BFKL equation at finite temperature

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We consider the Color Glass Condensate (CGC) at finite temperature, which would be relevant to the initial condition of relativistic heavy ion collisions and the energy loss of energetic partons in the quark-gluon plasma. In the weak source approximation, we derive the thermal BFKL equation.

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I. INTRODUCTION

The Relativistic Heavy Ion Collider (RHIC) at BNL has revealed various interesting natures of high energy QCD matter which would be the quark-gluon plasma (QGP). One of the findings is that the expanding matter can be described as a perfect fluid without dissipations [1, 2, 3, 4]. Even surprising would be the early thermalization. Hydrodynamic analyses suggest that the thermalization takes place in less than 1 fm/c after the collision [2]. In the future Large Hadron Collider (LHC) experiments at CERN, where higher energy collisions are possible, we expect an even earlier thermalization, that could be almost instantaneous after the collision moment [5]. It is important issues to elucidate the mechanism of the early thermalization [6, 7, 8, 9, 10, 11, 12, 13], as well as to determine the initial condition to be used as an input of hydrodynamical simulations [5, 14, 15, 16, 17, 18, 19]. Another discovery to be mentioned would be the jet quenching. QGP has an enormous stopping power for energetic partons, for which full theoretical understanding has not been achieved.

On the other hand, small $x$ physics in high energy hadron scatterings is one of the fascinating areas in QCD physics. The gluon distribution in the small $x$ region, which is responsible for the high energy hadronic scattering amplitude, is determined by the BFKL equation [20, 21, 22]. It is known, however, that the BFKL equation predicts too many gluons as we go to smaller $x$, resulting in unitarity violation of the scattering cross-section. We need some saturation mechanism to attenuate the growth of the small $x$ gluon number [3, 23, 24, 25, 26]. The Color Glass Condensate (CGC) is one of the formalisms which successfully implements the saturation [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. The idea of CGC is simple and clear. We assume the presence of recoilless large $x$ partons which travel at the speed of light and stay on the light-cone. These large $x$ partons act as static color sources, which radiate smaller $x$ partons. And the emitted gluons radiate further smaller $x$ partons. At some point, however, the population becomes so large that the recombination between gluons begins to be effective, resulting in the saturation in the small $x$ region. The state in the saturated regime, where the gluon field can be treated as classical, is called the Color Glass Condensate. Mathematically, the successive emission of smaller and smaller $x$ gluons is formulated as a Renormalization Group (RG) equation of the effective action. Within the formalism, the BFKL equation can be reproduced in the weak field regime. In the saturation regime, the evolution equation is replaced by the JIMWLK equation, which is a non-linear extension of the BFKL equation.

What we intend to do in this paper is to extend the CGC formalism to finite temperature and to derive the thermal BFKL equation. There are two reasons for doing that. (1) As we have mentioned, the matter reaches thermalization just after the collision, through some very complicated processes. Actually, the mechanism of the early thermalization is not fully understood so far. It may be that the mechanism should be verified within the CGC formalism itself [12, 13]. For our purpose, let us assume the early thermalization as an empirical fact. On the other hand, in the Bjorken picture of the ultra-relativistic nuclear collision [38], the large $x$ partons including valence partons can go through the other nucleus without damage, and remain almost on the light-cone. Then, just after the collision, the large $x$ partons would radiate smaller $x$ partons in the medium at finite temperature. This gluon distribution, as is determined by the thermal BFKL or JIMWLK equation, would give us the initial condition for the hydrodynamical expansion that follows. It may be that CGC is reconstructed at that moment, which is not necessarily identical with that being present before the collision [39]. In fact, in Refs. [40, 41, 42], it was shown that CGC can become an appropriate initial condition.

(2) The other motivation is related to the jet quenching [43, 44]. As is discussed in Ref. [44], the energetic parton going through QGP can be regarded as a collision of the parton and the QGP with the large center-of-mass energy. Just as in the high energy hadronic scattering where the hadrons exhibit the saturation, the QGP would
have saturated gluons as a consequence of radiation by the thermal quarks and gluons. Those extra emitted gluons could explain the observed large stopping ability of QGP. We notice that this argument is in a frame where the thermal bath of QGP is moving at high speed. In the rest frame of QGP, we have another picture, which should be equivalent to the physics in the QGP moving frame. Here the energetic parton radiates soft gluons inside the medium at finite temperature that is at rest. Those radiated soft gluons, the distribution of which could be determined by the thermal BFKL equation derived in the CGC formalism, would be responsible for the parton’s energy loss in QGP.

Motivated by these reasons, we will extend the CGC formalism to finite temperature to derive the thermal BFKL equation.

