CONSERVATIVE SUBGROUP SEPARABILITY FOR SURFACES WITH BOUNDARY.

MARK D. BAKER AND DARYL COOPER

Abstract. If $F$ is a surface with boundary, then a finitely generated subgroup without peripheral elements of $G = \pi_1 F$ can be separated from finitely many other elements of $G$ by a finite index subgroup of $G$ corresponding to a finite cover $\tilde{F}$ with the same number of boundary components as $F$.

Suppose $F$ is a compact, orientable surface with nonempty boundary. A non-trivial element of $\pi_1 (F)$ is peripheral if it is represented by a loop freely homotopic into $\partial F$. A covering space $p : \tilde{F} \to F$ is called conservative if $F$ and $\tilde{F}$ have the same number of boundary components: $|\partial F| = |\partial \tilde{F}|$.

**Theorem 0.1 (Main theorem).** Let $F$ be a compact, connected, orientable surface with $\partial F \neq \emptyset$ and $H \subset \pi_1 (F)$ a finitely generated subgroup. Assume that no element of $H$ is peripheral. Given a (possibly empty) finite subset $B \subset \pi_1 (F) \setminus H$, there exists a finite-sheeted cover $p : \tilde{F} \to F$ such that:

1. There is a compact, connected, $\pi_1$-injective subsurface $S \subset \tilde{F}$ such that $p_* (\pi_1 S) = H$.
2. $p_* (\pi_1 \tilde{F})$ contains no element of $B$.
3. $\tilde{F} \setminus S$ is connected and incl$_* : H_1 (S) \to H_1 (\tilde{F})$ is injective.
4. The covering is conservative.

This theorem, without (iii), is due to Masters and Zhang [4] and is a key ingredient in their proof that cusped hyperbolic 3-manifolds contain quasi-Fuchsian surface groups [4], [5]. Without (iii), (iv) the theorem is a special case of well-known theorems on subgroup separability of free groups [1] and surface groups [6], [7]. For a discussion of subgroup separability and 3-manifolds, see [3].

The proof in [4] uses the folded graph techniques due to Stallings, see [2]. The shorter proof below uses cut and cross-join of surfaces. A cover is called good if properties (i)-(ii) hold and very good if (i)-(iii) hold. The idea is to start with a good cover and then pass to a second cover which is very good. Then cross-join operations (defined below) are used to reduce the number of boundary components of a very good cover until it is conservative.

1. Constructing a Very Good Cover

We first explain a geometric condition on a cover which ensures it is good, and then use [1,3] to construct a very good cover.

Choose a basepoint $x$ in the interior of $F$ and suppose $p : \tilde{F} \to F$ is the cover corresponding to $H$. There is a compact, connected, incompressible subsurface $S$ in the interior of $\tilde{F}$ which is a retract of $\tilde{F}$ and which contains a lift $\tilde{x}$ of $x$. Each element $g \in \pi_1 (F, x)$ determines a unique lift $\tilde{x}(g) \in \tilde{F}$ of the basepoint $x$. The surface $S$ can be chosen large enough to contain $\{ \tilde{x}(b) : b \in B \}$. Then $p|_S : S \to F$ is a local homeomorphism. The work of M. Hall [1] and P. Scott [6] shows there is a finite cover $F' \to F$ such that $S$ lifts to an embedding in $F'$.

If $\pi : F' \to F$ is any cover and there is a lift of $p|_S$ to $\theta : S \to F'$ (thus $\pi \circ \theta = p|_S$) which is injective, we say $S$ lifts to an embedding in the cover $F'$.

---

Cooper was supported in part by NSF grant DMS-0706887 and CNRS.
The authors thank IHP for hospitality during the completion of this paper.
Proposition 1.1 (good cover). With the hypotheses of the main theorem, if \( \pi : F' \to F \) is any cover and \( S \) lifts to an embedding in \( F' \), then the cover is good.

