$-2E = 2, \ C = F = 0$ and arbitrary $f_j(t)$. As shown in figure 6, because of MI, the noisy backgrounds subsequently evolve into two vector chaotic fields, which are similar to ‘sea’ waves at different heights. In case (i), owing to $Q_2(t, 0) = (i/2)Q_1(t, 0)$, such a phenomenon is consistent with the rogue-wave excitation in the focusing NLS equation. For case (ii), such a chaotic field becomes more disorganized without distinct local structures like Peregrine solitons. Although the perturbation could also cause high-amplitude waves, such a chaotic field is obviously different from that in case (i). This implies that the integrability of equations (2.1) may be an important factor for the excitation of rogue waves in the form of Peregrine solitons, while the high-amplitude waves in non-integrable conditions are rather complex and difficult to obtain.
Figure 5. Numerical simulations of the $z-t$ evolution (shown through contour plots) of the fundamental rogue waves in figure 1 with $z_2 = 1/3$ without perturbation (a) and with 5% white noise perturbation (b). (Online version in colour.)

In the above two cases, the effective correlation built between the fields does not allow the maximal amplitude to be significantly larger than 3, i.e. they are not of the anomalously large-amplitude variety created by the mechanism proposed herein. However, in figure 7, considering the same integrable condition of model parameters as before, we select $f_1(t)$ and $f_2(t)$ to be different random variables (empirically, we find that even a small difference between the two is sufficient). Then, we see that some waves in the chaotic background clearly feature significant increase in their amplitudes. The parts indicated by red boxes in figure 7a are enlarged in the corresponding mesh plots for the two fields in figure 7b, revealing that a high peak-amplitude vector rogue wave is excited. Such a rogue wave is clearly distinguished from general higher-order rogue waves (superposition of fundamental rogue waves), arising in the form of suitable analytical solutions, e.g. in the realm of the single-component NLS model. Hence, this spontaneously emerging pattern within the chaotic background is a definitive manifestation of the type of wave pattern advocated in the present work. We attribute the key differences to the coherent coupling effect. Owing to the existence of coherent coupling terms in equations (2.1), the phase-dependent contribution (coherence) of plane waves in $Q_1$ and $Q_2$ plays an important role. Indeed, coherent coupling is chiefly responsible for the presence of a single energy/mass-type conservation law as per equation (2.6). If $C = F = 0$, then the two components individually conserve their
energy. In that case, this type of two-component interplay of mutual growth leading to pairwise cancellation at the level of equation (2.6), yet indefinite growth at the level of each individual component, is prohibited by the individual conservation laws. We thus argue that this mechanism and phenomenology is unique to this type of coherently coupled NLS model and thus is not encountered in the settings previously considered.

4. Conclusions/future work

In the present work, we have unveiled an unprecedented mechanism, to the best of our knowledge, regarding the formation of arbitrary amplitude rogue waves. Within this scenario, we have explained the role of coherent coupling terms, in conjunction with an unconventional (single) energy conservation law bearing indefinite sign between the energies of individual components. As a result, in a prototypical (within this class of features) NLS model inspired by the examination of multiple polarizations in a nonlinear fibre, we have shown that vector fundamental rogue waves (in both components) of such coupled equations could reach peak amplitudes of the order of a thousand times the background level. The fundamental solutions
we consider here are rational solutions of fourth degree. Their periodic extensions in the
$z$ (Kuznetsov–Ma soliton) and $t$ (Akhmediev breather) variables were also identified and
illustrated. Furthermore, we have numerically confirmed that such unusual, extremely high
peak-amplitude vector fundamental rogue waves are robust even under moderate perturbations.
Last but not least, we have confirmed that such vector fundamental rogue waves could be
excited in a chaotic background field, i.e. such excitations indeed spontaneously arise in
dynamical simulations against the backdrop of an unstable background. As far as we can tell,
the key mechanism elucidated here enables the significant eclipsing of the highest amplitude
of previously reported rogue waves. Such dynamics may not be (fully) observed in a physical
medium. If, for instance, considering Kerr media or water waves, the waves will inevitably break
and, as such, the expected dynamics cannot unfold. Nevertheless, our work naturally poses the
question of whether a setting can be engineered for the observation (and potential harnessing)
of this phenomenon. From a theoretical perspective numerous extensions of the ideas reported
herein can also be pursued, including the extension and numerical identification of such states
beyond the integrable limit considered herein, using, for example, recent ideas such as those
of [46]. Another extension of interest is to consider two-dimensional variants of the present model
and whether rogue waves of the present type can be identified in suitable generalizations of
settings related to the Davey–Stewartson and Benney–Roskes models where two-dimensional
rogue patterns were considered in [47].

