COMBINATORICS FOR GRADED CARTAN MATRICES OF THE IWAHORI-HECKE ALGEBRA OF TYPE A

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Abstract. A q-analogue of combinatorics concerning the Cartan matrix for the Iwahori-Hecke algebra of type A is investigated. We give several descriptions for the determinant of the graded Cartan matrix, which imply some combinatorial identities. A conjectural expression for the elementary divisors is also presented.

1. Introduction

Let $H_n(\zeta)$ be the Iwahori-Hecke algebra of type $A_{n-1}$ with the parameter $\zeta$ being a primitive $p$-th root of unity. The irreducible representations of $H_n(\zeta)$ are labeled by the set $P^{(p)}(n)$ of the $p$-regular partitions. The square matrix $C_n = ([P(\lambda) : D(\mu)])_{\lambda,\mu \in P^{(p)}(n)}$ is called the Cartan matrix of $H_n(\zeta)$, where $[P(\lambda) : D(\mu)]$ denotes the multiplicity of the irreducible module $D(\mu)$ in a composition series of the projective cover $P(\lambda)$ of $D(\lambda)$. As is well-known, the Cartan matrix can be expressed as $C_n = t D_n D_n$, where $D_n$ is the decomposition matrix of $H_n(\zeta)$.

There are several combinatorial expressions for the elementary divisors and the determinant of $C_n$. For example, when $p$ is prime, it is classically known that the elementary divisors of $C_n$ coincide with those of the Cartan matrix of the symmetric group $S_n$ at characteristic $p$, and they are given by

$$\left\{ \prod_{i \geq 1} (m_i!)_p \mid \lambda = (1^{m_1} 2^{m_2} \ldots) \in P^{(p)}(n) \right\}, \tag{1.1}$$

where $P^{(p)}(n)$ denotes the set of the $p$-class regular partitions of $n$ and $(k)_p$ is the $p$-part of $k$, i.e., $(k)_p = p^j$ for $k = ap^j$ with $p \nmid a$, and hence

$$\det C_n = \prod_{\lambda \in P^{(p)}(n)} \prod_{i \geq 1} (m_i!)_p. \tag{1.2}$$

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A bijection called the Glaisher correspondence \( \gamma : P_{(p)}(n) \to P^{(p)}(n) \) gives an alternative expression of the elementary divisors:

\[
\prod_{i \geq 1} (m_i!)_p = p^{\frac{\ell(\lambda) - \ell(\gamma(\lambda))}{p-1}}
\]

for \( \lambda = (1^{m_1} 2^{m_2} \ldots) \in P_{(p)}(n) \), where \( \ell(\lambda) \) denotes the length of the partition \( \mathbb{U}Y \).

The purpose of the present paper is to give a \( q \)-analogue of combinatorics concerning the Cartan matrix for \( H_n(\zeta) \), and to give some combinatorial identities for partitions. In particular we give a \( q \)-analogue of both sides of (1.3) and show that their products over \( P_{(p)}(n) \) coincide. It will turn out that these products equal the determinant of the \( q \)-Cartan matrix \( C_n(q) \) in the sense of Lascoux, Leclerc and Thibon \( \mathbb{L}T \).

We describe the composition of this article. In Section 2, we recall preliminary results on partitions. In Section 3, we define two weights \( w_E \) and \( w_H \) for partitions. Here \( w_E \) is a natural \( q \)-analogue of (1.1), while \( w_H \) is motivated by the work of Hill \( \mathbb{H} \) on block elementary divisors. We will prove that certain products of \( w_E \) and \( w_H \) coincide (Corollary 3.4). In Section 4, we define yet another weight \( w_G \) for partitions which comes from the Glaisher correspondence. Also we recall an expression of the block determinant of \( C_n(q) \) due to Tsuchioka (unpublished). The main result of this section (Theorem 4.1) reads

\[
\det C_n(q) = \prod_{\lambda \in P_{(p)}(n)} w_G(\lambda) = \prod_{\lambda \in P_{(p)}(n)} w_E(\lambda).
\]

Looking at the equality term-by-term, we obtain an interesting partition identity (Corollary 4.2). We also give alternative description for Tsuchioka’s expression of the block determinant (Theorem 4.3), and present another partition identity (Corollary 4.4). Theorem 4.1 and Theorem 4.3 are proved in Section 5 and Section 6 respectively. In Section 7, we focus on the case \( p = 2 \). In this case block splitting is easily described by means of \( H \)-abacus \( \mathbb{U}Y \).

Representation-theoretic meaning of the \( q \)-Cartan matrix \( C_n(q) \) is best understood if we consider the Khovanov-Lauda-Rouquier algebra \( \tilde{H}_n(\zeta) \) \( \mathbb{K}L1 \mathbb{K}L2 \) and \( \mathbb{R}o \). It is shown by Brundan and Kleshchev \( \mathbb{B}K2 \) that \( C_n(q) \) is the corresponding graded Cartan matrix. Section 8 is devoted to explaining these relationships. We conjecture that, when \( p \) is prime, the diagonal matrix with entries \( w_H \) has the same elementary divisors with the graded Cartan matrix (Conjecture 8.2).
2. Partitions

For a partition $\lambda$, we let $m_i(\lambda)$ denote the multiplicity of $i$ as its part, and we represent $\lambda$ as $(1^{m_1(\lambda)}2^{m_2(\lambda)}\ldots)$.

Let $p$ be a fixed integer greater than 1. A partition $\lambda = (1^{m_1}2^{m_2}\ldots)$ is said to be $p$-regular if $m_i < p$ for all $i$, and is said to be $p$-class regular if $m_i = 0$ for $i$ which is divisible by $p$. We let $P(n)$, $P^{(p)}(n)$ and $P_{(p)}(n)$ denote the set of the partitions of $n$, the $p$-regular partitions of $n$, and the set of the $p$-class regular partitions of $n$, respectively. It is well-known (see e.g. [And] that $P^{(p)}(n)$ and $P_{(p)}(n)$ have the same cardinality.

