Minimal length, maximal momentum and the entropic force law

Kourosh Nozari\textsuperscript{a,*}, Pouria Pedram\textsuperscript{b,†}, and M. Molkara\textsuperscript{c,‡}

\textsuperscript{a}Department of Physics, Faculty of Basic Sciences, University of Mazandaran, P. O. Box 47416-95447, Babolsar, Iran
\textsuperscript{b}Department of Physics, Science and Research Branch, Islamic Azad University, Tehran, Iran
\textsuperscript{c}Department of Physics, Central Tehran Branch, Islamic Azad University, Tehran, Iran

November 10, 2011

Abstract

Different candidates of quantum gravity proposal such as string theory, noncommutative geometry, loop quantum gravity and doubly special relativity, all predict the existence of a minimum observable length and/or a maximal momentum which modify the standard Heisenberg uncertainty principle. In this paper, we study the effects of minimal length and maximal momentum on the entropic force law formulated recently by E. Verlinde.

PACS: 04.60.-m
Keywords: Entropic Force; Quantum Gravity Phenomenology; Generalized Uncertainty Principle

1 Introduction

Various approaches to quantum gravity proposal support the idea that near the Planck scale, the standard Heisenberg uncertainty principle should be replaced by the so-called Generalized Uncertainty Principle (GUP) (see [1,2] and references therein). In fact, string theory, loop quantum gravity, non-commutative geometry, doubly special relativity and the TeV black hole physics all indicate the existence of a minimum observable length.

\textsuperscript{*}knozari@umz.ac.ir
\textsuperscript{†}p.pedram@sbu.ac.ir
\textsuperscript{‡}molkara.marziye@gmail.com
and/or a maximal momentum¹ in the high energy, quantum gravity era [3-5]. The existence of these natural cutoffs has phenomenologically interesting implications in all high energy physics problems (see for instance [6-9] and references therein). Although the order of magnitude of the corresponding quantum gravity corrections are generally very small, these are direct footprints of the Planck scale physics and contain some key attributes of ultimate quantum gravity scenario. The idea that a gravitational system could be regarded as a thermodynamical system has attracted a lot of attention recently (see [10] and references therein). In this viewpoint, E. Verlinde has reported his achievement on the issue of Entropic Force [11]. Verlinde suggested that gravity has an entropic origin. He postulated that the change of entropy near the holographic screen is linear in the displacement $\Delta x$, namely

$$\Delta S = 2\pi k_B \frac{mc}{\hbar} \Delta x$$  (1)

where $m$ is the mass of the test particle. The effective entropic force acting on the test particle due to the change in entropy obeys the first law of thermodynamics

$$\Delta W = T\Delta S = F\Delta x,$$  (2)

where $T$ is the temperature of the holographic screen. If one takes the Unruh temperature [12] experienced by an observer in an accelerated frame whose acceleration is $a$ to be

$$T = \frac{1}{2\pi k_B} \frac{\hbar a}{c},$$  (3)

then, as Vancea and Santos [13] have pointed out, the postulate (1) is essentially quantum in nature. They have used the entropic postulate to determine the quantum uncertainty in the laws of inertia and gravity. In addition, they have considered the most general quantum property of the matter represented by the uncertainty principle. Vancea and Santos have postulated an expression for the uncertainty of the entropy such that it is the simplest quantum generalization of the postulate of the variation of the entropy. This expression reduces to the variation of the entropy in the absence of uncertainties. In this way, they obtained the following generalization of relation (1)

$$\Delta S = 2\pi k_B \left( \frac{\Delta x}{l_c} + \frac{\Delta p}{mc} \right),$$  (4)

where $l_c = \frac{\hbar}{mc}$ is the Compton length. Then by using the Heisenberg uncertainty relation

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$  (5)

