Inverse-Scattering Theory and the Density Perturbations from Inflation

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(Dated: March 20, 2022)

Inflationary cosmology provides a successful paradigm for solving several problems, notably the generation of density perturbations which seed the formation of observed cosmic structure. We show how to use inverse scattering theory as the basis for the inflationary reconstruction program, the goal of which is to gain information about the physics which drives inflation. Inverse scattering theory provides an effective and well-motivated procedure, having a sound mathematical basis and being of sufficient generality that it can be considered the foundation for a non-parametric reconstruction program. We show how simple properties of the power spectrum translate directly into statements about the evolution of the background geometry during inflation.

PACS numbers: 98.80.Cq

A wealth of data drives much of the current activity in cosmology. Amongst these data, perhaps the cleanest signal from the primordial Universe is the power spectrum of density fluctuations, which is obtained from observations of fluctuations in the microwave background sky and from observations of the large-scale clustering of matter. Especially in the case of microwave background observations, the observed fluctuations give direct information on the “initial conditions” for the density perturbations which manifest in the gravitational clustering of matter in the Universe.

In an inflationary universe, these initial density perturbations arise as a relic of quantum fluctuations from the very earliest times. The inflationary evolution causes these modes to grow in amplitude, with a time-evolution that is determined completely by the evolution of the space-time geometry, encoded in the scale-factor $a(t)$. This evolution imprints a signature of the inflationary evolution on the spectrum of modes. Therefore, structure in the mode spectrum can be directly related to the evolution of $a(t)$ and hence to the nature of the stress-energy which drives inflation. Because the nature of inflation is quite mysterious, information of this sort is very important for cosmology.

Consider now the standard picture for quantum fluctuations and the generation of structure during inflation. The equation describing the evolution of gauge-invariant metric fluctuations can be taken in the form

$$\phi_k'' + \left[k^2 - q(\eta)\right] \phi_k = 0,$$

where primes indicate derivatives with respect to conformal time $\eta$, and $q(\eta)$ is an effective potential. The spatial dependence of the functions has been expanded in modes specified by three-momentum $k$, so that $\phi_k$ is a function of $\eta$ only. For scalar modes we have $q(\eta) = z''/z$, where $z(\eta)$ is a function determined by the evolving background; under quite general circumstances it is given by

$$z(\eta) = \frac{a(\eta)}{c_s(\eta) h(\eta)} \left[h^2(\eta) - h'(\eta) + K\right]^{1/2},$$

where $h(\eta) \equiv a'(\eta)/a(\eta)$ and $c_s(\eta)$ is the “adiabatic sound speed”. The curvature term $K$ rapidly becomes negligible as inflation proceeds and will be ignored for the remainder of this Letter. When inflation is driven by a scalar field, $c_s = 1$. For tensor modes the effective potential is given by $q_T(\eta) = a''(\eta)/a(\eta)$. In the infinite past, the mode population is that of a precisely specified vacuum state, which gives the condition

$$\phi_k \sim (2k)^{-\frac{1}{2}} e^{-ik\eta+i\gamma}, \quad \eta \to -\infty.$$ 

The argument of the exponential is determined by the positive-energy condition in the deep past, and the normalization is determined by the usual free-field normalization. The constant phase factor $\gamma$ is a matter of convention and is fixed later in order to simplify resulting expressions.

A given $k$-mode evolves to a good approximation as a free mode until it begins to feel the effective potential, for $\eta$ such that $k^2 \simeq q(\eta)$. When it crosses into the forbidden region (crosses outside the Hubble length), it begins to grow in amplitude. Deep within the forbidden region, it can be easily shown that the scalar modes behave like $\phi_k \sim A_k z(\eta)$, where $A_k$ is a function of $k$ only. The structure of $q(\eta)$ is imprinted on this function, which is directly related to the power spectrum of the fluctuations $\phi_k$,

$$P(k) = \frac{k^3}{2\pi^2} |A_k|^2.$$ 

This power spectrum is in turn directly related to the spectrum of density fluctuations or curvature fluctuations.

