Low-density homogeneous symmetric nuclear matter: disclosing dinucleons in coexisting phases

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Received: date / Revised version: date

Abstract. The effect of in-medium dineutron and deuteron bound states on self-consistent single-particle fields in Brueckner, Bethe and Goldstone theory is investigated in symmetric nuclear matter at zero temperature. To this end, dinucleon bound state occurrences in the \( ^1S_0 \) and \( ^3S_1^3D_1 \) channels are explicitly accounted for –within the continuous choice for the auxiliary fields– while imposing self-consistency in Brueckner-Hartree-Fock approximation calculations. Searches are carried out at Fermi momenta in the range \( 0 < k_F \leq 1.75 \text{ fm}^{-1} \), using the Argonne \( v_{18} \) bare nucleon-nucleon potential without resorting to the effective mass approximation. As a result, two distinct solutions meeting the self-consistency requirement are found with overlapping domains in the interval \( 0.130 \text{ fm}^{-1} < k_F \leq 0.285 \text{ fm}^{-1} \), corresponding to mass densities between \( 10^{11.4} \) and \( 10^{12.4} \text{ g cm}^{-3} \). Effective masses as high as three times the nucleon mass are found in the coexistence domain, in resemblance to those for Heavy Fermion superconductors identified in strongly correlated systems. Coherence lengths and condensation fractions of the dinucleon bound states tied with the two solutions are studied and discussed in relationship with the BCS pairing gap solutions.

PACS. 21.65.-f Nuclear matter – 21.45.Bc Two-nucleon system – 21.65.Mn Equations of state of nuclear matter

1 Introduction

Dinucleon formation, pairing, clustering and condensation in nuclear media have been subjects of considerable research during the past few decades motivated by their role in heavy-ion collisions, collapsing stars, weakly bound nuclear systems and critical phenomena [12]. Even in their simplest conceptions, nuclear matter models may become exceedingly complex systems featuring strongly-interacting Fermions over a broad range of conditions. Densities may vary from highly diluted systems, as in nuclear halos or envelopes of core-collapse supernovae, up to a few times the saturation density, as in the core of neutron stars. The isospin asymmetry can also vary from zero, as in symmetric nuclear matter, up to unity, as in neutron stars. The isospin asymmetry can also vary from zero, as in symmetric nuclear matter, up to unity, as in pure neutron matter thought to prevail in the interior of neutron stars. Temperature can also be subject to wide variations from zero up to a few hundreds of MeV.

Among the various approaches to study interacting nucleons systems, the Hartree-Fock approximation summarizes important information upon which higher order corrections can be incorporated [5]. In this regard the Brueckner-Hartree-Fock (BHF) approximation can be regarded as a reasonable starting point to investigate homogeneous infinite nuclear matter [3][4]. Despite the broad body of studies made on the single-particle (sp) spectrum in infinite nuclear matter, there is no retrievable reference known to the authors of this work, where dinucleon bound states are explicitly treated in conjunction with the self-consistent search for sp fields within the BHF approximation, particularly at very low densities. In this work we address this issue by providing an explicit account for dinucleon bound states in the \( ^1S_0 \) and \( ^3S_1^3D_1 \) channels, expressed as isolated poles on the real axis below the continuum threshold, while seeking self-consistency. We have avoided the use of the effective mass approximation to represent the sp fields, aiming to rule out potential sources of ambiguities in our findings. As a consequence we obtain two distinct solutions satisfying the self-consistency requirement in the range \( 0.13 \text{ fm}^{-1} < k_F \leq 0.285 \text{ fm}^{-1} \), i.e. matter densities between \( 1.5 \times 10^{-4} \) and \( 2 \times 10^{-3} \text{ fm}^{-3} \), corresponding to mass densities of \( 10^{11.4} \) and \( 10^{12.4} \text{ g cm}^{-3} \), respectively. To investigate and resolve these features, special measures were needed to control singularities implied by bound states. Considering that at very low densities hole-hole ladder contributions become weak [9] due to the limited phase space they act upon, the results we obtain for nuclear densities below \( 10^{-3} \text{ fm}^{-3} \) based only on particle-particle (pp) contributions become reasonably justified.

Nuclear matter properties have been extensively investigated in the framework of the Brueckner-Bethe-Goldstone (BBG) theory based on the hole-line expansion for the ground state energy [7]. Here the Goldstone diagrams are
grouped according to their number of hole lines, with each group summed up separately. The summation of the two-hole-line diagrams yields the usual BHF approximation, where the two-body scattering matrix in nuclear matter is calculated self-consistently with the sp energy spectrum. Although the sp potential is introduced only as an auxiliary quantity, its choice conditions the rate of convergence of the expansion for the binding energy. In ref. [8] it has been reported that the continuous choice [9,10,11] for the auxiliary potential provides better convergence over the so-called standard choice, where the sp potential is set to zero above the Fermi energy. Hence, we shall use the continuous choice to investigate the role of dinucleon bound states during the search of self-consistent sp potentials.

The composition and thermodynamics of nuclear matter with the inclusion of light clusters \((1 \leq A \leq 4)\) has been studied by Typel and collaborators [12]. In-medium cluster wavefunctions, based on a Schrödinger equation with medium modifications, are obtained by means of a variational method, constraining size and binding to their respective values in free space [13]. The sp dispersion relation is based on an effective mass approximation, where effective masses are assumed to decrease linearly with increasing density. An empirical energy shift arising from Pauli blocking is assumed for the dissolution of clusters, leading to full \(\alpha\) condensation at densities in the range \(10^{-4}\) fm\(^{-3}\) \(\lesssim \rho \lesssim 3 \times 10^{-2}\) fm\(^{-3}\), corresponding to Fermi momenta between 0.12 and 0.76 fm\(^{-1}\). No account for dineutrons as constituents of nuclear matter is made in these publications. As will be shown in the present study, dineutron bound states are actually present in symmetric nuclear matter over the energy range 0.06 fm\(^{-1}\) \(\lesssim k_F \lesssim 1\) fm\(^{-1}\).

This paper is organized as follows. In sec. 2 we outline the theoretical background upon which we address self-consistency in nuclear matter within the BHF approximation. In sec. 3 we succinctly describe special measures aimed to control numerical instabilities, particularly the treatment of dinucleon bound-state poles of the sp matrix on the real axis. We also address the existence of multiple roots in the energy denominator of the propagator and describe procedures to handle sharp integrands. In sec. 4 we present and discuss solutions for the self-consistent sp fields, associated nuclear matter binding and effective masses. In sec. 5 we discuss spatial properties of dinucleons as well as some immediate implications of the sp fields when applied in the context of pairing gap equations for superfluid states. Attention is given to pairing gaps, coherence lengths and implications for condensates. In sec. 6 we present a summary of the work, draw its main conclusions and outlook.

2 Framework

In BBG theory for symmetric nuclear matter the sp matrix depends on the density of the medium, characterized by the Fermi momentum \(k_F\), and a starting energy \(\omega\). To lowest order in the BHF approximation, when only two-body correlations are taken into account, the sp matrix satisfies

\[
g(\omega) = v + i\eta - h_1 - h_2\frac{Q}{\omega + i\eta - h_1 - h_2} g(\omega), \tag{1}\]

with \(v\) the bare interaction between nucleons, \(h_i\) the sp energy of nucleon \(i\) \((i = 1, 2)\), and \(Q\) the Pauli blocking operator. The solution to this equation enables the evaluation of the mass operator

\[
M(k; E) = \sum_{|p| \lesssim k_F} \langle \frac{1}{2}(k - p)|g_{K}(E + e_p)|\frac{1}{2}(k - p)\rangle, \tag{2}\]

where \(K\) is the total momentum of the interacting pair, \(K = k + p\), and

\[
e_p = \frac{p^2}{2m} + U(p), \tag{3}\]

the sp energy defined in terms of an auxiliary field \(U\). The nucleon mass \(m\) is taken as the average of proton and neutron masses. In the BHF approximation the sp potential is given by the on-shell mass operator,

\[
U(k) = \text{Re} M(k; e_k), \tag{4}\]

self-consistency requirement which can be achieved iteratively. In the continuous choice this condition is imposed at all momenta \(k\) [14].