A partition is called a $p$-core if it has no $p$-hooks. Let $\text{Core}_p(d)$ denote the set of all $p$-cores in $P(d)$, and let $c_p(d) = |\text{Core}_p(d)|$.

The set $P^{(p)}(n)$ (and $P_{(p)}(n)$) labels the the set of isomorphism classes of the irreducible representations of the Iwahori-Hecke algebra $H_n(\zeta)$ associated with the symmetric group with the parameter $\zeta$ being a primitive $p$-th root of 1. The set $\sqcup_{0 \leq d \leq \lfloor \frac{n}{p} \rfloor} \text{Core}_p(n - pd)$ labels the blocks of the Cartan matrix $C_n$ of $H_n(\zeta)$, where $\lfloor a \rfloor$ denotes the largest integer which is not greater than $a$.

For each $p$-core in $\text{Core}_p(n - pd)$, the size of the corresponding block matrix equals the cardinality of the set

$$M_{p-1}(d) = \{ \Delta = (\lambda^{(1)}, \ldots, \lambda^{(p-1)}) \mid \lambda^{(i)} \in P(d_i), \Sigma_{i=1}^{p-1} d_i = d \}$$

of $(p - 1)$-multipartitions of $d$. Put

$$Q_p(n) = \bigsqcup_{0 \leq d \leq \lfloor \frac{n}{p} \rfloor} M_{p-1}(d) \times \text{Core}_p(n - pd).$$

Then $\#P^{(p)}(n) = \#Q_p(n)$ as they both equal the size of the Cartan matrix $C_n$.

Put

$$\phi(x) = \prod_{n \geq 1} (1 - x^n).$$
Then we have $\sum_{n \geq 0} \#P(n)x^n = \frac{1}{\phi(x)}$ and the following formulas for generating functions:

$$\sum_{n \geq 0} \#P_p(n)x^n = \frac{\phi(x^p)}{\phi(x)}, \quad (2.1)$$

$$\sum_{n \geq 0} \#M_p(n)x^n = 1, \quad (2.2)$$

$$\sum_{n \geq 0} c_p(n)x^n = \frac{\phi(x^p)}{\phi(x)}. \quad (2.3)$$

We end this section with presenting a useful formula of generating functions. Let $A(n)$ and $B(n)$ be finite sets indexed by non-negative integers $n$, and $a(n)$, $b(n)$ their cardinalities, respectively. For a positive integer $p$, consider the set $Z(n) = \bigcup_{k \geq 0} A(k) \times B(n - pk)$. Then the generating function of $z(n) = \#Z(n)$ is given by

$$\sum_{n \geq 0} z(n)x^n = \left( \sum_{n \geq 0} a(n)x^{pn} \right) \left( \sum_{n \geq 0} b(n)x^n \right). \quad (2.4)$$

Note that equation (2.2) can be shown by this formula.

3. $p$-PART AND ELEMENTARY DIVISORS

Let $q$ be an indeterminate. For a positive integer $l$, we define $[p]_l$ as the $q$-integer of $p$ with $q^{2l}$ base;

$$[p]_l = \frac{1 - q^{2pl}}{1 - q^{2l}} = 1 + q^{2l} + \cdots + q^{2l(p-1)}. \quad (2.5)$$

If a positive integer $k$ is written as $k = ap^b$ with $p \nmid a$, then $(k)_p = p^b$ is called the $p$-part of $k$, and $(k)_{p'} = a$ is called the $p'$-part of $k$. Put

$$(k)_{[p]} = [p]_a [p]_{ap} \cdots [p]_{ap^{b-1}}, \quad (2.6)$$

which might be called the graded $p$-part of $k$.

For a positive integer $m$ with $p$-adic expansion

$$m = c_0 + c_1p + \cdots + c_rp^r \quad (0 \leq c_0, \ldots, c_r \leq p - 1, \ c_r \neq 0), \quad (2.7)$$

define

$$a_p(m) = r + 1. \quad (2.8)$$

The following two lemmas are verified by direct computations.
Lemma 3.1. Let \( m \in \mathbb{Z}_{\geq 1}, \ p \in \mathbb{Z}_{\geq 2} \). Then
\[
\prod_{j=1}^{m} [p]_{j}^{o_{p}([\frac{m}{j}])} = \prod_{j=1}^{m} [p]_{j}((j)_{[p]}).
\]

Lemma 3.2. Let \( m \in \mathbb{Z}_{\geq 1}, \ p \in \mathbb{Z}_{\geq 2} \). Then
\[
\prod_{j=1}^{m} (j)_{[p]} = \prod_{j=1}^{\left\lfloor \frac{m}{p} \right\rfloor} [p]_{j}((j)_{[p]}).
\]

For a partition \( \lambda \), we put
\[
w_{E}(\lambda) = \prod_{i \geq 1, \ p \mid i} \prod_{j=1}^{m_{i}(\lambda)} (j)_{[p]},
\]
\[
w_{H}(\lambda) = \prod_{j \geq 1} \prod_{i \geq 1, \ p \mid i} [p]_{j}^{o_{p}([\frac{m_{i}(\lambda)}{j}])}.
\]

For \((\underline{\mu}, \chi) = (\mu^{(1)}, \ldots, \mu^{(p-1)}, \chi) \in Q_{p}(n)\), define
\[
w_{H}(\underline{\mu}, \chi) = w_{H}(\mu^{(p-1)}).
\]

