Effects of thermal inflation on small scale density perturbations

Sungwook E. Hong, a Hyung-Joo Lee, b Young Jae Lee, b Ewan D. Stewart b,c and Heeseung Zoe d,e

a School of Physics, Korea Institute for Advanced Study, 85 Hoegiro, Seoul 130-722, Republic of Korea
b Department of Physics, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon 305-338, Republic of Korea
c Department of Physics, University of Auckland, 38 Princes Street, Auckland, New Zealand
d School of Basic Science, Daegu Gyeongbuk Institute of Science and Technology (DGIST), 333 Techno jungang-daero, Daegu 711-873, Republic of Korea
e Division of General Studies, Ulsan National Institute of Science and Technology (UNIST), 50 UNIST-gil, Ulsan 689-798, Republic of Korea

E-mail: swhong@kias.re.kr, ohsk111@kaist.ac.kr, noasac@kaist.ac.kr, jcap@profstewart.org, heezoe@dgist.ac.kr

Received April 1, 2015
Accepted May 13, 2015
Published June 3, 2015

Abstract. In cosmological scenarios with thermal inflation, extra eras of moduli matter domination, thermal inflation and flaton matter domination exist between primordial inflation and the radiation domination of Big Bang nucleosynthesis. During these eras, cosmological perturbations on small scales can enter and re-exit the horizon, modifying the power spectrum on those scales. The largest modified scale, $k_b$, touches the horizon size when the expansion changes from deflation to inflation at the transition from moduli domination to thermal inflation. We analytically calculate the evolution of perturbations from moduli domination through thermal inflation and evaluate the curvature perturbation on the constant radiation density hypersurface at the end of thermal inflation to determine the late time curvature perturbation. Our resulting transfer function suppresses the power spectrum by a factor $\sim 50$ at $k \gg k_b$, with $k_b$ corresponding to anywhere from megaparsec to subparsec scales depending on the parameters of thermal inflation. Thus, thermal inflation might be constrained or detected by small scale observations such as CMB distortions or 21cm hydrogen line observations.

Keywords: inflation, physics of the early universe, supersymmetry and cosmology, cosmological perturbation theory

ArXiv ePrint: 1503.08938
1 Introduction

Thermal inflation [1–9], a brief low energy inflation that occurs when thermal effects hold an unstable flat direction at the origin, occurs naturally in supersymmetric theories and is motivated to solve the moduli and gravitino problems [10–12]. Additionally, thermal inflation provides a mechanism for baryogenesis [13–20] and has implications for dark matter [3, 16, 19]. When thermal inflation is included, observable cosmology starts as usual with primordial inflation, which generates the density perturbations that go on to form galaxies, etc., but then has additional eras of moduli matter domination, thermal inflation and flaton matter domination inserted before the usual radiation domination of Big Bang nucleosynthesis (BBN).

During these thermal inflation eras, cosmological perturbations on small scales can enter and re-exit the horizon, modifying the power spectrum on those scales, while perturbations on larger scales, corresponding to cosmic microwave background (CMB) or large scale structure (LSS) observations, remain outside the horizon, preserving and only modestly redshifting their spectrum.

The observational impact of thermal inflation on the gravitational wave background was studied in [21, 22]. Thermal inflation wipes out any potentially observable gravitational waves from primordial inflation on solar system or smaller scales [21], but the first order phase transition at the end of thermal inflation generates gravitational waves [22, 23] with frequencies in the Hz range, though their amplitude may be too small for them to be observed in the near future [22].

In this paper, we study the impact of thermal inflation on small scale density perturbations with the aim of finding signatures of, or constraints on, thermal inflation. We note that a variety of small scale physics, such as ultracompact minihalos or primordial black holes [24–26], lensing dispersion of SNIa [27], CMB distortions [28, 30] and the 21cm hydrogen line at or prior to the era of reionization [32, 33] could be used as tools for studying
the effects of thermal inflation on the small scale power spectrum. Several prominent upcoming observations are designed for such small scale physics, for example the Primordial Inflation Explorer (PIXIE) \cite{34} and the Polarized Radiation Imaging and Spectroscopy Mission (PRISM) \cite{35, 36} for CMB distortions and the Square Kilometre Array (SKA) \cite{37} for the 21cm hydrogen line.

In section 2, we outline cosmology with thermal inflation and determine its characteristic scales. In section 3, we analytically calculate the thermal inflation transfer function which modifies the primordial power spectrum. In section 4, we summarize our results and briefly discuss the possibilities of observing the effects of thermal inflation. In this paper, we set $\hbar = c = 8\pi G = M_{\text{Pl}} = 1$.

2 Cosmology with thermal inflation

In cosmology with thermal inflation \cite{1, 2, 13–16, 22}, there are two different epochs of inflation. The first, primordial inflation \cite{38–42}, generates the primordial perturbations which are observed in the CMB and grow to form galaxies and the LSS. The second, thermal inflation, is brief and occurs after primordial inflation but before Big Bang nucleosynthesis, at a sufficiently low energy scale to solve the moduli problem, and may affect the perturbations on very small scales as we discuss in the following sections.

In figure 1, we illustrate the thermal inflation eras of moduli domination, thermal inflation and flaton domination. They are preceded by primordial inflation plus a possible post-inflationary era and followed by the radiation domination of BBN.
For definiteness, we consider a general class of supersymmetry breaking scenarios in which supersymmetry is broken in a hidden sector at a scale $M_s$ and transmitted to the observable sector via gravitational strength interactions, so that the supersymmetry breaking scale in the observable sector is $m_s \sim M_s^2/M_{Pl}$. We set $m_s \sim 10^3$ GeV.

In the early universe, the finite energy density, represented by the Hubble parameter $H$, breaks supersymmetry. When $H \gtrsim m_s$, the supersymmetry breaking by the finite energy density dominates over the vacuum supersymmetry breaking giving a moduli potential of the form

$$V(\Phi) \sim H^2 M_{Pl}^2 f\left(\frac{\Phi}{M_{Pl}}\right) \sim H^2 (\Phi - \Phi_1)^2 + \cdots.$$  \hspace{1cm} (2.1)

However, as the Hubble parameter drops through $H \sim m_{\Phi}$, the moduli potential switches to its vacuum form

$$V(\Phi) \sim M_s^4 g\left(\frac{\Phi}{M_{Pl}}\right) \sim m_{\Phi}^2 (\Phi - \Phi_2)^2 + \cdots$$  \hspace{1cm} (2.2)

with

$$m_{\Phi} \sim \frac{M_s^2}{M_{Pl}} \sim m_s \sim 10^3 \text{ GeV}$$ \hspace{1cm} (2.3)

and $\Phi_1 - \Phi_2 \sim M_{Pl}$. Thus, at $H \sim m_{\Phi}$, or $t \sim t_a$ in the notation of figure 1, the moduli start oscillating with Planckian amplitude, dominating the energy density of the universe. Furthermore, the moduli have relatively low masses and very weak interactions so this moduli oscillation may be sufficiently long-lived to have disastrous effects on, for example, BBN. During the era of moduli domination, the moduli abundance is

$$\frac{n_{\Phi}}{s} \sim \sqrt{\frac{M_{Pl}}{m_{\Phi}}} \sim 10^8,$$ \hspace{1cm} (2.4)

where $n_{\Phi}$ is the moduli number density and $s$ is the entropy density, but not to spoil BBN it should be [43]

$$\frac{n_{\Phi}}{s} \lesssim 10^{-15} \text{ to } 10^{-12}.$$ \hspace{1cm} (2.5)

This is called the moduli problem [10–12].