In momentum representation the BHF equation takes the explicit form

\[
\langle \kappa'|g_{K}(\omega)|\kappa\rangle = \langle \kappa'|v|\kappa\rangle + \int dq \langle \kappa'|v|q\rangle \times \theta(k_+ - k_F)\theta(k_- - k_F)\frac{(qg_{K}(\omega)|\kappa)}{\omega + i\eta - \frac{k^2}{2m} - \frac{k^2}{2m} - \Sigma}, \tag{5}\]

where \(\Sigma\) accounts for the particle-particle (pp) potential

\[
\Sigma(K, q; x) \equiv U(k_+) + U(k_-), \tag{6}\]

with

\[
k_{\pm}^2 = \frac{K^2}{4} + q^2 \pm qKx. \tag{7}\]

Here \(x = \hat{K} \cdot \hat{q}\), corresponding to the cosine of the angle between \(K\) and \(q\), with \(q\) the relative momentum of intermediate states.

3 Methods

A standard procedure to solve the non-linear system of Eqs. (1)(4) is by iterative feed-backing, where the auxiliary field \(U\) is initially unknown. Each cycle begins with the definition of an initial guess for \(U(k)\), which allows solving Eq. (1) for any \(\omega\) and \(K\). The series of cycles may be started with \(U_0(k) = 0\), to obtain all \(g\) matrix elements needed for the evaluation of \(M\) in Eq. (2). The real part of \(M\) defines a new sp field, \(U_1\), which becomes
the guess for a new cycle. The procedure is repeated until differences between consecutive solutions for $U$ meet some convergence criteria. Although this self-consistent scheme works well at normal densities, convergence becomes more difficult as $k_F$ diminishes below $\sim 0.8 \text{ fm}^{-1}$, feature manifested by instabilities in the evaluation of the on-shell mass operator. To attenuate these instabilities we cite refs. [15], for example, where self-consistency at low densities was reached by reducing the number of mesh points for the Fermi-motion integral in Eq. (2). With such numerical compromise it was possible to attenuate sudden fluctuations of the integrand, which at the time of the work had no identifiable cause. These solutions are used to calculate nucleon-nucleon ($NN$) effective interactions, i.e. off-shell $g$ matrix, to evaluate momentum-space optical model potentials for nucleon scattering off nuclei.

Aiming to obtain genuine self-consistent solutions for the sp mean fields over a wide density regime, particularly at low densities, a refinement of numerical techniques becomes crucial to control instabilities during the iterative process. In the following we describe the most relevant constructions toward this end.

3.1 Multiple roots

To solve Eq. (11) we take the angular average of the energy denominator,

$$\tilde{\Sigma}(K,q) = \frac{1}{2} \int_0^{\Delta} [U(k+) + U(k-)] \, dx ,$$

where $\Delta = \Delta(K,q) = \min\{1, \max[0, (K^2/4+q^2-k_F^2)/qK]\}$. Additionally, if the Pauli blocking operator is angle-averaged the multi-channel coupling among different total angular momentum states become disentangled. In ref. [16] this is referred to as ratio-of-average approximation, which in this study constitutes a starting point. After partial wave expansion, Eq. (11) for uncoupled channels reads

$$g(k',k;\omega) = v(k',k) + 2 \pi \int_0^{\infty} q^2 \, dq \, v(k',q) \times \left[ \frac{\Delta(K,q)}{\omega + i\eta} + \Sigma(K,q) \right] g(q,k;\omega) ,$$

where

$$E(K,q) \approx \frac{K^2}{4m} + \frac{q^2}{m} + \tilde{\Sigma}(K,q) .$$

A similar integral is obtained in the case of coupled states. The above integral equation can be solved numerically with the use of numerical quadrature and matrix inversion. However, caution is called for in the treatment of the zeros of the energy denominator. Indeed, the fact that $e_k = k^2/2m + U(k)$ is a growing function in $k$ does not necessarily imply the same trend for the two-particle energy $E(K,q)$. Hence, more than one root in the energy denominator of Eq. (11) cannot be ruled out.

To illustrate this point consider the angle-averaged potential $\Sigma(K,q)$ in Eq. (11) evaluated using a two-point Gaussian quadrature. Thus

$$E(K,q) \approx \frac{K^2}{4m} + \frac{q^2}{m} + U(p_+ + U(p_-) ,$$

where $p_{\pm} = (K^2/4 + q^2 \pm Kq/\sqrt{3})/2$. Single zeros (or none) in $E(K,q) - \omega$ are guaranteed if $E(K,q)$ is an ever growing function in $q$, namely $\partial E(K,q)/dq > 0$. Hence,

$$\frac{2q}{m} \left[ \frac{U'(p_+)}{p_+} + \frac{U'(p_-)}{p_-} \right] + \frac{K}{2\sqrt{3}} \left[ \frac{U'(p_+)}{p_+} - \frac{U'(p_-)}{p_-} \right] > 0 ,$$

where $U'$ stands for derivative of $U$ with respect to its argument. Assuming $U'(k)$ positive for all $k$, the validity of the above inequality depends on the sign of the third term. Only if $U(k)$ is quadratic in $k$ the above inequality is met, which is not the case in actual solutions.

Therefore, even if the genuine sp self-consistent fields result as monotonic functions in the momentum space, multiple roots cannot be ruled out in the energy denominator of the propagator. This aspect, in the context of the integral equation for the $g$ matrix, can be handled as outlined in Appendix A.

3.2 Fermi-motion integrals

The evaluation of the sp fields, $U(k) = Re M(k,e_k)$, involves the summation of on-shell $g$-matrix elements while keeping the momentum of one of the particles below the Fermi surface. Explicitly,

$$M(k,e_k) = \sum_\alpha n_\alpha \int_{k_0}^{K_F} p^2 dp \int_{-1}^{1} du \, g_{k+p}^{e_k}(\frac{\eta_{k+p}}{2}, \frac{\eta_{k+p}}{2};\omega) .$$

Here $u = \hat{k} \cdot \hat{p}$, the energy $\omega$ is evaluated on-shell ($\omega = e_k + e_p$), $\alpha$ denotes spin, isospin and angular momentum states, and $n_\alpha$ accounts for their degeneracy and for geometric factors.

For most NN states the integration within the Fermi sphere involves well behaved integrands, as given by the on-shell $g$. However, extra measures becomes unavoidable when considering the $^1S_0$ and $^3S_1-^3D_1$ channels. This is because during the evaluation of the momentum integral in Eq. (13), exceedingly large $g$-matrix elements stemming from NN bound states in these S and D channels. Indeed, any matrix element with starting energy near these bound states will lead to large contributions. To illustrate the occurrence of these states, let us define

$$D_\alpha(K;\omega) \equiv \det[1 - v_\alpha A_K(\omega)] ,$$

with $A_K(\omega)$ the pp propagator in Eq. (9) and $\alpha$ denoting a particular NN channel. In fig. 11 we plot $D_{\alpha}$ for $^1S_0$ (thin curves) and $^3S_1-^3D_1$ (thick curves) channels as a function of $(\omega - \omega_{tH})$, with $\omega_{tH}$ the threshold energy allowed by Pauli blocking. For this example we have chosen $k_F = 0.6 \text{ fm}^{-1}$. Solid, long- and short-dashed curves correspond to total pair momenta $K = 0, 0.05$ and $0.1 \text{ fm}^{-1}$, respectively. A singularity of $g_K(\omega)$ in channel $\alpha$ is identified by the zero
of \( D_\alpha(K;q) \). As observed, the deuteron channel exhibits bound states in all three cases, while for the \( ^1S_0 \) channel they occur over a narrower interval in \( \omega \). Hence, if during the evaluation of the \( dpdu \) integrals in Eq. (13), a configuration of total momentum and starting energy happens to be near (or at) one of these singular points, large (or undefined) contributions become unavoidable. This becomes a major source of numerical instabilities if not handled properly.

