A method of moments approach to pricing double barrier contracts driven by a general class of jump diffusions

Bjorn Eriksson† and Martijn Pistorius
Imperial College London,
Department of Mathematics,
South Kensington Campus,
London SW7 2AZ, UK
Email: \{b.eriksson08,m.pistorius\}@imperial.ac.uk

Abstract

We present the method of moments approach to pricing barrier-type options when the underlying is modelled by a general class of jump diffusions. By general principles the option prices are linked to certain infinite dimensional linear programming problems. Subsequently approximating those systems by finite dimensional linear programming problems, upper and lower bounds for the prices of such options are found. As numerical illustration we apply the method to the valuation of several barrier-type options (double barrier knockout option, American corridor and double no touch) under a number of different models, including a case with deterministic interest rates, and compare with Monte Carlo simulation results. In all cases we find tight bounds with short execution times. Theoretical convergence results are also provided.

Keywords: Method of Moments, Jump Diffusion, Lévy process, Polynomial interest rate, Linear Programming, Double Barrier option
1 Introduction

Barrier and barrier-type options are among the most widely and frequently traded exotic options, especially in the area of Foreign Exchange, which makes their valuation an important topic. For example, a double barrier option is cancelled depending on whether or not two levels have been crossed before maturity. Since the pay-off of a barrier option depends on the entire path of the underlying, it is clear that its valuation is more involved than that of a standard European type option.

A well-documented empirical observation is that financial returns data typically possess features such as asymmetry, heavy tails and excess kurtosis, which cannot be captured by the classical geometric Brownian motion model (GBM). Related is the well known fact that under the GBM model it is not possible to calibrate option prices to the volatility surface. One of the successful modifications that has been proposed is to introduce jumps in the evolution and work with Lévy models. Popular examples of such Lévy models are VG, CGMY, NIG, GH and KoBoL. This approach is classical by now and we refer to the standard references [16], [3] and [1] for further financial motivations for the use of jump models, background and references. In a separate development (see e.g. [4] and [9]) it was noted that commodity prices often display features such as mean-reversion and jumps that are clearly not captured by the geometric Brownian motion model, and it was proposed to employ jump-diffusion models to incorporate those effects.

The valuation of barrier options has attracted a good deal of attention and there exists currently a body of literature dealing with different aspects of pricing barrier options. In particular, for double barrier options, [10] and [15] developed a Laplace transform approach in the geometric Brownian motion setting. [17] derived semi-analytical expressions in a jump-diffusion setting with exponential jumps, also using a transform approach. [2] considered double no-touches in a setting with exponential jumps, allowing the process dynamics to change after a barrier is breached. [7] used eigenfunction expansions to price double barrier options in a CEV setting.

The mentioned papers exploit specific features of the model under consideration and can therefore not be readily generalized and applied to a different settings. A general approach, based on a characterization of the moments of the underlying process, was followed by [13] to price a class of exotic options. In a diffusion setting [13] derived upper and lower bounds for the price of exotic options in terms of semidefinite programs, and provided theoretical and numerical convergence results for these bounds. Before that, using linear programming, [11] developed a method of moments algorithm to calculate first exit time probabilities and moments of a diffusion.

In this paper we will follow a method of moments approach to price double barrier options in a general setting of a polynomial-type jump-diffusion. We will also allow the rate of discounting, which is typically taken to be constant, to be a function of time and underlying. We will now briefly describe the method of moments approach. The first step is to express the option price as an integral with respect to two measures, the (discounted) expected exit measure and the (discounted) expected occupation measure. The former describes the law of the
underlying at expiration or at crossing the barrier, while the latter described the law of the process until this moment. Restricting ourselves to pay-offs that are piecewise polynomial functions of the underlying, the value can then be expressed as a linear combination of moments of these two measures. The moments of those two measures are subsequently shown to satisfy an infinite dimensional linear system. To the price can thus be associated the two linear programming problems of minimization and maximization of the latter criterion over the spaces of measures. By adding conditions on the moments that guarantee that a given sequence is equal to the moments of a measure, one is led to an infinite dimensional linear programming problem or a semi-definite programming problem. By restricting to a finite number of moments we arrive at a finite dimensional linear programming problem or a semi-definite programming problem.

We will numerically illustrate this method for a American corridor and double no touch and double knock-out option under different models, by solving linear programming problems. In all cases we find tight bounds, with short execution times. We also provide a convergence proof to show that the values of the linear programming problems converge monotonically to the value of the option if the number of moments employed is increased.

The remainder of the paper is organized as follows. In Section 2 we specify the model and the problem setting. Section 3 is devoted to the method of moments, describes the algorithm and provides a convergence proof. Section 4 provides the implementation and numerical examples. Proofs are deferred to the Appendix.

2 Problem setting

Assume that the underlying $X$ evolves according to the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t + \lambda(t, X_t)dJ_t, \quad X_0 = x_0,$$

(1)

where $W$ is a Brownian motion and $J$ is a pure jump Lévy process (that is, a process with independent stationary increments without Gaussian component), and $\sigma, b,$ and $\lambda$ are given functions that will be specified below. In this setting we will value a barrier option of knock-out type with pay-off $h(T, X_T)$ at the maturity time $T$ if the underlying has not left a set $B$ before time $T$ and that pays a stream of payments $g(s, X_s)$ until the first moment $\tau_B$ that $X$ leaves $B$ or time $T,$ whichever comes earlier. Modelling the risk neutral discounting as a function $r = r(t, X_t)$ of $t$ and $X_t$ it follows by standard arbitrage pricing principles that the value $v$ of this contract is given by

$$v = E \left[ e^{-\alpha_T} h(T, X_T) I_{\{T < \tau_B\}} + \int_0^{\tau_B \wedge T} e^{-\alpha_s} g(s, X_s) ds \right],$$

(2)

where $\tau_B = \inf\{t \geq 0 : X_t \notin B\}$ and

$$\alpha_t = \int_0^t r(s, X_s) ds.$$

(3)
We will restrict ourselves to the case that the functions $h, g$ are piecewise polynomial functions, that is, for some partitions $\{C_i\}$ and $\{D_i\}$ of $[0,T] \times \mathbb{R}$,

$$
    h(t, x) = \sum_{i=1}^{k} h_i(t, x) I_{\{(t,x) \in C_i\}}, \quad g(t, x) = \sum_{i=1}^{l} g_i(t, x) I_{\{(t,x) \in D_i\}},
$$

