Maximizing the number of independent sets of fixed size in $K_n$-covered graphs

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Abstract
For some given graph $H$, a graph $G$ is called $H$-covered if each vertex in $G$ is contained in a copy of $H$. In this note, we determine the maximum number of independent sets of size $t \geq 3$ in $N$-vertex $K_n$-covered graphs and classify the extremal graphs. The result answers a question proposed by Chakraborti and Loh. The proof uses an edge-switching operation on hypergraphs which never increases the number of independent sets.

Keywords
edge-switching, extremal graph, $H$-covered, independent set

1 | INTRODUCTION

Let $H$ be a given graph. A graph $G$ is called $H$-covered if every vertex of $G$ is contained in at least one copy of $H$. Some natural extremal problems under $H$-covered condition were studied systematically by Chakraborti and Loh in [1]. They completely solved the problem of minimizing the number of edges in an $H$-covered graph with given number of vertices when $H$ is a clique or more generally when $H$ is a regular graph with degree at least about half its number of vertices. We write $i_t(G)$ and $k_t(G)$ for the number of independent sets and cliques of size $t$ in graph $G$, respectively. So minimizing $k_t(G)$ is equivalent to maximizing $i_t(G)$. But for $t > 2$, as pointed by Chakraborti and Loh in [1], the situation is quite different from $t = 2$, so they proposed the following question.

Problem 1.1 (Chakraborti and Loh [1]). It will be interesting to consider the problem of maximizing the number of independent sets of order $t > 2$ in an $n$-vertex $H$-covered graph.
The problem of determining the number of independent sets of fixed size under $H$-covered condition has been investigated in literatures. In what follows, let $G_1 \lor G_2$ be the join graph of $G_1$ and $G_2$, that is, the graph obtained by joining each vertex of $G_1$ to each vertex of $G_2$. For $H = K_{d, d}$, a star with $d + 1$ vertices, we call a graph with minimum degree at least $d$ a $K_{d, d}$-covered graph. Engbers and Galvin [3] showed that every $n$-vertex $K_{d, d}$-covered graph $G$ has $i_r(G) \leq i_r(K_{d,n-d})$, and the equality holds if and only if $G = D \lor \overline{K_{n-d}}$ for most of the cases of $t$ larger than $d$ and $n \geq 2d$, where $D$ is any graph on $d$ vertices and $\overline{K_{n-d}}$ is the empty graph on $n - d$ vertices. This result supports a conjecture of Galvin given in [5], and Engbers and Galvin further conjectured that this result holds for all positive integers $n, t, d$ with $n \geq 2d$ and $t \geq 3$. This conjecture was solved completely by Gan, Loh, and Sudakov in [6], in which they count cliques instead of independent sets in the complementary graph as Cutler and Radcliffe did in [2].

In this note, we consider the problem of maximizing the number of independent sets in a $K_n$-covered graph with a given number of vertices, our result answers Problem 1.1 completely when $H$ is a clique. It turns out that the extremal graphs for this problem have similar structures to the extremal graphs mentioned above. For given positive integers $n, k$ with $n \geq k$, write $S_{n,k} = K_k \lor K_{n-k}$, the join graph of the complete graph $K_k$ on $k$ vertices and the empty graph $K_{n-k}$ on $n - k$ vertices. The following is our main result.

**Theorem 1.2.** For any positive integers $n, N, t$ with $t \geq 3$ and $N \geq n$, every $K_n$-covered graph $G$ on $N$ vertices satisfies $i_t(G) \leq \binom{N-n+1}{t}$. Moreover, when $N \geq n + t - 1$, the graph $S_{N,n-1}$ is the unique extremal graph.

The following theorem given by Chakraborti and Loh [1] will be used in the proof of our main result.

**Theorem 1.3** (Chakraborti and Loh [1]). For any positive integers $q, n, t$ with $2 \leq t \leq n$ and any integer $N = qn + r$ with $0 \leq r \leq n - 1$, the graph consisting of two copies of $K_n$ sharing $n - r$ vertices, together with the disjoint union of $q - 1$ many $K_n$, has the least number of copies of $K_t$ among all $K_n$-covered graphs on $N$ vertices.

**Remark A.** Chakraborti and Loh remarked in [1] that they have some initial observations that the structure of the optimal graph for Problem 1.1 might be drastically different for $t > 2$.

We will use an edge-switching operation to edges of a hypergraph as our main tool in the proof of Theorem 1.2. In what follows, we give some standard definitions and notations. A *hypergraph* is a pair $H = (V, E)$, where $V$ is a set of elements called vertices, and $E$ is a collection of subsets of $V$ called edges. A hypergraph $H = (V, E)$ is called an $r$-uniform hypergraph, or $r$-graph, if each edge of $E$ has the same size $r$. In this article, all hypergraphs $H$ considered are simple, that is, $H$ contains no multiple edges. The *order* of $H$, denoted by $|H|$, is the number of vertices of $H$, and the *size* of $H$ is the number of edges of $H$. So a graph is a 2-uniform hypergraph by definition and we write graph for 2-graph for short. Given $S \subseteq V(H)$, the *degree* of $S$, denoted by $d_H(S)$, is the number of edges of $H$ containing $S$. The minimum $s$-degree $\delta_s(H)$ of $H$ is the minimum of $d_H(S)$ over all $S \subseteq V(H)$ of size $s$. We call $\delta_s(H)$ the *minimum degree* of $H$, that is, $\delta_s(H) = \min\{d_H(v) : v \in V(H)\}$.

