Spin correlation and entanglement detection in Cooper pair splitters by current measurements using magnetic detectors

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We analyze a model of double quantum dot Cooper pair splitter coupled to two ferromagnetic detectors and demonstrate the possibility of determination of spin correlation by current measurements. We use perturbation theory, taking account of the exchange interaction with the detectors, which leads to complex spin dynamics in the dots. This affects the measured spin and restricts the use of ferromagnetic detectors to the nonlinear current-voltage characteristic regime at the current plateau, where the relevant spin projection is conserved, in contrast to the linear current-voltage characteristic regime, in which the spin information is distorted. Moreover, we show that for separable states spin correlation can only be determined in a limited parameter regime, much more restricted than in the case of entangled states. We propose an entanglement test based on the Bell inequality.

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I. INTRODUCTION

Pairs of entangled particles provide the basis for modern applications in quantum cryptography, teleportation, and other topics in quantum information technology and quantum computation. Cooper pairs that naturally occur in the ground state of s-wave superconductors provide a continuous solid-state source of spatially separated spin-entangled electrons, which can be used as flying qubits in integrated and scalable on-chip quantum information systems. A substantial breakthrough has been achieved recently in the theoretical modeling of Cooper pair splitting (CPS)¹² and in experimental realizations¹² by the introduction of a double quantum dot (DQD), soon followed by the attainment of a splitting efficiency close to 1. An important step following a successful splitting of Cooper pairs is to verify experimentally whether the split electrons remain entangled. This turns out to represent a much greater challenge, as eight years after the first demonstration of Cooper pair splitting⁴ entanglement detection is still lacking in this system. Indeed, it is very difficult to find a suitable measurement scheme that would be both effective and relatively simple to realize experimentally.

Most of the proposed verification methods⁷–¹⁹ require the use of spin-sensitive detectors and higher-order cumulants, complex time-resolved measurements, or transfer of the spin state onto the polarization state of a pair of optical photons¹⁹, which are rather difficult experimental techniques. Some potentially simpler techniques based on dc current measurements²⁰–²¹ were proposed on theoretical grounds in the last years. However, some of these proposals¹¹–²⁰ neglect important physical aspects of the model, such as the Coulomb interaction, necessary to obtain a high splitting efficiency, or the exchange field-induced back action of the ferromagnetic detectors on the spin dynamics of the quantum dots, which can affect the results. An interesting recent proposal, also based on dc current measurements and the spin-orbit interaction, involves the use of a bent carbon nanotube CPS under strong magnetic field²². However, this technique has a disadvantage of using strong magnetic field, which can possibly modify the properties of the investigated ground state (as discussed in Ref. [²²]) and interfere with the measurements. To avoid these difficulties we propose and analyze a perfectly natural setup for entanglement detection in CPS, in the form of noncollinear ferromagnetic spin detectors attached to both QDs (Fig. 1). This solution is experimentally feasible now²³–²⁵ and has an additional advantage of involving simpler dc current measurements.

We develop a formalism²⁶–²⁸ that represents a systematic approach taking into account the spin dynamics in the QDs and the exchange interaction between the ferromagnetic leads and the QDs, issues not discussed in previous studies.²³–²⁵ We prove that the complex spin dynamics in the QDs does not prevent the extraction of spin information, since the measured spin projection is conserved during spin precession in the nonlinear current-voltage regime at characteristic dc current plateaus (see Fig. 2). It is in contrast to the linear regime, where it is distorted, which has been ignored in previous studies. We demonstrate that the spin correlation function can be determined by dc current measurements at current plateaus in the nonlinear regime only. The spin correlation functions contain all the information necessary for the determination of the properties of the investigated ground state; therefore, using them we are able to test the Clauser-Horne-Shimony-Holt (CHSH) Bell inequalities to discriminate between entangled and unentangled product states. We analyze the limitations of the entanglement detection scheme based on ferromagnetic detectors attached to the CPS and its sensitivity to various asymmetries.
II. THE MODEL

The Hamiltonian $H$ of the considered three-terminal system is defined as:

$$H = H_{\text{DQD}} + \sum_{\eta=L,R} (H_\eta + H_{T\eta}) + H_S + H_{TS}. \quad (1)$$

The first term is related to the single-level double quantum dot (DQD):

$$H_{\text{DQD}} = \sum_{\eta=\sigma=\uparrow,\downarrow} \varepsilon_{\eta\sigma} n_{\eta\sigma} + \sum_{\eta} U_{\eta} n_{\eta\uparrow} n_{\eta\downarrow} + \frac{1}{2} \sum_{\eta=\sigma=\upsilon} \eta_{\upsilon,\sigma} \hat{n}_{\upsilon,\sigma}, \quad (2)$$

where $n_{\eta\sigma}$ is the number operator of particles in the QD $\eta = L/R$ (left/right) with spin $\sigma$, energy $\varepsilon_{\eta\sigma}$, and $U$ is the Coulomb interaction operators in QDs and leads, respectively. The intradot Coulomb repulsion $U_{\eta}$ is infinite, which means that the dot can only be occupied by a single electron.