Proof. With the notation above, a based loop representing an element \( b \in B \) lifts to a path in \( F' \) that starts at the basepoint \( \tilde{x} \in Y = \theta(S) \) but ends at some other point \( \tilde{x}(b) \neq \tilde{x} \) in \( Y \). □

Addendum 1.2 (very good cover). There is a very good cover \( \tilde{F} \) of finite degree with \( |\partial \tilde{F}| \) is even.

Proof. We start with a good cover \( F' \) of \( F \) with finite degree and the subsurface \( S \subset F' \) described above and then construct a cover of \( F' \) with the required property. For notational elegance, we rename the first cover \( F' \) as \( F \). Let \( p : \tilde{F} \to F \) be the regular cover given by the kernel of the map of \( \pi_1F \) onto \( H_1(F; S; \mathbb{Z}/2) \). There is a lift \( \tilde{S} \) of \( S \) to this cover by construction. The connectedness of \( \tilde{F} \setminus \tilde{S} \) and the injectivity of \( \text{incl}_*: H_1(S) \to H_1(\tilde{F}) \) are shown in theorem 1.3 below. Finally, since \( \partial F \neq \emptyset \) and \( S \) has no peripheral elements, it follows that \( H_1(F; S; \mathbb{Z}/2) \neq 0 \) so the cover has even degree. Thus \( \chi(\tilde{F}) \) is even, therefore \( |\partial \tilde{F}| \) is even. □

The following allows us to lift a \( \pi_1 \)-injective subsurface to a regular cover where it is \( H_1 \)-injective and non-separating.

Theorem 1.3. Suppose \( F \) is a compact, connected, orientable surface, possibly with boundary, which contains a compact, connected, subsurface \( S \). Assume that no component of \( \text{cl}(F \setminus \overline{S}) \) is a disc or a boundary parallel annulus. Let \( p : \tilde{F} \to F \) be the cover corresponding to the kernel of the natural homomorphism of \( \pi_1F \) onto \( G = H_1(F; S; \mathbb{Z}/2) \). If \( \tilde{S}_0 \) is a connected component of \( p^{-1}(S) \) then \( \tilde{F} \setminus \tilde{S}_0 \) is connected. Hence \( \text{incl}_*: H_1(\tilde{S}_0) \to H_1(\tilde{F}) \) is injective.

Proof. We may assume \( S \neq F \). Define \( X = \text{cl}(\tilde{F} \setminus \tilde{S}_0) \). Let \( Y \) be a connected component of \( X \). We claim that \( p(Y) \supset S \). Otherwise \( p|_Y : Y \to \text{cl}(F \setminus \overline{S}) \). Since \( p(Y \cap \tilde{S}_0) \) is injective it follows that \( p|_Y \) is injective, thus \( Y \) is a lift of a component \( Z \) of \( \text{cl}(F \setminus \overline{S}) \).

If \( Z \cap S \) is connected, then since \( Z \) is not a disc or boundary parallel annulus, the image of \( H_1(Z) \) in \( G \) is not trivial. Thus \( Z \) does not lift to the \( G \)-cover, a contradiction.

Hence \( Z \cap S \) contains at least two distinct circle components \( B_1, B_2 \). There is a loop \( \alpha = \beta \cdot \gamma \subset F \) which is the union of two arcs connecting \( B_1 \) and \( B_2 \) one arc \( \beta \subset Z \) and one arc \( \gamma \subset S \). Since \( \alpha \) has non-zero algebraic intersection number with the boundary component \( B_1 \) of \( S \) it is a non-zero element of \( G \). It follows that the lift \( \tilde{\beta} \subset \tilde{S}_0 \) of \( \beta \) has endpoints in different components of \( p^{-1}(S) \), since otherwise \( \alpha \) would lift to a loop. But \( \partial \tilde{\beta} \subset \partial Y \subset \partial \tilde{S}_0 \) which is a contradiction. Thus \( p(Y) \supset S \).

