Designing of Architectural Shells on the Basis of Linear Hyperbolic Congruence Surfaces

A V Zamyatin\(^1\), Y A Kokareva\(^2\)

\(^1\)Engineering Geometry and Computer Graphic, Don State Technical University, 1, Gagarin sq., Rostov-on-Don 344000, Russian Federation
\(^2\)IT systems in constructions, Don State Technical University, 1, Gagarin sq., Rostov-on-Don 344000, Russian Federation

E-mail: kokareva.ya.a@gmail.com

Abstract. The possibilities of using the ruled surfaces of hyperbolic congruence for designing of architectural shells are considered in the article. The constructive design of selecting the surface from the hyperbolic congruence of straight lines with two proper directrices is given. In the general form, the extreme values of the distribution functions of the points on the directrices of congruences and conditions for the existence of torso generators on the surface are determined depending on the surface internal parameters. The dependence of the curvature radius of the principal non-zero direction along the torso generator on the position parameter of the point on the congruence ray is determined. The asymptotic lines for it are indicated. Specific examples show the solution of the inverse problems of applied geometry related to the determination of internal parameters of the surface depending on the given characteristics of the future shell: position-metric and differential problems.

1. Introduction

Modern architectural designs are distinguished by a wide variety of shapes, materials and constructive solutions used. The aesthetic component has long been one of the decisive factors in the approval of a project. The ruled surfaces of structures that can be realized in various ways are interesting solutions [1-5].

The works [6-11] showed the successful application of methods of synthetic and constructive geometry in the design of architectural shells of complex shape, including the compartments of ruled surfaces isolated from congruences of straight lines. Parametric equations of the surfaces of congruences of straight lines of the first order were obtained using the constructive-parametric method of forming [12,13] in the work [14].

2. Allocation of Problem

Design of surfaces for architectural purposes necessarily includes the solution of applied problems: the determination of positional, metric and differential characteristics of future surfaces that determine their design, optical, acoustic and other properties [15-22].

In this article we consider the inverse problem of applied geometry - the determination of the internal parameters of the surfaces of the hyperbolic congruence of straight lines depending on the specified shell characteristics (geometric, positional and differential).
3. Theoretical Information

The constructive scheme of obtaining surfaces by immersing the arbitrary curve in hyperbolic congruence of lines with two proper directrices is shown in figure 1. The crossing directrices of the congruence under consideration are located at a distance $h$ from each other at an angle $\phi$ along the axis OX. One of the directrices is combined with the axis OZ. Let's call it "directrix-1", and another - "directrix-2". Through each ordinary point of space there passes only one ray of congruence, which is a straight line. The rays of congruence passing through each point of the immersed line form a ruled surface, all the generatrices of which intersect both directrices.

![Figure 1. Constructive scheme of the formation of surfaces of hyperbolic congruence](image)

The first assemblage of curvilinear coordinates of the surface are the rectilinear rays of congruence, the second is a projective assemblage of immersive curves.

Parametric equations of the immersed curve:

$$
\begin{align*}
&x = x_k(w), \quad y = y_k(w), \quad z = z_k(w)
\end{align*}
$$

(1)

The parametric equations of the ruled surfaces of hyperbolic congruence with two proper directrices obtained by immersing the curve (1):

$$
\begin{align*}
x &= ht, \quad y = \frac{h y_k(w)t}{x_k(w)}, \\
z &= \frac{h [z_k(w) - y_k(w) \cot \phi]}{h - x_k(w)} (1 - t) + \frac{h y_k(w)}{x_k(w)} t \cot \phi,
\end{align*}
$$

(2)

where $t$ is the point location parameter on the congruence ray. At that, at $t = 0$ the point lays on directrix-1, and at $t = 1$ – on directrix-2; $w$ – point location parameter on the immersed straight line; $h$, $\phi$ – inner parameters of congruence, assigning, correspondently, the distance and angle between directrices, $h \neq 0$.

