A DYNAMICAL THOULESS FORMULA

JAMERSON BEZERRA, AO CAI, PEDRO DUARTE, CATALINA FREIJO, AND SILVIUS KLEIN

ABSTRACT. In this paper we establish an abstract, dynamical Thouless-type formula for affine families of GL(2, R) cocycles. This result extends the classical formula relating, via the Hilbert transform, the maximal Lyapunov exponent and the integrated density of states of a Schrödinger operator. Here, the role of the integrated density of states will be played by a more geometrical quantity, the fibered rotation number. As an application of this formula we present limitations on the modulus of continuity of random linear cocycles. Moreover, we derive Hölder-type continuity properties of the fibered rotation number for linear cocycles over various base dynamics.

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1. INTRODUCTION AND STATEMENTS

Thouless formula relates a mathematical physics object, the integrated density of states (IDS) of a Schrödinger operator, and a dynamical systems object, the Lyapunov exponent (LE) of the operator, via a singular integral operator, namely the Hilbert transform. It is named after the British condensed-matter physicist David J. Thouless, who formulated it in the context of the one dimensional Anderson model and proved it (not completely rigorously) in [32]. The result was later extended and proven rigorously by Avron and Simon [4], Craig and Simon [12] and others. Let us describe it more precisely.

Consider an invertible ergodic transformation $T: X \to X$ over a probability space $(X, \mu)$. Given a bounded and measurable observable $v: X \to \mathbb{R}$, let $v_n(x) := v(T^n x)$ for all $x \in X$ and $n \in \mathbb{Z}$.
Denote by $l^2(\mathbb{Z})$ the Hilbert space of square summable sequences of real numbers $(\psi_n)_{n \in \mathbb{Z}}$. The discrete ergodic Schrödinger operator with potential $n \mapsto \nu_n(x)$ is the operator $H(x)$ defined on $l^2(\mathbb{Z}) \ni \psi = \{\psi_n\}_{n \in \mathbb{Z}}$ by
\[
[H(x)\psi]_n := -(\psi_{n+1} + \psi_{n-1}) + \nu_n(x)\psi_n.
\]
(1.1)

Note that due to the ergodicity of the system, the spectral properties of the family of operators $\{H(x): x \in X\}$ are $\mu$-a.s. independent of the phase $x$.

Given an energy parameter $E \in \mathbb{R}$, the Schrödinger (or eigenvalue) equation $H(x)\psi = E\psi$ can be solved formally by means of the iterates of a certain dynamical system. More precisely, consider the associated Schrödinger cocycle $X \times \mathbb{R}^2 \ni (x,v) \mapsto (Tx,A_E(x)v) \in X \times \mathbb{R}^2$, where $A_E: X \to \text{SL}_2(\mathbb{R})$ is given by
\[
A_E(x) := \begin{bmatrix} \nu(x) - E & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \nu(x) & -1 \\ 1 & 0 \end{bmatrix} + E \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Let $A^n_E$ denote the $n$-th iterate of the cocycle, that is,
\[
A^n_E(x) = A_E(T^{n-1}x) \cdots A_E(Tx)A_E(x).
\]

Then the formal solution of the Schrödinger equation $H(x)\psi = E\psi$ is given by
\[
\begin{bmatrix} \psi_n \\ \psi_{n-1} \end{bmatrix} = A^n_E(x) \begin{bmatrix} \psi_0 \\ \psi_{-1} \end{bmatrix}.
\]
(1.2)

The average asymptotic growth rate of the iterates of the Schrödinger cocycle $A_E$ is called the maximal Lyapunov exponent, denoted by $L_1(A_E)$. Moreover, the integrated density of states $N(E)$ measures, in some sense, how many states correspond to energies below the level $E$. Thouless formula establishes the following relation between these two quantities:
\[
L_1(A_E) = \int \log |E - E'| dN(E'),
\]
where the integral above is in the Lebesgue-Stieltjes sense.

A version of the formula is also valid for (the slightly more general) one dimensional self-adjoint Jacobi operators, and it was subsequently extended in several directions: to band lattice Schrödinger operators by Craig and Simon [13], relating the sum of the nonnegative Lyapunov exponents to the IDS; to i.i.d. random non self-adjoint Jacobi operators by Goldsheid and Khoruzhenko [22]; to long-range quasi-periodic Schrödinger operators with trigonometric polynomial potentials by Haro and Puig [25]. Finally, a more general version of the formula for self-adjoint block Jacobi matrices with dynamically defined entries was obtained by Chapman and Stolz [11].

Thouless formula was initially employed by Craig and Simon [12, 13] to establish the log-Hölder continuity of the IDS.

Since it relates the LE to the IDS via a singular integral operator, Thouless formula can be used to transfer Hölder-type (e.g. Hölder or weak-Hölder) moduli
of continuity\textsuperscript{1} from one quantity to the other, see [23, Lemma 10.3] for a formal statement. For instance, for the classical Anderson model (where the potential \( \{ v_n \}_{n \in \mathbb{Z}} \) is an i.i.d. sequence of random variables), Le Page [30] established the Hölder continuity of the LE, which then implies the Hölder continuity of the IDS. This, in turn, can be used in a multiscale analysis scheme to establish the Anderson localization of the operator. This approach was also employed in other related contexts, see for instance [11, 18].

In the opposite direction, the formula can be used to establish limitations on the modulus of continuity of the LE, via the IDS. This method goes back to Halperin, whose argument was made rigorous by Simon and Taylor [31] and extended by Duarte, Klein and Santos [20] and more recently by Bezerra and Duarte [5].

Moreover, Thouless formula also plays a role in establishing the absolutely continuous spectrum of the almost Mathieu operator, see Avila [1].

Note that all of the aforementioned results are within the scope of lattice (or band lattice) Schrödinger or Jacobi operators. It turns out that the IDS \( N(E) \), a physical quantity, is (linearly) related to \( \rho(A_E) \), the fibered rotation number of the cocycle \( A_E \) (the exact linear relation depends on the scaling considered).

In this paper, instead of the one-parameter family \( E \mapsto A_E \) of Schrödinger cocycles, we consider general affine families of \( \text{GL}(2, \mathbb{R}) \) cocycles of the form \( A_t = A + tB \) where \( A: X \to \text{GL}(2, \mathbb{R}) \) and \( B: X \to \text{Mat}(2, \mathbb{R}) \). Under appropriate assumptions, to be formally introduced below, we establish the following abstract Thouless formula:

\[
L_1(A_t) = L_1(B) + \int_{\mathbb{R}} \log |t - s| \, d\rho(s), \quad \forall t \in \mathbb{C},
\]

where \( L_1 \) refers to the first Lyapunov exponent and \( d\rho \) is a density measure associated with the fibered rotation number of \( A_t \).

Moreover, we employ the above Thouless formula to establish sharp limitations on the modulus of continuity of the Lyapunov exponent of random linear cocycles, which improve on the result in [20]. Furthermore, we derive the Hölder-type continuity of the fibered rotation number for linear cocycles over various types of base dynamics.

1.1. The main assumptions. Let \( T: X \to X \) be a homeomorphism on a compact metric space \( X \) and let \( \mu \in \text{Prob}(X) \) be an ergodic \( T \)-invariant probability measure. Define

\[
\text{GL}_2^+(\mathbb{R}) := \{ A \in \text{Mat}_2(\mathbb{R}) : \det A > 0 \}
\]

\textsuperscript{1}Note that much weaker moduli of continuity, such as log-Hölder, cannot be transferred via the Hilbert transform. This can also be seen by recalling that the IDS is always log-Hölder continuous while the LE can be discontinuous, e.g. in the case of non-uniformly hyperbolic \( \text{SL}(2, \mathbb{R}) \) cocycles in the \( C^0 \) topology, see [7], or even in the case of quasiperiodic cocycles in the smooth topology, see [34, 35].
to be the group of 2 by 2 matrices with positive determinant.

Consider a continuous function $A : \mathbb{R} \times X \to \text{GL}_2^+(\mathbb{R})$ which we regard as a one parameter family $A_t : X \to \text{GL}_2^+(\mathbb{R})$ indexed by $t \in \mathbb{R}$. Let $F_t : X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$ be the cocycle defined by

$$F_t(x, v) := (Tx, A_t(x)v),$$

whose iterates are denoted by

$$A^n_t(x) := A_t(T^{n-1}x) \cdots A_t(Tx) A_t(x).$$

**Assumption 1 (Analyticity and Invertibility).** There are positive constants $R$ and $c$ such that for each $x \in X$, the function $R \ni t \mapsto A_t(x)$ admits an analytic extension to the complex strip $S_R := \{z \in \mathbb{C} : |\text{Im } z| \leq R\}$ with $|\det(A_t(x))| \geq c > 0$ for all $(t, x) \in S_R \times X$.

Many of the results below are stated for one-parameter families of matrices $\{A_t : t \in I\}$ which are smooth but not necessarily analytic, where the index set $I$ always stands for some interval $I \subset \mathbb{R}$.

**Definition 1.1.** A smooth curve of matrices $I \ni t \mapsto A_t \in \text{GL}_2^+(\mathbb{R})$ is said to be positively (resp. negatively) winding, if for all $t \in I$, the quadratic form $Q_{A_t} : \mathbb{R}^2 \to \mathbb{R}$,

$$Q_{A_t}(v) := (A_t v) \wedge (A_t v) = (\det A_t) v \wedge (A_t^{-1} \dot{A}_t v),$$

is positive (resp. negative) semidefinite, with one eigenvalue bounded away from 0. Here $\dot{A}_t := \frac{d}{dt} A_t$ and given any two vectors $v_1, v_2 \in \mathbb{R}^2$, $v_1 \wedge v_2 := \det(v_1, v_2) = \|v_1\| \|v_2\| \sin \angle(v_1, v_2)$.

Positive (negative) winding means that for every non-zero vector $v \in \mathbb{R}^2$ which is not a real eigenvector of any of the matrices $A_t^{-1} \dot{A}_t$, the curve $I \ni t \mapsto A_t v \in \mathbb{R}^2 \setminus \{0\}$ winds positively (resp. negatively) around the origin as $t$ runs in $I$.

**Definition 1.2.** A family of cocycles $A_t : X \to \text{GL}_2^+(\mathbb{R})$ is said to be positively (negatively) winding, if for every $x \in X$ the analytic curve $I \ni t \mapsto A_t(x)$ is positively (negatively) winding.

**Assumption 2 (Winding).** The family of cocycles $A_t$ is positively (or negatively) winding.

The first Lyapunov exponent of the cocycle, denoted by $L_1(A_t)$, measures the fiber exponential growth rate along the orbits of $F_t$. Since $(T, \mu)$ is ergodic, by J. Kingman sub-additive theorem [29], for $\mu$-almost every $x \in X$

$$L_1(A_t) = \lim_{n \to +\infty} \frac{1}{n} \log \|A^n_t(x)\|.$$
Throughout the manuscript we denote by \( \hat{v} \in \mathbb{P}^1 \) the projective point of a vector \( v \in \mathbb{R}^2 \setminus \{0\} \). Similarly we denote by \( \hat{A} \) the projective action of a matrix \( A \in \text{SL}_2(\mathbb{R}) \) and let \( \tilde{F}_t : X \times \mathbb{P}^1 \to X \times \mathbb{P}^1 \) denote the projective cocycle
\[
\tilde{F}_t(x, \hat{v}) := (Tx, \hat{A}_t(x)\hat{v}).
\]

For the sake of notational simplicity, many times we write \( A\hat{v} \) instead of \( \hat{A}\hat{v} \). The fibered rotation number of \( A_t \), denoted by \( \rho(A_t) \), is defined as the \( \mu \)-almost sure limit
\[
\rho(A_t) := \lim_{n \to +\infty} \frac{1}{\pi n} \angle(A^n_t(x)\hat{v}, \hat{v}), \tag{1.3}
\]
where \( (x, t, v) \in X \times I \times \mathbb{P}^1 \). In Section 2 we properly define the angle \( \angle(A^n_t(x)\hat{v}, \hat{v}) \) and show that \( I \ni t \mapsto \rho(A_t) \) is a well defined, non-decreasing and continuous function. Moreover up to an additive constant the fibered rotation number is independent of the choices made to define the angle \( \angle(A^n_t(x)\hat{v}, \hat{v}) \).

**Assumption 3** (Affine form). The one-parameter family \( A_t \) has the form \( A_t(x) = A(x) + tB(x) \) where \( A : X \to \text{GL}_2(\mathbb{R}) \) and \( B : X \to \text{Mat}_2(\mathbb{R}) \) are continuous functions.

The family of cocycles \( A_t \) is well defined for all \( t \in \mathbb{C} \), although, by Assumption 1, the matrices \( A_t(x) \) are possibly only invertible for \( t \in \mathcal{S}_R \).

We say that \( B \) has *dominated splitting* when there exists a continuous decomposition \( \mathbb{R}^2 = E_0(x) \oplus E_\infty(x) \) in lines \( E_0(x) \) and \( E_\infty(x) \), which is \( T \)-invariant, i.e., \( B(x)E_0(x) = E_0(Tx) \) and \( B(x)E_\infty(x) \subset E_\infty(Tx) \), for all \( x \in X \), and such that for some integer \( n_0 \), \( \|B^{n_0}(x)v_0\| > \|B^{n_0}(x)v_\infty\| \) for all \( x \in X \) and all unit vectors \( v_0 \in E_0(x) \) and \( v_\infty \in E_\infty(x) \). For a rank 1 cocycle \( B \), \( E_\infty(x) = \text{Ker}(B(x)) \).

**Assumption 4** (Dominated Splitting). The cocycle \( B \) has dominated splitting. In particular we have that \( L_1(B) > L_2(B) \geq -\infty \).

1.2. **Statements.** We can now state the main result of this paper and some of its consequences.

**Theorem 1.1.** With assumptions 1-4 fulfilled, for any \( t \in \mathbb{C} \),
\[
L_1(A_t) = L_1(B) + \int_\mathbb{R} \log|t - s|d\rho(s), \tag{1.4}
\]
where \( d\rho \) is the Lebesgue-Stieltjes measure associated with the fibered rotation number \( \rho(A_t) \).

**Remark 1.1.** The dominated splitting assumption is only used in Lemma 3.4 below through the following chain of implications
\[
B \text{ has dominated splitting} \Rightarrow A_t \text{ has dominated splitting } \forall \text{ large } t \Rightarrow d\rho \text{ has compact support}.
\]
If we can prove that $A_t$ has dominated splitting for all sufficiently large $t$, or else that the Lebesgue-Stieltjes measure $d\rho$ has compact support then Theorem 1.1 would hold with Assumption 4 replaced by the much weaker hypothesis $L_1(B) > L_2(B) \geq -\infty$.

Moreover, the compactness of supp$(d\rho)$ is a technical assumption required in the proof of Proposition 3.5. So it is possible that this proposition, and whence Theorem 1.1, still holds even if this support is not compact.

As a consequence of Theorem 1.1 we establish a limitation on the modulus of continuity of the Lyapunov exponent of random linear cocycles. Let $X := \{1, \ldots, \kappa\}^\mathbb{Z}$ be the space of sequences in $\kappa$ symbols, and let $T : X \to X$ be the Bernoulli shift on $X$ equipped with some Bernoulli probability measure $\mu = (p_1, \ldots, p_\kappa)^\mathbb{Z}$, where $p_1 + \cdots + p_\kappa = 1$ and $p_j > 0$ for $j = 1, \ldots, \kappa$.

A random or locally constant cocycle $A : X \to \text{SL}_2(\mathbb{R})$ is determined by a vector of $\kappa$ matrices $A = (A_1, \ldots, A_\kappa) \in \text{SL}_2(\mathbb{R})^\kappa$, via the formula $A(\omega) := A_{\omega_0}$, where $\omega = (\omega_j)_{j \in \mathbb{Z}} \in X$. We will use the notations $A$ and $A$ interchangeably.

The iterates of the cocycle $A$ are thus the multiplicative random process corresponding to the finitely supported measure

$$\mu(A) := \sum_{j=1}^{\kappa} p_j \delta_{A_j} \in \text{Prob}(\text{SL}_2(\mathbb{R})).$$

Denote by $H(\mu) := -\sum_{j=1}^{\kappa} p_j \log p_j$ the Shannon entropy of the measure $\mu(A)$. A simplified version of our result is as follows (see Corollary 5.4 for its more precise formulation). Throughout the manuscript we will use the notation $\text{SL}_2^*(\mathbb{R}) := \text{SL}_2(\mathbb{R}) \setminus \{-I, I\}$.

**Theorem 1.2.** Let $A = (A_1, \ldots, A_\kappa) \in \text{SL}_2^*(\mathbb{R})^\kappa$ be a random cocycle such that $L_1(A) > 0$ but $A$ is not uniformly hyperbolic. There is an open set $U$ of “directions” in $\{P = (P_1, \ldots, P_\kappa) : P_i^2 = 0, \text{tr} P_i = 0 \ \forall i \in \text{Mat}_2(\mathbb{R})^\kappa, \text{such that for all} \ P = (P_1, \ldots, P_\kappa) \in U, \text{if we denote by} \ A_t \text{ the random cocycle determined by the list} \ A_t := A(I + tP), \text{then the map} \ t \mapsto L_1(A_t) \text{is not} \ \alpha\text{-Hölder continuous near} \ t = 0 \ \text{provided that} \ \alpha > \alpha(A) := \frac{H(\mu)}{L_1(A)}.$

This result extends [5, Theorem A], where the same conclusion was achieved for a particular curve of cocycles obtained embedding $A$ into a family of Schrödinger cocycles over a Markov shift.

**Remark 1.2.** It is well known, starting with the work of Le Page [30], see also Duarte and Klein [15, 16] that under an irreducibility assumption, the maximal Lyapunov exponent is always Hölder continuous near cocycles $A$ with $L_1(A) > 0$. Moreover, in [18] it was shown that if such a cocycle $A$ is diagonalizable, then the Lyapunov exponent is at least weak-Hölder continuous in its vicinity, and if it is not diagonalizable (since then either $A$ or its inverse satisfy some irreducibility condition), the Lyapunov exponent is locally Hölder continuous.
Moreover, since uniform hyperbolicity is an open property, if a cocycle $A$ is uniformly hyperbolic, then the same holds for $A_t$ near $t = 0$. In this case the map $t \mapsto L_1(A_t)$ is analytic.

Therefore we have the following dichotomy for an irreducible cocycle $A$ with $L_1(A) > 0$. Near $t = 0$, the Lyapunov exponent of $A_t$ is either analytic or it is continuous with a strict limitation on the strength of its modulus of continuity, namely $\alpha$-Hölder with $\alpha \leq \alpha(A)$.