Choose some Riemannian metric on \( F \). This metric pulls back to one on \( \tilde{F} \) which is preserved by covering transformations. If \( X \) is not connected, let \( Y \) be a component of smallest area. It follows that \( Y \) contains some component \( \tilde{S}_1 \neq \tilde{S}_0 \) of \( p^{-1}(S) \) in its interior. However the cover is regular so there is a covering transformation \( \tau \) taking \( \tilde{S}_0 \) to \( \tilde{S}_1 \). Thus \( \tau \) takes components of \( \tilde{F} \setminus \tilde{S}_0 \) to components of \( \tilde{F} \setminus \tilde{S}_1 \). One of these components contains \( \tilde{S}_0 \) so the other ones are strictly contained in \( Y \) which contradicts that \( Y \) has minimal area. Hence \( X = Y \) is connected.

For the last conclusion, apply Mayer-Vietoris to \( \tilde{F} = \tilde{S}_0 \cup X \) with \( \tilde{S}_0 \cap X = \partial \tilde{S}_0 \cap \partial X \). Since \( X \) is connected, if the kernel of \( i_* : H_1(\tilde{S}_0) \to H_1(\tilde{F}) \) is nontrivial, then \( \partial X \subsetneq \partial \tilde{S}_0 \). This implies \( \partial X \cap \partial \tilde{F} = \phi \). Since \( p(X) \supset S \) it follows that \( \partial \tilde{F} = \phi \). But then \( \partial \tilde{S}_0 = \partial X \), hence the kernel is trivial. □

2. CROSS-JOINING COVERS

Suppose \( F \) is a surface and \( \alpha_1 \) and \( \alpha_2 \) are disjoint arcs properly embedded in \( F \). Let \( N(\alpha_i) \equiv \alpha_i \times [-1,1] \) be disjoint regular neighborhoods of the arcs \( \alpha_i \) in \( F \) such that \( \alpha_i \equiv \alpha_i \times 0 \) and \( N(\alpha_i) \cap \partial F = (\partial \alpha_i) \times [-1,1] \). The sets \( \alpha_i \times (0, \pm 1) \subset F \) are called the \( \pm \) sides of \( \alpha_i \).

Given a homeomorphism \( h : N(\alpha_1) \to N(\alpha_2) \) taking the + side of \( \alpha_1 \) to the + side of \( \alpha_2 \), the cross-join of \( F \) along \( (\alpha_1, \alpha_2) \) is the surface \( K \) defined as follows. The surface \( F^- = F \setminus (\alpha_1 \cup \alpha_2) \)
\(\alpha_2\) contains four subsurfaces \(\alpha_i \times (0, \pm 1)\). Let \(F^{cut}\) be the surface obtained by completing these subsurfaces to \(\alpha_i \times [0, \pm 1]\). Thus \(F^{cut}\) has two copies \(\alpha_i^+, \alpha_i^-\) of \(\alpha_i\) in \(\partial F^{cut}\) and identifying these copies suitably produces \(F\). The surface \(K\) is the quotient of \(F^{cut}\) obtained by using \(h\) to identify \(\alpha_1^-\) to \(\alpha_2^+\) and \(\alpha_1^+\) to \(\alpha_2^-\). Note that here we do not require \(F\) to be connected, so that \(\alpha\) and \(\beta\) might be in different components of \(F\).

There are two special cases of cross-join which will be used to change the number of boundary components of a surface:

**Lemma 2.1.** Suppose the compact surface \(F\) contains two disjoint properly embedded arcs \(\alpha\) and \(\beta\). In addition suppose:

1. either \(F\) is connected and the endpoints of \(\alpha, \beta\) lie on four distinct components of \(\partial F\)
2. or \(F\) is the union of two connected components \(A\) and \(B\) and \(\alpha \subset A\) has both endpoints on the same boundary component and \(\beta \subset B\) has endpoints on distinct boundary components.

Then a surface \(K\) obtained by cross-joining along these arcs has \(|\partial K| = |\partial F| - 2\). Furthermore \(\chi(K) = \chi(F)\) and \(K\) is connected.

