CONFORMALLY FLAT AFFINE HYPERSURFACES WITH SEMI-PARALLEL CUBIC FORM*

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Abstract  In this paper, we study locally strongly convex affine hypersurfaces with the vanishing Weyl curvature tensor and semi-parallel cubic form relative to the Levi-Civita connection of the affine metric. As a main result, we classify these hypersurfaces as not being of a flat affine metric. In particular, 2 and 3-dimensional locally strongly convex affine hypersurfaces with semi-parallel cubic forms are completely determined.

Key words  affine hypersurface; semi-parallel cubic form; Levi-Civita connection; conformally flat; warped product

2010 MR Subject Classification  53A15; 53B20; 53B25

1 Introduction

The classical equiaffine differential geometry is mainly concerned with the geometric properties and invariants of hypersurfaces in the affine space which are invariant under unimodular affine transformations. Let \( \mathbb{R}^{n+1} \) be the \((n+1)\)-dimensional real unimodular affine space. For any non-degenerate hypersurface immersion of \( \mathbb{R}^{n+1} \), it is well known how to induce the affine connection \( \nabla \), the affine shape operator \( S \) whose eigenvalues are called affine principal curvatures, and the affine metric \( h \). The classical Pick-Berwald theorem states that the cubic form \( C := \nabla h \) vanishes if and only if the hypersurface is a non-degenerate hyperquadric. In that sense, the cubic form plays the role of the second fundamental form for submanifolds of real space forms.

In recent decades, in a manner similar to that for the Pick-Berwald theorem, geometric conditions on the cubic form have been used to classify natural classes of affine hypersurfaces; see e.g., [4, 6, 7, 12, 14–17, 27, 29, 30]. Among these, one of the most interesting developments may be the classification of locally strongly convex affine hypersurfaces with \( \nabla C = 0 \), where \( \nabla \) is the Levi-Civita connection of the affine metric. In this regard, Dillen, Vrancken, et al. obtained the classifications for lower dimensions in [10, 13, 18, 25], and Hu, Li and Vrancken completed the classification for all dimensions as follows:

*Received June 10, 2022; revised May 20, 2023. This work was supported by the NNSF of China (12101194, 11401173).
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Theorem 1.1 ([20]) Let $M$ be an $n$-dimensional ($n \geq 2$) locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{\nabla}C = 0$. Then $M$ is either a hyperquadric (i.e., $C = 0$) or a hyperbolic affine hypersphere with $C \neq 0$. In the latter case, either

(i) $M$ is obtained as the Calabi product of a lower dimensional hyperbolic affine hypersphere with parallel cubic form and a point, or

(ii) $M$ is obtained as the Calabi product of two lower dimensional hyperbolic affine hyperspheres with parallel cubic form, or

(iii) $n = \frac{1}{2}m(m+1) - 1$, $m \geq 3$, $(M, h)$ is isometric to $\text{SL}(m, \mathbb{R})/\text{SO}(m)$, and $M$ is affinely equivalent to the standard embedding $\text{SL}(m, \mathbb{R})/\text{SO}(m) \hookrightarrow \mathbb{R}^{n+1}$, or

(iv) $n = m^2 - 1$, $m \geq 3$, $(M, h)$ is isometric to $\text{SL}(m, \mathbb{C})/\text{SU}(m)$, and $M$ is affinely equivalent to the standard embedding $\text{SL}(m, \mathbb{C})/\text{SU}(m) \hookrightarrow \mathbb{R}^{n+1}$, or

(v) $n = 2m^2 - m - 1$, $m \geq 3$, $(M, h)$ is isometric to $\text{SU}^*(2m)/\text{Sp}(m)$, and $M$ is affinely equivalent to the standard embedding $\text{SU}^*(2m)/\text{Sp}(m) \hookrightarrow \mathbb{R}^{n+1}$, or

(vi) $n = 26$, $(M, h)$ is isometric to $\text{E}_{6(-26)}/\text{F}_4$, and $M$ is affinely equivalent to the standard embedding $\text{E}_{6(-26)}/\text{F}_4 \hookrightarrow \mathbb{R}^{27}$.

As in [20, 21], we say that an affine hypersurface has the semi-parallel (resp. parallel) cubic form relative to the Levi-Civita connection of the affine metric if $\hat{\nabla}C = 0$ (resp. $\hat{\nabla}C = 0$), where $\hat{\nabla}$ is the curvature tensor of the affine metric, and the tensor $\hat{\nabla}C$ is defined by

$$\hat{\nabla}(X, Y) \cdot C = \hat{\nabla}_X \hat{\nabla}_Y C - \hat{\nabla}_Y \hat{\nabla}_X C - \hat{\nabla}_{[X,Y]} C$$

(1.1)

for tangent vector fields $X, Y$. Obviously, the parallelism of cubic form implies its semi-parallelism; the converse is not true, and we refer to Remark 3.2 for the counter-examples.

As a generalization of Theorem 1.1, one may naturally posit the following problem:

Problem 1.2 Classify all the $n$-dimensional locally strongly convex affine hypersurfaces with $\hat{\nabla}C = 0$.

Related to Problem 1.2, Hu and Xing [21] showed that such a surface is either a quadric or a flat affine surface. Recently, the present authors and Xing [24] gave an answer to Problem 1.2 in two cases for such hypersurfaces with at most one affine principal curvature of multiplicity one: one with no affine hyperspheres, the other with affine hyperspheres of constant scalar curvature. Based on the latter, the following conjecture was posed:

Conjecture 1.3 For any locally strongly convex affine hypersphere of dimension $n$, the semi-parallelism and parallelism of the cubic form are equivalent, i.e., $\hat{\nabla}C = 0$ iff $\hat{\nabla}C = 0$.

In this paper, we continue to study locally strongly convex affine hypersurfaces with $\hat{\nabla}C = 0$, and pay attention to the case where the Weyl curvature tensor vanishes identically. First, by investigating such affine hyperspheres, we can confirm Conjecture 1.3 for this case.

Theorem 1.4 Let $M^n$, $n \geq 3$ be a locally strongly convex affine hypersphere in $\mathbb{R}^{n+1}$ with $\hat{\nabla}C = 0$ and the vanishing Weyl curvature tensor. Then $M^n$ is affinely equivalent to either a hyperquadric, or the flat and hyperbolic affine hypersphere

$$x_1x_2 \cdots x_{n+1} = 1,$$

(1.2)

or the hyperbolic affine hypersphere with quasi-Einstein affine metric

$$(x_n^2 - x_1^2 - \cdots - x_{n-1}^2)^n x_{n+1}^2 = 1,$$

(1.3)
where \((x_1, \cdots, x_{n+1})\) are the standard coordinates of \(\mathbb{R}^{n+1}\).

From Theorem 1.4 and the fact that the Weyl curvature tensor vanishes automatically for \(n = 3\), we immediately obtain

**Corollary 1.5** Let \(M^3\) be a locally strongly convex affine hypersphere in \(\mathbb{R}^4\) with \(\hat{R} \cdot C = 0\). Then \(M^3\) is affinely equivalent to either a hyperquadric, or one of the two hyperbolic affine hyperspheres

\[
x_1x_2x_3x_4 = 1, \quad (x_3^2 - x_1^2 - x_2^2)^3x_4^2 = 1,
\]

where \((x_1, \cdots, x_4)\) are the standard coordinates of \(\mathbb{R}^4\).

