Phase transitions in cellular automata
for cargo transport and kinetically constrained traffic

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Abstract. A probabilistic cellular automaton for cargo transport is presented that generalizes the totally asymmetric exclusion process with a defect from continuous time to parallel dynamics. It appears as an underlying principle in cellular automata for traffic flow with non-local jumps for the kinetic constraint to drive as fast as possible. The exactly solvable model shows a discontinuous phase transition between two regions with different cargo velocities.

Key words: asymmetric exclusion process, matrix-product state

1 Introduction

Non-equilibrium phase transitions can rarely be calculated exactly, i.e. without need of approximations or fits of numerical data. One paradigmatic system where this is possible is the totally asymmetric simple exclusion process (TASEP) (see [1] and references therein). The model is defined on a 1d discrete lattice with sites being either empty (holes) or occupied by a single particle. A randomly chosen particle moves to the right at rate 1 provided that the site is empty. For open boundaries with particle input at rate \( \alpha \) and output at rate \( \beta \) this leads to three different phases: a low-density, a high-density and a maximum-current phase. For finite systems the process can be solved exactly by the matrix-product Ansatz (see [2] for a recent and exhaustive review) and the complete thermodynamic behavior that is relevant for understanding the phase diagram can be extracted from the asymptotics of this solution. On the ring the process has a uniform groundstate [1]. However the presence of a defect particle leads to a rich phase behavior [3]. In the defect TASEP usual particles move 10 \( \rightarrow \) 01 at rate 1, the single defect particle moves itself forward 20 \( \rightarrow \) 02 at rate \( \alpha \) and can be overtaken by usual particles 12 \( \rightarrow \) 21 with \( \beta \). The solution is formally related to the open-boundary case. There is one shock phase and three phases where it behaves once like a particle, once like a hole and once like a second-class particle. The second-class particle case corresponds to \( \alpha = \beta = 1 \). To the left the second-class particle looks like a hole (as seen from a particle) and to the right it looks like a particle (as seen from a hole). This case was studied in [4].
since it can be used to study the shock in the TASEP on the infinite line with
step-initial condition. The connection is that, due to its very special dynamics,
the defect 2 samples the random shock position on preferring configurations like
000021111.
The defect TASEP can alternatively be understood as a cargo transport process:
The defect is a usual particle carrying a cargo which can be handed over to a
particle behind. See [5] for a different definition of cargo in the same context.
These mechanisms play a fundamental role in the biology of intracellular trans-
port. Processive motors like kinesin transport cargo over long distances [6]. The
cargo can alternatively be interpreted as virus particles that use carrier particles
in order to attain the interior of a cell, see [5] for further references.
The Nagel-Schreckenberg model [7] is often referred to as the minimal model for
one-lane traffic-flow on a freeway. There it is essential to allow for faster and
slower cars to get a realistic flow-density relation: cars can move up to $v_{\text{max}}$
sites per time step. Further all cars are updated simultaneously according to a
parallel update and move independently with probability $p$. For $v_{\text{max}} = 1$ it is
equivalent to the TASEP with parallel dynamics. The steady state on the ring
shows nearest-neighbor correlations and has a simple pair-factorized form [8].
For open boundaries the matrix-product technique could be generalized to ob-
tain the exact steady state [9,10]. It can be interpreted as a pair-factorized state
as on the ring modulated by a matrix-product state [11].

In this article we introduce a generalization of the cargo-transport process to
discrete time with parallel updating and give its exact solution. Here we restrict
ourselves to light cargo, i.e. the case where the speed of a particle is not low-
ered by the presence of cargo. This appears naturally in the steady state of a
traffic cellular automaton [11]. We will see that non-local jump processes where
particles drive as fast as possible can lead for non-deterministic hopping to a
discontinuous phase transition on the ring.

2 The Cellular-Automaton Model and its Solution

Consider a periodic one-dimensional lattice with sites being either occupied by a
particle (in state $\tau = 1$) or empty ($\tau = 0$). One of the particles carries a cargo to
which we refer to as a defect ($\tau = 2$). The particles are updated simultaneously
and every particle (with or without cargo) moves forward with probability $p$. If
the site behind it is occupied, the cargo carrying particle can independently give
its cargo back at probability $\beta$. This simple dynamics is encoded in detail in the
transitions

\begin{align*}
10 & \rightarrow 01, \text{ at rate } p, \quad (1) \\
020 & \rightarrow x02, \text{ at rate } p, \quad (2) \\
120 & \rightarrow 210, \text{ at rate } \beta(1-p), \quad (3) \\
& \quad \rightarrow 102, \text{ at rate } (1-\beta)p, \quad (4) \\
& \quad \rightarrow 201, \text{ at rate } \beta p, \quad (5) \\
121 & \rightarrow 21x, \text{ at rate } \beta, \quad (6)
\end{align*}
with $x$ being either 0 or 1 indicating that the site can be either empty or occupied due to the parallel update. For example the evolution of the pattern 020 can be affected by a particle to the left moving itself forward. This is quite a general scenario that might apply to biological intracellular cargo transport. The parallel update reflects highly active transport where many particles move at the same time.

