Ultracold collision in the presence of synthetic spin-orbit coupling

Hao Duan,¹ Li You,¹ and Bo Gao²

¹State Key Laboratory of Low-Dimensional Quantum Physics, Department of Physics, Tsinghua University, Beijing 100084, China
²Department of Physics and Astronomy, Mailstop 111, University of Toledo, Toledo, Ohio 43606, USA
(Dated: January 24, 2013)

We present an analytic description of ultracold collision between two spin-\(\frac{1}{2}\) fermions with isotropic spin-orbit coupling of the Rashba type. We show that regardless of how weak the spin-orbit coupling may be, the ultracold collision at sufficiently low energies is significantly modified, including the ubiquitous Wigner threshold behavior. We further show that the particles are preferably scattered into the lower-energy helicity state due to the break of parity conservation, thus establishing interaction with spin-orbit coupling as one mechanism for the spontaneous emergence of handedness.

PACS numbers: 34.50.Cx, 67.85.Lm, 71.70.Ej, 05.30.Fk

Systems of cold atoms have become fertile laboratories for many-body and few-body physics largely because of the ability to tune and manipulate atomic interactions. The magnetic Feshbach resonance [1], for instance, has allowed precise tuning of scattering length to virtually arbitrary value, facilitating studies of strongly coupled many-body systems [2, 3] and also few-body systems in the universal regime [4–6].

A new class of manipulation of cold atoms has arisen recently under the general envelope of synthetic gauge fields, generated mainly through coherent laser-atom interactions [7]. Among various types, the synthetic spin-orbit coupling (SOC) [8–12] is of special interest as it simulates a type of coupling that is regarded as important in fractional quantum Hall effect and topological insulators [13, 14]. Despite a large body of recent works on SOC systems [13–28], many fundamental questions remain, as elementary as effects of SOC on the two-body scattering [26–28]. A recent experiment by Williams et al. [29] provides an early indication that such effects can be substantial, and essential for understanding interacting many-body, and few-body systems with SOC.

In this work, we present a general theoretical treatment of the scattering of two spin-\(\frac{1}{2}\) fermions with isotropic SOC of the Rashba type. We pick spin-\(\frac{1}{2}\) fermions for its direct relevance to electrons in condensed matter, and for the fact that it can be simulated by \(^6\)Li in its ground hyperfine state [10], or in the ultracold regime by pseudo spin-\(\frac{1}{2}\) fermions [9]. We choose isotropic coupling to isolate effects of SOC and effects of anisotropy. We show that in the presence of SOC, a non-Abelian gauge field that persists to infinite inter-particle separation, the scattering formulation has to be changed substantially, including the very definitions of fundamental quantities such as the scattering matrices and the flux. The formalism is solved analytically, in terms of scattering in the absence of SOC, by taking advantage of a length scale separation [30, 31]. The results, when compared with ones without SOC, show a substantially altered threshold behavior, different from the familiar Wigner behavior [32], and a preferential scattering into the lower-energy helicity state as a consequence of parity non-conservation. This preference implies that handedness can spontaneously emerge as a result of scattering with SOC.

We consider two identical particles with \(F_1 = F_2 = 1/2\). We use symbols \(F_1\) and \(F_2\) to distinguish them, for atoms, from electronic spin which has a well-defined separate meaning. In the absence of SOC, the interaction between two such particles can very generally be described by the Hamiltonian

\[
H = h_1 + h_2 + \hat{V}.
\]

Here \(h_i = p_i^2/2m\) is the single particle Hamiltonian in the absence of SOC, and \(\hat{V}\) is an interaction operator describing two effective central potentials, \(V^{(F=0)}(r)\) for the “singlet” states and \(V^{(F=1)}(r)\) for the “triplet” states. Without SOC, the total “spin” \(F = F_1 + F_2\) and the orbital angular momentum \(l\) are independently conserved.

The scattering is fully characterized by two sets of effective single-channel \(K\) matrices, \(\tan \delta^{F=0}_{\pm}\) for the singlet states and \(\tan \delta^{F=1}_{\pm}\) for the triplet states.

