A Bivariate Normal Inverse Gaussian Process with Stochastic Delay: Efficient Simulations and Applications to Energy Markets

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ABSTRACT
Using the concept of self-decomposable subordinators introduced by Gardini, Sabino, and Sasso, we build a new bivariate Normal Inverse Gaussian process that can capture stochastic delays. In addition, we also develop a novel path simulation scheme that relies on the mathematical connection between self-decomposable Inverse Gaussian laws and Lévy-driven Ornstein–Uhlenbeck processes with Inverse Gaussian stationary distribution. We show that our approach provides an improvement to the existing simulation scheme detailed in Zhang and Zhang, because it does not rely on an acceptance–rejection method. Eventually, these results are applied to the modelling of energy markets and to the pricing of spread options using the proposed Monte Carlo scheme and Fourier techniques.

1. Introduction and Preliminaries
The Black and Scholes (1973) model is probably the most popular stochastic model used to describe the dynamics of financial indices. Even though it is well-known that it is not able to capture many stylized facts, its simplicity and its flexibility often make it the standard choice for many financial applications. In the univariate setting, several models have been proposed to overcome its limits, relying, for example, on more general Lévy processes. However, in a multi-market setting, the Black–Scholes model is still a milestone due to the fact that alternatives can be less mathematically tractable and their calibration can be computationally demanding. An attempt to combine tractability and simple calibration for multivariate Variance Gamma (VG) processes, introduced by Madan and Seneta (1990) and for multivariate Normal Inverse Gaussian (NIG) processes, introduced by Barndorf-Nielsen (1998), has been proposed by Semeraro (2008) and Luciano and Semeraro (2010). The underlying idea of these studies is to build each marginal of the multivariate process via Brownian subordination: the resulting subordinator is the sum of an independent subordinator and a subordinator shared by all the components, both mutually independent.
Therefore, the construction has a straightforward financial interpretation in terms of idiosyncratic and systematic risks. An alternative approach to construct multidimensional Lévy processes, including VG and NIG processes, has been proposed by Ballotta and Bonfiglioli (2013), with the same logic of idiosyncratic and systematic risks. Such models are able to capture some empirical facts such as discontinuities in price trajectories, volatility smiles and non-normality in log-returns, whereas the joint dependence is driven by the common systematic component.

On the other hand, the impact of new information in one market might require some time to be propagated onto dependent markets; therefore, the aforementioned models cannot replicate any stochastic delay or any synaptic risk as named in Cufaro Petroni and Sabino (2020). Indeed, it is not so rare to observe that the impact on other related markets occurs after a stochastic time delay. Nowadays, a clear example is offered by the recent pandemic of COVID-19: as one can see in Figure 1, first cases appeared at the beginning of January 2020 in China leading to a big downward jump in Shanghai’s index after a flat period due to Chinese New Year celebrations, and subsequently, the virus spread all over the world. Italy registered first cases at the end of February, Brazil at the beginning of March leading to a general drop in the whole world economy.

To this end, recently in Gardini, Sabino, and Sasso (2020) we have shown how the notion of self-decomposability (sd) can be used to describe stochastic delays and to introduce synaptic risk in financial models. We recall that the law of a random variable (rv) $X$ is said to be sd (see Sato 1999; Cufaro Petroni 2008), if for every $a \in (0, 1)$ its characteristic function (chf) $\phi(u)$ can be represented as

$$\phi_X(u) = \phi_X(au)\chi_a(u),$$

(1)

Figure 1. Impact of COVID-19 pandemic in the indicated markets.
with \( \chi_a(u) \) also a chf. It means that we can always find two independent rv’s \( Y \) (with the same law of \( X \)) and \( Z_a \) such that, in distribution
\[
X = aY + Z_a,
\]
where of course, \( \chi_a(u) \) is the chf of \( Z_a \) (hereafter called the \( a \)-remainder of the law of \( X \)) whose law is infinitely divisible \((id)\), but not in general \( sd \) (see Sato 1999).

Based on these facts, Gardini, Sabino, and Sasso (2020) have introduced the concept of \( sd \)-subordinator which is the building block for the construction of correlated Lévy processes. In analogy to \( sd \) laws, such subordinator \( H = \{ (H_1(t), H_2(t)); t \geq 0 \} \) is defined as follows:
\[
H_2(t) = aH_1(t) + Z_a(t),
\]
where \( H_1 = \{ H_1(t); t \geq 0 \} \) and \( Z_a = \{ Z_a(t); t \geq 0 \} \) are independent processes and \( a \in (0, 1) \). Equation (2) is mathematically well-posed and has a clear interpretation: the stochastic time processes \( H_1 \) and \( H_2 \) run together with a stochastic delay \( Z_a \) that is controlled by \( a \) that simply plays the role of the instantaneous correlation at time \( t \) between \( H_1(t) \) and \( H_2(t) \). When \( a \) tends to 1 then \( H_1 \) and \( H_2 \) become essentially indistinguishable.\(^1\) By subordinating Brownian motions (BM) with such subordinators we construct a class of dependent multivariate stochastic processes which margins are Lévy processes. This means that we can extend the approaches of Semeraro (2008), Luciano and Semeraro (2010) and Ballotta and Bonfiglioli (2013) to cover stochastic delays while keeping mathematical tractability, easy calibration and clear financial interpretation.

This study can be considered the sequel of Gardini, Sabino, and Sasso (2020), but now our main focus is on bivariate \( sd \) Inverse Gaussian (IG) subordinators and on the construction of bivariate dependent NIG processes. The first contribution of this work is the derivation of closed form formulas for the linear correlation and the chf of the last processes. These results are instrumental for the calibration and the pricing of derivative contracts. Their valuation is often accomplished via Monte Carlo (MC) simulations. To this end, a second contribution of this study consists of a novel and efficient algorithm to generate the \( a \)-remainder of IG laws and therefore to simulate the skeleton of \( Z_a \) of the \( sd \) IG subordinators and of the bivariate NIG processes. As already observed among others in Taufer and Leonenko (2009), Sabino (2020b) and Sabino and Cufaro Petroni (2020), the transition law between \( t \) and \( t + \Delta t \) of a Lévy-driven Ornstein–Uhlenbeck (OU) process \( X = \{ X(t); t \geq 0 \} \) having a certain stationary law coincides with that of the \( a \)-remainder of such a law by setting \( a = e^{-\lambda \Delta t} \), where \( \lambda \) is the mean-reversion rate of \( X(t) \). Hence, the simulation of the \( a \)-remainder of a IG law is equivalent to the simulation of the skeleton of a IG-OU process, this last one having been illustrated in Zhang and Zhang (2008). We show that our proposal is more efficient than that of Zhang and Zhang (2008), because it does not rely on acceptance-rejection methods and therefore it also applicable to the stochastic volatility models of Barndorff-Nielsen (1997) and Andersson (2001). The only small difference it that since \( Z_a \) is a Lévy process, its simulation applies the same \( a \) at all times \( t \) while instead for a OU process, \( a = e^{-\lambda \Delta t} \) depends on the time step \( \Delta t \).
Finally, we illustrate the applicability of the proposed bivariate $sd$ -NIG processes to energy markets and in particular to the pricing of spread options via MC simulations and Fourier techniques.

The article is organized as follows: **Section 2** introduces $sd$ -NIG processes and their mathematical properties. In **Section 3** we describe the method to simulate the path of the process $Z_t$ and hence those of $H = \{(H_1(t), H_2(t)); t \geq 0\}$ defined in Equation (2). In **Section 4**, we apply the models described in **Section 2** to power and gas forward markets and to the pricing of spread options with MC and Fourier Techniques. **Section 5** concludes the paper with an overview of future inquiries and possible further applications.

