ON INTERPOLATION APPROXIMATION: CONVERGENCE RATES ON INTERPOLATION FOR FUNCTIONS OF LIMITED REGULARITY∗

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Abstract. The convergence rates of polynomial interpolations are generally estimated by their Lebesgue constants. However, these constants might be overestimated for functions of limited regularity for some special points of sets for functions of limited regularity. This paper aims to derive sharper convergence estimates; our method relies on the novel application of the Peano kernel theorem and Wainerman’s lemma. These new estimates enable us to show that the polynomial interpolations based on strongly normal point systems can achieve the optimal convergence rate, that is the rate of the best polynomial approximation. In addition, we study the convergence rates for Gauss-Jacobi and Jacobi-Gauss-Lobatto point system by using the asymptotics on Jacobi polynomials. It turns out that the polynomial interpolation based on the Gauss-Legendre, Legendre-Gauss-Lobatto, or at strongly normal point systems, has essentially the same approximation accuracy as those using the two Chebyshev point systems. This also verified the fact that Gauss and Clenshaw-Curtis quadrature formulas have essentially the same accuracy. Numerical results are in perfect coincidence with the estimates.

Key words. polynomial interpolation, Peano kernel, convergence rate, limited regularity, strongly normal point system, Gauss-Jacobi point, Jacobi-Gauss-Lobatto point, Chebyshev point.

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1. Introduction. Numerical analysis is built on a strong foundation of approximation theory. In almost every area of numerical analysis it is a fact that, sooner or later, the discussion comes down to approximation theory (Trefethen [57]).

A central problem in approximation theory is to construct accurate and efficient approximation to the targeting function by simple functions. The most useful class of simple functions are polynomials. There exist many investigations for the behavior of continuous functions approximated by polynomials. Weierstrass [73] in 1885 proved the well known result that every continuous function \( f(x) \) in \([-1, 1]\) can be uniformly approximated as closely as desired by a polynomial function. This result has both practical and theoretical relevance, especially in polynomial interpolation.

Polynomial interpolation is a fundamental tool in many areas of scientific computing. Lagrange interpolation is a classical technique for approximation of continuous functions. Let us denote by

\[ L_n[f](x) = \sum_{k=1}^{n} f(x_k) \ell_k(x), \]

where

\[ \ell_k(x) = \prod_{j=1, j \neq k}^{n} \frac{x - x_j}{x_k - x_j}, \]

the \( n \) distinct points in the interval \([-1, 1]\) and let \( f(x) \) be a function defined in the same interval. The \( n \)th Lagrange interpolation polynomial of \( f(x) \) is unique and given by the formula

\[ \ell_k^n(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x - x_k^n)}, \]

where \( \omega_n(x) = (x - x_1^n)(x - x_2^n) \cdots (x - x_n^n) \).

There is a well developed theory that quantifies the convergence or divergence of the Lagrange interpolation polynomials (Trefethen [59]). Two key notions for interpolation in a given set of points are that of the Lebesgue function

\[ \lambda_n(x) = \sum_{k=1}^{n} \left| \ell_k^n(x) \right| \]

and Lebesgue constant

\[ \Lambda_n = \max_{x \in [-1, 1]} \lambda_n(x), \]

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which are of fundamental importance (Cheney [9], Davis [12] and Szegő [54]). The Lebesgue constant can also be interpreted as the $\infty$-norm of the projection operator $L_n : C([-1, 1]) \to P_{n-1}$

$$\Lambda_n = \sup_f \frac{\|L_n[f]\|_\infty}{\|f\|_\infty},$$

where $P_{n-1}$ is the set of polynomials of degree less than or equal to $n - 1$.

The interpolation error can be bounded in terms of the Lebesgue constant by

$$\|L_n[f] - f\|_\infty \leq (1 + \Lambda_n)\|p^*_n - f\|_\infty,$$

where $p^*_n$ is the best polynomial approximation of degree $n - 1$. Thus, the Lebesgue constant $\Lambda_n$ indicates how good the interpolant $L_n[f]$ is in comparison with the best polynomial approximation $p^*_n$.

The study of the Lebesgue constant $\Lambda_n$ originated more than 100 years ago. Comprehensive reviews can be found in Brutman [8], Lubinsky [40], Trefethen [59, Chapter 15], etc. For an arbitrarily given system of points $\{x^{(n)}_1, x^{(n)}_2, \ldots, x^{(n)}_n\}_{n=1}^\infty$, Bernstein [2] and Faber [18] in 1914 obtained that

$$\Lambda_n \geq \frac{1}{12} \log n,$$

which, together with the uniform boundedness principle or Banach-CSteinhaus theorem, implies that there exists a continuous function $f(x)$ in $[-1, 1]$ for which the sequence $L_n[f] (n = 1, 2, \ldots)$ is not uniformly convergent to $f$ in $[-1, 1]$\(^1\). More precisely, Erdős [15] and Brutman [7] proved that

$$(1.6) \Lambda_n \geq \frac{2}{\pi} \log n + C \text{ for some constant } C \text{ (15)}; \quad \Lambda_n \geq \frac{2}{\pi} \left( \gamma_0 + \log \frac{4}{\pi} \right) + \frac{2}{\pi} \log n \text{ (17)},$$

where $\gamma_0 = 0.577\ldots$ is the Euler’s constant. In particular, for equidistant point system

$$\left\{ x^{(n)}_k = -1 + \frac{2k}{n-1} \right\}_{k=0}^{n-1},$$

Schönhage [51] showed that

$$\Lambda_n \sim \frac{2^n}{e(\log(n-1) + \gamma_0)(n-1)}, \quad n \to \infty.$$

Additionally, Trefethen and Weideman [61] established that

$$\frac{2^{n-3}}{(n-1)^2} \leq \Lambda_n \leq \frac{2^{n+2}}{n-1}, \quad n \geq 0.$$

Then generally, the set of equally spaced points is a bad choice for Lagrange interpolation (see Runge [48]).

However, for well chosen sets of points, the growth of $\Lambda_n$ may be extremely slow as $n \to \infty$:

\(^1\)Grünwald [24] in 1935 and Marcinkiewicz [42] in 1937, independently, showed that even for the Chebyshev points of first kind

$$x^{(n)}_k = \cos \left( \frac{2k - 1}{2n} \pi \right), \quad k = 1, 2, \ldots, n = 1, 2, \ldots,$$

there is a continuous function $f(x)$ in $[-1, 1]$ for which the sequence $L_n[f]$ is divergent everywhere in $[-1, 1]$. 
• Chebyshev point system of first kind $T_n = \left\{ x_k^{(n)} = \cos \left( \frac{2k - 1}{2n} \pi \right) \right\}_{k=1}^n$; An asymptotic estimate of $\Lambda_n(T_n)$ was given by Bernstein [11] as

$$\Lambda_n(T_n) \sim \frac{2}{\pi} \log n, \quad n \to \infty,$$

which is improved by Ehlich and Zeller [14], Rivlin [46] and Brutman [7] as

$$\frac{2}{\pi} \left( \gamma_0 + \log \frac{4}{\pi} \right) + \frac{2}{\pi} \log n < \Lambda_n(T_n) \leq 1 + \frac{2}{\pi} \log n, \quad n = 1, 2, \ldots.$$

• Chebyshev point system of second kind $U_n = \left\{ x_k^{(n)} = \cos \left( \frac{k}{n-1} \pi \right) \right\}_{k=0}^{n-1}$ (also called Chebyshev extreme or Clenshaw-Curtis points [58]): Ehlich and Zeller [14] proved that

$$\Lambda_n(U_n) = \left\{ \begin{array}{ll}
\Lambda_{n-1}(T_{n-1}), & n = 2, 4, 6, \ldots \\
\Lambda_{n-1}(T_{n-1}) - \alpha_n, & 0 \leq \alpha_n < \frac{1}{(n-1)^2}, \quad n = 3, 5, 7, \ldots.
\end{array} \right.$$

• The roots of Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ ($\alpha, \beta > -1$) denoted by $J_n$: The asymptotic estimate of $\Lambda_n(J_n)$ was found by Szegő [54] as

$$\Lambda_n(J_n) = \left\{ \begin{array}{ll}
O(n^{\gamma+\frac{1}{2}}), & \gamma > -\frac{1}{2}, \\
O(\log n), & \gamma \leq -\frac{1}{2}, \quad \gamma = \max\{\alpha, \beta\}.
\end{array} \right.$$

Comparing equations (1.7), (1.8) and (1.9) with (1.6), we see that the Lebesgue constant $\Lambda_n$ under two Chebyshev point systems and the Jacobi point system with $\gamma \leq -\frac{1}{2}$ achieves the optimal bound, which is of order $O(\log n)$.

