On the connections between Pell numbers and Fibonacci $p$-numbers

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Abstract: In this paper, we define the Fibonacci–Pell $p$-sequence and then we discuss the connection of the Fibonacci–Pell $p$-sequence with the Pell and Fibonacci $p$-sequences. Also, we provide a new Binet formula and a new combinatorial representation of the Fibonacci–Pell $p$-numbers by the aid of the $n$-th power of the generating matrix of the Fibonacci–Pell $p$-sequence. Furthermore, we derive relationships between the Fibonacci–Pell $p$-numbers and their permanent, determinant and sums of certain matrices.

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1 Introduction

The well-known Pell sequence $\{P_n\}$ is defined by the following recurrence relation:

$$P_{n+2} = 2P_{n+1} + P_n \text{ for } n \geq 0 \text{ in which } P_0 = 0 \text{ and } P_1 = 1.$$
There are many important generalizations of the Fibonacci sequence. The Fibonacci $p$-sequence \([22, 23]\) is one of them:

\[
F_p(n) = F_p(n - 1) + F_p(n - p - 1) \quad \text{for} \quad p = 1, 2, 3, \ldots \quad \text{and} \quad n > p
\]

in which $F_p(0) = 0$, $F_p(1) = \cdots = F_p(p) = 1$. When $p = 1$, the Fibonacci $p$-sequence \(\{F_p(n)\}\) is reduced to the usual Fibonacci sequence \(\{F_n\}\).

It is easy to see that the characteristic polynomials of the Pell sequence and Fibonacci $p$-sequence are $f_1(x) = x^2 - 2x - 1$ and $f_2(x) = x^{p+1} - x^p - 1$, respectively. We use these in the next section.

Let the $(n + k)$-th term of a sequence be defined recursively by a linear combination of the preceding $k$ terms:

\[
a_{n+k} = c_0a_n + c_1a_{n+1} + \cdots + c_{k-1}a_{n+k-1},
\]

in which $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [12], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix $A$ be defined by

\[
A = [a_{i,j}]_{k \times k} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1}
\end{bmatrix},
\]

then

\[
A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}
\]

for $n \geq 0$.

Several authors have used homogeneous linear recurrence relations to deduce miscellaneous properties for a plethora of sequences: see for example, [1,4,8–11,19–21,24]. In [5–7,14–16,22,23,25], the authors defined some linear recurrence sequences and gave their various properties by matrix methods.

In the present paper, we discuss connections between the Pell and Fibonacci $p$-numbers. Firstly, we define the Fibonacci–Pell $p$-sequence and then we study recurrence relation among this sequence, Pell and Fibonacci $p$-sequences. In addition, we obtain their generating matrices, Binet formulas, permanental, determinantal, combinatorial, exponential representations, and we derive a formula for the sums of the Fibonacci–Pell $p$-numbers.
2 Main results

Now we define the Fibonacci–Pell \( p \)-sequence \( \{ F_{n}^{P,p} \} \) by the following homogeneous linear recurrence relation for any given \( p (3, 4, 5, \ldots) \) and \( n \geq 0 \)

\[
F_{n+p+3}^{P,p} = 3 F_{n+p+2}^{P,p} - F_{n+p+1}^{P,p} - F_{n+p}^{P,p} + 2 F_{n+2}^{P,p} - F_{n+1}^{P,p} - F_{n}^{P,p},
\]

in which \( F_{n}^{P,p} = \cdots = F_{n+2}^{P,p} = 0 \) and \( F_{n+1}^{P,p} = 1 \).

First, we consider the relationship between the Fibonacci–Pell \( p \)-sequence which is defined above, Pell, and Fibonacci \( p \)-sequences.