## 2. Self-Decomposable NIG Process

The NIG process is constructed via the subordination of a BM with an IG process. There are different characterizations of the pdf of an IG law: on the one hand, we denote the notation using the parameter-setting $(\mu, \lambda)$, adopted for instance in Cont and Tankov (2003), with $IG_T(\mu, \lambda)$: within this setting $\mu > 0$ is the mean and $\lambda > 0$ is the shape parameter. On the other hand, we refer to the original notation in Barndorff-Nielsen (1997) with $IG_B(a, b)$: in this case $a > 0$ and $b > 0$ describe the scale and the shape of the distribution, respectively. In Appendix A we give some details on how to switch from one to the other. In general, the $IG_B$ notation is convenient to analyse sums of IG rv’s, whereas $IG_T$ is more convenient to work with expectations and chf.

Semeraro (2008), Luciano and Semeraro (2010) and Ballotta and Bonfiglioli (2013) proposed a simple technique to introduce dependence between Lévy processes: given three independent Lévy processes $X_1 = \{X_1(t); t \geq 0\}$, $X_2 = \{X_2(t); t \geq 0\}$ and $Z = \{Z(t); t \geq 0\}$ and $a_1, a_2 \in \mathbb{R}$ one can define the process $Y = \{(Y_1(t), Y_2(t)); t \geq 0\}$ by:

$$Y_1(t) = X_1(t) + a_1Z(t)$$

$$Y_2(t) = X_2(t) + a_2Z(t).$$

The processes $Y_1$ and $Y_2$ are clearly dependent, because of the common process $Z$. This idea can be applied to different types of processes, included subordinators. The economic interpretation is clear: $Z$ represents the *systematic risk* whereas $X_j$ models the *idiosyncratic risk*. Of course, this simple construction can be applied to obtain multivariate VG and NIG processes. Nevertheless, as mentioned in the introduction, these settings cannot cover stochastic-delay and what we call *synaptic risk*. In Gardini, Sabino, and Sasso (2020), we focused on the construction of bivariate $sd$ -VG processes, whereas in this sequel we focus on the bivariate version of $sd$ -NIG processes. Briefly, our approach consists of replacing the common and marginal-specific subordinators of Semeraro (2008), Luciano and Semeraro (2010) and Ballotta and Bonfiglioli (2013) with $sd$ -subordinators defined in (2).

### 2.1. Semeraro $sd$ -NIG Model

In this subsection, we illustrate the steps required to extend the model proposed by Semeraro (2008) in order to cope with stochastic delay relying on the $sd$ subordinators defined in (2).
Let \( I_j = \{I_j(t); t \geq 0\}, j = 1, 2 \) be independent subordinators, and \( H = \{(H_1(t), H_2(t)); t \geq 0\} \) be the sd subordinator defined in (2), independent of \( I_j \). Define the subordinator \( G = \{(G_1(t), G_2(t)); t \geq 0\} \) as:

\[
G_j(t) = I_j(t) + \alpha_j H_j(t), \quad j = 1, 2,
\]

where \( \alpha_j \in \mathbb{R}^+ \). Let \( \mu_j \in \mathbb{R}, \sigma_j \in \mathbb{R}^+ \) and \( W = \{(W_1(t), W_2(t)); t \geq 0\} \) a standard BM with independent components also independent of \( G \): we define the subordinated BM \( Y = \{(Y_1(t), Y_2(t)); t \geq 0\} \) as:

\[
Y_j(t) = \mu_j G_j(t) + \sigma_j W_j(G_j(t)), \quad j = 1, 2.
\]

We remark that when \( a \) in (2) tends to 1, there is no time delay and the synaptic risk coincides with the systematic risk as in the original approach of Semeraro (2008).

A bivariate NIG process with IG sd -subordinators can be defined starting from (3) in the following way. Assume \( \alpha_j = \gamma_j^2 \) and let \( I_j(t) \) and \( H_j(t) \) be distributed as follows:

\[
\begin{align*}
I_j(t) & \sim IG_T\left(\frac{A_j \gamma_j}{B}, A_j^2 t^2\right), \\
H_j(t) & \sim IG_T\left(\frac{A_j}{B}, A_j^2 t^2\right),
\end{align*}
\]

and hence we get:

\[
G_j(t) \sim IG_T\left(\frac{(A_j + A_j \gamma_j) \gamma_j t}{B}, (A_j + A_j \gamma_j)^2 t^2\right).
\]

Since \( G(t) \) is a stochastic time, it is customary to require that \( \mathbb{E}[G_j(t)] = t \): this condition can be easily fulfilled by imposing:

\[
A_j + A_j \gamma_j = \frac{B}{\gamma_j}.
\]

Consequently, denoting with \( k_j \) the variance of the subordinator \( G_j(t) \) at time \( t = 1 \), we have that:

\[
k_j := Var[G_j(1)] = \frac{1}{(A_j + A_j \gamma_j)^2} = \frac{\gamma_j^2}{B^2}.
\]

As observed in Luciano and Semeraro (2010), assuming \( B = 1 \) is not restrictive: hence we can set \( k_j = \gamma_j^2 \) and then \( k_j = \alpha_j \). After simple calculations, one can find that the expression of the (instantaneous) linear correlation coefficient at time \( t \) of the process \( Y = \{(Y_1(t), Y_2(t)); t \geq 0\} \) is:

\[
\rho(Y_1(t), Y_2(t)) = \frac{\mu_1 \mu_2 \alpha_1 \alpha_2 a A}{\sqrt{\sigma_1^2 + \mu_1^2 \alpha_1 \sqrt{\sigma_2^2 + \mu_2^2 \alpha_2}}).
\]

Compared to the formula of the linear coefficient in Semeraro (2008), the equation above has an additional parameter \( a \) that tunes the stochastic delay.

Finally, the chf of \( Y \) at time \( t \) is given by the following proposition.
Proposition 2.1. Denote $\phi(u; \mu, \lambda)$ the chf of a rv distributed according to a $IG_{p}(\mu, \lambda)$ law then the joint chf at time $t$ of the process $Y$ defined by Equation (4), where $H_{j}(t)$ and $I_{j}(t)$ are distributed as in (5) for $j = 1, 2$, is:

$$
\phi_{Y(t)}(u) = \phi_{H_{1}(t)}(u_{1}\mu_{1} + \frac{\sigma_{1}^{2}u_{1}}{2})\phi_{I_{2}(t)}(u_{2}\mu_{2} + \frac{\sigma_{2}^{2}u_{2}}{2})\phi_{Z_{a}(t)}(u_{2}\mu_{2} + \frac{\sigma_{2}^{2}u_{2}}{2}),
$$

where

$$
\phi_{H_{j}(t)}(u) = \phi(u; At, A^{2}t^{2}), \quad j = 1, 2,
$$

$$
\phi_{I_{j}(t)}(u) = \phi(u; A_{j}t, A_{j}^{2}t^{2}), \quad j = 1, 2,
$$

$$
\phi_{Z_{a}(t)}(u) = \frac{\phi(\mu A_{j}t, A_{j}^{2}t^{2})}{\phi(\mu A_{j}t, A_{j}^{2}t^{2})}.
$$

Proof. The proof follows the scheme we used to prove the Proposition 3.5 of Gardini, Sabino, and Sasso (2020).