The isotropic SOC of the Rashba type changes the single particle Hamiltonian from \(h_i = p_i^2/2m\) to

\[
\hat{h}_i = \frac{p_i^2}{2m} + \frac{\hbar C_{\text{so}}}{m} \sigma_1 \cdot \hat{p}_i, \tag{2}
\]

where \(\sigma\) denotes the Pauli spin matrix, \(C_{\text{so}}\) is a constant characterizing the strength of SOC. It has a dimension of a \(k\)-vector (inverse length), with its magnitude to be denoted by \(k_{\text{so}} \equiv |C_{\text{so}}|\). This single particle Hamiltonian is diagonalized by states \(|\pm, n_{\text{so}}\rangle_i |K_i\rangle\), where \(|K_i\rangle\) describes the translational motion, and is an eigenstate of \(p_i\) with an eigenvalue of \(\hbar K_i\). \(|\pm, n_{\text{so}}\rangle_i\) is a shorthand notation for \(|F_i = 1/2, M_i = \pm 1/2, n_{\text{so}}\rangle_i\), with \(n_{\text{so}}\) defining the direction of quantization. For isotropic SOC of the Rashba type, \(n_{\text{so}} = K_i/K_i\) for \(C_{\text{so}} > 0\), and \(n_{\text{so}} = -K_i/K_i\) for \(C_{\text{so}} < 0\). The “±” states have different energies as given by two distinctive dispersion relations, \(E_i = \hbar^2 K_i^2 / 2m \pm \hbar^2 k_{\text{so}}K_i/m\). We will generally
For two interacting particles with SOC, the conservation of the total momentum, \( \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \), allows the investigation of scattering and interaction in the center-of-mass frame. \( \mathbf{P} = 0 \), in which the relative motion is described by the Hamiltonian,

\[
H_{rel} = \frac{1}{2\mu} \mathbf{p}^2 + \hat{V} + \frac{\hbar C_{so}}{m} (\mathbf{\sigma}_1 - \mathbf{\sigma}_2) \cdot \mathbf{p} .
\]

Here \( \mu = m/2 \) is the reduced mass, and \( \mathbf{p} = (\mathbf{p}_1 - \mathbf{p}_2)/2 \) is the (canonical) momentum corresponding to the relative motion. In the center-of-mass frame, SOC changes the single dispersion relation, \( \epsilon = \hbar^2 k^2/2\mu \), applicable to all spin states without SOC, into three branches: \( \epsilon = (\hbar^2/2\mu)(k^2 + 2k_{so}k) \) for the \((+, n_{so})\), \((-+, n_{so})\), and \((-, n_{so})\) spin states, respectively. They define the asymptotic channels for interaction with SOC. The different dispersion relations for the \((+, +)\), \((+, -)\), \((-+, +)\), and \((-+, -)\) asymptotic states, which are related to each other by a parity operation (and exchange of particles), are direct consequences of the parity non-conservative nature of the SOC. The time-reversal symmetry is, however, still maintained.

With SOC, even the isotropic SOC under consideration here, the \( \mathbf{F} \) and \( \mathbf{l} \) are generally no longer independently conserved. For isotropic SOC, the total angular momentum \( \mathbf{F}_t = \mathbf{F} + \mathbf{l} \) is conserved. The wave function for each total angular momentum, \( \mathbf{F}_t, M_t \), can be expanded as

\[
\psi_{jF_tM_t}^{F_tM_t} = \sum_{\alpha} \Phi_{\alpha F_tM_t}^{F_tM_t} G_{\alpha j}^{F_tM_t}(r)/r .
\]

Here \( G_{\alpha j}^{F_tM_t}(r) \) describes the relative radial motion, and \( j \) is an index for different linearly independent solutions. The \( \Phi_{\alpha F_tM_t}^{F_tM_t} \) are channel functions, indexed by \( \alpha \), describing all degrees of freedom other than the relative radial motion. They are conveniently chosen here to be the \( \{F, l\} \) basis, in which the interaction in the absence of SOC is diagonal. The summation over \( \alpha \) namely the \( F \) and \( l \) combinations, is restricted both by the angular momentum conservation and by \( F + l = \text{even} \) as imposed by the symmetry under the exchange of particles. This leads to the following general channel structure for interaction with isotropic SOC. All \( \mathbf{F}_t \) odd states are described by single-channel problems with \( V^{(F_t = 1)}(r) \), corresponding to \( \{F = 1, l = F_t\} \). All \( \mathbf{F}_t \) even states, other than \( \mathbf{F}_t = 0 \), are described by three-channel problems, corresponding to \( \{F = 0, l = F_t\}, \{F = 1, l = F_t - 1\} \), and \( \{F = 1, l = F_t + 1\} \). The \( \mathbf{F}_t = 0 \) states are described by a two-channel problem with \( \{F = 0, l = 0\} \) and \( \{F = 1, l = 1\} \). The radial functions \( G_{\alpha j}^{F_t}(r) \) for \( \mathbf{F}_t = 1 \) even satisfy coupled-channel equations, which are given explicitly, for \( \mathbf{F}_t = 0 \) by

