Metric–Affine Gauge Theory of Gravity
I. Fundamental Structure and Field Equations

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Abstract

We give a self-contained introduction into the metric–affine gauge theory of gravity. Starting from the equivalence of reference frames, the prototype of a gauge theory is presented and illustrated by the example of Yang–Mills theory. Along the same lines we perform a gauging of the affine group and establish the geometry of metric–affine gravity. The results are put into the dynamical framework of a classical field theory. We derive subcases of metric–affine gravity by restricting the affine group to some of its subgroups. The important subcase of general relativity as a gauge theory of translations is explained in detail.

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Introduction

The notion of gauge symmetry is one of the cornerstones of theoretical physics. This is known to anybody who ever got in touch with the basics of modern quantum field theory. The three non–gravitational interactions are completely described by means of gauge theories in the framework of the standard model. Predictions of the standard model are experimentally verified with very good accuracy. Thus the concept of gauge symmetry should be contained in any future generalization of the standard model.

At least after the pioneering works of Utiyama [19], Sciama [16, 17], and Kibble [9], it was recognized that also gravitation can be formulated as a gauge theory. In this case, the relevant gauge symmetry is represented by the symmetry of spacetime itself. However, the hope that the formulation of gravity as a gauge theory could lead to a consistent quantum theory of gravity has not been fulfilled yet. Also the inclusion of supersymmetric gauge symmetries (“supergravity” [20]) and fundamental string-like objects (“string theories” [6]) has not changed this drastically.

This article is a self–contained introduction into the metric–affine gauge theory of gravity. The metric–affine (gauge theory of) gravity (MAG) is based on the assumption that affine transformations are gauge (symmetry) transformations of spacetime. It constitutes a general example of the gauging of an external symmetry group\(^1\). The material presented is not completely original. A recent review of metric–affine gravity with an exhaustive reference list is already available [7]. This article is claimed to be original in its kind of presentation of this subject: Starting from the independence of physical results of the choice of affine reference frames (to be defined below), we will develop the metric–affine theory from scratch. This is done in close analogy to the more familiar Yang–Mills theory. In our approach we try to elaborate on the idea behind the gauge procedure. This idea is essential for any gauge approach to gravity. It is hoped that this article makes the gauge framework of gravity accessible to everybody who wants to get started in this field.

The organization of this article is as follows: In Sec.1 we explain our view of what ingredients are the ones that define a gauge theory. This view is illustrated by the example of \(SU(N)\)-Yang Mills theory. In the same spirit, a gauging of the affine group is pursued in Sec.2. Then the emerging structures are embedded into the general framework of a classical field theory in Sec.3. In Sec.4 it is shown how to obtain general relativity as a special case from restricting MAG to a translational gauge theory. More general applications of MAG are left to a forthcoming paper [8].

\(^1\)Here an external symmetry group is understood as a symmetry group of spacetime.
1 Reference frames and gauge systems illustrated by means of $SU(N)$-Yang Mills theory

1.1 General remarks

The dynamical variables of a physical theory are usually expressed with respect to some reference frame\(^2\). Dynamical variables describe gauge systems, if there is some freedom in choosing a reference frame. This freedom is expressed by the possibility to transform a given reference frame into an equivalent one. Such a transformation is called a gauge transformation. Here, the equivalence of reference frames is defined by the symmetry of the physical theory: Equivalent reference frames are those which are connected by a symmetry transformation. The symmetry, in turn, is either postulated or deduced on empirical grounds, from the existence of corresponding conserved currents, e.g.,

Quite generally, we expect a change of the explicit form of the dynamical variables if we change the reference frame. The physically meaningful variables, i.e., the observables, are those which are independent of the reference frame. These variables are called gauge invariant, since they are invariant under any gauge transformation. Gauge transformations are often realized by means of transformations of a Lie group. This group, the gauge or local symmetry group, acts in an appropriate representation on the reference frame, inducing the gauge transformations of the dynamical variables.

As a rule, it is not possible to formulate a gauge theory in terms of gauge invariant variables right from the beginning. Therefore, dealing with gauge theories means dealing with unphysical degrees of freedom, since the freedom of choosing an arbitrary frame should be of no physical relevance. The task is to extract physically meaningful quantities from this, a difficulty which is present at both the classical and the quantum level.

But the gauge principle of choosing an arbitrary reference frame is not just a mathematical nuisance, it also exhibits physical beauty since it leads in a natural way to the introduction of gauge field potentials (=gauge connections) which mediate the interaction between matter. Gauge potentials are essential means for describing a reference frame. Thus they are as fundamental as the notion “reference frame” itself.

To put the arbitrariness of a reference frame at the basis of a gauge theory, as it will be done here, seems to be less familiar than the common definition of a gauge

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\(^2\)If we talk in the following about a “reference frame” we have not necessarily in mind a single reference frame at a single point. We think it is appropriate to also name a field of reference frames simply “reference frame”.

theory in terms of fibre bundle language, see [4, 18], e.g.. There, the gauge connection is viewed as the basic ingredient. The fact that the arbitrariness of a reference frame comes before the definition of a gauge connection may seem trivial. To understand the relationship between both approaches, it is sufficient to understand the basic definition of a linear connection, as explained in [1, 10], for example. In the following, the knowledge required is reformulated, adapted, and explained in view of a smooth introduction into metric–affine gravity.

To begin with, we will expound these introductory remarks in the next subsection by reviewing $SU(N)$-Yang-Mills theory (YM$_{SU(N)}$), probably the most prominent gauge theory. From this we will move on and develop, in close analogy, the gauging of the affine group $A(n, R)$, yielding the metric–affine theory of gravity.

### 1.2 The gauging of $SU(N)$

The gauging of the unitary groups $SU(N)$ is of fundamental importance in elementary particle physics. The standard model of strong and electroweak interactions relies on the gauging of $SU(3) \times SU(2) \times U(1)$, its simplest “grand unification” is described by the gauging of $SU(5)$. For a compact introduction into YM$_{SU(N)}$ we refer the reader to Ref. [2], Chap.8.

As the basic dynamical field variable we take a multiplet field $\psi = (\psi^1, ..., \psi^N)$ with complex components $\psi^i$. By splitting $\psi$ in this way into $N$ components, we have already assumed some reference frame $e_a$ within an $N$-dimensional complex representation space of $\psi$: $\psi = \psi^a e_a$. The $SU(N)$-matrices act in this representation space as linear transformations. YM$_{SU(N)}$ presupposes $SU(N)$-transformations as gauge transformations. That is, any frame $e'_a$ emerging from $e_a$ by an $SU(N)$-transformation yields an equivalent reference frame for expressing $\psi$ in $N$ components. The $SU(N)$-transformations can be generated by $N^2 - 1$ traceless hermitian $N \times N$ matrices. We write the $SU(N)$-transformations in the form $U = \exp(\frac{i}{2} \tau \cdot \theta)$ with the $\tau = (\tau_1, ... \tau_{N^2-1})$ $SU(N)$-group generators and $\theta = (\theta_1, ... \theta_{N^2-1})$ the corresponding group parameters, see Tab.1 for the cases $N = 2, 3$.

According to our conventions, the group generators act on the reference frame from the right, while they act on the coordinate functions of the fields from the left. The gauge transformations read

$$e'_a = e_b \left(\exp\left(\frac{i}{2} \tau \cdot \theta\right)\right)_a^b \iff \psi'^a = \left(\exp\left(-\frac{i}{2} \tau \cdot \theta\right)\right)_b^a \psi^b \quad (1.1)$$

or, infinitesimally,

$$\delta e_a = e_b \frac{i}{2} \left(\tau \cdot \theta\right)_a^b \iff \delta \psi^a = -\frac{i}{2} \left(\tau \cdot \theta\right)_b^a \psi^b. \quad (1.2)$$
Table 1: The standard generators of SU(N)-transformations (N = 2, 3)

|                | SU(2)                      | SU(3)                      |
|----------------|----------------------------|----------------------------|
| number of generators $\tau_i$ | 3                          | 8                          |
| standard representation       | Pauli matrices $(2 \times 2)$ | Gell-Mann matrices $(3 \times 3)$ |

The factor $\frac{i}{2}$ is conventional. We note that the active gauge transformation behavior of the field components $\psi^a$ on the right hand sides of (1.1), (1.2) are a consequence of the gauge invariance of the field $\psi$, i.e., the field $\psi$ itself remains unaffected by a change of the reference frame:

$$e_a \psi^a = \psi \equiv \psi' = e'_a \psi'^a.$$  \hfill (1.3)

This implies that also the operation of *some* differential $D$ expressing a “change” of the field $\psi$ must be invariant under gauge transformations,

$$e_a (D\psi)^a = D\psi = (D\psi)' = e'_a (D\psi)^a.'$$ \hfill (1.4)

We note that the reference frame $e_a$ is a function of spacetime, but not a reference frame with respect to some *tangent* space of the base manifold $M$ (more precisely, it is not a section of the frame bundle $LM$ associated to $M$.) It is a reference frame with respect to the representation space of $\psi$ which is a priori unrelated to the base manifold. This is why we speak of SU(N)-Yang-Mills as an *internal* gauge theory.

The gauge freedom of choosing an arbitrary reference frame comes also into play if we want to compare the field $\psi$ at two different spacetime-points. The total change $D\psi$ of $\psi$, while passing from one point $x$ to an infinitesimally neighboring point $\tilde{x} = x + dx$, is given by

$$D\psi = e_a (d\psi^a) + de_a \psi^a.$$ \hfill (1.5)

The first term on the right hand side is due to the change of $\psi$ with respect to an “unchanged” or “parallel” reference frame $e_a$ at $x$. This change is determined by the functions $\psi^a = \psi^a(x)$. The second term is due to the change of the reference frame while passing from $x$ to $\tilde{x}$. This change must be of the form of an infinitesimal SU(N)-transformation. It is unspecified so far. It remains with us to specify the term $de_a$. Let us write

$$de_a = -e_b \left( \frac{i}{2} \tau^b \cdot A \right)_a^b,$$ \hfill (1.6)

with an arbitrary one-form $A^a(x) = A^a_i(x) dx^i$. We may specify the term $de_a$ by choosing a particular function $A^a_i(x)$. 

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The meaning of equation (1.6) is the following: Given a frame field \( e_a \) we identify the frame \( e_a(x) + de_a(x) \) at \( x \) with the frame \( \tilde{e}_a = e_a(\tilde{x}) \) at \( \tilde{x} \). This is nothing else than the definition of parallel transport, a necessity in order to compare the field \( \psi \) at two different points. The differential operator \( D \) defined by (1.5) is called an \( SU(N) \)-covariant derivative, due to the property (1.4). We note that the specification (1.6) depends on the choice of the frame field \( \tilde{e}_a \), i.e., it is not gauge invariant.

The geometric meaning of the one-form \( A^a \) is that of a (gauge) connection, its physical meaning is that of a (gauge) potential. The action of the \( SU(N) \)-covariant derivative on the fields \( \psi \) is denoted here and in the following by

\[
\hat{D}\psi := e_a d\psi^a - e_a \left( \frac{i}{2} \tau \cdot A \right)^b_a \psi^b
\]

or, in components,

\[
\left( \hat{D}\psi \right)^a = d\psi^a - \left( \frac{i}{2} \tau \cdot A \right)^b_a \psi^b.
\]

A quite noticeable point is the following: The definition of a particular gauge potential of the form \( A^a \) does not fix both the parallel transport and the reference frames. This statement can be inferred from the equation

\[
e_a(\tilde{x}) = e_b(x) \left( \delta^b_a + \frac{i}{2} (\tau \cdot A)^b_a \right)
\]

as follows: In order to know what frame at \( x \) has to be identified with \( e_a(\tilde{x}) \), and this is what is meant by defining a parallel transport, we need to know both \( A(x) \) and \( e_a(x) \). Then we can deduce that the answer is \( e_b(x) \left( \delta^b_a + \frac{i}{2} (\tau \cdot A)^b_a \right) \). Vice versa, in order to know what reference at \( x \) is used, we need to know both \( A(x) \) and \( e_b(x) \left( \delta^b_a + \frac{i}{2} (\tau \cdot A)^b_a \right) \). Then we can deduce that the answer is \( e_a(x) \). Therefore, given the reference frames at different points, the gauge connection determines the actual parallel transport. Vice versa, given a specific parallel transport in a gauge invariant manner, e.g. in terms of curvature (=field strength) that might be implicitly defined by field equations, the prescription of a gauge connection fixes the reference frames at different points.

### 1.3 Field strengths and Lagrangian

In Yang-Mills theory the gauge connection becomes a dynamical variable. Corresponding kinematic terms on the Lagrangian level are built from the field strength two-form

\[
F^a = \hat{D}A^a = dA^a + \frac{1}{2} f^{a}_{bc} A^b \wedge A^c,
\]

where \( f^{a}_{bc} \) denotes the structure constants of the \( SU(N) \)-gauge group:

\[
[\tau_a, \tau_b] = f^c_{ab} \tau_c.
\]
The transformation behavior of the fields $A^a$ and $F^a$ is derived under the assumption that the covariant derivative $\hat{D}\psi$ transforms in the same homogeneous way as the field $\psi$ does,

$$\delta \psi^a = -\frac{i}{2} \left( \tau \cdot \theta \right)_b^a \psi^b \iff \delta \left( \hat{D}\psi \right)^a = -\frac{i}{2} \left( \tau \cdot \theta \right)_b^a \left( \hat{D}\psi \right)^b. \quad (1.12)$$

This condition is motivated by the gauge principle (1.3), (1.4). Inserting the explicit form (1.8) of $(\hat{D}\psi)^a$ into the right hand side of (1.12), together with the variation (1.1), yields the variation of the gauge potential $A^a$ as

$$\delta A^a = -d\theta^a - f^a_{\ bc} A^b \theta^c = -\hat{A} \theta^a. \quad (1.13)$$

Plugging this into (1.10), we obtain for the variation of the gauge field strength the homogeneous transformation behavior

$$\delta F^a = -f^a_{\ bc} F^b \theta^c. \quad (1.14)$$

The simplest gauge invariant term, which can be constructed from $A^a$, is the free Yang-Mills Lagrangian

$$L_{\text{free}} := \frac{1}{2} F^a \wedge * F^a. \quad (1.15)$$

We note that the Hodge star operator $^*$ appearing in (1.15) requires the presence of a metric on the base manifold. In contrast to this, the definition of the topological Yang-Mills Lagrangian [21]

$$L_{\text{top}} := \frac{1}{2} F^a \wedge F^a \quad (1.16)$$

does not require any metric structure at all.

