LIPSCHITZ CONSTANTS TO CURVE COMPLEXES

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Abstract. We determine the asymptotic behavior of the optimal Lipschitz constant for the systole map from Teichmüller space to the curve complex.

1. Introduction

Let $S = S_g$ be a closed surface of genus $g \geq 2$. We equip the Teichmüller space $\mathcal{T}(S)$ of $S$ with the Teichmüller metric, and equip the 1–skeleton $\mathcal{C}^{(1)}(S)$ of the complex of curves $\mathcal{C}(S)$ with its usual path metric $d_\mathcal{C}$.

In [6], Masur and Minsky study the systole map $\text{sys} : \mathcal{T}(S) \to \mathcal{C}^{(1)}(S)$, which assigns a hyperbolic metric one of its shortest curves, called a systole. They prove that $\text{sys}$ is $(K, C)$–coarsely Lipschitz for $K, C > 0$, meaning that, for all $X$ and $Y$ in $\mathcal{T}(S)$

$$d_\mathcal{C}(\text{sys}(X), \text{sys}(Y)) \leq K d_\mathcal{T}(X, Y) + C.$$ 

This is the starting point of their proof that $\mathcal{C}^{(1)}(S)$ is $\delta$–hyperbolic. (The constant $\delta$ has recently been shown to be independent of $g$ by Aougab [1], Bowditch [4], and Clay, Rafi, and Schleimer [5].)

In this paper we consider the optimal Lipschitz constant

$$\kappa_g = \inf\{K \geq 0 \mid \text{sys is } (K, C)\text{–coarsely Lipschitz for some } C > 0\}.$$ 

We write $F(g) \asymp H(g)$ to mean that $F(g)/H(g)$ is bounded above and below by two positive constants, and prove the following theorem.

Theorem 1.1. As $g \to \infty$ we have

$$\kappa_g \asymp \frac{1}{\log(g)}.$$ 

This is a sharp version of the closed case of Theorem 1.4 of [1], which provides a Lipschitz constant that is independent of $\chi(S)$. An analogous result holds when hyperbolic length is replaced with extremal length, see Proposition 4.9.

The upper bound on $\kappa_g$ is established by a careful version of Masur and Minsky’s proof that sys is coarsely Lipschitz. To establish the lower bound, we construct a

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sequence of pseudo-Anosov mapping classes whose translation lengths on $\mathcal{T}(S)$ and $\mathcal{C}^{(1)}(S)$ behave like $\log(g)/g$ and $1/g$, respectively.

2. A Lipschitz Constant.

Given the isotopy class $[f : S \to X]$ of a marked hyperbolic surface and the homotopy class of a curve $\alpha$, we write $\ell_X(\alpha)$ for the hyperbolic length of $\alpha$ in $[f : S \to X]$. Let $\text{sys}(X)$ denote the set of $\alpha$ in $\mathcal{C}(0)(S)$ for which $\ell_X(\alpha)$ is minimal. If $\alpha, \beta$ are in $\text{sys}(X)$, then the geometric intersection number $i(\alpha, \beta)$ is at most 1, and so the diameter of $\text{sys}(X)$ in $\mathcal{C}^{(1)}(S)$ is at most 2. We abuse notation and view $\text{sys}$ as a map from $\mathcal{T}(S)$ to $\mathcal{C}^{(1)}(S)$, although the image of $X$ is actually a subset of diameter at most 2. One may obtain a bona fide map via the Axiom of Choice.

Given a hyperbolic surface $X$ and a geodesic $\alpha$ on $X$, a collar neighborhood of $\alpha$ of width $r$ about $\alpha$ is an $r$–neighborhood whose interior is homeomorphic to an open annulus. We have the following lemma.

Lemma 2.1. Given a closed hyperbolic surface $X$, if $\alpha$ lies in $\text{sys}(X)$, then there is a collar neighborhood of $\alpha$ of width greater than $\ell_X(\alpha)/2$.

Proof. Consider a maximal–width collar neighborhood $N_w/2(\alpha)$ of width $w$. This has a self–tangency on its boundary. From this one can construct a curve $\gamma$ that runs a distance $w/2$ from one of the points of tangency to $\alpha$, then at most half–way around $\alpha$ a distance at most $\ell_X(\alpha)/2$, and then a distance $w/2$ to the second point of tangency. Since $\alpha$ is a systole, we have

$$\ell_X(\alpha) \leq \ell_X(\gamma) < w + \ell_X(\alpha)/2.$$  

So $w > \ell_X(\alpha)/2$ as required. $\square$

Recall that a pair of isotopy classes of curves fills $S$ if, whenever the curves are realized transversally, the complement of their union is a set of topological disks.

