M2-branes and AdS/CFT: A review

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1. Introduction

One day in 1998, while I was a Ph.D. student, Prof. Eguchi came to me in the tea room and asked if I had already read the paper by Maldacena about a new duality, which is now known as the AdS/CFT correspondence [1]. It was when another important paper by Witten [2] had appeared. I had just finished my first paper [3] on emissions from D1–D5 black holes, which was a project suggested by him. Though he did not mean anything special by that small conversation, it remained in my memory because I felt I was being treated as an independent researcher for the first time.

In 2009, I started working with Prof. Eguchi again at Yukawa Institute, where he was the Director at that time. We worked together, sometimes jointly with cosmologists, on organizing and running a series of conferences. That was a job requiring a different level of dedication to physics. I was influenced a lot by the eagerness with which he kept these activities running for many years, and also by the way he cared about the purpose and the real outcome of each of those events.

In 2019 we held a conference in Kyoto in memory of Prof. Eguchi. In this article, partly based on a talk given there, I will briefly review some of the important developments in the last decade in the theory of multiple M2-branes and AdS/CFT correspondence. I will illustrate how the large-N limit was studied and the correspondence checked by taking the superconformal index, the free energy on $S^3$, and the entropy of charged black holes as examples.

Most of the discussions are restricted to the Aharony–Bergman–Jafferis–Maldacena (ABJM) model [4] for $N$ M2-branes probing the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. In the three-dimensional (3D) $N = 2$ convention, it is a $U(N)_k \times U(N)_{-k}$ Chern–Simons theory with chiral multiplets $A_1, A_2$ in the bifundamental and $B_3, B_4$ in the anti-bifundamental representations, and a superpotential

$$W = -\frac{2\pi}{k} \text{tr} [A_a B_b A_c B_d] \epsilon^{ac} \epsilon^{bd}. \quad (1.1)$$

The gauge field, scalar, and auxiliary field in the two $U(N)$ vector multiplets will be denoted as $(A_\mu, \sigma, D)$ and $(\bar{A}_\mu, \bar{\sigma}, \bar{D})$, respectively. The model should be dual to the quantum supergravity on AdS$_4 \times S^7/\mathbb{Z}_k$. AdS$_4$ and $S^7/\mathbb{Z}_k$ have radii $L$ and $2L$, respectively, which are related to $N$ and the
11-dimensional Newton constant $G_{(11)}$ via
\[(2\pi \ell_P)^6 N = 384 L^6 \cdot \text{vol}(S^7/\mathbb{Z}_k), \quad 16\pi G_{(11)} = \frac{(2\pi \ell_P)^9}{2\pi}.
\]

2. Superconformal index
An important problem in AdS/CFT is to understand the spectrum of states of both sides. Although complete understanding is difficult, precise results can be obtained in supersymmetric theories by restricting attention to subsectors of states preserving supersymmetry (SUSY). One can argue that the index encoding the information of such states is independent of couplings which can vary continuously. Following earlier developments [5,6] and results [7], an exact formula for the superconformal index was derived for the ABJM model in Ref. [8]. The results were shown to agree perfectly with the index over supergravitons in AdS$_4 \times S^7/\mathbb{Z}_k$ in the large-$N$ limit.

2.1. Definition
The 3D $\mathcal{N} = 6$ superconformal symmetry of the ABJM model has conformal symmetry $SO(2, 3)$ and R-symmetry $SO(6)$ as bosonic subgroup. Let us denote by $\epsilon$ and $j_3$ the Cartan generators for $SO(2) \times SO(3) \subset SO(2, 3)$, and $h_1, h_2, h_3$ for $SO(6)$. Then one can find nilpotent supercharges $Q$ and $S$ satisfying
\[
\{Q, S\} = \epsilon - h_3 - j_3,
\]
and both commuting with $h_1, h_2$ and $\epsilon + j_3$. The superconformal index is defined by the trace
\[
I(x, y_1, y_2) \equiv \text{Tr} \left[ (-1)^F e^{-\beta'(Q, S) - \beta(\epsilon + j_3) - \gamma h_1 - \gamma_2 h_2} \right] = \text{Tr} \left[ (-1)^F e^{-(\beta + \beta')\epsilon - (\beta - \beta')j_3 + \beta h_3 - \gamma h_1 - \gamma_2 h_2} \right]
\]
over the Hilbert space of radial quantization. Here, $x \equiv e^{-\beta}$, $y_1 \equiv e^{-\gamma_1}$, and $y_2 \equiv e^{-\gamma_2}$. Note that it is independent of $\beta'$ since it only receives contributions from the states annihilated by $Q$ and $S$.

The index can be computed as a path integral of the theory on $S^1 \times S^2$ with the $S^1$ parametrized by Euclidean time $\tau \sim \tau + \beta + \beta'$. The presence of $j_3$ and the $h_i$ in the trace translates into twists in the periodicity of the fields. If one prefers to work with periodic fields, one can take account of them by turning on background $SO(6)$ gauge fields and off-diagonal metric components.

2.2. Computation
The index can be evaluated with the help of SUSY localization. The path integral
\[
I \equiv \int \mathcal{D}\text{(fields)} e^{-(\text{action})}
\]
is supersymmetric; namely, there is a supercharge $Q$ under which the measure $\mathcal{D}\text{(fields)}$ and the action are both invariant. As such, $I$ is invariant under modification of the action by terms of the form $\frac{1}{g^2} Q \Psi$, with $\Psi$ fermionic and $Q^2 \Psi = 0$. By choosing $\Psi$ suitably and taking the weak coupling limit $g^2 \to 0$, one can show the path integral is given exactly by the sum over contributions of saddle points, and that the contribution of each saddle point can be evaluated using Gaussian approximation.
For the ABJM superconformal index, the saddle points are labeled by integers \( n_i, \tilde{n}_i \) and periodic variables \( \alpha_i, \tilde{\alpha}_i \) \((i = 1, \ldots, N)\). They appear in the value of the flux and temporal holonomy as

\[
\sigma = \int_{S^2} \frac{F}{2\pi} = \text{diag}(n_1, \ldots, n_N), \quad \text{P exp} i \int_{S^1} A = \text{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_N}),
\]

\[
\tilde{\sigma} = \int_{S^2} \frac{\tilde{F}}{2\pi} = \text{diag}(\tilde{n}_1, \ldots, \tilde{n}_N), \quad \text{P exp} i \int_{S^1} \tilde{A} = \text{diag}(e^{i\tilde{\alpha}_1}, \ldots, e^{i\tilde{\alpha}_N}). \quad (2.3)
\]

The value of the action at this saddle point is

\[
e^{-S} = e^{i \sum (n_i \alpha_i - \tilde{n}_i \tilde{\alpha}_i)}. \quad (2.4)
\]

This is multiplied by two “determinants” to make up the contribution of a given saddle point. Note that, as it turns out, both determinants are invariant under a simultaneous shift of the \( 2N \) variables \( \alpha_i, \tilde{\alpha}_i \) by the same amount. The integration over \( \alpha_i, \tilde{\alpha}_i \) along this direction thus gives rise to the constraint \( \sum_i n_i = \sum_i \tilde{n}_i \).