**Proof.** We verify that \(K\) is connected. In the first case this follows since the arcs do not disconnect the boundary components on which they have endpoints; therefore \(F \setminus (\alpha \cup \beta)\) is connected. In the second case it follows because \(B \setminus \beta\) is connected, and every point in \(K\) is connected to a point in this subset by an arc. \(\square\)

Suppose \(p : \tilde{F} \to F\) is a (possibly not connected) covering of surfaces and \(\alpha\) is an arc properly embedded in \(F\). Suppose \(\bar{\alpha}_1\) and \(\bar{\alpha}_2\) are two distinct lifts of \(\alpha\) to \(\tilde{F}\); then they are disjoint. The map \(p\) provides a homeomorphism between small regular neighborhoods of these two arcs. Using this to cross-join produces a surface \(\tilde{F}'\) and since the identifications are compatible with \(p\) there is a covering map \(p' : \tilde{F}' \to F\).

An important special case is when \(\tilde{F}\) is a \((d + 1)\)-fold cover which is the disjoint union of a 1-fold cover \(F_1 \to F\) and some connected \(d\)-fold cover \(F_d \to F\). Then cross-joining an arc in \(F_1\) with one in \(F_d\) produces a cover of degree \(d + 1\).

To produce a new cover \(F'\) of \(F\) by a cross-join along two arcs in some cover \(\tilde{F}\) requires the arcs are disjoint from each other. If \(S\) embeds in \(\tilde{F}\) and these arcs are also disjoint from \(S\), then \(S\) lifts to an embedding in \(\tilde{F}'\), so the cover \(F'\) is good. We call the combination of these two properties the *disjointness condition*.

There is a metric condition, involving some arbitrary choice of Riemannian metric on \(F\), that ensures the disjointness condition is satisfied and therefore that the new cover is good. The next lemma provides a uniform upper bound on the lengths of the arcs we will use to cross-join in any cover of \(F\).

**Lemma 2.2** (short arcs). Suppose \(F\) is a compact, connected surface with a Riemannian metric such that the maximum distance between points in \(F\) is \(\ell\). If \(\tilde{F}\) is a finite cover of \(F\) then

1. If \(A\) and \(B\) are distinct components of \(\partial F\) then there is an arc \(\alpha\) in \(F\) connecting them and \(\text{length}(\alpha) \leq \ell\).
2. If some component \(A\) of \(\partial F\) has (at least) two pre-images in \(\partial \tilde{F}\) then there is an embedded arc \(\alpha\) in \(F\) of length at most \(2\ell\) which lifts to an arc with endpoints on distinct pre-images of \(A\).

**Proof.** The first claim is obvious. For the second claim, since every point in \(\tilde{F}\) is within a distance at most \(\ell\) of some point in \(p^{-1}(A)\) and \(\tilde{F}\) is connected, some point in \(\tilde{F}\) is within a distance at most \(\ell\) of points in two distinct components of \(p^{-1}(A)\). This gives an arc \(\beta\) in \(\tilde{F}\) of length at most \(2\ell\) which connects two distinct components of \(p^{-1}(A)\).
Let $\gamma : [0, 2R] \to \tilde{F}$ be a shortest arc connecting two distinct components of $p^{-1}(A)$ and parameterized by arc length. Then $R \leq \ell$. To complete the proof we show that $\gamma$ projects to an embedded arc in $F$. Observe that
\[ d_F(\gamma(t), p^{-1}(A)) = \min(t, 2R - t) \]
otherwise there is a shorter arc connecting two distinct components of $p^{-1}(A)$. It follows that
\[ d_F(p(\gamma(t)), A) = \min(t, 2R - t) \]
This means that the distance in $F$ of a point on $p \circ \gamma$ from $A$ is given by arc length along $p \circ \gamma$. It follows that $\alpha = p \circ \gamma$ is the required embedded arc. \qed

An arc of length at most $2\ell$ is called short. The next lemma provides a conservative cyclic cover with large diameter of a surface $F$. If a short arc in $F$ connects two distinct boundary components, then so does every covering translate of it. If $S$ lifts to the cover then there are many different translates of the short arc that are far from each other and far from the lift of $S$. In particular the disjointness condition is satisfied by suitable translates of a lifted short arc in this cover.

