An Example of Double Cross Coproducts with Non-trivial Left Coaction and Right Coaction in Strictly Braided Tensor Categories *

Shouchuan Zhang  Bizhong Yang
Department of Mathematics, Hunan University
Changsha 410082, P.R.China.  E-mail:z9491@yahoo.com.cn
Beishang Ren
Department of Mathematics, Guangxi Normal College
Nanning 530001, P.R.China.

Abstract

An example of double cross coproducts with both non-trivial left coaction and non-trivial right coaction in strictly braided tensor categories is given.
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0 Introduction and Preliminaries

The double cross coproducts in braided tensor categories have been studied by Y.Bespalov, B.Drabant and author in [2] [12]. However, hitherto any examples of double cross coproducts with both non-trivial left coaction and non-trivial right coaction in strictly braided tensor categories (i.e. the braiding is not symmetric) have not been found. Therefore Professor S.Majid asked if there is such example.

In this paper we first give the cofactorisation theorem of Hopf algebras in braided tensor categories. Using the cofactorisation theorem and Sweedler four dimensional Hopf algebra, we construct such example.

We denote the multiplication, comultiplication, evaluation $d$, coevaluation $b$, braiding and inverse braiding by

\[\begin{array}{c}
\bigtriangledown, \\
\bigtriangledown^{-1}, \\
\bigtriangleup, \\
\bigtriangleup^{-1}, \\
\bigotimes, \\
\bigotimes^{-1}
\end{array}\]

and \[\begin{array}{c}
\bigcirc, \\
\bigcirc^{-1}
\end{array}\]

respectively. For convenience, we denote the inverse of morphism $f$ by $\overline{f}$ if $f$ has an inverse.

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Since every braided tensor category is always equivalent to a strict braided tensor category by [12, Theorem 0.1], we can view every braided tensor category as a strict braided tensor category and use braiding diagrams freely.

1 The cofactorisation theorem of bialgebras in braided tensor categories

Throughout this section, we work in braided tensor category \((\mathcal{C}, \mathcal{C})\) and assume that all Hopf algebras and bialgebras are living in \((\mathcal{C}, \mathcal{C})\) unless otherwise stated. We give the cofactorisation theorem of bialgebras in braided tensor categories in this section.

We first recall the double bicrossproducts in [12]. Let \(H\) and \(A\) be two bialgebras in braided tensor categories and
\[
\alpha : H \otimes A \to A , \quad \beta : H \otimes A \to H, \\
\phi : A \to H \otimes A , \quad \psi : H \to H \otimes A
\]
morphisms in \(\mathcal{C}\).

\[
\Delta_D =: A H \blacklozenge \psi, \quad m_D =: A H \blacklozenge \alpha H
\]
and \(\epsilon_D = \epsilon_A \otimes \epsilon_H , \quad \eta_D = \eta_A \otimes \eta_H\). We denote \((A \otimes H, m_D, \eta_D, \Delta_D, \epsilon_D)\) by

\[
A_\phi \blacklozenge_\beta^\psi H,
\]
which is called the double bicrossproduct of \(A\) and \(H\).

When \(\phi\) and \(\psi\) are trivial, we denote \(A_\phi \blacklozenge_\beta^\psi H\) by \(A_\alpha \blacklozenge_\beta H\). When \(\alpha\) and \(\beta\) are trivial, we denote \(A_\phi \blacklozenge_\beta^\psi H\) by \(A^\phi \blacklozenge H\). We call \(A_\alpha \blacklozenge_\beta H\) a double cross product and denote it by \(A \blacklozenge H\) in short. We call \(A^\phi \blacklozenge H\) a double cross coproduct.