**Remark 1.6** Conjecture 1.3 is true for \(n = 2, 3, 4\), by the following three facts:

1. By Theorem 1.1 and Corollary 2.1 in [9], Theorem 1.4 confirms Conjecture 1.3 if either \(n = 3\), or \(M^n\) is conformally flat for \(n \geq 4\).
2. It has been shown in [24] that Conjecture 1.3 is true if either \(n = 2\), or \(M^n\) is a constant scalar curvature for \(n \geq 3\).
3. It was shown in Lemma 3.1 of [13] that if \(\hat{R} \cdot C = 0\) and \(n = 4\), then the Pick invariant \(J\) can only take four constant values at any point, which implies that \(J\), and thus the scalar curvature are constant. This, together with (2), confirms Conjecture 1.3 for \(n = 4\).

Second, by removing the restriction of affine hyperspheres, we give an answer to Problem 1.2 under the assumption that the Weyl curvature tensor vanishes identically.

**Theorem 1.7** Let \(M^n, n \geq 3\) be a locally strongly convex affine hypersurface in \(\mathbb{R}^{n+1}\) with \(\hat{R} \cdot C = 0\) and the vanishing Weyl curvature tensor. Denote by \(m\) the number of distinct eigenvalues of its Schouten tensor. Then \(m \leq 2\), and one of the two cases occurs:

(i) \(m = 1\), \(M^n\) is either a hyperquadric, or a flat affine hypersurface;

(ii) \(m = 2\), \(M^n\) is affinely equivalent to either (1.3), or one of the six quasi-umbilical affine hypersurfaces with the quasi-Einstein affine metric of nonzero scalar curvature, as is explicitly described in Theorem 4.3.

**Remark 1.8** \(m = 1\) in case (i) means that the affine metric is a constant sectional curvature. Related to this, we prove in Proposition 3.1 that any non-degenerate affine hypersurface satisfies that \(\hat{R} \cdot C = 0\), and that the affine metric is a constant sectional curvature if and only if it is either a hyperquadric or a flat affine hypersurface.

**Remark 1.9** The case (i) is partially classified in Remark 4.1, though its complete classification is still complicated and not solved. In fact, Antić-Li-Vrancken-Wang [3] recently classified the locally strongly convex affine hypersurfaces with constant sectional curvature for when the hypersurface admits at most one affine principal curvature of multiplicity one. Even for affine surfaces with a flat affine metric, the classification problem is still open.

**Remark 1.10** The examples in case (ii) are of the generalized Calabi compositions of a hyperquadric and a point in some special forms [1]. The construction method of such examples initially originates from Calabi [8], and has been extended and characterized by Antić, Dillen, Vrancken, Hu, Li, et al. in [1, 2, 11, 19, 22].

The rest of this paper is organized as follows. In Section 2, we briefly review the local theory of equiaffine hypersurfaces, and some notions and results of conformally flat manifolds and warped product manifolds. In Section 3, we study the properties of the hypersurfaces...
involving the eigenvalues and eigenvalue distributions of the Schouten tensor, the difference
tensor and the affine principal curvatures, and present the proof of Theorem 1.4. Based on these
properties and known results, in Section 4 we discuss all of the possibilities of the immersion,
and complete the proof of Theorem 1.7.

2 Preliminaries

In this section, we briefly review the local theory of equiaffine hypersurfaces. For more
details, we refer to the monographs [23, 26].

Let \( \mathbb{R}^{n+1} \) denote the standard \((n + 1)\)-dimensional real unimodular affine space that is
endowed with its usual flat connection \( D \) and a parallel volume form \( \omega \), given by the determi-
ant. Let \( F : M^n \to \mathbb{R}^{n+1} \) be an oriented non-degenerate hypersurface immersion. On such
a hypersurface, up to a sign, there exists a unique transversal vector field \( \xi \), called the affine
normal. A non-degenerate hypersurface equipped with the affine normal is called an (equi)affine
hypersurface, or a Blaschke hypersurface. Denote always by \( X, Y, Z, U \) the tangent vector fields
on \( M^n \). By the affine normal we can write that

\[
D_X F, Y = F, \nabla_X Y + h(X, Y)\xi, \quad \text{(Gauss formula)} \quad (2.1)
\]

\[
D_X \xi = -F, S X, \quad \text{(Weingarten formula)} \quad (2.2)
\]

which induce on \( M^n \) the affine connection \( \nabla \), a semi-Riemannian metric \( h \) called the affine
metric, the affine shape operator \( S \), and the cubic form \( C := \nabla h \). An affine hypersurface is said
to be locally strongly convex if \( h \) is definite, and we always choose \( \xi \), up to a sign, such that \( h \)
is positive definite. We call a locally strongly convex affine hypersurface quasi-umbilical (resp.
quasi-Einstein) if it admits exactly two distinct eigenvalues of \( S \) (resp. Ricci tensor of \( h \)), one
of which is simple.

Let \( \hat{\nabla} \) be the Levi-Civita connection of the affine metric \( h \). The difference tensor \( K \) is
defined by

\[
K(X, Y) := \nabla_X Y - \hat{\nabla}_X Y. \quad (2.3)
\]

We also write \( K_X Y \) and \( K_X = \nabla_X - \hat{\nabla}_X \). Since both \( \nabla \) and \( \hat{\nabla} \) have zero torsion, \( K \) is
symmetric in \( X \) and \( Y \). It is also related to the totally symmetric cubic form \( C \) by

\[
C(X, Y, Z) = -2h(K(X, Y), Z), \quad (2.4)
\]

which implies that the operator \( K_X \) is symmetric relative to \( h \). Moreover, \( K \) satisfies the
apolarity condition, namely, \( \text{tr} K_X = 0 \) for all \( X \).

The curvature tensor \( \hat{R} \) of the affine metric \( h \), \( S \) and \( K \) are related by the following Gauss
and Codazzi equations:

\[
\hat{R}(X, Y)Z = \frac{1}{2} [h(Y, Z)SX - h(X, Z)SY + h(SY, Z)X - h(SX, Z)Y] - [K_X, K_Y]Z, \quad (2.5)
\]

\[
(\hat{\nabla}_X K)(Y, Z) - (\hat{\nabla}_Y K)(X, Z) = \frac{1}{2} [h(Y, Z)SX - h(X, Z)SY - h(SY, Z)X + h(SX, Z)Y], \quad (2.6)
\]

