Using Tree Automata and Regular Expressions
to Manipulate Hierarchically Structured Data

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Abstract

Information, stored or transmitted in digital form, is often structured. Individual data records are usually represented as hierarchies of their elements. Together, records form larger structures. Information processing applications have to take account of this structuring, which assigns different semantics to different data elements or records. Big variety of structural schemata in use today often requires much flexibility from applications—for example, to process information coming from different sources. To ensure application interoperability, translators are needed that can convert one structure into another.

This paper puts forward a formal data model aimed at supporting hierarchical data processing in a simple and flexible way. The model is based on and extends results of two classical theories, studying finite string and tree automata. The concept of finite automata and regular languages is applied to the case of arbitrarily structured tree-like hierarchical data records, represented as “structured strings.” These automata are compared with classical string and tree automata; the model is shown to be a superset of the classical models. Regular grammars and expressions over structured strings are introduced.

Regular expression matching and substitution has been widely used for efficient unstructured text processing; the model described here brings the power of this proven technique to applications that deal with information trees. A simple generic alternative is offered to replace today’s specialised ad-hoc approaches. The model unifies structural and content transformations, providing applications with a single data type. An example scenario of how to build applications based on this theory is discussed. Further research directions are outlined.

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Additional Key Words and Phrases: data model, hierarchy, information structure, regular expression, tree automaton

1 Introduction

Information processing has always faced the need to take into account the structure of the data being processed. Structuring of information plays an important role in fostering automated, computerised data capture, storage, search, retrieval, and modification. For example, an unstructured bibliographic reference like ‘Bourbaki, N. Lie Groups and Lie Algebras’ requires either a human assistance or the use of heuristics to determine whether Lie is the name of the author or a part of

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the title. On the other hand, dividing that reference in two parts—‘author’ and ‘title’—from the very beginning would have solved this problem. The ‘author’ part can be further sub-structured into ‘last name’, ‘first initial’ and so on.

The structure of data records depends on the kind of information the records carry—for example, flights schedule, asset list, book, e-mail message, and so on. It is often necessary to convert data between different structured representations, along with more basic tasks such as retrieval of certain components of a record or addition of a new component.

Hierarchical (tree-like) way of organising information is very popular and convenient, because it allows aggregation of details at different granularity levels. Most of the data structures used today either are already hierarchical, or can be expressed using an information tree [1]. XML [2] is being increasingly used as the standard language for representing hierarchically structured data. Areas where structured information is actively utilised include:

- text processing (markup languages);
- information retrieval (document processing, query adaptation);
- compilers (syntax trees);
- library automation (bibliographic records);
- a wide variety of industrial applications.

This paper puts forward a simple and flexible formal data model for manipulating structured information. This model is built on top of a combination of results from finite automata and tree automata theories. The concept of automata and regular languages is applied to the case of arbitrarily structured tree-like hierarchical data records, represented as “structured strings.” The paper compares these automata with classical string and tree automata, showing that the theory presented here is a superset of the classical models: everything that can be done with finite string or tree automata can also be done in our model.

The data manipulation model suggested here is based around tree regular expressions, their matching and substitution. Regular expressions are widely used in non-structured text processing, serving as a core model in many text processing applications [3]. They are good for selecting fragments that match certain patterns in certain contexts. For example, regular expression ‘Figure +([0-9]+)’ selects all figure numbers in a text document (more precisely, it selects all decimal numbers that follow the word ‘Figure’, separated by at least one space).

This paper brings the power of regular expressions to serve hierarchical data manipulation. The notion of regular expressions is extended to information trees in a way that matches most applications’ needs. This offers a single, simple generic solution to many problems where different incompatible ad-hoc approaches have been used before. Although the model is described from a theoretical standpoint, its practical applications are considered, drawing on the experience of using regular expression matching techniques on plain text documents. There is room for further research in terms of both theory and applications.

The rest of the paper is organised as follows. The next section surveys related works and discusses approaches taken previously. Section 3 introduces the data model proposed for representation of information trees, and Section 4 provides its formal definition. Then, Section 5 defines finite tree automata that operate on information trees, and discusses properties of such automata. Section 6 introduces regular grammars and expressions and shows their equivalence to the finite tree automata. Section 7 presents an example application scenario to illustrate how the described theory can be used in practice. Section 8 concludes the paper and outlines further research directions.
2 Related works

2.1 Murata’s forest algebra and tree automata theory

The most recent research on formal tree-structured data models belongs to Makoto Murata, who applied and extended the theory of tree automata [4] to the problem of transformation of SGML/XML documents [5, 6, 7]. In [8] Murata offers a hierarchical data model based on tree automata theory and the work of Podelski on pointed trees [9]. However, Murata uses tree automata for purely representational purposes, as a means to formally define XML schema in his model, rather than as a main processing tool.

The theory of tree automata [4] studies classes of trees, called languages, in a way similar to the formal language theory. A tree language is defined by its syntax, which can, by analogy with the language theory, be described by either a tree grammar or a tree automaton. Murata shows the close relation between SGML or XML document syntax (usually referred to as Document Type Definition, or DTD) and the syntax of its tree representation. Because the classical theory only deals with tree languages with limited branching (the maximum number of branches that any node of any tree of a language may have is a constant determined by the syntax of that language), Murata had to extend the theory to handle unlimited branching, necessary to represent marked-up documents.

In [6], Murata suggests a data model for transformations of hierarchical documents. Although primarily intended to serve the SGML/XML community, the model is a generally applicable forest-based model. Murata’s extended tree automata are used for schema representation, parallel to DTDs in SGML and XML. The core of the model is a forest algebra, containing fourteen operators for selecting and manipulating document forests. This algebra can select document fragments based on patterns (conditions on descendent nodes) and contextual conditions (conditions on non-descendent nodes, such as ancestors, siblings, etc.). One of the strong points of Murata’s data model is that during transformation of documents, syntactical changes are tracked in parallel: operators apply not only to the document being processed, but also to its schema, so that at any stage in a transformation process, the schemata of the intermediate results are known.

2.2 XPath

A different approach to transformations of hierarchically organised documents is taken by the World Wide Web Consortium in their XPath [10] and XSLT [11] specifications. Although designed specifically for the XML document representation format, these recommendations are applicable to a wide variety of other types of documents that can be reasonably translated into XML. The XML Path Language (XPath) provides a common syntax and semantics for addressing parts of an XML document—functionality used by other specifications, such as XSL Transformations (XSLT). XPath also has facilities for manipulating strings, numbers, and Booleans, which support its primary purpose.

XPath’s model of an XML document is that of an ordered tree of nodes, where nodes can be of seven types. The multitude of types is needed to support various XML features, such as namespaces, attributes, and so on. XPath can operate on documents that come with or without a Document Type Definition (DTD). A DTD, when supplied, unlocks some functionality of the XPath processor, such as the ability to find unique IDs of document elements, or to use default attribute values.

XPath is an expression-based language. Expressions evaluate to yield objects of four basic types: node-set, Boolean, number (floating-point), and string. Expressions consist of string and numeric constants, variable references, unary and binary operators, function calls, and special tokens. The specification defines core function library that all XPath implementations must support. The library contains 7 node-set functions, 10 string, 5 Boolean, and 5 numeric functions. Some of the XPath operators and functions, such as +, -, floor, string-length, concat, are of general purpose nature and are typical of traditional programming languages.
3 Hierarchical data model: An informal introduction

The basic data model that we use in this paper was originally introduced in [1]. It was also shown there that all popular information structuring methods can be realised using tree-like structures and expressed by this model. We give here its brief description.

Informally, in the proposed model a document is represented as a finite ordered labelled tree. Each node of the tree is associated with a label: a string over an alphabet (see Figure 1). In the traditional terminology, the labels of leaf, or terminal, nodes of a document are called data elements. They carry the “actual” content of the document. The labels of internal (non-leaf) nodes are referred to as tags, whose purpose is to describe those data elements. When speaking of tags and data elements, we shall often make no distinction between nodes and their labels.

![Figure 1: Labelled tree over alphabet Σ](image)

Tags form the structure of a document, specifying the semantics of their underlying sub-trees (and, ultimately, the data elements). The sequence of tags from the root to a data element is called a tag path and fully identifies the properties and interpretation of that element. Sometimes tag paths are used as keys to extract data elements from documents.

This model has the following properties:

- Unlimited branching: although all trees are finite, the number of children a node may have can be arbitrarily large. By contrast, in the tree automata theory branching is limited and is determined by the tree’s ranked alphabet.

- Unlimited number of possible labels: labels are selected from an infinite set of strings over a finite alphabet.

- Tags and data elements, which are traditionally regarded as belonging to different domains, are built here uniformly from the same alphabet. This allows the use of the same operators and mechanisms for both leaves and internal nodes. Information contained in data elements and tags is freely interchangeable.

- This model can be used for string manipulations, whereby strings are represented as single-node trees.

The last property suggests that traditional string operators could probably be extended to the tree case. In other words, there is an opportunity to design a good tree algebra in such a way that restricting it to single node trees would result in a meaningful and convenient string algebra.

4 Model definition

This section presents a formal definition of the model described above. We start it by introducing the terminology used in the rest of the paper.
• $\Sigma$ is a finite set of symbols, called *alphabet*. For notational convenience, we assume that $\Sigma$ does not contain angle brackets and slash: $\langle, \rangle, / \notin \Sigma$.

• $\Sigma^*$ is the free monoid on $\Sigma$, or the set of all strings over $\Sigma$ together with the concatenation operator.

• $\varepsilon$ is the empty string; $\varepsilon \in \Sigma^*$.

All the examples given below are based on the alphabet of Latin letters.

### 4.1 String trees

We introduce *string trees* over an alphabet $\Sigma$ as strings with angle brackets such that (a) brackets match pairwise, and (b) each whole tree is enclosed in a pair of brackets, for example: $\langle ab\langle cde\langle ghij \rangle \rangle \rangle$. Visually similar to the tree representations found in classical literature [12] and in recent research [6], string trees bear one significant difference. Traditionally, each symbol marked a separate node; brackets contained all the children of the node marked by the symbol immediately preceding the opening bracket. In our model, each node is labelled by a *sequence* of symbols, enclosed in a pair of brackets.

The set $T(\Sigma)$ of string trees over $\Sigma$ is defined as the minimum subset of $(\Sigma \cup \{\langle, \rangle\})^*$ such that:

1. $\langle \rangle \in T(\Sigma)$ (the null tree)
2. $\langle a \rangle \in T(\Sigma)$ for any $a \in \Sigma$
3. $\langle xy \rangle \in T(\Sigma)$ for any $\langle x \rangle, \langle y \rangle \in T(\Sigma)$ (concatenation)
4. $\langle t \rangle \in T(\Sigma)$ for any $t \in T(\Sigma)$ (encapsulation).

It follows immediately from this definition that $T(\Sigma)$ forms a monoid with respect to concatenation (from 1 and 3) and that all strings from $\Sigma^*$, enclosed in a pair of angle brackets, are contained in $T(\Sigma)$.

Because concatenation in $T(\Sigma)$ is defined differently from the usual string concatenation in $(\Sigma \cup \{\langle, \rangle\})^*$, it will be denoted as a centered dot ($\cdot$). For example, if $u, v \in T(\Sigma)$, then $u \cdot v$ is their tree concatenation and $uv$ is their string concatenation. We could, of course, discard the outermost pair of angle brackets from trees in $T(\Sigma)$: they are present in all trees anyway. This would eliminate the difference between concatenations of trees and strings, making $T(\Sigma)$ a submonoid of $(\Sigma \cup \{\langle, \rangle\})^*$. However, this would also complicate automata and grammars on string trees.

