Classification problems for system of forms
and linear mappings

Vladimir V. Sergeichuk
Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine
sergeich@imath.kiev.ua

Abstract
A method is proposed that allow the reduction of many classification problems of linear algebra to the problem of classifying Hermitian forms. Over the complex, real, and rational numbers classifications are obtained for bilinear forms, pairs of quadratic forms, isometric operators, and selfadjoint operators.

Many problems of linear algebra can be formulated as problems of classifying the representations of a quiver. A \textit{quiver} is, by definition, a directed graph. A \textit{representation} of the quiver is given (see \cite{6}, and also \cite{2, 14}) by assigning to each vertex a vector space and to each arrow a linear mapping of the corresponding vector spaces. For example, the quivers

\[
\begin{array}{c}
1 \\
\end{array}
\quad \quad \quad
\begin{array}{c}
1 \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
2 \\
\end{array}
\quad \quad \quad
\begin{array}{c}
1 \\
\end{array}
\begin{array}{c}
\circ \\
\end{array}
\end{array}
\]

correspond respectively to the problems of classifying:

\begin{itemize}
  \item linear operators (whose solution is the Jordan or Frobenius normal form),
  \item pairs of linear mappings from one space to another (the matrix pencil problem, solved by Kronecker), and
\end{itemize}

This is the authors’ version of an article that was published in \textit{Izv. Akad. Nauk SSSR Ser. Mat.} 51 (no. 6) (1987) 1170–1190 in Russian and was translated into English in \textit{Math. USSR-Izv.} 31 (no. 3) (1988) 481–501. Theorem 2 in the article was formulated incorrectly. I correct it and add commentaries in footnotes.
• pairs of linear operators on a vector space (a classical unsolved problem).

The notion of quiver has become central in the theory of finite-dimensional algebras over a field: the modules over an algebra are in one-to-one correspondence with the representations of a certain quiver with relations—the Gabriel quiver of the algebra (see [6 19]). The theories of quadratic and Hermitian forms are well developed (see [15 23]).

We study systems of sesquilinear forms and linear mappings, regarding them as representations of a partially directed graph (assigning to a vertex a vector space, to an undirected edge a sesquilinear form, and to a directed edge a linear mapping); and we show that the problem of classifying such representations, over a skew field $K$ of characteristic $\neq 2$ reduces to the problems of classifying:

1° Hermitian forms over certain skew fields that are extensions of the center of $K$ and
2° representations of a certain quiver.

The quiver representations 2° are known in the case of the problems of classifying:

(i) bilinear or sesquilinear forms (see, for example, [7 17 18 22]),
(ii) pairs of symmetric, or skew symmetric, or Hermitian forms ([21 23 27 28 29 30]), and
(iii) isometric or selfadjoint operators on a space with nondegenerate symmetric, or skew symmetric, or Hermitian form ([9 10 12 13 22 23]).

We solve problems (i)–(iii) over a field $K$ of characteristic not 2 with involution (possibly the identity) up to classification of Hermitian forms over fields that are finite extensions of $K$. This yields a classification of bilinear forms and pairs of quadratic forms over the rationals since over finite extensions of the rationals classifications have been established for quadratic and Hermitian forms (see [15 23]).

We study systems of forms and linear mappings by associating with them selfadjoint representations of a category with involution. This method was suggested by Gabriel [7] for bilinear forms, and by Roiter [20] for systems of forms and linear mappings (see also [11 24]). Another approach to classification problems is proposed in [16 23], where quadratic and Hermitian forms are studied on objects of an additive category with involution.

The main results of this paper were previously announced in [25 26].
The author wishes to thank A. V. Roiter for his considerable interest and assistance.

1 Selfadjoint representations of a linear category with involution

In this section we prove what might be called a weak Krull-Schmidt theorem for selfadjoint representations of a linear category with involution. Vector spaces are assumed throughout to be right vector spaces.

By a linear category over a field $P$ is meant a category $\mathcal{C}$ in which for every pair of objects $u, v$ the set of morphisms $\text{Hom}(u, v)$ is a vector space over $P$ and multiplication of morphisms is bilinear. The set of objects in $\mathcal{C}$ will be denoted by $\mathcal{C}_0$, the set of morphisms by $\mathcal{C}_1$. We define the category $R(\mathcal{C})$ of representations of $\mathcal{C}$ over a skew field $K$ with center $P$ as follows. A representation is a functor $A$ from the category $\mathcal{C}$ to the category $\mathcal{V}$ of finite-dimensional vector spaces over $K$ having finite dimension

$$\dim(A) := \sum_{u \in \mathcal{C}_0} \dim(A_u) < \infty$$

and preserving linear combinations:

$$A_{a+\beta b} = A_{\alpha a} + A_{\beta b}, \quad \alpha, \beta \in \mathcal{C}_1, \quad a, b \in P.$$  

(The images of an object $u$ and an morphism $\alpha$ are denoted by $A_u$ and $A_\alpha$.)

A morphism of representations $f: A \to B$ is a natural transformation of functors, i.e., a set of linear mappings $f_u: A_u \to B_u, \quad u \in \mathcal{C}_0,$

such that

$$f_u A_{\alpha} = B_{\alpha} f_u, \quad \alpha: u \to v.$$  

Suppose now that $K$ has an involution $a \mapsto \overline{a}$; i.e., a bijection $K \to K$ satisfying

$$\overline{\overline{a}} = a, \quad \overline{a + b} = \overline{a} + \overline{b}, \quad \overline{ab} = \overline{b} \overline{a};$$

the involution can be the the identity if $K$ is a field. Following [20], we define an involution on each of the categories $\mathcal{C}$, $\mathcal{V}$, and $R(\mathcal{C})$: 

3
1. To each object $u \in C_0$ we associate an object $u^* \in C_0$, and to each morphism $\alpha: u \to v$ a morphism $\alpha^*: v^* \to u^*$ so that

$$u^{**} = u \neq u^*, \quad \alpha^{**} = \alpha,$$

$$(\alpha \beta)^* = \beta^* \alpha^*, \quad (\alpha a)^* = \alpha^* \bar{a}$$

for all $u \in C_0$, $\alpha, \beta \in C_1$, $a \in P$ (note that [11, 20] allow $u^* = u$).

2. To each space $V \in \mathcal{V}$ we associate the adjoint space $V^* \in \mathcal{V}$ of all semilinear forms $\varphi: V \to K$:

$$\varphi(x + y) = \varphi(x) + \varphi(y), \quad \varphi(x a) = \bar{a} \varphi(x)$$

$(x, y \in V; \ a \in K)$, and to each linear mapping $A: U \to V$ the adjoint linear mapping

$$A^*: V^* \to U^*, \quad A^* \varphi := \varphi A.$$

We identify $V$ and $V^{**}$.

3. To each representation $A \in R(C)$ we associate the adjoint representation $A^o \in R(C)$, where

$$A_u^o = A_u^{o*}, \quad A_\alpha^o = A_\alpha^*, \quad (u \in C_0, \ \alpha \in C_1);$$

and to each morphism $f: A \to B$ the adjoint morphism $f^o: B^o \to A^o$, where $f_u^o = f_u^{o*}$ $(u \in C_u)$. An isomorphism $f: A \to B$ of selfadjoint representations is called a congruence if $f^o = f^{-1}$.

In Section 2 we show that the problems of classifying the systems of sesquilinear forms and linear mappings over a skew field $K$ that satisfy certain relations with coefficients in the center of $K$ can be formulated as problems of classifying selfadjoint representations up to congruence. For the present we limit ourselves to examples.

**Example 1.**

$$C_0 = \{u, u^*\}, \quad C_1 = 1_u P \cup 1_{u^*} P \cup (\alpha P \oplus \alpha^* P),$$

where $\alpha, \alpha^*: u \to u^*$. A selfadjoint representation is given by a pair of adjoint linear mappings $A, A^*: U \to U^*$, assigned to the morphisms $\alpha$ and $\alpha^*$. The representation determines, in a one-to-one manner, a sesquilinear form

$$A(x, y) := A(y)(x)$$
on the space $U$; congruent representations determine equivalent forms.

**Example 2.**

\[ C_0 = \{ u, u^* \}, \quad C_1 = 1_u P \cup 1_{u^*} P \cup \alpha P, \]

where

\[ \alpha = \varepsilon \alpha^* : u \rightarrow u^*, \quad 0 \neq \varepsilon \in P. \]

A selfadjoint representation determines an $\varepsilon$-Hermitian form

\[ A(x, y) = \varepsilon A(y, x). \]

We show now how to obtain a classification, up to congruence, of the selfadjoint representations of a category $C$, starting with the knowledge of a complete system $\text{ind}(C)$ of its nonisomorphic direct-sum-indecomposable representations. To begin with, let us replace each representation in $\text{ind}(C)$ that is isomorphic to a selfadjoint representation by one that is actually selfadjoint, and denote the set of such by $\text{ind}_0(C)$. Denote by $\text{ind}_1(C)$ the set consisting of all representations in $\text{ind}(C)$ that are isomorphic to their adjoints (but not to a selfadjoint), together with one representation from each pair \{\(A, B\)\} \(\subset \text{ind}(C)\) such that $A$ is not isomorphic to $A^\circ$ but is isomorphic to $B^\circ$.

In addition, we divide the set $C_0$ into two disjoint subsets $S_0$ and $S_0^*$ such that each pair of adjoint objects $u, u^*$ has one member in $S_0$, the other in $S_0^*$.

By the *orthogonal sum* $A \perp B$ of two selfadjoint representations $A$ and $B$ we mean the selfadjoint representation obtained from $A \oplus B$ by specifying for each $v \in S_0$ the action of \[
\varphi + \psi \in A_v^* \oplus B_v^* = (A \oplus B)_v^*,
\]
on \[
a + b \in A_v \oplus B_v = (A \oplus B)_v
\]
as follows:

\[
(\varphi + \psi)(a + b) = \varphi(a) + \psi(b).
\]

For any representation $A$ we define a *selfadjoint representation* $A^\circ$, obtained from $A \oplus A^\circ$ by specifying in a similar fashion the action of \[
(A \oplus A^\circ)_v^* = A_{v^*} \oplus A_v^* = A_v^* \oplus A_v^{**},
\]
on \[
(A \oplus A^\circ)_v = A_v \oplus A_v^{**}, \quad v \in S_0.
\]

5
Taking into account the interchange of summands in (1), we have

\[ A_\alpha^+ = \begin{bmatrix} 0 & A_\alpha^0 \\ A_\alpha & 0 \end{bmatrix} \quad \text{for } \alpha: u \to v^*, \]

\[ A_\beta^+ = \begin{bmatrix} A_\beta & 0 \\ 0 & A_\beta^0 \end{bmatrix} \quad \text{for } \beta: u \to v, \]

\[ A_\gamma^+ = \begin{bmatrix} 0 & A_\gamma \\ A_\gamma^0 & 0 \end{bmatrix} \quad \text{for } \gamma: u^* \to v, \]

where \( u, v \in S_0 \).

For any selfadjoint representation \( A = A^\circ \) and selfadjoint automorphism \( f = f^\circ \) of \( A \), we define a selfadjoint representation \( A^f \) and an isomorphism

\[ \tilde{f}: A^f \to A, \quad \tilde{f} f^\circ = f, \tag{2} \]

by putting

\[ \tilde{f}_v = f_v, \quad \tilde{f}_{v^*} = 1 \quad (v \in S_0) \]

and

\[ A^f_v = A_v, \quad A^f_{\alpha} = \tilde{f}_{v}^{-1} A_{\alpha} \tilde{f}_u = 1 \quad (v \in C_0, \ \alpha: u \to v) \]

Now suppose \( K \) has characteristic \( \neq 2 \). We show in Lemma [1] that the set \( R \) of noninvertible elements of the endomorphism ring

\[ \Lambda = \text{End}(B), \quad B \in \text{ind}_0(C), \]

is the radical of \( \Lambda \). Therefore \( T(B) = \Lambda / R \) is a skew field with involution

\[ (f + R)^\circ = f^\circ + R. \]

For each element \( 0 \neq a = a^\circ \in T(B) \) choose a fixed automorphism \( f_a = f_a^\circ \in a \) (we can take \( f_a = (f + f^\circ)/2 \), where \( f \in a \)), and define \( B^a = B^{f_a} \). The set of representations \( B^a \) we call the orbit of the representation \( B \). For any Hermitian form

\[ \varphi(x) = x_0 a_1 x_1 + \cdots + x_r a_r x_r, \quad 0 \neq a_i = a_i^\circ \in T(\mathcal{N}), \]

we put

\[ B^{\varphi(x)} := B^{a_1} \perp \cdots \perp B^{a_r}. \]
Theorem 1. Over a field or skew field \( K \) of characteristic \( \neq 2 \), every self-adjoint representation of a linear category \( C \) with involution is congruent to an orthogonal sum

\[
A_1^+ \perp \cdots \perp A_m^+ \perp B_1^{\varphi_1(x)} \perp \cdots \perp B_n^{\varphi_n(x)},
\]

where \( A_i \in \text{ind}_1(C) \), \( B_j \in \text{ind}_0(C) \), and \( B_j \neq B_{j'} \) for \( j \neq j' \). The sum is uniquely determined by the original representation up to permutation of summands and replacement of \( B_j^{\varphi_j(x)} \) by \( B_j^{\psi_j(x)} \), where \( \varphi_j(x) \) and \( \psi_j(x) \) are equivalent Hermitian forms over the skew field \( T(B_j) \).

