Abstract

The standard approximation of a natural logarithm in statistical analysis interprets a linear change of $p$ in $\ln(X)$ as a $(1 + p)$ proportional change in $X$, which is only accurate for small values of $p$. I suggest base-$(1 + p)$ logarithms, where $p$ is chosen ahead of time. A one-unit change in $\log_{1+p}(X)$ is exactly equivalent to a $(1 + p)$ proportional change in $X$. This avoids an approximation applied too broadly, makes exact interpretation easier and less error-prone, improves approximation quality when approximations are used, makes the change of interest a one-log-unit change like other regression variables, and reduces error from the use of $\log(1 + X)$.

Keywords logarithms · regression · interpretation

1 The Traditional Interpretation of Logarithms

It is common practice in many statistical applications, especially in regression analysis, to transform variables using the natural logarithm $\ln(X)$. This can be done for statistical reasons, for example to fit an apparent functional form in the data or to reduce skew and the impact of positive outliers in the variable $X$. The logarithm transformation is also used for theoretical reasons, when theory dictates the model relates to proportional changes in $X$ rather than linear changes.

The standard interpretation of a log-transformed variable in a regression is that a linear increase of $p$ in $\ln(X)$ is equivalent to a $p \times 100\%$ increase in $X$. This is not literally true. A linear increase of $p$ in $\ln(X)$ is equivalent to an $(e^p - 1) \times 100\%$ increase in $X$. The standard interpretation relies on the approximation $e^p \approx 1 + p$, or equivalently $p \approx \ln(1 + p)$, which is fairly accurate for small values of $p$.

\[
\ln(X) + p \approx \ln(X) + \ln(1 + p) = \ln(X(1 + p))
\]

In this paper, I provide an very simple alternative approach to using and interpreting logarithms in the context of regression analysis, which solves three major problems with the standard approach.

The first major problem with the standard approach is that the approximation loses quality relatively quickly as $p$ grows. The error in approximation is equal to

\[1 + p - e^p\]
which is always negative, and grows more negative with \( p \), such that this approximation always understates
the proportional increase in \( X \) equivalent to a given linear increase in \( \ln(X) \). If \( \ln(X) \) is a treatment variable, the
approximation will always overstate its effect.

The quality of approximation is, subjectively, acceptable for small values of \( p \), but the error becomes large
within ranges of interest. For linear increases in \( \ln(X) \) of .1, .2, or .3, respectively, interpretations of these
changes as 10%, 20%, or 30% increases in \( X \) would underestimate the actual change in \( X \) by about .5, 2.1, and 5
percentage points, respectively.

This leads naturally to the second problem with the standard interpretation, which is sociological in nature.
The fact that the base-\( e \) approximation breaks down quickly as \( p \) increases does not appear to be universally
known, nor is there a standard maximum \( p \) for which the approximation is considered acceptable. It is not
uncommon to see papers describing 10% or 20% changes in a log-transformed variable, and it is doubtful that
these authors would willingly inject biases of .5 or 2.1 percentage points into their analysis for no reason.

Calculations that give exact interpretations using \( e^p \) or \( \ln(1 + p) \) are available but are not universally applied
even for larger percentage changes. Speculatively, this may be because the author assumes that approximation
error is too small to bother, because the additional calculation could confuse a reader, or because in some
fields it is not expected. It may even be the case that the researcher is not aware that the traditional
interpretation is an approximation. It is common in published studies and in econometric teaching materials
to see the traditional \( \ln(X) + p \approx \ln(X(1 + p)) \) approximation discussed without reference to its approximate
nature, implying by omission that it is an equality. At the undergraduate level, see for example Bailey (2017)
Section 7.2. At the graduate level, see Greene (2008) Section 4.7, specifically Example 4.3.

The third problem with the standard interpretation, when applied to variables on the right-hand side of
a regression, is that it requires nonstandard interpretation of regression coefficients. Nearly all regression
coefficients are understood in terms of one-unit changes in the associated predictor. Log-transformed variables
are an exception to this. A one-unit change in \( \ln(X) \) describes what would in most cases be an unrealistically
large increase in \( X \), and also would produce a large 71.8 percentage point error if the traditional approximation
were applied.