(4)

where $h_i, g_i$ are polynomials in $(t, x)$. Note that many contracts have a pay-off function that is of this form, including call and put options and straddles. We observe that we will then be able to express the value $v$ in terms of moments of certain probability measures, reducing the calculation of $v$ to the calculation of these moments. Further, we will assume that $X$ is a ‘polynomial’ process, that is, in eqs. (1) and (3)

$$
    b(t, x), \sigma^2(t, x), \lambda(t, x) \text{ and } r(t, x) \text{ are polynomials,}
$$

such that (11) admits a unique (weak) solution. Associated to $X$ is the infinitesimal generator that acts on functions $f$ in its domain as

$$
    Af = \frac{\partial f}{\partial t} + b \frac{\partial f}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2} + Bf,
$$

(5)

where $Bf$ is an integro-differential operator given by

$$
    Bf(t, x) = \int_{\mathbb{R}} \left[ f(x + \lambda(t, x)y) - f(t, x) - \lambda(t, x) \frac{\partial f}{\partial x}(t, x)y I_{\{|y| < 1\}} \right] \Lambda(dy),
$$

(6)

where $\Lambda$ denotes the Lévy jump measure of $X$. Note that the operator $A$ maps polynomials to polynomials, which is an essential property needed in the moment approach, as shown in the next section.

We next present some models that are included in our setting.

**Examples.**

- The classical geometric Brownian motion satisfies the SDE

$$
    dX = bX dt + \sigma X dW
$$

where $W$ denotes a one-dimensional Brownian motion, and has the infinitesimal generator

$$
    Af(x) = b x \frac{df}{dx} + \frac{x^2 \sigma^2}{2} \frac{d^2 f}{dx^2},
$$

(7)

- Lévy models (for an overview see e.g. [16] or [3]) For example, the variance gamma process (14) evolves according to $dX = b_1 dt + dZ$, where $b_1$ is a constant and $Z$ is a Lévy process with Lévy measure

$$
    \eta(dx) = \frac{C}{x} e^{-Mx} I_{\{x > 0\}} dx + \frac{C}{|x|} e^{-G|x|} I_{\{x < 0\}} dx,
$$

(8)

with $C, G, M$ positive constants, and its infinitesimal generator is specified by

$$
    Af(x) = b \frac{df(x)}{dx} + \int [f(x + y) - f(x)] \eta(dx).
$$

(9)

where $b = b_1 - \int_{-1}^{1} x \eta(dx)$. 


Additive processes with polynomial time-dependent coefficients (see e.g. [3] for background), obtained by taking $b$, $\sigma^2$ and $\lambda$ to be polynomials of $t$ only. For example, $X$ evolving according to $dX = b_1(t)dt + dZ$ for a Lévy process $Z$ and a polynomial $b_1(t)$.

Affine processes, obtained by taking $\lambda$ constant and $b$, $\sigma^2$ affine functions in $x$, independent of $t$ (see e.g. [5] for applications of affine models in finance). An example of an affine diffusion is the Cox Ingersoll Ross (CIR) model, which is a mean-reverting diffusion satisfying the SDE

$$dX = a(b - X)dt + \sigma \sqrt{X}dW, \quad a, b > 0,$$

with the infinitesimal generator

$$Af(x) = a(b - x) \frac{df}{dx} + \frac{\sigma^2 x^2}{2} \frac{d^2f}{dx^2}.$$  

### 3 Method of moments

Denoting by $\nu$ and $\mu$ the discounted exit location measure and the discounted occupation measure given by

$$\nu(A) = E[e^{-\alpha \tau}I_{\{(\tau, X_\tau) \in A\}}], \quad \mu(A) = E \left[ \int_0^\tau e^{-\alpha s}I_{\{(s, X_s) \in A\}}ds \right],$$

for Borel sets $A \in \mathcal{B}([0, T] \times \mathbb{R})$, the value $v$ of the contract can be expressed as

$$v = \int h(t, x)\nu(dt, dx) + \int g(t, x)\mu(dt, dx)$$

The measure $\mu$ describes the distribution of the process $(t, X_t)$ before the stopping time $\tau$ whereas the measure $\nu$ describes the distribution upon termination at $\tau$. For example, in the case of a up-and-out barrier option at level $B_u$, termination occurs if the barrier $B_u$ is crossed or the maturity $T$ is reached.

In view of the form (11) of $g$ and $h$, $v$ can be expressed in terms of the moments of $\mu$ and $\nu$, as follows:

$$v = \sum_{i,j} \sum_m d_{i,j}(m)\nu_{i,j}^{(m)} + \sum_{i,j} \sum_m b_{i,j}(m)\mu_{i,j}^{(m)},$$

where we denote by $m_{i,j} = \int t^i x^j m(dt, dx)$ the $ij$th moment of a measure $m$ and by $\nu^{(m)} = \nu(\cdot \cap C_m)$ and $\mu^{(m)} = \mu(\cdot \cap D_m)$ the restrictions of $\nu$ and $\mu$ to $C_m$ and $D_m$, and where $d_{i,j}(m)$ and $b_{i,j}(m)$ are some constants.

#### 3.1 The adjoint equation

The measures $\mu$ and $\nu$ are closely related to each other and to the generator of the underlying process $X$. Informally, for suitably regular $f$ and all bounded stopping times $\tau$ Dynkin’s lemma yields that

$$E[e^{-\alpha \tau}f(\tau, X_\tau)] - E[f(0, X_0)] = E \left[ \int_0^\tau e^{-\alpha (A_f - \tau f)}(t, X_t)dt \right],$$
where $Af$ is given in (15), which can be expressed in terms of the measures $\nu$ and $\mu$ as
\[
\int f(t, x)\nu(dt, dx) = f(0, x_0) + \int (Af - rf)(t, x)\mu(dt, dx).
\] (15)
The identity (15) is called the basic adjoint equation (See e.g. [11]). As noted before, a formal application of the generator shows that $A$ maps polynomials to polynomials. More specifically, by applying (15) to a monomial $f_{ij}(t, x) = t^ix^j$ we obtain the following infinite system of equations linking the moments of $\mu$ and $\nu$:
\[
\int t^ix^j\nu(dt, dx) - x^j_01_{i=0} = \sum_{k,l} c_{k,l}(i,j) \int t^kx^l\mu(dt, dx).
\] (16)
where $(Af_{ij} - rf_{ij})(t, x) = \sum_{k,l} c_{k,l}(i,j)t^kx^l$, or equivalently, in compact notation,
\[
\nu_{i,j} - x^j_01_{i=0} = \sum_{k,l} c_{k,l}(i,j)\mu_{k,l}.
\] (17)
The following result provides sufficient conditions to justify this informal analysis:

**Proposition 1** Suppose that for all $k = 0, 1, 2, \ldots$
\[
\int |y|^k(1 \wedge y^2)\Lambda(dy) + E \left[ \int_0^T e^{-\alpha t}|X_t|^k dt \right] < \infty.
\] (18)
Then eqn. (17) holds for all $i, j = 0, 1, 2, \ldots$.

