Holography and the Canonical Ensemble of Fermionic Strings

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Abstract

We show that the canonical ensemble in any of the six supersymmetric string theories, type IIA and IIB, type IB and type I', or heterotic $E_8 \times E_8$ and Spin(32)/$Z_2$, exhibits a strong version of holography: the growth of the number of degrees of freedom in the free energy at high temperatures is identical to that in a two-dimensional quantum field theory. We clarify the precise nature of the thermal duality phase transition in each case, confirming that it lies within the Kosterlitz-Thouless universality class. We show that, in the presence of Dbranes, and a consequent Yang-Mills gauge sector, the thermal ensemble of type II strings is infrared stable, with neither tachyons nor massless scalar tadpoles. Supersymmetry remains unbroken in the oriented closed string sector, but is broken by thermal effects in the full unoriented open and closed type I string theory. We identify an order parameter for an unusual phase transition in the worldvolume gauge theory signalled by the short distance behavior of the pair correlator of timelike Wilson loops. Note Added (Sep 2005).
1 Introduction

In this paper we will clarify the properties of the thermal canonical ensemble in the six different supersymmetric string theories—type IIA and type IIB, heterotic $E_8 \times E_8$ and Spin(32)/$\mathbb{Z}_2$, type I and type I'. The Polyakov path integral over worldsheet offers a first principles approach to the string canonical ensemble, originally pointed out in [1]. It is important to appreciate the significance of this observation. Recall that we only know how to formulate perturbative string theory in a given spacetime background: a “heat bath”, represented by the given spacetime metric and background fields, is forced upon us, implying that a self-consistent treatment of perturbative string thermodynamics must necessarily be restricted to the canonical ensemble of statistical mechanics. Thus, while an immense, and largely conjectural, literature exists on proposals for microcanonical ensembles of weakly-coupled strings [10, 11], the conceptual basis for such discussion is full of holes. Some of these pitfalls were already pointed out in [2]. Rather than engage in such conjectural debate, we will instead postpone treatment of the microcanonical ensemble until when we have consensus on a nonperturbative, and background independent, formalism for string/M theory. Matrix proposals for M theory may eventually offer such a possibility. It should be kept in mind that, strictly speaking, the microcanonical ensemble is what is called for when discussing the thermodynamics of the Universe [4]: the Universe is, by definition, an isolated closed system, and it is meaningless to invoke the canonical ensemble of the “fundamental degrees of freedom”. But there are many simpler questions in braneworld cosmology that might indeed be approachable within the framework of string thermodynamics in the canonical ensemble.

In what follows, we will both review, and make important corrections to, the standard worldsheet derivations for the canonical ensemble of the different supersymmetric string theories at finite temperature, following the approach outlined in Polchinski’s 1986 treatment of the canonical ensemble of closed bosonic strings.\(^2\) Given the enormous literature to date on string thermodynamics, even within the framework of the canonical ensemble, it may be helpful to clarify the precise corrections made in our analysis to the previous standard treatments. Polchinski gave the original derivation of the free energy of the canonical ensemble of free closed bosonic strings using the string path integral in [1]. However, in physically interpreting the correctly derived expression, he omits to mention that there are low temperature tachyonic thermal momentum modes in the spectrum at all temperatures starting from zero [7]. Thus, Polchinski’s oft-quoted identification of the first winding mode instability with the onset of a “Hagedorn” phase transition is suspect: the free energy is already ridden with tachyonic divergences long before the “Hagedorn” temperature has been reached [7]. This misleading identification has also been made independently in the papers by Kogan, and by Sathiapalan, listed in [12].

The canonical ensembles of the type II and heterotic strings were discussed in detail by Atick and Witten in [2]. They were the first authors to introduce thermal mode number dependent temperature-dependent phases in the expression for the free energy, except that the particular choice

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\(^{2}\) In the earlier papers [7, 8], we have pointed out a peculiar ambiguity in the Euclidean time prescription for finite temperature theories illustrated by its application to the canonical ensemble of free strings. To avoid the misconception that any of the results in our current paper are tied to the question of whether Euclidean time has the topology of a circle or that of an interval, we will refrain from any mention of the ambiguity in this paper. We emphasize that all of the derivations in the current paper make the standard assumption that Euclidean time has the topology of a circle, with inverse temperature identified as follows: $\beta = 2\pi r_{\text{circ}}$. 

We should also note that [2] makes an incorrect assertion about thermal duality: it is indeed true that the free energy of any sensible physical system should be a monotonically increasing function of temperature, but there is nothing “unphysical” about the generating functional of connected vacuum graphs being thermal self-dual. The Polyakov path integral yields $W(\beta)$, not $F(\beta)$, and $W(\beta)$ is indeed thermal self-dual in the case of the closed bosonic string theory: recall that a thermal duality transformation is simply a Euclidean timelike T-duality transformation. The free energy of the string ensemble is given by $F(\beta)=-W(\beta)/\beta$, which is, happily, a monotonically increasing function of temperature. Thermal duality has been extensively explored in the early days of string theory, notably by E. Alvarez and collaborators [10].\(^3\) Duality also makes an appearance in several of the papers listed in [12]. But the fundamental significance of a duality phase transition in the canonical string ensemble characterized by the analyticity in temperature of an infinite hierarchy of thermodynamic potentials appears not to have been noticed prior to our work [7]. It is this analytic behavior which provides the unambiguous signature of a phase transition belonging to the Kosterlitz-Thouless universality class [13], exemplified by our analysis in section 3 of the heterotic string ensemble. Finally, it should be emphasized that it is only the vacuum functional of string theory that is thermal self-dual in certain cases, namely, closed bosonic string theory. In the cases described in this paper, thermal duality acts within a pair of supersymmetric string theories, mapping the vacuum functional of one to the other. But in all cases, the free energy and remaining thermodynamic potentials, given by the basic thermodynamic identities listed in Eq. (1), are most certainly not thermal self-dual.

Our results for the canonical ensemble of the type I, or type $I'$, unoriented open and closed string theory with D-branes are entirely new. Also new is the observed connection between the spontaneously broken thermal duality, and the signal for a phase transition to a holographic phase at high temperatures: the growth with temperature in the free energy of the type $I'$ ensemble at high temperatures is that of a two-dimensional field theory. Our results bring to completion an unambiguous demonstration of the holographic behavior of all of the perturbative string theories: type I-I', heterotic, and type IIA-IIB, in addition to that of the closed bosonic string theory first demonstrated in [14], following the conjecture in [2]. The notion of a holographic principle has played

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\(^3\)These authors also attempt to “derive” the properties of the microcanonical ensemble from those of the canonical ensemble [10, 11] applying them to models for superstring cosmology. We have already noted that such attempts are conceptually flawed. The technical reason why the Legendre transform linking the microcanonical and canonical closed string ensembles cannot be carried out in closed form is that the integration over the moduli of the Riemann surfaces summed in the expression for the free energy cannot be carried out explicitly.
an active role in recent developments in String/M theory [37, 39]. But it should be emphasized that the holographic growth of the free energy at high temperature in string theory is a much more drastic reduction in the number of degrees of freedom: $F \propto T^2$ for the ten-dimensional string theories, not simply as fast as an area, $\propto T^9$, rather than as a volume, $\propto T^{10}$, as postulated in [37, 39]. Type I string thermodynamics was not discussed in [1, 2], although discussions have appeared elsewhere in the literature. To the best of our knowledge, all of these references neglect the errors we have pointed out in the standard references [1, 2]. Treatments which are restricted to the supergravity field theory limit—treating the Dbranes as semi-classical sources in a thermal bath of gravitons, dilatons, and nonabelian gauge fields, should be viewed as effective descriptions of low energy thermal string/M theory at finite temperature.

All of the analysis in this paper, and in [7, 8], is based on the one-loop amplitude in string theory, a loop contribution to the thermodynamic potentials that is independent of the strength of the string coupling constant [35]. The reader may wonder why we omit mention of tree-level contributions to the vacuum energy density. In the case of pure closed string theories, or for the closed string sector of the type I and type I$'$ open and closed string theories, there are none [14]: the underlying reason can be traced to reparametrization invariance of worldsheets with the topology of a sphere. On the other hand, in the presence of Dbranes, one indeed finds a nontrivial tree-level contribution to the vacuum energy density from the disk amplitude, namely, the Dbrane tension [31, 14, 35]. The reader can find a nice discussion of the tree-level contributions to the vacuum energy density in these references.

The thermodynamics of an ensemble of strings is not the same as that of an ensemble of infinitely many free particle species with Planck scale masses. While there is a well-established back-of-the-envelope argument deriving the existence of a Hagedorn divergence in particle ensembles at finite temperature described by a Boltzmann distribution [9] (see the review in the appendix of [7]), we must be careful not to apply this to the string ensemble. The partition function, $Z(\tau_i)$, on a strip of given shape and size, namely, with fixed values of the worldsheet moduli, $\tau_i$, indeed describes a conformally invariant two-dimensional field theory with an exponentially rising density of states. But the vacuum functional in string theory is given by a reparametrization invariant integral over strips of all possible shape and size. Remarkably, the physical state spectrum of string theory turns out to contain many fewer states than in the two-dimensional field theories described by the integrand of the one-loop vacuum amplitude as a consequence of reparametrization invariance [7, 35, 17].

