TRAPEZOIDAL TYPE INEQUALITIES RELATED TO $h$-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. A mapping $M(t)$ is considered to obtain some preliminary results and a new trapezoidal form of Fejér inequality related to the $h$-convex functions. Furthermore the obtained results are applied to achieve some new inequalities in connection with special means, random variable and trapezoidal formula.

1. Introduction

In 1906, the Hungarian mathematician L. Fejér [6] proved the following integral inequalities known in the literature as Fejér inequality:

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)dx,$$

where $f : [a,b] \to \mathbb{R}$ is convex and $g : [a,b] \to \mathbb{R}^+ = [0, +\infty)$ is integrable and symmetric to $x = \frac{a+b}{2}$ ($g(x) = g(a+b-x)$, $\forall x \in [a,b]$). For some other inequalities in connection with Fejér inequality see [8, 9, 11, 13, 14, 16] and references therein.

In 2006, the concept of $h$-convex functions related to the nonnegative real functions has been introduced in [15] by S. Varošanec. The class of $h$-convex functions is including a large class of nonnegative functions such as nonnegative convex functions, Godunova-Levin functions [7], s-convex functions in the second sense [2] and P-functions [5].

Definition 1.1. [15] Let $h : [0,1] \to \mathbb{R}^+$ be a function such that $h \neq 0$. We say that $f : I \to \mathbb{R}^+$ is a $h$-convex function, if for all $x, y \in I$, $\lambda \in [0,1]$ we have

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y).$$

Obviously, if $h(t) = t$, then all non-negative convex functions belong to the class of $h$-convex functions. Also if we take $h(t) = \frac{1}{t}$, $h(t) = t^s$, $s \in (0,1]$, and $h(t) = 1$ in (2) respectively, then Definition 1.1 reduces to definitions Godunova-Levin functions, 2010 Mathematics Subject Classification. Primary 26A51, 26D15, 52A01 Secondary 26A51.

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s-convex functions and P-functions respectively. To see Fejér inequality related to \( h \)-convex functions we refer the readers to [1].

By the Fejér trapezoidal inequality we mean the estimation of difference for right-middle part of (1). The Fejér trapezoidal inequality related to convex functions has been obtained in [3] as the following:

**Theorem 1.2.** Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be differentiable mapping on \( I^0 \), where \( a, b \in I \) with \( a < b \), and let \( g : [a, b] \rightarrow [0, \infty) \) be continuous positive mapping and symmetric to \( \frac{a+b}{2} \). If the mapping \( |f'| \) is convex on \([a, b]\), then the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} \int_a^b g(x)dx - \int_a^b f(x)g(x)dx \right| \leq \frac{(b-a)}{4} \left[ |f'(a)| + |f'(b)| \right] \int_0^1 \int_{\frac{1-t}{2}a + \frac{1+t}{2}b}^{\frac{1+t}{2}a + \frac{1-t}{2}b} g(x)dxdt.
\]

In this paper, motivated by above works and results we consider a mapping \( M(t) \) and obtain some introductory properties related to it. Also a new trapezoidal form of Fejér inequality is proved in the case that the absolute value of considered function is \( h \)-convex. Specially in the convex case, the obtained Fejér trapezoidal inequality is different from (3) with a new face. Furthermore some applications in connection with special means, random variable and trapezoidal formula are given.

2. **Main Results**

Related to a function \( g : [a, b] \rightarrow \mathbb{R} \) consider the mapping \( M : [0, 1] \rightarrow \mathbb{R} \) as the following:

\[
M(t) = \int_t^1 g(sa + (1-s)b)ds - \int_0^t g(sa + (1-s)b)ds.
\]

There exist some properties for the mapping \( M(t) \), compiled in the following lemma which are used to obtain our main results.