The proof is deferred to the Appendix.

**Remark.** Partial barrier or forward starting barrier options can also be included in this setting by slightly adapting the definitions. With $\tilde{\tau} = \inf\{t \in [T_0, T] : X_t \notin B\}$ it holds that
\[
E[e^{-\alpha \tilde{\tau}} f(\tilde{\tau}, X_\tilde{\tau})] - E[f(T_0, X_{T_0})] = E \left[ \int_{T_0}^{\tilde{\tau}} e^{-\alpha t}(Af - \alpha f)(t, X_t) dt \right]
\]
which leads to the adjoint equation
\[
\int fd\tilde{\nu} = \int fd\tilde{\nu}_0 + \int (Af - \alpha f)d\tilde{\mu}.
\]

### 3.1.1 Truncation

We restrict ourselves now to contracts that are knocked out if $X$ leaves a finite interval, so that $v$ is given by (2) with $h(t, x) = 0$ for $x \notin B := [b_-, b_+]$. If the minimum $\lambda$ of $\lambda(t, x)$ over $[0, T] \times [b_-, b_+]$ is strictly positive, it is possible to derive for such double knock out contracts a modification of the adjoint equations that is valid without integrability restrictions. To that end, note that a double knock-out option becomes worthless at the first time that a jump occurs of size larger than $L_+ := (b_+ - b_-)/\lambda$ or smaller than $L_- := (b_- - b_+)/\lambda$, since any such jump will take $X$ out of the interval $[b_-, b_+]$. As such jumps occur at a rate
\( \lambda_s = \Lambda(\mathbb{R} \setminus [L_-, L_+]) \), independent of the smaller size jumps and the diffusion part, it follows that the value \( v \) of the contract does not change if we replace \( \Lambda \) and \( r \) by
\[
\tilde{\Lambda} = \Lambda(\cdot \cap [L_-, L_+]), \quad \tilde{r} = r + \lambda_s,
\]
which corresponds to replacing the underlying \( X \) by the process \( \tilde{X} \) that is ‘killed’ when the first jump occurs with size larger than \( L_+ \) or smaller than \( L_- \). In summary, using \( \nu_{ij}, \mu_{ij} \) to denote the \( ij \)th moments of the exit and occupation measures \( \nu, \mu \) of the killed process \( \tilde{X} \), we have the following result (with a proof in the Appendix):

**Corollary 1** If \( B = [b_-, b_+] \) and \( \Delta > 0 \), then \( v \) is given by (12) where \( \nu_{ij} \) and \( \mu_{ij} \) solve the system of equations
\[
\nu_{ij} - x_j 1_{i=0} = \sum_{k,l} \tilde{c}_{k,l}(i,j) \mu_{k,l}
\]
where the coefficients \( \tilde{c}_{k,l}(i,j) \) are defined by
\[
\tilde{A} f_{ij} - \tilde{r} f_{ij} = \sum_{k,l} \tilde{c}_{k,l}(i,j) t^k x^l,
\]
with \( \tilde{A} \) defined in (5)–(6) with \( \Lambda \) replaced by \( \tilde{\Lambda} \).

### 3.2 Linear programs

By optimizing over the pair of measures that satisfies the adjoint equations, the value \( v \) can be bounded, as follows:
\[
\inf_{\nu,\mu} L(\nu,\mu) \leq v \leq \sup_{\nu,\mu} L(\nu,\mu)
\]
which concerns linear programs over the measures, since \( L \) is the linear functional of the moments of \( \nu \) and \( \mu \) given by
\[
L(\nu,\mu) := \sum_{i,j} \sum_m d_{i,j}(m) \nu_{i,j}^{(m)} + \sum_{i,j} \sum_m b_{i,j}(m) \mu_{i,j}^{(m)}
\]
and the infimum and the supremum are taken over the pairs of measures \((\nu,\mu)\) supported on \((0,T] \times \mathbb{R} \setminus B, [0,T] \times B)\) that satisfy the linear adjoint equations derived before. To formulate these optimization problems completely in terms of moment sequences, we need to express the condition that \( \mu \) and \( \nu \) be measures in terms of their moments, as in general there is no guarantee that any solution of the system (17) is the moment sequence of some measure. The problem to determine whether a given sequence is the moment sequence of some measure and, if so, whether this measure is uniquely determined (in which case the measure is called moment-determinate) has been extensively studied. It is known, see [19], that the Cramér condition
\[
\int_{\mathbb{R}} e^{c|x|} m(dx) < +\infty, \text{ for some } c > 0
\]
is a sufficient condition for a measure $m$ to be moment determinate. In particular, any measure with compact support is moment-determinate. Further, the following Hausdorff conditions are necessary and sufficient for a given sequence $m_i$ to correspond to a moments of a measure $m$ with support on the interval $[a, b]$ (see e.g. [8]):

$$
\sum_{j=0}^{n} \binom{n}{j} (-1)^j \tilde{m}_{j+k} \geq 0 \quad \forall n, k = 0, 1, 2, \ldots
$$

(21)

where the $\tilde{m}_i$ are linear combinations of the $m_j$, as follows:

$$
\tilde{m}_l = (b - a)^{-l} \sum_{i=0}^{l} \binom{l}{i} (-a)^{l-i} m_i.
$$

In fact, the $\tilde{m}_i$ are themselves the moments of a measure $\tilde{m}$ that is the affine transformation of $m$ supported on $[0, 1]$. That these conditions are necessary immediately follows by observing that $\int_{0}^{1} y^k (1 - y)^n \tilde{m}(dy)$ is non-negative and by expressing $\tilde{m}_i$ in terms of $m_j$. More generally, given an array $(m_{ij}, i, j = 0, 1, \ldots)$, the two dimensional Hausdorff-conditions (e.g. [18])

$$
\sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} (-1)^{i+j} \tilde{m}_{i+l,j+k} \geq 0. \quad \forall n, m, k, l = 0, 1, 2, \ldots
$$

