An attempt to prove an effective Siegel theorem  
Part One

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We describe a plan how to prove an effective Siegel theorem (about the exceptional Dirichlet character). The basic idea is quite simple: it is briefly outlined in Section 0. We completed a proof, but it is ridiculously long; more than 300 pages. Such a long paper probably contains several errors; the question is whether the errors are substantial or not. Even if there are errors, is the basic idea still good? In Sections 1-5 we give a very detailed plan for the case of negative discriminants. The missing details (mostly routine elementary estimations) are in Part Two. I am happy to send the pdf-file of Part Two to anybody who requests it by email.

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0. A nutshell summary

We begin with the case of negative discriminants; we want to prove

**Theorem 1:** For every negative fundamental discriminant \( \Delta < 0 \) with \(|\Delta| > e^{10^{100}}\),

\[
L(1, \chi_{\Delta}) \geq \frac{1}{(\log |\Delta|)^{146}}. \tag{0.1}
\]

**Brief outline of the proof.** Assume that there is a “bad” negative fundamental discriminant \( \Delta = -D < 0 \) (\( D = |\Delta| \) is always positive) such that

\[
L(1, \chi_{-D}) < \frac{1}{(\log D)^{146}} \quad \text{and} \quad D > e^{10^{100}}. \tag{0.2}
\]

By the class number formula and (0.2), the class-number \( h(-D) \) is relatively small:

\[
h(-D) < \sqrt{D}(\log D)^{-146}.
\]
Our goal is to derive a contradiction.

First we define a simple “sieve out” procedure; we distinguish two cases. Let \( p_0 \) denote the smallest prime that is not a divisor of the discriminant \( D \); clearly \( p_0 < 2 \log D \).

If \( \chi_D(p_0) = 1 \) then let

\[
Z_0 = Z'_0 = \prod_{\substack{p \leq D: \\ \chi_D(p) \neq -1}} p
\]

be the product of all primes \( p \leq D \) with \( \chi_D(p) = 1 \) or 0.

If \( \chi_D(p_0) = -1 \) then we have two possible choices for \( Z_0 \): either (3) (i.e., \( Z_0 = Z'_0 \)), or we include \( p_0 \) as an extra prime factor: \( Z_0 = Z''_0 = p_0Z'_0 \). One of these two choices will lead to a contradiction.

Let \( m \geq 1 \) be a squarefree integer relatively prime to \( Z_0 \), and let \( \ell \) be an integer in \( 1 \leq \ell < m \) relatively prime to \( m \). Given \( 0 < A_1 < A_2 \), let \( v_{m,\ell,a}(A_1, A_2) \) denote the vector

\[
v_{m,\ell,a}(A_1, A_2) = (e^{2\pi i((j_1D+a)\ell/m)}, e^{2\pi i((j_1+1)D+a)\ell/m}, e^{2\pi i((j_1+2)D+a)\ell/m}, \ldots, e^{2\pi i(j_2D+a)\ell/m})
\]

that plays a key role in the proof; here \( j = j_1 \) is the smallest integer with \( jD + a \geq A_1 \), \( j = j_2 \) is the largest integer with \( jD + a \leq A_2 \), and of course \( i = \sqrt{-1} \).

We need the usual inner product of complex vectors: if \( \psi = (a_1, \ldots, a_n) \) and \( \xi = (b_1, \ldots, b_n) \) with \( a_j, b_j \) complex numbers, then \( \langle \psi, \xi \rangle = \sum_{j=1}^n a_j b_j^* \) where \( b_j^* \) is the complex conjugate of \( b_j \). We write, as usual,

\[
\|\psi\| = \sqrt{\langle \psi, \psi \rangle}, \quad \text{so} \quad \|\psi\|^2 = \langle \psi, \psi \rangle
\]

is the square of the norm. For arbitrary complex numbers \( C_j \) we have

\[
\left\| \sum_{j=1}^L C_j \psi_j \right\|^2 = \text{Diagonal} + \text{OffDiagonal},
\]

where

\[
\text{Diagonal} = \sum_{j=1}^L |C_j|^2 \|\psi_j\|^2 \quad \text{and} \quad \text{OffDiagonal} = \sum_{j=1}^L \sum_{1 \leq k \leq L: k \neq j} C_j \overline{C_k} \langle \psi_j, \psi_k \rangle.
\]

We apply the quadratic identity (0.5) with the choice

\[
\psi_j = v_{m,\ell,a}(A_1, A_2) \quad \text{and} \quad C_j = C_{m,\ell,a} = \frac{\mu(m)}{\varphi(m)}
\]

We add up the equalities (0.5) with the choice of (0.6) for all \( 1 \leq a \leq D \) with weight \( \chi_D(a) \).
Later in the proof we apply a routine “smoothing” that I completely skip; here for simplicity we just choose $A_1 = 1$ and $A_2 = ND$.

Write

$$V_a(M; A_1, A_2) = \sum_{1 \leq m \leq M, 1 \leq \ell \leq m: \gcd(m, \ell Z_0) = 1} \frac{\mu(m)}{\varphi(m)} v_{m, \ell; a}(A_1, A_2).$$

By (0.4),

$$\|V_a(M; A_1, A_2)\|^2 = \sum_{j: A_1 \leq jD + a \leq A_2} \left| \sum_{1 \leq m \leq M: \gcd(Z_0, m) = 1} \frac{\mu(m)}{\varphi(m)} \sum_{1 \leq \ell \leq m: \gcd(\ell, m) = 1} e^{2\pi i (a + jD) \ell / m} \right|^2. \quad (0.7)$$

Using the well-known Ramanujan’s sum

$$\sum_{1 \leq \ell < m: \gcd(\ell, m) = 1} e^{2\pi i n \ell / m} = \frac{\mu(m') \varphi(m)}{\varphi(m')}$$

in (0.7), where $m' = m / \gcd(n, m)$ and $\varphi(m)$ is the Euler’s function, we have

$$\sum_{a = 1}^{D} \chi_{-D}(a) \|V_a(M; A_1, A_2)\|^2 =$$

$$= \sum_{a = 1}^{D} \chi_{-D}(a) \sum_{j: A_1 \leq jD + a \leq A_2} \left( \sum_{d \geq 1: d | a + jD} \mu(d) \sum_{1 \leq k \leq M/d: \gcd(k, (a + jD)Z_0) = 1} \left| \frac{\mu(k)}{\varphi(k)} \right| \right)^2$$

$$= \prod_{p | Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{N} + \text{negligible error term}, \quad (0.8)$$

where the last step requires some routine (mostly elementary) estimations, using the following choices of the key parameters $M$ and $N$: let

$$\log N = (\log D)^{15}, \quad M = N \exp \left( \frac{-2}{3} \sqrt{\log N} \right). \quad (0.9)$$

On the other hand, we have

$$\|V_a(M; A_1, A_2)\|^2 = \text{Diagonal}_a(M; A_1, A_2) + \text{OffDiagonal}_a(M; A_1, A_2), \quad (0.10)$$

where

$$\text{OffDiagonal}_a(M; A_1, A_2) = \sum_{d \geq 1: \gcd(d, Z_0) = 1} \sum_{d | m_1, m_2} \frac{\mu(m_1) \mu(m_2)}{\varphi(m_1) \varphi(m_2)}.$$
\[ e^{2\pi i a (\frac{t_1}{m_1} - \frac{t_2}{m_2})} \cdot \sum_{j: A_1 \leq j D + a \leq A_2} e^{2\pi i D j (\frac{t_1}{m_1} - \frac{t_2}{m_2})}. \]  

(0.11)

Note that

\[ \sum_{a=1}^{D} \chi_D(a) \| V_a(M; A_1, A_2) \|^2 = \sum_{a=1}^{D} \chi_D(a) \text{OffDiagonal}_a(M; A_1, A_2), \]  

(0.12)

since the diagonal part clearly cancels out. The critical sum (see the end of (0.11))

\[ \sum_{a=1}^{D} \chi_D(a) e^{2\pi i a (\frac{t_1}{m_1} - \frac{t_2}{m_2})} \cdot \sum_{j: A_1 \leq j D + a \leq A_2} e^{2\pi i D j (\frac{t_1}{m_1} - \frac{t_2}{m_2})} \]  

(0.13)

can be estimated by involving the Gauss sums corresponding to the exceptional character \( \chi_D \) (and some routine approximation; see Section 3). The explicit formula for these Gauss sums brings in the extra factor \( \chi_D(m_1) \chi_D(m_2) \) involving the exceptional character \( \chi_D \). The crucial fact is that now in (0.11) we have the product

\[ \mu(m_1) \mu(m_2) \cdot \chi_D(m_1) \chi_D(m_2), \]

which is \( \mu^2(m_1) \mu^2(m_2) \) for the overwhelming majority of the pairs \( (m_1, m_2) \) showing up in the sum in (0.11) (indeed, the assumption of small class number \( h(-D) \) implies that \( \mu(m) = \chi_D(m) \) for the overwhelming majority of square-free integers \( 1 \leq m \leq M \) with \( \gcd(m, Z_0) = 1 \)). Since \( \mu^2(m_1) \mu^2(m_2) = 1 \) or 0 is positive, (0.11) becomes a relatively simple Riemann sum (apart from a negligible error), and we can approximate (0.12) with the corresponding definite integral. To illustrate what is going on here, consider the sum

\[ \sum_{j=1}^{D^4} \chi_D(j) \sum_{k=1}^{M/j} \frac{\log k}{k} = \sum_{j=1}^{D^4} \chi_D(j) \int_{x=1}^{M/j} \frac{\log x}{x} \, dx + \text{negligible} = \]

\[ = \sum_{j=1}^{D^4} \chi_D(j) \frac{1}{2} \log^2(M/j) + \text{negligible} = \]

\[ = \sum_{j=1}^{D^4} \chi_D(j) \frac{1}{2} (\log^2 M - 2 \log M \log j + \log^2 j) + \text{negligible} = \]

\[ = - \log M \sum_{j=1}^{D^4} \chi_D(j) \log j + \text{negligible}, \]

(0.14)

where we used the following facts: \( \sum_{j=1}^{D^4} \chi_D(j) = 0, \)

\[ \sum_{j=1}^{D^4} \chi_D(j) \log^2 j = O(\sqrt{D} \log^3 D) \]
(Pólya–Vinogradov plus partial summation), log \( M \) is much larger than log \( D \) (see (0.9)), and finally we used (0.16) below.

By using (0.11)-(0.13) and an argument similar to (0.14), we eventually obtain

\[
\sum_{a=1}^{D} \chi_{-D}(a) \| V_a(M; A_1, A_2) \|^2 = \left( \sum_{j=1}^{D^4} \chi_{-D}(j) \log j \right) \cdot \text{RoutineFactor} =
\]

\[
= \prod_{p|Z_0} \left(1 - \frac{1}{p} \right)^2 \prod_{p|Z_0, \chi_{-D}(p)=1} \left(1 + \frac{1}{p-1} \right) \prod_{p|Z_0, \chi_{-D}(p)=-1} \left(1 - \frac{1}{p+1} \right) \cdot D \log \frac{M^2}{N}, \tag{0.15}
\]

where “RoutineFactor” does not distinguish between the primes \( p|Z_0 \) with \( \chi_{-D}(p) = 1, 0 \) or \(-1\) (the power \( D^4 \) is an accidental choice and we ignored the negligible error). Comparing (0.8) and (0.15) we obtain that they are not equal—a contradiction that proves (0.1). They are not equal, because (0.8) does not distinguish, but (0.15) does distinguish between the primes \( p|Z_0 \) with \( \chi_{-D}(p) = 1, 0 \) or \(-1\) due to the sum \( \sum_{j=1}^{D^4} \chi_{-D}(j) \log j \). Here the first factor \( \chi_{-D}(j) \) comes from the Gauss sum, and the second factor \( \log j \) is explained by the fact that (0.11) resembles a “double harmonic sum”, which is a Riemann sum for a logarithmic integral that we can evaluate explicitly (somewhat like (0.14)).

Furthermore, if the class number \( h(-D) \) is “substantially smaller” than \( \sqrt{D} \), then we have the good approximation

\[
\sum_{j=1}^{D^4} \chi_{-D}(j) \log j = -\frac{\pi}{6} \sqrt{D} \sum_{(a,b,c)} \frac{1}{a} + \text{negligible}, \tag{0.16}
\]

where \( \sum_{(a,b,c)} \frac{1}{a} \) means that we add up the reciprocals of the leading coefficients in the family of reduced, primitive, inequivalent binary quadratic forms of integer coefficients with discriminant \(-D < 0\).

Also, under the same condition, we have

\[
\sum_{(a,b,c)} \frac{1}{a} = \prod_{p|Z_0, \chi_{-D}(p)\neq-1} \frac{p+1}{p-1} \prod_{p|D} \frac{p-1}{p} + \text{negligible}. \tag{0.17}
\]

(0.16) and (0.17) explain why (0.15) does distinguish between the primes \( p|Z_0 \) with \( \chi_{-D}(p) = 1, 0 \) or \(-1\).

In the rest of the paper we work out the details of this brief outline. Sections 1-5 contain (almost) all basic ideas and ingredients of the proof of Theorem 1 (case of negative discriminants). Section 6 is an outline of the necessary changes in the case of positive discriminants (see Theorem 2 in Section 1 below). Part Two, including Sections 7-30, covers all the remaining details; they are mostly routine estimations that I could not simplify.

The basic idea is relatively simple and (almost) elementary, but the execution of the simple plan required dozens of elementary estimations (like partial summations), which
made the paper ridiculously long. Unfortunately, I don’t know how to cut the paper to a “reasonable size”. Neither the assumption of class number one, nor the Riemann Hypothesis seem to make a difference here.

If a paper is more than 300 pages long, then it is almost inevitable that there are mistakes. Are these mistakes substantial? Is the basic idea (see Section 0, or Sections 1-5) still good? I don’t know the answer to these questions.

Part One includes Sections 0-6. The reader who is interested in Part Two should send me an email, and I will send him/her the pdf-file. My email address is jbeck@math.rutgers.edu

1. Introduction

We assume that the reader is familiar with both the elements of multiplicative number theory and the elements of the theory of binary quadratic forms (=quadratic fields); see e.g. Davenport’s well-known book [Da] (see also the books [Bo-Sh] and [Za]). Suppose for simplicity that $\Delta$ is a fundamental discriminant (i.e., either $\Delta \equiv 1 \pmod{4}$ squarefree, or $\Delta/4 \equiv 3 \pmod{4}$ squarefree, or $\Delta/8$ odd squarefree integer; the fundamental discriminants are precisely the discriminants of the quadratic fields). The class number $h(\Delta)$ (=the number of equivalence classes of binary quadratic forms of discriminant $\Delta$) plays a key role in higher arithmetic, and also in the distribution of the primes in arithmetic progressions. The class number is strongly related to the well-known infinite series (real Dirichlet $L$-function at $s = 1$)

$$L(1, \chi_\Delta) = \sum_{n=1}^{\infty} \frac{\chi_\Delta(n)}{n},$$

(1.1)

where $\chi_\Delta$ is the real primitive Dirichlet character modulo $|\Delta|$, corresponding to the quadratic field of discriminant $\Delta$. (Note that $\chi_\Delta(n) = (\Delta/n)$ is also called the Kronecker symbol: it is totally multiplicative in $n$, $\chi_\Delta(p) = (\Delta/p)$ Legendre symbol for every prime $p \geq 3$, $\chi_\Delta(2) = 1$ or $-1$ or $0$ according to $\Delta \equiv 1$ or $5 \pmod{8}$ or $\Delta \equiv 0 \pmod{4}$, and $\chi_\Delta(-1) = 1$ or $-1$ according to $\Delta > 0$ or $< 0$.) The connection is Dirichlet’s famous (analytic) class number formula,

$$L(1, \chi_\Delta) = \frac{2\pi h(\Delta)}{w \sqrt{|\Delta|}},$$

(1.2)

for $\Delta < 0$ (here $w = 2$ if $\Delta < -4$, and $w = 6$ or $4$ if $\Delta = -3$ or $-4$), and

$$L(1, \chi_\Delta) = \frac{h(\Delta) \log \eta_\Delta}{\sqrt{\Delta}},$$

(1.3)

for $\Delta > 0$ (here $\eta_\Delta > 1$ denotes the fundamental unit and log is the natural logarithm).
Dirichlet also proved a remarkable finite class number formula: for $\Delta < 0$,

$$h(\Delta) = -\frac{w}{2} \frac{\sqrt{\Delta}}{2} \sum_{n=1}^{\frac{|\Delta|}{2}} n \chi_\Delta(n),$$  \hspace{1cm} (1.4)$$

and for $\Delta > 0$,

$$h(\Delta) = -\frac{1}{\log \eta_\Delta} \sum_{n=1}^{\Delta-1} \chi_\Delta(n) \log \sin(\pi n / \Delta).$$  \hspace{1cm} (1.5)$$

To find a good lower bound for $L(1, \chi_\Delta)$ is a famous open problem. The best known result is Siegel’s ineffective lower bound

$$L(1, \chi_\Delta) > \frac{C_0(\varepsilon)}{|\Delta|^\varepsilon},$$  \hspace{1cm} (1.6)$$

which holds for any $\varepsilon > 0$ with some positive constant factor $C_0(\varepsilon)$ depending only on $\varepsilon > 0$. Unfortunately, there is no way to compute the constant $C_0(\varepsilon)$ whose existence is asserted in Siegel’s proof (this explains the term ineffective).

Combining (1.3) and (1.6), for $\Delta < 0$ we have

$$h(\Delta) > C'_0(\varepsilon)|\Delta|^{\frac{1}{2} - \varepsilon},$$

where $C'_0(\varepsilon)$ is another ineffective constant depending only on $\varepsilon > 0$. The best known effective result, due to a combined effort of Goldfeld [Go2] and Gross–Zagier [Gr-Za] involving elliptic curves, is much weaker than (1.6): it is basically logarithmic instead of the correct roughly square root order of magnitude. As an illustration, we mention the following explicit estimation, due to J. Oesterlé, who substantially improved on Goldfeld’s implicit constants:

$$h(\Delta) > \frac{\log |\Delta|}{55} \prod_{p|\Delta} \left(1 - \frac{|2\sqrt{p}|}{p + 1}\right).$$  \hspace{1cm} (1.7)$$

(Goldfeld’s method does not seem to work for positive discriminants.)

The objective of this paper is to prove a new effective lower bound to $L(1, \chi_\Delta)$. First I discuss the case of negative discriminant $\Delta < 0$.

**Theorem 1** For every negative fundamental discriminant $\Delta < 0$ with $|\Delta| > e^{10^{100}}$,

$$L(1, \chi_\Delta) \geq \frac{1}{(\log |\Delta|)^{146}}.$$  \hspace{1cm} (1.8)$$

The lower bound (1.8) also holds for positive fundamental discriminants $\Delta > 0$, but the proof is somewhat more complicated (this explains why we formulate the two cases in separate theorems).
Theorem 2 For every positive fundamental discriminant $\Delta > e^{10^{100}}$ we have

$$L(1, \chi_\Delta) \geq \frac{1}{(\log \Delta)^{146}}.$$  \hspace{1cm} (1.9)

Remarks. The exponent 146 of $\log |\Delta|$ in (1.8) and (1.9) is certainly far from the truth. In this paper I don’t make a serious effort to find the best exponent. Instead I focus on presenting the idea of the (unfortunately very long) proof as clearly as possible.

The basic idea of the proof of Theorem 2 is the same as that of Theorem 1, but there are some substantial differences in the details. We explain the necessary modifications in Sections 6 and 15.

Proof of Theorem 1. Assume that there is a “bad” negative fundamental discriminant $\Delta = -D < 0$ ($D$ is always positive in the whole paper) such that

$$L(1, \chi_{-D}) < \frac{1}{(\log D)^{146}} \quad \text{and} \quad D > e^{10^{100}}.$$  \hspace{1cm} (1.10)

Our goal is to derive a contradiction from (1.10).

By (1.2) and (1.10),

$$h(-D) < \frac{\sqrt{D}}{(\log D)^{146}}.$$  \hspace{1cm} (1.11)

We need the following technical lemma.

Lemma 1.1 Let $R^{(0)}_\Delta[N]$ denote the number of squarefree integers $n$ in $1 \leq n \leq N$ which are represented by an arbitrary binary quadratic form of discriminant $\Delta < 0$. For all $N > |\Delta|$, \n
$$\sum_{\substack{p \leq N \text{ prime} \\ \chi_\Delta(p) = 1}} 1 \leq R^{(0)}_\Delta[N] \leq \frac{12h(\Delta)N}{|\Delta|^{1/2}}.$$  \hspace{1cm} \vspace{1cm}

Proof. Let $F_j(x,y) = a_jx^2 + b_jxy + c_jy^2$, $1 \leq j \leq h(\Delta)$ be the family of reduced, primitive, inequivalent binary quadratic forms of integer coefficients with

$$4a_jc_j = -\Delta + b_j^2 = |\Delta| + b_j^2, \quad a_j > 0, \quad c_j > 0,$$

where reduced means

$$-a_j < b_j \leq a_j \leq c_j \quad \text{with} \quad b_j \geq 0 \quad \text{if} \quad a_j = c_j.$$

These facts imply the useful inequalities $0 < a \leq \sqrt{|\Delta|/3}$ and $c_j \geq |\Delta|/(4a_j)$.

We rewrite $F_j$ as follows:

$$F_j(x,y) = a_jx^2 + b_jxy + c_jy^2 = \frac{(2a_jx + b_jy)^2 + |\Delta|y^2}{4a_j}.$$  \hspace{1cm} (1.12)
This implies that \(a_j\) is the first (=least) squarefree integer represented by the binary form \(F_j\), and \(c_j\) is the second squarefree integer represented by \(F_j\). Let \(N > c_j\); then we have

\[
\sum_{p \leq N \text{ prime } \, F_j \text{ represents } p} 1 \leq \sum_{n \leq N \text{ squarefree } \, F_j \text{ represents } n} 1 \leq \frac{1}{2} \sum_{(x,y) \in \mathbb{Z}^2; F_j(x,y) \leq N} 1 \leq \sum_{1 \leq y \leq \sqrt{4a_j N/|\Delta|}} \left\lfloor \frac{\sqrt{4a_j N}}{2a_j} \right\rfloor \leq 2 \left(1 + \sqrt{\frac{4a_j N}{|\Delta|}}\right) \left(1 + \frac{\sqrt{4a_j N}}{2a_j}\right) = \frac{4N}{\sqrt{|\Delta|}} + 4\sqrt{\frac{\sqrt{|\Delta|} N}{|\Delta|}} + 2\sqrt{N} + 2 = \frac{4N}{\sqrt{|\Delta|}} + 4\sqrt{\frac{N}{|\Delta|}} + 2\sqrt{N} + 2 \leq \frac{12N}{\sqrt{|\Delta|}},
\]

if \(N > |\Delta|\) (where, as usual, \([z]\) and \(\lfloor z\rfloor\) denote the upper and lower integral parts of a real number \(z\)). Since \(j\) runs in \(1 \leq j \leq h(\Delta)\), (1.12) implies Lemma 1.1. □

Lemma 1.1 implies that the primes \(\leq N\) are concentrated in the residue classes with \(\chi_D(a) = -1\), explaining why I refer to them as the “rich” residue classes modulo \(D\). The rest are the “poor” residue classes. (Here we assume that \(N\) is larger, but not much much larger than \(D\); the precise definition comes soon.) Since the “poor” half of the residue classes contain relatively few primes \(\leq N\), the “rich” half of residue classes contain on average twice as many primes \(\leq N\) as expected.