**Lemma 2.1.** Suppose that \( I \subseteq \mathbb{R} \) is an interval, \( a, b \in I^0 \) with \( a < b \) and \( g : [a, b] \rightarrow \mathbb{R} \) is an integrable function on \([a, b]\).
(i) If \( g \) is symmetric to \( \frac{a+b}{2} \), then

\[
M(t) = \begin{cases} 
2 \int_{t}^{1} \frac{1}{2} g(sa + (1-s)b) ds & 0 \leq t \leq \frac{1}{2}; \\
-2 \int_{\frac{1}{2}}^{t} g(sa + (1-s)b) ds & \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

(ii) For any \( t \in [0,1] \),

\[
M(t) + M(1-t) = 0. \quad (4)
\]

(iii) If \( g \) is a nonnegative function, then

\[
\begin{cases} 
M(t) \geq 0 & 0 \leq t \leq \frac{1}{2}; \\
M(t) \leq 0 & \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

(iv) The following inequalities hold.

\[
\int_{0}^{1} |M(t)| dt \leq \frac{1}{2} ||g||_{\infty},
\]

and

\[
\int_{0}^{1} |M(t)| dt \leq 2||g||_{q} \int_{0}^{1} |t - \frac{1}{2}|^{\frac{1}{q}} dt.
\]

(v) Let \( f : I^{{\circ}} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^{{\circ}} \), and \( g \) be a differentiable nonnegative mapping. If \( f' \in L[a,b] \), then the following equality holds:

\[
\frac{1}{b-a} \left( \frac{f(a) + f(b)}{2} \int_{a}^{b} g(x) dx - \int_{a}^{b} f(x) g(x) dx \right) = \frac{b-a}{2} \int_{0}^{1} M(t) f'(ta + (1-t)b) dt. \quad (5)
\]

Proof. (i) Using the change of variable \( x = sa + (1-s)b \) in the definition of \( M(t) \), for \( 0 \leq t \leq \frac{1}{2} \) we get

\[
M(t) = \frac{1}{b-a} \left[ \int_{a}^{ta+(1-t)b} g(x) dx - \int_{ta+(1-t)b}^{b} g(x) dx \right].
\]
where \( \frac{a+b}{2} \leq ta + (1-t)b \leq b \).

Since \( g \) is symmetric to \( \frac{a+b}{2} \),

\[
\int_a^{b} g(x) \, dx = \int_{\frac{a+b}{2}}^{b} g(x) \, dx,
\]

and so

\[
\int_a^{ta+(1-t)b} g(x) \, dx = \int_a^{\frac{a+b}{2}} g(x) \, dx + \int_{\frac{a+b}{2}}^{ta+(1-t)b} g(x) \, dx \quad (6)
\]

\[
= \int_{\frac{a+b}{2}}^{b} g(x) \, dx + \int_{\frac{a+b}{2}}^{ta+(1-t)b} g(x) \, dx.
\]

On the other hand

\[
\int_{\frac{a+b}{2}}^{b} g(x) \, dx = \int_a^{ta+(1-t)b} g(x) \, dx + \int_{ta+(1-t)b}^{b} g(x) \, dx \quad (7)
\]

Therefore applying (6) and (7) together imply that

\[
\frac{1}{b-a} \left[ \int_a^{ta+(1-t)b} g(x) \, dx - \int_{ta+(1-t)b}^{b} g(x) \, dx \right] = \frac{2}{b-a} \int_{\frac{a+b}{2}}^{ta+(1-t)b} g(x) \, dx.
\]

Hence

\[
M(t) = 2 \int_1^t \left( sa + (1 - s)b \right) ds,
\]

where \( 0 \leq t \leq \frac{1}{2} \).

With the same argument as above we can prove that

\[
M(t) = -2 \int_1^t \left( sa + (1 - s)b \right) ds,
\]

where \( \frac{1}{2} \leq t \leq 1 \).
(ii) For any \( t \in [0, 1] \),
\[
M(t) = \frac{1}{b - a} \left[ \int_{a}^{ta + (1-t)b} g(x) \, dx - \int_{(1-t)a + tb}^{a} g(x) \, dx \right] = \frac{1}{b - a} \left[ \int_{a}^{ta + (1-t)b} g(a + b - x) \, dx - \int_{(1-t)a + tb}^{a} g(a + b - x) \, dx \right] = \frac{1}{b - a} \left[ \int_{(1-t)a + tb}^{b} g(x) \, dx - \int_{a}^{(1-t)a + tb} g(x) \, dx \right] = -M(1 - t).
\]

(iii) It is easy consequence of assertion (i).