\[
\begin{pmatrix}
-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V^{(0)}(r) - \epsilon \\
\frac{i2\hbar^2 C_{so}}{m} \left( \frac{d}{dr} - \frac{1}{r} \right)
\end{pmatrix}
\begin{pmatrix}
\frac{d}{dr} + \frac{1}{r} \\
-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V^{(1)}(r) - \epsilon
\end{pmatrix}
\begin{pmatrix}
G_{F_t = 0\implies 0}^{F_t = 0\implies 0} \\
G_{F_t = 1\implies 1}^{F_t = 1\implies 1}
\end{pmatrix}
= 0 .
\]

The persistence of the SOC, corresponding to the off-diagonal terms in Eq. (3), to infinite separation requires re-definitions of scattering matrices. In particular, the \( K \)
matrix is now defined by
\[ \mathcal{J}^{F_1} = \mathcal{Y}^{F_1} K^{F_1}, \]
where
\[ \mathcal{J}^{F_1}_{i=0} = \begin{pmatrix} \frac{1}{\sqrt{2}} k_1 j_0(k_1 r) & -\frac{1}{\sqrt{2}} k_3 j_0(k_3 r) \\ i \frac{1}{\sqrt{2}} k_1 j_1(k_1 r) & -i \frac{1}{\sqrt{2}} k_3 j_1(k_3 r) \end{pmatrix}, \]
and
\[ \mathcal{Y}^{F_1}_{i=0} = \begin{pmatrix} \frac{1}{\sqrt{2}} k_1 y_0(k_1 r) & -\frac{1}{\sqrt{2}} k_3 y_0(k_3 r) \\ i \frac{1}{\sqrt{2}} k_1 y_1(k_1 r) & -i \frac{1}{\sqrt{2}} k_3 y_1(k_3 r) \end{pmatrix}. \]

Here \( k_1 = \sqrt{k_{s_{0}}^2 + k^2} - k_{s_{0}} \) and \( k_3 = \sqrt{k_{s_{0}}^2 + k^2} + k_{s_{0}} \), as illustrated in Fig. 1, and \( j_i(x) \) and \( y_i(x) \) are the spherical Bessel functions [34]. The \( \mathcal{J}^{F_1}_{i=0} \) and \( \mathcal{Y}^{F_1}_{i=0} \) are the exact regular and irregular analytic solutions of Eq. (3) in the absence of interaction, namely for \( V(0) = V(1) = 0 \). The two columns of the matrices correspond to solutions for the \((+,+)\) and the \((-,-)\) channels, respectively. Other scattering matrices such as the \( S \) matrix can be defined in a similar manner with their usual relationships maintained. For example, the \( S \) matrix is related to the \( K \) matrix by \( S^{F_1} = (I + iK^{F_1})(I - iK^{F_1})^{-1} \), where \( I \) represents the unit matrix. We note that in standard multichannel scattering theory without SOC (see, e.g., Ref. [33]), \( \mathcal{J}^{F_1} \) and \( \mathcal{Y}^{F_1} \) would have been diagonal.

From linear superposition of solutions that define the scattering matrices, a wave function satisfying scattering boundary condition can be constructed, from which all physical observables as related to scattering can be extracted [33]. One obtains, for instance,
\[ \sigma[(+,+) \rightarrow (+,+)] = \frac{2\pi}{k_1^2} \sum_{F_1=\text{even}} (2F_1 + 1) \left| S_{11}^{F_1} - 1 \right|^2, \]
and similarly for other cross sections. We emphasize that in deriving proper cross section formulas, it is crucial to recognize that in the presence of a gauge field, the flux or the current density has to be defined with the velocity operator (or the corresponding kinetic momentum) \( \mathbf{v} = \mathbf{r} = [r, H_{\text{rel}}]/\hbar = \mathbf{p}/\mu + \hbar \mathbf{C}_{s_{0}}(\sigma_1 - \sigma_2)/m \).