¹Note that we use the phrase maximal momentum instead of maximum measurable momentum. This is because while with lower bound for position fluctuations, one can rightfully claim that there is a minimum measurable distance, the way from an upper bound of momentum fluctuations to a maximum measurable momentum is not so clear. In fact, existence of an upper bound for momentum fluctuations just means that momentum measurements cannot be arbitrarily imprecise, but it says nothing about the measured momentum values (or momentum expectation values).
and replacing $\Delta p$ in (1) by $\Delta p = \frac{h}{2\Delta x}$, they obtained quantum correction of the Newton’s second law as follows

$$F(\Delta x) = ma + \frac{h}{2m} \left( \frac{ha}{c^2} - p \right)(\Delta x)^{-2}. \quad (6)$$

Recently Ghosh in Ref. [14] has generalized this relation by using the following generalized uncertainty principle

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2} \left[ 1 + \beta \left( (\Delta p)^2 + \langle p \rangle^2 \right) + 2 \beta \left( (\Delta p) \right)^2 + \langle p \rangle^2 \right], \quad (7)$$

where $p^2 = \sum_{i=1}^{3} p_i^2$, $\beta \sim \frac{1}{(M_P c)^2} = \frac{\ell_P^2}{2\hbar^2}$, $M_P$ = Planck mass, $M_P c^2$ = Planck energy $\approx 10^{19}$GeV. Within this framework, Ghosh has shown that the minimum length scale modifies the entropic force law. In the presence of a minimal observable length, the generalized uncertainty principle can be written as [1]

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left( 1 + \beta (\Delta p)^2 \right). \quad (8)$$

The minimum measurable length in this case is $\Delta x = \sqrt{\beta} \hbar$. In this way, Ghosh has found for corrected entropic force law the following generalized expression [14]

$$F = ma + \frac{h}{2mc^2(\Delta x)^2} (ah - pc^2) \left( 1 + \frac{\beta h^2}{4(\Delta x)^2} \right). \quad (9)$$

In what follows, we generalize this expression for the case that there are both a minimal observable length and a maximal momentum. In fact, the previous analysis has the shortcoming that test particle’s momentum can attain any values without upper bound. As we have stated, based on Doubly Special Relativity theories, test particle’s momentum has also an upper bound leading to the idea of a maximal momentum [15, 16]. In other words, at least the newly proposed doubly special relativity theories impose an upper bound on test particle’s momentum and this upper bound leads to a modification of the entropic force law.

## 2 Verlinde’s entropic force law with minimal length and maximal momentum

In the line of the mentioned progresses, here we are going to consider the effects of the minimal observable length and maximal momentum on the entropic force law. In other words, we add the existence of a maximal momentum as a new ingredient to the analysis performed by Ghosh [14]. In this respect, we obtain a general expression for entropic force law in the framework of the newly proposed GUP by Ali et al. [4, 5]. To do this end, we consider a general GUP which contains both a minimum observable length and a maximal momentum. The new ingredient of this GUP is realized through those terms.
that are related to the existence of $\langle p \rangle^2$ in equation (7). In this respect, we start with equation (7) expressed in the following form

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + \beta \left( (\Delta p)^2 + \langle p \rangle^2 \right) \right].$$

(10)

This relation can be expressed as follows

$$\Delta p = \frac{\Delta x}{\beta \hbar} \pm \frac{\Delta x}{\beta \hbar} \left( 1 - \frac{\beta \hbar^2}{(\Delta x)^2} - \frac{\beta^2 \hbar^2 \langle p \rangle^2}{(\Delta x)^2} \right)^{\frac{1}{2}}. \quad (11)$$

To have correct limiting result when $\beta \to 0$, we find

$$\Delta p = \frac{1}{2} \frac{\hbar}{\Delta x} \left( 1 + \frac{\beta \hbar^2}{4(\Delta x)^2} + \beta \langle p \rangle^2 \left( 1 + \frac{1}{2} \frac{\beta \hbar^2}{(\Delta x)^2} + \frac{1}{4} \frac{\beta^2 \hbar^2}{(\Delta x)^2} \langle p \rangle^2 \right) \right). \quad (12)$$

From the reality of $\Delta p$, we obtain

$$\Delta x = \hbar \sqrt{\beta \left( 1 + \beta \langle p \rangle^2 \right)^{\frac{1}{2}}}. \quad (13)$$