Let $N_H(S) = \{T : S \cup T \in E(H)\}$ and $N_H[S] = N_H(S) \cup \{S\}$. The *s-shadow* of a hypergraph $H$ is an $s$-uniform hypergraph $L$ on vertex set $V(H)$ and edge set $E(L)$ such that an $s$-set $S \in E(L)$ if and
only if there is an edge \( e \in E(H) \) containing \( S \). An independent set \( I \) in \( H \) is a set of vertices such that \( |I \cap e| \leq 1 \) for all \( e \in E(H) \). Let \( I_t(H) \) be the set of independent sets of size \( t \) in \( H \) and \( i_t(H) = |I_t(H)| \). Given a graph \( G \) and \( S \subseteq V(G) \), write \( G[S] \) for the subgraph induced by \( S \). Given two integers \( a, b \) with \( a < b \), write \([a, b]\) for the set \([a, a+1, \ldots, b]\) and \([b]\) for \([1, b]\). Given two sets \( A, B \), write \( A - B = A \setminus (A \cap B) \).

A \( K_n \)-covered graph \( G \) can be associated with an \( n \)-uniform hypergraph \( \mathcal{G} \) in a natural way. We can take \( V(\mathcal{G}) = V(G) \) and \( E(\mathcal{G}) = \{ e : e \subseteq V(G), |e| = n \} \), so that the 2-shadow of \( \mathcal{G} \) is a spanning \( K_n \)-covered subgraph of \( G \) and \( i_t(\mathcal{G}) \leq i_t(G) \). A graph \( G \) is called edge-critical \( K_n \)-covered if \( G \) is \( K_n \)-covered but for any edge \( e \in E(G) \), the graph \( G - e \) is not \( K_n \)-covered. For example, \( S_n \) is an edge-critical \( K_n \)-covered graph for \( n \geq 7 \).

For edge-critical \( K_n \)-covered graphs, we have the following observation.

**Observation 1.4.** Let \( G \) be an edge-critical \( K_n \)-covered graph and \( \mathcal{G} \) be its associated hypergraph. Then the following hold:

1. the 2-shadow of \( \mathcal{G} \) is isomorphic to \( G \);
2. \( i_t(\mathcal{G}) = i_t(G) \);
3. \( \delta(G) = n - 1 \).

**Proof.** (1) And (2) come directly from the definitions of \( G \) and \( \mathcal{G} \).

(3) We use an algorithm on \( G \). Define a vertex set \( S_0 = \emptyset \) and an edge set \( E_0 = \emptyset \). For \( j = 1, 2, \ldots \), we do the following.

(i) If \( S_{j-1} = V(G) \), terminate the algorithm.

(ii) Otherwise, pick a vertex \( v \notin S_{j-1} \) and choose a copy of \( K_n \) in \( G \) containing \( v \). Let \( V'_j \) and \( E'_j \) denote the set of vertices and edges of that copy of \( K_n \), respectively. Define \( S_j = S_{j-1} \cup V'_j \) and \( E_j = E_{j-1} \cup E'_j \).

Suppose the algorithm terminates after \( j = \tau \). It is obvious that \( S_0 \subseteq S_1 \subseteq \cdots \subseteq S_\tau = V(G) \) and \( E_0 \subseteq E_1 \subseteq \cdots \subseteq E_\tau \subseteq E(G) \). We first claim that \( E_\tau = E(G) \). Otherwise, there exists an edge \( e \in E(G) \setminus E_\tau \). Then \( G - e \) is still \( K_n \)-covered by the construction of \( E_\tau \). This is a contradiction to the edge-criticality of \( G \). By this claim and the construction of \( E_\tau \), each \( v \in S_\tau \setminus S_{\tau-1} \) is covered by exactly one copy of \( K_n \). This implies that \( d_G(v) = n - 1 \). Clearly, \( \delta(G) \geq n - 1 \). So we have \( \delta(G) = n - 1 \). \( \square \)

The rest of the note is arranged as follows. In Section 2, we give some lemmas and the proof of Theorem 1.2. We finish with some discussion and remarks in Section 3.