Thermal inflation, which is motivated to solve the moduli problem, is realized by an (almost) flat direction with negative mass-squared at the origin, called a flaton. Like the Standard Model Higgs field, the flaton potential near the origin is

$$V(\phi) = V_0 - \frac{1}{2} m_{\phi}^2 \phi^2 + \cdots.$$ \hspace{1cm} (2.6)

Unlike the Standard Model Higgs field, but like many scalar field directions in the Minimal Supersymmetric Standard Model, the flaton does not have a stabilizing $\phi^4$ term. Instead, higher order terms, or the renormalisation group running of the flaton mass squared, stabilize the flaton potential at a large field value $\phi_0 \gg m_{\phi}$. To have zero energy density at the minimum, we require $V_0 \sim m_{\phi}^2 \phi_0^2$, and so, for $m_{\phi} \ll \phi_0 \lesssim M_{Pl}$, we have

$$m_{\phi} \ll V_0^{\frac{1}{2}} \lesssim \sqrt{m_{\phi} M_{Pl}}.$$ \hspace{1cm} (2.7)

Taking

$$m_{\phi} \sim m_s \sim 10^3 \text{ GeV}$$ \hspace{1cm} (2.8)
gives
\[ 10^3 \text{GeV} \ll V_0^1 \lesssim 10^{11} \text{GeV}. \] (2.9)

At the finite temperature of the early universe, the flaton’s potential is modified by its (assumed) unsuppressed couplings to the thermal bath
\[ V(\phi, T) = V_0 + \frac{1}{2} (\sigma^2 T^2 - m_\phi^2) \phi^2 + \cdots, \] (2.10)

where \( \sigma \) is not small. When \( T \gtrsim m_\phi/\sigma \), corresponding to \( t \lesssim t_c \) in figure 1, the flaton is trapped at \( \phi = 0 \), leaving the potential energy \( V = V_0 \) which drives the thermal inflation.

At \( t \sim t_a \), the moduli dominate the universe, but with a comparable amount of radiation
\[ \rho_m(t_a) \sim \rho_r(t_a) \gg V_0. \] (2.11)

As the universe expands, the moduli and radiation are diluted
\[ \rho_m \propto a^{-3}, \quad \rho_r \propto a^{-4}. \] (2.12)

At \( t \sim t_b \), the moduli density drops below \( V_0 \)
\[ V_0 \sim \rho_m(t_b) \gg \rho_r(t_b) \] (2.13)

and thermal inflation begins. As the universe inflates, the temperature drops
\[ T \propto a^{-1} \] (2.14)

and the moduli are diluted to a safely small abundance, see eq. (2.24) below.

At \( t \sim t_c \), the temperature drops to the critical temperature
\[ T_c \sim \frac{m_\phi}{\sigma}, \] (2.15)

at which a first order phase transition ends thermal inflation [22] and the flaton rapidly rolls towards and oscillates about its minimum at \( \phi = \phi_0 \), leading to a flaton matter dominated era
\[ \rho_{\phi} \propto a^{-3}. \] (2.16)

The change in state at the end of thermal inflation perturbs the moduli potential by an amount
\[ \delta V(\Phi) \sim V_0 h \left( \frac{\Phi}{M_{\text{Pl}}} \right) \sim \frac{V_0}{M_{\text{Pl}}} \Phi + \cdots \] (2.17)

regenerating a moduli abundance
\[ \frac{n_\Phi}{n_\phi} \sim \frac{V_0 m_\phi}{m_\phi^3 M_{\text{Pl}}^2}. \] (2.18)

At \( t \sim t_d \), the flaton decays to radiation at a temperature [16]
\[ T_d \sim 10^2 \text{ to } 10^{-2} \text{GeV} \] (2.19)

and the standard cosmic history of radiation domination, BBN, etc., follows.
The moduli generated at \( t \sim t_a \) are diluted by thermal inflation to an abundance [2]

\[
\frac{n_\Phi}{s} \sim \frac{g_{ss}(T_c) T_c^3 T_d M_{Pl}^{1/2}}{V_0 m_{\Phi}^{1/2} m_{\nu}}
\sim 10^{-9} \left( \frac{g_{ss}(T_c)}{10^2} \right) \left( \frac{T_c}{10^3 \text{GeV}} \right)^3 \left( \frac{T_d}{\text{GeV}} \right) \left( \frac{10^3 \text{GeV}}{m_{\Phi}} \right)^{1/2} \left( \frac{10^7 \text{GeV}}{V_0^{1/4}} \right)^4,
\]

where \( g_{ss}(T) \) is the effective number of entropic degrees of freedom at temperature \( T \) [44]. In models of thermal inflation incorporating a mechanism for baryogenesis [13–16], entropy release by Affleck-Dine fields during thermal inflation may typically lead to an effective double thermal inflation [15, 16], further diluting the moduli by a factor

\[
\Delta_{AD} \sim \left( \frac{\rho_r(T_c)}{\rho_{\text{AD}}} \right)^{1/4} \sim \left( \frac{g_{ss}(T_c) T_c^4 m_{\nu}}{m_{LH_u}^2 (H_u/2)_{\text{EW}}} \right)^{1/4}
\sim 10^{-8} \left( \frac{g_{ss}(T_c)}{10^2} \right)^{1/4} \left( \frac{T_c}{10^3 \text{GeV}} \right)^3 \left( \frac{10^4 \text{GeV}}{m_{LH_u}} \right)^{1/2} \left( \frac{174 \text{GeV}}{H_u} \right)^{3/2} \left( \frac{m_{\nu}}{10^{-2} \text{eV}} \right)^{3/4}
\]

to

\[
\frac{n_\Phi}{s} \sim 10^{-17}.
\]

The moduli regenerated at \( t \sim t_c \) have abundance [2]

\[
\frac{n_\Phi}{s} \sim \frac{V_0 T_d}{m_{\Phi}^2 M_{Pl}^2} \sim 10^{-18} \left( \frac{T_d}{\text{GeV}} \right) \left( \frac{10^3 \text{GeV}}{m_{\Phi}} \right)^3 \left( \frac{V_0^{1/4}}{10^7 \text{GeV}} \right)^4.
\]

Thus for values of \( V^{1/4} \) in the middle of its expected range, eq. (2.9), the moduli can be diluted to, and regenerated with, a sufficiently small abundance to satisfy eq. (2.5), solving the moduli problem.

### 2.1 Characteristic scales

The thermal inflation eras, illustrated in figure 1, determine four characteristic scales, \( k_a, k_b, k_c \) and \( k_d \), where

\[
k_x \equiv a_x H_x
\]

and

\[
k_x \equiv a_x H_x
\]
corresponds to the comoving scale of the horizon at the era boundary \( t_x \). We express the \( k_x \) in terms of the effective thermal inflation parameters

\[
N_{xy} \equiv \ln \frac{a_y}{a_x},
\]

which measure the durations of the thermal inflation eras by their number of e-folds of expansion, and in turn estimate the \( N_{xy} \) in terms of the more fundamental thermal inflation parameters \( m_{\Phi}, V_0, T_c, T_d, \) etc.
2.1.1 $k_a$

During moduli domination, $t_a < t < t_b$, the energy density $\rho \propto a^{-3}$, so

$$k_a = \frac{a_a H_a}{a_b H_b} k_b = \left(\frac{a_a}{a_b}\right)^{-\frac{1}{2}} k_b = e^{\frac{1}{2} N_{ab}} k_b.$$  \hfill (2.28)

Using $\rho_a \sim m^2_\Phi M^2_{Pl}$ and $\rho_b \sim V_0$,

$$N_{ab} = \ln \frac{a_b}{a_a} = \frac{1}{3} \ln \frac{\rho_a}{\rho_b} \simeq \frac{1}{3} \ln \frac{m^2_\Phi M^2_{Pl}}{V_0}$$  \hfill (2.29)

$$\simeq 11 + \frac{2}{3} \ln \left[ \frac{m_\Phi}{10^3 \text{GeV}} \left( \frac{10^7 \text{GeV}}{V_0^{1/4}} \right)^2 \right].$$  \hfill (2.30)

2.1.2 $k_b$

During thermal inflation, $t_b < t < t_c$, the energy density $\rho = V_0$, so

$$k_b = e^{-N_{bc}} k_c.$$  \hfill (2.31)

At the beginning of thermal inflation, $t = t_b$,

$$\rho_r = \frac{\pi^2}{30} g_*(T_b) T^4_b$$  \hfill (2.32)

$$\sim \left(\frac{V_0}{m^2_\Phi M^2_{Pl}}\right)^{\frac{1}{2}} \rho_m \sim \left(\frac{V_0^2}{m_\Phi M^2_{Pl}}\right)^{\frac{1}{2}},$$  \hfill (2.33)

therefore

$$N_{bc} = \ln \frac{a_c}{a_b} = \ln \frac{T_b}{T_c} \simeq \frac{1}{6} \ln \frac{V^2_0}{g^{3/2}_*(T_b) T^6_c m_\Phi M^2_{Pl}}$$  \hfill (2.34)

$$\simeq 12 + \frac{1}{3} \ln \left[ \frac{10^2}{g_*(T_b)} \left( \frac{10^3 \text{GeV}}{T_c} \right)^3 \left( \frac{10^3 \text{GeV}}{m_\Phi} \right)^{\frac{1}{2}} \left( \frac{V_0^{1/4}}{10^7 \text{GeV}} \right)^4 \right].$$  \hfill (2.35)

As discussed above, entropy release during thermal inflation may typically add an extra $\sim 6$ $e$-folds, giving

$$N_{bc} \sim 18.$$  \hfill (2.36)

More general forms of double or multiple thermal inflation can in principle give even larger values of $N_{bc}$.

