Global existence for strong solutions of viscous Burgers equation. (1)
The bounded case

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We prove that the viscous Burgers equation \((\frac{\partial}{\partial t} - \Delta)u(t, x) + (u \cdot \nabla)u(t, x) = g(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d \quad (d \geq 1)\)
has a globally defined smooth solution in all dimensions provided the initial condition and the forcing term \(g\) are smooth and bounded together with their derivatives. Such solutions may have infinite energy. The proof does not rely on energy estimates, but on a combination of the maximum principle and quantitative Schauder estimates. We obtain precise bounds on the sup norm of the solution and its derivatives, making it plain that there is no exponential increase in time. In particular, these bounds are time-independent if \(g\) is zero. To get a classical solution, it suffices to assume that the initial condition and the forcing term have bounded derivatives up to order two.

Keywords: viscous Burgers equation, conservation laws, maximum principle, Schauder estimates.
Mathematics Subject Classification (2010): 35A01, 35B45, 35B50, 35K15, 35Q30, 35Q35, 35L65, 76N10.

1Laboratoire associé au CNRS UMR 7502
1 Introduction and scheme of proof

1.1 Introduction

The \((1 + d)\)-dimensional viscous Burgers equation is the following non-linear PDE,

\[
(\partial_t - \nu \Delta + u \cdot \nabla)u = g, \quad u|_{t=0} = u_0
\]  

for a velocity \(u = u(t, x) \in \mathbb{R}^d\) \((d \geq 1)\), \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\), where \(\nu > 0\) is a viscosity coefficient, \(\Delta\) the standard Laplacian on \(\mathbb{R}^d\), \(u \cdot \nabla u = \sum_{i=1}^d u_i \partial_i u\) the convection term, and \(g\) a continuous forcing term. Among other things, this fluid equation describes the hydrodynamical limit of interacting particle systems \([10, 7]\), is a simplified version without pression of the incompressible Navier-Stokes equation, and also (assuming \(g\) to be random) an interesting toy model for the study of turbulence \([1]\). The present study is purely mathematical: we show under the following set of assumptions on \(u_0\) and \(g\) that the Cauchy problem

\[
(\partial_t - \nu \Delta + u \cdot \nabla)u = g, \quad u|_{t=0} = u_0
\]  

has a unique, globally defined, classical solution in \(C^{1,2}\) (i.e. continuously differentiable in the time coordinate and twice continuously differentiable in the space coordinates), and provide explicit bounds for the supremum of \(u\) and its derivatives up to second order.

Assumptions.

\(i\) (initial condition) \(u_0 \in C^2\) and \(\nabla^2 u_0\) is \(\alpha\)-Hölder for every \(\alpha \in (0, 1)\); for \(\kappa = 0, 1, 2\),

\[
||\nabla^\kappa u_0||_\infty := \sup_{x \in \mathbb{R}^d} |\nabla^\kappa u_0(x)| < \infty;
\]

\(ii\) (forcing term) on every subset \([0, T] \times \mathbb{R}^d\) with \(T > 0\) finite, \(g\) is bounded and \(\alpha\)-Hölder continuous for every \(\alpha \in (0, 1)\); furthermore, \(g\) is \(C^{1,2}\) and \(t \mapsto ||\nabla^\kappa g_t||_\infty := \sup_{x \in \mathbb{R}^d} |\nabla^\kappa g_t(x)|, t \mapsto ||\partial_t g_t||_\infty := \sup_{x \in \mathbb{R}^d} |\partial_t g_t(x)|\) are locally integrable in time.

For convenience we redefine \(\bar{t} = \nu t, \bar{u} = \nu^{-1} u, \bar{g} = \nu^{-2} g\). The rescaled equation, \((\partial_{\bar{t}} - \Delta - \bar{u} \cdot \nabla)\bar{u} = \bar{g}\), has viscosity 1. We skip the tilde in the sequel. Our bounds blow up in the vanishing viscosity limit \(\nu \to 0\) (see Remarks after Theorem 1.1 for a precise statement).

Our approach is the following. We solve inductively the linear transport equations,

\[
u^{(-1)} := 0; \tag{1.3}
\]
If the sequence \((u^{(m)})_m\) converges in appropriate norms, then the limit is a fixed point of (1.4), hence solves the Burgers equation. Let \(\|u\|_\alpha\) denotes either the isotropic Hölder semi-norm on \(\mathbb{R}^d\), \(\|u\|_\alpha := \sup_{x,y \in \mathbb{R}^d} \frac{|u(x) - u(y)|}{|x-y|^\alpha}\), or the parabolic Hölder semi-norm on \(\mathbb{R} \times \mathbb{R}^d\), \(\|g\|_\alpha := \sup_{(s,x),(t,y) \in \mathbb{R} \times \mathbb{R}^d} \frac{|g(s,x) - g(t,y)|}{|s-t|^{\alpha/2}}\), (see section 4 for more on Hölder norms).

**Definition 1.1** Let, for \(c > 0\),

\[
K_0(t) := \|u_0\|_\infty + \int_0^t ds\|g_s\|_\infty \tag{1.5}
\]

\[
K_1(t) := \|\nabla u_0\|_\infty + \int_0^t ds\|\nabla g_s\|_\infty \tag{1.6}
\]

\[
K_2(t) := \|\nabla^2 u_0\|_\infty + \|u_0\|_\alpha\|\nabla u_0\|_\infty + \|g_0\|_\infty + \int_0^t ds \left(\|\nabla^2 g_s\|_\infty + \|\partial_s g_s\|_\infty\right) \tag{1.7}
\]

\[
K_{2+\alpha}(t) := \|\nabla^2 u_0\|_\alpha + \|g_0\|_{\alpha,[0,t] \times \mathbb{R}^d}, \quad \alpha \in (0,1) \tag{1.8}
\]

and

\[
K(t) := c^2 \left(K_0(t)^2 + K_1(t) + K_2(t)^{2/3} + K_{2+\alpha}(t)^{2/(3+\alpha)}\right) \tag{1.9}
\]

Note that \(K_0(t), K_1(t), K_2(t), K_{2+\alpha}(t), K(t) < \infty\) for all \(t \geq 0\) and \(\alpha \in (0,1)\) under the above Assumptions.

Our main result is the following.

**Theorem 1.1** For every \(\beta \in (0,\frac{1}{2})\), there exists an absolute constant \(c = c(d,\beta) \geq 1\), depending only on the dimension and on the exponent \(\beta\), such that the following holds.

(i) (uniform estimates)

\[
\|u_t^{(m)}\|_\infty \leq K_0(t), \quad \|\nabla u_t^{(m)}\|_\infty \leq K(t); \quad \|\partial_t u_t^{(m)}\|_\infty, \|\nabla^2 u_t^{(m)}\|_\infty \leq (cK(t))^{3/2} \tag{1.10}
\]

(ii) (short-time estimates) define \(v^{(m)} := u^{(m)} - u^{(m-1)}\) for \(m \geq 1\). If \(0 \leq t \leq T\) and \(t \leq m/cK(T)\), then

\[
\|v_t^{(m)}\|_\infty \leq cK_0(T)(cK(T)t/m)^m, \quad \|\nabla v_t^{(m)}\|_\infty \leq cK(T)(cK(T)t/m)^{\beta m}. \tag{1.11}
\]

Let us comment on these estimates.

1. The different powers in the expression of \(K(t)\) come from the dimension counting dictated by the Burgers equation: the diffusion term \(\Delta u\), the convection term \(u \cdot \nabla u\) and the forcing \(g\) are homogeneous if \(u\) scales like \(L^{-1}\), where \(L\) is a reference space scale, and \(g\) like \((LT)^{-1}\), where \(T\) is a reference time scale. Assuming parabolic scaling, \(K^{-1}(t)\) scales like time and plays the rôle of a reference time scale \(T(t)\) at time \(t\), leading to a time-dependent space scale \(L = L(t) \sim K^{-\frac{1}{4}}(t)\). The scaling of the other \(K\)-parameters is \(K_0 \sim T^{-\frac{1}{2}}\); \(K_1, K \sim T^{-1}\); \(K_2 \sim T^{-3/2}\); \(K_{2+\alpha} \sim T^{-(3+\alpha)/2}\).
2. The first uniform estimate

\[ \|u_t^{(m)}\|_\infty \leq K_0(t) \]  

(1.12)

follows from a straightforward application of the maximum principle to the transport equation (1.4).

3. (uniform estimates for the gradient). The function \( u^{(0)} \) satisfies the linear heat equation \( (\partial_t - \Delta)u^{(0)} = g \), whose explicit solution is \( u^{(0)}(t) = e^{t\Delta}u_0 + \int_0^t ds e^{(t-s)\Delta}g_s \). Thus

\[ \|\nabla u_t^{(0)}\|_\infty \leq \|\nabla u_0\|_\infty + \int_0^t ds \|\nabla g_s\|_\infty = K_1(t). \]  

(1.13)

Clearly \( K_1(t) \leq K(t) \). Estimates for further iterates \( u^{(1)}, u^{(2)}, \ldots \) involve \( K(t) \) instead of \( K_1(t) \).

4. Fix a time horizon \( T > 0 \) and consider the series \( S(t) := \sum_{m=0}^{+\infty} v_t^{(m)} = \sum_{m=0}^{+\infty} (u_t^{(m)} - u_t^{(m-1)}) \) for \( t \leq T \) (note that, by definition, \( v_t^{(0)} := u_t^{(0)} - u_t^{(-1)} = u_t^{(0)} \)). The short-time estimates (1.11) imply that \( S(t) \) is absolutely convergent. More precisely, letting \( m_0 := \lfloor cK(T)t \rfloor \) and \( \gamma := 1, \)

\[ \|u_t^{(m_0)}\|_\infty = \left\| \sum_{m=0}^{n} (u_t^{(m)} - u_t^{(m-1)}) \right\|_\infty \leq \|u_t^{(m_0)}\|_\infty + \sum_{m=m_0+1}^{+\infty} \|v_t^{(m)}\|_\infty \leq K_0(T) \left\{ 1 + c \sum_{m=m_0+1}^{+\infty} (cK(T)t/m)^{\gamma m} \right\} \]  

(1.14)

for all \( n \geq m_0 \). Let \( m > m_0 \) and \( x = 1 - cK(T)t/m \in [0, 1] \); using \( 1 - x \leq e^{-x} \), one gets \( (cK(T)t/m)^{\gamma m} = (1 - x)^{\gamma m} \leq e^{\gamma K(T)t} e^{-\gamma m} \) and

\[ \sum_{m=m_0+1}^{+\infty} (cK(T)t/m)^{\gamma m} \leq e^{\gamma K(T)t \sum_{m=m_0+1}^{+\infty} e^{-\gamma m}} \leq e^{\gamma / (e^\gamma - 1)} \]  

(1.15)

Hence \( \|u_t^{(m)}\|_\infty \leq K_0(T) \). In a similar way, letting \( \gamma := \beta \) this time, one shows that

\[ \|\nabla u_t^{(n)}\|_\infty = \left\| \sum_{m=0}^{n} (\nabla u_t^{(m)} - \nabla u_t^{(m-1)}) \right\|_\infty \leq K(T). \]  

(1.16)

These estimates are best when \( t = T \); one then retrieves the uniform estimates (1.10) up to some constant.

