LOCAL REGULARITY FOR QUASI-LINEAR PARABOLIC EQUATIONS IN NON-DIVERGENCE FORM

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Abstract. We consider viscosity solutions to non-homogeneous degenerate and singular parabolic equations of the $p$-Laplacian type and in non-divergence form. We provide local Hölder and Lipschitz estimates for the solutions. In the degenerate case, we prove the Hölder regularity of the gradient. Our study is based on a combination of the method of alternatives and the improvement of flatness estimates.

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1. Introduction

We are interested in the regularity of viscosity solutions of the following degenerate or singular parabolic equation in non-divergence form:

$$\partial_t u - |Du|^\gamma \left[ \Delta u + (p - 2) \left\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|} \right\rangle \right] = f \quad \text{in} \quad Q_1,$$

(1.1)
where $-1 < \gamma < \infty$, $1 < p < \infty$ and $f$ is a continuous and bounded function. Existence and uniqueness of solutions to (1.1) were proved in [13], where more general singular or degenerate parabolic equations were considered (see also [7, 32] and the references therein). In [13], Demengel established global Hölder regularity results for the solutions of the Cauchy-Dirichlet problem associated to (1.1), under the assumptions that $f$ is continuous and bounded in space and Hölder in time, and that the boundary data is Hölderian in space and Lipschitz in time.

In this work, we investigate the higher regularity of the solution $u$ to (1.1). We focus on interior regularity for the gradient, away from boundaries. Let us mention two special cases. The case $\gamma = 0$ corresponds to the normalized $p$-Laplacian

$$\Delta_p^N u := \Delta u + (p-2)\left\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|}\right\rangle,$$

and the regularity of the gradient was studied in [3, 22] using viscosity theory methods. The case $\gamma = p-2$ corresponds to the usual parabolic $p$-Laplace equations, and it was shown in [23] that bounded weak solutions and viscosity solutions are equivalent. From this equivalence, we get the Hölder regularity of the gradient for bounded $f$ using variational methods [14, 15, 28, 37]. Let us also mention that recently Parviainen and Vázquez [33] established an equivalence between the radial solutions of (1.1) and the radial solutions of the standard parabolic $\gamma + 2$-Laplace equation posed in a fictitious dimension. Notice that the regularity theory for (1.1) does not fall into the classical framework of fully nonlinear uniformly parabolic equations studied in [35, 36] due to the lack of uniform ellipticity and the presence of singularities.

In this paper, we provide local Lipschitz estimates for solutions to (1.1) in the whole range $-1 < \gamma < \infty$. For $\gamma > 0$, we prove the Hölder regularity of the gradient. Recently, for $\gamma \neq 0$, the homogeneous case $f = 0$ was treated by Imbert, Jin and Silvestre [19]. The case where $f$ depends only on $t$ can be handled using the results of [19], since $\tilde{u}(x, t) := u(x, t) - \int_0^t f(s) \, ds$ solves the homogeneous equation. If we assume more regularity on $f$, let us say $f \in C^{1,0}_{r,t}(Q_1)$, then one could adapt the argument of [19] by regularizing the equation and differentiating it, and then prove the Hölder continuity of the gradient of the solutions of (1.1) with a norm which will then depend on $||Df||_{L^\infty(Q_1)}$. Our study relies on a nonlinear method based on compactness arguments where we avoid differentiating the equation and assume only the continuity of $f$. There are different characterizations of pointwise $C^{1+\alpha, 1+\alpha/2}$ functions, and we will use the one relying on the rate of approximations by planes. The study is based on estimates which prove that the solution gets flatter and flatter, when zooming into the smaller scales. There are three key points: an improvement of flatness estimate, the method of alternatives and the intrinsic scaling technique. In the degenerate case $\gamma > 0$, in order to prove the Hölder regularity of the gradient, one has to choose a suitable scaling that takes into account the structure of the equation. Indeed, when the equation degenerates, the solutions locally generate their own scaling (“intrinsic scaling”) according to the values of the diffusion coefficients. The main idea behind the intrinsic scaling technique is to study the equation not on all parabolic cylinders, but rather...
on those whose ratio between space and time lengths depend on the size of the solution itself on the same cylinder, according to the regularity considered [14, 34]. Specifically, in order to prove Hölder regularity of the gradient, we consider the so called intrinsic cylinders defined by

\[ Q^\lambda_k(x_0, t_0) := B(x_0, r) \times (t_0 - \lambda^{-\gamma} r^2, t_0), \]

where the parameter \( \lambda > 0 \) behaves like sup\( |Du| \approx \lambda \) (see Sections 4 and 5).

Our strategy is to combine an improvement of flatness method with the method of alternatives (the Degenerate Alternative and the Smooth Alternative). This procedure defines an iteration that stops in the case where we reach a cylinder where the Smooth Alternative holds. More precisely, using an iteration process and compactness arguments, our aim is to prove that there exist \( \rho = \rho(p, n, \gamma) > 0 \) and \( \delta = \delta(p, n, \gamma) \in (0, 1) \) with \( \rho < (1 - \delta)^{\gamma + 1} \) such that one of the two following alternative holds:

- **Degenerate Alternative**: For every \( k \in \mathbb{N} \) there exists a vector \( l_k \) with \( |l_k| \leq C(1 - \delta)^k \) such that

\[
\text{osc}_{(x,t) \in Q^\lambda_k} (u(x, t) - l_k \cdot x) \leq r_k \lambda_k,
\]

where \( r_k := \rho^k \), \( \lambda_k := (1 - \delta)^k \) and \( Q^\lambda_{rk}(0) := B_{rk}(0) \times (-r_k^2\lambda_k^{-\gamma}, 0) \). That is, we have an improvement of flatness at all scales.

- **Smooth Alternative**: The previous iteration stops at some step \( k_0 \), that is, \( |l_{k_0}| \geq C(1 - \delta)^{k_0} \), and we can show that the gradient of \( u \) stays away from 0 in some cylinder and then use the known results for uniformly parabolic equations with smooth coefficients [29, 30].

Notice that the intrinsic scaling plays a role in the choice of the cylinders \( Q^\lambda_k \) in order to proceed with the iteration, and that if \( |l_k| \leq C(1 - \delta)^k \) for all \( k \), then \( |Du(0, 0)| = 0 \). Let us explain how these alternatives appear. The existence of the vector \( l_{k+1} \) in the iteration process can be reduced to the proof of an improvement of flatness (see Section 4) for the function

\[
w_k(x, t) = \frac{u(r_k x, r_k^2 \lambda_k^{-\gamma} t) - l_k \cdot r_k x}{r_k \lambda_k}.
\]

The function \( w_k \) solves

\[
\partial_t w_k - \left| Dw_k + \frac{l_k}{\lambda_k} \right|^\gamma \left[ \Delta w_k + (p - 2) \left\langle D^2 w_k \frac{Dw_k + l_k/\lambda_k}{|Dw_k + l_k/\lambda_k|}, \frac{Dw_k + l_k/\lambda_k}{|Dw_k + l_k/\lambda_k|} \right\rangle \right] = \tilde{f} \quad \text{in} \ Q_1,
\]

where \( \tilde{f}(x, t) := r_k \lambda_k^{-(\gamma + 1)} f(r_k x, r_k^2 \lambda_k^{-\gamma} t) \). This leads us to study the equation satisfied by the deviations of \( u \) from planes \( w(x, t) = u(x, t) - q \cdot x \),

\[
\partial_t w - \left| Dw + q \right|^\gamma \left[ \Delta w + (p - 2) \left\langle D^2 w \frac{Dw + q}{|Dw + q|}, \frac{Dw + q}{|Dw + q|} \right\rangle \right] = \tilde{f} \quad \text{in} \ Q_1, \quad (1.2)
\]
We see that $w_k$ satisfies (1.2) with $q = l_k/\lambda_k$. The proof of the improvement of flatness is based on compactness estimates and a contradiction argument. Unlike the case of the normalized $p$-Laplacian, the ellipticity coefficients of the equation (1.2) depend on $q$. To tackle this problem, we have to introduce the two alternatives: either we have a uniform bound on $|q|$ and we can run again our iteration, or $|q|$ is larger than some fixed constant. In this later case, using Lipschitz estimates in the space variable which are independent of $q$ (see Lemma 3.3), we can provide a strictly positive lower bound for the gradient of $u$ and finish the proof by using known results for uniformly parabolic equations with smooth coefficients. Our main result is the following.

**Theorem 1.1.** Let $0 \leq \gamma < \infty$ and $1 < p < \infty$. Assume that $f$ is a continuous and bounded function, and let $u$ be a bounded viscosity solution of (1.1). Then $u$ has a locally Hölder continuous gradient, and there exist a constant $\alpha = \alpha(p, n, \gamma)$ with $\alpha \in (0, \frac{1}{1+\gamma})$ and a constant $C = C(p, n, \gamma) > 0$ such that

$$|Du(x, t) - Du(y, s)| \leq C \left(1 + \|u\|_{L^\infty(Q_1)} + \|f\|_{L^\infty(Q_1)}\right) \left(|x - y|^{\alpha} + |t - s|^{\frac{\alpha}{2}}\right)$$

(1.3)

and

$$|u(x, t) - u(x, s)| \leq C \left(1 + \|u\|_{L^\infty(Q_1)} + \|f\|_{L^\infty(Q_1)}\right) |s - t|^{\frac{1+\alpha}{2}}.$$

(1.4)

The method of improvement of flatness was already used in the elliptic case [4, 8, 20] and in the uniformly parabolic case [3]. In these works, one ends up working with equations with ellipticity constants not depending on the slope, making the improvement of flatness working for all $k \in \mathbb{N}$. The method of alternatives is classical when studying the regularity for $p$-Laplacian type equations [14, 15, 28]. In the singular case $-1 < \gamma < 0$, we weren’t able to provide uniform (with respect to $q$) Lipschitz estimates for solutions to (1.2). The higher regularity of the gradient is still an open problem when $\gamma < 0$.

The paper is organized as follows. In Section 2 we fix the notations, gather some known regularity results that we will use later on, and reduce the problem by re-scaling. Section 3 is devoted to the study of the equation (1.2) and provides the needed compactness estimates. In Section 4, we provide the proof of the “improvement of flatness” property. In Section 5 we prove Theorem 1.1 proceeding by iteration and considering the two possible alternatives. Section 6 contains the proof of the Lipschitz regularity for solutions to (1.1) for $-1 < \gamma < \infty$. In Section 7 we prove the uniform Lipschitz estimates for solutions to (1.2) for $0 \leq \gamma < \infty$.

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2. Preliminaries and notations

In this section we fix the notation that we are going to use throughout the paper, recall the definitions of parabolic Hölder spaces and precise the definition of viscosity solutions that we adopt.
Notations. For \( x_0 \in \mathbb{R}^n, \ t_0 \in \mathbb{R} \) and \( r > 0 \) we denote the Euclidean ball
\[
B_r(x_0) = B(x_0, r) := \{ x \in \mathbb{R}^n \mid |x - x_0| < r \},
\]
and the parabolic cylinder
\[
Q_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0].
\]
We also define the re-scaled (or intrinsic) parabolic cylinders
\[
Q_r^\lambda(x_0, t_0) := B_r(x_0) \times (t_0 - r^2 \lambda^{-\gamma}, t_0],
\]
which are suitably scaled to reflect the degeneracy of the equation \((1.1)\). When \( x_0 = 0 \) and \( t_0 = 0 \) we omit indicating the centers in the above notations. For a set \( E \subset \mathbb{R}^{n+1} \), \( \partial_pE \) is its parabolic boundary. For the parabolic functional classes, we use the following notations. For \( \alpha \in (0, 1) \), we use the notation
\[
[u]_{C^{\alpha, \alpha/2}(Q_r)} := \sup_{(x, t), (y, s) \in Q_r, (x, t) \neq (y, s)} \frac{|u(y, s) - u(x, t)|}{|x - y|^{\alpha} + |t - s|^{\frac{\alpha}{2}}},
\]
\[
||u||_{C^{\alpha, \alpha/2}(Q_r)} := ||u||_{L^\infty(Q_r)} + [u]_{C^{\alpha, \alpha/2}(Q_r)}
\]
for Hölder continuous functions. The space \( C^{1+\alpha,(1+\alpha)/2}(Q_r) \) is defined as the space of all functions with a finite norm
\[
||u||_{C^{1+\alpha,(1+\alpha)/2}(Q_r)} := ||u||_{L^\infty(Q_r)} + ||Du||_{L^\infty(Q_r)} + [u]_{C^{1+\alpha,(1+\alpha)/2}(Q_r)},
\]
where
\[
[u]_{C^{1+\alpha,(1+\alpha)/2}(Q_r)} = \sup_{(x, t), (y, s) \in Q_r, (x, t) \neq (y, s)} \frac{|Du(x, t) - Du(y, s)|}{|x - y|^{\alpha} + |t - s|^{\frac{\alpha}{2}}}
\]
\[
+ \sup_{t \neq s} \frac{|u(x, t) - u(x, s)|}{|t - s|^{1+\alpha/2}}.
\]

In this paper \( C \) will denote generic constants which may change from line to line. If a more careful control over a constant is needed, we denote its dependence on certain parameters writing \( C(\text{parameters}) \).

