Integrable hierarchies associated to infinite families of Frobenius manifolds

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Abstract

We propose a new construction of an integrable hierarchy associated to any infinite series of Frobenius manifolds satisfying a certain stabilization condition. We study these hierarchies for Frobenius manifolds associated to $A_N$, $D_N$ and $B_N$ singularities. In the case of $A_N$ Frobenius manifolds our hierarchy turns out to coincide with the dispersionless KP hierarchy; for $B_N$ Frobenius manifolds it coincides with the dispersionless BKP hierarchy; and for $D_N$ hierarchy it is a certain reduction of the dispersionless 2-component BKP hierarchy. As a side product to these results we illustrate the enumerative meaning of certain coefficients of $A_N$, $D_N$ and $B_N$ Frobenius potentials.

Keywords: integrable hierarchies, KP hierarchy, Frobenius manifolds, singularity theory, WDVV equation

1. Introduction

The theory of Frobenius manifolds was introduced by B. Dubrovin in the early ‘90s as a general approach to Gromov-Witten theories and certain quantum field theories. An $N$–dimensional
Frobenius manifold can be defined (cf [[D2]]) via its potential $F_N = F_N(t_1, \ldots, t_N)$, s.t.

$$\eta_{\alpha, \beta} := \frac{\partial^3 F_N}{\partial t_1 \partial t_\alpha \partial t_\beta}$$

is a constant non-degenerate matrix and $F_N$ is subject to the following system of equations called the WDVV equations:

$$\sum_{\mu, \nu = 1}^{N} \frac{\partial^3 F_N}{\partial t_\mu \partial t_\nu \partial t_\alpha} \eta^{\mu, \nu} \frac{\partial^3 F_N}{\partial t_\nu \partial t_\gamma \partial t_\beta} = \sum_{\mu, \nu = 1}^{N} \frac{\partial^3 F_N}{\partial t_\mu \partial t_\nu \partial t_\gamma} \eta^{\mu, \nu} \frac{\partial^3 F_N}{\partial t_\nu \partial t_\beta \partial t_\sigma}.$$

that should hold for all $1 \leq \alpha, \beta, \gamma, \sigma \leq N$. Here $\eta^{\mu, \nu}$ is the inverse matrix for matrix $\eta_{\alpha, \beta}$.

Note that this definition of Frobenius manifolds already assumes that $t_1$ is a special variable. Important examples of Frobenius manifolds come from singularity theory, after the work of Saito and Saito [S1, S2]. In particular, Frobenius manifolds corresponding to ADE singularities have polynomial potentials $F_N$.

The connection between Frobenius manifolds and integrable hierarchies has been observed by many authors in various ways (cf [DVV, D2, FGM]). Due to the celebrated Witten conjectures, particular interest was attributed to Frobenius manifolds of ADE singularities [FSZ, FJR, LRZ].

It was a general idea of Dubrovin that Frobenius manifolds could be used as a tool to study integrable hierarchies. Dubrovin and Zhang proposed in [DZ1] a way to construct an integrable hierarchy associated to any Frobenius manifold. These integrable hierarchies are now called Dubrovin–Zhang hierarchies. Dubrovin–Zhang hierarchies of ADE Frobenius manifolds turned out to be equivalent to the corresponding Drinfeld-Sokolov hierarchies (cf [DLZ]). In particular, the Dubrovin–Zhang hierarchy of $A_N$-singularity Frobenius manifold appeared to be equivalent to the $(N + 1)$-reduction of the KP hierarchy and the Dubrovin–Zhang hierarchy of $D_N$-singularity Frobenius manifold appeared to be equivalent to the $(2N - 2, 2)$-reduction of the 2-component BKP hierarchy (cf [LWZ]).

We propose a new way to construct an integrable hierarchy associated to an infinite series of Frobenius manifolds (instead of just a single one, as in Dubrovin–Zhang case) satisfying certain stabilization conditions. The hierarchy we define has a rather simple form.

Fix a collection of numbers $R_{\alpha, \beta; \gamma_1, \ldots, \gamma_m} \in \mathbb{C}$ for $m, \alpha, \beta, \gamma_i \in \mathbb{Z}_{\geq 1}$ (for brevity we will sometimes write $R_\alpha$ for these coefficients in what follows), s.t. they are symmetric w.r.t. interchanging $\alpha$ and $\beta$, and under all permutations of $\gamma_i$’s. Consider an analytic function $f = f(t)$ depending on an infinite number of variables $t = (t_1, t_2, \ldots)$ and denote $\partial_i := \partial / \partial t_i$ for any $\alpha \in \mathbb{Z}_{\geq 1}$. Consider the following system of PDEs:

$$\partial_\alpha \partial_\beta f = \sum_{m \geq 0} \sum_{1 \leq \gamma_1, \ldots, \gamma_m} R_{\alpha, \beta; \gamma_1, \ldots, \gamma_m} \partial_1 \partial_{\gamma_1} f \cdot \ldots \cdot \partial_1 \partial_{\gamma_m} f.$$ (1.1)

It expresses arbitrary second order derivatives of $f$ via the special second order derivatives $\partial_1 \partial_\alpha f$ which are going to be used as the Cauchy data.

Lemma 3.2 of [NZ] implies (after setting $\hbar$ to zero) that the dispersionless KP hierarchy can be written in this way for an appropriate choice of the coefficients $R_\bullet$ (see section 6.2 for details; see also [DN, lemma 2]). In particular, in this case the coefficients $R_\bullet$ satisfy the
following condition:

\[ R_{\alpha, \beta; \gamma_1, \ldots, \gamma_m} = 0 \quad \text{unless} \quad \sum_{p=1}^{m} \gamma_p = \alpha + \beta, \]

which results in the summation over \( m \) and all \( \gamma_i \) in equation (1.1) being finite for every given pair \( \alpha, \beta \).

1.1. Construction

Let \( \{F_N\}_{N \geq N_{\text{min}}} \) be an infinite series of \( N \)-dimensional Frobenius manifold potentials with \( F_{N_{\text{min}}} \). Assume \( F_N \in \mathbb{C}[t_1, \ldots, t_N] \). Specific cases of such infinite series are given by e.g. \( F_N = F_{A_N} \) with \( N_{\text{min}} = 1 \), and \( F_N = F_{D_N} \) with \( N_{\text{min}} = 4 \); see below for more details on these specific examples.

The aim is to find coefficients \( R_* \) s.t.

(a) system of PDEs (1.1) is compatible,

(b) \( F_N \) is a solution to equation (1.1) for \( \alpha + \beta \ll N \),

which is possible if \( F_N \)'s satisfy a certain stabilization condition, see below. Here \( \alpha + \beta \ll N \) stands for 'sufficiently small \( \alpha + \beta \) compared to \( N \)'. More precisely, we can reformulate condition (b) as follows: \( \exists \kappa_1, \kappa_2 \in \mathbb{Q}, \forall N \ F_N \) is a solution of (1.1) for all \( \alpha \) and \( \beta \) satisfying \( \alpha + \beta \leq \kappa_1 N + \kappa_2 \).

If coefficients \( R_* \) satisfying (a) and (b) exist, they can be found from the series expansion of \( F_N \) as follows. By the definition of a Frobenius potential we have \( \partial_{\alpha} \partial_{\beta} F_N = \sum_{\gamma=1}^{N} \eta_{\alpha, \beta} \gamma \), where \( \eta \) is a flat metric of the Frobenius manifold. Assume that the coordinates \( \alpha \) are such that the latter sum only consists of one summand for every given \( \alpha \) (in most cases the flat coordinates can be chosen in such a way that \( \eta \) is antidiagonal, and this condition holds, cf [D2]).

Introducing the notation \( \tau := \partial_{\alpha} \partial_{\beta} F_N \) and substituting \( f = F_N \) in equation (1.1) we get:

\[ \partial_{\alpha} \partial_{\beta} F_N = \sum_{m} \sum_{\gamma_1, \ldots, \gamma_m} R_{\alpha, \beta; \gamma_1, \ldots, \gamma_m} \tau_{\gamma_1} \cdots \tau_{\gamma_m}, \]

for \( \alpha + \beta \ll N \) as in condition (b) above.

Now the numbers \( R_* \) are read off as the coefficients of series expansions of \( \partial_{\alpha} \partial_{\beta} F_N \). In particular, it is straightforward to see that

\[ R_{1, \beta; \gamma_1, \ldots, \gamma_m} = R_{\beta, 1; \gamma_1, \ldots, \gamma_m} = \delta_{m, 1} \delta_{\gamma_1, \beta}. \]

In order for equation (1.1) to make sense we need \( R_* \) to be independent of \( N \). Set

\[ R_{\alpha, \beta; \gamma_1, \ldots, \gamma_m} = \begin{cases} 
\frac{1}{m!} \left. \frac{\partial^{m+2} F_N}{\partial t_{\gamma_1} \cdots \partial t_{\gamma_m}} \right|_{t=0} & \text{if it is independent of } N, \\
0 & \text{otherwise.}
\end{cases} \]

The choice above amounts to a certain stabilization condition on \( \partial_{\alpha} \partial_{\beta} F_N \) that should hold after the change of the variables \( s_{\alpha} = t_{\alpha} \) and also the certain choice of allowed indices \( \alpha, \beta, \gamma_\alpha \). For such numbers \( R_* \) we show in proposition 2.1 that the compatibility condition of system (1.1) follows from the WDVV equation on \( F_N \), and thus we get a new dispersionless hierarchy.

Remark 1.1. Note that such a stabilization condition implies that for fixed \( \alpha \) and \( \beta \) the sum over \( \gamma_1, \ldots, \gamma_m \) in the rhs of (1.1) becomes finite (for a fixed \( m \)), i.e. \( R_{\alpha, \beta; \gamma_1, \ldots, \gamma_m} \) are all equal
to zero starting with sufficiently large $\gamma_i$'s. In particular, for $N$ sufficiently large for $\partial_\alpha \partial_\beta F_N$ to stabilize (for given $\alpha$ and $\beta$), if any of the $\gamma_i$ is larger than $N$, $R_{\alpha,\beta;\gamma_1,\ldots,\gamma_m}$ necessarily vanishes.

In the examples of $F_N = F_{A_N}$ and $F_N = F_{B_N}$ such a stabilization condition is just $\alpha + \beta \leq N + 1$. In the case of $F_N = F_{D_N}$ the stabilization condition is $\alpha + \beta \leq N$ and at most one of $\gamma_1, \ldots, \gamma_m$ is equal to $N$.

1.2. Main results

For the Frobenius manifolds of $A_N$ type the dispersionless hierarchy we construct coincides with the dispersionless KP hierarchy, rather than its reduction as it is in Dubrovin–Zhang case. It is important to note that this coincidence is proved in an easy and straightforward way without employing any complicated techniques. Namely, it turns out that our construction provides the Fay-identities form of the KP hierarchy.

For the Frobenius manifolds of $B_N$ type the dispersionless hierarchy we construct coincides with the dispersionless BKP hierarchy of [DJKM]. In $D_N$ case we get a reduction of dispersionless 2-component BKP hierarchy in one of the components only. Note again that Dubrovin–Zhang hierarchy coincides with 2-component BKP hierarchy reduced in both components.