(22)

where the $\tilde{m}_{i,j}$ are related to the $m_{i,j}$ by

$$
\tilde{m}_{k,l} = (b - a)^{-k}(d - c)^{-l} \sum_{i=0}^{k} \sum_{j=0}^{l} \binom{k}{i} \binom{l}{j} (-a)^{k-i}(-c)^{l-j} m_{i,j},
$$

are necessary and sufficient conditions to guarantee that there exists a measure $m$ supported on $[a, b] \times [c, d]$ such that $m_{ij} = \int_{a}^{b} \int_{c}^{d} x^i y^j m(dx, dy)$. See [18] for proofs and further background on problems of moments.

### 3.2.1 Unbounded support

In the case that the measure has unbounded support there also exists conditions to characterize a sequence of moments. These conditions are no longer linear but can be conveniently be formulated in terms of so-called moment and localizing matrices (their definitions are recalled in the appendix). For a sequence to be equal to the moments of some measure it is necessary and sufficient that these matrices are positive definite. See [12] or [6] for a proof of this fact.

### 3.3 Approximations and convergence

To be able to calculate lower and upper bounds for the value $v$ we approximate the optimization problems in (20) by restricting the total number of moments used to $N$. If $B$ is a finite interval, employing the moment conditions (21) and (22)
results in the following (finite) linear programming problems:

\[
\begin{align*}
\nu^{(N)}_\pm & := \max \min \left\{ \sum_{i,j} \sum_m d_{i,j}(m) \nu_{i,j}^{(m)} + \sum_{i,j} \sum_m b_{i,j}(m) \mu_{i,j}^{(m)} \right\} \\
\text{subject to} & \\
\bullet & \nu_{i,j} - x_{i=0}^j 1_{i=0} = \sum_{k,l} \tilde{c}_{k,l}(i,j) \mu_{k,l}, \ i + j \leq N, \ k + l \leq N \\
\text{with } & \nu = \sum_m \nu^{(m)}, \ \mu = \sum_m \mu^{(m)} \\
\bullet & \text{conditions (21)/(22) for } \nu_{i,j}^{(m)}, \ \mu_{i,j}^{(m)}, \ i + j \leq N
\end{align*}
\]

In the case that the set \( B \) is a half-line, the measures in question will not have bounded support and as a consequence in the above optimization problem the linear moment conditions (21)/(22) are replaced by the quadratic moment conditions described in Section 3.2.1. The resulting optimization problems are then semi-definite programming problems. [13] provided convergence results for this SDP approach in a diffusion setting for Asian, European and single barrier options. Restricting to the case that \( B \) is a finite interval we show that the values \( v^{(N)}_\pm \) of the linear programs converge:

**Proposition 2** Suppose that the system (19) has a unique solution and that \( B \) is a finite interval. Then

\[
v^{(N)}_\pm \uparrow v \quad \text{and} \quad v^{(N)}_\pm \downarrow v.
\]
as \( N \to \infty \).

**Remark.** The presented approach can in principle be extended to a multi-dimensional jump-diffusion \( X \) with polynomial coefficients. For example, if \( B \) is a hyper-cube the adjoint equations and the moment conditions take analogous forms. The limitation in practice will be the capacity of the LP and SDP solvers to deal with large size programs.

### 4 Numerical examples

For the numerical examples we have used Matlab and the LP solver lp_solve. The problems were set up in Matlab and then solved using the Matlab interface to lp_solve. The numerical outcomes were compared with Monte Carlo results, implemented in Matlab using the Euler scheme.

We will illustrate the method by valuing four different options.

#### 4.1 A double knockout barrier option driven by the Geometric Brownian motion.

In this benchmark example we consider a European double knock-out call option with underlying \( S_t \) assumed to evolve as a geometric Brownian motion. The value
\(v\) of such an option is given by

\[
v = e^{-rT}E[(S_T - K)^+ I_{\{\tau \geq T\}}] \quad \text{where} \quad \tau = \inf\{t \geq 0 : S_t \notin \{B_d, B_u\}\}.
\]

For the ease of notation we will now drop the discounting \(e^{-rT}\). As a geometric Brownian motion has continuous paths, we know that the time-space process \((t, S_t)\) will exit \([0, T] \times [B_d, B_u]\) either if \(S\) hits one of the barriers \(B_u\) or \(B_d\) or maturity is reached, so that the support for the exit location measure \(\nu\) is

\[
\Omega = \{(0, T) \times \{B_d\}\} \cup \{(T) \times \{B_d, B_u\}\} \cup \{(0, T) \times \{B_u\}\}.
\]

The set \(\Omega\) is partitioned into four parts with the restricted measures

- \(\nu^{(1)}\) and \(\nu^{(2)}\) with support on \([0, T]\)
- \(\nu^{(3)}\) and \(\nu^{(4)}\) with support on \([B_d, K]\) and \([K, B_u]\) respectively.

The expected occupation measure \(\mu\) is supported on the domain \([0, T] \times [B_d, B_u]\) – See also Figure 1 for an illustration.

![Diagram](image_url)

Figure 1: Domain for the measures of double knockout option driven by the Geometric Brownian motion.

Here the line-segments \([B_d, K]\) and \([K, B_u]\) were chosen in such a way that the pay-off function restricted to each of those line segments is zero or linear. Further note that the measures \(\nu^{(i)}\) can all be characterized by Hausdorff moment conditions, since they are supported on line-segments. The value of the option is then given by

\[
v = \int_{\Omega} (x - K)^+ I_{\{t=T\}} \nu(dt, dx) = \nu^{(4)}_1 - K \nu^{(4)}_0.
\]
Table 1: Numerical results for the Double knockout Barrier option with the underlying modelled by the Geometric Brownian motion. The option parameters are $B_u = 5$, $B_d = 1$, $K = 1.3$, $x_0 = 2$, $t_0 = 0$ and $T = 1$.