One of the determinants is the Faddeev–Popov determinant. The flux \( (n_i, \tilde{n}_i) \) generically breaks the gauge group \( U(N) \times U(N) \) to a subgroup \( \prod_i U(N_i) \times \prod_i U(\tilde{N}_i) \), with \( \sum_i N_i = \sum_i \tilde{N}_i = N \). The saddle point condition requires the holonomy to take values in this subgroup. Gauge-fixing the holonomy to also be diagonal gives rise to a factor \( \frac{1}{\text{Sym}} \cdot \Delta_{FP} \), where

\[
\text{Sym} = \prod_i N_i! \prod_i \tilde{N}_i!,
\]

\[
\Delta_{FP} = \prod_{i<j(n_i=n_j)} 2 \sin \left( \frac{\alpha_i - \alpha_j}{2} \right)^2 \prod_{i<j(\tilde{n}_i=\tilde{n}_j)} 2 \sin \left( \frac{\tilde{\alpha}_i - \tilde{\alpha}_j}{2} \right)^2. \quad (2.5)
\]

The other is the one-loop determinant arising from Gaussian integration over fluctuation of fields. It can be computed by Kaluza–Klein (KK) reducing the free theory of fluctuations along \( S^2 \). The resulting system can be regarded as a bunch of simple bosonic and fermionic harmonic oscillators with periodic Euclidean time. The determinant is its partition function,

\[
\Delta_{1\text{-loop}} = \left( \prod_{a:Fermi} 2 \sinh \frac{\beta \omega_a}{2} \right) / \left( \prod_{a:Bose} 2 \sinh \frac{\beta \omega_a}{2} \right), \quad (2.6)
\]

where we used the abbreviation

\[
\beta \omega_a \equiv \beta (\epsilon + j_3) + \beta' (\epsilon - h_3 - j_3) + \gamma_1 h_1 + \gamma_2 h_2 + \text{(gauge)} \quad (2.7)
\]

for the \( a \)th bosonic or fermionic oscillator. The term “(gauge)” represents the gauge charge; for example, it is \( \alpha_i - \tilde{\alpha}_j \) if the oscillator originates from the \((i,j)\) component of a bifundamental field. For later use, we rewrite it into a plethystic exponential,

\[
\Delta_{1\text{-loop}} = \exp \left[ - \beta \epsilon_0 + \sum_{n \geq 1} \frac{1}{n} f(x^n, y^n, e^{i\alpha_1}, e^{i\tilde{\alpha}_1}) \right], \quad (2.8)
\]

where the Casimir energy \( \epsilon_0 \) and the letter index \( f \) are defined by

\[
\beta \epsilon_0 = \sum_{B-F} \frac{\beta \omega_a}{2}, \quad f(x, y, e^{i\alpha_1}, e^{i\tilde{\alpha}_1}) \equiv \sum_{B-F} e^{-\beta \omega_a}. \quad (2.9)
\]
The KK reduction is performed using monopole harmonics (spherical harmonics for charged fields in the flux background). The quantities $\epsilon_0$ and $f$ therefore depend on the flux $n_i$ and $\tilde{n}_i$ as well, though this is not indicated explicitly.

The contribution to the index from saddle points with flux $(n_i, \tilde{n}_i)$ is thus given by

$$I \big|_{(n_i, \tilde{n}_i)} = \frac{1}{\text{Sym}} \int \prod_{i=1}^{N} \frac{d\alpha_i d\tilde{\alpha}_i}{(2\pi)^2} \exp \left[ i k \sum_{i=1}^{N} (n_i \alpha_i - \tilde{n}_i \tilde{\alpha}_i) - \beta \epsilon_0 + \sum_{n \geq 1} \frac{1}{n} f^{(n)} \right],$$ (2.10)

where the quantity Sym is defined in Eq. (2.5), and the letter index $f$ takes account of both the Faddeev–Popov and one-loop determinants:

$$f(x, y_1, y_2, e^{i\alpha_i}, e^{i\tilde{\alpha}_i}) = -\sum_{i \neq j} x^{i|n_i-n_j|} e^{-i(\alpha_i-\tilde{\alpha}_j)} + \sum_{i \neq j} x^{i|\tilde{n}_i-\tilde{n}_j|} e^{-i(\tilde{\alpha}_i-\alpha_j)} - \sum_{i \neq j} x^{i|\tilde{n}_i-\tilde{n}_j|} e^{-i(\tilde{\alpha}_i-\alpha_j)} + \sum_{i \neq j} f^{+}(x, y_1, y_2) x^{i|\tilde{n}_i-\tilde{n}_j|} e^{-i(\tilde{\alpha}_i-\alpha_j)} \right),$$

$$f^{+}(x, y_1, y_2) = \frac{1}{1-x^2} \left( x^2 y_1^2 y_2^2 - x^2 y_1^2 y_2^2 - x^2 y_1^2 y_2^2 \right),$$

$$f^{-}(x, y_1, y_2) = f^{+}(x, y_1, y_2^{-1}).$$ (2.11)

The Casimir energy is given by

$$\epsilon_0 = \sum_{i<j} |n_i - \tilde{n}_j| - \sum_{i<j} |n_i - n_j| - \sum_{i<j} |\tilde{n}_i - \tilde{n}_j|. \quad (2.12)$$

The full superconformal index $I(x, y_1, y_2, y_3)$ is given by the sum of Eq. (2.10) over different flux sectors with an additional weight $y_3^4 \sum n_i$. The new fugacity parameter $y_3$ counts the KK momentum along the M-theory circle, i.e. the Hopf fiber circle of $S^7/\mathbb{Z}_k$.

### 2.3. The large-$N$ limit

A nice way [9,10] to treat the integral over the $N+N$ variables $\alpha_i, \tilde{\alpha}_i$ in the limit is to express it in terms of the eigenvalue density functions $\rho(\alpha)$, $\tilde{\rho}(\tilde{\alpha})$ and their Fourier modes,

$$\rho(\alpha) \equiv \sum_{i=1}^{N} \delta(\alpha - \alpha_i) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \rho_n e^{-in\alpha_i}, \quad \rho_n \equiv \sum_{i=1}^{N} e^{-in\alpha_i},$$

$$\tilde{\rho}(\tilde{\alpha}) \equiv \sum_{i=1}^{N} \delta(\tilde{\alpha} - \tilde{\alpha}_i) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \tilde{\rho}_n e^{-in\tilde{\alpha}_i}, \quad \tilde{\rho}_n \equiv \sum_{i=1}^{N} e^{-in\tilde{\alpha}_i}. \quad (2.13)$$