In this paper, we will perform a first-principles analysis of the generating functional of connected one-loop vacuum string graphs, $W(\beta) \equiv \ln Z(\beta)$, following the approach given in [3, 1, 35]. The vacuum energy density can, of course, be directly inferred from $W(\beta)$. Let us recall the basic thermodynamic identities of the canonical ensemble [5]:

$$F = -W/\beta = V\rho, \quad P = -\left(\frac{\partial F}{\partial V}\right)_T, \quad U = T^2 \left(\frac{\partial W}{\partial T}\right)_V, \quad S = -\left(\frac{\partial F}{\partial T}\right)_V, \quad C_V = T \left(\frac{\partial S}{\partial T}\right)_V. \quad (1)$$

Note that $W(\beta)$ is an intensive thermodynamic variable without explicit dependence on the spatial volume. $F$ is the Helmholtz free energy of the ensemble of free strings, $U$ is the internal energy, and $\rho$ is the finite temperature effective potential, or vacuum energy density at finite temperature. $S$ and $C_V$ are, respectively, the entropy and specific heat of the thermal ensemble. The pressure of the
string ensemble simply equals the negative of the vacuum energy density, as is true for a cosmological constant, just as in an ideal fluid with negative pressure [6, 5]. The enthalpy, $H = U + PV$, the Helmholtz free energy, $F = U - TS$, and the Gibbs function, also known as the Gibbs free energy, is $G = U - TS + PV$. As a result of these relations, all of the thermodynamic potentials of the string ensemble have been given a simple, first-principles, formulation in terms of the path integral over worldsheats. Notice, in particular, that since $P = -\rho$, the one-loop contribution to the Gibbs free energy of the string ensemble vanishes identically!

Let us summarize our results. While the zero temperature type IIA or type IIB string is supersymmetric and tachyon-free, we will show that the thermal spectrum of either type II string theory contains an infinite tower of tachyonic physical states at infinitesimal temperature: this pathological behavior is already indicated by the preliminary analysis in [2], raising the puzzling question of whether one can describe a stable thermal ensemble of type II strings in flat noncompact 10d spacetime? It turns out, as unambiguously clarified by us in an accompanying work using the worldsheet renormalization group (RG) and the g-theorem [17, 18], that the type IIA vacuum at temperatures below $T_w = 1$, namely, the temperature at which the first of the winding modes turns tachyonic, is unstable, and the flow of the worldsheet renormalization group is towards the noncompact and supersymmetric zero temperature vacuum. How, then, does one describe a stable thermal ensemble of type IIA strings? The key to this puzzle lies in the Ramond-Ramond sector which has, thus far, not been taken into account. To understand why a nontrivial Ramond-Ramond sector might enable the elimination of tachyons from the thermal spectrum at all temperatures, let us turn to the heterotic and type I string ensembles.

As we show in section 2 and 3, the key feature necessary for a tachyon-free thermal ensemble that is also self-consistent with the thermal duality relations of String/M theory, lies in introduction of a temperature dependent Wilson line: an option available in any string theory with Yang-Mills gauge fields. As a consequence, it will turn out that both the heterotic $E_8 \times E_8$ and Spin(32)/$\mathbb{Z}_2$ theories, and the type I and type I' $O(32)$ theories, have a tachyon–free finite temperature ground state at all temperatures starting from zero, and with gauge group $SO(16) \times SO(16)$. From the perspective of the low energy finite temperature gauge-gravity theory [29], the presence of the temperature dependent Wilson line can be interpreted as quantization in the modified axial gauge, $A_0 = \text{constant}$, where the constant has been chosen to be temperature dependent. In fact, it is straightforward to write down the correct result for the vacuum functional at finite temperature if we recall some old results in the heterotic string literature. In [21, 23, 24, 25], the possible choices of spin structure for the worldsheet fermions consistent with both the (1, 0) superconformal invariance and modular invariance of the heterotic closed string theory were studied, providing a classification of ten-dimensional heterotic strings. There is a unique solution which is both nonsupersymmetric and tachyon-free, and it has nonabelian gauge group $O(16) \times O(16)$. In an interesting follow-up work [20], Ginsparg studied the compactification of the supersymmetric $E_8 \times E_8$ string on a circle, demonstrating, thereby, that by tuning both the radius and a possible Wilson line, it was possible to interpolate smoothly between the circle-compactified $E_8 \times E_8$ and Spin(32)/$\mathbb{Z}_2$ string theories. Related developments in this period were the interesting numerical studies [26, 27] of the effective potential in nonsupersymmetric ground states of the heterotic string, examining the effects of tuning the spatial radi, $r_i$, background antisymmetric field, $b_{ij}$, or Wilson lines, $A_i$ [19]. It should be mentioned that the nonsupersymmetric and tachyon-free $O(16) \times O(16)$ string also makes an appearance in recent work on the interpolation between the circle-compactified $E_8 \times E_8$ and Spin(32)/$\mathbb{Z}_2$ theories, in the context
of a conjectured strong-weak heterotic-type I duality in the absence of supersymmetry [28]. We will be able to make contact with some of these results in section 3, especially the analysis of Wilson lines that appears in the interpolation studied in [20].

The plan of this paper is as follows. Section 2 begins with a derivation of the free energy for the canonical ensemble of heterotic strings. We show that in the presence of a temperature-dependent Wilson line background, the thermal spectrum is tachyon-free at all temperatures starting from zero and has gauge symmetry $SO(16) \times SO(16)$. In section 2.2, we derive expressions for the hierarchy of thermodynamic potentials of the canonical ensemble, demonstrating both a holographic duality relation, and the signature of a Kosterlitz-Thouless duality transition [13]. We clarify that both the free energy and entropy are finite with no divergences. In particular, there is no evidence for a Hagedorn phase transition in the tachyon-free heterotic string ensemble.

In section 3 we introduce the type I and type II unoriented open and closed string theories with Dbranes. We begin with the closed oriented sector of this theory, computing the one-loop vacuum functional for an equilibrium ensemble of either type IIA or type IIB closed oriented strings at finite temperature, with a trivial Ramond-Ramond sector. In Section 3.2, we address the low temperature thermal instability of either type II ensemble in the absence of Dbranes. The instability of such an ensemble, and the worldsheet renormalization group flow to the supersymmetric infrared fixed point at zero temperature, is described in [17]. Thus, the closed oriented sector of the type I or type II theories does not break supersymmetry as a consequence of thermal effects. Section 3.3 describes the computation of the one-loop free energy for the remaining sectors of the type IB unoriented open and closed string theory. We derive the generating functional of one-loop vacuum string graphs, having noting that the contribution from worldsheets of torus topology vanishes. Remarkably, we find that the one-loop contribution to the free energy from the sum of the annulus, Mobius Strip, and Klein Bottle topologies also vanishes as a consequence of requiring tadpole cancellation for the unphysical Ramond-Ramond scalar, despite the fact that supersymmetry has been broken! It is most intriguing to find a nonsupersymmetric type I ground state, with supersymmetry broken by finite temperature effects, despite a vanishing one-loop vacuum energy density.

The holographic growth of the number of degrees of freedom in type I string theory at high temperatures is demonstrated in section 4.1, where we also evaluate the scaling behavior for the first few thermodynamic potentials, verifying that the duality phase transition is in the Kosterlitz-Thouless universality class [13]. Finally, in section 4.2, we find a plausible order parameter for a phase transition in the worldvolume gauge theory by looking for a signal in the short distance behavior of the pair correlator of closed timelike Wilson loops. We find a transition temperature at $T_C$, or at a temperature slightly below $T_C$ in the presence of an external electromagnetic field. We pause to remark that this continuous phase transition is the only possible phase transition compatible with the analyticity of string theory amplitudes and consequently, accessible within the worldsheet formalism of perturbative string theory. The intuition that a gas of strings can transition into a high temperature long string phase is an old piece of string folklore [14, 37, 38], although usually formulated within the context of a model for the microcanonical ensemble. It is possible that our results can be interpreted as evidence for such a phase transition. Conclusions and a brief discussion of open questions appear in section 5.
2 Heterotic Closed String Ensemble

The heterotic closed string theory is possibly the closest supersymmetric string analog of the closed bosonic string theory considered by Polchinski in [1], so let us begin with this case. Consider the ten-dimensional supersymmetric $E_8 \times E_8$ theory at zero temperature. The $\alpha' \to 0$ low energy field theory limit is 10D $N=1$ supergravity coupled to $E_8 \times E_8$ Yang-Mills gauge fields. What happens to the supersymmetric ground state of this theory at finite temperatures? Assuming that a stable thermal ensemble exists, the finite temperature heterotic ground state with nine noncompact spatial dimensions is expected to be tachyon-free, while breaking supersymmetry. Moreover, consistency with the low energy limit, which is a finite temperature gauge-gravity theory, implies that the thermal string spectrum must contain Matsubara-like thermal momentum modes. But the thermal spectrum is also likely to contain winding modes as expected in a closed string theory [1]. Most importantly, since we are looking for a self-consistent string ground state with good infrared and ultraviolet behaviour, it is important that the one-loop vacuum functional preserve the usual worldsheet symmetries of $(1,0)$ superconformal invariance and one-loop modular invariance. Finally, all of our considerations are required to be self-consistent with thermal duality transformation defined as a Euclidean timelike T-Duality transformation. Since the action of spatial target space dualities on the different supersymmetric string theories are extremely well-established, the finite temperature vacuum functional is required to interpolate between the following two spacetime supersymmetric limits: in the $\beta \to \infty$ limit we recover the vacuum functional of the supersymmetric $E_8 \times E_8$ heterotic string, while in the $\beta=0$ limit we must recover, instead, the vacuum functional of the supersymmetric Spin(32)/$\mathbb{Z}_2$ heterotic string. The reason is that the $E_8 \times E_8$ and Spin(32)/$\mathbb{Z}_2$ heterotic string theories are related by the Euclidean timelike T-duality transformation: $\beta_{E_8 \times E_8} \to \beta_{\text{Spin}(32)/\mathbb{Z}_2} = 4\pi^2 \alpha'/\beta_{E_8 \times E_8}$.