The ferromagnetic metal electrodes, acting as spin detectors, are treated as reservoirs of noninteracting fermions with momentum $k$ and spin $\alpha$:

$$H_\eta = \sum_{k,\alpha=\uparrow,\downarrow} \varepsilon_{\eta k} a_{\eta k\alpha}^\dagger a_{\eta k\alpha} \ . \quad (3)$$

The effective spin asymmetry, $\rho_{\eta\uparrow} \neq \rho_{\eta\downarrow}$, in the density of states $\rho_{\eta\alpha}$ at Fermi level in the electrodes can be described by spin polarization $p_\eta = (\rho_{\eta\uparrow} - \rho_{\eta\downarrow}) / (\rho_{\eta\uparrow} + \rho_{\eta\downarrow})$. In general, the magnetization directions $\hat{n}_L$ and $\hat{n}_R$ of the left and right leads, respectively, are noncollinear, $\hat{n}_L \neq \hat{n}_R$. To describe the spin conserving tunneling, taking account of rotation of the spin quantization axes, we have to include SU(2) rotation matrices $U_{\eta\sigma}^{n_\sigma}$, with elements $U_{\eta\sigma}^{n_\sigma} = \langle \eta\sigma | \eta,\sigma \rangle$, into the tunneling Hamiltonian:

$$H_{T\eta} = \sum_{k,\sigma=\uparrow,\downarrow} \left( V_{\eta} a_{\eta k\sigma}^\dagger U_{\eta,\alpha}^{n_\alpha} d_{\eta\sigma} + \text{H.c.} \right), \quad (4)$$

where $V_{\eta}$ denotes the tunneling amplitude between QD $\eta$ and ferromagnetic lead $\eta$; $d_{\eta\sigma}$ and $a_{\eta k\sigma}$ are the annihilation operators in QDs and leads, respectively.

The superconducting lead can be described by the mean-field BCS Hamiltonian:

$$H_S = \sum_{k,\sigma=\uparrow,\downarrow} \varepsilon_{S,\sigma} a_{S,\sigma}^\dagger a_{S,\sigma} + \Delta \sum_{k} \left( a_{-kS,\uparrow}^\dagger a_{kS,\downarrow} + \text{H.c.} \right), \quad (5)$$

where $\Delta$ is the pair potential and a reference electrochemical potential $\mu_S = 0$. The BCS Hamiltonian yields an $s$-wave superconductor, where each Cooper pair is in a spin singlet ground state $|S\rangle$, that can be generalized to an arbitrary ground state of Cooper pair given by:

$$|\varphi\rangle \equiv a_1 |\uparrow_L\uparrow_R\rangle + a_2 |\uparrow_L\downarrow_R\rangle + a_3 |\downarrow_L\uparrow_R\rangle + a_4 |\downarrow_L\downarrow_R\rangle, \quad \text{where} \quad \sum |a_\eta|^2 = 1. \quad (6)$$

The tunneling between the superconducting electrode and QD $\eta$ is given by:

$$H_{TS} = \sum_{k\sigma} \left( V_{S} a_{S,\sigma}^\dagger d_{\eta,\sigma} + \text{H.c.} \right). \quad (7)$$

The tunnel coupling strengths to the two ferromagnetic electrodes are expressed as $\Gamma_{\eta\uparrow/\downarrow} = (1 \pm p_\eta) \Gamma_{\eta}/2$, where $\Gamma_{\eta} = 2\pi \left( \langle \rho_{\eta\uparrow} + \rho_{\eta\downarrow} \rangle |V_{\eta}|^2 \right)$, and to the superconducting one as $\Gamma_{S\eta} = 2\pi \rho_S |V_{S}|^2$, where $\rho_S$ denotes density of states in superconducting leads.