**Theorem 1.1** (Factorisation theorem) (See [9, Theorem 7.2.3]) Let \(X\), \(A\) and \(H\) be bialgebras or Hopf algebras. Assume that \(j_A\) and \(j_H\) are bialgebra or Hopf algebra morphisms from \(A\) to \(X\) and \(H\) to \(X\) respectively. If \(\xi =: m_X(j_A \otimes j_H)\) is an isomorphism from \(A \otimes H\) onto \(X\) as objects in \(\mathcal{C}\), then there exist morphisms
\[
\alpha : H \otimes A \to A \quad \text{and} \quad \beta : H \otimes A \to H
\]
such that \(A_\alpha \blacklozenge_\beta H\) becomes a bialgebra or Hopf algebra and \(\xi\) is a bialgebra or Hopf algebra isomorphism from \(A_\alpha \blacklozenge_\beta H\) onto \(X\).
Proof. Set
\[ \zeta =: \begin{array}{c}
\begin{array}{c}
H \quad A \\
\scriptstyle uA \\
\scriptstyle uH \\
\end{array}
\end{array} \quad \alpha =: \begin{array}{c}
\begin{array}{c}
H \quad A \\
\scriptstyle \zeta \\
\scriptstyle A \\
\end{array}
\end{array} \quad \text{and} \quad \beta =: \begin{array}{c}
\begin{array}{c}
H \quad A \\
\scriptstyle \zeta \\
\scriptstyle H \\
\end{array}
\end{array} \]

We see
\[ \begin{array}{c}
\begin{array}{c}
H \quad H \quad A \\
\scriptstyle \zeta \\
\scriptstyle \zeta \\
\scriptstyle uA \\
\scriptstyle uH \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
H \quad H \quad A \\
\scriptstyle \zeta \\
\scriptstyle \zeta \\
\scriptstyle uA \\
\scriptstyle uH \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
H \quad H \quad A \\
\scriptstyle \zeta \\
\scriptstyle \zeta \\
\scriptstyle uA \\
\scriptstyle uH \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
H \quad H \quad A \\
\scriptstyle \zeta \\
\scriptstyle \zeta \\
\scriptstyle uA \\
\scriptstyle uH \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
H \quad H \quad A \\
\scriptstyle \zeta \\
\scriptstyle \zeta \\
\scriptstyle uA \\
\scriptstyle uH \\
\end{array}
\end{array}.
\]

Thus
\[ \begin{array}{c}
\begin{array}{c}
H \quad H \quad A \\
\scriptstyle \zeta \\
\scriptstyle \zeta \\
\scriptstyle uA \\
\scriptstyle uH \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
H \quad H \quad A \\
\scriptstyle \zeta \\
\scriptstyle \zeta \\
\scriptstyle uA \\
\scriptstyle uH \\
\end{array}
\end{array} \quad \ldots \ldots (1)
\]

Similarly we have
\[ \begin{array}{c}
\begin{array}{c}
H \quad A \\
\scriptstyle \zeta \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
H \quad A \\
\scriptstyle \zeta \\
\end{array}
\end{array} \quad \ldots \ldots (2)
\]

We also have
\[ \zeta(\eta \otimes \text{id}) = \text{id} \otimes \eta \quad \text{and} \quad \zeta(\text{id} \otimes \eta) = \eta \otimes \text{id} \quad \ldots \ldots (3)
\]

It is clear that \( \zeta \) is a coalgebra morphism from \( H \otimes A \) to \( A \otimes H \), since \( j_A, j_H \) and \( m_X \) all are coalgebra homorphisms. Thus we have
\[ \begin{array}{c}
\begin{array}{c}
H \quad A \\
\scriptstyle \zeta \\
\scriptstyle \zeta \\
\scriptstyle uA \\
\scriptstyle A \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
H \quad A \\
\scriptstyle \zeta \\
\scriptstyle \zeta \\
\scriptstyle uA \\
\scriptstyle A \\
\end{array}
\end{array} \quad \text{and} \quad (\varepsilon \otimes \varepsilon) \zeta = (\varepsilon \otimes \varepsilon). \quad \ldots \ldots (4) \]
We now show that \((A, \alpha)\) is an \(H\)-module coalgebra:

\[
\begin{align*}
    H A & \overset{\alpha}{\rightarrow} H A, \\
    A & \rightarrow A
\end{align*}
\]

by (1)

and \(\alpha(\eta \otimes id_A) = (id_A \otimes \epsilon) \zeta (\eta \otimes id_A) \overset{\text{by (3)}}{=} id_A\).

We see that \(\epsilon \circ \alpha = (\epsilon \otimes \epsilon) \zeta \overset{\text{by (4)}}{=} \epsilon \otimes \epsilon\) and

\[
\begin{align*}
    \begin{array}{c}
    H \overset{\alpha}{\rightarrow} H \overset{\alpha}{\rightarrow} H \\
    A \rightarrow A \rightarrow A
    \end{array}
    & \overset{\text{by (4)}}{=} \begin{array}{c}
    H \overset{\alpha}{\rightarrow} H \overset{\alpha}{\rightarrow} H \\
    A \rightarrow A \rightarrow A
    \end{array},
\end{align*}
\]

Thus \((A, \alpha)\) is an \(H\)-module coalgebra. Similarly, we can show that \((H, \beta)\) is an \(A\)-module coalgebra.