(It is interesting to point out that, by the well-known Brun–Titchmarsh–Selberg theorem, no residue class can contain more than \(2 + o(1)\) times as many primes as expected. So for the “rich” residue classes the average density is almost the same as the maximum density. Note, however, that in this paper we don’t use the Brun–Titchmarsh–Selberg theorem.)

We often use the prime number theorem, which in its simplest form states

\[
\pi(N) = \sum_{p \leq N} 1 = \frac{N}{\log N} + O \left( N \left( \log N \right)^{-2} \right),
\]

but at some point of the proof we need the following much more precise result (see e.g. [Da] or [Iw-Ko] or [Ka]):

\[
\left| \pi(N) - \int_2^N \frac{dx}{\log x} \right| = O(N e^{-\sqrt{\log N}}).
\]
We will also apply (usually in a weaker form) the following well-known asymptotic results related to the primes ($\gamma_0$ denotes Euler’s constant):

\[
\prod_{p \leq N} \left( 1 - \frac{1}{p} \right) = (1 + o(1)) \frac{e^{-\gamma_0}}{\log N},
\]

\[
\prod_{p \leq N} \left( 1 + \frac{1}{p} \right) = (1 + o(1)) \frac{6e^{\gamma_0} \log N}{\pi^2},
\]

\[
\prod_{p \leq N} p = e^{(1+o(1))N}.
\]

The proof of Theorem 1 consists of several steps. The first step is to construct a large number of almost orthogonal complex vectors in a high dimensional vector space. To take advantage of almost orthogonality, we apply a simple quadratic identity (see (1.17) below). This is a little bit similar to the basic idea of the Large Sieve, but here we work with a single modulus: we restrict ourselves to the reduced residue classes modulo $D$, where $-D$ is the “bad” discriminant.

We begin with a simple remark. At a later stage of the proof, estimating some error term, it would be helpful to have an estimation like

\[
\sum_{p \leq N: \chi_{-D}(p) = 1} \frac{1}{p} = o(1).
\]

Unfortunately, this is not necessarily true in general, due to the possible existence of very small primes $p$ with $\chi_{-D}(p) = 1$. Luckily, we can get around this problem: the weaker statement

\[
\sum_{D < p \leq N: \chi_{-D}(p) = 1} \frac{1}{p} = O(h(-D) \log N/\sqrt{D}) = o(1)
\]

suffices for our purposes. Note that this weaker statement is a consequence of Lemma 1.1, and $o(1)$ clearly holds under the condition (1.11), assuming $N$ is not too large. This motivates why we sieve out (among others) the primes $p \leq D$ with $\chi_{-D}(p) = 1$ or 0.

To define the “sieve out” procedure precisely, we distinguish two cases. Let $p_0$ denote the smallest prime that is not a divisor of the discriminant $D$; clearly

\[
p_0 < 2 \log D. \quad (1.13)
\]

Indeed, (1.13) follows from the fact

\[
\prod_{p \leq N} p = e^{(1+o(1))N}
\]

with the choice $N = \log D$. If $\chi_{-D}(p_0) = 1$ then let

\[
Z_0 = \prod_{p \leq D: \chi_{-D}(p) \neq -1} p \quad (1.14)
\]
be the product of all primes \( p \leq D \) with \( \chi_D(p) = 1 \) or 0. Note that \( p_0 | Z_0 \); moreover, if an integer \( m \) is relatively prime to \( Z_0 \), then \( m \) is automatically coprime to \( D \) (since the prime factors of \( D \) are listed in \( Z_0 \)).

If \( \chi_D(p_0) = -1 \) then we have two possible choices for \( Z_0 \): either (1.14), i.e.,

\[
Z_0 = Z'_0 = \prod_{p \leq D: \chi_D(p) \neq -1} p
\]

or we include \( p_0 \) as an extra prime factor:

\[
Z_0 = Z''_0 = p_0 Z'_0 = p_0 \prod_{p \leq D: \chi_D(p) \neq -1} p.
\]

(1.15)

We discuss the two possible choices \( Z_0 = Z'_0 \) and \( Z_0 = Z''_0 \) in a unified way.

Note that we are going to repeatedly use the following consequence of sieving out the primes \( p \leq D \) with \( \chi_D(p) = 1 \) or 0: if \( m \) is coprime to \( Z_0 \) and squarefree, then \( \mu(m) = \chi_D(m) \) for all \( 1 \leq m \leq D \).

Warning about the notation. It is important to remember that in the rest of the paper \( p, p_1, p_2, \ldots \) and \( q, q_1, q_2, \ldots \) always denote primes, and \( p_0 \) is reserved for the smallest prime that is not a divisor of the discriminant \( D \) (see (1.13)). Also \( c_1, c_2, \ldots \) denote effectively computable constants; in fact, from now on every constant will be effective. I use \( \gamma_0 = 0.5772 \ldots \) for the Euler’s constant (instead of the more common \( \gamma \)), \( \log \) denotes the natural logarithm (instead of \( \ln \)). The rest of the notation is more or less standard; e.g., \( \varphi(n) \) denotes the Euler’s function, \( \tau(n) \) is the number of divisors of \( n \), \( \mu(n) \) is the Möbius function, \( A | B \) means that the integer \( B \) is divisible by the integer \( A \), and \( A \nmid B \) means that \( B \) is not divisible by \( A \).

Let \( m \geq 1 \) be a squarefree integer relatively prime to \( Z_0 \), and let \( \ell \) be an integer in \( 1 \leq \ell < m \) relatively prime to \( m \). Given \( 0 < A_1 < A_2 \), let \( v_{m, \ell, a}(A_1, A_2) \) denote the vector

\[
v_{m, \ell, a}(A_1, A_2) = (e^{2\pi i j_1 D + a} \ell / m, e^{2\pi i (j_1 + 1) D + a} \ell / m, \ldots, e^{2\pi i (j_2 D + a) \ell / m})
\]

(1.16)

that plays a key role in the proof; here \( j = j_1 \) is the smallest integer with \( jD + a \geq A_1 \), \( j = j_2 \) is the largest integer with \( jD + a \leq A_2 \), and of course \( i = \sqrt{-1} \). Note that the dimension of \( v_{m, \ell, a}(A_1, A_2) \) is \( j_2 - j_1 + 1 = (A_2 - A_1) / D + O(1) \).

We need the usual inner product of complex vectors: if \( \psi = (a_1, \ldots, a_n) \) and \( \xi = (b_1, \ldots, b_n) \) with \( a_j, b_j \) complex numbers, then

\[
\langle \psi, \xi \rangle = \sum_{j=1}^n a_j \overline{b_j}
\]

where \( \overline{b_j} \) is the complex conjugate of \( b_j \). We write, as usual,

\[
\|\psi\| = \sqrt{\langle \psi, \psi \rangle}, \quad \text{so} \quad \|\psi\|^2 = \langle \psi, \psi \rangle
\]
is the square of the norm. For arbitrary complex numbers $C_j$ we have
\[ \left\| \sum_{j=1}^{L} C_j \psi_j \right\|^2 = \text{Diagonal} + \text{OffDiagonal}, \tag{1.17} \]
where
\[ \text{Diagonal} = \sum_{j=1}^{L} |C_j|^2 \|\psi_j\|^2 \]
and
\[ \text{OffDiagonal} = \sum_{j=1}^{L} \sum_{1 \leq k \leq L: k \neq j} C_j \overline{C}_k \langle \psi_j, \psi_k \rangle. \]

We apply the quadratic identity (1.17) with the choice
\[ \psi_j = v_{m,\ell,a}(A_1, A_2) \quad \text{and} \quad C_j = C_{m,\ell,a} = \frac{\mu(m)}{\varphi(m)} \tag{1.18} \]
for a fixed residue class $a$ modulo $D$. (Note that the alternative choice
\[ C_j = C_{m,\ell,a} = \frac{\mu(m)}{m} \tag{1.18'} \]
would be equally good.)

Later we are going to add up these equalities (i.e., (1.17) with the choice of (1.18))
for all $1 \leq a \leq D$ with weight $\chi_{-D}(a)$. The reason behind combining the Möbius function
with the real character $\chi_{-D}$ is the “similarity” between the two functions (restricted to
square-free integers) under the condition that $h(-D)$ is “small”. (I will return to this
guiding intuition after (2.10) below.) We will specify the parameters $0 < A_1 < A_2$ later in
Section 2; see (2.31).

2. More on our high-dimensional almost orthogonal vectors,
and some routine smoothing

Write
\[ \mathbf{V}_a(M; A_1, A_2) = \sum_{1 \leq m \leq M, \ 1 \leq \ell \leq m: \gcd(m, \ell, \mathbb{Z}_0) = 1} \frac{\mu(m)}{\varphi(m)} v_{m,\ell,a}(A_1, A_2). \]
Thus by (1.16),

\[
\|\mathbf{V}_a(M; A_1, A_2)\|^2 = \sum_{j: A_1 \leq jD + a \leq A_2} \left( \sum_{1 \leq m \leq M: \gcd(Z_0, m) = 1} \frac{\mu(m)}{\varphi(m)} \sum_{1 \leq \ell \leq m: \gcd(\ell, m) = 1} e^{2\pi i (a + jD)\ell/m} \right)^2.
\]  

(2.1)

I recall the so-called Ramanujan’s sum (see Theorem 272 in Hardy–Wright [Ha-Wr]):

\[
\sum_{1 \leq \ell < m: \gcd(\ell, m) = 1} e^{2\pi i \ell/m} = \frac{\mu(m')\varphi(m)}{\varphi(m')},
\]  

(2.2)

where \( m' = m / \gcd(n, m) \). As usual, \( \gcd \) denotes the greatest common divisor;

\[
\varphi(m) = m \prod_{p | m} \left(1 - \frac{1}{p}\right)
\]

denotes the Euler’s function; \( a | b \) means that \( b \) is divisible by \( a \); and \( \mu(n) \) stands for the Möbius function: \( \mu(1) = 1 \), and for \( n \geq 2 \), \( \mu(n) = (-1)^r \) if \( n = p_1 \cdots p_r \) with \( r \) distinct prime factors, and, finally, \( \mu(n) = 0 \) if \( n \geq 2 \) is not squarefree.

By using (2.2), we have

\[
\sum_{1 \leq m \leq M: \gcd(Z_0, m) = 1} \frac{\mu(m)}{\varphi(m)} \sum_{1 \leq \ell \leq m: \gcd(\ell, m) = 1} e^{2\pi i \ell/m} =
\]

\[
= \sum_{1 \leq m \leq M: \gcd(Z_0, m) = 1} \frac{\mu(m)}{\varphi(m)} \cdot \frac{\mu(m / \gcd(m, n))\varphi(m)}{\varphi(m / \gcd(m, n))} =
\]

\[
= \sum_{d \geq 1: \gcd(Z_0, d) = 1} \frac{\mu(d)\varphi(d)}{\varphi(m)} \sum_{1 \leq m \leq M: \gcd(Z_0, m) = 1} \frac{\mu^2(m)}{\varphi(m)}.
\]

Using this with \( n = a + jD \) in (2.1), we have

\[
\|\mathbf{V}_a(M; A_1, A_2)\|^2 =
\]

\[
= \sum_{j: A_1 \leq jD + a \leq A_2} \left( \sum_{d \geq 1: \gcd(Z_0, d) = 1} \frac{\mu(d)\varphi(d)}{\varphi(m)} \sum_{1 \leq m \leq M: \gcd(Z_0, m) = 1} \frac{\mu(m)}{\varphi(m)} \right)^2.
\]
\[
\sum_{j: A_1 \leq jD + a \leq A_2} \left( \sum_{d \geq 1: \gcd(Z_0, d) = 1 \atop d | a + jD} \mu(d) \sum_{1 \leq k \leq M/d: \gcd(k, (a + jD)Z_0) = 1} \left| \frac{\mu(k)}{\varphi(k)} \right| \right)^2 \quad (2.3)
\]

**Remark.** At first sight the big sum (2.3) may seem hopelessly complicated. We will show, however, that (2.3) is not so bad. For illustration note that the evaluation of the square

\[
\left( \sum_{d \geq 1: \gcd(Z_0, d) = 1 \atop d | a + jD} \mu(d) \sum_{1 \leq k \leq M/d: \gcd(k, (a + jD)Z_0) = 1} \left| \frac{\mu(k)}{\varphi(k)} \right| \right)^2
\]

at the end of (2.3) is quite simple if \( a + jD \) is a large prime: \( a + jD = p > M \). Indeed, then we have

\[
\sum_{d \geq 1: \gcd(Z_0, d) = 1 \atop d | p} \mu(d) \sum_{1 \leq k \leq M/d: \gcd(k, pZ_0) = 1} \left| \frac{\mu(k)}{\varphi(k)} \right| = \sum_{1 \leq m \leq M: \gcd(m, Z_0) = 1} \left| \frac{\mu(m)}{\varphi(m)} \right|.
\]

We use the fact

\[
\sum_{1 \leq m \leq M: \gcd(m, Z_0) = 1} \left| \frac{\mu(m)}{\varphi(m)} \right| = \prod_{p | Z_0} \left( 1 - \frac{1}{p} \right) (\log M + c) + \text{negligible}, \quad (2.4)
\]

where \( c \) is some explicit absolute constant. Note that (2.4) is the special case \( Q = 1 \) of the following lemma (in later applications we need the most general form of the lemma).

**Lemma 2.1** Assume that \( Z_0 \) and \( Q \geq 1 \) are relatively prime integers, then

\[
\left| \sum_{1 \leq k \leq L: \gcd(k, QZ_0) = 1} \frac{\mu^2(k)}{\varphi(k)} \prod_{q | QZ_0} \left( 1 - \frac{1}{q} \right) \left( \log L + \gamma_0 + 2\gamma^* - \gamma^{**} + \sum_{q | QZ_0} \frac{\log q}{q} \right) \right| \leq
\]

\[
\frac{10^4 \tau(Q) \log D \log L}{L^{1/4}} + \frac{10^4 (10 + \log Q)}{D^5} + \frac{4 \left( 10 + \log L + 2(\log D)^2 + (\log Q)^2 \right)}{\max \{ LD^{-6 \log D}, 1 \}},
\]

where \( \gamma_0 \) is the Euler’s constant, \( \gamma^*, \gamma^{**} \) are two absolute constants defined by the following convergent prime-series

\[
\gamma^* = \sum_p \frac{\log p}{p^2 - 1} \quad \text{and} \quad \gamma^{**} = \sum_p \frac{\log p}{p(p + 1)},
\]

and \( \tau(Q) = \sum_{d \mid Q} 1 \) is the number of divisors of \( Q \).
Probably the reader is wondering why we use the at first sight artificially complicated requirement $\gcd(k, QZ_0) = 1$ in Lemma 2.1 instead of simply writing $\gcd(k, Q) = 1$ with $Z_0|Q$. The reason is that in the error term of Lemma 2.1 we have the factor $\tau(Q)$ (and it is necessary to have it there), and

$$Z_0 = Z_0' = \prod_{p \leq D; \chi_D(p) \neq -1} p \quad \text{or} \quad Z_0 = Z_0'' = p_0Z_0'$$

can be exponentially large in terms of $D$. So if $Z_0$ is a divisor of $Q$, then $\tau(Q) \geq \tau(Z_0)$, where $\tau(Z_0)$ is possibly exponentially large. Such an extremely large factor $\tau(Q)$ in the error term would make Lemma 2.1 nearly useless.

We postpone the elementary proof of Lemma 2.1 to Section 30.

Let’s return to (2.1): an alternative way to evaluate it is to apply (1.17):

$$\|V_a(M; A_1, A_2)\|^2 = \text{Diagonal}_a(M; A_1, A_2) + \text{OffDiagonal}_a(M; A_1, A_2). \quad (2.5)$$

Note that

$$\langle v_{m, \ell_1, a}(A_1, A_2), v_{m, \ell_2, a}(A_1, A_2) \rangle = \sum_{A_1 \leq n \leq A_2: n \equiv a \pmod{D}} e^{2\pi i \left( \frac{\ell_1}{m_1} - \frac{\ell_2}{m_2} \right)} =$$

$$= e^{2\pi i a \left( \frac{\ell_1}{m_1} - \frac{\ell_2}{m_2} \right)} \sum_{j: A_1 \leq jD + a \leq A_2} e^{2\pi i D j \left( \frac{\ell_1}{m_1} - \frac{\ell_2}{m_2} \right)}, \quad (2.6)$$

and

$$\langle v_{m, \ell_1, a}(A_1, A_2), v_{m, \ell_1, a}(A_1, A_2) \rangle = \|v_{m, \ell_1, a}\|^2 = \sum_{A_1 \leq n \leq A_2: n \equiv a \pmod{D}} 1. \quad (2.7)$$

By (2.7) we have

$$\text{Diagonal}_a(M; A_1, A_2) = D_a(M; A_1, A_2) =$$

$$= \sum_{1 \leq m \leq M, 1 \leq \ell \leq m: \gcd(\ell Z_0, m) = 1} \left( \frac{\mu(m)}{\varphi(m)} \right)^2 \|v_{m, \ell_1, a}\|^2 =$$

$$= \sum_{1 \leq m \leq M: \gcd(Z_0, m) = 1} \varphi(m) \left( \frac{\mu(m)}{\varphi(m)} \right)^2 \|v_{m, \ell_1, a}\|^2 =$$

$$= \left( \sum_{A_1 \leq n \leq A_2: n \equiv a \pmod{D}} 1 \right) \sum_{1 \leq m \leq M: \gcd(Z_0, m) = 1} \frac{\mu(m)}{\varphi(m)}, \quad (2.8)$$
and by (2.6),

$$\text{OffDiagonal}_a(M; A_1, A_2) = OD_a(M; A_1, A_2) = \sum_{d \geq 1: \gcd(d, Z_0) = 1} OD_a(M; A_1, A_2; d), \quad (2.9)$$

where

$$OD_a(M; A_1, A_2; d) = \sum_{(m_1, \ell_1) \neq (m_2, \ell_2): 1 \leq m_1, m_2 \leq M \atop 1 \leq \ell_h \leq m_h, \gcd(\ell_h, m_h)=1, h=1,2 \atop \gcd(m_1, m_2)=d, \gcd(m_1 m_2, Z_0)=1} \frac{\mu(m_1) \mu(m_2)}{\phi(m_1) \phi(m_2)} \cdot e^{2\pi i a \left(\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2}\right)}.$$  \quad (2.10)

The proof of Theorem 1 is very long, but the basic idea is relatively simple, and it is explained in the first 5 sections.

Here is a brief intuitive explanation of why we combine the Möbius function $\mu(m)$ with exponential sums related to the reduced residue classes modulo $D$. The exponential sum related to the fixed modulus $D$ leads to a Gauss sum (see Section 3; especially Lemma 3.1), and the Gauss sum contains the real character $\chi_{-D}(m)$ as a factor. If $m$ is a squarefree integer, relatively prime to $D$, then $\chi_{-D}(m)$ and $\mu(m)$ are “typically equal”, assuming $m$ has no “small” prime divisor $p$ with $\chi_{-D}(p) = 1$ and $h(-D)$ is “substantially” smaller than $\sqrt{D}$. This “almost equality” of $\chi_{-D}(m)$ and $\mu(m)$ makes it possible to find a good estimation for the off-diagonal part (2.9)-(2.10), which is the hardest part of the proof of Theorem 1. I refer to this argument as the “almost equality of $\chi_{-D}(m)$ and $\mu(m)$”.

Sections 2 and 3 are routine but important preparatory sections. Sections 4 and 5 are the two most important sections; Lemma 5.4 and Lemma 5.5 are the two crucial lemmas. The rest are basically long but routine estimations.

The parameters $A_1, A_2$ indicate that the underlying interval is $A_1 \leq n \leq A_2$; more precisely, at the end of (2.10) we have the following exponential sum

$$\sum_{j: A_1 \leq jD+a \leq A_2} e^{2\pi i D j \left(\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2}\right)}.$$  \quad (2.11)

It is very useful to take a certain average here (where $A_1$ and $A_2$ are variables): we employ the standard trick of “smoothing” in Fourier analysis. The details go as follows.

I start with the well-known Dirichlet kernel

$$\sum_{k=-n}^{n} e^{ikx} = \frac{\sin((n + \frac{1}{2})x)}{\sin(x/2)}.$$  \quad (2.12)

Squaring (2.12), we obtain the Fejér kernel

$$\left(\frac{\sin((n + \frac{1}{2})x)}{\sin(x/2)}\right)^2 = \left(\sum_{k=-n}^{n} e^{ikx}\right)^2 = \sum_{\ell=-2n}^{2n} (2n + 1 - |\ell|) e^{i\ell x}.$$
= S_{2n}(x) + S_{2n-1}(x) + S_{2n-2}(x) + \ldots + S_1(x) + S_0(x), \tag{2.13}

where

\[ S_m(x) = \sum_{k=-m}^{m} e^{ikx}. \tag{2.14} \]

The sequence of coefficients \(2n+1-|\ell|\) of \(e^{i\ell x}\) in (2.13) represents the “roof function”

\[ f_2(y) = \begin{cases} \frac{1}{2} - \frac{|y|}{4}, & \text{if } |y| \leq 2; \\ 0, & \text{if } |y| > 2; \end{cases} \tag{2.15} \]

in the sense that

\[ \frac{2n+1-|\ell|}{(2n+1)^2} \cdot n = \frac{1}{2} - \frac{|y|}{4} + O(1/n) \quad \text{with } y = \ell/n \tag{2.16} \]

(here the factor \(n\) is explained by the renorming \(y = \ell/n\)). Note that \(f_2(y)\) is familiar from probability theory: it is the density function of the convolution—denoted by \(*\)—of the uniform distribution in \([-1, 1]\) with itself; formally,

\[ f_2(y) = f_1 * f_1(y) = \int_{-\infty}^{\infty} f_1(y-z)f_1(z) \, dz, \]

where

\[ f_1(y) = \begin{cases} \frac{1}{2}, & \text{if } |y| \leq 1; \\ 0, & \text{if } |y| > 1; \end{cases} \tag{2.17} \]

is the density function of the uniform distribution in \([-1, 1]\).

We rewrite (2.13) in the form

\[ \left( \frac{\sin(n+\frac{1}{2})x}{(2n+1)\sin(x/2)} \right)^2 = \sum_{k=0}^{2n} \frac{S_k(x)}{2k+1} \cdot \frac{2k+1}{(2n+1)^2} \tag{2.18} \]

with \(\sum_{k=0}^{2n} \frac{2k+1}{(2n+1)^2} = 1\), i.e., the weights \(\frac{2k+1}{(2n+1)^2}\), \(0 \leq k \leq 2n\) in (2.18) represent a discrete probability distribution.