In our study we consider transport processes involving Andreev reflection. By tracing out the degrees of freedom of the superconducting electrode we obtain the effective Hamiltonian of the DQD that takes account of the coupling to the superconducting lead:

$$H_{\text{eff}} = H_{\text{DQD}} - \sqrt{\frac{\Gamma_S}{2}} \left( a_1 d_{L\uparrow}^\dagger d_{R\uparrow}^\dagger + a_2 d_{L\uparrow}^\dagger d_{R\downarrow}^\dagger a_3 d_{L\downarrow}^\dagger d_{R\uparrow}^\dagger + a_4 d_{L\downarrow}^\dagger d_{R\downarrow}^\dagger + \text{H.c.} \right), \quad (7)$$

where $\Gamma_S = \sqrt{\Gamma_{S\uparrow} \Gamma_{S\downarrow}}$ and the second term describes the nonlocal proximity effect. The diagonalization of Hamiltonian (7) yields the coupling between two states, the empty state $|0\rangle$ and the two-particle state $|\varphi\rangle$, which results in the new Andreev bound eigenstates:

$$|\pm\rangle = w_{\pm} |0\rangle \mp w_{\pm} |\varphi\rangle, \quad (8)$$

with amplitudes $w_\pm = \sqrt{1/2 \mp \delta/4 \varepsilon A}$, where $\delta = \varepsilon_L - \varepsilon_R + U$ denotes the detuning parameter. The energies of states $|\pm\rangle$ in the diagonal basis are $E_\pm = \delta/2 \pm \varepsilon_A$, where $\varepsilon_A = \sqrt{\delta^2/4 + \Gamma_S^2/2}$. If we consider singlet pairing in the superconductor $|\varphi\rangle = |S\rangle$, then $a_2 = -a_3 = 1/\sqrt{2}$ and $a_1 = a_4 = 0$.

III. MASTER EQUATIONS

We consider spin-dependent electron transport to the lowest order in $\Gamma_0$, a regime known as the sequential tunneling limit, $\Delta >> k_B T > \Gamma_0$, easily reachable in current experiments, which allows us neglect quasiparticle excitations in the superconductor. The net tunneling rate to
and from ferromagnetic lead \( \eta \) depends on the direction of lead magnetization, which we describe by spinors \( m_{\eta}^{+} = \left( U^{\eta \dagger}_{\uparrow \uparrow}, U^{\eta \dagger}_{\uparrow \downarrow} \right) \) and \( m_{\eta}^{-} = \left( U^{\eta \dagger}_{\downarrow \downarrow}, U^{\eta \dagger}_{\downarrow \uparrow} \right) \).

Let us discuss in detail the formalism for \( |\varphi\rangle = |S\rangle \). The restriction \( \mu < (\delta + U)/2 \) for symmetric bias voltages \((\mu_{L} = \mu_{R} = \mu)\) allows us to neglect triplet states \( \rho \). Thus, we only consider six states: two states \(|\pm\rangle\) with occupancy probabilities \( p_{\pm} \), and four single-electron states \( |\sigma\rangle \) described by density matrices \( \rho_{\pm} = (\rho_{1\eta} / 2) I + S_{\xi_{\eta}}(\sigma_{X} + S_{\eta}(\sigma_{Y} + S_{\eta}(\sigma_{Z})), \rho_{1\eta} \) denotes the probability of the single electron occupancy of the QD \( \eta \), \( S_{\xi_{\eta}} = (S_{\xi_{\eta}} + S_{\eta} + S_{\eta} S_{\eta}) \) is the average spin vector in QD \( \eta \), and \( \sigma = (\sigma_{X}, \sigma_{Y}, \sigma_{Z}) \) is the Pauli matrix vector. A quantum dot state is characterized by a set of ten parameters, \( \{p_{\pm}, p_{1L}, p_{1R}, S_{\xi_{L}}, S_{\xi_{R}}, S_{\xi_{\eta}}, S_{\eta_{L}}, S_{\eta_{R}}, S_{\eta_{\eta}}\} \). Due to the normalization condition \( 1 = p_{-} + p_{+} + \sum_{\eta} p_{1\eta} \) only nine of them are independent.

The time evolution of the scalars \( p_{\pm} \) and density matrices \( \rho_{1\eta} \) is described by the following effective master rate equations:

\[
\frac{\hbar}{\eta} \frac{d\rho_{1\eta}}{dt} = \frac{i}{\hbar} \left[ \rho_{1\eta}, H_{1\eta} \right] - \frac{1}{2} \sum_{s,\sigma=\{-,\dagger\}} \left( f_{s,\sigma}^{f_{+},\sigma} \rho_{\sigma}^{f_{-},\sigma} \right) + \frac{1}{2} \sum_{s,\eta=\{\pm,\dagger\}} \left( f_{s,\eta}^{f_{+},\eta} \rho_{\eta}^{f_{-},\eta} \right)
\]