The \((+,+)\) and the \((-,-)\) states, despite having different canonical momenta, \( h k_1 \) and \( h k_3 \), respectively, have the same velocity of \( \hbar \sqrt{k_{s_{0}}^2 + k^2}/\mu \). The \((+,+)\) and the \((-,-)\) states have a different, the standard velocity of \( \hbar k/\mu \). This difference in the definition of flux is another key ingredient for a proper description of scattering in a gauge field.

The above formulation for two spin-\( \frac{1}{2} \) particles with SOC is very general, applicable for arbitrary energy and SOC coupling strength. In reality, both experimental realizations of SOC [8, 11] and the very validity of the Hamiltonian used to describe it, imply that we are most interested in a regime of SOC being weak, in the following sense. Let \( r_0 \) be the range of interaction without SOC, the energy scale associated with SOC, \( s_E \equiv \hbar^2 k_{s_{0}}^2/2\mu \), is generally much smaller that the energy scale associated with the shorter-range interactions, \((\hbar^2/2\mu)(1/r_0^2)\), which for atoms would be the van der Waals energy scale [35]. This criterion, which is equivalent to a length scale separation, \( 1/k_{s_{0}} \gg r_0 \), basically ensures that the SOC and other interactions are important in different regions and are not important simultaneously [30, 31]. Under such a condition, scattering in the presence of SOC can be solved in terms of scattering in the absence of SOC. Specifically, the \( K \) matrix, as defined by Eq. (4), in a region of \( r_0 \ll r \ll 1/k_{s_{0}} \), to inner solutions for which the SOC is negligible. This is conceptually similar to the multiscale quantum-defect treatment of two atoms in a trap [31].

For energies much greater than \( s_E \), we obtain the \( K \) matrix to be given by the \( K \) matrix in the \( \{F,l\} \) basis through a frame transformation. For \( F_1 = 0 \), e.g., we obtain
\[ K^{F_1=0} = U^{F_1=0} \begin{pmatrix} \tan \delta_{F=0} & 0 \\ 0 & \tan \delta_{F=1} \end{pmatrix} U^{F_1=0}^\dagger, \]
where \( U^{F_1=0} \) is a global unitary matrix
\[ U^{F_1=0} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}. \]
This result, together with similar results for other total angular momenta, has a very simple physical interpretation. It states that for energies much greater than the SOC energy scale, SOC has no effect on the scattering dynamics, except to facilitate the preparation and detection of particles in the helicity basis. In the absence of SOC, the same \( K \) matrix describes scattering in the helicity basis, and is applicable for all (positive) energies.

For energies comparable or smaller than \( s_E \), the length scale separation ensures that we are well into the region dominated by the \( s \) wave scattering, which is well characterized, for the vast majority of systems, by the universal behaviors of \( \tan \delta_{F=0} \sim -a_F/k_0 \) and \( \tan \delta_{F=1} \sim 0 \). In this case, we obtain
\[ K^{F_1=0} = -\frac{a_{F=0}}{k_3 + k_1} \begin{pmatrix} k_1^2 & -k_3 k_1 \\ -k_3 k_1 & k_3^2 \end{pmatrix}. \]

This result also represents the analytic solution of Eq. (4) for the pseudopotential model [30] of \( V(0) = \frac{4\pi\alpha F_0}{m} \delta(r) \partial \rho/\partial r \) and \( V(1) \equiv 0 \), which is equivalent to imposing the boundary conditions of \( G^{F_1=0}_{F_1=0} = G^{F_1=1}_{F_1=1} \sim 0 \) \( A(1 - a_F/r_0) \) and \( G^{F_1=0}_{F_1=1} \sim 0 \). The multiscale QDT approach contains the pseudopotential results [31]. It is more general and leaves room for future generalizations, including both the cases of non-universal behavior around \( a_F = 0 \) [32] and the case of much stronger SOC, the treatment of the latter would be similar to the treatment of hyperfine effects in atomic scattering [31].
The cross sections for ultracold collision with SOC with (solid line) and without SOC (dash-dot line), as a function of $k$. The result with SOC is guaranteed by the time-reversal symmetry to be valid at all energies. The result without SOC is guaranteed by the combination of time-reversal and parity conservations at all energies. The difference is due to the break of parity conservation by SOC.