Nevertheless, it is worth noting that if $f(x)$ has an absolutely continuous $(k-1)$st derivative $f^{(k-1)}$ on $[-1, 1]$ for some $k \geq 1$ and its $k$th derivative $f^{(k)}$ is of bounded variation $\text{Var}(f^{(k)}) < \infty$, Mastroianni and Szabados [41], Trefethen [59] and Xiang et al. [76] proved that

$$\|f - L_n[f]\|_\infty = O(n^{-k}),$$

where $L_n[f]$ is at the $n$ Chebyshev points of first or second kind, which has the same asymptotic order as

$$\|f - p^*_n\|_\infty = O(n^{-k})$$

for the best approximation $p^*_n$; see de la Vallée Poussin [62]. In particular, for $f(x) = |x|$, the error on the $L_n[f]$ at the above two Chebyshev point systems satisfies

$$\|f - L_n[f]\|_\infty \leq \frac{4}{\pi(n-1)}$$

(see [59, 76]), while

$$\|f - p^*_n\|_\infty \sim \frac{\beta}{n}, \quad 0.2801685 < \beta < 0.2801734$$

(see Bernstein [3] and Varga and Capenter [63]). Thus, the error estimate (1.5) may be overestimated for functions of limited regularity with some sets of special points.

Moreover, it has been observed, by Clenshaw-Curtis [10] and O’Hara and Smith [29], that $n$-point Gauss quadrature and $n$-point Clenshaw-Curtis quadrature have essentially the same accuracy, which has been showed recently by Trefethen [58, 59], Brass and Petras [6] and Xiang and Bornemann [75]. These two quadratures are derived from the interpolating polynomial $L_n[f]$ by

$$Q_n[f] = \int_{-1}^{1} L_n[f](x)dx,$$
based on the $n$ Gauss-Legendre and Clenshaw-Curtis points, respectively. This motivates us to conjecture that the interpolating polynomials based on the two point systems above have the same convergence rate. However, it can not be derived from (1.5).

In this paper, we present new and sharper convergence estimates of the interpolation error for functions of limited regularity. The key idea is to derive a Peano-type estimate for the interpolation error by the famous Peano kernel theorem [41]; on the other hand, application of Wainnerman’s lemma [72] allows us to bound the Peano-kernels whereby proving the ultimate estimate. Suppose $f(x)$ has an absolutely continuous $(r-1)$st derivative $f^{(r-1)}$ on $[-1, 1]$, and its $r$-th derivative $f^{(r)}$ is of bounded variation $\text{Var}(f^{(r)}) < \infty$. We will show that

$$
(1.11) \quad \|f - L_n[f]\|_{\infty} \leq \frac{\pi^{r}\text{Var}(f^{(r)})}{(n-1)(n-2)\cdots(n-r)} \mathcal{L}_n, \quad \mathcal{L}_n = \max_{1 \leq j \leq n} \|\ell_j^{(n)}\|_{\infty}.
$$

Comparing (1.11) with (1.5), we see that

$$
(1.12) \quad \|f - L_n[f]\|_{\infty} = O(\mathcal{L}_n)\|f - p_{n-1}^{*}\|_{\infty},
$$

and the Lebesgue constant $\Lambda_n = \max_{x \in [-1,1]} \sum_{k=1}^{n} \|\ell_k^{(n)}(x)\|$ is replaced by $\max_{1 \leq j \leq n} \|\ell_j^{(n)}\|_{\infty}$ in some sense.

Particularly, from (1.12), it directly follows that the interpolation $L_n[f]$ at the strongly normal point system (see Fejér [19] [21]) can achieve the optimal convergence rate $O(\|f - p_{n-1}^{*}\|_{\infty})$. The point system (1.1) is called strongly normal if for all $n$

$$
v_k(t) \geq c > 0, \quad k = 1, 2, \ldots, n, \quad t \in [-1, 1]
$$

for some positive constant $c$, where

$$
v_k(t) = 1 - (t - x_k) \frac{\omega''(x_k)}{\omega'_n(x_k)} \quad (21).
$$

Furthermore, $\|\ell_j^{(n)}\|_{\infty}$ can be explicitly estimated for Gauss-Jacobi or Jacobi-Gauss-Lobatto point systems, by using the asymptotics on Jacobi polynomials given by Szegő [54] and Sun [53] as follows:

- For the $n$ Gauss-Jacobi points:

$$
\mathcal{L}_n = O(n^{\max\{\gamma - \frac{3}{2}, 0\}}), \quad \gamma = \max\{\alpha, \beta\}.
$$

- For the $n$ Jacobi-Gauss-Lobatto points (the roots of $(1 - x^2)P_{n-2}^{(\alpha, \beta)}(x) = 0$):

$$
\mathcal{L}_n = \begin{cases} 
\left(n - \min\{0, \alpha + \frac{1}{2}, \beta + \frac{1}{2}\}\right), & -1 < \alpha, \beta \leq \frac{3}{2}, \\
O \left(n - \min\{2 + \alpha - \beta, 2 + \beta - \alpha, \frac{3}{2} - \alpha, \frac{3}{2} - \beta\}\right), & \text{otherwise}
\end{cases}
$$

From the above estimates, we see that the interpolation at the Gauss-Legendre or at the Legendre-Gauss-Lobatto point system, has essentially the same approximation accuracy compared with those at the two Chebyshev point systems. All of them satisfy that $\mathcal{L}_n = O(1)$. In addition, the convergence rate is attainable, which is illustrated by some functions of limited regularity.

It is particularly noteworthy that the interpolation polynomial $L_n[f]$, at the Gauss-Jacobi, Jacobi-Gauss-Lobatto or Gauss-Jacobi-Radau point system, can be efficiently evaluated by applying the second barycentric formula with the overall computational complexity $O(n)$. For more details on this topic, see Salzer [49], Henrich [30], Berrut and Trefethen [4], Higham [31, 32], Glaser et al. [23], Wang and Xiang [70], Bogaert et al. [5], Hale and Trefethen [28], Hale and Townsend [26], Trefethen [59], Wang et al. [71], etc. MATLAB routines can be found in CHEBFUN system [60] and Xiang and He [77].

The paper is organized as follows: In section 2, we present the error of $f(x) - L_n[f](x)$ for each fixed $x \in [0, 1]$ by using the Peano representation and the bounded variation. In
section 3, we introduce the interesting Wainerman’s lemma and deduce the error bound on \( \|f - L_n[f]\|_\infty \) by \( L_n \). We consider, in section 4, the estimates of \( \|\ell^{(n)}\|_\infty \) and derive convergence rates for the interpolating polynomials based on strongly normal point systems, Gauss-Jacobi and Jacobi-Gauss-Lobatto point systems sequentially. Numerical experiments also demonstrates the optimality of these theoretical convergence rates.

Throughout this paper, \( A \sim B \) means that there exist positive constants \( C_1 \) and \( C_2 \) such that \( C_1 B \leq A \leq C_2 B \).

For simplicity, in the following we abbreviate \( x_k^{(n)} \) as \( x_k \) and \( \ell_k^{(n)}(x) \) as \( \ell_k(x) \).

2. The Peano kernel theorem. There are two general methods for deriving strict error bounds (Dahlquist and Björck [1]). One applies the norms and distance formula together with the Lebesgue constants, which often overestimates the error. The other is due to the Peano kernel theorem.

Let \( L \) be a continuous linear functional on the continuous function space \( C([-1, 1]) \) satisfying \( L(f_1 + f_2) = L(f_1) + L(f_2) \) for any \( f_1, f_2 \in C([-1, 1]) \) and \( L(\alpha f) = \alpha L(f) \) for any scalar \( \alpha \).

In other words, we assume \( L(P_{-1}) = \{0\} \) for some \( r \in \{1, 2, \ldots\} \), where \( P_{r-1} \) denotes the set of polynomials with degree less than or equal to \( r - 1 \).

The Peano kernel theorem (Peano [44], see also Kowalewski [35], Schmidt [50] and Mises [43]) is the identity

\[
L(f) = \int_{-1}^{1} f^{(r)}(t)K_r(t)dt
\]

(2.1)

holding for all such functions \( f \in C^r([-1, 1]) \), where \( K_r(t) = \frac{1}{(r-1)!}L[(x-t)^{r-1}_+] \) and

\[
(x-t)_+^{r-1} = \begin{cases} (x-t)^{r-1}, & x \geq t \\ 0, & x < t \end{cases} \quad (r \geq 2), \quad (x-t)_+^0 = \begin{cases} 1, & x \geq t \\ 0, & x < t \end{cases} \quad (r = 1).
\]

For each fixed \( x \in [-1, 1] \), we consider the special functional \( L(f) = E_n[f](x) \), where \( E_n[f](x) \) is defined by

\[
E_n[f](x) = f(x) - \sum_{j=1}^{n} f(x_j)\ell_j(x) = f(x) - L_n[f](x)
\]

with \(-1 \leq x_n < x_{n-1} < \cdots < x_2 < x_1 \leq 1\). \( E_n[f] \) is a continuously linear functional since \( |E_n[f](x) - E_n[g](x)| \leq (1 + A_n)\|f - g\|_\infty \) for arbitrary \( f, g \in C([-1, 1]) \), and then by the Peano theorem [44] \( E_n[f] \) can be represented if \( f \in C^r([-1, 1]) \) for \( n \geq r \) as

\[
E_n[f](x) = \int_{-1}^{1} f^{(r)}(t)K_r(t)dt
\]

(2.3)

with

\[
K_r(t) = \frac{1}{(r-1)!}(x-t)_+^{r-1} - \frac{1}{(r-1)!} \sum_{j=1}^{n} (x_j - t)_+^{r-1} \ell_j(x).
\]

(2.4)

Particularly, (2.3) implies

\[
|E_n[f](x)| \leq \|f^{(r)}\|_\infty \int_{-1}^{1} |K_r(t)|dt \leq 2\|f^{(r)}\|_\infty \|K_r\|_\infty.
\]

Similar to the Peano kernel for quadrature [6], the kernel for interpolation satisfies the following properties.

