Comparing Graph Clusterings: 
Set partition measures vs. Graph-aware measures

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Abstract
In this paper, we propose a family of graph partition similarity measures that take the topology of the graph into account. These graph-aware measures are alternatives to using set partition similarity measures that are not specifically designed for graph partitions. The two types of measures, graph-aware and set partition measures, are shown to have opposite behaviors with respect to resolution issues and provide complementary information necessary to assess that two graph partitions are similar.

1 Introduction
An impressive number of graph clustering algorithms have been proposed, studied and compared over the past decades [4,10,17,19,21,23,25]. To identify better graph clustering techniques, one needs a way to score the techniques against one another. A typical method is to compare values of some similarity measure between ground truth partitions of given graphs and the partitions produced by the different algorithms on those graphs. However, the choice of the similarity measure used is crucial and has a huge impact on the conclusions made.

In graph clustering comparison studies [3,13,18,28], set partition similarities are used as accuracy measures. Typically, a member of the pair-counting family [2,11] such as Adjusted Rand Index, or of the Shannon information-based family [16,26,27] such as Adjusted Mutual Information is used to assess the superiority of a graph clustering algorithm over another. These measures are designed for comparing set partitions and not graph partitions specifically. We call them graph-agnostic as they ignore the graph structure.

In this paper, we propose a family of graph-aware measures for graph partition similarity with their adjusted forms. We compare the graph-aware with the graph-agnostic partition measures and demonstrate that the two types of measures offer complementary views of the typical clustering errors known as resolution errors and, therefore, should be used jointly before any proper conclusions can be made on the accuracy of a graph clustering algorithm.

The paper is organized as follows: In Section 2, we set the notation. Section 3 presents the most common and widely used families of set partition similarity measures. In Section 4, we define a family of graph-aware similarity measures,

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we prove a result on the complementarity of the two types of measures and we propose an adjustment for the family. Some experiments are presented in Section 5 to study the impact of the adjustments and to study the relation between graph-aware and graph-agnostic measures.

2 Notation

We define $G = (V, E)$ an graph where $V = \{1, 2, ..., n\}$ is the set of vertices and $E \subset \{(x, y) | x, y \in V, x < y\}$, the set of edges. All graphs considered are undirected. We use $V(G)$ to refer to the vertices of $G$ and $E(G)$ to refer to its edge set. If $A \subseteq V$, then $G_A$ denotes the subgraph of $G$ obtained by restricting $G$ to vertices in $A$.

Let $A = \{A_1, \cdots, A_{k_a}\}$ and $B = \{B_1, \cdots, B_{k_b}\}$ denote two partitions of $V$. The cardinality of the partitions $A$ and $B$ are $k_a$ and $k_b$ and the cardinalities of each of the parts are $|A_i| = a_i$ for $i = 1, \cdots, k_a$ and $|B_j| = b_j$ for $j = 1, \cdots, k_b$. Finally, the size of the pairwise intersections are $|A_i \cap B_j| = n_{ij}$.

Definition 1 A is a connected partition of $G$ if $A$ is a partition of $V$ and if all subgraphs $G_A_i$ are connected.

3 Graph-agnostic clustering comparison measures

Similarity measures between set partitions have been well studied [2,11,15,26]. The most widely used similarity measures lie in one of the two following families: pair-counting ($PC$) measures and mutual information ($MI$) based measures. In this section, we define the two families.

Let $P_A = \bigcup_{i=1}^{k_a} \{(x, y) \in A_i \times A_i | x < y\}$ denote the pairs of points lying in the same part of $A$. We define $P_B$ similarly, and we use an overline to denote the complement of a set: $\overline{P_A} = \{(x, y) \in (V \times V) | x < y \text{ and } (x, y) \notin P_A\}$. The two pair counting indices that were first proposed are the Rand Index ($R$) [20] and the Jaccard Index ($J$):

$$R(A, B) = \frac{|P_A \cap P_B| + |\overline{P_A} \cap \overline{P_B}|}{\binom{n}{2}}, \quad J(A, B) = \frac{|P_A \cap P_B|}{|P_A \cup P_B|}.$$ 

The key value of most pair counting similarity measures is $|P_A \cap P_B|$, the number of pairs belonging to the same parts in both partitions. The Rand index is an exception as it also includes the number of pairs belonging to different parts in both partitions. Different normalizations are used to ensure the measures are constraint to values in $[0, 1]$. Other members of the family of pair counting similarity measures are:

$$PC_f(A, B) = \frac{|P_A \cap P_B|}{f(|P_A|, |P_B|)},$$

where $f \in \{mn, gm, min, max\}$ which denote the mean, geometric mean, minimum or maximum function respectively.

