MATRIX LIBERATION PROCESS
II: RELATION TO ORBITAL FREE ENTROPY

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Dedicated to Professor Dan-Virgil Voiculescu on the occasion of his 70th birthday

ABSTRACT. We investigate the concept of orbital free entropy from the viewpoint of matrix liberation process. We will show that many basic questions around the definition of orbital free entropy are reduced to the question of full large deviation principle for the matrix liberation process. We will also obtain a large deviation upper bound for a certain family of random matrices that is an essential ingredient to define the orbital free entropy. The resulting rate function is made up into a new approach to free mutual information.

1. Introduction

This paper is a sequel to our previous one [29] on the matrix liberation process, and devoted to explaining how the matrix liberation process is connected to the orbital free entropy $\chi_{orb}$. Here, the negative of orbital free entropy may be regarded as a possible microstate approach to mutual information in free probability.

The key concept of free probability theory, initiated by Voiculescu in the early 80s, is the so-called free independence, which is a kind of statistical independence. Voiculescu then discovered around 1990 that the large $N$ limit of independent (suitable) random matrices produces freely independent non-commutative random variables. In the 90s, in order to understand the notion of free independence deeply, Voiculescu introduced and studied several notions of free entropy (the microstate and the microstate-free ones), which are both analogs of Shannon's entropy and expected to agree. Then, these notions of free entropy were further studied by Biane, Guionnet, Shlyakhtenko and many others from several viewpoints, including large deviation theory and optimal transportation theory. (See [31] for early history on free entropy.)

On the other hand, the information theory suggests us to introduce a free probability analog of mutual information that should characterize the freely independent situation as a unique minimizer. The main difficulty in such an attempt is the lack of free probability analog of relative entropy, and thus a completely new idea was (and probably still is) necessary. It was also Voiculescu [30] who first attempted to develop the theory of mutual information in free probability. His approach is based upon the liberation theory that he started to develop there with the microstate-free approach to free entropy. The most important concept in the liberation theory is the liberation process, a natural non-commutative probabilistic interpolation between given non-commutative random variables and their freely independent copies. Voiculescu’s idea of liberation theory is completely non-commutative in nature, and has no origin in the classical probability theory. Hence the liberation theory is quite attractive from the view point of noncommutative analysis.

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Almost a decade later, we introduced, in a joint work \cite{15} with Hiai and Miyamoto, the second candidate for mutual information in free probability, which we call the orbital free entropy, and its definition involves the adjoint actions of Haar-distributed unitary random matrices to the matrix space $M_N^*$ of $N \times N$ self-adjoint matrices and follows the basic idea of microstate approach to free entropy. (Some considerations looking for better variants of orbital free entropy were made by Biane and Dabrowski \cite{5}, and a direct generalization dropping the hyperfiniteness for given random multi-variables was then given by us \cite{27}.) The liberation process is exactly the large $N$ limit of the matrix liberation process introduced in \cite{29} and its ‘invariant measure’ (or its limit distribution as time goes to $\infty$) exactly arises as the ‘distribution’ of the adjoint actions of Haar-distributed unitary random matrices. Thus it is natural to consider the matrix liberation process for the conjectural unification between Voiculescu’s and our approaches to mutual information in free probability.

As a very first step, we proved in \cite{29}, following the idea of \cite{4}, the large deviation upper bound with a good rate function that completely characterizes the corresponding liberation process as a unique minimizer. The next ideal steps on this line of research should be: (1) proving the large deviation lower bound with the same rate function, (2) applying the contraction principle to the resulting large deviation upper/lower bounds at time $T = \infty$, and (3) identifying the resulting rate function with Voiculescu’s free mutual information.

In this paper, we will mainly work on item (2). As a consequence, we will clarify how the matrix liberation process might resolve several technical drawbacks around the definition of orbital free entropy. As another consequence, we will get a large deviation upper bound result by applying the established contraction principle at $T = \infty$ to the one for the matrix liberation process in our previous paper \cite{29}. We will then make the resulting rate function up into a new microstate-free candidate for free mutual information. Items (1) and (3) are left as sequels to this paper.

The precise contents of this paper are as follows. Sections 2 and 3 are preliminaries, and sections 4, 5 and 6 form the main body of this paper. The subsequent sections concern related materials.

In section 2, we will give one of the key technical lemmas. It is about the long time behavior of the large $N$ limit of the logarithm of the heat kernel on $U(N)$ divided by $N^2$. This seems to be of independent interest. Then we will give a slightly modified definition of orbital free entropy in section 3.

In section 4, building on the previous work \cite{29} we will prove that any large deviation upper or lower bound with speed $N^2$ for the matrix liberation process starting at given several deterministic matrices, say $\xi_{ij}(N)$, with limit joint distribution implies the corresponding one with the same rate for the corresponding random matrices $U_N^{(i)} \xi_{ij}(N) U_N^{(j)\ast}$ with independent Haar-distributed unitary random matrices $U_N^{(i)}$. This explicitly relates the matrix liberation process with the orbital free entropy. Combining this with the main result of \cite{29} we will obtain a large deviation upper bound for $U_N^{(i)} \xi_{ij}(N) U_N^{(j)\ast}$.

In section 5, we will investigate the resulting rate function for $U_N^{(i)} \xi_{ij}(N) U_N^{(j)\ast}$ in some detail; we will prove that it admits a unique minimizer, which is precisely given by freely independent copies of the initially given non-commutative random multi-variables. This fact supports the validity of full large deviation principle with speed $N^2$ and the same rate function for $U_N^{(i)} \xi_{ij}(N) U_N^{(j)\ast}$, because this unique minimizer property also follows from the conjectural full large deviation principle as well as the fact that the orbital free entropy completely characterizes the free independence (under the assumption of having matricial microstates). Moreover, this unique minimizer property suggests that the rate function can be regarded as a possible microstate-free candidate for free mutual information, and hence that the rate function ought have to have a coordinate-free fashion.

In section 6, we will give such a coordinate-free formulation. The coordinate-free formulation will be shown to be a quantity for a given finite family of subalgebras in a tracial $W^\ast$-probability
space, which satisfies a desired set of properties (see subsection 6.7) that any kind of free mutual information has to satisfy and, of course, Voiculescu’s one does.

In section 7, we will explain how the proofs given in the previous paper [29] also work well for several independent unitary Brownian motions with deterministic matrices (which are assumed to have the large $N$ limit joint distribution), and compare its consequences with the corresponding results on the matrix liberation process. In section 8, we will give an explicit description in terms of free cumulants for the conditional expectation of the (time-dependent) liberation cyclic derivative $E_{N(t)}(\pi_t(\Pi^t(\mathcal{D}(P))))$ (see section 4 for the notation), which is the most essential component of the rate function. The description is a complement to a rather ad-hoc computation made in section 5. Finally, in the appendix, we explain some basic facts on universal free products of unital $C^*$-algebras for the reader’s convenience.

Glossary.

- $\| \cdot \|$ denotes the operator norm.
- $M_N \supset M_N^{sa}$ denote the $N \times N$ complex matrices and the $N \times N$ self-adjoint matrices. For each $R > 0$, $(M_N^{sa})_R$ denotes the subset of $A \in M_N^{sa}$ with $\| A \|_\infty \leq R$.
- $\text{Tr}_N$ denotes the usual (i.e., non-normalized) trace on $M_N$, and $\text{tr}_N$ does its normalized one. We consider the Hilbert-Schmidt norm $\| A \|_{HS} := \sqrt{\text{Tr}(A^*A)}$ on $M_N$. It is known that $M_N^{sa}$ equipped with $\| \cdot \|_{HS}$ is naturally identified with the $N^2$-dimensional Euclidean space $\mathbb{R}^{N^2}$. Thus $M_N = M_N^{sa} + \sqrt{-1}M_N^{sa}$ equipped with $\| \cdot \|_{HS}$ is also naturally identified with the $2N^2$-dimensional Euclidean space $\mathbb{R}^{2N^2}$.
- $U(N)$ denotes the $N \times N$ unitary matrices equipped with the Haar probability measure $\nu_N$; n.b., the symbol $\nu_N$ differs from the one $\gamma_{U(N)}$ in [15], [27]. A Haar-distributed $N \times N$ random unitary matrix means a random variable with values in $U(N)$, whose probability distribution measure is exactly $\nu_N$.
- $\text{TS}(\mathcal{A})$ denotes the tracial states on a unital $C^*$-algebra $\mathcal{A}$. For a given subset $X$ of a $W^*$-algebra, we denote by $X^\sigma$ its closure in the $\sigma$-weak topology (i.e., the weak* topology induced from the predual). For a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ between unital $C^*$-algebras, $\pi^* : \text{TS}(\mathcal{B}) \to \text{TS}(\mathcal{A})$ denotes the dual map $\varphi \in \text{TS}(\mathcal{B}) \mapsto \varphi \circ \pi \in \text{TS}(\mathcal{A})$.
- For a random variable $X$ in the usual sense, $\mathbb{E}[X]$ denotes the expectation of $X$. Moreover, for a random variable $Y$ with values in a topological space, we write $\mathbb{P}(Y \in A) := \mathbb{E}[1_A(Y)]$ for any Borel subset $A$; this is the distribution measure of $Y$. Here $1_A$ denotes the indicator function of $A$.

Remark on Part I. We have investigated the matrix liberation process $\Xi^{lib}(N)$ starting at (deterministic) $\Xi(N) = (\xi_{ij}(N))_{i,j=1}^{n+1}$ with $\xi_{ij}(N) = (\xi_{ij}(N))_{i,j=1}^{m+1} \in (M_N^{sa})_{\rho(i)}$. Here, we remark that $r(i) = \infty$ is allowable; namely, each $\xi_{ij}(N)$ may be a countably infinite family of $N \times N$ self-adjoint matrices, and all the results given in part I still hold true in this more general situation without essential changes. In fact, we only need to change the metric $d$ on the continuous tracial states $TS\left(C_R^*(x_{\bullet}(\cdot))\right)$ (see subsection 4.3 below) as follows. Let $W_{\leq \ell}$ be all the words of length not greater than $\ell$ in indeterminates $x_{ij} = x_{ij}^*$ with $1 \leq i \leq n + 1$, $1 \leq j \leq \ell$ (remark this restriction on $j$, which guarantees that $W_{\leq \ell}$ is a finite set), and we define

\begin{equation}
(1.1) \quad d(\tau_1, \tau_2) = \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{1}{2^{m+\ell}} \max_{w \in W_{\leq \ell}} \sup_{(t_1, \ldots, t_\ell) \in [0, m]\ell} \left( |\tau_1(h(t_1, \ldots, t_\ell)) - \tau_2(h(t_1, \ldots, t_\ell))| \wedge 1 \right)
\end{equation}

for $\tau_1, \tau_2 \in TS\left(C_R^*(x_{\bullet}(\cdot))\right)$. Here, $w(t_1, \ldots, t_\ell)$ is constructed by substituting $x_{ij}^* (t_k)$ for $x_{ij}$ in a given word $w = x_{i_1 j_1} \cdots x_{i_\ell j_\ell}$ with $\ell' \leq \ell$. 

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Added in proof. We have further investigated the rate functions in this paper after the submission. As one of its simple consequences, we confirmed that $I_{\text{lib}, \infty}^\text{sh}(\tau) = I_{\text{lib}}^\text{sh}(\tau)$ certainly holds if $I_{\text{lib}}^\text{sh}(\tau) < +\infty$ (see subsection 4.6 for the notation). We will give those details elsewhere.

2. The long time behavior of the large $N$ limit of the heat kernel on $U(N)$

In this section, we will investigate the long time behavior of the large $N$ limit of the logarithm of the heat kernel on $U(N)$ by utilizing a recent work on the Douglas and Kazakov transition due to Thierry Lévy and Maïda [21] (based on Guionnet and Maïda’s work [14]) as well as Li and Yau’s classical work on parabolic kernels [22]. The consequence (Lemma 2.1) will play a key role in section 4 to establish the contraction principle at time $T = \infty$ for large deviation upper/lower bounds with speed $N^2$ for the matrix liberation process $\Xi^\text{lib}(N)$.

Consider $U(N)$ as a Riemannian manifold of dimension $N^2$ by the inner product on the corresponding Lie algebra $u(N) = \sqrt{-1}M_{N,\text{sa}}^2$:

$$\langle X | Y \rangle := -N \text{Tr}_N (XY), \quad X,Y \in u(N).$$

Let Ric be the Ricci curvature associated with this Riemannian structure. It is known, by e.g., [1, Lemma F.27], that

$$\text{Ric}(X,X) = \frac{N}{2} |X|^2 - \langle X | (1/N) \sqrt{-1} I_N \rangle^2 \geq 0$$

for every $X \in u(N)$.

Let $p_{N,t}(U)$ be the heat kernel on $U(N)$ with respect to this Riemannian structure as in [21, section 3.1]. Looking at the Fourier expansion of $p_{N,t}$ (see e.g., [21, Eq.(21)]) we observe that

$$\max_{U \in U(N)} p_{N,t}(U) = p_{N,t}(I_N)$$

holds for every $t > 0$. Recall that $p_{N,t}(U) = p_{N}(U,I_N,t/2)$, where $p_N(U,V,t), U,V \in U(N)$, $t > 0$, is a unique fundamental solution of the heat equation $\partial_u u = \Delta u$ with the Laplacian $\Delta$ on $U(N)$ equipped with the above Riemannian structure. See e.g., [10, p.135] for the notion of fundamental solutions of heat equations. It is well known, see e.g. [10, Theorem 1 in V.III.1], that $p_N$ is strictly positive. Since the Ricci curvature is non-negative as we saw before, we can apply Li–Yau’s theorem [22, Theorem 2.3] to $u(U,t) := p_{N}(U,I_N,t)$ and obtain that

$$p_N(I_N,I_N,t) \leq p_N(U,I_N,t) \exp(-N^2/4(1-\epsilon)t)$$

for every $t > 0$, $0 < \epsilon < 1$ and $U \in U(N)$, where $d_N(I_N,U)$ denotes the Riemannian distance between $I_N$ and $U$. Since $\max_{U \in U(N)} d_N(I_N,U) = N\pi$ (see e.g. the proof of [20, Proposition 4.1]), the above inequality with $t = T/2$ implies that

$$p_{N,T}(I_N) \leq p_{N,T}(I_N) \exp\left(-\frac{(N\pi)^2}{2(1-\epsilon)T}\right) \leq p_{N,T}(U) \exp\left(-\frac{d_N(I_N,U)^2}{2(1-\epsilon)T}\right)$$

for every $T > 0$, $0 < \epsilon < 1$ and $U \in U(N)$. Consequently, we have obtained that

$$\frac{1}{N^2} \log p_{N,T}(I_N) \leq \frac{1}{2} \log \frac{\pi^2}{2(1-\epsilon)T} \leq \frac{1}{N^2} \log p_{N,T}(U) \leq \frac{1}{N^2} \log p_{N,T}(I_N).$$
for every $t > 0$, $0 < \varepsilon < 1$ and $U \in U(N)$. By [21, Theorem 1.1], it is known that
\[
F(T) := \lim_{N \to \infty} \frac{1}{N^2} \log p_{N,T}(I_N) = \lim_{N \to \infty} \frac{1}{N^2} \log \left( \max_{U \in U(N)} p_{N,T}(U) \right)
\]
exists and defines a continuous function on $(0, +\infty)$. Thus, we have
\[
F(\varepsilon T) + \frac{1}{2} \log \varepsilon - \frac{\pi^2}{2(1 - \varepsilon)^2} \leq \lim_{N \to \infty} \frac{1}{N^2} \log p_{N,T}(U) \leq \lim_{N \to \infty} \frac{1}{N^2} \log p_{N,T}(U) \leq F(T)
\]
for every $T > 0$, $0 < \varepsilon < 1$ and $U \in U(N)$. In particular, we obtain that
\[
F(\varepsilon T) + \frac{1}{2} \log \varepsilon - \frac{\pi^2}{2(1 - \varepsilon)^2} \leq \lim_{N \to \infty} \frac{1}{N^2} \log \left( \min_{U \in U(N)} p_{N,T}(U) \right) \leq F(T)
\]
for every $T > 0$ and $0 < \varepsilon < 1$.

Assume that $T > \pi^2$ in what follows. We need the complete elliptic functions of the first kind and the second kind:
\[
K = K(k) := \int_0^1 \frac{ds}{\sqrt{(1 - s^2)(1 - k^2 s^2)}}, \quad E = E(k) := \int_0^1 \sqrt{1 - k^2 s^2} \, ds.
\]
With $T = 4K(2E - (1 - k^2)K)$, [21, Propositions 4.2, 5.2] show that
\[
F(T) = \frac{K(2E - (1 - k^2)K)}{6} + \frac{1}{2} \log \left( \frac{1}{4(2E - (1 - k^2)K)^2} (1 - k^2) \right) + \frac{2(1 + k^2)K}{3(2E - (1 - k^2)K)} + \frac{12(2E - (1 - k^2)K)^2}{(1 - k^2)^2 K^2}.
\]
Recall that
\[
K = \log \frac{4}{\sqrt{1 - k^2}} + o(1) = \frac{3}{2} \log 2 - \frac{1}{2} \log(1 - k) + o(1) \quad \text{as } k \to 1 - 0
\]
(see e.g. [8, p.11]). This immediately implies that $\lim_{k \to 1 - 0} (1 - k)^\alpha K = 0$ for any $\alpha > 0$. We also have $E = 1$ at $k = 1$. By the well-known formulas $dK/dk = (E - (1 - k^2)K)/(k(1 - k^2))$ and $dE/dk = (E - K)/k$, $0 < k < 1$ (see [8, p.282]), we have $d(2E - (1 - k^2)K)/dk = (1 - k^2)dK/dk$. It is clear that $K$ is increasing in $k$. Hence $T$ is an increasing function in $k$. Then, we observe that $T \to +\infty$ if and only if $k \to 1 - 0$. Moreover, we have
\[
F(T) = \frac{E}{3} + \frac{2(1 + k^2)}{3(2E - (1 - k^2)K)} K - \frac{3}{2} \log 2 + \frac{1}{2} \log(1 - k) + o(1)
\]
\[
= \frac{(E - 1)K}{3} + \frac{2((1 - k^2)K^2 - (1 - k^2)K - 2(E - 1)K)}{3(2E - (1 - k^2)K)K} + \frac{K - \frac{3}{2} \log 2 + \frac{1}{2} \log(1 - k)}{3(2E - (1 - k^2)K)^2} + o(1)
\]
as $k \to 1 - 0$. Since $dE/dk = (E - K)/k$, $0 < k < 1$ again, L'Hospital’s rule (see e.g. [26, Theorem 5.13]) enables us to confirm that $\lim_{k \to 1 - 0} (E - 1)/(1 - k)^{1/2} = 0$ and hence
\[
\lim_{k \to 1 - 0} (E - 1)K = \lim_{k \to 1 - 0} \left( \frac{E - 1}{(1 - k)^{1/2}} \cdot (1 - k)^{1/2} K \right) = 0.
\]
Consequently, we get $\lim_{T \to +\infty} F(T) = 0$. 


Taking the limit of (2.1) as $T \to +\infty$ we have

$$
\frac{1}{2} \log \varepsilon \leq \lim_{T \to +\infty} \lim_{N \to \infty} \frac{1}{N^2} \log \left( \min_{U \in U(N)} p_{N,T}(U) \right)
$$

\[ \leq \lim_{T \to +\infty} \lim_{N \to \infty} \frac{1}{N^2} \log \left( \max_{U \in U(N)} p_{N,T}(U) \right) = 0
\]

for all $0 < \varepsilon < 1$. Since $\varepsilon$ can arbitrarily be close to 1, we finally obtain the next lemma, which will play a key role in §4.

**Lemma 2.1.** With

$$L(T) := \lim_{N \to \infty} \frac{1}{N^2} \log \left( \min_{U \in U(N)} p_{N,T}(U) \right), \quad U(T) := \lim_{N \to \infty} \frac{1}{N^2} \log \left( \max_{U \in U(N)} p_{N,T}(U) \right) = F(T)
$$

we have

$$\lim_{T \to +\infty} L(T) = \lim_{T \to +\infty} U(T) = 0.
$$

### 3. Orbital free entropy revisited

Let $\Xi = (\Xi_i)_{i=1}^{n+1}$ with $\Xi_i = (\Xi_i(N))_{N \in \mathbb{N}}$ be a finite family of sequences of (deterministic) matrices such that each $\Xi_i(N)$ is a finite family of deterministic sequences of matrices of size $N$. We sometimes write $\Xi = (\Xi(N))_{N \in \mathbb{N}}$ with $\Xi(N) = ((\Xi_i(N))_{i=1}^{n+1})_n$. As in [29], we consider the universal $\mathcal{C}^\ast$-algebra $C_R^\ast(\mathbb{x}_\bullet) \otimes \mathcal{C}$ generated by $x_{ij} = x_{ij}^\ast$, $1 \leq i \leq n + 1, j \geq 1$, such that $\|x_{ij}\|_\infty \leq R$ for all $i, j$, into which the universal unital $\ast$-algebra $\mathcal{C}(\mathbb{x}_\bullet)$ generated by the $x_{ij} = x_{ij}^\ast$ is faithfully and norm-densely embedded. Similarly, we define $\mathcal{C}(\mathbb{x}_\bullet) \otimes \mathcal{C}$ for the first suffix $i$ of generators. These universal $\mathcal{C}^\ast$-algebras are constructed as universal free products of copies of $\mathcal{C}[-R,R]$. The above embedding properties are guaranteed by Proposition A.4. The $\ast$-homomorphism given by $x_{ij} \mapsto \xi_{ij}(N)$ enables us to define tracial states $\text{tr}^\Xi(N)$ as well as $\text{tr}^\Xi(N)$ to be the restriction of $\text{tr}^\Xi(N)$ to $C_R^\ast(\mathbb{x}_\bullet)$ faithfully by [6, Theorem 3.1 with Lemma A.1]. We also assume that each $\Xi_i$, $1 \leq i \leq n + 1$, has a limit distribution as $N \to \infty$; namely, there exists $a_0 \in \mathcal{T}(C_R^\ast(\mathbb{x}_\bullet))$ such that $\lim_{N \to \infty} \text{tr}^\Xi(N) = a_0$, in the weak* topology. (This is the minimum requirement for $\Xi$ to define $\chi_{orb}(\sigma | \Xi)$ below.) In what follows, we denote by $\text{tr}_{\text{orb}}(C_R^\ast(\mathbb{x}_\bullet))$ all the tracial states that arise in this way for a fixed $1 \leq i \leq n + 1$. We also define $\text{tr}_{\text{orb}}(C_R^\ast(\mathbb{x}_\bullet))$ similarly.

Let us introduce a variant of orbital free entropy, say $\chi_{orb}(\sigma | \Xi)$ for $\sigma \in \text{tr}(C_R^\ast(\mathbb{x}_\bullet))$, which is essentially the same as the old one in [15, section 4] for hyperfinite non-commutative random multi-variables.

Define $U = (U_i)_{i=1}^{n+1} \in U(N)^n \mapsto \text{tr}^\Xi(U) \in \text{tr}^\Xi(U) \in \text{tr}_{\text{orb}}(C_R^\ast(\mathbb{x}_\bullet))$ by $\Phi^\Xi(U) := \text{tr}_{\text{orb}} \circ \Phi^\Xi(U)$, where $\Phi^\Xi(U) : C_R^\ast(\mathbb{x}_\bullet) \to M_N(\mathbb{C})$ is a unique $\ast$-homomorphism sending $x_{ij}$ ($1 \leq i \leq n + 1$) to $U_i \xi_{ij}(N)U_i^\ast$ with $U = (U_i)_{i=1}^{n+1}$ and $x_{n+1,j}$ to $\xi_{n+1,j}(N)$, respectively. Consider an open neighborhood $O_{\sigma_0}(\delta)$, $m \in \mathbb{N}$, $\delta > 0$, at $\sigma$ in the weak* topology on $\text{tr}(C_R^\ast(\mathbb{x}_\bullet))$ defined to be all the $\sigma' \in \text{tr}(C_R^\ast(\mathbb{x}_\bullet))$ such that

$$||\sigma'(x_{i_1j_1} \cdots x_{i_pj_p}) - \sigma(x_{i_1j_1} \cdots x_{i_pj_p})|| < \delta.$$
whenever $1 \leq i_k \leq n + 1$, $1 \leq j_k \leq m$, $1 \leq k \leq p$ and $1 \leq p \leq m$. Then we define
\begin{equation}
\chi_{orb}(\sigma \mid \Xi(N) \mid X_{i,j}^r) := \lim_{N \to \infty} \frac{1}{N^2} \chi_{orb}(\sigma \mid \Xi(N) \mid \text{tr}_{U(N)}(\rho_{X_{i,j}^r}^{(N)})) \text{.}
\end{equation}

with $\log 0 := -\infty$. Remark that $\chi_{orb}(\sigma \mid \Xi) = -\infty$, if $\sigma$ does not agree with $\sigma_{0,i}$ on $C_r^*(x_{\sigma})$ for some $1 \leq i \leq n + 1$. This is a natural property; see [17, Proposition 3.1] as well as Remark 6.2.

