THE DYNAMIC PROPERTIES OF A GENERALIZED KAWAHARA EQUATION WITH KURAMOTO-SIVASHINSKY PERTURBATION

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Abstract. In this paper, we are concerned with the existence of solitary waves for a generalized Kawahara equation, which is a model equation describing solitary-wave propagation in media. We obtain some qualitative properties of equilibrium points and existence results of solitary wave solutions for the generalized Kawahara equation without delay and perturbation by employing the phase space analysis. Furthermore the existence of solitary wave solutions for the equation with two types of special delay convolution kernels is proved by combining the geometric singular perturbation theory, invariant manifold theory and Fredholm orthogonality. We also discuss the asymptotic behaviors of traveling wave solutions by means of the asymptotic theory. Finally, some examples are given to illustrate our results.

1. Introduction. The study on the theory of nonlinear evolution equations [1] has come a long way. Nonlinear equations are widely used to describe complex physical phenomena in various fields, such as fluid mechanics, optical fibers, solid state physics, plasma physics, chemical kinematics, chemical geochemistry, etc. [6, 7, 11, 15, 18]. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important therein.

The Kawahara equation occurs in the theory of magneto-acoustic waves in a plasma and in the theory of shallow water waves with surface tension. In order to balance the nonlinear effect, Kawahara took account of the higher order effect of dispersion, and established the following Kawahara equation in 1972, as a model equation describing solitary-wave propagation in media [24]

\[ u_t + \frac{3}{2} u u_x + \alpha u_{xxx} - \beta u_{xxxxx} = 0, \]

which is also referred as fifth-order Korteweg-de Vries (KdV) equation or singularly perturbed KdV equation [19], where \( \alpha \) and \( \beta \) may be either positive or negative and represent the effect of dispersion. Moreover \( u \) is a function of \( x, t \) in the
equation and \( u_x, u_t \) represent the partial derivatives with respect to the space and time coordinates respectively. The term \( uu_x \) denotes nonlinear convection and \( u_{xxx} \) represents the dispersion effect. The parameter \( \beta \) controls the fifth-order dispersion term and it is responsible for oscillations.

Molinet and Wang [30] investigated the Kawahara equation

\[
u_t + uu_x + u_{xxx} + \varepsilon u_{xxxxx} = 0,
\]

where \( \varepsilon > 0 \) is a small coefficient, and the terms \( u_{xxx} \) and \( u_{xxxxx} \) compete and cancel each other at frequencies of order \( \sqrt{\varepsilon} \). They established the limit behavior of the solutions for equation (2) as \( \varepsilon \) tends to 0. It was proven that the solutions converge in \( C([0,T];H^1(\mathbb{R})) \) by applying a standard dispersive approach [3]. Villagran [38] discussed the Kawahara equation

\[
u_t + uu_x + u_{xxx} + \eta u_{xxxxx} = 0,
\]

in which \( \eta \in \mathbb{R} \). He established the existence of local and global solutions, and proved the gain of regularity for the initial value problem associated with equation (3). The work was mainly motivated by Craig and Goodman [6].

Jia and Huo [21] dealt with a kind of fifth-order shallow water equation, which is described by

\[
u_t + \alpha u_{xxxxx} + \beta u_{xxx} + \gamma u_x + \mu(u^k)_x = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \quad k = 2, 3, 4, \cdots
\]

with the initial condition

\[u(0,x) = u_0(x) \in H^s(\mathbb{R}),\]

where \( \alpha \neq 0, \beta \) and \( \gamma \) are real numbers and \( \mu \) is a complex number. The equation (4) is regarded as a modified Kawahara equation. They established the local well-posedness of equation (4) for the initial condition in \( H^s(\mathbb{R}) \) with \( s > -\frac{7}{4} \), mainly using the method of Fourier restriction norm [29, 34].

Kwak [28] discussed equation (4) (posed on \( \mathbb{T} \)) under the condition \( \alpha = -1, \mu = -\frac{\mu}{3}, k = 3 \), i.e.,

\[
\left\{
\begin{aligned}
&u_t - u_{xxxxx} + \beta u_{xxx} + \gamma u_x - \frac{\mu}{3}(u^3)_x = 0, \\
&u(t,x) = u_0(x) \in H^s(\mathbb{T}),
\end{aligned}
\right.
\]

where \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \beta \geq 0, \gamma \in \mathbb{R}, \mu = \pm 1 \) and \( u \) is a real-valued unknown variable. They presented the global well-posedness results in \( L^2(\mathbb{T}) \). And the proof was based on the idea introduced in Takaoka-Tsutsumi’s works [32, 36]. The equation (5) can be generalized as follows

\[
u_t - u_{xxxxx} + \beta u_{xxx} + \gamma u_x - \frac{\mu}{3}(u^n)_x = 0, \quad n = 2, 3, \cdots
\]

For \( n = 1 \) and 2, the Kawahara equation has various kinds of applications, for instance in fluid mechanics and plasma physics. For \( n \geq 3 \), what interests us is the balance between the nonlinear effect and the scattering effect, which leads to the formation of solitary waves.

Much researches have been devoted to studying the Kawahara equations. There are a lot of methods including the symplectic Evans matrix theory [4], Laurent series [8], the standard dispersive approach [30], etc.

Recently geometric singular perturbation theory has been successfully used to prove the existence of traveling wave solution in evolution equations with small
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parameters. Generally, perturbations in dynamical systems can be divided into three types: periodic or quasi-periodic forcing, singular perturbation and regular perturbation. When the perturbation term is a quasi-periodic forcing, an infinite dimensional KAM theory is one effective method to deal with the case. This theory is an extension of the well-known classical KAM theory, which was established by Kolmogorov [26], Anorld [2] and Moser [31]. When a perturbed system can be reduced to a singularly perturbed system, the first question is about the existence of traveling wave solutions of the system. To deal with singular perturbations, an useful approach is the geometric singular perturbation theory [16], which can ensure the existence of the invariant manifold and the problem will be reduced to a regular perturbation system on the manifold. It has been successfully applied to analyze the perturbed Benjamin-Bona-Mahony equation [17, 35], Camassa-Holm equation [11], Keller-Segel system [12], Belousov-Zhabotinskii system [13] and perturbed KdV equations [33] etc.

Due to the existence of uncertainty or perturbation, certain relatively weak influences are unavoidable in solving real world problems. In 1993, Derks and Gils [9] proved the uniqueness of traveling waves for the perturbed KdV equation

\[ u_t + uu_x + u_{xxx} + \tau (u_{xx} + u_{xxxx}) = 0, \]

where \( \tau \) is a positive parameter and represents small perturbations to the equation, which determines the character of the solutions for the equation. The equation (7) is regarded as the classical KdV equation when the backward diffusion \( u_{xx} \) and dissipation \( u_{xxxx} \) terms are absent. Later Ogawa [33] studied the existence of solitary waves and periodic waves for equation (7). Motivated by Ogawa [33], a lot of works have been done on the study of the generalized BBM equation, i.e.

\[ (u^n)_t + (u^m)_x + u_{xxx} + \tau (u_{xx} + u_{xxxx}) = 0. \]

The authors of [17, 35] established the existence of solitary waves and periodic waves for the equation (8) by employing the geometric singular perturbation theory. Sun and Yu [35] dealt with the case of \( n = 3, m = 4 \), and Guo and Zhao considered the case of \( n = 3, m = 5 \). Moreover the monotonicity, the upper and lower bounds of the wave speed were proven by analyzing the ratio of Abelian integrals. The perturbation term \( u_{xx} + u_{xxxx} \) in equation (7) and (8) is called Kuramoto-Sivashinsky perturbation [20].

However geometric singular perturbation theory [16] has not been applied to study the Kawahara equation. The main interest of this paper is to explore the existence of solitary wave solutions for the following generalized Kawahara equation

\[ u_t + au_x + b(f \ast u)u^{n-1}u_x + ku_{xxx} + mu_{xxxxx} + \tau (u_{xx} + u_{xxxx}) = 0, \]

where \( a, b, k \) and \( m \) are arbitrary constants, \( n \) is an arbitrary positive constant, \( \tau \) is a small constant and \( b, m \neq 0 \). In the whole context, we consider three cases of \( (f \ast u)(x, t) \) as follows:

(i) without delay

\[ (f \ast u)(x, t) = u(x, t), \]

(ii) with the local delay

\[ (f \ast u)(x, t) = \int_{-\infty}^{t} f(t - s)u(x, s)ds, \]
(iii) with the nonlocal delay
\[
(f * u)(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} f(x - y, t - s)u(y, s)dyds.
\] (12)

The addition of convolution has practical significance for the discussion of the equations. It should be pointed out that some well known equations can be derived from the equation (9), such as, when \(a = m = \tau = 0, \ b = k = 1\), the equation (9) becomes the generalized KdV equation, i.e.,
\[
 u_t + u^{n+1}u_x + u_{xxx} = 0,
\] (13)
which was studied by Tao [37], Kenig [25] and Escauriaza [14] et al.

The rest of the paper is organized as follows. In section 2, we consider the generalized Kawahara equation without delay and perturbation by means of the phase space analysis. In section 3, the existence of solitary waves for the generalized Kawahara equation with local or nonlocal delay and perturbation will be established by using geometric singular perturbation theory, the linear chain trick, Implicit Function Theorem and the Fredholm theory. In section 4, we study the asymptotic behavior of the traveling waves for the generalized Kawahara equation by using the asymptotic theory. In section 5, some examples are given to illustrate our results.

2. The reduction of generalized Kawahara equation (9). Firstly we discuss the existence of solitary wave solutions for the generalized Kawahara equation (9) without delay and perturbation, which is described by
\[
 u_t + au_x + bu^n u_x + ku_{xxx} + mu_{xxxx} = 0.
\] (14)

It is proven that \((f * u)(x, t) \to u(x, t)\) when \(\tau \to 0\) in the paper [5, 11].

**Definition 2.1.** A traveling wave solution \(u(x, t) = \phi(x - ct) = \phi(\xi)\) of the equation is called a solitary wave solution if \(\lim_{\xi \to \pm \infty} \phi(\xi) = 0\), here \(c > 0\) is the wave speed.