Another family of measures used for measuring the similarity of partitions is the Shannon information based family. The entropy of a partition $A$ is defined as $H(A) = -\sum a_i \log a_i$, the joint entropy of $A$ and $B$ as $H(A, B)$ =
\[-\sum_{i,j} \frac{n_{ij}}{n} \log \frac{n_{ij}}{n}, \text{ and, the mutual information between two partitions is } I(A, B) = \sum_{i,j} \frac{n_{ij}}{a_i b_j/n^2} \text{. The mutual information between two partitions is the core value for comparing partitions with information-based measures. Similarly to the pair counting members, information based measures are normalized versions of the mutual information having the unit interval as image:}\]

\[MI_f(A, B) = \frac{I(A, B)}{f(H(A), H(B))},\]

where \(f\) is as for \(PC_f\). In \cite{27}, they show that \(MI_{max}\) is a true metric and argue that it should be favored over the other measures.

The two families of measures suffer from the problem of not having a constant baseline of 0 when the compared partitions are random and independent. For this reason, adjusted forms were proposed independently for the pair counting \cite{11} and information-based \cite{26} families. The adjustments consist of subtracting the expected value of the measure under a random model namely, the permutation model. The permutation model consists of the expected measure between random partitions \(A\) and \(B\) given \(a_i\) and \(b_j\) their marginals\footnote{The choice of random model is discussed in \cite{9}.}. The expectation values can be obtained empirically but closed forms exist for all but the Jaccard measure. Some measures collapse to having the same adjusted forms as in the case for the Rand Index and the \(PC_{mn}\). The adjusted Rand Index (ARI) is:

\[ARI(A, B) = \frac{|P_A \cap P_B| - |P_A||P_B|/\binom{n}{2}}{\frac{1}{2}(|P_A| + |P_B|) - |P_A||P_B|/\binom{n}{2}}.\]

Note that what is called the Adjusted Mutual Information, \(AMI(A, B)\), is the adjusted form of the measure \(MI_{max}\).

In \cite{22}, a broader family based on the generalized Tsallis \(q\)-entropy is proposed which unifies the two families. The authors analytically compute adjustments for this family which generalizes the adjustments that were derived independently for the two families. As a consequence of their work, it is justified to use one of \(AMI\) or \(ARI\) to compare the accuracy of clustering algorithms given a ground truth partition if the number of data points is large relative to the number of parts in the partition.

4 **Graph-aware clustering comparison measures**

The clustering comparison measures discussed in the previous section account for the nodes in a graph but ignore the edges. Should the similarity between partitions \(A\) and \(B\), shown in Figure 1 be the same on graph \(G_1\) as on graph \(G_2\)? When restricting to vertices, the two cases, \(G_1\) and \(G_2\), are indeed identical. However, when including the edges, the impact of placing vertex 8 in one part or the other is quite different on both graphs. In this section, we introduce a family of similarity measures for graph partitions that take edges into account and prove that both graph-agnostic and graph-aware measures are critical for effectively comparing graph partitions.
Figure 1: Partitions $A$ and $B$ are identical as set partitions on $G_1$ and $G_2$, and so their similarities are the same when using graph-agnostic measures regardless of the underlying graphs.