For $\beta = 0$ the cargo is attached to one special particle for all times and its dynamics is the same as for the other particles. This corresponds to the usual TASEP (with parallel update) and the single occupation $\tau = 2$ can be replaced by $\tau = 1$. The steady state for a lattice with $L + 1$ sites (with one of them occupied by the defect) is

$$P(\tau_1, \tau_2, \ldots, \tau_{L+1}) = \prod_{i=1}^{L+1} P(\{\tau_{i-1}, \tau_i\}), \quad (7)$$

thus it factorizes into symmetric two-site factors $P(\tau_{i-1}\tau_i) \equiv P(\{\tau_{i-1}, \tau_i\})$ with

$$P(00) = 1 - \rho - J/p, \quad (8)$$
$$P(10) = J/p, \quad (9)$$
$$P(11) = \rho - J/p. \quad (10)$$

Here $J$ is the particle current

$$J(\rho) = \frac{1 - \sqrt{1 - 4p\rho(1 - \rho)}}{2}. \quad (11)$$

We found that the steady state for $\beta > 0$ can be calculated exactly too. Here (7) is generalized to

$$P(2, \tau_1, \ldots, \tau_L) \propto \tilde{f}(\tau_1)f(\tau_1\tau_2) \ldots f(\tau_{L-1}\tau_L)\tilde{f}(\tau_L) \times \langle W| \prod_{i \geq 1} \tau_i D + (1 - \tau_i) E |V \rangle \quad (12)$$

This is a pair-factorized state (mainly the steady state for $\beta = 0$) modulated by a matrix-product state. Up to the normalization this is very related to the TASEP with open boundaries. The vectors $|W\rangle$ and $|V\rangle$ represent the defect and the matrices $D$ and $E$ represent particles and holes respectively. The operators obey the algebra

$$\langle W|E E = (1-p)\langle W|E, \quad (13)$$
$$\langle W|E D = (1-p)\langle W|D + p), \quad (14)$$
$$D E = (1-p)[D + E + p], \quad (15)$$
$$D|V\rangle = \frac{p(1-\beta)}{\beta}|V\rangle. \quad (16)$$
for details see [11]. Note that the relations (15) and (16) are quadratic and the other relations are cubic. Accordingly the dynamical rules in (15) are quadratic (for 10 and 12) and cubic (for 20) respectively. The thermodynamic particle current (11) obviously is unaffected by the cargo.

This steady state appears also in cellular-automaton models for traffic flow with non-local jumps under kinetic constraint. Consider the following process on a periodic one-dimensional lattice with sites being either occupied by one car (in state \( \tau = 1 \)) or empty (\( \tau = 0 \)). The update rules applied simultaneously to all cars (\( \equiv \) particles) are

\[
\begin{align*}
100 &\rightarrow 001, \quad \text{with probability } p, \\
101 &\rightarrow 01x, \quad \text{with probability } \beta,
\end{align*}
\]

where \( x \) denotes either a particle or hole. The maximum velocity \( v_{\text{max}} \) thus is two sites per time step instead of one in the usual TASEP and the kinetic constraint is that cars can not drive at reduced speed 1 if they could move at maximum speed. This leads under the parallel dynamics to a non-local repulsion between cars, so that finally in the thermodynamic limit only even gaps (0, 2, 4,... holes) have non-vanishing probability. Figure 1 a) shows schematically the allowed moves in a stationary configuration. For even number of holes the process is equivalent to

![Diagram](image-url)

**Fig. 1.** a) Allowed moves in a stationary configuration: only one odd gap between particles b) Equivalence of moves and according reduction of lattice units

the TASEP and for odd number of holes it is equivalent to the cargo-transport process. In this case a single hole in an environment of particles and hole pairs is formed that plays the role of cargo attached to varying particles. In figure 1 b) one sees that the particle movement of a single site is equivalent to backward movement of the 01 position. The 01 pair plays the role of the defect, having the characteristic ‘Janus face’, looking to the left like a hole and to the right like a particle. Usual holes are replaced by hole twins 00. To be precise, the probabilities (8,9) would be rewritten here as \( P(00) \equiv P(0000), \ P(01) \equiv P(001) \equiv P(0001) \). In the following we restrict ourselves in the terminology to the cargo-transport process.
3 The Phase Diagram

