Moyal-Weyl Star product and quasiconformal mapping relationship

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Abstract. The deformation quantization procedure can introduce the quantum effect by including the deformation of complex structures on a bi-dimensional Riemann surface Σ without boundary. This is the platform of bi-dimensional conformal models like string theory and two-dimensional gravity. Then, the star product can become a quasiconformal mapping on this surface. This makes the road toward a Teichmüller theory of the Moyal-Weyl star product through the quasiconformal mappings as lifting of universal covering surfaces of any Riemann surface.

1. Introduction
Deformation quantization was born with the paper [1] and grew out of the work of Weyl, Moyal and Vey. It has become a large research area covering several algebraic theories like the formal deformation of associative algebras and the more recent theory of operades, as well as geometric theories like the theory of symplectic and (more generally) Poisson manifolds, and physical theories like string theory and non commutative gauge theory.

The theory of deformation quantization has culminated with the recent work of Kontsevitch [7] proving that every Poisson manifold admits a non-trivial deformation quantization.

Two-dimensional conformal field theories on Riemann surfaces without boundaries are relevant models in string theory [11]. The dependence of these bidimensional conformal models on the background geometry turns out to be useful for the construction of effective actions for two-dimensional gravity [10]. This geometrical dependence is well exhibited using the Beltrami parametrization of the bidimensional world sheet metric of the bosonic string [8].

In the Riemannman manifold approach, Beltrami coefficients parametrize conformal classes of the metrics [2]. However, in the Riemannian surface formalism, they parametrize complex structures of the Riemann surface [8] and satisfy the ellipticity condition:

\[ \text{Sup}_Σ |μ| < 1, \; μ ∈ C^∞ (Σ) \]  

where, Σ is a Riemann surface.

2. Beltrami parametrization’s formalism
2.1. Riemann Surface
A Riemann surface Σ of genus \( g \) ≥ 1 is a connected topological manifold of real dimension two which is equipped with a reference complex structure say \( C_0 \). This is a maximal atlas of local
coordinates with a holomorphic change of coordinates:

\[(z, \overline{z}) \mapsto (z', \overline{z'} = \overline{f}(z))\]  

(2)

The natural tangent space basis is \[\partial \equiv \partial / \partial z, \partial / \partial \overline{z}].\]

Then, if the coordinates \(z\) and \(z'\) are related by a holomorphic (local) diffeomorphism they are said to belong to the same complex structure. However, if the transition functions \(z' = f(z, \overline{z})\) are not holomorphic, the two complex structures \((z)\) and \((\overline{z})\) belong to different atlas.

2.2. Quasiconformal mapping

Now, let us consider a Beltrami differential \(\mu(z, \overline{z})\) that is \(C^\infty(\Sigma)\) and satisfies

\[|\mu(z, \overline{z})| \leq 1.\]  

(3)

The set of Beltrami differentials on the Riemann surface \(\Sigma\) has a structure of topologically contractible complex Banach manifold. It parametrizes the set of all complex structures on \(\Sigma\). Indeed, to any Beltrami differential \(\mu\) there is associated a conformal structure \(C_\mu\) on \(\Sigma\) whose generic coordinate \(Z(z, \overline{z})\) is a \(C^\infty\)-diffeomorphism of the reference coordinates and satisfies the Beltrami equation:

\[\left(\overline{\partial} - \mu \partial\right)Z = 0.\]  

(4)

The diffeomorphism

\[z \mapsto Z(z, \overline{z}), \mu \neq 0\]  

(5)

is called a quasi-conformal transformation, that one can verify, becomes conformal for \(\mu = 0\). It can be seen as the transition function from the reference complex structure \(C_0\) to the complex structure \(C_\mu\) \((\mu = \overline{\partial}Z / \partial \overline{z})\).

On the other hand, let us consider a sense-preserving diffeomorphism \(f\).

The function

\[\partial_\alpha f = \partial f + e^{-2i\alpha}f \]  

(6)

is its derivative in the direction \(\alpha\).

The dilatation quotient of the diffeomorphism \(f\) is finite and is defined by

\[D_f = \frac{\max_\alpha |\partial_\alpha f|}{\min_\alpha |\partial_\alpha f|}\]  

(7)

where

\[\max_\alpha |\partial_\alpha f| = |\partial f| + |\overline{\partial} f|\]  

(8)

and

\[\min_\alpha |\partial_\alpha f| = |\partial f| - |\overline{\partial} f|.\]  

(9)

2.3. Theorem

Let \(f : D \rightarrow D'\) be a sense-preserving diffeomorphism satisfying the condition

\[D_f(z) \leq K\]  

(10)

for every \(z \in D\) (a domain of \(\Sigma\)). Then, \(f\) is a \(K\)-quasiconformal mapping.