Unfortunately our approach is not applicable directly to the Frobenius manifolds of $E_6$, $E_7$ and $E_8$ singularities, since they are not parts of some obvious infinite series of Frobenius manifolds. We hope that such a series (or multiple series) can be introduced, but it is a subject of future investigations.

On the way to prove the coincidence of the hierarchies explained above, we obtained the following interesting result. For every $m, \alpha, \beta, \gamma_1, \ldots, \gamma_m \in \mathbb{Z}_{\geq 1}$ denote by $\tilde{P}_{\alpha,\beta}(\gamma_1, \ldots, \gamma_m)$ the number of all partitions $i_1, \ldots, i_m$ of $i$ and $j_1, \ldots, j_m$ of $j$, s.t. $\forall k \ i_k + j_k = \gamma_k + 1$.

We have for all $\alpha + \beta \leq N + 1$ and $\kappa + \sigma \leq N$:

$$\left. \frac{\partial^{m+2} F_{A_N}}{\partial t_\alpha \partial t_\beta \partial t_{N+1-\gamma_1} \ldots \partial t_{N+1-\gamma_m}} \right|_{t=0} = (-1)^{m+1}(m-1)! \cdot \tilde{P}_{\alpha,\beta}(\gamma_1, \ldots, \gamma_m),$$

$$\left. \frac{\partial^{m+2} F_{D_N}}{\partial t_\alpha \partial t_\beta \partial t_{N-\gamma_1} \ldots \partial t_{N-\gamma_m}} \right|_{t=0} = (-1)^{m+1}(m-1)! \cdot \tilde{P}_{2\kappa-1,2\sigma-1}(2\gamma_1 - 1, \ldots, 2\gamma_m - 1).$$

These equalities provide an enumerative meaning of the respective coefficients of $F_{A_N}$ and $F_{D_N}$ potentials.

1.3. $h$–deformation and further development

Remark 1.2. It was observed in [NZ] that the full KP hierarchy can be obtained from the dispersionless KP hierarchy written in the form of system (1.1) by the substitution $\partial_k \rightarrow \partial_k^h$ for certain differential operators $\partial_k^h = \partial_k + O(h)$. This phenomenon was investigated deeply in the works of Takasaki and Takebe [TT1, TT2].

We hope to extend our dispersionless hierarchies to the full form in the same way. In order to do this we need to consider the $h$–deformations of the Frobenius manifold potentials $F_N$, called the higher genera potentials, with the help of Virasoro constraints. We hope to do this in subsequent works.

Remark 1.3. At the moment it seems to be quite hard to give a full classification of infinite series of Frobenius manifolds coupled with respective stabilization conditions which would then give compatible systems (1.2) as in the cases studied in the present paper. However, it
might be interesting to consider the other well-known series of Frobenius manifolds including the cases of extended affine Weyl groups (cf [DZ2, B2, Z2]). We plan to address this in subsequent works.

1.4. Organization of the paper

Section 2 is devoted to proving a key result stating that the WDVV equation implies the compatibility of system of PDEs (1.1) when the coefficients are coming from an infinite family of Frobenius potentials satisfying a stabilization condition.

In section 3 we recall the basic theory of Frobenius manifolds; then we recall the results due to Noumi–Yamada and Zuber on the form of Frobenius potentials associated to $A_N$, $D_N$ and $B_N$ singularities.

In section 4 we prove that Frobenius potentials associated to $A_N$, $D_N$ and $B_N$ singularities satisfy stabilization conditions of section 1.1.

In section 5 we touch upon the enumerative meaning of the coefficients of these potentials.

Section 6 is devoted to identifying the hierarchies resulting from the construction of section 1.1 applied for $A_N$ and $B_N$ Frobenius potentials with known integrable hierarchies.

Section 7 covers the same subject as section 6, just for the $D_N$ case, as it turns out that it is quite different to the $A_N$ and $B_N$ ones.

2. WDVV and compatibility of a system of PDEs

Assume that $M$ is an open full-dimensional subspace of $\mathbb{C}^N$. We say that it is endowed with a structure of Frobenius manifold if there is a function $F_N = F_N(t_1, \ldots, t_N)$, s.t. the following conditions hold (cf [D2]).

- The variable $t_1$ is special in the following sense:
  \[
  \frac{\partial F_N}{\partial t_1} = \frac{1}{2} \sum_{\alpha, \beta = 1}^N \eta_{\alpha, \beta} t_\alpha t_\beta,
  \]
  where $\eta_{\alpha, \beta}$ are components of a non-degenerate bilinear form $\eta$ (which does not depend on $t$’s). In what follows denote by $\eta^{\alpha, \beta}$ the components of $\eta^{-1}$.

- The function $F_N$ satisfies a large system of PDEs called the WDVV equations:
  \[
  \sum_{\mu, \nu = 1}^N \frac{\partial^3 F_N}{\partial t_\mu \partial t_\nu \partial t_\sigma} \eta^{\mu, \nu} \frac{\partial^3 F_N}{\partial t_\alpha \partial t_\beta \partial t_\gamma} = \sum_{\mu, \nu = 1}^N \frac{\partial^3 F_N}{\partial t_\mu \partial t_\nu \partial t_\tau} \eta^{\mu, \nu} \frac{\partial^3 F_N}{\partial t_\alpha \partial t_\beta \partial t_\sigma},
  \]
  which should hold for every given $1 \leq \alpha, \beta, \gamma, \sigma \leq N$.

- There is a vector field $E$ called the Euler vector field, s.t. modulo quadratic terms in $t_\bullet$ we have $E \cdot F_N = (3 - \delta) F_N$ for some fixed complex number $\delta$. We will assume $E$ to have the following simple form
  \[
  E = \sum_{k=1}^N d_k t_k \frac{\partial}{\partial t_k}
  \]
  for some fixed numbers $d_1, \ldots, d_N$. Moreover we set $d_1 = 1$. 

Given such a data \((M, F_N, E)\) one can endow every tangent space \(T_pM\) with a structure of commutative associative product \(\circ\) (depending on \(t\)) defined as follows:
\[
\frac{\partial}{\partial t_0} \circ \frac{\partial}{\partial t_j} = \sum_{k,\gamma=1}^{N} \frac{\partial^k F_N}{\partial t_0 \partial t_j \partial t_\gamma} \eta^{\gamma} \frac{\partial}{\partial t_\gamma}.
\]
It follows that \(\eta(a \circ b, c) = \eta(a, b \circ c)\) for any vector fields \(a, b, c\).

The following proposition is very important for what follows.

**Proposition 2.1.** Let the coefficients \(R_\ast\) be constructed as in section 1.1 from a series of Frobenius manifold potentials \(\{F_N\}_{N \geq N_{\text{min}}}\) which satisfy a stabilization condition. Then the system of PDEs (1.1) with such coefficients \(R_\ast\) is compatible.

**Proof.** We need to show that equalities \(\partial_\gamma(\partial_\alpha \partial_\beta f) = \partial_\beta(\partial_\alpha \partial_\gamma f)\) hold true if one substitutes the expressions inside the brackets with the rhs of (1.1).

If one of the indices is equal to 1 this follows from equation (1.3).

For any given \(\alpha, \beta, \gamma \geq 2\) we have
\[
\partial_\gamma(\partial_\alpha \partial_\beta f) = \sum_{k \geq 1} \sum_{j_1, \ldots, j_k} \sum_{j=1}^{k} \sum_{p=1}^{l} \sum_{q=1}^{l} \sum_{j_1, \ldots, j_l} R_{\gamma j_1 \ldots j_k} \partial_\alpha \partial_\beta f \cdot \partial_j \partial_\gamma f.
\]
Here we have applied (1.1) twice. All coefficients \(R_{\ast}\) here are either zero or can be recovered as coefficients in front of respective monomials in \(F_N\) for sufficiently large \(N\). For a given \(K \in \mathbb{Z}_{\geq 0}, I = (i_1, \ldots, i_k) \in (\mathbb{Z}_{\geq 1})^k\) and \(\gamma, \alpha, \beta, \kappa \in \mathbb{Z}_{\geq 1}\) denote
\[
\hat{\Omega}_{\gamma, \alpha, \beta, I, \kappa} := \sum_{h=0}^{K} \sum_{p=1}^{K-h+1} \sum_{q=1}^{\infty} \sum_{\nu=1}^{\infty} R_{\gamma i_1 \ldots i_k} R_{\alpha j_1 \ldots j_k} R_{\beta j_1 \ldots j_k} \cdot \Omega_{\nu, \gamma, \alpha, \beta, I, \kappa}.
\]
(2.2)

\[
\hat{\Omega}_{\gamma, \alpha, \beta, I, \kappa} := \sum_{h=0}^{K} \sum_{p=1}^{K-h+1} \sum_{q=1}^{\infty} \sum_{\nu=1}^{\infty} R_{\gamma i_1 \ldots i_k} R_{\alpha j_1 \ldots j_k} R_{\beta j_1 \ldots j_k} \cdot \Omega_{\nu, \gamma, \alpha, \beta, I, \kappa}.
\]
(2.3)

where we have used the symmetry of \(R_\ast\) in the second equality.

We can rewrite (2.1) as
\[
\partial_\gamma(\partial_\alpha \partial_\beta f) = \sum_{k \geq 0} \sum_{I \in (\mathbb{Z}_{\geq 1})^k} \sum_{\kappa \geq 1} \sum_{\nu=1}^{\Omega_{\gamma, \alpha, \beta, I, \kappa}} \partial_\alpha \partial_\beta f \cdot \partial_j \partial_\gamma f.
\]
(2.4)

Now for a given \(K \in \mathbb{Z}_{\geq 0}, J = (j_1, \ldots, j_k) \in (\mathbb{Z}_{\geq 1})^k, j_1 \leq \cdots \leq j_k\) and \(\gamma, \alpha, \beta, \kappa \in \mathbb{Z}_{\geq 1}\) denote
\[ \Omega_{\gamma;\alpha,\beta,J,K} := \sum_{\sigma \in S_K} \Omega_{\gamma;\alpha,\beta,J,K,\sigma} \]

\[ = \sum_{\nu \in \mathbb{N}} \sum_{h=0}^{\nu} (h+1)(K-h+1)R_{\nu;\alpha,\beta,J,K,\nu} \]

\[ \times R_{\nu;\gamma;J,K_1,\nu} R_{\nu;\gamma;J_2,K_2}, \quad (2.5) \]

With the help of this definition of \( \Omega_{\gamma;\alpha,\beta,J,K} \), we rewrite (2.4) as

\[ \partial_\kappa (\partial_\lambda \partial_\mu f) = \sum_{K \geq 0} \sum_{J = (j_1, \ldots, j_K)} \sum_{\nu \geq 1} \frac{\Omega_{\gamma;\alpha,\beta,J,K,K}^{\nu}}{|Aut(J)|} \prod_{a \in J} \partial_\mu \partial_\rho \partial_\eta f. \quad (2.6) \]

It remains to show that each \( \Omega_{\gamma;\alpha,\beta,J,K} \) is symmetric in \( \beta \) and \( \gamma \).