Note that $\rho_0 = \tilde{\rho}_0 = N$. As a simple exercise, let us rewrite the contribution of the zero-flux sector $I^{(0)}$ using these variables. We find that the result is a simple Gaussian integral,

$$I^{(0)} = \int \prod_{n \neq 0} d\rho_n d\tilde{\rho}_n \cdot \exp \left( \sum_{n \geq 1} \frac{1}{n} f^{(n)} \right),$$

$$f^{(n)} = -\left( \rho_n \tilde{\rho}_n \right) \begin{pmatrix} 1 & -f^{+}(n) \\ -f^{-}(n) & 1 \end{pmatrix} \begin{pmatrix} \rho_n \\ -\tilde{\rho}_n \end{pmatrix}. \quad (2.14)$$
which gives

\[ I^{(0)} = \prod_{n=1}^{\infty} \frac{(1 - x^{2n})^2}{(1 - x^n y_1^n)(1 - x^n y_2^n)(1 - x^n y_1^{-n})(1 - x^n y_2^{-n})}. \]  

(2.15)

The evaluation of the contributions of sectors with nonzero flux is apparently much harder. The idea employed in Ref. [8] is to divide the integration variables \( \alpha_i, \tilde{\alpha}_i \) into three groups. The first contains those \( \alpha_i \) or \( \tilde{\alpha}_i \) for which the corresponding flux \( (n_i \text{ or } \tilde{n}_i) \) is positive, and the second contains those corresponding to negative flux. As long as one looks at the sectors carrying \( O(N^0) \) momentum along the M-theory circle, these two groups have \( O(N^0) \) variables. All the rest, corresponding to zero flux, are in the third group. In the large-\( N \) limit one can apply the change of variables described in the previous paragraph to the third group, after which the integration measure becomes, schematically,

\[ \frac{1}{\text{Sym}} \int d(\alpha, \tilde{\alpha}) \implies \frac{1}{\text{Sym}'} \int d(\alpha, \tilde{\alpha})_+ d(\alpha, \tilde{\alpha})_- \prod_{n \neq 0} d\rho_n d\tilde{\rho}_n. \]  

(2.16)

A nice observation of Ref. [8] is that the integral over \( \rho_n, \tilde{\rho}_n \) is still Gaussian, and moreover the result takes the following factorized form:

\[ I \bigg|_{(n_i, \tilde{n}_i)} = I^{(0)} \int d(\alpha, \tilde{\alpha})_+ d(\alpha, \tilde{\alpha})_- \left( \text{function of } (\alpha, \tilde{\alpha})_+ \right) \cdot \left( \text{function of } (\alpha, \tilde{\alpha})_- \right). \]  

(2.17)

This implies that the full index \( I \) takes the factorized form

\[ I(x, y_1, y_2, y_3) = I^{(0)}(x, y_1, y_2) \cdot I_+(x, y_1, y_2, y_3) \cdot I_-(x, y_1, y_2, y_3), \]  

(2.18)

where \( I_+, I_- \) are positive and negative power series in \( y_3 \), respectively. Though we are left with a finite-dimensional integral, the computation becomes increasingly complicated as the flux increases.

### 3. Free energy on \( S^3 \)

Free energy measures the number of low-energy degrees of freedom. A supergravity analysis predicted [11] that the free energy for the system of \( N \) M2-branes should scale as \( N^{3/2} \) at large \( N \). This behavior was reproduced from the exact partition function of the ABJM model on \( S^3 \).

Generally, for a system of \( N \) M2-branes with near-horizon geometry \( \text{AdS}_4 \times Y \), the gravitational free energy is given by the classical action evaluated on the corresponding Euclidean background. Though it is naively infinite, after subtracting the power-law divergences by suitable counterterms [12–14] one can extract a finite positive value,

\[ F = \frac{\pi L^2}{2G(4)}. \]  

(3.1)

Here, \( L \) is the radius of \( \text{AdS}_4 \) and \( G(4) \) is the effective 4D Newton constant. As a function of \( N \) and the volume of \( Y \) (normalized so that its metric satisfies \( R_{mn} = 6g_{mn} \)), \( F \) becomes

\[ F = N^\frac{3}{2} \sqrt{\frac{2\pi^6}{27\text{vol}(Y)}}. \]  

(3.2)

For the ABJM model one has \( Y = S^7 / \mathbb{Z}_k \) and \( F = \frac{\sqrt{2\pi}}{3} k^{\frac{1}{2}} N^{\frac{3}{2}} \).
The path integral of the Euclidean ABJM model on $S^3$ was studied in Ref. [15]. It was shown that the saddle points are parametrized by $2N$ variables $\sigma_i, \tilde{\sigma}_i$ ($i = 1, \ldots, N$), and the scalar fields in the vector multiplets take constant values,

$$\sigma = \text{diag}(\sigma_1, \ldots, \sigma_N), \quad \tilde{\sigma} = \text{diag}(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N),$$

at the saddle points. The partition function is given by the following integral:

$$Z_{S^3} = \int d^N \sigma d^N \tilde{\sigma} e^{-F(\sigma, \tilde{\sigma})},$$

where we introduced $g_s \equiv 2\pi i/k$.

### 3.1. Large-$N$ limit: Traditional approach

A standard way to evaluate this integral is to use the idea of large-$N$ expansion [16]. Let us generalize the gauge group to $U(N_1) \times U(N_2)$ for a while and consider the limit $N_1, N_2, k \to \infty$ with the 't Hooft couplings $t_1 \equiv g_s N_1$ and $t_2 \equiv g_s N_2$ kept fixed. The free energy then has an expansion of the form

$$F = -\ln Z_{S^3} = \sum_{g=0}^{\infty} g_s^{2g-2} F_g(t_1, t_2).$$

The planar contribution $g_s^{-2} F_0(t_1, t_2)$ is given by the value of $F(\sigma_i, \tilde{\sigma}_i)$ at its extremum, where $\sigma_i, \tilde{\sigma}_i$ satisfy

$$0 = \frac{\partial F}{\partial \sigma_i} = \frac{\sigma_i}{g_s} - \sum_{j \neq i} \coth \frac{\sigma_i - \sigma_j}{2} + \sum_j \tanh \frac{\sigma_i - \tilde{\sigma}_j}{2},$$

$$0 = \frac{\partial F}{\partial \tilde{\sigma}_i} = -\frac{\tilde{\sigma}_i}{g_s} - \sum_{j \neq i} \coth \frac{\tilde{\sigma}_i - \tilde{\sigma}_j}{2} + \sum_j \tanh \frac{\tilde{\sigma}_i - \sigma_j}{2}.$$  \hspace{1cm} (3.6)

These equations are often interpreted as the condition for the equilibrium of forces acting on each eigenvalue. The forces between two $\sigma$s or two $\tilde{\sigma}$s are repulsive, whereas $\sigma_i$ and $\tilde{\sigma}_j$ attract each other.

