The path integrals on a curved manifold

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Abstract

For description of the quantum dynamics on a curved group manifold the path integrals in a space of the group parameters is offered. The formalism is illustrated by the $H$-atom problem.
1 Introduction

The classical particles motion may be described mapping the trajectory on the Lee group \(G\) manifold [1] imagining the particles coordinates as the elements of \(G\). Then, roughly speaking (e.g. [2]), the group combination law creates the particles classical trajectory. On the homogeneous and isotropic group manifold the motion of a particle is free, it moves with the constant velocity in an arbitrary point of the manifold. It is the crucial point of this approach since actually solves a quantum problem.

Indeed, this idea was used for description of the particles quantum motion [3] and it was shown for the essentially nonlinear Lagrangian

\[
L = \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu \tag{1.1}
\]

that the quasiclassical approximation is exact on the (semi)simple Lee group manifold.

But this slender mechanism of solution is destructed in presence of interaction potential \(v(x) = O(x^n), n > 2\), since it breaks the isotropy and homogeneity of the Lee groups manifolds [1].

We offer another approach to include the interaction potential. Firstly, we shall specify the topology of group manifold \(M_s\) through the group of the corresponding classical equation of motion. So, we shall assume that this problem has the classical solution.

But in the quantum theory one must expect that the quantum perturbations should lead to deformations of \(M_s\). This is the main problem: one must quantize the manifold \(M_s\) to describe the motion in the unhomogeneous and nonisotropic space.

Instead of this complicated problem we wish count the (quantum) fluctuations of the parameters of group manifold (GMP). It had shown in the paper [4] that this is possible (Sec.2) since conserves the integral probability.

It is remarkable that this “unitary” definition of the functional measure allows to consider independently the quantum excitations of each independent degree of the freedom of the invariant manifold \(M_s\), i.e. of each independent parameters of the classical trajectory.

In result, the GMP will be defined through the generators of \(G_s\). Considering the parameters of invariant hypersurface as the generalized coordinates and momenta of the “particles”, in the classical limit the motion of this “particles” must be free [4], i.e. theirs velocities are constants. Such choice of the “particles” coordinates allows to achieve the same effect as in the above discussed transformation to the homogeneous and isotropic (semi)simple Lee group manifold [1]. Moreover, even in quantum case one can get to the free “particles” motion rescaling the quantum sources.

Considering the quantum problem on the compact manifold one can demonstrate the absence of quantum corrections to the angular degrees of freedom. This is the new solution of the quantum integrability problem [4]. This result seems important since it demonstrates also the possibility of the partial quasiclassicality (i.e. of the partial integrability) of a quantum problem.

\(\text{We shall start the consideration from the flat space.}\)
It is known that the dynamical groups are more rich [8] and they cannot be reduced to the Lee groups. This forces us to consider the quantum problems in a phase space since without this groups one cannot solve a number of modern field-theory problems.

It is evident that the mapping on the dynamical group manifold has not a simple geometrical interpretation as we had discussed above for the Lee groups (the trajectory can not belong to the manifold of the dynamical group). But formally, as it will be seen, the description of quantum motion on the “dynamical groups manifolds” is practically the same: we will deal with the space of parameters of the dynamical group.

Therefore, the offered method is applicable as for the Lee groups, so for the dynamical groups since in both case we will work in the space of group parameters. It is better to say that we will consider the classical trajectories manifold. This allows to extend the approach on the field-theoretical problems.

To do the method clear we shall consider the \( H \)-atom problem. This problem has the additional (hidden) first integral in involution, the Laplace-Runge-Lentz (LRL) vector:

\[
\vec{A} = (\vec{l} \times \vec{p}) + \frac{\vec{x}}{|\vec{x}|},
\]

where \( \vec{p} \) is the momentum and \( \vec{l} \) is the angular momentum. It is easy seen that pare \((\vec{A}, \vec{l})\) form the algebra of \( O(4) \) Lee group. It allows to solve the \( H \)-atom problem algebraically [9]. Transforming the Schrödinger equation to the momentum space one can see the \( O(4) \) symmetry of this equation. This allows to solve the transformed equation and to explain the degeneracy of the energy terms [10].

The isomorphism between \( H \)-atom problem and harmonic oscillator model was shown in [11] using the path-integral formalism. One can interpret this result as the absence of necessity to handle the \( O(4) \) symmetry for the \( H \)-atom problem integrability.

For us the crucial point consist in the fact that the particles trajectories in the Coulomb potential are closed independently from the initial conditions. It is known that this is the result of conservation of the LRL vector [12].