We could prove in [15, Lemma 4.2] that $\chi_{orb}(\sigma \mid \Xi)$ depends only on the given $\sigma_{0,i}$, $1 \leq i \leq n + 1$, that is, it is independent of the choice of $\Xi$, when each tuple $(x_{ij}^r)^{(i)}$ produces a hyperfinite von Neumann algebra via the GNS construction associated with $\sigma_{0,i}$. However, we suspected that this is not always the case. Hence, in [27], in order to remove the dependency of $\Xi$ we took the supremum of $\chi_{orb}(\sigma \mid \Xi)$ all over the tuples $\sigma$ of multi-matrices in place of $\Xi(N)$ to define $\chi_{orb}(\sigma \mid X_1, \ldots, X_{n+1})$ (see the review below). Here, we will examine another simpler way of removing the dependency. So far, we have only assumed that each $\Xi$ has a limit distribution as $N \to \infty$, that is, $\lim_{N \to \infty} \text{tr}_{\Xi(N)}(\sigma_{0,i}) = \sigma_{0,i}$. In what follows, we need the stronger assumption that the whole $\Xi$ has a limit distribution as $N \to \infty$, that is, $\lim_{N \to \infty} \text{tr}_{\Xi(N)}(\sigma) = \sigma_0$.

We next recall the original orbital free entropy introduced in [27] (with a non-essential modification [28, Remark 3.3]) in the current setting. Let $\sigma : C_r^*(x_{\sigma}) \subset \mathcal{H}_{\sigma}$ be the GNS representation associated with $\sigma$. Set $X_{i,j}^r := \pi_{\sigma}(x_{ij})$, $1 \leq i \leq n + 1$, $1 \leq j \leq n + 2$, and then write $X_i = (X_{i,j}^r)_{j=1}^{n+2}$, $1 \leq i \leq n + 1$. Remark that the joint distribution of these $X_1, \ldots, X_{n+1}$ with respect to the tracial state on $\sigma_{0,i}(C_r^*(x_{\sigma}))^\prime$ induced from $\sigma$ is exactly $\sigma$. On the other hand, if we have uniformly norm-bounded non-commutative self-adjoint random multi-variables $X_1 = (X_{i,j}^r)_{j=1}^{(n+1)}$, $\ldots$, $X_{n+1} = (X_{i,j}^{(n+1)})_{j=1}^{(n+1)}$ in a $W^*$-probability space $(\mathcal{M}, \tau)$, i.e., $X_{i,j} = X_{i,j}^r$ and $R := \sup_{i,j} \|X_{i,j}\| < +\infty$, then we have a unique tracial state $\sigma_{X_i} \in TS(C_r^*(x_{\sigma}))$ naturally, that is, $\sigma_{X_i}(x_{i_1,j_1} \cdots x_{i_m,j_m}) = \tau(x_{i_1,j_1} \cdots x_{i_m,j_m})$ for example. For any $\mathbf{A} = (A_i)_{i=1}^{n+1}$ with $A_i = (A_{ij})_{j=1}^{(n+1)} \in ((M_{n+1})^\prime)^{(i)}$, $1 \leq i \leq n + 1$, we define
\begin{align*}
\chi_{orb}(X_1, \ldots, X_{n+1}; \mathbf{A}, N, m, \delta) &:= \log \nu_{\mathbf{A}}(\{U \in U(N)^n \mid \text{tr}_U(\rho_{X_{i,j}^r}^{(N)}) \in O_{m,\delta}(\sigma(X_{i,j}))\}) \\
\chi_{orb}(X_1, \ldots, X_{n+1}; \mathbf{A}, N, m, \delta) &:= \sup_{\mathbf{A}} \chi_{orb}(X_1, \ldots, X_{n+1}; \mathbf{A}, N, m, \delta) \\
\chi_{orb}(X_1, \ldots, X_{n+1}; m, \delta) &:= \lim_{N \to \infty} \frac{1}{N^2} \chi_{orb}(X_1, \ldots, X_{n+1}; N, m, \delta) \\
\chi_{orb}(X_1, \ldots, X_{n+1}) &:= \lim_{m \to \infty} \chi_{orb}(X_1, \ldots, X_{n+1}; m, \delta) \
\end{align*}
where \( \text{tr}_U^\Xi \) is defined in the same manner as the \( \text{tr}^{(N)}_U \) above. Note that the above definition clearly works even when \( r(i) = \infty \) for every \( 1 \leq i \leq n + 1 \).

The next proposition suggests which approximating sequences \( \Xi \) are suitable to define the orbital free entropy.

**Proposition 3.1.** We have

\[
\chi_{\text{orb}}(\sigma | \sigma_0) \leq \chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma),
\]

and equality holds when \( \sigma = \sigma_0 \).

**Proof.** Let \( \Xi = (\Xi(N))_{N \in \mathbb{N}} \) with \( \Xi_i(N) = (\xi_j(N))_{j=1}^{r(i)} \), \( 1 \leq i \leq n + 1 \), be as in definition (3.2). Clearly,

\[
\chi_{\text{orb}}(\sigma | \Xi; N, m, \delta) = \chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma; \Xi(N), N, m, \delta) \leq \chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma; N, m, \delta)
\]

holds for every \( N, m \) and \( \delta \). This immediately implies \( \chi_{\text{orb}}(\sigma | \Xi) \leq \chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma) \). Since \( \Xi \) has arbitrarily been chosen, we obtain \( \chi_{\text{orb}}(\sigma | \sigma_0) \leq \chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma) \).

We next prove the latter assertion. We may and do assume that \( \chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma) > -\infty \); otherwise the desired equality trivially holds as \( -\infty = -\infty \) by the first part. We can inductively choose an increasing sequence \( N_k \) in such a way that

\[
\tilde{\chi}_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma; k, 1/k) - \frac{1}{k} < \frac{1}{N_k} \chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma; N_k, k, 1/k)
\]

holds for every \( k \); hence

\[
\chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma) = \lim_{k \to \infty} \frac{1}{N_k} \chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma; N_k, k, 1/k).
\]

For each \( k \) one can choose \( A(N_k) = (A_i(N_k))_{i=1}^{n+1} \) with \( A_i(N_k) = (A_{ij}(N_k))_{j=1}^{r(i)} \in ((M_{n+1}^{sa})_R)^{r(i)} \), \( 1 \leq i \leq n + 1 \), in such a way that

\[
-\infty < \chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma; N_k, k, 1/k) - 1 < \chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma; A(N_k), N_k, k, 1/k).
\]

By definition, for each \( k \) there exists \( U(N_k) \in U(N_k)^n \) such that \( \text{tr}^{A(N_k)}_{U(N_k)} \in O_{k,1/k}(\sigma) \). With \( U(N_k) = (U_i(N_k))_{i=1}^{n+1} \) we define \( B(N_k) = (B_{ij}(N_k))_{i=1}^{n+1} \) by

\[
B_{ij}(N_k) := \begin{cases} 
U_i(N_k)A_{ij}(N_k)U_i(N_k)^* & (1 \leq i \leq n), \\
A_{n+1+j}(N_k) & (i = n + 1).
\end{cases}
\]

Let \( \Xi = (\Xi(N))_{N \in \mathbb{N}} \) with \( \Xi_i(N) = (\xi_j(N))_{j=1}^{r(i)} \), \( 1 \leq i \leq n + 1 \), be the one chosen at the beginning of this proof. (The existence of such a sequence follows from \( \chi_{\text{orb}}(X_1^\sigma, \ldots, X_{n+1}^\sigma) > -\infty \); see e.g. [17, Lemma 2.1].) Define \( \Xi' = (\Xi'(N))_{N \in \mathbb{N}} \) by

\[
\Xi'(N) := \begin{cases} 
B(N_k) & (N = N_k), \\
\Xi(N) & (\text{otherwise}).
\end{cases}
\]

Since

\[
\text{tr}^{\Xi'(N)}_{U(N_k)} = \text{tr}^{A(N_k)}_{U(N_k)} \in O_{k,1/k}(\sigma),
\]

it is easy to see that \( \text{tr}^{\Xi'(N)}_{U} \) converges to \( \sigma \) in the weak* topology on \( TS(C^*_R(x_\circ \diamond)) \). Since

\[
\text{tr}^{\Xi'(N_k)}_{U} = \text{tr}^{A(N_k)}_{(U_iU_i(N_k))_{i=1}^{n+1}}, \quad U = (U_i)_{i=1}^{n+1} \in U(N_k)^n
\]


for every $k$ and since $\nu_N$ is invariant under right-multiplication, we observe that
\[ \chi_{\text{orb}}(X_1^k, \ldots, X_{n+1}^k; A(N_k), N_k, k, 1/k) = \chi_{\text{orb}}(\sigma | \Xi'; N_k, k, 1/k) \]
for every $k$. Thus, for each $m \in \mathbb{N}$, $\delta > 0$, we have
\[ \chi_{\text{orb}}(\sigma | \Xi'; N_k, k, 1/k) \leq \chi_{\text{orb}}(\sigma | \Xi'; N_k, m, \delta) \]
for all sufficiently large $k$. Thus, for every $m \in \mathbb{N}$, $\delta > 0$, we obtain that
\[ \chi_{\text{orb}}(X_1^k, \ldots, X_{n+1}^k) = \lim_{k \to \infty} \frac{1}{N_k^2} \chi_{\text{orb}}(X_1^k, \ldots, X_{n+1}^k; N_k, k, 1/k) \]
\[ = \lim_{k \to \infty} \frac{1}{N_k^2} \left( \chi_{\text{orb}}(X_1^k, \ldots, X_{n+1}^k; N_k, k, 1/k) - 1 \right) \]
\[ \leq \lim_{k \to \infty} \frac{1}{N_k} \chi_{\text{orb}}(\sigma | \Xi'; N_k, m, \delta) \]
\[ \leq \lim_{N \to \infty} \frac{1}{N^2} \chi_{\text{orb}}(\sigma | \Xi'; m, \delta) \]
\[ = \chi_{\text{orb}}(\sigma | \Xi'; m, \delta). \]

Therefore, by taking the limit as $m \to \infty$, $\delta \searrow 0$ we have
\[ \chi_{\text{orb}}(X_1^k, \ldots, X_{n+1}^k) \leq \chi_{\text{orb}}(\sigma_0 | \Xi') \leq \chi_{\text{orb}}(\sigma | \sigma). \]

With the former assertion we are done. \(\square\)

Another natural choice of initial tracial state $\sigma_0$ is available; the tracial state is determined by making the resulting random multi-variables $X_1^{\sigma_0}$, $1 \leq i \leq n + 1$, freely independent. The $\chi_{\text{orb}}(\sigma | \sigma_0)$ with this choice of $\sigma_0$ is nothing but an unpublished variation of orbital free entropy due to Dabrowski, and the proposition below shows that it turns out to be the same as our original $\chi_{\text{orb}}(X_1^k, \ldots, X_{n+1}^k)$ in [27].

**Proposition 3.2.** When the $X_i^{\sigma_0}$, $1 \leq i \leq n + 1$, are freely independent, then $\chi_{\text{orb}}(\sigma | \sigma_0) = \chi_{\text{orb}}(X_1^k, \ldots, X_{n+1}^k)$.

**Proof.** By Proposition 3.1 we may and do assume $\chi_{\text{orb}}(X_1^k, \ldots, X_{n+1}^k) > -\infty$, and it suffices to prove
\[ \chi_{\text{orb}}(\sigma | \sigma_0) \geq \chi_{\text{orb}}(\sigma | \sigma) \]
\(= \chi_{\text{orb}}(X_1^k, \ldots, X_{n+1}^k)\).

To this end, let $\Xi = (\Xi(N))_{N=1}^\infty$ with $\Xi(N) = (\Xi_i(N))_{i=1}^{n+1}$, $\Xi_i(N) = (\xi_{ij}(N))_{j=1}^{r(i)}$, $1 \leq i \leq n + 1$, be such that $\lim_{N \to \infty} \text{tr}\Xi(N) = \sigma$ in the weak* topology. Choose an independent family of Haar-distributed unitary random matrices $V_i^{(N)}$, $1 \leq i \leq n$. It is known, see e.g. [16, Theorem 4.3.1], that $V_1^{(N)}, \ldots, V_n^{(N)}, \Xi(N)$ are asymptotically free almost surely as $N \to \infty$ and moreover that the subfamily $V_1^{(1)}, \ldots, V_n^{(1)}$ converges to a freely independent family of Haar unitaries in distribution almost surely as $N \to \infty$ too. Thus, thanks to the almost sure convergence, we can choose deterministic sequences $V_i(N)$, $1 \leq i \leq n$, from random sequences $V_1^{(i)}, \ldots, V_n^{(i)}$, $1 \leq i \leq n$ such that $V_1(N), \ldots, V_n(N), \Xi(N)$ converge to the same family of non-commutative random variables in distribution as $N \to \infty$. Define $\Xi'(N) = (\Xi'(N))_{N=1}^\infty$ with $\Xi'(N) = (\xi_{ij}'(N))_{j=1}^{r(i)}$, $\Xi_i'(N) = (\xi_{ij}'(N))_{j=1}^{r(i)}$ by
\[ \xi_{ij}'(N) := \begin{cases} V_i(N) \xi_{ij}(N) V_i(N)^* & (1 \leq i \leq n), \\ \xi_{i+1,j}(N) & (i = n + 1). \end{cases} \]

Then, the $\Xi_i'(N)$, $1 \leq i \leq n + 1$, are asymptotically free as $N \to \infty$. Therefore, we conclude that $\lim_{N \to \infty} \text{tr}\Xi'(N) = \sigma_0$ in the weak* topology. Remark that
\[ \text{tr}_{U}^{\Xi(N)} = U_{i}(U,V_i(N))_{i=1}^{n+1}, \quad U = (U_i)_{i=1}^{n} \in U(N)^n \]
holds for every $N$. Therefore, thanks to the invariance of $\nu_N$ under right-multiplication, we conclude, as in the proof of Proposition 3.1, that

$$\chi_{arb}(\sigma | \Xi) = \chi_{arb}(\sigma | \Xi') \leq \chi_{arb}(\sigma | \sigma_0).$$

Since $\Xi$ has arbitrarily been chosen, we are done. \qed

The above proof suggests that $\chi_{arb}(\sigma | \sigma_0)$ coincides with $\chi_{arb}(X_1^e, \ldots, X_{n+1}^e)$ for a large class of tracial states $\sigma_0 \in TS_{alb}(C_R^*(x_{*}))$.

4. ORBITAL FREE ENTROPY AND MATRIX LIBERATION PROCESS

Building on our previous work [29] we will clarify how some fundamental questions concerning the orbital free entropy $\chi_{arb}$ are precisely reduced to the conjectural large deviation principle for the matrix liberation process. Lemma 2.1 will play a key role in what follows.

4.1. Non-commutative coordinates. Let $C^*_R(x_{*}(\cdot)) \subset C^*_R(x_{*}(\cdot), v_{*}(\cdot))$ be the universal unital $C^*$-algebras generated by $x_{ij}(t) = x_{ij}(t)^*$, $1 \leq i \leq n + 1, j \geq 1, t \geq 0,$ and $v_{ij}(t), 1 \leq i \leq n, t \geq 0,$ with subject to $\|x_{ij}(t)\|_\infty \leq R$ and $v_{ij}(t)v_{ij}(t) = v_{ij}(t)v_{ij}(t)^* = 1 = v_{ij}(0)$. These universal $C^*$-algebras are constructed as universal free products of uncountably many $C[-R, R]$ and $C(T)$, and generators $x_{ij}(t)$ and $v_{ij}(t)$ are given by coordinate functions $f(t) = t \in t \in [-R, R]$ or $g(z) = z$ in $z \in T$ of component algebras. Proposition A.3 guarantees the inclusion of two universal $C^*$-algebras. Recall that $j$ may run over the natural numbers $N$ as we remarked at the end of section 1. The universal $*$-algebras $C(x_{*}(\cdot)) \subset C(x_{*}(\cdot), v_{*}(\cdot))$ generated by the same indeterminates $x_{ij}(t)$ and $v_{ij}(t)$ can naturally be regarded as norm-dense $*$-subalgebras of $C^*_R(x_{*}(\cdot), v_{*}(\cdot))$, respectively. Proposition A.4 guarantees this fact. For each $T \geq 0$, the correspondence $x_{ij} \mapsto x_{ij}(T), 1 \leq i \leq n + 1, j \geq 1$, defines a unique (injective) $*$-homomorphism $\pi_T : C^*_R(x_{*}) \to C^*_R(x_{*}(\cdot))$ with notation $C_R^*(x_{*})$ in section 3.

4.2. Time-dependent liberation derivative. We introduce the derivation

$$\delta^{(k)} : C(x_{*}(\cdot)) \to C(x_{*}(\cdot), v_{*}(\cdot)) \otimes_{alg} C(x_{*}(\cdot), v_{*}(\cdot)),$$

which sends each $x_{ij}(t)$ to

$$\delta_{ij} \mathbb{1}_{[0, s]}(s)x_{ij}(t)v_{ij}(t-s) \otimes v_{ij}(t-s) = v_{ij}(t-s) \otimes v_{ij}(t-s)^*x_{ij}(t)).$$

Then we write $\mathcal{D}_{s}^{(k)} := \delta \circ \delta^{(k)}, 1 \leq k \leq n, s \geq 0$, where $\theta$ denotes the flip-multiplication mapping $a \otimes b \mapsto ba$.

4.3. Continuous tracial states. A tracial state $\tau$ on $C_R^*(x_{*}(\cdot))$ is said to be continuous if $t \mapsto \pi_T(x_{ij}(t))$ is strongly continuous for every $1 \leq i \leq n + 1, j \geq 1$, where $\pi_T : C_R^*(x_{*}(\cdot)) \to C_T$ is the GNS representation associated with $\tau$. We denote by $TS^*(C_R^*(x_{*}(\cdot)))$ all the continuous tracial states. The space $TS^*(C_R^*(x_{*}(\cdot)))$ becomes a complete metric space endowed with metric $d$ defined by (1.1), which defines the topology of uniform convergence on finite time intervals.

4.4. Liberation process $\tau^*$ starting at a given time. We extend a given $\tau \in TS^*(C_R^*(x_{*}(\cdot)))$ to a unique $\tau^* \in TS^*(C_R^*(x_{*}(\cdot), v_{*}(\cdot)))$ in such a way that the $v_{ij}(t)$ are $*$-freely independent of $C_R^*(x_{*}(\cdot))$ and form a $*$-freely independent family of left-multiplicative free unitary Brownian motions under this extension $\tau$. This extension of tracial state can be constructed, via the GNS representation $\pi_{\tau} : C_R^*(x_{*}(\cdot)) \to C_T$, by taking a suitable reduced free product. We write

$$(\mathcal{N}(\tau) \subset \mathcal{M}(\tau)) := (\pi_{\tau}(C_R^*(x_{*}(\cdot)))'' \subset \pi_{\tau}(C_R^*(x_{*}(\cdot), v_{*}(\cdot))))''$$
on $\mathcal{H}_{\tau}$, where $\pi_{\tau} : C_R^*(x_{*}(\cdot), v_{*}(\cdot)) \to \mathcal{H}_{\tau}$ is the GNS representation associated with $\tau$. Write $x_{ij}^*(t) := \pi_{\tau}(x_{ij}(t))$ and $v_{ij}^*(t) := \pi_{\tau}(v_{ij}(t))$ and the canonical extension of $\tau$ to $\mathcal{M}(\tau)$ is still denoted by the same symbol $\tau$ for simplicity. We denote by $E_{\mathcal{N}(\tau)}(\tau)$ the $\tau$-preserving conditional expectation
from $\mathcal{M}(\tau)$ onto $\mathcal{N}(\tau)$, which is known to exist and to be unique as a standard fact on von Neumann algebras. Consider an ‘abstract’ non-commutative process in $C_R^*(\langle x \cdot \diamond \rangle)$$
t \mapsto x^\tau_{ij}(t) := \begin{cases} v_i^t((t-s) \lor 0)x_{ij}(s \land t)v_i^t((t-s) \lor 0)^* & (1 \leq i \leq n), \\ x^\tau_{n+1,j}(t) & (i = n+1) \end{cases}
and the corresponding ‘concrete’ non-commutative stochastic process in $\mathcal{M}(\tau)$
$$t \mapsto x^\tau_{ij}(t) := \pi_\tau(x^\tau_{ij}(t)) = \begin{cases} v_i^t((t-s) \lor 0)x^\tau_{ij}(s \land t)v_i^t((t-s) \lor 0)^* & (1 \leq i \leq n), \\ x^\tau_{n+1,j}(t) & (i = n+1). \end{cases}$$

By universality, this process $x^\tau_{ij}(t)$ clearly defines a tracial state $\tau^* \in TS^c(\mathcal{C}_R^*(\langle x \cdot \diamond \rangle))$.

By the $*$-homomorphism $\Gamma : C_R^*(\langle x \cdot \diamond \rangle) \to C_R^*(\langle x \cdot \diamond \rangle)$ sending each $x_{ij}$ to $x^\tau_{ij}(T)$, we obtain $\Gamma^*(\sigma_0) := \sigma_0 \circ \Gamma \in TS(\mathcal{C}_R^*(\langle x \cdot \diamond \rangle))$ for a given $\sigma_0 \in TS(\mathcal{C}_R^*(\langle x \cdot \diamond \rangle))$ and set $\sigma_0^lib := \Gamma^*(\sigma_0)^{lib} \in TS(\mathcal{C}_R^*(\langle x \cdot \diamond \rangle))$ ($\Gamma^*(\sigma_0)$ is defined in the same way as $\tau^*$ with $s = 0$), which we call the liberation process starting at $\sigma_0$ (precisely its empirical distribution).

4.5. **New description of $\tau^*$.** By universality, we have a unique unital $*$-homomorphism $\Pi^* : C_R^*(\langle x \cdot \diamond \rangle, \tau^* \langle \cdot \rangle) \to C_R^*(\langle x \cdot \diamond \cdot \rangle, \tau^* \langle \cdot \rangle)$ sending $x_{ij}$ and $v_i^t$ to $x^\tau_{ij}(t)$ and $v_i^t(t)$, respectively. By using this $*$-homomorphism we obtain a unique $*$-homomorphism
$$\pi_\tau \circ \Pi^* : C_R^*(\langle x \cdot \diamond \rangle, \tau^* \langle \cdot \rangle) \rightarrow C_R^*(\langle x \cdot \diamond \rangle, \tau^* \langle \cdot \rangle) \rightarrow \mathcal{M}(\tau)$$
where $\pi_\tau$ is the unique injective $*$-homomorphism $\Gamma : C_R^*(\langle x \cdot \diamond \rangle) \to C_R^*(\langle x \cdot \diamond \rangle)$ sending each $x_{ij}$ to $x^\tau_{ij}(T)$.

Then $\pi_\tau(\Pi^*(\mathcal{D}_k^{(k)}(P)))$, $P \in C(\mathcal{D}_k^{(k)}(\cdot))$, becomes the element of $\mathcal{M}(\tau)$ obtained by substituting $(x^\tau_{ij}(t), v_i^t(t))$ for $(x_{ij}(t), v_i^t(t))$ in $\mathcal{D}_k^{(k)}(P)$. Moreover, we have $\tau^* = \tau \circ \Pi^*$ on $C_R^*(\langle x \cdot \diamond \rangle)$

4.6. **Rate function.** To a given $\sigma_0 \in TS(\mathcal{C}_R^*(\langle x \cdot \diamond \rangle))$ we associate two functionals $I^lib_{\sigma_0}, I^lib_{\sigma_0,\infty} : TS^c(\mathcal{C}_R^*(\langle x \cdot \diamond \rangle)) \to [0, +\infty]$ as follows. For any $\tau \in TS^c(\mathcal{C}_R^*(\langle x \cdot \diamond \rangle))$, $P = P^* \in C(\mathcal{D}_k^{(k)}(\cdot))$ and $t \in [0, \infty]$ we first define

$$I^lib_{\sigma_0}(\tau, P) := \tau^*(P) - \sigma_0^lib(P) - \frac{1}{2} \sum_{k=1}^n \int_0^t \|E_{\mathcal{N}(\tau)}(\pi_\tau(\Pi^*(\mathcal{D}_k^{(k)}(P)))))\|_{L^2}^2 \, ds$$

with regard to $\tau$ as $\tau^\infty$ (since $\tau^*(P) = \tau(P)$ when $t$ is large enough), where $\| - \|_{L^2}$ denotes the 2-norm on the trivial $W^*$-probability space $(\mathcal{M}(\tau), \tau)$. We remark that the integrand in (4.1) agrees with that in [29] (though their representations are different at first glance), and moreover that the integration above is well defined even when $t = \infty$, because $\mathcal{D}_k^{(k)}P = 0$ when $s$ is large enough. Then we define

$$I^lib_{\sigma_0}(\tau) := \sup_{P = P^* \in C(\mathcal{D}_k^{(k)}(\cdot))} I^lib_{\sigma_0}(\tau, P), \quad I^lib_{\sigma_0,\infty}(\tau) := \sup_{P = P^* \in C(\mathcal{D}_k^{(k)}(\cdot))} I^lib_{\sigma_0,\infty}(\tau, P).$$

Each of the functionals $I^lib_{\sigma_0}, I^lib_{\sigma_0,\infty}$ is shown, in [29, Proposition 5.6, Proposition 5.7(3)] (n.b., their proofs work well even for the modification $I^lib_{\sigma_0,\infty}$ without any essential changes), to be well-defined, good rate function with unique minimizer. Moreover, the minimizer for both functionals is identified with the liberation process $\sigma_0^lib$ starting at $\sigma_0$ for both functionals. Remark that the proofs of [29, Proposition 5.6, Proposition 5.7(3)] do not use the assumption that $\sigma_0$ falls into $TS\sigma_0(\mathcal{C}_R^*(\langle x \cdot \diamond \rangle))$, and thus the functionals $I^lib_{\sigma_0,\infty}$ can be considered in the general setting. Remark that $I^lib_{\sigma_0,\infty} \leq I^lib_{\sigma_0}(\tau)$ obviously holds, but it is a question whether equality holds or not.