Here is an example. Suppose that \( p = 2 \) and \( \lambda = (1^{4}2^{3}45^{2}) \in P(24) \). Then we see that,
\[
w_{E}(\lambda) = \prod_{i=1,5} \prod_{j=1}^{m_{i}(\lambda)} (j)_{[p]}
\]
\[
= (1)_{[p]}(2)_{[p]}(3)_{[p]}(4)_{[p]} \times (1)_{[p]}(2)_{[p]}
\]
\[
= (2)_{[p]}(4)_{[p]} \times (2)_{[p]}
\]
\[
= [p]_{1}[p]_{1}[p]_{2} \times [p]_{1}
\]
\[
= [p]_{1}^{3}[p]_{2}.
\]
\[
w_{H}(\lambda) = \prod_{j \geq 1} \prod_{i=1,5} [p]_{j}^{o_{p}([\frac{m_{i}(\lambda)}{j}])}
\]
\[
= [p]_{1}^{o_{p}([\frac{1}{4}])} + o_{p}([\frac{1}{2}]) [p]_{2}^{o_{p}([\frac{1}{4}])} + o_{p}([\frac{3}{2}])
\]
\[
\times [p]_{3}^{o_{p}([\frac{1}{4}])} + o_{p}([\frac{3}{4}]) [p]_{4}^{o_{p}([\frac{1}{4}])} + o_{p}([\frac{5}{4}]) \ldots
\]
\[
= [p]_{1}^{3+2}[p]_{2}^{2+1}[p]_{3}^{1+0}[p]_{4}^{1+0}[p]_{5}^{0+0} \ldots
\]
\[
= [p]_{1}^{3}[p]_{2}^{3}[p]_{3}[p]_{4}.
\]

Remark that, when \( p \) is prime, it is classically known that \( w_{E}(\lambda)_{|q=1} = \prod_{i \geq 1} (m_{i}(\lambda)!)_{p} \) and that \( \{w_{E}(\lambda)_{|q=1} \mid \lambda \in P_{(p)}(n)\} \) gives the set of the
elementary divisors of the Cartan matrix of the symmetric group $\mathfrak{S}_n$ in characteristic $p$. Remark also that $w_H(\mu, \chi)|_{q=1}$ gives an expression of block elementary divisors of $C_n$ ([HI]).

**Theorem 3.3.** Let $p \in \mathbb{Z}_{\geq 2}$. Then, as multisets,

$$\{ w_E(\lambda) \mid \lambda \in P_{(p)}(n) \} = \{ w_H(\mu, \chi) \mid (\mu, \chi) \in Q_p(n) \}.$$

**Proof.** We write $r = \left\lfloor \frac{n}{p} \right\rfloor$ throughout this proof. Put $P_{(p)}(\leq r) = \bigsqcup_{0 \leq d \leq r} P_{(p)}(d)$. For a partition $\lambda = (1^{i_1}2^{i_2} \ldots) \in P_{(p)}(n)$, define an element of $P_{(p)}(\leq r)$ by

$$\alpha(\lambda) = (1^{\left\lfloor \frac{i_1}{p} \right\rfloor}2^{\left\lfloor \frac{i_2}{p} \right\rfloor} \ldots).$$

For $(\mu, \chi) = (\mu^{(1)}, \ldots, \mu^{(p-1)}, \chi) \in M_{p-1}(d) \times \text{Core}_p(n - pd) \subseteq Q_p(n)$ with $\mu^{(p-1)} = (1^{m_1}2^{m_2} \ldots)$, define $\beta(\mu, \chi)$ to be the $p$-class regular partition obtained from $\mu^{(p-1)}$ by removing all parts divisible by $p$, namely,

$$\beta(\mu, \chi) = \mu^{(p-1)}\setminus (p^{m_p}2^{m_2} \ldots).$$

In this way we obtain the maps

$$\alpha : P_{(p)}(n) \to P_{(p)}(\leq r),$$

$$\beta : Q_p(n) \to P_{(p)}(\leq r).$$

Moreover, we have

$$w_E(\lambda) = w_H(\alpha(\lambda)) \text{ (by Lemma 3.1 and Lemma 3.2)},$$

$$w_H(\mu, \chi) = w_H(\beta(\mu, \chi)).$$

Therefore, to prove the theorem, it is enough to show that

$$\forall \nu \in P_{(p)}(\leq r).$$

For $\nu = (1^{e_1}2^{e_2} \ldots)$. Then $\alpha^{-1}(\nu)$ consists of the elements of the form $(1^{m_1+e_i}2^{m_2+e_i} \ldots)$ with $0 \leq e_i \leq p-1$. Since $\sum_{i \geq 1} i(p^i e_i) = n$, it follows that $\alpha^{-1}(\nu)$ is in one-to-one correspondence with the set

$$A(n) = \{(1^{e_1}2^{e_2}, \ldots) \in P_{(p)}(n - p|\nu|) \mid 0 \leq e_i \leq p - 1 \text{ for } i \geq 1\}$$

Recall that the generating function of the set of $p$-class regular partitions is $\frac{\phi(x^p)}{\phi(x)}$ (the formula (2.1)). We have

$$\sum_{n \geq 0} \sum_{\nu \in A(n)} x^n = x^{p|\nu|} \prod_{i \geq 1; p|i; 0 \leq e_i < p} x^{i e_i} = x^{p|\nu|} \phi(x^p)^2 \phi(x) \phi(x^{p^2}). \quad (3.1)$$
It is easy to see that any element of the set $\beta^{-1}(\nu)$ is of the form

$$((\mu^{(1)}, \ldots, \mu^{(p-2)}), \nu \setminus p\tau), \chi)$$

for some $(\mu^{(1)}, \ldots, \mu^{(p-2)}) \in M_{p-2}(d - (|\nu| + pj))$, $\tau = (1^{t_1}2^{t_2} \ldots) \in P(j)$ and $\chi \in \text{Core}_p(n - dp)$. Therefore the set $\beta^{-1}(\nu)$ is in one-to-one correspondence with

$$B(n) = \bigcup_{0 \leq d \leq r} \bigcup_{j \geq 0} M_{p-2}(d - (|\nu| + pj)) \times P(j) \times \text{Core}_p(n - dp).$$

