THE MODULI SPACE OF POLYNOMIAL MAPS AND THEIR
HOLOMORPHIC INDICES: I. GENERIC PROPERTIES IN THE CASE OF
HAVING MULTIPLE FIXED POINTS

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Abstract. Following the author’s previous works, we continue to consider the problem
of counting the number of affine conjugacy classes of polynomials of one complex variable
when its unordered collection of holomorphic fixed point indices is given. The problem was
already solved completely in the case that the polynomials have no multiple fixed points, in
the author’s previous papers. In this paper, we consider the case of having multiple fixed
points, and obtain the formulae for generic unordered collections of holomorphic fixed point
indices, for each given degree and for each given number of fixed points.

1. Introduction

This paper is a continuation of the author’s previous works [12] and [13].
We first remind our setting from [12] and [13]. Let $\text{MP}_d$ be the family of affine conjugacy
classes of polynomial maps of one complex variable with degree $d \geq 2$, and $\mathbb{C}^d/\mathfrak{S}_d$ the set
of unordered collections of $d$ complex numbers, where $\mathfrak{S}_d$ denotes the $d$-th symmetric group.
We denote by $\Phi_d$ the map $\Phi_d : \text{MP}_d \to \tilde{\Lambda}_d \subset \mathbb{C}^d/\mathfrak{S}_d$
which maps each $f \in \text{MP}_d$ to its unordered collection of fixed-point multipliers. Here, fixed-
point multipliers of $f \in \text{MP}_d$ always satisfy certain relation by the fixed point theorem for
polynomial maps (see Proposition 1.2), which implies that the image of $\Phi_d$ is contained in
a certain hyperplane $\tilde{\Lambda}_d$ in $\mathbb{C}^d/\mathfrak{S}_d$. We also denote by $\text{MC}_d$ the family of monic centered
polynomials of one complex variable with degree $d \geq 2$, and by $\tilde{\Phi}_d : \text{MC}_d \to \tilde{\Lambda}_d$
the composite mapping of the natural projection $\text{MC}_d \to \text{MP}_d$ and $\Phi_d$. As mentioned in [12],
it is important to find the number of elements of each fiber of the maps $\Phi_d$ and $\tilde{\Phi}_d$, in the
study of algebraic properties of moduli of polynomial maps.

Motivated by some concerning results [7], [9], [3], [6], [4] and following some preliminary
works such as [8], [11], [1] and [2], we obtained, in [12] and [13], explicit formulae for finding
the number of elements of each fiber $\Phi_d^{-1}(\tilde{\lambda})$ and also the number of elements of each fiber
$\tilde{\Phi}_d^{-1}(\tilde{\lambda})$ for every $\tilde{\lambda} = \{\lambda_1, \ldots, \lambda_d\} \in \tilde{\Lambda}_d$ except when $\lambda_i = 1$ for some $i$. In this paper we
proceed to the next step, in which we consider the case where $\lambda_i = 1$ for some $i$. However in
this case the situation is much complicated, compared with the case where $\lambda_i \neq 1$ for every $i$.
In particular, if $\#\{i \mid 1 \leq i \leq d, \lambda_i = 1\} \geq 4$, then the fibers $\Phi_d^{-1}(\tilde{\lambda})$ and $\tilde{\Phi}_d^{-1}(\tilde{\lambda})$ can have
dimension greater than or equal to one. To overcome this difficulty, we consider holomorphic
fixed point indices in place of fixed-point multipliers, and modify the maps $\Phi_d, \tilde{\Phi}_d$ so that
the target space of the modified maps is the set of unordered collections of holomorphic
indices...
fixed point indices. Here, holomorphic fixed point index is in some sense a similar one to fixed-point multiplier, but gives more detailed information if the fixed point is multiple (see Definition 1.1). Under this modification, we obtain, in this paper, generic properties for the number of elements of a fiber of the modified maps.

In the rest of this section, we express the above mentioned explicitly and state the main theorem in this paper.

We first fix our notation, some of which are the same as in [12] and [13]. For \( d \geq 2 \), we put
\[
\text{Poly}_d := \{ f \in \mathbb{C}[z] \mid \deg f = d \},
\]
\[
\text{MC}_d := \left\{ f(z) = z^d + \sum_{k=0}^{d-2} a_k z^k \mid a_k \in \mathbb{C} \text{ for } 0 \leq k \leq d - 2 \right\}
\]
and
\[
\text{Aut}(\mathbb{C}) := \{ \gamma(z) = az + b \mid a, b \in \mathbb{C}, a \neq 0 \}.
\]

Since \( \gamma \in \text{Aut}(\mathbb{C}) \) naturally acts on \( f \in \text{Poly}_d \) by \( \gamma \cdot f := \gamma \circ f \circ \gamma^{-1} \), we can define its quotient
\[
\text{MP}_d := \text{Poly}_d / \text{Aut}(\mathbb{C})
\]
which we usually call the moduli space of polynomial maps of degree \( d \). Here, the equivalence class of \( f \in \text{Poly}_d \) in \( \text{MP}_d \) is called the affine conjugacy class of \( f \). An element of \( \text{MC}_d \) is called a monic centered polynomial of degree \( d \). We clearly have \( \text{MC}_d \subset \text{Poly}_d \), and denote by
\[
p : \text{MC}_d \to \text{MP}_d
\]
the natural projection \( \text{MC}_d \subset \text{Poly}_d \to \text{Poly}_d / \text{Aut}(\mathbb{C}) = \text{MP}_d \). The map \( p \) is surjective, and the action of the group \( \{ az \in \text{Aut}(\mathbb{C}) \mid a \in \mathbb{C}, a^{d-1} = 1 \} \cong \mathbb{Z}/(d-1)\mathbb{Z} \) on \( \text{MC}_d \) induces the isomorphism \( \mathbf{p} : \text{MC}_d / (\mathbb{Z}/(d-1)\mathbb{Z}) \cong \text{MP}_d \).

We put
\[
\text{Fix}(f) := \{ z \in \mathbb{C} \mid f(z) = z \}
\]
for \( f \in \text{Poly}_d \), where \( \text{Fix}(f) \) is not considered counted with multiplicity in this paper.

**Definition 1.1.** For \( f \in \text{Poly}_d \) and \( \zeta \in \text{Fix}(f) \),

1. the derivative \( f'(\zeta) \) is called the multiplier of \( f \) at a fixed point \( \zeta \).
2. We put
\[
\iota(f, \zeta) := \frac{1}{2\pi i} \oint_{|z-\zeta| = \epsilon} \frac{dz}{z - f(z)}
\]
for sufficiently small \( \epsilon > 0 \). The residue \( \iota(f, \zeta) \) is called the holomorphic index of \( f \) at a fixed point \( \zeta \).

Note that a fixed point \( \zeta \in \text{Fix}(f) \) is multiple if and only if \( f'(\zeta) = 1 \). Moreover if \( \zeta \in \text{Fix}(f) \) is not multiple, then we always have \( \iota(f, \zeta) = \frac{1}{1 - f'(\zeta)} \) by residue theorem. It is also well known that the holomorphic index \( \iota(f, \zeta) \) is invariant under the change of holomorphic coordinates even when \( f'(\zeta) = 1 \). Hence holomorphic index is a very similar object to multiplier. In particular, in the case \( f'(\zeta) \neq 1 \), these two give the equivalent information; however if \( f'(\zeta) = 1 \), holomorphic index gives more detailed information than multiplier (see section 12 in [10] for instance).

**Proposition 1.2 (Fixed Point Theorem).** For \( f \in \text{Poly}_d \) we have
\[
\sum_{\zeta \in \text{Fix}(f)} \iota(f, \zeta) = 0.
\]
Proposition 1.2 can easily be seen by the integration \( \frac{1}{2\pi i} \int_{|z|=R} \frac{dz}{z-f(z)} \) for sufficiently large real number \( R \).

Every \( f(z) \in \text{Poly}_d \) can be expressed in the form
\[
f(z) = z + \rho(z - \zeta_1)^{d_1} \cdots (z - \zeta_\ell)^{d_\ell},
\]
where \( d_1, \ldots, d_\ell \) are positive integers with \( d_1 + \cdots + d_\ell = d \), \( \rho \) is a non-zero complex number, and \( \zeta_1, \ldots, \zeta_\ell \) are mutually distinct complex numbers. In this expression, we have \( \#\text{Fix}(f) = \ell \) and \( \text{Fix}(f) = \{\zeta_1, \ldots, \zeta_\ell\} \). For such \( f(z) \), we put
\[
\text{mult}(f, \zeta_i) := d_i
\]
f for \( 1 \leq i \leq \ell \), which we usually call the fixed-point multiplicity of \( f \) at \( \zeta_i \).

**Definition 1.3.** For \( d \geq 2 \), we put
\[
\text{Mult}_d := \left\{ (d_1, \ldots, d_\ell) \left| \begin{array}{l}
\ell \geq 1, \ d_1, \ldots, d_\ell \in \mathbb{N}, \\
d_1 + \cdots + d_\ell = d, \\
1 \leq d_1 \leq \cdots \leq d_\ell
\end{array} \right. \right\}.
\]
Moreover for each \((d_1, \ldots, d_\ell) \in \text{Mult}_d\), we put
\[
\text{Poly}_d(d_1, \ldots, d_\ell) := \left\{ z + \rho(z - \zeta_1)^{d_1} \cdots (z - \zeta_\ell)^{d_\ell} \left| \begin{array}{l}
\rho, \zeta_1, \ldots, \zeta_\ell \in \mathbb{C}, \\
\rho \neq 0, \\
\zeta_1, \ldots, \zeta_\ell \\text{are mutually distinct}
\end{array} \right. \right\},
\]
\[
\text{MP}_d(d_1, \ldots, d_\ell) := \text{Poly}_d(d_1, \ldots, d_\ell) / \text{Aut}(\mathbb{C}),
\]
\[
\text{MC}_d(d_1, \ldots, d_\ell) := \text{Poly}_d(d_1, \ldots, d_\ell) \cap \text{MC}_d \quad \text{and}
\]
\[
\Lambda_d(d_1, \ldots, d_\ell) := \left\{ (d_1, m_1), \ldots, (d_\ell, m_\ell) \left| \begin{array}{l}
m_1, \ldots, m_\ell \in \mathbb{C}, \\
m_1 + \cdots + m_\ell = 0
\end{array} \right. \right\}.
\]
where \( \{(d_1, m_1), \ldots, (d_\ell, m_\ell)\} \) denotes the unordered collection of the pairs \((d_1, m_1), \ldots, (d_\ell, m_\ell)\).

We naturally have the following stratifications indexed by the set of fixed-point multiplicities \( \text{Mult}_d \):
\[
\text{Poly}_d = \bigsqcup_{(d_1, \ldots, d_\ell) \in \text{Mult}_d} \text{Poly}_d(d_1, \ldots, d_\ell),
\]
\[
\text{MP}_d = \bigsqcup_{(d_1, \ldots, d_\ell) \in \text{Mult}_d} \text{MP}_d(d_1, \ldots, d_\ell) \quad \text{and}
\]
\[
\text{MC}_d = \bigsqcup_{(d_1, \ldots, d_\ell) \in \text{Mult}_d} \text{MC}_d(d_1, \ldots, d_\ell),
\]
where \( \bigsqcup \) denotes the disjoint union. On each strata \( \text{MP}_d(d_1, \ldots, d_\ell) \) and \( \text{MC}_d(d_1, \ldots, d_\ell) \), we naturally have the maps
\[
\Phi_d(d_1, \ldots, d_\ell) : \text{MP}_d(d_1, \ldots, d_\ell) \to \Lambda_d(d_1, \ldots, d_\ell) 
\quad \text{and}
\]
\[
\hat{\Phi}_d(d_1, \ldots, d_\ell) : \text{MC}_d(d_1, \ldots, d_\ell) \to \Lambda_d(d_1, \ldots, d_\ell)
\]
by \( f \mapsto \{ \text{mult}(f, \zeta), \iota(f, \zeta) \} \mid \zeta \in \text{Fix}(f) \} \), by Proposition 1.2.