**Proposition 2.1.** \((\text{Peano representation})\) Let

\[
K_s(t) = \frac{1}{(s-1)!}(x-t)_+^{s-1} - \frac{1}{(s-1)!} \sum_{j=1}^{n} (x_j - t)_+^{s-1} \ell_j(x), \quad s = 1, 2, \ldots.
\]

(2.5)
Then for \( s \geq 2 \), the Peano kernel satisfies \( K_s(-1) = K_s(1) = 0 \) and can be rewritten as

\[
(2.6) \quad K_s(u) = \int_u^1 K_{s-1}(t)dt, \quad s = 2, 3, \ldots.
\]

**Proof.** From the definition of \( K_s \) in (2.5), it is easy to verify that \( K_s(-1) = K_s(1) = 0 \) by using \( \sum_{j=1}^n \ell_j(t) \equiv 1 \) for \( t \in [-1, 1] \). Furthermore, we find that

\[
(2.7) \quad \int_u^1 (x-t)^{s-2}dt = \begin{cases} 
0, & u > x \\
\frac{1}{(s-2)!} \int_u^x (x-t)^{s-2}dt = \frac{1}{(s-1)!} (x-u)^{s-1}, & u \leq x 
\end{cases} = \frac{1}{(s-1)!} (x-u)^{s-1}.
\]

Similarly by (2.7), we have

\[
(2.8) \quad \frac{1}{(s-2)!} \sum_{j=1}^n \int_u^1 (x_j-t)^{s-2} \ell_j(x)dt = \frac{1}{(s-2)!} \sum_{j=1}^m \ell_j(x) \begin{cases} 
0, & u > x_j \\
\int_u^{x_j} (x_j-t)^{s-2}dt, & u \leq x_j 
\end{cases} = \frac{1}{(s-1)!} \sum_{j=1}^n (x_j - u)^{s-1} \ell_j(x).
\]

Then from

\[
\int_u^1 K_{s-1}(t)dt = \frac{1}{(s-2)!} \int_u^1 (x-t)^{s-2}dt - \frac{1}{(s-2)!} \sum_{j=1}^n \int_u^1 (x_j-t)^{s-2} \ell_j(x)dt,
\]

we get \( \int_u^1 K_{s-1}(t)dt = K_s(u) \) by (2.7) and (2.8). \( \square \)

In the following, we consider functions of limited regularity as

**Suppose that \( f(t) \) has an absolutely continuous \((r-1)\)st derivative \( f^{(r-1)} \) on \([-1, 1] \)

(2.9) for some \( r \geq 1 \) with \( f^{(r-1)}(t) = f^{(r-1)}(-1) + \int_{-1}^t g(y)dy \), where \( g \) is absolutely integrable and of bounded variation \( \text{Var}(g) < \infty \) on \([-1, 1] \).

From Stein and Shakarchi [52, p. 130] and Tao [55, pp. 143-145], we see that a function \( G : [-1, 1] \rightarrow R \) is absolutely continuous if and only if it takes the form \( G(t) = \int_{-1}^t g(y)dy + C \) for some absolutely integrable \( g : [-1, 1] \rightarrow R \) and a constant \( C \). It is obvious that such \( g \) is not unique. Then in this paper, we suppose \( f(t) \) satisfies (2.9) and define

\[
V_r = \inf \left\{ \text{Var}(g) \left| f^{(r-1)}(t) = f^{(r-1)}(-1) + \int_{-1}^t g(y)dy \text{ for all } t \in [-1, 1] \text{ with } g \text{ being } \right. \right. \]

absolutely integrable and of bounded variation

**Remark 1.** Here, we use the condition "\( f^{(r-1)}(t) = f^{(r-1)}(-1) + \int_{-1}^t g(y)dy \)" instead of "\( f^{(r)} \) is of bounded variation \( V_r = \text{Var}(f^{(r)}) < \infty \)" in [58] [59]. If \( f^{(r)} \) is of bounded variation, then \( f^{(r+1)} \) exists almost everywhere and \( f^{(r+1)} \in L^1([-1, 1]) \) (see Lang [38] and Rudin [17]). Whereas, \( f^{(r)} \) in [58] [59] denotes an equivalent representation in the sense of almost everywhere. An example for \( f(x) = |t| \) is given in [58] [59], where \( f(t) \) is not differentiable at \( t = 0 \), but \( f' \) can be chosen as

\[
f'(t) = \begin{cases} 
1, & t > 0 \\
c, & t = 0 \\
-1, & t < 0
\end{cases},
\]

then \( \text{Var}(f') = \begin{cases} 
2, & |c| \leq 1 \\
|1+c| + |1-c|, & \text{otherwise}
\end{cases} \).
Using the new condition, we see that $|t|$ can be represented as $|t| = 1 + \int_{-1}^{t} g(y)dy$ with 

\[ g(y) = \begin{cases} 
1, & y > 0 \\
c, & y = 0 \\
-1, & y < 0 
\end{cases} \]

and $V_1 = 2$ is unique.

**Theorem 2.2.** Suppose $f(t)$ satisfies (2.9), then for $n \geq r$, we have

(2.10) \[ \|E_n[f]\|_{\infty} \leq V_r\|K_{r+1}\|_{\infty}. \]

**Proof.** The Peano kernel theorem implies that for each fixed $x \in [-1,1]$,

\[ E_n[f](x) = \int_{-1}^{1} f^{(s)}(t)K_s(t)dt, \quad s = 1,2,\ldots, r-1. \]

Then, following Brass and Petras [6] and integrating by parts, it yields

\[ E_n[f](x) = \int_{-1}^{1} f^{(r-1)}(t)K_{r-1}(t)dt = \int_{-1}^{1} g(t)K_r(t)dt. \]

Since $g$ can be written as $g = g_1 - g_2$ with $g_1$ and $g_2$ are monotonically increasing, and $\text{Var}(g) = \text{Var}(g_1) + \text{Var}(g_2)$ (see Lang [35] pp. 280-281). Without loss of generality, let us assume $g$ is monotonically increasing. Then by the second mean value theorem of integral calculus, it follows from $K_{r+1}(-1) = \int_{-1}^{1} K_r(t)dt = 0$ that there exists a $\xi \in [-1,1]$ such that

\[ E_n[f](x) = g(-1) \int_{-1}^{\xi} K_r(t)dt + g(1) \int_{\xi}^{1} K_r(t)dt = (g(1) - g(-1))K_{r+1}(\xi) = \text{Var}(g)K_{r+1}(\xi), \]

which leads to the desired result. \[ \square \]

**Lemma 2.3.** [6] Lemma 5.7.1 Assume that

\[ \sup_{-1 \leq t \leq 1} w(t)\sqrt{1-t^2} < \infty, \quad t_u(y) = \begin{cases} 
0, & y < u \\
1, & y \geq u 
\end{cases}. \]

Then, for every positive integer $\ell$ and every $u \in [-1,1]$, there is a $q_u \in \mathcal{P}_\ell$ satisfying

\[ q_u(y) \geq t_u(y) \quad \text{for all } y \in [-1,1] \]

and

\[ \int_{-1}^{1} [t_u(y) - q_u(y)] w(y)dy \geq -\frac{\pi}{\ell+1} \sup_{-1 \leq t \leq 1} w(t)\sqrt{1-t^2}. \]

**Lemma 2.4.**

(2.11) \[ |K_{s+1}(u)| \leq \frac{\pi}{n-s} \sup_{-1 \leq t \leq 1} |K_s(t)|. \]

**Proof.** In Lemma 2.3, letting $\ell = n-s-1$, $w(t) \equiv 1$, representing $q_u$ as $q_u(t) = q^{(s)}_{n-1}(t)$, and noting that $E_n[p_{n-1}] = 0$, by Theorem 2.2 we have

\[ 0 = E_n[p_{n-1}] = \int_{-1}^{1} p^{(s)}_{n-1}(t)K_s(t)dt = \int_{-1}^{1} q_u(t)K_s(t)dt. \]

Consequently, by Lemma 2.3 we get that

\[ |K_{s+1}(u)| = \int_{u}^{1} K_s(t)dt = \int_{-1}^{u} K_s(t)q_u(t)dt = \left| \int_{-1}^{1} K_s(t)[q_u(t) - q_u(t)]dt \right| \leq \frac{\pi}{n-s} \sup_{-1 \leq t \leq 1} |K_s(t)|. \]
From Theorem 2.2 and Lemma 2.4 we obtain that

