Repeated measurements from unitary evolution: avoiding the projection postulate

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Repeated measurements as typically occurring in two-time correlators rely on von Neumann’s projection postulate, telling how to restart the system after a measurement. We describe an alternative procedure where co-evolving quantum memories extract system information through entanglement, combined with a final readout of the memories according to Born’s rule. We apply this procedure to the calculation of the electron charge correlator in mesoscopic physics and the photon intensity correlator in quantum optics. While our approach to repeated quantum measurements deals with any system-memory coupling, we show that the limits of strong (weak) measurements are correctly reproduced at strong (weak) coupling.

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Traditionally, when repeating a measurement in quantum mechanics, we make use of von Neumann’s projection postulate [1] in order to know how to restart the system after a previous measurement, see Fig. 1(a); together with Born’s rule [2], these two elements residing outside the unitary evolution of quantum mechanics allow us to establish contact to our classical world. Furthermore, the von Neumann projection addresses the problem what correlator can be measured in quantum mechanics, resolving the question of how to order the operators in the expectation value. In this paper, we replace the von Neumann projection postulate in repeated measurements by quantum memories with which the system is entangled, see Fig. 1(b) for an illustration. The co-evolving unitary evolution of the system and quantum memories is read out simultaneously at the end using Born’s rule. Hence, we avoid the need for the von Neumann projection postulate during the measurement procedure, replacing it by the unitary co-evolution of entangled quantum memories. We demonstrate that this approach reproduces the same correlators as previously invoked by the von Neumann projection postulate, a projected correlator $S_P(\tau_1, \tau_2)$ in the case of strong system-detector coupling (implying a maximal system-memories entanglement), while a measurement with a weakly coupled detector [3] (and correspondingly a weak entanglement with the memories) generates a combination of symmetrized ($R_S(\tau_1, \tau_2)$) and antisymmetrized ($L_S(\tau_1, \tau_2)$) correlators weighted with different detector response functions [4, 5]. The appearance of non-symmetric correlators has been discussed recently within a general weak-measurement theory [6] and their different form under system- and detector projection in a spin system has been noted in Ref. 7.

Replacing the final-state projection in an individual measurement by the unitary co-evolution of the system with a detector and their concommitant entanglement is a concept that has been well developed over the past two decades [8–11]. In the context of repeated measurements, it is the statement about the system-restart (from the collapsed wave function) after the first measurement which is the crucial element of the von Neumann postulate; in this work, we carefully distinguish between the two types of projection, the final-state projection related to the application of Born’s rule and the von Neumann projection including the system restart from the collapsed wave function in repeated measurements. Within the present procedure, this von Neumann projection, either directly on the system for strong coupling or indirectly on the detector for weak coupling, is avoided and replaced by the entanglement with co-evolving quantum memories. The general idea of replacing the von Neumann projection postulate is a common element in the many worlds interpretation of quantum mechanics [12, 13]; in quantum information theory the inclusion of quantum memories and unitary co-evolution is known in the context of
of Stinespring’s dilation theorem [14]. Here, we adopt a more limited (and practical) view on these ideas in the context of repeated measurements for time correlators.

The general idea in replacing the von Neumann projection postulate is to start with a system state $|\psi\rangle = \sum_n \psi_n |n\rangle$ (in an eigenbasis of the observable) and a quantum memory in the initial state $|\phi_n^{(\text{in})}\rangle$ and have them transiently interact at time $\tau_1$ with the help of an externally controlled interaction (to be identified with a quantum detector). The system and memory then become entangled, $\sum_n \psi_n(\tau_1) |n\rangle |\phi_n^{(\text{in})}\rangle \rightarrow \sum_n \psi_n(\tau_1) |n\rangle |\phi_n^{(\text{out})}\rangle$, where $|\phi_n^{(\text{out})}\rangle$ denote memory states after interaction with the system in state $|n\rangle$ and we assume a negligible evolution of the system during the time of interaction (see Supplementary Material (SM), section E for an extended discussion). Evolving the system to the time $\tau_2$ and entangling it with a second memory, we obtain the state $\sum_{m,n} U_{mn}(\tau_2) \psi_n(\tau_1) |m\rangle |\phi_n^{(\text{out})}\rangle$, with $U_{mn}(\tau)$ the matrix elements of the system propagator $\hat{U}(\tau)$ and $\tau_{21} = \tau_2 - \tau_1$. We assume that the memory states $|\phi_n^{(\text{out})}\rangle$ preserve the system information gained at times $\tau_j$ during their subsequent evolution; applying the Born rule to the memory states at some later time $\tau_{\text{fin}} > \tau_2$ then provides us with the desired information on the system’s two-time correlator. In the following, we discuss this program on two examples, where the first is a detailed discussion of an electronic system within mesoscopic physics and the second a brief consideration on photons in a quantum optics setting. We analyze the cases of strong and weak coupling and derive the measurable correlators.

We consider a dynamical charge (a quantum dot (QD) attached to leads or an isolated double-dot) which we want to characterize by its two-time correlator; the last allows us with the desired information on the system’s two-time correlator. In the following, we discuss this program on two examples, where the first is a detailed discussion of an electronic system within mesoscopic physics and the second a brief consideration on photons in a quantum optics setting. We analyze the cases of strong and weak coupling and derive the measurable correlators.

![FIG. 2: Quantum dot system (QD) measured by a capacitively coupled quantum point contact (QPC): Single-electron pulses incident on the QPC from the left (at $\tau_{\text{in}}$) are either transmitted (with amplitude $t$) or reflected (amplitude $r$); the outgoing Lippmann-Schwinger wave functions describe flying qubits without own dynamics and serve as quantum memories. Two pulses separated in time by $\tau_2 - \tau_1$ are needed to measure the two-time correlator of the dot’s charge. After scattering at the QPC, the two electrons (flying qubits) are entangled with the quantum dot system and carry information on its dynamics. Simultaneous detection of the two scattered electrons at $\tau_{\text{fin}}$, e.g., a distance $v_F (\tau_2 - \tau_1)$ away with both positions on the right of the QPC, provides information on the two-time charge correlator.](image)

The single-electron pulses $|\phi_n^{(j)}\rangle$, $j = 1, 2$, incident on the QPC from the left are scattered to outgoing states $|\hat{S}_n|\phi_n^{(j)}\rangle$ with the scattering operator $\hat{S}_n$ depending on the state $|n\rangle$ of the QD. In a good measurement setup, the state $|\psi(\tau)\rangle = \sum_n \psi_n(\tau) |n\rangle$ of the dot remains constant during the time $\tau_{\text{scat}}$ of the individual scattering events. After the second scattering event $\tau_{\text{fin}} > \tau_2$, the wave function $|\psi(\tau_{\text{fin}})\rangle$ of the system is entangled with the states $|\phi_n^{(\text{scat})}\rangle$ of the scattered electrons and the total system plus memories wave function $|\Psi_f\rangle$ reads (for open systems, see [20] and SM, section D),

$$|\Psi_f\rangle = \sum_{l,m,n} U_{lm}(\tau_{f2}) U_{mn}(\tau_{21}) \psi_n(\tau_1) |l\rangle |\phi_n^{(1)}\rangle |\phi_n^{(2)}\rangle,$$

with $|\phi_n^{(j)}\rangle = t_n |\phi_n^{(j)}\rangle + r_n |\phi_n^{(j)}\rangle$ the scattered states of the $j$-th electron when the dot is in state $|n\rangle$ and $\tau_{f2} = \tau_{\text{fin}} - \tau_2$. We assume well separated single-electron pulses and an evolution of the scattered waves $|\phi_n^{(\text{scat})}\rangle$ emanating from the QPC at times $\tau_j$ that preserves the corresponding system information, in particular, $\langle \phi_n^{(2)} | \phi_n^{(1)} \rangle = 0$. Applying Born’s rule to the final state (1) we obtain the desired probabilities (or charge correlators $\langle \hat{q}_n^{(1)} \hat{q}_n^{(2)} \rangle$ in the detector)

$$P_{\alpha\beta} = \langle \Psi_f | \hat{q}_n^{(1)} \hat{q}_n^{(2)} | \Psi_f \rangle$$

(2)
with the (charge) operators $\hat{q}_\alpha^{(j)}$ providing the transmitted $(\alpha = t)$ or reflected $(\alpha = r)$ components of the $j$th electron, $\langle \hat{q}_\alpha^{(j)}|\hat{q}_\beta^{(j)}\rangle = \delta_{\alpha\beta}\delta_j\delta_\eta\langle \hat{q}_\beta^{(j)}\rangle$ (we define all charges in units of $e$). Making use of the unitarity condition $\sum_i U_{im}^* U_{im} = \delta_{mn}$ (rendering the evolution $U_{im}(\tau_{f2})$ in Eq. (1) irrelevant), we obtain the probabilities

$$P_{\alpha\beta}(\tau_{21}) = \sum_m \sum_n \delta_m^\alpha \delta_n^\beta \rho_{mn}(\tau_{21}) s_m^\alpha s_n^\beta(\tau_{21})^2$$

(3)

with the scattering amplitudes $s_m^\alpha = t_m$, $s_n^r = r_n$. The probabilities $P_{\alpha\beta}(\tau_{21})$ are the main result of this paper as they provide the desired information on the two-time correlator of the dot. However, as expressed in terms of evolution- and scattering amplitudes, it is difficult to appreciate the physical meaning and content of the expression (3). In the following, we analyze (3) for a simple two-state system with $n = 0, 1$ and the two cases of a strong and a weak measurement, unravelling its physical meaning and demonstrating its equivalence with previous results [4, 15].

A strong system–detector coupling generates a unique scattering outcome with $|0\rangle_0 = 1$, $|0\rangle_r = 0$ and $|1\rangle_0 = 0$, $|1\rangle_1 = 1$ and probabilities $P_{tt} = |U_{00}|^2$, $P_{tr} = |U_{10}|^2$, $P_{rt} = |U_{01}|^2$, $P_{rr} = |U_{11}|^2$. On the other hand, a strong measurement with a von Neumann projection at $\tau_1$, see Fig. 1(a), results in the projected correlator $S_{QQ}^{\alpha\beta}(\tau_1, \tau_2) =$ Tr$[\hat{Q}(\tau_2)\hat{Q}(\tau_1)\rho^\alpha(\tau_1)]$ with the dot’s charge operator $\hat{Q} = [1]/[1]$ (defined in units of $e$) measuring the charge in the dot and the projected single-particle density matrix $\rho^\alpha(\tau_1) = |\langle \psi(\tau_1) | \rangle_0^2|0\rangle\langle 0| + |\langle \psi(\tau_1) | \rangle_1^2|1\rangle\langle 1|$. Evaluating this expression, we easily find that $S_{QQ}^{tt}(\tau_1, \tau_2) = |U_{11}|^2$ and thus our result $P_{tt}$ as given by Eq. (3) reduces to the projected charge-charge correlator at strong coupling,

$$P_{tt}(\tau_{21}) = S_{QQ}^{tt}(\tau_1, \tau_2).$$

(4)

Hence, the perfect entanglement between the system and the memories arising due to strong coupling is equivalent to a von Neumann projection applied to the system. While no back action is apparent on the level of the system dynamics, the strong back action of this maximal entanglement with the memories manifests itself in a strong change of the system’s density matrix when tracing over the memories. The probabilities $P_{tt}$, $P_{rr}$, and $P_{tr}$ measured at other positions describe alternative correlators.