The organization of this paper is as follows. In Sec. II we present the gluon propagator in the light-front field theory at finite temperature. In Sec. III we discuss the CGC formalism at finite temperature. In Sec. IV we derive the thermal BFKL equation. Summary is given in Sec. V.

Throughout this paper, we use the same convention of the light-cone variable and the metric and so on as in Ref. [35].

II. LIGHT-FRONT FIELD THEORY AT FINITE TEMPERATURE

Let us begin with basic concepts of (conventional) thermal field theory [43, 46]. All the thermodynamic quantities can be derived from the partition function $Z = \text{Tr}[\exp(-\beta H)]$ with the Hamiltonian $H$ and the inverse temperature $\beta$, which can be rewritten in the path integral form. There are two formalisms available: The imaginary-time formalism and real-time formalism. In the imaginary-time formalism, the (anti-)periodic boundary condition with respect to the Euclidean time is imposed on the Bose (Fermion) fields. This results in the discrete Matsubara frequency and the summation over it is involved. In the real-time formalism, on the other, we deal with the real (Minkowski) time directly, at the cost of extending the theory to the complex time. The complex time path has some arbitrariness in the prescription for the pole in $1/p^+$. If we use an inappropriate prescription, we are in a trouble as the soft gluon emission becomes non-local: There appear smaller gluons in negative $x^-$ even if the color source lies in a positive $x^-$. This has a serious consequence especially for deriving the JIMWLK equation. We use the retarded prescription as in Ref. [35], in which all the radiated gluons come in positive $x^-$ definitely as it should be.

As is emphasized in Ref. [35], it is important to specify the prescription for the pole in $1/p^+$. If we use an inappropriate prescription, we are in a trouble as the soft gluon emission becomes non-local: There appear smaller gluons in negative $x^-$ even if the color source lies in positive $x^-$. This has a serious consequence especially for deriving the JIMWLK equation. We use the retarded prescription as in Ref. [35], in which all the radiated gluons come in positive $x^-$ definitely as it should be.

It is noted that there is still an arbitrariness for the pole prescription for the $G_{\mu\nu}(= G_{++})$ components even if $G_{\mu\nu}(= G_{++})$ is specified for which we use the retarded prescription. In this paper, we take the same prescription for the $G_{\mu\nu}$ components as for $G_{++}$, as is shown in Appendix A. In fact, Ref. [53] proposes a more symmetric prescription in which, for
instance, the $G_{i(+-)}^-$ component has the form of

$$iG_{i(+-)}^-(p) = \left[ \theta (-p^0) + n_B \left( |p^0| \right) \right]$$

$$\times \frac{p^i}{p^+ + i\epsilon} - G_0^+(p) \frac{p^i}{p^+ - i\epsilon},$$

while in our prescription we have;

$$iG_{i(+-)}^-(p) = \frac{p^i}{p^+ + i\epsilon} 2\pi \left[ \theta (-p^0) + n_B \left( |p^0| \right) \right] \delta (p^2)$$

$$= \left[ \theta (-p^0) + n_B \left( |p^0| \right) \right]$$

$$\times \frac{i}{p^+ + i\epsilon} (G_0(p) - G_0^+(p)).$$

In the symmetric prescription, the pole prescription is replaced by its Hermite conjugate when it is multiplied by $G_0^+$. It can be shown explicitly that in the symmetric prescription, there arises a contribution of the gluon distribution in negative $x^-$, while in our prescription, the radiated gluons appear only in positive $x^-$. Thus our prescription is the appropriate one for calculations of CGC.

### III. COLOR GLASS CONDENSATE AT FINITE TEMPERATURE

In this section, we discuss how the CGC formalism is extended to finite temperature. Let us begin with recapitulating the theory at zero temperature. CGC is the high gluon density matter that appears in the high energy limit of hadrons and is relevant to the hadron scatterings in the Regge kinematics. Specifically, we can imagine the high energy hadron-hadron (or nucleus-nucleus) collision or Deep Inelastic Scattering (DIS) where a high energy virtual photon collides with a hadron. The generating (or partition) functional for CGC is given by

$$Z[j] = \int D\rho W_\Lambda[\rho] Z_\Lambda^{-1}[\rho] \int^\Lambda D A^\mu_a \delta \left( A^\mu_a \right) e^{i S[A_a,\rho] - i \int j \cdot A},$$

which contains the generating functional for the soft gluon field $A^\mu$ at a fixed color charge density $\rho$;

$$Z_\Lambda[\rho, j] = Z_\Lambda^{-1}[\rho] \int^\Lambda D A^\mu_a \delta \left( A^\mu_a \right) e^{i S[A_a, \rho] - i \int j \cdot A}$$

with $Z_\Lambda[\rho] = Z_\Lambda[\rho, j = 0]$. The frozen color charge $\rho$ simulates the effect of the fast partons with momenta $|p^+| > \Lambda^\pm$, and the charge distribution is specified by the weight function $W_\Lambda[\rho]$ into which all the dynamics of the fast modes is encoded. Let us recall how we can calculate the gluon distribution function within the formalism. There are two ingredients:

(i) We solve the classical Yang-Mills equation to some specific charge distribution, assuming the presence of the strong gluon field at the small $x$ region of interest.