We shall often encounter simple cases of trees, consisting of just one label, such as $\langle cetus \rangle$, and their subset, single-symbol trees, such as $\langle a \rangle$. The following notation will be used:

• $\langle \Sigma \rangle$ is the set of all single-symbol trees from $T(\Sigma)$: $\{\langle a \rangle \mid a \in \Sigma\}$;

• $\langle \Sigma^* \rangle$ is, by analogy, the set of all single-string trees: $\{\langle s \rangle \mid s \in \Sigma^*\}$.

A matching pair of angle brackets can be thought of as a unary operator, which we called encapsulation (denoted as $\langle\rangle$). Note that encapsulation and concatenation operators are free from relations (apart from the associativity of concatenation), so they freely generate $T(\Sigma)$ from $\langle\Sigma\rangle$. Thus, $T(\Sigma)$ can be called a *free monoid with unary operator* on $\langle\Sigma\rangle$.

Note that there are two different structures denoted by $T(\Sigma)$: strings over $\Sigma \cup \{\langle, \rangle\}$ and trees over $\Sigma$. Strings possess just one binary operator, concatenation, which has no symbol. Trees have two operators, concatenation and encapsulation, denoted by $\cdot$ and $\langle\rangle$. In the rest of the paper, we shall be using this dual notation, where an element of $T(\Sigma)$ can be interpreted as a string or a tree depending on the context. The context is uniquely identified by the operator signs used. In the text, elements of $T(\Sigma)$ will always be called trees, to help distinguish them from arbitrary strings from the larger set $(\Sigma \cup \{\langle, \rangle\})^*$.
4.2 Reduced string trees

The above definition of string trees is, however, too broad for the informal data model described in Section 3. Let us consider the correspondence between the two.

A tree from \( T(\Sigma) \) can be uniquely represented as \( \langle s_0t_1s_1t_2s_2\cdots t_ns_n \rangle \), where \( n \geq 0 \), \( s_i \in \Sigma^* \), and \( t_i \in T(\Sigma) \). The root label of this tree is \( s_0s_1\cdots s_n \), and the children of the root node are the trees \( t_1, t_2, \ldots, t_n \).

It is easily noticeable that the same tree in the informal model can correspond to different trees in the formal model. For example, the following are two different versions of the tree depicted in Figure 2: \( \langle \text{name(first(Joe))})(\text{last(Bloggs)}) \rangle = \langle \text{na(fir}(\text{Joe})\text{st})(\langle \text{Bloggs}(\text{last)e}) \rangle \).

![Figure 2: A tree example](image)

However, a one-to-one correspondence can be achieved by the following commutativity relation in \( T(\Sigma) \):

\[
\langle a \rangle \cdot \langle t \rangle = \langle t \rangle \cdot \langle a \rangle \quad \text{for any} \ a \in \Sigma, \ t \in T(\Sigma).
\]

This relation defines the monoid of reduced trees. This monoid is an exact match for our informal data model, because the ordering of label symbols in relation with sub-trees no longer matters: all symbols can be collected in one part of the label and all sub-trees can be gathered in the other part.

Reduced trees can be constructed as the image of the following map \( r : T(\Sigma) \to T(\Sigma) \), recursively defined as follows:

\[
r(u_0 \cdot \langle t_1 \rangle \cdot u_1 \cdot \langle t_2 \rangle \cdot u_2 \cdot \ldots \cdot \langle t_n \rangle \cdot u_n) = u_0 \cdot u_1 \cdot \ldots \cdot u_n \cdot \langle r(t_1) \rangle \cdot \langle r(t_2) \rangle \cdot \ldots \cdot \langle r(t_n) \rangle,
\]

where \( n \geq 0 \), \( u_i \in \langle \Sigma^* \rangle \), and \( t_i \in T(\Sigma) \). This can also be written in string notation:

\[
r(\langle s_0t_1s_1t_2s_2\cdots t_ns_n \rangle) = \langle s_0s_1\cdots s_n r(t_1) r(t_2) \cdots r(t_n) \rangle,
\]

where \( n \geq 0 \), \( s_i \in \Sigma^* \), and \( t_i \in T(\Sigma) \). The binary operator (concatenation) in \( \text{Im} \ r \) is naturally defined as

\[
r(u) \cdot r(v) = r(u \cdot v).
\]

This operator is well-defined (that is, the result does not depend on the choice of \( u \) and \( v \)). Indeed, let us consider \( u, u', v, v' \) such that \( r(u) = r(u') \) and \( r(v) = r(v') \). Then, applying the definition of \( r \) to \( r(u \cdot v) \) and \( r(u' \cdot v') \), we get \( r(u \cdot v) = r(u) \cdot r(v) = r(u') \cdot r(v') = r(u' \cdot v') \). The associativity of concatenation in \( \text{Im} \ r \) follows immediately from the associativity in \( T(\Sigma) \). Thus, \( r(T(\Sigma)) \) is a monoid and \( r \) is a homomorphism from \( T(\Sigma) \) onto \( r(T(\Sigma)) \). Note that despite \( r(T(\Sigma)) \) being a subset of \( T(\Sigma) \), it is not a sub-monoid.

Although reduced trees do provide a better match for actual real-life hierarchical data structures, normal string trees can in fact be more useful because they give more flexibility in data manipulation. An actual transformation engine can easily convert from reduced trees to normal trees and back if it chooses to work with normal trees internally.
5 Finite automata

The notion of finite automata comes from different branches of computer science. A finite automaton is a machine that has a finite set of states, can accept input from a finite set of input symbols, and changes its state when input is applied. The new state depends on the current state and the input symbol.

In the formal language theory finite automata are used to describe sets of strings called regular languages. A string is accepted by an automaton if, having consumed all the symbols of the string one by one, the automaton ends up in a predefined final state. A set of all strings accepted by an automaton is called a regular set (or regular language). The automaton is said to recognise this language.

Similarly, in the tree automata theory, tree-like structures are operated on by automata which take symbols in tree nodes as inputs. There are two classes of tree automata. A “bottom-up” automaton starts at the leaves and moves upwards, while a “top-down” automaton descends from the root of the tree. Languages recognised by tree automata constitute the class of regular tree languages.

Let us now introduce finite automata that operate on string trees in the bottom-up manner. The following definition essentially presents a mixture of the corresponding notions from the formal language theory and tree automata theory.

**Definition 1.** A Non-deterministic Finite String Tree Automaton (NFSTA) over $\Sigma$ is a tuple $A = (\Sigma, Q, Q_f, q_0, \Delta)$, where

- $Q \supseteq \Sigma$ is a finite set, called set of states, or state set;
- $Q_f \subseteq Q$ is a set of final states;
- $q_0 \in Q$ is the initial state;
- $\Delta$ is a set of transition rules of the form $(q_1, q_2) \rightarrow q_3$, where $q_1, q_2, q_3 \in Q$.

$\Delta$ can also be thought of as a subset of $Q^3$, however interpreting it as a set of rules of the above form is more intuitive.

**Definition 2.** A Deterministic FSTA (DFSTA) is an NFSTA whose $\Delta$ contains at most one rule for each left hand side $(q_1, q_2)$. In a DFSTA, $\Delta$ can also be considered as a partial function $\Delta : Q \times Q \rightarrow Q$. A DFSTA whose $\Delta$-function is defined on all $Q \times Q$ is called complete.

The operation of a deterministic FSTA can be illustrated on our informal data model, introduced in Section 3 above, as follows. The automaton starts at the leaves. Each leaf is processed like traditional finite automata do. The automaton starts in the state $q_0$ and takes the first symbol from the leaf’s label. Because that symbol belongs to $\Sigma$, it also belongs to $Q$. The automaton finds the rule in $\Delta$ whose left-hand side matches the state and the input symbol. The right-hand side of that rule becomes the new state. Next input symbol is then taken from the label and the process continues. If at some stage no rule can be applied, the run is considered unsuccessful: the tree is not accepted.

When a leaf’s label is successfully processed, that leaf is cut off. The resulting state of the automaton is inserted into the leaf’s parent node label (the exact place in the label will be discussed later). As soon as all children of a node have been processed, the node itself becomes a leaf, and the automaton runs again. Note that the definition above allows an automaton to accept its own state as an input. If, after processing the root label, the automaton finishes in one of the “final states” ($Q_f$), its run is considered successful.

A non-deterministic FSTA is different from a DFSTA in that it can switch to different states given the same current state and input symbol. An NFSTA can, therefore, have different runs on the same tree. If at least one of the possible runs is successful, the tree is accepted.

Similarly to the string language and classical tree cases, deterministic and non-deterministic automata on string trees are equivalent: a language that is recognised by an NFSTA is recognised
by some DFST A and vice versa. The proof of this and some other basic facts about automata will be given in Section 5.4 after a more formal definition of an FST A run and recognisable string tree languages is presented.

In order to formally define the run of a string tree automaton, we need to find a way to associate current states with all the labels. To do this, we shall put the state in front of each label, separated by a special symbol, denoted as a slash (/). For example, \( \langle \text{abc} \rangle \) will be transformed to \( \langle q_0/\text{abc} \rangle \), after which the automaton will consume abc step by step, changing the state symbol before the slash. Thus, intermediate trees will all belong to \( T(Q \cup \{/\}) \). Note that \( T(\Sigma) \subset T(Q \cup \{/\}) \), because \( \Sigma \subseteq Q \).

**Definition 3.** Let \( A = (\Sigma, Q, Q_f, q_0, \Delta) \) be an FST A; assume for convenience that \( Q \) does not contain slash: \( / \notin Q \). Let also \( a, b, c \in Q \); \( r, l \in (Q \cup \{,), /\})^* \); and \( s \in Q^* \). The move relation \( \rightarrow \) between two trees from \( T(Q \cup \{/\}) \) is defined as follows:

\[(a) \quad l(s)r \rightarrow l(q_0/s)r \quad \text{(initial state assignment)}
(b) \quad l(a/bs)r \rightarrow l(c/s)r \quad \text{if } (a, b) \rightarrow c \in \Delta \quad \text{(horizontal move)}
(c) \quad l(a/)r \rightarrow lar \quad \text{(vertical move)}\]

\( \rightarrow \) is the reflexive and transitive closure of \( \xrightarrow{\rightarrow} \). A tree \( t \in T(\Sigma) \) is accepted by the automaton \( A \) if there exists \( q_f \in Q_f \) such that \( t \xrightarrow{\rightarrow} \langle q_f/\rangle \). An empty tree is therefore accepted if and only if \( q_0 \in Q_f \).

In the definition of the move relation, a tree from \( T(Q \cup \{/\}) \) contains automaton’s both input and state. The role of slash is actually to mark those labels to which step (a) has already been applied.

As described above, an FST A can make three different kinds of steps. Step (a)—initial state assignment—applies once to each leaf label (because \( s \) cannot contain angle brackets or slashes). Then, step (b)—horizontal move—transforms a label according to the rules from \( \Delta \) by consuming one symbol and changing the state. Finally, labels which have been fully processed by step (b) are cut off and their final states are inserted into their parent labels in step (c)—vertical move. Eventually, intermediate nodes lose their descendants and become leaves, making themselves available for step (a) and so on. The process stops at the root (or when there is no suitable rule in \( \Delta \)).

