Wormhole Thermodynamics at Apparent Horizons

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Abstract: In this paper, we discuss the thermodynamic properties of the evolving Lorentzian wormhole. For the shape function $b(r) = r_0^2/r$, it is shown that the wormhole spacetime admits two apparent horizons, the inner and the outer one. The inner horizon expands while the outer contracts with the passage of time. Corresponding to these horizons, we have three types of wormholes, regular, extreme and the naked wormholes. Moreover, it is shown that the Einstein field equations can be rewritten as a first law of thermodynamics $dE = TdS + \mathcal{W}dV$, at the apparent horizons of the wormhole, where $E = \rho V$, $T = \kappa / 2\pi$, $S = A / 4G$, $W = (\rho - P)/2$ and $V = \frac{4}{3}\pi b_0^3$ are the total matter energy, horizon temperature, wormhole entropy, work density and the volume of the wormhole respectively.

I. INTRODUCTION

Stephen Hawking showed \cite{1} that black holes emit thermal radiation corresponding to a temperature proportional to surface gravity and entropy proportional to the horizon area. The horizon temperature and entropy obey a simple differential relationship $-dE = TdS$, called the first law of black hole thermodynamics \cite{2}, where $E$ is the energy. Another significant development was made by Jacobson \cite{3} by deriving Einstein field equations from the proportionality of entropy to the horizon area together with the fundamental relation $\delta Q = TdS$, where $\delta Q$ and $T$ are the energy flux and Unruh temperature seen by an accelerated observer just inside the horizon. Padmanabhan \cite{4} made the major development by launching a general formalism for the spherically symmetric black hole spacetimes to understand the thermodynamics of horizons and showed that the Einstein field equations evaluated at event horizon can be expressed in the form of first law, $TdS = dE + \mathcal{W}dV$, of thermodynamics. Later on Padmanabhan et al and others \cite{5, 6} studied this approach for more general spacetime geometries and in various gravity theories. In the cosmological setup, Cai and his collaborators \cite{7, 8, 9, 10, 11} made the major development by showing that the Einstein field equations evaluated at the apparent horizon can also be expressed as the first law $TdS = dE + \mathcal{W}dV$ in various theories of gravity. This connection between gravity and thermodynamics has also been extended in the braneworld cosmology \cite{12}. More recently, using Clausius relation $\delta Q = TdS$, to the apparent horizon of a FRW universe, Cai et al are able to derive the modified Friedman equation by employing quantum corrected area-entropy formula \cite{13}. All these calculations indicate that the thermal interpretation of gravity is to be generic, so we have to investigate this relation for a more general spacetimes. Hence, in this paper, we extend this approach for evolving Lorentzian wormhole spacetime and showed by using the approach of \cite{13} that the field equations of the wormhole geometry can be expressed as a first law $TdS = dE + \mathcal{W}dV$, at the apparent horizon. We also study the dynamics of wormhole horizons. This paper is organized as follows: In the second section, we discuss the dynamics of wormhole horizons. In third section, we study the thermal interpretation of the field equations at wormhole apparent horizons. Finally, we present conclusion in the fourth section.

II. EVOLVING LORENTZIAN WORMHOLE

A simple generalization of Morris-Thorne (MT) wormhole \cite{14} to the time dependent background is given by the evolving Lorentzian wormhole \cite{15}

$$
\text{Id}^2 = -e^{2\Phi(t,r)}dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \frac{b(r)}{r(r)}} + r^2 d\Omega_2^2 \right].
$$

(1)

Here $d\Omega_2^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2$ is the line element of two-dimensional unit sphere, $b(r)$ and $\Phi(t,r)$ are the shape and potential functions respectively and $a(t)$ is the scale factor of the universe. It is clear from the metric (1) that if both $b(r) \rightarrow 0$, and $\Phi(t,r) \rightarrow 0$, the above metric reduces to the flat FRW metric. Furthermore, when $a(t) \rightarrow \text{const}$ and $\Phi(t,r) \rightarrow \Phi(r)$, it turns out the static MT wormhole \cite{14}. If one takes $a(t) = e^{\chi t}$, the metric (1) represents an inflating Lorentzian wormhole \cite{16}, where the arbitrary constant $\chi$ can be fixed by taking it a cosmological constant $\Lambda$.