**Remark 1.4.** For every \((d_1, \ldots, d_\ell) \in \text{Mult}_d\), we have
\[
\dim_{\mathbb{C}} \text{MP}_d(d_1, \ldots, d_\ell) = \dim_{\mathbb{C}} \text{MC}_d(d_1, \ldots, d_\ell) = \dim_{\mathbb{C}} \Lambda_d(d_1, \ldots, d_\ell) = \ell - 1.
\]

**Remark 1.5.** The maps \( \Phi_d(1, \ldots, 1) \) and \( \hat{\Phi}_d(1, \ldots, 1) \) are essentially the same as the maps
\( \Phi_d : \text{MP}_d \to \tilde{\Lambda}_d \) and \( \hat{\Phi}_d : \text{MC}_d \to \tilde{\Lambda}_d \) which were mainly considered in [12] and [13] if restricted on \( \tilde{V}_d := \{ \tilde{\lambda} = \{\lambda_1, \ldots, \lambda_d\} \in \tilde{\Lambda}_d \mid \lambda_i \neq 1 \ \text{for every} \ i \} \), because they are the same
under the correspondence \( \tilde{V}_d \ni \{\lambda_1, \ldots, \lambda_d\} \mapsto \left\{ \left(1, \frac{1}{1-\lambda_1}\right), \ldots, \left(1, \frac{1}{1-\lambda_d}\right) \right\} \in \Lambda_d(1,\ldots,1) \).

Hence in [12] and [13], we already have the formulae for finding \( \#\Phi_d(1,\ldots,1)^{-1}(\overline{m}) \) and \( \#\hat{\Phi}_d(1,\ldots,1)^{-1}(\overline{m}) \) for every \( \overline{m} \in \Lambda_d(1,\ldots,1) \). In this paper we consider \( \#\Phi_d(d_1,\ldots,d_\ell)^{-1}(\overline{m}) \) and \( \#\hat{\Phi}_d(d_1,\ldots,d_\ell)^{-1}(\overline{m}) \) for every \( (d_1,\ldots,d_\ell) \in \text{Mult}_d \) and for generic \( \overline{m} \in \Lambda_d(d_1,\ldots,d_\ell) \).

We now state the main theorem in this paper.

**Main Theorem.** Let \((d_1,\ldots,d_\ell)\) be an element of \(\text{Mult}_d\) with \(\ell \geq 2\). Then

1. for every \(\overline{m} = \{(d_1,m_1),\ldots,(d_\ell,m_\ell)\} \in \Lambda_d(d_1,\ldots,d_\ell)\), we have
   \[
   \#\Phi_d(d_1,\ldots,d_\ell)^{-1}(\overline{m}) \leq \frac{(d-2)!}{(d-\ell)!} \quad \text{and} \quad \#\hat{\Phi}_d(d_1,\ldots,d_\ell)^{-1}(\overline{m}) \leq \frac{(d-1)!}{(d-\ell)!}.
   \]
2. For \(\overline{m} = \{(d_1,m_1),\ldots,(d_\ell,m_\ell)\} \in \Lambda_d(d_1,\ldots,d_\ell)\), the implications \((c) \Leftrightarrow (b) \Rightarrow (a)\) always hold for the following three conditions:
   - \((a)\) \(\#\Phi_d(d_1,\ldots,d_\ell)^{-1}(\overline{m}) = \frac{(d-2)!}{(d-\ell)!}\)
   - \((b)\) \(\#\hat{\Phi}_d(d_1,\ldots,d_\ell)^{-1}(\overline{m}) = \frac{(d-1)!}{(d-\ell)!}\)
   - \((c)\) \((d_1,m_1),\ldots,(d_\ell,m_\ell)\) are mutually distinct, and \(\sum_{i \in I} m_i \neq 0\) holds for every \(\emptyset \neq I \subseteq \{1,\ldots,\ell\}\).

Moreover the implication \((a) \Rightarrow (c)\) also holds except in the case \(d = \ell = 3\).

**Remark 1.6.** As mentioned in the first page in [12], the map \(\Phi_3 : \text{MP}_3 \rightarrow \tilde{\Lambda}_3\) is bijective. Hence in the case \(d = \ell = 3\), the equality \(\#\Phi_3(1,1,1)^{-1}(\overline{m}) = 1\) holds for \(\overline{m} = \{(1,m_1),(1,m_2),(1,m_3)\} \in \Lambda_3(1,1,1)\) if and only if \(m_i \neq 0\) holds for every \(1 \leq i \leq 3\). Hence in this case, the implication \((a) \Rightarrow (c)\) does not hold since \((1,m_1),(1,m_2),(1,m_3)\) are not always mutually distinct.

The set of \(\overline{m} \in \Lambda_d(d_1,\ldots,d_\ell)\) satisfying the condition \((2c)\) in Main Theorem is Zariski open in \(\Lambda_d(d_1,\ldots,d_\ell)\); hence we have the following:

**Corollary.** Let \((d_1,\ldots,d_\ell)\) be an element of \(\text{Mult}_d\) with \(\ell \geq 2\). Then

1. \(\Phi_d(d_1,\ldots,d_\ell)\) is generically a \(\frac{(d-2)!}{(d-\ell)!}\)-to-one map.
2. \(\hat{\Phi}_d(d_1,\ldots,d_\ell)\) is generically a \(\frac{(d-1)!}{(d-\ell)!}\)-to-one map.

**Remark 1.7.** In the case \(\ell = 1\), we have \((d_1,\ldots,d_\ell) = (d)\) and also have \(\#\text{MP}_d(d) = \#\Lambda_d(d) = 1\). Hence in this case, the maps \(\Phi_d(d)\) and \(\hat{\Phi}_d(d)\) are trivially bijective.

We have three sections in this paper. The most frequently used tool for the proof of Main Theorem is linear algebra, especially Proposition 2.4. Section 2 is devoted to introduce some results in linear algebra including Proposition 2.4. On the other hand, the proof of Main Theorem itself is given in Section 3. Most steps in the proof of Main Theorem are analogies of the proofs of the main theorems in [12]; however in almost all the steps, its proof is much complicated, compared with the original one in [12]. Moreover in [12], there does not exist a counterpart for Proposition 3.6, which is the most crucial part in the proof of Main Theorem from the standpoint of technique.

**2. Preparation from Linear Algebra**

In this section, we remind our notations and propositions in linear algebra which were given in the latter half of Section 7 in [12]. These are also used very often in this paper throughout Section 3 in the proof of Main Theorem.
\textbf{Definition 2.1.} For non-negative integers \(n, b, k, h\) with \(n > k\) and \(b > h\), we denote by \(A_{n,k}^{b,h}(\alpha)\) the \((n-k,b-h)\) matrix whose \((i,j)\)-th entry is \((i+k-1)(j+h)\) for \(1 \leq i \leq n-k\) and \(1 \leq j \leq b-h\). Moreover we put \(A_{n,k}^b(\alpha) := A_{n,k}^{b,0}(\alpha)\) and \(A_{n,0}^b(\alpha) := A_{n,0}^{b,0}(\alpha)\).

By definition, the matrix \(A_{n,k}^{b,h}(\alpha)\) is obtained from the \((n,b)\) matrix

\[
A_n^b(\alpha) = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
\alpha & 1 & 0 & 0 & \cdots & 0 \\
\alpha^2 & 2\alpha & 1 & 0 & \cdots & 0 \\
\alpha^3 & 3\alpha^2 & 3\alpha & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha^{n-1} & (n-1)\alpha^{n-2} & (n-2)\alpha^{n-3} & (n-3)\alpha^{n-4} & \cdots & 1
\end{pmatrix}
\]

by cutting off the upper \(k\) rows and the left \(h\) columns.

\textbf{Definition 2.2.} For a positive integer \(b\), we denote by \(X_b\) the \((b,b)\) diagonal matrix whose \((i,i)\)-th entry is \(i\) for \(1 \leq i \leq b\). Moreover we denote by \(I_b\) the \((b,b)\) identity matrix, and by \(N_b\) the \((b,b)\) nilpotent matrix whose \((i,i+1)\)-th entry is 1 for \(1 \leq i \leq b-1\) and whose other entries are 0, i.e.,

\[
X_b = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b
\end{pmatrix}, \quad I_b = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix} \quad \text{and} \quad N_b = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
\]

\textbf{Proposition 2.3.} For positive integers \(n\) and \(b\), we have \(A_{n+1,1}^{b+1}(\alpha) = X_n \cdot A_n^b(\alpha) \cdot X_n^{-1}\).

\textit{Proof.} This can easily be verified by \(\binom{i}{j} = \frac{(i-1)!}{j!(i-j)!}\). \(\square\)

\textbf{Proposition 2.4.} Let \(r_1, \ldots, r_\ell, r\) be positive integers with \(r = r_1 + \cdots + r_\ell\). Then we have

\[
\det(A_r^{r_1}(\alpha_1), \ldots, A_r^{r_\ell}(\alpha_\ell)) = \prod_{1 \leq u < v \leq \ell} (\alpha_u - \alpha_v)^{r_u r_v}
\]

and

\[
\det(A_{r+1,1}^{r_1+1}(\alpha_1), \ldots, A_{r+1,1}^{r_\ell+1}(\alpha_\ell)) = \frac{r!}{r_1! \cdots r_\ell!} \prod_{1 \leq u < v \leq \ell} (\alpha_u - \alpha_v)^{r_u r_v}.
\]

\textit{Proof.} See the proof of Lemma 7.8 in [12]. Note that the latter equality is a direct consequence of the former one and Proposition 2.3. \(\square\)

3. PROOF

In this section, we prove Main Theorem by using the propositions in Section 2. We always assume the following throughout this section:

- \(\ell\) and \(d\) are integers greater than or equal to 2.
- \((d_1, \ldots, d_\ell)\) is an element of \(\text{Mult}_d\).
- \(\rho, \zeta_1, \ldots, \zeta_\ell\) are complex numbers with \(\rho \neq 0\).
- \(f(z) = z + \rho(z - \zeta_1)^{d_1} \cdots (z - \zeta_\ell)^{d_\ell}\). Hence \(f(z) \in \text{Poly}_d(d_1, \ldots, d_\ell)\) holds if and only if \(\zeta_1, \ldots, \zeta_\ell\) are mutually distinct.
- \(m = (m_1, \ldots, m_\ell)\) is an element of \(\mathcal{C}_d\) with \(m_1 + \cdots + m_\ell = 0\). Moreover for such \(m\), we put \(\overline{m} := \{(d_1, m_1), \ldots, (d_\ell, m_\ell)\}\). Hence we always have \(\overline{m} \in \Lambda_d(d_1, \ldots, d_\ell)\).
- \(m\) is assumed not to be equal to \((0, \ldots, 0)\), except in Propositions 3.1, 3.3 and 3.4.
We first prove the following proposition, which is the first step for the proof of Main Theorem, and is also an analogue of Key Lemma in Section 4 in \[12\] in the case of having multiple fixed points. However Key Lemma in Section 4 in \[12\] was much simpler than the following proposition.

