DUALITY AND TILTING FOR COMMUTATIVE DG RINGS

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Abstract. We consider commutative DG rings (with suitable finiteness conditions). For such a DG ring \( A \) we define the notions of perfect, tilting, dualizing and Cohen-Macaulay DG \( A \)-modules, generalizing the usual definitions for complexes over commutative noetherian rings. We investigate how these various kinds of DG modules interact with each other, and with DG ring homomorphisms. Notably, we prove that the canonical DG ring homomorphism \( A \to H^0(A) \) induces a bijection on the sets of isomorphism classes of tilting (resp. dualizing) DG modules. The functorial properties of Cohen-Macaulay DG modules established here are needed for our work on rigid dualizing complexes.

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0. Introduction

In this paper we consider \textit{commutative DG rings}. This is an abbreviation for strongly commutative nonpositive associative unital DG algebras over \( \mathbb{Z} \). Thus a commutative DG ring is a graded ring \( A = \bigoplus_{i \leq 0} A^i \), together with a differential \( d \) of degree 1, that satisfies the graded Leibniz rule. The multiplication satisfies \( b \cdot a = (-1)^{ij} \cdot a \cdot b \) for all \( a \in A^i \) and \( b \in A^j \), and \( a \cdot a = 0 \) if \( i \) is odd. By default all DG rings in the Introduction are assumed to be commutative. Rings are viewed as DG rings concentrated in degree 0 (and thus are assumed by default to be commutative).

Commutative DG rings come up in the foundations of \textit{derived algebraic geometry}, as developed by Toën-Vezzosi [TV] and Lurie [Lu1]; see also the expository article...
Ve]. Indeed, one incarnation of a derived stack is as a stack of groupoids on the site of commutative DG rings (with its étale topology). Some precursors of this point of view are the papers [Hi2], [Ke2], [KoSo] and [Be].

Another role of commutative DG rings is as resolutions of commutative rings. We wish to mention a particular instance, since it is closely related to the present paper. Rigid dualizing complexes over noetherian commutative rings are the foundation of a new approach to Grothendieck Duality on schemes and Deligne-Mumford stacks. See the papers [VeB], [YZ1], [YZ2], [Ye3], [Ye8], [Ye9]. The definition of rigid complex relies on the more primitive notion of square of a complex. Given a ring homomorphism $A \to B$, we choose a flat DG ring resolution $\tilde{B} \to B$ over $A$ (if $B$ is flat over $A$ we can take $\tilde{B} = B$). For any complex of $B$-modules $M$, its square is

$$\text{Sq}_{B/A}(M) := \text{RHom}_{\tilde{B}\otimes_A \tilde{B}}(B, M \otimes^L_A M).$$

The question of independence of $\text{Sq}_{B/A}(M)$ of the choice of resolution $\tilde{B}$ is very subtle; and in fact there was a mistake in the original proof in [YZ1] (which has been since corrected in [AILN] and [Ye5]).

The purpose of the present paper is to study perfect, tilting, dualizing and Cohen-Macaulay DG modules over commutative DG rings, trying to go as far as possible along the established theory for commutative rings. Our motivation comes from a concrete problem. In the course of writing the new paper [Ye8] – which corrects the mistakes in the earlier paper [YZ1], and extends it – we realized that we need Theorem 0.10 (dealing with Cohen-Macaulay DG modules). This is explained in Remark 8.9. The parts of the present paper leading to Section 8 set the stage for the definition of Cohen-Macaulay DG modules and the proof of Theorem 0.10.

Presumably the results in this paper shall find further applications, seeing that the interest in DG rings is on the rise, especially in connection with derived algebraic geometry. (Indeed, after writing an early version of this paper, we found that there is some overlap between our work on dualizing DG modules and Lurie’s [Lu2]; see Remark 7.26.)

Our work should be easily accessible to anyone with a working knowledge of the derived category of modules over a ring (e.g. from the book [Ve]). This is because the methods we use are basically the same; there are only slight modifications. The necessary tools to “upgrade” from rings to DG rings (such as K-injective resolutions in place of injective resolutions) are recalled in Section 1 of our paper. We do not resort at all the daunting technicalities of $E_\infty$ rings. Likewise, we do not touch simplicial methods or Quillen model structures.

Let us begin to describe our work in this paper. Consider a commutative DG ring $A = \bigoplus_{i \leq 0} A^i$. Its cohomology $H(A) = \bigoplus_{i \leq 0} H^i(A)$ is a commutative graded ring. We use the notation $\hat{A} := H^0(A)$. There is a canonical homomorphism of DG rings $A \to \hat{A}$. We say that $\hat{A}$ is cohomologically pseudo-noetherian if $\hat{A}$ is a noetherian ring, and $H^i(A)$ is a finite (i.e. finitely generated) $\hat{A}$-module for every $i$. (Note that this is weaker than the condition that $H(A)$ is noetherian.)

The category of DG $A$-modules is denoted by $\mathcal{M}(A)$. It is a DG category, and its derived category, gotten by inverting the quasi-isomorphisms, is denoted by $\mathcal{D}(A)$. There are full triangulated subcategories $\mathcal{D}^+(A), \mathcal{D}^-(A)$ and $\mathcal{D}^b(A)$ of $\mathcal{D}(A)$, made up of the DG modules $M$ with bounded below, bounded above and bounded cohomologies, respectively. The full subcategory of $\mathcal{D}(A)$ on the DG modules $M$,
whose cohomology modules \( H^i(M) \) are finite over \( \bar{A} \), is denoted by \( D_\ell(A) \). As usual, for any boundedness condition \( \ast \) we let \( D_\ell^\ast(A) := D_\ell(A) \cap D^\ast(A) \). If \( A \) is cohomologically pseudo-noetherian, then the categories \( D_\ell^\ast(A) \) are triangulated, and \( A \in D_\ell^\ast(A) \). In case \( A \) is a ring, then \( D(A) = D(\text{Mod} \ A) \), the derived category of \( A \)-modules.

In Section 1 we recall some facts on DG modules. We mention several kinds of resolutions of DG modules, and special attention is paid to semi-free resolutions. Section 2 is about various notions of cohomological dimension for DG modules and derived functors. In Section 3 we study the reduction functor \( D(A) \to D(\bar{A}) \), \( M \mapsto A \otimes_{\bar{A}} M \). We show that projective \( A \)-modules can be lifted to DG \( A \)-modules.

In Section 4 we discuss localization of a commutative DG ring \( A \) on \( \text{Spec} \ \bar{A} \). We introduce the Čech resolution \( C(M; a) \) of a DG \( A \)-module \( M \), associated to a covering sequence \( a = (a_1, \ldots, a_n) \) of \( \bar{A} \). In case there is a decomposition \( \text{Spec} \ \bar{A} = \prod_{i=1}^n \text{Spec} \ A_i \) into open-closed subsets, we show that there are canonically defined DG rings \( A_1, \ldots, A_n \) and a DG ring quasi-isomorphism \( A \to \prod_{i=1}^n A_i \), that in \( H^0 \) recovers the decomposition \( \bar{A} \cong \prod_{i=1}^n A_i \).

The topic of Section 5 is perfect DG modules. A DG \( A \)-module \( P \) is perfect if locally on \( \text{Spec} \ \bar{A} \) it is isomorphic, in the derived category, to a finite semi-free DG module. See Definition 5.4 for the precise formulation. Here is the first main result of this section (it is Theorem 5.11 there).

**Theorem 0.2.** Let \( A \) be a commutative DG ring, and let \( P \) be a DG \( A \)-module. The following two conditions are equivalent:

(i) \( P \) is perfect.

(ii) \( P \) is in \( D^\ast(\bar{A}) \), and the DG \( \bar{A} \)-module \( \bar{A} \otimes_{\bar{A}}^L P \) is perfect.

If \( A \) is cohomologically pseudo-noetherian, then these two conditions are equivalent to:

(iii) \( P \) is in \( D^\ast_{\ell}(\bar{A}) \), and it has finite projective dimension relative to \( D(A) \).

See Definition 2.4(1) regarding the projective dimension of a DG module.

Another result on perfect DG modules is the next theorem (repeated as Theorem 5.20). Recall that a DG module \( P \) is said to be a compact object of \( D(A) \) if the functor \( \text{Hom}_{D(A)}(P, -) \) commutes with infinite direct sums.

**Theorem 0.3.** Let \( A \) be a commutative DG ring, and let \( P \) be a DG \( A \)-module. The following three conditions are equivalent:

(i) \( P \) is perfect.

(ii) \( P \) is a compact object of \( D(A) \).

(iii) For any \( M, N \in D(A) \), the canonical morphism

\[
\text{RHom}_A(P, M) \otimes_{\bar{A}}^L N \to \text{RHom}_A(P, M \otimes_{\bar{A}}^L N)
\]

in \( D(A) \) is an isomorphism.

When \( A \) is a ring, the equivalence of conditions (i) and (ii) goes back to Rickard [Ri] and Neeman [Ne].

Section 6 is about tilting DG modules. A DG \( A \)-module \( P \) is said to be tilting if there is some DG module \( Q \) such that \( P \otimes_{\bar{A}}^L Q \cong A \) in \( D(A) \). The DG module \( Q \) is called a quasi-inverse of \( P \). The next theorem is repeated as Theorem 6.7.

**Theorem 0.4.** Let \( A \) be a commutative DG ring, and let \( P \) be a DG \( A \)-module. The following three conditions are equivalent:
(i) \( P \) is a tilting DG module.

(ii) The functor \( P \otimes^L_A − \) is an equivalence of \( \mathcal{D}(A) \).

(iii) \( P \) is a perfect DG module, and the adjunction morphism \( A \to \text{RHom}_A(P,P) \) in \( \mathcal{D}(A) \) is an isomorphism.

A combination of Theorems 0.4 and 0.3 yields the next result (repeated as Corollary 6.8).

Corollary 0.5. Let \( P \) be a tilting DG \( A \)-module. Then the DG module \( Q := \text{RHom}_A(P,A) \) is the quasi-inverse of \( P \).

As in [Ye2], we define the commutative derived Picard group \( \text{DPic}(A) \) to be the group whose elements are the isomorphism classes of tilting DG \( A \)-modules, and the multiplication is induced by \( − \otimes^L_A − \).

If \( A \to B \) is a homomorphism of DG rings, then the operation \( P \mapsto B \otimes^L_A P \) induces a group homomorphism \( \text{DPic}(A) \to \text{DPic}(B) \). The next result is Theorem 6.12 in the body of the paper.

Theorem 0.6. Let \( A \) be a commutative DG ring, such that \( \text{Spec} \overline{A} \) has finitely many connected components. Then the canonical group homomorphism

\[ \text{DPic}(A) \to \text{DPic}(\overline{A}) \]

is bijective.

The hypothesis of the theorem (namely that \( \text{Spec} \overline{A} \) has finitely many connected components) is often satisfied, e.g. when \( \overline{A} \) is noetherian, or when \( \overline{A} \) is the ring of continuous (resp. differentiable) functions \( X \to \mathbb{R} \), where \( X \) is a connected topological space (resp. a connected differentiable manifold).

It is known that the commutative derived Picard group of the ring \( \overline{A} \) has this structure:

\[ \text{DPic}(\overline{A}) \cong \mathbb{Z}^n \times \text{Pic}(\overline{A}) \]

where \( n \) is the number of connected components of \( \text{Spec} \overline{A} \), and \( \text{Pic}(\overline{A}) \) is the usual (commutative) Picard group. See [Ye2], [RZ] and [Ye4].

In Section 7 we talk about dualizing DG modules. Here \( A \) is a cohomologically pseudo-noetherian commutative DG ring. A DG \( A \)-module \( R \in \mathcal{D}^+(A) \) is called dualizing if it has finite injective dimension relative to \( \mathcal{D}(A) \), and the adjunction morphism \( A \to \text{RHom}_A(R,R) \) is an isomorphism. Note that when \( A \) is a ring, this is precisely the original definition found in [RD]; but for a DG ring there are several possible notions of injective dimension, and the correct one has to be used. See Definition 2.4(2) and Remark 2.6. Note also that \( R \) need not have bounded cohomology – see Corollary 7.3 and Example 7.25. For comparisons to dualizing DG modules, as defined previously in [Hi1] and [FIJ], see Example 7.22 and Proposition 7.16 respectively.

A cohomologically pseudo-noetherian commutative DG ring \( A \) is called tractable if there is a homomorphism \( \mathbb{K} \to A \) from a finite dimensional regular noetherian commutative ring \( \mathbb{K} \), such that the induced homomorphism \( \mathbb{K} \to A \) is essentially finite type.

The next result is a combination of Theorems 7.8 and 7.9. When \( A \) is a ring, this was proved by Grothendieck [RD] Sections V.3 and V.10. There is an analogous result in Lurie’s [La2], for \( E_\infty \) rings; see Remark 7.26 for a discussion.
Theorem 0.7. Let $A$ be a tractable cohomologically pseudo-noetherian commutative DG ring.

1. $A$ has a dualizing DG module.
2. The operation $(P, R) \mapsto P \otimes_R^L R$, for a tilting DG module $P$ and a dualizing DG module $R$, induces a simply transitive action of the group $\text{DPic}(A)$ on the set of isomorphism classes of dualizing $A$-modules.

In particular, if $\overline{A}$ is a local ring, then by Theorem 0.6 we have $\text{DPic}(A) \cong \mathbb{Z}$. Thus any two dualizing DG $A$-modules $R, R'$ satisfy $R' \cong R[m]$ for an integer $m$.

A combination of Theorems 0.6 and 0.7 yields (see Corollary 7.11):

Corollary 0.8. Let $A$ be a tractable cohomologically pseudo-noetherian commutative DG ring. Then the operation $R \mapsto \text{RHom}_A(\overline{A}, R)$ induces a bijection

$$\frac{\{\text{dualizing DG } A\text{-modules}\}}{\text{isomorphism}} \cong \frac{\{\text{dualizing DG } \overline{A}\text{-modules}\}}{\text{isomorphism}}.$$

Here is a result that is quite surprising. It relies on a theorem of Jørgensen [Jo], who proved it in the local case (i.e. when $\overline{A}$ is a local ring).

Theorem 0.9. Let $A$ be a tractable cohomologically pseudo-noetherian commutative DG ring. If $\overline{A}$ is a perfect DG $A$-module, and if $A$ has bounded cohomology, then the canonical homomorphism $A \to \overline{A}$ is a quasi-isomorphism.

This is repeated as Theorem 7.20 in the body of the paper. See Remark 7.24 for an interpretation of this theorem.

The final section of the paper is about Cohen-Macaulay DG Modules. The definition does not involve regular sequences of course; nor does it involve vanishing of local cohomologies as in [RD] (even though it could probably be stated in this language). Instead we use a fact discovered in [YZ3]: for a noetherian scheme $X$ with dualizing complex $R$, a complex $M \in D^b_*(\text{Mod} \mathcal{O}_X)$ is CM (in the sense of [RD], for the dimension function determined by $R$) iff $R\text{Hom}_X(M, R)$ is (isomorphic to) a coherent sheaf. In [YZ3] the CM complexes inside $D^b_*(\text{Mod} \mathcal{O}_X)$ were also called 

perverse coherent sheaves

With the explanation above, the next definition makes sense. Let $R$ be a dualizing DG $A$-module. A DG module $M \in D^0_0(A)$ is called $CM$ w.r.t. $R$ if $R\text{Hom}_A(M, R) \in D^0_0(A)$. Here $D^0_0(A)$ is the full subcategory of $D(A)$ consisting of DG modules with finite cohomology concentrated in degree 0; and we know that it is equivalent to the category $\text{Mod}_f \overline{A}$ of finite $\overline{A}$-modules.

The next theorem is repeated, in slightly greater generality, as Theorem 8.7.

Theorem 0.10. Let $f : A \to B$ be a homomorphism between tractable cohomologically pseudo-noetherian commutative DG rings, such that $H^0(f) : A \to B$ is surjective. Let $R_B$ be a dualizing DG $B$-module, and let $M, N \in D^0_0(B)$.

1. If $M$ is $CM$ w.r.t. $R_B$, and there is an isomorphism $\text{rest}_f(M) \cong \text{rest}_f(N)$ in $D(A)$, then $N$ is also $CM$ w.r.t. $R_B$.
2. If $M$ and $N$ are both $CM$ w.r.t. $R_B$, then the homomorphism

$$\text{rest}_f : \text{Hom}_{D(B)}(M, N) \to \text{Hom}_{D(A)}(\text{rest}_f(M), \text{rest}_f(N))$$

is bijective.
In the theorem, \( \text{rest}_f : \mathcal{D}(B) \to \mathcal{D}(A) \) is the restriction functor. As already mentioned, Theorem 0.10 is needed in [Ye5].

**Acknowledgments.** I wish to thank Peter Jørgensen, Bernhard Keller, Vladimir Hinich, Liran Shaul, James Zhang, Srikanth Iyengar, Amnon Neeman, Dennis Gaitsgory, Rishi Vyas, Matan Prasma, Jacob Lurie, Benjamin Antieau, John Palmieri and Pieter Belmans for helpful discussions.

1. **DG Modules and their Resolutions**

A DG ring (often called an associative unital DG algebra over \( \mathbb{Z} \)) is a graded ring \( A = \bigoplus_{i \in \mathbb{Z}} A^i \), with differential \( d \) of degree 1, satisfying the graded Leibniz rule

\[
d(a \cdot b) = d(a) \cdot b + (-1)^i \cdot a \cdot d(b)
\]

for \( a \in A^i \) and \( b \in A^j \). A homomorphism of DG rings is a degree 0 ring homomorphism that commutes with the differentials. Rings are viewed as DG rings concentrated in degree 0. For a DG ring \( A \), the cohomology \( H(A) = \bigoplus_{i \in \mathbb{Z}} H^i(A) \) is a graded ring.

The *opposite* of the DG ring \( A \) is the DG ring \( A^{\text{op}} \), that is the same graded abelian group as \( A \), with the same differential, but the multiplication is reversed in the graded sense, as follows. For elements \( a \in A^i \) and \( b \in A^j \), let us denote by \( a^{\text{op}} \) and \( b^{\text{op}} \) the same elements, but viewed in \( A^{\text{op}} \). Then

\[
a^{\text{op}} \cdot b^{\text{op}} := (-1)^{ij} \cdot (b \cdot a)^{\text{op}} \in (A^{\text{op}})^{i+j}.
\]

A left DG \( A \)-module is a graded left \( A \)-module \( M = \bigoplus_{i \in \mathbb{Z}} M^i \), with differential \( d \) satisfying the graded Leibniz rule. If \( M \) is a left DG \( A \)-module then \( H(M) = \bigoplus_{i \in \mathbb{Z}} H^i(M) \) is a left graded \( H(A) \)-module. Note that a right DG \( A \)-module is the same as a left DG \( A^{\text{op}} \)-module.

**Convention 1.1.** By default, in this paper all DG modules are left DG modules.

From Section 4 onwards our DG rings will be commutative, and for them the distinction between left and right DG modules becomes negligible.

**Definition 1.2.** Let \( A \) be a DG ring. The category of (left) DG \( A \)-modules, with \( A \)-linear homomorphisms of degree 0 that commute with differentials, is denoted by \( \text{DGMod}_A \), or by its abbreviation \( M(A) \). The derived category, gotten from \( \text{DGMod}_A \) by inverting quasi-isomorphisms, is denoted by \( \tilde{\mathcal{D}}(\text{DGMod}_A) \), or by the abbreviation \( \mathcal{D}(A) \).

For information on \( \mathcal{D}(A) \) see [BL, Section 10], [Ke1, Section 2] or [SP, Section 22.15, tag 09KV].

**Definition 1.3.** (1) A DG ring \( A = \bigoplus_{i \in \mathbb{Z}} A^i \) is called *nonpositive* if \( A^i = 0 \) for all \( i > 0 \).

(2) For a nonpositive DG ring \( A \) we write \( \tilde{A} := H^0(A) \), which is a ring. There is a canonical DG ring homomorphism \( A \to \tilde{A} \).

One of the important advantages of nonpositive DG rings is that the differential \( d \) of any DG \( A \)-module \( M \) is \( A^0 \)-linear. This implies that the two smart truncations

\[
\text{smt}^{>i}(M) := (\cdots \to 0 \to \text{Coker}(d|_{M^{i-1}}) \to M^{i+1} \to M^{i+2} \to \cdots)
\]

and

\[
\text{smt}^{\leq i}(M) := (\cdots \to M^{i-2} \to M^{i-1} \to \text{Ker}(d|_{M^{i-1}}) \to 0 \to \cdots)
\]
remain within $M(A)$; and there are functorial homomorphisms $M \to \text{smt}^l_i(M)$ and $\text{smt}^r_i(M) \to M$ in $M(A)$, inducing isomorphisms in $H^l_i$ and $H^r_i$ respectively. Note that these are the truncations $\tau_{\leq n}$ and $\tau_{\geq n}$ from [SP Section 12.13, tag 0118], that are variants of the truncations $\sigma_{>n}$ and $\sigma_{\leq n}$ from [RD Section I.7, page 69]. Warning: the two stupid truncations might fail to work in this context.

**Convention 1.6.** By default, in this paper all DG rings are nonpositive.

Recall that for a subset $S \subset \mathbb{Z}$, its infimum is $\inf(S) = \mathbb{Z} \cup \{\pm \infty\}$, where $\inf(S) = +\infty$ iff $S = \emptyset$. Likewise the supremum is $\sup(S) = \mathbb{Z} \cup \{\pm \infty\}$, where $\sup(S) = -\infty$ iff $S = \emptyset$. For $r, s \in \mathbb{Z} \cup \{\pm \infty\}$ and $t \in \mathbb{Z}$ the expressions $\pm r \pm t \in \mathbb{Z} \cup \{\pm \infty\}$ have obvious meanings, as do $r + s$ and $r \leq s$.

