Properties of Hubbard models with degenerate localised single-particle eigenstates

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April 20, 2012

Abstract

We consider the repulsive Hubbard model on a class of lattices or graphs for which there is a large degeneracy of the single-particle ground states and where the projector onto the space of single-particle ground states is highly reducible. This means that one can find a basis in the space of the single-particle ground states such that the support of each single-particle ground state belongs to some small cluster and these clusters do not overlap. We show how such lattices can be constructed in arbitrary dimensions. We construct all multi-particle ground states of these models for electron numbers not larger than the number of localised single-particle eigenstates. We derive some of the ground state properties, esp. the residual entropy, i.e. the finite entropy density at zero temperature.

1 Introduction

The physics of strongly-correlated Fermi systems is one of the most exciting branches of condensed matter theory. The most extreme case of strong correlations occurs in systems where the single-particle problem has a large degeneracy. Such systems with flat bands have been studied intensively during the last 20 years. One of the more prominent examples is the Hubbard model with a lowest flat band. The first examples were discussed by Tasaki and by the present author 20 years ago [1, 2, 3]. The aim of the authors was a rigorous, complete classification of the ground states. Among other results, it was shown that under certain conditions the Hubbard model has a unique ferromagnetic ground state. The final goal was the proof of the existence of metallic ferromagnetism in these and similar models.

If a translationally-invariant system has a flat band, not only the Bloch states but also the Wannier states are eigenstates of the single-particle Hamiltonian. This means that there are localised single-particle eigenstates. Some of the physical properties of these systems can be related to the existence of localised eigenstates. Most of the proofs used to derive exact results for ferromagnetism in Hubbard models with flat bands rely on the fact that such localised basis states exist. On the other hand, the existence of metallic ferromagnetism, which was always one of the goals, is not related to localised eigenstates. One class of models where one could have metallic ferromagnetism is constructed from the flat band models with additional hopping terms to lift the degeneracy of the degenerate band [4, 5]. Another class of models have partially flat bands and thus the localised Wannier states are not eigenstates [6, 7].

In the models mentioned so far the localised states, when they exist, overlap. To be precise, the representation of the projector onto the subspace of all degenerate single-particle ground states in position space, the single-particle density matrix, is irreducible. This was essential for the proofs in [3, 7, 6]. In [7] it was shown that if
and only if the single-particle density matrix is irreducible, the multi-particle ground state at a special density, where each of the single-particle eigenstates is filled with exactly one electron, is ferromagnetic and unique up to the usual \((2S+1)\) fold spin degeneracy.

Other models which recently caught some interest have strictly localised eigenstates in the sense that different localised single-particle eigenstates do not overlap, see e.g. Batista and Shastry [8], Dzerkho et al. [9], Maksymenko et al. [10], some of the examples by Schmidt et al. [11], and the references therein. These authors consider examples of decorated lattices in one or two dimensions. In these systems, the physical properties are strongly influenced by the existence of localised eigenstates. The system has a high degeneracy for the multi-particle ground states. The entropy density at zero temperature is finite. Since the single-particle eigenstates are localised the system is likely to be non-metallic. In some cases the ground states are Wigner crystals. The system is paramagnetic, not ferromagnetic.

The goal of the present paper is to provide a complete description of the class of lattices with degenerate single-particle eigenstates which fall into non-overlapping subsets. This is done using the projector onto the subspace of degenerate single-particle ground states in position space, the single-particle density matrix. The class of lattices with single-particle eigenstates which fall into non-overlapping subsets has a reducible single-particle density matrix, in contrast to the models discussed in [7, 6]. We discuss some of the properties of the Hubbard model on such lattices and we give a large class of examples of such lattices which can be constructed explicitly. The examples in [8, 9, 10] belong to this class as well. The construction is possible in arbitrary dimensions. We construct explicitly all ground states of these models for sufficiently low densities of states – the flat band must be at most half filled – and we calculate the entropy density at zero temperature. We prove that there is no long-range order in these models.

The paper is organised as follows: In Sect. 2 we define the class of lattices we are looking at. They are defined by some properties of the projector onto the single-particle ground states of the system. We give a general description how examples of such lattices can be constructed explicitly in arbitrary dimensions. In the Sect. 3 we state and proof our main result concerning the multi-particle ground states in such models. Sect. 4 contains a summary and an outlook.