To evaluate the mass operator we use the alternative set of variables \( q \) and \( K \) defined by \[ q = \frac{1}{2}(k^2 + p^2 - 2kpu)^{1/2}, \quad K = (k^2 + p^2 + 2kpu)^{1/2}. \] (15a) (15b)

Thus, \( p^2dpdu = (2Kq/k)dKdq \). Therefore the on-shell mass operator, Eq. (13), is expressed as

\[ M(k, e_k) = \sum_{\alpha} n_\alpha \int_{K_{\min}}^{K_{\max}} K dK \int_{q_{\min}}^{q_{\max}} \frac{2q}{k} dq \ g^\alpha_K(q, q; \omega). \] (16)

To be consistent with the angle-averaged pp energy, the starting energy \( \omega \) is taken as \( E(K, q) \) from Eq. (10). With this set of integration variables, for given \( k_F \), \( k \), and \( K \), the limits of integration on \( q \) become

\[ q_{\min} = |k - \frac{1}{2}K|, \]

\[ q_{\max} = \begin{cases} \min\left\{ k + \frac{1}{2}(R^2 - \frac{1}{4}K^2)^{1/2} \right\}, & \text{for } k \leq k_F, \\ (R^2 - \frac{1}{4}K^2)^{1/2}, & \text{for } k > k_F, \end{cases} \]

where \( R^2 = \frac{1}{4}(k^2 + k_2^2) \). In the case of \( K \), the integration ranges from \( K_{\min} = \max\{0, k - k_F\} \), up to \( K_{\max} = k + k_F \). In fig. 2 we illustrate the four possible shapes in the \((\frac{1}{2}K, q)\) plane where the integral on \( dKdq \) takes place, depending on \( k \) relative to \( k_F \).

The above choice of coordinates facilitates tracking and control of dinucleon singularities. Indeed, for constant \( K \) the eigenenergy \( \omega^*_K \) of an eventual dinucleon bound-state pole remains unchanged while the integration on \( q \) takes place. Additionally, considering the spectral representation of \( g \) when bound states occur \[ G, \] its behavior for \( \omega \) near the energy of the bound state becomes

\[ g^\alpha_K(\omega) \sim \frac{vQ|\psi^\alpha_K(\omega)|^2}{\omega - \omega^*_K}. \] (17)

Note that the numerator is bound. Expanding the denominator to first order in \( q \) around its zero, which we assume inside the interval of integration, the contribution around the singularity can be reduced to the form \( \int_{q_{\min}}^{q_{\max}} dQ \), which becomes finite. Therefore, integrals over momenta of the \( g \) matrix featuring dinucleon poles are not only finite but should be feasible. In actual calculations these singularities are controlled with the regularization

\[ g(\omega) \rightarrow g(\omega) \times \frac{(\Delta \omega)^2}{(\Delta \omega)^2 + \eta^2}, \]

where \( \Delta \omega = \omega - \omega^*_K \), and \( \eta \) is an infinitesimal. In this way contributions too close to the pole, namely \( |\Delta \omega/\eta| \rightarrow 0 \), become attenuated. On the other hand \( g \) is practically unchanged whenever \( |\Delta \omega/\eta| \gg 1 \). In the actual implementation of this technique we have used \( \eta = 0.1 \text{ MeV} \). These considerations in the handling of dineutron and deuteron bound states render a much needed stability in the evaluation of the Fermi-motion integrals.

Another aspect of particular relevance in the evaluation of \( M \) in Eq. (10) is the quadrature method to be used. Two elements enter into consideration here. On the one hand it is the varying range of integration along \( q \) for fixed \( K \) (c.f. fig. 2); on the other hand, it is the presence of
bound states responsible for steep variations of $g$. These features combined make inadequate the use of Gaussian-type quadratures designed for smooth integrands.

Aiming to a reliable precision in the evaluation of Fermi-motion integrals we have resorted to an adaptive trapezoidal quadrature considering gradually $2, 3, 5, \ldots, (2^n+1)$ knots. The scheme, described in Appendix B, is first applied to the (innermost) $q$ quadrature and then to the integration over $K$. The sequence is interrupted once the difference between two consecutive evaluations is bound to $2\%$. Given that in many cases the integrand is smooth and the spacing between consecutive knots diminishes exponentially, this criterion provides an overall accuracy in the $sp$ fields of the order of $0.5\%$. We have limited $n$ to eight, leading to 255 trapezoids over a given interval. Self-consistency of the calculated $U(k)$ is achieved when the maximum fluctuations of $U(k_i)$ over three consecutive cycles do not exceed 0.04 MeV, condition imposed at all $k_i$ where $U$ is evaluated. In this work we have used $k \leq 5.5$ fm$^{-1}$.

4 Results

Self-consistent solutions for $U(k)$ were obtained in the range $0 < k_F \leq 1.75$ fm$^{-1}$ using the Argonne $v_{18}$ bare internucleon potential [19]. We have included all $NN$ partial waves up to $J = 7\hbar$, the total angular momentum. This criterion was maintained at all densities in order to cross-check our findings near the saturation point, in addition to rule out any subtle variation of the calculated observables as caused by a discrete (sudden) change in the number of partial waves taken into account. With these specifications and the numerical considerations described in the previous section, each cycle at which $U$ is calculated involves between $10^5$ and $10^6$ matrix inversion operations, depending on the sharpness of the integrand.

4.1 Self-consistent solutions

Usually the search for self-consistent solutions to the $sp$ field by iterative feedback is stable enough to reach unique solutions, even starting out with zero field. We find this to be the case when the Fermi momentum is near and above $0.35$ fm$^{-1}$. The issue becomes more subtle at lower densities, where the iterative process yields ill-behaved solutions. This fact has led us to adopt different strategies for $0 < k_F < 0.35$ fm$^{-1}$ and $k_F \geq 0.35$ fm$^{-1}$. In the second case we have found unique self-consistent solutions at Fermi momenta $k_F = 0.35(0.05)1.75$ fm$^{-1}$, following the usual iterative procedures.

In contrast to this upper range, searches in the interval $0 < k_F < 0.35$ fm$^{-1}$, required a specific strategy due to the fact that two different self-consistent solutions could be obtained for the same value of $k_F$. These two genuine solutions meeting self-consistency will be referred to as co-existing solutions. To proceed systematically, a first class of solutions was obtained with ascending $k_F$, in steps of $\Delta k_F = 0.01$ fm$^{-1}$. At each $k_F$, the self-consistent loop starts with an $U(k)$ borrowed from the converged solution at the previous $k_F$. In this way, starting at $k_F = 0.01$ fm$^{-1}$, we obtain a class of self-consistent solutions characterized by a monotonically increasing binding energy per nucleon, $B/A$, as a function of $k_F$. We denote this class of solutions as $U_I = \{U_{k_F} \mid 0 \leq k_F \leq k_\beta\}$, with $k_\beta$ the maximum (critical) Fermi momentum at which self-consistency is achievable in ascending order. To resolve $k_\beta$, self-consistent solutions were explored with increasing $k_F$ in steps $\Delta k_F$ as small as 0.001 fm$^{-1}$, obtaining $k_\beta = 0.285$ fm$^{-1}$. Beyond this point the iteration loops indefinitely, with fluctuations in $U(k)$ unable to settle below 0.04 MeV. A more refined characterization of the solutions near and beyond $k_\beta$ may require a more specific approach.

A similar procedure was adopted for decreasing $k_F$, starting from the converged solution at $k_F = 0.35$ fm$^{-1}$ and going down in steps of 0.01 fm$^{-1}$. As in the previous case, the continuity of $B/A$ is monitored and we seek the minimum critical value of $k_F$, denoted by $k_\alpha$, at which self-consistency is reachable. This second class of solutions is denoted $U_{II} = \{U_{k_F} \mid k_F \geq k_\alpha\}$. We obtain $k_\alpha = 0.130$ fm$^{-1}$, with a resolution of 0.01 fm$^{-1}$.

In fig. 3 we show a surface plot of the calculated self-consistent solutions $U_{II}$, as functions of $k$ and the Fermi momentum $k_F$, with $k_F \geq 0.35$ fm$^{-1}$. In general terms they follow the same trend as those reported elsewhere [13], with $U(k)$ negative at $k = 0$, growing as $k$ increases.

