THE BROWN MEASURE OF A SUM OF TWO FREE NONSELFADJOINT RANDOM VARIABLES, ONE OF WHICH IS R-DIAGONAL

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Abstract. Suppose that $X_1$ and $X_2$ are two $*$-free (generally unbounded) random variables with Brown measures $\mu_{X_1}$ and $\mu_{X_2}$, respectively. Using properties of classical free additive convolutions, we develop a method for calculating $\mu_{X_1 + X_2}$ when $X_2$ is $R$-diagonal. This method determines a density relative to Lebesgue measure on an open set whose closure contains the support of $\mu_{X_1 + X_2}$. Effective calculations are possible in important cases.

Biane and Lehner were the first to make significant progress on the problem we consider, even in some cases in which neither $X_1$ nor $X_2$ is $R$-diagonal. Our examples overlap with theirs, but we emphasize the use of subordination functions. When $X_2$ is circular, $\mu_{X_1 + X_2}$ was studied earlier using two different approaches, one involving Hamilton-Jacobi equations, and another using standard free probability techniques. Our work extends the second approach.

1. Introduction

Consider a tracial $W^*$-probability space $(\mathcal{A}, \tau)$. Thus, $\mathcal{A}$ is a von Neumann algebra, and $\tau$ is a normal, faithful, tracial state on $\mathcal{A}$. In his extension of Lidskiï’s theorem [11, Theorem 1] to this context (and, indeed, to the larger context of semifinite von Neumann algebras), L. Brown [11] associated a kind of spectral distribution measure $\mu_X$ to possibly unbounded operators $X$ affiliated with $\mathcal{A}$. We use the notation $\mathcal{A}_\infty$ for the algebra consisting of all such operators, and we denote by $\text{Log}^+(\tau)$ the collection of those $X \in \mathcal{A}_\infty$ for which

$$\tau(\log^+ |X|) < +\infty.$$ 

Given $X \in \text{Log}^+(\tau)$, Brown shows that the function

$$\lambda \mapsto \tau(\log |X - \lambda|), \quad \lambda \in \mathbb{C},$$

is subharmonic and, in fact, it is the logarithmic potential of a Borel probability measure $\mu_X$ on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} \log^+ |z| \, d\mu_X(z) < +\infty.$$ 

In other words,

$$\tau(\log |X - \lambda|) = \int_{\mathbb{C}} \log |z - \lambda| \, d\mu_X(z), \quad \lambda \in \mathbb{C}.$$
The measure $\mu_X$ is obtained as

$$d\mu_X(\lambda) = \frac{1}{2\pi} \Delta_\lambda \tau(\log |X - \lambda|),$$

where $\Delta_\lambda = 4\partial_\lambda \partial_\lambda$ denotes the Laplace operator, applied in the sense of distributions to the locally integrable function $\tau(\log |X - \lambda|)$. The concept of the Brown measure overlaps the classical one of the scalar spectral measure for normal operators. Thus, supposing that $X \in \mathcal{A}$ is a normal operator (not necessarily in $\text{Log}^+(\mu)$) with spectral measure $E_X$, we set

$$\mu_X(\sigma) = \tau(E_X(\sigma))$$

for every Borel set $\sigma \subset \mathbb{C}$. This is also the Brown measure of $X$ if $X \in \text{Log}^+(\mu)$, so the notation is consistent.

Suppose now that $X_1, X_2 \in \text{Log}^+(\tau)$ are $\ast$-free (relative to $\tau$; see, for instance [26]) and $X = X_1 + X_2$. We wish to calculate the Brown measure $\mu_X$ in terms of data coming from $X_1$ and $X_2$. When $X_1$ and $X_2$ are selfadjoint, the resulting measure $\mu_X$ depends only on the measures $\mu_{X_1}$ and $\mu_{X_2}$, namely

$$\mu_X = \mu_{X_1} \boxplus \mu_{X_2},$$

where $\boxplus$ denotes the additive free convolution [24]. The nonselfadjoint situation is more difficult to deal with and progress has only been made in special situations. After one important case calculated by Haagerup and Schulz [16], Biane and Lehner [9] developed techniques that work in greater generality. In this work, we consider the case in which $X_2$ is $R$-diagonal (see [21] and [16]) but no further hypothesis is made about $X_1$. Previously, the case of a selfadjoint $X_1$ (and circular $X_2$) was studied in [19] and subsequently [28] using different approaches. We further develop the methods of [28] without assuming that $X_1$ is selfadjoint. The main result shows that there exists an open set $\Omega \subset \mathbb{C}$, whose closure contains the support of $\mu_X$, and such that $\mu_X$ has a real-analytic density on $\Omega$ relative to planar Lebesgue measure.

The main tool in our approach to $\mu_X$ is a result (proved in [22] for bounded operators, reformulated in that case in [15], and extended in [16] to unbounded operators) that identifies $\mu_{|X|}$ when both $X_1$ and $X_2$ are $R$-diagonal. More precisely, the symmetrization $\mu_{|X|}$ of $\mu_{|X|}$ is the usual free convolution of $\mu_{|X_1|}$ with $\mu_{|X_2|}$. This result has consequences, already observed in [9] and [10], for the case in which only $X_2$ is $R$-diagonal. This allows us to use the well understood mechanism of free convolution to gather information about $\mu_X$. We start in Section 2 by using the subordination properties of free convolution to establish a relation between the logarithmic potentials of $\mu_X$, $\mu_{X_1}$, and $\mu_{X_2}$ in case both $X_1$ and $X_2$ are selfadjoint. We also derive several useful consequences for the special case of symmetric distributions. The main results are derived in Section 3, including an explicit (provided that all subordination functions can be calculated) formula for the density of $\mu_X$. In Section 4 we describe a few cases in which the calculations can be performed in greater detail.

Brown measures have been useful in several ways. First, of course, is Brown’s theorem [11] extending Lidski˘ı’s theorem as well as [13, Theorem 2]. Subsequently, Brown measures were instrumental in parametrizing an important family of hyper-invariant subspaces for arbitrary operators in $\mathcal{A}$ [17]. Knowledge of $\mu_X$ is often predictive of the asymptotic behavior of the empirical eigenvalue distribution of
certain random matrix ensembles as their size tends to $+\infty$. For instance, the calculation of $\mu_X$ for selfadjoint $X_1$ ([19] and [28]) has a random matrix counterpart [10]. The results of the present paper have their own random matrix counterpart that is the subject of subsequent work [18].

2. Logarithmic integrals and subordination

As in the introduction, we work in the context of a tracial $W^*$-probability space $(\mathcal{A}, \tau)$. Given a selfadjoint operator $T \in \mathcal{A}$, we consider the functions

$$G_T(z) = \tau((z - T)^{-1}), \quad H_T(z) = \frac{1}{G_T(z)} - z,$$

defined for $z$ in the complex upper half-plane $\mathbb{C}^+$. We also employ the notation

$$G_\mu(z) = \int_{-\infty}^{z} d\mu(t)$$

for an arbitrary Borel probability measure $\mu$ on $\mathbb{R}$, so $G_T = G_{\mu_T}$. The function $G_T$ maps $\mathbb{C}^+$ to $-\mathbb{C}^+$. Unless $\mu_T$ is a point mass, $H_T$ maps $\mathbb{C}^+$ to itself. The following result, containing facts established in [25, 8] and [2, 3], describes the subordination property of free convolution. The notation $\overline{\mathbb{C}}^+$ indicates the closure of $\mathbb{C}^+ \cup \mathbb{R}$ of $\mathbb{C}$ in the complex plane, while $\overline{\mathbb{C}}^+ \cup \{\infty\}$ is the closure of $\mathbb{C}^+$ in the Riemann sphere.

**Theorem 2.1.** Suppose that $T_1, T_2 \in \mathcal{A}$ are selfadjoint operators free relative to $\tau$, and $T = T_1 + T_2$. If neither $T_1$ nor $T_2$ is a constant multiple of the identity operator, then there exist unique continuous functions $\omega_1, \omega_2 : \overline{\mathbb{C}}^+ \to \overline{\mathbb{C}}^+ \cup \{\infty\}$ that are analytic on $\mathbb{C}^+$ such that

$$(2.1) \quad G_T(z) = G_{T_1}(\omega_1(z)) = G_{T_2}(\omega_2(z)), \quad z \in \mathbb{C}^+,$$

and

$$(2.2) \quad \omega_1(z) + \omega_2(z) = z + \frac{1}{G_T(z)}, \quad z \in \mathbb{C}^+.$$

We recall a result due to Denjoy and Wolff ([12] and [27]; see also [23]) that applies to an arbitrary analytic function $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$, other than a conformal automorphism: the iterates

$$\varphi^{\circ n} = \underbrace{\varphi \circ \cdots \circ \varphi}_{n \text{ times}}$$

converge pointwise to a constant function, whose value (possibly in $\mathbb{R} \cup \{\infty\}$) is now called the Denjoy-Wolff point of $\varphi$. If $\varphi$ has a fixed point $\alpha \in \mathbb{C}^+$, then the Denjoy-Wolff point of $\varphi$ is $\alpha$, and

$$|\varphi'(\alpha)| < 1.$$
Theorem 2.2. Let \( T_1, T_2, T \) and \( \omega_1, \omega_2 \) be as in Theorem 2.1. Suppose also that the spectrum of one of the operators \( T_j \) contains at least three points. Given \( z \in \mathbb{C}^+ \), the function

\[
\varphi_z(w) = z + H_{T_2}(z + H_{T_1}(w)), \quad w \in \mathbb{C}^+,
\]

is analytic, maps \( \mathbb{C}^+ \) to itself, and is not a conformal automorphism. Moreover, the Denjoy-Wolff point of \( \varphi_z \) is precisely \( \omega_1(z) \). Similarly, the Denjoy-Wolff point of the function

\[
\psi_z(w) = z + H_{T_1}(z + H_{T_2}(w)), \quad w \in \mathbb{C}^+,
\]

is \( \omega_2(z) \). In particular, \( \omega_1(0) \) is the denjoy-Wolff point of \( H_{T_2} \circ H_{T_1} \) and \( \omega_2(0) \) is the denjoy-Wolff point of \( H_{T_1} \circ H_{T_2} \).

When both \( T_1 \) and \( T_2 \) have two-point spectra, \( \omega_1(z) \) is still a fixed point of \( \varphi_z \), which is a conformal automorphism in this case. The map \( \varphi_z \) has a unique fixed point unless it is the identity map, a situation that arises for at most one value of \( z \) for which \( \varphi_z \) is the identity map. The spectrum hypothesis in the statement above simply eliminates this rare occurrence.

We note a simple consequence.

Corollary 2.3. With the notation of Theorem 2.1, suppose that \( z \in \mathbb{R} \). Then \( \omega_1(z) \) belongs to \( \mathbb{C}^+ \) if and only if \( \omega_2(z) \) belongs to \( \mathbb{C}^+ \). When this occurs, \( G_T \) extends to \( z \) by continuity and the equations (2.1) and (2.2) are satisfied. In particular, we have \( H_{T_1}(\omega_1(0)) = \omega_2(0) \) and \( H_{T_2}(\omega_2(0)) = \omega_1(0) \) whenever \( \omega_1(0) \in \mathbb{C}^+ \).

