Lundberg-type inequalities

for non-homogeneous risk models

Qianqian Zhou\textsuperscript{a}, Alexander Sakhanenko\textsuperscript{a,b} and Junyi Guo\textsuperscript{a}

\textsuperscript{a}School of Mathematical Sciences, Nankai University, Tianjin, China;

\textsuperscript{b}Sobolev Institute of Mathematics, 4 Acad. Koptyug avenue, Novosibirsk, Russia

Abstract In this paper, we investigate the ruin probabilities of non-homogeneous risk models. By employing martingale method, the Lundberg-type inequalities of ruin probabilities of non-homogeneous renewal risk models are obtained under weak assumptions. Then we consider the application of our general results with special attention to new possibilities for the special non-homogeneous risk models. First, we study the upper bounds of ruin probabilities in periodic and quasi-periodic risk models with interest rates. Second, we present some examples to show that the ruin probability in non-homogeneous risk model may be fast decreasing which is impossible for the case of homogeneity.

Keywords: Non-homogeneous risk model; Martingale method; Ruin probability; Lundberg-type inequality
1 Introduction and some basic results

Ruin probability is a vital index of the robustness of an insurance company and it is also a useful tool for risk management. High probability of ruin means that the insurance company is not stable, and then the insurer should take corresponding measures to reduce risks. So the study of the probability of ruin is very essential for the insurers. The calculation of ruin probability is a classical problem in actuarial science. However, the exact value of ruin probability can only be calculated for exponential distribution or discrete distribution with finite values. But the upper bounds of run probabilities of risk models can be obtained. The insurer may employ the upper bound of ruin probability to evaluate the stability of insurance company.

Research on ruin probabilities of risk models is attracting an increasing attention since classic works of Lundberg [20] and Cramér [10,11]. After that, substantial amount of works, such as references [3,12–14,21], have been devoted to finding ruin probabilities and estimating ruin probabilities of homogeneous risk models.

Let $R(t), t \geq 0,$ be a risk reserve process with initial reserve $R(0) = u > 0$. We consider it as a general model for the time evolution of the reserves of an insurance company. Introduce the probability $\psi(u)$ of ultimate ruin, i.e., the probability that the reserve ever drops below zero:

$$\psi(u) := \mathbb{P}\left[ \inf_{t \geq 0} R(t) < 0 \mid R(0) = u \right]. \quad (1)$$

For the well known homogeneous risk models the most outstanding result about the behavior
of \( \psi(u) \) is the Lundberg inequality, which states that under appropriate assumptions (see \[10, 11\] and \[20\] for more details) the probability \( \psi(u) \) of ultimate ruin satisfies

\[
\psi(u) \leq e^{-Lu} \text{ for all } u \geq 0,
\]

(2)

The largest number \( L \) in (2) is called the adjustment coefficient or Lundberg exponent.

In the homogeneous risk models, inter-occurrence times and claim sizes are both assumed to be i.i.d. random variables. However, both factors are influenced by the economic environment. Then the usual assumptions on claim sizes and inter-occurrence times may be too restrictive for practical use. Thus relaxing the ”identically distributed” assumption imposed on claim sizes or (and) inter-occurrence times is necessary. In this case, risk model becomes to be non-homogeneous. In Castañer, et al. \[9\], and Lefèvre and Picard \[19\], they relaxed the ”identically distributed” assumption imposed on claim sizes. And in Bernackaitė and Šiaulys \[6\], Ignator and Kaishev \[16\] and Tuncel and Tank \[24\], the ”identically distributed” assumption imposed on inter-occurrence times is relaxed.

Without the condition of identically distribution on claim sizes and inter-occurrence times, there are serious difficulties in evaluating the probability of ruin. Hence, the related papers generally investigate the recursions for the finite time ruin probability under restrictive conditions. Such as, in Blaževićius, et al \[7\] the recursive formula of finite-time ruin probability of discrete time risk model with nonidentical distributed claims is obtained. In Răducan, et al \[22\], the recursive formulae for the ruin probability at or before a certain claim arrival instant of the non-homogeneous risk model are obtained. In this risk model, the claim sizes are independent non-homogeneous Erlang distributed and independent of the inter-occurrence times, which are assumed to be i.i.d. random variables following an Erlang
or a mixture of exponentials distribution. Răducan, et al [23] extended those recursive formulae to a more general case when the inter-occurrence times are i.i.d. nonnegative random variables following an arbitrary distribution.

Our aim is to study non-homogeneous risk model and obtain Lundberg inequality of (2) for the probability \( \psi(u) \) under more general assumptions.

We consider a class of risk reserve processes, which need not to be homogeneous, with the following properties.

(i) Process \( R(t) \) may have positive jumps only at random or non-random times \( T_1, T_2, \ldots \) such that

\[ \forall k = 1, 2, \ldots, T_{k+1} > T_k > T_0 := 0 \text{ and } T_n \to \infty \text{ a.s.} \]

(ii) Process \( R(t) \) is monotone on each time interval \([T_{k-1}, T_k), k = 1, 2, \ldots\), and \( R(0) = u > 0 \).

\textbf{Model A.} Assume that the \( kth \) claim \( Z_k \) occurs at time \( T_k \), i.e.,

\[ -Z_k := R(T_k) - R(T_k - 0) \leq 0, \quad \theta_k := T_k - T_{k-1} > 0, \quad k = 1, 2, \ldots \quad (3) \]

Suppose that on each interval \([T_{k-1}, T_k)\) the premium rate is \( p_k \), i.e.,

\[ \forall t \in [T_{k-1}, T_k), \quad R(t) - R(T_{k-1}) = p_k(t - T_{k-1}), \quad k = 1, 2, \ldots \quad (4) \]

Assume, also, that random vectors

\( (p_k, Z_k, \theta_k), \quad k = 1, 2, \ldots \)

are mutually independent. Then conditions (i) and (ii) hold and random variables

\[ Y_k = R(T_{k-1}) - R(T_k) = Z_k - X_k = Z_k - p_k \theta_k, \quad k = 1, 2, \ldots \]
are also mutually independent. Here we denote by $X_k = R(T_k - 0) - R(T_{k-1}) = p_k \theta_k$ the total premium collected by the insurer over the interval $[T_{k-1}, T_k)$.