2.1.3 $k_c$

During flaton domination, $t_c < t < t_d$, the energy density $\rho \propto a^{-3}$, so

$$k_c = \frac{a_c H_c}{a_d H_d} k_d = \left(\frac{a_c}{a_d}\right)^{-\frac{1}{2}} k_d = e^{\frac{1}{2} N_{cd}} k_d.$$  \hfill (2.37)

Using $\rho_c = V_0$ and $\rho_d = \rho_r(T_d)$,

$$N_{cd} = \ln \frac{a_d}{a_c} = \frac{1}{3} \ln \frac{\rho_c}{\rho_d} = \frac{1}{3} \ln \frac{30 V_0}{\pi^2 g_*(T_d) T^4_d}$$  \hfill (2.38)

$$\simeq 20 + \frac{1}{3} \ln \left[ \frac{10^2}{g_*(T_d)} \left( \frac{\text{GeV}}{T_d} \right)^4 \left( \frac{V_0^{1/4}}{10^7 \text{GeV}} \right)^4 \right].$$  \hfill (2.39)
2.1.4 $k_d$

At the beginning of radiation domination, $t = t_d$, the scale factor is \[ a_d = \frac{g_*^{1/3}(T_0)}{g_*^{1/3}(T_d)} \frac{T_0}{T_d} a_0 , \] (2.40)

where $a_0$ and $T_0$ are the current scale factor and temperature, respectively, and the Hubble parameter is

\[ H_d = \sqrt{\frac{\rho_d}{3M_{Pl}^2}} = \frac{\pi g_*^{1/2}(T_d)}{3\sqrt{10} M_{Pl}} , \] (2.41)

therefore eqs. (2.26), (2.40) and (2.41) give

\[ \frac{2\pi a_0}{k_d} \approx \frac{6\sqrt{10} g_*^{1/3}(T_d) M_{Pl}}{g_*^{1/2}(T_0) g_*^{1/2}(T_d) T_0 T_d} \] (2.42)

\[ \simeq 0.4 \left[ \frac{10^2 g_*^2(T_d)}{g_*^3(T_d)} \right]^{1/2} \text{GeV} \frac{T_d}{10^7 \text{GeV}} \text{pc.} \] (2.43)

2.2 The largest characteristic scale

Eqs. (2.28), (2.31) and (2.37) imply

\[ k_b < k_a, k_c , \] (2.44)
\[ k_d < k_c , \] (2.45)

thus either $k_b$ or $k_d$ will be the largest scale. Eqs. (2.31) and (2.37) give

\[ k_b = e^{-N_{bc} + \frac{1}{2} N_{cd} k_d} , \] (2.46)

therefore if

\[ N_{bc} > \frac{1}{2} N_{cd} , \] (2.47)

which we will assume, then $k_b$ will be the largest physical scale and hence the one that can be most easily observed. Using eqs. (2.38), (2.42) and (2.46),

\[ \frac{2\pi a_0}{k_b} \approx e^{N_{bc}} \left[ \frac{720\sqrt{3 \pi} g_*^3(T_d) M_{Pl}^3}{g_*^2(T_0) g_*^2(T_d) T_0 T_d V_0^{1/2}} \right]^{1/3} \] (2.48)

\[ \simeq 0.9 \left[ \frac{e^{N_{bc}} (\text{GeV})^{1/2}}{e^{18}} \right] \left( \frac{10^7 \text{GeV}}{V_0^{1/4}} \right) \text{pc.} \] (2.49)

Modes with $k < k_b$ remain outside the horizon throughout the thermal inflation eras, and so are not affected by thermal inflation, while those with $k > k_b$ enter the horizon during moduli domination, allowing their evolution to be modified. Hence, it is expected that there could be observable features of thermal inflation at $k \gtrsim k_b$.

Modes with $k > k_a$ enter the horizon before moduli domination and so probe that unknown era, and modes with $k > k_d$ reenter the horizon during flaton domination and so will be twice modified.

In the next section, we study the evolution of the density perturbations for modes

\[ k_b \lesssim k \ll k_a, k_d . \] (2.50)
3 Evolution of the density perturbations

During the moduli domination and thermal inflation eras, \( t_a < t < t_c \), we have moduli matter (m), thermal radiation (r) and vacuum energy (V), with

\[
\rho = \rho_m + \rho_r + V, \\
p = \frac{1}{3}\rho_r - V.
\]

(3.1) (3.2)

To describe the perturbations in moduli and radiation, we define the gauge invariant variables

\[
R_{\delta \rho_m} \equiv R - \frac{H}{\rho_m} \delta \rho_m, \\
R_{\delta \rho_r} \equiv R - \frac{H}{\rho_r} \delta \rho_r,
\]

(3.3) (3.4)

where \( R_{\delta \rho_m} \) is the curvature perturbation on constant moduli density hypersurfaces and \( R_{\delta \rho_r} \) is the curvature perturbation on constant radiation density hypersurfaces. Eq. A.20 gives

\[
\begin{align*}
\ddot{R}_{\delta \rho_m} + H \left( 2 + \frac{3}{2} \frac{\rho_m}{\rho_m + \frac{3}{2} q^2} \right) \dot{R}_{\delta \rho_m} & - \frac{1}{3} q^2 \left( \frac{\rho_m}{\rho_m + \frac{3}{2} q^2} \right) R_{\delta \rho_m} \\
& = - \frac{4}{3} \frac{\dot{\rho}_r}{\rho + \frac{3}{2} q^2} \left( H \dot{R}_{\delta \rho_r} - \frac{1}{3} q^2 R_{\delta \rho_r} \right), \\
\ddot{R}_{\delta \rho_r} + H \left( 1 + \frac{8}{3} \frac{\rho_r}{\rho + \frac{3}{2} q^2} \right) \dot{R}_{\delta \rho_r} & + \frac{1}{3} q^2 \left( 1 - \frac{8}{3} \frac{\rho_r}{\rho + \frac{3}{2} q^2} \right) R_{\delta \rho_r} \\
& = - \frac{2\rho_m}{\rho + \frac{3}{2} q^2} \left( H \dot{R}_{\delta \rho_m} - \frac{1}{3} q^2 R_{\delta \rho_m} \right),
\end{align*}
\]

(3.5) (3.6)

where \( q \equiv k/a \).