(iv) By the use of assertion (iii) We can obtain the following relations:
\[
\int_{0}^{1} |M(t)| \, dt = \int_{0}^{1} M(t) \, dt - \int_{0}^{1} M(t) \, dt = 2 \int_{0}^{1/2} \int_{t}^{1/2} g(sa + (1-s)b) \, ds \, dt \\
+ 2 \int_{1/2}^{1} \int_{1/2}^{t} g(sa + (1-s)b) \, ds \, dt \leq 2 \int_{0}^{1} \int_{1/2}^{t} \sup_{s \in [t, 1/2]} g(sa + (1-s)b) \, ds \, dt \\
+ 2 \int_{1/2}^{1} \int_{1/2}^{t} \sup_{s \in [1/2, t]} g(sa + (1-s)b) \, ds \, dt \leq 2 ||g||_{\infty} \int_{0}^{1} \left| t - \frac{1}{2} \right| \, dt = \frac{1}{2} ||g||_{\infty},
\]
which proves the first part (iv).

For the second part of (iv), we consider the following assertion which is not hard to prove:
\[
\int_{0}^{1} |M(t)| \, dt = 2 \int_{0}^{1/2} \left| \int_{t}^{1/2} g(sa + (1-s)b) \, ds \right| \, dt. \tag{8}
\]

Also using Hölder’s inequality we have
\[
\left| \int_{t}^{1/2} g(sa + (1-s)b) \, ds \right| \leq \left( \int_{t}^{1/2} \left| g(sa + (1-s)b) \right|^q \, ds \right)^{1/q} \left( \int_{t}^{1/2} \, ds \right)^{1/p} \leq \left( \int_{t}^{1/2} \left| g(sa + (1-s)b) \right|^q \, ds \right)^{1/q} \left( \int_{t}^{1/2} \, ds \right)^{1/p} \leq \left( \int_{t}^{1/2} \left| g(sa + (1-s)b) \right|^q \, ds \right)^{1/q} \left( \int_{t}^{1/2} \, ds \right)^{1/p} \leq ||g||_q \left| t - \frac{1}{2} \right|^{1/p}. \tag{9}
\]

Now applying (9) in (8) we get
\[
\int_{0}^{1} |M(t)| \, dt \leq 2 ||g||_q \int_{0}^{1} \left| t - \frac{1}{2} \right|^{1/p} \, dt.
\]

(v) This identity has been obtained in [14]. \( \Box \)
We can find more results related to the mapping $M(t)$, where the derivative of considered function is bounded or satisfies a Lipschitz condition.

**Remark 2.2.** Suppose that $f : I \to \mathbb{R}$ is a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$ and $g : [a, b] \to \mathbb{R}^+$ is a differentiable mapping. Assume that $f'$ is integrable on $[a, b]$ and there exist constants $m < M$ such that

$$-\infty < m \leq f'(x) \leq M < \infty \text{ for all } x \in [a, b].$$

Then (see [12])

$$\left| \frac{f(a) + f(b)}{2(b - a)} \int_a^b g(x)dx - \frac{1}{b - a} \int_a^b f(x)g(x)dx - \frac{m + M}{4} \int_0^1 M(t)dt \right| \leq \frac{(M - m)(b - a)}{4} \int_0^1 |M(t)|dt. \tag{10}$$

If in inequality (10) we assume that $g$ is symmetric to $\frac{a + b}{2}$, then from assertion (iv) of Lemma 2.1 we have

$$\left| \frac{f(a) + f(b)}{2(b - a)} \int_a^b g(x)dx - \frac{1}{b - a} \int_a^b f(x)g(x)dx - \frac{m + M}{4} \int_0^1 M(t)dt \right| \leq \frac{(M - m)(b - a)}{8} ||g||_\infty.$$