Here we call Model A is non-homogeneous renewal risk model and it is clear that Model A is more general than the classical compound Poisson and renewal risk models introduced by Sparre Andersen [1] in which the random vectors $(Z_k, \theta_k)$, $k = 1, 2, \ldots$, are assumed to be i.i.d. and the premium rate $p_k \equiv p > 0$ is fixed, positive and non-random.

Next we are going to study the ruin probability of model A.

**Theorem 1.** Assume that conditions (i) and (ii) hold and that the random variables

$$Y_k := R(T_{k-1}) - R(T_k), \ k = 1, 2, \ldots,$$

are mutually independent. Then for any $u > 0$, $h \geq 0$,

$$\psi(u) = \mathbb{P} \left[ \sup_{k \geq 1} S_k > u \right] \leq e^{-hu} \sup_{k \geq 1} \mathbb{E} e^{hS_k} = e^{-hu} \sup_{k \geq 1} \prod_{j=1}^{k} \mathbb{E} e^{hY_j},$$

where $S_k = Y_1 + Y_2 + \ldots + Y_k$.

Note that Theorem 1 allows us to obtain several simple generalizations of Lundberg inequality of (2) for non-homogeneous risk models. For example, the following assertions follow immediately from Theorem 1.

**Corollary 1.** Under the assumptions of Theorem 1 the following Lundberg inequality

$$\forall u > 0, \ \psi(u) \leq e^{-L(S_\bullet)u}$$

holds, where

$$L(S_\bullet) := \sup \{ h \geq 0 : \sup_{k \geq 1} \mathbb{E} e^{hS_k} \leq 1 \}.$$
Moreover, for all $u > 0$

$$\psi(u) \leq \inf_{h \in [0,L(Y_*)]} \{e^{-hu} \mathbb{E}^{e^{hY_1}}\} \leq e^{-L(Y_*)u} \mathbb{E}^{e^{L(Y_*)Y_1}} \leq e^{-L(Y_*)u}, \quad (8)$$

with $L(Y_*) := \sup\{h \geq 0 : \sup_{j \geq 1} \mathbb{E}^{e^{hY_j}} \leq 1\} \leq L(S_*)$.

Homogeneous renewal risk model is a special case of the model from Corollary 1 when random variables $Y_1, Y_2, \ldots$ are i.i.d.. In this case from (8) we have the classical Lundberg inequality (2) with the adjustment coefficient or Lundberg exponent $L = L(Y_1)$ given by

$$L(Y_1) := \sup\{h \geq 0 : \mathbb{E}^{e^{hY_1}} \leq 1\}. \quad (9)$$

It can be seen that our inequality (8) is a little better than Lundberg inequality in its classical form, which can be found in [3,14,21], because in our variant (8) of Lundberg inequality we do not exclude the cases when $\mathbb{E}^{e^{L(Y_1)Y_1}} < 1$ and/or when the expectation $\mathbb{E}Y_1 = -\infty$ does not exist.

It follows from (9) that $\mathbb{E}^{e^{hY_1}} > 1$ for all $h > L(Y_1)$. So, for i.i.d. random variables $Y_1, Y_2, \ldots$ we have that the right hand side in (6) is $+\infty$ for all $h > L(Y_1)$. Thus, for classical renewal risk models the adjustment coefficient $L(Y_1)$ is the natural boundary for possible values of the parameter $h$ in inequality (6) of Theorem 1. But for non-homogeneous risk models the situation may be significantly different because the optimal value $h = h(u)$ of the parameter $h$ in inequality (6) may be greater than $L(S_*)$ and may tends to $+\infty$ as $u \to \infty$. Moreover, in Examples 3 and 4 below we present random variables $Y_1, Y_2, \ldots$ corresponding to risk models such that

$$\psi(u) = o(e^{-Nu}) \text{ as } u \to \infty \text{ for all } N < \infty. \quad (10)$$
Moreover, in Example 3 we have:

\[ \forall u > 0, \quad \psi(2u) \leq e^{-u^{3/2}}. \tag{11} \]

So, very fast decreasing of ruin probabilities is possible in non-homogeneous cases.

Theorem 1 allows us also to obtain a generalization of the Lundberg inequality that for any \( u > 0 \)

\[ \psi(u) \leq Ce^{-Lu} \text{ with } C < \infty \text{ and } L > 0. \tag{12} \]

For example, for the periodic risk model with period \( l \) Theorem 1 yields immediately that

**Corollary 2.** Suppose that there exists an integer \( l \geq 1 \) such that for all \( n = 1, 2, \ldots \) random variables \( Y_{n+l} \) and \( Y_n \) are identically distributed. Then under assumptions of Theorem 1 inequality (12) holds with

\[ \sup_{k \geq 1} E e^{hS_k} = \max_{1 \leq k \leq l} E e^{hS_k} \text{ for each } h \in [0, L(S_l)], \]

where

\[ L(S_l) := \sup \{ h \geq 0 : E e^{hS_l} \leq 1 \}. \]

For the estimation of ruin probability of non-homogeneous renewal risk model, we found only a few works in which the types of inequalities (12) or (11) are proved. In Andrulyté, et al. [2], Kievinaitė & Šiaulys [17], and Kizinevič & Šiaulys [18], they consider the non-homogeneous renewal risk models, where claim sizes and inter-occurrence times are both independent but not necessarily identically distributed. We’ll show in Remarks 5 and 6 that their results are special cases of our Theorem 1. For example, it is not possible for them to obtain inequalities with properties (10) or (11). In the following, the non-homogeneous
renewal risk model with interest rate is also studied. Thus the results of the present paper are complementary to the results in Andrulytė, et al. [2], Kievinaitė & Šiaulys [17], and Kizinevič & Šiaulys [18].