Thouless formula (1.4) shows that the maps $t \mapsto L_1(A_t) - L_1(B)$ and $t \mapsto \rho(A_t)$ are obtained one from the other via the Hilbert transform (see for instance [21] for its definition). It is well known (see [23, Lemma 10.3]) that the Hilbert transform preserves Hölder-type continuity properties.

The Hölder (or weak-Hölder) continuity of the maximal Lyapunov exponents has been established for various types of linear cocycles. We list only a few of the more recent such results (for a more complete list of results, see the references therein). Lyapunov exponents of quasiperiodic cocycles (that is, linear cocycles over a torus translation) are Hölder or weak-Hölder continuous provided the translation frequency satisfies an appropriate arithmetic condition (e.g. a Diophantine condition) and the fiber action depends analytically on the base point (see [17]); results in other regimes (e.g. almost reducibility) are also available, see for instance [8]. Lyapunov exponents of random cocycles (i.e. locally constant cocycles over a Bernoulli or Markov shift) are Hölder continuous assuming a generic irreducibility condition (see [30] and [15, Chapter 5]) and weak-Hölder continuous without such an assumption (see [18]). Under appropriate conditions, Lyapunov exponents of linear cocycles over uniformly hyperbolic systems are Hölder continuous (see [19]). Similar continuity properties were also obtained for mixed random-quasiperiodic cocycles (see [10]).

**Proposition 1.1.** For each of the types of linear cocycles described above, under the specific assumptions ensuring the Hölder (resp. weak-Hölder) continuity of the Lyapunov exponent, the fibered rotation number $\rho(A_t)$, of a family of cocycles $A_t$ satisfying assumptions 1-4 above, is a locally Hölder (resp. locally weak-Hölder) continuous function.

It is natural to consider the problem of extending the results in this paper in other directions, as follows: relaxing Assumption 4 as explained in Remark 1.1; considering instead of affine one parameter families $A_t$, polynomial or even more general families; obtaining an analogue of Thouless formula in Theorem 1.1 for symplectic (higher dimensional) cocycles; using the approach in Theorem 1.2 to derive limitations on the modulus of continuity for other types of cocycles, such as mixed random-quasiperiodic cocycles, see [9, 6]. These extensions will be considered in separate projects.

In [24] A. Gorodetski and V. Kleptsyn have established the following result under a similar setting. Given a smooth family of positively winding cocycles
\{A_t : X \to \text{SL}(2, \mathbb{R})\}_{t \in I}, which is not uniformly hyperbolic for any \( t \in I \), there exist \( \Omega \subset X \) with full probability and a residual subset \( S \subset I \) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \log \|A^n_t(x)\| = L_1(A_t) \quad \forall x \in \Omega, \ t \in I
\]

\[
\liminf_{n \to \infty} \frac{1}{n} \log \|A^n_t(x)\| = 0 \quad \forall x \in \Omega, \ t \in S.
\]

This behavior complements the regularity dichotomy in Theorem 1.2. It would be interesting to explore the connection between the orbits analyzed in [24] and the fractal structure of the set of heteroclinic tangencies alluded to in Subsection 1.3 of [5].

Our results are also related to the work [14], where B. Deroin and R. Dujardin study holomorphic families \( \{A_\lambda\}_{\lambda \in \Lambda} \) of random (locally constant) \( \text{SL}(2, \mathbb{C}) \)-cocycles parametrized on a complex manifold \( \Lambda \). The connection becomes more clear when \( \Lambda \) has dimension 1, say \( \Lambda = \mathbb{C} \). In this case the Lyapunov exponent \( \lambda \mapsto L_1(A_\lambda) \) is a subharmonic function and its Laplacian (in the sense of distributions) is the so-called bifurcation current \( T_{\text{bif}} \). In [14, Theorem A] the authors characterize the bifurcation locus of the holomorphic family \( \{A_\lambda\}_{\lambda \in \Lambda} \) as being the support of the current \( T_{\text{bif}} \). For positively winding families, the bifurcation locus coincides with the set of parameters \( \lambda \in \Lambda \) such that \( A_\lambda \) is not uniformly hyperbolic. Moreover, [14, Theorem 3.5] can be used to prove that this bifurcation current matches the fibered rotation measure, i.e., \( T_{\text{bif}} = d\rho \). The Thouless formula (1.4) follows then from the Riesz representation theorem for subharmonic functions (see [26]). This connection provides an enlightening description of \( d\rho \) and its support in the special case where the homeomorphism \( T : X \to X \) is a Bernoulli shift in finitely many symbols.

The rest of this paper is organized as follows. In Chapter 2 we study the winding property and formally introduce the fibered rotation number. In Chapter 3 we provide the proof of our main result, the Thouless formula in Theorem 1.1. In Chapter 4 we present sufficient conditions ensuring the validity of each of the main assumptions besides the affine form (namely the invertibility, the winding property and the dominated splitting). In Chapter 5 we present applications of the main result, namely we establish Theorem 1.2 regarding the limitations on the modulus of continuity of random cocycles. In the Appendix, Chapter 6, we develop some linear algebra tools used in the rest of the work.

2. The winding property

In this section we establish some consequences of the winding property in Assumption 2. In Sections 3.1 and 3.2 the results are stated for general the one parameter differentiable family of maps \( A_t : X \to \text{GL}_2^+(\mathbb{R}) \) indexed by \( t \in I \). Section 3.3 is centred in the particular case where \( A_t = A + tB \).
2.1. General properties. The winding property means that for almost all \( \hat{v} \in \mathbb{P}^1 \), the projective curve \( I \ni t \mapsto \hat{A}_t \hat{v} \in \mathbb{P}^1 \), has non-vanishing derivative for almost every \( t \in I \), keeping the same orientation as \( t \) runs in \( I \). See Proposition 2.2 below.

**Proposition 2.1.** Given a unit vector \( v \in \mathbb{R}^2 \),

\[
\frac{d}{dt} [A_t \hat{v}] = \frac{d}{dt} \frac{A_t v}{\| A_t v \|} = \frac{(A_t v) \wedge (\dot{A}_t v)}{\| A_t v \|^2}
\]

where as before \( \dot{A}_t = \frac{d}{dt} A_t \).

**Remark 2.1.** To interpret this equality the reader should either regard right-hand-side derivative as a real number, because \( \mathbb{P}^1 \) is 1-dimensional, or else the left-hand-side as a vector, multiplying it by the unique unit and positive vector in \( T_{A_t v/\|A_t v\|} \mathbb{P}^1 \).

**Proof.** We want to establish an expression for the variation of the angle of the projective map \( t \mapsto \hat{A}_t \hat{v} \). Thus, since the metric considered in \( \mathbb{P}^1 \) is \( d(\hat{v}, \hat{w}) := |v \wedge w| / \|v\| \|w\| \), we get that,

\[
d(\hat{A}_t \hat{v}, \hat{A}_{t'} \hat{v}) = \frac{|A_t v \wedge A_{t'} v|}{\|A_t v\| \|A_{t'} v\|} = \frac{|A_t v \wedge (A_{t'} - A_t) v|}{\|A_t v\| \|A_{t'} v\|}
\]

by adding \( A_t v \) in the second term because \( A_t v \wedge A_{t'} v = 0 \). Dividing by \( |t - t'| \),

\[
\frac{d(\hat{A}_t \hat{v}, \hat{A}_{t'} \hat{v})}{|t - t'|} = \frac{|A_t v \wedge \frac{A_{t'} - A_t}{t'} v|}{\|A_t v\| \|A_{t'} v\|}
\]

and since the limit on the left hand side when \( t' \) goes to \( t \) is the absolute value of the derivative we obtain

\[
\left| \frac{d}{dt} \frac{A_t v}{\| A_t v \|} \right| = \frac{|(A_t v) \wedge (\dot{A}_t v)|}{\| A_t v \|^2}.
\]

The identity follows from simple geometric considerations on the oriented angle between \( A_t v \) and \( \dot{A}_t v \). \( \square \)

**Proposition 2.2.** Let \( I \ni t \mapsto A_t \in \text{GL}_+^1(\mathbb{R}) \) be an analytic curve with the winding property. For almost every \( \hat{v} \in \mathbb{P}^1 \) and almost every \( t \in I \), the map \( I \ni t \mapsto \hat{A}_t \hat{v} \in \mathbb{P}^1 \) has non-vanishing derivative and keeps the same orientation as \( t \) runs in \( I \).

**Proof.** By the winding assumption, \( Q_{A_t} \) is either positive or negative semi-definite for all \( t \). From now on we assume the winding is positive. By Definition 1.1, \( Q_{A_t}(v) := (\det A_t) v \wedge A_t^{-1} \dot{A}_t v = 0 \) if and only if \( v \) is an eigenvector of \( A_t^{-1} \dot{A}_t \), and since \( Q_{A_t} \) is a quadratic form, this eigenvector is unique. Therefore, if we denote

\[
\text{Eig}(A_t) := \left\{ \hat{v} \in \mathbb{P}^1 : v \text{ is an eigenvector of } A_t^{-1} \dot{A}_t \right\},
\]

then either \( Q_{A_t} \) is positive definite and \( \text{Eig}(A_t) = \emptyset \), or else \( Q_{A_t} \) is positive semi-definite and \( \text{Eig}(A_t) \) is singleton.
To follow we analyse the cases where \(\text{Eig}(A_t) \neq \emptyset\), otherwise the conclusion is obvious. Therefore for the degree two polynomial \(p(\lambda) := \det(I + \lambda A_t^{-1} \dot{A}_t)\), its discriminant is given by \(\Delta(t) := 4 \det(A_t^{-1} \dot{A}_t) - \text{tr}(A_t^{-1} \dot{A}_t)^2\) up to a sign, which is proved in Proposition 4.1 below. Because this function is analytic, we consider two cases

In the first case, \(\Delta(t) \neq 0\) and \(Z := \{t \in I: \Delta(t) = 0\}\) is a countable set consisting of isolated regular points. Since for all \(t \not\in Z\), by Proposition 2.4.

Proof.\[\frac{d}{dt} \frac{A_t v}{\|A_t v\|} = \|A_t v\|^{-2} Q_{A_t}(v) > 0\] and the conclusion follows.

In the second case \(\Delta(t) \equiv 0\), \(\text{Eig}(A_t) \neq \emptyset\) for all \(t \in I\), and there exists \(\varphi : I \to \mathbb{P}^1\) analytic such that \(\text{Eig}(A_t) = \{\varphi(t)\}\) for all \(t \in I\). Let \(V\) be the set of regular values of \(\varphi : I \to \mathbb{P}^1\). By Sard’s Theorem, \(V\) has full measure in \(\mathbb{P}^1\). For \(\hat{v} \in V\) the set \(Z_{\hat{v}} := \{t \in I: \hat{v} \in \text{Eig}(A_t)\}\) is a countable set consisting of isolated regular points. Since for all \(t \not\in Z_{\hat{v}}\),
\[
\frac{d}{dt} \frac{A_t v}{\|A_t v\|} = \|A_t v\|^{-2} Q_{A_t}(v) > 0,
\]
the map \(I \ni t \mapsto A_t \hat{v} \in \mathbb{P}^1\) has non-vanishing derivative for almost every \(t \in I\), keeping the same orientation as \(t\) runs in \(I\). \(\square\)

We want to see that the winding property is preserved under iterations of the cocycle. The following proposition will imply that the composition of positively (negatively) winding cocycles is also positively (negatively) winding.

**Proposition 2.3.** Given curves \(I \ni t \mapsto A_{i,t} \in \text{GL}_2^+(\mathbb{R})\), for \(i = 1, \ldots, n\), if each \(A_{i,t}\) is positively (negatively) winding then so is their product \(M_t := A_{n,t} \cdots A_{2,t} A_{1,t}\).

**Proof.** For simplicity consider \(n = 2\). Writing \(v_1 := A_{1,t} v\), we have
\[
(M_t v) \wedge (\dot{M}_t v) = (A_{2,t} A_{1,t} v) \wedge (\dot{A}_{2,t} A_{1,t} v + A_{2,t} \dot{A}_{1,t} v)
\]
\[
= (A_{2,t} v_1) \wedge (\dot{A}_{2,t} v_1) + (A_{2,t} A_{1,t} v) \wedge (A_{2,t} \dot{A}_{1,t} v)
\]
\[
\geq 0 + (\det A_{2,t}) (A_{1,t} v) \wedge (\dot{A}_{1,t} v) \geq 0.
\]
Moreover, it is positive if for example \(v\) is not an eigenvector of \(A_{1,t}^{-1} \dot{A}_{1,t}\). Thus \(M_t\) is positively winding. The general case follows by induction. \(\square\)

The previous argument shows a bit more. For the sake of simplicity we only state the following result for \(\text{SL}_2(\mathbb{R})\)-valued curves.

**Proposition 2.4.** Given curves \(I \ni t \mapsto A_{i,t} \in \text{SL}_2(\mathbb{R})\), for \(i = 1, \ldots, n\), define \(v_j(t) := A_{j,t} \cdots A_{1,t} v / \|A_{j,t} \cdots A_{1,t} v\|\), with the convention that \(v_0 = v\). Then...
for $M_t := A_{n,t} \cdots A_{2,t} A_{1,t},$

$$\frac{(M_t v) \wedge (\dot{M_t} v)}{\|M_t v\|^2} = \sum_{j=1}^n \frac{1}{\|A_{n,t} \cdots A_{j+1,t} v_j\|^2} \frac{(A_{j,t} v_{j-1}) \wedge (\dot{A}_{j,t} v_{j-1})}{\|A_{j,t} v_{j-1}\|^2},$$

$$= \sum_{j=1}^n \left( \frac{\|A_{j,t} \cdots A_{1,t} v\|}{\|M_t v\|} \right)^2 \frac{(A_{j,t} v_{j-1}) \wedge (\dot{A}_{j,t} v_{j-1})}{\|A_{j,t} v_{j-1}\|^2}.$$ 

**Remark 2.2.** The ratio $\frac{(M_t v) \wedge (\dot{M_t} v)}{\|M_t v\|^2}$ measures the rotation speed of $M_t v.$ Likewise the ratio $\frac{(A_{j,t} v_{j-1}) \wedge (\dot{A}_{j,t} v_{j-1})}{\|A_{j,t} v_{j-1}\|^2}$ measures the rotation speed of $\dot{A}_{j,t} \dot{w}$ when $w := v_{j-1}(t)$ is fixed.

**Corollary 2.5.** If $A_t : X \to \text{GL}_2^+(\mathbb{R})$ is a positively (negatively) winding family of cocycles then for every $n \in \mathbb{N}$ the family of iterated cocycles $A^n_t : X \to \text{GL}_2^+(\mathbb{R})$ is positively (negatively) winding.

2.2. **Fibered Rotation Number.** In this section we introduce the notion of fibered rotation number referred to in the main theorem. This concept was first introduced in [27] and further developed in [2, 24].

Let $\pi : \mathbb{R} \to \mathbb{P}^1$ denote the canonical covering map of $\mathbb{P}^1,$ which induces a diffeomorphism between $\mathbb{T}^1 := \mathbb{R}/\pi \mathbb{Z}$ and $\mathbb{P}^1.$ Consider a continuous family of cocycles $A_t : X \to \text{GL}_2^+(\mathbb{R}),$ with parameter $t \in I,$ over the continuous base map $T : X \to X.$ Each matrix $A_t(x)$ admits a lifting $\tilde{A}_t(x) : \mathbb{R} \to \mathbb{R}$ such that the following diagram commutes

$$\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\tilde{A}_t(x)} & \mathbb{R} \\
\downarrow \pi & & \downarrow \pi \\
\mathbb{P}^1 & \xrightarrow{\tilde{A}_t(x)} & \mathbb{P}^1
\end{array}$$

We gather these liftings in a single function $\tilde{G} : X \times I \times \mathbb{R} \to \mathbb{R},$ $\tilde{G}(x,t,v) := \tilde{A}_t(x)(v).$

Given two non-zero vectors $v, w \in \mathbb{R}^2,$ the angle $\angle(v, w)$ is well defined as an element in $\mathbb{R}/(2\pi \mathbb{Z}).$ We will use the notation $\angle(v, w)$ to represent a real argument of $\angle(v, w)$ so that

$$\angle(v, w) = \angle(v, w) + 2\pi \mathbb{Z}.$$ 

Notice that $\angle(v, w)$ can not be globally and continuously defined as a function of $(v, w) \in \mathbb{S}^1 \times \mathbb{S}^1,$ where $\mathbb{S}^1 := \{v \in \mathbb{R}^2 : \|v\| = 1\}.$

**Proposition 2.6.** Fix $t_0 \in I$ and a unit vector $v_0 \in \mathbb{S}^1.$ Then there exists $h : X \to \mathbb{R}$ such that
Proof. For each point \( x \in X \) we can take a radius \( r > 0 \) such that the ball \( B_r(x) := \{ z \in X : d(z, x) < r \} \) has boundary \( \partial B_r(x) := \{ z \in X : d(z, x) = r \} \) with zero measure, \( \mu(\partial B_r(x)) = 0 \), and a locally defined continuous function \( h_x : B_r(x) \to \mathbb{R} \) such that \( \angle(A_{t_0}(z) v_0, v_0) = h_x(z) + 2\pi \mathbb{Z} \) for all \( z \in B_r(x) \). Since \( X \) is compact we can cover \( X \) with a finite number of these balls \( B_1, \ldots, B_m \), where \( B_i = B_{r_i}(x_i) \) for \( i = 1, \ldots, m \). Writing \( h_i = h_{x_i} \), we define \( h(x) := h_1(x) \) if \( x \in B_1 \) and more generally \( h(x) := h_i(x) \) if \( x \in B_i \setminus (B_1 \cup \cdots \cup B_{i-1}) \) for some \( i = 2, \ldots, m \). This function \( h \) satisfies (1)-(3).

Proposition 2.7. The function \( \tilde{G} \), dependent on the arbitrary choices of the liftings \( \tilde{A}_t(x) \), can be made continuous in \( (t, v) \in I \times \mathbb{R} \), and measurable in \( (x, t, v) \in X \times I \times \mathbb{R} \) in a way that for some measurable set \( D \subset X \) with \( \mu(D) = 0 \), the function \( \tilde{G} \) is continuous over \( (X \setminus D) \times I \times \mathbb{R} \).