## 2 Classification of the single-particle problem

In this paper we consider a general fermionic Hubbard model

\[
H = H_{\text{hop}} + H_{\text{int}}
\]

where

\[
H_{\text{hop}} = \sum_{\langle x,y \rangle \in E, \sigma} t_{xy} c_{x\sigma}^\dagger c_{y\sigma}
\]

and

\[
H_{\text{int}} = \sum_{x \in V} U_x n_{x\uparrow} n_{x\downarrow}
\]

on a lattice or, more generally, on a connected graph \(G = (V,E)\) with a set of vertices \(V\) and edges \(E\) connecting the vertices. \(t_{xy}\) are the hopping matrix elements, \(U_x > 0\) is the local repulsive interaction. Two vertices \(x\) and \(y\) are connected by an edge \(e = \{x,y\}\) if and only if \(t_{xy} \neq 0\). In this section, we consider first the single-particle problem in order to define the class of models (or lattices) we are dealing with.

We consider the case where \(H_{\text{hop}}\) has a highly degenerate single-particle ground state with eigenenergy \(\varepsilon_d\). The degeneracy is \(N_d\). \(G\) does not need to be translationally invariant. In the case of a translationally-invariant lattice, we assume that the system has at least one degenerate energy band at the bottom of the spectrum.
Let $B = \{ \psi_i(x), i = 1 \ldots N_d \}$ be an arbitrary orthonormal basis in the subspace of the degenerate lowest eigenstates of the matrix $T = (t_{xy})_{x,y \in V}$. We assume that $t_{xy}$ are real, however a generalisation to complex $t_{xy}$ is straightforward. Complex $t_{xy}$ have also been discussed in the context of flat bands, see e.g. [12]. For real $t_{xy}$ we choose the basis $B$ to be real as well. The single-particle density matrix of these states is

$$\rho_{xy} = \sum_{i=1}^{N_d} \psi_i(x) \psi_i(y).$$

(4)

$\rho = (\rho_{xy})_{x,y \in V}$ is the projector onto the space spanned by the single-particle ground states in position space. We showed that if $\rho = (\rho_{xy})_{x,y \in V}$ is irreducible, the Hamiltonian has ferromagnetic multi-particle ground states and that special particle numbers $N_e = N_d$, the ferromagnetic ground state is unique up to the degeneracy due to the $SU(2)$ spin symmetry [7].

In this paper, we consider the case where the single-particle density matrix $\rho$ is highly reducible. $\rho$ should have the following properties:

1. $\rho$ is reducible. It can be decomposed into $N_r$ irreducible blocks $\rho_k$, $k = 1, \ldots, N_r$. $N_r$ should be an extensive quantity, i.e. $N_r \propto N_d \propto |V|$ so that in the thermodynamic limit the density of degenerate single-particle ground states and the density of irreducible blocks are both finite.

2. Let $V_k$ be the support of $\rho_k$, i.e. the set of vertices for which at least one element of $\rho_k$ does not vanish. $\rho_{k,xy} = 0$ if $x \notin V_k$ or $y \notin V_k$. One has $V_k \cap V_{k'} = \emptyset$ if $k \neq k'$ because of the fact that $\rho_k$ are irreducible blocks of the reducible matrix $\rho$ and $\bigcup_k V_k \subseteq V$.

3. We choose the $B$ such that the support of each basis states $\psi_i(x)$ is a subset of exactly one $V_k$. We denote the number of states belonging to the cluster $V_k$ as $v_k$. One has $\sum_k v_k = N_d$.

4. $v_{\text{max}} = \max_k \{v_k\}$ is $O(1)$, i.e. not an extensive quantity.

If $G$ represents a translationally-invariant lattice, only one or a few blocks belong to one elementary cell and the $v_k$ belong to classes where, within one class, all $v_k$ are the same due to translational invariance.

On each block, since $\rho_k$ is irreducible, the results obtained in [7] apply.

### 2.1 Lattices with such properties

The lattices in [8, 9, 10] have the properties mentioned above. We now give a more general construction for a large class of lattices in arbitrary dimensions which have these properties. Let us mention that these are only examples and that many other lattices with reducible $\rho_{xy}$ exist.