**Theorem 2.5.** Suppose \( f(t) \) satisfies (2.9), then for \( n \geq r + 1 \)

\[
\|E_n[f]\|_\infty \leq \frac{\pi^r V_r}{(n-1)(n-2)\cdots(n-r)} \|K_1\|_\infty.
\]

3. Wainerman’s lemma. In the following, we shall focus on the estimate of \( \|K_1\|_\infty \).

Notice that \( \sum_{j=1}^n \ell_j(t) \equiv 1 \) for \( t \in [-1, 1] \) and

\[
(3.1) \quad K_1(u) = (x-u)_+^0 - \sum_{j=1}^n (x_j - u)_+^0 \ell_j(x).
\]

If \( x_1 < u \leq 1 \), we have \( K_1(u) = 1 \) for \( u \leq x \), and \( K_1(u) = 0 \) for \( u > x \). While for \( -1 < u \leq x_n \), we have \( K_1(u) = 0 \) for \( u \leq x \), and \( K_1(u) = -1 \) for \( u > x \). Thus, in these cases we obtain

\[
(3.2) \quad |K_1(u)| \leq 1 \leq \max_{1 \leq j \leq n} \|\ell_j\|_\infty
\]

since \( \ell_j(x_j) = 1 \) for \( j = 1, 2, \ldots, n \).

Suppose that \( x_{m+1} < u \leq x_m \) for some positive integer \( m \), then for \( u \leq x \) we get

\[
(3.3) \quad K_1(u) = 1 - \sum_{j=1}^n (x_j - u)_+^0 \ell_j(x) = 1 - \sum_{j=1}^m \ell_j(x) = \sum_{j=m+1}^n \ell_j(x),
\]

while for \( u > x \) we have

\[
(3.4) \quad K_1(u) = -\sum_{j=1}^n (x_j - u)_+^0 \ell_j(x) = -\sum_{j=1}^m \ell_j(x).
\]

In particular, if \( x_1 = 1 \) or \( x_n = -1 \), \( K_1(u) \) can be estimated by (3.3), (3.4) or \( K_1(-1) = 0 \).

**Lemma 3.1.** (Wainerman’s lemma [72]) Suppose \( x_{m+1} < u \leq x_m \) for some positive integer \( m \), and let

\[
a_k(u) = \begin{cases} 
\sum_{j=1}^k \ell_j(u), & k = 1, 2, \ldots, m \\
\sum_{j=k}^n \ell_j(u), & k = m+1, m+2, \ldots, n 
\end{cases}
\]

and \( a_0(u) = a_{n+1}(u) \equiv 0 \). Then it follows for \( x_{m+1} < u < x_m \) that

\[
(3.5) \quad \text{sgn}(a_k(u)) = \text{sgn}(\ell_k(u)) = \begin{cases} 
(1)^{m-k}, & k = 1, 2, \ldots, m \\
(1)^{k-m-1}, & k = m+1, m+2, \ldots, n 
\end{cases}
\]

and for \( x_{m+1} < u \leq x_m \) that

\[
(3.6) \quad |a_k(u)| \leq |\ell_k(u)|, \quad k = 1, 2, \ldots, n,
\]

where \( \text{sgn} \) denotes the sign function.

**Proof.** The interesting result and its proof is published in Russian in [72]. For convenience and completeness, we present the proof here.

For \( x_{m+1} < u < x_m \), from the definition of \( \ell_k(t) \) we see that

\[
\text{sgn}(\ell_k(u)) = \text{sgn} \left( \frac{(u - x_1) \cdots (u - x_{k-1})(u - x_{k+1}) \cdots (u - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)} \right) = (-1)^{k-m} \text{sgn} \left( \frac{(u - x_1) \cdots (u - x_{k-1})(u - x_{k+1}) \cdots (u - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)} \right),
\]

Thus, we get

\[
\text{sgn}(a_k(u)) = \text{sgn}(\ell_k(u)) = \text{sgn} \left( \sum_{j=1}^k \ell_j(u) \right) = \text{sgn} \left( \sum_{j=1}^m \ell_j(u) \right) = (-1)^{m-k}.
\]

Since \( a_k(u) = \sum_{j=1}^k \ell_j(u) \), we have

\[
|a_k(u)| = \left| \sum_{j=1}^k \ell_j(u) \right| \leq \sum_{j=1}^k |\ell_j(u)| = |\ell_k(u)|.
\]
which directly leads to the desired result (3.5) for \( \text{sgn}(\ell_k(u)) \) based on \( k \leq m \) or \( k > m \), respectively.

In the following, we will show that \( \text{sgn}(a_k(u)) \) also satisfies (3.5).

**In the case** \( k \leq m \): Since

\[
a_k(x_j) = \sum_{i=1}^{k} \ell_i(x_j) = \begin{cases} 1, & j = 1, 2, \ldots, k \\
0, & j = k+1, k+2, \ldots, n 
\end{cases}
\]

then by the Rolle’s theorem it follows

\[
a'_k(y_j) = 0
\]

for some \( y_j \) satisfying \( x_{j+1} < y_j < x_j \) for \( j = 1, \ldots, k, k+1, \ldots, n-1 \).

\[
a_k(x_k) = 1 \quad a_k(x_{k-1}) = 1 \quad a_k(x_2) = 1 \quad a_k(x_1) = 1
\]

Note that \( a_k(t) \) is a polynomial of degree \( n-1 \), then \( a'_k(t) \) is a polynomial of degree \( n-2 \), which implies that \( y_j \) are the exact zeros of \( a'_k(t) \) and \( a'_k(t) \) has the form of

(3.7) \[
a'_k(t) = C(t - y_1) \cdots (t - y_{k-1})(t - y_k) \cdots (t - y_{n-1})
\]

for some non-zero constant \( C \). In addition, from (3.7) \( a'_k(t) \) has alternative sign between these roots. Then, by \( a_k(x_{k+1}) = 0 \) and \( a_k(x_k) = 1 \), it yields

\[
a'_k(t) > 0, \quad t \in (y_{k+1}, y_k-1)
\]

and

\[
\text{sgn}(a_k(t)) = 1, \quad t \in (x_{k+1}, x_k) \subset (y_{k+1}, y_k-1)
\]

since \( a(t) \) is strictly increasing in \( (x_{k+1}, x_k) \) and \( a_k(x_{k+1}) = 0 \).

By the alternative property of \( a'_k(t) \) between these roots, it deduces that \( \text{sgn}(a'_k(t)) = (-1)^{j-k} \) for \( t \in (y_{j+1}, y_j) \) and \( j > k \), particularly,

\[
\text{sgn}(a'_k(t)) = 1, \quad t \in (y_{k+1}, x_{k+1}) \subset (y_{k+1}, y_k-1)
\]

and

\[
\text{sgn}(a'_k(t)) = -1, \quad t \in (x_{k+2}, y_{k+1}) \subset (y_{k+2}, y_{k+1}),
\]

which, together with \( a_k(x_{k+1}) = a_k(x_{k+2}) = 0 \), derives \( \text{sgn}(a_k(t)) = -1 \) for \( t \in (x_{k+2}, x_{k+1}) \).

Similarly, applying

\[
\text{sgn}(a'_k(t)) = (-1)^{j-k}, \quad t \in (y_{j+1}, x_{j+1}); \quad \text{sgn}(a'_k(t)) = (-1)^{j-k+1}, \quad t \in (x_{j+2}, y_{j+1}),
\]

together with \( a_k(x_{j+2}) = a_k(x_{j+1}) = 0 \) for \( j > k \), we have \( \text{sgn}(a_k(t)) = (-1)^{j-k+1} \) for \( j > k \) and \( t \in (x_{j+2}, x_{j+1}) \) by induction. So we get \( a_k(u) = (-1)^{m-k} \).
In the case $k > m$: By

$$a_k(x_j) = \sum_{i=k}^{n} \ell_i(x_j) = \begin{cases} 0, & j = 1, 2, \ldots, k-1 \\ 1, & j = k, k+1, \ldots, n \end{cases},$$

applying similar arguments, we derive $a_k(u) = (-1)^{k-m-1}$ for $k > m$.

Furthermore, from (3.5) and the definition of $a_k(t)$, we see that immediately: for $k \leq m$ and $x_{m+1} < u < x_m$,

$$|a_k(u)| = |\ell_k(u) + a_{k-1}(u)| = |\ell_k(u)| - |a_{k-1}(u)| \leq |\ell_k(u)|,$$

and for $k > m$

$$|a_k(u)| = |\ell_k(u) + a_{k+1}(u)| = |\ell_k(u)| - |a_{k+1}(u)| \leq |\ell_k(u)|.$$

The special case of (3.6) when $u = x_m$ directly follows from $|a_k(x_m)| = |\ell_k(x_m)|$ by the definitions of $a_k(u)$ and $\ell_k(u)$.