Using the formulas (2.2), (2.3) and (2.4), we have

$$\sum_{n \geq 0} \sharp B(n) x^n = x^{|\nu|} \frac{1}{\phi(x)p^{p-2} \times \frac{1}{\phi(x)p^2} \times \frac{\phi(x)p}{\phi(x)}}. \quad (3.2)$$

This equals $\sum_{n \geq 0} \sharp A(n) x^n$ (3.1) and implies $\sharp A(n) = \sharp B(n)$. Therefore $\sharp \alpha^{-1}(\nu) = \sharp \beta^{-1}(\nu)$. \hfill $\square$

**Corollary 3.4.**

$$\prod_{\lambda \in P(p)(n)} w_E(\mu) = \prod_{(\mu, \chi) \in Q_p(n)} w_H(\mu, \chi).$$

### 4. Determinant formulas and the Glaisher correspondence

The elementary divisors and determinant of $C_n$ can be expressed by the Glaisher correspondence, which gives a bijection between the set of $p$-regular and $p$-class regular partitions as described below. Let $\lambda = (1^{m_1}2^{m_2} \ldots)$ be a partition of $n$. If $m_i \geq p$, then transform $\lambda$ as

$$(1^{m_1}2^{m_2} \ldots i^{m_i} \ldots (pi)^{m_p} \ldots) \xrightarrow{g_i} (1^{m_1}2^{m_2} \ldots i^{m_i-p} \ldots (pi)^{m_p+1} \ldots).$$

Repeat this procedure until all exponents will get to be less than $p$. The resulting partition $\lambda$ is $p$-regular.

Here is an example. Suppose that $p = 2$ and $\lambda = (1^9 3^5 3) \in P(2)(27)$. Then we have $\tilde{\lambda} = g_5g_4g_2^2g_1^4(\lambda)$:

$$\lambda \xrightarrow{g_5} (1^9 3^5 3) \xrightarrow{g_1} (1^7 2^3 3^5) \xrightarrow{g_1} (1^5 2^3 3^5) \xrightarrow{g_1} (1^3 2^3 3^5) \xrightarrow{g_1} (1 2^4 3^5) \xrightarrow{g_2} (1 2^2 2^4 3^5) \xrightarrow{g_4} (1 3^4 5^3) \xrightarrow{g_4} (1 3 5^3 8) \xrightarrow{g_5} (1 3 5 8 10) = \tilde{\lambda}. $$

Now attach weights to the Glaisher correspondence above. For a $p$-class regular partition $\lambda$, let the Glaisher weight $w_G(\lambda)$ be $\prod_{i \geq 1} [p^{d_i(\lambda)}]$, where $d_i(\lambda)$ is the number of occurrences of step $g_i$ in constructing $\lambda \mapsto \tilde{\lambda}$. In the previous example, we see that $w_G(\lambda) = [p^{4}] [p^{2}] [p^{1}] [p]$. 


For any $\lambda \in P(p)(n)$, we have an explicit formula

$$w_{G}(\lambda) = \prod_{a \geq 1} \prod_{b \geq 1} [p]_{ap^{b-1}}^{\left\lfloor \frac{m_{a}(\lambda)}{p^{b}} \right\rfloor}.$$

Under the specialization $q = 1$, it is not difficult to see that

$$w_{G}(\lambda)|_{q=1} = \prod_{i \geq 1} \prod_{b \geq 1} p^{\left\lfloor \frac{m_{i}(\lambda)}{p^{b}} \right\rfloor} = \prod_{i \geq 1} \prod_{j=1}^{m_{i}} (j)_{p} = w_{E}(\lambda)|_{q=1}$$

and that the left hand side is, by definition, equal to $p^{l(\lambda) - l(\tilde{\lambda})}$, where $l(\lambda)$ denotes the length of the partition $\lambda$ ([UY]). It is known that (4.1) gives the elementary divisors of the Cartan matrix $C_{n}$, and hence the product gives det $C_{n}$ (e.g. [LLT], [NT]).

The Cartan matrix $C_{n}$ is also related to the Gram matrix of the Shapovalov form for the basic representation of the affine Lie algebra $\hat{sl}_{p}$. Following Tsuchioka [Tsu], set

$$A_{j}(d) = \sum_{\lambda \in P(d)} \frac{m_{j}(\lambda)}{p-1} \prod_{i \geq 1} \left( p - 2 + m_{i}(\lambda) \right),$$

and define

$$\Delta_{p,n}(d) = \prod_{j \geq 1} [p]_{j}^{A_{j}(d)},$$

$$\Delta_{p,n} = \prod_{0 \leq d \leq \left\lfloor \frac{n}{p} \right\rfloor} \Delta_{p,n}(d)^{c_{p}(n-pd)}.$$  

As shown in [BK1], $\Delta_{p,n}(d)|_{q=1}$ equals the determinant of the block of $p$-weight $d$ of the Cartan matrix $C_{n}$, and it follows that $\Delta_{p,n}|_{q=1}$ gives the full determinant of $C_{n}$.