It is a standard exercise in probability theory to compute the higher convolution powers of the uniform distribution in \([-1, 1]\). A routine calculation gives

\[ f_3(y) = f_1 * f_1 * f_1(y) = \begin{cases} \frac{3-y^2}{8}, & \text{if } |y| \leq 1; \\ \frac{(3-|y|)^2}{16}, & \text{if } 1 \leq |y| \leq 3; \\ 0, & \text{if } |y| > 3. \end{cases} \tag{2.19} \]

Consider now the third power of (2.12):

\[ \left( \frac{\sin(n+\frac{1}{2})x}{\sin(x/2)} \right)^3 = \left( \sum_{k=-n}^{n} e^{ikx} \right)^3 = \sum_{\ell=-3n}^{3n} B^{(3)}_{\ell} e^{i\ell x}. \tag{2.20} \]
It is not too difficult to compute the coefficients $B^{(3)}_\ell$ of $e^{i\ell x}$ explicitly, but we don’t really need that. What we need is the analog of (2.15)-(2.16):

$$\frac{B^{(3)}_\ell}{(2n+1)^3} \cdot n = f_3(\ell/n) + O(1/n),$$  \hspace{1cm} (2.21)

where $f_3(y)$ is the function defined in (2.19). We also need the easy fact that the sequence $B^{(3)}_\ell$ is monotonic in the following sense:

$$B^{(3)}_{\ell_1} \geq B^{(3)}_{\ell_2} \geq 0 \text{ if } |\ell_1| \leq |\ell_2|.$$  \hspace{1cm} (2.22)

Therefore, returning to (2.20), we obtain the following analog of (2.18):

$$\left( \frac{\sin(n + \frac{1}{2})x}{(2n+1)\sin(x/2)} \right)^3 = \left( \frac{1}{2n+1} \sum_{k=-n}^{n} e^{ikx} \right)^3 = (2n+1)^{-3} \sum_{\ell=-3n}^{3n} B^{(3)}_\ell \cdot e^{i\ell x} = \sum_{k=0}^{3n} w^{(3,n)}_k \frac{S_k(x)}{2k+1},$$  \hspace{1cm} (2.23)

where $w^{(3,n)}_k$ are positive weights: $w^{(3,n)}_k \geq 0$, and $\sum_{k=0}^{3n} w^{(3,n)}_k = 1$, i.e., the coefficients $w^{(3,n)}_k$ form a discrete probability distribution. So (2.23) is an average of $\frac{S_k(x)}{2k+1}$, $0 \leq k \leq 3n$ (we divide by $2k+1$ since the sum $S_k(x)$ has $2k+1$ terms; see (2.14)).

In general, consider the $\kappa$th power of (2.12) where $\kappa \geq 2$ is an arbitrary integer:

$$\left( \frac{\sin(n + \frac{1}{2})x}{(2n+1)\sin(x/2)} \right)^\kappa = \left( \sum_{k=-n}^{n} e^{ikx} \right)^\kappa = \sum_{\ell=-\kappa n}^{\kappa n} B^{(\kappa)}_\ell \cdot e^{i\ell x}.$$  \hspace{1cm} (2.24)

For a general $\kappa \geq 2$ it is not easy to compute the coefficients $B^{(\kappa)}_\ell$ of $e^{i\ell x}$ explicitly, but we don’t really need that. We need less: we just need the analog of (2.21):

$$\frac{B^{(\kappa)}_\ell}{(2n+1)^\kappa} \cdot n = f_\kappa(\ell/n) + O(1/n),$$  \hspace{1cm} (2.25)

where

$$f_\kappa(y) = f_1 * f_1 * \cdots * f_1(y)$$  \hspace{1cm} (2.26)

is the $\kappa$th convolution power of $f_1$ (see (2.17)). It is not very hard to prove by induction on $\kappa$ the following generalization of (2.15) and (2.19) (see e.g. Rényi’s book [Re]):

$$f_\kappa(y) = \frac{1}{2^{\kappa(\kappa-1)!}} \sum_{j=0}^{[(\kappa+|y|)/2]} (-1)^j \binom{\kappa}{j} (\kappa + |y| - 2j)^{\kappa-1} \text{ if } |y| \leq \kappa,$$  \hspace{1cm} (2.27)
and 0 if $|y| > \kappa$. Note that $f_\kappa(y)$ is $(\kappa - 2)$-times differentiable, and consists of a few generalized parabola arcs of degree $\kappa - 2$ (due to the jumps of the lower integral part function $[(\kappa + y)/2]$ as $y$ runs in $0 \leq y \leq \kappa$) that smoothly fit together at the endpoints if $\kappa \geq 3$.

We also need the easy fact that the sequence $B_\ell^{(\kappa)}$ is monotonic in the following sense:

$$B_{\ell_1}^{(\kappa)} \geq B_{\ell_2}^{(\kappa)} \geq 0 \text{ if } |\ell_1| \leq |\ell_2|. \quad (2.28)$$

Therefore, returning to (2.24) we have

$$\left( \frac{\sin(n + \frac{1}{2})x}{(2n + 1)\sin(x/2)} \right)^\kappa = \left( \frac{1}{2n + 1} \sum_{k=-n}^{n} e^{ikx} \right)^\kappa = (2n + 1)^{-\kappa} \sum_{\ell=-\kappa n}^{\kappa n} B_\ell^{(\kappa)} e^{i\ell x} = \sum_{k=0}^{\kappa n} w_k^{(\kappa,n)} S_k \frac{1}{2k + 1} (x), \quad (2.29)$$

where $w_k^{(\kappa,n)}$ are positive weights: $w_k^{(\kappa,n)} \geq 0$ with $\sum_{k=0}^{\kappa n} w_k^{(\kappa,n)} = 1$ ("discrete probability distribution"). So (2.29) is an average of $\frac{S_k(x)}{2k + 1}$, $0 \leq k \leq \kappa n$.

Note that in (2.29) there is an underlying limit as $\kappa$ increases. Indeed, due to the Central Limit Theorem,

$$f_\kappa(y) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{\kappa/3}} e^{-\frac{\kappa y^2}{2 \pi}} + O(1/\kappa). \quad (2.30)$$

Here the factor $\sqrt{\kappa/3}$ is the "standard deviation". Indeed, in view of (2.26), what we are dealing with here is a sum of $\kappa$ independent random variables, where each component is uniformly distributed in $[-1, 1]$. This is why the variance is $\kappa \int_0^1 x^2 \, dx = \kappa/3$, and this is why (2.30) is a special case of the strong form of the Central Limit Theorem with explicit error term estimation.

Now we are ready to define an efficient average over the variables $A_1, A_2$ introduced in Section 1. Let $\kappa$ and $N$ be fixed positive integers (in the application later we choose $\kappa$ to be less than 10—note in advance that $\kappa = 8$ is a good choice—and $N$ is "large"), and let $k$ be an integral variable running in $0 \leq k \leq \kappa N$. Write

$$A_1 = A_1(a; k) = ((\kappa N - k) + a)D \quad \text{and} \quad A_2 = A_2(a; k) = ((\kappa N + k) + a)D. \quad (2.31)$$

By (2.31)

$$\{j : A_1(a; k) \leq jD + a \leq A_2(a; k)\} = \{j : \kappa N - k \leq j \leq \kappa N + k\} \quad (2.32)$$

is a set of $2k + 1$ consecutive integers (this explains the division by $2k + 1$ below). Write

$$W_a(M; \kappa; N) = \sum_{k=0}^{\kappa N} w_k^{(\kappa,N)} \left\| V_a(M; A_1(a; k), A_2(a; k)) \right\|^2 \cdot \frac{1}{2k + 1} =$$
\[
= \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k+1} \sum_{j: \kappa N - k \leq j \leq \kappa N + k} \left( \sum_{d \geq 1: \gcd(Z_0, d) = 1} \mu(d) \sum_{1 \leq k \leq M / d: \gcd(k, (a + jD)Z_0) = 1} \frac{\left| \mu(k) \right|}{\varphi(k)} \right)^2
\]

where \( w_k^{(\kappa, N)} \), \( 0 \leq k \leq \kappa N \) is the discrete probability distribution defined in (2.29), and in the last step we used (2.3) and (2.32).

By (2.5)-(2.10) and (2.31)-(2.33), we have

\[
W_a(M; \kappa; N) = \text{WDiag}_a(M; \kappa; N) + \text{WOOffDiag}_a(M; \kappa; N),
\]

where

\[
\text{WDiag}_a(M; \kappa; N) = \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{\text{Diagonal}_a(M; A_1(a; k), A_2(a; k))}{2k+1} = \\
= \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \cdot \frac{1}{2k+1} \left( \sum_{A_1(a; k) \leq n \leq A_2(a; k): n \equiv a (\text{mod} D)} 1 \right) \sum_{1 \leq m \leq M: \gcd(Z_0, m) = 1} \frac{\left| \mu(m) \right|}{\varphi(m)} = \\
= \left( \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \right) \sum_{1 \leq m \leq M: \gcd(Z_0, m) = 1} \frac{\left| \mu(m) \right|}{\varphi(m)} = \sum_{1 \leq m \leq M: \gcd(Z_0, m) = 1} \frac{\left| \mu(m) \right|}{\varphi(m)},
\]

and

\[
\text{WOOffDiag}_a(M; \kappa; N) = \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{\text{OffDiagonal}_a(M; A_1(a; k), A_2(a; k))}{2k+1} = \\
= \sum_{d \geq 1: \gcd(d, Z_0) = 1} \text{WOD}_a(M; \kappa; N; d),
\]

where

\[
\text{WOD}_a(M; \kappa; N; d) = \sum_{(m_1, \ell_1) \neq (m_2, \ell_2): 1 \leq m_1, m_2 \leq M} \frac{\mu(m_1)\mu(m_2)}{\varphi(m_1)\varphi(m_2)} \\
\cdot e^{2\pi i a(\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2})} \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \cdot \frac{1}{2k+1} \sum_{j: A_1(a; k) \leq jD + a \leq A_2(a; k)} e^{2\pi i j(\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2})}. \tag{2.37}
\]

By (2.32),

\[
\sum_{j: A_1(a; k) \leq jD + a \leq A_2(a; k)} e^{2\pi i j(\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2})} = e^{i\kappa N x} S_k(x) \quad \text{with} \quad x = 2\pi D(\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2}), \tag{2.38}
\]

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where $S_k(x)$ is the exponential sum defined in (2.14).

Therefore, by using (2.29) with $x = 2\pi D(\ell_1 - \ell_2)$, (2.37) and (2.38) imply

$$WOD_a(M; \kappa; N; d) = \sum_{(m_1, \ell_1) \neq (m_2, \ell_2): 1 \leq m_1, m_2 \leq M, 1 \leq \ell_1 \leq m_h, \gcd(\ell_h, m_h) = 1, h = 1, 2 \gcd(m_1, m_2) = d, \gcd(m_1, m_2, Z_0) = 1} \frac{\mu(m_1)\mu(m_2)}{\varphi(m_1)\varphi(m_2)}.$$}

With $\delta = \pm 1$, write

$$\Omega_{\delta; \kappa; N}(M) = \sum_{1 \leq a \leq D, \chi_D(a) = \delta} W_a(M; \kappa; N) = \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \sum_{j: \kappa N - k \leq j \leq \kappa N + k} \sum_{1 \leq a \leq D, \chi_D(a) = \delta} \mu(d) \sum_{d \mid a + j D, \gcd(d, Z_0,d) = 1} |\mu(k)| \varphi(k),$$

where in the last step we used (2.33). Similarly, write (see (2.31)-(2.37))

$$\Omega_{\delta; \kappa; N}(\text{Diag}; M) = \sum_{1 \leq a \leq D, \chi_D(a) = \delta} \text{WDia}_a(M; \kappa; N) = \frac{\varphi(D)}{2} \sum_{1 \leq m \leq M, \gcd(Z_0,m) = 1} \frac{|\mu(m)|}{\varphi(m)},$$

and

$$\Omega_{1; \kappa; N}(\text{OffDiag}; M) - \Omega_{-1; \kappa; N}(\text{OffDiag}; M) = \sum_{1 \leq a \leq D, \chi_D(a) = 1} \text{WOffDiag}_a(M; \kappa; N) - \sum_{1 \leq a \leq D, \chi_D(a) = -1} \text{WOffDiag}_a(M; \kappa; N) = \sum_{1 \leq a \leq D} \chi_D(a) \text{WOffDiag}_a(M; \kappa; N) = \sum_{(m_1, \ell_1) \neq (m_2, \ell_2): 1 \leq m_1, m_2 \leq M, 1 \leq \ell_1 \leq m_h, \gcd(\ell_h, m_h) = 1, h = 1, 2 \gcd(m_1, m_2) = d, \gcd(m_1, m_2, Z_0) = 1} \frac{\mu(m_1)\mu(m_2)}{\varphi(m_1)\varphi(m_2)}.$$
\[
\sum_{1 \leq a \leq D} \chi_D(a)e^{2\pi ia\left(\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2}\right)} e^{2\pi id\left(\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2}\right)\kappa N} \left(\frac{\sin \left(\frac{(2N + 1)\pi D}{\ell_1 m_1} - \frac{(2N + 1)\pi D}{\ell_2 m_2}\right)}{(2N + 1)\sin \left(\frac{\pi D}{\ell_1 m_1} - \frac{\pi D}{\ell_2 m_2}\right)}\right)^\kappa.
\]

(2.42)

By (2.34),
\[
\Omega_{\delta;\kappa;N}(M) = \Omega_{\delta;\kappa;N}(\text{Diag}; M) + \Omega_{\delta;\kappa;N}(\text{OffDiag}; M). \tag{2.43}
\]

Finally, for technical reasons to be explained later, we assume that \(N\) is also a variable, and we take the average as \(N\) runs in \(T \leq N < 2T\): let \((\delta = 1 \text{ or } -1)\)
\[
\overline{\Omega}_{\delta;\kappa;T}(M) = \frac{1}{T} \sum_{T \leq N < 2T} \Omega_{\delta;\kappa;N}(M). \tag{2.44}
\]

Similarly, by (2.41),
\[
\overline{\Omega}_{\delta;\kappa;T}(\text{Diag}; M) = \frac{1}{T} \sum_{T \leq N < 2T} \Omega_{\delta;\kappa;N}(\text{Diag}; M) = \frac{\varphi(D)}{2} \sum_{\substack{1 \leq m \leq M \colon 
\gcd(Z_0, m) = 1}} \left\lfloor \frac{\mu(m)}{\varphi(m)} \right\rfloor, \tag{2.45}
\]

and
\[
\overline{\Omega}_{\delta;\kappa;T}(\text{OffDiag}; M) = \frac{1}{T} \sum_{T \leq N < 2T} \Omega_{\delta;\kappa;N}(\text{OffDiag}; M). \tag{2.46}
\]

By (2.43),
\[
\overline{\Omega}_{\delta;\kappa;T}(M) = \overline{\Omega}_{\delta;\kappa;T}(\text{Diag}; M) + \overline{\Omega}_{\delta;\kappa;T}(\text{OffDiag}; M). \tag{2.47}
\]

Next we specify the key parameters \(M, \kappa, \text{ and } T\): let
\[
\kappa = 8, \quad \log T = (\log D)^{15}, \quad M = T \exp \left(-\frac{2}{3} \sqrt{\log T}\right), \tag{2.48}
\]

noting that \(N\) is an integer variable running in the interval \(T \leq N < 2T\).

We conclude Section 2 with a

**Brief summary of the proof of Theorem 1.** By (2.47),
\[
\overline{\Omega}_{1;\kappa;T}(M) - \overline{\Omega}_{-1;\kappa;T}(M) =
\]
\[
= \overline{\Omega}_{1;\kappa;T}(\text{OffDiag}; M) - \overline{\Omega}_{-1;\kappa;T}(\text{OffDiag}; M) + \overline{\Omega}_{1;\kappa;T}(\text{Diag}; M) - \overline{\Omega}_{-1;\kappa;T}(\text{Diag}; M) =
\]
\[
= \overline{\Omega}_{1;\kappa;T}(\text{OffDiag}; M) - \overline{\Omega}_{-1;\kappa;T}(\text{OffDiag}; M), \tag{2.49}
\]
i.e., the diagonal part cancels out. It is relatively easy to evaluate/estimate \(\overline{\Omega}_{\pm 1;\kappa;T}(M)\) by using (2.40), (2.44) and Lemma 2.1; see Lemma 5.4 later. Thus we obtain the following approximation of the left-hand side of (2.49):
\[
\overline{\Omega}_{1;\kappa;T}(M) - \overline{\Omega}_{-1;\kappa;T}(M) =
\]

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\[
\prod_{p|Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{T} + \text{negligible error term},
\]

where the “negligible error term” will be specified in Section 5.

On the other hand, by using (2.42) and (2.46), after a long chain of estimations we obtain the following approximation of the right-hand side of (2.49) (see Lemma 5.5 later):

\[
\prod_{p|Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{T} + \text{negligible error term},
\]

and again the “negligible error term” will be specified in Section 5. (Note that

\[
\prod_{p|Z_0} \left( 1 - \frac{1}{p} \right)^2 \left( 1 - \frac{1}{p_0} \right) \left( 1 - \frac{1}{p + 1} \right)
\]

see (1.14)-(1.15.).

Subtracting (2.51) from (2.50), by (2.49) we have

\[
\prod_{p|Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{T} \left( 1 - \prod_{p|Z_0} \left( 1 + \frac{1}{p - 1} \right) \prod_{p|Z_0} \left( 1 - \frac{1}{p + 1} \right) \right) =
\]

\[
= \text{negligible error term}.
\]

According to the definition of \(Z_0\) (see (1.14)-(1.15)), we distinguish two cases. If \(\chi_D(p) = 1\) then

\[
Z_0 = \prod_{p \leq D; \chi_D(p) \neq -1} p,
\]

and we can estimate (2.52) as follows:

\[
\text{negligible error term} = \prod_{p|Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{T} \left( 1 - \prod_{p|Z_0} \left( 1 + \frac{1}{p - 1} \right) \right) \leq
\]

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\[
\leq \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{T} \left( 1 - \left( 1 + \frac{1}{p_0 - 1} \right) \right) = \\
= - \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{T} \cdot \frac{1}{p_0 - 1} \leq \\
\leq - \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{T} \cdot \frac{1}{2 \log D}. \quad (2.53)
\]

If \( \chi_D(p_0) = -1 \) then we distinguish two subcases. If

\[
\prod_{p \mid Z_0 \atop \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \geq 1 + \frac{1}{4 \log D},
\]

then we choose

\[
Z_0 = \prod_{p \leq D: \atop \chi_D(p) \neq -1} \prod_{p \mid Z_0} p,
\]

and estimate (2.52) as follows:

\[
\text{negligible error term} = \\
= \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{T} \left( 1 - \prod_{p \mid Z_0 \atop \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \right) \leq \\
\leq \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{T} \left( 1 - \left( 1 + \frac{1}{4 \log D} \right) \right) = \\
= - \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{T} \cdot \frac{1}{4 \log D}. \quad (2.54)
\]

Finally, if \( \chi_D(p_0) = -1 \) and

\[
\prod_{p \mid Z_0 \atop \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) < 1 + \frac{1}{4 \log D},
\]

then we choose

\[
Z_0 = p_0 \prod_{p \leq D: \atop \chi_D(p) \neq -1} p,
\]

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and estimate (2.52) as follows:

$$\text{negligible error term} =$$

$$= \prod_{p \mid Z_0} \left(1 - \frac{1}{p}\right)^2 \cdot D \log \frac{M^2}{T} \left(1 - \prod_{p \mid Z_0, \chi_D(p) = 1} \left(1 + \frac{1}{p - 1}\right) \prod_{p \mid Z_0, \chi_D(p) = -1} \left(1 - \frac{1}{p + 1}\right)\right) \geq$$

$$\geq \prod_{p \mid Z_0} \left(1 - \frac{1}{p}\right)^2 \cdot D \log \frac{M^2}{T} \left(1 - \left(1 + \frac{1}{4 \log D}\right) \left(1 - \frac{1}{p_0 + 1}\right)\right) \geq$$

$$\geq \prod_{p \mid Z_0} \left(1 - \frac{1}{p}\right)^2 \cdot D \log \frac{M^2}{T} \cdot \frac{1}{4 \log D}, \quad (2.55)$$

since $p_0 < 2 \log D$.

Moreover, we use the well-known number-theoretic fact

$$\prod_{p \mid Z_0} \left(1 - \frac{1}{p}\right) \geq \prod_{p \leq D} \left(1 - \frac{1}{p}\right) \geq \frac{1}{3 \log D},$$

which implies

$$\prod_{p \mid Z_0} \left(1 - \frac{1}{q}\right)^2 \geq \frac{1}{10(\log D)^2}, \quad (2.56)$$

Combining (2.52)-(2.56), we obtain

$$\frac{D}{80(\log D)^3} \log \frac{M^2}{T} =$$

$$= \text{negligible error term}, \quad (2.57)$$

which is a contradiction, since $\frac{D}{80(\log D)^3} \log(M^2/T)$ is not negligible. More precisely, by using the choice (2.48) of the parameters, and the fact that $D > e^{10^{100}}$ is very large, a simple calculation shows that the left side of (2.57) is larger than the right side (=negligible error term), which contradicts the equality. (For more details, see the second half of Section 5.) This contradiction proves Theorem 1.

**Emphasizing the Simple Reason behind the proof of Theorem 1.** I conclude Section 2 by emphasizing the crucial difference between the left-hand side and the right-hand side of (2.49). In view of (2.51), the right-hand side distinguishes between the primes $p \mid Z_0$ with $\chi_D(p) = 1$, 0 or $-1$; on the other hand, in view of (2.50), the left-hand side does not distinguish between the primes $p \mid Z_0$ with $\chi_D(p) = 1$, 0 or $-1$ (here we ignore the negligible error term). I will return to this key fact in the Concluding Remark at the
end of Section 3 and Section 4. The details of the proof of Theorem 1 are complicated, so it is particularly important to see the simple reason why the method works.

The proof of Theorem 2 is similar: we start with the analog of (2.49) for positive discriminants, and study the left-hand side and the right-hand side of (2.49). Again the contradiction is based on the fact that, under the assumption that there is a “bad” positive fundamental discriminant $D$, the left side and the right side cannot be equal. They are not equal, because the right-hand side distinguishes between the primes $p | Z_0$ with $\chi_D(p) = 1$, 0 or $-1$; more precisely, we prove an analog of (2.50). On the other hand, the left-hand side does not distinguish between the primes $p | Z_0$ with $\chi_D(p) = 1, 0$ or $-1$ (here we ignore the negligible error term). This follows from an analog of (2.51).

3. Gauss sums and an application of smoothing

Let’s return to (2.49). By (2.42) we need to evaluate/estimate the sum

$$\left( \sum_{1 \leq a \leq D} \chi_D(a) e^{2\pi ia(\ell_1/n_1 - \ell_2/n_2)} \right) e^{2\pi iD(\ell_1/m_1 - \ell_2/m_2)\kappa N} \frac{\sin \left( (2N+1)\pi D(\ell_1/m_1 - \ell_2/m_2) \right)}{(2N+1) \sin \left( \pi D(\ell_1/m_1 - \ell_2/m_2) \right)}$$

where $\gcd(m_1m_2, Z_0) = 1$. If $(m_1, \ell_1) \neq (m_2, \ell_2)$ and $\gcd(m_1, m_2) = d$, then

$$\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2} = \frac{\ell}{n} \quad \text{with some } n \geq 1 \text{ and } \ell \text{ satisfying the properties}$$

$$m_1m_2d^{-2}|n \text{ and } n|\lcm(m_1, m_2) = m_1m_2/d, \text{ and } \gcd(\ell, n) = 1. \quad (3.1)$$

That is, $m_1m_2d^{-2}$ is a divisor of $n$, $n$ is a divisor of $m_1m_2/d$, and, finally, $\ell/n$ is an irreducible fraction. Here we used the assumption $\gcd(\ell_h, m_h) = 1, h = 1, 2$ (see (2.42)), and, as usual, $\lcm(a,b)$ denotes the least common multiple of the positive integers $a, b \geq 1$.

Let

$$D\ell \equiv s \pmod{n} \quad \text{and } 1 \leq |s| \leq n/2, \gcd(s, n) = 1. \quad (3.2)$$

Then we can estimate the “tail factor” in terms of $|s|$:

$$\left| e^{2\pi iD(\ell_1/m_1 - \ell_2/m_2)\kappa N} \frac{\sin \left( (2N+1)\pi D(\ell_1/m_1 - \ell_2/m_2) \right)}{(2N+1) \sin \left( \pi D(\ell_1/m_1 - \ell_2/m_2) \right)} \right| =$$

$$= \left| \left( \frac{\sin \left( (2N+1)\pi |s|/n \right)}{(2N+1) \sin \left( \pi |s|/n \right)} \right)^\kappa \right| \leq \min \left\{ 1, \left| \frac{n}{sN} \right|^\kappa \right\},$$

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since in the range $1 \leq |s| \leq n/2$ we have the trivial upper bound
\[
\left| \frac{\sin((2N + 1)\pi s/n)}{(2N + 1)\sin(\pi s/n)} \right| \leq \min \left\{ 1, \left| \frac{n}{sN} \right| \right\}.
\]

This means that the “tail factor” is negligible if $|s|$ is “large”; so the main contribution comes from the “small” values of $|s|$.