Here is a simple lemma, which can be applied to $I = I^lib_{\sigma_0}$ or $I = I^lib_{\sigma_0,\infty}$. Recall that $\pi_\tau : C_R^*(\langle x \cdot \diamond \rangle) \to C_R^*(\langle x \cdot \diamond \rangle)$ is the unique injective $*$-homomorphism sending each $x_{ij}$ to $x^\tau_{ij}(T)$. In the
lemma below, we will use the map $\pi^*_T : TS^*(C^*(x_{s*}(-))) \to TS^*(C^*(x_{s*}))$ induced from $\pi_T$, see the glossary in section 1.

Lemma 4.1. For any functional $I : TS^*(C^*(x_{s*}(-))) \to [0, +\infty]$, the new one $J : TS(C^*_R(x_{s*})) \to [0, +\infty]$ defined by

$$J(\sigma) = \lim_{\delta \to 0} \inf_{T \to \infty} \left\{ I(\tau) \mid \tau \in TS^*(C^*_R(x_{s*}(-))), \pi^*_T(\tau) \in O_{m,\delta}(\sigma) \right\}$$

$$= \sup_{m \in \mathbb{N}} \inf_{\delta > 0} \left\{ I(\tau) \mid \tau \in TS^*(C^*_R(x_{s*}(-))), \pi^*_T(\tau) \in O_{m,\delta}(\sigma) \right\}$$

for any $\sigma \in TS(C^*_R(x_{s*}))$ (with notation $O_{m,\delta}(\sigma)$ in the previous section) is a well-defined rate function, where $TS(C^*_R(x_{s*}))$ is endowed with the weak* topology and the infimum over the empty set is taken to be $+\infty$. Moreover, replacing $O_{m,\delta}(\sigma)$ with the closed neighborhood $F_{m,\delta}(\sigma)$ in the above definition of $J(\sigma)$ does not affect its value, where $F_{m,\delta}(\sigma)$ is all the $\sigma' \in TS(C^*_R(x_{s*}))$ such that

$$|\sigma'(x_{i_1j_1} \cdots x_{i_kj_k}) - \sigma(x_{i_1j_1} \cdots x_{i_kj_k})| \leq \delta$$

whenever $1 \leq i_k \leq n + 1$, $1 \leq j_k \leq m$, $1 \leq k \leq p$ and $1 \leq p \leq m$.

Proof. If $m_1 \leq m_2$ and $\delta_1 \geq \delta_2 > 0$, then $O_{m_1,\delta_1}(\sigma) \supseteq O_{m_2,\delta_2}(\sigma)$ so that

$$\lim_{T \to \infty} \inf_{\delta \to 0} \left\{ I(\tau) \mid \tau \in TS^*(C^*_R(x_{s*}(-))), \pi^*_T(\tau) \in O_{m_1,\delta_1}(\sigma) \right\}$$

$$\leq \lim_{T \to \infty} \inf_{\delta \to 0} \left\{ I(\tau) \mid \tau \in TS^*(C^*_R(x_{s*}(-))), \pi^*_T(\tau) \in O_{m_2,\delta_2}(\sigma) \right\}$$

Therefore, taking $\lim_{m_0 \to \infty, \delta_0 \to 0}$ in the definition of $J(\sigma)$ is actually well defined and coincides with taking the supremum all over $m \in \mathbb{N}$ and $\delta > 0$.

We then confirm that $J$ is lower semicontinuous. Assume that $\sigma_k \to \sigma$ in $TS(C^*_R(x_{s*}))$ as $k \to \infty$. Choose an arbitrary $0 \leq L < J(\sigma)$. Then there exist $m_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that

$$\lim_{T \to \infty} \inf_{\delta \to 0} \left\{ I(\tau) \mid \tau \in TS^*(C^*_R(x_{s*}(-))), \pi^*_T(\tau) \in O_{m_0,\delta_0}(\sigma_k) \right\} > L$$

Then, there exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, then $O_{m_0,\delta_0/2}(\sigma_k) \subseteq O_{m_0,\delta_0}(\sigma)$ and hence

$$J(\sigma_k) \geq \lim_{T \to \infty} \inf_{\delta \to 0} \left\{ I(\tau) \mid \tau \in TS^*(C^*_R(x_{s*}(-))), \pi^*_T(\tau) \in O_{m_0,\delta_0/2}(\sigma_k) \right\}$$

$$\geq \lim_{T \to \infty} \inf_{\delta \to 0} \left\{ I(\tau) \mid \tau \in TS^*(C^*_R(x_{s*}(-))), \pi^*_T(\tau) \in O_{m_0,\delta_0}(\sigma) \right\} > L,$$

where the first inequality follows from the fact that $\lim_{m \to \infty, \delta \to 0} \inf_{\delta} = \sup_{m,\delta}$ in the definition of $J(\sigma)$ as remarked before. Therefore, we obtain that $\lim_{k \to \infty} J(\sigma_k) \geq L$, which guarantees that $J$ is lower semicontinuous.

Since $O_{m,\delta}(\sigma) \subseteq F_{m,\delta}(\sigma)$, we have

$$\inf \left\{ I(\tau) \mid \tau \in TS^*(C^*_R(x_{s*}(-))), \pi^*_T(\tau) \in O_{m,\delta}(\sigma_k) \right\}$$

$$\geq \inf \left\{ I(\tau) \mid \tau \in TS^*(C^*_R(x_{s*}(-))), \pi^*_T(\tau) \in F_{m,\delta}(\sigma_k) \right\}$$

$$\geq \inf \left\{ I(\tau) \mid \tau \in TS^*(C^*_R(x_{s*}(-))), \pi^*_T(\tau) \in O_{m,\delta}(\sigma_k) \right\}$$

for every $m \in \mathbb{N}$ and $\delta > 0$. This implies the last assertion.

The above lemma clearly holds true even if $\lim_{T \to \infty}$ is replaced with $\lim_{T \to \infty}$ in the definition of $J$. We also remark that $TS(C^*_R(x_{s*}))$ is weak* compact, and hence $J$ is trivially a good rate function.
4.7. Matrix liberation process. Let $\Xi(N) = ((\xi_{ij}(N))_{i=1}^{r(i)}_{j=1}^{n+1})_{i=1}^{r(i)}$ with $\xi_{ij}(N) \in (M_{n+1}^N)_{i=1}^{n+1}$, be an approximation to a given $\sigma_0 \in T_{\text{sfda}}(C_R^*(x^\bullet))$. Let $U^{(i)}_{N}(t)$, $1 \leq i \leq n$, be independent, left-increment unitary Brownian motions on $U(N)$, and we define the matrix liberation process $\Xi^{lib}(N)(t) = ((\xi^{lib}_{ij}(N)(t))_{i=1}^{n+1})_{i=1}^{n+1}$, $t \geq 0$, starting at $\Xi(N)$ by

$$\xi^{lib}_{ij}(N)(t) := \begin{cases} U^{(i)}_{N}(t)\xi_{ij}(N)U^{(j)}_{N}(t)^* & (1 \leq i \leq n), \\ \xi_{i+n,j}(N) & (i = n + 1). \end{cases}$$

Then, via the $*$-homomorphism $\pi_{\Xi^{lib}(N)} : C_R^*(x^\bullet(\cdot)) \to M_N$ determined by $\pi_{\xi_{ij}(N)}(t) \to \xi^{lib}_{ij}(N)(t)$, $1 \leq i \leq n + 1, j \geq 1, t \geq 0$, we obtain a tracial state $\pi_{\Xi^{lib}(N)} := \pi_N \circ \pi_{\Xi^{lib}(N)}$, which falls into $TS^0(\langle x^\bullet(\cdot)\rangle)$. This tracial state is a random variable in $TS^0(\langle C^*_R(x^\bullet(\cdot))\rangle)$ in the ordinary sense, and hence we can consider the probability $\mathbb{P}(\pi_{\Xi^{lib}(N)} \in \Theta)$ of any Borel subset $\Theta \subseteq TS^0(\langle C^*_R(x^\bullet(\cdot))\rangle)$. By [29, Theorem 5.8] we already know that the sequence of probability measures $\mathbb{P}(\pi_{\Xi^{lib}(N)} \in \cdot)$ satisfies the large deviation upper bound with speed $N^2$ and the above rate function $I^{\ast}_P$.

4.8. Contraction principle at $T = \infty$. Let $U_N = (U^{(i)}_{N})_{i=1}^{n+1}$ be an $n$-tuple of independent $N \times N$ unitary random matrices distributed under the Haar probability measure $\nu_N$ on $U(N)$. The random tracial state $tr_{U_N}^{\Xi(N)} \in TS^{0}(C_R^*(x^\bullet))$ is defined in the same manner as in §3. A well-known, standard result on the heat kernel measure on $U(N)$ implies that $E[\tau^{\ast}_\pi(\pi_{\Xi^{lib}(N)}(a))]$ converges to $E[tr_{U_N}^{\Xi(N)}(a)]$ as $T \to \infty$ for every $a \in C_R^*(x^\bullet)$. The usual method to obtain the large deviation upper/lower bounds with speed $N^2$ for $\mathbb{P}(tr^{\Xi(N)}_{U_N} \in \cdot)$ from that for $\mathbb{P}(\pi_{\Xi^{lib}(N)} \in \cdot)$ in the same scale is to show that (a kind of) the exponential convergence of $\pi^{\ast}_T(\pi_{\Xi^{lib}(N)})$ to $tr_{U_N}^{\Xi(N)}$ as $T \to \infty$ (see e.g. [13, §4.2.2]). Nevertheless, we will be able to prove the next proposition by utilizing Lemma 2.1 without establishing the exponential convergence.

Proposition 4.2. Assume that the sequence of probability measures $\mathbb{P}(\pi_{\Xi^{lib}(N)} \in \cdot)$ satisfies the large deviation upper (lower) bound with speed $N^2$ and rate function $I^{\ast}\{\pi^{\ast}_T\}$ (resp. $I^{-}\{\pi^{\ast}_T\}$). Then $\mathbb{P}(tr^{\Xi(N)}_{U_N} \in \cdot)$ also satisfies the large deviation upper (resp. lower) bound with speed $N^2$ and the following rate function:

$$J^{\ast}(\sigma) := \lim_{m \to \infty} \lim_{\delta \to 0, T \to \infty} \inf\{I^{\ast}\{\tau\} \mid \tau \in TS^0(C_R^*(x^\bullet(\cdot)))\}, \pi^{\ast}_T(\tau) \in O_{m, \delta}(\sigma)\}$$

(resp. $J^{-}(\sigma) := \lim_{m \to \infty} \lim_{\delta \to 0, T \to \infty} \inf\{I^{-}\{\tau\} \mid \tau \in TS^0(C_R^*(x^\bullet(\cdot)))\}, \pi^{\ast}_T(\tau) \in O_{m, \delta}(\sigma)\}\}$

for every $\sigma \in TS^0(C_R^*(x^\bullet))$, where the infimum over the empty set is taken to be $+\infty$.

In particular, if the sequence of probability measures $\mathbb{P}(\pi_{\Xi^{lib}(N)} \in \cdot)$ satisfies the full large deviation principle with speed $N^2$, that is, the above large deviation upper and lower bounds with $I^{\ast} = I^{-}$, then $J^{\ast} = J^{-}$ and

$$\chi_{\text{orb}}(\sigma \mid \sigma_0) = \chi_{\text{orb}}(\sigma \mid \Xi) = -I(\sigma)$$

$$= \lim_{n \to \infty} \lim_{N \to \infty} -\frac{1}{N^2} \log \mu_0^{\Xi_N}(\{U \in U(N)^n \mid tr^{\Xi(N)}_U \in O_{m, \delta}(\sigma)\})$$

holds for every $\sigma \in TS^0(C_R^*(x^\bullet))$ and any choice of approximating sequence $\Xi = (\Xi(N))_{N \in \mathbb{N}}$ to $\sigma_0 \in T_{\text{sfda}}(C_R^*(x^\bullet))$.

Proof. Set

$$I^{\ast}_P(\sigma) := \inf\{I^\pm(\tau) \mid \tau \in TS^0(C_R^*(x^\bullet(\cdot)))\}, \pi^{\ast}_T(\tau) = \sigma, \sigma \in TS^0(C_R^*(x^\bullet)).$$
By the contraction principle (see e.g. [13, Theorem 4.2.1]), \( P(\pi^*(T(\tau_{\Xi^{\text{lib}}(N)})) \in \cdot) \) satisfies the large deviation upper (resp. lower) bound with speed \( N^2 \) and the rate function \( I^+_T \) (resp. \( I^-_T \). Write \( U_N(t) = (U_N(t))^n_{n=1}, \ t \geq 0 \), and define the random tracial state \( \rho^N_U(t) \) in the same manner as \( \rho^N_U \). Let \( L(T) \leq U(T) \) as well as \( \nu_{N,T} \) and \( \nu_N \) be as in the previous sections. Observe that

\[
P(\pi^*(T(\tau_{\Xi^{\text{lib}}(N)})) \in \cdot) = P(\rho^N_U(T) \in \cdot) = \nu^N_{N,T}(\{U \in U(N)^n \mid \rho^N_U \in \cdot\})
\]
as well as

\[
(4.2) \quad P(\rho^N_U \in \cdot) = \nu^N_N(\{U \in U(N)^n \mid \rho^N_U \in \cdot\}).
\]

Since

\[
\left( \min_{U \in U(N)} p_{N,T}(U) \right) \nu_N \leq \nu_{N,T} \leq \left( \max_{U \in U(N)} p_{N,T}(U) \right) \nu_N,
\]

we observe that

\[
\frac{n}{N^2} \log \min_{U \in U(N)} p_{N,T}(U) + \frac{1}{N^2} \log P(\rho^N_U \in \cdot) \leq \frac{1}{N^2} \log P(\pi^*(T(\tau_{\Xi^{\text{lib}}(N)})) \in \cdot)
\]

\[
\leq \frac{n}{N^2} \log \max_{U \in U(N)} p_{N,T}(U) + \frac{1}{N^2} \log P(\rho^N_U \in \cdot).
\]

Now, we will use the functions \( L(T), U(T) \) in \( T \) introduced in Lemma 2.1. If we assume the large deviation upper (resp. lower) bound for \( P(\pi^*(T(\tau_{\Xi^{\text{lib}}(N)})) \in \cdot) \), then

\[
nL(T) + \lim_{N \to \infty} \frac{1}{N^2} \log P(\rho^N_U \in \Lambda)
\]

\[
\leq \lim_{N \to \infty} \frac{1}{N^2} \log P(\pi^*(T(\tau_{\Xi^{\text{lib}}(N)})) \in \Lambda) \leq - \inf_{\sigma \in \Lambda} \{I^+_T(\sigma) \mid \sigma \in \Lambda\}
\]

for any closed \( \Lambda \subset TS(C^*_R(x_{\Theta})) \) (resp.

\[
nU(T) + \lim_{N \to \infty} \frac{1}{N^2} \log P(\rho^N_U \in \Gamma)
\]

\[
\geq \lim_{N \to \infty} \frac{1}{N^2} \log P(\pi^*(T(\tau_{\Xi^{\text{lib}}(N)})) \in \Gamma) \geq - \inf_{\sigma \in \Gamma} \{I^-_T(\sigma) \mid \sigma \in \Gamma\}
\]

for any open \( \Gamma \subset TS(C^*_R(x_{\Theta})) \). It follows by Lemma 2.1 that

\[
\lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{N^2} \log P(\rho^N_U \in O_{m,\delta}(\sigma)) \leq \inf_{\delta > 0} \lim_{N \to \infty} \inf_{\delta > 0} \inf_{\delta > 0} \inf_{T \to \infty} \{I^+_T(\sigma') \mid \sigma' \in F_{m,\delta}(\sigma)\}
\]

(resp. \( \lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{N^2} \log P(\rho^N_U \in O_{m,\delta}(\sigma)) \geq \inf_{\delta > 0} \lim_{N \to \infty} \inf_{\delta > 0} \inf_{T \to \infty} \{I^-_T(\sigma') \mid \sigma' \in O_{m,\delta}(\sigma)\}\)

for every \( \sigma \in TS(C^*_R(x_{\Theta})) \). Observe that

\[
\inf \{I^+_T(\sigma') \mid \sigma' \in \Theta\} = \inf \{I^+_T(\tau) \mid \tau \in TS^c(C^*_R(x_{\Theta}))\}, \pi^+_T(\tau) \in \Theta
\]

for any \( \Theta \subset TS(C^*_R(x_{\Theta})) \). By Lemma 4.1,

\[
\lim_{\delta \to 0} \lim_{T \to \infty} \inf \{I^+_T(\sigma') \mid \sigma' \in O_{m,\delta}(\sigma)\} = \lim_{\delta \to 0} \lim_{T \to \infty} \inf \{I^+_T(\sigma') \mid \sigma' \in F_{m,\delta}(\sigma)\}
\]

(resp. the same identity with replacing \( \lim_{T \to \infty} \) and \( I^+ \) with \( \lim_{T \to \infty} \) and \( I^- \), respectively) holds and defines a rate function. Since \( TS(C^*_R(x_{\Theta})) \) is weak* compact, we finally conclude by [13, Theorem 4.1.11, Lemma 1.2.18] that \( P(\rho^N_U \in \cdot) \) satisfies the large deviation upper (resp. lower) bound with speed \( N^2 \) and the rate function \( J^+ \) (resp. \( J^- \)).
For the last assertion, we first point out that

\[-J^-(\sigma) \leq \lim_{m,\delta \to \infty} \lim_{N \to \infty} \frac{1}{N^2} \log P(\text{tr}_U \Xi_N(\sigma)) \leq \lim_{m,\delta \to \infty} \lim_{N \to \infty} \frac{1}{N^2} \log P(\text{tr}_U \Xi_N(\sigma)) \leq -J^+(\sigma).\]  

(4.3)

Since \(I^+ = I^-\), we have \(-J^-(\sigma) \geq -J^+(\sigma)\) for every \(\sigma \in TS(C_R^* (x_{\bullet, \sigma}^*))\). Therefore, we conclude that equality holds in (4.3). This together with (4.2) immediately implies the last assertion. \(\square\)

It is plausible that the definition of the orbital free entropy \(\chi_{orb}(X_1, \ldots, X_{n+1})\) can still be defined independently of the choice of approximating sequence \(\Xi = (\Xi(N))_{N \in \mathbb{N}}\) (under the constraint that \(\text{tr}_U \Xi(N)\) converges to the joint distribution of the \(X_i\) without assuming the hyperfiniteness of each random multi-variable \(X_i\).

As mentioned before, we have already established that the sequence of probability measures \(P(\Xi; (N) \in \cdot)\) satisfies the large deviation upper bound with speed \(N^2\) and the rate function \(I_{orb}^\text{lib}\). Hence, we can prove the next corollary.

**Corollary 4.3.** The sequence of probability measures \(P(\text{tr}_U \Xi(N) \in \cdot)\) satisfies the large deviation upper bound with speed \(N^2\) and the rate function

\[I_{orb}^\text{lib}(\sigma) := \inf_{m,\delta} \lim_{N \to \infty} \inf \{I_{orb}^\text{lib}(\tau) \mid \tau \in TS(C_R^* (x_{\bullet, \sigma}^*)) \cap TS(C_R^* (x_{\bullet, \sigma}^*)) \},\]

where the infimum over the empty set is taken to be \(+\infty\). Moreover, \(\chi_{orb}(\sigma \mid \sigma_0) \leq -J_{orb}^\text{lib}(\sigma)\) holds for every \(\sigma \in TS(C_R^* (x_{\bullet, \sigma}^*))\).

**Proof.** The first assertion immediately follows from Lemma 4.1 and Proposition 4.2.

For the second assertion, we first observe that

\[\chi_{orb}(\sigma \mid \Xi) = \lim_{m,\delta \to \infty} \lim_{N \to \infty} \frac{1}{N^2} \log P(\text{tr}_U \Xi(N) \in O_{m,\delta}(\sigma)) \leq -J_{orb}^\text{lib}(\sigma)\]

for every \(\sigma \in TS(C_R^* (x_{\bullet, \sigma}^*))\). Since \(J_{orb}^\text{lib}\) is independent of the choice of approximation \(\Xi\) to \(\sigma_0\), we conclude that \(\chi_{orb}(\sigma \mid \sigma_0) \leq -J_{orb}^\text{lib}(\sigma)\) for every \(\sigma \in TS(C_R^* (x_{\bullet, \sigma}^*))\). \(\square\)

**Remark 4.4.** Several questions on the matrix liberation process \(\Xi^{lib}(N)\) toward the completion of developing the theory of orbital free entropy are in order.

(Q1) Show that \(J_{orb}^\text{lib}(\sigma) = 0\) implies that the \(X_i\) are freely independent. (This is a question about minimizers of \(J_{orb}^\text{lib}\).)

(Q2) Identify \(J_{orb}^\text{lib}(\sigma)\) with Voiculescu’s free mutual information \(i^\ast(W^* (X_i^\ast); \ldots; W^* (X_{n+1}^\ast))\) (at least when \(\sigma = \sigma_0\) or when the \(X_i\) are freely independent) if possible. Here each \(W^* (X_i^\ast)\) denotes the von Neumann subalgebra generated by \(X_i^\ast = (X_{ij}^\ast(i \to j))\).

(Q3) Prove a large deviation lower bound with speed \(N^2\) for the sequence of probability measures \(P(\Xi; (N) \in \cdot)\). It is preferable to identify its rate function with \(I_{orb}^\text{lib}\).

The affirmative answer to (Q2) shows \(\chi_{orb} \leq -i^\ast\). On the other hand, as we saw in Proposition 4.2, the affirmative complete answer to (Q3) enables one to define \(\chi_{orb}\) independently of the choice of approximating sequence at least when \(\sigma = \sigma_0\) or when \(\sigma_0\) is the ‘empirical distribution’ of a freely independent family as in (Q2). Also, the affirmative complete answers to both (Q2) and (Q3) show \(\chi_{orb} = -i^\ast\). Finally, the affirmative answer to (Q2) or (Q3) solves (Q1) in the affirmative; hence (Q1) is a test for both (Q2) and (Q3).
In this section, we will solve (Q1) of Remark 4.4 in the affirmative.

**Lemma 5.1.** The limit \( \sigma_0^n := \lim_{\tau \rightarrow \infty} \pi_\tau^* (\sigma_0^I) \) exists in \( TS(C_R^* (\mathcal{X}_{\bullet})) \), and we have

1. \( \sigma_0^n \) agrees with \( \sigma_0 \) on each \( C_R^* (\mathcal{X}_{\bullet}) \), \( i = 1, \ldots, n + 1 \);
2. the \( \mathcal{X}_n^I \), \( 1 \leq i \leq n + 1 \), are freely independent.

**Proof.** By construction it is clear that \( \pi_\tau^* (\sigma_0^I) \) agrees with \( \sigma_0 \) on \( C_R^* (\mathcal{X}_{\bullet}) \) for each \( 1 \leq i \leq n + 1 \). Hence (i) trivially holds. Thus it suffices to prove only (ii).

Let \( (\mathcal{M}, \tau) \) be a tracial \( W^* \)-probability space and \( \mathcal{N} \subset \mathcal{M} \) be a \( W^* \)-subalgebra. Let \( \{v_i(t)\}_{t=1}^n \) be a \( \ast \)-freely independent family of free left unitary Brownian motions in \( \mathcal{M} \) such that the family is \( \ast \)-freely independent of \( \mathcal{N} \). Set \( v_{n+1}(t) := 1 \) for all \( t \geq 0 \) for the ease of notations. In order to prove (ii), it suffices to prove that

\[
\|E (\tau (v_{n+1}(t))^\ast v_{n+1}(T)\Pi s(D(k)_n v_{n+1}(T)^\ast))\| \leq C \left[ \sup_{n \in \mathbb{N}} \|v_n\| \right] \]
for some constant $C = C(P) > 0$ depending only on $P$.

Proof. Iteratively performing the decomposition $Q = \sigma_0(Q)1 + Q^\circ$ with $Q^\circ = Q - \sigma_0(Q)1$ we observe that $P$ is a sum of a scalar and several monomials of the form:

$$Q^\circ_1 \cdots Q^\circ_m,$$

where $Q^\circ_\ell \in C(x_{i\ell})$ with $\sigma_0(Q^\circ_\ell) = 0$ such that $m \geq 1$ and $i_\ell \neq i_{\ell+1}$ $(1 \leq \ell \leq m - 1)$. Hence we may and do assume that $P = Q^\circ_1 \cdots Q^\circ_m$ in what follows, since any scalar term vanishes under $D_s^\ell$. We also observe that each $\delta_s^k \pi_T(Q^\circ_\ell)$, $1 \leq \ell \leq m$, becomes

$$\pi_T(Q^\circ_\ell) v_k(T-s) \otimes v_k(T-s)^* - v_k(T-s) \otimes v_k(T-s)^* \pi_T(Q^\circ_\ell) \quad (k = i_\ell, s \leq T),$$

(otherwise).