The physics at \( t \sim t_a \) is uncertain so we restrict ourselves to \( t_a \ll t < t_c \), in which case

\[
\rho_r \ll \rho_m
\]

(3.7)

and so eqs. (3.5) and (3.6) reduce to

\[
\begin{align*}
\ddot{R}_{\delta \rho_m} + H \left( 2 + \frac{3}{2} \frac{\rho_m}{\rho_m + \frac{3}{2} q^2} \right) \dot{R}_{\delta \rho_m} & - \frac{1}{3} q^2 \left( \frac{\rho_m}{\rho_m + \frac{3}{2} q^2} \right) R_{\delta \rho_m} = 0, \\
\ddot{R}_{\delta \rho_r} + H \dot{R}_{\delta \rho_r} & + \frac{1}{3} q^2 R_{\delta \rho_r} = F,
\end{align*}
\]

(3.8) (3.9)

where

\[
F = -2 \left( \frac{\rho_m}{\rho_m + \frac{3}{2} q^2} \right) \left( H \dot{R}_{\delta \rho_m} - \frac{1}{3} q^2 R_{\delta \rho_m} \right),
\]

(3.10)

which have solution

\[
\begin{align*}
R_{\delta \rho_m}(k, t) &= A_m(k, t_i) \left[ 1 + \frac{1}{3} H(t) \int_{t_i}^t \frac{q(k, t')^2 dt'}{H(t')^2} \right] + B_m(k, t_i) \frac{H(t)}{H(t_i)}, \\
R_{\delta \rho_r}(k, t) &= \frac{R_{\delta \rho_r}(k, t_i) \cos \int_{t_i}^t \frac{q(k, t') dt'}{\sqrt{3}} + \dot{R}_{\delta \rho_r}(k, t_i) \sqrt{3} \frac{q(k, t_i)}{q(k, t_i)} \sin \int_{t_i}^t \frac{q(k, t') dt'}{\sqrt{3}}}{\sqrt{3} \frac{q(k, t')}{q(k, t')}} F(k, t') + \int_{t_i}^t dt' \sqrt{3} \frac{q(k, t')}{q(k, t')} \sin \left( \frac{\int_{t_i}^t \frac{q(k, t'') dt''}{\sqrt{3}}}{\sqrt{3}} \right) F(k, t')
\end{align*}
\]

(3.11) (3.12)
with

\[ F(k, t) = \rho_m(t) \left[ \frac{1}{3} A_m(k, t_i) H(t) \int_{t_i}^{t} \frac{q(k, t')^2 dt'}{H(t')^2} + B_m(k, t_i) \frac{H(t)}{H(t_i)} \right], \tag{3.13} \]

where

\[ A_m(k, t_i) = \rho_m(t_i) \frac{\mathcal{R}_{\delta \rho_m}(k, t_i) + 2 H(t_i) \dot{\mathcal{R}}_{\delta \rho_m}(k, t_i)}{\rho_m(t_i) + \frac{2}{3} q(k, t_i)^2}, \tag{3.14} \]

\[ B_m(k, t_i) = \frac{2}{3} q(k, t_i) \mathcal{R}_{\delta \rho_m}(k, t_i) - 2 H(t_i) \dot{\mathcal{R}}_{\delta \rho_m}(k, t_i). \tag{3.15} \]

Defining the curvature perturbation on constant density hypersurfaces and the entropy perturbation

\[ \mathcal{R}_{\delta \rho} = \frac{\dot{\rho}_m}{\rho} \mathcal{R}_{\delta \rho_m} + \frac{\dot{\rho}_r}{\rho} \mathcal{R}_{\delta \rho_r} \simeq \mathcal{R}_{\delta \rho_m}, \tag{3.16} \]

\[ S_{sr} \equiv \mathcal{R}_{\delta \rho_r} - \mathcal{R}_{\delta \rho_m}, \tag{3.17} \]

respectively, eqs. (3.8) and (3.9) are equivalent to

\[ \ddot{\mathcal{R}}_{\delta \rho} + H \left( 2 + \frac{\rho_m}{\rho_m + \frac{2}{3} q^2} \right) \dot{\mathcal{R}}_{\delta \rho} - \frac{1}{3} q^2 \left( \frac{\rho_m}{\rho_m + \frac{2}{3} q^2} \right) \mathcal{R}_{\delta \rho} = 0, \tag{3.18} \]

\[ \ddot{S}_{sr} + H \dot{S}_{sr} + \frac{1}{3} q^2 S_{sr} = G, \tag{3.19} \]

where

\[ G = \frac{2}{3} q^2 \left( H \dot{\mathcal{R}}_{\delta \rho} - \frac{1}{3} q^2 \mathcal{R}_{\delta \rho} \right), \tag{3.20} \]

which have solution

\[ \mathcal{R}_{\delta \rho}(k, t) = A_m(k, t_i) \left[ 1 + \frac{1}{3} H(t) \int_{t_i}^{t} \frac{q(k, t')^2 dt'}{H(t')^2} \right] + B_m(k, t_i) \frac{H(t)}{H(t_i)}, \tag{3.21} \]

\[ S_{sr}(k, t) = \left[ \mathcal{R}_{\delta \rho_m}(k, t_i) - \mathcal{R}_{\delta \rho_m}(k, t_i) \right] \cos \int_{t_i}^{t} \frac{q(k, t') dt'}{\sqrt{3}} + \left[ \mathcal{R}_{\delta \rho_m}(k, t_i) - \mathcal{R}_{\delta \rho_m}(k, t_i) \right] \frac{\sqrt{3}}{q(k, t_i)} \sin \int_{t_i}^{t} \frac{q(k, t') dt'}{\sqrt{3}} + \int_{t_i}^{t} dt' \frac{\sqrt{3}}{q(k, t')} \sin \left( \int_{t'}^{t} \frac{q(k, t'') dt''}{\sqrt{3}} \right) G(k, t') \tag{3.22} \]

with

\[ G(k, t) = -\frac{1}{3} q(k, t)^2 \left[ \frac{1}{3} A_m(k, t_i) H(t) \int_{t_i}^{t} \frac{q(k, t')^2 dt'}{H(t')^2} + B_m(k, t_i) \frac{H(t)}{H(t_i)} \right]. \tag{3.23} \]

Thermal inflation ends when the temperature of the radiation drops to a critical temperature, given by eq. (2.15), triggering a first order phase transition converting the vacuum energy $V_0$ into flaton matter. The transition hypersurface matches to constant radiation density.
hypersurfaces before the transition and constant density hypersurfaces after the transition. Thus the late time perturbation is given by identifying

$$R_{\delta \rho} \left( k, t_f^+ \right) = R_{\delta \rho} \left( k, t_f^- \right).$$

(3.24)

Using eq. (3.12) and taking \( t_i < t_c < t_f \) gives

$$R_{\delta \rho} (k, t_i) = R_{\delta \rho} (r, t_i) \cos \left[ \int_{t_i}^{t_f} \frac{q(k, t) dt}{\sqrt{3}} + \frac{\sqrt{3} \dot{R}_{\delta \rho} (k, t_i)}{q(k, t_i)} \sin \int_{t_i}^{t_f} \frac{q(k, t) dt}{\sqrt{3}} \right]$$

$$+ \frac{1}{3} A_m (k, t_i) \int_{t_i}^{t_f} dt \frac{\sqrt{3} H(t) \rho_m(t)}{q(k, t)} \int_{t_i}^{t_f} \frac{q(k, t')^2 dt'}{H(t')^2} \sin \int_{t_i}^{t_f} \frac{q(k, t') dt'}{\sqrt{3}}$$

$$+ \frac{B_m (k, t_i)}{H(t_i)} \int_{t_i}^{t_f} dt \frac{\sqrt{3} H(t) \rho_m(t)}{q(k, t)} \sin \int_{t_i}^{t_f} \frac{q(k, t') dt'}{\sqrt{3}}. \quad (3.25)$$

The scale

$$k_b \equiv a(t_b) H(t_b),$$

(3.26)

discussed in section 2, is of central importance in this paper, and we precisely define \( t_b \), the boundary between moduli domination and thermal inflation, by

$$\ddot{a}(t_b) \equiv 0$$

(3.27)

or equivalently, using eq. (3.7),

$$\rho_m(t_b) \simeq 2V_0.$$

(3.28)