## 2 PROOF OF THEOREM 1.2

The proof uses a novel technique, called the edge-switching operation, the graph version has been used by Fan in [4] and Gao and Hou in [7] to cope with problems related to cycles in graphs. Here we give a hypergraph version of the edge-switching operation. For a hypergraph \( \mathcal{H} \), define \( f(\mathcal{H}) = \prod_{v \in V(\mathcal{H})} d_1(v) \) where \( d_1(v) \) is the degree of \( v \) in \( \mathcal{H} \).
Lemma 2.1 (Edge-switching lemma). Let $\mathcal{H}$ be a hypergraph (not necessarily uniform) on $N$ vertices without isolated vertices and $e_0$ be an edge of size $n$ in $E(\mathcal{H})$. Fix an order of elements in $e_0$, for example, denote $e_0 = (v_1, v_2, \ldots, v_n)$. Let $\{e_1, \ldots, e_k\}$ be the set of edges adjacent to $e_0$ and $n_i = e_i \cap e_0$ for $1 \leq i \leq k$. The edge-switching operation with respect to $e_0$ is defined as follows: for each $i \in [k]$, we replace the edge $e_i$ by the edge $e'_i = \{v_1, \ldots, v_n\} \cup (e_i - e_0)$. Let $\mathcal{H}'$ be the resulting hypergraph after the edge-switching operation. Then the following hold.

(a) $i_3(\mathcal{H}) \leq i_3(\mathcal{H}')$, the equality holds if and only if $\mathcal{H}' \cong \mathcal{H}$.

(b) $f(\mathcal{H}') \leq f(\mathcal{H})$, the equality holds if and only if $\mathcal{H}' \cong \mathcal{H}$.

Proof. By the definition of edge-switching operation, we have $e_0 \cup e_1 \cup \cdots \cup e_k = e_0 \cup e'_1 \cup \cdots \cup e'_k$ and $\{e'_1, \ldots, e'_k\}$ is the set of edges adjacent to $e_0$ in $\mathcal{H}'$. Let $S = e_0 \cup e_1 \cup \cdots \cup e_k$ and $\tilde{S} = V(\mathcal{H}) \setminus S$.

(a) We partition $I_3(\mathcal{H})$ according to the positions of the elements of an independent set into four subsets. Let

\begin{align}
T_1 &= \{I \in I_3(\mathcal{H}) : I \cap e_0 = \emptyset\}, \\
T_2 &= \{I \in I_3(\mathcal{H}) : |I \cap e_0| = 1 \text{ and } |I \cap \tilde{S}| = 2\}, \\
T_3 &= \{I \in I_3(\mathcal{H}) : |I \cap e_0| = 1 \text{ and } |I \cap \tilde{S}| = 1\}, \\
T_4 &= \{I \in I_3(\mathcal{H}) : |I \cap e_0| = 1 \text{ and } |I \cap \tilde{S}| = 0\},
\end{align}

and

\begin{align}
T'_1 &= \{I \in I_3(\mathcal{H}') : |I \cap e_0| = 0\}, \\
T'_2 &= \{I \in I_3(\mathcal{H}') : |I \cap e_0| = 1 \text{ and } |I \cap \tilde{S}| = 2\}, \\
T'_3 &= \{I \in I_3(\mathcal{H}') : |I \cap e_0| = 1 \text{ and } |I \cap \tilde{S}| = 1\}, \\
T'_4 &= \{I \in I_3(\mathcal{H}') : |I \cap e_0| = 1 \text{ and } |I \cap \tilde{S}| = 0\}.
\end{align}

Then $T_1, T_2, T_3, T_4$ form a partition of $I_3(\mathcal{H})$. We partition $I_3(\mathcal{H}')$ into four subsets $T'_1, T'_2, T'_3, T'_4$ in the same way. To show (a), it is sufficient to show that $|T_i| \leq |T'_i|$ for $i = 1, 2, 3, 4$. In the following proof, for $A \subseteq V(\mathcal{H})$, we write $\mathcal{H} - A$ for the hypergraph obtained from $\mathcal{H}$ by deleting the hyperedges $e$ included in $A$ and the vertices in $A$, that is, $V(\mathcal{H} - A) = V(\mathcal{H}) \setminus A$ and $E(\mathcal{H} - A) = \{e - A : e \in E(\mathcal{H})\}$; for a set $B$ of hyperedges not included in $\mathcal{H}$, write $\mathcal{H} \cup B$ for the hypergraph by adding the hyperedges of $B$.

For $i = 1$, if $I \in T_1$ then $I$ is also an independent set of $\mathcal{H}'$ by the definition of $\mathcal{H}'$, and vice versa. So we have $|T_1| = |T'_1|$.

For $i = 2$, an independent set $I \in T_2$ consists of one vertex in $e_0$ and an independent set in $I_2(\mathcal{H} - S)$. So $|T_2| = n|I_2(\mathcal{H} - S)|$. Clearly, $I_2(\mathcal{H} - S) = I_2(\mathcal{H}' - S)$. Therefore, $|T_2| = n|I_2(\mathcal{H}' - S)| = |T'_2|$. Now we let $\tilde{e}_i = e_i - (e_0 \cup e_1 \cup \cdots \cup e_{i-1})$ for $i \in [k]$. Then $\tilde{e}_1, \ldots, \tilde{e}_k$ are pairwise disjoint and form a partition of $(e_1 \cup \cdots \cup e_k) - e_0$. By the definition of $e'_i, e'_i - e_0 = e_i - e_0$ for $i \in [k]$. So $\tilde{e}_i = e'_i - (e_0 \cup e'_1 \cup \cdots \cup e'_{i-1})$ for $i \in [k]$.

