ERDŐS-RÉNYI LAW OF LARGE NUMBERS IN THE AVERAGING SETUP.

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Abstract. We extend the Erdős–Rényi law of large numbers to the averaging setup both in discrete and continuous time cases. We consider both stochastic processes and dynamical systems as fast motions whenever they are fast mixing and satisfy large deviations estimates. In the continuous time case we consider flows with large deviations estimates which allow a suspension representation and it turns out that fast mixing of corresponding base transformations suffices for our results.

1. Introduction

Let \( \xi_1, \xi_2, \ldots \) be a sequence of independent identically distributed (i.i.d.) random variables such that \( E\xi_1 = 0 \) and the moment generating function \( M(t) = Ee^{t\xi_1} \) exists. Denote by \( I \) the Legendre transform of \( \ln M \) and set \( X_n = \sum_{m=1}^{n} \xi_m \) for \( n \geq 1 \) and \( X_0 = 0 \). The Erdős-Rényi law of large numbers from \([16]\) says that with probability one,

\[
I(\beta) \lim_{n \to \infty} \max_{0 \leq m \leq \lfloor \frac{n-1}{\ln n} \rfloor} \frac{X_{m+\lfloor \frac{n}{\ln n} \rfloor} - X_m}{\ln n} = \beta
\]

for all \( \beta > 0 \) in some neighborhood of zero (actually, whenever \( I(\beta) < \infty \)). A related result was proved earlier in \([28]\) where the maximum in \( (1.1) \) is taken in \( m \) up to \( n \) and \( \ln n \) is replaced in the above numerator by \( \ln m \) so that this type of limits are sometimes called Erdős-Rényi-Shepp laws. There were numerous specifications and extensions of the Erdős-Rényi law for the last 40 years (see, for instance \([11]\) and references there) while in \([26]\) a corresponding limit law was derived. As the original version \( (1.1) \) most of these results were valid only in the one dimensional case, i.e. for random variables and not for random vectors. On the other hand, a functional form of \( (1.1) \) suggested in \([1]\) holds true for i.i.d. random vectors, as well. More recently papers on the Erdős-Rényi law appeared in the dynamical systems framework where extensions from i.i.d. to weakly dependent summands became necessary. In \([10]\) the Erdős-Rényi law was derived for functions of iterates of expanding maps of the interval while an extension of \( (1.1) \) to stationary α-mixing...
sequences was obtained in [12] and to functions of some nonuniformly expanding dynamical systems in [13]. In somewhat different direction an Erdős-Rényi law for Gibbs measure was derived earlier in [8].

In this paper we extend the Erdős-Rényi type results to the averaging setup which was not considered before generalizing all previous approaches to the problem (except for the case of nonconventional sums studied in [23]). We consider the slow motion $X_{\varepsilon}$ in both the discrete time case

$$(1.2) \quad X_{n+1}^{\varepsilon} = X_n^{\varepsilon} + \varepsilon B(X_{n}^{\varepsilon}, \xi_n), \quad X_0^{\varepsilon} = x$$

and in the continuous time case

$$(1.3) \quad \frac{dX_t^{\varepsilon}}{dt} = \varepsilon B(X_t^{\varepsilon}, \xi_t), \quad X_0^{\varepsilon} = x.$$  

Here the fast motion $\xi_n, n \in \mathbb{Z}$ or $\xi_t, t \in \mathbb{R}$ is a stationary stochastic process, in particular, it can be generated by a dynamical system $\xi_n = \xi_n(x) = f^n x, n \in \mathbb{Z}$ or $\xi_t = \xi_t(x) = f^t x, t \in \mathbb{R}$ preserving some probability measure $\mu$ which plays the role of probability on the corresponding space where $x$ lives. In the discrete time we assume that $\xi_n$ is exponentially fast $\alpha$-mixing while in the continuous time case, in order to enable applications to important classes of dynamical systems, we assume that $\xi_t$ can be represented via so called suspension construction over an exponentially fast $\alpha$-mixing discrete time stationary process.

We observe that (1.2) and (1.3) are generalizations of usual Cesàro averages of sums or integrals since if $B$ does not depend on the slow motion $X^\varepsilon$ then in the discrete time case

$$X_{[1/\varepsilon]}^{\varepsilon} = x + \varepsilon \sum_{0 \leq n < [1/\varepsilon]} B(\xi_n) \quad \text{and} \quad X_{1/\varepsilon}^{\varepsilon} = x + \varepsilon \int_0^{1/\varepsilon} B(\xi_t)dt$$

in the continuous time case.

If we fix an ergodic stationary measure $\mu$ of the process $\{\xi_n, n \in \mathbb{Z}_+\}$ or $\{\xi_t, t \in \mathbb{R}_+\}$ then by the ergodic theorem the limits

$$(1.4) \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} B(x, \xi_n) = \bar{B}(x) \quad \text{or} \quad \lim_{T \to \infty} \frac{1}{T} \int_0^T B(x, \xi_t)dt = \bar{B}(x)$$

exist $\mu$-almost surely (a.s.) but, of course, they depend on $\mu$. The averaging principle proved rigorously about 70 years ago (see [29] and references there) says that if $B$ is Lipschitz continuous in the first variable then (1.4) implies that

$$(1.5) \quad \lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T/\varepsilon} |X_t^{\varepsilon} - \bar{X}_t^{\varepsilon}| = 0$$

where $\bar{X}_t^{\varepsilon}$ is the averaged motion solving the equation

$$(1.6) \quad \frac{d\bar{X}_t^{\varepsilon}}{dt} = \varepsilon \bar{B}(\bar{X}_t^{\varepsilon}), \quad \bar{X}_0^{\varepsilon} = x.$$  

Since, in view of the above, the averaging principle can be considered as a generalization of the ergodic theorem, which in the probabilistic language is, essentially, a law of large numbers, it would be natural to ask whether the Erdős-Rényi law can be generalized to the averaging setup, as well. It should be clear from the beginning that the latter cannot be obtained in so general circumstances as the averaging principle itself since the Erdős-Rényi law strongly relies on large deviations which can be proved only for certain classes of processes.
In this paper we derive the Erdős-Rényi law type results in a functional form for the slow motion $X^n_t$ (both in discrete and continuous time cases) in place of sums of random variables in (1.1). When $B$ in (1.2) or in (1.3) does not depend on the first variable our results yield the functional form of the Erdős-Rényi law which was introduced in [1] in the particular case of sums of i.i.d. random vectors. When, in addition, $B$ is one dimensional this implies the Erdős-Rényi law in the form (1.1) but in a much more general situation.

This paper extends previous results on the Erdős-Rényi law in several directions. First, it was never considered before in the averaging setup. Secondly, its functional form appeared before only for sums of i.i.d. vectors. Thirdly, this law was never dealt with in the continuous time case. Finally, we require weaker $\alpha$-mixing and not $\psi$-mixing conditions which appeared in [12].

Our results are applicable to several types of stationary processes $\xi_t$. On the probabilistic side $\xi_t$ can be, in particular, a Markov chain satisfying an appropriate Doeblin condition or a nondegenerate diffusion process on a compact manifold. On the dynamical systems side we can take $\xi_t(x) = f^t x$ with $f$ being an Axiom A diffeomorphism or flow on a hyperbolic set (and could be considered also in a neighborhood of an attractor) while in the discrete time case additional options are possible such as mixing subshifts of finite type, expanding transformations and some maps of the interval.

The structure of this paper is the following. In the next section we present precisely our general setup and formulate main results under conditions which include both probabilistic and dynamical systems examples mentioned above. In Sections 3 and 4 we give proofs of our results in the discrete and continuous time cases, respectively. In Appendix we discuss applications to various specific classes of stochastic processes and dynamical systems and recall properties of rate functions of large deviations needed in the proofs of our results.

2. Preliminaries and main results

Let $M$ be a Polish (complete separable metric) space, $\mathbb{R}^d$ be a $d$-dimensional Euclidean space and a bounded Borel map $B : \mathbb{R}^d \times M \to \mathbb{R}^d$ satisfies

\begin{equation}
|B(x, y) - B(z, y)| \leq L_1|x - z|, \quad |B(x, y)| \leq L_1
\end{equation}

for some $L_1 > 0$, all $x, z \in \mathbb{R}^d$ and any $y \in M$. We consider also a stationary ergodic stochastic process $\xi_t$ with discrete $t \in \mathbb{Z}_+ = \{0, 1, \ldots\}$ or continuous $t \in \mathbb{R}_+ = \{s \geq 0\}$ time on a probability space $(\Omega, \mathcal{F}, P)$ with values in $M$. Our setup includes also a sequence $\mathcal{F}_{m,n} \subset \mathcal{F}$, $-\infty \leq m \leq n \leq \infty$ of $\sigma$-algebras such that $\mathcal{F}_{m,n} \subset \mathcal{F}_{m_1,n_1}$ whenever $m_1 \leq m$ and $n_1 \geq n$ which satisfies an exponentially fast $\alpha$-mixing condition (see, for instance, [5]),

\begin{equation}
\alpha(n) = \sup \{ |P(A \cap B) - P(A)P(B)| : A \in \mathcal{F}_{-\infty,k}, \ B \in \mathcal{F}_{k+n,\infty} \} \leq \kappa_1^{-1} e^{-\kappa_1 n}
\end{equation}

for some $\kappa_1 > 0$ and all $k, n \geq 0$.