**Theorem 4.1.** The following equalities hold as polynomials in $q$:

$$\Delta_{p,n} = \prod_{\lambda \in P(p)(n)} w_{G}(\lambda) = \prod_{\lambda \in P(p)(n)} w_{E}(\lambda).$$

The proof of Theorem 4.1 will be given in the next section. By comparing the exponents of $[p]_{j}$ in $\prod_{\lambda \in P(p)(n)} w_{G}(\lambda)$ and $\prod_{\lambda \in P(p)(n)} w_{E}(\lambda)$, we obtain the following formula.

**Corollary 4.2.** Let $j$ and $k$ be positive integers satisfying $p \nmid j$. Then

$$\sum_{\lambda \in P(p)(n)} \left\lfloor \frac{m_{j}(\lambda)}{p^{k}} \right\rfloor = \sum_{\lambda \in P(p)(n)} \sum_{i \geq 1} a_{p} \left( \left\lfloor \frac{m_{i}(\lambda)}{p^{k}j} \right\rfloor \right).$$
It will turn out that the polynomials in Theorem 4.1 coincide with the determinant of the graded Cartan matrix, and that the following theorem gives an expression for its block determinant (Theorem 8.1).

**Theorem 4.3.** Let \( p \in \mathbb{Z}_{\geq 2}, \ d \in \mathbb{Z}_{\geq 0} \) and \( j \in \mathbb{Z}_{\geq 1} \). Then
\[
A_j(d) = \sum_{(\mu^{(1)}, \ldots, \mu^{(p-1)}) \in M_{p-1}(d)} m_j(\mu^{(p-1)})
= \sum_{(\mu^{(1)}, \ldots, \mu^{(p-1)}) \in M_{p-1}(d)} \sum_{i \geq 1; \ p \nmid i} o_p \left( \left\lfloor \frac{m_i(\mu^{(p-1)})}{j} \right\rfloor \right).
\]
Therefore,
\[
\Delta_{p,n}(d) = \prod_{(\mu^{(1)}, \ldots, \mu^{(p-1)}) \in M_{p-1}(d)} w_H(\mu^{(p-1)}).\]

The proof of Theorem 4.3 will be given in Section 6. Though our proof of Theorem 4.1 is bijective and the proof of Theorem 4.3 is based on Theorem 4.1, it is also possible to prove Theorem 4.1 and Theorem 4.3 directly and independently by using generating functions.

The following corollary follows from the second equality in Theorem 4.3.

**Corollary 4.4.** Let \( j \in \mathbb{Z}_{\geq 1} \). Then
\[
\sum_{\lambda \in P(n)} m_j(\lambda) = \sum_{\lambda \in P(n)} \sum_{i \geq 1; \ p \nmid i} o_p \left( \left\lfloor \frac{m_i(\lambda)}{j} \right\rfloor \right).
\]

5. **Proof of Theorem 4.1**

To prove the first equality in Theorem 4.1 put
\[
N_{j,n} = \sum_{d \geq 0} c_p(n - dp)A_j(d)
= \sum_{d \geq 0} c_p(n - dp) \left( \sum_{\lambda \in P(d)} m_j(\lambda) \left( \prod_{i \geq 1} \left( p - 1 + \frac{m_i(\lambda)}{p} \right) \right) \right).
\]
Recall that \( c_p(k) = \# \text{Core}_p(k) \) and their generating function is given by \( \frac{\phi(x)^p}{\phi(x)} \). Note that
\[
\frac{m}{p-1} \binom{p - 2 + m}{m} = \frac{m}{p-1} \binom{(p-1)}{m} = \binom{(p-1)}{m-1},
\]
where the symbol “(( ))” stands for the number of combinations with repetitions. It is easy to see that
\[
\sum_{m \geq 0} \binom{p-1}{m} x^m = \frac{1}{(1-x)^{p-1}}.
\]

Then it is seen that the generating function of the sequence \( \{N_{j,n}\}_{n \geq 0} \) reads
\[
\sum_{n \geq 0} N_{j,n} x^n = \frac{\phi(x)^p}{\phi(x)} \times \frac{x^{jp}}{(1-x^{jp})^p} \times \prod_{i \neq j} \frac{1}{(1-x^{ip})^{p-1}}
\]
\[
= \frac{x^{jp}}{1-x^{jp}} \times \prod_{k \geq 1; p|k} \frac{1}{1-x^k}.
\]

The right-hand side is equal to the generating function
\[
\sum_{n \geq 0} \left( \sum_{k \geq 1} ^{\sharp} P_{(p)}(n - pkj) \right) x^n.
\]

Let \( j = ap^{b-1} \), where \( p \) does not divide \( a \), and \( b \geq 1 \). Then
\[
\sum_{k \geq 1} ^{\sharp} P_{(p)}(n - pkj) = \sum_{\lambda \in P_{(p)}(n)} \left\lfloor \frac{ma}{p^b} \right\rfloor,
\]
and this completes the proof of the first equality.

Next, we prove the second equality
\[
\prod_{\lambda \in P_{(p)}(n)} w_G(\lambda) = \prod_{\lambda \in P_{(p)}(n)} w_E(\lambda).
\]

Our proof is bijective. To this end, we reformulate the two weights \( w_G \) and \( w_E \) as follows. For \( \lambda = (1^{m_1}2^{m_2} \ldots) \in P_{(p)}(n) \) and \( i \geq 1; p \nmid i \), we associate a diagram
\[
D_i(\lambda) = \left\{ (j, k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} \mid 1 \leq k \leq \left\lfloor \frac{ma}{p^b} \right\rfloor, \ p^i | k \right\}.
\]

Here is an example. Let \( p = 2 \) and \( \lambda = (1^93^35^3) \in P_{(2)}(27) \). Then we see that
\[
D_1(\lambda) = \begin{array}{|c|c|c|}
\hline
 & & \\
\hline
 & & \\
\hline
 & & \\
\hline
\end{array},
\]
\[
D_5(\lambda) = \begin{array}{|c|}
\hline
\end{array}.
\]
and $D_i(\lambda) = \emptyset$ for other odd $i$.

