List-edge-colouring planar graphs with precoloured edges

Joshua Harrelson∗ Jessica McDonald† Gregory J. Puleo‡

Department of Mathematics and Statistics
Auburn University
Auburn, AL 36849

Abstract

Let $G$ be a simple planar graph of maximum degree $\Delta$, let $t$ be a positive integer, and let $L$ be an edge list assignment on $G$ with $|L(e)| \geq \Delta + t$ for all $e \in E(G)$. We prove that if $H$ is a subgraph of $G$ that has been $L$-edge-coloured, then the edge-precolouring can be extended to an $L$-edge-colouring of $G$, provided that $H$ has maximum degree $d \leq t$ and either $d \leq t - 4$ or $\Delta$ is large enough ($\Delta \geq 16 + d$ suffices). If $d > t$, there are examples for any choice of $\Delta$ where the extension is impossible.

1 Introduction

In this paper all graphs are simple.

An edge-colouring of $G$ is an assignment of colours to the edges of $G$ so that adjacent edges receive different colours; if at most $k$ colours are used we say it is a $k$-edge-colouring. The chromatic index of $G$, denoted $\chi'(G)$, is the minimum $k$ such that $G$ is $k$-edge-colourable. It is obvious that $\chi'(G) \geq \Delta$, where $\Delta := \Delta(G)$ is the maximum degree of $G$, and Vizing’s Theorem [12] says that $\chi'(G) \leq \Delta + 1$.

In this paper we are looking to edge-colour a graph $G$, but with the constraint that some edges have already been coloured and cannot be changed. In this scenario we have no control over the edge-precolouring – if the edge-precoloured subgraph is $H$, then it will certainly have at least $\chi'(H)$ colours, but it could have many more, perhaps even more than $\chi'(G)$ colours. If we are looking to extend the edge-precolouring to a $k$-edge-colouring of $G$, then we will certainly need that $k$ is at least the maximum degree of $G$, and that the edge-colouring of $H$ uses at most $k$ colours (i.e. is a $k$-edge-colouring). In general we consider the following question, first posed by Marcotte and Seymour [9]:

**Question 1.** Given a graph $G$ with maximum degree $\Delta$ and a subgraph $H$ of $G$ that has been $(\Delta + t)$-edge-coloured, can the edge-precolouring of $H$ be extended to a $(\Delta + t)$-edge-colouring of $G$?

∗jth0048@auburn.edu
†mcdonald@auburn.edu; Supported in part by NSF grant DMS-1600551.
‡gjp0007@auburn.edu
Marcotte and Seymour’s main result in [9] is a necessary condition for the answer to Question 1 to be “yes”; they prove that this condition is also sufficient when $G$ is a multifold (the condition is rather technical, so we do not state it here). Question 1 was shown to be NP-complete by Colbourn [4], and Marx [10] showed that this is true even when $G$ is a planar 3-regular bipartite graph. Since, as Holyer [7] showed, it is NP-complete to decide whether $\chi'(G) = \Delta(G)$ or not, the special case $t = 0$ of Question 1 is also NP-complete for general graphs. In this paper we focus on Question 1 for planar graphs. Before saying more about planar graphs in particular however, let us make several quick observations about Question 1 in general.

Firstly, if $t$ is huge – say at least $\Delta - 1$ – then the answer is $\text{yes}$, and moreover, the extension can be done greedily. This is because an edge in $G$ sees at most $2(\Delta - 1)$ other edges, and when $t \geq \Delta - 1$, this value is at most $\Delta + t - 1$. If the maximum degree of $H$ is $\Delta$ then this threshold for $t$ is actually sharp. To see this, consider the graph $G$ shown in Figure 1 formed by taking a copy of $K_{1,\Delta}$ with one edge coloured $\Delta$ and the rest uncoloured, and joining each leaf to $\Delta - 1$ distinct new vertices via edges coloured $1, 2, \ldots, \Delta - 1$. Then $G$ has maximum degree $\Delta$, as does its edge-precoloured subgraph. However, in order to extend the edge-precolouring to a $(\Delta + t)$-edge-colouring of $G$, we need $\Delta - 1$ new colours, which forces $t \geq \Delta - 1$.

Given the above paragraph, Question 1 is only interesting when the maximum degree of $H$, say $d$, is strictly less than $\Delta$. Here, we get a natural barrier to extension when $d > t$, via nearly the same example as above. Let $G$ be the graph shown in Figure 2 formed by taking an (uncoloured) copy of $K_{1,\Delta}$ and joining each leaf to $d < \Delta$ distinct new vertices, via edges coloured $1, 2, \ldots, d$. The resulting graph $G$ has maximum degree $\Delta$, and contains a precoloured subgraph $H$ with maximum degree $d$. However, in order to extend the edge-precolouring to $G$, we need $\Delta$ new colours, meaning that for a $(\Delta + t)$-edge-colouring of $G$, we need $d \leq t$.