The goal of inflationary reconstruction is to determine as much information as possible about the stress-energy which drove the inflationary expansion. The reconstruction program splits naturally into two clearly defined tasks. The first task is to solve the linear problem, inverting from the observed power spectrum to the function $q(\eta)$. The second task is to use the obtained $q(\eta)$...
to constrain the geometry and therefore to constrain the stress-energy tensor which drives inflation. In this Letter we show how to use inverse scattering theory to complete the first task; we then show how this information can be used in the second task, to constrain the physics of the inflationary epoch.

The basic problem of calculating density perturbations is most naturally posed as a problem in scattering theory for the wave equation (1). In the following we can restrict ourselves to the case of an effective radial Schrödinger equation for scattering from a central potential. In partial wave $\ell$ this equation is

$$\phi''(k, r) + \left[ k^2 - \frac{\ell(\ell + 1)}{r^2} - V(r) \right] \phi(k, r) = 0.$$ 

In order to avoid confusion in applying the formalism to the wave equations given above, we identify $r = -\eta$ and work solely in terms of the variable $r$.

The reason to concentrate on the partial wave equation at fixed $\ell$ is the following. It turns out to be most natural to split the potential in the form

$$q(r) = \frac{\nu^2(r)}{r^2} - \frac{1}{r^2} = \nu^2 - \frac{1}{r^2} + \frac{\nu^2(r) - \nu^2}{r^2},$$

where $\nu$ is a judiciously chosen constant. We identify $\ell(\ell + 1) = \nu^2 - 1/4$ and the remaining term as the central potential $V(r)$. In this way we collect the most singular part of the potential into the effective “centrifugal” term. The remainder of the discussion explains precisely how this process is carried out and what it means. Note also that there are no bound states in the problem, independent of the sign of $V(r)$, because of the positivity of $q(\eta)$.

The wave solution is purely an incoming wave for $r \to \infty$, which follows from the positive-energy vacuum condition in the deep past. Solutions which obey this condition are important in scattering theory and are called Jost solutions. Precisely, a Jost solution is the unique solution of the wave equation satisfying the condition $\phi(k, r) \sim \exp(ikr + \nu \ell \pi/2), r \to \infty$. Note that a Jost solution must become singular deep inside the forbidden region, as a consequence of the lack of reflection in its definition. In our case the Jost-like situation is actually the physically interesting solution, describing a positive-energy quantum mode from the deep past.

The behaviour of a Jost solution deep inside the forbidden region is described by a function of $k$,

$$F_\ell(k) = \lim_{r \to 0} \left[ e^{-i\ell \pi/2} \frac{1}{\Gamma(\ell + \frac{1}{2})} \left( \frac{k r}{2} \right)^\ell f_\ell(k, r) \right],$$

where $f_\ell(k, r)$ is the Jost solution for $k$. The function $F_\ell(k)$ is the so-called Jost function, and it encodes essentially all the information about the scattering theory. Given our setup of the problem, we have $z \sim c_0 (r_0/r)\ell$ for $r \to 0$, where $c_0$ is a $k$-independent constant. Therefore a direct relation holds between the Jost function $F_\ell(k)$ and the basic function of interest $A_\ell$,

$$A_\ell = \frac{\Gamma(\ell + \frac{1}{2})}{c_0 \Gamma(\frac{1}{2})} \left( \frac{2}{k} \right)^{\ell+1/2} F_\ell(k).$$

At this point it is possible to invoke the full machinery of inverse scattering theory, which allows us to reconstruct the potential $V(r)$ from the Jost function, by solving a linear integral equation (2). Rather than solve this numerical problem, in this Letter we instead show how the Jost function encodes information about the asymptotic behaviour of $V(r)$, and therefore the function $z(r)$, in a manner which is directly applicable to the cosmological inversion problem.