Definition of solutions. We adopt the same notion of viscosity solutions to \((1.1)\) as the one used in [13, 19]. For the existence and uniqueness of the solutions and the comparison principles for equations of type \((1.1)\), we refer the reader to [10, 13, 17, 19, 32]. For the very singular case \( \gamma < 0 \), the definition in the sense of Ohnuma-Sato [32] requires the introduction of a set of admissible test functions when the gradient of \( u \) is 0 whereas no special restrictions are needed in the degenerate case \( \gamma > 0 \). However, for \( \gamma \neq 0 \) the notion of solutions in the sense of Ohnuma-Sato and the one proposed by Demengel are equivalent (see Appendix of [13]). One can also show that the notion of solutions in [13] is equivalent to the one proposed by [23] (this was done in [4] for the elliptic case and it can be easily generalized to the parabolic case). In the proofs of the Hölder estimates in time we will rely on those comparison principles [13, Theorem 1]. We will also use the stability results
for (1.1) (see [13, Proposition 3] and [32, Theorem 6.1, 6.2]). Let us recall the definition of viscosity solutions [13].

**Definition 2.1.** A locally bounded and upper semi-continuous function \( u \) in \( Q_1 \) is called a viscosity subsolution of (1.1) if, for any point \((x_0, t_0) \in Q_1\), one of the following conditions holds,

i) Either for every \( \varphi \in C^2(Q_1) \), such that \( u - \varphi \) has a local maximum at \((x_0, t_0)\) and \( D\varphi(x_0, t_0) \neq 0 \) it holds

\[
\partial_t \varphi(x_0, t_0) - |D\varphi(x_0, t_0)|^2 \Delta_p^N \varphi(x_0, t_0) \leq f(x_0, t_0).
\]

ii) Or if there exists \( \delta_1 \) and \( \varphi \in C^2([t_0 - \delta_1, t_0 + \delta_1]) \), such that

\[
\begin{aligned}
\varphi(t_0) &= 0 \\
u(x_0, t_0) & \geq u(x_0, t) - \varphi(t) \quad \text{for all} \quad t \in [t_0 - \delta_1, t_0 + \delta_1] \\
\sup_{[t_0 - \delta_1, t_0 + \delta_1]} (u(x, t) - \varphi(t)) & \text{ is constant in a neighborhood of } x_0,
\end{aligned}
\]

then

\[
\varphi'(t_0) \leq f(x_0, t_0).
\]

A locally bounded and lower semi-continuous function \( u \) in \( Q_1 \) is called a viscosity supersolution of (1.1) if, for any point \((x_0, t_0) \in Q_1\), one of the following conditions holds,

i) Either for every \( \varphi \in C^2(Q_1) \), such that \( u - \varphi \) has a local minimum at \((x_0, t_0)\) and \( D\varphi(x_0, t_0) \neq 0 \) it holds

\[
\partial_t \varphi(x_0, t_0) - |D\varphi(x_0, t_0)|^2 \Delta_p^N \varphi(x_0, t_0) \geq f(x_0, t_0).
\]

ii) Or if there exists \( \delta_1 \) and \( \varphi \in C^2([t_0 - \delta_1, t_0 + \delta_1]) \), such that

\[
\begin{aligned}
\varphi(t_0) &= 0 \\
u(x_0, t_0) & \leq u(x_0, t) - \varphi(t) \quad \text{for all} \quad t \in [t_0 - \delta_1, t_0 + \delta_1] \\
\inf_{[t_0 - \delta_1, t_0 + \delta_1]} (u(x, t) - \varphi(t)) & \text{ is constant in a neighborhood of } x_0,
\end{aligned}
\]

then

\[
\varphi'(t_0) \geq f(x_0, t_0).
\]

A continuous function \( u \) is called a viscosity solution of (1.1), if it is both a viscosity subsolution and a viscosity supersolution.

**Normalization and scaling:** Without a loss of generality, we may assume in Theorem 1.1 that \( \|u\|_{L^\infty(Q_1)} \leq 1/2 \) and that \( \|f\|_{L^\infty(Q_1)} \leq \varepsilon_0 \), where \( \varepsilon_0 = \varepsilon_0(p, n, \gamma) > 0 \) will be chosen in Section 4. Indeed, we can use a nonlinear method to realize this (notice that contrary to the elliptic case, multiplying solutions by a constant does not yield a solution to a similar equation). For \( \gamma \geq 0 \), set

\[
\theta := \left( 2 \|u\|_{L^\infty(Q_1)} + \left( \frac{\|f\|_{L^\infty(Q_1)}}{\varepsilon_0} \right)^{\frac{1}{p+1}} + 1 \right)^{-1}.
\]
We may consider in $Q_1$ the function
$$u_\theta(x,t) := \theta u(x,\theta^\gamma t).$$

The function $u_\theta$ satisfies $\|u_\theta\|_{L^\infty(Q_1)} \leq 1/2$ and solves in $Q_1$
$$\partial_t u_\theta = |Du_\theta|^\gamma \Delta_N^p u_\theta + f_\theta$$
with
$$f_\theta(x,t) := \theta^{\gamma+1} f(x,\theta^\gamma t), \quad \|f_\theta\|_{L^\infty(Q_1)} \leq \varepsilon_0.$$

We will use the standard alternative characterization (see [30, Lemma 12.12]) of functions with Hölder continuous gradient. This one is more suitable when we prove regularity results for nonlinear equations using compactness methods. Indeed, we will show that $u$ can be approximated by planes with a good control on the rate of approximations. By removing a constant, we may assume that $u(0,0) = 0$.

**Known regularity results.** Here we gather some known regularity results that we will need later on. We start with the following result of [19].

**Theorem 2.2.** Let $-1 < \gamma < \infty$ and $1 < p < \infty$. Assume that $f \equiv 0$ and let $w$ be a viscosity solution to equation (1.1) in $Q_1$. For all $r \in (0,3/4)$, there exist constants $C_0 = C_0(p,n,\gamma) > 0$ and $\beta_1 = \beta_1(p,n,\gamma) > 0$ such that
$$\|w\|_{C^{1+\beta_1,1+2\beta_1/2}(Q_r)} \leq C_0(1 + \|w\|_{L^\infty(Q_1)}).$$

We will need the following result for uniformly parabolic equations with smooth coefficients depending on the gradient [29, Theorem 1.1] (see also [30, Lemma 12.13]).

**Theorem 2.3.** Define $\Omega_T := \Omega \times (-T,0)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and let $g \in C(\Omega_T) \cap L^\infty(\Omega_T)$. Let $w$ be a strong solution to the equation
$$\partial_t w - \sum_{i,m} a_{im}(Dw) \frac{\partial^2 w}{\partial x_i \partial x_m} = g,$$
where the coefficients $a_{im}(z)$ are differentiable with respect to $z$ in the set
$$\{(x,t) \in \Omega_T, |w(x,t)| \leq K, |Dw(x,t)| \leq K\}$$
and satisfy the following conditions:

i) $\lambda|\xi|^2 \leq \sum_{i,m} a_{im}(Dw) \xi_i \xi_m \leq \Lambda|\xi|^2$ for some $0 < \lambda \leq \Lambda$,

ii) $\max_{\Omega_T} \left| \frac{\partial a_{im}(Dw)}{\partial z_k} \right| \leq \mu_1$.

Assume also that $\|g\|_{L^\infty(\Omega_T)} \leq \mu_2$. Then there exists a constant $\bar{\alpha} := \bar{\alpha}(\lambda, K, \mu_1, \mu_2) > 0$ such that for any $Q' \subset \subset \Omega_T$ it holds
$$[Dw]_{C^{\bar{\alpha},\bar{\alpha}/2}(Q')} \leq \bar{C}_0,$$
where $\bar{C}_0 := \bar{C}_0(\lambda, K, \mu_1, \|g\|_{L^\infty(\Omega_T)}, \text{dist}(Q', \partial_p \Omega_T)) > 0$. 


Remark 2.4. We will apply the result of Theorem 2.3 in Section 5. The result of Theorem 2.3 was stated for strong solutions. A priori our solutions are only viscosity solutions. This can be remedied by approximating $g$ with smooth functions $g_\varepsilon$. Then the corresponding solutions $w_\varepsilon$ solve a uniformly parabolic equation with Hölder continuous coefficients. By standard regularity results (see [30, Theorem 14.10] and [30, Theorem 5.1]), we conclude that $w_\varepsilon$ are strong solutions and they converge locally uniformly towards $w$. The most important thing is that the $C^{1+\alpha,1+\frac{\alpha}{2}}$ norm of $w_\varepsilon$ does not depend on the regularity of $g_\varepsilon$.

3. Lipschitz estimates and study of the equation for deviation from planes

In this section we analyze the problem (1.2) and provide regularity estimates which will be needed in the next section. These regularity results are obtained by using standard techniques in the theory of viscosity solutions. We first provide local Hölder and Lipschitz estimates with respect to the space variable for viscosity solutions of (1.1).

Lipschitz and Hölder estimates for solutions to (1.1) Let us recall that, in the setting of viscosity solutions, there are essentially two approaches for proving Hölder regularity: either by Aleksandrov-Bakelman-Pucci (ABP) estimates, Krylov-Safonov estimates [24, 25] and Harnack type inequalities, or by the Ishii-Lions’s method [21]. The first method is suitable for uniformly parabolic equations (for example for $\gamma = 0$). We point out that Argiolas-Charro-Peral [2] provided an ABP estimate for solutions of (1.1) and that recently for $f = 0$, Parviainen and Vázquez [33] provided a Harnack estimate for solutions of (1.1).

More direct viscosity methods like the method proposed by Ishii and Lions apply under weaker ellipticity assumptions but do not seem to yield further regularity results beyond the Lipschitz regularity. In this work we make the choice to use this second method (see also [13, 19, 20] for further applications). For the Lipschitz estimates, we avoid the Bernstein method which would require a higher regularity of the source term $\overline{f}$ (see [6, 29, 30, 31]).

Lemma 3.1. Let $-1 < \gamma < \infty$, $1 < p < \infty$ and $f \in C(Q_1) \cap L^\infty(Q_1)$. Let $u$ be a bounded viscosity solution to equation (1.1). There exists a constant $C = C(p, n, \gamma, \beta) > 0$ such that for all $(x,t),(y,t) \in Q_{7/8}$, it holds

$$|u(x,t) - u(y,t)| \leq C \left( ||u||_{L^\infty(Q_1)} + ||u||_{L^\infty(Q_1)}^{1 - \frac{1}{p}} + ||f||_{L^\infty(Q_1)}^{1 - \frac{1}{p}} \right) |x - y|.$$  \hspace{1cm} (3.1)

In order to keep the paper easy to read, the proofs are postponed to Section 6.

Next, we use an extension of Kruzhkov’s regularity theorem in time (see [26]) to provide a uniform control on the Hölder norm with respect to the time variable. This method was used in [5, 19, 18] and is based on the interplay between the regularity in time and space. One could adapt the argument of [19] after taking into account the source term. Here we give the details for completeness.

Lemma 3.2. Assume that $-1 < \gamma < \infty$, $1 < p < \infty$ and $f \in C(Q_1) \cap L^\infty(Q_1)$. Let $u$ be a viscosity solution to (1.1) with $\text{osc}_{Q_1} u \leq A$. Then there exists a constant $C = \ldots$
\[ C(p, n, \gamma, A, \|f\|_{L^\infty(Q_1)}) > 0 \] such that
\[
\sup_{(x,t),(s) \in Q_{11/16}, s \neq t} \frac{|u(x, t) - u(x, s)|}{|t - s|^\nu} \leq C, \tag{3.2}
\]
where \( \nu := \min \left( \frac{1}{2}, \frac{1}{2+\gamma} \right) \).