Proof. If \( f, g : \mathbb{C}^+ \rightarrow \mathbb{C}^+ \) are analytic functions and \( f \circ g \) has a fixed point \( \alpha \in \mathbb{C}^+ \), then clearly \( g \circ f \) has the fixed point \( g(\alpha) \). The first claim follows from this observation applied to \( f(w) = z + H_{T_1}(w) \) and \( g(w) = z + H_{T_2}(w) \). The second claim follows from the continuity of \( \omega_j \) on \( \mathbb{C}^+ \) for \( j = 1, 2 \).

We focus next on selfadjoint operators \( T \) that belong to \( \text{Log}^+(\tau) \), so the function

\[
L_T(z) = \tau(\log(z - T)) = \int_{\mathbb{R}} \log(z - t) \, d\mu_T(t), \quad z \in \mathbb{C}^+,
\]

is defined and analytic. The principal value of the logarithm should be used in these formulas, that is, \( \exists \log(z - t) \in (0, \pi) \) for \( z \in \mathbb{C}^+ \) and \( t \in \mathbb{R} \). Eventually, we will only be interested in the logarithmic potential

\[
\Re L_T(z) = \tau(\log |z - T|) = \int_{\mathbb{R}} \log |z - t| \, d\mu_T(t).
\]

Observe that the complex derivative of \( L_T \) is

\[
L_T'(z) = G_T(z), \quad z \in \mathbb{C}^+.
\]

Lemma 2.4. Let \( T_1, T_2 \in \text{Log}^+(\tau) \) be free selfadjoint operators, let \( T = T_1 + T_2 \), and let \( \omega_1, \omega_2 \) be the subordination functions provided by Theorem 2.1. Then we have

\[
L_{T_1}(\omega_1(z)) + L_{T_2}(\omega_2(z)) = L_T(z) + \log(\omega_1(z) + \omega_2(z) - z), \quad z \in \mathbb{C}^+.
\]

Proof. Using (2.2) we can rewrite the identity above as

\[
L_{T_1}(\omega_1(z)) + L_{T_2}(\omega_2(z)) = L_T(z) - \log(G_T(z)).
\]
We show first that the difference between the two sides of (2.4) is a constant by verifying that the derivatives are the same:

\[ G_{T_1}(\omega_1(z))\omega_1'(z) + G_{T_2}(\omega_2(z))\omega_2'(z) = G_T(z) - \frac{G_T'(z)}{G_T(z)}, \]

Using (2.1), this can be rewritten as

\[ \omega_1'(z) + \omega_2'(z) = 1 - \frac{G_T'(z)}{G_T(z)^2}, \]

which is recognized as the identity obtained by differentiating (2.2). Thus, there exists a constant \( c \in \mathbb{C}^+ \) such that

\[ L_{T_1}(\omega_1(z)) + L_{T_2}(\omega_2(z)) = L_T(z) - \log(G_T(z)) + c, \quad z \in \mathbb{C}^+. \]

To identify the constant \( c \), we consider \( z = iy \) with \( y > 0 \) and subtract \( 2 \log(iy) \) from both sides to see that

\[ \tau \left( \log \frac{\omega_1(iy) - T_1}{iy} \right) + \tau \left( \log \frac{\omega_2(iy) - T_2}{iy} \right) = \tau \left( \log \frac{iy - T}{iy} \right) - \log(iyG_T(iy)) + c, \]

for \( y > 0 \). Now, let \( y \to +\infty \) and note that each of the four terms in this identity that involve \( y \) tends to zero. For instance,

\[ \tau \left( \log \frac{iy - T}{iy} \right) = \int_{\mathbb{R}} \log \left( 1 - \frac{t}{iy} \right) d\mu_T(t), \]

where the integrands are dominated by the integrable function \( \log(1+|t|) \) for \( y \geq 1 \), and tend to zero as \( y \to +\infty \). For the other terms, one uses the well-known limits

\[ \lim_{y \to \infty} \frac{\omega_1(iy)}{iy} = \lim_{y \to \infty} \frac{\omega_2(iy)}{iy} = \lim_{y \to \infty} iyG_T(iy) = 1. \]

For the remainder of this section, we restrict ourselves to selfadjoint operators \( T \) that have a symmetric distribution on \( \mathbb{R} \). This amounts to saying that \( T \) and \( -T \) have the same distribution, or that \( d\mu_T(t) = d\mu_T(-t) \). In terms of \( G_T \), symmetry is expressed by the identity

\[ -G_T(-\bar{z}) = G_T(z), \quad z \in \mathbb{C}^+, \]

where the bar indicates complex conjugation. Symmetry also yields

\[ G_T(z) = \frac{1}{2} \left[ \tau((z - T)^{-1}) + \tau((z + T)^{-1}) \right] = \frac{z\tau((z^2 + T^2)^{-1})}{z}, \quad z \in \mathbb{C}^+. \]

In particular, when \( z = iy \) is purely imaginary,

\[ G_T(iy) = \int_{\mathbb{R}} \frac{-iy d\mu_T(t)}{t^2 + y^2}, \]

(2.5) \[ H_T(iy) = \frac{\tau(T^2(T^2 + y^2)^{-1})}{-iy\tau((T^2 + y^2)^{-1})} = \frac{\int_{\mathbb{R}} t^2 d\mu_T(t)}{\int_{\mathbb{R}} t^2 + y^2}, \]

(2.6) \[ \int_{\mathbb{R}} \frac{-iy d\mu_T(t)}{t^2 + y^2}, \]
and
\[ \Re L_T(iy) = \frac{1}{2} \left[ \tau(\log |iy - T|) + \tau(\log |iy + T|) \right] \]
\[ = \frac{1}{2} \tau(\log(T^2 + y^2)) = \frac{1}{2} \int_{\mathbb{R}} \log(t^2 + y^2) \, d\mu_T(t). \]

(2.7)

The symmetry property is inherited by the subordination functions.

**Lemma 2.5.** Suppose that \( T_1, T_2 \in \text{Log}^+(\tau) \) are selfadjoint, have symmetric distributions, and are free. Let \( T = T_1 + T_2 \), and let \( \omega_1, \omega_2 \) be the subordination functions provided by Theorem 2.1. Then we have
\[ -\omega_j(-z) = \omega_j(z), \quad z \in \mathbb{C}^+, \quad j = 1, 2. \]

**Proof.** Since \(-T_1\) and \(-T_2\) are free and have the same distributions as \( T_1 \) and \( T_2 \), it follows that \(-T = (-T_1) + (-T_2)\) also has the same distribution as \( T \). If we set now
\[ \tilde{\omega}_j(z) = -\omega_j(-z), \quad z \in \mathbb{C}^+, \quad j = 1, 2, \]
we conclude that the equations (2.1) and (2.2) are also satisfied with \( \tilde{\omega}_j \) in place of \( \omega_j \). The lemma follows from the uniqueness of \( \omega_j \). \( \square \)

Given \( T \in \tilde{A} \) and \( p \in \mathbb{R} \), we use the notation
\[ m_p(T) = \tau(|T|^p) \in [0, +\infty] \]
for the \( p \)-th absolute moment of \( T \). If \( T \) is not injective, we set \( m_p(T) = \infty \) for \( p < 0 \). (Of course, it is possible that \( m_p(T) = \infty \) even when \( \ker T = 0 \).) We use a similar notation
\[ m_p(\mu) = \int_{\mathbb{C}} |z|^p \, d\mu \]
for an arbitrary probability measure \( \mu \) on \( \mathbb{C} \), and observe that, when \( T \) is normal,
\[ m_p(T) = m_p(\mu_T), \quad p \in \mathbb{R}. \]

Unless \( m_2(\mu) = 0 \), the Schwarz inequality shows that
\[ m_{-2}(\mu)m_2(\mu) \geq 1, \]
and equality occurs precisely when \( \mu \) is supported on a circle centered at the origin.

When \( \mu \) is a symmetric measure supported on the real line, equality occurs when
\[ \mu = \frac{1}{2}(\delta_a + \delta_{-a}) \]
for some \( a \in (0, +\infty) \). In particular, the inequality (2.8) is strict if the support of such a measure contains at least three points.

The following result is \[16\] Lemma 4.8. We provide a somewhat more transparent argument below.

**Lemma 2.6.** Suppose that \( T \in \tilde{A} \) is a symmetric selfadjoint operator whose distribution is not of the form \((\delta_a + \delta_{-a})/2\) for any \( a \in [0, +\infty) \). Then the function
\[ y \mapsto -iyH_T(iy), \quad y \in (0, +\infty), \]
is strictly increasing and its range is the open interval \((1/m_{-2}(T), m_2(T))\). If \( \mu = (\delta_a + \delta_{-a})/2 \), we have \(-iyH_T(iy) = a\) for every \( y > 0 \).
Proof. The limits at 0 and at $+\infty$ are easily calculated from \eqref{A3}, so we only verify that the function is increasing. The function $H_T$ maps $\mathbb{C}^+$ to itself,
\[
\lim_{y \to \infty} H_T(iy)/iy = 0,
\]
and
\[
-H_T(-z) = H_T(z), \quad z \in \mathbb{C}^+.
\]
It follows that $H_T$ has a Nevanlinna representation \[\text{1}\] of the form
\[
H_T(z) = \int_{\mathbb{R}} \frac{1 + tz}{t - z} \, d\rho(t)
\]
for some positive, finite, symmetric measure $\rho$ on $\mathbb{R}$ with support different from \{0\}. We can then calculate
\[
-iyH_T(iy) = \int_{\mathbb{R}} \frac{(1 + t^2)y^2}{y^2 + t^2} \, d\rho(t), \quad y > 0.
\]
Clearly, the integrand is a strictly increasing function of $y$ at points $t \neq 0$. The lemma follows. \qed

Remark 2.7. Suppose that $T \in \tilde{A}$ is a symmetric selfadjoint operator whose distribution is not of the form $(\delta_a + \delta_{-a})/2$ for any $a \in [0, +\infty)$. Denote by the Nevanlinna representation of $H_T$ as \eqref{2.9}. Then, Lemma 2.6 also implies
\[
\begin{array}{l}
(1) \quad \rho(\{0\}) = \lim_{y \to 0} -iyH_T(iy) = 1/m_{-2}(T), \\
(2) \quad \int_{\mathbb{R}} (1 + t^2) d\rho(t) = \lim_{y \to +\infty} -iyH_T(iy) = m_{2}(T).
\end{array}
\]

Proposition 2.8. Under the same assumption as Lemma 2.6, we have
\[
y^2 H_T'(iy) \leq \frac{m_2(T) - 1/m_{-2}(T)}{2} + iyH_T(iy), \quad y > 0.
\]
In particular, if $m_2(T)m_{-2}(T) \leq 3$, then $H_T'(iy) \leq 0$ for any $y > 0$.