If it happened that $H$ was edge-coloured efficiently (i.e. using at most $\chi'(H)$ colours), then our problem would be significantly reduced. In this special situation, one could use a completely new set of $\chi'(G - E(H))$ colours to extend to an edge-colouring of $G$ with at most the following number of colours (according to Vizing’s Theorem):

$$\chi'(G - E(H)) + \chi'(H) \leq \chi'(G) + \chi'(H) \leq \Delta + d + 2.$$  

That is, when $H$ has been edge-coloured efficiently, the answer to Question 1 is $\text{yes}$ whenever $d \leq t - 2$. Since extension can be impossible when $d > t$ (according to the
Figure 2: A graph $G$ with maximum degree $\Delta = 4$ and a precoloured subgraph of maximum degree $d = 2$. In order to extend the edge-precolouring to a $(\Delta + t)$-edge-colouring of $G$ we need $t \geq d$. (In fact, we actually prove a stronger result involving list edge colouring, stated at the end of this section as Theorem 5, but more exposition is required to properly state and contextualize the stronger result.)

Theorem 2. Let $G$ be a planar graph of maximum degree at most $\Delta$, let $t$ be a positive integer, and let $H$ be a subgraph of $G$ that has been $(\Delta + t)$-edge-coloured. If $H$ has maximum degree at most $d$, then the edge-precolouring can be extended to a $(\Delta + t)$-edge-colouring of $G$ provided that either:

1. $d \leq t - 4$, or
2. $t - 3 \leq d \leq t$ and

$$
\Delta \geq \begin{cases}
16 + d, & \text{if } d = t, \\
9 + d, & \text{if } d = t - 1, \\
8 + d, & \text{if } d = t - 2, \\
7 + d, & \text{if } d = t - 3.
\end{cases}
$$
Theorem 2 does not include the case \( t = 0 \), however the requirement of \( d \leq t \) means that would correspond to \( H \) being edgeless. Then the problem is not about precolouring at all, but simply about edge-colouring planar graphs as discussed above.

The case \( d = t = 1 \) of Theorem 5 was previously established by Edwards, Girão, van den Heuvel, Kang, Sereni and the third author [5], with the slightly stronger assumption of \( \Delta \geq 19 \). (Note that the restriction of our proof for Theorem 2 to this case provides a somewhat new proof; both arguments use global discharging, but we discharge in a different way). After the seminal work of Marcotte and Seymour [9], the vertex-version of the precolouring extension problem received much more attention than Question 1. Edwards et al. [5] re-initiated this study in their paper, with planar graphs being only one of the many families they considered. The main concern in [5] however is when \( H \) is a matching, and in order to guarantee extensions they often impose distance conditions on the edges in the precoloured matching. In particular, this means avoiding the issues with \( t \) being too small as exhibited in Figures 1 and 2. Specifically, in addition to the aforementioned result for \( d = t = 1 \), they showed that if \( H \) is an edge-precoloured matching in a planar graph \( G \) where edges are at distance at least 3 from one another, then any \( \Delta \)-edge-colouring on \( H \) can be extended to \( G \) provided \( \Delta \geq 20 \). More recently, Girão and Kang [6] studied extension from precoloured matchings in general graphs, proving that if \( H \) is a matching in a (not necessarily planar) graph \( G \) where edges are distance at least 9 from each other, then any \((\Delta+1)\)-edge-colouring on \( H \) can be extended to a \((\Delta + 1)\)-edge-colouring of \( G \).

As Edwards et al. [5] observed, extending an edge-colouring is closely related to list-edge-colouring. An edge list assignment on a graph \( G \) is a function \( L \) that assigns to each edge \( e \in E(G) \) a list of colours \( L(e) \). If \( L \) is an edge list assignment on a graph \( G \), an \( L \)-edge-colouring of \( G \) is an edge-colouring of \( G \) such that every edge \( e \) is given a colour from \( L(e) \). Note that a classical \( k \)-edge-colouring of \( G \) can be viewed as an \( L \)-edge-colouring for the list assignment \( L \) defined by \( L(e) = \{1, \ldots, k\} \) for all \( e \in E(G) \). A graph \( G \) is \( k \)-list-edge-colourable if it is \( L \)-edge-colourable for every edge list assignment \( L \) such that \(|L(e)| \geq k\) for all \( e \in E(G) \). The notorious List-Edge-Colouring Conjecture (attributed to many sources, some as early as 1975; see [8]) asserts that every \( G \) is \( \chi'(G) \)-list-edge-colourable. If this conjecture is true, then given the above discussion on the chromatic index of planar graphs, \( G \) should be \( \Delta \)-list-edge-colourable whenever \( \Delta \geq 7 \) (or perhaps 6). This has been verified when \( \Delta \geq 12 \).

**Theorem 3** (Borodin, Kostochka, and Woodall [2]). *If \( G \) is a planar graph with maximum degree \( \Delta(G) \geq 12 \), then \( G \) is \( \Delta(G) \)-list-edge-colourable.*

Borodin [1] proved a similar result; a short proof of this result was later obtained by Cohen and Havet [3].

**Theorem 4** (Borodin [1]). *If \( G \) is a planar graph with maximum degree \( \Delta(G) \geq 9 \), then \( G \) is \( (\Delta(G) + 1) \)-list-edge-colourable.*

In the present paper we have in fact proved the list-edge-colouring analog of Theorem 2. This stronger result is as follows.