**Proof.** We denote by \( C_{\text{Lip}} \) the Lipschitz constant obtained in Lemma 3.1 for the solution \( u \). We start by the case \( \gamma \geq 0 \). We aim to show that for all \( t_0 \in [-11/16)^2, 0) \) and \( \eta > 0 \), we can find positive constants \( M_1, M_2 \) such that the function
\[
\bar{v}(x, t) := u(0, t_0) + M_1(t - t_0) + \frac{M_2}{\eta}|x|^2 + \eta
\]
is a supersolution to (1.1) in \( B_{11/16} \times (t_0, 0) \) and satisfies \( u \leq \bar{v} \) on \( \partial_p(B_{11/16} \times (t_0, 0)) \). Using the boundedness and the spatial Lipschitz regularity of \( u \), we have for all \( x \in B_{11/16} \)
\[
u(\eta) \geq \gamma
\]
and for \( (x, t) \in \partial B_{11/16} \times [t_0, 0] \) we have
\[
u(u(x, t) - u(0, t_0) \leq 2 \|u\|_{L^\infty(Q_{11/16})} \leq 2 \|u\|_{L^\infty(Q_1)} \frac{16}{11} |x| \leq \left( \frac{32}{11} \right)^2 \frac{\|u\|^2_{L^\infty(Q_1)}}{\eta} |x|^2 + \eta.
\]
By taking
\[
u(M_2 = C_{\text{Lip}}^2 + \left( \frac{32}{11} \right)^2 \|u\|^2_{L^\infty(Q_1)},
\]
we have that \( u \leq \bar{v} \) on \( \partial_p(B_{11/16} \times (t_0, 0)) \). Next, taking
\[
u(M_1 = \eta^{-1}\gamma + 1) M_2^\gamma C(n, p) + \|f\|_{L^\infty(Q_1)},
\]
we get that the function \( \bar{v} \) satisfies in the viscosity sense
\[
u(\partial_t \bar{v} = M_1 = \eta^{-1}\gamma + 1) M_2^\gamma C(n, p) + \|f\|_{L^\infty(Q_1)}
\geq |D\bar{v}|^\gamma \Delta^\gamma \bar{v} + f.
\]
It follows that \( \bar{v} \) is a supersolution of (1.1) in \( B_{11/16} \times (t_0, 0) \), and by the maximum principle, we get that \( u(x, t) \leq \bar{v}(x, t) \) for \( (x, t) \in B_{11/16} \times [t_0, 0] \). Consequently, for any \( \eta > 0 \), we have
\[
u(u(0, t) - u(0, t_0) \leq \frac{C(p, n) M_2^{\gamma+1} (t - t_0)}{\eta^{\gamma+1}} + \|f\|_{L^\infty(Q_1)} \frac{1}{t - t_0} \frac{1}{\eta} + \eta.
\]
Taking \( \eta = (\|u\|^2_{L^\infty(Q_1)} + C_{\text{Lip}}^2)^{\frac{\gamma+1}{\gamma}} |t - t_0|^{\frac{\gamma}{\gamma+1}} \) in this inequality, we end up with
\[
u(u(0, t) - u(0, t_0) \leq C(p, n) (\|u\|^2_{L^\infty(Q_1)} + C_{\text{Lip}}^2)^{\frac{\gamma+1}{\gamma}} |t - t_0|^{\frac{\gamma}{\gamma+1}} + \|f\|_{L^\infty(Q_1)} |t - t_0|
\leq C(p, n, \gamma, A, \|f\|_{L^\infty(Q_1)}) |t - t_0|^{\frac{\gamma}{\gamma+1}}.
\]
The lower bound follows by comparing with similar barriers and we get the desired result.
For \( \gamma < 0 \), consider the function
\[
\bar{w}(x, t) := u(0, t_0) + M_1(t - t_0) + \frac{M_2}{\gamma + \frac{2}{\eta}} |x|^{\frac{\gamma + 2}{\gamma + 1}} + \eta^{\gamma + 2}.
\]
Using the boundedness and the spatial Lipschitz regularity of \( u \), we have for all \( x \in B_{11/16} \)
\[
uw(x, t_0) - u(0, t_0) \leq C_{Lip}|x| \leq C_{Lip} \left( \frac{|x|}{\eta} \right)^{\frac{\gamma + 2}{\gamma + 1}} + \eta^{\gamma + 2},
\]
and for \((x, t) \in \partial B_{11/16} \times [t_0, 0]\), we have
\[
uw(x, t) - u(0, t_0) \leq 2 \|u\|_{L^\infty(\mathcal{Q}_{11/16})} \leq 2 \|u\|_{L^\infty(\mathcal{Q}_1)} \frac{16}{11}|x| \leq \left( \frac{32}{11} \|u\|_{L^\infty(\mathcal{Q}_1)} \right)^{\frac{\gamma + 2}{\gamma + 1}} \left( \frac{|x|}{\eta} \right)^{\frac{\gamma + 2}{\gamma + 1}} + \eta^{\gamma + 2}.
\]
So taking
\[
M_2 = \left( C_{Lip} + \frac{32}{11} \|u\|_{L^\infty(\mathcal{Q}_1)} \right)^{\frac{\gamma + 2}{\gamma + 1}},
\]
we have that \( u \leq \bar{w} \) on \( \partial_p(B_{11/16} \times (t_0, 0]) \). Next, taking
\[
M_1 = \eta^{-\gamma - 2} M_2^{\gamma + 1} C(n, p) + \|f\|_{L^\infty(\mathcal{Q}_1)}
\]
we get that the function \( \bar{w} \) satisfies in the viscosity sense
\[
\partial_t \bar{w} = M_1 = \eta^{-\gamma - 1 + 2} M_2^{\gamma + 1} C(n, p) + \|f\|_{L^\infty(\mathcal{Q}_1)} \geq |D \bar{w} \partial^N_p \bar{w} + f.
\]
From the comparison principle we get that \( u(x, t) \leq \bar{w}(x, t) \) for \((x, t) \in B_{11/16} \times [t_0, 0]\). Hence, for any \( \eta > 0 \), we have
\[
uw(0, t_0) - u(0, t_0) \leq \frac{C(p, n) M_2^{\gamma + 1}(t - t_0)}{\eta^{\gamma + 2}} + \|f\|_{L^\infty(\mathcal{Q}_1)} (t - t_0) + \eta^{\gamma + 2}.
\]
Taking \( \eta = M_2^{\frac{\gamma + 1}{\gamma + 2}} |t - t_0|^{\frac{1}{\gamma + 2}} = (\|u\|_{L^\infty(\mathcal{Q}_1)} + C_{Lip}^{\frac{\gamma + 1}{\gamma + 2}} |t - t_0|^{\frac{1}{\gamma + 2}} \) in this inequality, we get
\[
uw(0, t) - u(0, t_0) \leq C(p, n) (\|u\|_{L^\infty(\mathcal{Q}_1)} + C_{Lip}^{\frac{\gamma + 1}{\gamma + 2}} |t - t_0|^{\frac{1}{\gamma + 2}} + \|f\|_{L^\infty(\mathcal{Q}_1)} |t - t_0| \leq C \left( p, n, \gamma, A, \|f\|_{L^\infty(\mathcal{Q}_1)} \right) |t - t_0|^{\frac{1}{2}}.
\]
The lower bound follows by comparing with similar barriers. The proof is complete once we recall the dependence of \( C_{Lip} \).

**Uniform Lipschitz estimates for deviation from planes** For \( \gamma \geq 0 \), we claim the following uniform Lipschitz estimates for solutions to \((1.2)\). In the singular case, it is not clear if it is possible to provide a similar result.

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Lemma 3.3. Assume that $0 \leq \gamma < \infty$, $1 < p < \infty$ and $\bar{f} \in L^{\infty}(Q_1) \cap C(Q_1)$. Let $w$ be a bounded viscosity solution of $(1.2)$. Then there exists a constant $C = C(p, n, \gamma)$ such that for all $(x, t), (y, t) \in Q_{3/4}$, it holds
\[
|w(x, t) - w(y, t)| \leq C \left(1 + ||w||_{L^{\infty}(Q_1)} + ||\bar{f}||_{L^{\infty}(Q_1)}\right) |x - y|.
\] (3.3)

The proof follows from classical long and tedious computations and it is postponed to Section 7. There are already many Lipschitz estimates in the literature for related equations. Here the main difficulty is to track the dependence on $q$ and to provide an estimate which does not depend on $|q|$, especially for $|q|$ large. These uniform estimates will play a crucial role in Section 5.

Fixing the constants. We denote by
\[
C_1 = C_1(p, n, \gamma) := C \left(1 + ||w||_{L^{\infty}(Q_1)} + ||\bar{f}||_{L^{\infty}(Q_1)}\right)
\]
the Lipschitz constant coming from Lemma 3.3 where we fix $||\bar{f}||_{L^{\infty}(Q_1)} = 2$ and $||w||_{L^{\infty}(Q_1)} = 4$. We set
\[
C_2 = C_2(p, n, \gamma) := 2C_1
\]
and
\[
A_1 = A_1(p, n, \gamma) := 1 + 2C_2.
\]
These constants will appear in Section 4 and Section 5.

4. First alternative and improvement of flatness

In this section we study a first alternative that corresponds to the Degenerate Alternative. In this case we show how to improve the flatness of the solution in a suitable inner cylinder and how one can iterate this improvement.

4.1. Approximation. First we state the improvement of flatness property for solutions to (1.1) and provide its proof. Our main task consists in showing a linear approximation result for solutions to equation (1.1). We proceed by contradiction, using the previous lemmas together with the result of [19] for the homogeneous equation associated to (1.1).

Lemma 4.1 (The approximation Lemma). Let $-1 < \gamma < \infty$ and $1 < p < \infty$. Let $u$ be a viscosity solution to (1.1) with $\text{osc} u \leq A_1 = A_1(p, n, \gamma)$. For every $\eta > 0$, there exists $\varepsilon_0(p, n, \gamma, \eta) \in (0, 1)$ such that if $||\bar{f}||_{L^{\infty}(Q_1)} \leq \varepsilon_0$, then there exists a solution $\bar{u}$ to
\[
\partial_t \bar{u} = |Du|^\gamma \Delta_p^N \bar{u} \quad \text{in} \quad Q_{11/16}
\]
(4.1)
such that
\[
||u - \bar{u}||_{L^{\infty}(Q_{5/8})} \leq \eta.
\]
Proof. We argue by contradiction. We assume that there exist a constant $\eta > 0$ and sequences $\varepsilon_k \to 0$, $f_k$ and $u_k$ such that $||f_k||_{L^{\infty}(Q_1)} \leq \varepsilon_k$, and $u_k$ are solutions in $Q_1$ of
\[
\partial_t u_k = |Du_k|^\gamma \Delta_p^N u_k + f_k,
\]
(11)
with \( \text{osc}_{Q_1} u_k \leq A_1 \) and such that
\[
\|u_k - \tilde{u}\|_{L^\infty(Q_{5/8})} > \eta \tag{4.2}
\]
whenever \( \tilde{u} \) is a viscosity solution to (4.1).

Combining the compactness estimates coming from Lemma 3.1 and Lemma 3.2 with the Arzelà-Ascoli’s Lemma, we can extract a subsequence of \((u_k)_k\) converging uniformly in \(Q_\rho\) for any \( \rho \in (0, 11/16) \) to a continuous function \( u_\infty \). Passing to the limit in the equation, we have that \( u_\infty \) solves in \( Q_{11/16} \)
\[
\partial_t u_\infty - |Du_\infty|^\gamma \left[ \Delta u_\infty + (p - 2) \frac{D^2 u_\infty}{|Du_\infty|^2} \right] = 0.
\]
We end up with a contradiction for (4.2), since for \( k \) large enough we have
\[
\|u_k - u_\infty\|_{L^\infty(Q_{5/8})} \leq \eta.
\]
\( \square \)

Combining the approximation lemma and the regularity result of Theorem 2.2, we can now state the following improvement of flatness lemma. Define
\[ Q_\rho^{(1-\delta)} := B_\rho \times (-\rho^2 (1 - \delta)^{-\gamma}, 0]. \]

**Lemma 4.2.** Let \( -1 < \gamma < \infty \) and \( 1 < p < \infty \). Let \( u \) be a viscosity solution to (1.1) such that \( \text{osc} u \leq A_1 = A_1(p, n, \gamma) \). There exist \( \varepsilon_0 = \varepsilon_0(p, n, \gamma) > 0 \), \( \rho = \rho(p, n, \gamma) > 0 \) and \( \delta = \delta(p, n, \gamma) \in (0, 1) \) with \( \rho < (1 - \delta)^{\gamma+1} \) such that if \( \|f\|_{L^\infty(Q_1)} \leq \varepsilon_0 \), then there exists a vector \( h \) with \( |h| \leq B = B(n, p, \gamma) \) such that
\[
\text{osc}_{(x,t) \in Q_\rho^{(1-\delta)}} (u(x, t) - h \cdot \cdot x) \leq \rho(1 - \delta).
\]

**Proof.** Let \( \tilde{u} \) be the viscosity solution to
\[
\partial_t \tilde{u} - |D\tilde{u}|^\gamma \Delta_{\rho}^{N} \tilde{u} = 0 \quad \text{in} \quad Q_{11/16}
\]
coming from Lemma 4.1. From the regularity result of Theorem 2.2, there exists \( C = C(p, n, \gamma) > 0 \) such that for all \( \mu \in (0, 5/8) \) there exists \( h \) with \( |h| \leq B = B(p, n, \gamma) \) such that
\[
\text{osc}_{(x,t) \in Q_\rho} (\tilde{u}(x, t) - h \cdot x) \leq C(p, n, \gamma)(1 + \|\tilde{u}\|_{L^\infty(Q_{5/8})})^\mu^{1+\beta_1}.
\]
We pick \( \mu_0 = \mu_0(p, n, \gamma) \in (0, 5/8) \) such that
\[
\text{osc}_{(x,t) \in Q_{\mu_0}} (\tilde{u}(x, t) - h \cdot x) \leq \frac{1}{4} \mu_0(1 - \delta)^{\gamma+2}
\]
for some \( \delta = \delta(p, n, \gamma) \in (0, 1) \). Thus there exist two constants \( \rho \) and \( \delta \) depending on \( p, n, \gamma \) such that
\[
\text{osc}_{(x,t) \in Q_\rho^{(1-\delta)}} (\tilde{u}(x, t) - h \cdot x) \leq \frac{1}{4} \rho(1 - \delta)
\]
with \( \rho = \mu_0(1 - \delta)^{\gamma + 1} < (1 - \delta)^{\gamma + 1} \). It follows from Lemma 4.1, that for \( \eta := \frac{1}{7}\rho(1 - \delta) \) there exists \( \varepsilon_0 \) such that if \( ||f||_{L^\infty(Q_1)} \leq \varepsilon_0 \), we have
\[
\text{osc}_{(x,t)\in Q_{\rho}^{1-\delta}} (u(x,t) - h \cdot x) \leq \text{osc}_{(x,t)\in Q_{\rho}} (u(x,t) - \bar{u}(x,t)) + \text{osc}_{(x,t)\in Q_{\rho}^{1-\delta}} (\bar{u}(x,t) - h \cdot x) \leq \eta + \frac{1}{2}\rho(1 - \delta) \leq \rho(1 - \delta).
\]
The choice of \( \eta \) determines the smallness of \( f \). \( \Box \)

4.2. Iteration and condition for alternatives. Here we study the situation when the Degenerate Alternative holds a certain number of times when considering a suitable chain of shrinking intrinsic cylinders.