2.1 Axial Gauge and the Euclidean Timelike Wilson Line

The generating functional of connected one-loop vacuum string graphs is given by the expression:

$$W_{\text{het.}} = L^9 (4\pi^2 \alpha')^{-9/2} \int \frac{d^2 \tau}{4\tau_2} (2\pi \tau_2)^{-9/2} |\eta(\tau)|^{-14} Z_{\text{het.}}(\beta).$$  \hspace{1cm} (2)

The canonical ensemble of oriented closed strings occupies the box-regularized spatial volume $L^9$. The thermodynamic limit is approached as follows: we take the limit $\alpha' \to 0$, $L \to \infty$, holding the dimensionless combination, $L^9 (4\pi^2 \alpha')^{-9/2}$, fixed. The function $Z_{\text{het.}}(\beta)$ contains the contributions from the rank $(17,1)$ Lorentzian self-dual lattice characterizing this particular ground state of the circle compactified $E_8 \times E_8$ heterotic string. Thus, we wish to identify a suitable interpolating expression for $W_{\text{het.}}$ valid at generic values of $\beta$, matching smoothly with the known vacuum functional of the supersymmetric $E_8 \times E_8$ string theory at zero temperature ($\beta=\infty$):

$$W_{\text{het.}} |_{T=0} = L^{10} (4\pi^2 \alpha')^{-5} \int \frac{d^2 \tau}{4\tau_2} (2\pi \tau_2)^{-5} \cdot \frac{1}{|\eta(\tau)|^{16}} \cdot \left[ \left( \frac{\Theta_3}{\bar{\eta}} \right)^4 - \left( \frac{\Theta_4}{\bar{\eta}} \right)^4 - \left( \frac{\Theta_2}{\bar{\eta}} \right)^4 \right]$$

$$\times \left[ \left( \frac{\Theta_3}{\eta} \right)^8 + \left( \frac{\Theta_4}{\eta} \right)^8 + \left( \frac{\Theta_2}{\eta} \right)^8 \right]^2 . \hspace{1cm} (3)$$
Thus, $Z_{\text{het}}(\beta)$ describes the thermal mass spectrum of $E_8 \times E_8$ strings.

It turns out that the desired result already exists in the heterotic string literature. The modular invariant possibilities for the sum over spin structures in the 10d heterotic string have been classified, both by free fermion and by orbifold techniques [21, 24, 23, 25], and there is a unique nonsupersymmetric and tachyon-free solution with gauge group $SO(16) \times SO(16)$. Recall the radius-dependent Wilson line background described by Ginsparg in [20] which provides the smooth interpolation between the heterotic $E_8 \times E_8$ and $SO(32)$ theories in nine dimensions. We have: $A = \frac{1}{2}(1, 0^7, -1, 0^7)$, $x = (\frac{2}{\alpha'})^{1/2} r_{\text{circ}}$. Introducing this background connects smoothly the 9D $SO(16) \times SO(16)$ string at generic radii with the supersymmetric 10d limit where the gauge group is enhanced to $E_8 \times E_8$. Note that the states in the spinor lattices of $SO(16) \times SO(16)$ correspond to massless vector bosons only in the noncompact limit. Generically, the $(17, 1)$-dimensional heterotic momentum lattice takes the form $E_8 \oplus E_8 \oplus U$. Here, $U$ is the $(1, 1)$ momentum lattice corresponding to compactification on a circle of radius $r_{\text{circ}} = x(\alpha'/2)^{1/2}$. A generic Wilson line corresponds to a lattice boost as follows [20]:

$$ (p; l_L, l_R) \rightarrow (p'; l_L', l_R') = (p + wxA; u_L - p \cdot A - \frac{wx}{2} A \cdot A, u_R - p \cdot A - \frac{wx}{2} A \cdot A) \quad . \quad (4) $$

$p$ is a 16-dimensional lattice vector in $E_8 \oplus E_8$. As shown in [20], the vacuum functional of the supersymmetric 9d heterotic string, with generic radius and generic Wilson line in the compact spatial direction, can be written in terms of a sum over vectors in the boosted lattice:

$$ W_{\text{SS}}(r_{\text{circ}}; A) = L^{10}(4\pi^2 \alpha')^{-5} \int \frac{d^2 \tau}{4\sqrt{2}} (2\pi \tau_2)^{-\frac{5}{2}} |\eta(\tau)|^{-16} \frac{1}{8 \eta^8} (\bar{\Theta}_3^4 - \Theta_4^4 - \bar{\Theta}_2^4) \times \left[ \frac{1}{\eta^{16}} \sum_{p' L, p' R} q^\frac{1}{2}(p^2 + l_{L}^2) q^\frac{1}{2} l_{R}^2 \right] . \quad (5) $$

$W_{\text{SS}}$ describes the supersymmetric heterotic string with gauge group $SO(16) \times SO(16)$ at generic radius. The partition function of the nonsupersymmetric but tachyon-free 9d string with gauge group $SO(16) \times SO(16)$ at generic radii is given by [24, 27, 21, 23, 25, 26]):

$$ Z_{\text{NS}}(r_{\text{circ}}) = \frac{1}{4} \left[ (\frac{\Theta_2}{\eta})^8 (\frac{\Theta_4}{\eta})^8 (\frac{\Theta_2}{\eta})^4 - (\frac{\Theta_2}{\eta})^8 (\frac{\Theta_3}{\eta})^8 (\frac{\Theta_4}{\eta})^4 - (\frac{\Theta_3}{\eta})^8 (\frac{\Theta_1}{\eta})^8 (\frac{\Theta_4}{\eta})^4 \right] \sum_{n, w} q^{\frac{1}{2} l_{L}^2} q^{\frac{1}{2} l_{R}^2} . \quad (6) $$

However, by identifying an appropriate interpolating function as in previous sections and appropriate background field, we can continuously connect this background to the supersymmetric $E_8 \times E_8$ string.

Since $x = (\frac{2}{\alpha'})^{1/2} \frac{\beta}{2\pi}$, from the viewpoint of the low-energy finite temperature gauge theory the timelike Wilson line is simply understood as imposing a modified axial gauge condition: $A^0 = \text{const.}$ The dependence of the constant on background temperature has been chosen to provide a shift in the mass formula that precisely cancels the contribution from low temperature $(n, 0)$ tachyonic modes. As before, we begin by identifying an appropriate modular invariant interpolating function:

$$ Z_{\text{het.}} = \sum_{n, w} \left[ (\frac{\Theta_2}{\eta})^8 (\frac{\Theta_3}{\eta})^8 (\frac{\Theta_4}{\eta})^4 - (\frac{\Theta_2}{\eta})^8 (\frac{\Theta_3}{\eta})^8 (\frac{\Theta_4}{\eta})^4 - (\frac{\Theta_3}{\eta})^8 (\frac{\Theta_1}{\eta})^8 (\frac{\Theta_4}{\eta})^4 \right] \frac{1}{q^{2 l_{L}^2} q^{2 l_{R}^2}} $$

$$ + \frac{1}{4} \sum_{n, w} e^{\pi i (n + 2w)} \left[ (\frac{\Theta_3}{\eta})^4 - (\frac{\Theta_2}{\eta})^4 - (\frac{\Theta_4}{\eta})^4 \right] \left[ (\frac{\Theta_3}{\eta})^4 + (\frac{\Theta_1}{\eta})^4 + (\frac{\Theta_2}{\eta})^4 \right] \frac{1}{q^{2 l_{L}^2} q^{2 l_{R}^2}} . $$
As in previous sections, the first term within square brackets has been chosen as the nonsupersymmetric sum over spin structures for a chiral type 0 string. This function appears in the sum over spin structures for the tachyon-free \( SO(16) \times SO(16) \) string given above. Notice that taking the \( x \to \infty \) limit, by similar manipulations as in the type II case, yields the partition function of the supersymmetric 10D \( E_8 \times E_8 \) string.

Consider accompanying the \( SO(17,1) \) transformation described above with a lattice boost that decreases the size of the interval [20]:

\[
e^{-\alpha_{00}} = \frac{1}{1 + |A|^2/4}.
\]

This recovers the \( \text{Spin}(32)/\mathbb{Z}_2 \) theory compactified on an interval of size \( 2/x \), but with Wilson line \( A = x \text{diag}(1^8,0^8) \) [20]. Thus, taking the large radius limit in the dual variable, and with dual Wilson line background, yields instead the spacetime supersymmetric 10D \( \text{Spin}(32)/\mathbb{Z}_2 \) heterotic string. It follows that the \( E_8 \times E_8 \) and \( SO(32) \) heterotic strings share the same tachyon-free finite temperature ground state with gauge symmetry \( SO(16) \times SO(16) \). The Kosterlitz-Thouless transformation at \( T_C = 1/2\pi \alpha'^{1/2} \) is a self-dual continuous phase transition in this theory.