Proof. In general it may not be possible to realize \( \tilde{G} \) as a globally continuous function, see [24, Remark A.4]. Fix \( t_0 \in \mathbb{R} \), \( v_0 \in S^1 \), \( x_0 \in \mathbb{R} \) such that \( \pi(x_0) = \hat{v}_0 \) and a measurable function \( h : X \to \mathbb{R} \) as in Proposition 2.6. For each \( x \in X \) let \( \tilde{A}_{t_0}(x) : \mathbb{R} \to \mathbb{R} \) be the unique lifting of \( A_{t_0}(x) : \mathbb{P}^1 \to \mathbb{P}^1 \) such that \( \tilde{A}_{t_0}(x)(x_0) - x_0 = h(x) \). Extending these liftings continuously in the variable \( t \in I \), we obtain a function \( \tilde{G}(x, t, v) := A_t(x) v - v \), continuous in \( (t, v) \in I \times \mathbb{R} \), and sharing with \( h(x) \) the same continuity points in \( X \). □

We define recursively \( \tilde{G}_n : X \times I \times \mathbb{R} \to \mathbb{R} \),
\[
\tilde{G}_n(x, t, v) := \tilde{G}\left(T^{n-1}x, t, \tilde{G}_{n-1}(x, t, v)\right)
\]
for \( n \geq 1 \) with \( \tilde{G}_0(x, t, v) := v \). Through this function we define the angle
\[
\angle(A^n_t(x) \pi(v), \pi(v)) := \tilde{G}_n(x, t, v) - v.
\]
The left-hand-side does not depend on the representative \( v \in \mathbb{R} \) of the projective point \( \pi(v) \in \mathbb{P}^1 \). Hence the expression \( \angle(A^n_t(x) v, v) = \angle(A^n_t(x) \hat{v}, \hat{v}) \) makes sense and defines a function on \( X \times I \times \mathbb{P}^1 \). For \( n = 1 \), this expression determines the family of functions \( \tilde{H}_t : X \times \mathbb{P}^1 \to \mathbb{R} \),
\[
\tilde{H}_t(x, \hat{v}) := \angle(A_t(x) \hat{v}, \hat{v}).
\]
Given \( s < t \), define for any \( (x, v) \in X \times \mathbb{P}^1 \),
\[
\angle(A^n_s(x) v, A^n_t(x) v) := \angle(A^n_s(x) v, v) - \angle(A^n_t(x) v, v).
\]

Proposition 2.8. The functions defined above satisfy:
(1) For each \( x \in X \), \( I \times \mathbb{P}^1 \ni (t, \hat{v}) \mapsto \angle(A^n_t(x) \hat{v}, \hat{v}) \), is a continuous map;
(2) \( \angle(A^n_t(x) \hat{v}, \hat{v}) = \sum_{j=0}^{n-1} \tilde{H}_t(F^j(x, \hat{v})) \), \( \forall t \in I, (x, \hat{v}) \in X \times \mathbb{P}^1 \).
The angle functions \((x, t, \hat{\nu}) \mapsto \angle(A^n_t(x) \hat{\nu}, \hat{\nu})\) are continuous in the complement of a product set \(D \times I \times \mathbb{P}^1\) where \(\mu(D) = 0\);

\(\angle(A^n_t(x) v, A^n_s(x) v) = \angle(A^n_t(x) v, A^n_s(x) v) + \pi \mathbb{Z}\);

\(\angle(A^n_t(x) v, A^n_s(x) v) \geq 0, \quad \text{for } t \geq s, \quad \text{if the family of cocycles } A_t \text{ is positively winding.}\)

**Proof.** Follows from the definitions and Proposition 2.7. The set in item (3) is the union \(D := \bigcup_{j \in \mathbb{Z}} T^{-j}(D_h)\), where \(D_h\) is the set of discontinuity points of the function \(h\) used in the proof of Proposition 2.7.

For item (5), notice that

\[
\angle(A^n_t(x) v, A^n_s(x) v) := \angle(A^n_t(x) v, v) - \angle(A^n_s(x) v, v) = \sum_{j=0}^{n-1} \tilde{H}_t(F^j(x, \hat{\nu})) - \tilde{H}_s(F^j(x, \hat{\nu}))
\]

and the function \(\tilde{H}_t(x, \hat{\nu})\) is non-decreasing in the variable \(t\), as a consequence of the winding property.

**Proposition 2.9.** For every \(t \in I\) there exist a number \(\rho \in \mathbb{R}\) and a measurable set \(\Omega_t \subset X\) with full measure, \(\mu(\Omega_t) = 1\), such that for all \(x \in \Omega_t\) and all \(\hat{\nu} \in \mathbb{P}^1\),

\[
\rho = \lim_{n \to +\infty} \frac{1}{\pi n} \angle(A^n_t(x) \hat{\nu}, \hat{\nu}).
\]

**Proof.** Follows from [24, Proposition A.1].

**Definition 2.1.** The previous limit \(\rho\) is called the fibered rotation number of the cocycle \(A_t\) and denoted by \(\rho(A_t)\).

**Proposition 2.10.** Given \(t \in I\), for any measure \(\nu \in \text{Prob}(X \times \mathbb{P}^1)\) such that \((\hat{F}_t)_*\nu = \nu\) and \(\pi_*\nu = \mu\), where \(\pi: X \times \mathbb{P}^1 \to X\) is the canonical projection \(\pi(x, \hat{\nu}) := x\), we have

\[
\rho(A_t) = \frac{1}{\pi} \int_{X \times \mathbb{P}^1} \angle(A_t(x) \hat{\nu}, \hat{\nu}) d\nu(x, \hat{\nu}).
\]

**Proof.** Assuming \((\hat{F}_t, \nu)\) is ergodic, the conclusion follows applying Birkhoff ergodic theorem to item (2) of Proposition 2.8 and Proposition 2.9. In general the space \(M\) of all measures \(\nu \in \text{Prob}(X \times \mathbb{P}^1)\) such that \((\hat{F}_t)_*\nu = \nu\) and \(\pi_*\nu = \mu\) is weak* compact and convex. Moreover the extremal points of \(M\) are the measures \(\nu \in M\) such that \((\hat{F}_t, \nu)\) is ergodic. When \(\nu\) is not ergodic, it admits an ergodic decomposition consisting of ergodic measures, i.e., extremal points of \(M\). Hence the stated identity must hold in this case as well.

**Proposition 2.11.** Assuming the family of cocycles \(A_t\) is positively winding then the function \(\rho : I \to \mathbb{R}, \ t \mapsto \rho(A_t)\), is continuous and non-decreasing.
Proof. Consider a convergent sequence \( t_n \to t \) in \( I \). For each \( n \in \mathbb{N} \) take \( \nu_n \in \text{Prob}(X \times \mathbb{P}^1) \) such that \( (F_{t_n})_*\nu_n = \nu_n \) and \( \pi_*\nu_n = \mu \). Denote by \( \eta \in \text{Prob}(X \times \mathbb{P}^1) \) any accumulation point of the sequence \( \nu_n \). We can easily check that \( (F_t)_*\eta = \eta \) and \( \pi_*\eta = \mu \). Since \( \varangle(A_{t_n}(x)\hat{v}, \hat{v}) \) converges almost uniformly to \( \varangle(A_t(x)\hat{v}, \hat{v}) \) on \( X \times \mathbb{P}^1 \) and by item (3) of Proposition 2.8 the set of discontinuity points of \( \varangle(A_t(x)\hat{v}, \hat{v}) \) has \( \mu \)-measure zero, we have for some sub-sequence \( n_i \),

\[
\lim_{i \to \infty} \rho(A_{t_{n_i}}) = \lim_{i \to \infty} \frac{1}{\pi} \int \varangle(A_{t_{n_i}}(x)\hat{v}, \hat{v}) \, d\nu_{n_i}(x, \hat{v}) = \frac{1}{\pi} \int \varangle(A_t(x)\hat{v}, \hat{v}) \, d\eta(x, \hat{v}) = \rho(A_t).
\]

Finally, the fact that this convergence holds for all sub-limits \( \eta \) of \( \nu_n \) implies that \( \lim_{n \to \infty} \rho(A_{t_{n_i}}) \) does indeed exist and is equal to \( \rho(A_t) \).

Given \( t_0 < t'_0 \), take a measurable set \( \Omega \subset X \), with full measure, i.e., \( \mu(\Omega) = 1 \), such that for all \( x \in \Omega \) and both \( t \in \{t_0, t'_0\} \),

\[
\rho(A_t) = \lim_{n \to \infty} \frac{1}{\pi n} \varangle(A^n_t(x)e_1, e_1).
\]

By the positive winding property

\[
\varangle(A^n_{t_0}(x)e_1, e_1) - \varangle(A^n_{t'_0}(x)e_1, e_1) = \varangle(A^n_{t_0}(x)e_1, A^n_{t'_0}(x)e_1) \geq 0
\]

Hence taking limits

\[
\rho(A_{t_0}) = \lim_{n \to \infty} \frac{1}{\pi n} \varangle(A^n_{t_0}(x)e_1, e_1) \geq \lim_{n \to \infty} \frac{1}{\pi n} \varangle(A^n_{t'_0}(x)e_1, e_1) = \rho(A_{t'_0}).
\]

Therefore \( t \mapsto \rho(A_t) \) is a non-decreasing function. \( \square \)

**Proposition 2.12.** Given any \( t, t' \in I \), the relative rotation number \( \rho(A_{t'}) - \rho(A_t) \), \( t' > t \) does not depend on the choice of the function \( h \).

In particular this relative rotation number is intrinsically defined.

**Proof.** See [24, Remark A.8]. \( \square \)

2.3. Homotopy properties. In this section we discuss what happen with the length of a the curve satisfying the winding property.

**Definition 2.2.** Given \( I \ni t \mapsto A_t \in \text{GL}_2^±(\mathbb{R}) \) smooth and a unit vector \( v \in S^1 \), the length (oriented angle) of \( A_t \, v / \|A_t \, v\| \in S^1 \) as \( t \) ranges in \( I \) is denoted by \( \ell_I(A_t, v) \). This also agrees with the length of the projective curve \( I \ni t \mapsto A_t \hat{v} \).

**Proposition 2.13.** Given a positively winding smooth curve \( I \ni t \mapsto A_t \in \text{GL}_2^±(\mathbb{R}) \) and \( \hat{v} \in \mathbb{P}^1 \),

\[
\ell_I(A_t, \hat{v}) = \int_I \frac{(A_t v) \wedge (\dot{A}_t v)}{\|A_t v\|^2} \, dt.
\]

**Proof.** Follows from Proposition 2.1. \( \square \)
Next lemma relates the asymptotic length of curves \( J \ni t \mapsto A^n t v \) with the Lebesgue-Stieltjes measure \( d\rho \) determined by the fibered rotation number \( \rho(A_t) \).

**Lemma 2.14.** For every \( J \subset I \), every \( \hat{v} \in \mathbb{P}^1 \) and \( \mu \)-a.e. \( x \in X \),
\[
\lim_{n \to \infty} \frac{1}{n\pi} \ell_J(A^n t (x) \hat{v}) = d\rho(J).
\]

**Proof.** By the winding property, for any \( \hat{v} \in \mathbb{P}^1 \) and \( t_0, t_1 \in I \) with \( t_0 < t_1 \) and \( x \in \Omega_{t_0} \cap \Omega_{t_1} \),
\[
\ell_{[t_0, t_1]}(A^n t (x) \hat{v}) = \angle(A^n t_1 (x) v, v) - \angle(A^n t_0 (x) v, v).
\]
Therefore this lemma follows from Proposition 2.9. \( \square \)

![Figure 1. Solutions of the equation \( \hat{A}_t \hat{v} = \hat{w}_a := \hat{A}_a \hat{v}, \ t \in [a, b] \).](image)

**Lemma 2.15.** Given \( v \in S^1 \) and \( I \ni t \mapsto A_t \in \text{GL}_2^+(\mathbb{R}) \) a smooth curve positively winding, the following are equivalent:

1. \( \ell_I(A_t v) \geq n \pi \);
2. \( \forall w \in S^1, \# \{ t \in I : \hat{A}_t \hat{v} = \hat{w} \} \geq n \).

Moreover, if \( I \) is closed, (1) or (2) hold, and \( \hat{w} \in \mathbb{P}^1 \) is one of the end points of the curve \( I \ni t \mapsto \hat{A}_t \hat{v} \), then the equation equation \( \hat{A}_t \hat{v} = \hat{w} \) has at least \( n + 1 \) solutions in \( t \in I \).

**Proof.** The direct implication (1) \( \Rightarrow \) (2) is clear. Assume (2), \( I = [a, b] \) and set \( w_t = A_t v \). By (2) there are at least \( n \) solutions \( a = t_1 < t_2 < \ldots < t_n \leq b \) of the equation \( \hat{A}_t \hat{v} = \hat{w}_a \), see Figure 1. For every small \( \varepsilon > 0 \), notice that the
equation $\hat{A} \hat{v} = \hat{w}_{t_n - \varepsilon}$ has $n - 1$ solutions for $t \in [a, t_n]$. In fact, we have one solution of this equation in each interval $[t_i, t_{i+1}]$. Hence using the assumption, for each $\varepsilon$, there exists $s(\varepsilon) \in (t_n, b]$ which is the $n$-th solution of the equation $\hat{A} \hat{v} = \hat{w}_{t_n - \varepsilon}$. Making $\varepsilon \to 0$ and using the winding property we find another solution $t_{n+1} := \lim_{s \to 0} s(\varepsilon) \in (t_n, b]$ of the equation $\hat{A} \hat{v} = \hat{w}_a$. This argument also shows that for any $c \in [t_{n+1}, b]$, the equation $\hat{A}_c \hat{v} = \hat{w}_c$ has at least $n + 1$ solutions in $[a, b]$. In particular, $l_f(A_t \hat{v}) \geq n \pi$. \hfill $\square$

**Corollary 2.16.** Given intervals $J \subset I$, $K \subset \mathbb{P}^1$, and $\hat{v} \in \mathbb{P}^1$, if for every $\hat{w} \in K$, the equation $\hat{A}_t \hat{v} = \hat{w}$ has at least one solution for some $t \in J$, then $l_f(A_t \hat{v}) \geq \text{length}(K)$.

**Proof.** The curve $J \ni t \mapsto A_t \hat{v}$ winds positively and by assumption passes through all the $\hat{w} \in K$. So, its length is larger or equal than $\geq \text{length}(K)$. \hfill $\square$

**Definition 2.3.** We say that a smooth curve $I \ni t \mapsto A_t \in \text{GL}^+_2(\mathbb{R})$ winds $n$ times around $\mathbb{P}^1$ if for all $\hat{v} \in \mathbb{P}^1$, $l_f(A_t \hat{v}) \geq n \pi$.

**Corollary 2.17.** Given $I \ni t \mapsto A_t \in \text{GL}^+_2(\mathbb{R})$ a smooth curve positively winding, the following are equivalent:

1. the curve $I \ni t \mapsto A_t \in \text{GL}^+_2(\mathbb{R})$ winds $n$ times around $\mathbb{P}^1$;
2. $\forall \hat{v}, \hat{w} \in \mathbb{P}^1, \# \{ t \in I : \hat{A}_t \hat{v} = \hat{w} \} \geq n$.

**Proof.** Follows from Lemma 2.15. \hfill $\square$

**Proposition 2.18.** Given positively winding smooth curves $I \ni t \mapsto A_{i,t} \in \text{GL}^+_2(\mathbb{R})$, $i = 1, \ldots, k$, if each $A_{i,t}$ winds $n_i \geq 0$ times around $\mathbb{P}^1$ then the composition curve $I \ni t \mapsto A_{n,t} \cdots A_{1,t}$ winds $n_1 + \cdots + n_k$ times around $\mathbb{P}^1$.

**Proof.** Using induction it is enough proving this statement for $k = 2$. For notational simplicity we will denote the positively winding smooth matrix curves as $A_t$ and $B_t$. Assume $A_t$ winds $n$ times around $\mathbb{P}^1$ and $B_t$ winds $m$ times around $\mathbb{P}^1$. We will prove that for any $\hat{v} \in \mathbb{P}^1$, $\ell_f(A_t B_t \hat{v}) \geq (n + m) \pi$. By Corollary 2.17, this will imply that $A_t B_t$ winds $n + m$ times around $\mathbb{P}^1$.

Let $I$ be an interval with end points $a < b$. Take $\hat{u} = \hat{B}_a \hat{v}$ to be the starting point of the curve $I \ni t \mapsto \hat{B}_t \hat{v}$ and let $\hat{w} = \hat{A}_a \hat{u}$ be the starting point of the curve $I \ni t \mapsto \hat{A}_t \hat{u}$. By the second statement in Lemma 2.15, the equation $\hat{B}_s \hat{v} = \hat{u}$ has at least $m + 1$ solutions $a = s_0 < s_1 < \cdots < s_m \leq b$ in $I$, while the equation $\hat{A}_t u = \hat{w}$ has also $n + 1$ solutions $a = t_0 < t_1 < \cdots < t_n \leq b$ in $I$. Let $\Gamma : I \times I \to \mathbb{P}^1$ be the continuous mapping $\Gamma(t, s) := \hat{A}_t \hat{B}_s \hat{v}$. The curve $\Gamma(t, t)$, with $t \in I$, is homotopic with fixed endpoints to the concatenation of the following three curves

- $\Gamma_1 : [a, t_n] \to \mathbb{P}^1$, $\Gamma_1(t) := \Gamma(t, a) = \hat{A}_t \hat{B}_a \hat{v} = \hat{A}_t \hat{u}$;
- $\Gamma_2 : [a, b] \to \mathbb{P}^1$, $\Gamma_2(s) := \Gamma(t_n, s) = \hat{A}_{t_n} \hat{B}_s \hat{v}$;
\( \Gamma_3: [t_n, b] \rightarrow \mathbb{P}_1, \Gamma_3(t) := \Gamma(t, b) = \hat{A}_t \hat{B}_b \hat{v}. \)

![Figure 2](image.png)

**Figure 2.** The concatenation of \( \Gamma_3 \ast \Gamma_2 \ast \Gamma_1 \) is homotopic to the diagonal curve \( \Gamma(t) := \hat{A}_t \hat{B}_t \hat{v}. \)

See Figure 2. Since all these curves are positively winding and the diagonal \( \Gamma(t, t) \) is homotopic to polygonal concatenation \( \Gamma_3 \ast \Gamma_2 \ast \Gamma_1 \) through a homotopy which fixes the endpoints

\[
\ell_{\mathbb{R}}(\Gamma(t, t)) = \ell_{[a, t_n]}(\Gamma_1(t)) + \ell_{[a, b]}(\Gamma_2(s)) + \ell_{[t_n, b]}(\Gamma_3(t)) \geq n \pi + m \pi + 0 = (n + m) \pi.
\]

This concludes the proof. \( \square \)

From now on we deal with affine families.