Next, we consider a weak system–detector coupling where the change in the scattering matrix for different charge states is small, resulting in a weak system–memory entanglement. We parametrize the scattering matrices for the system in states $|0\rangle$ and $|1\rangle$ by $t_0 = \sqrt{T} e^{i\theta}$, $r_0 = \sqrt{R} e^{i\chi}$, $t_1 = \sqrt{T - R} e^{i(\theta + \delta)}$, $r_1 = \sqrt{R + T} e^{i(\chi + \delta)}$, with small corrections $\delta T$, $\delta \theta$, and $\delta \chi$. We evaluate Eq. (3), e.g., for $P_{tt}$, expand the result to second order in $\delta T$, $\delta \theta$, and $\delta \chi$, and subtract the uncorrelated contribution $\langle \Psi_f|\hat{q}_{t1}^{(2)}|\Psi_f\rangle\langle \Psi_f|\hat{q}_{t1}^{(2)}|\Psi_f\rangle$ to find the irreducible component; the latter factorizes into system $[\ldots]$ and detector response

$$P_{tt}^{\text{irr}} = \delta T^2 \{ |\langle \bar{\psi}_1^* U_{11} \psi_1 + c.c. \rangle|/2 - |\langle \psi_1^{(2)} | \bar{\psi}_1^{(2)} \rangle|^2 \} + T\delta \theta(-\delta T)\{ |\langle \bar{\psi}_1^* U_{11} \psi_1 - c.c. \rangle|$$

(5)

where $\psi_1 = \psi_1(\tau_1)$, $\bar{\psi}_1 = \psi_1(\tau_2) = U_{10}\psi_0 + U_{11}\psi_1$ and we made use of the unitarity of the time evolution matrix. In order to find a physical interpretation of this result, we recalculate $P_{tt}$ from Eq. (2) perturbatively in the capacitive system–detector coupling $H_{sd} = e^2 q\hat{Q}/C$ using the interaction representation $(\hat{q}$ is the charge on the QPC) to find (the superscripts $(1), (2)$ refer to the memory to which the detector couples, see also SM, section A)

$$P_{tt}^{\text{irr}}(\tau_{21}) = \mathcal{I}\mathcal{S}_{det,t}^{(1)} \mathcal{I}\mathcal{S}_{det,t}^{(2)} \mathcal{R}\mathcal{S}_{QQ}^{\text{irr}}(\tau_1, \tau_2) + \mathcal{R}\mathcal{S}_{det,t}^{(1)} \mathcal{I}\mathcal{S}_{det,t}^{(2)} \mathcal{I}\mathcal{S}_{QQ}^{\text{irr}}(\tau_1, \tau_2).$$

(6)

The result involves the symmetrized and anti-symmetrized irreducible correlators (here, $\{\cdot\}^\text{sym}$ and $\{\cdot\}^\text{anti}$ are the usual (anti)-commutator)

$$\mathcal{R}\mathcal{S}_{QQ}^{\text{irr}}(\tau_1, \tau_2) = \{ \langle \hat{Q}(\tau_1), \hat{Q}(\tau_2) \rangle \} /2$$

(7)

$$\mathcal{I}\mathcal{S}_{QQ}^{\text{irr}}(\tau_1, \tau_2) = -i \{ [\langle \hat{Q}(\tau_1), \hat{Q}(\tau_2) \rangle] \} ,$$

(8)

with expectation values taken over the initial system state, $\langle \cdot \rangle = \langle \psi(\tau_{in}) | \cdot | \psi(\tau_{in}) \rangle$ and $\{ \cdot \}^\text{sym}$ refers to the irreducible part. The result (6) can be easily generalized to an expression for $P_{tt}^{\text{irr}}$ by replacing $t \rightarrow \alpha$ in the first and $t \rightarrow \beta$ in the second detector response function; for a linear system–detector coupling, its form as a product of detector and system response functions is generic.

The results (5) evaluated in the Schrödinger picture accounting for entanglement and (6) calculated via usual perturbation theory in the interaction picture are identical: Evaluating the expressions (7) and (8) for our QD, we find (see SM, Sec. A) that $\mathcal{R}\mathcal{S}_{QQ}^{\text{irr}}$ and $\mathcal{I}\mathcal{S}_{QQ}^{\text{irr}}$ reproduce the system factors in Eq. (5). Evaluating the detector response functions for our QPC, we find that $\mathcal{I}\mathcal{S}_{det,t}(-\delta T) = -\mathcal{I}\mathcal{S}_{det,t} = -\delta T$ and $\mathcal{R}\mathcal{S}_{det,t} = T\delta \theta$, $\mathcal{R}\mathcal{S}_{det,r} = R\delta \chi$. Hence the weak entanglement between the system and the memories arising due to weak coupling produces results that are equivalent to a von Neumann projection applied to the detector [21]. Note that while it is the time $\tau_1$ of von Neumann projection of the system or of the detector and the moment $\tau_2$ of the final state projection which determine the time instances probed in the correlator in the traditional strong and weak measurement scenarios, it is the scattering times $\tau_1$ and $\tau_2$ at the QPC where the memories entangle with the system which determine the time-difference $\tau_{21}$ in the two-time correlator within the new formulation.

In order to extract the separate system correlators $\mathcal{R}\mathcal{S}_{QQ}^{\text{irr}}$ and $\mathcal{I}\mathcal{S}_{QQ}^{\text{irr}}$ one may use either a detector with appropriate response functions (see below) or combine
different measurements involving detections to the left and right of the QPC,

\[
\mathcal{R}^{\text{crr}}_{\text{det,t}} = \frac{[\mathcal{R}^{\text{crr}}_{\text{det,r}} \mathcal{P}^{\text{irr}} - \mathcal{R}^{\text{crr}}_{\text{det,l}} \mathcal{P}^{\text{irr}}]}{\mathcal{D} \mathcal{S}_{\text{det,0}}^{\text{crr}}} \tag{9}
\]

\[
\mathcal{I}^{\text{crr}}_{\text{QQ}} = -[\mathcal{I}^{\text{crr}}_{\text{det,l}} \mathcal{P}^{\text{irr}} + \mathcal{I}^{\text{crr}}_{\text{det,0}} \mathcal{P}^{\text{irr}}]/\mathcal{D} \mathcal{S}_{\text{det,0}}^{\text{crr}} \tag{10}
\]

with \( \mathcal{D} = \mathcal{R}^{\text{crr}}_{\text{det,t,r}} - \mathcal{R}^{\text{crr}}_{\text{det,l,t}} \mathcal{I}^{\text{crr}}_{\text{det,0}} \), and for an arbitrary choice of \( \alpha = r, t \). Alternatively, the response functions in Eq. (6) can be tuned to deliver the individual system correlators \( \mathcal{R}^{\text{crr}}_{\text{QQ}} \) or \( \mathcal{I}^{\text{crr}}_{\text{QQ}} \). For a detector tuned to high transmission, e.g., a QPC with energetic (E) single-electron pulses \( E \gg V_0 \), the QPC barrier, and large on the scale of the system dynamics. The final state (1) is described by the scattering amplitudes \( Q_n = \sum_{\nu} U_{mn} \psi_n^\nu^* \psi_m^\nu \), the result (3) for the probabilities \( P_{\beta} \text{ still} \) provides the proper starting point. The QPC distinguishing these states is described by the scattering amplitudes \( t_n = \sqrt{T_n e^{i\theta_n}} \) and \( r_n = \sqrt{1 - T_n e^{i\chi_n}} \). Assuming a linear detector with \( T_n = T - Q_n \delta \theta, \theta_n = \theta + Q_n \delta \theta, \) and \( \chi_n = \chi + Q_n \delta \chi \), the weak coupling case is most easily treated (see SM, Sec. C I) and provides the result (6) with the proper charge operator \( \hat{Q} = \sum_n Q_n |n\rangle |n\rangle \) in (7) and (8).

The strong coupling case (see also SM, Sec. C II) is more difficult to treat, as a single electron is unable to distinguish between more than two charge states. Therefore, we expand our memory states to consist of two trains with \( J \) electrons each [10], probing the system state close to \( \tau_1 \) and \( \tau_2 \); the separation \( \delta \tau \) between electrons within a train must allow for their distinction while the duration \( J \delta \tau \) of the train is assumed to be small on the scale of the system dynamics. The final state (1) then involves the outgoing wave functions \( \Phi^{(1)}_{\alpha} = \prod_{j=1}^{J} \phi^{(j \alpha)}_{n_j} \) and \( \Phi^{(2)}_{\sigma} = \prod_{j=J+1}^{2J} \phi^{(j \sigma)}_{m_j} \), with the index \( j \) still enumerating the \( 2J \) memory states. Born’s final state projection then produces the probabilities \( P^{\mu \nu}_{\alpha \beta} \) for finding \( \mu (\nu) \) electrons of the first (second) train transmitted across \( \alpha, \beta = t, r \) the QPC,

\[
P^{\mu \nu}_{\alpha \beta} = \left( \begin{array}{c} J \\ \mu \end{array} \right) \left( \begin{array}{c} J \\ \nu \end{array} \right) \sum_n (s_n^\alpha)^\nu^* (s_n^\beta)^{J-\nu} U_{mn} \times (s_n^\alpha)^{\mu^*} (s_n^\beta)^{J-\mu} \psi_n \right|^2. \tag{12}
\]

with \( s_n^\alpha = r_n \) and \( s_n^\beta = t_n \). Introducing the detector response \( P_{\alpha} (\mu | n) = \left( \begin{array}{c} J \\ \mu \end{array} \right) \sum_n (s_n^\alpha)^\mu^* (s_n^\beta)^{J-\mu} \psi_n \) (the conditional probability to have \( \mu \) from \( J \) particles transmitted (\( \alpha = t \)) or reflected (\( \alpha = r \)) with the system in state \( |n\rangle \) and assuming a symmetric scatterer with \( t_n^r r_n + r_n^t t_n = 0 \), we can factorize Eq. (12) to exhibit separately the system and detector response,