This result can be described as a new, momentum-dependent minimal length scale. In this framework, the entropic force law in the presence of the minimal length, equation (9), generalizes to the following expression

$$F = ma + \frac{\hbar}{2mc^2(\Delta x)^2} (a \hbar - pc^2) \left( 1 + \frac{\beta \hbar^2}{4(\Delta x)^2} + \beta \langle p \rangle^2 \left( 1 + \frac{1}{2} \frac{\beta \hbar^2}{(\Delta x)^2} + \frac{1}{4} \frac{\beta^2 \hbar^2}{(\Delta x)^2} \langle p \rangle^2 \right) \right). \quad (14)$$

This relation represents the effects of both minimal observable length and the maximal momentum as quantum gravity features on the entropic force acting on a quantum test particle.

From another perspective, we note that quantum commutators originated in doubly special relativity which are consistent also with string theory and quantum black hole physics, ensure via the Jacobi identity that $[x_i, x_j] = 0 = [p_i, p_j]$. Under specific assumptions, these features lead to the following commutator [3–5]

$$[x_i, p_j] = i\hbar \left[ \delta_{ij} - \alpha \left( p \delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 \left( p^2 \delta_{ij} + 3 p_i p_j \right) \right], \quad (15)$$

where $\alpha = \frac{\alpha_0}{M_p c} = \frac{\alpha_0 \ell_p}{\hbar}$. Equation (15) in 1-dimension and up to $O(\alpha^2)$ terms gives

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle \right], \quad (16)$$

where constant $\alpha$ is related to $\beta$ through dimensional analysis with the expression $[\beta] = [\alpha^2]$. Like what we have done before, we obtain

$$\Delta p = \frac{1}{8\alpha^2} \left( \frac{2\Delta x}{\hbar} \pm \left( \frac{4(\Delta x)^2}{\hbar^2} - 16\alpha^2 (1 - 2\alpha \langle p \rangle) \right)^{\frac{1}{2}} \right). \quad (17)$$
By simplification, we get

$$\Delta p = \frac{\hbar}{2\Delta x} \left( 1 + \frac{\alpha^2\hbar^2}{(\Delta x)^2} \left( 1 - 2\alpha\langle p \rangle \right) \right) \left( 1 - 2\alpha\langle p \rangle \right).$$

(18)

Therefore, we obtain the following generalized entropic force law

$$F = ma + \frac{\hbar}{2mc^2(\Delta x)^2} \left( a\hbar - pc^2 \right) \left( 1 + \frac{\alpha^2\hbar^2}{(\Delta x)^2} \left( 1 - 2\alpha\langle p \rangle \right) \right) \left( 1 - 2\alpha\langle p \rangle \right).$$

(19)

The most significant difference is that in equation (14) the $\langle p \rangle$ term appears to be quadratic while in equation (19) there is linear term in momentum. We note that equation (19) (and essentially the formalism of the present paper) shows that the weak equivalence principle could be violated at the quantum gravity level. In fact, as Ali has shown in Ref. [17], by studying the Heisenberg equations of motions in the presence of GUP, the acceleration is no longer mass-independent because of the mass-dependence through the momentum $p$. Therefore, the equivalence principle is dynamically violated in the GUP framework. As Ali has stated in Ref. [17], this result agrees also with cosmological implications of the dark sector where a long-range force acting only between non-baryonic particles would be associated with a large violation of the weak equivalence principle [18]. The violation of the equivalence principle has been obtained also in the context of the string theory [19], where the extended nature of strings are subjected to tidal forces and do not follow the spacetime geodesics.

3 Entropic force from GUP modified Hamiltonian

In this section, we study the effects of the generalized uncertainty principle on the Hamiltonians of the quantum systems in quasi-space representation. In fact, when we consider energies beyond the Planck energy, the usual commutator relation between the position and momentum operators does not hold anymore. However, we can still write the Hamiltonian in terms of such operators. In this case, some extra terms should be added to the Hamiltonian. More precisely, the presence of a minimal length and/or a maximal momentum adds some terms proportional to $p^3$ and $p^4$ to the Hamiltonians of physical systems. In the following subsections, we first consider the case of the presence of a minimal length solely and find the modified version of the Newton’s law in two above mentioned scenarios. Then, for the case of both a minimal length and a maximal momentum, we obtain the relations for the uncertainty of the entropy and their corresponding entropic forces.