In the large-$N$ limit, the eigenvalues $\{\sigma_j\}$ and $\{\tilde{\sigma}_i\}$ will form continuous distributions along some intervals $C$ and $\tilde{C}$. Let $\rho(x)$ and $\tilde{\rho}(x)$ be their densities. Moreover, let us define the resolvent by

$$\omega(z) = g_s \sum_{j=1}^{N_1} \coth \frac{z - \sigma_j}{2} - g_s \sum_{j=1}^{N_2} \tanh \frac{z - \tilde{\sigma}_j}{2},$$

$$= t_1 \int_C dx \rho(x) \coth \frac{z - x}{2} - t_2 \int_{\tilde{C}} dx \tilde{\rho}(x) \tanh \frac{z - x}{2}.$$  \hspace{1cm} (3.7)

It turns out that the equations in Eq. (3.6) translate into the following discontinuity relations for $\omega(z)$:

$$\omega(z - i\epsilon) + \omega(z + i\epsilon) = 2z \quad \text{(for } z \in C),$$

\hspace{1cm} 6/17
Re\(z\), \quad \text{Im}\(z\)

\[ \omega(z + i\pi - i\epsilon) + e^{2z - \omega(z)} = 2z \quad (\text{for } z \in \tilde{C}), \]

which implies that \( f(z) \equiv e^{\omega(z)} + e^{2z - \omega(z)} \) is an entire function. By combining it with the boundary condition at infinity one can determine \( \omega(z) \) up to an arbitrary constant \( \kappa \),

\[ \omega = t_2 - t_1 + 2 \ln \frac{1}{2} \left( \sqrt{1 + (i\kappa - 2e^{t_1-t_2})e^{z} + e^{2z}} - \sqrt{1 + (i\kappa + 2e^{t_1-t_2})e^{z} + e^{2z}} \right). \]

The square roots produce two branch cuts \( C \) and \( \tilde{C} + i\pi \), and the left panel of Fig. 1 shows their form when \( t_1 - t_2 > 0 \) for a suitable choice of \( \kappa \). A useful fact is that the \( \kappa \)-derivative of the integral

\[ \frac{\partial}{\partial \kappa} \int \omega(z)dz = \int \frac{-i du}{\sqrt{(1 + (i\kappa - 2e^{t_1-t_2})u + u^2)(1 + (i\kappa + 2e^{t_1-t_2})u + u^2)}}, \]

where we have denoted \( e^z \equiv u \).

To find a relation between the planar free energy and 't Hooft couplings, we express them using contour integrals of \( \omega(z)dz \). First of all, one finds

\[ \oint_\alpha \omega(z)dz = 4\pi i t_1, \quad \oint_{\tilde{\alpha}} \omega(z)dz = -4\pi i t_2, \]

where \( \alpha, \tilde{\alpha} \) are the contours encircling \( C \) and \( \tilde{C} + i\pi \) as shown in Fig. 1. Also, notice that one can transport one \( \sigma \) eigenvalue from infinity to a point \( z_s \in C \) by integrating \( \frac{1}{g_s}(z - \omega(z)) \) along the contour \( \gamma \) shown in the figure. If the integral were not divergent, it would correspond to the change of the free energy under the shift of \( N_1 \) by one (or the shift of \( t_1 \) by \( g_s \)). It turns out that a simultaneous shift of \( t_1, t_2 \) corresponds to a finite contour integral,

\[ \frac{1}{2} \oint_\beta \omega(z)dz = \frac{\partial F_0}{\partial t_1} + \frac{\partial F_0}{\partial t_2} + i\pi(t_1 - t_2), \]

where the contour \( \beta \) is as shown in the figure.

We now restrict to the ABJM model and set \( t_1 = t_2 \equiv t \). For a large positive \( \kappa \), the four branch points of the elliptic integral are approximately at

\[ u \simeq -i\kappa + 2, \quad \frac{i}{\kappa + \frac{2}{\kappa^2}}, \quad \frac{i}{\kappa - \frac{2}{\kappa^2}}, \quad -i\kappa - 2, \]

Fig. 1. (Left) The branch cuts of \( \omega(z) \) and the contours \( \alpha, \tilde{\alpha}, \beta, \gamma \). (Right) A sketch of the numerical solution of Eq. (3.6) found in Ref. [17].

\[ \omega(z + i\pi - i\epsilon) + e^{2z - \omega(z)} = 2z \quad (\text{for } z \in \tilde{C}), \]
with $C$ running between the first two and $\tilde{C} + i\pi$ between the latter two. It is not difficult to extract, by evaluating the elliptic integrals and integrating with respect to $\kappa$, the leading large-\(\kappa\) behavior

$$t \sim \frac{i}{\pi} (\ln \kappa)^2, \quad \frac{\partial F_0}{\partial t} \sim i\pi \ln \kappa. \quad (3.13)$$

This implies $F_0 \sim -\frac{2}{3}(-i\pi t)^{3/2}$ and therefore $F \sim \frac{\sqrt{2} \pi}{3} k^{1/2} N^{3/2}$, which is in precise agreement with the prediction of supergravity.

The original work in Refs. [16,18] also studied non-planar corrections (higher orders of perturbative series in $g_s$) and instanton corrections by making use of the connection of the integral in Eq. (3.4) with the one for Chern–Simons theory on the lens space, which is in turn dual at large $N$ to topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$. The full perturbative series in Eq. (3.5) was computed in Ref. [19] using the holomorphic anomaly equation, and the result turned out to be given simply by an Airy function.

### 3.2. Large-\(N\) limit: Another approach

A different method for evaluating the integral in Eq. (3.4) was invented in Ref. [17], and it turned out to be very efficient for studying the large-\(N\) limit with $k$ fixed. It is partly based on the numerical result for the extremization of $F(\sigma_i, \tilde{\sigma}_i)$, which looks like the right panel of Fig. 1. It implied that the eigenvalue distribution is described by two functions $\rho(x), y(x)$ in such a way that the replacement

$$\sum_{i=1}^{N} \varphi(\sigma_i) \to N \int dx \rho(x) \varphi(N^\alpha x + iy(x)),$$

$$\sum_{i=1}^{N} \varphi(\tilde{\sigma}_i) \to N \int dx \rho(x) \varphi(N^\alpha x - iy(x)) \quad (3.14)$$

works for an arbitrary function $\varphi(x)$ in the large-\(N\) limit. By rewriting $F(\sigma_i, \tilde{\sigma}_i)$ using this rule, one finds it becomes a local functional of $\rho(x)$ and $y(x)$,

$$F[\rho, y] = \frac{k}{\pi} N^{1+\alpha} \int dx x \rho(x) y(x) + N^{2-\alpha} \int dx \rho(x)^2 f(2y(x)), \quad (3.15)$$

where $f(x)$ is a function of period $2\pi$ and $f(x) = \pi^2 - x^2$ for $|x| \leq \pi$. The balance of the two terms on the right-hand side requires $\alpha = \frac{1}{2}$, which immediately implies the $N^{3/2}$ scaling of the free energy. The initial assumption that the distributions of Re$(\sigma_i)$ and Re$(\tilde{\sigma}_i)$ are described by the same function $\rho(x)$ is also justified, because otherwise there would be terms of higher order in $N$ remaining on the right-hand side. It is now easy to extremize $F$ with respect to $\rho(x), y(x)$ under the condition $\int dx \rho(x) = 1$. The result reads

$$\rho(x) = \frac{1}{2x_*} (|x| \leq x_*), \quad x_* = \frac{\pi}{\sqrt{2} k}, \quad y(x) = \frac{\pi x}{2x_*}. \quad (3.16)$$

The value of $F[\rho, y]$ at this extremum is $F = \frac{\sqrt{2} \pi}{3} k^{1/2} N^{3/2}$, and thus the supergravity result was correctly reproduced again.