There is a discontinuous phase transition at $\rho_c$ the density

$$\rho_c = \frac{\beta(1-\beta)}{p-\beta^2}. \quad (17)$$

For $\rho < \rho_c$ and $\rho > \rho_c$ one finds different velocities of the defect which can be calculated through

$$v = p(1-\rho_+)(1-\beta\rho_-) - \beta\rho_- \quad (18)$$

Here $\rho_-$ and $\rho_+$ are the densities directly behind and in front of the defect. The dynamics of the defect is obtained from (16). It moves either forward with probability $p$ if it has a hole in front while at the same time there is no particle directly behind that simultaneously catches the cargo: first term in (18). Or it moves backwards with probability $\beta$ if it has a particle behind: second term in (18). The neighboring densities are in terms of the current $J(\rho)$ defined in (11):

$$\rho_- = \begin{cases} \frac{1}{p\rho - J} & \text{for } \rho < \rho_c, \\ \frac{J^2(\rho - J)^2}{p\rho - J} & \text{for } \rho > \rho_c, \end{cases}$$

$$1 - \rho_+ = \begin{cases} \frac{J}{p\rho} & \text{for } \rho < \rho_c, \\ \frac{J}{p^2(1-\beta)} \rho & \text{for } \rho > \rho_c. \end{cases} \quad (19, 20)$$

Fig. 2. The phase diagram shows the $\rho$-$\beta$ plane for $p = 3/4$. The thick line separates the two phases and on the dashed line the velocity of the defect changes its sign.
Note that there are many ways to express the results due to the relation \( J(1 - J) = p\rho(1 - \rho) \). In other words the square root in \( J \) appears in every power of \( J \) with a certain prefactor. Equations (19,20) yield

\[
v(\rho) = \begin{cases} 
1 - 2\rho & \text{for } \rho < \rho_c, \\
1 - \frac{2J}{\beta \rho} & \text{for } \rho > \rho_c.
\end{cases}
\]  (21)

The defect velocity vanishes for

\[
\rho_0 = \begin{cases} 
\frac{p - \beta}{p - \beta^2}, & \text{for } \beta < 1 - \sqrt{1 - p}, \\
1/2, & \text{for } \beta > 1 - \sqrt{1 - p}. 
\end{cases}
\]  (22)

and is positive for \( \rho > \rho_0 \) and negative for \( \rho < \rho_0 \). This leads to the phase diagram, depicted in figure 2. The formulae (17) and (22) can alternatively be

**Fig. 3.** Velocity of the defect for \( p = 3/4 \) for varying \( \beta \). *Continuous line:* result for \( \rho < \rho_c(\beta) \). The *broken lines* correspond to \( \beta = 0, \beta = 0.25, \beta = 0.5, \beta = 0.67 \) from top (*dashed*) to bottom (*dotted*) and are valid for \( \rho > \rho_c(\beta) \). The *segmented line* is the velocity of holes

interpreted in terms of a critical value \( \beta_c = \beta(\rho_c) \) and \( \beta_0 = \beta(\rho_0) \). This gives

\[
\beta_c = J(\rho_c)/(1 - \rho_c), \quad \beta_0 = J(\rho_c)/\rho_c.
\]  (23)

The value of \( \beta_c \) and \( \beta_0 \) respectively then is essentially the absolute velocity of holes and particles at the transition. Figure 3 shows the character of the defect and the discontinuous phase transition. For \( \beta = 0 \) its velocity is given by the
upper curve and equals the velocity of particles. In the zero-density limit the
velocity is independently of \( \beta \) equal to \( p \). For increasing \( \beta \) the second phase
appears: The velocity of the defect jumps at \( \rho = \rho_c(\beta) \) from the continuous
curve to the corresponding dashed curve. \( \rho_c \) increases with \( \beta \) until \( \beta = p \) where
\( \rho_c = 1 \), so that the system is completely in the second phase for all densities.
Note that for \( \beta = 1/2 \) one has \( \rho_c = 1/2 \). Finally in the limit of the fully occupied
lattice \( \rho = 1 \) and \( \rho_c \leq 1 \) the velocity equals \( \beta \) for \( \beta \leq p \). However for \( \rho_c > 1 \)
it can never increase the value of \( p \) which there is the corresponding velocity of
the holes given by the segmented line.

For the phase \( \rho < \rho_c \), which is purely present for \( \beta \geq p \), it is important to stress
that the quantities \( \rho_+ \), \( \beta \rho_- \) and \( v \) are independent of \( \beta \). The density profile is
symmetric around the defect and has an algebraic decay. This corresponds to
the second-class particle phase in the defect TASEP mentioned in the introduction.