Our starting point for the construction of a large class of lattices or graphs with these properties is an arbitrary lattice or graph $G = (V, E)$. $V$ is the set of vertices of $G$, $E$ is the set of edges of $G$. We consider only simple graphs, i.e. each edge is a set of exactly two vertices. If an edge $\{x, y\} \in E$ exists, the two vertices $x$ and $y$ are connected. For our construction, we decompose the vertex set $V$ into two disjoint subsets $V_1$ and $V_2$. As a special case, $V_2$ may be empty. To each vertex $x \in V_1$ we associate a complete graph $K_n$ with $n$ vertices, $n \geq 2$. A complete graph is a graph where each vertex is connected with each other vertex. $K_1$ is an edge, $K_3$ is a triangle, $K_4$ is a tetrahedron. These complete graphs form building blocks of the new lattice. We denote these subgraphs as $K_n(x)$. For a discussion of the Hubbard model on the complete graph the reader is referred to [13] and the references therein.

We now construct the graph $G = (V, E)$ as follows: the vertex set $V$ is $V = V_1 \cup V_2$ where $V_2 = \tilde{V}_2$ and

$$V_1 = \bigcup_{x \in V_1} V(K_n(x))$$

(5)
where $V(K_n(x))$ is the vertex set of the complete graph $K_n(x)$ and $E(K_n(x))$ is its edge set. The edge set $E$ is constructed as follows:

$$E = E_1 \cup E_2 \cup E_3 \cup E_4$$  

$$E_1 = \bigcup_{x \in \tilde{V}_1} E(K_n(x)) \quad (7)$$

$$E_2 = \{\{x,y\} \in \tilde{E} : x,y \in V_2\} \quad (8)$$

$$E_3 = \bigcup_{x \in \tilde{V}_1, y \in V_2, \{x,y\} \in \tilde{E}} \{\{z,y\} : z \in V(K_n(x))\} \quad (9)$$

$$E_4 = \bigcup_{x \in \tilde{V}_1, x' \in V_1, \{x,x'\} \in \tilde{E}} \{\{z,z'\} : z \in V(K_n(x')), z' \in V(K_n(x'))\}. \quad (10)$$

In words: if $x,x' \in \tilde{V}_1$ are nearest neighbours in $\tilde{G}$, we connect each vertex of $K_n(x)$ with each vertex of $K_n(x')$. If $x \in \tilde{V}_1, y \in V_2$ are nearest neighbours in $\tilde{G}$, we connect each vertex of $K_n(x)$ with $y$.

We associate with each edge in $E_i$ a hopping matrix element $t_i \geq 0$ and denote this new hopping matrix by $T$. We add to $T$ on-site energies $\bar{\epsilon}_x$ for the vertices in $V_2$. The eigenstates and the spectrum of this single-particle hopping matrix can be constructed from the eigenstates and the spectrum of the adjacency matrix $A(\tilde{G})$ of the original graph $\tilde{G}$, $A(\tilde{G}) = (a_{xy})_{x,y \in V(\tilde{G})}$ where $a_{xy} = 1$ if $\{x,y\} \in \tilde{E}$, 0 otherwise. To show this, we write

$$A(\tilde{G}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (11)$$

where $A_{ij}$ contains the matrix elements of $A(\tilde{G})$ connecting vertices of $\tilde{V}_i$ to $\tilde{V}_j$. We introduce the identity matrices $E_i = (\delta_{z,z'})_{z,z' \in \tilde{V}_i}$ on $V_i$ and the matrix $B = (b_{zx})_{z \in \tilde{V}_1, x \in \tilde{V}_1}$, $b_{zx} = 1$ if $z \in K_n(x)$ and $b_{zx} = 0$ otherwise. $B$ maps the vertices in $\tilde{V}_1$ to the vertices in $\tilde{V}_1$. Then, the new hopping matrix $T$ can be written as

$$T = \begin{pmatrix} t_1(BB' - E_1) + t_4BA_{11}B' & t_3BA_{12} \\ t_5A_{21}B' & t_2A_{22} + t_2E_2 \end{pmatrix} \quad (12)$$

Note that $B'B = nE_1$, where $E_1$ is the identity matrix on $V_1$. We now construct all eigenstates of $T$. One class of eigenstates of $T$ are

$$\psi = \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix}, \quad \psi_0 \in \text{kernel}(B') \quad (13)$$

with the eigenvalue $-t_1$. A basis in this eigenspace is formed by all eigenstates with eigenvalue -1 of $K_n(x)$ for all $x \in \tilde{V}_1$. $K_n$ has one eigenvalue $n - 1$ belonging to the eigenstate $\phi_0 = (1,1,\ldots,1)^T$ and $n - 1$ eigenvalues -1 belonging to eigenstates orthogonal to $\phi_0$, these are elements of the kernel of $B'$. The second class of eigenstates of $T$ are of the form

$$\psi = \begin{pmatrix} aB\psi_1 \\ \psi_2 \end{pmatrix} \quad \text{where} \quad A(\tilde{G}) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \tilde{a} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (14)$$