**Proposition 3.1.** Suppose that $\zeta_1, \ldots, \zeta_\ell$ are mutually distinct. Then the following two conditions (1) and (2) are equivalent:

1. The equalities $\iota(f, \zeta_i) = m_i$ hold for $1 \leq i \leq \ell$.
2. There exist $m_{i,k} \in \mathbb{C}$ for $1 \leq i \leq \ell$ and $1 \leq k \leq d_i - 1$ such that the equality

\[
\sum_{i=1}^{\ell} A_d^{d_i}(\zeta_i) \begin{pmatrix} m_i \\ m_{i,1} \\ \vdots \\ m_{i,d_i-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1/\rho \end{pmatrix}
\]

holds, where in the case $d_i = 1$, the column vector $^t(m_i, m_{i,1}, \ldots, m_{i,d_i-1})$ is assumed to be $(m_i)$.

**Proof.** Since $\deg(z - f(z)) = d \geq 2$, the equalities

\[
\frac{1}{2\pi \sqrt{-1}} \int_{|z| = R} \frac{z^k}{z - f(z)} \, dz = \begin{cases} 0 & (k = 0, 1, \ldots, d - 2) \\ -\frac{1}{\rho} & (k = d - 1) \end{cases}
\]

hold for sufficiently large real number $R$. On the other hand, by the residue theorem, we have

\[
\frac{1}{2\pi \sqrt{-1}} \int_{|z| = R} \frac{z^k}{z - f(z)} \, dz = \sum_{i=1}^{\ell} \frac{1}{2\pi \sqrt{-1}} \int_{|z - \zeta_i| = \epsilon} \frac{(z - \zeta_i) + \zeta_i}{z - f(z)} \, dz \\
= \sum_{i=1}^{\ell} \min\{k,d_i-1\} \sum_{h=0}^{k} \left( \begin{array}{c} k \\ h \end{array} \right) \cdot \zeta_i^{-h} \cdot \frac{1}{2\pi \sqrt{-1}} \int_{|z - \zeta_i| = \epsilon} \frac{(z - \zeta_i)^h}{z - f(z)} \, dz
\]

for sufficiently small positive real number $\epsilon$. Hence putting

\[
\iota_h(f, \zeta_i) := \frac{1}{2\pi \sqrt{-1}} \int_{|z - \zeta_i| = \epsilon} \frac{(z - \zeta_i)^h}{z - f(z)} \, dz
\]

for $h \geq 0$ and for sufficiently small $\epsilon > 0$, we have $\iota_0(f, \zeta_i) = \iota(f, \zeta_i)$, $\iota_h(f, \zeta_i) = 0$ for $h \geq d_i$, and also have

\[
\sum_{i=1}^{\ell} \sum_{h=0}^{d_i-1} \left( \begin{array}{c} k \\ h \end{array} \right) \cdot \zeta_i^{-h} \cdot \iota_h(f, \zeta_i) = \begin{cases} 0 & (k = 0, 1, \ldots, d - 2) \\ -\frac{1}{\rho} & (k = d - 1) \end{cases}
\]

for every $f(z) = z + \rho(z - \zeta_1)^{d_1} \cdots (z - \zeta_\ell)^{d_\ell} \in \text{Poly}(d_1, \ldots, d_\ell)$. Moreover by using matrix, we find that the equalities (3.2) are equivalent to the equality

\[
\sum_{i=1}^{\ell} A_d^{d_i}(\zeta_i) \begin{pmatrix} \iota_0(f, \zeta_i) \\ \iota_1(f, \zeta_i) \\ \vdots \\ \iota_{d_i-1}(f, \zeta_i) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1/\rho \end{pmatrix}
\]
Then the equality (3.1) is equivalent to the equality

\begin{equation}
\left( A_d^{d_1} (\zeta_1), A_d^{d_2} (\zeta_2), \ldots, A_d^{d_\ell} (\zeta_\ell) \right) \begin{pmatrix}
t_0(f, \zeta_1) \\
\vdots \\
t_{d_1} (f, \zeta_1) \\
\vdots \\
t_0(f, \zeta_\ell) \\
\vdots \\
t_{d_\ell} (f, \zeta_\ell)
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
-1/\rho
\end{pmatrix}.
\end{equation}

Hence the condition (1) implies the equality (3.1) by putting \(m_{i,k} = t_k(f, \zeta_i)\) for \(1 \leq i \leq \ell\) and \(1 \leq k \leq d_i - 1\), which verifies the implication (1) \(\Rightarrow\) (2).

On the other hand, since the square matrix \(\left( A_d^{d_1} (\zeta_1), A_d^{d_2} (\zeta_2), \ldots, A_d^{d_\ell} (\zeta_\ell) \right)\) is invertible by Proposition 2.4, the equalities (3.1) and (3.3) (or (3.4)) imply the equalities

\(t (m_i, m_{i,1}, \ldots, m_{i,d_i-1}) = t (t (f, \zeta_i), t_1 (f, \zeta_i), \ldots, t_{d_i-1} (f, \zeta_i))\)

for \(1 \leq i \leq \ell\), which verifies the implication (2) \(\Rightarrow\) (1).

\(\square\)

**Remark 3.2.** In the rest of this section, we often use expressions like the equality (3.3) in place of (3.4) for the simplicity of description as in the proof of Proposition 3.1.

Concerning Proposition 3.1, the following Propositions 3.3, 3.4 and 3.5 also hold.

**Proposition 3.3.** Suppose that \(\zeta_1, \ldots, \zeta_\ell\) are mutually distinct, and that the equality (3.1) holds for \(m_{i,k} \in \mathbb{C}\) with \(1 \leq i \leq \ell\) and \(1 \leq k \leq d_i - 1\). Then we have the following for \(1 \leq i \leq \ell\):

1. \(m_{i,d_i-1} \neq 0\) if \(d_i \geq 2\).
2. \(m_i \neq 0\) if \(d_i = 1\).

**Proof.** Without loss of generality, we may assume that \(i = 1\). Suppose \(d_1 \geq 2\) and \(m_{1,d_1-1} = 0\). Then the equality (3.1) is equivalent to the equality

\[A_d^{d_1-1} (\zeta_1) \begin{pmatrix}
m_1 \\
m_{1,1} \\
\vdots \\
m_{1,d_1-2}
\end{pmatrix} + \sum_{i=2}^\ell A_d^{d_i} (\zeta_i) \begin{pmatrix}
m_i \\
m_{i,1} \\
\vdots \\
m_{i,d_i-1}
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
-1/\rho
\end{pmatrix},\]

which implies

\[A_d^{d_1-1} (\zeta_1) \begin{pmatrix}
m_1 \\
m_{1,1} \\
\vdots \\
m_{1,d_1-2}
\end{pmatrix} + \sum_{i=2}^\ell A_d^{d_i} (\zeta_i) \begin{pmatrix}
m_i \\
m_{i,1} \\
\vdots \\
m_{i,d_i-1}
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
0
\end{pmatrix}.
\]

Since the square matrix \(\left( A_d^{d_1-1} (\zeta_1), A_d^{d_2-1} (\zeta_2), \ldots, A_d^{d_\ell-1} (\zeta_\ell) \right)\) is invertible by Proposition 2.4, we have \(t (m_i, m_{i,1}, \ldots, m_{i,d_i-1}) = t (0, 0, \ldots, 0)\) for every \(1 \leq i \leq \ell\). Hence the left-hand side of the equality (3.1) is equal to zero, which contradicts (3.1). We therefore have the contradiction, which implies the assertion (1).
Suppose \( d_1 = 1 \) and \( m_1 = 0 \) next. Then the equality (3.1) is, in this case, equivalent to
\[
\sum_{i=2}^{\ell} A_d^{d_i} (\zeta_i) \begin{pmatrix} m_i \\ m_{i,1} \\ \vdots \\ m_{i,d_i-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1/\rho \end{pmatrix}.
\]
Hence by a similar argument to the above, the invertibility of the square matrix
\[
\begin{pmatrix} A_d^{d_2} (\zeta_2), \ldots, A_d^{d_\ell} (\zeta_\ell) \end{pmatrix}
\]
leads to a contradiction, which implies the assertion (2). \( \square \)

**Proposition 3.4.** Suppose that \( f(z) \) is an element of \( \text{Poly}_d(d_1, \ldots, d_\ell) \). Then \( \iota (f, \zeta) \neq 0 \) holds for some \( \zeta \in \text{Fix}(f) \). Hence in the case \( m = (0, \ldots, 0) \), we have \( \Phi_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) = \tilde{\Phi}_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) = \emptyset \).

**Proof.** By the proof of Proposition 3.1 we always have the equality (3.3) for \( \Phi(d_1, \ldots, d_\ell) \) we always have the equality (3.3) for \( f(z) \in \text{Poly}_d(d_1, \ldots, d_\ell) \). Suppose \( \iota (f, \zeta_i) = 0 \) for every \( 1 \leq i \leq \ell \). Then the equality (3.3) is equivalent to the equality
\[
\sum_{i=1}^{\ell} A_{d-\ell+1,1}^{d_i} (\zeta_i) \begin{pmatrix} \iota_1(f, \zeta_i) \\ \vdots \\ \iota_{d_i-1}(f, \zeta_i) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},
\]
which implies
\[
\sum_{i=1}^{\ell} A_{d-\ell+1,1}^{d_i} (\zeta_i) \begin{pmatrix} \iota_1(f, \zeta_i) \\ \vdots \\ \iota_{d_i-1}(f, \zeta_i) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
\]
since \( \ell \geq 2 \). Since the square matrix \( \begin{pmatrix} A_{d-\ell+1,1}^{d_1} (\zeta_1), A_{d-\ell+1,1}^{d_2} (\zeta_2), \ldots, A_{d-\ell+1,1}^{d_\ell} (\zeta_\ell) \end{pmatrix} \) is invertible by Proposition 2.4, we have \( \iota (\iota_1(f, \zeta_i), \ldots, \iota_{d_i-1}(f, \zeta_i)) = (0, \ldots, 0) \) for every \( 1 \leq i \leq \ell \). Hence the left-hand side of the equality (3.3) must be equal to zero, which contradicts the equality (3.3). We therefore have \( \iota (f, \zeta_i) \neq 0 \) for some \( 1 \leq i \leq \ell \), which completes the proof of Proposition 3.4. \( \square \)

In the rest of this section, \( m = (m_1, \ldots, m_\ell) \) is always assumed not to be equal to \((0, \ldots, 0)\), as mentioned in the opening paragraph of this section.

**Proposition 3.5.** Suppose that \( \zeta_1, \ldots, \zeta_\ell \) are mutually distinct. If the equality
\[
(3.5) \quad \sum_{i=1}^{\ell} A_{d-1}^{d_i} (\zeta_i) \begin{pmatrix} m_i \\ m_{i,1} \\ \vdots \\ m_{i,d_i-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
\]
holds for \( m_{i,k} \in \mathbb{C} \) with \( 1 \leq i \leq \ell \) and \( 1 \leq k \leq d_i - 1 \), then there exists a unique non-zero complex number \( \rho \) such that the equality (3.1) holds.

**Proof.** Suppose that such \( \rho \) does not exist. Then the left-hand side of the equality (3.1) must be zero. Since the square matrix \( \begin{pmatrix} A_{d_1}^{d_i} (\zeta_1), A_{d_2}^{d_i} (\zeta_2), \ldots, A_{d_\ell}^{d_i} (\zeta_\ell) \end{pmatrix} \) is invertible by Proposition 2.4, we have \( \iota (m_i, m_{i,1}, \ldots, m_{i,d_i-1}) = (0, 0, \ldots, 0) \) for every \( 1 \leq i \leq \ell \), which contradicts \( (m_1, \ldots, m_\ell) \neq (0, \ldots, 0) \). Hence the contradiction assures the existence of \( \rho \) satisfying the equality (3.1). \( \square \)
We proceed to the next step, in which we exclude \( m_{i,k} \) for \( 1 \leq i \leq \ell \) and \( 1 \leq k \leq d_i - 1 \) from the equality (3.5). From the standpoint of technique, this step is the most crucial in the proof of Main Theorem. As can be verified in [12], this step does not exist in the case of having no multiple fixed points.

