Some surprises in the neutrino cross sections associated with neutrino spin

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Abstract

It is generally assumed that neutrino masses can be neglected to a high degree of approximation in cross section calculations. This assumption seems very reasonable since the neutrino masses are extremely small and the neutrinos are ultrarelativistic fermions at the energy scales of current experiments. Consequently, in cross section calculations in the Quantum Field Theory, the Standard Model neutrinos are frequently assumed to be described by 100% negative helicity states. This assumption is true in a sense that in the Standard Model processes the positive helicity states can be safely neglected for ultrarelativistic neutrinos. On the other hand, the assumption tacitly assert that the neutrino fields are completely longitudinally polarized, i.e., the contribution to the cross section coming from transverse polarization can be neglected. We show that this tacit assertion is not correct. Although the Standard Model cross section for a neutrino with positive helicity goes to zero as $m_\nu \to 0$, the cross section for a neutrino with transverse polarization remains finite in that limit. Thus the contribution coming from transverse polarization cannot be neglected even in the ultrarelativistic/zero-mass limit. We examine the consequences of this fact and deduce that it has some unexpected results in the neutrino cross sections.

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I. INTRODUCTION

According to the Standard Model (SM) of particle physics the neutrinos couple minimally to other SM particles only through $V - A$ type vertex and hence the interaction project out the left chiral component of the neutrino field. Consequently, all interacting neutrinos in the SM can be accepted to be left chiral which can be written mathematically as $\frac{1}{2}(1 - \gamma_5)u_\nu(p) = u_\nu(p)$ where $u_\nu(p)$ is the spinor for the neutrino. It is well known that massless fermions are completely longitudinally polarized. They are described by pure helicity states which coincide with chirality eigenstates. If the mass of the fermion is zero then its positive and negative helicity states coincide with right-handed and left-handed chirality eigenstates respectively. Since all SM neutrinos are left chiral, massless neutrinos must be described by 100% negative helicity states. On the other hand, as we know from experimental results obtained in Super-Kamiokande and Sudbury Neutrino Observatory that neutrinos oscillate and they cannot be massless. Although neutrinos are not massless they possess very tiny masses and hence they are ultrarelativistic at the energy scales of current experiments. Consequently, during cross section calculations it is generally assumed (except for some direct neutrino mass measurement experiments) that neutrino masses can be neglected and the neutrinos are described by 100% negative helicity states. Ignoring neutrino masses is an approximation which is believed to be valid with a high degree of accuracy for energies much greater than the neutrino mass. On the contrary, we will show in this paper that the approximation is not as accurate as expected even in the zero-mass limit.

The crucial point which is generally skipped in the literature is that the solutions of the free Dirac equation describing a general spin orientation have a discontinuity at the point $m = 0$. If we take the zero-mass limit of the spinor $u^{(s)}(p)$ describing a general spin orientation we do not get, in general, its value evaluated at $m = 0$, i.e., $\lim_{m \to 0} u^{(s)}(p) \neq u^{(s)}(p)|_{m=0}$. According to the seminal work of Wigner, strictly massless fermions are longitudinally polarized and described solely by helicity eigenstates. However if the fermion has a non-zero mass (no matter how small it is), then it is allowed to have an arbitrary spin orientation which is different from longitudinal direction. It is quite surprising that the transverse polarization does not disappear in the zero-mass limit but it vanishes instantly.

\footnote{In this paper only Dirac neutrinos and their SM interactions have been considered.}
at the point $m = 0$. This behavior is the origin of the discontinuity of the free Dirac solutions with general spin. If we restrict ourself to special type of Dirac solutions, namely helicity states we observe that the helicity states converge to the chirality eigenstates in the zero-mass limit and we do not encounter any discontinuity at $m = 0$. However, the zero-mass behavior observed from helicity states is not valid in general. Although the helicity states converge to the chirality eigenstates in the zero-mass limit, a spinor with arbitrary spin orientation does not necessarily result in a chirality eigenstate in that limit. For instance, the spinor with transverse polarization (relative to the direction of momentum) is always given by a mixed chirality eigenstate and hence does not converge to one of the chirality eigenstates left-handed or right-handed even in the zero-mass limit. The discontinuity of the free Dirac spinors at $m = 0$ induces a similar discontinuity in the cross sections. If we calculate the cross section for a neutrino with mass $m_\nu$ and then take its $m_\nu \to 0$ limit what we get is different from the cross section in which the neutrino is initially assumed to be massless, i.e., $\lim_{m_\nu \to 0} \sigma(m_\nu) \neq \sigma(0)$. As a result of this discontinuous behavior, neglecting neutrino masses in the cross section is not a good approximation even though neutrino masses are extremely small compared to the energy scale of the processes that we consider.

The organization of the paper is as follows. In section II we review the free Dirac spinors describing a general spin orientation and their discontinuous behavior at $m = 0$. In section III-A we present cross section calculations in some generic SM processes, assuming that neutrinos are massless. In section III-B the cross section calculations are performed for massive neutrinos but the mixing between different mass eigenstates is omitted for simplicity. In section III-C, a more realistic situation is considered where both neutrino masses and mixing are taken into account. In the conclusions section (section IV) we summarize the results that we obtain and discuss briefly some of its implications.