4. Solution of Position-Metric Problem

Let’s consider the problem of determination of congruence parameters from the given geometric and positional characteristics of the structural elements. According to the position of the submerged contour and the specified length and position of one of the directrices, determine the parameters $h$, $\phi$ of surfaces (2).

The solution of the problem is related to the problem of finding the extremes of a function that determines the position of points on the directrix and is closely related to the determination of the torso generators on the surface (2). Ruled surfaces at ordinary points have negative Gaussian curvature $K$ [23]. In the case when $K = 0$, the generator is called torso. The torso generator is characterized by the fact that all points of the surface lying on it have the same tangent plane and are parabolic.

Let’s consider the conditions under which the value of the Gaussian curvature for the surface (2) is zero.
\[
\frac{z_k(w) \tan \varphi - y_k(w)}{z_k'(w) \tan \varphi - y_k'(w)} = \frac{x_k(w) - h}{x_k'(w)} \tag{3}
\]

\[
\frac{x_k(w)}{x_k'(w)} = \frac{y_k(w)}{y_k'(w)} \tag{4}
\]

Analyzing the distribution functions of points on directrix-1 and directrix-2, we obtain expressions for the extrema of functions identical to (3) and (4), respectively. Hence it follows that the torso generators intersect the directrices at the points of maximum and minimum in pairs.

Further, the solution of the problem of the dependence of the internal congruence parameters on the metric characteristics of the given elements is reduced to finding the values of the argument \( w \) for known methods, at which the distribution functions reach extreme values:

\[
w_{\text{max/min}} = \psi(h, \varphi) \tag{5}
\]

Then we find the functional relationship between the internal arguments depending on the given size of the rectilinear element (rod, beam, column) and its design mark. The solution does not have a general problem, since it depends on the type of the submerged contour (1).

**Example 1.** The length \( l \) and lower design mark \( a_{\text{min}} \) of rectilinear vertical element, combined with axis \( OZ \) are assigned. The circle located in the plane parallel to the plane \( YOZ \), at a distance \( c \) from the origin, serves as the second contour:

\[
\left(0, \cos \varphi, \sin \varphi\right) = \left(R \cos w + y_0, y_0, z_0\right) = \left(R \sin w + z_0\right) \tag{6}
\]

Find dependencies \( h(l, a_{\text{min}}), \varphi(l, a_{\text{min}}) \).

Under given conditions, the distribution function of points on the directrix-1 has the form:

\[
f = -\frac{h}{h-c}\left[(R \sin w + z_0) - (R \cos w + y_0) \cot \varphi\right] \tag{7}
\]

From the condition that the derivative of the function (7) is equal to zero, we find the values (5) of the argument at which the function reaches its extreme values:

\[
w_1 = -\varphi, w_2 = -\varphi + \pi \tag{8}
\]

Adding the values (8) to the expression (7), and equating their difference to a value equal to \( l \), we find the dependance \( h(\varphi) \):

\[
h(\varphi) = \frac{lc \sin \varphi}{l \sin \varphi - 2R} \tag{9}
\]

Expression for the parameter \( \varphi \) is found, by equating the value of the function (7) to \( a_{\text{min}} \) taking into account the expression (10):

\[
\varphi = \arctan \left( \frac{y_0 \left(\sqrt{z_0^2 \left(l^2 \left(y_0^2 + z_0^2 - R^2\right) - 4R^2 a_{\text{min}} (l + a_{\text{min}})\right)} - y_0 R (l + 2a_{\text{min}})\right)}{z_0 \left(R (l + 2a_{\text{min}})\right) + y_0^2 + z_0^2} + \left(\frac{y_0 \left(\sqrt{z_0^2 \left(l^2 \left(y_0^2 + z_0^2 - R^2\right) - 4R^2 a_{\text{min}} (l + a_{\text{min}})\right)} - y_0 R (l + 2a_{\text{min}})\right)}{z_0 \left(R (l + 2a_{\text{min}})\right) + y_0^2 + z_0^2}\right) l \right) \tag{10}
\]
The figures 2a and 2b show the example of dependence (7) and obtained surface (2) under the following given values: $R = 0.8, y_0 = 1, z_0 = 1, c = 6, l = 6, a_{\text{min}} = 1$. Simulation values of parameters $h(l, a_{\text{min}}), \phi(l, a_{\text{min}})$, calculated under the formulas (9) and (10): $\phi = 1.6399, h = 8.1889$.