**Theorem 2.1.** Let \( P_{n} \), \( F_{3}^{P} \) \( (n) \) and \( F_{n}^{P,3} \) be the \( n \)-th Pell number, Fibonacci 3-number, and Fibonacci–Pell 3-numbers, respectively. Then, for \( n \geq 0 \)

\[
P_{n+2} = F_{n+5}^{P,3} + 2 F_{n+3}^{P,3} + F_{3}^{P} (n + 2) + F_{3}^{P} (n).
\]

**Proof.** The assertion may be proved by induction on \( n \). It is clear that

\[
P_{2} = F_{5}^{P,3} + 2 F_{3}^{P,3} + F_{3}^{P} (2) + F_{3}^{P} (0) = 2.
\]

Suppose that the equation holds for \( n \geq 1 \). Then we must show that the equation holds for \( n + 1 \).

Since the characteristic polynomial of Fibonacci–Pell \( p \)-sequence \( \{ F_{n}^{P,p} \} \), is

\[
g (x) = x^{p+3} - 3x^{p+2} + x^{p+1} + x^{p} - x^2 + 2x + 1
\]

and

\[
g (x) = f_{1} (x) f_{2} (x),
\]

where \( f_{1} (x) \) and \( f_{2} (x) \) are the characteristic polynomials of Pell sequence and Fibonacci \( p \)-sequence, respectively, we obtain the following relations:

\[
P_{n+6} = 3 P_{n+5} - P_{n+4} - P_{n+3} + P_{n+2} - 2 P_{n+1} - P_{n}
\]

and

\[
F_{3}^{P} (n + 6) = 3 F_{3}^{P} (n + 5) - F_{3}^{P} (n + 4) - F_{3}^{P} (n + 3) + F_{3}^{P} (n + 2) - 2 F_{3}^{P} (n + 1) - F_{3}^{P} (n)
\]

for \( n \geq 1 \). Thus, the conclusion is obtained. \( \Box \)

**Theorem 2.2.** Let \( P_{n} \) and \( F_{n}^{P,p} \) be the \( n \)-th Pell number and Fibonacci–Pell \( p \)-numbers. Then, for \( n \geq 0 \) and \( p \geq 3 \).

i. Let \( p \) be a positive integer, then

\[
P_{n} = F_{n+p+1}^{P,p} - F_{n+p}^{P,p} - F_{n}^{P,p}.
\]

ii. If \( p \) is odd, then

\[
P_{n} + P_{n+1} = F_{n+p+2}^{P,p} - F_{n+p}^{P,p} - F_{n+1}^{P,p} - F_{n}^{P,p}
\]

and

iii. If \( p \) is odd, then

\[
\sum_{i=0}^{n} (F_{i}^{P,p} + P_{i}) = F_{n+p+1}^{P,p}.
\]
Proof. Consider the Case ii. The assertion may be proved by induction on $n$. Then for $p = 3$, it is clear that $P_0 + P_1 = F_5^{P,3} - F_3^{P,3} - F_1^{P,3} - F_0^{P,3} = 1$. Suppose that the equation holds for $n > 0$. Then we must show that the equation holds for $n + 1$. Since the characteristic polynomial of the Pell sequence $\{P_n\}$, is

$$f_1(x) = x^2 - 2x - 1,$$

we obtain the following relations:

$$P_{n+6} = 3P_{n+5} - P_{n+4} - P_{n+3} + P_{n+2} - 2P_{n+1} - P_n$$

for $n \geq 1$. Now we consider the proof for the case $p > 3$. Suppose that the equation holds for $p = 2\alpha + 1$, $(\alpha \in \mathbb{N})$ and $n \geq 0$, it is clear that

$$P_n + P_{n+1} = F_{n+2\alpha+3}^{P,2\alpha+1} - F_{n+2\alpha+1}^{P,2\alpha+1} - F_{n+1}^{P,2\alpha+1} - F_n^{P,2\alpha+1}.$$ 