Defining

$$\alpha \equiv \frac{a}{a_b},$$

(3.29)

$$\kappa \equiv \frac{k}{k_b},$$

(3.30)

and using eqs. (3.7) and (3.28) gives

$$3H^2 \simeq V_0 + \rho_m = V_0 \left( 1 + \frac{2}{\alpha^3} \right)$$

(3.31)

and eq. (3.25) becomes

$$R_{\delta \rho} (\kappa, \alpha) = R_{\delta \rho} (\kappa, \alpha) \cos \left[ \kappa \int_{\alpha_i}^{\alpha_f} \frac{d\alpha}{\sqrt{\alpha(2 + \alpha^3)}} \right]$$

$$+ \frac{1}{\kappa} \left[ \frac{2 + \alpha^3}{\alpha_i} \right]^{\frac{1}{2}} \frac{dR_{\delta \rho} (\kappa, \alpha)}{d \ln \alpha} (\kappa, \alpha) \sin \left[ \kappa \int_{\alpha_i}^{\alpha_f} \frac{d\alpha}{\sqrt{\alpha(2 + \alpha^3)}} \right]$$

$$+ 6\kappa A_m (\kappa, \alpha) \int_{\alpha_i}^{\alpha_f} \frac{d\alpha}{\alpha^3} \int_{\alpha_i}^{\alpha} d\beta \left( \frac{\beta}{2 + \beta^3} \right)^{\frac{1}{2}} \sin \left[ \kappa \int_{\alpha}^{\alpha_f} \frac{d\gamma}{\sqrt{\gamma(2 + \gamma^3)}} \right]$$

$$+ 6 \frac{1}{\kappa} \left( \frac{\alpha^3}{2 + \alpha^3} \right)^{\frac{1}{2}} B_m (\kappa, \alpha) \int_{\alpha_i}^{\alpha_f} \frac{d\alpha}{\alpha^3} \sin \left[ \kappa \int_{\alpha}^{\alpha_f} \frac{d\beta}{\sqrt{\beta(2 + \beta^3)}} \right], \quad (3.32)$$

(3.32)
where

\[ A_m(\kappa, \alpha_i) = \frac{1}{1 + \frac{3}{2} \kappa^2 \alpha_i} \left[ \mathcal{R}_{\delta \rho m}(\kappa, \alpha_i) + \frac{1}{3} \left( 2 + \alpha_i^3 \right) \frac{d \mathcal{R}_{\delta \rho m}(\kappa, \alpha_i)}{d \ln \alpha} \right] , \quad (3.33) \]

\[ B_m(\kappa, \alpha_i) = \frac{1}{1 + \frac{3}{2} \kappa^2 \alpha_i} \left[ \frac{1}{3} \kappa^2 \alpha_i \mathcal{R}_{\delta \rho m}(\kappa, \alpha_i) - \frac{1}{3} \left( 2 + \alpha_i^3 \right) \frac{d \mathcal{R}_{\delta \rho m}(\kappa, \alpha_i)}{d \ln \alpha} \right] . \quad (3.34) \]

Taking \( t_i \ll t_b \) and \( q_i \ll H_i \), i.e. \( \alpha_i \ll 1 \) and \( \kappa^2 \alpha_i \ll 1 \), gives

\[ \mathcal{R}_{\delta \rho}(\kappa, \alpha_i) = \mathcal{R}_{\delta \rho}(\kappa, \alpha_i) \cos \left[ \kappa \int_{\alpha_i}^{\alpha_f} \frac{d \alpha}{\sqrt{\alpha(2 + \alpha^3)}} \right] + 6 \kappa A_m(\kappa, \alpha_i) \int_{\alpha_i}^{\alpha_f} \frac{d \alpha}{\alpha^3} \sin \left[ \kappa \int_{\alpha_i}^{\alpha_f} \frac{d \beta}{\sqrt{\beta(2 + \beta^3)}} \right] , \quad (3.35) \]

where

\[ A_m(\kappa, \alpha_i) = \mathcal{R}_{\delta \rho m}(\kappa, \alpha_i) + \frac{2}{3} \frac{d \mathcal{R}_{\delta \rho m}(\kappa, \alpha_i)}{d \ln \alpha} , \quad (3.36) \]

\[ B_m(\kappa, \alpha_i) = \frac{1}{3} \kappa^2 \alpha_i \mathcal{R}_{\delta \rho m}(\kappa, \alpha_i) - \frac{2}{3} \frac{d \mathcal{R}_{\delta \rho m}(\kappa, \alpha_i)}{d \ln \alpha} . \quad (3.37) \]

### 3.1 Thermal inflation transfer function

If we take an adiabatic initial condition

\[ S_{mr}(k, t_i) = \dot{S}_{mr}(k, t_i) = 0 \quad (3.38) \]

so that

\[ \mathcal{R}_{\delta \rho m}(k, t_i) = \mathcal{R}_{\delta \rho}(k, t_i) = \mathcal{R}_{\delta \rho}(k, t_i) , \quad (3.39) \]

\[ \dot{\mathcal{R}}_{\delta \rho m}(k, t_i) = \dot{\mathcal{R}}_{\delta \rho}(k, t_i) = \dot{\mathcal{R}}_{\delta \rho}(k, t_i) \quad (3.40) \]

neglect the decaying mode, \( B_m(k, t_i) = 0 \), and take \( t_i \to 0 \) and \( t_f \to \infty \), the effect of the thermal inflation era on the curvature perturbation can be expressed as the transfer function

\[ \mathcal{T}(\kappa) \equiv \frac{\mathcal{R}_{\delta \rho}(\kappa, \infty)}{\mathcal{R}_{\delta \rho}(\kappa, 0)} , \quad (3.41) \]

which, from eq. (3.35), is given by

\[ \mathcal{T}(\kappa) = \cos \left[ \kappa \int_0^\infty \frac{d \alpha}{\sqrt{\alpha(2 + \alpha^3)}} \right] + 6 \kappa \int_0^\infty \frac{d \gamma}{\gamma^3} \int_0^\gamma d \beta \left( \frac{\beta}{2 + \beta^3} \right)^{\frac{3}{2}} \sin \left[ \kappa \int_\gamma^\infty \frac{d \alpha}{\sqrt{\alpha(2 + \alpha^3)}} \right] . \quad (3.42) \]
\[ T(\kappa) \equiv k/k_b \]

![Figure 2](image.png)

**Figure 2.** The thermal inflation transfer function \( T(\kappa) \), where \( \kappa \equiv k/k_b \) and the characteristic scale \( k_b \) is defined by eq. (3.27) and estimated in eq. (2.48). The first peak is at \( (\kappa, T) \simeq (1.13, 1.21) \) and the first dip is at \( (3.11, 0.256) \). The analytic form is given in eq. (3.42) and the asymptotic behaviours are given in eqs. (3.43) and (3.45).

and is plotted in figure 2. On large scales, which remain outside the horizon, the transfer function asymptotes to one

\[
T(\kappa) \xrightarrow{\kappa \to 0} 1 + \nu_0 \kappa^2 + O(\kappa^4),
\]

where, using eq. (B.1),

\[
\nu_0 = \int_0^\infty d\alpha \left( \frac{\alpha}{2 + \alpha^3} \right)^{3/2} = \frac{2^{7/3} \pi^{3/2}}{3^{1/2} \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{1}{3}\right)} \simeq 0.3622.
\]

On small scales, which enter well into the horizon during moduli domination and thermal inflation, the transfer function oscillates due to the oscillation of the radiation perturbation inside the horizon

\[
T(\kappa) \xrightarrow{\kappa \to \infty} -\frac{1}{5} \cos(\nu_1 \kappa) + o(\kappa^{-n}),
\]

where we have used eq. (B.4) and

\[
\nu_1 = \int_0^\infty \frac{d\alpha}{\sqrt{\alpha(2 + \alpha^3)}} = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{4}\right)}{2^{1/3} 3^{1/2} \sqrt{\pi}} \simeq 2.2258.
\]
Figure 3. The thermal inflation transfer function \( T^2(\kappa) \).

The power spectrum after thermal inflation is

\[
P(k) = T^2(k/k_b) \times P_{\text{pri}}(k),
\]

where \( P_{\text{pri}}(k) \) is the power spectrum of the primordial inflation, see figure 3. Hence, the power spectrum is enhanced by a factor 1.46 at the first peak and, from eq. (3.45), suppressed by a factor 50 on small scales.

4 Discussion

In this paper, we presented the effect of thermal inflation on the linear evolution of small scale density perturbations. By introducing thermal inflation, the post-inflationary history is changed to include extra eras of moduli matter domination, thermal inflation and flaton matter domination, followed by the usual radiation domination, as in figure 1. Modes with \( k > k_b \) enter the horizon during moduli domination before exiting again during thermal inflation, and hence are modified relative to the modes with \( k < k_b \) which remain outside the horizon. At the end of thermal inflation, the radiation perturbation is converted into the final adiabatic (curvature) perturbation. The net effect of this for an adiabatic primordial perturbation is to suppress the perturbations by a factor \( \sim 50 \) on scales \( k \gg k_b \), see figures 2 and 3, and eq. (3.45).
A multicomponent, i.e. adiabatic (curvature) plus entropy (isocurvature), primordial perturbation leads to a more complicated, initial condition dependent, result given in eq. (3.35). Even for modes with \( k < k_b \), which remain outside the horizon, the case with thermal inflation can lead to different results compared to without thermal inflation as it is the subdominant radiation perturbation, rather than the dominant moduli matter perturbation, that is converted to the final adiabatic perturbation.

