Quantile regression for compositional covariates

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ABSTRACT

Quantile regression is a very important tool to explore the relationship between the response variable and its covariates. Motivated by mean regression with LASSO for compositional covariates proposed by Lin et al. (Biometrika 101 (4):785–97, 2014), we consider quantile regression with no-penalty and penalty function. We develop the computational algorithms based on linear programming. Numerical studies indicate that our methods provide the better alternative than mean regression under many settings, particularly for heavy-tailed or skewed distribution of the error term. Finally, we study the fat data using the proposed method.

1. Introduction

Compositional data are defined traditionally as constrained data, like proportions or percentages, with a fixed constant sum constraint (such as unit-sum constraint), which can find applications in a wide range of geology, sociology, economics, biology and so on (Shanmugam, Hron, and Templ 2018). Sometimes, we are especially interested in relative information, not the absolute values, such as geo-chemical compositions of rocks. Since the seminal work of Aitchison (1982), statistical methodologies have been proposed for compositional data analysis. However, owing to the special nature of compositional data, the usual linear regression model is inappropriate for our purposes. The linear log-con- trast model of Aitchison and Bacon-Shone (1984) is a very common method for regression to deal with compositional data. To the best of authors’ knowledge, statistical methods discussed mean regression. As we known, mean regression is not robust against outliers. For the testing problem, we address quantile regression in the paper.

Quantile regression is robust to outliers and heavy-tailed conditional error distributions. Moreover, it can provide a more complete picture than mean regression when the conditional distribution of the response variable is asymmetric. Hence, quantile regression has been applied in survival analysis, financial economics, investment analysis and so on (Koenker and Bassett 1978; Koenker and Geling 2001; Yu, Lu, and Stander 2003). Variable selection is an important issue in statistical modeling. In recent years, many different types of penalties have been introduced. Tibshirani (1996) proposed LASSO, which imposed the same penalty on every regression coefficient leading to excessive compression of larger coefficients. To tackle the problem, Fan and Li (2001) developed SCAD, which has three properties: unbiasedness, sparsity, continuity. Zou (2006) introduced the adaptive LASSO by using adaptive penalizing weights for different coefficients in the LASSO penalty, which meets above three properties. Besides, there are fused LASSO (Tibshirani et al. 2005), elastic net (Zou and Hastie 2005) and MCP (Zhang 2010). We also refer to Wu and Liu (2009).
For compositional data, Lin et al. (2014) studied variable selection in mean regression by LASSO. However, this method is sensitive to outliers. When the error follows the heavy-tailed distribution or asymmetric distribution, it works not well. In this paper, we develop a novel method to overcome these difficulties via combining quantile regression with the adaptive LASSO penalty, showing the new algorithms based on linear programme.

The paper is organized as follows. In Sec. 2, we introduce the proposed methods of quantile regression and its variable selection, and give a method for selecting the tuning parameter. In Sec. 3, we present the computational algorithms. Simulation studies and an empirical example are presented in Sec. 4. The article concludes in Sec. 5.

2. Quantile regression for compositional data

Let \((Y_i, X_i)\) be an observation collected from the subject \((i = 1, \ldots, n)\), where \(Y_i \in R\) is the response of interest, \(X_i = (X_{i1}, \ldots, X_{ip})^\top \in R^p\) is the \(p\)-dimensional covariates lying in the \((p - 1)\)-dimensional positive simplex \(S_{p-1} = \{(X_{i1}, \ldots, X_{ip})| X_{ij} > 0, \sum_{j=1}^p X_{ij} = 1\}\). \(Y = (Y_1, \ldots, Y_n)^\top\). We apply the log-ratio transformation of Aitchison (1982), and lead to the linear log-contrast model

\[
Y = Z\beta + \epsilon,
\]

where \(Z = \{ \log(x_{ij}/x_{ip}) \}\) is an \(n \times (p - 1)\) log-ratio matrix with the \(p\)th component as the reference component, \(\beta_p = (\beta_1, \ldots, \beta_{p-1})^\top\), \(\epsilon\) is an \(n\)-vector of independent error term.