Let $M = \bigoplus_{d \in \mathbb{Z}} M^d$ be a graded abelian group. We define

$$\inf(M) := \inf \{i \mid M^i \neq 0\}, \quad \sup(M) := \sup \{i \mid M^i \neq 0\}.$$  

The amplitude of $M$ is

$$\text{amp}(M) := \sup(M) - \inf(M) \in \mathbb{N} \cup \{\pm \infty\}.$$  

(For $M = 0$ this reads $\inf(M) = \infty$, $\sup(M) = -\infty$ and $\text{amp}(M) = -\infty$.) Thus $M$ is bounded (resp. bounded above, resp. bounded below) iff $\text{amp}(M) < \infty$ (resp. $\sup(M) < \infty$, resp. $\inf(M) > -\infty$).

For $d_0 \leq d_1$ in $\mathbb{Z} \cup \{\pm \infty\}$ we write

$$[d_0, d_1] := \{d \in \mathbb{Z} \cup \{\pm \infty\} \mid d_0 \leq d \leq d_1\}.$$  

This is referred to as an interval in $\mathbb{Z} \cup \{\pm \infty\}$. Given an interval $[d_0, d_1] \subset \mathbb{Z} \cup \{\pm \infty\}$, we denote by $D^{[d_0, d_1]}(A)$ the full subcategory of $D(A)$ consisting of DG modules $M$ whose cohomologies $H(M)$ are concentrated in this interval; namely $d_0 \leq \inf(H(M))$ and $\sup(H(M)) \leq d_1$. For $d \in \mathbb{Z}$ we write $D^d(A) := D^{[d, d]}(A)$. The subcategory $D^{[d_0, d_1]}(A)$ is additive, but not triangulated. Similarly we have the subcategory $M^{[d_0, d_1]}(A) \subset M(A)$, consisting of DG modules $M$ that are concentrated in the degree interval $[d_0, d_1]$; namely $d_0 \leq \inf(M)$ and $\sup(M) \leq d_1$.

**Definition 1.10.** Let $A$ be a DG ring.

1. A DG $A$-module $M$ is said to be **cohomologically bounded** (resp. **cohomologically bounded above**, resp. **cohomologically bounded below**) if the graded module $H(M)$ is bounded (resp. bounded above, resp. bounded below). We denote by $D^b(A)$, $D^+(A)$ and $D^-(A)$ the full subcategories of $D(A)$ consisting of DG modules $M$ that are cohomologically bounded, cohomologically bounded above and cohomologically bounded below, respectively.

2. The full subcategory of $D(A)$ consisting of the DG modules $M$, whose cohomology modules $H^i(M)$ are finite over $A$, is denoted by $D^f(A)$.

3. For any boundedness condition $\star$ we write $D^\star_f(A) := D^f(A) \cap D^\star(A)$.

The categories $D^\star_f(A)$ are triangulated. If $A$ is left noetherian, then the categories $D^\star_f(A)$ are also triangulated.

Given a DG $A$-module $M$, its shift by an integer $i$ is the DG module $M[i]$, whose $j$-th graded component is $M[i]^j := M^{i+j}$. Elements of $M[i]^j$ are denoted by $m[i]^j$, with $m \in M^{i+j}$. The differential of $M[i]$ is this:

$$d_{M[i]}(m[i]) := (-1)^i \cdot d_M(m[i]).$$
The left action of $A$ on $M[i]$ is also twisted by $±1$, as follows:

$$a \cdot m[i] := (-1)^{ki} \cdot (a \cdot m)[i]$$

for $a \in A^k$. The right action remains untwisted:

$$m[i] \cdot a := (m \cdot a)[i].$$

See [Ye5] Section 1 for a detailed study of the shift operation, including an explanation of the unexpected sign that appears in the left action.

We now recall some resolutions of DG $A$-modules. A DG module $N$ is called acyclic if $H(N) = 0$. A DG $A$-module $M$ is called $K$-projective (resp. $K$-injective), if for any acyclic DG $A$-module $N$, the DG $\mathbb{Z}$-module $Hom_A(M, N)$ (resp. $Hom_A(N, M)$) is also acyclic. The DG $A$-module $M$ is called $K$-flat if for any acyclic DG $A^{op}$-module $N$, the DG $\mathbb{Z}$-module $N \otimes_A M$ is acyclic. It is easy to see that $K$-projective implies $K$-flat.

For a cardinal number $r$ (possibly infinite) we denote by $M^\oplus r$ the direct sum of $r$ copies of $M$. Recall that a DG $A$-module $P$ is a free DG module if $P \cong \bigoplus_{i \in \mathbb{Z}} A[-i]^\oplus r_i$, where $r_i$ are cardinal numbers. We say that $P$ is a finite free DG module if $\sum_i r_i < \infty$ (for some such isomorphism).

**Definition 1.11.** Let $P$ be a DG $A$-module. A semi-free filtration of $P$ is an ascending filtration $\{\nu_j(P)\}_{j \in \mathbb{Z}}$ by DG $A$-submodules $\nu_j(P) \subset P$, such that $\nu_{-1}(P) = 0$, $P = \bigcup_j \nu_j(P)$, and each $\text{gr}_j^P := \nu_j(P)/\nu_{j-1}(P)$ is a free DG $A$-module.

**Definition 1.12.** Let $P$ be a DG $A$-module, with semi-free filtration $\{\nu_j(P)\}_{j \in \mathbb{Z}}$.

1. The filtration $\{\nu_j(P)\}_{j \in \mathbb{Z}}$ is said to have length $d$ if

   $$d = \inf \{ j \in \mathbb{N} | \nu_j(P) = P \} \in \mathbb{N} \cup \{\infty\}.$$

2. The filtration $\{\nu_j(P)\}_{j \in \mathbb{Z}}$ is called pseudo-finite if each free DG $A$-module $\text{gr}_j^P$ is finite, and

   $$\lim_{j \to \infty} \sup(\text{gr}_j^P) = -\infty.$$

3. The filtration $\{\nu_j(P)\}_{j \in \mathbb{Z}}$ on $P$ is called finite if it is pseudo-finite and has finite length.

**Definition 1.13.** A DG $A$-module $P$ is called a semi-free (resp. pseudo-finite semi-free, resp. finite semi-free) DG module if it admits a semi-free (resp. pseudo-finite semi-free, resp. finite semi-free) filtration. The semi-free length of $P$ is defined to be the minimal length of any of its semi-free filtrations.

It is important to note that a semi-free DG module need not be bounded above. If $P$ is a semi-free DG module then it is $K$-projective.

The next proposition gives another characterization of pseudo-finite semi-free DG modules. The graded ring gotten from $A$ by forgetting the differential is denoted by $A^\natural$. Likewise for DG modules. Recall that our DG rings are always nonpositive.

**Proposition 1.14.** Let $P$ be a DG $A$-module.

1. $P$ is pseudo-finite semi-free iff there are $i_1 \in \mathbb{Z}$ and $r_i \in \mathbb{N}$, such that

   $$P^\natural \cong \bigoplus_{i \leq i_1} A^\natural[-i]^\oplus r_i$$

   as graded $A^\natural$-modules.
(2) $P$ is finite semi-free if and only if there is an isomorphism of graded $A^\bullet$-modules as in item (1) above, and $i_0 \in \mathbb{Z}$, such that $r_i = 0$ for all $i < i_0$.

Proof. Given an isomorphism $P^\bullet \cong \bigoplus_{i \leq i_1} A^\bullet[-i]^\oplus_{r_i}$, define

$$\nu_j(P) := \bigoplus_{i_1 - j \leq i \leq i_1} A^\bullet[-i]^\oplus_{r_i} \subset P.$$ 

This is a pseudo-finite semi-free filtration, of length $\leq i_1 - i_0$ in the finite case. The converse is clear, and so is item (2). \qed

Remark 1.15. Suppose $A$ is a ring. A DG $A$-module $P$ is pseudo-finite semi-free if it is a bounded above complex of finite free $A$-modules. Now according to [SGA 6] or [SP, Definition 15.52.1, tag 064Q], a DG $A$-module $M$ is called pseudo-coherent if it is quasi-isomorphic to some pseudo-finite semi-free DG module $P$. This explains the name “pseudo-finite”.

Definition 1.16. A DG $A$-module $P$ is said to be generated in the degree interval $[d_0, d_1] \subset \mathbb{Z} \cup \{\pm \infty\}$ if for any interval $[e_0, e_1]$ and any $N \in M[e_0, e_1](A^{op})$, we have

$$N \otimes_A P \in M[e_0+d_0, e_1+d_1](\mathbb{Z}).$$

Proposition 1.17. Let $P$ be a semi-free DG $A$-module, and let $[d_0, d_1]$ be an interval. Assume that $\bar{A} \neq 0$. The following two conditions are equivalent.

(i) $P$ is generated in the degree interval $[d_0, d_1]$.

(ii) There is an isomorphism of graded $A^\bullet$-modules

$$P^\bullet \cong \bigoplus_{d_0 \leq i \leq d_1} A^\bullet[-i]^\oplus_{r_i}$$

for some cardinal numbers $r_i$.

Proof. (i) $\Rightarrow$ (ii): Take a semi-free filtration $\{\nu_j(P)\}_{j \in \mathbb{Z}}$ of $P$. By splitting the surjections $\nu_j(P)^{\bullet} \twoheadrightarrow \text{gr}^j(P)^{\bullet}$ in the category of graded $A^\bullet$-modules, we see that there is such an isomorphism, with $i$ running over all $\mathbb{Z}$. But $\bar{A} \in M[0,0](A^{op})$, and hence the graded $\mathbb{Z}$-module

$$\bar{A} \otimes_A P \cong \bigoplus_{i} \bar{A}[-i]^\oplus_{r_i}$$

has to be concentrated in the range $[d_0, d_1]$. Therefore $r_i = 0$ outside this range.

(ii) $\Rightarrow$ (i): For any $N \in M(A^{op})$ we have

$$(N \otimes_A P)^{\bullet} \cong N^\bullet \otimes_A P^\bullet \cong \bigoplus_{d_0 \leq i \leq d_1} N^\bullet[-i]^\oplus_{r_i}. \qed$$

Definition 1.18. Let $A$ be a nonpositive DG ring.

(1) $A$ is called cohomologically left noetherian if the graded ring $H(A)$ is left noetherian.

(2) $A$ is called cohomologically left pseudo-noetherian if it satisfies these two conditions:

(i) The ring $\bar{A} = H^0(A)$ is left noetherian.

(ii) For every $i$ the left $\bar{A}$-module $H^i(A)$ is finite (i.e. finitely generated).
We shall be mostly interested in cohomologically left pseudo-noetherian DG rings. Of course if $A$ is cohomologically left noetherian, then it is cohomologically left pseudo-noetherian. These conditions are equivalent when $A$ is cohomologically bounded.

As already mentioned, starting from Section 4, our DG rings will be commutative. There the adjective “left” will be dropped from the noetherian conditions on DG rings.

As usual, by semi-free resolution of a DG module $M$ we mean a quasi-isomorphism $P \to M$ in $\mathcal{M}(A)$, where $P$ is semi-free. Likewise we talk about $K$-injective resolutions $M \to I$.

**Proposition 1.19.** Let $M$ be a DG $A$-module.

1. There is a semi-free resolution $P \to M$ such that $\text{sup}(P) = \text{sup}(H(M))$.
2. If the DG ring $A$ is cohomologically left pseudo-noetherian, and if $M \in D^-(A)$, then there is a pseudo-finite semi-free resolution $P \to M$ such that $\text{sup}(P) = \text{sup}(H(M))$.
3. There is a $K$-injective resolution $M \to I$ such that $\text{inf}(I) = \text{inf}(H(M))$.

**Proof.**

1. See [Ke1, Theorem 3.1] or [SP, Section 12.13, tag 09KK], noting that $\text{sup}(A) = 0$ (if $A$ is nonzero).

2. The ring $\bar{A}$ is left noetherian, the graded $\bar{A}$-module $H(M)$ is bounded above, and each $H^i(M)$ is a finite left $\bar{A}$-module. So there is a resolution
\[
\cdots \to Q^{-1} \to Q^0 \to H(M) \to 0
\]
in the category of graded $H(A)$-modules, where each $Q^{-j}$ is a graded free $H(A)$-module, with finitely many basis elements in each degree, and $\text{sup}(Q^{-j}) \leq \text{sup}(H(M))$. By [Ke1, Theorem 3.1] there is a semi-free resolution $P \to M$, with a semi-free filtration $\{\nu_j(P)\}$, such that $H(\text{gr}^P_j(P)) \cong Q^{-j}[j]$ for all $j \geq 0$. So $P$ is a pseudo-finite semi-free DG $A$-module, with $\text{sup}(P) = \text{sup}(H(M))$.

3. See [Ke1, Theorem 3.2] or [SP, Section 22.14, tag 09KQ]. Note that any graded $H(A)$-module $H(M)$ can be embedded in a product of shifts of the graded injective module $H(I)$, where $I$ is the $K$-injective DG $A$-module $I := \text{Hom}_{Z}(A, Q/Z)$; and that $\text{inf}(I) = 0$ (if $A$ is nonzero). □

**Remark 1.20.** In an earlier version of this paper, our definition of a cohomologically noetherian DG ring $A$ included the condition that $A$ is cohomologically bounded. In this version (Definition 1.18) we dropped the boundedness condition. This is possible because we now have better technical control of resolutions etc. See also Remark 7.23, where the analogy to adic rings is discussed. The name “pseudo-noetherian” was chosen due to the close relation to pseudo-finite semi-free resolutions; see Proposition 1.19(2) and Remark 1.15.

### 2. Cohomological Dimension

We continue with the conventions of Section 1, namely our DG rings are nonpositive, and the DG modules are acted upon from the left. The notions sup, inf and $[d_0, d_1]$ were introduced in formulas (1.9) and (1.7).

**Definition 2.1.** Let $A$ and $B$ be DG rings, and let $E \subset D(A)$ be a full subcategory.
(1) Let \( F : E \to D(B) \) be an additive functor, and let \([d_0, d_1] \subset \mathbb{Z} \cup \{\pm \infty\}\) be an interval. We say that \( F \) has cohomological displacement at most \([d_0, d_1]\) if
\[
\sup(H(F(M))) \leq \sup(H(M)) + d_1
\]
and
\[
\inf(H(F(M))) \geq \inf(H(M)) + d_0
\]
for every \( M \in E \).

(2) Let \( F : E^{op} \to D(B) \) be an additive functor, and let \([d_0, d_1] \subset \mathbb{Z} \cup \{\pm \infty\}\) be an interval. We say that \( F \) has cohomological displacement at most \([d_0, d_1]\) if
\[
\sup(H(F(M))) \leq -\inf(H(M)) + d_1
\]
and
\[
\inf(H(F(M))) \geq -\sup(H(M)) + d_0
\]
for every \( M \in E \).

(3) Let \( F \) be as in item (1) or (2). The cohomological displacement of \( F \) is the smallest interval \([d_0, d_1] \subset \mathbb{Z} \cup \{\pm \infty\}\) for which \( F \) has cohomological displacement at most \([d_0, d_1]\). If \( d_0 \in \mathbb{Z} \) (resp. \( d_1 \in \mathbb{Z} \), resp. \( d_0, d_1 \in \mathbb{Z} \)) then \( F \) is said to have bounded below (resp. bounded above, resp. bounded) cohomological displacement.

(4) Let \([d_0, d_1]\) be the cohomological displacement of \( F \). The cohomological dimension of \( F \) is \( d := d_1 - d_0 \in \mathbb{N} \cup \{\infty\} \). If \( d \in \mathbb{N} \), then \( F \) is said to have finite cohomological dimension.

Note that if \( E' \subset E \), and \( F \) has cohomological displacement at most \([d_0, d_1]\), then \( F|_{E'} \) also has cohomological displacement at most \([d_0, d_1]\).

Example 2.2. Consider a commutative ring \( A = B \), and let \( E := D(A) \). For the covariant case (item (1) in Definition 2.1) take a nonzero projective module \( P \), and let \( F := \text{Hom}_A(P \oplus P[1], -) \). Then \( F \) has cohomological displacement \([0, 1]\). For the contravariant case (item (2)) take a nonzero injective module \( I \), and let \( F := \text{Hom}_A(-, I \oplus I[1]) \). Then \( F \) has cohomological displacement \([-1, 0]\). In both cases the cohomological dimension of \( F \) is 1.

Remark 2.3. Suppose \( A \) and \( B \) are rings. If \( E = D^+(A) \) and \( F = RF_0 \) (or \( E = D^-(A) \) and \( F = LF_0 \)) for some additive functor \( F_0 : \text{Mod} A \to \text{Mod} B \), then the cohomological dimension of \( F \) is the usual cohomological dimension of \( F_0 \).

Assume that \( E = D(A) \) and \( F \) is a triangulated functor, with cohomological displacement \([d_0, d_1]\). The functor \( F \) has bounded below cohomological displacement iff it is way-out right, in the sense of [RD Section I.7]. The relation to the numbers \( n_1, n_2 \) appearing in [RD, Section I.7] is \( d_0 = n_1 - n_2 \). Likewise \( F \) has bounded above cohomological displacement iff it is way-out left.

The cohomological dimension in this definition is the same as [PSY, Definition 2.6]. The relation to the numbers \( n, s \) appearing in [PSY, Definition 2.6] is \( d_0 = s - n \) and \( d_1 = s \).

Definition 2.4. Let \( A \) be a DG ring, let \( M \in D(A) \), and let \([d_0, d_1]\) be an interval in \( \mathbb{Z} \cup \{\pm \infty\} \) of length \( d := d_1 - d_0 \).
Continuing with the setup of Example 2.2, the DG module $\text{Example 2.5}$.

A

Remark 2.6. Let us write $<\text{att}>$ for either of the attributes projective, injective or flat.

Similar notions of $<\text{att}>$ dimension appeared in [Ap, Section 2], where they were called “functorial $<\text{att}>$ dimension”; and in [YZ1] Section 1. Both references considered only $E = D(A)$.

When $A$ is a ring and $E = D^0(A)$, then we recover the usual definition of $<\text{att}>$ dimension in ring theory. Furthermore, in the ring case, $M$ has finite $<\text{att}>$ dimension relative to $D^0(A)$ if it isomorphic in $D(A)$ to a bounded complex of $<\text{att}>$ $A$-modules. This implies that $M$ has finite $<\text{att}>$ dimension relative to $D(A)$. We do not know if anything like this is true for a DG ring (except for perfect DG modules – see Section 5).

Proposition 2.7. Let $M \in D^{[i_0,i_1]}(A)$ for some interval $[i_0, i_1] \subset \mathbb{Z} \cup \{\pm \infty\}$. Then $M$ has projective displacement at most $[-i_1, \infty]$, injective displacement at most $[i_0, \infty]$ and flat displacement at most $[-\infty, i_1]$, all relative to $D(A)$.

Proof. First let’s assume that $M \neq 0$ and $i_1 < \infty$. We know that $M$ admits a semi-free resolution $P \to M$ with $\text{sup}(P) = \text{sup}(H(M)) \leq i_1$. For any $N \in D(A)$ and $N' \in D(A^{op})$ we have $\text{RHom}_A(M, N) \cong \text{Hom}_A(P, N)$ and $N' \otimes_A^L M \cong N' \otimes_A P$. Hence the bounds on projective and flat displacements.

Now let’s assume that $M \neq 0$ and $i_0 > -\infty$. By Proposition 1.19 there is a K-injective resolution $M \to I$ with $\text{inf}(I) = \text{inf}(H(M)) \geq i_0$. For any $N \in D(A)$ we have $\text{RHom}_A(N, M) \cong \text{Hom}_A(N, I)$, and hence the bound on injective displacement.

The range of generation of a DG module was introduced in Definition 1.16.

Proposition 2.8. Let $M \in D(A)$. Assume there is an isomorphism $P \cong M$ in $D(A)$, where $P$ is a DG $A$-module generated in the degree range $[d_0, d_1]$. 
(1) If $P$ is $K$-flat, then $M$ has flat displacement at most $[d_0, d_1]$, and flat dimension at most $d_1 - d_0$.

(2) If $P$ is semi-free, then $M$ has projective displacement at most $[d_0, d_1]$, and flat dimension at most $d_1 - d_0$.

Proof. (1) Take any $N \in D(A^{op})$. After applying smart truncation, we can assume that $\sup(N) = \sup(H(N))$ and $\inf(N) = \inf(H(N))$. Now $N \otimes_A P \cong N \otimes_A P$, and the bounds on $N \otimes_A P$ are as claimed.

(2) Take any $N \in D(A)$. As above, we can assume that $\sup(N) = \sup(H(N))$ and $\inf(N) = \inf(H(N))$. We know that

$$R\text{Hom}_A(M, N) \cong \text{Hom}_A(P, N).$$

The bounds on the DG module $\text{Hom}_A(P, N)$ are as claimed, by Proposition I.17.

The next theorem is a variation of the opposite (in the categorical sense) of [RD, Proposition I.7.1], the “Lemma on Way-Out Functors”. The canonical homomorphism $A \to \bar{A}$ lets us view any $A$-module as a DG $A$-module.