\[
(\hat{\nabla}_X S)Y - (\hat{\nabla}_Y S)X = K(SX, Y) - K(SY, X). \quad (2.7)
\]
Here, by definition, $[K_X, K_Y]Z = K_X K_Y Z - K_Y K_X Z$, and
\[
\hat{R}(X, Y)Z = \hat{\nabla}_X \hat{\nabla}_Y Z - \hat{\nabla}_Y \hat{\nabla}_X Z - \hat{\nabla}_{[X,Y]} Z,
\]
\[
(\hat{\nabla}_X K)(Y, Z) = \hat{\nabla}_X (K(Y,Z)) - K(\hat{\nabla}_X Y, Z) - K(Y, \hat{\nabla}_X Z),
\]
\[
(\hat{\nabla}_X S)Y = \hat{\nabla}_X (SY) - S\nabla_X Y.
\]
Contracting the Gauss equation (2.5) twice we have that
\[
\chi = H + J,
\]
where $J = \frac{1}{n(n-1)}h(K, K)$, $H = \frac{1}{n} \text{tr } S$, $\chi = \frac{r}{n(n-1)}$ and $r$ are the Pick invariant, affine mean curvature, normalized scalar curvature and scalar curvature of $h$, respectively. Recall the following Ricci identity:
\[
(\hat{R}(X, Y) \cdot K)(Z, U) = \hat{R}(X, Y)K(Z, U) - K(\hat{R}(X, Y)Z, U) - K(Z, \hat{R}(X, Y)U).
\]
$M^n$ is called an affine hypersphere if $S = H \text{id}$. Then it follows from (2.7) that $H$ is constant if $n \geq 2$. $M^n$ is said to be a proper (resp. improper) affine hypersphere if $H$ is nonzero (resp. zero). Moreover, a locally strongly convex affine hypersphere is said to be parabolic, elliptic or hyperbolic according to whether $H = 0$, $H > 0$ or $H < 0$, respectively. For affine hyperspheres, the Gauss and Codazzi equations reduce to
\[
\hat{R}(X, Y)Z = H[h(Y, Z)X - h(X, Z)Y] - [K_X, K_Y]Z,
\]
\[
(\hat{\nabla}_X K)(Y, Z) = (\hat{\nabla}_Y K)(X, Z).
\]
We collect the following two results for later use:

**Theorem 2.1** (cf. Theorem 1.1 and Corollary 2.1 of [9]) Let $M^n$, $n \geq 3$ be a locally strongly convex affine hypersphere in $\mathbb{R}^{n+1}$ with the constant scalar curvature $r$. Then $Jr \leq 0$ and the traceless Ricci tensor $\text{Ric}$ satisfies that
\[
\|\text{Ric}\|_h^2 \leq -\frac{(n+1)(n-2)}{n+2}Jr,
\]
where $\| \cdot \|_h$ denotes the tensorial norm with respect to $h$. This equality sign holds identically if and only if $M^n$ has the parallel cubic form and the vanishing Weyl curvature tensor, and if and only if one of three cases occurs:

(i) $J = 0$, $M^n$ is affinely equivalent to a hyperquadric;

(ii) $J \neq 0$, $r = 0$, $M^n$ is affinely equivalent to (1.2);

(iii) $J \neq 0$, $r < 0$, $M^n$ is affinely equivalent to (1.3).

**Theorem 2.2** (cf. Theorem 1 of [1]) Let $M^{m+1}$, $m \geq 2$ be a locally strongly convex affine hypersurface of the affine space $\mathbb{R}^{m+2}$ such that its tangent bundle is an orthogonal sum with respect to the affine metric $h$ of two distributions: a one-dimensional distribution $\mathcal{D}_1$ spanned by a unit vector field $T$, and an $m$-dimensional distribution $\mathcal{D}_2$ such that
\[
K(T, T) = \lambda_1 T, \quad K(T, X) = \lambda_2 X, \quad ST = \mu_1 T, \quad SX = \mu_2 X, \quad \forall X \in \mathcal{D}_2.
\]
Then either $M^{m+1}$ is an affine hypersphere such that $K_T = 0$, or it is affinely congruent to one of the following immersions:

1. $f(t, x_1, \cdots, x_m) = (\gamma_1(t), \gamma_2(t)g_2(x_1, \cdots, x_m))$ for $\gamma_1, \gamma_2$ such that $\epsilon \gamma_1' \gamma_2' \gamma_2'' - \gamma_1'' \gamma_2' < 0$;
\( f(t, x_1, \cdots, x_m) = \gamma_1(t)C(x_1, \cdots, x_m) + \gamma_2(t)e_{m+1} \) for \( \gamma_1, \gamma_2 \) such that

\[
\text{sgn}(\gamma_1' \gamma_2 - \gamma_1' \gamma_2') = \text{sgn}(\gamma_1') \neq 0;
\]

\( f(t, x_1, \cdots, x_m) = C(x_1, \cdots, x_m) + \gamma_2(t)e_{m+1} + \gamma_1(t)e_{m+2} \) for \( \gamma_1, \gamma_2 \) such that

\[
\text{sgn}(\gamma_1' \gamma_2^\prime - \gamma_1' \gamma_2^\prime') = \text{sgn}(\gamma_1') \neq 0.
\]

Here \( g_2 : \mathbb{R}^m \to \mathbb{R}^{m+1} \) is a proper affine hypersphere centered at the origin with the affine mean curvature \( \epsilon \), and \( C : \mathbb{R}^m \to \mathbb{R}^{m+2} \) is an improper affine hypersphere given by \( C(x_1, \cdots, x_m) = (x_1, \cdots, x_m, p(x_1, \cdots, x_m), 1) \) with the affine normal \( e_{m+1} \).

Finally, we review some notions and results regarding conformally flat manifolds and warped product manifolds. The Schouten tensor of \((1, 1)\)-type, on a Riemannian manifold \((M, h)\) of dimension \( n \geq 3 \), is a self-adjoint operator relative to \( h \) defined by

\[
P = \frac{Q}{n-2} - \frac{r}{2(n-1)(n-2)} I,
\]

where \( Q, I \) and \( r \) are the Ricci operator, identity operator and scalar curvature, respectively. Note that \((M, h)\) is conformally flat, which means that a neighborhood of each point can be conformally immersed into the Euclidean space \( \mathbb{R}^n \). For \( n \geq 4 \), \((M, h)\) is conformally flat if and only if the Weyl curvature tensor defined below vanishes:

\[
W(X, Y)Z = \hat{R}(X, Y)Z - (h(Y, Z)PX - h(X, Z)PY + h(PY, Z)X - h(PX, Z)Y).
\]

For \( n = 3 \), it is well known that \( W = 0 \) identically, and that \((M, h)\) is conformally flat if and only if the Schouten tensor is a Codazzi tensor. If \( W = 0 \), the Riemannian curvature tensor can be rewritten by

\[
\hat{R}(X, Y)Z = h(Y, Z)PX - h(X, Z)PY + h(PY, Z)X - h(PX, Z)Y.
\]

\((M, h)\) of dimension \( n \geq 3 \) is of constant sectional curvature if and only if \( W = 0 \) and its metric is Einstein, where the Einstein metric means that the eigenvalues of the operator \( P \) or \( Q \) are single.