Lemma 4.3 (Improvement of flatness). Let \(-1 < \gamma < \infty\) and \(1 < p < \infty\). Let \( u \) be a viscosity solution to (1.1) such that \( \text{osc}_{Q_1} u \leq 1 \). Assume that \( ||f||_{L^\infty(Q_1)} \leq \varepsilon_0 \), where \( \varepsilon_0 = \varepsilon_0(p,n,\gamma) > 0 \), is the constant appearing in Lemma 4.2. Then there exist \( \rho = \rho(p,n,\gamma) > 0 \) and \( \delta = \delta(p,n,\gamma) \in (0,1) \) with \( \rho < (1 - \delta)^{\gamma + 1} \) such that, if for every nonnegative integer \( k \) it holds
\[
\left\{ \begin{array}{l}
\exists l_i \text{ with } |l_i| \leq C_2(1 - \delta)^i \text{ such that } \text{osc}_{(x,t)\in Q_{\rho}^{(1-\delta)^i}} (u(x,t) - l_i \cdot x) \leq \rho^i(1 - \delta)^i \\
\text{for } i = 0, \ldots, k,
\end{array} \right.
\]
then there exists a vector \( l_{k+1} \) such that
\[
\text{osc}_{(x,t)\in Q_{\rho}^{(1-\delta)^{k+1}}} (u(x,t) - l_{k+1} \cdot x) \leq \rho^{k+1}(1 - \delta)^{k+1}
\]
and
\[
|l_{k+1} - l_k| \leq C_3(1 - \delta)^k,
\]
with \( C_3 = C_3(p,n,\gamma) > 0 \).

Proof. Let \( \rho, \delta, B \) and \( \varepsilon_0 \) be the constants coming from Lemma 4.2 and let
\[
C_3 := B + C_2.
\]
For \( j = 0 \) we take \( l_0 = 0 \), and the result follows from Lemma 4.2, since \( \text{osc}_{Q_1} u \leq 1 < A_1 \). Suppose that the result of the Lemma holds true for \( j = 0, \ldots, k \). We are going to prove it for \( j = k + 1 \). Let
\[
w_k(x,t) := \frac{u(\rho^kx,\rho^{2k}(1 - \delta)^{-k}t) - l_k \cdot \rho^kx}{\rho^{k}(1 - \delta)^{k}},
\]
and denote \( \bar{f}(x,t) := \rho^{k}(1 - \delta)^{-k}f(\rho^kx,\rho^{2k}(1 - \delta)^{-k}t) \). By assumption, we have that \( \text{osc}_{Q_1} w_k \leq 1 \) and \( |l_k| \leq C_2(1 - \delta)^k \). Let \( q = \frac{l_k}{(1 - \delta)^k} \). Now the function \( \bar{v}(x,t) := w_k(x,t) + q \cdot x \) satisfies
\[
\text{osc}_{Q_1} \bar{v} \leq 1 + 2|q| \leq 1 + 2C_2 \leq A_1.
\]
Moreover, \( \bar{v} \) solves in \( Q_1 \)
\[
\partial_t \bar{v} = |D\bar{v}|^{\gamma} \Delta_p^N \bar{v} + \bar{f},
\]
with 
\[ \|f\|_{L^\infty(Q_1)} \leq \varepsilon_0 < 1. \]
Here we used that \( \rho(1-\delta)^{-(\gamma+1)} < 1 \). Therefore, by Lemma 4.2 we have that there exists \( h \) with \( |h| \leq B = B(p,n,\gamma) \) such that
\[ \text{osc}_{(x,t) \in Q_1^{(1-\delta)}}(\bar{v}(x,t) - h \cdot x) \leq \rho(1-\delta). \]

Going back to \( u \), we have
\[ \text{osc}_{(x,t) \in Q_1^{(1-\delta)}}(u(\rho^k x, \rho^{2k}(1-\delta)^{\gamma}t) - \rho^k(1-\delta)^k h \cdot x) \leq \rho^{k+1}(1-\delta)^{k+1}. \]

Scaling back, we get that
\[ \text{osc}_{(x,t) \in Q_1^{(1-\delta)^{k+1}}} (u(x,t) - l_{k+1} \cdot x) \leq \rho^{k+1}(1-\delta)^{k+1}, \]
where
\[ l_{k+1} := (1-\delta)^k h \]
satisfies \( |l_{k+1} - l_k| \leq (B + C_2)(1-\delta)^k = C_3(1-\delta)^k. \)

\[ \square \]

5. Handling the two alternatives and proof of the main theorem

In this section, we assume that \( \gamma \geq 0 \). We prove the Hölder continuity of \( Du \) at the origin and the improved Hölder regularity of \( u \) with respect to the time variable. Then the result follows by standard translation and scaling arguments. The Hölder regularity with respect to the space variable is a direct consequence of the following lemma after scaling back from \( u_\theta \) to \( u \).

**Theorem 5.1.** Let \( 0 \leq \gamma < \infty, 1 < p < \infty \) and let \( u \) be a viscosity solution to (1.1) with \( \text{osc}_{Q_1} u \leq 1 \). Let \( \varepsilon_0 \) be the constant coming from Lemma 4.2 and assume that \( \|f\|_{L^\infty(Q_1)} \leq \varepsilon_0 \).

Then there exist \( \alpha = \alpha(p,n,\gamma) \in \left(0, \frac{1}{\gamma+1}\right) \) and \( C = C(p,n,\gamma) > 0 \) such that
\[ |Du(x,t) - Du(y,s)| \leq C(|x-y|^\alpha + |t-s|^{\alpha/2}) \]
and
\[ |u(x,t) - u(x,s)| \leq C|t-s|^{(1+\alpha)/2}. \]

**Proof.** Let \( \rho \) and \( \delta \) be the constants coming from Lemma 4.2. Let \( k \) be the minimum nonnegative integer such that the condition 4.3 does not hold. We can conclude from Lemma 4.3 that for any vector \( \xi \) with \( |\xi| \leq C_2(1-\delta)^{k} \), it holds
\[ |u(t,x) - \xi \cdot x| \leq C(|x|^{1+\tau} + |t|^{\frac{1+\tau}{2-\gamma}}) \quad \text{for} \quad (x,t) \in Q_1 \setminus Q_1^{(1-\delta)^{k+1}}, \]
where \( \tau := \frac{\log(1-\delta)}{\log(\rho)} \) and \( C = \frac{1+C_2+C_3(1-\delta)^{-1}}{\rho(1-\delta)}. \) Now we treat differently the following two cases.
**First case:** If $k = \infty$, then the regularity result holds with
\[
\alpha = \min(1, \tau) = \min \left( 1, \frac{\log(1 - \delta)}{\log \rho} \right) \in \left( 0, \min \left( 1, \frac{1}{1 + \gamma} \right) \right).
\]
Indeed, for all $k \in \mathbb{N}$, there exists $l_k \in \mathbb{R}^n$ with $|l_k| \leq C_2 (1 - \delta)^k$ such that
\[
\operatorname{osc}_{(y, t) \in Q^{(1-\delta)^k}_k} (u(y, t) - l_k \cdot y) \leq \rho^k (1 - \delta)^k,
\]
and the conclusion follows using the characterization of functions with Hölder continuous gradient [30].

**Second case:** If $k < \infty$, then it follows from Lemma 4.2 that for all $i = 0, \ldots, k$, we have the existence of vectors $l_i$ such that
\[
\operatorname{osc}_{(y, t) \in Q^{(1-\delta)^i}_k} (u(y, t) - l_i \cdot y) \leq \rho^i (1 - \delta)^i, \tag{5.2}
\]
with
\[
|l_i| \leq C_2 (1 - \delta)^i \quad \text{for } i = 0, \ldots, k - 1,
\]
\[
|l_k - l_{k-1}| \leq C_3 (1 - \delta)^{k-1},
\]
\[
|l_k| \geq C_2 (1 - \delta)^k.
\]
It follows that
\[
\operatorname{osc}_{(y, t) \in Q^{(1-\delta)^k}_k} (u(y, t) - l_k \cdot y) \leq \rho^k (1 - \delta)^k \tag{5.3}
\]
and
\[
C_2 (1 - \delta)^k \leq |l_k| \leq (C_3 + C_2)(1 - \delta)^{k-1}. \tag{5.4}
\]
Consider for $(x, t) \in Q_1$ the function
\[
v(x, t) := \frac{u(\rho^k x, \rho^2k(1 - \delta)^{-k\gamma} t)}{\rho^k (1 - \delta)^k}.
\]
Then $v$ satisfies
\[
\partial_t v = |Dv|^\gamma \Delta^N_p v + \tilde{f},
\]
where
\[
\tilde{f}(x, t) := \rho^k (1 - \delta)^{-k(1+\gamma)} f(\rho^k x, \rho^2k(1 - \delta)^{-k\gamma} t).
\]
Notice that we can write
\[
v(x, t) = v(x, t) - q \cdot x + q \cdot x := w(x, t) + q \cdot x,
\]
where $q := \frac{l_k}{(1 - \delta)^k}$ and
\[
w(x, t) := \frac{u(\rho^k x, \rho^2k(1 - \delta)^{-k\gamma} t) - l_k \cdot \rho^k x}{\rho^k (1 - \delta)^k}.
\]
Observe that by assumption, $w$ satisfies $\operatorname{osc}_{Q_1} w \leq 1$ and solves in $Q_1$.
\[ \partial_t w - |Dw + q|^\gamma \left[ \Delta w + (p-2) \left( \frac{D^2 w}{|Dw + q|^\gamma} \cdot \frac{Dw + q}{|Dw + q|} \right) \right] = f \]

where, due to \( \rho < (1 - \delta)^{\gamma + 1} \), we have \( ||f||_{L^\infty(Q_1)} \leq \varepsilon_0 \leq 1 \). From Lemma 3.3, we have a uniform Lipschitz bound for \( w \):

\[ |Dw(x, t)| \leq C_1 \quad \text{for} \quad (x, t) \in Q_{3/4}. \]

It follows that for \( (x, t) \in Q_{3/4} \), we have

\[ |Dv(x, t)| = |Dw(x, t) + q| \geq |q| - |Dw(x, t)| \geq C_1, \quad (5.5) \]

where we used that from (5.4) we have \( |q| = \left| \frac{l_k}{(1-\delta)^k} \right| \geq C_2 = 2C_1 \). Moreover, using the upper bound on \( |l_k| \) coming from (5.4), we have

\[ ||v||_{L^\infty(Q_1)} = \left| w + \frac{l_k}{(1-\delta)^k} \right| \leq ||v||_{L^\infty(Q_1)} + \frac{l_k}{(1-\delta)^k} \]

\[ \leq 2 + (C_3 + C_2)(1 - \delta)^{-1} := C_4(p, n, \gamma). \quad (5.6) \]

From the local Lipschitz estimate coming from Lemma 3.1 and the estimate (5.6), it follows that

\[ ||Dv||_{L^\infty(Q_{3/4})} \leq C(p, n, \gamma) \left( 1 + ||v||_{L^\infty(Q_1)} + ||f||_{L^\infty(Q_1)} \right) \leq C_5(p, n, \gamma). \quad (5.7) \]

Combining the estimates (5.5) and (5.7), we notice that \( v \) solves a uniformly parabolic equation with ellipticity constants depending only on \( p, n, \gamma \) and that this equation is smooth in the gradient variables. It follows that \( v \) has a Hölder continuous gradient with a Hölder norm and a Hölder exponent depending only on \( p, n, \gamma \). Indeed, from Theorem 2.3, we get that \( v \in C^{1+\tilde{\alpha},(1+\tilde{\alpha})/2}_{\text{loc}}(Q_{3/4}) \) for some \( \tilde{\alpha} = \tilde{\alpha}(p, n, \gamma) > 0 \) and

\[ ||Dv||_{C^{\alpha}(\omega)} \leq C \left( p, n, \gamma, \text{dist}(\omega, \partial_{\rho}Q_{3/4}) \right) \]

for any \( \omega \subseteq Q_{3/4} \). Let \( 0 < \alpha \leq \min \left( \tilde{\alpha}, \frac{\log(1-\delta)}{\log(p)} \right) \). Hence, there exists a vector \( l \in \mathbb{R}^n \) such that in \( Q^{1-\delta}_\rho \) we have

\[ |Dv(x, t) - l| \leq C(|x|^{\alpha} + |t|^{\frac{\alpha}{2}}) \]

and

\[ |v(x, t) - v(x, 0)| \leq C|t|^{\frac{\alpha}{2}}. \]

Coming back to \( u \), it follows that in \( Q^{1-\delta k+1}_{\rho^k+1} \), it holds

\[ |Du(y, s) - (1 - \delta)^k l| \leq C(\rho^{-k\alpha}(1 - \delta)^k|y|^\alpha + (1 - \delta)^k(\rho^{-2}(1 - \delta)^\gamma)^{\frac{k}{2}}|s|^{\frac{\alpha}{2}}) \]

\[ \leq C(|y|^{\alpha} + |s|^{\frac{\alpha}{2}}) \]

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and
\[
|u(y, s) - u(y, 0)| \leq C \rho^k (1 - \delta)^k |s|^{\frac{k+\alpha}{2}} (\rho^{-2(1 - \delta)} \gamma)^{\frac{k(1+\alpha)}{2}}
\leq C |s|^{\frac{k+\alpha}{2}}. \tag{5.8}
\]

Here we used that \( \rho^{-\alpha}(1 - \delta) \leq 1 \) due to \( 0 < \alpha \leq \frac{\log(1 - \delta)}{\log \rho} \).

Consequently, combining these estimates with (5.2), we have showed that for
\[
0 < \alpha = \min \left( \bar{\alpha}, \frac{\log(1 - \delta)}{\log \rho} \right),
\]
there exists a constant \( C = C(p, n, \gamma) \) such that, for any \( r \leq \frac{1}{2} \), there exists a vector \( V = V(r) \) such that
\[
|u(x, t) - u(0, 0) - V \cdot x| \leq Cr^{1+\alpha} \tag{5.9}
\]
whenever \( |x| + \sqrt{|t|} \leq r \). The regularity of \( Du \) follows then from [30, Lemma 12.12].