Put
\[
\mathfrak{D}(n) = \mathfrak{D}(n, p) = \{ (\lambda; i, j, k) \mid \lambda \in P_{(p)}(n), \ p \nmid i, \ (j, k) \in D_i(\lambda) \}.
\]

We consider two tableaux $G$ and $E$ on $\mathfrak{D}(n)$, namely
\[
G, \ E : \mathfrak{D}(n) \to \mathbb{Z}_{\geq 1}.
\]
For $c = (\lambda; i, j, k) \in \mathfrak{D}(n)$, define $G(c) = ip^j$ and $E(c) = k/p^j$.

In the previous example, $G$ and $E$ are tabulated on $D_1(\lambda)$, respectively, as
\[
G(D_1(\lambda)) = \begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 4
\end{array},
\]
\[
E(D_1(\lambda)) = \begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 4
\end{array}.
\]

Here $G(\lambda; 1, j, k)$ (resp. $E(\lambda; 1, j, k)$) is written in the $(j, k)$-position.

The claim of the theorem is equivalent to the identity
\[
\prod_{c \in \mathfrak{D}(n)} [p]_{G(c)} = \prod_{c \in \mathfrak{D}(n)} [p]_{E(c)}. \tag{5.1}
\]

We prove (5.1) by constructing an involution
\[
\theta : \mathfrak{D}(n) \to \mathfrak{D}(n)
\]
such that $E \circ \theta = G$. Take an element $c = (\lambda; i, j, k) \in \mathfrak{D}(n)$. By definition, $pk \leq m_i$ and $p^j|k$. Therefore there exists a $p$-class regular partition $\mu$ such that $\lambda = \mu + (i^p k)$, where “+” denotes the concatenation (union) of two Young diagrams. Let $k$ be decomposed as $k = i'p^{i+j'}$, where $i' = (k)p$ is the $p'$-part of $k$, and $p^{i+j'} = (k)_p$ is the $p$-part of $k$. Hence we can write
\[
c = (\mu + (i'p^{i+j'}); i, j, i'p^{i+j'}).
\]

Define $\theta$ by
\[
\theta(c) = (\mu + (i'p^{i+j'}); i', j', i'p^{i+j'}). \tag{5.1}
\]
Namely, $\theta$ interchanges $i$ with $i'$, and $j$ with $j'$. It can be seen that $\alpha$ is a map from $\mathfrak{D}(n)$ to itself, and is an involution. It is also easy to see that, for $c = (\lambda; i, j, k)$,
\[
E(\theta(c)) = ip^j = G(c).
\]
This proves the formula.

6. **Proof of Theorem 4.3**

**Proof of the first equality.** For each \( j \), we have
\[
\sum_{d \geq 0} \sum_{\mu \in M_{p-1}(d)} m_j(\mu^{(p-1)}) x^d = \sum_{d \geq 0} \sum_{k=0}^{d} \sum_{(\Delta\mu) \in M_{p-2}(d-k) \times P(k)} m_j(\mu) x^d
\]
\[
= \frac{1}{\phi(x)^{p-2}} \times \frac{1}{\phi(x)} \times \frac{x^j}{(1-x^j)} = \frac{x^j}{(1-x^j)\phi(x)^{p-1}},
\]
\[
\sum_{d \geq 0} \sum_{\lambda \in P(d)} \frac{m_j}{p-1} \prod_{i \geq 1} \left( \frac{(p-1)}{m_i} \right) x^d = \frac{x^j}{(1-x^j)^p} \prod_{i \geq 1, i \neq j} \frac{1}{(1-x^i)^{p-1}}
\]
\[
= \frac{x^j}{(1-x^j)\phi(x)^{p-1}}.
\]

Hence the formula follows. \( \square \)

**Proof of the second equality.**

By Theorem 3.3 and Theorem 4.1, we have
\[
\prod_{(\Delta\lambda) \in Q_p(n)} w_H(\lambda^{(p-1)}) = \prod_{\lambda \in P(n)} w_E(\lambda) = \Delta_{p,n}
\]
By comparing the exponent of \([p]_j\), we have
\[
\sum_{0 \leq d \leq \lfloor \frac{n}{p} \rfloor} c_p(n - pd\delta_d) = 0,
\]
for each \( n \), where
\[
\delta_d = \sum_{\lambda \in M_{p-1}(d)} \left( A_j(d) - \sum_{i \geq 1; p|i} a_p \left[ \frac{m_i(\lambda^{(p-1)})}{j} \right] \right) = 0.
\]
By letting \( n = pd' \), we have
\[
\delta_{d'} + \sum_{0 \leq d < d'} c_p(n - pd)\delta_d = 0
\]
as \( c_p(0) = 1 \). By induction on \( d' \), we have \( \delta_{d'} = 0 \), namely,
\[
A_j(d) = \sum_{\lambda \in M_{p-1}(d)} \sum_{i \geq 1; p|i} a_p \left[ \frac{m_i(\lambda^{(p)})}{j} \right].
\]
\( \square \)
7. Block version for $p = 2$

When $p = 2$, the Glaisher correspondence turns out to be a bijection from the set of the odd partitions to the set of the strict partitions. In this case we can refine Theorem 4.1 to block-wise products. We write $SP(m) = P^{(2)}(m)$, $OP(m) = P_{(2)}(m)$ and $OSP(m) = OP(m) \cap SP(m)$. We use the following diagram representing a strict partition, which is called the 4-bar abacus in [BO] and the H-abacus in [UY]. For example, the H-abacus of $\lambda = (2, 3, 7, 9)$ is shown below.

```
1  3
2
4  5  7
6
8 9 11
```

Namely, for a strict partition we put a set of beads on the assigned positions. Two beads do not occupy the same position. From the H-abacus of the given strict partition $\lambda$, we obtain the H-core $\lambda^H$ by moving and removing the beads as follows:

1. Move a bead one position up along the leftmost runner.
2. Remove a bead at the position 2.
3. Move a bead one position up along the runner of 1 or of 3.
4. Remove the two beads at the positions 1 and 3, simultaneously.