In fig. 3 we show surface plots for the self-consistent fields $U(k)$ for solutions I and II in the range $0.05$ fm$^{-1} \leq k_F \leq 0.35$ fm$^{-1}$. These solutions define distinct surfaces at densities corresponding to $k_\alpha \leq k_F \leq k_\beta$. The upper sheet, representing $U_I$, features a moderate repulsion at $k = 0$, with a shallow ditch in the vicinity $k \sim 4k_F/3$. On the other hand the sheet for $U_{II}$ starts with negative mean fields at $k = 0$, showing a moderate decrease up to $k \sim k_F$ and then followed by a steep increase to reach $U(k) \rightarrow 0$. In both cases the slope of the self-consistent fields become negative at the Fermi surface, feature responsible for nucleon effective masses greater than the bare nucleon mass, to be discussed ahead.
A closer view of the coexisting solutions is presented in fig. 5, where in panel (a) we plot sp energies as functions of the ratio $k/F$ in the coexistence region, and in panel (b) the corresponding sp potentials. Dashed and solid curves represent solutions for $k/F = k_\alpha$ and $k/F = k_\beta$, respectively. Dotted curves represent consecutive solutions for $k/F = 0.14(0.02)0.28$ fm$^{-1}$. Vertical arrows connect coexisting solutions at $k_\alpha$ and $k_\beta$, respectively. The difference between Fermi energies of solutions I and II is 1.6 MeV at $k_\alpha$, and 5.6 MeV at $k_\beta$. We also notice that none of the solutions for $U(k)$ (lower panel) exhibit a parabolic behavior in the whole momentum range. As a matter of fact the slope of $U(k)$ for solution I exhibits a sudden increase from negative to near-zero at $k/F \approx 1.15$, while for solution II the increase goes from near-zero to positive slope. These features rule out the validity of the effective mass approximation at low densities.

The disclosure of the two distinct phases I and II for the sp spectrum constitutes an unexpected outcome of the calculations we have performed. To gain insight into possible causes behind this behavior, in fig. 6 we plot the total single particle potentials $U_I$ and $U_{II}$, together with the partial contributions from $^3S_1-^3D_1$ and $^1S_0$ channels. Solid and dashed curves represent solutions in phase I and II, respectively. In this illustration we have selected $k/F = 0.26$ fm$^{-1}$, a Fermi momentum reasonably close to $k_\beta$ but not at the edge of the overlap. The most prominent difference between phases I and II appears in the deuteron channel, passing from a correlated Fermi gas (FG) behavior in phase I to an attractive (condensing) state in phase II, with an energy decrease of about 6 MeV near the Fermi surface ($k/F \approx 1$). This is in contrast with the moderate increase, by about 1 MeV, observed in the $^1S_0$ channel. This feature points to the deuteron as the driving constituent responsible of the occurrence of phase II beyond $k_\alpha$.

The strategy we have adopted to obtain self-consistent solutions, starting out iterative loops with a converged solution at a nearby $k/F$, has proven essential to elucidate the coexisting solutions. Actually if such a measure were not taken the calculated $U(k)$ might exhibit piecewise zigzagging instabilities, preventing any relaxation toward convergence. In the case of solution I, successive solutions were obtained with increasing $k/F$ up to $k_\beta$. When the solution at $k/F = k_\beta$ was used as starting guess for $k/F = k_\beta + \delta$, with $\delta$ some small increment, the requirement we have established for stable solutions could not be met after sixty iterative loops. However, if the increment is large enough the self-consistent solution settles over that of $U_{II}$. An analogous process is followed to obtain solution II, but generating solutions with decreasing $k/F$ up to $k_\alpha$. Here again, the use of $U(k)$ at $k_\alpha$ to start iterations for $k/F = k_\beta - \delta$, would not relax if $\delta$ were small and positive, but it would settle over $U_I$ if $\delta$ were large enough. In this respect, the procedure to obtain the solutions in the coexistence interval $[k_\alpha, k_\beta]$ resembles that of hysteresis.

The present study has been repeated all through the range $k/F \lesssim 0.3$ fm$^{-1}$, this time assuming valid the effective mass approximation while solving Eqs. H in subject to dinucleon bound state occurrences and self-consistency. We have been able to identify phase I, over a more limited range, when the quadratic approximation of $U(k)$ is imposed over the range $0 \leq k \leq 2k_F$. However, solutions for phase II become unstable, preventing actual
self-consistency. A more dedicated study on merits and shortcomings of the effective mass approximation shall be presented elsewhere.

4.1.1 Binding energy

The fact that two different solutions for \( U(k) \) satisfy self-consistency at a given Fermi momentum between \( k_a \) and \( k_\beta \), implies two accessible sp states for nucleons in the medium. If these were the only degrees of freedom of the system, then solution II would be the one accountable for the ground state. However, pairing under coexisting solutions would change the scenario, leading to more subtle issues regarding actual sp occupations on each phase, an aspect that needs to be investigated in a more specific work. Having these shortcomings in mind, we have proceeded to evaluate the binding energies per nucleon of the system considering the two phases separately. This simplifying assumption would help us to set bounds to the results.

In fig. 4 we plot the binding energy per nucleon for symmetric nuclear matter, \( B_1/A \) and \( B_{II}/A \), as obtained from solutions \( U_I \) and \( U_{II} \), respectively. Here each small dot denotes an actual calculation, while the continuous curves represent suitable parametrizations. We note that the two solutions exhibit distinct behaviors as functions of \( k_F \) in their respective domains, without intercepting each other in the range \([k_a, k_\beta] \). While solution I resembles a correlated FG in metastable state, solution II represents a condensed medium featuring a minimum at the point of nuclear saturation. From the inset for low densities we also note that solution I departs from an uncorrelated FG above \( k_a \), the Fermi momentum where solution II begins. This departure features a moderate repulsiveness relative to a free FG.

To facilitate the analysis of solutions I and II we have found simple parametrizations of the mean potential energy

\[
u(k_F) = \frac{3}{k_F^2} \int_0^{k_F} k^2 dk U(k).
\]

Specifically

\[
u_I(x) = a_2 x^2 + a_3 x^3 + a_6 x^6,
\]

\[
u_{II}(x) = b_1 x + b_2 x^2 + b_7 x^7,
\]

leading to accurate representations of the calculated binding energy per nucleon

\[
B_i/A = \frac{3}{10} \frac{k_F^2}{m} + \frac{1}{2} u_i(k_F),
\]

with \( i = I \) and \( II \). The coefficients \( a \) and \( b \), summarized in Table 1, were obtained from a least-square fitting procedures for the calculated \( B_i/A \) values as a function of \( k_F \).

The standard deviations associated with each set of coefficients are 0.01 MeV and 0.06 MeV, respectively. In the case of solution \( B_{II}/A \) we obtain a saturation energy of \(-16.8 \text{ MeV at } k_F = 1.53 \text{ fm}^{-1} \), with an incompressibility \( K_{\infty} = 213 \text{ MeV} \).

The coexistence of solutions I and II in the range \( k_a \leq k_F \leq k_\beta \) define three distinct regimes according to \( k_F \), namely: (i) a diluted phase in the form of an interacting FG up to \( k_a \); (ii) a mixed phase with coexisting sp states between \( k_a \) and \( k_\beta \); and (iii) a condensing phase above \( k_\beta \). The low-density behavior of \( B/A \) in phase I, featuring a repulsive FG, points to dominance of the kinetic contribution over its interaction term. Being this the case, if no condensation takes place, the gas behavior up to \( k_F \sim 0.13 \text{ fm}^{-1} (\rho \sim 10^{-4} \text{ fm}^{-3}) \), prevents homogeneous symmetric nuclear matter from spontaneous collapse.

A survey of available literature have resulted in scarce information on zero-temperature and low-density behavior of symmetric nuclear matter \( (k_F \lesssim 0.4 \text{ fm}^{-1} \), or \( \rho \lesssim 0.004 \text{ fm}^{-3}) \) when the bare NN interaction is used. In ref. [20] for instance, the authors have made explicit the fact that a polynomial extrapolation is used for densities below \( 0.02 \text{ fm}^{-3} \) (i.e. \( k_F < 0.67 \text{ fm}^{-1} \)). Extrapolations of similar nature, roughly from the same density, have been made in refs. [21][22] in the context of density-dependent effective interactions based on BHF model and applied to nucleon-nucleon scattering. In a recent publication [23] aimed to the study of the phase structure of symmetric nuclear matter in the extended Nambu-Jona-Lasinio model, fig. 1 for \( B/A \) as a function of the density suggests FG behavior at \( \rho \lesssim 0.007 \text{ fm}^{-3} \), although no comments are made by the authors on this feature.