5. (short-time estimates) Bounds (1.11) are of order \( O((Ct)^{\gamma m} / (m!)^\gamma) \), \( \gamma = 1 \) or \( \beta \), and obtained by \( m \) successive integrations. For linear equations, or equations with bounded, uniformly Lipschitz coefficients, successive integrations typically yield \( O((Ct)^m / m!) \). The Burgers equation, on the other hand is strongly non-linear. While using precise Schauder estimates to obtain the gradient bound in (1.11), one stumbles into the condition \( \beta < \frac{1}{2} \) at the very end of section 3 which apparently cannot be improved.

6. (blow-up of the above estimates in the vanishing viscosity limit) Undoing the initial rescaling, we obtain \( \gamma \)-dependent estimates,

\[ \|u_t\|_\infty \leq K_0(t), \quad \|\nabla u_t\|_\infty \leq \nu^{-1} K(t), \quad \|\partial_t u_t\|_\infty \leq \nu^{-1} K(t)^{3/2}, \quad \|\nabla^2 u_t\|_\infty \leq \nu^{-2} K(t)^{3/2} \]  

(1.17)
with \( K_0(t), K_1(t) \) as in (1.5), (1.6), \( K_2(t) := v|\nabla^2 u_0|_\infty + |u_0|_\infty |\nabla u_0|_\infty + |g_0|_\infty + \int_0^t ds(v|\nabla^2 g_s|_\infty + ||\partial_s g_s||_\infty), \) \( K_{2+s}(t) := v|\nabla^2 u_0|_\infty + \sup_{\alpha \in [0,1]} ||g_\alpha||_\infty \) and \( K(t) := K_0(t)^2 + vK_1(t) + (vK_2(t))^{2/3} + (v^{1+\alpha}K_{2+s}(t))^{2/(3+\alpha)}. \) Thus the derivative bounds \( ||\nabla^\kappa u_t||_\infty, \kappa = 1, 2 \) and \( ||\partial_t u||_\infty \) blow up at different rates when \( v \to 0. \)

From the above theorem, one deduces easily that the solution of the Burgers equation is smooth on \( \mathbb{R}_+ \times \mathbb{R}^d \) provided (i) \( u_0 \) is smooth and its derivatives are bounded; (ii) \( g \) is smooth and its derivatives are bounded on \([0, T] \times \mathbb{R}^d \) for all \( T \):

**Corollary 1.2** Assume \( u_0 \) and \( g \) are smooth, and \( ||\nabla^\kappa u_0||_\infty < \infty \) (\( \kappa = 0, 1, 2, \ldots \)), \( ||\partial^\kappa F^\kappa g||_\infty < C(\mu, \kappa, T) \), \( \mu, \kappa = 0, 1, 2, \ldots \) for every \( t \leq T \). Then the Burgers equation (1.1) has a unique smooth solution \( u \) such that \( ||\partial^\kappa F^\kappa u_t||_\infty < C'(\mu, \kappa, T) \) for every \( \mu, \kappa \) and \( t \leq T \). In particular, \( C'(\mu, \kappa, T) = C'(\mu, \kappa) \) is uniform in time if \( g = 0 \).

We do not prove this corollary, since it results from standard extension to higher-order derivatives of the initial estimates of section 2, and an equally standard iterated use of Schauder estimates to derivatives of Burgers equation.

Our results extend without any modification to nonlinearities of the type \( F(u) \cdot \nabla u \) with smooth matrix-valued coefficient \( F \) if \( F \) is sublinear, and even (with different scalings and exponents for the \( K \)-constants) to the case when \( F \) has polynomial growth at infinity.

Let us compare with the results available in the literature. The one-dimensional case \( d = 1 \) or the irrotational \( d \)-dimensional case with \( g = \nabla f \) of gradient form, is exactly solvable through the Cole-Hopf transformation \( u = \nabla \log \phi \) which reduces it to a scalar, linear PDE \( \partial_t \phi = v\Delta \phi + f \phi; \) note also that \( \log \phi \) is a solution of the KPZ (Kardar-Parisi-Zhang) equation. In that case the equation is immediately shown to be well-defined for every \( t > 0 \) under our hypotheses, and estimates similar to ours are easily obtained; specifically in \( d = 1 \), an invariant measure is known to exist if \( g \) is e.g. a space-time white noise [3]. For periodic solutions on the torus in one dimension, the above results extend to the vanishing viscosity limit [5]. The reader may refer e.g. to [4] for a more extended bibliography.

So our result is mostly interesting for \( d \geq 2 \); as mentioned above, our scheme of proof extends to more general non-linearities of the form \( F(u) \cdot \nabla u \), for which the equation is not exactly solvable in general. In this setting, the classical result is that due to Kiselev and Ladyzhenskaya [8]. The authors consider solutions in Sobolev spaces and use repeatedly energy estimates. They work on a bounded domain \( \Omega \) with Dirichlet boundary conditions, but their results extend with minor modifications to the case \( \Omega = \mathbb{R}^d \). If \( u_0 \in \mathcal{H}^s \) with \( s > d/2 \), then \( ||u_0||_{\mathcal{H}^s} < \infty \) by Sobolev’s imbedding theorem. Then the maximum principle gives \( ||u_t||_{\mathcal{H}^1} \leq ||u_0||_{\mathcal{H}^s} \) as long as the solution is classical; this key estimate allows one to bootstrap and get bounds for higher-order Sobolev spaces which increase exponentially in time, e.g. \( ||u_t||_{\mathcal{H}^1} = O(e^{||u_0||_{\mathcal{H}^s}^2 t}) \), as follows from the proof of Lemma 3 in [8]. Compared to these estimates, ours present two essential improvements: (i) we do not assume any decrease of the data at spatial infinity, so that they do not necessarily belong to Sobolev spaces; (ii) more importantly perhaps, our bounds do not increase exponentially in time; in the case the right-hand side \( g \) vanishes identically, they are even uniform in time, \( K_0(t), K(t) \leq C \) where \( C \) is a constant depending only on the initial condition.
1.2 Scheme of proof

Recall that we solve inductively the following linear transport equations, see (1.4),

\[ u^{(0)} = 0; \]  
(1.18)

\[ (\partial_t - \Delta + u^{(m-1)} \cdot \nabla) u^{(m)} = g, \quad u^{(m)}|_{t=0} = u_0 \quad (m \geq 0). \]  
(1.19)

Under the first set of assumptions, standard results on linear equations show that \( u^{(m)}, m \geq 0 \) is \( C^{1,2} \). Assume we manage to prove locally uniform convergence of \( u^{(m)}, \nabla u^{(m)}, \nabla^2 u^{(m)} \) when \( m \to \infty \). Then there exists \( u \in C^{1,2} \) such that locally uniformly \( u^{(m)} \to u \), \( \nabla u^{(m)} \to \nabla u \), \( \nabla^2 u^{(m)} \to \nabla^2 u \) and \( \partial_t u^{(m)} \to \partial_t u \). Hence \( \partial_t u^{(m)} = u^{(m)} - u^{(m-1)} \cdot \nabla u^{(m)} + g \) converges locally uniformly to \( \Delta u - u \cdot \nabla u + g \), and \( \partial_t u = \lim_{m \to \infty} \partial_t u^{(m)} = \Delta u - u \cdot \nabla u + g \). In other words, the limit \( u \) is a \( C^{1,2} \) solution of the Burgers equation.

The key point in our scheme is to prove locally uniform convergence of \( u^{(m)} \) and \( \nabla u^{(m)} \), and to show uniform bounds in Hölder norms for second order derivatives \( \nabla^2 u^{(m)}, \partial_t u^{(m)} \); a simple argument (see below) yields then the convergence of second order derivatives, allowing to apply the above elementary argument. The basic idea is to rewrite \( u \) as \( \sum_{m=0}^{\infty} v^{(m)} \), with \( v^{(m)} := u^{(m)} - u^{(m-1)} \) and to show that the series is convergent, uniformly in space and locally uniformly in time.

In the sequel we fix a constant \( c \geq 1 \) such that Theorem 1.1 holds and let

\[ \bar{K}_0(t) := cK_0(t), \quad \bar{K}_1(t) := cK_1(t), \quad \bar{K}(t) := cK(t) \]  
(1.20)

to simplify notations.

The proof relies on two main ingredients: \textit{a priori estimates} coming from the maximum principle; and \textit{Schauder estimates}. Schauder estimates are difficult to find in a precise form suitable for the kind of applications we have in view, so the reader will find in the appendix a precise version of these estimates, see Proposition 4.5, following a multi-scale proof introduced by X.-J. Wang. These imply in particular the following.

\textbf{Lemma 1.3} Let \( 0 \leq t \leq T \). Then

\[ \| \partial_t u^{(m)} \|_{\alpha, [0, T] \times \mathbb{R}^d}, \| \nabla^2 u^{(m)} \|_{\alpha, [0, T] \times \mathbb{R}^d} \leq \bar{K}(T)^{(3+\alpha)/2}. \]  
(1.21)

\textbf{Lemma 1.3} is proved in section 3, at the same time as Theorem 1.1.

We now use a classical result about Hölder spaces: let \( C^\alpha(Q) \), with \( Q \subset \mathbb{R} \times \mathbb{R}^d \) compact, be the Banach space of \( \alpha \)-Hölder functions on \( Q \) equipped with the norm \( \| u \|_\alpha := \| u \|_{\infty, Q} + \| u \|_{0, Q} \). Then the injection \( C^{\alpha'}(Q) \subset C^\alpha(Q) \) is compact for every \( \alpha' < \alpha \). In particular, Lemma 1.3 implies the existence of a subsequence \( (u^{(m_n)}) \) such that \( \nabla^2 u^{(m_n)} \to u \) in \( C^{\alpha'} \)-norm. On the other hand, as discussed in Remark 4 above, \( u^{(m)} \to u \) and \( \nabla u^{(m)} \to \nabla u \) in the sup norm for some \( u \in C^{0,1} \). Hence \( u \) is twice continuously differentiable in the space variables, and \( \nabla^2 u = v \). Now every subsequence \( \nabla^2 u^{(m_n)} \) converges to the same limit, \( \nabla^2 u \). Hence \( \nabla^2 u^{(m)} \to \nabla^2 u \) in \( C^{\alpha'} \). In a similar way, one proves that \( u \) is continuously differentiable in the time variable, and \( \partial_t u = \lim_{m_n \to \infty} \partial_t u^{(m_n)} \) in \( C^{\alpha'} \). In particular, \( u \in C^{1,2} \), and the arguments given at the very beginning of the present subsection show that \( u \) is a classical solution of the Burgers equation. Note that we may reach the same conclusion even if we do not know that the series \( \| \nabla u^{(m+1)} - \nabla u^{(m)} \|_{0, Q} \) converges. Actually the bound on
Theorem 2.1 (initial estimates) exclude this trivial case, so that equation is simply 0. The case Galilean transformation where $|||v|||_\infty$.