Substituting \(u(x, t) = \phi(\xi) = \phi(x - ct)\) into equation (14), then we have the solitary wave equation
\[
(a - c)\phi' + b\phi^n \phi' + k\phi''' + m\phi'''' = 0,
\] (15)
where \(\prime = \frac{d}{dx}\). Integrating equation (15) once to yield
\[
(a - c)\phi + \frac{b}{n + 1} \phi^{n+1} + k\phi'' + m\phi''' = 0,
\] (16)
which is equivalent to the following system
\[
\begin{cases}
 \phi' = \phi_1, \\
 \phi_1' = \phi_2, \\
 \phi_2' = \phi_3, \\
 \phi_3' = -\frac{1}{m}[(a - c)\phi + \frac{b}{n + 1} \phi^{n+1} + k\phi_2].
\end{cases}
\] (17)

From the analysis of the phase space, we could easily obtain the following results when the constant \(n\) is odd and even respectively.

**Theorem 2.2.** When \(n\) is odd, system (17) has two equilibrium points \(E_1(0, 0, 0, 0)\) and \(E_2\left(\frac{n}{\sqrt{(c-a)(n+1)}}, 0, 0, 0\right)\) in the \((\phi, \phi_1, \phi_2, \phi_3)\) phase plane, and the following results hold
(i) If \( a > c, \ m > 0 \), \( E_1 \) is a center and \( E_2 \) is a saddle, and system (17) has a homoclinic orbit to the critical point \( E_2 \).

(ii) If \( a > c, \ m < 0 \), \( E_1 \) is a saddle and \( E_2 \) is a center, and system (17) has a homoclinic orbit to the critical point \( E_1 \).

(iii) If \( a < c, \ m > 0 \), \( E_1 \) is a saddle and \( E_2 \) is a center, and system (17) has a homoclinic orbit to the critical point \( E_1 \).

(iv) If \( a < c, \ m < 0 \), \( E_1 \) is a center and \( E_2 \) is a saddle, and system (17) has a homoclinic orbit to the critical point \( E_2 \).

**Theorem 2.3.** When \( n \) is even and \( b(c - a) > 0 \), system (17) has three equilibrium points \( E_1(0, 0, 0, 0) \), \( E_2 \left( \sqrt[3]{\frac{(c-a)(n+1)}{b}}, 0, 0, 0 \right) \) and \( E_3 \left( -\sqrt[3]{\frac{(c-a)(n+1)}{b}}, 0, 0, 0 \right) \) in the \((\phi, \phi_1, \phi_2, \phi_3)\) phase plane, and the following results hold

(i) If \( b > 0, \ c > a, \ m > 0 \), \( E_1 \) is a saddle, \( E_2 \) and \( E_3 \) are centers. The system (17) has periodic orbits to the equilibrium point \( E_1 \) which is connected by two homoclinic orbits.

(ii) If \( b > 0, \ c > a, \ m < 0 \), \( E_1 \) is a center, \( E_2 \) and \( E_3 \) are saddles. The system (17) has two heteroclinic orbits which are connected by two equilibria \( E_2 \) and \( E_3 \).

(iii) If \( b < 0, \ c < a, \ m > 0 \), \( E_1 \) is a center, \( E_2 \) and \( E_3 \) are saddles. The system (17) has two heteroclinic orbits which are connected by two equilibria \( E_2 \) and \( E_3 \).

(iv) If \( b < 0, \ c < a, \ m < 0 \), \( E_1 \) is a saddle, \( E_2 \) and \( E_3 \) are centers. The system (17) has periodic orbits to the equilibrium point \( E_1 \) which is connected by two homoclinic orbits.

**Remark 1.** Here we consider the above two cases: \( a > c \) and \( a < c \). For the third case \( a = c \), system (17) has only one high order equilibrium point which possesses complex properties. Since this paper is focused on the existence of solitary wave solutions, we don’t make the detailed explanation.

3. Solitary waves for generalized Kawahara equation (9).

3.1. Local delay and K-S perturbation. In this section, we consider the generalized Kawahara equation (9) with local delay and perturbation, where \( b > 0, c > a, m > 0 \) and \( n \) is even. The other cases can be discussed similarly. The analysis is based on geometric singular perturbation theory and Implicit Function Theorem.

The corresponding convolution in the equation (9) is defined by (11), in which the kernel \( f : [0, +\infty) \to [0, +\infty) \) satisfies the following normalization assumption,

\[
\int_{0}^{+\infty} f(t) dt = 1 \quad \text{and} \quad tf(t) \in L^1((0, \infty), \mathbb{R}).
\]

It is notable that the normalization assumption on \( f \) ensures that the uniform non-negative steady-state solutions are unaffected by the delay. The two special kernels are defined by

\[
f(t) = \frac{1}{\tau e^{-\frac{t}{\tau}}} \quad \text{and} \quad f(t) = \frac{t}{\tau^2} e^{-\frac{t}{\tau}},
\]

which are regarded as weak distributed delay and strong distributed delay respectively. We mainly consider the local strong distributed delay here.

Substituting \( u(x, t) = \phi(\xi) = \phi(x - ct) \) into equation (9), then we have the solitary wave equation

\[
(a - c)\phi' + b\eta\phi^{n-1}\phi' + k\phi''' + m\phi'''' + \tau(\phi'' + \phi''') = 0,
\]

(18)
where \( \dot{\eta} = \frac{d}{d\xi} \), and

\[
\eta(\xi) = (f * u)(x, t) = \int_{-\infty}^{t} f(t - s) u(x, s) ds = \int_{-\infty}^{t} t - s \frac{t - s}{\tau^2} e^{-\frac{t - s}{\tau}} u(x, s) ds = -\int_{0}^{\infty} \frac{z}{\tau^2} e^{-\frac{z}{\tau}} u(x, t - z) dz = \int_{0}^{+\infty} \frac{z}{\tau^2} e^{-\frac{z}{2}} \phi(x - c(t - z)) dz = \int_{0}^{+\infty} \frac{t}{\tau^2} e^{-\frac{t}{2}} \phi(\xi + ct) dt.
\]

Differentiating \( \eta \) with respect to \( \xi \), we obtain

\[
\frac{d\eta}{d\xi} = \int_{0}^{+\infty} \frac{t}{\tau^2} e^{-\frac{t}{2}} \frac{1}{\tau} \phi(\xi + ct) dt = \frac{1}{c} \int_{0}^{+\infty} \frac{t}{\tau} e^{-\frac{t}{\tau}} \phi(\xi + ct) dt = \frac{1}{c} \int_{0}^{+\infty} \frac{t}{\tau} e^{-\frac{t}{\tau}} d\phi = \frac{1}{c\tau} \left( \int_{0}^{+\infty} \frac{t}{\tau} e^{-\frac{t}{\tau}} \phi(\xi + ct) dt - \int_{0}^{+\infty} \frac{1}{\tau} e^{-\frac{t}{\tau}} \phi(\xi + ct) dt \right),
\]

i.e.,

\[
\frac{d\eta}{d\xi} = \frac{1}{c\tau}(\eta - \zeta),
\]

where

\[
\zeta = \int_{0}^{+\infty} \frac{1}{\tau} e^{-\frac{t}{\tau}} \phi(\xi + ct) dt.
\]

Differentiating both sides of \( \zeta \) with respect to \( \xi \), we get

\[
\frac{d\zeta}{d\xi} = \frac{1}{c\tau}(\zeta - \phi).
\]

Integrating equation (18) once, we obtain

\[
(a - c)\phi + bF + k\phi'' + m\phi'' + \tau(\phi' + \phi'') = 0,
\]

where

\[
F(\xi) = \int_{-\infty}^{\xi} \eta(s)\phi^{n-1}(s)\phi'(s) ds.
\]

The equation (19) is equivalent to the slow system of four first-order equations

\[
\begin{cases}
\phi' = \phi_1, \\
\phi_1' = \phi_2, \\
\phi_2' = \phi_3, \\
\phi_3' = \phi_4 - \frac{1}{m}(a - c)\phi + bF + k\phi + \tau(\phi_1 + \phi_3), \\
\phi_4' = \eta - \zeta, \\
\phi_5' = \zeta - \phi,
\end{cases}
\]
where \( \dot{\tau} = \frac{d}{d\xi} \) and it is a singularly perturbed system. Obviously system (20) degenerates to system (17) when \( \tau \to 0 \). It can be transformed into an equivalent problem by changing the time scale when \( \tau > 0 \). Let \( \xi = \tau z \), then system (20) is changed into the following fast system

\[
\begin{align*}
\dot{\phi} &= \tau \phi_1, \\
\dot{\phi}_1 &= \tau \phi_2, \\
\dot{\phi}_2 &= \tau \phi_3, \\
\dot{\phi}_3 &= -\frac{\tau}{m} [(a-c)\phi + bF + k\phi_2 + \tau(\phi_1 + \phi_3)], \\
\dot{\eta} &= \eta - \zeta, \\
\dot{\zeta} &= \zeta - \phi.
\end{align*}
\]

(21)

where \( \dot{\tau} = \frac{d}{d\xi} \). While the slow system (20) and the fast system (21) are equivalent when \( \tau > 0 \), the different time scales give rise to different limiting systems. Letting \( \tau \to 0 \) in the system (20), we obtain the reduced system

\[
\begin{align*}
\phi' &= \phi_1, \\
\phi'_1 &= \phi_2, \\
\phi'_2 &= \phi_3, \\
\phi'_3 &= -\frac{1}{m} [(a-c)\phi + bF + k\phi_2 + \tau(\phi_1 + \phi_3)], \\
0 &= \eta - \zeta, \\
0 &= \zeta - \phi.
\end{align*}
\]

(22)

Thus the flow of system (22) is constrained to the set

\[
M_0 = \{(\phi, \phi_1, \phi_2, \phi_3, \eta, \zeta) \in \mathbb{R}^6 : \eta = \zeta = \phi \}.
\]

If \( M_0 \) is normally hyperbolic, then we can use the geometric singular perturbation theory of Fenichel [16, 22] to obtain a four-dimensional invariant manifold \( M_\tau \) for the flow when \( 0 < \tau \ll 1 \), which implies the persistence of the slow manifold as well as the stable and unstable foliations. Thus the dynamics in the vicinity of the slow manifold are completely determined by the one on the slow manifold.

In fact, the linearized matrix of the fast system (21) restricted to \( M_0 \) is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{c} \\
-\frac{1}{c} & 0 & 0 & 0 & 0 & \frac{1}{c}
\end{pmatrix},
\]

which has six eigenvalues: 0, 0, 0, 0, \( \frac{1}{c}, \frac{1}{c} \), the number of eigenvalues with zero real part is equal to the dimension of \( M_0 \). Therefore, the manifold \( M_0 \) is normally hyperbolic.

