Molecules that conduct electrical current [1–4], when embedded in a junction between two metal electrodes [5], can become the active element in a circuit, such as a rectifier [6, 7] or a memory element [8]. Alternatively, molecules deposited on a metal substrate can be probed by a scanning tunneling microscope to study their diffusion [9], conformation changes or a memory element [8]. Strong coupling between electronic and vibrational degrees of freedom plays a critical role for the molecule’s functional properties [15, 16]. The electron-phonon (e-ph) coupling is thus specified by two parameters, \( \Gamma \) and \( \epsilon \). Strong coupling eliminates the fluctuating-sign problem of the lowest-order potential [37, 38] and the displacement exponentially modulates the tunneling barrier [39]. The anharmonicity is required at the displaced molecule feels a realistic Morse potential, while magnetic molecules in contact with a superconductor. The Kondo screening of the local moment [28, 29], while magnetic molecules on normal metal surfaces, the low-temperature spectra exhibit zero-bias anomalies due to the Kondo effect and the electron-phonon coupling [10, 11], dissociation [12], and chemical reactions [13, 14]. In magnetic molecules on normal metal surfaces, the low-temperature spectra exhibit zero-bias anomalies due to the competition between the Kondo effect and the electron-phonon coupling [10, 11], dissociation [12], and chemical reactions [13, 14].

In this work we study a realistic model of a deformable magnetic molecule in contact with a superconductor. The vibrational mode is described using the anharmonic Morse potential \( \epsilon_0 = b/2m + V_{\text{Morse}} \). We define the harmonic frequency as \( \omega_0 = \sqrt{b/2m} \), and the displacement and momentum operators as \( \hat{x} = (\hat{a} + \hat{a}^\dagger) \sqrt{1/2m\omega_0} \) and \( \hat{p} = i(\hat{a}^\dagger - \hat{a}) \sqrt{m\omega_0/2} \), where \( \hat{a} \) and \( \hat{a}^\dagger \) are the phonon ladder operators.

The shape of the potential is then fully described by two parameters, \( D_e \) and \( \omega_0 \). The harmonic potential is recovered in the limit \( D_e \to \infty \), keeping \( \omega_0 \) constant. In the coupling part \( H_{\text{coup}} = V(x) \sum_{k,\sigma} (c_{k,\sigma}^\dagger d_\sigma + \text{h.c.}) \), the tunneling term is exponentially modulated by the molecular displacement:

\[
V(x) = V_0 \exp \left[ -g(\hat{a} + \hat{a}^\dagger) \right],
\]

where \( g > 0 \) is the e-ph coupling constant. The hybridization strength at zero displacement is characterized by \( \Gamma_0 = \pi \rho V_0^2 \), where \( \rho \) is the density of states in the band in the normal state. In all numerical calculations presented in this work, we set \( U = 1, \Delta = 0.04, \) and \( \omega_0 = 0.01 \).

In the absence of e-ph coupling \( g = 0 \) and close to the particle-hole symmetric point \( \delta \sim 0 \), the ground state of the main ABS peaks. The side-peak features visible for generic model parameters even for moderate, experimentally relevant e-ph coupling strength. Because the peak width is due only to thermal broadening and experimental noise, this setup permits very precise determination of the phonon frequency renormalization due to the e-ph coupling and the e-e interactions.

We describe the system with the impurity model \( H = H_{\text{band}} + H_{\text{mol}} + H_{\text{osc}} + H_{\text{coup}} \). Here \( H_{\text{band}} = \sum_{k,\sigma} \epsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} + \Delta \sum_{k,\sigma} (c_{k,\sigma}^\dagger c_{-k,-\sigma}^\dagger + \text{h.c.}) \), where \( c_{k,\sigma} \) are the conduction-band electron operators with momentum \( k \), spin \( \sigma \) and energy \( \epsilon_k \), and \( \Delta \) is the superconducting gap. \( H_{\text{mol}} = \epsilon(n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow \) is the molecule Hamiltonian, where \( n_\sigma = d_\sigma^\dagger d_\sigma \), \( \epsilon \) is the on-site energy, and \( U \) the e-e repulsion. The quantity \( \delta = \epsilon + U/2 \) measures the deviation of the system from the particle-hole (p-h) symmetric point.