Now we show that conditions \((M1)\)–\((M4)\) in [12, p37] hold. By (3), we easily know that \((M1)\) holds. Next we show that \((M2)\) holds.

\[
\begin{align*}
    \begin{array}{c}
    H \overset{\alpha}{\rightarrow} H \overset{\alpha}{\rightarrow} H \\
    A \rightarrow A \rightarrow A
    \end{array}
    & \overset{\text{by (4)}}{=} \begin{array}{c}
    H \overset{\alpha}{\rightarrow} H \overset{\alpha}{\rightarrow} H \\
    A \rightarrow A \rightarrow A
    \end{array},
\end{align*}
\]

Thus \((M2)\) holds. Similarly, we can get the proofs of \((M3)\) and \((M4)\). Consequently, \(A \alpha \bowtie \beta H\) is a bialgebra or Hopf algebra by [12, Corollary 1.8]. It suffices to show that \(\zeta\) is a bialgebra.
morphism from $A_\alpha \bowtie \beta H$ to $X$. Let $D = A_\alpha \bowtie \beta H$. Since

$$
\begin{array}{c}
\text{Diagram (4)}
\end{array}
$$

we have that $\xi$ is a bialgebra morphism from $A_\alpha \bowtie \beta H$ to $X$ by [12, Lemma 2.5].

**Theorem 1.2 (Co-factorisation theorem)** Let $X$, $A$, and $H$ be bialgebras or Hopf algebras. Assume that $p_A$ and $p_H$ are bialgebra or Hopf algebra morphisms from $X$ to $A$ and $X$ to $H$, respectively. If $\xi = (p_A \otimes p_H) \Delta_X$ is an isomorphism from $X$ onto $A \otimes H$ as objects in $C$, then there exist morphisms:

$$
\phi : A \rightarrow H \otimes A \quad \text{and} \quad \psi : H \rightarrow H \otimes A
$$

such that $A^\phi \bowtie \psi H$ becomes a bialgebra or Hopf algebra and $\xi$ is a bialgebra or Hopf algebra isomorphism from $X$ to $A^\phi \bowtie \psi H$.

**Proof.** Set

$$
\begin{array}{c}
\text{Diagram 1.2}
\end{array}
$$

We can complete the proof by turning upside down the diagrams in the proof of the preceding theorem.

From now on, we always consider Hopf algebras over field $k$ and the diagram

$$
\begin{array}{c}
\text{Diagram 1.3}
\end{array}
$$

always denotes the ordinary twisted map: $x \otimes y \rightarrow y \otimes x$. Our diagrams only denote homomorphisms between vector spaces, so two diagrams can have the additive operation.

Let $H$ be an ordinary bialgebra and $(H_1, R)$ an ordinary quasitriangular Hopf algebra over field $k$. Let $f$ be a bialgebra homomorphism from $H_1$ to $H$. Then there exists a bialgebra $B$, written as $B(H_1, f, H)$, living in $(H_1, M, C^R)$. Here $B(H_1, f, H) = H$ as algebra, its counit is $\epsilon_H$, 5
and its comultiplication and antipode are

\[
\begin{align*}
B &\overset{\Delta_H}{\to} B \\
\downarrow & \\
H &\overset{\mathcal{R}}{\to} H
\end{align*}
\]

(see [8, Theorem 4.2]), respectively. In particular, when \( H = H_1 \) and \( f = id_H \), \( B(H_1, f, H) \) is a braided group, called the braided group analogue of \( H \) and written as \( H \).

\( R \) is called a weak \( R \)-matrix of \( A \otimes H \) if \( R \) is invertible under convolution with

\[
\begin{align*}
A \otimes H &\overset{R}{\to} A \otimes H \\
\downarrow & \\
A \otimes H &\overset{R}{\to} A \otimes H
\end{align*}
\]

is a quasitriangular structure of \( D \) and every quasitriangular structure of \( D \) is of this form ([8, Theorem 2.9]), where \( R = \sum R' \otimes R'' \), etc.