For $i = 3$, an independent set $I \in T_3$ consists of one vertex in $e_0 - e_j$ and an independent set in $I_2((\mathcal{H} - S) \cup \{\tilde{e}_j\}) \setminus I_2(\mathcal{H} - S)$ for some $1 \leq j \leq k$. For each $j \in [k]$, let

\begin{equation}
L_j = \{I \in T_3 : |I \cap e_0| = 1 \text{ and } |I \cap \tilde{e}_j| = 1\}.
\end{equation}
Then $L_1, \ldots, L_k$ form a partition of $T_3$. Similarly, an independent set $I \in T'_3$ consists of one vertex in $e_0 - e'_j$ and an independent set in $I_2((\mathcal{H}' - S) \cup \{\bar{e}_j\}) \setminus I_2(\mathcal{H}' - S)$ for some $1 \leq j \leq k$. Define

$$L'_j = \{I \in T'_3 : |I \cap e_0| = 1 \text{ and } |I \cap \bar{e}_j| = 1\}$$

for each $j \in [k]$. Then $L'_1, \ldots, L'_k$ form a partition of $T'_3$. Since $\mathcal{H} - S = \mathcal{H}' - S$, we have $I_2((\mathcal{H}' - S) \cup \{\bar{e}_j\}) = I_2((\mathcal{H} - S) \cup \{\bar{e}_j\})$. And since $|e_0 - e_j| = |e_0 - e'_j| = n - n_j$, we have $|L'_j| = |e_0 - e_j||I_2((\mathcal{H}' - S) \cup \{\bar{e}_j\})| = |e_0 - e'_j||I_2((\mathcal{H} - S) \cup \{\bar{e}_j\})| = |L'_j|$ for each $j \in [k]$. Therefore, $T'_3 = \sum_{j=1}^k |L'_j| = \sum_{j=1}^k |L'_j| = |T'_3|$.

For $i = 4$, an independent set $I \in T_4$ consists of one vertex in $e_0 - (e_j + e_e)$ and an independent set in $I_2(\mathcal{H}[\bar{e}_j \cup \bar{e}_\ell] \cup \{\bar{e}_j, \bar{e}_\ell\})$ for some $1 \leq j < \ell \leq k$, where $\mathcal{H}[\bar{e}_j \cup \bar{e}_\ell] \cup \{\bar{e}_j, \bar{e}_\ell\}$ means the hypergraph obtained by adding the two edges $\bar{e}_j, \bar{e}_\ell$ to the induced subgraph $\mathcal{H}[\bar{e}_j \cup \bar{e}_\ell]$. Let

$$M_{j\ell} = \{I \in T_4 : |I \cap e_0| = 1 \text{ and } |I \cap (\bar{e}_j \cup \bar{e}_\ell)| = 2\},$$

for $1 \leq j < \ell \leq k$. Then $\{M_{j\ell} : 1 \leq j < \ell \leq k\}$ forms a partition of $T_4$. Similarly, an independent set $I \in T'_4$ consists of one vertex in $e_0 - (e'_j + e'_\ell)$ and an independent set in $I_2(\mathcal{H}'[\bar{e}_j \cup \bar{e}_\ell] \cup \{\bar{e}_j, \bar{e}_\ell\})$. Define

$$M'_{j\ell} = \{I \in T'_4 : |I \cap e_0| = 1 \text{ and } |I \cap (\bar{e}_j \cup \bar{e}_\ell)| = 2\},$$

for $1 \leq j < \ell \leq k$. Then $\{M'_{j\ell} : 1 \leq j < \ell \leq k\}$ forms a partition of $T'_4$ too. By the definition of $\mathcal{H}'$, $\mathcal{H}'[\bar{e}_j \cup \bar{e}_\ell] = \mathcal{H}[\bar{e}_j \cup \bar{e}_\ell]$. So $I_2(\mathcal{H}'[\bar{e}_j \cup \bar{e}_\ell] \cup \{\bar{e}_j, \bar{e}_\ell\}) = I_2(\mathcal{H}[\bar{e}_j \cup \bar{e}_\ell] \cup \{\bar{e}_j, \bar{e}_\ell\})$. But $|e_0 - (e'_j + e'_\ell)| = n - \max\{n_j, n_\ell\} \geq |e_0 - (e_j + e_\ell)|$. So $\|M_{j\ell}\| \leq \|M'_{j\ell}\|$ for any pair $1 \leq j < \ell \leq k$. Therefore,

$$|T_4| = \sum_{1 \leq j < \ell \leq k} |M_{j\ell}| \leq \sum_{1 \leq j < \ell \leq k} |M'_{j\ell}| = |T'_4|,$$

the equality holds if and only if $e_0 \cap e_j \subseteq e_0 \cap e_\ell$ or $e_0 \cap e_\ell \subseteq e_0 \cap e_j$ for any pair $1 \leq j < \ell \leq k$. So we can rearrange the order of $[e_1, \ldots, e_k]$, that is, there is a permutation $\pi$ on $[k]$, such that $e_0 \cap e_{\pi(1)} \subseteq \cdots \subseteq e_0 \cap e_{\pi(k)}$, which implies that $\mathcal{H}' \cong \mathcal{H}$. This completes the proof of (a).