The whole set of gauge potentials $A^a$ can be divided into equivalence classes of gauge related potentials. These are the gauge orbits. Two elements of the same gauge orbit can always be related by a gauge transformation. Performing a gauge transformation on one element of a gauge orbit yields another element of the same gauge orbit. Different gauge potentials belonging to the same gauge orbit correspond to different choices of reference frames. This is evident for gauge invariant quantities that are constructed from the potential $A^a$: They assume the same value for each choice of potential of the same gauge orbit.

2 Gauging the affine group

As a next step we will gauge the affine group $A(n, R) = T^n \rtimes GL(n, R)$, i.e., the semidirect product of the translation group and the group of general linear
transformations. The Poincaré group, the group of motions in SR, is a special case therefrom. The gauging of a group stands out from a mere mathematical procedure as long as we believe that the corresponding gauge transformations are symmetry transformations of a physical system at hand. For example, it is experimentally well established and generally accepted that Poincaré transformations are symmetry transformations with respect to physical systems embodied in Minkowski spacetime. That is, observers in Minkowski spacetime detect the same physics, as long as they use reference frames that are related to each other by Poincaré transformations. Consequently, in order to describe this variety of possible reference frames, one is led to introduce appropriate gauge fields, i.e. to gauge the Poincaré group.

Returning to the more general affine group, we have indications (but no conclusive evidence) for assuming invariance of physical systems under the action of the entire affine group. General affine invariance adds dilation and shear invariance as physical symmetries to Poincaré invariance, and both of these symmetries are of physical importance. Dilation invariance is a crucial component of particle physics in the high energy regime. Shear invariance was shown to yield representations of hadronic matter, the corresponding shear current can be related to hadronic quadrupole excitations. From this it is speculated that the invariance under affine transformations played an essential part at an early stage of the universe, such that todays Poincaré invariance might be a remnant of affine invariance after some symmetry breaking mechanism.\textsuperscript{3} From this point of view it is important to pursue a gauging of the affine group in order to see what kind of theory emerges. It is expected that one obtains a very general framework, encompassing theories like GR, Poincaré gauge theory, and conformal gravity.

Proceeding in close analogy to $SU(N)$-YM, we first have to specify the reference frame which we will use to describe our physical system. For $\text{YM}_{SU(N)}$ we considered the physical fields to be expressed in a special unitary frame, unrelated to the frame bundle $LM$ of the (spacetime) base manifold $M$. Now we will concentrate on physical fields $\psi$ expressed in an affine frame related to the frame bundle $LM$. Therefore we will call the resulting gauge theory an external one.

\subsection{Affine geometry}

An affine frame is introduced on the base manifold as follows (for a rigorous treatment with more details one should consult Ref.[10], Chap.3): Viewing the (real) base manifold $M$ as a differentiable manifold, we can establish at any point $x \in M$ a tangent space $T_x M$. The collection of all tangent spaces $T_x M$ forms the tangent bundle $TM$. We enlarge any $T_x M$ to an affine tangent space $A_x M$ by allowing to freely translate elements of $T_x M$ to different points $p \in A_x M$. The collection of all affine tangent spaces $A_x M$ forms the affine bundle $AM$. An affine frame of $M$

\textsuperscript{3}We recommend Ref. [7] for details on this subject.
Figure 1: Some arbitrary affine frames of affine tangent spaces $A_xM$ and $A_{\bar{x}}M$.

at $x$ is a pair $(e_a, p)$ consisting of a linear frame $e_a \in L_xM$ and a point $p \in A_xM$, see Fig.1. The origin of $A_xM$ is that point $o_x \in A_xM$ for which the affine frame $(e_a, o_x) \in A_xM$ reduces to the linear frame $e_a \in L_xM$.

Until further notice in Sec. 2.4, we assume that no particular origin has been chosen. The transformation behavior of an affine frame $(e_a, p)$ under an affine transformation $(\Lambda, \tau)$ with $\tau = \tau^a \in T^n \simeq \mathbb{R}^n$ and $\Lambda = \Lambda^a_b \in GL(n, \mathbb{R})$ reads

$$(e, p) \xrightarrow{\Lambda, \tau} (e', p') = (e\Lambda, p + \tau) = (e_a \Lambda^b_a, p + \tau^a e_a). \quad (2.1)$$

The affine group acts transitively on the affine tangent spaces $A_xM$: Any two affine frames of some $A_xM$ can be related by a unique affine transformation. By picking one particular affine frame, one can thus establish a one-to-one correspondence between affine transformations and affine frames of $A_xM$. However, a priori no affine frame is “preferred”.

We introduce a generalized affine connection as a prescription $(\Gamma^{(L)}, \Gamma^{(T)})$ which maps infinitesimally neighboring affine tangent spaces $A_xM$, $A_{\bar{x}}M$, where $\bar{x} = x + dx$, by an $A(n, R)$-transformation onto each other. The generalized affine connection consists of a $GL(n, R)$-valued one-form $\Gamma^{(L)}$ and an $R^n$-valued one form $\Gamma^{(T)}$, both of which generate the required $A(n, R)$-transformation. To make this mapping precise, we have to choose bases $(e_a, p) = (e_a, p)(x)$ and $(\bar{e}_a, \bar{p}) = (e_a, p)(\bar{x})$ in both affine tangent spaces. We note again that the points $p$ and $\bar{p}$ are arbitrary in the sense
that they do not represent an origin of $A_x M$ and $A_{\tilde{x}} M$, respectively. The two affine tangent spaces get now related by an affine transformation according to the prescription

\begin{align}
dp &= \Gamma^{(T)a} e_a, \\
dea &= \Gamma^{(L)b} e_b.
\end{align}

Equations (2.2), (2.3) have to be interpreted as follows, compare Fig.2: First, the point $p = p(x) \in A_x M$ is mapped onto the point

\[ (p + dp)(\tilde{x}) = \tilde{p} + dp(\tilde{x}) = \tilde{p} + \Gamma^{(T)a}(\tilde{x}) \tilde{e}_a \in A_{\tilde{x}} M \]

by means of the translational part $\Gamma^{(T)}$ of the generalized affine connection. Second, the frame $e_a(x)$ at $p(x)$ is mapped onto the frame

\[ (e_a + dea)(\tilde{x}) = \tilde{e}_a + dea(\tilde{x}) = \tilde{e}_a + \Gamma^{(L)b}(\tilde{x})\tilde{e}_b \in A_{\tilde{x}} M \]

at $(p + dp)(\tilde{x})$ by means of the linear part of the generalized affine connection. This completes the affine transformation of $A_x M$ onto $A_{\tilde{x}} M$. It is immediately clear that the generalized affine connection is gauge dependent, i.e. dependent on the bases chosen. Under an infinitesimal $A(n,R)$-transformation, expressed by functions $\varepsilon^a$, $\varepsilon^b$ which change the bases ($e_a, p$) at $x$ and $\tilde{x}$ according to

\[ \delta p = \varepsilon^a e_a, \quad \delta e_a = \varepsilon^b e_b, \]

the generalized affine connection transforms according to

\begin{align}
\delta \Gamma^{(T)a} &= -\varepsilon^b \Gamma^{(T)b} - d\varepsilon^a - \Gamma^{(L)a}_b \varepsilon^b, \\
\delta \Gamma^{(L)b}_a &= -d\varepsilon_a - \Gamma^{(L)c}_a \varepsilon^c + \Gamma^{(L)c}_a \varepsilon^b.
\end{align}

This result we just quote from the literature (see e.g. [7], p.23, where the gauge variation is given in its more general finite form) since a “physicist’s” derivation of it will be given in the following section.

So far, the notion of affine parallel transport was defined as an $A(n,R)$-transformation between affine tangent spaces of infinitesimally neighboring points $x$ and $\tilde{x}$. For finitely separated points $x_0$ and $x_1$, one has to consider curves $\tau = x_t$, $0 \leq t \leq 1$ on $M$ that connect $x_0$ and $x_1$. Then parallel transport from $x_0$ to $x_1$ is defined along the curve $\tau$, resulting in an $A(n,R)$-transformation from $A_{x_0} M$ to $A_{x_1} M$. This affine transformation in general does depend on the curve $\tau$ chosen since parallel transport may not be integrable.
Figure 2: Affine parallel transport between infinitesimally neighboring points $x$ and $\tilde{x}$. Under affine parallel transport from $x$ to $\tilde{x}$ the image of $(e_a, p)(x)$ is obtained by first translating $(\tilde{e}_a, \tilde{p})(\tilde{x})$ to $(\tilde{e}_a, \tilde{p} + dp)(\tilde{x})$ (dotted frame) and, secondly, linear transforming $(\tilde{e}_a, \tilde{p} + dp)(\tilde{x})$ into $(e_a + de_a, \tilde{p} + dp)(\tilde{x})$. The translation is defined by $\Gamma^{(T)}$ and the linear transformation by $\Gamma^{(L)}$. 

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2.2 Affine frames and physical fields

Gauging the affine group presupposes that any physical field \( \psi \), to be expressed with respect to some affine frame, can be expressed with respect to any affine frame, i.e., a physical field \( \psi \) is invariant under arbitrary \( A(n, R) \) transformations. But what does it mean to express a field in an affine frame? To begin with, we clearly need a suitable \( GL(n, R) \)-representation that acts on the fields \( \psi \). In order to obtain a certain representation, we first have to specify the vector space in which \( \psi \) assumes its values. Then we have to specify within this vector space a certain basis, i.e. a certain reference frame. A vector field, for example, is a field which is to be expressed in a linear frame \( e_a \) as introduced in the previous subsection 2.1. In contrast to this, spinor fields cannot be expressed in such a linear frame. In fact, spinor representations of \( GL(n, R) \) turn out to be infinite-dimensional \cite{13} and thus require an infinite number of basis vectors. The notion of an affine frame is thus tied to the representations of the matter fields which are to be expressed componentwise in its linear part, i.e. in the linear frame. Consequently, we should enlarge the notion of an affine frame to include all \( GL(n, R) \)-representations needed.

However, this is not necessary in order to arrive at a gravity theory: What distinguishes an external gauge theory (in this case the gauge theory of the affine group) from an internal one is that reference frames of an affine bundle are later to be identified by a “soldering” with elements of the frame bundle \( LM \) of the \( n \)-dimensional base manifold. This soldering mediates the transition from internal structures to external structures and is essential in order to project geometric gauge structures on the base manifold to induce gravity. It seems to be unclear how a soldering of arbitrary affine frames corresponding to arbitrary \( GL(n, R) \)-representations could take place. The problem is to convert by the soldering process a frame of dimension different than \( n \) to a linear frame of dimension \( n \).\footnote{This issue is similar to the compactification of higher-dimensional supergravity or string theories.} However, in constructing a gravity theory a soldering cannot be avoided. A gauge theory without soldering remains an internal one, exhibiting only internal geometric structures. Therefore it is our assumption that there are physical fields to be described by affine frames which are bases of affine tangent spaces. These affine frames are the ones we will work with for constructing a gauge theory of the affine group.

We still have to clarify the meaning of a point \( p \) which makes a linear frame \( e_a \) to an affine frame \( (e_a, p) \). Since no origin is chosen a priori in an affine tangent space, at the beginning we have no relation between a point \( p \in \mathbb{A}_xM \) and a point \( x \in M \). Since our physical fields are defined in an affine tangent space, rather than on the base manifold, we have to associate with them at this stage a point of an affine tangent space \( \mathbb{A}_xM \) rather than a point \( x \) of the base manifold. (At first sight this might seem a bit awkward.) Thus we would associate a point to a field, i.e. a point where the field is supposed to be located. The prescription would then be \( \psi \rightarrow \psi^{(p)} \).
with some \( p \in A_x M \) rather than by \( \psi \rightarrow \psi(x) \) with some \( x \in M \). Since any affine frame of \( A_x M \) should be suitable to express \( \psi(p) \), one should expand \( \psi(p) \), for any fixed \( p \), according to \( \psi(p) = \psi^a(p) e_a(p) \), with \( e_a(p) = (e_a, p) \) an affine frame of \( A_x M \).

In words: The field \( \psi \) at \( p \in A_x M \) is expressed by the affine frame \( (e_a, p) \in A_x M \). Later, the soldering will take place in a way such that \( \psi(p) = \psi^a(p) e_a(p) \) reduces to the familiar expression \( \psi = \psi^a(x)e_a \).

### 2.3 The gauge procedure

Let us now begin with the gauging of the affine group. We denote the generator of translations by \( P_a \) and the generator of general linear transformations by \( L^a b \). These generators act on the affine frame \( e_a(p) \) from the right, their action on the component functions of the fields is from the left. Together with the group parameters \( \varepsilon^a(x) \) and \( \varepsilon_a^b(x) \) we can write the \( (n, R) \) transformations in close analogy to (1.1), (1.2) as

\[
e^{(p')} = e^{(p)} \left( \exp(\varepsilon^c P_c + \varepsilon^d L^c d) \right)_{b}^{a} \iff \psi^{a(p')} = \left( \exp(-\varepsilon^c P_c - \varepsilon^d L^c d) \right)_{b}^{a} \psi^{b(p)},
\]

or, in infinitesimal form,

\[
\delta e^{(p')} = \left( \varepsilon^c P_c \right)_{a}^{b} + \left( \varepsilon^d L^c d \right)_{a}^{b} \iff \delta \psi^{a(p')} = - \left[ \left( \varepsilon^c P_c \right)_{b}^{a} + \left( \varepsilon^d L^c d \right)_{b}^{a} \right] \psi^{b(p)}.
\]

The term \( e^{(p)}_{b} \left( \varepsilon^c P_c \right)_{a}^{b} \) represents the covariant components of the difference vector \( \delta \bar{p} \) belonging to the shift of the base point of the affine frame (or, equivalently, the shift of the base point of the field \( \psi \)). Correspondingly, the term \( - \left( \varepsilon^c P_c \right)_{a}^{b} \psi^{b(p)} \) represents the contravariant components of the difference vector \( \delta \bar{p} \). The active gauge transformation behavior of the field component \( \psi^a \) was obtained from the gauge invariance of the field \( \psi \):

\[
e^{(p)} e^{a(p)} = \psi^{(p)} \equiv \psi^{(p')} e^{a(p')} = \psi^{(p')} \psi^{a(p')},
\]

or, dropping for convenience here and in the following the explicit indices referring to the point \( p \),

\[
e_a \psi^a = \psi \equiv \psi' = e_a' \psi^{a'}.
\]

Again, as in the case of \( SU(N) \)-YM, we want to compare the field \( \psi \) at different spacetime points. The total change \( D\psi \) of \( \psi \), while passing from one point \( x \) to an infinitesimally neighboring point \( \bar{x} = x + dx \), is given by

\[
D\psi = e_a(d\psi^a) + de_a \psi^a.
\]

The first term on the right hand side is due to the change of \( \psi \) with respect to an “unchanged” or “parallel” affine reference frame \( e_a \) at \( x \). This change is determined
by the functions \( \psi^a = \psi^a(x) \). The second term is due to the change of the affine reference frame while passing from \( x \) to \( \tilde{x} \). This change must be of the form of an infinitesimal \( A(n, R) \)-transformation. It is unspecified so far. It remains with us to specify the term \( de_a \). For the specification we use a generalized affine connection \( (\Gamma^T, \Gamma^L) \), as introduced in the last section, and write

\[
de_a = e_b (\Gamma^T c P_c)_a b + e_b (\Gamma^L d L^c d)_a b .
\]

(2.15)

Again, the term \( e_b (\Gamma^T c P_c)_a b \) represents the shift of the base point in terms of contravariant components.

