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Analytical and numerical solution for wave reflection from a porous wave absorber

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Abstract. In this paper, wave reflection from a porous wave absorber is investigated theoretically and numerically. The equations that we used are based on shallow water type model. Modification of motion inside the absorber is by including linearized friction term in momentum equation and introducing a filtered velocity. Here, an analytical solution for wave reflection coefficient from a porous wave absorber over a flat bottom is derived. Numerically, we solve the equations using the finite volume method on a staggered grid. To validate our numerical model, comparison of the numerical reflection coefficient is made against the analytical solution. Further, we implement our numerical scheme to study the evolution of surface waves pass through a porous absorber over varied bottom topography.

1. Introduction

Porous wave absorber offers an excellent solution for protecting harbours against the action of incoming waves or resonance phenomena. Prediction of the reflection and transmission from porous absorber plays a significant role in the assessment of the wave conditions in a harbour. A numerical wave model is valuable when it comes to a quick investigation to account this phenomenon. Several researchers have investigated the advantages of porous structure on reducing wave amplitude, such as in [1], [2], [3], [4], [5] and [6]. They studied submerged type porous structure but here we focus on emerged porous structure. Several researchers have investigated the advantages of a porous breakwater on reducing wave amplitude using an experimental approach such as [7] and [8]. The mathematical model often to use is a model from potential theory. Some researchers who studied this problem using potential theory are [9] and [10]. But the difficulty arises when we work with potential theory due to the complexity of the
Analytical and numerical solution for wave reflection from a porous wave absorber equation. Using potential theory makes the equations difficult to solve analytically or numerically. In this research, we propose a modification of shallow water equation involving the existent of porous structure to study the transmission and reflection wave with investigating the effect of incoming waves and various wave absorber characteristics. Working with shallow water type model makes us solve the equation relatively easier than using potential theory.

There are six sections in this paper. We start with an introduction. The governing equations are briefly presented in the second section. In the third section, the analytical solution for wave reflection and transmission coefficients are derived. In the following section, the numerical methods were described. A staggered finite volume method is introduced. In the fifth section, the numerical results are presented and compared with analytical solutions. Conclusions are outlined in the last section.

There are porosity and friction, which are significant in wave absorbers. Porosity affects the depth of the absorbing region, while friction reduces the wave energy. The equations inside the porous wave absorber are modification of a shallow water type model. The adjustment is by adding friction term $c_f$ and filtered velocity $u' = \frac{u}{n}$ written in:

$$\eta_t + \left(\frac{d}{n}u_x\right)_x = 0,$$

$$\frac{1}{n} u_t + g \eta_x + f \omega \frac{u}{n} = 0,$$

in which $f$ is friction coefficient and $\omega$ is radian frequency.

Further, we recapitulate the governing equations for the whole domain are as follows:

$$\eta_t + \left(\frac{d}{N}u\right)_x = 0,$$

2. Governing equation

Here, we consider wave elevation in a domain as depicted in Figure 1. For that purpose, we denote a domain with porous media as $\Omega_2 = 0 \leq x < L$, whereas the region without porous structure is denoted by $\Omega_1$ for $x < 0$. The governing equation that we used in $\Omega_1$ is shallow water equations over a flat bottom $d$, read as:

$$\eta_t + du_x = 0,$$  \hspace{1cm} (1)

$$u_t + g \eta_x = 0,$$  \hspace{1cm} (2)

in which $\eta$ and $u$ are surface elevation and fluid velocity, respectively. The equations inside the porous wave absorber are modification of a shallow water type model. The adjustment is by adding friction term $c_f$ and filtered velocity $u' = \frac{u}{n}$ written in:

$$\eta_t + \frac{1}{n} u_t + g \eta_x + f \omega \frac{u}{n} = 0,$$  \hspace{1cm} (4)

in which $f$ is friction coefficient and $\omega$ is radian frequency.

Further, we recapitulate the governing equations for the whole domain are as follows:

$$\eta_t + \left(\frac{d}{N}u\right)_x = 0,$$  \hspace{1cm} (5)
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\[ \frac{1}{N} u_t + gn_x + F_l \frac{\omega}{N} u = 0, \]  
(6)

with piecewise constant functions of \( N \) and \( F \) as follows

\[ N = \begin{cases} 
1, & \text{if } x \in \Omega_1, \\
n, & \text{if } x \in \Omega_2 
\end{cases} \]  
(7)

and

\[ F_l = \begin{cases} 
0, & \text{if } x \in \Omega_1, \\
f, & \text{if } x \in \Omega_2. 
\end{cases} \]  
(8)