\[
\frac{\hbar}{\eta} \frac{dp_{\pm}}{dt} = \left( - f_{s,\eta}^{f_{-},\sigma} \rho_{\sigma}^{f_{+},\eta} \right) + \frac{1}{2} \sum_{s,\sigma=\{-,\dagger\}} \left( f_{s,\sigma}^{f_{+},\sigma} \rho_{\sigma}^{f_{-},\sigma} \right)
\]

where the square brackets \([\text{\large{-}}]\) denote the commutator/anticommutator, and the tunneling amplitude spinor \( \gamma_{\eta\sigma} = \omega_{\eta} \Gamma_{\eta}^{\dagger} m_{\eta\sigma} \). The additional factor \( \sqrt{2} \) in \( \gamma_{\eta\sigma} \) is a consequence of the participation of singlet state \(|S\rangle\) in the given process. The symbols \( \sigma \) and \( \eta \) denote the spin opposite to \( \sigma \) and the ferromagnetic lead opposite to \( \eta \), respectively. We use spinor \( \gamma_{\eta\sigma} \) for the description of processes changing the occupancy of the DQD between empty and single, and \( \gamma_{\eta\sigma} \) for processes switching between single and double. The tunneling amplitude spinor \( \gamma_{\eta\sigma} \) can indicate the entanglement of a singlet state. The tunneling of one electron, with spin \( \sigma \), of an \(|S\rangle\) pair from QD \( \eta \) to ferromagnetic electrode \( \eta \) causes the collapse of the two-particle wave function; thus the next Cooper pair electron in QD \( \eta \) has the opposite spin, described by spinor \( m_{\eta\sigma} \). A similar effect occurs in electron tunneling in the opposite direction, i.e., from ferromagnetic electrode \( \eta \) to QD \( \eta \). In the adopted formalism we use the following notation for the Fermi distribution functions \( f_{\eta}^{f_{+},\sigma} \): \( f_{\eta}^{f_{+},\sigma} = f_{\eta}^{f_{+},\sigma} \left( E_{\eta} - \xi_{\eta} \right) \), \( f_{\eta}^{f_{-},\sigma} = f_{\eta}^{f_{-},\sigma} \left( \xi_{\eta} - E_{\eta} \right) \), where \( f^{-} = 1 - f^{+} \) and the third subscript indicates the change in the DQD occupation, + between double and single and \(-\) between single and empty.

The Hamiltonian

\[
H_{1N} = \frac{\hbar}{4\pi} P \int d\xi \sum_{\sigma,\sigma=\{-,\dagger\}} \frac{(1 - 2f_{\eta}^{f_{+},\sigma}(\xi)) \gamma_{\eta\sigma}^{f_{-},\sigma}}{\varepsilon_{\eta} - E_{\eta} - \xi} \tag{10}
\]

describes virtual particle exchange processes resulting in an effective exchange field \( B_{\eta} \) and spin precession for a single electron states \( |\sigma\rangle \) around the direction of \( B_{\eta} \). Here \( P \) denotes the Cauchy principal value. Using the relation \( S_{\eta} = (1/2) \text{Tr} \left[ \rho_{1\eta} \sigma_{3} \right] \) it can be shown that the expression \( d\rho_{1\eta}/dt = (i/\hbar) [\rho_{1\eta}, H_{1\eta}] \) is equivalent to the Bloch equation:

\[
d\vec{S}_{\eta}/dt = \vec{S}_{\eta} \times \vec{B}_{\eta}, \tag{11}
\]

which describes the spin precession around the effective field \( B_{\eta} = B_{\eta} \vec{n}_{\eta} \), where

\[
B_{\eta} = \frac{\Gamma_{\eta} - \Gamma_{\eta}^{*} P}{\eta} \int d\xi \sum_{\sigma,\sigma=\{-,\dagger\}} \frac{\omega_{\eta}^{2} f_{\eta}^{f_{-},\sigma}(\xi)}{\varepsilon_{\eta} - E_{\eta} - \xi}. \tag{12}
\]

The effective exchange field results not only in a torque of the accumulated spin, but also in a spin splitting of the dot level, similar to Zeeman splitting. In the weak-coupling regime it cannot be resolved, since the splitting is proportional to the coupling strength and must be dropped in first-order transport calculation. The in the case of two-electron states \(|\varphi\rangle\), spin splitting and precession can be neglected due to weak coupling to ferromagnetic electrodes, \( \Gamma_{\eta} \gg \eta \), in the considered system.