**Theorem 2.9.** Let $A$ and $B$ be nonpositive DG rings, let $F, G: D(A) \to D(B)$ be triangulated functors, and let $\eta: F \to G$ be a morphism of triangulated functors. Assume that $\eta_M: F(M) \to G(M)$ is an isomorphism for every $M \in \text{Mod} \bar{A}$.

(1) The morphism $\eta_M$ is an isomorphism for every $M \in D^-(A)$.

(2) If $F$ and $G$ have bounded above cohomological displacements, then $\eta_M$ is an isomorphism for every $M \in D^-(A)$.

(3) If $F$ and $G$ have finite cohomological dimensions, then $\eta_M$ is an isomorphism for every $M \in D(A)$.

Proof. (1) The proof is by induction on $j := \text{amp}(H(M))$. If $j = 0$ then $M$ is isomorphic to a shift an object of $\bar{A}$, so $\eta_M$ is an isomorphism. If $j > 0$, then using smart truncation we obtain a distinguished triangle $M' \to M \to M'' \to$ in $D(A)$ such that $\text{amp}(H(M'')) < j$ and $\text{amp}(H(M')) < j$. Since $\eta_{M''}$ and $\eta_{M'}$ are isomorphisms, so is $\eta_M$.

(2) Here we assume that $F$ and $G$ have cohomological displacements at most $[-\infty, d_1]$ for some integer $d_1$. Take any $M \in D^-(A)$. In order to prove that $\eta_M$ is an isomorphism it suffices to show that

$$H^i(\eta_M): H^i(F(M)) \to H^i(G(M))$$

is bijective for every $i \in \mathbb{Z}$.

Fix an integer $i$. Let $M' \to M \to M'' \to$ be a distinguished triangle such that $\text{sup}(H(M')) \leq i - d_1 - 2$ and $\text{inf}(H(M'')) \geq i - d_1 - 1$. This can be obtained using smart truncation.

The cohomologies of $F(M')$ and $G(M')$ are concentrated in the degree range $\leq i - 2$. The distinguished triangle induces a commutative diagram of $A$-modules with exact rows:

$$
\begin{array}{cccccc}
H^i(F(M')) & \to & H^i(F(M)) & \to & H^i(F(M'')) & \to & H^{i+1}(F(M')) \\
\downarrow_{H^i(\eta_M)} & & \downarrow_{H^i(\eta_M)} & & \downarrow_{H^i(\eta_{M''})} & & \downarrow_{H^{i+1}(\eta_{M'})} \\
H^i(G(M')) & \to & H^i(G(M)) & \to & H^i(G(M'')) & \to & H^{i+1}(G(M'))
\end{array}
$$
The four terms involving $M'$ are zero. Since $M''$ has bounded cohomology, we know by part (1) that $H^i(\eta_{M''})$ is an isomorphism. Therefore $H^i(\eta_M)$ is an isomorphism.

(3) Here we assume that $F$ and $G$ have finite cohomological dimensions. So $F$ and $G$ have cohomological displacements at most $|d_0, d_1|$ for some $d_0 \leq d_1$ in $\mathbb{Z}$. Take any $M \in D(A)$, and fix $i \in \mathbb{Z}$. We want to show that $H^i(\eta_M)$ is an isomorphism. Using smart truncations of $M$ we obtain a distinguished triangle $M' \to M \to M'' \to$ in $D(A)$, such that $\sup(H(M')) \leq i - d_0 + 1$ and $\inf(H(M'')) \geq i - d_0 + 2$. The cohomologies of $F(M')$ and $G(M'')$ are concentrated in degrees $\geq i + 2$. We have a commutative diagram of $\mathcal{A}$-modules with exact rows:

$$
\begin{array}{cccc}
H^{i-1}(F(M'')) & \longrightarrow & H^i(F(M')) & \longrightarrow & H^i(F(M)) & \longrightarrow & H^i(F(M'')) \\
H^{i-1}(\eta_{M''}) & \downarrow & H^i(\eta_{M'}) & \downarrow & H^i(\eta_M) & \downarrow & H^i(\eta_{M''}) \\
H^{i-1}(G(M'')) & \longrightarrow & H^i(G(M')) & \longrightarrow & H^i(G(M)) & \longrightarrow & H^i(G(M''))
\end{array}
$$

The four terms involving $M''$ are zero here. Since $M'$ has bounded above cohomology, we know by part (2) that $H^i(\eta_{M'})$ is an isomorphism. Therefore $H^i(\eta_M)$ is an isomorphism.

\begin{theorem}
Let $A$ and $B$ be nonpositive DG rings, let $F, G : D(A) \to D(B)$ be triangulated functors, and let $\eta : F \to G$ be a morphism of triangulated functors. Assume that $A$ is cohomologically left pseudo-noetherian, and $\eta_A : F(A) \to G(A)$ is an isomorphism.

(1) If $F$ and $G$ have bounded above cohomological displacements, then $\eta_M : F(M) \to G(M)$ is an isomorphism for every $M \in D^-_1(A)$.

(2) If $F$ and $G$ have finite cohomological dimensions, then $\eta_M : F(M) \to G(M)$ is an isomorphism for every $M \in D_1(A)$.
\end{theorem}

Cohomological displacement and cohomological dimension of a functor were defined in Definition 2.1. This theorem is another variant (cf. Theorem 2.9) of the opposite (in the categorical sense) of [RD] Proposition I.7.1.

\begin{proof}
Step 1. Consider a finite free DG $A$-module $P$, i.e. $P \cong \bigoplus_{k=1}^r A[-i_k]$ in $M(A)$ for some $i_1, \ldots, i_r \in \mathbb{Z}$. Because the functors $F, G$ are triangulated, and $\eta_A$ is an isomorphism, it follows that $\eta_P$ is an isomorphism.

Step 2. Now let $P$ be a finite semi-free DG $A$-module, with finite semi-free filtration $\{\nu_j(P)\}$ of length $j_1$ (see Definition 1.12). We prove that $\eta_P$ is an isomorphism by induction on $j_1$. For $j_1 = 0$ this is step 1. Now assume $j_1 \geq 1$. Write $P' := \nu_{j_1-1}(P)$ and $P'' := \gr^{\nu_j}_{j_1}(P)$, so there is a distinguished triangle $P' \to P \to P'' \to$ in $D(A)$. According to step 1 and the induction hypothesis, the morphisms $\eta_{P''}$ and $\eta_{P'}$ are isomorphisms. Hence $\eta_P$ is an isomorphism.

Step 3. Here we assume that $F$ and $G$ have cohomological displacements at most $[\neg \infty, d_1]$ for some integer $d_1$. Take any $M \in D^-_1(A)$. In order to prove that $\eta_M$ is an isomorphism it suffices to show that

$$
H^i(\eta_M) : H^i(F(M)) \to H^i(G(M))
$$

is bijective for every $i \in \mathbb{Z}$.

We may assume that $M$ is nonzero. Let $i_1 := \sup(H(M))$, which is an integer. There exists a pseudo-finite semi-free resolution $P \to M$ such that $sup(P) = i_1$; see
Proposition 1.19. We will prove that $H^i(\eta_P)$ is an isomorphism for every $i$. Fix a pseudo-finite semi-free filtration $\{\nu_j(P)\}$ of $P$.

Take an integer $j$, and define $P^t := \nu_j(P)$ and $P'' := P/\nu_j(P)$. So there is a distinguished triangle $P' \to P \to P'' \xrightarrow{\Delta}$. The DG module $P''$ is concentrated in the degree range $\leq i_1 - j - 1$, and hence so is its cohomology. Thus the cohomologies of $F(P''')$ and $G(P''')$ are concentrated in the degree range $\leq i_1 - j - 1 + d_1$, or in other words $H^i(F(P''')) = H^i(G(P''')) = 0$ for all $i > i_1 - j - 1 + d_1$. On the other hand the DG module $P'$ is finite semi-free. The distinguished triangle above induces a commutative diagram of $A$-modules with exact rows:

$$
\begin{array}{c}
H^{i-1}(F(P''')) & \to H^i(F(P''')) & \to H^i(F(P')) & \to H^i(F(P''')) \\
\downarrow H^{i-1}(\eta_{P'''}) & \downarrow H^i(\eta_{P'''}) & \downarrow H^i(\eta_P) & \downarrow H^i(\eta_{P'''}) \\
H^{i-1}(G(P''')) & \to H^i(G(P''')) & \to H^i(G(P')) & \to H^i(G(P'''))
\end{array}
$$

For any $i > i_1 + d_1 - j$ the modules in this diagram involving $P''$ are zero. By step 2 we know that $H^i(\eta_{P''})$ is an isomorphism for every $i$. Therefore $H^i(\eta_{P'})$ is an isomorphism for every $i > i_1 + d_1 - j$. Since $j$ can be made arbitrarily large, we conclude that $H^i(\eta_P)$ is an isomorphism for every $i$.

Step 4. Here we assume that $F$ and $G$ have finite cohomological dimensions. So $F$ and $G$ have cohomological displacements at most $[d_0, d_1]$ for some $d_0 \leq d_1$ in $\mathbb{Z}$. This step is very similar to the proof of Theorem 2.9(3).

Take any $M \in D_{i_1}(A)$. Fix $i \in \mathbb{Z}$. We want to show that $H^i(\eta_M)$ is an isomorphism. Using smart truncations of $M$, we obtain a distinguished triangle $M' \to M \to M'' \xrightarrow{\Delta}$ in $D_{i_1}(A)$, such that $M' \in D_{[-\infty, i_1-d_0+1]}(A)$ and $M'' \in D_{[i_1-d_0+1, \infty]}(A)$. We get a commutative diagram like (2.11). The modules $H^{i-1}(F(M'''))$, $H^i(F(M'''))$, $H^{i-1}(G(M'''))$ and $H^i(G(M'''))$ are zero. Since $M' \in D_{i_1}(A)$, step 3 says that $H^i(\eta_{M'})$ is an isomorphism. Therefore $H^i(\eta_M)$ is an isomorphism.

The next theorem is a variant of the opposite of [RD, Proposition I.7.3]. For a boundedness condition $\star$, we denote by $\dashv \star$ the opposite boundedness condition. Thus if $D^\star$ denotes either $D$, $D^\ast$, $D^-$ or $D^b$, then $D^{\dashv \star}$ denotes either $D$, $D^\ast$, $D^-$ or $D^b$ respectively.

**Theorem 2.12.** Let $A$ and $B$ be nonpositive DG rings, and let $F : D(A)^{\op} \to D(B)$ be a triangulated functor. Assume that $A$ and $B$ are cohomologically left pseudo-noetherian, and $F(A) \in D^+_B$.

1. If $F$ has bounded below cohomological displacement, then $F(M) \in D^+_B$ for every $M \in D^-_A$.
2. If $F$ has finite cohomological dimension, then $F(M) \in D^{-\ast}_B$ for every $M \in D^+_A$.

**Proof.** The proof is very similar to that of Theorem 2.10. We just outline the necessary changes.

Steps 1-2. $P$ is a finite semi-free DG $A$-module. By induction on the length $j_1$ of a semi-free filtration, we prove that $F(P) \in D^+_B$.

Step 3. Here $F$ has cohomological displacement $[d_0, \infty]$ for some $d_0 \in \mathbb{Z}$, and $M \in D^{-\ast}_A$. Let $P \to M$ be a pseudo-finite semi-free resolution, with $\sup(P) = i_1$. 

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Take any $j \in \mathbb{Z}$, and consider the distinguished triangle $P' \to P \to P'' \xrightarrow{\Delta}$. There is an exact sequence of $B$-modules

$$(2.13) \quad H^{i-1}(F(P')) \to H^i(F(P')) \to H^i(F(P)) \to H^i(F(P'')).$$

By step 2 we know that $H^i(F(P')) \in \text{Mod}_i \bar{B}$ for every $i$. If $i \leq d_0 - i_1 + j$ then $H^{i-1}(F(P')) = H^i(F(P'')) = 0$. Therefore $H^i(F(P)) \in \text{Mod}_i \bar{B}$. But $j$ can be made arbitrarily large. This proves that $F(M) \in D_i(B)$. But on the other hand we know that $H^i(F(M)) = 0$ for all $i < -i_1 + d_0$. We conclude that $F(M) \in D_i^+(B)$.

Step 4. Here $F$ has cohomological displacement at most $[d_0, d_1]$ for some $d_0 \leq d_1$ in $\mathbb{Z}$, $M \in D_i(A)$, and $i \in \mathbb{Z}$. We truncate $M$ to obtain a distinguished triangle $M' \to M \to M'' \xrightarrow{\Delta}$ in $D_i(A)$, such that $M' \in D_i^-[\infty,j+1](A)$ and $M'' \in D_i^{[j+1,\infty]}(A)$ for $j := i - d_1 - 3$. We get an exact sequence like $(2.13)$. The modules $H^{i-1}(F(M''))$ and $H^i(F(M'))$ are zero, and $H^i(F(M')) \in \text{Mod}_i \bar{B}$ by step 3. Therefore $H^i(F(M)) \in \text{Mod}_i \bar{B}$. The condition on the boundedness of $H(F(M))$ is established like in step 3. □

3. Reduction and Lifting

Recall Conventions 1.1 and 1.6 by default, all DG modules are left DG modules, and all DG rings are nonpositive. In this section we study the canonical DG ring homomorphism $A \to \bar{A}$, and the corresponding reduction functor

$$D(A) \to D(\bar{A}), \ M \mapsto \bar{A} \otimes^L_A M.$$ 

We do not make any finiteness assumptions on the cohomology modules $H^i(M)$.

A triangulated functor $F$ is called conservative if for any object $M$, $F(M) = 0$ implies $M = 0$; or equivalently, if for any morphism $\phi$, $F(\phi)$ is an isomorphism implies $\phi$ is an isomorphism. Cf. [KaSc] Section 1.4. The following result is analogous to the Nakayama Lemma (cf. Remark 7.23).

**Proposition 3.1.** Let $A$ be a DG ring. The functor

$$\bar{A} \otimes^L_A - : D^-(A) \to D^-(\bar{A})$$

is conservative.

**Proof.** Take a nonzero $M \in D^-(A)$, and let $i_1 := \sup(H(M))$. We can find a K-flat resolution (e.g. a semi-free resolution) $P \to M$ over $A$ such that $\sup(P) = i_1$. Then $\bar{A} \otimes^L_A M \cong \bar{A} \otimes_A P$, and (by the “K"unneth trick”)

$$H^{i_1}(\bar{A} \otimes^L_A M) \cong H^{i_1}(\bar{A} \otimes_A P) \cong \bar{A} \otimes_A H^{i_1}(P) \cong H^{i_1}(M)$$

is nonzero. Hence $\bar{A} \otimes^L_A M$ is nonzero. □

Given a homomorphism $\phi : P \to Q$ in $\text{M}(A)$, we denote by $\text{cone}(\phi)$ the corresponding cone, which is also an object of $\text{M}(A)$.

**Lemma 3.2.** Suppose $\phi : P \to Q$ is a homomorphism in $\text{M}(A)$, where $P$ and $Q$ are pseudo-finite (resp. finite) semi-free DG modules. Then $\text{cone}(\phi)$ is a pseudo-finite (resp. finite) semi-free DG module.

**Proof.** Clear from Proposition 1.14. □

**Proposition 3.3.** Let $M \in D^-(A)$. We write $\bar{M} := \bar{A} \otimes^L_A M \in D^-(\bar{A})$. 


(1) If $\bar{M}$ is isomorphic in $D(\bar{A})$ to $\bar{A}^{\oplus r}$ for some cardinal number $r$, then $M$ is isomorphic in $D(A)$ to $A^{\oplus r}$.

(2) If $\bar{M}$ is isomorphic in $D(\bar{A})$ to a semi-free DG $\bar{A}$-module $\bar{P}$ of semi-free length $d$, then $M$ is isomorphic in $D(A)$ to a semi-free DG $A$-module $P$ of semi-free length $d$.

Observe that the DG $\bar{A}$-module $\bar{P}$ in (2) above is nothing but a bounded complex of free $A$-modules; cf. Proposition 1.14. The semi-free length was introduced in Definition 1.13.

**Proof.** Step 1. In view of Proposition 3.1 we can assume that $\bar{A}$ and $M$ are nonzero. Define $i_1 := \text{sup}(H(M))$. By replacing $M$ with a suitable resolution of it, we can assume that $M$ is a K-flat DG $A$-module satisfying $\text{sup}(M) = i_1$. After that we can also assume that $M = A \otimes_A M$. The Künneth formula says that

$$H^i(\bar{M}) \cong H^i(\bar{A} \otimes_A M) \cong \bar{A} \otimes_A H^i(M) \cong H^i(M)$$

as $\bar{A}$-modules. Therefore $\text{sup}(H(\bar{M})) = \text{sup}(\bar{M}) = i_1$.

We are given an isomorphism $\phi: \bar{P} \to M$ in $D(\bar{A}^0)$, where $\bar{P}$ is a bounded complex of free $A$-modules. Since $\bar{P}$ is K-projective, we can assume that the isomorphism $\phi: \bar{P} \to \bar{M}$ in $D(\bar{A})$ is in fact a quasi-isomorphism in $M(\bar{A})$. The proof continues by induction on $j := \text{amp}(\bar{P}) \in \mathbb{N}$. Note that in item (1) we have $i_1 = 0$, $\bar{P} = A^{\oplus r}$ and $j = 0$.

Step 2. In this step we assume that $j = 0$. This means that the only nonzero term of $\bar{P}$ is in degree $i_1$, and it is the free module $\bar{P}^{i_1} \cong A^{\oplus r_1}$ for some cardinal number $r_1$. In other words, $\bar{P} \cong \bar{A}[-i_1]^{\oplus r_1}$ in $M(\bar{A})$. So $H^{i_1}(M) \cong \bar{P}^{i_1}$, and $H^j(M) = 0$ for all $i \neq i_1$. Recall the isomorphism of $\bar{A}$-modules $H^{i_1}(\bar{M}) \cong H^{i_1}(M)$ from Step 1. There are canonical surjections $\bar{M}^{i_1} \to H^{i_1}(M)$ and $\bar{M}^{i_1} \to H^{i_1}(\bar{M})$. We can write the quasi-isomorphism $\bar{\phi}$ as $\bar{\phi}: \bar{A}[-i_1]^{\oplus r_1} \to \bar{M}$.

Consider the nonderived reduction functor $F: M(A) \to M(\bar{A})$, $F(-) := \bar{A} \otimes_A -$. Let $P := A[-i_1]^{\oplus r_1}$, a free DG $A$-module satisfying $F(P) \cong \bar{P}$. There exists a homomorphism $\phi: P \to M$ in $M(A)$ that lifts the quasi-isomorphism $\bar{\phi}: \bar{P} \to \bar{M}$, namely $\phi = F(\bar{\phi})$. Now the DG modules $P$ and $M$ are K-flat, so $\phi = LF(\bar{\phi})$. Since $\phi$ is an isomorphism, and since the functor $LF$ is conservative, we conclude that $\phi$ is an isomorphism. This proves item (1).

Step 3. Here we suppose that $j \geq 1$. Let $i_2 := \text{sup}(\bar{P})$, which is of course $\geq i_1$. Say $\bar{P}^{i_2} \cong \bar{A}^{\oplus r_2}$ for some natural number $r_2$. Define DG modules $\bar{P}' := \bar{A}[-i_2]^{\oplus r_2}$ and $P' := A[-i_2]^{\oplus r_2}$; these satisfy $\bar{P}' \cong A \otimes_A P'$. The inclusion $\bar{P}^{i_2} \subset P$ is viewed as a DG module homomorphism $\bar{\alpha}: \bar{P}' \to \bar{P}$. We also have a quasi-isomorphism $\bar{\phi}: \bar{P} \to \bar{M}$ and an equality $\bar{M} = \bar{A} \otimes_A M$ in $M(\bar{A})$. In this way we obtain a homomorphism $\phi': P' \to M$, $\psi := \bar{\phi} \circ \bar{\alpha}$. Because $P'$ is a free DG $A$-module, there is a homomorphism $\psi: P' \to M$ in $M(A)$ lifting $\bar{\psi}$, namely $\psi = F(\bar{\psi})$, where $F$ is the functor $\bar{A} \otimes_A -$.

Let $M'' \in M(A)$ be the cone of $\psi$, so there is a distinguished triangle

$$P' \xrightarrow{\phi'} M \xrightarrow{\chi} M'' \to P'[1] \quad (3.4)$$

in $D(A)$. Define $\bar{M}'' := F(M'')$ and $\bar{\chi} := F(\chi)$, which are an object and a morphism in $M(\bar{A})$, respectively. Since all three DG modules in this triangle are K-flat, it
follows that
\[ \bar{P}' \xrightarrow{\phi} \bar{M} \xrightarrow{\bar{\alpha}} \bar{M}'' \to \bar{P}'[1] \]
is a distinguished triangle in \( D(\bar{A}) \). On the other hand, let \( \bar{P}'' \) be the cokernel of the inclusion \( \bar{\alpha} : \bar{P}' \to \bar{P} \). So there is a distinguished triangle
\[ \bar{P}' \xrightarrow{\bar{\alpha}} \bar{P} \xrightarrow{\bar{\beta}} \bar{P}'' \to \bar{P}'[1] \]
in \( D(\bar{A}) \). Consider the diagram of solid arrows in \( D(\bar{A}) \):
\[
\begin{array}{ccc}
\bar{P}' & \xrightarrow{\bar{\alpha}} & \bar{P} \\
\downarrow & & \downarrow \\
\bar{P}' & \xrightarrow{\bar{\phi}''} & \bar{M} \\
\end{array}
\]
The square on the left is commutative, and therefore it extends to an isomorphism of distinguished triangles. So there is an isomorphism \( \bar{\phi}'' : \bar{P}'' \to \bar{M}'' \) in \( D(\bar{A}) \).