We obtain

$$T\psi = \begin{pmatrix} t_1(n - 1)aB\psi_1 + t_4anBA_{11}\psi_1 + t_3BA_{12}\psi_2 \\ t_5A_{21}an\psi_1 + t_2A_{22}\psi_2 + t_2\psi_2 \end{pmatrix} \quad (15)$$

$$4$$
where we made use of $BB = nE_1$. We now let $a = n^{-1/2}$, $t_3 = t_4/a$, $t_2 = t_3/a$, and $\bar{t}_2 = t_1 (n - 1)$. Then, we obtain $T \psi = [t_1 (n - 1) + t_3 a] \psi$. We choose $t_1$ and $t_4$ so that $t_1 n + t_4 a_{\text{min}} > 0$ where $a_{\text{min}}$ is the lowest eigenvalue of $A(\bar{G})$. Since $a_{\text{min}} \geq -d_{\text{max}}(\bar{G})$, which is the maximal degree of $\bar{G}$, it is sufficient to choose $t_4 < t_1 n / d_{\text{max}}(\bar{G})$. Then the ground state of $T$ has a lowest eigenvalue $-t_1$ with degeneracy $N_d = (n - 1) |V_1|$ and the ground states are given by $|\psi\rangle$. By construction, there are $|V|$ eigenstates with eigenvalues above $-t_1$ and since $N_d + |\bar{V}| = |V|$ the construction yields all eigenstates of $T$.

This construction is valid even if $V_2$ is empty.

Let us remark that if $\bar{G}$ is a translationally-invariant lattice with $r$ energy bands and $G$ is translationally invariant as well, then $G$ has $r + n - 1$ energy bands and the $n - 1$ additional energy bands are flat and degenerate with eigenenergy $-t_1$.

This construction yields a system in which the matrix $\rho$ decomposes into $|\bar{V}|$ blocks, one for each $x \in \bar{V}_1$. The $v_x$ are all the same and their value is $n - 1$.

The construction can be generalized, we may choose different values of $n$ for different $x \in \bar{V}$.

3 Ground state properties

We are now ready to state our main results for the Hubbard model with a lowest single-particle eigenenergy $\varepsilon_d$ which is $N_d$-fold degenerate and for which the projector onto the eigenspace of $\varepsilon_d$ fulfills the properties of Sect. 2.

Theorem. For Hubbard models with a lowest single-particle eigenenergy $\varepsilon_d$ which is $N_d$-fold degenerate and for which the projector onto the eigenspace of $\varepsilon_d$ fulfills the properties of Sect. 2 the following results hold for $N_e \leq N_d$:

1. The ground state energy is $\varepsilon_d N_e$.
2. Let $A_x$ be an arbitrary local operator, i.e. an arbitrary combination of the four creation and annihilation operators $c_{x,\sigma}^\dagger$ and $c_{x,\sigma}$. The correlation function $\rho_{A_x A_y} = \langle A_x A_y \rangle - \langle A_x \rangle \langle A_y \rangle$ has a finite support for any fixed $x$ and vanishes if $x$ and $y$ are out of different clusters $V_k$. The system has no long-range order.
3. The system is paramagnetic.
4. The entropy at zero temperature $S(c)$ is an extensive quantity, $S(c) = O(N_e)$. It increases as a function of $c = N_e / N_d$ from 0 for $c = 0$ to some maximal value $S_{\text{max}} = \sum_k [(v_k - 1) \ln 2 + \ln (v_k + 2)]$ and then decays to $S(1) = \sum_k \ln (v_k + 1)$.