$I_{1}$ and $H_{1}$ are IG processes and hence their chf`s at time $t$ can be computed starting from the chf expression of an IG rv, which is reported in Appendix A, whereas $Z_{a}(t)$ is the $a$-remainder of $H_{1}(t)$ and then its chf can be easily computed relying on Equation (1). The obtained chf`s of $H_{j}(t)$, $I_{j}(t)$ and $Z_{a}(t)$ are those of Equations (8).

Let $\phi_{Y(t)}(u)\mathbb{E}[e^{iu_{1}Y_{1}(t) + iu_{2}Y_{2}(t)}]$ be the chf of the process $Y$ defined in (4) at time $t$: conditioning on $G_{1}(t)$ and $G_{2}(t)$ and recalling that $W_{1}(t)$ and $W_{2}(t)$ are independent we get:

$$
\phi_{Y(t)}(u) = \mathbb{E}
\left[
\phi\left(u_{1}\mu_{1} + \frac{\sigma_{1}^{2}u_{1}}{2}, G_{1}(t)\right)e^{iu_{2}\mu_{2} + \frac{\sigma_{2}^{2}u_{2}}{2}G_{2}(t)}G_{1}(t)\right].
$$

Substitute in the previous equation the expression of $G_{j}(t)$, given by (3), for $j = 1, 2$: by the property of the expected value for the product of independent rv`s, since $I_{j}(t)$, $H_{1}(t)$ and $Z_{a}(t)$ are mutually independent processes, we finally get the result of the Equation (7).

□

2.2. Semeraro–Luciano’s Sd-NIG Model

In this subsection, we extend the model of Luciano and Semeraro (2010), and we build bivariate NIG processes with stochastic delays relying on the sd subordinator $H = \{(H_{1}(t), H_{2}(t)): t \geq 0\}$ defined in (2). Unlike the previous model, we consider a standard BM, $W = \{(W_{1}^{p}(t), W_{2}^{p}(t)): t \geq 0\}$ with correlated margins in order to obtain higher correlations in log-returns.

Let $I_{j} = \{I_{j}(t): t \geq 0\}$, $j = 1, 2$, be independent subordinators and let $H = \{(H_{1}(t), H_{2}(t)): t \geq 0\}$ be a sd subordinator independent of $I_{j}$. We define the process $Y^{p} = \{Y_{1}^{p}(t), Y_{2}^{p}(t)): t \geq 0\}$ as:
\[
Y^p(t) = \left( \begin{array}{l}
\mu_1 I_1(t) + \sigma_1 W_1(I_1(t)) + \alpha_1 \mu_1 H_1(t) + \sqrt{\alpha_1 \sigma_1} W^p_1(H_1(t)) \\
\mu_2 I_2(t) + \sigma_2 W_2(I_2(t)) + \alpha_2 \mu_2 H_2(t) + \sqrt{\alpha_2 \sigma_2} \left( W^p_2(aH_1(t)) + \tilde{W}(Z_a(t)) \right)
\end{array} \right),
\]

(9)

where \( W = \{(W_1(t), W_2(t))\} \) is standard a BM with independent components, \( W^p = \{(W^p_1(t), W^p_2(t))\} \) is standard a BM such that \( \mathbb{E}[dW^p_1(t)dW^p_2(t)] = \rho dt \) and \( \tilde{W} = \{\tilde{W}(t); t \geq 0\} \) is another standard BM independent of \( W \) and \( W^p \).

A bivariate version of NIG process with sd -subordinators can be obtained letting \( H_j(t) \) and \( I_j(t) \) for \( j = 1, 2 \) be distributed as in the previous section. Moreover, the expression of the chf of the process \( Y^p \) at time \( t \) is given by the following proposition.

**Proposition 2.2.** The joint chf \( \phi_{Y^p(t)}(u) \) of the process \( Y^p = \{(Y^p_1(t), Y^p_2(t)); t \geq 0\} \) at time \( t \) defined in (9) is given by:

\[
\phi_{Y^p(t)}(u) = \phi_{H_1(t)}(u_1 \mu_1 + i \frac{\sigma^2_1}{2} u_1^2) \phi_{H_2(t)}(u_2 \mu_2 + i \frac{\sigma^2_2}{2} u_2^2) \phi_{Z_a(t)}(u_2 \mu_2 \alpha_2 + i \frac{\sum a_2 \alpha_2}{2} u_2^2),
\]

where \( \mu = [\alpha_1 \mu_1, \alpha_2 \mu_2] \) and

\[
\Sigma = \begin{bmatrix}
\alpha_1 \sigma_1^2 & \sqrt{\alpha_1 \alpha_2} \sigma_1 \sigma_2 \\
\sqrt{\alpha_1 \alpha_2} \sigma_1 \sigma_2 & \alpha_2 \sigma_2^2
\end{bmatrix},
\]

where \( \phi_{H_1(t)} \), \( \phi_{H_2(t)} \) and \( \phi_{Z_a(t)} \) were defined in Proposition 2.1.

**Proof.** The proof retraces the idea we used in the proof of Proposition 2.1, recalling, in addition, that for \( u \in \mathbb{R}^2 \) the chf \( \phi(u) \) of a multivariate normal rv with mean vector \( \mu \) and covariance matrix \( \Sigma \) is given by:

\[
\phi(u) = \exp \left( i u^T \mu - \frac{1}{2} u^T \Sigma u \right).
\]

It is easy to show, by direct computation or by using the chf of Proposition 2.2, that the linear correlation coefficient at time \( t \) is given by:

\[
\rho_{Y^p(t)} = \frac{\alpha_1 \mu_2 \alpha_2 + \rho A \sigma_1 \sigma_2 \sqrt{\alpha_1 \alpha_2}}{\sqrt{\sigma_1^2 + \mu_1^2 \alpha_1 \sigma_2^2 + \mu_2^2 \alpha_2}}.
\]

(10)

Once again, \( a \) can be seen as the parameter that activates stochastic delay.

### 2.3. Ballotta–Bonfiglioli’s Sd -NIG Model

The construction of bivariate Lévy processes proposed by Ballotta and Bonfiglioli (2013) is slightly different from that of Semeraro (2008) and Luciano and Semeraro (2010), because the dependence is not introduced at the level of the subordinators but rather directly on the subordinated processes. Nevertheless, we can also extend this approach to include stochastic delay.
The construction of the bivariate process with stochastic delay proceeds as follows. Let $H = \{(H_1(t), H_2(t)); t \geq 0\}$ be the $sd$ subordinator as in (2); define subordinated $BM$ $R = \{(R_1(t), R_2(t)); t \geq 0\}$, for $j = 1, 2$, with drift $\beta_{R_j} \in \mathbb{R}$ and diffusion $\gamma_{R_j} \in \mathbb{R}^+$, as:

$$R_1(t) = \beta_{R_1}H_1(t) + \gamma_{R_1}W(H_1(t))$$

$$R_2(t) = \beta_{R_2}H_2(t) + \gamma_{R_2}(W(aH_1(t)) + \tilde{W}(Z_a(t))),$$

(11)

where $W = \{W(t); t \geq 0\}$ and $\tilde{W} = \{\tilde{W}(t); t \geq 0\}$ are standard independent $BM$ 's. Consider $W = \{(W_1(t), W_2(t)); t \geq 0\}$ be a standard $BM$ with independent margins, $G = \{(G_1(t), G_2(t); t \geq 0\}$ a subordinator independent of $W$ and consider $X = \{(X_1(t), X_2(t)); t \geq 0\}$ be a subordinated $BM$ with drift $\beta_j \in \mathbb{R}$ and diffusion $\gamma_j \in \mathbb{R}^+$, defined as follows:

$$X_j(t) = \beta_jG_j(t) + \gamma_jW_j(G_j(t)).$$

(12)

Finally, combining previous processes, we can define the process $Y = \{(Y_1(t), Y_2(t)); t \geq 0\}$ as:

$$Y(t) = (Y_1(t), Y_2(t)) = (X_1(t) + a_1R_1(t), X_2(t) + a_2R_2(t)),$$

(13)

where $a_j \in \mathbb{R}$.