The rest of the paper is organized as follows. In Section 2 we introduce the non-homogeneous renewal risk model with interest rate and present a more general Theorem 2 with some corollaries and examples. In Section 3, by employing main results periodic and quasi-periodic risk models with interest rates, which are also automatically non-homogeneous, are studied. The examples with properties (10) or (11) are also given. Almost all proofs are gathered in the last section.

Later on we regularly use the fact that expectations $E e^{hS} \in (0, \infty]$ are everywhere defined for all random variables $S$ and all real $h \geq 0$ but may take value $+\infty$. By this reason all inequalities of the form $P[A] \leq \text{const} \cdot E e^{hS}$ make sense even we omit, for brevity, the assumption $E e^{hS} < \infty$.

Note also that for the probability $\psi(u) \leq 1$ the inequality $\psi(u) \leq \psi^*(u) \leq \infty$ means that $\psi(u) \leq \min\{\psi^*(u), 1\} \leq 1$. Later on we will use an agreement that

$$E + \text{const} = \infty \text{ and } \text{const}/E = 0 \text{ when } E = \infty.$$ 

All limits in this paper are taken with respect to $n \to \infty$ unless the contrary is specified. And we use random variables $Y_1, Y_2, \ldots$ only when they are mutually independent.
2 Main results

2.1 Non-homogeneous risk model with interest rate.

Now we consider a more general class of risk reserve processes $R(t)$ which, together with properties (i) and (ii) from introduction, satisfy the following two assumptions.

(iii) For some non-random $r_1, r_2, \ldots$

$$\forall n \geq 1, \ R(T_k) \geq (1 + \alpha_k)R(T_{k-1}) - Y_k \text{ and } \alpha_k \geq r_k \geq 0. \quad (13)$$

(iv) Random variables $Y_k^* := \frac{Y_k}{1 + \alpha_k}, \ k = 1, 2, \ldots$, are mutually independent.

**Model B.** Instead of (11) suppose that on each interval $[T_{k-1}, T_k)$ the surplus accumulates as follows

$$R(t) = (1 + \alpha_k(t))R(T_{k-1}) + (1 + \beta_k(t))p_k(t - T_{k-1}), \ k = 1, 2, \ldots,$$

where $(1 + \alpha_k(t))R(T_{k-1})$ is the accumulated value of $R(T_{k-1})$ from $T_{k-1}$ to $t$ under interest rate $\alpha_k(t)$ and $(1 + \beta_k(t))p_k(t - T_{k-1})$ is the accumulated value of premiums collected from $T_{k-1}$ to $t$ under premium rate $p_k$ and interest rate $\beta_k(t)$.

Assume again that claims $Z_k$ arrive to an insurer only at moments of time $T_k$, i.e. (3) holds. Suppose now that processes $\alpha_k(t) \geq 0$ and $\beta_k(t) \geq 0$ are non-decreasing, and random vectors

$$ (p_k, Z_k, \theta_k, \alpha_k(T_k), \beta_k(T_k)), \ k = 1, 2, \ldots, $$

are mutually independent. Then conditions (i)–(iv) hold and random variables

$$Y_k = (1 + \alpha_k(T_k))R(T_{k-1}) - R(T_k) = Z_k - X(T_k) = Z_k - (1 + \beta_k(T_k))p_k\theta_k, \ k = 1, 2, \ldots, \quad (14)$$
are also mutually independent.

It is clear that the presented model is more general than Model A.

Introduce notations:

\[ \forall k \geq 1, \ v_k := \prod_{j=1}^{k} \frac{1}{1 + r_j} \text{ and } S^*_k := \sum_{j=1}^{k} v_{j-1} Y^*_j, \]  

(15)

where \( v_0 := 1 \).

**Theorem 2.** Under assumptions (i)--(iv), for any \( u > 0 \) and any \( h \geq 0 \) the following inequality

\[ \psi(u) \leq e^{-hu} \sup_{k \geq 1} E e^{hS^*_k} = e^{-hu} \sup_{k \geq 1} \prod_{j=1}^{k} E e^{hv_j Y^*_j}, \]  

(16)

of ruin probability of Model B holds.

In addition, for all \( u > 0, h \geq 0 \) and \( n \geq 1 \)

\[ \psi(u, T_n) \leq e^{-hu} \sup_{1 \leq k \leq n} E e^{hS^*_k} = e^{-hu} \sup_{1 \leq k \leq n} \prod_{j=1}^{k} E e^{hv_j Y^*_j}, \]  

(17)

where

\[ \psi(u, T_n) := P \left[ \inf_{0 \leq t \leq T_n} R(t) < 0 \left| R(0) = u \right. \right] \text{ for } T_n < \infty. \]  

(18)

**Remark 1.** As an analogue of Corollary for the model without interest rate, it is easy to see that all the assertions of Corollary also hold for the model with interest rate with

\[ L(S^*_\bullet) = L(S^*_\bullet) \text{ and } L(Y^*_\bullet) = L, \]

where

\[ L(S^*_\bullet) := \sup \left\{ h \geq 0 : \sup_{k \geq 1} E e^{hS^*_k} \leq 1 \right\} \geq L := \sup \left\{ h \geq 0 : \sup_{j \geq 1} E e^{hv_j Y^*_j} \leq 1 \right\}. \]
Theorem 2 also yields the following result.

**Corollary 3.** Suppose that for each $k \geq 1$

$$Y_k = b_k \xi_k \text{ and } b_k v_{k-1} \leq 1,$$

where random vectors $\{(\alpha_k, \xi_k), k = 1, 2, \ldots, \}$ are i.i.d. In this case under assumptions (i)-(iii) inequality (8) holds with

$$L(Y_\cdot) = L(Y^*_1) := \sup \{h \geq 0 : \mathbb{E} e^{hY^*_1} \leq 1 \}.$$

In particular, for any $u > 0$

$$\psi(u) \leq e^{-\kappa u} \text{ if } \mathbb{E} e^{\kappa Y^*_1} = 1. \quad (19)$$

Thus, when $b_n = (1 + r)^n$ and $v_n = (1 + r)^{-n} < 1$ we obtain two generalizations (8) with $L(Y_\cdot) = L(Y^*_1)$ and (19) of the famous Lundberg inequality for the case when $|Y_n| = (1 + r)^n|\xi_n| \to \infty$ almost surely and with high speed.