The displaced molecule feels a realistic Morse potential of the form

\[
V_{\text{Morse}} = D_e \left[ 1 - \exp \left( -b\hat{x} \right) \right]^2,
\]

where \( D_e \) is the well depth and \( b \) controls its width. The oscillator Hamiltonian is thus \( H_{\text{osc}} = \hat{p}^2/2m + V_{\text{Morse}} \). We define the harmonic frequency as \( \omega_0 = b/2m \), and the displacement and momentum operators as \( \hat{x} = (\hat{a} + \hat{a}^\dagger) \sqrt{1/2m\omega_0} \) and \( \hat{p} = i(\hat{a}^\dagger - \hat{a}) \sqrt{m\omega_0/2} \).
system depends on the relative values of the Kondo temperature $T_K$ and the BCS gap $\Delta$. In the limit $T_K \gg \Delta$, the local moment is screened by non-paired conducting electrons. The ground state is then a spin singlet (S) many-body Kondo state. In the opposite limit of $T_K \ll \Delta$, the formation of the Cooper pairs leaves no low-energy electrons available to screen the impurity spin. The ground state is then a spin doublet (D) unscreened many-body state. For

$$\omega > \Gamma_0 = \Gamma(\omega, 0),$$

the enhancement of the hybridization is approximated by the e-ph coupling \[39, 42–44\], see also Fig. 3(d)-(f), and, furthermore, the effective energy spacing is renormalized by the e-ph coupling \[39, 42–44\], see also Fig. 3(d)-(f).

We now consider the effect of the e-ph coupling ($g > 0$) on the subgap states. For small coupling the hybridization is renormalized as $\Gamma_0 \to \tilde{\Gamma} = \Gamma(g, (x))$, where $(x)$ is the expectation value of the displacement that is negative (the molecule approaches the surface) and linear in $g$ [see Fig. 3(g)-(i)], thus the enhancement of the hybridization is approximately quadratic. For large coupling this mean-field approximation is no longer accurate, but the trend toward stronger effective $\Gamma$ remains. In addition, if the phonon frequency is smaller that the BCS gap, $\omega_0 < \Delta$, entirely new features (vibronic ABS sidepeaks) appear in the subgap part of the molecule spectral function, as we show in the following.

The full problem is solved using the numerical renormalization group \[40, 41\] with extensions for superconducting bands \[30, 32\]. This technique treats all interactions (e-ph coupling, e-e repulsion, BCS pairing) on equal footing. The calculations have been performed for the discretization parameter $\Lambda = 4$ and are fully converged with respect to the phonon cutoff and the number of states kept in the truncation.

We first consider the evolution of the subgap states as a function of the hybridization strength $\Gamma_0$, keeping the e-ph coupling $g$ constant, see Fig. 1(a). This is motivated by the experimental realization of the singlet-doublet crossing, which is tuned by small changes in the molecule-substrate coupling \[36\]. The calculation has been performed for generic parameter values, in particular away from the p-h symmetric point. The level crossing between the doublet and singlet ground states D0 and S0 occurs at $\Gamma_0 = \Gamma_0$. All other excited subgap states, one series of D states (D1, D2, ...) and another of S states (S1, S2, ...), are phonon induced. For a harmonic potential and very weak e-ph coupling, these vibronic states would form ladders with equidistant spacing $\omega_0$. For the anharmonic Morse potential, however, the exact solution for the vibrational energy levels includes a quadratic correction term \[37\] and, furthermore, the effective energy spacing is renormalized by the e-ph coupling \[39, 42–44\], see also Fig. 3(d)-(f).

The subgap part of the spectral function, Fig. 1(b), demonstrates that in addition to the main spectral peak (S0-D0 transitions), other phonon-induced side-peaks (S0-D1, D0-S1, etc. transitions) are also present and have sufficient spectral weight to be observable. The subfigures Fig. 1(b1-b3) show spectral curves at three characteristic parameter regimes; first side peaks are clearly resolved in all cases. The ratio between the spectral weight of the side peaks and that of the main ABS peaks (i.e., the visibility), Fig. 1(c), is maximal near the level crossing. We also find that the visibility is largest at the p-h symmetric point ($\delta = 0$) and is reduced somewhat away from
as a function of \( S \) sector [see Fig. 3(e)]. The deformation of the molecules phonon mode softens in the \( D \) sector, while it hardens in the \( S \) for intermediate anharmonicity where, as a function of \( g \), it softens. We also observe remarkably large differences in the \( D \) and \( S \) sectors, which are the most pronounced for intermediate anharmonicity where, as a function of \( g \), the phonon mode softens in the \( D \) sector, while it hardens in the \( S \) sector [see Fig. 3(e)]. The deformation of the molecules as a function of \( g \) is shown in Fig. 3(f) [notice that the horizontal ranges are different]. In the harmonic limit the e-ph coupling leads to unphysically strong deformation already at small values of \( g \), while for anharmonic potential, the displacement of the molecule is constrained by the repulsive part of the Morse potential. The insets represent the fluctuations of the effective phonon frequency in- 

ternal excited states exhibit an unexpected feature: with increasing e-ph coupling, the effective phonon frequency increases (the phonon mode hardens). The renormalization of the phonon frequency has been noted in previous studies of impurity models with vibration modes \([39, 42–44]\), where the phonon mode softens. This is the case both for the coupling to charge (Anderson-Holstein model) and for the coupling to the center-of-mass modes. The phonon hardening is a characteristic feature of the anharmonic potential and results from the displacement of the oscillator to the part of the potential with higher second derivative, while the softening results from the electron tunneling fluctuations \([45]\) and occurs generically. Both tendencies are present in our model and the dominant effect depends on the value of the parameter \( D_e \).