As detailed by Ballotta and Bonfiglioli (2013) and Gardini, Sabino, and Sasso (2020), for any chosen distribution for the margin $Y_j(t)$, for example a NIG distribution, it is possible to impose convolution conditions on $X_j(t)$ and $R_j(t)$ such that their linear combination has the same given distribution of $Y_j(t)$. The following proposition shows how to build a bivariate $BM$ process with stochastic delays and gives the closed form expression for its chf.

**Proposition 2.3.** Consider an IG subordinator $H = \{(H_1(t), H_2(t)); t \geq 0\}$ such that $H_1(t) \sim IG_T\left(t, \frac{\nu}{\nu_k}\right)$, $H_2(t)$ defined in Equation (2) and $R_j(t)$ given by (11). Let then $X$ be a subordinated $BM$, defined as in (12), via an $IG$ process $G$ such that $G_j(t) \sim IG_T\left(t, \frac{\nu_j}{\nu_k}\right)$, for $j = 1, 2$.

Then $Y_j(t)$ in (13) are distributed according to a NIG law and the joint chf of the process $Y$ at time $t$ is

$$\phi_{Y(t)}(u_1, u_2) = \phi\left(\beta_1u_1 + \frac{i}{2}u_1^2\gamma_1^2; t, \frac{\nu_1}{\nu_k}\right)\phi\left(\beta_2u_2 + \frac{i}{2}u_2^2\gamma_2^2; t, \frac{\nu_2}{\nu_k}\right)\xi(a \circ u),$$

(14)

where $\phi(u; \mu, \lambda)$ is the chf of a $IG_T(\mu, \lambda)$ distributed rv, $a = (a_1, a_2)$, $u = (u_1, u_2)$ and $\circ$ is the Hadamard product. Finally $\xi(u)$ is given by:

$$\xi(w) = \phi\left(w_1\beta_{R_1} + w_2\beta_{R_2}a + \frac{i}{2}\left(w_1^2\gamma_{R_1}^2 + 2w_1w_2\gamma_{R_1}\gamma_{R_2}a + w_2^2\gamma_{R_2}^2\right); t, \frac{\nu_k}{\nu_k}\right)$$

$$\phi\left(w_2\beta_{R_2} + \frac{i}{2}w_2^2\gamma_{R_2}^2; t, \frac{\nu_k}{\nu_k}\right),$$

(15)
Proof. Relying on properties of the IG distribution in Appendix A, it is easy to check that marginal distributions of the $Y$ process at time $t$ have a NIG law. Since $X_1(t), X_2(t)$ and $R(t) = (R_1(t), R_2(t))$ are mutually independent we have that

$$
\phi_{Y(t)}(u_1, u_2) = \mathbb{E}[e^{iu_1 X_1(t)}] \mathbb{E}[e^{iu_2 X_2(t)}] = \mathbb{E}[e^{iu_1 R_1(t) + iu_2 R_2(t)}].
$$

The computation consists of two steps: firstly we compute the chf $\mathbb{E}[e^{iu_1 R_1(t) + iu_2 R_2(t)}]$ of the joint process $R = \{ (R_1(t), R_2(t)); t \geq 0 \}$ at time $t$ defined in (11). This can be done by conditioning with respect $H_1(t)$ and $Z_{a}(t)$, relying upon the independence of $W(t)$ and $\tilde{W}(t)$ and recalling the expression of the chf of a $IG_T(t, \ell^2, v^2)$ rv, which is given in Appendix A, and the chf of its $a$-remainder, obtained by applying the Equation (1). By direct computation we obtain that the chf of $R(t)$ has the form shown in Equation (15) valuated at $w = a \circ u$.

Secondly, we observe that first two terms of the right hand side of the Equation (16) are the chf’s of subordinated BM’s where subordinators are IG processes and hence their expressions are given by:

$$
\mathbb{E}[e^{iu_j X_j(t)}] = \phi \left( \beta_j u_j + \frac{i}{2} u_j^2 \gamma_j^2; t, \frac{t^2}{v_j} \right),
$$

where $\phi(u; \mu, \lambda)$ denotes the chf of a rv with $IG_T(\mu, \lambda)$ law. Combining Equations (15)–(17) we finally obtain (14).

The linear correlation coefficient of a bivariate sd -NIG process at time $t$ can be directly computed and it is given by:

$$
\rho_{Y(t)} = \frac{a_1 a_2 \left( \beta_{R_1} \beta_{R_2} \nu_R + \gamma_{R_1} \gamma_{R_2} \right)}{\sqrt{\sigma_1^2 + \mu_1^2 \alpha_1 \sqrt{\sigma_2^2 + \mu_2^2 \alpha_2}}},
$$

As expected, if $a = 1$ we retrieve the original expression of correlation coefficient obtained by Ballotta and Bonfiglioli (2013).

3. Simulation Algorithm

Simulating the paths of the model dynamics defined in Section 2 can be accomplished by simulating BM’s on a stochastic time grid generated by the relative IG sd subordinators. These subordinators are only marginally IG, in order to get the joint trajectories one has to simulate the skeleton of $Z_{a} = \{ Z_{a}(t); t \geq 0 \}$ in (2) and therefore must have a way to obtain random samples distributed according to the law of the $a$-remainder $Z_{a}$ of an IG distribution.

The methodology that we propose in this section is based on the close relation between sd laws and Lévy-driven OU processes. Following the naming convention in Barndorff-Nielsen and Shephard (2001) we say that a Lévy -driven OU process $X = \{ X(t); t \geq 0 \}$ is an IG-OU process if its stationary law is an IG$\beta$ distribution with scale parameter $\delta$ and shape parameter $\gamma$. Now a well known result (see for instance Cont and Tankov (2003) or Sato (1999)) is that a given one-dimensional distribution $D$ always is the stationary law of a suitable Lévy-driven OU process if and only if $D$ is sd. As shown by Halgreen (1979), the IG law is sd and can be taken as the stationary distribution of a fully-fledged OU process.
We recall that a Lévy-driven OU process is defined as,

\[ X(t) = X(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-u)}dL(u), \]

where \( L = \{L(t); t \geq 0\} \) is a Lévy process and \( \lambda > 0 \). In addition, as observed in Barndorff-Nielsen and Shephard (2001), \( X = \{X(t); t \geq 0\} \) is stationary if and only if the chf \( \Phi_X(u) \) of its marginal distribution is of the form

\[ \Phi_X(u) = \Phi_X(ue^{-\lambda t})\chi_a(u, t), \]

where \( \chi_a(u, t) \) is the chf of the second term of (19). On the other hand, due to the definition of \( sd \), the last equation means that \( \chi_a(u, t) \) is the chf of the \( a \)-remainder of the stationary law if one sets \( a = e^{-\lambda t} \). We can then write

\[ X(t) = X(0)e^{-\lambda t} + Z_{e^{-\lambda t}}(t). \quad (20) \]

Note that the parameter \( e^{-\lambda t} \) is now time dependent and the law of \( Z_{e^{-\lambda t}}(t) \) coincides with that of \( Z_a(t) \) with \( a = e^{-\lambda t} \) only at a given time \( t \); indeed \( Z_{e^{-\lambda t}}(t) \) is not a Lévy but rather an additive process. Nevertheless, in practice the simulation of the skeleton of a IG-OU process relies on the generation of a rv that is distributed according to the law of the \( a \)-remainder of the stationary distribution setting \( a = e^{-\lambda t} \).