Then we use the theorem on invariant manifolds of Fenichel [16] due to Jones [22].

**Lemma 3.1.** For the system

\[
\begin{align*}
x'(t) &= f(x, y, \epsilon), \\
y'(t) &= \epsilon g(x, y, \epsilon),
\end{align*}
\]

(23)
where \( x \in \mathbb{R}^n, y \in \mathbb{R}^l \) with \( n, l \geq 1 \) in general and \( \varepsilon \) is a real parameter, \( f, g \) are \( C^\infty \) on the set \( V \times I \) where \( V \in \mathbb{R}^{n+l} \) and \( I \) is an open interval containing 0. If when \( \varepsilon = 0 \), the system has a compact, normally hyperbolic manifold of critical points \( M_0 \), which is contained in the set \( \{ f(x, y, 0) = 0 \} \). Then for any \( 0 < r < +\infty \), if \( \varepsilon > 0 \), but sufficiently small, there exists a manifold \( M_\varepsilon \) such that the following hold.

(I) \( M_\varepsilon \) is locally invariant under the flow of system (23);

(II) \( M_\varepsilon \) is \( C^r \) in \( x, y \) and \( \varepsilon \);

(III) \( M_\varepsilon = \{ (x, y) : x = h^\varepsilon(y) \} \) for some \( C^r \) function \( h^\varepsilon(y) \) and \( y \) in some compact \( K \);

(IV) there exist locally invariant stable and unstable manifolds \( W^s(M_\varepsilon) \) and \( W^u(M_\varepsilon) \) that lie within \( O(\varepsilon) \), and are diffeomorphic to \( W^s(M_0) \) and \( W^u(M_0) \).

According to Lemma 3.1, there exists a four-dimensional manifold \( M_\tau, O(\tau) \) close and diffeomorphic to \( M_0 \) for \( \tau > 0 \), which can be written as

\[
M_\tau = \{ (\phi, \phi_1, \phi_2, \phi_3, \eta, \zeta) \in \mathbb{R}^5 : \eta = \zeta + g(\phi, \phi_1, \phi_2, \phi_3, \tau), \zeta = \phi + h(\phi, \phi_1, \phi_2, \phi_3, \tau) \},
\]

where the function \( g \) and \( h \) are smooth functions defined on a compact domain, and we have

\[
g(\phi, \phi_1, \phi_2, \phi_3, 0) = 0, \quad h(\phi, \phi_1, \phi_2, \phi_3, 0) = 0.
\]

Thus the functions \( g \) and \( h \) can be expanded into the form of a Taylor series with respect to the delay \( \tau \),

\[
g(\phi, \phi_1, \phi_2, \phi_3, \tau) = \tau g_1(\phi, \phi_1, \phi_2, \phi_3) + \tau^2 g_2(\phi, \phi_1, \phi_2, \phi_3) + \cdots,
\]

\[
h(\phi, \phi_1, \phi_2, \phi_3, \tau) = \tau h_1(\phi, \phi_1, \phi_2, \phi_3) + \tau^2 h_2(\phi, \phi_1, \phi_2, \phi_3) + \cdots. \tag{24}
\]

Substituting

\[
\eta = \zeta + g(\phi, \phi_1, \phi_2, \phi_3, \tau) \quad \text{and} \quad \zeta = \phi + h(\phi, \phi_1, \phi_2, \phi_3, \tau)
\]

into the slow system (20), we get

\[
c\tau \left( 1 + \frac{\partial g}{\partial \phi} + \frac{\partial h}{\partial \phi_1} \right) \phi_1 + \left( \frac{\partial g}{\partial \phi_1} + \frac{\partial h}{\partial \phi_2} \right) \phi_2 + \left( \frac{\partial g}{\partial \phi_2} + \frac{\partial h}{\partial \phi_3} \right) \phi_3 + \left( \frac{\partial g}{\partial \phi_3} + \frac{\partial h}{\partial \phi_3} \right) \phi_4 \right)
\]

\[
= \left( \frac{1}{m} \right) \left[ (a-c) \phi + bF + k \phi + \tau \phi + \phi_3 \right] \tag{25}
\]

Comparing the coefficients of \( \tau \) and \( \tau^2 \), we have

\[
g_1 = c \phi_1, \quad g_2 = 2c^2 \phi_2, \quad h_1 = c \phi_1, \quad h_2 = c^2 \phi_2. \tag{26}
\]

Then the slow system restricted to \( M_\tau \) is given by

\[
\begin{align*}
\phi' &= \phi_1, \\
\phi_1' &= \phi_2, \\
\phi_2' &= \phi_3, \\
\phi_3' &= -\frac{1}{m} [(a-c) \phi + bF + k \phi + \tau \phi + \phi_3] + \int_{-\infty}^{\xi} \phi^{n-1} \phi'(g + h) ds + k \phi_2 + \tau \phi_1 + \phi_3 \tag{27}
\end{align*}
\]

where \( g \) and \( h \) is given by (24) and (26). Obviously, the system (27) is simplified to the system (17) when \( \tau = 0 \). In the following, we prove that there exists the homoclinic orbit around the equilibria \( E_1 \), and the equation (26) has a solitary wave
solution around the equilibria point $E_1 (0, 0, 0, 0)$. The system (27) can be written as
\[
\begin{aligned}
\phi' &= \phi_1, \\
\phi_1' &= \phi_2, \\
\phi_2' &= \phi_3, \\
\phi_3' &= \Phi(\phi, \phi_1, \phi_2, \phi_3, c, \tau).
\end{aligned}
\]  
(28)

Note that
\[
\Phi(\phi, \phi_1, \phi_2, \phi_3, c, 0) = -\frac{1}{m}[(a - c)\phi + \frac{b}{n + 1}\phi^{n+1} + k\phi_2].
\]

From the above discussion, we know that the system (28) has traveling wave fronts when $\tau = 0$. Therefore, in the $(\phi, \phi_1, \phi_2, \phi_3)$ phase plane, it can be characterized as the graph of some function $\omega$, which means when $\tau = 0$, $\phi_2 = \omega(\phi, c)$. According to the stable manifold theorem, for sufficiently small $\tau > 0$, we can still characterize the stable manifold at
\[
\left(\sqrt[\frac{n}{2}]\frac{(c-a)(n+1)}{b}, 0, 0, 0\right)
\]
as the graph of some function $\phi_2 = \omega_1(\phi, c, \tau)$ where $\omega_1\left(\sqrt[\frac{n}{2}]\frac{(c-a)(n+1)}{b}, c, \tau \right) = 0$. Furthermore, based on continuous dependence of solution trajectories on parameters, the manifold must still cross the line $\phi = \frac{1}{2}\sqrt[\frac{n}{2}]\frac{(c-a)(n+1)}{b}$ somewhere provided $\tau$ is sufficiently small.

Similarly, let $\phi_2 = \omega_2(\phi, c, \tau)$ be the function for the unstable manifold at the origin. It satisfies $\omega_2(0, c, \tau) = 0$, and it must also cross the line $\phi = \frac{1}{2}\sqrt[\frac{n}{2}]\frac{(c-a)(n+1)}{b}$ somewhere for suitable sufficiently small $\tau$. Hence one has
\[
\omega_1(\phi, c, 0) = \omega_2(\phi, c, 0) = \omega(\phi, c).
\]  
(29)

For the unperturbed problem, fix a value of $c = c^*$, so that the equation of corresponding front in the phase plane is $\phi_2 = \omega(\phi, c^*)$. In order to show that a homoclinic connection exists in the perturbed problem ($\tau > 0$), we only need to prove that there exists a value of $c = c(\tau)$, so that the manifold $\omega_1$ and $\omega_2$ cross the line $\phi = \frac{1}{2}\sqrt[\frac{n}{2}]\frac{(c-a)(n+1)}{b}$ at a point.

Next we will use the implicit function theorem [27] to prove there exists a unique wave speed $c = c(\tau)$.

**Lemma 3.2.** Let $F$ be a real-valued continuously differentiable function defined in a neighborhood of $(X_0, Y_0) \in \mathbb{R}^2$. Suppose that $F$ satisfies the two conditions $F(X_0, Y_0) = Z_0$, $\frac{\partial F}{\partial Y}(X_0, Y_0) \neq 0$. Then there exist open intervals $U$ and $V$, with $X_0 \in U$, $Y_0 \in V$, and a unique function $F : U \to V$ satisfying
\[
F[X, F(X)] = Z_0, \text{ for all } X \in U,
\]
and this function $F$ is continuously differentiable with
\[
\frac{dY}{dX}(X_0) = F'(X_0) = -\frac{\partial F}{\partial X}(X_0, Y_0) / |\frac{\partial F}{\partial Y}(X_0, Y_0)|.
\]