\[
P^{\mu \nu}_{\alpha \beta} = \sum_{n,n',m} U_{mn} \psi_n^\mu^* \psi_m^\nu^* R_{\alpha} (\mu, n', n') P_{\beta} (\nu | m), \tag{13}
\]

with \( R_{\alpha} (\mu, n', n') = \sqrt{P_{\alpha} (\mu | n') P_{\alpha} (\mu | n')} e^{i(\theta_n - \theta_{n'})} \). For a strong coupling correlator in the reflected channel \( S_{q_{\gamma q_{\gamma}}}(\mu, n', n) \) are suppressed. The charge-charge correlator in the reflected channel \( S_{q_{\gamma q_{\gamma}}}(\mu, n', n) \) can be written as \( S_{q_{\gamma q_{\gamma}}}(\mu, n', n) \) and assuming a linear detector with \( R_{\alpha} = Q_n \delta \chi \), we obtain the final result

\[
S_{q_{\gamma q_{\gamma}}} = (J \delta R)^2 \sum_{m,n} Q_m Q_n |U_{mn} \psi_n^\mu|^2 = (J \delta R)^2 \mathcal{S}_{\text{QQ}}^{\text{crr}}, \tag{14}
\]

using the relation \( \sum_{\mu} P_{\mu} (\mu | n) = J R_n = J Q_n \delta \chi \). The result (14) generalizes the relation (4) to a system with many (non-degenerate) charge states [22]. The above analysis can be generalized to open systems (see SM, Sec. D) and non-instantaneous entanglement (see SM, Sec. E).

The study of higher-order correlators is straightforward, e.g., to measure a third-order charge correlator, we send three electrons scattering from the QPC at times \( \tau_1 < \tau_2 < \tau_3 \) and obtain the probabilities

\[
P_{\alpha \beta \gamma} = \left( \begin{array}{c} J \\ \gamma \end{array} \right) \left( \begin{array}{c} J \\ \mu \end{array} \right) \left( \begin{array}{c} J \\ \nu \end{array} \right) \sum_{m,n} s_n^\alpha U_{lm} (\tau_3 \tau_2) s_m^\mu U_{mn} (\tau_2 \tau_1) s_n^\nu \psi_n \right|^2. \tag{15}
\]

describing electrons transmitted across \( (\alpha, \beta, \gamma = t) \) or reflected from \( (\alpha, \beta, \gamma = r) \) the QPC. For weak coupling, its irreducible part can be recast in the form

\[
P^{\mu \nu}_{\alpha \beta} = \sum_{\sigma \sigma'} S^{(1) \sigma \sigma'}_{\text{det,0}} S^{(2) \sigma' \sigma \sigma} S^{(3) \sigma \sigma' \sigma \sigma} \mathcal{S}_{\text{QQ}}^{\text{crr}} (\tau_1, \tau_2, \tau_3) \tag{16}
\]

with \( \sigma = -\sigma \), the detector responses \( S^{(1) \sigma \sigma'}_{\text{det,0}} = \mathcal{R} S^{(1) \sigma \sigma'}_{\text{det,0}} \) and \( S^{(2) \sigma' \sigma \sigma} = \mathcal{I} S^{(2) \sigma' \sigma \sigma} \), and the third-order correlators

\[
\mathcal{S}_{\text{QQ}}^{\text{crr}} = \gamma_{c \sigma} \sigma \mu \nu \left[ (\hat{Q} (\tau_1), [(\hat{Q} (\tau_2), \hat{Q} (\tau_3)]) \right] \right] \bar{\psi}_{\nu}, \tag{17}
\]

with the constants \( c_+ = 1/2 \) and \( c_- = -i \) and \( \{\cdot, \cdot\} = \{\cdot, \cdot\} \) resp. \( \{\cdot, \cdot\} = \{\cdot, \cdot\} \) (note that (anti-)symmetrized charges in \( \mathcal{S}_{\text{QQ}}^{\text{crr}} \) (encoded in \( \sigma \) relate to opposite detector response functions (encoded by \( \bar{\sigma} \)). This result agrees with the one in Ref. [23] obtained with the help of the von Neumann projection postulate and shows that only Keldysh time-ordered charge correlators are measurable.

A similar analysis can be done for any repeated measurement scheme where the von Neumann projection postulate was used. Below, we discuss another generic example, photonic correlations in quantum optics. Traditionally, photon correlations are measured via photoinization which destroys the photon and thus is projective by its very nature. According to Glauber [2], the
correlations are given by a specific time- and normal-ordered sequence of creation and annihilation operators; his result has later been formally derived with the help of the von Neumann projection postulate [3]. Here, we consider a measurement setup with two qubits placed at positions $r_1$ and $r_2$ serving as quantum memories. The latter are entangled with the photon field at times $\tau_1$, $\tau_2$ with $\tau_1 < \tau_2$ through their transient coupling (with strengths $d_1$, $d_2$) during a time interval $\Delta \tau$. We determine the probability $P_{ee}$ to find the two qubits excited at a later time $\tau_f$, see SM, Sec. F, and find the result $P_{ee} = (d_1^2 d_2^2 \Delta \tau^4/\hbar^4)(T_K : \hat{I}(r_2, \tau_2) \hat{I}(r_1, \tau_1) :)_\text{ph}$ with $\hat{I}$ denoting intensities, $\langle \cdots \rangle_{\text{ph}}$ the expectation value over the photon field, and proper time- and anti-time- as well as normal-ordering (denoted by $T_K$ and $:\cdots :$, respectively) of the field operators. This result is in line with the classic results from photoionization theory [2, 3].

In conclusion, we have demonstrated, by way of example, that the concept of von Neumann projection invoked in deriving the outcome of repeated measurements can be replaced by the unitary co-evolution and controlled entanglement with quantum memories. Our scheme allows to describe both strong and weak measurement on the same footing. As a practical outcome, we could rederive the results (4) and (6) for strong and weak measurements within the fully unitary evolution of quantum mechanics, emphasizing their generic nature.

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[20] For an open system, we go over to density matrices, $\hat{\rho}_f = \sum_{n,n',m,m'} \rho_{mn,m'n'}(\tau_1) \hat{P}_{nm} \hat{P}_{mn'}$, with $\hat{P}_{mn,m'n'}(\tau_1) = \hat{U}(\tau_2) \hat{U}_{mn}(\tau_2) \hat{\rho}(\tau_1) \hat{U}_{m'n'}(\tau_2) \hat{U}^\dagger(\tau_2)$, $\hat{U}_{mn} = \hat{P}_m \hat{P}_n$, and $\hat{P}_n$ the projector to the charge $n$ sector.
[21] For degenerate charge states, the relation $S_{\text{det}_1} = (J\delta R)^2$ still holds true with $S_{\text{det}_1}$ calculated using the projected density matrix $\hat{\rho}^P(\tau_1) = \sum_{Q} \hat{P}_{Q} \hat{\rho}(\tau_1) \hat{P}_{Q}$ and the charge projector $\hat{P}_{Q} = \sum_{\{n|Q=-Q\}} |n\rangle\langle n|$, see SM, Sec. C.
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Supplementary Material

A. PERTURBATION THEORY

When the system–detector coupling is weak we can treat the interaction perturbatively. We make use of the interaction representation and describe the detector by the Hamiltonian $H_{\text{ad}} = \epsilon^2 q \tilde{Q}/C$, giving rise to an increase in the scattering potential $\tilde{v} = \epsilon^2 \tilde{q}/C$ if the dot is charged (state [1]). Given the initial state of the system and memories (i.e., the electrons in the detector), $|\Psi\rangle = |\psi\rangle \otimes |\phi_1\rangle \otimes |\phi_2\rangle$, we study the irreducible component of the probability $P_{tt}$ for joint transmission, i.e.,

$$P_{tt}^{\text{irr}} = \langle \Psi | \hat{U}^\dagger_{\text{d}}(\tau_f, \tau_n) \hat{q}^{(1)}(\tau_f \tau_f) \hat{U}_{\text{d}}(\tau_f, \tau_n) |\Psi\rangle$$

(A.1)

with the time evolution operator $\hat{U}_{\text{d}}(\tau_f, \tau_n) = T \exp(-i \int_{\tau_n}^{\tau_f} d\tau' H_{\text{ad}}(\tau')/\hbar)$. Evaluating (A.1) to lowest relevant order in the coupling, we find

$$P_{tt}^{\text{irr}} = \left( -\frac{1}{\hbar^2} - \frac{i}{\hbar^2} \int_{\tau_n}^{\tau_f} d\tau' \int_{\tau_n}^{\tau_f} d\tau'' \langle \Psi | [(\hat{q}^{(1)}(\tau_f \tau_f), \hat{H}_{\text{ad}}(\tau')], \hat{H}_{\text{ad}}(\tau'')] |\Psi\rangle \right)$$

where we made sure that the first electron impinges on the QPC at the earlier time $\tau''$. For a slow system dynamics and exploiting that $\tilde{v}(\tau')|\phi_1\rangle \neq 0$ only for $\tau \approx \tau$, we can replace $\hat{Q}(\tau') \rightarrow \hat{Q}(\tau_1)$ and $\hat{Q}(\tau') \rightarrow \hat{Q}(\tau_2)$. Inserting the definitions of $\mathcal{R}^{\text{irr}}_{\text{Q}Q}(\tau_1, \tau_2)$ and $\mathcal{I}^{\text{irr}}_{\text{Q}Q}(\tau_1, \tau_2)$ as given in Eqs. (7) and (8) and defining the detector response functions

$$\mathcal{I}^{\text{irr}}_{\text{det},t} = \frac{-i}{\hbar} \int_{\tau_n}^{\tau_f} d\tau \langle \phi_1 | [\hat{q}^{(1)}(\tau_f \tau_f), \tilde{v}(\tau)] |\phi_1\rangle, \quad (A.3)$$

$$\mathcal{R}^{\text{irr}}_{\text{det},t} = -\frac{1}{2\hbar} \int_{\tau_n}^{\tau_f} d\tau \langle \phi_1 | [\hat{q}^{(1)}(\tau_f \tau_f), \tilde{v}(\tau)] |\phi_1\rangle, \quad (A.4)$$

we arrive to Eq. (6). We are left with determining the detector response functions (A.3) and (A.4). Starting from the matrix element $\langle \phi_1 | U_{\text{d}}^\dagger(\hat{q}^{(1)} \otimes U) |\phi_1\rangle$ and expanding $U = U_0 U_d$ (with $U (U_0)$ describing the full (free) dynamics) to lowest order in $\tilde{v}$ we can rewrite the term

$$\frac{-i}{\hbar} \int_{\tau_n}^{\tau_f} d\tau \langle \phi_1 | \hat{q}^{(1)}(\tau_f \tau_f) \tilde{v}(\tau) |\phi_1\rangle = \langle \phi_1 | e^{i \hat{H}_{\text{ad}}(\tau_f - \tau_n)/\hbar} e^{-i \hat{H}_{\text{ad}}(\tau_f - \tau_n)/\hbar} |\phi_1\rangle$$

$$- \langle \phi_1 | \hat{q}^{(1)}(\tau_f \tau_f) |\phi_1\rangle = t_{\phi}^d t_{\phi} = t_{\phi}^d$$