In the discrete time case we rely on the following approximation condition

\begin{equation}
\zeta(n) = E \sup_{x,k} |B(x, \xi_k) - E(B(x, \xi_k) | \mathcal{F}_{k-n,k+n})| \leq \kappa_2^{-1} e^{-\kappa_2 n}
\end{equation}

for some $\kappa_2 > 0$ and all $n \geq 0$. 
Next, set \( \tilde{B}(x) = EB(x, \xi_0) \), \( \tilde{Z}_t = \tilde{X}_{t/\varepsilon}^\gamma \), \( B_t(y) = B(\tilde{Z}_t, y) \), \( \tilde{B}_t = EB_t(\xi_0) \) and \( G_t(y) = B_t(y) - \tilde{B}_t \) where \( \tilde{X}_t^\gamma \) satisfies (1.6), and so

\[
\frac{d\tilde{Z}_t}{dt} = \tilde{B}(\tilde{Z}_t).
\]

Define

\[
Y_{t,r}(u) = \frac{1}{r} \sum_{0 \leq j \leq r} G_t(\xi_j), \quad u \in [0, 1], \quad r \in \mathbb{N} = \{1, 2, \ldots\}.
\]

Denote the space of continuous curves \( \gamma : [0, 1] \to \mathbb{R}^d \) by \( C([0, 1], \mathbb{R}^d) \) and assume that for any \( \gamma \in C([0, 1], \mathbb{R}^d) \) and \( t \in [0, T] \) the limit

\[
\lim_{r \to \infty} \ln E \exp(r \int_0^1 (\gamma_u, G_t(\xi_{[ru]})))du = \int_0^1 \Pi_t(\gamma_u)du
\]

exists, where \( \Pi_t(b) \), \( b \in \mathbb{R}^d \) is a convex twice differentiable function such that \( \nabla_b \Pi_t(b)|_{b=0} = 0 \) and the Hessian matrix \( \nabla_b^2 \Pi_t(b)|_{b=0} \) is positively definite. Here \( (\cdot, \cdot) \) denotes the inner product. Let

\[
I_t(\beta) = \sup_b ((b, \beta) - \Pi_t(b))
\]

and for any \( \gamma \in C([0, 1], \mathbb{R}^d) \) set

\[
S_t(\gamma) = \int_0^1 I_t(\gamma_s)ds, \quad \gamma_s = \frac{d\gamma_s}{ds}
\]

if \( \gamma \) is absolutely continuous and \( S_t(\gamma) = \infty \) for otherwise. It follows from the above (see, for instance, Section 7.4 in [17]) that \( Y_{t,r} \) satisfies large deviations estimates in the form that for any \( a, \delta, \lambda > 0 \) and every \( \gamma \in C([0, 1], \mathbb{R}^d) \), \( \gamma_0 = 0 \) there exists \( r_0 > 0 \) such that for \( r \geq r_0 \),

\[
P\{\rho(Y_{t,r}, \gamma) < \delta\} \geq \exp(-r(S_t(\gamma) + \lambda)) \quad \text{and} \quad P\{\rho(Y_{t,r}, \Phi_t(a)) \geq \delta\} \leq \exp(-r(a - \lambda))
\]

where \( \rho(\gamma, \eta) = \sup_{s \in [0, 1]} |\gamma_s - \eta_s| \) and \( \Phi_t(a) = \{ \gamma \in C([0, 1], \mathbb{R}^d) : \gamma_0 = 0, S_t(\gamma) \leq a \} \).

Since \( S_t \) is a lower semi-continuous functional then each \( \Phi_t(a) \) is a closed set and, moreover, it is compact for any finite \( a \). Indeed, \( |\Pi_t(b)| \leq 2L_1 |b| \) by (2.1) and (2.6) which implies by (2.7) that \( I_t(\beta) = \infty \) provided \( |\beta| > 2L_1 \) (take \( b = a\beta/|\beta| \) in (2.7) and let \( a \to \infty \)). Hence, \( |\gamma_s| \leq 2L_1 \) for Lebesgue almost all \( s \in [0, 1] \) if \( \gamma \in \Phi_t(a) \), and so the latter set is bounded and equicontinuous which by the Arzelà-Ascoli theorem implies its compactness.

In the discrete time case (1.2) for any \( \varepsilon > 0, t \in [0, T], u \in [0, 1] \) and \( N \in \mathbb{N} \) set

\[
V_{t,N}^\varepsilon(u) = \frac{X_t^\varepsilon - X_t^{\varepsilon/\varepsilon} - X_t^{\varepsilon/\varepsilon}}{\varepsilon b_t(\varepsilon, N) - uB_t}
\]

where

\[
b_t(\varepsilon, N, u) = c_{t}(t,N)u \ln \frac{1}{\varepsilon}, \quad b_t(\varepsilon, N) = b_t(\varepsilon, N, 1), \quad \tau(t, N) = [Nt/T]T/N
\]

and \( c_t \) is a function on \([0, T]\) such that \( 0 < \hat{c}^{-1} \leq c_t \leq \hat{c} < \infty \) for some \( \hat{c} \).
2.1. Theorem. Assume that the conditions \((2.2), (2.3)\) and \((2.7)\) hold true. Then \(V^ε,N_t(u)\) defined above satisfies

\[
\lim_{N \to \infty} \limsup_{\epsilon \to 0} \sup_{0 \leq t \leq T} \rho \left( V^ε,N_t, \Phi_{T(t,N)}(c^{-1}_{T(t,N)}) \right) = 0 \quad \text{a.s.}
\]

and

\[
\lim_{N \to \infty} \limsup_{\epsilon \to 0} \sup_{\gamma \in \Phi(1/c)} \inf_{0 \leq t \leq T} \rho \left( V^ε,N_t, \gamma \right) = 0 \quad \text{a.s.}
\]

2.2. Corollary. Suppose that \(B(x,y) = B(y)\) does not depend on the first variable. Then \(G_t, Y_{t,r}, I_t\) and \(S_t\) in \((2.3), (2.6)\) do not depend on \(t\), and so \(\Phi_t(a) = \Phi(a)\) does not depend on \(t\), as well. Let \(c_t \equiv c > 0\) be a constant then \(V^ε,N_t = V^ε_t\) does not depend on \(N\). Set \(W^c_t = \cup_{0 \leq t \leq T} V^c_t\). Then for any \(c > 0\),

\[
\lim_{\epsilon \to 0} H(W^c_t, \Phi(1/c)) = 0 \quad \text{a.s.}
\]

where \(H(\Gamma_1, \Gamma_2) = \inf \{ \delta > 0 : \Gamma_1 \subset \Gamma_\delta, \Gamma_\delta \subset \Gamma_1 \}\) is the Hausdorff distance between sets of curves with respect to the uniform metric \(\rho\) (and \(\Gamma_\delta = \{ \gamma : \rho(\gamma, \Gamma) < \delta \}\) is the \(\delta\)-neighborhood of \(\Gamma\)).

2.3. Corollary. Suppose that \(d = 1\) and set \(c_t = \frac{1}{\pi(\beta)}\) where \(\beta_t > \beta > 0\) and \(\beta_t = \sup \{ \beta > 0 : I_t(\beta) < \infty \}\). Then

\[
\lim_{N \to \infty} \limsup_{\epsilon \to 0} V^ε,N_t(1) = \beta \quad \text{a.s.}
\]

where \(\lim_{N \to \infty} \limsup_{\epsilon \to 0} = \lim_{N \to \infty} \limsup_{\epsilon \to 0} = \lim_{N \to \infty} \liminf_{\epsilon \to 0}\). In particular, if \(d = 1\) and \(B(x,y) = B(y)\) does not depend on the first variable then \(I_t \equiv I\) does not depend on \(t\) and \((2.13)\) holds true with \(c_t \equiv \frac{1}{\pi(\beta)}\) for all \(0 < \beta < \beta_0 = \sup \{ \beta > 0 : I(\beta) < \infty \}\).

We observe that the proof of Theorem 2.1 and of Corollary 2.2 require certain time discretization which cannot be achieved relying on some continuity properties since \(I_t\) and \(S_t\) are only lower semi continuous in \(t\). By this reason we had to introduce \(\tau(t,N)\) and to have the second limit as \(N \to \infty\). In fact, if \(|\beta|\) is small enough then \(I_t(\beta)\) is continuous in \(t\) but in order to use this we would have to consider only curves \(\gamma\) with uniformly small speeds \(|\gamma_t|\) which would not be a natural restriction. Though we work in a substantially more general averaging setup the strategy of our proof of Theorem 2.1 resembles previous works, in particular, [12] and [13] but observe that we rely only on \(\alpha\)-mixing and not on a stronger \(\psi\)-mixing assumed in the above papers. Large deviations estimates for hyper-geometrically fast \(\alpha\) and \(\phi\)-mixing stationary sequences were derived in [3] and [6] while existence of such processes follows from Theorem 2 in [4]. We observe also that Corollary 2.3 does not require full strength of large deviations in the form \((2.9)\) and it suffices to have here usual level one large deviations estimates for \(Y_{t,r}(1)\) in the form

\[
\limsup_{\tau \to \infty} \frac{1}{\eta} \ln P \{ Y_{t,r}(1) \in K \} \leq -\inf_{b \in K} I(b) \quad \text{and}
\]

\[
\liminf_{\tau \to \infty} \frac{1}{\eta} \ln P \{ Y_{t,r}(1) \in U \} \geq -\inf_{b \in U} I(b)
\]

for any closed \(K\) and open \(U\) subsets of real numbers. Our method will still go through with minor modifications if the exponentially fast decay in \((2.2)\) and \((2.3)\) is replaced by a stretched exponential one, i.e. by \(\exp(-\kappa n^\delta)\) for some \(\kappa, \delta > 0\).

Next, we deal with the continuous time case. In addition to a stationary ergodic process \(\xi_t, t \in \mathbb{R}_+\) on a probability space \((\Omega, \mathcal{F}, P)\) with a path shift operator
\[ \theta : \Omega \to \Omega \] we consider now an embedded discrete time process \( \eta_k, k \in \mathbb{Z}_+ \) related to \( \xi_t(\omega) = \xi_0(\theta^t \omega) \) by means of measurable maps \( \varphi : \Omega \to \Omega \subset \Omega, \) \( \theta : \hat{\Omega} \to \hat{\Omega} \) and a measurable function \( \zeta : \hat{\Omega} \to \mathbb{R}_+ \) such that \( \varphi^{-1}(\hat{\omega}) \subset \{ \theta^t \hat{\omega} : 0 \leq t < \zeta(\hat{\omega}) \} \) and for any \( \omega = \varphi(\omega) \) and \( k \geq 0, \)

\[
\hat{\theta}^k(\omega) = \theta^k \omega, \quad \eta_k(\hat{\omega}) = \eta_0(\hat{\theta}^k \omega) \text{ and } \eta_0(\hat{\omega}) = \xi_0(\hat{\omega}).
\]

Set \( \hat{P} = \varphi P \) and \( \hat{\mathcal{F}} = \varphi \mathcal{F}. \) We assume that there exists a \( \theta \)-invariant probability measure \( Q \) on \( (\hat{\Omega}, \hat{\mathcal{F}}) \) equivalent to \( \hat{P} \) and such that

\[
L_2^{-1} \leq \frac{dQ}{d\hat{P}} \leq L_2
\]

for some \( L_2 > 0. \) Thus, \( \eta_k, k \geq 0 \) is a stationary process with respect to \( Q. \) For each \( \omega = \varphi(\omega) \) and \( j \geq 0 \) set

\[
\hat{B}(x, \omega) = \hat{B}(x, \omega) = \int_0^{\zeta(\omega)} \hat{B}(x, \xi_u(\omega)) du.
\]