Next we focus on the critical sum
\[
\sum_{1 \leq a \leq D} \chi_D(a)e^{2\pi ia\ell/n} = \sum_{1 \leq a \leq D} \chi_D(a)e^{2\pi ia\ell},
\]
where $|s|$ in (3.2) is “small”. The idea is to involve Gauss sums. In the following crucial lemma we use the notation introduced in (3.1)-(3.2).

**Lemma 3.1** Let $n$ and $\ell$ be integers such that $\gcd(\ell D, n) = 1$. Then
\[
\left| \sum_{1 \leq a \leq D} \chi_D(a)e^{2\pi ia\ell/n} + i\chi_D(s)\chi_D(n)\sqrt{D} \right| \leq \frac{2\pi D|s|}{n},
\]
where $s \equiv D\ell \ (mod \ n)$, $1 \leq |s| \leq n/2$, $\gcd(s, n) = 1$, assuming $\gcd(s, D) = 1$.

If $\gcd(s, D) \geq 2$, then $\chi_D(s) = 0$, and we have the simpler approximation
\[
\left| \sum_{1 \leq a \leq D} \chi_D(a)e^{2\pi ia\ell/n} \right| \leq \frac{2\pi D|s|}{n}.
\]

**Remark.** We emphasize the fact that the error term is “small” if $|s|$ is “small” relative to $n$.

**Proof of Lemma 3.1.** The smallest possible value of $|s|$ is 1—this case deserves a special study. Let $s = 1$; since $D$ and $n$ are relatively prime, there exists an $\ell^* = \ell^*_n$ satisfying
\[
D\ell^* \equiv 1 \ (mod \ n) \quad \text{and} \quad 1 \leq \ell^* < n,
\]
that is, $\ell^*$ is the multiplicative inverse of $D$ (mod $n$). (3.3) is equivalent to $D\ell^* - 1 = nj^*$ for some integer $1 \leq j^* = j^*_n < D$, or equivalently, $D|nj^* + 1$ for some $1 \leq j^* = j^*_n < D$.

So $j^* = j^*_n$ and $D$ are relatively prime. Thus we have
\[
e^{2\pi ia\ell^*/n} = e^{2\pi iaD\ell^*/(Dn)} = e^{2\pi ia(D\ell^*/Dn + 1/Dn)} = e^{2\pi ia(D\ell^*/D + 1/D)} = e^{2\pi ia(2j^*_n/Dn + 1/Dn)} = e^{2\pi ia(Dn)/D} = e^{2\pi ia/Dn}. \tag{3.4}
\]
If $n$ is “large” then the last factor is almost 1:
\[
e^{2\pi ia/(Dn)} = 1 + O(1/n).
\]
This leads us to the evaluation of the exponential sum

\[ \Theta(j^*) = \sum_{1 \leq a \leq D} \chi_D(a)e^{2\pi ij^*/D}. \]  

(3.5)

Notice that (3.5) is a Gauss sum

\[ G(\chi; K) = \sum_{k=1}^{K} \chi(k)e^{2\pi ik/K}, \]  

(3.6)

where \( \chi \) is a primitive Dirichlet character (mod \( K \)). We apply the following classical result due to Gauss.

**Lemma 3.2** For all integers \( K \) and \( r \),

\[ \sum_{k=1}^{K} \chi(k)e^{2\pi ikr/K} = \chi(r)G(\chi; K), \]  

(3.7)

where

\[ G(\chi; K) = i\sqrt{K} \text{ if } \chi(-1) = -1. \]  

(3.8)

Applying Lemma 3.2 in (3.5),

\[ \Theta(j^*) = \chi_D(j^*)i\sqrt{D}. \]  

(3.9)

Since \( D|nj^* + 1 \), we have \( \chi_D(j^*) = -\chi_D(n) \), and using it in (3.9),

\[ \Theta(j^*) = -\chi_D(n)i\sqrt{D}. \]  

(3.10)

Next we switch from the special case \( s = 1 \) to the general case of arbitrary \( s \). Let

\[ D\ell \equiv s \mod n \]  

and \( 1 \leq |s| \leq n/2, \gcd(s, n) = 1, \)  

(3.11)

then with \( \ell^* = \ell_n^* \) defined in (3.3) we have

\[ s \equiv Ds\ell^* \mod n, \]  

(3.12)

so \( \ell \equiv s\ell^* \mod n \), and

\[ e^{2\pi ia\ell/n} = e^{2\pi ias\ell^*/n} = e^{2\pi iD\ell^*/(Dn)} = e^{2\pi i(D\ell^*/D - s + D/n)} = e^{2\pi iasj^*/D} \cdot e^{2\pi ias/(Dn)}. \]  

(3.13)
We need the simple estimation
\[ |e^{2\pi i a/(Dn)} - 1| \leq \frac{2\pi |s|}{n}. \] (3.14)
By using (3.14), we have \((\delta = 1 \text{ or } -1)\)
\[ \left| \sum_{1 \leq a \leq D} e^{2\pi i a \ell/n} - \sum_{1 \leq a \leq D} e^{2\pi i a s_j^*/D} \right| \leq \frac{\pi D |s|}{n}. \] (3.15)
Write
\[ \Theta(\ell; \delta) = \sum_{1 \leq a \leq D, \chi_D(a) = \delta} e^{2\pi i a \ell/D} \] (3.16)
where \(\ell = sj^*\) and \(\delta = 1 \text{ or } -1\). By Lemma 3.2,
\[ \sum_{1 \leq a \leq D} \chi_D(a) e^{2\pi i as_j^*/D} = \Theta(sj^*; +1) - \Theta(sj^*; -1) = \]
\[ = \chi_D(sj^*)i\sqrt{D} = -\chi_D(s)\chi_D(n)i\sqrt{D}, \] (3.17)
if \(\gcd(D, s) = 1\), and
\[ \sum_{1 \leq a \leq D} \chi_D(a) e^{2\pi i as_j^*/D} = 0 \] (3.17')
if \(\gcd(D, s) \geq 2\).
Finally notice that (3.15), (3.17) and (3.17') imply Lemma 3.1.\(\Box\)

We are going to apply Lemma 3.1 in (2.42). First note that
the real part of the product \(e^{2\pi \kappa Ns/n} \left( -i\chi_D(s)\chi_D(n)\sqrt{D} \right) = \]
\[ = \chi_D(s)\chi_D(n)\sqrt{D} \sin(2\pi \kappa Ns/n). \] (3.18)
By (2.40)
\[ \Omega_{1;\kappa;N}(M) - \Omega_{-1;\kappa;N}(M) = \text{ real}, \]
so we have
\[ \text{real} = \Omega_{1;\kappa;N}(M) - \Omega_{-1;\kappa;N}(M) = \]
\[ = \Omega_{1;\kappa;N}(\text{OffDiag}; M) - \Omega_{-1;\kappa;N}(\text{OffDiag}; M) = \]
\[ = \text{real part of } (\Omega_{1;\kappa;N}(\text{OffDiag}; M) - \Omega_{-1;\kappa;N}(\text{OffDiag}; M)) = \]
\[ = 2OD_{\kappa;N}(M) + \text{Error}(\kappa; N; M), \] (3.19)
where, by applying Lemma 3.1 in (2.42), using (3.18), and also using Lemma 3.3 (to be formulated below), the first term in the last line of (3.19) is

\[ OD_{\kappa;N}(M) = \frac{\sqrt{D}}{2} \sum_{1 \leq d \leq M; \gcd(d, Z_0) = 1} \varphi(d) \sum_{d_1|d} \cdot \prod_{p|d_1} \frac{p-2}{p-1} \sum_{(m_1, m_2): 1 \leq m_1, m_2 \leq M; \gcd(m_1, m_2) = d, \gcd(m_1, m_2, Z_0) = 1} \mu(m_1)\mu(m_2) / \varphi(m_1)\varphi(m_2) \cdot \chi_{-D}(\frac{m_1 m_2 d_1}{d^2}). \]

\[ \cdot \left( 2 \sum_{1 \leq s \leq n = \frac{m_1 m_2 d_1}{2d^2}; \gcd(s, n) = 1} \chi_{-D}(s) \sin(2\pi \kappa N s/n) \left( \frac{\sin((2N+1)\pi s/n)}{(2N+1)\sin(\pi s/n)} \right)^\kappa \right). \]

(3.20)

with \( s, n, \ell \) coming from

\[ \frac{\ell_1}{m_1} - \frac{\ell_2}{m_2} = \frac{\ell}{n} \text{ with } n = \frac{m_1 m_2 d_1}{d^2}, \]

and \( s \equiv D\ell \pmod{n}, 1 \leq |s| \leq n/2, \gcd(s, n) = 1. \)

In (3.20) we used the following simple but important lemma.

**Lemma 3.3** Let \( m_1 \geq 1, m_2 \geq 1 \) be squarefree integers, and let \( d = \gcd(m_1, m_2) \) be their greatest common divisor. Let \( d_1 \geq 1 \) be an arbitrary divisor of \( d \), and write \( n = m_1 m_2 d_1^{-2} d_1 \). Let \( \ell \) be an integer with \( 1 \leq \ell < n, \gcd(\ell, n) = 1 \). Let \( \mathcal{A}(n, \ell) \) denote the number of pairs \((\ell_1, \ell_2)\) of integers such that \( 1 \leq \ell_i \leq m_i, \gcd(\ell_i, m_i) = 1, i = 1, 2 \), and

\[ \frac{\ell_1}{m_1} - \frac{\ell_2}{m_2} = \frac{\ell}{n} \text{ or } \frac{\ell}{n} - 1. \]

Then

\[ \mathcal{A}(n, \ell) = \varphi_2(d_1)\varphi(d/d_1) = \varphi(d) \prod_{p|d_1} \frac{p-2}{p-1}, \]

where

\[ \varphi(m) = m \prod_{p|m} \left( 1 - \frac{1}{p} \right) \quad \text{and} \quad \varphi_2(m) = m \prod_{p|m} \left( 1 - \frac{2}{p} \right). \]

I postpone the proof of Lemma 3.3 to the end of Section 3.

The second term in the last line of (3.19) is the error:

\[ |\text{Error}(\kappa; N; M)| \leq 2 \sum_{1 \leq d \leq M} d \sum_{d_1|d} \sum_{(m_1, m_2): 1 \leq m_1, m_2 \leq M; \gcd(m_1, m_2) = d} \frac{1}{\varphi(m_1)\varphi(m_2)}. \]
\[ \tau(D) \sum_{1 \leq s \leq \frac{m_1m_2d_1}{2d^2}} \frac{2\pi Ds}{n} \cdot \min \left\{ 1, \left( \frac{n}{sN} \right)^\kappa \right\}, \] \quad (3.22)

since in the range \( 1 \leq s \leq n/2 \) we have the trivial upper bound

\[ \left| \frac{\sin((2N + 1)\pi s/n)}{(2N + 1)\sin(\pi s/n)} \right| \leq \min \left\{ 1, \left| \frac{n}{sN} \right| \right\}. \quad (3.23) \]

I estimate the error term (3.22) from above. This is the first application of the efficient average ("smoothing") introduced in Section 2.

**Lemma 3.4** If \( \kappa \geq 3 \) then

\[ |\text{Error}(\kappa; N; M)| < 1. \]

**Proof of Lemma 3.4.** We distinguish two cases in (3.22) depending on whether

\[ 1 \leq \frac{n}{sN} \text{ or } 1 > \frac{n}{sN}, \]

which implies

\[ |\text{Error}(\kappa; N; M)| \leq E_1 + E_2, \quad (3.24) \]

where (let \( m'_i = m_i/d, i = 1, 2 \) and \( n = m'_1m'_2d_1 \))

\[ E_1 = 2 \sum_{1 \leq d \leq M} \frac{d}{\varphi^2(d)} \sum_{d_1 | d} \sum_{(m'_1, m'_2): 1 \leq m'_1, m'_2 \leq M/d} \frac{1}{\varphi(m'_1)\varphi(m'_2)} \cdot 2\pi \tau(D)D \left( \sum_{1 \leq s \leq \frac{m'_1m'_2d_1}{sN}} \frac{s}{m'_1m'_2d_1} \right), \quad (3.25) \]

and

\[ E_2 = 2 \sum_{1 \leq d \leq M} \frac{d}{\varphi^2(d)} \sum_{d_1 | d} \sum_{(m'_1, m'_2): 1 \leq m'_1, m'_2 \leq M/d} \frac{1}{\varphi(m'_1)\varphi(m'_2)} \cdot 2\pi \tau(D)D \left( \sum_{\frac{m'_1m'_2d_1}{sN} < s \leq \frac{m'_1m'_2d_1}{2}} \frac{1}{N} \cdot \left( \frac{m'_1m'_2d_1}{sN} \right)^{\kappa - 1} \right). \quad (3.26) \]

Using the trivial upper bound

\[ \sum_{1 \leq s \leq \frac{m'_1m'_2d_1}{sN}} \frac{s}{m'_1m'_2d_1} \leq \frac{1}{m'_1m'_2d_1} \left( \frac{m'_1m'_2d_1}{N} \right)^2 = \frac{m'_1m'_2d_1}{N^2} \]

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in (3.25), we have

\[
E_1 \leq \frac{4\pi \tau(D) D}{N^2} \sum_{1 \leq d \leq M} \sum_{d | d} \frac{dd_1}{\varphi^2(d)} \sum_{(m'_1, m'_2): 1 \leq m'_1, m'_2 \leq M/d} \frac{m'_1 m'_2}{\varphi(m'_1) \varphi(m'_2)}. \tag{3.27}
\]

Here we need (and repeatedly use later) the following well-known fact from elementary number theory:

\[
\frac{n}{\varphi(n)} \leq 10 \log \log n \tag{3.28}
\]

for all \(n \geq 100\). By (3.28),

\[
\sum_{(m'_1, m'_2): 1 \leq m'_1, m'_2 \leq M/d} \frac{m'_1 m'_2}{\varphi(m'_1) \varphi(m'_2)} \leq \left( \sum_{1 \leq m \leq M/d} \frac{m}{\varphi(m)} \right)^2 \leq 10^4 \left( \frac{M}{d} \log \log M \right)^2. \tag{3.29}
\]

By using (3.29) in (3.27),

\[
E_1 \leq \frac{4\pi \tau(D) D}{N^2} \sum_{1 \leq d \leq M} \sum_{d | d} \frac{dd_1}{\varphi^2(d)} \cdot \frac{10^4 M^2 (\log \log M)^2}{d^2} \leq
\]

\[
\leq \frac{2\pi \tau(D) D}{N^2} \cdot \frac{10^4 \left( \frac{M}{d} \log \log M \right)^2 \left( \sum_{d \geq 1} \frac{\tau(d)}{\varphi^2(d)} \right)}{d^2} \leq
\]

\[
\leq \frac{10^7 \tau(D) DM^2 (\log \log M)^2}{N^2}, \tag{3.30}
\]

since the infinite series in (3.30) is clearly convergent (note that \(\tau(n) = n^{o(1)}\);

\[
\sum_{d \geq 1} \frac{\tau(d)}{\varphi^2(d)} \leq 100.
\]

To estimate \(E_2\), we use a standard power-of-two decomposition in (3.26):

\[
E_2 \leq 4\pi \tau(D) D \sum_{1 \leq d \leq M} \frac{d}{\varphi^2(d)} \sum_{(m'_1, m'_2): 1 \leq m'_1, m'_2 \leq M/d} \frac{1}{\varphi(m'_1) \varphi(m'_2)} \cdot
\]

\[
\sum_{d_1 | d} \left( \sum_{j \geq 1} \frac{m'_1 m'_2 d_1}{sN} 2^{-j-1} \leq \frac{m'_1 m'_2 d_1}{sN} 2^j \right)^{\kappa-1} \leq
\]

\[
\sum_{d_1 | d} \left( \sum_{j \geq 1} \frac{m'_1 m'_2 d_1}{sN} 2^{-j-1} \leq \frac{m'_1 m'_2 d_1}{sN} 2^j \right)^{\kappa-1} \leq
\]

\[
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\]
\[
\leq 4\pi \tau(D) D \sum_{1 \leq d \leq M} \frac{d}{\varphi^2(d)} \sum_{(m_1', m_2') \atop 1 \leq m_1', m_2' \leq M/d} \frac{1}{\varphi(m_1') \varphi(m_2')} \cdot \sum_{d_1 | d} \left( \sum_{j \geq 1} \frac{m_1' m_2' d_1 2^j}{N} \cdot \frac{1}{N}, 2^{-(\kappa - 1)(j - 1)} \right) \leq \]

\[
\leq \frac{4\pi \tau(D) D}{N^2} \sum_{1 \leq d \leq M} \frac{dd_1}{\varphi^2(d)} \sum_{(m_1', m_2') \atop 1 \leq m_1', m_2' \leq M/d} \frac{m_1' m_2'}{\varphi(m_1') \varphi(m_2')} \cdot \left( \sum_{j \geq 1} 2^{-(\kappa - 2)(j - 1) + 1} \right) \leq \]

\[
\leq \frac{16\pi \tau(D) D}{N^2} \sum_{1 \leq d \leq M} \frac{d^2 \tau(d)}{\varphi^2(d)} \left( \sum_{1 \leq n \leq M/d} \frac{n}{\varphi(n)} \right)^2 \leq \]

\[
\leq \frac{5 \cdot 10^5 \tau(D) D(M \log \log M)^2}{N^2} \sum_{1 \leq d \leq M} \frac{\tau(d)}{\varphi^2(d)} \leq \]

\[
\leq \frac{5 \cdot 10^7 \tau(D) D M^2 (\log \log M)^2}{N^2}, \quad (3.31)
\]

assuming \(\kappa \geq 3\) (which guarantees the convergence of the series \(\sum_{j \geq 1} 2^{-(\kappa - 2)j}\)). In the last steps of the argument in (3.31) we used (3.28) the same way as we did in (3.29).

Finally notice that (3.24), (3.30) and (3.31) imply Lemma 3.4:

\[
|\text{Error}(\kappa; N; M)| \leq \frac{10^8 \tau(D) D M^2}{N^2} (\log \log M)^2 < 1,
\]

where the last step is a trivial calculation using the values of the parameters (see (2.48)):

\[
\kappa = 8, \quad \log T = (\log D)^{15}, \quad T \leq N < 2T, \quad M = T \exp \left( -\frac{2}{3} \sqrt{\log T} \right), \quad \log D > 10^{100}.
\]

\(\square\)

As we promised, we include the proof of Lemma 3.3.

**Proof of Lemma 3.3.** Let \(B(n, \ell)\) denote the number of pairs of residue classes \((\ell_1 \pmod{m_1}, \ell_2 \pmod{m_2})\) such that \(1 \leq \ell_h \leq m_h, \gcd(\ell_h, m_h) = 1, h = 1, 2,\) and there exist integers \(\ell^{(1)}\) and \(\ell^{(2)}\) with the property

\[
\ell^{(1)} \equiv \ell_1 \pmod{m_1}, \quad \ell^{(2)} \equiv \ell_2 \pmod{m_2},
\]

and

\[
\frac{\ell^{(1)}}{m_1} - \frac{\ell^{(2)}}{m_2} = n.
\]

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Clearly \( \mathcal{A}(n, \ell) \leq \mathcal{B}(n, \ell) \).

Next I prove the upper bound

\[
\mathcal{B}(n, \ell) \leq \varphi_2(d_1) \varphi(d/d_1) = \varphi(d) \prod_{p|d_1} \frac{p-2}{p-1}, \tag{3.32}
\]

where \( n = m_1' m_2' d_1 \).

If

\[
\frac{\ell}{n} = \frac{\ell^{(1)}}{m_1} - \frac{\ell^{(2)}}{m_2} = \frac{\ell^{(3)}}{m_1} - \frac{\ell^{(4)}}{m_2}
\]

holds for some integers \( \ell^{(1)}, \ell^{(2)}, \ell^{(3)}, \ell^{(4)} \) with \( 1 \leq \ell^{(h)} \leq m_h, \gcd(\ell^{(h)}, m_h) = 1, h = 1, 2 \) and \( 1 \leq \ell^{(h+2)} \leq m_h, \gcd(\ell^{(h+2)}, m_h) = 1, h = 1, 2 \), then \( (\ell^{(1)} - \ell^{(3)}) m_2' = (\ell^{(2)} - \ell^{(4)}) m_1' \), where \( m_1' = m_h/d, h = 1, 2 \). Since \( m_1' \) and \( m_2' \) are coprime, we have \( \ell^{(1)} - \ell^{(3)} = j m_1' \) and \( \ell^{(2)} - \ell^{(4)} = j m_2' \) with the same integer \( j \). It follows that the residue class \( j \pmod{d} \) uniquely determines both residue classes

\[
\ell^{(1)} - \ell^{(3)} \pmod{m_1} \text{ and } \ell^{(2)} - \ell^{(4)} \pmod{m_2}. \tag{3.33}
\]

We now study what restrictions apply to the residue class \( j \pmod{d} \).

First, let \( p \) be a prime divisor of \( d_1 \) (I recall that \( n = m_1' m_2' d_1 \)). The equalities \( \ell^{(3)} m_2' = \ell^{(1)} m_2' - j m_1' m_2' \) and \( \ell^{(4)} m_1' = \ell^{(2)} m_1' - j m_1' m_2' \) imply

\[
\ell^{(3)} m_2' \equiv \ell^{(1)} m_2' - j m_1' m_2' \pmod{p} \text{ and } \ell^{(4)} m_1' \equiv \ell^{(2)} m_1' - j m_1' m_2' \pmod{p}.
\]

Note that \( \ell^{(3)} m_2' \not\equiv 0 \pmod{p} \), since otherwise \( p|\ell^{(3)} \), implying that both \( \ell^{(3)} \) and \( m_1' \) are divisible by \( p \), which contradicts the fact that they are coprime. Similarly, \( \ell^{(4)} m_1' \not\equiv 0 \pmod{p} \).

Furthermore, \( \ell^{(3)} m_2' - \ell^{(4)} m_1' \) is not divisible by \( p \), since otherwise we could simplify the fraction

\[
\frac{\ell}{n} = \frac{\ell^{(3)}}{m_1} - \frac{\ell^{(4)}}{m_2} = \frac{\ell^{(3)} m_2' - \ell^{(4)} m_1'}{m_1' m_2' d}
\]

by \( p \), which is a contradiction.

This means that for \( j \) we have two forbidden residue classes \( \pmod{p} \):

\[
j \not\equiv \ell^{(1)}/m_1' \pmod{p} \text{ and } j \not\equiv \ell^{(2)}/m_2' \pmod{p}.
\]

On the other hand, if \( q \) is a prime divisor of \( d/d_1 \), then the same argument gives that there is only one forbidden residue class \( \pmod{q} \).