**Theorem 5.** *Let \( G \) be a planar graph of maximum degree at most \( \Delta \), let \( L \) be an edge list assignment on \( G \) with \(|L(e)| \geq \Delta + t \) for all \( e \in E(G) \), where \( t \) is a positive integer, and let \( H \) be a subgraph of \( G \) that has been \( L \)-edge-coloured. If \( H \) has maximum degree at
most $d$, then the edge-precolouring can be extended to an $L$-edge-colouring of $G$ provided that either:

1. $d \leq t - 4$, or
2. $t - 3 \leq d \leq t$ and

\[
\Delta \geq \begin{cases} 16 + d, & \text{if } d = t, \\ 9 + d, & \text{if } d = t - 1, \\ 8 + d, & \text{if } d = t - 2, \\ 7 + d, & \text{if } d = t - 3. \end{cases}
\]

We again omit the case $t = 0$, however the required $d \leq t$ condition means that $H$ is edgeless and hence the best result is that of Theorem 3 above. Theorem 5 does have something meaningful to say when $H$ edgeless however: the case $t = 1$ and $d = 0$ gives Theorem 4 precisely.

The following section contains some technical results needed for our proof of Theorem 5, which comprises Section 3. The final section of this paper, Section 4, is about pushing Theorem 5 beyond planar graphs. We show that requiring $G - E(H)$ to be planar is sufficient, and in fact “planar” can be replaced by “non-negative Euler characteristic”.

2 Technical Lemmas

In this section, we gather some technical lemmas that will be needed for the proof of Theorem 5.

**Theorem 6** (Borodin, Kostochka, Woodall [2]). Let $G$ be a bipartite graph and let $L$ be an edge list assignment on $G$. If $|L(xy)| \geq \max\{\deg(x), \deg(y)\}$ for every edge $xy \in E(G)$, then $G$ is $L$-edge-colourable.

Edwards et al. [5] applied Theorem 6 to obtain a precolouring extension result for bipartite graphs (Theorem 15 of [5]), which we will use as part of our proof. While the result as stated in [5] only applies to classical edge-precolouring, a list-edge-colouring version can be obtained using essentially the same proof:

**Theorem 7.** Let $G$ be a bipartite multigraph, and let $L$ be an edge list assignment on $G$ with $|L(e)| \geq \Delta + t$ for all $e \in E(G)$. Let $H$ be a subgraph of $G$ that has been $L$-edge-coloured. If $H$ has maximum degree at most $d$, then the edge-precolouring can be extended to an $L$-edge-colouring of $G$ provided that $t \geq d$.

**Proof.** Let $G' = G - E(H)$. For each edge $e \in E(G')$, let $L'(e)$ be obtained from $L(e)$ by removing all colours used on the edges of $H$ incident to $e$. Let $xy$ be an arbitrary edge of $G'$. Now

\[
|L'(xy)| \geq |L(xy)| - \deg_H(x) - \deg_H(y) \geq \Delta + t - \deg_H(x) - \deg_H(y).
\]

Since $t \geq d \geq \Delta(H)$, this implies that

\[
|L'(xy)| \geq \Delta - \deg_H(x), \quad \text{and} \quad |L'(xy)| \geq \Delta - \deg_H(y).
\]
On the other hand,

\[
\deg_{G'}(x) = \deg_{G}(x) - \deg_{H}(x) \leq \Delta - \deg_{H}(x), \quad \text{and} \quad \\
\deg_{G'}(y) = \deg_{G}(y) - \deg_{H}(y) \leq \Delta - \deg_{H}(y).
\]

Thus, \(|L'(xy)| \geq \max\{\deg_{G'}(x), \deg_{G'}(y)\}\), and this holds for all \(xy \in E(G')\). By Theorem \([6]\) it follows that \(G'\) is \(L'\)-edge-colourable, and any \(L'\)-edge-colouring of \(G'\) gives the desired \(L\)-edge-colouring of \(G\). \(\square\)

In what follows and in the main argument, given a graph \(G\), we define \(V_i(G) = V_i\) as the set of all vertices \(v \in V(G)\) with \(\deg(v) = i\), and we define \(V_{[a,b]}(G) = V_{[a,b]}\) as \(\cup_{i \in [a,b]} V_i\).

**Lemma 8.** Let \(G\) be a graph of maximum degree at most \(\Delta\), and let \(L\) be an edge list assignment on \(G\) with \(|L(e)| \geq \Delta + t\) for all \(e \in E(G)\). Let \(H\) be a subgraph of \(G\) with maximum degree at most \(d\). Suppose that \(H\) has been \(L\)-edge-coloured, and that this extends to an \(L\)-edge-colouring of \(G - e\) for all \(e \in E(G) \setminus E(H)\), but not to \(G\).