We assume that there exist sub \( \sigma \)-algebras \( \mathcal{F}_{n, m} \subset \hat{\mathcal{F}} \) on \( \hat{\Omega} \) satisfying (2.2) with \( Q \) in place of \( P \) and such that for some \( \kappa_3, L_3 > 0 \) and all \( n \geq 0, \)

\[
\sup_{x,j} \hat{E}[\hat{B}(x, \cdot) \circ \theta^j - \hat{E}(\hat{B}(x, \cdot) \circ \theta^j | \mathcal{F}_{j-n-1}] \leq \kappa_3^{-1} e^{-\kappa_3 n},
\]

\[
L_3^{-1} \leq \zeta \leq L_3 \quad \text{and} \quad \sup_{j \geq 0} \hat{E}[\zeta \circ \theta^j - \hat{E}(\zeta \circ \theta^j | \mathcal{F}_{j-n-1}] \leq \kappa_3^{-1} e^{-\kappa_3 n},
\]

where \( \hat{E} \) is the expectation with respect to \( \hat{P}. \) Our proof will also go through if in the approximation conditions (2.14) and (2.15) we take the expectation \( E_Q \) with respect to \( Q \) in place of \( \hat{E}. \) We assume also an upper large deviations bound for \( \sigma_n = \sum_{i=0}^{n-1} \zeta \circ \theta^i \) in the form that for any \( \delta > 0 \) there exists \( \kappa_3 > 0 \) such that

\[
Q\{ \frac{1}{n} \sigma_n \geq \bar{\zeta}(1 + \delta) \} \leq \kappa_3^{-1} e^{-\kappa_3 n}
\]

where \( \bar{\zeta} = E_Q \zeta. \)

Now, let \( X_t^\varepsilon \) be defined by (1.3), \( \hat{X}_t^\varepsilon \) be defined by (1.4) and, again, \( \hat{Z}_t = \hat{X}_{t/\varepsilon^2}, \hat{B}_t(y) = B(Z_t, y), \hat{B}_i = E\hat{B}_i(\xi_0) \) and \( G_t(y) = B_t(y) - B_i. \) Set

\[
Y_{t,r}(u) = \frac{1}{r} \int_0^r G_t(s) ds, \quad u \in [0, 1].
\]

Assume that

\[
\lim_{r \to \infty} \frac{1}{r} \ln \exp(r \int_0^1 (\gamma_u, G_t(\xi_{ru})) du) = \int_0^1 \Pi_t(\gamma_u) du
\]

exists for any \( \gamma \in C([0, 1], \mathbb{R}^d) \) with \( Y_{t,r} \) defined by (2.20), where, again, \( \Pi_t(b), b \in \mathbb{R}^d \) is a convex twice differentiable function such that \( \nabla \Pi_t(b)|_{b=0} = 0 \) and the Hessian matrix \( \nabla^2 \Pi_t(b)|_{b=0} \) is positively definite. Then, again, by (17) the large deviations estimates (2.19) hold true with \( S_t(\gamma) \) defined by (2.7) and (2.8).

2.4. Theorem. Assume that the conditions (2.2), (2.6) and (2.16)–(2.19) hold true. Then \( V_t^\varepsilon \) defined by

\[
V_t^\varepsilon, N(u) = \frac{X_{t/\varepsilon + b_t(\varepsilon, N, u)}^\varepsilon - X_{t/\varepsilon}^\varepsilon}{\varepsilon b_t(\varepsilon, N)} - u \hat{B}_t
\]

satisfies (2.10) and (2.11).
2.5. **Corollary.** Corollaries 2.2 and 2.3 remain true under the conditions of Theorem 2.4 as well.

The proof of the upper bound (2.10) proceeds in the continuous time case similarly to Theorem 2.1 since it uses essentially only stationarity of the process \( \psi_i \) and the large deviations bounds which are our assumptions in the above setup. On the other hand, the proof of the lower bound (2.11) requires additional ingredients in the continuous time case since in order to accommodate important classes of dynamical systems we do not impose strong mixing conditions on the process \( \xi_i \) itself but only on the base discrete time process \( \eta_k \) (via the family of \( \sigma \)-algebras \( \mathcal{F}_{m,n} \)). To the best of our knowledge the Erdős-Rényi law of large numbers type results were not obtained before in any continuous time framework. If the fast motion \( \xi_i \) is, say, a nondegenerate diffusion process on a compact manifold then it is exponentially fast \( \psi \)-mixing with respect to \( \sigma \)-algebras generated by itself (see, for instance [7]) and in this case Theorem 2.4 is easy to derive from Theorem 2.1. Our setup is adapted, in particular, to important classes of dynamical systems such as Axiom A flows which may exhibit arbitrarily slow mixing but, on the other hand, can be represented by means of the above suspension construction over an exponentially fast \( \psi \)-mixing base transformation (see [2]).

### 3. Discrete time case

#### 3.1. Basic estimates.

Taking into account (1.2) write

\[
(3.1) \quad \varepsilon^{-1}(X^\varepsilon_{[t/\varepsilon]}-[b_t(\varepsilon,N),u]-X^\varepsilon_{[t/\varepsilon]}) = \sum_{[t/\varepsilon]<n\leq[t/\varepsilon]+[b_t(\varepsilon,N),u]} B(X^\varepsilon_n,\xi_n) = \sum_{[t/\varepsilon]<n\leq[t/\varepsilon]+[b_t(\varepsilon,N),u]} B_t(\xi_n) + \Psi^{(1)}_{\varepsilon,N}(t,u) + \Psi^{(2)}_{\varepsilon,N}(t,u)
\]

where

\[
\Psi^{(1)}_{\varepsilon,N}(t,u) = \sum_{[t/\varepsilon]<n\leq[t/\varepsilon]+[b_t(\varepsilon,N),u]} (B(X^\varepsilon_n,\xi_n) - B(X^\varepsilon_n,\xi_n))
\]

and

\[
\Psi^{(2)}_{\varepsilon,N}(t,u) = \sum_{[t/\varepsilon]<n\leq[t/\varepsilon]+[b_t(\varepsilon,N),u]} (B(X^\varepsilon_n,\xi_n) - B_t(\xi_n)).
\]

By (2.4),

\[
(3.2) \quad |\Psi^{(1)}_{\varepsilon,N}(t,u)| \leq L_1 b_t(\varepsilon,N) \Psi^{(3)}_{\varepsilon,N}
\]

where by the averaging principle (1.5) with probability one uniformly in \( N \),

\[
(3.3) \quad \Psi^{(3)}_{\varepsilon,N} = \sup_{0\leq n\leq T(\varepsilon^{-1})} |X^\varepsilon_n - \bar{X}^\varepsilon_n| \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Furthermore, by (1.4), (1.6) and (2.4),

\[
(3.4) \quad |\Psi^{(2)}_{\varepsilon,N}(t,u)| \leq L_1 b_t(\varepsilon,N) \Psi^{(4)}_{\varepsilon,N}
\]

where

\[
(3.5) \quad \Psi^{(4)}_{\varepsilon,N} = \sup_{0\leq t\leq T} \sup_{|t/\varepsilon|<n\leq t/\varepsilon+[b_t(\varepsilon,N)]} |X^\varepsilon_n - \bar{X}^\varepsilon_{t/\varepsilon}| \leq L_1 \varepsilon b_t(\varepsilon,N) \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

It follows from (3.1)–(3.5) that (2.10) and (2.11) will hold true provided we establish these limits for

\[
\hat{V}^{\varepsilon,N}_t(u) = \frac{1}{b_t(\varepsilon,N)} \sum_{[t/\varepsilon]<n\leq[t/\varepsilon]+[b_t(\varepsilon,N),u]} (a_t(\xi_n) - \bar{B}_t), \quad t \in [0,T], \quad u \in [0,1]
\]

with
in place of \( V_{i}^{ε,N}(u) \). Next, observe that
\[
(3.6)
|B_{t}(y - \bar{B}_{τ(t,N)}) + |\bar{B}_{τ(t,N)}| \leq 2L_{1}|\bar{Z}_{t} - \bar{Z}_{τ(t,N)}| \leq 2L_{1}^{2}|t - τ(t,N)| \leq 2L_{1}^{2}TN^{-1}.
\]
It follows from here that (2.10) and (2.11) will hold true provided we obtain these limits for
\[
(3.7)
W_{i}^{ε,N}(u) = \frac{1}{b_{t}(ε,N)} \sum_{t \leq t \leq t + [b_{t}(ε,N,u)]} (B_{τ(t,N)}(ξ_{n}) - \bar{B}_{τ(t,N)}),
\]
t in \([0,T] \), \( u \in [0,1] \) in place of \( V_{i}^{ε,N}(u) \).

3.2. The upper bound. For \( k = 0, ..., N, l = 0, 1, ..., [T/ε] \) and \( u \in [0,1] \) set
\[
\hat{W}_{k,l}(u) = b_{kT/N}(ε,N) \sum_{t \leq t \leq t + [b_{t}(ε,N,u)]} (B_{kT/N}(ξ_{n}) - \bar{B}_{kT/N}).
\]
In order to prove (2.10) for \( W_{i}^{ε,N} \) given by (3.7) in place of \( V_{i}^{ε,N} \) it suffices to show that with probability one,
\[
(3.8)
\lim_{N \to \infty} \sup_{ε \to 0} \max_{0 \leq k \leq N} \max_{0 \leq l \leq [T/ε]} \rho(\hat{W}_{k,l}(u), \Phi_{kT/N}(1/c_{kT/N})) = 0.
\]
For any \( ε, δ > 0 \) and each integer \( N \geq 1 \) define the event
\[
Γ(δ,ε,N) = \{ \max_{0 \leq k \leq N} \max_{0 \leq l \leq [T/ε]} \rho(\hat{W}_{k,l}(u), \Phi_{kT/N}(1/c_{kT/N})) > δ \}.
\]
Then
\[
(3.9)
P(Γ(δ,ε,N)) \leq \sum_{0 \leq k \leq N} \sum_{0 \leq l \leq [T/ε]} P(\hat{Γ}_{k,l}(ε,N))
\]
where
\[
\hat{Γ}_{k,l}(ε,N) = \{ \rho(\hat{W}_{k,l}(u), \Phi_{kT/N}(1/c_{kT/N})) > δ \}.
\]
Recall, that each \( \Phi_{k}(a) \) is a compact set. It follows that for any \( t \geq 0 \) and \( a, δ > 0 \) there exists \( σ = σ_{k,a,δ} \) such that (cf. 11),
\[
(3.10)
\Phi_{k}(a + σ) \subset U_{δ}(\Phi_{k}(a)) = \{ x : \rho(x, \Phi_{k}(a)) < δ \}.
\]
Indeed, if for some \( t \geq 0 \) and \( a, δ > 0 \) the sets \( Q_{σ} = \Phi_{k}(a + σ) \setminus U_{δ}(\Phi_{k}(a)) \neq \emptyset \) for all \( σ > 0 \) then \( ∩_{σ > 0} Q_{σ} \neq \emptyset \) since \( Q_{σ} \), \( σ > 0 \) are compact and \( Q_{σ} \supseteq Q_{σ'} \) when \( σ > σ' \). But if \( γ_{0} \in ∩_{σ > 0} Q_{σ} \) then \( γ_{0} \notin U_{δ}(\Phi_{k}(a)) \) which contradicts the fact that \( γ_{0} \notin U_{δ}(\Phi_{k}(a)) \).