(ii) The gluon distribution function is obtained by averaging the classical solution $A^\mu(x) = (x^-, x^-)$ $[\rho]$ we have obtained over $\rho$ with the weight function $W_\Lambda[\rho]$.

We note that the classical solution $A^\mu(x)[\rho]$ we need is a static, time-independent one, which is necessary for computing the gluon distribution function.

If we lower the cutoff $\Lambda^\pm$ by integrating out the semi-fast gluons $a^\mu \left( b \Lambda^\pm < |p^+| < \Lambda^\pm \right)$, their dynamics is renormalized into $W_\Lambda[\rho]$ to give the RG equation for $W_\Lambda[\rho]$. Convoluted with the classical solution, this RG equation for $W_\Lambda[\rho]$ finally gives us the evolution equation for the gluon distribution function, that is, the BFKL and JIMWLK equations. The RG equation for $W_\Lambda[\rho]$ reads

$$\frac{\partial W_\Lambda[\rho]}{\partial \tau}$$

$$= \alpha_s \left\{ \frac{1}{2} \frac{\delta^2}{\delta \rho_a(x) \delta \rho_b(y)} \right\} W_\tau \chi_{xy} - \frac{\delta}{\delta \rho_a(x)} W_\tau \sigma_x \right\},$$

where

$$\chi_{\alpha\beta}(x, y) \equiv \langle \delta \rho_\alpha(x) \delta \rho_\beta(y) \rangle,$$

and $\tau = \ln(P^+/\Lambda^\pm) = \ln(1/x)$ is the rapidity variable. $\hat{\sigma}(x)$ and $\hat{\chi}(x, y)$ are the one- and two-point functions, respectively, of the induced color charge which is generated by the semi-fast gluon fluctuation and is defined by

$$\delta J^\mu_a(x) \equiv - \left. \frac{\delta S}{\delta A^\mu_a(x)} \right|_{\Lambda \to +a}$$

$$\approx - \left. \frac{\delta^2 S}{\delta A^\mu_a(x) \delta A^\nu_b(y)} \right|_{\Lambda} a^\nu_b(y)$$

$$- \frac{1}{2} \left. \frac{\delta^3 S}{\delta A^\mu_a(x) \delta A^\nu_b(y) \delta A^\lambda_c(z)} \right|_{\Lambda} a^\nu_b(y) a^\lambda_c(z).$$

with $\mu = \epsilon$. The derivation of the RG equation is reduced to computation of $\hat{\sigma}(x)$ and $\hat{\chi}(x, y)$.

Now we consider the extension to finite temperature of the formalism. First, we note that the classical solution at finite temperature is identical with that at zero temperature: The thermal effect does not alter the classical solution within the classical approximation. This can be seen as follows. The thermal effect in finite temperature is provided by the same one as at zero temperature: The thermal effect does not alter the classical time contour on which the theory lives. As we noticed, the classical solution needed in the CGC formalism is the time-independent one. Thus, the classical solution at finite temperature is provided by the same one as at zero temperature.

All the thermal fluctuations are contained in the correlators $\hat{\sigma}(x)$ and $\hat{\chi}(x, y)$, the evaluation of which is all we have to do. The action in the real time formalism at
finite temperature is given by

\[ S[A, \rho] = S_{YM} + S_W, \]

\[ S_{YM} = -\int_C d^4x \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a, \]  

\[ S_W[A^-, \rho] = \frac{i}{gN_c} \int d^3x T [\rho(\mathbf{x}) W_C[A^-](\mathbf{x})] \]  

with the time-ordered Wilson line

\[ W_C(\mathbf{x}) = T_C \exp \left\{ ig \int_C dz A^-(z, \mathbf{x}) \right\}, \]

where \( C \) denotes the complex time contour given by \( C = C_+ \cup C_- \cup C_{\text{vertical}} \), for which the last piece is eventually decoupled from the theory \cite{43}. With this action, we can calculate the induced charge \( \delta \rho \) defined in Eq. (6), which reads

\[ \delta \rho_a(x) = \delta \rho_a^{(1)}(x) + \delta \rho_a^{(2)}(x), \]

for which the last piece is eventually calculated.