Remark. Theorem 1 in fact holds for any ordinary (i.e., nonlinear) category \( C \) with involution, so long as we understand by a representation a functor \( A : C \to \mathcal{V} \) that has finite dimension \( \dim(A) = \sum \dim(A_u) \). The ring \( K \) can be replaced by any finite-dimensional quasi-Frobenius algebra \( F \) with involution over a field of characteristic \( \neq 2 \) (a representation assigns to an object a finitely generated module over \( F \)). A finite-dimensional algebra \( F \) is quasi-Frobenius if the regular module \( F_F \) is injective; over such an algebra the finitely generated modules \( M \) and \( M^{**} \) can still be identified.

Theorem 1 reduces the classification, up to congruence, of the self-adjoint representations of the category \( C \), assuming known the representations \( \text{ind}_1(C) \) and the orbits of the representations \( \text{ind}_0(C) \), to the classification of Hermitian forms over the skew fields

\[ T(B), \quad B \in \text{ind}_0(C). \]

If \( K \) is a finite-dimensional over its center \( Z \), then \( T = T(B) \) is finite-dimensional over \( Z \) under the natural imbedding of \( Z \) in the center of \( T \), and the involution on \( T \) extends the involution on \( Z \).

Suppose, for example, that \( K \) is a real closed field; i.e.,

\[ 1 < (K_{\text{alg}} : K) < \infty, \]

where \( K_{\text{alg}} \) is the algebraic closure of \( K \). Then its characteristic is 0, \( K_{\text{alg}} = K(\sqrt{-1}) \), and \( K \) has only the identity involution: the stationary subfield relative to involution must coincide with \( K \) (see [3, Chap. VI, §2, nos. 1, 6 and Exercise 22(d)]). By the theorem of Frobenius [4], \( T \) is equal to either \( K \), or \( K_{\text{alg}} \), or the algebra \( \mathbb{H} \) of quaternions over \( K \). By the law of inertia
If \( T = K \) or \( T = K_{alg} \) with nonidentity involution, or \( T = \mathbb{H} \) with the standard involution
\[
a + bi + cj + dk \mapsto a - bi - cj - dk,
\]
then a Hermitian form over \( T \) is equivalent to exactly one form of the form
\[
x_1^0x_1 + \cdots + x_i^0x_i - x_{i+1}^0x_{i+1} - \cdots - x_r^0x_r.
\]
If \( T = \mathbb{H} \) with nonstandard involution, then every Hermitian form
\[
\varphi(x) = x_1^0a_1x_1 + \cdots + x_r^0a_rx_r \quad (a_i = a_i^0 \neq 0)
\]
over \( T \) is equivalent to the form
\[
x_1^0x_1 + \cdots + x_r^0x_r
\]
since
\[
a_i = b_i^2 = b_i^0b_i,
\]
where \( b_i \in K(a_i) \) for \( a_i \notin K \) and \( b_i \in K(d) \) for \( a_i \in K \). Here \( d = d^c \notin K \); the existence of \( d \) follows from [4] (Chap. VIII, §11, Proposition 2); and \( K(a_i) \) and \( K(d) \) are algebraically closed fields with the identity involution.

Suppose \( K \) is a finite field. Then \( T \) is also a finite field, over which a Hermitian form reduces uniquely to the form
\[
x_1^0tx_1 + x_2^0x_2 + \cdots + x_r^0x_r,
\]
where \( t \) is equal to 1 for nonidentity involution on \( T \), and \( t \) is equal to 1 or a fixed nonsquare for the identity involution ([5, Chap. 1, §8]).

Thus, applying Theorem 1, we obtain the following assertion, a special case of which is the law of inertia for quadratic and Hermitian forms.

**Theorem 2.** Let \( K \) be one of the following fields or skew fields of characteristic not 2:

(a) An algebraically closed field with the identity involution.
(b) An algebraically closed field with nonidentity involution.
(c) A real closed field or the algebra of quaternions over a real closed field.
Then over \( K \) every selfadjoint representation of a linear category \( C \) with involution is congruent to an orthogonal sum, uniquely determined up to permutation of summands, of representations of the following form (where \( A \in \text{ind}_1(C) \) and \( B \in \text{ind}_0(C) \)):

(a) \( A^+, B \).

(b) \( A^+, B, B^{-1} \) \((-1 \in \text{Aut}(B))\).

(c) \( A^+, B^t \), where \( t = 1 \) if \( T(B) \) is an algebraically closed field with the identity involution or the algebra of quaternions with nonstandard involution, and \( t = 1 \) otherwise.

(d) \( A^+, B^t \), where \( t \) is equal to 1 for nonidentity involution on the field \( T(B) \), and is equal to 1 or a fixed nonsquare in \( T(B) \) for the identity involution, while for each \( B \) the orthogonal sum has at most one summand \( B^t \) with \( t \neq 1 \).

Remark 1. A similar assertion can be made for representations over a field \( K \) of algebraic numbers since over skew fields that are finite central extensions of such a \( K \) the classification of Hermitian forms is known (see [5, Chap. 1, § 8]).

Remark 2. It can be shown that over an algebraically closed field of characteristic \( \neq 2 \), or over a real closed field, a system of tensors of valence \( \geq 2 \) decomposes uniquely, to isomorphism of summands, into a direct sum of indecomposable subsystems. For a system of valence 2 this hollows from Theorem 2 (see Section 2) Over an algebraically closed field of characteristic 2, on the other hand, even the number of summands depends on particular decomposition: the symmetric bilinear forms \( x_1y_1 + x_2y_2 + x_3y_3 \) and \( x_1y_2 + x_2y_1 + x_3y_3 \) are equivalent, but the form \( x_1y_2 + x_2y_1 \) is indecomposable.

The rest of this section is devoted to the proof of Theorem \(^1\)

\(^1\)This theorem was formulated in [V. V. Sergeichuk, Classification problems for systems of forms and linear mappings, Math. USSR-Izv. 31 (no. 3) (1988) 481–501] incorrectly in the case of the algebra \( \mathbb{H} \) of quaternions over a real closed field. Formulating it, I erroneously thought that all \( T(B) = \mathbb{H} \) if \( K = \mathbb{H} \). I wrote that the summands have the form (a) or (b) if \( K = \mathbb{H} \) and the involution on \( \mathbb{H} \) is nonstandard or standard, respectively.
Lemma 1. The radical of the endomorphism ring of a direct-sum-indecomposable representation is a nilpotent ideal and consists of all non-invertible endomorphisms.

Proof. For every representation $B$, any endomorphism $f$ of $B$ satisfies Fitting’s lemma:

$$B = \text{Im}(f^d) \oplus \text{Ker}(f^d), \quad d = \dim(B),$$

where $\text{Im}(f^d)$ and $\text{Ker}(f^d)$ are the restrictions of $B$ to the subspaces $\text{Im}(f_v^d)$ and $\text{Ker}(f_v^d)$ for all $v \in C_0$.

Suppose $B$ is direct-sum-indecomposable, and let $R$ be its set of noninvertible endomorphisms. Then $f^d = 0$ for all $f \in R$. If

$$\text{Im}(f) = \text{Im}(fg) \quad (f, g \in R),$$

then

$$\text{Im}(f) = \text{Im}(fg^d) = 0,$$

i.e., $f = 0$. Consequently,

$$f_1 \cdots f_d = 0, \quad (f_1 + f_2)^d = 0$$

for all $f_i \in R$, and $R$ is a nilpotent ideal of the endomorphism ring and coincides with the radical. \qed

Lemma 2. Let $A$ be a selfadjoint representation of a category $C$ over a skew field $K$ of characteristic $\neq 2$ that is indecomposable into an orthogonal sum but possesses a nontrivial decomposition into a direct sum. Then $A$ is congruent to $B^+$, where $B \in \text{ind}_1(C)$.

Proof. 1°. Let $f = \pm f^c$ be an endomorphism of $A$. By Fitting’s lemma,

$$A = \text{Im}(f^d) \oplus \text{Ker}(f^d), \quad d = \dim(A).$$

But $A$ is indecomposable into an orthogonal sum. Therefore $f$ is either invertible or nilpotent.

2°. Since $A$ is direct-sum-decomposable, there exists a nontrivial idempotent $e = e^o \in \text{End}(A)$, the projection onto an indecomposable direct summand. From 1° it follows that $ee^o$ is nilpotent. Consider the selfadjoint endomorphism $h = p(ee^o)$, where

$$p(x) = 1 + a_1 x + a_2 x^2 + \cdots$$

(4)
is an infinite series with coefficients in the prime subfield of $K$ such that $p(x)^2 = 1 - x$. The series exists, since the characteristic $\neq 2$.

Consider the idempotent $f = h^{-1}eh$. It satisfies

$$
ff^o = h^{-1}eh^2e^o h^{-1} = h^{-1}e(l - ee^o)e^o h^{-1}
= h^{-1}(ee^o - ee^o)h^{-1} = 0.
$$

If we take a new $e$ equal to $f^o$, then $e^o e = 0$ and for the new $f = h^{-1}eh$ we find, in addition to $ff^o = 0$, that

$$
f^o f = he^o h^{-2}eh = he^o (1 - ee^o)^{-1} eh
= he^o (1 + ee^o)eh = 0.
$$

Consequently, $f + f^o$ is idempotent. By $1^o$, $f + f^o$ is a trivial idempotent. If $f + f^o = 0$, then $f^o = -f$, a contradiction to $1^o$. Therefore $f + f^o = 1$ and $A = D^+$, where the representation $D = \text{Im}(f)$ is direct-sum-indecomposable (recall that $e$ is the projection onto an indecomposable direct summand).

The representation $D$ cannot be isomorphic to one that is selfadjoint. Indeed, suppose

$$
\varphi: D \rightarrow B = B^o
$$

is an isomorphism. Then

$$
\varphi \oplus (\varphi^o)^{-1}: D^+ \rightarrow B^+
$$

and

$$
\psi: B^+ \rightarrow B \perp B^{-1} \quad (-1 \in \text{Aut}(B))
$$

are congruences, where

$$
\psi_v = \begin{bmatrix} 1/2 & 1 \\ -1/2 & 1 \end{bmatrix}, \quad \psi_{v^*} = \psi_v^{-1} \quad (v \in S_0).
$$

This contradicts the assumption that $A = D^+$ is indecomposable into an orthogonal sum.

Consequently, the representation $D$ is isomorphic either to $B$ or to $B^o$, where $B \in \text{ind}_1(C)$. Then $D^+$ is congruent either to $B^+$ or to $(B^o)^+$. But $(B^o)^+$ is congruent to $B^+$. Therefore $A = D^+$ is congruent to $B^+$; and the lemma is proved. \qed
Remark. Over a skew field of characteristic 2 the lemma is false, but a weaker version holds: if \( A \) is the representation of the lemma, then \( A \cong B \oplus B^\circ \), where \( B \) is indecomposable. Indeed, let

\[
g = e + e^\circ + ee^\circ \in \text{End}(A),
\]

where \( e = e^2 \) is the projection onto the direct summand \( B \) of least dimension \( \sum_u \dim(B_u) \). If \( g \) is noninvertible, it is nilpotent (step 1\(^\circ\) in the proof of the lemma) and \( h = 1 + g \) is invertible; which implies, since

\[
eh = e + e + ee^\circ + ee^\circ = 0,
\]

that \( e = 0 \). Therefore \( g \) must be invertible, and

\[
A = (e + e^\circ + ee^\circ)A = eA + e^\circ A \cong B \oplus B^\circ.
\]

Lemma 3. Let \( A \) and \( B \) be two selfadjoint representations,

\[
f = f^\circ \in \text{Aut}(A), \quad g = g^\circ \in \text{Aut}(B).
\]

Then \( A^f \) is congruent to \( B^g \) if and only if \( g = h^\circ fh \) for some isomorphism \( h: B \to A \).

Proof. Suppose \( \varphi: B^g \to A^f \) is a congruence. Define the isomorphism

\[
h = (\tilde{f}^\circ)^{-1} \varphi \tilde{g}^\circ: B \to A,
\]

where \( \tilde{f} \) and \( \tilde{g} \) are the isomorphisms of form \((2)\). Using the relations

\[
f = \tilde{f} \tilde{f}^\circ, \quad g = \tilde{g} \tilde{g}^\circ, \quad \varphi^\circ \varphi = 1,
\]

we find \( h^\circ fh = g \).

Conversely, if \( g = h^\circ fh \), then

\[
\varphi = \tilde{f}^\circ h(\tilde{g}^\circ)^{-1}: B^g \to A^f
\]

is a congruence. \( \square \)

Lemma 4. Suppose a representation \( A \) is selfadjoint and direct-sum-indecomposable. Then \( A \) is congruent to a representation \( B^f \), where \( B \in \text{ind}_0(C) \) and \( f = f^\circ \in \text{Aut}(B) \).
Proof. By definition of the set $\text{ind}_0(C)$, there exists an isomorphism $h : B \to A$ for some $B \in \text{ind}_0(C)$. By Lemma 3, $B^{h \circ h}$ is congruent to $B$. \qed

**Lemma 5.** Over a skew field $K$ of characteristic not 2, the representations

$$B^{f_1} \perp \cdots \perp B^{f_n} \quad \text{and} \quad B^{g_1} \perp \cdots \perp B^{g_n}$$

$(B \in \text{ind}_0(C), \ f_i = f_i^\circ \in \text{Aut}(B), \ g_i = g_i^\circ \in \text{Aut}(B))$ are congruent if and only if the Hermitian forms

$$\sum x_i^\circ (f_i + R)x_i, \quad \sum x_i^\circ (g_i + R)x_i$$

are equivalent over the skew field $T(B) = \text{End}(B)/R$.