The Hölder regularity of \( u \) in time follows from (5.2),(5.3),(5.4) and (5.8). Indeed, for \( i = 0, \ldots k \), we have
\[
\operatorname{osc}_{(y,t) \in Q^{(1-\delta)^i}_{\rho^i}} (u(y, t) - u(0, 0)) \leq \operatorname{osc}_{(y,t) \in Q^{(1-\delta)^i}_{\rho^i}} (u(y, t) - l_i \cdot y) + \operatorname{osc}_{(y,t) \in Q^{(1-\delta)^i}_{\rho^i}} l_i \cdot y
\leq \rho^i (1 - \delta)^i + 2|l_i| \rho^i \tag{5.10}
\leq [1 + 2(C_2 + C_3)(1 - \delta)^{-1}] \rho^i (1 - \delta)^i
\leq C(p, n, \gamma) \rho^i (1 - \delta)^i.
\]
Combining the estimate (5.10) with (5.8), we obtain that for \(-1/4 \leq t \leq 0\), it holds
\[
|u(0, t) - u(0, 0)| \leq C(p, n, \gamma) |t|^{\frac{1+\alpha}{2}}.
\]

\[\square\]

**Remark 5.2.** One could use another alternative to improve the Hölder regularity in time knowing the Hölder regularity of the spatial derivative, based on a parabolic maximum principle and barriers. Indeed, one can modify the arguments used in the proof of Lemma 3.2 or use the argument in [19, proof of Theorem 4.5]. For related works, we refer to [1, 19]. It would be useful to know when it is possible to provide the local regularity in time without using the regularity in space (and without assuming much regularity on the initial data) and how it would then imply the higher regularity in space. This was done for the Cauchy problem for a class of parabolic equations in [9].

6. **Proofs of the local Hölder and Lipschitz regularity for solutions to (1.1)**

In this section we provide Hölder and Lipschitz estimates for viscosity solutions to (1.1). These estimates are valid for the degenerate and the singular case. We assume that \(-1 <
γ < ∞ and 1 < p < ∞. The proof follows roughly the same lines as the one in [3, 4, 13, 19]. We aim at proving that the maximum
\[
\max_{(x,t)\in Q_r} (u(x,t) - u(y,t) - \varphi(|x-y|))
\]
is non-negative, choosing in a first step Φ(s) = L\sigma^β with β ∈ (0, 1), to obtain a Hölder bound, and in a second step, Φ(s) = L(s - κ_0s^\nu), to improve the Hölder bound into a Lipschitz one. To do this, we argue by contradiction and we use in a crucial way the strict concavity behavior of Φ near 0 to take profit of the strict ellipticity of the equation as usual in Ishii-Lions’ method. We first show C^0,β estimates for all β ∈ (0, 1), then we check that this implies the Lipschitz continuity estimates.

6.1. Local Hölder estimates.

**Lemma 6.1.** Let −1 < γ < ∞ and 1 < p < ∞. Let u be a bounded viscosity solution to equation (1.1). For any β ∈ (0, 1), there exists a constant C = C(p, n, γ, β) > 0 such that for all x, y ∈ B_{15/16} and t ∈ (−(15/16)^2, 0], it holds
\[
|u(x, t) - u(y, t)| \leq C \left( ||u||_{L^\infty(Q_1)} + ||u||_{L^\infty(Q_3)} + ||f||_{L^\infty(Q_3)} \right) |x - y|^\beta. \tag{6.1}
\]

**Proof.** We fix x₀, y₀ ∈ B_{15/16}, t₀ ∈ (−(15/16)^2, 0). For suitable constants L₁, L₂ > 0, we define the auxiliary function
\[
\Phi(x, y, t) := u(x, t) - u(y, t) - L₂\varphi(|x - y|) - \frac{L₁}{2} |x - x₀|^2 - \frac{L₁}{2} |y - y₀|^2 - \frac{L₁}{2} (t - t₀)^2,
\]
where \(\varphi(s) = s^\beta\). We want to show that \(\Phi(x, y, t) \leq 0\) for \((x, y) \in \overline{B_{15/16}} \times \overline{B_{15/16}}\) and \(t \in [−(15/16)^2, 0]\). We point out that the role of the term \(−\frac{L₁}{2} |x - x₀|^2\) is to localize, while the term \(−L₂\varphi(|x - y|)\) is concerned with the Hölder continuity. The main idea of the proof relies on the concavity of \(\varphi\) to create a large negative term in the viscosity inequalities. We argue by contradiction. We assume that \(\Phi\) has a positive maximum at some point \((\bar{x}, \bar{y}, \bar{t}) \in B_{15/16} \times B_{15/16} \times [−(15/16)^2, 0]\) and we are going to get a contradiction for \(L₂, L₁\) large enough. The positivity of the maximum of \(\Phi\) implies that \(\bar{x} \neq \bar{y}\). Using the boundedness of \(u\) we can choose
\[
L₁ \geq \frac{140 \text{osc}_{Q₁} u}{\min (d ((x₀, t₀), \partial Q_{15/16}), d ((y₀, t₀), \partial Q_{15/16}))^2},
\]
so that
\[
|\bar{y} - y₀| + |\bar{x} - x₀| + |\bar{t} - t₀| \leq 2√\frac{2 \text{osc}_{Q₁} u}{L₁} \leq \frac{d ((x₀, t₀), \partial Q_{15/16})}{2},
\]
and hence \(\bar{x}\) and \(\bar{y}\) are in \(B_{15/16}\) and \(\bar{t} \in (−(15/16)^2, 0)\).

The remaining of the proof is divided into 3 steps. We write down the viscosity inequalities and get suitable matrices inequalities from the Jensen-Ishii’s lemma (Step 1). Then we estimate the difference of the obtained terms, make use of the concavity of \(\varphi\) (Step 2) and finally we obtain a contradiction and conclude (Step 3).
**Step 1.** By Jensen-Ishii’s lemma (see [12, Theorem 8.3]), there exist
\[(\sigma, \tilde{\zeta}_x, X) \in P_2^+ \left( u(\bar{x}, i) - \frac{L_1}{2} |\bar{x} - x_0|^2 - \frac{L_1}{2} (i - t_0)^2 \right), \]
\[(\sigma, \tilde{\zeta}_y, Y) \in P_2^- \left( u(\bar{y}, i) + \frac{L_1}{2} |\bar{y} - y_0|^2 \right), \]
which we can rewrite as
\[(\sigma + L_1(i - t_0), a_1, X + L_1I) \in P_2^+ u(\bar{x}, i), \]
\[(\sigma, a_2, Y - L_1I) \in P_2^- u(\bar{y}, i), \]
where \(\tilde{\zeta}_x = \tilde{\zeta}_y\) and
\[a_1 = L_2\varphi'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} + L_1(\bar{x} - x_0) = \tilde{\zeta}_x + L_1(\bar{x} - x_0), \]
\[a_2 = L_2\varphi'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} - L_1(\bar{y} - y_0) = \tilde{\zeta}_y - L_1(\bar{y} - y_0). \]

Assuming that \(L_2 > \frac{L_1^{2^\beta - 1}}{\beta}\), we have
\[
\begin{cases}
2L_2|\bar{x} - \bar{y}|^{\beta - 1} & \geq |a_1| \geq L_2\varphi'(|\bar{x} - \bar{y}|) - L_1|\bar{x} - x_0| \geq \frac{L_2^2}{\beta} |\bar{x} - \bar{y}|^{\beta - 1} \\
2L_2|\bar{x} - \bar{y}|^{\beta - 1} & \geq |a_2| \geq L_2\varphi'(|\bar{x} - \bar{y}|) - L_1|\bar{y} - y_0| \geq \frac{L_2^2}{\beta} |\bar{x} - \bar{y}|^{\beta - 1}.
\end{cases}
\tag{6.2}
\]

Thanks to Jensen-Ishii’s lemma [11, Theorem 12.2], we can take \(X, Y \in S^n\) such that for any \(\tau > 0\) such that \(\tau Z < I\), it holds
\[-\frac{2}{\tau} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} Z^\tau & -Z^\tau \\ -Z^\tau & Z^\tau \end{pmatrix}, \tag{6.3}
\]
where
\[Z = L_2\varphi''(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \otimes \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} + \frac{L_2\varphi'(|\bar{x} - \bar{y}|)}{|\bar{x} - \bar{y}|} \left( I - \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \otimes \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \right) \]
\[= L_2\beta |\bar{x} - \bar{y}|^{\beta - 2} \left( I + (\beta - 2) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \otimes \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \right) \]
and
\[Z^\tau = (I - \tau Z)^{-1} Z. \]

We fix \(\tau = \frac{1}{2L_2\beta |\bar{x} - \bar{y}|^{\beta - 2}}\) so that we have
\[Z^\tau = (I - \tau Z)^{-1} Z = 2L_2\beta |\bar{x} - \bar{y}|^{\beta - 2} \left( I - \frac{2}{3 - \beta} \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \otimes \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \right). \]

With this choice for \(\tau\), we have for \(\xi = \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \)
\[\langle Z^\tau \xi, \xi \rangle = 2L_2\beta |\bar{x} - \bar{y}|^{\beta - 2} \left( \frac{\beta - 1}{3 - \beta} \right) < 0. \tag{6.4} \]
Applying the inequality (6.3) to any vector \((\xi, \xi)\) with \(|\xi| = 1\), we get that \(X - Y \leq 0\) and
\[
||X||, ||Y|| \leq 4L_2\beta |\bar{x} - \bar{y}|^{\beta-2}.
\]  
(6.5)

**Step 2.** For \(\eta \neq 0\), denoting \(\hat{\eta} = \frac{\eta}{|\eta|}\) and
\[
\mathcal{A}(\eta) := I + (p - 2)\hat{\eta} \otimes \hat{\eta},
\]
the viscosity inequalities read as
\[
L_1(\bar{t} - t_0) + \sigma - f(\bar{x}, \bar{t}) \leq |a_1|^\gamma \text{tr}(\mathcal{A}(a_1)(X + L_1 I))
\]
\[
- \sigma + f(\bar{y}, \bar{t}) \leq -|a_2|^\gamma \text{tr}(\mathcal{A}(a_2)(Y - L_1 I)).
\]
Adding the two inequalities and using that \(|\bar{t} - t_0| \leq 2\), we get
\[
0 \leq 2(L_1 + \|f\|_{L^\infty(Q_{1}))} + |a_1|^\gamma \text{tr}(\mathcal{A}(a_1)(X - Y)) + |a_1|^\gamma \text{tr}((\mathcal{A}(a_1) - \mathcal{A}(a_2))Y)
\]
\[
+ (|a_1|^\gamma - |a_2|^\gamma) \text{tr}(\mathcal{A}(a_2)Y) + L_1 \left[ |a_1|^\gamma \text{tr}(\mathcal{A}(a_1)) + |a_2|^\gamma \text{tr}(\mathcal{A}(a_2)) \right].
\]
(6.6)

In order to estimate \((i_1)\), we use the fact that all the eigenvalues of \(X - Y\) are non positive and that at least one of the eigenvalues of \(X - Y\) that we denote by \(\kappa_{io}(X - Y)\) is negative and smaller than \(8L_2\beta |\bar{x} - \bar{y}|^{\beta-2} \left( \frac{\beta - 1}{3 - \beta} \right)\). Indeed, applying the matrix inequality (6.3) to the vector \((\xi, -\xi)\) where \(\xi := \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}\) and using (6.4), we obtain
\[
((X - Y)\xi, \xi) \leq 4(Z^T\xi, \xi) \leq 8L_2\beta |\bar{x} - \bar{y}|^{\beta-2} \left( \frac{\beta - 1}{3 - \beta} \right) < 0.
\]
(6.7)

Noticing that the eigenvalues of \(\mathcal{A}(a_1)\) belong to \([\min(1, p - 1), \max(1, p - 1)]\) and using (6.2) and (6.7), we end up with the estimate
\[
|a_1|^\gamma \text{tr}(\mathcal{A}(a_1)(X - Y)) \leq |a_1|^\gamma \sum_i \kappa_i(\mathcal{A}(a_1))\kappa_i(X - Y)
\]
\[
\leq |a_1|^\gamma \min(1, p - 1)\kappa_{io}(X - Y)
\]
\[
\leq |a_1|^\gamma \min(1, p - 1)8L_2\beta |\bar{x} - \bar{y}|^{\beta-2} \left( \frac{\beta - 1}{3 - \beta} \right)
\]
\[
\leq C \left( L_2\beta |\bar{x} - \bar{y}|^{\beta-1} \right) 8L_2\beta |\bar{x} - \bar{y}|^{\beta-2} \left( \frac{\beta - 1}{3 - \beta} \right).
\]

In order to estimate \((i_2)\), we decompose
\[
\mathcal{A}(a_1) - \mathcal{A}(a_2) = (\hat{a}_1 \otimes \hat{a}_1 - \hat{a}_2 \otimes \hat{a}_2)(p - 2) = [(\hat{a}_1 - \hat{a}_2) \otimes \hat{a}_1 - \hat{a}_2 \otimes (\hat{a}_2 - \hat{a}_1)](p - 2)
\]
and get
\[
\text{tr}((\mathcal{A}(a_1) - \mathcal{A}(a_2))Y) \leq n \|Y\| \|\mathcal{A}(a_1) - \mathcal{A}(a_2)\| \leq 2n |p - 2| \|Y\| |\hat{a}_1 - \hat{a}_2|.
\]

Using that \(|a_1 - a_2| \leq 4L_1\) and the estimate (6.2), we get
\[ |\hat{a}_1 - \hat{a}_2| = \left| \frac{a_1}{|a_1|} - \frac{a_2}{|a_2|} \right| \leq \max \left( \left| \frac{a_2 - a_1}{|a_2|} \right|, \left| \frac{a_2 - a_1}{|a_1|} \right| \right) \leq \frac{16L_1}{\beta L_2 |\bar{x} - \bar{y}|^{\beta-1}}. \]

Recalling that \( ||Y|| = \max_{\xi} |\langle Y \xi, \xi \rangle| \leq 4L_2 \beta |\bar{x} - \bar{y}|^{\beta-2} \), it follows that

\[ |a_1|^\gamma |\text{tr}((A(a_1) - A(a_2)) Y)| \leq |a_1|^\gamma 128n |p - 2| L_1 |\bar{x} - \bar{y}|^{-1} \leq C \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^\gamma |p - 2| L_1 |\bar{x} - \bar{y}|^{-1}. \]

Now we estimate the term (i3). Notice that \(|a_2|/|a_1| \leq 16\) and \(|a_1 - a_2| \leq 4L_1\). Using the mean value theorem and the estimate (6.2), we get that

\[ ||a_1|^\gamma - |a_2|^\gamma|| \leq C |a_1 - a_2| |a_1|^\gamma 17^{\gamma-1} \leq C L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma-1} \]

if \( \gamma \geq 1 \)

\[ \leq |a_1 - a_2|^\gamma \leq (4L_1)^\gamma \]

if \( 0 \leq \gamma \leq 1 \)

\[ \leq C |a_1 - a_2|^\gamma (|a_1|^\gamma - |a_2|^\gamma) \leq C L_1 |a_1 - a_2|^\gamma \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{-\gamma} \]

if \(-1 < \gamma < 0\)

where \(0 < \kappa < 1\).