With (2.15) we obtain the explicit expression for the \( A(n, R) \)-covariant derivative \( \Gamma_D \psi \), i.e.

\[
\Gamma_D \psi := e_a d\psi^a + e_b (\Gamma^T c P_c)_a b \psi^a + e_b (\Gamma^L d L^c d)_a b \psi^a ,
\]

(2.16)

or, in components,

\[
(\Gamma_D \psi)^a = d\psi^a + (\Gamma^T c P_c)_b a \psi^b + (\Gamma^L d L^c d)_b a \psi^b .
\]

(2.17)

Next we derive the transformation behavior of the gauge connection \( \equiv \Gamma^T a, \Gamma^L b \) under the condition that the covariant derivative \( \Gamma_D \psi \) transforms in the same homogeneous way as the field \( \psi \) does,

\[
\delta\psi^a = - (\varepsilon^c P_c)_b a \psi^b - (\varepsilon^d L^c d)_b a \psi^b \quad \Leftrightarrow \quad \delta a (\Gamma_D \psi)^a = - (\varepsilon^c P_c)_b a (\Gamma_D \psi)^b - (\varepsilon^d L^c d)_b a (\Gamma_D \psi)^b .
\]

(2.18)

(2.19)

We insert into (2.19) the explicit form (2.17) of \( (\Gamma_D \psi)^a \) and the variation (2.11). This yields, after some algebra, the gauge variations of \( \Gamma^T a \) and \( \Gamma^L a b \),

\[
\delta\Gamma^T a = - \varepsilon^b a \Gamma^T b - d\varepsilon^a a - \Gamma^L b \varepsilon^b a \quad \Leftrightarrow \quad \delta a \Gamma^T a = - \varepsilon^b a \Gamma^T b - \Gamma^L d \varepsilon^a a ,
\]

(2.20)

\[
\delta\Gamma^L a b = - d\varepsilon^a b - \Gamma^L b \varepsilon^a c + \Gamma^L c \varepsilon^b a \quad \Leftrightarrow \quad \delta a \Gamma^L b a = - D \varepsilon^a b ,
\]

(2.21)

where we introduced the \( GL(n, R) \)-covariant derivative \( \Gamma^L a \).

This, in principle, completes the gauging of the affine group. Demanding the equivalence of affine frames has led to the introduction of an \( A(n, R) \)-gauge connection with translational part \( \Gamma^T \) and linear part \( \Gamma^L \). In the corresponding physical theory this connection will become a true dynamical field with its own kinetic terms featuring in the Lagrangian.
Figure 3: Establishing an (infinitesimal) one-to-one correspondence between points of an affine tangent space and points of a manifold: The point $o_x$ is soldered to the base manifold by its identification with $\tilde{x}$. It also corresponds to a point $P_x \in A_x M$ which is the image of $o_x$ under the action of $\Gamma(T)$ during affine parallel transport from $\tilde{x}$ to $x$. Therefore the point $\tilde{x} \in M$ corresponds to the point $p_\tilde{x} \in A_x M$.

2.4 The breaking of translational invariance

So far we haven’t used the fact that each affine tangent space represents a flat affine model space which is to be seen as a first order approximation of the base manifold. This means in particular that an affine tangent space $A_x M$ should represent the flat first order approximation of the base manifold at $x$. Hence there should be a one-to-one correspondence between points in the neighborhood of $x$ and points of $A_x M$. Such a correspondence is established by choosing an origin in $A_x M$, i.e. by choosing a point $o_x \in A_x M$ which is to be identified with $x \in M$, together with the definition of a generalized affine connection:

Suppose we take a point $\tilde{x} \in M$ which is neighboring to $x$. To both $x$ and $\tilde{x}$ there corresponds an origin $o_x$, $o_\tilde{x}$ in $A_x M$, $A_\tilde{x} M$, respectively. Having also a generalized affine connection to our disposal, we can identify the point $o_\tilde{x}$ with a point $o_{\tilde{x}} + \Gamma(T)^a(\tilde{x})\tilde{e}_a \in A_\tilde{x} M$. This also implies an identification of $x \in M$ with $o_x + \Gamma(T)^a(x)e_a \in A_x M$. Vice versa, the point $\tilde{x} \in M$ corresponds to $p_\tilde{x} = o_x + \Gamma(T)^a(x)e_a \in A_x M$, see Fig.3. Extending this one-to-one correspondence to finitely separated points $x_0$ and $x_1$ leads to the notion of the development of a curve.
on $M$ into an affine tangent space, compare Ref.[10], p.130: We consider curves $	au = x_t$, $0 \leq t \leq 1$, on $M$ and choose origins $o_{x_t}$ in each affine tangent space $A_{x_t}M$.\footnote{Choosing in each affine tangent space $A_{x}M$ one point establishes a point field. Each point field corresponds to a section of the affine bundle $AM$, cf.[10], p.131.} Affine parallel transport from any $A_{x_t}M$ to $A_{x_0}M$ maps any $o_{x_t}$ into $A_{x_0}M$. The images $\tau^*(x_t)$ of all $o_{x_t}$ under affine parallel transport along $\tau$ on $A_{x_0}M$ constitute a curve in $A_{x_0}$. This curve is called the development $\tau^*$ of $\tau \in M$ into the affine tangent space $A_{x_0}M$. Then the desired one-to-one correspondence is given by associating $x_t \in M$ to $\tau^*(x_t) \in A_{x_0}M$.

Quite generally, the choice of an origin $o_x$ reduces an affine frame $(o_x, e_a) \in A_xM$ to a linear frame $e_a \in L_xM$. This constitutes the soldering of the affine tangent space to the base manifold. But having chosen an origin in each affine tangent space we have also broken the translational invariance: Under the action of the affine group $A(n,R)$ on an affine tangent space only $GL(n,R)$-transformations leave the origin invariant. How does this circumstance affect the gauge principle of choosing affine reference frames at will?

The choice of an origin does not prevent us from performing translations within an affine tangent space. Moreover, it allows us to locally interpret translations within affine tangent spaces as diffeomorphisms on the manifold $M$ and vice versa, as we will explain now: First we suppose that a vector field $u$ is given. The vector field $u$ induces, at least locally, a diffeomorphism on $M$ by the flow of its integral curves. We concentrate on a point $x_0$ of one of these integral curves. It can be translated to a point $x_1$ of the same integral curve by using the diffeomorphism generated by $u$ (“$x_0$ is dragged along $u$ to $x_1$”). The part of the integral curve in between $x_0$ and $x_1$ represents a curve $\tau$ on $M$ which can be lifted to a development $\tau^* \in A_{x_0}M$. The curve $\tau^*$ contains the origin $o_{x_0}$ and also the image $\tau^*(x_1)$ of $o_{x_1}$ under affine parallel transport from $x_1$ to $x_0$. Thus the translation of $x_0$ to $x_1$ on $M$ along $u$ induces a translation of $o_{x_0}$ to $\tau^*(x_1)$ in $A_{x_0}M$. This applies to all points $x \in M$ ($x_0$ was arbitrary), such that a diffeomorphism on $M$ does generate a translation in $AM$, indeed. Vice versa we can start from a translation in $AM$ defined by two point fields $s_0$ and $s_1$ (i.e. by two sections of $AM$), that is, we regard $s_0$ to be translated to $s_1$. Then we choose a smooth family $s_t$ ($0 \leq t \leq 1$) of sections such that $s_{(t=0)} = s_0$, $s_{(t=1)} = s_1$. The family $s_t$ generates in each affine tangent space $A_xM$ a curve $\tau^*$ which can be taken as the development of a curve $\tau$ in $M$. Then the vector field that is tangent to all such curves generates the diffeomorphism corresponding to the translation from $s_0$ to $s_1$.

We summarize this subsection: By introducing origins in $AM$, i.e. by soldering $AM$ to $M$, we lost translational invariance in $AM$ but gained a local one-to-one correspondence \textit{between translations} in $AM$ \textit{and diffeomorphisms} on $M$. It doesn’t make sense anymore to speak about translational invariance in $AM$, nevertheless, we can introduce translational invariance by demanding diffeomorphism invariance.
instead. We will continue to work with this modified notion of translation invariance but still keep the quantities introduced by the gauging of the whole affine group. In particular we will keep the translational gauge potential $\Gamma^{(T)}$. The diffeomorphisms itself, as horizontal transformations in their active interpretation, cannot be gauged according to the usual gauge principle and thus do not furnish their own gauge potential.

In this way the $A(n, R)$-invariance of affine frames in $AM$ splits by the soldering into (i) diffeomorphism (or translational) invariance on $M$ and (ii) $GL(n, R)$-gauge invariance of linear frames.

### 2.5 Diffeomorphism invariance and Lie derivatives

Since we want to talk about translation invariance of a physical system, we have to know how to actually perform a translation and how to measure its effect on the physical system. A passive translation of a geometric object $O$, with the translation defined by pointing from a point $p$ with coordinates $x$ to a point $p + dp$ with coordinates $\tilde{x}$, means taking the value of $O$ at $p$ in the translated coordinate system $\tilde{x}$ of $p + dp$. This is opposed to an active translation, where the value of the actively translated $O$ is taken at $p + dp$ in the coordinate system $\tilde{x}$. Both (passive) translations of the coordinate system $\tilde{x}$ to $O$ or (active) translations of $O$ to the coordinate system $\tilde{x}$, with subsequent comparison to the original value of $O$ at $p$ or $p + dp$, respectively, are generated by Lie-derivatives.

As the generator of translations we will take the $GL(n, R)$-gauge-covariant Lie-derivative $L$. Its action on $gl(n, R)$-valued $p-$forms $\Psi_{a\ldots}^b\ldots$ with respect to a vector $\varepsilon = \varepsilon^i \partial_i$ reads

$$L_\varepsilon \Psi_{a\ldots}^b\ldots = \varepsilon^i \left[ D \Gamma^{(L)} \Psi_{a\ldots}^b\ldots \right] + \Gamma^{(L)} D \varepsilon^i \Psi_{a\ldots}^b\ldots.$$  \hspace{1cm} (2.22)

The operator $L_\varepsilon$ maps tensors into tensors, i.e. it is, as its name suggests, gauge covariant and thus independent of the orientation of linear frames at different points. Therefore it is independent of the linear part of the affine gauge transformations, a property we want to require for a proper translation generator. Only the covariant Lie-derivative generates translations which are independent of the choice of linear reference frames.

The action of $L$ on a vector field $\psi^a$ (representing a covariant, tensor–valued

---

[6] As should be clear from the context, the indices $a, b, \ldots$ that appear in $\Psi_{a\ldots}^b\ldots$ denote Lie-Algebra indices rather than form indices. In particular, if $\Psi_{a\ldots}^b\ldots$ represents a tensor, i.e. a tensor valued $p-$form, then the expression $\Psi_{a\ldots}^b\ldots$ has to be understood as the tensor components of a tensor $\Psi$ according to the expansion $\Psi = \Psi_{a\ldots}^b\ldots e^a \otimes \ldots \otimes e^b \otimes \ldots$. Thus, the complete expansion of the tensor valued $p-$form $\Psi$ reads, in a holonomic basis e.g. $\Psi = \Psi_{i_1\ldots i_p a\ldots}^{b\ldots} dx^{i_1} \wedge \ldots \wedge dx^{i_p} e^a \otimes \ldots \otimes e^b \otimes \ldots$.  