**Proposition 3.6.** Suppose that \( \zeta_1, \ldots, \zeta_{\ell} \) are mutually distinct. Then the following two conditions (1) and (2) are equivalent:

1. There exist \( m_{i,k} \in \mathbb{C} \) for \( 1 \leq i \leq \ell \) and \( 1 \leq k \leq d_i - 1 \) such that the equality (3.5) holds.
2. The equality

\[
A_{\ell-2}^{d-2}(0) \left( \prod_{i=1}^{\ell} (-\zeta_i I_{d-2} + N_{d-2})^{d_i-1} \right) (X_{d-2})^{-1} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{d-2} \\ \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{d-2} \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_{d-1} \end{pmatrix} = 0
\]

holds.

In Proposition 3.6, note that in the case \( \ell = 2 \), the equality (3.6) always holds since it can be considered to be an equality in 0-dimensional \( \mathbb{C} \)-vector space \( \mathbb{C}^0 \).

Proposition 3.6 is obtained from Lemma 3.7 below by substituting \( q = \ell' = \ell \), \( d_i' = d_i - 1 \), \( \alpha_i = \zeta_i \) and \( m_i' = m_i \) for \( 1 \leq i \leq \ell \). Lemma 3.7 is also utilized later in the proofs of Proposition 3.13 and Lemma 3.22.

**Lemma 3.7.** Let \( \ell', q, d_1', \ldots, d_q' \) be non-negative integers with \( q \geq 1 \), \( \ell' \geq 2 \) and \( d_1' + \cdots + d_q' = d - \ell' \). Moreover let \( \alpha_1, \ldots, \alpha_q \) be mutually distinct complex numbers, and \( m_1', \ldots, m_q' \) complex numbers with \( m_1' + \cdots + m_q' = 0 \). Then the following two conditions (1) and (2) are equivalent:

1. There exist \( m_{u,k} \in \mathbb{C} \) for \( 1 \leq u \leq q \) and \( 1 \leq k \leq d_u' \) such that the equality

\[
\sum_{u=1}^{q} A_{d_u' - 1}^{d_u' + 1}(\alpha_u) \begin{pmatrix} m_{u,1}' \\ \vdots \\ m_{u,d_u' - 1} \end{pmatrix} = 0
\]

holds.
2. The equality

\[
A_{\ell'-2}^{d-2}(0) \left( \prod_{u=1}^{q} (-\alpha_u I_{d-2} + N_{d-2})^{d_u'} \right) (X_{d-2})^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{q} \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{q} \end{pmatrix} \begin{pmatrix} m_1' \\ \vdots \\ m_q' \end{pmatrix} = 0
\]

holds.

**Proof.** Note first that the first row of the equality (3.7) is the same as \( m_1' + \cdots + m_q' = 0 \), which is assumed in Lemma 3.7. Hence the equality (3.7) is equivalent to the equality

\[
\sum_{u=1}^{q} A_{d_u' - 1,1}^{d_u' + 1}(\alpha_u) \begin{pmatrix} m_{u,1}' \\ \vdots \\ m_{u,d_u'} \end{pmatrix} = 0.
\]
Moreover since
\[
\sum_{u=1}^{q} A_{d_{d-1,1}^u} (\alpha_u) \begin{pmatrix} m_{u,1}' \\ \vdots \\ m_{u,d_u}' \end{pmatrix} = \sum_{u=1}^{q} \begin{pmatrix} m_{u,1}' \\ \vdots \\ m_{u,d_u}' \end{pmatrix} + A_{d_{d-1,1}^u} (\alpha_u) \begin{pmatrix} m_{u,1}' \\ \vdots \\ m_{u,d_u}' \end{pmatrix} = \sum_{u=1}^{q} X_{d-2} A_{d_{d-2}^u} (\alpha_u) \begin{pmatrix} m_{u,1}' \\ \vdots \\ m_{u,d_u}' \end{pmatrix} + A_{d_{d-1,1}^u} (\alpha_u) \begin{pmatrix} m_{u,1}' \\ \vdots \\ m_{u,d_u}' \end{pmatrix}
\]
by Proposition 2.3, the equality (3.9) is equivalent to
\[
\sum_{u=1}^{q} A_{d_{d-2}^u} (\alpha_u) \begin{pmatrix} m_{u,1}' \\ \vdots \\ m_{u,d_u}' \end{pmatrix} + (X_{d-2})^{-1} \begin{pmatrix} m_{1}' \\ \vdots \\ m_{q}' \end{pmatrix} = 0.
\]

To proceed further the proof of Lemma 3.7 we make use of the following lemma.

**Lemma 3.8.** Let \( \ell', q, d_1', \ldots, d_q'; \alpha_1, \ldots, \alpha_q \) be as in Lemma 3.7. Then the linear map
\[
A_{\ell'-2}^{d-2} (0) \prod_{u=1}^{q} (-\alpha_u I_{d-2} + N_{d-2})^{d_u}: \mathbb{C}^{d-2} \to \mathbb{C}^{\ell'-2}
\]
is surjective. Moreover the basis of its kernel consists of the column vectors of \( A_{d_{d-2}^u} (\alpha_1), \ldots, A_{d_{d-2}^u} (\alpha_q) \).

**Proof of Lemma 3.8.** We first show the surjectivity of the map (3.11). Note that \( A_{\ell'-2}^{d-2} (0) = (I_{\ell'-2}, O) \), where \( O \) is the zero matrix of size \((\ell' - 2, d - \ell')\). Hence the map \( A_{\ell'-2}^{d-2} (0): \mathbb{C}^{d-2} \to \mathbb{C}^{\ell'-2} \) is surjective. If \( \alpha_u \neq 0 \), then the map \(-\alpha_u I_{d-2} + N_{d-2}: \mathbb{C}^{d-2} \to \mathbb{C}^{d-2}\) is bijective. If, for instance, \( \alpha_1 = 0 \), then \( \alpha_u \neq 0 \) holds for \( 2 \leq u \leq q \), and the map \( A_{\ell'-2}^{d-2} (0)(N_{d-2})^{d_u} = (O_1, I_{\ell'-2}, O_2): \mathbb{C}^{d-2} \to \mathbb{C}^{\ell'-2} \) is surjective, where \( O_1 \) and \( O_2 \) are the zero matrices of sizes \((\ell' - 2, d_1')\) and \((\ell' - 2, d - \ell' - d_1') = (\ell' - 2, \sum_{u=1}^{q} d_u')\) respectively. Hence in every case, the map (3.11) is surjective.

Since the square matrix \((A_{\ell'-2}^{d-2} (\alpha_1), \ldots, A_{\ell'-2}^{d-2} (\alpha_q))\) is invertible by Proposition 2.4, the column vectors of \( A_{d_{d-2}^u} (\alpha_1), \ldots, A_{d_{d-2}^u} (\alpha_q) \) are linearly independent since \( \ell \geq 2 \), and span a \( \sum_{u=1}^{q} d_u' = d - \ell' \) dimensional linear subspace in \( \mathbb{C}^{d-2} \). On the other hand, since the map (3.11) is surjective, its kernel is a \((d - 2) - (\ell' - 2) = d - \ell' \) dimensional linear subspace in \( \mathbb{C}^{d-2} \). Hence to complete the proof of Lemma 3.8 we only need to check the equality
\[
A_{\ell'-2}^{d-2} (0) \prod_{u=1}^{q} (-\alpha_u I_{d-2} + N_{d-2})^{d_u} A_{d_{d-2}^u} (\alpha_v) = O
\]
for every \( 1 \leq v \leq q \). Without loss of generality, it suffices to show
\[
A_{\ell'-2}^{d-2} (0) \prod_{u=1}^{q} (-\alpha_u I_{d-2} + N_{d-2})^{d_u} A_{d_{d-2}^u} (\alpha_1) = O.
\]

Since \(-\alpha_1 I_{d-2} + N_{d-2})^{d_1} = \sum_{h=0}^{d_1} \binom{d_1}{h} (-\alpha_1)^{d_1-h}(N_{d-2})^{h}, \) the \((i, i + h)\)-th entry of \(-\alpha_1 I_{d-2} + N_{d-2})^{d_1} \) is \( \binom{d_1}{h} (-\alpha_1)^{d_1-h} \) for \( 0 \leq h \leq \min\{d_1', d - 2 - i\} \), and its other entries are...
zero. Hence for \(1 \leq i \leq d - 2 - d'_1\) and \(1 \leq j \leq d'_1\), we have

\[
\left[ \text{the } (i, j)\text{-th entry of } (-\alpha_1 I_{d-2} + N_{d-2})^{d'_1} A_{d-2}^{d'_1} (\alpha_1) \right]
= \sum_{h=0}^{d'_1} \left( \left[ \text{the } (i, i+h)\text{-th entry of } (-\alpha_1 I_{d-2} + N_{d-2})^{d'_1} \right] \cdot \left[ \text{the } (i+h, j)\text{-th entry of } A_{d-2}^{d'_1} (\alpha_1) \right] \right)
= \sum_{h=0}^{d'_1} \left\{ \left( \frac{d'_1}{h} \right) (-\alpha_1)^{d'_1-h} \cdot \binom{i+h-1}{j-1} \alpha_1^{i+h-j} \right\}
= (-\alpha_1)^{d'_1+i-j} \sum_{h=0}^{d'_1} \left( \frac{d'_1}{h} \right) \binom{i+h-1}{j-1} (-1)^{i+h-j}
= (-\alpha_1)^{d'_1+i-j} \left[ \frac{1}{(j-1)!} \left( \frac{d}{dx} \right)^{j-1} \left\{ \sum_{h=0}^{d'_1} \left( \frac{d'_1}{h} \right) x^{i+h-1} \right\} \right]_{x=-1}
= 0
\]

since \(j-1 < d'_1\). On the other hand, since \(\prod_{u=2}^{q} (-\alpha_u I_{d-2} + N_{d-2})^{d_u}\) is a polynomial of \(N_{d-2}\) with degree \(d'_2 + \cdots + d'_q = d - \ell' - d'_1\), its \((i, j)\) entry is zero for \(j-i > d - \ell' - d'_1\). Hence for \(1 \leq i \leq \ell' - 2\) and \(d - 2 - d'_1 < j \leq d - 2\), we have \(j-i > d - \ell' - d'_1\), which implies that the \((i, j)\) entry of \(A^{d'_2-2}(0) \prod_{u=2}^{q} (-\alpha_u I_{d-2} + N_{d-2})^{d_u}\) is zero.

To summarize, for every \(1 \leq i \leq \ell' - 2\) and \(1 \leq j \leq d'_1\), we have

\[
\left[ \text{the } (k, j)\text{-th entry of } (-\alpha_1 I_{d-2} + N_{d-2})^{d'_1} A_{d-2}^{d'_1} (\alpha_1) \right] = 0
\]

for \(1 \leq k \leq d - 2 - d'_1\), and also have

\[
\left[ \text{the } (i, k)\text{-th entry of } A^{d'_2-2}(0) \prod_{u=2}^{q} (-\alpha_u I_{d-2} + N_{d-2})^{d_u} \right] = 0
\]

for \(d - 2 - d'_1 < k \leq d - 2\). We therefore have the equality (3.12), which completes the proof of Lemma 3.8. \(\square\)

We return to the proof of Lemma 3.7.