4.1 Non-adjusted graph-aware similarity measures

A connected graph partition $A$ can be described in two ways. The most intuitive one is to consider $A$ as a partition of the graph’s vertices, as we did in Section 3. Another way, is with a binary classification of the edges: the endpoints of the edges being either in the same part or in different parts of the partition. Hence, a connected partition $A$ of a graph $G$ induces a binary edge classification:

$$b_A : E(G) \rightarrow \{0, 1\}$$

$$(i, j) \mapsto \begin{cases} 
1 & \exists A_r \in A \text{ with } i, j \in A_r, \\
0 & \text{otherwise.}
\end{cases}$$

Not all binary edge classifications correspond to a connected graph partition. However, the set of binary edge classifications $\{0, 1\}^E$, where $E = E(G)$, can be grouped into equivalence classes w.r.t. the graph partition they induce.

**Definition 2** Let $b \in \{0, 1\}^E$ and consider $G' = (V, b^{-1}(1))$ a subgraph of $G$ formed of all class-one edges of $b$. We say $G'$ is the subgraph induced by the classification $b$.

Let $b_1, b_2 \in \{0, 1\}^E$. The binary classification $b_1$ is in relation with $b_2$ if the two subgraphs induced by the classifications have identical connected components on $G$. In that case, we write $b_1 \equiv_G b_2$.

The quotient set $\{0, 1\}^E / \equiv_G$ divides the set of binary edge classifications into equivalence classes $[b]$ where members of a class all induce the same connected partition on $G$. For each class $[b]$ we define its representative class member, $\overline{b}^G$, as the binary classification having the largest number of class-one edges in that class: $\overline{b}^G(i, j) = \max\{b(i, j) : b \in [b]\}$. For each $b \in \{0, 1\}^E$, there exists a connected graph partition $A$ such that $\overline{b}^G = b_A$. Clearly, if $A$ is a connected partition of $G$, $\overline{b}_A^G = b_A$. See Figure 2 for an illustration.

This partition description opens the door to new ways of measuring the similarity between two graph partitions using the corresponding edge classifications.
Binary edge classification \( b \) on \( G \)

Class-1 edges: \( b_1(1) \)

Equivalence class representative: \( b \)

Figure 2: From left to right: edge classification \( b \), the class-1 edges corresponding to \( b \), and \( b \)'s class representative \( b \).

With a fixed arbitrary ordering of the edges, the binary classifications are considered as binary vectors in \( \{0, 1\}^{|E|} \). We use \( |b| \) to denote the L1-norm of the vectors and so expressions such as \( |b_1^{-1}(1)| \) and \( |b_1^{-1}(1)^c \cap b_2^{-1}(1)| \) are replaced by \( |b| \) and \( |b_1 \cdot b_2| \).

Given two binary edge classifications \( b_1, b_2 \in \{0, 1\}^{|E|} \), we define their similarity as

\[
S_G(b_1, b_2) = S^*(\overline{b_1}, \overline{b_2}) = S^*(b_A, b_B),
\]

where \( S^* \) is a pre-determined similarity measure on binary vectors and \( A \) and \( B \) are the two connected graph partitions associated to \( b_1 \) and \( b_2 \). There are four core values on which all binary similarity measures are based: 

- \( a_{00} \), \( a_{01} \), \( a_{10} \) and \( a_{11} \) where \( a_{ij} \) denotes the number of elements on which \( b_A \) takes value \( i \) and \( b_B \) takes value \( j \):
  - \( a_{11} = |b_A \cdot b_B| \)
  - \( a_{01} = |(1 - b_A) \cdot b_B| \)
  - \( a_{10} = |b_A \cdot (1 - b_B)| \)
  - \( a_{00} = |(1 - b_A) \cdot (1 - b_B)| \)

In the graph context, those four values can be expressed as:

| \( b_A/b_B \) | 1 | 0 |
|----------------|----------------|----------------|
| 1              | \( P_A \cap P_B \cap E \) | \( P_A \cap P_B^{c} \cap E \) |
| 0              | \( P_A^{c} \cap P_B \cap E \) | \( P_A^{c} \cap P_B^{c} \cap E \) |

Well-known binary similarity measures include accuracy, the F-score, Cosine similarity, Simpson, Braun & Banquet, etc. See [3, 24] for an exhaustive list of measures. It turns out that those measures relate to the pair counting measures described in Section 3 obtained by restricting the counts to pairs of vertices sharing an edge, see Table 1.