Hence we may and do restrict our consideration to the case $s \leq T$, and obtain that

$$Z^k(s) := E_{\mathcal{N}(\tau)}(\pi_T((D_s^k) \pi_T(P)))) = \sum_{\ell=1}^m [Z^k_\ell(s), (Q^\circ_\ell)_s],$$

where $Z^k_\ell(s)$ is defined to be 0 when $i_\ell \neq k$; otherwise to be

$$\begin{align*}
\left\{ \begin{array}{ll}
E_{\mathcal{N}(\tau)}(w_{i_1,i_2+1}(Q^\circ_{i_2+1})_{s} \cdots w_{i_{m-1},i_m}(Q^\circ_m)_s w_{i_m,i_1}(Q^\circ_1)_s w_{i_1,i_2} \cdots (Q^\circ_{i_{m-1}})_s w_{i_{m-1},i_1}) & (i_m \neq i_1), \\
E_{\mathcal{N}(\tau)}(w_{i_1,i_1+1}(Q^\circ_{i_1+1})_{s} \cdots w_{i_{m-1},i_m}(Q^\circ_m)_s w_{i_m,i_1}(Q^\circ_1)_s w_{i_1,i_2} \cdots (Q^\circ_{i_{m-1}})_s w_{i_{m-1},i_1}) & (i_m = i_1),
\end{array} \right.
\end{align*}$$

and we write $w_{i_{\ell'},\ell'} := v^\ell(T-s)^* v^\ell(T-s) (1 \leq i \neq i' \leq n + 1)$. (n.b. $v_{i+1}^\ell(t) := 1$ for all $t \geq 0$ and $(Q)_s := \pi_T(\sigma_0(Q))$ for $Q \in C(x_{i\bullet})$. By [29, Proposition 5.7(1),(2)], which still holds for $\mathcal{D}^\infty_{i\infty}$ without any essential changes, $\mathcal{D}^\infty_{i\infty}(\tau) < +\infty$ guarantees that $\bar{\tau}(Q)_s = \sigma_0(Q)$ for all $Q \in C(x_{i\bullet})$ with each fixed $i = 1, \ldots, n + 1$. Hence the first case $i_m \neq i_1$ can be treated essentially in the same way as in the proof of Lemma 5.1. Namely, when $i_m \neq i_1$ (and $i_\ell = k$), we have, for any $y \in \mathcal{N}(\tau)$ (see subsection 4.4 for this notation),

$$\begin{align*}
\tau(y Z^k_\ell(s)) &= \tau(y w_{i_1,i_2+1}(Q^\circ_{i_2+1})_{s} \cdots w_{i_{m-1},i_m}(Q^\circ_m)_s w_{i_m,i_1}(Q^\circ_1)_s w_{i_1,i_2} \cdots (Q^\circ_{i_{m-1}})_s w_{i_{m-1},i_1}) \\
&\quad + \tau(y w_{i_1,i_1+1}(Q^\circ_{i_1+1})_{s} \cdots w_{i_{m-1},i_m}(Q^\circ_m)_s w_{i_m,i_1}(Q^\circ_1)_s w_{i_1,i_2} \cdots (Q^\circ_{i_{m-1}})_s w_{i_{m-1},i_1}),
\end{align*}$$

and obtain that

$$Z^k_\ell(s) = \begin{align*}
\bar{\tau}(w_{i_1,i_2+1}) E_{\mathcal{N}(\tau)}((Q^\circ_{i_2+1})_{s} \cdots w_{i_{m-1},i_m}(Q^\circ_m)_s w_{i_m,i_1}(Q^\circ_1)_s w_{i_1,i_2} \cdots (Q^\circ_{i_{m-1}})_s w_{i_{m-1},i_1}) \\
+ E_{\mathcal{N}(\tau)}((w_{i_1,i_1+1})^2(Q^\circ_{i_1+1})_{s} \cdots w_{i_{m-1},i_m}(Q^\circ_m)_s w_{i_m,i_1}(Q^\circ_1)_s w_{i_1,i_2} \cdots (Q^\circ_{i_{m-1}})_s w_{i_{m-1},i_1})
\end{align*}$$

with $(w_{i_1,i_1'})^2 := w_{i_1,i'} - \bar{\tau}(w_{i_1,i'})$. Making the same computation for the second term and iterating this procedure until $w_{i_{\ell-2},i_{\ell-1}}$, we finally arrive at the following formula: $Z^k_\ell(s)$ is the sum of $\bar{\tau}(w_{i_1,i_{\ell-1}})$ times

$$E_{\mathcal{N}(\tau)}((w_{i_1,i_2})^\ell(Q^\circ_{i_2})_{s} \cdots (w_{i_{\ell-1},i_\ell})^\ell(Q^\circ_\ell)_s w_{i_\ell,i_{\ell+1}}(Q^\circ_{i_{\ell+1}})_s w_{i_{\ell+1},i_{\ell+2}} \cdots (Q^\circ_{i_{\ell-1}})_s w_{i_{\ell-1},i_\ell})$$

over all $j = \ell, \ldots, m, 1, \ldots, \ell - 2$ (where we read $m + 1$ as 1). Therefore, we have obtained that

$$\|Z^k_\ell(s)\|_\infty \leq (2^{m-1} - 1) \left( \sup_{1 \leq j \leq m} \|Q^\circ_j\| \right)^{m-1} e^{(s-T)/2}.$$
since \(\|\langle w_{i,i'}\rangle\|_\infty \leq 2\) and \(0 \leq \tilde{\tau}(w_{i,i'}) = \tilde{\tau}(v_i^e(T-s)^*v_i^e(T-s)) \leq e^{(s-T)/2}\) with \(i \neq i'\) \(\) (see (5.1) for a similar computation). Hence we get

\[
(3.3) \quad \|\{Z(t_i^e)\}_s, (Q(t_i^e))_s\|_\infty \leq C_1 e^{(s-T)/2}
\]

with a positive constant \(C_1\) depending only on \(P\) and \(\ell\).

We then consider the case \(i_m = i_1\) (and \(s \leq T\)). This case is a bit complicated, but can still be treated similarly as above. In fact, if \(i_{m-1} \neq i_2\), then

\[
Z(t_i^e) = \tilde{\tau}(\{Q_i^e Q_{i+1}^e\}_s) E_N(\tau)(w_{i_1,i_2+1}(Q_{i+1}^e) \cdots w_{i_{m-1},i_2} \cdots (Q_{i}^e) w_{i_{m-2},i_1}) + E_N(\tau)(w_{i_1,i_2+1}(Q_{i+1}^e) \cdots w_{i_{m-1},i_2} \cdots (Q_{i}^e) w_{i_{m-2},i_1})\]

since \(w_{i_{m-1},i_2} w_{i_{m-2},i_2} = w_{i_{m-1},i_2}\). Thus, we apply the previous procedure to the first and the second terms, respectively, and conclude

\[
\|\{Z(t_i^e)\}_s, (Q(t_i^e))_s\|_\infty \leq \left\{ (2^{m-3} - 1) + (2^{m-2} - 1) \right\} \left( \sup_{1 \leq j \leq m} \|Q(t_j^e)\|_\infty \right)^{m-1} e^{(s-T)/2}.
\]

Iterating this procedure in the cases \(i_m = i_1, i_{m-1} = i_2\) and \(i_{m-2} \neq i_3\), we can estimate \(\|Z(t_i^e)\|_\infty\) by \(e^{(s-T)/2}\) times a positive constant only depending on \(P\) except the case when \(i_m = i_1, i_{m-1} = i_2, \ldots, i_{k+1} = i_{k-1}\) \((\text{i.e.}, m\) is odd and \(\ell = (m+1)/2\)). In the remaining case, we can easily observe that

\[
Z(t_i^e) = \sigma_0(\{Q_m^e Q_1^e\})_s \sigma_0(\{Q_{m-1}^e Q_2^e\})_s \cdots \sigma_0(\{Q_{i+1}^e Q_{i}^e\})_s + Z(t_i^e)\]

with an element \(Z(t_i^e)\) \(\in N(\tau)\) whose operator norm \(\|Z(t_i^e)\|_\infty\) is not greater than \(e^{(s-T)/2}\) times a positive constant only depending on \(P\). Then we have

\[
(3.4) \quad \|\{Z(t_i^e)\}_s, (Q(t_i^e))_s\|_\infty = \|\{Z(t_i^e)\}_s, (Q(t_i^e))_s\|_\infty \leq 2\|Z(t_i^e)\|_\infty \|Q(t_i^e)\|_\infty \leq C_2 e^{(s-T)/2}.
\]

with a positive constant \(C_2\) depending only on \(P\) and \(\ell\).

Consequently, the expansion (5.2) of \(Z(t_i^e)\) together with the above norm estimates (5.3), (5.4) shows the desired norm estimate. \(\square\)

A more explicit description on \(E_N(\tau)(\pi_\tau(\Pi^e(\mathcal{P}))\) is possible based on the combinatorial techniques introduced by Speicher (see e.g. Nica–Speicher [23] as a standard textbook). See section 8.

With the above lemmas we will prove that the rate function \(J(t_n)\) admits a unique minimizer, and moreover, we will explicitly compute the minimizer. Moreover, we will also prove that the modification \(J(t_n)\) of \(J(t_n)\) by replacing \(J(t_n)\) with \(J(t_n)\), i.e.,

\[
J(t_n, \infty) = \lim_{t_n \to \infty} \lim_{t \to \infty} \inf \{J(t_n, \infty) | \tau \in TS^e(C_R(x_n, *)), \pi^{\tau}(\tau) \in O(n, b)(\sigma)\}
\]

admits the same unique minimizer.

**Theorem 5.3.** For any \(\sigma \in TS(C_R(x_n, *))\) the following are equivalent:

(1) \(\sigma = \sigma_0^e\)

(2) \(J(t_n^e) = 0\)

(3) \(J(t_n^e) = 0\)

**Proof.** (1) \(\Rightarrow\) (2): Since \(J(t_n^e) = 0\) and moreover since \(\pi^{\tau}(\sigma_0^e) \rightarrow \sigma_0^e\) as \(T \to +\infty\) by Lemma 5.1, we have \(J(t_n, \infty) = 0\).

(2) \(\Rightarrow\) (3): Trivial because \(0 \leq J(t_n, \infty) \leq J(t_n^e)\), which follows from \(0 \leq J(t_n, \infty) \leq J(t_n^e)\).
Thus we can choose a sequence $0 < T_1 < T_2 < \cdots < T_m \nearrow +\infty$ as $m \nearrow +\infty$ and $\tau_{m,n} \in TS(C^*(\mathcal{H}_{m,n}(-)))$ for each $m \in \mathbb{N}$ such that $\pi^{\text{fr}}_{T_m}(\tau_{m,n}) \in O_{m,1/m}(\sigma)$ and $\pi^{\text{lib}}_{\sigma_{0,\infty}}(\tau_{m,n}) < 1/m$ for every $m \in \mathbb{N}$. For each $P = P^* \in C(\mathcal{A})$ we have

\[
|\sigma(P) - \sigma^0(P)| \leq \left| \sigma(P) - \pi^{\text{fr}}_{T_m}(\tau_{m,n})(P) \right| + \left| \pi^{\text{fr}}_{T_m}(\tau_{m,n})(P) - \sigma^0(\pi^{\text{fr}}_{T_m}(\tau_{m,n}))(P) \right| + \left| \sigma^0(\pi^{\text{fr}}_{T_m}(\tau_{m,n}))(P) - \sigma^0(P) \right|
\]

\[
+ \frac{2\pi^{\text{lib}}_{\sigma_{0,\infty}}(\tau_{m,n}) \sum_{k=1}^{\infty} \left\| E_{\mathcal{N}(\tau)}(\pi^{\text{fr}}_{T_m}(\tau_{m,n}))(P) \right\|_{2}^{2} ds}
\]

by [29, Lemma 5.3] that still holds true for $\pi^{\text{lib}}_{\sigma_{0,\infty}}$ without any essential changes. Now, we use Lemma 5.2 to get

\[
\sum_{k=1}^{\infty} \int_{0}^{\infty} \left\| E_{\mathcal{N}(\tau)}(\pi^{\text{fr}}_{T_m}(\tau_{m,n}))(P) \right\|_{2}^{2} ds \leq C \int_{0}^{T_m} e^{s-T_m} ds = C(1-e^{-T_m}) \leq C
\]

for all $m$ with a constant $C > 0$ only depending on $P$. Consequently, we obtain that

\[
|\sigma(P) - \sigma^0(P)| \leq |\sigma(P) - \pi^{\text{fr}}_{T_m}(\tau_{m,n})(P)| + |\pi^{\text{fr}}_{T_m}(\tau_{m,n})(P) - \sigma^0(\pi^{\text{fr}}_{T_m}(\tau_{m,n}))(P)| + \frac{2C}{m}
\]

whose right-hand side converges to 0 as $m \to \infty$ thanks to $\pi^{\text{fr}}_{T_m}(\tau_{m,n}) \in O_{m,1/m}(\sigma)$ (that guarantees that $\sigma = \lim_{m \to \infty} \pi^{\text{fr}}_{T_m}(\tau_{m,n})$ in $TS(C^*(\mathcal{H}_{m,n}(-)))$) and Lemma 5.1. Hence we conclude that $\sigma = \sigma^0$. \hfill \Box

Thanks to the standard Borel-Cantelli argument (see e.g. the proof of [29, Corollary 5.9]) the above proposition together with Corollary 4.3 implies that $\tau_{\pi^{(N)}_{\mathcal{A}}}$ converges to $\sigma^0$ almost surely as $N \to \infty$. This is nothing less than a consequence of the asymptotic freeness of independent Haar-distributed unitary random matrices. On the other hand, the corresponding result for the matrix liberation process [29, Corollary 5.9] was not known prior to it.

We would also like to point out that both $\pi^{\text{lib}}_{\sigma_{0,\infty}}$, $\pi^{\text{fr}}_{\sigma_{0,\infty}}$ can be regarded as a kind of mutual information in free probability, since they characterize the free independence as a unique minimizer (see the third paragraph of section 1). Thus it is natural to reformulate the functionals $\pi^{\text{lib}}_{\sigma_{0,\infty}}$, $\pi^{\text{fr}}_{\sigma_{0,\infty}}$ as well as their sources $\pi^{\text{lib}}_{\sigma_{0,\infty}}$, $\pi^{\text{fr}}_{\sigma_{0,\infty}}$ in a coordinate-free fashion. This will be done in the next section.

6. A COORDINATE-FREE APPROACH: A NEW KIND OF FREE MUTUAL INFORMATION

Let $(\mathcal{M}, \tau)$ be a tracial $W^*$-probability space. We consider unital $C^*$-subalgebras $\mathcal{A}_i \subset \mathcal{M}$, $1 \leq i \leq n+1$, and define a kind of free mutual information $i^*(A_1; \ldots; A_n : A_{n+1})$, without appealing to any kind of (matricial) microstates, whose definition comes from the rate functions discussed so far.

6.1. Universal algebras. Let $\mathfrak{A} := \star \bigwedge_{i=1}^{n+1} \mathcal{A}_i$ be the universal free product $C^*$-algebra. Let $\mathfrak{A}(t)$, $t \geq 0$, be copies of $\mathfrak{A}$, and define $\mathfrak{A}(\mathbb{R}_+)$ to be the universal free product $C^*$-algebra $\star_{t \geq 0} \mathfrak{A}(t)$. (Here we write $\mathbb{R}_+ = [0, +\infty)$.) We denote by $\lambda_i : \mathcal{A}_i \to \mathfrak{A}$ and $\rho_t : \mathfrak{A} \to \mathfrak{A}(t) \subset \mathfrak{A}(\mathbb{R}_+)$ the canonical $*$-homomorphisms, which are known to be injective, see the appendix for an explicit reference about this fact. Write $\rho_{t,\lambda} := \rho_t \circ \lambda_i : \mathcal{A}_i \to \mathfrak{A}(\mathbb{R}_+)$. By Lemma A.1, $\mathfrak{A}(\mathbb{R}_+)$ with
**-homomorphisms \( \rho_{t,i} \) can naturally be identified with the universal free product of the copies of \( A_i \), \( 1 \leq i \leq n + 1 \), over \( \mathbb{R}_+ \).

### 6.2. Time-dependent liberation derivatives

Let \( \mathcal{P} \) be the \( * \)-subalgebra of \( \mathfrak{A} \) algebraically generated by \( \lambda_i(A_i) \), \( 1 \leq i \leq n + 1 \). Consider the \( * \)-subalgebra \( \mathcal{P}(\mathbb{R}_+) \) of \( \mathfrak{A}(\mathbb{R}_+) \) algebraically generated by \( \rho_i(\mathcal{P}) \), \( t \geq 0 \). Remark that \( \lambda_i(A_i) \), \( 1 \leq i \leq n + 1 \), and \( \rho_{t,i}(A_i) \), \( 1 \leq i \leq n + 1 \), \( t \geq 0 \), are algebraically free families of \( * \)-subalgebras, and the resulting \( \mathcal{P} \) and \( \mathcal{P}(\mathbb{R}_+) \) are naturally identified with the algebraically free products of the \( \lambda_i(A_i) \), \( 1 \leq i \leq n + 1 \), and of the \( \rho_{t,i}(A_i) \), \( 1 \leq i \leq n + 1 \), \( t \geq 0 \), respectively. See Proposition 4.4.

We extend \( \mathfrak{A}(\mathbb{R}_+) \) to \( \mathfrak{A}(\mathbb{R}_+) \) by taking its universal free product with the universal \( C^* \)-algebra generated by \( u_i(t) \), \( 1 \leq i \leq n \), \( t \geq 0 \) with subject to \( u_i(t)u_i(t)^* = 1 \) and \( u_i(0) = 1 \). This procedure is justified by Proposition 3.3. Consider the derivation \( \Delta_{k} : \mathcal{P}(\mathbb{R}_+) \to \mathfrak{A}(\mathbb{R}_+) \otimes_{\text{alg}} \mathfrak{A}(\mathbb{R}_+) \), \( 1 \leq k \leq n \), sending each \( \rho_{t,i}(x) \) with \( x \in A_i \) to

\[
\delta_{i,k}1_{[0,2]}(s)\rho_{t,k}(x)(u_k(t-s)\otimes u_k(t-s)^* - u_k(t-s)\otimes u_k(t-s)^*\rho_{t,k}(x))
\]

(\( n.b. \), the algebraic freeness among the \( \rho_{t,i}(A_i) \) makes every \( \Delta_{k} \) well-defined). Therefore, with the flip-multiplication map \( \theta : \mathfrak{A}(\mathbb{R}_+) \otimes_{\text{alg}} \mathfrak{A}(\mathbb{R}_+) \to \mathfrak{A}(\mathbb{R}_+) \) sending \( a \otimes b \) to \( ba \), we obtain the cyclic derivative \( \nabla^{(k)} := \theta \circ \Delta^{(k)} : \mathcal{P}(\mathbb{R}_+) \to \mathfrak{A}(\mathbb{R}_+) \).

### 6.3. Continuous tracial states

Differently from the previous sections we will use symbols \( \varphi, \psi, \) etc., instead of \( \tau \) for tracial states on \( \mathfrak{A}(\mathbb{R}_+) \), etc., in order to avoid any confusion of symbols.

A tracial state \( \varphi \in TS(\mathfrak{A}(\mathbb{R}_+)) \) is said to be continuous, if \( t \to \varphi(\rho_t(x)) \) is strongly continuous for every \( x \in \mathfrak{A} \), where \( \pi_\varphi : \mathfrak{A}(\mathbb{R}_+) \to \mathcal{H}_\varphi \) denotes the GNS representation associated with \( \varphi \). In what follows, we denote by \( TS^c(\mathfrak{A}(\mathbb{R}_+)) \) all the continuous tracial states on \( \mathfrak{A}(\mathbb{R}_+) \).

**Lemma 6.1.** For a given \( \varphi \in TS(\mathfrak{A}(\mathbb{R}_+)) \) the following are equivalent:

(i) \( \varphi \) is continuous.

(ii) For every \( m \in \mathbb{N} \) and every \( x_1, \ldots, x_m \in \mathfrak{A} \) the function

\[
(t_1, \ldots, t_m) \to \varphi(\rho_{t_1}(x_1) \cdots \rho_{t_m}(x_m))
\]

is continuous.

(iii) For every \( m \in \mathbb{N} \) and every \( x_k \in A_i, 1 \leq i_k \leq n + 1, 1 \leq k \leq m \), the function

\[
(t_1, \ldots, t_m) \to \varphi(\rho_{t_1,i_1}(x_1) \cdots \rho_{t_m,i_m}(x_m))
\]

is continuous.

(iv) For every \( 1 \leq i \leq n + 1 \), there exists a \( C^* \)-generating set \( \mathcal{X}_i \) consisting of self-adjoint elements in \( A_i \) such that for every \( m \in \mathbb{N} \) and every \( x_j \in \mathcal{X}_j, 1 \leq i_j \leq n + 1, 1 \leq j \leq m \), the function

\[
(t_1, \ldots, t_m) \to \varphi(\rho_{t_1,i_1}(x_1) \cdots \rho_{t_m,i_m}(x_m))
\]

is continuous.

**Proof.** Since \( \|\rho_t(x)\|_\infty = \|x\|_\infty \) for every \( x \in \mathfrak{A} \) and since the \( \rho_t(\mathfrak{A}) \) over \( t \geq 0 \) generate \( \mathfrak{A}(\mathbb{R}_+) \) as a \( C^* \)-algebra, the proof of [29, Lemma 2.1] works for showing that item (i) \( \Leftrightarrow \) item (ii) without any essential changes. Item (ii) \( \Rightarrow \) item (iii) is trivial. The standard approximation argument using the norm density of the unital \( * \)-algebra algebraically generated by \( \lambda_i(A_i) \) in \( \mathfrak{A} \) shows that item (iii) \( \Rightarrow \) item (ii). Item (iii) \( \Leftrightarrow \) item (iv) is also confirmed similarly by using the norm density of the unital \( * \)-algebra algebraically generated by \( \mathcal{X}_i \) in \( A_i \).

We extend each \( \varphi \in TS^c(\mathfrak{A}(\mathbb{R}_+)) \) to a unique \( \hat{\varphi} \in TS(\mathfrak{A}(\mathbb{R}_+)) \) in such a way that the \( u_i(t) \)'s are \( * \)-freely independent of \( \mathfrak{A}(\mathbb{R}_+) \) and form a \( * \)-freely independent family of left-multiplicative free unitary Brownian motions under this extension \( \hat{\varphi} \). It is not difficult to see that \( \varphi \) is ‘continuous’,
that is, both $t \mapsto \pi_{\phi}(\rho_t(x))$ with $x \in A$ and $t \mapsto \pi_{\phi}(\rho_t(x))$ with $x \in A_i$, $t \geq 0$ to

$$\rho_{t,n+1}(x) := \begin{cases} u_i((t-s) \vee 0) \rho_{t,n+1}(x) & (1 \leq i \leq n), \\ \rho_{t,n+1}(x) & (i = n+1) \end{cases}$$

and keeping each $u_i(t)$ as it is. Note that each $\rho_{t,i}$ clearly defines a unital $*$-homomorphism from $A$ to $\mathfrak{A}(R_+)$ for every $1 \leq i \leq n+1$, and moreover, by universality, those $\rho_{t,i}$ give rise to a unital $*$-homomorphism $\rho_t^s : A \to \mathfrak{A}(R_+)$. Observe that $\Lambda^* \circ \rho_t^s$ holds for every $s, t \geq 0$. We define $\varphi^s := \tilde{\varphi} \circ \Lambda^*$ on $\mathfrak{A}(R_+)$. Since

$$\tilde{\varphi} \circ \Lambda^*(\rho_{t,i}(x_1) \cdots \rho_{t,n+1}(x_m)) = \tilde{\varphi}(\rho_{t_1,i_1}(x_1) \cdots \rho_{t_n,i_n}(x_m)),$$

we observe, by (6.1), that $\varphi^s$ is a continuous tracial state.

By the $*$-homomorphism $\Gamma : \mathfrak{A}(R_+) \to \mathfrak{A}$ sending each $\rho_{t,i}(x) \in A_i$ to $\lambda_i(x)$ we construct $\Gamma^*(\sigma_0) := \sigma_0 \in \operatorname{TS}^*(\mathfrak{A}(R_+))$ with a given $\sigma_0 \in \operatorname{TS}(\mathfrak{A})$ and set $\sigma_0^\lambda := \Gamma^*(\sigma_0) \in \operatorname{TS}^*(\mathfrak{A}(R_+))$.

### 6.4. The new free mutual information.

For a given $\sigma_0 \in \operatorname{TS}(\mathfrak{A})$ let us define two functionals $T_{\sigma_0}^{\text{lib}}, T_{\sigma_0}^{\text{lib}} : \operatorname{TS}^*(\mathfrak{A}(R_+)) \to [0, +\infty)$ as follows. Let $\varphi \in \operatorname{TS}^*(\mathfrak{A}(R_+))$ be arbitrarily given. Let $E_{\sigma,\varphi}^2$ denote the $\varphi$-preserving conditional expectation from $\sigma_0$ onto $\sigma_0^\lambda$ of words like $\rho_t(x)$, where the double commutants are taken on $\mathfrak{H}_\varphi$. For any $P = P^* \in \mathfrak{P}(R_+)$ and $t \in [0, \infty)$ we define

$$T_{\sigma_0}^{\text{lib}}(\varphi, P) = \varphi^s(P) - \sigma_0^\lambda(P) - \frac{1}{2} \sum_{k=1}^{\infty} \int_0^t \| E_{\sigma,\varphi}^2(\pi_{\phi}(\Lambda^*(\nabla_s^{(k)}P))) \|^2_{\varphi,2} \, ds$$

with regarding $\varphi$ as $\varphi^\infty$ (since $\varphi^s(\varphi^s(P) = \varphi(P)$ when $t$ is large enough). We observe that $s \mapsto \| E_{\sigma,\varphi}^2(\pi_{\phi}(\Lambda^*(\nabla_s^{(k)}P))) \|^2_{\varphi,2}$ is piecewise continuous in $s$ and becomes zero when $s$ is large enough thanks to $P \in \mathfrak{A}(R_+)$. These two facts guarantee that $T_{\sigma_0}^{\text{lib}}(\varphi, P)$ is well defined for every $t$ possibly with $t = \infty$. Then we define

$$T_{\sigma_0}^{\text{lib}}(\varphi) = \sup_{P = P^* \in \mathfrak{P}(R_+)} T_{\sigma_0}^{\text{lib}}(\varphi, P), \quad T_{\sigma_0}^{\text{lib}}(\varphi) = \sup_{P = P^* \in \mathfrak{P}(\mathfrak{A}(R_+))} T_{\sigma_0}^{\text{lib}}(\varphi, P).$$

Clearly, $T_{\sigma_0}^{\text{lib}}(\varphi) \geq T_{\sigma_0}^{\text{lib}}(\varphi)$ holds, and it is a question again whether equality holds or not.