Figure 7. Numerical excitation of the fundamental rogue waves. The initial condition is a plane wave perturbed by 2%
random noise with $2A = -B = -C = D = -2E = F = 2$, $a_1 = 2$, $a_2 = 1$ and $f_1(t) \neq f_2(t)$. (a) The amplitude evolution;
(b) enlarged three-dimensional plots of the high peak-amplitude vector rogue-wave profiles highlighted by the red boxes in
(a). (Online version in colour.)
Authors’ contributions. W.R.S., L.L. and P.G.K. contributed to the study of the mechanism. W.R.S. obtained the vector rogue-wave solutions of ultra-high peak amplitude. L.L. did the numerical work. P.G.K. did the physical analysis. All authors wrote the paper and gave final approval for publication.

Data accessibility. All the mathematical results are in analytic form and are reproducible.

Competing interests. We declare we have no competing interests.

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Appendix A

The fundamental solutions (2.2) with the parameters $a_1 = 2, a_2 = 1, z_1 = 3 + i, z_3 = 0$ and $z_4 = 3$ are given as:

$$Q_1(z,t) = -2e^{6iz} \frac{G_1}{F}, \quad Q_2(z,t) = -e^{6iz} \frac{G_2}{F},$$

$$G_1 = 36z_2^4t^4 + 288\sqrt{3}z_2^3t^4 - 144\sqrt{3}z_2^3t^4 + 2952z_2^2t^4 + 144\sqrt{3}t^4 + 4896\sqrt{3}z_2t^4 - 1728z_2t^4 + 16020t^4 + 24\sqrt{3}z_2^3t^4 + 288z_2^3t^3 + 384\sqrt{3}z_2^3t^3 + 144\sqrt{3}t^3 + 456\sqrt{3}t^3 + 288z_2t^3 + 2736\sqrt{3}t^3 + 864z_2^2t^2 - 144izz_2t^2 + 12z_2t^2 - 1152iz\sqrt{3}z_2t^2 + 6912z^2\sqrt{3}z_2t^2 - 36\sqrt{3}z_2^2t^2 + 384480z^2t^2 - 3456\sqrt{3}z_2^2t^2 + 70848z^2z_2t^2 - 11808izz_2t^2 - 864 - 576i\sqrt{3}z_2^2t^2 + 72i\sqrt{3}z_2^2t^2 + 456z_2^2t^2 + (444 - 1296i)t^2 + (15552 - 64080i)t^2 - (864 + 576i)\sqrt{3}t^2 + 3456z^2\sqrt{3}t^2 + 72i\sqrt{3}t^2 + 41472z^2z_2t^2 + 6912izz_2t^2 + 117504z^2\sqrt{3}z_2^2t^2 - (612 + 288i)\sqrt{3}z_2^2t^2 + (3456 - 19584i)z^2z_2t^2 + 216z_2t^2 - 48i\sqrt{3}zz_2^4 + 288z^2\sqrt{3}z_2^4t + 3456z^2z_2^4t - 576izz_2^4t - 36z_2^2t + 32832z^2t^2 - 1728z^2z_2^2t + 288izz_2^2t - 768i\sqrt{3}zz_2^2t + 4608\sqrt{3}z_2^2t - 5472izt - 912i\sqrt{3}zt + 5472z^2\sqrt{3}t - 3456z_2^2zt + (36 - 216i)z_2t + (2592 + 576i)z_2zt - 216i\sqrt{3}z_2zt + 2592z_2\sqrt{3}z_2t + 23606880z^4 + 5184z^4 + 1278iz^4 - 24izz^4 + 1 + (186624 - 768960i)z^3 - (10368 + 6912i)\sqrt{3}z^3 - 13824i\sqrt{3}z^3 + 720\sqrt{3}z^3z_2^3 - 72i\sqrt{3}zz_2^3 + 41472z^4\sqrt{3}z^3 - 6\sqrt{3}z^3 - (58752 + 64656i)z^2 - (576 - 2592i)\sqrt{3}z^2 - 20736\sqrt{3}z^3z_2^2 + 425088z^4z_2^2 - 141696iz(z^3z_2^2 - (10368 - 6912i)\sqrt{3}z^3z_2^2 - 6912z^2z_2^2z_2^2 - 816izz_2^2z_2^2 + (576 + 2592i)z^2\sqrt{3}z_2^2z_2^2 + 144z^2\sqrt{3}z_2^2z_2^2 - 34z_2^2 - (2592 + 888i)z + 20736z^2\sqrt{3} + 144z\sqrt{3} - 248832z^4z_2 + 82944izz^3z_2^2 - (12240 + 10368i)\sqrt{3}z^3z_2^2z_2^2 + 4320z^2z_2^2z_2^2 - 144izz_2^2z_2^2 + z_2^2 + 705024z^4\sqrt{3}z_2^2 + (41472 - 235008i)z^3\sqrt{3}z_2z_2 - (102 + 36i)\sqrt{3}z_2z_2 - 1, \quad F = 36t^4z_2^4 + 288\sqrt{3}t^4z_2^4 + 144\sqrt{3}t^4z_2^4 + 2952z_2^4t^2 + 4896\sqrt{3}t^4z_2^2 - 1728t^4z_2^2 + 144\sqrt{3}t^4z_2^4 + 16020t^4 + 24\sqrt{3}t^4z_2^4 + 288z^2z_2^4 + 384\sqrt{3}z^2z_2^4 + 144z^2z_2^4 + 288z_2^4z_2^2 + 456\sqrt{3}z_2^4z_2^2 + 2736\sqrt{3}z_2^4z_2^2
\[
G_2 = 36i z_2^4 + 288i \sqrt{3} z_2 t^4 + 2952i z_2^4 t + 10 020i t^4 + 44i \sqrt{3} t^4 - 1728i z_2 t^4 + 4896i z_2^3 + 24i \sqrt{3} z_2 t^3 + 288i z_2^3 t + 348i \sqrt{3} z_2 t^3 + 2736i t^3 + 456i \sqrt{3} t^3 - 288i z_2 t^3 + 864i z_2^2 t^2 + 12i z_2^2 t^2 + 144i z_2^2 t^2 + 6912i z_2^2 t^2 + 1152i \sqrt{3} z_2 t^2 + 384480i z_2^3 t^2 + 70 848i \sqrt{3} z_2^3 t^2 - 3456i \sqrt{3} z_2^3 t^2 - 24i z_2^3 t^2 - (576 + 864i) \sqrt{3} z_2^3 t^2 + 11 808z_2^3 t^2 - (72 + 72i) \sqrt{3} z_2^3 t^2 + (1296 + 4764i) t^2 + (64 080 + 15 552i) t^2 + 3456iz_2^2 \sqrt{3} t^2 - (72 - 72i) \sqrt{3} t^2 + (576 - 864i)z_2 t^2 + 11 5704i z_2^3 t^2 + (19 584 + 3456i) z_2 t^3 + 288 \sqrt{3} z_2^2 t + 288i z_2^2 t + 24i z_2^2 t + (7 68 960 + 186 624i) z_2^2 t - 11 52i \sqrt{3} z_2^2 t^2 + 41 472i z_2^2 t^2 + 13 824i \sqrt{3} z_2^2 t + (46 656 - 1 10 592i) z_2^2 t - (288 - 2592i) z_2^2 t + 460i z_2^2 \sqrt{3} z_2 t + 768z_2 \sqrt{3} z_2 t + (4104 + 1368i) t + (5472 + 49 248i) t + 5472i z_2 \sqrt{3} t + 912z_2 \sqrt{3} t - 3456iz_2 t + 10 368i z_2 t + 864 \sqrt{3} z_2 t + 2 306 880i z_2^4 + 5184iz_2^4 t + 1728z_2^4 t - iz_2^4 + 24z_2^4 t + (7 68 960 + 186 624i) z_2^4 t - 11 52i \sqrt{3} z_2^4 t + 10 020i \sqrt{3} t^4 + (77 52 + 2592i) z - (144 - 576i) \sqrt{3} z + 20736i i z_2^2 z + (6912 + 10 368i) \sqrt{3} z_2^2 z + 14 169 63z_2^2 z - (2592 - 1440i) \sqrt{3} z_2^2 z - 34iz_2^2 + 48z_2^2 z + (36 - 12i) \sqrt{3} z_2^2 z + (144 + 576i)z_2^2 \sqrt{3} z - i - 248 832i z_2 z + 82 944z_2 z + 17 280i z_2 z - 70 5024iz_2^4 \sqrt{3} z_2 + (2 35 008 + 41 472i) \sqrt{3} z_2 - (10 020 - 19 584i) z^2 \sqrt{3} z_2.
\]

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