Definition 2.2 Hence the result by the maximum principle.

Lemma 2.1 (Gronwall lemma) Let $\phi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, resp. $\bar{\phi} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be the solution of the transport equation $(\partial_t - \Delta + b \cdot \nabla - c)\phi = f$, resp. $(\partial_t - \Delta + \tilde{b} \cdot \nabla - \tilde{c})\bar{\phi} = \tilde{f}$, with same initial condition, $\phi|_{t=0} = \phi|_{t=0}$; the coefficients $c = c(t, x), \tilde{c} = \tilde{c}(t, x) \in M_{d \times d}(\mathbb{R})$ are matrix-valued, and $b, \tilde{b}, c, \tilde{c}$ are assumed to be bounded and continuous. Let $v := \phi - \bar{\phi}$. Then

$$||v||_\infty \leq \int_0^t ds A(s, t) ||\bar{b}_s - b_s||_\infty ||\nabla \phi_s||_\infty + \int_0^t ds A(s, t) ||\bar{c}_s - c_s||_\infty ||\phi_s||_\infty + \int_0^t ds A(s, t) ||\bar{f}_s - f_s||_\infty,$$

where $||.||_\infty$ is the supremum over $\mathbb{R}^d$ of the operator norm in $M_{d \times d}(\mathbb{R})$, and $A(s, t) = \exp \int_s^t ||\bar{c}_r||_\infty dr$.

Proof. By subtracting the PDEs satisfied by $\phi$ and $\bar{\phi}$, one gets

$$(\partial_t - \Delta + \tilde{b} \cdot \nabla - \tilde{c})v = -(\tilde{b} - b) \cdot \nabla \phi + (\tilde{f} - f) + (\tilde{c} - c)\phi. \tag{2.2}$$

Hence the result by the maximum principle. \hfill \Box

Definition 2.2 Let $t_{init} := \inf \{t > 0; t\bar{K}(t) = 1\}$.

By hypothesis, $t_{init} > 0$. If $u_0 \equiv 0$ and $g \equiv 0$, then $t_{init} = +\infty$ and the solution of Burgers’ equation is simply 0. The case $u_0 = \text{Cst}, \nabla g = 0$ reduces to the previous one by the generalized Galilean transformation $x \mapsto x + \int_0^t g(s)ds, u \mapsto u - a$ with $a(t) = u_0 + \int_0^t g_s ds$. We henceforth exclude this trivial case, so that $t_{init} \in (0, +\infty)$.

Theorem 2.1 (initial estimates) Let $t \leq t_{init}$. Then the following estimates hold:

(i)

$$||u_t^{(m)}||_\infty \leq K_0(t_{init}), \quad ||\nabla u_t^{(m)}||_\infty \leq K(t_{init}); \quad ||\partial_t u_t^{(m)}||_\infty, ||\nabla^2 u_t^{(m)}||_\infty \leq \bar{K}(t_{init})^{3/2}. \tag{2.3}$$

Furthermore,

$$||\partial_t u_t^{(m)}||_\alpha, ||\nabla^2 u_t^{(m)}||_\alpha \leq C\bar{K}(t_{init})^{(3+\alpha)/2} \tag{2.4}$$

with $C = C(d, \alpha)$. 


(ii) let $m \geq 1$, then

$$||v_t^{(m)}||_\infty \leq \bar{K}_0(t_{init})(\bar{K}(t_{init})t/m)^m, \quad ||\nabla v_t^{(m)}||_\infty \leq \bar{K}(t_{init})(\bar{K}(t_{init})t/m)^m. \quad (2.5)$$

Remarks.

1. Let $T \leq t_{init}$, then (2.3), (2.4), and (2.5) remain true for $t \leq T$ if one replaces $K_0(t_{init}), \bar{K}_0(t_{init}), K(t_{init}), \bar{K}(t_{init})$ by $K_0(T), \bar{K}_0(T), K(T), \bar{K}(T)$. Hence Theorem 1.1 is proved for $t \leq t_{init}$ (actually with $\beta = 1$).

2. The value of $t_{init}$ depends on the choice of $c$. We provide in the course of the proof a rather explicit minimal value of $c$ for which (2.3), (2.4), and (2.5) hold. Further estimates in the next section may require a larger value of $c$.

3. From Hölder interpolation estimates (see Lemma 4.2), one also has a bound for lower-order Hölder norms,

$$||u^{(m)}||_\alpha \leq K_0(t_{init})^{1-\alpha} \bar{K}(t_{init})^\alpha + K_0^{1-\alpha/2}(t_{init}) \bar{K}^{3\alpha/4}(t_{init}), \quad (2.6)$$

and, for fixed $s \leq t_{init}$,

$$||\nabla u^{(m)}_s||_\alpha \leq K_0^{1-\alpha}(t_{init}) \bar{K}(t_{init})^{3\alpha/2}. \quad (2.7)$$

Proof. Let us abbreviate $K_0(t_{init}), \bar{K}_0(t_{init}), K_1(t_{init}), \bar{K}_1(t_{init}), K(t_{init}), \bar{K}(t_{init})$ to $K_0, \bar{K}_0, K_1, \bar{K}_1, K, \bar{K}$.

(i) We first prove estimates (i) by induction, assuming them to be proved for $m - 1$. Note first that (2.3) holds true for $m = 0$ with $c = 1$, see eq. (1.13); as for (2.4),

$$||\nabla^2 u_t^{(0)}||_Y \leq ||\nabla^2 u_0||_Y + \int_0^t ds ||\nabla^2 e^{sA}g(t-s)||_Y$$

$$\leq K_2^{1-\gamma/\alpha}(t_{init}) K_2^{\gamma/\alpha}(t_{init}) + t_{init}^{(\alpha-\gamma)/2} K_2^{\alpha/2}(t_{init})$$

$$\leq C(d, \alpha, \gamma) \bar{K}^{(3+\gamma)/2}, \quad \gamma < \alpha \quad (2.8)$$

as follows from Hölder interpolation inequalities (see Lemma 4.2 and Corollary 4.4). Time variations of $\nabla^2 u_t^{(0)}$ scale similarly, yielding $||\nabla^2 u_t^{(0)}||_Y, 0 \leq t_t \leq \bar{K}^{(3+\gamma)/2}$ (see Lemma 4.3, eq. (4.8), and Corollary 4.4). Note that similarly, $||\nabla u_t^{(0)}||_Y, 0 \leq t_t \leq \bar{K}^{(2+\gamma)/2}$. The estimate for $||u_t^{(m)}||_\infty$ is a direct consequence of the gradient principle. Then $\nabla u_t^{(m)}$ satisfies the gradient equation

$$(\partial_t - \Delta + u^{(m-1)} \cdot \nabla + \nabla u^{(m-1)}) \nabla u^{(m)} = \nabla g, \quad (2.9)$$

where $\nabla u^{(m-1)}(t, x)$ is viewed as the $d \times d$ matrix $(\partial_j u_t^{(m-1)})_{jk}$ acting on the vector $(\partial_k u_t)_k$.

Note that

$$||\nabla u^{(m-1)}(t, x)|| \leq \sqrt{\text{Tr}(\nabla u^{(m-1)}(t, x)(\nabla u^{(m-1)}(t, x))^*)} = ||\nabla u^{(m-1)}(t, x)||_2. \quad (2.10)$$

By the maximum principle,

$$||\nabla u_t^{(m)}||_\infty \leq A(0, t) ||\nabla u_0||_\infty + \int_0^t ds A(s, t) ||\nabla g_s||_\infty, \quad (2.11)$$
where \( A(s, t) := \exp \int_0^t ||\nabla u_r^{(m-1)}||_\infty dr \) is the exponential amplification factor of Lemma 2.1.

By induction hypothesis and Definition 2.2, \( A(s, t) \leq A(0, t_{\text{init}}) \leq e^{cK} \leq e \), hence (provided \( c^2 \geq e \))

\[
\|\nabla u_r^{(m)}\|_\infty \leq eK_1 \leq K. \tag{2.12}
\]

To bound \( \nabla^2 u_r^{(m)} \) we differentiate once more,

\[
( \partial_t - \Delta + u^{(m-1)} \cdot \nabla + \nabla u^{(m-1)}) \nabla^2 u^{(m)} = \nabla^2 g - \nabla^2 u^{(m-1)} \nabla u^{(m)}, \tag{2.13}
\]

where \( \nabla u^{(m-1)} \) is viewed this time as the \( d^2 \times d^2 \) matrix \( (\partial_{ij} g_{k})_{ij} \in \mathbb{R}^{d}, \) and has matrix norm \( ||\nabla u^{(m-1)}(t, x)||_{M_{\tilde{\mathcal{C}}^2_1}(\mathbb{R})} \leq C_d ||\nabla u^{(m-1)}(t, x)||_{\infty} \), yielding an amplification factor \( \tilde{A}(s, t) := \exp \int_s^t ||\nabla u_r^{(m-1)}||_{M_{\tilde{\mathcal{C}}^2_1}(\mathbb{R})} ||dr \leq C_d' \).

By the maximum principle,

\[
||\nabla^2 u_r^{(m)}||_\infty \leq C_d' \left( ||\nabla^2 u_0||_\infty + C_d \int_0^t ds \left( ||\nabla^2 g_s||_\infty + ||\nabla^2 u^{(m-1)}||_\infty ||\nabla u^{(m)}||_\infty \right) \right)
\]

\[
\leq C_d' \left( ||\nabla^2 u_0||_\infty + \int_0^t ds ||\nabla^2 g_s||_\infty + t_{\text{init}} \tilde{K}^{3/2}K \right)
\]

\[
\leq C_d'(K_2(t_{\text{init}}) + \tilde{K}^{3/2}) \leq C_d'(c^{-3} + c^{-1})\tilde{K}^{3/2} \leq \tilde{K}^{3/2} \tag{2.14}
\]

provided \( c \geq 2 \max(1, C_d') \).