We then introduce two functionals $T_{\sigma_0}^{\text{lib}}, T_{\sigma_0}^{\text{lib}} : \operatorname{TS}(\mathfrak{A}) \to [0, +\infty]$ as before. To this end, we have to endow $\operatorname{TS}(\mathfrak{A})$ with the weak* topology. Let $\sigma \in \operatorname{TS}(\mathfrak{A})$ be arbitrarily given. Let $\mathcal{O}(\sigma)$ be the open neighborhoods at $\sigma$ in the weak* topology on $\operatorname{TS}(\mathfrak{A})$. Then we define

$$T_{\sigma_0}^{\text{lib}}(\sigma) := \sup_{\sigma \in \mathcal{O}(\sigma)} \inf_{t \geq 0} \{ T_{\sigma_0}^{\text{lib}}(\varphi) \mid \varphi \in \operatorname{TS}^*(\mathfrak{A}(R_+)), \rho_t^s(\varphi) \in \sigma \}$$

and also $T_{\sigma_0}^{\text{lib}}(\sigma)$ in the same manner as above with replacing $T_{\sigma_0}^{\text{lib}}(\varphi)$ with $T_{\sigma_0}^{\text{lib}}(\varphi)$. Here the infimum over the empty set is taken to be $+\infty$ as usual. Remark that the supremum over $\mathcal{O}(\sigma)$ coincides with the limit over a neighborhood basis at $\sigma$. We also remark that $\mathcal{O}(\sigma)$ can be replaced with the smaller neighborhood basis consisting of

$$O_{W,\delta}(\sigma) := \{ \sigma' \in \operatorname{TS}(\mathfrak{A}) \mid |\sigma'(W) - \sigma(W)| < \delta \text{ for all } W \in W \}$$

all over the finite collections $W$ of words $W$ like $\lambda_i(a_1) \cdots \lambda_m(a_m)$ with $a_i \in A_i$ and $\delta > 0$, since all the linear combinations of words form a norm dense $*$-subalgebra of $\mathfrak{A}$. 
Definition 6.1. Thanks to the universality of $\mathfrak{A}$, we have a unique $*$-homomorphism $\Upsilon : A \to M$ sending each $\lambda_i(x_i)$ to $x_i$ with $x_i \in A_i \subset M_i$, $1 \leq i \leq n+1$. Then we define

$${\mathcal J}_0^{\text{lib}}(A_1; \ldots ; A_n : A_{n+1}) : = \mathcal J^{\text{lib}}_{\sigma_0}(\Upsilon) \geq \mathcal J^{\text{lib}}_{\sigma_{0,\infty}}(\Upsilon) =: \mathcal J^{\text{lib}}_{\sigma_{0,\infty}}(A_1; \ldots ; A_n : A_{n+1}).$$

Moreover, we write

$$i^*(A_1; \ldots ; A_n : A_{n+1}) : = \mathcal J^{\text{lib}}_{\sigma_{0,\infty}}(A_1; \ldots ; A_n : A_{n+1}).$$

These quantities will be shown to satisfy that (i) characterizing free independence, (ii) invariance under taking closure $\bar{\mathfrak{A}}$, and (iii) the monotonicity in $A_i$. Hence they can be understood as a kind of mutual information in free probability. Here is a remark on the choice of $\sigma_0$.

Remark 6.2. If $\mathcal J^{\text{lib}}_{\sigma_0}(A_1; \ldots ; A_n : A_{n+1})$ is finite, then $\lambda_i^*(\sigma_0)$ must agree with $\tau$ on $A_i$ for every $1 \leq i \leq n+1$.

Proof. Assume that $\lambda_i^*(\sigma_0)$ does not agree with $\tau$ for some $i$. Namely, there is an element $x_i \in A_i$ such that $\sigma_0(\lambda_i(x_i)) \neq \tau(x_i)$. Remark that $\tau(x_i) = \Upsilon^*(\tau)(\lambda_i(x_i))$. Then we can choose an open neighborhood $O \in \mathcal{O}(\Upsilon^*(\tau))$ in such a way that $\sigma(\lambda_i(x_i)) \neq \sigma_0(\lambda_i(x_i))$ for every $\sigma \in O$. As in the proof of [29, Proposition 5.7] we have

$$\tau(\rho_{\tau}(\varphi)(\lambda_i(x_i)) - \sigma_0(\lambda_i(x_i))) = T^{\text{lib}}(\varphi, \rho_{\tau,i}(x_i)) \leq T^{\text{lib}}_{\sigma_0,\infty}(\varphi)$$

for all $\varphi \in \mathbb{R}$ and $T \geq 0$. It follows that $T^{\text{lib}}_{\sigma_0,\infty}(\varphi) = +\infty$ as long as $\rho_{\tau}(\varphi) \in O$. It follows that $\mathcal J^{\text{lib}}_{\sigma_0}(A_1; \ldots ; A_n : A_{n+1}) = \mathcal J^{\text{lib}}_{\sigma_0}(\Upsilon^*(\tau)) \geq \mathcal J^{\text{lib}}_{\sigma_0,\infty}(\Upsilon^*(\tau)) = +\infty$. $\square$

Consequently, we will assume that $\lambda_i^*(\sigma_0)$ agrees with $\tau$ on $A_i$ for every $1 \leq i \leq n+1$ throughout the rest of this section. In particular, the natural two choices of $\sigma_0$ are $\Upsilon^*(\tau)$ and the so-called free product state $\lambda_{i+1}^*(\lambda_i^{-1})^*(\tau)$.

6.5. Relation to the matrix liberation process. Assume that each $A_i$, $1 \leq i \leq n+1$, is generated by a self-adjoint random multi-variable $X_i := (X_{ij})_{i=1}^{\infty}$ as in section 3, that is, $A_i = C^*(X_i)$. Assume further that $R := \sup_{ij} \|X_{ij}\|_2 < +\infty$. Then we have two unique surjective unital $*$-homomorphisms $\Phi : C^*_R(X_\circ) \to \mathfrak{A}$ and $\Psi : C^*_R(x_\circ(\cdot), v_\circ(\cdot)) \to \mathfrak{A}(\mathbb{R}_+)$ sending $x_{ij}$, $x_{ij}(t)$ and $v_{ij}(t)$ to $\lambda_i(\lambda_j(x_{ij}))$, $\rho_{ij}(\lambda_i(x_{ij}))$ and $u_{ij}(t)$, respectively. Clearly, $\Psi(C^*_R(x_\circ(\cdot), v_\circ(\cdot))) = \mathfrak{A}(\mathbb{R}_+)$ and $\Psi(x_{ij}(t)) = \rho_{ij}(\Phi(x_{ij}(t)))$ hold. In particular, the latter implies that $\Psi \circ \tau_0 = \rho_0 \circ \Phi$.

For the reader’s convenience we summarize the notations of algebras and maps that we have introduced so far. The algebras and the maps between them are:

$$\begin{array}{cccc}
C^*_R(x_\circ) & \xrightarrow{\psi} & C^*_R(x_\circ(\cdot), v_\circ(\cdot)) & \xrightarrow{\psi} & C^*_R(x_\circ(\cdot), v_\circ(\cdot)) \\
\Phi & \downarrow & \Psi & \downarrow & \Psi \\
A_i & \xrightarrow{\lambda_i} & \mathfrak{A} & \xrightarrow{\rho_{ij}} & \mathfrak{A}(\mathbb{R}_+) & \xrightarrow{\lambda_{i+1}^*} & \mathfrak{A}(\mathbb{R}_+)
\end{array}$$

The liberation cyclic derivatives $\nabla^{(k)}_s$ (see subsection 4.2) and the maps $\Pi^*$ (see subsection 4.5) on the upper line of the above diagram correspond to $\nabla^{(k)}_s$ (see subsection 6.2) and $\Lambda^*$ (see subsection 6.3) on the lower line, respectively. Moreover, the spaces of (continuous) tracial states and the
dual maps between them are:

\[ TS(C^*_R(x \circ \bullet (\cdot))) \xrightarrow{\Phi^*} TS^a(C^*_R(x \circ \bullet (\cdot))) \xrightarrow{\pi^*} TS^a(C^*_R(x \circ \bullet (\cdot), v_\bullet (\cdot))) \]

\[ TS(A) \xrightarrow{\lambda^*_t} TS(A) \xrightarrow{\varphi^*} TS^a(\mathbb{R}(\mathbb{R}_+)) \xrightarrow{\varphi \circ \tilde{\varphi}} TS^a(C^*_R(x \circ \bullet (\cdot), v_\bullet (\cdot))). \]

Lemma 6.3. For any \( \varphi \in TS^a(\mathbb{R}(\mathbb{R}_+)) \) we have \( \Psi^*(\varphi) := \varphi \circ \Psi \in TS^a(C^*_R(x \circ \bullet (\cdot))) \) and \( \Psi^*(\tilde{\varphi}) = \Psi^*(\varphi)^- \) holds for every \( s \geq 0 \). Moreover, for any \( P \in \mathbb{C}(x \circ \bullet (\cdot)) \), we have

\[ \| E_Q(\varphi)(\lambda^*(\nabla_{\tilde{\varphi}}^k(\Psi(P))))\|_{\varphi,2} = \| E_N(\Psi^*(\varphi) - (\Lambda^*(\tilde{\varphi}^k)P))\|_{\Psi^*(\varphi)^-;2} \]

for every \( 1 \leq k \leq n \) and \( s \geq 0 \).

Proof. Observe that

\[ \Psi^*(\varphi)(x_{i,j}(t_1) \cdots x_{m,n}(t_m)) = \varphi(\rho_{1,i}(X_{i,j})) \cdots \rho_{m,i}(X_{m,n})) \]

which implies that \( \Psi^*(\varphi) \) falls in \( TS^a(C^*_R(x \circ \bullet (\cdot))) \) by [29, Lemma 2.1] and Lemma 6.1. Moreover, we have

\[ \Psi^*(\tilde{\varphi})(a_1v_1(t_1)^i \cdots a_nv_n(t_m)^i) = \tilde{\varphi}(\Psi(a_1)u_i(t_1)^i \cdots \Psi(a_n)u_i(t_m)^i) \]

for any \( a_k \in C^*_R(x \circ \bullet (\cdot)), 1 \leq k \leq n \), \( t_k \geq 0 \) and \( \epsilon_k = \pm 1 \). Since \( \Psi(C^*_R(x \circ \bullet (\cdot))) = \mathbb{R}(\mathbb{R}_+) \), we conclude that the \( v_i(t) \) are freely independent of \( C^*_R(x \circ \bullet (\cdot)) \) and form a freely independent family of left-multiplicative free unitary Brownian motions under \( \Psi^*(\tilde{\varphi}) \). Therefore, we conclude that \( \Psi^*(\tilde{\varphi}) = \Psi^*(\varphi)^- \). We observe that

\[
\Psi(\pi^*(x_{i,j}(t))) = \Psi(x_{i,j}(t))
= \begin{cases} 
\Psi(u_i((t-s) \land 0)x_{i,j}(s \land t)v_i((t-s) \land 0)^i) 
= u_i((t-s) \land 0)\rho_{s+t,i}(X_{i,j})u_i((t-s) \land 0)^i & (1 \leq i \leq n), \\
\Psi(x_{n+1,j}(t)) = \rho_{n+1,i}(X_{i,j}) & (i = n+1) \\
\rho_i^e(X_{i,j}) = \Lambda^*(\rho_i^e(X_{i,j})) = \Lambda^*(\Psi(x_{i,j}(t))) 
\end{cases}
\]

implying that \( \Psi \circ \pi^* = \Lambda^* \circ \Psi \) on \( C^*_R(x \circ \bullet (\cdot)) \). Therefore, we obtain that

\[ \Psi^*(\varphi^s) = \tilde{\varphi} \circ \Lambda^* \circ \Psi = \tilde{\varphi} \circ \Psi \circ \pi^* = \Psi^*(\tilde{\varphi}) \circ \pi^* = \Psi^*(\varphi)^- \circ \pi^* = \Psi^*(\varphi)^s. \]
Choose an arbitrary monomial \( P = x_{i_1,j_1}(t_1) \cdots x_{i_m,j_m}(t_m) \in \mathbb{C}<x_\bullet \diamond (\cdot)> \). By definition we have \( \Psi(P) = \rho_{i_1,i_1}(X_{i_1,j_1}) \cdots \rho_{i_m,i_m}(X_{i_m,j_m}) \). We observe that
\[
\Pi^*(\mathcal{O}(t_0) P) = \sum_{i_1 = k} \Pi^*([v_k(t_1 - s)^x x_{i_1,j_1} x_{i_1,j_1}(t_1 + 1) \cdots x_{i_1,j_1}(t_1 - 1) x_0(t_1 - s), x_{i_1,j_1}(s)])
\]
\[
= \sum_{i_1 = k} [v_k(t_1 - s)^x x_{i_1,j_1}(t_1 + 1) \cdots x_{i_1,j_1}(t_1 - 1) x_0(t_1 - s), x_{i_1,j_1}(s)],
\]
(6.3) \( \Lambda^*(\nabla^k(\Psi(P))) = \sum_{i_1 = k} \Lambda^*([u_k(t_1 - s)^x \rho_{i_1,i_1}(X_{i_1,j_1}) x_{i_1,j_1}(t_1 + 1) \cdots x_{i_1,j_1}(t_1 - 1) u_0(t_1 - s), \rho_{i_1,i_1}(X_{i_1,j_1})])
\]
\[
= \sum_{i_1 = k} \Lambda^*([u_k(t_1 - s)^x \rho_{i_1,i_1}(X_{i_1,j_1}) x_{i_1,j_1}(t_1 + 1) \cdots x_{i_1,j_1}(t_1 - 1) u_0(t_1 - s), \rho_{i_1,i_1}(X_{i_1,j_1})]).
\]
Since \( \Psi(\varphi)^\sim = \Psi(\tilde{\varphi}) \) and since \( \Psi(x_{ij}(t)) = \rho_{t_0,i_0}(X_{ij}) \) and \( \Psi(v_0(t)) = u_0(t) \), we observe that the joint distribution of the \( x_{ij}(t) \) and the \( v_0(t) \) under \( \Psi(\varphi)^\sim \) coincides with that of the \( \rho_{t_0,i_0}(X_{ij}) \) and the \( u_0(t) \) under \( \tilde{\varphi} \). Moreover, \( \mathcal{N}(\Psi(\varphi)^\sim) \) is generated by the \( \pi_{X_{ij}(t)}(x_{ij}(t)) \) and also \( \mathcal{Q}(\tilde{\varphi}) \) is by the \( \pi_{X_{ij}(t)}(X_{ij}) \). These together with the definitions of \( x_{ij}^0(t) \) and \( \rho_{t_0,i_0}(X_{ij}) \) imply the desired 2-norm equality. □

**Proposition 6.4.** With \( \Phi^*(\sigma_0) := \sigma_0 \circ \Phi \in TS(C^t_\mathbb{R}(x_\bullet \diamond)) \) we have
\[
\mathcal{I}^{lib}_{\Phi^*(\sigma_0)}(\Psi^*(\varphi)) = \mathcal{I}^{lib}_{\Phi^*(\sigma_0)}(\Psi^*(\varphi)) = \mathcal{I}^{lib}_{\sigma_0}(\varphi).
\]
for any \( \varphi \in TS(\mathcal{E}(\mathcal{R}(\mathbb{R}_+))) \). Moreover, \( \Phi^*(TS^r(\mathcal{E}(\mathcal{R}(\mathbb{R}_+)))) \) is an essential domain of both the functionals \( \mathcal{I}^{lib}_{\Phi^*(\sigma_0)}, \mathcal{I}^{lib}_{\Phi^*(\sigma_0)}, \) that is, the functionals take \( +\infty \) outside it.

**Proof.** We first remark the following facts:

- \( \Psi^*(\varphi)^{!}(P) = \Psi^*(\varphi^{!}(P)) = \varphi^{!}(\Psi^*(P)) \) for any \( P \in \mathbb{C}(x_\bullet \diamond) \).
- If \( \sigma_0(\varphi) = \sigma_0 \), then \( \sigma_0(\Psi(\varphi)) = \varphi \circ \Psi \circ \sigma_0 = \varphi \circ \rho_0 \circ \Phi = \Phi^*(\sigma_0) \). Thus, \( \Phi^*(\sigma_0)^{lib}(P) = \sigma_0^{lib}(\Psi^*(P)) \) for any \( P \in \mathbb{C}(x_\bullet \diamond) \).

Thus, (the last equality in Lemma 6.3 states that
\[
\mathcal{I}^{lib}_{\Phi^*(\sigma_0)}(\Psi^*(\varphi), P) = \mathcal{I}^{lib}_{\sigma_0}(\varphi, P, \Psi(P))
\]
holds for any \( P \in \mathbb{C}(x_\bullet \diamond) \). Note that \( \Psi(\mathbb{C}(x_\bullet \diamond)) \subset \mathcal{Q}(\mathbb{R}) \). Hence the above identity at least gives
\[
\mathcal{I}^{lib}_{\Phi^*(\sigma_0)}(\Psi^*(\varphi)) \leq \mathcal{I}^{lib}_{\sigma_0}(\varphi), \quad \mathcal{I}^{lib}_{\Phi^*(\sigma_0)}(\Psi^*(\varphi)) \leq \mathcal{I}^{lib}_{\sigma_0}(\varphi).
\]
To show the reverse inequality in both, it suffices to prove:
\[
(\diamond) \text{ For any } Q = Q^* \in \mathcal{E}(\mathbb{R}_+) \text{ there is a sequence } Q_k = Q_k^* \in \mathbb{C}(x_\bullet \diamond) \text{ such that } \mathcal{I}^{lib}_{\sigma_0}(\tau, Q_k) \rightarrow \mathcal{I}^{lib}_{\sigma_0}(\tau, Q) \text{ for all } t \in [0, \infty) \text{.}
\]
Remark that \( Q \) is a finite sum of monomials, say \( W = \rho_{t_{1_1,i_1}}(x_{1_1,j_1}) \cdots \rho_{t_{m,i_m}}(x_{m,j_m}) \) with \( x_{j} \in A_{ij} \).

Since the unital +-subalgebra \( A_{ij} \) algebraically generated by \( (X_{ij})_{ij}^{(j)} \) is norm-dense in \( A_{ij} \), we can choose norm-bounded sequences \( x_{j}^{(p)} \) in \( A_{ij,0} \) in such a way that \( x_{j}^{(p)} \rightarrow x_{j} \) in norm as \( p \rightarrow \infty \) for every \( 1 \leq \ell \leq m \). Since \( \Psi(x_{j}(t)) = \Psi(\rho_{t,j}(X_{ij})) \) and \( \rho_{t,j} \) is a unital +-homomorphism, \( W_p := \rho_{t_{1_1,i_1}}(x_{1_1,j_1}) \cdots \rho_{t_{m,i_m}}(x_{m,j_m}) \) falls into \( \mathbb{C}(x_\bullet \diamond) \) and converges to \( W \) in norm as \( p \rightarrow \infty \). Moreover, using expression (6.3) we can easily see that both \( \Lambda^*(\nabla^k(W_p)) \rightarrow \Lambda^*(\nabla^k(W)) \) and
\( \Lambda^*(\nabla sW^*) \rightarrow \Lambda^*(\nabla sW^*) \) in norm and uniformly in \( s \) as \( p \rightarrow \infty \). Since all the maps involved are linear, we have proved the desired assertion (\( \star \)) by taking, if necessary, the (operator-theoretic) real part of the approaching sequence that we have obtained. Hence, we complete the proof of the first part of the proposition.

We will then prove the second part of the proposition. Choose \( \psi \in TS^c(C_R^*(x_{\infty} (\cdot))) \) with \( I_{\Phi}^{\text{lib}}(\sigma_0,\infty) (\psi) < +\infty \). By (the proof of) [29, Proposition 5.7] we have \( \pi^*_t(\psi) = \Phi^*(\sigma_0) \) on \( C_R^*(x_{\infty}) \), the unital \( C^* \)-subalgebra generated by the \( x_{ij} \), \( j \geq 1 \), with fixing \( i \), for each \( 1 \leq i \leq n+1 \). Denote by \( \Phi_t \) the restriction of \( \Phi : C_R^*(x_{\infty}) \rightarrow \mathfrak{A} \) to each \( C_R^*(x_{ij}) \). Since \( \Phi_t : C_R^*(x_{\infty}) \rightarrow \lambda_i(A_t) \) is a surjective \( * \)-homomorphism, we obtain a bijective unital \( * \)-homomorphism \( \lambda_i(A_t) \cong C_R^*(x_{ij})/\text{Ker}(\Phi_t) \) sending \( \lambda_i(x_{ij}) \) to \( x_{ij} + \text{Ker}(\Phi_t) \) for \( j \geq 1 \). Consider the GNS representation \( \pi_\psi : C_R^*(x_{\infty} (\cdot)) \hookrightarrow H_\psi \). For any \( y \in \text{Ker}(\Phi_t) \) we have
\[
\psi(\pi_t(y)^* \pi_t(y)) = \pi^*_t(\psi)((y^* y) = \Phi^*(\sigma_0)(y^* y) = \sigma_0(\Phi_t(y)^* \Phi_t(y)) = 0,
\]
and hence \( \pi_t(\pi_t(y)) = 0 \) thanks to the trace property of \( \psi \). Therefore, by the \( C^* \)-algebraic freeness among the \( \rho_k : (A_t : \mathfrak{A}) \cong C_R^*(x_{ij})/\text{Ker}(\Phi_t) \), \( \rho_k(x_{ij}) \mapsto \lambda_i(X_{ij}) \mapsto x_{ij} + \text{Ker}(\Phi_t) \) as remarked before, we obtain a unique unital \( * \)-homomorphism from \( \mathfrak{A}(\mathbb{R},\mathfrak{A}) \) to \( B(H_\psi) \) sending each \( \rho_k(x_{ij}) \) to \( \pi_\psi(\pi_t(x_{ij})) = \pi_\psi(x_{ij}(t)) \). Then the pull-back of \( \psi \) by this \( * \)-homomorphism defines a tracial state \( \varphi \) on \( \mathfrak{A}(\mathbb{R},\mathfrak{A}) \), under which the \( \rho_k(x_{ij}) \) have the same joint distribution as that of the \( x_{ij}(t) \) under \( \psi \). This means that \( \Psi^*(\varphi) = \psi \) and the continuity of \( \varphi \) follows thanks to Lemma 6.1. Hence we are done.

Corollary 6.5. In the same setting as in Proposition 6.4 we have
\[
(6.4) \quad J_{\emptyset}^{\text{lib}}(\Phi^*(\sigma)) = J_{\emptyset}^{\text{lib}}(\sigma), \quad J_{\emptyset}^{\text{lib}}(\sigma,\infty)(\Phi^*(\sigma)) = J_{\emptyset,\infty}^{\text{lib}}(\sigma)
\]
for any \( \sigma \in TS(\mathfrak{A}) \). In particular, the following are equivalent:

(1) \( A_n, 1 \leq i \leq n+1, \) are freely independent.

(2) \( J_{\emptyset}^{\text{lib}}(A_1; \ldots; A_n : A_{n+1}) = 0 \).

(3) \( J_{\emptyset,\infty}^{\text{lib}}(A_1; \ldots; A_n : A_{n+1}) = 0 \).

Moreover,
\[
(6.5) \quad \chi_{\text{orb}}(X_1, \ldots, X_{n+1}) \leq -J_{\emptyset}^{\text{lib}}(A_1; \ldots; A_n : A_{n+1}) \leq -J_{\emptyset,\infty}^{\text{lib}}(A_1; \ldots; A_n : A_{n+1}),
\]
at least when \( \sigma_0 \) is either \( \Upsilon^*(\tau) \) or \( \bigstar_{i=1}^{n+1}(\lambda_i^{-1})^*(\tau) \).