Finally, the complex \( \bar{P}'' \) is a bounded complex of finite free \( \bar{A} \)-modules, of amplitude \( j - 1 \geq 0 \). According to the induction hypothesis (step 2 for \( j > 1 \), and step 1 for \( j = 1 \)) there is an isomorphism \( \phi'' : P'' \xrightarrow{\sim} M'' \) in \( D(A) \) for some finite semi-free DG \( A \)-module \( P'' \). From \( \Phi(3.4) \) we obtain a distinguished triangle
\[ P'' \xrightarrow{\phi''} M \xrightarrow{\psi} P'' \xrightarrow{\sim} P'[1] \]
in \( D(A) \). We can assume that \( \gamma \) is a homomorphism in \( M(A) \). Turning this triangle we get a distinguished triangle
\[ P''[-1] \xrightarrow{-\gamma[-1]} P' \xrightarrow{\phi} M \xrightarrow{\Delta} . \]

Define \( P \) to be the cone on the homomorphism \( -\gamma[-1] : P''[-1] \to P' \). Then \( P \cong M \) in \( D(A) \), and by Lemma \( \Phi(3.2) \) \( P \) is a finite semi-free DG \( A \)-module. \( \square \)

We learned the next proposition from B. Antieau and J. Lurie. Cf. \( \Phi(3.1) \) for a more general statement.

**Proposition 3.5.** Let \( \bar{P} \) be a projective \( \bar{A} \)-module. Then there exists a DG \( A \)-module \( P \) with these properties:

(i) \( P \) is a direct summand, in \( D(A) \), of a direct sum of copies of \( A \).

(ii) \( \bar{A} \otimes_A P \cong \bar{P} \) in \( D(\bar{A}) \).

**Proof.** Say \( \bar{P} \) is a direct summand, in \( \text{Mod} \, \bar{A} \), of a free \( \bar{A} \)-module \( \bar{F} \). So \( \bar{P} \) is the image of an idempotent endomorphism \( \phi : \bar{F} \to \bar{F} \). Let \( F \) be the direct sum in \( M(A) \) of copies of \( A \), as many as there are copies of \( \bar{A} \) in \( \bar{F} \). There is a canonical surjection \( F \to \bar{F} \). Choose any homomorphism \( \phi : F \to F \) in \( M(A) \) lifting \( \phi \). Note that many such \( \phi \) exist; and they aren’t necessarily idempotents. Let \( P \) be the homotopy colimit construction on \( \phi \). Namely we let
\[
\Phi : \bigoplus_{i \in \mathbb{N}} F \to \bigoplus_{i \in \mathbb{N}} F
\]
be the homomorphism
\[
\Phi(a_0, a_1, a_2, ...) := (a_0, a_1 - \phi(a_0), a_2 - \phi(a_1), ...)
\]
in \( M(A) \), and then we define \( P := \text{cone}(\Phi) \in M(A) \). Because \( P \) is K-flat over \( A \), we see that
\[
\bar{A} \otimes_A P \cong \bar{A} \otimes_A P \cong \text{cone}(\Phi),
\]
where
\[ \Phi : \bigoplus_{i \in \mathbb{N}} \bar{F} \to \bigoplus_{i \in \mathbb{N}} \bar{F} \]
is the homotopy colimit construction on \( \bar{\phi} \). An easy calculation, using the fact that \( \bar{\phi} \) is an idempotent with image \( \bar{P} \), shows that \( \text{cone}(\Phi) \cong \bar{P} \) in \( D(\bar{A}) \). This proves (ii).

As for (i): say \( \bar{Q} \) is the other direct summand of \( \bar{F} \) in \( \text{Mod} \bar{A} \). We can lift it to a DG \( \bar{A} \)-module \( Q \) as above. Now \( \bar{P} \oplus \bar{Q} \cong \bar{F} \) in \( D(\bar{A}) \). By Proposition 3.3(1) we deduce that \( P \oplus Q \cong F \) in \( D(A) \). \( \square \)

4. Localization of Commutative DG Rings

In this section we specialize to DG rings satisfying a commutativity condition. Such a DG ring \( A \) can be localized on \( \text{Spec} \bar{A} \).

**Definition 4.1.** Let \( A = \bigoplus_{i \in \mathbb{Z}} A^i \) be a DG ring.

1. \( A \) is called **weakly commutative** if \( b \cdot a = (-1)^{ij} \cdot a \cdot b \) for all \( a \in A^i \) and \( b \in A^j \).
2. \( A \) is called **strongly commutative** if it is weakly commutative, and \( a \cdot a = 0 \) for all \( a \in A^i \) with \( i \) odd.

If \( A \) is weakly commutative, then any left DG \( A \)-module \( M \) can be viewed as a right DG \( A \)-module. The formula for the right action is this:

\[ m \cdot a := (-1)^{ij} \cdot a \cdot m, \]

for \( a \in A^i \) and \( m \in M^j \).

**Convention 4.3.** We use the abbreviation “commutative DG ring” to mean “strongly commutative nonpositive DG ring”. See Definitions 1.3 and 4.1.

By default, all DG rings from here on in the paper are commutative (unless explicitly stated otherwise). In particular all rings are commutative by default.

Since our DG rings are now commutative, we can talk about cohomologically pseudo-noetherian DG rings, instead of cohomologically left pseudo-noetherian ones (Definition 1.18).

**Remark 4.4.** In [YZ1], and in earlier versions of this paper, we used the name “super-commutative” for what we now call “strongly commutative”. The book [ML] uses the term “strictly commutative”.

Of course when 2 is invertible in the ring \( A^0 \) (e.g. in characteristic 0), there is no difference between weakly commutative and strongly commutative DG rings.

The reader may wonder why we prefer to work with strongly commutative DG rings, and not with the weakly commutative ones. One reason is this: the weakly commutative polynomial ring \( \mathbb{Z}[x] \), where the variable \( x \) has degree \(-1\), is not flat over \( \mathbb{Z} \), since \( x^2 \neq 0 \) but \( 2 \cdot x^2 = 0 \). However the strongly commutative polynomial ring \( \mathbb{Z}[x] = \mathbb{Z} \oplus \mathbb{Z} \cdot x \) is flat, and even free, over \( \mathbb{Z} \).

**Proposition 4.5.** Let \( A \) be a commutative DG ring. The action of \( A^0 \) on \( M(A) \) makes \( D(A) \) into an \( \bar{A} \)-linear category.

**Proof.** For any \( a \in A^{-1} \) and any \( \phi \in \text{Hom}_A(M, N)^0 \), the homomorphism \( d(a) \cdot \phi = d(a \cdot \phi) \) vanishes in the homotopy category \( K(A) \). Therefore \( K(A) \) is an \( \bar{A} \)-linear category, and hence so is its localization \( D(A) \). \( \square \)
Definition 4.6. Let $A$ be a commutative DG ring. We denote by $\pi: A \to \bar{A} = H^0(A)$ the canonical homomorphism. Given a multiplicatively closed subset $S$ of $\bar{A}$, the set $\bar{S} := \pi^{-1}(S) \cap A^0$ is a multiplicatively closed subset of $A^0$. Define the ring $A^0_S := \bar{S}^{-1} \cdot A^0$ (this is the usual localization), and the DG ring $A_S := A^0_S \otimes_{A^0} A$. There is a canonical DG ring homomorphism $\lambda_S: A \to A_S$.

If $S = \{s^i\}_{i \in \mathbb{N}}$ for some element $s \in A$, then we also use the notation $A_s := A_S$.

There is the usual localization $\bar{A}_S = S^{-1} \cdot \bar{A}$ of the ring $\bar{A}$ w.r.t. $S$. We get a graded ring $H(A)_S := A_S \otimes_{\bar{A}} H(A)$. There are graded ring homomorphisms $H(A) \to H(A)_S$ and $H(\lambda_S): H(A) \to H(A_S)$.

Proposition 4.7. Let $A$ be a commutative DG ring and $S \subset \bar{A}$ a multiplicatively closed subset.

1. There is a unique isomorphism of graded $H(A)$-rings $H(A)_S \cong H(A)_S$.
2. For any DG $A$-module $M$ there is a unique isomorphism of graded $H(A_S)$-modules

$$H(A_S \otimes_A M) \cong H(A)_S \otimes_{H(A)} H(M)$$

that is compatible with the homomorphisms from $H(M)$.
3. If $A$ is cohomologically pseudo-noetherian, then so is $A_S$.

Proof. Items (1-2) are true because the ring homomorphism $\lambda_S: A^0 \to A^0_S$ is flat. Item (3) is an immediate consequence of (1-2).

Definition 4.8. Let $B$ be a commutative ring, and let $b = (b_1, \ldots, b_n)$ be a sequence of elements of $B$. We call $b$ a covering sequence of $B$ if $\sum_{i=1}^n B \cdot b_i = B$.

Consider the spectrum $\text{Spec} B$. For an element $b \in B$ we identify the principal open set $\{p \in \text{Spec} B \mid b \notin p\}$ with the scheme $\text{Spec} B_b$, where $B_b$ is the localization of $B$ w.r.t. $b$. Clearly a sequence $b = (b_1, \ldots, b_n)$ in $B$ is a covering sequence iff

$$\text{Spec } B = \bigcup_{i=1}^n \text{Spec } B_{b_i}.$$  

Let $A$ be a commutative DG ring, and let $a = (a_1, \ldots, a_n)$ be a covering sequence of $\bar{A}$. For any strictly increasing sequence $i = (i_0, \ldots, i_p)$ of length $p$ in the integer interval $[0, n]$, i.e. $1 \leq i_0 < \cdots < i_p \leq n$, we define the ring

$$C(A^0; a)(i) := A^0_{a_{i_0}} \otimes_{A^0} \cdots \otimes_{A^0} A^0_{a_{i_p}},$$

where $A^0_{a_0}, \ldots, A^0_{a_n}$ are the localizations from Definition 4.6. Next, for any $p \in [0, n-1]$ we let

$$C^p(A^0; a) := \bigoplus_i C(A^0; a)(i),$$

where the sum is on all strictly increasing sequences $i$ of length $p$. Finally we define the DG $A^0$-module

$$C(A^0; a) := \bigoplus_{p=0}^{n-1} C^p(A^0; a).$$

The differential $C^p(A^0; a) \to C^{p+1}(A^0; a)$ is $\sum_{i,k} (-1)^k \cdot \lambda_{i,k}$, where $i$ runs over the strictly increasing sequences of length $p + 1$, $k \in [0, p + 1]$, $\partial_k(i)$ is the sequence
obtained from $i$ by omitting $i_k$, and
\[ \lambda_{i,k} : C(A^0; \mathfrak{a})(\partial_k(i)) \to C(A^0; \mathfrak{a})(i) \]
is the canonical ring homomorphism.

**Definition 4.12.** Let $A$ be a commutative DG ring, and let $\mathfrak{a} = (a_1, \ldots, a_n)$ be a covering sequence of $\bar{A}$. For a DG $A$-module $M$, the Čech DG module of $M$ is the DG $A$-module
\[ C(M; \mathfrak{a}) := C(A^0; \mathfrak{a}) \otimes_{A^0} M. \]

There is a canonical DG module homomorphism $c_M : M \to C(M; \mathfrak{a})$, sending $m \in M$ to $\sum_i 1_i \otimes m \in C(M; \mathfrak{a})$, where $1_i \in A^n_0 \subset C^0(A^0; \mathfrak{a})$.

**Proposition 4.13.** Let $\mathfrak{a} = (a_1, \ldots, a_n)$ be a covering sequence of $\bar{A}$, and let $M$ be a DG $A$-module. Then the homomorphism $c_M : M \to C(M; \mathfrak{a})$ is a quasi-isomorphism.

**Proof.** Since $C(A; \mathfrak{a})$ is a K-flat DG $A$-module, and $C(M; \mathfrak{a}) \cong C(A; \mathfrak{a}) \otimes_A M$, we see that $C(-; \mathfrak{a})$ is a triangulated functor from $D(A)$ to itself. The homomorphism $c_M$ is a quasi-isomorphism if and only if it is an isomorphism in $D(A)$.

The cohomological dimension of the functor $C(-; \mathfrak{a})$ is finite: it is at most $n - 1$. According to Theorem 2.9(3) it suffices to check that $c_M$ is a quasi-isomorphism for $M \in \text{Mod} \bar{A}$. But in this case $C(M; \mathfrak{a}) \cong \bar{C}(\mathfrak{a}) \otimes_{\bar{A}} M$, so $C(M; \mathfrak{a})$ is the usual Čech complex for the covering of $\text{Spec} \bar{A}$ determined by the sequence $\mathfrak{a}$. In geometric language (cf. [Ha, Section III.4]), writing $X := \text{Spec} \bar{A}$ and $U_i := \text{Spec} \bar{A}_{a_i}$, and letting $\mathcal{M}$ denote the quasi-coherent $O_X$-module corresponding to $M$, we have $M \cong \Gamma(X, \mathcal{M})$ and $C(M; \mathfrak{a}) \cong C(\{U_i\}; \mathcal{M})$. By [SP, Lemma 25.2.1, tag 01X9], the homomorphism $c_M : M \to C(M; \mathfrak{a})$ is a quasi-isomorphism. \hfill $\Box$

**Remark 4.14.** Actually the DG $A$-module $C(A; \mathfrak{a})$ has more structure. There is a cosimplicial commutative ring $C_{\text{cos}}(A^0; \mathfrak{a})$, whose degree $p$ piece is
\[ C_{\text{cos}}^p(A^0; \mathfrak{a}) := \prod_i C(A^0; \mathfrak{a})(i), \]
where $i = (i_0, \ldots, i_p)$ are weakly increasing sequences in $[1, n]$. The Čech DG module $C(A^0; \mathfrak{a})$ is the standard normalization of $C_{\text{cos}}(A^0; s)$, and as such it has a structure of noncommutative central DG $A^0$-ring (which is concentrated in non-negative degrees). Hence $C(A; \mathfrak{a})$ is a noncommutative DG ring, and $c_A : A \to C(A; \mathfrak{a})$ is a DG ring quasi-isomorphism. See [PSY, Section 8].

**Definition 4.15.** Let $B$ be a commutative ring, and let $\mathfrak{e} = (e_1, \ldots, e_n)$ be a sequence of elements of $B$. We call $\mathfrak{e}$ an idempotent covering sequence if each $e_i$ is an idempotent element of $B$, $e_i \cdot e_j = 0$ for $i \neq j$, and $1 = \sum_{i=1}^n e_i$.

Suppose $\mathfrak{e} = (e_1, \ldots, e_n)$ is an idempotent covering sequence of $B$. Of course $\mathfrak{e}$ is a covering sequence in the sense of Definition 4.8. The scheme $\text{Spec} B_{e_i}$ is an open-closed subscheme of $\text{Spec} B$, and
\[ \text{Spec} B = \coprod_{i=1}^n \text{Spec} B_{e_i}. \]

There is equality of rings $B = \prod_{i=1}^n B_{e_i}$, and equality of $B$-modules $B = \bigoplus_{i=1}^n B \cdot e_i$. 


Proposition 4.16. Let $A$ be a DG ring, and let $(e_1, \ldots, e_n)$ be an idempotent covering sequence of $\tilde{A} = H^0(A)$. For any $i$ we have the localized DG ring $A_i := A_{e_i}$ as in Definition 4.16, and a DG ring homomorphism $\lambda_i := \lambda_{e_i} : A \to A_i$. Then the DG ring homomorphism

$$
\lambda := (\lambda_1, \ldots, \lambda_n) : A \to A_1 \times \cdots \times A_n
$$

is a quasi-isomorphism.

Proof. Let’s write $\tilde{A}_i := \tilde{A}_{e_i} = H^0(A)_{e_i}$, so $\tilde{A} = \prod_{i=1}^n \tilde{A}_i$. Using Proposition 4.7(1) we obtain canonical graded ring isomorphisms

$$
H(A) \cong \prod_{i=1}^n (\tilde{A}_i \otimes \tilde{A}) \cong \prod_{i=1}^n H(A_i) \cong H(\prod_{i=1}^n A_i).
$$

The composition of these isomorphisms is exactly $H(\lambda)$. □

Corollary 4.17. With $A_1, \ldots, A_n$ as in Proposition 4.16, the restriction functors

$$
\text{rest}_{\lambda_i} : D(A_i) \to D(A)
$$

induce an equivalence of triangulated categories

$$
\prod_{i=1}^n D(A_i) \to D(A).
$$

Proof. This is standard. Cf. [YZ1, Proposition 1.4]. □

Definition 4.18. Let $A$ be a commutative DG ring, and let $e \in \tilde{A} = H^0(A)$ be an idempotent element. Consider the localized DG ring $A_e$ corresponding to $e$, as in Definition 4.6. The triangulated functor

$$
E : D(A) \to D(A), \quad E(M) := A_e \otimes_A M,
$$

is called the idempotent functor corresponding to $e$.

Definition 4.19. Let $A$ be a commutative DG ring, and assume that the scheme Spec $\tilde{A}$ has finitely many connected components.

1. The connected component decomposition of $\tilde{A}$ is the canonical ring isomorphism $\tilde{A} \cong \prod_{i=1}^n \tilde{A}_i$, such that each Spec $\tilde{A}_i$ is connected.
2. The corresponding idempotent covering sequence $e = (e_1, \ldots, e_n)$ of $\tilde{A}$ is called the connected component idempotent covering sequence.
3. The corresponding DG ring quasi-isomorphism $\lambda : A \to \prod_{i=1}^n A_i$ is called the connected component decomposition of $A$.
4. The corresponding functors $E_1, \ldots, E_n$ are called the connected component idempotent functors of $A$.

Example 4.20. Here are a few typical examples of a commutative ring $B$ whose prime spectrum has finitely many connected components:

1. $B$ is a noetherian ring.
2. $B$ is a local ring.
3. $B$ is the ring of continuous (resp. differentiable) functions $X \to \mathbb{R}$, where $X$ is a connected topological space (resp. a connected differentiable manifold).

Proposition 4.21. Suppose $(e_1, \ldots, e_n)$ is an idempotent covering sequence of $\tilde{A}$. Let $E_1, \ldots, E_n$ be the corresponding idempotent functors.

1. We have $E_i \circ E_j \cong E_i$, $E_i \circ E_j = 0$ for $i \neq j$, and $\sum_{i=1}^n E_i \cong 1_{D(A)}$.
2. Under the equivalence of categories in Corollary 4.17, $D(A_i)$ is the essential image of $E_i$. 
**Proposition 4.22.** Let $f : A \to B$ be a DG ring homomorphism, and let $M, N \in \mathcal{D}(B)$. We write $f := H^0(f)$ and $F := \text{rest}_f : \mathcal{D}(B) \to \mathcal{D}(A)$. Assume that the ring homomorphism $\bar{f} : \bar{A} \to \bar{B}$ is surjective. Let $(e_1, \ldots, e_n)$ be an idempotent covering sequence of $\bar{B}$, let $E_1, \ldots, E_n$ be the corresponding idempotent functors of $\mathcal{D}(B)$, and write $M_i := E_i(M)$ and $N_i := E_i(N)$. Then for any $i \neq j$ we have

$$\text{Hom}_{\mathcal{D}(A)}(F(M_i), F(N_j)) = 0.$$ 

Therefore we get a canonical isomorphism of $\bar{A}$-modules

$$\text{Hom}_{\mathcal{D}(A)}(F(M), F(N)) \cong \bigoplus_{i=1}^n \text{Hom}_{\mathcal{D}(A)}(F(M_i), F(N_i)).$$

**Proof.** For any $i$ choose some element $a_i \in \bar{A}$ such that $\bar{f}(a_i) = e_i$. Consider the noncommutative rings $\text{End}_{\mathcal{D}(B)}(M_i)$ and $\text{End}_{\mathcal{D}(A)}(F(M_i))$. There is a commutative diagram (of noncommutative rings)

$$
\begin{array}{c}
\bar{A} \\
\downarrow f
\end{array} \longrightarrow
\begin{array}{c}
\text{End}_{\mathcal{D}(A)}(F(M_i)) \\
\downarrow F
\end{array}
\begin{array}{c}
\bar{B} \\
\downarrow \bar{f}
\end{array} \longrightarrow
\begin{array}{c}
\text{End}_{\mathcal{D}(B)}(M_i)
\end{array}
$$

Cf. Proposition 4.5.

Take two distinct indices $i, j$. We know that $e_i \cdot 1_{M_i} = 1_{M_i}$ in $\text{End}_{\mathcal{D}(B)}(M_i)$, and $e_i \cdot 1_{M_j} = 0$ in $\text{End}_{\mathcal{D}(B)}(M_j)$. Therefore $a_i \cdot 1_{F(M_i)} = 1_{F(M_i)}$ in $\text{End}_{\mathcal{D}(A)}(F(M_i))$, and $a_i \cdot 1_{F(M_j)} = 0$ in $\text{End}_{\mathcal{D}(A)}(F(M_j))$.

Consider any morphism $\phi : F(M_i) \to F(N_j)$ in $\mathcal{D}(A)$. Then

$$\phi = 1_{F(M_j)} \circ \phi \circ 1_{F(M_i)} = 1_{F(M_j)} \circ \phi \circ (a_i \cdot 1_{F(M_i)}) = (a_i \cdot 1_{F(M_j)}) \circ \phi \circ 1_{F(M_i)} = 0.$$ 

□

## 5. Perfect DG Modules

Recall that all DG rings are now commutative by default (Convention 4.3). In particular all rings are commutative. For a DG ring $A$ we write $\bar{A} := H^0(A)$.