4.2 Properties of graph-aware and graph-agnostic measures

Different algorithms produce partitions of different sizes and many are known to suffer from the resolution issue [7, 12]. It is therefore interesting to understand how the measures behave on partitions of various resolutions.

A partition \( B \) is said to be a refinement of a partition \( A \), denoted \( B < A \), if each part of \( B \) is a subset of a part of \( A \). In that case, we also say that
variables $X$ and $q = \frac{a_{00} + a_{11}}{2a_{00}}$, consider Lemma 1 graph. The fact that there is an underlying partition $G$ necessarily a connected partition of the random graphs. Let $A$ some community structure. We will use $G$ respect to partition refinements or coarsenings if the underlying graph has $A$ is a $\leq |$ community structure. We will use $G$ to study a simple family of graphs having community structure. Graphs in $G(n, k_1, k_2, A)$ have $n$ vertices split into a partition $A$: $k_1$ edges are randomly placed between pairs of vertices in same parts of $A$ and $k_2$ are randomly placed between pairs of vertices in different parts, $k_1 \leq |A|$ and $k_2 \leq |A|$. Note that with this random process, $A$ is not necessarily a connected partition of the random graphs. Let $p = k_1/|A|$ and $q = k_2/|A|$, to simplify the notation, we write $G_A \sim G(n, p, q, A)$ to denote $G_A \sim G(n, k_1 = p/|A|, k_2 = q/|A|, A)$. Moreover, we write $G_A$ to emphasize the fact that there is an underlying partition $A$ in the random generation of the graph.

Lemma 1 Consider $G_A \sim G(n, p, q, A)$ with $B_1 > A$ a coarsening of $A$ and $B_2 < A$, a refinement of $A$. Then

(i) $\mathbb{E}_{G_A}\left[PC_{mn}(A, B_1; G_A)\right] \geq PC_{mn}(A, B_1)$, if $p \geq q$.
(ii) $\mathbb{E}_{G_A}\left[PC_{mn}(A, B_2; G_A)\right] \leq PC_{mn}(A, B_2)$, for all $p, q$ values.

proof Let $a = |A|$, $x_1 = |B_1 \setminus A|$, $x_2 = |A \setminus B_2|$, $X_1 = |B_1 \setminus A \cap E|$ and $X_2 = |A \setminus B_2 \cap E|$. $X_1$ and $X_2$ are two independent hypergeometric random variables: $X_1 \sim Hyper(x_1, q, a)$ and $X_2 \sim Hyper(x_2, p, a)$ with $\mathbb{E}(X_1) = qx_1$ and $\mathbb{E}(X_2) = px_2$. Note that $|A \cap E| = pa$, $|B_1 \cap E| = pa + x_1$, $|B_2 \cap E| = pa - X_2, A \cap B_1 = B_1$, $A \cap B_2 = B_2$.

(i) Let $Z = \frac{pa}{pa \cdot 1/2X_2}$. Since, $Z > 0, 1/Z$ is a convex function of $Z$. We have

$\mathbb{E}(Z) \geq \mathbb{E}(1/Z)^{-1} = \left[\frac{pa + 1/2qx_1}{pa}\right]^{-1} \geq PC_{mn}(A, B_1)$, if $p \geq q$.

(ii) Again, we use the convex function trick with $Z = \frac{pa}{pa - 1/2X_2}$, $Z > 0$.

$\mathbb{E}\left\{\frac{pa - X_2}{pa - 1/2X_2}\right\} = 2 - \mathbb{E}(Z) \leq 2 - \left[\frac{a - 1/2x_2}{a}\right]^{-1} = PC_{mn}(A, B_2)$. 