**Proposition 2.19.** Given an affine curve \( M_t := A + tB \in \text{GL}_2^+(\mathbb{R}) \) satisfying the winding property, for any \( v \in \mathbb{R}^2 \) such that \( Bv \neq 0 \) we have \( \ell_{\mathbb{R}}(M_t v) = \pi. \)

**Proof.** If \( Bv \neq 0 \) then by the winding property \( A_t v \wedge \hat{A}_t v > 0 \) for every \( t \in \mathbb{R}. \) Notice that \( Av \neq 0 \) because \( A \in \text{GL}_2^+(\mathbb{R}). \) Hence the straight-line \( M_t v = Av + tBv \) induces a simple closed curve in \( \mathbb{P}_1 \) that begins and ends at \( \hat{B} \hat{v}. \) This implies that \( \ell_{\mathbb{R}}(M_t v) = \text{length}(\mathbb{P}_1) = \pi. \) \( \square \)

**Proposition 2.20.** Let \( M_t^n := (A_n + tB_n) \cdots (A_1 + tB_1), \) where \( A_i \in \text{GL}_2^+(\mathbb{R}) \) and \( A_i + tB_i \) is positively winding for \( i = 1, \ldots, n. \) Given \( v \in \mathbb{R}^2 \setminus \{0\}, \) if \( B_n B_{n-1} \cdots B_1 v \neq 0 \) then \( \ell_{\mathbb{R}}(M_t^n v) = n \pi. \)

**Proof.** The proof is made by induction over \( n. \)

The case \( n = 1 \) follows by Proposition 2.19.
Assume now that the induction hypothesis holds for a product of \( n - 1 \) factors. Then if \( M^{n-1}_t := (A_{n-1} + t B_{n-1}) \cdots (A_1 + t B_1) \), we have \( \ell_2(M^{n-1}_t v) = (n - 1) \pi \). For \( M^n_t := (A_n + t B_n) M^{n-1}_t \) we define \( \Gamma : [\rightarrow -\infty, +\infty[^2 \rightarrow \mathbb{P}^1 \) by

\[
\Gamma(t, s) := (A_n + t B_n) M^{n-1}_s \hat{v}.
\]

Consider the product matrices \( B^{n-1} = B_{n-1} \cdots B_1 \) and \( B^n = B_n B^{n-1} \). By assumption this matrix has either rank 1 or 2.

If \( \text{rank}(B_n) = 2 \) then \( \Gamma \) extends continuously to \( \hat{\Gamma} : [-\infty, +\infty[^2 \rightarrow \mathbb{P}^1 \) and the closed curve \( \hat{\mathbb{R}} \ni t \mapsto \hat{\Gamma}(t, t) = \hat{M}^n_t \hat{v} \in \mathbb{P}^1 \) is homotopic to the concatenation of two closed curves, one of degree 1, \( \Gamma_1 : \mathbb{R} \ni t \mapsto \Gamma(t, -\infty) \in \mathbb{P}^1 \), and the other of degree \( n - 1 \), \( \Gamma_2 : \mathbb{R} \ni s \mapsto \Gamma(+\infty, s) \). Note that writing \( C_n(t) := A_n + t B_n \), we have

\[
\Gamma_1(t) := \Gamma(t, -\infty) = \hat{C}_n(t) \lim_{t \to -\infty} \hat{M}^{n-1}_s \hat{v} = \hat{C}_n(t) \hat{B}^{n-1}_n \hat{v},
\]

the last equality holds because \( B^{n-1} v \neq 0 \). Likewise, since \( B_n \) is invertible,

\[
\Gamma_2(s) := \Gamma(+\infty, s) = \lim_{t \to +\infty} \hat{C}_n(t) \hat{M}^{n-1}_s \hat{v} = \hat{B}_n \hat{M}^{n-1}_s \hat{v}.
\]

By Proposition 2.19, \( \text{deg}(\Gamma_1) = 1 \), while by induction hypothesis we have \( \text{deg}(\Gamma_2) = n - 1 \). Hence by continuity of \( \hat{\Gamma} \) in the square \([-\infty, +\infty[^2 \), \( \hat{M}^n_t \hat{v} = \Gamma(t, t) \) is homotopic to the concatenation of \( \Gamma_1(t) \) with \( \Gamma_2(s) \) which implies that \( \text{deg}_{t \in \mathbb{R}}(\hat{M}^n_t) = 1 + (n - 1) = n \). Thus we have \( \ell_2(M^n_t v) = n \pi \).

Consider now the case \( \text{rank}(B_n) = 1 \). By assumption, we have that \( \hat{B}^{n}_t \hat{v} = \lim_{s \to \pm\infty} \hat{B}_n \hat{M}^{n-1}_s \hat{v} \) is well-defined. By induction hypothesis the map \( \hat{\mathbb{R}} \to \mathbb{P}^1 \), \( s \mapsto \hat{M}^{n-1}_s \hat{v} \), is a closed curve of degree \( n - 1 \). Hence, there are exactly \( n - 1 \) elements \( s^*_i \in ]-\infty, \infty[ \) such that \( B_n M^{n-1}_s v = 0 \) for \( i = 1 \ldots n - 1 \) and we can extend \( \Gamma(t, s) \) continuously to the set \([-\infty, +\infty[^2 \setminus \{ (\pm\infty, s^*_i) : i = 1 \ldots n - 1 \} \).

We claim that the closed curve \( \hat{\mathbb{R}} \ni t \mapsto \Gamma(t, t) = \hat{M}^n_t \hat{v} \) is homotopic to the concatenation of two closed curves, one of degree 1 and the other of degree \( n - 1 \). This implies that \( \hat{\mathbb{R}} \ni t \mapsto \Gamma(t, t) \) has degree \( n \), and whence \( \ell_2(M^n_t v) = n \pi \).

The first of these curves, \( \Gamma_1 : \hat{\mathbb{R}} \to \mathbb{P}^1 \), is the same as above

\[
\Gamma_1(t) := \Gamma(t, -\infty) = \hat{C}_n(t) \lim_{s \to -\infty} \hat{M}^{n-1}_s \hat{v} = \hat{C}_n(t) \hat{B}^{n-1}_n \hat{v}.
\]

It is a well defined and continuous curve with degree 1.

The second curve \( \Gamma_2 : \hat{\mathbb{R}} \to \mathbb{P}^1 \) cannot be \( \Gamma_2(s) := \Gamma(+\infty, s) \), because of the discontinuities at \( s = s^*_i \). Fix a small number \( \varepsilon > 0 \) and choose \( t^* \in ]-\infty, +\infty[ \) large enough so that the curve \( \Gamma_1|_{[t^*, +\infty[} \) has length bounded by \( \varepsilon \) and \( |s^*_i| < t^* \) for \( i = 1, \ldots, n - 1 \). Define

\[
\Gamma_3 : [-\infty, -t^*] \to \mathbb{P}^1 \text{ by } \Gamma_3(t) := \Gamma_1|_{[-\infty, -t^*]}(-t, -\infty) = \Gamma(-t, -\infty),
\]

\[
\Gamma_4 : [-\infty, +\infty] \to \mathbb{P}^1 \text{ by } \Gamma_4(s) := \Gamma(t^*, s) = \hat{C}_n(t^*) \hat{M}^{n-1}_s \hat{v}.
\]
and
\[ \Gamma_5: [t^*, +\infty] \to \mathbb{P}^1 \text{ by } \Gamma_5(t) := \Gamma(t, +\infty). \]

Finally let \( \Gamma_2 \) be the concatenation of \( \Gamma_3, \Gamma_4 \) and \( \Gamma_5 \) in this order. Let \( \tilde{\Gamma}_i \) be the liftings of these curves for \( i = 1, \ldots, 5 \). On one hand the procedure for the case of \( n = 1 \) implies that \( \ell(R(\tilde{\Gamma}_1)) = \pi \). On the other hand, by the assumption on \( t^* \) we get that \( \ell_{[-\infty, -t^*]}(\tilde{\Gamma}_3) < \varepsilon \) and \( \ell_{[t^*, +\infty]}(\tilde{\Gamma}_5) < \varepsilon \), because they both match the same arc of \( \Gamma_1 \). Finally, by the induction hypothesis, \( \ell(R(\tilde{\Gamma}_4)) = \ell(R(M_i^n v) = (n - 1)\pi) \).

Since \( \Gamma(t, t) \) is homotopic to the concatenation of the curves \( \Gamma_1, \Gamma_3, \Gamma_4 \) and \( \Gamma_5 \), we get
\[ \pi - 2\varepsilon + (n - 1)\pi \leq \ell(R(M_i^n v) \leq \pi + 2\varepsilon + (n - 1)\pi. \]

Because the closed curve \( R \ni t \mapsto M_i^n \hat{v} \in \mathbb{P}^1 \) does not depend on \( \varepsilon \) and its length is a multiple of \( \pi \), we conclude that \( \ell(R(M_i^n v) = n\pi) \).

\[ \square \]

2.4. Trace derivative. In this subsection we establish some properties about the derivatives of the trace of winding matrix curves.

**Proposition 2.21.** Let \( I \ni t \mapsto M_i \in \text{SL}_2(\mathbb{R}) \) be a smooth curve with the winding property. For any \( t \in \mathbb{R} \), if \( |tr(M_i)| < 2 \) then \( \frac{d}{dt} [tr(M_i)] \neq 0 \).

**Proof.** Exclusively for the purpose of this proof we introduce the following non-oriented angle between non-collinear vectors \( v, w \in \mathbb{R}^2 \), defined by
\[ \angle(v, w) := \arccos \left( \frac{v \cdot w}{\|v\|\|w\|} \right) \in [0, \pi], \]
i.e., in terms of some Euclidean product in \( \mathbb{R}^2 \).

**Lemma 2.22.** Given \( A \in \text{SL}_2(\mathbb{R}) \) elliptic, i.e., \( |tr(A)| < 2 \), there exists a smooth measure \( \mu_A \in \text{Prob}(\mathbb{S}^1) \) such that
\[ \arccos \left( \frac{1}{2} trA \right) = \int_{\mathbb{S}^1} \angle(A v, v) \, d\mu_A(v). \]

Moreover this integral does not depend on the Euclidean product in \( \mathbb{R}^2 \).

**Proof.** Write \( A = M R_\alpha M^{-1} \) for some \( M \in \text{SL}_2(\mathbb{R}) \) and where \( R_\alpha \) is the angle \( \alpha \) rotation with \( tr(A) = 2 \cos \alpha \). Let \( \hat{M} : \mathbb{S}^1 \to \mathbb{S}^1 \) be the projective action induced by \( M \) on \( \mathbb{S}^1 \), i.e., \( \hat{M}(v) := M v / \|M v\| \). Next define \( \mu_A = \hat{M}_*m \) where \( m \) denotes the normalized Riemannian measure on \( \mathbb{S}^1 \). Then \( \hat{A} : \mathbb{S}^1 \to \mathbb{S}^1 \) is a circle homeomorphism which preserves the measure \( \mu_A \), i.e., \( \hat{A}_* \mu_A = \mu_A \). The angle \( \angle \) is a metric on \( \mathbb{S}^1 \) where the circle \( \mathbb{S}^1 \) has diameter \( \pi \) and length \( 2\pi \) but in general the rotation angle \( \angle(A v, v) \) is not constant for this metric. Assume \( \alpha \) is irrational \( \text{ mod } 2\pi \). Then by the unique ergodicity of \( \hat{A} \),
\[ \int_{\mathbb{S}^1} \angle(A v, v) \, d\mu_A(v) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \angle(A^{i+1} v, A^i v) = \alpha. \]
measures the rotation number of $\hat{A} : \mathbb{S}^1 \to \mathbb{S}^1$. For $\alpha$ rational (mod $2\pi$) the result follows by continuity. □

If $|\text{tr}(M_{t_0})| < 2$ then $M_t$ is elliptic for all $t$ in a small interval around $t_0$. Hence, for such $t$, $\text{tr}(M_t) = 2 \cos \theta(t)$ where

$$\theta(t) = \int_{\mathbb{S}^1} \angle(M_t v, v) \, d\mu_t(v)$$

and $\mu_t \in \text{Prob}(\mathbb{S}^1)$ is the unique probability measure invariant under the rotating action of $M_t$. Because $M_t$ is elliptic, the oriented angle from $v$ to $M_t v$ is either always positive (for all $v$) or else always negative. By the positive winding assumption, the curve $M_t v / \|M_t v\|$ rotates anti-clock wisely around the origin with positive speed. Hence, by ellipticity,

$$\frac{d}{dt} \left[ \angle(M_t v, v) \right]_{t=t_0} \neq 0$$

with a constant sign independent of the unit vector $v$.

Next choose an Euclidean product in $\mathbb{R}^2$ that makes $M_{t_0}$ an orthogonal rotation and consider the associated angle $\angle(\cdot, \cdot)$. The map $v \mapsto \angle(M_{t_0} v, v)$ is constant equal to $\theta(t_0)$, and because of this

$$\theta'(t_0) = \int_{\mathbb{S}^1} \frac{d}{dt} \left[ \angle(M_t v, v) \right]_{t=t_0} \, d\mu_{t_0}(v) \neq 0.$$  

Therefore, since $0 < \theta(t_0) < \pi$, one has $\sin \theta(t_0) > 0$ and

$$\frac{d}{dt} \left[ \text{tr}(M_t) \right]_{t=t_0} = 2 \sin \theta(t_0) \theta'(t_0) \neq 0,$$

which concludes the proof of Proposition 2.21. □

**Definition 2.4.** A smooth function $f : \mathbb{R} \to \mathbb{R}$ is called log-concave, respectively strictly log-concave if

$$f(t)f''(t) - (f'(t))^2 \leq 0 \quad \text{resp.} \quad f(t)f''(t) - (f'(t))^2 < 0 \quad \forall \ t \in \mathbb{R}.$$  

**Remark 2.3.** If all zeros of the function $f$ are isolated then $f$ is log-concave if and only if $\log|f|$ is a concave function. Likewise, $f$ is strictly log-concave if and only if $\log|f|$ is a strictly concave function. Notice that

$$\frac{d^2}{dt^2} \log|f(t)| = \frac{f(t)f''(t) - (f'(t))^2}{f(t)^2}.$$  

A strictly log-concave function is always a Morse function, i.e., all its critical points are non-degenerate, and take strictly positive (negative) values at local maxima (minima).
Proposition 2.23. Consider a curve of the form

\[ M_t^n = (A_n + t B_n) \cdots (A_2 + t B_2) (A_1 + t B_1) \]

where each factor \( A_j + t B_j \) takes values in \( \text{SL}_2(\mathbb{R}) \) and is positively winding. Then

1. Given \( v, w \in S^1 \), if \( \langle B_n \cdots B_1 v, w \rangle \neq 0 \) then \( \mathbb{R} \ni t \mapsto \langle M_t^n v, w \rangle \) is a strictly log-concave polynomial of degree \( n \) with \( n \) simple real roots.
2. If \( B_n \cdots B_1 \neq 0 \) then \( \mathbb{R} \ni t \mapsto \text{tr}(M_t^n) \) is a strictly log-concave polynomial of degree \( n \) with \( n \) simple real roots.

Proof. Let \( B^n := B_n \cdots B_1 \) and choose unit vectors \( v, w \in \mathbb{R}^2 \) such that \( \langle B^n v, w \rangle \neq 0 \). Then the function \( f(t) := \langle M_t^n v, w \rangle \) is strictly log-concave. Indeed, notice that \( f(t) \) is a polynomial of degree \( n \) with leading coefficient \( \langle B^n v, w \rangle \neq 0 \).

By Proposition 2.20, \( \ell_{\mathbb{R}}(M_t^n v) = n \pi \) and hence the polynomial \( f(t) = \langle M_t^n v, w \rangle \) must have \( n \) distinct roots \( t_1 < t_2 < \ldots < t_n \) which correspond to the values of the parameter \( t \in \mathbb{R} \) where \( M_t^n v \) crosses the line \( w^\perp \).

Hence \( f(t) = \langle B^n v, w \rangle \prod_{j=1}^n (t - t_j) \) and

\[
\log|f(t)| = \log|\langle B^n v, w \rangle| + \sum_{j=1}^n \log|t - t_j|
\]

is strictly concave because it is a sum of the \( n \) strictly concave functions \( \log|t - t_j| \).

This shows that \( f(t) \) is strictly log-concave. This concludes the proof of item (1).

To prove (2), let \( \{e_1, e_2\} \) be an orthonormal basis such that \( \langle B^n e_i, e_i \rangle \neq 0 \) for \( i = 1, 2 \), and consider the (half) trace function \( f: \mathbb{R} \to \mathbb{R} \),

\[
f(t) := \frac{\text{tr}(M_t^n)}{2} = \frac{1}{2} \langle M_t^n e_1, e_1 \rangle + \frac{1}{2} \langle M_t^n e_2, e_2 \rangle = \frac{f_1(t) + f_2(t)}{2},
\]

where each \( f_i(t) := \langle M_t^n e_i, e_i \rangle \) is a strictly log-concave polynomial of degree \( n \) by item (1), for \( i = 1, 2 \).

Let \( t_1 < t_2 < \ldots < t_n \) be the roots of \( f_1(t) \) and \( s_1 < s_2 < \ldots < s_n \) be the roots of \( f_2(t) \). These roots are interlaced in the sense that

\[
\max\{t_{i-1}, s_{i-1}\} < \min\{t_i, s_i\} \quad \forall i = 2, \ldots, n.
\]

Otherwise we would have \( t_{i-1} < t_i \leq s_{i-1} < s_i \) or \( s_{i-1} \leq s_i < t_{i-1} \leq t_i \). Keep in mind that the frame \( \{M_t^n e_1, M_t^n e_2\} \) is moving anti-clockwise while both its vectors maintain a positive orientation. In the first case, as \( t \) varies from \( t = t_{i-1} \) to \( t = t_i \) the vector \( M_t^n e_1 \) crosses twice the line \( e_2^\perp \) while the second vector \( M_t^n e_2 \) is kept from crossing \( e_2^\perp \).

Hence \( \angle(M_{t_{i-1}} e_1, M_{t_i} e_1) > \pi \) while \( \angle(M_{t_{i-1}} e_2, M_{t_i} e_2) < \pi \). This is impossible because at some intermediate time the positive orientation of the frame would break. The second case is completely analogous.