**Remark 2.** Two special cases of inequality (19) are obtained in Corollaries 3.1 and 3.2 of Cai [8] when

$$Y_k = X_k - Z_k \text{ or } Y_k = (1 + \alpha_k)X_k - Z_k$$

with $r_k \equiv 0$. Earlier a simpler case with

$$\alpha_k = r \geq 0 \text{ and } Y_k = (1 + r)X_k - Z_k,$$

is considered in Yang [23]. Underline that the mutual independence of non-negative random variables $X_k, Z_k$ and $\alpha_k$ is essential for the proofs in [8, 23].
2.2 General remarks

Remark 3. Theorem 1 is an evident special case of Theorem 2 when $\alpha_k = r_k = 0$ for all $k = 1, 2, \ldots$. Hence, all corollaries from Theorem 2 which are presented below, may be considered also as corollaries from Theorem 1.

Remark 4. Theorems 1 and 2 and all their corollaries remains valid also for non-homogeneous discrete-time risk models when claims arrive at non-random times $T_1, T_2, \ldots$.

Remark 5. In [2, 17], the authors use the trivial inequality

$$\forall u > 0, \forall h \geq 0, \mathbb{P}[\sup_{k \geq 1} S_k > u] \leq \sum_{k=1}^{\infty} \mathbb{P}[S_k > u] \leq e^{-hu} \sum_{k=1}^{\infty} \mathbb{E}e^{hS_k}$$

instead of our sharper estimate (6). It is clear that the results in [2, 17] will be improved automatically by using our estimate (6).

Remark 6. In the proof of Theorem 4 in [18] it is shown (under several additional assumptions) that for any $u > 0$

$$\psi(u) \leq \inf_{h \in [0, L(Y_n)]} \{e^{-hu} \sup_{i \geq 1} \mathbb{E}e^{hY_i}\}. \quad (20)$$

It is clear that the inequality (8) is better than (20). It also follows from Example 3 below that our estimate (6) from Theorem 1 may be qualitatively better than the estimate (20).

Remark 7. Note that all independent random variables $\{Y_n\}$ (with all possible distributions) may appear in Theorem 1 since we everywhere may consider Model A with values

$$Z_n = Y_n^+ := \max\{Y_n, 0\}, \quad X_n = p_n \theta_n = Y_n^- := \max\{-Y_n, 0\}, \quad p_n = 1.$$

For this reason, in Examples 1, 2, 4 below we do not construct risk models in which may appear the random variables $\{Y_n\}$ which we investigate in these examples.
Similarly, all random variables \( \{Y^*_n\} \) (with all possible distributions) may appear in Theorem 2 with all possible real numbers \( r_n \geq 0 \). Indeed, we may consider Model B with

\[
Z_n = (1 + r_n) \max \{Y^*_n, 0\}, \quad X_n(T_n) = p_n \theta_n = \max \{-Y^*_n, 0\}, \quad p_n = 1,
\]

and with \( \alpha_n(T_n) = \beta_n(T_n) = r_n \geq 0 \) for all \( n = 1, 2, \ldots \).

3 Applications to special models

In this section, we mainly introduce the applications of our main results. By employing our main results some special non-homogeneous renewal risk models are studied. First, the periodic and quasi-periodic risk model with interest rate is investigated. And then some examples are presented to show that the probability of ruin in non-homogeneous risk models may be fast decreasing which is impossible in homogeneous case.

3.1 Periodic and quasi-periodic risk model

Asmussen & Rolski [4] (see also Asmussen & Albrecher [3]) studied a kind of risk process which happens in a periodic environment. For the Lundberg-type inequality (12) they obtained that adjustment coefficient \( L \) is the same as for the standard time-homogeneous Poisson risk process obtained by averaging the parameters over a period. In Corollary 2 we have found the similar property for periodic risk models with period \( l \) under assumptions of Theorem 1. Now we present two more general results under conditions of Theorem 2.

**Corollary 4.** Suppose that there exist integers \( l, m \geq 1 \) and a real number \( L^* \geq 0 \) such
that for any $n \geq m$

$$Ee^{L^*(S_{n+l}^*-S_n^*)} \leq 1. \quad (21)$$

Then under assumptions (i)–(iv) inequality (16) holds for each $h \in [0, L^*]$ with

$$\sup_{k \geq 1} Ee^{hS_k^*} = \max_{1 \leq k \leq l+m-1} Ee^{hS_k^*}. \quad (22)$$

**Theorem 3.** Suppose that there exist an integer $l \geq 1$ and a real number $q_l > 0$ such that for all $n = 1, 2, \ldots$ random variables $Y_{n+l}^*$ and $q_lY_n^*$ are identically distributed. Assume also that for each $n \geq 1$

$$q_nv_l \leq 1 \text{ and } r_{n+l} = r_n. \quad (23)$$

And denote

$$L(S_l^*) := \sup \{ h \geq 0 : Ee^{hS_l^*} \leq 1 \}. \quad (24)$$

Then under assumptions (i)–(iv) inequality (16) holds with

$$\sup_{k \geq 1} Ee^{hS_k^*} \leq \max_{0 \leq k < l} Ee^{hS_k^*}, \ \forall h \in [0, L(S_l^*)]. \quad (25)$$

Moreover, if $q_nv_l = 1$ then (22) is also true with $m = 1$ for all $h \in [0, L(S_l^*)]$.

Note that the value in the right hand side of (22) may be less than 1. On the other hand, the right hand side of (25) may not be less than 1 since $Ee^{hS_0^*} = e^0 = 1$. Thus, Corollary 4 may give sharper estimates than Theorem 3 because condition (21) may be stronger than the assumptions in Theorem 3.