Also by the use of Hölder’s inequality the following relation holds:

$$\left| \frac{f(a) + f(b)}{2(b - a)} \int_a^b g(x)dx - \frac{1}{b - a} \int_a^b f(x)g(x)dx - \frac{m + M}{4} \int_0^1 M(t)dt \right| \leq \frac{(M - m)(b - a)}{2} ||g||_q \int_0^1 |t - \frac{1}{2}|^{\frac{q}{2}} dt. \tag{11}$$

**Definition 2.3.** [10] A function $f : [a, b] \to \mathbb{R}$ is said to satisfy Lipschitz condition on $[a, b]$ if there is a constant $K$ so that for any two points $x, y \in [a, b]$,

$$|f(x) - f(y)| \leq K|x - y|.$$

**Remark 2.4.** Suppose that $f : I \to \mathbb{R}$ is a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$ and $g : [a, b] \to \mathbb{R}^+$ is a differentiable mapping. Assume that $f'$ is integrable on $[a, b]$ and satisfies a Lipschitz condition for some $K > 0$. Then (see [12])

$$\left| \frac{f(a) + f(b)}{2(b - a)} \int_a^b g(x)dx - \frac{1}{b - a} \int_a^b f(x)g(x)dx - \frac{1}{2} f'\left(\frac{a + b}{2}\right) \int_0^1 M(t)dt \right| \leq K \frac{(b - a)}{2} \int_0^1 |t - \frac{1}{2}| |M(t)|dt. \tag{11}$$
In inequality (11) if we assume that $g$ is symmetric to $\frac{a+b}{2}$, then from assertion (i) of Lemma 2.1 we get
\[
\left| \frac{f(a) + f(b)}{2(b-a)} \int_a^b g(x)dx - \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{(b-a)}{2} f'\left(\frac{a+b}{2}\right) \int_0^1 M(t)dt \right| 
\]
\[
\leq K(b-a) \int_0^1 \int_t^{\frac{1}{2}} |t - \frac{1}{2}| |g(sa + (1-s)b)|dsdt.
\]
Also we have
\[
\left| \frac{f(a) + f(b)}{2(b-a)} \int_a^b g(x)dx - \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{(b-a)}{2} f'\left(\frac{a+b}{2}\right) \int_0^1 M(t)dt \right| 
\]
\[
\leq K(b-a) ||g||_{\infty} \int_0^1 (t - \frac{1}{2})^2 dt = \frac{K(b-a)}{12} ||g||_{\infty}.
\]

The following is the main result of the paper.

**Theorem 2.5.** Suppose that $f : I \to \mathbb{R}$ is a differentiable mapping on $I^o$, $a, b \in I^o$ with $a < b$ and $g : [a, b] \to \mathbb{R}^+$ is a differentiable mapping symmetric to $\frac{a+b}{2}$. If $|f'|$ is a $h$-convex mapping on $[a, b]$, Then
\[
\left| \frac{f(a) + f(b)}{2} \int_a^b g(x)dx - \int_a^b f(x)g(x)dx \right| \leq \ (b-a)(|f'(a)| + |f'(b)|) \int_a^b \int_0^{\frac{a+b}{2}} g(x)[h(t) + h(1-t)]dtdx.
\]