Similarly, \( \partial_t u^{(m)} \) satisfies the transport equation

\[
( \partial_t - \Delta + u^{(m-1)} \cdot \nabla ) \partial_t u^{(m)} = \partial_t g - \partial_t u^{(m-1)} \cdot \nabla u^{(m)}, \tag{2.15}
\]

hence

\[
||\partial_t u_r^{(m)}||_\infty \leq ||\nabla^2 u_0||_\infty + ||u_0||_\infty ||\nabla u_0||_\infty + ||g_0||_\infty + \int_0^t ds ||\partial_s g_s||_\infty + t_{\text{init}} \tilde{K}^{3/2}K
\]

\[
\leq K_2(t_{\text{init}}) + \tilde{K}^{3/2} \leq (c^{-3} + c^{-1})\tilde{K}^{3/2} \leq \tilde{K}^{3/2} \tag{2.16}
\]

provided \( c \geq 2 \).

Finally, we must prove the Hölder estimate (2.4): for that, we use the integral representation

\[
\nabla^2 u_r^{(m)} = \nabla^2 G_r^{(m)} - \int_0^t \nabla^2 e^{(t-s)\Delta} \left( (u_r^{(m-1)} \cdot \nabla) u_r^{(m)} \right) ds. \tag{2.17}
\]

By Lemma 4.2 considering \( \alpha \)-Hölder norms on \( [0, t_{\text{init}}] \times \mathbb{R}^d \),

\[
||(u_r^{(m-1)} \cdot \nabla) u_r^{(m)} ||_y \leq ||u_r^{(m-1)}||_\infty ||\nabla u_r^{(m)}||_y + ||\nabla u_r^{(m)}||_\infty ||u_r^{(m-1)}||_y \leq K_0 K^{1-\gamma} \tilde{K}^{3/2} + K K_0^{1-\gamma} \tilde{K}^{3/2} \leq \tilde{K}^{(3+\gamma)/2} \tag{2.18}
\]

Thus by Lemma 4.3

\[
||\nabla^2 u_r^{(m)} - \nabla^2 u_r^{(m)} ||_\infty \leq ||\nabla^2 u_0^{(0)} - \nabla^2 u_0^{(0)} ||_\infty + \int_t^{t'} (t-s)^{\frac{\alpha-1}{2}} ||(u_r^{(m-1)} \cdot \nabla) u_r^{(m)}||_\infty ds \leq (t-t')^{\alpha/2} \tilde{K}^{(3+\alpha)/2} \tag{2.19}
\]
for $t' < t$, and (choosing any $\gamma \in (\alpha, 1)$)
\[
\|\nabla^2 u^{(m)}_t\|_\alpha \leq \|\nabla^2 u^{(0)}_t\|_\alpha + C'(d, \alpha, \gamma) \tilde{K}^{(3+\gamma)/2} \int_0^{t_\text{init}} (t-s)^{-1+(\gamma-\alpha)/2} ds \lesssim \tilde{K}^{(3+\alpha)/2},
\]
(2.20)
hence the result for $\|\nabla^2 u^{(m)}\|_\alpha$. Similarly,
\[
\|\nabla u^{(m)}_t - \nabla u^{(m)}_{t'}\|_\alpha \lesssim \|\nabla u_t^{(0)} - \nabla u_{t'}^{(0)}\|_\alpha + \int_{t'}^{t} (t-s)^{(\alpha-1)/2} \|\nabla (u^{(m-1)}_s \cdot \nabla u^{(m)}_s)\|_\alpha \gamma ds
\]
\[
\lesssim (t-t')^{\alpha/2} \tilde{K}^{(2+\alpha)/2} + (t-t')^{(\alpha+1)/2} \tilde{K}^{(3+\alpha)/2}
\]
\[
\lesssim (t-t')^{\alpha/2} \tilde{K}^{(2+\alpha)/2} + (t-t')^{\alpha/2} \tilde{K}_\text{init}^{(3+\alpha)/2} \lesssim (t-t')^{\alpha/2} \tilde{K}^{(2+\alpha)/2},
\]
(2.21)
hence (using Hölder interpolation inequalities once more) $\|\nabla u^{(m)}\|_\alpha \lesssim \tilde{K}^{(2+\alpha)/2}$. From the previous bounds follows immediately $\|\partial_t u^{(m)}\|_\alpha \lesssim \|\nabla^2 u^{(m)}\|_\alpha + \|\nabla (u^{(m-1)} \cdot \nabla u^{(m)})\|_\alpha \lesssim \tilde{K}^{(3+\alpha)/2}$.

(ii) Apply Lemma 2.1 with $\phi = \tilde{b} = u^{(m-1)}$, $b = u^{(m-2)}$, $\tilde{\phi} = u^{(m)}$, $f = \tilde{f} = g$ and $c = \tilde{c} = 0$. It comes out
\[
\|v^{(m)}_t\|_\infty \lesssim \int_0^t ds \|v^{(m-1)}_s\|_\infty \|\nabla u^{(m-1)}_s\|_\infty.
\]
(2.22)
Thus, using the induction hypothesis,
\[
\|v^{(m)}_t\|_\infty \lesssim \int_0^t ds K_0(\tilde{K} s/(m-1))^{m-1} K \leq K_0(\tilde{K} t/m)^m (1-\frac{1}{m})^{-(m-1)}(K/\tilde{K}) \leq K_0(\tilde{K} t/m)^m, \quad m \geq 2
\]
(2.23)
for $c$ large enough, and
\[
\|v^{(1)}_t\|_\infty \lesssim \int_0^t ds \|u^{(0)}_s\|_\infty \|\nabla u^{(0)}_s\|_\infty \lesssim K_0 K t \leq K_0(\tilde{K} t).
\]
(2.24)
Consider now as in (i) the gradient of the transport equations of index $m-1, m$,
\[
(\partial_t - \Delta + u^{(n-1)} \cdot \nabla + u^{(m-1)} \nabla) \nabla u^{(m)} = \nabla g, \quad n = m-1, m
\]
(2.25)
and apply Lemma 2.1 with $\phi = \nabla u^{(m-1)}$, $\tilde{\phi} = \nabla u^{(m)}$, $b = u^{(m-2)}$, $\tilde{b} = u^{(m-1)}$ and $c = \nabla u^{(m-2)}$, $\tilde{c} = \nabla u^{(m-1)}$. Using the induction hypothesis, one gets
\[
\|\nabla v^{(m)}_t\|_\infty \leq \int_0^t ds A(s, t) \|v^{(m-1)}_s\|_\infty \|\nabla^2 u^{(m-1)}_s\|_\infty + \int_0^t ds A(s, t) \|\nabla v^{(m-1)}_s\|_\infty \|\nabla u^{(m-1)}_s\|_\infty
\]
\[
\lesssim e \int_0^t ds (K_0 \tilde{K}^{3/2} + \tilde{K} K)(\tilde{K} s/(m-1))^{m-1}
\]
\[
\lesssim e(1-\frac{1}{m})^{-(m-1)}(\tilde{K} t/m)^m (\tilde{K} \tilde{K}^{3/2} + \tilde{K}) \leq e(1-\frac{1}{m})^{-(m-1)}(e^{-\frac{1}{2}} + e^{-1})\tilde{K}^{(3+\alpha)/2}
\]
\[
\lesssim \tilde{K}^{(3+\alpha)/2}, \quad m \geq 2
\]
(2.26)
and
\[
\|\nabla v^{(1)}_t\|_\infty \lesssim \int_0^t ds \left( \|u^{(0)}_s\|_\infty \|\nabla^2 u^{(0)}_s\|_\infty + \|\nabla u^{(0)}_s\|_\infty^2 \right)
\]
\[
\lesssim e(K_0 \tilde{K}^{3/2} + K^2) t \leq \tilde{K}(\tilde{K} t)
\]
(2.27)
for $c$ large enough.
\[\square\]
3 Proof of main theorem

By Remark 1 following Theorem 2.1, we may now restrict to times larger than \( t_{\text{init}} \). We fix a time horizon \( T > t_{\text{init}} \) and distinguish two regimes: a short-time regime, \( t \leq m/\bar{K}(T) \); and a long-time regime, \( t > m/\bar{K}(T) \). Clearly the short-time regime does not exist for \( m = 0 \); as already noted before (see comments after Theorem 1.1), this case is trivial and estimates (1.10), proven in the course of Theorem 2.1 in the initial regime, extend without any modification to arbitrary time. So we assume henceforth that \( m \geq 1 \).

Theorem 1.1 follows immediately from an estimate for \( u^{(m)}, \nabla u^{(m)} \) valid over the whole region \( t \in [t_{\text{init}}, T] \) and another estimate for \( v^{(m)}, \nabla v^{(m)} \) valid only in the short-time regime. These are proved by induction.

Theorem 3.1 (estimates for \( u^{(m)} \) and \( \nabla u^{(m)} \)) Let \( m \geq 1 \) and \( t \in [t_{\text{init}}, T] \). Then
\[
\|u^{(m)}_t\|_\infty \leq K_0(T), \quad \|\nabla u^{(m)}_t\|_\infty \leq K(T); \quad \|\partial \partial u^{(m)}_t\|_\infty, \|\nabla^2 u^{(m)}_t\|_\infty \leq \bar{K}(T)^{3/2}.
\]
Furthermore,
\[
\|\partial \partial u^{(m)}_t\|_{l_{2,\alpha}}, \|\nabla^2 u^{(m)}_t\|_{l_{2,\alpha}} \leq \bar{K}(T)^{(3+\alpha)/2}.
\]

**Proof.** As already noted, the inequality \( \|u^{(m)}_t\|_\infty \leq K_0(T) \) follows immediately from the maximum principle, so we consider only the bound for the gradient and higher-order derivatives in (3.1). We prove it by induction on \( m \), assuming it to be true for \( m - 1 \). We abbreviate \( K_0(T), K(T), \bar{K}(T) \) to \( K_0, K, \bar{K} \).