Lemma 2.2. Given $\alpha$ and $\beta$ in $\mathcal{C}(0)(S)$ that fill the surface $S$, we have

$$i(\alpha, \beta) \geq 2g - 1.$$  

Proof. The union $\alpha \cup \beta$ is a graph on $S$ with $i(\alpha, \beta)$ vertices and $2i(\alpha, \beta)$ edges. The complement is a union of $F \geq 1$ disks. Therefore

$$2g - 2 = -\chi(S) = -i(\alpha, \beta) + 2i(\alpha, \beta) - F = i(\alpha, \beta) - F \leq i(\alpha, \beta) - 1.$$  

So $i(\alpha, \beta) \geq 2g - 1$ as required. $\square$

We need Wolpert’s inequality \cite{12} describing change in lengths in terms of the Teichmüller distance.

Lemma 2.3 (Wolpert, Lemma 3.1 of \cite{12}). Given $X, Y \in \mathcal{T}(S)$ and a curve $\alpha$ on $S$ we have

$$\ell_Y(\alpha) \leq e^{d_{\mathcal{T}}(X,Y)} \ell_X(\alpha).$$  

Our upper bound on $\kappa_g$ now follows from the following proposition.
Proposition 2.4. For \( g \geq 2 \) and all \( X, Y \in \mathcal{T}(S_g) \) we have
\[
d_{\mathcal{C}}(\text{sys}(X), \text{sys}(Y)) \leq \frac{2}{\log(g - \frac{1}{2})} d_{\mathcal{T}}(X, Y) + 2.
\]

Lemma 2.5. If \( d_{\mathcal{T}}(X, Y) \leq \log(g - 1/2) \), then \( d_{\mathcal{C}}(\text{sys}(X), \text{sys}(Y)) \leq 2 \).

Proof: Suppose that \( d_{\mathcal{T}}(X, Y) \leq \log(g - 1/2) \). Write \( \alpha = \text{sys}(X) \) and \( \beta = \text{sys}(Y) \), and, without loss of generality, assume that
\[
\ell_X(\alpha) \leq \ell_Y(\beta).
\]
According to Lemma 2.1, we have
\[
i(\alpha, \beta) \ell_Y(\beta) < \ell_Y(\alpha).
\]
On the other hand, Lemma 2.3 implies that
\[
\ell_Y(\alpha) \leq e^{\log(g - 1/2)} \ell_X(\alpha) = (g - 1/2) \ell_X(\alpha) = \frac{2g - 1}{2} \ell_X(\alpha).
\]
Combining these two inequalities yields
\[
i(\alpha, \beta) < \frac{2\ell_Y(\alpha)}{\ell_Y(\beta)} \leq \frac{(2g - 1)\ell_X(\alpha)}{\ell_Y(\beta)} \leq 2g - 1.
\]
By Lemma 2.2, \( \alpha \) and \( \beta \) cannot fill the surface \( S \), and hence
\[
d_{\mathcal{C}}(\text{sys}(X), \text{sys}(Y)) = d_{\mathcal{C}}(\alpha, \beta) \leq 2.
\]
This proves the claim. \( \square \)

Proof of Proposition 2.4 Now, given any two points \( X \) and \( Y \) in \( \mathcal{T}(S) \), let \( n \) be the nonnegative integer such that
\[
n \log(g - 1/2) \leq d_{\mathcal{T}}(X, Y) < (n + 1) \log(g - 1/2).
\]
Let \( X = X_0, \ldots, X_{n+1} = Y \) be a chain in \( \mathcal{T}(S) \) with
\[
d_{\mathcal{T}}(X_{k-1}, X_k) \leq \log(g - 1/2)
\]
for each \( 1 \leq k \leq n + 1 \). By the triangle inequality and (2.5), we have
\[
d_{\mathcal{C}}(\text{sys}(X), \text{sys}(Y)) \leq \sum_{k=1}^{n+1} d_{\mathcal{C}}(\text{sys}(X_{k-1}), \text{sys}(X_k))
\leq 2(n + 1)
\leq \frac{2}{\log(g - 1/2)} d_{\mathcal{T}}(X, Y) + 2
\]
as required. \( \square \)