\[ \delta \rho_a^{(1)}(x) = -2ig \mathcal{F}^{\alpha\beta}_{ac}(\mathbf{x}) a^\alpha c(x) \]

\[ -g \rho_a(\mathbf{x}) \left[ \int_{-\infty}^{\infty} dy^{(+) -} \left| y^+ \right| a^{\alpha c -}(y^+, \mathbf{x}) + \frac{i}{2} \int_{-\infty}^{\infty} dy^{(- -)} a^{\alpha c -}(y^-, \mathbf{x}) \right], \]

\[ \delta \rho_a^{(2)}(x) = g^2 f_{abc} [\partial^+ a^b_\alpha(x)] a^c_\alpha(x) \]

\[ - \frac{g^2}{N_c} \rho_b(x) \left[ \int_{-\infty}^{\infty} dy^{(+) d} \int_{-\infty}^{\infty} dz^{(+) d} a^{\alpha c -}(y^{(+) d}, \mathbf{x}) d^{d -} (z^{(+) d}, \mathbf{x}) \right] \]

\[ \times \left\{ \theta(x^+ - y^{(+) d}) \theta(y^{(+) d} - z^{(+) d}) \text{Tr}(T^a T^c T^d T^b) + \theta(y^{(+) d} - z^{(+) d}) \theta(z^{(+) d} - x^+) \text{Tr}(T^a T^b T^c T^d) \right\} \]

\[ - \int_{-\infty}^{\infty} dy^{(- -)} \int_{-\infty}^{\infty} dz^{(- -) d} a^{\alpha c -}(y^{(- -) d}, \mathbf{x}) d^{d -} (z^{(- -) d}, \mathbf{x}) \theta(z^{(- -) d} - x^+) \text{Tr}(T^a T^b T^c T^d) \]

\[ + \int_{-\infty}^{\infty} dy^{(- -)} \int_{-\infty}^{\infty} dz^{(- -) d} a^{\alpha c -}(y^{(- -) d}, \mathbf{x}) d^{d -} (z^{(- -) d}, \mathbf{x}) \theta(z^{(- -) d} - y^{(- -) d}) \text{Tr}(T^a T^b T^c T^d) \]

\[ - \int_{-\infty}^{\infty} dy^{(- -)} \int_{-\infty}^{\infty} dz^{(- -) d} a^{\alpha c -}(y^{(- -) d}, \mathbf{x}) d^{d -} (z^{(- -) d}, \mathbf{x}) \theta(x^+ - z^{(- -) d}) \text{Tr}(T^a T^b T^c T^d) \].

It is noted that the terms that involve the integration over the time variable on \( C_- \) (i.e., \( y^{(+) -} \) and \( z^{(- -)} \)) represent the thermal effects, while the other terms are already present at zero temperature.

Having obtained the charge fluctuation, we can now express \( \hat{\sigma}(\mathbf{x}) \) and \( \hat{\chi}(x, y) \) in terms of the full Feynman propagator \( G^{\mu\nu}_{\alpha\beta}(x, y) \) of the semi-fast gluons. To the lowest order of \( \alpha_s \), we have

\[ \hat{\sigma}^\alpha(\mathbf{x}) = \hat{\sigma}^\alpha(\mathbf{x}) + \hat{\sigma}^\alpha(\mathbf{x}), \]

where

\[ \hat{\sigma}^\alpha_\text{i}(\mathbf{x}) = g^2 f_{abc} [\partial^+ a^b_\alpha(x)] a^c_\alpha(x) \]

\[ = -g^2 \partial^+ \text{Tr} T^a G^{\alpha\beta}_{\alpha\beta}(x^+, \mathbf{x}; y^+, y^0) \big|_{x=y}, \]
\[ \hat{\sigma}^2(x) = -i \frac{g^2}{N_c} \rho^b(x) \left[ \int_{-\infty}^{\infty} dy^+ \int_{-\infty}^{\infty} dz^+_+ G^{++}_{(+)}(z^+_+, x; y^+, \bar{x}) 
 \times \left\{ \theta(x^+ - y^+) \theta(y^+) \theta(z^+_+) \text{Tr}(T^a T^b T^c) + \theta(y^+ - z^+_+) \theta(z^+_+) \text{Tr}(T^a T^b T^c) \right\} + \theta(y^+ - x^+) \theta(z^+_+) \text{Tr}(T^a T^b T^c) \right) 
 - \int_{-\infty}^{\infty} dy^- \int_{-\infty}^{\infty} dz^- G^{--}_{(-)}(z^-,-, x; y^-,-, \bar{x}) \theta(z^-,-, x; y^-,-, \bar{x}) \text{Tr}(T^a T^b T^c) 
 + \int_{-\infty}^{\infty} dy^- \int_{-\infty}^{\infty} dz^- G^{--}_{(-)}(z^-,-, x; y^-,-, \bar{x}) \theta(z^-,-, y^-,-, \bar{x}) \right\}, \] (16)