To be precise, suppose that we write $z(r) = (r_0/r)^\ell c(r)$, where $c(r)$ is assumed to be regular and to satisfy $c(\infty) = 1$. The potential is then given by

$$V(r) = \frac{z''}{z} - \frac{\ell(\ell + 1)}{r^2} = \frac{c''}{c} - \frac{2 \ell c'}{r c}.$$ 

By direct calculation, we can show that $F_\ell(0) = c(0)$. This is a consequence of the special form for $V(r)$. A more difficult result expresses the behaviour of $F_\ell(k)$ for $k \to 0$ in terms of the large $r$ behaviour of $V(r)$. To this end we define the length-scale $r_1$ and the exponent $\delta$ by

$$V(r) \sim \frac{1}{r^2} \left( \frac{r_1}{r} \right)^{\delta}, \quad r \to \infty.$$ 

The small $k$ behaviour for $\ell \to 0$ has been calculated by Klaus (3). Extending Klaus’ calculation to general $\ell$, we obtain the $k \to 0$ asymptotic

$$F_\ell(k) = 1 + e^{-i\ell \pi/2} \frac{1}{\Gamma(\ell + \frac{1}{2})} \frac{\Gamma\left(\ell + \frac{1}{2}\right)}{\Gamma\left(\ell + \frac{1}{2}\right)} (kr_1)^\delta + \cdots,$$ 

(2)

where $A(\ell, \delta) = A_1(\ell, \delta) + A_2(\ell, \delta)$ with

$$A_1(\ell, \delta) = 2^{\ell-1/2} \frac{\Gamma\left(\ell + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left[ 1 + \frac{\Gamma(1 + \delta)}{\Gamma(\frac{1}{2})\Gamma(2 + \ell + \delta)} \right].$$

$$A_2(\ell, \delta) = 2^{\ell+1/2} \frac{\Gamma\left(\ell - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma(1 + \delta)}{\Gamma(\frac{1}{2})\Gamma(2 + \ell + \delta)} - 1.$$ 

This particular calculation assumes $0 < \delta < 1$, which turns out to be the generic case for inflationary models; other cases can be handled in a similar manner. Given this result, we see that knowledge of the Jost function for $k$ near zero translates into knowledge of $V(r)$ and $z(r)$ for large $r$. This result is all we will need for the remainder of the discussion.

Before proceeding let us examine the meaning of the various ingredients introduced thus far. First consider the form chosen for the potential. In power-law inflation, where the scale-factor is assumed to grow like $a(t) \sim t^p$, the function $\nu(r)$ is a constant, $\nu = 3/2 + 1/(p-1)$. The fact that the power-law case corresponds precisely to a free wave equation is essentially a statement of the exact
scale-invariance of the wave equation in that case. The usual spectral index and the partial wave index are related by \( n = 3 - 2\ell \). The Harrison-Zeldovich case with \( n = 1 \) corresponds to partial wave \( \ell = 1 \). When \( \ell \) is different from zero, the power-spectrum has a pure power-law scaling form with an anomalous scaling exponent. It is interesting to note that the Harrison-Zeldovich case corresponds to an anomalous exponent, although from another point of view it is “natural”.

Only deviations from scale-invariance in the wave equation can give rise to deviations from power-law behaviour in the spectrum of density perturbations and to a nontrivial Jost function. Clearly the Jost function is identically one in the free case. Note that the parameter \( \ell \) need not be an integer; nothing in the analysis is affected by this.

Next consider the meaning of the limit \( r \to 0 \) which defines the Jost function and the spectrum of density perturbations. Obviously the detailed structure of the potential in a neighbourhood of the origin depends on the way in which inflation ends; if the details of this process become important, then we lose the ability to make robust predictions about the spectrum of density perturbations. However, this is simply a statement defining the range of \( k \) modes of interest. In practice we observe a range of modes, which we assume were “well-inflated”, so that they are not sensitive to such details. A typical complementarity relation holds, so that this range of \( k \) modes corresponds to a range of \( r \). In this spirit, the range of \( k \) modes of interest corresponds to small \( k \) and therefore to large \( r \); these are the modes which spent a significant amount of time outside the horizon.