**Example 1.** Let \( \Sigma = \{a, b\} \). The following automaton \( A = (\Sigma, Q, Q_f, q_0, \Delta) \) accepts only trees whose labels (all of them) are composed of the same letter, either \( a \) or \( b \).

\[
\begin{align*}
Q &= \{a, b, q_0\} \\
Q_f &= Q \\
\Delta &= \begin{cases} 
(q_0, a) &\rightarrow a, \\
(a, a) &\rightarrow a, \\
(q_0, b) &\rightarrow b, \\
(b, b) &\rightarrow b
\end{cases}
\end{align*}
\]

Consider the tree \( \langle\langle a\rangle a\langle\langle a\rangle a\rangle\rangle \). This tree is accepted by \( A \), as illustrated below by (slightly abbreviated) one of its possible runs:
It is intuitively understandable that this automaton does not accept trees containing mixed symbols: because of the way symbols are propagated, a mixed tree would eventually “resolve” to the point where a leaf would contain different symbols. A move of the automaton on that leaf would require a rule with \((a, b)\) or \((b, a)\) in its left-hand side, but \(\Delta\) contains no such rule. A complete proof of this statement is not significant for the further discussion and is therefore omitted.

### 5.1 Generalised automata

Like with deterministic automata being a case of more general non-deterministic automata, the latter can be reasonably generalised even further. In an NFS\(TA\), non-determinism is only present in step (b)—horizontal move. Therefore, it seems natural to extend non-determinism to the two other steps as well:

- Initial state assignment (a) can be generalised by allowing a set of possible initial states \(Q_0\), rather than a single initial state \(q_0\).
- Vertical move (c) can be governed by a set of rules \(\gamma\), which (non-deterministically) map one state to another during the move.

It happens that such generalisation does not increase expressive power: all three kinds of string tree automata recognise the same class of languages, as will be shown in Section 5.4. Because generalised automata are somehow cumbersome to deal with, and because they are not as useful as their more restrictive counterparts are, in our further study we shall be primarily dealing with “simple” FST\(As\). However, the concept of a generalised automaton will be indispensable for proving the equivalence of automata and tree regular grammars in Section 6.1.

**Definition 4.** A Generalised Non-deterministic Finite String Tree Automaton (GNFSTA) over an alphabet \(\Sigma\) is a tuple \(A = (\Sigma, Q, Q_f, Q_0, \Delta, \gamma)\), where

- \(Q \supseteq \Sigma\) is a finite set, called set of states, or state set;
- \(Q_f \subseteq Q\) is a set of final states;
- \(Q_0 \subseteq Q\) is a set of initial states;
- \(\Delta\) is a set of “horizontal” transition rules of the form \((q_1, q_2) \rightarrow q_3\), where \(q_1, q_2, q_3 \in Q\);
- \(\gamma\) is a set of “vertical” transition rules of the form \(q_1 \rightarrow q_2\), where \(q_1, q_2 \in Q\).

It is often more convenient to treat \(\gamma\) as a function \(\gamma : Q \rightarrow 2^Q\), so that \(\gamma(q)\) denotes the set of states \(q\) can be transformed to during a vertical move. This is the notation that will be used in the rest of the paper.

**Definition 5.** Let \(A = (\Sigma, Q, Q_f, Q_0, \Delta, \gamma)\) be a GNFSTA; as usual, we assume that \(/ \notin Q\). Let also \(a, b, c \in Q\); \(r, l \in (Q \cup \{(, ), /\})^*\); and \(s \in Q^*\). The move relation \(\rightarrow\) between two trees from \(T(Q \cup \{/\})\) is defined as follows:

| (a) | \(l(s)r\) | \(l(a/s)r\) if \(a \in Q_0\) (initial state assignment) |
| (b) | \(l(a/bs)r\) | \(l(c/s)r\) if \((a, b) \rightarrow c \in \Delta\) (horizontal move) |
| (c) | \(l(a/)r\) | \(lbr\) if \(b \in \gamma(a)\) (vertical move) |

As we can see from this definition, an empty tree is accepted if and only if \(Q_0 \cap Q_f \neq \emptyset\).
5.2 Locality

As follows from the informal description of the operation of a bottom-up tree automaton, the actions done in one branch of a tree are independent from the actions performed in another branch. Intuitively, the order in which individual leaves are processed should not matter until their branches are folded up to a common ancestor.

By definition, an automaton’s run is sequential; there is no parallelism allowed. Thus, even a deterministic string tree automaton can produce different runs on the same tree. At each point there may be a choice of multiple labels the next step can be applied to. However, this fact does not really qualify as non-determinism, because this choice does not affect the success of the run.

**Proposition 1.** Consider a non-final tree in a successful GNFSTA run. Then for any of its leaves a step can be applied to that leaf that belongs to a (probably different) successful run.

*Proof.* Let \( A = (\Sigma, Q, Q_f, Q_0, \Delta, \gamma) \) be a GNFSTA that accepts tree \( t_0 \). Let \( t \in T(Q \cup \{/\}) \) be a non-final tree in the run of \( A \) on \( t_0 \): \( t_0 \xrightarrow{q} t \xrightarrow{q_f} \langle q_f \rangle \), where \( n \geq 1 \), \( q_f \in Q_f \). Now, if \( t \) consists of only one label, the statement is trivial. If \( t \) contains more than one label, it can be represented as \( t = \langle x(w)y \rangle \), where \( w \in (Q \cup \{/\})^* \) is one of its leaves. We want to prove that for any such representation, \( t \xrightarrow{q_f} \langle q_f \rangle \) for some \( v \) such that:

\[
\langle x(w)y \rangle \xrightarrow{q_f} \langle w' \rangle, \quad \text{where} \quad w' \in (Q \cup \{/\})^*, \quad \text{or} \quad v = q, \quad \text{where} \quad q \in Q.
\]

This is proven by induction on \( n \).

**Basis** \( n = 3 \). The smallest number of steps for a tree consisting of more than one label is three:

\[
\langle \langle q \rangle \rangle \xrightarrow{q} \langle q_0/q \rangle \xrightarrow{q_f} \langle q_f \rangle.
\]

In this case, \( v = q \) and the statement is true.

**Induction** Suppose the statement is true for \( n \) steps; we need to prove it for \( n + 1 \) steps. Since \( t \xrightarrow{n+1} \langle q_f \rangle \), there exists \( t' \) such that \( t \xrightarrow{q_f} t' \). Consider all possible steps that can be applied to \( t \). By the move relation definition, the step from \( t \) to \( t' \) can be done on the label that is either \( \langle w \rangle \), or wholly contained within \( x \) or \( y \). This gives us four possibilities for this step (and thus for \( t' \)):

\[
\langle x(w)y \rangle \xrightarrow{q_f} \langle x'w'y \rangle, \quad \langle x(w)y \rangle \xrightarrow{q_f} \langle x(w'y) \rangle,
\]

\[
\langle x(w)y \rangle \xrightarrow{q_f} \langle x'(w)y \rangle, \quad \langle x(w)y \rangle \xrightarrow{q_f} \langle xqy \rangle.
\]

If \( t' \) is one of those listed in the right column, then the induction holds immediately. Otherwise, we assume that \( t' = \langle x'(w)y \rangle \) (the other case is fully analogous).

Applying the inductive hypothesis to \( t' \), we get \( t' \xrightarrow{q_f} \langle x'vy \rangle \xrightarrow{q_f} \langle q_f \rangle \). The move from \( t' \) to \( \langle x'vy \rangle \) is done via one of the three GNFSTA steps with \( l = \langle x' \rangle \) and \( r = y \). Let \( l = \langle x \rangle \); then the same step will apply to \( t \):

\[
t = \langle x(w)y \rangle \xrightarrow{q_f} \langle xvy \rangle.
\]

Now, by analogy, consider the step from \( t \) to \( t' \). During that step, \( l \) is a prefix of \( \langle x \rangle \) and \( r = z(w)y \), where \( z \) is an arbitrary string from \( (Q \cup \{/,\})^* \). Let \( r = zvy \); then the same step will apply to the tree \( \langle xvy \rangle \): \( \langle xvy \rangle \xrightarrow{q_f} \langle x'vy \rangle \). Thus, we get:

\[
t \xrightarrow{q_f} \langle xvy \rangle \xrightarrow{q_f} \langle x'vy \rangle \xrightarrow{q_f} \langle q_f \rangle,
\]

which proves the inductive statement. \( \square \)
5.3 Automata with “pure” states

A salient difference between string tree automata and the traditional finite state machines is that an FSTA’s state set is a superset of the “input” alphabet Σ. In other words, in FSTAs all input symbols can serve as what corresponds to states in traditional automata—for example, they can occur in right hand sides of rules from Δ.

This came out of the fact that in string tree automata, “traditional” states become input symbols during vertical moves. Therefore, there is not much conceptual difference between them; that’s why we often call states state symbols. An FSTA has to accept both sets as its input. Then, to make life easier and notation simpler, we allowed our FSTAs to use both sets as states as well. As a side effect, this has permitted in certain cases simple and elegant implementations, such as the one from Example 1.

However, there is nothing special about using input symbols as states. In fact, any (GN)FSTA can be trivially rewritten so as not to use them. This rewrite preserves determinism of the automaton; i.e. whether it is deterministic, non-deterministic, or generalised.

Indeed, let \( A = (\Sigma, Q, Q_f, Q_0, \Delta, \gamma) \) be an FSTA. Consider a set of “complementary” symbols \( \overline{\Sigma} \) such that there is a unique symbol \( \overline{a} \in \overline{\Sigma} \) for each \( a \in \Sigma \), and \( \overline{\Sigma} \cap Q = \emptyset \). Let the new state set \( Q' \) be \( Q \cup \overline{\Sigma} \). Let us also extend the complementation operator to \( Q \) so that \( \overline{\overline{q}} = q \) if \( q \in Q \setminus \Sigma \). Then, for each rule from \( \Delta \) we add one or two rules to (initially empty) \( \Delta' \) as follows:

\[
\Delta' = \bigcup_{(q,r) \to p \in \Delta} \left\{ \begin{array}{l}
(q, r) \to \overline{p}, \\
(\overline{q}, \overline{r}) \to \overline{p}
\end{array} \right\}
\]

(Note that when \( r \not\in \Sigma \), the set under the union sign is effectively a single-element set.) For a GNFSTA, we also build the new automaton’s \( \gamma' \) as

\[
\gamma'(\overline{q}) = \overline{\gamma(q)} \quad \text{for all} \quad q \in Q; \\
\gamma'(a) = \emptyset \quad \text{for all} \quad a \in \Sigma.
\]

This procedure simply creates a duplicate set of input symbols, which can be used as states, and then replaces input symbols with their duplicates in every place where they were used as states.

Note the following properties of the automaton \( A' = (\Sigma, Q', Q'_f, Q'_0, \Delta', \gamma') \):

- no rule (horizontal or vertical) can produce a symbol from \( \Sigma \);
- no rule takes a symbol from \( \Sigma \) as its state symbol;
- neither the starting nor the final state sets contain symbols from \( \Sigma \).

Such automata will be called automata with pure states.

Simple substitution by FSTA definition reveals that \( L(A') = L(A) \). Also, nothing in this procedure changes the deterministic properties of the automaton.

5.4 Equivalence of deterministic and non-deterministic automata

Now, we are ready to show that deterministic, non-deterministic, and generalised automata recognise the same class of languages. For this, it is sufficient to show that for any GNFSTA there exists an equivalent (i.e., recognising the same language) DFSTA. Moreover, as follows from the previous section, GNFSTAs can be assumed to be automata with pure states. The overview of the proof is presented in Figure 3.