For this purpose, we construct the auxiliary function
\[
M(c, \tau) = \omega_1\left(\frac{1}{2}\sqrt[\frac{n}{2}]\frac{(c-a)(n+1)}{b}, 0, c, \tau\right) - \omega_2\left(\frac{1}{2}\sqrt[\frac{n}{2}]\frac{(c-a)(n+1)}{b}, 0, c, \tau\right).
\]
Now we only need to verify $\frac{\partial M}{\partial c}|_{(c^*,0)} \neq 0$. It is easy to know
\[
\frac{d\phi_3}{d\phi_1} = \frac{\phi_3'}{\phi_1'} = \frac{\Phi(\phi, \phi_1, \phi_2, \phi_3, c, \tau)}{\phi_2}.
\]
Then
\[
\frac{d}{d\phi} \left( \frac{\partial \omega_1}{\partial c}(\phi, \phi_1, \phi_2, \phi_3, c^*, 0) \right)
= \frac{\partial}{\partial c} \left( \frac{\partial \omega_1}{\partial \phi}(\phi, c, 0) \right) \bigg|_{c=c^*}
= \frac{\partial}{\partial c} \left( \frac{\Phi(\phi, \phi_1, \omega_1(\phi, c, 0), c, 0)}{\omega_1(\phi, c, 0)} \right) \bigg|_{c=c^*}
= \frac{\partial}{\partial c} \left( -\frac{1}{m} \frac{(a-c)\phi + \frac{b}{n+1}\phi^{n+1} + k\phi_2}{\omega_1(\phi, c, 0)} \right) \bigg|_{c=c^*}
= \frac{1}{m} \left[ \left( c-a \right) \phi - \frac{b}{n+1}\phi^{n+1} - k\phi_2 \right] \frac{\partial \omega_1}{\partial c}(\phi, c^*, 0) - \frac{\phi}{\omega_1(\phi, c^*, 0)} \right].
\]
Let
\[
H = -\int_{\xi}^{\frac{1}{2} \sqrt{\frac{(c-a)(n+1)}{b}}} \frac{\left( (c-a) s - \frac{b}{n+1} s^{n+1} - k\phi_2 \right)}{\omega_1^2(s, c^*, 0)} ds.
\]
Integrating it from $\phi = \frac{1}{2} \sqrt{\frac{(c-a)(n+1)}{b}}$ to $\phi = \sqrt{\frac{(c-a)(n+1)}{b}}$, we have
\[
\frac{\partial \omega_1}{\partial c} \left( \frac{1}{2} \sqrt{\frac{(c-a)(n+1)}{b}}, c^*, 0 \right) = -\frac{1}{m} \frac{s}{\omega_1(s, c^*, 0)} \int_0^{\frac{1}{2} \sqrt{\frac{(c-a)(n+1)}{b}}} e^H d\xi. \tag{30}
\]
Similarly, we have
\[
\frac{d}{d\phi} \left( \frac{\partial \omega_2}{\partial c}(\phi, \phi_1, \phi_2, \phi_3, c^*, 0) \right)
= -\frac{1}{m} \left[ \left( c-a \right) \phi - \frac{b}{n+1}\phi^{n+1} - k\phi_2 \right] \frac{\partial \omega_1}{\partial c}(\phi, c^*, 0) + \frac{\phi}{\omega_1(\phi, c^*, 0)} \right].
\]
Integrating it from 0 to $\phi = \frac{1}{2} \sqrt{\frac{(c-a)(n+1)}{b}}$, we yield
\[
\frac{\partial \omega_2}{\partial c} \left( \frac{1}{2} \sqrt{\frac{(c-a)(n+1)}{b}}, c^*, 0 \right) = \frac{1}{m} \frac{s}{\omega_1(s, c^*, 0)} \int_0^{\frac{1}{2} \sqrt{\frac{(c-a)(n+1)}{b}}} e^H d\xi. \tag{31}
\]
From (30) and (31), one has
\[
\frac{\partial M}{\partial c}(c^*, 0) = \frac{\partial \omega_1}{\partial c} \left( \frac{1}{2} \sqrt{\frac{(c-a)(n+1)}{b}}, c^*, 0 \right) - \frac{\partial \omega_2}{\partial c} \left( \frac{1}{2} \sqrt{\frac{(c-a)(n+1)}{b}}, c^*, 0 \right)
= -\frac{1}{m} \frac{s}{\omega_1(s, c^*, 0)} \int_0^{\sqrt{\frac{(c-a)(n+1)}{b}}} e^H d\xi < 0. \tag{32}
\]
Thus we obtain the following main result.

Theorem 3.3. Assume that $b > 0$, $c > a$, $m > 0$ and $n$ is even, the generalized KdV equation (9) with the local strong general delay kernel exists a solitary wave solution for any sufficiently small $\tau > 0$. 
Remark 2. The existence of the solitary wave solution can be similarly established if the local delay kernel is a weak kernel, i.e., \( f(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} \).

3.2. Nonlocal delay and K-S perturbation. In this section, we consider the generalized Kawahara equation (9) with nonlocal delay and perturbation, where \( b > 0, \ c > a, \ m > 0 \) and \( n \) is even. The other cases can be discussed similarly. We mainly use geometric singular perturbation theory, the Fredholm theory and the linear chain trick. The corresponding convolution in equation (9) is described by (12), in which the kernel \( f : [0, +\infty) \to [0, +\infty) \) satisfies the following normalization assumption,
\[
\int_{-\infty}^{t} \int_{-\infty}^{+\infty} f(x, t) dx dt = 1.
\]
It is notable that the normalization assumption on \( f \) ensures that the uniform non-negative steady-state solutions are unaffected by the delay. The two special kernels are defined by
\[
f(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \quad \text{and} \quad f(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \frac{t}{\tau^2} e^{-\frac{t}{\tau}},
\]
where \( \tau > 0 \) in each case. The first of two kernels is sometimes called the weak generic delay kernel and the second one is regarded as the strong general delay kernel. The two kernels have been frequently used, and we mainly consider the nonlocal strong delay here.

3.2.1. The normally hyperbolic slow manifold \( M_0 \) and \( M_{\varepsilon} \).

In the following we define that
\[
v(x, t) = (f * u)(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{t-s}{\tau^2} u(y, s) dy ds. \tag{33}
\]
By a simple computation, we have
\[
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \frac{1}{\tau} w - \frac{1}{\tau} v, \tag{34}
\]
where
\[
w(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{1}{\tau} e^{-\frac{t-s}{\tau^2}} u(y, s) dy ds. \tag{35}
\]
Similarly, we obtain
\[
\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{1}{\tau} (u - w). \tag{36}
\]
Substituting (34) into (36), one has
\[
\frac{\partial^2 v}{\partial t^2} = 2 \frac{\partial^3 v}{\partial t \partial x^2} - \frac{\partial^4 v}{\partial x^4} + \frac{2}{\tau} \left( \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} \right) + \frac{1}{\tau^2} (u - v). \tag{37}
\]
Then the equation (9) can be written as
\[
\begin{aligned}
\left\{ \begin{array}{l}
u_t + au_x + bvu^{n-1}u_x + ku_{xxx} + mu_{xxxx} + \tau (u_{xx} + u_{xxxx}) = 0, \\
v_{tt} = 2v_{xx} - v_{xxxx} + \frac{2}{\tau} (v_{xx} - v_t) + \frac{1}{\tau^2} (u - v).
\end{array} \right. \tag{38}
\end{aligned}
\]
Let
\[
\phi(\xi) = u(x, t), \ \varphi(\xi) = v(x, t), \ \xi = x - ct,
\]
where $c > 0$ is wave speed, and we obtain the solitary wave equations

\[
\begin{align*}
&(a - c)\phi' + b\phi^n\phi' + k\phi'' + m\phi''' + \tau(\phi'' + \phi''') = 0, \\
&\phi''' + 2c\phi'' + c^2\phi' - \frac{2}{\tau}(\phi'' + c\phi') - \frac{1}{\tau^2}(\phi - \varphi) = 0,
\end{align*}
\]  

where $\phi' = \frac{d}{d\xi}$. Integrate the first equation of (39) once to yield

\[
\begin{align*}
&(a - c)\phi + b\int_{-\infty}^{\xi}\phi^n\phi' ds + k\phi'' + m\phi''' + \tau(\phi' + \phi''') = 0, \\
&\phi''' + 2c\phi'' + c^2\phi' - \frac{2}{\tau}(\phi'' + c\phi') - \frac{1}{\tau^2}(\phi - \varphi) = 0.
\end{align*}
\]  

In order to seek the solitary wave solution of the system (40), we convert (40) into an eight-dimensional system, which is given as

\[
\begin{align*}
&\phi' = \phi_1, \\
&\phi_1' = \phi_2, \\
&\phi_2' = \phi_3, \\
&\phi_3' = -\frac{1}{m}[(a - c)\phi + b\int_{-\infty}^{\xi}\phi^n\phi' ds + k\phi_2 + \tau(\phi_1 + \phi_3)], \\
&\phi' = \varphi_1, \\
&\varphi_1' = \varphi_2, \\
&\varphi_2' = \varphi_3, \\
&\varphi_3' = -2c\varphi_3 - c^2\varphi_2 + \frac{2}{\tau}(\varphi_2 + c\varphi_1) + \frac{1}{\tau^2}(\phi - \varphi).
\end{align*}
\]  

Let $\varepsilon = \sqrt{\tau}$ and define new variables

\[
\begin{align*}
u_1 = \phi, &\quad u_2 = \phi_1, &\quad u_3 = \phi_2, &\quad u_4 = \phi_3, \\
v_1 = \varphi, &\quad v_2 = \varepsilon\varphi_1, &\quad v_3 = \varepsilon^2\varphi_2, &\quad v_4 = \varepsilon^3\varphi_3.
\end{align*}
\]

Then the system (41) can be transformed into the slow system

\[
\begin{align*}
u_1' &= u_2, \\
u_2' &= u_3, \\
u_3' &= u_4, \\
u_4' &= -\frac{1}{m}[(a - c)u_1 + bG + ku_3 + \tau(u_2 + u_4)], \\
\varepsilon v_1' &= v_2, \\
\varepsilon v_2' &= v_3, \\
\varepsilon v_3' &= v_4, \\
\varepsilon v_4' &= -2c\varepsilon v_4 - c^2\varepsilon^2 v_3 + 2(v_3 + c\varepsilon v_2) + u_1 - v_1.
\end{align*}
\]  

where $' = \frac{d}{d\xi}$ and

\[
G(\xi) = \int_{-\infty}^{\xi} v_1(s)u_1^{n-1}(s)u_1'(s)ds.
\]
System (42) is a singularly perturbed system. Let $\xi = \varepsilon z$, the system (42) can be reformulated into the fast system

$$
\begin{align*}
\dot{u}_1 &= \varepsilon u_2, \\
\dot{u}_2 &= \varepsilon u_3, \\
\dot{u}_3 &= \varepsilon u_4, \\
\dot{u}_4 &= -\frac{\varepsilon}{m}[(a-c)u_1 + b \int_{-\infty}^{\varepsilon} v_1(s) u_1^{n-1}(s)u_1'(s)ds + ku_3 + \varepsilon^2(u_2 + u_4)], \\
\dot{v}_1 &= v_2, \\
\dot{v}_2 &= v_3, \\
\dot{v}_3 &= v_4, \\
\dot{v}_4 &= -2\varepsilon v_4 - \varepsilon^2 v_3 + 2(v_3 + \varepsilon v_2) + u_1 - v_1,
\end{align*}
$$

(43)

where $' = \frac{d}{dt}$. The slow system (42) and the fast system (43) are equivalent when $\varepsilon > 0$. In the slow system (42), the flow is confined to the set

$$
M_0 = \{(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4) \in \mathbb{R}^8 : u_1 = v_1, v_2 = v_3 = v_4 = 0\}.
$$

The linearization of the system restricted to $M_0$ is given by the matrix

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & -1 & 0 & 2 & 0
\end{pmatrix},
$$

which has eight eigenvalues: 0, 0, 0, 0, 1, 1, -1, -1, the number of eigenvalues with zero real part is equal to the dimension of $M_0$ and there are two stable and two unstable normal directions. Therefore, the manifold $M_0$ is normally hyperbolic.