Then we must show that the equation holds for $p = 2\alpha + 3$, $(\alpha \in \mathbb{N})$. For $n = 0$, it is clear that

$$P_0 + P_1 = F_{2\alpha+5}^{P,2\alpha+1} - F_{2\alpha+3}^{P,2\alpha+1} - F_1^{P,2\alpha+1} - F_0^{P,2\alpha+1} = 1.$$ 

The assertion may be proved again by induction on $n$. Assume that the equation holds for $n > 0$. Then we must show that the equation holds for $n + 1$. Since the characteristic polynomial of the Pell sequence $\{P_n\}$, is

$$f_1(x) = x^2 - 2x - 1,$$

we obtain the following relations:

$$P_{n+2\alpha+6} = 3P_{n+2\alpha+5} - P_{n+2\alpha+4} - P_{n+2\alpha+3} + P_{n+2} - 2P_{n+1} - P_n$$

for $n \geq 1$. Thus, the conclusion is obtained.

There is a similar proof for Case i and Case iii. \hfill \Box

By the recurrence relation (1), we have

$$[F_{n+p}^{P,p}]
\begin{bmatrix}
3 & -1 & -1 & 0 & \ldots & 0 & 0 & 1 & -2 & -1 \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
F_{n+p}^{P,p} \\
F_{n+p+1}^{P,p} \\
F_{n+p+2}^{P,p} \\
F_{n+p+3}^{P,p} \\
\vdots \\
F_{n}^{P,p} \\
F_{n+1}^{P,p} \\
F_{n+2}^{P,p} \\
F_{n+3}^{P,p} \\
\end{bmatrix} = 
\begin{bmatrix}
F_{n+p}^{P,p} \\
F_{n+p+1}^{P,p} \\
F_{n+p+2}^{P,p} \\
F_{n+p+3}^{P,p} \\
\vdots \\
F_{n}^{P,p} \\
F_{n+1}^{P,p} \\
F_{n+2}^{P,p} \\
F_{n+3}^{P,p} \\
\end{bmatrix}$$

for the Fibonacci–Pell $p$-sequence $\{F_n^{P,p}\}$. Letting
the companion matrix $D_p = [d_{ij}]_{(p+3)\times(p+3)}$ is said to be the Fibonacci–Pell $p$-matrix. For more details on the companion type matrices, see [17,18]. It can be readily established by mathematical induction that for $p \geq 3$ and $n \geq 3p - 1$,

$$D_p^n = \begin{bmatrix}
F_{n+p+2}^{p} & F_p(n-p+1) - F_{n+p+1}^{p} & F_p(n-p+2) - F_{n+p+2}^{p} & F_p(n-p+3) ~ \cdots \\
F_{n+p+1}^{p} & F_p(n-p) - F_{n+p}^{p} & F_p(n-p+1) - F_{n+p+1}^{p} & F_p(n-p+2) ~ \cdots \\
F_{n+p}^{p} & F_p(n-p-1) - F_{n+p-1}^{p} & F_p(n-p) - F_{n+p-1}^{p} & F_p(n-p+1) ~ \cdots \\
F_n^{p} & F_p(n-2p - 1) - F_{n-1}^{p} & F_p(n-2p) - F_{n-1}^{p} & F_p(n-2p+1) ~ \cdots 
\end{bmatrix},$$