It follows that

\[ ||a_1|^\gamma - |a_2|^\gamma|| \text{tr}(A(a_2) Y)| \leq n ||Y|| \||A(a_2)|| \||a_1|^\gamma - |a_2|^\gamma|| \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma-1} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

\[ \leq C L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-2} (1 + |p - 2|) L_1 \left( L_2^2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma} \]

In order to estimate (i4), we use the estimate (6.2), and get

\[ L_1 (|a_1|^\gamma \text{tr}(A(a_1)) + |a_2|^\gamma \text{tr}(A(a_2))) \leq 2L_1 n \max(1, p - 1) C(\beta) \left( L_2 |\bar{x} - \bar{y}|^{\beta-1} \right)^{\gamma}. \]

Finally, gathering the previous estimates and plugging them into (6.6), we get

\[ 0 \leq 4L_1 + 2 ||f||_{L^\infty(Q_1)} + 2L_1 n \max(1, p - 1) C(\beta) \left( L_2 |\bar{x} - \bar{y}|^{\beta-1} \right)^\gamma \]

\[ + C \min(1, p - 1) \left( L_2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right) \leq L_2 |\bar{x} - \bar{y}|^{\beta-2} \left( \frac{\beta - 1}{3 - \beta} \right) \]

\[ + C \left( L_2 \beta |\bar{x} - \bar{y}|^{\beta-1} \right)^\gamma |p - 2| L_1 |\bar{x} - \bar{y}|^{-1} \]

\[ + \text{right hand term of } (6.8). \]

Choosing \( L_2 \) large enough

\[ L_2 \geq C(L_1 + L_1^\frac{1}{\gamma} + ||f||_{L^\infty(Q_1)} + ||f||_{L^\infty(Q_1)} + ||u||_{L^\infty(Q_1)} + ||f||_{L^\infty(Q_1)}), \]
we end up with
\[ 0 \leq \frac{\min(1, p - 1)\beta(\beta - 1)}{1000(3 - \beta)}L_2|\bar{x} - \bar{y}|^{\beta - 2} < 0, \]
which is a contradiction. Hence, \( \Phi(x, y, t) \leq 0 \) for \((x, y) \in \overline{B_{15/16}} \times \overline{B_{15/16}} \) and \( t \in [-15/16]^2, 0 \). This concludes the proof since for \( x_0, y_0 \in B_{15/16} \) and \( t_0 \in (-15/16, 0) \), we have \( \Phi(x_0, y_0, t_0) \leq 0 \) and we get
\[ |u(x_0, t_0) - u(y_0, t_0)| \leq L_2|x_0 - y_0|^\beta. \]
Remembering the dependence of \( L_2 \) (see (6.9)), we get the desired result. \( \square \)

### 6.2. Local Lipschitz estimates.

**Lemma 6.2.** Let \(-1 < \gamma < \infty \) and \(1 < p < \infty \). Let \( u \) be a bounded viscosity solution to equation (1.1). For all \( r \in (0, \frac{7}{8}) \), and for all \( x, y \in \overline{B_r} \) and \( t \in [-r^2, 0] \), it holds
\[ |u(x, t) - u(y, t)| \leq \bar{C} \left( ||u||_{L^\infty(Q_1)} + ||u||_{L^\infty(Q_1)^2} + ||f||_{L^\infty(Q_1)} \right) |x - y|, \]
where \( \bar{C} = \bar{C}(p, n, \gamma) > 0 \).

**Proof.** In the sequel we fix \( r = 7/8 \) and we fix \( x_0, y_0 \in B_r, t_0 \in (-r^2, 0) \). For positive constants \( L_1, L_2 \), we consider the function
\[ \Phi(x, y, t) = u(x, t) - u(y, t) - L_2\varphi(|x - y|) - \frac{L_1}{2}|x - x_0|^2 - \frac{L_1}{2}|y - y_0|^2 - \frac{L_1}{2}(t - t_0)^2, \]
where \( \varphi \) is defined below. We want to show that \( \Phi(x, y, t) \leq 0 \) for \((x, y) \in \overline{B_r} \times \overline{B_r} \) and \( t \in [-r^2, 0] \). This time we take
\[ \varphi(s) = \begin{cases} s - s^\nu \kappa_0 & 0 \leq s \leq s_1 := \left( \frac{1}{4\nu\kappa_0} \right)^{1/(\nu - 1)} \\ \varphi(s_1) & \text{otherwise}, \end{cases} \]
where \( 2 > \nu > 1 \) and \( \kappa_0 > 0 \) is taken so that \( s_1 > 2 \) and \( \nu \kappa_0 s_1^{\nu - 1} \leq 1/4 \). Then \( \varphi \) is smooth in \((0, s_1)\) and for \( s \in (0, s_1) \) we have
\[ \begin{aligned} \varphi'(s) &= 1 - \nu s^{\nu - 1} \kappa_0, \\ \varphi''(s) &= -\nu(\nu - 1)s^{\nu - 2} \kappa_0. \end{aligned} \]
Next, observe that with these choices we have \( \varphi'(s) \in \left[ \frac{3}{4}, 1 \right] \) and \( \varphi''(s) < 0 \) when \( s \in (0, 2] \).

We proceed by contradiction assuming that \( \Phi \) has a positive maximum at some point \((\bar{x}, \bar{y}, \bar{t}) \in \overline{B_r} \times \overline{B_r} \times [-r^2, 0] \) and we are going to get a contradiction for \( L_2, L_1 \) large enough and for a suitable choice of \( \nu \). As in the proof of the Hölder estimate, we notice that \( \bar{x} \neq \bar{y} \) and for \( L_1 \geq C ||u||_{L^\infty(Q_1)} \), we have that \( \bar{x} \) and \( \bar{y} \) are in \( B_r \) and \( \bar{t} \in (-r^2, 0) \). Moreover, from Lemma 6.1 we know that \( u \) is locally Hölder continuous, and that for any \( \beta \in (0, 1) \) there exists a constant \( C_H > 0 \)
\[ C_H := C \times \left( ||u||_{L^\infty(Q_1)} + ||u||_{L^\infty(Q_1)} + ||f||_{L^\infty(Q_1)} \right) \]
such that
\[ |u(x, t) - u(y, t)| \leq C_H|x - y|^\beta \]
for \( x, y \in B_r, t \in (-r^2, 0) \).
Using this estimate and adjusting the constants (by choosing $2L_1 \leq C_H$), we have that

$$L_1 |\bar{y} - y_0| , L_1 |\bar{x} - x_0| \leq C_H |\bar{x} - \bar{y}|^{\beta/2}. \quad (6.11)$$

From the Jensen-Ishii’s lemma, we have the existence of

$$(\sigma + L_1(\bar{t} - t_0), a_1, X + L_1I) \in \overline{P^{\alpha,+}} u(\bar{x}, \bar{t}),$$

$$(\sigma, a_2, Y - L_1I) \in \overline{P^{\alpha,-}} u(\bar{y}, \bar{t}),$$

where

$$a_1 = L_2 \varphi'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} + L_1(\bar{x} - x_0),$$

$$a_2 = L_2 \varphi'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} - L_1(\bar{y} - y_0).$$

Recalling that $\varphi' \geq \frac{3}{4}$, then if $L_2 \geq 4C_H$, we have

$$2L_2 \geq |a_1|, |a_2| \geq L_2 \varphi'(|\bar{x} - \bar{y}|) - C_H |\bar{x} - \bar{y}|^{\beta/2} \geq \frac{L_2}{2}. \quad (6.12)$$

Moreover, by Jensen-Ishii’s lemma, for any $\tau > 0$, we can take $X, Y \in S^n$ such that

$$- [\tau + 2||Z||] \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} Z & -Z' \\ -Z & Z' \end{pmatrix} + \frac{2}{\tau} \begin{pmatrix} Z^2 & -Z^2 \\ -Z^2 & Z^2 \end{pmatrix}, \quad (6.13)$$

where

$$Z = L_2 \varphi''(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \otimes \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} + L_2 \varphi'(|\bar{x} - \bar{y}|) \left( I - \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \otimes \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \right)$$

and

$$Z^2 = L_2^2 \left( \varphi'(|\bar{x} - \bar{y}|) \right)^2 \left( I - \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \otimes \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \right) + L_2^2 \left( \varphi''(|\bar{x} - \bar{y}|) \right)^2 \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \otimes \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}.$$

Simple computations give

$$||Z|| \leq L_2 \frac{\varphi'(|\bar{x} - \bar{y}|)}{|\bar{x} - \bar{y}|}, \quad (6.14)$$

$$||Z^2|| \leq L_2^2 \left( \varphi''(|\bar{x} - \bar{y}|) + \frac{\varphi'(|\bar{x} - \bar{y}|)}{|\bar{x} - \bar{y}|} \right)^2, \quad (6.15)$$

and for $\xi = \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}$, we have

$$\langle Z\xi, \xi \rangle = L_2 \varphi''(|\bar{x} - \bar{y}|) < 0, \quad \langle Z^2\xi, \xi \rangle = L_2^2 \left( \varphi''(|\bar{x} - \bar{y}|) \right)^2.$$
We take $\tau = 4L_2 \left( |\varphi''(\bar{x} - \bar{y})| + \frac{|\varphi'(|\bar{x} - \bar{y}|)|}{|\bar{x} - \bar{y}|} \right)$ and we observe that for $\xi = \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}$,

$$
\langle Z\xi, \xi \rangle + \frac{2}{\tau} \langle Z^2\xi, \xi \rangle = L_2 \left( \varphi''(|\bar{x} - \bar{y}|)^2 + \varphi''(|\bar{x} - \bar{y}|) \right) \leq \frac{L_2}{2} \varphi''(|\bar{x} - \bar{y}|) < 0.
$$

(6.16)

From (6.13), we deduce that $X - Y \leq 0$ and $\|X\|, \|Y\| \leq 2 ||Z|| + \tau$. Moreover, applying the matrix inequality (6.13) to the vector $(\xi, -\xi)$ where $\xi := \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}$ and using (6.16), we obtain

$$
\langle (X - Y)\xi, \xi \rangle \leq 4 \left( \langle Z\xi, \xi \rangle + \frac{2}{\tau} \langle Z^2\xi, \xi \rangle \right) \leq 2L_2\varphi''(|\bar{x} - \bar{y}|) < 0.
$$

(6.17)

This implies that at least one of the eigenvalue of $X - Y$ that we denote by $\lambda_{i_0}$ is negative and smaller than $2L_2\varphi''(|\bar{x} - \bar{y}|)$. Writing the viscosity inequalities and adding them, we have

$$
0 \leq 2(L_1 + ||f||_{L^\infty(q_1)}) + |a_1|\gamma \underbrace{\text{tr}(A(a_1)(X - Y))}_1 + |a_1|\gamma \underbrace{\text{tr}(A(a_1) - A(a_2))Y)}_2
$$

$$
+ |a_1|\gamma \underbrace{\text{tr}(A(a_2)Y)}_3 + L_1 \left[ |a_1|\gamma \underbrace{\text{tr}(A(a_1))}_4 + |a_2|\gamma \underbrace{\text{tr}(A(a_2))}_4 \right].
$$

(6.18)

The eigenvalues of $A(a_1)$ belong to $[\min(1, p - 1), \max(1, p - 1)]$. Using (6.17), it follows that we can estimate (1) by

$$
\text{tr}(A(a_1)(X - Y)) \leq \sum_i \lambda_i(A(a_1))\lambda_i(X - Y)
$$

$$
\leq \min(1, p - 1)\lambda_{i_0}(X - Y)
$$

$$
\leq 2 \min(1, p - 1)L_2\varphi''(|\bar{x} - \bar{y}|).
$$

As in the proof of the Hölder estimate, we estimate (2) by

$$
\text{tr}(A(a_1) - A(a_2))Y) \leq 2n |p - 2||Y|||\hat{a}_1 - \hat{a}_2|.
$$

With the new choice of $\varphi$, we have

$$
|\hat{a}_1 - \hat{a}_2| = \left| \frac{a_1}{|a_1|} - \frac{a_2}{|a_2|} \right| \leq \max \left( \left| \frac{a_2 - a_1}{|a_2|} \right|, \left| \frac{a_2 - a_1}{|a_1|} \right| \right) \leq \frac{8C_H}{L_2} |\bar{x} - \bar{y}|^{\beta/2},
$$

where we used (6.11) and (6.12). Using (6.13)–(6.15), we have

$$
||Y|| \leq 2||Z\xi, \xi|| + \frac{4}{\tau} \langle Z^2\xi, \xi \rangle \leq 4L_2 \left( \frac{|\varphi'(|\bar{x} - \bar{y}|)|}{|\bar{x} - \bar{y}|} + |\varphi''(|\bar{x} - \bar{y}|)| \right).
$$

Hence, remembering that $|\bar{x} - \bar{y}| \leq 2$ and $|a_1|\gamma \leq CL_2^2$, we end up with

$$
|a_1|\gamma |\text{tr}(A(a_1) - A(a_2))Y)| \leq CL_2^2 |p - 2| C_H\varphi'(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}|^{-1+\beta/2}
$$

$$
+ CL_2^2 |p - 2| C_H|\varphi''(|\bar{x} - \bar{y}|)|.
$$
Using the mean value theorem and the estimates (6.11) and (6.12), we have
\[
||a_1||^\gamma - ||a_2||^\gamma \leq \frac{|a_1 - a_2|}{|a_1|} |a_1|^{\gamma-1} \leq CC_H L_2^{-1} |\bar{x} - \bar{y}|^{\beta/2} \quad \text{if } \gamma \geq 1
\]
\[
\leq |a_1 - a_2|^\gamma \leq (C_H |\bar{x} - \bar{y}|^{\beta/2})^\gamma \quad \text{if } 0 < \gamma \leq 1
\]
\[
\leq |a_1 - a_2|^\gamma (|a_1|^{\gamma-\kappa} + |a_2|^{\gamma-\kappa}) \leq CL_2^{\gamma-\kappa} (C_H |\bar{x} - \bar{y}|^{\beta/2})^\kappa \quad \text{if } -1 < \gamma \leq 0
\]
where $0 < \kappa < 1$.