Suppose first the condition (1) in Lemma 3.7. Then there exist \(m_{u,k} \in \mathbb{C}\) for \(1 \leq u \leq q\) and \(1 \leq k \leq d'_u\) such that the equality (3.10) holds. Multiplying \(A^{d'_2-2}(0) \prod_{u=2}^{q} (-\alpha_u I_{d-2} + N_{d-2})^{d_u}\) to both sides of the equality (3.10) from the left, we have the equality (3.8) by Lemma 3.8.

Suppose next the condition (2) in Lemma 3.7. Then the vector

\[
(X_{d-2})^{-1} \begin{pmatrix} \alpha_1 & \cdots & \alpha_q \\ \vdots & \ddots & \vdots \\ \alpha_1^{d-2} & \cdots & \alpha_q^{d-2} \end{pmatrix} \begin{pmatrix} m'_1 \\ \vdots \\ m'_q \end{pmatrix}
\]
is contained in the kernel of the linear map $A_{d-2}(0) \prod_{u=1}^q (-\alpha_u I_{d-2} + N_{d-2})^{d'_u}$. Hence by Lemma 3.8, there exist $m'_{u,k} \in \mathbb{C}$ for $1 \leq u \leq q$ and $1 \leq k \leq d'_u$ such that the equality

$$\begin{pmatrix} (X_{d-2})^{-1} & \alpha_1 & \cdots & \alpha_q \\ \alpha_1 & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \alpha_q^{d-2} \\ \alpha_1^{d-2} & \cdots & \alpha_q^{d-2} & \alpha_q \end{pmatrix} \begin{pmatrix} m'_1 \\ \vdots \\ m'_q \\ m'_{u,1} \\ \vdots \\ m'_{u,d'_u} \end{pmatrix} = \sum_{u=1}^q A_{d-2}^{d'_u} (\alpha_u) \begin{pmatrix} m'_1 \\ \vdots \\ m'_q \\ m'_{u,1} \\ \vdots \\ m'_{u,d'_u} \end{pmatrix}$$

(3.13)

holds. Putting $m_{u,k} := -km'_{u,k}$ for $1 \leq u \leq q$ and $1 \leq k \leq d'_u$, we find that the equality (3.13) is the same as the equality (3.10), which is also equivalent to (3.7). Hence the implication (2) $\Rightarrow$ (1) is proved, which completes the proof of Lemma 3.7. \( \Box \)

In the rest of the proof of Main Theorem, we continue to consider an analogue of each step in Sections 4 and 6 in [12] and in Section 5 in [13], in the case of having multiple fixed points. However in our case, its proofs are much complicated, compared with the case of having no multiple fixed points.

Based on the propositions above, we make the following definition:

**Definition 3.9.** We put

$$\mathcal{T}(m) := \left\{ (\zeta_1, \ldots, \zeta_\ell) \in \mathbb{C}^\ell \mid \text{The equality (3.6) holds} \right\},$$

$$\mathcal{S}(m) := \left\{ (\zeta_1, \ldots, \zeta_\ell) \in \mathcal{T}(m) \mid \zeta_1, \ldots, \zeta_\ell \text{ are mutually distinct} \right\},$$

$$\mathcal{B}(m) := \mathcal{T}(m) \setminus \mathcal{S}(m),$$

$$\mathcal{G}(m) := \{ \sigma \in \mathfrak{S}_\ell \mid (d_{\sigma(i)}, m_{\sigma(i)}) = (d_i, m_i) \text{ for every } 1 \leq i \leq \ell \}$$

and

$$\mathcal{Y}(m) := \left\{ \{I_1, \ldots, I_q\} \mid q \geq 1, \emptyset \neq I_u \subseteq \{1, \ldots, \ell\} \text{ for every } 1 \leq u \leq q, \\
I_1 \sqcup \cdots \sqcup I_q = \{1, \ldots, \ell\}, \\
\sum_{i \in I_u} m_i = 0 \text{ for every } 1 \leq u \leq q \right\},$$

where $\mathfrak{S}_\ell$ denotes the $\ell$-th symmetric group, and $I_1 \sqcup \cdots \sqcup I_q$ denotes the disjoint union of $I_1, \ldots, I_q$. Moreover we put

$$\mathcal{E}(I) := \left\{ (\zeta_1, \ldots, \zeta_\ell) \in \mathbb{C}^\ell \mid i, j \in I \Rightarrow \zeta_i = \zeta_j \right\} \text{ for } I \in \mathcal{Y}(m).$$

Combining Propositions 3.1, 3.5 and 3.6 we obviously have the following:

**Proposition 3.10.** The equality

$$\mathcal{S}(m) = \left\{ (\zeta_1, \ldots, \zeta_\ell) \in \mathbb{C}^\ell \mid \text{There exists a unique non-zero complex number } \rho \text{ such that } f(z) = z + \rho(z - \zeta_1)^{d_1} \cdots (z - \zeta_\ell)^{d_\ell} \in \text{Poly}_d(d_1, \ldots, d_\ell) \text{ holds} \right\}$$

holds.

The group $\text{Aut}(\mathbb{C})$ naturally acts on $\mathbb{C}^\ell$ by $\gamma \cdot (\zeta_1, \ldots, \zeta_\ell) := (\gamma(\zeta_1), \ldots, \gamma(\zeta_\ell))$. By Proposition 3.10 we obviously have the following:

**Proposition 3.11.** Let $P : \text{Poly}_d \rightarrow \text{MP}_d$ be the natural projection. Then

1. we can define the surjection $\bar{\pi}(m) : \mathcal{S}(m) \rightarrow (\Phi_d(d_1, \ldots, d_\ell) \circ P)^{-1}(\bar{m})$ by

$$\bar{\pi}(m) : (\zeta_1, \ldots, \zeta_\ell) \mapsto f(z) := z + \rho(z - \zeta_1)^{d_1} \cdots (z - \zeta_\ell)^{d_\ell},$$

where $\rho$ is defined to be a unique non-zero complex number such that the equalities $\nu(f, \zeta_i) = m_i$ hold for $1 \leq i \leq \ell$.

2. $\mathcal{S}(m)$ is invariant under the action of $\text{Aut}(\mathbb{C})$ on $\mathbb{C}^\ell$. 
(3) The actions of $\text{Aut}(\mathbb{C})$ on $\tilde{S}(m)$ and on $(\Phi_d(d_1, \ldots, d_\ell) \circ P)^{-1}(m)$ commute with the map $\tilde{\pi}(m)$.

(4) The group $\mathcal{G}(m)$ acts on $\tilde{S}(m)$ by the permutation of coordinates. Moreover for $\zeta, \zeta' \in \tilde{S}(m)$, the equality $\tilde{\pi}(m)(\zeta) = \tilde{\pi}(m)(\zeta')$ holds if and only if there exists $\sigma \in \mathcal{G}(m)$ such that $\sigma \cdot \zeta = \zeta'$.

Note that the action of $\mathcal{G}(m)$ on $\tilde{S}(m)$ and the action of $\text{Aut}(\mathbb{C})$ on $\tilde{S}(m)$ commute. Hence the action of $\mathcal{G}(m)$ on $\tilde{S}(m)$ naturally induces the action of $\mathcal{G}(m)$ on $\tilde{S}(m)/\text{Aut}(\mathbb{C})$. By Proposition 3.11 we naturally have the following:

**Proposition 3.12.** Under the same notation as in Proposition 3.11, the map $\tilde{\pi}(m)$ induces the surjection $\tilde{\pi}'(m) : \tilde{S}(m)/\text{Aut}(\mathbb{C}) \to \Phi_d(d_1, \ldots, d_\ell)^{-1}(m)$, which also induces the bijection $\tilde{\pi}''(m) : \tilde{S}(m)/\text{Aut}(\mathbb{C}) \to \Phi_d(d_1, \ldots, d_\ell)^{-1}(m).$

On the other hand, we have the following for $\tilde{B}(m)$:

**Proposition 3.13.** We have

$$\tilde{B}(m) = \bigcup_{\mathbb{I} \in \mathcal{Y}(m)} \tilde{\mathbb{E}}(\mathbb{I}).$$

**Proof.** We prove $\tilde{B}(m) \supseteq \bigcup_{\mathbb{I} \in \mathcal{Y}(m)} \tilde{\mathbb{E}}(\mathbb{I})$ first. For arbitrary $\mathbb{I} \in \mathcal{Y}(m)$ and $\zeta = (\zeta_1, \ldots, \zeta_\ell) \in \tilde{\mathbb{E}}(\mathbb{I})$, we put $\mathbb{I} = \{I_1, \ldots, I_q\}$ and $\alpha_u := \zeta_i$ for $i \in I_u$ for each $1 \leq u \leq q$. Then we have

$$\sum_{i=1}^\ell m_i \zeta_i = \sum_{u=1}^q \sum_{i \in I_u} m_i \zeta_i = \sum_{u=1}^q (\sum_{i \in I_u} m_i) \zeta_i^k = 0 \text{ for every } k \in \mathbb{N},$$

which implies

$$\left(\begin{array}{cccc}
\zeta_1 & \cdots & \zeta_\ell \\
\vdots & \ddots & \vdots \\
\zeta_1^{d-2} & \cdots & \zeta_\ell^{d-2} \\
\end{array}\right) \left(\begin{array}{c}
m_1 \\
\vdots \\
m_\ell \\
\end{array}\right) = \left(\begin{array}{c}
0 \\
\vdots \\
0 \\
\end{array}\right),$$

and hence $\zeta \in \tilde{T}(m)$. Moreover since $(m_1, \ldots, m_\ell) \neq (0, \ldots, 0)$, there exists $I \in \mathbb{I}$ with $\# I \geq 2$, which implies $\zeta \notin \tilde{S}(m)$. We therefore have $\zeta \in \tilde{B}(m)$ for every $\zeta \in \bigcup_{\mathbb{I} \in \mathcal{Y}(m)} \tilde{\mathbb{E}}(\mathbb{I})$, which assures $\tilde{B}(m) \supseteq \bigcup_{\mathbb{I} \in \mathcal{Y}(m)} \tilde{\mathbb{E}}(\mathbb{I})$.