where

$$D_p^* = \begin{bmatrix}
F_p(n) & F_p(n-p+3) + F_p(n-p) + F_p(n-p-1) + \cdots + F_p(n-2p+3) - F_{n+p+2}^{p} & -F_{n+p+1}^{p} \\
F_p(n-1) & F_p(n-p+2) + F_p(n-p-1) + F_p(n-p-2) + \cdots + F_p(n-2p+2) - F_{n+p}^{p} & -F_{n+p}^{p} \\
F_p(n-2) & F_p(n-p+1) + F_p(n-p-2) + F_p(n-p-3) + \cdots + F_p(n-2p+1) - F_{n+p-1}^{p} & -F_{n+p-1}^{p} \\
\cdots & \cdots & \cdots & \cdots \\
F_p(n-p-1) & F_p(n-2p+2) + F_p(n-2p-1) + F_p(n-2p-2) + \cdots + F_p(n-3p+2) - F_{n+1}^{p} & -F_{n}^{p} \\
F_p(n-p-2) & F_p(n-2p+1) + F_p(n-2p-2) + F_p(n-2p-3) + \cdots + F_p(n-3p+1) - F_{n-1}^{p} & -F_{n-1}^{p} 
\end{bmatrix}.$$
**Proof.** It is clear that $x^{p+3} - 3x^{p+2} + x^{p+1} + x^p - x^2 + 2x + 1 = (x^{p+1} - x^p - 1)(x^2 - 2x - 1)$. In [13], it was shown that the equation $x^{p+1} - x^p - 1 = 0$ does not have multiple roots for $p > 1$. It is easy to see that the roots of the equation $x^2 - 2x - 1 = 0$ are $1 + \sqrt{2}$ and $1 - \sqrt{2}$. Since $(1 + \sqrt{2})^{p+1} - (1 + \sqrt{2})^p - 1 \neq 0$ and $(1 - \sqrt{2})^{p+1} - (1 - \sqrt{2})^p - 1 \neq 0$, the equation $x^{p+3} - 3x^{p+2} + x^{p+1} + x^p - x^2 + 2x + 1 = 0$ does not have multiple roots for $p \geq 3$. \qed

Let $\alpha_1, \alpha_2, \ldots, \alpha_{p+3}$ be the roots of the equation $x^{p+3} - 3x^{p+2} + x^{p+1} + x^p - x^2 + 2x + 1 = 0$ and let $V_p$ be a $(p + 3) \times (p + 3)$ Vandermonde matrix as follows:

$$V_p = \begin{bmatrix}
(\alpha_1)^{p+2} & (\alpha_2)^{p+2} & \cdots & (\alpha_{p+3})^{p+2} \\
(\alpha_1)^{p+1} & (\alpha_2)^{p+1} & \cdots & (\alpha_{p+3})^{p+1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{p+3} \\
1 & 1 & \cdots & 1
\end{bmatrix}.$$  

Assume that $V_p(i, j)$ is a $(p + 3) \times (p + 3)$ matrix derived from the Vandermonde matrix $V_p$ by replacing the $j$-th column of $V_p$ by $W_p(i)$, where, $W_p(i)$ is a $(p + 3) \times 1$ matrix as follows:

$$W_p(i) = \begin{bmatrix}
(\alpha_1)^{n+p+3-i} \\
(\alpha_2)^{n+p+3-i} \\
\vdots \\
(\alpha_{p+3})^{n+p+3-i}
\end{bmatrix}.$$  

**Theorem 2.4.** Let $p$ be a positive integer such that $p \geq 3$ and let $(D_p)^n = d_{i,j}^{(p,n)}$ for $n \geq 1$, then

$$d_{i,j}^{(p,n)} = \frac{\det V_p(i, j)}{\det V_p}.$$  

**Proof.** Since the equation $x^{p+3} - 3x^{p+2} + x^{p+1} + x^p - x^2 + 2x + 1 = 0$ does not have multiple roots for $p \geq 3$, the eigenvalues of the Fibonacci–Pell $p$-matrix $D_p$ are distinct. Then, it is clear that $D_p$ is diagonalizable. Let $A_p = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{p+3})$, then we may write $D_pV_p = V_pA_p$. Since the matrix $V_p$ is invertible, we obtain the equation $(V_p)^{-1}D_pV_p = A_p$. Therefore, $D_p$ is similar to $A_p$; hence, $(D_p)^nV_p = V_p(A_p)^n$ for $n \geq 1$. So we have the following linear system of equations:

$$\left\{ \begin{array}{l}
d_{i,1}^{(p,n)}(\alpha_1)^{p+2} + d_{i,2}^{(p,n)}(\alpha_1)^{p+1} + \cdots + d_{i,p+3}^{(p,n)} = (\alpha_1)^{n+p+3-i} \\
d_{i,1}^{(p,n)}(\alpha_2)^{p+2} + d_{i,2}^{(p,n)}(\alpha_2)^{p+1} + \cdots + d_{i,p+3}^{(p,n)} = (\alpha_2)^{n+p+3-i} \\
\vdots \\
d_{i,1}^{(p,n)}(\alpha_{p+3})^{p+2} + d_{i,2}^{(p,n)}(\alpha_{p+3})^{p+1} + \cdots + d_{i,p+3}^{(p,n)} = (\alpha_{p+3})^{n+p+3-i}.
\end{array} \right.$$  

Then we conclude that

$$d_{i,j}^{(p,n)} = \frac{\det V_p(i, j)}{\det V_p}$$  

for each $i, j = 1, 2, \ldots, p + 3$. \qed
Thus by Theorem 2.4 and the matrix \((D_p)^n\), we have the following useful result for the Fibonacci–Pell \(p\)-numbers.

**Corollary 2.1.** Let \(p\) be a positive integer such that \(p \geq 3\) and let \(F_n^{F,p}\) be the \(n\)-th element of the Fibonacci–Pell \(p\)-sequence, then

\[
F_n^{F,p} = \frac{\det V_p (p + 3, 1)}{\det V_p}
\]

and

\[
F_n^{F,p} = -\frac{\det V_p (p + 2, p + 3)}{\det V_p}
\]

for \(n \geq 1\).

It is easy to see that the generating function of the Fibonacci–Pell \(p\)-sequence \(\{F_n^{F,p}\}\) is as follows:

\[
g(x) = x^{p+2} \frac{1 - 3x + x^2 + x^3 - x^{p+1} + 2x^{p+2} + x^{p+3}}{1 - 3x + x^2 + x^3 - x^{p+1} + 2x^{p+2} + x^{p+3}},
\]

where \(p \geq 3\).

Then we can give an exponential representation for the Fibonacci–Pell \(p\)-numbers by the aid of the generating function with the following Theorem.

**Theorem 2.5.** The Fibonacci–Pell \(p\)-numbers \(\{F_n^{F,p}\}\) have the following exponential representation:

\[
g(x) = x^{p+2} \exp \left( \sum_{i=1}^{\infty} \frac{(x)^i}{i} \left(3 - x - x^2 + x^p - 2x^{p+1} - x^{p+2} \right)^i \right),
\]

where \(p \geq 3\).

**Proof.** Since

\[
\ln g(x) = \ln x^{p+2} - \ln \left(1 - 3x + x^2 + x^3 - x^{p+1} + 2x^{p+2} + x^{p+3}\right)
\]

and

\[
-\ln \left(1 - 3x + x^2 + x^3 - x^{p+1} + 2x^{p+2} + x^{p+3}\right) = -\left[-x \left(3 - x - x^2 + x^p - 2x^{p+1} - x^{p+2}\right) - \frac{1}{2} x^2 \left(3 - x - x^2 + x^p - 2x^{p+1} - x^{p+2}\right)^2 - \ldots \right]
\]

\[
-1 \frac{1}{i} x^i \left(3 - x - x^2 + x^p - 2x^{p+1} - x^{p+2}\right)^i \ldots
\]

it is clear that

\[
g(x) = x^{p+2} \exp \left( \sum_{i=1}^{\infty} \frac{(x)^i}{i} \left(3 - x - x^2 + x^p - 2x^{p+1} - x^{p+2} \right)^i \right)
\]

and by a simple calculation, we obtain the conclusion.
Let $K (k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K (k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$ 

**Theorem 2.6.** (Chen and Louck [3]) The $(i, j)$ entry $k_{i,j}^{(n)} (k_1, k_2, \ldots, k_v)$ in the matrix $K^n (k_1, k_2, \ldots, k_v)$ is given by the following formula:

$$k_{i,j}^{(n)} (k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \left( \frac{t_1 + \cdots + t_v}{t_1, t_2, \ldots, t_v} \right) k_{t_1}^{(1)} \cdots k_{t_v}^{(v)} \tag{2}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$, $(t_1, t_2, \ldots, t_v) = \frac{(t_1 + \cdots + t_v)!}{t_1! \cdots t_v!}$ is a multinomial coefficient, and the coefficients in (2) are defined to be 1 if $n = i - j$.