We apply Proposition 4.3 on the parabolic ball \( Q^{(j)} = [t - M^j, t] \times \bar{B}(x, M^j/2) \), with \( M^j := \frac{1}{2}\bar{K}(T)^{-1} \). Note that, by definition, \( t - M^j \geq t_{\text{init}} - \frac{1}{2}\bar{K}(t_{\text{init}})^{-1} \geq \frac{1}{2}t_{\text{init}} > 0 \). We consider first the bound (4.16) for the gradient,
\[
\|\nabla u^{(m)}_{t, Q^{(j)}}\| \leq R_b^{-1} K^{-1/2} \|g\| \|R^{(m-1)}\|_{l_{2,\alpha}} + R_b^{-1} K_0 \left( \bar{K}^{-1/2} R_b^{-1} \|u^{(m-1)}\|_{l_{2,\alpha}} + \bar{K}^{1/2} \right).
\]

The multiplicative factor \( R_b^{-1} \) is bounded by \( 1 + (2\bar{K}^{-1/2}) \|u^{(m-1)}\|_{l_{2,\alpha}} \). On the other hand, by Hölder interpolation inequalities (see Lemma 4.2),
\[
\|u^{(m-1)}\|_{l_{2,\alpha}} \leq K^{(m-1)} K_0^{1-\alpha} + \bar{K}^{3\alpha/4} K_0^{1-\alpha/2} \\
\leq (1 + c^{3\alpha/4} (K_0^2/K^\alpha)^{1/4}) K_0^{1-\alpha} \\
\leq (1 + c^{\alpha/4}) K_0^{1-\alpha} \leq (1 + c^{\alpha/4}) c^{2\alpha-2} K^{1+\alpha/2}.
\]

Hence
\[
\|\nabla u^{(m)}_{t, Q^{(j)}}\| \leq \bar{K}^{-1/2} K_{2+\alpha}(T) + K_0^{2-\alpha} \cdot c^{\alpha/2} K^{2\alpha} K_0^{3-2\alpha} + \bar{K}^{1/2} K_0 \\
\leq c^{-1/2} K + c^{(1+\alpha)/2} K_0^{\alpha-1/2} K_0^{3-2\alpha} + c^{1/2} K
\]
which is \( \leq K \) for \( c \) large enough.

Bounds for higher-order derivatives \( \|\partial u^{(m)}_{t, Q^{(j)}}\|_{l_{2,\alpha}}, \|\nabla^2 u^{(m)}_{t, Q^{(j)}}\|_{l_{2,\alpha}} \) follow from (4.18) instead, contributing an extra \( M^{-1/2} \approx \bar{K}^{1/2} \) multiplicative factor. They hold true for \( c \) large enough. Finally, (4.19)
yields
\[
\|\partial_t u^{(m)}\|_{\mathcal{A}, Q^{(j-1)}} + \|\nabla^2 u^{(m)}\|_{\mathcal{A}, Q^{(j-1)}} \leq \|g\|_{\mathcal{A}, Q^{(po)}} + K_0 \left( \|u^{(m-1)}\|_{\mathcal{A}, Q^{(po)}} + \tilde{K}^{1+\alpha/2} \right)
\]
\[
\leq K_{2+\alpha}(T) + K_0 \cdot c \left( \frac{K}{1+2(1+\alpha)} \right) K^{1+\alpha/2} + c^{-1} \tilde{K}^{(3+\alpha)/2}
\]
from which
\[
\|\nabla^2 u^{(m)}\|_{\mathcal{A}, [t_{\text{init}}, T] \times \mathbb{R}^d} \leq \sup_{(t,x) \in [t_{\text{init}}, T] \times \mathbb{R}^d} \|\nabla^2 u^{(m)}\|_{\mathcal{A}, Q^{(j-1)}(t,x)} + M^{-1/2} \|\nabla^2 u^{(m)}\|_{\mathcal{A}, \mathbb{R}^d} \leq \tilde{K}^{(3+\alpha)/2},
\]
and similarly for \(\|\partial_t u^{(m)}\|_{\mathcal{A}, [t_{\text{init}}, T] \times \mathbb{R}^d}\).

We take the opportunity to derive from (4.17) a bound for \(\nabla u^{(m)}\) (also valid for \(\|\nabla u^{(m)}\|_{\mathcal{A}, [t_{\text{init}}, T] \times \mathbb{R}^d}\)) that will be helpful in the next theorem,
\[
\|\nabla u^{(m)}\|_{\mathcal{A}, Q^{(j-1)}} \leq \tilde{K}^{-1/2} \left( 1 + \tilde{K}^{-1+\alpha/2} \|u^{(m-1)}\|_{\mathcal{A}, Q^{(po)}} \right) \|g\|_{\mathcal{A}} + K_0 \tilde{K}^{(1+\alpha)/2} \left( 1 + \tilde{K}^{-1+\alpha/2} \|u^{(m-1)}\|_{\mathcal{A}, Q^{(po)}} + (\tilde{K}^{-1+\alpha/2} \|u^{(m-1)}\|_{\mathcal{A}, Q^{(po)}}) \right)
\]
\[
\leq \tilde{K}^{1+\alpha/2}
\]
(3.8)
since (from (3.4)) \(\|u^{(m-1)}\|_{\mathcal{A}, Q^{(po)}} \leq \tilde{K}^{1+\alpha/2}\).

\[\Box\]

**Theorem 3.2 (short-time estimates for \(v^{(m)}\) and \(\nabla v^{(m)}\))** Let \(m \geq 1\) and \(t \in [t_{\text{init}}, \min(T, m/\tilde{K}(T))]\).

Then \(\|v^{(m)}_t\|_\infty \leq \tilde{K}_0(T)(\tilde{K}(T)t/m)^{m}\), \(\|\nabla v^{(m)}_t\|_\infty \leq \tilde{K}_0(T)(\tilde{K}(T)t/m)^{\beta m}\).

**Proof.** We abbreviate as before \(K_0, \tilde{K}_0(T), K(T), \tilde{K}(T), K, \tilde{K}\) to \(K_0, \tilde{K}_0, K, \tilde{K}\) and prove simultaneously the bounds on \(\|v^{(m)}_t\|_\infty\) and \(\|\nabla v^{(m)}_t\|_\infty\), assuming them to be true for \(m - 1\).

(i) (bound for \(v^{(m)}_t\)) As in the proof of Theorem 2.1 (ii), the case \(m = 1\) is essentially trivial: namely, using Lemma 2.1 we have for \(t \leq \tilde{K}^{-1}\)
\[
\|v^{(1)}_t\|_\infty \leq \int_0^t ds \|u^{(0)}_s\|_\infty \|\nabla u^{(0)}_s\|_\infty \leq K_0 K t \leq \tilde{K}_0(\tilde{K} t).
\]
(3.10)

So now we restrict to \(m \geq 2\).

Assume first \(t \leq (m - 1)/\tilde{K}\), so that \(t\) is in the short-time regime for \(u^{(m-1)}\). By Lemma 2.1 (see proof of Theorem 2.1 (ii)),
\[
\|v^{(m)}_t\|_\infty \leq \int_0^t ds \|v^{(m-1)}_s\|_\infty \|\nabla u^{(m-1)}_s\|_\infty
\]
\[
\leq \int_0^t ds \tilde{K}_0(\tilde{K}s/(m - 1))^{m-1} K \leq (\tilde{K}t/(m - 1))^{m} \tilde{K}_0(K/\tilde{K})
\]
\[
\leq c^{-1} \tilde{K}_0(\tilde{K}t/(m - 1))^{m} \leq \frac{1}{2} \tilde{K}_0(\tilde{K}t/m)^m
\]
(3.11)
for \(c\) large enough.
For $s, t \in [(m - 1)/\bar{K}, m/\bar{K}]$, one uses instead $\|v_s^{(m-1)}\|_{\infty} \leq \|u_t^{(m-1)}\|_{\infty} + \|u_s^{(m-2)}\|_{\infty} \leq 2K_0$ and obtains

$$
\|v_t^{(m)}\|_{\infty} \leq \int_0^{(m-1)/\bar{K}} ds \|v_s^{(m-1)}\|_{\infty} \|
abla u_s^{(m-1)}\|_{\infty} + \int_{(m-1)/\bar{K}}^{m/\bar{K}} ds \|v_s^{(m-1)}\|_{\infty} \|
abla u_s^{(m-1)}\|_{\infty}
$$

$$
\leq \frac{1}{2} \bar{K}_0(\bar{K}t/m)^m + \bar{K}^{-1} \cdot 2K_0K
$$

$$
\leq \bar{K}_0(\bar{K}t/m)^m \quad (3.12)
$$

for $c$ large enough.

(ii) (bound for $\nabla v^{(m)}_{t}$) We start from the observation (see (2.2)) that $v^{(m)}$ satisfies the transport equation $(\partial_t - \Delta + u^{(m-1)} \cdot \nabla) v^{(m)} = -v^{(m-1)} \cdot \nabla u^{(m-1)}$ and apply Schauder estimates on $Q^{(j-1)}$ as in the proof of Theorem 3.1 with $b = u^{(m-1)}$ and $f := v^{(m-1)} \cdot \nabla u^{(m-1)}$. In the course of the proof of Theorem 3.1 and in (i), we obtained $\|u^{(m-1)}\|_{\infty, Q^{(j)}} \leq K_0$ and

$$
\|u^{(m-1)}\|_{\alpha, Q^{(j-1)}} \leq \bar{K}^{1+1/2} \cdot \bar{K}_0(\bar{K}t/m)^m, \quad \|\nabla u^{(m-1)}\|_{\alpha, Q^{(j-1)}} \leq \bar{K}^{1+1/2} \cdot \bar{K}_0(\bar{K}t/m)^m. \quad (3.13)
$$

Furthermore, from Hölder interpolation inequalities (see Lemma 4.2) and induction hypothesis,

$$
\|v^{(m-1)}\|_{\alpha, Q^{(j-1)}} \leq \bar{K}_0^{1-\alpha} \bar{K}^\alpha (\bar{K}t/(m - 1))^{\beta(m-1)}. \quad (3.14)
$$

Hence (using once again the induction hypothesis)

$$
\|f\|_{\alpha, Q^{(j-1)}} \leq \|v^{(m-1)}\|_{\alpha, Q^{(j-1)}} \|\nabla u^{(m-1)}\|_{\infty, Q^{(j-1)}} + \|v^{(m-1)}\|_{\infty, Q^{(j-1)}} \|\nabla u^{(m-1)}\|_{\alpha, Q^{(j-1)}}
$$

$$
\leq (\bar{K}t/(m - 1))^{\beta(m-1)}(\bar{K}_0^{1-\alpha} \bar{K}^\alpha K + \bar{K}_0 \bar{K}^{1+1/2})
$$

$$
\leq c^{-1} \bar{K}^{3+1/2} (\bar{K}t/(m - 1))^{\beta(m-1)} \quad (3.15)
$$

A priori we should now use the Schauder estimate (4.17) to bound $\|\nabla v^{(m)}\|_{\alpha, Q^{(j-2)}}$; as in the proof of Theorem 3.1 $R^{-1}_b \leq 2$, so

$$
\|\nabla v^{(m)}\|_{\infty, Q^{(j-2)}} \leq \bar{K}^{-(1+1/2)} \|f\|_{\alpha} + \bar{K}^{1/2} \bar{K}_0 \left[ 1 + (\bar{K}^{1-1/2} \|u^{(m-1)}\|_{\alpha})^2 \right] (\bar{K}t/m)^{\beta(m-1)}
$$

$$
\leq \bar{K}^{-(1+1/2)} \|f\|_{\alpha} + \bar{K}^{1/2} \bar{K}_0(\bar{K}t/m)^{\beta(m-1)}. \quad (3.16)
$$

The second term in (3.16) is bounded by $c^{-1} \bar{K}(\bar{K}t/m)^{\beta m}$, in agreement with the desired bound (3.9), but not the first one, which is bounded by $c^{-1} \bar{K}(\bar{K}t/(m - 1))^{\beta(m-1)}$.

