Approximate filtered back-projection algorithm for plane curves in tomography problems

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Abstract. In previous works by the author and colleagues, it was proposed in some tomography problems to convert the fan-shaped projection measurement scheme into a system of parallel rays by deforming reconstructed image. The deformation of the tomogram for each direction of observation will be different, but the one-to-one nature of these deformations allows one to return back to the original coordinate system. Further, this method is generalized to a family of plane curvilinear trajectories that allow one-to-one transitions to parallel rays. For each back projection, the image is modulated by a known function following from the path differential of a given trajectory. Illustrations of the operation of this algorithm are given on examples of parabolic, sinusoidal and fan-beam paths of rays. The applicability of the algorithm in problems of visualization of discontinuities, in problems of ROI-tomography (local tomography) is shown.

1. Introduction

The problems of reconstructive tomography are actively penetrating into various areas of scientific research, including diagnostics of gas, plasma, and industrial flaw detection research. Accordingly, the mathematical apparatus for these problems is developing at an accelerated pace: linear and nonlinear, vector and tensor tomography. The most developed such apparatus and algorithms for its implementation, when it is possible to record the radiation passing through the object along straight line trajectories, when the effects of refraction or diffraction are negligible.

The two-dimensional integral Radon transform (RT - Radon transform) allows, from a set of integrals of an unknown function along straight lines on a plane, to determine this function inside in a unit circle [1]. In practical applications, this transform has found wide application in physical tomography, X-ray tomography in medicine, as well as in a wide range of image processing problems [2]. The Radon transform makes it possible to automatically select and emphasize straight lines in photographs (using the Hough transform based on RT [3]), and the need to search structures other than straight lines poses the problem of generalizing RT to curvilinear trajectories.

In recent years, more and more attention has been paid to specific curvilinear ray trajectories - parabolas, hyperbolas, circular arcs [4-8]. In these works, various methods of curve parameterization are often used; in some cases, some special symmetry of the desired image is required, for example, vertical, as is used for the elliptic Radon transform [9]. The new field of Compton scatter tomography also leads to problems using arcs of circles [10-11].

In this paper, we propose a method for the approximate reconstruction of an image by curvilinear integrals, where a class of curves is chosen that are one-to-one related to straight lines. Geometric
transformations of such curves into straight lines, accompanied by the corresponding transformations of the reconstructed function (‘by deformation’), allows one to construct algorithms for approximate solutions, similar to those in ordinary rectilinear tomography.

2. Theory

An algorithm for converting curvilinear trajectories into rectilinear ones is investigated by the method used earlier for the filtering and backprojection (FOP) algorithm in fan-beam tomography [12-13]. This fan-beam tomography algorithm is used further as a test one, which can be used to check all stages of the proposed algorithm implementation in the formulation of curvilinear tomography.

As one knows, the Radon inversion formula in one of the most common form in applications looks like this [14]:

\[ g(x, y) = -\frac{1}{2\pi^2} \int_0^\pi d\beta \int_{-\infty}^{\infty} f_\beta(s) ds \]  \hspace{1cm} (1)

The same formula for fan-beam geometry of data collection transforms into a similar expression [6]:

\[ g(x, y) = -\frac{1}{4\pi^2} \int_0^{2\pi} d\beta \int_{-\infty}^{\infty} f_\beta(s) ds \]  \hspace{1cm} (2)

Here \( s_0 \) is the projection of the point (\( x, y \)) onto the S axis (figure 1), for the fan-beam system, \( S_m \) is the coordinate of the intersection of the ray tangent to the unit circle with the S axis, the parameters \( Q \) and \( s_0 \) are given by the formulas:

\[ Q(\beta) = 1 + (r/D) \cos(\beta - \varphi), \quad s_0 = (r/Q) \sin(\beta - \varphi), \quad \varphi = \beta + \gamma. \]  \hspace{1cm} (3)

In these formulas (\( r, \varphi \)) are the coordinates of the point B (\( x, y \)) in the polar coordinate system, D is the distance of the emitter F from the origin of coordinates O (figure 1). The projections \( f_\beta(s) \) used in the Radon inversion formula are modified by the formula \( f_\beta(s) = f_\beta(s) \cos(\gamma) \). For convenience, the recorded data from the real detector Det can be transferred to the origin of coordinates on the S axis parallel to the detector. For parallel geometry projections, all angles \( \gamma = 90^0 \) and the factor \( Q = 1 \), since in this case \( D \) tends to infinity.