For Riemannian manifolds \((B, g_B), (M_1, g_1)\) and a positive function \( f : B \to \mathbb{R} \), the product manifold \( M := B \times M_1 \), equipped with the metric \( h = g_B \otimes f^2 g_1 \), is called a warped product manifold with the warped function \( f \), denoted by \( B \times_f M_1 \). If \( E \) and \( U, V \) are independent vector fields of \( B \) and \( M_1 \), respectively, then (cf. (2) of [5]) the sectional curvatures of \( M \) satisfy that

\[
K^M_{EU} = -\frac{H_f(E, E)}{fh(E, E)}, \quad K^M_{UV} = \frac{1}{f^2} K^M_{UV} - \frac{h(\hat{\nabla} f, \hat{f} f)}{f^2},
\]

where \( \hat{\nabla} \) is the Levi-Civita connection of \((M, h)\), \( K^M \) and \( K^{M_1} \) are the sectional curvatures of \( M \) and \( M_1 \), respectively, \( \hat{\nabla} f \) is the gradient of the function \( f \), and \( H_f(X, Y) = h(\hat{\nabla} X \hat{\nabla} f, Y) \) is the Hessian of \( f \). Related to the conformally flat manifold, we recall

**Theorem 2.3** (cf. Theorem 3.7 of [5]) Let \( M = B \times_f M_1 \) be a warped product with \( \dim B = 1 \) and \( \dim M_1 \geq 2 \). Then \( M \) is conformally flat if and only if, up to a reparametrization of \( B, (M_1, g_1) \) is a space of constant curvature and \( f \) is any positive function.
3 Conformally Flat Affine Hypersurfaces with $\hat{R} \cdot C = 0$

From this section on, when we say that an affine hypersurface has the semi-parallel cubic form, this always means that $\hat{R} \cdot C = 0$, or equivalently, that $\hat{R} \cdot K = 0$. Then, by the Ricci identity of $K$, we have that

$$\hat{R}(X,Y)K(Z,U) = K(\hat{R}(X,Y)Z,U) + K(Z,\hat{R}(X,Y)U).$$

(3.1)

In fact, by (2.4) and the Ricci identities of $C$ and $K$, the equivalence above follows from the following formula:

$$(\hat{R} \cdot C)(U,V,X,Y,Z) = (\hat{R}(U,V) \cdot C)(X,Y,Z)$$

$$= -C(X,Y,\hat{R}(U,V)Z) - C(X,\hat{R}(U,V)Y,Z) - C(\hat{R}(U,V)X,Y,Z)$$

$$= 2[h(K_XY,\hat{R}(U,V)Z) + h(K_X\hat{R}(U,V)Y,Z) + h(K_Y\hat{R}(U,V)X,Z)]$$

$$= -2h(\hat{R}(U,V)K_XY - K_X\hat{R}(U,V)Y - K_Y\hat{R}(U,V)X,Z)$$

$$= -2h((\hat{R}(U,V) \cdot K)(X,Y),Z).$$

(3.2)

First, under the assumption that the affine metric is of constant sectional curvature, we extend the result of Theorem 6.2 in [21], stating that a locally strongly convex affine surface has the semi-parallel cubic form if and only if it is either a quadric or a flat affine surface, from the surface to the non-degenerate affine hypersurface, as follows:

**Proposition 3.1** Let $M^n, n \geq 2$ be a non-degenerate affine hypersurface of $\mathbb{R}^{n+1}$. Then $M^n$ has the semi-parallel cubic form and an affine metric of constant sectional curvature if and only if $M^n$ is either a hyperquadric or a flat affine hypersurface.

**Proof** The “if part” follows from (3.1) and the Pick-Berwald theorem.

Now, we prove the “only if part”. Assume that the affine metric is of constant sectional curvature $c$, then

$$\hat{R}(X,Y)Z = c(h(Y,Z)X - h(X,Z)Y).$$

(3.3)

Let us choose an orthonormal basis $\{e_1, \cdots, e_n\}$ such that $h(e_i, e_j) = \epsilon_i \delta_{ij}$ and $\epsilon_i = \pm 1$. Then, from (3.1), we get that

$$h(e_i, \hat{R}(e_i, e_j)K(e_i, e_i)) = 2h(e_i, K(\hat{R}(e_i, e_j)e_i, e_i)),$$

which, together with (3.3), implies that, for $i \neq j$,

$$0 = h(e_i, \hat{R}(e_i, e_j)K_{e_i}e_i - 2K_{e_i}\hat{R}(e_i, e_j)e_i) = 3\epsilon_i h(K_{e_i}e_i, e_j).$$

Therefore, either $c = 0$, or $c \neq 0$ and $K_{e_i}e_i = \epsilon_i e_i$ for all $i$. In the latter case, the apolarity condition shows that $c_j = \text{tr} K_{e_j} = 0$ for each $j$, and thus $K_{e_i}e_i = 0$ for all $i$. As $K$ is symmetric, we have that $K = 0$ identically. By the Pick-Berwald theorem, the conclusion follows. \hfill $\square$

**Remark 3.2** To see the examples whose cubic forms are semi-parallel but not parallel, we refer to Remark 6.2 in [21] for flat surfaces, and Theorem 4.1 in [3] for the flat quasi-umbilical affine hypersurfaces.

From now on, let $F : M^n \to \mathbb{R}^{n+1}, n \geq 3$ be a locally strongly convex affine hypersurface with $\hat{R} \cdot C = 0$ and the vanishing Weyl curvature tensor. Then we have (3.1) and (2.12). Denote by $\{e_1, \cdots, e_n\}$ the local orthonormal frame of $M^n$, where $e_i$ are the eigenvector fields of the
Schouten tensor $P$ with corresponding eigenvalues $\nu_i$, $i = 1, \cdots, n$. Then, we see from (2.12) that
\begin{equation}
\hat{R}(e_i, e_j)Z = (\nu_i + \nu_j)(h(e_j, Z)e_i - h(e_i, Z)e_j)
\end{equation}
for any tangent vector $Z$. Taking $X = e_i, Y = e_j, u = e_k, Z = e_\ell$ in (3.1), we have that
\begin{equation}
(v_i + v_j)(K_{e_i, e_k, e_\ell})e_i = (v_i + v_j)(\delta_{jk}K_{e_i, e_\ell} - \delta_{ik}K_{e_j, e_\ell} + \delta_{j\ell}K_{e_i, e_k} - \delta_{i\ell}K_{e_j, e_k}).
\end{equation}
For $k = \ell = i \neq j$ in (3.5), we obtain that
\begin{equation}
(v_i + v_j)(2K_{e_i, e_j} + h(K_{e_i, e_j})e_i - h(K_{e_i, e_j})e_j) = 0.
\end{equation}
Taking the inner product of (3.6) with $e_i$, we deduce that
\begin{equation}
(v_i + v_j)h(K_{e_i, e_i, e_j}) = 0, \ \forall \ i \neq j.
\end{equation}
Interchanging the roles of $e_i$ and $e_j$, similarly we have that
\begin{equation}
(v_j + v_i)h(K_{e_j, e_i, e_i}) = 0, \ \forall \ i \neq j.
\end{equation}
Taking the inner product of (3.6) with $e_j$, by (3.8), we see that
\begin{equation}
(v_i + v_j)h(K_{e_i, e_i, e_i}) = 0, \ \forall \ i \neq j.
\end{equation}
Together with (3.7), we obtain from (3.6) that
\begin{equation}
(v_i + v_j)K_{e_i, e_j} = 0, \ \forall \ i \neq j.
\end{equation}
On the other hand, setting $k = i, \ell = j$ in (3.5), and by (3.7) and (3.8), we deduce that
\begin{equation}
(v_i + v_j)(K_{e_i, e_i, e_j}) = 0, \ \forall \ i \neq j.
\end{equation}
For more information, we denote by $\nu_1, \cdots, \nu_m$ the $m$ distinct eigenvalues for $P$ of multiplicities $(n_1, n_2, \cdots, n_m)$. Let $\mathcal{D}(\nu_i)$ be the eigenvalue distribution of eigenvalue $\nu_i$ for $i = 1, \cdots, m$. Then, we can prove the following two lemmas:

**Lemma 3.3** It holds that
(i) if $\nu_i \neq 0$ and $n_i \geq 2$, then $h(K(u, v), w) = 0, \ \forall \ u, v, w \in \mathcal{D}(\nu_i)$;
(ii) if $\nu_i^2 \neq \nu_j^2$, then $K(u, v) = 0, \ \forall \ u, v \in \mathcal{D}(\nu_i) \oplus \mathcal{D}(\nu_j)$.