It follows (using that $|\bar{x} - \bar{y}| \leq 2$)
\[
||a_1||^\gamma - ||a_2||^\gamma \leq n ||Y|| ||A(a_2)|| ||a_1||^\gamma - ||a_2||^\gamma \quad \text{(6.19)}
\]
\[
\leq CL_2 \left( |\bar{x} - \bar{y}|^{1+\beta/2} + |\varphi''(\bar{x} - \bar{y})| \right) (1 + |p - 2|) C_H L_2^{-1} \quad \text{if } \gamma \geq 1
\]
\[
\leq CL_2 \left( |\varphi''(\bar{x} - \bar{y})| + |\varphi''(\bar{x} - \bar{y})| \right) (1 + |p - 2|) C_H^{\nu} |\bar{x} - \bar{y}|^{\nu - 2} \quad \text{if } 0 < \gamma \leq 1
\]
\[
\leq CL_2^{\gamma-\kappa} \left( |\varphi''(\bar{x} - \bar{y})| + |\varphi''(\bar{x} - \bar{y})| \right) (1 + |p - 2|) |\bar{x} - \bar{y}|^{\nu - 2} C_H^\kappa \quad \text{if } -1 < \gamma \leq 0.
\]

Finally, we have
\[L_1 ||a_1||^\gamma ||A(a_1)|| + ||a_2||^\gamma ||A(a_2)|| \leq 2CL_2 L_1 n \max(1, p - 1).
\]

Gathering the previous estimates with (6.18) and recalling the definition of $\varphi$, we get
\[
0 \leq 2(L_1 + ||f||_{L^\infty(Q_1)}) + CL_2 L_1 \max(1, p - 1)
\]
\[
+ CL_2 |p - 2| C_H |\bar{x} - \bar{y}|^{1+\beta/2} + CL_2 |p - 2| C_H |\bar{x} - \bar{y}|^{\nu - 2}
\]
\[
- 2CL_2 \min(1, p - 1) L_2 (\nu - 1) \nu \kappa_0 |\bar{x} - \bar{y}|^{\nu - 2} + \text{right hand term of (6.19)}.
\]

Taking $\nu = 1 + \frac{\beta}{2}$, recalling the dependence of $C_H$ and choosing $L_2$ large
\[
L_2 \geq C \left( ||u||_{L^\infty(Q_1)} + ||u||_{L^\infty(Q_1)}^{1+\gamma} + ||f||_{L^\infty(Q_1)}^{1+\gamma} \right),
\]
we get that
\[
0 \leq - \frac{\min(1, p - 1) \nu (\nu - 1) \kappa_0}{1000} L_2 |\bar{x} - \bar{y}|^{\nu - 2} < 0,
\]
which is a contradiction. It follows that $\Phi(x, y, t) \leq 0$ for $(x, y, t) \in \overline{B_r \times B_r} \times [-r^2, 0]$. The desired result follows since for $x_0, y_0 \in B_r$, $t_0 \in (-r^2, 0)$, we have $\Phi(x_0, y_0, t_0) \leq 0$, so that
\[
|u(x_0, t_0) - u(y_0, t_0)| \leq L_2 \varphi(|x_0 - y_0|) \leq L_2 |x_0 - y_0|.
\]

**7. Proof of the uniform Hölder and Lipschitz estimates**

In this section we provide a proof for Lemma 3.3. Assume that $0 \leq \gamma < \infty$ and consider bounded solutions $w$ to
\[
\partial_t w - |Dw + q|^\gamma \left[ \Delta w + (p - 2) \left( D^2 w \frac{Dw + q}{|Dw + q|^2}, \frac{Dw + q}{|Dw + q|} \right) \right] = \bar{f} \quad \text{in } Q_1. \quad (7.1)
\]
Noticing that \( h(x, t) := w(x, t) + q \cdot x \) is a solution of (1.1). It follows from Lemma 6.2 that \( w \) is Lipschitz continuous with respect to the space variable. Moreover, for \( x, y \in B_{7/8} \) and \( t \in (- (7/8)^2, 0] \), it holds
\[
|w(x, t) - w(y, t)| \leq |h(x, t) - h(y, t)| + |q||x - y|
\]
\[
\leq \left( |q| + C(||h||_{L^\infty(Q_1)} + ||f||_{L^\infty(Q_1)} + ||h||_{L^\infty(Q_1)}^2) \right)|x - y|.
\]
\[(7.2)\]
Hence, if \( |q| \geq \Gamma_0 := 2 + ||w||_{L^\infty(Q_1)} + ||f||_{L^\infty(Q_1)} \), then for \( (x, t), (y, t) \in Q_{7/8} \), we have
\[
|w(x, t) - w(y, t)| \leq C(p, n, \gamma)|q||x - y|.
\]
\[(7.3)\]
We will improve this estimate and provide uniform estimates for deviation from planes with \( |q| > \Gamma_0 \). The case \( |q| < \Gamma_0 \) follows from (7.2). In order to prove uniform Lipschitz estimates with respect to \( q \), we first need to prove uniform Hölder estimates.

### 7.1. Local uniform Hölder estimates.

**Lemma 7.1.** Let \( w \) be a bounded viscosity solution to equation (7.1) with \( |q| > \Gamma_0 \). There exist a constant \( \beta = \beta(p, n, \gamma) \in (0, 1) \) and a constant \( C = C(p, n, \gamma) > 0 \) such that for all \( x, y \in B_{13/16} \) and \( t \in (- (13/16)^2, 0] \), it holds
\[
|w(x, t) - w(y, t)| \leq C(1 + ||w||_{L^\infty(Q_1)} + ||f||_{L^\infty(Q_1)})|x - y|^\beta.
\]
\[(7.4)\]
**Proof.** We fix \( x_0, y_0 \in B_{13/16} \), \( t_0 \in (- (13/16)^2, 0) \). For suitable constants \( L_1, L_2 > 0 \), we define the auxiliary function
\[
\Phi(x, y, t) := w(x, t) - w(y, t) - L_2 |x - y|^{\beta} - L_1^\frac{1}{2}|x - x_0|^2 - L_1^\frac{1}{2}|y - y_0|^2 - \frac{L_1}{2}(t - t_0)^2.
\]
We want to show that \( \Phi(x, y, t) \leq 0 \) for \( (x, y) \in \overline{B_{13/16}} \times \overline{B_{13/16}} \) and \( t \in [- (13/16)^2, 0] \). We argue by contradiction. We assume that \( \Phi \) has a positive maximum at some point \( (\bar{x}, \bar{y}, \bar{t}) \in \overline{B_{13/16}} \times \overline{B_{13/16}} \times [- (13/16)^2, 0] \) and we are going to get a contradiction for \( L_2, L_1 \) large enough. The positivity of the maximum of \( \Phi \) implies that \( \bar{x} \neq \bar{y} \). Choosing \( L_1 \geq C||w||_{L^\infty(Q_1)} \), we have that \( \bar{x} \) and \( \bar{y} \) are in \( B_{13/16} \) and \( \bar{t} \in (- (13/16)^2, 0) \). We proceed as in the proof of Lemma 6.1. By Jensen-Ishii’s lemma, there exist
\[
(\sigma + L_1(\bar{t} - t_0), a_1, X + L_1I) \in \overline{\mathcal{P}}_{a, +} w(\bar{x}, \bar{t}),
\]
\[
(\sigma, a_2, Y - L_1I) \in \overline{\mathcal{P}}_{a, -} w(\bar{y}, \bar{t}).
\]
Assuming that \( L_2 \geq CL_1 \), we have
\[
\begin{aligned}
2L_2\beta|\bar{x} - \bar{y}|^{\beta - 1} &\geq |a_1| \geq L_2\beta|\bar{x} - \bar{y}|^{\beta - 1} - L_1|\bar{x} - x_0| \geq L_2^\frac{1}{2}\beta|\bar{x} - \bar{y}|^{\beta - 1}, \\
2L_2\beta|\bar{x} - \bar{y}|^{\beta - 1} &\geq |a_2| \geq L_2\beta|\bar{x} - \bar{y}|^{\beta - 1} - L_1|\bar{y} - y_0| \geq L_2^\frac{1}{2}\beta|\bar{x} - \bar{y}|^{\beta - 1}.
\end{aligned}
\]
\[(7.5)\]
We can take \( X, Y \in \mathcal{S}^n \) such that it holds
\[
- \frac{2}{\tau} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} Z^\tau & -Z^\tau \\ -Z^\tau & Z^\tau \end{pmatrix},
\]
\[(7.6)\]
where
\[
Z^\tau = (I - \tau Z)^{-1}Z = 2L_2\beta |\bar{x} - \bar{y}|^{\beta-2} \left( I - \frac{2 - \beta}{3 - \beta} \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \otimes \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \right).
\]

We have for \( \xi = \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \),
\[
\langle Z^\tau \xi, \xi \rangle = 2L_2\beta |\bar{x} - \bar{y}|^{\beta-2} \left( \frac{\beta - 1}{3 - \beta} \right) < 0.
\]

Applying the inequality (7.6) to any vector \((\xi, \xi)\) with \(|\xi| = 1\), we get that \(X - Y \leq 0\) and
\[
||X||, ||Y|| \leq 4L_2\beta |\bar{x} - \bar{y}|^{\beta-2}.
\]

Moreover, using the positivity of the maximum of \(\Phi\) and the Lipschitz regularity of \(w\) (see (7.3)), we have for \(0 < \beta \leq \frac{3}{4}\),
\[
2\beta L_2 |\bar{x} - \bar{y}|^{\beta-1} \leq 2\beta \frac{|w(\bar{x}, \bar{t}) - w(\bar{y}, \bar{t})|}{|\bar{x} - \bar{y}|} \leq 2\beta C(p, n, \gamma)|q| \leq \frac{|q|}{2}, \quad (7.7)
\]

Setting \(\eta_1 = a_1 + q, \eta_2 = a_2 + q\), we get by using (7.5) and (7.7), that
\[
2|q| \geq |\eta_1| \geq |q| - |a_1| \geq \frac{|q|}{2} \geq 2L_2\beta |\bar{x} - \bar{y}|^{\beta-1},
\]
\[
2|q| \geq |\eta_2| \geq |q| - |a_2| \geq \frac{|q|}{2} \geq 2L_2\beta |\bar{x} - \bar{y}|^{\beta-1}. \quad (7.8)
\]

Writing the viscosity inequalities and adding them, we get
\[
0 \leq 2|\eta_1|^{-\gamma}(L_1 + ||\tilde{f}||_{L^\infty(Q_1)}) + \underbrace{\text{tr}(A(\eta_1)(X - Y))}_{(i_1)} + \underbrace{\text{tr}((A(\eta_1) - A(\eta_2))Y)}_{(i_2)}
\]
\[
+ |\eta_1|^{-\gamma}(|\eta_1|^{-\gamma} - |\eta_2|^{-\gamma}) \text{tr}(A(\eta_2)Y) + L_1 \left[ \text{tr}(A(\eta_1)) + |\eta_2|^{-\gamma}|\eta_1|^{-\gamma} \text{tr}(A(\eta_2)) \right]. \quad (7.9)
\]

We estimate \((i_1)\) as in the proof of Lemma 6.1
\[
\text{tr}(A(\eta_1)(X - Y)) \leq \min(1, p - 1)8L_2\beta |\bar{x} - \bar{y}|^{\beta-2} \left( \frac{\beta - 1}{3 - \beta} \right).
\]

In order to estimate \((i_2)\), we use that \(|\eta_1 - \eta_2| \leq 4L_1\) and the estimate (7.8), so that
\[
|\eta_1 - \eta_2| = \left| \frac{\eta_1}{|\eta_1|} - \frac{\eta_2}{|\eta_2|} \right| \leq \max \left( \frac{|\eta_2 - \eta_1|}{|\eta_2|}, \frac{|\eta_2 - \eta_1|}{|\eta_1|} \right) \leq \frac{16L_1}{\beta L_2 |\bar{x} - \bar{y}|^{\beta-1}}.
\]