**Lemma 2.3** (big covers). Suppose $F$ is a compact connected surface with $k \geq 2$ boundary components and which contains a compact, connected, incompressible subsurface $S \subset \text{int}(F)$ with $F \setminus S$ connected. Given $n > 0$ there is a conservative finite cyclic cover $\tilde{F} \to F$ of degree bigger than $n$ and a lift, $\tilde{S}$, of $S$ to $\tilde{F}$. Furthermore $\tilde{F} \setminus \tilde{S}$ is connected.

**Proof.** Let $Y$ be the surface obtained from $F \setminus \text{int}(S)$ by gluing a disc onto each component of $\partial S$. Then $Y$ is a connected surface with $k$ boundary components and there is a natural isomorphism of $H_1(F)/H_1(S)$ onto $H_1(Y)$. Choose a prime $p > \max(k, n)$. Because $Y$ is connected, there is an epimorphism from $H_1(Y)$ onto $\mathbb{Z}/p$ which sends one component of $\partial Y$ to $k - 1$ and all the other $(k - 1)$ components of $\partial Y$ to $-1$. Now $(k - 1)$ is coprime to $p$ because $2 \leq k < p$. Therefore this defines a conservative cyclic $p$-fold cover $\tilde{Y}$ of $Y$. It also determines a conservative cyclic $p$-fold cover of $F$ such that $S$ lifts. Since $\tilde{Y}$ is connected it follows that $\tilde{F} \setminus \tilde{S}$ is connected. \qed

3. **Proof of main theorem**

In this section all covers are of finite degree. Given a cover $p : \tilde{F} \to F$ the excess number of boundary components $E(p)$ for this cover is defined as $E(p) = |\partial F| - |\partial \tilde{F}|$. By 1.2 there is a very good cover $p : \tilde{F} \to F$ with $|\partial \tilde{F}|$ is even. If $E(p) = 0$ the theorem is proved.

We first use 2.3 to replace a very good cover by another very good cover with the same excess and where there are lifts of a short arc that are far apart. Then we change the cover with a cross-join that reduces the excess. To apply 2.3 requires that $F \setminus S$ is connected. We must verify this property continues to hold after the cross-join so the process can be repeated. First observe that the cyclic cover produced by 2.3 leaves $F \setminus S$ connected.

In each case (except the last one) we will use one of the two cross-joins described in 2.1 to produce a new connected cover $F'$ of $F$ and a lift $\tilde{S}$ of $S$. Since the cross-join arcs are disjoint from $S$ they also determine a connected cover of $F \setminus S$. Thus $F' \setminus \tilde{S}$ is connected, as required. This implies $\text{incl}_* : H_1(\tilde{S}) \to H_1(F')$ is injective, so the new cover is also very good.

**Case when $|\partial F| = 1$.**

By 2.2 there is a properly embedded, short arc, $\alpha$, in $F$ which is covered by an arc $\beta$ with endpoints on two distinct boundary circles of $\partial \tilde{F}$. There is a conservative cyclic cover of $\tilde{F}$, which we also denote by $\tilde{F}$, to which $S$ lifts with diameter much larger than the length of $\beta$ and the diameter of $S$. Thus there is a lift of $\beta$ which is disjoint from $S$. Since the cover is conservative every lift of $\beta$ connects (the same pair of) distinct boundary components.
Cross-join \((\tilde{F}, \beta)\) with \((F, \alpha)\) to obtain a cover \(F'\) with one fewer boundary circle than \(\tilde{F}\). There is a lift of \(\tilde{S}\) to \(F'\) and \(F' \setminus \tilde{S}\) is connected. Repeat the process until the cover has only one boundary component. This completes the proof when \(|\partial F| = 1|\).