In order to compare the degree of the anharmonicity, the bare (non-renormalized) potential profiles are shown in the top panel of Fig. 3. In Fig. 3(a)-(c) the energy dependence of the Andreev states shows the S-D transition as a function of the e-ph coupling \( g \) for different values of \( D_e \). The renormalized phonon frequencies, Fig. 3(d)-(f), indicate that for \( D_e = 0.1 \), the potential is strongly anharmonic and the phonon mode hardens, while for \( D_e = 5.0 \), in the harmonic limit, it softens. We also observe remarkably large differences in the \( D \) and \( S \) sectors, which are the most pronounced for intermediate anharmonicity where, as a function of \( g \), the phonon mode softens in the \( D \) sector, while it hardens in the \( S \) sector [see Fig. 3(e)]. The deformation of the molecules as a function of \( g \) is shown in Fig. 3(f) [notice that the

FIG. 2. (Color online) (a) Energy of the subgap singlet (S) and doublet (D) states as a function of the electron-phonon coupling \( g \). (b) Subgap part of the spectral function. Parameters are \( \Gamma_0 = 0.1 \), \( D_e = 0.1 \), and \( \delta = 0 \).

FIG. 3. (Color online) Top panel: bare (non-renormalized) Morse potential for different parameter of the anharmonicity. Bottom panel: (a,b,c) Energy of the subgap singlet (S) and doublet (D) states versus electron-phonon coupling \( g \) for different parameter of the anharmonicity. Parameters: \( \Gamma_0 = 0.1 \), \( \delta = 0 \). (d,e,f) Effective phonon frequencies of D and S states. (g,h,i) Displacement and fluctuations (insets) of the displacement. (j,k,l) Visibility of first vibronic side-peak.
effective potential is significantly shifted due to the e-ph coupling to its new equilibrium position and we find that the effective potential is not the same for S and D states. This implies that the oscillator parameters are renormalized differently in the two spin sectors due to different electron tunneling rates in the unscreened D and Kondo screened S many-body states. In the weak coupling (small-$g$) regime the effect of the phonons on the subgap part of the spectral function can be understood using the Frank-Condon principle [47], which states that the transition between the vibrational states is more likely to happen if the vibrational wave functions overlap more significantly. The overlap of the effective vibronic part of the wave functions, $\phi(x) = \sqrt{\rho(x)}$, is proportional to the transition probability between the states. For the lowest-lying singlet $S_0$ and doublet $D_0$ states, and the excited singlet $S_1$ and doublet $D_1$ states, the overlaps are represented in Fig. 4(d). Within the Franck-Condon approximation the overlap ratio $\langle S_0|D_1 \rangle^2/\langle S_0|D_0 \rangle^2$ is proportional to the visibility and provides good agreement with the actual values, Fig. 3(j), for small electron-phonon couplings $g$. However, since the Franck-Condon principle is based on the Born-Oppenheimer approximation, polaronic effects are neglected and, therefore, it cannot explain the features in the spectral function for e-ph interaction close to or beyond the level crossing. The sudden increase in the overlap between the $S_0$ and $D_1$ states for $g \approx 0.115$, coincides with the transition of the $D_1$ state into the continuum, where the excited $D_1$ state strongly mixes with the direct product states consisting of the $S_0$ state and an additional continuum low-energy quasiparticle, thus its vibrational properties are essentially those of the $S_0$ state, which explains the perfect overlap.

We have shown that due to the deformability of the molecule additional vibronic states occur inside the BCS gap, which are clearly visible in the subgap part of the spectral function. We find that generically the intensity of the side peak reaches a maximum close to the ground-state level crossing, but that the peak intensity depends on the system parameters: it increases near the particle-hole symmetric point, for large $D_s$, and for large e-ph coupling $g$. The increase in the visibility for weak e-ph coupling can be described using the Franck-Condon principle.

To test the predictions of this work, we propose as candidate systems planar macrocyclic molecules which form coordination complexes with weak metal-ligand bonds, so that the magnetic ions support low frequency "rattling" modes, while appropriate surfaces are elemental BCS superconductors.

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