**Theorem 1.** For any GNFSTA with pure states over \( \Sigma \) there exists an equivalent DFSTA over \( \Sigma \).

This statement can be proven using the same technique as in the classical (string) automata theory. We only give an informal sketch of the proof here, referring the reader to Appendix A for the complete proof.
Automata with impure states

| Generalised    | GNFSTA, impure states ⇔ GNFSTA, pure states |
| Non-deterministic | NFSTA, impure states ⇔ NFSTA, pure states |
| Deterministic  | DFSTA, impure states ⇔ DFSTA, pure states |

Figure 3: Equivalence of different kinds of string tree automata

Let $\mathcal{A}$ be a GNFSTA $(\Sigma, Q, Q_f, Q_0, \Delta, \gamma)$ with pure states. We construct DFSTA $\mathcal{A}' = (\Sigma', Q', Q'_f, q'_0, \Delta')$, using the set of all subsets of $Q$ as its state set. All possible outcomes of rules from $\Delta$ with identical left hand side will then be lumped together into one set; this set (which is a single state in the new automaton) will be assigned to the corresponding deterministic rule in $\Delta'$.

Firstly, however, one must remember that, according to the FSTA definition, the state set must contain all the symbols from the alphabet. We could satisfy this by letting $Q' = 2^Q \cup \Sigma$, but this would complicate the proof unnecessarily. Instead, we let the new alphabet be the set of all single-element sets, containing symbols from the original alphabet: $\Sigma' = \{\{a\} | a \in \Sigma\} = \bigcup_{a \in \Sigma} \{\{a\}\}$. Then we note that the natural bijection between $\Sigma$ and $\Sigma'$ implies a one-to-one correspondence between $T(\Sigma)$ and $T(\Sigma')$, so the two automata $\mathcal{A}$ and $\mathcal{A}'$ operate in fact on the same trees (with renamed alphabet symbols).

Now, let $Q' = 2^Q$; $Q'_f = \{q' \in Q' | q' \cap Q_f \neq \emptyset\}$; $q'_0 = Q_0$. Because $\mathcal{A}'$ does not have vertical rules, their functionality has to be incorporated in horizontal rules. To see how this can be done, consider the following example that shows excerpts from a GNFSTA run on a tree fragment:

\[
\begin{align*}
&\text{ab c} \quad \text{def} \quad \text{ab c} \\
&\text{q}_1/ \quad q_02/\text{ab q}_2\text{c}
\end{align*}
\]

Here, $q_02$ is one of the starting states, and $q_2 \in \gamma(q_1)$. Imagine an arbitrary leaf label in a GNFSTA run on some tree (looking at $t_2$ as an example). Suppose it’s ready for a horizontal move (i.e., contains /). What symbols can it be composed of?

- The symbol on the left hand side of the / can be either an initial state, or a result of some horizontal rule.
- The symbols on right hand side of the / can be either original input symbols from $\Sigma$ (such as $\text{abc}$), or results of some vertical rule (such as $q_2$, which was produced by the rule $q_1 \rightarrow q_2$ from $\gamma$).

If vertical rules were not allowed, then $q_1$ would have squeezed into $t_2$ in the place of $q_2$. Thus, new horizontal rules from $\Delta'$ have to apply $\gamma$ to all right hand side symbols that result from vertical moves, prior to looking up an appropriate horizontal rule from $\Delta$. Roughly speaking, $\Delta'(p, q) = \Delta(p, \gamma(q))$, if $q$ came via a vertical move from a subordinate node; and $\Delta'(p, q) = \Delta(p, q)$, if $q$ is an input symbol that originally belonged to the node being processed.

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5.5 Finite string tree automata and classical automata

Due to their historical background, FSTAs share much in common with classical string and tree automata. In a sense, FSTAs can be thought of as a generalisation of classical automata: an FSTA can be applied wherever a string or tree automaton is used. The aim of this section is to discuss the relationships between different kinds of automata.

5.5.1 String automata

Finite automata (FA), also called finite state machines (FSM), are widely used in automata and formal language theories. In this paper, we often call them “finite string automata” to explicitly distinguish them from tree automata. A finite automaton is a tuple $(\Sigma, Q, Q_f, q_0, \delta)$, where $\Sigma$ is an alphabet, $Q$ is a finite state set, $Q_f \subseteq Q$ is a set of final states, $q_0 \in Q$ is the initial state, and $\delta$ is a map from $Q \times \Sigma$ to $2^Q$. A finite automaton works on strings from $\Sigma^*$, starting in the state $q_0$ and applying rules from $\delta$ to its current state and the next symbol from the input string to determine its next state. The language recognised by an automaton is the set of strings that it accepts.

The move of an FA is, therefore, defined exactly like the combination of steps (a) and (b) of the FSTA move (see Definition 3). This leads us to the following statement.

**Proposition 2.** If $L$ is the language recognised by a finite automaton $(\Sigma, Q, Q_f, q_0, \delta)$, then $(L)$ is recognised by the FSTA $(\Sigma, Q \cup \Sigma_f, q_0, \delta)$, where $\delta$ is naturally extended so that $\delta(q_1, q_2) = \emptyset$ for all $q_1, q_2 \in Q$.

As we remember, $(L) = \{ t \in T(\Sigma) \mid t = \langle s \rangle \text{ for some } s \in L \}$. Indeed, for single-label trees this statement follows immediately from the automata definitions. Trees containing more than one label are not accepted by this FSTA for the following reason. Suppose $t \in T(\Sigma)$ has more than one label. Then there exist a pair of labels $l_1$ and $l_2$ such that $l_1$ contains $l_2$: $l_1 = \langle x_1l_2y \rangle$. When $l_2$ has been fully processed, it becomes $\langle q_2 \rangle$, where $q_2 \in Q$ by definition of the FA (because $q_0 \in Q$ and $\text{Im} \ \delta \subseteq 2^Q$). A vertical move then injects $q_2$ into $l_1$: $\langle xl_2y \rangle$. The automaton will stop at $q_2$ because there is no suitable rule in $\delta$.

A number of finite automata can be combined together to form an FSTA that accepts different string languages in labels depending on their position in the tree, where position is determined by the topology of the tree and by the information in other labels. This will be discussed in more details in Section 6.1.

5.5.2 Tree automata

In this paper, these are usually referred to as “classical tree automata” (CTA). The following introduction is largely borrowed from [13].

Trees in the classical tree automata theory are called terms and are composed of ranked symbols. A ranked alphabet is a finite set of symbols, in which each symbol is associated with a whole non-negative number, called arity. The arity of a symbol $f$ is denoted as $\text{Arity}(f)$. Symbols of arity $0, 1, 2, \ldots, p$ are called constants, unary, binary, \ldots, $p$-ary symbols, respectively.

The set $\hat{T}(\mathcal{F})$ of terms over the ranked alphabet $\mathcal{F}$ is the smallest set defined by:

- $f \in \hat{T}(\mathcal{F})$ for any constant $f \in \mathcal{F}$; and
- if $f \in \mathcal{F}$, $p = \text{Arity}(f) \geq 1$, and $t_1, \ldots, t_p \in \hat{T}(\mathcal{F})$, then $f(t_1, \ldots, t_p) \in \hat{T}(\mathcal{F})$. 

Fortunately, distinguishing between these two cases is very easy, because our GNFSTA is an automaton with pure states. Indeed, in such an automaton $\gamma$ is not allowed to produce symbols from $\Sigma$, which is where all the original input symbols come from. Thus, if a symbol in a label belongs to $\Sigma$, it must be an original one; otherwise, the symbol has to be a result of a vertical rule. Appendix A gives a full proof why automaton $\mathcal{A}'$ built as described above is indeed equivalent to $\mathcal{A}$. 


enumerate the rules in \( \hat{\Delta} \), indexing them from 1 to \( Q \). The term \( +(1, + (1, 2)) \) represents the following tree:

\[
\begin{array}{c}
+ \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\downarrow \\
2
\end{array}
\]

A term \( \hat{t} \in \hat{T}(\mathcal{F}) \) may be viewed as a finite ordered labelled tree, the leaves of which are labelled with constants and the internal nodes are labelled with symbols of positive arity. The number of children a node has must be equal to the arity of the node’s symbol.

Consequently, a term from \( \hat{T}(\mathcal{F}) \) can be represented as a string tree over the un-ranked alphabet \( \Sigma = \mathcal{F} \). This is illustrated by the following recursively defined injective map \( \tau : \hat{T}(\mathcal{F}) \rightarrow T(\Sigma) \):

- \( \tau(a) = a \) (for a constant symbol \( a \));
- \( \tau(f(t_1, \ldots, t_p)) = (\tau(t_1) \cdots \tau(t_p)) \).

For instance, the term \( +(1, + (1, 2)) \) pictured above would map to \( +(1, +(1, 2)) \).

A non-deterministic classical finite tree automaton (NCFTA) over \( \mathcal{F} \) is a tuple \( A = (Q, \mathcal{F}, Q_f, \Delta) \). \( Q \) is an alphabet, consisting of constant symbols called states; \( Q_f \subseteq Q \) is a set of final states. \( \Delta \) is a set of rules of the following type: \( f(q_1, \ldots, q_n) \rightarrow q \), where \( n = \text{Arity}(f) \), \( q, q_1, \ldots, q_n \in Q \).

Like our string tree automata, a classical bottom-up tree automaton also starts at the leaves and moves upwards, associating a state with each subterm. If the direct subterms \( u_1, \ldots, u_n \) of term \( t = f(u_1, \ldots, u_n) \) are assigned states \( q_1, \ldots, q_n \) respectively, then the term \( t \) will be assigned some state \( q \) given that \( f(q_1, \ldots, q_n) \rightarrow q \in \Delta \). The move relation \( \overrightarrow{t} \) is thus defined in \( \hat{T}(\mathcal{F} \cup Q) \); its full formal definition is outside the scope of this paper and can be found in [13].

Tree languages recognised by classical automata happen to be recognisable by string tree automata, as demonstrated by the following proposition.

**Proposition 3.** Let \( \hat{A} \) be an NCFTA \( (\hat{Q}, \hat{F}, \hat{Q}_f, \hat{\Delta}) \), recognising language \( \hat{L} \subseteq \hat{T}(\mathcal{F}) \), and let \( \Sigma \) be the (non-ranked) set of symbols of the alphabet \( \mathcal{F} \). Then there exists an NFSTA \( A = (\Sigma, Q, Q_f, q_0, \Delta) \) that recognises the language \( L = \tau(\hat{L}) \).

**Proof.** The automaton \( A \) is constructed step-by-step, by taking transition rules from \( \hat{\Delta} \) and populating \( Q \) and \( \Delta \) as described below.

Let \( \Delta_0 = \emptyset \), \( Q_f = \hat{Q}_f \), and \( Q_0 = \hat{Q} \cup \{q_0\} \), where \( q_0 \) is a new state symbol (\( q_0 \not\in \hat{Q} \)). Let us enumerate the rules in \( \hat{\Delta} \), indexing them from 1 to \( n \).

For each \( i \) from 1 to \( n \) we do the following: assuming that the \( i \)-th rule in \( \hat{\Delta} \) is \( f_i(\hat{q}_{i1}, \ldots, \hat{q}_{ip_i}) \rightarrow \hat{q}_i \), \( p_i \geq 0 \), let

\[
Q_i = Q_{i-1} \cup \{q_{i1}, \ldots, q_{ip_i}\}, \quad \Delta_i = \Delta_{i-1} \cup \left\{ \begin{array}{c}
\{ (q_0, f_i) \rightarrow q_{i1} \}, \\
\{ (\hat{q}_{i1}, \hat{q}_{i1}) \rightarrow q_{i2} \}, \\
\vdots \\
\{ (\hat{q}_{ip_i}, \hat{q}_{ip_i}) \rightarrow q_{iq_i} \}
\end{array} \right\},
\]

where \( q_{ij} (1 \leq i \leq n, 1 \leq j \leq p_i) \) are new unique states (\( q_{ij} \not\in Q_{i-1} \)).