The scale \( k_b \) is theoretically estimated in eq. (2.48), but the uncertainties are large, with anything from megaparsec to subparsec scales being reasonable, and no robust upper or lower bound. Current and future observations could provide stronger constraints, or an observational signature of thermal inflation. For example, the lack of small scale suppression in the primordial power spectrum reconstructed from the recent Planck results [46] suggests \( k_b \gtrsim 1 \text{ Mpc}^{-1} \).

The suppression of the power spectrum would reduce the number density of dark matter clumps of mass

\[
M \lesssim \frac{4 \pi}{3} \rho_m(t_0) \left( \frac{2 \pi a_0}{3 k_b} \right)^3 \sim 10^{11} M_\odot \left( \frac{k_b}{\text{Mpc}^{-1}} \right)^{-3},
\]

where the scale \( 3k_b \) is estimated from figure 3. Depending on the value of \( k_b \), this might be able to be seen from 21cm hydrogen line observations such as the Square Kilometre Array (SKA), see for example [37], or gamma-ray observations of WIMP-annihilation in, or gravitational microlensing of, ultracompact minihalos, see for example [26].

Silk damping might be relevant when radiation perturbations enter the horizon during either \( t_a < t < t_c \) or \( t > t_d \). For \( t_a < t < t_c \), we estimate the Silk damping scale to be much smaller than our characteristic scale \( k_b \) [47]. For \( t > t_d \), modes near \( k_b \) can be affected by Silk damping. For \( 50 \text{ Mpc}^{-1} \lesssim k_b \lesssim 10^4 \text{ Mpc}^{-1} \), the dissipation of modes with \( k \sim k_b \) would generate a unique pattern of CMB distortions, different from [29–31, 45], due to the suppression and oscillations of the thermal inflation transfer function. Future observations such as the Primordial Inflation Explorer (PIXIE) [34] or the Polarized Radiation Imaging and Spectroscopy Mission (PRISM) [35, 36] could be used to probe these distortions [45]. For \( 10^4 \text{ Mpc}^{-1} \lesssim k_b \lesssim 10^5 \text{ Mpc}^{-1} \), the suppression of the power spectrum could diminish any effect of the release of energy by Silk damping on Big Bang nucleosynthesis [47].

Acknowledgments

The authors thank Donghui Jeong, Ido Ben-Dayan, Wan-il Park, Richard Easther, Chang Sub Shin, Ki-Young Choi, Kyungjin Ahn, Bayram Tekin, Dong-han Yeom and Kyujin Kwak for useful discussions at various stages of this work. HZ thanks KAIST, APCTP and METU for their hospitality. This work was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (N01110095 and N01130488).

A Review of perturbations in a multicomponent system

The scalar parts of the perturbed metric and energy momentum tensor are [48]

\[
\ddot{\tilde{s}}^2 = (1 + 2A) dt^2 - 2B_i dt dx^i - \left[ (1 + 2\mathcal{R}) a^2(t) \delta_{ij} + 2C_{ij} \right] dx^i dx^j \quad (A.1)
\]

and

\[
\tilde{T}_{\mu\nu} = \tilde{\rho} \tilde{u}_\mu \tilde{u}_\nu - \tilde{p} (\tilde{g}_{\mu\nu} - \tilde{u}_\mu \tilde{u}_\nu) + \tilde{\pi}_{\mu\nu}, \quad (A.2)
\]
where $\tilde{u}^0 = 1/a$ and
\[
\tilde{p} = \rho + \delta \rho, \quad \tilde{u}^i = \frac{1}{a^2} \partial_i v, \quad \tilde{u}^0 = \frac{1}{a^2} \partial_0 v, \quad \tilde{\pi}_{\mu\nu} = \partial_\mu \partial_\nu \pi.
\]

The Einstein equation gives
\[
3H \left( \dot{R} - HA \right) + q^2 \left[ R - H \left( \dot{C} - 2HC - B \right) \right] = \frac{1}{2} \delta \rho, \quad (A.4)
\]
\[
\dot{R} - HA = -\dot{H} \left( v + B \right), \quad (A.5)
\]
\[
\frac{d}{dt} \left( \dot{C} - 2HC - B \right) + H \left( \dot{C} - 2HC - B \right) - R - A = \pi, \quad (A.6)
\]
\[
\frac{d}{dt} \left( \dot{R} - HA \right) + 3H \left( \dot{R} - HA \right) - \dot{H}A = -\frac{1}{2} \delta p + \frac{1}{3} q^2 \pi, \quad (A.7)
\]

where $q \equiv k/a$. Taking the derivatives of eqs. (A.4) and (A.5) gives
\[
\dot{\delta \rho} + 3H \left( \delta \rho + \delta p \right) + 2q^2 \dot{H} \left( v + B \right) - 2\dot{H} \left[ 3\dot{R} - q^2 \left( \dot{C} - 2HC - B \right) \right] = 0, \quad (A.8)
\]
\[
\frac{d}{dt} \left[ \dot{H} \left( v + B \right) \right] + 3H\dot{H} \left( v + B \right) - \frac{1}{2} \delta p + \frac{1}{3} q^2 \pi + \dot{H}A = 0 \quad (A.9)
\]
corresponding to $\nabla \cdot T = 0$. Decomposing into components
\[
\delta \rho = \sum_X \delta \rho_X, \quad (\rho + p) v = \sum_X (\rho_X + p_X) v_X, \quad \delta \rho = \sum_X \delta p_X, \quad \pi = \sum_X \pi_X \quad (A.10)
\]
gives
\[
\sum_X \left\{ \delta \rho_X + 3H \left( \delta \rho_X + \delta p_X \right) - q^2 \left( \rho_X + p_X \right) \left( v_X + B \right) + (\rho_X + p_X) \left[ 3\dot{R} - q^2 \left( \dot{C} - 2HC - B \right) \right] \right\} = 0, \quad (A.11)
\]
\[
\sum_X \left\{ \frac{d}{dt} \left[ (\rho_X + p_X) \left( v_X + B \right) \right] + 3H \left( \rho_X + p_X \right) \left( v_X + B \right) + \delta p_X - \frac{2}{3} q^2 \pi_X + (\rho_X + p_X) A \right\} = 0 \quad (A.12)
\]
corresponding to $\sum_X \nabla \cdot T_X = 0$.

For components which couple only gravitationally, we have $\nabla \cdot T_X = 0$ separately
\[
\delta \rho_X + 3H \left( \delta \rho_X + \delta p_X \right) - q^2 \left( \rho_X + p_X \right) \left( v_X + B \right) + (\rho_X + p_X) \left[ 3\dot{R} - q^2 \left( \dot{C} - 2HC - B \right) \right] = 0, \quad (A.13)
\]
\[
\frac{d}{dt} \left[ (\rho_X + p_X) \left( v_X + B \right) \right] + 3H \left( \rho_X + p_X \right) \left( v_X + B \right) + \delta p_X - \frac{2}{3} q^2 \pi_X + (\rho_X + p_X) A = 0. \quad (A.14)
\]