The current in electrode \( \eta \) is given by the following equation:

\[
I_{\eta} = \frac{\epsilon}{\hbar} \sum_{\sigma,\eta} \left( f_{\eta}^{f_{+,\eta}} S_{\eta}^{\eta} S_{\eta}^{\eta} - f_{\eta}^{f_{-,\eta}} S_{\eta}^{\eta} S_{\eta}^{\eta} \right) + f_{\eta}^{f_{+,\eta}} S_{\eta}^{\eta} S_{\eta}^{\eta} + f_{\eta}^{f_{-,\eta}} S_{\eta}^{\eta} S_{\eta}^{\eta} \tag{13}
\]

where occupancy probabilities can be obtained from the stationary solution of Eq. (10). The studied model provides 100% efficiency of Cooper pair splitting; thus, \( I_{L} = I_{R} \) and the total current \( I = I_{L} + I_{R} \). In the case of nonmagnetic electrodes \((p_{\eta} = 0)\) our model is equivalent to that presented in Ref. [38]. In particular magnetization configurations we can denote the total current as \( I_{L,R}^{\eta \eta} \), where \( \downarrow \eta \) indicates \( \eta \), while \( \downarrow \eta \) indicates \( -\eta \). For example, \( I_{L,R}^{\eta \eta} \) describes a reversal of the magnetization direction of electrode \( \eta \), \( -\eta \) indicates \( -\eta \). This implies \( m_{R}^{\eta} \Rightarrow m_{R}^{\eta} \) and \( m_{R}^{\eta} \Rightarrow -m_{R}^{\eta} \). The other configurations, \( I_{L,R}^{\eta \eta} \) and \( I_{L,R}^{\eta \eta} \), are defined similarly. Our aim is to determine the spin correlation of the ground state \(|\varphi\rangle\) of the superconductor by the measurement of spin-dependent currents in different configurations of the electrode magnetizations. We seek evidence that Cooper pairs that occupy the QDs are still in a quantum entangled state for \(|\varphi\rangle = |S\rangle\).
I
leads in a symmetric system: $\Gamma_L = \Gamma_R = \Gamma$, $\epsilon_L = \epsilon_R$, $p_L = p_R = p = 0.9$, $\delta = 0$, $k_B T = 0.01 U$, and $\Gamma_S = 0.5 U$. In the inset, $I_{\uparrow \uparrow \uparrow R}$ and $I_{\downarrow \downarrow \downarrow R}$, versus the spin polarization $p$ for two plateaus (A, B).

IV. COLLINEAR CONFIGURATIONS

Let us first consider the currents in the case of collinear magnetizations of the ferromagnetic electrodes, $\hat{n}_L = \pm \hat{n}_R$. Currents $I_{\sigma L \sigma R}$ and $I_{\bar{\sigma} L \bar{\sigma} R}$ for parallel (P) and antiparallel (AP) magnetizations, respectively, are plotted versus voltage $\mu$ in Fig. 2. Two characteristic plateaus are observed in the plot: (A) with $|\eta \sigma\rangle$ and $|\bar{\eta} \bar{\sigma}\rangle$ as the only states participating in transport, and (B) with state $|+\rangle$ available as well (see Ref. [30] for details). Close to $\mu = 0$ only states $|\eta \sigma\rangle$ are occupied and the system is in the Coulomb blockade regime. The P configuration current $I_{\sigma L \sigma R}$ is much smaller than the AP configuration current $I_{\bar{\sigma} L \bar{\sigma} R}$ and decreases with increasing spin polarization $p_L = p_R = p$, as shown in Fig. 2 inset. The AP configuration current $I_{\bar{\sigma} L \bar{\sigma} R}$ is independent of $p$ and equal to the current $I_0$ in a system with nonmagnetic electrodes ($p = 0$), $I_{\bar{\sigma} L \bar{\sigma} R} = I_0$. This is related to the fact that the Cooper pairs are in singlet states $|S\rangle$ and the AP alignment of electrode magnetizations better suits the antiferromagnetic order of the singlet state. We can try to use this sensitivity of the current to magnetization configuration to determine the spin correlation of the Cooper pairs directly from electric current measurements.

V. SPIN DYNAMICS

In this paper we propose a method for entanglement detection using noncollinear ferromagnetic electrodes as effective spin detectors. However, such noncollinear ferromagnetic detectors can affect the state of a quantum dot in a system results in a complex spin dynamics in the QD. Virtual particle exchange processes between ferromagnetic electrode $\eta(\eta = \{L, R\})$ and QD lead to an effective exchange field $\hat{B}_\eta$ [Eq. (12)], which in the sequential tunneling limit causes
potential conductance which excludes the use of ferromagnetic leads as spin de-

The two-spin correlation η in QD precesses across the direction of Bη, parallel to the magnetization direction nη of electrode η, and its projection S_{Zη} on this direction is conserved (Fig. 3). Thus, the quantity relevant to the measurement is not affected, in contrast to the spin components S_{Yη}, perpendicular to the plane spanned by the two magnetizations, and S_{Xη}, which do change with the field amplitude Bη.