We prove $\tilde{B}(m) \subseteq \bigcup_{\mathbb{I} \in \mathcal{Y}(m)} \tilde{\mathbb{E}}(\mathbb{I})$ next. For $\zeta = (\zeta_1, \ldots, \zeta_\ell) \in \tilde{B}(m)$, we put

$$\mathbb{I}(\zeta) := \left\{ I : \begin{array}{l}
\emptyset \neq I \subseteq \{1, \ldots, \ell\}, \\
 i, j \in I \Rightarrow \zeta_i = \zeta_j, \\
i \in I, j \in \{1, \ldots, \ell\} \setminus I \Rightarrow \zeta_i \neq \zeta_j
\end{array} \right\},$$

where $\zeta = (\zeta_1, \ldots, \zeta_\ell)$, $d'_u = \sum_{i \in I_u} (d_i - 1)$ and $\alpha_u := \zeta_i$ for $i \in I_u$ for each $1 \leq u \leq q$. Then by definition, we have $I_1 \Pi \cdots I_q = \{1, \ldots, \ell\}$, $\sum_{u=1}^q d'_u = d - \ell$, the mutual distinctness of $\alpha_1, \ldots, \alpha_\ell$ and $\sum_{i=1}^\ell m_i \zeta_i^k = \sum_{u=1}^q (\sum_{i \in I_u} m_i) \zeta_i^k = \sum_{u=1}^q (\sum_{i \in I_u} m_i) \alpha_u^k$ for every $k \in \mathbb{N}$. Moreover since $\zeta \notin \tilde{S}(m)$, we have $q < \ell$. Hence for $\zeta = (\zeta_1, \ldots, \zeta_\ell) \in \tilde{B}(m)$, we have

$$0 = A^{d-2}_{\ell-2}(0) \left( \prod_{i=1}^\ell (-\zeta_i d_{i-2} + N_{d-2})^{d_i-1} \right) (X_{d-2})^{-1} \left( \begin{array}{ccc}
\zeta_1 & \cdots & \zeta_\ell \\
\vdots & \ddots & \vdots \\
\zeta_1^{d-2} & \cdots & \zeta_\ell^{d-2} \\
\end{array}\right) \left(\begin{array}{c}
m_1 \\
\vdots \\
m_\ell \\
\end{array}\right),$$

and

$$= A^{d-2}_{\ell-2}(0) \left( \prod_{u=1}^q (-\alpha_u d_{d-2} + N_{d-2})^{d'_u} \right) (X_{d-2})^{-1} \left( \begin{array}{ccc}
\alpha_1 & \cdots & \alpha_q \\
\vdots & \ddots & \vdots \\
\alpha_1^{d-2} & \cdots & \alpha_q^{d-2} \\
\end{array}\right) \left(\sum_{i \in I_u} m_i \right).$$
Hence by applying Lemma 3.14 in the case $\ell' = \ell$ and $m'_u = \sum_{i \in I_u} m_i$ for $1 \leq u \leq q$, we have the equality
\[
\sum_{u=1}^{q} A_{d-1}^{d'_{u}+1} (\alpha_u) \begin{pmatrix}
\sum_{i \in I_u} m_i \\
m_{u,1} \\
\vdots \\
m_{u,d_u}
\end{pmatrix} = 0
\]
for some $m_{u,k} \in \mathbb{C}$ with $1 \leq u \leq q$ and $1 \leq k \leq d'_{u}$. Since $\sum_{u=1}^{q} (d'_{u} + 1) = d - \ell + q \leq d - 1$, the invertibility of the square matrix $\begin{pmatrix} A_{d-\ell+q}^{d'_{1}+1} (\alpha_1), \ldots, A_{d-\ell+q}^{d'_{q}+1} (\alpha_q) \end{pmatrix}$ implies $t(\sum_{i \in I_u} m_i, m_{u,1}, \ldots, m_{u,d_u}) = t (0, \ldots, 0)$ for every $1 \leq u \leq q$. We therefore have $\mathcal{I}(\zeta) \in \mathcal{I}'(m)$ and also have $\zeta \in \tilde{E}(\mathcal{I}(\zeta))$, which completes the proof of $\tilde{B}(m) \subseteq \bigcup_{I \in \mathcal{I}'(m)} \tilde{E}(I)$. \qedsymbol

By Propositions 3.11(2) and 3.13 we have the following:

**Proposition 3.14.** The subsets $\tilde{T}(m), \tilde{S}(m), \tilde{B}(m)$ and $\tilde{E}(I)$ for $I \in \mathcal{I}'(m)$ are invariant under the action of $\text{Aut}(\mathcal{C})$ on $\mathcal{C}^\ell$.

**Proof.** For each $I \in \mathcal{I}'(m)$, the subset $\tilde{E}(I)$ is trivially invariant under the action of $\text{Aut}(\mathcal{C})$ on $\mathcal{C}^\ell$, which implies the invariance of $\tilde{B}(m)$ by Proposition 3.13. Moreover $\tilde{S}(m)$ and $\tilde{T}(m)$ are also invariant under the action of $\text{Aut}(\mathcal{C})$ on $\mathcal{C}^\ell$ by Proposition 3.11(2) and $\tilde{T}(m) = \tilde{S}(m) \Pi \tilde{B}(m)$. \qedsymbol

We denote by $\mathbb{P}^{\ell-2}$ the complex projective space of dimension $\ell - 2$. Note that in the case $\ell = 2$, the 0-dimensional projective space $\mathbb{P}^{\ell-2} = \mathbb{P}^0$ consists of one point. We naturally have the isomorphism
\[
\left( \mathcal{C}^\ell / \text{Aut}(\mathcal{C}) \right) \setminus \{ \overline{0} \} \cong \mathbb{P}^{\ell-2} \quad \text{by} \quad (\zeta_1, \ldots, \zeta_{\ell-1}, \zeta_{\ell}) \mapsto (\zeta_1 - \zeta_{\ell} : \cdots : \zeta_{\ell-1} - \zeta_{\ell}),
\]
where $\overline{0} := \{(\zeta_1, \ldots, \zeta) \in \mathcal{C}^\ell \mid \zeta \in \mathbb{C} \}$ denotes the equivalence class of 0 in $\mathcal{C}^\ell$ under the action of $\text{Aut}(\mathcal{C})$ on $\mathcal{C}^\ell$. Note that we always have $\mathcal{I}_0 := \{\{1, \ldots, \ell\}\} \in \mathcal{I}'(m)$ and $\tilde{E}(\mathcal{I}_0) = \overline{0}$.

**Definition 3.15.** We put
\[
\begin{pmatrix}
\psi_1(\zeta) \\
\vdots \\
\psi_{\ell-2}(\zeta)
\end{pmatrix} := A_{d-2}^{d'_{\ell-2}} (0) (N_{d-2})^{d_{\ell-1}} \left( \prod_{i=1}^{\ell-1} (-\zeta_i I_{d-2} + N_{d-2})^{d_{i-1}} \right)
\]
\[
\cdot (X_{d-2})^{-1} \begin{pmatrix}
\zeta_1 & \cdots & \zeta_{\ell-1} \\
\vdots & \ddots & \vdots \\
\zeta_{d-2}^{\ell-2} & \cdots & \zeta_{\ell-1}^{\ell-1}
\end{pmatrix} \begin{pmatrix}
m_1 \\
m_2 \\
\vdots \\
m_{\ell-1}
\end{pmatrix}
\]
\[
T(m) := \left\{ (\zeta_1, \ldots, \zeta_{\ell-1}) \in \mathbb{P}^{\ell-2} \mid \psi_1(\zeta) = \cdots = \psi_{\ell-2}(\zeta) = 0 \right\},
\]
\[
S(m) := \left\{ (\zeta_1, \ldots, \zeta_{\ell-1}) \in T(m) \mid \zeta_1, \ldots, \zeta_{\ell-1} \text{ and 0 are mutually distinct} \right\},
\]
\[
B(m) := T(m) \setminus S(m) \quad \text{and} \quad \mathcal{I}(m) := \left\{ I_1, \ldots, I_q \right\} \quad \text{for} \quad q \geq 2, \quad \emptyset \neq I_u \subset \{1, \ldots, \ell\} \text{ for every } 1 \leq u \leq q,
\]
\[
\sum_{i \in I_u} m_i = 0 \text{ for every } 1 \leq u \leq q.
\]
Moreover we put
\[
E(I) := \left\{ (\zeta_1, \ldots, \zeta_{\ell-1}) \in \mathbb{P}^{\ell-2} \mid i, j \in \mathcal{I} \Rightarrow \zeta_i = \zeta_j, \text{ where } \zeta_\ell = 0 \right\} \text{ for } I \in \mathcal{I}(m).
\]
Note that \( \psi_1(\zeta), \ldots, \psi_{\ell-2}(\zeta) \) is obtained from the left-hand side of the equality (3.6) by substituting \( \zeta = 0 \). Also note that \( \mathcal{I}(m) \) is obtained from \( \mathcal{I}'(m) \) by excluding exactly one element \( \mathbb{I}_0 = \{1, \ldots, \ell\} \in \mathcal{I}'(m) \). Moreover for every \( \mathbb{I} \in \mathcal{I}(m) \), we have \#\( \mathbb{I} \geq 2 \), which implies \( E(\mathbb{I}) \neq \emptyset \). Hence under the isomorphism \( (\mathbb{C}^k/\text{Aut}(\mathbb{C})) \setminus \{ \mathbb{I} \} \cong \mathbb{P}^{\ell-2} \), we clearly have

\[
\left( \mathcal{T}(m)/\text{Aut}(\mathbb{C}) \right) \setminus \{ \mathbb{I} \} \cong T(m), \quad \mathcal{S}(m)/\text{Aut}(\mathbb{C}) \cong S(m), \quad \left( \hat{B}(m)/\text{Aut}(\mathbb{C}) \right) \setminus \{ \mathbb{I} \} \cong B(m),
\]

\[
\left( \hat{E}(\mathbb{I})/\text{Aut}(\mathbb{C}) \right) \setminus \{ \mathbb{I} \} \cong E(\mathbb{I}) \quad \text{for} \quad \mathbb{I} \in \mathcal{I}(m) \quad \text{and} \quad B(m) = \bigcup_{\mathbb{I} \in \mathcal{I}(m)} E(\mathbb{I})
\]

by Propositions 3.13 and 3.14.

**Proposition 3.16.** \( \psi_k(\zeta) \) is a homogeneous polynomial of \( \zeta_1, \ldots, \zeta_{\ell-1} \) with degree \( d - \ell + k \) for \( 1 \leq k \leq \ell - 2 \).

*Proof.* Direct calculation easily verifies the proposition. \( \square \)

By Proposition 3.12 and the isomorphism \( \mathcal{S}(m)/\text{Aut}(\mathbb{C}) \cong S(m) \), we have the following:

**Proposition 3.17.** Under the isomorphism \( \mathcal{S}(m)/\text{Aut}(\mathbb{C}) \cong S(m) \), the surjection \( \pi'(m) : \mathcal{S}(m)/\text{Aut}(\mathbb{C}) \to \Phi_d(d_1, \ldots, d_\ell)^{-1}(\mathbb{m}) \) induces the surjection

\[
\pi(m) : S(m) \to \Phi_d(d_1, \ldots, d_\ell)^{-1}(\mathbb{m}).
\]

Moreover the action of \( \mathcal{S}(m) \) on \( \mathcal{S}(m)/\text{Aut}(\mathbb{C}) \) induces the action of \( \mathcal{S}(m) \) on \( S(m) \), and the bijection \( \varpi'(m) : \left( \mathcal{S}(m)/\text{Aut}(\mathbb{C}) \right) /\mathcal{S}(m) \cong \Phi_d(d_1, \ldots, d_\ell)^{-1}(\mathbb{m}) \) induces the bijection

\[
\varpi(m) : S(m)/\mathcal{S}(m) \cong \Phi_d(d_1, \ldots, d_\ell)^{-1}(\mathbb{m}).
\]

**Definition 3.18.** We put

\[
\Sigma(m) := \left\{ (\zeta_1 : \cdots : \zeta_{\ell}) \in \mathbb{P}^{\ell-1} \mid (\zeta_1, \ldots, \zeta_{\ell}) \in \mathcal{S}(m) \quad \text{and} \quad d_1\zeta_1 + \cdots + d_\ell\zeta_\ell = 0 \right\}
\]

and

\[
\tilde{\Sigma}(m) := \left\{ (\zeta_1, \ldots, \zeta_{\ell}) \in \mathcal{S}(m) \mid \varpi(m)(\zeta_1, \ldots, \zeta_{\ell}) \in \text{MC}_d(d_1, \ldots, d_\ell) \right\},
\]

where \( \varpi(m) \) is the map defined in Proposition 3.11(1).