(b) Without loss of generality, assume that $n_1 \geq n_2 \geq \cdots \geq n_k$. For simplicity, we write $d_i(v)$ and $d'_i(v)$ for the degrees of $v$ in $\mathcal{H}$ and $\mathcal{H}'$, respectively. So, we have

$$f(\mathcal{H}) = \prod_{v \in V(\mathcal{H})} d_i(v) = \prod_{v \in V_0} d_i(v) \prod_{v \in V(\mathcal{H}) \setminus e_0} d_i(v)$$

and

$$f(\mathcal{H}') = \prod_{v \in V(\mathcal{H}')} d'_i(v) = \prod_{v \in V_0} d'_i(v) \prod_{v \in V(\mathcal{H}') \setminus e_0} d'_i(v).$$

Since $d_i(v) = d'_i(v)$ for each $v \in V(\mathcal{H}) \setminus e_0 = V(\mathcal{H}') \setminus e_0$, to show $f(\mathcal{H}') < f(\mathcal{H})$, it is sufficient to show $\prod_{v \in e_0} d'_i(v) < \prod_{v \in e_0} d_i(v)$. Define $g(x_1, \ldots, x_n) = \prod_{i=1}^n x_i$. Then $d_i(v) = d'_i(v)$ for each $v \in V(\mathcal{H}) \setminus e_0$, so $d'_i(v)$ is obtained by removing $e_0$ from $d_i(v)$, which implies that $d'_i(v) < d_i(v)$ for each $v \in e_0$. Therefore, $\prod_{v \in e_0} d'_i(v) < \prod_{v \in e_0} d_i(v)$, which completes the proof of (b).
Then \( \prod_{v \in \mathcal{E}} d_1(v) = g(d_1(v_{\pi(1)}), \ldots, d_1(v_{\pi(n)})) \), where \( \pi \) is a permutation of \([n]\) such that \( k + 1 \geq d_1(v_{\pi(1)}) \geq \cdots \geq d_1(v_{\pi(n)}) \geq 1 \). Similarly, we have \( \prod_{v \in \mathcal{E}} d'_1(v) = g(d'_1(v_1), \ldots, d'_1(v_n)) = g(k + 1, \ldots, k + 1, k, \ldots, k, \ldots, 1, \ldots, 1) \) where the number of \( k + 1 \) is \( n_k \), the number of \( i \) is \( n_{i-1} - n_i \) for \( 2 \leq i \leq k \) and the number of \( 1 \) is \( n - n_1 \). Let \( y^0_i = d_1(v_{\pi(i)}) \) and \( \alpha^0_i = d'_1(v_i) - y^0_i \) for \( 1 \leq i \leq n \), that is, \( (\alpha^0_1, \ldots, \alpha^0_n) = (d'_1(v_1), \ldots, d'_1(v_n)) - (y^0_1, \ldots, y^0_n) \). Let

\[
\zeta_{ij} = \begin{cases} 
1 & \text{if } v_{\pi(i)} \in e_j, \\
0 & \text{if } v_{\pi(i)} \notin e_j.
\end{cases}
\] (10)

Then, for each \( m \in [n] \), \( \sum_{i=1}^{m} \zeta_{ij} \leq m \), and furthermore, \( \sum_{i=1}^{m} \zeta_{ij} \leq n_j \) if \( m \geq n_j \). We claim that \( \sum_{i=1}^{m} y^0_i \leq \sum_{i=1}^{m} d'_1(v_i) \). It is easy to check for \( m < n_k \) or \( m \geq n_k \). Suppose \( n_k \leq m < n_1 \), then there exists \( 2 \leq j \leq k \) such that \( n_j \leq m < n_{j-1} \). By the double-counting argument, we have

\[
\sum_{i=1}^{m} y^0_i = \sum_{i=1}^{m} \left( \sum_{s=1}^{k} \zeta_{is} + 1 \right) = \sum_{s=1}^{k} \sum_{i=1}^{m} \zeta_{is} + m \leq m(j - 1) + \sum_{s=1}^{k} n_s + m = m(j - 1) + \sum_{s=1}^{k} n_s + m = \sum_{i=1}^{m} d'_1(v_i).
\] (11)

Thus,

\[
\sum_{i=1}^{j} \alpha^0_i \geq 0 \text{ for each } j \in [n] \quad \text{and} \quad \sum_{i=1}^{n} \alpha^0_i = 0.
\] (12)