**Proof** If $\nu_i \neq 0$ and $n_i \geq 2$, then, by taking $\nu_j = \nu_i$ and $e_i = u$ in (3.9), we have $h(K(u, u), u) = 0$ for any unit vector $u \in \mathcal{D}(\nu_i)$. Then, the conclusion (i) follows from the symmetric property of $K$.

If $\nu_i^2 \neq \nu_j^2$, it holds that
\begin{equation}
(v_i + v_j)(\nu_i - \nu_j) \neq 0, \ m \geq 2.
\end{equation}
Then, by (3.9), we have that $h(K(u, u), u) = 0$ for any vector $u \in \mathcal{D}(\nu_i)$, which, together with the symmetric property of $K$, implies that $K(u, u) \notin \mathcal{D}(\nu_i)$. Furthermore, it follows from (3.10) and (3.11) that
\begin{equation}
K(u, v) = 0, \ K(u, u) = K(v, v) \notin \mathcal{D}(\nu_i) \oplus \mathcal{D}(\nu_j)
\end{equation}
for any unit vectors $u \in \mathcal{D}(\nu_i), v \in \mathcal{D}(\nu_j)$. If $m = 2$, then (3.13) immediately implies that $K(u, u) = K(v, v) = 0$. If $m \geq 3$, for arbitrary $\nu_k$, different from $\nu_i$ and $\nu_j$, then by (3.12),
either \( \nu_k + \nu_l \neq 0 \) or \( \nu_k + \nu_j \neq 0 \) holds. In either case, by (3.7) and the arbitrariness of \( \nu_k \) we see from (3.13) that \( K(u, u) = K(v, v) = 0 \). Therefore, by the symmetric property of \( K \) we have (ii).

**Lemma 3.4** The number \( m \) of distinct eigenvalues of the Schouten tensor \( P \) is at most 2. In particular, if \( m = 2 \), denoting by \( \nu_1, \nu_2 \) the two distinct eigenvalues of \( P \), then one of their multiplicity must be 1, and

\[
\nu_2 = -\nu_1 \neq 0. \tag{3.14}
\]

**Proof** First, for \( m \geq 2 \), we claim that there must exist two distinct eigenvalues, \( \nu_i \) and \( \nu_j \), of \( P \) such that \( \nu_i + \nu_j = 0 \). Otherwise, we have that \( \nu_i^2 \neq \nu_j^2 \) for any \( \nu_i \neq \nu_j \), which, together with Lemma 3.3 (ii), implies that \( K = 0 \) identically. Then, the Pick-Berwald theorem implies that the hypersurface is a hyperquadric, and is thus of constant sectional curvature. This means that \( m = 1 \), which stands in contradiction to \( m \geq 2 \).

By the claim above, for \( m \geq 2 \), we denote by \( \nu_1, \nu_2 \) the two distinct eigenvalues of \( P \) such that \( \nu_1 + \nu_2 = 0 \), and thus \( \nu_1 \nu_2 = 0 \).

Next, we prove that \( m \leq 2 \). Otherwise, if \( m \geq 3 \), then, for arbitrary \( \nu_i \), differently that for \( \nu_1 \) and \( \nu_2 \), we have that \( (\nu_i + \nu_1)(\nu_i + \nu_2) \neq 0 \), and thus,

\[
\nu_i^2 \neq \nu_1^2, \quad \nu_i^2 \neq \nu_2^2.
\]

It follows from Lemma 3.3 (ii) that

\[
\begin{align*}
K(w, u) &= K(w, v) = 0, \\
K(u, u) &= K(v, v) = K(w, w) = 0
\end{align*} \tag{3.15}
\]

for any vectors \( u \in \mathcal{D}(\nu_i), v \in \mathcal{D}(\nu_2), w \in \mathcal{D}(\nu_4) \). Therefore, the arbitrariness of \( \nu_i \) implies that \( K(u, v) = 0 \). Together with (3.15) and the symmetric property of \( K \), we have that \( K = 0 \) identically. As before, it follows from the Pick-Berwald theorem that \( m = 1 \), which stands in contradiction to \( m \geq 3 \). Hence \( m \leq 2 \).

Finally, for \( m = 2 \), let \( \nu_1, \nu_2 \) be the two distinct eigenvalues of \( P \) with multiplicities \( (n_1, n_2) \). From the analysis above, we have (3.14). It is sufficient to prove that either \( n_1 = 1 \) or \( n_2 = 1 \) holds. On the contrary, assume that \( n_1 \geq 2 \) and \( n_2 \geq 2 \). Denote by \( v_1^i, \ldots, v_{n_i}^i \), the orthonormal eigenvector fields of \( P \), which span \( \mathcal{D}(\nu_i) \) for \( i = 1, 2 \). Then, as \( \nu_i \neq 0 \) and \( n_i \geq 2 \), by Lemma 3.3 (i) and (3.11), we have that

\[
\begin{align*}
&h(K(v_p^i, v_q^i), v_i^j) = 0, \\
&K(v_p^i, v_p^i) = K(v_q^i, v_q^i), \quad \forall \ p, q, \ell
\end{align*} \tag{3.16}
\]

for \( i = 1, 2 \). It follows from the apolarity condition that

\[
0 = \sum_{\ell=1}^{n_1} h(K_u v_1^\ell, v_1^\ell) + \sum_{q=1}^{n_2} h(K_u v_q^2, v_q^2) = \sum_{q=1}^{n_2} h(K_u v_q^2, v_q^2) = n_2 h(K_u v_q^2, v_q^2)
\]

for any \( u \in \mathcal{D}(\nu_1) \). Therefore, \( h(K_u v, v) = 0 \) for any \( u \in \mathcal{D}(\nu_1), v \in \mathcal{D}(\nu_2) \). Similarly, we have that \( h(K_v u, u) = 0 \). Then, by the symmetric property of \( K \), we have that \( K(u, v) = 0 \). Together with (3.16), we further obtain that \( K = 0 \) identically. As before, this stands in contradiction to the fact that \( m = 2 \). Lemma 3.4 has been proven. \( \Box \)

Next, based on Lemma 3.4, we pay attention to the case of \( m = 2 \). We always denote by \( \nu_1, \nu_2 \) the two distinct eigenvalues of \( P \) with multiplicities \( 1, n - 1 \), respectively. Let \( T \) be the
unit eigenvector field of the eigenvalue \( \nu_1 \), and let \( \{X_1, \cdots, X_{n-1}\} \) be any orthonormal frame of \( \mathcal{D}(\nu_2) \). By (3.14), (3.10), (3.11) and Lemma 3.3 (i), we see that
\[
K(X_i, X_i) = K(X_j, X_j), \\
K(X_i, X_j) = 0, \quad \forall \ i \neq j, \\
h(K_{X_i} X_j, X_k) = 0, \quad \forall \ i, j, k.
\]

(3.17)

It follows from the apolarity condition that
\[
\sum_{j=1}^{n-1} h(K(X_j, X_j), X_i) = 0,
\]
\[
\sum_{j=1}^{n-1} h(K(X_j, X_j), T) = -(n-1) h(K(X_i, X_i), T), \quad \forall \ i.
\]

Summing the above, as \( m = 2 \) and thus \( K \neq 0 \), we can assume that
\[
K_T T = \lambda_1 T, \quad K_T X_i = \lambda_2 X_i, \quad K_X X_j = \lambda_2 \delta_{ij} T,
\]
\[
PT = \nu_1 T, \quad PX_i = \nu_2 X_i, \quad i, j = 1, \cdots, n-1,
\]
\[
(n-1)\lambda_2 = -\lambda_1 \neq 0, \quad \nu_2 = -\nu_1 \neq 0.
\]

(3.18)

Combining the last formula with (2.11) and the fact that \( \text{tr} Q = r \), we obtain that
\[
Q T = 0, \quad Q X_i = \frac{r}{n-1} X_i, \quad i = 1, \cdots, n-1,
\]
\[
\nu_2 = -\nu_1 = \frac{r}{2(n-1)(n-2)} \neq 0,
\]

(3.19)

which implies that the affine metric is quasi-Einstein with nonzero scalar curvature.