Proof. We will first prove two identities (6.4), which enables us to derive the equivalence of (1) - (3) from Theorem 5.3 immediately. In the current setting, an open neighborhood basis at \( \sigma \) in \( TS(\mathfrak{A}) \) should be given as a collection of \( O_{m,\delta}(\sigma) \), where \( O_{m,\delta}(\sigma) \) is all the \( \sigma' \in TS(\mathfrak{A}) \) such that
\[
|\sigma|\rho_i(X_{ij1}) \cdots \rho_i(X_{ijp})| = |\lambda_i(X_{ij1}) \cdots \lambda_i(X_{ijp})| < \delta
\]
whenever \( 1 \leq i_k \leq n+1, 1 \leq j_k \leq m, 1 \leq k \leq p \) and \( 1 \leq p \leq m \). Thus, sup\( \cup_{\sigma' \in O(\sigma)} \) and \( \rho_i^* (\varphi) \in O \) can/should be replaced with \( \text{lim}_{m,\delta} \) and \( \rho_i^* (\varphi) \in O_{m,\delta}(\sigma) \), respectively. By definition we observe that
\[
\pi^*_t(\Psi^*(\varphi))((x_{ij1} \cdots x_{ijk})) - \Phi^*(\sigma)(x_{ij1} \cdots x_{ijk})
\]
\[
= [\rho^*_i(\varphi)(\lambda_i(X_{ij1}) \cdots \lambda_i(X_{ijp})) - \sigma(\lambda_i(X_{ij1}) \cdots \lambda_i(X_{ijp})))].
\]
Hence \( \pi^*_t(\Psi^*(\varphi)) \in O_{m,\delta}(\Phi^*(\sigma)) \) if and only if \( \rho_i^* (\varphi) \in O_{m,\delta}(\sigma) \). Moreover, \( \Psi^*(TS^c(C_R^*(x_{\infty}))) \) is an essential domain for the functionals by Proposition 6.4. Therefore, the main identities in Proposition 6.4 imply two identities (6.5).

Since
\[
\Phi^*(\Upsilon^*(\tau))(x_{ij1} \cdots x_{ijn,m}) = \tau(X_{ij1} \cdots X_{ijn,m}),
\]
\[
\Phi^*(\bigstar_{i=1}^{n+1}(\lambda_i^{-1})^*(\tau))(x_{ij1} \cdots x_{ijn,m}) = \bigstar_{i=1}^{n+1}(\lambda_i^{-1})^*(\tau)(\lambda_i(X_{ij1}) \cdots \lambda_i(X_{ijn,m})),
\]

\( \Lambda^* \rightarrow \Lambda^* \) is a surjective \( * \)-homomorphism.
Corollary 4.3 together with Propositions 3.1, 3.2 implies inequality (6.5).

□

Remarks 6.6. (1) The part characterizing free independence by \( \pi \) as \( \mathcal{J}_{\sigma_0}^{lib}(A_1; \ldots; A_n : A_{n+1}) \) may be independent of \( \sigma_0 \), at least under some constraint. However, this question is untouched yet due to the lack of techniques to discuss ‘minimal paths’ of tracial states under the functionals.

6.6. Invariance under weak closure. Corollary 6.5 suggests that \( \mathcal{J}_{\sigma_0}^{lib}(A_1; \ldots; A_n : A_{n+1}) \) as well as \( \mathcal{J}_{\sigma_0}^{lib}(A_1; \ldots; A_n : A_{n+1}) \) are \( W^* \)-invariants, that is, they are unchanged if each \( A_i \) is replaced with its \( \sigma \)-weak closure \( \tilde{A}_i \). This is indeed the case, as we will see below. The proof is rather technical, but the idea behind it is simple.

Let us denote by \( \mathfrak{N} \) and \( \mathfrak{N}(\mathbb{R}_+) \subset \mathfrak{N}(\mathbb{R}_+) \) the \( C^* \)-algebras corresponding to \( \mathfrak{A} \) and \( \mathfrak{A}(\mathbb{R}_+) \subset \mathfrak{A}(\mathbb{R}_+) \) when each \( A_i \) is replaced with \( \tilde{A}_i := \mathfrak{A}_{\sigma_0}^+ \). Observe that the original \( \mathfrak{A}(\mathbb{R}_+) \subset \mathfrak{A}(\mathbb{R}_+) \) are naturally embedded into \( \mathfrak{N} \). See Proposition A.3. Notations \( \lambda_i, \rho_i, \rho_i \) of morphisms are used simultaneously in what follows. To this end, we need several technical, purely operator algebraic facts (Lemmas 6.7-6.9).

The first lemma seems a folklore among operator algebraists, but we do give its proof because it plays a key role in the discussion below.

Lemma 6.7. Let \( A \) be a \( \sigma \)-weakly dense, unital \( C^* \)-subalgebra of a \( W^* \)-algebra \( \mathfrak{P} \) and \( \varphi \) be a normal state on \( \mathfrak{P} \). Then \( \pi : A \cap \mathfrak{P} \) be a unital \( \ast \)-representation with a distinguished vector \( \xi_0 \in \mathfrak{H} \) such that \( \xi_0 \) is separating for \( \pi(A) \) and that \( (\pi(a)\xi_0)_{\mathfrak{H}} = \varphi(a) \) holds for every \( a \in A \). Then there is a unique normal unital \( \ast \)-representation \( \tilde{\pi} : \mathfrak{M} \cap \mathfrak{H} \) extending \( \pi \) such that \( \tilde{\pi}(M) = \pi(M) \). □

Proof. Let \( (\mathcal{H}_\psi, \pi_\psi, \xi_\psi) \) be the GNS triple of \( (\mathfrak{M}, \varphi) \). Set \( K := \pi(\mathfrak{A})\xi_0 \), a reducing subspace for \( \pi(\mathfrak{A}) \). Observe, by the uniqueness of GNS representations, that the restriction of \( \pi \) to \( K \) with \( \xi_0 \) is a realization of \( (\mathcal{H}_\psi, \pi_\psi, A_\psi) \). Since \( \xi_0 \) is separating for \( \pi(\mathfrak{A}) \), \( \pi \) is quasi-equivalent to \( \varphi_\psi \) by [19, Theorem 10.3.3(ii)]. This means that there exists a normal unital, bijective \( \ast \)-homomorphism \( \rho : \pi_\psi(\mathfrak{M}) = \pi(\mathfrak{A}) \rightarrow \pi(\mathfrak{A}) \) sending \( \pi_\psi(a) \) to \( \pi(a) \) for every \( a \in A \). Thus, \( \tilde{\pi} := \rho \circ \pi_\psi : \mathfrak{M} \rightarrow \pi(\mathfrak{A}) \) is the desired \( \ast \)-homomorphism.

We need the next two state extension properties. The proofs crucially use the previous lemma with the universality of universal free products.

Lemma 6.8. Any \( \sigma_0 \in TS(\mathbb{A}) \) with \( \lambda_i(\sigma_0) = \tau \) on \( A_i \) for all \( 1 \leq i \leq n + 1 \) has a unique extension \( \sigma_0 \in TS(\mathfrak{M}) \) with \( \lambda_i(\sigma_0) = \tau \) on \( M_i \) for all \( 1 \leq i \leq n + 1 \).

Proof. Let \( (\mathcal{H}_{\sigma_0}, \pi_{\sigma_0}, \xi_{\sigma_0}) \) be the GNS triple of \( (\mathfrak{A}, \sigma_0) \). Since \( \sigma_0 \) is tracial, \( \xi_{\sigma_0} \) must be separating for \( \pi_{\sigma_0} \). In particular, \( \xi_{\sigma_0} \) is separating for each \( \pi_{\sigma_0}(\lambda_i(\mathfrak{A})) \). Let \( \pi_{\sigma_0, i} := \pi_{\sigma_0} \circ \lambda_i : A_i \rightarrow \mathcal{H}_{\sigma_0} \). Then we have \( \pi_{\sigma_0, i}(a)\xi_{\sigma_0}(\xi_0)_{\mathfrak{H}} = \sigma_0 \circ \lambda_i(a) = \lambda_i(\sigma_0)(a) = \tau(a) \) for every \( a \in A_i \). Thus, the previous lemma shows that there exists a unique normal extension \( \pi_{\sigma_0, i} : M_i := \mathfrak{A}_{\sigma_0}^+ \rightarrow \mathcal{H}_{\sigma_0} \) such that \( \pi_{\sigma_0, i}(M_i) = \pi_{\sigma_0}(\lambda_i(\mathfrak{A})) \) and \( \pi_{\sigma_0, i} \restriction A_i = \pi_{\sigma_0, i} \). By the universality of universal free products, there exists a unique \( \ast \)-homomorphism \( \pi_{\sigma_0} : \mathfrak{M} \rightarrow \mathcal{H}(\mathfrak{H}_{\sigma_0}) \) such that \( \pi_{\sigma_0} \circ \lambda_i = \pi_{\sigma_0, i} \). The final question is normal for every \( 1 \leq i \leq n + 1 \). By construction, it is clear that \( \pi_{\sigma_0} \restriction \mathfrak{A} = \sigma_0 \). Set \( \sigma_0 := (\pi_{\sigma_0}(\cdot)\xi_{\sigma_0}(\xi_0)_{\mathfrak{H}})_{\mathcal{H}_{\sigma_0} \in TS(\mathfrak{M})} \). Trivially, \( \sigma_0 \restriction \mathfrak{A} = \sigma_0 \). For each \( x_k \in M_{k+1} \), \( 1 \leq k \leq m \), by the Kaplansky density theorem, one can choose a net \( a_k^{(x)} \in A_i \) (with a common index set) such that \( \|a_k^{(x)}\|_\infty \leq \|x_k\|_\infty \) and \( a_k^{(x)} \rightarrow x_k \) in the \( \sigma \)-strong* topology on \( M_{k+1} \). Since each \( \pi_{\sigma_0,i} \) is normal on \( M_{k+1} \), we observe that

\[
\pi_{\sigma_0}(\lambda_i(a_1^{(x)} \cdot \lambda_m(a_m^{(x)})) = \pi_{\sigma_0, i_1}(a_1^{(x)}) \cdots \pi_{\sigma_0, i_m}(a_m^{(x)})
\]
= \bar{\pi}_{\sigma_0,i_1}(a^{(\kappa)}_{i_1}) \cdots \bar{\pi}_{\sigma_0,i_m}(a^{(\kappa)}_{i_m})
\rightarrow \bar{\pi}_{\sigma_0,i_1}(x_1) \cdots \bar{\pi}_{\sigma_0,i_m}(x_m) = \bar{\pi}_{\sigma_0}(\lambda_{i_1} \cdots \lambda_{i_m}(x_m))
and hence \bar{\sigma}_0(\lambda_{i_1}(x_1) \cdots \lambda_{i_m}(x_m)) = \lim_n \bar{\sigma}_0(\lambda_{i_1}(a^{(\kappa)}_{i_1}) \cdots \lambda_{i_m}(a^{(\kappa)}_{i_m})).
Since the \lambda_i(M_i) generate \mathfrak{M} as a C\sp*\-algebra, we conclude that \bar{\sigma}_0 is a unique extension of \sigma_0. Moreover, \lambda^*_i(\bar{\sigma}_0)(x) = \bar{\sigma}_0(\lambda_i(x)) = \lim_n \sigma_0(\lambda_i(a_{n})) = \lim_n \lambda^*_i(\sigma_0)(a_{n}) = \lim_n \tau(a_n) = \tau(x) for every x \in M_i with approximation a_n \to x as above.

Lemma 6.9. Any \varphi \in TS^t(\mathfrak{M}(\mathbb{R}_+)) with \rho^{*}_{t,i}(\varphi) = \tau on \mathcal{A}_i, for all t \geq 0 and 1 \leq i \leq n + 1 has a unique extension \bar{\varphi} \in TS^t(\mathfrak{M}(\mathbb{R}_+)) with \rho^{*}_{t,i}(\bar{\varphi}) = \tau on M_i for all t \geq 0 and 1 \leq i \leq n + 1.
Proof. Let (\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi) be the GNS triple of (\mathfrak{M}(\mathbb{R}_+), \varphi). The same argument as in the previous lemma shows that there exists a *-representation \bar{\pi}_\varphi : \mathfrak{M}(\mathbb{R}_+) \hookrightarrow \mathcal{H}_\varphi such that \bar{\pi}_\varphi \circ \rho_{t,i} : M_i \to B(\mathcal{H}_\varphi) is normal as well as that \bar{\pi}_\varphi \circ \rho_{t,i} | \mathcal{A}_i = \pi_\varphi \circ \rho_{t,i} holds for every t \geq 0 and 1 \leq i \leq n + 1. Define \tilde{\varphi} := (\bar{\pi}_\varphi(\cdot) \cdot \xi_\varphi(\cdot))_{\mathcal{H}_\varphi} \in TS^t(\mathfrak{M}(\mathbb{R}_+)). Remark that \rho^{*}_{t,i}(\tilde{\varphi}) = \tau on M_i holds for every t \geq 0 and 1 \leq i \leq n + 1. By the uniqueness of GNS representations, the triple (\mathcal{H}_\tilde{\varphi}, \bar{\pi}_\varphi, \xi_\varphi) is identified with the GNS triple of (\mathfrak{M}(\mathbb{R}_+), \varphi). Namely, we may and do assume that \pi_\varphi = \bar{\pi}_\varphi, \mathcal{H}_\varphi = \mathcal{H}_\tilde{\varphi} and \xi_\varphi = \xi_\tilde{\varphi}.
Since the given \varphi is continuous, the mapping t \mapsto \pi_\varphi(\rho_{t,i}(a)) is strongly continuous for every a \in \mathcal{A}_i. We claim that this is the case even when a \in \mathcal{A}_i is replaced with an arbitrary x \in M_i. By the Kaplansky density theorem, we can choose a net a_n \in \mathcal{A}_i in such a way that \|a_n\|_\infty \leq \|x\|_\infty and \|a_n - x\|_{r,2} := \sqrt{\tau((a_n - x)^*(a_n - x))} \to 0. We have
\|\pi_\varphi(\rho_{t,i}(a_n - x))\|_{\mathcal{H}_\varphi} = \sqrt{\rho^{*}_{t,i}(\varphi)((a_n - x)^*(a_n - x))}
= \sqrt{\tau((a_n - x)^*(a_n - x))} = \|a_n - x\|_{r,2}.
For any \eta \in \mathcal{H}_\varphi and any \varepsilon > 0, there is a Y' \in \pi_\varphi(\mathfrak{M}(\mathbb{R}_+))^\prime such that \|\eta - Y'\xi_\varphi\|_{\mathcal{H}_\varphi} < \varepsilon (n.b., \xi_\varphi is separating for \pi_\varphi(\mathfrak{M}(\mathbb{R}_+)), and the existence of such a Y' is guaranteed). Then
\|\pi_\varphi(\rho_{t,i}(a_n - x))\|_{\mathcal{H}_\varphi} \leq 2\|x\|_\infty \|\eta - Y'\xi_\varphi\|_{\mathcal{H}_\varphi} + \|Y'\|_\infty \|\pi_\varphi(\rho_{t,i}(a_n - x))\|_{\mathcal{H}_\varphi} \xi_\varphi \|_{\mathcal{H}_\varphi}
\leq 2\|x\|_\infty \varepsilon + \|Y'\|_\infty \|a_n - x\|_{r,2},
and hence
\lim_{t \to 0} \sup \|\pi_\varphi(\rho_{t,i}(a_n - x))\|_{\mathcal{H}_\varphi} = 0.
Then, we can see that t \mapsto \pi_\varphi(\rho_{t,i}(x)) is strongly continuous for every x \in M_i. It follows thanks to Lemma 6.1 (iii) that \tilde{\varphi} is continuous.

Here is an important remark obtained from the above proof.

Remark 6.10. We keep the notations \varphi, \tilde{\varphi}, etc., of the previous lemma. If a bounded net a^{(\kappa)} in \mathcal{A}_i converges to x in M_i in \|\cdot\|_{r,2} or equivalently, in the \sigma\textendash strong topology on M_i, then
\lim_{\kappa} \sup_{t \geq 0} \|\pi_\varphi(\rho_{t,i}(a^{(\kappa)} - x))\|_{\mathcal{H}_\varphi} = 0
for every \xi \in \mathcal{H}_\varphi, that is, the convergence \pi_\varphi(\rho_{t,i}(a^{(\kappa)})) \to \pi_\varphi(\rho_{t,i}(x)) in the strong operator topology is uniform for t \geq 0.

Lemma 6.11. For any \varphi \in TS^t(\mathfrak{M}(\mathbb{R}_+)) with \rho^{*}_{t,i}(\varphi) = \tau on \mathcal{A}_i, for all t \geq 0 as well as \lambda^*_i(\sigma_0) = \tau on \mathcal{A}_i, for all 1 \leq i \leq n + 1, we have \mathcal{I}^{\text{th}}_{\sigma_0}(\varphi) = \mathcal{I}^{\text{th}}_{\bar{\sigma}_0}(\varphi) as well as \mathcal{I}^{\text{th}}_{\sigma_0,\infty}(\varphi) = \mathcal{I}^{\text{th}}_{\bar{\sigma}_0,\infty}(\varphi) with the notations in the previous lemmas.
Proof. The same pattern as in the proof of Proposition 6.4 (and Lemma 6.3) works well by replacing the norm convergence $x^{(p)}_t \to x_t$ with a bounded net convergence $a^{(i)}_t \to x_t$ in the $\sigma$-strong* topology with the help of Remark 6.10.

Here is the desired statement. Namely, the next proposition tells us that taking the $\sigma$-weak closure does not give any effect to $\mathcal{J}^{lib}_{\lambda_0}$ as well as $\mathcal{J}^{lib}_{\lambda_0,\infty}$. This is analogous to [30, Remarks 10.2].

**Proposition 6.12.** With the notations as in the previous lemmas we have

$$\mathcal{J}^{lib}_{\lambda_0}(A_1; \ldots; A_n : A_{n+1}) = \mathcal{J}^{lib}_{\lambda_0}(M_1; \ldots; M_n : M_{n+1}),$$

$$\mathcal{J}^{lib}_{\lambda_0,\infty}(A_1; \ldots; A_n : A_{n+1}) = \mathcal{J}^{lib}_{\lambda_0,\infty}(M_1; \ldots; M_n : M_{n+1})$$

as long as $\lambda^*_i(\sigma_0) = \tau$ on $\mathcal{A}_i$ for all $1 \leq i \leq n + 1$.

Proof. For the ease of notations we will write $\sigma := \Upsilon^*(\tau) \in TS(\mathfrak{A})$ and $\hat{\sigma} := \hat{\Upsilon}^*(\tau) \in TS(\mathfrak{M})$, where $\Upsilon : \mathfrak{A} \to \mathfrak{M}$ and $\hat{\Upsilon} : \mathfrak{M} \to \mathfrak{M}$ the unital +-homomorphisms sending each $\lambda_i(a)$ with $a \in \mathcal{A}_i$ to $a$ and $\lambda_i(x)$ with $x \in \mathcal{M}_i$ to $x$, respectively. In particular, $\hat{\Upsilon}$ is an extension of $\Upsilon$, and hence $\hat{\sigma}$ is an extension of $\sigma$ too.

We denote by $W$ a word whose letters from the $\lambda_i(\mathcal{A}_i)$ and also by $\bar{W}$ a word whose letters from the $\lambda_i(\mathcal{M}_i)$. According to this notation, we will also denote by $\mathcal{W}$ a finite collection of words $W$ and by $\mathcal{W}$ a finite collection of words $W$. These play parts of parameters to define neighborhood base of the weak* topologies on $TS(\mathfrak{A})$ and $TS(\mathfrak{M})$, respectively.

Let $T \geq 0$, $\delta > 0$, and $\psi \in TS^c(\mathfrak{M}(\mathbb{R}_+))$ be arbitrarily chosen. Denote by $\psi$ the restriction of $\psi$ to $\mathfrak{A}(\mathbb{R}_+)$, which clearly falls into $TS^c(\mathfrak{A}(\mathbb{R}_+))$. By construction, it is easy to see that $\mathcal{T}^b_{\lambda_0}(\psi) \leq \mathcal{T}^b_{\lambda_0}(\psi)$ holds in general. Hence

\[
\inf_{\mathcal{T}^b_{\lambda_0}(\psi) \mid T} \{T^b_{\lambda_0}(\varphi) \mid \varphi \in TS^c(\mathfrak{A}(\mathbb{R}_+)), \rho^T_{\lambda_0}(\varphi) \in O_{W,\lambda}(\sigma)\}
\]

\[
\leq \inf_{\mathcal{T}^b_{\lambda_0}(\psi) \mid T} \{T^b_{\lambda_0}(\varphi) \mid \varphi \in TS^c(\mathfrak{M}(\mathbb{R}_+)), \rho^T_{\lambda_0}(\varphi) \in O_{W,\lambda}(\hat{\sigma})\}
\]

\[
\leq \inf_{\mathcal{T}^b_{\lambda_0}(\psi) \mid T} \{T^b_{\lambda_0}(\varphi) \mid \varphi \in TS^c(\mathfrak{M}(\mathbb{R}_+)), \rho^T_{\lambda_0}(\varphi) \in O_{W,\lambda}(\hat{\sigma})\},
\]

where we use that $\rho^T_{\lambda_0}(\psi) \in O_{W,\lambda}(\sigma) \Leftrightarrow \rho^T_{\lambda_0}(\psi) \in O_{W,\lambda}(\hat{\sigma})$, since every $W \in W$ falls into $\mathfrak{A}$ (and hence $\sigma(W) = \hat{\sigma}(W)$ and $\psi(\rho(W)) = \psi(\rho(W)))$. Taking the $\lim_{T \to \infty}$ of the above inequality, we get

\[
\lim_{\bar{\lambda} \to \infty} \inf_{\mathcal{T}^b_{\lambda_0}(\psi) \mid T} \{T^b_{\lambda_0}(\varphi) \mid \varphi \in TS^c(\mathfrak{A}(\mathbb{R}_+)), \rho^T_{\lambda_0}(\varphi) \in O_{W,\lambda}(\sigma)\}
\]

\[
\leq \lim_{\bar{\lambda} \to \infty} \inf_{\mathcal{T}^b_{\lambda_0}(\psi) \mid T} \{T^b_{\lambda_0}(\varphi) \mid \varphi \in TS^c(\mathfrak{M}(\mathbb{R}_+)), \rho^T_{\lambda_0}(\varphi) \in O_{W,\lambda}(\hat{\sigma})\}
\]

\[
\leq \sup_{\bar{\lambda} \to \infty} \lim_{\bar{\lambda} \to \infty} \inf_{\mathcal{T}^b_{\lambda_0}(\psi) \mid T} \{T^b_{\lambda_0}(\varphi) \mid \varphi \in TS^c(\mathfrak{M}(\mathbb{R}_+)), \rho^T_{\lambda_0}(\varphi) \in O_{W,\lambda}(\hat{\sigma})\} = \mathcal{J}^{lib}_{\lambda_0}(\hat{\sigma}).
\]

Since $(W, \delta)$ is arbitrary, $\mathcal{J}^{lib}_{\lambda_0}(A_1; \ldots; A_n : A_{n+1}) = \mathcal{J}^{lib}_{\lambda_0}(\sigma) \leq \mathcal{J}^{lib}_{\lambda_0}(\hat{\sigma}) = \mathcal{J}^{lib}_{\lambda_0}(M_1; \ldots; M_n : M_{n+1})$. The same assertion also holds with the same proof even if $\mathcal{J}^{lib}_{\lambda_0}$ and $\mathcal{J}^{lib}_{\lambda_0}$ are replaced with $\mathcal{J}^{lib}_{\lambda_0,\infty}$ and $\mathcal{J}^{lib}_{\lambda_0,\infty}$, respectively. We remark that the discussion in this paragraph uses only inclusion relation $\mathcal{A}_i \subset M_i$, $1 \leq i \leq n + 1$. This remark will be summarized into the corollary following this proposition.

We will then prove the reverse inequality. To this end, we may assume that $\mathcal{J}^{lib}_{\lambda_0}(A_1; \ldots; A_n : A_{n+1}) = \mathcal{J}^{lib}_{\lambda_0}(\sigma) < +\infty$; otherwise the reverse inequality trivially holds as $-\infty = -\infty$ by the first part of this proof. Let $(W, \delta)$ is arbitrarily given. For each $W \in W$, we can choose a word $W$ in such a way that

\[
|\sigma(W) - \hat{\sigma}(W)| < \frac{\delta}{3}, \quad \sup_{T \geq 0} |\rho^T_{\lambda_0}(\varphi)(W) - \rho^T_{\hat{\lambda}_0}(\varphi)(W)| < \frac{\delta}{3}
\]

\[
|\sigma(W) - \hat{\sigma}(W)| < \frac{\delta}{3}, \quad \sup_{T \geq 0} |\rho^T_{\lambda_0}(\varphi)(W) - \rho^T_{\hat{\lambda}_0}(\varphi)(W)| < \frac{\delta}{3}
\]
whenever \( \varphi \in TS^i(\mathfrak{A}(\mathbb{R}_+)) \) satisfies that \( \rho^*_{t,i}(\varphi) = \tau^{on} \). It is also an interesting question whether or not \( \sigma_0 \) and \( \sigma_0,\infty \) are independent of the choice of \( \sigma_0 \).

\[
\lim_{t \to \infty} |X(a)| = |X(a)| + |X(b)| - 1
\]

We denote by \( \rho_t \) for every \( t \geq 0 \) and \( 1 \leq i \leq n+1 \). By the above consideration we observe that \( \varphi \in O_{W,\delta}(\sigma) \). Therefore, we conclude that

\[
\inf \{ T^{lib}_{\sigma_0}(\varphi) \mid \varphi \in TS^i(\mathfrak{A}(\mathbb{R}_+)), \rho^*_{t,i}(\varphi) \in O_{W,\delta}(\sigma) \}
\]

which implies the desired inequality since \( (W, \delta) \) is arbitrary. The discussion so far in this paragraph also works again when \( J^{lib}_{\sigma_0} \) and \( J^{lib}_{\sigma_0,\infty} \) are replaced with \( J^{lib}_{\sigma_0,\infty} \) and \( J^{lib}_{\sigma_0,\infty} \), respectively. Hence we are done. \( \square \)

As remarked in the above proof, we have essentially proved the next monotonicity fact too.