Let \( m \) be the smallest index such that \( \alpha^0_m \neq 0 \). Since \( \sum_{i=1}^{m} \alpha^0_i \geq 0 \), we have \( \alpha^0_m > 0 \). Let \( s \) be the smallest index such that \( \alpha^0_s < 0 \). Such \( s \) exits since \( \sum_{i=1}^{n} \alpha^0_i = 0 \), and \( m < s \). Let

\[
y^1_i = \begin{cases} 
y^0_i & \text{if } i \neq m, s, \\
y^0_m + 1, & \text{if } i = m, \\
y^0_s - 1, & \text{if } i = s.
\end{cases}
\] (13)

Since \( m < s \), \( y^0_s \geq y^0_s \). So \( g(y^1_1, \ldots, y^1_n) < g(y^0_1, \ldots, y^0_n) \). We claim that \( y^1_1 \geq \cdots \geq y^1_n \). Otherwise, we have \( y^0_m + 1 = y^1_m \geq y^1_m + 1 = y^0_{m-1} + 1 \) or \( y^0_s - 1 = y^1_s \leq y^1_{s+1} - 1 \). Without loss of generality, assume the former holds. Then \( y^0_m = y^0_{m-1} \). Thus \( d'_1(v_m) = y^0_m + \alpha^0_m > y^0_m = y^0_{m-1} = d'_1(v_{m-1}) \), a contradiction. Define \( \alpha^1_i = d'_1(v_i) - y^1_i \) for
\[ i \in [n], \text{ that is, } (\alpha_1, ..., \alpha_n) = (d'_1(v_1), ..., d'_n(v_n)) - (y_1^1, ..., y_n^1). \] By the definition of \( m \) and \( s \), we can easily check that

\[ \sum_{j=1}^{n} \alpha_j^i \geq 0 \text{ for each } j \in [n], \quad \sum_{i=1}^{n} \alpha_i^j = 0, \quad \text{and } \sum_{i=1}^{n} |\alpha_i^j| = \sum_{i=1}^{n} |\alpha_i^0| - 2. \quad (14) \]

So we can continue the process to obtain sequences \((y_1^2, ..., y_n^2), (\alpha_1^1, ..., \alpha_n^1), (y_1^3, ..., y_n^3), (\alpha_1^2, ..., \alpha_n^2)\) such that \( g(y_1^t, ..., y_n^t) < g(y_1^{t-1}, ..., y_n^{t-1}) \) for \( 1 \leq i \leq t \). We will stop at step \( t \) if \( (\alpha_1^t, ..., \alpha_n^t) = (0, ..., 0) \), such \( t \) exits since \( \sum_{i=1}^{n} |\alpha_i^0| \) is finite and decreases by 2 in each iteration. Therefore,

\[ \prod_{v \in E_0} d'_i(v) = g\left(y_1^t, ..., y_n^t\right) < g\left(y_1^0, ..., y_n^0\right) = \prod_{v \in E_0} d_i(v) \quad (15) \]

for \( t > 0 \). If \( t = 0 \) then \( \alpha_i^0 = 0 \) for each \( i \in [n] \). So \( d_1(v_{\pi(0)}) = y_1^0 = d'_1(v_i) \) for \( i \in [n] \). Therefore, \( \prod_{v \in E_0} d'_i(v) = \prod_{v \in E_0} d_i(v) \) if and only if \( \mathcal{H}' \cong \mathcal{H} \). This completes the proof of (b). \( \square \)

A hypergraph \( \mathcal{H} \) is called stable under edge-switching if the hypergraph obtained from the edge-switching operation with respect to any edge \( e \) of \( \mathcal{H} \) is isomorphic to \( \mathcal{H} \). For example, the \( n \)-uniform hypergraph associated with the \( K_n \)-covered graph \( S_{N,n-1} \) is stable under edge-switching operation.

**Lemma 2.2.** Let \( \mathcal{H} \) be a stable hypergraph. If \( \mathcal{H} \) is connected then \( \Delta_1(H) = |E(H)| \).

**Proof.** Let \( x \) be the vertex with maximum 1-degree in \( V(\mathcal{H}) \) and let \( e_1, ..., e_t \) be all the edges containing \( x \). If \( t = |E(\mathcal{H})| \) then we are done. Now suppose \( t < |E(\mathcal{H})| \). Since \( \mathcal{H} \) is connected, there is an edge \( e_{i+1} \in E(\mathcal{H}) \) such that \( x \notin e_{i+1} \) and \( e_{i+1} \cap e_j = \emptyset \) for some \( j \in [t] \). Without loss of generality, assume \( e_{i+1} \cap e_i = \emptyset \). For each \( i \in [t-1] \), since \( x \in e_i \) and \( x \notin e_{i+1} \), we have \( e_{i+1} \cap e_i \subseteq e_i \cap e_i \) because \( \mathcal{H} \) is stable under edge-switching. Thus for each vertex \( y \in e_{i+1} \cap e_i \), we have \( d_i(y) \geq t + 1 > d_i(x) \), a contradiction. \( \square \)

**Remark B.** Note that \( f(\mathcal{H}) \) is finite for any finite hypergraph \( \mathcal{H} \). So, by (b) of Lemma 2.1, we will obtain a stable hypergraph from \( \mathcal{H} \) after finite times of edge-switching operations with respect to edges of \( \mathcal{H} \).