3. Wave reflection and transmission coefficient

In this section, we derive wave reflection and transmission coefficient from a porous wave absorber. In this study, we will limit ourselves to the study of small wave amplitude in a shallow water area. Consider a harmonic wave with certain frequency \( \omega \) in porous domain read as:

\[ \eta(x, t) = F(x) \exp(-i\omega t), \]  
(9)

\[ u(x, t) = G(x) \exp(-i\omega t). \]  
(10)

Substituting equations (9, 10) into (3, 4) will yield:

\[ F_{xx} + \frac{\omega^2}{gd}(1 - if)F = 0, \]  
(11)

\[ G(x) = \frac{gn}{\omega} \frac{1}{i + f} F_x. \]  
(12)

Solution of equation (11) is \( F(x) = a_1 e^{-i\kappa x} + a_2 e^{i\kappa x} \), where the wave number in porous media \( \kappa \) follows the following dispersion relation:

\[ \kappa^2 = \frac{\omega^2}{gd}(1 - if). \]  
(13)

A formulation for surface elevation is then:

\[ \eta(x, t) = a_1 e^{i(\kappa x - \omega t)} + a_2 e^{-i(\kappa x + \omega t)}, \]  

where \( a_1 \) is the right running waves which is our transmitted waves and \( a_2 \) is the left running waves that is our reflected wave after hitting a wall. Differentiate \( F(x) \) with respect to \( x \) and substitute the result into (12) will give us

\[ G(x) = \frac{gn}{\omega} \frac{ik}{i + f} \left( a_1 e^{-i\kappa x} - a_2 e^{i\kappa x} \right). \]  
(14)

Next, substituting (14) into Eqn. (10) will yield:

\[ u(x, t) = \sqrt{g \frac{n}{d}} \frac{1}{\sqrt{1 - if}} \left( a_1 e^{-i(\kappa x - \omega t)} - a_2 e^{i(\kappa x + \omega t)} \right). \]  
(15)

Consider an incoming wave from free water area propagates to the right. After passing through the porous media and hit a seawall, the wave is fully reflected. So, in the upstream free region \( \Omega_1 \) there will be right running wave \( e^{-i(\kappa x - \omega t)} \) and left running
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wave $e^{i(kx+\omega t)}$, where $k$ denotes wave number in free region. The right running wave has amplitude $a_r$ and left running wave has amplitude $a_i$. In general, surface elevation in the whole domain can be formulated as:

$$
\eta(x, t) = e^{-i\omega t} \begin{cases} 
    a_i e^{-ikx} + a_r e^{ikx}, & \text{in } x < 0 \\
    a_1 e^{-i\kappa x} + a_2 e^{i\kappa x}, & \text{in } 0 \leq x < L 
\end{cases}$$

(16)

with $k$ follows dispersion relation $\frac{\omega^2}{d} = k^2$. And the relation between $\kappa$ and $k$ is $\kappa^2 = k^2(1 - if)$.

In free region, traveling waves of the linear SWE has to satisfy the following relation

$$
u_k \frac{d^2 \eta}{dx^2} - i \kappa \frac{d \eta}{dx} + \eta \frac{d \nu_k}{dx} + \frac{\nu_k}{2} \frac{d^2 \eta}{dt^2} = 0$$

Further, the coefficients $a_1, a_2$ and $a_r$ can be found by matching $\eta$ and $d\eta$ at the interface and applying hard wall boundary. Continuity of surface elevation $\eta$ and horizontal flux $d\eta$ at $x = 0$, will yield

$$a_1 + a_2 = a_i + a_r,$$

(18)

and

$$\epsilon(a_1 - a_2) = a_i - a_r,$$

(19)

with $\epsilon = \frac{n}{\sqrt{1-\nu}L}$. The hard wall boundary condition at $x = L$ for $n \neq 0$ will yield

$$\epsilon(a_1 e^{-i\kappa L} - a_2 e^{i\kappa L}) = 0,$$

(20)

From equations (18) and (19) we can express $a_1$ and $a_2$ in terms of $a_r$:

$$a_1 = \frac{2e^{i\kappa L}}{(1 - \epsilon)e^{i\kappa L} + (1 + \epsilon)e^{-i\kappa L}a_r},$$

(21)

$$a_2 = \frac{2e^{-i\kappa L}}{(1 - \epsilon)e^{i\kappa L} + (1 + \epsilon)e^{-i\kappa L}a_r}.$$  

(22)

Substituting the results above for $a_1$ and $a_2$ into equations (20), will give us one equations in two variables $a_i$ and $a_r$. Eliminating $a_i$ and $a_r$ will give us wave reflection coefficients of the porous structure:

$$K_R = \frac{|a_r|}{|a_i|} = \frac{|(1 - \epsilon)e^{i\kappa L} + (1 + \epsilon)e^{-i\kappa L}|}{|(1 + \epsilon)e^{i\kappa L} + (1 - \epsilon)e^{-i\kappa L}|}.$$  

(23)