In the linear response regime, µ ≪ kBT, the differential conductance \(G_{\eta}(\nu)\) is independent of the field Bη. This is because the spin \(\hat{S}_η\) in QD η precesses across the direction of Bη, parallel to the magnetization direction nη of electrode η, and its projection S_{Zη} on this direction is conserved (Fig. 3). Thus, the quantity relevant to the measurement is not affected, in contrast to the spin components S_{Yη}, perpendicular to the plane spanned by the two magnetizations, and S_{Xη}, which do change with the field amplitude Bη.

The two-spin correlation \(\eta\) shows that the current \(I_{\eta}(\tau)\) plotted versus \(\tau\) in Fig. 3(b), is affected by the spin precession, which excludes the use of ferromagnetic leads as spin detectors in this limit. As shown in Fig. 3(a), apart from \(S_{Xη}\) and \(S_{Yη}\), also the spin component \(S_{Zη}\) in the direction \(n_\eta\) varies with the exchange field \(B_\eta\). This is caused by the possibility of tunneling in the reverse direction, from ferromagnetic electrodes to quantum dots.

The plots for the nonlinear (on plateaux), Fig. 3 and linear response regimes, Fig. 4 have different scales of exchange field \(B_\eta\). In the linear response regime the electron spin in a QD is more sensitive to the field \(B_\eta\), because of the lower current and the related longer dwell time, the average time spent by an electron on the quantum dot. Therefore, saturation is observed for much lower values of \(B_\eta\) with respect to the plateaux.

We prove that at the observed current plateaux the field \(B_\eta\) does not interfere with the reading of spin in the electrodes. Thus, the measured current \(I_{\eta}(\tau)\) (where \(\sigma_\eta\) indicates the magnetization direction \(n_\eta\) of ferromagnetic electrode η) can be used for determining spin correlations and testing the Bell inequalities.

VI. SPIN CORRELATION

Since the current \(I_{\eta}(\tau)\) depends on the magnetization direction, we can try to extract spin information from it. The two-spin correlation \(C_{\eta}^p\) can be calculated from the equation \(C_{\eta}^p = \text{Tr}(\otimes \rho)\), where \(\sigma_\eta = \sigma \cdot n_\eta\) and \(\rho \equiv |\phi\rangle \langle \phi|\) denotes the two-particle density matrix.

To extract spin information from the direct current we propose the following function:

\[
C_{\eta}^p = \frac{I_{\eta}(\tau) + I_{\eta}(\tau) - I_{\eta}(\tau) - I_{\eta}(\tau)}{I_{\eta}(\tau) + I_{\eta}(\tau) + I_{\eta}(\tau) + I_{\eta}(\tau)},
\]

and test its correspondence to the spin-spin correlation function \(C_{\eta}^p\). Equation (14) is analogous to the correlation function defined for the number of coincidences.

Let us first consider the symmetric case (i.e. \(p_L = p_R = p, \Gamma_L = \Gamma_R = \Gamma, \) and \(\delta = 0\)). In this regime we are able to reproduce the spin correlations for any state \(|\phi\rangle\) up to a spin polarization-dependent amplitude \(\Im(p)\):

\[
C_{\eta}^p = \Im(p) C_{\eta}^p.
\]

This, however, can only be done at the two plateaux, in the symmetric case \(\Im(p) = 2p^2/(3 - p^2)\) and \(\Im(p) = 2p^2/(2 - p^2)\) for plateaux A and B, respectively. This is main result of our study, since by studying \(C_{\eta}^p\) from Eq. (15) we can obtain information on the spin correlations in our system, and detect entanglement of split Cooper pairs.