The jacobian of \(f\) :

\[J_f = |\partial f|^2 - |\overline{\partial} f|^2\]  

(11)
is positive.

Then, we can define the quotient

\[ \mu(z, \bar{z}) = \frac{\partial f}{\partial f} \]  

(12)

Since \( f \) is continuous, the function \( \mu \) is Borel-measurable and then we have

\[ |\mu(z, \bar{z})| \leq \frac{K - 1}{K + 1} < 1. \]  

(13)

Then, a sense-preserving diffeomorphism \( f \) satisfying the Beltrami equation

\[ \left( \partial - \mu \partial \right) f = 0 \]  

(14)

is a quasiconformal mapping.

3. Moyal Star product on a Riemann surface

3.1. Symplectic structure

An even dimensional manifold \( M^{2n} \) endowed with a closed 2-form \( \omega; d\omega = 0 \) is called a symplectic manifold and the structure given by \( \omega \) is a symplectic structure.

Locally we have

\[ \omega = \omega_{ij} dx^i dx^j, \quad i, j = 1, 2, \ldots, 2n \]  

(15)

where \( \det(\omega_{ij}) \neq 0 \) and \( (x^i)_{i=1,2,\ldots,2n} \) is a local coordinates system on \( M^{2n} \).

The Poisson bracket of two observables \( f \) and \( g \) is defined by:

\[ \{f, g\} = \omega^{ij} \partial_i f \partial_j g \]  

(16)

Consider the following Beltrami equations

\[ \bar{\partial} f = \mu_f \partial f \]  

(17)

\[ \bar{\partial} g = \mu_g \partial g. \]  

(18)

Then, the P. B. is given by

\[ \{f, g\}(z, \bar{z}) = \partial f \bar{\partial} g - \bar{\partial} f \partial g \]  

(19)

One can see that the P. B. vanishes at the region \( \mu_f = \mu_g \).

3.2. Moyal-Weyl Star product on \( \Sigma \)

Let \( f \) and \( g \) be two functions defined on a Riemann surface \( \Sigma \) which is equipped with a reference complex structure that is parametrized by a system of coordinates \((z, \bar{z})\).

The star product of \( g \) and \( f \) is a function:

\[ w(z, \bar{z}) = f(z, \bar{z}) \ast g(z, \bar{z}) \]  

(20)

\[ \overline{w}(z, \bar{z}) = \overline{f}(z, \bar{z}) \ast \overline{g}(z, \bar{z}) \]  

(21)

such that

\[ \ast \equiv \exp \left[ i\hbar \left( \bar{\partial} \partial - \partial \bar{\partial} \right) \right]. \]  

(22)

The associativity of the star product is satisfied while it does not commute.
$$f_1 * f_2 * f_3 = f_1 * (f_2 * f_3)$$  \hspace{1cm} (23) $$f_1 * f_2 * f_3 \neq f_2 * f_3 * f_1$$  \hspace{1cm} (24) 

Here we choose the operator of the star product to generate the Poisson bracket in the canonical form.

Usually, the star product is interpreted as an expansion of power series of $\hbar$. Our star product may be expanded by $\hbar$ or by the Beltrami coefficients:

$$w = w^{(0)} + \hbar w^{(1)} + \hbar^2 w^{(2)} + ...$$  \hspace{1cm} (25) $$\tilde{w} = \mu_f \tilde{w}_f^{(1)} + \mu_g \tilde{w}_g^{(1)} + \mu_f^2 \tilde{w}_f^{(2)} + \mu_g^2 \tilde{w}_g^{(2)} + ...$$  \hspace{1cm} (26) 

and

$$\mu_w = \mu_w^{(0)} + \hbar \mu_w^{(1)} + \hbar^2 \mu_w^{(2)} + ...$$  \hspace{1cm} (27) 

Hence, the star product is given as a superposition of $w^{(n)}$ or of that $\tilde{w}^{(n)}$ ($n \in \mathbb{Z}$, $0 \leq n \leq +\infty$).

This indicates a deep relationship between the Beltrami coefficients and $\hbar$ in the star product.

4. Some quasiconformal mapping’s star product

The Moyal-Weyl star product can become a quasiconformal mapping. Indeed, let us consider the affine mappings defined by:

$$f(z, \bar{z}) = a_1 z + b_1 \bar{z} + c_1$$  \hspace{1cm} (28) $$g(z, \bar{z}) = a_2 z + b_2 \bar{z} + c_2$$  \hspace{1cm} (29) 

$$a_i, b_i \in \mathbb{C}$$

$$|a_i| > |b_i|, \quad i = 1, 2$$

Then, the star product for these two functions is given by:

$$w = f * g$$  \hspace{1cm} (30) $$= fg + i\hbar (a_1 b_2 - a_2 b_1)$$  \hspace{1cm} (31) $$= fg - i\hbar (\mu_f - \mu_g) a_1 a_2$$  \hspace{1cm} (32) 

where the criterion for the quasiconformal mapping restricts the Beltrami differentials to satisfy the following conditions

$$0 \leq |\mu_f| \leq 1, \quad 0 \leq |\mu_g| \leq 1.$$  \hspace{1cm} (33) 

Moreover, one can verify that $w$ is convergent for $\mathbb{C}^2 = \mathbb{C}_{\mu_f} \otimes \mathbb{C}_{\mu_g}$.