**Corollary 6.13.** If \( B_i \subseteq A_i \) be a unital C*-subalgebra (possibly \( W^*-\)subalgebra) for each \( 1 \leq i \leq n+1 \), then

\[
J^{\text{lib}}_{\sigma_0}(B_1; \cdots; B_n : B_{n+1}) \leq J^{\text{lib}}_{\sigma_0}(A_1; \cdots; A_n : A_{n+1}),
\]

where \( \sigma_0 \) on the left-hand side should be understood as the restriction of \( \sigma_0 \) to the universal C*-algebra obtained from the \( B_i \).

### 6.7. Summary of basic properties

We have established the next properties of \( i^* \) so far.

- \( i^*(A_1; \cdots; A_n : A_{n+1}) = i^*(W^*(A_1); \cdots; W^*(A_n) : W^*(A_{n+1})) \).
- If \( B_i \subseteq A_i \), then \( i^*(B_1; \cdots; B_n : B_{n+1}) \leq i^*(A_1; \cdots; A_n : A_{n+1}) \).
- \( i^*(A_1; \cdots; A_n : A_{n+1}) = 0 \) if and only if \( A_1, \ldots, A_{n+1} \) are freely independent.

Here \( W^*(A_i) \) and \( W^*(X_i) \) denote the von Neumann subalgebras generated by \( A_i \) and \( X_i \), respectively. An important question is whether or not \( i^* = i^* \). It is also an interesting question whether or not \( J^{\text{lib}}_{\sigma_0} \) and \( J^{\text{lib}}_{\sigma_0,\infty} \) are independent of the choice of \( \sigma_0 \).
7. Unitary Brownian motions

Let \( \Xi(N) \) and \( U_N^{(i)}(t), 1 \leq i \leq n \) be as in subsection 4.7, that is, \( \Xi(N) \) is a countable family of deterministic \( N \times N \) self-adjoint matrices and the \( U_N^{(i)}(t) \) are independent, left-increment unitary Brownian motions on \( U(N) \). For the ease of notation, we number the elements of \( \Xi(N) \) as \( \xi_i(N) \) rather than \( \xi_i \). In this section, we will explain how the proofs in [29] work well for the \( U_N^{(i)}(t) \) together with \( \Xi(N) \) and compare their consequences on the matrix liberation process \( \Xi^{lib}(N) \) with the corresponding results on the \( U_N^{(i)}(t) \) together with \( \Xi(N) \).

7.1. Malliavin derivatives of unitary Brownian motions.

We begin with the SDE representation of \( U_N^{(i)}(t) \): Let \( B^{(i)}_{\alpha\beta}(t), 1 \leq \alpha, \beta \leq N, 1 \leq i \leq n \), be the \( nN^2 \) independent Brownian motions on the real line with natural filtration \( \mathcal{F}_t \). Consider the system of SDEs in the \( 2nN^2 \)-dimensional Euclidean space \( (M_N)^n \):

\[
(7.1) \quad dX^{(i)}(t) = \frac{\sqrt{-1}}{\sqrt{N}} \sum_{1 \leq \alpha, \beta \leq N} C_{\alpha\beta} X^{(i)}(t) dB^{(i)}_{\alpha\beta}(t) - \frac{1}{2} X^{(i)}(t) dt \quad (1 \leq i \leq n),
\]

where \( C_{\alpha\beta}, 1 \leq \alpha, \beta \leq N, \) form an orthonormal basis of the Euclidean space \( M_N \). This system of SDEs are linear, and thus each system of them admits a unique strong solution after fixing initial \( X^{(i)}(0) \). The unitary Brownian motions \( U_N^{(i)}(t), 1 \leq i \leq n, \) are constructed as a unique strong solution \( X^{(i)}(t) \) of the system (7.1) under initial condition \( X^{(i)}(0) = I \).

**Lemma 7.1.** Let \( D^{(k,\alpha,\beta)} \) be the Malliavin derivative along the Brownian motion \( B^{(k)}_{\alpha\beta} \). Then

\[
D^{(k,\alpha,\beta)} U_N^{(i)}(t) = \delta_{k,i} \frac{1}{\sqrt{N}} C_{\alpha\beta} U_N^{(k)}(s) \left( \frac{1}{\sqrt{N}} C_{\alpha\beta} U_N^{(k)}(s) \right),
\]

\[
D^{(k,\alpha,\beta)} U_N^{(i)}(t)^* = \delta_{k,i} \frac{1}{\sqrt{N}} C_{\alpha\beta} \left( U_N^{(k)}(s) U_N^{(k)}(s)^* \right)^t
\]

for almost every \( t \geq 0 \).

**Proof.** We also consider the system of SDEs

\[
(7.2) \quad dY^{(i)}(t) = -\frac{\sqrt{-1}}{\sqrt{N}} \sum_{1 \leq \alpha, \beta \leq N} Y^{(i)}(t) C_{\alpha\beta} dB^{(i)}_{\alpha\beta}(t) - \frac{1}{2} Y^{(i)}(t) dt \quad (1 \leq i \leq n).
\]

For a given \( X \in M_N \), it is easy to see that \( X^{(i)}(t) := U_N^{(i)}(t)X \) and \( Y^{(i)}(t) := Xu_N^{(i)}(t)^* \) satisfy the systems (7.1), (7.2) of SDEs, respectively. Thus, the unique strong solutions of the system of SDEs (7.1), (7.2) with initial condition \( X^{(i)}(0) = X, Y^{(i)}(0) = X \) must be \( U_N^{(i)}(t)X, Xu_N^{(i)}(t)^* \). Thus, \( U_N^{(i)}(t)X, Xu_N^{(i)}(t)^* \) are both linear in the variable \( X \), and hence their gradients (or ‘Jacobian matrix’) in \( X \) become the linear transformations \( L_{U_N^{(i)}(t)} \) and \( Ru_N^{(i)}(t)^* \), on \( M_N \), respectively, where \( LA_X := AX, RB_X := XB \) for \( A, B, X \in M_N \). By a standard fact on Malliavin derivatives for strong solutions of SDEs [24, Theorem 2.2.1; Eq.(2.59)] it follows that

\[
D^{(k,\alpha,\beta)} U_N^{(i)}(t) = \delta_{k,i} \frac{1}{\sqrt{N}} C_{\alpha\beta} \left( \frac{\sqrt{-1}}{\sqrt{N}} U_N^{(k)}(t) U_N^{(k)}(s) \right) \frac{1}{\sqrt{N}} C_{\alpha\beta} U_N^{(k)}(s),
\]

\[
D^{(k,\alpha,\beta)} U_N^{(i)}(t)^* = \delta_{k,i} \frac{1}{\sqrt{N}} C_{\alpha\beta} \left( \frac{\sqrt{-1}}{\sqrt{N}} U_N^{(k)}(t)^* U_N^{(k)}(s)^* \right) \frac{1}{\sqrt{N}} C_{\alpha\beta} U_N^{(k)}(s)^* U_N^{(k)}(t)^*.
\]
Hence we are done.

By the linearity and the Leibniz rule of $D^{(k;\alpha,\beta)}_s$ we have, for a monomial $W$ in $U_N^{(i)}(t)$, $U_N^{(j)}(t)^*$ and $\xi_j(N)$,

$$D^{(k;\alpha,\beta)}_s tr_N(W) = \sum_{s \leq t} \left( tr_N \left( W_1 \left( \sqrt{-1} U_N^{(k)}(t) U_N^{(k)}(s)^* \left( \frac{1}{\sqrt{N}} C_{\alpha \beta} \right) U_N^{(k)}(s) \right) W_2 \right) + \sum_{s \leq t} \left( tr_N \left( W_3 \left( - \sqrt{-1} U_N^{(k)}(s)^* \left( \frac{1}{\sqrt{N}} C_{\alpha \beta} \right) U_N^{(k)}(t)^* \right) W_4 \right) \right) \right) \right). \quad (7.3)$$

With these remarks it is a straightforward task to modify the proof of the large deviation upper bound for the matrix liberation process in [29] to the case of unitary Brownian motions with deterministic time. The consequence is as follows.

7.2. Non-commutative derivations. We assume the norm constraint $\|\xi_j(N)\|_\infty \leq R$ for all $j \geq 1$, and moreover that $\Xi(N)$ has a limit distribution as $N \to \infty$. Thus we consider the universal $C^*$-algebras $C_R^*(x\xi_j(N)) \subset C_R^*(x\xi_j(N)) \subset C_R^*(x\xi_j(N))$ generated by $x_j = x_j^*$, $j \geq 1$, and $u_i(t), v_i(t)$, $1 \leq i \leq n$, $t \geq 0$, with subject to $\|x_j\|_\infty \leq R$ and $u_i(t) u_i(t)^*=v_i(t)^* v_i(t)=u_i(0)=v_i(0)=1$, $1 \leq i \leq n$, $t \geq 0$. Remark that the universal s-algebra $C(x\xi_j(N))$ generated by the same indeterminates with the same algebraic constraints (and without the norm constraint) is naturally embedded into $C_R^*(x\xi_j(N))$ as a norm-dense s-subalgebra. By formula (7.3) we introduce derivations $\delta^{(k)}_s : C(x\xi_j(N)) \to C(x\xi_j(N)) \otimes_{alg} C(x\xi_j(N))$ determined by

$$\delta^{(k)}_s \left( u_i(t) \right) := \delta^{(k)}_s \left( u_i(t) (u_i(t)^* \otimes u_i(s)) \right),$$

$$\delta^{(k)}_s \left( u_i(t)^* \right) := \delta^{(k)}_s \left( - u_i(t)^* (u_i(t)^* \otimes u_i(s) u_i(t)) \right),$$

$$\delta^{(k)}_s (x_j) := 0.$$

(In fact, one can easily check $(u^{(k)}_s u_t)^* - u_t^* (u^{(k)}_s u_t)^* = 0$ for example, and hence the above definition works well.) With the linear mapping $\theta : a \otimes b \mapsto ba$ we define cyclic derivatives $D^{(k)}_s : \Theta(x\xi_j(N)) \to \Theta(x\xi_j(N))$, and we denote by $P(\xi_j(N), U_N^{(\cdot)}(\cdot))$ the specialization of a given $P \in C(x\xi_j(N))$ with $x_j = \xi_j(N)$ and $u_i(t) = \xi_j(N) t$, then formula (7.3) admits a ‘compact’ expression

$$D^{(k;\alpha,\beta)}_s tr_N \left( P(\xi_j(N), U_N^{(\cdot)}(\cdot)) \right) = tr_N \left( \left( D^{(k)}_s \right) P(\xi_j(N), U_N^{(\cdot)}(\cdot)) \left( \frac{1}{\sqrt{N}} C_{\alpha \beta} \right) \right) \quad \text{for any } P \in C(x\xi_j(N)) \text{ with } x_j = \xi_j(N).$$

Thus, the Clark-Ocone formula (see e.g., [18, Proposition 6.11] for any dimension and [24, subsection 1.3.4] for 1 dimension) shows that

$$E \left[ tr_N \left( P(\xi_j(N), U_N^{(\cdot)}(\cdot)) \right) \right] = E \left[ tr_N \left( \left( D^{(k)}_s \right) P(\xi_j(N), U_N^{(\cdot)}(\cdot)) \left( \frac{1}{\sqrt{N}} C_{\alpha \beta} \right) \right) \right] \quad \text{for any } P \in C(x\xi_j(N)) \text{ with } x_j = \xi_j(N).$$

7.3. Continuous tracial states. A tracial state $\varphi$ on $C^*_R(\xi_j(N))$ (or $C^*_R(x\xi_j(N), u\xi_j(N))$) is said to be continuous if $t \to u_i(t)^* \varphi(u_i(t))$ is strongly continuous (resp. $t \to \varphi(u_i(t))$) for every $1 \leq i \leq n$, where $\varphi : C^*_R(\xi_j(N)) \to H_\varphi$ (resp. $\varphi : C^*_R(\xi_j(N)) \to H_\varphi$) is the GNS representation associated with $\varphi$. We then denote by $TS^*(C^*_R(\xi_j(N), u\xi_j(N))$) all the continuous tracial states on $C^*_R(\xi_j(N))$ and $C^*_R(x\xi_j(N), u\xi_j(N))$, respectively. Set $x_j(t) := x_j$, $t \geq 0$, for each $j$ for the ease of notation.
below. Then, the same facts as [29, Lemmas 2.1,2.2] holds and the metric $d$ on $TS^c(C^*_R(x_0,u_*(\cdot)))$ can be defined in the exactly same manner as (1.1) by considering words in $x_j(t)$ and $u_i(t), u_i(t)^*$ in place of $x_{i,j}(t_1)\cdots x_{i,m}(t_m)$ for $w(t_1,\ldots,t_m)$. We remark that $\tau((x_j(s) - x_i(t))^2)$ in [29, Lemma 2.2(2)] should be replaced with $\varphi((u_i(s) - u_j(t))^2(u_i(s) - u_j(t))) = 2(1 - \text{Re } \varphi(u_j(t))^* u_i(t)))$ in this context.

7.4. Rate function. By universality, we have the $\ast$-homomorphism
\[ \Pi^\ast : C^*_R(x_0,u_*(\cdot)) \to C^*_R(x_0,u_*(\cdot),u_*(\cdot)) \]
for each $s \geq 0$, which sends each $u_i(t)$ to $u_i(t)$ and keeping each $x_j$ as it is, where
\[ u_i(t) := v_i((t-s) \vee 0)u_i(t \wedge t), \quad 1 \leq i \leq n, t \geq 0. \]
We can extend each $\varphi \in TS^c(C^*_R(x_0,u_*(\cdot),u_*(\cdot)))$ to a unique $\hat{\varphi} \in TS^c(C^*_R(x_0,u_*(\cdot),u_*(\cdot)))$ in such a way that the $v_i(t)$ are freely independent of $C^*_R(x_0,u_*(\cdot))$ and form a freely independent family of left-multiplicative free unitary Brownian motions under $\hat{\varphi}$. For each $\varphi \in TS^c(C^*_R(x_0,u_*(\cdot)))$ we define $\varphi^\ast := \hat{\varphi} \circ \Pi^\ast \in TS^c(C^*_R(x_0,u_*(\cdot)))$, $s \geq 0$, and also write
\[ (N(\varphi)) \subset M(\varphi) := (\pi_\varphi(C^*_R(x_0,u_*(\cdot))))^\ast \subset \pi_\varphi(C^*_R(x_0,u_*(\cdot),u_*(\cdot)))^\ast \]
on $H_\varphi$, where $\pi_\varphi : C^*_R(x_0,u_*(\cdot),v_*(\cdot)) \subset H_\varphi$ is the GNS representation associated with $\varphi$. We fix a distribution of the $x_j$ by $\sigma_0 \in TS(C^*_R(x_0))$. Let $\sigma_0^{BM}$ be $\varphi^\ast$ with $\varphi \in TS^c(C^*_R(x_0,u_*(\cdot)))$ such that the restriction of $\varphi$ to $C^*_R(x_0)$ is $\sigma_0$. Such a continuous tracial state $\varphi^\ast$ is uniquely determined; in fact, it is the joint distribution of the $x_j$’s and the $v_i(t)$’s such that the $v_i(t)$ form a freely independent family of left-multiplicative free unitary Brownian motions and are freely independent of the $x_j$’s, and moreover that the distribution of the $x_j$’s is $\sigma_0$. For any $\varphi \in TS^c(C^*_R(x_0,u_*(\cdot)))$, $P = P_\ast \in C(x_0,u_*(\cdot))$ and $t \in [0,\infty]$ we define
\[ I_{\sigma_0,t}^{BM}(\varphi,P) := \varphi^\ast(P) - \sigma_0^{BM}(P) - \frac{1}{2} \sum_{k=1}^n \int_0^t \| E_{N(x)}(\pi_\varphi(\Pi^\ast(\Delta^t(\varphi,P))))\|_2^2 \, ds \]
with regarding $\varphi$ as $\varphi^\ast$. Then we introduce two functionals $I_{\sigma_0}^{BM}$, $I_{\sigma_0,\infty}^{BM} : TS^c(C^*_R(x_0,u_*(\cdot))) \to [0,\infty]$ defined by
\[ I_{\sigma_0}^{BM}(\varphi) := \sup_{P = P_\ast \in C(x_0,u_*(\cdot))} I_{\sigma_0,t}^{BM}(\varphi,P), \quad I_{\sigma_0,\infty}^{BM}(\varphi) := \sup_{P = P_\ast \in C(x_0,u_*(\cdot))} I_{\sigma_0,\infty}^{BM}(\varphi,P) \]
for $\varphi \in TS^c(C^*_R(x_0,u_*(\cdot)))$.

7.5. Consequences. Here is the main consequence of this section.

**Theorem 7.2.** Assume that $\sigma_0 \in TS(C^*_R(x_0))$ is the limit distribution of $\Xi(N)$ as $N \to \infty$. We denote by $P \in C^*_R(x_0,u_*(\cdot)) \rightarrow (\Xi(N), U^{\ast}_N(\cdot)) \in M_N$ the $\ast$-homomorphism sending $u_i(t)$ and $x_j$ to $U(t)$ and $\xi_i(N)$, respectively. Let $\varphi_{BM}(N) \in TS^c(C^*_R(x_0,u_*(\cdot)))$ be the random tracial state sending $P \in C^*_R(x_0,u_*(\cdot))$ to $\text{tr}_N(P(\xi(N),U^{\ast}_N(\cdot)))$. Then we have the following large deviation upper bound:
\[ \lim_{N \to \infty} \frac{1}{N^2} \log P(\varphi_{BM}(N) \in \Lambda) \leq -\inf_{\varphi \in \Lambda} \{ I_{\sigma_0}^{BM}(\varphi) \} \]
for every closed $\Lambda \subset TS^c(C^*_R(x_0,u_*(\cdot)))$. Moreover, both $I_{\sigma_0}^{BM} \geq I_{\sigma_0,\infty}^{BM}$ are good rate functions and admit the same unique minimizer $\sigma_0^{BM}$.

Proving that the rate functions are good along the line of the proof of [29, Proposition 5.6] needs the formula
\[ E_{N(x)}(\Pi^\ast(\Delta^t(\varphi,(u_i(t_1) - u_i(t_2))^2(u_i(t_1) - u_i(t_2)))) = \delta_{k,i} \sqrt{-1} e^{-\frac{1}{2} t(t_1 \wedge t_2 - s)} \cdot 1_{t_1 \lt t_2 \lt t_1 + t_2}(s) \cdot (u_k(t_1 \wedge t_2) u_k(s)^* - u_k(s) u_k(t_1 \wedge t_2)^*). \]
Similarly to [29, Corollary 5.9] the standard Borel–Cantelli argument shows the next corollary.

**Corollary 7.3.** Keep the same setting as in Theorem 7.2. Let \( \sigma_0^{\text{frBM}} \in T S^c(C_R^s(x_0, u_\bullet(\cdot))) \) be constructed in such a way that the distribution of the \( x_j \) is \( \sigma_0 \) under \( \sigma_0^{\text{frBM}} \) and also that the \( u_i(t) \) form a freely independent family of left-multiplicative free unitary Brownian motions and are freely independent of the \( x_j \) under \( \sigma_0^{\text{frBM}} \). Then \( d(\varphi^{\text{frBM}}(P), \sigma_0^{\text{frBM}}) \to 0 \) almost surely as \( N \to \infty \).

This is a precise statement about the almost sure convergence as continuous process for an independent family of unitary Brownian motions together with deterministic matrices, and seems to have been missing so far, even though the almost sure strong convergence for its time marginals was already established by Collins, Dahlqvist and Kemp [11].

7.6. **Haar-distributed unitary random matrices.** As in section 4, using Lemma 2.1 we can derive a large deviation upper bound for an independent family of \( N \times N \) Haar-distributed unitary random matrices \( U_N^{(i)} \), \( 1 \leq i \leq n \), with deterministic matrices \( \Xi(N) \) from Theorem 7.2. The resulting rate function is given as in Lemma 4.1. Let \( C_R^s(x_0, u_\bullet) \) be the universal \( C^* \)-algebra generated by \( x_j, j \geq 1 \), and \( u_i, 1 \leq i \leq n \), with subject to \( \|x_j\|_\infty \leq R \) and \( u_i^* u_i = u_i u_i^* = 1 \).

We denote by \( P \in C_R^s(x_0, u_\bullet) \to P(\Xi(N), U_N^{(i)}) \in M_N \) the \( \ast \)-homomorphism sending \( x_j \) and \( u_i \) to \( \xi_j(N) \) and \( U_N^{(i)} \), respectively. Then we have the random tracial state \( \varphi^{\Xi(N)}(P) \in T S(C_R^s(x_0, u_\bullet)) \to C \) defined by \( \varphi^{\Xi(N)}(P) := \text{tr}_N(P(\Xi(N), U_N^{(i)})) \) for \( P \in C_R^s(x_0, u_\bullet) \). Namely, let \( \tau \in C_R^s(x_0, u_\bullet) \to C_R^s(x_0, u_\bullet(\cdot)) \) be the \( \ast \)-homomorphism sending \( x_j \) and \( u_i \) to \( x_j \) and \( u_i(T) \), respectively, as before. Then we have the large deviation upper bound for the probability measures \( \mathbb{P}(\varphi^{\text{frBM}}(\cdot)) \) with speed \( N^2 \) and the rate function

\[
\psi \in T S(C_R^s(x_0, u_\bullet)) \to \lim_{m \to 0} \lim_{T \to \infty} \inf \{ I_{\sigma_0}^{\text{frBM}}(\varphi) \mid \varphi \in T S^c(C_R^s(u_\bullet(\cdot), x_0)) \}, \tau^\ast(\varphi) = 0, \psi(\varphi) = 0 \in [0, +\infty],
\]

where as before the infimum over the empty set is taken as \(+\infty\) and \( \Omega_{m, \delta}(\psi) \) is the open neighborhood consisting of all the \( \chi \in T S(C_R^s(x_0, u_\bullet)) \) such that \( |\chi(w) - \psi(w)| < \delta \) for all words \( w \) in \( x_j, u_i, u_i^* \) \((j \leq m, 1 \leq i \leq n)\) of length not greater than \( m \).

We remark that Cabanal Duvillard and Guionnet [9, Corollary 4.2] have also obtained a large deviation upper bound for the \( U_N^{(i)} \) with seemingly different rate function based on self-adjoint matrix Brownian motions.

7.7. **Relation to the matrix liberation process.** We will compare Theorem 7.2 with [29, Theorem 5.8]. To this end, we re-number \( \xi_j(N) \) and \( x_j \) as \( \xi_j(N) \) and \( x_j \), respectively. Let \( \pi_{\text{lib}} : C_R^s(x_0, u_\bullet(\cdot)) \to C_R^s(x_0, u_\bullet(\cdot)) \) be the \( \ast \)-homomorphism sending \( x_j(t) \) to \( u_i(t)x_j(t)u_i^* \). This induces a continuous map \( \pi_{\text{lib}} : T S(C_R^s(x_0, u_\bullet(\cdot))) \to T S^c(C_R^s(x_0(\cdot))) \) defined by \( \pi_{\text{lib}}(\varphi) := \varphi \circ \pi_{\text{lib}} \). We observe that \( \pi_{\text{lib}}(\varphi^{\Xi(N)}(\cdot)) = \tau_{\Xi(N)}^{\text{lib}}(\cdot) \). Therefore, the contraction principle in large deviation theory implies the large deviation upper bound for \( \mathbb{P}(\tau_{\Xi(N)}^{\text{lib}}(\cdot)) \) in the same scale with the good rate function:

\[
\tau \in T S^c(C_R^s(x_0(\cdot))) \to I_{\sigma_0}^{\text{lib}}(\tau) := \inf \{ I_{\sigma_0}^{\text{frBM}}(\varphi) \mid \varphi \in T S^c(C_R^s(u_\bullet(\cdot), x_0)), \pi_{\text{lib}}(\varphi) = \tau \} \in [0, +\infty],
\]

where the infimum over the empty set is taken as \(+\infty\). Therefore, we have two large deviation upper bounds with (seemingly different) rate functions for \( \mathbb{P}(\tau_{\Xi(N)}^{\text{lib}}(\cdot)) \).

Let \( \tau \in T S^c(C_R^s(x_0(\cdot))) \) be given. Consider an arbitrary \( \varphi \in T S^c(C_R^s(x_0, u_\bullet(\cdot))) \) with \( \pi_{\text{lib}}(\varphi) = \tau \). It is not difficult to show that

\[
\varphi^\ast(\pi_{\text{lib}}(P)) = \tau^\ast(P), \quad E_N(\varphi^\ast(\mathbb{P}(\mathcal{D}_N^{(k)}(\pi_{\text{lib}}(P)))) = E_N(\varphi^\ast(\mathbb{P}(\mathcal{D}_N^{(k)}(P))))
\]
for every $P \in \mathbb{C}(x_{\ast} \cdot \cdot \cdot)$ and every $s \geq 0$. Therefore, $I_{\sigma_0, t}^{\text{lib}}(\tau, P) = I_{\sigma_0, t}(\varphi, \pi_{\text{lib}}(P))$ for every $P \in \mathbb{C}(x_{\ast} \cdot \cdot \cdot)$ and every $t \geq 0$, and hence

\begin{equation}
I_{\sigma_0, t}^{\text{lib}}(\tau) \leq I_{\sigma_0, \infty}^{\text{lib}}(\tau), \quad I_{\sigma_0, \infty}^{\text{lib}}(\tau) \leq I_{\sigma_0, t}^{\text{lib}}(\tau),
\end{equation}

where $I_{\sigma_0, \infty}^{\text{lib}}(\tau) := \inf\{I_{\sigma_0, t}^{\text{lib}}(\varphi) \mid \varphi \in T^s(C_R(\mathfrak{u}_{\ast}(\cdot), \mathfrak{x}_0)), \pi_{\text{lib}}(\varphi) = \tau\}$. Therefore, the current approach using unitary Brownian motions directly gives an improved large deviation upper bound for the matrix liberation process, though the description of the resulting rate function is ‘indirect’.