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zero-form of rank one) is given by (2.22) as a special case. It reduces to
\[
L_x \psi^a = \varepsilon^j D_j \psi^a.
\] (2.23)
This equation can be interpreted actively or passively. For infinitesimal \( \varepsilon \) the coordinate transformation corresponding to the variation (2.23) is explicitly given by \( x^i \to x^i + \varepsilon^i \). For an affine variation, compare the right hand side of (2.11), we obtain
\[
\delta \psi^a = \varepsilon^j D_j \psi^a - (\varepsilon_c^d L^c_{d})_{a}^{b} \psi^b.
\] (2.24)
The action of the Lie-derivative \( L \) on a frame is not explicitly defined by (2.23). We derive the corresponding expression for the case of a holonomic frame simply by hand: Under an infinitesimal coordinate transformation \( x^i \to x^i + \varepsilon^i \) the holonomic frame \( e_i = \frac{\partial}{\partial x^i} \) transforms to \( e'_i = \frac{\partial}{\partial (x^i + \varepsilon^i)} \). Therefore
\[
e'_i - e_i = - e_j \partial_i \varepsilon^j.
\] (2.25)
This variation corresponds to the action of the “ordinary” Lie-derivative \( L_{\varepsilon} \Psi = \varepsilon \mathcal{D} \Psi + d(\varepsilon) \Psi \). To make the variation (2.25) \( GL(n,R) \)-covariant we replace the ordinary derivative \( \partial_i \) by the \( GL(n,R) \)-covariant derivative \( D_i \) and get
\[
L_{\varepsilon} e_i = \delta e_i = - e_j D_j \varepsilon^j.
\] (2.26)
This is the translational part of the affine transformation behavior of a holonomic frame \( e_i \). For an affine variation, compare the left hand side of (2.11), we obtain
\[
\delta e_i = - e_j D_i \varepsilon^j + e_j (\varepsilon_c^d L^c_{d})_{i}^{j}.
\] (2.27)
The corresponding formula for the holonomic coframe \( dx^i \) reads
\[
\delta dx^i = D_j \varepsilon^j dx^j + (\varepsilon^d L^c_{d})_{j}^{i} dx^j = D_i \varepsilon^i + (\varepsilon^d L^c_{d})_{j}^{i} dx^j.
\] (2.28)

### 2.6 Anholonomic frames

One may wonder if it is possible to choose a frame \( e_\alpha \) which transforms under \( A(n,R) \)-transformations according to
\[
\delta e_\alpha = e_\beta (\varepsilon^d L^c_{d})_{\alpha}^{\beta},
\] (2.29)
i.e. which is automatically translation invariant. The answer to this question is positive, and we will show in the following how such a frame \( e_\alpha \) can be constructed from a holonomic frame \( e_i \): We define the frame \( e_\alpha \) by
\[
e_\alpha = \delta'^{\alpha}_i e_i + E_\alpha
\] (2.30)
with a vector-valued quantity $E_\alpha$ which is unspecified so far. The $A(n, R)$-transformation behavior of $E_\alpha$ is deduced from the transformation behavior of $e_i$ and $\epsilon_{\alpha}$, i.e. from (2.27) and (2.29). Using the implicit definition of $E_\alpha$, (2.30), one finds easily
\[
\delta E_\alpha = \epsilon_\beta D_\alpha \epsilon^\beta + E_\beta (\epsilon_c^d L_c^\beta)_{\alpha} \beta.
\]
(2.31)
The corresponding $A(n, R)$-transformation behavior of the one-form $A^\alpha$, dual to $E_\alpha$, is given by, compare (2.28),
\[
\delta A^\alpha = -D \epsilon^\alpha - (\epsilon_c^d L_c^\beta)_{\beta} \epsilon^\alpha A^\beta =: -D \epsilon^\alpha - \epsilon_\beta \alpha A^\beta,
\]
(2.32)
where we introduced the shorthand notation $\epsilon_\beta \alpha := (\epsilon_c^d L_c^\beta)_{\beta} \alpha$. We recognize (2.32) as the transformation behavior of the translation part of an $A(n, R)$-connection, see the corresponding formula (2.20). This identifies $A^\alpha$, or $E_\alpha$, as an translation potential $\Gamma^{(T)}_\alpha$ of metric affine gravity, $A^\alpha \equiv \Gamma^{(T)}_\alpha$. Its absorption (2.30) into an anholonomic frame $e_\alpha$, or its dual counterpart
\[
\delta \theta^\alpha = \epsilon_\beta \alpha \theta^\beta = \epsilon_\beta \alpha \theta^\beta,
\]
(2.33)
allows for automatic translation invariance: The translation part $\delta_t$ of the affine gauge transformations on the frame $e_\alpha$ and the coframe $\theta^\alpha$ vanishes automatically, $\delta_t e_\alpha = 0$ and $\delta_t \theta^\alpha = 0$. For completeness we also note the affine transformation behavior of anholonomic field (vector) components $\psi^\alpha$ referring to a translation invariant frame $e_\alpha$, compare also (2.11),
\[
\delta \psi^\alpha = -\epsilon_c^d L_c^\beta)_{\beta} \psi^\beta = -\epsilon_\beta \alpha \psi^\beta.
\]
(2.34)
The explicit expression for the $A(n, R)$-covariant derivative $\Gamma^D \psi$, as encountered in (2.17), becomes
\[
(\Gamma^D \psi)^\alpha = d \psi^\alpha + (\Gamma^c_c^d L_c^\beta)_{\beta} \psi^\beta =: d \psi^\alpha + \Gamma_\beta \alpha \psi^\beta,
\]
(2.35)
with $\Gamma_\beta \alpha := (\Gamma^c_c^d L_c^\beta)_{\beta} \alpha$.

Let us pause for a moment in order to summarize: The translation potential of the affine gauge approach to gravity, originally introduced as $\Gamma^{(T)}_\alpha$ in (2.10), can be used for the construction of translation invariant frames. This step is not mandatory but will turn out to be quite convenient. The bases $e_\alpha$, $\theta^\alpha$ turn by this procedure from mere arbitrary reference frames to independent physical quantities since they encapsulate the translation potential. Therefore they have to be determined by the dynamics of the physical theory.

Now we choose two neighboring points $x$ and $\tilde{x} = x + dx$ on $M$. In order to recognize the geometric meaning of the coframe $\theta^\alpha$, we inspect, on the level of the affine tangent space, the point $a_x + \theta^\alpha e_\alpha$, compare Fig.4. By the definition (2.33) of
Figure 4: Geometric interpretation of the orthonormal coframe $\vartheta^\alpha$ by means of the vector–valued one–form $\vartheta = \vartheta^\alpha e_\alpha$, see in this context also Fig.2.

For $\vartheta^\alpha$ we have

$$
(o_x + \vartheta^\alpha e_\alpha)(\tilde{x}) = o_x + (\delta^\alpha_i dx^i e_\alpha)(\tilde{x}) + (\Gamma^{(T)}_\alpha e_\alpha)(\tilde{x}).
$$

(2.36)

The term

$$
(\delta^\alpha_i dx^i e_\alpha)(\tilde{x}) = (dx^i e_i)(\tilde{x})
$$

(2.37)

denotes the difference vector between the origins $o_x$ and $o_{\tilde{x}}$, which corresponds to the difference between $x$ and $\tilde{x}$ on the manifold. From the definition and interpretation of the translational part of the generalized affine connection $\Gamma^{(T)}_\alpha$, compare Eq.(2.2) and the discussion below, we see that the term $(\Gamma^{(T)}_\alpha e_\alpha)(\tilde{x})$ constitutes the difference vector between the origin $o_{\tilde{x}}$ and the point

$$
(o_x + \vartheta^\alpha e_\alpha)(\tilde{x}) = o_{\tilde{x}} + (\Gamma^{(T)}_\alpha e_\alpha)(\tilde{x}),
$$

(2.38)

which is the image of $o_x$ under the action of the (translational part of the) generalized affine connection in direction $\tilde{x}$. Therefore the term $\vartheta^\alpha e_\alpha$ acquires the meaning of the translational part of a so-called Cartan connection $(\vartheta^\alpha, \Gamma^\alpha_\beta)$, compare [1]: Applied to an origin $o_x$ on the manifold it represents the difference vector between $o_x$ and its image under the action of the generalized affine connection. We stress that affine parallel transport of a linear frame from $x$ to $\tilde{x}$ does generally not yield a linear frame at $o_{\tilde{x}}$. It yields a linear frame at the point given by (2.38). The translation gap between $o_{\tilde{x}}$ and (2.38) is the origin of torsion, which measures the non-integrability of this gap.
2.7 Introducing a metric: Orthonormal frames

During the gauging of the affine group $A(n, R)$, we didn’t mention a metric structure $g$ on the base manifold $M$ at all. The reason for this, as might be obvious, is that the $A(n, R)$-gauge process is simply unrelated to a metric: We started from a general differentiable manifold $M$ without any predefined structure. Then we introduced affine frames on $M$ which allowed to define the notion of affine gauging. The gauge process itself led to the introduction of an affine $GL(n, R)$-connection on $M$, and this is all we ended up with.

The purpose of this subsection is to demonstrate how a dynamical metric can be introduced by the affine gauging scheme. All we will start from is a flat affine manifold with predefined (flat) metric structure. A special example of this would be a Minkowski space. The idea of this approach is to turn the flat, metric–affine manifold into a manifold with dynamical metric structure by demanding $A(n, R)$–gauge invariance. However, it will turn out that the full affine gauge invariance is actually too large to do this. The contained general linear invariance has to be restricted to an orthonormal invariance.

We begin with a flat $n$-dimensional manifold $M$. In view of special relativity, we could specify $M$ to be the $n$-dimensional Minkowski space $M_n$, though this specification is actually not necessary. The flatness of $M$ implies that a presupposed metric $g$ in $M$ can be written in (pseudo–)cartesian coordinates $x^i$ as

$$g = o_{ij} \, dx^i \otimes dx^j, \quad o_{ij} = \text{diag}(-1, 1, ..., 1). \quad (2.39)$$

The orthonormal coframe $dx^i$ is suitable to represent standard measurement devices for measuring length and time intervals in $M$. Of course, it should also be possible to use any other coframe to perform measurements, the actual form of which depending primarily on the dynamical state of the observer. However, the advantage to use an orthonormal coframe, like $dx^i$, is the immediate construction of measurements with results displayed in terms of the orthonormal form of the metric (2.39). This means that the result of a measurements is as if obtained by an inertial observer using (pseudo–)cartesian coordinates, and this is how we usually want to interpret them. This interpretation is no artificial constraint but should be seen as the establishment of a “standard language” to be used by different observers that are located at different points.

Now we demand $A(n, R)$-gauge invariance. The gauging is accomplished by the introduction of the affine gauge connection $\Gamma_a^{(L)b}$, $\Gamma^{(T)a}$. In the following the translation potential $\Gamma^{(T)a}$ will turn out to be the key ingredient: It allows to choose in $M$ the (in general) anholonomic translation invariant coframes $\vartheta^a$ of the last subsection as reference frame. The corresponding transition of the reference coframe $dx^i$ is written as

$$dx^i \rightarrow \vartheta^a = \delta^a_i \, dx^i + \Gamma^{(T)a}. \quad (2.40)$$
In order to know how to perform and interpret measurements after the gauge process, we look what happened to the flat metric (2.39) during the gauge process. We note that pointwise we can always find a gauge transformation, determined by the inhomogeneous transformation behavior (2.32), in order to make \( A^\alpha \) vanish. Therefore, at one point \( p \in M \) and in a special gauge \( \ast \),

\[
 g|_p \overset{\ast}{=} o_{ij} \, dx^i|_p \otimes dx^j|_p = o_{\alpha\beta} \, \delta^\alpha_j \delta^\beta_i \, dx^i|_p \otimes dx^j|_p, \tag{2.41}
\]

\[
 \overset{\ast}{=} o_{\alpha\beta} \, \vartheta^\alpha|_p \otimes \vartheta^\beta|_p. \tag{2.42}
\]

i.e., the process of affine gauging can be compensated at one point by choosing a special gauge, leading back to the flat metric (2.39), (2.41). In this gauge it seems to be irrelevant whether one should use the holonomic \( dx^i \) or the anholonomic \( \vartheta^\alpha \) as measuring devices. They both constitute orthonormal coframes at \( p \). Turning to a general gauge we have to perform an \( A(n, R) \)-transformation on the metric. It is clear that for the metric in the form (2.41) the inhomogeneous transformation behavior of the coframe \( dx^i \), (2.28), constitutes a major drawback: To make \( g \) invariant under gauge transformation one needs a corresponding inhomogeneous, thus non-tensorial, transformation behavior of the metric components \( o_{ij} \). This complication is unsatisfactory. One can do better by using the metric \( g \) in the form (2.42). Gauge transforming (2.42) by using the homogeneous transformation behavior (2.33) of \( \vartheta^\alpha \) yields

\[
 \delta g = \delta o_{\alpha\beta} \, \vartheta^\alpha \otimes \vartheta^\beta + o_{\alpha\beta} \, \delta \vartheta^\alpha \otimes \vartheta^\beta + o_{\alpha\beta} \, \vartheta^\alpha \otimes \delta \vartheta^\beta \tag{2.43}
\]

\[
 = (\delta o_{\alpha\beta} + \varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha}) \, \vartheta^\alpha \otimes \vartheta^\beta. \tag{2.44}
\]

Therefore we obtain the result that the metric is gauge invariant if the metric components \( o_{\alpha\beta} = \text{diag}(-1, 1, ...1) \) transform like tensor components,

\[
 \delta o_{\alpha\beta} = -\varepsilon_{\alpha\gamma} o_{\gamma\beta} - \varepsilon_{\beta\gamma} o_{\alpha\gamma} = -\varepsilon_{\alpha\beta} - \varepsilon_{\beta\alpha}, \tag{2.45}
\]

as they should.

Now suppose we do not require gauge invariance under the full affine group but under a restriction of it which satisfies \( \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \) (e.g. the Poincaré group or the translation group as a trivial example). We will call such a group affine-orthogonal. Then affine-orthogonal gauge invariance of the metric implies \( \delta o_{\alpha\beta} = 0 \), i.e. the coframe \( \vartheta^\alpha \) keeps its orthonormality under any affine-orthogonal gauge transformations. This subcase is important since then the metric acquires the form

\[
 g = o_{\alpha\beta} \, \vartheta^\alpha \otimes \vartheta^\beta \tag{2.46}
\]

and the coframe \( \vartheta^\alpha \) becomes orthonormal in any gauge at every point. Formula (2.46) is the generalization of the Minkowski metric (2.39), thus generalizing the notion of the “standard language”, i.e. the inertial observer using pseudo-cartesian coordinates in flat spacetime, to an affine-orthogonal invariant spacetime. For such
a spacetime the coframe $\vartheta^\alpha$ represents the appropriately generalized standard measuring devices. Equation (2.46) couples the metric to the coframe $\vartheta^\alpha$, in general anholonomic, which encapsulates the translation potential. Thus the interpretation of the translation potential as an independent physical quantity implies the transition of the flat metric (2.39) to an independent physical quantity itself, which is induced by translation invariance. In other words:

\begin{center}
Gauging an affine-orthogonal group by starting from a flat spacetime with a given metric structure will lead to a gravitational theory where the coframe $\vartheta^\alpha = \delta^\alpha_i \, dx^i + \Gamma^{(T)\alpha}_{\beta\gamma}$ induces a metric $g$ as an independent dynamical quantity.
\end{center}

3 Metric-Affine Gravity as a classical field theory

After the rather formal gauging of the affine group in the last section we will now explain how to formulate MAG, i.e. the gauge theory of the affine group with a metric supplemented, as a physically meaningful field theory. Therefore we will recapitulate in the following the Lagrangian formulation of a classical field theory and discuss how it leads to the equations of motion of MAG. We will also derive the Noether identities of MAG in order to discover the conserved quantities corresponding to the affine gauge invariance postulated.