The models satisfying conditions of Corollary 4 or Theorem 3 will be called quasi-periodic. For us it is essential that for $l > 1$ all such risk models are automatically non-homogeneous.
Models satisfying assumptions of Theorem 3 with \( q_l = 1 \) are naturally called periodic. Periodic model from Corollary 2 is a special case of Theorem 3 with \( q_l = v_l = 1 = m \) when \( S_k^* = S_k \) and (22) is true.

**Example 1.** Let \( Y_1, Y_2, \ldots \) be independent normal random variables with

\[
Y_n \sim N(a_n, 1) \text{ and } a_n + a_{n+1} \leq a_1 + a_2 = -1, \ n = 1, 2, \ldots
\]  

(26)

It is easy to calculate that

\[
E e^{hY_n} = e^{h a_n + h^2/2} \quad \text{and} \quad E e^{hS_n} = E e^{h \sum_{i=1}^{n} a_n + h^2/2}, \ n = 1, 2, \ldots
\]  

(27)

Hence,

\[
L(S_2) = \sup \{ h \geq 0 : E e^{hS_2} \leq 1 \} = \sup \{ h \geq 0 : e^{-h + h^2} \leq 1 \} = 1.
\]

Thus, we have from (26) and (27) with \( h = 1 \) that

\[
E e^{S_{n+2} - S_n} = e^{a_{n+1} + a_{n+2} + 1} \leq e^0 = 1, \ \forall n \geq 0.
\]  

(28)

Comparing (21) and (28) we obtain that random variables \( \{Y_n\} \) from (26) satisfy all conditions of Corollary 4 with \( l = 2, \ m = 1 \) and \( L^* = 1 \). So, we have from (16) and (22) with \( h = L^* = 1 \) that for any \( u > 0 \)

\[
\psi(u) \leq \max \{ E e^{S_1}, E e^{S_2} \} e^{-u} = e^{(a_1 + 1/2) + u},
\]  

(29)

because

\[
\max \{ E e^{S_1}, E e^{S_2} \} = \max \{ e^{a_1 + 1/2}, 1 \} = e^{\max \{a_1 + 1/2, 0\}}.
\]

Inequality (29) allows us to make several conclusions about risk models with random variables \( \{Y_n\} \) from (26). First, if \( a_1 \leq -1/2 \), we can obtain the Lundberg inequality (2)
with $L = L^* = 1$. Second, if $a_1 > -1/2$, we can prove the generalization \((12)\) of the Lundberg inequality with $C = e^{a_1+1/2} > 1$. Third, in case of $a_1 \geq 0$ we have

$$\sup_{i \geq 1} E e^{hY_i} \geq E e^{hY_1} = e^{a_1 h + h^2/2} \geq e^{h^2/2} \geq e^0 = 1, \forall h \geq 0.$$ 

Thus, it follows from \((7)\) that in this case $L(Y_\bullet) = 0$ and, hence, inequality \((20)\) allows us to obtain only trivial estimate $\psi(u) \leq 1$. So, (see Remark \(6)\) Theorem 4 from \([18]\) does not work in this case, whereas, our results yield the estimate \((12)\) with $C = e^{a_1+1/2} < \infty$.

**Example 2.** (see \([17, Example 1]\)) Suppose $Y_1, Y_2, \ldots$ are independent random variables such that:

- $Y_i$ are uniformly distributed on interval $[0, 2]$ for $i \equiv 1 \pmod{3}$;
- $Y_i$ are uniformly distributed on interval $[-1, 0]$ if $i \equiv 2 \pmod{3}$;
- $F_{Y_i}(x) = 1_{(\infty, -2)}(x) + e^{-x-2}1_{[-2, \infty)}(x)$ when $i \equiv 0 \pmod{3}$.

It is a periodic risk model with $l = 3$. We can calculate that for all $0 < h < 1$

$$E e^{hY_2} = \frac{1 - e^{-2h}}{2h} = e^{-2h} E e^{hY_1} < 1, \quad E e^{hY_3} = \frac{e^{-2h}}{1 - h}.$$ 

Then for $h_0 = 2/3$

$$E e^{h_0 S_3} = \frac{(1 - e^{-2h_0})^2}{(2h_0)^2(1 - h_0)} < 1, \quad E e^{h_0 Y_1} = \frac{e^{2h_0} - 1}{2h_0} < 2.2.$$ 

Thus, we have from Corollary \(2\) that

$$\psi(u) \leq \max\{E e^{S_1}, E e^{S_2}, E e^{S_3}\} e^{-h_0 u} < 2.2 e^{-\frac{2}{3}u}.$$ 

So, we have proved that

$$\psi(u) \leq \psi_1^*(u) := \max\{1, 2.2 e^{-\frac{2}{3}u}\} \leq e^{(1 - \frac{2}{3}u)^-}, \forall u \geq 0. \quad (30)$$
Remind that in [17] the following bound is obtained

\[ \psi(u) \leq \psi_1^*(u) := \min \{1, 1502e^{-0.01269u}\}, \quad \forall u \geq 0. \quad (31) \]

It is clear that estimate (30) is more accurate than (31). For example,

\[ \psi_1^*(576) < e^{-381} < 10^{-165} \text{ whereas } \psi_1^*(576) = 1. \]

### 3.2 Two special models with fast decreasing ruin probabilities

Here we are going to present examples of risk models with property (10) which is impossible in homogeneous case.

**Example 3.** Consider again independent random variables \( Y_1, Y_2, \ldots \) with normal distributions from Example 1. First, suppose that they are i.i.d. with \( N(-1/2, 1) \). In this case condition (26) holds and \( L(Y_1) = \sup \{ h \geq 0 : e^{-h/2 + h^2/2} \leq 1 \} = 1 \). Thus we have the Lundberg inequality (2) with \( L = 1 \) and it is the upper boundary for values \( h \) which we use in inequality (6).