For a DG ring $A$ and an element $s \in \bar{A}$, the localization $A_s$ was defined in Definition 4.6. The notion of covering sequence of $\bar{A}$ was introduced in Definition 4.8, and finite semi-free DG modules were introduced in Definition 1.13.

For a ring $A$, a complex of $A$-modules $M$ is called a **perfect complex** if it is isomorphic in $\mathcal{D}(A) = \mathcal{D}(\text{Mod} A)$ to a bounded complex of finite projective $A$-modules. See [SGA 6] or [SP, Definition 15.59.1, tag 0657].

**Definition 5.1.** Let $A$ be a commutative DG ring, and let $M$ be a DG $A$-module. We say that $M$ is **perfect** if there is a covering sequence $s = (s_1, \ldots, s_n)$ of $\bar{A}$, and for every $i$ there is an isomorphism $A_{s_i} \otimes_A M \cong P_i$ in $\mathcal{D}(A_{s_i})$, for some finite semi-free DG $A_{s_i}$-module $P_i$.

The next proposition says that when $A$ is a ring, this definition agrees with the usual definition.
Proposition 5.2. Assume A is a commutative ring. A DG A-module M is perfect, in the sense of definition 5.1, iff it is isomorphic in D(A) to a bounded complex of finite projective A-modules.

Proof. This is standard; cf. [SP, Lemma 12.44.11, tag 066Y]. □

Proposition 5.3. Let A be a commutative DG ring, and let M be a perfect DG A-module.

1. If N is another DG A-module, and M ∼= N in D(A), then N is perfect too.
2. Let A → B be a homomorphism of DG rings. Then B ⊗_A M is a perfect DG B-module.

By part (1), it does not matter which representative DG B-module we choose for B ⊗_A M. Thus part (2) makes sense.

Proof. (1) If M ∼= N in D(A), then A_s ⊗_A M ∼= A_s ⊗_A N in D(A_s).

(2) Let f : A → B denote the DG ring homomorphism. Define t_i := f(s_i) and N := B ⊗_A M. Then (t_1, ..., t_n) is a covering sequence of B. Q_i := B_{t_i} ⊗_{A_{t_i}} P_i is a finite semi-free DG B_{t_i}-module, and B_{t_i} ⊗_B N ∼= Q_i in D(B_{t_i}). □

Lemma 5.4. Let M be a perfect DG A-module.

1. M belongs to D^-(A).
2. If A is cohomologically pseudo-noetherian, then M belongs to D^-_f(A).

Proof. Step 1. Assume M ∼= P, where P is a finite semi-free DG A-module. We know that A ∈ D^-(A), and that A ∈ D^-_f(A) in the cohomologically pseudo-noetherian case. Now P is obtained from A by finitely many shifts and cones, and hence P also belongs to D^-(A), and to D^-_f(A) in the cohomologically pseudo-noetherian case.

Step 2. Let s_1, ..., s_n ∈ A and P_i ∈ D(A_s_i) be as in Definition 5.1. We know that for each i, A_{s_i} ⊗_A H(M) ∼= H(P_i) as graded modules over A_{s_i}. Step 1 tells us that P_i ∈ D^-(A_{s_i}), and that P_i ∈ D^-_{f_i}(A_{s_i}) in the cohomologically pseudo-noetherian case. From the faithfully flat ring homomorphism A → A_{s_i}, we deduce that H(M) is bounded above (cf. Proposition 1.1). In the cohomologically pseudo-noetherian case, descent implies that each H^j(M) is finite over A. Cf. [SP, Lemma 15.52.14, tag 066D], noting that an A module is finite iff it is 0-pseudo-coherent. □

Let L, M, N ∈ D(A). There is a canonical morphism

(5.5) ψ_{L,M,N} : RHom_A(L, M) ⊗_A^L N → RHom_A(L, M ⊗_A^L N)

in D(A), which is functorial in the three arguments. If we choose a K-projective resolution L ∼= L, and a K-flat resolution N ∼= N, then the morphism ψ_{L,M,N} is represented by the homomorphism

(5.6) \tilde{\psi}_{L,M,N} : \text{Hom}_A(\tilde{L}, M) ⊗_A \tilde{N} → \text{Hom}_A(\tilde{L}, M ⊗_A \tilde{N})

in M(A), where

\tilde{\psi}_{L,M,N}(α ⊗ n)(l) := (-1)^{jk} \cdot α(l) ⊗ n

for α ∈ \text{Hom}_A(\tilde{L}, M)^i, n ∈ \tilde{N}^j and l ∈ \tilde{L}^k.
Lemma 5.7. Let $L, M, N \in \mathcal{D}(A)$, and assume $L$ is perfect. Then the morphism $\psi_{L,M,N}$ in formula (5.5) is an isomorphism.

**Proof.** Step 1. Assume $L \cong \tilde{L}$ in $\mathcal{D}(A)$, where $\tilde{L}$ is a finite semi-free DG $A$-module. Choose a K-flat resolution $\tilde{N} \cong N$. Then the homomorphism $\tilde{\psi}_{L,M,N}$ in (5.6) is in fact bijective.

Step 2. Let $s = (s_1, \ldots, s_n)$ be a covering sequence of $\tilde{A}$, and for every $i$ let us write $A_i := A_{s_i}$. We assume that there are isomorphisms $A_i \otimes_A L \cong \tilde{L}_i$ in $\mathcal{D}(A_i)$, such that $\tilde{L}_i$ is a finite semi-free DG $A_i$-module; cf. Definition 5.1. In this step we assume that $M \cong A_i \otimes_A M$ in $\mathcal{D}(A)$ for some index $i$. Let’s write $L_i := A_i \otimes_A L$, $M_i := A_i \otimes_A M$ and $N_i := A_i \otimes_A N$. Then, and using adjunction with respect to the homomorphism $A \to A_i$, we get isomorphisms

\[
\text{RHom}_A(L, M) \otimes^L_A N \cong \text{RHom}_A(L, M_i) \otimes^L_A N_i
\]

and

\[
\text{RHom}_A(L, M \otimes^L_A N) \cong \text{RHom}_A(L, M_i \otimes^L_A N_i)
\]

in $\mathcal{D}(A)$. By step 1, the morphism

\[
\psi_{L_i, M_i, N_i} : \text{RHom}_{A_i}(L_i, M_i) \otimes^L_{A_i} N_i \to \text{RHom}_{A_i}(L_i, M_i) \otimes^L_{A_i} N_i
\]

is an isomorphism.

Step 3. We keep the covering sequence $s = (s_1, \ldots, s_n)$ from step 2. Since the Čech resolution $c_N : M \to C(M; s)$ is a quasi-isomorphism (Proposition 4.13), it suffices to prove that $\psi_{L,M',N}$ is an isomorphism, where $M' := C(M; s)$.

The DG $A^0$-module $C(A^0; s)$ is filtered by degree:

\[
\mu^k(C(A^0; s)) := \bigoplus_{j \geq k} C^j(A^0; s).
\]

This is a decreasing filtration of finite length: $\mu^0(C(A^0; s)) = C(A^0; s)$, and $\mu^n(C(A^0; s)) = 0$. Now by definition

\[
C(M; s) = C(A^0; s) \otimes_{A^0} M,
\]

so we get an induced filtration of finite length $\{\mu^k(C(M; s))\}_{k \in \mathbb{Z}}$ on the DG module $C(M; s)$, with

\[
(5.8) \quad \mu^k(C(M; s)) := \mu^k(C(A^0; s)) \otimes_{A^0} M.
\]

For every $k$ the filtration gives rise to an exact sequence of DG $A$-modules, that becomes a distinguished triangle

\[
(5.9) \quad \mu^{k+1}(C(M; s)) \to \mu^k(C(M; s)) \to \gr^k_{\mu}(C(M; s)) \to 
\]

in $\mathcal{D}(A)$. Thus to prove that $\psi_{L,M',N}$ is an isomorphism, it suffices to prove that $\psi_{L,M'',N}$ is an isomorphism, where

\[
(5.10) \quad M'_k := \gr^k_{\mu}(C(M; s)) \cong C^k(A^0; s)[-k] \otimes_{A^0} M.
\]

But $M'_k$ is a finite direct sum of shifts of the DG modules

\[
M''_k := C(A^0; s)(i) \otimes_{A^0} M;
\]

see formula (4.10). Thus we reduce the problem to proving that $\psi_{L,M'',N}$ is an isomorphism. Because $M''_k$ satisfies the assumption in step 2, we are done. \qed
Theorem 5.11. Let $A$ be a commutative DG ring, and let $M$ be a DG $A$-module. The following two conditions are equivalent:

(i) $M$ is perfect.
(ii) $M$ belongs to $D^{-}(A)$, and the DG $\bar{A}$-module $\bar{A} \otimes_{A}^{L} M$ is perfect.

If $A$ is cohomologically pseudo-noetherian, then these two conditions are equivalent to:

(iii) $M$ is in $D_{f}^{-}(A)$, and it has finite projective dimension relative to $D(A)$.

When $A$ is a ring, i.e. $A = \bar{A}$, this is a standard result.

Proof. (i) $\Leftrightarrow$ (ii): According to Lemma 5.4 we have $M \in D^{-}(A)$. Write $\bar{M} := \bar{A} \otimes_{A}^{L} M$. Consider a covering sequence $(s_{1}, \ldots, s_{n})$ of $\bar{A}$. Let us write $A_{i} := A_{s_{i}}$, $M_{i} := A_{i} \otimes_{A} M$ and $A_{i} := \text{H}^{0}(A_{s_{i}})$. For every $i$ there is an isomorphism $\bar{A}_{i} \otimes_{A} M \cong A_{i} \otimes_{A}^{L} M_{i}$ in $D(A_{s_{i}})$. Using Proposition 3.3 we see that $M_{i}$ is isomorphic in $D(A_{i})$ to a finite semi-free DG $A_{i}$-module iff $\bar{A}_{i} \otimes_{A}^{L} M_{i}$ is isomorphic in $D(A_{i})$ to a finite semi-free DG $\bar{A}_{i}$-module.

(i) $\Rightarrow$ (iii): Here $A$ is cohomologically pseudo-noetherian. Lemma 5.4 says that $M \in D_{f}^{-}(A)$. To prove that $M$ has finite projective dimension relative to $D(A)$, we have to bound $\text{H}(\text{RHom}_{A}(M, N))$ in terms of $\text{H}(N)$ for any $N \in D(A)$. Choose a covering sequence $(s_{1}, \ldots, s_{n})$ of $\bar{A}$ and finite semi-free DG $A_{s_{i}}$-modules $P_{i}$ as in Definition 5.1 where $A_{i} := A_{s_{i}}$. Let $d_{0} \leq d_{1}$ be integers such that each $P_{i}$ is generated in the degree range $[d_{0}, d_{1}]$ (see Proposition 1.17). Using Lemma 5.7 for the isomorphism $\cong^{\dag}$, and adjunction, we obtain isomorphisms

\[
\text{A}_{i} \otimes_{A} \text{RHom}_{A}(M, N) \cong^{\dag} \text{RHom}_{A}(\text{A}_{i} \otimes_{A} M, \text{A}_{i} \otimes_{A} N) \cong \text{RHom}_{A_{i}}(\text{A}_{i} \otimes_{A} M, \text{A}_{i} \otimes_{A} N) \cong \text{Hom}_{A_{i}}(P_{i}, \text{A}_{i} \otimes_{A} N)
\]

in $D(A)$. This proves that

\[
\sup(\text{H}(\text{RHom}_{A}(M, N))) \leq -d_{0} + \sup(\text{H}(N))
\]

and

\[
\inf(\text{H}(\text{RHom}_{A}(M, N))) \geq d_{1} + \inf(\text{H}(N)).
\]

We conclude that the projective dimension of $M$ relative to $D(A)$ is $\leq d_{1} - d_{0}$.

(iii) $\Rightarrow$ (ii): Here again $A$ is cohomologically pseudo-noetherian. Let’s write $\bar{M} := \bar{A} \otimes_{A}^{L} M$. Because $M \in D_{f}^{-}(A)$, we can find a pseudo-finite semi-free resolution $P \to M$ in $M(A)$. Thus $\bar{M} \cong \bar{A} \otimes_{A} P$ belongs to $D_{f}^{-}(\bar{A})$.

For every $N \in D(\bar{A})$ we have, by adjunction,

\[
\text{RHom}_{A}(\bar{M}, N) \cong \text{RHom}_{A}(\bar{M}, N).
\]

This shows that the projective dimension of $\bar{M}$ relative to $D(\bar{A})$ is finite. But this just means that the complex $\bar{M}$ has finite projective dimension over the ring $\bar{A}$. In particular $\bar{M}$ belongs to $D_{f}^{-}(\bar{A})$. The usual syzygy argument shows that there is a quasi-isomorphism $\bar{P} \to \bar{M}$ in $\text{M}(\bar{A})$, for some bounded complex of finite projective $\bar{A}$-modules $\bar{P}$. But locally on $\text{Spec} \bar{A}$ each $\bar{P}^{i}$ is a free $\bar{A}$-module; and hence $\bar{M}$ is perfect.

Remark 5.12. Possibly we can remove the pseudo-noetherian hypothesis in condition (iii) of Theorem 5.11. The new condition would be this:
(iii') The DG module \(M\) is pseudo-coherent and has finite flat dimension relative to \(D^b(A)\).

This would require a detailed study of pseudo-coherent DG \(A\)-modules. Cf. [SP, Definition 15.59.1, tag 0657] and [SP, Lemma 15.59.2, tag 0658].

Recall that a DG \(A\)-module \(M\) is called a compact object of \(D(A)\) if for any collection \(\{N_z\}_{z \in Z}\) of DG \(A\)-modules, the canonical homomorphism

\[
\bigoplus_{z \in Z} \text{Hom}_{D(A)}(M, N_z) \rightarrow \text{Hom}_{D(A)}\left(M, \bigoplus_{z \in Z} N_z\right)
\]

is bijective. (In general this is only injective.) It is known that for a ring \(A\), compact and perfect are the same (see [RI, Section 6], [Ne, Example 1.13], or [SP, Proposition 12.45.3, tag 07LT]). It turns out that this is also true for a DG ring.

First we need to know that being compact is a local property on \(\text{Spec } \bar{A}\). This is very similar to arguments found in [Ne].

**Lemma 5.14.** Let \(A\) be a commutative DG ring, let \(M\) be a DG \(A\)-module, and let \((s_1, \ldots, s_n)\) be a covering sequence of \(\bar{A}\). The following conditions are equivalent.

(i) \(M\) is a compact object of \(D(A)\).

(ii) For every \(i\) the DG \(A_{s_i}\)-module \(A_{s_i} \otimes_A M\) is a compact object of \(D(A_{s_i})\).

**Proof.** (i) \(\Rightarrow\) (ii): This is the easy implication. We write \(A_i := A_{s_i}\) and \(M_i := A_i \otimes_A M\). Let \(F_i : D(A_i) \rightarrow D(A)\) be the restriction functor. It commutes with all direct sums. Given a collection \(\{N_z\}_{z \in Z}\) in \(D(A_i)\), we have canonical isomorphisms

\[
\text{Hom}_{D(A_i)}(M_i, \bigoplus_{z \in Z} N_z) \cong \text{Hom}_{D(A)}(M, F_i(\bigoplus_{z \in Z} N_z))
\]

\[
\cong \text{Hom}_{D(A)}(M, \bigoplus_{z \in Z} F_i(N_z)) \cong \bigoplus_{z \in Z} \text{Hom}_{D(A)}(M, F_i(N_z))
\]

\[
\cong \bigoplus_{z \in Z} \text{Hom}_{D(A)}(M_i, N_z).
\]

We use the adjunction for \(F_i\) and the fact that \(M\) is compact. The conclusion is that \(M_i\) is compact.

(ii) \(\Rightarrow\) (i): For any DG \(A\)-module \(N\) we have the Čech resolution \(c_N : N \rightarrow C(N; s)\) from Definition [4.12] and Proposition [4.13]. Because

\[
C(N; s) = C(A^0; s) \otimes_{A^0} N \cong C(A; s) \otimes_A N,
\]

this functor commutes with all direct sums. Thus the canonical homomorphism

\[
\bigoplus_{z \in Z} C(N_z; s) \rightarrow C\left(\bigoplus_{z \in Z} N_z; s\right)
\]

is an isomorphism in \(M(A)\). Using (5.15) we obtain a commutative diagram of \(A\)-modules

\[
\bigoplus_{z} \text{Hom}_{D(A)}(M, N_z) \quad \quad \quad \quad \text{Hom}_{D(A)}(M, \bigoplus_{z} N_z)
\]

\[
\cong \bigoplus_{z} \text{Hom}_{D(A)}(M, C(N_z; s)) \quad \quad \quad \text{Hom}_{D(A)}(M, \bigoplus_{z} C(N_z; s))
\]

where the vertical arrows are bijections. So it suffices to prove that the lower horizontal arrow is a bijection.
Consider the finite length filtration \( \{ \mu^k(C(N_z; s)) \}_{k \in \mathbb{Z}} \) on the DG module \( C(N_z; s) \), as in formula (5.8). Passing to the associated distinguished triangles, and using induction on \( k \), as was done in the proof of Lemma 5.7 we reduce the problem to the verification that

\[
(5.18) \quad \bigoplus_z \text{Hom}_{D(A)}(M, A_i \otimes_A N_z) \to \text{Hom}_{D(A)} \left( M, \bigoplus_z A_i \otimes_A N_z \right)
\]

is a bijection, where

\[
A_i = A_{s_{i_0}}^0 \otimes_A \cdots \otimes_A A_{s_{i_k}}^0
\]

for some strictly increasing sequence \( i = (i_0, \ldots, i_k) \) in the integer interval \([1, n]\). Let’s write \( A' := A_{s_{i_0}} \). Adjunction for the DG ring homomorphism \( A \to A' \) allows us to replace (5.17) with the homomorphism

\[
(5.17) \quad \bigoplus_z \text{Hom}_{D(A')}(A' \otimes_M A, A_i \otimes_A N_z) \to \text{Hom}_{D(A')} \left( A' \otimes_M A, \bigoplus_z A_i \otimes_A N_z \right).
\]

But we are assuming that \( A' \otimes_M A \) is compact in \( D(A') \); so (5.18) is bijective. \( \square \)

**Lemma 5.19.** If \( M \) is a compact object of \( D(A) \), then it belongs to \( D^- (A) \).

**Proof.** This is an argument from [Ri], slightly improved in the proof of [Sp, Proposition 12.45.3, tag 07LT].

Suppose \( \{ N_z \}_{z \in \mathbb{Z}} \) is a collection of DG \( A \)-modules. Given a morphism \( \psi : M \to \bigoplus_{z \in \mathbb{Z}} N_z \) in \( D(A) \), there is a finite subset \( Z_0 \subset \mathbb{Z} \) such that \( \psi \) factors through \( \bigoplus_{z \in Z_0} N_z \). So for any \( z \notin Z_0 \), the component \( \psi_z : M \to N_z \) of \( \psi \) is zero.

For every \( k \geq 0 \) consider the smart truncation \( \text{smt}^{\geq k}(M) \) from (1.4). There is a canonical surjective homomorphism \( \phi_k : M \to \text{smt}^{\geq k}(M) \) in \( M(A) \), and we know that \( H^l(\phi_k) \) is an isomorphism for all \( l \geq k \). Consider the homomorphism

\[
\phi : M \to \bigoplus_{k \in \mathbb{N}} \text{smt}^{\geq k}(M), \quad \phi := \sum \phi_k
\]

in \( M(A) \). Let \( \psi := \text{Q}(\phi) \); so the \( k \)-th component of \( \psi \) is

\[
\psi_k := \text{Q}(\phi_k) : M \to \text{smt}^{\geq k}(M).
\]

As explained in the paragraph above, there is an integer \( k_0 \) such that \( \psi_{k_0 + 1} = 0 \). Therefore

\[
H^l(\psi_{k_0 + 1}) = H^l(\phi_{k_0 + 1}) : H^l(M) \to H^l(\text{smt}^{\geq k_0 + 1}(M))
\]

is zero for all \( l \). We see that \( H^l(M) = 0 \) for all \( l \geq k_0 + 1 \). \( \square \)

**Theorem 5.20.** Let \( A \) be a commutative DG ring, and let \( L \) be a DG \( A \)-module. The following three conditions are equivalent:

(i) \( L \) is a perfect DG \( A \)-module.

(ii) \( L \) is a compact object of \( D(A) \).

(iii) For any \( M, N \in D(A) \), the canonical morphism

\[
\psi_{L,M,N} : \text{RHom}_A(L, M) \otimes_A^L N \to \text{RHom}_A(L, M \otimes_A^L N)
\]

in \( D(A) \) is an isomorphism.

See the text just after formula (5.5) for a description of the morphism \( \psi_{L,M,N} \) in condition (iii).
Proof. (i) ⇒ (ii): Since a finite semi-free DG module is clearly compact, this follows from Lemma 5.14.