3.1 Lagrangian formulation

In order to derive or describe the features of a (quantum) field theory, it is customary to put the theory at hand into a Lagrangian or a Hamiltonian form. Both formulations have their own advantages and are, for a broad class of applications, equivalent. In those cases one can (more or less easily) shift back and forth between Lagrangian and Hamiltonian formulation.

The Hamiltonian choice seems to be more natural when it comes to quantization because it provides the canonical commutation and anticommutation relations. Since the discovery of the Feynman path integral, powerful Lagrangian quantization schemes got developed, though. But when it comes to the actual calculation of $S$-matrix elements one still can not refrain from using the Hamiltonian.

In MAG we are far from actually calculating $S$-matrix elements. Therefore we will use the Lagrangian formulation together with all its advantages it has to offer:

- Symmetries are easily employed. A Poincaré invariant Lagrange function, for example, will lead to a Poincaré invariant theory.
• Conservation laws corresponding to symmetries can be obtained straightforwardly by the Noether procedure.

• The Lagrangian formulation is a covariant one, avoiding a 3 + 1-split of spacetime which, in MAG, is usually cumbersome to use.

The strategy is to first use the Lagrangian formulation to decide on an appropriate Lagrangian. Later it still will be possible to pass to the Hamiltonian, if needed.

On top of the Lagrangian formulation of a field theory stands the definition of an action $W$ as the integral of a Lagrange density $L$,

$$ W = \int L(\Phi^{(i)}, d\Phi^{(i)}) , $$ \hspace{1cm} (3.1)

where the form-fields $\Phi^{(i)}$ (the index $i$ numbers the different fields) are supposed to define the physical system of the field theory. The integration usually extends over a compact submanifold of spacetime, using certain boundary conditions. The fields $\Phi^{(i)}$ are not specified so far, they might represent matter fields, gauge fields, or geometric functions like for example coordinate functions. In (3.1) we made the assumption that no second or higher order derivatives of the fields $\Phi^{(i)}$ feature in the Lagrangian. This assumption of a first order Lagrangian already excludes general relativity in its original second order formulation. However, we note that all field theories used in current theories of elementary particle physics rely on Lagrangians of the form (3.1).

The next assumption ("Hamilton’s principle of least action") is that the dynamics of the physical system is completely described by the Euler-Lagrange equations, i.e. the equations of motion, which result from extremizing the action under arbitrary variations $\Phi^{(i)} \rightarrow \Phi^{(i)} + \delta \Phi^{(i)}$, with the prescription that the variations $\delta \Phi^{(i)}$ have to vanish at the boundary of the integration domain.

$$ \delta W = 0 \implies d \frac{\partial L}{\partial d\Phi^{(i)}} - (-1)^{p} \frac{\partial L}{\partial \Phi^{(i)}} = 0 $$ \hspace{1cm} (3.2)

The label $p$ denotes the degree of the form $\Phi^{(i)}$. Some remarks are in order:

1. The assumption of a first order Lagrangian leads to field equations of no higher differential order than two.

2. Assuming a real Lagrange density $L$ yields as many field equations as there are fields $\Phi^{(i)}$.

3. In the course of the implementation of internal or external gauge symmetries it might be necessary to replace the exterior derivative appearing in (3.1) by a covariant derivative in order to arrive at gauge covariant field equations. Equation (3.2), as it stands, is valid for uncoupled fields $\Phi^{(i)}$ in Minkowski spacetime.
3.2 General model building

In MAG we want to describe pure gravity and its coupling to matter. The matter fields, denoted in the following by $\Psi$, are supposed to be represented by vector- or spinor-valued $p$-forms. This is motivated by the observations, taken from quantum field theory in Minkowski spacetime, that (i) every particle species transforms irreducibly under the Poincaré group and (ii) the most general irreducible representation of the Poincaré group is either a tensor, a spinor, or a direct product of both. However, in MAG we go beyond Poincaré invariance, assuming that matter fields might not only undergo Poincaré transformations but also the more general linear transformations. In this case more general spinor–representations than the Poincaré–representations have to be constructed. Such representations of the matter fields corresponding to the affine group must exist. Otherwise it does not make sense to demand $A(n,R)$-invariance of a non-vacuum field theory, since the corresponding gauge transformations of the matter fields cannot be defined. The construction of the spinor representations of fermionic matter fields in MAG, the so-called manifields, is illustrated in [7] Chap.4. These representations turn out to be infinite dimensional, due to the non-compactness of the gauge (sub-)group $GL(n,R)$. The restriction of $GL(n,R)$ to $SO(1,n-1)$ reduces the manifield representations to the familiar spinor representations. Right now we are interested in a general metric–affine theory and will thus assume that the matter fields are described in terms of manifields.

The actual gauging of the affine group introduced, in addition to the matter fields $\Psi$, the gravitational gauge potentials $\Gamma^{(T)}$ and $\Gamma^{(L)}$. We will comply to the common practice and will use as gauge potential the translation invariant $\vartheta^{\alpha}$ in place of the translational part $\Gamma^{(T)}$ of the affine connection, simply because it has the immediate interpretation as a reference (co-)frame. Expanded in a holonomic frame, the components of $\vartheta^{\alpha}$ and $\Gamma^{(T)}$ differ just by a Kronecker symbol, as is clear from the definition of $\vartheta^{\alpha}$ in (2.33). Also the homogeneous transformation behavior of $\vartheta^{\alpha}$ will turn out to be quite convenient. For the action of the $GL(n,R)$-gauge potential we will often write the shorthand notation $\Gamma_{\alpha}^{\beta}$ instead of $(\Gamma^{(L)}_{\alpha}c^{L}_{d})_{\alpha}^{\beta}$.

In order to make the quantities $(\vartheta^{\alpha},\Gamma_{\alpha}^{\beta})$ true dynamical variables with own degrees of freedom, one has to add to the minimally coupled matter Lagrangian $L_{\text{mat}}$ a gauge Lagrangian $V$ of the form (3.1). Then the construction of a metric–affine gravity theory can be summarized by the following six-step-procedure:

1. Specify the external gauge group to be used. This is either the affine group, in order to obtain the full theory, or one of its subgroups, to obtain special cases (Poincaré gauge theory, General Relativity,...)

2. Derive the corresponding gauge potentials by the principle of gauge invariance.

3. Construct from the gauge potentials and their first derivatives a gauge invari-
Lagrangian $V$ of the form (3.1). In general this will require the introduction of a covariant derivative which, in turn, will lead to (self-)couplings between the gauge potentials.

4. If matter is to be incorporated provide a representation for the matter fields \( \Psi \) with respect to the gauge group.

5. Construct a gauge invariant matter Lagrangian \( L_{\text{mat}} \). Similar to step 3, this will require the introduction of a covariant derivative which, in turn, couples the gauge potentials to the matter fields.

6. Write down the total Lagrangian \( L = L_{\text{mat}} + V \) and derive the Euler-Lagrange equations. These equations of motions are to be discussed subsequently.

This six-step-procedure is still very general and leaves quite some freedom in executing each step. We already performed a gauging of the affine group \( A(n,R) \), step 1, and derived corresponding gauge potentials \( (\vartheta^\alpha, \Gamma^{\alpha\beta}) \), step 2.

It was shown in Sec.2.7 that the gauging of the affine group does not determine a metric structure. Only by gauging a restricted affine-orthonogal group one might “inheritate” a metric structure. For the full affine group we have to assume a metric of the general form

\[
g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta
\]  

(3.3)

with coefficients \( g_{\alpha\beta} \) which are independent of the coframe \( \vartheta^\alpha \). The introduction of a metric into MAG is mandatory since we are interested in a realistic macroscopic gravity theory that contains General Relativity in some limit. The gauge potentials \( (\vartheta^\alpha, \Gamma^{\alpha\beta}) \) were introduced by the gauge process by demanding \( A(n,R) \)-invariance of the physical theory. It would be neat if also the metric could be deduced by a similar physical procedure, e.g. by the gauge principle or some symmetry breaking mechanism. However, for the time being, it simply seems to be unclear how to do this in a convincing manner. Future developments in this context seem possible. In order to proceed, we have to be satisfied by introducing the independent metric components \( g_{\alpha\beta} \) by hand, i.e. we will just add \( g_{\alpha\beta} \) to the list of gauge fields of metric–affine gravity. It is not a gauge field in the usual sense, though.

We move on to step 3, i.e., to the construction of a gauge invariant gauge field Lagrangian \( V \) from \( g_{\alpha\beta}, \vartheta^\alpha, \Gamma^{\alpha\beta} \), and their first derivatives. The general ansatz reads

\[
V = V(g_{\alpha\beta}, \vartheta^\alpha, \Gamma^{\alpha\beta}, dg_{\alpha\beta}, d\vartheta^\alpha, d\Gamma^{\alpha\beta}).
\]  

(3.4)

Neither \( \Gamma^{\alpha\beta} \) nor \( dg_{\alpha\beta}, d\vartheta^\alpha, d\Gamma^{\alpha\beta} \) transform homogeneously under \( A(n,R) \)-transformations. To make \( V \) gauge invariant one first has to require that the gauge connection \( \Gamma^{\alpha\beta} \) features in \( V \) only via the covariant derivative \( D \). Then the gauge covariant extensions of \( dg_{\alpha\beta}, d\vartheta^\alpha, d\Gamma^{\alpha\beta} \) read (we drop in the following the explicit label \( \Gamma^{(L)} \) on top of the \( GL(n,R) \)-covariant derivative \( D \))
nonmetricity \( Q_{\alpha \beta} := -Dg_{\alpha \beta}, \) \( (3.5) \)
torsion \( T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma^\alpha_{\beta \gamma} \wedge \vartheta^\beta, \) \( (3.6) \)
curvature \( R_{\alpha}{}^\beta := "D\Gamma_{\alpha}{}^\beta" = d\Gamma_{\alpha}{}^\beta - \Gamma^\gamma_{\alpha \gamma} \wedge \Gamma^\beta_{\gamma}, \) \( (3.7) \)

Here we introduced the nonmetricity one-form \( Q_{\alpha \beta}, \) the torsion two-form \( T^\alpha, \) and the curvature two-form \( R_{\alpha}{}^\beta. \) Thus the general form of an \( A(n, R) \)-gauge invariant gauge Lagrangian \( V \) becomes

\[
V = V(g_{\alpha \beta}, \vartheta^\alpha, Q_{\alpha \beta}, T^\alpha, R_{\alpha}{}^\beta). \tag{3.8}
\]

This, in principle, completes step 3. We summarize the building blocks of the gauge Lagrangian \( V \) in Table 2. The explicit form of \( V \) depends on the particular physical model to be chosen. To carry through step 4 in full generality we simply assume

| Potential | Field strength | Bianchi identity |
|-----------|----------------|------------------|
| metric \( g_{\alpha \beta} \) | \( Q_{\alpha \beta} = -Dg_{\alpha \beta} \) | \( DQ_{\alpha \beta} = 2R_{(\alpha}{}^\mu g_{\beta)\mu} \) |
| coframe \( \vartheta^\alpha \) | \( T^\alpha = D\vartheta^\alpha \) | \( DT^\alpha = R_{\mu}{}^\alpha \wedge \vartheta^\mu \) |
| connection \( \Gamma_{\alpha}{}^\beta \) | \( R_{\alpha}{}^\beta = d\Gamma_{\alpha}{}^\beta - \Gamma^\gamma_{\alpha \gamma} \wedge \Gamma^\beta_{\gamma} \) | \( DR_{\alpha}{}^\beta = 0 \) |

Table 2: The gauge potentials of MAG together with their corresponding field strengths and Bianchi identities. Denoting the metric components \( g_{\alpha \beta} \) as “gauge potential” is more for historic and conventional reasons. They constitute not an \( A(n, R) \)-gauge potential in the mathematical sense.

the existence of appropriate representations of the matter fields \( \Psi \) which are to be incorporated. Of course, the choice of the matter fields \( \Psi \) depends also on the particular model.

For the matter Lagrangian \( L_{\text{mat}} \) of step 5 we make the ansatz

\[
L_{\text{mat}} = L_{\text{mat}}(g_{\alpha \beta}, \vartheta^\alpha, \Gamma_{\alpha}{}^\beta, \Psi, d\Psi). \tag{3.9}
\]

Therefore we allow \( \Psi \) to couple to the geometric structure \( (g_{\alpha \beta}, \vartheta^\alpha, \Gamma_{\alpha}{}^\beta) \) of space-time. This constitutes the coupling to gravity. Kinetic terms of the gauge potentials, i.e. those with \( d\vartheta^\alpha \) and \( d\Gamma_{\alpha}{}^\beta \), are forbidden in \( L_{\text{mat}} \). They belong into the gauge Lagrangian \( V \). Again, similar to step 3, the connection \( \Gamma_{\alpha}{}^\beta \) has to feature in \( L_{\text{mat}} \) only via the covariant derivative \( D\Psi \) to make \( L_{\text{mat}} \) gauge invariant. This leads to the form

\[
L_{\text{mat}} = L_{\text{mat}}(g_{\alpha \beta}, \vartheta^\alpha, \Psi, D\Psi). \tag{3.10}
\]
of the matter Lagrangian and completes step 5.

Finally we turn to step 6, i.e. to the...