Finally, let \( Q = Q_n, \Delta = \Delta_n \). The language, recognised by the resulting NFSTA \( A = (\Sigma, Q, Q_f, q_0, \Delta) \), is exactly (up to the mapping \( \tau \)) the language recognised by the original NCFTA, which can be proven by applying the relevant definitions.

\( \square \)
5.5.3 Vertical move and classical string automata

As already discussed, the horizontal move of a string tree automaton is an exact copy of the step of a traditional string automaton. Let us now investigate the vertical move from this point of view.

Consider string $abc$ and tree $(\langle\langle a \rangle b \rangle c)$. The tree is arranged so that one vertical move is required for an FSTA to consume each symbol, bringing the analogy between an FSTA’s vertical movement and a step of a string automaton. Note the seemingly excessive extra pair of angle brackets in the middle; they actually simplify things, as will be shown below. Despite the fact that the actual moves are quite different—for example, an FSTA has to start with its initial state at each symbol, while a conventional FA only starts with $q_0$ at the beginning of a string—any FA can be implemented as an FSTA working with “vertical” tree representations of strings.

Let $\Sigma$ be an alphabet. We shall call a tree $t \in T(\Sigma)$ a vertical tree representation of a string $s \in \Sigma^*$, if

$$s = a_1 \cdots a_n \quad (a_1, \ldots, a_n \in \Sigma, \ n \geq 0) \quad \text{and} \quad t = (\langle\langle \cdots \langle a_1 \rangle a_2 \rangle \cdots a_n \rangle)_{n+1}.$$

The vertical tree representation can be obtained as the image of the function $\omega : \Sigma^* \to T(\Sigma)$, recursively defined as

- $\omega(\varepsilon) = \langle\rangle$, and
- $\omega(sa) = \langle \omega(s) \rangle \cdot \langle a \rangle$ (where $a \in \Sigma$, $s \in \Sigma^*$).

**Proposition 4.** For any $FA$ over $\Sigma$ there exists an FSTA over $\Sigma$ that recognises the vertical tree representation of the FA’s language.

**Sketch of proof.** Let $FA = (\Sigma, Q, Q_f, q_0, \delta)$ be a finite string automaton, recognising language $L(FA) \subseteq \Sigma^*$. We want to construct a finite string tree automaton $A$ that recognises the language $\omega(L(FA)) \subseteq T(\Sigma)$.

To build $A$, we need an additional set of complementary states $\bar{Q}$ such that there is a unique state $\bar{q} \in \bar{Q}$ for each $q \in Q$, where $\bar{q} \notin Q, \bar{q} \notin \Sigma$. We also need a new unique initial state $q_0'$. Each step of $FA$ will map to four steps of $A$: horizontal, vertical, initial state assignment, and horizontal again. The first horizontal move will each time produce a complementary state; vertical move will deliver it one level up; and the following two steps will convert this complementary state to its counterpart, preparing for the next horizontal move. This is illustrated below by a run of some FA on the string $ab$ and a corresponding run of an FSTA on $\omega(ab) = (\langle\langle a \rangle b \rangle)$:

$$
\begin{array}{c|c|c|c}
(q_0, ab) & (q_1, b) & (q_2, \varepsilon) \\
\hline
\langle\langle a \rangle b \rangle & \bar{a} & \bar{\varepsilon}
\end{array}
$$

Let $A = (\Sigma, Q', Q'_f, q'_0, \Delta)$, where

$$Q' = Q \cup \bar{Q} \cup \{q'_0\} \cup \Sigma,$$

$$Q'_f = \begin{cases} 
Q_f \cup \{q'_0\} & \text{if} \ q_0 \notin Q_f, \\
Q_f & \text{if} \ q_0 \in Q_f.
\end{cases}$$

and the rule set $\Delta$ is built as follows:

- for each rule $(q, a) \to p$ from $\delta$ we add $(q, a) \to \bar{p}$ to $\Delta$;
- for each state $q \in Q$ we add $(q'_0, \bar{q}) \to q$ to $\Delta$;

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\[ \Delta = \bigcup_{(q, a) \to \bar{p} \in \delta} \{(q', p) \} \cup \bigcup_{q \in Q} \{(q', 0) \} \cup \{(q'_0, q'_0) \to q_0\} \]

Let \( L(A) \) be the language recognised by \( A \). To prove the equivalence of the two automata, we must show that

(i) \( s \in L(FA) \) implies that \( \omega(s) \in L(A) \); and

(ii) for any \( t \in L(A) \) there exists \( s \in L(FA) \) such that \( \omega(s) = t \).

Statement (i) can be proven by considering the run of \( FA \) on \( s \) and building the corresponding run on \( \omega(s) \), replacing each original step with four new as illustrated above. The result is then shown to be a valid successful run of \( A \).

To prove statement (ii), consider the sequence of tree sets \( T_n \), composed of trees that produce a successful run in \( n \) steps:

\[ T_n = \{ t \in T(Q' \cup \{ / \}) | t \xrightarrow{\Delta} (q'_f), q'_f \in Q'_f \} \]

Induction on \( n \) with 4 step increments shows that all trees from \( L(A) \) are contained in \( T_{4k+1} \) \((k = 0, 1, \ldots)\) and that each tree from \( L(A) \cap T_{4k+1} \) can be represented as \( \omega(s) \), where \( s \in L(FA) \).

5.5.4 Boolean closure

According to the formal language theory, the class of recognisable (regular) string languages is closed under union, under intersection, and under complementation [14]. The same applies to the classical tree languages [13]. Quite naturally, string tree languages are no exception and also exhibit the same Boolean closure properties. The proof from classical literature can be easily transferred to our case. Thus, only the main idea of the proof is presented here.

To prove that languages \( L_1 \cup L_2 \) and \( L_1 \cap L_2 \) are recognisable, we take the FSTAs that recognise \( L_1 \) and \( L_2 \) and unite their state and rule sets. The final state sets for the new automata are taken as the union and intersection, respectively, of the original final state sets.

To prove that and \( T(\Sigma) \setminus L \) is recognisable, we consider a complete FSTA that recognises \( L \). Any FSTA can be made complete, as shown in Corollary 1 in Appendix A. Complementing its final state sets produces the automaton that recognises \( T(\Sigma) \setminus L \).

5.6 Discussion

We have compared regular string trees with regular strings and classical trees. The most important conclusion is that the former can “grow” in two directions: horizontally and vertically, combining the properties of the latter two. Classical trees, for instance, can regularly grow downwards, but the degree of each node is fixed and depends on the node’s symbol (label). In string trees, a node can have a regular set of child trees.

The noted similarities between traditional regular sets and horizontal and vertical arrangements of nodes in regular string trees hint that usual regular properties are likely to hold. Pumping, for example, applies to trees growth both in width and in depth.

6 Regular languages, grammars, and expressions

In this section, we discuss an alternative approach to the definition of recognisable tree languages, based on the concept of grammars. Again, this goes in parallel with, and derives from, the classical theories.

6.1 Regular tree grammars

In contrast with an automaton, which is an accepting device, a grammar is a generating device. A grammar defines set of rules, which generate objects (strings or trees) from a pre-defined starting
point. The language defined by a grammar is the set of all objects that can be generated using the rules of the grammar.

A string tree grammar is very similar to an extended context-free grammar (an extended CFG is a CFG which allows regular expressions, rather than simple sequences, in the right hand sides of its rules; ECFGs have the same expressive power as normal CFGs). Before proceeding with the tree grammar definition, we need to briefly introduce “string-regular” expressions, which we shall be using extensively.

A regular expression (RE) \[14\] is a mechanism of formal language theory that describes regular languages. A regular expression over \(\Sigma\) defines a (regular) subset of \(\Sigma^*\), using symbols from \(\Sigma\) and three regular operators. The set \(RE(\Sigma)\) of regular expressions over \(\Sigma\) is defined as follows:

- \(\varepsilon \in RE(\Sigma)\);
- \(\Sigma \subseteq RE(\Sigma)\);
- if \(r_1, r_2 \in RE(\Sigma)\), then \(r_1 r_2 \in RE(\Sigma)\) (concatenation);
- if \(r_1, r_2 \in RE(\Sigma)\), then \(r_1 \mid r_2 \in RE(\Sigma)\) (union);
- if \(r \in RE(\Sigma)\), then \(r^* \in RE(\Sigma)\) (iteration, or Kleene star).

The language defined by a RE \(r\) is denoted as \([r]\).

**Definition 6.** A regular string tree grammar (RSTG) is a tuple \((\Sigma, N, S, R)\), where:

- \(\Sigma\) is a finite alphabet,
- \(N\) is a finite set of non-terminal symbols \((N \cap \Sigma = \emptyset)\),
- \(S \in N\) is an axiom, or starting non-terminal, and
- \(R\) is a set of production rules of the form \(n \rightarrow \langle e \rangle\), where \(n \in N\), \(e \in RE(\Sigma \cup N)\).

The only difference between RSTGs and extended context-free grammars is the pair of angle brackets in the right hand sides of production rules. They are there to indicate that every application of a production rule inserts a pair of angle brackets into the generated string.

The derivation relation \(\overrightarrow{G}\), associated to a regular tree grammar \(G = (\Sigma, N, S, R)\), is defined on pairs of strings from \((\Sigma \cup N \cup \{\langle, \rangle\})^*\):

\[
\begin{align*}
    u \overrightarrow{G} v & \iff \begin{cases} 
    u = lx, \quad X \rightarrow e \in R & \text{and} \\
    v = l(x)r, \quad x \in [e],
\end{cases}
\end{align*}
\]

where \(u, v, l, r \in (\Sigma \cup N \cup \{\langle, \rangle\})^*\), \(X \in N\), \(x \in (\Sigma \cup N)^*\), \(e \in RE(\Sigma \cup N)\).

**Theorem 2.** A string tree language is recognisable (by a finite string tree automaton) if and only if it is generated by a regular tree grammar.

**Proof. Grammar → automaton:** Given some regular tree grammar \(G = (\Sigma, N, S, R)\), we show how to build a generalised finite tree automaton with pure states \(A = (\Sigma, Q, Q_f, Q_0, \Delta, \gamma)\) which recognises \(L(G)\)—the language generated by \(G\).

Let \(k = |R|\) be the number of production rules in \(R\). Each rule \(r_i \in R\) has the form \(n_i \rightarrow \langle e_i \rangle\), where \(n_i\) is a non-terminal from \(N\) and \(e_i\) is a regular expression over \(\Sigma \cup N\). According to the automata theory, for each regular expression there exists a finite (string) automaton that recognises the same language. Let \(FA_i\) be a finite automaton equivalent to \(e_i\). Consider \(FA_i = (\Sigma \cup N, Q_i, q_{0i}, Q_f, \delta_i)\) for \(i = 1, \ldots, k\), where:

- \(Q_i\) is a finite set of states, disjoint with \(\Sigma \cup N\);
- \(q_{0i}\) is the initial state, \(q_{0i} \in Q_i\);
• $Q_{fi}$ is the set of final states, $Q_{f} \subseteq Q_i$;

• $\delta_i$ is a map from $Q_i \times (\Sigma \cup N)$ to $2^Q$; or—equivalently—a set of rules of the form $(q, a) \rightarrow p$, where $q, p \in Q$, $a \in \Sigma \cup N$.

Since all these automata are independent (apart from having a common alphabet $\Sigma \cup N$), we can assume that their state sets $Q_1, \ldots, Q_k$ are pairwise disjoint.