We improve the results of Zhang and Zhang (2008) relative to IG-OU processes and we derive an efficient algorithm to simulate the \( a \)-remainder of the \( IG_B(\delta; \gamma) \), that is the building block for the generation of the trajectory of the process \( Z_a(t) \).

**Theorem 3.1** (Zhang and Zhang 2008). The rv

\[ Z^\Delta_a = \int_0^\Delta e^{-\lambda(\Delta-u)}dL(u), \quad a = e^{-\lambda \Delta}, \quad \Delta > 0, \]

can be represented as

\[ Z^\Delta_a \overset{d}{=} W_0^\Delta + \sum_{i=1}^{\tilde{N}_a} W_i^\Delta, \]

where \( W_0^\Delta \sim IG_B(\delta(1 - e^{-\lambda \Delta}), \gamma) \), \( \tilde{N}_a \) is a Poisson-distributed rv with parameter \( \delta(1 - e^{-\lambda \Delta}) \gamma \) and \( W_i^\Delta \) are independent rv’s with pdf:

\[ f_{W_i^\Delta}(w) = \frac{\gamma^{-1}}{\sqrt{2\pi}} w^{-\frac{3}{2}} \left( e^{\lambda \Delta} - 1 \right)^{-1} \left( e^{-\lambda \Delta} w - e^{-\lambda \Delta} w e^{\lambda \Delta} \right) \chi_{[w > 0]}(w). \quad (21) \]

Assuming for simplicity \( \Delta = 1 \), we can then rely on Theorem 3.1 to conceive the simulation procedure of two correlated IG rv’s with linear correlation coefficient \( a \), and hence of the \( sd \) subordinators of (2) simply setting \( \lambda = -\log a \). We get:

\[ Z_a \overset{d}{=} W_0 + \sum_{i=1}^{\tilde{N}} W_i, \]

where \( W_0 \sim IG(\delta(1 - a^2), \gamma) \) and \( \tilde{N} \sim \text{Poisson}(\delta(1 - a^2) \gamma) \).
Obtaining random samples according to IG and Poisson laws is relatively easy, whereas the simulation of $W_i$ is non-standard and can be generated using the acceptance-rejection algorithm proposed by Zhang and Zhang (2008) observing that:

$$f_W(w) \leq c \cdot \Gamma\left(\frac{1}{2}; \frac{1}{2} w^2\right),$$

where $c = \frac{1}{2} \left(1 + e^{\lambda w}\right)$ and $\Gamma(\alpha, \beta)$ denote the law of a gamma rv with shape $\alpha > 0$ and rate $\beta > 0$.

Although Zhang and Zhang (2008) illustrated a more accurate solution to reduce the expected number of iterations before acceptance $c$, acceptance-rejection algorithms might be slow and then sometimes inadequate for real time applications. This situation is exacerbated if the software implementation relies on interpreted languages like MATLAB, Python, or R. In the following, we detail a simple and more efficient way to draw a random variate from the pdf $f_{W_2}(w)$ without relying on acceptance-rejection methods.

Assuming once again $\Delta = 1$ and $\lambda = -\log a$, Equation (21) becomes:

$$f_W(w) = \frac{y^{-1}}{\sqrt{2\pi}} w^{\frac{1}{2}} \left(a^{-\frac{1}{2}} - 1\right)^{-1} \left(e^{-\frac{1}{2} y^2 w} - e^{-\frac{1}{2} y^2} \right) 1_{\{w > 0\}}(w).$$

We recall that a rv is distributed according to a Gamma law with shape $\alpha > 0$ and rate $\beta > 0$ if its pdf is:

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ is the Euler Gamma function. Knowing that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and observing that:

$$\int_1^\infty e^{-\frac{1}{2} y w} \frac{y^2}{2} w dy = e^{-\frac{1}{2} y^2 w} - e^{-\frac{1}{2} y^2},$$

we can write:

$$f_W(w) = \int_1^\infty \left(\frac{y^{\frac{1}{2}}}{\sqrt{2\pi}}\right) w^{\frac{1}{2}} e^{-\frac{1}{2} y^2 w} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{y^2}{2} w dy$$

$$= \int_1^\infty f_Y(y) \cdot f_Y\left(w|\alpha = \frac{1}{2}, \beta = \frac{y^2}{2}\right) dy.$$  

This means that $f_W(w)$ is a mixture of a Gamma law $\Gamma\left(\alpha = \frac{1}{2}, \beta = \frac{y^2}{2}\right)$ and a law whose pdf and cdf are, respectively:

$$f_Y(y) = \frac{y^{\frac{1}{2}}}{2(a^{-\frac{1}{2}} - 1)} 1_{\{1 \leq y \leq a\}}$$

$$F_Y(y) = \frac{y^{\frac{1}{2}} - 1}{a^{-\frac{1}{2}} - 1} 1_{\{1 \leq y \leq a\}}.$$
The simulation of \( Z_a \) and of the rv \( Y \) distributed according to the law with cdf \( F_Y(y) \) is straightforward as is summarized in Algorithms 1 and 2, respectively.

**Algorithm 1** Simulation of \( Z_a \)

1: Simulate \( W_0 \sim IG(\delta(1 - \sqrt{a}), y) \)
2: Simulate \( \tilde{N} \sim Poisson(\delta(1 - \sqrt{a})y) \)
3: Simulate \( W_i, i = 1 \ldots \tilde{N} \) using Algorithm 2
4: Set \( Z_a = \sum_{i=0}^{\tilde{N}} W_i \)

In Table 1, we compare the theoretical values of the first five moments of \( Z_a \) against those obtained by MC simulations using Algorithm 1. We observe that the precision of the algorithms is high for different values of \( a \in (0, 1) \). In Figure 2, we draw the probability density function of two correlated rv \( X, Y \sim IG(\delta, y) \) and their scatter plot for two different values of \( a \).

**Algorithm 2** Simulation of \( W_i, \tilde{N} \)

1: Simulate \( U_i \sim U([0, 1]) \)
2: Compute \( Y_i = (1 + (a^{-\frac{1}{2}} - 1)U_i)^2 \)
3: Simulate \( W_i \) from a \( \Gamma(\frac{1}{2}, \frac{1}{2}y^2Y_i) \)

The proposed algorithm is approximately 10 times faster than the one presented by Zhang and Zhang (2008), as one can see from results reported in Table 2. This time complexity analysis was implemented on a PC having an Intel Core i5-10210U 2.11 GHz processor and all codes are written in MATLAB.

The simulation of the \( a \)-remainder of an IG law provides the generation of the joint trajectories of the sd subordinator \( H = \{(H_1(t), H_2(t); t \geq 0)\} \), and therefore those of the models presented in Section 2. The application of these MC schemes will be shown in next section.

Finally, we remark that our path-generation procedure is also applicable to the stochastic volatility models based on IG-OU processes proposed by Barndorff-Nielsen (1997) and extended by Andersson (2001).

### 4. Financial Application

In this section, we use the bivariate Lévy processes illustrated in Section 2 to model power and gas forward markets.