3.1 GUP with a minimal length

Let us consider the Hamiltonian of a quantum system in the presence of a GUP which implies a minimal length in quasi-space representation [6–9] i.e.

$$H = \frac{p^2}{2m} + \beta \frac{p^4}{3m} + V(x).$$

(20)
From equation (2) we find
\[
\frac{p}{m} \delta p + \beta \frac{4}{3} \frac{p^3}{m} \delta p + F \delta x = T \delta S.
\] (21)

Now we can follow two different procedures for finding the uncertainty of the entropy. First, let us consider the Ghosh proposal, namely [14]
\[
\delta S_G = 2 \pi k_B \left( \frac{\delta x}{l_c} + \frac{\delta p}{mc} \right),
\] (22)

where \( l_c = \frac{\hbar}{mc} \) is the particle’s Compton length. By using \( \delta p = \frac{\hbar}{2\delta x} \left( 1 + \frac{\beta \hbar^2}{4(\delta x)^2} \right), \) (see Ref. [14]) we have
\[
F_G = ma + \frac{\hbar}{2m} \left( \frac{\hbar a}{c^2} - p - \frac{4}{3} \beta p^3 \right) \left( 1 + \frac{\beta \hbar^2}{4(\delta x)^2} \right) (\delta x)^{-2},
\] (23)

which differs with Ghosh result [14].

The second procedure is due to Vancea and Santos (VS) [13]. The idea is writing the uncertainty of the entropy as \( \delta S = 2 \pi k_B \left( \frac{\delta x}{l_c} + \frac{\delta K}{mc^2} \right) \) where \( K \) is the kinetic part of the Hamiltonian. For our GUP, we have
\[
\delta S_{VS} = 2 \pi k_B \left( \frac{\delta x}{l_c} + \frac{p \delta p}{mc^2} + \frac{4}{3} \beta p^3 \delta p \right),
\] (24)

which using \( \delta p = \frac{\hbar}{2\delta x} \left( 1 + \frac{\beta \hbar^2}{4(\delta x)^2} \right) \) results in
\[
F_{VS} = ma + \frac{\hbar p}{2m} \left( \frac{\hbar a}{mc^2} - 1 \right) \left( 1 + \frac{4}{3} \beta p^2 \right) \left( 1 + \frac{\beta \hbar^2}{4(\delta x)^2} \right) (\delta x)^{-2}.
\] (25)

Comparing equations (23) and (25), we see that these two approaches are not equivalent. In fact the difference of these forces is proportional to the acceleration as follows
\[
F_G - F_{VS} = \frac{\hbar a}{2m^2 c^3} \left( mc - p - \frac{4}{3} \beta p^3 \right) \left( 1 + \frac{\beta \hbar^2}{4(\delta x)^2} \right) (\delta x)^{-2}.
\] (26)

This difference is due to different assumptions adopted for \( \delta S_G \) and \( \delta S_{VS} \).

3.2 GUP with a minimal length and a maximal momentum

The modified Hamiltonian in the presence of both a minimal length and a maximal momentum can be written as [3–5]
\[
H = \frac{p^2}{2m} - \alpha \frac{p^3}{m} + 5\alpha^2 \frac{p^4}{m} + V(x).
\] (27)
We note that due to the presence of the cubic term of the particles’ momentum, the time reversal invariance violates as a result of the GUP with maximal momentum. In this situation, the Hamiltonian is not certainly the physical energy of the system under consideration. Nevertheless, the Heisenberg equation of motion is still valid.

