Standard cosmological evolution in the \( f(R) \) model to Kaluza–Klein cosmology

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Abstract

In this paper, using \( f(R) \) theory of gravity we explicitly calculate cosmological evolution in the presence of a perfect fluid source in four- and five-dimensional space–time in which this cosmological evolution in self-creation is presented by Reddy \textit{et al} (2009 Int. J. Theor. Phys. \textbf{48} 10). An exact cosmological model is presented using a relation between Einstein’s gravity field equation components due to a metric with the same component from \( f(R) \) theory of gravity. Some physics and kinematical properties of the model are also discussed.

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1. Introduction

In recent years, there has been considerable interest in alternative theories of gravitation. The observation that the universe appears to be accelerating at present times has caused one of the greatest problems for modern cosmology. High-precision data from the type Ia supernova, cosmic microwave background and large-scale structure seem to hint that the universe is presently dominated by an unknown form of energy, dubbed dark energy \cite{1–6}. One obvious contender for the role of dark energy is Einstein’s cosmological constant, but particle physics failed to predict the correct density. We refer the reader to \cite{7}.

Recently, a modification of general relativity itself was suggested to explain this accelerating universe \cite{8–18}. For a review, see e.g. \cite{19–21}. The assumption is that the Ricci scalar of the Einstein–Hilbert action is replaced by adding a perturbative function \( h(R) \) to the Einstein–Hilbert action \cite{22}. Recently, it has been claimed that, in all theories that behave as a power of \( R \) at large or small \( R \), standard cosmological evolution cannot be obtained \cite{23,24}. These models have raised much recent interest, perhaps due to their simple nature. The presence of ghosts and stabilities have been studied \cite{25–28}.

Studies of higher dimensional models are also important because of the underlying idea that the cosmos in its early stage of evolution might have had a higher dimensional era \cite{31}. The extra space reduces to a volume with the passage of time, which is beyond the ability of experimental observation at the moment \cite{31}. Our motivation to consider the \( f(R) \) model in five-dimensional (5D) space–time in the presence of a perfect fluid is that the same problem is considered in self-creation cosmology (SCC), in which the corresponding Mach’s principle (MP) is incorporated in SCC by assuming that the inertial masses of fundamental particles are dependent upon their interaction with a scalar field \( \phi \) coupled to the large-scale distribution of matter in a similar fashion as Brans–Dicke theory (BD). We refer the reader to \cite{29–31}. However, instead, with recourse to the \( f(R) \) model, perhaps because of their simple nature, and on modifying the part of geometry instead of matter in the equation of Einstein, we obtain the same result. In this paper, we have obtained cosmological evolution in the presence of a perfect fluid source representing disordered radiation in 4D and 5D space–time \cite{31}. However, cosmological evolution in 5D, in \( f(R) \) models in the presence of a perfect fluid source, has not been investigated.

2. Theoretical framework

For convenience, we consider a class of modified gravity in which we add a perturbative function \( \epsilon h(R) \) to the Einstein–Hilbert action, where \( \epsilon \) is a small parameter. We consider an action as

\[
S = \int \sqrt{-g} \left[ \frac{R + \epsilon h(R)}{2} + k L_m \right] d^4x. \tag{1}
\]
Here $R$ is a Ricci scalar and $L_m$ is the matter Lagrangian. The field equation, using the metric approach, can be derived from action as

$$G_{\mu\nu} = -\epsilon \left[ G^{\mu\nu} + g^{\mu\nu} \Box - \nabla_\mu \nabla_\nu + \frac{g^{\mu\nu}}{2} \left( R - \frac{h(R)}{\varphi(R)} \right) \right] \times \varphi(R) + k T_{\mu\nu}.$$  \hspace{1cm} (2)

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are Einstein and stress–energy tensors, respectively, $\Box \equiv \nabla_\mu \nabla^\mu$ and $\varphi(R) = dh(R)/dR$. Although this theory can be written as scalar tensor theory \cite{14,32}, we shall not use the conformal transformation. We complete all our calculations in the Jordan frame, given by the action above. We consider the metric given by

$$ds^2 = dt^2 - a^2(t)(dr^2 + r^2 d\Omega^2).$$  \hspace{1cm} (3)

Here space–time is assumed to be of flat Friedmann–Robertson–Walker (FRW) type, and the components of the Ricci tensor can be written in terms of the scale factor, $a(t)$, as

$$R'_i = 3 \frac{\dot{a}}{a},$$  \hspace{1cm} (4)

$$R'_r = R'_\theta = R'_\phi = \frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} + \frac{3\dot{a}^2}{a^2},$$  \hspace{1cm} (5)

where the overdot denotes differentiation with respect to cosmic time. The components of the Einstein tensor in terms of the scale factor, $a(t)$, in the flat FRW line element, can be found as follows.