II. ZERO-MASS DISCONTINUITY OF THE DIRAC SPINORS

Let us review shortly the free Dirac spinors describing a general spin orientation. Assume that in the rest frame of the fermion, its spin is quantized along the direction defined by the

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2 The explicit expressions for Dirac spinors describing a general spin orientation and their behavior in the zero-mass limit can be found in Ref. [4] in detail.
unit vector $\vec{n}$. Then, in the rest frame we can write the following eigenvalue equations

$$(\vec{n} \cdot \vec{S}) u^{(\uparrow)}_{RF} = + \frac{1}{2} u^{(\uparrow)}_{RF}, \quad (\vec{n} \cdot \vec{S}) u^{(\downarrow)}_{RF} = - \frac{1}{2} u^{(\downarrow)}_{RF};$$

where $\vec{S} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ is the non-relativistic 4×4 spin matrix. The eigenvectors (rest spinors) which correspond to the eigenvalues $+1/2$ and $-1/2$ are called spin-up ($\uparrow$) and spin-down ($\downarrow$) spinors respectively. The spinor for a moving fermion can be obtained by applying a Lorentz boost to the spinor at rest. Suppose that $S'$ frame is moving along negative $z$-axis with relative speed $v$ with respect to the rest frame of the fermion $S$. Then the observer in the $S'$ frame sees a moving fermion with four-momentum $p^\mu = (E, \vec{p}) = (E, 0, 0, p_z)$ and four-spinors

$$s^\mu = L^\mu_\nu (s')_{RF} = \left( \frac{\vec{p} \cdot \vec{n}}{m}, \vec{n} + \frac{\vec{p} \cdot \vec{n}}{m(E + m)} \vec{p} \right),$$

where $L^\mu_\nu$ is the Lorentz transformation tensor and $(s')_{RF} = (0, \vec{n})$ is the spin vector defined in the rest frame of the fermion. Without loss of generality, choose $\vec{n} = \sin \theta \, \hat{x} + \cos \theta \, \hat{z}$, i.e., $\vec{n}$ is in the $z$-$x$ plane which makes an angle $\theta$ (polar angle) with respect to the $z$-axis. Then according to an observer in $S'$, the spin-up ($\uparrow$) and spin-down ($\downarrow$) spinors describing a general spin orientation are given by

$$u^{(\uparrow)}(p) = \cos \left( \frac{\theta}{2} \right) u^{(+)}(p) + \sin \left( \frac{\theta}{2} \right) u^{(-)}(p)$$

$$u^{(\downarrow)}(p) = \cos \left( \frac{\theta}{2} \right) u^{(-)}(p) - \sin \left( \frac{\theta}{2} \right) u^{(+)}(p)$$

where $u^{(+)}(p)$ and $u^{(-)}(p)$ represent positive and negative helicity spinors which correspond to the special spin orientation (special orientation of the spin quantization axis) $\vec{n} = \frac{\vec{p}}{|p|}$ or equivalently $\theta = 0$. It is obvious from equations (1) or (3) and (4) that the spin-up and spin-down spinors are interchanged under the transformation $\vec{n} \to -\vec{n}$ or equivalently $\theta \to \pi + \theta$. Finally, let us stress the simple but crucial point which is at the heart of the analysis presented in this paper. The angle $\theta$ that appears in equations (3) and (4) is not a dynamical variable. It does not depend on the relative velocity between the frames $S$ and $S'$. It is the angle measured in the frame in which the particle is at rest. Hence, $\theta$ is not affected by relativistic aberration. The angle $\theta$ resembles the term "proper time" which is a frame-independent quantity. Due to this resemblance, we will call it "proper angle".
We should also remind the fact that when we talk about the spin orientation of a moving fermion, we mean the orientation of the spin quantization axis $\vec{n}$ in the rest frame of the particle. Therefore the spinors $\mathbf{(3)}$ and $\mathbf{(4)}$ for a general spin orientation, describe a fermion which in its rest frame its spin is quantized along $\vec{n} = \sin \theta \hat{x} + \cos \theta \hat{z}$.

Now we are ready to discuss the discontinuous behavior of the Dirac spinors at $m = 0$. Since the expressions $\mathbf{(3)}$ and $\mathbf{(4)}$ for spinors with general spin orientation are obtained by means of a Lorentz transformation from the rest frame of the fermion $S$ to a moving frame $S'$, they remain valid for every value of the relative speed $v$ that satisfies $|v - c| < \epsilon$ where $\epsilon$ is infinitesimal. Consequently, the zero-mass ($m \to 0$) or ultrarelativistic ($v \to c$) limit of the spinors $u^{(\uparrow)}(p)$ and $u^{(\downarrow)}(p)$ exists and given by

$$\lim_{m \to 0} u^{(\uparrow)}(p) = \cos \left( \frac{\theta}{2} \right) u^{(R)}(p) + \sin \left( \frac{\theta}{2} \right) u^{(L)}(p)$$

$$\lim_{m \to 0} u^{(\downarrow)}(p) = \cos \left( \frac{\theta}{2} \right) u^{(L)}(p) - \sin \left( \frac{\theta}{2} \right) u^{(R)}(p)$$