In the model (1), the reference component selection is not easy. As Lin et al. (2014), let \(\beta_p = -\sum_{j=1}^{p-1} \beta_j\), the expression can be rewritten as

\[
Y = Z\beta + \epsilon, \quad \sum_{j=1}^p \beta_j = 0,
\]

where \(Z = \{ \log(x_{ij}) \}_{n \times p} = (Z_1, \ldots, Z_n)^\top\), \(\beta = (\beta_1, \ldots, \beta_p)^\top\). Note that the intercept is not included in the model, since it can be eliminated by centering the response and predictor variables. So, the estimator of quantile regression is to minimize the following objective function

\[
\arg \min_{\beta} \sum_{i=1}^n \rho_\tau(Y_i - Z_i^\top \beta),
\]

s.t. \(\sum_{j=1}^p \beta_j = 0\),

where \(\rho_\tau(u) = u(\tau - I(u < 0))\) is called the check function. \(\tau \in (0, 1)\) is the quantile, and \(I(\cdot)\) is an indicative function. \(\beta = (\beta_1, \ldots, \beta_p)^\top\) is the unknown parameter vector.

Now, we consider quantile regression with the adaptive LASSO, which is motivated by the LASSO penalty mean regression proposed by Lin et al. (2014). As we known, the adaptive LASSO penalty function is a generalization of the LASSO penalty via adaptive weights. Hence, we consider the constrained optimization problem

\[
\arg \min_{\beta} \left( \sum_{i=1}^n \rho_\tau(Y_i - Z_i^\top \beta) + \lambda \sum_{j=1}^p w_j |\beta_j| \right),
\]

s.t. \(\sum_{j=1}^p \beta_j = 0\),
where \( w_j = \frac{1}{|\beta_j|^\kappa} \), and \( \kappa > 0 \). \( \hat{\beta} \) is the solution of model (3). Here, \( \lambda \) is the tuning parameter. As the suggestion of Zou (2006), we set \( \kappa = 1 \) in our paper.

In the problem (4), the tuning parameter is very important because the penalty method depends on the choice of it. We can use the BIC criterion to select the parameter \( \lambda \) (Wang, Li, and Tsai 2007), which is defined as

\[
\text{BIC}(\lambda_n) = \log \left( \sum_{i=1}^{n} \rho_i(Y_i - Z_i^\top \beta) \right) + \frac{\log n}{n} \times df
\]

where \( df \) is the number of nonzero coefficients. The optimal regularization parameter \( \lambda_{\text{opt}} = \arg\min_{\lambda_n} \text{BIC}(\lambda_n) \).

3. Theoretical properties

In this section, we first establish the asymptotic properties of quantile regression (3), which can be re-expressed as

\[
\arg\min_{\beta} \sum_{i=1}^{n} \rho_i(Y_i - Z_i^\top \beta),
\]

s.t. \( C^\top \beta = 0 \),

where \( C \) is a row \( p \)-vector of ones. To facilitate asymptotic properties of our proposed estimator, we need assume the following technical conditions.

(A1) \( \{e_i\} \) are independent distributed, with \( r \)th quantile zero and a continuous, positive density \( f_i(\cdot) \) in a neighborhood of zero. The distribution functions \( F_i(\cdot) \) are absolutely continuous, and \( F_i(0) = \tau \).

(A2) There exist positive definite matrices \( D_0 \) and \( D_1(\tau) \) such that

(i) \( \lim_{n \to \infty} \frac{\sum_{i=1}^{n} Z_i Z_i^\top}{n} = D_0 \)

(ii) \( \lim_{n \to \infty} \frac{\sum_{i=1}^{n} f_i(0) Z_i Z_i^\top}{n} = D_1 \)

(iii) \( \lim_{n \to \infty} \max_{i=1, \ldots, n} \frac{\|Z_i\|}{\sqrt{n}} = 0 \).

These Conditions are regular in the literature of quantile regression (Koenker and Bassett 1978).

Theorem 1. Under conditions A1 and A2, we have

\[
\sqrt{n}(\hat{\beta} - \beta) \overset{D}{\to} N(0, \tau(1 - \tau)(E - M)D_1^{-1}D_0D_1^{-1}(E - M^\top))
\]

where \( M = D_1^{-1}C(C^\top D_1^{-1}C)^{-1}C^\top \) and \( E \) is the identity matrix.