Proof. Obviously,

$$B^{f_1} \perp \cdots \perp B^{f_n} = D^f,$$

where

$$D := B \perp \cdots \perp B, \quad f := \text{diag}(f_1, \ldots, f_n).$$

By Lemma 3, $D^f$ is congruent to $D^g$ for some

$$g = \text{diag}(g_1, \ldots, g_n)$$

if and only if $g = h^\circ f h$, where $h = [h_{ij}], \ h_{ij} \in \text{End}(B)$.

In particular, if $g_i \in f_i + R$, then $D^g$ is congruent to $D^f$. Indeed, let $h = p(r)$, where $p(x)$ is the series (1) and

$$r = \text{diag}(r_1, \ldots, r_n), \quad r_i = f_i^{-1}(f_i - g_i) \in R$$

(by Lemma 1, the matrix $r$ is nilpotent). Then

$$r_i^\circ f_i = (1 - g_i f_i^{-1}) f_i = f_i r_i,$$

$$h^\circ f h = p(r^\circ) f h = f h^2 = f (1 - r) = g.$$

Consequently, all the matrices in the set

$$\text{diag}(f_1 + R, \ldots, f_n + R)$$

give congruent representations. Thus, $D^g$ is congruent to $D^f$ if and only if

$$\text{diag}(b_1, \ldots, b_n) = [c_{ij}]^\circ \text{diag}(a_1, \ldots, a_n)[c_{ij}],$$

13
where

\[ a_i = f_i + R, \quad b_i = g_i + R, \quad c_{ij} = h_{ij} + R; \]

i.e., if and only if the Hermitian forms

\[ \sum x_i^* a_i x_i, \quad \sum x_i^* b_i x_i \]

are equivalent over \( T(B) \). \( \square \)

Proof of Theorem 1.

1°. If

\[ M_1 \oplus \cdots \oplus M_t \xrightarrow{f} N \oplus \cdots \oplus N \xrightarrow{g} M_1 \oplus \cdots \oplus M_t \]

are two homomorphisms of direct sums of indecomposable representations of the category \( \mathcal{C} \), with \( N \) nonisomorphic to any of the representations \( M_1, \ldots, M_t \), then the endomorphism \( h := fg \) is nilpotent. Indeed, \( f, \ g, \) and \( h \) can be written as matrices:

\[ f = \begin{bmatrix} f_{ij} \end{bmatrix}, \quad g = \begin{bmatrix} g_{jk} \end{bmatrix}, \quad h = \begin{bmatrix} h_{ik} \end{bmatrix}, \]

where

\[ f_{ij} : M_j \rightarrow N, \quad g_{jk} : N \rightarrow M_j, \]

and

\[ h_{ik} = \sum_j f_{ij} g_{jk} : N \rightarrow N. \]

Since the set \( R \) of non-invertible elements of the ring \( \text{End}(N) \) is a nilpotent ideal (Lemma 1), it suffices to show that \( f_{ij} g_{ik} \in R \). Suppose that, on the contrary, \( f_{ij} g_{ik} \) is invertible. Then so is \( g_{ik} f_{ij} \) (since it is not nilpotent); and therefore \( f_{ij} \) is an isomorphism, contradicting the assumption \( M_j \not\cong N \).

2°. By Lemmas 2 and 4, every selfadjoint representation is congruent to a representation \( A \) of the form (3). Let

\[ C = C_1^+ \perp \cdots \perp C_k^+ \perp D_1^{\psi_1(x)} \perp \cdots \perp D_l^{\psi_l(x)} \]  

be another representation of the same form, and \( f : A \rightarrow C \) a congruence. Since the representations \( A \) and \( C \) are isomorphic, so are their indecomposable direct summands (the Krull-Schmidt theorem [1] Chap. I, Theorem (3.6)] for the additive category \( R(\mathcal{C}) \)). In view of the isomorphism

\[ A \cong \bigoplus_{i=1}^m (A_i \oplus A_i^\circ) \oplus \bigoplus_{j=1}^n (B_j \oplus \cdots \oplus B_j) \]
(see (3) and (2)) and the definition of the sets \( \text{ind}_0(\mathcal{C}) \) and \( \text{ind}_1(\mathcal{C}) \), we find that \( m = k \) and \( n = l \), and that, reindexing if necessary,

\[
A_i = C_i, \quad B_j = D_j, \quad B_j^{\varphi_j(x)} \simeq B_j^{\psi_j(x)}.
\]

Write the congruence \( f : A \to C \) as a matrix

\[
f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} : S \perp B_{n}^{\varphi_n(x)} \to T \perp B_{n}^{\psi_n(x)},
\]

where \( S \) and \( T \) are the sums (3) and (5) without the last summand. From \( f^\circ f = 1 \) it follows that

\[
f_{12}^\circ f_{12} + f_{22}^\circ f_{22} = 1.
\]

Since \( f_{12}^\circ f_{12} \) is nilpotent (see \( 1^\circ \)), we can define the homomorphism

\[
g = f_{22} p(f_{12}^\circ f_{12})^{-1} : B_{n}^{\varphi_n(x)} \to B_{n}^{\psi_n(x)},
\]

where \( p(x) \) is the series (4). Since

\[
g^\circ g = p(f_{12}^\circ f_{12})^{-1}(1 - f_{12}^\circ f_{12})p(f_{12}^\circ f_{12})^{-1} = 1
\]

and

\[
B_{n}^{\varphi_n(x)} \simeq B_{n}^{\psi_n(x)},
\]

g is a congruence. By Lemma \([5]\) the Hermitian forms \( \varphi_n(x) \) and \( \psi_n(x) \) are equivalent. A similar argument gives equivalence of each of the forms \( \varphi_j(x) \) and \( \psi_j(x) \) \( (1 \leq j < n) \).

\[\square\]

2 Applications to linear algebra

In this section we apply Theorem 1 to some classical classification problems.

Let \( \mathcal{C} \) be a linear category with involution over a field \( P \). To specify the category \( \mathcal{C} \), it suffices to list:

(i) a set \( S_0 \in \mathcal{C}_0 \) of objects of the category such that

\[
S_0 \cup S_0^\ast = \mathcal{C}_0, \quad S_0 \cap S_0^\ast = \emptyset,
\]

\[
S_0 \cup S_0^\ast = \mathcal{C}_0, \quad S_0 \cap S_0^\ast = \emptyset,
\]
(ii) a set \( S_1 \in C_1 \) of generating morphisms, such that every morphism in the category is representable as a linear combination of products of morphisms in
\[
S_1 \cup S_1^* \cup \{1_u \mid u \in C_0\},
\]
(iii) a set \( S_2 \) of defining relations for \( C \):
\[
\sum \alpha_{i_1} \cdots \alpha_{i_t} a_i = 0,
\]
\((a_i \in P, \quad \alpha_{ij} \in S_1 \cup S_1^* \cup \{1_u \mid u \in C_0\})\),
such that multiplication of morphisms in \( C \) is completely determined by the bilinearity property and the relations \( S_2 \cup S_2^* \), where \( S_2^* \) consists of the adjoints of the relations in \( S_2 \):
\[
\sum \alpha_{i_1}^* \cdots \alpha_{i_t}^* a_i = 0.
\]
Let us agree, further, that the set \( S_1 \) does not contain any morphisms of the form \( \alpha : v^* \to u^* \ (u, v \in S_0) \)—since these can be replaced by the adjoint morphisms \( \alpha^* : u \to v \).
If the sets \( S_0 \) and \( S_1 \) are finite, \( S_2 \) are also be taken to be finite. Such categories, called finitely generated, can be conveniently presented by graphs in the following two ways:

- By a quiver \( \overline{S} \) with the set of vertices \( \overline{S}_0 := S_0 \cup S_0^* \), the set of arrows \( \overline{S}_1 := S_1 \cup S_1^* \), and the set of defining relations \( \overline{S}_2 := S_2 \cup S_2^* \). Such a quiver is called a quiver with involution of the category \( C \) (see [20]).

- By a graph \( S \) with the set of vertices \( S_0 \), the set of edges \( S_1 \), and the set of defining relations \( S_2 \). Each morphisms in \( S_1 \) of the form
\[
\alpha : u \to v^*, \quad \beta : u \to v, \quad \gamma : u^* \to v \quad (u, v \in S_0)
\]
is represented, respectively, by edges of the form
\[
\alpha : u \to v, \quad \beta : u \to v, \quad \gamma : u \leftrightarrow v.
\]
Such a graph, with nondirected, directed, and doubly directed edges, we call a doubly oriented graph (dograph for short) of the category \( C \).\(^2\)

\(^2\)Such a graph is called a discheme (directed scheme) in [24] and in [V. V. Sergeichuk, Classification problems for systems of forms and linear mappings, Math. USSR-Izv. 31 (no. 3) (1988) 481–501].
For example:

In what follows, a representation of the category $C$ will be specified not on the whole set $C_0 \cup C_1$, but on the subset $S_0 \cup S_1$ (being completely determined by its values on the subset); and we shall speak, correspondingly, not of a representation of the category $C$ but of a representation of the quiver $S$.

Thus, a representation $A$ of the quiver $S$ over a skew field $K$ is a set of finite-dimensional vector spaces $A_v$ ($v \in S_0$) and linear mappings $A_\alpha : A_u \to A_v$ ($\alpha : u \to v$) satisfying the relations $S_2$ (with the $\alpha \in S_1$ replaced by the $A_\alpha$). A selfadjoint representation is completely determined by its values on the set $S_0 \cup S_1$, i.e., by a set of finite-dimensional vector spaces $A_v$ ($v \in S_0$) and linear mappings $A_\alpha$ ($\alpha \in S_1$) of the form $A_u \to A_v$ for $\alpha : u \to v$, satisfying the relations $S_2$ (with $\alpha \in S_1$ and $\alpha^* \in S_1^*$ replaced by $A_\alpha$ and $A_\alpha^*$). Such a set will be called a representation $A$ of the dograph $S$ (see [24]).

A linear mapping $A : U \to V^*$ will be identified with the sesquilinear form $A : V \times U \to K$, $A(v, u) := A(u)(v)$ (their matrices coincide, with the understanding that in the adjoint space we choose the adjoint basis). Recall that by a sesquilinear form is meant a mapping $A : V \times U \to K$ such that

$$A(va + v'a', u) = \bar{a}A(v, u) + \bar{a}'A(v', u),$$

$$A(v, ua + u'a') = A(v, u)a + A(v, u')a'$$

for all $v, v' \in V$, $u, u' \in U$, and $a, a' \in K$.

With this identification, a representation $A$ of a dograph $S$ is a set of vector spaces $A_v$ ($v \in S_0$), and linear mappings and sesquilinear forms $A_\alpha$ ($\alpha \in S_1$) of the form $A_v \times A_u \to K$ for $\alpha : u \to v$, $A_u \to A_v$ for $\alpha : u \to v$, and $A_v^* \times A_u^* \to K$ for $\alpha : u \leftrightarrow v$ (in other words, the $A_\alpha$ are doubly covariant, mixed, or doubly contravariant tensors on the spaces $A_u$ and $A_v$). For two representations $A$ and $B$ of a dograph there is also a natural translation of the notions of

- congruence $f : A \to B$—a set of nonsingular linear mappings $f_v : A_v \to B_v$ ($v \in S_0$) taking the $A_\alpha$, into the $B_\alpha$ ($\alpha \in S_1$)—and of
• **orthogonal sum:**

\[(A \perp B)_x = A_x \oplus B_x, \quad x \in S_0 \cup S_1.\]

**Example 1.** The problems of classifying, up to congruence, the representations over a skew field \(K\) of the dographs

\[v \bigcirc\alpha \]
\[\alpha \bigcirc v \bigcirc \beta \quad \alpha = \varepsilon \alpha^*, \quad \beta = \delta \beta^*, \quad (7)\]
\[\alpha \bigcirc v \bigcirc \beta \bigcirc \gamma \quad \beta = \alpha^* \beta \alpha, \quad \beta = \varepsilon \beta^*, \quad (8)\]
\[\alpha \bigcirc v \bigcirc \beta \bigcirc \gamma \quad \beta \alpha = \alpha^* \beta, \quad \beta = \varepsilon \beta^*, \quad \gamma \beta = 1_v, \quad \beta \gamma = 1_{v^*}, \quad (9)\]

(where \(\varepsilon\) and \(\delta\) are elements of the center of \(K\), \(\varepsilon \bar{\varepsilon} = \delta \bar{\delta} = 1\)), are the problems of classifying, respectively:

- sesquilinear forms over \(K\),
- pairs of forms, the first form is \(\varepsilon\)-Hermitian and the second is \(\delta\)-Hermitian,
- isometric operators on a space with nondegenerate \(\varepsilon\)-Hermitian form
  (an operator \(A\) is *isometric* for a form \(F(x, y)\) if \(F(Ax, Ay) = F(x, y)\)),
  and
- selfadjoint operators on a space with nondegenerate \(\varepsilon\)-Hermitian form
  (an operator \(A\) is *selfadjoint* for a form \(F(x, y)\) if \(F(x, Ay) = F(Ax, y)\)).