Then we can give other combinatorial representations than for the Fibonacci–Pell $p$-numbers by the following Corollary.

**Corollary 2.2.** Let $F_n^{P,p}$ be the $n$-th Fibonacci–Pell $p$-number for $n \geq 1$. Then

i. \quad $F_n^{P,p} = \sum_{(t_1, t_2, \ldots, t_v)} \left( \frac{t_1 + t_2 + \cdots + t_{p+3}}{t_1, t_2, \cdots, t_{p+3}} \right) 3^{t_1} (-2)^{t_{p+2}} (-1)^{t_2+t_3+t_{p+3}},$

where the summation is over nonnegative integers satisfying

$$t_1 + 2t_2 + \cdots + (p + 3) t_{p+3} = n - p - 2.$$

ii. \quad $F_n^{P,p} = - \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_{p+3}}{t_1 + t_2 + \cdots + t_{p+3}} \times \left( \frac{t_1 + t_2 + \cdots + t_{p+3}}{t_1, t_2, \cdots, t_{p+3}} \right) 3^{t_1} (-2)^{t_{p+2}} (-1)^{t_2+t_3+t_{p+3}},$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p + 3) t_{p+3} = n + 1$.

**Proof.** If we take $i = p + 3$, $j = 1$ for the Case i. and $i = p + 2$, $j = p + 3$ for the Case ii. in Theorem 2.6, then we can directly see the conclusions from $(D_p)^n$.

Now we consider the relationship between the Fibonacci–Pell $p$-numbers and the permanent of a certain matrix which is obtained using the Fibonacci–Pell $p$-matrix $(D_p)^n$.

**Definition 2.1.** A $u \times v$ real matrix $M = [m_{i,j}]$ is called a contractible matrix in the $k$-th column (respectively, row) if the $k$-th column (respectively, row) contains exactly two non-zero entries.

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix $M$. If $M$ is contractible in the $k$-th column such that $m_{i,k} \neq 0$, $m_{j,k} \neq 0$ and $i \neq j$, then the $(u - 1) \times (v - 1)$ matrix $M_{ij;k}$ obtained
from $M$ by replacing the $i$-th row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the $j$-th row. The $k$-th column is called the contraction in the $k$-th column relative to the $i$-th row and the $j$-th row.

In [2], Brualdi and Gibson obtained that $\text{per} (M) = \text{per} (N)$ if $M$ is a real matrix of order $\alpha > 1$ and $N$ is a contraction of $M$.

Now we concentrate on finding relationships among the Fibonacci–Pell $p$-numbers and the permanents of certain matrices which are obtained by using the generating matrix of the Fibonacci–Pell $p$-numbers. Let $E_{m,p}^{F,P} = [e_{i,j}]$ be the $m \times m$ super-diagonal matrix, defined by

$$e_{i,j} = \begin{cases} 3 & \text{if } i = \tau \text{ and } j = \tau \text{ for } 1 \leq \tau \leq m, \\ 1 & \text{if } i = \tau \text{ and } j = \tau + p \text{ for } 1 \leq \tau \leq m - p \\ & \text{and } i = \tau + 1 \text{ and } j = \tau \text{ for } 1 \leq \tau \leq m - 1, \\ -1 & \text{if } i = \tau \text{ and } j = \tau + 1 \text{ for } 1 \leq \tau \leq m - 1, \\ -2 & \text{if } i = \tau \text{ and } j = \tau + p + 1 \text{ for } 1 \leq \tau \leq m - p - 1, \\ 0 & \text{otherwise}. \end{cases}$$

for $m \geq p + 3$. Then we have the following Theorem.