The $K$ matrix of Eq. (11) gives the following set of cross sections for ultracold collision with SOC

$$
\sigma[(+,+)\rightarrow(+,+)] = \sigma[(-,-)\rightarrow(+,+)] = 8\pi a_F^2 k_1^2 \left( \frac{1}{k_3 + k_1} \right)^2, \quad (12a)
$$

$$
\sigma[(-,-)\rightarrow(-,-)] = \sigma[(+,-)\rightarrow(-,+)] = \frac{k_2^2}{(k_3 + k_1)^2 + a_F^2 (k_3^2 + 4 k_1^2)}, \quad (12b)
$$

In comparison, the cross sections in the absence of SOC, determined by the $K$ matrix of Eq. (1) in the helicity basis, are given in the $s$ wave region by

$$
\sigma[(+,+)\rightarrow(+,+)] = \sigma[(+,+)\rightarrow(-,-)] = \sigma[(-,-)\rightarrow(+,+] = \sigma[(-,-)\rightarrow(-,-)] = \frac{2\pi a_F^2}{1 + a_F^2 k_3^2}, \quad (13)
$$

which all follow the Wigner threshold behavior [32] of $\sigma \sim \text{const.}$

Equations (12)-(13) are the main results of this work. They represent the universal behaviors followed by the vast majority of spin-$\frac{1}{2}$ fermionic systems in the ultracold regime. The strength of SOC only affects length and energy scaling, and with proper scaling, different systems differ from each other only in a single dimensionless parameter of $\eta_{so} \equiv k_3 a_F k_3$, with $\eta_{so} = \infty$ corresponding to the unitarity limit.

We focus here on two aspects of physics contained in these results. (a) The SOC has substantially modified the threshold behavior, from the Wigner threshold law of $\sigma \sim \text{const.}$ for all cross sections, to

$$
\sigma[(+,+)\rightarrow(+,+)] = \sigma[(-,-)\rightarrow(+,+)] \sim \frac{\pi a_F^2}{2k_3^2} k^4, \quad (14a)
$$

$$
\sigma[(-,-)\rightarrow(-,-)] = \sigma[(+,-)\rightarrow(-,+)] \sim 8\pi a_F^2 k, \quad (14b)
$$

implying that the $(+,+)$ interaction is dominated by inelastic collision into the $(-,-)$ channel, while $(-,-)$ interaction is dominated by elastic collision. (b) Particles are preferably scattered into the lower-energy helicity state, the “$-$” state, as reflected by $\sigma[(+,-)\rightarrow(-,-)]$ being always greater than $\sigma[(+,+)\rightarrow(+,+)]$. More specifically

$$
\frac{\sigma[(+,+)\rightarrow(-,-)]}{\sigma[(+,-)\rightarrow(+,+)]} = \frac{k_3^2}{k_1^2} = \left( \frac{\sqrt{1 + (k/k_3)^2} + 1}{\sqrt{1 + (k/k_3)^2} - 1} \right)^2 > 1, \quad (15)
$$

and diverges as $1/k^4$ around the threshold. Equation (15) for the ratio of inelastic cross sections is applicable not only in the ultracold region, but at arbitrary energy as a result of time-reversal symmetry [37]. Its implication is best understood by noting that in the absence of SOC, the two inelastic cross sections are strictly equal at all energies as guaranteed by the combination of time-reversal and parity conservations. The two universal ratios are compared in Fig. 2. In an ultracold sample with SOC, the $(+,+) \rightarrow (-,-)$ state has a finite cross section to be converted into $(+,+)$, the $(+,-)$ and $(-,+) \rightarrow (-,-)$ interactions are negligible, and the $(-,-)$ state interacts mostly elastically, namely remains in $(-,-)$. Independent of the initial statistical distribution, such a system will evolve into a steady state made of mostly particles in the lower-energy helicity “$-$” state. In other words, a system with a preferred chirality would develop spontaneously through interaction.

In conclusion, we have developed a general formalism for the scattering of two spin-$\frac{1}{2}$ particles in the presence of isotropic SOC. We believe it to be the first rigorous formulation for scattering in a non-Abelian gauge field. We have derived the universal analytic results in the ultracold regime and discussed their implications. Many of the concepts introduced are generally applicable, and provide important guidance for investigations of other spin systems and anisotropic SOC. The theory is part of an essential foundation for understanding interacting many-body and few-body systems with SOC.
This work was supported by NSFC (No. 91121005 and No. 11004116), MOST 2013CB922000 of the National Key Basic Research Program of China, and the research program 2010THZO of the Tsinghua University.