Following Cont and Tankov (2003), we assume that each forward price dynamics is driven by an exponential Lévy process based on \( Y = \{(Y_1(t), Y_2(t)); t \geq 0\} \) derived in Section 2. The forward price \( F_j(t), j = 1, 2 \) at time \( t \) can be defined as follow:

\[
F_j(t) = F_j(0)e^{\omega_j t + Y_j(t)},
\]

where \( \omega_j \) is the drift correction required for risk-neutral arguments such that

\[
\omega_j = -\varphi_j(-i),
\]

where \( \varphi_j(u) \) is the characteristic exponent of the process \( Y_j \) at time \( t \).
Table 1. Moments comparison using \(N_{\text{sim}} = 10^5\) for \(\delta = 5\) and \(\gamma = 1.5\). \(T\) stands for the values of the theoretical n-th moment, whereas \(N\) stands for the MC-based estimations.

| \(\mathbb{E}[Z^2_i]\) | \(T\) | \(N\) |
|------------------|------|------|
| (a) \(a = 0.1\) |      |      |
| \(\mathbb{E}[Z^1_i]\) | 3.00 | 3.00 |
| \(\mathbb{E}[Z^2_i]\) | 10.47| 10.48|
| \(\mathbb{E}[Z^3_i]\) | 42.17| 42.26|
| \(\mathbb{E}[Z^4_i]\) | 194.72| 195.49|
| \(\mathbb{E}[Z^5_i]\) | 1021.84| 1029.41|
| (b) \(a = 0.5\) |      |      |
| \(\mathbb{E}[Z^1_i]\) | 1.67 | 1.67 |
| \(\mathbb{E}[Z^2_i]\) | 3.89 | 3.89 |
| \(\mathbb{E}[Z^3_i]\) | 11.91| 11.90|
| \(\mathbb{E}[Z^4_i]\) | 45.58| 45.46|
| \(\mathbb{E}[Z^5_i]\) | 209.90| 208.97|
| (c) \(a = 0.7\) |      |      |
| \(\mathbb{E}[Z^1_i]\) | 1.00 | 1.00 |
| \(\mathbb{E}[Z^2_i]\) | 1.76 | 1.76 |
| \(\mathbb{E}[Z^3_i]\) | 4.56 | 4.59 |
| \(\mathbb{E}[Z^4_i]\) | 15.77| 15.89|
| \(\mathbb{E}[Z^5_i]\) | 67.94| 68.66|
| (d) \(a = 0.9\) |      |      |
| \(\mathbb{E}[Z^1_i]\) | 0.33 | 0.33 |
| \(\mathbb{E}[Z^2_i]\) | 0.39 | 0.40 |
| \(\mathbb{E}[Z^3_i]\) | 0.85 | 0.86 |
| \(\mathbb{E}[Z^4_i]\) | 2.66 | 2.68 |
| \(\mathbb{E}[Z^5_i]\) | 10.71| 10.72|

In order to calibrate the derived models, we use the two-steps procedure adopted by Luciano and Semeraro (2010), Ballotta and Bonfiglioli (2013) and Gardini, Sabino, and Sasso (2020). It is worthwhile noticing that the marginal distributions do not depend on the parameters required to model the dependence structure. For this reason, the set of marginal parameters \(\theta^*\) is obtained by solving the following optimization problem:

\[
\theta^* = \arg\min_{\theta} \sum_{i=1}^{n} (C_i^\theta(K, T) - C_i)^2 ,
\]

where \(\{C_i\}_{i=1}^{n}\) are the values of \(n\) quoted vanilla products and \(\{C_i^\theta(K, T)\}_{i=1}^{n}\) are the relative model prices. In the second step, we estimate the remaining dependence structure parameters. This can be done by matching the theoretical correlation matrix derived in Section 2 with the numerical one obtained by historical quotations. The choice of fitting the dependence structure on historical quotations is motivated by the fact that derivative contracts written on more than one underlying asset, which should be used for market calibration purposes, are not liquid.

Once we calibrated the whole set of parameters, spread options written on future prices, which payoff is given by:
can be priced. It is customary to reserve the name Cross-Border or Spark-Spread option if the futures are relative to power or gas markets, respectively. In all experiments we use the MC technique with \( N_{\text{sim}} = 10^6 \) simulations and the Fourier-based method presented by Caldana and Fusai (2016). This method provides a good approximation for spread-options prices and it is simpler to implement than the one proposed by Hurd and Zhou (2009), because it requires only one Fourier inversion.

The numerical investigation is split into two parts: in the first one we use \( sd\) -NIG processes to model German and French power forward markets, whereas in the second part we focus on German power and natural gas forward markets.

All these markets are very correlated in particular, the German and French power forward markets exhibit an extremely high log-returns correlation. This is due to the structure of the electricity network that connects the two countries and to the fact that electricity cannot be stored. Therefore, if the price of electricity rises in Germany we can observe an increase of electricity prices in France as well. The log-returns correlation between German power and natural gas forward markets is still positive, but lower than that of the previous case. This depends on the percentage of installed capacity depending
on natural gas (in 2020, 13.9% in Germany) and, moreover, gas can be stored. For the sake of concision we introduce the following notation:

- (SSD – NIG): sd -NIG model presented in Section 2.1.
- (LSSD – NIG): sd -NIG model presented in Section 2.2.
- (BBSD – NIG): sd -NIG model presented in Section 2.3.

4.1. Application to German and French Power Markets

In order to calibrate the proposed sd-NIG models, we rely upon derivative contracts written on each of the two forward prices and upon the joint historical time series of forward quotations. The data-set\(^2\) is composed as follow:

- Forward quotations from 25 April 2017 to 12 November 2018 of Calendar 2019 power forward. Calendar power forward in Germany and France are stated respectively with DEBY and F7BY.
- Call Options on power forward 2019 quotations for both countries with settlement date 12 November 2018. We used strikes in a range of \(\pm 10 \text{[EUR/MWh]}\) around the settlement price of the forward contract.
- We assume a risk-free rate \(r = 0.015\).
- The historical correlation observed between markets is \(\rho_{mkt} = 0.94\).

We denote \((\theta_1, \theta_2)\) parameters related to the French and German power forward markets respectively. Defining the error \(\epsilon_i\) as

\[
\epsilon_i = \frac{C_i^\theta(K, T) - C_i}{C_i},
\]

where \(C_i^\theta(K, T)\) is the value of the \(i\)-th Call option obtained by the model and \(C_i\) is its market price, the picture at the bottom of Figure 3 shows that all models provide a good fit for quoted market options because \(\epsilon\) is negligible. In Figure 3, the picture at the top shows that the SSD-NIG model overprices Cross-Border options: this is because the fitted model correlation is low, as shown by the value \(\rho_{mod}\) in Table 4, so one should avoid using this model for pricing. For LSSD-NIG model the situation is better, but it is not really able to capture the prevailing market correlation. Fortunately, the BBSD-NIG model can replicate the market correlation and then can be used to price Cross-Border options. Fitted common parameters are shown in Table 3, whereas the dependence parameters for SSD-NIG, LSSD-NIG and BBSD-NIG models are shown in Tables 4–6, respectively. The value of \(a\) is shown in Table 7. We observe that the parameter \(a\) is very close to one, as expected. Indeed this result has a very natural economic interpretation: the European electricity network is strongly connected and a certain price signal in either the German or French market is propagated without stochastic delay. Finally, in Table 8, we compare values of Cross-Border options priced using the FFT method proposed by Caldana and Fusai (2016) and the MC scheme that we proposed in Section 3. Option prices provided by both algorithms are very close and this allows us to use indistinctly the FFT or the MC method.
In this section, we present numerical results obtained applying our models to German power forward market (DE) and to natural gas forward market (NCG). These two markets are positively correlated, but the log-return correlation is lower than the one between power futures.