In order to get an integrated bound of order $(\bar{K}t/m)^{\beta m}$ for the first term, we need a refinement of Proposition 4.5. We let

$$
\bar{v}^{(m)}(t', x') := v^{(m)}(t', x') + \int_{t'}^{t} f(s, x) ds, \quad (t', x') \in Q^{(j-1)} \quad (3.17)
$$

so that $\bar{v}^{(m)}$ satisfies the modified transport equation

$$
(\partial_{t'} - \Delta + v^{(m-1)} \cdot \nabla) \bar{v}^{(m)}(t', x') = \bar{f}(t', x') \quad (3.18)
$$
with
\[ \tilde{f}(t', x') := f(t', x') - f(t', x). \] (3.19)

This introduces the following modifications. First, letting \( \bar{B}^{(j-1)} := B(x, M^{j-1/2}) \),
\[ \|v^{(m)} - \tilde{v}^{(m)}\|_{\infty, Q^{(j-1)}} \leq \int_{J_{M-1}} ds\|f(s)\|_{\infty, \bar{B}^{(j-1)}} \leq \tilde{K}_0(\tilde{K}t/m)^{\beta m} \] (3.20)
as follows from (3.11), (3.12). Thus \( \|v^{(m)}\|_{\infty, Q^{(j-1)}} \leq \tilde{K}_0(\tilde{K}t/m)^{\beta m} \) is bounded like \( \|v^{(m)}\|_{\infty, Q^{(j-1)}} \).

Second (see (4.25)), \( \tilde{f}(t', x') - \tilde{f}(t, x) = f(t', x') - f(t', x) \) involves values of \( f \) only at time \( t' \).

We now go through the proof of Proposition 4.5 writing \( \tilde{v}^{(m)} \) as the sum of a series \( \tilde{v}^{(m)}_0 + \sum_{k=1}^{\infty} (\tilde{v}^{(m)}_k - \tilde{v}^{(m)}_{k+1}) \). Instead of (4.26), we get from the maximum principle
\[ \sup_{Q^{(j-2)-}} \|\tilde{v}^{(m)}_{k+1} - \tilde{v}^{(m)}_k\| \leq M^{(j-2)(1+\alpha/2)} \left( \int_{t-M^{j-2-k}}^t ds\|f(s)\|_{\alpha, \bar{B}^{(j-2-k)}} + \|u^{(m-1)}\|_{\alpha} \sup_{Q^{(j-2)}} \nabla \tilde{v}^{(m)} \right), \] (3.21)
where \( \int_{t-M^{j-2-k}}^t (\cdot ) ds := M^{-(j-2-k)} \int_{t-M^{j-2-k}}^t (\cdot ) ds \) is the average over the time interval \( [t - M^{j-2-k}, t] \). We have proved above that \( \|f(s)\|_{\alpha, \bar{B}^{(j-1)}} \leq c^{-1} K^{(3+\alpha)/2}(Kt/(m-1))^{\beta(m-1)} \); thus (by explicit computation)
\[ \int_{t-M^{j-2-k}}^t ds\|f(s)\|_{\alpha, \bar{B}^{(j-2-k)}} \leq c^{-1} K^{(3+\alpha)/2}(Kt/(m-1))^{\beta(m-1)} a_k, \] (3.22)
with \( a_k := M^{k/j-t^{\beta(m-1)/2}-1}(Kt/m)^{\beta(m-1)+1} \). Let \( k_0 := \inf\{k \geq 0; M^{j-2-k} < t/m\} \); the case \( k = 0 \) corresponds to \( t \) close to its maximal value \( m/\tilde{K} \), hence \( M^{j-2-k_0} \approx t/m \).

For \( k > k_0, a_k \approx 1 \), as follows from Taylor’s formula; bounding all \( a_k, k \geq 1 \) by \( 1 \) would yield the estimate (3.16). However, for \( k \leq k_0, a_k \leq M^{k/j-t/2} \), which is a much better bound for \( k_0 - k \) large. Summarizing, the only change in the right-hand side of (4.30) is that \( \|f\|_{\alpha} \) may be replaced by
\[ \sum_{k=0}^{\infty} M^{-\alpha k/2} \int_{t-M^{j-2-k}}^t ds\|f(s)\|_{\alpha, \bar{B}^{(j-2-k)}} \leq c^{-1} K^{(3+\alpha)/2}(Kt/(m-1))^{\beta(m-1)}(A_1 + A_2), \] (3.23)
where
\[ A_1 := \sum_{k=k_0}^{\infty} M^{-\alpha k/2} \leq (Kt/m)^{\alpha/2} \] (3.24)
and similarly
\[ A_2 := \sum_{k=0}^{k_0-1} M^{-\alpha k/2} M^{k-j} \frac{t}{m} \leq M^{\alpha(1-\alpha/2)}(Kt/m) \approx (Kt/m)^{\alpha/2}. \] (3.25)

All together, with respect to the rougher bound (3.16), we have gained a small multiplicative factor of order \( A_1 + A_2 \leq (Kt/m)^{\beta} \), with \( \beta := \alpha/2 \). Thus
\[ \|\nabla \tilde{v}^{(m)}\|_{\infty, Q^{(j-2)}} \leq c^{-1} K(Kt/(m-1))^{\beta(m-1)}(Kt/m)^{\beta} + c^{-1} K(Kt/m)^{\beta m} \leq c^{-1} K(Kt/m)^{\beta m}. \] (3.26)
4 Hölder estimates

We prove in this section elementary Hölder estimates, together with a precise form of the Schauder estimates which is crucial in the proof of Theorem [1.1] in section 3.

**Definition 4.1 (Hölder semi-norms)** Let $\gamma \in (0, 1)$.

1. $f_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is $\gamma$-Hölder continuous if $\|f_0\|_\gamma := \sup_{x,x' \in \mathbb{R}^d} \frac{|f_0(x) - f_0(x')|}{|x - x'|^{\gamma}} < \infty$.

2. $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ is $\gamma$-Hölder continuous if $\|f\|_\gamma := \sup_{(t,x),(t',x') \in \mathbb{R}_+ \times \mathbb{R}^d} \frac{|f(t,x) - f(t',x')|}{|t - t'|^{\gamma}} < \infty$.

Note that $\|\cdot\|_\gamma$ is only a semi-norm since $\|1\|_\gamma = 0$. We also define Hölder semi-norms for functions restricted to $Q_0 \subset \mathbb{R}_+ \times \mathbb{R}^d$ or $Q \subset \mathbb{R}^d$ compact, with the obvious definitions,

$$\|f_0\|_{\gamma, Q_0} := \sup_{x,x' \in Q_0} \frac{|f_0(x) - f_0(x')|}{|x - x'|^{\gamma}}, \quad \|f\|_{\gamma, Q} := \sup_{(t,x),(t',x') \in Q} \frac{|f(t,x) - f(t',x')|}{|x - x'|^{\gamma} + |t - t'|^{\gamma}}.$$  \hspace{1cm} (4.1)

**Remark.** For $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$, we use in this article either the parabolic Hölder semi-norm $\|f\|_{\alpha, Q_0}$ or the isotropic Hölder semi-norm $\|f(t)\|_{\alpha, Q_0}$ for $t \in \mathbb{R}_+$ fixed. The distinction is really important in the proof of Theorem [5,2](ii). Clearly, $\|f(t)\|_{\alpha, Q_0} \leq \|f\|_{\alpha, t \times Q_0}$ if $I$ is some time interval containing $t$.

**Lemma 4.2 (Hölder interpolation estimates)**

1. (on $\mathbb{R}^d$) Let $Q_0 \subset \mathbb{R}$ be a convex set, and $u_0 : Q_0 \rightarrow \mathbb{R}$ such that $\|u_0\|_{\infty, Q_0}, \|\nabla u_0\|_{\infty, Q_0} < \infty$. Then

$$\|u_0\|_{\alpha, Q_0} \leq \|u_0\|_{\alpha, Q_0}^{\frac{1-\alpha}{\alpha}} \|\nabla u_0\|_{\infty, Q_0}^{\alpha}, \quad \alpha \in (0, 1).$$ \hspace{1cm} (4.2)

2. (on $\mathbb{R}_+ \times \mathbb{R}^d$) Let $Q \subset \mathbb{R}_+ \times \mathbb{R}^d$ be a convex set, and $u : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\|u\|_{\infty, Q}, \|\nabla u\|_{\infty, Q}, \|\partial_t u\|_{\infty, Q} < \infty$. Then

$$\|u\|_{\alpha, Q} \leq 2 \left( \|u\|_{\infty, Q}^{1-\alpha} \|\nabla u\|_{\infty, Q}^{\alpha} + \|u\|_{\infty, Q}^{1-\alpha/2} \|\partial_t u\|_{\infty, Q}^{\alpha/2} \right), \quad \alpha \in (0, 1).$$ \hspace{1cm} (4.3)

**Proof.** (see [9]) we prove (ii). Let $X = (t, x)$ and $X' = (t', x')$ in $Q$, then

$$|u(X) - u(X')| = \left| \int_0^1 \frac{d}{d\tau} u((1 - \tau)X + \tau X') d\tau \right| \leq |t - t'| \|\partial_t u\|_{\infty, Q} + |x - x'| \|\nabla u\|_{\infty, Q} \leq 2 \max (|t - t'|, |\partial_t u|_{\infty, Q}, |x - x'|) \|\nabla u\|_{\infty, Q}. \hspace{1cm} (4.4)$$

On the other hand, $|u(X) - u(X')| \leq 2\|u\|_{\infty}$. Hence

$$|u(X) - u(X')| \leq 2 \max \left( \|u\|_{\infty, Q}^{1-\alpha/2} \|\partial_t u\|_{\infty, Q}^{\alpha/2}, \|u\|_{\infty, Q}^{1-\alpha} \|\nabla u\|_{\infty, Q}^{\alpha} \right). \hspace{1cm} (4.5)$$