The H-cores are thus characterized by the stalemates, which constitute the set

$$HC = \{ \emptyset, (1, 5, \ldots, 4m - 3, 4m + 1), (3, 7, \ldots, 4m - 1, 4m + 3) \mid m \geq 0 \}.$$ 

For example, the H-core of the above $\lambda = (2, 3, 7, 9)$ is $\lambda^H = (3)$. Notice that the number of nodes in every H-core is a triangular number, $m(m + 1)/2$, and conversely, for any triangular number $r$, there is a unique H-core with $r$ nodes. Thus there is a unique bijection between $HC$ and the set of 2-cores

$$\{ \Delta_0 = \emptyset, \Delta_m = (1, 2, \ldots, m) \mid m \geq 1 \}$$

that preserves the number of nodes. In fact, the bijection is obtained by applying unfolding, which is defined as taking the hook lengths of the main diagonal in the Young diagram. Namely, we have

$$HC = \{ \Delta^u_m \mid m \geq 0 \},$$

where $\lambda^u$ stands for the unfolding of $\lambda$. For example, $\Delta^u_4 = (3, 7)$. 

We need the $H$-quotient $\lambda^H[1]$ for a strict partition $\lambda$. Draw the H-abacus of the strict partition $\lambda$, and read out a 0-1 sequence as follows. First look at the runner of 3 starting from the bottom. If the number is circled, then attach 0, and attach 1 otherwise. In the above example, $\lambda = (2, 3, 7, 9)$, this 0-1 sequence is ...100. Next look at the runner of 1 starting at the top. If the number is circled, then attach 1 and attach 0 otherwise. In the above example, the 0-1 sequence is 0010... . Concatenate two 0-1 sequences to get a two-side infinite 0-1 sequence. In the above example we have ...1000010... . From this Maya diagram we define a partition $\lambda^H[1]$ by counting 0’s on the left of each 1. In the example, we have the partition $\lambda^H[1] = (4)$. It should be noticed that, for every fixed H-core $\lambda^H$, the map $\lambda \mapsto \lambda^H[1]$ is a bijection from $OSP(4d + |\lambda^H|)$ to $P(d)$. (See for example [O].)

**Theorem 7.1.** For any non-negative integer $d$,

$$\prod_{\lambda} w_G(\lambda) = \prod_{\lambda} w_E(\lambda),$$

where the products of the both sides run over all odd partitions $\lambda$ of $2d$ such that $\tilde{\lambda}^H = \emptyset$. Moreover, they equal the block determinant of the graded Cartan matrix of 2-weight $d$.

**Proof.** This first claim is easily verified by noticing that the involution $\theta$ does not change the H-core of $\tilde{\lambda}$. In fact, according to the decomposition $\lambda = \mu + (i^{pk})$ in the proof of Theorem 4.1 $\tilde{\lambda}^H = \tilde{\mu}^H$.

As for the second half we show

$$\left\{ w_E(\lambda) \mid \lambda \in OP(2d), \tilde{\lambda}^H = \emptyset \right\} = \left\{ w_H(\mu, \emptyset) \mid (\mu, \emptyset) \in Q_2(d) \right\}.$$

as multisets.

This can be shown in a similar way to that of Theorem 3.3. Put $r = \lfloor \frac{n}{2} \rfloor$ and recall the two maps

$$\alpha : OP(n) \to OP(\leq r)$$

$$\beta : Q_2(n) \to OP(\leq r).$$

Restrict these maps to the subsets $A'(d) = \{ \lambda \in OP(2d) \mid \tilde{\lambda}^H = \emptyset \}$ and $B'(d) = \{ \mu \in P(d) \mid (\mu, \emptyset) \in Q_2(d) \}$, respectively, and keep the same notation

$$\alpha : A'(d) \to OP(\leq d)$$

$$\beta : B'(d) \to OP(\leq d)$$

Here we identify the pair $(\mu, \emptyset) \in Q_2(d)$ with a single partition $\mu \in P(d)$, and write $w_H(\mu)$ in place of $w_H(\mu, \emptyset)$. 
It is easily seen that \( w_E(\lambda) = w_H(\mu) \) if \( \alpha(\lambda) = \beta(\mu) \). Therefore it is enough to show that
\[
\sharp \alpha^{-1}(\nu) = \sharp \beta^{-1}(\nu).
\]
for all \( \nu \in OP(\leq d) \).

Fix \( \nu = (1^n_1 2^n_2 \ldots) \). Then \( \alpha^{-1}(\nu) \) consists of the elements of the form \((1^{2n_1+e_1} 2^{2n_2+e_2} \ldots)\) with \( e_i = 0 \) or \( 1 \). Since \( \sum_{i \geq 1} i(2n_i + e_i) = 2d \), it follows that \( \alpha^{-1}(\nu) \) is in one-to-one correspondence with the set \( OSP(2d - 2|\nu|) \).