Table 1: From classification measures to graph-aware clustering measures.

| Accuracy | $\frac{a_{00} + a_{11}}{2a_{00}}$ | RI($G$) | $\frac{|P_A \cap P_B \cap E| + |P_A \cap P_B \cap E|}{|E|}$ |
|----------|-------------------------------|---------|-----------------------------------------------|
| F-score ($\beta = 1$) | $\frac{2a_{00} + a_{01} + a_{10}}{a_{00}}$ | $PC_{mn}(A, B; G)$ | $\frac{1}{2}(|P_A \cap E| + |P_B \cap E|)$ |
| Cosine | $\frac{\sqrt{(a_{00} + a_{10})(a_{00} + a_{01})}}{a_{00}}$ | $PC_{gm}(A, B; G)$ | $\frac{1}{\sqrt{|P_A \cap E| |P_B \cap E|}}$ |
| Simpson | $\frac{\min((a_{00} + a_{10}, (a_{00} + a_{01}))}{a_{00}}$ | $PC_{min}(A, B; G)$ | $\frac{1}{|P_A \cap E| |P_B \cap E|}$ |
| Braun&Banquet | $\frac{\max((a_{00} + a_{10}, (a_{00} + a_{01}))}{a_{00}}$ | $PC_{max}(A, B; G)$ | $\frac{1}{|P_A \cap E| |P_B \cap E|}$ |

$A$ is a coarsening of $B$. The following result demonstrates that the graph-aware and the non-adjusted pair-counting measures behave differently with respect to partition refinements or coarsenings if the underlying graph has some community structure. We will use $G(n, k_1, k_2, A)$, a variant of Girvan and Newman model $[3][10]$ to study a simple family of graphs having community structure. Graphs in $G(n, k_1, k_2, A)$ have $n$ vertices split into a partition $A$: $k_1$ edges are randomly placed between pairs of vertices in same parts of $A$ and $k_2$ are randomly placed between pairs of vertices in different parts, $k_1 \leq |A|$ and $k_2 \leq |A|$. Note that with this random process, $A$ is not necessarily a connected partition of the random graphs. Let $p = k_1/|A|$ and $q = k_2/|A|$, to simplify the notation, we write $G_A \sim G(n, p, q, A)$ to denote $G_A \sim G(n, k_1 = p/|A|, k_2 = q/|A|, A)$. Moreover, we write $G_A$ to emphasize the fact that there is an underlying partition $A$ in the random generation of the graph.
This lemma shows how the graph-aware and graph-agnostic similarity measures compare to one another given refinements or coarsenings of the ground truth partition of a graph. The following result is very important for understanding the degradation of similarities given different types of perturbations (groupings or splittings) of the ground truth partition.

Theorem 1 Consider \( G_A \sim G(n, p, q, A) \) with \( B_1 > A \) a coarsening of \( A \) and \( B_2 < A \), a refinement of \( A \) such that \(|P_A|^2 < |P_{B_1}| \cdot |P_{B_2}|\). Then

(i) \( PC_{mn}(A, B_1) < PC_{mn}(A, B_2) \).

(ii) \( \mathbb{E}_G[PC_{mn}(A, B_1; G_A)] > \mathbb{E}_G[PC_{mn}(A, B_2; G_A)] \), if \( p > q |P_A| \cdot |P_{B_2}| \).

proof We use the same notation as for the proof of Lemma 1.

(i) Follows directly from the condition \(|P_A|^2 < |P_{B_1}| \cdot |P_{B_2}|\).

(ii) In the previous proof, we showed that

\[
\mathbb{E}[PC_{mn}(A, B_1; G_A)] \geq Z_1 \quad \text{and} \quad Z_2 \geq \mathbb{E}[PC_{mn}(A, B_2; G_A)],
\]

where \( Z_1 = \frac{pa}{pa + 1/2qx} \) and \( Z_2 = \frac{a - x_2}{a - 1/2x_2} \). We only need to show that \( Z_1 > Z_2 \) whenever \( px_2 > qx_1 \):

\[
Z_1 = \frac{pa}{pa + 1/2qx} > \frac{pa}{pa + 1/2px} > \frac{a - x_2}{(a + 1/2x_2) - x_2} = Z_2.
\]