3. PSEUDO-ANOSOV MAPS
Given a pseudo-Anosov homeomorphism \( f : S \to S \), we let \( \lambda(f) \) denote the dilatation of \( f \). We recall a few facts about pseudo-Anosov homeomorphisms, and refer the reader to the listed references for more detailed discussions.
3.1. **Asymptotic translation length.** Given a homeomorphism \( f : S \to S \), the asymptotic translation length of \( f \) on \( \mathcal{C}(1)(S) \) is defined by

\[
\ell_{\mathcal{C}}(f) = \liminf_{j \to \infty} \frac{d_{\mathcal{C}}(\alpha, f^j(\alpha))}{j},
\]

where \( \alpha \) is any simple closed curve. This is easily seen to be independent of \( \alpha \). When \( f \) is pseudo-Anosov, Masur and Minsky proved \( f \) has a quasi-invariant geodesic axis, and so this limit infimum is in fact a limit. Moreover, there is a \( C > 0 \) depending only on the genus of \( S \) such that

\[
\ell_{\mathcal{C}}(f) \geq C,
\]

see [6] or Corollary of 1.5 [3]. It follows from the definition that \( \ell_{\mathcal{C}}(f^k) = k \ell_{\mathcal{C}}(f) \).

One can similarly define the asymptotic translation length of \( f : S \to S \) acting on \( \mathcal{T}(S) \). A pseudo-Anosov \( f \) has an axis in \( \mathcal{T}(S) \) (see [2]), and the asymptotic translation length is just the translation length \( \ell_{\mathcal{T}}(f) \). In fact, Bers’ proof of Thurston’s classification theorem shows that

\[
\ell_{\mathcal{T}}(f) = \log(\lambda(f)).
\]

The following lemma allows us to use asymptotic translation lengths to bound optimal Lipschitz constants.

**Lemma 3.2.** For any pseudo-Anosov \( f : S_g \to S_g \) we have

\[
\kappa_g \geq \frac{\ell_{\mathcal{C}}(f)}{\log(\lambda(f))}.
\]

**Proof.** If \( K, C > 0 \) are such that \( \text{sys} \) is \((K, C)\)-coarsely Lipschitz, then, for any \( X \) in \( \mathcal{T}(S) \), we have

\[
\frac{\ell_{\mathcal{C}}(f)}{\log(\lambda(f))} = \lim_{j \to \infty} \frac{d_{\mathcal{C}}(\text{sys}(X), f^j(\text{sys}(X)))}{d_{\mathcal{T}}(X, f^j(X))} = \lim_{j \to \infty} \frac{d_{\mathcal{C}}(\text{sys}(X), \text{sys}(f^j(X)))}{d_{\mathcal{T}}(X, f^j(X))} \leq \lim_{j \to \infty} \frac{Kd_{\mathcal{T}}(X, f^j(X)) + C}{d_{\mathcal{T}}(X, f^j(X))} \leq K.
\]

Since \( \kappa_g \) is the infimum of these \( K \), the lemma is proven. \( \square \)

3.3. **Invariant train tracks for pseudo-Anosov maps.** For more on train tracks, we refer the reader to [10], whose notation we adopt.

Given a pseudo-Anosov map \( f : S \to S \), let \( \tau \) denote an invariant train track. So \( \tau \) carries \( f(\tau) \), written \( f(\tau) \prec \tau \), and a carrying map sends vertices of \( f(\tau) \) to vertices of \( \tau \). Let \( P_\tau \) denote the polyhedron of measures on \( \tau \), viewed either as the space of weights on the branches \( B \) of \( \tau \) satisfying the switch conditions (a cone in \( \mathbb{R}^B_{\geq 0} \)), or a subset of the space \( \mathcal{ML}(S) \) of measured laminations on \( S \).