where $u^{(R)}(p) = \lim_{m \to 0} u^{(+)}(p) = u^{(+)}(p) |_{m=0}$ and $u^{(L)}(p) = \lim_{m \to 0} u^{(-)}(p) = u^{(-)}(p) |_{m=0}$ are the right-handed and left-handed chirality eigenstates. On the other hand, the Lorentz group is non-compact and the parameter space of the Lorentz group does not contain the point $v = c$. Therefore, we cannot perform a Lorentz transformation to the rest frame of a massless particle. In other words, massless particles do not have a rest frame. Consequently, the expressions $\mathbf{(5)}$ and $\mathbf{(6)}$ become invalid for strictly massless particles. In the case of massless fermions, we should employ the little group analysis of Wigner $\mathbf{[1]}$. According to Wigner massless particles are described by $E(2)$-like little group and that their spin orientations other than parallel or antiparallel to the direction of momentum are not allowed. Hence, massless fermions must be completely longitudinally polarized and described by pure helicity states which coincide with chirality eigenstates. We observe from equations $\mathbf{(5)}$ and $\mathbf{(6)}$ that the zero-mass limit of the spinors for a general spin orientation are not equal to a chirality eigenstate unless $\theta = 0$ or $\pi$. Therefore the spinors $u^{(\uparrow)}(p)$ and $u^{(\downarrow)}(p)$ have a discontinuity at $m = 0$ that is $\lim_{m \to 0} u^{(s)}(p) \neq u^{(s)}(p) |_{m=0}$ for $\theta \neq 0, \pi$ where $u^{(s)}(p) |_{m=0}$ is either $u^{(R)}(p)$ or $u^{(L)}(p)$. In the case of SM neutrinos $u^{(s)}(p) |_{m=0} = u^{(L)}(p)$. 

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III. NEUTRINO CROSS SECTION FOR GENERAL SPIN ORIENTATION

A. Massless case

In the SM of particle physics, neutrinos interact through the weak interaction. Hence any SM process which contains the neutrinos involve \( W \) or/and \( Z \) boson exchange. The former generates charged current and the latter generates neutral current neutrino interactions. In both of the cases, the interaction is proportional to the left chirality projection operator \( \hat{L} = \frac{1}{2}(1 - \gamma_5) \). Hence, the neutrinos must be left-handed chiral in interactions. This fact is always true in the SM, independent to whether the neutrinos are massless or not. Assume that neutrinos are strictly massless. In this case, the neutrinos must also be described by a pure negative helicity state. This is evident since massless fermions are completely longitudinally polarized, and that their positive and negative helicity eigenstates coincide with right-handed and left-handed chirality eigenstates. Possibly because of this reason, sometimes the terms "left-handed" and "negative helicity" are used interchangeably in the literature for massless neutrinos, although there are some differences in their meaning. However one should be very careful in the case of massive neutrinos and does not use these terms instead of each other even though neutrino masses are extremely small.\(^3\)

Let us first assume that neutrinos are strictly massless and their flavor and mass eigenstates coincide. Consider single neutrino production and absorption processes \( ab \rightarrow \nu c \) and \( \nu a' \rightarrow b' c' \) where \( a, b, c, a', b', c' \) are charged fermions. The tree-level amplitudes for these processes can be written in the form

\[
M = g \left( J_{\nu C}^\alpha J'_{C \alpha} \right) \big|_{m_\nu = 0} \tag{7}
\]

where \( J_{\nu C}^\alpha \) is the charged current that contains the neutrino field and \( J'_{C \alpha} \) is the charged current for charged fermions and \( g \) is some constant. In writing equation (7) we assume that the \( W \) propagator can be approximated as

\[
\frac{(g_{\mu \nu} - q_{\mu} q_{\nu}/m_w^2)}{q^2 - m_w^2} \approx -\frac{g_{\mu \nu}}{m_w^2}.
\]

The explicit form of the

\(^3\) Sometimes the terms "left-handed" and "right-handed" are used for the eigenstates of the helicity instead of chirality. This is a matter of convention but the important thing is not to confuse the eigenstates of the helicity and chirality. In this paper we use the terms "left-handed" and "right-handed" for the eigenstates of the chirality and "negative helicity" and "positive helicity" for the eigenstates of the helicity.
charged neutrino current is given by
\[ J_C^\alpha = \left[ \bar{u}_\nu \gamma^\alpha \hat{L} u_\ell \right] \]
for the process \( ab \to \nu c \) (8)
\[ J_C^\alpha = \left[ \bar{u}_\ell \gamma^\alpha \hat{L} u_\nu \right] \]
for the process \( \nu a' \to b' c' \) (9)

where \( \hat{L} = \frac{1}{2} (1 - \gamma_5) \) is the left chirality projection operator. The unpolarized cross section is proportional to the squared amplitude which is averaged over initial and summed over final spins. In the case of single neutrino production, the sum over final neutrino spin gives just one term with \( s_\nu = -1 \) that corresponds to negative helicity. Hence, for \( m_\nu = 0 \) the produced neutrinos are completely longitudinally polarized⁴ and described by a state with 100% negative helicity. In the case of single neutrino absorption, we do not perform an average over initial neutrino spins and omit the factor of \( 1/2 \) coming from spin average of initial neutrinos. Omitting initial neutrino spin average is based on the assumption that all neutrinos in the SM are described by completely longitudinally polarized negative helicity states. This assumption is obviously true for massless neutrinos. The neutrinos which enter neutrino absorption processes should be produced through some production processes. If the neutrinos are strictly massless, then all produced neutrinos through a SM process are indeed completely longitudinally polarized and described by a state with 100% negative helicity. Nevertheless, as we will see in the next subsection surprisingly, the assumption underestimates the cross section for neutrinos with non-zero mass even though neutrino masses are very tiny.