![Figure 2](image_url)

**Figure 2.** Example of finding the internal congruence parameters depending on the required sizes of structural elements: a - the extremes of a function that determines the position of points on the directrix, b – example of surface (2) with circle (6).

5. **Finding of Differential Characteristics of the Surface**

The differential characteristics of the surface (2) are determined by its first and second quadratic form, resulting in a consequence of the triviality of the solution [15], [23]. We only note that, like for any ruled surface, the coefficient of the second quadratic form, which includes the second partial derivatives with respect to the parameter $t$, is zero.

In this paper, we have considered the determination of the torso generators on the surface (2) - the dependences (3) and (4). Next we show the definition of the tangent planes passing through the torsion generators and the dependence of the radius of curvature along one of the principal directions along the torso generator of the parameter $t$.

The dependence of the radius of curvature $r_2$ of the non-zero principal direction along the torsion guide is determined from the expression for the average curvature, taking into account the fact that the curvature along the torso generator is 0. From where

$$r_2 = \frac{1}{k_2} = \frac{EG - F^2}{-2FM + GL}$$

Dependence (11) will have the asymptote $t = 0$ or $t = 1$ depending on the considered torso generator.

**Example 2.** A circle (6) is immersed in the hyperbolic congruence of straight lines. It is necessary to find the torso generators of the obtained surface, the equations of the tangent planes passing through them, dependence of the radius of curvature along the torso generator along the second principal
direction, and also solve the inverse problem: determine the values of the congruence parameters under the given vector of the tangent plane passing through the torso line.

Taking into account the equations (6), (3) and (4), we find the values of the parameter \( w \) corresponding to the torso generators:

\[
w = 0, w = \pi, w = -\varphi, w = -\varphi + \pi.
\]

Adding the obtained values of the parameter \( w \) to the equations (2), we can get the parametrical equations of torso generators.

Let's consider the finding of the tangent plane for the torso generator \( w = -\varphi \). Using the equation of the plane passing through the point on the immersed circle \( (w = -\varphi, t = c / h) \), we find the coefficients of the general equation of the plane:

\[
A = -\frac{h^2 R(z_0 \sin \varphi - y_0 \cos \varphi - R)}{c(c - h)}, B = -\frac{h^2 R \cos \varphi}{c}, C = -B \tan \varphi, D = -hA
\]  

From the expressions (12) it follows that the desired tangent plane passes through the second directrix.

Analyzing the equations of the tangent planes passing through the other three torso generators, we can come to the conclusion that the given surface lies inside the tetrahedron whose faces touch the surface along the torsion generators, and whose vertices are extreme points on the directrixes of the congruence.

Conditions for finding the congruence parameters from a given standard vector \((A, B, C)\) of the tangent plane passing through a given torso generator, and hence also through the point \((w = -\varphi, t = c / h)\), taking into account the dependencies (12), are expressed as:

\[
\varphi = \arctan \left( \frac{C}{B} \right),
\]

\[
D = -Ac - B( -R \cos \varphi + y_0 ) - C( -R \sin \varphi + z_0 ),
\]

\[
h = -D / A.
\]  

Figures 3a and 3b show the example of the surface (2) with torsion generators and dependencies (11) at the following specified values: \( R = 0.7, y_0 = 0, z_0 = -0.5, c = 4, A = 0.5, B = 1, C = -0.5 \). Simulation values of parameters (18): \( \varphi = 0.4636, D = -3.03, h = 6.06 \).