In this paper, we shall use the ansatz $\Phi(t,r) = 0$, and $b(r) = r_0^2 / r$, where $r_0$ is a finite radius of the wormhole’s throat. The former assumption is motivated from two facts; the potential function must be finite quantity for all values of $r$, for the static case. Moreover, it is also one of the solutions of Einstein field equations for the wormhole spacetime \cite{14, 17}. This ansatz for the shape function clearly obeys flare-out condition at the throat; $b(r_0) = r_0$, and $b'(r_0) < 1$ and $b(r) < r$. It also fulfills another requirement
of asymptotic flatness. Thus, taking the above ansatz we have the following evolving wormhole metric
\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \frac{r_0^2}{r^2}} + r^2 d\Omega^2 \right]. \] (2)

One can rewrite the above metric in the spherical form
\[ ds^2 = h_{ab} dx^a dx^b + \tilde{r}^2 d\Omega^2, \quad a, b = 0, 1 \] (3)
where \( \tilde{r} = a(t)r \) and \( x^0 = t, \ x^1 = r \) and the two dimensional metric
\[ h_{ab} = \text{diag} \left( -1, a(t)^2 \left( 1 - \frac{r_0^2}{\tilde{r}^2} \right)^{-1} \right). \] (4)

Let us now consider the Einstein field equations
\[ G_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3 \] (5)
where \( G_{\mu\nu} \) is the Einstein tensor and \( T_{\mu\nu} \) is the energy-momentum tensor of the matter fields. Due to the spherical symmetry of the wormhole geometry (3), the stress energy tensor \( T_{\mu\nu} \) must be diagonal. The simplest one is that of a perfect fluid described by a time dependent energy density \( \rho(t) \) and pressure \( p(t) \)
\[ T^{\mu\nu} = (\rho + p) u^\mu u^\nu + pg^{\mu\nu}, \] (6)
where \( u^\mu = (1, 0, 0, 0) \) is the comoving four velocity of the fluid. The energy conservation condition \( T^{\mu}_\nu = 0 \), admits \( \dot{\rho} + 3H(\rho + p) = 0 \). Solving the Einstein field equation (5), in the background of wormhole geometry (2), one can get the Friedman-like field equations
\[ H^2 - \frac{\tilde{r}_A^2}{3r^4a^2} = \frac{8\pi G}{3} \rho, \] (7)
\[ \dot{H} + \frac{r_0^2}{a^2r^4} = -4\pi G (\rho + p). \] (8)

Here \( H = \dot{a}/a \) is the Hubble parameter and dot refers to the cosmic time derivative. The apparent horizon of the wormhole geometry can be evaluated by using relation \( h_{ab} \partial_a \tilde{r} \partial_b \tilde{r} = 0 \), which after simplification yields
\[ H^2 \tilde{r}_A^4 - \tilde{r}_A^2 + a^2 \tilde{r}_0^2 = 0. \] (9)

It can be seen from Eq. (9) that when \( r_0 = 0 \), namely a flat FRW universe, the wormhole apparent horizon \( \tilde{r}_A \), has the same value as the Hubble horizon, \( \tilde{r}_A = 1/H \). The Hubble parameter in terms of the wormhole apparent radius is \( H^2 = 1/\tilde{r}_A^2 - a^2 \tilde{r}_0^2/r^4 \), and its time derivative admits
\[ \dot{H} = -\frac{\dot{\tilde{r}}_A}{H \tilde{r}_A^3} \left( 1 - \frac{2a^2 \tilde{r}_0^2}{\tilde{r}_A^2} \right) - \frac{a^2 \tilde{r}_0^2}{\tilde{r}_A^2}. \] (10)

The apparent horizons of the wormhole metric (3) are the roots of the Eq. (9) which admits
\[ \tilde{r}^2_{A+} = \frac{1 + \sqrt{1 - 4H^2a^2 \tilde{r}_0^2}}{2H^2}, \quad \tilde{r}^2_{A-} = \frac{1 - \sqrt{1 - 4H^2a^2 \tilde{r}_0^2}}{2H^2}. \] (11)