Now let us consider another simple process the neutrino scattering from a charged fermion, \( \nu a \to \nu a \). Depending on the type of the charged fermion \( a \), the process may contain only neutral or both neutral and charged neutrino currents. If we consider the most general case, the tree-level amplitude for the process can be written in the form
\[ M = g \left( J_N^\alpha J_N^{\alpha'} + J_C^\alpha J_C^{\alpha'} \right) \big|_{m_\nu=0} \]
where \( J_N^\alpha \) and \( J_N^{\alpha'} \) are the neutral currents for the neutrino and charged fermion respectively. \( J_C^\alpha \) is the charged neutrino current defined, similar to (8) and (9). For completeness, let us write the explicit form of the neutral neutrino current:
\[ J_N^\alpha = \left[ \bar{u}_\nu \gamma^\alpha \hat{L} u_\nu \right] \]

⁴ The statement "completely longitudinally polarized" is used in the meaning that the only possible spin orientation is the one which is parallel or anti-parallel to the direction of momentum.
Here, $u_{\nu_i}$ ($u_{\nu_f}$) represents the spinor for initial (final) neutrino field. Similar to the single neutrino absorption, we do not perform an average over initial neutrino spins and omit the factor of 1/2 coming from spin average of initial neutrinos.

B. Massive case without mixing

Now assume that neutrino masses are not strictly zero although they are extremely small. We also assume that flavor and mass eigenstates of the neutrino coincide, i.e., we ignore the mixing. Then, the polarized cross section for a process where the neutrino has a spin orientation defined by the proper angle $\theta$, can be obtained by inserting the spinors (3) or (4) into the relevant squared amplitudes and performing the phase space integration. We additionally assume that the energy scale of the process is much greater than the mass of the neutrino, $E >> m_\nu$. Then, it is a very good approximation to use the expressions obtained in the $m_\nu \rightarrow 0$ limit. Therefore, during calculations, the zero-mass limit of the spinors (5) and (6) can be used instead of (3) and (4).

If we insert the spinors $u(\uparrow)(p)$ and $u(\downarrow)(p)$ describing a general spin orientation (spin orientation defined by the proper angle $\theta$) into charged neutrino current and take the $m_\nu \rightarrow 0$ limit, we obtain

$$\lim_{m_\nu \rightarrow 0} J^{(1)\alpha}_C = \sin \left( \frac{\theta}{2} \right) (J^\alpha_C |_{m_\nu=0})$$

$$\lim_{m_\nu \rightarrow 0} J^{(4)\alpha}_C = \cos \left( \frac{\theta}{2} \right) (J^\alpha_C |_{m_\nu=0})$$

where $J^\alpha_C |_{m_\nu=0}$ is the charged current for massless neutrinos defined in (8) or (9). While calculating the spin-up and spin-down neutrino currents in the above equations, we make use of the following identities: $\hat{L} \left\{ \lim_{m_\nu \rightarrow 0} u(+) (p) \right\} = \hat{L} u^{(R)}(p) = 0$ and $\hat{L} \left\{ \lim_{m_\nu \rightarrow 0} u(-) (p) \right\} = \hat{L} u^{(L)}(p) = u^{(L)}(p)$. We also use the continuity of the helicity states at $m_\nu = 0$: $\lim_{m_\nu \rightarrow 0} u^{(+,-)}(p) = u^{(-,+)}(p) |_{m_\nu=0} = u^{(R,L)}(p)$. The squared amplitude for single neutrino production or absorption processes $ab \rightarrow \nu c$ or $\nu a' \rightarrow b'c'$ discussed in the previous subsection is then found to be

$$\lim_{m_\nu \rightarrow 0} |M^{(\lambda)}|^2 = \frac{(1 - \lambda \cos \theta)}{2} \left( |M|^2 |_{m_\nu=0} \right)$$

where $|M|^2 |_{m_\nu=0}$ is the squared amplitude for massless neutrinos and $\lambda = +1$ corresponds to spin-up ($\uparrow$) and $\lambda = -1$ corresponds to spin-down ($\downarrow$) polarization. We observe from (14)
that the squared amplitude and consequently the cross section has a discontinuity at \( m_\nu = 0 \). For instance, if we choose \( \theta = \pi/2 \) (transverse polarization) \( m_\nu \to 0 \) limit of the cross section gives half of the cross section for massless neutrinos: \( \lim_{m_\nu \to 0} \sigma^{(\lambda)}|_{\theta=\pi/2} = \frac{1}{2} (\sigma |_{m_\nu=0}) \). The cross section for transverse polarization remains finite in the \( m_\nu \to 0 \) limit but it vanishes instantly at the point \( m_\nu = 0 \). Let us examine the zero-mass behavior of the cross section when the neutrinos are described by helicity states. The negative (positive) helicity corresponds to the choice \( \lambda = -1 \) and \( \theta = 0 \) (\( \lambda = +1 \) and \( \theta = 0 \)). We see from (14) that the cross section for positive helicity goes to zero and the cross section for negative helicity goes to \( \sigma |_{m_\nu=0} \) as \( m_\nu \to 0 \). Hence, if we restrict ourselves to special spin orientations namely helicity states, we do not encounter any discontinuity at \( m_\nu = 0 \). However, the zero-mass continuity observed from helicity states is misleading and does not hold true in general as has been clearly shown above.