Therefore, we have

\[
\varphi_2(d_1) \varphi(d/d_1) = \varphi(d) \prod_{p|d_1} \frac{p-2}{p-1} \tag{3.34}
\]

available residue classes of \( j \pmod{d} \). Combining (3.33) and (3.34), (3.32) follows.
Summarizing, for \( m_1 \neq m_2 \) we have

\[
\sum_{n,\ell} A(n, \ell) \leq \sum_{n,\ell} B(n, \ell) \leq \sum_{n=m'_1 m'_2 d_1} \sum_{\ell} \varphi_2(d_1) \varphi(d/d_1) = \\
= \sum_{d_1 | d} \varphi(m'_1 m'_2) \varphi(d_1) \cdot \varphi_2(d_1) \varphi(d/d_1) = \sum_{d_1 | d} \varphi(m'_1 m'_2) \varphi(d_1) \cdot \varphi(d) \prod_{p | d_1} \frac{p-2}{p-1} = \\
= \varphi(m'_1 m'_2) \varphi(d) \sum_{d_1 | d \ p | d_1} (p-2) = \varphi(n) \prod_{p | d} (1 + p - 2) = \\
= \varphi(m'_1 m'_2) \varphi^2(d) = \varphi(m_1) \varphi(m_2). \quad (3.35)
\]

On the other hand,

\[
\sum_{n,\ell} A(n, \ell) = \varphi(m_1) \varphi(m_2), \quad (3.36)
\]

since the right-hand side of (3.36) is the number of all possible differences

\[
\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2},
\]

assuming \( m_1 \neq m_2 \).

Combining (3.35) and (3.36), we have with \( n = m'_1 m'_2 d_1 \),

\[
\varphi(m_1) \varphi(m_2) = \sum_{n,\ell} A(n, \ell) \leq \sum_{n,\ell} B(n, \ell) \leq \sum_{n=m'_1 m'_2 d_1} \sum_{\ell} \varphi_2(d_1) \varphi(d/d_1) \leq \varphi(m_1) \varphi(m_2). \quad (3.37)
\]

Since

\[
A(n, \ell) \leq B(n, \ell) \leq \varphi_2(d_1) \varphi(d/d_1),
\]

(3.37) upgrades the inequality to equality:

\[
A(n, \ell) = \varphi_2(d_1) \varphi(d/d_1).
\]

Finally, in the trivial case \( m_1 = m_2 \) we have \( m_1 = m_2 = d \) and \( n = d_1 \). Then we have to exclude the case \( d_1 = 1 \) (since \((m_1, \ell_1) \neq (m_2, \ell_2)\)) in (3.35), and proceed as follows:

\[
\sum_{(d_1, \ell): \ d_1 | d, \ d_1 \neq 1} A(d_1, \ell) \leq \sum_{(d_1, \ell): \ d_1 | d, \ d_1 \neq 1} B(d_1, \ell) \leq \sum_{d_1 | d: \ d_1 \neq 1} \sum_{\ell} \varphi_2(d_1) \varphi(d/d_1) = \\
= \sum_{d_1 | d: \ d_1 \neq 1} \varphi(d_1) \cdot \varphi_2(d_1) \varphi(d/d_1) = \left( \sum_{d_1 | d} \varphi(d_1) \cdot \varphi_2(d_1) \varphi(d/d_1) \right) - \varphi(d) = \varphi^2(d) - \varphi(d). \quad (3.38)
\]
On the other hand,
\[ \sum_{(d_1, \ell) : d_1 \mid d, d_1 \neq 1} A(d_1, \ell) = \varphi^2(d) - \varphi(d), \quad (3.39) \]
since the right-hand side of (3.39) is the number of all possible differences
\[ \frac{\ell_1}{d} - \frac{\ell_2}{d} \]
with \( \ell_1 \neq \ell_2 \). Again combining (3.38) and (3.39), we can upgrade the inequality to equality, and the proof of Lemma 3.3 is complete. \( \square \)

**Concluding Remark of Section 3.** Let us return to (3.19)-(3.20). The message of Lemma 3.4 is that \( \text{Error}(\kappa; N; M) \) is negligible compared to \( OD_{\kappa; N}(M) \), so \( 2OD_{\kappa; N}(M) \) is the dominating part of
\[ \Omega_{1; \kappa; N}(\text{OffDiag}; M) - \Omega_{-1; \kappa; N}(\text{OffDiag}; M). \]

At the end of Section 2 I made the claim that
\[ \overline{\Omega}_{1; \kappa; T}(\text{OffDiag}; M) - \overline{\Omega}_{-1; \kappa; T}(\text{OffDiag}; M) \]
(which is simply the average of
\[ \Omega_{1; \kappa; N}(\text{OffDiag}; M) - \Omega_{-1; \kappa; N}(\text{OffDiag}; M) \]
as \( N \) runs in \( T \leq N < 2T \); see (2.46)) distinguishes between the primes \( p \mid Z_0 \) with \( \chi_D(p) = 1, 0 \) or \(-1\). This claim is intuitively well justified by the last line in (3.20)
\[ \sum_{1 \leq s \leq \frac{n}{2} - \frac{m_1 + m_2 + d_1}{2d_2^2} : \gcd(s, n) = 1} \chi_D(s) \sin(2\pi \kappa N s/n) \left( \frac{\sin((2N + 1)\pi s/n)}{(2N + 1) \sin(\pi s/n)} \right)^\kappa, \]
due to the appearance of the character \( \chi_D(s) \).

Note in advance that the evaluation of the sum (3.20) will eventually involve, or rather lead to, a sum like \( \sum_{j=1}^{D^4} \chi_D(j) \log j \), which will become a factor in the dominating part of (3.20) (note that the power \( D^4 \) here is accidental; any not too small power of \( D \) would do). The reason behind it is that the sum in (3.20) resembles a “double harmonic sum”, and also the main term in Lemma 2.1 is the logarithm function. This explains why (3.20) resembles a Riemann sum for a logarithmic integral that we can evaluate explicitly. The explicitly evaluated logarithmic integral can be well illustrated with the example
\[ \sum_{j=1}^{D^4} \chi_D(j) \left( \log \frac{M^2}{Nj} \right)^2 = \sum_{j=1}^{D^4} \chi_D(j) \left( \log \frac{M^2}{N} \right)^2 + \]
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\[
+ \sum_{j=1}^{D^4} \chi_{-D}(j)(\log j)^2 - 2 \log \frac{M^2}{N} \sum_{j=1}^{D^4} \chi_{-D}(j) \log j =
\]
\[
= -2 \log \frac{M^2}{N} \sum_{j=1}^{D^4} \chi_{-D}(j) \log j + \text{negligible.} \quad (3.40)
\]

The fact that the sum \( \sum_{j=1}^{D^4} \chi_{-D}(j)(\log j)^2 \) represents a negligible contribution here follows from the Pólya–Vinogradov inequality and partial summation.

Note that the harmonic sum

\[
\sum_{n=1}^{M} \frac{1}{n} = \log M + O(1) \quad \text{if} \quad M \geq 1 \quad \text{and} \quad 0 \quad \text{if} \quad M < 1
\]

is actually associated with the function \( \log_+ x \) instead of \( \log x \) (\( \log_+ x = \log x \) if \( x \geq 1 \) and 0 if \( 0 < x < 1 \)). On the other hand, in (3.40) we work with \( \log x \) instead of \( \log_+ x \). The intuitively plausible assumption that “\( \log_+ x \) and \( \log x \) are basically the same” requires a precise proof applied in our particular case. Unfortunately, this part of the proof turns out to be annoyingly cumbersome. This kind of technical problems explain why the paper is so long.

Let’s return to (3.40). If the class number \( h(-D) \) is “substantially smaller” than \( \sqrt{D} \), then we have the good approximation

\[
\sum_{j=1}^{D^4} \chi_{-D}(j) \log j = -\frac{\pi}{6} \sqrt{D} \sum_{(a,b,c)} \frac{1}{a} + \text{negligible}
\]

(see Lemma 14.2), where \( \sum_{(a,b,c)} \frac{1}{a} \) means that we add up the reciprocals of the leading coefficients \( a = a_j \) in the family of reduced, primitive, inequivalent binary quadratic forms of integer coefficients \( ax^2 + bxy + cy^2 \) with discriminant \( b^2 - 4ac = -D < 0 \), \( a > 0 \), \( c > 0 \), \( 1 \leq j \leq h(-D) \).

Moreover, under the same condition, we have another good approximation (see Lemma 14.3):

\[
\sum_{j=1}^{h(-D)} \frac{1}{a_j} = \prod_{p | Z_0} \prod_{\chi_{-D}(p) \neq -1} \frac{p + 1}{p - 1} \frac{p - 1}{p} + \text{negligible.}
\]

Notice that the product

\[
\prod_{p | Z_0} \prod_{\chi_{-D}(p) \neq -1} \frac{p + 1}{p - 1} \frac{p - 1}{p}
\]

clearly distinguishes between the primes \( p | Z_0 \) with \( \chi_{-D}(p) = 1, 0 \) or \( -1 \).

On the other hand, (2.50) does not distinguish between the primes \( p | Z_0 \) with \( \chi_{-D}(p) = 1, 0 \) or \( -1 \). This is how we obtain a contradiction, which proves Theorem 1.
The proof of (2.50) is not trivial (see Sections 4, 7 and 8). What is trivial is the case of primes; see the Remark after (2.3): the factor $\prod_{p \mid Z_0} \left(1 - \frac{1}{p}\right)$ in (2.4) does not distinguish between the primes $p \mid Z_0$ with $\chi_D(p) = 1, 0$ or $-1$. Since the primes play a key role in the proof of (2.50), it is not surprising that (2.50) does not distinguish between the primes $p \mid Z_0$ with $\chi_D(p) = 1, 0$ or $-1$.

Note in advance that the case of positive discriminants (Theorem 2) is somewhat different. Instead of the sum $\sum_{j=1}^{D^4} \chi_D(j) \log j$, we make use of the other sum $\sum_{j=1}^{D^4} \chi_D(j)(\log j)^2$. (For $D > 0$ the sum $\sum_{j=1}^{D^4} \chi_D(j) \log j$ turns out to be “negligible” if $L(1, \chi_D)$ is “small”.) We have the analog approximation

$$\sum_{j=1}^{D^4} \chi_D(j)(\log j)^2 = -\frac{\pi^2}{6} \sqrt{D} \prod_{p \mid Z_0} \frac{p + 1}{p - 1} \prod_{p \mid D} \frac{p - 1}{p} + \text{negligible}$$

if $L(1, \chi_D)$ is “small”. Again the key observation is that the product on the right-hand side clearly distinguishes between the primes $p \mid Z_0$ with $\chi_D(p) = 1, 0$ or $-1$.

### 4. The method of the proof (I)

As I said at the beginning of Section 2, the basic idea of the proof of Theorem 1 is to evaluate/estimate the two sides of the equality (2.49) with the choice (2.48) of the key parameters. First we evaluate the left side of (2.49) by using (2.40) and (2.44); see Lemma 5.4. This is the subject of Section 4 and the beginning of Section 5.

Then we evaluate the right side of (2.49) by using (2.42) and (2.45), and we obtain a different expression. This is the subject of the middle part of Section 5; see Lemma 5.5. Finally, at the end of Section 5 we derive a contradiction, which completes the proof of Theorem 1.

The details go as follows. We want to find a good estimation for the difference

$$\overline{\Omega}_{1;\kappa;T}(M) - \overline{\Omega}_{-1;\kappa;T}(M).$$

The main difficulty is how to estimate (2.40).

By (2.41)-(2.43) and (3.19),

$$\Omega_{1;\kappa;N}(M) - \Omega_{-1;\kappa;N}(M) = \Omega_{1;\kappa;N}(\text{OffDiag}; M) - \Omega_{-1;\kappa;N}(\text{OffDiag}; M) =$$

$$= 2OD_{\kappa;N}(M) + \text{Error}(\kappa; N; M). \quad (4.1)$$
By Lemma 3.4, for $\kappa \geq 3$ we have

$$|\text{Error}(\kappa; N; M)| \leq \frac{10^7 \tau(D)DM^2}{N^2} \log M (\log \log M)^4.$$ (4.2)

We also need the special case $Q = 1$ in Lemma 2.1:

$$\left| \sum_{1 \leq m \leq M; \gcd(Z_0, m) = 1} \frac{\mu(m)}{\varphi(m)} - \prod_{q | Z_0} \left( 1 - \frac{1}{q} \right) \left( \log M + c' + \sum_{q | Z_0} \frac{\log q}{q} \right) \right| \leq \frac{10^4 \log D \log M}{M^{1/4}} + \frac{10^5}{D^3} + \frac{4 \left( 10 + \log M + 2(\log D)^2 \right)}{\max \{MD^{-6 \log D}, 1 \}},$$ (4.3)

where $c' = \gamma_0 + 2\gamma^* - \gamma^{**}$. Note that trivially

$$\sum_{q | Z_0} \frac{\log q}{q} \leq \sum_{q \leq D} \frac{\log q}{q} \leq (\log D)^2.$$ (4.4)

By (2.40),

$$\Omega_{1;\kappa;N}(M) - \Omega_{-1;\kappa;N}(M) =$$

$$= \sum_{k=0}^{\kappa N} w_k^{(\kappa,N)} \frac{1}{2k+1} \sum_{j: \kappa N - k \leq j \leq \kappa N + k} \sum_{\gcd(a,j) = 1} \chi_D(a)S^2(M; a + jD) =$$

$$= \sum_{n=p^r} (N; M) + \sum_{n \neq p^r} (N; M),$$ (4.5)

where

$$\sum_{n=p^r} (N; M), \sum_{n \neq p^r} (N; M) \text{ and } S(M; a + jD)$$

are defined in the following way. The first sum in the last line of (4.5) is restricted to the primepowers (of course every prime is a primepower):

$$\sum_{n=p^r} (N; M) = \sum_{k=0}^{\kappa N} w_k^{(\kappa,N)} \frac{1}{2k+1} \sum_{(\kappa N - k)D < n < (\kappa N + k+1)D: \gcd(n,p^r) = 1} \chi_D(p^r)S^2(M; p^r),$$ (4.6)

the second sum in the last line of (4.5) is restricted to the rest of the integers:

$$\sum_{n \neq p^r} (N; M) = \sum_{k=0}^{\kappa N} w_k^{(\kappa,N)} \frac{1}{2k+1} \sum_{(\kappa N - k)D < n < (\kappa N + k+1)D: \gcd(n,p^r) = 1} \chi_D(n)S^2(M; n),$$ (4.7)
and, finally, \( S(M; n) \) is defined as

\[
S(M; n) = \sum_{d \geq 1 : \gcd(Z_0, d) = 1} \mu(d) \sum_{1 \leq k \leq M/d : \gcd(k, nZ_0) = 1} \frac{\mid \mu(k) \mid}{\varphi(k)}.
\] (4.8)

The evaluation of \( S(M; p) \) is quite easy. Indeed, in the case \( n = p \) prime we have

\[
S(M; p) = \sum_{d \geq 1 : d \mid p} \mu(d) \sum_{1 \leq k \leq M/d : \gcd(k, pZ_0) = 1} \frac{\mid \mu(k) \mid}{\varphi(k)} = \sum_{1 \leq k \leq M} \frac{\mid \mu(k) \mid}{\varphi(k)} \quad \text{if} \quad p > M,
\] (4.9)

and

\[
S(M; p) = \sum_{d \geq 1 : d \mid p} \mu(d) \sum_{1 \leq k \leq M/d : \gcd(k, pZ_0) = 1} \frac{\mid \mu(k) \mid}{\varphi(k)} - \sum_{1 \leq k \leq M/p : \gcd(k, pZ_0) = 1} \frac{\mid \mu(k) \mid}{\varphi(k)} \quad \text{if} \quad 1 < p \leq M.
\] (4.10)

Using (4.9)-(4.10) and (4.3)-(4.4) (note that \( M > D^{9 \log D} \)),

\[
\left| \sum_{n=p^r} (N; M) - \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k+1} \sum_{\kappa N-k \leq p < \kappa N+k+1} \chi_D(p) \cdot \prod_{q \mid Z_0} \left( 1 - \frac{1}{q} \right)^2 \left( \log M + c' + \sum_{q \mid Z_0} \frac{\log q}{q} \right)^2 \right| \leq
\]

\[
\leq 1 + \sum_{k=\kappa N-M}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k+1} M (\log M)^2 \leq 1 + \frac{M (\log M)^2}{N - M} < 2,
\] (4.11)

where we used (2.48), and the simple fact that—roughly speaking—the overwhelming majority of primepowers are primes. More precisely, we used the statement that the number of integers of the form \( a^b \leq x \), where \( a \geq 2, b \geq 2 \) are integers, is clearly less than

\[
x^{1/2} + x^{1/3} + x^{1/5} + x^{1/7} + x^{1/11} + \ldots = \sum_{2 \leq p \leq \log x} x^{1/p} \leq 2 \sqrt{x}
\]

if \( x \geq 100 \).

Since we have very good estimations for the number of primes in long intervals, the estimations become (basically) trivial if we turn \( N \) into a variable running in the long interval \( T \leq N < 2T \) and take the average as follows:

\[
\left| \frac{1}{T} \sum_{T \leq N < 2T} \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k+1} \sum_{\kappa N-k \leq p < \kappa N+k+1} \chi_D(p) \right|.
\]
\[
\prod_{q \mid Z_0} \left(1 - \frac{1}{q}\right)^2 \left(\log M + c' + \sum_{q \mid Z_0} \frac{\log q}{q}\right)^2 + \prod_{q \mid Z_0} \left(1 - \frac{1}{q}\right)^2 \frac{D(\log M)^2}{\log T} \leq \\
\leq \frac{10D \log D(\log M)^2}{(\log T)^2} + \frac{10D \log D \log M}{\log T} + 100D(\log M)^2L(1, \chi_D) \leq \\
\leq 30D \log D, \quad (4.12)
\]

where we used the Prime Number Theorem, Lemma 1.1 with (1.11), (4.4), (2.48) and (1.10). (The choice \( \kappa = 8 \) implies that \( w_k^{(\kappa, N)} \), \( 0 \leq k \leq \kappa N \) is a “reasonable” probability distribution, explicitly described in Section 2.)

Write
\[
\sum_{n=p^r} (T; M) = \frac{1}{T} \sum_{T \leq N < 2T} \sum_{n=p^r} (N; M). \quad (4.13)
\]

Combining (4.11)-(4.13), we obtain

**Lemma 4.1** We have
\[
\left| \sum_{n=p^r} (T; M) + \prod_{q \mid Z_0} \left(1 - \frac{1}{q}\right)^2 \frac{D(\log M)^2}{\log T} \right| \leq \\
\leq 30D \log D + 2.
\]

\( \square \)

The estimation of the second sum \( \sum_{n \neq p^r} (N; M) \) in (4.7) is based on Lemmas 4.2-3-4 below. To estimate \( \sum_{n \neq p^r} (N; M) \), we focus on the end-sum \( S(M; n) \); see (4.8). Lemma 2.1 implies the trivial upper bound
\[
|S(M; n)| \leq 20\tau(n) \log M, \quad (4.14)
\]

which will be constantly used below.

In Sections 4 and 7-8 we employ the following unusual notation: for every integer \( n \geq 1 \), let \( n^* \) denote the largest squarefree divisor of \( n \). That is, \( n^* = n \) holds if \( n \) is squarefree, and in general,
\[
n^* = p_1 \cdots p_r \text{ if } n = p_1^{a_1} \cdots p_r^{a_r} \text{ with } a_i \geq 1, \ 1 \leq i \leq r
\]
is the prime factorization of \( n \).

Write
\[
M' = MD^{-9 \log D}. \quad (4.15)
\]

We can rewrite the sum (4.8) as follows.
Lemma 4.2 We have

\[
S(M; n) = \mu(n^*) \left( \prod_{q|n^*Z_0} \left( 1 - \frac{1}{q} \right) \sum_{1 \leq d < n^*/M'} \frac{\mu(d)}{d} \right) \cdot \left( \log(n^*/d_2) - \log M - c' - \sum_{q|n^*Z_0} \frac{\log q}{q} \right) +
\]

\[
+ \sum_{n^*/M \leq d_1 \leq n^*/M'} \mu(d_1) \sum_{1 \leq k \leq M/d_1 : \gcd(k, n^*Z_0) = 1} \frac{|\mu(k)|}{\varphi(k)} + \text{Error}(M; n),
\]

where

\[
|\text{Error}(M; n)| \leq \frac{10^7 \tau^2(n) \log n}{D^2}.
\]

Proof. By (4.8)

\[
S(M; n) = \sum_{d \geq 1 : \gcd(Z_0, d) = 1} \mu(d) \sum_{1 \leq k \leq M/d : \gcd(k, n^*Z_0) = 1} \frac{|\mu(k)|}{\varphi(k)} =
\]

\[
= \sum_{1 \leq d \leq M' : \frac{M}{d} \in n^*} \mu(d) \sum_{1 \leq k \leq M/d : \gcd(k, n^*Z_0) = 1} \frac{|\mu(k)|}{\varphi(k)} + \sum_{M' < d \leq M : \frac{M}{d} \in n^*} \mu(d) \sum_{1 \leq k \leq M/d : \gcd(k, n^*Z_0) = 1} \frac{|\mu(k)|}{\varphi(k)} =
\]

\[
= \prod_{q|n^*Z_0} \left( 1 - \frac{1}{q} \right) \sum_{1 \leq d < M' : \frac{M}{d} \in n^*} \mu(d) \left( \log M - \log d + c' + \sum_{q|n^*Z_0} \frac{\log q}{q} \right) +
\]

\[
+ \sum_{M' < d \leq M : \frac{M}{d} \in n^*} \mu(d) \sum_{1 \leq k \leq M/d : \gcd(k, n^*Z_0) = 1} \frac{|\mu(k)|}{\varphi(k)} + \text{Error}(M; n), \quad (4.16)
\]

where in the last step we applied Lemma 2.1, which implies the upper bound

\[
|\text{Error}(M; n)| \leq 10^6 \tau(n) \sum_{1 \leq d \leq M' : \frac{M}{d} \in n^*} \left( \frac{1 + \log(M/d)}{(M/d)^{1/4}} + \frac{\log n}{D^5} \right) \leq \frac{10^7 \tau^2(n) \log n}{D^2}, \quad (4.17)
\]

since by (4.15), \( M/d \geq D^9 \log D \) if \( 1 \leq d \leq M' \).
The Möbius function has the following well-known property: \( \sum_{d \mid k} \mu(d) = 0 \) for all \( k \geq 2 \)—it will play a crucial role in the argument. The first application of this fact goes as follows. Suppose that \( n^* \geq 2 \); we have with \( d_1 d_2 = n^* \)

\[
\sum_{1 \leq d \leq M'} \mu(d) = \left( \sum_{d \mid n^*} \mu(d) \right) - \sum_{d_1 > M'} \mu(d_1) =
\]

\[
= - \sum_{1 \leq d < n^*/M'} \mu(n^*/d_2) = -\mu(n^*) \sum_{1 \leq d < n^*/M'} \mu(d_2), \quad (4.18)
\]

since

\[
\sum_{d \mid n^*} \mu(d) = \prod_{p \mid n^*} (1 - 1) = 0
\]

if \( n^* \geq 2 \).

Similarly, if \( n \geq 2 \) is not a primepower, i.e., \( n^* \geq 2 \) is not a prime, then

\[
- \sum_{1 \leq d \leq M'} \mu(d) \log d = - \left( \sum_{d \mid n^*} \mu(d) \log d \right) + \sum_{d_1 > M'} \mu(d_1) \log d_1 =
\]

\[
= \sum_{1 \leq d < n^*/M'} \mu(n^*/d_2) \log(n^*/d_2) = \mu(n^*) \sum_{1 \leq d < n^*/M'} \mu(d_2) \log(n^*/d_2), \quad (4.19)
\]

since

\[
- \sum_{d \mid n^*} \mu(d) \log d = \sum_{p \mid n^*} \log p \sum_{d \mid n^* \atop d \atop \perp p} \mu(d) = 0
\]

if \( n^* \geq 2 \) is not a prime.