**Example 2.** The problem of classifying the representations of a group \(G\) by isometries of a nondegenerate \(\varepsilon\)-Hermitian form is presented by the dograph \(\overline{\nabla}\), with the arrow \(\alpha\) replaced by arrows \(\alpha_1, \ldots, \alpha_n\) (these being generators

\[^{3}\text{The edge } \gamma \text{ and the relations } \gamma \beta = 1_v, \beta \gamma = 1_{v^*} \text{ ensure the nonsingularity of forms assigned to } \beta.\]
of $G$, and the relation $\beta = \alpha^* \beta \alpha$ replaced by the relations $\beta_i = \alpha_i^* \beta_i \alpha_i$ (1 \leq i \leq n) and the defining relations of $G$ (see [23 Chap. 7, no. 2.6]).

The rest of the paper has to do with the representations of the dographs (6)–(9) over a field $K$ of characteristic not 2 (these, as well as the representations of some other dographs, were announced in [25, 26]. Without loss of generality, we assume that $\varepsilon, \delta \in \{-1, 1\}$ in the case of the identity involution on $K$, and $\varepsilon = \delta = 1$ in the case of nonidentity involution, an $\varepsilon$-Hermitian form can be made Hermitian by multiplying by $1 + \varepsilon$ if $\varepsilon \neq -1$, and by $a - \bar{a} \neq 0$ if $\varepsilon \neq -1$).

For any polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \in K[x]$
we define the polynomials

$$f^\vee(x) := \bar{a}_{n-1}(\bar{a}_n x^n + \cdots + \bar{a}_1 x + \bar{a}_0) \quad \text{if } a_n \neq 0,$$
$$\check{f}(x) := \bar{a}_0 x^n + \bar{a}_1 x^{n-1} + \cdots + \bar{a}_n.$$ 

By the adjoint of the matrix $A = [a_{ij}]$, we mean the matrix $A^* = [\bar{a}_{ji}]$ (this being the matrix of the adjoint operator on the adjoint bases).

Every square matrix over $K$ is similar to a direct sum of Frobenius blocks

$$\Phi = \begin{bmatrix}
0 & 0 & -c_n \\
1 & \ddots & \\
& \ddots & 0 & -c_2 \\
& & 1 & -c_1
\end{bmatrix},$$

whose characteristic polynomials

$$\chi_\Phi = x^n + c_1 x^{n-1} + \cdots + c_n$$

are integer powers of polynomials $p_\Phi(x)$ that are irreducible over $K$. For each Frobenius block $\Phi$, denote by $\sqrt[\varepsilon]{\Phi}$, $\Phi_\varepsilon$, and $\Phi_{(\varepsilon)}$ ($\varepsilon = \pm 1$, $\varepsilon = 1$ for nonidentity involution on $K$) fixed nonsingular matrices satisfying, respectively, the conditions\footnote{The matrix $\sqrt[\varepsilon]{\Phi}$ was denoted by $\hat{\Phi}$ in [V. V. Sergeichuk, Classification problems for systems of forms and linear mappings, Math. USSR-Izv. 31 (no. 3) (1988) 481–501].}

$$\sqrt[\varepsilon]{\Phi} = (\sqrt[\varepsilon]{\Phi})^* \Phi,$$
$$\Phi_\varepsilon = \Phi^* \varepsilon, \quad \varepsilon \Phi_\varepsilon = \varepsilon (\Phi_\varepsilon \Phi)^*,$$
$$\Phi_{(\varepsilon)} = \varepsilon \Phi_{(\varepsilon)} = \Phi^* \Phi_{(\varepsilon)}.$$ (10) (11) (12)
Each of these matrices may do not exist for some \( \Phi \); existence conditions and explicit forms of these matrices will be established in Section 3.

The following lemma will be employed in the construction of the set \( \text{ind}_0(S) \).

**Lemma 6.** Let \( S \) be a dograph. If a representation \( A \) of the quiver \( S \) is isomorphic to a selfadjoint representation, then there exist a selfadjoint representation \( B \) and an isomorphism \( h : A \to B \) such that \( h_v = 1 \) for all vertices \( v \) of \( S \).

**Proof.** Let \( f : A \to C = C^\circ \) be an isomorphism. Define \( B = B^\circ \) and a congruence \( g : A \to B \) as follows:

\[
g_u := f_u^{-1}, \quad g_u^* := f_u^*, \quad B_u := A_u, \quad B_u^* := A_u^*
\]

for each vertex \( u \) of \( S \), and

\[
B_\alpha := g_v C_\alpha g_u^{-1}
\]

for each arrow \( \alpha : u \to v \). Then \( h := gf : A \to B \) is the desired isomorphism. \( \square \)

### 2.1 Classification of sesquilinear forms

**Lemma 7.** Let \( p(x) = p^\vee(x) \) be an irreducible polynomial of degree \( 2r \) or \( 2r + 1 \). Then every stationary element of the field

\[
K(\kappa) = K[x]/p(x)K[x]
\]

with the involution

\[
f(\kappa)^\circ := \bar{f}(\kappa^{-1})
\]

is uniquely representable in the form \( q(\kappa) \), where

\[
q(x) = \bar{a}_r x^{-r} + \cdots + a_0 + \cdots + a_r x^r
\]

(\( a_0 = \bar{a}_0, a_1, \ldots, a_r \in K \)), and when \( \deg(p(x)) = 2r \) the following hold:

(a) \( a_r = 0 \) if the involution on \( K \) is the identity.

(b) \( a_r = \bar{a}_r \) if the involution on \( K \) is nonidentity and \( p(0) \neq 1 \).
(c) $a_r = -\bar{a}_r$ if the involution on $K$ is nonidentity and $p(0) = 1$.

**Proof.** Suppose $\deg(p(x)) = 2r + 1$. The elements

$$\kappa^{-r}, \ldots, 1, \ldots, \kappa^r$$

are linearly independent over $K$. Therefore all elements of the form

$$a_{-r}\kappa^{-r} + \cdots + a_0 + \cdots + a_r\kappa^r$$

are distinct. They are stationary if and only if $a_{-i} = \bar{a}_i$ for all $i = 0, 1, \ldots, r$.

Suppose $\deg(p(x)) = 2r$ and the involution on $K$ is the identity. Then the stationary elements of the form

$$a_{r-1}\kappa^{-r+1} + \cdots + a_0 + \cdots + a_{r-1}\kappa^{r-1}$$

are distinct and form a vector space of dimension $r$ over $K$. But this is the dimension over $K$ of the whole stationary subfield of the field $K(\kappa)$, and therefore the subfield and the vector space coincide.

Suppose $\deg(p(x)) = 2r$ and the involution on $K$ is nonidentity. The equality $p(x) = p^\vee(x)$ implies that $\alpha\bar{\alpha} = 1$, where $\alpha = p(0)$. Taking any $a \neq \bar{a} \in K$ and putting

$$\delta = \begin{cases} 1 + \bar{\alpha} & \text{if } \alpha \neq 1, \\ a - \bar{a} & \text{if } \alpha = 1, \end{cases}$$

we find that $\delta\alpha = \bar{\delta}$. The function $\pi(x) := \delta x^{-r}p(x)$ has the form

$$\pi(x) = c_{-r}x^{-r} + \cdots + c_rx^r \quad (c_{-i} = \bar{c}_i).$$

Using the equalities $c_r = \delta$ and $\delta\alpha = \bar{\delta}$, we find that $c_r \neq \bar{c}_r$ if $\alpha \neq 1$, and $c_r \neq -\bar{c}_r$ if $\alpha = 1$.

Let $q(x)$ be a function of the form (15). If $q(\kappa) = 0$, then $q(x) = a\pi(x)$, $a = \bar{a} \in K$, and in view of conditions (b) and (c) of the lemma this is possible only if $q(x) = 0$. Consequently, the stationary elements $q(\kappa)$ are distinct and form a vector space of dimension $2r$ over the stationary subfield $K_0$ of $K$. But this is the dimension over $K_0$ of the whole stationary subfield of $K(\kappa)$.

Define the *skew sum* of two matrices $A$ and $B$ as follows:

$$[A \setminus B] = \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}.$$
Theorem 3. Let $K$ be a field of characteristic not two with involution (the involution can be the identity). For any sesquilinear form on a finite-dimensional vector space over $K$, there exists a basis, in which the matrix of the form is a direct sum of matrices of the three types:

(i) a singular Jordan block $J_n(0)$;

(ii) $A_\Phi = [\Phi - I]$, where $\Phi$ is a nonsingular Frobenius block for which $\sqrt{\Phi}$ does not exist; the block $\Phi$ and the identity matrix $I$ have the same size, and

(iii) $\sqrt[\ast]{\Phi}q(\Phi)$, where $q(x)$ is a nonzero function of the form \(^{15}\).

The summands are determined to the following extent:

**Type** (i) uniquely.

**Type** (ii) up to replacement of the block $\Phi$ by the block $\Psi$ with $\chi_\Psi(x) = \chi_\Phi(x)$.

**Type** (iii) up to replacement of the whole group of summands

$$\sqrt[\ast]{\Phi}q_1(\Phi) \oplus \cdots \oplus \sqrt[\ast]{\Phi}q_s(\Phi)$$

with the same $\Phi$ by

$$\sqrt[\ast]{\Phi}q'_1(\Phi) \oplus \cdots \oplus \sqrt[\ast]{\Phi}q'_s(\Phi)$$

in which each $q'_i(x)$ is a nonzero function of the form \(^{15}\) and the Hermitian forms

$$q_1(\kappa)x_1^\circ x_1 + \cdots + q_s(\kappa)x_s^\circ x_s \quad (16)$$

$$q'_1(\kappa)x_1^\circ x_1 + \cdots + q'_s(\kappa)x_s^\circ x_s \quad (17)$$

are equivalent over the field $K[\kappa] = K[x]/\Phi K[x]$ defined in \(^{13}\) with the involution \(^{14}\).

In particular, if $K$ is an algebraically closed field with the identity involution, then the summands of type (iii) can be taken equal to $\sqrt{\Phi}$. If $K$ is an algebraically closed field with nonidentity involution, or a real closed field, then the summands of type (iii) can be taken equal to $\pm \sqrt{\Phi}$. In these cases the summands are then uniquely determined by the sesquilinear form.

---

\(^5\)See also [R.A. Horn, V.V. Sergeichuk, Canonical matrices of bilinear and sesquilinear forms, Linear Algebra Appl. 428 (2008) 193–223; arXiv:0709.2408].
Proof. We will study representations of the dograph (6):

\[ v \overset{\alpha}{\rightarrow} \]

1° Let us describe \( \text{ind}(S) \). The dograph \( S \) defines the quiver

\[ \overline{S} : \quad v \overset{\alpha}{\rightarrow} \overset{\alpha^*}{\rightarrow} \overset{v^*}{\rightarrow} \]  \hspace{1cm} (18)

The representations of this quiver, as well as morphisms of the representations, will be specified by pairs of matrices. A representation is a matrix pair \( (A_\alpha, A_{\alpha^*}) \) of the same size with entries in \( K \). A morphism \( g : (A_\alpha, A_{\alpha^*}) \to (B_\alpha, B_{\alpha^*}) \) is a matrix pair \( g = [G_v, G_{v^*}] \) (for morphisms we use square brackets) such that

\[ G_{v^*} A_\alpha = B_\alpha G_v, \quad G_{v^*} A_{\alpha^*} = B_{\alpha^*} G_v. \]  \hspace{1cm} (19)

The adjoint of a representation is given by

\[ (A_\alpha, A_{\alpha^*})^\circ := (B_\alpha, B_{\alpha^*}), \]

where \( B_\alpha := A_{\alpha^*}^* \) and \( B_{\alpha^*} := A_\alpha^* \)

As shown by Kronecker (the matrix pencil problem; see [13, Chap. XII]), the set \( \text{ind}(\overline{S}) \) consists of the representations

\[ (N_1, N_2), \quad (N_2^*, N_1^*), \quad (\Phi, I_n), \quad (I, J_n(0)), \]  \hspace{1cm} (20)

where \( \Phi \) is an \( n \times n \) Frobenius block and

\[ N_1 := \begin{bmatrix} 1 & 0 & 0 \\ \vdots & \ddots & \ddots \\ 0 & 1 & 0 \end{bmatrix}, \quad N_2 := \begin{bmatrix} 0 & 1 & 0 \\ \vdots & \ddots & \ddots \\ 0 & 0 & 1 \end{bmatrix}. \]  \hspace{1cm} (21)

2°. We describe \( \text{ind}_0(\overline{S}) \) and \( \text{ind}_1(\overline{S}) \). By (19),

\[ (\Psi, I_n) \simeq (\Phi, I_n)^\circ = (I_n, \Phi^*) \]

if and only if \( \Psi \) is similar to \( \Phi^{*-1} \), i.e., if and only if

\[ \chi_\Psi(x) = \det(x I_n - \Phi^{*-1}) = \det(-\Phi^{*-1}) \cdot x^n \cdot \det(x^{-1} - \Phi^*) = \chi_\Phi^\vee(x). \]
Suppose the representation \((\Phi, I_n)\) is isomorphic to a selfadjoint representation. By Lemma 6 there exists an isomorphism
\[ h = (I, H) : (\Phi, I) \rightarrow (A, A^*) \]
By (19), \(A = H\Phi\) and \(A^* = H\). Then \(A = A^*\Phi\), and by (10) we can take
\[ h = (I, \sqrt[2]{\Phi}) : (\Psi, I) \rightarrow (\sqrt[2]{\Phi}, \sqrt[2]{\Phi}^*) \]
(22)
Consequently, the set \(\text{ind}_0(S)\) consists of the representations \(M_\Phi = (\sqrt[2]{\Phi}, \sqrt[2]{\Phi}^*)\). The set \(\text{ind}_1(S)\) consists of the representations \((N_1, N_2)\) and \((\Phi, I)\), where \(\Phi\) is a Frobenius block for which \(\sqrt[2]{\Phi}\) does not exist; and if \(\Phi\) is nonsingular, then it is determined up to replacement by the Frobenius block with characteristic polynomial \(\chi_\Phi(x)\).