Then by the geometric singular perturbation theory, there exists an invariant manifold $M_\varepsilon$ for sufficiently small $\varepsilon > 0$. $M_\varepsilon$ is closed to $M_0$ as well as diffeomorphism.

$$
M_\varepsilon = \{(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4) \in \mathbb{R}^8 : v_1 = g(u_1, u_2, u_3, u_4, \varepsilon) + u_1, \\
v_2 = h(u_1, u_2, u_3, u_4, \varepsilon), v_3 = p(u_1, u_2, u_3, u_4, \varepsilon), v_4 = q(u_1, u_2, u_3, u_4, \varepsilon)\},
$$

where $g$, $h$, $p$ and $q$ are smooth functions of $\varepsilon$, and satisfy

$$
g(u_1, u_2, u_3, u_4, 0) = h(u_1, u_2, u_3, u_4, 0) = p(u_1, u_2, u_3, u_4, 0) = q(u_1, u_2, u_3, u_4, 0) = 0.
$$

Thus the functions $g$, $h$, $p$ and $q$ can be expanded into the form of a Taylor series with respect to the delay $\varepsilon$

$$
g(u_1, u_2, u_3, u_4, \varepsilon) = \varepsilon g_1(u_1, u_2, u_3, u_4) + \varepsilon^2 g_2(u_1, u_2, u_3, u_4) + \cdots, \\
h(u_1, u_2, u_3, u_4, \varepsilon) = \varepsilon h_1(u_1, u_2, u_3, u_4) + \varepsilon^2 h_2(u_1, u_2, u_3, u_4) + \cdots, \\
p(u_1, u_2, u_3, u_4, \varepsilon) = \varepsilon p_1(u_1, u_2, u_3, u_4) + \varepsilon^2 p_2(u_1, u_2, u_3, u_4) + \cdots, \\
q(u_1, u_2, u_3, u_4, \varepsilon) = \varepsilon q_1(u_1, u_2, u_3, u_4) + \varepsilon^2 q_2(u_1, u_2, u_3, u_4) + \cdots.
$$

(44)

Substituting

$$
v_1 = g + u_1, \ v_2 = h, \ v_3 = p \text{ and } v_4 = q,
$$
into the slow system, we obtain
\[
\left\{ \begin{array}{l}
\varepsilon[(1 + \frac{\partial g}{\partial u_1})u_2 + \frac{\partial g}{\partial u_2}u_3 + \frac{\partial g}{\partial u_3}u_4 + \frac{\partial g}{\partial u_4}(\frac{1}{m}) [(a - c)u_1 + bG + ku_3 + \varepsilon^2(u_2 + u_4)] = h, \\
\varepsilon[\frac{\partial h}{\partial u_1}u_2 + \frac{\partial h}{\partial u_2}u_3 + \frac{\partial h}{\partial u_3}u_4 + \frac{\partial h}{\partial u_4}(\frac{1}{m}) [(a - c)u_1 + bG + ku_3 + \varepsilon^2(u_2 + u_4)] = p, \\
\varepsilon[\frac{\partial p}{\partial u_1}u_2 + \frac{\partial p}{\partial u_2}u_3 + \frac{\partial p}{\partial u_3}u_4 + \frac{\partial p}{\partial u_4}(\frac{1}{m}) [(a - c)u_1 + bG + ku_3 + \varepsilon^2(u_2 + u_4)] = q, \\
\varepsilon[\frac{\partial q}{\partial u_1}u_2 + \frac{\partial q}{\partial u_2}u_3 + \frac{\partial q}{\partial u_3}u_4 + \frac{\partial q}{\partial u_4}(\frac{1}{m}) [(a - c)u_1 + bG + ku_3 + \varepsilon^2(u_2 + u_4)] = -2\varepsilon q - \varepsilon^2 q^2 + 2(p + \varepsilon h) - g.
\end{array} \right.
\]
Comparing the coefficients of \( \varepsilon \) and \( \varepsilon^2 \), we get
\[
g_1(u_1, u_2, u_3, u_4) = 0, \quad g_2(u_1, u_2, u_3, u_4) = 2cu_2 + 2u_3, \\
h_1(u_1, u_2, u_3, u_4) = u_2, \quad h_2(u_1, u_2, u_3, u_4) = 0, \\
p_1(u_1, u_2, u_3, u_4) = 0, \quad p_2(u_1, u_2, u_3, u_4) = u_3, \\
q_1(u_1, u_2, u_3, u_4) = 0, \quad q_2(u_1, u_2, u_3, u_4) = 0.
\]
thus
\[
g(u_1, u_2, u_3, u_4, \varepsilon) = (2cu_2 + 2u_3)\varepsilon^2 + o(\varepsilon^2), \quad h(u_1, u_2, u_3, u_4, \varepsilon) = u_2\varepsilon + o(\varepsilon^2), \\
p(u_1, u_2, u_3, u_4, \varepsilon) = u_3\varepsilon^2 + o(\varepsilon^2), \quad q(u_1, u_2, u_3, u_4, \varepsilon) = o(\varepsilon^2).
\]
Next we contribute to studying the manifold \( M_\varepsilon \), the slow system restricted to \( M_\varepsilon \) is given by
\[
\left\{ \begin{array}{l}
u_1' = u_2, \\
u_2' = u_3, \\
u_3' = u_4, \\
u_4' = -\frac{1}{m}[(a - c)u_1 + \frac{b}{n + 1}u_{n+1} + b \int_{-\infty}^{\xi} g u_1' u_1' ds + ku_3 + \tau(u_2 + u_4)],
\end{array} \right.
\]
where \( g \) is given by (44) and (45). Obviously, the system (46) is simplified to system (17) when \( \varepsilon = 0 \).

3.2.2. The Fredholm theory and solitary wave solutions.
In the following, we will prove that there exists a homoclinic orbit around the equilibria point \( E_1(0, 0, 0, 0) \), then the equation has a solitary wave solution. Set
\[
\begin{align*}
u_1 &= u_0 + \varepsilon^2 \bar{\phi} + \cdots, \\
u_2 &= \bar{u}_0 + \varepsilon^2 \bar{\phi}_1 + \cdots, \\
u_3 &= \bar{u}_0 + \varepsilon^2 \bar{\phi}_2 + \cdots, \\
u_4 &= \bar{u}_0 + \varepsilon^2 \bar{\phi}_3 + \cdots.
\end{align*}
\]
Substituting transformation (47) into system (46) and comparing the coefficients of \( \varepsilon^2 \), the differential equation system determining \( \bar{\phi} \) and \( \bar{\phi}_i \) is
\[
\frac{d\Psi(\xi)}{d\xi} + P(\xi)\Psi(\xi) = Q(\xi),
\]
where
\[
\Psi(\xi) = \begin{pmatrix} \bar{\phi}(\xi) \\ \bar{\phi}_1(\xi) \\ \bar{\phi}_2(\xi) \\ \bar{\phi}_3(\xi) \end{pmatrix}, \quad P(\xi) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{1}{m}[a - c + bu_0^2] & 0 & \frac{k}{m} & 0 \end{pmatrix}.
\]
and

\[ Q(\xi) = \left(0, 0, 0, \frac{b}{m} \int_{-\infty}^{\xi} (2c\hat{u}_0 + 2\tilde{u}_0)u_0' ds - \frac{\hat{u}_0 + \tilde{u}_0}{m}\right)^T. \]

Next we shall show that system (48) has a solution satisfying

\[ \bar{\phi}(\pm \infty) = 0, \quad \bar{\phi}_1(\pm \infty) = 0, \quad \bar{\phi}_2(\pm \infty) = 0, \quad \bar{\phi}_3(\pm \infty) = 0. \]

Let

\[ l = \frac{d}{d\xi} + P(\xi), \]

and \( L^2 \) denotes the space of square integral functions, with inner product

\[ \int_{-\infty}^{+\infty} (X(\xi), Y(\xi)) d\xi, \]

where

\[ X(\xi) = (\bar{\phi}(\xi), \bar{\phi}_1(\xi), \bar{\phi}_2(\xi), \bar{\phi}_3(\xi))^T, \quad Y(\xi) = (\hat{\phi}(\xi), \hat{\phi}_1(\xi), \hat{\phi}_2(\xi), \hat{\phi}_3(\xi))^T, \]

and \((\cdot, \cdot)\) represents the Euclidean inner product on \( \mathbb{R}^4 \). From the Fredholm theory, we know that system (48) will have a solution if and only if

\[ \int_{-\infty}^{+\infty} (X(\xi), Q(\xi)) d\xi, \]

for all functions \( X(\xi) \in \mathbb{R}^4 \) in the kernel of the adjoint of the operator \( l \). It is easy to verify that the adjoint operator \( l^* \) is given by

\[ l^* = -\frac{d}{d\xi} + P^T(\xi), \]

where

\[ P^T(\xi) = \begin{pmatrix} 0 & 0 & 0 & \frac{a-c}{m} + \frac{bn_0}{m} \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \frac{k}{m} \\ 0 & 0 & -1 & 0 \end{pmatrix}. \]

In order to compute \( \text{Ker} l^* \), we find that all \( X(\xi) \) satisfying \( l^* X(\xi) = 0 \), i.e.,

\[ \frac{dX(\xi)}{d\xi} = P^T(\xi) X(\xi), \quad (49) \]

the general solution of the equation (49) is difficult to find because the matrix \( P^T(\xi) \) is nonconstant. However, we are only looking for solutions satisfying \( X(\pm \infty) = 0 \), and in fact, the only such solution is the zero solution. Recall that \( u_0(\xi) \) is the solution of the unperturbed problem and although we have no explicit express for it, we do know that \( u_0(\xi) \) tends to zero as \( \xi \to -\infty \). Letting \( \xi \to -\infty \), the matrix becomes a constant matrix

\[ \begin{pmatrix} 0 & 0 & 0 & \frac{a-c}{m} \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \frac{k}{m} \\ 0 & 0 & -1 & 0 \end{pmatrix}, \]

and the eigenvalues \( \lambda \) satisfying

\[ \lambda^4 + \frac{k}{m} \lambda^2 + \frac{a-c}{m} = 0. \]
Due to the conditions $b > 0$, $c > a$, $m > 0$ and $n$ is even, there are two real nonzero eigenvalues (one is positive, the other one is negative) and two pure imaginary eigenvalues. Thus the only solution satisfying $X(\pm \infty) = 0$ is the zero solution.