Let \(A = V_{[a_0,a]}\) and \(B = V_{[b_0,\Delta]}\), where \(a_0, a, b_0\) are positive integers with \(a_0 \geq t + 1\), \(b_0 > a\), and \(a + b_0 \geq \Delta + t + 1\). Let \(X\) be the bipartite subgraph of \(G - E(H)\) induced by the bipartition \((A, B)\). If every vertex \(u \in A\) has the property that

\[
\deg_{X}(u) \geq \deg_{G}(u) - d,
\]

then

\[
(t + 1 - d)|A| \leq \sum_{i=b_0}^{\Delta} (a + i - 1 - (\Delta + t))|V_i|.
\]

Moreover, if \(a_0 > t + 1\) and \(a + b_0 \geq \Delta + t + 1\) then the above inequality is strict.

**Proof.** Say that an induced subgraph \(J \subseteq X\) is bad if

- \(\deg_J(u) \geq \deg_G(u) - t\) for all \(u \in A \cap V(J)\), and
- \(\deg_J(v) \geq a + \deg_G(v) - (\Delta + t)\) for all \(v \in B \cap V(J)\).

Notice that for all \(u \in A, v \in B\),

\[
\deg_G(u) - t \geq a_0 - t \geq 1 \quad (2)
\]

and

\[
a + \deg_G(v) - (\Delta + t) \geq a + b_0 - (\Delta + t) \geq 1, \quad (3)
\]

so that if a bad induced subgraph exists, it has no isolated vertices, and in particular has at least one edge. We will first show that \(X\) has no bad induced subgraph, and then show that this implies the desired claim.

Suppose that \(X\) has a bad induced subgraph \(J\). Let \(G' = G - E(J)\). Since \(E(J)\) is nonempty, \(G'\) is a proper subgraph of \(G\), so by assumption, the edge-precolouring on \(H\) extends to an \(L\)-edge-colouring \(\varphi\) of \(G'\). We derive a contradiction by showing we can further extend to an \(L\)-edge-colouring of \(G\). To this end, let \(L^J\) be the edge list assignment on \(J\) defined as follows: for each edge \(uv \in E(J)\), \(L^J(uv)\) is the set of colours
from $L(uv)$ that do not appear on any $G'$-edge adjacent to $uv$. Observe that for each $uv \in E(J)$, we have
\[ |L^J(uv)| \geq \Delta + t - \deg_G(u) - \deg_G(v) + \deg_J(u) + \deg_J(v). \]
Since $J$ is bad, we have $\deg_J(u) \geq \deg_G(u) - t$, so that
\[ |L^J(uv)| \geq \Delta - \deg_G(v) + \deg_J(v) \geq \deg_J(v), \]
and likewise $\deg_J(v) \geq \deg_G(v) + a - (\Delta + t)$ so that
\[ |L^J(uv)| \geq a - \deg_G(u) + \deg_J(u) \geq \deg_J(u). \]
Hence, for every $uv \in E(J)$, we have $|L^J(uv)| \geq \max\{d(u), d(v)\}$. By Theorem 6, $J$ is $L^J$-edge-colourable. Now any proper $L^J$-edge-colouring of $J$, combined with the $L$-edge-colouring $\varphi$ of $G'$, yields a proper $L$-edge-colouring of $G$ that extends the edge-precolouring of $H$ as desired; contradiction.

Hence, $X$ contains no bad induced subgraph, and so every induced subgraph $J$ of $X$ contains a vertex violating the definition of a “bad” subgraph. By iteratively removing these vertices and counting the edges removed when each vertex is deleted, we see that
\[ |E(X)| \leq \sum_{u \in A} [\deg_G(u) - t - 1] + \sum_{v \in B} [a + \deg_G(v) - (\Delta + t) - 1] \]
\[ \leq \sum_{u \in A} [(\deg_X(u) + d) - t - 1] + \sum_{v \in B} [a + \deg_G(v) - (\Delta + t) - 1] \]
\[ = |E(X)| + \sum_{u \in A} [d - t - 1] + \sum_{i = b_0}^{\Delta} (a + i - (\Delta + t) - 1) |V_i|. \]
Rearranging the last inequality yields
\[ (t + 1 - d) |A| \leq \sum_{i = b_0}^{\Delta} (a + i - 1 - (\Delta + t)) |V_i|, \]
which is the desired conclusion. If we additionally know that $a_0 > t + 1$ and $a + b_0 > \Delta + t + 1$, then inequalities 2 and 3 become strict. Hence each $u \in A$ and $v \in B$ is contributing a positive amount to the right-hand-side of 4. Since the last vertex removed is isolated, this is an overcount, and hence we get a strict inequality.\[ \square \]

3 Proof of Theorem 5

For fixed values of $\Delta, t, d$, we choose a counterexample $(G, H)$ where the quantity $3|E(G)| + |V_{[2,t+1]}(G)|$ is as small as possible.

Claim 1. The edge-precolouring on $H$ can be extended to an $L$-edge-colouring of $G - e$ for any $e \in E(G) \setminus E(H)$.