Now, we are ready to prove the following:

**Lemma 3.5** If \( m = 2 \), then the number \( \sigma \) of distinct affine principal curvatures is at most 2. Moreover, there exists a local orthonormal frame, still denoted by \( \{T, X_1, \cdots, X_{n-1}\} \), such that there hold (3.18), (3.19) and
\[
ST = \mu_1 T, \quad SX_i = \mu_2 X_i, \quad i = 1, \cdots, n-1,
\]
\[
\mu_2 = 2\nu_2 + \lambda_2^2, \quad \mu_1 + \mu_2 = -2n\lambda_2^2.
\]

(3.20)

**Proof** From the analysis above we still have the freedom to rechoose the orthonormal frame of \( \mathcal{D}(\nu_2) \) such that (3.18) and (3.19) hold. Therefore, we can reselect the orthonormal frame of \( \mathcal{D}(\nu_2) \) if necessary, still denoted by \( \{X_1, X_2, X_3, \cdots, X_{n-1}\} \), such that
\[
ST = \mu_1 T + \tau_1 X_1, \quad SX_1 = \tau_1 T + \mu_2 X_1 + \tau_2 X_2.
\]

(3.21)

Then, it follows from (3.18), (3.4) and the Gauss equation (2.5) that
\[
0 = h(\tilde{R}(X_1, T)T, X_2) = \frac{1}{2} \tau_2,
\]
\[
0 = h(\tilde{R}(X_1, X_2)X_2, T) = \frac{1}{2} \tau_1,
\]

which, together with (3.21), implies that both \( \mathcal{D}(\nu_1) \) and \( \mathcal{D}(\nu_2) \) are the invariant subspaces of \( S \). Therefore, we can rechoose the local orthonormal frame on \( \mathcal{D}(\nu_2) \), still denoted by \( \{X_1, \cdots, X_{n-1}\} \), such that
\[
ST = \mu_1 T, \quad SX_{i-1} = \mu_i X_{i-1}, \quad i = 2, \cdots, n.
\]
Then, we see from (3.18), (3.4) and the Gauss equation (2.5) that
\[ 0 = \nu_1 + \nu_2 = h(\hat{R}(X_{i-1}, T)T, X_{i-1}) = \frac{1}{2}(\mu_1 + \mu_i) + n\lambda^2, \quad \forall \ i \geq 2, \]
\[ 2\nu_2 = h(\hat{R}(X_{i-1}, X_{j-1})X_{j-1}, X_{i-1}) = \frac{1}{2}(\mu_i + \mu_j) - \lambda^2, \quad \forall \ i \neq j \geq 2, \]
which imply that
\[ \mu_i = -\mu_1 - 2n\lambda^2 = 2\nu_2 + \lambda^2, \quad i = 2, \cdots, n. \]
Therefore, we have that \( \sigma \leq 2 \) and (3.20).

Finally, we conclude this section by proving Theorem 1.4.

**Completion of Proof Theorem 1.4** Let \( F: M^n \to \mathbb{R}^{n+1} \), \( n \geq 3 \) be a locally strongly convex affine hypersphere with \( \hat{R} \cdot C = 0 \) and the vanishing Weyl curvature tensor. Then, we see from Lemma 3.4 that the number \( m \) of distinct eigenvalues of the Schouten tensor \( P \) is at most 2.

If \( m = 1 \), by \( W = 0 \) we see from (2.12) that \( M^n \) is of constant sectional curvature. Then, it follows from Proposition 3.1 and the Main theorem of [28] that \( M^n \) is affinely equivalent to either a hyperquadric, or the flat and hyperbolic affine hypersphere (1.2).

If \( m = 2 \), based on Lemma 3.5, by the fact that \( H = \mu_1 = \mu_2 \) is constant, we see from (3.20), (3.19) and (2.8) that
\[ H = -n\lambda^2 < 0, \]
\[ r = n(n-1)\chi = \frac{(n^2 - 1)(n-2)}{n} H < 0, \]
\[ J = -\frac{n+2}{n^2} H > 0, \]
and thus \( r \) is a negative constant. Defining the traceless Ricci tensor \( \widetilde{\text{Ric}} = \text{Ric} - \frac{r}{n} h \), we deduce from (3.19) and (3.22) that all of the components of \( \text{Ric} \) vanish except for
\[ \widetilde{\text{Ric}}(T, T) = -\frac{r}{n}, \quad \widetilde{\text{Ric}}(X_1, X_1) = \cdots = \widetilde{\text{Ric}}(X_{n-1}, X_{n-1}) = \frac{r}{n(n-1)}. \]
Combining this with (3.22), we have that
\[ \| \widetilde{\text{Ric}} \|^2_h = \frac{r^2}{n(n-1)} = -\frac{(n+1)(n-2)}{n+2} Jr, \] (3.23)
where \( \| \cdot \|_h \) denotes the tensorial norm with respect to \( h \).

In summary, \( M^n \) is a hyperbolic affine hypersphere with constant negative scalar curvature and \( J \neq 0 \), which further satisfies the formula (3.23). Then, it follows from Theorem 2.1 that \( M^n \) is affinely equivalent to the hyperbolic affine hypersphere (1.3).

\[ \square \]

### 4 Proof of Theorem 1.7

Let \( M^n, n \geq 3 \) be a locally strongly convex affine hypersurface in \( \mathbb{R}^{n+1} \) with \( \hat{R} \cdot C = 0 \) and the vanishing Weyl curvature tensor. Denote by \( m \) (resp. \( \sigma \)) the number of distinct eigenvalues of its Schouten tensor (resp. affine shape operator). Then, by Lemma 3.4 we have that \( m \leq 2 \).

If \( m = 1 \), by \( W = 0 \), we see from (2.12) that \( M^n \) is of constant sectional curvature. Then, it follows from Proposition 3.1 that \( M^n \) is either a hyperquadric, or a flat affine hypersurface.
Remark 4.1 If \( M^n \) is a flat affine hypersurface with \( C \neq 0 \), it follows from Theorems 1.1 and 1.2 of [3] and Theorem 1.4 that \( M^n \) is affinely equivalent to either (1.2) if \( \sigma = 1 \), or one of the three flat quasi-umbilical affine hypersurfaces (1), (2) or (7) explicitly described in Theorem 4.1 of [3] if \( \sigma = 2 \). However, for \( \sigma \geq 3 \), \( M^n \) is a flat affine hypersurface with at least two affine principal curvatures of multiplicity one, whose classification is still complicated and has not been involved until now.