3°. We describe the orbits of the representations in \(\text{ind}_0(S)\). Let \(g = [G_1, G_2] \in \text{End}(M_\Phi)\) and \(h\) be the isomorphism (22). Then
\[ h^{-1}gh = [G_1, \sqrt[2]{\Phi}^{*^{-1}} G_2 \sqrt[2]{\Phi}^*] : (\Phi, I) \rightarrow (\Phi, I); \]
that is,
\[ G_1 = \sqrt[2]{\Phi}^{*^{-1}} G_2 \sqrt[2]{\Phi}^*, \quad \Phi G_1 = G_1 \Phi. \]
Since a matrix that commutes with a Frobenius block is a polynomial in this block, we have
\[ G_1 = f(\Phi), \quad f(x) \in K[x], \]
\[ G_2 = \sqrt[2]{\Phi}^* f(\Phi) \sqrt[2]{\Phi}^{*^{-1}} = f(\sqrt[2]{\Phi} \Phi \sqrt[2]{\Phi}^{*^{-1}}) = f(\Phi^{*^{-1}}). \]
Consequently, the ring \(\text{End}(M_\Phi)\) consists of matrix pairs
\[ g_f = [f(\Phi), f(\Phi^{*^{-1}})], \quad f(x) \in K[x], \]
with involution
\[ g_f^0 = [\bar{f}(\Phi^{-1}), \bar{f}(\Phi^*)]. \]
By Lemma 11 its radical \(R\) consists of the pairs \(g_f\) for which \(f(x) \in p_\Phi(x)K[x]\). Hence the field \(T(M_\Phi) = \text{End}(M_\Phi)/R\) can be identified with the field \(K[\kappa] = K[x]/p_\Phi(x)\) with involution \(f(\kappa)^0 = \bar{f}(\kappa^{-1})\).

Under this identification, a stationary element \(q(\kappa) \neq 0\) of the field \(K[\kappa]\) (where \(q(x)\) is a function of form (15)) corresponds to the coset in the quotient
ring \( \text{End}(M_\Phi)/R \) that contains the selfadjoint automorphism \([q(\Phi), q(\Phi^{*-1})]\).

By \([2]\), the representations
\[
M_\Phi^{q(r)} = (\sqrt{\Phi}q(\Phi), \sqrt{\Phi}^{*-1}q(\Phi))
\]
constitute the orbit of the representation \(M_\Phi\).

\(4^\circ\). We now apply Theorems \([1]\) and \([2]\). Each selfadjoint representation \((A, A^*)\) of the quiver \((18)\) corresponds, in a one-to-one manner, to the representation of the dograph \((6)\) given by the matrix \(A\). In particular, the representation \((A, B)^+\) of the quiver \((18)\) corresponds to the representation \([A \setminus B^*]\) (see \((2.1)\)) of the dograph. From Theorem \([1]\) and items \(2^\circ\) and \(3^\circ\) above, it follows that every representation of the dograph \((6)\) is congruent to an orthogonal sum of representations of the form \([N_1 \setminus N_2^*], [\Phi \setminus I]\) if \(\sqrt{\Phi}\) does not exist, and \(\sqrt{\Phi}f(\Phi)\).

Let us prove that the representations \([N_1 \setminus N_2^*]\) and \([J_n(0) \setminus I_n]\) are congruent to a singular Jordan block. We show that each matrix \([N_1 \setminus N_2^*]\) or \([J_m(0) \setminus I_m]\) can be obtained by simultaneous permutations of rows and columns of a singular Jordan block. The units of \(J_m(0)\) are disposed at the places \((1, 2), (2, 3), \ldots, (n-1, n)\); it suffices to prove that there is a permutation \(f\) on \(\{1, 2, \ldots, n\}\) such that
\[
(f(1), f(2)), (f(2), f(3)), \ldots, (f(n-1), f(n))
\]
are the positions of the unit entries in \([N_1 \setminus N_2^*]\) if \(n = 2m - 1\) or in \([J_m(0) \setminus I_m]\) if \(n = 2m\). This becomes clear if we arrange the positions of the unit entries in the \((2m - 1) \times (2m - 1)\) matrix
\[
[N_1 \setminus N_2^*] =
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]
as follows:
\[
(m, 2m - 1), (2m - 1, m - 1), (m - 1, 2m - 2), (2m - 2, m - 2), \ldots, (2, m + 1), (m + 1, 1),
\]
\[25\]
and the positions of the unit entries in the $2m \times 2m$ matrix $[J_m(0) \setminus I_m]$ as follows:

$$(1, m+1), (m+1, 2), (2, m+2),$$

$$(m+2, 3), \ldots, (2m-1, m), (m, 2m).$$

This proves the first assertion of Theorem 3 (concerning existence of a basis). The remaining assertions follow from Theorems 1 and 2.

**Remark.** It can be shown that over an algebraically closed field of characteristic 2, there exists for any bilinear form a basis in which its matrix is a direct sum

$$[\Phi_1 \setminus I] \oplus \cdots \oplus [\Phi_r \setminus I] \oplus \sqrt[n]{\Psi_1} \oplus \cdots \oplus \sqrt[n]{\Psi_t} \oplus J_{n_1}(0) \oplus \cdots \oplus J_{n_s}(0),$$

where the $\Phi_i$ and $\Psi_j$ are nonsingular Jordan blocks and $\Phi_i \neq \Psi_j$ for all $i, j$. This direct sum is uniquely determined by the bilinear form up to permutation of the summands and replacement of the eigenvalue $\lambda$ in a block $\Phi_i$ by $\lambda^{-1}$. The matrix $\sqrt[n]{\Psi}$ exists if and only if the matrix $\Psi$ is of odd size with eigenvalue 1.

### 2.2 Classification of pairs of Hermitian forms

**Lemma 8.** Let $K$ be a field with the identity involution, and suppose $A = \varepsilon A$ and $A\Phi = \delta(A\Phi)^*$ for a nonsingular matrix $A$ and a Frobenius block $\Phi$. Then either $\varepsilon = 1$ or $\delta = 1$. If $\chi_\Phi = x^n$, then $\varepsilon = 1$ for $n$ odd and $\delta = 1$ for $n$ even.

**Proof.** Let $A = [a_{ij}]$ be $n$-by-$n$. Since multiplication by a Frobenius block moves the columns of this matrix to the left, we have $A\Phi = [a_{i,j+1}]$ (the entries $a_{i,n+1}$ are defined by this equality). The relations

$$A = \varepsilon A^*, \quad A\Phi = \delta(A\Phi)^*$$

can then be written

$$a_{ij} = \varepsilon a_{ji}, \quad a_{i,j+1} = \delta a_{j,i+1}. \quad (23)$$

This statement was proved in [V.V. Sergeichuk, The canonical form of the matrix of a bilinear form over an algebraically closed field of characteristic 2, Math. Notes 41 (1987) 441–445.]
Consequently,

\[ a_{ij} = \varepsilon \delta a_{i-1,j+1} = (\varepsilon \delta)^{-i} b_{i+j}, \quad b_2, \ldots, b_{2n} \in K. \]

Putting \( i = j \) in (23), we find that \( b_{2i} = 0 \) if \( \varepsilon \neq 1 \), and \( b_{2i+1} = 0 \) if \( \delta \neq 1 \). Since \( A \neq 0 \), this implies either \( \varepsilon = 1 \) or \( \delta = 1 \). If \( \chi \Phi(x) = x^n \), then the formula \( A\Phi = [b_{i+j+1}] \) implies

\[ b_{n+2} = b_{n+3} = \cdots = 0; \]

and since \( A = [b_{i+j}] \) is nonsingular, this means \( b_{n+1} \neq 0 \), and therefore \( \varepsilon = 1 \) for \( n \) odd, \( \delta = 1 \) for \( n \) even.

For any matrices \( A, B, C, D \) we define

\[ (A, B) \oplus (C, D) = (A \oplus C, B \oplus D), \quad (A, B)C = (AC, BC). \]

**Theorem 4.** Let \( F_1 \) and \( F_2 \) be \( \varepsilon \)- and \( \delta \)-Hermitian forms, respectively, in a finite dimensional vector space over a field \( K \) of characteristic \( \neq 2 \) (\( \varepsilon = \pm 1, \delta = \pm 1, \varepsilon \geq \delta, \) and \( \varepsilon = \delta = 1 \) for nonidentity involution on \( K \)). Then there exists a basis in which the pair \( (F_1, F_2) \) is given by a direct sum of matrix pairs of the following types:

(i) \( ([N_1 \setminus \varepsilon N_1^*], [N_2 \setminus \delta N_2^*]) \), where \( N_1 \) and \( N_2 \) are defined in (21).

(ii) \( ([I_n \setminus \varepsilon I_n], [\Phi \setminus \delta \Phi^*]) \), where \( \Phi \) is an \( n \times n \) Frobenius block such that \( \Phi_{\delta} \) (see (11)) does not exist if \( \varepsilon = 1 \).

(iii) \( A_{\Phi}^{f(x)} := (\Phi_{\delta}, \Phi_{\delta} \Phi)f(\Phi) \), where \( \varepsilon = 1, 0 \neq f(x) = \tilde{f}(\delta x) \in K[x] \), and \( \deg(f(x)) < \deg(p_{\Phi}(x)) \).

(iv) \( ([J_n(0) \setminus \varepsilon J_n(0)^*], [I_n \setminus (J_n)]) \), where \( \delta = -1, \) and \( n \) is odd if \( \varepsilon = 1 \).

(v) \( B^n_{\alpha} := \begin{pmatrix} 0 & 1 & 0 \\ \delta & 1 \\ \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \delta & \vdots \\ \vdots & \ddots & 1 \\ a & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \)

(24)
where the matrices are \( n \text{-by-} n \), \( \varepsilon = 1 \), \( 0 \neq a = \bar{\alpha} \in K \), and \( n \) is even if \( \delta = -1 \).

The summands are determined to the following extent:

**Type (i)** uniquely.

**Type (ii)** up to replacement of \( \Phi \) by \( \Psi \) with \( \chi(x) = \pm \bar{\chi}(\varepsilon \delta x) \).

**Type (iii)** up to replacement of the whole group of summands

\[
A_{\Phi}^{f_1(x)} \oplus \cdots \oplus A_{\Phi}^{f_s(x)}
\]

with the same \( \Phi \) by

\[
A_{\Phi}^{g_1(x)} \oplus \cdots \oplus A_{\Phi}^{g_s(x)}
\]

such that the Hermitian forms

\[
f_1(\omega)x_1^\circ x_1 + \cdots + f_s(\omega)x_s^\circ x_s,
\]
\[
g_1(\omega)x_1^\circ x_1 + \cdots + g_s(\omega)x_s^\circ x_s
\]

are equivalent over the field \( K[\omega] = K[x]/p_\Phi K[x] \) with involution \( f(\omega)^\circ = \bar{f}(\delta \omega) \).

**Type (iv)** uniquely.

**Type (v)** up to replacement of the whole group of summands

\[
B_n^{a_1} \oplus \cdots \oplus B_n^{a_s}
\]

with the same \( n \) by

\[
B_n^{b_1} \oplus \cdots \oplus B_n^{b_s}
\]

such that the Hermitian forms

\[
a_1 x_1^\circ x_1 + \cdots + a_s x_s^\circ x_s,
\]
\[
b_1 x_1^\circ x_1 + \cdots + b_s x_s^\circ x_s
\]

are equivalent over \( K \).

**Proof.** We will study representations of the dograph (1):

\[
S : \quad \alpha \bigcirc \bigcirc \beta \quad \alpha = \varepsilon \alpha^*, \quad \beta = \delta \beta^*.
\]
1° Let us describe \( \text{ind}(\mathcal{S}) \). The do-graph \( \mathcal{S} \) defines the quiver \( \mathcal{S} \):

\[
\begin{array}{c}
\alpha \\
\alpha^* \\
\beta \\
\beta^*
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\uparrow \\
\uparrow
\end{array}
\begin{array}{c}
v \\
v^*
\end{array}, \quad \alpha = \varepsilon \alpha^*, \quad \beta = \delta \beta^*.
\]

The representations of this quiver will be specified by pairs of matrices \((A_\alpha, A_\beta)\) of the same size; then \( A_\alpha^* = \varepsilon A_\alpha \) and \( A_\beta^* = \delta A_\beta \). The adjoint representation is given by

\[
(A_\alpha, A_\beta)^\circ = (\varepsilon A_\alpha^*, \delta A_\beta^*).
\]

The set \( \text{ind}(\mathcal{S}) \) consists of the representations

\[(N_1, N_2), \ (N_1^*, N_2^*), \ (I_n, \Phi), \ (J_n(0), I_n)\]

(which we prefer now to the set \((20)\)).