Figure 5(a) shows the spin correlation \(C_{LR}(\theta)\) as a function of the angle \(\theta_{LR}\) between the electrode magnetization directions for a singlet state \(|\phi\rangle = |S\rangle\). Interestingly, although the current \(I_{\eta}(\theta)\), plotted in Fig. 5(b), does not follow simple \(N_{\eta}(\theta)\) coincidence predictions, the spin correlator \(C_{LR}(\theta) = -\Im(p) \cos(\theta_{LR})\) behaves as predicted by quantum theory \(C_{LR}(\theta_{LR}) = -\cos(\theta_{LR})\). The \(\theta_{LR}\) and \(p\) dependence of the currents \(I_{\eta}(\theta)\) in Fig. 5(b) is described by the following expressions:

\[
I_{\eta}(\theta) = \frac{p^2 - 1}{2p^2 \cos(\theta_{LR}) - p^2 + 3},
\]

\[
I_{\eta}(\theta) = \frac{1 - p^2}{2p^2 \cos(\theta_{LR}) + p^2 - 3},
\]

and

\[
I_{\eta}(\theta) = \frac{p^2 - 1}{2p^2 \cos(\theta_{LR}) - p^2 + 2},
\]

\[
I_{\eta}(\theta) = \frac{1 - p^2}{2p^2 \cos(\theta_{LR}) + p^2 - 2},
\]

for plateaux A and B, respectively.

We can use our model for the determination of the spin correlation of electrons in a Cooper pair naturally occurring in the superconductor. In general, the ground state \(|\phi\rangle\) of the superconductor can be an entangled state or a separable state. The essential difference between these two kinds of quantum states is that, in contrast to an entangled state, particles in a separable state are independent of each other. In the case of separable pure states (product states) each dot in our system has a well-defined spin in the state \(|\phi\rangle\). In a two-spin product state the spin correlation function \(C_{LR}^p = \cos(\alpha_L) \cos(\alpha_R)\) depends only on the angles \(\alpha_L = \alpha_R = \theta\) between the magnetization direction \(n_\eta\) and the spin direction at QD η in the state \(|\phi\rangle\). In a symmetric system (i.e., for \(p_L = p_R = p, \Gamma_L = \Gamma_R, \\delta = 0\)) we can also determine the spin correlation for separable states in the DQD by measuring the current, Eq. (13).

Let us consider the case when \(\alpha_L = -\alpha_R = \theta\). The spin correlation function \(C_{LR}^p\) obtained by measurement of the current Eq. (13) is plotted versus \(\theta\) in Fig. 6(a). Shown in Fig. 6(b), the currents \(I_{\eta}(\theta)\) and \(I_{\eta}(\theta)\), for parallel and antiparallel magnetizations, respectively,
plotted versus angle $\theta_{LR}$ between the magnetizations, plotted versus the angle $\theta_{LR}$ between the magnetizations of the leads ($I_{LR}$) for two configurations of ferromagnetic electrode magnetizations, plotted versus the angle $\theta_{LR}$ between the magnetizations $\hat{n}_\theta$ in a symmetric system for $p = 0.95$ at two plateaus (A - solid, B - dotted lines).

Figure 5. (Color online) (a) Spin correlation function $C_{LR}^p$ for the singlet state $|\varphi\rangle = |S\rangle$, (b) currents $I_{ll\tau R}$ and $I_{ll\tau L}$ for two configurations of ferromagnetic electrode magnetizations, plotted versus the angle $\theta_{LR}$ between the magnetizations directions $\hat{n}_\theta$ in a symmetric system for $p = 0.95$ at two plateaus (A - solid, B - dotted lines).

Figure 6. (Color online) (a) Spin correlation function $C_{LR}^p$ for the product state $|\varphi\rangle$ and (b) currents $I_{ll\tau R}$, dotted, $I_{ll\tau L}$, solid, and $I_{ll\tau R}$, dashed line, for three configurations of the magnetic leads in the symmetric system at the plateau B, plotted versus angle $\theta$ for $\alpha_L = -\alpha_R = \theta$, $p = 0.9$, and $k_B T = 0.01 U$.

The spin correlator $C_{LR}^p = \Im(p) \cos^2(\theta)$ behaves as predicted by quantum theory $C_{LR}^p = \cos^2(\theta)$.

VII. ENTANGLEMENT DETECTION BY TESTING BELL INEQUALITIES

The Bell inequalities concern measurements of separated particles that interacted before the separation. Assuming a local realism, certain constraints must hold on the relationships between the correlations between successive measurements of the particles in various possible measurement settings. We use the CHSH version of the Bell inequality with the correlator $Q^p$ defined by means of spin correlation functions as:

$$Q^p = |C_{LR}^p + C_{LR'}^p + C_{LR''}^p - C_{LR'}^p| \leq 2. \quad (18)$$

When the inequality is not fulfilled the particles are in an entangled state.

Using the fact that we can determine the spin correlation by the current measurements, Eq. (15), we can test the Bell inequality using the current measurements as well. The CHSH correlator in our system has the following form:

$$Q^p = \Im(p)Q^p, \quad (19)$$

where we have substituted $C_{LR}^p$ in Eq. (18) with $C_{LR}^p$ defined by Eq. (14).