Using the form (7) of the infinitesimal generator of the Geometric Brownian motion, the basic adjoint equation for this problem can be seen to be

$$B_u^m \nu_n^{(1)} + B_d^m \nu_n^{(2)} + T^n \nu_m^{(3)} + T^n \nu_m^{(4)} - n \mu_{n-1,m} - \left( (b m + \frac{\sigma^2}{2} m (m - 1)) \right) \mu_{n,m} = \nu_0^m x_0^m.$$

This is valid for all $n, m$ such that $n + m \leq N$, when we are using all moments up to degree $N$. To complete the setup of the problem we add the LP moment conditions for the measures $\nu^{(i)}$ and $\mu$ with support as given above.

The numerical results for two given sets of parameter values are given in Table 1. We can see that we get fast convergence to the exact solution, which was calculated using the formula from [15].

### 4.2 A double knockout barrier option driven by the Variance gamma process.

In this example we consider again a double knock-out option but now driven by a Variance Gamma process, which as described in Section 2. Since the Variance Gamma process is a finite activity jump process, it will not hit the barrier but jump across it. As a consequence the exit location measure $\nu$ is supported on

$$\Omega = \{[0, T] \times [B_u, \infty)\} \cup \{\{T\} \times [B_d, B_u]\} \cup \{[0, T] \times (-\infty, B_d]\}.$$

In order to be able to calculate the value $v$ of the option using the LP moment conditions, we will adjust the Lévy measure $\eta$ as described in Section 3.1 to achieve bounded support. In this case we observe that any jump with absolute size larger than $L = B_u - B_d$ will trigger an immediate knock-out. We note that the probability that no such a jump occurs before maturity is $p_* = e^{-\lambda_* T}$ where $\lambda_* = \eta(\mathbb{R} \setminus [-L, L])$ with $\eta$ the Variance Gamma Lévy measure given in (6). We
thus truncate the Lévy measure \( \eta \) by restricting it to absolute jump-sizes smaller \( L \):

\[
\tilde{\eta}(dx) = I_{\{|x|<L\}} \eta(dx).
\]

The value of the option in terms of moments of these measures is then

\[
v = p_\ast \int_\Omega (x-K)^+ I_{\{t=T\}} \nu(dt, dx) = p_\ast [\nu^{(4)}_1 - K \nu^{(4)}_0].
\]

where the support of the truncated exit location measure is

\[
\tilde{\Omega} = \{[0,T] \times [B_u, B_u + L]\} \cup \{[T] \times [B_d, B_u]\} \cup \{[0,T] \times [B_d - L, B_d]\}
\]

and the four restrictions of \( \nu \) are

\[
\begin{align*}
\nu^{(1)} &\text{ supported on } [0,T] \times [B_u, B_u + L] \\
\nu^{(2)} &\text{ supported on } [0,T] \times [B_d - L, B_d] \\
\nu^{(3)} &\text{ and } \nu^{(4)} \text{ supported on } [B_d, K] \text{ and } [K, B_u]
\end{align*}
\]

The domain of the truncated measures is shown in Figure 2. Denoting by

\[
c(k) = \int_{-L}^{L} y^k k(y)dy
\]

the moments of the truncated Lévy measure \( \tilde{\eta} \) and taking note of the form (9) the infinitesimal generator, we find the basic adjoint equation

\[
\nu^{(1)}_{n,m} + \nu^{(2)}_{n,m} + T \nu^{(3)}_m + \nu^{(4)}_m - n \mu_{n-1,m} - bm \mu_{n,m-1} - \sum_{k=1}^{m} \binom{m}{k} c(k) \mu_{n,m-k} = 0^n x_0^m
\]

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Case 1  \( G = 8 \)  \( M = 12 \)  \( p_* = 1.0000 \)

| Degree of moment | 7 | 8 | 9 | 10 |
|------------------|---|---|---|----|
| Upper Bound      | 0.5045 | 0.5030 | 0.5022 | 0.5017 |
| Lower Bound      | 0.4946 | 0.4983 | 0.4987 | 0.4994 |
| Relative Error   | 0.13% | 0.09% | 0.05% | 0.07% |
| Cpu Time         | 1.262s | 3.245s | 6.236s | 17.936s |
| Monte Carlo      | 0.5002 | Std Error 0.0005 |

Case 2  \( G = 4 \)  \( M = 10 \)  \( p_* = 1.0000 \)

| Degree of moment | 6 | 7 | 8 | 9 |
|------------------|---|---|---|----|
| Upper Bound      | 0.5158 | 0.5151 | 0.5135 | 0.5115 |
| Lower Bound      | 0.4857 | 0.4886 | 0.4943 | 0.4958 |
| Relative Error   | 0.39% | 0.17% | 0.24% | 0.19% |
| Monte Carlo      | 0.5027 | Std Error 0.0008 |

Case 3  \( G = 8 \)  \( M = 8 \)  \( p_* = 1.0000 \)

| Degree of moment | 5 | 6 | 7 | 8 |
|------------------|---|---|---|----|
| Upper Bound      | 0.5133 | 0.5078 | 0.5049 | 0.5033 |
| Lower Bound      | 0.4682 | 0.4894 | 0.4917 | 0.4957 |
| Relative Error   | 1.71% | 0.14% | 0.20% | 0.04% |
| Monte Carlo      | 0.4993 | Std Error 0.0006 |

Case 4  \( G = 3 \)  \( M = 6 \)  \( p_* = 0.9998 \)

| Degree of moment | 6 | 7 | 8 | 9 |
|------------------|---|---|---|----|
| Upper Bound      | 0.5277 | 0.5237 | 0.5197 | 0.5182 |
| Lower Bound      | 0.4672 | 0.4720 | 0.4745 | 0.4772 |
| Relative Error   | 1.32% | 1.24% | 1.39% | 1.27% |
| Monte Carlo      | 0.5041 | Std Error 0.0011 |

| Case | Degree of moment | Upper Bound | Lower Bound | Relative Error | Monte Carlo | Std Error |
|------|------------------|-------------|-------------|----------------|-------------|-----------|
| 1    | 7, 8, 9, 10      | 0.5045, 0.5030, 0.5022, 0.5017 | 0.4946, 0.4983, 0.4987, 0.4994 | 0.13%, 0.09%, 0.05%, 0.07% | 0.5002 | 0.0005 |
| 2    | 6, 7, 8, 9      | 0.5158, 0.5151, 0.5135, 0.5115 | 0.4857, 0.4886, 0.4943, 0.4958 | 0.39%, 0.17%, 0.24%, 0.19% | 0.5027 | 0.0008 |
| 3    | 5, 6, 7, 8      | 0.5133, 0.5078, 0.5049, 0.5033 | 0.4682, 0.4894, 0.4917, 0.4957 | 1.71%, 0.14%, 0.20%, 0.04% | 0.4993 | 0.0006 |
| 4    | 6, 7, 8, 9      | 0.5277, 0.5237, 0.5197, 0.5182 | 0.4672, 0.4720, 0.4745, 0.4772 | 1.32%, 1.24%, 1.39%, 1.27% | 0.5041 | 0.0011 |