(ii) ⇒ (i): Assume L is compact in D(A). Consider the DG A-module \( \bar{L} := \hat{A} \otimes_A^L L \). Adjunction shows that

\[
\text{Hom}_{D(A)}(\bar{L}, M) \cong \text{Hom}_{D(A)}(L, F(M))
\]

functorially for \( M \in D(\bar{A}) \). Here \( F \) is the forgetful functor, that commutes with all direct sums. Thus \( \bar{L} \) is a compact object of \( D(\bar{A}) \). Now by [Ri, Section 6], [Ne, Example 1.13] or [SP, Proposition 15.60.3, tag 07LT] the DG \( \bar{A} \)-module \( \bar{L} \) is perfect, in the sense that it is quasi-isomorphic to a bounded complex of finite projective \( \bar{A} \)-modules \( \bar{P} \). But locally on \( \text{Spec} \bar{A} \) each \( \bar{P} \) is a free module. Thus \( \bar{L} \) is perfect in the sense of Definition 5.1. By the lemma above we know that \( L \in D^-(A) \). The implication (ii) ⇒ (i) in Theorem 5.11 says that \( L \) is perfect.

(i) ⇒ (iii): This is Lemma 5.7.

(iii) ⇒ (ii): Take any collection of DG A-modules \( \{N_z\}_{z \in \mathbb{Z}} \), and define \( N := \bigoplus_{z \in \mathbb{Z}} N_z \). By assumption, the any \( z \) the morphism \( \psi_{L, A, N} : \text{RHom}_A(L, A) \otimes_A^L N_z \rightarrow \text{RHom}_A(L, N_z) \) is an isomorphism. Since derived tensor products commute with all direct sums, we get an isomorphism

\[
\phi : \text{RHom}_A(L, A) \otimes_A^L N \xrightarrow{\cong} \bigoplus_{z \in \mathbb{Z}} \text{RHom}_A(L, N_z).
\]

Now the functor \( H^0 \) also commutes with all direct sums. So we get a commutative diagram of \( A \)-modules

\[
\begin{array}{ccc}
H^0(\text{RHom}_A(L, A) \otimes_A^L N) & \xrightarrow{H^0(\psi_{L, A, N})} & H^0(\text{RHom}_A(L, N)) \\
\bigoplus_{z \in \mathbb{Z}} \text{Hom}_{D(A)}(L, N_z) & \xrightarrow{\text{can}} & \text{Hom}_{D(A)}(L, N)
\end{array}
\]

in which the vertical arrows are isomorphisms. Our assumption says that \( H^0(\psi_{L, A, N}) \) is an isomorphism. Therefore the bottom arrow (marked “can”) is an isomorphism too. But this is the morphism (5.13). \( \square \)

6. Tilting DG Modules

Recall that all DG rings here are commutative (Convention 4.3). In particular all rings are commutative.

Definition 6.1. Let \( A \) be a commutative DG ring. A DG \( A \)-module \( P \) is called a tilting DG module if there exists some DG \( A \)-module \( Q \) such that \( P \otimes_A^L Q \cong A \) in \( D(A) \).

The DG module \( Q \) in the definition is called a quasi-inverse of \( P \). Due to symmetry of the operation \( - \otimes_A^L - \), \( Q \) is also tilting. If \( P_1 \) and \( P_2 \) are tilting then so is \( P_1 \otimes_A^L P_2 \); this because of the associativity of \( - \otimes_A^L - \). Hence the next definition makes sense.
**Definition 6.2.** The commutative derived Picard group of $A$ is the abelian group $\text{DPic}(A)$, whose elements are the isomorphism classes, in $D(A)$, of tilting DG $A$-modules. The product is induced by the operation $-\otimes^L_A -$, and the unit element is the class of $A$.

For a ring $B$ we have the following result, due (independently) to the author and Rouquier-Zimmermann. As usual, $\text{Pic}(B)$ denotes the (commutative) Picard group of $A$.

**Theorem 6.3 ([RZ], [Ye2] Proposition 3.5]).** Let $B$ be a commutative ring, with connected component decomposition $B = \prod_{i=1}^n B_i$. Then there is a canonical group isomorphism

$$\text{DPic}(B) \cong \text{Pic}(B) \times \mathbb{Z}^n,$$

characterized as follows: the homomorphism $\text{Pic}(B) \to \text{DPic}(B)$ comes from the inclusion $\text{Mod} B \to D(B)$, and the homomorphism $\mathbb{Z}^n \to \text{DPic}(B)$ sends $(i_1, \ldots, i_n)$ to the class of the tilting DG module $B[i_1] \times \cdots \times B[n]$.

**Remark 6.4.** To keep strict adherence to the notation of [Ye2], we should write $\text{DPic}_B(B)$, to indicate that we are looking at $B$-central two-sided tilting DG $B$-modules.

The proof of [Ye2] Proposition 3.5] relies on [Ye2] Theorem 2.6], and the proof of the latter has an implicit assumption that the ring $B$ is noetherian. A correct proof, for any commutative ring whose spectrum has finitely many connected components, is in [Ye4] Theorem 1.9).

**Lemma 6.5.** Let $f : A \to B$ be a homomorphism of DG rings.

1. For any $M, N \in D(A)$ there is an isomorphism

$$(B \otimes^L_A M) \otimes^L_B (B \otimes^L_A N) \cong B \otimes^L_A (M \otimes^L_A N)$$

in $D(A)$.

2. If $P$ is a tilting DG $A$-module, then $B \otimes^L_A P$ is a tilting DG $B$-module.

3. If $f$ is a quasi-isomorphism and $Q$ is a tilting DG $B$-module, then $\text{rest}_f(Q)$ is a tilting DG $A$-module.

**Proof.** (1) Choose K-flat resolutions $\tilde{M} \to M$ and $\tilde{N} \to N$ over $A$. Then $B \otimes_A \tilde{M}$ and $B \otimes_A \tilde{N}$ are K-flat over $B$, $\tilde{M} \otimes_A \tilde{N}$ is K-flat over $A$, and

$$(B \otimes^L_A M) \otimes^L_B (B \otimes^L_A N) \cong (B \otimes_A \tilde{M}) \otimes_B (B \otimes_A \tilde{N})$$

$$\cong B \otimes_A (\tilde{M} \otimes_A \tilde{N}) \cong B \otimes^L_A (M \otimes^L_A N).$$

(2) Let $P, Q \in D(A)$ be such that $P \otimes^L_A Q \cong A$. By (1) we have

$$(B \otimes^L_A P) \otimes^L_B (B \otimes^L_A Q) \cong B.$$

(3) Say $Q_1, Q_2 \in D(B)$ satisfy $Q_1 \otimes^L_B Q_2 \cong B$. Let $P_i := \text{rest}_f(Q_i) \in D(A)$. By the equivalence for DG algebra quasi-isomorphisms (see [YZ1 Proposition 1.4]) we have

$$P_1 \otimes^L_A P_2 \cong \text{rest}_f(Q_1 \otimes^L_B Q_2) \cong \text{rest}_f(B) \cong A.$$

**Proposition 6.6.** Let $f : A \to B$ be a homomorphism of commutative DG rings.
(1) There is a group homomorphism
\[ \text{DPic}(f) : \text{DPic}(A) \to \text{DPic}(B) \]
with formula \( P \mapsto B \otimes_A L_A P \).

(2) If \( f \) is a quasi-isomorphism then \( \text{DPic}(f) \) is bijective.

Proof. (1) This follows from parts (1) and (2) of the lemma above.

(2) Part (3) of Lemma 6.5 shows that in case \( f \) is a quasi-isomorphism, the function \( Q \mapsto \text{rest}_f(Q) \) is an inverse of \( \text{DPic}(f) \). □

**Theorem 6.7.** Let \( A \) be a commutative DG ring and \( P \in \text{D}(A) \). The following three conditions are equivalent.

(i) \( P \) is a tilting DG \( A \)-module.

(ii) The functor \( P \otimes_A - \) is an equivalence of \( \text{D}(A) \).

(iii) \( P \) is a perfect DG \( A \)-module, and the adjunction morphism \( A \to \text{RHom}_A(P,P) \) in \( \text{D}(A) \) is an isomorphism.

Proof. (i) \( \Rightarrow \) (ii): Let \( Q \) be a quasi-inverse of \( P \). Then the functor \( G(M) := Q \otimes_A M \) is a quasi-inverse of the functor \( F(M) := P \otimes_A M \).

(ii) \( \Rightarrow \) (i): The functor \( F := P \otimes_A - \) is essentially surjective on objects, so there is some \( Q \in \text{D}(A) \) such that \( F(Q) \cong A \). Then \( Q \) is a quasi-inverse of \( P \).

(ii) \( \Rightarrow \) (iii): Consider the auto-equivalence \( F := P \otimes_A - \in \text{D}(A) \). Since \( A \) is compact and \( P = F(A) \), it follows that \( P \) is compact. Now according to Theorem 5.20, perfect is the same as compact.

For any \( M \in \text{D}(A) \), the adjunction morphism \( A \to \text{RHom}_A(M,M) \) is an isomorphism iff the canonical graded ring homomorphism
\[ c_M : \text{H}(A) \to \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\text{D}(A)}(M,M[k]) \]
is bijective. The equivalence \( F \) induces a commutative diagram of graded rings
\[
\begin{array}{ccc}
\text{H}(A) & \xrightarrow{c_M} & \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\text{D}(A)}(M,M[k]) \\
\downarrow & & \downarrow \text{c}_{F(M)} \\
\bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\text{D}(A)}(F(M),F(M)[k]) & \xrightarrow{F} & \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\text{D}(A)}(F(M),F(M)[k])
\end{array}
\]
in which the horizontal arrow is an isomorphism. Now take \( M := A \). Because \( c_A \) is an isomorphism, so is \( c_P \).

(iii) \( \Rightarrow \) (i): Define \( Q := \text{RHom}_A(P,A) \in \text{D}(A) \). The implication (i) \( \Rightarrow \) (iii) of Theorem 5.20 shows that
\[ Q \otimes_A P = \text{RHom}_A(P,A) \otimes_A P \cong \text{RHom}_A(P,P) \cong A \]
in \( \text{D}(A) \). So \( P \) is tilting, with quasi-inverse \( Q \). □

**Corollary 6.8.** Let \( P \) be a tilting DG \( A \)-module. Then the DG module \( Q := \text{RHom}_A(P,A) \) is the quasi-inverse of \( P \).

Proof. This was shown at the end of the proof of the theorem; it relies on Theorem 5.20. □
Corollary 6.9. Let $P$ be a tilting DG $A$-module.

1. The functor $P \otimes A^! -$ is an equivalence of $D(A)$, it has finite cohomological dimension, and it preserves $D^+(A)$, $D^-(A)$ and $D^b(A)$.

2. If $A$ is cohomologically pseudo-noetherian, then the auto-equivalence $P \otimes A^! -$ preserves the subcategory $D_f(A)$.

Proof. (1) The theorem says that $P \otimes A^! -$ is an equivalence, and that $P$ is perfect. Let $s_1, \ldots, s_n \in \bar{A}$ and $P_1, \ldots, P_n$ be as in Definition 5.1. Let $d_0 \leq d_1$ be integers such that each $P_i$ is generated in the degree range $[d_0, d_1]$. Then the functor $F := P \otimes A^! -$ has cohomological displacement at most $[d_0, d_1]$ relative to $D(A)$. The claim about $D^+(A)$ is now clear.

(2) Use Theorem 2.12(2), noting that $F(A) = P \in D^-_f(A)$, by Theorems 6.7 and 6.11.

Proposition 6.10. Let $A$ be a commutative DG ring, and let $(e_1, \ldots, e_n)$ be an idempotent covering sequence of $\bar{A} = \Pi^0(A)$. For any $i$ we have the localized DG ring $A_i := A_{e_i}$ as in Definition 4.6, and the DG ring homomorphism $\lambda_i : A \to A_i$. The group homomorphisms

$$DPic(\lambda_i) : DPic(A) \to DPic(A_i)$$

induce a group isomorphism

$$DPic(A) \cong \prod_{i=1}^n DPic(A_i).$$

Proof. (1) According to Proposition 4.16 there is a DG ring quasi-isomorphism $\lambda : A \to \prod_{i=1}^n A_i$. By Proposition 6.6(2) there is a group isomorphism

$$DPic(\lambda) : DPic(A) \cong DPic\left( \prod_{i=1}^n A_i \right).$$

And there is an obvious group isomorphism

$$DPic\left( \prod_{i=1}^n A_i \right) \cong \prod_{i=1}^n DPic(A_i).$$

□

Here is a result about Picard groups, that is probably not new, yet we could not find it in the literature.

Proposition 6.11. Consider a commutative ring $B$, with a nilpotent ideal $b$, and let $\bar{B} := B/b^{j+1}$. Then the group homomorphism $Pic(B) \to Pic(\bar{B})$ is bijective.

Proof. For any $j \in \mathbb{N}$ let $B_j := B/b^{j+1}$. So $B_0 = \bar{B}$, and $B_j = B$ for $j \gg 0$. We will prove that the group homomorphism $Pic(B_{j+1}) \to Pic(B_j)$ is bijective for every $j$. The group homomorphism $Pic(B_{j+1}) \to Pic(B_j)$ is bijective for every $j$.

On the topological space $X := \text{Spec} B_0$ we have, for every $j$, a sheaf of rings $B_j$, that is the sheafification of the ring $B_j$. More precisely, if we let $X_j := \text{Spec} B_j$, then $B_j = \mathcal{O}_{X_j}$, and the canonical map $X_j \to X_0 = X$ is a homeomorphism of topological spaces. There is a canonical group isomorphism $Pic(B_j) \cong H^1(X, B_j^\times)$.

Let $I_j := \text{Ker}(B_{j+1} \to B_j)$. The short exact sequence of sheaves of abelian groups

$$0 \to I_j \to B_{j+1}^\times \to B_j^\times \to 0$$
on $X$ gives rise to an exact sequence of abelian groups

$$H^1(X, I_j) \to H^1(X, B^X_{j+1}) \to H^1(X, B^X_j) \to H^2(X, I_j).$$

But $I_j$ is a quasi-coherent $B_j$-module, so $H^q(X, I_j) = 0$ for all $q > 0$. Thus the homomorphism $H^1(X, B^X_{j+1}) \to H^1(X, B^X_j)$ is bijective. \hfill \Box

The next theorem is an analogue of Proposition 6.11. Recall the canonical DG ring homomorphism $A \to \breve{A}$.

**Theorem 6.12.** Let $A$ be a commutative DG ring, such that $\text{Spec } \breve{A}$ has finitely many connected components. Then the canonical group homomorphism

$$DPic(A) \to DPic(\breve{A})$$

is bijective.

**Proof.** Let us denote by $\pi : A \to \breve{A}$ the canonical DG ring homomorphism. We begin by proving that the homomorphism $DPic(\pi)$ is injective. Suppose $P$ is a tilting DG $A$-module such that $\breve{A} \otimes_A^L P \cong A$ in $D(\breve{A})$. By Corollary 6.9(1) we know that $P \in D^-(A)$. Then Proposition 3.3(1) says that $P \cong A$ in $D(\breve{A})$.

Now let us prove that $DPic(\pi)$ is surjective. Using Proposition 6.10 we may assume that $\text{Spec } \breve{A}$ is connected. By Theorem 6.3 any element of $DPic(\breve{A})$ is represented by $P[i]$ for some rank 1 projective $\breve{A}$-module $P$ and some integer $i$. By Proposition 3.5 there exists a DG $A$-module $P$, whose cohomology is nonpositive, and $\breve{A} \otimes_A^L P \cong P$ in $D(\breve{A})$. Let $Q \in \text{Mod } \breve{A}$ be a quasi-inverse of $P$. By the same reasoning, there exists $Q \in D^-(A)$ such that $\breve{A} \otimes_A^L Q \cong Q$ in $D(\breve{A})$. Now by Lemma 6.5(1) we have

$$\breve{A} \otimes_A^L (P \otimes_A^L Q) \cong P \otimes_A^L Q \cong \breve{A}$$

in $D(\breve{A})$. Therefore, by Proposition 3.3(1) we know that $P \otimes_A^L Q \cong A$ in $D(\breve{A})$. This shows that $P$ is a tilting DG $A$-module. And clearly the image of the class $P[i]$ under $DPic(\pi)$ is the class of $P[i]$ in $DPic(\breve{A})$. \hfill \Box

**Corollary 6.13.** In the situation of Theorem 6.12, let $n$ be the number of connected component of $\text{Spec } \breve{A}$. Then there is a canonical group isomorphism

$$DPic(A) \cong \mathbb{Z}^n \times Pic(\breve{A}).$$

**Proof.** Combine Theorems 6.12 and 6.3. \hfill \Box

**Definition 6.14.** Let $A$ be a commutative DG ring, such that $\text{Spec } \breve{A}$ has finitely many connected components. We define $DPic^0(A)$ to be the subgroup of $DPic(A)$ that corresponds to $Pic(\breve{A})$, under the canonical group isomorphism of Corollary 6.13.

**Corollary 6.15.** If $\breve{A}$ is a local ring, then $DPic(A) \cong \mathbb{Z}$.

**Proof.** We know that $Pic(\breve{A})$ is trivial, and $\text{Spec } \breve{A}$ is connected. Now use Corollary 6.13. \hfill \Box

**Remark 6.16.** Proposition 6.11 has an alternative proof, that is analogous to that of Theorem 6.12; injectivity of the group homomorphism $Pic(B) \to Pic(\breve{B})$ can be proved by the Nakayama Lemma; and surjectivity can be proved using lifting of idempotents.

However, we do not know a cocycle description of the group $DPic(A)$, so we cannot produce a cohomological proof of Theorem 6.12 that mimics the proof of...
Proposition 6.11. Such an approach might be possible in the context of derived algebraic geometry, where cocycles should involve infinity categories in some way.

7. Dualizing DG Modules

Recall that a commutative DG ring is, according to Convention 4.3, nonpositive and strongly commutative. In this section we only look at cohomologically pseudo-noetherian commutative DG rings (Definition 1.18). For a DG ring $A$, we write $\overline{A} := \text{H}(A)$.

Here is a generalization of Grothendieck’s definition of dualizing complex. When $A$ is a ring, this is identical to the definition in [RD, Section V.2].

**Definition 7.1.** Let $A$ be a cohomologically pseudo-noetherian commutative DG ring. A DG $A$-module $R$ is called a dualizing DG $A$-module if it satisfies these three conditions:

(i) $\text{H}(R)$ is bounded below, and each $\text{H}^i(R)$ is a finite $\overline{A}$-module.

(ii) $R$ has finite injective dimension relative to $\text{D}(A)$.

(iii) The adjunction morphism $A \rightarrow \text{RHom}_A(R, R)$ in $\text{D}(A)$ is an isomorphism.

In other words, condition (i) says that $R \in \text{D}^{+, f}(A)$; condition (ii) says that the functor $\text{RHom}_A(\cdot, R)$ has finite cohomological dimension relative to $\text{D}(A)$, as in Definitions 2.1 and 2.4(2); and condition (iii) says that the canonical graded ring homomorphism

$$\text{H}(A) \rightarrow \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\text{D}(A)}(R, R[k])$$

is bijective.

**Proposition 7.2.** Suppose $R$ is a dualizing DG module over $A$. Let $\text{D}^*$ denote either $\text{D}$, $\text{D}^b$, $\text{D}^+$ or $\text{D}^-$; and correspondingly let $\text{D}^{-*}$ denote either $\text{D}$, $\text{D}^b$, $\text{D}^-$ or $\text{D}^+$.

1. For any $M \in \text{D}^+(A)$ we have $\text{RHom}_A(M, R) \in \text{D}^{-, f}(A)$.
2. For any $M \in \text{D}(A)$ the adjunction morphism

$$M \rightarrow \text{RHom}_A(\text{RHom}_A(M, R), R)$$

in $\text{D}(A)$ is an isomorphism.
3. The functor

$$\text{RHom}_A(\cdot, R) : \text{D}^+(A)^{\text{op}} \rightarrow \text{D}^{-*}(A)$$

is an equivalence of triangulated categories.

**Proof.** (1) The functor $F := \text{RHom}_A(\cdot, R)$ from $\text{D}(A)$ to itself has finite cohomological dimension, and $F(A) \cong R \in \text{D}^+(A)$. So we can apply Theorem 2.12(2).

(2) There is a morphism of triangulated functors $\eta : 1_{\text{D}(A)} \rightarrow F \circ F$. Both functors have finite cohomological dimensions, and $\eta_A$ is an isomorphism. We can apply Theorem 2.10(2).

(3) Combine items (1) and (2). \qed

**Corollary 7.3.** Let $A$ be a cohomologically pseudo-noetherian commutative DG ring, with dualizing DG module $R$. The following two conditions are equivalent:

(i) The DG ring $A$ is cohomologically bounded.

(ii) The DG module $R$ is cohomologically bounded.
Proof. This is by Proposition 7.2(1), since $R \cong \text{RHom}_A(A, R)$ and $A \cong \text{RHom}_A(R, R)$. \hfill \Box

In Example 7.25 we demonstrate the unbounded option.

Given a homomorphism $f : A \to B$ of commutative DG rings, we denote by $\bar{f} := H^0(f)$ the induced ring homomorphism.

**Definition 7.4.** Let $f : A \to B$ be a homomorphism between cohomologically pseudo-noetherian commutative DG rings. We say that $f$ is a cohomologically pseudo-finite homomorphism if $\bar{f} : \bar{A} \to \bar{B}$ is a finite ring homomorphism, i.e. $\bar{f}$ makes $\bar{B}$ into a finite $\bar{A}$-module.

Clearly if $f$ is cohomologically finite, then rest makes $D^\ast_f(B)$ into $D^\ast_f(A)$, where $\ast$ is either $+,-,b$ or blank.

**Proposition 7.5.** Let $f : A \to B$ be a cohomologically pseudo-finite homomorphism between cohomologically pseudo-noetherian DG rings.