The curvilinear ray trajectories considered in this work represent three types of families: sinusoids, parabolas and, as a test example with a well-known analytical solution (2), a bundle of fan-beam rays. In figure 2, all the curves are described by the equation \( y = s + \psi(x) \), where for the fan-beam the function \( \psi = kx, \quad k = s / (D + x) = \tan \gamma \) (a); for the waves in figure 2, (b) the function \( \psi(x) = a \cos (2\pi T^{-1} x) \); for a parabola, the family of curves is described by the function \( \psi(x) = (x-x_0)^2 \), where \( x_0 \) specifies the position of the vertex of the parabola (c).

![Figure 1. Fan-beam imaging geometry. A radiation source F generates fan-beams recorded by the detector Det.](image-url)
Figure 2. Examples of curve families for which there is a one-to-one transition to parallel horizontal lines. Types of families: (a) waves, amplitude $a = 0.1$; period $T = 1$; (b) parabolas, $a = 0.2$, $x_0 = 0$; (c) fan-beam, $D = 2R$. Curves: 1 - paths of integration, 2 - part of the path differential along the ray, $w(x)$.

Further, the projection recorded system is considered stationary, and the function $g_{\beta}(x, y)$ is rotated, $\beta$ is the clockwise image rotation angle, which corresponds to the counterclockwise rotation of the coordinate system $(p, s)$ attached to the object by the same angle $\beta$. Thus, for each fixed angle $\beta$, the Radon transform in the horizontal direction $s = \text{const}$ is represented as a curvilinear integral:

$$f_{\beta}(s) = \int g_{\beta}(x, y= s + \psi(x)) \, dL = \int q_{\beta}(x, y) \, dx,$$

$$dL = w(x) dx, \quad w(x) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$  \hspace{1cm} (4)

In formula (4), the function $q_{\beta}(x, y)$ denotes the product of a deformed tomogram $g^d$, which is different for each direction $\beta$, and a part of the path differential $w$.

For each of the families under consideration, we write down the corresponding formulas for the variable part of the path differential $w(x)$:

a) fan-beam: $w(x) = \sqrt{1 + \frac{dy}{dx}^2} \, dx = \sqrt{1 + \frac{y^2}{D^2}} \, dx = \frac{1}{\cos y}; \quad \tan(y) = \frac{y}{D+x} = \frac{s}{D};$

b) waves: $w(x) = \sqrt{1 + \left(2\pi T - 1\right)^2 \cdot \left(\sin(2\pi T - 1)\right)^2};$

c) parabolas: $w(x) = \sqrt{1 + 4 \cdot \left(a \cdot (x - x_0)^2\right)}$.

By analogy with the transition from the "parallel" Radon inversion (1) to the fan-shaped (2), proposed in [12-13], a method is proposed for the geometric solution of a similar problem for the families of curvilinear ray trajectories mentioned above. The geometric approach consists in a preliminary transition from the desired image $g(x, y)$ to its deformed version $g^d(p, s)$, in which curvilinear rays transform into parallel rays $s = \text{const}$. A horizontal straight line, to which the trajectory is anchored, is drawn through the point of intersection of this curve with the axis $p = 0$. In this case, in the forward problem, to calculate integrals over curves, it is sufficient to calculate them along parallel straight lines from the deformed image and take into account the contribution to the integral of the path differential $dL$ by the weight function $w(x)$.