**Proof.** From the definition of $M(t)$, assertions of Lemma 2.1 and $h$-convexity of $|f'|$ we have
\[
\left| \frac{f(a) + f(b)}{2} \int_a^b g(x)dx - \int_a^b f(x)g(x)dx \right| = \frac{(b-a)^2}{2} \left| \int_0^1 M(t)f'(ta + (1-t)b)dt \right| 
\]
\[
\leq \frac{(b-a)^2}{2} \left\{ \int_0^{\frac{1}{2}} |M(t)||f'|(ta + (1-t)b)dt + \int_{\frac{1}{2}}^1 |M(t)||f'|(ta + (1-t)b)dt \right\} 
\]
\[
= \frac{(b-a)^2}{2} \left\{ \int_0^{\frac{1}{2}} M(t)|f'|(ta + (1-t)b)dt - \int_{\frac{1}{2}}^1 M(t)|f'|(ta + (1-t)b)dt \right\} 
\]
\[
\leq \frac{(b-a)^2}{2} \left\{ 2 \int_0^{\frac{1}{2}} \int_t^{\frac{1}{2}} g(sa + (1-s)b)(h(t)|f'(a)| + h(1-t)|f'(b)|)dsdt 
+ 2 \int_{\frac{1}{2}}^1 \int_t^{\frac{1}{2}} g(sa + (1-s)b)(h(t)|f'(a)| + h(1-t)|f'(b)|)dsdt \right\}.
\]
Replacing (14) and (15) in (13) implies that

\[
\left| \frac{f(a) + f(b)}{2} \int_a^b g(x)dx - \int_a^b f(x)g(x)dx \right|
\]

\[
\leq (b - a)^2 \left\{ \int_0^{\frac{b-a}{2}} \int_0^{\frac{b-x}{b-a}} g(sa + (1 - s)b)(h(t)|f'(a)| + h(1 - t)|f'(b)|)dt ds \\
+ \int_{\frac{b-a}{2}}^1 \int_0^{\frac{b-x}{b-a}} g(sa + (1 - s)b)(h(t)|f'(a)| + h(1 - t)|f'(b)|)dt dx \right\}.
\]

Using the change of variable \( x = sa + (1 - s)b \) we get

\[
\left| \frac{f(a) + f(b)}{2} \int_a^b g(x)dx - \int_a^b f(x)g(x)dx \right| \leq (b - a) \left\{ \int_a^{\frac{a+b}{2}} \int_0^{\frac{x-a}{b-a}} g(x)(h(t)|f'(a)| + h(1 - t)|f'(b)|)dt dx \\
+ \int_{\frac{a+b}{2}}^b \int_0^{\frac{x-a}{b-a}} g(x)(h(t)|f'(a)| + h(1 - t)|f'(b)|)dx dt \right\}.
\]

Since the function \( g \) is symmetric to \( \frac{a+b}{2} \), then

\[
\int_{\frac{a+b}{2}}^b \int_0^{\frac{b-x}{b-a}} g(x)(h(t)|f'(a)| + h(1 - t)|f'(b)|)dt dx = \int_a^{\frac{a+b}{2}} \int_0^{\frac{x-a}{b-a}} g(x)(h(t)|f'(a)| + h(1 - t)|f'(b)|)dx dt.
\]

Also it is not hard to see that

\[
\int_0^{\frac{x-a}{b-a}} h(1 - t) = \int_{\frac{b-x}{b-a}}^1 h(t)dt.
\]

Replacing (14) and (15) in (13) implies that

\[
\left| \frac{f(a) + f(b)}{2} \int_a^b g(x)dx - \int_a^b f(x)g(x)dx \right|
\]

\[
\leq (b - a)(|f'(a)| + |f'(b)|) \left\{ \int_a^{\frac{a+b}{2}} g(x)\left[ \int_0^{\frac{x-a}{b-a}} h(t)dt + \int_{\frac{b-x}{b-a}}^1 h(t)dt \right]dx \right\}
\]

\[
= (b - a)(|f'(a)| + |f'(b)|) \int_a^{\frac{a+b}{2}} \int_0^{\frac{x-a}{b-a}} g(x)(h(t) + h(1 - t))dt dx.
\]

\[\square\]
Remark 2.6. We can obtain another form of (5) in Lemma 2.1 by the use of (4). In fact we get
\begin{equation}
\frac{1}{b-a} \left( \frac{f(a) + f(b)}{2} \int_a^b g(x) dx - \int_a^b f(x) g(x) dx \right) = \frac{b-a}{2} \int_0^1 M(1-t)f'(tb + (1-t)a) dt.
\end{equation}