We now build our GNFSTA $A = (\Sigma, Q, Q_f, Q_0, \Delta, \gamma)$ as follows:

$$
\begin{align*}
Q &= \Sigma \cup N \cup Q_1 \cup \cdots \cup Q_k \\
Q_0 &= \{q_{01}, \ldots, q_{0k}\} \\
\Delta &= \delta_1 \cup \cdots \cup \delta_k \\
\gamma &= \bigcup_{i=1}^{k} \bigcup_{q \in Q_i} \{q \rightarrow n_i\} \\
Q_f &= \gamma^{-1}(\{s\}) = \bigcup_{i: s \rightarrow \langle e_i \rangle \in R} Q_{fi}
\end{align*}
$$

To prove the equivalence of $G$ and $A$, we show for any tree $t \in T(\Sigma \cup N)$ that it is generated by $G$ if and only if it is accepted by $A$. The induction is done on the number of labels in $t$.

**Basis** $n = 1$. Then $t = \langle s \rangle$, where $s \in (\Sigma \cup N)^*$. 

$\implies$: If $t$ is generated by $G$, then there is a rule $S \rightarrow \langle e_i \rangle$ in $R$ such that $s \in [e_i]$. By definition of $A$, $s$ is accepted by $FA_i = (\Sigma \cup N, Q_i, q_{0i}, Q_{fi}, \delta_i)$, where $Q_i \subseteq Q$, $q_{0i} \in Q_0$, $Q_{fi} \subseteq Q_f$, $\delta_i \subseteq \Delta$. Therefore, $\langle s \rangle$ is accepted by $A$ as follows:

$$
\langle s \rangle \xrightarrow{\gamma} \langle q_{0i} / s \rangle \xrightarrow{\gamma} \langle q_f / \rangle
$$

for some $q_f \in Q_{fi} \subseteq Q_f$.

$\impliedby$: If $\langle s \rangle$ is accepted by $A$, then there is a run

$$
\langle s \rangle \xrightarrow{\gamma} \langle q_{0i} / s \rangle \xrightarrow{\gamma} \langle q_{1i} / s_1 \rangle \xrightarrow{\gamma} \cdots \xrightarrow{\gamma} \langle q_f / \rangle,
$$

where $q_0 \in Q_0$, $s = a_1s_1$, $s_1 = a_2s_2$, $\ldots$; $(q_{j-1}, a_j) \rightarrow q_j \in \Delta$; and $q_f \in Q_f$. By definition of $Q_0$, $q_0 = q_{0i} \in Q_i$ for some $i$. Because $Q_1, \ldots, Q_k$ are pairwise disjoint (and by definition of $\Delta$), the rule $(q_0, a_1) \rightarrow q_1$ belongs to $\delta_i$, which implies that $q_1 \in Q_i$. The same reasoning can then be applied to $q_1, q_2, \ldots$, showing that all the horizontal rules applied belong to the same $\delta_i$ and $q_f \in Q_{fi}$. Thus, $s$ is accepted by the finite string automaton $FA_i$. (If $s$ is empty, then $q_0 \in Q_f$, therefore $q_0 \in Q_{fi}$ for some $i$, so $s$ is accepted by $FA_i$.)

Notice that $q_f$ belongs to both $Q_{fi}$ and $Q_f$, which by definition of $Q_f$ implies that the rule $S \rightarrow \langle e_i \rangle$ belongs to $R$, where $e_i$ is the regular expression equivalent to $FA_i$. Thus, the grammar $G$ generates $\langle [e_i] \rangle$, and in particular, $\langle s \rangle$.

**Induction** By inductive hypothesis, we assume that any tree $t' \in T(\Sigma \cup N)$ that has $m$ labels ($m \geq 1$) is generated by $G$ if and only if it is accepted by $A$. We want to show that the same holds true for any tree $t$ with $m+1$ labels. Let us select a leaf label in $t$: $t = l(s)r$, where $s \in (\Sigma \cup N)^*$. 

$\implies$: If $t$ is produced by $G$, then there is a tree $t' = lnr$ also produced by $G$ such that $n \in N$ and there is a rule $r_i : n \rightarrow \langle e_i \rangle$ in $R$ such that $s \in [e_i]$. Thus, by definition of $A$,

$$
\begin{align*}
l(s)r &\xrightarrow{\gamma} l(q_{0i}/s)r \xrightarrow{\gamma} l(q_f)/r
\end{align*}
$$

for some $q_f \in Q_{fi}$. Then we notice that $\gamma$ contains the rule $q_f \rightarrow n$, which implies $l(q_f)/r \xrightarrow{\gamma} lnr$.

By induction, $lnr = t'$ is accepted by $A$, therefore $t$ is accepted as well.

$\impliedby$: If $t = l(s)r$ is accepted by $A$, then by Proposition 1 (locality) there is a successful run of $A$ on $t$:

$$
\begin{align*}
l(s)r &\xrightarrow{\gamma} l(q_{0i}/s)r \xrightarrow{\gamma} l(q_{1i}/s_1)r \xrightarrow{\gamma} \cdots \xrightarrow{\gamma} l(q_{2i}/r) \xrightarrow{\gamma} lq'/r \xrightarrow{\gamma} \langle q_f / \rangle,
\end{align*}
$$

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where \( q_0 \in Q_0; \ s = a_1s_1, s_1 = a_2s_2, \ldots; (q_{j-1}, a_j) \rightarrow q_j \in \Delta; \ q_z \rightarrow q^f \in \gamma; \) and \( q_f \in Q_f. \) By analogy with the inductive basis, we can see that \( q_0 = q_{0i} \) for some \( i; \) all rules \((q_{j-1}, a_j) \rightarrow q_j\) belong to \( \delta_i \) for the same \( i; \) \( q_z \in Q_{fj}; \) and \( q^f = n_i \) (the non-terminal in the left hand side of the \( i \)-th rule in \( R \)). Thus, \( s \) is accepted by \( F A_i, \) equivalent to the regular expression \( e_i \) in the rule \( n_i \rightarrow \langle e_i \rangle \in R. \)

Consider the tree \( l_{n_i}r. \) It is accepted by \( A \) (because \( ln_i r = lq^f r \) and \( lq^f r \) is accepted), therefore by the inductive hypothesis \( ln_i r \) is generated by \( G. \) Then, all trees \( l([e_i])r \) are also generated by \( G; \) and this includes \( t = l(s)r, \) because \( s \in [e_i]. \)

**Automaton \( \rightarrow \) grammar:** Given a finite tree automaton with pure states \( A = (\Sigma, Q, Q_f, q_0, \Delta), \) we want to build a regular tree grammar \( G = (\Sigma, N, S, R) \) that recognises \( L(A). \)

Let \( N = Q \cup \Sigma \cup \{ S \}, \) where \( S \) is a symbol not in \( Q \) or \( \Sigma. \) For each \( n_i \in N \cup \{ S \}, \) let us build an automaton \( F A_i = (Q, Q \cup \Sigma, \{ n_i \}, q_0, \Delta), \) whose input symbols are all the states of \( A, \) and whose states are the “pure states” \( (Q \cup \Sigma) \) of \( A. \) The transition rules and the initial state for each \( F A_i \) are taken directly from \( A. \) Let also \( F A_S = (Q, Q \cup \Sigma, Q_f, q_0, \Delta). \) All these automata only differ in their final state set.

According to the automata theory, for each \( F A_i \) there exists an equivalent regular expression \( e_i \) over \( \Sigma^*. \) Since \( Q \subset \Sigma \cup N, \) each \( e_i \) is a valid regular expression over \( (\Sigma \cup N)^* \) as well. The same applies to \( e_S, \) equivalent to \( F A_S. \)

Now, let \( R = \bigcup \{ n_i \rightarrow \langle e_i \rangle \} \cup \{ S \rightarrow \langle e_S \rangle \}. \) The resulting grammar \( G = (\Sigma, N, S, R) \) is a regular string tree grammar, which recognises the language of automaton \( A. \)

### 6.2 Vertical concatenation

Although the two tree operators, concatenation and encapsulation, are sufficient to build all possible string trees out of basic elements, it may occasionally be useful to link trees at places other than their roots. For example, a tree can be built from root downwards by repeatedly attaching other trees at its leaves. In tree automata theory this type of tree linking is done by replacing a symbol in the first tree by the root of the second tree (Figure 4). This is readily transferable to string trees.

![Vertical concatenation](image)

**Figure 4:** Concatenation through variable \( x \)

**Definition 7.** Vertical concatenation of two trees \( u, v \in T(\Sigma) \) through symbol \( x \in \Sigma, \) denoted \( u \cdot_x v, \) is the tree derived from \( u \) by replacing all occurrences of \( x \) in it with \( v. \)

Vertical concatenation can also be defined recursively as follows:

\[
\begin{align*}
\langle \rangle & \cdot_x t = \langle \rangle \\
\langle a \rangle & \cdot_x t = \langle a \rangle, \quad \text{for } a \neq x \\
\langle x \rangle & \cdot_x t = \langle t \rangle
\end{align*}
\]

where \( a, x \in \Sigma; \ u, v, t \in T(\Sigma). \)

Note that vertical concatenations have no neutral element, because they always insert at least a pair of angle brackets: \( (axb) \cdot_x (x) = \langle a(x)b \rangle. \) By analogy with tree automata theory, symbols which are used for vertical concatenated will be called **variables** to help us distinguish them from other symbols in the alphabet. Normally, symbols which can occur in the actual trees being processed are not used as variables; the latter are chosen from some disjoint set.
6.3 Regular tree expressions

Combination of string regular expressions and classical tree regular expressions yields the following definition.

**Definition 8.** The set \( \text{RSTE}(\Sigma) \) of regular string tree expressions over \( \Sigma \) is defined as follows:

- \( \langle \rangle \in \text{RSTE}(\Sigma) \);
- \( \Sigma \subset \text{RSTE}(\Sigma) \);
- if \( r_1, r_2 \in \text{RSTE}(\Sigma) \), then \( r_1 | r_2 \in \text{RSTE}(\Sigma) \) (union);
- if \( r_1, r_2 \in \text{RSTE}(\Sigma) \), then \( r_1 \cdot r_2 \in \text{RSTE}(\Sigma) \) (horizontal concatenation);
- if \( r_1, r_2 \in \text{RSTE}(\Sigma) \) and \( x \in \Sigma \), then \( r_1 \cdot x \cdot r_2 \in \text{RSTE}(\Sigma) \) (vertical concatenation);
- if \( r \in \text{RSTE}(\Sigma) \), then \( r^* \in \text{RSTE}(\Sigma) \) (horizontal iteration);
- if \( r \in \text{RSTE}(\Sigma) \) and \( x \in \Sigma \), then \( r^*x \in \text{RSTE}(\Sigma) \) (vertical iteration).

Note that encapsulation, not mentioned above explicitly, is also a regular operator: \( \langle r \rangle = \langle x \rangle \cdot x \cdot r \) for \( x \in \Sigma \).

The language described by an RSTE \( r \) is defined analogously to the classical theories and is denoted as \( [r] \). Regular string tree expressions have the same expressive power as regular string tree grammars and finite automata.

6.3.1 Regular expression matching and substitution

Regular expressions play a major role in many text processing tools, such as ‘sed’, ‘awk’, ‘perl’, typically found on Unix systems [3]. They allow matching and selecting pieces of text that can then be re-combined to produce desired results. First, the input text is matched against a regular expression. When a match is found, sub-expressions of that regular expression are associated with the corresponding fragments of text that they match. These fragments can then be extracted simply by referring to the desired sub-expressions.