Observe that
\[
\hat{W}_{k,0,N}(u) = Y_{kT,N-1,N}^{-1}\ln(1/ε)(u)
\]
with \( Y_{t,r} \) defined by (2.6). Hence, choosing \( σ > 0 \) satisfying (3.10) for \( a = c_{kT/N}^{-1}, t = kT/N \) and \( δ > 0 \) we obtain employing the upper bound of large deviations from (2.9) that
\[
(3.11)
P(\hat{Γ}_{k,l}(ε,N)) \leq P(\rho(\hat{W}_{k,l}(u), \Phi_{kT/N}(c_{kT/N}^{-1} + σ)) > δ)
\leq \exp(-b_{kT/N}(ε)(c_{kT/N}^{-1} + σ - λ)) \leq ε^{1+δσ_{kT/N}εd_{k,N}}
\]
where \( ε \leq ε_{0} \) and \( ε_{0} > 0 \) is chosen so small that (3.11) holds true for some \( λ < σ \), and so \( δ = δ_{k,δ} = σ - λ > 0 \) while \( d_{k,N} = c_{kT/N}^{-1} + δ_{k,δ} \). Here \( δ \) depends on \( k \) and \( δ \) so we take \( δ_{δ} = \min_{0 \leq k \leq N} δ_{k,δ} \). Now, by (3.9) and (3.10),
\[
(3.12)
P(Γ(δ,ε,N)) \leq T \sum_{0 \leq k \leq N} ε^{δ_{k,δ}c_{kT/N}d_{k,N}} \leq TNε^{1+δεδ}\hat{δ}_{k,δ}-1
\]
where, recall, $\tilde{c}^{-1} \geq \xi_1 \geq \tilde{c} > 0$.

Choose the sequence $\varepsilon_n = n^{-\frac{2}{m}}$. Then (3.12) together with the Borel-Cantelli lemma yield that for $P$-almost all $\omega$ there exists $n_\delta(\omega) < \infty$ such that $\omega \notin \Gamma_{2\delta}(\varepsilon_n, N)$ for all $n \geq n_\delta(\omega)$. It follows that with probability one,

$$\limsup_{n \to \infty} \max_{0 \leq k \leq N} \max_{0 \leq l \leq \lfloor T/\varepsilon_n \rfloor} \rho(W_{k,l}^{\varepsilon,N}, \Phi_{kT/N}(c_{kT/N}^{-1})) \leq 2\delta. \tag{3.13}$$

Now observe that if $\varepsilon_n \leq \varepsilon < \varepsilon_{n-1}$ then 

$$|b_t(\varepsilon, N, u) - b_t(\varepsilon_n, N, u)| \leq 2\tilde{c}^2 \sigma_5 \ln \frac{n}{n-1} \to 0 \text{ as } n \to \infty.$$ 

This together with (3.13) gives that with probability one,

$$\limsup_{\varepsilon \to 0} \max_{0 \leq k \leq N} \max_{0 \leq l \leq \lfloor T/\varepsilon \rfloor} \rho(W_{k,l}^{\varepsilon,N}, \Phi_{kT/N}(c_{kT/N}^{-1})) \leq 2\delta$$

for any $N \in \mathbb{N}$. Since $\delta > 0$ is arbitrary we obtain (3.3) even without $\limsup_{N \to \infty}$ yielding (2.10).

3.3. The lower bound. In view of (3.3) (3.10) it suffices to establish (2.11) for $W_t^{\varepsilon,N}$ in place of $V_t^{\varepsilon,N}$. Next, we have

$$P\{\inf_{0 \leq l \leq T} \rho(W_t^{\varepsilon,N}, \gamma) \geq \delta\} \leq P\{\cap_{0 \leq l \leq T/N} \{\rho(W_t^{\varepsilon,N}, \gamma) \geq \delta\}\}$$

$$= P\{\cap_{0 \leq l \leq T/N} \{\rho(\tilde{W}_{0,l}^{\varepsilon,N}, \gamma) \geq \delta\}\}$$

$$\leq P\{\cap_{0 \leq j \leq T/N} \{\rho(\tilde{W}_{0,j}^{\varepsilon,N}, \gamma) \geq \delta\}\} \overset{\text{def}}{=} I_\varepsilon. \tag{3.14}$$

Now, we are going to rely on mixing and approximation assumptions (2.2) and (2.3). Set

$$\tilde{W}_t^{\varepsilon,N}(u) = E(\tilde{W}_0^{\varepsilon,N}(u) \mid F_{-\lfloor \frac{1}{4}\ln^2 \varepsilon \rfloor, l + \lfloor \frac{1}{4}\ln^2 \varepsilon \rfloor}), \ u \in [0, 1].$$

Then by (2.2) for all $\varepsilon > 0$ small enough,

$$\max_{0 \leq l \leq \lfloor T/\varepsilon \rfloor} E \sup_{u \in [0,1]} |\tilde{W}_t^{\varepsilon,N}(u) - \tilde{W}_t^{\varepsilon,N}(u)| \leq \exp(-\frac{K_2}{4}\ln^2 \varepsilon) \tag{3.15}$$

where we use that for any random vector $\Xi$ and $\sigma$-algebras $\mathcal{G} \subset \mathcal{H}$,

$$E|\Xi - E(\Xi \mid \mathcal{G})| \leq E|\Xi - E(\Xi \mid \mathcal{H})| + E|E(\Xi \mid \mathcal{G}) - E(\Xi \mid \mathcal{H})| \leq 2E|\Xi - E(\Xi \mid \mathcal{G})|.$$ 

Set

$$J_\varepsilon = P\{\cap_{0 \leq j \leq T/N} \{\rho(\tilde{W}_0^{\varepsilon,N}, \gamma) \geq \delta/2\}\}.$$ 

Then by (3.15) and the Chebyshev inequality,

$$I_\varepsilon \leq J_\varepsilon + \sum_{0 \leq j \leq T/N} P\{\rho(\tilde{W}_0^{\varepsilon,N}, \gamma) \geq \delta/2\}$$

$$\leq J_\varepsilon + 2T/N \varepsilon^{-1}\delta^{-1}\exp(-\frac{K_2}{4}\ln^2 \varepsilon). \tag{3.16}$$

Next, we use (2.2) which yields easily by induction that for any events $A_1, \ldots, A_k$ such that $A_i \in \mathcal{F}_{m_i, n_i}$, where $m_i \leq n_i < n_i + l_i \leq m_{i+1}$ for all $i = 1, \ldots, k$ with $m_{k+1} = \infty$,

$$P(\cap_{1 \leq i \leq k} A_i) \leq \prod_{1 \leq i \leq k} P(A_i) + \sum_{1 \leq i \leq k-1} \alpha(l_i). \tag{3.17}$$

Indeed, (3.17) follows for $k = 2$ directly from (2.2). If (3.17) holds true for $k - 1$ in place of $k$ then applying (2.2) to $A = \cap_{1 \leq i \leq k-1} A_i$ and $B = A_k$ we derive (3.17). Now, taking into account that the random vectors $\tilde{W}_{0,j}^{\varepsilon,N}$ are
explained in Appendix, if
$$(3.19)$$
\[ | \delta | \leq | \beta | < 1 \]
then
$$(3.20)$$
\[ \limsup_{n \to \infty} \rho(W_{0,j}[\ln^2 \varepsilon], \gamma) \geq \delta / 2 \] + \varepsilon^{-1} \exp\left(-\frac{K_1}{4} \ln^2 \varepsilon \right).

Using (3.15) and the Chebyshev inequality again we have that
\[ P\{ \rho(W_{0,j}[\ln^2 \varepsilon], \gamma) \geq \delta / 4 \} \]
\[ + P\{ \rho(W_{0,j}[\ln^2 \varepsilon], \gamma) \geq \delta / 4 \} \leq P\{ \rho(W_{0,j}[\ln^2 \varepsilon], \gamma) \geq \delta / 4 \} + \varepsilon^{-1} \exp\left(-\frac{K_2}{2} \ln^2 \varepsilon \right).

Since $\gamma \in \Phi_0^{1/c_0}$ then $I_0(\gamma_u) < \infty$ for Lebesgue almost all $u \in [0,1]$ and, as explained in Appendix, if $I_0(\beta) < \infty$, $\beta \in \mathbb{R}^d$ then $I_0(a\beta) < I_0(\beta)$ for $0 < a < 1$. Hence, if we define
\[ \eta_u = (1 - \delta(8 \sup_{v \in [0,1]} |\gamma_u|^{-1})) \gamma_u, \quad u \in [0,1] \]
then
$$\rho(\gamma, \eta) \leq \delta / 8 \quad \text{and} \quad S_0(\eta) \leq S_0(\gamma) - a \leq \frac{1}{c_0} - a$$
for some $a$.

Next, we write
$$(3.21)$$
$P\{ \rho(W_{0,j}[\ln^2 \varepsilon], \gamma) \geq \delta / 4 \} \leq P\{ \rho(W_{0,j}[\ln^2 \varepsilon], \eta) \leq \delta / 8 \} = 1 - P\{ \rho(W_{0,j}[\ln^2 \varepsilon], \eta) > \delta / 8 \}.$$

Now, taking into account stationarity of the process $\xi_t$ and that $W_{0,j}[\ln^2 \varepsilon] = Y_{0,c_0,\ln^2 \varepsilon}$ (with the latter defined by (2.9)) and relying on the lower large deviations bound in (2.9) we obtain that for any $\delta, \lambda > 0$ there exists $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$,
\[ P\{ \rho(W_{0,j}[\ln^2 \varepsilon], \eta) < \delta \} = P\{ \rho(W_{0,j}[\ln^2 \varepsilon], \eta) < \delta \} \]
\[ \geq \exp\left(-b_0(\varepsilon, N)(1/c_0 + a + \lambda)\right) = \varepsilon^{1-c_0a} \]
where we choose $\lambda > 0$ so small that $\lambda = a - \lambda > 0$. By (3.21) and (3.22) we obtain
$$(3.23)$$
\[ \prod_{0 \leq j < (T/\varepsilon)[\ln^2 \varepsilon]} P\{ \rho(W_{0,j}[\ln^2 \varepsilon], \gamma) \geq \delta / 4 \} \leq (1 - \varepsilon^{1-c_0})^{(T/\varepsilon)[\ln^2 \varepsilon]^{-1}}. \]

Taking $\varepsilon_n = \frac{1}{n}$, it follows from the Borel-Cantelli lemma together with the estimates (3.14), (3.10), (3.18), (3.19) and (3.23) that with probability one
$$(3.24)$$
\[ \lim_{n \to \infty} \sup_{0 \leq t \leq (T/\varepsilon)^{N}} \rho(W_{t}^{1/N}, \gamma) \leq \delta. \]