If \( \{e_i\} \) are independent and identically distributed, we get

\[
\sqrt{n}(\hat{\beta} - \beta) \overset{D}{\to} N\left(0, \frac{\tau(1 - \tau)}{f^2(0)} (E - M)D_0^{-1}(E - M^\top) \right).
\]

where \( M_0 = D_0^{-1}C(C^\top D_0^{-1}C)^{-1}C^\top \).
In addition to the conditions in Theorem 1, we further assume that

\[ \beta_{10}^T = (\beta_{10}^T, \beta_{20}^T)^T \text{ where } \beta_{10} \text{ is a } s \times 1 \text{ nonzero column vector, } \beta_{20}^T = 0. \]

The corresponding parameter estimator is \( \hat{\beta}_{0}^{AQR} = (\hat{\beta}_{10}^{AQR, T}, \hat{\beta}_{20}^{AQR, T})^T \).

**Theorem 2.** In addition to the conditions in Theorem 1, we further assume that \( \{e_i\} \) are independent and identically distributed. If \( \sqrt{n} \lambda_n \to 0, \ n^{(1+\kappa)/2} \lambda_n \to \infty, \) for \( C^T \beta_0 = 0, \) we have

1. **Sparsity:** \( \hat{\beta}_{20}^{AQR} = 0. \)
2. **Asymptotic normality:** \( \sqrt{n}(\hat{\beta}_{10}^{AQR} - \beta_{10}) \overset{D}{\to} N(0, \frac{t(1-\tau)^2}{f(0)}(E - M_{11})D_{11}^{-1}(E - M_{11})^T). \)

where \( D_{11} \) and \( M_{11} \) are the top-left \( s \)-by-\( s \) submatrix of \( D_0 \) and \( M_0 \) separately.

## 4. Algorithms

From (3) and (4), they are the constrained optimization problems. First of all, we deal with quantile regression with the adaptive LASSO. Here, we give the computational algorithm, which introduces some slack variables replacing the objective function with an equality constraint so that (4) can be transformed into a linear programming problem.

Let \( u_i = \max(0, Y_i - Z_i^T \beta), \ v_i = \max(0, -(Y_i - Z_i^T \beta)), \ \beta_j^+ = \max(0, \beta_j), \ \beta_j^- = \max(0, -\beta_j). \ \beta_j = \beta_j^+ - \beta_j^- \) and \( |\beta_j| = \beta_j^+ + \beta_j^- . \ w = (w_1, ..., w_p)^T \). By the expressions of slack variables, (4) can be re-expressed as

\[
\arg\min_\beta \frac{1}{n} \sum_{i=1}^{n} \rho_\tau(Y_i - Z_i^T \beta) + \lambda \sum_{j=1}^{p} w_j |\beta_j| \quad (7)
\]

subject to \( C^T \beta_0 = 0, \)

where \( C \) is a matrix with \( s \times q \) and \( C^T \beta_0 = 0. \)

Here \( u = (u_1, ..., u_n)^T \) and \( v = (v_1, ..., v_n)^T. \ \beta^+ = (\beta_1^+, ..., \beta_p^+)^T \) and \( \beta^- = (\beta_1^-, ..., \beta_p^-)^T. \ I_n \) denotes the \( n \)-vector of ones. \( A = (\lambda w^T, \lambda w^T, \tau I_n^T, (1 - \tau)I_n^T) \) and \( \gamma = ((\beta^+)^T, (\beta^-)^T, u^T, v^T)^T. \)

The constrained condition is

\( Y - Z \beta = u - v. \)

Elementary calculations show that

\[
\begin{bmatrix}
Z & -Z & E_n & -E_n \\
\end{bmatrix}
\begin{bmatrix}
\beta^+ \\
u_j \\
\end{bmatrix}
= Y
\]

where \( E_n \) denotes the \( n \times n \) identity matrix.
Let

\[ B = \begin{bmatrix} I_p^\top & -I_p^\top & 0_n^\top & -0_n^\top \\ Z & -Z & E_n & -E_n \end{bmatrix} \]

and

\[ H = (0, Y^\top)^\top . \]

where \( 0_n \) denotes the \( n \)-vector of zeros. Combined with \( \sum_{j=1}^p \beta_j = 0 \), the constrained optimization problem (4) can be transformed into a linear programming problem

\[
\min \, A\gamma \\
\text{s.t. } B\gamma = H
\]

Similarly, quantile regression without penalty model (3) can also transform into a linear programming problem.
\[
\min A_1 \gamma_1 \\
\text{s.t. } B_1 \gamma_1 = H_1
\]

where \( A_1 = (0_n^T, 0_n^T, \tau I_n^T, (1 - \tau)I_n^T) \), \( B_1 = B \), \( \gamma_1 = \gamma \), and \( H_1 = H \).