For
\[
\delta p_X = \frac{\dot{p}_X}{\dot{\rho}_X} \delta \rho_X, \quad (A.15)
\]
\[
\pi_X = 0, \quad (A.16)
\]
combining eqs. (A.13) and (A.14) and using eqs. (A.4) to (A.7) gives
\[
\frac{d^2}{ds^2} \left( R - \frac{H}{\rho} \delta \rho_X \right) + H \left( 2 - 3 \frac{\dot{\rho}_X}{\rho} \right) \frac{d}{ds} \left( R - \frac{H}{\rho} \delta \rho_X \right) + q^2 \frac{\dot{\rho}_X}{\rho} \left( R - \frac{H}{\rho} \delta \rho_X \right) \\
= \frac{1}{3} \left( \frac{1 + 3 \frac{\dot{\rho}_X}{\rho}}{\rho + p + \frac{2}{3} q^2} \right) \left[ \rho \frac{d}{ds} \left( R - \frac{H}{\rho} \delta \rho \right) + q^2 (\rho + p) \left( R - \frac{H}{\rho} \delta \rho \right) + 3H^2 \frac{\dot{\rho}}{\rho} \left( \frac{\delta \rho}{\rho} - \frac{\delta \rho}{\rho} \right) \right] \\
\text{(A.17)}
\]

with
\[
R - \frac{H}{\rho} \delta \rho = \sum_X \frac{\dot{\rho}_X}{\rho} \left( R - \frac{H}{\rho} \delta \rho_X \right), \\
\text{(A.18)}
\]
\[
H \left( \frac{\delta \rho}{\rho} - \frac{\delta \rho}{\rho} \right) = - \sum_X \left( \frac{\dot{\rho}_X}{\rho} - \frac{\dot{\rho}_X}{\rho} \right) \left( R - \frac{H}{\rho} \delta \rho_X \right). \\
\text{(A.19)}
\]

For two components, X and Y, eq. (A.17) gives
\[
\ddot{R} - \frac{H}{\rho} \delta \rho_X + H \left[ 3 - \left( 1 + 3 \frac{\dot{\rho}_X}{\rho} \right) \left( \frac{\rho_Y + p_Y + \frac{2}{3} q^2}{\rho + p + \frac{2}{3} q^2} \right) \right] \ddot{R} \delta \rho_X - \frac{1}{3} q^2 \left[ 1 - \left( 1 + 3 \frac{\dot{\rho}_X}{\rho} \right) \left( \frac{\rho_Y + p_Y + \frac{2}{3} q^2}{\rho + p + \frac{2}{3} q^2} \right) \right] \dot{R} \delta \rho_X \\
= - (1 + 3 \frac{\dot{\rho}_X}{\rho}) \left( \frac{\rho_Y + p_Y}{\rho + p + \frac{2}{3} q^2} \right) \left( H \ddot{R} \delta \rho_Y - \frac{1}{3} q^2 \ddot{R} \delta \rho_Y \right), \\
\text{(A.20)}
\]

where
\[
R \delta \rho_X \equiv R - \frac{H}{\rho} \delta \rho_X. \\
\text{(A.21)}
\]

B Mathematical formulae

Integrating by parts:
\[
\begin{align*}
6 \int_0^\infty \frac{d\gamma}{\gamma^3} \int_0^\gamma d\beta \left( \frac{\beta}{2 + \beta^3} \right)^{\frac{3}{2}} \int_0^\infty \frac{d\alpha}{\sqrt{\alpha(2 + \alpha^3)}} & = 3 \int_0^\infty \frac{d\gamma}{\gamma^{1/2} (2 + \gamma^{3/2})} \int_0^\infty \frac{d\alpha}{\sqrt{\alpha(2 + \alpha^3)}} - 3 \int_0^\infty \frac{d\gamma}{\gamma^{5/2} (2 + \gamma^{3/2})} \int_0^\gamma d\beta \left( \frac{\beta}{2 + \beta^3} \right)^{\frac{3}{2}} \\
& = \frac{1}{2} \left[ \int_0^\infty \frac{d\alpha}{\sqrt{\alpha(2 + \alpha^3)}} \right]^2 + \int_0^\infty d\alpha \left( \frac{\alpha}{2 + \alpha^3} \right)^{\frac{3}{2}}, \\
\text{(B.1)}
\end{align*}
\]

where we have used
\[
\int \frac{3 \gamma^{1/2} d\gamma}{(2 + \gamma^{3/2})^{5/2}} = \frac{\gamma^{3/2}}{(2 + \gamma^{3/2})^{1/2}} \\
\text{(B.2)}
\]

and
\[
\int \frac{3 d\gamma}{\gamma^{5/2} (2 + \gamma^{3/2})^{1/2}} = - \frac{(2 + \gamma^{3})^{1/2}}{\gamma^{3/2}}. \\
\text{(B.3)}
\]
Integrating by parts:

\[
6 \kappa \int_0^\infty \frac{d\gamma}{\gamma^3} \sin \left[ \kappa \int_\gamma^\infty \frac{d\alpha}{\sqrt{\alpha(2 + \alpha^3)}} \right] \int_0^\gamma d\beta \left( \frac{\beta}{2 + \beta^3} \right)^{\frac{3}{2}}
\]

\[
= -\frac{6}{5} \cos \left[ \kappa \int_0^\infty \frac{d\alpha}{\sqrt{\alpha(2 + \alpha^3)}} \right] - 9 \int_0^\infty \frac{d\gamma}{\sqrt{\gamma(2 + \gamma^3)}} \cos \left[ \kappa \int_\gamma^\infty \frac{d\alpha}{\sqrt{\alpha(2 + \alpha^3)}} \right]
\]

\[
\times \left[ \int_0^\gamma d\beta \left( \frac{\beta}{2 + \beta^3} \right)^{\frac{3}{2}} - 3 \left( \frac{2 + \gamma^3}{\gamma^3} \right) \int_0^\gamma d\beta \left( \frac{\beta^{9/2}}{(2 + \beta^3)^{5/2}} \right) \right]
\]

\[
= -\frac{6}{5} \cos \left[ \kappa \int_0^\infty \frac{d\alpha}{\sqrt{\alpha(2 + \alpha^3)}} \right]
\]

\[
+ \frac{18}{\kappa} \int_0^\infty d\gamma \sin \left[ \kappa \int_\gamma^\infty \frac{d\alpha}{\sqrt{\alpha(2 + \alpha^3)}} \right] \left[ \left( \frac{\gamma}{2 + \gamma^3} \right)^{\frac{3}{2}} - \frac{9}{\gamma^3} \int_0^\gamma d\beta \frac{\beta^{9/2}}{(2 + \beta^3)^{5/2}} \right]. \tag{B.4}
\]

References

[1] D.H. Lyth and E.D. Stewart, *Cosmology with a TeV mass GUT Higgs*, Phys. Rev. Lett. 75 (1995) 201 [hep-ph/9502417] [inSPIRE].

[2] D.H. Lyth and E.D. Stewart, *Thermal inflation and the moduli problem*, Phys. Rev. D 53 (1996) 1784 [hep-ph/9510204] [inSPIRE].

[3] K. Yamamoto, *Saving the Axions in Superstring Models*, Phys. Lett. B 161 (1985) 289 [inSPIRE].

[4] K. Yamamoto, *Phase Transition Associated With Intermediate Gauge Symmetry Breaking in Superstring Models*, Phys. Lett. B 168 (1986) 341 [inSPIRE].

[5] K. Enqvist, D.V. Nanopoulos and M. Quirós, *Cosmological Difficulties for Intermediate Scales in Superstring Models*, Phys. Lett. B 169 (1986) 343 [inSPIRE].

[6] O. Bertolami and G.G. Ross, *Inflation as a Cure for the Cosmological Problems of Superstring Models With Intermediate Scale Breaking*, Phys. Lett. B 183 (1987) 163 [inSPIRE].

[7] J.R. Ellis, K. Enqvist, D.V. Nanopoulos and K.A. Olive, *Comments On Intermediate Scale Models*, Phys. Lett. B 188 (1987) 415 [inSPIRE].

[8] J.R. Ellis, K. Enqvist, D.V. Nanopoulos and K.A. Olive, *Can Q Balls Save the Universe From Intermediate Scale Phase Transitions?*, Phys. Lett. B 225 (1989) 313 [inSPIRE].

[9] L. Randall and S.D. Thomas, *Solving the cosmological moduli problem with weak scale inflation*, Nucl. Phys. B 449 (1995) 229 [hep-ph/9407248] [inSPIRE].

[10] G.D. Coughlan, W. Fischler, E.W. Kolb, S. Raby and G.G. Ross, *Cosmological Problems for the Polonyi Potential*, Phys. Lett. B 131 (1983) 59 [inSPIRE].