and
\[ \hat{\chi}(x, y) = \hat{\chi}_1(x, y) + \hat{\chi}_2(x, y), \] (17)
where
\[ \hat{\chi}_1(x, y) = 4i g^2 \mathcal{F}^{+i}(x) G_{ij}^{++}(x, y) \mathcal{F}^{-j}(y), \] (18)

\[ \frac{1}{g^2} \hat{\chi}_2(x, y) = \mathcal{F}^{+i}(x) \left\{ -2 \int dw^+ G^{--}_{(-)}(x^+, x; w^+, y) \langle w^+ | PV \frac{1}{i \partial y} | y^+ \rangle + i \int dw^+ G^{--}_{(-)}(x^+, x; w^+, y) \right\} \rho(y) 
 + \rho(x) \left\{ 2 \int dz^+_+ \langle x^+ | PV \frac{1}{i \partial z_+} | z^+ \rangle G^{--}_{(-)}(z^+_+, x; y^+, \bar{y}) + i \int dz^+_+ G^{--}_{(-)}(z^+_+, x; y^+, \bar{y}) \right\} \mathcal{F}^{+j}(y) 
 + \rho(x) \left\{ i \int dz^+_+ \int dw^+ \langle x^+ | PV \frac{1}{i \partial z_+} | z^+ \rangle G^{--}_{(-)}(z^+_+, x; w^+, \bar{y}) \rangle \langle w^+ | PV \frac{1}{i \partial y} | y^+ \rangle 
 + \frac{1}{2} \int dz^+_+ \int dw^+ \langle x^+ | PV \frac{1}{i \partial z_+} | z^+ \rangle G^{--}_{(-)}(z^+_+, x; w^+, \bar{y}) \rangle \langle w^+ | PV \frac{1}{i \partial y} | y^+ \rangle 
 - \frac{1}{2} \int dz^+_+ \int dw^+ G^{--}_{(-)}(z^+_+, x; w^+, \bar{y}) \rangle \langle w^+ | PV \frac{1}{i \partial y} | y^+ \rangle 
 + \frac{i}{4} \int dz^+_+ \int dw^+ G^{--}_{(-)}(z^+_+, x; w^+, \bar{y}) \rangle \langle w^+ | PV \frac{1}{i \partial y} | y^+ \rangle \right\} \rho(y) \bigg|_{y = x^+ + \epsilon}. \] (19)

The diagrammatic representations are the same as given in Ref. [35], although the internal time variables can now take values on \( C_+ \), giving rise to the additional terms corresponding to the thermal effects.

**IV. BFKL EQUATION AT FINITE TEMPERATURE**

It has been shown that the CGC formalism at zero temperature reproduces the BFKL equation in the weak source approximation [33]. In the present section, we derive the thermal BFKL equation within the finite temperature CGC discussed in the previous section, using the same approximation. All we have to do is to evaluate \( \hat{\sigma} \) and \( \hat{\chi} \) in the approximation, for which we need the free propagator \( G^{++}_{0,0}(x, y) \) presented in Appendix A. The discussion in this section is proceeded in parallel with that given in Sec. 5 of Ref. [35].
A. Evaluation of $\hat{\sigma}$ in the weak source limit

We first consider $\hat{\sigma}_1$. Because of the color trace in Eq. \[15\], the vacuum contribution disappears. The contribution linear in $\rho$ that has logarithmic enhancement is given by a direct insertion of $\rho$, which reads

$$
\hat{\sigma}_1(\vec{x}) = -g^2 \int \frac{d^4z}{C} \int \frac{d^4u}{C} \times \sigma^\mu_0^{-}(x-z) \Pi^C(z,u) \partial^\nu_0 G^{0^{-}}(u-y) \bigg|_{y=x}, \tag{20}
$$

where

$$
\Pi^C_\mu_\nu(z,u) \equiv \left. \frac{\delta^2 S_W}{\delta A_\mu_\nu(z) \delta A_\nu_\mu(u)} \right|_A = \frac{1}{2} \rho_{\mu\nu} \delta(\vec{z} - \vec{u}) \left\{ \theta_C(z^+ - u^+) - \theta_C(u^+ - z^+) \right\} \tag{21}
$$