(A.5)

where, in the last equation, we have accounted for the scattering properties of the QPC in the different charge states of the dot. Expanding $t_1$ in $\delta T$ and $\delta \theta$, we find that

$$\frac{-i}{\hbar} \int_{\tau_n}^{\tau_f} d\tau \langle \phi_1 | \hat{q}^{(1)}(\tau_f \tau_f) \tilde{v}(\tau) |\phi_1\rangle = -\frac{1}{2} \delta T + i \delta \theta$$

(A.6)

and we arrive at the final results $\mathcal{I}^{\text{irr}}_{\text{det},t} = -\delta T$ and $\mathcal{R}^{\text{irr}}_{\text{det},t} = T \delta \theta$.

B. DETECTOR PROPERTIES

A quick overview is provided by the example of a $\delta$-function scatterer: Expressing the strength of the scatterer $\hbar^2 \lambda/m$ for the two charge states by $\lambda_0 = \lambda$ and $\lambda_1 = \lambda + \delta \lambda$, an incoming state with wave vector $k$ is transmitted with amplitude $t_k = k/(k + i \lambda_n)$, $n \in \{0, 1\}$. Expanding the transmission $T_n = k^2/(k^2 + \lambda_n^2)$ and phase $\theta_n = -\arctan(\lambda_n/k)$, we find the modifications $\delta T$ and $\delta \theta = \delta \chi$ (for a symmetric scatterer) in the scattering characteristic of the QPC upon changing the dot

$$\delta T \approx -\frac{2 k^2 \delta^2}{(k^2 + \lambda^2)^2} \lambda,$$

(B.1)

$$\delta \theta = \delta \chi = -\frac{2 k^2 \delta^2}{(k^2 + \lambda^2)^2} \lambda$$

(B.2)

In the limit of a large incoming energy, i.e., $k \gg \lambda$, we find $\delta T \approx -2 \lambda \delta \lambda/k^2$ and $\delta \theta = \delta \chi \approx -\delta \lambda/k$ and hence $\delta \theta, \delta \chi \gg \delta T$; with $T \approx 1$ and $R \approx 1/k^2 \ll 1$, we have $|T|, |\delta T| \gg |R|, |\delta \chi|$ and therefore $|\mathcal{I}^{\text{irr}}_{\text{det},t}| \gg |\mathcal{R}^{\text{irr}}_{\text{det},t}| \gg |\mathcal{I}^{\text{irr}}_{\text{det},t}|$. For a small incoming energy $k \ll \lambda$, we obtain $\delta T \approx -2 \lambda \delta \lambda/k^2$ and $|\delta \theta| \approx -k \delta \lambda/k^2$ and using $T \approx 2/k^2$ and $R \approx 1$ we find $|\mathcal{I}^{\text{irr}}_{\text{det},t}| \gg |\mathcal{R}^{\text{irr}}_{\text{det},t}| \gg |\mathcal{I}^{\text{irr}}_{\text{det},t}|$. When $k \approx \lambda$, all response functions are of the same order.
Alternatively, we can consider a single-electron transistor (SET) with the level position $k_{\text{res},n}$ affected by the capacitive coupling and depending on the dot’s charge state $|n\rangle$, i.e., $k_{\text{res},0} = k_{\text{res}}$ and $k_{\text{res},1} = k_{\text{res}} + \delta k_{\text{res}}$. The transmission coefficient is given by $t_n = i\gamma/(k_{\text{res},n}^2 + \gamma^2)$, where $\gamma$ is the level width. Again expanding $T_n = \gamma^2/((k-k_{\text{res},n})^2 + \gamma^2)$ and $\tan \theta_n = (k-k_{\text{res},n})/\gamma$ for small $\delta k_{\text{res}}$, we find

$$
\delta T = \frac{2(k-k_{\text{res}})(\gamma^2 \delta k_{\text{res}})}{[(k-k_{\text{res}})^2 + \gamma^2]^2},
$$

$$
\delta \theta = \frac{\gamma \delta k_{\text{res}}}{(k-k_{\text{res}})^2 + \gamma^2}.
$$

For incoming electrons on resonance with the level, i.e., $|k-k_{\text{res}}| \ll \gamma$, we obtain $\delta T \approx -2(k-k_{\text{res}})\delta k_{\text{res}}/\gamma^2$ and $\delta \theta = \delta \chi \approx -\delta k_{\text{res}}/\gamma$, such that $\delta \theta, \delta \chi \gg \delta T$ and using $T \approx 1$ and $R \approx (k-k_{\text{res}})^2/\gamma^2$ we find that $|\mathcal{L}_{\text{S}_{\text{det},l}}| \gg |\mathcal{R}_{\text{S}_{\text{det},r}}| \gg |\mathcal{L}_{\text{S}_{\text{det},l}}|$. On the other hand, for off-resonant electrons $\delta T 
approx -2\gamma^2 \delta k_{\text{res}}/(k-k_{\text{res}})^3$ and $\delta \theta = \delta \chi \approx -\gamma \delta k_{\text{res}}/(k-k_{\text{res}})^2$, such that $\delta \theta, \delta \chi \gg \delta T$ and using $T \approx 2\gamma^2/(k-k_{\text{res}})^2$ and $R \approx 1$, we find $|\mathcal{L}_{\text{S}_{\text{det},r}}| \gg |\mathcal{R}_{\text{S}_{\text{det},r}}| \gg |\mathcal{L}_{\text{S}_{\text{det},l}}|$. When $|k-k_{\text{res}}| \approx \gamma$, all response functions are of the same order.

A more realistic description for the quantum point contact (QPC) is achieved by considering a parabolic scattering potential $V_n(x) = V_n - kx^2/2$ where the offset $V_n$ is the QPC barrier height when the dot is in the charge state $|n\rangle$. Here, we assume a quasi-classical description and consider the two limits of electrons with energy $E \gg V_n$ resp. $E \ll V_n$, see Fig. B.1.

![QPC modeled by a parabolic potential.](image)

Using the Kemble formula [1], we obtain the transmission $T_n = 1/(1 + \exp[-2\pi \sqrt{mL^2/\hbar^2}(E - V_n)/\sqrt{|V_n|}])$, where we have chosen $V(\pm L/2) = 0$. For weak coupling, $\delta V = V_1 - V_0 \ll V_0$ we obtain the shift (we define the energy scale $E_L = \hbar^2/2mL^2$)

$$
\delta T = \frac{\pi E + V_0}{4V_0} T_E R_E \frac{\delta V}{\sqrt{E_L V_0}},
$$

which is suppressed exponentially for $E \gg V_0$ and $E \ll V_0$ due to an exponentially small reflection or transmission.

The change in phase at large energies $E \gg V_0$ is determined by the transmission phase accumulated in the region $[-L/2, L/2]$; within a quasi-classical description, this is given by ($\varepsilon \equiv E/V_n$)

$$
\theta_n = \frac{1}{\hbar} \int_{-L/2}^{L/2} dx \sqrt{2m(E - V_n(x))}
= \frac{1}{2} \frac{V_n}{E_L} \left[\sqrt{\varepsilon} - (\varepsilon - 1) \log(\varepsilon - 1)^{1/2}ight]
+ (\varepsilon - 1) \log(1 + \sqrt{\varepsilon})].
$$

Expanding this result for small $\delta V$, we obtain the change in phase $\delta \theta = \delta \chi$,

$$
\delta \theta = -\frac{1}{3} \sqrt{\frac{V_0}{E L V_0}} \frac{\delta V}{\sqrt{E L V_0}}.
$$

Given the exponential suppression of $\delta T$ at large energies $E \gg V_0$, we find that $\delta T \ll |\delta \theta| = |\delta \chi|$ and for large transmission we have $|\mathcal{L}_{\text{S}_{\text{det},l}}| \gg |\mathcal{R}_{\text{S}_{\text{det},r}}| \gg |\mathcal{L}_{\text{S}_{\text{det},l}}|$. In the opposite regime of small energies $E \ll V_0$, we determine the change in phase (within quasi-classics) from the phase of the reflection amplitude,

$$
\chi_n = \frac{2}{\hbar} \int_{-L/2}^{L/2} dx \sqrt{2m(E - V_n(x))},
$$

where the reversal point $x_0 < 0$ is characterized by $V(x_0) = E$. To leading order in $\delta V$ we find that

$$
\delta \chi \approx -\frac{1}{3} \left(\frac{E}{V_0}\right)^{3/2} \frac{\delta V}{\sqrt{E_L V_0}}.
$$

Once more, it follows that $|\delta \theta|, |\delta \chi| \ll \delta T$ due to the exponential suppression of $T$ and the response functions respect the order $|\mathcal{R}_{\text{S}_{\text{det},r}}| \gg |\mathcal{L}_{\text{S}_{\text{det},l}}| \gg |\mathcal{R}_{\text{S}_{\text{det},l}}|$. At intermediate energies, the response functions are of similar magnitude. Summarizing, we find that a scatterer with large transmission is characterized by the response functions satisfying $|\mathcal{R}_{\text{S}_{\text{det},r}}| \gg |\mathcal{L}_{\text{S}_{\text{det},r}}| \gg |\mathcal{R}_{\text{S}_{\text{det},l}}|$ while at small transmission $|\mathcal{R}_{\text{S}_{\text{det},r}}| \gg |\mathcal{L}_{\text{S}_{\text{det},r}}| \gg |\mathcal{R}_{\text{S}_{\text{det},l}}|$.