If \( m = 2 \), by Lemma 3.5, we have that \( \sigma \leq 2 \). For \( \sigma = 1 \), i.e., \( M^n \) is an affine hypersphere, Theorem 1.4 shows that \( M^n \) is affinely equivalent to (1.3).

For the last case, \( m = \sigma = 2 \), we see from Lemma 3.5 that \( M^n \) is a quasi-umbilical affine hypersurface with a quasi-Einstein affine metric of nonzero scalar curvature. Moreover, for the local orthonormal frame \( \{ T, X_1, \cdots, X_{n-1} \} \), it holds that

\[
K_T T = \lambda_1 T, \quad K_T X_i = \lambda_2 X_i, \quad K_X X_j = \lambda_2 \delta_{ij} T,
\]

\[
ST = \mu_1 T, \quad SX_i = \mu_2 X_i, \quad i, j = 1, \cdots, n - 1,
\]

where

\[
\mu_2 - \lambda_2^2 = \frac{r}{(n - 1)(n - 2)} \neq 0, \quad \mu_1 + \mu_2 = -2n\lambda_2^2,
\]

\[
\lambda_1 = -(n - 1)\lambda_2 \neq 0, \quad \mu_1 - \mu_2 \neq 0.
\]

By (3.18) and (3.19), we further obtain from (2.12) that

\[
\hat{R}(X_i, X_j) X_k = \frac{r}{(n - 1)(n - 2)} (\delta_{jk} X_i - \delta_{ik} X_j),
\]

\[
\hat{R}(X_i, T) T = \hat{R}(T, X_i) X_i = 0, \quad \forall \ 1 \leq i, j, k \leq n - 1.
\]

Remark 4.2 For \( \sigma = m = 2 \), \( M^n \) is affinely equivalent to one of the three classes of immersions in Theorem 2.2. In fact, it follows from (4.1) that \( M^n \) satisfies the conditions of Theorem 2.2. In the proof of Theorem 2.2 in [1], it was shown in Lemma 3 that if \( \lambda_2 = 0 \), then \( K_T = 0 \) and \( M^n \) is an affine hypersphere. In our situation, by \( M^n \) not an affine hypersphere, we can exclude this possibility in Theorem 2.2.

Next, we give more information about the three classes of warped product immersions in Theorem 2.2. Denote by \( \mathcal{D}(\mu_2) = \text{span}\{X_1, \cdots, X_{n-1}\} \) the eigenvalue distribution of \( S \) corresponding to \( \mu_2 \). In the proof of Theorem 2.2 in [1], by (4.2) it was shown that

\[
\hat{\nabla}_T T = 0, \quad \hat{\nabla}_X T = -\alpha X, \quad T(\alpha) = \alpha^2,
\]

\[
X(\alpha) = X(\mu_1) = X(\mu_2) = X(\lambda_2) = 0, \quad \forall \ X \in \mathcal{D}(\mu_2),
\]

\[
T(\lambda_2) = (n + 1)\lambda_2 \alpha + \frac{1}{2} (\mu_1 - \mu_2),
\]

\[
T(\mu_2) = (\mu_2 - \mu_1)(\alpha - \lambda_2),
\]

and that \( M^n \) is locally a warped product \( \mathbb{R} \times f M_2 \), where the warping function \( f \) is determined by \( \alpha = -T(\ln f) \), and \( \mathbb{R} \) and \( M_2 \) are, respectively, integral manifolds of the distributions \( \text{span}\{T\} \) and \( \mathcal{D}(\mu_2) \), and \( M_2 \) is an affine hypersphere. The same proof implies that the projection of the difference tensor on \( \mathcal{D}(\mu_2) \) is the difference tensor of the components \( M_2 \) given by

\[
L^2(X, Y) = K(X, Y) - \lambda_2 h(X, Y) T, \quad \forall \ X, Y \in \mathcal{D}(\mu_2).
\]

Then, it follows from (4.1) that \( L^2 = 0 \). Hence \( M_2 \) is a hyperquadric with constant sectional curvature \( c \).
By the warped product structure, if not otherwise stated, we always take the local coordinates \( \{ t, x_1, \cdots, x_{n-1} \} \) on \( M^n \) such that \( \frac{\partial}{\partial t} = T, \) span\( \{ \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{n-1}} \} = \mathcal{D}(\mu_2). \) Then all functions \( \mu_i, \lambda_i, \alpha, r \) and \( f \) depend only on \( t. \) Denoting that \( \frac{\partial}{\partial t} (\cdot) = (\cdot)' \), we have that \( \alpha = -f'/f. \) It follows from (2.13), (4.3) and \( \alpha' = \alpha^2 \) in (4.4) that
\[
c = f'^2 + \frac{r}{(n-1)(n-2)} f^2, \quad f'' = 0. \tag{4.6}
\]
Furthermore, we can get, from the equations above for \( f \) and \( \alpha, \) that, up to a translation and a direction of the parametric \( t, \)
\[
f = 1, \quad \alpha = 0; \quad \text{or} \quad f = t, \quad \alpha = -\frac{1}{t}, \tag{4.7}
\]
where locally we take that \( t > 0. \) Together with (4.2) and (4.4) we can check that \( \mu_2 - \lambda_2^2 \) if \( f = 1, \) and \( \mu_2 - \lambda_2^2 \) if \( f = t. \) It follows from (4.2) and (4.6) that
\[
c = \begin{cases}
\frac{r}{(n-1)(n-2)} = \mu_2 - \lambda_2^2, & \text{if } f = 1, \\
1 + \frac{r}{(n-1)(n-2)} t^2 = 1 + (\mu_2 - \lambda_2^2) t^2, & \text{if } f = t,
\end{cases} \tag{4.8}
\]
where the scalar curvature \( r \) is nonzero in either case.

The Pick-Berwald theorem states that the cubic form or difference tensor vanishes if and only if the hypersurface is a non-degenerate hyperquadric. For the locally strongly convex case, for later use, we recall from [23] pages 104–105 the following three hyperquadrics revised for dimension \( n - 1: \)

1. The ellipsoid described in \( \mathbb{R}^n \) by
   \[x_1^2 + \cdots + x_n^2 = c^{-(n+1)/n}, \quad c > 0.\]
   This has constant sectional curvature \( c \) and the affine normal \(-cF_1, \) where in local coordinates we denote this immersion by \( F_1(x_1, \cdots, x_{n-1}). \)

2. The hyperboloid described in \( \mathbb{R}^n \) by
   \[x_n = (x_1^2 + \cdots + x_{n-1}^2 + (c)^{(n+1)/n})^{1/2}, \quad c < 0.\]
   This has constant sectional curvature \( c \) and the affine normal \(-cF_2, \) where in local coordinates we denote this immersion by \( F_2(x_1, \cdots, x_{n-1}). \)

3. The elliptic paraboloid described in \( \mathbb{R}^n \) by
   \[x_n = \frac{1}{2}(x_1^2 + \cdots + x_{n-1}^2).\]
   The affine metric is flat. This is the only parabolic affine hypersphere with constant sectional curvature.