Suppose now that condition (26) takes the form

\[ Y_n \sim N(a_n, 1) \text{ with } a_n = (1 - 2n)/4, \quad n = 1, 2, \ldots. \quad (32) \]

In this case \( \sum_{i=1}^{n} a_i = -n^2/4 \) and we obtain from (27) that for any \( h \geq 0 \) and each \( n = 1, 2, \ldots \)

\[ E e^{hS_n} = e^{-hn^2/4 + nh^2/2} = e^{h^3/4 - h(n-h)^2/4} \leq e^{h^3/4}. \]

Hence, we have from (16) that for any \( u > 0 \) and \( h \geq 0 \)

\[ \psi(u) \leq e^{-hu} \sup_{n \geq 1} E e^{hS_n} \leq e^{-hu + h^3/4}. \quad (33) \]
With $h = 2\sqrt{u/3}$ we obtain from (33) that for any $u > 0$

$$\psi(u) \leq e^{-4(u/3)^{3/2}} = e^{-cu^{3/2}}$$

where $c^2 = 16/27$.

So, we obtain an example of a risk model with property (11) mentioned in the introduction.

**Example 4.** (see [17, Example 2]) Suppose that $Y_1, Y_2, \ldots$ are independent random variables with

$$P[Y_n = 1] = \frac{1}{n+1}, \quad P[Y_n = -1] = 1 - P[Y_n = 1], \quad n = 1, 2, \ldots.$$  

In this case for all $h$

$$\mathbb{E} e^{hY_n} = e^h \frac{1}{n+1} + e^{-h} \left(1 - \frac{1}{n+1}\right) = 1 + \frac{(1-e^{-h})(e^h - n)}{n+1}.$$  

So, $\mathbb{E} e^{hY_n} \leq 1$ if and only if $n \geq e^h$. Hence, for each $h > 0$

$$\sup_{n \geq 1} \mathbb{E} e^{hS_n} = \mathbb{E} e^{hS_m} \iff m + 1 \geq e^h \geq m. \quad (34)$$

Thus, with $m$ from (34)

$$\sup_{n \geq 1} \mathbb{E} e^{hS_n} = \mathbb{E} e^{hS_m} < (e^h)^m \leq e^{he^h},$$

and, using (6), we obtain

$$\forall h, \ u > 0, \ \psi(u) \leq e^{-hu} \sup_{n \geq 1} \mathbb{E} e^{hS_n} \leq e^{-hu+he^h}. \quad (35)$$

With $h = \log(u/2) > 0$ we have from (35) that

$$\forall u > 2, \ \psi(u) \leq \psi_2^*(u) := e^{-(u/2)\log(u/2)} = \left(\frac{2}{u}\right)^{u/2}. \quad (36)$$
So, we obtained another example of a risk model with property (10).

Remind that in [17] the following bound is given

\[ \forall u \geq 0, \psi(u) \leq \psi^*_2(u) := \min \left\{ 1, 178e^{-\frac{u}{20}} \right\}. \]

It is clear that estimate (36) is more accurate than (37). For example,

\[ \psi^*_2(103) < 10^{-88} \text{ in (36), whereas } \psi^*_2(103) = 1 \text{ in (37)}. \]

4 Proofs

4.1 Key Lemma

Before proving our main results, we first introduce the following key lemma.

**Lemma 1.** If random variables \( X_1, X_2, \ldots \) are mutually independent, then for any real number \( w \), any \( h \geq 0 \) and any \( n \geq 1 \)

\[ P[\max_{1 \leq k \leq n} W_k > w] \leq e^{-hw} \max_{1 \leq k \leq n} E e^{hW_k}, \]  

where \( W_k = X_1 + X_2 + \ldots + X_k \). In addition, for any \( w \) and any \( h \geq 0 \)

\[ P[\sup_{k \geq 1} W_k > w] \leq e^{-hw} \sup_{k \geq 1} E e^{hW_k}. \]

**Proof.** If \( M_n(h) := \max_{1 \leq k \leq n} E e^{hW_k} = \infty \) then the inequality (38) is obvious. So, suppose that \( 0 < M_n(h) < \infty \) and note that in this case the following sequence

\[ \mu_k = \frac{e^{hW_k}}{E e^{hW_k}}, \quad k = 0, 1, 2, \ldots, n, \]  

with \( \mu_0 = 1 \),
is a martingale. This fact is evident, since for all $k \geq 0$

$$
\mu_k = \mu_{k-1} \frac{e^{hX_k}}{E[e^{hX_k}]} \quad \text{and} \quad E\left[ \frac{e^{hX_k}}{E[e^{hX_k}]} \middle| \mu_1, \ldots, \mu_{k-1} \right] = 1.
$$

Thus, by maximal inequality for martingale

$$
\forall x > 0, \quad P\left[ \max_{1 \leq k \leq n} \frac{e^{hW_k}}{E[e^{hW_k}]} > x \right] = P\left[ \max_{1 \leq k \leq n} \mu_k > x \right] \leq \frac{E\mu_n}{x} = \frac{1}{x}.
$$

Hence, with $x = e^{hw}/M_n(h)$ we obtain that

$$
P\left[ \max_{1 \leq k \leq n} W_k > w \right] = P\left[ \max_{1 \leq k \leq n} \frac{e^{hW_k}}{M_n(h)} > x = \frac{e^{hw}}{M_n(h)} \right] \leq P\left[ \max_{1 \leq k \leq n} \frac{e^{hW_k}}{E[e^{hW_k}]} > x \right] \leq \frac{1}{x} = e^{-hw} M_n(h).
$$

So, inequality (38) is proved.

Note that $\max_{1 \leq k \leq n} W_n \uparrow \sup_{k \geq 1} W_k$. Hence

$$
P\left[ \sup_{k \geq 1} W_k > w \right] = \lim_{n \to \infty} P\left[ \max_{1 \leq k \leq n} W_k > w \right] \leq e^{-hw} \sup_{n \geq 1} M_n(h) = e^{-hw} \sup_{k \geq 1} E[e^{hW_k}].
$$

Thus, Lemma 1 is proved.