Finally, since \( f(t) \) is a convex combination of \( f_1(t) \) and \( f_2(t) \), the function \( f(t) \) has at least \( n \) zeros, one between \( t_i \) and \( s_i \) for every \( i = 1, \ldots, n \). See Figure 3. Arguing as above we derive that being a polynomial of degree \( n \), \( f(t) \) must be strictly log-concave.
By Remark 2.3 and Proposition 2.21, the graph of \( f(t) := \text{tr}[A^n_t(x)] \) completely crosses \( n \) times the open horizontal strip \( S := \{(t, s) \in \mathbb{R}^2 : -2 < s < 2\} \), with local maxima and minima outside \( S \).

3. Proof of the main theorem

In this section we prove Theorem 1.1.

Assume \( T : X \to X \) is a homeomorphism on a compact metric space \( X \) that preserves an ergodic measure \( \mu \in \text{Prob}(X) \) and let \( A_t : X \to \text{GL}_2^+(\mathbb{R}) \) be a family of cocycles of the form \( A_t(x) = A(x) + tB(x) \) indexed in \( t \in \mathbb{R} \) and satisfying the assumptions 1-4.

By Assumption 4 there exists a continuous invariant decomposition \( \mathbb{R}^2 = E_0(x) \oplus E_\infty(x) \), where the sub-bundle \( E_0 \) is associated with the top (finite) Lyapunov exponent and \( E_\infty \) is associated with the second Lyapunov exponent, possibly \( -\infty \). The sub-bundles \( E_0 \) and \( E_\infty \) determine continuous functions \( \hat{e}_0 : X \to \mathbb{P}^1 \) and \( \hat{e}_\infty : X \to \mathbb{P}^1 \) respectively. Consider also the adjoint cocycle \( B^*(x) := B(T^{-1}x)^t \) over the base map \( T^{-1} : X \to X \), which shares with \( B \) the same Lyapunov exponents \( L_1(B^*) = L_1(B) > L_2(B) = L_2(B^*) \). Let \( \hat{e}_0^* : X \to \mathbb{P}^1 \) and \( \hat{e}_\infty^* : X \to \mathbb{P}^1 \) denote the corresponding continuous functions associated with its dominated splitting decomposition.

Lemma 3.1. For every \( x \in X \), if \( v \notin \hat{e}_\infty(x) \) and \( w \notin \hat{e}_\infty^*(x) \) then

\[
L_1(B) = \lim_{n \to \infty} \frac{1}{n} \log | \langle B^n(x)v, w \rangle |.
\]
Proof. We relate the directions \( \hat{e}_0(x), \hat{e}_\infty(x), \hat{e}_0^*(x) \) and \( \hat{e}_\infty^*(x) \) with the singular vectors of the matrices \( B^n(x) \) and \( (B^*)^n(x) \). See the definitions in Subsection 6.1. To simplify notations we set

\[
\overline{v}_n(x) := \overline{v}(B^n(x)) \quad \text{and} \quad \overline{v}_n(x) := v(B^n(x))
\]
as well as

\[
\overline{v}_n(x) = \overline{v}^*(B^n(T^{-n}x)) = \overline{v}((B^*)^n(x))
\]
\[
\overline{v}_n^*(x) = \overline{v}^*(B^n(T^{-n}x)) = \overline{v}((B^*)^n(x))
\]

Since \( L_1(B) > L_2(B) \), all these four sequences converge \( \mu \)-almost surely respectively to \( \hat{e}_0^*(x), \hat{e}_\infty(x), \hat{e}_0(x) \) and \( \hat{e}_\infty^*(x) \). Moreover

\[
\hat{e}_0^*(x) = \hat{e}_\infty(x)^\perp \quad \text{and} \quad \hat{e}_\infty^*(x) = \hat{e}_0(x)^\perp.
\]
See Chapter 4 of [33]. Given unit vectors \( v, w \in S^1 \),

\[
v = \langle v, \overline{v}_n(x) \rangle \overline{v}_n(x) + \langle v, \overline{v}_n(x) \rangle \overline{v}_n(x),
\]
and whence

\[
B^n(x) v = \|B^n(x)\| \langle v, \overline{v}_n(x) \rangle \overline{v}_n(x) + m(B^n(x)) \langle v, \overline{v}_n(x) \rangle \overline{v}_n^*(x)
\]
which implies that \( \langle B^n(x) v, w \rangle \) is equal to

\[
\langle v, \overline{v}_n(x) \rangle \langle w, \overline{v}_n^*(x) \rangle \|B^n(x)\| + \langle v, \overline{v}_n(x) \rangle \langle w, \overline{v}_n(x) \rangle m(B^n(x))
\]
Note that as \( n \) large, the first term dominates.

Since \( v \neq \hat{e}_\infty(x) = \hat{e}_0(x)^\perp \),

\[
\lim_{n \to +\infty} \frac{1}{n} \log \|v, \overline{v}_n(x)\| = \lim_{n \to +\infty} \frac{1}{n} \log \|v, \hat{e}_0(x)^\perp\| = 0.
\]

Analogously, since \( w \neq \hat{e}_\infty^*(x) = \hat{e}_0(x)^\perp \),

\[
\lim_{n \to +\infty} \frac{1}{n} \log \|w, \overline{v}_n^*(x)\| = \lim_{n \to +\infty} \frac{1}{n} \log \|w, \hat{e}_0(x)^\perp\| = 0.
\]

Taking absolute values, logarithms, dividing by \( n \), and the limit as \( n \to +\infty \), we have \( \mu \)-almost surely

\[
\lim_{n \to +\infty} \frac{1}{n} \log \|B^n(x) v, w\| = \lim_{n \to +\infty} \frac{1}{n} \log \|B^n(x)\| = L_1(B),
\]
which concludes the proof. \( \square \)

Lemma 3.2. For every \( x \in X \) there exists a countable set of (bad) directions \( \mathcal{B}_x \subset S^1 \) such that for any \( v \in S^1 \setminus \mathcal{B}_x, w \in S^1 \) and \( n \in \mathbb{N} \) the function \( f(t) := \langle A^n_t(x) v, w \rangle \) is a polynomial of degree \( n \).
Proof. Consider $E : X \to \text{Mat}_2(X)$ defined by $E(x) := A(x)^{-1} B(x)$. The winding assumption implies that $E(x)$ has either complex, non real eigenvalues or else it has a single real eigenvalue with multiplicity two, for all $x \in X$. See Propositions 6.6 and 6.7 in Subsection 6.2 of the Appendix.

Given $x \in X$ consider the countable set,

$$
\mathcal{B}_x := \{ v \in S^1 : \text{for some } j \geq 0, A^j(x) v \text{ is an eigenvector of } E(T^j x) \}.
$$

If $v \in \mathcal{B}_x^c$ then the smooth curve $A_t(T^j x) = A(T^j x) + tB(T^j x)$ is positively winding (for all $j \geq 0$) and whence by Proposition 2.3, see also Proposition 2.20, the curve $A^j_t(x)$ is positively winding and $f(t) := \langle A^j_t(x) v, w \rangle$ is a polynomial of degree $n$. □

Consider the Lebesgue-Stieltjes measure $d\rho$ associated with the continuous non-decreasing function (see Proposition 2.11) $\rho(t) := \rho(A_t, A_{t_0})$.

Lemma 3.3. If the cocycle $A_t$ has dominated splitting over some open interval $I \subseteq \mathbb{R}$ then the function $\rho(t)$ is constant on $I$.

Proof. Let $I \subseteq \mathbb{R}$ be an open interval. Assume that $A_t$ has dominated splitting for all $t \in I$ and consider the unstable or dominating direction $\hat{e}_u(x, t) \in \mathbb{P}^1$ of $A_t$. By splitting domination, the map $\hat{e}_u : X \times I \to \mathbb{P}^1$ is continuous. Given $t > s$ in $I$, $x \in \Omega_t \cap \Omega_s$ (see Proposition 2.9 for the definition of the sets $\Omega_t$ and $\Omega_s$) and some appropriate $\hat{v} \in \mathbb{P}^1$,

$$
\rho(A_t) - \rho(A_s) = \lim_{n \to \infty} \frac{1}{\pi n} \angle(A^n_t(x) v, v) - \frac{1}{\pi n} \angle(A^n_s(x) v, v)
$$

$$
= \lim_{n \to \infty} \frac{1}{\pi n} \angle(A^n_t(x) v, A^n_s(x) v)
$$

$$
= \lim_{n \to \infty} \frac{1}{\pi n} \angle(\hat{e}_u(T^n x, t), \hat{e}_u(T^n x, s)) = 0,
$$

because the angle $\angle(\hat{e}_u(x, t), \hat{e}_u(x, s))$ is uniformly bounded, which proves that $\rho$ is constant over $I$. □

Lemma 3.4. The cocycle $A_t$ has dominated splitting outside some compact interval $[-a, a]$. In particular, the support of the measure $d\rho$ is compact.

Proof. Any continuous family of cones $\{C_x\}_{x \in X}$ adapted to the cocycle $B$, which has dominated splitting, is shared by the cocycles $tB$ and $A_t = A + tB$ for all $t$ with large enough absolute value, i.e., $|t| > a$. It follows that $A_t$ has dominated splitting for all $|t| > a$ and whence by Lemma 3.3, $\text{supp}(d\rho) \subseteq [-a, a]$. □

Proof of Theorem 1.1. By Lemma 3.4 the non-decreasing function $\rho(t) = \rho(A_t)$ has compact support contained in some interval $[-a, a]$. 

Fixing $t \in \mathbb{C}$, for $\mu$-almost every $x \in \Omega$ (depending on $t$)
\[
L_1(A_t) = \lim_{n \to \infty} \frac{1}{n} \log \|A_t^n(x)\|
= \lim_{n \to \infty} \frac{1}{n} \log \max_{i,j=1,2} |\langle A_t^n(x) e_i, e_j \rangle| 
= \max_{i,j=1,2} \left[ \lim_{n \to \infty} \frac{1}{n} \log |\langle A_t^n(x) e_i, e_j \rangle| \right],
\]
where $\{e_1, e_2\}$ is any basis of $\mathbb{R}^2$.

On the other hand we are going to prove that for $\mu$-almost $x \in X$, taking an appropriate basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ (depending on $x$), for all $i,j = 1,2$ and $t \in \mathbb{C} \setminus \mathbb{R}$,
\[
\lim_{n \to \infty} \frac{1}{n} \log |\langle B_t^n(x) e_i, e_j \rangle| = L_1(B) + \int \log |t - s| \, d\rho(s).
\]

Hence the identity (1.4) holds for all $t \in \mathbb{C} \setminus \mathbb{R}$.

Indeed this enough by the following argument. The function $t \mapsto L_1(A_t)$ is upper semi-continuous and since
\[
u_n: \mathbb{C} \to \mathbb{R}, \quad \nu_n(t) := \frac{1}{n} \int X \log \|A_t^n(x)\| \, d\mu(x)
\]
is a family of subharmonic functions, uniformly bounded from above. Note that $\{\nu_{2j}\}_{j \geq 1}$ is a convergent and decreasing subsequence of subharmonic functions, so the limit function $L_1(A_t) = \lim_{n \to +\infty} \nu_n(t)$ is a subharmonic function. The right-hand-side in (1.4) is also a subharmonic function since it is an average of the subharmonic functions $v_s(t) := \log |t - s|$. Finally, because these two subharmonic agree Lebesgue almost everywhere, they must coincide everywhere, see [13, Theorem 1.1]. This proves that (1.4) holds for all $t \in \mathbb{C}$.

To finish the proof of Theorem 1.1 we establish the previous claim.

Consider the full measure set $\Omega = \cap_{\beta \in \mathbb{Q}} \Omega_\beta \subset X$, where each $\Omega_\beta$ is the full measure set in Proposition 2.9 associated with $\beta \in \mathbb{Q}$. Taking $x \in \Omega$, and an orthonormal basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ consisting of vectors which do not match the dominated directions $\hat{e}_\infty(x)$ and $\hat{e}_\infty^*(x)$ of the cocycle $B$ and its adjoint $B^*$ at $x$, by Lemma 3.1 we have
\[
L_1(B) = \lim_{n \to +\infty} \frac{1}{n} \log |\langle B_t^n(x) e_i, e_j \rangle| \quad \forall i,j = 1,2.
\]
We can also take this basis $\{e_1, e_2\}$ outside the countable set $\mathcal{B}_x$ of Lemma 3.2 so that the functions $f_{i,j}(t) := |\langle B_t^n(x) e_i, e_j \rangle|$ are polynomials of degree $n$ by item (1) of Proposition 2.23. By the same item, each of these functions has exactly $n$ roots, denoted by $t_1(i,j) < t_2(i,j) < \cdots < t_n(i,j)$. Hence $\langle B_t^n(x) e_i, e_j \rangle$ is the leading coefficient of $f_{i,j}(t)$ and we have
\[
\lim_{n \to \infty} \frac{1}{n} \log |f_{i,j}(t)| = \lim_{n \to \infty} \frac{1}{n} \log |\langle B^n(x) e_i, e_j \rangle \prod_{k=1}^{n} (t - t_k(i,j))| \\
= \lim_{n \to \infty} \left[ \frac{1}{n} \log |\langle B^n(x) e_i, e_j \rangle| + \frac{1}{n} \sum_{k=1}^{n} \log |t - t_k(i,j)| \right] \\
= L_1(B) + \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log |t - t_k(i,j)|.
\]

Therefore, it is enough proving now that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log |t - t_k| = \int_{\mathbb{R}} \log |t - s| \, d\rho(s) \quad \forall t \in \mathbb{C} \setminus \mathbb{R}
\]  

(3.1)

for which we need the following.

**Proposition 3.5.** For any \( x \in \Omega \), \( \{e_1, e_2\} \) as above, \( i, j = 1, 2 \) and any continuous function \( \varphi : \mathbb{R} \to \mathbb{R} \), if \( t_k = t_k(i,j) \), \( k = 1, \ldots, n \) are the \( n \) roots of the polynomial equation \( \langle A^n_t(x) e_i, e_j \rangle = 0 \) then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(t_k) = \int_{\mathbb{R}} \varphi(s) \, d\rho(s).
\]

**Proof.** Fix \( \varepsilon > 0 \) and let us prove that for all large enough \( n \),

\[
\left| \frac{1}{n} \sum_{k=1}^{n} \varphi(t_k) - \int_{\mathbb{R}} \varphi(s) \, d\rho(s) \right| < 2\varepsilon.
\]

The measure \( d\rho \) is supported on a compact interval \([-a, a]\). Because \( \varphi \) is uniformly continuous on \([-a, a]\) there exists \( \delta > 0 \) such that for any decomposition \(-a = \beta_0 < \beta_1 < \cdots < \beta_{m-1} < \beta_m = a \) of the interval \([-a, a]\) with diameter \( \max_{1 \leq l \leq m} |\beta_l - \beta_{l-1}| < \delta \), any Riemannian sum

\[
\sum_{l=1}^{m} \varphi(s_l) (\rho(\beta_l) - \rho(\beta_{l-1}))
\]

with \( s_l \in [\beta_{l-1}, \beta_l] \) for all \( l = 1, \ldots, m \), satisfies

\[
\left| \sum_{l=1}^{m} \varphi(s_l) (\rho(\beta_l) - \rho(\beta_{l-1})) - \int_{\mathbb{R}} \varphi(s) \, d\rho \right| < \varepsilon.
\]

Fix \( \delta > 0 \) and the decomposition \(-a = \beta_0 < \beta_1 < \cdots < \beta_{n-1} < \beta_n = a \) with \( \beta_l \in \mathbb{Q} \), for \( l = 1, \ldots, m \), and diameter less than \( \delta \) as above. Let \( L := \)
max\{∥\varphi(x)\| : x ∈ [-a, a]\} and take 0 < \eta < \varepsilon/(2mL). Since \(x ∈ Ω ⊂ \bigcap_{i=1}^{n} Ω_{β_i}\), by Lemma 2.14 there exists \(n_0 ∈ \mathbb{N}\) such that \(2n_0^{-1} < \varepsilon/(2mL)\) and for all \(n ≥ n_0\),

\[
\left| dρ([β_{l-1}, β_l]) - \frac{1}{\pi n} \angle(A^{n}_{β_{l}}(x), A^{n}_{β_{l-1}}(x) v) \right| < \eta \quad ∀ l = 1, \ldots, m.
\]

Since from \(t_{l-1}\) to \(t_l\) the curve \(t ↦ A^{n}_{t_l}(x)v\) gives one turn around the projective space, i.e. \(\angle(A^{n}_{t_l}(x)v, A^{n}_{t_{l-1}}(x)v) = \pi\), setting

\[N_l := \#\{k ∈ \{1, \ldots, n\} : t_k ∈ [β_{l-1}, β_l]\}, \quad l = 1, \ldots, m,
\]

we have that \(|π(N_l - 1) - \angle(A^{n}_{β_{l}}(x)v, A^{n}_{β_{l-1}}(x)v)| < 2\pi\), which implies that

\[
\left| dρ([β_{l-1}, β_l]) - \frac{N_l - 1}{n} \right| < \eta + \frac{2}{n_0} \quad ∀ l = 1, \ldots, m.
\]

Hence

\[
\frac{1}{n} \sum_{k=1}^{n} \varphi(t_k) = \sum_{l=1}^{m} \frac{N_l - 1}{n} \left( \frac{1}{N_l - 1} \sum_{t_k ∈ [β_{l-1}, β_l]} \varphi(t_k) \right)
\]

where by continuity of \(\varphi\) we can find, for each \(l = 1, \ldots, m\), \(t^*_l ∈ [β_{l-1}, β_l]\) such that

\[
\varphi(t^*_l) = \frac{1}{N_l - 1} \sum_{t_k ∈ [β_{l-1}, β_l]} \varphi(t_k)
\]

and

\[
\frac{1}{n} \sum_{k=1}^{n} \varphi(t_k) = \sum_{l=1}^{m} \frac{N_l - 1}{n} \varphi(t^*_l).
\]

Therefore

\[
\left| \frac{1}{n} \sum_{k=1}^{n} \varphi(t_k) - \sum_{l=1}^{m} (ρ(β_l) - ρ(β_{l-1})) \varphi(t^*_l) \right|
\]

is bounded by

\[
\sum_{l=1}^{m} \left| \frac{N_l - 1}{n} - dρ([β_{l-1}, β_l]) \right| \left| \varphi(t^*_l) \right| \leq \left( \eta + \frac{2}{n_0} \right) mL < \varepsilon
\]

and then

\[
\left| \frac{1}{n} \sum_{k=1}^{n} \varphi(t_k) - \int \varphi(s) \, dρ(s) \right| \leq \left| \frac{1}{n} \sum_{k=1}^{n} \varphi(t_k) - \sum_{l=1}^{m} (ρ(β_l) - ρ(β_{l-1})) \varphi(t^*_l) \right|
\]

\[
+ \left| \sum_{l=1}^{m} (ρ(β_l) - ρ(β_{l-1})) \varphi(t^*_l) - \int \varphi(s) \, dρ(s) \right|
\]

\[
< \varepsilon + \varepsilon = 2\varepsilon.
\]

which establishes the wanted convergence. □
Given \( \delta > 0 \) and \( t \in \mathbb{R} \), consider the family \( \varphi_{t,\delta} : \mathbb{R} \to \mathbb{R} \) of continuous functions defined by

\[
\varphi_{t,\delta}(s) := \log|t + i\delta - s|.
\]

By Proposition 3.5 we have for all \( t \in \mathbb{R} \) and \( \delta > 0 \),

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \log|t + i\delta - t_k| = \int_{\mathbb{R}} \log|t + i\delta - s|d\rho(s).
\]

Therefore (3.1) holds for all \( t \in \mathbb{C} \setminus \mathbb{R} \).