![Table 1. Best-fit coefficients for \( u(k_F) \).](image)

| \( p \) | \( a_p \) (MeV fm\(^{-3}\)) | \( p \) | \( b_p \) (MeV fm\(^{-3}\)) |
|---|---|---|---|
| 2 | -13.911780 | 1 | -5.826045 |
| 3 | 99.7568282 | 2 | -40.534406 |
| 6 | -311.050742 | 7 | 0.006716 |
4.1.2 Effective masses and dinucleon binding

In this section we examine the density dependence of the effective mass and dinucleon binding energies associated to each phase. To this end we show in panel (a) of fig. 8 the total effective mass relative to the bare nucleon mass, \( m^*/m \), as a function of the Fermi momentum. Phase-I results are plotted up to \( k_F \) while results for phase II appear for \( k_F \geq k_m \). In this case we evaluate

\[
m^* = \left[ 1 + m \frac{\partial U(k)}{k} \right]^{-1},
\]

at \( k = k_F \). In panel (b) we plot dinucleon binding energies in states \(^4\text{S}_1-^3\text{D}_1\) (circles) and \(^4\text{S}_0\) (squares), in units of deuteron binding in free space, \( E_d = -2.224 \text{ MeV} \). The dinucleon binding energy \( E \) is given by

\[ E = \omega - \omega_{\text{th}}, \]

with \( \omega \) the eigenenergy and \( \omega_{\text{th}} \) the pp energy at the Fermi surface. Here we restrict to cases where \( K = 0 \), i.e., pairs with no translational motion. For reference purposes, in panel (c) we plot again the binding energy per nucleon associated with solutions I and II. The upper scale denotes density, \( \rho = \frac{2k^2_F}{3\pi^2} \). We first observe that \( m^*/m \) for solution I grows monotonically with increasing \( k_F \), reaching a maximum \( m^*/m \approx 3 \), close to the upper edge of the coexistence interval \( (k_{F}) \). In turn, \( m^*/m \) for solution II starts out near unity at \( k_m \), reaching a maximum \( m^*/m \approx 1.8 \), at \( k_F = 0.55 \text{ fm}^{-1} \). Beyond \( k_F = 0.9 \text{ fm}^{-1} \), \( m^*/m \) decreases monotonically, taking values from 0.88 to 0.79 for Fermi momenta from 1.3 to 1.5 \text{ fm}^{-1} \), respectively, consistent with quotes found in the literature. In the domain of coexistence the two solutions yield dressed nucleons with distinct effective masses, reaching values as high as \( m^*/m \approx 3 \), for solution I and \( m^*/m \approx 1 \), for solution II.

Panel (b) of fig. 8 show qualitative similarities between deuteron and dineutron binding energies, although differences point to weaker binding in the case of the \(^4\text{S}_0\) channel. Solution I for deuteron pairs shows increasing binding energy as the density grows up to \( k_m \), reaching up to twice that in free space. The same solution but for dineutrons exhibits increasing binding up to nearly 0.3 \( E_d \), although no bound state is found in this channel for \( k_F \) below 0.06 \text{ fm}^{-1} \) (Mott transition). In the case of solution II, deuteron pairs are found for \( k_F \geq k_m \) but get dissolved beyond \( k_F \approx 1.325 \text{ fm}^{-1} \). In turn, dineutron pairs appear over a narrower range in \( k_F \); 0.23–1.05 \text{ fm}^{-1}. The strongest bindings are \( E/E_d \approx 1.2 \) and 0.3, for deuteron and dineutron pairs, respectively. These maxima occur at \( k_F \approx 0.7 \text{ fm}^{-1} \). We also notice that the \(^4\text{S}_0\) and \(^3\text{S}_1-^3\text{D}_1\) binding energies tied with solution II display shallow extrema. These oscillating patterns are in contradistinction with the smooth behavior displayed over \( k_F \) by conventional S and D solutions of pairing gap equations in nuclear matter [24]. We have found no interpretation as to why such pairing gaps and dinucleon binding energies, two closely related properties, differ in their patterns over \( k_F \).

A simple argument of plausibility to admit effective masses greater than bare masses in a very diluted medium comes from considering two nucleons interacting via an attractive square well. If the depth of the potential is \( V_0 \) and its radius is \( R \), then the condition to hold at least one bound state is \( mV_0R^2 > \pi^2/4 \). In the context of two interacting neutrons in free space this condition should be regarded as barely missed, as inferred from the low-energy behavior of the phase shifts in the \(^1\text{S}_0\) channel and scattering angle \( \theta_{\text{cm}} = -18.5 \text{ fm} \). If we assume unaffected the bare interaction by the medium, then dineutrons become possible provided \( m^* \) increases enough so as to allow \( m^*V_0R^2 > \pi^2/4 \). This trend is observed in panel (b) of fig. 8 where the Fermi momenta at which dineutrons get unbound are indicated with small black arrows on the horizontal axis. Here \( k_F = 0.06, 0.23 \) and 1.025 \text{ fm}^{-1}, corresponding to \( m^*/m \approx 1.06, 1.25 \) and 1.15, respectively. For deuterons, instead, bound states occurrences appear in correspondence with \( m^* \gtrsim 0.87m \). In this case bound states dissolve near \( k_F = 1.325 \text{ fm}^{-1} \). These features were cross-checked for solution I with the aid of a simple computer code applied to a square well potential where the trend of \( m^*/m \) in panel (b) of fig. 8 can be reproduced by feeding in \( E/E_d \) for the deuteron from panel (b).

4.2 Dinucleons

Singularity of the \( g \) matrix in the real axis below the Fermi surface, in the \(^1\text{S}_0\) and \(^3\text{S}_1-^3\text{D}_1\) channels, are unequivocal signs for in-medium dinucleon and deuteron bound states, respectively. As stated before, the effects of these structures on the sp potentials are explicitly accounted for. In this section we explore some features of these Cooper pair like solutions [25], representing the formation of single pairs in the presence of homogeneous nuclear matter.
4.2.1 Bound states in BHF

Let $E_b$ be the eigenenergy corresponding to a bound state of two interacting nucleons in the nuclear medium. Then it can easily be shown \[3\] that

$$\lim_{\eta \rightarrow 0} \ln g(E_b + i\eta) = vQ|\psi\rangle\langle \psi|Qv ,$$

with $Q$ the Pauli blocking operator and $|\psi\rangle$ the corresponding eigenstate (dependence on $K$ is implicit). On the other hand, the wave equation associated with Eq. (1) when $\omega$ matches the eigenenergy $E_b$ reads

$$(\hat{h}_1 + i\hat{h}_2 + Q\psi)(|\psi\rangle = E_b|\psi\rangle .$$

In free space, where $(\hat{h}_1 + \hat{h}_2)$ becomes the kinetic energy of the two nucleons and $Q$ the identity, this equation reduces to the Schrödinger equation in stationary state.

After projecting in momentum space $(|q\rangle)$ we obtain for the wavefunction

$$|q\rangle = \frac{\langle q|QvQ|\psi\rangle}{E_b - \epsilon_+ - \epsilon_-} ,$$

where $\epsilon_+ = \epsilon(k_\perp)$, consistent with the notation in Eq. (7).

In the case of pairs without translational motion, where $K = 0$, the numerator reduces to

$$\langle q|QvQ|\psi\rangle = \Theta(q - k_F)|q\rangle\langle q|Qv|\psi\rangle .$$

Thus, the wavefunction can be obtained after $\langle q|QvQ|\psi\rangle$ is extracted from Eq. (22), a procedure we have achieved numerically solving the BHF equation for small $\eta$ and extrapolating to $\eta \rightarrow 0$. The solution in momentum space is then Fourier transformed to get its coordinate space representation.

4.2.2 Eigenfunctions

As mentioned before, wavefunctions are calculated in momentum space. If $\psi(q)$ denotes the wavefunction for a bound state of orbital angular momentum $L$, its coordinate space representation is given by

$$\psi(r) = \sqrt{\frac{2}{\pi}} \int_{q}^{\infty} q^2 dq \ j_L(qr)\psi(q) .$$

The lower integration bound $q$ corresponds to that allowed by Pauli blocking. In what follows we shall consider pairs with their center of mass at rest ($K = 0$), so that $q = k_F$.