There are three cases depending upon the roots. (a) Two distinct real roots \( (1 - 4H^2a^2 \tilde{r}_0^2 > 0) \) refer as a usual wormhole geometry, (b) two repeated real roots \( (1 - 4H^2a^2 \tilde{r}_0^2 = 0) \) called as the ‘extreme wormhole’ geometry, (c) no real roots \( (1 - 4H^2a^2 \tilde{r}_0^2 < 0) \) imply the ‘naked wormhole’. If we assume that \( 0 < \tilde{r}_0^2 \ll 1 \), and neglecting \( O(\tilde{r}_0^4) \), it is possible to simplify the expressions for \( \tilde{r}_{A+} \) and \( \tilde{r}_{A-} \), which give
\[ \tilde{r}^2_{A+} = \frac{1}{H^2} - a^2 \tilde{r}_0^2, \quad \tilde{r}^2_{A-} = a^2 \tilde{r}_0^2. \] (12)

It is evident from equation (12) that the outer apparent horizon will contract while the inner horizon will expand with the passage of time. These coincide at the extreme case \( \tilde{r}_{A+} = \tilde{r}_{A-} = \tilde{r}_A = 1/\sqrt{2}H \). It is also interesting to note that the wormhole horizons satisfy
\[ \tilde{r}^2_{A+} + \tilde{r}^2_{A-} = \frac{1}{H^2}. \quad \tilde{r}^2_{A+} - \tilde{r}^2_{A-} = \frac{a^2 \tilde{r}_0^2}{H^2}. \] (13)

In the extreme case \( \tilde{r}_{A-} = \tilde{r}_{A+} \), the quantity inside the square root, \( 1 - 4H^2a^2 \tilde{r}_0^2 \), in (11) vanishes. In this case, the wormhole parameters satisfy \( \dot{a}^2 = \frac{1}{4\tilde{r}_0^2} \), which upon integration gives \( a(t) = \frac{t^2}{2\tilde{r}_0^2} \), where the constant of integration is assumed to be zero. It shows that the wormhole is expanding uniformly if \( a(t) > 0 \) and contracting if \( a(t) < 0 \). Also the naked wormhole is obtained if the discriminant \( 1 - 4H^2a^2 \tilde{r}_0^2 < 0 \), which yields \( \dot{a} > \frac{1}{2\tilde{r}_0} \).

### III. WORMHOLE THERMODYNAMICS

In this section, we discuss the thermodynamic properties of wormhole at wormhole horizons. We assume that the entropy associated with the outer horizon \( \tilde{r}_{A+} \), of the wormhole is proportional to the wormhole horizon area analogous to the black hole entropies. So the entropy of the wormhole becomes
\[ S = \frac{\pi \tilde{r}^2_{A+}}{G} = \frac{\pi}{G} \left( 1 + \frac{\sqrt{1 - 4H^2a^2 \tilde{r}_0^2}}{2H^2} \right). \] (14)

The surface gravity is defined as
\[ \kappa = \frac{1}{2\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b \tilde{r}), \] (15)
where \( h \) is the determinant of metric \( h_{ab} \) (4). The direct calculation of the surface gravity from Eq. (15) at the wormhole horizon \( \tilde{r}_{A+} \) yields
\[ \kappa = -\frac{\tilde{r}_{A+}}{2} \left( \dot{H} + 2H^2 - \frac{a^2 \tilde{r}_0^2}{\tilde{r}_{A+}^2} \right), \] (16)
\[ = -\frac{1}{\tilde{r}_{A+}^2} \left( 1 - \frac{\dot{\tilde{r}}_{A+}}{2H \tilde{r}_{A+}} \right) \left( 1 - \frac{2a^2 \tilde{r}_0^2}{\tilde{r}_{A+}^2} \right). \] (17)

The factor \( -\frac{1}{\tilde{r}_{A+}^2} \left( 1 - \frac{\dot{\tilde{r}}_{A+}}{2H \tilde{r}_{A+}} \right) \), in (17) is the general expression for the surface gravity of FRW universe while
the second factor \(1 - \frac{2a^2r_0^2}{r_{A+}^2}\) has been appeared due to wormhole geometry. This expression for the surface gravity reduces to the expression for FRW universe when \(r_0\) vanishes. The horizon temperature \(T = \kappa/2\pi\), of the wormhole is given by

\[
T = -\frac{1}{2\pi}\left(1 - \frac{\dot{r}_{A+}}{2H\tilde{r}_{A+}}\right)\left(1 - \frac{2a^2r_0^2}{r_{A+}^2}\right). \tag{18}
\]

Now our purpose is to rewrite the Friedman-like equation (8) for the wormhole as a first law of thermodynamics.