4. AFFINE FAMILIES OF \( \text{GL}^+_2 \)-COCYCLES

In this section we provide some easily verifiable sufficient conditions for the Assumptions 1, 2 and 4. We assume that \( A : X \to \text{GL}^+_2(\mathbb{R}) \) and \( B : X \to \text{Mat}_2(\mathbb{R}) \) are continuous functions such that

\[
A_t := A + tB = A(I + tE) \quad \forall t \in \mathbb{R}
\]

where \( E : X \to \text{Mat}_2(\mathbb{R}) \) denotes the function \( E := A^{-1}B \).

4.1. Conditions for invertibility. To ensure the invertibility of \( A_t \) we have the following criterion.

**Proposition 4.1.** If there exists a constant \( r > 0 \) such that \( E^2(x) = 0 \) or \( \Delta_{E(x)} := 4 \det E(x) - (\text{tr} E(x))^2 \geq r > 0, \quad \forall x \in X \)

then the cocycle \( A_t \) satisfies Assumption 1, i.e., there exist positive constants \( c \) and \( R \) such that \( |\det A_t(x)| \geq c > 0 \) for all \( (x, t) \in X \times S_R \).

**Proof.** The following argument relies on the conclusions of Lemma 6.4. If \( E^2 = 0 \) then \( \text{tr} E = \text{tr} E^2 = \det E = 0 \) and \( \det(I + tE) \equiv 1 \), so that there is nothing to prove. Otherwise \( \Delta_E \geq r > 0 \) which implies that \( \det(E) \geq r/4 > 0 \).

Taking \( \ell := \max\{\det E(x) : x \in X\} \) and \( \delta > 0 \) small enough so that \( 4 \ell^2 \delta^2 < r \) and using Lemma 6.4 we have for \( |\text{Im} t| < \delta \)

\[
|\det(I + tE)| \geq |\det E| \left[ -\delta^2 + \frac{\Delta_E}{4(\det E)^2} \right] \geq \frac{r}{4} \left( \frac{r}{4\ell^2} - \delta^2 \right) > 0.
\]

This concludes the proof with \( R = \delta \) and \( c = \frac{r}{4} \left( \frac{r}{4\ell^2} - \delta^2 \right) \).

For Assumption 1 to hold with \( R = \infty \), the polynomials \( \det(A_t(x)) \) must be constant. A special case of interest is the following:

**Proposition 4.2.** For all \( x \in X \), the following are equivalent

1. \( A_t(x) \in \text{SL}_2(\mathbb{C}) \) for all \( t \in \mathbb{C} \);
2. \( A(x) \in \text{SL}_2(\mathbb{C}) \) and \( E(x)^2 = 0 \).

**Proof.** Using Lemma 6.4, \( A(I + tE) \in \text{SL}_2(\mathbb{C}) \) for all \( t \in \mathbb{C} \) if and only if \( \text{tr} E = \det E = 0 \) which occurs if and only if \( E^2 = 0 \).
4.2. Conditions for winding. Next proposition describe how to obtain families of invertible cocycles with the winding property. Define the seminorm \( \Xi : \text{Mat}_2(\mathbb{R}) \to [0, +\infty) \),
\[
\Xi(E) := \max \{ |e_{11} - e_{22}|, 2|e_{12}|, 2|e_{21}| \},
\]
for \( E = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \) and consider the following sets where \( \Delta : \text{Mat}_2(\mathbb{R}) \to \mathbb{R} \) is the function in Lemma 6.4.
\[
\Gamma_+ := \{ E \in \text{Mat}_2(\mathbb{R}) : \Xi(E) > 0, \Delta_E \geq 0 \text{ and } e_{21} \leq 0 \leq e_{12} \}
\]
\[
\Gamma_- := \{ E \in \text{Mat}_2(\mathbb{R}) : \Xi(E) > 0, \Delta_E \geq 0 \text{ and } e_{12} \leq 0 \leq e_{21} \}.
\]

**Proposition 4.3.** For any continuous function \( E : X \to \Gamma_\pm \), the family of cocycles \( A_t := A + tAE \) takes values in \( \text{GL}_2^+(\mathbb{R}) \) and satisfies Assumptions 1-3.

**Proof.** Follows from the propositions 6.6, 6.7 and 6.8. \( \square \)

We now give a similar criterion to obtain \( \text{SL}_2(\mathbb{R}) \) cocycles with the winding property. Consider the sets
\[
N_+ := \{ E \in \text{Mat}_2(\mathbb{R}) : \Xi(E) > 0, E^2 = 0 \text{ and } e_{12} \leq 0 \leq e_{21} \},
\]
\[
N_- := \{ E \in \text{Mat}_2(\mathbb{R}) : \Xi(E) > 0, E^2 = 0 \text{ and } e_{21} \leq 0 \leq e_{12} \},
\]
whose union can be characterized as the following conic surface.

**Proposition 4.4.** For any matrix \( E \in N_- \cup N_+ \) there exist unique real numbers \( r \neq 0 \) and \( \theta \in [0, 2\pi[ \) such that
\[
E = \begin{bmatrix} -r \cos \theta \sin \theta & r \cos^2 \theta \\ -r \sin^2 \theta & r \cos \theta \sin \theta \end{bmatrix}.
\]

**Proposition 4.5.** For any continuous function \( E : X \to N_\pm \), the family of cocycles \( A_t := A + tAE \) takes values in \( \text{SL}_2(\mathbb{R}) \) and satisfies Assumptions 1-3.

**Proof.** Notice that by Proposition 4.2, we have \( E \in N_\pm \) if and only if \( E \in \Gamma_\pm \) and \( I + tE \in \text{SL}_2(\mathbb{R}) \). The conclusion follows then from Proposition 4.3. \( \square \)

4.3. Conditions for dominated splitting. We give some sufficient conditions for Assumption 4 and the weaker alternative hypothesis \( L_1(B) > -\infty \).

**Proposition 4.6.** Let \( X \) be a compact metric space and let \( T : X \to X \) be a homeomorphism preserving a probability measure \( \mu \in \text{Prob}(X) \). Given \( B : X \to \text{Mat}_2(\mathbb{R}) \) continuous such that \( \text{rank } B(x) = 1 \) and \( B(Tx)B(x) \neq 0 \), for every \( x \in X \), then \( B \) has dominated splitting and \( L_1(B) > -\infty = L_2(B) \).
Proof. Any rank 1 matrix $M \in \text{Mat}_2(\mathbb{R})$ can be written as $M = v w^t$ where $v, w \in \mathbb{R}^2$ are column vectors, which means that the action of $M$ on $\mathbb{R}^2$ is described by $Mu = v \langle w, u \rangle$. Given a continuous function $B : X \to \text{Mat}_2(\mathbb{R})$ with rank 1 values, there exist continuous maps $v, w : X \to \mathbb{R}^2$ such that $B(x) = v_x w^t_x$, for all $x \in X$. We can assume that $\|v_x\| = 1$ for all $x \in X$, so that $\|B(x)\| = \|w_x\|$. With this notation

$$B^n(x) u = B(T^{n-1}x) \cdots B(Tx) B(x) u$$

In particular, since $B(Tx) B(x) = \langle w_x, \cdot \rangle \langle w_{Tx}, v_x \rangle v_{Tx} \neq 0$, we get that $|\langle w_{Tx}, v_x \rangle| > 0$ for all $x \in X$. By compactness of $X$, this function admits a positive lower bound $c > 0$. Therefore, by Birkhoff’s Theorem, for $\mu$-almost every point $x \in X$

$$L_1(B) = \lim_{n \to \infty} \frac{1}{n} \log \|B^n(x)\|$$

$$= \frac{1}{n - 1} \sum_{j=1}^{n-1} \log |\langle w_{T^jx}, v_{T^{j-1}x} \rangle|$$

$$= \int_X \log |\langle w_{Tx}, v_x \rangle| \, d\mu(x) \geq \log c > -\infty.$$

Using (4.1), we can easily determine the Oseledets decomposition $\mathbb{R}^2 = E_0(x) \oplus E_\infty(x)$ of the cocycle $B$. The subspace $E_0(x)$ is the linear span of $v_{T^{-1}x}$, associated with the first Lyapunov exponent, while $E_\infty(x) = w^t_x$ is associated with $L_2(B) = -\infty$. Finally, because $\|B(x)|_{E_0(x)}\| = |\langle w_x, v_{T^{-1}x} \rangle| \geq c > 0$ and $\|B(x)|_{E_\infty(x)}\| = 0$ the cocycle $B$ has dominated splitting.

The following is a sufficient condition for $L_1(B) > -\infty$.

**Proposition 4.7.** Let $X$ be a compact metric space and let $T : X \to X$ be a homeomorphism preserving a probability measure $\mu \in \text{Prob}(X)$. Given $B : X \to \text{Mat}_2(\mathbb{R})$ continuous such that

1. $\text{rank } B(x) = 1$ for $\mu$-almost all $x \in X$,
2. $\int \log \|B(x)\| \, d\mu(x) < \infty$,
3. $\int \log \|B(Tx) B(x)\| \, d\mu(x) > -\infty$

then $L_1(B) > -\infty = L_2(B)$.

**Proof.** Similar to the proof of Proposition 4.6. \qed

**Definition 4.1.** We say that a smooth family of cocycles $\{A_t : X \to \text{GL}_2^+(\mathbb{R})\}_{t \in I}$ is strictly positively winding over an interval $J \subset I$, if there exist $c > 0$, $n_0 \in \mathbb{N}$
such that for every $n \geq n_0$, $\hat{v} \in \mathbb{P}^1$, $t \in J$, $x \in X$,
\[
\frac{(A^n_t(x)v) \wedge (\frac{d}{dt} A^n_t(x)v)}{\|A^n_t(x)v\|^2} \geq c.
\]
Analogously we define strictly negatively winding. Strictly winding means either strictly positively or else strictly negatively winding.

**Remark 4.1.** Schrödinger families are always strictly positively winding with $n_0 = 2$ over any compact interval of energies.

**Proposition 4.8.** For any continuous function $E : X \to \mathbb{N}_\pm$ such that
\[
E(Tx) A(x) E(x) \neq 0 \quad \forall x \in X,
\]
the family of cocycles $A_t := A + t AE$ takes values in $\text{SL}_2(\mathbb{R})$ and satisfies Assumptions 1-4. Moreover, for every compact interval $I \subset \mathbb{R}$ the family $\{A_t\}_{t \in I}$ is strictly winding with $n_0 = 2$.

**Proof.** The first statement follows from Proposition 4.6. To get the lower bound on the winding speed we use Proposition 2.4. For the sake of concreteness we assume that $E$ takes values in $N_+$ which leads to a positive winding family $A_t$. Let $\hat{e}(x) \in \mathbb{P}^1$ be the direction of $\text{Ker}(E(x))$, which is also the range of $E(x)$. By hypothesis $\hat{e}(Tx) \neq A(x) \hat{e}(x)$, for all $x \in X$. Hence, by continuity, and compactness of $X$, there exists $r > 0$ such that $A(x) B(\hat{e}(x), r) \cap B(\hat{e}(Tx), r) = \emptyset$, for all $x \in X$, where $B(\hat{v}, r)$ denotes the ball of radius $r$ centered at $\hat{v}$ in $\mathbb{P}^1$. By the positive winding property there exists $\beta > 0$ such that
\[
(A_t(x)v) \wedge (\hat{A}_t(x)v) = v \wedge E(x)v \geq \beta
\]
for all $(t, x) \in \mathbb{R} \times X$ and every unit vector $v$ with $\hat{v} \notin B(\hat{e}(x), r)$. Let
\[
C = \max_{(t,x) \in I \times X} \|A_t(x)\|.
\]
Fixing $x \in X$ and a unit vector $v \in \mathbb{R}^2$, we write $A_{j,t} := A_t(T^{j-1}x)$ and $v_j = v_j(t) := A_j^j(x)v/\|A_j^j(x)v\|$, for all $j \in \mathbb{N}$. Then since every summand in the conclusion of formula provided in Proposition 2.4 is non-negative, the last two terms are enough to get the desired positive lower bound. In fact since either $v_{n-1}(t) \notin B(\hat{e}(T^{n-1}x), r)$ or else $v_n(t) \notin B(\hat{e}(T^nx), r)$ we have
\[
\frac{(A^n_t(x)v) \wedge (\frac{d}{dt} A^n_t(x)v)}{\|A^n_t(x)v\|^2} = \frac{(A_{n,t}v_{n-1}) \wedge (\hat{A}_{n,t}v_{n-1})}{\|A_{n,t}v_{n-1}\|^2} + \frac{1}{\|A_{n,t}v_{n-1}\|^2} \frac{(A_{n-1,t}v_{n-2}) \wedge (\hat{A}_{n-1,t}v_{n-2})}{\|A_{n-1,t}v_{n-2}\|^2} + \cdots
\]
\[
\geq \max \{C^{-2} \beta, C^{-4} \beta\} = C^{-4} \beta
\]
which concludes the proof with $c := C^{-4}\beta$. \hfill \Box

From now on until the end of this section we focus on the random case. Let $X := \{1, \ldots, \kappa\}^\mathbb{Z}$ be the space of sequences in $\kappa$ symbols, and let $T : X \to X$ be the Bernoulli shift in $X$ equipped with some Bernoulli probability measure $\mu = (p_1, \ldots, p_\kappa)^\mathbb{Z}$, where $p_1 + \cdots + p_\kappa = 1$ and $p_j > 0$ for $j = 1, \ldots, \kappa$.

A random or locally constant cocycle $A : X \to \text{SL}_2(\mathbb{R})$ is determined by a vector of $\kappa$ matrices $A = (A_1, \ldots, A_\kappa) \in \text{SL}_2(\mathbb{R})^\kappa$, via the formula $A(\omega) := A_{\omega_0}$, where $\omega = (\omega_j)_{j \in \mathbb{Z}}$. This information is usually gathered in the form of a finitely supported measure

$$\mu(A) := \sum_{j=1}^\kappa p_j \delta_{A_j} \in \text{Prob}(\text{SL}_2(\mathbb{R})).$$

Assume also that $A \in \text{SL}_2^*(\mathbb{R})^\kappa$, where $\text{SL}_2^*(\mathbb{R}) := \text{SL}_2(\mathbb{R}) \setminus \{ \pm I \}$.

**Definition 4.2.** $N_\pm(A)$ is the space of $E = (E_1, \ldots, E_\kappa) \in \text{Mat}_2(\mathbb{R})^\kappa$ such that respectively $E_j \in N_+$ for all $j$ and $E_j \in N_-$ for all $j$, and $E_i A_j E_j \neq 0$ for all $i, j$.

**Proposition 4.9.** Given $A \in \text{SL}_2^*(\mathbb{R})^\kappa$, the set $N_\pm(A)$ is open, dense and full measure in $N_\pm(A)^\kappa$.

**Proof.** By Proposition 4.4, $N_- \cup N_+$ is a conic surface: its elements are determined by the sign $\pm$, the direction of the kernel and their norm. Let $\hat{v}_1, \ldots, \hat{v}_\kappa$ represent the kernels of the matrices $E_j$ in $E$. Then the condition $E_i A_j E_j \neq 0$ translates to $\hat{v}_i \neq \hat{A}_j \hat{v}_j$. Since $A_j \neq \pm I$, each equation $\hat{v}_i = \hat{A}_j \hat{v}_j$ determines a codimension 1 submanifold of the $2\kappa$-dimensional manifold $(N_\pm)^\kappa$. Therefore $N_\pm(A)$ is the complement of a finite union of regular hypersurfaces, which shows that it is open, dense and full measure in $(N_\pm)^\kappa$. \hfill \Box

Given $A = (A_1, \ldots, A_\kappa) \in \text{SL}_2(\mathbb{R})^\kappa$ and $E = (E_1, \ldots, E_\kappa) \in N_\pm(A)$ we may consider the parameterized line of random cocycles

$$A_t = A(I + tE) := (A_1(I + tE_1), \ldots, A_\kappa(I + tE_\kappa)).$$

**Proposition 4.10.** If $A \in \text{SL}_2^*(\mathbb{R})^\kappa$ and $E \in N_\pm(A)$, then the family of random cocycles $A_t := A(I + tE)$ takes values in $\text{SL}_2(\mathbb{R})$ and satisfies Assumptions 1-4. Moreover $A_t$ is strictly winding with $n_0 = 2$.

**Proof.** Since $E_i A_j E_j \neq 0$ for all $i, j$, the conclusion follows by Proposition 4.8. \hfill \Box

5. Applications

In this section we provide a list of applications of the main theorem.
5.1. **Regularity of the rotation number.** Just like the IDS, the fibered rotation number has a minimal modulus of continuity in the same spirit of Craig and Simon [13].

**Proposition 5.1.** The function \( \rho \) is log-Hölder continuous. More precisely, for any \( t \in \mathbb{R} \) and any \( s \in \mathbb{R} \) with \( |t - s| < \epsilon < 1 \), there exist a constant \( C = C(t, \epsilon) \) which tends to zero as \( \epsilon \to 0 \), such that

\[
|\rho(t) - \rho(s)| \leq \frac{C}{\log \frac{1}{|t-s|}}.
\]

**Proof.** Since the support of \( d\rho \) is bounded, we have

\[
0 \leq \int_{|t-s| \geq 1} \log |t-s| d\rho(s) < +\infty,
\]

Since the cocycle \( A_t \) is uniformly bounded for \( t \) in the support of \( d\rho \) and \( L_1(B) > -\infty \), it follows by (1.4) that

\[
\int_{|t-s| < 1} \log \frac{1}{|t-s|} d\rho(s) = L_1(B) - L_1(A_t) + \int_{|t-s| \geq 1} \log |t-s| d\rho(s) < +\infty.
\]

Thus we have

\[
C(t, \epsilon) := \max \left\{ \int_t^{t+\epsilon} \log \frac{1}{|t-s|} d\rho(s), \int_{t-\epsilon}^t \log \frac{1}{|t-s|} d\rho(s) \right\} < +\infty.
\]

Finally, observe that \( C(t, \epsilon) \geq |\rho(t) - \rho(s)| \log \frac{1}{|t-s|} \) when \( |t-s| < \epsilon \).