Now, from $\delta W = T \delta S$ we find

$$\frac{p}{m} \delta p - 3 \alpha \frac{p^2}{m} \delta p + 20 \alpha^2 \frac{p^3}{m} \delta p + F \delta x = T \delta S. \quad (28)$$

To proceed further, we need to choose the functional form of $\delta S$. Following the Ghosh’s proposal $\delta S_G = 2 \pi k_B \left( \frac{\delta x}{l_c} + \frac{\delta \rho}{mc} \right)$ and $\delta p = \frac{\hbar}{2 \delta x} \left( 1 + \frac{\beta h^2}{4(\delta x)^2} \right)$, we obtain

$$F_G = ma + \frac{\hbar}{2m} \left( \frac{ha}{c^2} p + 3 \alpha p^2 - 20 \alpha^2 p^3 \right) \left( 1 + \frac{\beta h^2}{4(\delta x)^2} \right)(\delta x)^{-2}. \quad (29)$$

Alternatively, using Vancea and Santos proposal for the uncertainty of the entropy we find

$$\delta S_{VS} = 2 \pi k_B \left( \frac{\delta x}{l_c} + \frac{p \delta p}{mc^2} - 3 \alpha \frac{p^2 \delta p}{mc^2} + 20 \alpha^2 \frac{p^3 \delta p}{mc^2} \right). \quad (30)$$

Therefore, we obtain the corresponding entropic force where we have both a minimal length and a maximal momentum

$$F_{VS} = ma + \frac{\hbar p}{2m} \left( \frac{ha}{mc^3} - 1 \right) \left( 1 - 3 \alpha p + 20 \alpha^2 p^2 \right) \left( 1 + \frac{\beta h^2}{4(\delta x)^2} \right)(\delta x)^{-2}. \quad (31)$$

For this case we have

$$F_G - F_{VS} = \frac{\hbar^2 a}{2mc^3} \left( mc - p + 3 \alpha p^2 - 20 \alpha^2 p^3 \right) \left( 1 + \frac{\beta h^2}{4(\delta x)^2} \right)(\delta x)^{-2}. \quad (32)$$

Again, the difference is due to different assumptions adopted for $\delta S_G$ and $\delta S_{VS}$.

### 4 Summary

Gravity seems to have an entropic origin and a gravitational system could be regarded as a thermodynamical system. This idea comes from a thermodynamical interpretation of gravitational field equations. Based on the Verlinde’s conjecture, the change of entropy near the holographic screen is linear in the displacement of the test particle about holographic screen. On the other hand, all approaches to quantum gravity proposal support the idea of existence of a minimal observable length of the order of string or Planck length. In addition, theories such as doubly special relativity lead to existence of an upper bound of test particle’s momentum. In this respect, these theories realize a maximal momentum.
too. While with lower bound for position fluctuations, one can rightfully claim that there is a minimum measurable distance, existence of an upper bound for momentum fluctuations just means that momentum measurements cannot be arbitrarily imprecise. For this reason we used only the phrase maximal momentum in our setup. It is reliable to expect that finite resolution of the spacetime points and also an upper bound on the test particle’s momentum affect the formulation of the entropic force law. The effect of the finite resolution of spacetime points (through existence of a minimal observable length) on the Verlinde’s entropic force law was studied by Ghosh [14]. Here we focused mainly on the simultaneous effects of both minimal length and maximal momentum on the formulation of the entropic force law. We generalized the entropic force law via a phenomenological interpretation of quantum gravity proposal which contains both a minimal observable length and a maximal momentum. This generalization could be important in the interpretation of entropic nature of gravity at Planck scale. We note that the formalism presented in this paper shows that the weak equivalence principle could be violated at the quantum gravity level. In fact, in the presence of the GUP, the test particles’ acceleration is no longer mass-independent because of the mass-dependence through the momentum $p$. So, the equivalence principle is dynamically violated in the GUP framework. Finally due to the presence of the cubic term of the particles’ momentum, the time reversal invariance violates as a result of the GUP with maximal momentum. In this situation, the Hamiltonian is not certainly the physical energy of the system under consideration. Nevertheless, the Heisenberg equation of motion is still valid.

Acknowledgments

The work of Kourosh Nozari is supported financially by the Research Council of the University of Mazandaran.