As in the case of zero temperature, we can find that $k^+$ could be replaced by $p^+$ or $(k^+ + p^+)/2$, by noting the symmetry of the integrand under the simultaneous exchange $\vec{k} \leftrightarrow -\vec{p}$ and $p^+ \leftrightarrow -p^-$. If we take the strip restriction on $p^- (\Lambda^- < |p^-| < \Lambda^-/b)$, the second and third terms in the curly brackets in Eq. \[22\] disappear due to the delta-function of $p^-$. After a straightforward calculation similar to that at zero temperature, we obtain

$$
\hat{\sigma}_1(\vec{x}) = -\frac{i}{2} g^2 \int \frac{dp^-}{2\pi} \int \frac{d^3\vec{p}^-}{(2\pi)^3} e^{-i\vec{x} \cdot \vec{p}} \rho(\vec{q}) \left\{ -2G^\mu_0^{0^{-}}(p^+, \vec{p}) \left[ \frac{\text{PV} 1}{p} \right] k^+ G^{i^{-}}_0(p^-, \vec{k}) + 2\pi i \delta(p^-) G^{0^{-}}_0(p^+, \vec{p}) k^+ G^{i^{-}}_0(p^-, \vec{k}) - 2\pi i \delta(p^-) G^{i^{-}}_0(p^-, \vec{p}) k^+ G^{0^{-}}_0(p^+, \vec{k}) \right\} . \tag{22}
$$

We note that at zero temperature, the integral over $p^-$ gives rise to the logarithmic enhancement factor $\ln(1/b)$, while in the present case at finite temperature, the integral cannot be performed analytically.

One comment is in order. We could calculate $\hat{\sigma}(\vec{x})$ in the $x^-$ representation, explicitly showing the longitudinal structure. As we have noted, it is possible to verify that $\hat{\sigma}(\vec{x})$ has contributions only at positive $x^-$ in the non-symmetric prescription presented in Appendix [A] while if we use the symmetric prescription mentioned in Sec. [11] it turns out that there appear contributions at both positive and negative $x^-$. Thus our non-symmetric prescription is valid for calculations of CGC.

Similarly, we can compute $\sigma^\nu_2(q_\perp)$ to be

$$
\sigma^\nu_2(q_\perp) = 2N_c a_s \int_{\Lambda^-}^{\Lambda^-/b} dp^- \frac{1}{p} \left( 1 + 2n_B \left( \frac{p^-}{\sqrt{2}} \right) \right) \times \rho^\nu(q_\perp) \int \frac{d^2p_\perp}{(2\pi)^2} \frac{1}{p^2_\perp}, \tag{25}
$$

which cancels the tadpole term $1/p^2_\perp$ in $\sigma_1$. Combining
the two pieces, we finally obtain
\[
\sigma_a^{(0)}(q_\perp) = \frac{N_c \alpha_s}{2} \frac{1}{p^+} \left( 1 + 2n_B \left( \frac{p^-}{\sqrt{2}} \right) \right) \rho^a(q_\perp) \int_{\Lambda^-}^{\Lambda^+/b} \left( \frac{2p^2}{(2\pi)^2} \right) \rho^b \left( \frac{p^-}{\sqrt{2}} \right).
\]

**B. Evaluation of \( \hat{\chi} \) in the weak source limit**

We start with \( \hat{\chi}_1 \) given in Eq. (18), which, in lowest order in \( \rho \), reads
\[
\hat{\chi}_1^{ab}(x,y) = 4g^2 \mathcal{F}_{ac}^i(\vec{x})G^{ij}_0(x,y)\mathcal{F}^{ji}_b(\vec{y})|_{y^+=x^+ + \tau}.
\]
As in the zero temperature case, we can manipulate \( G^{ij}_0(++)(x,y) \) to be
\[
G^{ij}_0(++)(x,y) = -\delta^{ij} \int_{\Lambda^-}^{\Lambda^+/b} \frac{dp^+}{2\pi} \frac{1}{2p^+} \left( 1 + 2n_B \left( \frac{p^+ + \sqrt{p^2}}{\sqrt{2}} \right) \right),
\]
where we have imposed the strip restriction on the \( p^+ \) integration, though it is possible to show that the same result can be obtained when we take the restriction on \( p^- \). Up to the leading logarithmic accuracy (LLA), we can replace \( p^2 \) in the Bose distribution by \( Q^2_\perp \) which is some typical transverse momentum. Then the factor in the parentheses can be factorized out of the \( p^\perp \) integral. Furthermore, as we will see later, we assume the temperature is comparable with \( P^-/x = p^- \) with \( P^- \) being the nucleon light-cone energy, which is much larger than \( p^+ \) so that the term \( p^+ \) in the argument of the Bose distribution can be neglected. It is noted that the \( p^+ \) term is also much less than the term \( Q^2_\perp/2p^+ \approx p^- \) in the argument for the on-shell excitation. After a simple change of variable, we obtain
\[
G^{ij}_0(++)(x,y) = -\delta^{ij} \int_{\Lambda^-}^{\Lambda^+/b} \frac{dp^+}{2\pi} \frac{1}{2p^+} \left( 1 + 2n_B \left( \frac{p^+}{\sqrt{2}} \right) \right) \times \delta^2(x_- - y_-),
\]
which gives us
\[
\hat{\chi}_1^{ab}(x,y) = 2g^2 \int_{\Lambda^-}^{\Lambda^+/b} \frac{dp^-}{2\pi} \frac{1}{2p^-} \left( 1 + 2n_B \left( \frac{p^-}{\sqrt{2}} \right) \right) \times \delta^2(x_- - y_-) \mathcal{F}_{ac}^i(x)\mathcal{F}_{cb}^i(y).
\]