### C. GENERAL CHARGE STATES

#### I. Weak measurement

The generalization of Eq. (6) in the main text to the case of arbitrary charge states $|n\rangle$ (with charge $Q_n$) is straightforward. Parametrizing the scattering matrix as $t_n = \sqrt{T_n} e^{i \theta_n}$ and $r_n = \sqrt{T_n - T_n^*} e^{i \chi_n}$ with $T_n = T + \delta T_n$, $\theta_n = \theta + \delta \theta_n$, and $\chi_n = \chi + \delta \chi_n$, an expansion of Eq. (3) in $\delta T_n, \delta \theta_n,$ and $\delta \chi_n$ results in (see Eq. (5))

$$
P_{tt} = T^2 + \sum_n |\psi_n|^2 T \delta T_n + \sum_{m,n} U_{mn} \psi_m^* |T \delta T_m
+ \sum_{n,m} R_{C_{mn}} \delta T_n \delta T_m + \sum_{n,m} I_{C_{mn}} T \delta \theta_n \delta T_m
$$

(C.1)
with the matrix elements
\[ R_{mn} = \langle \bar{\psi}_m | U_{mn} | \bar{\psi}_n + \text{c.c.} \rangle / 2, \]
\[ I_{mn} = i \langle \bar{\psi}_m^* | U_{mn} | \bar{\psi}_n + \text{c.c.} \rangle, \]
and \( \psi_n = \psi_n (\tau_1), \psi_m = \psi_m (\tau_2) = \sum_n U_{mn} \psi_n. \) Assuming a linear detector with \( \delta T_n = -Q_n \delta \tau, \delta \theta_n = Q_n \delta \theta, \) and \( \delta \chi_n = Q_n \delta \chi, \) we obtain (with \( Q = \sum_n Q_n |n\rangle \langle n| \))
\[ P_{11} = T^2 - \langle \hat{Q} (\tau_1) \rangle T \delta T - \langle \hat{Q} (\tau_2) \rangle T \delta T \]
\[ + R S_{QQ} (\tau_1, \tau_2) \delta T^2 - T S_{QQ} (\tau_1, \tau_2) \delta T \delta T, \]
with the expectation values \( \langle \hat{Q} (\tau_j) \rangle = \sum_n Q_n |\psi_n (\tau_j)\rangle^2 \)
and the correlators
\[ R S_{QQ} (\tau_1, \tau_2) = \frac{1}{2} \{ \langle \hat{Q} (\tau_1) \rangle \langle \hat{Q} (\tau_2) \rangle \} = \sum_{m,n} Q_{mn} R_{mn}, \]
\[ I S_{QQ} (\tau_1, \tau_2) = -i \{ \langle \hat{Q} (\tau_1) \rangle \langle \hat{Q} (\tau_2) \rangle \} = \sum_{m,n} Q_{mn} I_{mn}, \]
with expectation values defined with respect to \( | \psi (\tau_n) \rangle \).
The assumption of a linear detector is crucial in arriving at the correlators (C.5) and (C.6). Subtracting the uncorrelated contribution \( \langle \psi_f | \bar{\psi}_1 (\tau_1) \langle \psi_f | \bar{\psi}_2 (\tau_2) \rangle = (T - \langle \hat{Q} (\tau_1) \rangle T - \langle \hat{Q} (\tau_2) \rangle T) \delta T, \) we arrive at Eq. (6) with the appropriate charge operator defined above entering the correlators \( R_{\text{crr}}^{QQ} (\tau_1, \tau_2) = R_{QQ} (\tau_1, \tau_2) - \langle \hat{Q} (\tau_1) \rangle \langle \hat{Q} (\tau_2) \rangle \) and \( I_{\text{crr}}^{QQ} (\tau_1, \tau_2) = I_{QQ} (\tau_1, \tau_2). \) The other probabilities \( P_{\text{crr}}^{\alpha \beta} \) follow analogously.

II. Strong measurement

A single electron is unable to distinguish between more than two charge states. For a strong measurement, we consider two memory arrays with trains of \( J \) electrons each, probing the system state close to \( \tau_1 \) and \( \tau_2. \) The separation \( \delta \tau \) between electrons within a train is sufficiently large to allow for their distinction while the overall duration \( J \delta \tau \) of the train is assumed to be small on the scale of the system dynamics in order to be able to measure the system before it changes.

Assuming the system not to change during the passage of one train of electrons, the final state of system and memory is of the same form as in Eq. (1) with the outgoing individual memory states \( | \phi_m^{(1)} \rangle \) and \( | \phi_m^{(2)} \rangle \) replaced by the outgoing train memory states \( | \phi_n^{(1)} \rangle = \prod_{j=1}^J | \phi_n^{(j)} \rangle \) and \( | \phi_n^{(2)} \rangle = \prod_{j=J+1}^J | \phi_n^{(j)} \rangle. \) Making use of Born’s rule, we obtain the probabilities \( P_{\alpha \beta}^{\mu \nu} (\tau_1) \) for finding \( \mu \) electrons of the first (second) train transmitted across \( (\alpha, \beta = t) \) or reflected from \( (\alpha, \beta = r) \) the QPC,
\[ P_{\alpha \beta}^{\mu \nu} = \left( \begin{array}{cc} J & J' \\ \mu & \nu \end{array} \right) \sum_n \left( \begin{array}{cc} s_m \rangle \langle r_m^\alpha \psi_m^{\mu} \langle s_n \rangle \langle r_n^\nu \psi_n^{\nu} \end{array} \right)^2, \]
with \( s_n^\alpha = r_n \) and \( s_n^\nu = t_n. \) Here, we made use of the distinguishability of the individual electrons. Introducing the conditional probability (a detector property)
\[ P_\alpha (\mu | n) = \left( \begin{array}{c} J \\ \mu \end{array} \right) | s_n^\alpha \rangle \langle r_n^\nu | n \rangle, \]
for having \( \mu \) of \( J \) particles transmitted \( (\alpha = t) \) or reflected \( (\alpha = r) \) with the system in state \( | n \rangle \) and assuming a symmetric scatterer with \( t_n^\alpha r_n^\alpha + r_n^\alpha t_n^\alpha = 0, \) we can separate the system- and detector response in the above equation,
\[ P_{\alpha \beta}^{\mu \nu} = \sum_{n,n',m} U_{mn} \psi_n U_{mn'}^* \psi_{n'}^{\nu} R_\alpha (\mu | n, n') P_\beta (\nu | m) \]
with \( R_\alpha (\mu | n, n') = \sqrt{P_\alpha (\mu | n) P_\alpha (\mu | n')} e^{i J (\theta_n - \theta_{n'})} \) and \( \theta_n \) the phases of the transmission amplitudes \( t_n. \) Note that the conditional probabilities \( P_\alpha (\mu | n) \) depend only on the charge \( Q_n \) of the state \( | n \rangle \) (and not on the state \( | n \rangle \) itself). We then have to distinguish two cases: (i) all charge states \( | n \rangle \) are non-degenerate, i.e., \( Q_n \neq Q_{n'} \) for \( n \neq n', \) (ii) there are degenerate charge states.

In the non-degenerate case (i), for a strong measurement, the probability distributions \( P_\alpha (\mu | n) \) and \( P_\alpha (\mu | n') \) for different \( n \neq n' \) do not overlap as functions of \( \mu \) and therefore the ‘off-diagonal’ elements \( R_\alpha (\mu | n, n') \) are suppressed. We then can simplify \( P_{\alpha \beta}^{\mu \nu} = \sum_{n,m} | U_{mn} \psi_n |^2 P_\alpha (\mu | n) P_\beta (\nu | m) \). The charge-charge correlator in the reflected channel takes the form
\[ S_{q_\alpha q_\beta} = \langle \Psi_f | \sum_{j=1}^J i \sum_{j=J+1}^J \langle \psi_m^{(j)} | \psi_m^{(j')} \rangle \rangle = \sum_{\mu, \nu} P_{\mu \nu} \]
and assuming a linear detector with \( R_n = Q_n \delta R \), we obtain the final result (see Eq. (14))
\[ S_{q_\alpha q_\beta} = (J \delta R)^2 \sum_{m,n} Q_m Q_n | U_{mn} \psi_n |^2 = (J \delta R)^2 S_{QQ} (\tau_1, \tau_2), \]
where we used the relation \( \int \mu \rho \hat{P} (\tau_1) = J R_n = J Q_n \delta R \)
\[ S_{QQ} (\tau_1, \tau_2) = \text{Tr} [ \hat{Q} (\tau_2) \hat{Q} (\tau_1) \hat{\rho} (\tau_1) ] \]
with the projected density matrix \( \hat{\rho} (\tau_1) = \sum_n \rho (\tau_1) P_n \) and the projection operators \( P_n = | n \rangle \langle n |. \)

In the degenerate case (ii) the distribution functions separate only for different charge states (i.e., \( n \) and \( n' \)) with \( Q_n \neq Q_{n'} \) and hence \( R_\alpha (\mu | n, n') \sim 0, \) while for degenerate charge states with \( Q_n = Q_{n'} \), \( R_\alpha (\mu | n, n') = P_\alpha (\mu | n) \) the probabilities \( P_{\alpha \beta}^{\mu \nu} \) then are given by
\[ P_{\alpha \beta}^{\mu \nu} = \sum_{m,n} \sum_{n'|Q_{n'}=Q_n} U_{mn} \psi_n U_{mn'}^* \psi_{n'}^{\nu} P_\alpha (\mu | n) P_\beta (\nu | m). \]

For the charge-charge correlator in the reflected channel \( S_{q_\alpha q_\beta} \) we find
\[ S_{q_\alpha q_\beta} = (J \delta R)^2 \sum_{m,n} Q_m Q_n \sum_{n'|Q_{n'}=Q_n} U_{mn} \psi_n U_{mn'}^* \psi_{n'}^{\nu} \]
\[ = (J \delta R)^2 S_{QQ} (\tau_1, \tau_2), \]
with \( \hat{\rho}^f(\tau_1) = \sum_Q \hat{P}_Q \hat{\rho}(\tau_1) \hat{P}_Q \) the projected density matrix and the projection operator \( \hat{P}_Q = \sum_{n|Q_n=Q} |n\rangle \langle n| \). The results (C.10) and (C.12) generalize the relation (4) in the main text to systems with many charge states.