All three of these problems can be solved by simply changing the base of the logarithm\(^2\) Selecting a percentage
increase \( p \times 100\% \) ahead of time and using \( \log_{1+p}(X) \) in place of \( \ln(X) \) means that a one-unit change in
\( \log_{1+p}(X) \) is exactly equivalent to a \( p \times 100\% \) increase in \( X \). There is no need for approximation, and the
exact interpretation can be written directly into a regression table, solving the first problem and much of the
second. The use of base 1 + \( p \) can be restated as \( \ln(X)/\ln(1+p) \), framing the method in the easily-understood
terms of linearly rescaling the variable by the constant 1/\( \ln(1+p) \). The third problem is also solved because
the relevant increase in \( \log_{1+p}(X) \) is 1, in line with other variables in the regression.

Based on these results, I recommend the use of alternate logarithmic bases, or the “linear rescaling” approach,
when using logarithms in statistical analysis, especially in regression. The benefits are most apparent when
applied to variables on the right-hand side of a regression (the predictors/independent variables), but there
are also benefits on the left-hand side (the outcome). Additionally, the use of linear rescaling helps ease some
problems related to the use of \( \ln(1 + X) \) with \( X \) variables that contain values of zero.

2 Linearly Rescaling Logarithms

2.1 Exact Interpretation of Logarithms

For any base \( b \), a linear increase of \( p \) in a logarithm \( \log_b(X) \) is equivalent to a proportional change in \( X \)
of \( bp \), or a percentage increase of \( (bp - 1) \times 100\% \). Researchers could report exact interpretation of linear
logarithmic increases using this formula. However, this practice is far from universal.

\(^2\)The approximation error expression also demonstrates that \( e \) does not hold any special property in regards to
reducing approximation error, and is actually a poor choice of logarithmic base if the traditional approximation is to
be used. Many other bases, like 2.6, produce nearly identical errors for small \( p \) and then dominate \( e \) afterwards. Base
2.35 is attractive in other ways: errors for base 2.35 are no larger in absolute value than .014 all the way up to \( p = .43 \),
and considerably improve on base \( e \) above that, although unfortunately performance relative to base-\( e \) is worst around
\( p = .1 \). If a researcher insists on the traditional approximation, I at least recommend the use of base-2.6 logarithm,
which is a clear improvement on \( e \), or perhaps base-2.35 for something less sensitive to \( p \).

\(^3\)After a literature search, I was unable to find previous studies making this same recommendation. However, given
the long history of logarithms in regression, it seems unlikely that nobody has thought of the insight in this paper
before. So I will not claim that this method is novel, but just that it is currently not widely known or applied.
Another approach to exact interpretation of linear logarithmic increases is to take advantage of the following feature of $b^p - 1$:

$$(1 + p)^1 - 1 = p$$

That is, a linear increase of 1 in $\log_{1+p}(X)$ is exactly equivalent to a percentage increase of $p \times 100\%$ (or a proportional increase of $1 + p$) in $X$.

This means that a researcher can select ahead of time the percentage increase in $X$ that they are interested in, for example 10% (although any other percentage would work as well), and then use $\log_{1.1}(X)$ in their analysis instead of $\ln(X)$. Then, a one-unit increase in $\log_{1.1}(X)$ can be exactly interpreted as a 10% increase in $X$.

Further, because of the change of base formula, $\log_{1.1}(X)$ can be calculated as $\ln(X)/\ln(1.1)$. $\ln(1.1)$ is a constant, and so the researcher can achieve exact interpretation of the change by scaling the variable they’re already using ($\ln(X)$) by a constant. Researchers using any estimation method should already be aware of the implications of scaling by a constant in that method, so the linear rescaling approach to changing the base should be understandable by both researchers and readers of research. The use of the change-of-base formula also allows another clear demonstration of what the choice of logarithm base does for interpretations of proportional change:

$$\frac{\ln(X(1 + p))}{\ln(1 + p)} = \frac{\ln(X) + \ln(1 + p)}{\ln(1 + p)} = \frac{\ln(X)}{\ln(1 + p)} + 1$$

There are several benefits to linear rescaling:

- It produces exact interpretations of linear increases in logarithms.
- It is, arguably, conceptually more simple than using $b^p - 1$ or $\log_b(1 + p)$ to adjust the result after estimation, and avoids the introduction of another calculation where error may occur.
- The change of interest is one log unit, which is how most other variables are understood.
- The exact interpretation can be written directly onto a regression table rather than relying on supplemental calculations, as will be shown in the following sections.