Moreover, we also can obtain the wave transmission coefficient that is achieved from the ratio between $a_1$ and $a_i$:

$$K_T = \frac{|a_1|}{|a_i|} = \frac{2e^{i\kappa L}}{(1 + \epsilon)e^{i\kappa L} + (1 - \epsilon)e^{-i\kappa L}}.$$  

(24)
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Transmission and reflection coefficient are both depend on the characteristic of porous media such as porosity $n$, friction $f$, and length $L$ of the porous media. In this absorber porous structure, the incident wave scattered because it enters a porous media. Formula (23) is tested for the following limiting case. Case $n \to 0$, the porous media becomes a solid wall and we obtain $K_R \to 1$ which mean perfect reflection and $K_T \to 0$ no transmission. Further, for the case $n \to 1$ and $f \to 0$, in which the porous media becomes a free region, the dispersion relation (13) reduces to the well-known dispersion relation for gravity wave: $\omega^2 gk = kd$, as we expect.

Taking parameter values $\omega = 3\pi$, $d = 3$, $g = 9.81$, $n = 0.8$, $f = 0.18$, dispersion relation (13) will give us a complex value wave number $\kappa = 1.7443 - 0.1557i$. A monochromatic wave $\exp^{-i(\kappa x - \omega t)}$ with negative imaginary part $\Im(\kappa)$ will undergo amplitude reduction, see Figure 2. Let $K_T = |\eta(x, t)| = \exp \Im(\kappa)x$, the term $K_T$ denotes amplitude reduction of incident wave as a function of $x$, the horizontal length of an emerged porous breakwater. It also denotes the ratio between wave transmission amplitude and incident wave amplitude or wave transmission coefficient. It is clear that the profile of wave transmission coefficient depends strongly on the complex wave number $\kappa$, and hence on the dispersion relation (13). Parameters involved in (13) are wave frequency $\omega$, gravitational acceleration $g$, porosity $n$, friction coefficient $f$, and water depth $d$.

Next, we study wave reflection and transmission coefficient concerning porous structure parameter such as porosity $n$, wavelength $L$, and friction coefficient $f$. Here, we analyze the dependence of $K_R$ and $K_T$ on those parameters. Figure 4 shows curves of $K_R$ and $K_T$ with respect to non-dimensional variable $kL$ for several values of $n$. For all computations follow we take $g = 9.8$ and $f = 1$. We observe that the longer porous media does not directly mean larger $K_R$, after a certain length of the porous structure, the wave reflection stay steady in a certain value follows this formula:

$$\lim_{x \to \infty} K_R = \frac{1 - \epsilon}{1 + \epsilon}.$$
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Figure 3. Curves of $K_R$ w.r.t $kL$ for fixed values of the friction coefficient $f = 1$.

Figure 4. (Left) Curves of $K_R$ w.r.t $kL$ for fixed values of the friction coefficient $f = 1$. (Right) Curves of $K_T$ w.r.t. $kL$ for for fixed values of the friction coefficient $f = 1$.

For relatively small value of $kL$, we observe an oscillating behavior. The same behavior of $K_R$ curve is also found by W. Sulisz [11] and I. Magdalena et.al [12]. We can also conclude that smaller $n$ will yield smaller $K_T$.

4. A staggered finite volume method

In this section, a numerical finite volume method on a staggered grid will be implemented for simulating an incident wave passing through an emerged porous media. We will use the numerical computations to confirm the analytical results. Consider equation for waves pass through a porous structure (5, 6) in domain $[0, L]$. We discretize the porous domain in a staggered way $0 = x_0, x_1, \ldots, x_{Nx+1} = L$. Mass conservation (5) is approximated at a cell centred at $x_i$, whereas momentum conservation (6) is approximated at a cell centred at $x_{i+1/2}$, see Figure 6. Approximate equations are then

\[
\frac{\eta_{i}^{n+1} - \eta_{i}^{n}}{\Delta t} + \frac{d}{n} \frac{u_{i+1/2}^{n} - u_{i-1/2}^{n}}{\Delta x} = 0
\] (25)
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Figure 5. (Left) Curves of $K_R$ w.r.t $n$ for fixed values of $kL = 1$. (Right) Curves of $K_T$ w.r.t. $n$ for fixed values of $kL = 1$.