Proof. We first construct a suitable basis. The different clusters $V_k$ are completely decoupled as long as ground states are considered. We therefore discuss first a single cluster. According to [7], it is possible to choose a single-particle basis $B_k$ (not orthonormal) for the space of degenerate single-particle ground states on the cluster $V_k$ with the following properties:

1. For each basis state $\psi_i$ there exists a vertex set $V_k(i)$ so that the support of $\psi_i$ is a subset of $V_k(i)$.
2. For each basis state $\psi_i$ there exists a unique $x_i \in V_k(i)$ such that $\psi_i(x_i) > 0$ and $\psi_i(x_j) = 0$ for all $i \neq j$, $j = 1, \ldots, |B_k|$. The set $V_k(i) \setminus \{x_i, i = 1, \ldots, |B_k|\}$ is not empty.
Since the set \( V_k \backslash \{ x_i, i = 1, \ldots, |B_k| \} \) is not empty, these states overlap and are not orthogonal. Let \( V_k = |B_k| \).

Using such a basis \( B_k \), we can put electrons into the different basis states with the condition that one single-particle basis state contains at most one electron. In a state with two electrons in a basis state \( \psi \), we have a double occupancy on \( x_i \). Since all other basis states vanish on \( x_i \), it is not possible to get rid of that double occupancy due to some linear combinations of these states. Therefore, any state with a doubly-occupied \( \psi \) has a non-vanishing interaction energy and cannot be a ground state. Note that the absence of doubly occupied \( \psi \) is a necessary but not a sufficient condition for a ground state, since it does not exclude double occupancies on the lattices sites in \( V_k \backslash \{ x_i, i = 1, \ldots, |B_k| \} \). This construction was used in [7] to show that if there are \( V_k \) electrons on the cluster, they all have the same spin \( S_k = V_k / 2 \) and that the degeneracy of the ground state on the cluster is \( 2S_k + 1 = V_k + 1 \). The trivial case is one electron on the cluster, where the degeneracy is \( 2V_k \). For electron numbers \( n_k \) with \( 1 < n_k < V_k \) it may be difficult to calculate the ground state degeneracies. But for \( V_k \leq 2 \) we have a complete description of all ground states in the cluster \( V_k \).

It is trivial to generalise this argument to the entire lattice using the basis \( \bigcup_k B_k \). The states in the different \( B_k \) can be filled independently. The lattices studied in [8] [9] [10] all belong to the class with \( V_k \leq 2 \). In [10], it was assumed that the construction above yields all of the ground states and numerical results were presented to confirm that. Our argument is a rigorous proof of this statement.

We now come to the four statements in the theorem. The first point is trivial. The ground state energy is \( \epsilon_{d} N_e \), since states with that energy minimise both the kinetic energy and the interaction. The ground states have no doubly-occupied sites.

For the proof of the next statements, we use a grand-canonical formulation. Let

\[
Z(z, \{ a_x, x \in V \}) = \langle z^{N_e} \exp(\sum_x a_x A_x) \rangle
\]

be the generating function for correlation functions containing the operators \( A_x \). \( N_e \) is the number operator. \( \langle \cdot \rangle \) denotes the ground state expectation value for arbitrary electron numbers \( \leq N_d \). Since the system decomposes into clusters, the generating function can be written as

\[
Z(z, \{ a_x, x \in V \}) = \prod_k Z(z, \{ a_x, x \in V_k \}, \rho_k)
\]

where

\[
Z(z, \{ a_x, x \in V_k \}, \rho_k) = \langle z^{N_e} \exp(\sum_x a_x A_x) \rangle_k
\]

\( \langle \cdot \rangle_k \) denotes the ground state expectation value on the cluster \( V_k \). We have

\[
\langle A_x \rangle = \frac{\partial}{\partial a_x} \ln Z(z, \{ a_x, x \in V \})|_{\alpha_x = 0 \forall x}
\]

\[
\langle A_x A_y \rangle = \frac{\partial^2}{\partial a_x \partial a_y} \ln Z(z, \{ a_x, x \in V \})|_{\alpha_x = 0 \forall x}
\]

Since \( \ln Z(z, \{ a_x, x \in V \}) = \sum_k \ln Z(z, \{ a_x, x \in V_k \}, \rho_k) \) one has \( \langle A_x A_y \rangle = \langle A_x \rangle \langle A_y \rangle \) if \( x \) and \( y \) are out of different clusters \( V_k \). Thus, \( \rho_{A_x A_y} \) vanishes if \( x \) and \( y \) are out of different clusters. Since this statement holds for any \( z \), it holds as well for a fixed particle number \( N_e \).