Let \( N_1 \) be s.t. for our fixed \( \alpha \) and \( \beta \) the stabilization condition of section 1.1 holds for all \( R_{\alpha,\beta,*} \) appearing in (2.5). Due to remark 1.1, the sum over \( \nu \) in (2.5) is actually finite, and (2.5) can be rewritten as

\[ \Omega_{\gamma;\alpha,\beta,J,K} = \sum_{I_1, J_2 = (1, \ldots, K)} \sum_{\nu = 1}^{N_1} (|I_1| + 1)(|I_2| + 1)!R_{\nu;\alpha,\beta,J_1,J_2,K_2}. \quad (2.7) \]

Now let \( N_2 \geq N_1 \) be s.t. the stabilization condition of section 1.1 holds for all \( R_{\gamma;*,*} \) appearing in (2.8) for our fixed \( \gamma \) and all \( 1 \leq \nu \leq N_1 \). Since the terms with \( \nu > N_1 \) vanish in any case, we can write (2.8) as follows:

\[ \Omega_{\gamma;\alpha,\beta,J,K} = \sum_{I_1, J_2 = (1, \ldots, K)} \sum_{\nu = 1}^{N_2} (|I_1| + 1)(|I_2| + 1)!R_{\nu;\alpha,\beta,J_1,J_2,K_2}, \quad (2.8) \]

where all \( R_{\gamma,*} \)’s which are parts of non-vanishing terms satisfy the stabilization condition.

The following expression is symmetric in \( \beta \) and \( \gamma \) due to the WDVV equation for \( F_{N_1} \):

\[ \frac{\partial^{|J|}}{\prod_{a \in J} \partial_\mu} \left( \sum_{\mu, \nu} \sum_{\sigma = 1}^{N_2} \frac{\partial^3 F_{N_2}}{\partial_\mu \partial_\nu \partial_\sigma} f^{\mu, \nu, \sigma} \right) \bigg|_{t=0} \]

\[ = \sum_{I_1, J_2 = (1, \ldots, K)} \left( \sum_{\nu = 1}^{N_2} \frac{\partial^{|I_1|}}{\prod_{a \in J_1} \partial_\mu} \frac{\partial^3 F_{N_2}}{\partial_\mu \partial_\nu} \bigg|_{t=0} \right) \]

\[ \times \left( \sum_{\nu = 1}^{N_2} \frac{\partial^{|I_2|}}{\prod_{a \in J_2} \partial_\mu} \frac{\partial^3 F_{N_2}}{\partial_\mu \partial_\nu} \bigg|_{t=0} \right). \]

The latter form of this expression due to equation (1.2), cf (1.4), explicitly coincides with the rhs of (2.9) and thus with \( \Omega_{\gamma;\alpha,\beta,J,K} \) which implies that \( \Omega_{\gamma;\alpha,\beta,J,K} \) is symmetric in \( \beta, \gamma \).

In the next section we show how one can obtain the data above starting from the associative commutative product \( \circ \) and a pairing \( \eta \) in the case of \( A \) and \( D \) singularities.
3. Frobenius structures of $A_N$, $D_N$ and $B_N$ singularities

The $A_N$ and $D_N$ type singularities are defined via the following polynomials:

$$f_{A_N} = \frac{x^{N+1}}{N+1} + y^2, \quad f_{D_N} = \frac{x^{N-1}}{N-1} + xy^2.$$ 

One associates to them the so-called unfoldings $\Lambda_W : \mathbb{C}^2 \times \mathbb{C}^N \to \mathbb{C}$

$$\Lambda_{A_N} = \frac{x^{N+1}}{N+1} + y^2 + \sum_{k=1}^{N} v_k x^{k-1}, \quad \Lambda_{D_N} = \frac{x^{N-1}}{N-1} + xy^2 + \sum_{k=1}^{N-1} v_k x^{k-1} + v_N y,$$

that depend on additional parameters $v = (v_1, \ldots, v_N) \in M_W := \mathbb{C}^N$.

Let us introduce the Frobenius manifold structure on $M_W$. To do this for every fixed $v \in M_W$ consider the following quotient-ring:

$$A_v := \mathbb{C}[x, y] / \left( \frac{\partial \Lambda_W}{\partial x} \frac{\partial \Lambda_W}{\partial y} \right).$$

It is endowed with the quotient-ring product structure, and the classical singularity theory arguments assure that $A_v$ is an $N$-dimensional $\mathbb{C}$-vector space. Let $c_{ab}(v)$ stand for the structure constants of this product in the basis $[\partial \Lambda_W / \partial v_1], \ldots, [\partial \Lambda_W / \partial v_N]$, namely,

$$A_N : \frac{\partial \Lambda_W}{\partial v_k} = x^{k-1}, \quad 1 \leq k \leq N,$$

$$D_N : \frac{\partial \Lambda_W}{\partial v_k} = x^{k-1}, \quad 1 \leq k \leq N - 1, \quad \frac{\partial \Lambda_W}{\partial v_N} = y.$$

The product $\circ : T_v M \otimes T_v M \to T_v M \otimes \mathbb{C}[v_1, \ldots, v_N]$ is now defined by

$$\frac{\partial}{\partial v_a} \circ \frac{\partial}{\partial v_b} := \sum_{k=1}^{N} c_{ab}(v) \frac{\partial}{\partial v_k}.$$

Obviously, $\partial / \partial v_1$ is the unit of this product. In particular, we have for $A_N$

$$\frac{\partial}{\partial v_a} \circ \frac{\partial}{\partial v_b} = \frac{\partial}{\partial v_{a+b-1}} \quad \forall a + b \leq N + 1, \tag{3.1}$$

$$\frac{\partial}{\partial v_a} \circ \frac{\partial}{\partial v_{N+2-a}} = -\sum_{k=2}^{N} (k-1)v_k \frac{\partial}{\partial v_{k-1}}. \tag{3.2}$$

Introduce the non-degenerate $\mathbb{C}[v]$-bilinear pairing $\eta : T_v M \otimes T_v M \to \mathbb{C}[v_1, \ldots, v_N]$ by

$$\eta \left( \frac{\partial}{\partial v_a}, \frac{\partial}{\partial v_b} \right) := c_N^{ab}, \quad W = A_N,$$

$$\eta \left( \frac{\partial}{\partial v_a}, \frac{\partial}{\partial v_b} \right) := c^{N-1}_{ab}, \quad W = D_N.$$

The pairing we introduce is in fact the well-known residue pairing.

**Theorem 3.1 (cf [D1, S1, ST]).** The data $(M, \circ, \eta)$ is a Frobenius manifold. In particular, there is a choice of the coordinates $t_a = t_a(v)$ (which are called the flat coordinates), s.t. in the basis $\partial / \partial t_1, \ldots, \partial / \partial t_N$ we have
• the pairing $\eta$ is constant,
• there is a Frobenius manifold potential $F_W = F_W(t_1, \ldots, t_N)$, s.t.

$$\frac{\partial}{\partial t_\alpha} \sum_{\beta} \frac{\partial F_W}{\partial t_\beta} = \sum_{\gamma} \frac{\partial^2 F_W}{\partial t_\alpha \partial t_\beta} \eta_{\alpha \beta} \frac{\partial}{\partial t_\beta}$$

For the cases of $A_N$ and $D_N$ singularities, the flat coordinates of the theorem above were investigated by Noumi and Yamada in [NY]. We use their result in what follows applying the certain rescaling of the coordinates that makes the formulae simpler. They also gave the formula for the potentials $F_W$.

### 3.1. $A_N$ and $D_N$ Frobenius manifold potentials

Let $W$ be either $A_N$ or $D_N$. The potential $F_W$ is a polynomial in $t_1, \ldots, t_N$ with rational coefficients subject to the quasi-homogeneity condition $E_W \cdot F_W = (3 - \delta_W)F_W$ with

$$E_{A_N} = \sum_{\alpha=1}^{N} \frac{N + 2 - \alpha}{N + 1} t_\alpha \frac{\partial}{\partial t_\alpha}, \quad \delta_{A_N} = \frac{N - 1}{N + 1},$$

$$E_{D_N} = \sum_{\alpha=1}^{N-1} \frac{N - \alpha}{N - 1} t_\alpha \frac{\partial}{\partial t_\alpha} + \frac{N}{2(N - 1)} \eta_{\alpha} \frac{\partial}{\partial t_N}, \quad \delta_{D_N} = \frac{N - 2}{N - 1}.$$  

The pairing $\eta$ reads

$$\eta_{\alpha, \beta} = \delta_{\alpha + \beta + 1}$$

for $W = A_N$,

$$\eta_{\alpha, \beta} = \begin{cases} 1 & \text{when } \alpha = \beta = N, \\ \delta_{\alpha + \beta} & \text{otherwise.} \end{cases}$$

for $W = D_N$.

We see that for all these $W$ for any $\alpha \in \{1, \ldots, N\}$ there exists a unique integer $\bar{\alpha} \in \{1, \ldots, N\}$ such that $\eta_{\alpha, \bar{\alpha}} = 1$.

For $W = A_N, D_N$ Noumi–Yamada gave the formulae for the potential $F_W(t_1, \ldots, t_N)$ in the following way. They introduce functions $\psi^{(2)}_\alpha \in \mathbb{Q}[v_1, \ldots, v_N]$ depending of the unfolding variables $v_k$ as above, s.t. for all $1 \leq \alpha \leq N$ the following equations hold:

$$\frac{\partial F_W}{\partial t_{\alpha}} = \psi^{(2)}_\alpha(t_1, \ldots, t_N),$$

$$t_\alpha = \psi^{(1)}_{\bar{\alpha}}(v_1, \ldots, v_N).$$

It is only reasonable to consider the potential $F_W$ in flat coordinates $t_\alpha$ and therefore it is important to invert the above formula of [NY] in order to express $v_k = v_k(t)$.

### 3.2. $A_N$ case

We have $\bar{\alpha} = N + 1 - \alpha$ and

$$\psi^{(2)}_\gamma(v) := \sum_{\alpha_1, \ldots, \alpha_N \geq 0} (-1)^{\gamma + 1} \prod_{k=0}^{[\alpha] - 1 - r} \prod_{\alpha_k = N(N + 1) + 1 - \gamma}^{N} \frac{v_k^{\alpha_k}}{\alpha_k!},$$

where $|\alpha| = \sum_{k=1}^{N} \alpha_k$. 


The inverted formulae were given by Buryakin [B1] from the study of open Gromov–Witten theories:

\[
v_{\gamma} = \sum_{\alpha_1, \ldots, \alpha_N \geq 0 \atop \sum_{k=1}^{N} (N+2-k)\alpha_k = N+2-\gamma} \frac{(|\alpha| + \gamma - 2)!}{(\gamma - 1)!} \prod_{k=1}^{N} \frac{t_{\alpha_k}^{\gamma}}{\alpha_k!}.
\] (3.3)

Note that the condition \(\sum_{k=1}^{N} (N+2-k)\alpha_k = N+2-\gamma\) precisely ensures that \(v_{\gamma}\) is quasi-homogeneous (w.r.t. the Euler field \(E_{\mathbb{A}_N}\)) and its weight is equal to the weight of \(t_{\gamma}\).