$$G'_i = 3 \left( \frac{\dot{a}}{a} \right)^2,$$  \hspace{1cm} (6)

$$G'_r = G'_\theta = G'_\phi = - \left( \frac{2\ddot{a} + \dot{a}^2}{a^2} \right).$$  \hspace{1cm} (7)

The components of d’Alembertian in the flat FRW line element are

$$\nabla_i \nabla^i = \frac{\ddot{a}^2}{a^2},$$  \hspace{1cm} (8)

$$\Box = \frac{\ddot{a}^2}{a^2} + 3 \frac{\dot{a}^2}{a^2},$$  \hspace{1cm} (9)

$$\nabla_i \nabla^i = \nabla_i \nabla^\theta = \nabla_i \nabla^\phi = \frac{\dot{a}}{a} \frac{\partial}{\partial t}.$$  \hspace{1cm} (10)

The energy–momentum tensor, $T_{\mu\nu}$, for perfect fluid distribution is given by

$$T_{\mu\nu} = (p + \rho) u_{\mu} u_{\nu} - pg_{\mu\nu}.$$  \hspace{1cm} (11)

Here, $p$ is the isotropic pressure, $\rho$ is the energy density and $u_{\mu}$ represents the four velocity of the fluid. Corresponding to the line element given by \cite{3}, the four velocity vector $u_{\mu}$ satisfies the equation

$$g_{\mu\nu} u_{\mu} u_{\nu} = 1.$$  \hspace{1cm} (12)

In a co-moving coordinate system, the component of the Einstein tensor with the help of the field equation, using the metric approach, can be derived from action (1). In field equation (2), if $h(R)/\varphi(R) = 0$ as $R \to 0$, i.e.

$$\lim_{R \to 0} \frac{h(R)}{\varphi(R)} \to 0,$$  \hspace{1cm} (13)

we can neglect $h(R)/\varphi(R)$, and then we have

$$G^\mu_\nu = -\epsilon \left( \delta^\mu_\nu - \nabla^\nu \nabla_\mu \right) \varphi(R) + T^\nu_\mu,$$  \hspace{1cm} (14)

$$G'_i = -\epsilon \left( 3 \frac{\dot{a}}{a} \frac{\partial}{\partial t} \right) \varphi(k\rho),$$  \hspace{1cm} (15)

$$G'_r = G'_\theta = G'_\phi = -\epsilon \left( \frac{\partial^2}{\partial t^2} + 2 \frac{\dot{a}}{a} \frac{\partial}{\partial t} \right) \varphi(R) + k\rho,$$  \hspace{1cm} (16)

where $\rho$ and $p$ are components of $T^\mu_\nu$. Furthermore, from contracting field equation \cite{16}, we find that

$$R = 3\epsilon \Box \varphi.$$  \hspace{1cm} (17)

From combination of equations (6) and (7) with \cite{15} and \cite{16}, one obtains

$$3 \left( \frac{\dot{a}}{a} \right)^2 = -\epsilon \left( 3\dot{a} \frac{\partial}{\partial t} \right) \varphi + k\rho,$$  \hspace{1cm} (18)

$$2\ddot{a} + \dot{a}^2 = \epsilon \left( \frac{\partial^2}{\partial t^2} + 2 \frac{\dot{a}}{a} \frac{\partial}{\partial t} \right) \varphi - k\rho.$$  \hspace{1cm} (19)

Field equations \cite{17}–\cite{19} are three independent equations in four unknown variables $a$, $\varphi$, $\rho$ and $p$. Hence to obtain a determinate solution and correspondence with the standard Einstein cosmology solution, one has to assume a physical or mathematical condition. We choose the scale factor as

$$a(t) = t^\gamma,$$  \hspace{1cm} (20)

and the equation of state as

$$\rho = 3p,$$  \hspace{1cm} (21)

which represents disordered radiation in 4D space. Now, using equations \cite{20} and \cite{21}, field equations \cite{17}–\cite{19} yield an exact solution for $a$, $\varphi$, $\rho$, $p$. According to equations \cite{18}–\cite{20}, we can choose $\varphi$ as

$$\varphi = a \ln t + c,$$  \hspace{1cm} (22)

where $a$ and $c$ are arbitrary constants. Substituting equations \cite{22}, \cite{18} and \cite{19} into equation \cite{17}, the power-law solution can be identified as

$$\gamma = \frac{1}{2} + \frac{\alpha \epsilon}{4}.$$  \hspace{1cm} (23)

By substituting equation \cite{23} into equations \cite{18} and \cite{21}, we arrive at

$$\rho = \frac{3}{4kt^2} (1 + \alpha \epsilon),$$  \hspace{1cm} (24)

$$p = \frac{1}{4kt^2} (1 + \alpha \epsilon),$$  \hspace{1cm} (25)

$$H = \frac{2 + \alpha \epsilon}{4t}.$$  \hspace{1cm} (26)
where, with $\epsilon = 0$, solutions above correspond to epochs of radiation domination. The energy density $\rho$ and the isotropic pressure $p$ tend to zero as time increases indefinitely. Also, Hubble’s parameter, $H$, tends to zero as $t \to \infty$. The positive value of the deceleration parameter $q$ shows that the model decelerates in the standard way.