If $1/n \leq \varepsilon \leq 1/(n - 1)$ then $[tn] - [t/\varepsilon] \leq T + 1$ and
\[ |b_0(\varepsilon, N, u) - b_0(1/n, N, u)| \leq c_0 \ln\left(\frac{n}{n - 1}\right) \to 0 \quad \text{as} \quad n \to \infty. \]

This together with (3.14) and (3.24) yields
$$(3.25)$$
\[ \limsup_{\varepsilon \to 0} \inf_{0 \leq t \leq T} \rho(W_{t}^{\varepsilon}, \gamma) \leq \delta \quad \text{a.s.} \]
Since $\Phi_0(1/c_0)$ is a compact set we can choose there a $\delta$-net $\gamma_1, \gamma_2, \ldots, \gamma_{k(\delta)}$ and then with probability one \(\text{2.20}\) will hold true simultaneously for all $\gamma_i$, $i = 1, \ldots, k(\delta)$ in place of $\gamma$ there. It follows then that with probability one

$$
\limsup_{\varepsilon \to 0} \sup_{\gamma \in \Phi_0(1/c_0)} \inf_{0 \leq t \leq T} \rho(W_t^{\varepsilon, N}, \gamma) \leq 2\delta
$$

and since $\delta > 0$ is arbitrary we obtain \(\text{2.11}\) for $W_t^{\varepsilon, N}$ in place of $V_t^{\varepsilon, N}$ which, as explained at the beginning of this subsection gives \(\text{2.11}\) and completes the proof of Theorem 2.1.

\[\square\]

3.4. Proof of Corollaries 2.2 and 2.3. Under the conditions of Corollary 2.2 there is no dependence on $t$ of $S_t = S$ in \(\text{2.8}\) and we consider

$$
\Phi(1/c) = \{\gamma \in C([0, 1], \mathbb{R}^d) : \gamma_0 = 0, S(\gamma) \leq 1/c\}
$$

in \(\text{2.11}\). Thus there is no dependence on $N$ of quantities in \(\text{2.10}\) and \(\text{2.11}\), so that the limit in $N$ is not relevant now. It follows from \(\text{2.10}\) that all limit points as $\varepsilon \to 0$ of curves from $\mathcal{V}^\varepsilon_0$ belong to the compact set $\Phi(1/c)$. Now observe that \(\text{2.11}\) means that with probability one any $\gamma \in \Phi(1/c)$ is a limit point as $\varepsilon \to 0$ of curves from $\mathcal{V}^\varepsilon_0$ which yields \(\text{2.12}\).

In order to derive Corollary 2.3 observe that \(\text{2.10}\) implies, in particular, that for any continuous function $f$ on the space of curves $[0, 1] \to \mathbb{R}^d$ with probability one,

$$
\lim_{N \to \infty} \limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq T} f(V_t^{\varepsilon, N}) \leq \limsup_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{\gamma \in \Phi(1/c)} f(\gamma).
$$

Now set $c_t = 1/I_t(\beta)$ assuming that $I_t(\beta) < \infty$ for all $t \in [0, 1]$. Since now $d = 1$ we can define $f(\gamma) = \gamma(1)$, $\gamma(u) = \gamma_u$. Then

$$
\limsup_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{\gamma \in \Phi(1/c)} f(\gamma) = \limsup_{N \to \infty} \sup_{0 \leq t \leq T} \{\gamma(1) : \gamma \in \Phi(1/c)\} = \beta.
$$

Indeed, by convexity of each rate function $I_t$ for any $\gamma \in \Phi_s(I_s(\beta))$,

$$
I_s(\beta) \geq S_s(\gamma) = \int_0^1 I_s(\dot{\gamma}(u))du \geq I_s(\int_0^1 \dot{\gamma}(u)du) = I_s(\gamma(1))
$$

and by monotonicity of $I_s$ (see Appendix), $\beta \geq \gamma(1)$. On the other hand, take $\gamma(u) = u\beta$, $u \in [0, 1]$, then $S_s(\gamma) = I_s(\beta)$ for all $s \in [0, T]$ and $\gamma(1) = \beta$ implying \(\text{3.28}\). Observe also that, in particular, if $\gamma \in \Phi_0(I_0(\beta))$ then by \(\text{2.11}\) with probability one,

$$
\lim_{N \to \infty} \limsup_{\varepsilon \to 0} \inf_{0 \leq t \leq T} |V_{t,T}(\gamma) - \beta| = 0
$$

which together with \(\text{3.27}\) and \(\text{3.28}\) yields \(\text{2.13}\).

\[\square\]

4. Continuous time case

4.1. Basic estimates. In view of \(\text{1.3}\) we have

$$
\varepsilon^{-1}(X_{t/\varepsilon}^{\varepsilon} + b(\varepsilon, N, u)X_{t/\varepsilon}^{\varepsilon}) = \int_{t/\varepsilon}^{t/\varepsilon + b(\varepsilon, N, u)} B(X_s^{\varepsilon, \xi_s})ds \\
= \int_{t/\varepsilon}^{t/\varepsilon + b(\varepsilon, N, u)} B_t(\xi_s)ds + \Psi_1(\varepsilon, N, t, u) + \Psi_2(\varepsilon, N, t, u)
$$
where
\[ \Psi_{\varepsilon,N}^{(1)}(t, u) = \int_{t/\varepsilon}^{t/\varepsilon+b_t(\varepsilon,N,u)} (B(X_s^\varepsilon, \xi_s) - B(\bar{X}_s^\varepsilon, \bar{\xi}_s)) ds \]
and
\[ \Psi_{\varepsilon,N}^{(2)}(t, u) = \int_{t/\varepsilon}^{t/\varepsilon+b_t(\varepsilon,N,u)} (B(\bar{X}_s^\varepsilon, \xi_s) - B_1(\xi_s)) ds. \]

Similarly to (3.2) and (3.3) by (2.1) and the averaging principle (1.5),
\[ \left| \Psi_{\varepsilon,N}^{(1)}(t, u) \right| \leq L_1 T \ln(1/\varepsilon) \sup_{0 \leq t \leq T / (\varepsilon - \ln \varepsilon)} |X_t^\varepsilon - \bar{X}_t^\varepsilon| \to 0 \text{ as } \varepsilon \to 0. \]

Similar to (3.4) and (3.5) by (2.1) and the averaging principle (1.5),
\[ \left| \Psi_{\varepsilon,N}^{(2)}(t, u) \right| \leq L_2 T^2 \varepsilon \ln^2(1/\varepsilon) \to 0 \text{ as } \varepsilon \to 0. \]

It follows from (4.1) - (4.3) that (2.10) and (2.11) will hold true in the continuous time case provided these limits are verified for
\[ \hat{V}_t^{\varepsilon,N}(u) = 1/b_t(\varepsilon,N,\varepsilon) \int_{t/\varepsilon}^{t/\varepsilon+b_t(\varepsilon,N,u)} (B_t(\xi_s) - \bar{B}_t) ds, \quad t \in [0, T], \; u \in [0, 1] \]
in place of \( V_t^{\varepsilon,N}(u) \). Since in the continuous time case we have the same estimates as in (3.6) it follows that, again, (2.10) and (2.11) will hold true here provided they are obtained for
\[ \hat{W}_t^{\varepsilon,N}(u) = 1/b_t(\varepsilon,N,\varepsilon) \int_{t/\varepsilon}^{t/\varepsilon+b_t(\varepsilon,N,u)} (B_{t(\tau,N)}(\xi_s) - \bar{B}_{t(\tau,N)}) ds, \quad t \in [0, T], \; u \in [0, 1] \]
in place of \( V_t^{\varepsilon,N}(u) \).

4.2. The upper bound. Set
\[ \hat{W}_{k,\varepsilon,N}^{\varepsilon,N}(u) = b_{kT/N}^{-1}(\varepsilon,N) \int_{t/\varepsilon}^{t/\varepsilon+b_{kT/N}(\varepsilon,N,u)} (B_{kT/N}(\xi_s) - \bar{B}_{kT/N}) ds. \]

In order to prove (2.10) for \( W_{t}^{\varepsilon,N} \) in place of \( V_t^{\varepsilon,N} \) it suffices to obtain (3.8) for \( \hat{W}_{k,\varepsilon,N}^{\varepsilon,N} \) defined by (4.3). The proof of this proceeds almost verbatim as for the discrete time in Section 3.2 using now the continuous time case upper large deviations bounds for normalized integrals (2.20) taking into account that \( \hat{W}_{k,0}^{\varepsilon,N}(u) = Y_{kT^{N-1},\varepsilon} \ln(1/\varepsilon)(u) \).

4.3. The lower bound. Set
\[ \hat{W}_t^{\varepsilon}(u) = \hat{W}_t^{\varepsilon}(u, \omega) = b_0^{-1}(\varepsilon) \int_{t/\varepsilon}^{t/\varepsilon+b_0(\varepsilon,u)} (B_0(\xi_s(\omega)) - \bar{B}_0) ds, \quad u \in [0, 1] \]
where \( b_0(\varepsilon, u) = b_0(\varepsilon, N, u) \) and \( b_0(\varepsilon) = b_0(\varepsilon, N) \) do not depend on \( N \). Clearly, for any \( \gamma \in C([0,1], \mathbb{R}^d) \),
\[ \inf_{0 \leq t \leq T} \rho(W_{t}^{\varepsilon,N}, \gamma) \leq \inf_{0 \leq t \leq T / N} \rho(W_{t/N}^{\varepsilon,N}, \gamma) = \inf_{0 \leq t \leq T / N} \rho(\hat{W}_{t}^{\varepsilon}, \gamma). \]

Introduce,
\[ U^{\varepsilon}(k, \hat{\omega}) = b_0^{-1}(\varepsilon) \sum_{0 \leq n \leq k} \hat{B}_0 \circ \theta^n(\hat{\omega}) = b_0^{-1}(\varepsilon) \int_0^{\sigma_k(\hat{\omega})} B_0(\xi_s(\hat{\omega})) ds \]
where $\hat{B}_0(\omega) = \hat{B}(\hat{Z}_0, \hat{\omega})$, $\sigma_k = \sum_{i=0}^{k-1} \varsigma \circ \theta^i$ and $\hat{B}$ is the same as in (2.17). Set

$$v^\varepsilon(u, \omega) = \max\{j \geq 0 : \sigma_j(\hat{\omega}) \leq b_0(\varepsilon, u)\}.$$ 

Observe that if $\sigma_j(\hat{\omega}) \leq \lfloor \varepsilon \rfloor \leq \sigma_{\lfloor \varepsilon \rfloor + 1}(\hat{\omega})$ then

$$\sup_{0 \leq u \leq 1} |\hat{W}^\varepsilon_i(u, \omega) - U^\varepsilon(U^\varepsilon(u, \theta^i\hat{\omega}), \theta^i\hat{\omega})| \leq 4b_0^{-1}(\varepsilon) L_1 L_3.$$ 

It follows that for any $\delta > 8b_0^{-1}(\varepsilon) L_1 L_3$,

$$\{\hat{\omega} : \inf_{0 \leq t \leq T/N} \rho(\hat{W}^\varepsilon_i(\hat{\omega}), \gamma) \geq \delta\}$$

$$\subset \left\{\min_{0 \leq t \leq T/N} \rho(U^\varepsilon \circ \theta^i, \gamma) > \delta/2\right\} \cup \{\hat{\omega} : \sigma_{\lfloor \varepsilon \rfloor + 1}(\hat{\omega}) \geq T/\varepsilon N\}$$

where $U^\varepsilon(u) = \hat{U}^\varepsilon(u, \hat{\omega}) = U^\varepsilon(U^\varepsilon(u, \hat{\omega}), \hat{\omega})$. It follows from (2.19) that

$$\hat{P}\{\sigma_{\lfloor \varepsilon \rfloor + 1}(\hat{\omega}) \geq T/\varepsilon N\} \leq \kappa^{-1} \exp(-\kappa/\varepsilon N)$$

for some $\kappa = \kappa_T > 0$.