Now, using the Moyal-Weyl star product, the deformation quantization can be introduced as a phase factor to the normal product of two diffeomorphisms defined on the Riemann surface $\Sigma$. Indeed, let us consider the two functions:

$$f(z, \bar{z}) = e^{i\alpha_1 z} e^{i\beta_1 \bar{z}}$$  \hspace{1cm} (34) $$g(z, \bar{z}) = e^{i\alpha_2 z} e^{i\beta_2 \bar{z}}$$  \hspace{1cm} (35) 

$$\alpha_i, \beta_i \in \mathbb{C}, \quad \alpha_i \neq 0$$  \hspace{1cm} (36)
Then, the first terms of the power series of the star product in terms of $\bar{\hbar}$ are given by:

$$w(0) = fg = e^{i(\alpha_1 + \alpha_2)z} e^{i(\beta_1 + \beta_2)\bar{z}}$$

$$w(1) = i \{f, g\} = -i (\alpha_1 \beta_2 - \alpha_2 \beta_1) = i (\mu_f - \mu_g)$$

and then

$$w = e^{-i \hbar(\alpha_1 \beta_2 - \alpha_2 \beta_1)} fg$$

with

$$\mu_w = \frac{\alpha_1 \mu_f + \alpha_2 \mu_g}{\alpha_1 + \alpha_2}$$

Hence, the effect of the deformation quantization is introduced as the phase factor to the product $fg$. $w$ is absolutely convergent at $0 \leq \hbar \leq \infty$.

More generally, let us consider a series of quasiconformal mappings:

$$f_j = e^{i \alpha_j z} e^{i \beta_j \bar{z}}, \ j = 1, 2, ...$$

Then, the star product is given by

$$w = f_1 * f_2 * ... * f_n$$

where the Beltrami equations for such quasiconformal mapping are given by

$$\bar{\partial} f_j = \mu_j \partial f_j$$

Then, one can find that the Beltrami differential of the star product is

$$\mu_w = \frac{\sum_{j=1}^n \beta_j}{\sum_{j=1}^n \alpha_j}$$

and this later is holomorphic with respect to these Beltrami differential:

$$\frac{\partial w}{\partial \mu_j} = 0$$

It is an interesting fact that this effect disappears and the P. B. vanishes identically when this P.B. is defined on the same complex structure $C_{\mu_f=\mu_g}$.

In another point of view, if the star product is considered as a functional of $\mu_f$ and $\mu_g$, it can be expressed with the help of the Cauchy theorem of the case of several complex variables as follows:

$$w(\mu_f, \mu_g) = \int_{C_1} \frac{d\eta_1}{2i\pi} \int_{C_2} \frac{d\eta_2}{2i\pi} \frac{w(\eta_1, \eta_2)}{(\eta_1 - \mu_f) (\eta_2 - \mu_g)}$$

The $C_i$ are the appropriate integration paths inside $\mathcal{C}$.

Hence, we can write it as a functional of the Beltrami differentials as follows:

$$w(\mu_{f_1}, \mu_{f_2}, ..., \mu_{f_n}) = \prod_{j=1}^n \left( \int_{C_j} \frac{d\eta_j}{2i\pi} \right) \frac{w(\eta_1, \eta_2, ..., \eta_n)}{(\eta_1 - \mu_{f_1}) (\eta_2 - \mu_{f_1}) ... (\eta_n - \mu_{f_n})}$$

and

$$\frac{\partial^{m_1+...+m_n} w(\mu_1, ..., \mu_n)}{\partial^{m_1} \mu_{f_1} \partial^{m_2} \mu_{f_2} ... \partial^{m_n} \mu_{f_n}} = (m_1!...m_n!) \prod_{j=1}^n \left( \int_{C_j} \frac{d\eta_j}{2i\pi} \right) \frac{w(\eta_1, \eta_2, ..., \eta_n)}{(\eta_1 - \mu_{f_1})^{m_1+1} ... (\eta_n - \mu_{f_n})^{m_n+1}}.$$
5. Conclusion.
It is an interesting fact that the deformation of complex structures disappears when the poisson bracket is defined on the same complex structure.
This can shed some light on the deep relationship between the deformation of complex structures and the deformation quantization on a bi-dimensional Riemann surface.

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