![Figure 3](image-url)
6. Conclusion

Thus, the principles of determination of the internal parameters of surfaces (2) depending on the specified sizes of structural elements and required differential characteristics are considered in this article, which shows the possibility of practical application of ruled surfaces of the hyperbolic congruence of straight lines in the design of architectural shells.

References

[1] Santoso K 2004 *Wide-Span Cable Structures* (Berkeley: University of California)

[2] Krivoshapko S N 2015 Suspension cable structures and roofs of erections *J. Construction of Unique Buildings and Structures* 7(34) 51

[3] Krivoshapko S N 2013 On opportunity of shell structures in modern architecture and building *J. Structural Mechanics of Engineering Constructions and Buildings* 1 51

[4] Krivoshapko S N and Mamieva I A 2012 *Analytic surfaces in the architecture of buildings, structures and products* (Moscow: Knjiyny dom “LIBROKOM”)

[5] Korotich A V 2015 Innovative solutions of architectural shells: alternative to traditional building construction *J. Academicheski Vestnik URALNIIPROEKT RAASN* 4 70

[6] Mihajlenko V E, Obuhova V S and Podgornyj A L 1972 *Forming of cover in architecture* (Kiev: Budivel'nik)

[7] Podgornyj A L 1969 *J. Applied Geometry and Engineering Graphics* VIII 17

[8] Podgornyj A L 1976 *J. Applied Geometry and Engineering Graphics* XXII 18

[9] Podgornyj A L and Nesvidomin V N 2007 J. Geometric and Computer Modelling 17 35

[10] Nesvidomin V N 2008 *Computer models of synthetic geometry (Manuscript)* (Kyiv: Kyiv National University of Building and Architecture)

[11] Sanger R G 1939 *Synthetic projective geometry* (New York-London: McCraw-Hill Book Company)

[12] Skidan I A and Zhurba N V 1993 *J. Applied Geometry and Engineering Graphics* 55 35

[13] Skidan I A 2001 *Proc. of Tavria State Agrotechnological University. J. Applied Geometry and Engineering Graphics* is 13 vol 13 (Melitopol’: Tavria State Agrotechnological University) pp 21–28

[14] Kokareva Ya A 2011 *Analytic and computer aided modeling of surfaces of first order linear congruences* (Makivka: Donbass National Academy of Civil Engineering and Architecture)

[15] Golovanov N N 2002 *Geometric modeling* (Moscow: Fizmatlit)

[16] Zamyatin A V and Suhomlinova V V 2012 Algorithm of calculation of the second reflexions on the basis of geometrical model *J. Naukovedenie SOAJN*03(2012)12–88 Retrieved from http://naukovedenie.ru/sbornik12/12-88.pdf

[17] Zamyatin A V and Suhomlinova V V 2012 Algorithm of calculation of the first reflexions on the basis of geometrical model *J. Naukovedenie SOAJN*03(2012)12–89 Retrieved from http://naukovedenie.ru/sbornik12/12-89.pdf

[18] Svidrak I G and Galkina N S 2014 Problem modelling surface *J. Scientific Messenger of LNU of Veterinary Medicine and Biotechnologies* 2-4 171

[19] Korotkiy V A and Usmanova E A 2015 Architectural shell on the closed circuit *Bulletin of the South Ural State University. Series: Construction engineering and architecture* 2 47

[20] Vlasov V Z 1949 *General theory of shells and its applications in engineering* (Leningrad: Gostechizdat)

[21] Voloshinov D V 2010 *Constructive Geometric Modeling. Theory, Practice, Automation* (Saarbrucken: Lambert Academic Publishing)

[22] Kokareva Ya A 2015 *J. Inženernyj vestnik Dona* JIVD04(2015)3355 Retrieved from ivdon.ru/ru/magazine/archive/n4y2015/3355

[23] Kagan V F 1947 *Fundamentals of the theory of surfaces in tensor presentation (Part 1)* (Moscow: Gostechizdat, OGIZ)