The thermal duality transition in this theory is in the universality class of the Kosterlitz-Thouless transition [13]: namely, the partial derivatives of the free energy to arbitrary order are analytic functions of temperature. The duality transition interchanges thermal momentum modes of the \( E_8 \times E_8 \) theory with winding modes of the \( \text{Spin}(32)/\mathbb{Z}_2 \) theory, and vice versa. Note that the vacuum functional, the Helmholtz and Gibbs free energies, the internal energy, and all subsequent thermodynamic potentials, are both finite and tachyon-free.

### 2.2 Holography and the Duality Phase Transition

In the closed bosonic string theory, the generating functional for connected one-loop vacuum string graphs is invariant under the thermal duality transformation: \( W(T) = W(T^2_c/T) \), with self-dual temperature, \( T_c = 1/2\pi \alpha'^{1/2} \). As already noted by Polchinski [14], we can infer the following thermal duality relation which holds for both the Helmholtz free energy, \( F(T) = -T \cdot W(T) \), and the effective potential, \( \rho(T) = -T \cdot W(T)/V \) of the closed bosonic string:

\[
F(T) = \frac{T^2}{T^2_c} F\left(\frac{T^2_c}{T}\right), \quad \rho(T) = \frac{T^2}{T^2_c} \rho\left(\frac{T^2_c}{T}\right).
\]

In the case of the heterotic string, the thermal duality relation instead relates, respectively, the free energies of the \( E_8 \times E_8 \) and \( \text{Spin}(32)/\mathbb{Z}_2 \) theories:

\[
F(T)_{E_8 \times E_8} = \frac{T^2}{T^2_c} F\left(\frac{T^2_c}{T}\right)_{\text{Spin}(32)/\mathbb{Z}_2}, \quad \rho(T)_{E_8 \times E_8} = \frac{T^2}{T^2_c} \rho\left(\frac{T^2_c}{T}\right)_{\text{Spin}(32)/\mathbb{Z}_2}.
\]
Consider the high temperature limit of this expression:

$$
\lim_{T \to \infty} \rho(T)_{E_8 \times E_8} = \lim_{T \to \infty} \frac{T^2}{T^2_C} \rho(T)_{\text{Spin}(32)/Z_2} = \lim_{(T^2_C/T) \to 0} \frac{T^2}{T^2_C} \rho(T)_{\text{Spin}(32)/Z_2} = \frac{T^2}{T^2_C} \rho(0)_{\text{Spin}(32)/Z_2}
$$

where $\rho(0)$ is the cosmological constant, or vacuum energy density, at zero temperature. Note that it is finite. Thus, at high temperatures, the free energy of either heterotic theory grows as $T^2$. In other words, the growth in the number of degrees of freedom at high temperature in the heterotic string ensemble is only as fast as in a two-dimensional field theory. This is significantly slower than the $T^{10}$ growth of the high temperature degrees of freedom expected in the ten-dimensional low energy field theory.

Notice that the prefactor, $\rho(0)/T^2_C$, in the high temperature relation is unambiguous, a consequence of the normalizability of the generating functional of one-loop vacuum graphs in string theory [1]. It is also background dependent: it is computable as a continuously varying function of the background fields upon compactification to lower spacetime dimension [26]. The relation in Eq. (11) is unambiguous evidence of the holographic nature of perturbative heterotic string theory: there is a drastic reduction in the degrees of freedom in string theory at high temperature, a conjecture first made in [2].

Starting with the duality invariant expression for the string effective action functional, we can derive the thermodynamic potentials of the heterotic string ensemble. The Helmholtz free energy follows from the definition below Eq. (2), and is clearly finite at all temperatures, with no evidence for either divergence or discontinuity. The internal energy of the heterotic ensemble takes the form:

$$
U \equiv -\left( \frac{\partial W}{\partial \beta} \right)_V
= L^9(4\pi^2\alpha')^{-9/2} \frac{1}{2} \int_{\mathcal{F}} \frac{|d\tau|^2}{4\pi^2} (2\pi\tau)^{-9/2} \left| \eta(\tau) \right|^{-16} \frac{4\pi\tau_2}{\beta} \sum_{n,w\in\mathbb{Z}} \left( w^2 x^2 - \frac{n^2}{x^2} \right) \cdot q^{12} l^4 \cdot Z_{[SO(16)]^2},
$$

where $Z_{[SO(16)]^2}$ denotes the sums over spin structures appearing in Eq. (7). Notice that $U(\beta)$ vanishes precisely at the self-dual temperature, $T_c=1/2\pi\alpha'^{1/2}$, $x_c=\sqrt{2}$, where the internal energy contributed by winding sectors cancels that contributed by momentum sectors. Note also that the internal energy changes sign at $T = T_C$. Hints of this behaviour are already apparent in the numerical analyses of the one-loop effective potential given in [26, 27].

It is easy to demonstrate the analyticity of infinitely many thermodynamic potentials in the vicinity of the critical point. It is convenient to define:

$$
[d\tau] \equiv \frac{1}{2} L^9(4\pi^2\alpha')^{-9/2} \left[ \frac{|d\tau|^2}{4\pi^2} (2\pi\tau_2)^{-9/2} \left| \eta(\tau) \right|^{-16} \cdot Z_{[SO(16)]^2} e^{2\pi i w \tau_1} \right]
$$

$$
y(\tau_2; x) \equiv 2\pi\tau_2 \left( \frac{n^2}{x^2} + w^2 x^2 \right).
$$

Denoting the $m$th partial derivative with respect to $\beta$ at fixed volume by $W_{(m)}$, $y_{(m)}$, we note that the higher derivatives of the vacuum functional take the simple form:

$$
W_{(1)} = \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y(-y_{(1)})}
$$
\[ W_{(2)} = \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y} (-y_{(2)} + (-y_{(1)})^2) \]
\[ W_{(3)} = \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y} (-y_{(3)} - y_{(1)}y_{(2)} + (-y_{(1)})^3) \]
\[ \cdots = \] 
\[ W_{(m)} = \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y} (-y_{(m)} - \cdots + (-y_{(1)})^m) \quad (14) \]

Referring back to the definition of \( y \), it is easy to see that both the vacuum functional and, consequently, the full set of thermodynamic potentials, are analytic in \( x \). Notice also that third and higher derivatives of \( y \) are determined by the momentum modes alone:
\[ y_{(m)} = (-1)^m n^2\frac{(m + 1)!}{x^{m+2}}, \quad m \geq 3 \quad (15) \]

For completeness, we give explicit results for the first few thermodynamic potentials:
\[ F = -\frac{1}{\beta} W_{(0)}, \quad U = -W_{(1)}, \quad S = W_{(0)} - \beta W_{(1)}, \quad C_V = \beta^2 W_{(2)}, \cdots \quad (16) \]

The entropy is given by the expression:
\[ S = \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y} \left[ 1 + 4\pi\tau_2(-\frac{n^2}{x^2} + w^2x^2) \right], \quad (17) \]

For the specific heat at constant volume, we have:
\[ C_V = \sum_{n,w} \int_{\mathcal{F}} [d\tau] e^{-y} \left[ 16\pi^2\tau_2^2\left(-\frac{n^2}{x^2} + w^2x^2\right)^2 - 4\pi\tau_2(3\frac{n^2}{x^2} + w^2x^2) \right]. \quad (18) \]

Explicitly, the Helmholtz free energy takes the form:
\[ F(\beta) = -\frac{1}{2\beta} L^9(4\pi^2\alpha')^{-9/2} \int_{\mathcal{F}} [d\tau]\frac{|d\tau|^2}{4\pi\tau_2^2} (2\pi\tau_2)^{-9/2} |\eta(\tau)|^{-16} \sum_{n,w} Z_{[SO(16)]^2} q^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} q^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} \] 
\[ = -\frac{1}{2} L^9(4\pi^2\alpha')^{-9/2} \int_{\mathcal{F}} [d\tau]\frac{|d\tau|^2}{4\pi\tau_2^2} |\eta(\tau)|^{-16} \sum_{n,w} \left[ 1 + 4\pi\tau_2(-\frac{n^2}{x^2} + w^2x^2) \right] Z_{[SO(16)]^2} q^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} q^{\frac{1}{2} \frac{1}{2} \frac{1}{2}}. \quad (19) \]

while for the entropy of the heterotic string ensemble, we have the result:
\[ S(\beta) = -\frac{1}{2} L^9(4\pi^2\alpha')^{-9/2} \int_{\mathcal{F}} [d\tau]\frac{|d\tau|^2}{4\pi\tau_2^2} |\eta(\tau)|^{-16} \sum_{n,w} \left[ 1 + 4\pi\tau_2(-\frac{n^2}{x^2} + w^2x^2) \right] Z_{[SO(16)]^2} q^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} q^{\frac{1}{2} \frac{1}{2} \frac{1}{2}}. \quad (20) \]

The thermodynamic potentials of the heterotic ensemble are finite normalizable functions at all temperatures starting from zero. In summary, the heterotic string ensemble displays a continuous phase transition at the self-dual temperature, mapping thermal winding modes of the \( E_8 \times E_8 \) theory to thermal momentum modes of the \( \text{Spin}(32)/\mathbb{Z}_2 \) theory, and vice versa, unambiguously identifying a phase transition of the Kosterlitz-Thouless type [13, 7].