□

**Lemma 4.3** Let $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ be $\alpha$-Hölder. Then

$$\|\nabla (e^{\Delta} u_0)\|_{\infty} \leq C(d, \kappa, \alpha) \|\partial_\alpha u_0\|_{\kappa} \quad (\kappa \geq 1); \hspace{1cm} (4.6)$$

$$\|\nabla^2 (e^{\Delta} u_0)\|_{\gamma} \leq C' (d, \gamma, \alpha) \|\partial_\alpha u_0\|_{\alpha} \quad (\gamma \in (0, 1)); \hspace{1cm} (4.7)$$

$$\|e^{\Delta} u_0 - e^{\Delta} u_0\|_{\infty} \leq C''(d, \alpha) (t - t')^{\alpha/2} \|u_0\|_{\alpha} \quad (\alpha \in (0, 1), t > t' > 0). \hspace{1cm} (4.8)$$
Proof. (4.7) follows by Lemma 4.2 from the bounds (4.6) with $\kappa = 2, 3$. Thus let us first prove (4.6). The regularizing operator $e^{i\Lambda}$ is defined by convolution with respect to the heat kernel $p_t$. By translation invariance, it is enough to bound the quantity $I(\varepsilon) := \nabla^{s-1}(e^{i\Lambda}u_0)(0) - \nabla^{s-1}(e^{i\Lambda}u_0)(0)$ in the limit $\varepsilon \to 0$. The quantities in (4.6) are invariant through the substitution $u_0 \to u_0 - u_0(0)$, so we assume that $u_0(0) = 0$. We may also assume $|\varepsilon| \ll \sqrt{t}$. Let $A := e^{\beta t(1-\beta)/2}$ with $\beta = (1 - \alpha)/d$; note that $|\varepsilon| \ll A \ll \sqrt{t}$. We split the integral into three parts, $I(\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)$, with

\[ I_1(\varepsilon) := \int_{|x| < A} dx \nabla^{s-1} p_t(x)(u_0(x) - u_0(0)), \quad I_2(\varepsilon) := \int_{|x| > A} dx (\nabla^{s-1} p_t(x) - \nabla^{s-1} p_t(x + \varepsilon))(u_0(x) - u_0(0)) \]

\[ I_3(\varepsilon) = \left( \int_{|x| > A} dx - \int_{|x - \varepsilon| > A} dx \right) \nabla^{s-1} p_t(x + \varepsilon)(u_0(x) - u_0(0)). \]

We use $|u_0(x) - u_0(x + \varepsilon)| \leq ||u_0||_a |\varepsilon|^\alpha$ in the first integral, and get

\[ I_1(\varepsilon) \leq ||u_0||_a A^d \int_r A^{-(s-k-1)/2} |\varepsilon|^{\alpha} = ||u_0||_a e^{(\alpha-k)/2} |\varepsilon|. \]

For the second integral, we use $|\nabla^{s-1} p_t(x) - \nabla^{s-1} p_t(x + \varepsilon)| \leq \frac{|\varepsilon|}{2} p_t(x)$ and $|u_0(x) - u_0(0)| \leq ||u_0||_a |\varepsilon|^\alpha$, yielding the same estimate. Finally, (4.8) may be obtained through the use of the fractional derivative $u_0 \mapsto \int d\xi |\xi|^{s-\alpha} e^{i(x-y)\xi} u_0(y)$, hence

\[ I_3(\varepsilon) \leq ||u_0||_a A^{d-1} |\varepsilon|^{(s-k-1)/2} A^\alpha \leq ||u_0||_a A^{d+1} e^{(\alpha-k)/2} |\varepsilon|. \]

Finally, (4.8) may be obtained through the use of the fractional derivative

\[ |\nabla|^\alpha : u_0 \mapsto \left( |\nabla|^\alpha u_0 : x \mapsto \int d\xi |\xi|^{s-\alpha} e^{i(x-y)\xi} u_0(y) \right), \]

namely,

\[ |(e^{i\Lambda}u_0 - e^{i\Lambda}u_0)(x)| = \left| \int_r ds \int dy \partial_s p_s(x - y)u_0(y) \right| = \left| \int_r ds \int dy \Delta p_s(x - y)u_0(y) \right| \leq \int_r ds \left| \nabla |^{2-\alpha} p_s(x - y) \right| ||u_0||_a \leq (r^{\alpha/2} - (t')^{\alpha/2}) ||u_0||_a. \]

\[ \tag{4.12} \]

\[ \square \]

**Corollary 4.4** Let $g : [0, t] \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function such that $(g_s)_{s \in [0, t]}$ are uniformly $\alpha$-Hölder, and $\gamma < \alpha$. Then $s \mapsto ||\nabla^2(e^{(t-s)\Lambda}g_s)||_y$ is $L^1_{loc}$ and, for $0 < t' < t$,

\[ \int_r^t ds \left| \nabla^2(e^{(t-s)\Lambda}g_s)||_y \right| \leq C'(d, \gamma, \alpha)(t - t')^{(\alpha - \gamma)/2} \sup_{s \in [t', t]} ||g_s||_a. \]

\[ \tag{4.13} \]

We may now state a very precise version of Schauder estimates, inspired by [12]. We fix a constant $M > 1$, e.g. $M = 2$ for a dyadic scale decomposition.

**Proposition 4.5** Let $v$ solve the linear parabolic PDE

\[ (\partial_t - \Delta)u(t, x) = b(t, x) \cdot \nabla u(t, x) + f(t, x) \]

\[ \tag{4.14} \]
on the "parabolic ball" $Q^{(j)} = Q^{(j)}(t_0, x_0) := \{(t, x) \in \mathbb{R} \times \mathbb{R}^d; t_0 - M^j \leq t \leq t_0, x \in \bar{B}(x_0, M^{j/2})\}$, with given initial-boundary value $u\mid_{\partial_{\text{par}} Q^{(j)}} = \nu$, where $\partial_{\text{par}} Q^{(j)} := \{(t_0 - M^j) \times B(x_0, M^{j/2}) \cup \{(t_0 - M^j, t_0) \times \partial B(x_0, M^{j/2})\}$ is the parabolic boundary of $Q^{(j)}$. If $\nu$ (hence $u$) is bounded and

$$\|f\|_{\alpha, Q^{(j)}} := \sup_{(t, x), (t', x') \in Q^{(j)}} \frac{|f(t, x) - f(t', x')|}{|x - x'|^{\alpha} + |t - t'|^{\alpha/2}} < \infty$$

for some $\alpha \in (0, 1)$, and similarly $\|b\|_{\alpha, Q^{(j)}} < \infty$, then

$$\sup_{Q^{(j-1)}} |\nabla u| \leq M^{j/2} R^{-1}_b \left( M^{j/2} \|f\|_{\alpha, Q^{(j)}} + (M^{j/2} R^{-1}_b \|b\|_{\alpha, Q^{(j)}} + M^{-j}) \sup_{Q^{(j)}} |u| \right), \tag{4.16}$$

$$\|\nabla u\|_{\alpha, Q^{(j-1)}} \leq R_b^{(-(1+\alpha)/2)} \left( M^{j/2}(1 + R^{-1}_b \|b\|_{\alpha, Q^{(j)}}) \|f\|_{\alpha, Q^{(j)}} + (R^{-1}_b \|b\|_{\alpha, Q^{(j)}} + M^{j/2} R^{-2}_b \|b\|_{\alpha, Q^{(j)}} + (M^{-j})^{1+\alpha}) \sup_{Q^{(j)}} |u| \right), \tag{4.17}$$

$$\sup_{Q^{(j-1)}} |\partial_t u|, \sup_{Q^{(j-1)}} |\nabla^2 u| \leq R_b \left( M^{j/2} \|f\|_{\alpha, Q^{(j)}} + (M^{j/2} R^{-1}_b \|b\|_{\alpha, Q^{(j)}} + M^{-j}) \sup_{Q^{(j)}} |u| \right), \tag{4.18}$$

and for every $\alpha' > \alpha$,

$$\|\partial_t u\|_{\alpha', Q^{(j-1)}}, \|\nabla^2 u\|_{\alpha', Q^{(j-1)}} \leq M^{-j/2} R^{-1}(1+\alpha'/2) R^{-1}_b \left( M^{j/2} \|f\|_{\alpha, Q^{(j)}} + (R^{-1}_b \|b\|_{\alpha, Q^{(j)}} + M^{j/2} R^{-2}_b \|b\|_{\alpha, Q^{(j)}} + (M^{-j})^{1+\alpha}) \sup_{Q^{(j)}} |u| \right), \tag{4.19}$$

where $R_b := (1 + M^{j/2} \|b(t_0, x_0)\|)^{-1}$.

**Proof.** Let $\tilde{u}(\tilde{t}, \tilde{x}) := u(M^{j/2}\tilde{t}, M^{j/2}\tilde{x})$, $\tilde{b}(\tilde{t}, \tilde{x}) := M^{j/2}b(M^{j/2}\tilde{t}, M^{j/2}\tilde{x})$, $\tilde{f}(\tilde{t}, \tilde{x}) := M^{j/2}f(M^{j/2}\tilde{t}, M^{j/2}\tilde{x})$. The PDE $(\partial_t - \Delta)u = b \cdot \nabla u + f$ on $Q^{(j)}$ reduces to an equivalent PDE $(\partial_{\tilde{t}} - \tilde{\Delta})\tilde{u} = \tilde{b} \cdot \nabla \tilde{u} + \tilde{f}$ on a parabolic ball $\tilde{Q}$ of size unity. Assume (leaving out for sake of conciseness the powers of $R_b = (1 + \|b(t_0, x_0)\|)^{-1}$) that we have proved an inequality of the type

$$\sup_{\tilde{Q}^{(j-1)}} |\nabla^k \tilde{u}| \leq \left( \|\tilde{f}\|_{\alpha} + (\|\tilde{b}\|_{\alpha}^\beta + 1) \sup_{\tilde{Q}} |\tilde{u}| \right), \tag{4.20}$$

resp.

$$\|\nabla^k \tilde{u}\|_{\alpha, \tilde{Q}^{(j-1)}} \leq \left( \|\tilde{f}\|_{\alpha} + (\|\tilde{b}\|_{\alpha}^\beta + 1) \sup_{\tilde{Q}} |\tilde{u}| \right). \tag{4.21}$$

By rescaling, we get

$$\sup_{Q^{(j-1)}} |\nabla^k u| \leq (M^{-j/2})^k \left( M^{j(1+\alpha/2)} \|\tilde{f}\|_{\alpha} + ((M^{j/2})^{1+\alpha} \|\tilde{b}\|_{\alpha}^\beta + 1) \sup_{\tilde{Q}} |\tilde{u}| \right), \tag{4.22}$$

resp.
\[ \| \nabla^k u \|_{\alpha, Q^{j-1}} \leq (M^{-j/2})^{k+\alpha} \left( M^{k(1+\alpha/2)} \| f \|_\alpha + \left( (M^{j/2})^{1+\alpha} \| b \|_\alpha \right)^2 + 1 \right) \sup_{Q^{j-0}} |u| . \] (4.23)

This gives the correct scaling factors in (4.164), (4.174), (4.184), (19). Thus we may assume that \( j = 0 \). In the sequel we write for short \( \| \cdot \|_\alpha \) instead of \( \| \cdot \|_{\alpha, Q^{j-0}} \) and \( \| \cdot \|_{\infty} \) instead of \( \sup_{Q^{j-0}} | \cdot | \).