3. Numerical simulation

Thus, the modified filtering and back projection (FBP) algorithm consists of the following steps.

1) Filtration of deformed projections $f_{\beta}(s)$ with a conventional filter in tomography (for example, Shepp-Logan [14]).

2) At the stage of back projection, the values of the filtered projection along the horizontal line are divided by the weight $w(p)$ at the points of this line $(p, s)$.

3) Inverse deformation is then applied to the corrected 2D back projection.

4) The image obtained for a given angle is rotated back by the same angle $\beta$, and interpolated from the grid $(p, s)$ to the grid of the reconstructed tomogram $(x, y)$.
After steps (1-4) for all projections, summation over the angles is performed to calculate the external integral in formula (1), which gives the desired tomogram \( g(x, y) \).

Note that the weighting function \( Q^2(\beta) \) in the inversion formula for the fan-beam (2) just carries out the inverse deformation of the filtered two-dimensional strip, which in this algorithm is carried out by bilinear interpolation from the grid \((p, s)\). Thus, the geometric interpretation of the reconstruction algorithm in fan-beam tomography is fully confirmed by the analytical formula of the inverse Radon transform (2) for fan-beam rays. We also emphasize that the proposed MFBP algorithm at the filtering stage is based on the well-known central slice Fourier theorem for the image of the reconstructed tomogram [14], which is valid in this case for the function \( q_\beta(x, y) \). This leads to the possibility of filtering projections by approximations to the inverse Fourier transform of the modulus of frequency \( |\nu| \) (for example, by the Shepp-Logan filter). However, the product \((g(x, y) w(x))\) results in convolution in the frequency domain, not in the spatial domain. Apparently, in addition to the main central frequency of the tomogram \(|\nu|=0\), additional frequencies from such a convolution also penetrate into the projection which can lead to artifacts in the image. Thus, the proposed algorithm should be considered approximate for curvilinear ray trajectories (except for the fan-beam observation scheme).

The following are some results of numerical modeling that illustrate the quality of reconstruction of tomograms from their wave projections using the MFBP algorithm.

The TM-247 model (figure 3) shows the type of deformations arising for curvilinear tomography on smooth models, usually inherent in the problems of tomographic diagnostics of gas or plasma flows.

![Figure 3](image)

**Figure 3.** Reconstruction of the TM-247 model by a new algorithm MFBP for curvilinear tomography along sine trajectories, using parallel projections from deformed tomograms. Curve parameters: \( a = 0.05, T = 1.0; S_{\text{max}} = 1.05; \) error norm RMS = 5.4%.

Here are the deformed projections along sinusoids (sinogram), the abscissa shows the values of the observation angles, from 0 to 360 degrees (figure 3, a). The ordinate is the values along the grid on the detector S. In figure 3, (b-e) are the exact model tomogram (b), its deformation (c), reconstruction by the MFBP algorithm (d), central cross-sections along the axes for the exact and reconstructed tomograms (e). Description of curves: 1 and 4 - horizontal sections for \( y = 0 \), exact tomogram and reconstructed,
solid lines 2 and 4 - vertical section \( x = 0 \); curve 3 is the first horizontal projection, \( \beta = 0 \). The tomogram size is 129x129, the number of views is 181, the number of detectors on each projection is \( K = 129 \), the reconstruction error is RMS = 5.4%.

![Figure 4](image)

**Figure 4.** Reconstruction of the discontinuous model TM-257 along wave trajectories, using the MFBP algorithm. Curve parameters: \( a = 0.05 \), \( T = 1.0 \); \( S_{\text{max}} = 1.05 \); error RMS = 12.5%.

Figure 4 illustrates the reconstruction of a complex discontinuous model (TM-257). Shown: the sinogram of the model (a), the exact tomogram (b), its wave deformation (c) and the result of the MFBP algorithm (d), with the Shepp-Logan filter and back projection along curvilinear trajectories, using inverse deformation. The parameters of the wave and the detection system are the same as in figure 3.