2°. We describe \( \text{ind}_0(\mathcal{S}) \) and \( \text{ind}_1(\mathcal{S}) \). It is obvious that

\[(I, \Psi) \simeq (I, \Phi)^\circ = (\varepsilon I, \delta \Phi^*)\]

if and only if \( \Psi \) is similar to \( \varepsilon \delta \Phi^* \), i.e., if and only if

\[
\chi_\Psi(x) = \pm \bar{\chi}_\Phi(\varepsilon \delta x).
\]

Suppose \((I, \Phi)\) is isomorphic to a selfadjoint representation. By Lemma \(\mathfrak{G}\) there exists an isomorphism

\[
h = [I, H] : (I, \Phi) \rightarrow (A, B) = (A, B)^\circ.
\]

Then

\[
A = H, \quad B = H \Phi, \quad A = \varepsilon A^*, \quad B = \delta B^*;
\]

i.e.,

\[
A = \varepsilon A^*, \quad A \Phi = \delta (A \Phi)^*.
\]

Since \( \varepsilon \geq \delta \), we have by Lemma \(\mathfrak{H}\) that \( \varepsilon = 1 \), and by \((11)\),

\[
h = [I, \Phi_\delta] : (I, \Phi) \rightarrow (\Phi_\delta, \Phi_\delta \Phi).
\]

Similarly, if \((J_n(0), I_n) \simeq (A, B) = (A, B)^\circ\),

\[
(25)
\]

29
then
\[ B = \delta B^*, \quad BJ_n(0) = \varepsilon (BJ_n(0))^*; \]
by Lemma 8, \( \varepsilon = 1 \), and \( n \) is even if \( \delta = -1 \). It is easily verified that
\[ (J_n(0), I_n) \simeq B_n, \]
where \( B_n = B^1_n \) is of the form (24).

Consequently, the set \( \text{ind}_0(S) \) is empty if \( \varepsilon = -1 \), and consists of the representations
\[ A_\Phi = (\Phi_\delta, \Phi_\delta \Phi) \]
and \( B_n \) (where \( n \) is even when \( \delta = -1 \)) if \( \varepsilon = 1 \).

The set \( \text{ind}_1(S) \) consists of the following representations:

- \( (N_1, N_2) \)
- \( (I, \Phi) \), where \( \Phi_\delta \) does not exist if \( \varepsilon = 1 \), and \( \chi_\Phi(x) \) is determined up to replacement by \( \bar{\chi}_\Phi(\varepsilon \delta x) \).
- \( (J_n(0), I_n) \), where \( \delta = -1 \), and \( n \) is odd if \( \varepsilon = 1 \).

3\textdegree. We describe the orbits of the representations in \( \text{ind}_0(S) \). Let
\[ g = [G_1, G_2] \in \text{End}(A_\Phi), \]
and \( h \) be the isomorphism (25). Then
\[ h^{-1}gh = [G_1, \Phi_\delta^{-1}G_2 \Phi_\delta]: (I, \Phi) \to (I, \Phi); \]
i.e.,
\[ G_1 = \Phi_\delta^{-1}G_2 \Phi_\delta, \quad \Phi G_1 = G_1 \Phi. \]
Since \( G_1 \) commutes with \( \Phi \), we have
\[ G_1 = f(\Phi), \quad f(x) \in K[x], \]
and by (11),
\[ G_2 = \Phi_\delta f(\Phi) \Phi_\delta^{-1} = f(\Phi_\delta \Phi_\delta^{-1}) = f(\delta \Phi^*). \]
Consequently, the ring \( \text{End}(A_\Phi) \) consists of the matrix pairs
\[ g_f = [f(\Phi), f(\delta \Phi^*)], \quad f(x) \in K[x], \]
with involution  
\[ g_f^\circ = [\bar{f}(\delta \Phi), f(\Phi)^*]. \]
Hence the field  
\[ T(A_\Phi) = \text{End}(A_\Phi)/R \]
can be identified with the field  
\[ K[\omega] = K[x]/p_\Phi(x)K[x], \]
with involution  \( f(\omega)^\circ = \bar{f}(\delta \omega). \) The set of representations  
\[ A_{\Phi}^{(\omega)} = A_\Phi f(\Phi), \]
where  
\[ 0 \neq f(x) = \bar{f}(\delta x) \in K[x], \quad \text{deg}(f(x)) < \text{deg}(p_\Phi(x)), \]
is the orbit of the representation  \( A_\Phi. \)

Similarly,  \( T(B_n) \) can be identified with the field  \( K, \) and the set of representations of the form  \( B_n a, \) where  \( 0 \neq a = \bar{a} \in K, \) is the orbit of the representation  \( B_n. \)

4°. From 2°, 3°, and Theorem 1, the proof of Theorem 4 now follows. \( \square \)

2.3 Classification of isometric operators

**Theorem 5.**

7Let  \( A \) be an isometric operator on a finite-dimensional vector space with nondegenerate \( \varepsilon \)-Hermitian form  \( F \) over a field  \( K \) of characteristic not 2. Then there exists a basis in which the pair  \( (A, F) \) is given by a direct sum of matrix pairs of the following types:

(i)  \( (\Phi \oplus \Phi^* - I_n, [I_n \setminus \varepsilon I_n]), \) where  \( \Phi \) is a nonsingular  \( n \times n \) Frobenius block for which  \( \Phi_{(\varepsilon)} \) (see 12) does not exist.

(ii)  \( A_{\Phi}^{q(x)} = (\Phi, \Phi(\varepsilon)q(\Phi)), \) where  \( q(x) \neq 0 \) is of the form 15.

The summands are determined to the following extent:

**Type (i)** up to replacement of  \( \Phi \) by  \( \Psi \) with  \( \chi_\Phi(x) = \chi_\Psi(x). \)

\footnote{See also [V.V. Sergeichuk, Canonical matrices of isometric operators on indefinite inner product spaces, Linear Algebra Appl. 428 (2008) 154–192; arXiv:0710.0933.}
Type (ii) \( \text{up to replacement of the whole group of summands} \)
\[
A_{\Phi}^{q_1(x)} \oplus \cdots \oplus A_{\Phi}^{q_s(x)}
\]

with the same \( \Phi \) by
\[
A_{\Phi}^{q_1'(x)} \oplus \cdots \oplus A_{\Phi}^{q_s'(x)}
\]
such that the Hermitian forms
\[
q_1(\omega)x_1^\omega x_1 + \cdots + q_s(\omega)x_s^\omega x_s,
\]
\[
q'_1(\omega)x_1^\omega x_1 + \cdots + q'_s(\omega)x_s^\omega x_s
\]
are equivalent over the field \( K[\kappa] = K[x]/p_\Phi K[x] \) with involution
\( f(\kappa)^o = \bar{f}(\kappa^{-1}). \)

Proof. We will study representations of the dograph (8):

\[
S: \quad \alpha \circ \begin{array}{c} \beta \\ \gamma \end{array} \quad \beta = \alpha^* \beta \alpha, \quad \beta = \varepsilon \beta^*,
\gamma \beta = 1_v, \quad \beta \gamma = 1_{v^*}.
\]

1° Let us describe \( \text{ind}(\overline{S}) \). The dograph \( S \) defines the quiver \( \overline{S} \):

\[
\alpha \circ \begin{array}{c} \beta \\ \gamma \end{array} \circ \begin{array}{c} \alpha^* \\ v^* \end{array} \quad \beta = \alpha^* \beta \alpha, \quad \beta = \varepsilon \beta^*,
\gamma \beta = 1_v, \quad \beta \gamma = 1_{v^*},
\gamma^* \beta^* = 1_v, \quad \beta^* \gamma^* = 1_{v^*}.
\]

The representations of this quiver will be specified by triples of square matrices \((A_\alpha, A_\beta, A_{\alpha^*})\) of the same size, where \( A_\beta \) is nonsingular and
\[
A_\beta = A_{\alpha^*} A_\beta A_\alpha,
\]
and then
\[
A_{\beta^*} = \varepsilon^{-1} A_\beta, \quad A_\gamma = A_\beta^{-1}, \quad A_{\gamma^*} = \varepsilon A_\beta^{-1}.
\]
The adjoint representation is given by
\[
(A, B, C)^o = (C^*, \varepsilon B^*, A^*).
\]
Every representation of the quiver is isomorphic to one of the form \((A, I, A^{-1})\). The set \(\text{ind}(\mathcal{S})\) consists of the representations \((\Phi, I, \Phi^{-1})\), where \(\Phi\) is a Frobenius block.

2°. We describe \(\text{ind}_0(\mathcal{S})\) and \(\text{ind}_1(\mathcal{S})\). It is obvious that

\[
(\Psi, I, \Psi^{-1}) \simeq (\Phi, I, \Phi^{-1})^\circ = (\Phi^*, I, \Phi^*)
\]

if and only if \(\Psi\) is similar to \(\Phi^*\), i.e., if and only if \(\chi_\Psi(x) = \chi_{\Phi^*}(x)\).

Suppose \((\Phi, I, \Phi^{-1})\) is isomorphic to a selfadjoint representation. By Lemma \(\text{[3]}\) there exists an isomorphism

\[
h = [I, H]: (\Phi, I, \Phi^{-1}) \rightarrow (A, B, A^*), \quad B = \varepsilon B^*.
\]

Then

\[
A = \Phi, \quad B = H, \quad A^*H = H\Phi^{-1}, \quad B = \varepsilon B^*;
\]

i.e.,

\[
A = \Phi, \quad B = \varepsilon B^* = \Phi^* B \Phi.
\]

By \(\text{[12]}\),

\[
h = [I, \Phi(x)]: (\Phi, I, \Phi^{-1}) \rightarrow (\Phi, \Phi(x), \Phi^*).
\]

Consequently, the set \(\text{ind}_0(\mathcal{S})\) consists of the representations

\[
A_\Phi = (\Phi, \Phi(x), \Phi^*).
\]

The set \(\text{ind}_1(\mathcal{S})\) consists of the representations \((\Phi, I, \Phi^{-1})\), in which \(\Phi\) is a Frobenius block such that \(\Phi(x)\) does not exist and \(\chi_{\Phi}(x)\) is determined up to replacement by \(\chi_{\Phi^*}(x)\).

3°. We describe the orbits of the representations in \(\text{ind}_0(\mathcal{S})\). Let

\[
g = [G_1, G_2] \in \text{End}(A_\Phi).
\]

Then

\[
\Phi G_1 = G_1 \Phi, \quad \Phi(x) G_1 = G_2 \Phi(x), \quad \Phi^* G_2 = G_2 \Phi^*.
\]

Since \(G_1\) commutes with the Frobenius block, we have

\[
G_1 = f(\Phi) \quad (f(x) \in K[x]), \quad G_2 = \Phi(x) f(\Phi) \Phi(x)^{-1} = f(\Phi^* x^{-1}).
\]

Consequently, the algebra \(\text{End}(A_\Phi)\) consists of the matrix pairs

\[
[f(\Phi), f(\Phi^* x^{-1})], \quad f(x) \in K[x],
\]

33
with involution

\[ [f(\Phi), f(\Phi^*)]^\circ = [\bar{f}(\Phi^{-1}), f(\Phi)^*]. \]

The field \( T(A_\Phi) \) can be identified with the field

\[ K[\kappa] = K[x]/p_\Phi K[x] \]

with involution \( f(\kappa)^\circ = \bar{f}(\kappa^{-1}) \).

Let \( q(\kappa) \) (where \( q(x) \neq 0 \) is of the form (15)) be a stationary element of this field. The representations

\[ A_{\Phi}^{q(\kappa)} = (\Phi, \Phi_{(\epsilon)}q(\Phi)) \]

constitute the orbit of the representation \( A_\Phi \).

4°. From 2°, 3°, and Theorem 1, the proof of Theorem 5 now follows. \( \square \)

### 2.4 Classification of selfadjoint operators

**Theorem 6.** Let \( A \) be a selfadjoint operator on a finite-dimensional vector space with nondegenerate \( \varepsilon \)-Hermitian form \( F \) over a field \( K \) of characteristic not 2 (\( \varepsilon = \pm 1; \varepsilon = 1 \) for nonidentity involution on \( K \)). Then there exists a basis in which the pair \( (A, F) \) is given by a direct sum of matrix pairs of the following types:

1. \( (\Phi \oplus \Phi^*, [I_n \setminus \varepsilon I_n]) \), where \( \Phi \) is an \( n \times n \) Frobenius block and if \( \varepsilon = 1 \) then \( \Phi_1 \) (see (11)) does not exist.

2. \( A_{\Phi}^{f(x)} = (\Phi, \Phi_{(\epsilon)}f(\Phi)) \), where \( \varepsilon = 1, 0 \neq f(x) = \bar{f}(x) \in K[x] \), and
   \( \deg(f(x)) < \deg(p_\Phi(x)) \).