### 3.3. \(D_N\) case

We have

\[\overline{\alpha} = N - \alpha, \quad 1 \leq \alpha \leq N - 1, \quad \overline{\alpha} = N.\]

and

\[
\psi_{\gamma}^{(1)} = \sum_{\alpha_1, \ldots, \alpha_{N-1} \geq 0 \atop \sum_{k=1}^{N-1} (N-k)\alpha_k = N-\gamma} \frac{(-1)^{|\alpha|-1} |\alpha|-2}{\prod_{k=0}^{N-1} (2\gamma - 1 + 2k(N - 1))} \prod_{k=0}^{N-1} \frac{v_{\alpha_k}^{2\gamma}}{\alpha_k!}, \quad 1 \leq \gamma \leq N - 1,
\]

\[
\psi_{\gamma}^{(1)} = v_{\gamma}.
\]

where \(|\alpha| = \sum_{k=1}^{N-1} \alpha_k\).

In order to introduce \(\psi_{\gamma}^{(2)}\) let us define the following combinatorial coefficients:

\[
A_{\gamma, \alpha}^{(1)} := (-1)^{|\alpha|-1} \prod_{k=0}^{N-1} (2\gamma - 1 + 2k(N - 1)),
\]

\[
A_{\gamma, \alpha}^{(2)} := (-1)^{|\alpha|-2} \prod_{k=0}^{N-1} (2\gamma - 1 + 2k(N - 1)), \quad 1 \leq \gamma \leq N - 2,
\]

\[
A_{N-1, \alpha}^{(2)} := 2.
\]

Then

\[
\psi_{\gamma}^{(2)}(v) := \sum_{\alpha_1, \ldots, \alpha_{N-1} \geq 0 \atop \sum_{k=1}^{N-1} (N-k)\alpha_k = 2(N-1)+1-\gamma} A_{\gamma, \alpha}^{(2)} \prod_{k=1}^{N} \frac{v_{\alpha_k}^{2\gamma}}{\alpha_k!} + \sum_{\alpha_1, \ldots, \alpha_{N-1} \geq 0 \atop \sum_{k=1}^{N-1} (N-k)\alpha_k = N-1-\gamma} \frac{A_{\gamma, \alpha}^{(2)}}{2} \prod_{k=1}^{N} \frac{v_{\alpha_k}^{\gamma}}{\alpha_k!}, \quad 1 \leq \gamma \leq N - 1,
\]

\[
\psi_{N}^{(2)}(v) := v_{\gamma}^N.
\]
3.4. $B_N$ Frobenius manifold potential

This Frobenius manifold does not correspond to a deformation theory of a hypersurface singularity. This makes its definition more involved. It was shown by Zuber in [Z1] that the following equation holds

$$ F_{B_N}(t_1, \ldots, t_N) = F_{A_{2N-1}}(t_1, 0, t_2, 0, t_3, \ldots, t_N). $$ \hfill (3.4)

We use the above equation as the definition of $F_{B_N}$. It follows that $\eta_{\alpha, \beta} = \delta_{\alpha+\beta, N+1}$ and

$$ E_{B_N} = \sum_{\alpha=1}^{N} \frac{N+1-\alpha}{N} t_\alpha \frac{\partial}{\partial t_\alpha}, \quad \delta_{B_N} = \frac{N-1}{N}. $$

4. Stabilization of $A_N$, $B_N$, and $D_N$ potentials

In this section we discuss in details the structure of $A_N$, $B_N$ and $D_N$ Frobenius manifold potentials. In particular, we prove the respective stabilization statements in theorem 4.1, proposition 4.4 and theorem 4.9.

4.1. $A_N$ case

**Theorem 4.1.** For any $N_2 > N_1 \geq 1$ and $\alpha, \beta$, s.t. $1 \leq \alpha, \beta \leq N_1$, $\alpha + \beta \leq N_1 + 1$ we have

$$ \left. \frac{\partial^2 F_{A_{N_1}}}{\partial t_\alpha \partial t_\beta} \right|_{\forall \gamma ; t_{N_1+1-\gamma} = t_\gamma} = \left. \frac{\partial^2 F_{A_{N_2}}}{\partial t_\alpha \partial t_\beta} \right|_{\forall \gamma ; t_{N_2+1-\gamma} = t_\gamma}, $$

understood as an equality of polynomials in $s_\ast$.

**Proof.** In this proof we have to make use of both flat coordinates $t_\ast$ and unfolding coordinates $v_\ast$. Denote by $c'_{ab} = c'_a(v)$ the structure constants in the basis $\partial/\partial v_\ast$. Consider also the basis change matrices

$$ \Psi^a_a := \frac{\partial t_\alpha}{\partial v_a}, \quad \Psi^a_\alpha := \frac{\partial v_a}{\partial t_\alpha} $$

where we use the greek and latin letters for $t$ and $v$ coordinates respectively. We have $\sum_a \Psi^a_a \Psi^a_\beta = \delta^\beta_\beta$.

**Lemma 4.2.** The matrices $\Psi^a_\alpha$ and $\Psi^a_\beta$ stabilize. Namely, for $v_a^{(N)} = v_a^{(N)}(t)$ being the expression of unfolding coordinates via flat coordinates for $A_N$, we have

$$ \left. \frac{\partial v_a^{(N_1)}}{\partial t_\alpha} \right|_{\forall \gamma ; t_{N_1+1-\gamma} = t_\gamma} = \left. \frac{\partial v_a^{(N_2)}}{\partial t_\alpha} \right|_{\forall \gamma ; t_{N_2+1-\gamma} = t_\gamma}, $$

for $1 \leq \alpha, a \leq N_1$ and $N_1 < N_2$.

**Proof.** By using equation (3.3) we have
\[
\frac{\partial v^{(N)}}{\partial t_{\gamma}}|_{t_{\gamma+1}=s_{\gamma}} = \sum_{\alpha_{1}, \ldots, \alpha_{N} \geq 0} \frac{(|\alpha| + a - 1)!}{(a - 1)!} \prod_{k=1}^{N} t_{\alpha_{k}}^{(N+1-\gamma)}(a_{k})^{|\alpha|} \sum_{\gamma_{1}+\ldots+\gamma_{N}=\delta-a} (m + a - 1)! \alpha_{m}!
\]

It is now straightforward to see that the last expression we obtained does not depend on \(N\), what concludes the proof. \(\square\)

For any \(\gamma\) and \(N \geq 1\) we have

\[
c_{\alpha,\beta,N+\gamma} := \frac{\partial^{\gamma} F_{\alpha\beta,N}}{\partial t_{\alpha} \partial t_{\beta} \partial t_{\gamma}} = \sum_{r=1}^{N} \sum_{a,b=1}^{N} \Psi_{r}^{N+\gamma} \Psi_{a}^{\alpha} \Psi_{b}^{\beta} e_{ab}.
\]

The change of coordinates \(t_{\alpha} = t_{\alpha}(v)\) is quasi-homogeneous. In particular, it follows that \(\Psi_{a}^{\alpha} = 0\) unless \(\alpha \leq a\). Therefore for any \(K \leq N\) and \(\alpha + \beta \leq K + 1\) we have

\[
c_{\alpha,\beta,N+\gamma} = \sum_{r=1}^{N} \sum_{1 \leq a,b \leq N} \Psi_{r}^{N+\gamma} \Psi_{a}^{\alpha} \Psi_{b}^{\beta} e_{ab} = \sum_{1 \leq a,b \leq N} \Psi_{a+b-1}^{\alpha} \Psi_{a}^{\alpha} \Psi_{b}^{\beta},
\]

where we have used equation (3.1) to get the second equality.

By using the lemma above and the quasi-homogeneity of the change of the variables \(t_{\alpha} = t_{\alpha}(v)\) we have for all \(\alpha + \beta \leq N_{1} + 1\):

\[
c_{\alpha,\beta,N_{1}+\gamma} = c_{\alpha,\beta,N_{1}+\gamma}^{(N_{1})} \bigg|_{t_{\gamma+1}=s_{\gamma}} = c_{\alpha,\beta,N_{1}+\gamma}^{(N_{1})} \bigg|_{t_{\gamma+1}=s_{\gamma}},
\]

which concludes the proof of the theorem. \(\square\)

By using the quasi-homogeneity condition of \(F_{\alpha\beta,N}\) we have for any \(\alpha + \beta \leq N + 1\) the equality

\[
\frac{\partial^{\gamma} F_{\alpha\beta,N}}{\partial t_{\alpha} \partial t_{\beta}} = \sum_{r=1}^{N} \frac{1 + \gamma}{\alpha + \beta + N + 1 - \gamma} \prod_{a,b=1}^{N} \frac{\partial^{\gamma} F_{\alpha\beta,N}}{\partial v_{a+b-1} \partial v_{\alpha} \partial v_{\beta}}.
\]

Consider the quotient-ring

\[
A_{N} := \mathbb{Q}[t] \otimes \mathbb{Q}[[z_{1}^{1}, z_{2}^{1}]] / (z_{1}^{1} - z_{1}^{-1}, z_{2}^{1} - z_{2}^{-1}, \ldots, z_{1}^{1} - z_{1}^{-(N+1)}).
\]

Namely, this is the finite rank \(\mathbb{Q}[t]\)-module generated by polynomials in \(z_{1}^{-1}\) and \(z_{2}^{-1}\) with the total degree not exceeding \(N + 1\).

**Proposition 4.3.** Denote \(\partial_{\alpha} := \partial / \partial t_{\alpha}\) and \(F = F_{\alpha\beta,N}\). In the ring \(A_{N}\) we have

\[
(z_{1} - z_{2}) \exp \left( \sum_{\alpha,\beta \geq 1} z_{1}^{\alpha} z_{2}^{\beta} \partial_{\alpha} \partial_{\beta} F \right) = (z_{1} - z_{2}) \left( \sum_{\alpha \geq 1} z_{1}^{\alpha} \partial_{\alpha} F - \sum_{\beta \geq 1} z_{2}^{\beta} \partial_{\beta} F \right) + \sum_{\alpha,\beta \geq 1} z_{1}^{\alpha} z_{2}^{\beta} \partial_{\alpha} \partial_{\beta} F. \quad (4.1)
\]
Note that this statement only concerns the second derivatives $\partial_{\alpha} \partial_{\beta} F$ s.t. $\alpha + \beta \leq N + 1$.

**Proof.** Because of the special role of variable $t_1$ in a Frobenius manifold potential, it is easy to see that the desired equality holds in the rank 2 submodule $Q(t_1, t_2, \ldots, t_N, z_1^{-1}, z_2^{-1}) \subset A_N$.