**Case when \(|\partial F| \geq 2|\).**

First we show how to make \(E(p)\) even by performing a cross-join if needed. This first step will increase the number of boundary components.

Suppose \(E(p)\) is odd. By \([1, 2]\) \(|\partial F|\) is even, so \(|\partial F|\) is odd. We can make \(E(p)\) even by cross-joining \((\tilde{F}, \tilde{\alpha})\) and \((F, \alpha)\) to obtain a cover \(p' : F' \rightarrow F\). To perform the cross-join choose a short embedded arc \(\alpha \subset F\) with endpoints on two distinct circles \(C, C'\) of \(\partial F\). Choose a lift of \(\tilde{\alpha} \subset \tilde{F}\), with endpoints on two preimages \(\tilde{C}, \tilde{C}'\). By the big cover lemma \([2, 3]\) we can choose \(\tilde{\alpha}\) disjoint from \(S\) in \(\tilde{F}\). Then \(F'\) is the cross-join of \((F, \alpha)\) and \((\tilde{F}, \tilde{\alpha})\). The surface \(S\) lifts to \(F'\) and \(F' \setminus S\) is connected by \([2, 1]\).

Here is the outline of the rest of the proof. If \(|E(p)| \neq 0\) then it is even. We proceed as follows using suitable cross-joins to construct new coverings. If there are two different components \(C, C' \subset \partial F\) which both have more than one pre-image in \(\partial \tilde{F}\) then we find a short arc \(\alpha\) in \(F\) connecting \(C\) and \(C'\) and cross-join \(\tilde{F}\) to itself along two suitable lifts of \(\alpha\) in \(\tilde{F}\). This reduces the excess by 2. After finitely many steps we obtain a cover so that at most one component \(C \subset \partial F\) has more than one pre-image. A single cyclic cross-join (defined below) is done simultaneously to reduce the excess to zero. Here are the details.

Suppose \(A\) and \(B\) are distinct circles in \(\partial F\) which both have (at least) two distinct pre-images \(\tilde{A}_i, \tilde{B}_i\) for \(i = 1, 2\) in \(\partial \tilde{F}\). Choose a short arc \(\gamma\) in \(F\) with endpoints on \(A\) and \(B\). Let \(\alpha_i\) be a lift of \(\gamma\) with one endpoint on \(\tilde{A}_i\) and \(\beta_i\) a lift with an endpoint on \(\tilde{B}_i\). Inductively we assume that \(\tilde{F} \setminus S\) is connected. Replace \(\tilde{F}\) by a large cyclic conservative cover such that these arcs are all far apart and far from \(S\). Thus there is a cover obtained by cross-joining along any pair of distinct arcs chosen from this set of four and \(S\) lifts to this cover.

We claim that there is a pair of these arcs which have endpoints on four distinct boundary components of \(\tilde{F}\). It follows from lemma \([2, 4]\) that cross-joining along this pair reduces the excess by 2 and \(S\) lifts to the cover \(F'\) so produced. Furthermore, since \(\tilde{F} \setminus S\) is connected it follows that \(F' \setminus S\) is connected by \([2, 1]\).

If \(\alpha_1\) and \(\alpha_2\) do not both have endpoints on the same lift \(\tilde{B}\) of \(B\) the pair \((\alpha_1, \alpha_2)\) works. Similarly if \(\beta_1\) and \(\beta_2\) do not both have endpoints on the same lift \(\tilde{A}\) of \(A\) the pair \((\beta_1, \beta_2)\) works. The remaining case is (after relabelling) \(\alpha_1\) and \(\alpha_2\) both have endpoints on a component \(\tilde{B} \neq \tilde{B}_2\) which covers \(B\) and \(\beta_1, \beta_2\) both have endpoints on some component \(\tilde{A} \neq \tilde{A}_2\) which covers \(A\). Then \(\alpha_2\) connects \(\tilde{A}_2\) to \(\tilde{B} \neq \tilde{B}_2\) and \(\beta_2\) connects \(\tilde{B}_2\) to \(\tilde{A} \neq \tilde{A}_2\). Thus the pair \((\alpha_2, \beta_2)\) works.