Proof. We can calculate
\[
H_T'(z) = \int_{\mathbb{R}} \frac{1 + t^2}{(t - z)^2} \, d\rho(t).
\]
Hence, for $z = iy$, we have
\[
H_T'(iy) = \int_{\mathbb{R}} \frac{t^2 - y^2}{(t^2 + y^2)^2} (1 + t^2) \, d\rho(t).
\]
By Remark 2.7, we have
\[
y^2 H_T'(iy) = \int_{\mathbb{R}} \frac{y^2 (t^2 - y^2)}{(t^2 + y^2)^2} (1 + t^2) \, d\rho(t)
\]
\[
= 2 \int_{\mathbb{R}} \frac{y^2 t^2}{(t^2 + y^2)^2} (1 + t^2) \, d\rho(t) - \int_{\mathbb{R}} \frac{y^2}{t^2 + y^2} (1 + t^2) \, d\rho(t)
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R} \setminus \{0\}} (1 + t^2) d\rho(t) + iyH_T(iy)
\]
\[
= \frac{m_2(T) - 1/m_{-2}(T)}{2} + iyH_T(iy).
\]
Since $iyH_T(iy) \leq -1/m_{-2}(T)$, it follows that
\[
y^2 H_T'(iy) \leq \frac{m_2(T) - 3/m_{-2}(T)}{2} \leq 0
\]
provided that $m_2(T)m_{-2}(T) \leq 3$. \hfill \Box

The following result is used in Section 3 to determine the supports of Brown measures.

**Proposition 2.9.** Let $T_1, T_2 \in \mathcal{A}$ be nonzero, free, selfadjoint elements that have symmetric distributions, and let $\omega_1, \omega_2$ be the subordination functions provided by Theorem 2.1.

1. If $\omega_1(0) \in \mathbb{C}^+$, then $\omega_2(0) \in \mathbb{C}^+$. When this occurs, the closed intervals $[1/m_{-2}(T_1), m_2(T_1)]$ and $[1/m_{-2}(T_2), m_2(T_2)]$ have a common point. Unless $1/m_{-2}(T_1) = m_2(T_1)$, the common point is in $(1/m_{-2}(T_1), m_2(T_1))$.

2. We cannot have $\omega_1(0) = \omega_2(0) = \infty$.

3. We have $\omega_1(0) = \omega_2(0) = 0$ if and only if $\mu_{T_1} \{ \{0\} \} + \mu_{T_2} \{ \{0\} \} \geq 1$. When this occurs, we have $G_{T_1+T_2}(0) = \infty$.

4. If $\omega_1(0) = 0$ and $\omega_2(0) = \infty$, we have

$$m_2(T_2) \leq \frac{1}{m_{-2}(T_1)}.$$

In particular, $m_2(T_2) < +\infty$ and $m_{-2}(T_1) < +\infty$.

**Proof.** The first assertion in (1), and the equality $\omega_2(0) = H_{T_1}(\omega_1(0))$, follow from Corollary 2.3 and its proof. For the second assertion, we multiply this equality by $-\omega_1(0)$ and apply Lemma 2.6 to conclude that

$$-\omega_1(0)\omega_2(0) = -\omega_1(0)H_{T_1}(\omega_1(0))$$

$$= -\omega_2(0)H_{T_2}(\omega_2(0))$$

belongs to the the intervals $[1/m_{-2}(T_j), m_2(T_j)]$ since $\omega_j(0)$ is purely imaginary.

We prove (2) by contradiction. Suppose that $\omega_1(0) = \omega_2(0) = \infty$, and rewrite (2.2) as

$$\frac{\omega_2(iy)}{\omega_1(iy)} = \frac{iy}{\omega_1(iy)} - 1 + \frac{1}{\omega_1(iy)G_{T_1}(\omega_1(iy))}, \quad y > 0,$$

and recall the fact that

$$\lim_{t \to \infty} \frac{1}{iG_{T_1}(it)} = 1.$$

Letting $y \downarrow 0$, we see that $\lim_{y \downarrow 0} \omega_2(iy)/\omega_1(iy) = 0$. By symmetry, we also have $\lim_{y \downarrow 0} \omega_1(iy)/\omega_2(iy) = 0$, and these last two equalities cannot hold simultaneously.

The equality $\omega_1(0) = \omega_2(0) = 0$ means that 0 is the Denjoy-Wolff point of the maps $H_{T_1} \circ H_{T_2}$ and $H_{T_2} \circ H_{T_1}$. Since both of these maps preserve the positive imaginary line, we deduce that $H_{T_j}$ extends to 0 such that $H_{T_j}(0) = 0$. Moreover, the Julia-Carathéodory derivative of $H_{T_j}$ at zero is

$$\lim_{y \downarrow 0} \frac{H_{T_j}(iy)}{iy} = \frac{1}{\mu_j(\{0\})} - 1.$$

Thus, applying the chain rule for functions of one real variable, we see that 0 is the Denjoy-Wolff point of $H_{T_1} \circ H_{T_2}$ precisely when

$$\left( \frac{1}{\mu_1(\{0\})} - 1 \right) \left( \frac{1}{\mu_2(\{0\})} - 1 \right) \leq 1,$$

and this is equivalent to the inequality stated in (3). The fact that $G_{T_1+T_2}(0) = \infty$ follows from (2.2).
Finally, suppose that $\omega_1(0) = 0$ and $\omega_2(0) = \infty$. Multiplying (2.2) by $\omega_1$ or $\omega_2$, we obtain the identities
\[
\omega_1(iy)\omega_2(iy) = iy\omega_1(iy) + \omega_1(iy)H_{T_1}(\omega_1(iy)) = iy\omega_2(iy) + \omega_2(iy)H_{T_2}(\omega_2(iy)), \quad y > 0.
\]

Observe that $iy\omega_2(iy) \leq 0$ to conclude that
\[
-1/m_{-2}(T_1) \leq -m_2(T_2),
\]
thus establishing (4).

**Remark 2.10.** In the preceding result, it is useful to separate the cases in which one or both of the measures $\mu_{T_1}$ are of the form $(\delta_a + \delta_{-a})/2$ for some $a > 0$. Denote by $J_T$ or $J_{\mu_T}$ the interval $[1/m_{-2}(T), m_2(T)]$.

1. Suppose that the intervals $J_{T_1}$ and $J_{T_2}$ have interior points. Then one of the following mutually exclusive possibilities arises:
   (a) $\mu_{T_1}([0,1]) + \mu_{T_2}([0,1]) \geq 1$, in which case $\omega_1(0) = \omega_2(0) = 0$.
   (b) $\mu_{T_1}([0,1]) + \mu_{T_2}([0,1]) < 1$ but $J_{T_1}$ and $J_{T_2}$ have common interior points.
   In this case, $\omega_1(0)$ and $\omega_2(0)$ are finite and different from zero.
   (c) $J_{T_1}$ is on the right of $J_{T_2}$, that is, $0 < m_2(T_2) \leq 1/m_{-2}(T_1) < +\infty$, in which case $\omega_1(0) = 0$ and $\omega_2(0) = \infty$.
   (d) $0 < m_2(T_1) \leq 1/m_{-2}(T_2) < +\infty$, in which case $\omega_1(0) = \infty$ and $\omega_2(0) = 0$.

2. Suppose that $J_{T_1}$ has interior points but $J_{T_2}$ does not. In this case, $J_{T_2} = \{a_2\}$, $\mu_{T_2} = (\delta_{a_2} + \delta_{-a_2})/2$, and $a_2 > 0$ because $T_2$ was assumed to be nonzero. Then there are three mutually exclusive possibilities:
   (a) $a_2$ is an interior point of $J_{T_1}$, in which case $\omega_1(0)$ and $\omega_2(0)$ are finite and nonzero.
   (b) $J_{T_1}$ is on the right of $a_2$, that is, $0 < m_2(T_2) = a_2 \leq 1/m_{-2}(T_1) < +\infty$, in which case $\omega_1(0) = 0$ and $\omega_2(0) = \infty$.
   (c) $0 < m_2(T_1) \leq a_2 = 1/m_{-2}(T_2) < +\infty$, in which case $\omega_1(0) = \infty$ and $\omega_2(0) = 0$.
   A symmetric statement holds if $J_{T_1}$ is a singleton and $J_{T_2}$ has interior points.

3. Neither $J_{T_1}$ nor $J_{T_2}$ has interior points, so $J_{T_1} = \{a_1\}$ and $J_{T_2} = \{a_2\}$ with $a_1, a_2 > 0$. The alternatives, which are also easily verified by explicit calculation, are as follows:
   (a) $a_1 = a_2$, in which case $\omega_1(0) = \omega_2(0) = ia_1$.
   (b) $a_1 > a_2$, in which case $\omega_1(0) = 0$ and $\omega_2(0) = \infty$.
   (c) $a_1 < a_2$, in which case $\omega_1(0) = \infty$ and $\omega_2(0) = 0$.

**Remark 2.11.** Suppose $\omega_1(0) = 0$ and $\omega_2(0) = \infty$. The last calculation in the proof of Proposition 2.9 also implies the identity
\[
\lim_{y \downarrow 0} (iy\omega_2(iy)) = m_2(T_2) - \frac{1}{m_{-2}(T_1)},
\]
or, equivalently,
\[
\lim_{y \to 0} \frac{1}{i y \omega_2(iy)} = \frac{m_2(T_1)}{1 - m_2(T_2)m_2(T_1)}.
\]
This is the Julia-Carathéodory derivative of \(-1/\omega_2\) at zero. The Julia-Carathéodory derivative of \(\omega_1\) at zero can also be calculated by noting that
\[
\frac{\omega_1(iy)}{iy} = 1 + \frac{H_{T_2}(iy)}{iy} = 1 + \frac{1}{iy\omega_2(iy)} \omega_2(iy) H_{T_2}(\omega_2(iy)).
\]

Let now \(y \to 0\) and recall that \(\omega_2(iy) \to \infty\) to obtain
\[
\lim_{y \to 0} \frac{\omega_1(iy)}{iy} = 1 + \frac{m_2(T_2)m_2(T_1)}{1 - m_2(T_2)m_2(T_1)} = \frac{1}{1 - m_2(T_2)m_2(T_1)},
\]
where we use once again Lemma 2.6.