We also have with \( d_1 = n^*/d \),

\[
\sum_{M' < d \leq M_{\ast}} \mu(d) \sum_{1 \leq k \leq M_{\ast} / d} \frac{|\mu(k)|}{\varphi(k)} =
\]

\[
= \sum_{M' < n^*/d_1 \leq M_{\ast}} \mu(n^*/d_1) \sum_{1 \leq k \leq M_{\ast} / d_1} \frac{|\mu(k)|}{\varphi(k)} =
\]

\[
= \mu(n^*) \sum_{n^*/M \leq d_1 \leq n^*/M_{\ast}} \mu(d_1) \sum_{1 \leq k \leq M_{\ast} / d_1} \frac{|\mu(k)|}{\varphi(k)}. \quad (4.20)
\]
Note that in (4.20) we have
\[ 1 \leq k \leq \frac{M d_1}{n^*} < \frac{M}{n^*} \cdot \frac{n^*}{M'} = \frac{M}{M'} = D^{9 \log D}. \]  

By using (4.18), (4.19) and (4.20), we can rewrite (4.16) as follows:
\[ S(M; n) = \mu(n^*) \left( \prod_{q \mid n^* Z_0} \left( 1 - \frac{1}{q} \right) \sum_{1 \leq d_2 < n^*/M'} \mu(d_2) \cdot \left( \log(n^*/d_2) - \log M - c' - \sum_{q \mid n^* Z_0} \frac{\log q}{q} \right) + \sum_{n^*/M \leq d_1 \leq n^*/M'} \mu(d_1) \sum_{1 \leq k \leq M d_1/n^* : \gcd(k, n^* Z_0) = 1} \frac{|\mu(k)|}{\varphi(k)} + \text{Error}(M; n). \]  

Combining (4.17) and (4.22), Lemma 4.2 follows. □

Given an arbitrary positive integer \( H \), we define a factoring \( n = P_H^-(n) P_H^+(n) \) of every integer \( n \geq 1 \) into a product of two of its divisors \( P_H^-(n) \) and \( P_H^+(n) \) by splitting the prime factors \( p \) of \( n \), counted with multiplicity, into two groups depending on whether \( p \leq H \) or \( p > H \). More precisely, let \( P_H^-(n) \) be the product of the prime factors of \( n \) which are \( \leq H \), and let \( P_H^+(n) \) be the product of the prime factors of \( n \) which are \( > H \) (the prime factors are taken with multiplicity, and the empty product is 1).

We choose
\[ H = H(N) = N^{1/\sqrt{\log N}} = e^{\sqrt{\log N}}, \]  
which implies \( H > D^{1200 \log D} \) (4.23′)

(see (2.48)). Note in advance that we will repeatedly use the “almost equality of \( \chi_{-D}(P_H^+(n)) \) and \( \mu(P_H^+(n)) \)” (see the argument after (2.10)).

By definition (see (2.48) and (4.15))
\[ H = e^{\sqrt{\log N}} \geq \frac{5\kappa N D}{2M'}, \]  
which implies
\[ H \geq \frac{n^*}{M'}, \]  
and so we can rewrite (4.22) in terms of \( \ell = P_H^-(n) \) as follows:
\[ S(M; n) = \mu(n^*) \left( \prod_{q \mid \ell Z_0} \left( 1 - \frac{1}{q} \right) \prod_{p \mid n^* : p > H} \left( 1 - \frac{1}{p} \right). \]  

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\[
\sum_{1 \leq d_2 < n^*/M'} \mu(d_2) \left( \log(n^*/d_2) - \log M - c' - \sum_{q | \ell Z_0} \frac{\log q}{q} - \sum_{p | n^*: p > H} \frac{\log p}{p} \right) + \\
\sum_{n^*/M \leq d_1 \leq n^*/M'} \mu(d_1) \sum_{1 \leq k \leq M d_1 / n^*: \gcd(k, \ell Z_0) = 1} \frac{|\mu(k)|}{\varphi(k)} + \text{Error}(M; n),
\]

where in the last step we used the fact

\[
\sum_{1 \leq k \leq M d_1 / n^*: \gcd(k, n^* D) = 1} \frac{|\mu(k)|}{\varphi(k)} = \sum_{1 \leq k \leq M d_1 / n^*: \gcd(k, n^* D) = 1} \frac{|\mu(k)|}{\varphi(k)}
\]

if \( d_1 \leq n^*/M' \) (since \( H > D^{9 \log D} \); see (4.21) and (4.23)).

Motivated by (4.21) and (4.25), we call a real number \((\ell, M)\)-bad (where \( \ell = P_H(n) \)) if either \( x \) has the form

\[
x = M'd = \frac{Md}{D^{9 \log D}} \quad \text{for some divisor } d | \ell,
\]

or \( x \) has the form

\[
x = \frac{Md}{r} \quad \text{for some divisor } d | \ell \text{ and some integer } 1 \leq r \leq D^{9 \log D}.
\]

By definition, there are at most \( \tau(\ell)(1 + D^{9 \log D}) \) \((\ell, M)\)-bad numbers.

Let’s return to (4.25): with \( \ell = P_H(n) \) we have

\[
S(M; n) = S_1(M; n) + \text{Error}_1(M; n),
\]

where

\[
S_1(M; n) = \mu(n^*) \left( \prod_{q | \ell Z_0} \left( 1 - \frac{1}{q} \right) \sum_{1 \leq d_2 < n^*/M'} \mu(d_2) \cdot \left( \log(n^*/d_2) - \log M - c' - \sum_{q | \ell Z_0} \frac{\log q}{q} \right) + \sum_{n^*/M \leq d_1 \leq n^*/M'} \mu(d_1) \sum_{1 \leq k \leq M d_1 / n^*: \gcd(k, \ell Z_0) = 1} \frac{|\mu(k)|}{\varphi(k)} \right)
\]

and

\[
|\text{Error}_1(M; n)| \leq \sum_{1 \leq d_2 < n^*/M'} \left( \left( 1 - \prod_{p | n^*: p > H} \left( 1 - \frac{1}{p} \right) \right) \log N + \sum_{p | n^*: p > H} \frac{\log p}{p} \right) + 
\]

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Since $n \leq 3\kappa ND$ and $H = e^{\sqrt{\log N}}$, we have

$$1 - \prod_{p\mid n^*: p > H} \left(1 - \frac{1}{p}\right) \leq \sum_{p\mid n^*: p > H} \frac{1}{p} < \frac{\log N}{e^{\sqrt{\log N}}},$$

Using (4.17) and (4.31) in (4.30), we have

$$|\text{Error}_1(M; n)| \leq \tau(n) \frac{2(\log N)^2}{e^{\sqrt{\log N}}} + \frac{10\tau^2(n) \log n}{D^2} + 10\frac{10}{7} \cdot 10^{-7}.$$  (4.32)

Clearly

$$S^2(M; n) = S^2_1(M; n) + 2\text{Error}_1(M; n)S_1(M; n) + \text{Error}_2^1(M; n).$$  (4.33)

Write

$$\sum_{n^*\neq n^r}(N; M) = \sum_{k=0}^{\kappa N} (\kappa, N) \frac{1}{2k + 1} \sum_{(\kappa N - k)D < n < (\kappa N + k + 1)D: n \text{ is not a primepower}} \chi_D(n)S_1^2(M; n).$$  (4.34)

Next we prove the following lemma.

**Lemma 4.3** We have

$$\left| \sum_{n^*\neq n^r}(N; M) - \sum_{n^*\neq n^r}^{(1)}(N; M) \right| \leq$$

$$\leq \frac{10^6 D (\log N)^7}{e^{\sqrt{\log N}}} + \frac{10^{20} (\log N)^9}{D} + \frac{10^{24} (\log N)^{17}}{D^3}.$$

**Proof.** Let’s return to (4.7): combining (4.8), (4.14), (4.28)-(4.34), we have

$$\left| \sum_{n^*\neq n^r}(N; M) - \sum_{n^*\neq n^r}^{(1)}(N; M) \right| \leq$$

$$\leq \frac{80(\log N)^4}{e^{\sqrt{\log N}}} \sum_{k=0}^{\kappa N} (\kappa, N) \frac{1}{2k + 1} \sum_{n: (\kappa N - k)D < n < (\kappa N + k + 1)D} \tau^2(n) +$$

$$+ \frac{40 \cdot 10^7 \log M}{D^2} \sum_{k=0}^{\kappa N} (\kappa, N) \frac{1}{2k + 1} \sum_{n: (\kappa N - k)D < n < (\kappa N + k + 1)D} \tau^3(n) \log n +$$

$$+ \frac{80(\log N)^4}{e^{\sqrt{\log N}}} \sum_{k=0}^{\kappa N} (\kappa, N) \frac{1}{2k + 1} \sum_{n: (\kappa N - k)D < n < (\kappa N + k + 1)D} \tau^2(n) +$$

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\[
+ \frac{2 \cdot 10^{14}}{D^4} \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k+1} \sum_{n: \ (\kappa N-k)D < n \leq \kappa N+k+1} \tau^4(n)(\log n)^2. \tag{4.35}
\]

We need the well-known asymptotic results

\[
\frac{1}{L} \sum_{n=1}^{L} \tau^2(n) = O \left( (\log L)^3 \right), \quad \frac{1}{L} \sum_{n=1}^{L} \tau^3(n) = O \left( (\log L)^7 \right),
\]

\[
\frac{1}{L} \sum_{n=1}^{L} \tau^4(n) = O \left( (\log L)^{15} \right), \quad \text{and in general, } \frac{1}{L} \sum_{n=1}^{L} \tau^k(n) = O \left( (\log L)^{2^k-1} \right) \tag{4.36}
\]

for an arbitrary but fixed \( k \geq 1 \), where the implicit constants are effectively computable (note that the surprising pattern of power-of-two minus one in the exponents of \( \log L \) is best possible). Moreover, for later applications I mention two more related inequalities

\[
\sum_{1 \leq d \leq M} \frac{\tau^2(d)}{\varphi(d)} \leq 10^6(\log M)^4 \quad \text{and} \quad \sum_{1 \leq k \leq M^2} \frac{\tau(k)}{k} \leq 100(\log M)^2. \tag{4.36'}
\]

Working with explicit constants in (4.36), it is easy to estimate (4.35), and thus we obtain

\[
\left| \sum_{n \neq p^r} (N; M) - \sum_{n \neq p^r}^{(1)} (N; M) \right| \leq \frac{10^6 D (\log N)^7}{e \sqrt{\log N}} + \frac{10^{20} D (\log N)^9}{D^2} + \frac{10^{24} D (\log N)^{17}}{D^4}, \tag{4.37}
\]

which completes the proof of Lemma 4.3. \( \square \)

I recall the Liouville function

\[
\lambda(n) = \lambda(p_1^{r_1}p_2^{r_2} \cdots p_s^{r_s}) = (-1)^{r_1+r_2+\cdots+r_s}.
\]

Applying the notation \( n = P_H^+(n) P_H^+(n) \) and the Liouville function, we have

\[
\sum_{n \neq p^r}^{(1)} (N; M) = \sum_{\lambda=\chi_D}^{(1)} (N; M) + \sum_{\lambda \neq \chi_D}^{(1)} (N; M), \tag{4.38}
\]

where

\[
\sum_{\lambda=\chi_D}^{(1)} (N; M) = \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k+1} \sum_{\ell \geq 1: \ P_H^+(\ell)=1} \chi_D(\ell). \cdot \sum_{(\kappa N-k)D < n \leq (\kappa N+k+1)D: \gcd(n,D)=1} \lambda(P_H^+(n)) S_1^2(M; n), \tag{4.39}
\]

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and
\[ \sum_{\lambda \neq \chi_{-D}}^{(1)} (N; M) = \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k+1} \sum_{\ell \geq 1: P_H^{+}(\ell) = 1} \chi_{-D}(\ell). \]

\[ \sum_{(\kappa N - k)D < n < (\kappa N + k + 1)D: \gcd(n, D) = 1} \left( \chi_{-D}(P_H^{+}(n)) - \lambda(P_H^{+}(n)) \right) S_1^2(M; n). \]  \hfill (4.40)

Note that, if \( P_H^{+}(n) = \ell \geq 2 \) then \( n \) is certainly not a prime, and \( \chi_{-D}(\ell) \neq 0 \) implies that the condition \( \gcd(D, n) = 1 \) holds automatically, since \( P_H^{+}(\ell) = 1 \) and \( H > D \) (see (4.23)).

Next we handle \( N \) as a variable, and take the average as \( N \) runs in an interval \( T \leq N < 2T \). That is, roughly speaking, from now on \( T \) will play the role of \( N \). As a byproduct, we slightly modify (4.23) and (4.24): let
\[ H = H(T) = T^{1/\sqrt{\log T}} = e^{\sqrt{\log T}}, \]  \hfill (4.41')

which implies \( H \geq \frac{5\kappa T D}{M'} \) and \( H > D^{1200 \log D} \). \hfill (4.41'')

To finish the estimation of \( \Omega_{1; \kappa; T}(M) - \Omega_{-1; \kappa; T}(M) \), we need the following technical lemma.

**Lemma 4.4** We have
\[
\frac{1}{T} \sum_{T \leq N < 2T} \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k+1} \sum_{\ell \geq 2: \gcd(D, \ell) = 1} \lambda(n) S_1^2(M; n) +
\]
\[
\sum_{(\kappa N - k)D < n < (\kappa N + k + 1)D: \gcd(n, D) = 1} \lambda(n) S_1^2(M; n) \quad \text{subject to } P_H^{+}(n) = \ell, \text{ } n \text{ is not a primepower}
\]
\[
+ \frac{1}{T} \sum_{T \leq N < 2T} \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k+1} \sum_{(\kappa N - k)D < n < (\kappa N + k + 1)D: \gcd(n, D) = 1} \lambda(n) S_1^2(M; n) \quad \text{subject to } P_H^{+}(n) = 1, \text{ } n \text{ is not a primepower}
\]
\[
- \prod_{q | Z_0} \left( 1 - \frac{1}{q} \right)^2 \frac{D(\log T - \log M)^2}{\log T} \leq 2 \cdot 10^3 D \log D.
\]

The proof of Lemma 4.4 is elementary but technical. We just mention the key ingredient—a routine sieve lemma—at the beginning of the next section (see Lemma 5.1), and postpone the details of the proof of Lemma 4.4 to Sections 7-8.
**Concluding Remark of Section 4.** Notice that Lemmas 4.1 and 4.4 do not distinguish between the primes $q \mid Z_0$ with $\chi_D(q) = 1$, 0 or $-1$.

5. The method of the proof (II)

Beside the simple but crucial fact $\sum_{d \mid k} \mu(d) = 0$ for all $k \geq 2$, the proof of Lemma 4.4 is based on the next lemma, which is a routine application of the simplest “sieve method” in number theory.

**Lemma 5.1** If $\log x > 2e^2 \log H (\log \log H + 10)$ then we can estimate the following Liouville sum from above:

$$\left| \sum_{1 \leq n \leq x} \frac{\lambda(n)}{P_H^{-}(n) = 1} \right| \leq x \exp \left( -\frac{\log x}{2 \log H} + 1 \right) + x \log x \exp \left( -\frac{\log x}{15} \right) + \frac{x}{H},$$

where $\exp(y) = e^y$.

**Proof.** If $\lambda(n) \neq \mu(n)$ and $P_H^{-}(n) = 1$, then $n$ can be written in the form $n = p^2 s$ with some prime $p > H$. Thus we have

$$\left| \sum_{1 \leq n \leq x} \frac{\lambda(n)}{P_H^{-}(n) = 1} - \sum_{1 \leq n \leq x} \frac{\mu(n)}{P_H^{-}(n) = 1} \right| \leq \sum_{p > H} \frac{x}{p^2} < \sum_{m > H} \frac{x}{m^2} < \frac{x}{H}. \tag{5.1}$$

To study the restricted Möbius sum in (5.1), we use the inclusion-exclusion principle:

$$\sum_{1 \leq n \leq x} \frac{\mu(n)}{P_H^{-}(n) = 1} = \sum_{1 \leq m \leq x} \frac{\mu^2(m)}{P_H^{-}(m) = 1} \sum_{1 \leq r \leq x/m} \frac{\mu(r)}{}. \tag{5.2}$$

We need the following well-known result that I put in the form of another lemma.

**Lemma 5.2** The Möbius sum

$$\mathcal{M}(L) = \sum_{1 \leq n \leq L} \mu(n)$$

has the following upper bound for all $L \geq 2$:

$$|\mathcal{M}(L)| \leq L e^{-\sqrt{\log L}/10}. \tag{5.3}$$
Remark. It is worth to point out that Lemma 5.2 is a deep result in analytic number theory (see e.g. [Ka]). It is based on the fact that the Dirichlet series
\[ \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left( 1 - \frac{1}{p^s} \right) = \frac{1}{\zeta(s)} \]
is the reciprocal of the Riemann’s zeta function \( \zeta(s) \), and so the proof of (5.3) can be carried out along the same lines as that of the classical analytic proof of the Prime Number Theorem, due to Hadamard and de la Vallee Poussin.

Returning to the proof of Lemma 5.1, by Lemma 5.2 we have
\[
\left| \sum_{1 \leq r \leq x/m} \mu(r) \right| \leq \frac{x}{m} e^{-\sqrt{\log \sqrt{x}/10}} \quad \text{if} \quad 1 \leq m \leq \sqrt{x}.
\]
(5.4)

If \( \sqrt{x} < m \leq x \) then we just use the trivial bound
\[
\left| \sum_{1 \leq r \leq x/m} \mu(r) \right| \leq \frac{x}{m}.
\]
(5.5)

Applying (5.4)-(5.5) in (5.2), we have
\[
\left| \sum_{1 \leq n \leq x} \mu(n) \right| \leq \sum_{1 \leq m \leq \sqrt{x}} \frac{x}{m} e^{-\sqrt{\log x/10\sqrt{2}}} + \sum_{\sqrt{x} < m \leq x} \frac{x \mu^2(m)}{m} \leq x \log x \frac{1}{e^{\sqrt{\log x/15}}} + x \sum_{\sqrt{x} < m \leq x} \frac{\mu^2(m)}{m}.
\]
(5.6)

To estimate the last sum in (5.6), we use the obvious fact that for every integer \( k \geq 1 \),
\[
\sum_{m > H^k} \frac{\mu^2(m)}{m} \leq \frac{1}{(k+1)!} \left( \sum_{1 < p \leq H} \frac{1}{p} \right)^{k+1} + \frac{1}{(k+2)!} \left( \sum_{1 < p \leq H} \frac{1}{p} \right)^{k+2} + \frac{1}{(k+3)!} \left( \sum_{1 < p \leq H} \frac{1}{p} \right)^{k+3} + \ldots.
\]
(5.7)
It is well-known that
\[ \sum_{p \leq H} \frac{1}{p} \leq \log \log H + 10, \]
so if \( k \geq e^2(\log \log H + 10) \), then for every \( j > k \) we have
\[ \frac{1}{j!} \left( \sum_{1 < p \leq H} \frac{1}{p} \right)^j \leq \left( \frac{e(\log \log H + 10)}{j} \right)^j \leq e^{-j}. \]

Using this in (5.7), we have
\[ \sum_{m > H^k} \frac{\mu^2(m)}{m} \leq \sum_{j > k} e^{-j} < e^{-k} \quad \text{for} \quad k \geq e^2(\log \log H + 10). \quad (5.8) \]

Note that
\[ \sqrt{x} \geq H^k \quad \text{for} \quad k = \left\lceil \frac{\log x}{2 \log H} \right\rceil. \quad (5.9) \]

Combining (5.1), (5.6) and (5.8)-(5.9), Lemma 5.1 follows. □

The deduction of Lemma 4.4 from Lemma 5.1 is a routine but rather long and cumbersome estimation. We postpone it to Sections 7-8.

Let’s return to (4.39): write
\[
\sum_{\lambda = \chi_D \lambda = \chi_D} (T; M) = \frac{1}{T} \sum_{T \leq N < 2T} \sum_{\lambda = \chi_D} (N; M) =
\]
\[
= \frac{1}{T} \sum_{T \leq N < 2T} \sum_{k=0}^{\kappa N} \frac{1}{w_k^{(\kappa, N)}} \frac{1}{2k + 1} \sum_{(\kappa N - k)D < n < (\kappa N + k + 1)D; \gcd(n, D) = 1} \lambda(n) S_1^2(M; n) +
\]
\[
+ \frac{1}{T} \sum_{T \leq N < 2T} \sum_{k=0}^{\kappa N} \frac{1}{w_k^{(\kappa, N)}} \frac{1}{2k + 1} \sum_{\ell \geq 2; \, P_{H}(\ell) = 1} \chi_D(\ell) \frac{\lambda(\ell)}{\lambda(\ell)}. \cdot
\]
\[
\sum_{(\kappa N - k)D < n < (\kappa N + k + 1)D; \gcd(n, D) = 1} \lambda(n) S_1^2(M; n). \quad (5.10)
\]

Combining (5.10) with Lemma 4.4, we obtain,
\[
\left| \sum_{\lambda = \chi_D} (T; M) - \prod_{q|Z_0} \left( 1 - \frac{1}{q} \right)^2 \frac{D(\log T - \log M)^2}{\log T} \right| \leq
\]

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Next we go to (4.40): write
\[
\sum_{\lambda \neq \chi_D}^{(1)} (T; M) = \frac{1}{T} \sum_{T \leq N < 2T} \sum_{\lambda \neq \chi_D}^{(1)} (N; M) = \frac{1}{T} \sum_{T \leq N < 2T} \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k + 1} \sum_{\ell \geq 1: P_H^\pm(\ell) = 1} \chi_D(\ell).
\]

\[
\sum_{(\kappa N - k)D < n < (\kappa N + k + 1)D: \gcd(n, D) = 1} (\chi_D(P_H^+(n)) - \lambda(P_H^+(n))) \overline{S}_1^2(M; n).
\]

The following lemma involves $L(1, \chi_D)$, which is small by hypothesis.