**Theorem 2.5** Suppose $f(t)$ satisfies (2.9), then for $n \geq r + 1$

$$\|E_n[f]\|_\infty \leq \frac{\pi^r V_r}{(n-1)(n-2)\cdots(n-r)} L_n, \quad L_n = \max_{1 \leq j \leq n} \|\ell_j\|_\infty. \tag{3.8}$$

In the next section, we shall focus on estimates of $L_n$ for special points of sets.

**4. Estimates of $\|\ell_j\|_\infty$ and convergence rates on $\|f - L_n[f]\|_\infty$.** For any convergent quadrature derived from polynomial interpolation at the grid points (1.1) for the following integral

$$\int_{-1}^{1} f(x)w(x)dx = \int_{-1}^{1} f(x)d\sigma(x)$$

for each $\sigma(x)$ of bounded variation and any analytic function $f(x)$ on $[-1, 1]$, the clustering of the $n$ points has a limiting Chebyshev distribution

$$\mu(t) = \frac{1}{\pi} \int_{-1}^{t} \frac{1}{\sqrt{1 - x^2}} dx$$

(see Krylov [38] Theorem 7, p. 263); that is, the clustering will be asymptotically the same: on $[-1, 1]$, $n$ points will be distributed with density (per unit length)

$$\frac{n}{\pi \sqrt{1 - x^2}}$$

as $n$ tends to infinity (see Hale and Trefethen [27] and Trefethen [56]).

Moreover, optimal point systems for polynomial interpolation often appears to be clustering near endpoints $\pm 1$ (see Z. Ditzian and V. Totik [19] and [38]). (The Gauss-Jacobi type point systems have this proposition.) The density of the zeros of orthogonal polynomials has been extensively studied in Erdős and Turán [16], Gatteschi [22] and Szegő [54].

**4.1. Strongly normal point systems.** One of the proofs of Weierstrass approximation theorem using interpolation polynomials was first presented by Fejér [19] in 1916 based on the Chebyshev point system of first kind $\{x_k = \cos \left(\frac{2k-1}{2n}\pi\right)\}_{k=1}^{n}$; If $f \in C([-1, 1])$, then there is a unique polynomial $H_{2n-1}(f, t)$ of degree at most $2n-1$ such that $\lim_{n \to \infty} \|H_{2n-1}(f) - f\|_\infty = 0$, where $H_{2n-1}(f, t)$ is determined by

$$H_{2n-1}(f, x_k) = f(x_k), \quad H_{2n-1}^2(f, x_k) = 0, \quad k = 1, 2, \ldots, n. \tag{4.1}$$
This polynomial is known as the Hermite-Fejér interpolation polynomial.

The convergence result has been extended to general Hermite-Fejér interpolation of \( f(x) \) at nodes (1.1) by Grünwald [25] in 1942, upon strongly normal point systems introduced in Fejér [20]. Given, respectively, the function values \( f(x_1), f(x_2), \ldots, f(x_n) \) and derivatives \( d_1, d_2, \ldots, d_n \) at these grids, the general Hermite-Fejér interpolation polynomial \( H_{2n-1}(f) \) has the form of

\[
H_{2n-1}(f, t) = \sum_{k=1}^{n} f(x_k) h_k(t) + \sum_{k=1}^{n} d_k b_k(t),
\]

where \( h_k(t) = v_k(t) (\ell_k(t))^2, b_k(t) = (t - x_k) (\ell_k(t))^2 \) and

\[
v_k(t) = 1 - (t - x_k) \frac{\omega''_n(x_k)}{\omega'_n(x_k)} \quad \text{(see Fejér [21]).}
\]

The point system (1.1) is called strongly normal if for all \( n \)

\[
v_k(t) \geq c > 0, \quad k = 1, 2, \ldots, n, \quad t \in [-1, 1]
\]

for some positive constant \( c \). The point system (1.1) is called normal if for all \( n \)

\[
v_k(t) \geq 0, \quad k = 1, 2, \ldots, n, \quad t \in [-1, 1].
\]

Fejér [20] (also see Szegö [54] p. 339) showed that for the zeros of Jacobi polynomial \( P_n^{(\alpha, \beta)}(t) \) of degree \( n \) (\( \alpha > -1, \beta > -1 \))

\[
v_k(t) \geq \min\{-\alpha, -\beta\} \quad \text{for } -1 < \alpha \leq 0, -1 < \beta \leq 0, k = 1, 2, \ldots, n \text{ and } t \in [-1, 1].
\]

Then the point system is strongly normal for \( \max\{\alpha, \beta\} < 0 \) and normal for \( \max\{\alpha, \beta\} \leq 0 \).

While for the Legendre-Gauss-Lobatto point system (the roots of \( (1-t^2)P_{n-2}^{(1,1)}(t) = 0 \)),

\[
v_k(t) \geq 1, \quad k = 1, 2, \ldots, n, \quad t \in [-1, 1] \quad \text{(Fejér [21]).}
\]

This result is extended to Jacobi-Gauss-Lobatto point system (the roots of \( (1-t^2)P_{n-2}^{(\alpha, \beta)}(t) = 0 \)) and Jacobi-Gauss-Radau point system (the roots of \( (1-t)P_{n-1}^{(\alpha, \beta)}(t) = 0 \) or \( (1+t)P_{n-1}^{(\alpha, \beta)}(t) = 0 \)) by Vértesi [63]: for all \( k \) and \( t \in [-1, 1] \),

\[
v_k(t) \geq \min\{2 - \alpha, 2 - \beta\} \quad \text{for } \{x_k\} \cup \{-1, 1\} \text{ with } 1 \leq \alpha \leq 2 \text{ and } 1 \leq \beta \leq 2,
\]

\[
v_k(t) \geq \min\{2 - \alpha, -\beta\} \quad \text{for } \{x_k\} \cup \{1\} \text{ with } 1 \leq \alpha \leq 2 \text{ and } -1 < \beta \leq 0,
\]

\[
v_k(t) \geq \min\{-\alpha, 2 - \beta\} \quad \text{for } \{x_k\} \cup \{-1\} \text{ with } -1 < \alpha \leq 0 \text{ and } 1 \leq \beta \leq 2.
\]

Then the Jacobi-Gauss-Lobatto point system is strongly normal for \( 1 \leq \alpha < 2 \) and \( 1 \leq \beta < 2 \), while the Jacobi-Gauss-Radau point system for \( 1 \leq \alpha < 2 \) and \( -1 \leq \beta < 0 \), and \( -1 < \alpha < 0 \) and \( 1 \leq \beta < 2 \), respectively.

It is worth noticing that if the point system is strongly normal, then \( v_i(t) \geq c > 0 \) for all \( i = 1, 2, \ldots, n \) and \( t \in [-1, 1] \), and

\[
1 \equiv \sum_{i=1}^{n} h_i(t) = \sum_{i=1}^{n} v_i(t) \ell_i^2(t) \geq c \sum_{i=1}^{n} \ell_i^2(t)
\]

(see [20]) and then

\[
\|\ell_i\|_{\infty} \leq \frac{1}{\sqrt{c}}, \quad i = 1, 2, \ldots, n.
\]
Theorem 4.1. Suppose $f(t)$ satisfies (2.9) and $\{x_j\}_{j=1}^n$ is a strongly normal point system, then for $n \geq r + 1$

\begin{equation}
\|E_n[f]\|_\infty \leq \frac{\pi^r V_r}{\sqrt{e(n-1)(n-2)\cdots(n-r)}}.
\end{equation}

Following de la Vallée Poussin [62], the error bound indicates $\|f - L_n[f]\|_\infty = O(\|f - P_{n-1}[f]\|_\infty)$ for the interpolant at a strongly normal point system for a function of limited regularity with $V_r < \infty$ for some $r \geq 1$.

To check the error bounds in Theorem 4.1 numerically, we consider two limited regularity functions: $f(x) = |x|$ ($V_1 < \infty$) and $f(x) = |x|^3$ ($V_3 < \infty$). All $(\alpha, \beta)$ are generated by $\text{rand}(1,2)$ except for $(\alpha, \beta) = (-0.5, -0.5)$, $(\alpha, \beta) = (0,0)$, $(\alpha, \beta) = (1,1)$ or $(\alpha, \beta) = (1.5, 1.5)$. Particularly, we used $-\text{rand}(1,2)$ in Figs. 4.1-4.2 for strongly normal Gauss-Jacobi point systems, while $\text{rand}(1,2) + 1$ in Figs. 4.3-4.4 for strongly normal Jacobi-Gauss-Lobatto point systems. In Figs. 4.5-4.6, we used $(2\text{rand}(1), -\text{rand}(1))$ (1st row, the roots of the Jacobi-Gauss-Radau point system in (4.8)) and $(-\text{rand}(1), 2\text{rand}(1))$ (2nd row, the roots of the Jacobi-Gauss-Radau point system in (4.9)) for strongly normal Jacobi-Gauss-Radau point systems, respectively.