Proof of Claim. Let any $e \in E(G) \setminus E(H)$ be given, and let $G' = G - e$. Note that $(G', H)$ satisfies the hypotheses of the theorem with $\Delta, t, d$. Exactly two vertices in $G'$ have lower
Moving from \((G, H)\) to \((G', H')\) in the proof of Claim 4.

degrees than in \(G\), so \(|V_{[2,t+1]}(G')|\) may be as large as \(|V_{[2,t+1]}(G)| + 2\). However, since \(G'\) has one edge less than \(G\), we still get that

\[
3 |E(G')| + |V_{[2,t+1]}(G')| < 3 |E(G)| + |V_{[2,t+1]}(G)|.
\]

Hence, by our choice of counterexample, the edge-precolouring of \(H\) extends to an \(L\)-edge-colouring of \(G'\).

**Claim 2.** If \(uv \in E(G) \setminus E(H)\), then \(\text{deg}_G(u) + \text{deg}_G(v) \geq \Delta + t + 2\).

*Proof of Claim.* By Claim 1, the edge-precolouring of \(H\) can be extended to an \(L\)-edge-colouring \(\varphi\) of \(G - uv\). The edge \(uv\) sees at most \(\text{deg}_G(u) + \text{deg}_G(v) - 2\) different colours in \(\varphi\), so since \((G, H, t)\) is a counterexample, it must be that \(\text{deg}_G(u) + \text{deg}_G(v) - 2 \geq \Delta + t\).

**Claim 3.** If \(v \in V_{[1,t+1]}\), then every edge incident to \(v\) in \(G\) is also in \(H\).

*Proof of Claim.* Assume for contradiction that \(v \in V_{[1,t+1]}\) and \(v\) is incident to an edge not in \(H\), say \(uv\). By Claim 2, we know that \(\text{deg}_G(u) + \text{deg}_G(v) \geq \Delta + t + 2\). However, since \(\text{deg}_G(v) \leq t + 1\), this implies that \(\text{deg}_G(u) \geq \Delta + 1\), a contradiction.

**Claim 4.** \(V_{[2,t+1]} = \emptyset\).

*Proof of Claim.* Suppose not, and take \(v \in V_{[2,t+1]}\). By Claim 3, every edge \(uv\) incident to \(v\) must lie in \(H\).

Let \(G'\) and \(H'\) be the graphs obtained from \(G\) and \(H\), respectively, by deleting \(v\) and, for each \(u \in N_G(v)\), adding a new vertex \(v_u\) adjacent only to \(u\). We precolour each edge \(uv_u\) with the same colour received by the edge \(uv\) in the precolouring of \(H\). See Figure 3. Observe that the edge-precolouring of \(H'\) extends to \(G'\) if and only if the edge-precolouring of \(H\) extends to \(G\).

Now \(G'\) has the same number of edges as \(G\), and has one fewer vertex in \(V_{[2,t+1]}\). As \(\Delta(G') \leq \Delta\) and \(\Delta(H') \leq d\), our choice of counterexample implies that the edge-precolouring of \(H'\) extends to \(G'\), but this means that the edge-precolouring of \(H\) extends to \(G\) as well.

**Claim 5.** Every vertex of \(G\) is either a leaf incident to an edge in \(H\), or of degree at least \(t + 2\).

*Proof.* This follows by combining Claim 3 and Claim 4.

![Figure 3: Moving from \((G, H)\) to \((G', H')\) in the proof of Claim 4.](image)


Let $F_m$ be the set of faces in $G$ with exactly $m$ vertices on its boundary having degree 3 or higher in $G$.

**Claim 6.** $F_0 = F_1 = F_2 = \emptyset$.

**Proof of Claim.** Suppose that $f \in F_0 \cup F_1 \cup F_2$; we will show a contradiction. We know that $V_2 = \emptyset$ by Claim 5 since $t \geq 1$. So, if the boundary of $f$ contains a cycle, then it contains at least three vertices of degree at least three, yielding a contradiction. Thus, the boundary of $f$ contains no cycle. This means that $G$ is a forest, and $f$ is its one face. In particular, $G$ is bipartite. By Theorem 7, this implies that the precolouring of $H$ extends to all of $G$, contradicting our choice of $G$ as a counterexample. \hfill ∎

We now introduce a discharging argument. To each vertex in $G$ assign an initial charge of $\alpha(v) = 3 \deg_G(v) - 6$. To each face in $G$ assign an initial charge of $\alpha(f) = -6$. We also define an additional structure $P$ (a “global pot”) and assign to it an initial charge of $\alpha(P) = 0$. We discharge along the following rules:

(a) For each $m$, every face $f \in F_m$ takes $\frac{6}{m}$ from each vertex of degree 3 or higher on its boundary.

(b) Every vertex $v \in V_1$ takes 3 from its neighbor.

In the special case where $t = d + \ell$ for $\ell \in \{0, 1, 2, 3\}$, we also add the following rules:

(c) For every vertex $v \in V_i$, where $i \in \{t + 2, \ldots, t + 5 - \ell\}$:
   
   $v$ takes $t + 6 - \ell - i$ from $P$.

(d) For every vertex $v \in V_j$, where $j \in \{\Delta - 3 + \ell, \ldots, \Delta\}$:
   
   $v$ gives $\frac{q(j)(q(j) + 1)}{2(\ell + 1)}$ to $P$, where $q(j) = j - \Delta + 4 - \ell$.

While it is not immediately obvious, discharging rules (c) and (d) never apply to the same vertex, due to the following claim.