Though this method is efficient, it is not very obvious how to go beyond the strict large-\(N\) limit. Another powerful method, called the “Fermi gas” approach, to study the model systematically at large $N$ with $k$ kept fixed, was introduced in Ref. [20]. It is based on a reformulation of the integral.
in Eq. (3.4) as the partition function of a 1D gas of $N$ non-interacting fermions with a non-trivial Hamiltonian. The combination of this approach with other methods from topological Boolean algebra and topological strings led to a very detailed understanding of the structure of non-perturbative corrections [21–26].

3.3. Generalization

The check of AdS/CFT via comparison of free energy on $S^3$ can be generalized to the cases with less SUSY. An explicit formula is known for the free energy of general $\mathcal{N} \geq 2$ Chern–Simons matter theories [27,28]. For $N$ M2-branes at the tip of some Calabi–Yau 4-fold cone $X$, the worldvolume dynamics is described by $U(N)^p$ Chern–Simons matter theories with matters satisfying certain conditions. By adopting an ansatz similar to Eq. (3.14), one can show that the free energy for such theories scales as $N^{3/2}$, and obtain a local functional of $\rho(x)$ and $y_1(x), \ldots, y_p(x)$ [29].

A new issue arises from the fact that, for general $\mathcal{N} = 2$ theories of vector and chiral multiplets, the Lagrangian on $S^3$ at the starting point has arbitrariness in the assignment of R-charges to chiral multiplets. By extremizing the functional of $\rho(x), y_a(x)$ one therefore ends up with a function of the matter R-charges. As proposed in Refs. [27,29] and proved in Ref. [30], the correct assignment corresponding to the R-charge of $\mathcal{N} = 2$ superconformal symmetry is the one which maximizes the free energy. As an illustrative exercise, let us break the SUSY of the ABJM model to $\mathcal{N} = 2$ by assigning arbitrary R-charges $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ to the chiral fields $A_1, A_2, B_3, B_4$, with the constraint

$$\sum_{a=1}^{4} \Delta_a = 2.$$  (3.17)

We also turn on the R-charge $\Delta_m$ for the monopole operator $T$ carrying a unit flux. By deriving the free energy functional and extremizing it with respect to $\rho(x), y(x)$, one obtains

$$F = \frac{N^3 4\sqrt{2\pi}}{3k^2} \sqrt{(k\Delta_1 - \Delta_m)(k\Delta_2 - \Delta_m)(k\Delta_3 + \Delta_m)(k\Delta_4 + \Delta_m)}.$$  (3.18)

It has a flat direction under which $(\Delta_1, \ldots, \Delta_4, \Delta_m)$ shifts by $(\delta, \delta, -\delta, -\delta, k\delta)$, which is a reflection of the fact that the operators $A_1, A_2, B_3, B_4, T$ carry $U(1)$ gauge charges corresponding to $\text{Tr}(A_\mu - \tilde{A}_\mu)$. By extremizing with respect to the other non-flat directions with the constraint in Eq. (3.17), one recovers $F = \frac{\sqrt{2\pi}}{k^2} N^3$ again.

On the gravity side, we need to compute the volume of the 7D Sasaki–Einstein space $Y$ which is the base of the cone $X$. If $X$ is toric, there is a useful technique to compute the volume of $Y$ as a solution to a minimization problem [31]. By definition, $X$ has $U(1)^4$ symmetry, and one can regard $X$ as a $T^4$ fibration over a convex polyhedral cone $C$ inside $\mathbb{R}^4$. Its Kähler form is given by

$$\omega = \sum_{i=0}^{3} dx_i \wedge d\varphi_i,$$  (3.19)

with $x_i$ the coordinates on the base and $\varphi_i \sim \varphi_i + 2\pi$ on the fiber. We denote by $\vec{v}_a$ the inward-pointing normal vector to the $a$th facet of $C$. The Calabi–Yau condition implies that one may assume $v_{a0} = 1$ for all $a$. Note that the components $v_{ai}$ are all integers, since $v_a$ also specifies the 1-cycle of $T^4$ which shrinks above the $a$th facet.

There is a distinguished isometry $\sum b_i \partial s_i$, called the Reeb vector, which is paired up with the radial vector field under a chosen complex structure of $X$. As shown in Ref. [31], $\vec{b}$ contains some
information on the (Kähler- but not necessarily Ricci-flat) metric of $X$ which is actually enough to
determine the volume of $Y$ and all of its 5-cycles. Consider a hyperplane

$$
\sum_{i=0}^{3} b_{i} x_{i} = \frac{1}{2},
$$

(3.20)

which intersects $C$ to make a finite polytope $\Delta_{b}$. Then

$$
\text{vol}(Y) = 128 \pi^{4} \text{vol}(\Delta_{b}).
$$

(3.21)

$Y$ is Sassaki–Einstein when $\vec{b}$ is chosen to minimize $\text{vol}(\Delta_{b})$ under the condition $b_{0} = 4$.

As an example, for $X = \mathbb{C}^{4}/\mathbb{Z}_{k}$ one can take

$$
(\tilde{v}_{1}, \tilde{v}_{2}, \tilde{v}_{3}, \tilde{v}_{4}) = \left( \begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & k \\
\end{array} \right), \quad \tilde{b} = 2 \cdot \left( \begin{array}{cc}
\Delta_{1} + \Delta_{3} \\
\Delta_{1} + \Delta_{4} \\
\Delta_{1} - \Delta_{m} \\
\end{array} \right).
$$

(3.22)

The parametrization of $\tilde{b}$ by $\Delta$s can be obtained by matching the R-charges of M5-branes wrapping
various 5-cycles with those of gauge-invariant operators in ABJM. One then finds

$$
\text{vol}(Y) = \frac{\pi^{4} k^{3}}{48(k \Delta_{1} - \Delta_{m})(k \Delta_{2} - \Delta_{m})(k \Delta_{3} + \Delta_{m})(k \Delta_{4} + \Delta_{m})},
$$

(3.23)

where we used Eq. (3.17). Note that the above result reproduces Eq. (3.18) via Eq. (3.2) before extremization. Generalization of this correspondence was studied in Refs. [32,33], though it is not simple because the numbers of parameters on the gauge and gravity sides do not agree in general.