3. The Calculation of \(\mu_X\)

With \((A, \tau)\) as before, we consider \(*\)-free operators \(X_1, X_2 \in \text{Log}^+(\tau)\) such that \(X_2\) is \(R\)-diagonal. We exclude the trivial case in which either \(X_1\) or \(X_2\) is a scalar multiple of the identity. We also assume that there exists a Haar unitary operator \(U \in A\) that is \(*\)-free from \(\{X_1, X_2\}\). This is not a true restriction as we can always enlarge the algebra \(A\) without affecting the measure \(\mu_X\), where \(X = X_1 + X_2\). For every \(\lambda \in \mathbb{C}\), the selfadjoint operators
\[
|X - \lambda| = |(X_1 - \lambda) + X_2|
\]
and
\[
T^{(\lambda)} = |X_1 - \lambda + U^* X_2| = |U(X_1 - \lambda) + X_2|
\]
have the same distribution. Now, \(U(X_1 - \lambda)\) is \(R\)-diagonal and, according to [21, 15, 16],
\[
\tilde{\mu}_T^{(\lambda)} = \mu_1^{(\lambda)} \boxplus \mu_2,
\]
where
\[
(3.1) \quad \mu_1^{(\lambda)} = \tilde{\mu}_{|X_1 - \lambda|} \text{ and } \mu_2 = \tilde{\mu}_{|X_2|}.
\]
Denote by \(\omega_1^{(\lambda)}\) and \(\omega_2^{(\lambda)}\) the subordination functions associated with this free convolution. Thus,
\[
G_{\mu_1^{(\lambda)}}(\omega_1^{(\lambda)}(z)) = G_{\mu_2^{(\lambda)}}(\omega_2^{(\lambda)}(z)) = G_{\tilde{\mu}_{|X_1 - \lambda|}}(z),
\]
and
\[
\omega_1^{(\lambda)}(z) + \omega_2^{(\lambda)}(z) = z + \frac{1}{G_{\tilde{\mu}_{|X_1 - \lambda|}}(z)}, \quad z \in \mathbb{C}^+.
\]

We now introduce several subsets of the complex plane \(\mathbb{C}\) that are related with the the values of \(\omega_1^{(\lambda)}(0)\) and \(\omega_2^{(\lambda)}(0)\).
Definition 3.1. Let $X_1, X_2 \in \text{Log}^+(\tau) \setminus \mathbb{C}$ be free, let $X = X_1 + X_2$, suppose that $X_2$ is $R$-diagonal, and let $\omega_1^{(\lambda)}, \omega_2^{(\lambda)}$ be the subordination functions arising from the free convolution $\mu_{[X_1 - \lambda]} = \mu_{[X_1 - \lambda]} \oplus \mu_{[X_2]}$. We define subsets of $\mathbb{C}$ as follows:

$$S = \{ \lambda : \omega_1^{(\lambda)}(0) = \omega_2^{(\lambda)}(0) = 0 \},$$

$$F_1 = \left\{ \lambda : m_2(|X_2|) \leq \frac{1}{m_{-2}(|X_1 - \lambda|)} \right\},$$

$$F_2 = \left\{ \lambda : m_2(|X_1 - \lambda|) \leq \frac{1}{m_{-2}(|X_2|)} \right\},$$

$$F = F_1 \cap F_2,$$

$$\Omega = \mathbb{C} \setminus (S \cup F_1 \cup F_2).$$

Thus, the sets $S, F_1 \setminus F, F_2 \setminus F, \Omega$ and $\Omega$ form a partition of $\mathbb{C}$. Remark 2.10 allows for a different description of these sets in terms of the values of $\omega_1^{(\lambda)}(0)$ and $\omega_2^{(\lambda)}(0)$. For the following statement, $\ker(Y) \in \mathcal{A}$ denotes the orthogonal projection onto the null space of $Y$. Also, note that $\mu_{[Y]}$ is of the form $(\delta_a + \delta_{-a})/2$, $a > 0$, precisely when $|Y| = a$ or, equivalently, when $Y/a$ is unitary. If $Y$ is also $R$-diagonal, the unitary operator $Y/a$ is a Haar unitary.

Lemma 3.2. Suppose that $X_1, X_2 \in \text{Log}^+(\tau) \setminus \mathbb{C}$ are *-free and $X_2$ is $R$-diagonal.

1. We have $S = \{ \lambda : \tau(\ker(X_1 - \lambda)) + \tau(\ker(X_2)) \geq 1 \}$. In particular, $S$ is a finite set, disjoint from $F_1 \cup F_2$. If $S$ is not empty then $F_2 = \emptyset$.

2. $F$ consists of those $\lambda \in \mathbb{C}$ such that $|X_1 - \lambda| = |X_2| = |\|X_2\||$. Thus, $F$ contains at most two points. $F$ is contained in the boundary of $\Omega$. For $\lambda \in F$, we have $\omega_1^{(\lambda)}(0) \in \mathbb{C}^+$ and $\omega_2^{(\lambda)}(0) \in \mathbb{C}^+$.

3. The set $F_1 \setminus F$ consists of those $\lambda \in \mathbb{C}$ for which $\omega_1^{(\lambda)}(0) = 0$ and $\omega_2^{(\lambda)} = \infty$. In particular, $F_1$ is empty if $m_2(|X_2|) = +\infty$. If $F_1 \neq \emptyset$ then it is a closed set.

4. The set $F_2 \setminus F$ consists of those $\lambda \in \mathbb{C}$ for which $\omega_1^{(\lambda)}(0) = \infty$ and $\omega_2^{(\lambda)}(0) = 0$. In particular, $F_2$ is empty if $m_2(|X_1|) = +\infty$. In general, $F_2$ is $\emptyset$, $\{\tau(X_1)\}$, or a closed disk centered at $\tau(X_1)$.

5. We have $\lambda \in \Omega$ if and only if $\lambda \notin F$ and neither $\omega_1^{(\lambda)}(0)$ nor $\omega_2^{(\lambda)}(0)$ belongs to $\{0, \infty\}$. The sets $\Omega$ and $\Omega \cup S$ are open.

Proof. The first assertion in (1) follows from Proposition 2.10(3). If $S$ is not empty, then $\ker(X_2) \neq 0$. This implies $m_{-2}(|X_2|) = +\infty$, and hence $m_{2}(|X_1 - \lambda|) = 0$ for $\lambda \in F_2$. Since $X_1$ is not a scalar multiple of the identity, there is no such $\lambda$, in other words, $F_2 = \emptyset$.

Suppose now that $F \neq \emptyset$ and $\lambda_0 \in F$. The definition of the sets $F_1$ and $F_2$, combined with (2.8), shows that

$$\frac{1}{m_{-2}(|X_2|)} \leq m_2(|X_2|) \leq \frac{1}{m_{-2}(|X_1 - \lambda|)},$$

and

$$\frac{1}{m_{-2}(|X_1 - \lambda|)} \leq m_2(|X_1 - \lambda|) \leq \frac{1}{m_{-2}(|X_2|)}.$$ 

These inequalities imply

$$m_2(|X_2|) = \frac{1}{m_{-2}(|X_2|)} = m_2(|X_1 - \lambda|) = \frac{1}{m_{-2}(|X_1 - \lambda|)},$$

(3.2)
and thus $|X_2| = |X_1 - \lambda|$ is a scalar multiple of the identity. To see that $F$ is contained in the boundary of $\Omega$, suppose that $\lambda_0 \in F$, so $|X_1 - \lambda_0| = |X_2| = \|X_2\|$. Thus, $\ker X_2 = 0$ and so $S$ is empty. Also, $X_1 - \lambda_0 = aU$, where $a = \|X_2\|$ and $U$ is unitary. For $\lambda$ close to $\lambda_0$, we have

$$m_{-2}(|X_1 - \lambda|) = \frac{1}{a^2} \int_{|\zeta| = 1} \frac{d\mu_U(\zeta)}{|\zeta + \lambda_0 - \lambda|^2},$$

and this function of $\lambda$ has a positive Laplacian in a neighborhood of $\lambda_0$. It follows that there exist points $\lambda$ that are arbitrarily close to $\lambda_0$ such that $1/m_{-2}(|X_1 - \lambda|) < a^2 = m_{2}(|X_2|)$. Such points $\lambda$ do not belong to $F_1$. The inequality (2.8), combined with (3.2), then shows that

$$m_2(|X_1 - \lambda|) \geq \frac{1}{m_{-2}(|X_1 - \lambda|)} > \frac{1}{a^2} = \frac{1}{m_{-2}(|X_2|)},$$

so $\lambda$ does not belong to $F_2$ either. This completes the proof of (2).

The first two assertions in (3) follow from Remark 2.10. To show that $F_1$ is closed, it suffices to verify that the function $\lambda \mapsto m_{-2}(|X_1 - \lambda|)$ is upper semicontinuous. If this function is not identically $+\infty$, upper semicontinuity follows because

$$m_{-2}(|X_1 - \lambda|) = \inf_{\varepsilon > 0} \tau(|X_1 - \lambda|^{-2} + \varepsilon),$$

and the function $\lambda \mapsto \tau(|X_1 - \lambda|^{-2} + \varepsilon)$ is continuous for every $\varepsilon > 0$.

The first two assertions in (4) also follow from Remark 2.10. The fact that $F_2$ is a closed disk follows because

$$m_2(|X_1 - \lambda|) = \|X_1 - \lambda\|_2^2$$

$$= \|(X_1 - \tau(X_1)) - (\lambda - \tau(X_1))\|_2^2$$

$$= \|X_1 - \tau(X_1)\|_2^2 + |\lambda - \tau(X_1)|^2.$$

Thus, provided $m_2(X_2) < +\infty$, $F_2$ is nonempty precisely when $m_2(|X_1 - \tau(X_1)|) \leq 1/m_{-2}(X_2)$, in which case $F_2$ is a closed disk (possibly of zero radius).

Finally, (5) follows from Remark 2.10 and from the fact that $F_1 \cup F_2$ and $F_1 \cup F_2 \cup S$ are closed.

Remark 3.3. The argument above shows that $F$ is empty unless $X_2$ is a scalar multiple of a Haar unitary, and $X_1$ is normal with spectrum contained in a circle of radius $\|X_2\|$. Consider the case in which $X_2$ is a Haar unitary and $X_1$ is normal. Then $F$ consists of the centers of those circles of radius one which contain the spectrum of $X_1$. It the spectrum of $X_1$ contains three or more points, or if it consists of two points at distance 2, there is exactly one such point. We illustrate in Figure 3.1 the support of the Brown measure when the $X_1$ is distributed like the $3 \times 3$ diagonal matrix with spectrum $\{1, i, -1\}$. In this case, $F$ is a singleton and $F_2$ is a disk. In Figure 3.2, $X_1$ has spectrum $\{1, -1\}$, so $F = \{0\}$ in both cases. For the first image, $X_1$ is distributed like the $3 \times 3$ diagonal matrix with entries $1, 1, -1$ and $F_2$ is a disk. For the second one (also calculated and illustrated in Figure 1), $X_1$ is distributed like the $2 \times 2$ diagonal matrix with entries $1, -1$, and $F = F_2 = \{0\}$. If $X_1$ is normal and its spectrum consists of just two points with distance in $(0, 2)$, the set $F$ contains two points. $\lambda_2 \neq \lambda_1$ and the center of the disk $F_2$ is $(\lambda_1 + \lambda_2)/2$. Figure 3.3 illustrates the Brown measure when $X_1$ is distributed like the $2 \times 2$ diagonal matrix with spectrum $\{0, \sqrt{2}\}$. The pictures are obtained from simulations of random matrices that approximate $X_1 + X_2$. If $X_2$ is Haar
unitary and $X_1$ is unitary, we always have $0 \in F$, and $\tau(X_1)$ is the center of $F_2$; see also [9, Figure 6] for another illustration.