Now we are ready to give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** We prove the theorem by induction on \( t \). The most difficult part is to tackle the base case \( t = 3 \). This phenomenon (most of the difficulty of \( t \geq 3 \) lying in the case \( t = 3 \)) has been observed before in these kinds of problems (e.g., Gan et al. have done in [6]).

The base case: \( t = 3 \).

We use induction on \( n \).
For $n = 1$, it is a trivial case. For $n = 2$, $S_{N,1} \cong K_{N-1,1}$. We show by induction on $N$ that $i_3(G) \leq \binom{N-1}{3}$ for any $K_2$-covered graph $G$ on $N$ vertices, and $S_{N,1}$ is the unique extremal graph if $N \geq 4$. For $N \leq 4$, one can easily check the truth of the statement by case analysis. Now assume $N \geq 5$ and the result is true for all $K_2$-covered graphs of order less than $N$. Let $G$ be a $K_2$-covered graph of order $N$. Without loss of generality, assume $G$ is edge-critical (otherwise, we may choose a minimum $K_n$-covered spanning subgraph $G'$ of $G$ to replace $G$. Clearly, $G'$ is edge-critical of order $N$ and $i_3(G') \geq i_3(G)$). By (3) of Observation 1.4, $\delta(G) = 1$. If $G$ is 1-regular, then we can calculate directly that $i_3(G) = 8\binom{N/2}{3} < \binom{N-1}{3}$. Now assume $G$ is not 1-regular. Then we can choose $x \in V(G)$ such that $d_G(x) = 1$ and $x$ has a neighbor $y$ of degree larger than one. Thus $G - x$ is also $K_2$-covered. Thus by the induction hypothesis, we have

$$i_3(G) = i_3(G - x) + i_2(G - N_G[x]) \leq \binom{N-2}{3} + \binom{N-2}{2} = \binom{N-1}{3},$$

the equality holds if and only if $G - x \cong S_{N-1,1}$ and $G - N_G[x] \cong K_{N-2}$, that is, $G \cong S_{N,1}$.

Now assume $n > 2$ and, for any $K_{n-1}$-covered graph $G'$ of order $N \geq n - 1$, we have $i_3(G') \leq \binom{N-n+2}{3}$, and the equality holds if and only if $G' \cong S_{N,n-2}$ for $N \geq n + 1$. Assume that $G$ is a $K_n$-covered graph on $N$ vertices. It can be easily checked that the statement is true by case analysis when $N \leq n + 1$. Suppose $i_3(G) \geq \binom{N-n+1}{3}$ when $N \geq n + 2$. We also assume $G$ is edge-critical here (otherwise, we may choose an edge-critical spanning subgraph of $G$ to replace it).

Claim 1. $G$ is connected.

Proof. Suppose to the contrary that $G$ is disconnected. Since $G$ is $K_n$-covered, we have $n - 1 \leq d_G(v) \leq N - n - 1$ for each $v \in V(G)$ and so $N \geq 2(d_G(v) + 1) \geq 2n$. For $i \in \{0, 1, 2, 3\}$, let $\tau_i$ be the set of unordered triples $\{u, v, w\} \subseteq V(G)$ such that $G[\{u, v, w\}]$ has exactly $i$ edges. Let us count the number of pairs $(T, v)$ with $T \in \tau_1 \cup \tau_2$ and $v \in T$ with $d_{G[T]}(v) = 1$ by a double counting argument. Note that for a fixed $T \in \tau_1 \cup \tau_2$, there are exactly two vertices in $T$ of degree 1 in $G[T]$ and for a fixed $v \in V(G)$, there are $d_G(v)(N - 1 - d_G(v))$ members $T$ in $\tau_1 \cup \tau_2$. Hence we have

$$|\tau_1| + |\tau_2| = \frac{1}{2} \sum_{v \in V(G)} d_G(v)(N - 1 - d_G(v)) \geq \frac{(n-1)(N-n)N}{2},$$

where the inequality holds since $N \geq 2n$. Since $G$ is $K_n$-covered, every vertex in $G$ is in at least $\binom{n-1}{2}$ copies of triangles. So $k_3(G) \geq \frac{N}{3}\binom{n-1}{2}$. Therefore,
\[ i_3(G) = \binom{N}{3} - |\mathbf{r}| - |\mathbf{r}'| - k_3(G) \]
\[ \leq \binom{N}{3} - \frac{(n-1)(N-n)N}{2} - \frac{N}{3} \left( \binom{n}{2} \right) \]
\[ = \frac{N(N-n)(N-2n)}{6} \]
\[ < \frac{(N-n+1)(N-n)(N-n-1)}{6} \]
\[ = \binom{N-n+1}{3}, \]  
(18)
a contradiction to the assumption. So \( G \) is connected.