**D. OPEN SYSTEMS**

The considerations in the main text deal with an isolated system described by eigenstates \(|n\rangle\). Here, we generalize the main result Eq. (3) to open systems and outline how to obtain the weak coupling results. We describe the open system through its density matrix with \( \hat{\rho}(\tau = \tau_n) = \hat{\rho}_0 \). We introduce the projection operators \( \hat{P}_n \) projecting on the system states with \( n \) charges in the QD. The projection operators satisfy \( \sum_n \hat{P}_n = 1 \) and allow us to express the density matrix as \( \hat{\rho}(\tau) = \sum_{n,n'} \hat{P}_n \hat{\rho}(\tau) \hat{P}_{n'} \). Note that the projection operators \( \hat{P}_n \) are introduced as a mathematical convenience and are not related to the physical von Neumann projection postulate. Starting from the initial system–memory density matrix \( \hat{\rho}_0 \otimes |\phi_{in}^{(1)}\rangle\langle\phi_{in}^{(1)}| \otimes |\phi_{in}^{(2)}\rangle\langle\phi_{in}^{(2)}| \) at time \( \tau_n \), we proceed similarly as in the case of isolated systems by conditioning the time evolution of the memory states on the corresponding system states and obtain the final density matrix at time \( \tau_f \)

\[
\hat{\rho}_f = \sum_{n,n',m,m'} \hat{U}(\tau_{f2}) \hat{P}_m \hat{U}^\dagger(\tau_{21}) \hat{P}_n \hat{\rho}(\tau_1) \hat{P}_n \hat{U}^\dagger(\tau_{21}) \hat{P}_m \hat{U}(\tau_{f2}) \otimes |\phi_{in}^{(1)}\rangle\langle\phi_{in}^{(1)}| \otimes |\phi_{m}^{(2)}\rangle\langle\phi_{m}^{(2)}|,
\]

with the system density matrix \( \hat{\rho}(\tau_1) \) at time \( \tau_1 \). Note that here, the outgoing states \( |\phi_{m}^{(2)}\rangle \) resp. \( |\phi_{n}^{(1)}\rangle \) are conditioned on the charge state \( n^{(r)} \), resp. \( m^{(l)} \), of the system at times \( \tau_1 \) resp. \( \tau_2 \). We define the probabilities \( P_{\alpha\beta} \) as

\[
P_{\alpha\beta} = \text{Tr}[\hat{q}_\alpha^\dagger \hat{q}_\beta^\dagger \hat{\rho}_f],
\]

with the trace taken over both the system and the memory states. Calculating this expression with the final density matrix Eq. (D.1), we obtain

\[
P_{\alpha\beta} = \sum_{n,n',m} \text{Tr}[S_{nm} \hat{U}(\tau_{21}) \hat{P}_m \hat{\rho}(\tau_1) \hat{P}_n \hat{U}^\dagger(\tau_{21}) S_{nm}^\dagger \hat{\rho}(\tau_1) \hat{P}_n \hat{U}(\tau_{21})],
\]

This result is the direct generalization of our main result (3) to the case of open systems. It is straightforward to evaluate the weak-coupling limit for a linear detector: using the same parametrization for the scattering matrix and expanding in \( \delta T, \delta \theta, \delta \chi \), we obtain the generalized version of Eq. (6) with the charge operator given by \( \hat{Q} = \sum_n \hat{P}_n \) and the expectation values in Eqs. (7) and (8) given through \( \langle \hat{A} \rangle = \text{Tr}_{n_{\text{sys}}} \langle \hat{A} \hat{\rho}_0 \rangle \). The generalization to open systems of strong measurements by trains of electrons is possible as well but involves a lengthy calculation which we do not pursue here.

**E. FINITE-WIDTH MEMORY WAVE-PACKETS**

In the main text we have assumed an instantaneous entanglement between the system and the memory states, requiring that both the width \( \tau_{wp} \) of the wave-packet and the scattering time \( \tau_{sca} \) at the QPC satisfy \( \tau_{wp}, \tau_{sca} \ll \tau_{sys} \), where \( \tau_{sys} \) denotes the characteristic time scale of the system. Here, we allow for a spread in time of the detector’s electron wave function and drop the condition \( \tau_{wp} \ll \tau_{sys} \), i.e., we assume that \( \tau_{wp} \lesssim \tau_{sys} \) while the scattering event itself remains fast, \( \tau_{sca} \ll \tau_{sys} \). In general terms, this corresponds to a measurement which probes the system sharply (\( \tau_{sca} \ll \tau_{sys} \)) during some finite time (\( \tau_{wp} \lesssim \tau_{sys} \); longer measurement times \( \tau_{wp} > \tau_{sys} \) do not provide meaningful results).

Let us suppose that the \( j \)-th wave-packet incident on the QPC around \( \tau_j \) is described by the wave function \( f^{(j)}(\tau) \) which is normalized (\( \int d\tau |f^{(j)}(\tau)|^2 = 1 \)) and peaked at the time \( \tau_j \). Assuming instantaneous scattering, we obtain the final state

\[
|\tilde{\psi}_f \rangle = \int d\tau_1 f^{(j)}(\tau_1) \int d\tau_2 f^{(2j)}(\tau_2) \times \sum_{l,m,n} U_{lm}(\tau_{f2}) U_{mn}(\tau_{21}) \psi_n(\tau_1) |\phi_{n}^{(1)}(\tau_1)\rangle |\phi_{m}^{(2)}(\tau_2)\rangle,
\]

with \( \tau_{f2} = \tau_f - \tau_2 \) and \( \tau_{21} = \tau_2 - \tau_1 \) and the outgoing memory states \( |\phi_{m}^{(2)}(\tau_2)\rangle \) that have been scattered at the QPC at time \( \tau_f \). Making use of \( |\phi_{n}^{(1)}(\tau_f)\rangle |\phi_{m}^{(2)}(\tau_f)\rangle = \delta(\tau_f - \tau_0) (|\phi_{n}^{(1)}\rangle|\phi_{m}^{(2)}\rangle) \), we obtain the smeared probabilities

\[
\hat{P}_{\alpha\beta}(\tau_2) = \int d\tau_1 \int d\tau_2 |f^{(j)}(\tau_1)|^2 |f^{(2j)}(\tau_2)|^2 P_{\alpha\beta}(\tau_2),
\]

with \( P_{\alpha\beta} \) given by Eq. (3). The finite width wave-packets enter as integration kernels. While for sharp wave-packets, the entanglement between system and memories (i.e., the ’measurement’) takes place at times \( \tau_1 \) and \( \tau_2 \), for broader wave-packets, the entanglement arises within a finite time \( \tau_{wp} \) around \( \tau_1 \) and \( \tau_2 \) with distribution functions \( |f^{(j)}(\tau_1)|^2 \) and \( |f^{(j)}(\tau_2)|^2 \). As a consequence, for the case of strong coupling where the entanglement gives rise to projective measurements, the times of projection are not fixed but distributed with the distribution functions above.

Note that the result (E.2) is only valid for negligible scattering time \( \tau_{sca} \) in the QPC. If the scattering time \( \tau_{sca} \) is finite compared to the system time \( \tau_{sys} \), the effect of interaction cannot be accounted for by the scattering matrices \( S_n \) of the QPC depending on the system state, but the interaction during the scattering has to be treated in more detail.

The limit of \( \tau_{sca} \ll \tau_{sys} \) considered above has also implications for the resulting back-action: While the back-action for \( \tau_{sca} \ll \tau_{sys} \) only consists of dephasing, i.e., a suppression of off-diagonal elements of the system’s den-
sity matrix, for finite $\tau_{\text{esc}}$, the back-action of the measurement on the system goes beyond pure dephasing and alters the system’s dynamics.

**F. PHOTONS**

We derive two theoretical schemes to measure the two-time photon correlator, one using qubits as memories as described in the main text and a second scheme involving ionized states. While the first approach is closer in spirit to the argumentation in the main text, the subsequent discussion is more in line with the traditional photoionization measurement scheme.

**I. Qubit excitation**

We determine the photon correlator with the help of two qubits placed at positions $r_1$ and $r_2$ serving as quantum memories. The latter interact with the photon field via the Hamiltonian (we drop the diamagnetic term)

$$\hat{H}_{\text{int}} = \sum_{i=1,2} \int d^3r \frac{e}{2mc} [\hat{p}_i \cdot \hat{A}(r) + \hat{A}(r) \cdot \hat{p}_i]$$

with the photon field $\hat{A}(r) = \hat{A}^+(r) + \hat{A}^-(r)$,

$$\hat{A}^+(r) = \sum_{q,s} \left( \frac{\hbar c^2}{\sqrt{\omega q}} \right)^{1/2} \hat{a}_{q,s} \hat{c}_{q,s} e^{iqr},$$

and $\hat{A}^-(r) = \hat{A}^{\dagger}(r)$. Choosing the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ and using the dipole approximation, we arrive at

$$\hat{H}_{\text{int}} = \frac{e}{mc} \sum_{i=1,2} \hat{p}_i \cdot \hat{A}(r_i).$$

Expressing the Hamiltonian in the basis $|g_i\rangle$, $|e_i\rangle$ of the qubits, introducing the operators $\sigma_i^+ = |e_i\rangle\langle g_i|$ and $\sigma_i^- = |g_i\rangle\langle e_i|$, and adopting the rotating wave-approximation, we find

$$\hat{H}_{\text{int}} = \sum_{i=1,2} = \sigma_i^+ [d_i^+ \cdot \hat{A}^+(r_i)] + \sigma_i^- [d_i^\dagger \cdot \hat{A}^-(r_i)].$$

with the dipole elements $d_i = (e/mc)(e_i|\hat{p}_i|g_i)$. Finally, we assume time dependent couplings, e.g., via tuning the qubits in and out of resonance with the photon field, and consider an axial polarization $d_i(\tau)$ of the qubits along $x$ to arrive at the tunable Jaynes-Cummings type Hamiltonian

$$\hat{H}_{\text{int}} = \sum_{i=1,2} d_i(\tau) \sigma_i^+ \hat{A}^+(r_i) + d_i(\tau) \sigma_i^- \hat{A}^-(r_i).$$