The summands are determined to the following extent:

**Type (i)** up to replacement of \( \Phi \) by \( \Psi \) with \( \chi_{(\Phi)}(x) = \bar{\chi}_{(\Phi)}(x) \).

**Type (ii)** up to replacement of the whole group of summands

\[ A_{\Phi}^{f_1(x)} \oplus \cdots \oplus A_{\Phi}^{f_s(x)} \]

with the same \( \Phi \) by

\[ A_{\Phi}^{g_1(x)} \oplus \cdots \oplus A_{\Phi}^{g_t(x)} \]
such that the Hermitian forms
\[
f_1(\omega)x_1^0x_1 + \cdots + f_s(\omega)x_s^0x_s, \\
g_1(\omega)x_1^0x_1 + \cdots + g_s(\omega)x_s^0x_s
\]
are equivalent over the field \( K[\omega] = K[x]/p_\Phi K[x] \) with involution \( f(\omega)^\circ = \bar{f}(\omega). \)

**Proof.** We will study representations of the dograph (9):

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\beta\alpha = \alpha^*\beta, & \beta = \varepsilon\beta^*, & \gamma\beta = 1_v, \quad \beta\gamma = 1_{v^*}.
\end{array}
\]

1° Let us describe \( \text{ind}(\overline{S}) \). The dograph \( S \) defines the quiver \( \overline{S} : \)

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\beta\alpha = \alpha^*\beta, & \beta = \varepsilon\beta^*, & \gamma\beta = 1_v, \quad \beta\gamma = 1_{v^*}, \\
\gamma^*\beta^* = 1_v, & \beta^*\gamma^* = 1_{v^*}.
\end{array}
\]

The representations of this quiver will be specified by triples of square matrices \( (A_\alpha, A_\beta, A_{\alpha^*}) \) of the same size, where \( A_\beta \) is nonsingular and

\[
A_\beta A_\alpha = A_{\alpha^*} A_{\beta^*}.
\]

The adjoint representation is given by

\[
(A, B, C)^\circ = (C^*, \varepsilon B^*, A^*).
\]

Every representation of the quiver is isomorphic to one of the form \( (A, I, A) \). The set \( \text{ind}(\overline{S}) \) consists of the representations \( (\Phi, I, \Phi) \), where \( \Phi \) is a Frobenius block.

2°. We describe \( \text{ind}_0(\overline{S}) \) and \( \text{ind}_1(\overline{S}) \). It is obvious that

\[
(\Psi, I, \Psi) \simeq (\Phi, I, \Phi)^\circ = (\Phi^*, \varepsilon I, \Phi^*)
\]

if and only if \( \Psi \) is similar to \( \Phi^* \), i.e., if and only if \( \chi_\Psi(x) = \bar{\chi}_\Phi(x) \).

Suppose \( (\Phi, I, \Phi) \) is isomorphic to a selfadjoint representation. By Lemma 6 there exists an isomorphism

\[
h = [I, H] : (\Phi, I, \Phi) \to (A, B, A^*), \quad B = \varepsilon B^*.
\]
Then
\[ A = \Phi, \quad B = H, \quad A^*H = H\Phi, \quad B = \varepsilon B^*; \]
i.e.,
\[ B = \varepsilon B^*, \quad B\Phi = \Phi^*B = \varepsilon(B\Phi^*).\]
By Lemma 8, \( \varepsilon = 1 \) and we can take \( B = \Phi_1 \) (see (11)).

Consequently, the set \( \text{ind}_0(S) \) is empty if \( \varepsilon = -1 \), and consists of the representations
\[ A_\Phi = (\Phi, \Phi_1, \Phi) \]
if \( \varepsilon = 1 \). The set \( \text{ind}_1(S) \) consists of the representations \((\Phi, I, \Phi)\), in which \( \Phi \) is a Frobenius block such that \( \chi_\Phi(x) \) is determined up to replacement by \( \bar{\chi}_\Phi(x) \) and if \( \varepsilon = 1 \) then \( \Phi_1 \) does not exist.

3°. We describe the orbits of the representations in \( \text{ind}_0(S) \). Let
\[ g = [G_1, G_2] \in \text{End}(A_\Phi). \]
Then
\[ \Phi G_1 = G_1\Phi, \quad \Phi_1 G_1 = G_2\Phi_1, \quad \Phi^* G_2 = G_2\Phi^*. \]
Since \( G_1 \) commutes with the Frobenius block, we have
\[ G_1 = f(\Phi) \quad (f(x) \in K[x]), \]
\[ G_2 = \Phi_1 f(\Phi)\Phi_1^{-1} = f(\Phi_1\Phi_1^{-1}) = f(\Phi^*). \]

Consequently, the algebra \( \text{End}(A_\Phi) \) consists of the matrix pairs
\[ [f(\Phi), f(\Phi^*)], \quad f(x) \in K[x], \]
with involution
\[ [f(\Phi), f(\Phi^*)]^\circ = [\bar{f}(\Phi), f(\Phi^*)]. \]
The field \( T(A_\Phi) \) can be identified with the field
\[ K[\omega] = K[x]/p_\Phi(x)K[x] \]
with involution \( f(\omega)^\circ = \bar{f}(\omega) \). The set of representations
\[ A^r_\Phi = (\Phi, \Phi_1 f(\Phi)), \]
where
\[ 0 \neq f(x) = \bar{f}(x) \in K[x] \]
and
\[ \deg(f(x)) < \deg(p_\Phi(x)), \]
constitute the orbit of the representation \( A_\Phi \).

4°. From 2°, 3°, and Theorem 1, the proof of Theorem 6 now follows. \( \square \)
3 The matrices $\sqrt[\ast]{\Phi}$, $\Phi_{\varepsilon}$, and $\Phi(\varepsilon)$

Let $K$ be a field of characteristic not 2. In this section we obtain the existence conditions and forms for the matrices $\sqrt[\ast]{\Phi}$, $\Phi_{\varepsilon}$, and $\Phi(\varepsilon)$ defined in (10)–(12) by the equalities:

$$
\begin{align*}
\sqrt[\ast]{\Phi} &= (\sqrt[\ast]{\Phi})^* \Phi, \\
\Phi_{\varepsilon} &= \Phi_{\varepsilon}^*, \\
\Phi_{\varepsilon^2} &= \varepsilon(\Phi_{\varepsilon^2})^*, \\
\Phi(\varepsilon) &= \varepsilon\Phi_{\varepsilon}^* = \Phi^*\Phi(\varepsilon)^* \Phi;
\end{align*}
$$

$(\varepsilon = \pm 1, \varepsilon = 1$ for nonidentity involution on $K$).

In the case of nonidentity involution on $K$, we choose a fixed nonzero element

$$
k = -\bar{k} \neq 0;
$$

we can take $k = a - \bar{a}$ with any $a \neq \bar{a} \in K$.

By $\Phi$ we denote an $n \times n$ Frobenius block, and by

$$
\chi(x) := p(x)^s = \alpha_0 x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n, \tag{27}
$$

$$
\mu(x) := p(x)^{n-1} = \beta_0 x^t + \beta_1 x^{t-1} + \cdots + \beta_t \tag{28}
$$

$(\alpha_0 = \beta_0 = 1)$ we denote the characteristic polynomial of $\Phi$ and its maximal divisor.

Let

$$f(x) = \gamma_0 x^m + \gamma_1 x^{m-1} + \cdots + \gamma_m \in K[x].$$

A sequence

$$(a_q, a_{q+1}, \ldots, a_r)$$

of elements of $K$ will be called $f$-recurrent if

$$\gamma_0a_{l+m} + \gamma_1a_{l+m-1} + \cdots + \gamma_ma_l = 0$$

$(q \leq l \leq r - m)$; the sequence is completely determined, assuming $\gamma_0 \neq 0 \neq \gamma_m$, by any fragment of length $m$. The sequence will be called strictly $\chi$-recurrent if it is $\chi$-recurrent but not $\mu$-recurrent (see (27) and (28)).

**Lemma 9.** The following two conditions on a matrix $A$ are equivalent:

(a) $A = \Phi^* A \Phi$ and $A$ is nonsingular.
(b) \( A = [a_{j-i}] \), where the sequence \((a_{1-n}, \ldots, a_{n-1})\) is strictly \(\chi\)-recurrent, with \(\chi(x) = \chi^\vee(x)\).

**Proof.** \((a) \implies (b)\). Suppose the matrix \( A = [a_{ij}] \) satisfies condition (a). Then
\[
A\Phi^{-1}A^{-1} = \Phi^*,
\]
and
\[
\chi(x) = \det(xI - \Phi^* - I) = \det(-\Phi^* - I) \cdot x^n \cdot \det(x^{-1}I - \Phi^*) = \chi^\vee(x).
\]

Since
\[
\Phi^* A \Phi = \Phi^*[a_{i,j+1}] = [a_{i+1,j+1}]
\]
(the entries \(a_{i,n+1}\) and \(a_{n+1,j}\) are defined by this equality), we have \(a_{ij} = a_{i+1,j+1}\), so that the matrix entries depend only on the difference of the indices; i.e., \( A = [a_{j-i}] \). That the sequence \((a_{1-n}, \ldots, a_{n-1})\) is \(\chi\)-recurrent follows from the equality \( A\Phi = [a_{j-i+1}] \). Furthermore, the recurrence is strict; otherwise, we should have
\[
(0, \ldots, 0, \beta_0, \ldots, \beta_t)A = 0
\]
(see (28)), contradicting the assumption that \( A \) is nonsingular.

\((a) \iff (b)\). Suppose (b) is satisfied. Then
\[
\Phi^* A \Phi = \Phi^*[a_{j-i+1}] = [a_{j-i}] = A.
\]

We show now that \( A \) is nonsingular. Suppose that, on the contrary, its rows
\[
v\Phi^{n-1}, v\Phi^{n-2}, \ldots, v, \quad \text{where} \quad v = (a_{1-n}, \ldots, a_0),
\]
are linearly dependent. Then \(vf(\Phi) = 0\) for some polynomial \( f(x) \neq 0 \) of degree less than \( n \). Since \(v\chi(\Phi) = 0\), we have \(vp(\Phi)^r = 0\), where \( p(x)^r \) is the greatest common divisor of the polynomials \( f(x) \) and \( \chi(x) \). But then
\[
v\Phi^i\mu(\Phi) = (0, \ldots, 0, \beta_0, \ldots, \beta_t, 0, \ldots, 0)A = 0
\]
\((0 \leq i < n - t; \text{ see (28)}\); so the sequence \((a_{1-n}, \ldots, a_{n-1})\) is \(\mu\)-recurrent, contradicting condition (b).

\(\square\)

**Theorem 7.** Existence conditions for the \( n \times n \) matrix \( \sqrt[n]{\Phi} \) are:
(A1) $\chi(x) = \chi^\vee(x)$.  

(A2) $p(x) \neq x + (-1)^{n-1}$ in the case of the identity involution.

With these conditions satisfied, we can take  

$$\sqrt[n]{\Phi} = [a_{j-1}],$$

where the sequence $(a_{1-n}, \ldots, a_{n-1})$ is $\chi$-recurrent, and is defined by the fragment  

$$(a_{-m}, \ldots, a_{m-1}) = (\bar{a}, 0, \ldots, 0, a)$$  

(29)

of length either $n$ or $n + 1$, in which

(a) $a = 1$ if $n = 2m$, except for the case $p(x) = x + \alpha$ with $\alpha^{n-1} = -1$;

(b) $a = k$ (see (26)) if $n = 2m$, $p(x) = x + \alpha$, $\alpha^{n-1} = -1$, and also if $n = 2m - 1$, $p(x) = x + 1$;

(c) $a = \chi(-1)$ if $n = 2m - 1$, $p(x) \neq x + 1$.

Proof. 1°. If the matrix $A = \sqrt[n]{\Phi}$ exists, then conditions (A1) and (A2) must be satisfied. Indeed, in view of the relations

$$A = A^* \Phi = \Phi^* A \Phi$$

(see (10)) and Lemma 9, condition (A1) is satisfied, and the entries of the matrix

$$[a_{j-1}] = A = A^* \Phi = [\bar{a}_{i-j-1}]$$

form a strictly $\chi$-recurrent sequence

$$(a_{1-n}, \ldots, a_{n-1}) = (\bar{a}_{n-2}, \ldots, \bar{a}_{0}, a_{0}, \ldots, a_{n-1}).$$  

(30)

This sequence is completely determined by the fragment

$$(\bar{a}_{m-1}, \ldots, a_{0}, a_{0}, \ldots, a_{m-1})$$  

(31)

of length $2m$, equal either to $n$ or to $n + 1$.

Now suppose condition (A2) is not satisfied; i.e., that the involution is the identity and

$$p(x) = x + (-1)^{n-1}.$$
Then the vector (31) is \(\mu(x) = (x + (-1)^{n-1})^{n-1}\)-recurrent. For \(n = 2m\) this is obvious; and for \(n = 2m - 1\) it follows from the property
\[
\alpha_i = \beta_i + \beta_{i-1} = \beta_i + \beta_{n-i}, \quad 0 < i < n,
\]
of the binomial coefficients \(\alpha_i\) and \(\beta_i\) (see (27) and (28)), since
\[
2[\beta_0a_{m-1} + \beta_1a_{m-2} + \cdots + \beta_{n-2}a_{m-3} + \beta_{n-1}a_{m-2}]
= (\beta_0 + 0)a_{m-1} + (\beta_1 + \beta_{n-1})a_{m-2} + (\beta_2 + \beta_{n-2})a_{m-3}
+ \cdots + (\beta_{n-1} + \beta_1)a_{m-2} + (0 + \beta_0)a_{m-1}
= \alpha_0a_{m-1} + \alpha_1a_{m-2} + \cdots + \alpha_{n-1}a_{m-1} = 0 \quad (32)
\]
in view of the \(\chi\)-recurrence of (31). But then its \(\mu\)-recurrent extension coincides with (30), contradicting the strict \(\chi\)-recurrence of (30).