4 Limits

Table 1 shows the limit of small hopping probabilities: \( p \equiv dt, \beta \equiv \tilde{\beta} dt \) with
\( dt \rightarrow 0 \). Note in comparison that the velocity of normal particles is always \( J/\rho \)
which gives \( p(1 - \rho) + p^2(1 - \rho)^2 + \ldots \). This limit has been studied in the traffic
picture in \([12,13]\) and corresponds to the defect TASEP. As mentioned before,
in the limit \( \beta = 0 \) the defect moves only forward and loses its role as a defect.
Therefore the steady state is the same as for the TASEP. Thus one has the
expressions given in table 2. In comparison the results from \((19,20,21)\) for

\[
\begin{array}{c|c|c|c}
\rho & v & \frac{1}{1 - \rho_+} & \frac{1}{\rho -} \\
\hline
\rho < \rho_c & (1 - 2\rho)dt & (1 - \rho)^2 \left[ 1 + 2p(1 - \rho)dt \right] & \rho^2/\beta \left[ 1 -(1 - \rho)(1 - 3p)dt \right] \\
\rho > \rho_c & (1 - \tilde{\beta} - \rho)dt & (1 - \tilde{\beta})(1 - \rho) \left[ 1 + (\tilde{\beta} + \rho(1 - \rho))dt \right] & \rho \left[ 1 -(1 - \rho)(1 - \tilde{\beta} - \rho)dt \right] \\
\end{array}
\]

**Table 1.** Continuous-time limit \( p \equiv dt, \beta \equiv \tilde{\beta} dt \): Values to the order \( O(dt) \) of
velocity, empty-space density in front and particle density behind the defect in
the two phases. Here \( \rho_c \sim \beta [1 - \beta (1 - \beta) dt] \)

\[
\begin{array}{c|c|c|c}
\rho & v_P^{(10)} & \frac{1}{\rho} & v_P^{(11)} \\
\hline
\rho < \rho_c & \frac{J}{\rho} & \frac{1 - \rho_+}{\rho} & \frac{1- J}{\rho p} \\
\rho > \rho_c & \frac{J}{\rho p} & \frac{1}{\rho} & \frac{1- J}{\rho p} \\
\end{array}
\]

**Table 2.** Limit of the discrete-time TASEP \( \beta = 0 \)
\( \rho > \rho_c \) are rewritten and expanded around the TASEP value:

\[
\rho_-(\beta) = \frac{P(11)}{\rho - \beta P(10)} = \frac{P(11)}{\rho} \left[ 1 + \frac{P(10)}{\rho - \beta + \ldots} \right]
\]

(24)

\[
1 - \rho_+(\beta) = \frac{p - p(1 - \beta)}{\rho} = \frac{P(10)}{\rho} \left[ 1 - \frac{1 - \rho}{p} \beta - \ldots \right]
\]

(25)

\[
v(\beta) = \frac{J - \beta p}{\rho - \beta P(10)} = \frac{J}{\rho} - \left( 1 - \frac{J^2}{pp^2} \right) \beta - \ldots
\]

(26)

One sees that \( \rho_- \) is mainly the same as for \( \beta = 0 \) but the density to which the numerator is addressed is reduced by backward moving so that \( \rho_- \) is increased. The same holds for the velocity \( v \). \( 1 - \rho_+ \) is even the same as for \( \beta = 0 \) up to a scale which is, using (17) and (22), given by \( \rho_c/\rho_0 \). For \( p = 1 \) (24)-(26) yield the results displayed in table 3. For \( \rho < 1/2 \) all particles are separated and move deterministically as in the TASEP. For \( \rho > 1/2 \) the effect of \( \beta \) on the velocity remains for all possible values and the phase transition disappears.

5 Conclusions

A cellular automaton for cargo transport was introduced that generalizes the (continuous-time) defect TASEP. The parallel update is often more realistic in describing active many-particle transport and makes the link between deterministic and random-sequential dynamics. The point of interest was a single defect, i.e. a particle carrying light cargo in an environment of particles and holes on a periodic 1d lattice. Particles move forward with probability \( p \) and if a particle is directly behind the particle that carries the cargo it may catch the cargo with probability \( \beta \). We found a discontinuous phase transition between two phases with different cargo velocities. Successively increasing \( \beta \) lowers its velocity only until \( \beta = p \). Then a saturation effect appears where the velocity becomes independent of \( \beta \). The same holds for the stationary state of a cellular automaton for traffic where the ‘cargo’ corresponds to small headway that is formed dynamically. It is attached to one car from behind. If the subsequent car comes close enough it will catch it up. The fact that the cargo process appears in a
seemingly unrelated non-local jump process underlines its universal role. The exact matrix-product state and its cubic algebra holds also for the two-species case where multiple cargo is present. It is also trivially generalized to case where cargo lowers the speed of the particle to $\alpha < p$. For $p = \beta$ and in the presence of several second-class particles it serves also as a model system for the study of shocks on the infinite line.

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