**Proposition 5.2.** Let \( A_t : X \to \text{SL}_2(\mathbb{R}) \), \( A_t(x) = A(x)(I + tE(x)) \), be a family of cocycles under the assumptions of Theorem 1.1. For any open interval \( I \subset \mathbb{R} \), the following are equivalent:

1. \( t \mapsto L_1(A_t) \) is analytic on \( I \);
2. \( t \mapsto \rho(A_t) \) is constant on \( I \).

**Proof.** Follows from Thouless formula.

By Goldstein and Schlag [23], under the assumptions of Theorem 1.1 a **good enough** regularity of the Lyapunov exponent \( L_1(A_t) \) transfers over to the same regularity for the rotation number \( \rho(A_t) \). As mentioned in Section 1, this good enough regularity of the LE is indeed available for a wide class of linear cocycles, which then implies similar continuity properties for the corresponding fibered rotation number and establishes Proposition 1.1.
5.2. A Johnson-type theorem. A classical result of R. Johnson [28] states that given a family of Schrödinger cocycles $A_E$, over some ergodic transformation, $A_E$ is uniformly hyperbolic if and only if $E$ lies inside a gap of the spectrum of the corresponding Schrödinger operator, i.e., the IDS is locally constant around $E$. In [2] Avila, Bochi and Damanik considered a continuous cocycle $A \in C^0(X, \text{SL}_2(\mathbb{R}))$ homotopic to a constant and the winding family of cocycles $R_\theta A$, where $R_\theta$ denotes the rotation by angle $\theta$. In the spirit of Johnson’s theorem they have proved that $R_\theta A$ is uniformly hyperbolic if and only if the fibered rotation number $\rho(R_\theta A)$ is locally constant around $\theta$. See [2, Proposition C.1]. More recently, in [24, Theorem A.9] Gorodetski and Kleptsyn have generalized this result to a context that basically matches our own. The following is a corollary of their theorem A.9.

**Theorem 5.1** (Gorodetski, Kleptsyn). Let $A_t : X \to \text{SL}_2(\mathbb{R})$, $A_t(x) = A(x)(I + tE(x))$, be a family of cocycles under the assumptions of Theorem 1.1 and such that

$$A(x) \text{Ker}(E(x)) \neq \text{Ker}(E(Tx)) \quad \forall x \in X. \quad (5.1)$$

For any open interval $I \subset \mathbb{R}$, the following are equivalent:

1. the cocycle $A_t$ is uniformly hyperbolic for $t \in I$;
2. $t \mapsto \rho(A_t)$ is constant on $I$.

**Proof.** The implication (1)$\Rightarrow$(2) follows by Lemma 3.3.

For the converse implication, (2)$\Rightarrow$(1), consider the family of cocycles $A_t^2(x) = A_t(Tx) A_t(x)$. For the sake of simplicity we assume that $A_t$ is positively winding. Assumption (5.1) implies that $E(Tx)A(x)E(x) \neq 0$. Hence, by Proposition 4.8, there exists $c > 0$ such that

$$\frac{d}{dt} A_t(x)v = (A_t^2(x)v) \wedge \left( \frac{d}{dt} A_t^2(x)v \right) \geq c \quad \forall (x, t, v) \in X \times \mathbb{R} \times \mathbb{S}^1.$$

This ensures the strict increasing assumption of Theorem A.9 in [24], and the conclusion (1) follows from this theorem.

The need for assumption (5.1) is justified by the next example.

**Example 5.2.** Choose $E \in \mathbb{N}_+$ and a hyperbolic matrix $H \in \text{SL}_2(\mathbb{R})$ such that $H$ and $E$ share a common eigen-direction. Consider the random cocycle generated by the vector $A = (H, I)$ with probabilities $(\frac{1}{2}, \frac{1}{2})$. The family of cocycles $A_t := A(I + tE)$ satisfies:

1. Assumption (5.1) does not hold;
2. $A_t$ is not uniformly hyperbolic. It contains the parabolic matrix $I + tE$;
3. $\rho(A_t)$ is constant. There is a common direction fixed by all $A_t$;
5.3. **Typical regularity of the Lyapunov exponent.** This last section focuses on the regularity of the Lyapunov exponent of random, locally constant, $\text{SL}_2$ valued cocycles. Each of these cocycles $A : X \to \text{SL}_2(\mathbb{R})$ is determined by a matrix vector $A \in \text{SL}_2^*(\mathbb{R})^\kappa$ as well as a probability vector $(p_1, \ldots, p_\kappa)$. Recall that $\text{SL}_2^*(\mathbb{R}) := \text{SL}_2(\mathbb{R}) \setminus \{-I, I\}$, as well as Definition 4.2 where the set $N_{\pm}(A)$ of ‘good’ directions was introduced.

For each $A = (A_1, \ldots, A_\kappa) \in \text{SL}_2(\mathbb{R})^\kappa$, we denote by $\Gamma(A)$ the semigroup generated by the matrices $\{A_1, \ldots, A_\kappa\}$. Given a hyperbolic matrix $B \in \text{SL}_2(\mathbb{R})$ we write $\hat{u}(B) \in \mathbb{P}^1$ and $\hat{s}(B) \in \mathbb{P}^1$ to denote respectively the unstable and stable eigen-directions of $B$.

We say that $A \in \text{SL}_2(\mathbb{R})^\kappa$ has a heteroclinic tangency if there exist matrices $B, C, A \in \Gamma(A)$ such that $A$ and $B$ are hyperbolic and $C \hat{u}(B) = \hat{s}(A)$. In this case we also say that $(B, C, A)$ is a tangency for $A$. Recall Definition 4.1.

**Theorem 5.3.** Let $(A_t : X \to \text{SL}_2(\mathbb{R}))_{t \in I}$ be a smooth family of cocycles such that $A_t$ is strictly positively winding over $I$, for some $t_0 \in I$, $A_{t_0}$ is irreducible and has a heteroclinic tangency. If $\alpha > \frac{H(\mu)}{L_1(A_{t_0})}$, then the fibered rotation number $I \ni t \mapsto \rho(A_t)$ is not $\alpha$-Hölder continuous around $t = t_0$.

The proof of Theorem 5.3 is presented below.

**Corollary 5.3.** Let $(A_t : X \to \text{SL}_2(\mathbb{R}))_{t \in I}$ be a smooth family of cocycles such that $A_t$ is strictly positively winding over $I$, for some $t_0 \in I$, $A_{t_0}$ is not uniformly hyperbolic. If $\alpha > \frac{H(\mu)}{L_1(A_{t_0})}$, then the fibered rotation number $I \ni t \mapsto \rho(A_t)$ is not $\alpha$-Hölder continuous around $t = t_0$.

**Proof.** By assumption, $A_0 = A$ is not uniformly hyperbolic. Hence by [5, Propositions 7.8 and 7.12] there exist $t_0$ arbitrary close to $0$ such that $A_{t_0}$ admits a heteroclinic tangency and is irreducible. By Theorem 5.3 the fibered rotation number $\rho(A_t)$ is not $\alpha$-Hölder continuous around $t_0$. \hfill $\square$

Next result is a reformulation of Theorem 1.2.

**Corollary 5.4.** Let $A = (A_1, \ldots, A_\kappa) \in \text{SL}_2^*(\mathbb{R})^\kappa$ be a random cocycle such that $L_1(A) > 0$ but $A$ is not uniformly hyperbolic. For all $E \in N_{\pm}(A)$ and $\alpha > \frac{H(\mu)}{L_1(A)}$, the family of random cocycles $A_t := A(I + tE)$ has Lyapunov exponent $t \mapsto L_1(A_t)$ which is not $\alpha$-Hölder continuous near $t = 0$.

**Proof.** As explained in Subsection 1.2, by [23, Lemma 10.3] and Theorem 1.1, the Lyapunov exponent $L_1(A_t)$ has the same regularity as the fibered rotation number. The conclusion follows by Corollary 5.3. \hfill $\square$

**Definition 5.1.** Given $\gamma > 0$, we say that a pair of matrices $(B, A) \in \text{SL}_2(\mathbb{R})^2$ has a $\gamma$-matching if there exists a pair of points $\hat{e}_1, \hat{e}_2 \in \mathbb{P}^1$ such that

(1) $AB\hat{e}_1 = \hat{e}_2$;
Remark 5.2. We can use Proposition 6.3 to conclude that 
\[
\phi_i \in \mathbb{P}^1.
\]

Remark 5.1. Notice that given \( \hat{e}_1 \in \mathbb{P}^1 \), we can define \( \hat{e}_2 \) so that (1)-(2) hold, but then (3) will only be satisfied for very particular choices of \((B, A)\). Given 1-parameter smooth families \( A_t, B_t \in \text{SL}_2(\mathbb{R}) \), the correct way to find \( \gamma \)-matchings \((B_t, A_t)\) is to fix first \( \hat{e}_1, \hat{e}_2 \in \mathbb{P}^1 \) and then solve the equation \( B_t \hat{e}_1 = A_t^{-1} \hat{e}_2 \) in \( t \), looking for parameters \( t \) such that both norms \( \|B_t e_1\| \) and \( \|A_t^{-1} e_2\| \) are large of order \( e^\gamma \).

For families of cocycles we have the following definition of matching.

Definition 5.2. Given a family of cocycles \( \{A_t\}_t \), \( \gamma > 0 \), \( k \geq 2 \) and \( t_0 \in I \) we say that a sequence \( \omega \in X \) is a \((\gamma, k, t_0)\)-matching if there exists \( 1 \leq m < k \) such that \((A_{t_0}^m(\omega), A_{t_0}^{k-m}(T^m \omega))\) has a \( \gamma \)-matching.

Remark 5.2. When \((A_{t_0}^m(\omega), A_{t_0}^{k-m}(T^m \omega))\) connects the vectors \( \hat{e}_1, \hat{e}_2 \) of the canonical basis of \( \mathbb{R}^2 \), the above definition of matching agrees with the definition of matching in Section 6 of [5] with \( \gamma := \log \delta^{-1} \).

Proposition 5.5. There exists \( \gamma_0 \) and constant \( c_* > 0 \) such that for every positively smoothly strictly winding family of cocycles \( \{A_t\}_t \) if \( \omega \in X \) is a \((\gamma, k, t_0)\)-matching, then \( A_t^k(\omega) \) winds once around \( \mathbb{P}^1 \) as \( t \) ranges in \([t_0 - 4c_*^{-1} e^{-\gamma}, t_0 + 4c_*^{-1} e^{-\gamma}]\).

Proof. Consider \( 1 \leq m < k \) and directions \( \hat{e}_1, \hat{e}_2 \in \mathbb{P}^1 \) which are connected by \((A_{t_0}^m(\omega), A_{t_0}^{k-m}(T^m \omega))\). Set \( J_0 := [t_0 - 3c_*^{-1} e^{-\gamma}, t_0 + 3c_*^{-1} e^{-\gamma}] \)

For any pair of vectors \( \hat{v}, \hat{w} \in \mathbb{P}^1 \) satisfying
\[
d(\hat{v}, \hat{w}(A_{t_0}^m(\omega))) \geq e^{-\gamma} \quad \text{and} \quad d(\hat{w}, \hat{w}(A_{t_0}^{k-m}(T^m \omega))) \geq e^{-\gamma},
\]
we can use Proposition 6.3 to conclude that \( d(A_{t_0}^m(\omega) \hat{v}, A_{t_0}^{k-m}(T^k \omega) \hat{w}) \leq 3 e^{-\gamma} \).

If we define \( f_+, f_- : J_0 \to \mathbb{P}^1 \) by,
\[
f_+(t) := A_t^m(\omega) \hat{v} \quad \text{and} \quad f_-(t) := A_t^{-m}(T^k \omega) \hat{w},
\]
then by the strict winding hypothesis, Definition 4.1, which we can assume to be positive, we have that there exists \( c_* > 0 \) such that \( f'(t) < -c_* < 0 < c_* < f'(t) \), for every \( t \in I \). Since \( d(f_+(t_0), f_-(t_0)) \leq 3 e^{-\gamma} \), there exists \( t_* \in J_0 \) such that
\[
A_{t_*}^m(\omega) \hat{v} = f_+(t_*) = f_-(t_*) = A_{t_*}^{-m}(T^k \omega) \hat{w}.
\]

Using Corollary 2.16, this implies that for every \( \hat{v} \notin B(\psi(A_{t_0}^m(\omega)), e^{-\gamma}) \),
\[
\ell_f(A_t^k(\omega) \hat{v}) \geq \pi - e^{-\gamma}.
\]
Writing $J_1 := J_- \cup J \cup J_+$, where $J_- := [t_0 - \frac{7}{2} c_\gamma^{-1} e^{-\gamma}, t_0 - 3c_\gamma^{-1} e^{-\gamma}]$ and $J^+ := [t_0 + 3c_\gamma^{-1} e^{-\gamma}, t_0 + \frac{7}{2} c_\gamma^{-1} e^{-\gamma}]$ and using again the strict positive winding, we have
\[
\ell_{J_1}(A_t^k \hat{v}) \geq \ell_{J_+}(A_t^k \hat{v}) + \ell_{J_-}(A_t^k \hat{v}) + \ell_{J}(A_t^k \hat{v}) \\
\geq e^{-\gamma} + \pi - e^{-\gamma} = \pi.
\]
In other words, for every $\hat{v} \notin B(y(A_t^m(\omega), e^{-\gamma}))$, the curve $t \in J_1 \mapsto A_t^k(\omega) \hat{v}$ gives one turn around $\mathbb{P}^1$. Thus, by Corollary 2.17 for every $\hat{w} \in \mathbb{P}^1$ and $\hat{v} \notin B(y(A_t^m(\omega)), e^{-\gamma})$, the equation $A_t^{-m}(T^k \omega) \hat{w} = \hat{v}$ has a solution for some $t \in J_1$ which implies again by Corollary 2.16 that
\[
\ell_{J_1}(A_t^{-m}(T^k \omega) \hat{w}) \geq \pi - e^{-\gamma}.
\]
Taking $J_* := [t_0 - 4c_\gamma^{-1} e^{-\gamma}, t_0 + 4c_\gamma^{-1} e^{-\gamma}]$ and using the same argument as above we conclude that for every $\hat{w} \in \mathbb{P}^1$,
\[
\ell_{J_*}(A_t^{-m}(T^k \omega) \hat{w}) \geq \pi.
\]
Therefore, for every $\hat{v}, \hat{w} \in \mathbb{P}^1$, the equation
\[
A_t^{-m}(T^k \omega) \hat{w} = \hat{v} \iff A_t^k(\omega) \hat{v} = \hat{w},
\]
has a solution for some $t \in J_*$ one and more application of Corollary 2.17 concludes the proof.

Let $\{A_t\}_{t \in I}$ be a strictly positively winding smooth family of cocycles and $J$ be a sub-interval of $I$. We denote by $\Sigma(\gamma, k, J)$ the set of sequences $\omega \in X$ which are $(\gamma, k, t)$-matching for some $t \in J$. For each $\delta > 0$ we write $J_\delta := J + [-\delta, \delta]$. 

\textbf{Proposition 5.6.} Given $\omega \in X$, $n \geq 1$, for every $\hat{v} \in \mathbb{P}^1$ if $\delta := 4c_\gamma^{-1} e^{-\gamma}$,
\[
\ell_{J_\delta}(A_t^{nk}(\omega) \hat{v}) \geq \pi \sum_{j=0}^{n-1} \chi_{\Sigma(\gamma, k, J)}(T^{jk} \omega).
\]

\textbf{Proof.} If $T^{jn} \omega \in \Sigma(\gamma, k, t_0)$ for some $t_0 \in J$ and $0 \leq j \leq n - 1$, then by Proposition 5.5, $A_t^k(T^{jn} \omega)$ winds once around $\mathbb{P}^1$ as $t$ ranges in the interval $[t_0 - 4c_\gamma^{-1} e^{-\gamma}, t_0 + 4c_\gamma^{-1} e^{-\gamma}] \subset J_\delta$. In particular, for every $\hat{v} \in \mathbb{P}^1$,
\[
\ell_{J_\delta}(A_t^{k}(T^{jn} \omega) \hat{v}) \geq \pi.
\]
Combining Proposition 2.18 with Corollary 2.17 and Lemma 2.15, the stated inequality follows. \hfill \square

\textbf{Corollary 5.7.} For any interval $J \subseteq I$ and $\gamma > 0$, if $\delta := 4c_\gamma^{-1} e^{-\gamma}$,
\[
d \rho(J_\delta) \geq \frac{1}{k} \mu(\Sigma(\gamma, k, J)) , \quad \forall k \in \mathbb{N}.
\]

\textbf{Proof.} Apply Birkhoff’s ergodic theorem together with Lemma 2.14 and Proposition 5.6. \hfill \square

Let $\lambda := \inf_{|t-t_0| \leq \delta} L(A_t)$ and assume $A_{t_0}$ has a heteroclinic tangency.
Thus, using (5.2) we get

\[ \lambda \]

Remark 5.3. Proposition 5.8 says that for every \( \omega \) smooth families of cocycles.

\[ \Box \]

Lemma 8.1 uses the strict positive winding property through Proposition 7.13, this being the only place where the fact that the cocycle comes from a Schrödinger cocycle is used. All other statements hold for general strictly positively winding smooth families of cocycles.

Proposition 5.8. For every \( \hat{v}, \hat{w} \in \mathbb{P}^1 \) and \( 0 < \beta \ll \lambda \), there exist constants \( C^*, C, c > 0 \), an infinite subset \( \mathbb{N}' \subset \mathbb{N} \), a sequence \( (t_i)_{i \in \mathbb{N}'} \) with \( |t_i - t_0| \leq C^* e^{-t_i/3} \), and a sequence of measurable sets \( M_i \subset X \) such that defining

\[ I_i := [t_i - C e^{-i(\lambda-\beta)}, t_i + C e^{-i(\lambda-\beta)}], \]

for every \( \omega \in M_i \) we can find \( t_i^* \in I_i \) such that

1. \( A_{t_i^*}^{2l^3+l}(\omega) \hat{v} = \hat{w} \);
2. \( e^{(\lambda-\beta)l^3} \leq \|A_{t_i^*}^{l}(\omega) v\| \leq \|A_{t_i^*}^{l}(\omega)\| \leq e^{(\lambda+\beta)l^3} \);
3. \( e^{(\lambda-\beta)l^3} \leq \|A_{t_i^*}^{-l}(T^{2l^3+l} \omega) w\| \leq \|A_{t_i^*}^{-l}(T^{2l^3+l} \omega)\| \leq e^{(\lambda+\beta)l^3} \);
4. \( \|A_{t_i^*}^{2l^3+l}(\omega) v\| \leq e^{3l^3} \).