3 Type I and Type II Open and Closed String Theories

The Type I and Type II string theories can have both open and closed string sectors, and the vacuum can contain D-branes: sources for Ramond-Ramond charge, with worldvolume Yang-Mills fields [14, 35]. It is therefore helpful to consider them in a unified treatment. We will begin with the pure oriented closed string sector common to all of these theories.
3.1 Closed Oriented Superstring Sector

We begin with a discussion of the pure type II oriented closed string one-loop vacuum functional. A thermal duality transformation mapping the IIA string to the IIB string simply maps IIA winding to IIB momentum modes, and vice versa. In the absence of a Ramond-Ramond sector, the expression for the normalized generating functional of connected one-loop vacuum graphs takes the form:

\[ W_{II} = L^9(4\pi^2\alpha')^{-9/2} \int_{\mathcal{F}} \frac{d^2\tau}{4\tau_2^2} (2\pi\tau_2)^{-4} |\eta(\tau)|^{-14} Z_{II}(\beta) \quad , \]

where the spatial volume \( V = L^9 \), while the dimensionless (scaled) spatial volume is \( L^9(4\pi^2\alpha')^{-9/2} \). The inverse temperature is given by \( \beta = 2\pi r_{\text{circ}} \). Notice that in the \( \alpha' \to 0 \) limit one can simultaneously take the size of the “box” to infinity while keeping the rescaled volume fixed. This defines the approach to the thermodynamic limit. The function \( Z_{\text{II orb.}}(\beta) \) is the product of contributions from worldsheet fermions and bosons, \( Z_F Z_B \), and is required to smoothly interpolate between finite temperature and the spacetime supersymmetric zero temperature limit. The spectrum of thermal modes will be unambiguously determined by modular invariance. The spacetime supersymmetry breaking projection, \((-1)^{N_F}\), is modified by the introduction of phases in the interpolating function. Such phases can depend on thermal mode number. They must be chosen compatible with the requirement that spacetime supersymmetry is restored in the zero temperature limit of the interpolating function. We comment that temperature dependent phases in the free energy were first proposed by Atick and Witten in [2]. The unique modular invariant interpolating function satisfying these requirements is:

\[ Z_{II}(\beta) = \frac{1}{2\eta\bar{\eta}} \left( \sum_{\varphi,\mu \in \mathbb{Z}} q^{\frac{1}{2}(\frac{\varphi}{2} + \mu x)^2} \bar{q}^{\frac{1}{2}(\frac{\varphi}{2} - \mu x)^2} \right) \{ (|\Theta_3|^8 + |\Theta_4|^8 + |\Theta_2|^8) - \pi i (\Theta_3^4 \bar{\Theta}_4^4 + \Theta_4^4 \bar{\Theta}_3^4) \} \quad , \]

where \( x = (\frac{2}{\alpha'})^{1/2} r_{\text{circ}} \). The world-sheet fermions have been conveniently complexified into left- and right-moving Weyl fermions. As in the superstring, the spin structures for all ten left- and right-moving fermions, \( \psi^\mu, \bar{\psi}^\mu, \mu = 0, \cdots, 9 \), are determined by those for the world-sheet gravitino associated with left- and right-moving N=1 superconformal invariances.

To understand our result for the correct interpolating function, first recall the expression for \( Z_{SS} \)— the zero temperature, spacetime supersymmetric, limit of our function given by the ordinary GSO projection:

\[ Z_{SS} = \frac{1}{4 \eta \bar{\eta}^2} \left( (\Theta_3^4 - \Theta_4^4 - \bar{\Theta}_3^4 + \bar{\Theta}_4^4) \right) \quad , \]

Notice that the first of the relative signs in each round bracket preserves the tachyon-free condition. The second relative sign determines whether spacetime supersymmetry is preserved in the zero temperature spectrum. Next, notice that \( Z_{SS} \) can be rewritten using theta function identities as follows:

\[ Z_{SS} = \{ |\Theta_3|^8 + |\Theta_4|^8 + |\Theta_2|^8 \} + \{ (\Theta_3^4 \bar{\Theta}_4^4 + \Theta_4^4 \bar{\Theta}_3^4) - (\Theta_3^4 \bar{\Theta}_4^4 + \Theta_4^4 \bar{\Theta}_3^4 + \Theta_3^4 \bar{\Theta}_4^4 + \Theta_2^4 \bar{\Theta}_4^4) \} \quad . \]
Either of the two expressions within square brackets is modular invariant. The first may be recognized as the nonsupersymmetric sum over spin structures for the type 0 string [14].

Thus, the interpolating function captures the desired zero temperature limits of both the IIA and IIB strings: for large $\beta_{IIA}$, terms with $w_{IIA} \neq 0$ decouple in the double summation since they are exponentially damped. The remaining terms are resummed by a Poisson resummation, thereby inverting the $\beta_{IIA}$ dependence in the exponent, and absorbing the phase factor in a shift of argument. The large $\beta_{IIA}$ limit can then be taken smoothly providing the usual momentum integration for a non-compact dimension, plus a factor of $\tau^{-1/2}$ from the Poisson resummation. This recovers the supersymmetric zero temperature partition function.

As regards thermal duality, small $\beta_{IIA}$ maps to large $\beta_{IIB} = \beta_{IIA}^2/\beta_{IIA}$, also interchanging the identification of momentum and winding modes, $(n, w)_{IIA} \rightarrow (n' = w, w' = n)_{IIB}$. We can analyze the limit of large dual inverse temperature as before, obtaining the zero temperature limit of the dual IIB theory. At any intermediate temperature, all of the thermal modes contribute to the vacuum functional with a phase that takes values $(\pm 1)$ only. Note that the spacetime fermions of the zero temperature spectrum now contribute with a reversed phase, evident in the first term in Eq. (22), as required by the thermal boundary conditions. In summary, the vacuum functional of the IIA string maps precisely into the vacuum functional of the IIB string under a thermal duality transformation.

3.2 Tachyonic Thermal Momentum and Thermal Winding Modes

We now point out a peculiarity of the type IIA and IIB canonical ensemble at finite temperature. First, recall some pertinent facts from quantum field theory. It is generally assumed to be the case [15] that in a field theory known to be perturbatively renormalizable at zero temperature, the new infrared divergences introduced by a small variation in the background temperature can be self-consistently regulated by a suitable extension of the renormalization conditions, at the cost of introducing a finite number of additional counter-terms. The zero temperature renormalization conditions on 1PI Greens functions are conveniently applied at zero momentum, or at fixed spacelike momentum in the case of massless fields. Consider a theory with one or more scalar fields. Then the renormalization conditions must be supplemented by stability constraints on the finite temperature effective potential [15]:

$$\frac{\partial V_{\text{eff}}(\phi_{\text{cl}})}{\partial \phi_{\text{cl}}} = 0, \quad \frac{\partial^2 V_{\text{eff}}(\phi_{\text{cl}})}{\partial \phi_{\text{cl}}^2} \equiv G^{-1}(k) \big|_{k=0} \geq 0.$$  \hspace{1cm} (25)

Here, $V_{\text{eff}}(\phi_{\text{cl}}) = -\Gamma(\phi_{\text{cl}})/V\beta$, where $\Gamma$ is the effective action functional, or sum of connected 1PI vacuum diagrams at finite temperature. In the absence of nonlinear field configurations, and for perturbation theory in a small coupling about the free field vacuum, $|0\rangle$, $\phi_{\text{cl}}$ is simply the expectation value of the scalar field: $\phi_{\text{cl}} = \langle 0 | \phi(x) | 0 \rangle$. The first condition holds in the absence of an external source at every extremum of the effective potential. The second condition states that the renormalized masses of physical fields must not be driven to imaginary values at any non-pathological and stable minimum of the effective potential.

The conditions in Eq. (25) are, in fact, rather familiar to string theorists. Weakly coupled superstring theories at zero temperature are replete with scalar fields and their vacuum expectation
values, or moduli, parameterize a multi-dimensional space of degenerate vacua. Consider the effect of an infinitesimal variation in the background temperature. Such an effect will necessarily break supersymmetry, and it is well-known that in the presence of a small spontaneous breaking of supersymmetry the dilaton potential will generically develop a runaway direction, signalling an instability of precisely the kind forbidden by the conditions that must be met by a non-pathological ground state [16]. This is worrisome. Namely, while the zero temperature effective potential may be correctly minimized with respect to the renormalizable couplings in the potential and, in fact, vanishes in a spacetime supersymmetric ground state, one or more of the scalar masses could be driven to imaginary values in the presence of an infinitesimal variation in background temperature. Such a quantum field theory would be simply unacceptable both as a self-consistent effective field theory in the Wilsonian sense, and also as a phenomenological model for a physical system [15]. The same conclusion would hold for any weakly coupled superstring theory with these pathological properties. Fortunately, as is convincingly demonstrated in the follow-up work [17], the type IIA and type IIB thermal ensembles we have described in the previous sub-section with trivial Ramond-Ramond sector are indeed unstable. Most importantly, the flow of the worldsheet renormalization group (RG) is in the direction towards the supersymmetric zero temperature vacuum of the type IIA, or type IIB, string theory.