It can be seen that for curvilinear wave tomography characteristic wave structures appear in the polar coordinate system - along the radius and along the angle (figure 4, d). These artifacts appear at the stage of back projection, during the transition from rectilinear projections to deformed projections.

The result of the Fourier transform of the first (for \( \beta = 90^\circ \)) projection in figure 4, (e) (solid line 2) is also interesting. It coincides with the central slice of the two-dimensional Fourier transform of the deformed tomogram, i.e. function \( q_{\beta}(x, y) \) from formula (4), circles (1) in the figure. This result confirms the central slice theorem for a deformed tomogram.

The so-called “local tomography” or “lambda tomography” has found application in the problems of image processing, discontinuities reconstruction, and obtaining images of only a selected interesting part of image (ROI tomography) [15-22]. In the methods of local tomography, in contrast to the global one described by the inverse Radon transform, in order to reconstruct the image at each individual point, it is necessary to know the integrals only along the straight lines that pass through the interesting given point. Such an image can be obtained, for example, if the filtering procedure is canceled in the back projection operator (the inner integral in (1)), and instead of filtering, the back projection of projections is used directly. The resulting image is sometimes called the summary image [15] and it differs from the true tomogram by the convolution of this tomogram with the \( 1/r \) function, which leads to the demonstration of only low-frequency components of the tomogram.
In [16], it was also proposed to work without convolution, and for back projection, use the second derivative of convolution (equivalent to multiplying the projection spectrum by the square of frequency), which enhances high frequencies and enhances contrast of edges and discontinuities in the image. Here, in the numerical application of this approach, the concept of locality is slightly extended to a small vicinity of the point of recording on the detector in order to calculate the second derivative of the projection. In [17], both mentioned versions of local tomography are summed up with weights that bring them to approximately the same scale, and the two components of this expression are written in the form of direct and inverse $\Lambda$-operators. In [18-19], a generalization of the method of local tomography with separation of discontinuities for problems of two-dimensional vector tomography is carried out.

![Figure 5](image.png)

**Figure 5.** Reconstruction of the TM-247 model, $a = 0.05$, $T_0 = 1$, wave rays; artifacts have the form of regular wave structures in a polar coordinate system, along the radius and angle.

Vainberg’s image (a); summary image (b).

Figure 5 demonstrates the results of applying the methods of double differentiation (a) and the summary image (b) for the wave scheme of projection measurements. The mathematical model used is the same as in figure 3. This model is smooth, without discontinuities, with the parabolic dependence of the amplitude of a function on an elliptical support. It should be noted that its first and second derivatives have jumps at the edges of ellipses, and this may contribute to the development of artifacts. Local second-derivative tomography image shows wave artifacts that also appeared on the results of global reconstruction (figures 3 and 4). There are practically no artifacts in the summary image. In the future, it is planned to further develop approximate MFBP algorithms with an iterative decrease of artifacts on the reconstructed tomogram. It is also planned to investigate the algorithm for deconvolution of the summary image in order to eliminate the tomogram blurring by convolution of the $1/r$ type (figure 5, b).

**4. Conclusion**

In this work, a method is developed for reducing curvilinear tomography problems to tomography on parallel straight lines for a number of families of curves. Subsequent processing is associated with the deformation of the unknown tomogram, different in each direction, and the use of a modified filtering and backprojection method (MFBP).

The numerical simulation carried out on mathematical models shows that the accuracy of the tomogram reconstruction now depends not only on the number of views and the number of detectors in each direction, but also on the structure of the rays - their period, the range of gradient changes in the region where the tomogram is specified. An illustration of the Fourier central slice theorem for a deformed back projection shows the possibility of usage Fourier analysis in new algorithms for solving problems of the generalized Radon transform.
Note that the method of geometric deformation described in this work can be used to generalize the inversion of the exponential Radon transform, for three-dimensional ray tomography, as well as for the development of local curvilinear ROI tomography algorithms.