Although the carrying map is not unique, \( f \) induces a canonical linear inclusion \( f_* : P_\tau \subset P_\tau \). There is a unique eigenray in \( P_\tau \) spanned by the stable lamination, and the corresponding eigenvalue is the dilatation \( \lambda(f) \). In fact, this is the unique eigenray in all of \( \mathbb{R}^B_{\geq 0} \) with eigenvalue greater than one.
Theorem 3.4. If \( \tau \) is an invariant train track for a pseudo-Anosov homeomorphism \( f : S \to S \) with transition matrix \( A \), then \( \lambda(f) \) is the spectral radius of \( A \). \( \square \)

The dilatation \( \lambda(f) \) is also the spectral radius of the matrix that defines the map

\[
\mathbb{R}_{\geq 0}^B \to \mathbb{R}_{\geq 0}^B,
\]

induced by \( f \). Furthermore, given any \( f \)-invariant subspace \( V \) of \( P_\tau \), the dilatation is the spectral radius of the matrix (with respect to any basis) defining the map \( V \to V \) induced by \( f \). If the matrix is a nonnegative integral matrix \( A \), there is an associated directed graph, a **digraph**, with vertices the basis vectors, and \( A_{ij} \) edges from the \( i^{th} \) basis vector to the \( j^{th} \) basis vector.

3.5. **Basic Nesting Lemma and lower bound for asymptotic translation length.**

A maximal train track \( \tau \) is **recurrent** if there is some \( \mu \) in \( P_\tau \) that has positive weights on every branch. The set of such \( \mu \) will be denoted \( \text{int}(P_\tau) \). A maximal train track \( \tau \) is **transversely recurrent** if every branch intersects some closed curve that intersects \( \tau \) efficiently. A train track that is both recurrent and transversely recurrent is called birecurrent.

For a maximal train track \( \tau \), Masur and Minsky observed that if \( \alpha \) is a curve in \( \text{int}(P_\tau) \) and a curve \( \beta \) is disjoint from \( \alpha \), then \( \beta \) is in \( P_\tau \), see Observation 4.1 of [6]. From this they deduce the following proposition.

**Proposition 3.6.** If \( \tau \) is a maximal birecurrent invariant train track for a pseudo-Anosov \( f : S \to S \) and \( r \geq 1 \) is such that \( f^r(P_\tau) \subset \text{int}(P_\tau) \), then

\[
\ell_C(f) \geq 1/r. \quad \square
\]

We call an \( r \) satisfying the conditions of Proposition 3.6 a **mixing number** for \( f \) and \( \tau \). In the next section, we construct a family of pseudo-Anosov maps \( \{ \phi_g : S_g \to S_g \} \) and maximal birecurrent invariant train tracks \( \tau_g \) with mixing numbers \( 2g - 1 \).

4. **Lower bound on \( \kappa_g \).**

We build a family of pseudo-Anosov maps \( \{ \phi_g : S_g \to S_g \} \) for which the asymptotic translation lengths on \( \mathcal{F}(S_g) \) are on the order of \( \log g / g \) while the asymptotic translation lengths on \( \mathcal{C}^{(1)}(S_g) \) are bounded below by a linear function of \( g \). The lower bound on \( \kappa_g \) in Theorem 1.1 follows from this and Lemma 3.2. Our construction is similar to Penner’s [3], but the asymptotic behavior is different.

Let \( g \geq 4 \) and consider the genus \( g \) surface \( S = S_g \) with curves

\[
\Omega = \Omega_g = \{a_0, \ldots, a_{g-2}, b_0, \ldots, b_{g-2}, c_0, \ldots, c_{g-2}, d_0, \ldots, d_{g-2}\}
\]
as indicated in Figure 4 when \( g = 9 \). For a curve \( x \) in \( \Omega \), let \( T_x \) be the left–handed Dehn twist in \( x \). Let \( \rho = \rho_g \) be the symmetry of order \( g - 1 \) obtained by rotating \( S_g \) clockwise by \( 2\pi/(g-1) \), and let

\[
\phi = \phi_g = \rho_g \circ T_{a_0} \circ T_{b_1} \circ T_{c_0} \circ T_{d_0}^{-1}.
\]
Figure 4.1. The pseudo-Anosov \( \phi_0 \)

Observe that the only nonzero intersection numbers among curves in \( \Omega \) are
\[
i(d_j, a_j) = i(d_j, a_{j+1}) = i(d_j, b_j) = i(d_j, b_{j+1}) = 1 \quad \text{and} \quad i(d_j, c_j) = 2
\]
for \( j \in \{0, \ldots, g-2\} \), where indices are taken modulo \( g-1 \). Smoothing intersection points as indicated in Figure 4.2, we produce a maximal train track \( \tau = \tau_g \). Each of the curves in \( \Omega \) is carried by \( \tau \), proving that \( \tau \) is recurrent, and these curves are elements of \( P_{\tau} \). Moreover, each of the curves can be pushed off \( \tau \) to meet it efficiently, proving that \( \tau \) is transversely recurrent. Let \( P_{\Omega} \subset P_{\tau} \) be the subspace of measures carried by \( \tau \) that lie in the span of \( \Omega \). Because no two curves of \( \Omega \) put nonzero weights on the same set of branches, the set \( \Omega \) is a basis for \( P_{\Omega} \).