The conditions required in the theorem above are easily satisfied when \( B_1 \) and \( B_2 \) are perturbations of \( A \). One of them states that the coarsening perturbation of \( A \) must be as important as the refinement perturbation: the geometric mean between \(|P_{B_1}|\) and \(|P_{B_2}|\) must be greater than \(|P_A|\). The second condition requires the ratio between \( p \) and \( q \) — the intra and inter-edge densities — to be larger as \( B_1 \) gets coarser. A consequence of this result is that none of the two measures \( PC_{mn}(A, B) \), \( PC_{mn}(A, B; G) \) directly captures how ‘close’ partition \( B \) is to the ground truth partition \( A \). Instead, \( PC_{mn}(A, B) \) measures how close \( B \) is to being a refinement of \( A \), whereas \( PC_{mn}(A, B; G) \) measures the opposite, how close \( B \) is to being a coarsening of \( A \). For this reason, when used together, the graph-aware and graph-agnostic measures give indications on the containment of parts of \( B \) and \( A \). Getting high values with respect to both measures indicates that the partitions are indeed similar.

### 4.3 Adjusting the graph-aware measures

The expected value of the graph-aware measures of two independent partitions does not take a constant value. In fact, the expectation depends on the graph topology. Here, we propose an adjustment that does not depend on the graph topology but that considerably reduces the baseline, i.e., the expected similarity of random partitions. Recall that an adjusted similarity measure is obtained from a similarity measure by subtracting the expected value and re-normalizing properly:

\[
\text{AdjSim}(A, B) = \frac{\text{Sim}(A, B) - \mathbb{E}[\text{Sim}(A, B)]}{1 - \mathbb{E}[\text{Sim}(A, B)]},
\]
where the expectation is computed over all partitions \( \mathbf{A}, \mathbf{B} \) from some random model. The permutation model used as the random model to adjust pair-counting measures on the set \( V \) is not suitable here as it does not yield connected graph partitions. A simple random model that can be used for graph partitions assumes constant values of the number of internal edges: two partitions are drawn randomly with a fixed number of internal edges \(|b_A|\) and \(|b_B|\) respectively. We call this model the fix-intra-edges random model. The rough approximation we make under this model, is the following:

\[
\mathbb{E}[|b_A \cdot b_B| : |b_A|, |b_B|, G] \approx |b_A| \cdot |b_B|^2 / |E|.
\]

From the approximation above, adjustments to each graph-aware measures defined in Table 1 can be obtained. We omit the computation details, and we give the resulting adjusted measures:

\[
APC_f(\mathbf{A}, \mathbf{B}; G) = |b_A \cdot b_B| - \frac{|b_A| |b_B|}{|E|} - \frac{|b_A| |b_B|}{|E|}.
\]

Just as it is the case for set measures, the adjusted graph-aware Rand Index \( ARI(\cdot; G) \) is the same as one of the adjusted graph-aware pair counting measures:

\[
ARI(\mathbf{A}, \mathbf{B}; G) = APC_{\text{min}}(\mathbf{A}, \mathbf{B}; G) = |b_A \cdot b_B| - \frac{|b_A| |b_B|}{|E|} - \frac{|b_A| |b_B|}{|E|}.
\]

5 Experiments

To test graph algorithms’ reliability, different benchmarks have been developed to generate graphs and their associated ground truth partitions \([1, 5, 6, 10, 14]\). The LFR model \([1, 14]\) was designed to reproduce certain topological properties observed in real-world networks: the size of the communities is power-law distributed, and so is the node degree. A typical way to assess the superiority of a partitioning algorithm over another is to use a family of LFR graphs that range from clear partition structure (low inter-part edge density) to practically no partition structure (high inter-part edge density) and to plot the similarities between the output partitions and the true graph partitions of this family against the inter-part edge density \( \mu \). This produces a similarity curve for each algorithm and conclusions are made based on these curves: higher curves imply better algorithms \([28]\).

In this section, we study the impact of the adjustment on the graph-aware measures and we illustrate the usefulness of Theorem 1 on data, i.e., the complementarity of graph-aware and graph-agnostic measures.

5.1 Adjusted graph-aware measures

To study the expected value of the adjusted graph-aware measures on random partitions, we need to generate random connected partitions of graphs. We use two different generation processes.

**Generation Process 1:** Fix \( k \), the size of the partition. From a random vertex, generate a depth-first search tree that spans \( G \) and delete \( k - 1 \) random
edges from the tree. The remaining \( n - k \) edges of the tree are considered class-1 edges, yielding a binary vector \( b \) where \(|b| = n - k\). We then get its associated connected graph partition \( U \) using its class representative: \( b_A = \overline{b}^G \).