Table 2: Numerical results for the Double knockout barrier option driven by the Variance Gamma process. The option parameters are in all cases, \( B_u = 1, B_d = -1 \) and \( K = -0.3 \), the model parameters are \( b = 0.2, x_0 = 0, T = 1 \) and \( C = 0.5 \). The truncation size is \( L = 2 \) in all cases, and \( p_* \) is the probability that no jump of absolute size larger than 2 occurs before \( T \).

for all \( n, m \) such that \( m + n \leq N \). As before, to complete the LP problem we add the appropriate LP moment conditions for the measures \( \nu^{(i)} \) and \( \mu \) with support as given above.

Numerical results can be found in the Table 2. The relative error was calculated using the arithmetic mean of the upper and lower bounds and the Monte Carlo outcome (taking the latter as the ‘true’ result). Studying the results we see that we get tight bounds within 8 or 9 moments. Beyond 10 or 11 moments we experienced instabilities with the LP solver.

In Case 1 the execution times are shown, which should be compared to the execution time for the Monte Carlo simulation that was around 52 minutes. As we employed a basic Euler scheme for the Monte Carlo simulation the speed of convergence of the Monte Carlo simulation could be improved by using more specialised Monte Carlo schemes and also by changing to a compiling programming language. However considering the time difference, it should be clear that the method of moments will still be considerably faster.
4.3 The American Corridor under a CIR model with constant interest rate

An American corridor is a contract traded in the foreign exchange markets that pays a continuous rate until either the underlying leaves the corridor or maturity is reached, whichever comes earlier (see for example [20] or [21] for background). The value of this contract is given by

\[ v = E \left[ \int_0^\tau e^{-rt} dt \right] = \int_\Theta \mu(dt, ds) = \mu_{0,0} \quad \text{where} \]

\[ \tau = T \wedge \inf\{t \geq 0 : S_t \notin [B_d, B_u]\}, \]

with \( \Theta = \{(0, T] \times [B_d, B_u]\} \). We will model the underlying as a Cox Ingersoll Ross (CIR) process, evolving according to the SDE (10). Since the CIR model is continuous the supports of the different measures are given as follows:

- \( \nu^{(1)} \) and \( \nu^{(2)} \) supported on \([0, T]\)
- \( \nu^{(3)} \) supported on \([B_d, B_u]\)
- \( \mu \) is supported on \([0, T] \times [B_d, B_u]\)

In view of the form of the infinitesimal generator for the CIR process (11) we can now assemble the basic adjoint equation for this problem,

\[ B_u^m \nu_n^{(1)} + B_d^m \nu_n^{(2)} + T^n \nu_n^{(3)} - n \mu_{n-1,m} + \]

\[ (am + r) \mu_{n,m} - \left( abm + \frac{\sigma^2}{2} m(m-1) \right) \mu_{n,m-1} = t^0 x^n_0 \]

for all \( m, n \) such that \( m + n \leq N \), and add appropriate LP moment conditions as before.

The results are reported in Table 3. We observe that tight bounds are achieved, and that in cases 1 and 2 the upper bounds are accurate for 9-10 moments, with relative errors 0.16\% and 0.12\%. We also see that the speed of convergence varies with the particular parameter values.

4.4 Double No Touch option under the exponential Variance Gamma process with a non constant interest rate

A double no touch option pays one unit at maturity if the underlying has not crossed either of the barriers \( B_d \) or \( B_u \). Its value can be expressed as

\[ v = E[e^{-\alpha \tau} I_{\{\tau \geq T\}}] \quad \text{where} \]

\[ \tau = \inf\{t \geq 0 : S_t \notin [B_d, B_u]\}. \]

Letting the underlying \( S_t \) be an exponential Variance Gamma process we note that the stopping time \( \tau \) is equivalent to

\[ \tau = \inf\{t \geq 0 : X_t \notin [\log(B_d), \log(B_u)]\} \]
Table 3: Numerical results for the American Corridor model by the Cox Ingersoll Ross model. The problem parameters are $B_d = 0.5$, $B_u = 1.5$, $x_0 = 1$ and $T = 1$, for the model parameters $a = 0.5$ and $b = 1$ are fixed.

where $X_t$ is a Variance Gamma process. Under no-arbitrage pricing the process $e^{-\alpha t}S_t = e^{-\alpha t} + X_t$ needs to be a martingale which is equivalent to the requirement that

$$b(t) + \int_{-\infty}^{\infty} (e^x - 1)\eta(dx) = r(t)$$

so that $b(t)$ is determined by our choice of $r(t)$. For the Variance Gamma process

$$c = \int_{-\infty}^{\infty} (e^x - 1)\eta(dx) = C \left( \log \left( \frac{G}{1+G} \right) + \log \left( \frac{M}{1-M} \right) \right)$$

With these points in mind we find,

$$(A - r(t))f(t, x) = \frac{\partial f}{\partial t} + b(t)\frac{\partial f}{\partial x} + \int [f(t, x + y) - f(t, x)]\eta(dx) - r(t)f(t, x)$$

$$= \frac{\partial f}{\partial t} + (r(t) - c)\frac{\partial f}{\partial x} + \int [f(t, x + y) - f(t, x)]\eta(dx) - r(t)f(t, x)$$

We have chosen to study interest rates of the type

$$r(t) = r_b + r_s t^2.$$

Since there is no need too split the exit location measure at maturity, the domain of $\nu$ only needs to be split into three parts $K_1 = [0, T] \times [0, B_d]$, $K_2 = [0, T] \times [B_u, \infty)$ and $K_3 = \{T\} \times [B_d, B_u]$, so that in this case the value $v$ of the option can be expressed in terms of moments as follows

$$v = E[e^{-\alpha \tau} I(\tau \geq T)] = \int_{\Omega} I(t = T)\nu(dt, ds)$$

$$= \int_{K_3} \nu^{(3)}(ds) = \nu_0^{(3)}$$
where as before $\nu^{(i)} = \nu(\cdot \cap K_i)$. In terms of moments the basic adjoint equation is then given by

$$
\nu^{(1)}_{n,m} + \nu^{(2)}_{n,m} + T^n \nu^{(3)}_m - t_0^n x_0^m = n \mu_{n-1,m} - r_b \mu_{n,m} - r_s \mu_{n+2,m} - (r_b - c) m \mu_{n,m-1} + r_s m \mu_{n+2,m-1} + \sum_{k=1}^{m \choose k} c(k) \mu_{n,m-k},
$$

where $c(k)$ is given in (23).