**Claim 7.** If $t = d + \ell$ for some $\ell \in \{0, 1, 2, 3\}$, then $\Delta - 3 + \ell > t + 5 - \ell$.

**Proof of Claim.** We get the desired inequality if and only if $\Delta + 2\ell > 8 + t$. If $\ell = 0$, then we have $d = t$, so the hypothesis of Theorem 5 yields

$$\Delta + 2\ell = \Delta \geq 16 + d = 16 + t > 8 + t.$$ 

If $\ell \in \{1, 2, 3\}$ we may rewrite hypothesis of Theorem 5 as

$$\Delta \geq 10 + d - \ell = 10 + (t - \ell) - \ell = 10 + t - 2\ell,$$

so

$$\Delta + 2\ell \geq 10 + t > 8 + t.$$

\hfill ∎
Using Euler’s formula for planar graphs, the sum of initial charges is at most $-12$:

$$\alpha(P) + \sum_{v \in V(G)} \alpha(v) + \sum_{f \in F(G)} \alpha(f) = 0 + \sum_{v \in V(G)} (3\deg_G(v) - 6) + \sum_{f \in F(G)} (-6)$$

$$= 6|E(G)| - 6|V(G)| - 6|F(G)| \leq 6(-2) = -12. \tag{5}$$

For each graph element $x$ (either a vertex, a face, or the global pot), let $\alpha'(x)$ denote the final charge of $x$. Since each discharging rule conserves the total charge, we see that $\sum_x \alpha'(x) = \sum_x \alpha(x) = -12$. We will achieve our desired contradiction by showing that the final charge of each element is nonnegative.

First consider a face $f$. By Claim 6 $f \in F_m$ for $m \geq 3$. So according to discharging rule (a) (the only rule affecting $f$),

$$\alpha'(f) = (-6) + m\left(\frac{5}{m}\right) = 0.$$

Now consider the global pot $P$. We know $\alpha(P) = 0$ and that the charge of $P$ is unaffected when $d \leq t - 4$, so the following claim precisely amounts to showing showing that $\alpha'(P) > 0$ when $t - 3 \leq d \leq t$.

**Claim 8.** If $t = d + \ell$ for some $\ell \in \{0, 1, 2, 3\}$, then

$$\sum_{i=t+2}^{t+5-\ell} (t + 6 - i)|V_i| < \sum_{j=\Delta-3+\ell}^{\Delta} \frac{q(j)(q(j) + 1)}{2(\ell + 1)} |V_j|. \tag{6}$$

**Proof of Claim.** For each $k \in \{0, \ldots, 3 - \ell\}$, define $A_k = V_{[t+2,t+5-\ell-k]}$ and $B_k = V_{[\Delta-3+\ell+k,\Delta]}$ and let $X_k$ be the bipartite subgraph of $G - E(H)$ induced by the partition $(A_k, B_k)$. We will show we can apply Lemma 8 for each value of $k$, and then we will sum the resulting inequalities to get our desired result. For fixed $k$, this means we want to apply Lemma 8 with parameter choices

$$a_0 = t + 2, \quad a = t + 5 - \ell - k,$$

$$b_0 = \Delta - 3 + \ell + k,$$

and hence to do so we must verify that $a_0 \geq t + 1$ (true) and that $a + b_0 \geq \Delta + t + 1$, which is true since

$$(t + 5 - \ell - k) + (\Delta - 3 + \ell + k) = t + 2 + \Delta.$$

In fact, since both these inequalities hold strictly, we will apply the strict version of Lemma 8. Of course, there are several other hypotheses we must check. In particular, we must verify that $b_0 > a$, which is equivalent to showing that $\Delta > t + 8 - 2\ell$. Since $t = d + \ell$, we get this inequality by Claim 7. By Claim 1 we can therefore apply Lemma 8 for $k$ provided that every vertex $u \in A_k$ has the property that

$$\deg_{X_k}(u) \geq \deg_G(u) - d.$$

Consider such a vertex $u$ with incident edge $uv$ in $E(G) \setminus E(H)$. Since $u \in A_k$, and by Claim 2 we know that

$$\deg_G(v) \geq \Delta + t + 2 - \deg_G(u) \geq \Delta + t + 2 - (t + 5 - \ell - k) = \Delta - 3 + \ell + k.$$
This means, by definition of $X_k$, that the edge $uv$ is in $X_k$. So we get $\deg_{X_k}(u) \geq \deg_G(u) - \deg_H(u) \geq \deg_G(u) - d$, as desired.