Next, we deal with the other event in the right hand side of (4.9). Set

$$v^\varepsilon_M(u, \omega) = \max\{j \geq 0 : \sigma_{j,M}(\hat{\omega}) \leq b_0(\varepsilon, u)\}$$

where

$$\sigma_{j,M} = \sum_{i=0}^{j-1} \varsigma_{i,M}, \quad \sigma_{0,M} = 0 \quad \text{and} \quad \varsigma_{i,M} = E(\varsigma \circ \theta^i| F_{i-M,i+M}).$$

Put also $\tilde{B}_{n,M} = \hat{E}(\hat{B}_0 \circ \theta^n| F_{n-M,n+M})$ and define

$$\tilde{U}^\varepsilon_{i,M}(u) = \tilde{U}^\varepsilon_{i,M}(u, \hat{\omega}) = b_0^{-1}(\varepsilon) \sum_{1 \leq i \leq l + v^\varepsilon_M(u, \hat{\omega})} \tilde{B}_{n,M}(\hat{\omega}).$$

Then

$$|\tilde{U}^\varepsilon(u, \theta^i\hat{\omega}) - \tilde{U}^\varepsilon_{i,M}(u, \hat{\omega})| \leq L_1 b_0^{-1}(\varepsilon) |v^\varepsilon(u, \hat{\omega}) - v^\varepsilon_M(u, \hat{\omega})|$$

$$+ b_0^{-1}(\varepsilon) \sum_{1 \leq i \leq l + v^\varepsilon_M(u, \hat{\omega})} |\hat{B}_0 \circ \theta^n(\hat{\omega}) - \tilde{B}_{n,M}(\hat{\omega})|.$$ 

Next, we estimate the right hand side of (4.11). Observe that for all $u \in [0, 1], v^\varepsilon_M(u, \hat{\omega}) - v^\varepsilon(u, \hat{\omega}) \geq 2\} \subset \bigcup_{k: \varsigma_k(\hat{\omega}) \leq b_0(\varepsilon)} \{\hat{\omega} : |\sigma_k(\hat{\omega}) - \sigma_{k,M}(\hat{\omega})| > L_3^{-1}\}. $$

Hence, by (2.17) and the Chebyshev's inequality,

$$\hat{P}\{\sup_{0 \leq u \leq 1} |v^\varepsilon_M(u, \cdot) - v^\varepsilon(u, \cdot)| \geq 2\} \leq \sum_{k: \varsigma_k(\hat{\omega}) \leq b_0(\varepsilon)} \hat{P}\{|\sigma_k - \sigma_{k,M}(\hat{\omega})| > L_3^{-1}\} \leq L_3 \sum_{k: \varsigma_k(\hat{\omega}) \leq b_0(\varepsilon)} \sum_{0 \leq j \leq L_1 b_0(\varepsilon)} \hat{E}|\varsigma_j - \varsigma_{j,M}| \leq L_3 b_0^2(\varepsilon) \kappa_3^{-1} e^{-\kappa_3 M}.$$ 

By (2.16) we also have that

$$\sum_{1 \leq n \leq l + L_3 b_0(\varepsilon)} \hat{E}|\hat{B}_0 \circ \theta^n - \tilde{B}_{n,M}| \leq L_3 b_0(\varepsilon) \kappa_3^{-1} e^{-\kappa_3 M},$$

and so

$$\hat{P}\{|\hat{U}^\varepsilon \circ \theta^i, \tilde{U}^\varepsilon_{i,M}\} \geq \delta/4\} \leq \hat{P}\{\sup_{0 \leq u \leq 1} |v^\varepsilon(u) - v^\varepsilon_M(u)| \geq L_1 b_0(\varepsilon) \delta/8\} + \hat{P}\{\sum_{1 \leq i \leq l + L_3 b_0(\varepsilon)} |\hat{B}_0 \circ \theta^n - \tilde{B}_{n,M}| \geq b_0(\varepsilon) \delta/8\}$$

$$\leq (L_3^2 b_0^2(\varepsilon) + 8/\delta) \kappa_3^{-1} e^{-\kappa_3 M}$$

provided $b_0(\varepsilon) \geq 16\delta^{-1} L_1^{-1}$.
Observe that $\sigma_{j,M}$ is $\mathcal{F}_{-M,j+M}$-measurable and
$$\{v_M^+ (u, \hat{\omega}) = k\} = \{\sigma_{k,M} \leq b_0(\varepsilon, u)\} \cap \{\sigma_{k+1,M} > b_0(\varepsilon, u)\}$$
which is $\mathcal{F}_{-M,k,M+1}$-measurable. Since always $v_M^+(u, \hat{\omega}) \leq L_3 b_0(\varepsilon)$ we obtain that $\hat{U}_M^\varepsilon$ is $\mathcal{F}_{-M,k,M+L_3 b_0(\varepsilon)+1}$-measurable. Now we choose $M = [M(\varepsilon)] = [\ln^2 \varepsilon]$ and obtain by (4.15) that
$$\begin{align}
\hat{P}\{\min_{0 \leq l \leq \frac{T}{\kappa_3}} \rho(\hat{U}_M^\varepsilon \circ \theta^l, \gamma) > \delta/2\} \\
\leq \hat{P}\{\min_{0 \leq l \leq \frac{T}{\kappa_3}} \rho(\hat{U}_M^\varepsilon(\omega) , \gamma) > \delta/4\} \\
+ T(6\varepsilon N_\gamma M(\varepsilon))^{-1} L_3 (L_3 b_0^2(\varepsilon) + 8/\delta) \kappa_3^{-1} e^{-\kappa_3 M(\varepsilon)}. 
\end{align}$$
Introduce the event
$$C_\gamma^\varepsilon = \{\rho(\hat{U}_M^\varepsilon(\omega), \gamma) > \delta/4\}$$
which is $\mathcal{F}_{l-M(\varepsilon), l+M(\varepsilon)+L_3 b_0(\varepsilon)+1}$-measurable. Then by (2.2) for the probability $Q$ in the same way as in (3.17) and (3.18) it follows that,
$$\begin{align}
\hat{P}\{\bigcap_{0 \leq l \leq \frac{T}{\kappa_3}} C_\gamma^\varepsilon\} &\leq L_2 Q\{\bigcap_{0 \leq l \leq \frac{T}{\kappa_3}} C_\gamma^\varepsilon\} \\
&\leq L_2 \prod_{0 \leq l \leq \frac{T}{\kappa_3}} Q(C_\gamma^\varepsilon) + L_2 \varepsilon^{-1} a([M(\varepsilon)/2])
\end{align}$$
provided $\varepsilon > 0$ is small enough.

Now,
$$Q(\hat{\Omega} \setminus C_\gamma^\varepsilon) \geq Q(\hat{C}_\gamma^\varepsilon) - L_2 L_3 (L_3 b_0^2(\varepsilon) + 16\delta^{-1}) \kappa_3^{-1} e^{-\kappa_3 M(\varepsilon)}$$
where $\hat{C}_\gamma^\varepsilon = \{\rho(\hat{U}_M^\varepsilon \circ \theta^l, \gamma) \leq \delta/8\}$ and we use (4.16) with $\delta/2$ in place of $\delta$. Since $\theta$ preserves the measure $Q$,
$$Q(\hat{C}_\gamma^\varepsilon) = Q(\hat{C}_0^\gamma) \geq L_2^{-1} \hat{P}(\hat{C}_0^\gamma).$$
By (4.8) for all $\hat{\omega}$,
$$\sup_{0 \leq u \leq 1} |\hat{U}_M^\varepsilon (u, \hat{\omega}) - \hat{W}_0^\varepsilon (u, \hat{\omega})| \leq \frac{6L_1 L_2}{b_0(\varepsilon)}.$$  
Set $D^\varepsilon = \{\hat{\omega} : \rho(\hat{W}_0^\varepsilon (\hat{\omega}), \gamma) \leq \delta/16\}$ then
$$\hat{P}(\hat{C}_0^\gamma) \geq \hat{P}(D^\varepsilon) \quad \text{provided} \quad \delta \geq 96 L_1 L_3 b_0^{-1}(\varepsilon).$$
Set $\hat{D}^\varepsilon = \{\omega : \rho(\hat{W}_0^\varepsilon (\omega), \gamma) \leq \delta/32\}$. Recal that if $\hat{\omega} = \varphi \omega$ then $\omega = \varphi \circ \hat{\omega}$ for some $0 \leq s < \zeta(\hat{\omega})$, and so $\xi_t(\omega) = \xi_{t+s}(\hat{\omega})$ for any $t \geq 0$. This together with (2.1) and (2.15) yields
$$\sup_{0 \leq u \leq 1} |\hat{W}_0^\varepsilon (u, \omega) - \hat{W}_0^\varepsilon (u, \hat{\omega})| \leq 4 L_1 L_3 b_0^{-1}(\varepsilon).$$
Hence, if $\varepsilon$ is small enough then
$$\varphi \hat{D}^\varepsilon \subset D^\varepsilon.$$  
In the same way as in the discrete time case we argue that since $\gamma \in \Phi_0^1/\alpha_0$ then the curve
$$\eta_u = (1 - \delta(72 \sup_{v \in [0,1]} |\gamma_u|)^{-1}) \gamma_u, \ u \in [0,1]$$
satisfies
\begin{equation}
\rho(\gamma, \eta) \leq \frac{\delta}{t^2} \quad \text{and} \quad S_0(\eta) \leq S_0(\gamma) - a \leq \frac{1}{c_0} - a
\end{equation}
for some \( a > 0 \). This relies, again, on the strict monotonicity of the rate function of large deviations in the domain where it is finite (see Appendix). Next, we apply the lower large deviations bound in (4.23) to \( Y_{0,0,|\ln \varepsilon|}(u) = \hat{W}_0(u) \) obtaining that for any \( \lambda, \delta > 0 \) there exists \( \varepsilon_0 > 0 \) such that whenever \( \varepsilon \leq \varepsilon_0 \),
\begin{equation}
\hat{P}(D^\varepsilon) \geq P(D^\varepsilon) \geq \exp(-b_0(\varepsilon)(\frac{1}{c_0} - a + \lambda)) = \varepsilon^{1-c_0a}
\end{equation}
where we choose \( \lambda > 0 \) so small that \( \hat{a} = a - \lambda > 0 \).