5. Numerical studies

5.1. Simulations

As Lin et al. (2014), we generate the covariate data in the following way. We first generate an \( n \times p \) data matrix \( O = (o_{ij}) \) from a multivariate normal distribution \( N_p(\mu, \Sigma) \), and then obtain the covariate matrix \( X = (x_{ij}) \), where \( x_{ij} = \exp(o_{ij})/\sum_{k=1}^p \exp(o_{ik}) \). Here \( \mu = (\mu_1, \ldots, \mu_p)^T \). We repeat 500 times for each setting. The error term \( \varepsilon \) is generated from five distributions.

Case 1. \( \varepsilon \sim N(0, 1) \).
Case 2. \( \varepsilon \sim t(3) \), which is symmetric and heavy-tail distribution.
Case 3. \( \varepsilon \sim \text{pareto}(2, 1) \), which is the heavy-tail distribution.
Case 4. \( \varepsilon \sim \text{gpd}(0.2, 0.1.2) \), which is the skewed distribution.
Case 5. \( \varepsilon \sim \text{gev}(0.2, 3, 1.5) \), which is the extreme value distribution, and the skewed distribution.

In Example 1, we examine the performance of mean regression (MR) and quantile regression (QR, \( \tau = 0.5 \)). In Example 2, we conduct the Monte Carlo comparisons for variable selection.

Example 1. Let

\[
\mu_j = \begin{cases} 
  \log(0.5 \ast p), & j = 1, 2, 3 \\
  0, & \text{others}
\end{cases}
\]

and \( \Sigma = \rho^{i-j} \) with \( \rho = 0.2 \). We set \( n = \{50, 100, 200, 500\} \), and generate the responses according to model (2) with \( \beta = (1, -0.8, 0.6, -1.5, -0.5, 1.2)^T \). We evaluate the performance through the following two criteria:

(1) \( b_j = \frac{1}{500} \sum_{m=1}^{500} |\hat{\beta}_j^{(m)} - \beta_j| \)
(2) \( L_1 = \frac{1}{500} \sum_{m=1}^{500} \sum_{j=1}^p |\hat{\beta}_j^{(m)} - \beta_j| \)

Here \( \hat{\beta}_j^{(m)} \) is the estimator of \( \beta_j \) based on the m-th sample. We compare the performance of quantile regression (QR) with mean regression (MR).

Table 1 summarizes the simulation results. We can draw the following conclusions:

(1) When the error distribution follows the normal distribution, MR is slightly better than QR. As the sample size increases, the differences between them are decreasing.
(2) When the error distribution is the heavy-tailed or skewed, QR performs better than MR since these distributions have outliers.

Example 2. Let

\[
\mu_j = \begin{cases} 
  \log(0.5 \ast p), & j = 1, \ldots, 5 \\
  0, & \text{others}
\end{cases}
\]

and \( \beta = (1, -0.8, 0.6, 0.0, -1.5, -0.5, 1.2, 0, \ldots, 0)^T \). We set \( (n, p) = \{(50, 10), (100, 10), (100, 20), (200, 20)\} \). To summarize the variable selection results and evaluate estimation accuracy, we consider the following criteria:
From Tables 2 and 3, we can get the following comments:

1. From $L_1$, QR-ALA is better than MR-LA and MR-ALA, especially for the heavy-tail or skewed distribution. Even if $N(0, 1)$, QR-ALA is still slightly better, which implies that QR-ALA is more accurate. The performances of three methods increase gradually with $n$.

2. For variable selection, QR-ALA and MR-ALA outperform than MR-LA, which is more inclined to set zero coefficients to nonzero since FP is very large. When the error term follows the normal distribution, MR-ALA is better than QR-ALA. However, when the error term follows other distributions, QR-ALA is superior to MR-ALA, which clearly indicates that the proposed method is more efficient.