Let us explain why the type IIA or type IIB thermal ensemble is pathological in the absence of a Ramond-Ramond sector, or of Yang-Mills gauge fields. To check for potential tachyonic instabilities in the expressions given in Eq. (22), consider the mass formula in the (NS,NS) sector for world-sheet fermions, with $I_L^2 = I_R^2$, and $N=\bar{N}=0$:

$$\text{(mass)}^2_{nw} = \frac{2}{\alpha'} \left[ -1 + \frac{2\alpha' \pi^2 n^2}{\beta^2} + \frac{\beta^2 w^2}{8\pi^2 \alpha'} \right]. \quad (26)$$

This is the only sector that contributes tachyons to the thermal spectrum. Recall that $\beta_C^2 = 4\pi^2 \alpha'$ is the self-dual point. A nice check is that the momentum mode number dependence contained in the phase factors does not impact the spacetime spin-statistics relation in the NS-NS sector: all potentially tachyonic states are spacetime scalars as expected. Notice that the $n=w=0$ sector common to both type II string theories contains a potentially tachyonic state whose mass is now temperature dependent. However, it corresponds to an unphysical tachyon. The potentially tachyonic physical states are the pure momentum and pure winding states, $(n,0)$ and $(0,w)$, with $N=\bar{N}=0$. As in the closed bosonic string analysis, we can compute the temperatures below, and beyond, which these modes become tachyonic in the absence of oscillator excitations. Each momentum mode, $(\pm n, 0)$, is tachyonic up to some critical temperature, $T_n^2 = 1/2\pi^2 \alpha'$, after which it turns marginal (massless). Conversely, each winding mode $(0, \pm w)$, is tachyonic beyond some critical temperature, $T_w^2 = w^2/8\pi^2 \alpha'$. The thermal spectrum of pure type IIA or IIB closed oriented strings is unstable at all temperatures starting from zero.

The source of the thermal instability discussed here should not be confused with the gravitational instability of flat spacetime found in the finite temperature field theory analysis of [30]. Nor is it to be confused with the famous Jeans instability of gravitating matter in the limit of infinite spatial volume [2]. Our considerations have been limited to the internal consistency conditions of the free closed string thermal spectrum. In particular, the ensemble of closed strings does not gravitate at this order in perturbation theory.
We should now remind the reader that in the presence of Dbranes, the type IIA and IIB strings acquire an open string sector and, consequently, nonabelian gauge fields in the massless spectrum. Dbranes are BPS sources, and half of the supersymmetries of the type II theory are broken in their presence. As we will show in section 3, in the type I’ theory with D8branes, the presence of nonabelian gauge fields enables invoking a Wilson line background. In the presence of a temperature dependent (timelike) Wilson line, it is a simple matter to remove all of the thermal tachyons from the spectrum of the type II theory. This mechanism has a precise analog in the case of both the closed heterotic string ensemble, as well as in the type I’ ensemble of open and closed strings as demonstrated in the following sections.

4 Canonical Ensemble of Type I’ Strings

4.1 Cancellation of Ramond-Ramond Scalar Tadpole

In the previous section, we have seen that the two distinct ten-dimensional supersymmetric $E_8 \times E_8$ and Spin(32)/$Z_2$ heterotic string theories have a common tachyon-free ground state at finite temperature in the presence of a temperature dependent timelike Wilson line. The gauge group at finite temperatures is $SO(16) \times SO(16)$. Let us now consider what happens to the type I and type I’ open and closed string theories at finite temperature.

It will be convenient to begin with the timelike T-dual type I’ string theory with 16 D8branes on each of two orientifold planes, separated by $\beta$ in the Euclidean time direction [14, 8]. From the perspective of the original type IB string theory with gauge group $O(32)$, we have turned on a temperature-dependent timelike Wilson line, $A_0=\frac{1}{2}((1)^8, 0^8)$, in the worldvolume of the space-filling D9branes. This breaks the gauge symmetry to $O(16) \times O(16)$. In the type I’ picture, the counting of zero length open strings connecting a pair of D8branes at finite temperature is as follows: we have a total of $(2 \cdot 8 \cdot 7) \cdot 2$ massless states from zero length open strings connecting a pair of D8branes on the same orientifold plane. In addition, there are $8 \cdot 2$ photons from zero length open strings connecting each D8brane to itself. Open strings connecting branes on distinct orientifold planes are ordinarily massive, resulting in a breaking of the gauge group from $O(32)$ to $O(16) \times O(16)$ at finite temperature.

Consider the free energy, $F(\beta)$, of a gas of free type I’ strings in this ground state. $F$ is obtained from the generating functional for connected vacuum string graphs, $F(\beta)=-W(\beta)/\beta$. The one-loop free energy receives contributions from worldsurfaces of four different topologies [14, 35]: torus, annulus, Mobius Strip, and Klein Bottle. The torus is the sum over closed oriented worldsheets and the result is, therefore, identical to that derived in section 2 for the type IIB string theory, refer to Eqs. (21) and (22). Notice that the closed oriented string sector of the type IB theory does not distinguish between the Dbrane worldvolume and the bulk spacetime orthogonal to the branes, since the closed oriented strings live in all ten dimensions of spacetime. Nor does this sector have any knowledge about the Yang-Mills sector, or of the timelike Wilson line. The comments we have made earlier regarding worldsheet renormalization group (RG) flow in the direction towards the noncompact supersymmetric vacuum apply for this sector of the theory: although we begin with an, a priori, non-vanishing thermal contribution to the vacuum energy, RG flow takes us back to the supersymmetric infra-red stable fixed point. Thus, the torus contribution to the vacuum energy
vanishes, and supersymmetry is not broken by thermal effects in this sector of the unoriented type I open and closed string theory.

The one-loop contribution to $W$ from the remaining three worldsheet topologies is given by the Polyakov path integral summing surfaces with two boundaries, with a boundary and a crosscap, or with two crosscaps [14]. The boundaries and crosscaps are localized on the orientifold plane but the worldsheet itself can extend into the transverse Dirichlet directions. $W(\beta)|_{\beta=\infty}$ is required to agree with the vacuum functional of the supersymmetric $O(32)$ type I' string at zero temperature. We also require a tachyon-free thermal spectrum which retains the good ultraviolet and infrared behavior of the supersymmetric zero temperature limit. As in the case of the supersymmetric vacuum, consistency of the finite temperature type I' string vacuum requires cancellation of the tadpole for the unphysical Ramond-Ramond scalar [14]. Such a scalar potential could appear as the Poincare dual of an allowed ten-form potential in the unoriented type I string theory, but notice that the associated 11-form field strength is required to be identically zero in this ten-dimensional theory.

On the other hand, the usual cancellation between closed string NS-NS and R-R exchanges resulting from spacetime supersymmetry must not hold except in the zero temperature limit. This is in precise analogy with the finite temperature analysis given earlier for closed string theories and is achieved by introducing temperature dependent phases in the vacuum functional [2]. We will insert an identical temperature-dependent phase for the contributions to $F(\beta)$ from worldsheets with each of the three remaining classes of one-loop graphs—annulus, Mobius Strip, and Klein Bottle, in order to preserve the form of the RR scalar tadpole cancellation in the zero temperature ground state. The result for the free energy at one-loop order, with $N=16$ D8branes on each orientifold plane, takes the form:

$$F = -\beta^{-1} L^9 (4\pi^2 \alpha')^{-9/2} \int_0^\infty \frac{dt}{8t} e^{-\beta^2 t/2\pi\alpha'} \frac{(2\pi t)^{-9/2}}{\eta(it)^8} \sum_{n\in\mathbb{Z}} \left[ 2^{-8} N^2 Z[0] - e^{i\pi n} Z[1] \right] \times q^{4\alpha' t^2 n^2/\beta^2}$$

We emphasize that this expression is uniquely singled out by the requirement of tadpole cancellation for the dualized RR scalar. It also interpolates smoothly between the supersymmetric zero temperature limit, with gauge group $O(32)$, and the finite temperature result with gauge group $O(16)\times O(16)$. Notice that the momentum modes have no winding mode counterparts in the open string thermal spectrum because of the absence of thermal duality. The Matsubara frequency spectrum is given by the eigenvalues for timelike momentum: $p_0=2n\pi/\beta$, with $n\in\mathbb{Z}$. Finally, as an important consistency check, notice that there are no new tadpoles in the expression for $W$ due to finite temperature excitations alone.

The variable $q=e^{-2\pi t}$ is the modular parameter for the cylinder, and we have used the identifications: $2t_K=t_C$, and $2t_M=t_C$, to express the Mobius strip and Klein bottle amplitudes in terms of the cylinder’s modular parameter. As in [14], the subscripts [0], [1], denote, respectively, NS-NS and R-R closed string exchanges. Thus,

$$Z[0] = \left( \frac{\Theta_{00}(it; 0)}{\eta(it)} \right)^4 - \left( \frac{\Theta_{10}(it; 0)}{\eta(it)} \right)^4$$
$$Z[1] = \left( \frac{\Theta_{01}(it; 0)}{\eta(it)} \right)^4 - \left( \frac{\Theta_{11}(it; 0)}{\eta(it)} \right)^4$$
where the (00), (10), (01), and (11), denote, respectively, (N S-NS), (R-NS), (NS-R), and (R-R), 
boundary conditions on worldsheet fermions in the closed string sector [14]. We emphasize that, 
alogous to the supersymmetric zero temperature limit, cancellation of the tadpole for the un-
physical Ramond-Ramond scalar requires $N=16$. It can be verified that the dilaton tadpole is also 
absent as a consequence of this constraint. Remarkably, we have found a nonsupersymmetric type 
I vacuum with supersymmetry broken by thermal effects, but without the appearance of a dilaton 
tadpole.