The main downside of this approach is that it requires the choice of the percentage increase beforehand. In practice, this is unlikely to be a major hurdle, as researchers often report only a single percentage increase value anyway, typically a preselected value like 10%, based on what a reasonable observable change in $X$ would be.

Additionally, if selecting a percentage increase beforehand is not realistic, or if multiple percentage increases are desired, exact interpretation is still available, as in the traditional method, using $(1 + b)^p - 1$. Linear rescaling also improves the process of adjusting the estimate, at least on the right-hand side of a regression (see Section 2.2).

Further, if an approximation is used instead of an exact interpretation, linear rescaling often produces better approximations than the traditional approximation. For example, if $\log_{1.1}(X)$ has been used and the researcher wants to approximate a 20% increase in $X$ using a two-unit increase in $\log_{1.1}(X)$, they will actually see the effects of a $1.1^2 = 1.21$, or 21%, increase, rather than 20%. This is an error of one percentage point, compared to the traditional approach, which produces an error of 2.1 percentage points for a .2 increase.

Figure 1 examines the error in approximating different percentage increases with a traditional approximation, using a base-$e$ logarithm where a $p$-unit increase in $\ln(X)$ is taken to be a $p \times 100\%$ increase in $X$. I contrast the traditional approximations with approximations from the linear rescaling approach using two different bases: a base-1.1 logarithm, where a $p$-unit increase in $\log_{1.1}(X)$ is taken to be a $p \times 10\%$ increase in $X$, and a base-1.4 logarithm, where a $p$-unit increase in $\log_{1.4}(X)$ is taken to be a $p \times 40\%$ increase in $X$.

The traditional base-$e$ method slightly outperforms the base-1.1 linear rescaling method, by a miniscule degree, up to a linear change of .048 (although the linear rescaling method could outperform the traditional method for any linear change by selecting a different base than the ones shown in the graph). After .048, approximation with base-1.1 linear rescaling dominates the traditional approximation, especially near $p = .1$. Both the traditional and base-1.1 approximations considerably outperform base-1.4 for small $p$, but this is to be expected - base-1.4 is to be used when the change of interest is 40%. In the region of $p = .4$, the base-1.4 logarithm considerably outperforms the other two, and approximation errors with base-1.4 are relatively
Linear Rescaling to Accurately Interpret Logarithms

2.2 Linear Rescaling on the Right Hand Side

The benefits of linear rescaling are clearest when applied to a logarithmic transformation on the right-hand side of a regression. Considering the model:

$$ Y = \beta_0 + \beta_1 \ln(X) + \varepsilon $$

the interpretation of $\hat{\beta}_1$ is often given in a format similar to “a 10% increase in $X$ is associated with a $0.1 \times \hat{\beta}_1$ increase in $Y$.” Or for an exact interpretation, “a 10% increase in $X$ is associated with a $\ln(1.1) \times \hat{\beta}_1$ = $0.0953 \times \hat{\beta}_1$ increase in $Y$.”

Under linear rescaling for a 10% increase in $X$, instead the model is:

$$ Y = \beta_0 + \beta_1 \frac{\ln(X)}{\ln(1.1)} + \varepsilon $$

in which the interpretation of $\hat{\beta}_1$ is the simpler “a 10% increase in $X$ is associated with a $\hat{\beta}_1$ increase in $Y$.”

The interpretation is simple enough that it can be written directly into a regression table, as in Table 1 Column 1, rather than relying on additional in-text calculations. The row heading in the table itself “$X$ (10% Increase)” is able to convey that a 10% change in $X$ is associated with a $\hat{\beta}_1$ change in $Y$, and the table note provides more detail. The label “log$_{1.1}$(X) (10% increase)” may be preferred.

If the researcher is interested in multiple percentage changes, adjustment to get exact interpretation for each percentage change is straightforward under linear rescaling, because of the change-of-base formula. If log$_{1.1}$(X) has been used, but the researcher wants an exact interpretation for the linear change equivalent to a 20% increase in $X$, the researcher can multiply the coefficient $\hat{\beta}_1$ by $\ln(1.2)/\ln(1.1)$. This will work for any regression method where scaling a predictor by $c$ has the result of scaling its coefficient by $1/c$.