The numerical results are presented in Table 4. In all cases we see tight bounds nicely agreeing with the Monte Carlo simulation result. We also observe that the upper bound is very accurate already for a small number of moments. For example, for 7 moments, the relative errors of the upper bounds in the three different cases are 0.032%, 0.076% and 0.069%, respectively.

### Table 4: Numerical results for the double no touch option with barriers at $B_u = 2$ and $B_d = 0.5$ and maturity $T = 1$, driven by an exponential Variance Gamma process with parameters $C = 0.5$, $G = 8$ and $M = 12$, and with $S_0 = 1$. Note that the upper bounds are very accurate already for 6-7 moments, with relative errors less than 0.1%.

| Case | $r_b = 0.05$ | $r_s = 0.05$ |
|------|---------------|---------------|
| Degree of moment | 6 | 7 | 8 | 9 |
| Upper Bound | 0.9356 | 0.9355 | 0.9355 | 0.9355 |
| Lower Bound | 0.8453 | 0.8757 | 0.9042 | 0.9143 |
| Relative Error | 4.79% | 3.17% | 1.64% | 1.10% |
| Monte Carlo | 0.9352 | Std Error | 0.0002 |

| Case | $r_b = 0.05$ | $r_s = 0.1$ |
|------|---------------|---------------|
| Degree of moment | 6 | 7 | 8 | 9 |
| Upper Bound | 0.9203 | 0.9201 | 0.9200 | 0.9200 |
| Lower Bound | 0.8196 | 0.8533 | 0.8836 | 0.8957 |
| Relative Error | 5.38% | 3.56% | 1.91% | 1.26% |
| Monte Carlo | 0.9194 | Std Error | 0.0002 |

| Case | $r_b = 0.1$ | $r_s = 0.1$ |
|------|---------------|---------------|
| Degree of moment | 7 | 8 | 9 | 10 |
| Upper Bound | 0.8752 | 0.8752 | 0.8752 | 0.8752 |
| Lower Bound | 0.7980 | 0.8319 | 0.8449 | 0.8565 |
| Relative Error | 6.68% | 4.73% | 2.84% | 2.09% |
| Monte Carlo | 0.8746 | Std Error | 0.0002 |

### 5 Conclusion

We have presented a method of moments approach that can be used to price double barrier-type options driven by ‘polynomial’ jump-diffusions, allowing for a non-constant (deterministic or stochastic) interest rate. An infinite-dimensional linear program was derived, which was then approximated, depending on the choice of moment conditions, either by a sequence of LP problems, or by a sequence of SDP problems. Although the SDP-type problems may be theoretically more appealing as the SDP method naturally handles measures with unbounded support, further development of stable SDP solvers would be needed for this method.
to be truly usable in practice. Since, on the other hand, the LP solvers are in
a more advanced state of development and several (commercial) LP solvers are
available capable of solving (large scale) LP problems, we focussed on the LP
approach. We formulated the approximating programs as LP problems by using
truncation, and provided theoretical convergence results for this approach. We
illustrated the method with numerical examples, using the Matlab interface of the
solver lp_solve, and compared the outcomes with Monte Carlo simulation results.
We found that accurate results with tight upper and lower bounds were obtained
with a small number of moments in most of the examples, and observed that the
algorithm was significantly faster than Monte Carlo simulation.
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A Proofs

A.1 Proof of Proposition 2

Since the number of equations grows with \( N \), it follows that \( v^{(N)}_\tau \) is monotone increasing, since the minimum taken over a smaller set of elements is larger.

Note that \( v \) is a finite linear combination of moments. Because of the fact that the support of the different measures is compact, if follows that each moment is bounded, so that \( v \) is bounded and \( v^{(N)}_\tau \) is the minimization of a linear function over a bounded set. Thus, the minimum \( v^{(N)}_\tau \) is finite and attained at a vector \( q^*_N = (q^*_i)^N \); that satisfies the corresponding linear system of equations. Thus, for each fixed \( i \) there exists a \( q^*_i \) such that, for \( N \) along a subsequence, \( q^*_N \to q^*_i \). In fact, by a diagonal argument it follows that there exists a subsequence \( \tilde{N} \) such that, as \( \tilde{N} \to \infty \),

\[
q^*_{\tilde{N}} \to q^*_i \quad \text{for all } i.
\]

Clearly, \( q^* \) satisfies the infinite system and thus under the assumption that there exists a unique sequence that solves the infinite system, it follows that \( q^* \) must be equal to \( (\mu, \nu) \). Moreover, \( \mu \) and \( \nu \) are the unique measures corresponding to these moments, as they are both moment-determinate. Thus, \( v = L(q^*) \) and \( v^{(N)}_\tau \uparrow L(q^*) \).

The proof of the convergence of the sequence \( v^{(N)}_\tau \) is similar and omitted. \( \square \)

A.2 Proof of Proposition 1

We will show the following lemma:

**Lemma 1** For any bounded stopping time \( \tau \) and \( f \in C^{1,2} \) with

\[
\mathbb{E} \left[ \int_0^\tau e^{-\alpha t} \left[ \sigma^2 \frac{\partial f}{\partial x} \right]^2 (t, X_t) \right] + \mathbb{E} \left[ \int_0^\tau e^{-\alpha t} |g(t, X_t, y)| \Lambda(dy)dt \right] < +\infty, \quad (24)
\]

with

\[
g(t, x, y) = f(t, x + y \lambda(t, x)) - f(t, x) - \frac{\partial f}{\partial x}(t, x) \lambda(t, x) y 1_{|y| < 1},
\]

eqn. (15) holds true.