This means that the Fredholm orthogonality condition trivially holds

$$
\int_{-\infty}^{+\infty} (X(\xi), Q(\xi)) d\xi = \int_{-\infty}^{+\infty} (0, Q(\xi)) d\xi = 0,
$$

and the solutions of (48) exist, which satisfy

$$
\widetilde{\phi}(\pm \infty) = 0, \quad \widetilde{\phi}_1(\pm \infty) = 0, \quad \widetilde{\phi}_2(\pm \infty) = 0, \quad \widetilde{\phi}_3(\pm \infty) = 0.
$$

Therefore a homoclinic connection exists around the equilibrium point $E_1(0, 0, 0, 0)$ for sufficiently small $\varepsilon > 0$. Furthermore, the equation (9) with nonlocal delay and perturbation exists a solitary wave solution.

**Theorem 3.4.** Assume that $b > 0$, $c > a$, $m > 0$ and $n$ is even, the generalized Kawahara equation (9) with the nonlocal strong general delay kernel exists a solitary wave solution for any sufficiently small $\tau > 0$.

**Remark 3.** The existence of the solitary wave solution can be similarly established if the nonlocal delay kernel is a weak kernel, i.e., $f(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} e^{-\frac{t}{\tau}}$.

**Remark 4.** This paper mainly discusses the existence of solitary wave solutions for the case $b > 0$, $c > a$, $m > 0$ and $n$ is even. In fact, we could obtain the existence of solitary (traveling) wave solutions for other cases of the generalized Kawahara equation (9) with local or nonlocal delay by similar discussions, and we omit the details here.

4. **Asymptotic behavior.** In this section, we obtain the description of the asymptotic behavior of the traveling waves by using the asymptotic theory. There are many literatures focused on the asymptotic theory [10, 13].

4.1. **Analysis for the reduced equation (16).** Let $\Phi(\xi) = \phi(\xi)$ be the traveling wave solution of solitary wave equation (16) for the generalized Kawahara equation (9) without delay and perturbation. Differentiate the equation (16) with respect to $\xi$ and denote $\Phi' = \tilde{\phi}(\xi)$,

we obtain

$$
(a - c)\tilde{\phi} + b\phi^p \tilde{\phi} + k\phi''' + m\phi'''' = 0.
$$

(50)

From section 2, we know $\lim_{\xi \to \pm \infty} \phi(\xi) = 0$, then the limiting system of (50) as $\xi \to \pm \infty$ is

$$
(a - c)\tilde{\phi}_\pm + k\tilde{\phi}_\pm'' = 0,
$$

(51)

which is equivalent to

$$
\begin{cases}
\tilde{\phi}_1' = \tilde{\phi}_1, \\
\tilde{\phi}_2' = \tilde{\phi}_2, \\
\tilde{\phi}_3' = \frac{-1}{m} [(a - c)\tilde{\phi}_2'' + k\tilde{\phi}_2].
\end{cases}
$$

(52)

We rewrite system (52) as

$$
Z'_1 = S_1Z_1,
$$

(53)
where
\[
Z_1 = \begin{pmatrix}
\tilde{\phi}_\pm(\xi) \\
\tilde{\phi}_{1\pm}(\xi) \\
\tilde{\phi}_{2\pm}(\xi) \\
\tilde{\phi}_{3\pm}(\xi)
\end{pmatrix}, \quad S_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{-a-c}{m} & 0 & -\frac{k}{m} & 0
\end{pmatrix}.
\]

One can easily obtain that the general solution of the system (53) has the form
\[
(\tilde{\phi}_\pm(\xi), \tilde{\phi}_{1\pm}(\xi), \tilde{\phi}_{2\pm}(\xi), \tilde{\phi}_{3\pm}(\xi))^T = \sum_{i=1}^4 \alpha_i d_i e^{\lambda_i \xi}, \quad (54)
\]
where \(d_i (i = 1, 2, 3, 4)\) are eigenvectors of the matrix \(S_1\) in (53) with \(\lambda_i\) as corresponding eigenvalues, \(\alpha_i\) are arbitrary constants. We can calculate the solutions \(\lambda_i (i = 1, 2, 3, 4)\) of the characteristic equation, which are respectively
\[
\lambda_1 = -\frac{k}{2m} + \sqrt{\frac{k^2 - 4m(a-c)}{4m^2}} > 0, \quad \lambda_2 = -\frac{k}{2m} - \sqrt{\frac{k^2 - 4m(a-c)}{4m^2}} < 0,
\]
\[
\lambda_3 = i\sqrt{\frac{k}{2m} + \sqrt{\frac{k^2 - 4m(a-c)}{4m^2}}}, \quad \lambda_4 = -i\sqrt{\frac{k}{2m} + \sqrt{\frac{k^2 - 4m(a-c)}{4m^2}}}.
\]

Since
\[
(\tilde{\phi}_-(\xi), \tilde{\phi}_{1-}(\xi), \tilde{\phi}_{2-}(\xi), \tilde{\phi}_{3-}(\xi))^T \to (0, 0, 0, 0)^T, \quad \text{as} \ \xi \to -\infty,
\]
and from (54), we get that \(\alpha_2 = \alpha_3 = \alpha_4 = 0\), and
\[
(\tilde{\phi}_-(\xi), \tilde{\phi}_{1-}(\xi), \tilde{\phi}_{2-}(\xi), \tilde{\phi}_{3-}(\xi))^T = \alpha_1 d_1 e^{\lambda_1 \xi}.
\]
Thus we deduce the following asymptotic behavior as \(\xi \to -\infty\)
\[
\tilde{\phi}_-(\xi) = \mu_1 (\gamma_1 + o(1)) e^{\lambda_1 \xi}, \quad (55)
\]
where \(\gamma_1\) are constants and \(\mu_1\) cannot be zero simultaneously. If the first component of eigenvector \(d_1\) is zero, then the matrix \(S_1\) implies that the other components are zero, which implies the \(\gamma_1 \neq 0\).

Again since
\[
(\tilde{\phi}_+(\xi), \tilde{\phi}_{1+}(\xi), \tilde{\phi}_{2+}(\xi), \tilde{\phi}_{3+}(\xi))^T \to (0, 0, 0, 0)^T, \quad \text{as} \ \xi \to +\infty,
\]
and from (54), we get that \(\alpha_1 = \alpha_3 = \alpha_4 = 0\) and
\[
(\tilde{\phi}_+(\xi), \tilde{\phi}_{1+}(\xi), \tilde{\phi}_{2+}(\xi), \tilde{\phi}_{3+}(\xi))^T = \alpha_2 d_2 e^{\lambda_2 \xi}.
\]
Thus we deduce the following asymptotic behavior as \(\xi \to +\infty\)
\[
\tilde{\phi}_+(\xi) = \mu_2 (\gamma_2 + o(1)) e^{\lambda_2 \xi}, \quad (56)
\]
where \(\gamma_2\) are constants and \(\mu_2\) cannot be zero simultaneously. Then we claim that \(\gamma_2 \neq 0\) similarly as \(\xi \to -\infty\).

**Theorem 4.1.** Assume that \(b > 0\), \(c > a\), \(m > 0\) and \(n\) is even, then there exist positive constants \(K\) and \(L\) such that the generalized Kawahara equation (9) without delay and perturbation has a traveling wave solution \(\Phi(\xi)\) with the following asymptotic properties
\[
\Phi(\xi) = (K + o(1)) e^{\lambda_1 \xi}, \quad \text{as} \ \xi \to -\infty,
\]
and
\[
\Phi(\xi) = -(L + o(1)) e^{\lambda_2 \xi}, \quad \text{as} \ \xi \to +\infty.
\]
4.2. Analysis for the equation (9) with local delay and K-S perturbation.

Let \( \Phi(\xi) = (\phi(\xi), \eta(\xi), \zeta(\xi))^T \) be the traveling wave solution of the following solitary wave equations for the generalized Kawahara equation (9) with local delay and K-S perturbation, i.e.,

\[
\begin{align*}
(a - c)\phi + \frac{b}{n + 1} \eta \phi' + k \phi'' + m \phi''' + \tau (\phi' + \phi'') &= 0, \\
ct \eta' &= \eta - \zeta, \\
ct \zeta' &= \zeta - \phi.
\end{align*}
\]

Differentiating the equation (57) with respect to \( \xi \) and denoting 
\[
\Phi'(\xi) = (\tilde{\phi}(\xi), \tilde{\eta}(\xi), \tilde{\zeta}(\xi))^T,
\]
we obtain

\[
\begin{align*}
(a - c)\tilde{\phi} + \frac{b}{n + 1} \tilde{\eta} \phi' + k \tilde{\phi}' + m \tilde{\phi}'' + \tau (\tilde{\phi}' + \tilde{\phi}'') &= 0, \\
ct \tilde{\eta}' &= \tilde{\eta} - \tilde{\zeta}, \\
ct \tilde{\zeta}' &= \tilde{\zeta} - \tilde{\phi}.
\end{align*}
\]

From section 3, we know \( \lim_{\xi \to \pm \infty} \phi(\xi) = 0 \), the limiting system of (58) as \( \xi \to \pm \infty \) is

\[
\begin{align*}
(a - c)\tilde{\phi} + k \tilde{\phi}' + m \tilde{\phi}'' + \tau (\tilde{\phi}' + \tilde{\phi}'') &= 0, \\
ct \tilde{\eta}' &= \tilde{\eta} - \tilde{\zeta}, \\
ct \tilde{\zeta}' &= \tilde{\zeta} - \tilde{\phi}.
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
\tilde{\phi}' &= \tilde{\phi}_{1\pm}, \\
\tilde{\phi}_{1\pm} &= \tilde{\phi}_{2\pm}, \\
\tilde{\phi}_{2\pm} &= \tilde{\phi}_{3\pm}, \\
\tilde{\phi}_{3\pm} &= -\frac{1}{m}[(a - c)\tilde{\phi} + k \tilde{\phi}' + \tau (\tilde{\phi}' + \tilde{\phi}')] , \\
\tilde{\eta}' &= \frac{1}{ct} (\tilde{\eta} - \tilde{\zeta}), \\
\tilde{\zeta}' &= \frac{1}{ct} (\tilde{\zeta} - \tilde{\phi}).
\end{align*}
\]