Let \( G = (V, E) \) be the associated hypergraph with \( G \). Since \( G \) is connected, \( G \) is connected. Now we apply the edge-switching operation to \( G \) until we get a stable hypergraph \( G_0 \). Let \( G_0 \) be the 2-shadow of \( G_0 \). Then \( G_0 \) is \( K_n \)-covered since \( G_0 \) is \( n \)-uniform. Since the edge-switching operation does not affect the connectivity, \( G_0 \) is connected too. By Lemma 2.2, \( G_0 \) has a vertex \( x \) of 1-degree \(|E(G_0)|\), or equivalently, \( x \) has degree \(|V(G)| - 1 = N - 1 \) in \( G_0 \). Then there is no independent set \( I \in I_3(G_0) \) containing \( x \). Clearly, \( G_0 - x \) is \( K_{n-1} \)-covered. By the induction hypothesis, we have \( i_3(G_0 - x) \leq \binom{N-n+1}{3} \). By Lemma 2.1,

\[ i_3(G) = i_3(G) \leq i_3(G_0) = i_3(G_0) = i_3(G_0 - x) \leq \binom{N-n+1}{3}, \]  
(19)

the equality holds if and only if \( G \cong G_0 \) (or equivalently, \( G \cong G_0 \)) and \( G_0 - x \cong S_{N-1,n-2} \), that is, \( G \cong S_{N,n-1} \). This completes the proof of the base case.

**Induction step:** \( t > 3 \).

Now assume \( t \geq 4 \) and \( i_{t-1}(G) \leq \binom{|V(G)|-n+1}{t-1} \) for any \( K_n \)-covered graph \( G \) on at least \( n \) vertices, and when \(|V(G)| \geq n + t - 1 \), the equality holds if and only if \( G \cong S_{|V(G)|,n-1} \). Let \( G \) be a \( K_n \)-covered graph on \( N \geq n \) vertices. We show by induction on \( N \) that \( i_t(G) \leq \binom{N-n+1}{t} \), and when \( N \geq n + t - 1 \) the equality holds if and only if \( G \cong S_{N,n-1} \).

For \( N \leq n + t - 1 \), it can be easily checked that the result is true. Now assume \( N \geq n + t \) and the result holds for all \( K_n \)-covered graphs of order at most \( N - 1 \). Let \( G \) be a \( K_n \)-covered graph on \( N \) vertices and we may assume \( G \) is edge-critical for the same reason as aforementioned. By (3) of Observation 1.4, \( \delta(G) = n - 1 \). Choose \( v \in V(G) \) with \( d_G(v) = n - 1 \) and let \( S = \{ u : u \in N_G[v] \} \) and \( d_G(u) = n - 1 \). Since \( G \) is edge-critical, \( S \) is a clique contained in a unique copy of \( K_n \) in \( G \). So \( G - S \) is still \( K_n \)-covered. Let \( s = |S| \) and let \( A = \{ I \in I_t(G) : I \cap S = \emptyset \} \) and \( B = \{ I \in I_t(G) : |I \cap S| = 1 \} \). Then \(|A| = i_t(G - S) \) and \(|B| \leq s \cdot i_{t-1}(G - S) \). So
\[
i_t(G) = |A| + |B| \\
\leq i_t(G - S) + s \cdot i_{t-1}(G - S) \\
\leq \binom{N - s - n + 1}{t} + s \cdot \binom{N - s - n + 1}{t - 1} \\
\leq \binom{N - s - n + 1}{t} + \sum_{i=0}^{s-1} \binom{N - s - n + 1 + i}{t - 1} \\
= \binom{N - n + 1}{t},
\]

the equality holds if and only if \( s = 1 \) and \( G - S \cong S_{N-1,n-1} \), that is, \( G \cong S_{N,n-1} \).

3 | DISCUSSIONS AND REMARKS

In this note, we completely resolve Problem 1.1 when \( H = K_n \). From the result of Engbers and Galvin [3], and Gan, Loh, and Sudakov [6], we know that the optimal \( K_{t,d} \)-covered graphs \( G \) of order \( N \) maximizing \( i_t(G) \) have a structure of the form \( D \cup K_{n-d} \) with \( |D| = d \). It will be interesting if one can show for which graph \( H \), the optimal \( H \)-covered graphs \( G \) of order \( N \) maximizing \( i_t(G) \) have a structure of the form \( D \cup K_{N-|H|+1} \) with \( |D| = |H| - 1 \).

Remark. Stijn Cambie told us that not all extremal graphs are of the form \( D \cup K_{N-|H|+1} \) for the above question, and he gave an example for \( N = 8 \) and \( H = C_6 \) with the extremal \( H \)-covered graphs having no construction like \( D \cup K_{N-|H|+1} \).

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