For any fixed $k$, we can now apply Lemma 8 to get

$$\ell + 1 |A_k| < \sum_{j=\Delta-3+\ell+k}^{\Delta} (q(j) - k)|V_j|, \quad (7)$$

since $t + 1 - d = \ell + 1$ by the hypothesis of Claim 8, and since, for our choices of parameters,

$$a + j - 1 - (\Delta + t) = (t + 5 - \ell - k) + j - 1 - (\Delta + t) = j - \Delta + 4 - \ell - k = q(j) - k.$$ 

Dividing (7) by $(\ell + 1)$ and summing over all $k$ yields

$$\sum_{k=0}^{3-\ell} |A_k| < \left( \frac{1}{\ell + 1} \right) \sum_{k=0}^{3-\ell} \sum_{j=\Delta-3+\ell+k}^{\Delta} (q(j) - k)|V_j|. \quad (8)$$

The left-hand-side of (8) is

$$\sum_{k=0}^{3-\ell} |V_{[t+2,t+5-\ell-k]}| = |V_{[t+2,t+5-\ell]}| + |V_{[t+2,t+4-\ell]}| + \cdots + |V_{[t+2,t+2]}| = (4 - \ell)|V_{t+2}| + \cdots + 2|V_{t+4-\ell}| + |V_{t+5-\ell}|$$

$$= \sum_{i=t+2}^{t+5-\ell} (t + 6 - \ell - i)|V_i|,$$ 

matching the left-hand side of (6). It remains only to show that the right-hand-side of (8) equals the right-hand side of (6). To this end, note that

$$j \geq \Delta - 3 + \ell + k \iff k \leq j - \Delta + 3 - \ell = q(j) - 1,$$

and so

$$\sum_{k=0}^{3-\ell} \sum_{j=\Delta-3+\ell+k}^{\Delta} (q(j) - k)|V_j| = \sum_{j=\Delta-3+\ell}^{\Delta} \left( \sum_{k=0}^{(q(j) - 1)} (q(j) - k) \right) |V_j|.$$ 

Now the bracketed sum can be rewritten as

$$\sum_{k=0}^{q(j)-1} (q(j) - k) = q(j) + (q(j) - 1) + (q(j) - 2) + \cdots + 1 = \frac{q(j)(q(j)+1)}{2},$$

which is precisely what we needed to prove. \hfill \Box

We have now shown $\alpha'(P) > 0$, so it remains only to consider the final charge of an arbitrary vertex $v$. If $v \in V_1$, then only discharging rule (b) affects $v$, and we get

$$\alpha'(v) = (-3) + 3 = 0.$$ 

By Claim 3 we may now assume that $\deg_G(v) \geq t + 2$. 

11
Suppose $v$ lies on the boundary of $x$ distinct faces and is incident to $y$ leaves. We know that $x$ is no more than $\deg_G(v) - y$, so $x + y \leq \deg_G(v)$. We also know that $y \leq d$, by Claim 5 and by definition of $d$. By doubling the first inequality and adding the result to the second inequality we get
\[ 2x + 3y \leq 2\deg_G(v) + d. \] (9)
Since $F_0, F_1, F_2 = \emptyset$ by Claim 6, each of the $x$ distinct faces incident to $v$ has at least 3 vertices of degree at least 3 on their boundary. This means that each of these $x$ faces takes charge at most 2 from $v$, according to discharging rule (a). Each of the $y$ leaves incident to $v$ takes exactly 3 from $v$, according to discharging rule (b). Hence by inequality (9), after applying discharging rules (a) and (b) (but before considering discharging rules (c) or (d)), the charge of $v$ is at least
\[ 3\deg_G(v) - 6 - (2x + 3y) \geq \deg_G(v) - 6 - d. \] (10)
Note that since $d \leq t$, the additional discharging rules (c) and (d) are applied precisely when $d \geq t - 3$. If $d \leq t - 4$, then we do not apply them, and by inequality (10),
\[ \alpha'(v) \geq \deg_G(v) - 6 - d \geq \deg_G(v) - 6 - (t - 4) = \deg_G(v) - (t + 2) \geq 0. \]
We may now assume that $t = d + \ell$ for $\ell \in \{0, 1, 2, 3\}$. Let $p$ denote the total charge transferred from $P$ to $v$ according to discharging rules (c) and (d); note that $p$ may be positive, negative, or zero. In all cases, by inequality (10), we have that
\[ \alpha'(v) \geq \deg_G(v) - 6 - d + p. \] (11)
If neither discharging rule (c) nor (d) applies to $v$, then we know that $t + 5 - \ell < \deg_G(v)$ and therefore (11) says that
\[ \alpha'(v) \geq (t + 5 - \ell + 1) - 6 - d + (0) = (t - d) - \ell = 0, \]
as desired.

Now suppose that discharging rule (c) applies to $v$ (and hence (d) does not, according to Claim 7). In this situation, (11) implies that
\[ \alpha'(v) \geq \deg_G(v) - 6 - d + (t + 6 - \ell - \deg_G(v)) = 0. \]
Finally, we may assume that discharging rule (d) applies to $v$ (and hence (c) does not, according to Claim 7). In this case, we have $t = d + \ell$, where $\ell \in \{0, 1, 2, 3\}$, and $F\deg_G(v) \in \{\Delta - 3 + \ell, \ldots, \Delta\}$. By (11),
\[ \alpha'(v) \geq \deg_G(v) - 6 - d - \left(\frac{(\deg_G(v) - (\Delta - 4 + \ell))(\deg_G(v) - (\Delta - 5 + \ell))}{2(\ell + 1)}\right). \]
Writing $\deg_G(v)$ as $\Delta - h + \ell$, where $h \in \{\ell, \ldots, 3\}$, we can rewrite this lower bound as
\[ \alpha'(v) \geq \Delta - h + \ell - 6 - d - \left(\frac{(4 - h)(5 - h)}{2(\ell + 1)}\right) = \Delta - d - \left(6 + h - \ell + \frac{(4 - h)(5 - h)}{2(\ell + 1)}\right). \]
Table 1 computes the bracketed quantity for each permissible combination of $\deg_G(v)$ and $\ell$. For each possible value of $\ell$, the hypothesis of Theorem 5 ensures that this lower bound is always nonnegative.