Figure 6. Illustration of staggered grid.

\[
\frac{1}{n} \frac{u_{n+1}^{i+1/2} - u_{n}^{i+1/2}}{\Delta t} + g \frac{\eta_{i+1}^{n+1} - \eta_{i}^{n+1}}{\Delta x} + f \frac{u_{n+1}^{i+1/2}}{n} = 0.
\] (26)

The above approach is known as the finite volume method on a staggered grid. This discretization for free region is described extensively in S.R. Pudjaprasetya and I. Magdalena [13] and I. Magdalena et al. [14]. In this setting, values of $\eta$ will be computed at every full grid points $x_i$, with $i = 1, 2, ..., Nx$ using mass conservation (25). Velocity $u$ will be computed at every staggered grid points $x_{i+1/2}$, with $i = 1, 2, ..., Nx - 1$ using momentum equation (26). Note that the friction term $f u_{n+1/2}$ is calculated implicitly in order to avoid more restricted stability condition. Implementing Von Neumann stability analysis, we obtain Courant-Friedrichs-Lewy condition for (25,26) which is $\sqrt{g d \Delta t / \Delta x} \leq 1$, where $d$ is the flat bottom depth.

Further, for simulating the gravity waves in free water area, the approximate equations are just (25) and (26) with $n = 1$ and $f = 0$. The resulting scheme is free from numerical damping error, see S.R. Pudjaprasetya and I. Magdalena [13] for details.

5. Numerical simulation

In this section, we will implement the above scheme to simulate wave interaction with a porous structure. For simulation, we take a computational domain $-30 < x < 30$. We take $g = 9.81$ and a constant depth $d = 10$. Along the left and right boundaries, we
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apply absorbing boundary. The initial condition is the following hump:
\[
\eta(x, 0) = 1e^{-(0.5x-15)^2}
\]
\[
u(x, 0) = \sqrt{g/d} \eta(x, 0).
\]

Along the right boundary, we apply a hard wall.

We first test the no porous case \( n = 1 \) and without friction \( f = 0 \), for which equations (3) and (4) reduce to the shallow water equations without porous structure. Numerical simulation of (25) and (26) will yield the wave travels to the right undisturbed in shape and hit the wall, it is fully reflected by the wall, as we expect, see Figure 7 (Left). Next, we will simulate the evolution of a wave initially being in a free water

\[\text{Figure 7. (Left) Wave propagates in a free domain and hit a wall in the right. (Right) Wave interactions with an emerged porous media.}\]

region. It travels to the right and enters a porous region. In the porous region, the wave moves further to the right and hit the wall. The porous structure with parameters \( n = 0.8, f = 0.18 \) is installed in \( 10 < x < 30 \). The numerical simulation is given in Figure 7 (Right). When the waves hit the emerged porous media, it will split into reflected and transmitted waves. In the porous media, transmitted wave is reduced. When the

\[\text{Figure 8. (Left) Wave propagates pass through porous media and hit a wall in the right.}\]
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Wave travels further to the right and hit the wall, the waves reflect again. The first and second reflected wave can be seen in Figure 7 (Right). Moreover, we implement our numerical scheme to general case where the porous absorber is not on the flat bottom but on a sloping bottom, see Figure 8.

5.1. Comparison with Analytical Solution

In this section, we will make a comparison of our numerical result with the analytical wave reflection coefficient formula for various variables such as porosity, length of the porous structure, and friction coefficient. Further, we will show that our numerical reflected wave profile confirms the analytical $K_R$ formula. For numerical computations, we take the parameters used in Section 3, and we use $\Delta x = 0.1$, and $\Delta t = \Delta x/\sqrt{\gamma d}$. The surface profile is plotted in Figure 9. We use Healy's formula to calculate reflection wave: $K_R = \frac{a_{\text{max}} - a_{\text{min}}}{a_{\text{max}} + a_{\text{min}}}$, as stated in [15]. Wave reflection and transmission coefficient,

![Graphs showing analytical and numerical comparisons](image_url)

Figure 9. Solid line is the analytical wave reflection coefficient $K_R$ from equation (24). Dash curves are surface elevation in porous media. Left upper: $n=0.45$, right upper: $n=0.55$, left lower: $n=0.65$, right lower: $n=0.75$

(23) and (24) depend on porosity $n$, friction coefficient $f$, length of the porous structure $L$, and wave number $k$, and $\kappa$. We make another comparison, the dependence of $K_T$ with $n$ and $f$. For the computation, we take parameter $\omega = 3\pi$ and $L = 10$. We plot the curve of wave transmission coefficient $K_T$ with respect to porosity $n$ for several values of $f$. From Figure 10, we conclude that for a certain porosity $n$, larger friction coefficient, will lead to smaller wave transmission coefficient. The numerical results are in good agreement with the analytical result, especially for $n$ greater than 0.8.
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![Figure 10. Comparison between wave reflection coefficient from numerics and analytic as function of porosity $n$ for a certain friction coefficient $f$. Solid lines: analytical result, solid lines: numerical result.](image)

6. Conclusions

We have derived dispersion relation that explain diffusive mechanism of waves because of the porous structure. Further, analytical solution for wave reflection coefficients is obtained. To solve our equation numerically, we applied Finite volume method on a staggered grid that is free from damping error. We show that our numerical wave reflection confirms the analytical solution.

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