Now, by (4.7), (4.9), (4.10) and (4.22) for \( \varepsilon \) small enough,
\begin{equation}
P\{\inf_{0 \leq t \leq T} \rho(W_{t}^{\varepsilon,N}, \gamma) > 2\delta\} \leq \hat{P}\{\inf_{0 \leq t < T/N} \rho(\hat{W}_t, \gamma) > \delta\}
\end{equation}
\begin{equation}
\leq \hat{P}\{\min_{0 \leq t < T/(2\varepsilon N)} \rho(\hat{W}_t, \gamma) > \delta/2\} + \kappa^{-1} \exp(-\kappa/\varepsilon N)
\end{equation}
for some \( \kappa > 0 \). Next, by (4.16)–(4.19), (4.21), (4.25) and (4.26),
\begin{equation}
P\{\inf_{0 \leq t \leq T} \rho(W_{t}^{\varepsilon,N}, \gamma) > 2\delta\}
\end{equation}
\begin{equation}
\leq L_2(1 - e^{-c_0\varepsilon} + L_4\delta^{-1}e^{-\kappa_4\ln^2\varepsilon} + L_4\varepsilon L_4\delta^{-1}e^{-\kappa_4\ln^2\varepsilon})^{-1}
\end{equation}
for some \( \kappa_4, L_4 > 0 \) independent of \( \varepsilon \).

Taking \( \varepsilon_n = 1/n \) it follows from the Borel-Cantelli lemma that with probability one,
\begin{equation}
\limsup_{n \to \infty} \inf_{0 \leq t \leq T} \rho(W_{t}^{1/n}, \gamma) \leq 2\delta.
\end{equation}
If \( 1/n \leq \varepsilon < \frac{1}{n-1} \) then using (5.25) we conclude again that (4.28) implies, in fact, that with probability one,
\begin{equation}
\limsup_{\varepsilon \to 0} \inf_{0 \leq t \leq T} \rho(W_{t}^{\varepsilon}, \gamma) \leq 2\delta.
\end{equation}

Concluding in the same way as in the discrete time case of Section 5.2.3 by choosing a \( \delta \)-net in \( \Phi_0(1/c_0) \) and taking into account that \( \delta > 0 \) is arbitrary we obtain (2.11) for \( W_{t}^{\varepsilon,N} \) in place of \( V_{t}^{\varepsilon,N} \) which, as explained at the beginning of this section gives (2.11) and completes the proof of Theorem 2.4.

4.4. Proof of Corollary 2.5. The proof of Corollary 2.5 proceeds, essentially, in the same way as the proofs of Corollaries 2.2 and 2.3 relying on properties of large deviations rate functionals for the continuous time case (see Appendix).

5. Appendix

5.1. Applications. The main applications in the discrete time case of Theorem 2.4 concern Markov chains and some classes of dynamical systems such as Axiom A diffeomorphisms, expanding transformations and topologically mixing subshifts of finite type. We will restrict ourselves to several main setups to which our results are applicable rather than trying to describe most general situations. First, let \( \xi_n, n \geq 0 \) be a time homogeneous Markov chain on a Polish state space \( M \) whose transition probability \( P(x, \Gamma) = P\{\xi_1 \in \Gamma|\xi_0 = x\} \) satisfies
\begin{equation}
\kappa \nu(\Gamma) \leq P(x, \Gamma) \leq \kappa^{-1} \nu(\Gamma)
\end{equation}
for some $\kappa > 0$, a probability measure $\nu$ on $M$ and any Borel set $\Gamma \subset M$. Then $\xi_n$, $n \geq 0$ is exponentially fast $\psi$-mixing with respect to the family of $\sigma$-algebras $\mathcal{F}_{m,n} = \sigma\{\xi_k, m \leq k \leq n\}$ generated by the process (see, for instance, [19]). The strong Doeblin type condition (5.1) implies geometric ergodicity

$$\|P(n, x, \cdot) - \mu\| \leq \beta^{-1} e^{-\beta n}, \beta > 0$$

where $\| \cdot \|$ is the variational norm, $P(n, x, \cdot)$ is the $n$-step transition probability and $\mu$ is the unique invariant measure of $\{\xi_n, n \geq 0\}$ which makes it a stationary process.

In this situation the limit (2.6) exists (see Lemma 4.3 in Ch.7 of [17]) and $\exp(\Pi_{t}(b))$ turns out to be the principal eigenvalue of the positive operator

$$Qf(x) = E_{x}f(\xi_{1}) \exp((b,G_{t}(\xi_{1})))$$

where $E_{x}$ is the expectation provided $\xi_{0} = x$ (see [20], [21] and references there). It is well known (see [25], [19], [18] and references there) that $\Pi_{t}(b)$ is convex and differentiable in $b$. Furthermore, the Hessian matrix $\nabla_{b}^{2}\Pi_{t}(b)|_{b=0}$ is positively definite if and only if for each $b \in \mathbb{R}^{d}$, $b \neq 0$ the limiting variance

$$\sigma_{b}^{2} = \lim_{n \to \infty} n^{-1}E(\sum_{i=0}^{n}(b,G_{t}(\xi_{i})))^{2}$$

(5.2)

is positive. The latter holds true unless there exists a representation $(b,G_{t}(\xi_{n})) = g(\xi_{n}) - g(\xi_{n-1})$, $n = 1, 2, \ldots$ for some bounded Borel function $g$ (see [18]).

In the discrete time dynamical systems case we consider $\xi_{n} = \xi_{n}(x) = f^{n}x, n \geq 0$ where $f : M \to M$ is a $C^{2}$ Anosov diffeomorphism on a hyperbolic set or a topologically mixing subshift of finite type or a $C^{2}$ expanding transformation. Here $\xi_{n}$, $n \geq 0$ is considered as a stationary process on the probability space $(M, \mathcal{F}, \mu)$ where $\mathcal{F}$ is the Borel $\sigma$-algebra and $\mu$ is a Gibbs measure constructed by a Hölder continuous function (see [2]). Then the process $\xi_{n}$ is exponentially fast $\psi$-mixing (see [2]) with respect to the family of (finite) $\sigma$-algebras generated by cylinder sets in the symbolic setup of subshifts of finite type or with respect to the corresponding $\sigma$-algebras constructed via Markov partitions in the Axiom A and expanding cases.

Existence of the limit (2.6) and its form was proved in [21]. Here $\Pi_{t}(b)$ turns out to be the topological pressure for the function $(b,G_{t}) + \varphi$ where $\varphi$ is the potential of the corresponding Gibbs measure. The differentiability properties of $\Pi_{t}(b)$ in $b$ are well known and, again, the Hessian matrix $\nabla_{b}^{2}\Pi_{t}(b)|_{b=0}$ is positively definite if and only if for each $b \in \mathbb{R}^{d}$, $b \neq 0$ the limiting variance (5.2) is positive where the expectation should be taken with respect to the corresponding Gibbs measure (see [27], [18], [20], [22] and references there). The latter holds true unless there exists a coboundary representation $(b,G_{t}) = g \circ f - g$ for some bounded Borel function $g$.

Next, we discuss the continuous time case. Here $\xi_{t}$, $t \geq 0$ can be a nondegenerate random evolution on a compact manifold $M$, in particular, a nondegenerate diffusion there. The existence and the form of the limit (2.6) in this case is shown in [24]. By discretizing time the problem is reduced to the discrete time process $\xi_{n}$, $n \geq 0$ which is exponentially fast $\psi$-mixing and in this case the continuous does not pose additional difficulties provided we consider the $\sigma$-algebras $\mathcal{F}_{m,n}$ generated by $\xi_{i}$, $m \leq i \leq n$. We observe that this case fits in our general continuous time scheme taking the projection $\varphi$ to be the identity map, $\tilde{\Omega} = \Omega$ and $\varsigma(\tilde{\omega}) \equiv 1$.

If the information about the process comes only at some random times then we
arrive at a more general setup of Theorem 2.4 though its main motivation comes from dynamical systems as described below.

We deal now with continuous time dynamical systems, namely, with a $C^2$ Axiom A flows $f^t : M \to M$ on a hyperbolic set considered with a Gibbs measure built by a Hölder continuous function. Using Markov partitions such a flow can be represented by means of a suspension construction (see [7]) with a transformation $\theta : \hat{M} \to M$ on the bases of elements of the Markov partition and a roof function $\varsigma$ so that $f^{\varsigma(\hat{x})}\hat{x} = \theta\hat{x}$ for each $\hat{x} \in \hat{M}$. Here $M$ is identified with the space $\hat{M} = \{(s, \hat{x}) : \hat{x} \in \hat{M}, 0 \leq s < \varsigma(\hat{x})\}$ and $f^t(s, \hat{x}) = (s + t, \hat{x})$ for $s + t < \varsigma(\hat{x})$. Furthermore, $\theta^n, n \geq 0$ on $\hat{M}$ turns out to be an exponentially fast $\psi$-mixing discrete time dynamical system preserving a Gibbs measure $Q$ constructed by a Hölder continuous function while $f^t$ preserves the measure $P$ such that

$$
\int g(s, \hat{x})dP(s, \hat{x}) = (1/\bar{\varsigma}) \int_{\hat{M}} \int_0^{\varsigma(\hat{x})} g(s, \hat{x})d\varsigma dQ(\hat{x})
$$

where $\bar{\varsigma} = \int \varsigma dQ$. The roof function $\varsigma$ turns out to be Hölder continuous and bounded away from zero and infinity. Large deviations estimates for sums $\sum_{i=1}^n \varsigma(\theta^i)$ follow from [20] and existence of the limit in (2.6) and its form follow from [21] and [24]. Again, $\Pi_t(b)$ is the topological pressure of the flow $f^t$ for the function $(b, G_x) + \varphi$, with $\varphi$ being the potential of the corresponding Gibbs measure, and the differentiability properties of $\Pi_t(b)$ in $b$ are well known (see, for instance, [27] and [9]). Similarly to the discrete time case the Hessian matrix is positively definite if and only if all limiting variances

$$(5.3) \quad \sigma_b^2 = \lim_{T \to \infty} T^{-1} \int (\int_0^T (b, G_x \circ f^u) du)^2 dP$$

are positive when $b \neq 0$ which holds true unless there exists a coboundary representation $(b, G_x) = g \circ f^t - g$ for some $t$ and a bounded Borel function $g$.