The third point follows from the second if we take for \( A_x \) the local spin-operators. To be more explicit, let us calculate the expectation value of the total spin. It can be written as \( \langle \hat{S}^2 \rangle = \sum_k \langle \hat{S}_k^2 \rangle \) where \( \hat{S}_k = \sum_{x \in V_k} \hat{S}_x \) is the spin operator on the cluster \( k \). A trivial upper bound for \( \langle \hat{S}_k^2 \rangle \) is \( \frac{3}{2} V_k (V_k + 2) \). A trivial lower bound for \( \langle \hat{S}_k^2 \rangle \) is \( \frac{3}{8} N_k \), where \( N_k \) is the number of electrons on the cluster \( V_k \). Therefore, we obtain \( \frac{3}{4} N_e \leq \langle \hat{S}^2 \rangle = \sum_k \langle \hat{S}_k^2 \rangle \)
The grand canonical partition function is a product of the partition functions of these clusters. To calculate the entropy at zero temperature, let us now calculate the grand canonical partition function. Since the lattice decomposes into finite clusters, so that the boundary has no effect on the result.

We come now to the fourth point of the theorem. We will calculate the entropy density by calculating the ground state degeneracy for all possible boundary conditions. This is possible in our case since the problem decomposes into a set of clusters.

We have proven by Aizenman and Lieb [14]. They pointed out that there is a problem when interchanging the state degeneracy of the system. The fact that this can be done for lattice models with finite range interactions has been proven by Aizenman and Lieb [14]. The lower limit for $p_j^\nu$ is the number of fully-polarised states with $j$ electrons on a cluster with $\nu$ states.

From $Z$ we obtain the grand canonical potential $\Omega$. The general from of $\Omega$ is

$$\Omega = -\beta^{-1} \sum_k \ln Z(z, \rho_k).$$

The entropy is

$$S(z) = -\frac{\partial \Omega}{\partial T} = \sum_k \ln Z(z, \rho_k) - z \ln z \sum_k \frac{d}{dz} \ln Z(z, \rho_k).$$

Let us introduce $c(z) = N_c(z)/N_r$. Since $N_c(z) = \frac{\partial \Omega}{\partial \mu}$ we obtain $c(z) = \frac{1}{N_r} \sum_k N_k(z)$ where

$$N_k(z) = \frac{z}{Z(z, \rho_k)} \frac{\partial Z(z, \rho_k)}{\partial z}$$

is the number of particles on the cluster $V_k$. One has $S(z) = \sum_k S_k(z)$, $S_k(z) = \ln Z(z, \rho_k) - N_k(z) \ln z$.

c(z) is a strongly monotonically increasing function of $z$. For the derivative of the entropy, we obtain

$$\frac{dS}{dz} = -\ln z \left[ \sum_k \frac{d \ln Z(z, \rho_k)}{dz} + z \sum_k \frac{d^2 \ln Z(z, \rho_k)}{dz^2} \right]$$

$$= -\ln z \left[ 1 + z \frac{d}{dz} \right] \frac{1}{z} \sum_k N_k(z)$$

$$= -N_r \ln z \frac{dc}{dz}.$$
Since \( \frac{dS}{dz} > 0 \), the only maximum of \( S(z) \) occurs at \( z = 1 \). The value is \( S(z = 1) = \sum_k \ln Z(1, \rho_k) \). For small values of \( z \) we have

\[
Z(z, \rho_k) = 1 + 2\nu_k z + O(z^2)
\]

and therefore

\[
S(z) = 2z(1 - \ln z)N_d(1 + O(z))
\]

In the limit \( z \to \infty \), at the maximal density, the degeneracy in the cluster \( \rho_k \) is \( \nu_k + 1 \). The total degeneracy is \( \Pi_k (\nu_k + 1) \) and the entropy is \( S(z \to \infty) = \sum_k \ln (\nu_k + 1) \). Therefore, the entropy increases monotonically to its maximum at \( z = 1 \), \( S(z = 1) = \sum_k \ln Z(1, \rho_k) \) and then decays monotonically to \( S(z \to \infty) = \sum_k \ln (\nu_k + 1) \). Using \( \rho_j^\nu \geq (j + 1) \binom{\nu}{j} \) we obtain the lower bound for \( S(z = 1) \) in point 4 of the theorem. Since \( c(z) \) is strongly monotonically increasing, these properties hold for \( S(c) \) as well.