Moreover, \( \mu(M_i) \geq (1/2 - \beta)^2 e^{-l(H(\mu)+\beta)} \).

Proof. This proposition is a reformulation of [5, Lemma 8.1], which holds for positively winding SL\(_2(\mathbb{R})\)-cocycles. As indicated in [5, Figure 1], the proof of Lemma 8.1 uses the strict positive winding property through Proposition 7.13, this being the only place where the fact that the cocycle comes from a Schrödinger cocycle is used. All other statements hold for general strictly positively winding smooth families of cocycles.

Remark 5.3. Proposition 5.8 says that for every \( l \in \mathbb{N}' \) and every \( \omega \in M_i \), \( (A_{t_i}^{l}(\omega), A_{t_i}^{3l^3}(T^{3l^3+1} \omega)) \) has a \([l^3\beta]\)-matching. In particular,

\[ M_i \subset \Sigma((l - \beta) l^3, 2l^3 + l, I_i). \]

Proof of Theorem 5.3. Given \( \alpha > H(\mu)/L_1(A_{\delta_\alpha}) \), choose \( \delta > 0 \) small enough so that \( \lambda := \inf_{|t_i - t_0| \leq \delta} L_1(A_{\delta}) \) satisfies \( \alpha \lambda - H(\mu) > 0 \) and then take \( \beta > 0 \) small so that \( \alpha \lambda - H(\mu) > 2 \beta + \alpha \beta \). This implies that

\[ -H(\mu) - \beta + \alpha (\lambda - \beta) > \beta. \tag{5.2} \]

From Proposition 5.8 take \( l \in \mathbb{N}' \), consider the set \( M_i \subset X \) and the interval \( I_i \subset \mathbb{R} \) therein of length \( |I_i| = 2 C e^{-i(\lambda-\beta)} \). Defining \( \gamma_i := l^3 (\lambda - \beta) \), by Remark 5.3 we have \( M_i \subset \Sigma(\gamma_i, 2l^3 + l, I_i) \). Let \( \delta_i := 6 c_{\alpha}^{-1} e^{-\gamma_i} \) be the constant associated with \( \gamma_i > 0 \) in Corollary 5.7 and set \( \tilde{I}_i := I_i + [-\delta_i, \delta_i] \), so that \( |\tilde{I}_i| \sim |I_i| \sim e^{-\gamma_i} \).

By Corollary 5.7 and Proposition 5.8

\[ d \rho \left( \tilde{I}_i \right) \geq \frac{1}{2l^3 + l} \mu \left( \Sigma(\gamma_i, 2l^3 + l, I_i) \right) \geq \frac{1}{2l^3 + l} \mu(M_i) \geq \frac{1}{(2l^3 + l)} (1/2 - \beta)^2 e^{-l(H(\mu)+\beta)}. \]

Thus, using (5.2) we get

\[ \frac{d \rho \left( \tilde{I}_i \right)}{|\tilde{I}_i|^\alpha} \geq e^{l(-H(\mu)-\beta+\alpha(\lambda-\beta))} \geq e^{\beta l}. \]
Taking $l \to \infty$ we conclude that $\rho$ can not be $\alpha$-Hölder continuous. $\square$

Given $\beta > 0$, the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called $\beta$-log-Hölder continuous if there exists $C < \infty$ such that for all $t, t' \in I$,

$$|f(t) - f(t')| \leq C \frac{1}{\log^\beta (|t - t'|^{-1})}. $$

**Theorem 5.4.** Consider the random linear cocycle generated by the matrices $A = (C, D)$

$$C := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad D := \begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix} \quad (5.3)$$

with probability vector $(\frac{1}{3}, \frac{2}{3})$. For any $E \in N_\pm(A)$, the Lyapunov exponent function $t \mapsto L_1(A(I + tE))$ is not $\beta$-log-Hölder continuous around $t = 0$ for any $\beta > 3$.

**Proof.** In [20] it was shown that there exists a 1-parameter curve of cocycles $A_t$ passing through $A = (C, D)$ at $t = 0$, along which $L_1(A_t)$ has a ‘nasty’ modulus of continuity. The meaning of ‘nasty’ is clarified below. The strategy was to conjugate $A$ to a Schrödinger cocycle over a mixing Markov shift, so that, up to conjugation, $A_t$ could be viewed as a family of Schrödinger cocycles, where $t$ is the system’s energy. It was proved in [20, Theorem 1] that the IDS of the corresponding Schrödinger operator is not $\beta$-log-Hölder continuous for any $\beta > 2$. Recall that, the IDS is the fibered rotation number of this positively winding Schrödinger family. The loss of regularity comes from the existence of many matching configurations. See Lemma 1 and Proposition 8 in [20].

Given $E \in N_\pm(A)$, by Proposition 4.10 the cocycle $A_t := A(I + tE)$ satisfies assumptions 1-4 of Theorem 1.1 and whence also the Thouless formula (1.4). An adaption of the argument in [20] based on the Lemma 1 mentioned above, gives the following lemma. Actually some technical aspects of the proof get simplified because there is no need anymore to conjugate the original cocycle to a Schrödinger one over a mixing Markov shift.

**Lemma 5.9.** The function $\mathbb{R} \ni t \mapsto \rho(A_t)$ is not $\beta$-log-Hölder continuous for any $\beta > 2$.

From the Thouless formula (1.4) it follows that $L_1(A_t)$ is essentially the Hilbert transform of the fibered rotation number $\rho(A_t)$. The Hilbert transform and its inverse are examples of singular integral operators. M. Goldstein and W. Schlag [23, Lemma 10.3] proved that any singular integral operator on a space of functions preserves certain modulus of continuity, which include the Hölder and weak-Hölder but not the $\beta$-log-Hölder modulus of continuity. In a recent paper, Avila, Last, Shamis and Zhou have improved this result showing that for any $\beta > 2$, if $L_1(A_t)$ is $\beta$-log-Hölder then $\rho(A_t)$ is $(\beta - 1)$-log-Hölder , see Proposition 2.2 and
Corollary 2.3 in [3]. Whence, in view of the previous lemma, $L_1(A_t)$ can not be $\beta$-log-Hölder continuous for any $\beta > 3$. 

6. Appendix: linear algebra facts

6.1. Projective analysis. Given $A \in \text{Mat}_2(\mathbb{R})$ denote by $\{\overline{v}(A), \underline{v}(A)\}$ an orthonormal set of singular directions of $A$, defined by the relations

$$(A^t A) \overline{v}(A) = \|A\|^2 \overline{v}(A) \quad \text{and} \quad (A^t A) \underline{v}(A) = m(A) \underline{v}(A),$$

where $m(A)$ denotes the co-norm of $A$, $m(A) := \|A^{-1}\|^{-1}$ if $A$ is invertible and otherwise $m(A) := 0$. Define also

$$\overline{v}^*(A) := \overline{v}(A^t) \quad \text{and} \quad \underline{v}^*(A) := \underline{v}(A^t),$$

so that

$$A \overline{v}(A) = \|A\| \overline{v}^*(A) \quad \text{and} \quad A \underline{v}(A) = m(A) \underline{v}^*(A).$$

For the sake of simplicity we use the same notation for the singular vectors and its projectivization. For each pair of vectors $v, w \in \mathbb{R}^2$ we write

$$d(\hat{v}, \hat{w}) := \left| \frac{v \wedge w}{\|v\| \|w\|} \right| = \sin \angle(v, w),$$

for the usual distance in $\mathbb{P}^1$.

**Lemma 6.1.** Take $A \in \text{SL}_2(\mathbb{R})$. For each $\hat{v} \in \mathbb{P}^1$,

$$d(\hat{v}, \hat{v}(A)) = \sqrt{\frac{\|Av\|^2 - \|A\|^{-2}}{\|A\|^2 - \|A\|^{-2}}}. $$

In particular,

$$\frac{1}{\|A\|} \sqrt{\|Av\|^2 - \|A\|^{-2}} \leq d(\hat{v}, \hat{v}(A)) \leq \frac{\|Av\|}{\|A\|}. $$

**Proof.** Let $v = a \overline{v}(A) + b \underline{v}(A)$ be a unit vector, i.e., $a^2 + b^2 = 1$, with $a > 0$. Then $a = d(\hat{v}, \hat{v}(A))$ and $Av = a \|A\| \overline{v}^*(A) + b \|A\|^{-1} \underline{v}^*(A)$, which implies that

$$\|Av\|^2 = a^2 \|A\|^2 + (1 - a^2) \|A\|^{-2}. $$

Solving in $a = d(\hat{v}, \hat{v}(A))$ yields the conclusion. \qed

**Lemma 6.2.** For any $\hat{v} \in \mathbb{P}^1$,

$$d(A\hat{v}, \hat{v}^*(A)) \leq \frac{1}{d(\hat{v}, \hat{v}(A))\|A\|^2}. $$

**Proof.** Using the setup of Lemma 6.1’s proof,

$$d(\hat{v}, \hat{v}^*(A)) = \frac{|Av \wedge \overline{v}^*(A)|}{\|Av\|} \leq \sqrt{1 - a^2} \leq \frac{1}{d(\hat{v}, \hat{v}(A))\|A\|^2}$$

where in the last inequality we use Lemma 6.1. \qed
In the next proposition we use Definition 5.1.

**Proposition 6.3.** There exists $\gamma_0 > 0$ such that for every $\gamma \geq \gamma_0$, if $(B, A) \in \text{SL}_2(\mathbb{R})^2$ is a $\gamma$-matching and $\hat{v}, \hat{w} \in \mathbb{P}^1$ satisfy

\[
d(\hat{v}, \hat{v}(B)) \geq e^{-\gamma} \quad \text{and} \quad d(\hat{w}, \hat{v}^*(A)) \geq e^{-\gamma},
\]

then

\[
d(B \hat{v}, A^{-1} \hat{w}) \leq 3 e^{-\gamma}.
\]

**Proof.** Assume $(B, A)$ connects $\hat{e}_1, \hat{e}_2 \in \mathbb{P}^1$. By Lemma 6.2 and the matching assumption,

\[
d(B \hat{v}, \hat{v}^*(B)) \leq \frac{1}{d(\hat{v}, \hat{v}(B))\|B\|^2} \leq e^{-\gamma}
\]

and

\[
d(A^{-1} \hat{w}, \hat{v}(A)) \leq \frac{1}{d(\hat{w}, \hat{v}^*(A))\|A\|^2} \leq e^{-\gamma}.
\]

Using additionally Lemma 6.1,

\[
d(\hat{v}^*(B), B \hat{e}_1) \leq \frac{1}{d(\hat{e}_1, \hat{v}(B))\|B\|^2} \leq \frac{\|B\|}{\|B\| \sqrt{\|Be_1\|^2 - \|B\|^2}}
\]

\[
\leq e^{-\gamma} \frac{1}{\sqrt{e^{2\gamma} - e^{-2\gamma}}} \leq 2 e^{-2\gamma},
\]

and similarly,

\[
d(\hat{v}(A), A^{-1} \hat{e}_2) \leq 2 e^{-2\gamma},
\]

for every $\gamma \geq \gamma_0$, for some $\gamma_0$ sufficiently large.

Thus, by triangular inequality, and the fact that $B \hat{e}_1 = A^{-1} \hat{e}_2$,

\[
d(B \hat{v}, A^{-1} \hat{w}) \leq d(B \hat{v}, \hat{v}^*(B)) + d(\hat{v}^*(B), B \hat{e}_1)
\]

\[
+ d(A^{-1} \hat{e}_2, \hat{v}(A)) + d(\hat{v}(A), A^{-1} \hat{w})
\]

\[
\leq 2 (e^{-\gamma} + 2 e^{-2\gamma}) \leq 3 e^{-\gamma},
\]

for every $\gamma \geq \gamma_0$. \hfill \Box

### 6.2. Conditions to ensure Assumptions 1-3.

**Lemma 6.4.** For any matrix $E \in \text{Mat}_2(\mathbb{R})$ and $t \in \mathbb{R}$,

\[
\det(I + tE) = 1 + t \text{tr}(E) + t^2 \det(E).
\]

In particular, if $E \in \text{GL}_2^+(\mathbb{R})$,

\[
\det(I + tE) = (\det E) \left[ t + \frac{\text{tr}E}{2 \det E} \right]^2 + \frac{\Delta_E}{4 (\det E)^2},
\]

where $\Delta_E := 4(\det E) - (\text{tr}E)^2$. 

Proof. Direct computation. □

Consider matrices $A, E \in \text{Mat}_2(\mathbb{R})$. In the rest of this appendix we write $A_t := A (I + t E)$ and $Q_{A_t}(v) := (A_t v) \wedge (\dot{A}_t v)$. Define $J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $E^\sharp := (J E + (J E)^t)/2 = (J E - E^t J)/2$.

Proposition 6.5. For all $v \in \mathbb{R}^2$, $Q_{A_t}(v) = (\det A) (v \wedge E v) = (\det A) (v^t E^\sharp v)$.

Proof. Since $(A E v) \wedge (A E v) = 0$, $Q_{A_t}(v) = (A_t v) \wedge (\dot{A}_t v) = (A(I + tE)v) \wedge (A E v) = (A v) \wedge (A E v) = (\det A) (v \wedge (E v)) = (\det A) (v, J E v) = (\det A) (v^t E^\sharp v)$. □

Consider the function $\Delta : \text{Mat}_2(\mathbb{R}) \to \mathbb{R}$ in Lemma 6.4.

Proposition 6.6. The following are equivalent:

1. The quadratic form $Q_{A_t}$ is positive (resp. negative) definite;
2. $\Delta_E > 0$ and $e_{12} < 0 < e_{21}$ (resp. $\Delta_E > 0$ and $e_{21} < 0 < e_{12}$);
3. $E$ has no real eigenvalues and the solutions of the linear O.D.E. $\dot{X} = EX$ wind positively (resp. negatively) around the origin.

Proof. Simple calculations give $E^\sharp = \begin{bmatrix} e_{21} & -\frac{e_{21} - e_{11}}{2} \\ -\frac{e_{21} - e_{11}}{2} & -e_{12} \end{bmatrix}$ and $\det E^\sharp = \frac{1}{4} \Delta_E$ (6.1) from which the equivalence (1)$\iff$ (2) follows. For the equivalence (1)$\iff$ (3) notice that $v \wedge (E v) = 0$ if and only if $v$ is an eigendirection of $E$ associated with some real eigenvalue. Therefore, the quadratic form $Q_{A_t}$ is definite (positive or negative) if and only if $E$ has no real eigenvalues and this happens exactly when the discriminant $-\Delta_E$ of the polynomial $\det(I + \lambda E)$ is strictly negative. Finally notice that since the curves $A e^{tE} v$ and $A (I + t E) v$ are tangent at $t = 0$, they wind in the same direction. □

Proposition 6.7. The following are equivalent:

1. The quadratic form $Q_{A_t}$ is positive (resp. negative) semi-definite but not definite;
2. $\Delta_E = 0$, $e_{12} \leq 0 \leq e_{21}$ and $e_{12} < e_{21}$ (resp. $\Delta_E = 0$, $e_{21} \leq 0 \leq e_{12}$ and $e_{12} > e_{21}$);
3. $E$ has a double real eigenvalue and the solutions of the linear O.D.E. $\dot{X} = EX$ wind positively (resp. negatively) around the origin.
Proof. Same argument as in the proof of Proposition 6.6. \qed

Consider the seminorm $\Xi$ and the following sets introduced in Subsection 4.2.

$$\Gamma_+ := \{ E \in \text{Mat}_2(\mathbb{R}) : \Xi(E) > 0, \Delta_E \geq 0 \text{ and } e_{21} \leq 0 \leq e_{12} \}$$

$$\Gamma_- := \{ E \in \text{Mat}_2(\mathbb{R}) : \Xi(E) > 0, \Delta_E \geq 0 \text{ and } e_{12} \leq 0 \leq e_{21} \}$$

The intersection of these sets is empty, i.e., $\Gamma_- \cap \Gamma_+ = \emptyset$.

**Proposition 6.8.** Using the previous definitions:

$$\Gamma_+ = \{ E \in \text{Mat}_2(\mathbb{R}) : Q_{I+tE} \text{ is positive definite or semi-definite } \}$$

$$\Gamma_- = \{ E \in \text{Mat}_2(\mathbb{R}) : Q_{I+tE} \text{ is negative definite or semi-definite } \}$$

Proof. Notice that

$$\Delta_E = 4 (e_{11} e_{22} - e_{21} e_{21}) - (e_{11} + e_{22})^2 = -4 e_{12} e_{21} - (e_{11} - e_{22})^2$$

so that if $\Delta_E = 0$ and $e_{12} \leq 0 \leq e_{21}$ then

$$E \in \Gamma_+ \iff \Xi(E) > 0 \iff e_{12} < e_{21} \iff Q_{I+tE} \text{ is positive semi-def.}$$

Similarly, if $\Delta_E > 0$ and $e_{12} \leq 0 \leq e_{21}$ then

$$E \in \Gamma_+ \iff \Xi(E) > 0 \iff e_{12} < e_{21} \iff Q_{I+tE} \text{ is positive definite.}$$

An entirely analogous argument works for $\Gamma_-$. \qed

**Example 6.1.** In the case of Schrödinger matrices,

$$S(x-t) := \begin{bmatrix} x-t & -1 \\ 1 & 0 \end{bmatrix},$$

we can write

$$S(x-t) = \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix} \left( I + t \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) =: A(I + tE).$$

with $E \in \Gamma_+$.

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Faculty of Mathematics and Computer Science, Nicolaus Copernicus University (UMK), Poland.
Email address: jdouglas@impa.br

Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro (PUC-Rio), Brazil
Email address: godcaiao@gmail.com

Departamento de Matemática and CMAF-CIO, Faculdade de Ciências, Universidade de Lisboa, Portugal
Email address: pmduarte@fc.ul.pt

Instituto de Matemática e Estatística, Universidade de São Paulo, Brazil
Email address: catalinafreijo@gmail.com

Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro (PUC-Rio), Brazil
Email address: silviusk@mat.puc-rio.br