The longitudinal polarization of the neutrino is usually defined as follows

\[
P_{\text{long}} = \frac{\sigma^{(+)} - \sigma^{(-)}}{\sigma^{(+)} + \sigma^{(-)}}
\]  

(15)

where \( \sigma^{(+)} \) and \( \sigma^{(-)} \) are the cross sections for positive and negative helicity neutrinos. \( P_{\text{long}} \) was calculated for various SM processes in the literature (for example, see Ref. [7]). It was shown that \( P_{\text{long}} \) goes to \(-1\) as the neutrino mass approaches zero. Indeed as we have discussed in the previous paragraph, according to the squared amplitude (14), \( \lim_{m_\nu \to 0} \sigma^{(+)} = 0 \Rightarrow \lim_{m_\nu \to 0} P_{\text{long}} = -1 \). However, it is not correct to conclude from this result that the neutrinos become completely longitudinally polarized and described by 100% negative helicity states in the \( m_\nu \to 0 \) limit. This is evident since the helicity basis is not the only basis that spans the Hilbert space of the spin states. A transversely polarized state is given by the superposition of positive and negative helicity states and vanishing of the positive helicity does not require the transverse polarization to be zero. As we have discussed, although the cross section for positive helicity goes to zero as \( m_\nu \to 0 \), the cross section for transverse polarization does not go to zero, instead it approaches half of the cross section for massless neutrinos in that limit. Therefore, the quantity \( P_{\text{long}} \) defined in (15) is not the genuine measure of the longitudinal polarization. It measures only the asymmetry between positive and negative helicity states. If we define the quantity which we call the degree of transverse polarization by

\[
P_{\text{trans}} = \frac{\sigma^{(T)}}{\sigma^{(+)} + \sigma^{(-)}}
\]  

(16)
we deduce that \( \lim_{m_\nu \to 0} P_{\text{trans}} = 1/2 \). Here, \( \sigma^{(T)} \) represents the cross section for either spin-up \((\lambda = +1 \text{ and } \theta = \pi/2)\) or spin-down \((\lambda = -1 \text{ and } \theta = \pi/2)\) state of the transverse polarization.

The polarized cross section for neutrino scattering process \( \nu a \to \nu a \) can be calculated in a similar manner. If we insert the spinors for a general spin orientation into neutral neutrino current and take the \( m_\nu \to 0 \) limit, we obtain

\[
\lim_{m_\nu \to 0} J^{(\lambda_i, \lambda_f)}_N = \left[ \frac{1 - \lambda_i \cos \theta_i}{2} \right]^{1/2} \left[ \frac{1 - \lambda_f \cos \theta_f}{2} \right]^{1/2} (J^\alpha_N |_{m_\nu=0}) \tag{17}
\]

where \( \lambda_i \) and \( \theta_i \) \((\lambda_f \text{ and } \theta_f)\) belong to the initial state (final state) neutrino and \( J^\alpha_N |_{m_\nu=0} \) is the neutral current for massless neutrinos defined in \(11\). The squared amplitude for neutrino scattering process \( \nu a \to \nu a \) is then found to be

\[
\lim_{m_\nu \to 0} |M^{(\lambda_i, \lambda_f)}|^2 = \left( \frac{1 - \lambda_i \cos \theta_i}{2} \right) \left( \frac{1 - \lambda_f \cos \theta_f}{2} \right) (|M|^2 |_{m_\nu=0}) \tag{18}
\]

where \( |M|^2 |_{m_\nu=0} \) is the squared amplitude for massless neutrinos. The cross section of the neutrino-electron scattering for polarized initial state neutrinos with general spin orientation and unpolarized final state neutrinos, was calculated in Ref.\[8\]. To obtain the cross section for unpolarized final state neutrinos we should sum the squared amplitude over \( \lambda_f \), which gives:

\[
\lim_{m_\nu \to 0} |M^{(\lambda_i)}|^2 = \sum_{\lambda_f = +1, -1} \left\{ \lim_{m_\nu \to 0} |M^{(\lambda_i, \lambda_f)}|^2 \right\} = \left( \frac{1 - \lambda_i \cos \theta_i}{2} \right) (|M|^2 |_{m_\nu=0}) \tag{19}
\]

This squared amplitude coincides with the result of Ref.\[8\] with only one difference that \( m_\nu \to 0 \) limit in the left-hand side of \(19\) appears in our calculations but it is absent in Ref.\[8\]. Instead, spin-dependent squared amplitude was evaluated at \( m_\nu = 0 \), i.e., according to \[8\]: \( |M^{(\lambda_i)}|^2 |_{m_\nu=0} = \left( \frac{1 - \lambda_i \cos \theta_i}{2} \right) (|M|^2 |_{m_\nu=0}) \). It seems the authors assumed that the spinors for a general spin orientation have a continues behavior in the massless limit, i.e., they assumed \( \lim_{m \to 0} u^{(s)}(p) = u^{(s)}(p) |_{m=0} \). We also would like to draw reader’s attention to the following point. We see from equations \(14\), \(18\) and \(19\) that the perpendicular component (relative to momentum direction) of the spin three-vector \( \vec{n} \) does not appear in the squared amplitudes. Recall that we choose \( \vec{n} = \sin \theta \hat{x} + \cos \theta \hat{z} \) and \( \vec{p} = p\hat{z} \). Therefore \( \vec{n} = (n_\perp, 0, n_\parallel) \) where \( n_\perp = \sin \theta \) and \( n_\parallel = \cos \theta \). Then equation \(19\) can be written as \( \lim_{m_\nu \to 0} |M^{(\lambda_i)}|^2 = \left( \frac{1 - \lambda_i n_\perp}{2} \right) (|M|^2 |_{m_\nu=0}) \). However, the disappearance of \( n_\perp \)
in the squared amplitude does not imply that the squared amplitude is independent from \( n_\perp \). This is obvious because we have a condition between \( n_\perp \) and \( n_i \) obtained from the normalization of the spin four-vector \( s^\mu s_\mu = -1 \Rightarrow \bar{n} \cdot \bar{n} = n_\perp^2 + n_i^2 = 1 \). Therefore we have one independent parameter representing the orientation of the spin. One may decide to choose \( n_\perp \) or \( n_i \) as an independent parameter or for instance, the proper angle \( \theta \) as we did in this paper. Regardless of which parameter we choose, the cross section for transverse polarization evaluated in the \( m_\nu \to 0 \) limit gives half of the cross section for massless neutrinos: \( n_\perp = 1 \Rightarrow n_i = 0 \Rightarrow \lim_{m_\nu \to 0} \sigma = \frac{1}{2} (\sigma \mid_{m_\nu=0}) \). Thus, the production, absorption and scattering probability of the neutrinos with transverse polarization cannot be neglected.