**C. Thermal BFKL equation**

Now we derive the thermal BFKL equation, using the RG equation (3). As we have mentioned, contrary to zero temperature, the enhancement factor \( \ln(1/b) \) is not manifest at finite temperature, for which we have to be careful. If we make the factor explicit, Eq. (3) is expressed as
\[
\frac{\partial W_\tau[\rho]}{\partial \tau} = \lim_{b \to 1} \frac{1}{\ln b} \int d^2 p \frac{1}{2} \frac{\delta^2}{\delta \rho_\perp(x) \delta \rho_\perp(y)} \left[ W_\tau \chi_{xy} - \frac{\delta}{\delta \rho_\perp(x)} \left( W_\tau \chi_\tau \right) \right].
\]
We are interested in the evolution of the diagonal element of the two-point function \( \langle pp \rangle_\tau \),
\[
\langle p_a(k_\perp) p_a(-k_\perp) \rangle_\tau \approx k^2 \langle |\mathcal{F}_a^i(k_\perp)|^2 \rangle_\tau \equiv \varphi(x,k^2_\perp),
\]
which is called the unintegrated gluon distribution. By multiplying Eq. (32) by \( \int d^3 \vec{x} d^3 \vec{y} \exp \left( -i k_\perp \cdot (x + y) \right) \rho(\vec{x}) \rho(\vec{y}) \), and by using Eqs. (26) and (31) for \( \sigma^{(0)} \) and \( \chi^{(0)} \), we obtain
\[
\frac{\partial}{\partial \tau} \varphi(x,k^2_\perp) = \lim_{b \to 1} \frac{1}{\ln b} \int d^2 p \frac{1}{2} \frac{\delta^2}{\delta \rho_\perp(x) \delta \rho_\perp(y)} \left[ W_\tau \chi_{xy} - \frac{\delta}{\delta \rho_\perp(x)} \left( W_\tau \chi_\tau \right) \right],
\]
which is the thermal BFKL equation.
regimes will be translated to (i) \( T \ll \Lambda^- \), (ii) \( T \sim \Lambda^- \) and (iii) \( T \gg \Lambda^- \) after all the calculation. In the regime (i), the thermal distribution function is exponentially small in the integral shell so that the thermal effect plays no role up to LLA and we find that the BFKL equation reduces to that at zero temperature. In the regime (ii), the distribution function is substantial at the upper boundary. The thermal effect contributes to the logarithmic enhancement. This is so because the logarithmic factor comes from the upper boundary of the integral. In the regime (iii), the temperature is much higher than the shell energy. In this case, we find that there appears a power enhancement rather than the logarithmic one. This means that the BFKL equation which resums the leading logarithmic terms makes no sense. Physically, the eikonal approximation breaks down in this regime.

In the following, we restrict ourselves to the regime (ii). Although the evolution in the regime (iii) is interesting, it is beyond the scope of this work.

Let us now extract the logarithmic factor. For that purpose, we need to be careful. As is seen in Eq. (34), it is beyond the scope of this work.

Finally, we comment on a different thermal BFKL equation that has been derived in Ref. [57]. In their equation, the thermal modification appears in the kernel in transverse coordinates, while our equation involves the overall Bose prefactor. The difference is due to the fact that they assume the thermalization for modes in the t-channel at the center-mass frame, while in our framework it is for modes in the s-channel. Their thermal BFKL equation would be useful for investigating Pomerons at finite temperature as argued in Ref. [57], and ours would be relevant to the initial state of heavy ion collisions and the jet quenching.