2.3 Linear Rescaling on the Left Hand Side

The benefits of linear rescaling are less clear on the left-hand side of the regression, since the proportional change of interest cannot be exactly chosen ahead of time. Still, there are benefits.
Consider Table 1 Column 2, which uses the model

\[
\frac{\ln(Y)}{\ln(1.1)} = \beta_0 + \beta_1 X + \varepsilon
\]

The regression estimate is \( \hat{\beta}_1 = 2.001 \). By itself, this does not easily lead to exact interpretation of the coefficient.

At this point, the researcher can approximate the effect of 2.001 as a 2.001 \times 10\% \approx 20\% increase. As in Section 2.1, this will lead to less approximation error than in the traditional method provided that \( \hat{\beta}_1 \) is not too far from 1.

There is also the option to provide an exact interpretation, where a one-unit increase in \( X \) is associated with a \( b^{\hat{\beta}_1} \) proportional change in \( Y \). This is not much different from the process for getting exact interpretation using the traditional approach, although it may be somewhat easier to understand if \( p \) is a more natural object to think about than \( e \).

A third option is to rerun the model with a different logarithmic base such that the \( \hat{\beta}_1 \) will be near 1. Then, \( \hat{\beta}_1 \) can be very accurately interpreted as a proportional increase of \( b \) in \( Y \). However, this is both laborious and would result in the coefficient of interest being oddly located in the logarithmic base.

One note about left-hand side use is that the traditional approximation is known to perform particularly poorly, and exact interpretation is especially important, when the logarithm is on the left-hand side and a predictor of interest is binary (Halvorsen & Palmquist, 1980). In theoretical terms this is because the derivative does not exist. In practical terms this is because, if the coefficient on the binary variable is large, the researcher cannot naturally select a linear change small enough for the approximation to perform well. Easy access to exact interpretation, and improved approximation when used, are especially important in this case.

### 2.4 Linear Rescaling on Both Sides

In the case of the log-log model

\[
\ln(Y) = \beta_0 + \beta_1 \ln(X) + \varepsilon
\]

linear rescaling on both the left and right-hand sides using the same bases will have no effect on \( \hat{\beta}_1 \) or on its interpretation, and offers no major improvement over the traditional method, unless there is a reason to want different bases on the left and right-hand sides, or if there is interest in interpreting the coefficients on control variables with a linearly-rescaled left-hand side.

There are some minor expositional benefits. Linear rescaling could be used here for consistency with other models that are not log-log. Rescaling can also make clear to an audience unfamiliar with log-log models how they can be interpreted. For example, it’s not uncommon in a log-log model to still report a result like “a 10\% increase in \( X \) is associated with a \( \hat{\beta}_1 \times .1\% \) change in \( Y \)” Herz & Mejer (2016) is just one example of this. If the author wants the reader to think in terms of a percentage increase of a particular size in this way, linear rescaling can make that interpretation explicit on the regression table, as in Column 3 of Table 1.
2.5 Linear Rescaling and Zeroes

Researchers often want to apply a logarithmic transformation to a variable that can take values of zero. There are two common approaches to this: \( \ln(1 + X) \), and the asymptotic hyperbolic sine transformation \( \text{asinh}(X) = \ln(X + \sqrt{X^2 + 1}) \). Exact calculations for elasticity interpretations using the \( \text{asinh}(X) \) transformation are described in Bellemare & Wichman (2020). Both \( \ln(1 + X) \) and \( \text{asinh}(X) \) reduce skew and accept values of zero. However, if the researcher wishes to maintain a proportional-change interpretation (at values other than zero), there are problems with any sort of ad-hoc transformation like this, including a sensitivity to the scale of \( X \) and the fact that the zero-censored variable is treated as uncensored. For a left-hand side variable, poisson regression or a censoring model are likely to be superior to an ad-hoc transformation. However, the use of an ad-hoc transformation is still a concern on the right-hand side or for researchers who want to use standard linear regression for other reasons.

In this section, I will assume that the researcher’s goal is interpret a linear change in \( \ln(1 + X) \) in terms of a proportional change in \( X \) (rather than a proportional change in \( 1 + X \)). In this case, exact interpretation is particularly important, whether performed using linear rescaling or \( e^p \), but there are several details that still separate the two approaches.