### 3.1 Examples

For clusters with \( \nu_k = 1 \) one has

\[
Z(z, \rho_k) = 1 + 2z.
\]

For \( \nu_k = 2 \) one obtains

\[
Z(z, \rho_k) = 1 + 4z + 3z^2
\]

since there is only one state on the cluster \( \nu_k \) with no particles, four states with one particle and three states with two particles. For \( \nu_k = 3 \) one has \( 2\nu_k = 6 \) states with one electron and \( \nu_k + 1 = 4 \) states with three electrons. Two electrons on the cluster can form a triplet state or a singlet state. One gets \( 3\nu_k = 9 \) triplet states and between 0 and 2 singlet states. In the basis \( B_k \) none of the basis states can be doubly occupied. Therefore one has only three different pairs which could form a singlet. But, since \( \rho_k \) is irreducible, at most two different pairs without a doubly occupied site can be constructed. Therefore, for \( \nu_k = 3 \) we obtain

\[
Z(z, \rho_k) = 1 + 6z + (9 + s_k)z^2 + 4z^3
\]

where \( s_k \) is the number of possible singlets on the cluster \( \rho_k \), which can be 0, 1, or 2. If \( \nu_k \leq 3 \) for all \( k \), the total partition function is thus

\[
Z(z) = (1 + 2z)^{N_{1,0}}(1 + 4z + 3z^2)^{N_{2,0}}
\]

\[
(1 + 6z + 9z^2 + 4z^3)^{N_{3,0}}
\]

\[
(1 + 6z + 10z + 4z^3)^{N_{3,1}}
\]

\[
(1 + 6z + 11z^2 + 4z^3)^{N_{3,2}}
\]

where \( N_{\nu,s} \) is the number of clusters with \( \nu_k = \nu \) and \( s \) possible singlet states. From (32) one obtains the grand canonical potential

\[
\frac{\Omega}{N_r} = -\beta^{-1}(n_{1,0} \ln(1 + 2z) + n_{2,0} \ln(1 + 4z + 3z^2))
\]

\[
+ n_{3,0} \ln(1 + 6z + 9z^2 + 4z^3)
\]

\[
+ n_{3,1} \ln(1 + 6z + 10z^2 + 4z^3)
\]

\[
+ n_{3,2} \ln(1 + 6z + 11z^2 + 4z^3)
\]
where \( n_{v,s} = N_{v,s}/N_r \), \( n_{1,0} + n_{2,0} + n_{3,0} + n_{3,1} + n_{3,2} = 1 \).

Let us mention that all of the one-dimensional lattices treated in [10] belong to this class with \( n_{2,0} = 1 \), i.e. \( N_{1,0} = 0 \) and \( N_{3,s} = 0 \) for \( s = 0, 1, 2 \).

Inverting \( c(z) \) we obtain \( z \) as a function of \( c \) and then \( S(c) \). There are two cases where \( c(z) \) can be inverted easily, the case where all \( v_k = 1 \) and the case where all \( v_k = 2 \). In the first case one obtains a linear equation for \( z(c) \), in the second case a quadratic. The one-dimensional models in [10] are a special case for \( v_k = 2 \). The result given there is valid for any model where all clusters have \( v_k = 2 \). For the case \( v_k \leq 3 \), ([33], one obtains

\[
c = \frac{2z}{1+2z} + \frac{4z + 6z^2}{1 + 4z + 3z^2} + \frac{n_{3,0}}{1 + 6z + 9z^2 + 4z^3} + \frac{n_{3,1}}{1 + 6z + 10z^2 + 4z^3} + \frac{n_{3,2}}{1 + 6z + 12z^2 + 4z^3}.
\]

One has \( c \leq n_1 + 2n_2 + 3(n_{3,0} + n_{3,1} + n_{3,2}) = 1 + n_2 + 2(n_{3,0} + n_{3,1} + n_{3,2}) \), where the upper limit is reached for \( z \to \infty \). In this limit, the entropy takes the value

\[
S = N_r(n_{1,0} + 2n_{2,0} + 3(n_{3,0} + n_{3,1} + n_{3,2})).
\]

The maximum

\[
S = N_r(n_{1,0} \ln 3 + n_{2,0} \ln 8 + n_{3,0} \ln 20 + n_{3,1} \ln 21 + n_{3,2} \ln 22)
\]

of the entropy occurs at

\[
c = \frac{2}{3} n_{1,0} + \frac{5}{4} n_{2,0} + \frac{9}{5} n_{3,0} + \frac{38}{21} n_{3,1} + \frac{5}{4} n_{3,2}.
\]