5.2. Some properties of rate functions. We collect here few properties of rate functions of large deviations which are essentially well known but hard to find in major works on large deviations. First, observe that if $\Pi(b), b \in \mathbb{R}^d$ is a twice differentiable function such that $\Pi(0) = 0, \nabla_b \Pi(b)|_{b=0} = 0$ then $\Pi(b) = o(|b|), and so

$$(5.4) \quad I(\beta) = \sup_b ((b, \beta) - \Pi(b)) > 0$$

unless $\beta = 0$. Indeed, by the above

$$I(\beta) \geq \delta |\beta|^2 - \Pi(\delta \beta) > 0$$

if $\delta > 0$ is small enough. Curiously, positivity of the rate function is not discussed in several books on large deviations without which upper large deviations bounds do not make much sense.

Next, assume, in addition, that $\Pi$ is convex and has a positively definite at zero Hessian matrix $\nabla_b^2 \Pi(b)|_{b=0}$. Then $\Pi(b) \geq 0$ for all $b \in \mathbb{R}^d$ and for some $\delta_1, \delta_2 > 0,

$$(5.5) \quad \Pi(b) \geq \delta_1 |b| \quad \text{provided} \quad |b| > \delta_2.$$
Next, under the above conditions on $\Pi$ suppose that $I(\beta) < \infty$ for some $\beta \neq 0$. Then
\begin{equation}
I((1 + \delta)\beta) > I(\beta) \quad \text{for any } \delta > 0.
\end{equation}
Indeed, for any $\varepsilon > 0$ there exists $b_{\beta, \varepsilon}$ such that
\begin{equation}
(b_{\beta, \varepsilon}, \beta) - \Pi(b_{\beta, \varepsilon}) \geq I(\beta) - \varepsilon.
\end{equation}
Since $\Pi(b_{\beta, \varepsilon}) \geq 0$ we have
\begin{equation}
I((1 + \delta)\beta) \geq (1 + \delta)(b_{\beta, \varepsilon}, \beta) - \Pi(b_{\beta, \varepsilon}) \geq I(\beta) + \delta(I(\beta) - \varepsilon) - \varepsilon > I(\beta)
\end{equation}
provided $\varepsilon < \delta(1 + \delta)^{-1}I(\beta)$ yielding (5.6).

In the Erdős-Rényi law type results it is important to know where a rate function $I(\beta)$ is finite. This issue is hidden inside the functional form of Theorems 2.1 and 2.4 but appears explicitly in Corollary 2.3 and in the classical form (1.1). The discussion on finiteness of rate functions is hard to find in books on large deviations though without studying this issue lower bounds there do not have much sense. We start with the rate functional $J(\nu)$ of the second level of large deviations for occupational measures
\begin{equation}
\zeta_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\xi_k} \quad \text{or} \quad \zeta_t = \frac{1}{t} \int_0^t \delta_{\xi_s} ds
\end{equation}
in the discrete or continuous time cases, respectively, where $\delta_x$ denotes the unit mass at $x$ (see [20]). Explicit formulas for $J(\nu)$ are known when $\xi_k$ is a Markov chain whose transition probability satisfies (5.1) and when $\xi_k = f^k x$ with $f$ being an Axiom A diffeomorphism, expanding transformation or subshift of finite type. In the former case (see [14]),
\begin{equation}
J(\nu) = - \inf_{u > 0, \text{continuous}} \int \ln(Pu \mu) d\nu
\end{equation}
and in the latter case (see [20]),
\begin{equation}
J(\nu) = \begin{cases} 
- \int \varphi d\mu - h_\nu(f) & \text{if } \nu \text{ is } f^1 \text{-invariant,} \\
\infty & \text{otherwise}
\end{cases}
\end{equation}
where $h_\nu(f)$ is the Kolmogorov–Sinai entropy of $f$ with respect to $\nu$ and $\varphi$ is the potential of the corresponding Gibbs measure $\mu$ playing the role of probability here.

In the continuous time case these functionals have explicit forms for diffusions $\xi_t$ (see [13]),
\begin{equation}
J(\nu) = - \inf_{u > 0, \text{is } C^2} \int \frac{Lu}{u} d\nu,
\end{equation}
where $L$ is the corresponding generator, and for Axiom A flows $\xi_t = f^t x$ where (see [20]),
\begin{equation}
J(\nu) = \begin{cases} 
- \int \varphi d\mu - h_\nu(f^1) & \text{if } \nu \text{ is } f^1 \text{-invariant,} \\
\infty & \text{otherwise}
\end{cases}
\end{equation}
with the same notations as in (5.9).
Necessary and sufficient conditions for finiteness of $J(\nu)$ in the Markov chain and diffusion cases are given in [14] while in the above dynamical systems cases $J(\nu) < \infty$ for any $f$-invariant measure $\nu$. If

\begin{equation}
\Pi(b) = \lim_{n \to \infty} \frac{1}{n} \ln E \exp \left( \sum_{j=0}^{n-1} (b, G(\xi_j)) \right)
\end{equation}

in the discrete time case or

\begin{equation}
\Pi(b) = \lim_{t \to \infty} \frac{1}{t} \ln E \exp \left( \int_0^t (b, G(\xi_s))ds \right)
\end{equation}

in the continuous time case, where $\xi_t$ is a stationary process as above on a compact space $M$ and $G \not\equiv 0$ is a continuous vector function with $EG(\xi_0) = 0$, then by the contraction principle (see, for instance, [15]) the rate function $I(\beta)$ given by (5.4) can be represented as

\begin{equation}
I(\beta) = \inf \{ J(\nu) : \int Gd\nu = \beta \}
\end{equation}

where the infimum is taken over the space $\mathcal{P}(M)$ of probability measures on $M$.

Set

$$ \Gamma = \{ \beta \in \mathbb{R}^d : \exists \nu \in \mathcal{P}(M) \text{ such that } \int Gd\nu = \beta \text{ and } J(\nu) < \infty \} $$

and let $\text{Co}(\Gamma)$ be the interior of the convex hull of $\Gamma$. Then

\begin{equation}
I(\beta) < \infty \text{ for any } \beta \in \text{Co}(\Gamma).
\end{equation}

Indeed, any $\beta \in \text{Co}(\Gamma)$ can be represented as $\beta = p_1 \beta_1 + p_2 \beta_2$ with $\beta_1, \beta_2 \in \Gamma$, $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$. Then $\beta_1 = \int Gd\nu_1$, $\beta_2 = \int Gd\nu$, and so $\int Gd\nu = \beta$ for $\nu = p_1 \nu_1 + p_2 \nu_2$. Since $J(\nu_1), J(\nu_2) < \infty$ then by convexity of $J$ we have that $J(\nu) \leq p_1 J(\nu_1) + p_2 J(\nu_2) < \infty$, and so (5.15) holds true.

When $d = 1$, i.e. when $G$ is (not vector) function we can give another description of the domain where $I(\beta) < \infty$. In this case set

\begin{equation}
\beta_+ = \sup \{ \beta : \beta \in \Gamma \} \text{ and } \beta_- = \inf \{ \beta : \beta \in \Gamma \}.
\end{equation}

Then by (5.15), $I(\beta) < \infty$ for any $\beta \in (\beta_-, \beta_+)$. It is possible to extract from [12] that under $\psi$-mixing,

\begin{equation}
\beta_+ = \lim_{n \to \infty} \frac{1}{n} \text{ess sup} \sum_{j=0}^{n-1} G(\xi_j) \text{ and } \beta_- = \lim_{n \to \infty} \frac{1}{n} \text{ess inf} \sum_{j=0}^{n-1} G(\xi_j).
\end{equation}

Since Axiom A flows are not $\psi$-mixing, in general, we will give another proof for this case.

Let

\begin{equation}
\beta_+ = \lim_{t \to \infty} \frac{1}{t} \text{sup}_x \int_0^t G \circ f^s ds \text{ and } \beta_- = \lim_{t \to \infty} \frac{1}{t} \text{inf}_x \int_0^t G \circ f^s ds = - \lim_{t \to \infty} \frac{1}{t} \text{sup}_x ( - \int_0^t G \circ f^s ds )
\end{equation}

The limits in (5.17) exist since $a(t) = \sup_x \int_0^t G \circ f^s ds$ is subadditive $a(t + s) \leq a(t) + a(s)$. Since $G$ is a continuous function on a compact space $M$ we can find $x_t$ such that $a(t) = \int_0^t G \circ f^s(x_t) ds$. Consider the family of occupational measures

$$ \nu_t = \frac{1}{t} \int_0^t \delta_{f^s x_t} ds. $$
Then any weak limit $\tilde{\nu}$ of $\nu_t$ as $t \to \infty$ is an $f^t$-invariant measure and $\int g d\tilde{\nu} = \beta_+^*$. It follows that $\beta_+^* \leq \beta_+$, where $\beta_+$ is given by (5.16). On the other hand, $a(t) \geq t^{-1} \int_0^t G \circ f^s(x)ds$, and so for any $x$,

$$
\int G d\tilde{\nu} = \limsup_{t \to \infty} \frac{1}{t} \int_0^t G \circ f^s(x)ds.
$$

Hence, $\int G d\tilde{\nu} \geq \int G d\nu$ for any $f^t$-invariant probability measure $\nu$. Hence, $\beta_+^* = \beta_+$ and similarly we obtain that $\beta_-^* = \beta_-^*$.

Even in the classical i.i.d. case of the Cramér theorem which is relevant to the original form (1.1) of the Erdős-Rényi law finiteness of the rate function is rarely discussed in details. Here, we provide a simple argument. Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables such that $E\xi_1 = 0$ and $\Pi(b) = \ln E e^{b\xi_1} < \infty$ for all real $b$. Set

$I(\beta) = \sup_b (b\beta - \Pi(b))$, $\beta_+ = \|\xi_1^+\|_\infty = \text{ess sup} \xi_1$ and $\beta_- = -\|\xi_1^-\|_\infty = \text{ess inf} \xi_1$.

Then

$$(5.19) \quad I(\beta) < \infty \text{ for any } \beta \in (\beta_-, \beta_+) \text{ and } I(\beta) = \infty \text{ if } \beta \notin [\beta_-, \beta_+].$$

Indeed, if $0 \leq \beta < \beta_+$ then $P\{\xi_1 > \beta\} = p_\beta > 0$. Hence,

$$(5.20) \quad I(\beta) = -\inf_{b \geq 0} \ln(E e^{-b\xi_1}) \leq -\ln p_\beta < \infty$$

and similarly for $0 \geq \beta > \beta_-$. On the other hand, if $\beta > \beta_+$ then

$$(5.21) \quad I(\beta) = -\inf_{b \geq 0} \ln(E e^{-b\xi_1}) \geq -\inf_{b \geq 0} (\beta_+ - \beta) = \infty$$

and similarly for $\beta < \beta_-$. 

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