The zero-mass discontinuity that we have discussed has important implications on neutrino physics. It makes a significant distinction between the cases in which neutrinos are exactly massless and neutrinos have non-zero but very tiny masses. In the former case, all SM neutrinos are described by completely longitudinally polarized negative helicity states. Therefore, the factor \( 1/2 \) due to spin average of initial state neutrinos is omitted for processes where neutrinos take part in the initial state. However, in the later case it is not possible anymore to assume that neutrinos are completely longitudinally polarized. This is obvious because, the production cross section and hence the production probability of the neutrinos with transverse spin orientation through SM processes cannot be neglected. Therefore some part of the neutrinos in the SM is transversely polarized. Consequently, the spin average of initial state neutrinos in a process cannot be neglected and the cross section is reduced due to this spin average. Our reasoning can be presented in detail as follows: Consider a process in which the neutrinos take part in the initial state. For example it might be the neutrino absorption or scattering process. In order to calculate the unpolarized total cross section we have to average over initial and sum over final state spins. Some of the initial state neutrinos are transversely polarized. Therefore for these neutrinos, spin average is performed over spin-up and spin-down states of the transverse polarization (FIG.1). Then in the \( m_\nu \to 0 \) limit, the unpolarized cross section gives

\[
\lim_{m_\nu \to 0} \sigma^{(\text{unpol})} = \frac{1}{2} \sum_{\lambda_i = +1, -1} \lim_{m_\nu \to 0} \sigma^{(\lambda_i)} = \frac{1}{2} (\sigma \mid_{m_\nu=0})
\]

(20)

where we use \( \lim_{m_\nu \to 0} \sigma^{(\lambda_i = +1)} = \lim_{m_\nu \to 0} \sigma^{(\lambda_i = -1)} = \frac{1}{2} (\sigma \mid_{m_\nu=0}) \) for \( \theta = \pi/2 \) (transverse polarization). We see from equation (20) that the unpolarized cross section is reduced by a factor of \( 1/2 \) compared to the cross section for massless neutrinos. Hence, for transversely
polarized initial state neutrinos we obtain an average factor of 1/2. However, not all initial state neutrinos are transversely polarized. Some others are longitudinally polarized. Since the cross section for neutrinos with positive helicity is zero in the zero-mass limit, longitudinally polarized initial neutrino states consist of 100% negative helicity states. In this case we do not perform an average over initial neutrino spins and the unpolarized cross section is equal the cross section for massless neutrinos:

\[ \lim_{m_\nu \to 0} \sigma^{(\text{unpol})} = \lim_{m_\nu \to 0} \sigma^{(-)} = (\sigma \mid_{m_\nu=0}). \] (21)

We have deduced from the above analysis that if we consider transversely polarized part of the initial neutrinos, then 50% of them are spin-up and other 50% are spin-down. We should then perform an average over initial spins which gives a factor of 1/2. On the other hand, if we consider longitudinally polarized part of the initial neutrinos, then 100% of them are negative helicity and none of them are positive helicity. Then we do not perform an average and instead of 1/2 we get a factor of 1. Hence, an important question arises: By which factor is the cross section reduced? In order to give an answer to this question, let us consider the following gedankenexperiment. Assume that neutrinos are detected in a particle detector via the absorption process \( \nu a' \to b'c' \). Without loss of generality, also assume that all the detected neutrinos, are produced via the production processes \( ab \to \nu c \). FIG.1 represents a schematic diagram for this gedankenexperiment. We will use the subscript ”1” to denote the production process \( ab \to \nu c \) and subscript ”2” to denote the absorption process \( \nu a' \to b'c' \).

If the neutrinos are produced in a particle accelerator then the total number of produced neutrinos is given by

\[ N_1 = \sigma_1^{(\text{unpol})} L_1, \]

where \( \sigma_1^{(\text{unpol})} \) is the unpolarized total cross section and \( L_1 \) is the integrated luminosity. If we assume that all of the produced neutrinos have a fix spin orientation defined by the proper angle \( \theta \), then the number of produced neutrinos with this spin orientation is given by

\[ N_1^{(\lambda)}(\theta) = \sigma_1^{(\lambda)}(\theta) L_1 = \frac{1 - \lambda \cos \theta}{2} \left( \sigma_1 \mid_{m_\nu=0} \right) L_1 \] (22)

where we take \( m_\nu \to 0 \) limit and make use of equation (14). We observe from (22) that the total number of produced neutrinos is independent from the proper angle \( \theta \):

\[ N_1 = N_1^{(\lambda=+1)}(\theta) + N_1^{(\lambda=-1)}(\theta) = \left( \sigma_1 \mid_{m_\nu=0} \right) L_1. \] (23)