In what follows we are going to use the expression

$$P := \sum_{a, b = 1}^{N} (\alpha + \beta) \frac{\partial^2 F_{A_N}}{\partial t_a \partial t_b} z_1^{-\alpha} z_2^{-\beta}.$$

$$= - \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right) \sum_{a, b = 1}^{N} \frac{\partial^2 F_{A_N}}{\partial t_a \partial t_b} z_1^{-\alpha} z_2^{-\beta}.$$

Denote $P_{\alpha, \beta} := [z_1^{-\alpha} z_2^{-\beta}] P$. With the help of the above equality it is easy to see that the statement of the proposition is equivalent to the following one:

$$P_{\alpha+1, \beta} = \sum_{\delta = 1}^{\beta-1} \delta P_{\alpha, \beta-\delta} = P_{\alpha, \beta+1} - \sum_{\delta = 1}^{\alpha-1} \delta P_{\alpha-\delta, \beta}.$$

In coordinates this is equivalent to

$$\sum_{\gamma = 1}^{N} (N + 2 - \gamma) t_1 \sum_{a, b \in \emptyset} \frac{\partial t_{N+1-\gamma}}{\partial v_a v_b} \left[ \frac{\partial v_a}{\partial t_{\alpha+1}} \frac{\partial v_b}{\partial t_{\beta}} - \frac{\partial v_a}{\partial t_{\alpha}} \frac{\partial v_b}{\partial t_{\beta+1}} \right]$$

$$= \sum_{\gamma = 1}^{N} (N + 2 - \gamma) t_1 \sum_{a, b} \frac{\partial t_{N+1-\gamma}}{\partial v_a v_b} \left[ \sum_{\delta} t_{N+1-\delta} \frac{\partial v_a}{\partial t_{\alpha-\delta}} \frac{\partial v_b}{\partial t_{\beta}} \right].$$

which should hold for all $\alpha + \beta \leq N$. This equality can be checked combinatorially via equation (3.3), but we are going to use a more geometrical approach.

Denote by $\Lambda := \Lambda_{A_N}(x, t)$ the unfolding of $A_N$ singularity written in the flat coordinates. Set

$$\phi_\alpha := \frac{\partial \Lambda}{\partial t_1} = \sum_{k=1}^{N} \frac{\partial v_k}{\partial t_1} x^{k-1} \in \mathbb{C}[x] \otimes \mathbb{C}[t_1, \ldots, t_N].$$

These functions satisfy the following recursive relation: $x \phi_\alpha = \phi_{\alpha+1} + \sum_{\sigma=0}^{\alpha-1} t_{N+1+\sigma-\alpha} \phi_\sigma$.

In terms of these functions we have

$$\sum_{a + b = p} \frac{\partial v_a}{\partial t_1} \frac{\partial v_b}{\partial t_2} = [x^{p-2}] (\phi_\alpha \cdot \phi_\beta).$$
where the product in the bracket is just $x$-polynomial product. By using this observation and the recursive relations on $\phi_\star$ we have

$$
\sum_{a+b+p=q} \left( \frac{\partial v_a}{\partial t_{a+1}} \frac{\partial v_b}{\partial t_{b+1}} - \frac{\partial v_a}{\partial t_{a}} \frac{\partial v_b}{\partial t_{b+1}} \right) = [x^{p-2}] \left( \phi_{a+1} \phi_{b+1} - \phi_a \phi_{b+1} \right)
$$

$$
= [x^{p-2}] \left( (x\phi_a - \sum_{\sigma=0}^{a-1} \sum_{\alpha=0}^{\beta-1} t_{N+1-(\alpha-\sigma)} \phi_{\beta} - \phi_a (x\phi_{\beta} - \sum_{\sigma=0}^{\beta-1} t_{N+1-(\beta-\sigma)} \phi_{\beta}) \right)
$$

$$
= [x^{p-2}] \left( -\sum_{\beta=1}^{\alpha} t_{N+1-\delta} \phi_{\alpha-\delta} \phi_{\beta} + \sum_{\delta=1}^{\beta} t_{N+1-\delta} \phi_{\alpha} \phi_{\beta-\delta} \right)
$$

$$
= \sum_{a+b+p=q} \left[ \sum_{\beta=1}^{\alpha} t_{N+1-\delta} \phi_{\alpha-\delta} \phi_{\beta} - \sum_{\delta=1}^{\beta} t_{N+1-\delta} \phi_{\alpha} \phi_{\beta-\delta} \right].
$$

This completes the proof. □

4.2. $B_N$ case

The following stabilization proposition is straightforward in proof but nontrivial in its statement.

**Proposition 4.4.** For any $N_2 > N_1 \geq 1$ and $\alpha, \beta$, s.t. $1 \leq \alpha, \beta \leq N_1$, $\alpha + \beta \leq N_1 + 1$ we have

$$
\frac{\partial^2 F_{B_N}}{\partial t_\alpha \partial t_\beta} \bigg|_{t_{N_1+1-\gamma} = \gamma} = \frac{\partial^2 F_{B_N}}{\partial t_\alpha \partial t_\beta} \bigg|_{t_{N_2+1-\gamma} = \gamma},
$$

understood as an equality of polynomials in $s_\star$.

**Proof.** This follows immediately from the definition of $F_{B_N}$ and theorem 4.1. □

The $B_N$ Frobenius manifolds were introduced via the $A_N$ Frobenius manifolds. In what follows it will be useful to build up the connection of the $B_N$ Frobenius manifolds to the $D_N$ Frobenius manifolds too.

**Proposition 4.5.** We have

$$
F_{B_N}(t_1, \ldots, t_N) = F_{D_N+1}(t_1, t_2, \ldots, t_N, 0). \quad (4.2)
$$

**Proof.** For $D_{N+1}$ Frobenius manifold setting $t_{N+1} = 0$ is equivalent to setting $v_{N+1} = 0$. One shows easily that for $A_{2N-1}$ Frobenius manifold setting all $t_{2a} = 0$ is equivalent to setting all $v_{2a} = 0$.

We compare the functions $\psi^{(r)}$ for both cases. In this proof we denote by $A_{\psi^{(r)}_a}$ and $D_{\psi^{(r)}_a}$ the respective $\psi$-functions of $A_{2N-1}$ and $D_{N+1}$ respectively. It follows immediately from the definition that we have

$$
A_{\psi^{(1)}_{2a-1}}(v_1, 0, v_2, 0, \ldots, v_{2N-1}) = D_{\psi^{(1)}_a}(v_1, v_2, \ldots, v_{N-1}, 0)
$$

for all $1 \leq a \leq N - 1$.

Comparing the $\psi^{(2)}$-functions we should take care of the involution on both sides. It remains to note that $A_{\psi^{(2)}_{2N-2b+1}} = D_{\psi^{(2)}_{N+1-b}}$ which completes the proof. □
**Corollary 4.6.** The following formula expresses the dependence of $D_N$ coordinate $v$ on the flat coordinate $t$

$$v_b = \sum_{\alpha_1, \ldots, \alpha_N \geq 0} \frac{(|\alpha| + 2b - 3)!^{N-1}}{(2b - 2)! \prod_{k=1}^{N-1} \alpha_k!} \sum_{k=1}^{N-1} (N-k) \alpha_k = N - b,$$  

(4.3)

$$v_N = t_N,$$  

(4.4)

where $|\alpha| = \sum_{k=1}^{N-1} \alpha_k$.

**Proof.** This follows immediately from the above proposition and formula (3.3). □

**Remark 4.7.** One could also give a geometric proof of the isomorphism of two $N$-dimensional Frobenius submanifolds of $A_{2N-1}$ and $D_{N+1}$. However in what follows we need the flat structures of both submanifolds to agree. Namely we indeed want to fix not only the isomorphism class of both Frobenius submanifolds but the potentials too.

**Remark 4.8.** The construction of Dubrovin attributes a Frobenius manifold to a Weyl group. In that sense Frobenius manifolds theory could not distinguish between $B_N$ and $C_N$ root systems whose Weyl groups coincide. The two submanifolds above are just two ways of how to find the same Weyl group as a subgroup of $A_\bullet$ and $D_\bullet$ Weyl groups.

Studying the solutions to ‘open’ WDVV equation it was found in [BB1] that potentials $F_{A_{2N-1}}$ and $F_{D_{N+1}}$ can be accompanied with the ‘open’ potentials $F'_{A_{2N-1}}$ and $F'_{D_{N+1}}$ being some new functions of $t_\bullet$ and one additional variable $s$. The open potential $F'_{A_{2N-1}}$ is a polynomial in $s$ and the open potential $F'_{D_{N+1}}$ is a Laurent polynomial with an order two pole in $s$. This made us hope that ‘open’ theories could distinguish between $B_N$ and $C_N$ root systems. Unfortunately the equality of the above proposition holds for the open potentials too.

### 4.3. $D_N$ case

For any fixed $N$ denote by $v_1^{(N)}(t)$ the polynomial expressing the $v_1$ coordinate of $D_N$ via $t_1, \ldots, t_{N-1}$. The formulae of Noumi–Yamada show that

$$\frac{\partial F_{D_N}}{\partial t_\gamma} = A_N^{(N)}(\gamma) + B_N^{(N)}(\gamma) \cdot s, \quad 1 \leq \gamma \leq N - 1,$$  

(4.5)

$$\frac{\partial F_{D_N}}{\partial t_N} = v_1^{(N)}(t) \cdot s,$$  

(4.6)

with $A_N^{(N)}, B_N^{(N)} \in \mathbb{Q}[t_1, \ldots, t_{N-1}]$. Namely, these functions do not depend on $t_N$.

**Theorem 4.9.** For any $N_2 > N_1 \geq 4$ we have

$$\frac{\partial v_1^{(N_1)}(t)}{\partial t_\beta} \bigg|_{\gamma \tau_i = 0_{N_1-1}} = \frac{\partial v_1^{(N_2)}(t)}{\partial t_\beta} \bigg|_{\gamma \tau_i = 0_{N_2-1}}, \quad \forall \beta < \min(N_1, N_2),$$

$$\frac{\partial A_N^{(N_1)}(t)}{\partial t_\beta} \bigg|_{\gamma \tau_i = 0_{N_1-1}} = \frac{\partial A_N^{(N_2)}(t)}{\partial t_\beta} \bigg|_{\gamma \tau_i = 0_{N_2-1}}, \quad \forall \beta + \alpha < \min(N_1, N_2)$$

understood as an equality of polynomials in $s$. 


Proof. For a fixed $N$ using equation (4.3) we get
\[
\frac{\partial v_1(t)}{\partial r_\beta} = \sum |\alpha|! \prod_{k=1}^{N-1} \frac{\partial}{\partial \alpha_k}, \quad 1 \leq \beta \leq N - 1, \quad |\alpha| = \sum_{k=1}^{N-1} \alpha_k,
\]
where the summation is taken over all $\alpha_1, \ldots, \alpha_{N-1} \geq 0$ satisfying $\sum_{k=1}^{N-1} (N-k)\alpha_k = \beta - 1$. The last equation can be rewritten as $\sum_{k=1}^{N-1} k\alpha_{N-k} = \beta - 1$. After taking the involution $\bar{\alpha} := N - \alpha$ one notes that the derivatives we compute only depend on $N$ via the number of summands. However for every fixed $\beta$ the numbers $k$, s.t. $k \geq \beta$ only contribute to the solution set with $\alpha_k$ because we should have $\alpha_k \geq 0$. Once the solution set $\{\alpha_k\}$ is obtained for some $N$, it contributes to all the higher ones, but with shifted indices. The shift is exactly the $D_N$ involution $\Xi := N - x$.