The proposition is a direct consequence of this lemma, since under the condition (18) the integrability conditions are satisfied for each monomial \( t^ix^j \).

**Proof of Lemma** Applying (a general form of) Itô’s lemma to the stochastic process \( e^{-\alpha t}f(t, X_t) \) (which is justified as \( f \in C^{1,2} \)) shows that

\[
e^{-\alpha t}f(t, X_t) - f(0, X_0) = M_t + \int_0^t e^{-\alpha s}(Af - rf)(s, X_s)ds \quad (25)
\]

where \( Af \) is given in (5) and \( M_t \) is the local martingale given by

\[
M_t = \int_0^t e^{-\alpha t} \sigma(t, X_t) dW_t + \int_{[0,t] \times \mathbb{R}} e^{-\alpha t} g(t, X_t, y) \phi(dy, dt),
\]

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A.3 Proof of Corollary 1

Since the jump-diffusion $\tilde{X}$ with drift $b$, volatility $\sigma$, Lévy measure $\tilde{\Lambda}$ and discounting $\tilde{r}$ satisfies (18), Proposition 1 yields that the measures $\tilde{\nu}$ and $\tilde{\mu}$ corresponding to $\tilde{X}$ satisfy

$$
\tilde{\nu}_{i,j} - \tau_0^j 1_{i=0} = \sum_{k,l} \tilde{c}_{k,l}(i,j)\tilde{\mu}_{k,l} \quad i,j = 0,1,2,\ldots.
$$

where

$$(\tilde{A}f_{ij} - \tilde{r}f_{ij})(t,x) = \sum_{k,l} \tilde{c}_{k,l}(i,j)t^k x^l,$$

with $f_{ij} = t^i x^j$. To complete the proof we will now show that

$$
v = \int h(t,x)\tilde{\nu}(dt,dx) + \int g(t,x)\tilde{\mu}(dt,dx). \quad (26)
$$

Denoting by $\rho$ the first time that a jump of $J$ of size smaller than $L_-$ or larger than $L_+$ and let

$$
\tilde{\tau} = \inf\{ t \geq 0 : \tilde{X}_t \notin B \}.
$$

Then, if $\rho > \tau$ it holds that $X_{\rho \wedge \tau} = \tilde{X}_{\rho \wedge \tau}$ for all $t \geq 0$ and in particular

$$
\tau = \tilde{\tau} \quad \text{and} \quad X_{\tilde{\tau}} = \tilde{X}_{\tilde{\tau}} = \tilde{X}_{\tilde{\tau}}.
$$

Also, since it is assumed that $h(t,x) = 0$ for $x \notin B$, we have that $h(\tau, X_{\tau}) = h(\tau, X_{\tilde{\tau}})1_{\{\rho > \tau\}}$. Taking note of these observations, it follows that

$$
E[e^{-\alpha \tau} h(\tau, X_{\tau})] = E[e^{-\alpha \tau} h(\tau, X_{\tilde{\tau}})1_{\{\rho > \tau\}}] = E[e^{-\alpha \tau - \tilde{\lambda} \tau} h(\tau, X_{\tilde{\tau}})] = E[e^{-\alpha \tau - \tilde{\lambda} \tilde{\tau}} h(\tilde{\tau}, X_{\tilde{\tau}})]
$$

where $\tilde{\lambda} := \tilde{\Lambda}(\mathbb{R}\setminus[L_-,L_+])$ and we used that $\rho$ follows an exponential distribution with mean $\tilde{\lambda}^{-1}$, independent of $\tilde{X}$. Similarly,

$$
E \left[ \int_0^\tau e^{\alpha s} g(s, X_{S-}) ds \right] = E \left[ \int_0^\tau e^{\alpha s} g(s, X_{S-}) 1_{\{s \leq \rho\}} ds \right] = E \left[ \int_0^\tau e^{-\alpha s} g(s, \tilde{X}_{S-}) 1_{\{s \leq \rho\}} ds \right] = E \left[ \int_0^\tau e^{-\alpha s - \tilde{\lambda}s} g(s, \tilde{X}_{S-}) ds \right] = E \left[ \int_0^\tau e^{-\tilde{\alpha} s} g(s, \tilde{X}_{S-}) ds \right].
$$
The two identities imply that (26) holds true, and the proof is complete. □

B Semi-definite moment conditions

The moment matrices are defined as follows:

Moment Matrices Let

$$(x^\alpha, |\alpha| \leq k) = (1, x_1, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_1^{k-1} x_2, \ldots, x_n^k), \quad (27)$$

be the usual basis of polynomials in $n$ variables with degree at most $k$.

Given a series of moments of a measure $m = \{m_\alpha, \alpha \in \mathbb{N}^n\}$ let $\hat{m} = \{\hat{m}_i, i \in \mathbb{N}\}$ be that sequence ordered in accordance with (27). The Moments matrix $M_k(m)$ is then defined as

$$M_k(m)(1, i) = M_k(m)(i, 1) = \hat{m}_{i-1}, \text{ for } i = 1, \ldots, k + 1,$$

$$M_k(m)(1, j) = m_\alpha \text{ and } M_k(m)(i, 1) = m_\beta \Rightarrow M_k(m) = m_{\alpha+\beta}$$

where $M_k(m)(i, j)$ is the $(i, j)$-entry of the matrix $M_k(m)$.

Localising Matrices Given a polynomial $q$ with coefficients $(q_\alpha)$ in the basis (27). If $\beta(i, j)$ is the $\beta$ subscript of the $(i, j)$-entry of the moment matrix $M_k(m)$ then the localising matrix is defined by

$$M_k(q, m)(i, j) = \sum_\alpha q_\alpha m_{\beta(i, j)+\alpha}$$

The choice of the function $q$ depends on the support of the measure. For example, in the one-dimensional setting we have three cases,

1. Support on $[a, b]$, with function $q = (b - x)(x - a)$
2. Support on $[a, \infty)$, with function $q = (x - a)$
3. Support on $(-\infty, a]$, with function $q = (a - x)$

In terms of moment and localizing matrices the characterization is then as follows: Given $m = (m_0, m_1, \ldots, m_{2r})$ the condition that $M_r(m)$ and $M_{r-1}(q, m)$ are positive semi-definite are sufficient conditions for the elements of $m$ to be the first $2r + 1$ moments of a measure supported on the appropriate interval.