Now using (16) in the proof of Theorem 2.5 implies another form of (12).
\begin{equation}
\left| \frac{f(a) + f(b)}{2} \int_a^b g(x) dx - \int_a^b f(x) g(x) dx \right| \leq (b-a) \left( |f'(a)| + |f'(b)| \right) \int_0^{a+b} g(x)[h(t) + h(1-t)] dt dx.
\end{equation}

Corollary 2.7. With the assumptions of Theorem 2.5, if the function $|f'|$ is $s$-convex on $[a, b]$, then
\begin{equation}
\left| \frac{f(a) + f(b)}{2} \int_a^b g(x) dx - \int_a^b f(x) g(x) dx \right| \leq \frac{(b-a)}{1+s} \left( |f'(a)| + |f'(b)| \right) \int_a^{a+b} g(x) \left( \left( \frac{x-a}{b-a} \right)^{1+s} - \left( \frac{b-x}{b-a} \right)^{1+s} + 1 \right) dx.
\end{equation}

Corollary 2.8. With the assumptions of Theorem 2.5, if the function $|f'|$ is convex on $[a, b]$, then
\begin{equation}
\left| \frac{f(a) + f(b)}{2} \int_a^b g(x) dx - \int_a^b f(x) g(x) dx \right| \leq (|f'(a)| + |f'(b)|) \int_a^{a+b} g(x) (x-a) dx.
\end{equation}

Equivalently
\begin{equation}
\left| \frac{f(a) + f(b)}{2} \int_a^b g(x) dx - \int_a^b f(x) g(x) dx \right| \leq (|f'(a)| + |f'(b)|) \int_a^b g(x) (b-x) dx.
\end{equation}

Also if we consider $g \equiv 1$, then we recapture the following result obtained in [4].
\begin{equation}
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}.
\end{equation}
Remark 2.9. Inequalities (12), (17), (18), (19) and (20) are new type in the literature for the class of Fejér trapezoidal inequality related to $h$-convex, $s$-convex and convex functions respectively.

3. Application

3.1. Special Mean. In the literature, the following means for real numbers $a, b \in \mathbb{R}$ are well known:

\[
A(a, b) = \frac{a + b}{2} \quad \text{arithmetic mean},
\]

\[
L_n(a, b) = \left[ \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right]^{\frac{1}{n}} \quad \text{generalized log–mean, } n \in \mathbb{R}, \ a < b.
\]

Consider
\[
\begin{aligned}
f(x) &= x^n, \quad x > 0 \text{ and } n \in (-\infty, -1) \cup (-1, 0) \cup [1, \infty); \\
h(t) &= t^k, \quad k \leq 1 \text{ and } k \neq -1, -2; \\
g(x) &\equiv 1.
\end{aligned}
\]

Theorem 2.5 implies the following inequalities:

\[
\left| \frac{a^n + b^n}{2} (b-a) - \frac{1}{n+1} \left[ b^{n+1} - a^{n+1} \right] \right| \leq n(b-a) \left( |a|^{n-1} + |b|^{n-1} \right)
\]

\[
\times \int_a^{a+b} \int_0^{x-a} \left[ k^k + (1-t)^k \right] dt \, dx = \frac{n(b-a)}{k+1} \left( |a|^{n-1} + |b|^{n-1} \right)
\]

\[
\times \int_a^{a+b} \left[ \left( \frac{x-a}{b-a} \right)^{k+1} - \left( \frac{b-x}{b-a} \right)^{k+1} + 1 \right] \, dx = \frac{n}{(k+1)(k+2)(b-a)^k} \left( |a|^{n-1} + |b|^{n-1} \right)
\]

\[
\times \left[ (x-a)^{k+2} - (b-x)^{k+2} + (b-a)^{k+1}(k+2)x \right]_a^{a+b}
\]

\[
= \frac{n}{(k+1)(k+2)(b-a)^k} \left( |a|^{n-1} + |b|^{n-1} \right) (b-a)^{k+2} \left[ \frac{1}{2^{k+1}} + \frac{1}{2} \right]
\]

\[
= \frac{n(b-a)^2}{(k+1)(k+2)} \left( |a|^{n-1} + |b|^{n-1} \right) \left[ \frac{1}{2^{k+1}} + \frac{1}{2} \right].
\]