4.2 Proof of Theorem 1

It follows from (i) and (ii) that

$$
\inf_{t \geq 0} R(t) = \inf_{k \geq 1} \inf_{t \in [T_{k-1}, T_k]} R(t) = \inf_{k \geq 1} \min\{R(T_{k-1}), R(T_k)\} = \inf_{k \geq 1} R(T_k).
$$

Next, we have from [3] that

$$
R(T_k) = R(T_0) - \sum_{j=1}^{k} Y_j = u - S_k, \quad k = 0, 1, 2, \ldots,
$$

4.2 Proof of Theorem 1

It follows from (i) and (ii) that

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\inf_{t \geq 0} R(t) = \inf_{k \geq 1} \inf_{t \in [T_{k-1}, T_k]} R(t) = \inf_{k \geq 1} \min\{R(T_{k-1}), R(T_k)\} = \inf_{k \geq 1} R(T_k).
$$

Next, we have from [3] that

$$
R(T_k) = R(T_0) - \sum_{j=1}^{k} Y_j = u - S_k, \quad k = 0, 1, 2, \ldots,
$$
where \( S_0 = 0 \). Thus, \( \inf_{t \geq 0} R(t) = u - \sup_{k \geq 0} S_k \), and hence, by (11) for any \( u > 0 \)

\[
\psi(u) = P[u - \sup_{k \geq 0} S_k < 0] = P[\sup_{k \geq 0} S_k > u]. \tag{41}
\]

Since \( S_0 = 0 \), we have from (41) that for any \( u > 0 \)

\[
\psi(u) = P[\sup_{k \geq 1} S_k > u] = \psi_{\infty}(u). \tag{42}
\]

From (42) and estimate (39) of Lemma 1 we obtain (6).

### 4.3 Proof of Theorem 2

For each \( k = 1, 2, \ldots \), introduce notations:

\[
v_k^* := \prod_{j=1}^{k} \frac{1}{1 + \alpha_j} \quad \text{and} \quad S_k^{**} := \sum_{j=1}^{k} v_j^* Y_j = \sum_{j=1}^{k} v_{j-1}^* Y_j^*, \tag{43}
\]

with \( v_0^* := 1 \).

**Lemma 2.** If \( v_0 = v_0^* = 1 \), \( S_0^{**} = S_0^* = 0 \) and \( \alpha_k \geq r_k \geq 0 \) for all \( k \geq 0 \), then for any \( n \geq 0 \)

\[
\max_{0 \leq k \leq n} S_k^{**} \leq \max_{0 \leq k \leq n} S_k^*. \tag{44}
\]

**Proof.** Since \( \alpha_j \geq r_j \geq 0 \) for \( j = 1, 2, \ldots \), we have that

\[
\forall j \geq 1, \ c_j := \prod_{i=1}^{j} \frac{1 + r_i}{1 + \alpha_i} = \frac{1 + r_j}{1 + \alpha_j} \geq c_{j-1}. \tag{45}
\]

Thus, real numbers \( \{c_j\} \) have the following property

\[
\forall j \geq 1, \ 1 = c_0 \geq c_1 \geq \cdots \geq c_{j-1} \geq c_j > 0. \tag{46}
\]
Next, from (43) and (45) we have for any \( j = 1, 2, \ldots \) that
\[
Y_j^*v_{j-1}^* = Y_j^* \prod_{i=1}^{j-1} \frac{1}{1 + r_i} \prod_{i=1}^{j-1} \frac{1 + r_i}{1 + \alpha_i} = Y_j^*v_{j-1}c_{j-1} = (S_j^* - S_{j-1}^*)c_{j-1}.
\] (47)

Now, substituting (47) into (43) we obtain that for all \( k \leq n \)
\[
S_k^{**} = \sum_{j=1}^{k} v_{j-1}^*Y_j^* = \sum_{j=1}^{k} (S_j^* - S_{j-1}^*)c_{j-1} = c_{k-1}S_k^* + \sum_{j=1}^{k-1} (c_{j-1} - c_j)S_j^* \\
\leq c_{k-1}M_n^* + \sum_{j=1}^{k-1} (c_{j-1} - c_j)M_n^* = c_0M_n^* = M_n^* := \max_{1 \leq j \leq n} S_j^*.
\]

where we also use (46). So, inequality (44) is proved.

**Proof Theorem 2.** Multiplying (13) by \( v_k^* \) we obtain for any \( k \geq 1 \) that
\[
v_k^*R(T_k) \geq v_{k-1}^*R(T_{k-1}) - v_k^*Y_k = v_{k-1}^*R(T_{k-1}) - v_{k-1}^*Y_k.
\]

Hence, by induction for any \( k \geq 1 \)
\[
v_k^*R(T_k) \geq v_0^*R(T_0) - \sum_{j=1}^{k} v_j^*Y_j = u - S_k^{**}.
\]

This fact and Lemma 2 imply for any \( n \geq 0 \) that
\[
\min_{0 \leq k \leq n} v_k^*R(T_k) \geq u - \max_{0 \leq k \leq n} S_k^{**} \geq u - \max_{0 \leq k \leq n} S_k^*.
\] (48)

On the other hand, equality (40) again follows from (i) and (ii) for all \( n \geq 0 \). This fact together with (18) and (48) with \( v_k^* > 0 \) imply that for any \( u > 0 \) and each \( n = 0, 1, 2, \ldots \)
\[
\psi(u, T_n) = P[ \inf_{0 \leq t \leq T_n} R(t) < 0 ] = P[ \min_{0 \leq k \leq n} R(T_k) < 0 ] = P[ \min_{0 \leq k \leq n} v_k^*R(T_k) < 0 ] \\
\leq P[u - \max_{0 \leq k \leq n} S_k^* < 0 ] = P[ \max_{0 \leq k \leq n} S_k^* > u ].
\]

Since \( S_0 = 0 \), for any \( u > 0 \) and each \( n = 0, 1, 2, \ldots \) we have that
\[
\psi(u, T_n) \leq P[ \max_{1 \leq k \leq n} S_k^* > u ].
\]
Now the desired inequality (17) follows from Lemma 1 since \( \{S_k^*\} \) are sums of independent random variables \( \{v_{j-1}Y_j^*\} \).

Taking limit in (17) as \( n \to \infty \) we obtain the inequality (16).