[11] T. Banks, D.B. Kaplan and A.E. Nelson, *Cosmological implications of dynamical supersymmetry breaking*, Phys. Rev. D 49 (1994) 779 [hep-ph/9308292] [inSPIRE].

[12] B. de Carlos, J.A. Casas, F. Quevedo and E. Roulet, *Model independent properties and cosmological implications of the dilaton and moduli sectors of 4 − D strings*, Phys. Lett. B 318 (1993) 447 [hep-ph/9308325] [inSPIRE].

[13] E.D. Stewart, M. Kawasaki and T. Yanagida, *Affleck-Dine baryogenesis after thermal inflation*, Phys. Rev. D 54 (1996) 6032 [hep-ph/9603324] [inSPIRE].
[14] D.-h. Jeong, K. Kadota, W.-I. Park and E.D. Stewart, Modular cosmology, thermal inflation, baryogenesis and predictions for particle accelerators, JHEP 11 (2004) 046 [hep-ph/0406136] [inSPIRE].
[15] G.N. Felder, H. Kim, W.-I. Park and E.D. Stewart, Preheating and Affleck-Dine leptogenesis after thermal inflation, JCAP 06 (2007) 005 [hep-ph/0703275] [inSPIRE].
[16] S. Kim, W.-I. Park and E.D. Stewart, Thermal inflation, baryogenesis and axions, JHEP 01 (2009) 015 [arXiv:0807.3607] [inSPIRE].
[17] M. Kawasaki and K. Nakayama, Late-time Affleck-Dine baryogenesis after thermal inflation, Phys. Rev. D 74 (2006) 123508 [hep-ph/0608335] [inSPIRE].
[18] G. Lazarides, C. Panagiotakopoulos and Q. Shafi, Baryogenesis and the Gravitino Problem in Superstring Models, Phys. Rev. Lett. 56 (1986) 557 [inSPIRE].
[19] K. Yamamoto, A Model for Baryogenesis in Superstring Unification, Phys. Lett. B 194 (1987) 390 [inSPIRE].
[20] R.N. Mohapatra and J.W.F. Valle, Late Baryogenesis in Superstring Models, Phys. Lett. B 186 (1987) 303 [inSPIRE].
[21] L.E. Mendes and A.R. Liddle, Thermal inflation and the stochastic gravitational wave background, gr-qc/9811040 [inSPIRE].
[22] R. Easther, J.T. Giblin Jr., E.A. Lim, W.-I. Park and E.D. Stewart, Thermal Inflation and the Gravitational Wave Background, JCAP 05 (2008) 013 [arXiv:0801.4197] [inSPIRE].
[23] A. Kosowsky, M.S. Turner and R. Watkins, Gravitational radiation from colliding vacuum bubbles, Phys. Rev. D 45 (1992) 4514 [inSPIRE].
[24] B.J. Carr, The Primordial black hole mass spectrum, Astrophys. J. 201 (1975) 1 [inSPIRE].
[25] A.S. Josan, A.M. Green and K.A. Malik, Generalised constraints on the curvature perturbation from primordial black holes, Phys. Rev. D 79 (2009) 103520 [arXiv:0903.3184] [inSPIRE].
[26] T. Bringmann, P. Scott and Y. Akrami, Improved constraints on the primordial power spectrum at small scales from ultracompact minihalos, Phys. Rev. D 85 (2012) 125027 [arXiv:1110.2484] [inSPIRE].
[27] I. Ben-Dayan and T. Kalaydzhyan, Constraining the primordial power spectrum from SNIa lensing dispersion, Phys. Rev. D 90 (2014) 083509 [arXiv:1309.4771] [inSPIRE].
[28] J. Chluba and R.A. Sunyaev, The evolution of CMB spectral distortions in the early Universe, Mon. Not. Roy. Astron. Soc. 419 (2012) 1294 [arXiv:1109.6552] [inSPIRE].
[29] R. Khatri, R.A. Sunyaev and J. Chluba, Does Bose-Einstein condensation of CMB photons cancel μ distortions created by dissipation of sound waves in the early Universe?, Astron. Astrophys. 540 (2012) A124 [arXiv:1110.0475] [inSPIRE].
[30] J. Chluba, R. Khatri and R.A. Sunyaev, CMB at 2x2 order: The dissipation of primordial acoustic waves and the observable part of the associated energy release, Mon. Not. Roy. Astron. Soc. 425 (2012) 1129 [arXiv:1202.0057] [inSPIRE].
[31] R. Khatri and R.A. Sunyaev, Forecasts for CMB μ and i-type spectral distortion constraints on the primordial power spectrum on scales 8 \lesssim k \lesssim 10^4 Mpc^{-1} with the future Pixie-like experiments, JCAP 06 (2013) 026 [arXiv:1303.7212] [inSPIRE].
[32] A. Cooray, 21-cm Background Anisotropies Can Discern Primordial Non-Gaussianity, Phys. Rev. Lett. 97 (2006) 261301 [astro-ph/0610257] [inSPIRE].
[33] Y. Mao, M. Tegmark, M. McQuinn, M. Zaldarriaga and O. Zahn, How accurately can 21 cm tomography constrain cosmology?, Phys. Rev. D 78 (2008) 023529 [arXiv:0802.1710] [inSPIRE].
A. Kogut, D.J. Fixsen, D.T. Chuss, J. Dotson, E. Dwek et al., *The Primordial Inflation Explorer (PIXIE): A Nulling Polarimeter for Cosmic Microwave Background Observations*, JCAP 07 (2011) 025 [arXiv:1105.2044] [inSPIRE].

PRISM collaboration, P. Andre et al., *PRISM (Polarized Radiation Imaging and Spectroscopy Mission): A White Paper on the Ultimate Polarimetric Spectro-Imaging of the Microwave and Far-Infrared Sky*, arXiv:1306.2259 [inSPIRE].

PRISM collaboration, P. André et al., *PRISM (Polarized Radiation Imaging and Spectroscopy Mission): An Extended White Paper*, JCAP 02 (2014) 006 [arXiv:1310.1554] [inSPIRE].

S. Furlanetto, A. Lidz, A. Loeb, M. McQuinn, J. Pritchard et al., *Cosmology from the Highly-Redshifted 21 cm Line*, arXiv:0902.3259 [inSPIRE].

E.B. Gliner, *Algebraic Properties of the Energy-momentum Tensor and Vacuum-like States of Matter*, Zh. Eksp. Teor. Fiz. 49 (1966) 542 [Sov. Phys. JETP 22 (1966) 378].

E.B. Gliner, *The vacuum-like state of a medium and Friedman cosmology*, Sov. Phys. Dokl. 15 (1970) 559.

A.H. Guth, *The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems*, Phys. Rev. D 23 (1981) 347 [inSPIRE].

A.D. Linde, *A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems*, Phys. Lett. B 108 (1982) 389 [inSPIRE].

A. Albrecht and P.J. Steinhardt, *Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking*, Phys. Rev. Lett. 48 (1982) 1220 [inSPIRE].

M. Kawasaki, K. Kohri and T. Moroi, *Big-Bang nucleosynthesis and hadronic decay of long-lived massive particles*, Phys. Rev. D 71 (2005) 083502 [astro-ph/0408426] [inSPIRE].

E.W. Kolb and M.S. Turner, *The Early Universe*, Westview Press, Boulder U.S.A. (1994).

J. Chluba, A.L. Erickcek and I. Ben-Dayan, *Probing the inflaton: Small-scale power spectrum constraints from measurements of the CMB energy spectrum*, Astrophys. J. 758 (2012) 76 [arXiv:1203.2681] [inSPIRE].

Planck collaboration, P.A.R. Ade et al., *Planck 2015 results. XX. Constraints on inflation*, arXiv:1502.02114 [inSPIRE].

D. Jeong, J. Pradler, J. Chluba and M. Kamionkowski, *Silk damping at a redshift of a billion: a new limit on small-scale adiabatic perturbations*, Phys. Rev. Lett. 113 (2014) 061301 [arXiv:1403.3697] [inSPIRE].

H. Kodama and M. Sasaki, *Cosmological Perturbation Theory*, Prog. Theor. Phys. Suppl. 78 (1984) 1 [inSPIRE].