We map the \( H \)-atom problem initially defined in the plain Cartesian space into the space of the group parameters (Sec.2). The trajectory can be defined for \( O(4) \) group by the pare \((A, l)\) and by the true anomaly \( \phi_t \), or for \( O(3) \) group by the pare \((h, l)\), where \( h \) is the classical Hamiltonian, and by the eccentric anomaly \( \phi_e \). We will choose the second definition and will describe the quantum fluctuations of \((h, l, \phi_e)\).

We will show the integrability of \( H \)-atom problem demonstrating the sufficiency of angular corrections cancelation. For this we will use the trajectory closeness only, and presence of the LRL vector will not be used explicitly, i.e. the presence of \( O(4) \) dynamical group will be taken into account latently.

\section{The dynamics in a curved space}

We shall start from the flat Cartesian coordinates. The amplitude with energy \( E \) has the form:

\[
A(x_1, x_2; E) = i \int_0^\infty dT_+ e^{iET_+} \int_{x_1(0) = x_1}^{x_1(T_+) = x_2} D x_+ e^{iS_{C+}(x_+)(x_+)} ,
\]

(2.1)
where $x$ is the 2-dimensional vector. The action

\[ S_{C^+}(T)(x) = \int_{C^+(T)} dt \left( \frac{1}{2} \dot{x}^2 + \frac{1}{|x|} \right) \]  

and the measure are defined on the Mills’ contour [13]:

\[ C_{\pm}(T) : t \to t \pm i\epsilon, \quad \epsilon \to +0, \quad 0 \leq t \leq T. \]  

Indroduction of Mills’ time contour is necessary to ensure the convergence of path integral (2.1).

We will calculate the probability

\[ R(E) = \int dx_1 dx_2 |A(x_1, x_2)|^2, \]  

to introduce the unitary definition of path-integral measure [14]. Inserting (2.1) into (2.4) we find that

\[ R(E) = \int_0^\infty dT_+ dT_- e^{iE(T_+-T_-)} \int_{x_+(0)=x_-(0)}^{x_+(T_+)=x_-(T_-)} Dxe^{iS_{C^-}(x)} \]  

is described by the closed-path integral.

The total action

\[ S_{C^-}(x) = S_{C^+(T_+)}(x_+) - S_{C^-(T_-)}(x_-) \]  

contains the sum of two actions, where $S_{C^+(T_+)}(x_+)$ describes forward in time motion and $-S_{C^-(T_-)}(x_-)$ backward one. Note that $S_{C^-}$ is defined on the complex conjugate contour $C^*(T) = C_+(T)$ and, since $x_+(t)$ and $x_-(t)$ are in our formalism the independent trajectories, $S^-$ is the complex quantity. This allows to define the measure $Dx$: extracting the linear over $e(t) = (x_+ - x_-)(t)/2$ term in the exponent we explicitly find in result of integration over $e(t)$ that the path integral differential measure over $x(t) = (x_+ + x_-)(t)/2$ is $\delta$-like.

The path integral of (2.5) type was considered in [5]. Performing the transformation to the cylindrical coordinates we find that the trajectory winds the surface arbitrary times since we map the noncompact space on the compact one. One can choose the principal domain for the angular variables and calculate sum over repeated contributions.

So, in the cylindrical coordinates the result looks as follows:

\[ R(E) = 2\pi \int_0^\infty dT \exp \{ -i \int_{C(T)} dt (\dot{j}_r(t)\dot{c}_r(t) + \dot{j}_\phi(t)\dot{c}_\phi(t)) \} \times \int D^{(2)} M(r, \phi) \exp \{ -iV_T(x, e_C) \}, \]  

where

\[ \int_C = \int_{C^+} + \int_{C^-} \]  

and the Dirac’s δ-like measure

\[
D^{(2)}M(r, \phi) = \delta(E - H_T(r, \phi)) \prod_t r^2(t)dr(t)d\phi(t) \times
\]

\[
\times \delta(\ddot{r} - \dot{\phi}^2r + v'(r) - j_r)\delta(\partial_t(\dot{\phi}r^2) - j_\phi)
\]

\]

\[
(H_T \text{ is the classical Hamiltonian at time moment } T). \text{ Note that we are not able to shift the time contours } C_\pm \text{ on the real axis since of the Green functions singularities.}
\]

The symbol “hat” in (2.7) meance the differentiation over corresponding quantity:

\[
\exp\{-i \int_{C(T)} dt(\dot{j}_r(t)\dot{e}_r(t) + \dot{j}_\phi(t)\dot{e}_\phi(t))\} \equiv \exp\{-i \int_{C(T)} dt(\frac{\delta}{\delta j_r(t)}\frac{\delta}{\delta e_r(t)} + \frac{\delta}{\delta j_\phi(t)}\frac{\delta}{\delta e_\phi(t)})\}
\]

\]

\[
(2.10)
\]

Here \( j_r \) and \( j_\phi \) are the radial and angular perturbation sources, and \( e_r \) and \( e_\phi \) are the corresponding auxiliary fields. At the end of calculations one must take this quantities equal to zero. This means that the quantum exitation sources are switched on adiabaticaly and the equations of motion can be solved perturbatively, expanding over \( j_r \) and \( j_\phi \).