4. Entropy of charged black holes

According to AdS/CFT, any classical solution with AdS asymptotics should be described as an ensemble of states in the corresponding CFT. Construction of black holes in AdS spacetime was known to be considerably harder than in flat spacetime, but an analytic solution for asymptotically AdS4 static Bogomol’nyi–Prasad–Sommerfield (BPS) black holes with magnetic (and electric) charges was found in Ref. [34]. A natural question is whether the dual CFTs correctly account for their entropy as the degeneracy of states.

4.1. Black hole solutions and their entropy

The black hole solutions were found in 4D $\mathcal{N} = 2$ supergravity with $n$ Abelian vector multiplets and gauging [35]. The bosonic fields in this theory are the metric $g_{\mu\nu}(x)$, $(n + 1)$ gauge fields $A_{\mu}^{A}(x)$ ($A = 0, \ldots, n$) and $n$ complex scalars $z^{i}(x)$ ($i = 1, \ldots, n$) which parametrize a special Kähler manifold $\mathcal{M}$. There is a rank-$(2n + 2)$ holomorphic vector bundle over $\mathcal{M}$, and the Kähler potential of $\mathcal{M}$ is expressed in terms of its section $\Omega \equiv (X^{A}(z), F_{A}(z))$ and its conjugate $\bar{\Omega} \equiv (\bar{X}^{A}(\bar{z}), \bar{F}_{A}(\bar{z}))$ as

$$
K(z, \bar{z}) = - \ln (i(\Omega, \bar{\Omega})),
$$

(4.1)

where $\langle \Omega, \bar{\Omega} \rangle \equiv F_{A} \bar{X}^{A} - X^{A} \bar{F}_{A}$ is the duality-invariant bilinear product. The covariant derivatives of $\Omega, \bar{\Omega}$ with respect to $z^{i}, \bar{z}^{i}$ are

$$
\nabla_{i} \Omega \equiv \partial_{i} \Omega + \partial_{i} K \cdot \Omega = (\nabla_{i} X^{A}, \nabla_{i} F_{A}), \quad \nabla_{i} \bar{\Omega} \equiv \partial_{i} \bar{\Omega} = 0,
$$

where $\mathcal{D}_{i} \equiv \partial_{i} + \partial_{i} K \cdot \Omega$ is the covariant derivative.
The duality group

\[ \nabla_i \tilde{\Omega} \equiv \partial_i \tilde{\Omega} + \partial_i K \cdot \tilde{\Omega} = (\nabla_i \tilde{X}^\Lambda, \nabla_i \tilde{F}_\Lambda), \quad \nabla_i \tilde{\Omega} \equiv \partial_i \tilde{\Omega} = 0, \]  

(4.2)

and the Kähler metric on \( \mathcal{M} \) is

\[ g_{ij} = -\frac{\langle \nabla_i \Omega, \nabla_j \Omega \rangle}{\langle \Omega, \Omega \rangle}. \]  

(4.3)

The condition \( \langle \Omega, \nabla_i \Omega \rangle = 0 \) implies the existence of the prepotential \( F(X) \), which is a homogeneous function of degree 2 in \( X^\Lambda \) satisfying \( F^\Lambda(z) = \frac{\partial F}{\partial X^\Lambda}(X(z)) \). It also implies there is a symmetric matrix \( N_{\Lambda \Sigma}(z, \bar{z}) \) such that

\[ F_\Lambda = N_{\Lambda \Sigma} X^\Sigma, \quad \nabla_i \tilde{F}_\Lambda = N_{\Lambda \Sigma} \nabla_i \tilde{X}^\Sigma. \]  

(4.4)

The first few terms in the supergravity action \([36]\) read

\[ S = \frac{1}{16\pi G_N} \int \left( -d\text{vol} \cdot R + N_{\Lambda \Sigma} F^{+\Lambda} \wedge F^{+\Sigma} + \tilde{N}_{\Lambda \Sigma} F^{-\Lambda} \wedge F^{-\Sigma} + \cdots \right), \]  

(4.5)

where \( F^{\pm\Lambda} \) is the imaginary (anti-)self-dual part of the field strengths \( F^\Lambda = dA^\Lambda \) satisfying \( *F^{\pm\Lambda} = \pm i F^{\pm\Lambda} \). We define the magnetic and electric charges \( q \equiv (q^\Lambda, q_\Lambda) \) of spherically symmetric solutions by

\[ q^\Lambda = \int_{S^2} \frac{F^\Lambda}{4\pi}, \quad q_\Lambda = \int_{S^2} \frac{G_\Lambda}{4\pi} \left( G^\Lambda \equiv N_{\Lambda \Sigma} F^{+\Sigma} + \tilde{N}_{\Lambda \Sigma} F^{-\Sigma} \right). \]  

(4.6)

The duality group \( Sp(2n + 2, \mathbb{R}) \) rotates the vectors \( \Omega \) and \( q \) in the same way.

Static extremal black holes with flat asymptotics were studied in Ref. \([36]\). It was found that, for spherically symmetric solutions of the form

\[ ds^2 = e^{2U(r)} dt^2 - e^{-2U(r)} \left( dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right), \quad z^i = z^i(r), \]  

(4.7)

the BPS condition can be cast into a flow equation,

\[ r^2 \frac{dU}{dr} = e^U |Z|, \quad r^2 \frac{dz^i}{dr} = 2 e^U g^{ij} \frac{\partial}{\partial z^j} |Z|. \]  

(4.8)

Here, \( g^{ij}(z, \bar{z}) \) is the inverse metric on \( \mathcal{M} \) and \( Z(z, \bar{z}) = e^{K/2} (\Omega, q) \) is the central charge of the black hole with charge \( q \). These imply that the value of the scalars \( z^i \) at the horizon \( r = 0 \) should extremize \( |Z(z, \bar{z})| \), and also that the entropy of the black hole is given by

\[ S_{\text{BH}} = \frac{1}{4G_N} \text{(horizon area)} = \frac{\pi |Z|^2}{G_N} \]  

(4.9)

at its extremum, which is therefore a function of the charge \( q \) only.

To discuss black holes with AdS asymptotics, one needs to move to gauged supergravity. In 4D \( \mathcal{N} = 2 \) supergravity, gauging amounts to assigning \( U(1)^{n_H+1} \) charges to fields according to their \( SU(2)_R \) charges. We denote the couplings as \( g \equiv (g^\Lambda, g_\Lambda) \), though the discussions of concrete theories are often restricted to those with \( g^\Lambda = 0 \).