The second statement follows immediately from proposition 4.5 and theorem 4.1. \hfill \Box

5. Enumerative meaning of the coefficients of $A_N$ and $D_N$ potentials

Proposition 4.3 allows us to make a statement about the combinatorial meaning of the $F_{\lambda\lambda'}$ coefficients.

For every positive $i, j, m$ denote by $P_{ij}(\gamma_1, \ldots, \gamma_m)$ the number of all partitions $i_1, \ldots, i_m$ of $i$ and $j_1, \ldots, j_m$ of $j$, s.t. $\forall k \ i_k + j_k = \gamma_k + 1$.

Corollary 5.1. For every $\alpha + \beta \leq N + 1$ and $m \geq 1$ we have
\[
\frac{\partial^{m+2} F_{\lambda\lambda'}}{\partial t_\alpha \partial t_\beta \partial t_{N+1-\gamma_1} \cdots \partial t_{N+1-m}} = (-1)^{m-1}(m-1)! \cdot P_{\alpha\beta}(\gamma_1, \ldots, \gamma_m).
\]

Proof. This follows from proposition 4.3 in the same way as it is proved in lemma 3.2 in [NZ]. We repeat it here for completeness. Recall that $\partial_1 \partial_\alpha F = t_{N+1-\alpha} = s_\alpha$. By proposition 4.3 we have in $A_N$ (in the notation of that proposition):
\[
\sum_{\alpha, \beta \geq 1} z_1^{-\alpha} z_2^{-\beta} \partial_\alpha \partial_\beta F = \log \left[ 1 - \sum_{\alpha \geq 1} (z_1^{-\alpha} - z_2^{-\alpha}) s_\alpha \right]
\]
\[
= \log \left[ 1 + \sum_{\alpha \geq 1} \frac{z_1^{-\alpha} - z_2^{-\alpha}}{z_1^{-\alpha} - z_2^{-\alpha}} s_\alpha \right]
\]
\[
= \log \left[ 1 + \sum_{p \geq 1} \left( \sum_{i+j=1} \frac{z_1^{-i} z_2^{-j}}{i+j=p+1} s_p \right) \right]
\]
\[
= \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \sum_{i,j \geq 1} S_{i,j} \left( \sum_{j_1+\cdots+j_m=\bar{j}} S_{i_1+\cdots+i_m+j} \right)
\]
The rest follows by comparing the coefficients of $z_1^{-\alpha} z_2^{-\beta}$ on both sides of equation. \hfill \Box

Corollary 5.2. For any $\alpha + \beta \leq N$ and $m \geq 1$ we have
\[
\frac{\partial^{m+2} F_{D_N}}{\partial t_\alpha \partial t_\beta \partial t_{N-\gamma_1} \cdots \partial t_{N-\gamma_m}} = (-1)^{m-1}(m-1)! \cdot P_{2\alpha - 1, 2\beta - 1}(2\gamma_1 - 1, \ldots, 2\gamma_m - 1).
\]

Proof. This follows immediately from proposition 4.5 and the above corollary. \hfill \Box
6. A and B hierarchies

In this section we present the integrable hierarchy associated to the series of $A_N$ and $B_N$ Frobenius manifolds. Theorem 6.2 beneath shows that it coincides with the KP hierarchy.

6.1. Dispersionless hierarchy of type A

For any $\alpha, \beta \geq 1$, s.t. $\alpha + \beta \leq N + 1$ set

$$R^{(A_N)}_{\alpha, \beta; \gamma_1 + \cdots + \gamma_m} := \frac{1}{m!} \partial_{t_1} \partial_{t_2} \cdots \partial_{t_{N+1-\gamma_1}} \cdots \partial_{t_{N+1-\gamma_m}} |_{t=0}.$$ 

It follows by stabilization theorem 4.1 that the following quantities are well-defined

$$R^{A}_{\alpha, \beta; \gamma_1 + \cdots + \gamma_m} := R^{(A_{\alpha+\beta+1})}_{\alpha, \beta; \gamma_1 + \cdots + \gamma_{m}},$$

giving us an infinite set of rational numbers. One notes immediately that $R^{A}_{\alpha, \beta; \gamma_1 + \cdots + \gamma_m}$ is only non-zero when $\alpha + \beta - k = \gamma_1 + \cdots + \gamma_m$, which is essentially the quasi-homogeneity condition.

Consider the infinite system of PDEs on $f = f(t_1, t_2, \ldots)$, equation (1.1):

$$\partial_{\alpha} \partial_{\beta} f = \sum_{m \geq 1} \sum_{\gamma_1 + \cdots + \gamma_m = \alpha + \beta - m} R^{A}_{\alpha, \beta; \gamma_1 + \cdots + \gamma_m} \partial_{t_1} \cdots \partial_{t_{N+1}} f.$$ (6.1)

This is a Cauchy-type system of PDEs expressing any second order derivatives of $\partial_{\alpha} \partial_{\beta} f$ via the special second order derivatives $\partial_{t_1} \cdots \partial_{t_{N+1}} f$.

It follows from proposition 2.1 that the system of PDEs (6.1) is compatible.

6.2. KP hierarchy

We are going to introduce KP hierarchy in a Hirota form as a system of equations on the function $\tau = \tau(t_1, t_2, \ldots)$, and also in the form of Fay identities as a system of equations on $F = h^2 \log(\tau)$. We skip many important details of the general theory that do not play any particular role in our exposition. They can be found for example in [DN, NZ].

For any function $\tau = \tau(t_1, t_2, \ldots)$ and formal variables $z, h$ denote

$$\tau(t \pm [z^{-1}]) := e^{\pm hD(z)} \cdot \tau(t), \quad D(z) := \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k.$$ 

It is straightforward to note that the action of $\exp(D(z))$ is just the change of variables $\{t_k\}_{k=1}^{\infty} \mapsto \{t_k + h z^{-k}/k\}_{k=1}^{\infty}$. KP hierarchy in Hirota form is the following equality in the ring of formal power series in $t, t'$:

$$\text{res} \left( e^{\xi(t' - t) \cdot \tau(t' - [z^{-1}]) - \tau(t + [z^{-1}])} dz \right) = 0,$$

where $\xi(t, z) := \sum_{n \geq 1} t_n z^n$.

In what follows we need to consider the KP hierarchy in terms of $F = h^2 \log(\tau)$. Consider another differential operator

$$\Delta(z) := \frac{\exp(hD(z)) - 1}{h} = D(z) + O(h).$$
Then Hirota equation above is equivalent to the following equation in the ring of formal power series in $z_1^{-1}, z_2^{-1}$, called Fay identity [T, section 2]:

$$\exp(\Delta(z_1)\Delta(z_2)F) = 1 - \frac{\Delta(z_1)\partial_2 F - \Delta(z_2)\partial_1 F}{z_1 - z_2}.$$ 

The following proposition is crucial for our exposition.

**Proposition 6.1 (Lemma 3.2 in [NZ]).** Fay identities on the function $F = \hbar^2 \log \tau$ are equivalent to the following system of equations

$$\partial_i \partial_j F = \sum_{m \geq 1} (-1)^{m-1} \sum_{\gamma_1 + \cdots + \gamma_m = i + j - m} \tilde{P}_{ij}(\gamma_1, \ldots, \gamma_m) \times \partial_{\gamma_1} \partial_{\gamma_2} \cdots \partial_{\gamma_m} F,$$

where $\partial_k^\hbar$ are the differential operators defined by the equality

$$\Delta(z) = \sum_{k \geq 1} z_{-k}^{k} \partial_k^\hbar. \quad (6.2)$$

In particular, we have $\partial_k^\hbar = \partial_k + O(\hbar)$.

Assume $F = \sum_{\ell \geq 0} F_\ell \hbar^\ell$. It follows immediately from the proposition above that if $F$ is subject to Fay identities, then the function $F_0$ satisfies

$$\partial_i \partial_j F_0 = \sum_{m \geq 1} (-1)^{m-1} \sum_{\gamma_1 + \cdots + \gamma_m = i + j - m} \tilde{P}_{ij}(\gamma_1, \ldots, \gamma_m) \times \partial_{\gamma_1} \partial_{\gamma_2} \cdots \partial_{\gamma_m} F_0.$$

This system of equations is called the dispersionless limit of the KP hierarchy.

6.3. Identification

**Theorem 6.2.** The system of PDEs (6.1) coincides with the dispersionless KP hierarchy after the change of variables $t_k \mapsto t_k / k$.

Full KP hierarchy is obtained from the system of PDEs (6.1) via the substitution $\partial_k \mapsto \partial_k^\hbar$ (where $\partial_k^\hbar$ is defined via (6.2)).

**Proof.** It follows immediately from proposition 4.3 and corollary 5.1.

At the same time the full KP hierarchy is obtained from its dispersionless limit via the substitution $\partial_k \mapsto \partial_k^\hbar$, which completes the proof. \qed

6.4. BKP hierarchy

This hierarchy was introduced in [DJKM] via the Lax form (see also [N1, N2] for another context). We present it here via the bilinear identity form on the function $\tau = \tau(t)$, for $t = \{t_1, t_2, t_3, \ldots\}$, following [T].

Consider the operators

$$D^B(z) := \sum_{n \geq 0} \frac{z^{-2n-1}}{2n + 1} \partial_{2n+1}, \quad \Delta^B(z) := \exp(2\hbar D(z)) - 1.$$
Consider an infinite set of rational numbers
\[ \tau(t \pm 2[z^{-1}]) := e^{i2ht\partial^0z}, \quad \zeta^B(t, z) := \sum_{n \geq 0} t_{2n+1}z^{2n+1}. \]

The BKP hierarchy is the following equation
\[
\text{res} \left( e^{i\partial^0z} \tau(t' - 2[z^{-1}]) \tau(t + 2[z^{-1}]) \frac{dz}{z} \right) = \tau(t') \tau(t).
\]

The corresponding Fay identity for \( F = \hbar^2 \log \tau \) reads
\[
(z_1 + z_2 - \partial_1 h \Delta^B(z_1) \Delta^B(z_2) F - \partial_1 (\Delta^B(z_1) F + \Delta^B(z_2) F)) \exp(\Delta^B(z_1) \Delta^B(z_2) F) = \frac{z_1 + z_2}{z_1 - z_2} (z_1 - z_2 - \partial_1 (\Delta^B(z_1) F - \Delta^B(z_2) F)). \quad \text{(BKP)}
\]

The dispersionless limit is
\[
\left( 1 - \frac{\partial_1 (2D^B(z_1) + 2D^B(z_2) F_0)}{z_1 + z_2} \right) e^{2D^B(z_1) 2D^B(z_2) F_0} = 1 - \frac{\partial_1 (2D^B(z_1) - 2D^B(z_2) F_0)}{z_1 - z_2}.
\]

(BKP-dl)

6.5. Dispersionless hierarchy of type B

Consider an infinite set of rational numbers
\[
R^{B\gamma}_{\alpha, \beta; \gamma_1, \ldots, \gamma_k} := \left. \frac{\partial^{m+2} F_{\gamma \gamma}}{\partial t_{\alpha} \partial t_{\beta} \partial t_{\gamma 1-\gamma_1} \cdots \partial t_{N+1-\gamma_k}} \right|_{t=0}.
\]

It follows by proposition 4.4 that the following quantities are well-defined
\[
R^B_{\alpha, \beta; \gamma_1, \ldots, \gamma_k} := R^{B\gamma}_{\alpha, \beta; \gamma_1, \ldots, \gamma_k}.
\]

One notes immediately that \( R^B_{\alpha, \beta; \gamma_1, \ldots, \gamma_k} \) is only non-zero when \( \alpha + \beta - 1 = \gamma_1 + \cdots + \gamma_k \), which is essentially the quasi-homogeneity condition.