For a singlet state $|S\rangle$ the inequality is maximally violated for example, when $\theta_{LR} = \theta_{LR} = \theta_{LR'} = \pi/4$ and $\theta_{LR'} = 3\pi/4$; then, $Q^p = 2\sqrt{2}$.

In the next step we determine the system parameter limits within which the Bell inequality can be violated by a singlet state $|\varphi\rangle = |S\rangle$. We obtain the minimum spin polarization $p$ of the leads necessary for the violation of the CHSH inequality for plateaus A and B, $p > \sqrt{2(2\sqrt{2} - 1)} \approx 0.885$ and $p > \sqrt{2\sqrt{2} - 2} \approx 0.91$, respectively.

VIII. RESULTS FOR ASYMMETRIC SYSTEM

Now let us consider a case with a number of asymmetries: $p_L \neq p_R$, $\kappa = (\Gamma_L - \Gamma_R)/(\Gamma_L + \Gamma_R) \neq 0$, $\delta \neq 0$, and some difference between the dot energy levels, $\Delta \varepsilon = \varepsilon_L - \varepsilon_R$. We find that the quantity $C_{LR}^p$ defined in Eq. (14) still describes the spin correlation, $C_{LR}^p = \Im C_{LR}^p$, for all maximally entangled Bell states: $|\Psi^{\pm}\rangle \equiv \frac{1}{\sqrt{2}} (|L_L R_R\rangle \pm |L_R R_L\rangle)$ and $|\Phi^{\pm}\rangle \equiv \frac{1}{\sqrt{2}} (|L_L R_R\rangle \pm |L_R R_L\rangle)$.

The correlation function $C_{LR}^p$ does not depend on $\Delta \varepsilon$, and the other asymmetries mentioned above only affect the amplitude $\Im$ of the correlator. For $p_L \neq p_R$, $\kappa = 0$ and $\delta = 0$, knowing that $\Im(p_L, p_R) > \sqrt{2}/2$, we can specify the conditions to be fulfilled by the spin polarizations of the leads $(p_L, p_R)$ for quantum entanglement to be...
of the proposed method is found to grow with $\kappa$. For the extreme value $\kappa = 1$ the requirement for spin polarization $p$ is minimal, and corresponds to the condition $p > 1/\sqrt{2} \approx 0.84$ established in Refs. \cite{11, 19}. Analogous results at plateau B are in Figs. \ref{fig7}(d) and \ref{fig7}(e). The parameters range in which entanglement detection is possible at plateau B is smaller with respect to plateau A. The influence of the detuning parameter $\delta$ on the detection of state $|S\rangle$ has a different character for the plateau A [Fig. \ref{fig7}(c)] and plateau B [Fig. \ref{fig7}(f)]. Unfortunately, for separable states $|\varphi\rangle$ in an asymmetric system, $C_{LR}^{\eta} \neq 3 C_{LR}^{\varphi}$, which implies distorted spin information. To exclude that inequality \cite{18} might be unfulfilled by separable states we have analyzed the corresponding CHSH correlator $Q^{\eta}$, and found that separable states can only violate the CHSH inequality in a very restricted range of parameters in a strongly asymmetric DQD system when the spin polarization $p_{R}$ in one of the leads is close to 1, $(1 - p_{L}) \lesssim 3 \times 10^{-3}$; this, however, is difficult to achieve experimentally. Thus, beyond this restricted regime the correlator $Q^{\eta}$ can be used for detecting maximally entangled states.

\section{IX. CONCLUSIONS}

We have studied theoretically the role of ferromagnetic electrodes connected to two QDs of a CPS to act as spin detectors converting spin information directly into a charge current. We have derived effective master equations describing transport in the system with the exchange interaction and the related spin dynamics taken into account. Despite the complexity of the spin dynamics, the conservation of the relevant spin projection allows for the determination of spin correlations from current measurements and the detection of entanglement by testing the Bell inequality. The spin correlation of maximally entangled states is insensitive to various asymmetries in the system parameters. In the case of separable states, symmetry conservation is required for the determination of the spin correlation.

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We show that the simple relationship between the spin polarizations of the leads $|\varphi\rangle = |\mathcal{S}\rangle$ predicted by Kawabata et al. generally cannot be satisfied in more realistic models. This results from the fact that in our system electrons tunnel directly to the ferromagnetic electrodes, and from a consistent treatment of the Coulomb interaction $U$ of the QDs that also leads to spin precession. A. P. Mackenzie and Y. Maeno, Phys. Mod. Phys. 75, 657 (2003).

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