We can now calculate the logarithmic integral of $\mu_X$ in terms of $\mu_{X_1}, \mu_{X_2}$, and the values of $\omega_j^{(\lambda)}(0)$ for $\lambda \in \Omega$.

**Proposition 3.4.** Suppose that $\lambda \in \mathbb{C}\backslash S$. Then:

1. If $F_1$ is not empty, we have $m_{-2}(|X_1 - \lambda|) < +\infty$ and $\tau(\log(|X - \lambda|)) = \tau(\log |X_1 - \lambda|)$ for every $\lambda \in F_1$.
2. If $F_2$ is not empty, we have $m_{-2}(|X_2|) < +\infty$, and $\tau(\log(|X - \lambda|)) = \tau(\log |X_2|)$ for every $\lambda \in F_2$.
3. If $\lambda \in \Omega$, we have

$$
\tau(\log(|X - \lambda|)) = \frac{1}{2}\tau(\log(|X_1 - \lambda|^2 - \omega_1^{(\lambda)}(0)^2)) + \frac{1}{2}\tau(\log(|X_2|^2 - \omega_2^{(\lambda)}(0)^2))
- \log(-i\omega_1^{(\lambda)}(0) - i\omega_2^{(\lambda)}(0)).
$$

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Proof. Taking real parts, and setting \( z = iy \), \( y > 0 \), in (2.3), we obtain
\[
\int_{\mathbb{C}} \log |t + iy| \, d\mu_T(t) = \int_{\mathbb{C}} \log |t + \omega_1^{(\lambda)}(iy)| \, d\mu_T_1(t) + \int_{\mathbb{C}} \log |t + \omega_2^{(\lambda)}(iy)| \, d\mu_T_2(t) - \log |\omega_1^{(\lambda)}(z) + \omega_2^{(\lambda)}(z) - z|.
\]
In the case of symmetric measures, we can use (2.7) to rewrite this as
\[
\frac{1}{2} \int_{\mathbb{R}} \log(t^2 + y^2) \, d\mu_T(t) = \frac{1}{2} \int_{\mathbb{R}} \log(t^2 - \omega_1^{(\lambda)}(iy)^2) \, d\mu_T_1(t) + \frac{1}{2} \int_{\mathbb{R}} \log(t^2 - \omega_2^{(\lambda)}(iy)^2) \, d\mu_T_2(t) - \log(-i\omega_1^{(\lambda)}(iy) - i\omega_2^{(\lambda)}(iy) - y),
\]
where we used that fact that \(-i\omega_j^{(\lambda)}(iy) = |\omega_j^{(\lambda)}(iy)| \geq y\), hence \(\omega_j^{(\lambda)}(iy)^2 = -|\omega_j^{(\lambda)}(iy)|^2\). We apply this to \( \mu = \tilde{\mu}_{|X-\lambda|}, \mu_1 = \tilde{\mu}_{|X_1-\lambda|}, \) and \( \mu_2 = \tilde{\mu}_{X_2} \), and use the fact that
\[
\int_{\mathbb{R}} \log(t^2 + y^2) \, d\tilde{\mu}(t) = \int_{\mathbb{R}} \log(t^2 + y^2) \, d\mu(t)
\]
for an arbitrary Borel measure on \( \mathbb{R} \), to obtain
\[
\frac{1}{2} \int_{\mathbb{R}} \log(t^2 + y^2) \, d\mu_{|X-\lambda|}(t) = \frac{1}{2} \int_{\mathbb{R}} \log(t^2 - \omega_1^{(\lambda)}(iy)^2) \, d\mu_{|X_1-\lambda|}(t) + \frac{1}{2} \int_{\mathbb{R}} \log(t^2 - \omega_2^{(\lambda)}(iy)^2) \, d\mu_{|X_2|}(t) - \log(-i\omega_1^{(\lambda)}(iy) - i\omega_2^{(\lambda)}(iy) - y).
\]
Equivalently,
\[
\frac{1}{2} \tau(\log(|X - \lambda|^2 + y^2) = \frac{1}{2} \tau(\log(|X_1 - \lambda|^2 - \omega_1^{(\lambda)}(iy)^2))
\]
\[
+ \frac{1}{2} \tau(\log(|X_2|^2 - \omega_2^{(\lambda)}(iy)^2))
\]
\[
- \log(-i\omega_1^{(\lambda)}(iy) - i\omega_2^{(\lambda)}(iy) - y).
\]
Part (3) of the statement follows now by letting \( y \downarrow 0 \) in (3.4). Next, suppose that \( \lambda \in F_1 \). Subtract \( \log(-i\omega_2^{(\lambda)}(iy))\) from the last two lines of (3.4) to see that
\[
\frac{1}{2} \tau(\log(|X - \lambda|^2 + y^2) = \frac{1}{2} \tau(\log(|X_1 - \lambda|^2 - \omega_1^{(\lambda)}(iy)^2))
\]
\[
+ \frac{1}{2} \tau\left( \log\left( \frac{|X_2|^2}{-\omega_2^{(\lambda)}(iy)^2} + 1 \right) \right) - \log \frac{-i\omega_1^{(\lambda)}(iy)}{-i\omega_2^{(\lambda)}(iy)} + 1 - \frac{y}{-i\omega_2^{(\lambda)}(iy)}.
\]
Now, let \( y \downarrow 0 \) and observe that the last two terms above tend to 0; note that \( m_2(X_2) < +\infty \) because \( \lambda \in F_1 \). The resulting equation yields (1). To verify (most of) (2), simply switch the roles of \( |X_1 - \lambda| \) and \( |X_2| \) in the preceding argument. \( \square \)
Remark 3.5. For points $\lambda \in S$, there are two situations to consider. If $\tau(\ker(X_1 - \lambda)) + \tau(\ker X_2) > 1$, then (see \cite{7})

$$
\tau(\ker(X - \lambda)) = \tau(\ker(X_1 - \lambda)) + \tau(\ker X_2) - 1 > 0,
$$

and it follows that $\tau(\log(|X - \lambda|)) = -\infty$. According to \cite{7}, the only points $\lambda$ such that $\tau(\ker(X - \lambda)) > 0$ belong to $S$. If $\tau(\ker(X_1 - \lambda)) + \tau(\ker X_2) = 1$, it may happen that $\tau(\log(|X - \lambda|))$ is finite. In any case, the values of $\tau(\log(|X - \lambda|))$ for $\lambda$ in a finite set do not affect the calculation of $\mu_X$.

We are now ready to prove the main result of this section.

Theorem 3.6. Let $X_1, X_2 \in \log^+(\tau) \setminus \mathbb{C}$ be two free variables such that $X_2$ is R-diagonal, set $X = X_1 + X_2$, and let $\Omega$ be the set introduced in Definition 3.1. Then the support of the Brown measure $\mu_X$ is contained in the closure of $\Omega$. Moreover, $\mu_X|\Omega$ is absolutely continuous relative to area measure and it has a real-analytic density in that open set.

Proof. Since the set $S$ and the boundaries of $F_1$ and $F_2$ are contained in the closure of $\Omega$, we need to show that $\mu_X$ is equal to zero on the interiors of $F_1$ and $F_2$. We prove the equivalent statement that the logarithmic integral $\tau(\log(|X - \lambda|))$ is a harmonic function on these interiors. For the set $F_2$, this follows from Proposition 3.4(2). Indeed, $\tau(\log(|X - \lambda|))$ is constant on $F_2$. Now, Proposition 3.4(1) states that $m_{-2}(|X_1 - \lambda|) \leq \frac{1}{m_{2}(|X_2|)} < +\infty$ for every $\lambda \in F_1$. According to \cite[Theorem 4.5]{28}, this implies that $\mu_{X_1}(F_1) = 0$, and hence $\tau(\log(|X_1 - \lambda|))$ is a harmonic function for $\lambda$ in the interior of $F_1$. Since, according to the same proposition, $\tau(\log(|X - \lambda|)) = \tau(\log(|X_1 - \lambda|))$ for $\lambda \in F_1$, the desired conclusion about the support of $\mu_X$ follows.

To conclude the proof, we show that $\mu_X$ has a real-analytic density on the set $\Omega$. The formula established in Proposition 3.4(3) shows that it suffices to prove that $\omega_j^{(\lambda)}(0)$ are real-analytic functions of $\lambda \in \Omega$. For $\lambda \in \Omega$, $\omega_1^{(\lambda)}(0)$ is the unique fixed point of the map $\psi_\lambda = H_{\mu_2} \circ H_{\mu_{X_1 - \lambda}} : \mathbb{C}^+ \to \mathbb{C}^+$ and this map is not a conformal automorphism. Therefore $|\psi_\lambda'(\omega_1^{(\lambda)}(0))| < 1$ for every $\lambda \in \Omega$. We observe next that the map $\lambda \mapsto H_{\mu_{X_1 - \lambda}}(iy)$ is real-analytic for every $y > 0$. Indeed, \cite[62]{20} shows that

$$
(3.5) \quad H_{\mu_{X_1 - \lambda}}(iy) = \frac{\tau((\lambda - X_1)^* (\lambda - X_1) + y^2)^{-1})}{-iy\tau((\lambda - X_1)^* (\lambda - X_2) + y^2)^{-1})}.
$$

It follows that the map $(\lambda, w) \mapsto \psi_\lambda(w)$ is a real-analytic map on $\Omega \times i\mathbb{R}_+$. The real analyticity of $\omega_j^{(\lambda)}$ follows now from the implicit function theorem because $\partial_y(iy - \psi_\lambda(iy)) = i(1 - \psi_\lambda'(iy)) \neq 0$ for $iy = \omega_1^{(\lambda)}(0)$. The fact that $\omega_j^{(\lambda)}(0)$ is real-analytic follows the same way if we replace $\psi_\lambda$ by $H_{\mu_{X_1 - \lambda}} \circ H_{\mu_{X_2}}$.