To achieve reliable precision in the evaluation of $\psi(r)$ we found it crucial to control the behavior of $\psi$ near the Fermi surface. To this end we constructed auxiliary functions

$$\psi_0(q) = Aq^L \exp[-R(q - k_F)] \Theta(q - k_F) ,$$

with $\Theta$ the Heaviside step function to suppress momentum components below the Fermi surface. Parameters $A$ and $R$ are adjusted to match the exact $\psi(q)$ just above the surface, i.e. $q \rightarrow k_F^+$. With this construction we evaluate

$$\psi(r) = \sqrt{\frac{2}{\pi}} \int_{k_F}^{\infty} q^2 dq \ j_L(qr)(\psi - \psi_0) + \psi_0(r) ,$$

where

$$\psi_0(r) = \sqrt{\frac{2}{\pi}} \int_{k_F}^{\infty} q^2 dq \ j_L(qr)\psi_0(q) .$$

For the latter we proceed analytically, where specific results are presented in Appendix C. Integrals are evaluated over a finite intervals in $r$ and $q$, respectively. These intervals are chosen so that

$$\int_{0}^{R_{\max}} |\psi(r)|^2 dr = \int_{k_F}^{q_{\max}} |\psi(q)|^2 q^2 dq ,$$

is met with reasonable precision. Typically $q_{\max} \approx 15$ fm$^{-1}$, whereas $R_{\max}$ could go as high as 1500 fm.

In fig. 9 we plot radial probability densities, $r^2|\psi(r)|^2$, for in-medium S-wave dinucleon bound states as functions of the relative distance $r$. Solid (dashed) curves denote solutions for phase I (II) in the case of $k_F = 0.25$ fm$^{-1}$. Panels (a) and (c) show results for channels $^1S_0$ and $^3S_1$, respectively. Panel (b) displays $^1S_0$ and $^3S_1$--$^3D_1$ bound-state waves in phase I and II, respectively. The dash-dotted curve in panel (c) represents the deuteron in free space.

Fig. 9. Radial probability density, $r^2|\psi(r)|^2$, for in-medium S-wave dinucleon bound states as functions of the relative distance r. Solid (dashed) curves denote solutions for phase I (II) in the case of $k_F = 0.25$ fm$^{-1}$. Panels (a) and (c) show results for channels $^1S_0$ and $^3S_1$, respectively. Panel (b) displays $^1S_0$ and $^3S_1$--$^3D_1$ bound-state waves in phase I and II, respectively. The dash-dotted curve in panel (c) represents the deuteron in free space.
in phase I. This indicates an enlargement of dineutrons in phase II relative to those in phase I, to be discussed later. The same feature is observed for deuterons in panel (c), where the free-space solution (solid black curve) is included. In this case we observe a similarity between the probability density of deuterons in free space with that of dineutrons in phase I. This suggests that deuteron maintain their size as long as they remain in phase I. These spatial aspects of in-medium dinucleon solutions shall further be discussed in the following section.

In fig. 11 we show contour plots for S-wave radial probabilities $r^2|\Psi(r)|^2$ as functions of the relative distance $r$ and Fermi momentum $k_F$. The left hand-side panel shows the $1S_0$ solutions in phase I, with $0 \leq k_F \leq 0.26$ fm$^{-1}$, together with those in phase II for $k_F \geq 0.26$ fm$^{-1}$. In the right hand-side panel the $1S_0$ solution in phase II has been replaced by the corresponding $3S_1$ solution. It is interesting to note how the solutions merge at the boundary $k_F = 0.26$ fm$^{-1}$ (horizontal line), illustrating the way the $3S_1$ solution in phase II overtakes that for $1S_0$, with the latter pushed further away as inferred from the left hand-side panel.

4.2.3 Spatial properties

Cooper-like wavefunctions appear to have unique spatial properties, especially regarding their large size relative to the mean internucleon distance in the nuclear medium. This feature is already suggested in figs. 9 and 10 where the bound state wavefunctions are observed to decay slowly. Actually, these wavefunctions behave as $\sim \cos(k_F r)/r^2$ for large radii, feature which poses convergence difficulties in the evaluation of mean radii $r_m \equiv \langle r \rangle$ or root-mean-square (rms) radii $\langle r^2 \rangle^{1/2}$. In momentum space, in turn, we cannot make straightforward use of the customary identity 

$$\langle r^2 \rangle = \frac{\int |\Psi_{\text{pair}}(r)|^2 r^3 dr}{\int |\Psi_{\text{pair}}(r)|^2 r^2 dr} = \frac{\int_0^{\infty} (\frac{2^3}{2!})q^2 dq}{\int_0^{\infty} q^2 dq},$$

due to the discontinuity of the wavefunction $\psi(q)$ at the Fermi surface. Actually, the above identity is valid as long as the wavefunction is bound in coordinate space, while in momentum space derivatives are continuous at all momenta. Such is not the case for $\psi(q)$, exhibiting a cut-off below $q = \bar{q}$, which occurs at $k_F$ for pairs with no translational motion ($K = 0$).

To overcome the convergence difficulties posed in coordinate space evaluations, we propose to consider the Laplace transform of the density $r^2|\Psi(r)|^2 \equiv r^2\rho$. This transform reads

$$F(s) = \mathcal{L}\{r^2\rho(r)\} = \int_0^{\infty} e^{-sr} |\Psi(r)|^2 r^2 dr. \quad (29)$$

Clearly, $F(0) = \mathcal{N}$, the volume integral of the density. The small-$s$ behavior yields

$$F(s) = \mathcal{N} \left(1 - \langle r \rangle s + \frac{1}{2} \langle r^2 \rangle s^2 - \cdots\right),$$

from which we construct the alternative function $f(s)$ given by

$$f(s) \equiv \frac{1}{s} \left[1 - \frac{F(s)}{\mathcal{N}}\right] = \langle r \rangle - \frac{1}{2} \langle r^2 \rangle s + \cdots . \quad (30)$$

Since $\mathcal{N}$ is finite, $F(s)$ becomes always defined due to the exponential in the integrand. Thus, from an analysis of $f(s)$ at small $s$ we can extract the mean radii $\langle r \rangle$ and the mean-square-radii $\langle r^2 \rangle$. A linear regression of Eq. (30) for $f$ provides us with accurate fits, with correlations equal to unity out to six significant figures.

In fig. 11 we show results for the calculated mean radii $\langle r \rangle$ and rms radii $\langle r^2 \rangle^{1/2}$ as functions of the Fermi momentum $k_F$. Solutions for states $1S_0$, $3S_1$ and $3D_1$ are labeled as (1), (2) and (3), respectively. Empty and filled symbols denote results for phases I and II, respectively. Upper panels show $\langle r \rangle$ (a) and $\langle r^2 \rangle^{1/2}$ (b) in fm units. Lower panels show results for the ratios $\langle r \rangle/L$ (c) and $\langle r^2 \rangle^{1/2}/L$ (d), with $L = \rho^{-1/3}$, the mean internucleon separation at density $\rho$. We note that both mean radii and rms radii for dineutrons (1) are larger than those for deuterons (2 and 3). What is also evident is that S-wave bound states experience a sudden increase of size in the transit from phase I to phase II. The rate of the change is more pronounced for neutrons than for dineutrons. Additionally, observing panels (c) and (d) we note that deuteron bound states in phase I are much smaller than the internucleon separation $L$, while in phase II they remain bound with sizes several times the internucleon separation.

Another feature worth noting from panel (a) is that the size of deuterons in phase I remain roughly constant as a function of the density ($k_F \leq 0.285$ fm$^{-1}$), with sizes similar to that in free space. In phase II, instead, they become fairly large objects with mean radii above 30 fm. In this respect deuterons change from very compact bosons in phase I to very extended ones in phase II. This is in contrast to dineutrons, where their sizes outpass the internucleon separation in both phases.
The occurrence of dimuon bound states in nuclear medium is closely related with nuclear pairing phenomena, mechanism responsible for the formation of Cooper pairs and emergence of superfluid and superconducting states of matter \[12\]. Hence, it is of interest to explore some immediate implications of phases I and II for the sp potentials in the context of pairing gap equations. To this end we have solved the Bardeen-Cooper-Schrieffer (BCS) gap equations at zero temperature. These gap equations in triplet coupled states lead to coupled equations displaying an explicit angular dependence of the kernel, problem which has been addressed in ref. \[28\] in the context of pure neutron matter. However, an important simplification to this problem is obtained when the anisotropic kernel is angle averaged \[29\]. Within this approximation the gap functions for states of orbital angular momentum L take the form

\[
\Delta_L(k) = -\frac{2}{\pi} \int_0^\infty k'^2 dk' \sum_{L'} v_{LL'}(k,k') \frac{\Delta_{L'}(k')}{2E(k')},
\]  

where the quasiparticle energy reads

\[
E(k)^2 = (e_k - \mu)^2 + D(k)^2,
\]

\[
D(k)^2 = \sum_{L} \Delta_L(k)^2 .
\]