Since \( \Omega \) is \( \rho \)-invariant, we may assume that \( \tau \) is. Furthermore, one has that \( T_{a_j}(\tau), T_{b_j}(\tau), T_{c_j}(\tau), \) and \( T_{d_j}^{-1}(\tau) \) are carried by \( \tau \) for any \( j \), as in [9]. In fact, we have \( f(P_{\Omega}) \subset P_{\Omega} \) for any \( f \) in \( \{ \rho, T_{d_j}, T_{a_j}, T_{b_j}, T_{c_j} \mid 0 \leq j \leq g-1 \} \). It follows that \( \phi(P_{\Omega}) \subset P_{\Omega} \) and, as in [8], \( \phi \) is pseudo-Anosov. Let \( A \) denote the matrix for the
action of $\phi$ on $P_2$ in terms of the basis $\Omega$. This is a Perron–Frobenius matrix whose associated digraph $G_g$ is shown in Figure 4.3 in the case $g = 9$. The vertices are labeled by the corresponding elements of $\Omega$, and multiple edges are represented by an edge labeled with the multiplicity. An important feature is that $G$ has exactly one self-loop, at the vertex $a_1$.

**Figure 4.3.** The digraph $G_9$. 
First we bound the translation length on \( \mathcal{E}^{(1)}(S) \) from below.

**Proposition 4.4.** For every \( g \geq 4 \),
\[
\ell_{\mathcal{E}}(\phi_g) \geq \frac{1}{2g-1}.
\]

**Proof.** By Proposition 3.6, it is enough to show that \( r = 2g - 1 \) is a mixing number for \( \phi \) and \( \tau \). We show this in two steps.

We first show that, for any \( \mu \in P_\tau \), there is an \( s \leq g \) so that \( \phi^s(\mu) = ta_1 + \mu' \) for some \( t > 0 \) and \( \mu' \in P_\tau \). Observe that \( \mu \) has positive intersection number with some curve \( a_j \) or \( d_j \). Indeed, if we push all of the \( a_j \) and \( d_j \) off of \( \tau \) in both directions so as to meet it efficiently, then the union of these curves intersects every branch. Next, set \( s_0 = g - 1 - j \), so that \( 1 \leq s_0 \leq g - 1 \). Then \( \mu_{s_0} = \phi^{s_0}(\mu) \) has positive intersection number with either \( a_0 \) or \( d_0 \). From this we have
\[
T_{a_0}T_{d_0}^{-1}(\mu_{s_0}) = \mu_{s_0} + i(\mu_{s_0}, d_0)d_0 + i(\mu_{s_0} + i(\mu_{s_0}, d_0))a_0 \geq \mu_{s_0} + i(\mu_{s_0}, d_0)d_0 + i(\mu_{s_0}, d_0)i(\mu_{s_0}, d_0)a_0 \\
= \mu_{s_0} + i(\mu_{s_0}, d_0)d_0 + (i(\mu_{s_0}, a_0) + i(\mu_{s_0}, d_0)i(\mu_{s_0}, d_0))a_0 \\
= \mu_{s_0} + i(\mu_{s_0}, d_0)d_0 + (i(\mu_{s_0}, a_0) + i(\mu_{s_0}, d_0))a_0.
\]

Applying \( \rho T_{b_1} T_{c_0} \) to this is the same as applying \( \phi \) to \( \mu_{s_0} \) since \( T_{a_0} \) commutes with \( T_{b_1} T_{c_0} \). Therefore
\[
\phi^{s_0+1}(\mu) = \phi(\mu_{s_0}) = ta_1 + \mu'
\]
where
\[
s = s_0 + 1, \\
t = i(\mu_{s_0}, a_0) + i(\mu_{s_0}, d_0) > 0, \quad \text{and} \\
\mu' = \rho T_{b_1} T_{c_0} (\mu_{s_0} + i(\mu_{s_0}, d_0)d_0) \in P_\tau.
\]

The second step is to show that, for any \( k \geq g - 1 \), we have \( \phi^k(a_1) \in \text{int}(P_\tau) \). This follows from the fact that, for any \( k \geq g - 1 \), there is a path of length \( k \) from \( a_1 \) to any other vertex \( x \in \Omega \), see Figure 4.3.