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References
[1] Translation of Radon’s 1917 paper in Deans [3].
[2] Natterer F 1986 The mathematics of computerized tomography (Stuttgart: Wiley)
[3] Deans S R 1983 The Radon transform and some of its applications (New York: John Wiley)
[4] Nikitin V V, Andersson F, Carlsson M and Duchkov A A 2017 Fast hyperbolic Radon transform represented as convolutions in log-polar coordinates Computers & Geosciences 105 21–33
[5] Tasinkevych J and Trots I 2014 Circular Radon transform inversion technique in synthetic aperture ultrasound imaging: an ultrasound phantom evaluation Archives of Acoustics 39 569–82
[6] Ambartsumian G and Kuchment P 2006 A Range Description for the Planar Circular Radon Transform SIAM Journal on Mathematical Analysis 38 681–92
[7] Eller M, Hoskins P and Kunyansky L 2020 Microlocally accurate solution of the inverse source problem of thermoacoustic tomography Inverse Problems 36 085012 (21pp)
[8] Hariharan K and Raajan N R 2020 Performance enhanced hyperspectral and multispectral image fusion technique using ripple type-II transform and deep neural networks for multimedia applications Multimedia Tools and Applications 79 3561–70
[9] Moon S and Heo J 2016 Inversion of the elliptical Radon transform arising in migration imaging using the regular Radon transform Journal of Mathematical Analysis and Applications 436 138–148
[10] Tarpa C, Cebeiro J, Nguyen M K, Rollet G and Morvidone M A 2020 Analytic Inversion of a Radon Transform on Double Circular Arcs With Applications in Compton Scattering Tomography IEEE Transactions on Computational Imaging 6 958–67
[11] Rahman M A, Zhu Y, Clarkson E, Kupinski M A, Frey E C and Jha A K 2020 Fisher information analysis of list-mode SPECT emission data for joint estimation of activity and attenuation distribution Inverse Problems 36 84002
[12] Pickalov V V, Kazantzef D I, Ayupova N B and Golubyatnikov V P 2005 Considerations on iterative algorithms for fan-beam tomography scheme 4th World Congress in Industrial Process Tomography (Aizu, Japan) vol 2 p 687-90.
[13] Kazantzef D and Pickalov V 2017 New iterative reconstruction methods for fan-beam tomography Inverse Problems in Science and Engineering 26 773–91
[14] Kak A C, Slaney M 1988 Principles of computerized tomographic imaging (New York: IEEE Press)
[15] Vainshtein B K 1973 Three-dimensional electron microscopy of biological macromolecules Soviet Physics Uspekhi 16 185
[16] Vainberg E I, Kazak I A and Faingoiz M L 1985 X-ray computerized back projection tomography with filtration by double differentiation. Procedure and information features Soviet J. Nondest. Test. 21 106–13
[17] Faridani A, Finch D V, Ritman E L and Smith K T 1997 Local tomography II SIAM Journal on Applied Mathematics 57 1095–127
[18] Derevtsov Y E, Pickalov V V and Schuster T 2008 Application of local operators for numerical reconstruction of the singular support of a vector field by its known ray transforms Journal of Physics: Conference Series 135 12035–1
[19] Derevtsov E Y and Pickalov V V 2011 Reconstruction of vector fields and their singularities from ray transforms Numerical Analysis and Applications 4 21–35

[20] Noo F, Clackdoyle R and Pack J D 2004 A two-step Hilbert transform method for 2D image reconstruction Physics in Medicine and Biology 49 3903–23

[21] Anastasio M A, Zou Y, Sidky E Y and Pan X 2007 Local cone-beam tomography image reconstruction on chords J. Opt. Soc. Am. A 24 1569–79

[22] Webber J W, Quinto E T and Miller E L 2020 A joint reconstruction and lambda tomography regularization technique for energy-resolved x-ray imaging Inverse Problems 36 74002