Remark that the above inequalities between two kinds of rate functions guarantee that $I_{\sigma_0}^{\text{lib}} \geq I_{\sigma_0, \infty}^{\text{lib}}$ also have a unique minimizer, which is given by $\sigma_0^{\text{lib}}$. Remark that this fact on the rate functions $I_{\sigma_0}^{\text{lib}} \geq I_{\sigma_0, \infty}^{\text{lib}}$ holds even when $\sigma_0$ does not fall into $TS_{\text{fda}}(C^*(x_{\ast} \cdot \cdot \cdot))$.

8. Conditional expectations of liberation cyclic derivatives

We will give a technical result on liberation cyclic derivatives $\mathcal{D}_s^{(k)}, 1 \leq k \leq n$, for future work. The most non-trivial component of the rate functions $I_{\rho_s}^{\text{lib}}, I_{\rho_s, \infty}^{\text{lib}}$ is $E_{\mathcal{N}(\tau)}(\pi_s(\Pi^s(\mathcal{D}_s^{(k)} P)))$, which will be described in terms of free cumulants when $P$ is a monomial. In what follows, we use the notations in section 4.

We first introduce some terminology: Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and $a_1, \ldots, a_n \in \mathcal{A}$ be arbitrarily chosen. For a ‘block’ $V = (i_1 < \cdots < i_k)$ of $[n] = \{1, \ldots, n\}$, we define $\text{id}(V[a_1, \ldots, a_n]) := a_{i_1} \cdots a_{i_k}$ (i.e., the word obtained by arranging $a_{i_1}, \ldots, a_{i_k}$ in order). For a partition $\pi = (V_1, \ldots, V_m)$ of $[n]$, we define

$$C(\varphi; \pi)[a_1, \ldots, a_n] := \sum_{k=1}^{m} \left( \prod_{1 \leq \ell < k} \varphi(V_{\ell})[a_1, \ldots, a_{\ell}] \right) \text{id}(V_k)[a_{\ell+1}, \ldots, a_n],$$

where $\varphi(V_{\ell})[a_1, \ldots, a_{\ell}]$ is defined as in [23, Lecture 11]; namely, we have $\varphi(V_{\ell})[a_1, \ldots, a_{\ell}] = \varphi(\text{id}(V_{\ell}))[a_1, \ldots, a_{\ell}]$.

**Proposition 8.1.** Write

$$w_{t} := v_{t_{s}}((t_{s} - s)_{+}) e_{t_{s}}((t_{s} - s)_{+}), \quad 1 \leq t \leq n,$$

with $i_0 := i_n$ and $(t - s)_{+} := 0 \vee (t - s)$. Then, we have

$$E_{\mathcal{N}(\tau)}(\pi_\pi(\Pi^s(\mathcal{D}_s^{(k)} x_{i_1j_1}(t_1) \cdots x_{i_nj_n}(t_n)))) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_{\pi} w_{t_1, \ldots, w_{t_n}} \pi_\pi(\mathcal{D}_s^{(k)} C(\tau; K(\pi))[x_{i_1j_1}(s \wedge t_1), \ldots, x_{i_nj_n}(s \wedge t_n)]),$$

where $\mathcal{NC}(n)$ denotes the non-crossing partitions of $[n]$, $\kappa_{\pi}$ the free cumulant associated with $\pi$, and $K : \mathcal{NC}(n) \rightarrow \mathcal{NC}(n)$ the Kreweras complementation map; see [23, Lecture 11].

**Proof.** Write $P = x_{i_1j_1}(t_1) \cdots x_{i_nj_n}(t_n)$ for simplicity. Let $y \in C_R^{\text{lib}}(x_{\ast} \cdot \cdot \cdot)$ be arbitrarily chosen. Then we compute

$$\tilde{\pi}(E_{\mathcal{N}(\tau)}(\pi_\pi(\Pi^s(\mathcal{D}_s^{(k)} P)))) \pi_\pi(y) = \tilde{\pi}(\Pi^s(\mathcal{D}_s^{(k)} P)y),$$

where we use the same symbol $\tilde{\pi}$ as a different meaning on each side; see subsection 4.D. By a direct computation using the trace property, we have

$$\tilde{\pi}(\Pi^s(\mathcal{D}_s^{(k)} P)y) = \sum_{s \geq t_1 \ldots s \geq t_n} \tilde{\pi}(w_{t_1, t_2} x_{i_1j_1}(s \wedge t_1) x_{i_2j_2}(s \wedge t_2) \cdots x_{i_nj_n}(s \wedge t_n)) y$$

$$= \sum_{s \geq t_1 \ldots s \geq t_n} \tilde{\pi}(w_{t_1, t_2} x_{i_1j_1}(s \wedge t_1) y) x_{i_2j_2}(s \wedge t_2) \cdots w_{t_n, t_1} x_{i_nj_n}(s \wedge t_n).$$
each of whose terms is the \( \tilde{\tau} \)-value of the monomial obtained from \( \Pi^s(P) \) by replacing \( x_{ij}(s \land t_e) \) with \( [x_{ij}(s \land t_e), y] \). By \cite[Theorem 14.4]{23} we obtain that

\[
\sum_{i_s = k}^{\infty} \sum_{s \leq t_e} \tau(w_1 x_{i_1j_1}(s \land t_1) \cdots w_n x_{i_nj_n}(s \land t_n))
\]

\[
= \sum_{i_s = k}^{\infty} \sum_{s \leq t_e} \kappa_{\pi}[w_1, \ldots, w_n] \tilde{\tau}_{\Pi^s(P)}[x_{i_1j_1}(s \land t_1), \ldots, x_{i_nj_n}(s \land t_n)]
\]

\[
= \sum_{\pi \in NC(n)} \kappa_{\pi}[w_1, \ldots, w_n] \left( \sum_{i_s = k}^{\infty} \sum_{s \leq t_e} \tau_{\Pi^s(P)}[x_{i_1j_1}(s \land t_1), \ldots, x_{i_nj_n}(s \land t_n)] \right)
\]

When \( K(\pi) = \{V_1, \ldots, V_m\} \) with \( \ell \in V_p \) \( (1 \leq p \leq m) \), we have

\[
\sum_{i_s = k}^{\infty} \sum_{s \leq t_e} \tau(w_1 x_{i_1j_1}(s \land t_1) \cdots w_n x_{i_nj_n}(s \land t_n))
\]

\[
= \sum_{i_s = k}^{\infty} \sum_{s \leq t_e} \tau\left(\prod_{1 \leq q \leq m, q \neq p} \tau(V_q)[x_{i_1j_1}(s \land t_1), \ldots, x_{i_nj_n}(s \land t_n)] \right) \times \tau(V_p)[x_{i_1j_1}(s \land t_1), \ldots, x_{i_nj_n}(s \land t_n)]
\]

\[
= \sum_{i_s = k}^{\infty} \sum_{s \leq t_e} \tau\left(\prod_{1 \leq q \leq m, q \neq p} \tau(V_q)[x_{i_1j_1}(s \land t_1), \ldots, x_{i_nj_n}(s \land t_n)] \right) \times \tau(V_p)[x_{i_1j_1}(s \land t_1), \ldots, x_{i_nj_n}(s \land t_n)]
\]

If \( V_p = \{s_1 < \cdots < s_{t_e}\} \) with \( s_{t_e} = \ell \), then

\[
\tau(V_p)[x_{i_1j_1}(s \land t_1), \ldots, x_{i_nj_n}(s \land t_n)] = \tau([x_{s_{t_e-1}j_{s_{t_e-1}}(s \land t_{s_{t_e-1}}) \cdots x_{s_{t_e-1}j_{s_{t_e-1}}(s \land t_{s_{t_e-1}})} x_{i_1j_1}(s \land t_1) \ldots x_{i_nj_n}(s \land t_n)] y),
\]

which together with the definition of \( D^{(k)} \) implies that

\[
\sum_{i_s = k}^{\infty} \sum_{s \leq t_e} \tau\left(\prod_{1 \leq q \leq m, q \neq p} \tau(V_q)[x_{i_1j_1}(s \land t_1), \ldots, x_{i_nj_n}(s \land t_n)] \right) \times \tau([x_{s_{t_e-1}j_{s_{t_e-1}}(s \land t_{s_{t_e-1}}) \cdots x_{s_{t_e-1}j_{s_{t_e-1}}(s \land t_{s_{t_e-1}})} x_{i_1j_1}(s \land t_1) \ldots x_{i_nj_n}(s \land t_n)] y)
\]

\[
= \tilde{\tau}((D^{(k)} C; K(\pi))[x_{i_1j_1}(s \land t_1), \ldots, x_{i_nj_n}(s \land t_n)]) \pi_\pi(y).
\]

Hence we conclude that

\[
\tilde{\tau}(E_{\Pi}(\tau)[w_1, \ldots, w_n]) \pi_\pi(y)
\]

\[
= \sum_{\pi \in NC(n)} \kappa_{\pi}[w_1, \ldots, w_n] \tilde{\tau}(\pi_\pi[D^{(k)} C; K(\pi)][x_{i_1j_1}(s \land t_1), \ldots, x_{i_nj_n}(s \land t_n)]) \pi_\pi(y).
\]

Hence we are done. \( \square \)

It is interesting to compute \( \kappa_{\pi}[w_1, \ldots, w_n] \) in the above explicitly.
Appendix A. Universal free products of unital C*-algebras

The concept of universal free products in the category of unital C*-algebras has been studied in detail by several hands, including Blackadar [6], Pedersen [25] and others. However, almost all existing works deal with only universal free products of two unital C*-algebras. We have used universal free products of uncountably many unital C*-algebras crucially (even in [29] without any references). Hence, we will collect a few facts on universal free products of arbitrary number of unital C*-algebras with explicit explanations for the reader’s convenience. However, we do not claim any credit to the materials in this appendix, because they all seem to be known among specialists.

Let $A_i, i \in I$, be unital C*-algebras. Consider their universal free product $\ast_{i \in I} A_i$ with canonical unital $*$-homomorphisms $\lambda_i : A_i \to \ast_{i \in I} A_i$, $i \in I$, which is characterized by the universality asserting that for any family $\pi_i : A_i \to B$ of unital $*$-homomorphisms into a common unital C*-algebra, then there exists a unital $*$-homomorphism $\pi : \ast_{i \in I} A_i \to B$ such that $\pi \circ \lambda_i = \pi_i$ for all $i \in I$. Note that the injectivity of each $\lambda_i$ is guaranteed by universality. Then $\lambda_i(a)$ with $a \in A_i$ is sent to the corresponding element in the $i$th free product component $\ast_{i \in I} A_i$ on the right-hand side when $i \in I$.

Lemma A.1. For any disjoint decomposition $I = \bigsqcup_{j \in J} I_j$ of $I$ into non-empty subsets, we consider the universal free product C*-algebras $\ast_{i \in I_j} A_i$, $j \in J$. Then $\ast_{i \in I} A_i \cong \ast_{j \in J} (\ast_{i \in I_j} A_i)$ naturally, that is, each $\lambda_i(a)$ with $a \in A_i$ is sent to the corresponding element in the $i$th free product component $\ast_{i \in I} A_i$ on the right-hand side when $i \in I$.

Proof. This follows from the universality of the involved universal free product C*-algebras. □

Lemma A.2. For each finite subset $F \subseteq I$, we consider the universal free product C*-algebra $\mathfrak{A}_F := \ast_{i \in F} A_i$, with setting $\mathfrak{A}_{\emptyset} := C1$. Then the following hold true:

1. If $F_1 \subseteq F_2$, then the canonical unital $*$-homomorphism $\mathfrak{A}_{F_1} \to \mathfrak{A}_{F_2} \ast_{F_2 \setminus F_1} = \mathfrak{A}_{F_2}$ via Lemma A.1 is injective.

2. $\ast_{i \in I} A_i \cong \lim_{\mathcal{F}_F} \mathfrak{A}_F$ naturally (see e.g. [19, Proposition 11.4.1(i)] for the latter), that is, the isomorphism sends each $\lambda_i(a)$ with $a \in A_i$ to the corresponding one in $\mathfrak{A}_F$ with $i \in F$.

Proof. (1) follows from Blackadar’s result [6, Theorem 3.1]. (2) follows from [6, Theorem 3.1] and [19, Proposition 11.4.1(ii)] for example. □

Proposition A.3. Let $B_i \subseteq A_i$, $i \in I$, be unital C*-subalgebras. Then the universal free product C*-algebra $\ast_{i \in I} B_i$ is naturally embedded into $\ast_{i \in I} A_i$. Namely, $\ast_{i \in I} B_i$ can be identified with the C*-subalgebra generated by the $\lambda_i(B_i)$ and the canonical unital $*$-homomorphisms from $B_i$ into $\ast_{i \in I} A_i$ is given by the restriction of $\lambda_i$ to $B_i$.

Proof. Write $\mathfrak{B}_F := \ast_{i \in F} B_i$ for each finite subset $F \subseteq I$ with $\mathfrak{B}_{\emptyset} := C1$. By the iterative use of Pedersen’s result [25, Theorem 4.2] with the help of Lemma A.1 we can see that $\mathfrak{B}_F \hookrightarrow \mathfrak{A}_F$ naturally. Then, by e.g. [19, Proposition 11.4.1(ii)] we have a natural unital injective $*$-homomorphism from $\lim_{\mathcal{L}_F} \mathfrak{B}_F$ into $\lim_{\mathcal{L}_{\mathfrak{A}_F}} \mathfrak{A}_F$ by means of inductive limits. Thus the desired assertion follows thanks to Lemma A.2(2).

□

Proposition A.4. Let $\ast_{i \in I}^b A_i$ be the free product of the $\lambda_i(A_i)$, $i \in I$, in the category of unital $*$-algebras, in which we regard each $A_i$ as a unital $*$-subalgebra. Let $\lambda : \ast_{i \in I} A_i \to \ast_{i \in I}^b A_i$ be the unique $*$-homomorphism sending $a \in A_i \subseteq \ast_{i \in I} A_i$ to $\lambda_i(a) \in \ast_{i \in I}^b A_i$, whose existence is guaranteed by universality. Then $\lambda$ must be injective. Namely, the $*$-algebra algebraically generated by the $\lambda_i(A_i)$ in $\ast_{i \in I} A_i$ can be identified with $\ast_{i \in I}^b A_i$.

Proof. We have to show that if $a \in \ast_{i \in I}^b A_i$ satisfies $\lambda(a) = 0$, then $a = 0$. To this end we will use the reduced free product construction, see e.g. [32], following Avitzour’s idea [2, Proposition 2.3].
Let $a \in \bigstar_{i \in I} A_i$ be given. Then $a$ is nothing but a linear combination of words whose letters from the $A_i$. For each $i \in I$ we let $A_{i0}$ be the unital $C^*$-subalgebra of $A_i$ generated by the letters from $A_i$ (with fixed $i$) appearing in the words in the linear combination description of $a$. Since there are only finitely many letters for each $i \in I$, $A_{i0}$ must be separable. By Proposition A.3 we may and do regard $\bigstar_{i \in I} A_{i0}$ as a unital $C^*$-algebra of $\bigstar_{i \in I} A_i$ naturally, and $\lambda(a)$ falls into $\bigstar_{i \in I} A_{i0}$. Hence we may and do regard each $A_i$ as a separable unital $C^*$-algebra.

We claim that for each $i \in I$ there exists a faithful state $\omega_i$ on $A_i$. Since $A_i$ is separable, it faithfully acts on a separable Hilbert space, say $\pi : A_i \to K$. See [12, Theorem 1.9.12]. Then we choose a dense sequence of non-zero vectors $\xi_n \in K$ and set $\omega_i(a) := \sum_{n=1}^{\infty} \frac{1}{n} \langle \pi(a)\xi_n | \xi_n \rangle_K$ for $a \in A_i$. This clearly defines a faithful state.

Consider the reduced $C^*$-free product $(\mathcal{F}, \omega) = \bigstar_{i \in I} (A_i, \omega_i)$ with canonical $*$-homomorphisms $\gamma_i : A_i \to \mathcal{F}$. See e.g. [32]. By universality, we have a unique $*$-homomorphism $\gamma : \bigstar_{i \in I} A_i \to \mathcal{F}$ such that $\gamma \circ \lambda_i = \gamma_i$ for every $i \in I$. Write

$$\bigstar_{i \in I} A_i = C_1 + \sum_{m \geq 1} \sum_{i_k \neq i_{k+1}} A_{i_1} \cdots A_{i_m}$$

with $A_{i_1} := \text{Ker}(\omega_i)$, where $A_{i_1} \cdots A_{i_m}$ denotes all the linear combinations of words $a_{i_1} \cdots a_{i_m}$ with $a_{i_k} \in A_{i_k}$. According to this representation we write

$$a = a_1 + \sum_{m \geq 1} \sum_{i_k \neq i_{k+1}} a(i_1, \ldots, i_m),$$

where $a^\circ(i_1, \ldots, i_m)$ is an element in $A_{i_1}^\circ \cdots A_{i_m}^\circ$. Remark that $a(i_1, \ldots, i_m) = 0$ for all but except finitely many $(i_1, \ldots, i_m)$. We denote by $a^\circ(i_1, \ldots, i_m)^\circ$ in the spacial (or minimal) $C^*$-tensor product $A_1 \otimes \cdots \otimes A_m$ the corresponding elements obtained by changing each word $a_{i_1}^\circ \cdots a_{i_m}^\circ$ appearing in $a^\circ(i_1, \ldots, i_m)$ to a simple tensor $a_{i_1}^\circ \otimes \cdots \otimes a_{i_m}^\circ \in A_1 \otimes \cdots \otimes A_m$. By universality of algebraic tensor products sitting inside $A_1 \otimes \cdots \otimes A_m$ (which is simply confirmed by the iterative use of a well-known fact, see e.g. [19, Proposition 11.18] or a more direct statement [7, Corollary 3.1]), we observe that $a^\circ(i_1, \ldots, i_m)^\circ = 0$ implies $a^\circ(i_1, \ldots, i_m) = 0$.

Assume that $\lambda(a) = 0$. Since $\pi_a(\gamma(\lambda(a))) \xi_\omega = \alpha \xi_\omega + \sum_{m \geq 1} \sum_{i_k \neq i_{k+1}} \pi_{\omega}(\gamma(\lambda(a(a^\circ(i_1, \ldots, i_m)))) \xi_\omega,$

where $(H_\omega, \pi_\omega, \xi_\omega)$ is the GNS triple of $(\mathcal{F}, \omega)$. By the free independence among the $\lambda_i(A_i)$, we can easily see that $\alpha \xi_\omega$ and the $\pi_{\omega}(\gamma(\lambda(a(a^\circ(i_1, \ldots, i_m)))) \xi_\omega$ are mutually orthogonal in $H_\omega$. In particular, $\alpha$ as well as all the $\pi_{\omega}(\gamma(\lambda(a(a^\circ(i_1, \ldots, i_m)))) \xi_\omega$ must be 0. Let $(H_\omega, \pi_\omega, \xi_\omega)$ be the GNS triple of $(A_i, \omega_i)$. Then, it is easy to see that the norm of each $\pi_{\omega}(\gamma(\lambda(a(a^\circ(i_1, \ldots, i_m)))) \xi_\omega$ is the same as that of

$$(\pi_{\omega_1} \otimes \cdots \otimes \pi_{\omega_m})(a^\circ(i_1, \ldots, i_m)^\circ)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}),$$

which must be 0 too. Since $\omega$ is faithful, so is $\pi_{\omega_1}$ and hence the tensor product representation $\pi_{\omega_1} \otimes \cdots \otimes \pi_{\omega_m} : A_{i_1} \otimes \cdots \otimes A_{i_m} \rtimes H_{\omega_1} \otimes \cdots \otimes H_{\omega_m}$ too (see e.g. [19, Theorem 11.1.3]). We conclude that $a^\circ(i_1, \ldots, i_m)^\circ = 0$ so that $a^\circ(i_1, \ldots, i_m) = 0$. Consequently, $a$ must be 0.

References

[1] G. Anderson, A. Guionnet and O. Zeitouni, An Introduction to Random Matrices. Cambridge Studies in Advanced Mathematics, 118. Cambridge University Press, Cambridge, 2009.

[2] D. Avitzour, Free products of $C^*$-algebras. Trans. Amer. Math. Soc., 271 (1982), 423–435.
Ph. Biane, Free Brownian motion, free stochastic calculus and random matrices. Free Probability Theory, Fields Institute Communications, 12 (1997), 1–19.

[4] Ph. Biane, M. Capitaine and A. Guionnet, Large deviation bounds for matrix Brownian motion. Invent. math., 152 (2003), 483–459.

[5] Ph. Biane and Y. Dabrowski, Concavification of free entropy, Adv. Math., 234 (2013), 667–696.

[6] B. Blackadar, Weak expectations and nuclear $C^*$-algebras. Indiana Univ. Math. J., 26 (1978), 1021–1026.

[7] N.P. Brown and N. Ozawa, $C^*$-algebras and Finite-dimensional Approximations. Graduate Studies in Mathematics, 88, American Mathematical Society, Providence, RI, 2008.

[8] P.F. Byrd and M.D. Friedman, Handbook of Elliptic Integrals for Engineers and Scientists. Springer-Verlag Berlin Heidelberg New York, 1971.

[9] T. Cabanal Duvillard and A. Guionnet, Large deviations upper bounds for the laws of matrix-valued processes and non-commutative entropies. Ann. Probab., 29 (2001), 1205–1261.

[10] I. Chavel, Eigenvalues in Riemannian Geometry. Pure and applied mathematics, Academic Press, 1984.

[11] B. Collins, A. Dahlqvist and T. Kemp, The spectral edge of unitary Brownian motion. Probab. Theory Relat. Fields, 170 (2018), 49–93.

[12] K.R. Davidson, $C^*$-algebras by Example. Fields Inst. Monographs, 6. Amer. Math. Soc., 1996.

[13] A. Dembo and O. Zeitouni, Large Deviations Techniques and Applications. Springer, 1998.

[14] A. Guionnet and M. Maïda, Character expansion method for the first order asymptotics of a matrix integral. Probab. Theory Relat. Fields, 132 (2005), 539–578.

[15] F. Hiai, T. Miyamoto and Y. Ueda, Orbital approach to microstate free entropy, Internat. J. Math., 20 (2009), 227–273.

[16] F. Hiai and D. Petz, The Semicircle Law, Free Random Variables and Entropy. Mathematical Surveys and Monographs, Vol. 77. Amer. Math. Soc., Providence, 2000.

[17] F. Hiai and Y. Ueda, Orbital free pressure and its Legendre transform. Comm. Math. Phys., 334 (2015), 275–300.

[18] Y. Hu, Analysis on Gaussian Spaces. World Scientific, 2016.

[19] R.V. Kadison and J.R. Ringrose, Fundamentals of the Theory of Operator Algebras, Vol.2. Graduate studies in mathematics, 15. Amer. Math. Soc., Providence, RI, 1997.

[20] T. Lévy and M. Maïda, Central limit theorem for the heat kernel measure on the unitary group. J. Funct. Anal., 259 (2010), 3163–3204.

[21] T. Lévy and M. Maïda, On the Douglas–Kazakov phase transition. ESAIM: Proc., 51 (2015), 89–121.

[22] P. Li and S.T. Yau, On the parabolic kernel of the Schrödinger operator. Acta Math., 156 (1986), 153–201.

[23] A. Nica and R. Speicher, Lectures on the Combinatorics of Free Probability. London Mathematical Society Lecture Notes Series, 335. Cambridge University Press, 2006.

[24] D. Nualart, Malliavin Calculus and its Related Topics, 2nd edn., Springer, Berlin (2006).

[25] G.K. Pedersen, Pullback and pushout constructions in $C^*$-algebra theory. Jour. Funct. Anal., 167 (1999), 243–344.

[26] W. Rudin, Principles of Mathematical Analysis, Third Edition. McGraw-Hill, 1976.

[27] Y. Ueda, Orbital free entropy, revisited, Indiana Univ. Math. J., 63 (2014), 551–577.

[28] Y. Ueda, A remark on orbital free entropy. Arch. Math., 108 (2017), 629–638.

[29] Y. Ueda, Matrix liberation process I: Large deviation upper bound and almost sure convergence. J. Theor. Probab., 32 (2019), 806–847.

[30] D. Voiculescu, The analogue of entropy and of Fisher’s information measure in free probability theory VI: Liberation and mutual free information, Adv. Math., 146 (1999), 101–166.

[31] D. Voiculescu, Free entropy. Bull. London Math. Soc., 34 (2002), 257–278.

[32] D.-V. Voiculescu, K.-J. Dykema and A. Nica, Free Random Variables, CRM Monograph Series 1. Amer. Math. Soc., Providence, RI, 1992.