5.2. Application

In this section, to illustrate the usefulness of the proposed procedure, we apply the proposed method in the dataset fat, which contains many physical measurements of 252 males can be found in R package” UsingR”. Body.fat is the response variable. The following X-variables are
used as covariates: neck (circumference), chest (circumference), abdomen (circumference), hip (circumference), thigh (circumference), knee (circumference), ankle (circumference), bicep (circumference), forearm (circumference) and wrist (circumference). Actually, the covariates can be regarded as composition because the “size” of the body is not relevant, only the relations (ratios) between the different body measurements contain the important information. We transform covariates into compositional data. As the suggestion of Shanmugam, Hron, and Templ (2018),

\[ Y = \log\left(\frac{\text{body}}{\text{fat}}/100\right) \]

There are 251 observations after removing suspicious observations. Here, the ten-fold cross-validation method is used to select the tuning parameter. To evaluate the performance of MR-ALA and QR-ALA, we divide the data set into test set and training set, 9 copies as the training set, and 1 copy as the test set at random. We repeat 100 simulations and use NMSE to compare two methods. NMSE is defined by

\[ \text{NMSE} = \frac{\sum (Y_i - \hat{Y}_i)^2}{\sum (Y_j - \bar{Y})^2} \]

where \( \bar{Y} \) is the mean of the response variable, \( \hat{Y}_i \) is the predictive value of the test data set using the model obtained from the training set.

As Table 4, NMSE of QR-ALA is less than MR-ALA whether it is the raw data or transformed compositional data, which QR-ALA is better than that MR-ALA since there are outliers in the dataset fat. It is surprised that the performances of the two methods with compositional data are
better than the corresponding models with original data, which implies this transform may be necessary and meaningful in application.

6. Discussion

In this paper, we study quantile regression with compositional data, and propose penalized quantile regression with the adaptive LASSO penalty function. Due to linear programming, the proposed of the algorithm works not well when dimension $p$ is much larger than sample size $n$. This problem may be achieved by ADMM (Yu and Lin 2017). We will study it in our future research.

Appendix

Proof of Theorem 1. Let

$$G_n(\delta) = \sum_{i=1}^{n} \rho_{\tau}\left(e_i - \frac{Z_i^T\delta}{\sqrt{n}} \right) - \rho_{\tau}(e_i),$$

where $e_i = Y_i - Z_i^T\beta$. $G_n(\delta)$ is obviously convex and is minimized at $\hat{\delta}_n = \sqrt{n}(\hat{\beta}_{nc} - \beta)$. $\hat{\beta}_{nc}$ is the no constrained estimations, that is $\hat{\beta}_{nc} = \text{argmin}\rho_{\tau}(Y_i - Z_i^T\beta)$. Following Knight (1998), the limiting distribution of $\hat{\delta}_n$ is determined by the limiting behavior of the function $G_n(\delta)$. If there exists $G(\delta)$, $G_n(\delta) \to G(\delta)$, we have

$$\text{argmin} G_n(\delta) \xrightarrow{D} \text{argmin} G(\delta).$$

Using Knight’s identity,

$$\rho_{\tau}(h - t) - \rho_{\tau}(h) = -t\psi_{\tau}(h) + \int_{0}^{t} (I(h \leq s) - I(h \leq 0))ds,$$

where $\psi_{\tau}(h) = \tau - I(h < 0)$. $G_n(\delta)$ can be written as

$$G_n(\delta) = G_n^{(1)}(\delta) + G_n^{(2)}(\delta)$$

where

$$G_n^{(1)}(\delta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i^T\delta\psi_{\tau}(e_i),$$

$$G_n^{(2)}(\delta) = \sum_{i=1}^{n} \int_{0}^{v_{ni}} (I(e_i \leq s) - I(e_i \leq 0))ds$$

$$= \sum_{i=1}^{n} G_n^{(2)}(\delta).$$

Here $v_{ni} = Z_i^T\delta/\sqrt{n}$. By the Lindeberg–Feller central limit theorem and Condition A2, we obtain

$$G_n^{(1)}(\delta) \xrightarrow{D} - \delta^T W, \ W \sim N(0, \tau(1 - \tau)D_0).$$

(A.1)

The second term $G_n^{(2)}(\delta)$, we write

$$G_n^{(2)}(\delta) = \sum_{i=1}^{n} E(G_n^{(2)}(\delta)) + \sum_{i=1}^{n} (G_n^{(2)}(\delta) - E(G_n^{(2)}(\delta))).$$