The initial state at time $\tau_0 = 0$ of the combined qubit-photon system is described by the density matrix

$$\hat{\rho}(0) = |g_1\rangle\langle g_1| \otimes |g_2\rangle\langle g_2| \otimes \hat{\rho}_{\text{ph}}(0)$$

with the density matrix of the photon field $\hat{\rho}_{\text{ph}}$ and the two qubits in their ground state $|g_i\rangle$. Going over to the Dirac representation, the unitary evolution $\hat{U}_{\text{dir}}(\tau_f) = T \exp[-i \int_0^{\tau_f} d\tau \hat{H}_{\text{int}}(\tau)/\hbar]$ takes the density matrix to the final time $\tau_f$,

$$\hat{\rho}_f = \hat{U}_{\text{dir}}(\tau_f) \hat{\rho}(0) \hat{U}^*_{\text{dir}}(\tau_f),$$

with $\hat{H}_{\text{int}}(\tau)$ adapted to the Dirac representation. The probability $P_{ee}$ to find the two detectors in their excited states $|e_i\rangle$,

$$P_{ee} = \text{Tr}[\hat{\rho}_f^{(ee)}(\tau_f) \hat{\rho}_f(\tau_f)],$$

provides us with the desired information about the photon correlation; here, $\hat{\rho}_f^{(ee)} = \hat{\sigma}_1^+ \hat{\sigma}_2^+$ are the projectors on the excited state of the i-th detector. Calculating the final density matrix $\hat{\rho}_f$ in lowest order perturbation theory, we obtain the normal ordered expression ($\Omega_i$ denote the transition frequencies of the qubits)

$$P_{ee} = \frac{1}{h^4} \prod_{i=1,2} \int_0^{\tau_f} d\tau_i' d\tau_i'' d\tau_i'(\tau_i',\tau_i'') e^{i\Omega_i(\tau_i'-\tau_i'')} \times \langle \{ \mathcal{T}_\text{r} \hat{A}^-(r_1,\tau_i') \hat{A}^+(r_2,\tau_i') \hat{A}^-(r_2,\tau_i'') \hat{A}^+(r_1,\tau_i'') \} \rangle. \tag{F.9}$$

Next, we adopt a quasi-monochromatic approximation with the photon field carrying frequencies $\omega$ within a narrow interval $\delta \omega$ around $\tilde{\omega}$. The (free) time evolution of the field operators, e.g., the annihilation operator $\hat{a}_{q,s}$ in (F.2), then can be approximated by

$$e^{-i\omega_\tilde{\omega} \tau'} = e^{-i\omega_\tilde{\omega} \tau} e^{-i\omega_\tilde{\omega}(\tau'-\tau)} = e^{-i\omega_\tilde{\omega} \tau} e^{-[\omega + (\omega_\tilde{\omega} - \tilde{\omega})](\tau'-\tau)} \approx e^{-i\omega_\tilde{\omega} \tau} e^{-i\tilde{\omega}(\tau'-\tau)}, \tag{F.11}$$

where we have dropped the phase $(\omega_\tilde{\omega} - \tilde{\omega})(\tau'-\tau)$; since $\tau' - \tau \lesssim \Delta \tau$, the narrow frequency band $\delta \omega$ defining the quasi-monochromatic approximation restricts the time window for the interaction to $\Delta \tau \ll 1/\delta \omega$. Making use of this approximation in (F.10), we find that

$$P_{ee} = \prod_{i=1,2} |d_i|^2 \int_{\tau_i}^{\tau_i+\Delta \tau} d\tau_i' d\tau_i'' e^{i(\Omega_i-\tilde{\omega})(\tau_i'-\tau_i'')} \times \langle \{ \hat{A}^-(r_1,\tau_i') \hat{A}^+(r_2,\tau_i') \hat{A}^-(r_2,\tau_i'') \hat{A}^+(r_1,\tau_i'') \} \rangle. \tag{F.12}$$
The time integrals are trivial and the expansion for small time intervals $\Delta \tau (\Omega_0 - \omega) \ll 1$ (small detuning of the qubits away from the photonic carrier frequency $\omega$) produces the result

$$P_{ee} = \frac{[d_1^2 |d_2|^2] \Delta \tau^4}{\hbar^4} \times \langle \hat{A}_x(r_1, \tau_1) \hat{A}_x(r_2, \tau_2) \hat{A}_x^*_x(r_2, \tau_2) \hat{A}_x^*_x(r_1, \tau_1) \rangle \tag{F.13}$$

Repeating the measurement for different directions of the qubit polarizations, we can construct the final result $P_{ee} = \{(d_1^2 |d_2|^2) \Delta \tau^4 / \hbar^4\}(\tau_K : \hat{I}(r_2, \tau_2) \hat{I}(r_1, \tau_1) \rangle$ in terms of field intensities $\hat{I}(r_1, \tau_i) = \sum \delta \hat{A}_x^*(r_1, \tau_1) \hat{A}_x^*(r_1, \tau_1)$. The photonic correlator appearing in (F.13) coincides with the well known results of Glauber [2] and of Kelley and Kleiner [3].

II. Photoionization

We consider a photon field described by $\hat{A}$ as described above interacting with the detector via (F.3). As our photon detectors, we consider two atoms coupled to individual 1D waveguides, e.g., carbon nanotubes, see Fig. F.1, channeling the ionized electrons to the measuring device at positions $R_1$ and $R_2$. While before the memory states stored the quantum information in superpositions of forward-/back-scattered electrons or ground-/excited-state qubits, here, the two memory states involve an electron localized on the atom or an ionized electron in the wire. We restrict the discussion below dealing with the waveguides along $x$ to one dimension as the transverse dimensions $y, z$ can be integrated over trivially.

![FIG. F.1: Photon detection via ionization of two atoms positioned at $r_1$ and $r_2$. The excited electrons are channeled through 1D waveguides to the detectors at $R_1$ and $R_2.$](image)

1. Single-photon absorption by one atom

We start with an analysis of a single photo-excitation of one electron by one photon at position $x_1$ and the detector at $X_1$, with both coordinates taken along the waveguide. The electronic state at time $\tau_i = 0$ is denoted by $|\Phi_b\rangle$ and the initial density matrix reads

$$\hat{\rho}(0) = |\Phi_b\rangle \langle \Phi_b | \otimes \hat{\rho}_{ph}(0). \tag{F.14}$$

We consider the unitary evolution within the interaction representation with the evolution operator given as $\hat{U}_d(\tau) = T \exp\left[-i \int_0^\tau d\tau' \hat{H}_{\text{int}}(\tau') / \hbar\right]$. The density matrix at a later time $\tau_f$ is given by

$$\hat{\rho}(\tau_f) = \hat{U}_d(\tau_f) \hat{\rho}(0) \hat{U}_d^\dagger(\tau_f). \tag{F.15}$$

Expanding the evolution operator, i.e., $\hat{U}_d(\tau) = 1 - i \int_0^\tau d\tau' \hat{H}_{\text{int}}(\tau') / \hbar + \ldots$, the leading contribution to the density matrix describing photon absorption is

$$\delta \hat{\rho}(\tau_f) = \frac{1}{\hbar^2} \int_{0}^{\tau_f} \int_{0}^{\tau_f} d\tau / \int_{0}^{\tau_f} d\tau' \hat{H}_{\text{int}}(\tau) \hat{\rho}(0) \hat{H}_{\text{int}}(\tau') \tag{F.16}$$

$$= \left( \frac{e}{\hbar mc} \right)^2 \int_{0}^{\tau_f} d\tau / \int_{0}^{\tau_f} d\tau' \hat{p}_x(\tau) \hat{\rho}(0) \hat{p}_x(\tau) \otimes \hat{A}_x(x_1, \tau) \hat{\rho}(0) \hat{A}_x(x_1, \tau).$$

At time $\tau_f$, the photo-excited electron is detected at the position $X_1$ along the waveguide, i.e., we measure the electron number $n = \hat{\Psi}^\dagger(X_1) \hat{\Psi}(X_1)$. The corresponding expectation value is given by (with $\langle \cdots \rangle$ denoting the average over photonic degrees of freedom)

$$\langle \hat{n} \rangle = \text{Tr} \left[ \hat{\rho}(\tau_f) \delta \hat{\rho}(\tau_f) \right] \tag{F.17}$$

$$= \int_{0}^{\tau_f} d\tau / \int_{0}^{\tau_f} d\tau' \left( \frac{e}{\hbar mc} \right)^2 \langle \Phi_b | \hat{p}_x(\tau') \hat{n}(\tau_f) \hat{p}_x(\tau) | \Phi_b \rangle$$

$$\times \langle \hat{A}_x(x_1, \tau_f) \hat{A}_x(x_1, \tau) \rangle$$

$$= \int_{0}^{\tau_f} d\tau / \int_{0}^{\tau_f} d\tau' \mathcal{K}(\tau, \tau') \langle \hat{A}_x(x_1, \tau_f) \hat{A}_x(x_1, \tau) \rangle$$

with $\mathcal{K}(\tau, \tau')$ the response function describing the photo-excitation of the electron at the position $x_1$ and its subsequent detection (at time $\tau_f$) in the position $X_1$. The available electronic states are described by the bound state $\phi_b(x - x_1)$ of width $w$ and with energy $\epsilon_b < 0$ and the freely propagating states $\phi_b(x) = e^{i k x}$ with energies $\epsilon_k = h^2 k^2 / 2m$ (having in mind equal transverse wavefunctions $\chi(x, y)$ for both the bound and free states). The detector response function then is given by

$$\mathcal{K}(\tau, \tau') = \left( \frac{e}{\hbar mc} \right)^2 \langle \Phi_b | \hat{p}_x(\tau) \hat{n}(\tau_f) \hat{p}_x(\tau) | \Phi_b \rangle$$

$$= \left( \frac{e}{\hbar mc} \right)^2 \int_{0}^{\infty} dk dk' \sigma(k) \sigma(k')$$

$$\times \langle \Phi_b | \hat{p}_x(\tau') | \Phi_b \rangle \langle \hat{n}(\tau_f) | \Phi_b \rangle \langle \Phi_b | \hat{p}_x(\tau) | \Phi_b \rangle$$

with the density of states $\sigma(k)$. The matrix elements are given by

$$\langle \Phi_k | \hat{p}_x(\tau) | \Phi_b \rangle = \hbar k \phi_b(k) e^{i k x_1} e^{i (\epsilon_b - \epsilon_k) \tau / \hbar} \tag{F.18}$$

$$\text{where} \phi_b(k) \text{denotes the Fourier transform of } \phi_b(x), \text{ and}$$

$$\langle \Phi_b | \hat{n}(\tau_f) | \Phi_k \rangle = e^{-i k x_1} e^{i \epsilon_x \tau_f / \hbar} e^{-i k x_1} e^{-i \epsilon_k \tau_f / \hbar}. \tag{F.20}$$
Inserting this expression into Eq. (F.24), we find
\[ q(\tau) = \frac{e}{m\hbar} \int_0^\infty dk \sigma(k) \hbar k \times \phi_b(k) e^{-i(\varepsilon_n - \varepsilon_0)(\tau_f - \tau)/\hbar} e^{ik(X_1 - x_1)} \] (F.21)

describing an electron emitted at the position \( x_1 \) and time \( \tau \) and measured in the detector at \( X_1 \) at a later time \( \tau_f \). For a linear dispersion \( \varepsilon_k \) (with velocity \( v \)), this amplitude is roughly given by the bound state amplitude shifted by \( v(\tau_f - \tau) \) and evaluated at \( X_1 \), \( q(\tau) \propto \phi_b[X_1 - x_1 - v(\tau_f - \tau)] \). The kernel
\[ K(\tau, \tau') = q(\tau) q^*(\tau') \] (F.22)
therefore is finite for times \( \tau, \tau' \) in a time window of width \( w/v \) centered at \( \tau_0 = \tau_f - (X_1 - x_1)/v \).