Hence we get

\[
\left| \frac{a^n + b^n}{2} - \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right| \leq \frac{n(b-a)}{(k+1)(k+2)} \left( |a|^{n-1} + |b|^{n-1} \right) \left[ \frac{1}{2^{k+1}} + \frac{1}{2} \right],
\]
which implies that
\[
\left| A(a^n, b^n) - L_n^a(a,b) \right| \leq \frac{n(b-a)}{(k+1)(k+2)}L(|a|^{n-1}, |b|^{n-1}) \left[ \frac{1}{2k} + k \right]. \quad (21)
\]

If we consider \( k = 1 \) in (21), then we recapture the following result.

**Corollary 3.1** (Proposition 3.1 in [4]). Let \( a, b \in \mathbb{R}, a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then the following inequality holds:
\[
\left| A(a^n, b^n) - L_n^a(a,b) \right| \leq \frac{n(b-a)}{4}A(|a|^{n-1}, |b|^{n-1}).
\]

### 3.2. Random Variable.

Suppose that for \( 0 < a < b \) and \( g : [a, b] \to \mathbb{R}^+ \) is a continuous probability density function i.e.
\[
\int_a^b g(x)dx = 1,
\]
which is symmetric to \( \frac{a+b}{2} \). Also for \( \lambda \in \mathbb{R} \), suppose that the \( \lambda \)-moment
\[
E_\lambda(X) = \int_a^b x^\lambda g(x)dx,
\]
is finite.

From (12) and the fact that for any \( a \leq x \leq \frac{a+b}{2} \) we have \( 0 \leq \frac{x-a}{b-a} \leq \frac{1}{2} \), the following inequality holds.
\[
\left| \frac{f(a) + f(b)}{2} \int_a^b g(x)dx - \int_a^b f(x)g(x)dx \right| \leq (b-a)(|f'(a)| + |f'(b)|) \quad (22)
\]
\[
\times \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} g(x)[h(t) + h(1-t)]dt dx = \frac{b-a}{2}(|f'(a)| + |f'(b)|)
\]
\[
\times \int_0^{\frac{1}{2}} [h(t) + h(1-t)]dt,
\]
where from the fact that \( g \) is symmetric and \( \int_a^b g(x)dx = 1 \), we have \( \int_a^{\frac{a+b}{2}} g(x)dx = \frac{1}{2} \).

**Example 3.2.** If we consider
\[
\begin{align*}
  f(x) &= \frac{1}{\lambda}x^\lambda, \quad x > 0, \lambda \in (-\infty, 0) \cup (0, 1] \cup [2, +\infty); \\
  h(t) &= t^k, \quad k \in (-\infty, -1) \cup (-1, 1]; \\
  g(x) &\equiv 1.
\end{align*}
\]
Then $|f'|$ is $h$-convex (see Example 7 in [15]) and so from (22) we have
\[
\left| \frac{a^\lambda + b^\lambda}{2\lambda} - E_\lambda(X) \right| \leq \frac{\lambda(b - a)}{2(k + 1)} \left( a^{\lambda-1} + b^{\lambda-1} \right),
\]

since
\[
\left| \frac{a^\lambda + b^\lambda}{2\lambda} - E_\lambda(X) \right| \leq \frac{\lambda(b - a)}{2} \left( a^{\lambda-1} + b^{\lambda-1} \right) \int_0^1 \left[ t^k + (1 - t)^k \right] dt
\]
\[
= \frac{\lambda(b - a)}{2(k + 1)} \left( a^{\lambda-1} + b^{\lambda-1} \right).
\]

If $\lambda = 1$, $E(X)$ is the expectation of the random variable $X$ and from above inequality we obtain the following bound
\[
\left| \frac{a + b}{2} - E(X) \right| \leq \frac{b - a}{k + 1},
\]

and in the case that $k = 1$, we recapture the following known bound
\[
\left| \frac{a + b}{2} - E(X) \right| \leq \frac{b - a}{2}.
\]