4.2. Application to German Power Market and NCG gas Market

Figure 3. Percentage errors and cross-border option prices.

Table 3. Fitted marginal parameters for German and French power markets.

| Model | $\mu_1$ | $\mu_2$ | $\sigma_1$ | $\sigma_2$ | $\alpha_1$ | $\alpha_2$ |
|-------|---------|---------|-----------|-----------|-----------|-----------|
| SSD   | 0.64    | 0.40    | 0.31      | 0.32      | 0.02      | 0.03      |
| LSSD  | 0.64    | 0.40    | 0.31      | 0.32      | 0.02      | 0.03      |
| BBSD  | 0.64    | 0.40    | 0.31      | 0.32      | 0.02      | 0.03      |

Table 4. SSD.

| Parameter | Value |
|-----------|-------|
| $A$       | 40.15 |
| $B$       | 1.00  |
| $\alpha$  | 0.99  |
| $\rho_{mod}$ | 0.05 |

Table 5. LSSD.

| Parameter | Value |
|-----------|-------|
| $A$       | 40.15 |
| $B$       | 1.00  |
| $\rho$    | 0.99  |
| $\alpha$  | 0.99  |
| $\rho_{mod}$ | 0.88 |
Table 6. BBSD.

| Parameter  | Value | Parameter  | Value |
|------------|-------|------------|-------|
| $\beta_1$ | -0.001| $\beta_{R_2}$ | 0.800 |
| $\beta_2$ | 0.013 | $\gamma_{R_1}$ | 0.448 |
| $\gamma_1$ | 0.002 | $\gamma_{R_2}$ | 0.50  |
| $\gamma_2$ | 0.103 | $\nu_{R}$ | 0.025 |
| $\nu_1$  | 1.007 | $\alpha$  | 0.99  |
| $\nu_2$  | 0.091 | $\rho_{mod}$ | 0.94  |
| $\beta_{R_1}$ | 0.554 |       |       |

Table 7. Values for the $a$ parameter of the three models.

| Model | $a$ |
|-------|-----|
| SSD   | 0.99 |
| LSSD  | 0.99 |
| BBSD  | 0.99 |

Table 8. CrossbbBorder option prices comparison between the three models. Option prices are obtained using both FFT and MC methods. $\Delta$ is the difference between prices.

| $K$ | SSD-NIG | SSD-NIG | SSD-NIG | SSD-NIG |
|-----|---------|---------|---------|---------|
|     | FFT     | MC      | $\Delta$ | FFT     | MC      | $\Delta$ | FFT     | MC      | $\Delta$ |
| 0.0 | 6.61    | 6.59    | (0.02)   | 5.01    | 5.00    | (0.01)   | 4.95    | 4.95    | (0.00)   |
| 0.5 | 6.26    | 6.24    | (0.02)   | 4.54    | 4.53    | (0.01)   | 4.46    | 4.46    | (0.00)   |
| 1.0 | 5.92    | 5.90    | (0.02)   | 4.08    | 4.07    | (0.01)   | 3.98    | 3.97    | (0.01)   |
| 1.5 | 5.59    | 5.57    | (0.02)   | 3.63    | 3.62    | (0.01)   | 3.50    | 3.49    | (0.01)   |
| 2.0 | 5.27    | 5.25    | (0.02)   | 3.20    | 3.19    | (0.01)   | 3.03    | 3.03    | (0.00)   |
| 2.5 | 4.96    | 4.94    | (0.02)   | 2.79    | 2.78    | (0.01)   | 2.58    | 2.58    | (0.00)   |
| 3.0 | 4.67    | 4.65    | (0.02)   | 2.41    | 2.40    | (0.01)   | 2.16    | 2.16    | (0.00)   |
| 3.5 | 4.38    | 4.36    | (0.02)   | 2.06    | 2.04    | (0.02)   | 1.78    | 1.78    | (0.00)   |
| 4.0 | 4.11    | 4.09    | (0.02)   | 1.74    | 1.73    | (0.01)   | 1.44    | 1.44    | (0.00)   |
| 4.5 | 3.85    | 3.82    | (0.03)   | 1.46    | 1.45    | (0.01)   | 1.15    | 1.15    | (0.00)   |
| 5.0 | 3.59    | 3.57    | (0.02)   | 1.22    | 1.20    | (0.02)   | 0.91    | 0.90    | (0.01)   |
| 5.5 | 3.36    | 3.33    | (0.03)   | 1.01    | 1.00    | (0.01)   | 0.71    | 0.71    | (0.00)   |
| 6.0 | 3.13    | 3.11    | (0.02)   | 0.83    | 0.82    | (0.01)   | 0.56    | 0.56    | (0.00)   |
| 6.5 | 2.91    | 2.89    | (0.02)   | 0.69    | 0.68    | (0.01)   | 0.44    | 0.44    | (0.00)   |
| 7.0 | 2.71    | 2.68    | (0.03)   | 0.57    | 0.56    | (0.01)   | 0.34    | 0.34    | (0.00)   |
| 7.5 | 2.51    | 2.49    | (0.02)   | 0.47    | 0.46    | (0.01)   | 0.27    | 0.27    | (0.00)   |
| 8.0 | 2.33    | 2.31    | (0.02)   | 0.39    | 0.38    | (0.01)   | 0.21    | 0.21    | (0.00)   |
| 8.5 | 2.16    | 2.14    | (0.02)   | 0.32    | 0.32    | (0.00)   | 0.17    | 0.17    | (0.00)   |
| 9.0 | 2.00    | 1.97    | (0.03)   | 0.27    | 0.26    | (0.01)   | 0.14    | 0.13    | (0.01)   |
| 9.5 | 1.84    | 1.82    | (0.02)   | 0.22    | 0.22    | (0.00)   | 0.11    | 0.11    | (0.00)   |
| 10.0 | 1.70   | 1.68    | (0.02)   | 0.19    | 0.18    | (0.01)   | 0.09    | 0.09    | (0.00)   |
| 10.5 | 1.57   | 1.55    | (0.02)   | 0.16    | 0.15    | (0.01)   | 0.07    | 0.07    | (0.00)   |
| 11.0 | 1.44   | 1.42    | (0.02)   | 0.13    | 0.13    | (0.00)   | 0.06    | 0.06    | (0.00)   |
| 11.5 | 1.33   | 1.31    | (0.02)   | 0.11    | 0.11    | (0.00)   | 0.05    | 0.05    | (0.00)   |
| 12.0 | 1.22   | 1.20    | (0.02)   | 0.09    | 0.09    | (0.00)   | 0.04    | 0.04    | (0.00)   |

The data-set$^3$ we relied upon is the following:

- Forward quotations from 1 July 2019 to 9 September 2019 relative to the Month January 2020 for the Power Forward in Germany and the Gas NCG Forward.
- Call Options on power forward NCG with settlement date 9 September 2019. As done before, we use strike prices $K$ in a range of $\pm 10\text{[EUR/MWh]}$ around the settlement price of the forward contract.
• We assume a risk-free rate \( r = 0.015 \).
• The historical correlation between log-returns is \( \rho_{mkt} = 0.54 \).