4.2 Holography in the Open and Closed String Ensemble

The expression for the generating functional of one-loop vacuum string graphs given in Eq. (27) 
explicitly violates thermal duality and so, unlike the case of closed string gases, there is no analogous 
thermal self-duality relation holding at $T_C$. Nevertheless, we will find concrete evidence of a high 
temperature holographic phase by direct inspection of the modular integration. Notice that the high 
temperature behavior of $F(\beta)$ can be unambiguously identified because the dominant contribution 
in this limit comes from the $t \to 0$ regime. A modular transformation on the argument of the theta 
functions, $t \to 1/t$, puts the integrand in a suitable form for term-by-term expansion in powers of 
e$^{-1/t}$; this enables term-by-term evaluation of the modular integral. Isolating the $t \to 0$ asymptotics 
in the standard way [14], we obtain a most unexpected result:

\[
\lim_{\beta \to 0} F = -\lim_{\beta \to 0} \beta^{-1} L^9(4\pi^2 \alpha')^{-9/2} \int_0^\infty \frac{dt}{8t} e^{-\beta^2 t/2 \pi \alpha'} \left( \frac{2\pi t}{\eta(i/t)^8} \right)^{-1/2} \left[ 2^{-9} N^2 (Z_{[0]}(i/t) - e^{i\pi n} Z_{[1]}(i/t)) + 2^{-3} N e^{i\pi n} Z_{[1]}(i/t) \right] \times q^{4\alpha' \pi^2 n^2/\beta^2} 
\]

where $\rho_{\text{high}}$ is a constant independent of temperature. Thus, the canonical ensemble of type I' strings 
is also holographic, displaying the $T^2$ growth in free energy at high temperatures characteristic of 
a two-dimensional field theory!

It is helpful to verify the corresponding scaling relations for the first few thermodynamic poten-
tials. The internal energy of the free type I' string ensemble takes the form:

\[
U = -\left( \frac{\partial W}{\partial \beta} \right)_V 
= \frac{1}{2} L^9(4\pi^2 \alpha')^{-9/2} \int_0^\infty \frac{dt}{8t} e^{-\beta^2 t/2 \pi \alpha'} \left( \frac{2\pi t}{\eta(it)^8} \right)^{-9/2} \cdot \sum_{n \in \mathbb{Z}} Z_{\text{open}} \frac{4\pi t}{\beta} \left( \frac{\beta^2}{4\pi^2 \alpha'} - \frac{4\alpha' \pi^2 n^2}{\beta^2} \right) q^{4\alpha' \pi^2 n^2/\beta^2}, 
\]

where $Z_{\text{open}}$ is the factor in square brackets in the expression in Eq. (27). Unlike the heterotic string 
ensemble, $U(\beta)$ no longer vanishes at the self-dual temperature, $T_c=1/2\pi \alpha'^{1/2}$, since there are no 
winging modes in the thermal spectrum.

The analyticity of infinitely many thermodynamic potentials in the vicinity of the critical point
can be demonstrated as for the heterotic string ensemble. We define:

\[
[dt] \equiv \frac{1}{2} L^9 (4\pi^2 \alpha')^{-9/2} \left[ \frac{dt}{8t} e^{-\beta^2 t/2\alpha'} \left( \frac{2\pi t}{\eta(it)} \right)^{-9/2} \cdot Z_{\text{open}} \right], \quad y(t; \beta) \equiv 2\pi t \left( \frac{\beta^2}{4\pi^2 \alpha'} + \frac{4\alpha' \pi^2 n^2}{\beta^2} \right),
\]

and denote the \(m\)th partial derivative with respect to \(\beta\) at fixed volume by \(W_{(m)}, y_{(m)}\). The higher derivatives of the generating functional now take the simple form:

\[
W_{(1)} = \sum_{n \in \mathbb{Z}} \int_0^\infty [dt] e^{-y}(-y_{(1)}) \\
W_{(2)} = \sum_{n \in \mathbb{Z}} \int_0^\infty [dt] e^{-y}(-y_{(2)} + (-y_{(1)})^2) \\
W_{(3)} = \sum_{n \in \mathbb{Z}} \int_0^\infty [dt] e^{-y}(-y_{(3)} - y_{(1)} y_{(2)} + (-y_{(1)})^3) \\
\vdots \\
W_{(m)} = \sum_{n \in \mathbb{Z}} \int_0^\infty [dt] e^{-y}(-y_{(m)} - \cdots + (-y_{(1)})^m) .
\]

Referring back to the definition of \(y\), it is easy to see that the generating functional and, consequently, the full set of thermodynamic potentials is analytic in \(\beta\). Notice that third and higher derivatives of \(y\) are determined by the momentum modes alone:

\[
y_{(m)} = (-1)^m n^2 \frac{(m + 1)!}{\beta^{m+2}}, \quad m \geq 3 .
\]

Explicit expressions for the first few thermodynamic potentials are as follows:

\[
F = -\frac{1}{\beta} W_{(0)}, \quad U = -W_{(1)}, \quad S = W_{(0)} - \beta W_{(1)}, \quad C_V = \beta^2 W_{(2)}, \cdots .
\]

The entropy is given by the expression:

\[
S = \sum_{n \in \mathbb{Z}} \int_0^\infty [dt] e^{-y} \left[ 1 + 4\pi t \left( \frac{\beta^2}{4\pi^2 \alpha'} - \frac{4\alpha' \pi^2 n^2}{\beta^2} \right) \right],
\]

For the specific heat at constant volume, we have:

\[
C_V = \sum_{n \in \mathbb{Z}} \int_0^\infty [dt] e^{-y} \left[ 16\pi^2 t^2 \left( \frac{\beta^2}{4\pi^2 \alpha'} - \frac{4\alpha' \pi^2 n^2}{\beta^2} \right)^2 - 4\pi t \left( \frac{\beta^2}{4\pi^2 \alpha'} + \frac{3\alpha' \pi^2 n^2}{\beta^2} \right) \right].
\]

In summary, the free type \(I'\) string ensemble displays holographic behavior at high temperatures characterized by a free energy scaling as \(\beta^{-2}\)!

### 4.3 An Order Parameter for a Gauge Theory Transition?

In this subsection, we will examine a plausible order parameter for a phase transition in the gauge sector of the theory. The calculation that follows relies on the well-known fact that the sub-string-scale dynamics of string theory can be probed by D0branes: pointlike topological string solitons
whose mass scales as $1/g$, in the presence of external background fields [31, 32, 35]. In the free string limit the pointlike D0branes behave like analogs of infinitely massive heavy quarks: semiclassical, theoretical probes of the confinement regime. It is well known that an order parameter signalling the thermal deconfinement phase transition in a nonabelian gauge theory is the expectation value of a closed timelike Wilson loop. We therefore look for a signal of a thermal phase transition in the short distance behavior of the pair correlator of timelike Wilson loops in finite temperature string theory. We will use the fact that the loop pair correlation function in string theory has a simple worldsheet representation in terms of the path integral with fixed boundary conditions [33, 34, 35].

The short distance behavior of this correlation function is dominated by the shortest open strings, namely, the gauge sector of the massless spectrum.

Since we wish to focus on a property of the gauge theory on the worldvolume of type I D9branes, we require both fixed boundaries of the cylinder to lie on the same orientifold plane, assumed to be a spatial distance $r$ apart in the $X^9$ direction. Consider turning on an external constant electric field, $F_{09}$. The pair correlator of closed timelike Wilson loops computes the Minkowskian time propagator of a pair of electric sources lying in the worldvolume of D9branes with external electric field, and with fixed spatial separation $r$. In the $T_0$-dual type I’ picture, the pointlike sources are interpreted as being in relative motion in the direction $X^0$ transverse to their spatial separation. As shown in [34, 35], they experience an attractive binding potential of the form $u^4_4/\alpha_4^{4}/r^9$ at zero temperature, where the dimensionless parameter $u$ is defined as $u=\tanh^{-1}F^{09}$. A systematic expansion in field strength to all orders gives the small velocity, short distance, corrections to the leading field-dependent binding interaction between the sources [34, 35]. The Wilson loops represent the Euclidean worldlines of a pair of closely separated semiclassical sources in slow motion.