1. If $R_A$ is a dualizing DG $A$-module, then $R_B := \text{RHom}_A(B, R_A)$ is a dualizing DG $B$-module.
2. If $f$ is a quasi-isomorphism and $R_B$ is a dualizing DG $B$-module, then $R_A := \text{rest}_f(R_B)$ is a dualizing DG $A$-module.

Proof. (1) The proof is almost the same as in the case of a ring; see [RD, Proposition V.2.4]. Viewing $B$ as an object of $D^-(f)(A)$, Proposition 7.2(1) tells us that $R_B \in D^+_f(A)$; and hence $R_B \in D^+_f(B)$. For any $N \in \text{D}(B)$ we have

$$\text{RHom}_B(N, R_B) \cong \text{RHom}_A(N, R_A)$$

by adjunction, and hence the injective dimension of $R_B$ relative to $\text{D}(B)$ is at most the injective dimension of $A$ relative to $\text{D}(A)$, which is finite. And finally

$$\text{RHom}_B(R_B, R_B) \cong \text{RHom}_A(\text{RHom}_A(B, R_A), R_A) \cong B$$

by Proposition 7.2(2). These isomorphisms are actually inside $\text{D}(B)$, and they are compatible with the canonical morphism from $B$.

(2) This is because here $\text{rest}_f : \text{D}(B) \to \text{D}(A)$ is an equivalence, preserving boundedness and finiteness of cohomology. \hfill \Box

In commutative ring theory, many good properties of a ring $A$ can be deduced if it is “not far” from a “nice” ring $\mathbb{K}$ (such as a field). This is the underlying reason for the next definition.

Recall that a ring homomorphism $A \to B$ is called essentially finite type if it can be factored as $A \to B_\text{ft} \to B$, where $A \to B_\text{ft}$ is finite type (i.e. $B$ is finitely generated as $A$-ring), and $B_\text{ft} \to B$ is a localization.

**Definition 7.6.** A cohomologically pseudo-noetherian commutative DG ring $A$ is called tractable if there is a homomorphism $\mathbb{K} \to A$ from a finite dimensional regular noetherian ring $\mathbb{K}$, such that $\mathbb{K} \to \hat{A}$ is essentially finite type.

**Lemma 7.7.** Let $A$ be a cohomologically noetherian commutative DG ring, let $\mathbb{K}$ be a noetherian commutative ring, and suppose there is a homomorphism $\mathbb{K} \to A$
such that the induced homomorphism $\mathbb{K} \to \bar{A}$ is essentially finite type. Then there is a commutative diagram of commutative DG rings

\[
\begin{array}{ccc}
\mathbb{K} & \xrightarrow{\pi} & \bar{A} \\
\downarrow & & \downarrow \\
A_{\text{eft}} & \xrightarrow{g} & A_{\text{loc}}
\end{array}
\]

such that $\pi$ is the canonical homomorphism; $f$ and $g$ are quasi-isomorphisms; and $\mathbb{K} \to A_{\text{eft}}^0$ is essentially finite type.

**Proof.** Let $S := \pi^{-1}(\bar{A}^\times) \cap A^0$, namely $s \in S$ iff $\pi(s)$ is invertible in the ring $\bar{A}$. Define the DG ring $A_{\text{loc}} := (S^{-1} \cdot A^0) \otimes_{A^0} A$. Then $\pi$ factors via $f$, and $f$ is a quasi-isomorphism.

Since $\mathbb{K} \to A$ is essentially finite type, there is a polynomial ring $\mathbb{K}[t]$ in finitely many variables of degree 0, and a homomorphism $h : \mathbb{K}[t] \to \bar{A}$ which is essentially surjective, i.e. it is surjective after a localization. Thus, letting $T := h^{-1}(\bar{A}^\times) \subset \mathbb{K}[t]$, and defining $A_{\text{eft}}^0 := T^{-1} \cdot \mathbb{K}[t]$, the homomorphism $h_T : A_{\text{eft}}^0 \to \bar{A}$ is surjective.

Now the ring $A_{\text{eft}}^0$ is noetherian. The homomorphism $h_T : A_{\text{eft}}^0 \to \bar{A}$ factors via a homomorphism $g^0 : A_{\text{eft}}^0 \to A_{\text{loc}}^0$, and the composed homomorphism $A_{\text{eft}}^0 \to H^0(A_{\text{loc}})$ is surjective. Since the the modules $H^i(A_{\text{loc}})$ are finite over $A_{\text{eft}}^0$, we can extend $A_{\text{eft}}^0$ to a DG ring $A_{\text{eft}}$, and simultaneously extend $g^0$ to a quasi-isomorphism $g : A_{\text{eft}} \to A_{\text{loc}}$, by inductively introducing finitely many new variables (free ring generators) in negative degrees. The process is the same as in the proof of [YZ1, Proposition 1.7(2)].

**Theorem 7.8.** Let $A$ be a tractable cohomologically pseudo-noetherian commutative DG ring. Then $A$ has a dualizing DG module.

**Proof.** Consider the diagram of homomorphisms in Lemma 7.7. Since $A \to A_{\text{loc}}$ is a quasi-isomorphism, and $A_{\text{eft}}^0 \to A_{\text{eft}} \to A_{\text{loc}}$ are cohomologically pseudo-finite, it suffices (by Proposition 7.5) to show that $A_{\text{eft}}^0$ has a dualizing DG module (which is the same as a dualizing complex over this ring, in the sense of [RD]). But the ring homomorphism $\mathbb{K} \to A_{\text{eft}}^0$ can be factored into $\mathbb{K} \to \mathbb{K}[t] \to B \to A_{\text{eft}}^0$, where $\mathbb{K}[t]$ is a polynomial ring in $n$ variables, $\mathbb{K}[t] \to B$ is surjective, and $B \to A_{\text{eft}}^0$ is a localization. Thus, using [RD, Theorem V.8.3] and Proposition 7.5(1) above, the DG module

$$A_{\text{eft}}^0 \otimes_B R\text{Hom}_{\mathbb{K}[t]}(B, \Omega_{\mathbb{K}[t]/\mathbb{K}}[n])$$

is a dualizing DG module over $A_{\text{eft}}^0$.

**Theorem 7.9.** Let $A$ be a cohomologically pseudo-noetherian commutative DG ring, and let $R$ be a dualizing DG module over $A$.

1. If $P$ is a tilting DG module, then $P \otimes_{\mathbb{A}}^L R$ is a dualizing DG module.
2. If $R'$ is a dualizing DG module, then $P := R\text{Hom}_{\mathbb{A}}(R, R')$ is a tilting DG module, and $R' \cong P \otimes_{\mathbb{A}}^L R$ in $\text{D}(A)$.
3. If $P$ is a tilting DG module, and if $R \cong P \otimes_{\mathbb{A}}^L R$ in $\text{D}(A)$, then $P \cong A$ in $\text{D}(A)$.

The strategy of the proof is the same as in [RD].

**Proof.** (1) Assume $P$ is a tilting DG module, and let $R' := P \otimes_{\mathbb{A}}^L R$. According to Corollary 6.9 the functor $P \otimes_{\mathbb{A}}^L -$ is an auto-equivalence of $\text{D}(A)$, it has finite
cohomological dimension, and it preserves $D^+_I(A)$. Therefore the DG module $R'$ is dualizing.

(2) Define the objects $P := \text{RHom}_A(R, R')$ and $P' := \text{RHom}_A(R', R)$, and the functors $D := \text{RHom}_A(-, R)$, $D' := \text{RHom}_A(-, R')$, $F := P \otimes^L_A -$ and $F' := P' \otimes^L_A -$. We know that the functors $D, D', D' \circ D, D \circ D'$ have finite cohomological dimensions relative to $D(A)$; the DG modules $P, P' \in D^+_I(A)$; and the functors $F, F'$ have bounded above cohomological displacements relative to $D(A)$. For any $M \in D(A)$ there is a canonical morphism

$$\text{RHom}_A(R, R') \otimes^L_A M \to \text{RHom}_A(\text{RHom}_A(M, R), R'),$$

so we get a morphism of triangulated functors $\eta : F \to D' \circ D$. By definition $\eta_A$ is an isomorphism, and Theorem 2.10(1) says that $\eta_M$ is an isomorphism for every $M \in D^+_I(A)$. Likewise there is an isomorphism $\eta'_M : F'(M) \to (D \circ D')(M)$ for every $M \in D^+_I(A)$.

Let us calculate $P \otimes^L_A P'$:

$$P \otimes^L_A P' \cong F(P') \cong (D' \circ D)(P')$$

$$\cong (D' \circ D)(F'(A)) \cong (D' \circ D \circ D')(A) \cong A.$$

This proves $P$ is tilting. And

$$P \otimes^L_A R \cong F(R) \cong (D' \circ D)(D(A)) \cong D'(A) \cong R'.$$

(3) If $P$ is tilting and $R \cong P \otimes^L_A R$, then

$$A \cong \text{RHom}_A(R, R) \cong \text{RHom}_A(P \otimes^L_A R, R)$$

$$\cong^* \text{RHom}_A(P, \text{RHom}_A(R, R)) \cong \text{RHom}_A(P, A),$$

where the isomorphism $\cong^*$ is by adjunction. But then

$$P \cong A \otimes^L_A P \cong \text{RHom}_A(P, A) \otimes^L_A P \cong^! \text{RHom}_A(P, P) \cong^† A,$$

where the isomorphism $\cong^!$ is by a combination of Theorems 6.7 and 5.20 and the isomorphism $\cong^†$ is by Theorem 6.7.

Corollary 7.10. Assume $A$ has some dualizing DG module. The formula $R \mapsto P \otimes^L_A R$ induces a simply transitive action of the group $\text{DPic}(A)$ on the set of isomorphism classes of dualizing DG $A$-modules.

Proof. Clear from the theorem.

Corollary 7.11. Assume $A$ has some dualizing DG module (e.g. $A$ is tractable). The formula $R \mapsto \text{RHom}_A(\tilde{A}, R)$ induces a bijection

$$\{\text{dualizing DG } A\text{-modules}\} \xrightarrow{\text{isomorphism}} \{\text{dualizing DG } \tilde{A}\text{-modules}\}.$$

Proof. By Corollary 7.10 the actions of the groups $\text{DPic}(A)$ and $\text{DPic}(\tilde{A})$ on these two sets, respectively, are simply transitive. And by Theorem 6.12 the group homomorphism $\text{DPic}(A) \to \text{DPic}(\tilde{A})$ induced by $P \mapsto \tilde{A} \otimes^L_A P$ is bijective. Thus it suffices to prove that the function induced by $R \mapsto \text{RHom}_A(\tilde{A}, R)$ is equivariant for the action of $\text{DPic}(A)$. Here is the calculation:

$$\text{RHom}_A(\tilde{A}, P \otimes^L_A R) \cong^† P \otimes^L_A \text{RHom}_A(\tilde{A}, R) \cong (\tilde{A} \otimes^L_A P) \otimes^L_A \text{RHom}_A(\tilde{A}, R).$$
The isomorphism \(\cong^\dagger\) comes from Lemma 7.12 below, noting that the tilting DG module \(P\) satisfies condition \((\ast)\) of the lemma, since it is perfect. \(\square\)

Stretching Definition 1.16 a bit, we say that a DG \(A\)-module \(\tilde{N}\) has bounded below generation if it is generated in the degree range \([d_0, \infty)\) for some integer \(d_0\). This means that for any \(M \in M^{[e_0, \infty]}(A)\) we have \(M \otimes_A \tilde{N} \in M^{[e_0 + d_0, \infty]}(A)\).

**Lemma 7.12.** Let \(L \in \mathcal{D}_-^-(A)\) and \(M, N \in \mathcal{D}^+(A)\). Assume that \(N\) satisfies this condition:

\[(\ast)\quad \text{There is a covering sequence } (s_1, \ldots, s_n) \text{ of } \tilde{A}, \text{ and for every } i \text{ there is an isomorphism } A_{s_i} \otimes_A N \cong \tilde{N}_i \text{ in } \mathcal{D}(A_{s_i}), \text{ where } \tilde{N}_i \text{ is a K-flat DG } A_{s_i}-\text{module with bounded below generation.}\]

Then the canonical morphism

\[
\psi_{L, M, N} : \mathcal{RHom}_A(L, M) \otimes_{\mathcal{A}} N \to \mathcal{RHom}_A(L, M \otimes_{\mathcal{A}} N)
\]

in \(\mathcal{D}(A)\) is an isomorphism.

**Proof.** Step 1. Here we assume that \(N \cong \tilde{N}\) in \(\mathcal{D}(A)\), where \(\tilde{N}\) is a K-flat DG \(A\)-module of bounded below generation. Using smart truncation if needed, we can assume that the DG \(B\)-module \(M\) is bounded below. Let \(\bar{L} \to L\) be a pseudo-finite semi-free resolution over \(A\) (see Proposition 1.19). The morphism \(\psi_{L, M, N}\) is represented by the homomorphism

\[
\tilde{\psi}_{L, M, N} : \text{Hom}_A(\bar{L}, M) \otimes_A \tilde{N} \to \text{Hom}_A(\bar{L}, M \otimes_A \tilde{N})
\]

in \(M(A)\). Because the semi-free DG \(A\)-module \(\bar{L}\) is bounded above and has finitely many basis elements in each degree, and both \(M\) and \(M \otimes_A \tilde{N}\) are bounded below, we see that \(\tilde{\psi}_{L, M, N}\) is bijective.

Step 2. Here \(N\) satisfies condition \((\ast)\). We claim that the obvious morphisms

\[
(\mathcal{RHom}_A(L, M) \otimes_{\mathcal{A}} N) \otimes_{\mathcal{A}^0} A_{s_i}^0 \to \mathcal{RHom}_A(L, M) \otimes_{\mathcal{A}} \tilde{N}_i \tag{7.13}
\]

and

\[
\mathcal{RHom}_A(L, M \otimes_{\mathcal{A}} N) \otimes_{\mathcal{A}^0} A_{s_i}^0 \to \mathcal{RHom}_A(L, M \otimes_{\mathcal{A}} \tilde{N}_i) \tag{7.14}
\]

in \(\mathcal{D}(A)\), that come from the given isomorphisms \(A_{s_i} \otimes_A N \cong \tilde{N}_i\), are isomorphisms. That \((7.13)\) is an isomorphism is trivial. As for the morphism \((7.14)\): condition \((\ast)\) implies that \(M \otimes_{\mathcal{A}} N\) belongs to \(\mathcal{D}^+(A)\). Since \(A_{s_i}^0\) is a K-flat DG module over \(A^0\) generated in degree 0, we can use Step 1.

Step 3. Now we are in the general situation. Let \(\psi_i\) be the morphism gotten from \(\psi_{L, M, N}\) by the localization \(A_{s_i}^0 \otimes_{\mathcal{A}^0} -\), so \(\psi_i\) goes from the first object in \(7.13\) to the first object in \(7.14\). Since \(\tilde{A} \to \prod_i \tilde{A}_{s_i}\) is faithfully flat, it suffices to prove that all the \(\psi_i\) are isomorphisms. But by step 2 it suffices to show that

\[
\psi_{L, M, \tilde{N}_i} : \mathcal{RHom}_A(L, M) \otimes_{\mathcal{A}} \tilde{N}_i \to \mathcal{RHom}_A(L, M \otimes_{\mathcal{A}} \tilde{N}_i)
\]

is an isomorphism. Since \(\tilde{N}_i\) is a K-flat DG \(A\)-module with bounded below generalization, we can use step 1. \(\square\)

**Corollary 7.15.** If the ring \(A\) is local, then any two dualizing DG \(A\)-modules \(R\) and \(R'\) satisfy \(R' \cong R[m]\) for some integer \(m\).
Proof. By Corollary 6.15 we have $\text{DPic}(A) \cong \mathbb{Z}$, generated by the class of $A[1]$. Now use Corollary 7.10. □

**Proposition 7.16.** Let $A$ be a cohomologically noetherian cohomologically bounded DG ring, and let $R$ be a dualizing DG $A$-module. Then $R$ is dualizing in the sense of [FLJ] Definition 1.8.

Proof. There are four conditions in [FLJ] Definition 1.8. Condition (1) – the existence of resolutions – is trivial in our commutative situation. Condition (2) says that if $M \in D^b_f(A)$ then $\text{RHom}_A(M, R) \in D^b_f(A)$; and this is true by Proposition 7.2(1). Condition (3) requires that for any $M \in D^b_f(A)$, letting $N$ be either $M$ or $M \otimes_A R$, the adjunction morphisms

$$N \rightarrow \text{RHom}_A(\text{RHom}_A(N, R), R)$$

are isomorphisms. Now by Corollary 7.3 we know that $R \in D^b_f(A)$. A combination of Proposition 2.7 and Theorem 2.12(1) tells us that $M \otimes_A^L R \in D^+_f(A)$. Thus in both cases $N \in D^+_f(A)$, and according to Proposition 7.2(2) the morphism in question is an isomorphism. Condition (4) is part of condition (3) in the commutative situation. □

**Definition 7.17.** A cohomologically noetherian cohomologically bounded commutative DG ring $A$ is called Gorenstein if the DG module $A$ has finite injective dimension relative to $D(A)$.

**Proposition 7.18.** Let $A$ be a cohomologically noetherian cohomologically bounded DG ring. The following conditions are equivalent:

(i) $A$ is Gorenstein.

(ii) The DG $A$-module $A$ is dualizing.

Proof. Since conditions (i) and (iii) of Definition 7.1 are automatic for $R := A$, this is clear. □

**Remark 7.19.** For DG rings that are not cohomologically bounded, a comparison like in Proposition 7.16 does not seem to work nicely.

Corollary 7.15 is very similar to [FLJ] Theorem III; but of course the assumptions are not exactly the same.

We do not know of a reasonable definition of Gorenstein DG ring in the cohomologically unbounded case.

Here is a rather surprising result, that was pointed out to us by Jørgensen.

**Theorem 7.20.** Let $A$ be a tractable cohomologically noetherian cohomologically bounded commutative DG ring. If $\tilde{A}$ is a perfect DG $A$-module, then $A \rightarrow \tilde{A}$ is a quasi-isomorphism.

Example 7.25 shows that the assumption that $A$ is cohomologically bounded is really needed.

Proof. We will prove that $\tilde{A}_p \otimes_{\tilde{A}} \text{H}^i(A) = 0$ for every $i < 0$ and every $p \in \text{Spec } \tilde{A}$.

Fix such $i$ and $p$. Because the assertion is invariant under DG ring quasi-isomorphisms, we may assume, by Lemma 7.7 (replacing $A$ with $A_{\text{et}}$), that $A^0$ is a noetherian ring. Consider the ring $A^0_p := \tilde{S}^{-1} \cdot A^0$, where $\pi : A \rightarrow \tilde{A}$ is the canonical homomorphism, $S := \tilde{A} - p$, and $\tilde{S} := \pi^{-1}(S) \cap A^0$. Next define the DG
There is a further analogy between “dualizing objects” in the two scenarios:

Of course, this observation is not new (cf. [Lu1], [Lu2], [TV] and [AG]).

We refer to this as the DG vs. adic analogy. This analogy restricts to the “degenerate cases” in these scenarios:

(dg) The DG scenario: a cohomologically pseudo-noetherian commutative DG ring $A$, with reduction $\bar{A} := H^0(A)$.

(ad) The adic scenario: a noetherian commutative ring $A$, $a$-adically complete w.r.t. an ideal $a$, with reduction $\bar{A} := A/a$.

We refer to this as the DG vs. adic analogy. This analogy restricts to the “degenerate cases” in these scenarios:

(dg) The cohomology $H(A)$ is bounded.

(ad) The defining ideal $a$ is nilpotent.

Of course, this observation is not new (cf. [Lu1], [Lu2], [TV] and [AG]).

The DG vs. adic analogy holds also for “finite homomorphisms”:

(dg) A cohomologically pseudo-finite homomorphism $f : A \to B$ between cohomologically pseudo-noetherian commutative DG rings (Definition 7.4).

(ad) A formally finite or pseudo-finite homomorphism $f : A \to B$ between $a$-adically complete noetherian commutative rings, as in [Ye1] and [AJL2] respectively.

There is a further analogy between “dualizing objects” in the two scenarios:

(dg) A dualizing DG module $R$ over a cohomologically pseudo-noetherian commutative DG ring $A$ (Definition 7.1).
(ad) A \textit{t-dualizing complex} $R$ over an adically complete noetherian commutative ring $A$, as in [Ye1] and [AJL1].

The analogies above raise two questions:

1. Is there a DG analogue of the \textit{c-dualizing complex} of [AJL1]?
2. Is there a DG analogue of the GM Duality of [AJL1] and the MGM Equivalence of [PSY]?

\textbf{Remark 7.24.} Recall that a noetherian ring $A$ of finite Krull dimension is regular (i.e. all its local rings $A_p$ are regular) if and only if it has finite global cohomological dimension.

Now suppose $A$ is a cohomologically pseudo-noetherian commutative DG ring. By “Krull dimension” we could mean that of $\bar{A}$, but “regular local ring” has no apparent meaning here. Hence we propose this definition: $A$ is called regular if it has finite global cohomological dimension. By this we mean that there is a natural number $d$, such that every $M \in D(A)$ has projective dimension at most $\text{amp}(H(M)) + d$ relative to $D(A)$; see Definition 2.4. According to Theorem 5.11, we see that any $M \in D^b_f(A)$ is perfect.

Now assume that $A$ is a regular commutative DG ring, but with bounded cohomology. Then, taking $M := A$, Theorem 7.20 says that $A \to \bar{A}$ is a quasi-isomorphism. The conclusion is that \textit{the only regular DG rings with bounded cohomology are the regular rings} (up to quasi-isomorphism).