**Theorem 2.7.** For $m \geq p + 3$,

$$\text{per} E_{m,p}^{F,P} = F_{m+p}^{P,p}.$$

**Proof.** Let us consider matrix $E_{m,p}^{F,P}$ and let the equation hold for $m \geq p + 3$. Then we show that the equation holds for $m + 1$. If we expand the $\text{per} E_{m,p}^{F,P}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$\text{per} E_{m+1,p}^{F,P} = 3 \text{ per} E_{m,p}^{F,P} - \text{per} E_{m-1,p}^{F,P} - \text{per} E_{m-2,p}^{F,P} + \text{per} E_{m-p,p}^{F,P} - 2 \text{ per} E_{m-p-1,p}^{F,P} - \text{per} E_{m-p-2,p}^{F,P}.$$

Since

$$\text{per} E_{m,p}^{F,P} = F_{m+p}^{P,p},$$
$$\text{per} E_{m-1,p}^{F,P} = F_{m+p+1}^{P,p},$$
$$\text{per} E_{m-2,p}^{F,P} = F_{m+p}^{P,p},$$
$$\text{per} E_{m-p,p}^{F,P} = F_{m+2}^{P,p},$$
$$\text{per} E_{m-p-1,p}^{F,P} = F_{m+1}^{P,p},$$
$$\text{per} E_{m-p-2,p}^{F,P} = F_{m}^{P,p},$$

we easily obtain that $\text{per} E_{m+1,p}^{F,P} = F_{m+p+3}^{P,p}$. So the proof is complete. \(\square\)
Let $F_{m,p}^{F,P} = [f_{i,j}]$ be the $m \times m$ matrix, defined by

$$f_{i,j} = \begin{cases} 
3 & \text{if } i = \tau \text{ and } j = \tau \text{ for } 1 \leq \tau \leq m - p, \\
1 & \text{if } i = \tau \text{ and } j = \tau + p \text{ for } 1 \leq \tau \leq m - p, \\
& \text{and } i = \tau + 1 \text{ and } j = \tau \text{ for } 1 \leq \tau \leq m - p - 1, \\
-1 & \text{if } i = \tau \text{ and } j = \tau + 1 \text{ for } 1 \leq \tau \leq m - p, \\
& \text{and } i = \tau \text{ and } j = \tau + 2 \text{ for } 1 \leq \tau \leq m - p - 2, \\
-2 & \text{if } i = \tau \text{ and } j = \tau + 1 \text{ for } 1 \leq \tau \leq m - p - 1, \\
0 & \text{otherwise}
\end{cases},$$

for $m \geq p + 3$. Then we have the following Theorem.

**Theorem 2.8.** For $m \geq p + 3$,

$$\text{per } F_{m,p}^{F,P} = F_{m+2,p}^{P}.$$  

**Proof.** Let us consider matrix $F_{m,p}^{F,P}$ and let the equation hold for $m \geq p + 3$. Then we show that the equation holds for $m + 1$. If we expand the per $F_{m,p}^{F,P}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$\text{per } F_{m+1,p}^{F,P} = 3 \text{ per } F_{m,p}^{F,P} - \text{ per } F_{m-1,p}^{F,P} + \text{ per } F_{m-2,p}^{F,P} - 2 \text{ per } F_{m-p-1,p}^{F,P} - \text{ per } F_{m-p-2,p}^{F,P}.$$  