In Figure 3(a), we show the adjusted and non-adjusted similarity measures between ground truth partitions obtained from the LFR model and random partitions generated according to Process 1. As we can see, the adjusted measures are much closer to a 0-baseline for independent partitions. Only four curves are shown as adjusted functions since \( ARI(\cdot;G) \) and \( APC_{min}(\cdot;G) \) collapse to the same function. We see that the adjusted measure \( APC_{min}(\cdot;G) \) has much higher baseline and variance compared to the others.

**Generation Process 2:** The second random graph partition generation consists in randomly selecting \( k \) edges of \( G \) as class-1 edges, i.e., randomly select a binary vector in \( \{b \in \{0,1\}^E : |b| = k\} \). Then, we get its associated connected graph partition \( U \) using its class representative: \( b_A = \overline{b}^G \).

![Figure 3: Similarity measurements between a ground truth graph partition of size 78 and random partitions having (a) pre-determined sizes, (b) pre-determined number of internal edges. |E(G)| = 4000, \( \mu = 0.1 \) which yields a ground truth partition with 3590 internal edges. The measures were smoothed on windows of size 5 in (a) on size 250 in (b), shaded regions indicate the standard deviations.](image)

In Figure 3(b), we see the similarities between a graph’s communities and some random partitions containing a pre-determined number of internal edges. From the plot, we see that the adjustment reduces the baseline of the measures considerably except for the \( APC_{min}(\cdot;G) \). However, this plot demonstrates that the approximation we are using, \( \mathbb{E}[|b_A \cap b_B| : |b_B| = |b_B| \cdot |b_A|/|E|] \), for a fix \( b_A \), is wrong. The quantity \( |b_A| \cdot |b_B|/|E| \) is a good estimation for graphs with no community structure: trees, complete graphs or Erdős-Renyi random graphs. For graphs with community structure, the approximation underestimates the true expectation so the adjusted measures are still above the desired 0-baseline. This statement is illustrated in Figure 4. The similarity measure \( ARI(\cdot;G) \) is

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2 The generation parameters are given in Appendix B.
**Figure 4:** Graph-aware adjusted Rand Index between random partitions and ground truth partition for graphs $G_\mu$ with various community structure strengths (small $\mu$ implies strong community structure). 10 independent graphs were generated for each $\mu$-value, shaded regions indicate the standard deviations.

computed between random partitions and the ground truth partitions of LFR graphs having various level of community structure: low $\mu$-values indicate low inter-partition edge density, so strong community structure. As one can see, for graphs with strong community structure, the adjustment estimation is worse than on weak community structure graphs. The issue is that a good estimation of $E_B(\ |b_A \cap b_B| : |b_B|)$ given $G$ and $A$ is still an open question.

### 5.2 Adjusted graph-aware vs. graph-agnostic measures

We limit our comparisons to the graph-aware and agnostic variants of the Rand Index and the adjusted mutual information. Theorem 1 shows that the unadjusted versions of the measures penalize refinements and coarsenings in opposite ways. In Figure 5, we present empirical evidences that the same is true for the adjusted versions of the measures. We compare the similarity curves of two graph partition algorithms: the first level and the last level of a hierarchy of partitions obtained with the Louvain method [25]. The algorithms compared correspond to two different resolutions, one being a refinement of the other. As one can see, the graph-agnostic and graph-aware measures yield contradicting conclusions. According to the graph-agnostic measures $ARI$, the finer partitions are more similar to the ground truth partitions, therefore, a better choice of algorithm. Note that we obtain the same conclusion when using $AMI$. When using the graph-aware measure $ARI(\cdot, G)$, the conclusion is the opposite: the coarser partitions are closer to the ground truth partitions. This is a good
illustration of the fact that graph-agnostic similarities measure how close a partition is to being a refinement of the ground truth partition while graph-aware captures how close a partition is to being a coarsening of the ground truth partition. It is therefore not possible to assess the superiority of any of the two algorithms compared in Figure 5 when using both types of measures: one produces a refinement and the other a coarsening of the ground truth partition.