The general principle underlying the proof of the Schauder estimates in [12] is the following. One rewrites \( u \) as the sum of the series \( u = u_0 + \sum_{k=1}^{+\infty} (u_{k+1} - u_k) \), where \( u_k, k \geq 0 \) is the solution on \( Q^{(k)} \) of the ’frozen’ PDE

\[ (\partial_t - \Delta) u_k(t, x) = b(t_0, x_0) \cdot \nabla u_k(t, x) + f(t_0, x_0) \] (4.24)

with initial-boundary condition \( u_k \big|_{\partial_{\text{par}} Q^{(k-1)}} = u \big|_{\partial_{\text{par}} Q^{(k-1)}} \). We split the proof into several steps.

(i) (estimates for \( |u_{k+1} - u_k| \)) One first remarks that \( u_k - u \) solves on \( Q^{(k)} \) the heat equation

\[ (\partial_t - \Delta - b(t_0, x_0) \cdot \nabla) (u_k - u) = (b(t_0, x_0) - b) \cdot \nabla u + (f(t_0, x_0) - f) \] (4.25)

with zero initial-boundary condition \( (u_k - u) \big|_{\partial_{\text{par}} Q^{(k-1)}} = 0 \), implying by the maximum principle

\[ \sup_{Q^{(k-1)}} |u_{k+1} - u_k| \leq \sup_{Q^{(k-1)}} |u_{k+1} - u| + \sup_{Q^{(k-1)}} |u_k - u| \leq M^{-k(1+\alpha/2)} \left( \| f \|_\alpha + \| b \|_\alpha \sup_{Q^{(k-1)}} |\nabla u| \right). \] (4.26)

(ii) (estimates for higher-order derivatives of \( u_0 \)) Recall \( u_0 \) is a solution of the heat equation \( (\partial_t - \Delta - b(t_0, x_0) \cdot \nabla) u_0 = f(t_0, x_0) \) with initial-boundary condition \( u_0 \big|_{\partial_{\text{par}} Q^{(0)}} = u \big|_{\partial_{\text{par}} Q^{(0)}} \).

Assume first \( |b(t_0, x_0)| \leq 1 \). As follows from standard estimates,

\[ \| \nabla u_0 \|_{\alpha, Q^{-1}}, \sup_{Q^{-1}} |\partial_t u_0|, \sup_{Q^{-1}} |\nabla^2 u_0|, \| \nabla^2 u_0 \|_{\alpha, Q^{-1}} \leq \sup_{\partial_{\text{par}} Q^{(0)}} |u| \leq \| u \|_{\infty}. \] (4.27)

If \( |b(t_0, x_0)| \gg 1 \), then one makes the Galilean transformation \( x \mapsto x - b(t_0, x_0) t \) to get rid of the drift, after which the boundary of \( Q^{(0)} \) lies at distance \( R = O(1/|b(t_0, x_0)|) \) instead of \( O(1) \) of \( (t_0, x_0) \); thus, in general,

\[ \| \nabla u_0 \|_{\alpha, Q^{-1}} \leq R_b^{-1} \| u \|_{\infty}, \quad \sup_{Q^{-1}} |\partial_t u_0|, \sup_{Q^{-1}} |\nabla^2 u_0| \leq R_b^{-1} \| u \|_{\infty}, \quad \| \nabla^2 u_0 \|_{\alpha, Q^{-1}} \leq R_b^{-1} \| u \|_{\infty}. \] (4.28)

(iii) (estimates for higher-order derivatives of \( u_{k+1} - u_k \)) Similarly to (ii), we note that \( u_{k+1} - u_k \) is a solution on \( Q^{(k-1)} \) of the heat equation \( (\partial_t - \Delta - b(t_0, x_0) \cdot \nabla) (u_{k+1} - u_k) = 0 \). Thus (by parabolic rescaling)

\[ \sup_{Q^{(k-2)}} |\partial_t (u_{k+1} - u_k)|, \sup_{Q^{(k-2)}} |\nabla^2 (u_{k+1} - u_k)| \leq M^k R_b^{-1} \sup_{Q^{(k-1)}} |u_{k+1} - u_k|, \]

\[ \| \nabla^2 (u_{k+1} - u_k) \|_{\alpha', Q^{(k-2)}} \leq (M^k)^{1+\alpha'/2} R_b^{-1} \sup_{Q^{(k-1)}} |u_{k+1} - u_k| \] (4.29)

is bounded using (i) in terms of \( R_b, \| b \|_\alpha, \| f \|_\alpha \) and \( \sup_{Q^{(k-1)}} |\nabla u| \).
(iv) (Schauder estimates for higher-order derivatives of $u$) Summing up the estimates in (i), (ii), (iii), one obtains

$$\sup_{Q^{(k)}} |\partial_t u|, \sup_{Q^{(k)}} |\nabla^2 u| \leq R_b^{-1} \left( \|f\|_s + \|b\|_s \sup_{Q^{(k)}} |\nabla u| + \|u\|_\infty \right).$$

(4.30)

To bound $\|\nabla^2 u\|_{\alpha, Q^{(k)}}$, one may take $(t, x, t', x') \in Q^{(k)}$ of $(t, x) \in Q^{(k)}$, or equivalently $\|\partial_t u\|_{\alpha, Q^{(k)}}$, one first notes that one may choose $(t_0, x_0)$ to be the mid-point $(\frac{1}{2}(t + t'), \frac{1}{2}(x + x'))$. Letting $k_0$ be the largest integer $k$ such that $(t, x), (t', x') \in Q^{(k)}$, so that $d_{\text{par}}(t, x, t', x') \approx M^{-k_0/2}$, one has $|\nabla^2 u(t, x) - \nabla^2 u(t', x')| \leq I_1 + I_2 + I_3 + I_4$, with

$$I_1 = |\nabla^2 u_0(t, x) - \nabla^2 u_0(t', x')| \leq R_b^{-1} \|u\|_\infty d_{\text{par}}(t, x, t', x')^{\alpha};$$

(4.31)

$$I_2 = \sum_{k=1}^{k_0} |\nabla^2 (u_k - u_{k-1})(t, x) - \nabla^2 (u_k - u_{k-1})(t', x')|$$

$$\leq R_b^{1-(\alpha'/2)} d_{\text{par}}(t, x, t', x')^{\alpha'} \left( \sum_{k=1}^{k_0} (M^{k/2})^{\alpha'-\alpha} \right) \left( \|f\|_s + \|b\|_s \sup_{Q^{(k)}} |\nabla u| \right)$$

$$\leq d_{\text{par}}(t, x, t', x')^{\alpha} R_b^{-(1+\alpha'/2)} \left( \|f\|_s + \|b\|_s \sup_{Q^{(k)}} |\nabla u| \right);$$

(4.32)

and

$$I_3 = \sum_{k \geq k_0} |\nabla^2 (u_{k+1} - u_k)(t, x)|, I_4 = \sum_{k \geq k_0} |\nabla^2 (u_{k+1} - u_k)(t', x')|$$

(4.33)

are bounded as in (4.30) with a supplementary multiplicative factor $O(M^{-k_0/2}) \approx d_{\text{par}}(t, x, t', x')^{\alpha}$. Hence

$$\|\partial_t u\|_{\alpha, Q^{(k)}} \leq R_b^{-1} \left( \|f\|_s + \|b\|_s \sup_{Q^{(k)}} |\nabla u| + \|u\|_\infty \right).$$

(4.34)

By standard Hölder interpolation inequalities [9],

$$\sup_{Q^{(k)}} |\nabla u| \leq \|\nabla^2 u\|^{1/(2+\alpha)/2}_{\alpha, Q^{(k)}} \|u\|^{(1+\alpha)/(2+\alpha)}_{\alpha, Q^{(k)}} \leq R_b^{1+\alpha'}/\|b\|_s \|u\|_\infty$$

(4.35)

for every $\epsilon > 0$. Choosing $\epsilon^{2+\alpha} \approx R_b^{1+\alpha'/2}/\|b\|_s$ yields (4.19). Similarly (letting formally $\alpha' = 0$ in the above inequalities),

$$\sup_{Q^{(k)}} |\nabla u| \leq R_b \|b\|_s^{-1} \sup_{Q^{(k)}} |\nabla^2 u| + R_b^{-1} \|b\|_s \|u\|_\infty,$$

(4.36)

yielding (4.18).

In order to obtain the bound (4.17), we proceed initially in the same way, with the only difference that one may take $\alpha' = \alpha$ in (4.32) since one gets a series $\sum_{k=1}^{k_0} M^{-k/2}$ of order $O(1)$. Thus (4.34) becomes

$$\|\nabla u\|_{\alpha, Q^{(k)}} \leq R_b^{-1} \left( \|f\|_s + \|b\|_s \sup_{Q^{(k)}} |\nabla u| + \|u\|_\infty \right).$$

(4.37)
One now uses (4.36) to bound $\nabla u$, to the result that

$$
\|\nabla u\|_{\alpha, Q^{-1}} \leq R_b^{-(1+\alpha)/2} \left( \|f\|_\alpha + R_b^{-1}\|b\|_\alpha \left( \|f\|_\alpha + \left( R_b^{-1}\|b\|_\alpha^2 + 1 \right) \|u\|_\infty \right) \right) + \|u\|_\infty)
$$

$$
\leq R_b^{-(1+\alpha)/2} \left( \left( 1 + R_b^{-1}\|b\|_\alpha \|f\|_\alpha + \left( R_b^{-1}\|b\|_\alpha (R_b^{-1}\|b\|_\alpha^2 + 1) + 1 \right) \|u\|_\infty \right) \right).
$$

(4.38)

yielding (4.17) after rescaling.

(v) Finally, using the estimates (4.18) and the interpolation inequality

$$
\sup_{Q^{-1}} |\nabla u| \leq \|u\|_\infty + \sup_{Q^{-1}} |\nabla^2 u|
$$

(4.39)

yields (4.16).

□

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