Here \(\mu\) is the chemical potential and \(e_k = k^2/2m + U(k)\), corresponds to the BHF sp spectrum. In this case solutions I and II for \(U(k)\) are treated independently, an issue that needs to be reexamined for more realistic calculations. The matrix elements \(v_{LL'}(k,k')\) of the bare interaction in a channel of total spin \(S\) and isospin \(T\) are given by

\[
v_{LL'}(k,k') = \int_0^\infty r^2 dr j_{L}(kr) v_{LL'}(r) j_{L'}(k'r). \]

The corresponding normal (\(n\)) and anomalous (\(\kappa\)) density distributions are given by

\[
n(k) = \frac{1}{2} \left[ 1 - e_k - \mu \right], \quad \kappa(k) = \frac{\Delta(k)}{2E(k)},
\]

respectively. Selfconsistency for the chemical potential is imposed through

\[
\frac{\langle N \rangle}{V} = \frac{k_F^3}{3\pi^2} = \frac{1}{\pi^2} \int k^2 dk n(k) .
\]

We have developed computer codes to solve BCS gap equations \[c.f. Eqs. \(31-36\)\] following the method introduced by Baldo et al. \[30\]. The approach relies on the introduction of a reduced interaction which disconnects the low and high momentum components of the interaction but reproduces exactly the gap function in the low momentum regime.

In fig. 12 we show results for pairing gaps \(\Delta_F = \Delta(k_F)\), in symmetric nuclear matter for \(^1S_0\) and \(^3S_1-3D_1\) states. Empty and filled circles denote the use of BHF potentials in phase I and II, respectively. The inset shows the corresponding chemical potentials at low densities. Gap calculations in the overlapping domain of phases I and II have been performed in a simplified way, namely, pairing in phase I do not take into account phase II and vice versa. This simplification is intended to explore the most immediate consequences of the coexisting sp solutions. A consistent treatment of the two phases, together with \(pp\), \(nn\) and \(pn\) pairing, goes beyond the focus of this work.

Overall we observe that pairing gaps as function of the Fermi momentum, when \(k_F \geq 0.3\) fm\(^{-1}\), behave similarly to those published elsewhere \[1\]. The \(^1S_0\) state in symmetric nuclear matter yields non-vanishing pairing gap up to \(k_F \approx 1.25\) fm\(^{-1}\), with a peak \(\Delta_F = 2.93\) MeV, at \(k_F = 0.85\) fm\(^{-1}\). Similarly, pairing in the \(^3S_1-3D_1\) channel occurs up to \(k_F \approx 1.8\) fm\(^{-1}\), with a peak \(\Delta_F = 8.1\) MeV, at \(k_F = 1.0\) fm\(^{-1}\). For \(k_F \lesssim 0.3\) fm\(^{-1}\), however, the coexisting phases for the sp potentials yield different energy.
gaps. These features contrast with the continuous low-density solutions for symmetric nuclear matter based on Gogny’s interaction [26].

Regarding the low density behavior of the chemical potential shown in the inset of fig. 12, we note that for the deuteron channel $\mu \rightarrow -1.1$ MeV, close to half the binding energy of the deuteron in free space. This is an expected result as it is well known that gap equations at low densities reduce to the Schrödinger equation in free space, where the anomalous density distribution $\kappa$ takes the place of the wavefunction.

### 4.3.1 Coherence length

In BCS theory the coherence length $\xi$ sets a length scale for the distance between the two constituents forming Cooper pairs. In terms of the anomalous density distribution, this is given by

$$\xi = \frac{\int_0^\infty (\frac{dk}{\sqrt{2}}) k^2 dk}{\int_0^\infty k^2 dk}.$$ 

Its evaluation follows the procedure introduced in sec. 3.2.26 based on Laplace transforms. In fig. 13 we plot the coherence length as function of $k_F$ for the $^1S_0$ and $^3S_1$-$^3D_1$ channels. Phases I and II are denoted with empty and filled circles, respectively. The solid curve accounts for the internucleon separation $\rho^{-1/3}$. The $\pm$ symbol denote $\xi$ in channel $^1S_0$ while the solid curve represents mean internucleon separation $\rho^{-1/3}$.

### 4.3.2 Condensate fraction

As a last assessment of pairing solutions based on phases I and II for the sp spectra, we evaluate the condensate fraction of Cooper pairs. Here again we consider $^1S_0$ and...
condensate in channels \(^{3}\text{S}_0\) and \(^{3}\text{S}_1 - ^{3}\text{D}_1\) channels separately, disregarding the pairing competition from the two phases. In terms of the normal and anomalous density distributions the condensate fraction of paired nucleons is given by \([33]\)

\[
\frac{\rho_{\text{con}}}{\rho} = \frac{\int_{k_0}^{\infty} k^2 (k) k^2 dk}{\int_{0}^{\infty} n(k) k^2 dk}.
\] (38)

In fig. 15 we plot the condensate fraction \(\rho_{\text{con}}/\rho\) for channels \(^{3}\text{S}_0\) and \(^{3}\text{S}_1 - ^{3}\text{D}_1\) as functions of the Fermi momentum. Plus symbols and circles denote the use of sp spectra in phase I and II, respectively. We observe nearly total deuteron condensate (above 90\%) at low densities, in phase I, accounting for Bose-Einstein condensation. At the same time dineutron pairing emerges at \(k_F = 0.04\) fm\(^{-1}\), reaching maximum condensate fraction of 67\% at \(k_B\), the upper edge of phase I. This fraction is lower than that for deuterons in phases I and II. In particular, the condensate fraction for deuterons in phase II decreases monotonically from 0.8 at \(k_F = k_o\), vanishing at \(k_F = 1.8\) fm\(^{-1}\). In the case of the \(^{1}\text{S}_0\) channel, instead, maximum condensate fraction of about 0.33 is reached at \(k_F = 0.5\) fm\(^{-1}\), vanishing at \(k_F = 1.25\) fm\(^{-1}\).

As mentioned earlier, the results discussed above help us to visualize the trend of NN pairing when each phase is considered separately. One should keep in mind that pairing on different channels and phases is a competing process, where both the number of protons and neutrons remain constant. Hence, a proper treatment of pairing would require to take into account simultaneously \(pp\), \(nn\) and \(pn\) pairing on phases I and II. Such a treatment, although important for realistic purposes, goes beyond the scope of this work.

5 Summary and outlook

Within the BHF approximation in BBG theory for symmetric nuclear matter at zero temperature, we have investigated the role of dinucleon structures over a wide range of densities (10\(^{-7}\) fm\(^{-3}\) \(\leq \rho \leq 0.36\) fm\(^{-3}\)), with emphasis placed on the low-density regime. To this end we have calculated self-consistent sp potentials at Fermi momenta up to 1.75 fm\(^{-1}\) using the continuous choice. The study is based on the Argonne \(v_{14}\) bare NN potential considering all partial waves up to \(J = 7\) without three-body forces. The actual momentum dependence of the sp potential resulting from the evaluation of the mass operator has been retained. An explicit treatment was given to the occurrence of dinucleon bound states in the \(^{1}\text{S}_0\) and \(^{3}\text{S}_1 - ^{3}\text{D}_1\) channels during the evaluation of the mass operator.

As a result, two distinct families of solutions for the sp self-consistent potentials have been disclosed. The first class, \(U_I\), defines a surface in the \((k,k_F)\) plane at Fermi momenta below \(k_B = 0.285\) fm\(^{-1}\). The second class, \(U_{II}\) defines a surface at Fermi momenta above \(k_B = 0.130\) fm\(^{-1}\). In the range \([k_\alpha,k_\beta]\) both solutions satisfy the self-consistency requirement, representing coexisting solutions. Solution I behaves as a correlated FG with slight repulsion relative to its non-interacting counterpart. Solution II represents a cohesive system up to the saturation point at \(k_F = 1.53\) fm\(^{-1}\), where we obtain \(B/A = -16.9\) MeV for its binding energy per nucleon. Each solution leads to distinct behaviors in their corresponding effective masses. While the correlated gas phase shows an increasing effective mass, reaching \(m^* \approx 3m\) near \(k_\beta\), the condensed phase starts out with near-bare-mass sp at \(k_\alpha\), reaching a maximum of 1.8m at \(k_F \approx 0.55\) fm\(^{-1}\), and decreasing to about 0.8 m at saturation density. The emergence of massive sp solutions is a peculiarity of our results, in formal resemblance to Heavy Fermions (electrons) observed in unconventional superconductors \([34,35]\). Work in progress using alternative realistic NN interactions also yields coexisting solutions with large effective masses, confirming the robustness of these findings.