5.3 Impacts on Comparison Study Conclusion

In 2017, [28] provided an exhaustive comparison study of graph clustering algorithms on artificial LFR networks. The families of graphs generated for the study are strongly structured, hence the large majority of their conclusions hold regardless of the choice of measure: adjusted or not, graph-aware or not. However, in cases where the sizes of the partitions differ significantly, where one of the algorithm underestimates while the other overestimates the number of clusters with respect to the ground truth partition, the choice of measure does have an impact.

In Figure 6, we show the similarity curves of the FastGreedy algorithm [4], the Leading eigenvector algorithm [17] and the Louvain method on one of the LFR family used in the study [3]. The choice of measure in that case impacts the conclusion: the graph-agnostic measure shows that FastGreedy outperforms Leading Eigenvector, whereas the graph-aware measure shows the opposite. This can partly be explained by the fact that Fast Greedy underestimates the number of clusters, as shown in Figure 6 (bottom plot), and therefore produces better coarsenings of the ground truth partitions than Leading Eigenvector. Again,
Figure 6: Comparing the similarity curves of three partition algorithms on LFR graphs. The graph-aware and graph-agnostic have contradicting conclusions for the algorithms Leading Eigenvector and Fast Greedy. The right plot indicates that Leading Eigenvector produces finer partitions than Fast Greedy. The Louvain algorithm outperforms the other two. 100 independent graphs were generated for each $\mu$-value, shaded regions indicate the standard deviations.

according to this experiment, it is not possible to claim which of the two algorithms has higher accuracy. However, it is possible to claim that the Louvain method outperforms the other two algorithms on this family of graphs since it has higher similarity curves with respect to both measures.

6 Conclusion

In this paper, we proposed an adjusted form of graph-aware similarity measures for graph partitions. We demonstrated that the graph-aware measures and graph-agnostic measures are complementary in that they behave differently with respect to refinements and coarsenings of partitions. As a consequence, both measures should be used jointly to assess similarity of graph partitions, using a single measure can lead to wrong conclusions in the study of graph partitioning algorithms.

The estimation used for the adjustment of the graph-aware measures is not tight for structured graphs. Hence, finding a better estimation of the adjustment under the fix-intra-edges random model that incorporates information about the graph’s topology would increase the adjusted graph-aware measures’ reliability. This is a topic for future work.

A Adjustment for graph-aware measures

To compute the adjustment of the graph-aware measures, note that $f(|b_A|, |b_B|)$ is a constant under the conditional expectation so we have

$$E[PC_f(A, B; G) : |b_A|, |b_B|, G] = \frac{|b_A| \cdot |b_B|}{|E| \cdot f(|b_A|, |b_B|)}.$$
For the Rand Index, since \(|(1 - b_A) \cdot (1 - b_B)| = |E| - (|b_A| + |b_B|) + |b_A \cdot b_B|\), we can write \(RI(A, B, G) = 1 - \frac{|b_A| + |b_B|}{|E|} + \frac{2|b_A \cdot b_B|}{|E|^2}\) and so
\[
E[RI(A, B; G) : |b_A|, |b_B|, G] = 1 - \frac{|b_A| + |b_B|}{|E|} + \frac{2|b_A \cdot b_B|}{|E|^2}.
\]
The rest is obtained using the definition
\[
\frac{Sim(A, B) - E[Sim(A, B)]}{1 - E[Sim(A, B)]}.
\]

B Graph generation parameters

Experiments on Figures 3, 4, 5 and 6 were obtained by generating LFR graphs using the first set of parameters of Table 2, while the graph generation for Figure 7 used the second set of parameters of the table. Auto indicates that the value is automatically obtained by the generation algorithm.

Table 2: Graph generation parameters for LFR algorithm.

| N   | Degree | Community sizes |
|-----|--------|-----------------|
|     | Max    | Avg  | Exp. | Min | Max | Exp. |
| 1000| 8      | 8    | -1   | 10  | 15  | -1   |
| 233 | 23     | 20   | -2   | Auto| 23  | -1   |

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