From these two steps, we have
\[
\phi^{2g-1}(\mu) = \phi^{2g-1-s}(\phi^s(\mu)) \\
= \phi^{2g-1-s}(ta_1 + \mu') \\
= t\phi^{2g-1-s}(a_1) + \phi^{2g-1-s}(\mu').
\]

The iterate \( s \) from step one satisfies \( 2g - 1 - s \geq g - 1 \). By step two, we know that the right–hand side lies in \( \text{int}(P_\tau) \). It follows that \( \phi^{2g-1}(P_\tau) \subset \text{int}(P_\tau) \) and so \( 2g - 1 \) is a mixing number for \( \phi \) and \( \tau \).

**4.5. Bounds on dilatations.**

**Lemma 4.6.** For \( g > 4 \), the mapping classes \( \phi_g \) satisfy
\[
\frac{\log(4g - 4)}{2g - 2} \leq \log(\lambda(\phi_g)) \leq \frac{\log(10g - 21)}{g - 2}.
\]
Proof. The lower bound holds for any Perron–Frobenius digraph with a self–loop, thanks to work of Tsai (Proposition 2.4 of [11]), and so we prove only the upper bound.

For any \( j \leq g - 2 \), inspection reveals that the number of directed edge–paths in \( G_g \) of length \( j \) emanating from each of 
\[
 a_0, a_1, b_0, b_1, c_0, d_{g-2}, \text{ and } d_0
\]
to be
\[
(10j - 6), 5j, (10j - 1), 5j, (10j - 6), (10j - 11), \text{ and } (5j - 1),
\]
respectively—see Figure 4.3. For any other vertex \( v \) of \( G_g \), there is a unique edge–path starting at \( v \) and ending at one of the vertices listed above, and every shorter edge–path is an initial segment of this one. It follows that the number of edge–paths of length \( g - 2 \) starting at any vertex is maximized at one of the vertices listed above, and is hence at most \( 10g - 21 \).

Let \( A_g \) be the incidence matrix of \( G_g \). The maximum row sum of \( A_g^{g-2} \) is precisely the maximum number of edge–paths starting at any vertex, and is hence at most \( 10g - 21 \). Applying this to \( A_g^{g-2} \) we have
\[
\log(\lambda(\phi_g)) = \frac{\log(\lambda(\phi_g^{g-2}))}{g-2} = \frac{\log(10g - 21)}{g-2} \leq \log\left(\frac{10g - 21}{g-2}\right).
\]

Alternatively, one may calculate the characteristic polynomial \( P_{G_g}(x) \) of \( G_g \) by observing that the mapping classes \( \phi_g \) are the monodromies of fibrations of a single 3–manifold. In fact, all of the fibers lie in a single cone on a fibered face of the Thurston norm ball, and one can use the Teichmüller polynomial to calculate the \( P_{G_g}(x) \) by specializing a single polynomial. See [7]. The polynomial is
\[
P_{G_g} = x^{4g-4} - x^{4g-5} - x^{2g-1} - 10x^{2g-2} - x^{2g-3} - x + 1,
\]
and one may estimate \( \lambda(\phi_g) \) by noting that it equals the maximum modulus of the roots of \( P_{G_g} \), which is estimable due to the special form of \( P_{G_g} \). Though more involved, this argument yields the better upper bound of
\[
\log(\lambda(\phi_g)) \leq \frac{3\log(4g - 4)}{(4g - 4)}.
\]

4.7. The main theorem. We can now assemble the proof of the main theorem.

Proof of Theorem 1.1. Proposition 2.4 implies that
\[
\kappa_g \leq \frac{2}{\log(g - \frac{1}{2})} \approx \frac{1}{\log(g)}.
\]