Repeating this process a finite number of times reduces the excess by an even number until either \(|\partial \tilde{F}| = |\partial F|\) or else there is a unique component \(C\) of \(\partial F\) with more than one pre-image. In the last case the excess is even so there is an odd number of pre-images \(p^{-1}(C) = \{C_0, ..., C_{2k}\}\).

Refer to figure \([1]\). Choose a component \(A\) of \(\partial \tilde{F}\) that does not cover \(C\). This is possible because \(|\partial F| \geq 2\). Let \(\beta\) be a short arc in \(F\) with endpoints on \(p(A)\) and \(C\). For each \(i\) there is a lift \(\beta_i\) of \(\beta\) with one endpoint on \(C_i\) and the other on \(A\). As before we may assume all these lifts are far apart and far from \(S\). Orient each arc \(\beta_i\), so it points from \(A\) to \(C_i\) and call the left side + and the right side −. Now cross-join cyclically as follows. Cut \(\tilde{F}\) along the union of these arcs and join the − side of \(\beta_i\) to the + side of \(\beta_{i+1}\), with all integer subscripts taken mod \(2k + 1\).

The resulting cover has a single pre-image of \(\tilde{C}\). Indeed, each \(C_i\) has been cut at one point to give an interval \(D_i = [t^+_i, t^-_i]\) where the label \(i\) denotes an endpoint of \(\beta_i\) and \(t^+_i\) is on the ± side of \(\beta_i\). These intervals are then glued by identifying \(t^-_i\) in \(D_i\) to \(t^+_{i+1}\) in \(D_{i+1}\). The result is obviously connected: a single circle.

To analyse the preimage of \(p(A)\) the circle \(A\) was cut at \(2k + 1\) points to produce \(2k + 1\) subarcs \(E_i = [u^+_i, u^-_{i+1}]\) where \(u^\pm_i\) is on the ± side of \(\beta_i\). Then \(E_i\) is glued to \(E_{i+2}\) by identifying \(u^+_{i+1}\) with
The preimage of $A$.

Figure 1. Cyclic cross-joining, $2k + 1 = 5$ illustrated

$u_{i+2}^+$ (see figure 1). Since there are $2k + 1$ intervals and the $i$th one is glued to the $(i + 2)$th one the result is connected because 2 is coprime to $2k + 1$. This gives the required conservative cover completing the proof of the main theorem.

References

[1] M. Hall, Jr. Coset representations in free groups. Trans. Amer. Math. Soc., 67:421–432, 1949.
[2] I. Kapovich and A. Myasnikov. Stallings foldings and subgroups of free groups. J. Algebra, 248(2):608–668, 2002.
[3] D. Long and A. W. Reid. Surface subgroups and subgroup separability in 3-manifold topology. Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2005. 25o Colóquio Brasileiro de Matemática. [25th Brazilian Mathematics Colloquium].
[4] J. D. Masters and X. Zhang. Closed quasi-Fuchsian surfaces in hyperbolic knot complements. Geom. Topol., 12(4):2095–2171, 2008.
[5] J. D. Masters and X. Zhang. Quasi-Fuchsian Surfaces In Hyperbolic Link Complements. ArXiv e-prints, Sept. 2009.
[6] P. Scott. Subgroups of surface groups are almost geometric. J. London Math. Soc. (2), 17(3):555–565, 1978.
[7] P. Scott. Correction to: “Subgroups of surface groups are almost geometric” [J. London Math. Soc. (2) 17 (1978), no. 3, 555–565; MR0494062 (58 #12996)]. J. London Math. Soc. (2), 32(2):217–220, 1985.