In the traditional tools, regular expressions are represented in a parenthesised infix form. Round brackets in expressions play a dual role: they group regular operators and also identify sub-expressions that will be used for text extraction. These sub-expressions are then referred to by their numbers (counted left-to-right according to their opening brackets). For example, expression ‘(press|push|hit|strike) space (key|bar)’, applied to the phrase ‘push space bar’, would result in a successful match, selecting ‘push’ and ‘bar’ as fragments 1 and 2, respectively.

Below are a few examples of string tree regular expressions. We assume that the variables used in the expressions cannot occur in actual trees; in other words, variables (denoted below as \( X, Y, Z \)) are unique symbols, added to the input alphabet \( \Sigma \). In our notation, operators have the following priorities:

\[
\begin{align*}
&* &*x &\quad \text{(highest priority)} \\
&\cdot &\cdot x &\quad \text{(lowest priority)}
\end{align*}
\]

We also use the symbol \( \alpha \) as a convenient shorthand for the union of all single symbol trees without variables.

The expression \( (\alpha|\langle X \rangle)^*x X \) matches any tree. This expression will be denoted as \( \Theta \), assuming that the variable \( X \) is not used anywhere in the expression that contains \( \Theta \), as it may cause unwanted interference. In other words, the scope of \( X \) in \( \Theta \) is restricted to \( \Theta \). An expression that matches any tree with some variable(s) will also be handy, e.g.: \( \Theta_{XY} = (\alpha|\langle X \rangle|\langle Y \rangle|\langle Z \rangle)^*Z \), where \( Z \) has a restricted scope.
Figure 5 shows a regular expression that selects a subtree located tightly between two subtrees labelled left and right (in this order) somewhere below the root of the input tree. The requested subtree is selected by the second pair of round brackets. The first pair is needed for grouping operators and (as a side effect) selects the parent tree of the one we are looking for.

What if the same regular sub-expression matches more than one fragment? This is largely an implementation issue and is up to a particular application. Let us consider potential solutions.

To begin with, we need to separate two cases when an expression can match more than one fragment: ambiguity and multiplicity. Ambiguity happens when a sub-expression can match one out of several alternatives, for example, in an expression like $r^* \cdot (r) \cdot r^*$. This is the same as $r^*$—a concatenation of zero or more trees described by $r$. The $(r)$ in the middle can match any tree in the sequence—the first one, the last one, or a tree somewhere in between. Multiplicity happens when the same sub-expression matches several fragments simultaneously, as in $(r)^*$. This describes the same sequence as the previous example, but this time $(r)$ matches all the trees that compose that sequence. By definition of iteration, there are in fact infinitely many copies of $r$ in that expression, each copy matching at most one tree; when these copies are represented in the formula by a single sub-expression, it happens to match all those trees simultaneously.

Ambiguity is traditionally solved by imposing longest-match or shortest-match rules, or by forbidding ambiguous regular expressions. In case of multiplicity, usually only the first or the last matching fragment is selected. However, as opposed to the text case, a tree processing application has the benefit of hierarchical structure: several fragments can be combined into one object simply by adding another level of hierarchy. Thus, a regular sub-expression that can potentially exhibit multiplicity may be associated with a tree whose subtrees are the fragments matched.

7 Potential applications of the string tree automata theory

The data model described in this paper enables processing of structured information in the same way as it is done today with non-structured textual data. In many text processing utilities and applications, data records are matched against one or more string regular expressions. Some parts of these expressions are marked. When a match occurs, each marked part is associated with the piece of text that matched that part within the regular expression. These pieces of text, extracted from the data record, are then used according to the application’s needs. They can be combined together and with other pieces (e.g., string literals) using concatenation, or they can be processed further as simple strings. The same basic processing scenario applies to our hierarchical data model (as illustrated in Figure 6).

An information processing application which uses the proposed data model needs to implement
a set of operations on trees that can be used in formulae (the right box in Figure 6). These operations will likely include the basic tree operators (encapsulation, concatenation) and some traditional string functions (character translation, table lookup, etc.) Note that many of these functions can be expressed using regular expression matching, concatenation, and string literals—thus, they do not extend the model, but merely act as convenient shortcuts. Tree processing primitives (such as sorting of children) can also be included in the application’s operational model.

The application may also employ some execution model which specifies how (in what order, on what conditions) tree operators are invoked. For example, the execution model can provide variables for storing trees. In one of envisaged scenarios, a tree transformation utility may treat the transformation it is performing as a sequence of rules, each consisting of a condition and an action. The utility would execute rules sequentially by checking the condition and, if necessary, doing the corresponding action. Actions would normally consist of concatenating trees taken from variables and constant expressions, and storing them back into variables.

As suggested by the practices of using string regular expressions in text manipulation tools, in such scenarios regular expressions on trees can play a dual role:

- to serve as a rule condition by telling whether a particular tree matches a pattern or not;
- to “extract” parts of a tree.

Types of applications that can benefit from this data model include:

- stand-alone processing tools (generic or specialised), such as HTML or XML processors or MARC (Machine Readable Catalogue [15]) converters;
- programming languages that include hierarchical data types;
- information retrieval;
- query languages.

8 Conclusions

The “string tree” data model introduced in this paper provides a simple algebraic notion for the hierarchical information structures. The model is based on two classical works: theory of automata and formal languages; and tree automata theory. The classical notions are extended and combined together to provide a powerful solution for today’s information processing needs. The resulting model combines the expressive power of its parents: strings, trees, finite automata, and regular languages defined by the classical theories can be expressed in the proposed model. It is shown that many of the properties of classical models apply here as well. Therefore, it is reasonable to expect that even those techniques that are used in classical theories but not considered explicitly in this paper, can be formulated and re-used in terms of the proposed model.
The most important conclusion is that processing of tree-like hierarchical data can benefit from the power of regular expressions in the same way that simple text processing has benefited from them to date. It is shown that regular expression matching on trees can be done by finite tree automata, which fit into linear memory and space constraints.

The formal description of the proposed data model supplies a basis for building custom, problem-oriented, as well as generic solutions for data retrieval and processing. These solutions can build upon additional operators introduced on trees.

The model offered is simple, consisting of only three operators: concatenation, encapsulation, and regular expression matching. The regular algebra contains five operators. Many processing functions, including frequently used convenience operators (such as, for example, extraction of n-th subtree) can be expressed using the basic set of operators without the need to extend the formal model. This puts the proposed model in a favourable position with respect to existing solutions, which are often over-complicated. Currently, different bits of tree manipulation functionality (such as execution model, string functions, tree operators, extraction of sub-records) are often bundled together and depend on each other. This makes existing models cumbersome and inflexible. Also, if, for instance, such model is incorporated into a programming language to provide it with a “tree” data type, that language is likely to already offer string and numeric processing. This results in duplicate functionality.

Finally, the model described in this paper provides a unified representation of both information trees and character strings, which makes it suitable as a single data type for processing of both structure and content of hierarchical documents. A simple scenario is presented showing how this model can be used in information processing applications. This scenario is patterned after the current usage of traditional regular expressions in unstructured text processing.

Further research can be centred along the following lines:

- **Use of non-string data types** (e.g., numeric and Boolean), both in the data being processed and in the operational model. The present data model is capable of processing, say, numeric data if the application’s operational model supports it (i.e., it contains arithmetic and conversion operators). However, feeding numeric data back to the regular expression matching engine is not obvious. This may be needed, for example, to extract a subtree by its number, where this number is determined dynamically—much like indexing an array by a variable. Of some relevance here is research on XML that has been investigating how data typing can be incorporated into XML schemata [16].

- **Presentation of regular expressions.** Tree regular expressions written in infix form (as in Figure 5 on page 21) look more complex and may be more difficult to compose than string regular expressions. On the other hand, this notation is also compact and close to the conventional syntax. Another possible representation may be that of the regular tree grammar, which has the advantage of being similar to SGML/XML DTD format.

### A Proof of equivalence of deterministic and generalised tree automata

This Appendix contains the proof of Theorem 1 on page 11 that states that any generalised non-deterministic finite string tree automaton (GNFSTA) with “pure” states has an equivalent deterministic automaton (DFSTA). The basic principle behind this proof is the same as the one used in the classical automata theory to prove the equivalence of deterministic and non-deterministic automata, namely subset construction. For an arbitrary GNFSTA, we build an equivalent DFSTA whose state set is the set of all subsets of the original GNFSTA’s state set. A brief sketch of this proof is presented in Section 5.4, where the theorem is introduced.

The requirement that the original GNFSTA be an automaton with “pure” states allows us to distinguish original tree’s input symbols from states generated during the automaton’s run. As shown in Section 5.3, any GNFSTA can be converted to an automaton with pure states.
In the next section, we introduce two auxiliary items: a map $\Gamma$ and a relation $\prec$, which will be used in the proof. Then, the following section proves a lemma which incorporates most of the work. After that, the validity of the main result of Theorem 1 follows almost immediately from the lemma.

### A.1 Preliminaries

Let $A$ be a GNFSTA with pure states over $\Sigma$: $A = (\Sigma, Q, Q_f, Q_0, \Delta, \gamma)$. As suggested above (and in the sketch of the proof in Section 5.4), we want to construct a suitable DFSTA $A' = (\Sigma', Q', Q'_f, q'_0, \Delta')$, where $Q' = 2Q$, and prove their equivalence. First, however, a few auxiliary concepts need to be introduced.

Let $T = T(Q \cup \{/\})$, $T' = T(2Q \cup \{/\})$. These sets contain trees on which the move relations $\rightarrow$ and $\xrightarrow{}$ are defined. That is, all intermediate trees which compose runs of $A$ and $A'$ belong to $T$ and $T'$, respectively.

Because $A$ has pure states, its “vertical move” function $\gamma$ produces empty set on all symbols from $\Sigma$. Let us redefine $\gamma$ on $\Sigma$ and then extend it to $2Q$ as follows:

$$\gamma(a) = \{a\} \text{ for all } a \in \Sigma;$$

$$\gamma(\{q_1, \ldots, q_n\}) = \gamma(q_1) \cup \cdots \gamma(q_n).$$

We now define function $\Gamma: T' \rightarrow T'$, which applies $\gamma$ to all the input symbols in intermediate trees for $A'$. That is, if a label in the tree is yet “untouched” by the automaton (does not contain $/$), $\gamma$ is applied to all symbols of the label. If a label is partly processed, $\gamma$ is only applied to the part on the right hand side of the slash $(/)$.

Remembering that symbols in $T'$ are sets of symbols of $T$, we define $\Gamma$ using string representations of trees from $T'$ as follows. Let $a$ denote a single state of $T'$: $a \in 2Q$. Let $s$ denote an arbitrary sub-string of a tree from $T'$, which does not start with $/$: $s \in (2Q \cup \{\}, /)^{*}$, $s \neq /x$. Then, let

$$\Gamma(\varepsilon) = \varepsilon$$

$$\Gamma((s) = (\Gamma(s)$$

$$\Gamma(a/s) = \gamma(a)\Gamma(s)$$

$$\Gamma(a/s) = a/\Gamma(s)$$

Note that by this definition, $\Gamma(xy) = \Gamma(x)\Gamma(y)$, if $y$ does not start with a slash. Also, if a tree $t'$ is not just an arbitrary tree from $T'$, but an intermediate stage of the automaton $A'$, each / in $t'$ must contain a state from $Q'$ immediately on its left. This implies, in particular, that for any representation $t' = l'(s')r'$, it is true that $\Gamma(t') = \Gamma(l')(\Gamma(s'))\Gamma(r')$.