4.4 Proof of Corollary 4

The main idea here is that for all \( n \geq m + l \) random variables \( \Delta_{n,l}^* := S_n^* - S_{n-l}^* \) and \( S_{n-l}^* \) are independent. Hence, for any \( h \in [0, L^*] \) we have from (21) that

\[
E e^{h\Delta_{n,l}^*} \leq \left( E e^{L^*\Delta_{n,l}^*} \right)^{h/L^*} \leq 1 \quad \text{and} \quad E e^{hS_n^*} = E e^{h\Delta_{n,l}^*} \cdot E e^{hS_{n-l}^*} \leq E e^{hS_{n-l}^*}. \tag{49}
\]

Using induction with respect to \( n \) it is not difficult to obtain from (49) that for any \( n \geq m + l \)

\[E e^{hS_n^*} \leq \max_{1 \leq k \leq m+l-1} E e^{hS_k^*}.
\]

Hence

\[
\max_{1 \leq k \leq m+l-1} E e^{hS_k^*} \leq \sup_{n \geq 1} E e^{hS_n^*} \leq \max_{1 \leq k \leq m+l-1} E e^{hS_k^*}.
\]

So, equality (22) is proved. The rest of the assertion of Corollary 4 comes from Theorem 2.

4.5 Proof of Theorem 3

It is clear that each integer \( n \geq 1 \) may be represented in the following way

\[1 \leq n = il + k \quad \text{for some} \quad i \geq 0 \quad \text{and some} \quad 1 \leq k \leq l. \tag{50}\]

**Lemma 3.** Under assumptions of Theorem 3 for all \( i = 0,1,2,\ldots \) and \( k = 1,2,\ldots \), the following random variable

\[
\Delta_{i,k} := S_{i+k}^* - S_{il}^* = \sum_{j=1}^k v_{ik+j-1}Y_{ik+j}^* \tag{51}
\]
is identically distributed with \((q_{vl})^i S^*_k\). In particular, for any \(h \in [0, L(S^*_l)]\)

\[
\sup_{n \geq 1} E e^{h S^*_n} \leq \max_{1 \leq k \leq l} \sup_{i \geq 0} E e^{h(q_{vl})^i S^*_k}, \tag{52}
\]

where \(L(S^*_l)\) is defined in (24).

**Proof.** It follows from condition (23) that

\[
v_{j-1} = \prod_{i=1}^{j-1} \frac{1}{1 + r_i} = \prod_{i=1}^{l} \frac{1}{1 + r_i} \cdot \prod_{i=l+1}^{j-1} \frac{1}{1 + r_i} = v_l \cdot \prod_{i=1}^{j-l-1} \frac{1}{1 + r_i} = v_l v_{j-l-1}.
\]

Hence, for all \(n = 1, 2, \ldots\), random variables \(v_{n+l-1} Y^*_{n+l}\) and \((q_{vl}) v_{n-1} Y^*_n\) are identically distributed because \(Y^*_n\) and \(q_{vl} Y^*_n\) are identically distributed by the assumption of Theorem 3.

Using induction with respect to \(i\) it is not difficult to obtain that \(v_{i+k+l} Y^*_{i+k+l}\) and \((q_{vl}) v_{i-1} Y^*_i\) are identically distributed for all \(i = 0, 1, 2, \ldots\) and \(j = 1, 2, \ldots\). As a result, sums over \(j\) from 1 to \(l\) of these independent random variables are identically distributed. So, using definition (51), we have the first assertion of the lemma that \(\Delta_{i,k}\) and \((q_{vl})^{i} \Delta_{0,k} = (q_{vl})^{i} S^*_k\) are identically distributed.

Now note that \(S^*_l = \sum_{m=0}^{i-1} \Delta_{m,l} + \Delta_{i,k}\). Hence

\[
E e^{h S^*_l} = \prod_{m=0}^{i-1} E e^{h \Delta_{m,l}} \cdot E e^{h \Delta_{i,k}} = \prod_{m=0}^{i-1} E e^{h(q_{vl})^m S^*_l} \cdot E e^{h(q_{vl})^{i} S^*_k}. \tag{53}
\]

Since \(0 \leq h(q_{vl})^m \leq h \leq L(S^*_l) < \infty\), for any \(h \in [0, L(S^*_l)]\) we have that

\[
E e^{h(q_{vl})^m S^*_l} \leq (E e^{L(S^*_l) S^*_l} h(q_{vl})^m / L(S^*_l)) \leq 1
\]

by definition of \(L(S^*_l)\). From this inequality and (53) we obtain

\[
\forall h \in [0, L(S^*_l)], \forall i \geq 0, \forall k \geq 1, \ E e^{h S^*_l} \leq E e^{h(q_{vl})^{i} S^*_k}.
\]

The latter yields (52) if only we remind that each integer \(n \geq 1\) may be represented in the form (50).
Thus the lemma is proved.

**Proof of Theorem 3.** If \( qlv_1 = 1 \) then for any \( h \in [0, L(S_l^*)] \) we have from (52) that

\[
\sup_{k \geq 1} E e^{hS_k^*} \leq \max_{1 \leq k \leq l} E e^{hS_k^*}.
\]

So, the second assertion of Theorem 3 is proved.

To prove the first one note that for \( (qlv_1)^i \leq 1 \) and \( 1 \leq k \leq l \) we have

\[
E e^{h(qlv_1)^iS_k^*} \leq (E e^{hS_k^*})^{(qlv_1)^i} \leq \max_{0 \leq k \leq l} E e^{hS_k^*} = \max_{0 \leq k < l} E e^{hS_k^*},
\]

because \( S_0^* = 0 \) and \( E e^{hS_0^*} = 1 \). Inequality (54) together with (52) implies (25) under assumptions of Theorem 3.

Remind that Corollary 1 immediately follows from Theorem 1. Corollary 2 is a partial case of Theorem 3, whereas Corollary 3 simply follow from Theorem 2. All examples and lemmas are proved after their statements. Thus, all results of the paper are proved.

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