Lemma 3.2 applied to the sequence \( \phi_g : S_g \to S_g \) above, together with Proposition 4.4 and the upper bound in Lemma 4.6 implies
\[
\kappa_g \geq \frac{\ell(\phi_g)}{\log(\lambda(\phi_g))} \geq \frac{1}{\log(10g - 21)/(g-2)} \times \frac{1}{\log(g)}. \quad \square
\]
4.8. **Extremal length.** Masur and Minsky [6] use extremal length rather than hyperbolic length to define the map \( \mathcal{T}(S) \to \mathcal{C}^{(1)}(S) \). Recall that the extremal length of a curve \( \alpha \) with respect to \( X \) in \( \mathcal{T}(S) \) is \( \text{Ext}_X(\alpha) = 1/\text{mod}_X(\alpha) \), where \( \text{mod}_X(\alpha) \) is the supremum of conformal moduli for embedded annuli with core curves homotopic to \( \alpha \). The set of curves with smallest extremal length,

\[ \text{sys}_{\text{Ext}}(X) = \{ \alpha \in \mathcal{C}^{(1)}(S) \mid \text{Ext}_X(\alpha) \leq \text{Ext}_X(\beta) \text{ for all } \beta \in \mathcal{C}^{(0)}(S) \} \]

is finite. As with hyperbolic length, the set \( \text{sys}_{\text{Ext}}(X) \) has diameter bounded above by a constant \( c = c(S) \) (Lemma 2.4 of [6]), and again we view \( \text{sys}_{\text{Ext}} \) as a map \( \mathcal{T}(S) \to \mathcal{C}^{(1)}(S) \). This map is also coarsely Lipschitz, and we let \( \kappa^\text{Ext}_g \) denote the optimal Lipschitz constant for \( \text{sys}_{\text{Ext}} : \mathcal{T}(S_g) \to \mathcal{C}^{(1)}(S_g) \).

**Proposition 4.9.** We have \( \kappa_g = \kappa^\text{Ext}_g \) for all \( g \). In particular, \( \kappa^\text{Ext}_g \approx \frac{1}{\log(g)} \).

**Proof:** Suppose \( \alpha \) in \( \text{sys}(X) \). The collar neighborhood of width \( \ell_X(\alpha)/2 \) from Lemma 2.1 provides a conformal annulus of definite modulus (depending on \( \ell_X(\alpha) \)), and hence \( \text{Ext}_X(\alpha) < L' \) for some \( L' = L'(S) \). Now let \( \beta \) lie in \( \text{sys}_{\text{Ext}}(X) \), so that \( \text{Ext}_X(\beta) \leq L' \). By Lemma 2.5 of [6], \( d(\alpha, \beta) \leq 2L' + 1 \). From this we deduce

\[ |\text{sys}(X) - \text{sys}_{\text{Ext}}(X)| < 2L' + 1. \]

Therefore, if one of \( \text{sys} \) or \( \text{sys}_{\text{Ext}} \) is \( (K,C) \)-coarsely Lipschitz, then, by the triangle inequality, the other is \( (K,C + 2(2L' + 1)) \)-coarsely Lipschitz. The proposition follows. \( \square \)

**References**

[1] Aougab, T. *Uniform hyperbolicity of the graph of curves*, arXiv:1212.3160.
[2] Bers, L. *An extremal problem for quasiconformal mappings and a Theorem of Thurston*, Acta Math. 141 (1978), 73–98.
[3] Bowditch, B. *Tight geodesics in the curve complex*, Invent. Math. 171, (2008) 2, 281–300.
[4] Bowditch, B. *Uniform hyperbolicity of the curve graphs*, preprint, 2012.
[5] M. T. Clay, K. Rafi, S. Schleimer, *Uniform hyperbolicity and geometric topology*, in preparation.
[6] Masur, H. and Minsky, Y. *Geometry of the complex of curves I: Hyperbolicity*. Invent. Math. 138, 103–149 (1999).
[7] McMullen, C. T. *Polynomial invariants for fibered 3–manifolds and Teichmüller geodesics for foliations*, Ann. Sci. École Norm. Sup. (4), 33 (4): 519–560, 2000.
[8] Penner, R. *Bounds on least dilatations*, Proc. Amer. Math. Soc. 113 (1991) 2, 443-450.
[9] Penner, R. *A construction of pseudo-Anosov homeomorphisms*. Trans. Amer. Math. Soc. 31 (1988), 170–197.
[10] Penner, R. and Harer, J. *Combinatorics of train tracks*. Annals of Mathematics Studies, 125. Princeton University Press, Princeton, NJ, 1992.
[11] Tsai, C-Y. *The asymptotic behavior of least pseudo-Anosov dilatations*. Geom. Topol. 13 (2009), 4, 2253–2278.
[12] S. Wolpert, *The length spectra as moduli for compact Riemann surfaces*, Ann. of Math. 109 (1979), 323–351.

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