2°. If conditions (A1) and (A2) are satisfied, then the matrix \(\sqrt[\ast]{\Phi}\) exists. Indeed, let us verify that the vector (29) is strictly \(\chi\)-recurrent.

- Suppose \(n = 2m\). Since (29) is of length \(n\), it suffices to verify that it is not \(\mu\)-recurrent. If \(\deg(\mu(x)) < n - 1\), this is obvious. If \(\deg(\mu(x)) = n - 1\), then the polynomial \(\mu(x)\) is of the form \((x + \alpha)^{n-1}\), and therefore
\[
a + \beta_{n-1} \tilde{a} = a + \alpha^{n-1} \tilde{a} \neq 0.
\]

- Suppose \(n = 2m - 1\). Since (29) is of length \(n + 1\), it suffices to verify that it is \(\chi\)-recurrent, i.e., that \(a + \alpha_n \tilde{a} = 0\) (see (27)). Condition (A1) implies that \(\alpha_n = \bar{\alpha}_n^{-1}\), and so, since
\[
\chi^\gamma(x) = \bar{\alpha}_n^{-1}x^n \bar{\chi}(x^{-1}),
\]
that
\[
\chi(-1) = -\alpha_n \chi(-1).
\]
If \(\chi(-1) = 0\), then
\[
\chi(x) = (x + 1)^n, \quad k + \alpha_n \tilde{k} = 0.
\]
Thus, the vector (29) is strictly \(\chi\)-recurrent, and its \(\chi\)-recurrent extension has, in view of (A1), the form (30). Consequently,
\[
A = [a_{j-i}] = A^\ast \Phi.
\]
By Lemma 9, the matrix \(A\) is nonsingular, and it can be taken to be \(\sqrt[\ast]{\Phi}\). \(\square\)
Theorem 8. Existence conditions for the $n \times n$ matrix $\Phi_\varepsilon$ are:

(B1) $\chi(x) = \varepsilon^n \bar{\chi}(\varepsilon x)$.

(B2) $\chi(x) \notin \{x^2, x^4, x^6, \ldots\}$ if $\varepsilon = -1$.

With these conditions satisfied, we can take

$$
\Phi_\varepsilon = [\varepsilon^i a_{i+j}],
$$

where the sequence $(a_2, a_3, \ldots, a_{2n})$ is $\chi$-recurrent, and is defined by the fragment

$$(a_2, \ldots, a_{n+1}) = \begin{cases} 
(1, 0, \ldots, 0) & \text{if } \Phi \text{ is nonsingular}, \\
(0, \ldots, 0, 1) & \text{if } \Phi \text{ is singular}. 
\end{cases} \quad (33)
$$

Proof. 1°. Suppose $\Phi_\varepsilon$ exists. Then

$$
\Phi = \Phi_\varepsilon^{-1} (\varepsilon x^*) \Phi_\varepsilon
$$

(see (11)), and this gives condition (Bl):

$$
\chi(x) = \deg(xI - \varepsilon \Phi^*) = \varepsilon^n \bar{\chi}(\varepsilon x).
$$

Condition (B2) follows from (11) and Lemma 8.

2°. Suppose conditions (B1) and (B2) are satisfied. The matrix $\Phi_\varepsilon = [\varepsilon^i a_{i+j}]$, defined in the statement of Theorem 8, is nonsingular. Let us verify that it satisfies (11).

If $\Phi$ is singular, this is obvious. Suppose $\Phi$ is nonsingular. Then the $\chi$-recurrence of the sequence $(a_2, a_3, \ldots, a_{2n})$ implies that

$$
\Phi_\varepsilon \Phi = [\varepsilon^i a_{i+j+1}],
$$

and so relations (11) can be written in the form

$$
\varepsilon^i a_{i+j} = \varepsilon^j \bar{a}_{j+i}, \quad \varepsilon^i a_{i+j+1} = \varepsilon \varepsilon^j \bar{a}_{j+i+1},
$$

i.e.,

$$
a_t = \varepsilon^i \bar{a}_t, \quad 2 \leq t \leq 2n. \quad (34)
$$

We argue now by induction. Relation (34) certainly holds for $t \leq n + 1$ (see (33)). Assuming it holds for $t < n + l$ ($l \geq 2$), we must verify it for
t = n + l. And indeed, using the $\chi$-recurrence of the sequence $(a_2, \ldots, a_{2n})$ and equalities (27) and (B1), we find that

$$a_{n+l} = -\alpha_1 a_{n+l-1} - \cdots - \alpha_n a_l = -\varepsilon^n \alpha_n \varepsilon^l a_l = \varepsilon^n \alpha_n \varepsilon^l a_{n+l}.$$  

\[ \square \]

**Theorem 9.** Existence conditions for the $n \times n$ matrix $\Phi(\varepsilon)$ are:

(C1) $\chi(x) = \chi^\vee(x)$.

(C2) If the involution on $K$ is the identity and $\varepsilon = (-1)^n$, then $\deg(p(x)) > 1$ (see (27)).

With these conditions satisfied, we can take

$$\Phi(\varepsilon) = [a_{j-l}],$$

where the sequence $(a_{1-n}, \ldots, a_{n-1})$ is $\chi$-recurrent, and is defined by the fragment $v = (a_{-m}, \ldots, a_m)$ of length either $n$ or $n + 1$, that equals to

(a) $(\varepsilon \alpha_n - 1, 0, \ldots, 0, \alpha_n - \varepsilon)$ if $n = 2m$, $\alpha_n \neq \varepsilon$ (see (27));

(b) $(\alpha_1, -1, 0, \ldots, 0, -1, \alpha_1)$ $(v = (\alpha_1, -2, \alpha_1)$ for $n = 2)$ if $n = 2m$, $\varepsilon = 1$, and the involution on $K$ is the identity;

(c) $(-k, 0, \ldots, 0, k)$ (see (26)) if $n = 2m$, $\alpha_n = 1$, and the involution is nonidentity, and also if $n = 2m + 1$, $p(x) = x + \alpha$, $\alpha^{n-1} = -1$;

(d) $(\varepsilon, 0, \ldots, 0, 1)$ if $n = 2m + 1$, in any other case besides $p(x) = x + \alpha$, $\alpha^{n-1} = -1$.

**Proof.** 1°. If the matrix $A = \Phi(\varepsilon)$ exists, then conditions (Cl) and (C2) are satisfied. Indeed, in view of the relations (12) and Lemma 9, condition (Cl) is satisfied, and the entries of the matrix

$$A = [a_{j-l}] = \varepsilon A^*$$

form a strictly $\chi$-recurrent sequence

$$(a_{1-n}, \ldots, a_{n-1}) = (\varepsilon a_{n-1}, \ldots, \varepsilon a_0 = a_0, \ldots, a_{n-1}) \quad (35)$$
Suppose condition (C2) is not satisfied. By (Cl),
\[ p(x) = p^\vee(x) = x \pm 1, \]
and the fragment \((\varepsilon a_m, \ldots, a_m)\) of length either \(n\) or \(n + 1\), of the vector (35) is \(\mu\)-recurrent. This is obvious if \(n = 2m + 1\) since \(\varepsilon = -1\); and if \(n = 2m\), it follows from (32) as applied to the fragment (replace \(m\) in (32) by \(m + 1\)). But then the vector (35) is also \(\mu\)-recurrent, and we have a contradiction.

2°. If conditions (Cl) and (C2) are satisfied, then \(\Phi_{(\varepsilon)}\) exists. To show this, let us verify that the vector \(v\) of Theorem 9 is strictly \(\chi\)-recurrent and of the form
\[ (\varepsilon a_m, \ldots, \varepsilon a_0 = a_0, \ldots, a_m). \]

- The vector in (a) is \(\chi\)-recurrent, since its length is \(n + 1\) and, by (Cl), \(\alpha_n\bar{\alpha}_n = 1\).

- The vector in (b) is \(\chi\)-recurrent, since for the identity involution conditions (Cl) and (C2) imply
\[ \chi(1) = \alpha_n^{-1}\chi(1) \neq 0, \quad \alpha_n = 1, \quad \alpha_{n-1} = \alpha_1. \]
The vector is not \(\mu\)-recurrent, since \(t \leq n - 2\) (by (28) and (C2)) and \(\beta_t = 1\) (by the equality \(p(x) = p^\vee(x)\) and (C2)).

- If \(n = 2m + 1\), \(p(x) = x + \alpha\), and \(\alpha^{n-1} = -1\) (see (c)), then the involution is nonidentity: otherwise
\[ p(x) = p^\vee(x) = x \pm 1, \]
contradicting the equality \(\alpha^{n-1} = -1\).

- The vector in (d) is not \(\mu\)-recurrent, in view of (C2).

Now let (35) be the \(\chi\)-recurrent extension of the vector \(v\). Then the matrix \(A = [a_{j-1}]\) is equal to \(\varepsilon A^*\), and by Lemma 9 it can be taken for \(\Phi_{(\varepsilon)}\). \(\square\)

References

[1] H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968.
[2] I. N. Bernstein, I. M. Gel’fand, V. A. Ponomarev, Coxeter functors and
Gabriel’s theorem, Uspekhi Mat. Nauk 28 (no. 2) (1973) 19–33; English
transl. in Russian Math. Surveys 28 (no. 2) (1973) 17–32.

[3] N. Bourbaki, Algèbre, Chaps. 6,7, Actualités Sci. Indust., nos. 1179,
Hermann, Paris, 1952.

[4] N. Bourbaki, Algèbre, Chaps. 8,9, Actualités Sci. Indust., nos. 1261,
1272, Hermann, Paris, 1958, 1959.

[5] J. A. Dieudonné, La géométrie des groupes classiques, 3me éd., Springer,
1971.

[6] P. Gabriel, Unzerlegbare Darstellungen I, Manuscripta Math. 6 (1972)
71–103.

[7] P. Gabriel, Appendix: degenerate bilinear forms, J. Algebra 31 (1974)
67–72.

[8] F. R. Gantmacher, The Theory of Matrices, Vols. 1,2, Chelsea, New
York, 1959.

[9] B. Huppert, Isometrien von Vektorräumen. I, Arch. Math. (Basel) 35
(1980) 164–176.

[10] B. Huppert, Isometrien von Vektorräumen. II, Math. Z. 175 (1980) 5–20.

[11] S. A. Kruglyak, Representations of involutive quivers, Preprint, Akad.
Nauk Ukrain. SSR, Inst. Mat., Kiev, 1984 = Manuscript No. 7266-84,
deposited at VINITI, 1984, (Russian); R. Zh. Mat. 1985, 24A367.

[12] D. W. Lewis, The isometry classification of Hermitian forms over divi-
sion rings, Linear Algebra Appl. 43 (1982) 345–272.

[13] J. Milnor, On isometries of inner product spaces, Invent. Math. 8 (1969)
83–97.

[14] L. A. Nazarova, Representations of quivers of infinite type, Izv. Akad.
Nauk SSSR. Ser. Mat. 37 (1973) 752–791: English transl. in Math. USSR
Izv. 7 (1973) 749–792.

[15] O. T. O’Meara, Introduction to Quadratic Forms, Springer-Verlag, 1971.
[16] H.-G. Quebbemann, W. Scharlau, M. Schulte, Quadratic and Hermitian forms in additive and abelian categories, *J. Algebra* 59 (1979) 264–289.

[17] C. Riehm, The equivalence of bilinear forms, *J. Algebra* 31 (1974) 45–66.

[18] C. Riehm, M. Shrader-Frechette, The equivalence of sesquilinear forms, *J. Algebra* 42 (1976) 495–530.

[19] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lect. Notes Math. 1099, Springer, 1984.

[20] A. V. Roiter, Bocses with involution, in: *Representations and Quadratic Forms*, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1979, 124–128 (in Russian).

[21] R. Scharlau, Paare alternierender Formen, *Math. Z.* 147 (1976) 13–19.

[22] R. Scharlau, Zur Klassification von Bilineaformen und von Isometrien über Körpern, *Math. Z.* 178 (1981) 359–373.

[23] W. Scharlau, *Quadratic and Hermitian Forms*, Springer-Verlag, 1985.

[24] V. V. Sergeichuk, Representations of simple involutive quivers, in: *Representations and quadratic forms*, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1979, pp. 127–148 (in Russian).

[25] V. V. Sergeichuk, Representation of oriented schemes, in: *Linear algebra and the theory of representations*, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1983, 110–134 (in Russian).

[26] V. V. Sergeichuk, *Classification problems for systems of linear mappings and sesquilinear forms* (Russian) Preprint, Kiev University, 1983, 60 p. = Manuscript No. 196 Uk-D84, deposited at the Ukrainian NIINTI, 1984; R. Zh. Mat. 1984, 7A331.

[27] F. Uhlig, A recurring theorem about pairs of quadratic forms and extensions: a survey, *Linear Algebra Appl.* 25 (1979) 219–237.

[28] F. Uhlig, A rational canonical pair form for a pair of symmetric matrices over an arbitrary field $F$ with char $F \neq 2$ and applications to finest simultaneous block diagonalizations, *Linear Multilinear Algebra* 8 (1979/80) 41–67.
[29] W. C. Waterhouse, Pairs of quadratic forms, *Invent. Math.* 37 (no. 2) (1976) 157–164.

[30] I. M. Yaglom, Quadratic and skew-symmetric bilinear forms in a real symplectic space, *Trudy Sem. Vector. Tensor Analiz.* 8 (1950) 364–381 (Russian).