Consider the infinite system of PDEs on \( f = f(t_1, t_2, \ldots) \), equation (1.1):
\[
\partial_\alpha \partial_\beta f = \sum_{k \geq 1} \sum_{\gamma_1 + \cdots + \gamma_k = \alpha + \beta - 1} R^B_{\alpha, \beta; \gamma_1, \ldots, \gamma_k} \partial_{\gamma_1} f \cdots \partial_{\gamma_k} f.
\]

(6.3)

This is a system of PDEs expressing any second order derivatives of \( \partial_\alpha \partial_\beta f \) via the special second order derivatives \( \partial_1 \partial_\nu f \).

It follows from proposition 2.1 that the system of PDEs (6.3) is compatible.

**Theorem 6.3.** The system of PDEs (6.3) coincides with the dispersionless BKP hierarchy (BKP-dl) after the change of variables \( t_1 \mapsto t_{2k-1}/(2k-1) \) and the substitution \( f \mapsto 2F_0 \).

**Proof.** Note that \( D^B(z) = D(z)|_{z^k=0} \). Denote
The proof follows now by corollary 5.2.

Equation (BKP-dl) reads

\[ 4D^B(z_1)D^B(z_2)F_0 = \log \left( \frac{1 - g^{\text{BKP}}(z_1, z_2) \cdot F_0}{1 - g^{\text{KP}}(z_1, z_2) \cdot F_0} \right) \]

\[ = \left[ \log \left( 1 - g^{\text{KP}}(z_1, z_2) \cdot 2F_0 \right) \right] - \log \left( 1 - g^{\text{KP}}(z_1, z_2) \cdot 2F_0 \right) \bigg|_{z_1^2 = z_2^2 = 0}. \]

Expanding rhs in series, one gets the following identity:

\[ \sum_{i,j \geq 1 \atop i,j \notin \mathbb{Z}} \frac{(-1)^{m-1}}{m} \sum_{i,j \geq 1 \atop i,j \notin \mathbb{Z}} z_1^{-i} z_2^{-j} \sum_{\gamma_1=1 \atop \gamma_2 \in \mathbb{Z}} \gamma_1 \sigma_1 + j - m \gamma \sigma_1 \cdot \gamma \sigma_1 + j - m \]

\[ = 2 \sum_{i,j \geq 1 \atop i,j \notin \mathbb{Z}} \frac{(-1)^{m-1}}{m} \sum_{i,j \geq 1 \atop i,j \notin \mathbb{Z}} z_1^{-i} z_2^{-j} \sum_{\gamma_1=1 \atop \gamma_2 \in \mathbb{Z}} \gamma_1 \sigma_1 + j - m \gamma \sigma_1 \cdot \gamma \sigma_1 + j - m \]

\[ \sum_{k=1}^{\infty} 2 \partial_\sigma \partial_\sigma F_0. \]

The proof follows now by corollary 5.2.

7. D hierarchy

Consider the series of rational numbers \( R^{(D_N,1)} \) and \( R^{(D_N,2)} \) defined as follows. For any fixed \( \alpha, \beta \geq 1 \) set

\[ R^{(D_N)^{\alpha \beta}}_{\alpha \beta \gamma_1 \ldots \gamma_m} := \frac{\partial^{m+1} F_{D_N}}{m! \partial_{\alpha} \partial_{\beta} \partial_{\gamma_1} \ldots \partial_{\gamma_m}} \bigg|_{t_0 = 0}, \quad 1 \leq \sigma_k \leq N, \]

where we use the notation \( N := N \) and \( \kappa := N - \kappa \) for \( \alpha < N \).

Recall that dependence of \( F_{D_N} \) on the variable \( t_N \) is very special (see section 4.3). We have \( \partial_\alpha \partial_\beta F_{D_N} = \partial_\alpha A^{(N)}_{\beta} + t_{\beta} \partial_\alpha B_{\beta} \) and \( \partial_\beta F_{D_N} = t_N v_{\beta}^{(N)} \). The variable \( t_N \) has non-zero weight, therefore it follows that for \( \alpha, \beta < N \), and \( \gamma_k < N \) we have

\[ R^{(D_N)^{\alpha \beta}}_{\alpha \beta \gamma_1 \ldots \gamma_m} = \frac{\partial^{m+1} A^{(N)}_{\beta}}{m! \partial_{\alpha} \partial_{N-\gamma_1} \ldots \partial_{N-\gamma_m}} \bigg|_{t_0 = 0}, \]

\[ R^{(D_N)^{\alpha \beta}}_{\alpha \beta \gamma_1 \ldots \gamma_m} = \frac{\partial^{m+1} v_{\beta}}{m! \partial_{\alpha} \partial_{N-\gamma_1} \ldots \partial_{N-\gamma_m}} \bigg|_{t_0 = 0}. \]
By theorem 4.9 the following quantities are well-defined

$$
R^{(D,1)}_{\alpha,\beta;\gamma_1,\ldots,\gamma_m} = R^{(D,\alpha+\beta-1)}_{\alpha,\beta;\gamma_1,\ldots,\gamma_m},
$$

$$
R^{(D,2)}_{\alpha;\gamma_1,\ldots,\gamma_m} = R^{(D,\alpha-1)}_{\alpha;\gamma_1,\ldots,\gamma_m}.
$$

It follows immediately from the weights counting that $R^{(D,1)}_{\alpha,\beta;\gamma_1,\ldots,\gamma_m}$ is only non-zero when $\gamma_1 + \cdots + \gamma_m = \alpha + \beta - 1$ and $R^{(D,2)}_{\alpha;\gamma_1,\ldots,\gamma_m}$ is only non-zero when $\gamma_1 + \cdots + \gamma_m = \alpha - 1$.

For a function $f = f(t_0, t_1, t_2, \ldots)$ consider the system of PDEs

$$
\partial_\alpha \partial_\beta f = \sum_{m \geq 1, \gamma_1, \ldots, \gamma_m} R^{(D,1)}_{\alpha,\beta;\gamma_1,\ldots,\gamma_m} \prod_{k=1}^m \partial_\gamma f,
$$

(7.1)

$$
\partial_\alpha \partial_\beta f = \partial_\beta \partial_\alpha f \cdot \sum_{m \geq 1, \gamma_1, \ldots, \gamma_m} R^{(D,2)}_{\alpha;\gamma_1,\ldots,\gamma_m} \prod_{k=1}^m \partial_\gamma f,
$$

(7.2)

for all $\alpha, \beta \geq 2$.

Denote $p_k := \partial_1 \partial_k f$. The first PDE's of the system above read

$$
\partial_2 \partial_1 f = \frac{1}{12} p_1^3 - \frac{1}{2} p_2 p_1 + p_3,
$$

$$
\partial_2 \partial_2 f = -\frac{1}{2} p_2^2 + \frac{1}{4} p_3 p_2 - \frac{1}{2} p_1 p_3 + p_4,
$$

$$
\partial_2 \partial_3 f = \frac{1}{4} p_2 p_3 + \frac{1}{4} p_4 p_3 - \frac{1}{2} p_1 p_4 p_2 + p_5,
$$

$$
\partial_2 \partial_4 f = \frac{1}{80} p_4^2 - \frac{1}{8} p_1 p_4^2 + \frac{3}{4} p_1 p_2 p_4 + \frac{1}{4} p_3^2 p_2 - \frac{1}{2} p_1 p_4 - \frac{3}{2} p_2 p_3 + p_5
$$

and

$$
\partial_3 \partial_3 f = \frac{1}{2} p_1,
$$

$$
\partial_3 \partial_4 f = \frac{1}{2} p_3 + \frac{1}{2} p_2,
$$

$$
\partial_3 \partial_5 f = \frac{1}{8} p_4 + \frac{1}{2} p_1 p_2 + \frac{1}{2} p_3,
$$

$$
\partial_3 \partial_6 f = \frac{1}{16} p_4 + \frac{3}{8} p_2 p_2 + \frac{1}{4} p_3^2 + \frac{1}{2} p_2 p_3 + \frac{1}{2} p_4.
$$

Proposition 2.1 implies that this system of PDEs is compatible.

7.1. 2-component BKP

Consider the function $\tau = \tau(t, \bar{t})$, for $t = \{t_1, t_2, t_3, \ldots\}$ and $\bar{t} = \{\bar{t}_1, \bar{t}_2, \bar{t}_3, \ldots\}$ being two sets of independent variables.

Together with the operators $D^B(z)$ and $\Delta^B(z)$ from section 6.4 we will need two similar operators acting on the additional set of variables $t$:

$$
D^B(z) := \sum_{n \geq 0} \frac{z^{-2n-1}}{2n+1} \partial_{2n+1}, \quad \Delta^B(z) := \frac{\exp(2\hbar D^B(z)) - 1}{\hbar},
$$

where $\hbar$ is the Planck constant.
where $\hat{\partial}_t := \partial / \partial t$. Denote
\[ \tau(t \pm 2z^{-1}, \bar{t}) := e^{2i\theta B\bar{K}(z)} \cdot \tau(t, \bar{t}), \quad \tau(t, \bar{t} + 2z^{-1}) := e^{i2\theta B\bar{K}(z)} \cdot \tau(t, \bar{t}). \]

Set also $\xi(t, z) := \sum_{n \geq 0} t^{2n+1} z^{2n+1}$.

The 2-component BKP hierarchy is the following equation
\[
\text{res} \left( e^{i(t' - t) z} \tau(t' - 2z^{-1}, \bar{t}') \tau(t + 2z^{-1}, \bar{t}) \frac{dz}{z} \right) = \text{res} \left( e^{i(t' - t) z} \tau(t' - 2z^{-1}, \bar{t}') \tau(t + 2z^{-1}, \bar{t}) \frac{dz}{z} \right).
\]

This equation coincides with the BKP hierarchy by putting $\bar{t} = t$.