Consider a one-unit increase in \( \ln(1 + X)/\ln(1 + p) \). This is equivalent to a proportional change of \( 1 + p \) in \( 1 + X \), or an absolute increase of \( p(1 + X) \). If \( X \) increases by \( p(1 + X) \), what proportional increase is that equivalent to?

\[
X + p(1 + X) = X(1 + p + \frac{p}{X}) \tag{2}
\]

A one-unit linear increase in \( \ln(1 + X)/\ln(1 + p) \) interpreted as a \( 1 + p \) proportional change in \( X \) will get the proportional change wrong by \( \frac{p}{X} \).

Similarly, a linear increase of \( p \) in \( \ln(1 + X) \) is a proportional increase of \( e^p \) in \( 1 + p \). As above,

\[
X + (e^p - 1)(1 + X) = X(e^p + \frac{e^p - 1}{X}) \tag{3}
\]

A linear increase of \( p \) in \( \ln(1 + X) \) interpreted as a proportional increase of \( e^p \) in \( X \) will get the proportion wrong by \( \frac{e^p - 1}{X} \).

For a given \( p \), since \( p \leq e^p - 1 \) \( \forall p \geq 0 \), the linear rescaling approach will always outperform the traditional approach, and by a greater margin as \( p \) increases. However, this is due to the fact that for a given \( p \), the traditional method describes a proportional change of \( e^p \geq 1 + p \). For a given proportional change, for example comparing a linear increase of \( 1 \) under linear rescaling to a linear increase of \( \ln(1 + p) \) in the traditional method, both methods perform identically.

There are still several points to recommend linear rescaling here. First, linear rescaling is an improvement if researchers using the traditional method first select a \( p \) of interest and then calculate \( e^p \), rather than selecting \( e^p \) directly (since, as above, \( p \leq e^p - 1 \) \( \forall p \geq 0 \)). Second, a bias of \( p/X \) may be easier to reason about than \( (e^p - 1)/X \).

The comparison so far assumes that the researcher using the traditional approach uses the exact interpretation, where a linear increase of \( p \) is a proportional increase of \( e^p \). If they instead interpret a linear increase of \( p \) as a proportional change of \( 1 + p \) in \( X \) using the \( e^p \approx 1 + p \) approximation, the error will be

\[
e^p - (1 + p) + \frac{e^p - 1}{X}
\]

There are two problems here. First, the fact that linear rescaling and the traditional method perform identically for a given proportional increase doesn’t matter, as by using the approximation the researcher has chosen to fix \( p \), under which linear rescaling outperforms the traditional method. Second, the use of the approximation adds the traditional approximation error \( e^p - (1 + p) \) on top of \( \frac{e^p}{X} \), further increasing error relative to the linear rescaling method, and making the error grow even faster in \( p \).

For either the linear rescaling or traditional approaches, the recommendation from Bellemare & Wichman (2020) to scale \( X \) upwards in reference to \( \text{asinh}(X) \), elaborated upon in Aihounotun & Henningsen (2019) to determine optimal scaling values, is also implied by these results for \( \ln(1 + X) \), as the interpretation error declines proportionally with larger absolute values of \( X \).
2.6 Exact interpretation with zeroes

As an aside (since it does not relate to linear rescaling in particular), a researcher using either the linear rescaling or traditional method could use one of the error formulas in Equations 2 or 3 to adjust their proportional-change interpretation, or decide whether the error is small enough to ignore, for a given \( p \) and \( X \).

This may be especially useful in the calculation of elasticities, since it allows proportional changes in both \( Y \) and \( X \) to be recovered from proportional changes in \( 1 + Y \) and \( 1 + X \).

For example, in the log-log model

\[
\frac{\ln(1 + Y)}{\ln(1 + p_Y)} = \beta_0 + \beta_1 \frac{\ln(1 + X)}{\ln(1 + p_X)} + \varepsilon
\]

a \( 1 + p_X \) proportional change in \( 1 + X \) is associated with a \((1 + p_Y)^{\beta_1}\) proportional change in \( 1 + Y \). For the specific values \( X = X_0 \) and \( Y = Y_0 \), this means that a \( 1 + p_X + \frac{p_X}{X_0} \) proportional change in \( X \) is associated with a \((1 + p_Y)^{\beta_1} + \frac{(1+p_Y)^{\beta_1}-1}{Y_0} \) proportional change in \( Y \). If desired, \( p_X \) and \( X_0 \) can be selected ahead of time so that \( 1 + p_X + \frac{p_X}{X_0} \) is a round number. Similar calculations follow for the \( \ln(1 + X) \)-linear and linear-\( \ln(1 + X) \) cases. The only task remaining at that point is calculation of standard errors for this nonlinear function of \( \beta_1 \). The delta method is one acceptable approach, at least in large samples.\(^4\)

3 Example Applications

In this section I refer to several published studies that use natural logarithm transformations in regression analysis, and discuss how those papers might have been different using linear rescaling.