Since

$$\text{per } F_{m,p}^{F,P} = F_{m+2}^{P},$$

$$\text{per } F_{m-1,p}^{F,P} = F_{m+1}^{P},$$

$$\text{per } F_{m-2,p}^{F,P} = F_{m}^{P},$$

$$\text{per } F_{m-p,p}^{F,P} = F_{m-p+2}^{P},$$

$$\text{per } F_{m-p-1,p}^{F,P} = F_{m-p+1}^{P},$$

$$\text{per } F_{m-p-2,p}^{F,P} = F_{m-p}^{P},$$

we easily obtain that $\text{per } F_{m+1,p}^{F,P} = F_{m+3}^{P}$. So the proof is complete. \( \square \)

Assume that $G_{m,p}^{F,P} = [g_{i,j}]$ be the $m \times m$ matrix, defined by

$$G_{m,p}^{F,P} = \begin{bmatrix} 
1 & \cdots & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & \\
0 & \cdots & F_{m-1,p}^{F,P} & \\
0 & \cdots & \cdots & \\
0 & \cdots & \cdots & 
\end{bmatrix}, \text{ for } m > p + 3,$$  

then we have the following results.
**Theorem 2.9.** For \( m > p + 3 \),

\[
\text{per } G_{m,p}^F = \sum_{i=0}^{m+1} F_{i,p}^P.
\]

**Proof.** If we extend \( \text{per } G_{m,p}^F \) with respect to the first row, we write

\[
\text{per } G_{m,p}^F = \text{per } G_{m-1,p}^F + F_{m-1,p}^P.
\]

Thus, by the results and an inductive argument, the proof is easily seen. \( \square \)

A matrix \( M \) is called convertible if there is an \( n \times n \) \((1,-1)\)-matrix \( K \) such that

\[
\text{per } M = \det (M \circ K),
\]

where \( M \circ K \) denotes the Hadamard product of \( M \) and \( K \).

Now we give relationships among the Fibonacci–Pell \( p \)-numbers and the determinants of certain matrices which are obtained by using the matrices \( E_{m,p}^F, F_{m,p}^F \) and \( G_{m,p}^F \). Let \( m > p + 3 \) and let \( R \) be the \( m \times m \) matrix, defined by

\[
R = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & 1 & 1 & \cdots & 1 & 1 \\
1 & -1 & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & 1 & -1 & 1 & 1 \\
1 & \cdots & 1 & 1 & -1 & 1
\end{bmatrix}.
\]

**Corollary 2.3.** For \( m > p + 3 \),

\[
\det \left( E_{m,p}^F \circ R \right) = F_{m+p+2,p}^P,
\]

\[
\det \left( F_{m,p}^F \circ R \right) = F_{m+p+2,p}^P,
\]

and

\[
\det \left( G_{m,p}^F \circ R \right) = \sum_{i=0}^{m+1} F_{i,p}^P.
\]

**Proof.** Since \( \text{per } E_{m,p}^F = \det \left( E_{m,p}^F \circ R \right), \text{ per } F_{m,p}^F = \det \left( F_{m,p}^F \circ R \right) \) and \( \text{ per } G_{m,p}^F = \det \left( G_{m,p}^F \circ R \right) \) for \( m > p + 3 \), by Theorem 2.7, Theorem 2.8 and Theorem 2.9, we have the conclusion. \( \square \)

Now we consider the sums of the Fibonacci–Pell \( p \)-numbers. Let

\[
S_n = \sum_{i=0}^{n} F_{i,p}^P
\]

for \( n \geq 0 \) and let \( U_{F,p} \) and \( (U_{F,p})^n \) be the \((p + 4) \times (p + 4)\) matrix such that

\[
U_{F,p} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & & & & \\
\vdots & & \ddots & & & \\
0 & & & \ddots & D_p & \\
0 & & & & & 0
\end{bmatrix},
\]

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If we use induction on \( n \), then we obtain

\[
(U_{F,P})^n = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
S_{n+p+1} & S_{n+p} & \vdots & & & (D_p)^n \\
S_n & S_{n-1} & & & \\
& & & & \\
& & & & \\
& & & & & 
\end{bmatrix}
\]

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