In the picture at the bottom of Figure 4, we observe that all models provide a good fitting of quoted market options because the relative error \( \varepsilon_i \) is small. The picture at the top of Figure 4 shows that the SSD-NIG model overprices the Spark-Spread option due to the fact that fitted model correlation is close to zero, as shown by the value \( \rho_{mod} \) in Table 10. In contrast, LSSD-NIG and BBSD-NIG models provide a lower price and catch the right level of market correlation as shown in Tables 11 and 12. We can conclude that both LSSD-NIG and BBSD-NIG models can be used to price Spark-Spread options. Table 9 shows fitted common parameters whereas dependence parameters for SSD-NIG, LSSD-NIG and BBSD-NIG models are shown in Tables 10–12: the value of \( a \), the \( sd \) parameter, which aims to model the stochastic delay, is shown in Table 13. The value is still close to

![Figure 4. Percentage errors and Spark-Spread option prices.](image)

**Table 9.** Fitted marginal parameters for German and French power markets.

| Model | \( \mu_1 \) | \( \mu_2 \) | \( \sigma_1 \) | \( \sigma_2 \) | \( a_1 \) | \( a_2 \) |
|-------|-------------|-------------|-------------|-------------|---------|---------|
| SSD   | 0.37        | 0.20        | 0.44        | 0.33        | 0.09    | 0.07    |
| LSSD  | 0.37        | 0.20        | 0.44        | 0.33        | 0.09    | 0.07    |
| BBSD  | 0.37        | 0.20        | 0.44        | 0.33        | 0.09    | 0.07    |

**Table 10.** SSD.

| Parameter | Value |
|-----------|-------|
| \( A \)   | 11.27 |
| \( B \)   | 1.00  |
| \( a \)   | 0.99  |
| \( \rho_{mod} \) | 0.03 |
one but it is smaller than that estimated for the power forward markets. From the expressions of the linear correlation coefficient reported in Equations (6), (10) and (18), it is easy to see that a change in the value of $a$ has an impact on the value of the correlation coefficient and it is a fact that even a small change in correlation has a high impact on the spread option price. On the other hand, unlike electricity, natural gas can be stored and therefore the impact on the power market can be moderated and delayed, for example, using storage contracts or other types of OTC derivatives. If the gas price suddenly rises, then it is not rare to observe that the power price is not immediately effected.

### 5. Conclusions

Using the concept of self-decomposable subordinators introduced by Gardini, Sabino, and Sasso (2020), we showed how some recently proposed multivariate Lévy models can be extended to include what we name synaptic risk. Based on this methodology, in this paper we built a new bivariate version of a Normal Inverse Gaussian process which captures stochastic delays and at the same time preserves tractability. Moreover, we derived closed form expressions for their characteristic functions and linear correlation coefficients. These results are instrumental to apply calibration and derivative pricing methods based on Fourier techniques.
Nevertheless, in many real applications, Monte Carlo simulations are required for complex derivative contracts pricing. Based on the observations in Taufer and Leonenko (2009) and Sabino (2020) about the strong mathematical connection between self-decomposable laws and Lévy-driven Ornstein-Uhlenbeck processes, we developed a new efficient algorithm to generate the a-remainder of the Inverse Gaussian law. This algorithm is instrumental for the simulation of the desired bivariate Normal Inverse Gaussian process with stochastic delays and can also be used to generate the skeleton of a OU process with an Inverse Gaussian marginal law which describes the evolution of the stochastic volatility according to the models of Barndorff-Nielsen (1997) and Andersson (2001). The just mentioned algorithm is more efficient than the one proposed by Zhang and Zhang (2008), because it is not based on acceptance-rejection methods: for this reason it can be adopted for real time simulations and pricing.

Eventually, we applied these results to the modelling of energy markets: using the two-steps calibration technique proposed by Luciano and Semeraro (2010), all presented models have been calibrated on vanilla products and on historical quotations and, finally, commonly traded derivative contracts, such as Cross-Border or Spark-Spread options, have been efficiently priced using both Monte Carlo simulations and the Fourier method proposed by Caldana and Fusai (2016).

Finally, many complex energy derivatives, such as swing and storage contracts, have an American exercise style and are often priced using the Least Squares Monte Carlo method. To this end, it is convenient to generate prices backward in time and to rely on the time reversal simulation approaches presented in Pellegrino and Sabino (2015) and Sabino (2020). Therefore, the extension to multivariate NIG and VG processes can be a subject of future inquires.

### 5.1. IG Laws Parametrization

The characterization of the pdf of an IG law is not unique. For example, Cont and Tankov (2003) proposed a parameters setting in \((\mu, \lambda)\), that we denoted by \(IG_T(\mu, \lambda)\) where \(\mu > 0\) is the mean and \(\lambda > 0\) is the shape parameter. Within this setting the pdf of an Inverse Gaussian law is given by:

\[
f_Z(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda(x - \mu)^2}{2\mu^2x}\right\}
\]

and its chf is:

\[
\phi_Z(u) = \exp\left\{\frac{\lambda}{\mu} \left[1 - \sqrt{1 - \frac{2i\mu u}{\lambda}}\right]\right\}.
\]

Moreover let be \(X \sim IG_T(\mu, \lambda)\) then we have that:

\[
\mathbb{E}[X] = \mu, \quad \text{Var}[X] = \frac{\mu^3}{\lambda}.
\]

The original parameter setting of a IG law proposed by Barndorff-Nielsen (1997) is denoted with \(IG_B(a, b)\), where \(a\) can is the scale parameter and \(b\) represents the shape of the distribution. Its probability density function is given by:
\[
f_Z(x; a, b) = \frac{a}{\sqrt{2\pi}} \exp(ab)x^{-3/2} \exp\left(-\frac{1}{2}(a^2x^{-1} + b^2x)\right)
\]  
(27)

and the chf has the following form:

\[
\phi_Z(u) = \exp\left\{-a\left(\sqrt{-2iu + b^2} - b\right)\right\}.
\]  
(28)

If \(X \sim IG_B(a, b)\) then we have that:

\[
\mathbb{E}[X] = \frac{a}{b}, \quad \text{Var}[X] = \frac{a}{b^3}.
\]

Both parametrizations can be adopted and it is possible to switch from one to the other by observing that:

\[
\mu = \frac{a}{b}, \quad \lambda = a^2.
\]  
(29)

We report some very useful properties of the IG law.

- Let be \(X \sim IG_B(a_1, b)\) and \(Y \sim IG_B(a_2, b)\) and let \(X\) and \(Y\) be independent. Then:

\[
cX \sim IG_B\left(ca_1, \frac{b}{c}\right), \quad X + Y \sim IG_B(a_1 + a_2, b).
\]

- Let be \(X \sim IG_T(\mu_0w_1, \lambda_0w_1^2)\) and \(Y \sim IG_T(\mu_0w_2, \lambda_0w_2^2)\) and let \(X\) and \(Y\) be independent. Then:

\[
cX \sim IG_T(c\mu_0w_1, c\lambda_0w_1^2), \quad X + Y \sim IG_T(\mu_0(w_1 + w_2), \lambda_0(w_1 + w_2)^2),
\]

for any \(c > 0\).

**Notes**

1. This follows from the fact that the chf of \(Z_a(t)\) tends to 1 as \(a \to 1\) and hence \(Z_a(t)\) is the constant random variable with value 0. Therefore, \(H_2(t) = H_1(t), \forall t \geq 0\) by construction.
2. Data Source: [www.eex.com](http://www.eex.com).
3. Data Source: [www.eex.com](http://www.eex.com) and [www.theice.com](http://www.theice.com).

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