### 3.3 Field equations

Having to our disposal the full Lagrangian $L = L_{\text{mat}} + V$, which is of the form $L = L(\Phi, d\Phi)$, we may apply Hamilton’s principle of least action to arrive at the equations of motion in the form (3.2), with the exterior derivative $d$ replaced by the $GL(n, R)$-covariant derivative $D$. Taking in (3.1), (3.2) for the general field $\Phi$ successively the fields $\Psi, \ g_{\alpha\beta}, \ \vartheta^\alpha, \ \Gamma_{\alpha\beta}$, we obtain the following equations of motion:

\[ D \left( \frac{\partial L}{\partial (d\Psi)} \right) - (-1)^p \frac{\partial L}{\partial \Psi} = 0, \]  
(MATTER) (3.11)

\[ D \left( \frac{\partial V}{\partial Q_{\alpha\beta}} \right) + \frac{\partial V}{\partial g_{\alpha\beta}} = -\frac{\delta L_{\text{mat}}}{\delta g_{\alpha\beta}}, \]  
(ZEROTH) (3.12)

\[ D \left( \frac{\partial V}{\partial T^\alpha} \right) + \frac{\partial V}{\partial \vartheta^\alpha} = -\frac{\delta L_{\text{mat}}}{\delta \vartheta^\alpha}, \]  
(FIRST) (3.13)

\[ D \left( \frac{\partial V}{\partial R_{\alpha\beta}} \right) + \frac{\partial V}{\partial \Gamma_{\alpha\beta}} = -\frac{\delta L_{\text{mat}}}{\delta \Gamma_{\alpha\beta}}. \]  
(SECOND) (3.14)

We already separated the gauge field equations of motion (3.12)-(3.14) into a contribution due to $V$ (on the left hand side) and a contribution due to $L_{\text{mat}}$ (the material currents on the right hand side). In the following we will condense our notation a bit and define shorter symbols for the partial derivative terms in (3.12)-(3.14). These definitions, together with the corresponding physical nomenclature, are summarized in Table 3. Using this shorter notation we can rewrite the field equations as

\[ \frac{\delta L}{\delta \psi} = 0, \]  
(MATTER) (3.15)

\[ DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta}, \]  
(ZEROTH) (3.16)

\[ DH_{\alpha} - E_{\alpha} = \Sigma_{\alpha}, \]  
(FIRST) (3.17)

\[ DH^{\alpha}_{\beta} - E^{\alpha}_{\beta} = \Delta^{\alpha}_{\beta}. \]  
(SECOND) (3.18)

### 3.4 Noether identities

In this section we sketch the derivation of the conservation or Noether identities which result from the assumed $A(n, R)$-invariance of the Lagrangian $L$. The $A(n, R)$-
| Definition of symbol | Nomenclature | Form–Degree |
|----------------------|--------------|-------------|
| $\sigma^\alpha{}^\beta := 2 \frac{\delta {L}_{\text{mat}}}{\delta g^\alpha{}^\beta}$ | metrical energy–momentum current of matter | $n$ |
| $\Sigma_\alpha := \frac{\delta {L}_{\text{mat}}}{\delta \vartheta^\alpha} = \frac{\partial {L}_{\text{mat}}}{\partial \vartheta^\alpha}$ | canonical energy–momentum current of matter | $n - 1$ |
| $\Delta^{\alpha}{}_{\beta} := \frac{\delta {L}_{\text{mat}}}{\delta \Gamma^{\alpha}{}_{\beta}}$ | hypermomentum current of matter | $n - 1$ |
| $m^{\alpha}{}^\beta := 2 \frac{\partial V}{\partial g^\alpha{}^\beta}$ | metrical energy–momentum of the gauge field | $n$ |
| $E_\alpha := \frac{\partial V}{\partial \vartheta^\alpha}$ | canonical energy–momentum of the gauge field | $n - 1$ |
| $E^{\alpha}{}_{\beta} := \frac{\partial V}{\partial \Gamma^{\alpha}{}_{\beta}}$ | hypermomentum of the gauge field | $n - 1$ |
| $M^{\alpha}{}^\beta := -2 \frac{\partial V}{\partial d g^\alpha{}^\beta} = -2 \frac{\partial V}{\partial Q^\alpha{}^\beta}$ | metrical gauge field momentum | $n - 1$ |
| $H_\alpha := -\frac{\partial V}{\partial d \vartheta^\alpha} = -\frac{\partial V}{\partial T^\alpha}$ | gauge field momentum | $n - 2$ |
| $H^{\alpha}{}_{\beta} := -\frac{\partial V}{\partial d \Gamma^{\alpha}{}_{\beta}} = -\frac{\partial V}{\partial R^\alpha{}^\beta}$ | gauge field hypermomentum | $n - 2$ |

Table 3: A collection of the relevant physical quantities of MAG.

invariance was broken down to diffeomorphism invariance on $M$ and $GL(n,R)$-invariance. Both types of invariance are independent of each other. Therefore we consider them separately.

### 3.4.1 Translation invariance and first Noether identities

Translations are generated by the gauge covariant Lie-derivative $L$, as was explained in Sec.2.5. A general variation of the Lagrangian $L$ is given by

$$
\delta L = \delta g^\alpha{}^\beta \frac{\partial L}{\partial g^\alpha{}^\beta} + \delta Q^\alpha{}^\beta \wedge \frac{\partial L}{\partial Q^\alpha{}^\beta} + \delta \vartheta^\alpha \wedge \frac{\partial L}{\partial \vartheta^\alpha} + \delta T^\alpha \wedge \frac{\partial L}{\partial T^\alpha} + \delta R^\alpha{}^\beta \wedge \frac{\partial L}{\partial R^\alpha{}^\beta} \\
+ \delta \Psi \wedge \frac{\partial L}{\partial \Psi} + \delta (D \Psi) \wedge \frac{\partial L}{\partial D \Psi}.
$$

(3.19)
This implies the form of the variation of $L$ under an infinitesimal translation $\varepsilon$:

$$
\mathcal{L}_\varepsilon L = (\mathcal{L}_\varepsilon g_{\alpha\beta}) \frac{\partial L}{\partial g_{\alpha\beta}} + (\mathcal{L}_\varepsilon Q_{\alpha\beta}) \wedge \frac{\partial L}{\partial Q_{\alpha\beta}} + (\mathcal{L}_\varepsilon \vartheta^\alpha) \wedge \frac{\partial L}{\partial \vartheta^\alpha} \\
+ (\mathcal{L}_\varepsilon T^\alpha) \wedge \frac{\partial L}{\partial T^\alpha} + (\mathcal{L}_\varepsilon R_{\alpha}^\beta) \wedge \frac{\partial L}{\partial R_{\alpha}^\beta} \\
+ (\mathcal{L}_\varepsilon \Psi) \wedge \frac{\partial L}{\partial \Psi} + (\mathcal{L}_\varepsilon D\Psi) \wedge \frac{\partial L}{\partial D\Psi}.
$$

(3.20)

The condition for translation invariance of $L$ reads

$$
\mathcal{L}_\varepsilon L = 0.
$$

(3.21)

The total Lagrangian $L$ is the sum of the gauge Lagrangian $V$ and the matter Lagrangian $L_{\text{mat}}$:

$$
L = V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_{\alpha}^\beta) + L_{\text{mat}}(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_{\gamma}^\beta, \Psi, d\Psi).
$$

(3.22)

Both parts are independent a priori. Therefore we investigate the invariance condition (3.21) for both $V$ and $L_{\text{mat}}$ independently. The necessary algebraic calculations are elementary but a bit lengthy. We will omit the explicit steps and give immediately the main results, but refer to [7] Chap.5, for more details.

The invariance condition

$$
\mathcal{L}_\varepsilon V = 0
$$

leads to the first Noether identity for the gauge Lagrangian $V$,

$$
D \delta V \delta^\alpha = (e_\alpha[T^\beta]) \wedge \delta V \delta^\beta + (e_\alpha[R^\beta]) \wedge \delta V \delta_{\beta}^{\gamma} - (e_\alpha[Q_{\beta\gamma}]) \delta V \delta_{g_{\beta\gamma}},
$$

(3.24)

together with the explicit expression for the canonical energy momentum $E_\alpha$,

$$
E_\alpha = e_\alpha[V + (e_\alpha[T^\beta]) \wedge H_{\beta} + (e_\alpha[R^\beta]) \wedge H_{\beta}^{\gamma} + \frac{1}{2}(e_\alpha[Q_{\beta\gamma}])M^{\beta\gamma}.
$$

(3.25)

For the matter Lagrangian $L_{\text{mat}}$ we get from the condition

$$
\mathcal{L}_\varepsilon L_{\text{mat}} = 0
$$

(3.26)

the first Noether identity

$$
D \Sigma_\alpha = (e_\alpha[T^\beta]) \wedge \Sigma_{\beta} + (e_\alpha[R^\beta]) \wedge \Delta^\beta_{\alpha} - \frac{1}{2}(e_\alpha[Q_{\beta\gamma}])\sigma^{\beta\gamma} \\
+ (e_\alpha[D\Psi]) \delta L \delta^{\beta} \delta\Psi + (-1)^p(e_\alpha[D\Psi]) \wedge D \delta L \delta\Psi \\
\cong (e_\alpha[T^\beta]) \wedge \Sigma_{\beta} + (e_\alpha[R^\beta]) \wedge \Delta^\beta_{\alpha} - \frac{1}{2}(e_\alpha[Q_{\beta\gamma}])\sigma^{\beta\gamma},
$$

(3.27)
with the explicit form of the canonical energy momentum tensor $\Sigma_{\alpha}$,

$$\Sigma_{\alpha} = e_{\alpha} \mid L_{\text{mat}} - (e_{\alpha} \mid D\Psi) \wedge \frac{\partial L_{\text{mat}}}{\partial D\Psi} - (e_{\alpha} \mid \Psi) \wedge \frac{\partial L_{\text{mat}}}{\partial \Psi}. \quad (3.28)$$

The symbol “$\cong$” in equation (3.27) denotes a “weak” identity which holds only if the matter field equations $\frac{\delta L}{\delta \Psi} = 0$ are fulfilled. Only for special relativity, i.e. for vanishing nonmetricity, torsion, and curvature, we recover from (3.27) the familiar energy momentum conservation law $d\Sigma_{\alpha} = 0$ (when written in pseudo-cartesian coordinates). The “field strength $\times$ current”-terms on the right hand side of (3.27) express the energy conservation of one united system (matter coupled to geometry).

### 3.4.2 General linear invariance and second Noether identities

Now we focus on variations of the Lagrangian $L$ induced by general linear transformations. An infinitesimal $GL(n, R)$-transformation reads

$$\Lambda_{\alpha}^\beta = \delta_{\alpha}^\beta + \epsilon_{\alpha}^\beta. \quad (3.29)$$

The variations of the geometric quantities $g_{\alpha\beta}, \vartheta^\alpha, \Gamma_{\alpha}^\beta$ under the transformation (3.29) were derived in (2.45), (2.33), and (2.21):

$$\delta g_{\alpha\beta} = -\epsilon_{\alpha\beta} - \epsilon_{\beta\alpha}, \quad \delta \vartheta^\alpha = \epsilon_{\beta}^\alpha \vartheta^\beta, \quad \delta \Gamma_{\alpha}^\beta = -D \epsilon_{\alpha}^\beta. \quad (3.30)$$

For the variation of the matter field we write $\delta \Psi = \epsilon_{\alpha}^\beta L_{\alpha\beta} \Psi$. Then the invariance condition

$$\delta V = 0, \quad \delta \equiv \text{infinitesimal } GL(n, R) \text{ transformation}, \quad (3.31)$$

implies the second Noether identity for the gauge Lagrangian $V$:

$$D \frac{\delta V}{\delta \Gamma_{\alpha}^\beta} + \vartheta^\alpha \wedge \frac{\delta V}{\delta \vartheta^\beta} - 2g_{\beta\gamma} \frac{\delta V}{\delta g_{\alpha\beta}} = 0, \quad (3.32)$$

which can be written more explicitly as (compare table 3),

$$m_{\alpha}^\beta = \vartheta^\alpha \wedge E_\beta + Q_{\beta\gamma} \wedge M_{\alpha}^{\alpha\gamma} - T^\alpha \wedge H_\beta - R^\alpha_\gamma \wedge H_\beta^\gamma + R^\beta_\gamma \wedge H^\alpha_\gamma. \quad (3.33)$$

The second Noether identity for the matter Lagrangian $L_{\text{mat}}$ is derived from the invariance condition $\delta L_{\text{mat}} = 0$ and turns out to be

$$D \Delta_{\alpha}^\beta + \vartheta^\alpha \wedge \Sigma_{\beta} - g_{\beta\gamma} \sigma_{\alpha}^{\alpha\gamma} = -(L_{\alpha}^\beta \Psi) \wedge \left(\frac{\delta L}{\delta \Psi}\right) \quad (3.34)$$

$$\cong 0. \quad (3.35)$$

The $GL(n, R)$-invariance, reflected by the second Noether identities, implies a redundancy carried by either the metric or the coframe: Adding up (3.32) and (3.34), we
find immediately that one of the field equations, ZEROTH or FIRST, is redundant, provided the MATTER- and SECOND-field equation are fulfilled. This can be traced back to the observation that the field equations of a $GL(n, R)$-invariant theory should not fix the choice of the coframe $\vartheta^\alpha$: $GL(n, R)$-invariance just represents the freedom to choose any coframe. This allows for quite some flexibility in solving the field equations, since one might solve MATTER, ZEROTH, and SECOND by using some convenient gauge for the coframe $\vartheta^\alpha$. Then FIRST is automatically fulfilled. Vice versa, one might fix the metric coefficients to a certain gauge, to the orthonormal one, e.g., and solve under this prerequisite MATTER, ZEROTH, and SECOND.

3.5 Subcases of MAG by reducing the affine group

The established framework of MAG is fairly general. In order to obtain more limited gravity models, one might use at least two different methods:

- Invoke restrictions on the geometry on the level of the Lagrangian. This includes the possibility of enforcing constraints (e.g. constraints of vanishing nonmetricity, torsion, or curvature) via the method of Lagrange multipliers. By means of this elegant method restrictions are imposed on the full $A(n, R)$-gauge model, but the geometrical and physical variables are kept which we introduced so far.

- Restrict MAG right from the beginning by restricting the gauge group and, accordingly, by dropping some gauge variables of the set $(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_{\alpha\beta})$.

The second possibility is discussed in this subsection.