Now we consider a massive detector which is composed of a huge number of atoms. The produced neutrinos can interact with the electrons and nucleons (or quarks) of the detector
through the process $\nu a' \rightarrow b' c'$ and the detection occurs. For simplicity we assume that all of the produced neutrinos are passing through the detector. Then, the number of detected neutrinos with spin orientation defined by the proper angle $\theta$ can be written as

$$N_2^{(\lambda)}(\theta) = N_1^{(\lambda)}(\theta) P^{(\lambda)}(\theta) = \sigma_1^{(\lambda)}(\theta) \sigma_2^{(\lambda)}(\theta) L_1 L_2. \quad (24)$$

Here $P^{(\lambda)}(\theta) = \sigma_2^{(\lambda)}(\theta) L_2$ is the detection probability of a single polarized neutrino and $L_2$ is a constant that depends on the parameters of the detector. For instance, $L_2$ depends on the number of electrons or nucleons per unit volume, the fiducial mass of the detector, etc. Since the details of the detector are irrelevant to our analysis, we do not consider its explicit form as a function of detector parameters and assume that it is just a constant. According to equation (14), the zero-mass limit of the production and absorption cross sections can be written as $\lim_{m_\nu \rightarrow 0} \sigma_{1,2}^{(\lambda)}(\theta) = \frac{1}{2} (1 - \lambda \cos \theta) (\sigma_{1,2} |_{m_\nu = 0})$. The total number of detected neutrinos is then

$$N_2^{(\lambda=+1)}(\theta) + N_2^{(\lambda=-1)}(\theta) = \left[ \sin^4 \left( \frac{\theta}{2} \right) + \cos^4 \left( \frac{\theta}{2} \right) \right] N_2. \quad (25)$$

where $N_2 = (\sigma_1 |_{m_\nu = 0}) (\sigma_2 |_{m_\nu = 0}) L_1 L_2$ is the total number of detected neutrinos in case all produced neutrinos are massless. In the left hand side of (25), the limit $m_\nu \rightarrow 0$ is implemented but not shown.
In the above analysis we assume that all the neutrinos are produced having the same spin orientation with respect to the direction of momentum, i.e., with a same proper angle $\theta$. Specifically if we assume that all produced neutrinos are transversely polarized ($\theta = \pi/2$) then total number of detected neutrinos is $N_2/2$. On the other hand, if all produced neutrinos are longitudinally polarized ($\theta = 0$) then total number of detected neutrinos is $N_2$. However in a real situation, the produced beam is comprised from neutrinos with different spin orientations. Hence, we should consider every possible spin orientations and an average over different spin orientations has to be performed. It is easy to show that the average of the trigonometric expressions in the square parentheses yields \[ \langle \left[ \sin^4 \left( \frac{\theta}{2} \right) + \cos^4 \left( \frac{\theta}{2} \right) \right] \rangle = 2/3. \]

Here we should note that a statistical weight of $\sin \theta$ is used during the average. Therefore different from other standard model fermions, spin average of initial state neutrino in a SM process yields a factor of $2/3$ instead of $1/2$. Hence the total cross section is reduced by this factor compared to the case in which neutrinos are described by $100\%$ negative helicity states. Here we should emphasize that the standard model fermions other than neutrinos carry electric and/or color charge and they interact dominantly through vector type coupling. Since the vector coupling does not provide a preferred spin orientation, all different orientations of their spin three-vector $\vec{n}$ are equally probable unless they are intentionally produced polarized. Therefore, for initial state electrons, quarks, etc. the average over proper angle $\theta$ is omitted. On the other hand, the average over spin-up and spin-down states is performed and yields a factor of $1/2$.

Let’s summarize what we have done so far: We have deduced that due to zero-mass discontinuity in the cross section the cases in which neutrinos are exactly massless and neutrinos have non-zero but very tiny masses, have completely different implications. Therefore, contrary to the previously accepted opinion in the literature, it is not a good approximation to neglect neutrino masses during cross section calculations even though neutrino masses are very small and the energy scale of the processes are much greater than the neutrino mass. We have deduced a surprising result that the total cross section of the process where a neutrino takes part in the initial state is reduced by a factor of $2/3$ due to spin average.

As far as we know, this fact has been overlooked in the literature. In the previous studies on this subject, the spin average of initial state neutrinos was omitted for processes where neutrinos take part in the initial state.

The total neutrino (anti-neutrino) cross sections have been measured in plenty number
of experiments since the famous experiments of Cowan and Reines [9, 10]. In all these experiments the measured cross sections seem not to be reduced by the factor 2/3. They confirm the fact that neutrino states are almost 100% negative helicity. Possibly because of the experimental verification of the neutrino helicity, theoretical predictions have not been examined in much detail by previous studies. However, as we have deduced, a straightforward calculation taking into account the existing zero-mass discontinuity of the free Dirac spinors yields a discrepancy between quantum field theory predictions and the experimental results. One possible solution to this problem might be provided by adding a new simple hypothesis to established axioms of quantum field theory [11]. The scope of this paper is limited; we do not aim to discuss possible solutions to the discrepancy. Our purpose is just to reveal the surprising consequences of the zero-mass discontinuity of the Dirac spinors on neutrino cross sections.