Summing up the propositions above, we naturally have the following:

**Proposition 3.19.**

1. The natural map \( \Sigma(m) \to S(m) \) defined by \( (\zeta_1 : \cdots : \zeta_{\ell}) \mapsto (\zeta_1 - \zeta_{\ell} : \cdots : \zeta_{\ell-1} - \zeta_{\ell}) \) is well-defined and bijective.
2. The group \( \mathcal{S}(m) \) acts on \( \Sigma(m) \) by the permutation of coordinates. Hence we have the bijection \( \Sigma(m)/\mathcal{S}(m) \cong S(m)/\mathcal{G}(m) \cong \Phi_d(d_1, \ldots, d_\ell)^{-1}(\mathbb{m}) \).
3. The map \( \tilde{\Sigma}(m) \to \tilde{\Phi}_d(d_1, \ldots, d_\ell)^{-1}(\mathbb{m}) \) defined by \( (\zeta_1, \ldots, \zeta_{\ell}) \mapsto f(z) = z + (z - \zeta_1)^{d_1} \cdots (z - \zeta_{\ell})^{d_\ell} \) is well-defined and surjective.
4. The group \( \mathcal{S}(m) \) acts on \( \tilde{\Sigma}(m) \) freely by the permutation of coordinates. Moreover we have the bijection \( \tilde{\Sigma}(m)/\mathcal{S}(m) \cong \tilde{\Phi}_d(d_1, \ldots, d_\ell)^{-1}(\mathbb{m}) \). Hence we have \( \#\tilde{\Phi}_d(d_1, \ldots, d_\ell)^{-1}(\mathbb{m}) = \#\tilde{\Sigma}(m)/\#\mathcal{S}(m) \).
5. The natural projection \( \Sigma(m) \to \Sigma(m) \) defined by \( (\zeta_1, \ldots, \zeta_{\ell}) \mapsto (\zeta_1 : \cdots : \zeta_{\ell}) \) is well-defined and surjective. Moreover it is a \((d - 1)\)-to-one map. Hence we have \( \#\tilde{\Sigma}(m) = (d - 1)\cdot\#\Sigma(m) \).

*Proof.* For every \( (\zeta_1, \ldots, \zeta_{\ell}) \in \mathcal{S}(m) \), there exists a unique non-zero complex number \( \rho \) such that \( f(z) = z + \rho(z - \zeta_1)^{d_1} \cdots (z - \zeta_{\ell})^{d_\ell} \in \text{Poly}_d(d_1, \ldots, d_\ell) \) holds and that the equalities \( \psi_i(f, \zeta_i) = m_i \) hold for \( 1 \leq i \leq \ell \), by Proposition 3.10. Putting \( \zeta_i' = \zeta_i - b \) for \( b := (d_1\zeta_1 + \cdots + d_\ell\zeta_{\ell})/n \).\( \square \)
Proof. For $1 \leq i \leq \ell - 2$, we have
\[
\begin{vmatrix}
\frac{\partial \psi_1}{\partial \zeta_1} & \cdots & \frac{\partial \psi_1}{\partial \zeta_{\ell-2}} \\
\vdots & \ddots & \vdots \\
\frac{\partial \psi_2}{\partial \zeta_1} & \cdots & \frac{\partial \psi_2}{\partial \zeta_{\ell-2}}
\end{vmatrix}_{\zeta_{\ell-1}=1} \neq 0
\]
for every $(\zeta_1 : \cdots : \zeta_{\ell-1}) \in S(m)$. Proposition 3.20. In the case $\ell \geq 3$, we have
\[
d_{\ell}(\zeta_1, \ldots, \zeta_\ell)/d, \text{ we still have } (\zeta_1', \ldots, \zeta_\ell') \in \tilde{S}(m) \text{ and also have } (\zeta_1' : \cdots : \zeta_\ell') \in \Sigma(m). \text{ Here, note that } \begin{aligned}
z + \rho(z - \zeta_1')^{d_1} \cdots (z - \zeta_\ell')^{d_\ell} \text{ and } z + \rho a^{-(d-1)}(z - a\zeta_1')^{d_1} \cdots (z - a\zeta_\ell')^{d_\ell} \text{ are affinely conjugate for } a \in \mathbb{C} \setminus \{0\}. \end{aligned}
\]
It follows that $(a\zeta_1', \ldots, a\zeta_\ell')$ belongs to $\tilde{\Sigma}(m)$ if and only if $\rho a^{-(d-1)} = 1$. Since $\rho \neq 0$, we always have $\# \{ a \in \mathbb{C} \mid \rho a^{-(d-1)} = 1 \} = d - 1$, which implies the assertion (5). The rests are almost obvious by the propositions already obtained. \qed

\[
A_{\ell-2}^{d-2}(0) (N_{d-2})^{d_{\ell-1}} \left( \prod_{j=1}^{\ell-1} (-\zeta_j I_{d-2} + N_{d-2})^{d_{j-1}} \right) \left( \begin{array}{c}
\zeta_1 \\
\vdots \\
\zeta_{\ell-1}
\end{array} \right) \left( \begin{array}{c}
m_1 \\
\vdots \\
m_{\ell-1}
\end{array} \right)
\]

\[
A_{\ell-2}^{d-2}(0) (N_{d-2})^{d_{\ell-1}} \left( \prod_{j=1}^{\ell-1} (-\zeta_j I_{d-2} + N_{d-2})^{d_{j-1}} \right) \left( \begin{array}{c}
\zeta_1 \\
\vdots \\
\zeta_{\ell-1}
\end{array} \right) \left( \begin{array}{c}
m_1 \\
\vdots \\
m_{\ell-1}
\end{array} \right)
\]

\[
A_{\ell-2}^{d-2}(0) (N_{d-2})^{d_{\ell-1}} \left( \prod_{j=1}^{\ell-1} (-\zeta_j I_{d-2} + N_{d-2})^{d_{j-1}} \right) \left( \begin{array}{c}
\zeta_1 \\
\vdots \\
\zeta_{\ell-1}
\end{array} \right) \left( \begin{array}{c}
m_1 \\
\vdots \\
m_{\ell-1}
\end{array} \right)
\]

\[
A_{\ell-2}^{d-2}(0) (N_{d-2})^{d_{\ell-1}} \left( \prod_{j=1}^{\ell-1} (-\zeta_j I_{d-2} + N_{d-2})^{d_{j-1}} \right) \left( \begin{array}{c}
\zeta_1 \\
\vdots \\
\zeta_{\ell-1}
\end{array} \right) \left( \begin{array}{c}
m_1 \\
\vdots \\
m_{\ell-1}
\end{array} \right)
\]
Moreover since
\[-\zeta_j I_d - 2 + N_{d-2} \cdot t(1, \zeta_1, \ldots, \zeta_i^k, \ldots, *) = t(\zeta_i - \zeta_j, (\zeta_i - \zeta_j)\zeta_i, \ldots, (\zeta_i - \zeta_j)\zeta_i^{k-1}, *, \ldots , *) \]
for \(1 \leq k \leq d - 3\), we have
\[
A_{\ell-2}^{d-2}(0) (N_{d-2})^{d_{\ell-1}-1} \left( \prod_{j=1}^{\ell-1} (-\zeta_j I_d - 2 + N_{d-2})^{d_j-1} \right) \left( \begin{array}{c} \zeta_1 \\ \vdots \\ \zeta_{\ell-3} \end{array} \right) = \zeta_i^{d_{\ell-1}-1} \prod_{j=1}^{\ell-1} (\zeta_i - \zeta_j)^{d_j-1} \left( \begin{array}{c} 1 \\ \vdots \\ \zeta_{\ell-3} \end{array} \right).
\]
Hence we have

**Lemma 3.21.** If \(d_i = 1\), then
\[
\left( \frac{\partial \psi_i}{\partial \zeta_i} \right) = m_i \zeta_i^{d_i-1} \left( \prod_{1 \leq j \leq \ell-1, j \neq i} (\zeta_i - \zeta_j)^{d_j-1} \right) \left( \begin{array}{c} 1 \\ \vdots \\ \zeta_{\ell-3} \end{array} \right).
\]
Moreover for \((\zeta_1 : \cdots : \zeta_{\ell-1}) \in S(m)\), we have \(m_i \zeta_i^{d_i-1} \prod_{1 \leq j \leq \ell-1, j \neq i} (\zeta_i - \zeta_j)^{d_j-1} \neq 0\).

**Proof.** By the assertion (2) in Proposition 3.3 and the conditions \(d_i = 1\) and \((\zeta_1 : \cdots : \zeta_{\ell-1}) \in S(m)\) imply \(m_i \neq 0\). The rest of the assertions are obvious by the argument above. \(\square\)

We consider the case \(d_i \geq 2\) next. If \(d_i \geq 2\), then we have \(\prod_{j=1}^{\ell-1}(\zeta_i - \zeta_j)^{d_j-1} = 0\), which implies
\[
\left( \frac{\partial \psi_i}{\partial \zeta_i} \right) = -(d_i - 1) A_{\ell-2}^{d-2}(0) (N_{d-2})^{d_{\ell-1}-1} \left( \prod_{1 \leq j \leq \ell-1, j \neq i} (-\zeta_j I_d - 2 + N_{d-2})^{d_j-1} \right)
\cdot (-\zeta_i I_d - 2 + N_{d-2})^{d_i-2} (X_{d-2})^{-1} \left( \begin{array}{c} \zeta_1 \\ \vdots \\ \zeta_{\ell-1} \\ \zeta_i^{d_i-2} \\ \zeta_{\ell-2} \end{array} \right) \left( \begin{array}{c} m_1 \\ \vdots \\ m_{\ell-1} \end{array} \right).
\]
We put
\[
\left( \begin{array}{c} \xi_1 \\ \vdots \\ \xi_{d-2} \end{array} \right) := (N_{d-2})^{d_{i-1}-1} \left( \prod_{1 \leq j \leq \ell-1, j \neq i} (-\zeta_j I_d - 2 + N_{d-2})^{d_j-1} \right)
\cdot (-\zeta_i I_d - 2 + N_{d-2})^{d_i-2} (X_{d-2})^{-1} \left( \begin{array}{c} \zeta_1 \\ \vdots \\ \zeta_{\ell-1} \\ \zeta_i^{d_i-2} \\ \zeta_{\ell-2} \end{array} \right) \left( \begin{array}{c} m_1 \\ \vdots \\ m_{\ell-1} \end{array} \right).
\]
Then for \((\zeta_1 : \cdots : \zeta_{\ell-1}) \in S(m)\), we have \(t(\psi_1(\zeta), \ldots, \psi_{\ell-2}(\zeta)) = t(0, 0, \ldots, 0)\), which is equivalent to
\[
A_{\ell-2}^{d-2}(0) (-\zeta_i I_d - 2 + N_{d-2}) \cdot t(\xi_1, \ldots, \xi_{d-2}) = 0.
\]
Hence we have \(\xi_k = \xi_k^{d-k-1}\) for \(1 \leq k \leq \ell - 1\), and therefore we have
\[
\left( \frac{\partial \psi_i}{\partial \zeta_i} \right) = -(d_i - 1) A_{\ell-2}^{d-2}(0) \left( \begin{array}{c} \xi_1 \\ \vdots \\ \xi_{d-2} \end{array} \right) = -(d_i - 1) \xi_1 \left( \begin{array}{c} 1 \\ \vdots \\ \xi_{\ell-3} \end{array} \right).
\]
Lemma 3.22. In the equality (3.14), we have $\xi_1 \neq 0$.