To begin with, we note that for $n \geq 2$ the affine group $A(n, R) = T^n \ltimes GL(n, R)$ can be further decomposed according to the group isomorphism

$$GL(n, R) \approx [T \ltimes SL(n, R)] \times R^+.$$  \hspace{1cm} (3.36)

Here, $T$ denotes the time reflection $T \in GL(n, R)$ with its defining property $\det T = -1$. The special linear group $SL(n, R)$ consists of the set of elements of $GL(n, R)$ with unit determinant. In view of the local gauge procedure we are not concerned about global issues and focus on the Lie-algebras of the gauge groups rather than on the gauge groups themselves. Hence, disregarding the discontinuous time reflection $T$, the group isomorphism (3.36) leads to the Lie-algebra isomorphism

$$gl(n, R) \approx sl(n, R) \times R^+.$$ \hspace{1cm} (3.37)

The analogous splitting of the $GL(n, R)-$generators $L^a_{\ b}$, which were first introduced in (2.10), yields traceless linear transformations $\mathcal{E}^a_{\ b}$ and dilations $L^c_{\ c}$:

$$L^a_{\ b} = \mathcal{E}^a_{\ b} + \delta^a_{\ b} L^c_{\ c} / n.$$ \hspace{1cm} (3.38)
For a further separation we need to introduce a metric $g$, defined in the affine tangent spaces, in order to raise and lower indices. Then the traceless linear transformations $\mathcal{F}^a_b \rightarrow \mathcal{F}^a_b = g_{ac} \mathcal{F}^c_b$ decompose into their skew symmetric parts $L_{[ab]}$ (Lorentz rotations) and their traceless symmetric parts $\mathcal{F}_{(ab)} = L_{(ab)} - g_{ab} L^c_c/n$ (shears):

$$L_{ab} = L_{[ab]} + \mathcal{F}_{(ab)} + g_{ab} L^c_c/n.$$  \hfill (3.39)

The corresponding decomposition of the Lie-algebra $gl(n, R)$ is given by

$$gl(n, R) = [so(n) \oplus \mathbb{R}] \times \mathbb{R}^+,$$  \hfill (3.40)

where the Lie-algebra $sl(n, R)$ got split into its maximal compact subalgebra $so(n)$ and its noncompact part $\mathbb{R}$.

Hence, if a metric is present, general linear invariance splits into Lorentz invariance, shear invariance, and dilation invariance. Together with the translation invariance we thus have decomposed the general affine invariance into four different types of invariances. These can be separately combined to yield different gauge models as subcases of MAG.

### 3.5.1 Full MAG

This is the theory we explained so far. Here, the complete affine invariance is postulated, leading to the introduction of the translation gauge potential $\vartheta^\alpha$ and the $GL(n, R)$-gauge connection $\Gamma_\alpha^{\beta}$. Both quantities imprint an affine parallel transport on the spacetime-manifold which is described by the field strengths curvature and torsion. A metric structure is not given a priori but later introduced by hand. The full spectrum of physical quantities is available, as listed in table 3.

It is remarkable to note that both, the translation potential and the reference frame related to the $GL(n, R)$-gauge invariance, are represented by the coframe $\vartheta^\alpha$: If expanded in holonomic coordinates as $\vartheta^\alpha = e_i^\alpha dx^i$, the $n^2$ components $e_i^\alpha$ are expected to be of physical significance in their role as translation potentials. On the other hand, the $n^2$-parameter $GL(n, R)$-invariance tells us that no admissible choice of the $e_i^\alpha$ is preferred. Does this mean that in the case of $GL(n, R)$-invariance the translation potential is of no physical significance?

If one were to replace $GL(n, R)$-invariance by Lorentz invariance ($\frac{1}{2} n(n-1)$ parameters), then $n^2 - \frac{1}{2} n(n-1) = \frac{1}{2} n(n+1)$ functions of the $n^2$ components $e_i^\alpha$ would survive. In Lorentz invariant theories these remaining $\frac{1}{2} n(n+1)$ functions provided by the coframe can be taken as components $g_{ij}$ of a metric. In this case the coframe $\vartheta^\alpha$ acquires the role of an orthonormal coframe in accordance with the discussion of Sec. 2.7.

Dealing with the complete $GL(n, R)$-invariance, $\frac{1}{2} n(n+1)$ metric components $g_{ij}$ have to be introduced by hand. However, once the existence of a metric is
postulated, one can reexpress the metric components \( g_{ij} \) in terms of the components \( e_{i}^{\alpha} \) of the coframe \( \vartheta^{\alpha} \):

\[
g = g_{ij} \, dx^i \otimes dx^j \quad \Rightarrow \quad g_{ij} \equiv o_{\alpha\beta} \, e_{i}^{\alpha} e_{j}^{\beta} \, dx^i \otimes dx^j,
\]

(3.41)

The star * indicates that for the full \( GL(n,R) \)–invariance the equality holds only in a specific gauge. Equation (3.42) expresses the possibility of always finding a gauge such that the coframe becomes orthonormal. This determines the coframe. Vice versa, one can start from a (dynamical) coframe and view eq. (3.42) as establishing a gauge in which the metric is determined by an orthonormal coframe. This is the physical meaning which is given to the translation potential in a \( GL(n,R) \)–invariant theory: In a \( GL(n,R) \)–invariant theory the coframe \( \vartheta^{\alpha} \) does determine a metric in a specific gauge. However, we stress that this interpretation comes after the postulation of the existence of a metric \( g \). It also explains the redundancy of either metric or coframe that was established in the discussion of the Noether identities of MAG in the previous section.

### 3.5.2 MAG with restricted connection

In a next step one can keep translation and Lorentz invariance while dropping either shear or dilation invariance. A subsequent gauging of the remaining invariances leads to the introduction of the translation gauge potential \( \vartheta^{\alpha} \) and a gauge connection \( \Gamma_{\alpha\beta} \) which is no longer \( gl(n,R) \)–valued but restricted in the following sense: In the expansion \( \Gamma_{\alpha\beta} = (\Gamma_{c}^{[L]} L_{d})_{\alpha\beta} \) the generator \( L_{c}^{d} \) has to be replaced by \( L_{[ab]} + \mathcal{P}_{(ab)} \) or \( L_{[ab]} + g_{ab} L_{c}/n \), compare (3.39). The gauge group is still non-orthogonal, such that, as in the case of the full MAG, metric components \( g_{ij} \) have to be introduced by hand and the coframe \( \vartheta^{\alpha} \) cannot be taken as a gauge independent orthonormal coframe. The resulting geometry of spacetime is characterized by a Riemannian background and the restricted gauge connection \( \Gamma_{\alpha\beta} \).

The dropping of the shear invariance leads to what is known as gravity with Weyl-invariance. The second case which drops dilation and keeps shear invariances represents a gauge theory of volume preserving linear transformations. Its corresponding \( SL(n,R) \)–symmetry plays an important role in the group theoretical classification of hadrons [14, 15].

### 3.5.3 Affine-orthogonal gravity: Poincaré gauge theory

Leaving aside both shear and dilation invariance, we are left with Poincaré invariance. Gauging the Poincaré group yields the translation potential \( \vartheta^{\alpha} \) and the Lorentz connection \( \Gamma_{\alpha\beta} = (\Gamma_{c}^{[L]} L_{d})^{\alpha\beta} \). The \( \frac{1}{2}n(n-1) \) gauge parameters of Lorentz invariance reduce the \( n^2 \)–components of the coframe \( \vartheta^{\alpha} \) to \( \frac{1}{2}n(n+1) \) physical degrees.
of freedom. These can be taken to define a metric, i.e. a Riemannian background, on the spacetime manifold. This follows again from the discussion of Sec. 2.7. No additional metric components have to be supplemented. The geometry of spacetime is that of a Riemannian manifold with independent linear Lorentz connection, i.e., it is characterized by torsion and Lorentz curvature. This is the framework of the Poincaré gauge theory, the Einstein–Cartan theory is a well–known example within this class of theories.

3.5.4 Translational gauging

A gauging of the translation group leads to the introduction of the translation potential $\vartheta^\alpha$. However, only a non–dynamical linear connection $\Gamma^\alpha_{\beta \gamma}$ with vanishing curvature $R^\alpha_{\beta \gamma}$ is introduced. If the existence of a metric is assumed the coframe $\vartheta^\alpha$ induces a Riemannian background on the spacetime manifold. The resulting geometry is a so-called Weitzenböck geometry. No independent linear connection is present. This case is explained in detail in the next chapter by the teleparallel version of general relativity.

We order the gauge models introduced so far by displaying their corresponding geometries in Tab. 4

3.5.5 Gravity without gauging

The original second order approach to gravity, i.e. the occurrence of second derivatives of the fundamental variable in the Lagrangian, presupposes the metric $g$ as fundamental variable (Einstein’s GR). In this case the theory is also built in a way such that it possesses certain types of invariances (coordinate invariance, local Lorentz invariance). However, these invariances are not subject to a gauge procedure. In particular, no gauge potentials are introduced. Thus, if no matter is included, the metric is the only dynamical variable of the theory, leading to a Riemannian spacetime with no additional geometric structure. We will subsequently show how to obtain this case from a pure gauging of the translation group.
|                      | local gauge group | potentials introduced | metric $g$  | torsion $T^\alpha$ | curvature $R^\alpha_\beta$ | nonmetricity $Q_{\alpha\beta}$ |
|----------------------|-------------------|-----------------------|-------------|-------------------|-----------------------------|-------------------------------|
| **Full MAG**         | $T^n \ltimes GL(n,R)$ | $\vartheta^\alpha (\mathbb{R}^n)$ | introduced by hand | yes | yes | yes |
| MAG with restricted connection: dilations | $T^n \ltimes [SO(1,n-1) \times \mathbb{R}^+]$ | $\vartheta^\alpha (\mathbb{R}^n)$ | introduced by hand | yes | yes | yes |
| MAG with restricted connection: shears | $T^n \ltimes SL(n,R)$ | $\Gamma^\alpha_{\beta} (\mathfrak{sl}(n,r))$ | introduced by hand | yes | yes | yes |
| Affine-orthogonal gravity | $T^n \ltimes SO(1,n-1)$ | $\vartheta^\alpha (\mathbb{R}^n)$ | defined by $\vartheta^\alpha$ | yes | yes | no |
| Translational gauging | $T^n$ | $\vartheta^\alpha (\mathbb{R}^n)$ | defined from $\vartheta^\alpha$ | no | no | no |

Table 4: The geometries induced by various subcases of MAG. Parentheses include the Lie-algebra in which the corresponding quantity assumes its values. The nonmetricity $Q_{\alpha\beta} = -\frac{1}{n}Dg_{\alpha\beta}$ is not Lie-algebra valued. It splits into tracefree shear and trace parts according to $Q_{\alpha\beta} = \mathcal{Q}_{\alpha\beta} + Qg_{\alpha\beta}$, where $Q := Q^{\alpha}_{\alpha}/n$ denotes the Weyl covector. In the case of translational gauging, both torsion and curvature may exist. However, in contrast to the other cases, they are not independent of each other and always, in some manner, derived from the Riemannian background which is determined by the coframe $\vartheta^\alpha$. 

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4 Pure translation invariance and the reduction from MAG to teleparallelism and GR

Having successfully gauged the affine group, it is straightforward to specialize from MAG to GR in order to recover familiar ground. But first we will give some physical evidence in favor of the translation group being the gauge group of GR.

4.1 Motivation

Suppose we start from SR. The invariance of the action \( W = \int L_{\text{mat}}(\Psi, d\Psi) \) of an isolated material system under rigid spacetime translations yields, by the application of the Noether theorem, a conserved energy-momentum current three-form

\[
\Sigma_j := \frac{\delta L_{\text{mat}}}{\delta x^j} = \frac{1}{3!} \Sigma_{klmj} dx^k \wedge dx^l \wedge dx^m , \quad d \Sigma_j = 0 . \tag{4.1}
\]

(One obtains the conventional energy-momentum tensor \( T_{ij} \) from \( \Sigma_j \) by means of \( T_{ij} = \epsilon_{iklm} \Sigma_{klmj} \).) The corresponding charge \( M := \int d^3 x \Sigma_0 \) is conserved in time. In other words: Rigid translational invariance, in a classical field-theoretical context, implies the conservation of mass-energy which itself is the source of Newton-Einstein gravity.

We can compare this to internal gauge theories. In electrodynamics one finds from rigid \( U(1) \)-invariance of an action \( W = \int L_{\text{mat}}(\Psi, d\Psi) \) a conserved electric current \( J_{\text{Max}} \) with corresponding electric charge \( Q \), which is the source of Maxwell’s theory. Also in Yang-Mills theories, one starts from the rigid symmetry of a Lagrangian \( L_{\text{mat}} \), implying via Noether’s theorem a conserved current \( J \), the “isotopic spin”, with corresponding charges. This is illustrated in the upper half of Fig.5. Subsequent gauging of the rigid symmetry leads to the introduction of a gauge potential \( A \), accounting for the freedom to choose at any point reference frames modulo gauge transformations. Gauge invariance of the Lagrangian is achieved by replacing exterior derivatives of the matter field by gauge covariant ones,

\[
d \longrightarrow \hat{D} := d + A , \quad L_{\text{mat}}(\Psi, d\Psi) \longrightarrow L_{\text{mat}}(\Psi, \hat{D}_\Psi) . \tag{4.2}
\]

This is called minimal coupling of the matter field to the new gauge interaction. By construction, the gauge potential in the Lagrangian couples to the conserved
current one started with – and the original conservation law, in case of a non-Abelian symmetry, gets modified and is only gauge covariantly conserved,

\[ dJ = 0 \quad \rightarrow \quad \hat{A} \, dJ = 0 \, , \quad J = \partial L_{\text{mat}} / \partial A . \quad (4.3) \]

The physical reason for this modification is that the gauge potential itself contributes a piece to the current, that is, the gauge field is charged itself in the non-Abelian case.

Let us come back to gravity. Having the conserved energy momentum tensor, which is a consequence of rigid translation invariance, we expect to switch on the gravitational interaction by gauging the translation group. The details are given in the following section.