Combining the expressions given above, we obtain a representation for the power spectrum,

\[
P(k) = \frac{|\Gamma(\ell + \frac{1}{2})|^2}{4\pi^2} k^2 \left( \frac{k r_0}{2} \right)^{-2\ell} \left| \frac{F_i(k)}{F_i(0)} \right|^2. \tag{3}
\]

This form for the power spectrum illustrates several ideas in the scattering theory approach. The partial wave index \( \ell \) is directly related to the anomalous scaling exponent in the leading power-law behaviour, and the momentum scale for the anomalous part of the scaling relation is set by \( r_0^{-1} \). These observations completely specify the meaning of the parameters \( \ell \) and \( r_0 \) in this formalism. All deviations from scaling are contained in the shape function, \( |F_i(k)/F_i(0)| \).

The usual form of the power-spectrum given in the literature is non-dimensional, being a power-spectrum for a non-dimensionalized curvature scalar \( \mathcal{R} \). Relating the definition of \( \mathcal{R} \) to the objects given here, we find

\[
P_{\mathcal{R}}(k) = 2\pi G P(k).
\]

This yields an object with no engineering dimensions. Note that we have set \( \hbar \) and \( c \) to unity everywhere, so that \( G \) has units of length-squared. At this point it is possible to interpret a large class of power-spectra in a model-independent way. Next we illustrate this approach by example.

As an example, consider the power spectrum in Fig. 1. This power spectrum was computed numerically from an inflation model with a \( \phi^2 \) inflaton potential. Computations of power spectra for other single-field models show that the properties of this power-spectrum are generic for a large class of such models and it serves as a good illustration. The behaviour as \( k \) approaches zero allows us to directly determine the parameters \( \ell \) and \( r_0 \). The global behaviour is fit to the \( k \to 0 \) asymptotic form given by the combination of Eqns. 4 and 5. For fitting, one must specify a range of \( k \) values in which the asymptotic result is assumed to hold. Using the full range of data for the power spectrum, which is displayed in Fig. 1, we obtain \( \ell \simeq 1.02496, \delta \simeq 0.103, \) and \( r_1 \simeq 6.43 \times 10^{-15} h^{-1} \text{Mpc} \text{sec}/\text{km} \). Using a significantly more restricted set of the data, with \( k < 0.0001 h \text{Mpc}^{-1} \), we obtain \( \ell \simeq 1.02603, \delta \simeq 0.118, \) and \( r_1 \simeq 1.39 \times 10^{-13} h^{-1} \text{Mpc} \text{sec}/\text{km} \).

With this information we reconstruct the large \( r \) behaviour for \( V(r) \) and therefore \( z(r) \). In Fig. 2 we compare the reconstructed function \( r^2 z''(r)/z(r) = r^2 V(r) + \ell(\ell + 1) \) to the input which gave the power spectrum of Fig. 1. Note that both reconstructions agree well with the input and with each other for large \( r \), as they should. A small additive error results from lack of precision in the determination of \( \ell \), which determines the pedestal for each of the functions. The input function rolls over for small \( r \), which indicates the end of inflation; of course, this rollover is not captured by the asymptotic reconstruction. Note further that the amplitude of the power spectrum has no effect on these results; the amplitude simply determines the scale \( r_0 \). In this case \( r_0 \) is roughly \( 5 \times 10^{5} G^{1/2} \). The fact that \( r_0 \) is much larger than the Planck length reflects the fine-tuning necessary to obtain a viable inflationary model.
FIG. 2: Comparison of reconstructed scattering potential to the input. The input was tabulated during computation of the power spectrum.

To close the argument, we relate the large $r$ asymptotic for $V(r)$ to the early time behaviour of the Hubble expansion. If we assume a past asymptotic for the Hubble expansion in the form

$$H(t) = pt^{-1} \left[ 1 + \left( \frac{t}{t_0} \right)^b + \ldots \right], \quad t \to 0,$$

then a short calculation shows that

$$V(r) = \frac{1}{r^2} \left( \frac{r_1}{r} \right)^{b/(p-1)} + \ldots, \quad r \to \infty,$$

where $r_1$ can be written in terms of $t_0$, though the relation is not needed here. Applying this result to the example at hand, we obtain $b \simeq 4.2 - 4.5$. The large power is consistent with the idea that the evolution of the scale factor for our $\phi^4$ model is very close to power-law inflation for most of the inflationary epoch; this is why the power spectrum is close to a simple scaling law.