Effective calculation of the density of $\mu_X$ depends on finding sufficiently explicit expressions for the subordination functions $\omega_j^{(\lambda)}(0)$ and for their derivatives. Recall that $\mu_X$ is equal to

$$
\frac{1}{2\pi} \Delta_X \tau(\log(|X - \lambda|)) = \frac{2}{\pi} \partial_X \partial_\lambda \tau(\log(|X - \lambda|)).
$$

Inside $\Omega$, these differential operators can be applied in the classical sense to yield the density of $\mu_X$ relative to area measure. We begin with the calculation of $2\partial_\lambda \tau(\log(|X - \lambda|))$, using Proposition 3.4(3).
Lemma 3.7. With the notation of Proposition 3.4, we have

$$2\partial_\lambda \tau(\log(|X - \lambda|)) = \tau((\lambda - X_1)^*(|\lambda - X_1|^2 - \omega_1^{(\lambda)}(0)^2)^{-1}), \quad \lambda \in \Omega.$$  

Proof. Differentiating the identity in Proposition 3.4(3), we obtain

$$2\partial_\lambda \tau(\log(|X - \lambda|)) = \partial_\lambda \tau(\log(|X_1 - \lambda|^2 - \omega_1^{(\lambda)}(0)^2)) + \partial_\lambda \tau(\log(|X_2|^2 - \omega_2^{(\lambda)}(0)^2))$$

$$- 2\partial_\lambda \log(-i\omega_1^{(\lambda)}(0) - i\omega_2^{(\lambda)}(0)).$$

The three terms on the right can be written as

\begin{align}
(3.6) \quad \partial_\lambda \tau(\log(|X_1 - \lambda|^2 - \omega_1^{(\lambda)}(0)^2)) &= \partial_\lambda \tau(\log((\lambda - X_1)^*(|\lambda - X_1|^2 - \omega_1^{(\lambda)}(0)^2)^{-1})
- 2\omega_1^{(\lambda)}(0)(\partial_\lambda \omega_1^{(\lambda)}(0))\tau((|\lambda - X_1|^2 - \omega_1^{(\lambda)}(0)^2)^{-1}),
\end{align}

and

\begin{align}
(3.7) \quad \partial_\lambda \tau(\log(|X_2|^2 - \omega_2^{(\lambda)}(0)^2)) &= -2\omega_2^{(\lambda)}(0)(\partial_\lambda \omega_2^{(\lambda)}(0))\tau((|X_2|^2 - \omega_2^{(\lambda)}(0)^2)^{-1}),
\end{align}

where we have used the differentiation rule under the trace pointed out in [13] and [16]. In order to simplify the result, we note that

$$\omega_1^{(\lambda)}(0) + \omega_2^{(\lambda)}(0) = \frac{1}{G_{\mu_j}^{(\lambda)}(\omega_j^{(\lambda)}(0))},$$

by Corollary 2.3. Thus (3.9) yields

$$- \frac{2}{\omega_1^{(\lambda)}(0) + \omega_2^{(\lambda)}(0)} = 2\omega_1^{(\lambda)}(0)\tau(|\lambda - X_1|^2 - \omega_1^{(\lambda)}(0)^2)^{-1})$$

$$= 2\omega_2^{(\lambda)}(0)\tau(|X_2|^2 - \omega_2^{(\lambda)}(0)^2)^{-1})$$

for \(j = 1, 2\). Combining this with (3.7) and (3.8). □

We proceed next to the second derivative.

Lemma 3.8. For every \(\lambda \in \Omega\), we have

\begin{align}
(3.10) \quad \frac{1}{2\pi} \Delta_\lambda \tau(\log|X - \lambda|) &= \frac{-\omega_1^{(\lambda)}(0)^2\tau((|\lambda - X_1|^2 - \omega_1^{(\lambda)}(0)^2)^{-1})(|\lambda - X_1|^2 - \omega_1^{(\lambda)}(0)^2)^{-1})}{\pi}
+ \frac{2}{\omega_1^{(\lambda)}(0)(\partial_\lambda \omega_1^{(\lambda)}(0))}\tau((|\lambda - X_1|)^*(|\lambda - X_1|^2 - \omega_1^{(\lambda)}(0)^2)^{-2}).
\end{align}

Proof. We write the result of Lemma 3.7 as

$$2\partial_\lambda \tau(\log|X - \lambda|) = \tau(Z_\lambda^*(Z_\lambda^*Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-1}),$$
where $Z_\lambda = \lambda - X_1$ satisfies $\mathcal{V}_\lambda Z_\lambda^* = 1$ and $\mathcal{V}_\lambda Z_\lambda = 0$. We apply $\mathcal{V}_\lambda$ to this identity and obtain

$$
2\mathcal{V}_\lambda \partial_\lambda \tau(\log(|X - \lambda|))
= \tau((Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-1})
- \tau((Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-1}(Z_\lambda - 2\omega_1^{(\lambda)}(0)\mathcal{V}_\lambda \omega_1^{(\lambda)}(0))(Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-1})
+ 2\omega_1^{(\lambda)}(0)(\mathcal{V}_\lambda \omega_1^{(\lambda)}(0))\tau(Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-2}).
$$

The equality $Z_\lambda(Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-1} = (Z_\lambda Z_\lambda^* - \omega_1^{(\lambda)}(0)^2)^{-1} Z_\lambda$, and the trace identity, allow us to replace the second term in the last expression by

$$
\tau((Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-1}(Z_\lambda Z_\lambda^* - \omega_1^{(\lambda)}(0)^2)^{-1} Z_\lambda Z_\lambda^*),
$$

and thus

$$
\mathcal{V}_\lambda \partial_\lambda \tau(\log(|X - \lambda|))
= \tau((Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-1})
- \tau((Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-1}(Z_\lambda Z_\lambda^* - \omega_1^{(\lambda)}(0)^2)^{-1} Z_\lambda Z_\lambda^*)
+ 2\omega_1^{(\lambda)}(0)(\mathcal{V}_\lambda \omega_1^{(\lambda)}(0))\tau(Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-2})
= \tau((Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-1})
- \tau((Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-1} Z_\lambda Z_\lambda^* (Z_\lambda Z_\lambda^* - \omega_1^{(\lambda)}(0)^2)^{-1})
+ 2\omega_1^{(\lambda)}(0)(\mathcal{V}_\lambda \omega_1^{(\lambda)}(0))\tau((Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-2})
= \tau((Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-1} Z_\lambda Z_\lambda^* (Z_\lambda Z_\lambda^* - \omega_1^{(\lambda)}(0)^2)^{-1})
+ 2\omega_1^{(\lambda)}(0)(\mathcal{V}_\lambda \omega_1^{(\lambda)}(0))\tau((Z_\lambda^* Z_\lambda - \omega_1^{(\lambda)}(0)^2)^{-2}).
$$

Canceling $Z_\lambda Z_\lambda^*$ yields the stated formula. 

The formula in Lemma 3.8 involves $\mathcal{V}_\lambda \omega_1^{(\lambda)}(0)$, which we proceed to calculate next. As in the proof of Theorem 3.6, $\psi_\lambda'$ denotes the derivative of $\psi$ for fixed $\lambda$, while $\mathcal{V}_\lambda \psi_\lambda(2y)$ denotes the $\mathcal{V}$ derivative for fixed $y$.

**Lemma 3.9.** With the notation $\psi_\lambda = H_{\bar{\mu}_1|X_1} \circ H_{\bar{\mu}|X_1 - \lambda}$, we have

$$
\mathcal{V}_\lambda \omega_1^{(\lambda)}(0) = \frac{(\mathcal{V}_\lambda \psi_\lambda)(\omega_1^{(\lambda)}(0))}{1 - \psi_\lambda'(\omega_1^{(\lambda)}(0))}, \quad \lambda \in \Omega.
$$

**Proof.** Apply $\mathcal{V}_\lambda$ to the equation $\omega_1^{(\lambda)}(0) = \psi_\lambda(\omega_1^{(\lambda)}(0))$. 

The derivative $(\mathcal{V}_\lambda \psi_\lambda)(\omega_1^{(\lambda)}(0))$ can be made more explicit.

**Lemma 3.10.** For $\lambda \in \Omega$, we have

$$
(\mathcal{V}_\lambda \psi_\lambda)(\omega_1^{(\lambda)}(0)) = -H'_{\mu_2}(\omega_2^{(\lambda)}(0)) \frac{\tau((\lambda - X_1)(\omega_1^{(\lambda)}(0)^2 - |\lambda - X_1|^2)^{-2})}{\omega_1^{(\lambda)}(0)\tau((\omega_1^{(\lambda)}(0)^2 - |\lambda - X_1|^2)^{-1})^2},
$$

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and

\[ H'_{\mu_2}(\omega_2^{(\lambda)}(0)) = \frac{\tau((\omega_2^{(\lambda)}(0))^2 + |X_2|^2)(\omega_2^{(\lambda)}(0)^2 - |X_2|^2)^{-2}}{\tau(\omega_2^{(\lambda)}(0))^2 - |X_2|^2)^{-1}} - 1. \]

Proof. Since the function \( z \mapsto H_{\mu_2}(z) \) is analytic in \( \mathbb{C}^+ \), we have

\[ (\nabla \lambda \psi)(z) = H'_{\mu_2}(H_{\mu_1}^{-1} - \lambda)(z)(\nabla \lambda H_{\mu_1}^{-1} - \lambda)(z). \]

We proceed to calculate the two derivatives in the right hand side, recalling that \( H_\mu(z) = (1/G_\mu(z)) - z \) and that \( \mu_2 \) is a symmetric measure:

\[
H'_{\mu_2}(z) = -\frac{G'_{\mu_2}(z)}{(G_{\mu_2}(z))^2} - 1 = \int_\mathbb{R} \frac{1}{(z-t)^2} d\mu_2 \left( \int_\mathbb{R} \frac{1}{z-t} d\mu_2 \right)^2 - 1
\]

\[
= \frac{1}{2} \int_\mathbb{R} \left[ \frac{1}{(z-t)^2} + \frac{1}{(z+t)^2} \right] d\mu_2 \left( \int_\mathbb{R} \frac{1}{z-t} d\mu_2 \right)^2 - 1 = \int_\mathbb{R} \frac{z^2 + t^2}{(z^2 - t^2)^2} d\mu_2 \left( \int_\mathbb{R} \frac{z}{z^2 - t^2} d\mu_2 \right)^2 - 1
\]

\[
= \frac{\tau((z^2 + |X_2|^2)(z^2 - |X_2|^2)^{-2})}{\tau(z^2 - |X_2|^2)^{-1}} - 1.
\]

We substitute now \( H_{\mu_1}^{-1}(\omega_1^{(\lambda)}(0)) = \omega_2^{(\lambda)}(0) \) to obtain

\[
(3.11) \quad H'_{\mu_2}(H_{\mu_1}^{-1}(\omega_1^{(\lambda)}(0))) = \frac{\tau((\omega_2^{(\lambda)}(0))^2 + |X_2|^2)(\omega_2^{(\lambda)}(0)^2 - |X_2|^2)^{-2})}{\tau(\omega_2^{(\lambda)}(0))^2 - |X_2|^2)^{-1}} - 1.
\]

Next,

\[
(\nabla \lambda H_{\mu_1}^{-1})(z) = \frac{-\nabla \lambda G_{\mu_1}^{-1}(z)}{(G_{\mu_1}^{-1}(z))^2} = \frac{-\nabla \lambda \tau(z^2 - |X_1 - \lambda|^2)^{-1}}{\tau(z^2 - |X_1 - \lambda|^2)^{-1}}
\]

\[
= \frac{-\tau((\lambda - X_1)(z^2 - |X_1 - \lambda|^2)^{-2})}{z \cdot \tau((z^2 - |X_1 - \lambda|^2)^{-1})},
\]

and substituting \( \omega_1^{(\lambda)}(0) \) for \( z \),

\[
(3.12) \quad (\nabla \lambda H_{\mu_1}^{-1})(\omega_1^{(\lambda)}(0)) = -\frac{\tau((\lambda - X_1)(\omega_1^{(\lambda)}(0)^2 - |X_1 - \lambda|^2)^{-2})}{\omega_1^{(\lambda)}(0)\tau((\omega_1^{(\lambda)}(0)^2 - |X_1 - \lambda|^2)^{-1})}.
\]

The stated formula is obtained by multiplying (3.11) and (3.12). \( \square \)

Since \( \tau(\log(|X - \lambda|)) \) is subharmonic, the Laplacian \( \nabla \lambda \delta \tau(\log(|X - \lambda|)) \) must be a nonnegative function in \( \Omega \). It may be worthwhile to observe that the formula for this function that follows from the preceding lemmas does obviously yield a real-valued function in \( \Omega \).