### V. SUMMARY

We have derived the thermal BFKL equation within the formalism of Color Glass Condensate (CGC). CGC at finite temperature would be relevant to the initial state of ultra-relativistic heavy ion collisions and to understand the anomalous cross-section of energetic partons passing through the quark-gluon plasma (QGP). We have discussed the extension of CGC to finite temperature and found that the classical solution is not modified at finite temperature and that all thermal effects are included in the induced charge correlators \( \hat{\sigma}(x) \) and \( \hat{\chi}(x, y) \) appearing in the renormalization group equation for the weight function \( W_A(\rho) \). In the weak source approximation, we have derived the thermal BFKL equation by explicitly evaluating the charge correlators in the real-time formalism. We have specified the pole prescription for \( 1/p^+ \) in the free gluon propagator, as is presented in Appendix A. We have employed the retarded prescrip-
tion for the $G_{(+)}$ and $G_{(-)}(^*G_{(+)}^*)$ components, as well as the non-symmetric prescription for the $G_{(+)}$ and $G_{(-)}$ components. The prescription is justified by the fact that the gluon radiations occur definitely at positive $x^\perp$.

We note that there are the three temperature regimes: (i) $T \ll P^-/x$, (ii) $T \sim P^-/x$ and (iii) $T \gg P^-/x$. In the regime (i), thermal effects disappear up to LLA and the evolution equation reduces to the vacuum BFKL equation. In the regime (iii), the temperature is too high. We have a power enhancement rather than the logarithmic one so that the evolution is described by an equation different from the BFKL-type equation which resums the leading logarithmic terms. This regime is beyond the scope of the present paper. In the regime (ii), the temperature matches the light-cone energy of soft gluons we are thinking of, giving rise to the logarithmic enhancement. Thus, in this regime, it is meaningful to consider the thermal BFKL equation. The resulting BFKL equation indicates that the thermal effect shows up as the Bose enhancement of the soft gluon emissions due to the surrounding thermal gluons. We have concluded that at finite temperature, the growth of small $x$ gluons can be more rapid and the saturation regime can be reached sooner than at zero temperature.

To complete, the full non-linear JIMWLK equation at finite temperature will be derived in the future.

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APPENDIX A: REAL-TIME THERMAL GLUON PROPAGATOR IN LIGHT-FRONT FIELD THEORY

In this appendix, we write down explicitly the real-time thermal gluon propagator in light-front field theory with the light-cone gauge ($A^+ = 0$). Note the retarded prescription for $G^{i-}$ and $G^{-i}$ components.

\[
\begin{align*}
\iG_{0(+)}^{-i}(p) &= \frac{p^i}{p^+ + i\epsilon} \left[ iG_0(p) + 2\pi n_B (|p^0|) \delta (p^2) \right], \\
\iG_{0(+)}^{-i}(p) &= \frac{p^i}{p^+ - i\epsilon} \left[ iG_0(p) + 2\pi n_B (|p^0|) \delta (p^2) \right], \\
\iG_{0(-)}^{-i}(p) &= \frac{p^i}{p^+ + i\epsilon} \left[ \frac{2\pi}{2\pi} \left[ \theta (p^0) + n_B (|p^0|) \right] \delta (p^2) \right], \\
\iG_{0(-)}^{-i}(p) &= \frac{p^i}{p^+ - i\epsilon} \left[ -iG_0^*(p) + 2\pi n_B (|p^0|) \delta (p^2) \right], \\
\iG_{0(+)}^{i+}(p) &= \frac{p^i}{p^+ + i\epsilon} \left[ iG_0(p) + 2\pi n_B (|p^0|) \delta (p^2) \right], \\
\iG_{0(-)}^{i+}(p) &= \frac{p^i}{p^+ + i\epsilon} \left[ iG_0(p) + 2\pi n_B (|p^0|) \delta (p^2) \right], \\
\iG_{0(+)}^{i+}(p) &= \frac{p^i}{p^+ - i\epsilon} \left[ \frac{2\pi}{2\pi} \left[ \theta (p^0) + n_B (|p^0|) \right] \delta (p^2) \right], \\
\iG_{0(-)}^{i+}(p) &= \frac{p^i}{p^+ - i\epsilon} \left[ -iG_0^*(p) + 2\pi n_B (|p^0|) \delta (p^2) \right].
\end{align*}
\]

(A1)

where

\[
\begin{align*}
G_0(p) &= \frac{1}{p^2 + i\epsilon}, \\
n_B(x) &= \frac{1}{e^{\beta x} - 1}, \\
PV \frac{1}{p^+} &= \frac{1}{2} \left( \frac{1}{p^+ - i\epsilon} + \frac{1}{p^+ + i\epsilon} \right).
\end{align*}
\]
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