We have studied dinucleon structures associated with phases I and II for the sp fields. These bound states exhibit unconventional features, with mean radii as large as \(\sim 100\) fm. In the case of dineutrons, their size is always greater than the internucleon separation, with a sudden increase in the transit from phase I to phase II. Deuterons, in turn, are very confined in phase I but the wavefunction outranges the internucleon separation in phase II. These findings were contrasted with those obtained from BCS pairing states, where BCS gap equations were solved considering sp spectra for phases I and II, independently. Additionally, no consideration was given to \(nn\), \(pp\) and \(pn\) competing pairing. Nearly total deuteron condensate is obtained in phase I, decreasing monotonically in phase II up to \(k_F = 1.8\) fm\(^{-1}\), where \(^{3}\text{S}_1 - ^{3}\text{D}_1\) pairs dissolve. In the case of \(^{1}\text{S}_0\) pairing, the condensate fraction in phase I reaches 67\% at the upper edge, while in phase II the condensate fraction reaches \(\sim 1/3\) at \(k_F \sim 0.5\) fm\(^{-1}\). These results need to be reassessed considering the two phases in conjunction with competing pairing channels.

The disclosure of coexisting solutions in symmetric nuclear matter at low densities constitutes a major finding in this work. A crucial role was played by the explicit handling of dinucleon bound states together with...
refined methods to obtain self-consistent solutions for the
sp fields. We have shown that the low-density solutions
for the sp potentials \( U(k) \) are not quadratic in \( k \) as they
exhibit a sudden change of slope at \( k/k_F \approx 1.15 \). This fea-
ture rules out the reliability of using the effective mass ap-
proximation for the sp fields at low densities. In any case,
the results reported here demonstrate that self-consistency
within the BHF approximation, accounting for dinucleon
bound states, is feasible without the need for introducing
a functional form to the sp fields.

The study presented here, where emphasis has been
placed on pp correlations with explicit treatment of dinu-
cleon bound states, is basic in the sense that multi-particle
and clustering were not taken into account, although they
are known to be important in low-density clustering mod-
els. Additionally, three-body forces are known to be
relevant to account for the saturation point of nu-
clear matter \([35,37]\). However, beyond such simplifying assumptions as embodied by the
BHF framework to treat nuclear correlations, it is reason-
able to conceive that the model retains leading-order fea-
tures in diluted nuclear environments expected to linger at
low temperatures and isospin asymmetry. In this sense
the findings reported in this work may provide additional
conceptual tools to better understand the physics of di-
nucleated nuclear matter in the context of surface-sensitive
nuclear reactions, the coupling to continuum in modeling
nuclear open quantum systems \([39]\) or dense star environ-
ments. From a more general perspective, it is reasonable
to expect that phase-coexistence in homogeneous nuclear
matter breaks down as temperature and isospin asymme-
try reaches some critical values. Identifying the conditions
under which coexistence persists constitutes an interesting
extension of this work.

H.F.A. thanks colleagues at CEA, Bruyères-le-Châtel, France,
for their warm hospitality during his stay where part of this
work took place. The authors are indebted to M. Baldo for
useful discussions. Funding from CEA/DAM/DIF is acknowl-
dged. This research has been funded in part by FONDECYT
under grant No 1120396.

A Integral equation with multiple roots on
the real axis

A common technique to solve integral equations in mo-
mmentum space is by discretizing the momentum in the
range \( q : 0 \to \infty \), reducing the problem to a matrix form.
Thus, the equation

\[
t(k', k) = v(k', k) + \int_0^{\infty} \frac{F(k', q)}{\omega + i\eta - E(q)} t(q, k) \, dq,
\]

is reduced to the matrix equation

\[
t_{ij} = v_{ij} + \sum_{k=1}^{N} \frac{F_{ik} w_k}{\omega - E_k} t_{kj} - i\pi \sum_{p=N+1}^{N+K} \frac{F_{ip}}{|E'(q_p)|} t_{pj} \quad (A.1)
\]

Here we are assuming that there are \( K \) solutions to \( E(q) = \omega \), with \( q \) above the minimum value allowed by Pauli
blocking, leading to the set of roots \( \{ q^{(1)}, \ldots, q^{(K)} \} \). The first sum in Eq. (A.1) represents the principal-value integral,
for which we apply Gaussian-Legendre quadrature to inte-
grate between consecutive roots, up to infinity.

B Adaptive trapezoidal quadrature

Consider the evaluation of the integral

\[
I \equiv \int_0^1 f(x) \, dx,
\]

with \( f(x) \) any finite function in the range \([0, 1]\). This func-
tion may also have a finite number of narrow peaks. Let
us define

\[
I_0 = \frac{1}{2} [f(0) + f(1)] \quad (B.2)
\]

Additionally, let us consider the following sequence of sets of points in the interval \((0, 1)\): \( s_1 = \{1/2\} \), \( s_2 = \{1/4, 3/4\} \),
\( s_3 = \{1/8, 3/8, 5/8, 7/8\} \), etc. Note that the union of the first
\( N \) sets yields a uniform distribution of points, with no
repetition of its elements. If the elements of the union are
sequenced in ascending order, the separation between con-
secutive elements is \( 1/2^N \). With this construction it can be
verified that a trapezoidal quadrature can be represented by
the recurrence

\[
I_n = \frac{1}{2} I_{n-1} + \frac{1}{2} \sum_{x_j \in s_n} f(x_j) \quad (B.3)
\]

where \( I_0 \) is given by Eq. (B.2). In actual evaluations the it-
eration is interrupted at some \( n = M \) when a convergence
criteria for \( |I_M - I_{M-1}| \) is met. If narrow peaks occur, the
minimum \( n \) should be such that \( 1/2^n \) becomes smaller
than the smallest width. The advantage of this method is
that the historical reckoning of \( f(x) \) is not discarded if a
narrower mesh is needed. This is an important considera-
tion whenever a single evaluation of \( f \) is time-consuming.

C Evaluation of \( \psi_0 \) for S and D waves

Let us define

\[
\psi_0^{(L)}(q) = A q^L \exp[-R(q - q)|\Theta(q - q)] \quad (C.1)
\]

Denoting \( z = qR \), and \( x = qr \), the Fourier transform for
\( \psi_0^{(L)} \) for S waves becomes

\[
\psi_0^{(0)}(r) = \sqrt{\frac{2}{\pi}} \frac{A q^3}{x(2 z^2 + x^2)} \left[ (2 z + z^2 + x^2) x \cos x + (z^2 + z^2 + x^2 - x^2) \sin x \right] \quad (C.2)
\]

The asymptotic behavior \( r \to \infty \) is identified for \( x \gg 1 \),
case in which we get

\[
\psi_0^{(0)}(r) \approx A q \sqrt{\frac{2}{\pi}} \cos(qr) \quad (C.3)
\]
In the case of D-waves we obtain
\[
\psi_0^{(2)}(r) = \frac{2}{\pi} \sqrt{\frac{-Ag^3}{x^4(x^2 + z^2)^4}} \left\{ x^8 + 3x^6(z^2 + 3z - 5) + 3x^4(z^3 + 7z^2 - z - 16) + x^2z^4(z^2 + 15z + 15) + 3z^8(z^4 + 1)x\cos(x) + [x^3(z - 6) - 3z^6(z + 1) + x^3(z^3 - 15z - 15) + 3z^6(z^3 - 4z^2 - 11z + 5) + 3z^4(z^3 - 2z^2 - 15z - 15)]\sin(x) \right\}.
\] (C.4)
This result yields for large distances \((x \gg 1)\)
\[
\psi_0^{(2)}(r) \approx -Ag^3 \sqrt{\frac{2}{\pi}} \frac{\cos(\bar{q}r)}{r^4}.
\]

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