The action of the perturbations generating operator (2.10) on the weight functional

\[
-V_T(x; e_c) = S_{C_+}(x + e_c) - S_{C_-}(x - e_c) + \int_{C(T)} dt(\ddot{\vec{x}} - \vec{x}/|\vec{x}|^3)e_c
\]

\]

\[
(2.11)
\]

generates the asymptotic series\(^3\).

The vector \( \vec{e}_c \) has the components:

\[
e_{c,1} = e_r \cos \phi - re_\phi \sin \phi, \quad e_{c,2} = e_r \sin \phi + re_\phi \cos \phi.
\]

\]

\[
(2.12)
\]

and it was assumed that \( V_T(x, e_c) \) in (2.11) was written in the cylindrical coordinates.

As was explained in the Sec.1 we will deal with motion in the tangent space. Inserting

\[
1 = \int DpDl \prod_t \delta(p - \dot{r})\delta(l - \dot{\phi}r^2)
\]

\]

\[
(2.13)
\]

into (2.11) we introduce the motion in the phase space with the Hamiltonian

\[
H_j = \frac{1}{2}p^2 + \frac{l^2}{2r^2} - \frac{1}{r} - j_r r - j_\phi \phi.
\]

\]

\[
(2.14)
\]

The Dirac’s measure becomes four dimensional:

\[
D^{(4)}M(r, \phi, p, l) = \delta(E - H_T(r, p, l)) \prod_t dr(t)d\phi(t)dp(t)dl(t) \times
\]

\[
\times \delta(\dot{r} - \frac{\partial H_j}{\partial p})\delta(\dot{\phi} - \frac{\partial H_j}{\partial l})\delta(\dot{p} + \frac{\partial H_j}{\partial r})\delta(\dot{l} + \frac{\partial H_j}{\partial \phi}).
\]

\]

\[
(2.15)
\]

\(^3\)Note that only the potential \( v(x) = O(x^s), \ s \leq 2 \), the Dirac’s measure is free from the source of quantum exitations \( j \). In this case the quasiclassical approximation is exact. Starting from Cartesian flat coordinates frame we can map this model on the arbitrary manyfold. In result, considering the Fourier-transform of the functional δ-function, we will find explicitely the model Lagrangian (1.1). The more complicate case will be considered.
Rescaling the auxiliary field $e$, we take away the $r^2$ term from the measure.

Now we will map the problem into group parameters space. Instead of $(p, l, r, \phi)$ we will introduce the motion into $(h, l, \theta, \phi)$ space where

$$h = \frac{1}{2}p^2 + \frac{l^2}{2r^2} - \frac{1}{r}$$

is the classical Hamiltonian and

$$\theta = \int dr \{2h - \frac{l^2}{2r^2} + \frac{1}{r}\}^{-1/2}$$

is the “time” variable. This transformation is not canonical and

$$D^{(4)} M(h, l, \theta, \phi) = \delta(E - h(T)) \prod_t dh(t) dl(t) d\theta(t) \phi(t) \times$$

$$\times \delta(\dot{r} - \frac{l}{r_c^2}) \delta(\dot{j} - j_c) \delta(\dot{\phi} - \frac{l}{r_c^2}) \delta(\dot{\theta} - 1)$$

where $r_c = r_c(h, l; \theta)$ is the solution of eq. (2.17) and $p_c(r)$ is the solution of eq. (2.16).

Shifting now the time contours $C_\pm$ on the real axis we find:

$$R(E) = 2\pi \int_0^\infty dT \exp\{-i \int_{C(T)} dt \{\dot{r}(t)\dot{r}(t) + \dot{\phi}(t)\dot{\phi}(t)\}\} \times$$

$$\times \int D^{(4)} M(h, l, \theta, \phi) \exp\{-iV_T(r_c, e_C)\},$$

where

$$V_T(r_c, e_C) = S_\phi(r_c) + \int_0^T dt \{(r_c + e_r)^2 + r_c^2 e_\phi^2\}^{-1/2} -$$

$$-((r_c - e_r)^2 + r_c^2 e_\phi^2)^{-1/2} + 2e_re_c^{-2}$$

and $S_\phi(r_c)$ defines the nonintegrable phase factor. The quantization of this factor determines the bound state energy (see below). Such factor will appear in the case if the phase of amplitude can not be fixed (as, for instance, in the Aharonov-Bohm case).