**Corollary 3.11.** If \( \lambda \in \Omega \) satisfying \( -\omega_2^{(\lambda)}(0)H_{\mu_2}(\omega_2^{(\lambda)}(0)) \geq m_{2HX_2})-1/m_{-2HX_2}), \) then the second term in Lemma 3.8 is non-negative. In particular, if \( m_{2HX_2})-1/m_{-2HX_2}) \leq 3 \) holds, then the second term in Lemma 3.8 is non-negative for any \( \lambda \in \Omega \) and the density formula is strictly positive.
Proof. By Proposition 2.8 we have
\[ \omega_2^{(\lambda)}(0) H_{\mu_2'}(\omega_2^{(\lambda)}(0)) \leq \frac{m_2(|X_2|) - 1/m_{-2}(|X_2|)}{2} + \omega_2^{(\lambda)}(0) H_{\mu_2}(\omega_2^{(\lambda)}(0)) \leq 0. \]
Hence, then second term in Lemma 3.8 is non-negative by Lemma 3.8 and Lemma 3.10. By the same proposition, if \( m_2(|X_2|)m_{-2}(|X_2|) \leq 3 \), we have \( H_{\mu_2}(iy) \leq 0 \) for all \( y > 0 \). Hence, the second term in Lemma 3.8 is non-negative for any \( \lambda \in \Omega \) and the density formula is strictly positive. \( \Box \)

4. Examples

Most examples below were already known. We show how the techniques developed above may lead to effective calculation. Examples 4.1 and 4.5 are in [10], where some of the techniques we use were first introduced. Example 4.2 illustrates the fact that the addition of freely independent variables does not lead to a new convolution operation on probability measures on \( C \). More precisely, the Brown \( \mu_{X_1+X_2} \) does not depend solely on \( \mu_{X_1} \) and \( \mu_{X_2} \) for general \( \ast \)-free variables \( X_1 \) and \( X_2 \), even when \( X_2 \) is \( R \)-diagonal.

The calculations require determining first the set \( \Omega \) and setting up the fixed point equation for \( \omega_1^{(\lambda)}(0) \). In case the result is sufficiently explicit (for instance, in Examples 4.1, 4.3), one can calculate the density of \( \mu_{X_1+X_2} \) by further differentiation of the expression in Lemma 3.7. Otherwise, we use Lemma 3.8 and calculate \( \partial_\lambda \omega_1^{(\lambda)}(0) \) in order to determine the second term in the formula provided therein.

Example 4.1. Brown measure of an \( R \)-diagonal operator. Observe that the measure \( \mu_{X_1+X_2} \) requires knowledge of \( \mu_{X_1+X_2} \), in addition to information about \( X_1 \). The particular case in which \( X_1 = 0 \) reflects the fact that the Brown measure of an arbitrary \( R \)-diagonal operator \( X \) is determined by \( \mu_\cdot X \). An actual calculation was first done in [15] (in the bounded case) and [16] (in general). That calculation can also be derived from our approach, as we indicate briefly.

Fix an arbitrary \( R \)-diagonal operator \( X \), and write it as \( X = X_1 + X_2 \), where \( X_1 = 0 \) and \( X_2 = X \). According to Definition 3.1 we have \( S = \{0\} \),
\[ F_1 = \{ \lambda : |\lambda| \geq m_2(X) \}, \quad F_2 = \{ \lambda : |\lambda| \leq 1/m_{-2}(X) \}, \]
and thus
\[ \Omega = \{ \lambda : 1/m_{-2}(X) \leq |\lambda| \leq m_2(X) \}. \]
Furthermore,
\[ \mu_1^{(\lambda)} = \frac{1}{2}(\delta_{|\lambda|} + \delta_{-|\lambda|}), \quad \text{and} \quad \mu_2 = \tilde{\mu}_X, \]
so
\[ H_{\mu_1^{(\lambda)}}(z) = -\frac{|\lambda|^2}{z}, \]
and
\[ \psi_\lambda(z) = H_{\mu_2}(H_{\mu_1^{(\lambda)}}(z)) = \frac{1}{G_{\mu_2}(-|\lambda|^2/z)} + \frac{|\lambda|^2}{z}. \]
For \( \lambda \in \Omega \), \( \omega_1^{(\lambda)}(0) \) is the fixed point of the function \( \psi_\lambda \). Using the notation
\[ M(z) = \int_R \frac{zt}{t - zt} d\mu_{X \cdot X}(t) \]
for the moment generating function of $\mu_{X'X}$, the equation $\psi_{\lambda}(\omega_{1}(\lambda)) = \omega_{1}(\lambda)$ is rewritten as

$$\frac{1}{|\lambda|^2 - \omega_{1}(\lambda)} = \frac{1 + M(\omega_{1}(\lambda)/|\lambda|^4)}{|\lambda|^2},$$

which implies

\begin{equation}
M(\omega_{1}(\lambda)/|\lambda|^4) = \frac{\omega_{1}(\lambda)}{|\lambda|^2 - \omega_{1}(\lambda)}.
\end{equation}

Next we use Lemma 3.7 and (4.1) to calculate

$$2\partial_\lambda \tau(\log(|X - \lambda|)) = \frac{\bar{\lambda}}{|\lambda|^2 - \omega_{1}(\lambda)} = \frac{\bar{\lambda} + M(\omega_{1}(\lambda)/|\lambda|^4)}{|\lambda|^2}.$$

This equation can be simplified using the $S$-transform of $\mu_{X'X}$, which we denote by $S$, which satisfies

$$S(M(z)) = \frac{M(z) + 1}{M(z)}.$$

Combined with (4.1), this yields

$$S(M(\omega_{1}(\lambda)/|\lambda|^4)) = \frac{1}{|\lambda|^2},$$

or, equivalently,

$$S^{(-1)}(1/|\lambda|^2) = M(\omega_{1}(\lambda)/|\lambda|^4).$$

Thus,

$$2\partial_\lambda \tau(\log(|X - \lambda|)) = \frac{1 + S^{(-1)}(1/|\lambda|^2)}{\lambda},$$

and applying $2\partial_\lambda/\pi$ we obtain the formula

$$\frac{(S^{(-1)})'(1/|\lambda|^2)}{2\pi|\lambda|^4}$$

for the density of $\mu_X$ at $\lambda \in \Omega$. This, of course, agrees with [10] Theorem 4.17. (See [20] for a more detailed exposition of this derivation of the formula for $\mu_X$.)

**Example 4.2. Nilpotent plus Haar unitary.** For this example, we assume that $X_1$ is distributed like an $n \times n$ complex matrix $A$ and $X_2$ is a Haar unitary. Later, we specialize to $n = 2$ and $A$ nilpotent. (The case in which $n = 2$ and $A$ is selfadjoint with eigenvalues $\pm 1$ was considered in [9]) As in the preceding example, we have $H_{\mu_2}(z) = -1/z$. Suppose that $\lambda \in \mathbb{C}$, and denote by $P_\lambda(z) = \det(z - |\lambda - A|^2)$, and by $a_j(\lambda), j = 1, \ldots, n$ the eigenvalues of $|\lambda - A|$. We have

$$P_\lambda(z) = \prod_{j=1}^{n} (z - a_j(\lambda)^2) = z^n - t(\lambda)z^{n-1} + \cdots + (-1)^{n-1}s(\lambda)z + (-1)^nd(\lambda),$$

where $t(\lambda)$ and $d(\lambda)$ are the trace and determinant of $|\lambda - A|^2$, respectively, while $s(\lambda)$ is the sum of all products of $n - 1$ distinct $a_j(\lambda)$. Then

$$G_{\mu_1(\lambda)}(z) = \frac{1}{2n} \sum_{j=1}^{n} \left[ \frac{1}{z - a_j(\lambda)} + \frac{1}{z + a_j(\lambda)} \right] = \frac{z}{n} \sum_{j=1}^{n} \frac{1}{z^2 - a_j(\lambda)^2} = \frac{zP_\lambda(z)}{nP_\lambda(z^2)},$$

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therefore
\[ H_{\mu_1(\lambda)}(z) = \frac{n P_\lambda(z^2) - z^2 P_\lambda'(z^2)}{z P_\lambda'(z^2)}, \]
and
\[ H_{\mu_2}(H_{\mu_1(\lambda)}(z)) = \frac{z P_\lambda'(z^2)}{z^2 P_\lambda'(z^2) - n P_\lambda(z^2)} = \frac{n z^{2n-1} - (n-1)t(\lambda) z^{2n-3} + \cdots + (-1)^{n-1} s(\lambda) z}{t(\lambda) z^{2n-2} + \cdots + (-1)^{n-1} n d(\lambda)}. \]
Observe that 0 and \( \infty \) are fixed points of this function for every \( \lambda \). However, \( \infty \) is the Denjoy-Wolff point only when \( t(\lambda) \leq n \), demonstrating again that \( F_2 \) is either empty of a closed disk (possibly of zero radius). Also, 0 is the Denjoy-Wolff point when \( s(\lambda) \leq nd(\lambda) \), and the boundary of \( F_1 \) is described by the real-algebraic equation \( s(\lambda) = nd(\lambda) \). When neither of these inequalities is satisfied, \( \omega_{\lambda}^{(1)}(0) \) is the unique solution in \( \mathbb{C}^+ \) of a polynomial equation of degree \( 2n - 2 \).

In the special case in which \( n = 2 \), the above formula specializes to
\[ (H_{\mu_2} \circ H_{\mu_1(\lambda)})(z) = -\frac{2 z^3 - t(\lambda) z}{t(\lambda) z^2 - 2 d(\lambda)}. \]
We have
\[ \omega_{\lambda}^{(1)}(0) = i \sqrt{\frac{t(\lambda) - 2d(\lambda)}{t(\lambda) - 2}}. \]
when \( t(\lambda) > 2 \) and \( t(\lambda) > 2d(\lambda) \). The measure \( \mu_{X_1+X_2} \) can be calculated quite explicitly when \( A \) is a nilpotent operator. Suppose that
\[ A = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix} \]
for some \( \varepsilon > 0 \). (Every nonzero nilpotent in \( M_2(\mathbb{C}) \) is unitarily equivalent to such an operator \( A \).) In this case, we have
\[ t(\lambda) = 2|\lambda|^2 + \varepsilon^2, \quad d(\lambda) = |\lambda|^4, \]
and thus \( \omega_{\lambda}^{(1)}(0) \) belongs to \( \mathbb{C}^+ \) provided that \( 2|\lambda|^2 + \varepsilon^2 > 2 \) and \( 2|\lambda|^2 + \varepsilon^2 > |\lambda|^4 \). The second condition amounts to
\[ |\lambda| < \sqrt{\frac{1 + \sqrt{1 + 2\varepsilon^2}}{2}}, \]
while the first condition is vacuous if \( \varepsilon > \sqrt{2} \) and it amounts to
\[ |\lambda| > \sqrt{\frac{2 - \varepsilon^2}{2}}. \]
for \( \varepsilon \leq \sqrt{2} \). Without any further calculation, we may conclude that the support of \( \mu_{X_1+X_2} \) is an annulus (or disk for \( \varepsilon < \sqrt{2} \)) that depends on \( \varepsilon \). In particular, \( \mu_{X_1+X_2} \) does not depend just on \( \mu_{X_1} \) and \( \mu_{X_2} \), in contrast with the case of free selfadjoint variables \( X_1 \) and \( X_2 \). We also observe that as \( \varepsilon \downarrow 0 \) the support of \( \mu_{X_1+X_2} \) converges, as may be expected, to the unit circle.