∗lyou@mail.tsinghua.edu.cn
†bo.gao@utoledo.edu

[1] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. 82, 1225 (2010).
[2] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
[3] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 80, 1215 (2008).
[4] T. Kraemer, M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nägerl, and R. Grimm, Nature 440, 315 (2006).
[5] E. Braaten and H.-W. Hammer, Physics Reports 428, 259 (2006).
[6] C. H. Greene, Physics Today 63, 40 (2010).
[7] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, Rev. Mod. Phys. 83, 1523 (2011).
[8] Y.-J. Lin, K. Jiménez-García, and I. B. Spielman, Nature 471, 83–6 (2011).
[9] P. Wang, Z.-Q. Yu, Z. Fu, J. Miao, L. Huang, S. Chai, H. Zhai, and J. Zhang, Phys. Rev. Lett. 109, 095301 (2012).
[10] L. W. Cheuk, A. T. Sommer, Z. Hadzibabic, T. Yefsah, W. S. Bakr, and M. W. Zwierlein, Phys. Rev. Lett. 109, 095302 (2012).
[11] J.-Y. Zhang, S.-C. Ji, Z. Chen, L. Zhang, Z.-D. Du, B. Yan, G.-S. Pan, B. Zhao, Y.-J. Deng, H. Zhai, S. Chen, and J.-W. Pan, Phys. Rev. Lett. 109, 115301 (2012).
[12] C. Qu, C. Hamner, M. Gong, C. Zhang, and P. Engels, arXiv:1301.0658.
[13] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
[14] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
[15] T.-L. Ho and S. Zhang, Phys. Rev. Lett. 107, 150403 (2011).
[16] J. P. Vyasanakere and V. B. Shenoy, Phys. Rev. B 83, 094515 (2011).
[17] Z. F. Xu, R. Lü, and L. You, Phys. Rev. A 83, 053602 (2011).
[18] J. P. Vyasanakere, S. Zhang, and V. B. Shenoy, Phys. Rev. B 84, 014512 (2011).
[19] M. Gong, S. Tewari, and C. Zhang, Phys. Rev. Lett. 107, 195303 (2011).
[20] Z.-Q. Yu and H. Zhai, Phys. Rev. Lett. 107, 195305 (2011).
[21] H. Hu, L. Jiang, X.-J. Liu, and H. Pu, Phys. Rev. Lett. 107, 195304 (2011).
[22] Z. F. Xu and L. You, Phys. Rev. A 85, 043605 (2012).
[23] B. M. Anderson, G. Juzeliūnas, V. M. Galitski, and I. B. Spielman, Phys. Rev. Lett. 108, 235301 (2012).
[24] Z. F. Xu, Y. Kawaguchi, L. You, and M. Ueda, Phys. Rev. A 86, 033628 (2012).
[25] J. P. Vyasanakere and V. B. Shenoy, New Journal of Physics 14, 043041 (2012).
[26] X. Cui, Phys. Rev. A 85, 022705 (2012).
[27] P. Zhang, L. Zhang, and Y. Deng, Phys. Rev. A 86, 053608 (2012).
[28] P. Zhang, L. Zhang, and W. Zhang, Phys. Rev. A 86, 042707 (2012).
[29] R. A. Williams, L. J. LeBlanc, K. Jimenez-Garca, M. C. Beeler, A. R. Perry, W. D. Phillips, and I. B. Spielman, Science 335, 314 (2012).
[30] B. Gao, E. Tiesinga, C. J. Williams, and P. S. Julienne, Phys. Rev. A 72, 042719 (2005).
[31] Y. Chen and B. Gao, Phys. Rev. A 75, 053601 (2007).
[32] E. P. Wigner, Phys. Rev. 73, 1002 (1948).
[33] B. Gao, Phys. Rev. A 54, 2022 (1996).
[34] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., NIST Handbook of Mathematical Functions (NIST and Cambridge University Press, Cambridge, 2010).
[35] B. Gao, Phys. Rev. A 80, 012702 (2009).
[36] K. Huang and C. N. Yang, Phys. Rev. 105, 767 (1957).
[37] L. D. Landau and E. M. Lifshitz, Quantum Mechanics (Pergamon Press, Oxford, 1977).