Recalling that \(||Y|| = \max_{\xi} |\langle Y\xi, \hat{\xi} \rangle| \leq 4L_2\beta |\bar{x} - \bar{y}|^{\beta-2}\), it follows that
\[
\text{tr}((A(\eta_1) - A(\eta_2))Y) \leq C \min(p - 2, L_1) |\bar{x} - \bar{y}|^{-1}.
\]
Finally, gathering the previous estimates and plugging them into (7.8), we get that
\[ |\eta_1| - |\eta_2| \leq C \frac{|\eta_1 - \eta_2|}{|\eta_1|} |\eta_1| \leq \gamma C L_1 |\eta_1|^{-1} \]
\[ \leq |\eta_1 - \eta_2|^\gamma \leq (4L_1)^\gamma \]
It follows that
\[ \frac{|\eta_1|^\gamma - |\eta_2|^\gamma}{|\eta_1|^\gamma} \leq n|\eta_1|^{-\gamma} \|Y\| \|\mathcal{A}(\eta_2)\| |\eta_1|^\gamma - |\eta_2|^\gamma | \]
\[ \leq C |\bar{x} - \bar{y}|^{-1} (1 + |p - 2|) L_1 \quad \text{if } \gamma \geq 1 \]
\[ \leq C L_2^{1-\gamma} \beta^{1-\gamma} |\bar{x} - \bar{y}|^{\beta - 2 + \gamma(1 - \beta)} (1 + |p - 2|) L_1^\gamma \quad \text{if } \gamma \in [0, 1]. \]

We estimate (i_4) by
\[ L_1(\text{tr}(\mathcal{A}(\eta_1)) + |\eta_2|^\gamma |\eta_1|^{-\gamma} \text{tr}(\mathcal{A}(\eta_2))) \leq C L_1 n \max(1, p - 1). \]

Finally, gathering the previous estimates and plugging them into (7.9) and recalling that $|\eta_1| \geq 1$, we get
\[ 0 \leq 4L_1 + 2 \|\bar{f}\|_{L^{\infty}(Q_1)} + C L_1 n \max(1, p - 1) + C \min(1, p - 1) L_2 \beta |\bar{x} - \bar{y}|^{\beta - 2} \left( \frac{\beta - 1}{3 - \beta} \right) \]
\[ + C n |p - 2| L_1 |\bar{x} - \bar{y}|^{-1} + \text{right hand term of } (7.10). \]
Choosing $L_2$ large enough
\[ L_2 \geq C (1 + L_1 + \|\bar{f}\|_{L^{\infty}(Q_1)}) \geq C (1 + \|w\|_{L^{\infty}(Q_1)} + \|\bar{f}\|_{L^{\infty}(Q_1)}), \]
we end up with
\[ 0 \leq \frac{\min(1, p - 1) \beta (\beta - 1)}{1000 (3 - \beta)} L_2 |\bar{x} - \bar{y}|^{\beta - 2} < 0, \]
which is a contradiction. This concludes the proof since for $(x_0, t_0), (y_0, t_0) \in Q_{13/16}$, we have $\Phi(x_0, y_0, t_0) \leq 0$ and we get
\[ |u(x_0, t_0) - u(y_0, t_0)| \leq L_2 |x_0 - y_0|^\beta. \]
Remembering the dependence of $L_2$ (see (7.11)), we get the desired result. \[ \square \]

### 7.2. Local uniform Lipschitz estimates.

**Lemma 7.2.** Let $w$ be a bounded viscosity solution to equation (7.1) with $|q| > C T_0$. For all $r \in (0, \frac{1}{4})$, and for all $x, y \in \overline{B_r}$ and $t \in [-r^2, 0]$, it holds
\[ |w(x, t) - w(y, t)| \leq \tilde{C} \left( 1 + \|w\|_{L^{\infty}(Q_1)} + \|\bar{f}\|_{L^{\infty}(Q_1)} \right) |x - y|, \]
where $\tilde{C} = \tilde{C}(p, n, \gamma) > 0$.  

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Proof. We proceed as in the proof of Lemma 6.2. We fix \( r = 3/4 \) and we fix \( x_0, y_0 \in B_r, t_0 \in (-r^2, 0) \). For positive constants \( L_1, L_2, \) we define the function

\[
\Phi(x, y, t) := w(x, t) - w(y, t) - L_2 \varphi(|x - y|) - \frac{L_1}{2} |x - x_0|^2 - \frac{L_1}{2} |y - y_0|^2 - \frac{L_1}{2} (t - t_0)^2,
\]

where

\[
\varphi(s) = \begin{cases} 
    s - s' \kappa_0 & 0 \leq s \leq s_1 := \left( \frac{1}{\nu \kappa_0} \right)^{1/(\nu - 1)} \\
    \varphi(s_1) & \text{otherwise},
\end{cases}
\]

where \( 2 > \nu > 1 \) and \( \kappa_0 > 0 \) is taken so that \( s_1 > 2 \) and \( \nu \kappa_0 s_1^{\nu - 1} \leq 1/4 \). We want to show that \( \Phi(x, y, t) \leq 0 \) for \( (x, y) \in \overline{B_r} \times \overline{B_r} \) and \( t \in [-r^2, 0] \). We proceed by contradiction assuming that \( \Phi \) has a positive maximum at some point \( (\bar{x}, \bar{y}, \bar{t}) \in \overline{B_r} \times \overline{B_r} \times [-r^2, 0] \) and we are going to get a contradiction for \( L_2, L_1 \) large enough and for a suitable choice of \( \nu \).

As in the proof of the Hölder estimate, we notice that \( \bar{x} \) and \( \bar{y} \) are in \( B_r \) and \( \bar{t} \in (-r^2, 0) \). Moreover, from Lemma 7.1, we know that \( w \) is locally Hölder continuous, and that there exist a constant \( C_H > 0 \)

\[
\bar{C}_H := \bar{C} \times \left( 1 + \|u\|_{L^\infty(Q_1)} + \|\bar{f}\|_{L^\infty(Q_1)} \right)
\]

and a constant \( \beta = \beta(p, n, \gamma) \in (0, 1) \) such that

\[
|w(x, t) - w(y, t)| \leq \bar{C}_H |x - y|^{\beta} \quad \text{for } x, y \in B_r, t \in (-r^2, 0).
\]

Using this estimate, we have that

\[
L_1 |\bar{y} - y_0|, L_1 |\bar{x} - x_0| \leq \bar{C}_H |\bar{x} - \bar{y}|^{\beta/2}.
\]

From the Jensen-Ishii’s lemma, we have the existence of

\[
(\sigma + L_1 (\bar{t} - t_0), a_1, X + L_1 I) \in P_{\sigma^+}^2 u(\bar{x}, \bar{t}),
\]

\[
(\sigma, a_2, Y - L_1 I) \in P_{\sigma^-}^2 u(\bar{y}, \bar{t}),
\]

where

\[
a_1 = L_2 \varphi'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} + L_1 (\bar{x} - x_0),
\]

\[
a_2 = L_2 \varphi'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} - L_1 (\bar{y} - y_0).
\]

Recalling that \( \varphi' \geq \frac{3}{4} \), then if \( L_2 \) is large (\( L_2 \geq 4\bar{C}_H \)), we have

\[
2L_2 \geq |a_1|, |a_2| \geq L_2 \varphi'(\bar{x} - \bar{y}) - \bar{C}_H |\bar{x} - \bar{y}|^{\beta/2} \geq \frac{L_2}{2}.
\]
Denoting \( \eta_1 = a_1 + q, \eta_2 = a_2 + q \), we have for \( |q| \geq L_2 + 2 \)
\[
3|q| \geq |
\eta_1| \geq |q| - |a_1| \geq \frac{|q|}{2} \geq \frac{L_2}{2},
\]
\[
3|q| \geq |
\eta_2| \geq |q| - |a_2| \geq \frac{|q|}{2} \geq \frac{L_2}{2}
\]
\[
|
\eta_1 - \eta_2| \leq 2 \bar{C}_H |\vec{x} - \vec{y}|^{\beta/2}.
\]  
Moreover, by Jensen-Ishii’s lemma, for any \( \tau > 0 \), we can take \( X, Y \in \mathcal{S}^n \) such that
\[
- [\tau + 2 \|Z\|] \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + \frac{2}{\tau} \begin{pmatrix} Z^2 & -Z^2 \\ -Z^2 & Z^2 \end{pmatrix}.
\]  
As in the proof of Lemma 6.2, we take \( \tau = 4L_2 \left( |\varphi''(|\vec{x} - \vec{y}|)| + \frac{|\varphi'(|\vec{x} - \vec{y}|)|}{|\vec{x} - \vec{y}|} \right) \) and we observe that for \( \xi = \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|} \),
\[
\langle Z\xi, \xi \rangle + \frac{2}{\tau} \langle Z^2\xi, \xi \rangle \leq \frac{L_2}{2} \varphi''(|\vec{x} - \vec{y}|) < 0.
\]  
We also have that \( X - Y \leq 0, \|X\|, \|Y\| \leq 2 \|Z\| + \tau \), and that at least one of the eigenvalue of \( X - Y \) that we denote by \( \lambda_0 \) is negative and smaller than \( 2L_2 \varphi''(|\vec{x} - \vec{y}|) \). Writing the viscosity inequalities and adding them, we have
\[
0 \leq 2(L_1 + \|\bar{f}\|_{L^{\infty}(Q_1)})|\eta_1|^{-7} + \underbrace{\text{tr}(\mathcal{A}(\eta_1)(X - Y))}_{(1)} + \underbrace{\text{tr}(\mathcal{A}(\eta_1) - \mathcal{A}(\eta_2))Y}_{(2)} + \underbrace{|\eta_1|^{-7}(|\eta_1|^{-7} - |\eta_2|^{-7}) \text{tr}(\mathcal{A}(\eta_2)Y)}_{(3)} + \underbrace{L_1[ \text{tr}(\mathcal{A}(\eta_1)) + |\eta_2|^{-7}|\eta_1|^{-7} \text{tr}(\mathcal{A}(\eta_2))]}_{(4)}.
\]  
We estimate (1) by
\[
\text{tr}(\mathcal{A}(\eta_1)(X - Y)) \leq \sum_i \lambda_i(\mathcal{A}(\eta_1))\lambda_i(X - Y) \leq 2 \min(1, p - 1)L_2 \varphi''(|\vec{x} - \vec{y}|).
\]  
As previously, we estimate (2) by
\[
|\text{tr}(\mathcal{A}(\eta_1) - \mathcal{A}(\eta_2))Y| \leq C|p - 2| \bar{C}_H \varphi''(|\vec{x} - \vec{y}|) |\vec{x} - \vec{y}|^{-1 + \beta/2} + C|p - 2| \bar{C}_H \varphi''(|\vec{x} - \vec{y}|).
\]  
We have also
\[
|\eta_1|^{-7}|\eta_1|^{-7} - |\eta_2|^{-7}|| \text{tr}(\mathcal{A}(\eta_2)Y) | \leq n \|Y\| \|\mathcal{A}(\eta_2)\| |\eta_1|^{-7} - |\eta_2|^{-7}|
\leq C \left( |\vec{x} - \vec{y}|^{-1 + \beta/2} + |\varphi''(|\vec{x} - \vec{y}|)| \right) (1 + |p - 2|) \bar{C}_H \text{ if } \gamma \geq 1
\leq CL_2^{1-\gamma} \left( \frac{\varphi''(|\vec{x} - \vec{y}|)}{|\vec{x} - \vec{y}|} + |\varphi''(|\vec{x} - \vec{y}|)| \right) (1 + |p - 2|) \bar{C}_H^{\gamma} |\vec{x} - \vec{y}|^{\frac{\gamma}{2}} \text{ if } 0 < \gamma \leq 1.
\]  
Finally, we have
\[
|\eta_1|^{-7}L_1[ |\eta_1|^{-7} \text{tr}(\mathcal{A}(\eta_1)) + |\eta_2|^{-7} \text{tr}(\mathcal{A}(\eta_2)) ] \leq 2CL_1 n \max(1, p - 1).
\]
Gathering the previous estimates with (7.16), recalling the definition of \( \varphi \) and using that
\[ |\eta_1| \geq |q|/2 \geq 1, \]
we get
\[
0 \leq C(L_1 + ||f||_{L^\infty(Q_1)}) + CL_1 n \max(1, p - 1) + C |p - 2| C_H |\bar{x} - \bar{y}|^{-1+\beta/2}
+ C |p - 2| C_H |\bar{x} - \bar{y}|^{\nu-2} - 2 \min(1, p - 1)(\nu - 1)\nu \kappa_0 L_2 |\bar{x} - \bar{y}|^{\nu-2}
+ \text{right hand term of } (7.17).
\]
Taking \( \nu = 1 + \frac{\beta}{2} \), recalling the dependence of \( C_H \) and choosing \( L_2 \) large
\[ L_2 \geq C \left( 1 + ||u||_{L^\infty(Q_1)} + ||\bar{f}||_{L^\infty(Q_1)} \right), \]
we get that
\[
0 \leq -\frac{\min(1, p - 1)\nu(\nu - 1)\kappa_0}{1000} L_2 |\bar{x} - \bar{y}|^{\nu-2} < 0,
\]
which is a contradiction. Hence
\[
|u(x_0, t_0) - u(y_0, t_0)| \leq L_2 \varphi(|x_0 - y_0|) \leq L_2|x_0 - y_0|.
\]

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