**Lemma 5.3** We have
\[
\left| \sum_{\lambda \neq \chi_D}^{(1)} (T; M) \right| \leq 10^8 (\log T)^5 D \sum_{H < q \leq T^2: \chi_D(q) = 1} \frac{1}{q} \leq 10^{11} (\log T)^6 D \cdot L(1, \chi_D).
\]

**Proof.** Let $J(\kappa N; k)$ denote the interval $(\kappa N - k)D < x < (\kappa N + k + 1)D$.

By (5.12),
\[
\left| \sum_{\lambda \neq \chi_D}^{(1)} (T; M) \right| \leq \frac{2}{T} \sum_{T \leq N < 2T} \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k + 1} \sum_{n \in J(\kappa N; k): \gcd(D, n) = 1} \chi_D(P_H^+(n)) \overline{S}_1^2(M; n). \tag{5.13}
\]

If $\chi_D(P_H^+(n)) \neq \lambda(P_H^+(n))$ and $\gcd(D, n) = 1$, then there is a prime $q > H$ with $\chi_D(q) = 1$ such that $q|n$. Using this observation, and the trivial upper bound
\[
\overline{S}_1^2(M; n) \leq 400 \tau^2(n) (\log M)^2 \tag{5.14}
\]
in (5.13), we have
\[
\left| \sum_{\lambda \neq \chi_D}^{(1)} (T; M) \right| \leq 400(\log M)^2 \frac{2}{T} \sum_{T \leq N < 2T} \sum_{k=0}^{\kappa N} w_k^{(\kappa, N)} \frac{1}{2k + 1} \sum_{n \in J(\kappa N; k): \gcd(D, n) = 1} \chi_D(P_H^+(n)) \overline{S}_1^2(M; n) \leq \tau^2(n) \leq 52.
\[
\leq 400(\log M)^2 \left( \frac{2}{\kappa T} \sum_{1 \leq n \leq 5\kappa TD: \gcd(D,n)=1, \chi_D(P_H(n)) \neq \lambda(P_H(n))} \tau^2(n) \right) \frac{1}{T} \sum_{T \leq N < 2T} \sum_{k=0}^{\kappa N} w_k^{(\kappa,N)} = \\
= \frac{800(\log M)^2}{\kappa T} \sum_{1 \leq n \leq 5\kappa TD: \gcd(D,n)=1, \chi_D(P_H(n)) \neq \lambda(P_H(n))} \tau^2(n) \leq \\
\leq \frac{800(\log M)^2}{\kappa T} \sum_{H < q \leq 5\kappa TD: 1 \leq m \leq 5\kappa TD/q} \sum_{\chi_D(q)=1} \tau^2(mq) \leq \\
\leq \frac{800(\log M)^2}{\kappa T} \sum_{q > H: \chi_D(q)=1} \tau^2(q) \sum_{1 \leq m \leq 5\kappa TD/q} \tau^2(m) \leq \\
\leq \frac{4000(\log M)^2}{\kappa T} \sum_{H < q \leq 5\kappa TD: \chi_D(q)=1} 10^3 \frac{\kappa T D}{q} (\log(T^2)) \leq 10^9 (\log T)^5 D \sum_{H < q \leq T^2: \chi_D(q)=1} \frac{1}{q}. \tag{5.15}
\]

Applying Lemma 1.1 in (5.15), we easily have
\[
\left| \sum_{\lambda \neq \chi_D} (1) (T; M) \right| \leq 10^9 (\log T)^5 D \sum_{H < q \leq T^2: \chi_D(q)=1} \frac{1}{q} \leq \\
\leq 10^9 (\log T)^5 D \cdot 100 \log T \cdot L(1, \chi_D) = 10^{11} (\log T)^6 D \cdot L(1, \chi_D), \tag{5.16}
\]
completing the proof of Lemma 5.3. \hfill \Box

Let’s return to (4.38): write
\[
\sum_{n \neq p^r}^{(1)} (T; M) = \frac{1}{T} \sum_{T \leq N < 2T} \sum_{n \neq p^r}^{(1)} (N; M).
\]

Combining (4.38)-(4.40), (5.10)-(5.12) and Lemma 5.3, we have
\[
\left| \sum_{n \neq p^r}^{(1)} (T; M) - \prod_{q \mid Z_0} \left( 1 - \frac{1}{q} \right)^2 \frac{D(\log T - \log M)^2}{\log T} \right| \leq \\
\leq 2 \cdot 10^5 D \log D + 10^{11} (\log T)^6 D \cdot L(1, \chi_D). \tag{5.17}
\]

Next write
\[
\sum_{n \neq p^r}^{(1)} (T; M) = \frac{1}{T} \sum_{T \leq N < 2T} \sum_{n \neq p^r}^{(1)} (N; M).
\]
By (5.17) and Lemma 4.3,

\[
\left| \sum_{n \neq p^r} (T; M) - \prod_{q \mid Z_0} \left( 1 - \frac{1}{q} \right)^2 D \frac{(\log T - \log M)^2}{\log T} \right| \leq 3 \cdot 10^3 D \log D. \tag{5.18}
\]

Now we are ready to estimate the difference

\[
\Omega_{1;\kappa;T}(M) - \Omega_{-1;\kappa;T}(M)
\]

(see (2.44)) as we promised at the beginning of Section 4. Combining (5.18) with (4.5) and Lemma 4.1, we obtain

\[
\left| \Omega_{1;\kappa;T}(M) - \Omega_{-1;\kappa;T}(M) - \prod_{q \mid Z_0} \left( 1 - \frac{1}{q} \right)^2 D (\log T - 2 \log M) \right| \leq 3 \cdot 10^3 D \log D + 30D \log D + 2 \leq 4 \cdot 10^3 D \log D, \tag{5.19}
\]

where the main term comes from

\[
\frac{(\log T - \log M)^2 - (\log M)^2}{\log T} = \log T - 2 \log M.
\]

To emphasize its importance, we rewrite (5.19) as a lemma.

**Lemma 5.4** We have

\[
\left| \Omega_{1;\kappa;T}(M) - \Omega_{-1;\kappa;T}(M) - \prod_{q \mid Z_0} \left( 1 - \frac{1}{q} \right)^2 D (\log T - 2 \log M) \right| \leq 3 \cdot 10^3 D \log D.
\]

\[\square\]

The next lemma will lead to a different approximation of

\[
\left| \Omega_{1;\kappa;T}(M) - \Omega_{-1;\kappa;T}(M) \right|.
\]

**Lemma 5.5** We have (see (3.20))

\[
\left| OD_{\kappa;N}(M) + \frac{D}{2} \log \frac{M^2}{N} \right|.
\]

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\[
\prod_{q \mid Z_0 \chi_D(q) = 1} \left( 1 - \frac{1}{q} \right) \prod_{q \mid Z_0 \chi_D(q) = 0} \left( 1 - \frac{1}{q} \right)^2 \prod_{q \mid Z_0 \chi_D(q) = -1} \left( 1 - \frac{1}{q} \right)^2 \left( 1 - \frac{1}{q+1} \right) \leq \\
\leq 10^{22} D (\log D)^{11} + \log \frac{M^2}{N} \cdot \frac{10^5 D}{(\log D)^3}.
\]

Unfortunately the proof of Lemma 5.5 is very long (starting from Sections 9, it is the subject of more than a dozen sections).

Note that Lemma 5.4 represents the evaluation of the left side of (2.49) via (2.40) and (2.44), and Lemma 5.5 represents the evaluation of the dominating part of the right side of (2.49) via (2.42) and (2.45).

**Completing the proof of Theorem 1 via contradiction.** We recall (3.19):

\[
\Omega_{1; \kappa; N}(M) - \Omega_{-1; \kappa; N}(M) = \\
= \Omega_{1; \kappa; N}(\text{OffDiag}; M) - \Omega_{-1; \kappa; N}(\text{OffDiag}; M) = \\
= 2OD_{\kappa; N}(M) + \text{Error}(\kappa; N; M).
\]

By Lemma 3.4,

\[
|\text{Error}(\kappa; N; M)| < 1.
\]

Thus we have

\[
\left| \Omega_{1; \kappa; N}(\text{OffDiag}; M) - \Omega_{-1; \kappa; N}(\text{OffDiag}; M) - 2OD_{\kappa; N}(M) \right| < 1. \quad (5.20)
\]

We want to estimate

\[
\overline{\Omega}_{1; \kappa; T}(\text{OffDiag}; M) - \overline{\Omega}_{-1; \kappa; T}(\text{OffDiag}; M)
\]

where \(\overline{\Omega}_{\pm 1; \kappa; T}(\text{OffDiag}; M)\) is the average

\[
\overline{\Omega}_{\pm 1; \kappa; T}(\text{OffDiag}; M) = \frac{1}{T} \sum_{T \leq N < 2T} \Omega_{\pm 1; \kappa; N}(\text{OffDiag}; M).
\]

Since

\[
\log \frac{M^2}{N} = 2 \log M - \log N,
\]

by Lemma 5.5 and (5.20) we have

\[
\left| \overline{\Omega}_{1; \kappa; T}(M) - \overline{\Omega}_{-1; \kappa; T}(M) - D(\log T - 2 \log M) \right|.
\]
\[
\prod_{q|Z_0, \chi_D(q) = 1} \left( 1 - \frac{1}{q} \right) \prod_{q|Z_0, \chi_D(q) = 0} \left( 1 - \frac{1}{q} \right)^2 \prod_{q|Z_0, \chi_D(q) = -1} \left( 1 - \frac{1}{q} \right)^2 \left( 1 - \frac{1}{q + 1} \right) \leq \\
\leq 2 \cdot 10^{22} D \log D^{11} + \log \frac{M^2}{N} \cdot 2 \cdot 10^5 D \left( \log D \right)^3 + 2D + 1. \quad (5.21)
\]

Subtracting (5.21) from Lemma 5.4, \( \Omega_{\pm 1; \kappa; T}(M) \) cancels out, and we have

\[
\left| \prod_{p|Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot D \log \frac{M^2}{T} \left( 1 - \prod_{p|Z_0, \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \prod_{p|Z_0, \chi_D(p) = -1} \left( 1 - \frac{1}{p + 1} \right) \right) \right| \leq \\
\leq 10^4 D \log D^{2} + 2 \left( 10^{22} D \log D^{11} + \log \frac{M^2}{N} \cdot 10^5 D \left( \log D \right)^3 \right) + 2D + 1. \quad (5.22)
\]

According to the definition of \( Z_0 \) (see (1.14)-(1.15)), we now distinguish two cases \( (\chi_D(p_0) = 1 \text{ or } -1) \). If \( \chi_D(p_0) = 1 \), then

\[
Z_0 = \prod_{p \leq D: \chi_D(p) \neq -1} p,
\]

and thus we have the estimation

\[
\prod_{p|Z_0} \left( 1 - \frac{1}{p} \right)^2 \left( 1 - \prod_{p|Z_0, \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \prod_{p|Z_0, \chi_D(p) = -1} \left( 1 - \frac{1}{p + 1} \right) \right) = \\
= \prod_{p|Z_0} \left( 1 - \frac{1}{p} \right)^2 \left( 1 - \prod_{p|Z_0, \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \right) = \\
= \prod_{p|Z_0} \left( 1 - \frac{1}{p} \right)^2 \left( 1 - \left( 1 + \frac{1}{p_0 - 1} \right) \prod_{p|Z_0, \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \right) \leq \\
\leq - \prod_{p|Z_0} \left( 1 - \frac{1}{p} \right)^2 \frac{1}{p_0 - 1} \prod_{p|Z_0, \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) =
\]

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\[
= - \frac{1}{p_0 - 1} \left( 1 - \frac{1}{p_0} \right)^2 \prod_{p | \mathbb{Z}_0 \atop \chi_D(p) = 1} \left( 1 - \frac{1}{p} \right) \prod_{p | D} \left( 1 - \frac{1}{p} \right)^2 \leq
\]
\[
\leq - \frac{1}{2 + 2 \log D} \prod_{p | \mathbb{Z}_0 \atop \chi_D(p) = 1} \left( 1 - \frac{1}{p} \right) \cdot \frac{2^2(D)}{D^2} \leq
\]
\[
\leq - \frac{1}{2 + 2 \log D} \cdot \frac{1}{400 \log D (\log \log D)^2},
\]

(5.23)

since
\[
\left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{p - 1} \right) = 1,
\]

and
\[
\prod_{q | \mathbb{Z}_0} \left( 1 - \frac{1}{q} \right) \geq \prod_{p \leq D} \left( 1 - \frac{1}{q} \right) \geq \frac{1}{4 \log D},
\]

where we used the well-known number-theoretic facts \(\frac{n}{\varphi(n)} \leq 10 \log \log n\) (see (3.28)) and
\[
\prod_{p \leq n} \left( 1 - \frac{1}{p} \right) = (1 + o(1)) \frac{e^{-\gamma_0}}{\log n},
\]

and also used that \(D > e^{10^{100}}\) is very large.

If \(\chi_D(p_0) = -1\) then we distinguish three subcases. If
\[
\prod_{p | \mathbb{Z}_0 \atop \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \geq 2,
\]

then again we choose
\[
\mathbb{Z}_0 = \prod_{p \leq D; \atop \chi_D(p) \neq -1} p,
\]

and have the estimation
\[
\prod_{p | \mathbb{Z}_0} \left( 1 - \frac{1}{p} \right)^2 \left( 1 - \prod_{p | \mathbb{Z}_0 \atop \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \prod_{p | \mathbb{Z}_0 \atop \chi_D(p) = -1} \left( 1 - \frac{1}{p + 1} \right) \right) =
\]
\[
= \prod_{p | \mathbb{Z}_0} \left( 1 - \frac{1}{p} \right)^2 \left( 1 - \prod_{p | \mathbb{Z}_0 \atop \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \right) \leq
\]

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\[
\leq - \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot \frac{1}{2} \prod_{p \mid Z_0} \left( 1 + \frac{1}{p - 1} \right) = \\
= - \frac{1}{2} \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right) \prod_{p \mid D} \left( 1 - \frac{1}{p} \right)^2 = \\
= - \frac{1}{2} \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right) \cdot \frac{\varphi^2(D)}{D^2} \leq \\
\leq - \frac{1}{2} \cdot \frac{1}{400 \log D (\log \log D)^2}, \tag{5.24}
\]

where in the last steps we repeated the estimations in (5.23).

If
\[
2 > \prod_{p \mid Z_0, \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \geq 1 + \frac{1}{4 \log D},
\]

then again we choose
\[
Z_0 = \prod_{p \leq D, \chi_D(p) \neq -1} p,
\]

and have the estimation
\[
\prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right)^2 \left( 1 - \prod_{p \mid Z_0, \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \prod_{p \mid Z_0, \chi_D(p) = -1} \left( 1 - \frac{1}{p + 1} \right) \right) = \\
= \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right)^2 \left( 1 - \prod_{p \mid Z_0, \chi_D(p) = 1} \left( 1 + \frac{1}{p - 1} \right) \right) \leq \\
\leq \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right)^2 \left( 1 - \left( 1 + \frac{1}{4 \log D} \right) \right) = \\
= - \prod_{p \mid Z_0} \left( 1 - \frac{1}{p} \right)^2 \cdot \frac{1}{4 \log D} = \\
= - \prod_{p \mid Z_0, \chi_D(p) = 1} \left( 1 - \frac{1}{p} \right)^2 \cdot \frac{\varphi^2(D)}{D^2} \cdot \frac{1}{4 \log D} \leq
\]

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\[ \leq -\frac{1}{2^2} \cdot \frac{1}{100(\log \log D)^2} \cdot \frac{1}{4 \log D}, \quad (5.25) \]

since in this subcase

\[ \prod_{\substack{p \mid Z_0 \\ \chi_D(p) = 1}} \left(1 - \frac{1}{p}\right) = \prod_{\substack{p \mid Z_0 \\ \chi_D(p) = 1}} \left(1 + \frac{1}{p - 1}\right)^{-1} \geq \frac{1}{2}. \]

Finally, if \( \chi_D(p_0) = -1 \) and

\[ \prod_{\substack{p \mid Z_0 \\ \chi_D(p) = 1}} \left(1 + \frac{1}{p - 1}\right) < 1 + \frac{1}{4 \log D}, \]

then we choose

\[ Z_0 = p_0 \prod_{\substack{p \leq D: \\ \chi_D(p) \neq -1}} p, \]

and have the estimation

\[ \prod_{p \mid Z_0} \left(1 - \frac{1}{p}\right)^2 \left(1 - \prod_{\substack{p \mid Z_0 \\ \chi_D(p) = 1}} \left(1 + \frac{1}{p - 1}\right) \cdot \left(1 - \frac{1}{p_0 + 1}\right)\right) \geq \]

\[ \geq \prod_{p \mid Z_0} \left(1 - \frac{1}{p}\right)^2 \left(1 - \left(1 + \frac{1}{4 \log D}\right) \left(1 - \frac{1}{p_0 + 1}\right)\right) \geq \]

\[ \geq \prod_{p \mid Z_0} \left(1 - \frac{1}{p}\right)^2 \cdot \frac{1}{4 \log D} \geq \]

\[ \geq \frac{1}{2^2} \cdot \frac{1}{100(\log \log D)^2} \cdot \frac{1}{4 \log D}, \quad (5.26) \]

since \( p_0 < 2 \log D \) (and we also repeated the estimations in (5.25)).

Combining (5.22)-(5.26), we obtain

\[ \frac{1}{10^3(\log D)^2(\log \log D)^2} \log \frac{M^2}{T} \leq \]

\[ \leq 10^4(\log D)^2 + 2 \left(10^{22}(\log D)^{11} + \frac{10^5 \log M}{(\log D)^3}\right) + 3. \quad (5.27) \]

Using \( \log T = (\log D)^{15}, M = T \exp \left(-\frac{2}{3} \sqrt{\log T}\right), \) and \( \log D > 10^{100}, \) we see that the left-hand side of (5.27) is clearly larger than the right-hand side (since \( (\log D \log \log D)^2 \) is
much smaller than \((\log D)^3\), which is a contradiction. This contradiction proves Theorem 1.

It remains to prove Lemma 2.1 (see Section 30), Lemma 4.4 (see Sections 7-8), and Lemma 5.5 (it starts from Section 9).

The proof of Lemma 2.1 is based on the well-known elementary fact

\[
\left| \sum_{1 \leq m \leq x} \frac{1}{m} - \log x - \gamma_0 \right| \leq \frac{5}{x},
\]

where \(\gamma_0 = 0.5772\ldots\) is Euler’s constant. The proof of Lemma 4.4 is based on the “sieve lemma” Lemma 5.1 (combined with Lemma 5.2) and the Prime Number Theorem.

Finally, the long proof of Lemma 5.5 uses the following key ingredients:

Fact (1): the Pólya–Vinogradov inequality for character sums;

Fact (2): the sum of prime-reciprocals starting from \(p > D\) and ending at \(x\)

\[
\sum_{D < p \leq x \atop \chi_{-D}(p) = 1} \frac{1}{p}
\]

is “small” if the class number \(h(-D)\) is substantially smaller than \(\sqrt{D}\) and \(x\) is not too large;

Fact (3): the proof technique of the well-known average result for the divisor function

\[
\sum_{1 \leq n \leq x} \tau(n) = x \log x + (2\gamma_0 - 1)x + O(\sqrt{x}),
\]

and other similar average type arguments/estimations based on partial summation;

Fact (4): let

\[
S(\Delta; k) = \sum_{1 \leq n \leq k} \chi_\Delta(n),
\]

then for every negative fundamental discriminant \(\Delta < 0\),

\[
\frac{1}{|\Delta|} \sum_{k=1}^{|\Delta|-1} S(\Delta; k) = h(\Delta)
\]

(note that Fact (4) is a corollary of the class number formula (1.4));

Fact (5): the “logarithmic” character sum \(\sum_{j=1}^{D^4} \chi_{-D}(j) \log j\) can be estimated as follows:

\[
\left| \sum_{j=1}^{D^4} \chi_{-D}(j) \log j + \frac{\pi}{6} \sqrt{D} \sum_{(a,b,c)} \frac{1}{a} \right| \leq h(-D)(6 \log D + 30) + 21(\log D)^2 D^{1/6} + \sqrt{D} \left( 10^3 \left( \frac{h(-D)}{\sqrt{D}} \right)^{3/2} + \frac{10^3 h(-D)}{\sqrt{D}} \right)
\]

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(see Lemma 14.2), where $\sum_{(a,b,c)} \frac{1}{a}$ means that we add up the reciprocals of the leading coefficients $a = a_j$ in the family of reduced, primitive, inequivalent binary quadratic forms of integer coefficients $ax^2 + bxy + cy^2$ with discriminant $b^2 - 4ac = -D < 0$, $a > 0$, $c > 0$, $1 \leq j \leq h(-D)$ (note that Fact (5) is based on

$$\sum_{j=1}^{D^4} \chi_{-D}(j) \log j = -\frac{\sqrt{D}}{\pi} L'(1, \chi_{-D}) + \text{negligible}$$

if $h(-D)$ is small, where $L'(1, \chi_{-D})$ is the derivative of the Dirichlet L-function $L(s, \chi_{-D})$ at $s = 1$);

and to estimate the critical sum $\sum_{(a,b,c)} \frac{1}{a}$ in Fact (5), we need

Fact (6):

$$0 \leq \prod_{p \mid \mathbb{Z}_0, \chi_{-D}(p) \neq -1} \frac{p+1}{p-1} \prod_{p \mid D} \frac{p-1}{p} - \sum_{j=1}^{h(-D)} \frac{1}{a_j} \leq$$

$$\leq \frac{10^3 h(-D)(\log D)^4}{\sqrt{D}} + \frac{10^4}{(\log D)^3} + \frac{1}{D}$$

(see Lemma 14.3).

Facts (5) and (6) enter the proof of Lemma 5.5 in the following way. The sum $OD_{\kappa;N}(M)$ (see (3.20)) resembles a “double harmonic sum”, and also the main term in Lemma 2.1 is the logarithm function. This explain why (3.20) resembles a Riemann sum for a logarithmic integral that we can evaluate explicitly. Thus we obtain—via long estimations—the following good approximation of (3.20) (see (19.12) later):

$$OD_{\kappa;N}(M) = OD_{\kappa;N}(M; \text{Core}; \bullet\bullet\bullet; \varepsilon) + \text{negligible},$$

where

$$OD_{\kappa;N}(M; \text{Core}; \bullet\bullet\bullet; \varepsilon) =$$

$$= \sqrt{D} \sum_{1 \leq d \leq M: \gcd(d, \mathbb{Z}_0) = 1} \frac{\mu^2(d)}{\varphi(d)} \sum_{1 \leq r \leq D^8: \gcd(r, d \mathbb{Z}_0) = 1} \frac{\mu(r) \chi_{-D}(r)}{r \varphi(r)} \sum_{(r_1, r_2): \frac{r_1 r_2 = r}{r_1, r_2 \in \mathbb{Z}}}$$

$$\cdot \sum_{|d_1|} \chi_{-D}(d_1) \prod_{p | d_1} \frac{p-2}{p-1} \sum_{v | r d_1} \mu(v) \chi_{-D}(v) \sum_{j=1}^{D^4} \chi_{-D}(j) \sum_{\ell \in \mathbb{Z}} \varepsilon \cdot f \left((1 + \varepsilon)^\ell\right) \cdot$$

$$\cdot \left(c_{13}(rd) \left(\log \left(\frac{M^2 d_1 (1 + \varepsilon)^\ell}{2\pi N d^2 v j}\right)\right)^3 + c_{14}(rd) \left(\log \left(\frac{M^2 d_1 (1 + \varepsilon)^\ell}{2\pi N d^2 v j}\right)\right)^2 +$$

$$+ c_{15}(rd) \left(\log \left(\frac{M^2 d_1 (1 + \varepsilon)^\ell}{2\pi N d^2 v j}\right)\right) + c_{16}(rd) \right) \cdot \delta_{1,0} \left\{ \frac{M^2 d_1}{2\pi N d^2 v} \geq C^* \right\}.$$

(5.28)
where (see (9.7))

$$f(y) = \sin(\kappa y) \left( \frac{\sin((1 + \frac{1}{2N})y)}{(2N + 1) \sin(y/2N)} \right)^\kappa$$

if $|y| \leq \pi N$, and $f(y) = 0$ if $|y| > \pi N$, and

$$\sum_{\varepsilon \in \mathbb{Z}} \varepsilon \cdot f((1 + \varepsilon)^\ell) = \frac{\pi}{2} + \text{negligible}$$

(see (12.7), (12.12), (12.15) and Lemma 14.1); moreover $c_i(rd)$, $13 \leq i \leq 16$ are constants depending only on $rd$ (see (19.4)-(19.5)), $C^* = e^{(\log D)^3}$, and $\delta_{1,0}\{\cdots\}$ is a 0-1 valued “cutoff function” defined as

$$\delta_{1,0}\{\text{true}\} = 1 \quad \text{and} \quad \delta_{1,0}\{\text{false}\} = 0.$$

Equation (5.28) is the first main step in the long proof of Lemma 5.5.