On the other hand, any element of the set \( \beta^{-1}(\nu) \) is of the form \( \nu + 2\tau \) for some \( \tau = (1^{t_1} 2^{t_2} \ldots) \in P(d - |\nu|/2) \). It is already mentioned that the two sets \( OSP(2d - 2|\nu|) \) and \( P(d - |\nu|/2) \) have the same cardinality. \( \square \)

8. **Graded Cartan matrices**

In 1996, Lascoux, Leclerc and Thibon \([LLT]\) presented an algorithm for computing the global crystal basis for the basic representation \( L(\Lambda_0) \) of \( U_q(\widehat{\mathfrak{sl}}_p) \), where \( q \) is an indeterminate. The basic representation of \( U_q(\widehat{\mathfrak{sl}}_p) \) is realized as the highest irreducible component of the Fock space

\[
\mathfrak{F} = \bigoplus_{\lambda \in P} \mathbb{Q}(q)\lambda
\]

where \( P \) denotes the set of all partitions. Let \( d_{\lambda\mu}(q) \) be determined by

\[
G(\mu) = \sum_{\lambda \in P} d_{\lambda\mu}(q)\lambda,
\]

where \( \{ G(\mu) \mid \mu \in P(p) = \bigcup_{n \geq 0} P(p)(n) \} \) is Kashiwara’s lower global crystal basis for the basic representation of \( U_q(\widehat{\mathfrak{sl}}_p) \). Define the matrices

\[
D_n(q) = (d_{\lambda\mu}(q))_{\lambda \in P(n), \mu \in P(p)(n)};
C_n(q) = t D_n(q) D_n(q).
\]

Lascoux, Leclerc and Thibon \([LLT]\) conjectured that \( D_n(1) = (d_{\lambda\mu}(1)) \) is the decomposition matrix of the Iwahori-Hecke algebra \( H_n(\zeta) \) of type \( A_{n-1} \) with \( \zeta \) being a primitive \( p \)-th root of unity, and this conjecture was proved by Ariki \([Ar]\).

A representation-theoretic background of \( D_n(q) \) is given by the Khovanov-Lauda-Rouquier algebra \( \widehat{H}_n(\zeta) \) associated with the symmetric group, introduced independently by Khovanov and Lauda \([KLL, KL2]\) and Rouquier \([Ro]\). The algebra \( \widehat{H}_n(\zeta) \) is a \( \mathbb{Z} \)-graded algebra and is isomorphic to the Iwahori-Hecke algebra \( H_n(\zeta) \) as a non-graded algebra. It is shown by Brundan and Kleshchev \([BK2]\) that \( D_n(q) \) is nothing but the “graded decomposition matrix” for \( \widehat{H}_n(\zeta) \). As a consequence, it
follows that $C_n(q)$ is the corresponding graded Cartan matrix (see also [HM, Theorem 2.17]). More precisely,

$$C_n(q) = \left( \sum_{k \in \mathbb{Z}} [\tilde{P}(\lambda) : \tilde{D}(\mu)(k)]q^k \right)_{\lambda, \mu \in P^{(n)}}$$

Here, $\tilde{D}(\mu)(k)$ is the graded simple module of $\tilde{H}_n(\zeta)$ which corresponds to $D(\mu)$ (the simple $H_n(\zeta)$-module of label $\mu$) by the grade forgetting functor, $\langle k \rangle$ indicates the grading shift, and $\tilde{P}(\lambda)$ denotes the projective cover of $\tilde{D}(\lambda)(0)$. We remark here that $C_n(0) = E$, the unit matrix [LLT, Theorem 6.8].

Using the description of the Fock space in [CJ, Theorem 3], Tsuchioka [Tsu] obtained a formula for the block determinants of the Gram matrix of the Shapovalov form for the basic $U_q(\hat{\mathfrak{sl}}_p)$-module by a similar argument in [BK1].

As is well-known, the blocks are labeled by the $p$-cores. A partition $\lambda$ is said to have weight $d$ if $d$ successive removals of $p$-hooks from $\lambda$ achieve a $p$-core. On the other hand, the block of weight $d$ of the Cartan matrix corresponds to the Gram matrix of the Shapovalov form for the weight space of weight $w\Lambda_0 - d\delta$, where $w$ is an element of the Weyl group and $\delta$ is the fundamental imaginary root of $\hat{\mathfrak{sl}}_p$ (see for example [BH]).

We remark that the Shapovalov determinant for $L(\Lambda_0)$ is determined up to the even powers of $q$, and we normalize the determinant so that it equals 1 at $q = 0$.

**Theorem 8.1** (Tsuchioka). Let $d$ be the weight of the block. Then the corresponding block determinant of the graded Cartan matrix $C_n(q)$ is given by

$$\prod_{j \geq 1} [p]_j^{A_j(d)} ,$$

where

$$A_j(d) = \sum_{\lambda \in P(d)} \frac{m_j(\lambda)}{p-1} \prod_{i \geq 1} \left( p - 2 + m_i(\lambda) \right)$$

as before.

Tsuchioka’s formula gives a natural graded analogue of the block determinant formula for the Cartan matrix $C_n$ by Brundan and Kleshchev [BK1]. As an analogue of the expression for the elementary divisors for $C_n$, we conjecture the following:
Conjecture 8.2. Let $p$ be a prime.

(i) The elementary divisors of the block of $p$-weight $d$ of the graded Cartan matrix over the ring $\mathbb{Q}[q, q^{-1}]$ coincide with the elementary divisors of the diagonal matrix with entries

$$\{ w_H(\mu^{(p-1)}) \mid \mu = (\mu^{(1)}, \ldots, \mu^{(p-1)}) \in M_{p-1}(d) \}.$$ 

(ii) The elementary divisors of the graded Cartan matrix $C_n(q)$ over the ring $\mathbb{Q}[q, q^{-1}]$ coincide with the elementary divisors of the diagonal matrix with entries

$$\{ w_E(\lambda) \mid \lambda \in P_{(p)}(n) \},$$

and also that of the diagonal matrix with entries

$$\{ w_G(\lambda) \mid \lambda \in P_{(p)}(n) \}.$$ 

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