The system (60) can be rewritten as

\[
Z' = S_2 Z_2,
\]

where

\[
Z_2 = \begin{pmatrix}
\tilde{\phi}_{1\pm}(\xi) \\
\tilde{\phi}_{2\pm}(\xi) \\
\tilde{\phi}_{3\pm}(\xi) \\
\tilde{\eta}_{1\pm}(\xi) \\
\tilde{\zeta}_{1\pm}(\xi)
\end{pmatrix},
\]

\[
S_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-\frac{a - c}{m} & -\frac{\tau}{m} & -\frac{k}{m} & -\frac{\tau}{m} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{ct} & 0 \\
-\frac{1}{ct} & 0 & 0 & 0 & 0 & \frac{1}{ct}
\end{pmatrix}.
\]
Obviously the eigenvalues of the matrix $S_2$ in (61) are

$$A_5 = A_6 = \frac{1}{c^2} > 0,$$

and the remaining $A_i (i = 1, 2, 3, 4)$ are the solutions of the characteristic equation

$$A^4 + \frac{\tau}{m} A^3 + \frac{k}{m} A^2 + \frac{\tau}{m} A + \frac{a - c}{m} = 0.$$  \hspace{1cm} (62)

From Viete Theorem and the conditions $b > 0, \ c > a, \ m > 0$, we have

$$A_1 + A_2 + A_3 + A_4 = -\frac{\tau}{m} < 0, \quad A_1 A_2 A_3 A_4 = \frac{a - c}{m} < 0.$$

By a direct calculation, for sufficiently small $\tau > 0$, we find that the equation has a pair imaginary root, a positive real root and a negative real root. Then we have the following three cases.

**Case 1**: $\text{Re} A_1 = \text{Re} A_2 > 0$, $A_3 > 0$, $A_4 < 0$, $A_5 = A_6 > 0$.

Since

$$(\tilde{\phi}_-(\xi), \tilde{\phi}_1-(\xi), \tilde{\phi}_2-(\xi), \tilde{\phi}_3-(\xi), \tilde{\eta}_-(\xi), \tilde{\zeta}_-(\xi))^T \to (0, 0, 0, 0, 0)^T, \text{ as } \xi \to -\infty,$$

we obtain

$$(\tilde{\phi}_-(\xi), \tilde{\phi}_1-(\xi), \tilde{\phi}_2-(\xi), \tilde{\phi}_3-(\xi), \tilde{\eta}_-(\xi), \tilde{\zeta}_-(\xi))^T = \sum_{i=1}^{3} \beta_i s_i e^{\Lambda_i \xi}$$

\hspace{1cm} + \sum_{i=5}^{6} \beta_i s_i \{I + (S - \Lambda_i I)\xi\} e^{\Lambda_i \xi}.

Then we deduce the following asymptotic behavior as $\xi \to -\infty$

$$\left(\begin{array}{c}
\tilde{\phi}_-(\xi) \\
\tilde{\eta}_-(\xi) \\
\tilde{\zeta}_-(\xi)
\end{array}\right) = \left(\begin{array}{c}
\sum_{i=1}^{3} \mu_i (\gamma_i + o(1)) e^{\Lambda_i \xi} + \sum_{i=5}^{6} \mu_i \{(\gamma_i + o(1)) + (\gamma_{ii} + o(1))\xi\} e^{\Lambda_i \xi} \\
\sum_{i=1}^{3} \mu_i (\bar{\gamma}_i + o(1)) e^{\Lambda_i \xi} + \sum_{i=5}^{6} \mu_i \{(\bar{\gamma}_i + o(1)) + (\bar{\gamma}_{ii} + o(1))\xi\} e^{\Lambda_i \xi} \\
\sum_{i=1}^{3} \mu_i (\tilde{\gamma}_i + o(1)) e^{\Lambda_i \xi} + \sum_{i=5}^{6} \mu_i \{(\tilde{\gamma}_i + o(1)) + (\tilde{\gamma}_{ii} + o(1))\xi\} e^{\Lambda_i \xi}
\end{array}\right),$$

where $\gamma_{ii}, \bar{\gamma}_{ii}, \gamma_{ii}, \bar{\gamma}_{ii}, \gamma_{ii} \neq 0 (i = 1, \cdots, 6)$ are constants and $\mu_i$ cannot be zero simultaneously.

Again since

$$(\tilde{\phi}_+(\xi), \tilde{\phi}_1+(\xi), \tilde{\phi}_2+(\xi), \tilde{\phi}_3+(\xi), \tilde{\eta}_+(\xi), \tilde{\zeta}_+(\xi))^T \to (0, 0, 0, 0, 0)^T, \text{ as } \xi \to +\infty,$$

we obtain

$$(\tilde{\phi}_+(\xi), \tilde{\phi}_1+(\xi), \tilde{\phi}_2+(\xi), \tilde{\phi}_3+(\xi), \tilde{\eta}_+(\xi), \tilde{\zeta}_+(\xi))^T = \beta_4 s_4 e^{\Lambda_4 \xi}.$$

Then we deduce the following asymptotic behavior as $\xi \to +\infty$

$$\left(\begin{array}{c}
\tilde{\phi}_+(\xi) \\
\tilde{\eta}_+(\xi) \\
\tilde{\zeta}_+(\xi)
\end{array}\right) = \left(\begin{array}{c}
\mu_4 (\gamma_4 + o(1)) e^{\Lambda_4 \xi} \\
\mu_4 (\bar{\gamma}_4 + o(1)) e^{\Lambda_4 \xi} \\
\mu_4 (\tilde{\gamma}_4 + o(1)) e^{\Lambda_4 \xi}
\end{array}\right),$$

where $\gamma_4, \bar{\gamma}_4, \tilde{\gamma}_4 \neq 0$ are constants and $\mu_4$ cannot be zero simultaneously.

**Case 2**: $\text{Re} A_1 = \text{Re} A_2 < 0$, $A_3 > 0$, $A_4 < 0$, $A_5 = A_6 > 0$. 
Then we deduce the following asymptotic behavior as \( \xi \to -\infty \)

\[
\begin{pmatrix}
\tilde{\phi}_-(\xi) \\
\tilde{\eta}_-(\xi) \\
\tilde{\zeta}_-(\xi)
\end{pmatrix} = 
\begin{pmatrix}
\mu_3(\gamma_3 + o(1))e^{\Lambda_3\xi} + \sum_{i=5}^{6} \mu_i\{(\gamma_i + o(1)) + (\gamma_{ii} + o(1))\}e^{\Lambda_i\xi} \\
\mu_3(\tilde{\gamma}_3 + o(1))e^{\Lambda_3\xi} + \sum_{i=5}^{6} \mu_i\{(\tilde{\gamma}_i + o(1)) + (\tilde{\gamma}_{ii} + o(1))\}e^{\Lambda_i\xi} \\
\mu_3(\tilde{\gamma}_3 + o(1))e^{\Lambda_3\xi} + \sum_{i=5}^{6} \mu_i\{(\tilde{\gamma}_i + o(1)) + (\tilde{\gamma}_{ii} + o(1))\}e^{\Lambda_i\xi}
\end{pmatrix}.
\]

Again since (64), we obtain

\[
(\tilde{\phi}_+(\xi), \tilde{\phi}_1+(\xi), \tilde{\phi}_2+(\xi), \tilde{\phi}_3+(\xi), \tilde{\phi}_4+(\xi), \tilde{\chi}_+(\xi), \tilde{\xi}_+(\xi)) = 
\sum_{i=1}^{2} \beta_i s_i e^{\Lambda_i\xi} + \beta_4 s_4 e^{\Lambda_4\xi}.
\]

Then we deduce the asymptotic behavior as \( \xi \to +\infty \)

\[
\begin{pmatrix}
\tilde{\phi}_+(\xi) \\
\tilde{\eta}_+(\xi) \\
\tilde{\chi}_+(\xi)
\end{pmatrix} = 
\begin{pmatrix}
\sum_{i=1}^{2} \mu_i(\gamma_i + o(1))e^{\Lambda_i\xi} + \mu_4(\gamma_4 + o(1))e^{\Lambda_4\xi} \\
\sum_{i=1}^{2} \mu_i(\tilde{\gamma}_i + o(1))e^{\Lambda_i\xi} + \mu_4(\tilde{\gamma}_4 + o(1))e^{\Lambda_4\xi} \\
\sum_{i=1}^{2} \mu_i(\tilde{\gamma}_i + o(1))e^{\Lambda_i\xi} + \mu_4(\tilde{\gamma}_4 + o(1))e^{\Lambda_4\xi}
\end{pmatrix}.
\]

**Case 3:** \( \text{Re}\Lambda_1 = \text{Re}\Lambda_2 = 0, \Lambda_3 > 0, \Lambda_4 < 0, \Lambda_5 = \Lambda_6 > 0. \)

The asymptotic behavior as \( \xi \to -\infty \) is the same as **Case 2**. And the asymptotic behavior as \( \xi \to +\infty \) is the same as **Case 1**.

**Theorem 4.2.** Assume that \( b > 0, c > a, m > 0 \) and \( n \) is even, there exist positive constant \( K_i (i = 1, 2, \cdots, 6) \) and \( L_j (j = 1, 2, 3) \) such that the generalized Kawahara equation (9) with local delay and perturbation has a traveling wave solution \( \Phi(\xi) \) with the following asymptotic properties

\[
\Phi(\xi) = \begin{pmatrix}
(K_1 + o(1))e^{\Lambda_1\xi} + (K_2 + o(1))\xi e^{\Lambda_2\xi} \\
(K_3 + o(1))e^{\Lambda_3\xi} + (K_4 + o(1))\xi e^{\Lambda_4\xi} \\
(K_5 + o(1))e^{\Lambda_5\xi} + (K_6 + o(1))\xi e^{\Lambda_6\xi}
\end{pmatrix}, \quad \text{as} \quad \xi \to -\infty,
\]

and

\[
\Phi(\xi) = \begin{pmatrix}
-(L_1 + o(1))e^{\Lambda_1\xi} \\
-(L_2 + o(1))e^{\Lambda_2\xi} \\
-(L_3 + o(1))e^{\Lambda_3\xi}
\end{pmatrix}, \quad \text{as} \quad \xi \to +\infty,
\]

where \( \Lambda_1, \Lambda_3 \) are given in (62).