We proceed to calculate the density of \( \mu_{X_1+X_2} \), using the restatement of 3.7 as
\[ 2 \delta_\lambda \tau(\log(|X - \lambda|)) = \tau_2((\lambda - A)^* (|\lambda - A|^2 - \omega_{\lambda}^{(1)}(0)^2)^{-1}), \]
where $\text{tr}_2$ denotes the normalized trace on $M_2(\mathbb{C})$, and

$$\omega_1^{(\lambda)}(0)^2 = -\frac{2|\lambda|^2 + \varepsilon^2 - 2|\lambda|^4}{2|\lambda|^2 + \varepsilon^2 - 2}$$

for $\lambda$ in the annulus (or punctured disk)

$$\Omega_\varepsilon = \left\{ \lambda \in \mathbb{C} : \sqrt{\left(\frac{2 - \varepsilon^2}{2}\right)_+} < |\lambda| < \sqrt{\frac{1 + \sqrt{1 + 2\varepsilon^2}}{2}} \right\}.$$

Since

$$|\lambda - A|^2 - \omega_1^{(\lambda)}(0)^2 = \begin{bmatrix} |\lambda|^2 - \omega_1^{(\lambda)}(0)^2 & -\varepsilon\lambda \\ -\varepsilon\lambda & |\lambda|^2 + \varepsilon^2 - \omega_1^{(\lambda)}(0)^2 \end{bmatrix},$$

we have

$$(|\lambda - A|^2 - \omega_1^{(\lambda)}(0)^2)^{-1} = \frac{1}{D_\varepsilon(\lambda)} \begin{bmatrix} |\lambda|^2 + \varepsilon^2 - \omega_1^{(\lambda)}(0)^2 & \varepsilon\lambda \\ -\varepsilon\lambda & |\lambda|^2 - \omega_1^{(\lambda)}(0)^2 \end{bmatrix},$$

where

$$D_\varepsilon(\lambda) = (|\lambda|^2 - \omega_1^{(\lambda)}(0)^2)^2 - \varepsilon^2 \omega_1^{(\lambda)}(0)^2$$

is the determinant of $|\lambda - A|^2 - \omega_1^{(\lambda)}(0)^2$. Thus,

$$= \frac{1}{D_\varepsilon(\lambda)} \begin{bmatrix} \varepsilon\lambda^2 & \varepsilon\lambda \\ -\varepsilon(\lambda^2 + \varepsilon^2 - \omega_1^{(\lambda)}(0)^2) & -\varepsilon^2\lambda + \varepsilon\lambda(\lambda^2 - \omega_1^{(\lambda)}(0)^2) \end{bmatrix},$$

and we find that

$$2\partial_\lambda \tau(\log(|X - \lambda|)) = \bar{X}_\varepsilon f_\varepsilon(|\lambda|^2),$$

where $f(z)$ is a rational function of $z$ such that

$$f(|\lambda|^2) = \frac{|\lambda|^2 - \omega_1^{(\lambda)}(0)^2}{(|\lambda|^2 - \omega_1^{(\lambda)}(0)^2)^2 - \varepsilon^2 \omega_1^{(\lambda)}(0)^2}.$$ 

It follows that, in the annulus $\Omega_\varepsilon$, the measure $\mu_{X_1 + X_2}$ has density

$$\frac{1}{\pi} \gamma_\lambda(\bar{X} f(\lambda \bar{X})) = \frac{1}{\pi} \left[ f(|\lambda|^2) + |\lambda|^2 f'(|\lambda|^2) \right],$$

where $f'$ is the usual derivative of $f$. The integral of this density relative to area measure is

$$2 \int \sqrt{(1 + \sqrt{1 + 2\varepsilon^2})/2} \frac{r^2 f(r^2) + r^2 f'(r^2)r}{\sqrt{(2 - \varepsilon^2)/2}_+} d\rho = \int \frac{(1 + \sqrt{1 + 2\varepsilon^2})/2}{\sqrt{(2 - \varepsilon^2)/2}_+} (f(\rho) + \rho f'(\rho)) d\rho$$

$$= \rho f(\rho)|_{(1 + \sqrt{1 + 2\varepsilon^2})/2}^{(1 + \sqrt{1 + 2\varepsilon^2})/2}.$$

This quantity is easily evaluated to be 1 if we observe that $\omega_1^{(\lambda)}(0) = 0$ when $|\lambda| = \sqrt{(1 + \sqrt{1 + 2\varepsilon^2})/2}$. Thus, $\mu_\lambda$ is absolutely continuous.
Example 4.3. Addition to a free Haar unitary. As in the preceding example, $X_2$ is a Haar unitary operator, but $X_1$ is an arbitrary variable *-free from $X_2$. As seen above, $H_{\mu_2}(z) = H_{\mu_1|X_2}(z) = -1/z$, so $\omega_1^{(\lambda)}(0)$ is the Denjoy-Wolff point of the map $w \mapsto \psi_{\lambda}(w) = -1/H_{\mu_1}(w)$. In particular, for $\lambda \in \Omega$, $\omega_1^{(\lambda)}(0)$ is the unique solution $w \in \mathbb{C}^+$ of the equation

$$w = -\frac{1}{H_{\mu_1}(w)}.$$

We use this equation (rather than Lemma 3.10) in order to calculate $\overline{\partial}_A \omega_1^{(\lambda)}(0)$ and the second term in the formula of Lemma 3.8. Using the notation

$$Y(\lambda) = (\omega_1^{(\lambda)}(0)^2 - |X_1 - \lambda|^2)^{-1},$$

it is easily verified that

$$G_{\mu_1}(\omega_1^{(\lambda)}(0)) = \omega_1^{(\lambda)}(0)\tau(Y(\lambda)).$$

Recalling that $H_{\mu_1}(w) = (1/G_{\mu_1}(w)) - w$, the fixed point equation reduces to

$$\omega_1^{(\lambda)}(0)^2 = 1 + \frac{1}{\tau(Y(\lambda))},$$

and thus

$$2\omega_1^{(\lambda)}(0)\overline{\partial}_A \omega_1^{(\lambda)}(0) = -\frac{\overline{\partial}_A \tau(Y(\lambda))}{Y} = \frac{\tau(\overline{\partial}_A Y(\lambda)) Y(\lambda)^2}{\tau(Y(\lambda))}.$$

Solving for $\overline{\partial}_A \omega_1^{(\lambda)}(0)$ yields

$$\overline{\partial}_A \omega_1^{(\lambda)}(0) = \frac{\tau((\lambda - X_1)Y(\lambda)^2)}{2\omega_1^{(\lambda)}(0)\text{Var}(Y(\lambda))},$$

where $\text{Var}(Y(\lambda)) = \tau(Y(\lambda)^2) - \tau(Y(\lambda))^2$. The second term in Lemma 3.8 becomes

$$\frac{1}{\pi} \frac{|\tau((\lambda - X_1)Y(\lambda)^2)|^2}{\text{Var}(Y(\lambda))},$$

and this shows that the density of $\mu_{X_1 + X_2}$ is positive in the entire open set $\Omega$.

Example 4.4. Addition to a circular operator. Consider now a circular operator $X_2$ with variance $t$, and an arbitrary variable $X_1$, *-free from $X_2$. In this case, $\tau(\ker(X_2)) = 0$, $m_2(|X_2|) = t$ and $1/m_{-2}(|X_2|) = 0$. Hence,

$$\Omega = \{\lambda : m_{-2}(|X_1 - \lambda|) > 1/t\}.$$

Since $\mu_2$ is the semicircular distribution with variance $t$, we have

$$1 = \frac{1}{G_{\mu_2}(z)} + tG_{\mu_2}(z) = z.$$
Using equations (2.1) and (2.2), we obtain
\[ -tG_{\mu_2}(\omega_1^{(\lambda)}(0)) = -tG_{\mu_2}(\omega_2^{(\lambda)}(0)) = \frac{1}{G_{\mu_2}(\omega_2^{(\lambda)}(0))} - \omega_2^{(\lambda)}(0) = \omega_1^{(\lambda)}(0), \]
which can be rewritten as
\[ (4.3) \tau(\omega_1^{(\lambda)}(0)^2 - |X_1 - \lambda|^2)^{-1} = \frac{1}{t}. \]
Denote \( Y(\lambda) = (\omega_1^{(\lambda)}(0)^2 - |X_1 - \lambda|^2)^{-1} \). Applying implicit differentiation, we have
\[ \partial_\lambda \omega_1^{(\lambda)}(0) = \frac{\tau((\lambda - X_1)Y(\lambda)^2)}{2\omega_1^{(\lambda)}(0) \cdot \tau(Y(\lambda))^2}. \]
The second term in Lemma 3.8 becomes
\[ \frac{1}{\pi} \tau((\lambda - X_1)Y(\lambda)^2)^2. \]
This formula was proved in [28, Theorem 4.2] (see also [10] for a special case and in [5] for the unbounded case). It is shown in [5, Section 7] that \( \mu_{X_1+X_2} \) is absolutely continuous in \( \mathbb{C} \) and the density is bounded by \( 1/\pi t \).

**Example 4.5. Addition to a circular Cauchy operator.** Suppose that \( c_1 \) and \( c_2 \) are \(*\)-free standard circular operators, set \( X_2 = c_1 c_2^{-1} \), and let \( X_1 \) be an arbitrary variable that is \(*\)-free from \( X_2 \). It is shown in [17] that \( \tilde{\mu}_{|X_2|} \) is the standard Cauchy distribution, and hence \( H_{\mu_2}(z) = i \) is constant. The fixed point equation shows that \( \omega_1^{(\lambda)}(0) = i \) for every \( \lambda \in \mathbb{C} \), in particular, \( \partial_\lambda \omega_1^{(\lambda)}(0) = 0 \). Lemma 3.8 shows now that the density of \( \mu_{X_1+X_2} \) is equal to
\[ \frac{1}{\pi} \tau(\lambda - X_1)^2 + 1)^{-1}(|\lambda - X_1|^2 + 1)^{-1}, \quad \lambda \in \mathbb{C}. \]
This formula was first proved in [17, Corollary 4.6].

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