Another thing that will be needed for the proof is a relation between $T$ and $T'$, which is induced by the simple “belongs to” relation between a set and its element. Let $t \in T$ and $t' \in T'$. We say that $t$ is an instance of $t'$ (denoted $t \triangleleft t'$), if:

(a) $t$ and $t'$ have the same structure, i.e. symbols $\langle , \rangle$, and / occupy the same positions in both trees; and

(b) each $Q$-symbol in $t$ belongs to the set of symbols in the same position in $t'$.

Or, more formally,

$$\varepsilon \triangleleft \varepsilon$$

$$as \triangleleft a's' \iff a \in a' \text{ and } s \triangleleft s'$$

for any

$$\langle s \triangleleft \langle s' \iff s \triangleleft s'$$

$$\langle s \triangleleft r' \iff s \triangleleft s'$$

$$/s \triangleleft /s' \iff s \triangleleft s'$$

(\text{where } \iff \text{ denotes logical equivalence}). An immediate corollary is that $xy \triangleleft x'y'$ is equivalent to $x \triangleleft x'$, $y \triangleleft y'$ for any strings $x$, $y$, $x'$, $y'$ taken from the appropriate string sets.
A.2 Equivalence of DFSTA and GNFSTA

Given a generalised non-deterministic finite string tree automaton $A = (\Sigma, Q, Q_f, Q_0, \Delta, \gamma)$ with pure states, let us define DFSTA $A' = (\Sigma', Q', Q'_f, q'_0, \Delta')$ as follows:

$$\Sigma' = \{ \{ a \} \mid a \in \Sigma \}; \quad Q' = 2^Q; \quad q'_0 = Q_0; \quad Q'_f = \{ q' \in Q' \mid q' \cap Q_f \neq \emptyset \};$$

$$\Delta'(a', b') = \{ c \in Q \mid (a, b) \rightarrow c \in \Delta, a \in a', b \in \gamma(b') \} \quad \text{for all } a', b' \in Q',$$

where $\gamma$ is understood in the extended sense. As follows from the definition of $\Sigma'$, the tree monoids $T(\Sigma)$ and $T(\Sigma')$ are identical up to renaming of symbols: symbols from $\Sigma$ map to the corresponding single element sets in $\Sigma'$. To prove the equivalence of $A$ and $A'$, all we need to do is to prove that for any $t \in T(\Sigma)$ and its counterpart $t' \in T(\Sigma')$, $t$ belongs to $L(A)$ if and only if $t'$ belongs to $L(A')$.

To do this, firstly consider the set of all intermediate trees, reachable from $t$ in $n$ steps under $\rightarrow$, and, analogously, the set of all trees reachable from $t'$ via $\rightarrow$. We shall prove that the first set contains exactly all instances of $\Gamma$-images of trees from the second set.

**Lemma 1.** Let $Steps^n_A(t)$ denote such sets: $Steps^n_A(t) = \{ v \in T(Q_A \cup \{ / \}) \mid t \xrightarrow{t} v \}$. Then for any $v \in Steps^n_A(t)$, $v$ is an instance of $\Gamma$-image of some $v' \in Steps^n_A(t')$ (v $\ll\Gamma(v')$), and for any $v' \in Steps^n_A(t')$ all of the instances of $\Gamma(v')$ belong to $Steps^n_A(t)$.

**Proof.** Induction on $n$.

**Basis** $n = 0$. Then $Steps^0_A(t) = \{ t \}$, $Steps^0_A(t') = \{ t' \}$, $\Gamma(t') = t'$ (because $t'$ only contains symbols from $\Sigma'$), and $t \ll t'$ by the definition of $\ll$.

**Induction** By the inductive hypothesis, the statement is assumed to be true for $Steps^n$. We need to prove that:

(i) for any $v \in Steps^{n+1}_A(t)$ there exists $v' \in Steps^{n+1}_A(t')$ such that $v \ll\Gamma(v')$;

(ii) for any $v' \in Steps^{n+1}_A(t')$ and $v \in T(Q \cup \{ / \})$, if $v \ll\Gamma(v')$, then $v \in Steps^{n+1}_A(t)$.

Let us start by proving (i). Since $v \in Steps^{n+1}_A(t)$, there exists $u \in Steps^n_A(t)$ such that $u \xrightarrow{t} v$. Therefore, by the inductive hypothesis there exists $u' \in Steps^n_A(t')$ such that $u \ll\Gamma(u')$. Consider all possible moves from $u$ to $v$.

(a) $u = l(s)r$, $v = l(q_0/s)r$, where $q_0 \in Q_0$.

Since $u \ll\Gamma(u')$, we can write $u' = l'(s')r'$, where $l \ll\Gamma(l')$, $s \ll\Gamma(s')$, and $s'$ does not contain / . Consider now $v' = l'(q_0/s')r'$. By the initial state assignment move of $A'$, $u' \xrightarrow{a/b} v'$ (remembering that $q'_0 = Q_0$). At the same time, $\Gamma(v') = \Gamma(l')(q_0/\Gamma(s'))(r')$; since $q_0 \in Q_0$, we get that $v \ll\Gamma(v')$.

(b) $u = l(a/b)s$, $v = l(c/s)r$, where $(a, b) \rightarrow c \in \Delta$.

Then, $u'$ can be represented as $u' = l'(a'/b's')r'$, where $a \in a'$, $b \in \gamma(b')$, and $l, s, r$ are instances of $\Gamma(l')$, $\Gamma(s')$, $\Gamma(r')$ respectively. From definition of $A'$ it immediately follows that $\Delta'(a', b') = c' \cup c$. Let $v' = l'(c'/s')r'$, then $u' \xrightarrow{a/b} v'$ and $v \ll\Gamma(v')$.

(c) $u = l(q_1/r)$, $v = l(q_2/r)$, where $q_1 \rightarrow q_2 \in \gamma$ or, in other words, $q_2 \in \gamma(q_1)$.

Then, $u' = l'(q'_1/r')$, where $q_1 \in q'_1$, $l \ll\Gamma(l')$, $r \ll\Gamma(r')$. Consider $v' = l'(q'_2/r')$ by the vertical move of $A'$, $u' \xrightarrow{a/b} v'$. Now, $q_2 \in \gamma(q_1) \subseteq \gamma(q'_1)$. Because $r$ cannot start with a / , neither can $r'$, therefore $\Gamma(v') = \Gamma(l')(q'_1/\Gamma(r'))$, and consequently, $v \ll\Gamma(v')$.

Let us now prove (ii). Since $v' \in Steps^{n+1}_A(t')$, there exists $u' \in Steps^n_A(t')$ such that $u' \xrightarrow{t} v'$. Therefore, by the inductive hypothesis for any $u \in T(Q \cup \{ / \})$, $u \ll\Gamma(u')$ implies that $u \in Steps^n_A(t)$. We need to show that $v \ll\Gamma(v')$ implies $v \in Steps^{n+1}_A(t)$. Let us take an arbitrary $v$ such that $v \ll\Gamma(v')$, and consider all possible moves from $u'$ to $v'$.

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Lemma 1, being in this case $\langle q_0 \rangle$.

Proof. By definition of $\Gamma$, in this case $v \leq \langle q_0 \rangle$, we can write $v$ as $v = l(q_0/s)r$, where $l, s, r$ are instances of $\Gamma(l'), \Gamma(s'), \Gamma(r')$, and $q_0 = Q_0$. Let $u = l(s)r$.

By the properties of $\Gamma$ and the $\leq$-relation, $u \leq \Gamma(u')$, therefore by the inductive hypothesis $u \in \text{Steps}_{\Delta}^n(t)$. On the other hand, $u \mapsto v$ by the initial state assignment move of $A$. Consequently, $v \in \text{Steps}_{\Delta}^{n+1}(t)$.

(b) $u' = l'(a'/b's')r'$, $v = l'(c'/s')r'$, and $\Delta'(a', b') = c'$.

We know that $v < \Gamma(v') = \Gamma(l'(c'/\Gamma(s'))\Gamma(r'))$, which implies that $v = l(c/s)r$ for some $l, c, r$ such that $l, s, r$ are instances of $\Gamma(l'), \Gamma(s'), \Gamma(r')$, and $c \in c'$. Since $c \in \Delta'(a', b')$ and by definition of $\Delta'$ (which we used to build our $A'$), there exist $a \in a'$, $b \in \gamma(b')$, and rule $(a, b) \rightarrow c \in \Delta$. Let $u = l(\alpha/b)s$. Using definition and properties of $\Gamma$, we get $\Gamma(u') = \Gamma(l'(a'/\gamma(b'))\Gamma(s'))\Gamma(r')$, so that $u < \Gamma(u')$. Using the inductive hypothesis and the fact that $u \mapsto v$ by the horizontal move of $A$, we can observe that $v \in \text{Steps}_{\Delta}^{n+1}(t)$.

(c) $u' = l'(q'/r')r'$, $v' = l'q'r'$, and $q' \in Q'$.

Since $r'$ cannot start with a $/$, $v < \Gamma(v') = \Gamma(l'(c'/\gamma(q'))\Gamma(r'))$. Then, $v = l(q_2)r$, where $l < \Gamma(l'), r < \gamma(r')$, and $q_2 \in \gamma(q')$. Because $\gamma$ of a set is the union of $\gamma$-images of all elements of that set, there must be $q_2 \in q'$ such that $q_2 \in \gamma(q_1)$. Now let $u = l(q_2)r$. Because $\Gamma(u') = \Gamma(l'(q'/\gamma(q'))\Gamma(r')$ and $q_2 \in q'$, we get $u < \Gamma(u')$ and use the inductive hypothesis. Also, $q_2 \in \gamma(q_1)$ implies that $u \mapsto v$ by the vertical move of $A$. Therefore, $v \in \text{Steps}_{\Delta}^{n+1}(t)$.

Now, it only remains to show that a final state is either reachable or non-reachable simultaneously from both $t$ and $t'$. In other words, we want to prove that $\langle q_f \rangle \in \text{Steps}_{\Delta}^n(t)$ if and only if $\langle q_f \rangle \in \text{Steps}_{\Delta}^n(t')$, where $q_f \in Q_f$, $q_f \in Q_f'$.

Proof. $\Longrightarrow$: Suppose $\langle q_f \rangle \in \text{Steps}_{\Delta}^n(t)$ for some $q_f \in Q_f$ and $n \in \mathbb{N}$. By Lemma 1, there is $v' \in \text{Steps}_{\Delta}^n(t')$ such that $\langle q_f \rangle < \Gamma(v')$. Therefore, we can say that $\Gamma(v') = \langle q_f \rangle$, where $q' \geq q_f$. By definition of $\Gamma$, in this case $v' = \Gamma(v')$. Now, $q' \cap Q_f$ is non-empty (contains $q_f$), which means that $q' \in Q_f'$.

$\Longleftarrow$: Suppose $v' = \langle q_f' \rangle \in \text{Steps}_{\Delta}^n(t')$ for some $q_f' \in Q_f'$ and $n \in \mathbb{N}$. Again, $\Gamma(v') = v'$. Now, $q_f'$ being in $Q_f'$ implies that $q_f' \cap Q_f$ is non-empty; that is, there is $q$ such that $q \in q_f'$, and $q \in Q_f$. Let $v = \langle q_f \rangle - a$ tree from $T(Q \cup \{f\})$. Since $q \in q_f'$, we can see that $v < v' = \Gamma(v')$. Then, by Lemma 1, $v \in \text{Steps}_{\Delta}^n(t)$; and, as we already know, $q \in Q_f$.

Corollary 1. For any FSTA there exists an equivalent complete FSTA, i.e. such that any pair of states has a matching rule in $\Delta$.

Proof. The automaton $A'$ built in the above proof is complete.

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