Thus far we have discussed the power-spectrum for scalar fluctuations. We now briefly show how to relate the scalar and tensor spectral indices using the behaviour of the equation of state.

The stress-energy driving inflation can be parameterized by an equation of state function $w(\eta)$, defined by $p = w \rho$. Recall that one of the important predictions of inflation is a relation between the spectral indices for scalar and tensor fluctuation modes. During inflation the equation of state is close to $w = -1$. As inflation gradually ends, the equation of state drifts away from $-1$.

Assume a behaviour of the form

$$w(\eta) = -1 + (\eta_0/\eta)^m + \ldots, \quad \eta \to -\infty.$$

We consider the simplest case for $z(\eta)$, equivalent to assuming that inflation is driven by a scalar-field. In this case it is easy to show that the following relation holds:

$$\frac{a'}{a} = \frac{z'}{z} - \frac{1}{2} \frac{w'}{1 + w},$$

Using this relation, a short calculation allows us to identify the partial wave index for tensor perturbations as $\ell_T = \ell - m/2$, where $\ell$ is the value for scalar perturbations. Therefore we obtain a statement of the scalar-tensor relation in terms of the parameter $m$,

$$n_T = n_S - 1 + m.$$

These spectral indices are defined as outlined in the above, in terms of the small $k$ behaviour of the power spectra, where the Jost function contributions are negligible. For power-law inflation, $\ell = \ell_T$, and we obtain the exact result $n_T = n_S - 1$.

The procedure we have described is essentially non-parametric. No particular model for the stress-energy need be assumed, and geometric information is extracted directly from the power spectrum. This analysis does not depend on slow-roll parametrizations or assumptions, as distinct from previous treatments [4, 7]. Equation (4) provides a very clear separation of the contributions to the power spectrum, and this separation enables the procedure. Furthermore, the relations of scattering theory provide a very convenient general framework for posing the density-perturbation problem in inflationary cosmology.

We have seen that inflationary reconstruction is essentially a problem in inverse scattering theory. Direct relations were used to invert from the power spectrum to asymptotic quantities of interest, which describe the background evolution through most of the inflationary epoch. Although the full machinery of the Gelfand-Levitan equation [4] was not introduced here, we are currently investigating its use for future analyses. For example, the Gelfand-Levitan equation can be applied to the analysis of any features which do not represent small deviations from scaling, such as might occur in multi-field scenarios or other dynamically complicated models.

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[1] C.B. Netterfield et al., ApJ 571, 604 (2002); N.W. Halver-son et al., ApJ 568, 38 (2002); A. Benoit et al., AA 399, L19 (2003); T.J. Pearson et al., ApJ 591, 556 (2003); C.L. Bennett et al. ApJS 148, (2003); Chao-lin Kuo et al., ApJ 600, 32 (2004). For a review of CMB anisotropies, see W. Hu and S. Dodelson, Ann. Rev. Astron. and Astrophys. 40, 171 (2002).

[2] e.g., W.J. Percival et al., MNRAS 327, 1297 (2001); K. Abazajian et al., Astron.J. 128, 502, (2004).

[3] V.F. Mukhanov, H.A. Feldman, and R.H. Brandenberger,
Phys. Rep. 215, 203 (1992).

[4] K. Chadan and P. Sabatier, Inverse Problems in Quantum Scattering Theory (Springer-Verlag, 1977).

[5] M. Klaus, J. Math. Phys. 27, 2720 (1988).

[6] E. Lidsey, A.R. Liddle, E.W. Kolb, E.J. Copeland, T. Barreiro, and M. Abney, Rev. Mod. Phys. 69, 373 (1997).

[7] H.M. Hodges and G.R. Blumenthal, Phys. Rev. D 42, 3329 (1990); E.J. Copeland, E.W. Kolb, A.R. Liddle, J.E. Lidsey, Phys. Rev. Lett. 71, 219 (1993); M.S. Turner, Phys. Rev. D 48, 3502 (1993).