If we rescale the auxiliary field $e_c \rightarrow r_c e_c$ and introduce a new time variable $dt \rightarrow r_c dt$ the $H$-atom problem is solved in the quasiclassical approximation exactly since it becomes isomorphic to the harmonic oscillators case. This solution was firstly demonstrated in [11].

But we wish to demonstrate another solution. For this purpose we need only one fact: the periodicity of $V_T$. Firstly, taking into account eq. (2.20) we can put out the angular quantum fluctuations over $\phi$. This means that

$$e_\phi = 0, \quad j_\phi = 0.$$
In result,

$$V_T(r_c, e_C) = S_o(r_c) + \int_0^T dt \left[ \frac{1}{r_c + e_r} - \frac{1}{r_c - e_r} + 2e_r r_c^{-2} \right]$$

(2.22)

and

$$D^{(4)}M(h, l, \theta, \phi) = \delta(E - h(T)) \prod_t dh(t) dl(t) d\theta(t) d\phi(t) \delta(\dot{h} - p_c(r_c) j_r) \times$$

$$\times \delta(\dot{\theta} - 1 + \frac{\partial r_c}{\partial h} j_r) \delta(\dot{\phi} - \frac{l}{r_c^2}) \delta(\dot{l})$$

(2.23)

Note that now $l$ and $\phi$ are classical quantities and $h, \theta$ are quantum.

We can use now the closeness of classical trajectory $r_c$. Deviding the integral over $\theta$ on the periods ($\theta$ is proportional to eccentric anomaly) we easely find the canelation of quantum corrections to $\theta$:

$$j_r = 0, \quad e_r = 0.$$  

(2.24)

This gives the quasiclassical solutin of the $H$-atom problem:

$$R(E) = 2\pi \int_0^\infty dT \int D^{(4)}M(h, l, \theta, \phi)e^{-iS_o(r_c)},$$

(2.25)

where

$$D^{(4)}M(h, l, \theta, \phi) = \delta(E - h(T)) \prod_t dh(t) dl(t) d\theta(t) d\phi(t) \delta(\dot{h} - 1) \delta(\dot{\phi} - \frac{l}{r_c^2}) \delta(\dot{l}),$$

(2.26)

In result, using the identity [14]:

$$\sum_{-\infty}^{+\infty} e^{iS_1(E)} = 2\pi \sum_{-\infty}^{+\infty} \delta(S_1(E) - 2\pi n),$$

(2.27)

where $S_1(E)$ is the action over one classical period $T_1$ ($S_o = nS_1$):

$$\frac{\partial S_1(E)}{\partial E} = T_1(E),$$

(2.28)

we find from (2.23) the valid exression:

$$R(E) = \pi \Omega \sum_n \delta(E + 1/2n^2)$$

(2.29)

where $\Omega$ is the phase space volume:

$$\Omega = \int dl_0 d\phi_0 d\theta_0.$$  

(2.30)

This infinite coefficient should be canceled by the normalization condition of $R(E)$.  

3 Concluding remarks

In this and in the previous paper [7] there was given the attempt to demonstrate the dynamical mechanism of depression of quantum corrections to the quasiclassical approximation. It is noticeable that the mechanism of depression is due to the compactness of the classical trajectories manifold.

The idea based on the observation that known integrable quantum-mechanical problems can be solved quasiclassically [5]. This mechanism seems natural remembering also the stochastic nature of quantum trajectories.

Note also that the “level” of integrability of the problem shifts singularities in the interaction constant complex plane. Indeed, for the nonintegrable case the singularities are located at the origin [16]. This picture shows the close connection with the threshold singularities of the amplitude [17]. In the semi-integrable case the singularities at origin are canceled [15] and the main (rightest) singularities are located at the finite negative values of interaction constant. And, at the end, the singularities of the integrable systems are located infinitely far from the origin.

This qualitative conclusions confirms the main result of this paper and of [7] that the integrability of quantum systems meonce the stability of the classical trajectory manifolds against quantum excitations.

It is known that the case of totally integrable systems is very rare in the Nature. But our secondary result is the observation that a quantum problem can be integrable over part of the degrees of freedom. This fact should have the important consequences in the field theory.

The partial integrability of the quantum system means the stability of the manifold over some part of the classical degrees of freedom. In quantum field theory absence of quantum corrections leads to absence of radiation [8] and the partial integrability means the impotence of the system concerning radiation of the corresponding degrees of freedom.

Two opposite experimental fact, the confinement of the color charge and the observation of color jets, forced to think that the Yang-Mills theory is partly integrable. The proof of stability of the non-Abelian centre of group manifold can solve this problem. It ensure impossibility to observe the free color charge leaving main part of the theory quantum.

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\footnote{This statement simply follows from the $S$-matrix unitarity condition, or can be shown by the explicit calculations using the generalization of LSZ reduction formula [18]. It will be demonstrated later also.}
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