Proof. Suppose that $\xi_1 = 0$ holds. Then we have $\xi_k = 0$ for $1 \leq k \leq \ell - 1$, which implies that the vector

$$(X_{d-2})^{-1} \begin{pmatrix} \zeta_1 & \cdots & \zeta_{\ell-1} \\ \vdots & \ddots & \vdots \\ \zeta_{d-2} & \cdots & \zeta_{d-2} \\ \zeta_{\ell-1} & \cdots & \zeta_{\ell-1} \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_{\ell-1} \end{pmatrix}$$

is contained in the kernel of the linear map

$$A_{d-2}^{d-2}(0) (N_{d-2})^{d-1} \left( \prod_{1 \leq j \leq \ell-1, \ j \neq i} (-\zeta_j I_{d-2} + N_{d-2})^{d_j-1} \right) \left( -\zeta_i I_{d-2} + N_{d-2} \right)^{d_i-2}.$$ Hence by applying Lemma 3.7 in the case $\ell' = \ell + 1$, $q = \ell$, $d' = d_j - 1$ for $j \in \{ j \in \mathbb{Z} \mid 1 \leq j \leq \ell, \ j \neq i \}$, $d_i = d_i - 2$, $\alpha_i = \zeta_i$ for $1 \leq i \leq \ell - 1$, $\alpha_\ell = 0$ and $m_i' = m_i$ for $1 \leq i \leq \ell$, we have the equality

$$\sum_{1 \leq j \leq \ell-1, \ j \neq i} A_{d-1}^{d_j-1}(\zeta_j) \begin{pmatrix} m_j \\ m_{j,1} \\ \vdots \\ m_{j,d_j-1} \end{pmatrix} + A_{d-1}^{d_i-1}(0) \begin{pmatrix} m_\ell \\ m_{\ell,1} \\ \vdots \\ m_{\ell,d_\ell-1} \end{pmatrix} + A_{d-1}^{d_i-2}(\zeta_i) \begin{pmatrix} m_i \\ m_{i,1} \\ \vdots \\ m_{i,d_i-2} \end{pmatrix} = 0$$

for some $m_{j,k} \in \mathbb{C}$ with

$$(j,k) \in \left\{ (j,k) \in \mathbb{Z}^2 \mid \begin{array}{l} 1 \leq j \leq \ell, \\
 j \neq i \Rightarrow 1 \leq k \leq d_j - 1, \\
 j = i \Rightarrow 1 \leq k \leq d_i - 2 \end{array} \right\}.$$ Since $\left( \sum_{1 \leq j \leq \ell, \ j \neq i} d_j \right) + (d_i - 1) = d - 1$, the square matrix

$$\left( A_{d-1}^{d_1}(\zeta_1), \ldots, A_{d-1}^{d_i-1}(\zeta_i), \ldots, A_{d-1}^{d_{\ell-1}}(\zeta_{\ell-1}), A_{d-1}^{d_\ell}(0) \right)$$

is invertible by Proposition 2.4. We therefore have $(m_1, \ldots, m_\ell) = (0, \ldots, 0)$, which contradicts the assumption $(m_1, \ldots, m_\ell) \in \mathbb{C}^\ell \setminus \{0\}$. Hence the contradiction assures $\xi_1 \neq 0$. \hfill \Box

By Lemmas 3.21 and 3.22, we have

Lemma 3.23. For every $(\zeta_1 : \cdots : \zeta_{\ell-1}) \in S(m)$ and for every $1 \leq i \leq \ell - 2$, there exists a non-zero complex number $c_i$ such that the equality

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial \zeta_i} \\ \vdots \\ \frac{\partial \psi_{\ell-2}}{\partial \zeta_i} \end{pmatrix} = c_i \begin{pmatrix} 1 \\ \zeta_i \\ \vdots \\ \zeta_{\ell-3} \end{pmatrix}$$

holds.

Hence by Lemma 3.23, we have

$$\det \begin{pmatrix} \frac{\partial \psi_1}{\partial \zeta_1} & \cdots & \frac{\partial \psi_{\ell-1}}{\partial \zeta_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_1}{\partial \zeta_{\ell-2}} & \cdots & \frac{\partial \psi_{\ell-1}}{\partial \zeta_{\ell-2}} \end{pmatrix} = \prod_{i=1}^{\ell-2} c_i \cdot \prod_{1 \leq i < j \leq \ell-2} (\zeta_j - \zeta_i) \neq 0,$$

which completes the proof of Proposition 3.20. \hfill \Box

By Propositions 3.16 and 3.20, we have the following:
Proposition 3.24. We always have \( \#S(m) \leq \frac{(d-2)!}{(d-\ell)!} \). Moreover the equality \( \#S(m) = \frac{(d-2)!}{(d-\ell)!} \) holds if and only if \( J(m) = \emptyset \).

Proof. First we consider the case \( \ell = 2 \). In this case, the equality (3.6) always holds, which implies \( \tilde{T}(m) = \mathbb{C}^2 \) and \( \tilde{S}(m) = \{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 \mid \zeta_1 \neq \zeta_2 \} \). Hence we always have \( T(m) = S(m) = \mathbb{P}^0 \), and therefore have \( \#S(m) = 1 = \frac{(d-2)!}{(d-\ell)!} \). On the other hand, since \( m_1 + m_2 = 0 \) and \( (m_1, m_2) \neq (0,0) \), we have \( m = (m_1, -m_1) \) with \( m_1 \in \mathbb{C} \setminus \{0\} \), which implies that \( J(m) = \emptyset \) always holds.

We consider the case \( \ell \geq 3 \) next. In this case, \( S(m) \) is a discrete set by Proposition 3.20. Moreover for every \( \zeta \in S(m) \), the intersection multiplicity of \( \psi_1(\zeta), \ldots, \psi_{\ell-2}(\zeta) \) at \( \zeta \) is 1. Hence by a similar argument to the proof of Proposition 6.2 in [12], we have \( \#S(m) \leq \prod_{k=1}^{\ell-2} \deg \psi_k = \frac{(d-2)!}{(d-\ell)!} \). Moreover since \( B(m) = T(m) \setminus S(m) \) and \( B(m) = \bigcup_{1 \in \tilde{S}(m)} E(1) \), the equality \( \#S(m) = \frac{(d-2)!}{(d-\ell)!} \) holds if and only if \( J(m) = \emptyset \) holds, which can also be obtained by a similar argument to the proof of Proposition 6.6 in [12].

Based on the propositions above, we complete the proof of Main Theorem.

Proof of Main Theorem.

First we consider the case \( (m_1, \ldots, m_\ell) = 0 \). In this case, we have \( \Phi_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) = \tilde{\Phi}_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) = \emptyset \) by Proposition 3.24. Moreover the condition (2c) in Main Theorem is not satisfied. Hence in this case, Main Theorem holds. In the rest of the proof we assume \( (m_1, \ldots, m_\ell) \neq 0 \).

Note first that

- the condition \( \sum_{i \in I} m_i \neq 0 \) for every \( 0 \neq I \subset \{1, \ldots, \ell\} \) is equivalent to \( J(m) = \emptyset \).
- \( (d_1, m_1), \ldots, (d_\ell, m_\ell) \) are mutually distinct if and only if \( \#G(m) = 1 \).

Hence the condition (2c) in Main Theorem is equivalent to “\( J(m) = \emptyset \) and \( \#G(m) = 1 \)”.

We consider \( \#\tilde{\Phi}_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) \) first. By the assertions (1), (4) and (5) in Proposition 3.19 we always have

\[
\#\tilde{\Phi}_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) = \frac{(d-1) \cdot \#S(m)}{\#G(m)}.
\]

Hence by Proposition 3.24 we have

\[
\#\tilde{\Phi}_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) \leq \frac{(d-1)!}{(d-\ell)!}.
\]

Moreover the equality \( \#\tilde{\Phi}_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) = \frac{(d-1)!}{(d-\ell)!} \) holds if and only if \( \#S(m) = \frac{(d-2)!}{(d-\ell)!} \) and \( \#G(m) = 1 \), which is equivalent to the condition “\( J(m) = \emptyset \) and \( \#G(m) = 1 \)” by Proposition 3.24. Hence we have the implication (b) \( \iff \) (c) in Main Theorem (2).

We consider \( \#\Phi_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) \) next. By Propositions 3.17 and 3.24 we have

\[
\Phi_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) \cong S(m)/G(m) \quad \text{and} \quad \#S(m) \leq \frac{(d-2)!}{(d-\ell)!},
\]

which implies the inequality

\[
\#\Phi_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) \leq \frac{(d-2)!}{(d-\ell)!}.
\]

On the other hand, the isomorphism \( \varphi : MC_d/(\mathbb{Z}/(d-1)\mathbb{Z}) \cong MP_d \) implies the inequality \( \#\tilde{\Phi}_d(d_1, \ldots, d_\ell)^{-1}(\overline{m})/d-1 \leq \#\Phi_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) \). Hence we have

\[
\frac{\#\tilde{\Phi}_d(d_1, \ldots, d_\ell)^{-1}(\overline{m})}{d-1} \leq \#\Phi_d(d_1, \ldots, d_\ell)^{-1}(\overline{m}) \leq \frac{(d-2)!}{(d-\ell)!},
\]
which assures the implication (b) ⇒ (a) in Main Theorem (2).

Last of all, we show the implication (a) ⇒ (c) in Main Theorem (2), except in the case
\(d = \ell = 3\). Since \(\Phi_d(d_1, \ldots, d_\ell)^{-1}(\mathcal{S}) \cong S(m)/\mathcal{G}(m)\) and \(\#S(m) \leq \frac{(d-2)!}{(d-\ell)!}\), the condition
\[\#\Phi_d(d_1, \ldots, d_\ell)^{-1}(\mathcal{S}) = \frac{(d-2)!}{(d-\ell)!}\]
is satisfied if and only if \(\#S(m) = \frac{(d-2)!}{(d-\ell)!}\) holds, and the action
of \(\mathcal{G}(m)\) on \(S(m)\) is trivial. Moreover by Proposition \[5.24\] \(\#S(m) = \frac{(d-2)!}{(d-\ell)!}\) holds if and only if \(\mathcal{I}(m) = \emptyset\) holds. Hence the implication (a) ⇒ (c) does not hold if and only if there exists \(m\) such that Condition (\(\ast\)) is satisfied.

**Condition (\(\ast\))** : \(\#S(m) = \frac{(d-2)!}{(d-\ell)!}\), \(\#\mathcal{G}(m) \geq 2\) and the action of \(\mathcal{G}(m)\) on \(S(m)\) is trivial.

Here, note that if \(\#\mathcal{G}(m) \geq 2\), then there exist \(1 \leq i < j \leq \ell\) such that \((i, j) \in \mathcal{G}(m)\).

In the case \(\ell \geq 4\), we have \(\sigma \cdot \zeta \neq \zeta\) for every \(\sigma = (i, j) \in \mathcal{G}(m)\) and \(\zeta \in S(m)\). If \(\ell = 2\), we always have \(\#\mathcal{G}(m) = 1\). Hence in the case \(\ell \neq 3\), Condition (\(\ast\)) is not satisfied for every \(m\).

We consider the case \(\ell = 3\) next. In this case, we always have \(\#\mathcal{G}(m) \leq 2\). Hence under Condition (\(\ast\)), we have \(\#\mathcal{G}(m) = 2\), and may assume that \(\mathcal{G}(m)\) consists of the identity and \(\sigma := (1, 2)\). Under this assumption, the action of \(\mathcal{G}(m)\) on \(S(m)\) is trivial if and only if \(\sigma \cdot \zeta = \zeta\) holds for every \(\zeta = (\zeta_1 : \zeta_2) \in S(m)\). However for \(\zeta = (\zeta_1 : \zeta_2) \in S(m)\), the equality \(\sigma \cdot \zeta = \zeta\) holds if and only if \((\zeta_2 : \zeta_1) = (\zeta_1 : \zeta_2)\) holds, which is also equivalent to \(\zeta = (1 : -1)\).

Hence \(S(m)\) must be equal to \(\{(1 : -1)\}\).

Summing up the above mentioned, Condition (\(\ast\)) implies \(\ell = 3\) and \(\#S(m) = \frac{(d-2)!}{(d-3)!} = 1\), which also implies \(d = 3\). Hence except in the case \(d = \ell = 3\), Condition (\(\ast\)) is not satisfied for every \(m\), and the implication (a) ⇒ (c) holds.

To summarize, we have completed the proof of Main Theorem. \(\square\)

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