References

1. A. Kempf, G. Mangano and R. B. Mann, Hilbert space representation of the minimal length uncertainty relation, Phys. Rev. D 52 (1995) 1108.

2. K. Nozari and P. Pedram, Minimal length and bouncing particle spectrum, Europhys. Lett. 92 (2010) 50013.

3. P. Pedram, On the modification of the Hamiltonians’ spectrum in gravitational quantum mechanics, Erouphys. Lett. 89 (2010) 50008.

4. A. F. Ali, S. Das and E. C. Vagenas, Discreteness of space from the generalized uncertainty principle, Phys. Lett. B 678 (2009) 497.

5. S. Das, E. C. Vagenas and A. F. Ali, Discreteness of space from GUP II: Relativistic wave equations, Phys. Lett. B 690 (2010) 407.
6. S. Das and E. C. Vagenas, *Universality of quantum gravity corrections*, Phys. Rev. Lett. **101** (2008) 221301. See also S. Das and E. C. Vagenas, Phys. Rev. Lett. **104** (2010) 119002.

7. K. Nozari, *Some aspects of Planck scale quantum optics*, Phys. Lett. B **629** (2005) 41.
   K. Nozari and B. Fazlpour, *Generalized uncertainty principle, modified dispersion relations and the early universe thermodynamics*, Gen. Rel. Grav. **38** (2006) 1661.
   M. R. Setare, *Corrections to the Cardy-Verlinde formula from the generalized uncertainty principle*, Phys. Rev. D **70** (2004) 087501.
   M. R. Setare, *The Generalized Uncertainty Principle and Corrections to the Cardy-Verlinde Formula in SAdS5 Black Holes*, Int. J. Mod. Phys. A **21** (2006) 1325.
   B. Vakili, *Cosmology with minimal length uncertainty relations*, Int. J. Mod. Phys. D **18** (2009) 1059.
   B. Vakili, P. Pedram and S. Jalalzadeh, *Late time acceleration in a deformed phase space model of dilaton cosmology*, Phys. Lett. B **687** (2010) 119.

8. S. Das and E. C. Vagenas, *Phenomenological implications of the generalized uncertainty principle*, Can. J. Phys. **87** (2009) 233.

9. P. Pedram, *A class of GUP solutions in deformed quantum mechanics*, Int. J. Mod. Phys. D **19** (2010) 2003.

10. T. Padmanabhan, *Thermodynamical aspects of gravity: New insights*, Rep. Prog. Phys. **73** (2010) 046901.

11. E. P. Verlinde, *On the origin of gravity and the laws of Newton*, JHEP **1104** (2011) 029, [arXiv:1001.0785].

12. W. G. Unruh, *Notes on black hole evaporation*, Phys. Rev. D **14** (1976) 870.

13. I. V. Vancea and M. A. Santos, *Entropic law of force, emergent gravity and the uncertainty principle*, [arXiv:1002.2454].

14. S. Ghosh, *Planck scale effect in the entropic force law*, [arXiv:1003.0285].

15. J. Magueijo and L. Smolin, *Lorentz invariance with an invariant energy scale*, Phys. Rev. Lett. **88** (2002) 190403.

16. J. Magueijo and L. Smolin, *String theories with deformed energy-momentum relations and a possible nontachyonic bosonic string*, Phys. Rev. D **71** (2005) 026010.

17. A. F. Ali, *Minimal Length in Quantum Gravity, Equivalence Principle and Holographic Entropy Bound*, Class. Quant. Grav. **28** (2011) 065013.

18. J. A. Keselman, A. Nusser and P. J. E. Peebles, *Cosmology with Equivalence Principle Breaking in the Dark Sector*, Phys. Rev. D **81** (2010) 063521.
19. P. F. Mende, *String Theory at Short Distance and the Principle of Equivalence*, arXiv:hep-th/9210001; R. Lafrance and R. C. Myers, *Gravity’s Rainbow: Limits for the applicability of the equivalence principle*, Phys. Rev. D 51 (1995) 2584; L. J. Garay, *Quantum gravity and minimum length*, Int. J. Mod. Phys. A 10 (1995) 145.