**Lemma 1.3** Under the above discussion, then

(i) \( \pi_A : D \to A \) and \( \pi_H : D \to H \) are bialgebra or Hopf algebra homomorphisms, respectively. Here \( \pi_A \) and \( \pi_H \) are trivial action, that is, \( \pi_A(h \otimes a) = \epsilon(h)a \) for any \( a \in A, h \in H \).

(ii) \( B(D, \pi_A, A) = A \) and \( B(D, \pi_H, H) = H \).

(iii) \( \pi_A : D \to A \) and \( \pi_H : D \to H \) are bialgebra or Hopf algebra homomorphisms, respectively.

**Proof.** (i) It is clear.

(ii) It is enough to show \( \Delta_B = \Delta_A \) since \( B = A \) as algebras, where \( B = B(D, \pi_A, A) \). See
Thus $\Delta_B = \Delta_A$. Similarly, we have $B(D, \pi_H, H) = H$.

(iii) See

and $\epsilon \circ \pi_H = \epsilon$. Thus $\pi_H$ is a coalgebra homomorphism.

Since the multiplications in $D$ and $H$ are the same as in $D$ and $H$, respectively, we have that $\pi_H$ is an algebra homomorphism by (i). Similarly, we can show that $\pi_A$ is a bialgebra homomorphism. \(\square\)

We now investigate the relation among braided group analogues of quasitriangular Hopf algebras $A$ and $H$ and their double crossed coproduct $D = A \bowtie R H$.

**Theorem 1.4** Under the above discussion, let $\xi = (\pi_A \otimes \pi_H)\Delta_D$. Then
and $\xi$ is surjective, where $ad$ denotes the left adjoint action of $H$.

(ii) Furthermore, if $A$ and $H$ are finite-dimensional, then $\xi$ is a bijective map from $D$ onto $A \otimes H$. That is, in braided tensor category $(D,M,C^{RD})$, there exist morphisms $\phi$ and $\psi$ such that

$$D \cong A^\phi \bowtie^\psi H$$

( as Hopf algebras )

and the isomorphism is $\left(\pi_A \otimes \pi_H\right)\Delta_D$.

(iii) If $H$ is commutative or $V = R$, then $\xi = id_D$.

Proof. (i)

(ii) By the proof of (i), $\xi$ is bijective and

$$\xi = id_D.$$
Applying the cofactorization theorem \[ \text{1.2} \] we complete the proof of (ii).

(iii) follows from (i).

Remark. Under the assumption of Theorem \[ \text{1.4} \] if we set

\[ A \phi' \bowtie \psi' H = A \bowtie^R H \text{ by } [3, \text{Lemma 1.3}] \]. We now see the relation among \( \phi, \psi, R, \phi' \) and \( \psi' \):

\[ \phi \text{ by proof of Th } \text{1.2} = \zeta (id \otimes \eta) = \text{ by Th } \text{1.3 (iii)} \]
and \( \psi \) by proof of Th 1.2, \( \zeta (\eta \otimes \text{id}) = \) 

by Th 1.3 (iii)
Furthermore, \( \psi = \psi' \) when \( H \) is commutative or \( R = V \). In this case, \( \Delta H = A = A \otimes R H \) as Hopf algebras living in braided tensor category \( (D\mathcal{M}, C^{R_D}) \).

2 An example

In this section, using preceding cofactorisation theorem, we give an example of double cross coproducts with both non-trivial left coaction and non-trivial right coaction in strictly braided tensor categories.

Let \( H^* = Hom(H, k) \) be the dual of finite-dimensional Hopf algebra \( H \). \( H^* \) can become a Hopf algebra under convolution (Theorem 9.1.3). That is, for any \( f, g \in H^*, h, h' \in H \),

\[
(f \ast g)(h) = \sum_{(h)} f(h_1)g(h_2), \Delta H^*(f)(h \otimes h') = f(hh'), S_{H^*}(f)(h) = f(S(h)).
\]
Assume \{e_{x_i} \mid i = 1,2,\cdots,n\} is the dual basis of \{x_i \mid i = 1,2,\cdots,n\}. Define \(d_H = \left\{ \begin{array}{ll} H^* \otimes H & \to k \\ f \otimes h & \to f(h) \end{array} \right\} \) and \(b_H = \left\{ \begin{array}{ll} k & \to H \otimes H^* \\ 1 & \to \sum_{i=1}^n x_i \otimes e_{x_i} \end{array} \right\} \). \(d_H\) and \(b_H\) are called evaluation and coevaluation of \(H\), respectively. It is clear that

\[
H^* = H^* H^* = H H^*
\]

Note the multiplication and comultiplication of \(H^*\) exactly are anti-multiplication and anti-comultiplication in \([8, \text{Proposition 2.4}]\).