For the second main step, we use the explicit forms of the constants $c_i(rd)$, $13 \leq i \leq 16$, and employ many more routine estimations (see Sections 19-24). Thus we are able to rewrite (5.28) in the following form (see (12.27)):

$$OD_{\kappa;N}(M) = OD_{\kappa;N}(M; \text{Core}; \bullet \bullet \bullet \bigtriangledown; 1; \text{DominatingPart}) + \text{negligible},$$

where

$$OD_{\kappa;N}(M; \text{Core}; \bullet \bullet \bullet \bigtriangledown; 1; \text{DominatingPart}) =
\prod_{p} p^{3(p + 4)} \left( \frac{1}{(p + 1)^3} \right) \cdot \left( \frac{6}{\pi^2} \right)^4 \prod_{q | Z_0} \frac{q}{q + 4} \cdot \frac{\pi}{2} \cdot \left( \sum_{j=1}^{D^4} \chi_D(j) \log j \right) \cdot
\sqrt{D} \sum_{1 \leq r \leq D^a; \gcd(r, Z_0) = 1} \mu(r) \chi_D(r) \tau(r) \cdot \prod_{q | r} \frac{q}{q + 4} \cdot \sum_{r_3 | r} \mu(r_3) \chi_D(r_3) \cdot
\sum_{1 \leq d_4 \leq M; \gcd(d_4, rd) = 1} \frac{\mu^2(d_4) \chi_D(d_4)}{\varphi(d_4)} \prod_{p | d_4} \frac{p(p - 2)}{(p + 4)(p - 1)} \sum_{1 \leq d_3 \leq M/d_4; \gcd(d_3, rd) = 1} \frac{\mu(d_3)}{\varphi(d_3)} \prod_{q | d_3} \frac{q(q - 2)}{(q + 4)(q - 1)} \cdot
\sum_{1 \leq d_2 \leq M/(d_4 d_3); \gcd(d_2, rd) = 1} \frac{\mu^2(d_2)}{\varphi(d_2)} \prod_{p | d_2} \frac{p}{p + 4} \cdot \left( 2 \log d_4 + 4 \log d_3 + 4 \log d_2 \right) \log \frac{M^2}{N} - 4 \log d_4 \log d_3 - 8 \log d_3 \log d_2 - 4 \log d_4 \log d_2 - 4(\log d_2)^2 + \text{negligible.} \quad (5.29)$$
The evaluation/estimation of the complicated sum (5.29) is the third main step in the proof of Lemma 5.5. We repeatedly use the simple fact that \( \mu(r)\chi_D(r) = \mu^2(r) \) if \( r \leq D \) and \( \gcd(r, Z_0) = 1 \), and also use the less trivial facts

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0,
\]

\[
\sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n} = -1,
\]

see the *Guiding Intuition* at the end of Section 12. Thus we obtain—again via a long chain of estimations that prove the error term to be *negligible*—the last approximation (see Lemma 13.2 and Lemma 14.1):

\[
OD_{\kappa,N}(M) =
\]

\[
= \sqrt{D} \pi \left( \sum_{j=1}^{D^4} \chi_D(j) \log j \right) \log \frac{M^2}{N} \cdot \frac{6}{\pi^2} \prod_{q \mid Z_0} \frac{(q-1)^2}{q(q+1)} + \text{negligible}, \tag{5.30}
\]

Finally, to evaluate the critical sum \( \sum_{j=1}^{D^4} \chi_D(j) \log j \) in (5.30), we use Facts (5) and (6), and Lemma 5.5 follows. (The proofs of Facts (5) and (6) are in Section 29.)

We may say (with some gross oversimplification) that the rest of the proof is just routine elementary calculations/estimations using the listed ingredients in the outlined way.

6. Theorem 2: the case of positive discriminants (I)

**Outline of the proof of Theorem 2.** Let \( D > 0 \) denote a positive fundamental discriminant violating Theorem 2. The proof is similar to the proof of Theorem 1, but it requires several modifications. The first change comes from the fact that in the proof of Lemma 1.1 a positive definite binary quadratic form has positive values only. In sharp contrast, a binary quadratic form of positive discriminant is indefinite, and has both positive and negative values. This is why we replace the elementary Lemma 1.1 with the following more sophisticated lemma (its proof uses the Pólya–Vinogradov inequality). Let

\[
R(\Delta; n) = \sum_{i=1}^{h(\Delta)} R(\Delta; n, F_i) \tag{6.1}
\]
denote the total number of primary representations of a given integer \( n \geq 1 \) by a representative set of binary quadratic form \( F(x, y) = ax^2 + bxy + cy^2 \) of discriminant \( b^2 - 4ac = \Delta > 0 \). By Dirichlet’s theorem

\[
R(\Delta; n) = \sum_{m \mid n} \chi_\Delta(m).
\] (6.2)

**Lemma 6.1** For every fundamental discriminant \( \Delta > 0 \) and integer \( N \geq 1 \) we have

\[
\left| \sum_{n=1}^{N} R(\Delta; n) - L(1, \chi_\Delta) N \right| \leq 4\sqrt{N}\Delta^{1/4}\sqrt{\log \Delta}.
\]

**Proof.** By (6.1) and (6.2),

\[
\sum_{n=1}^{N} R(\Delta; n) = \sum_{m=1}^{N} \sum_{n \mid m} \chi_\Delta(m) = \sum_{1 \leq m \leq \sqrt{N}\Delta^{1/4}\sqrt{\log \Delta}} \chi_\Delta(m) \sum_{1 \leq k \leq N/m} \chi_\Delta(m),
\]

and combining this with the Pólya–Vinogradov inequality (see Lemma 9.3), we obtain

\[
\left| \sum_{n=1}^{N} R(\Delta; n) - \sum_{1 \leq m \leq \sqrt{N}\Delta^{1/4}\sqrt{\log \Delta}} \chi_\Delta(m) \frac{N}{m} \right| \leq \sqrt{N}\Delta^{1/4}\sqrt{\log \Delta} + \sum_{1 \leq k < \sqrt{N}\Delta^{1/4}\sqrt{\log \Delta}} \sqrt{\Delta}\log \Delta \leq 2\sqrt{N}\Delta^{1/4}\sqrt{\log \Delta}.
\] (6.3)

Clearly

\[
\sum_{1 \leq m \leq \sqrt{N}\Delta^{1/4}\sqrt{\log \Delta}} \chi_\Delta(m) \frac{N}{m} = L(1, \chi_\Delta) N - N \sum_{m > \sqrt{N}\Delta^{1/4}\sqrt{\log \Delta}} \chi_\Delta(m). \] (6.4)

Using partial summation and the Pólya–Vinogradov inequality,

\[
\left| \sum_{m > \sqrt{N}\Delta^{1/4}\sqrt{\log \Delta}} \chi_\Delta(m) \frac{N}{m} \right| =
\]

\[
= \left| \sum_{n > \sqrt{N}\Delta^{1/4}\sqrt{\log \Delta}} \left( \sum_{\sqrt{N}\Delta^{1/4}\sqrt{\log \Delta} < m \leq n} \chi_\Delta(m) \right) \left( \frac{1}{n} - \frac{1}{n+1} \right) \right| \leq
\]

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\[
\frac{2\sqrt{\Delta \log \Delta}}{\sqrt{N} \Delta^{1/4} \sqrt{\log \Delta}} = \frac{2\Delta^{1/4} \sqrt{\log \Delta}}{\sqrt{N}}.
\]

Combining (6.3)-(6.5), Lemma 6.1 follows. (Note that we didn’t use the assumption \(\Delta > 0\).) \(\square\)

Through the proof of Theorem 2 Lemma 6.1 plays the same role as that of Lemma 1.1 in the proof of Theorem 1.

The second change is due to the fact \(\chi_D(-1) = 1\) for \(D > 0\), which implies that the interval \([A_1, A_2]\) in Section 1 is replaced by the symmetric interval \([-A, A]\) (i.e., \(A_2 = -A_1 > 0\)). Then for \(D > 0\) the analog of (2.39) is

\[
WOD_a(M; \kappa; N; d) = \sum_{(m_1, \ell_1) \neq (m_2, \ell_2): 1 \leq m_1, m_2 \leq M, 1 \leq \ell_h \leq m_h, \gcd(\ell_h, m_h) = 1, h = 1, 2, \gcd(m_1, m_2) = d, \gcd(m_1, m_2, Z_0) = 1} \mu(m_1)\mu(m_2) \varphi(m_1)\varphi(m_2) \cdot e^{2\pi i \left(\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2}\right)} \left(\frac{\sin \left((2N + 1)\pi D \left(\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2}\right)\right)}{(2N + 1) \sin \left(\pi D \left(\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2}\right)\right)}\right)^\kappa,
\]

and the analog of (2.40) is (\(\delta = \pm 1\))

\[
\Omega_{\delta; \kappa; N}(M) = \sum_{1 \leq a \leq D, \chi_D(a) = \delta} W_a(M; \kappa; N) = \\
\sum_{k=0}^{\kappa N} w_{k}^{(\kappa, N)} \frac{1}{2k + 1} \sum_{j: -k \leq j \leq k} \sum_{1 \leq a \leq D, \chi_D(a) = \delta} \left(\sum_{\substack{d \geq 1: \gcd(Z_0, d) = 1 \\text{and} \ d | a + jD}} \mu(d) \sum_{1 \leq k \leq M / d: \gcd(k, (a + jD)Z_0) = 1} \frac{|\mu(k)|}{\varphi(k)}\right)^2.
\]

There is no change in (2.41), but the analog of (2.42) is the following:

\[
\Omega_{\delta; \kappa; N}(\text{OffDiag}; M) = \sum_{1 \leq a \leq D, \chi_D(a) = \delta} \text{WOOffDiag}_a(M; \kappa; N) = \\
\sum_{(m_1, \ell_1) \neq (m_2, \ell_2): 1 \leq m_1, m_2 \leq M, 1 \leq \ell_h \leq m_h, \gcd(\ell_h, m_h) = 1, h = 1, 2, \gcd(m_1, m_2) = d, \gcd(m_1, m_2, Z_0) = 1} \mu(m_1)\mu(m_2) \varphi(m_1)\varphi(m_2).
\]
Now we are ready to formulate the analog of the key equality (2.49). Again the proof is based on the fact that, under the existence of a “bad” positive discriminant $D > 0$, the left-hand side and the right-hand side of the analog of (2.49) are not equal. They are not equal for the same reason as in the proof of Theorem 1: we prove the analogs of (2.50) and (2.51), and the underlying intuition is that the right side of the analog of (2.49) does, and the left side does not distinguish between the primes $q | Z_0$ with $\chi_D(q) = 1, 0$ or $-1$.

The third change is that in Lemma 3.2 the Gauss sum becomes real: $G(\chi_D; D) = \sqrt{D}$ for $D > 0$. It follows that for $D > 0$ we have to modify Lemma 3.1 by removing the factor $i = \sqrt{-1}$. Therefore, the analog of the critical sum (3.20) goes as follows:

$$OD_{\kappa; N}(M) = \frac{\sqrt{D}}{2} \sum_{1 \leq d \leq M: \mu^2(d) = 1} \varphi(d) \sum_{d_1 \div d} \cdot \prod_{p \mid d_1} \frac{p-2}{p-1} \sum_{(m_1, m_2): 1 \leq m_1, m_2 \leq M} \frac{\mu(m_1)\mu(m_2)}{\varphi(m_1)\varphi(m_2)} \cdot \chi_D\left(\frac{m_1 m_2 d_1}{d^2}\right).$$

$$\cdot \left( \sum_{1 \leq s \leq \frac{\ell_1}{m_1} - \frac{\ell_2}{m_2} = \frac{\ell}{n} \text{ with } n = \frac{m_1 m_2 d_1}{d^2}, \text{ and } s \equiv D\ell \pmod{n}, 1 \leq |s| \leq n/2, \gcd(s, n) = 1, \right)$$

with $s, n, \ell$ coming from

$$\frac{\ell_1}{m_1} - \frac{\ell_2}{m_2} = \frac{\ell}{n} \text{ with } n = \frac{m_1 m_2 d_1}{d^2},$$

(6.9)

There will be some change in Section 9. Lemma 9.6 states that for every negative fundamental discriminant $\Delta < 0$,

$$\frac{1}{|\Delta|} \sum_{k=1}^{|\Delta|} S(\Delta; k) = h(\Delta),$$

(6.11)

where

$$S(\Delta; k) = \sum_{1 \leq j \leq k} \chi_\Delta(j)$$

(6.12)
is the character-sum. In the proof of the case of negative discriminants we heavily used the
fact that the average of the character-sum $h(-D)$ for the “bad discriminant” was $o(\sqrt{D})$
(since $h(-D) = o(\sqrt{D})$ was the hypothesis of Theorem 1).

The good news is that in the case of positive discriminants we have the following
stronger form of (6.11), which makes the analog arguments simpler: for every positive
fundamental discriminant $\Delta > 0$,

$$\frac{1}{\Delta} \sum_{k=1}^{\Delta} S(\Delta; k) = 0,$$

(6.13)

where

$$S(\Delta; k) = \sum_{1 \leq j \leq k} \chi_{\Delta}(j)$$

is the character-sum. Indeed, since $\chi_{\Delta}(-1) = 1$ for $\Delta > 0$, we have

$$S(\Delta; \Delta - k) = \sum_{j=1}^{\Delta} \chi_{\Delta}(j) - S(\Delta; k - 1) = -S(\Delta; k - 1),$$

and so

$$\sum_{k=1}^{\Delta} S(\Delta; k) = \frac{1}{2} \sum_{k=1}^{\Delta} (S(\Delta; k - 1) + S(\Delta; \Delta - k)) = 0,$$

proving (6.13).

The next major change is that $\sum_{n=1}^{ND} \chi_{D}(n) \log n$ is not about constant times $\sqrt{D}$ for
the bad positive discriminant $D > 0$ with “large” $N$. Indeed, we have

$$\sum_{n=1}^{ND} \chi_{D}(n) \log n = O(L(1, \chi_{D}) \sqrt{D}) + O(D/N),$$

(6.14)

which implies that (6.14) is negligible in the sense

$$\sum_{n=1}^{ND} \chi_{D}(n) \log n = o(\sqrt{D})$$

(6.15)

if $L(1, \chi_{D}) = o(1)$ (and $N \geq D > 0$).

To prove (6.14), we need

**Lemma 6.2** For every fundamental discriminant $\Delta > 0$,

$$\lim_{N \to \infty} \sum_{n=1}^{N\Delta} \chi_{\Delta}(n) \log n = -L'(0, \chi_{\Delta}),$$

(6.16)

where of course $L'(0, \chi_{\Delta})$ denotes the derivative of the $L$-function $L(s; \chi_{\Delta})$ at $s = 0$ (note
that $L(s; \chi_{\Delta})$ is regular on the entire complex plane).
Moreover, we can estimate the speed of convergence as follows:

\[
\left| \sum_{n=1}^{N\Delta} \chi_{\Delta}(n) \log n + L'(0, \chi_{\Delta}) \right| \leq \frac{2\Delta}{N}.
\] (6.16')

We postpone the proof of Lemma 6.2 to the end of Section 15.

To find a connection between \( L'(0, \chi_D) \) and \( L'(1, \chi_D) \), we use the Functional Equation of \( L(s, \chi_D) \) for the “bad” discriminant \( D > 0 \):

\[
L(1-s, \chi_D) \Gamma \left( \frac{1-s}{2} \right) \left( \frac{D}{\pi} \right)^{(1-s)/2} = L(s, \chi_D) \Gamma \left( \frac{s}{2} \right) \left( \frac{D}{\pi} \right)^{s/2}.
\]

In fact, we use the following equivalent form:

\[
L(1-s, \chi_D) = L(s, \chi_D) \Gamma \left( \frac{s}{2} \right) \frac{1}{\Gamma((1-s)/2)} \left( \frac{D}{\pi} \right)^{s-\frac{1}{2}}.
\] (6.17)

Note that the reciprocal of the gamma function \( 1/\Gamma(s) \) is regular on the entire complex plane, and has a simple zero at \( s = 0 \), so differentiating (6.17) at \( s = 1 \), we have

\[
-L'(0, \chi_D) = L'(1, \chi_D) \Gamma \left( \frac{1}{2} \right) \frac{1}{\Gamma(0)} \left( \frac{D}{\pi} \right)^{1/2} + \n\]

\[
+L(1, \chi_D) \Gamma' \left( \frac{1}{2} \right) \Gamma^{-1}(0) \left( \frac{D}{\pi} \right)^{1/2} - L(1, \chi_D) \Gamma \left( \frac{1}{2} \right) \frac{d}{ds} \left. \frac{1}{\Gamma(s)} \right|_{s=0} \left( \frac{D}{\pi} \right)^{1/2} + \n\]

\[
+L(1, \chi_D) \Gamma \left( \frac{1}{2} \right) \frac{1}{\Gamma(0)} \left( \frac{D}{\pi} \right)^{1/2} \log(D/\pi) = \n\]

\[
= -L(1, \chi_D) \Gamma \left( \frac{1}{2} \right) \frac{d}{ds} \left. \frac{1}{\Gamma(s)} \right|_{s=0} \left( \frac{D}{\pi} \right)^{1/2},
\] (6.18)

since the rest of the terms contain the factor \( 1/\Gamma(0) = 0 \).

Now (6.18) gives (6.14). But this “idiosyncrasy” of the case of positive discriminants actually helps us; we will take advantage of it.

Since the first derivative of (6.17) turned out to be “negligible”, we study the second derivative of (6.17) at \( s = 1 \): ignoring the terms that contain the factor \( 1/\Gamma(0) = 0 \), we have

\[
L''(0, \chi_D) = -L'(1, \chi_D) \Gamma \left( \frac{1}{2} \right) \frac{d}{ds} \left. \frac{1}{\Gamma(s)} \right|_{s=0} \left( \frac{D}{\pi} \right)^{1/2}
\]

\[
-L(1, \chi_D) \Gamma' \left( \frac{1}{2} \right) \frac{d}{ds} \left. \frac{1}{\Gamma(s)} \right|_{s=0} \left( \frac{D}{\pi} \right)^{1/2} + L(1, \chi_D) \Gamma \left( \frac{1}{2} \right) \frac{d^2}{ds^2} \left. \frac{1}{\Gamma(s)} \right|_{s=0} \left( \frac{D}{\pi} \right)^{1/2}
\]

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\[-L(1, \chi_D)\Gamma \left(\frac{1}{2}\right) \frac{d}{ds} \frac{1}{\Gamma(s)} \bigg|_{s=0} \left(\frac{D}{\pi}\right)^{1/2} \log(D/\pi) . \] (6.19)

Since \(\frac{d}{ds} \frac{1}{\Gamma(s)} \bigg|_{s=0} = 1\) and \(\Gamma(1/2) = \sqrt{\pi}\), by (6.19),

\[\left| L''(0, \chi_D) + L'(1, \chi_D)\sqrt{D}\right| \leq 100L(1, \chi_D)\sqrt{D}\log D. \] (6.20)

Note that for positive discriminants \(D > 0\) Goldfeld gave an explicit formula for \(L'(1, \chi_D)\) (see equations (12), (13) and (14) in [Go]) that goes as follows:

\[L'(1, \chi_D) = \frac{\pi^2}{6} \sum_{(a,b,c)} \left(\frac{1}{a} + Q(a,b,c)\right) + O(h(D)/\sqrt{D}) + O(L(1, \chi_D)\log D), \] (6.21)

where \(Q(a, b, c) \geq 0\) is a constant that depends only on the quadratic irrational

\[-\frac{b + \sqrt{D}}{2a} = -\frac{b + \sqrt{b^2 - 4ac}}{2a}, \] (6.22)

where \(\sum_{(a,b,c)}\) is taken over all reduced, primitive, inequivalent binary quadratic forms \(ax^2 + bxy + cy^2\) of discriminant \(D > 0\) (so there are \(h(D)\) triples \((a, b, c)\)). The implicit constants in the \(O\)-notation in (6.21) are effectively computable, and also Goldfeld gave an explicit form of \(Q(a, b, c)\) in terms of the continued fraction expansion of the number (6.22).

In a related paper, Goldfeld and Schinzel [Go-Sch] introduced and studied the other sum

\[\sum_{(A,B,C)} \frac{1}{A}, \] (6.23)

where \(\sum_{(A,B,C)}\) is taken over all binary quadratic forms \(Ax^2 + Bxy + Cy^2\) of discriminant \(D > 0\) such that

\[-A < B \leq A < \frac{1}{4}\sqrt{D}. \] (6.24)

(In sharp contrast with the case \(D < 0\), if the fundamental unit of the real quadratic field \(\mathbb{Q}(\sqrt{D})\) is “large” and the class number \(h(D)\) is “small”, then many different triples \((A, B, C)\) satisfying (6.24) are equivalent, and belong to the same class represented by one triple \((a, b, c)\).) The argument of the paper [Go-Sch] (using the theory of continued fraction) gives that

\[\sum_{(a,b,c)} \left(\frac{1}{a} + Q(a,b,c)\right) = \sum_{(A,B,C)} \frac{1}{A} + \text{negligible} = \prod_{p|Z_0} \prod_{\chi_D(p) \neq -1} \frac{p+1}{p-1} \prod_{p|D} \frac{p-1}{p} + \text{negligible}, \]

if \(L(1, \chi_D)\) is small (which is true for our “bad” discriminant \(D > 0\)).
More precisely, [Go-Sch] yields the following result (an analog of Lemma 14.2 later).

**Lemma 6.3** For our “bad” fundamental discriminant \( D > 0 \) we have

\[
\left| \prod_{p \mid \mathbb{Z}_0, \chi_D(p) \neq -1} \frac{p+1}{p-1} \prod_{p \mid D} \frac{p-1}{p} - \sum_{(a,b,c)} \left( \frac{1}{a} + Q(a,b,c) \right) \right| \leq 10^3 L(1, \chi_D) + \frac{10^4}{(\log D)^3}.
\]

I recall that for \(-D < 0\) the sum

\[
\sum_{j = 1}^{D^4} \chi_{-D}(j) \log j
\]

played a key role in the proof of Theorem 1 (see Lemma 13.2 later). For \( D > 0 \) the sum \((6.25)\) is “negligible” in the sense

\[
\sum_{j = 1}^{D^4} \chi_D(j) \log j = o(\sqrt{D})
\]

(see \((6.14)-(6.15)\)), and the other sum

\[
\sum_{j = 1}^{D^4} \chi_D(j)(\log j)^2
\]

will play the same key role.

We need the following analog of Lemma 6.2 for the second derivative.

**Lemma 6.4** For every fundamental discriminant \( \Delta > 0 \),

\[
\lim_{N \to \infty} \sum_{n = 1}^{N \Delta} \chi_{\Delta}(n)(\log n)^2 = L''(0, \chi_{\Delta}),
\]

where \( L''(0, \chi_{\Delta}) \) denotes the second derivative of the \( L \)-function \( L(s; \chi_{\Delta}) \) at \( s = 0 \).

Moreover, we can estimate the speed of convergence:

\[
\left| \sum_{n = 1}^{N \Delta} \chi_{\Delta}(n)(\log n)^2 - L''(0, \chi_{\Delta}) \right| \leq \frac{8\Delta \log(N \Delta)}{N}.
\]
We postpone the proof to the end of Section 15.

Combining (6.28), (6.21), (6.20) and Lemma 6.3, we conclude Section 6 with

**Lemma 6.5** For our “bad” fundamental discriminant \( D > 0 \) we have

\[
\left| \sum_{j=1}^{D^4} \chi_D(j)(\log j)^2 + \frac{\pi^2}{6} \sqrt{D} \prod_{p|D} \frac{p+1}{p-1} \prod_{p \not| D} \frac{p-1}{p} \right| =
\]

\[
= O(L(1, \chi_D)\sqrt{D} \log D) + O(h(D)) + O(\sqrt{D}(\log D)^{-3}),
\]

where the implicit constants are effectively computable.

We complete the outline of the proof of Theorem 2 in Section 15.

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