For the examples constructed in Sect. 2 the construction of this basis is easy. We simply use the states with the properties \( \psi_i(i) = 1, \psi_i(n) = -1, \psi_j(j) = 0 \) for \( j \neq i, n, i = 1, \ldots, n - 1 \). The partition function is

\[
Z(z) = \prod_{x \in V_i} Z(z, K_n(x)) = \prod_{x \in V_n} Z(z, K_n)^{N_n}.
\]

It is sufficient to consider a single \( K_n \). Let \( V(K_n) = \{1, \ldots, n\} \). The cases \( n \leq 3 \) correspond to the cases \( v_k \leq 2 \) already discussed above. \( K_4 \) has \( v_k = 3 \). For \( K_4 \) it is possible to construct two pairs of non-overlapping single-particle ground states. Using the basis introduced above the two pairs are \( \{\psi_1, \psi_2 - \psi_3\} \) and \( \{\psi_1 - \psi_2, \psi_3\} \).

Therefore, we have \( N_{3,0} = N_{3,1} = 0 \) in that case, and \( Z(z, K_4) = 1 + 6z + 11z^2 + 4z^3 \). For larger values of \( n \) the number of non-trivial cases increases rapidly. In principle it is possible to completely describe the multi-particle states as well. We do not discuss these cases here.

The most important point in the discussion of these systems and of the multi-particle ground states is that although this construction allows for many different examples of solvable systems in arbitrary dimensions (since in the above construction \( \tilde{G} \) may be an arbitrary lattice in arbitrary dimensions), the ground state properties for \( T = 0 \) and \( N_c \leq N_d \) and the contribution of the ground states to the low temperature properties of the system are that of a collection of zero-dimensional systems. The properties do not depend on the dimension of the lattice, but only on the number of different subgraphs of type \( K_n \) the lattice contains.
4 Summary and Outlook

This paper yields a complete classification of all Hubbard models for which the degeneracy \( N_d \) of the single-particle ground states is some finite fraction of the number of lattice sites and for which the projector onto this subspace is highly reducible, i.e. where the number of irreducible submatrices \( N_r \) of this projector is some finite fraction of the number of lattice sites. Each subspace lives on a local cluster and different clusters do not overlap. We show how lattices with these properties can be constructed in arbitrary dimensions and we derive some properties of the ground states of such models for electron numbers \( N_e \leq N_d \). Examples of such lattices in one and two dimensions were previously presented by Batista and Shastry [8], Maksymenko et al. [10], and others, see also the references therein. Maksymenko et al. [10] gave a rather complete discussion of the Hubbard model on some one-dimensional lattices of this type.

The important point is that the ground states properties for such models do not depend on the details of the lattice or on its dimensionality, but only on the properties of the local clusters. Global properties like the entropy at \( T = 0 \) can be calculated. The behaviour of the entropy is similar for all of these lattices. The entropy density as a function of the density of particles grows from 0 to some maximum and then decays to some finite value at \( N_e = N_d \). Thus, the one-dimensional lattices in [10] are ideal prototypes of all these models, and no essentially new physics occurs in the higher-dimensional models. Maksymenko et al. [10] discussed the case where the degeneracy within the cluster is lifted. In that case the system still has a large ground state degeneracy, finite entropy density, etc. In that case, the ground states still are located on the small local clusters and the dimensionality of the lattice remains unimportant. This may of course change if one lifts the degeneracy by some small perturbation so that the lowest bands are no longer strictly flat. For the discussion of the stability of ferromagnetism the situation then becomes much more difficult, see [4, 5]. We expect that with such perturbations, the dimensionality of the lattice becomes important as well and that the analysis will be much more difficult. Nevertheless, in that case new and interesting physics may occur.

Maksymenko et al. [10] discussed not only the Hubbard model but also the Heisenberg model on such lattices. We expect that their results for the two one-dimensional models can easily be generalised to the class of lattices described here as well. We expect that, as for the Hubbard model, one obtains no new interesting physics.

Another class of models which are closely related to spin systems with antiferromagnetic exchange interactions are bosonic Hubbard models with flat bands on similar lattices, see e.g. [11, 15] and the references therein. Bosonic Hubbard models can also be discussed on the lattices presented here with similar results.

In this paper, we only discussed the \( T = 0 \) properties of these models. For \( T > 0 \), the situation becomes more complicated. The detailed structure of the lattice and the dimensionality become important, since the other single-particle eigenstates [14] depend on the detailed lattice properties.

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