The set of Fay identities equivalent to the equation above were derived in [T, equations 4.9–4.12]. They read
\[
\begin{aligned}
(2z + 1 - h\partial_t \Delta^B(z_1) \Delta^B(z_2) F - \partial_t (\Delta^B(z_1) F + \Delta^B(z_2) F)) \exp(\Delta^B(z_1) \Delta^B(z_2) F) \\
\quad = \frac{2z + 1}{2} \left( z_1 - z_2 - \partial_t (\Delta^B(z_1) F - \Delta^B(z_2) F) \right).
\end{aligned}
\]

(2BKP-1)

(\[ z_1 + z_2 - h\partial_t \Delta^B(z_1) \Delta^B(z_2) F - \partial_t (\Delta^B(z_1) F + \Delta^B(z_2) F) \exp(\Delta^B(z_1) \Delta^B(z_2) F) \\
\quad = \frac{2z + 1}{2} \left( z_1 - z_2 - \partial_t (\Delta^B(z_1) F - \Delta^B(z_2) F) \right).
\]

(2BKP-2)

(\[ z_1 - h\partial_t \Delta^B(z_1) \Delta^B(z_2) F - \partial_t (\Delta^B(z_1) F + \Delta^B(z_2) F) \exp(\Delta^B(z_1) \Delta^B(z_2) F) \\
\quad = z_1 - \partial_t (\Delta(z_1) - \Delta(z_2)) F.
\]

(2BKP-3)

(\[ z_2 - h\partial_t \Delta^B(z_1) \Delta^B(z_2) F - \partial_t (\Delta^B(z_1) F + \Delta^B(z_2) F) \exp(\Delta^B(z_1) \Delta^B(z_2) F) \\
\quad = z_2 - \partial_t (\Delta(z_2) - \Delta(z_1)) F.
\]

(2BKP-4)

One notes immediately that (2BKP-1) coincides with (2BKP-2) after the interchange $t$ and $t'$.

In the case of 2-component BKP hierarchy we do not have any result similar to proposition 6.1 above. However it is not hard to derive the dispersionless form of the Fay-type identities above.

Assume $\tau(t, \bar{t}) = \hbar^2 F$ with $F = \sum_{g \geq 0} \hbar^g F_g(t, \bar{t})$. Dispersionless limit of the Fay-type identities above reads:
\[
\begin{aligned}
\left( 1 - \frac{\partial_t (2D^B(z_1) + 2D^B(z_2)) F_0}{z_1 + z_2} \right) e^{2D^B(z_1) 2D^B(z_2) F_0} = \left( 1 - \frac{\partial_t (2D^B(z_1) - 2D^B(z_2)) F_0}{z_1 - z_2} \right), \\
(2BKP-1dl)
\end{aligned}
\]

(\[ 1 - \frac{\partial_t (2D^B(z_1) + 2D^B(z_2)) F_0}{z_1 + z_2} \right) e^{2D^B(z_1) 2D^B(z_2) F_0} = \left( 1 - \frac{\partial_t (2D^B(z_1) - 2D^B(z_2)) F_0}{z_1 - z_2} \right), \\
(2BKP-2dl)\]

(\[ 1 - \frac{\partial_t (2D^B(z_1) + 2D^B(z_2)) F_0}{z_1 + z_2} \right) e^{2D^B(z_1) 2D^B(z_2) F_0} = \left( 1 - \frac{\partial_t (2D^B(z_1) - 2D^B(z_2)) F_0}{z_1 - z_2} \right), \\
(2BKP-2dl)\]

(\[ 1 - \frac{\partial_t (2D^B(z_1) + 2D^B(z_2)) F_0}{z_1 + z_2} \right) e^{2D^B(z_1) 2D^B(z_2) F_0} = \left( 1 - \frac{\partial_t (2D^B(z_1) - 2D^B(z_2)) F_0}{z_1 - z_2} \right), \\
(2BKP-2dl)\]
\[
(z_1 - \partial_1(2D^B(z_1) + 2D^B(z_2))F_0) e^{2D^B(z_1)2D^B(z_2)F_0} = z_1 - \partial_1(2D^B(z_1) - 2D^B(z_2))F_0,
\]
(2BKP-3dl)

\[
(z_2 - \partial_1(2D^B(z_1) + 2D^B(z_2))F_0) e^{2D^B(z_1)2D^B(z_2)F_0} = z_2 - \partial_1(2D^B(z_1) - 2D^B(z_2))F_0.
\]
(2BKP-4dl)

\[7.2. \text{Identification}\]

The connection between our hierarchy for the \(D_N\) series of Frobenius manifolds and the 2-component BKP hierarchy requires taking a certain reduction in \(t\) variable on the BKP side.

In particular, we assume the Fay-type identities above to be reduced by setting \(\bar{\partial}_1 = z_1 - 1\bar{\partial}_1\), and then taking only the first order in \(z_1\) of the respective identity where \(\bar{\partial}_1\) appears (i.e. taking the coefficient in front of \(z_1\) for (2BKP-2dl) and the coefficient in front of \(z_1\) for (2BKP-3dl) and (2BKP-4dl)).

That is, the solution \(F_0\) of the reduced hierarchy should satisfy equation (2BKP-1dl) and also

\[D^B(z_1) \cdot \bar{\partial}_1 2F_0 = \sum_{m \geq 1} \partial_1^m \bar{\partial}_1 \partial_1^m (\partial_1^m D^B(z_1) 2F_0)^{m-1}.\]
(7.3)

while equations (2BKP-2dl) and (2BKP-4dl) hold trivially in this reduction.

\textbf{Theorem 7.1.} The system of PDEs (7.1) and (7.2) coincides with the discussed above reduction of the 2-component BKP hierarchy after the change of variables \(t_k \mapsto t_{2k-1}/(2k-1)\), \(t_0 \mapsto t_1\) and substitution \(f \mapsto 2F_0\).

\textbf{Proof.} Note that equation (2BKP-1dl) coincides with equation (BKP-dl). It follows immediately from corollary 5.2 and theorem 6.3 that the system (7.1) coincides after the discussed substitution with equation (2BKP-1dl).

We show that equation (7.2) coincides with equation (7.3) after the discussed substitution. This is essentially a question about the rational numbers \(R^{(a,2)}_{\gamma_1,\ldots,\gamma_k}\).

Applying the discussed substitution to equation (7.3) we have

\[
\sum_{a \geq 1} z_1^{-(2a-1)} \partial_1^a \bar{\partial}_1 f = \sum_{m \geq 1} \partial_1^m \bar{\partial}_1 f - \sum_{b \geq 1} z_1^{-(2b-1)} \partial_1^b \partial_1 f_{m-1} \left( \sum_{b \geq 1} z_1^{-(2b-1)} \partial_1^b \partial_1 f \right)^{m-1}
\]
\[
\Leftrightarrow \partial_1^a \bar{\partial}_1 f = \partial_1^{m-2} \bar{\partial}_1 f \sum_{a \geq 1} \partial_1^a \partial_1 \gamma f \ldots \partial_1 \partial_1 \gamma f,
\]

where the last summation is taken over all \(2a - 1 = m + 1 + \sum_{j=1}^m (2\gamma_j - 1)\) that hold if and only if \(a = 1 + \sum_{j=1}^m \gamma_j\). This coincides exactly with the expansion of \(\partial \gamma_1/\partial t_a\) in the \(s_{\gamma_k} = t_{N-\gamma_k}\) variables, cf equation (4.3). This completes the proof. \(\square\)
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Data availability statement

No new data were created or analysed in this study.

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References

[B1] Buryak A 2020 Extended r-spin theory and the mirror symmetry for the $A_{r-1}$-singularity Mosc. Math. J. 20 475–93
[BB1] Basalaev A and Buryak A 2020 Open Saito theory for $A$ and $D$ singularities Int. Math. Res. Not. rmz381
[B2] Bertola M 2000 Frobenius manifold structure on orbit space of Jacobi groups: Part II Differ. Geom. Appl. 13 213–33
[D1] Dubrovin B 1993 Differential geometry of the space of orbits of a Coxeter group (arXiv: hep-th/9303152) (posted 27 Mar 1993, accessed 15 Jan 2021)
[D2] Dubrovin B 1996 Geometry of 2d topological field theories Lect. Notes Math. 120–348
[DN] Dubrovin B A and Natanzon S M 1989 Real theta-function solutions of the Kadomtsev–Petviashvili equation Math. USSR Izv. 32 269
[DLZ] Dubrovin B and Zhang Y 2001 Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants (arXiv: math/0108160v1) (posted 23 Aug 2001, accessed 15 Jan 2021)
[DLZ2] Dubrovin B and Zhang Y 1998 Extended affine Weyl groups and Frobenius manifolds Compositio Mathematica 111 167–219
[DLZ3] Dubrovin B, Liu S-Q and Zhang Y 2008 Frobenius manifolds and central invariants for the Drinfeld–Sokolov bihamiltonian structures Adv. Math. 219 780–837
[DJKM] Date E, Jimbo M, Kashiwara M and Miwa T 1982 Transformation groups for soliton equations: IV. A new hierarchy of soliton equations of KP-type Phys. D 4 343–65
[FGM] Frenkel E, Givental A and Milanov T 2010 Soliton equations, vertex operators, and simple singularities Funct. Anal. Other Math. 3 47–63
[FSZ] Faber C, Shadrin S and Zvonkine D 2006 Tautological relations and the r-spin Witten conjecture Ann. Sci. Ec. Norm. Super. 4 621–58
[FJR] Fan H, Jarvis T and Ruan Y 2013 The Witten equation, mirror symmetry, and quantum singularity theory Ann. Math. 178 1–106
[LRZ] Liu S-Q, Ruan Y and Zhang Y 2015 BCFG Drinfeld–Sokolov hierarchies and FJRW-theory Invent. math. 201 711–72
[LWZ] Liu S-Q, Wu C-Z and Zhang Y 2011 On the Drinfeld–Sokolov hierarchies of D type Int. Math. Res. Not. 2011 1952–96
[N1] Natanzon S 1992 Differential equations on the Prym theta function. A realness criterion for two-dimensional, finite-zone, potential Schrödinger operators Funktsional. Anal. Prilozhen. 26 17–26 (Russian)
[N2] Natanzon S M 1992 Funct. Anal. Appl. 26 13–20 (Engl. transl.)
[N3] Natanzon S M 1995 Real nonsingular finite zone solutions of soliton equations Trans. Am. Math. Soc. Series 2 170 (1995): 153–84
[NZ] Natanzon S and Zabrodin A 2016 Formal solutions to the KP hierarchy J. Phys. A: Math. Theor. 49 20
[NY] Noumi M and Yamada Y 1997 Notes on the flat structures associated with simple and simply elliptic singularities Proc. of the Taniguchi Symp. 373–83
[S1] Saito K 1983 Period mapping associated to a primitive form Publ. Res. Inst. Math. Sci. 19 1231–64 Kyoto University
[S2] Saito M 1989 On the structure of Brieskorn lattices Ann. Inst. Fourier Grenoble. 1 27–72
[ST] Saito K and Takahashi A 2008 From primitive forms to Frobenius manifolds Proc. Symp. Pure Math. 78 31–48
[T] Takasaki K 2011 Differential Fay identities and auxiliary linear problem of integrable hierarchies Adv. Stud. Pure Math. 61 387–441
[TT1] Takasaki K and Takebe T 1995 Integrable hierarchies and dispersionless limit Rev. Math. Phys. 07 743
[TT2] Takasaki K and Takebe T 1999 Quasiclassical limit of KP hierarchy, W-symmetries, and free fermions J. Math. Sci. 94 1635–41
[Z1] Zuber J-B 1994 On Dubrovin topological field theories Mod. Phys. Lett. A 09 749–60
[Z2] Zuo D 2007 Frobenius manifolds associated to $B_l$ and $D_l$, revisited Int. Math. Res. Not. 2007 rnm020