In order to explain the mechanism of gauging, we think of adding \( n_H \) hypermultiplets whose scalars \( y^m (m = 1, \ldots, 4n_H) \) parametrize a quaternionic space \( \mathcal{M}_H \). There is a principal \( SU(2) \) bundle over
\( \mathcal{M}_H \) with connection \( V^a = V^a_m(y)dy^m \) \((a = 1, 2, 3) \) such that the triplet of Kähler forms of \( \mathcal{M}_H \) is proportional to its curvature 2-form. All the fields with \( SU(2)_R \) charges are then coupled to \( V^a \). For example, the SUSY transformation rule for the gravitino \( \psi^A_\mu(x) \) reads

\[
\delta \psi^A_\mu = \partial_\mu \epsilon^A + \frac{1}{4} \omega^{ab}_\mu \gamma^{ab} \epsilon^A - i \frac{1}{2} \partial_\mu y^{m}(y) (\sigma^a)^A_B e^B + \cdots. \tag{4.10}
\]

If \( \mathcal{M}_H \) has a \( U(1)^{n+1} \) isometry and we want to gauge it, we covariantize the derivatives,

\[
\partial_\mu y^m(x) \rightarrow \partial_\mu y^m(x) + A^A_\mu (x) \cdot k^m(y(x)), \tag{4.11}
\]

where \( k^m(y) \) is the Killing vector field on \( \mathcal{M}_H \) for the \( \Lambda \)th \( U(1) \). At the same time, an analogue of the \( U(1)^{n+1} \) hyperKähler moment map \( P_A(y) \) takes part in the modification,

\[
\partial_\mu y^m(x) V^a_m(y(x)) \rightarrow \partial_\mu y^m(x) V^a_m(y(x)) + A^A_\mu (x) \cdot P_A(y(x)). \tag{4.12}
\]

This procedure works even for the case with empty \( \mathcal{M}_H \) and constant \( P_A(y) = g_\Lambda \delta^{a3} \), and thus couples the \( U(1)^{n+1} \) gauge fields to fields with \( SU(2)_R \) charges. The gravitino SUSY transformation rule now becomes

\[
\delta \epsilon (\psi^A d\epsilon^\mu) = d\epsilon^A + \frac{1}{4} \omega^{ab} \gamma^{ab} \epsilon^A - i \frac{1}{2} g_\Lambda A^A \cdot (\sigma^3^A)_B e^B + \cdots. \tag{4.13}
\]

Note that the coupling \( g_\Lambda \) determines the quantization rule of the charges,

\[
g^\Lambda \equiv \frac{1}{2g_\Lambda} \mathbb{Z}, \quad g_\Lambda = 2g_N G_{\mathbb{Z}}. \tag{4.14}
\]

A peculiar feature of the black hole solutions of Ref. [34] with magnetic charge is that, with a spherical symmetric ansatz, the gauge field \( A^\Lambda \) and spin connection \( \omega^{ab} \) both take the form \( \sim \cos \theta d\varphi \) so that the second and the third terms in Eq. (4.13) cancel each other. This occurs when the twisting condition

\[
g_\Lambda q^\Lambda = -1 \tag{4.15}
\]

is satisfied. The cancellation among contributions from various connections is reminiscent of the construction of topologically twisted theories. The observation of this fact led to the identification of the dual description [37,38]. Black hole solutions with \( g_\Lambda q^\Lambda = 0 \) are also known, but they require some rotation to be free of a naked singularity [39,40].

For BPS black hole solutions in gauged supergravity, there is an attractor flow equation similar to Eq. (4.8) which allows one to obtain the black hole entropy without working out the metric explicitly [41]. It implies that the value of \( \varphi \) at the horizon extremizes

\[
R^2 = -i \frac{q_\Omega}{q_\Omega} \frac{\partial X^\Lambda}{q_\Omega} = -i \frac{q_\Omega X^\Lambda - q^\Lambda \mathcal{F}_\Lambda}{g_\Lambda X^\Lambda}, \tag{4.16}
\]

and the value of \( R \) at the extremum gives the horizon radius. Thanks to the homogeneity of the right-hand side, one can think of extremizing the numerator as a function of \( X^\Lambda \) keeping the denominator fixed.

As an example, let us consider \( \mathcal{N} = 8 \) maximal gauged supergravity truncated to \( \mathcal{N} = 2 \), which is relevant to the ABJM model at \( k = 1 \). Its prepotential and couplings are given by

\[
\mathcal{F}(X) = -2i \sqrt{X^0 X^1 X^2 X^3}, \quad g_0 = g_1 = g_2 = g_3 \equiv g = \frac{1}{\sqrt{2L}}, \tag{4.17}
\]

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where $L$ is the AdS$_4$ radius. In terms of integer charges

$$n^\Lambda = -2g q^\Lambda, \quad e^\Lambda = \frac{q^\Lambda}{2g G_N},$$

(4.18)

the twisting condition in Eq. (4.15) becomes $\sum_\Lambda n^\Lambda = 2$, and the entropy is given by

$$S_{BH} = -i \left( e^\Lambda X^\Lambda + \frac{L^2}{2 G_N} \cdot n^\Lambda \frac{\partial F}{\partial X^\Lambda} \right),$$

(4.19)

extremized as a function of $X^\Lambda$ under the constraint $\sum_\Lambda X^\Lambda = 2\pi$. Note that the extremum value has to be real and positive in order for the solution to have a smooth horizon. This puts an independent condition on $(n^\Lambda, e^\Lambda)$.

### 4.2. Microscopic theory

The microscopic theory for these black holes is a 3D $\mathcal{N} = 2$ supersymmetric theory on $S^2 \times \mathbb{R}$ with a topological twist by a unit background $U(1)_R$ flux through $S^2$. The matter R-charge has to be integer due to Dirac quantization, but there are infinite choices for its assignments if the theory has flavor symmetry. The path integral of the theory with periodic time (i.e. on $S^1 \times S^2$) is called the twisted index [42]. For the theories with conserved flavor charges $J^a$, one can turn on the constant $\sigma$ and $A_\tau$ components of the corresponding vector multiplet in the background. In a Hamiltonian description, the twisted index computes the trace over the Hilbert space $\mathcal{H}$,

$$I = \text{Tr}_\mathcal{H} \left[ (-1)^F e^{-\beta \left( H - i \sum_a A^a_j J^a \right)} \right].$$

(4.20)

It is a function of the complexified flat connections $\Delta_a = \beta(\bar{A}^a + i \sigma^a)$ only, because the supercharge $Q$ satisfies $Q^2 = H - \sum_a \sigma^a J^a$.