In closing to this subsection, we would like to draw reader’s attention to another surprising consequence of the zero-mass discontinuity of the Dirac spinors. Throughout this paper, all calculations have been carried out considering only Dirac neutrinos. It is assumed that Dirac and Majorana neutrino cross sections coincide in the $m_\nu \to 0$ limit [8, 12]. This fact is based on the assumption that both Dirac and Majorana spinors become completely left-handed chiral in the $m_\nu \to 0$ limit. However, as we have discussed in section II, a free Dirac spinor with arbitrary spin orientation does not necessarily result in a chirality eigenstate in the zero-mass limit. Therefore, contrary to expectations, Dirac and Majorana cross sections can lead to different results even though the limit $m_\nu \to 0$ is performed.

C. Massive case with mixing

We have so far ignore the mixing between different mass eigenstates of the neutrino. However, in a realistic situation the neutrinos interact through weak interaction in flavor eigenstates which are given by a superposition of the mass eigenstates. The mixing equation is given by $\nu_{\ell L} = \sum_{i=1}^{3} U_{\ell i} \nu_{i L}$ where $U_{\ell i}$ is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix element [13, 14]. Here we use the subscript $\ell$ to denote the flavor and subscript $i$ to denote the mass eigenstates. Therefore, the scattering process for $\nu_{\ell}$ consist of separate processes for mass eigenstates $\nu_{i}$, $i = 1, 2, 3$. The cross section calculations are then performed for neutrino mass eigenstates and the contributions coming from different mass eigenstates
are added. According to the minimal extension of the SM with massive neutrinos, the scattering amplitude for $\nu_i$ is almost same with the amplitude for a neutrino without mixing. The only difference is that the charged neutrino current picks up an extra factor $U_{\ell i}$. It is obvious that the surprising result that we encounter in the previous subsection is also true when we consider the processes $ab \rightarrow \nu_ic$, $\nu_id' \rightarrow b'c'$ and $\nu_ia \rightarrow \nu_ia$ where the neutrinos are taken to be in the mass eigenstate. The neutrino mixing does not solve the problem, on the contrary, the problem becomes worse than it was before. Since various different spin orientations of the neutrino contribute to the cross section, we can conceive the flavor eigenstate as a superposition of the mass eigenstates where each mass eigenstate may have an arbitrary spin orientation. Then, the spin state of the flavor eigenstate becomes ambiguous. One may assume the flavor neutrino has a mixed spin state, in the sense that, each of its constituent mass eigenstates has a different spin orientation. Let us consider the single neutrino production or absorption processes discussed in the previous subsections. If we sum the squared amplitudes that belong to individual mass eigenstates we expect to obtain the squared amplitude for the flavor eigenstate: $|M_\ell|^2 = \sum_i |M_i|^2$. According to equation (14) the sum over mass eigenstates gives:

$$\lim_{m_\nu \to 0} \sum_i |M_i^{(\lambda_i)}|^2 = \sum_i \left[U_{\ell i} \frac{1 - \lambda_i \cos \theta_i}{2}\right] (|M_\ell|^2|_{m_\nu=0})$$

where $(|M_\ell|^2|_{m_\nu=0})$ is the squared amplitude for the flavor neutrino evaluated at $m_\nu = 0$. In case all spin orientations of the mass eigenstates are equal ($\lambda_1 = \lambda_2 = \lambda_3; \theta_1 = \theta_2 = \theta_3$), we obtain the expected result:

$$\lim_{m_\nu \to 0} \sum_i |M_i^{(\lambda_i)}|^2 = \frac{1 - \lambda \cos \theta}{2} (|M_\ell|^2|_{m_\nu=0}) = \lim_{m_\nu \to 0} |M_\ell^{(\lambda)}|^2$$

where we use the unitarity of the PMNS matrix. However, we do not have any reasonable explanation for the choice $\lambda_1 = \lambda_2 = \lambda_3; \theta_1 = \theta_2 = \theta_3$. In general, spin orientations of different mass eigenstates can be different.

**IV. CONCLUSIONS**

The helicity states have a continuous behavior in the massless limit. When we take $m \to 0$ limit, a helicity state converge to one of the chirality eigenstate and becomes completely left-handed or right-handed chiral. The zero-mass behavior observed from helicity states can
make one think that massless limit is always smooth. However, this behavior is specific to helicity states and is not valid in general. Massless limit has some subtleties in the case of spinors with general spin orientations. The angle which defines the spin orientation of a fermion is an invariant quantity by definition. Hence, the spin orientation of a fermion does not necessarily becomes parallel or anti-parallel to the momentum direction and does not necessarily result in a chirality eigenstate in the zero-mass limit. This behavior makes free Dirac solutions discontinues at \( m = 0 \). We explore the consequences of this zero-mass discontinuity of the Dirac spinors and show that it has surprising consequences for neutrino cross sections.

The most challenging consequence of the zero-mass discontinuity is that it yields a discrepancy between theoretical predictions and the experimental results. We call this discrepancy the neutrino helicity problem. The theoretical predictions of the cross section for massive neutrinos with general spin orientation have been discussed for decades. In this respect, many of the calculations presented in this paper is not totally novel; the new idea of the paper, lies in the reinterpretation of the dependence of the cross section on the spin three-vector \( \vec{n} \). Although the resultant discrepancy is very disturbing, we decide to present our results since we think that they are concrete predictions of the theory. The neutrino helicity problem points out that something is wrong in the assumptions used in the theory. The polarized cross section calculation technique for a general spin orientation is a conventional method which is used successfully for other fermions. Indeed, top quark spin polarization has been measured for various spin orientations and it was found to be consistent with the theoretical predictions \cite{15,16}. Therefore, the problem should be associated with the neutrino nature.

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