Moreover,

|                  | MR-ALA | QR-ALA |
|------------------|--------|--------|
| Original data    | 0.426  | 0.376  |
| Compositional data | 0.424  | 0.353  |
\[ \sum_{i=1}^{n} E\left( G_{ni}^{(2)}(\delta) \right) = \sum_{i=1}^{n} \int_{0}^{\tau_{ni}} (F_{i}(0+s) - F_{i}(0)) ds \]
\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{0}^{\tau_{ni}} \left( F_{i} \left( 0 + \frac{t}{\sqrt{n}} \right) - F_{i}(0) \right) dt \]
\[ = n^{-1} \sum_{i=1}^{n} \int_{0}^{\tau_{ni}} \sqrt{n} \left( F_{i} \left( 0 + \frac{t}{\sqrt{n}} \right) - F_{i}(0) \right) dt \]
\[ = n^{-1} \sum_{i=1}^{n} \int_{0}^{\tau_{ni}} f_{i}(0) dt + o(1) \]
\[ = (2n)^{-1} \sum_{i=1}^{n} f_{i}(0) \delta^T Z_{i} Z_{i}^T \delta + o(1) \]
\[ \Rightarrow \frac{1}{2} \delta^T D_{1} \delta, \quad n \to \infty. \]

By condition A2(iii), we get
\[ E\left( \sum_{i=1}^{n} \left( G_{ni}^{(2)}(\delta) - E\left( G_{ni}^{(2)}(\delta) \right) \right) \right)^2 \]
\[ = \sum_{i=1}^{n} E\left( G_{ni}^{(2)}(\delta) - E\left( G_{ni}^{(2)}(\delta) \right) \right)^2 \]
\[ \leq \sum_{i=1}^{n} E\left( G_{ni}^{(2)}(\delta) \right)^2 \]
\[ \leq \sum_{i=1}^{n} \frac{Z_{i}^T \delta}{\sqrt{n}} E\left( G_{ni}^{(2)}(\delta) \right) \]
\[ \leq \frac{2}{\sqrt{n}} \max_{i=1, \ldots, n} |Z_{i}^T \delta| \sum_{i=1}^{n} E\left( G_{ni}^{(2)}(\delta) \right) \]
\[ \Rightarrow 0. \]

So,
\[ G_{ni}^{(2)}(\delta) = \frac{1}{2} \delta^T D_{1} \delta + o_p(1). \quad \text{(A.2)} \]

Together with (A.1) and (A.2),
\[ G_{n}(\delta) \xrightarrow{D} G(\delta) = -\delta^T W + \frac{1}{2} \delta^T D_{1} \delta. \]

Hence,
\[ \sqrt{n} \left( \hat{\beta}_{mc} - \beta \right) = \delta_{n} \xrightarrow{D} \delta = D_{1}^{-1} W. \]

Furthermore,
\[ \sqrt{n} \left( \hat{\beta}_{mc} - \beta \right) \xrightarrow{D} N(0, \tau(1 - \tau)D_{1}^{-1} D_{0} D_{1}^{-1}). \]

Under the constrained condition \( C^T \beta = 0 \) and \( C^T (\beta + \delta/\sqrt{n}) = 0 \). Let
\[ L(\delta, \lambda) = G(\delta) - \lambda C^T \delta \]
\[ = -\delta^T W + \frac{1}{2} \delta^T D_{1} \delta - \lambda C^T \delta \]

We observe that
\[ G_{n}(\delta) - \lambda C^T \delta \xrightarrow{D} L(\delta, \lambda) \]

It can be easily shown that
\[ \frac{\partial L(\delta, \lambda)}{\partial \delta} = -W + D_1 \delta - \lambda C = 0. \]

Furthermore,
\[ \hat{\delta}_c = D_1^{-1}(W + \lambda C) = \hat{\delta} + \lambda D_1^{-1}C. \]

By \( C^T \hat{\delta}_c = 0 \), we have \( \lambda = -(C^T D_1^{-1}C)^{-1}C^T \hat{\delta} \).
\[ \sqrt{n}(\hat{\beta} - \beta) \overset{p}\rightarrow \hat{\delta}_c = (E - M) \hat{\delta}. \]

Here, \( M = D_1^{-1}C(C^T D_1^{-1}C)^{-1}C^T \). Combining the limiting distribution of \( \hat{\delta} \), we can prove this theorem.

**Proof of Theorem 2.** We use the similar argument following Theorem 1 with Theorem 3 in Wu and Liu (2009), we omit the details.

**Acknowledgments**

The authors thank the editor and one referee for their constructive suggestions that led to an improved paper.

**Funding**

The research was supported by the Natural Science Foundation of Jiangsu Province (Grants No. BK20200854).

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