The corresponding adic ring $A$, under the DG vs. adic analogy of Remark 7.23, has a nilpotent defining ideal $a$. If $A$ is regular, then it cannot have nonzero nilpotent elements. Therefore $a = 0$ here, and $A \to \bar{A}$ is bijective.

\textbf{Example 7.25.} Take a field $K$, and let $A := K[t]$, a polynomial ring in a variable $t$ of degree $-2$. We view $A$ as a DG ring with zero differential, so $H(A) \cong A$; and it is cohomologically noetherian. The DG ring homomorphism $K \to A$ is cohomologically pseudo-finite. Hence the DG $A$-module $R := \text{Hom}_K(A,K)$ is dualizing. This DG module is not bounded below.

Note that here $\bar{A} \cong K$ is a perfect DG $A$-module. We can easily produce a finite semi-free resolution of $\bar{A}$. The DG module $A^{\leq -2}$, which is both the stupid and the smart truncation of $A$ at $-2$, is free, since $A^{\leq -2} \cong A[2]$ as DG $A$-modules. Let $\phi : A^{\leq -2} \to A$ be the inclusion, and let $P := \text{cone}(\phi)$. There is an obvious quasi-isomorphism $P \to A$.

The adic analogue is the ring of formal power series $A := K[[t]]$, with ideal of definition $a := (t)$. The corresponding t-dualizing complex is $R := \text{Hom}^\text{cont}_K(A,K)$, which is an artinian $A$-module of infinite length.

\textbf{Remark 7.26.} Our definition of dualizing DG modules, Definition 7.1, might seem an almost straightforward generalization of Grothendieck’s original definition in [RD]. However there are at least two subtle points: (a) Finding the correct notion of injective dimension in the DG case (condition (ii) of Definition 7.1). (b) Allowing a dualizing DG module to have unbounded above cohomology in the DG case (condition (i) of Definition 7.1). Cf. Corollary 7.3 and Remark 7.23.

All results in this section, up to and including Corollary 7.10, might also seem to be straightforward generalizations of Grothendieck’s corresponding results in [RD]. But the technical difficulties (mainly when $A$ is cohomologically unbounded) cannot be neglected.
We should mention that Theorem 7.8 can be made a bit stronger, by replacing the condition that $A$ is tractable with the weaker condition that there is a homomorphism $K \to A$, where $K$ is a noetherian ring with a dualizing complex, and $K \to \bar{A}$ is essentially finite type. Theorem 7.20 can be similarly strengthened.

Some earlier papers, notably [Hi1] and [FIJ], had adopted other definitions of dualizing DG modules; see Example 7.22 and Proposition 7.16 respectively. These definitions are not consistent with our definition in general, and there does not appear to be a well-developed theory for them.

In [Lu2, Definition 4.2.5], Lurie gives a definition of dualizing $E_\infty$ module over an $E_\infty$ ring. Now any DG ring $A$ can be viewed as an $E_\infty$ ring, and DG $A$-modules can be viewed $E_\infty$ $A$-modules. Under this correspondence, at least when $A$ is cohomologically bounded, a dualizing DG $A$-module (in the sense of Definition 7.1) becomes a dualizing module in the sense of [Lu2]. Thus our results in this section, up to and including Corollary 7.10, could be viewed as “special instances” of Lurie’s more general statements. Still, a rigorous proof that our results are consequences of the corresponding results in [Lu2] (e.g. deducing our Theorem 7.8 from [Lu2] Theorem 4.3.14), or deducing our Theorem 7.9 from [Lu2] Proposition 4.2.9) might be nontrivial, and possibly longer than our own direct proofs!

Our Corollary 7.11 appears to be totally new. We could not find anything resembling it in Lurie’s papers, nor elsewhere in the literature. Likewise for Theorem 7.20 (except for Jørgensen’s original local result).

Remark 7.27. A result that is noticeably missing from our paper is a DG analogue of [Lu2] Theorem 4.3.5. Translated to the DG terminology, it states that if the ring $\bar{A}$ admits a dualizing DG module, then the DG ring $A$ admits a dualizing DG module. We do not know whether this result can be proved within the DG framework; this is a question that we find interesting.

Note however that the corresponding result in the adic case, namely when $A$ is a complete $\mathfrak{a}$-adic ring extension of $\bar{A}$ (cf. Remark 7.23), was proved a long time ago by Faltings [Fa]. The proof of the nilpotent case in [Fa] is quite easy; but the passage to the complete adic case is somewhat involved there. The proof can be greatly simplified by first proving existence of a $t$-dualizing complex $R'_A$ over $A$, and then applying derived completion to obtain a $c$-dualizing complex $R_A := \Lambda_\mathfrak{a}(R'_A)$. See [Ye1] [AJL1], [AJL2] and [PSY] for information on derived completion, and on dualizing complexes over adic rings.

8. COHEN-MACAULAY DG MODULES

In this last section we work with cohomologically pseudo-noetherian commutative DG rings (see Convention 4.3 and Definition 1.18).

Let $A$ be such a DG ring. Recall that $\bar{A} = H^0(A)$, and $D^0(A)$ is the full subcategory of $D(A)$ consisting of the DG modules $M$ such that $H^i(M) = 0$ for all $i \neq 0$. Inside $D^0(A)$ we have $D^0_i(A) = D_i(A) \cap D^0(A)$.

Proposition 8.1. Consider the canonical DG ring homomorphism $\pi : A \to \bar{A}$. The functor
\[ Q \circ \text{rest}_\pi : \text{Mod} \bar{A} \to D^0(A) \]
is an equivalence. It restricts to an equivalence
\[ Q \circ \text{rest}_\pi : \text{Mod} \bar{A} \to D^0_i(A). \]
Proof. Smart truncation shows that any object of $D^0(A)$ is isomorphic to an object of $\text{Mod} \bar{A}$. Finiteness of $\bar{A}$-modules is preserved. It remains to show that $Q \circ \text{rest}_\pi$ is a fully faithful functor.

So take $M,N \in \text{Mod} \bar{A}$, and let $\bar{M} \to M$ be a semi-free resolution over $A$ with $\sup(\bar{M}) \leq 0$. Then

$$\text{Hom}_{D(A)}(M,N) \cong H^0(\text{Hom}_A(\bar{M},N)) \cong \text{Hom}_\bar{A}(H^0(\bar{M}),N) \cong \text{Hom}_\bar{A}(M,N).$$

\[
\square
\]

**Definition 8.2.** Let let $R$ be a dualizing DG $A$-module. A DG module $M \in D^b(A)$ is called Cohen-Macaulay with respect to $R$ if

$$\text{RHom}_A(M,R) \in D^b(A).$$

In other words, the condition is that $\text{RHom}_A(M,R)$ is isomorphic, in $D(A)$, to an object of $\text{Mod} \bar{A}$. As usual “Cohen-Macaulay” is abbreviated to “CM”. Let us denote by $D^b(A)_{\text{CM},R}$ the full subcategory of $D^b(A)$ consisting of DG modules that are CM w.r.t. $R$.

**Remark 8.3.** Observe that the functor $\text{RHom}_A(\cdot, R)$ gives rise to a duality between $D^b(A)_{\text{CM},R}$ and $D^b(A)$. And the latter is equivalent to $\text{Mod} \bar{A}$. Therefore $D^b(A)_{\text{CM},R}$ is an artinian abelian category.

If $A \to \bar{A}$ is not a quasi-isomorphism, then $A$ does not belong to $D^0(A)$, and therefore $R$ is not a CM DG module w.r.t. itself. This is unlike the ring case.

For a comparison to Cohen-Macaulay modules and Grothendieck’s notion of Cohen-Macaulay complexes, see [YZ3, Theorem 6.2] and [YZ4, Section 7].

The groups $\text{DPic}^0(A) \subset \text{DPic}(A)$ were introduced in Definitions 6.2 and 6.14.

**Proposition 8.4.** Let $P$ be a tilting DG $A$-module. The following are equivalent:

(i) The auto-equivalence $P \otimes^L_A -$ of $D(A)$ preserves the subcategory $D^b(A)$.
(ii) The class of $P$ is in $\text{DPic}^0(A)$.

**Proof.** (i) $\Rightarrow$ (ii): Let $\bar{P} := \bar{A} \otimes^L_\bar{A} P$. By assumption it belongs to $D^0(A)$. But then the class of $\bar{P}$ is in $\text{Pic}(\bar{A}) = \text{DPic}^0(\bar{A})$, so by definition the class of $P$ is in $\text{DPic}^0(A)$.

(ii) $\Rightarrow$ (i): Here $\bar{P} := \bar{A} \otimes^L_\bar{A} P$ is isomorphic to an invertible $\bar{A}$-module, so $\bar{P} \otimes^L_\bar{A} -$ preserves $D^0(\bar{A})$. Now take any $M \in D^0(\bar{A})$. By Proposition 8.1 we can assume that $M \in \text{Mod}_\bar{A}(\bar{A})$. Then

$$P \otimes^L_A M \cong \bar{P} \otimes^L_{\bar{A}} \bar{A} \otimes^L_\bar{A} M \cong \bar{P} \otimes^L_{\bar{A}} M \in \text{DPic}^0(\bar{A}).$$

We see that $H^i(P \otimes^L_A M) = 0$ for all $i \neq 0$. 

\[
\square
\]

Recall the connected component idempotent functors of a DG ring from Definition 14.19.

**Theorem 8.5.** Let $f : A \to B$ be a cohomologically pseudo-finite homomorphism between cohomologically pseudo-noetherian commutative DG rings. Assume that $A$ and $B$ have dualizing DG modules $R_A$ and $R_B$ respectively, and that $B$ is nonzero. Let $E_1, \ldots, E_n$ be the connected component decomposition functors of $B$. 

(1) There are unique integers $k_1, \ldots, k_n$ such that, letting
\[
R_B' := \bigoplus_{i=1}^n E_i(R_B)[k_i] \in \mathcal{D}(B),
\]
the class of the tilting DG $B$-module $\text{RHom}_A(R_B', R_A)$ is inside $\text{DPic}^0(B)$.

(2) Let $M \in \mathcal{D}^b_! (B)$. The following conditions are equivalent:
(i) $M$ is CM w.r.t. to $R_B'$.
(ii) $\text{rest}_f(M)$ is CM w.r.t. to $R_A$.

Note that $R_B'$ and $R_B'' := \text{RHom}_A(B, R_A)$ are dualizing DG $B$-modules, so
\[
\text{RHom}_A(R_B', R_A) \cong \text{RHom}_B(R_B', R_B'')
\]
is a tilting DG $B$-module (by Theorem 7.9), and hence item (1) makes sense.

**Proof.** (1) Let $R_B''$ be as above. By the classifications in Corollaries 7.10 and 6.13 there are unique $k_1, \ldots, k_n \in \mathbb{Z}$, and a tilting DG $B$-module $Q$ whose class in $\text{DPic}^0(B)$ is unique, such that $R_B'' \cong Q \otimes_B R_B'$. Let $P := \text{RHom}_B(Q, B)$, which by Corollary 6.8 is the quasi-inverse of $Q$. Then $R_B' \cong P \otimes_B R_B''$. Using adjunction we get isomorphisms
\[
\text{RHom}_A(R_B', R_A) \cong \text{RHom}_B(R_B', R_B'') \cong \text{RHom}_B(P, \text{RHom}_B(R_B''', R_B''')) \cong \text{RHom}_B(P, B) \cong Q
\]
in $\mathcal{D}(B)$. This proves (1).

(2) By Proposition 8.4, the functor $Q \otimes_B -$ preserves $\mathcal{D}^b_! (B)$. Take any $M \in \mathcal{D}^b_! (B)$.
Then, using Lemma 7.12 we get isomorphisms
\[
\text{RHom}_A(M, R_A) \cong \text{RHom}_B(M, R_B'') \cong \text{RHom}_B(M, Q \otimes_B R_B') \cong \text{RHom}_B(M, R_B') \otimes_B Q.
\]
This gives (i) $\Rightarrow$ (ii). The converse is very similar. \qed

**Corollary 8.6.** Let $f : A \to B$ be a cohomologically finite homomorphism between tractable DG rings. Assume Spec $B$ is connected. The following are equivalent for $M \in \mathcal{D}^b_! (B)$ :
(i) $M$ is CM w.r.t. some dualizing DG $B$-module.
(ii) $\text{rest}_f(M)$ is CM w.r.t. some dualizing DG $A$-module.

**Proof.** The implication (ii) $\Rightarrow$ (i) comes from Theorem 8.5(2) – and does not require Spec $B$ to be connected.

For the implication (i) $\Rightarrow$ (ii), let $R_B$ be a dualizing DG $B$-module such that $M$ is CM w.r.t. it. Take any dualizing DG $A$-module $R_A$. Let $k \in \mathbb{Z}$ be such that $\text{RHom}_A(R_B[k], R_A)$ is inside $\text{DPic}^0(B)$. See Theorem 8.5(1). Then by Theorem 8.5(2) we know that $\text{rest}_f(M)$ is CM w.r.t. $R_A[-k]$. \qed

**Theorem 8.7.** Let $f : A \to B$ be a homomorphism between cohomologically pseudo-noetherian commutative DG rings, such that $H^0(f) : \hat{A} \to \hat{B}$ is surjective. Assume $A$ and $B$ have dualizing DG modules. Let $R_B$ be a dualizing DG $B$-module and let $M, N \in \mathcal{D}^b_! (B)$.

(1) If $M$ is CM w.r.t. $R_B$, and there is an isomorphism $\text{rest}_f(M) \cong \text{rest}_f(N)$ in $\mathcal{D}(A)$, then $N$ is also CM w.r.t. $R_B$. 


(2) If $M$ and $N$ are CM w.r.t. $R_B$, then the homomorphism 
\[ \text{rest}_f : \text{Hom}_{D(B)}(M, N) \to \text{Hom}_{D(A)}(\text{rest}_f(M), \text{rest}_f(N)) \]

is bijective.

**Proof.** (1) We may assume that $\bar{B} \neq 0$. Let $E_1, \ldots, E_n$ be the connected component decomposition functors of $B$. Write $F := \text{rest}_f$, $M_i := E_i(M)$ and $N_i := E_i(N)$. Choose some dualizing DG $A$-module $R_A$, and let $k_1, \ldots, k_n$ be the integers from Theorem 8.5(1). Thus the dualizing DG $B$-module $R_B := \bigoplus_{i=1}^n E_i(R_B)[k_i]$ satisfies this: the tilting DG $B$-module $Q := \text{RHom}_A(R_B', R_A)$ is inside $\text{DPic}^0(B)$. Define $M' := \bigoplus_i M_i[k_i]$ and $N' := \bigoplus_i N_i[k_i]$.

We are given that $M$ is CM w.r.t. $R_B$. Using the equivalence of Corollary 4.17 we see that $M'$ is CM w.r.t. $R_B'$. Thus by Theorem 8.5(2) the DG $A$-module $F(M')$ is CM w.r.t. $R_A$. Proposition 4.22 implies that $F(M') \cong F(N')$ in $D(A)$, and hence $F(N')$ is CM w.r.t. $R_A$. Using Theorem 8.5(2) once more we conclude that $N'$ is CM w.r.t. $R_B'$; and hence $N$ is CM w.r.t. $R_B$.

(2) Let us write $R_B'' := \text{RHom}_A(B, R_A)$, $D_B'' := \text{RHom}_B(-, R_B')$ and $D_A := \text{RHom}_A(-, R_A)$. Define $M' := \bigoplus_i M_i[k_i]$ and $N' := \bigoplus_i N_i[k_i]$ as above, so these are CM w.r.t. $R_B''$. There are isomorphisms

\[ \text{Hom}_{D(B)}(M, N) \cong^1 \text{Hom}_{D(B)}(M', N') \]

\[ \cong^2 \text{Hom}_{D(B)}(D_B''(N'), D_B''(M')) \]

\[ \cong^3 \text{Hom}_{\text{Mod } B}(D_B''(N'), D_B''(M')) \]

\[ \cong^4 \text{Hom}_{\text{Mod } \bar{A}}(F(D_B''(N')), F(D_B''(M'))) \]

\[ \cong^5 \text{Hom}_{\text{Mod } \bar{A}}(D_A(F(N')), D_A(F(M'))) \]

\[ \cong^{2,3} \text{Hom}_{D(A)}(F(M'), F(N')) \]

\[ \cong^6 \text{Hom}_{D(A)}(F(M), F(N)) \]

They are gotten as follows: the isomorphism $\cong^1$ is by Corollary 4.17 and Proposition 4.21; the isomorphism $\cong^2$ is by Proposition 7.2(3); the isomorphism $\cong^3$ is by Proposition 8.1; the isomorphism $\cong^4$ is because $\mathbb{H}^p(f) : A \to B$ is surjective; the isomorphism $\cong^5$ is because $F \circ D_B'' \cong D_A \circ F$ as functors; and isomorphism $\cong^6$ is due to Proposition 4.22. The composition of all these isomorphisms is $F$. \qed

**Corollary 8.8.** Let $f : A \to B$ be a homomorphism between cohomologically pseudo-noetherian DG rings, such that $\mathbb{H}^0(f) : A \to B$ is bijective. Let $R_A$ be a dualizing DG $A$-module, and define $R_B := \text{RHom}_A(B, R_A)$. Then the functor

\[ \text{rest}_f : D^b_{\text{CM}}(B)_{\text{CM}:R_B} \to D^b_{\text{CM}}(A)_{\text{CM}:R_A} \]

is an equivalence.

**Proof.** In view of the last theorem, it suffices to show that $\text{rest}_f$ is essentially surjective on objects. Take any $M \in D^b_{\text{CM}}(A)_{\text{CM}:R_A}$. Then $N := \text{RHom}_A(M, R_A)$ is (isomorphic to) a module in $\text{Mod}_{\bar{A}}$. Because

\[ \text{rest}_{\mathbb{H}^p(f)} : \text{Mod}_{\bar{A}} \to \text{Mod}_{\bar{A}} \]

is an equivalence, there is $N' \in \text{Mod}_{\bar{A}}$ that is sent to $N$. Therefore the DG module

\[ M' := \text{RHom}_A(N', R_B) \in D^b_{\text{CM}}(B)_{\text{CM}:R_B} \]
satisfies \( \text{rest}_f(M') \cong M \). \qed

**Remark 8.9.** Here is a quick explanation of the role of CM DG modules in [Ye5]. Suppose \( K \to A \to B \) are ring homomorphisms, \( M \in \text{D}(A) \) and \( N \in \text{D}(B) \). Under suitable assumptions we want to have a canonical isomorphism

\[
\varpi : \text{Sq}_{A/K}(M) \otimes_A \text{Sq}_{B/A}(N) \xrightarrow{\simeq} \text{Sq}_{B/K}(M \otimes_K N)
\]

in \( \text{D}(B) \), that we call the **cup product**. This isomorphism was already constructed in [YZ1, Theorem 4.11]; but unfortunately this part of [YZ1] also contained a serious mistake.

The construction in [Ye5] goes like this. We choose a semi-free DG ring resolution \( K \to \tilde{A} \) of \( K \to A \), and then a semi-free DG ring resolution \( \tilde{A} \to \tilde{B} \) of \( A \to B \). So

\[
\text{Sq}_{A/K}(M) = \text{RHom}_{\tilde{A} \otimes_K \tilde{A}}(A, M \otimes_K \tilde{A})
\]

etc. The construction goes through a few “standard moves” (adjunction formulas mostly), until we arrive at the following situation. Consider the surjective DG ring homomorphism

\[
f : \tilde{B} \otimes_K \tilde{B} \to \tilde{B} \otimes_A \tilde{B},
\]

and the DG modules

\[
K := N \otimes_A \text{Sq}_{A/K}(M)
\]

and

\[
L := \text{RHom}_{\tilde{B} \otimes_K \tilde{B}}(\tilde{B} \otimes_A \tilde{B}, (M \otimes_A \text{Sq}_{A/K}(M) \otimes_A \text{Sq}_{A/K}(M))
\]

in \( \text{D}(\tilde{B} \otimes_A \tilde{B}) \). The “standard moves” give us a canonical isomorphism

\[
\chi : \text{rest}_f(K) \xrightarrow{\simeq} \text{rest}_f(L)
\]

in \( \text{D}(\tilde{B} \otimes_A \tilde{B}) \); but what we need to continue the construction is a canonical isomorphism \( \bar{\chi} : K \xrightarrow{\simeq} L \) in \( \text{D}(\tilde{B} \otimes_A \tilde{B}) \) such that \( \text{rest}_f(\bar{\chi}) = \chi \).

The only conceivable hope was that something like Theorem 8.7 should appear. Fortunately, in the situation where we require the cup product, the ring \( K \) is a regular noetherian ring; \( K \to A \) is essentially finite type; \( A \to B \) is essentially Gorenstein (i.e. essentially finite type, flat, and the fibers are Gorenstein rings); \( M \) is a rigid dualizing complex over \( A \) relative to \( K \); and \( N \) is a tilting complex over \( B \) (and hence it is a relative dualizing complex for \( A \to B \)). These assumptions imply that \( K \) is a dualizing DG module over the DG ring \( \tilde{B} \otimes_A \tilde{B} \), and therefore it is a CM DG module w.r.t. itself. Now Theorem 8.7 says that there exists a unique isomorphism \( \bar{\chi} : K \xrightarrow{\simeq} L \) satisfying \( \text{rest}_f(\bar{\chi}) = \chi \).

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