Additionally, we implemented the interpolation by the barycentric algorithms from [77] with $m = 1$ for Gauss-Jacobi point systems under $\alpha > -1$ and $\beta > -1$, while for Jacobi-Gauss-Lobatto point systems under $\alpha > 0$ and $\beta > 0$. In other cases, we used the barycentric algorithms together with

\begin{align*}
P_{N-1}(x) &= \tilde{P}_{N-2}(x) + \frac{f(1) - P_{N-2}(x)}{2\omega(-1)}\omega(x), \quad \omega(x) = (x - x_1)\cdots(x - x_{N-1}) \\
P_N(x) &= P_{N-1}(x) - \frac{f(1) - P_{N-1}(x)}{2\omega(-1)}\omega(x)(x - 1)
\end{align*}

for the Jacobi-Gauss-Lobatto point systems with $\alpha \cdot \beta \leq 0$ at the $N$ nodes, and

\begin{align*}
P_N(x) &= \tilde{P}_{N-1}(x) + \frac{f(1) - P_{N-2}(x)}{2\omega(-1)}\omega(x) \\
\tilde{P}_N(x) &= \tilde{P}_{N-1}(x) + \frac{f(1) - P_{N-2}(x)}{2\omega(-1)}\omega(x)
\end{align*}

for the Gauss-Jacobi-Radau point systems at the $N$ nodes, respectively, where $\tilde{P}_{N-2}(x)$ and $\tilde{P}_{N-1}(x)$, calculated by the barycentric algorithms in [77], are the interpolant of $f$ at the $N - 2$ and $N - 1$ Gauss-Jacobi points of $P_{N-2}^{(\alpha, \beta)}(t) = 0$ and $P_{N-1}^{(\alpha, \beta)}(t) = 0$, respectively.

From Figs. 4.1-4.6, we see that these convergence rates are in conformity to the estimates and attainable.

4.2. General Gauss-Jacobi point systems. Let

\begin{equation}
-1 < x_n < x_{n-1} < \cdots < x_2 < x_1 < 1
\end{equation}

be the roots of the Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$ ($\alpha, \beta > -1$) and $x_k = \cos(\theta_k)$. Then from Szegő [54], it follows

\begin{equation}
P_n^{(\alpha, \beta)}(t) = (-1)^nP_n^{(\beta, \alpha)}(-t) \tag{5.4 (4.1.3))}
\end{equation}

\begin{equation}
\max_{-1 \leq t \leq 1} |P_n^{(\alpha, \beta)}(t)| = \begin{cases}
\binom{n + q}{n} \sim n^q, & q = \max\{\alpha, \beta\} \geq -\frac{1}{2} \\
|P_n^{(\alpha, \beta)}(t')| \sim n^{-\frac{1}{2}}, & q = \max\{\alpha, \beta\} < -\frac{1}{2}
\end{cases} \tag{54 (7.32.2))}
\end{equation}

where $x'$ is one of the two maximum points, and for $t = \cos(\theta)$ and any fixed constant $c$ with $0 < c < 1$,

\begin{equation}
P_n^{(\alpha, \beta)}(\cos(\theta)) = \begin{cases}
O(n^\alpha), & 0 \leq \theta \leq cn^{-1} \\
\theta^{-\alpha - \frac{1}{2}}O\left(n^{-\frac{1}{2}}\right), & cn^{-1} \leq \theta \leq \frac{\pi}{2}
\end{cases} \tag{54 Theorem 7.32.2)},
\end{equation}

$\text{rand}(m,n)$ returns an $m$-by-$n$ matrix containing pseudorandom values drawn from the standard uniform distribution on the open interval $(0,1)$. 

Fig. 4.1. $\|f(u) - L_0[f](u)\|_\infty$ at $u = -1 : 0.001 : 1$ with $n = 10 : 10 : 1000$ at the strongly normal Gauss-Jacobi point systems for $f(x) = |x|$, respectively.

Fig. 4.2. $\|f(u) - L_0[f](u)\|_\infty$ at $u = -1 : 0.001 : 1$ with $n = 10 : 10 : 1000$ at the strongly normal Gauss-Jacobi point systems for $f(x) = |x|^3$, respectively.

\[ \theta_k = n^{-1} [k \pi + O(1)] \]  \hspace{1cm} (54 \ (8.9.1)),

\[ |P_n^{(a,b)}(\cos(\theta_k))| \sim k^{-a-\frac{2}{2} n^{a+\frac{2}{2}}}, \quad 0 < \theta_k \leq \frac{\pi}{2} \]  \hspace{1cm} (54 \ (8.9.2)).

Moreover, expression (4.17) can be extended to

\[ |P_n^{(a,b)}(\cos(\theta_k))| \sim k^{-a-\frac{2}{2} n^{a+\frac{2}{2}}}, \quad 0 < \theta_k \leq c_1 \pi \]  \hspace{1cm} (4.18)
for any fixed $c_1$ with $0 < c_1 < 1$ ([67, (4.6)]).

Based on these identities, the estimates on $\ell_k(t)$ have been extensively studied in Kelzon [33, 34], Vértesi [66, 68], Sun [53], Prestin [45], Kvernadze [37], Vecchia et al. [69], etc.

**Lemma 4.2.** ([53] (also see [37])) For $t \in [-1, 1]$, let $x_m$ be the root of the Jacobi polynomial
Fig. 4.5. $|f(u) - L_n[f](u)|_\infty$ at $u = -1 : 0.001 : 1$ with $n = 10 : 10 : 1000$ at the strongly normal Jacobi-Gauss-Radau point systems for $f(x) = |x|$ (1st row by the roots of Jacobi-Gauss-Radau $(1 - x)^{p_{\alpha,\beta}}(x)$, and 2nd row by the roots of Jacobi-Gauss-Radau $(1 + x)^{p_{\alpha,\beta}}(x)$), respectively.

Fig. 4.6. $|f(u) - L_n[f](u)|_\infty$ at $u = -1 : 0.001 : 1$ with $n = 10 : 10 : 1000$ at the strongly normal Jacobi-Gauss-Radau point systems for $f(x) = |x|^3$ (1st row by the roots of Jacobi-Gauss-Radau $(1 - x)^{p_{\alpha,\beta}}(x)$, and 2nd row by the roots of Jacobi-Gauss-Radau $(1 + x)^{p_{\alpha,\beta}}(x)$), respectively.

$P_n^{(\alpha,\beta)}$ which is closest to $t$. Then we have

$$
    \ell_k(t) = \begin{cases} 
    O \left( |k - m|^{-1} + |k - m|^{-4} \right), & k \neq m \\
    O(1), & k = m 
    \end{cases}, \quad \gamma = \max \{ \alpha, \beta \},
$$

for $k = 1, 2, \ldots, n$. 

$\text{Fig. 4.6.}$ $\|f(u) - L_n[f](u)\|_\infty$ at $u = -1 : 0.001 : 1$ with $n = 10 : 10 : 1000$ at the strongly normal Jacobi-Gauss-Radau point systems for $f(x) = |x|^3$ (1st row by the roots of Jacobi-Gauss-Radau $(1 - x)^{p_{\alpha,\beta}}(x)$, and 2nd row by the roots of Jacobi-Gauss-Radau $(1 + x)^{p_{\alpha,\beta}}(x)$), respectively.
Proof. In [53], the proof of Lemma 4.2 is given only for $0 \leq \theta_k \leq \frac{\pi}{3}$ or $k = m$. That proof can be readily extended to $0 \leq \theta_k \leq \frac{2\pi}{3}$ due to (4.18). We complement the proof for $\frac{2\pi}{3} < \theta_k < \pi$ and $k \neq m$ next.

From (4.13) and (4.18), we see that

\begin{equation}
\left| P_n^{(\alpha, \beta)}(\cos(\theta_k)) \right| \sim (n - k + 1)^{-\beta - \frac{3}{2}n^{\beta + 2}}, \quad \frac{2\pi}{3} < \theta_k < \pi \quad ([53] (9)).
\end{equation}

Then for $0 \leq t = \cos(\theta) \leq 1$ with $0 \leq \theta \leq cn^{-1}$ and $\frac{2\pi}{3} < \theta_k < \pi$, it follows by (4.15) and (4.20) that

$\ell_k(t) = O\left(\frac{n^\alpha}{(n - k + 1)^{\beta + \frac{3}{2}n^{\beta + 2}}}\right) = O\left(\frac{(n - k + 1)^{\beta + \frac{3}{2}}}{n^{\beta + 2 - \alpha}}\right) = O\left(n^{\alpha - \frac{1}{2}}\right)$.