3.3. Trapezoidal Formula. Consider the partition $(P)$ of interval $[a, b]$ as $a = x_0 < x_1 < x_2 < ... < x_n = b$. The quadrature formula is
\[
\int_a^b f(x)g(x)dx = T(f, g, P) + E(f, g, P),
\]
where
\[
T(f, g, P) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \int_{x_i}^{x_{i+1}} g(x)dx,
\]
is the trapezoidal form and $E(f, g, P)$ is the associated approximation error.

For each $i \in \{0, 1, ..., n - 1\}$ consider interval $[x_i, x_{i+1}]$ of partition $(P)$ of interval $[a, b]$. Suppose that all conditions of Theorem 2.5 are satisfied on $[x_i, x_{i+1}]$. Then
\[
\left| \frac{f(x_i) + f(x_{i+1})}{2} \int_{x_i}^{x_{i+1}} g(x)dx - \int_{x_i}^{x_{i+1}} f(x)g(x)dx \right| \leq (x_{i+1} - x_i) \left[ |f'(x_i)| + |f'(x_{i+1})| \right] \int_{x_i}^{x_{i+1}} \int_0^{x_{i+1} - x_i} g(x)[h(t) + h(1 - t)]dt dx,
\]
For each \( i \in \{0, 1, ..., n - 1\} \). Then using inequality \( (23) \), summing with respect to \( i \) from \( i = 0 \) to \( i = n - 1 \) and using triangle inequality we obtain

\[
\left| T(f, g, P) - \int_a^b f(x)g(x)dx \right| \\
= \left| \sum_{i=0}^{n-1} \left[ \frac{f(x_i) + f(x_{i+1})}{2} \int_{x_i}^{x_{i+1}} g(x)dx - \int_{x_i}^{x_{i+1}} f(x)g(x)dx \right] \right| \\
\leq \sum_{i=0}^{n-1} \left[ \frac{f(x_i) + f(x_{i+1})}{2} \int_{x_i}^{x_{i+1}} g(x)dx - \int_{x_i}^{x_{i+1}} f(x)g(x)dx \right] \\
\leq \sum_{i=0}^{n-1} \left( x_{i+1} - x_i \right) \left[ |f'(x_i)| + |f'(x_{i+1})| \right] \int_{x_i}^{x_{i+1}} \int_0^{x_{i+1} - x_i} g(x)[h(t) + h(1 - t)]dtdx.
\]

So we get the error bound:

\[
|E(f, g, P)| \leq \sum_{i=0}^{n-1} \left( x_{i+1} - x_i \right) \left[ |f'(x_i)| + |f'(x_{i+1})| \right] \\
\times \int_{x_i}^{x_{i+1}} \int_0^{x_{i+1} - x_i} g(x)[h(t) + h(1 - t)]dtdx. 
(24)
\]

**Corollary 3.3.** If we consider \( h(t) = t^k \) in \( (24) \) then:

\[
|E(f, g, P)| \leq \frac{1}{k+1} \sum_{i=0}^{n-1} \left( x_{i+1} - x_i \right) \left[ |f'(x_i)| + |f'(x_{i+1})| \right] \\
\times \int_{x_i}^{x_{i+1}} \left[ \left( \frac{x_{i+1} - x}{x_{i+1} - x_i} \right)^{k+1} - \left( \frac{x - x_i}{x_{i+1} - x_i} \right)^{k+1} + 1 \right] g(x)dx. 
(25)
\]

If \( k = 1 \) and \( g \equiv 1 \) in \( (25) \), then we recapture the inequality obtained in Proposition 4.1 in \([4]\):

\[
|E(f, P)| \leq \frac{1}{8} \sum_{i=0}^{n-1} \left( |f'(x_i)| + |f'(x_{i+1})| \right) (x_{i+1} - x_i)^2.
\]

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