In the following, we will make use of the kernel's particular shape in order to simplify the expression (F.17). Indeed, the latter can be factorized into two time integrals of the form
\[ \int_0^{\tau_f} d\tau \ e^{i(\varepsilon_n - \varepsilon_0) \tau} \] (F.23)
with \( \pm \omega_q \) referring to the positive/negative frequency parts of the vector field's \( \mathbf{q} \) components \( \hat{A}^\pm(\mathbf{q}, \tau) \) and \( k \) referring to the \( k \) component in the response amplitude. The physically relevant absorption processes involve the negative frequency part of \( \hat{A} \) (the rapid oscillations of the positive frequency parts suppress the time integral for large \( \tau_f \)) and only the normal-ordered term
\[ \langle \hat{n} \rangle = \int_0^{\tau_f} d\tau d\tau' K(\tau, \tau') \left[ \hat{A}_x^+(x_1, \tau') \hat{A}_x^+(x_1, \tau) \right] \] (F.24)
survives. The time integrals in (F.24) are restricted to an interval of width \( w/v \) around \( \tau_0 \). Assuming a quasi-monochromatic photon field with frequencies \( \omega \) only within an narrow interval \( \delta \omega \) around \( \bar{\omega} \), we make use of the approximation (F.11) (with \( \tau \sim \tau_0, \tau' \sim \tau_0 + \delta \tau \)) to approximate
\[ \hat{A}_x^+(x_1, \tau = \tau_0 + \delta \tau) \approx \hat{A}_x^+(x_1, \tau = \tau_0) e^{i\bar{\omega} \delta \tau}. \] (F.25)

Inserting this expression into Eq. (F.24), we find
\[ \langle \hat{n} \rangle = \int_0^{\tau_f} d\tau d\tau' K(\tau, \tau') e^{-i\bar{\omega}(\tau - \tau_0)} e^{i\bar{\omega}(\tau' - \tau_0)} \times \left[ \hat{A}_x^+(x_1, \tau_0) \hat{A}_x^+(x_1, \tau_0) \right] \] (F.26)
and extending the integration boundaries to infinity, we obtain
\[ \langle \hat{n} \rangle = \left| \frac{2\pi \sigma(\bar{\omega}) \phi_b(\bar{\omega})}{\hbar c} \right|^2 \left[ \hat{A}_x^+(x_1, \tau_0) \hat{A}_x^+(x_1, \tau_0) \right] \] (F.27)
with \( \bar{\omega} = 2m\sqrt{\omega + \varepsilon_0}/\hbar \). Defining the \((x,\text{polarized})\) electric field intensity as \( I_x(x_1, \tau_0) = E_x^+(x_1, \tau_0) E_x^+(x_1, \tau_0) \),

the result (F.27) can be written in the form (we explicitly write normal ordering \( : \cdots : \) for emphasis and assume a narrow bandwidth of the photon field)
\[ \langle \hat{n} \rangle = C \langle : \hat{I}_x(x_1, \tau_0) : \rangle \] (F.28)
with \( C = |2\pi \sigma(\bar{\omega}) \phi_b(\bar{\omega})/\hbar \bar{\omega}|^2 \).

### 2. Two-photon absorption by two atoms

Next, we analyze the two-photon absorption by two atoms coupled to individual waveguides. The initial density matrix is given by
\[ \hat{\rho}(0) = |\Phi^{(1)}_b\rangle \langle \Phi^{(1)}_b| \otimes |\Phi^{(2)}_b\rangle \langle \Phi^{(2)}_b| \otimes \hat{\rho}_{\text{ph}}(0) \] (F.29)
and the interaction Hamiltonian reads \( H_{\text{int}} = H^{(1)}_{\text{int}} + H^{(2)}_{\text{int}} \). The unitary evolution of this density matrix will generate a variety of terms involving no ionization, ionization of one or both atoms during any time between the initial and final times \( \tau_i \) and \( \tau_f \), both diagonal and off-diagonal in these events. Of all these terms, the final state projection selects the term describing two-photon processes as given by
\[ \delta \hat{\rho}(\tau_f) = \frac{1}{\hbar^2} \int_0^{\tau_f} d\tau d\tau' d\bar{\tau} d\bar{\tau}' \times T\left[ H^{(1)}_{\text{int}}(\bar{\tau}) H^{(2)}_{\text{int}}(\bar{\tau}') \right] \hat{\rho}(0) T\left[ H^{(1)}_{\text{int}}(\bar{\tau}') H^{(2)}_{\text{int}}(\bar{\tau}) \right] \] (F.30)
with the time-ordering and anti-time-ordering operators \( T \) and \( \bar{T} \). A straightforward calculation along the lines described above provides us with the expression for the correlator \( \langle \hat{n}_1 \hat{n}_2 \rangle = \text{Tr}[\hat{n}_1(\tau_f) \hat{n}_2(\tau_f) \delta \hat{\rho}(\tau_f)] \),
\[ \langle \hat{n}_1 \hat{n}_2 \rangle = \int_0^{\tau_f} d\tau d\tau' d\bar{\tau} d\bar{\tau}' K^{(1)}(\tau, \bar{\tau}) K^{(2)}(\tau', \bar{\tau}') \times \langle \bar{T}\left( A_x(r_1, \bar{\tau}) A_x(r_2, \bar{\tau}') \right) \rangle \times \langle \bar{T}\left( A_x(r_2, \tau') A_x(r_1, \tau) \right) \rangle \] (F.31)
with the response kernels \( K^{(1,2)} \) for the two waveguides peaking around the times \( \tau_1 = \tau_f - (X_1 - x_1)/v \) and \( \tau_2 = \tau_f - (X_2 - x_2)/v \), respectively. The dominance of the absorption processes enforces normal ordering and use of the quasi-monochromatic approximation (see Eq. (F.25)) produces the final result
\[ \langle \hat{n}_1 \hat{n}_2 \rangle = C^2 \langle T_K : \hat{I}_x(r_2, \tau_2) \hat{I}_x(r_1, \tau_1) : \rangle \] (F.32)
with \( T_K \) ordering creation operators in a time-ordered way and annihilation operators in an anti-time-ordered way. The correlator \( \langle \hat{n}_1 \hat{n}_2 \rangle \) picks up weight only from processes where electrons are detected in each of the two wires and therefore coincides with the probability \( P_{ww} \) for the simultaneous measurement of electrons. Repeating the above analysis for other polarizations, we obtain access to the full intensity correlator \( P_{ww} \propto \langle T_K : \hat{I}(r_2, \tau_2) \hat{I}(r_1, \tau_1) : \rangle \).
The photonic correlator appearing on the right-hand side of the result (F.32) agrees with the result of the measurement scheme involving qubits as well as with standard textbook results, see, e.g., Ref. 4. In contrast to the derivation above based on the unitary evolution of quantum mechanics, previous analyses relied on the subsequent projection of the photonic field at times \( \tau_1 \) and \( \tau_2 \), e.g., see equation (4.8) in Ref. 3. Similarly, the derivation in Ref. 4 introduces a von Neumann projection already after the first ionization event, discarding all terms in the density matrix that do not describe an ionized atom at the time \( \tau_1 \). The system then is restarted from the corresponding projected term and is further evolved to the time \( \tau_2 \) where the second atom gets ionized. On the other hand, within our formulation, the first measurement generates an entangled state described by a density matrix that keeps all terms, which then is further propagated via unitary time evolution first to \( \tau_2 \) where a second ionization takes place, and then to the final time \( \tau_f \). The projection to the component \( \hat{\rho} \) in (F.30) describing both ionization events is done only at the very end.

These two schemes, subsequent discarding of terms in the density matrix via consecutive projection as in Ref. 4 versus single projection at the end as described above, formally and philosophically treat the two subsequent ionization events in a different way. A subtle issue is the treatment of the normalization of the projected density matrix after the first measurement at time \( \tau_1 \), \( \hat{\rho} \rightarrow \hat{P}_i \hat{\rho} \hat{P}_i / \text{Tr}[\hat{P}_i] \), as demanded by the von Neumann projection postulate. This normalization by \( \text{Tr}[\hat{P}_i] = \hat{P}_i \) with \( \hat{P}_i \) the projector to the first measurement, has been suppressed in the discussion of Ref. 4, rendering the identification of the first projection more difficult. Suppressing the division by \( \text{Tr}(\hat{P}_i) \) is formally justified, since the final result after the second measurement at time \( \tau_2 \) is given by the product of probabilities \( \text{Tr}(\hat{P}_i) \hat{P}(\hat{P}(2|1) \text{ for the first } \hat{P}(1)) \text{ and (conditional) second event } \hat{P}(2|1) \), with \( \hat{P}(2|1) \) containing the factor \( 1/\text{Tr}(\hat{P}_i) \) from the normalization of the density matrix at time \( \tau_1 \) (note that this statement is independent of the use of the coherent state basis, i.e., Glauber’s \( \hat{P} \) representation).

Nevertheless, we emphasize that the classical formula for the joint probability \( \text{P}(2,1) = \text{P}(1) \text{P}(2|1) \), forming the basis for such a treatment and which in our case reads \( \text{P}(1) \text{P}(2|1) = \langle \psi | \psi \rangle \langle \psi | \text{U}_1 | \text{P} \rangle \langle \psi | \text{U}_1 | \text{P} \rangle \) with \( |0\rangle \) and \( |1\rangle \) the states just before and after the first measurement, is not described by a unitary system evolution.

The formal possibility to ignore the normalization of the density matrix in the restart after the first (strong) measurement, an explicitly non-unitary step, renders the difference between the two schemes, traditional von Neumann projection versus entanglement and final state projection, less clear. The novelty and usefulness of the new measurement scheme based on entanglement and unitary evolution then lies i) in restoring the fully unitary description of the two-time measurement process, and ii) in unifying the treatment of the measurement process for any system–detector coupling, be it weak or strong. Unfortunately, in photodetection, the measurement is always a strong projective one (i.e., the system–detector entanglement is always strong) and we do not have a tuning parameter at our disposal where we can interpolate between strong and weak measurement as in the microscopic setting or in measuring photonic correlations with two qubits initialized away from the ground state.

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