While for $cn^{-1} \leq \theta \leq \frac{\pi}{2}$ and $\frac{2\pi}{3} < \theta_k < \pi$, it follows by (4.15)-(4.18) and (4.20) that

$\ell_k(t) = O\left(\frac{(m\pi/n)^{-\alpha - \frac{1}{2}n^{-\beta}}}{(n - k + 1)^{\beta - \frac{3}{2}n^{\beta + 2}}}\right) = O\left(\frac{1}{m^{\alpha}n^{\beta - \alpha}}\right) = \begin{cases} O\left(n^{\alpha - \frac{1}{2}}\right), & \alpha > -\frac{1}{2} \\ O\left(n^{-1}\right), & -1 < \alpha \leq -\frac{1}{2}. \end{cases}$

Thus for $0 \leq t \leq 1$, we have $\ell_k(t) = O\left(n^{-1} + n^{\alpha - \frac{1}{2}}\right)$ for $k \neq m$, which leads to the desired result due to that $k - m \sim n$ in the case $\frac{2\pi}{3} < \theta_k < \pi$.

Similarly, by (4.13) together with the above analysis, we get for $-1 \leq t \leq 0$ that

$\ell_k(t) = O\left(|k - m|^{-1} + |k - m|^{\beta - \frac{1}{2}}\right), \quad k \neq m.$

These together lead to the desired result (4.19) for $k \neq m$. \(\square\)

Theorem 4.3. Suppose $f(t)$ satisfies (2.9) and $\{x_j\}_{j=1}^n$ are the roots of the Jacobi polynomial $P_n^{(\alpha, \beta)}(t)$, then for $n \geq r + 1$

\begin{equation}
\|E_n[f]\|_\infty = O\left(n^{-r + \max\{0, \gamma - \frac{1}{2}\}}\right), \quad \gamma = \max\{\alpha, \beta\}
\end{equation}

Proof. From Lemma 4.2, we see that $L_n = O\left(n^{\max\{0, \gamma - \frac{1}{2}\}}\right)$, which together with Theorem 3.2 yields the desired result. \(\square\)

Remark 2. Theorem 4.3 implies

$\|f - L_n[f]\|_\infty \leq O(\|f - P_{n+1}^\ast\|_\infty)$

at the roots of the Jacobi polynomial $P_n^{(\alpha, \beta)}(t)$ for $-1 \leq \alpha, \beta \leq \frac{1}{2}$ if $f(t)$ satisfies (2.9). Then the interpolations at the $n$-point Gauss-Legendre points and at the $n$-point Chebyshev points of first kind or second kind have essentially the same accuracy. All of them can achieve the optimal convergence rate $O(\|f - P_{n+1}^\ast\|_\infty)$. Consequently, the corresponding quadrature Gauss, Clenshaw-Curtis and Fejér first rule have equally the same performance [74].

Here, we used Figs. 4.7-4.8 to illustrate the convergence rates for general Gauss-Jacobi point systems, where $(\alpha, \beta)$ are obtained by rand(1, 2) (1st row) and mrand(1, 2) with $\|\text{mrand}(1, 2)\|_\infty > m - 1$ for $m = 2, 3, 4$ (2nd row), respectively. From these figures, we see that the convergence rates are attainable too, which are in accordance with the estimates. Then the convergence rates at the Gauss-Jacobi point systems are optimal.

4.3. General Jacobi-Gauss-Lobatto point systems. Let

\begin{equation}
-1 = x_{n+1} < x_n < x_{n-1} < \cdots < x_2 < x_1 < x_0 = 1
\end{equation}

be the roots of $(1 - t^2)P_n^{(\alpha, \beta)}(t) = 0$ $(\alpha, \beta > -1), \; x_k = \cos(\theta_k)$ and

$\omega(t) = (t - x_0)(t - x_1) \cdots (t - x_n)(t - x_{n+1}), \quad \ell_k(t) = \frac{\omega(t)}{(t - x_k)\omega'(x_k)}.$
Then

\[(4.23) \quad \ell_0(t) = \frac{1 + t}{2} \cdot \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(1)}, \quad \ell_{n+1}(t) = \frac{1 - t}{2} \cdot \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(-1)},\]

and

\[(4.24) \quad \ell_k(t) = \frac{(1 - t^2)P_n^{(\alpha,\beta)}(t)}{(t - x_k)(1 - x_k^2)P_n^{(\alpha,\beta)'(x_k)}}, \quad k = 1, 2, \ldots, n.\]
In the next, we shall concentrate on estimates of $\ell_k(t)$ for $k = 0, 1, 2, \ldots, n + 1$.

- On the estimate of $\ell_0(t)$: (i) In the case $0 \leq t \leq 0$, setting $t = \cos(\theta)$ for $0 \leq \theta \leq \frac{\pi}{2}$, and using

\[ P_n^{(\alpha, \beta)}(1) = \left( \begin{array}{c} n + \alpha \\ n \end{array} \right) \sim n^\alpha \]  

we find that from (4.15) and (4.23) for $0 \leq \theta \leq \frac{\pi}{2}$,

\[ \ell_0(t) = \begin{cases} O(1), & 0 \leq \theta \leq cn^{-1} \\ O\left(\theta^{-\alpha-\frac{1}{2}}n^{-\frac{1}{2}}n^{-\alpha}\right) = O\left((n\theta)^{-\alpha-\frac{1}{2}}\right) = O\left(n^{-\min\{0,\alpha+\frac{1}{2}\}}\right), & cn^{-1} \leq \theta \leq \frac{\pi}{2}. \end{cases} \]

(ii) In the case $-1 \leq t \leq 0$, letting $t = -\cos(\theta)$ for $0 \leq \theta \leq \frac{\pi}{2}$ and applying

\[ P_n^{(\alpha, \beta)}(-\cos(\theta)) = (-1)^n P_n^{(\beta, \alpha)}(\cos(\theta)) \]  

and $1 - \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right)$ and $\frac{\theta}{2}(\theta) \leq \sin(\theta) \leq \theta$, together with (4.15) and (4.23), we have

\[ \ell_0(t) = \begin{cases} O\left(\theta^2 P_n^{(\beta, \alpha)}(\cos(\theta))n^{-\alpha}\right) & 0 \leq \theta \leq cn^{-1} \\ O\left(\theta^{-\beta-\frac{1}{2}}n^{-\frac{1}{2}}n^{-\alpha}\right) = O\left(n^{-\min\{2+\alpha-\beta,\alpha+\frac{1}{2}\}}\right), & cn^{-1} \leq \theta \leq \frac{\pi}{2}. \end{cases} \]

These together yield

\[ ||\ell_0||_\infty = O\left(\frac{1}{n^{\min\{0,2+\alpha-\beta,\alpha+\frac{1}{2}\}}}\right). \]

- Similarly, we have

\[ ||\ell_{n+1}||_\infty = O\left(\frac{1}{n^{\min\{0,2+\beta-\alpha,\beta+\frac{1}{2}\}}}\right). \]

- For $k = 1, 2, \ldots, n$, let $x_m$ be the nearest to $t \in [0, 1]$ and $t = \cos(\theta)$. From (4.24), we have for $k \neq m$ that

\[ \ell_k(t) = \frac{\sin^2 \theta P_n^{(\alpha, \beta)}(\cos \theta)}{(\cos \theta - \cos \theta_k) \sin^2 \theta_k P_n^{(\alpha, \beta)}(\cos \theta_k)} \]

\[ = \frac{\sin^2 \theta P_n^{(\alpha, \beta)}(\cos \theta)}{2 \sin \left(\frac{\theta - \theta_k}{2}\right) \sin \left(\frac{\theta + \theta_k}{2}\right) \sin^2 \theta_k P_n^{(\alpha, \beta)}(\cos \theta_k)}. \]

\[ (4.27) \]

In the case $0 \leq \theta \leq cn^{-1}$ and $0 \leq \theta_k \leq \frac{2\pi}{nf}$: From (4.15)-(4.18), it follows

\[ (4.28) \ell_k(\cos \theta) = O\left(\frac{n^{-2}n^\alpha}{|k-m||k+m|n^{-2}k^2n^{-2}k^{-\alpha-\frac{1}{2}}n^{\alpha+2}}\right) = O\left(\frac{k^{\alpha-\frac{1}{2}}}{|k-m||k+m|}\right). \]

Define

\[ h_1(u) = \frac{u^{\alpha-\frac{1}{2}}}{u^2 - m^2} \quad \text{for } m + 1 \leq u \leq n; \quad h_2(u) = -\frac{u^{\alpha-\frac{1}{2}}}{u^2 - m^2} \quad \text{for } 1 \leq u \leq m - 1. \]

Then by an elementary proof and noting that $m \leq cn$ for $0 < c_1 < 1$, we get

\[ \max_{m+1 \leq u \leq n} h_1(u) = \begin{cases} h_1(m + 1) = O\left(m^{\alpha-\frac{1}{2}}\right), & -1 < \alpha \leq \frac{5}{2} \\ \max \{h_1(m + 1), h_1(u)\} = O\left(\max \left\{m^{\alpha-\frac{1}{2}}, n^{\alpha-\frac{1}{2}}\right\}\right), & \alpha > \frac{5}{2} \end{cases} \]

and

\[ \max_{1 \leq u \leq m - 1} h_2(u) = \begin{cases} h_2(y_0) = O\left(m^{\alpha-\frac{1}{2}}\right), & -1 < \alpha \leq \frac{1}{2} \quad (y_0 = m \sqrt{\frac{\alpha-\frac{1}{2}}{n-\frac{1}{2}}}), \\ h_2(m - 1) = O\left(m^{\alpha-\frac{1}{2}}\right), & \alpha > \frac{1}{2} \end{cases} \]
which together establishes that

\begin{equation}
\ell_k(\cos \theta) = \begin{cases} 
O(1), & -1 < \alpha \leq \frac{3}{2}, \\
O\left(\frac{n^{\alpha-\frac{3}{2}}}{m}\right), & \alpha > \frac{3}{2}.
\end{cases}
\end{equation}