Let \(A = H^* \text{cop}^\sim\), \(\tau = \begin{array}{l}\end{array}\) and \([b] = \begin{array}{l}\end{array}\). Thus \(\tau\) is a skew pairing and the Drinfeld Double \(D(H) = A \bowtie_r H\) (see [4] [5]). Furthermore, \([b]\) is a quasitriangular structure of \(D(H)\) by [10, Theorem 10.3.6].

**Lemma 2.1** Let \(H\) be a finite-dimensional Hopf algebra with \(\dim H > 1\). Then Drinfeld double \((D(H), [b])\) is not triangular.

**Proof.** Assume that \(x_i^s\)s are a basis of \(H\) with \(x_1 = 1_H\). Set \(x = x_2\) and see

\[
\begin{array}{l}
x \otimes \epsilon \otimes \epsilon \otimes e_x = 0 \\
x \otimes \epsilon \otimes e \otimes e_x = 0
\end{array}
\]

which implies that \([b]\) is not triangular. \(\square\)
Let us recall Sweedler’s four dimensional Hopf algebra $H_4$. That is, $H_4$ is a Hopf algebra generated $g$ and $x$ with relations

$$g^2 = 1, \quad x^2 = 0, \quad xg = -gx$$

and $\Delta(x) = x \otimes 1 + g \otimes x$, $\Delta(g) = g \otimes g$, $\epsilon(x) = 0$, $\epsilon(g) = 1$, $S(x) = xg$, $S(g) = g$. Let $\{e_1, e_g, e_x, e_{gx}\}$ denote the dual basis of $\{1, g, x, gx\}$.

**Example 2.2** Let $H$ be Sweedler’s four dimensional Hopf algebra over field $k$ with $\text{char } k \neq 2$. Let $D = D(H)$. Thus $D =: D \triangleright \triangleright [\mathbf{b}] D$ is quasitriangular, but it is not triangular by Lemma 2.1. Considering [3, Theorem 2.5], $B$ has a quasitriangular structure $R_B$, defined in preceding Theorem 1.4 with $U = V = 1 \otimes 1$, and $R_B$ never is triangular. Thus $(B, M, C_{R_B})$ is a strictly braided tensor category by [10, Theorem 10.4.2 (3)]. It follows from Theorem 1.4 (ii) that $D \triangleright \triangleright [\mathbf{b}] D \cong D \triangleright \triangleright \psi D$ for some $\phi$ and $\psi$. Furthermore, $D \triangleright \triangleright \psi D$ is a double cross coproduct. We shall show that both left coaction $\phi$ and right coaction $\psi$ are non-trivial.

**Proof.**

$$
\begin{array}{c}
D \\
\gamma \gamma \\
\hline
I \\
\gamma \\
D & D
\end{array}
$$
Compute:

\[ A = A \]

and \( \lambda =: \)

Also,

\[ D \]

\[ D \]

\[ D \]
and

\[
A \otimes \eta = e_{q_x} \otimes \eta
\]

\[
x \otimes \epsilon_{g_x} \otimes e_x =
\]
Let $u$ and $v$ denote the first term and the second term, respectively.

$$u = \ldots$$
\[ g \cdot \frac{1}{3} \cdot g + e_{gx} + 1 = 1 \]

and

\[ x = v \]

\[ e_{gx} \]
\[
\begin{align*}
&g x + e_{gx} = 1 + e_x \\
&x 1 + e_{gx} = 1 + e_x \\
&x 1 + e_{gx} = 1 + e_x \\
&x 1 + e_{gx} = 1 + e_x \\
&= 1.
\end{align*}
\]
Thus \( e_{gx} \otimes \eta = 2 \), but \( \eta D e_{gx} \otimes \eta = 0 \).

Consequently, \( \phi \) is not trivial.
For convenience, we omit $g$ in the following diagrams:
and

\[ e_x \otimes g = x \otimes \text{id} \otimes A \otimes H = xHA \otimes H \]
Thus $\psi$ is not trivial. $\square$

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