**Case 1:** If \( \text{Re}\Lambda_1 = \text{Re}\Lambda_2 > 0, \Lambda_1 \) may be \( \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_5, \) and \( \Lambda_3 \) may be \( \Lambda_4. \)

**Case 2:** If \( \text{Re}\Lambda_1 = \text{Re}\Lambda_2 < 0, \Lambda_1 \) may be \( \Lambda_3, \Lambda_5, \) and \( \Lambda_3 \) may be \( \Lambda_1, \Lambda_2, \Lambda_4. \)

**Case 3:** If \( \text{Re}\Lambda_1 = \text{Re}\Lambda_2 = 0, \Lambda_1 \) may be \( \Lambda_3, \Lambda_5, \) and \( \Lambda_3 \) may be \( \Lambda_4. \)

**Remark 5.** We could obtain the asymptotic properties of solitary wave solutions for the generalized Kawahara equation (9) with no local delay and perturbation by
similar discussions. It prompts us to pay attention to two quartic equations and we omit the details here.

5. Illustrating examples. As applications, we present several examples to illustrate our main results in Theorem 3.3 and Theorem 4.2 by utilizing Matlab.

According to the physical and practical meanings of the parameters $a$, $b$, $k$, $m$, $n$, $\tau$ in the generalized Kawahara equation (9) and Theorem 3, we choose the parameter values that satisfy $b > 0$, $c > a$, $m > 0$, $c > 0$, $0 < \tau \ll 1$ and $n$ is even. In addition, $a$ and $k$ are arbitrary constants.

Example 1: Consider the following generalized Kawahara equation with local delay and perturbation

$$u_t + 2u_x + (f * u)u_x + 4u_{xxx} + 10u_{xxxx} + 10^{-6}(u_{xx} + u_{xxxx}) = 0, \quad (65)$$

where $\tau = 10^{-6}$, $m = 10$, $k = 4$, $a = 2$, $c = 4$, $b = 1$, $n = 2$, and

$$(f * u)(x, t) = \int_{-\infty}^{t} f(t - s)u(x, s)ds = \int_{-\infty}^{t} \frac{t - s}{\tau^2} e^{-\frac{t - s}{\tau}} u(x, s)ds, \quad \tau = 4.$$

By a direct calculation, we obtain

$$\frac{\partial M}{\partial c}(c^*, 0) = -\frac{s}{10\omega_1(s, c^*, 0)} \int_{-\infty}^{\sqrt{6}} e^H d\xi < 0.$$

By Theorem 3.3, the equation (65) has a solitary wave solution.

Next we calculate the eigenvalues of the matrix $S_2$ in (61) as follows

$$\Lambda_1 = 1.58 \times 10^{-8} + 0.8306009563i, \quad \Lambda_2 = 1.58 \times 10^{-8} - 0.8306009563i,$$

$$\Lambda_3 = 0.5384216542, \quad \Lambda_4 = -0.5384217858, \quad \Lambda_5 = \Lambda_6 = 2.5 \times 10^5.$$

This is the Case 1 in Theorem 4.2, therefore there exist positive constants $K_i (i = 1, \cdots, 6)$ and $L_j (j = 1, 2, 3)$ such that the generalized Kawahara system (9) with local delay and perturbation has a traveling wave solution $\Phi(\xi)$ with the following asymptotic properties

$$\Phi(\xi) = \begin{pmatrix} (K_1 + o(1))e^{\Lambda_1\xi} + (K_2 + o(1))\xi e^{2.5 \times 10^5\xi} \\ (K_3 + o(1))e^{\Lambda_1\xi} + (K_4 + o(1))\xi e^{2.5 \times 10^5\xi} \\ (K_5 + o(1))e^{\Lambda_1\xi} + (K_6 + o(1))\xi e^{2.5 \times 10^5\xi} \end{pmatrix}, \quad \text{as} \quad \xi \rightarrow -\infty,$$

where $\Lambda_1$ may be $1.58 \times 10^{-8} + 0.8306009563i$, $1.58 \times 10^{-8} - 0.8306009563i$, $0.5384216542$, $2.5 \times 10^5$. And

$$\Phi(\xi) = \begin{pmatrix} -(L_1 + o(1))e^{-0.5384217858\xi} \\ -(L_2 + o(1))e^{-0.5384217858\xi} \\ -(L_3 + o(1))e^{-0.5384217858\xi} \end{pmatrix}, \quad \text{as} \quad \xi \rightarrow +\infty.$$

In order to present the results directly and clearly, we chose $\Lambda_1 = 0.5384216542$ and gave simulations. The graph of the traveling wave solution $\Phi(\xi)$ is shown in Figure 1.

Example 2: Consider the following generalized Kawahara equation with local delay and perturbation

$$u_t + 2u_x + (f * u)u_x + 4u_{xxx} + 10u_{xxxx} + 10^{-6}(u_{xx} + u_{xxxx}) = 0, \quad (66)$$

asymptotic properties
where $\tau = 10^{-6}$, $m = 10$, $k = 4$, $a = 2$, $c = 10^6$, $b = 1$, $n = 2$, and

$$(f * u)(x,t) = \int_{-\infty}^{t} f(t-s)u(x,s)ds = \int_{-\infty}^{t} \frac{t-s}{\tau^2} e^{-\frac{t-s}{\tau}} u(x,s)ds,$$

By a direct calculation, we obtain

$$\frac{\partial M}{\partial c}(c^*,0) = -\frac{s}{10\omega_1(s,c^*,0)} \int_0^{\sqrt{3\times(10^6-2)}} e^H d\xi < 0.$$ 

By Theorem 3.3, the equation (66) has a solitary wave solution.

Next we calculate the eigenvalues of the matrix $S_2$ in (61) as follows:

$$\Lambda_1 = -2.4936752 \times 10^{-8} + 17.788409513908885i,$$

$$\Lambda_2 = -2.4936752 \times 10^{-8} - 17.788409513908885i,$$

$$\Lambda_3 = 17.777162657280726, \quad \Lambda_4 = -17.777162707407236, \quad \Lambda_5 = \Lambda_6 = 1.$$ 

This is the Case 2 in Theorem 4.2, therefore there exist positive constants $K_i (i = 1, \cdots, 6)$ and $L_j (j = 1, \cdots, 6)$ such that the generalized Kawahara system (9) with local delay and perturbation has a traveling wave solution $\Phi(\xi)$ with the following asymptotic properties

$$\Phi(\xi) = \begin{pmatrix} (K_1 + o(1))e^{\Lambda_1\xi} + (K_2 + o(1))\xi e^{\xi} \\ (K_3 + o(1))e^{\Lambda_2\xi} + (K_4 + o(1))\xi e^{\xi} \\ (K_5 + o(1))e^{\Lambda_3\xi} + (K_6 + o(1))\xi e^{\xi} \end{pmatrix}, \quad \text{as} \quad \xi \to -\infty,$$
where Λ_i may be $17.777162657280726$, 1. And

$$\Phi(\xi) = \begin{pmatrix} -(L_1 + o(1))e^{\Lambda_i \xi} \\ -(L_2 + o(1))e^{\Lambda_j \xi} \\ -(L_3 + o(1))e^{\Lambda_j \xi} \end{pmatrix}, \quad \text{as} \quad \xi \to +\infty,$$

where Λ_j may be $-17.777162707407236$, $-2.4936752 \times 10^{-8} + 17.78840951390885i$, $-2.4936752 \times 10^{-8} - 17.78840951390885i$.

Similarly we chose Λ_i = 1, Λ_j = $-17.777162707407236$ and gave simulations.

The graph of the traveling wave solution Φ(ξ) is shown in Figure 2.

**Example 3**: Consider the following generalized Kawahara equation with local delay and perturbation

$$u_t - u_x + (f \ast u)u_x + 2u_{xxx} + 2u_{xxxx} + 10^{-4}(u_{xx} + u_{xxxx}) = 0,$$

where τ = $10^{-4}$, m = 2, k = 2, a = $-1$, c = 1, b = 1, n = 2, and

$$\int_{-\infty}^{t} f(t-s)u(x,s)ds = \int_{-\infty}^{t} \frac{t-s}{\tau^2} e^{-\frac{(t-s)}{\tau^2}} u(x,s)ds,$$

$$= \int_{-\infty}^{t} 10^8(t-s)e^{-10^8(t-s)} u(x,s)ds,$$

By a direct calculation, we obtain

$$\frac{\partial M}{\partial c}(c^*, 0) = -\frac{s}{2\omega_1(s, c^*, 0)} \int_{0}^{\sqrt{s}} e^{Hd}d\xi < 0.$$
By Theorem 3.3, the equation (67) has a solitary wave solution. Next we calculate the eigenvalues of the matrix $S_2$ in (61) as follows:

$\Lambda_1 = i$, $\Lambda_2 = -i$, $\Lambda_3 = 1$, $\Lambda_4 = -1$, $\Lambda_5 = \Lambda_6 = 10^4$.

This is the Case 3 in Theorem 4.2, therefore there exist positive constants $K_i (i = 1, \cdots, 6)$ and $L_j (j = 1, 2, 3)$ such that the generalized Kawahara system (9) with local delay and K-S perturbation has a traveling wave solution $\Phi(\xi)$ with the following asymptotic properties:

$$
\Phi(\xi) = \begin{pmatrix}
(K_1 + o(1))e^{\Lambda_1 \xi} + (K_2 + o(1))\xi e^{10^4 \xi} \\
(K_3 + o(1))e^{\Lambda_2 \xi} + (K_4 + o(1))\xi e^{10^4 \xi} \\
(K_5 + o(1))e^{\Lambda_3 \xi} + (K_6 + o(1))\xi e^{10^4 \xi}
\end{pmatrix}, \quad \text{as } \xi \to -\infty,
$$

where $\Lambda_i$ may be $1$, $10^4$. And

$$
\Phi(\xi) = \begin{pmatrix}
-(L_1 + o(1))e^{-\xi} \\
-(L_2 + o(1))e^{-\xi} \\
-(L_3 + o(1))e^{-\xi}
\end{pmatrix}, \quad \text{as } \xi \to +\infty.
$$

Similarly we chose $\Lambda_i = 1$ and gave simulations. The graph of the traveling wave solution $\Phi(\xi)$ is shown in Figure 3.

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