### 3. Kaluza–Klein cosmology in $f(R)$ theory of gravity

Once more we consider action (1) together with field equation (2) that yields to variation of action (1) for the line element of 5D Kaluza–Klein space–time given by

$$ds^2 = dt^2 - a^2(t) \left( dx^2 + dy^2 + dz^2 \right) - A^2(t) dy^2.$$

In Kaluza–Klein space–time, the components of the Ricci tensor and the Ricci scalar can be written in terms of the scale factors $a(t)$ and $A(t)$. As a result, we find

$$R_{tt} = \frac{3\dot{a}}{a} + \frac{\dot{A}}{A},$$

$$R_{\varphi\varphi} = -\frac{3\dot{a}A}{a} - \dot{A}a,$$

$$R_{xx} = R_{yy} = R_{zz} = -2(\dot{a})^2 - \frac{\dot{a}A}{A} - a\ddot{a},$$

$$R = 6\left( \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \right) + 6\frac{\dot{A}a}{A} + 2\frac{\dot{A}a}{A},$$

The components of the Einstein tensor in terms of the scale factors $a(t)$ and $A(t)$, in the Kaluza–Klein space–time line element, can be found as follows:

$$G^t_t = 3\left( \frac{\dot{a}}{a} \right)^2 + \frac{3\dot{a}A}{aA},$$

$$G^\varphi_\varphi = -3\left( \frac{\dot{a}}{a} \right)^2 - 3\frac{\ddot{a}}{a},$$

$$G^t_\varphi = -\left( \frac{\dot{a}}{a} \right)^2 - 2\frac{\ddot{a}}{a} - 2\frac{\dot{a}A}{aA} - \frac{\dot{A}a}{A}.$$

Dolambrian and its components to the Kaluza–Klein line element are as follows:

$$\nabla_t \nabla^t = \frac{\dot{a}^2}{a},$$

$$\nabla_\varphi \nabla^\varphi = \frac{A}{A} \frac{\partial}{\partial t},$$

$$\nabla_x \nabla^x = \nabla_\varphi \nabla^\varphi = \frac{\dot{a}}{a} \frac{\partial}{\partial t},$$

$$\Box = \frac{\ddot{a}^2}{a} + \left( \frac{3\dot{a} + \dot{A}}{aA} \right) \frac{\partial}{\partial t}.$$
We require that the model decelerates in the standard way, so that \( q \) must be bigger than zero. Therefore, we find an extra condition on \( \gamma \) as

\[
0 < \gamma < 1,
\]
and then

\[
n \neq -3.
\]

For considering the stability of this model of \( f(R) \), we must obtain the second derivation of \( f(R) \) with respect to \( R \). It is well known that, for an arbitrary \( f(R) \) model, if \( d^2 f(R)/dR^2 > 0 \), then that model is stable [25, 28]. Therefore, using equations (22) and (43), we obtain

\[
\frac{d^2 f(R)}{dR^2} = \frac{\epsilon}{dR} = \frac{3\gamma^2}{16\epsilon((n+3)\gamma-1)}.
\]

It is clearly seen that \( R = \infty \) as \( t \) tends to zero, i.e.

\[
\lim_{t \to 0} R \to \infty.
\]

This means that this model has initial singularity. Thus, the 5D Kaluza–Klein cosmological line element corresponding to the above solutions can be written in \( \epsilon = 0 \) by

\[
ds^2 = dt^2 - \left(t^{\frac{1}{n}}\right)^2 \left(dx^2 + dy^2 + dz^2 + \xi^2 t^{\gamma(n-1)} \right) d\phi^2.
\]

### 4. Some physical properties of the model

Equations (49)–(54) represent an exact 5D Kaluza–Klein cosmology in the framework of \( f(R) \) theory of gravity in the presence of a perfect fluid source. We observe that the \( f(R) \) theory of gravity has initial singularity, equation (62), and we have seen that with \( n \) defined by equations (60) and (61), the requirement of stability is satisfied. For equation (64), the physical and kinematical variables that are important in cosmology are \( \rho, p, q \) and \( H \). The energy density \( \rho \) and the isotropic pressure \( p \) tend to zero as time increases indefinitely. For this model Hubble’s parameter tends to zero as \( t \to \infty \). The positive value of the deceleration parameter \( q \) shows, for both 4D and 5D, deceleration in the standard way.

### 5. Discussion

A modification of gravity has been suggested in the form of \( R + e\phi(R) \) theories, in the presence of a perfect fluid source. The 5D Kaluza–Klein and 4D line element in the \( f(R) \) theory of gravity have stability and initial singularity. The model is expanding, non-rotating and decelerates in the standard way.

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