### 4.2 Einsteinian teleparallelism: Translation gauge potential and Lagrangian

We start from a special-relativistic and rigidly translation invariant field theory\(^7\). In particular we assume a Minkowski spacetime, pseudo-Cartesian coordinates \(x^i\), and

\(^7\)The following presentation was inspired by the paper of Cho [3].
parallel transport to be defined. The latter assumption implies the existence of a linear connection $\Gamma^\alpha{}_{\beta}$ which, in view of Minkowski spacetime, produces vanishing curvature and can globally gauged to zero. We define a matter Lagrangian $L_{\text{mat}} = L_{\text{mat}}(\Psi, d\Psi, dx^i)$, where we accounted for an explicit dependence on the coordinate differentials $dx^i$. An explicit dependence on the coordinate functions $x^i$ is already forbidden by rigid translation invariance. The transition to a locally translation invariant theory is conveniently accomplished by using the translation invariant and orthonormal coframe $\vartheta^\alpha$ of Sec. 2.7. This introduces the translation potential $\Gamma^{(T)}{}^\alpha$ via

$$\vartheta^\alpha = \delta^\alpha_i dx^i + \Gamma^{(T)}{}^\alpha. \quad (4.4)$$

The transformation behavior of $\vartheta^\alpha$, $dx^i$, and $\Gamma^{(T)}$ under infinitesimal translations $x^i \to x^i + \varepsilon^i$ is obtained from (2.33), (2.28), and (2.20) as

$$\delta \vartheta^\alpha = 0, \quad \delta (dx^i) = \varepsilon^i, \quad \delta \Gamma^{(T)}{}^\alpha = -d\varepsilon^\alpha. \quad (4.5)$$

The coupling of $\Gamma^{(T)}{}^\alpha$ to the matter fields is a bit unfamiliar: Wherever the holonomic basis $dx^i$ occurs explicitly in $L_{\text{mat}}(\Psi, d\Psi, dx^i)$, it is replaced by

$$Dx^\alpha := \delta^\alpha_i dx^i + \Gamma^{(T)}{}^\alpha = \vartheta^\alpha. \quad (4.6)$$

The corresponding field strength $T^\alpha := d\vartheta^\alpha$ can be used to construct a kinetic supplementary term for $\vartheta^\alpha$ to the Lagrangian. The double role of $\vartheta^\alpha$ as both, a dynamical gauge potential and an orthonormal frame (defining a new metric via $g = o_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$), explains the transition from Minkowski space to a dynamical spacetime, which is due to translational invariance.

For the kinetic term we make the quadratic ansatz $V = d\vartheta^\alpha \wedge H_\alpha$, i.e. $H_\alpha$ is linear in $d\vartheta^\alpha$. What would be a good choice for $H_\alpha$? Eyeing at Yang-Mills theory, we are tempted to put $H_\alpha = \frac{1}{2\ell^2} * d\vartheta^\alpha$, with $\ell =$ Planck length. But we would like to end up with a locally Lorentz invariant theory. The Lagrangian $V = \frac{1}{2\ell^2} * d\vartheta^\alpha \wedge d\vartheta_\alpha$ is rigidly but not locally Lorentz invariant, though. In order to achieve local Lorentz invariance, i.e. the freedom to choose at any point of spacetime reference frames modulo Lorentz transformations, we could gauge the Lorentz group. This would introduce a dynamical linear Lorentz connection $\Gamma$ and, on the level of the Lagrangian $V$, lead to the replacement of ordinary exterior derivatives by Lorentz covariant ones:

$$d \rightarrow d + \Gamma \wedge, \quad V = \frac{1}{2\ell^2} * d\vartheta^\alpha \wedge d\vartheta_\alpha \rightarrow V = \frac{1}{2\ell^2} * D\vartheta^\alpha \wedge D\vartheta_\alpha = \frac{1}{2\ell^2} * T^\alpha \wedge T_\alpha. \quad (4.8)$$

This is what we will not do here. A gauging of both translation and Lorentz group would lead to a framework larger than that of GR. In fact, local Lorentz invariance can be achieved without the introduction of a dynamical Lorentz connection $\Gamma$. This demand determines an appropriate Lagrange function $V$ as follows:
The most general term $V$ quadratic in $d\theta^\alpha$ is obtained by choosing $H_\alpha$ as

$$H_\alpha = \frac{1}{2\ell^2} \{a_1 (1) d\theta_\alpha + a_2 (2) d\theta_\alpha + a_3 (3) d\theta_\alpha\}.$$  \hspace{1cm} (4.9)

The pieces $(1) d\theta^\alpha$ correspond to the irreducible pieces $(1) T^\alpha$ of the torsion, compare Table 5:

$$
\begin{align*}
(1) d\theta^\alpha &:= d\theta^\alpha - (2) d\theta^\alpha - (3) d\theta^\alpha, \\
(2) d\theta^\alpha &:= \frac{1}{3} \vartheta^\alpha \wedge (e_\beta \wedge d\theta^\beta), \\
(3) d\theta^\alpha &:= -\frac{1}{3} \{\vartheta^\alpha \wedge (d\theta^\beta \wedge \vartheta_\beta)\}. \\
\end{align*}
\hspace{1cm} (4.10)
$$

|   | explicit expression | number of components | name    |
|---|---------------------|----------------------|---------|
| $(1) T^\alpha$ | $T^\alpha - (2) T^\alpha - (3) T^\alpha$ | 16 | TENTOR |
| $(2) T^\alpha$ | $\frac{1}{3} \vartheta^\alpha \wedge (e_\beta \wedge T^\beta)$ | 4 | TRATOR |
| $(3) T^\alpha$ | $-\frac{1}{3} \{\vartheta^\alpha \wedge (T^\beta \wedge \vartheta_\beta)\}$ | 4 | AXITOR |

Table 5: Irreducible decomposition of the torsion $T^\alpha = (1) T^\alpha + (2) T^\alpha + (3) T^\alpha$ under the Lorentz group $SO(1,3)$

The postulate of local Lorentz invariance leads to a solution for the constant and real parameters $a_I$ in the following way: Infinitesimal Lorentz rotations are expressed by

$$\delta \theta^\alpha = \varepsilon^{\alpha\beta} \vartheta_\beta,$$

where $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$ are the antisymmetric Lorentz group parameters. It is easy to check that the gauge Lagrangian $V = d\theta^\alpha \wedge H_\alpha$, with $H_\alpha$ given by (4.9), is invariant under rigid Lorentz rotations, $\delta V = 0$. The general expression for $\delta V$ reads

$$\delta V = \left(\frac{\partial V}{\partial \vartheta^\alpha} + d \frac{\partial V}{\partial d\theta^\alpha}\right) \wedge \delta \theta^\alpha - d \left(\frac{\partial V}{\partial d\theta^\alpha} \wedge \delta \theta^\alpha\right).$$  \hspace{1cm} (4.11)

Hence we have $\delta V = 0$ for rigid Lorentz rotations. However, for local Lorentz rotations with spacetime-dependent group parameters $\varepsilon_{\alpha\beta}(x)$, we find from (4.11) the offending term as

$$\delta_{\text{(local)}} V = -d \frac{\partial V}{\partial d\theta^\alpha} \wedge \vartheta^\beta.$$  \hspace{1cm} (4.12)
In order to achieve local Lorentz invariance, this term has to vanish, modulo an exact form. Using the Leibniz rule, we obtain

\[ d\varepsilon^{\alpha\beta} \wedge \frac{\partial V}{\partial d\vartheta^\alpha} \wedge \vartheta^\beta = -\varepsilon^{\alpha\beta} d \left( \frac{\partial V}{\partial d\vartheta^\alpha} \wedge \vartheta^\beta \right) + d \left( \varepsilon^{\alpha\beta} \frac{\partial V}{\partial d\vartheta^\alpha} \wedge \vartheta^\beta \right). \] (4.13)

The second term on the r.h.s. is already exact. From the first term we get as condition for local Lorentz invariance of \( V \)

\[ \frac{\partial V}{\partial d\vartheta}[^\alpha \wedge \vartheta^\beta] = \text{exact form}. \] (4.14)

We plug in the explicit expression for \( V \) and obtain, after some algebra,

\[
2l^2 \frac{\partial V}{\partial d\vartheta}[^\alpha \wedge \vartheta^\beta] = \left( \frac{1}{3}a_1 - \frac{1}{3}a_3 \right) d\eta_{\alpha\beta} - \left( \frac{2}{3}a_3 + \frac{1}{3}a_1 \right) d\vartheta[^\alpha \wedge \vartheta^\beta] + \left( \frac{1}{6}a_1 - \frac{1}{6}a_2 - \frac{1}{3}a_3 \right) (e^\gamma \left[ d\vartheta^\gamma \right]) \wedge \eta_{\alpha\beta}.
\] (4.15)

The last two terms are made vanishing by choosing

\[ a_3 = -\frac{1}{2}a_1, \quad a_2 = -2a_1. \] (4.16)

Then we obtain

\[ 2l^2 \frac{\partial V}{\partial d\vartheta}[^\alpha \wedge \vartheta^\beta] = \frac{a_1}{2} d\eta_{\alpha\beta}. \] (4.17)

The constant \( a_1 \) can be absorbed by a suitable choice of the coupling constant \( \ell \) in \( V, \) see (4.9). According to the usual conventions, we put \( a_1 = -1, \) i.e. \( V \) is locally Lorentz invariant for parameters

\[ a_1 = -1, \quad a_2 = 2, \quad a_3 = \frac{1}{2}. \] (4.18)

Thus

\[ V_{||} = \frac{1}{2l^2} d\vartheta^\alpha \wedge ^* \left( -^{(1)} d\vartheta^\alpha + 2^{(2)} d\vartheta^\alpha + \frac{1}{2}^{(3)} d\vartheta^\alpha \right). \] (4.19)

The total Lagrangian reads

\[ L_{\text{tot}} = V_{||} + L_{\text{mat}}(\Psi, d\Psi, \vartheta^\alpha). \] (4.20)

It is locally Lorentz invariant only if \( d\Psi \) transforms covariantly under the Lorentz group. This happens for scalar fields or gauge fields like the Maxwell field – and is a further assumption to be made in order to avoid a gauging of the Lorentz group.

The field equation \( \delta L_{\text{tot}}/\delta \vartheta^\alpha = 0 \) becomes

\[ dH_\alpha - E_\alpha = \Sigma_\alpha, \] (4.21)
where, as before, $\Sigma_\alpha = \delta L_{\text{mat}}/\delta \vartheta^\alpha$ denotes the material canonical energy-momentum current and

$$E_\alpha = (e_\alpha \lfloor d\vartheta^\beta \rfloor \wedge H_\beta - \frac{1}{2} e_\alpha \lfloor (d\vartheta^\beta \wedge H_\beta)\rfloor$$

$$= \frac{1}{2} \left[ (e_\alpha \lfloor d\vartheta^\beta \rfloor \wedge H_\beta - d\vartheta^\beta \wedge (e_\alpha \lceil H_\beta \rceil) \right]$$

(4.22)

the energy-momentum current of the gauge field.

### 4.3 Transition to GR

If the Lagrangian (4.19) is substituted into the field equation (4.21), it can be seen that the antisymmetric part of the left hand side of (4.21) vanishes,

$$\vartheta_{[\beta} \wedge dH_{\alpha]} - \vartheta_{[\beta} \wedge E_{\alpha]} = 0.$$  (4.23)

Therefore the right hand side has to be symmetric, too. Again, we recognize that only scalar matter fields or gauge fields, such as the electromagnetic field, are allowed as material sources, whereas matter carrying spin cannot be consistently coupled in such a framework.

The object of anholonomy $d\vartheta^\alpha$ describes a Riemannian geometry of spacetime. The corresponding Levi-Civita (or Christoffel) connection $\{\Gamma_{\alpha\beta}\}$, referring to the metric $g = o_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$, cf. (2.46), can be derived from Cartan’s (first) structure equation

$$d\vartheta^\alpha = -\{\Gamma_{\alpha\beta}\} \wedge \vartheta^\beta.$$  (4.24)

Solving (4.24) for $\{\Gamma_{\beta\alpha}\}$ yields

$$\{\Gamma_{\alpha\beta}\} = \frac{1}{2} \left( e_\alpha \lfloor d\vartheta_\beta - e_\beta \lfloor d\vartheta_\alpha \rfloor - (e_\alpha \lfloor e_\beta \lfloor d\vartheta_\gamma \rfloor \wedge \vartheta_\gamma ) \right).$$  (4.25)

The corresponding Riemannian curvature is given by

$$\{\{\Gamma_{\alpha\beta}\} = d\{\Gamma_{\alpha\beta}\} - \{\Gamma_{\alpha\gamma}\} \wedge \{\Gamma_{\gamma\beta}\}.$$  (4.26)

However, parallel transport is still determined by the nondynamical and trivial linear connection $\Gamma_{\alpha\beta}$ introduced before the gauge process. It vanishes in a certain gauge, $\Gamma_{\alpha\beta} = 0$, i.e. a teleparallelismus is imprinted on the Riemannian background. If the Riemannian background is nontrivial this implies the existence of nontrivial torsion:

$$T^\alpha = D\vartheta^\alpha = d\vartheta^\alpha - \{\Gamma_{\beta\alpha}\} \wedge \vartheta^\beta.$$  (4.27)

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In other words: Meaningful teleparallel theories do not presuppose spinning matter as a source for nontrivial torsion, in contrast to what is sometimes stated in the literature [11].

Doesn’t all this look like general relativity? We use (4.25) and (4.26) to replace, on the Lagrangian level, the variable $d\vartheta^\alpha$ by $\Gamma^\alpha_{\alpha\beta}$: Using these equations one can prove the quite remarkable identity

$$\frac{1}{2} \{ R^\alpha_{\beta\gamma} \wedge \eta_{\alpha\beta} - \ell^2 V_\parallel = d(\vartheta^\alpha \wedge *d\vartheta_\alpha) ,$$

with $V$ given by (4.19). Therefore one finds that the kinetic term $V_\parallel$, with the parameters $a_I$ as chosen above, is equal to the Hilbert-Einstein action modulo an exact form. Replacing $V$ in the action $S$ by means of (4.28) leads, via $\delta L_{\text{tot}}/\delta \vartheta^\alpha = 0$, to Einstein’s equation

$$\frac{}{G_\alpha := 1}{\eta_{\alpha\beta\gamma} \wedge R^\beta_{\gamma\gamma} = \ell^2 \Sigma_\alpha .}$$

But remember, since $\vartheta^{(\alpha \wedge \Sigma_{\beta)} = 0$, this is only valid for spinless matter or for gauge matter. Nevertheless, in such a way, we arrive at GR in its original form. Shifting back and forth from the variable pair $(\vartheta^\alpha, \Gamma^\alpha_{\alpha\beta})$ to $(\vartheta^\alpha, d\vartheta^\alpha)$ means shifting back and forth from original GR to its teleparallel equivalent GR_\parallel.

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