In the case of the ABJM model, the twist is labeled by the R-charges $n_1, \ldots, n_4 \in \mathbb{Z}$ of the chiral multiplets $A_1, A_2, B_3, B_4$, obeying the constraint $\sum_\alpha n_\alpha = 2$. The model has $U(1)^3$ flavor symmetry generated by $J^a - J^4 (a = 1, 2, 3)$, where $J^a$ phase-rotates the $a$th chiral multiplet. It is therefore convenient to regard $I$ as a function of flat connections $\Delta_1, \ldots, \Delta_4$ obeying the constraint

$$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 0 \mod 2\pi \mathbb{Z}.$$  

(4.21)

Thanks to SUSY localization, the path integral can be reduced to an integral over saddle points labeled by integers $m_i, \tilde{m}_i$ and periodic variables $u_i, \tilde{u}_i (i = 1, \ldots, N)$. They appear in the value of the vector multiplet field at the saddle point as

$$\text{P exp} i \int_{S^1} (A + i \sigma d\tau) = \text{diag}(e^{iu_1}, \ldots, e^{iu_N}), \quad \int_{S^2} \frac{F}{2\pi} = \text{diag}(m_1, \ldots, m_N),$$

$$\text{P exp} i \int_{S^1} (\tilde{A} + i \tilde{\sigma} d\tau) = \text{diag}(e^{i\tilde{u}_1}, \ldots, e^{i\tilde{u}_N}), \quad \int_{S^2} \frac{\tilde{F}}{2\pi} = \text{diag}(\tilde{m}_1, \ldots, \tilde{m}_N).$$

(4.22)

The system also has fermionic zero modes $\xi, \tilde{\xi}$ which are paired with $u^*_i, \tilde{u}^*_i$ under the supersymmetry. Consequently, after localization one is left with an integral over the $u$ along some contour, not over the cylinder. With $x_i \equiv e^{iu_i}, \tilde{x}_i \equiv e^{i\tilde{u}_i}$, and $y_a \equiv e^{i\Delta_a}$, one can express the index as

$$I = \frac{1}{(N!)^2} \sum_{m_i, \tilde{m}_i} \prod_{i=1}^N \frac{dx_i}{2\pi i x_i} \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} e^{k m_i \tilde{m}_i \frac{x_i}{\tilde{x}_i} - k \tilde{m}_i m_i} \prod_{i \neq j}^N \left( 1 - \frac{x_i}{x_j} \right) \left( 1 - \frac{\tilde{x}_i}{\tilde{x}_j} \right).$$

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The contour integral is performed following the Jeffrey–Kirwan (JK) residue prescription [43], which was first used in the study of SUSY –localized path integrals in 2D [44,45] and 1D [46]. It goes roughly as follows. For each pole (intersections of singular hyperplanes) $p$ of the integrand, there is a matter field responsible for each of the hyperplanes. Label $p$ by the charges $Q_p = (\bar{q}_1, \bar{q}_2, \ldots)$ of those matters under the Cartan of the gauge group. (In the present problem there are additional singularities at $x_i, \tilde{x}_i = 0$ or $\infty$. They are labeled according to the Chern–Simons couplings [42].) The JK-residue prescription begins by choosing a reference charge $\tilde{\eta}$ arbitrarily. Then one decides whether to pick up the residue of a pole $p$ according to whether the cone spanned by the charge vectors in $Q_p$ includes $\tilde{\eta}$ or not. The end result is independent of the initial choice of $\tilde{\eta}$.

For the index of the ABJM model in Eq. (4.23), there is a suitable choice of $\tilde{\eta}$ such that one only has to evaluate the residue of the pole at $x_i = \tilde{x}_i = 0$. Then the terms in $I$ with $m_i$ very large (or $\tilde{m}_i$ negatively very large) can be discarded because there would not be a pole at $x_i = 0$ (or $\tilde{x}_i = 0$). As a result, one only has to sum over $m_i \leq M$ and $\tilde{m}_i \geq -M$ for some $M$, which can be performed easily before integrating over $x_i, \tilde{x}_i$. One is then left with an integral over $x_i, \tilde{x}_i$, and the integrand has poles at the solution of a Bethe ansatz like

$$x_i^k = \prod_{j=1}^N \frac{(1 - \tilde{x}_j/x_i y_1)(1 - \tilde{x}_j/x_i y_2)}{(1 - y_3 \tilde{x}_j/x_i)(1 - y_4 \tilde{x}_j/x_i)}, \quad \tilde{x}_i^k = \prod_{i=1}^N \frac{(1 - \tilde{x}_i/x_j y_1)(1 - \tilde{x}_i/x_j y_2)}{(1 - y_3 \tilde{x}_i/x_j)(1 - y_4 \tilde{x}_i/x_j)}.$$ (4.24)

These are actually the equations for the extremum of the potential,

$$\tilde{W} \equiv \frac{k}{2} \sum_{i=1}^N (\bar{u}_i^2 - u_i^2) - \sum_{i,j=1}^N \sum_{a=1}^4 \epsilon_a \text{Li}_2(e^{(\bar{u}_j - u_i - \epsilon_a \Delta a)}) - 2\pi \sum_{i=1}^N (\tilde{c}_i \bar{u}_i - c_i u_i).$$ (4.25)

Here, $\text{Li}_n(x) = \sum_{k \geq 1} x^k / k^n$ is the polylogarithm function, and $\epsilon_a = (+1, +1, -1, -1)$; $c_i, \tilde{c}_i$ are integers which arise from the multi-valuedness of the log function.

The extremization of $\tilde{W}$ was studied in Ref. [37]. It was found that by using an ansatz similar to Eq. (3.14),

$$\sum_{i=1}^N \varphi(u_i) \to N \int dx \rho(x) \varphi(iN^ax + y(x)),$$

$$\sum_{i=1}^N \varphi(\bar{u}_i) \to N \int dx \rho(x) \varphi(iN^ax + \bar{y}(x)),$$ (4.26)

one can rewrite $\tilde{W}$ into a local functional of $\rho(x)$ and $\bar{y}(x) - y(x)$, and moreover the local functional takes the same form as the one for the free energy (of the $\mathcal{N} = 2$ deformed theory with general R-charge assignments) on $S^3$. It turned out that the variational problem has a consistent solution only...
for $\sum_a \Delta_a = 2\pi$, and the value of $\tilde{W}$ and $I$ at the solution are given (for $k = 1$) by

$$\ln I(n_a; \Delta_a) = i \sum_{a=1}^{4} n_a \frac{\partial \tilde{W}}{\partial \Delta_a}, \quad \tilde{W} = iN^\frac{3}{2} \cdot \frac{2\sqrt{2}}{3} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}. \quad (4.27)$$

Now recall that $I(n_a; \Delta_a)$ is the trace over the states of the twisted theory labeled by $n_a$, with the weight $e^{i \sum_{a} e_a \Delta_a}$ for the states with flavor charge $J^a = e_a$. The number $d(n_a; e_a)$ of states with charge $e_a$ should therefore be related to $I$ by Legendre transformation,

$$\ln d(n_a; e_a) = \sum_{a=1}^{4} \left( -ie_a \Delta_a + in_a \frac{\partial \tilde{W}}{\partial \Delta_a} \right), \quad (4.28)$$

where the right-hand side should be extremized as a function of $\Delta_a$ with the constraint $\sum_a \Delta_a = 2\pi$. In view of Eq. (3.1), we see that the black hole entropy in Eq. (4.19) has been reproduced precisely by the microscopic theory.

Ever since I became a student of Prof. Eguchi, I used to feel tense every time we had discussions on physics. It always led me to commit to physics more seriously and helped me grow. The same feeling still comes back now whenever I remember him.

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