It is straightforward to extend this result to the finite temperature correlation function and, for clarity, we begin by focussing on the static limit, setting $u = 0$. The path integral expression for the pair correlator, $W_2$, at finite temperature takes the form [33, 34, 35, 8]:

$$W_2(r, \beta) = \lim_{r \to 0} \int_0^\infty dt e^{-r^2t/2\pi \alpha'} \sum_{n \in \mathbb{Z}} q^{4n^2n'/\beta^2} \left[ \left( \frac{\Theta_{00}(it; 0)}{\eta(it)} \right)^4 - \left( \frac{\Theta_{10}(it; 0)}{\eta(it)} \right)^4 \right. \left. - e^{i\pi n} \left( \frac{\Theta_{01}(it; 0)}{\eta(it)} \right)^4 - \left( \frac{\Theta_{11}(it; 0)}{\eta(it)} \right)^4 \right].$$

The static pair potential at short distances is extracted from the dimensionless amplitude $W_2$ as follows. We set $W_2=\lim_{r \to \infty} \int_{-T}^{+T} d\tau V[r(\tau), \beta]$, inverting this relation to express $V[r, \beta]$ as an integral over the modular parameter $t$. Consider a $q$ expansion of the integrand, valid in the limit $r \leq 2\pi \alpha'$, $t \to \infty$, where the shortest open strings dominate the modular integral. Retaining the leading terms in the $q$ expansion and performing explicit term-by-term integration over the worldsheet modulus, $t$, [34, 35], isolates the following short distance interaction [8]:

$$V(r, \beta) = (8\pi^2 \alpha')^{-1/2} \int_0^\infty dt e^{-r^2t/2\pi \alpha'} t^{1/2} \times \sum_{n \in \mathbb{Z}} (16 - 16e^{i\pi n}) q^{4n^2n'/\beta^2} + \cdots$$

$$= 2^4 \frac{1}{r(1 + \frac{r_{min}^2 \beta^2}{\beta^2})^{1/2}} + \cdots$$

$$(38)$$
where we have dropped all but the contribution from the \( n=1 \) thermal mode in the last step. We have expressed the result in terms of the characteristic minimum distance scale probed in the absence of external fields, \( r_{\text{min}} = 2\pi \alpha'^{1/2} \), and the closed string’s self-dual temperature \( T_C \). Consider the crossover in the behavior of the interaction as a function of temperature: at low temperatures, with \( (r/r_{\text{min}})\beta \gg \beta_C \), we can expand in a power series. The leading correction to the inverse power law is \( O(1/\beta^2 r^3) \). At high temperatures with \( (r/r_{\text{min}})\beta \ll \beta_C \), the potential instead approaches a constant independent of \( r \). We note the characteristic signal of the onset of the holographic high temperature phase where the order parameter approaches a constant independent of \( r \). This is reminiscent of the usual signal for confinement, but we should emphasize that we are studying the short distance behavior of the Wilson loop correlator. The short distance regime is not usually accessible in standard gauge theory calculations; inferring the low energy field theory result from the full string theory calculation has enabled exploration of this phenomenon.

Notice that at the crossover between the two phases the potential is a precise inverse power law. Notice also that the overall coefficient of the potential is independent of the spacetime dimensionality of the gauge fields: the factor of \( 2^4 \) is related to the critical dimension of the type I’ string theory—not the dimensionality of the worldvolume of the Dbrane in question.

It is rather interesting to compare these two observations with an old conjecture in the gauge theory literature. Peskin has argued [36] that when the deconfining phase transition in a renormalizable gauge theory is second order or is, more generally, a continuous phase transition without discontinuity in the free energy, the heavy quark potential must necessarily take the scale-invariant form, \( C/r \), with \( C \) a constant independent of spacetime dimension. As mentioned above, the short distance asymptotics of the Wilson loop correlator is dominated by a result in gauge theory alone. Moreover, perturbative string theory is fully renormalizable, and its low energy limit is a renormalizable gauge theory. Finally, the phase transition we have observed is a continuous phase transition as follows from the analyticity properties demonstrated earlier.

The pair potential between semiclassical point sources is indeed found to take the scale-invariant inverse power form at criticality. And the coefficient of the potential is independent of the dimensionality of the worldvolume of the Dbrane, in other words, the spacetime dimension of the gauge theory. Thus, our results are found to be in satisfying accord with the intuition in Peskin’s conjecture [36]. But it should be noted that the original conjecture was not made in the context of thermal deconfinement. Nor was there a well-developed notion of the short distance regime of the “potential”. Hopefully, future work will bring further insight into these issues.

It is rather straightforward to include in this result the dependence on an external electromagnetic field. From our previous works [34, 35, 8], the result above is modified by the simple replacement: \( r_{\text{min}} \rightarrow 2\pi \alpha'^{1/2} u \), where \( F^{09} = \tanh^{-1} u \) is the electric field strength. The transition temperature in the presence of an external field, \( T_d = u T_C \), is consequently lower than the closed string’s self-dual temperature.

5 Conclusions

As emphasized by us in [7], while a point-particle field theory with an exponential growth in the density of states is indeed expected to exhibit a Hagedorn phase transition, there is no sign of such a
transition in the canonical ensemble of free strings. Thus, by the usual procedure of taking the $\alpha'=0$ low energy field theory limit, we can also infer the absence of such a transition in the zero coupling limit of the corresponding low energy field theory. In particular, we see no signal of a Hagedorn phase transition at zero coupling in the nonabelian gauge theories at finite temperature obtained in the $\alpha'=0$ limit of our string theory calculation: the free energy of the gauge theory as inferred from the string theory result is both finite, and free of divergences, at all temperatures. Is this property unique to the anomaly-free gauge-gravity theories explored in infrared-finite perturbative string theories, or does it hold more generally for nonabelian gauge theory? It would be nice to test this result from an independent investigation of the Hagedorn transition in the zero coupling limit of gauge theories. Full insight into the significance of the order parameter for the thermal phase transition in the finite temperature gauge theory demonstrated in section 4.3 also remains open for future work.

Our results provide concrete evidence of the holographic nature of perturbative string theory at high temperatures [2, 37, 14, 39]: the growth of the free energy at high temperatures is that of a two-dimensional field theory. We have identified the duality phase transition in the canonical ensemble of free strings in each case—heterotic $E_8 \times E_8$ and Spin(32)/$\mathbb{Z}_2$, type I and type I', or type IIA and IIB, as belonging to the universality class of the Kosterlitz-Thouless transition [13], characterized by the analyticity in temperature of an infinite hierarchy of thermodynamic potentials. Notice that the precise manifestation of the thermal duality transition is different in each case, and that it acts like a map between a pair of supersymmetric string theories. Our results for the canonical ensemble of type I strings are especially intriguing. As demonstrated unambiguously in an accompanying paper using the worldsheet renormalization group and g-theorem [17, 18], we have shown that the nonsupersymmetric ground state described in section 4 has both vanishing one-loop vacuum energy, and vanishing dilaton tadpole. Supersymmetry is spontaneously broken by thermal effects, except in the closed oriented sector of the theory.

We would like to reiterate a point originally made by us in [8]: since a thermal duality transformation is nothing other than a Euclidean timelike T-duality transformation, and the action of spacelike T-dualities among the six ten-dimensional perturbative string theories is extremely well-established [14], it is not possible to modify the precise role of thermal duality transformations as used in this paper without also violating Lorentz invariance. We offer this as a cautionary comment on recent conjectural applications of thermal duality.

An important general conclusion from this analysis is that, from a physics standpoint, and in the absence of Dbrane sources, the type II superstring theories and their corresponding low-energy limits: pure supergravity theories with 32 supercharges, are a somewhat artificial truncation of the more physical type II theories with Dbranes. These are string theories with 16 or fewer supercharges, and they have both Yang-Mills fields and gravity in common with the type I and heterotic string theories. This insight has played a key role in my renewed focus on theories with 16 supercharges in recent years. Recovering the hidden symmetry algebra of supergravity theories with 32 supercharges as a special limit of the algebra of the theories with 16 supercharges and Yang-Mills degrees of freedom, is a key element of the proposal for M Theory given in [40].

Acknowledgements: I thank C. Bachas, A. Dhar, J. Distler, M. Fukuma, P. Ginsparg, J. Harvey, G. Horowitz, S. Kachru, H. Kawai, D. Kutasov, M. Peskin, J. Polchinski, G. Shiu, B. Sundborg, H.
Tye, and E. Witten for early comments. This research was supported in part by the award of grant NSF-PHY-9722394 by the National Science Foundation under the auspices of the Career program. This paper was updated at the Aspen Center for Physics, following the appearance of [17]. I am grateful to Hassan Firouzjahi for pointing out a minor, but embarrassing, typo in the discussion of the thermal tachyon spectrum of the pure type II closed string ensemble in an earlier version of this work. I would like to thank S. Abel, K. Dienes, H. Firouzjahi, M. Kleban, and E. Mottola for their questions and interest, and Scott Thomas for the invitation to present this work at the Cosmic Acceleration workshop.

Note Added (Sep 2005): Many of the points made in this paper are either extraneous, or incorrect in the details, although the broad conclusions do stand. Namely, that there is no self-consistent type II superstring ensemble, in the absence of a Yang-Mills gauge sector. The fact that both heterotic and type I theory have equilibrium canonical ensembles free of thermal tachyons; a crucial role is played by the temperature dependent Wilson line wrapping Euclidean time. In addition to topics covered in hep-th/0105244, we have the discussion from hep-th/0208112 of the type IB-I’ open and closed string ensembles, including the evidence for an order parameter for the unusual duality phase transition in this theory [8]. A step forward is the correction that the thermal tachyon spectrum in the type II theories covers the full temperature range, down to $T = 0$. Note that the pressure of the heterotic and type I string ensembles equals the negative of the vacuum energy density, incorrectly stated in my previous papers; I thank H. Firouzjahi for pointing this out. The remaining errors have to do with thermal mode number dependent phases that are not modular invariant, which I only became aware of in hep-th/0506143.

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