The first study I look at, which is from economics, is Eren, Onda, & Unel (2019). This study looks at the impact of foreign direct investment (FDI) on entrepreneurship in the United States, covering business-creation, business-destruction, and self-employment rates as outcome variables. The authors use a natural logarithmic transformation of FDI, and the headline result specified in the abstract is “A 10% increase in FDI decreases the average monthly rate of business creation and destruction by roughly 4 and 2.5% (relative to the sample mean), respectively,” where these figures refer to results in states without Right-to-Work laws. Despite the percentage interpretation given for the effects, business creation and destruction are not log-transformed, so that \( \hat{\beta}_1 \) is associated with a \( (1 + p_Y)^{\hat{\beta}_1} \) proportional change in \( Y \). If desired, \( p_X \) and \( X_0 \) can be selected ahead of time so that \( 1 + p_X + \frac{p_X}{X_0} \) is a round number. Similar calculations follow for the \( \ln(1 + X) \)-linear and linear-\( \ln(1 + X) \) cases. The only task remaining at that point is calculation of standard errors for this nonlinear function of \( \beta_1 \). The delta method is one acceptable approach, at least in large samples.\(^4\)

This study makes use of the traditional approximation. The result for business creation comes from a regression of business creation rates on \( \ln(FDI) \) (lagged two years), where the coefficient on \( \ln(FDI) \) is -1.083. They interpret this as a 10% increase in FDI reducing the business creation rate by .1083 (or perhaps they round the effect to .11 before proceeding, this is not clear), which is roughly 4% of the mean (.1083/2.889 = .0375 or .11/2.889 = .038). Under linear rescaling with a base of 1.1, the coefficient on \( \log_{1.1}(FDI) \) would be .1032, which indicates an effect roughly 3.5% of the mean (.1032/2.889 = .0357), keeping in mind that the abstract reports its other effect to the half-percent level of precision. In this case, the use of the approximation made the effect seem larger than it was. The correction is not enormous, but still there is no particular reason the correct result could not have been in the original study. Traditional exact interpretation or linear rescaling would have avoided this problem. However, linear rescaling in this case offers the additional benefit over traditional exact approximation that, if the dependent variable had been divided by its mean, the coefficient on \( \log_{1.1}(FDI) \) would have been the value of interest – .0357 directly, and no further calculation would have been necessary. This ability to put the value of interest on the table also applies to the 2.5% result, although in this case the substantive reported conclusion does not change at the half-percent level of precision (the effect to two decimal places drops from to 2.55% to 2.43%).

The second study is from public health. Kim & Leigh (2020) look at the effect of wages on obesity rates, finding that low wages increase body mass and the prevalence of obesity. This study uses traditional exact interpretation, so linear rescaling cannot change its results, but it may change how they are presented. Their...
instrumental variables estimate with BMI as a dependent variable reads “the coefficient on ln-wage was statistically significant \((p < 0.01)\) and its value was \(-3.3\). Standard errors appear in parentheses. This coefficient suggests that a 10% increase in wages is associated with 0.32 decline in BMI.” The \(0.32\) appears to be a slight miscalculation and can be derived from \(\ln(1.1) \times (-3.3) = -0.3145\), which should round to \(-0.31\), not \(-0.32\). Under linear rescaling with a base of 1.1, this additional calculation would not need to appear in the text, and the coefficient on \(\log_{1.1}(\text{Wage})\) would be rounded to \(-0.3145\) or \(-0.315\), which would also avoid the slight miscalculation. The authors also report the effect of a 100% increase in wages as a 2.29 decline in BMI \((\ln(2) \times (-3.3) = -2.29)\). To achieve this with linear rescaling, the authors either could rerun analysis using a log base of 2, or, more realistically, retain the calculation in the text for this additional result, adjusting the 10% estimate by \(-0.3145 \times \ln(2)/\ln(1.1) = -2.29\). This simplified presentation for the BMI results could be similarly applied in analysis of the obesity rate dependent variable.