For this we follow the procedure already developed in [18] in which the field equations of FRW universe have been expressed as a first law at the apparent horizon. So first we evaluate the Friedman-like equation (8) at the apparent horizon \(\tilde{r}_{A+}\) which turns out

\[
\left(1 - \frac{2a^2r_0^2}{r_{A+}^2}\right)d\tilde{r}_{A+} = 4\pi GH\tilde{r}_{A+}^3(\rho + p)dt. \tag{19}
\]

Now we multiply to the above equation by a factor \(1 - \frac{\tilde{r}_{A+}}{2H\tilde{r}_{A+}}\) and arranging the terms, we get

\[
-\frac{1}{2\pi}\left(1 - \frac{\dot{r}_{A+}}{2H\tilde{r}_{A+}}\right)\left(1 - \frac{2a^2r_0^2}{r_{A+}^2}\right)d\left(\frac{\pi\tilde{r}_{A+}^2}{G}\right) = -4\pi H\tilde{r}_{A+}^3\left(1 - \frac{\dot{r}_{A+}}{2H\tilde{r}_{A+}}\right)(\rho + p)dt. \tag{20}
\]

From equation (20), one can immediately identify that the term on the left hand side is \(TdS\), where \(S = A/4G = 4\pi\tilde{r}_{A+}^2/4G\), is the entropy of the wormhole. So the above equation reduces to

\[
TdS = -4\pi H\tilde{r}_{A+}^3\left(1 - \frac{\dot{r}_{A+}}{2H\tilde{r}_{A+}}\right)(\rho + p)dt. \tag{21}
\]

Now we consider the total matter-energy \(E = \rho V\), surrounded by the apparent horizon \(\tilde{r}_{A+}\) of the wormhole. Taking the differential of \(E\) and using the energy conservation relation, we get

\[
dE = 4\pi\tilde{r}_{A+}^2\rho d\tilde{r}_{A+} - 4\pi\tilde{r}_{A+}^3H(\rho + p)dt. \tag{22}
\]

Using Eqs. (21) and (22), we finally get

\[
dE = TdS + WdV, \tag{23}
\]

where \(W = (\rho - p)/2\) is the work density. The expression (23) is the unified first law of thermodynamics in the cosmological setup [19]. Summarizing, by taking total matter energy density within the apparent horizon of the wormhole, the Friedman-like equation can be expressed as a thermodynamic identity. It is important to note that one can associate the notions of temperature and entropy with the apparent horizons of the wormhole analogous to the apparent horizon of FRW universe. A trivial calculation shows that the similar thermal interpretation of the field equations also hold at the inner horizon \(\tilde{r}_{A-}\) of the wormhole.

**IV. CONCLUSION**

In this paper we have shown that the Friedman-like equation of the wormhole geometry can be expressed as a first law, \(dE = TdS + WdV\), at the apparent horizon of the wormhole geometry. Here \(E = \rho V\) is the total energy of matter inside the horizons, \(W = (\rho - p)/2\) and \(V\) are the work density and volume inside the horizon respectively. In this approach, the entropy \(S\) is assumed to be quarter of the wormhole apparent horizon area. In the cosmological setting, indeed, the Friedman equations of the FRW universe can be expressed as a thermal identity at apparent horizon [7, 18]. Our results indicate that the dynamic apparent horizon has a wider range of applications when one associates it with the notions of temperature and entropy. It is also interesting to investigate the properties of the wormhole geometry at extreme case. In case \(p = -\rho\), one gets the standard form of the first law \(dE = TdS - pdV\). In addition, we also studied the apparent horizons of the evolving wormhole by assuming a particular shape function, \(b(r) = r_0^2/r\), of the wormhole. Under this assumption, it is found that the evolving wormhole contains two apparent horizons. The outer wormhole horizon will contract while the inner will expand. Both horizon will coincide at the extreme case. One question may arise: whether we can express the field equations of the wormhole geometry in the extended theories of gravity. The work in this direction is under investigation.

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