**In the case** \(0 \leq \theta \leq cn^{-1}\) and \(\frac{2\pi}{3} < \theta_k < \pi\): Similarly, from (4.15) and (4.20) we have

\begin{align}
\ell_k(\cos \theta) &= O\left(\frac{n^{-2}\alpha}{m^{\alpha-\frac{3}{2}}(n-k+1)^{\frac{3}{2}+\frac{1}{2}}(n-k+1)^{\beta}}\right) \\
&= O\left(\frac{n^{-2}\alpha}{m^{\alpha-\frac{3}{2}}(n-k+1)^{\beta}}\right) \\
&= O\left(n^{-\min(2+\beta-\alpha,\frac{3}{2}-\alpha)}\right).
\end{align}

**In the case** \(cn^{-1} \leq \theta \leq \frac{\pi}{2}\) and \(0 \leq \theta_k \leq \frac{2\pi}{3}\): By (4.27), together with (4.15)-(4.18), we obtain

\begin{equation}
\ell_k(\cos \theta) = O\left(\frac{m^{\frac{3}{2}-\alpha}k^{\alpha-\frac{3}{2}}}{k-m||k+m||}\right) = m^{\frac{3}{2}-\alpha}O\left(\frac{k^{\alpha-\frac{3}{2}}}{k-m||k+m||}\right)
\end{equation}

which establishes that

\begin{equation}
\ell_k(\cos \theta) = \begin{cases} 
O(1), & -1 < \alpha \leq \frac{5}{2}, \\
O\left(n^{\max(0,\alpha-\frac{3}{2})}\right), & \alpha > \frac{5}{2}.
\end{cases}
\end{equation}

**In the case** \(cn^{-1} \leq \theta \leq \frac{\pi}{2}\) and \(\frac{2\pi}{3} < \theta_k < \pi\): From (4.15)-(4.18), (4.20) and (4.27), we see that

\begin{equation}
\ell_k(\cos \theta) = m^{\frac{3}{2}-\alpha}O\left(\frac{n^{-\min(0,\beta+\frac{1}{2})}}{n^{2+\beta-\alpha}}\right) = \begin{cases} 
O\left(n^{-\min(0,\beta+\frac{1}{2})}\right), & -1 < \alpha \leq \frac{3}{2}, \\
O\left(n^{-\min(2+\beta-\alpha,\frac{3}{2}-\alpha)}\right), & \alpha > \frac{3}{2}.
\end{cases}
\end{equation}

Thus for \(t \in [0,1]\), we get

\begin{equation}
\|\ell_k\|_\infty = \begin{cases} 
O\left(n^{-\min(0,\beta+\frac{1}{2})}\right), & -1 < \alpha \leq \frac{3}{2}, \\
O\left(n^{-\min(2+\beta-\alpha,\frac{3}{2}-\alpha)}\right), & \alpha > \frac{3}{2}.
\end{cases}
\end{equation}

Similarly, for \(t \in [-1,0]\), by \(P_n^{(\beta,\alpha)}(-t) = (-1)^n P_n^{(\alpha,\beta)}(t)\) we get that

\begin{equation}
\|\ell_k\|_\infty = \begin{cases} 
O\left(n^{-\min(0,\alpha+\frac{1}{2})}\right), & -1 < \beta \leq \frac{3}{2}, \\
O\left(n^{-\min(2+\alpha-\beta,\frac{3}{2}-\beta)}\right), & \beta > \frac{3}{2},
\end{cases}
\end{equation}

which together with (4.34) leads to that for \(t \in [-1,1]\)

\begin{equation}
\|\ell_k\|_\infty = \begin{cases} 
O\left(n^{-\min(0,\alpha+\frac{3}{2},\beta+\frac{1}{2})}\right), & -1 < \alpha, \beta \leq \frac{3}{2}, \\
O\left(n^{-\min(2+\alpha-\beta,2+\beta-\alpha,\frac{3}{2}-\alpha,\frac{3}{2}-\beta)}\right), & \text{otherwise}
\end{cases}
\end{equation}

**Theorem 4.4.** Suppose \(f(t)\) satisfies (2.9) and \(\{x_j\}_{j=0}^{n+1}\) are the roots of \((1-t^2) P_n^{(\alpha,\beta)}(t)\), then for \(n \geq r+1\)

\begin{equation}
E_n[f] = \begin{cases} 
O\left(n^{-\min(0,2+\alpha-\beta,2+\beta-\alpha,\alpha+\frac{3}{2},\beta+\frac{1}{2})}\right), & -1 < \alpha, \beta \leq \frac{3}{2}, \\
O\left(n^{-\min(2+\alpha-\beta,2+\beta-\alpha,\alpha+\frac{3}{2},\beta+\frac{1}{2},\frac{3}{2}-\alpha,\frac{3}{2}-\beta)}\right), & \text{otherwise}
\end{cases}
\end{equation}
Particularly, we have

\[(4.38)\quad \|E_n[f]\|_\infty = O\left(\frac{1}{n^r}\right), \quad -\frac{1}{2} \leq \alpha, \beta \leq \frac{3}{2}.
\]

**Remark 3.** Theorem 4.4 together with Theorem 4.1 implies

\[\|f - L_n[f]\|_\infty = O(\|f - p^*_n\|_\infty)\]

at the roots of \((1 - t^2)P_n^{(\alpha, \beta)}(t)\) for \(-\frac{1}{2} \leq \alpha, \beta \leq \frac{3}{2}\) or \(1 \leq \alpha, \beta < 2\).

Figs. 4.9-4.10 show the convergence rates for \(f(x) = |x|\) or \(f(x) = |x|^3\) at the Jacobi-Gauss-Lobatto point systems, respectively, where each \((\alpha, \beta)\) is generated by \(2\text{rand}(1, 2) - 0.5\).

![Fig. 4.9.](image)

**Fig. 4.9.** \(\|f(u) - L_n[f](u)\|_\infty\) at \(u = 1 : 0.001 : 1\) with \(n = 10 : 10 : 1000\) at the Jacobi-Gauss-Lobatto point systems for \(f(x) = |x|\), respectively.

5. **Final remarks.** The results in section 4 indicate the fact that the interpolations, for functions of limited regularity, at strongly normal point systems, including Gauss-Jacobi point systems with \(-1 < \alpha, \beta < \frac{1}{2}\) and Jacobi-Gauss-Lobatto point systems with \(-\frac{1}{2} \leq \alpha, \beta \leq \frac{3}{2}\) or \(1 \leq \alpha, \beta < 2\), have the same convergence order compared with the best polynomial approximation of the same degree. Numerical experiments give in line with the estimates.

It is interesting to note that numerical experiments also show that the same occurs for analytic or smooth functions at these point systems. Here we illustrate the phenomena by entire function \(f(x) = e^x\), i.e., analytic throughout the complex plane, \(f(x) = 1/(1 + 25x^2)\), which is analytic in a neighborhood of \([-1, 1]\) but not throughout the complex plane, and \(f(x) = e^{-1/x^2}\), which is not analytic in a neighborhood of \([-1, 1]\) but is infinitely differentiable in \([-1, 1]\).

In Figs. 5.1-5.3, the left columns are computed by zeros of Gauss-Jacobi polynomial \(P_n^{(\alpha, \beta)}(x)\), the middles by Jacobi-Gauss-Lobatto \((1 - x^2)P_n^{(\alpha, \beta)}(x)\), while the rights by Jacobi-Gauss-Radau \((1 - x)P_{n-1}^{(\alpha, \beta)}(x)\) (first three cases) or \((1 + x)P_{n-1}^{(\alpha, \beta)}(x)\), respectively. From these figures, we see that the interpolations at these point systems including the Gauss-Legendre and Legendre-Gauss-Lobatto, achieve essentially the same approximation accuracy compared with those at the two Chebyshev point systems too.
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Fig. 5.2. $\|f(u) - L_n[f](u)\|_\infty$ at $u = -1 : 0.001 : 1$ with $n = 10 : 10 : 1000$ at Gauss-Jacobi point systems for $f(x) = \frac{1}{1 + 25x^2}$, respectively.

Fig. 5.3. $\|f(u) - L_n[f](u)\|_\infty$ at $u = -1 : 0.001 : 1$ with $n = 10 : 10 : 1000$ at Jacobi-Gauss-Radau point systems for $f(x) = e^{-1/x^2}$, respectively.

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