The final study I will mention is an example of how the complex calculations necessary to produce exact interpretations of logarithms using the traditional method can lead to error. Lin, Teymourian, & Tursini (2018) look at the effect of the natural logarithm of sugar and processed food imports on the prevalence of obesity and overweight. In one model the coefficient on logged imports is \(0.085\), which is interpreted as “10% increase in import is associated with approximately 0.004 increase in average BMI.” However, the effect should instead be of a \(0.0081\) increase in BMI using exact interpretation, or \(0.0085\) using the traditional approximation, an effect more than twice as large. Similar errors are made in another table, where a coefficient of \(0.004\) is again interpreted as having about half the appropriate effect size for a 10% increase. They also provide an interpretation of a 50% increase, which is about 40% the appropriate size. If the authors had used linear rescaling, they would not have needed to produce these calculations, and there would have been no potential for this error to occur. Instead, the effect size of interest would have automatically been produced by the statistics software.

4 Conclusion

Despite their wide use across many fields, both in the context of regression and in other statistical applications, logarithms are frequently misinterpreted to greater or lesser degrees. This is partly due to the standard interpretation of logarithms, which relies on an approximation that produces non-negligible errors outside of a fairly narrow range of percentage increases. It is possible to skip the approximation and instead accurately interpret a linear increase \(p\) in \(\ln(X)\) as a proportional increase of \(e^p\) in \(X\). However, this practice is not universal, and carries a chance of error.

The interpretation of logarithms can be improved by changing the base of the logarithm to \(1 + p\), where \(1 + p\) is a proportional change of interest. The change of base can be achieved by linearly rescaling \(\ln(X)\) to \(\ln(X)/\ln(1 + p)\). This rescaling offers a way of producing exact interpretations of logarithms, or in some cases approximations with less error than the traditional approximation.

Linear rescaling can be easier for a reader to understand on a regression table. The one-unit log increase that accompanies a given percentage increase can be understood in the same way as changes in untransformed variables. In order to interpret the coefficient the reader does not need to search for additional calculations in the text, nor does the author need to perform and provide them.

Crucially, rescaling is as easy to perform and explain as the traditional approximation, and does not in most cases require the additional post-analysis calculations that usually go along with exact interpretations under the traditional method. All three of the studies covered in Section 3 contained either errors or misrepresentations of their results because of these post-analysis calculations. I did not select these studies because I knew they had problems or anticipated any; it just so happens that the first three studies I selected as good candidates for demonstrating the method all had problems that could have been avoided with linear rescaling.

Ease of use for the researcher is important, because it may help sidestep some of the reasons why researchers do not already report exact results, and may help avoid calculation errors with traditional exact interpretation. Researchers should in general be producing exact interpretations of logarithms and be aware of the extent of error in the traditional approximation. Because it is as easy as the traditional approximation, linear rescaling is an attractive way of providing exact results.
References

Aihounton, Ghislain BD, and Arne Henningsen. 2019. “Units of Measurement and the Inverse Hyperbolic Sine Transformation.” IFRO Working Paper.

Bailey, Michael A. 2017. Real Econometrics: The Right Tools to Answer Important Questions. Oxford University Press.

Bellemare, Marc F, and Casey J Wichman. 2020. “Elasticities and the Inverse Hyperbolic Sine Transformation.” Oxford Bulletin of Economics and Statistics 82 (1): 50–61.

Eren, Ozkan, Onda, Masayuki, and Bulent Unel. 2019. “Effects of FDI on Entrepreneurship: Evidence from Right-to-work and Non-right-to-work States.” Labour Economics 58: 98–109.

Greene, William H. 2008. Econometric Analysis. Pearson Education.

Halvorsen, Robert, and Raymond Palmquist. 1980. “The Interpretation of Dummy Variables in Semilogarithmic Equations.” American Economic Review 70 (3): 474–75.

Herz, Benedikt, and Malwina Mejer. 2016. “On the Fee Elasticity of the Demand for Trademarks in Europe.” Oxford Economic Papers 68 (4): 1039–1061.

Kim, DaeHwan, and John Paul Leigh. 2010. “Estimating the Effects of Wages on Obesity.” Journal of Occupational and Environmental Medicine 52 (5): 495–500.

Lin, Tracy Kuo, Teymourian, Yasmin, and Maitri Shila Tursini. 2018. “The Effect of Sugar and Processed Food Imports on the Prevalence of Overweight and Obesity in 172 Countries.” Globalization and health 14 (1): 1–14.