We have proved that $\alpha'(x) \geq 0$ for every graph element $x$, and this completes the proof of Theorem 5.
\[
\deg_G(v) = \Delta - 3 + \ell \\
\deg_G(v) = \Delta - 2 + \ell \\
\deg_G(v) = \Delta - 1 + \ell \\
\deg_G(v) = \Delta - 0 + \ell
\]

\[
\begin{array}{c|c|c|c|c}
\ell = 0 & \ell = 1 & \ell = 2 & \ell = 3 \\
\hline
\Delta - d - 10 & \Delta - d - 17/2 & \Delta - d - 22/3 & \Delta - d - 25/4 \\
\Delta - d - 11 & \Delta - d - 17/2 & \Delta - d - 7 & * \\
\Delta - d - 13 & \Delta - d - 9 & * & * \\
\Delta - d - 16 & * & * & *
\end{array}
\]

**Table 1**: Lower bounds on \(\alpha'(v)\) when discharging rule (d) applies. Starred entries are impossible due to \(\deg_G(v) \leq \Delta\).

## 4 Beyond planarity

In the proof of Theorem 5 our initial charges sum to at most \(-12\), and after discharging the vertices, faces, and global pot all have nonnegative charge. In fact, when we examine inequality \([5]\), we see that the sum of initial charges is at most \(-6\varepsilon\), where \(\varepsilon\) is the Euler characteristic of the plane. Hence our argument works for any surface of positive Euler characteristic; namely \(G\) may be embedded on the plane or projective plane. Moreover, this embedding requirement need not concern the edges of the precoloured \(H\): imagine applying Theorem 5 to the graph obtained by replacing every edge \(e = uv\) in \(H\) with a pair of edges \(e_u = uu'\) and \(e_v = vv'\) where \(u', v'\) are new leaves, and \(e_u\) and \(e_v\) retain the precolouring (and lists) of \(e\). Given these observations, we can strengthen Theorem 5 by removing the assumption that "\(G\) is planar" and replacing it by the somewhat milder "\(G - E(H)\) can be embedded in a surface of positive Euler characteristic".

**Acknowledgements**

We are indebted to an anonymous referee whose insightful comments strengthened our main result.

**References**

[1] O. V. Borodin, *A generalization of Kotzig's theorem and prescribed edge coloring of planar graphs*, Mat. Zametki 48 (1990), no. 6, 22–28, 160, (in Russian). MR 1102617

[2] O. V. Borodin, A. V. Kostochka, and D. R. Woodall, *List edge and list total colourings of multigraphs*, J. Combin. Theory Ser. B 71 (1997), no. 2, 184–204. MR 1483474

[3] Nathann Cohen and Frédéric Havet, *Planar graphs with maximum degree \(\Delta \geq 9\) are \((\Delta+1)\)-edge-choosable—a short proof*, Discrete Math. 310 (2010), no. 21, 3049–3051. MR 2677668

[4] Charles J. Colbourn, *The complexity of completing partial Latin squares*, Discrete Appl. Math. 8 (1984), no. 1, 25–30. MR 739595
[5] Katherine Edwards, António Girão, Jan van den Heuvel, Ross J. Kang, Gregory J. Puleo, and Jean-Sébastien Sereni, *Extension from precoloured sets of edges*, 2016. arXiv:1407.4339.

[6] António Girão and Ross J. Kang, *Precolouring extension of vizing’s theorem*, 2016.

[7] Ian Holyer, *The NP-completeness of edge-coloring*, SIAM J. Comput. 10 (1981), no. 4, 718–720. MR 635430

[8] Tommy R Jensen and Bjarne Toft, *Graph coloring problems*, vol. 39, John Wiley & Sons, 2011.

[9] O. Marcotte and P. D. Seymour, *Extending an edge-coloring*, J. Graph Theory 14 (1990), no. 5, 565–573. MR 1073098

[10] Dániel Marx, *NP-completeness of list coloring and precoloring extension on the edges of planar graphs*, J. Graph Theory 49 (2005), no. 4, 313–324. MR 2197234

[11] Daniel P. Sanders and Yue Zhao, *Planar graphs of maximum degree seven are class I*, J. Combin. Theory Ser. B 83 (2001), no. 2, 201–212. MR 1866396

[12] V. G. Vizing, *On an estimate of the chromatic class of a p-graph*, Diskret. Analiz No. 3 (1964), 25–30. MR 0180505

[13] Limin Zhang, *Every planar graph with maximum degree 7 is of class 1*, Graphs Combin. 16 (2000), no. 4, 467–495. MR 1804346