Finally, armed with the above preparations, by Remark 4.2 and the computations carried out in [3] for the three classes of immersions in Theorem 2.2, we can prove the following theorem, which, together with the previous cases, completes the proof of Theorem 1.7:

**Theorem 4.3** Let \( M^n, n \geq 3 \) be a locally strongly convex affine hypersurface in \( \mathbb{R}^{n+1} \) with \( R \cdot C = 0 \) and the vanishing Weyl curvature tensor. Assume that both the numbers of distinct eigenvalues of its Schouten tensor and affine shape operator are 2. Then \( M^n \) is a quasi-umbilical affine hypersurface with a quasi-Einstein metric of nonzero scalar curvature \( r. \)
Moreover, \((M^n, h)\) is locally isometric to the warped product \(\mathbb{R}_+ \times f M_2\), where the warped function is \(f(t) = 1\) or \(t\), \(M_2\) is of constant sectional curvature \(c\), and \(M^n\) is affinely equivalent to one of the following six hypersurfaces:

1. The immersion \((\gamma_1(t), \gamma_2(t) F_1(x_1, \ldots, x_{n-1}))\), where
   \(f(t) = 1\), and \(c = \frac{r}{(n-1)(n-2)}\) is positive constant,
   \(\gamma_1\) and \(\gamma_2\) are determined by
   \[
   \gamma_1' = \gamma_2^{-1}, \quad \gamma_2 = (c_1 \cos(\sqrt{(n+1)ct}) + c_2 \sin(\sqrt{(n+1)ct})) \frac{1}{\sqrt{\pi}},
   \]
   where \(c_1, c_2\) are constants such that \(\gamma_2 > 0\).

2. The immersion \((\gamma_1(t), \gamma_2(t) F_2(x_1, \ldots, x_{n-1}))\), where
   \(f(t) = 1\), and \(c = \frac{r}{(n-1)(n-2)}\) is negative constant,
   \(\gamma_1\) and \(\gamma_2\) are determined by
   \[
   \gamma_1' = \gamma_2^{-1}, \quad \gamma_2 = (c_1 e^{-\sqrt{(n+1)ct}} + c_2 e^{-\sqrt{(n+1)ct}}) \frac{1}{\sqrt{\pi}},
   \]
   where \(c_1, c_2\) are constants such that \(\gamma_2 > 0\).

3. The immersion \((\gamma_1(t), \gamma_2(t) F_1(x_1, \ldots, x_{n-1}))\), where
   \(f(t) = t\), and \(c = 1 + \frac{r}{(n-1)(n-2)} t^2 \neq 1\) is positive constant,
   \(\gamma_1\) and \(\gamma_2\) are determined by
   \[
   \gamma_1' = t^{n+1} \gamma_2^{-n}, \quad \gamma_2 = k(t)^{1/(n+1)},
   \]
   where \(k(t)\) is a positive solution to the linear differential equation
   \[
   t^2 k''(t) - (n+1)tk'(t) + (n+1)ck(t) = 0.
   \]

4. The immersion \((\gamma_1(t), \gamma_2(t) F_2(x_1, \ldots, x_{n-1}))\), where
   \(f(t) = t\), and \(c = 1 + \frac{r}{(n-1)(n-2)} t^2 \) is negative constant,
   \(\gamma_1\) and \(\gamma_2\) are determined by
   \[
   \gamma_1' = t^{n+1} \gamma_2^{-n}, \quad \gamma_2 = k(t)^{1/(n+1)},
   \]
   where \(k(t)\) is a positive solution to the linear differential equation
   \[
   t^2 k''(t) - (n+1)tk'(t) + (n+1)ck(t) = 0.
   \]

5. The immersion \((\gamma_1(t)x_1, \ldots, \gamma_1(t)x_{n-1}, \frac{1}{2} \gamma_1(t) \sum_{i=1}^{n-1} x_i^2 + \gamma_2(t), \gamma_1(t))\), where
   \(f(t) = t\), \(c = 1 + \frac{r}{(n-1)(n-2)} t^2 = 0\),
   \(\gamma_1\) and \(\gamma_2\) are determined by
   \[
   \gamma_1 = (\frac{n+1}{n+2} t^{n+2} + c_1) \frac{1}{\sqrt{\pi}}, \quad \gamma_2' = \frac{n+1}{n+2} \gamma_1' \ln t - \frac{\gamma_1}{(n+2)t},
   \]
   where \(c_1\) is a constant.

6. The immersion \((x_1, \ldots, x_{n-1}, \frac{1}{2} \sum_{i=1}^{n-1} x_i^2 - \frac{1}{n+2} \ln t, \frac{1}{n^2+2} t^{n+2})\), where \(f(t) = t\), \(c = 1 + \frac{r}{(n-1)(n-2)} t^2 = 0\).

**Remark 4.4** By Theorem 2.3, the warped product structure in Theorem 4.3 implies that all of the examples above, especially for the 3-dimensional case, are conformally flat.
Proof Based on the previous analysis, we will discuss the three classes of immersions in Theorem 2.2 by taking $m = n - 1$.

Case 1 It was shown in [1] that if

$$\mu_2^2 + (\alpha - \lambda_2)^2 \neq 0, \quad \mu_2 - \lambda_2^2 + \alpha^2 \neq 0, \quad (4.9)$$

then $M^n$ is affinely equivalent to the immersion $(\gamma_1(t), \gamma_2(t))g_2(x_1, \cdots, x_{n-1})$, where $g_2 : M_2 \to \mathbb{R}^n$ is a proper affine hypersphere with the difference tensor $L^2$ defined by (4.5). It follows from the analysis above and (4.7)-(4.9) that $L^2 = 0$, and $g_2$ has constant sectional curvature $c \neq 0$, the affine mean curvature $c$ and the affine normal $-cg_2$. Thus $g_2 = F_1$ if $c > 0$, or $g_2 = F_2$ if $c < 0$. As $r \neq 0$, (4.8) implies that $c \neq 1$ if $f = t$. Moreover, by the computations of the immersion on pages 292–294 of [3], we take that $\lambda = -c$ in (4.3) of [3], and deduce that

$$\gamma'_1 = f^{2n+1}g^{-n}, \quad \gamma_2 = k(t)^{1/(n+1)},$$

$$f^2k''(t) - (n + 1)f'k'(t) + (n + 1)ck(t) = 0,$$  \( (4.10) \)

where $k(t)$, and thus $\gamma_2$ are positive functions.

If $f = 1$, we see from (4.10) that $\gamma'_1 = \gamma^{-n}_2$ and $\gamma_2 = k(t)^{1/(n+1)}$, where

$$k''(t) + (n + 1)ck(t) = 0.$$

Solving this equation, we obtain that

$$\gamma_2 = \begin{cases} (c_1 e^{\sqrt{-n+1}ct} + c_2 e^{-\sqrt{-n+1}ct}) \frac{1}{\sqrt{n+1}}, & \text{if } c < 0, \\ (c_1 \cos(\sqrt{(n+1)ct}) + c_2 \sin(\sqrt{(n+1)ct})) \frac{1}{\sqrt{n+1}}, & \text{if } c > 0, \end{cases} \quad (4.11)$$

where the constants $c_1, c_2$ are chosen such that $\gamma_2 > 0$. We obtain the immersion (1) if $c > 0$, and the the immersion (2) if $c < 0$.

If $f = t$, we see from (4.2), (4.8) and (4.10) that $\gamma'_1\gamma_2^n = t^{n+1}$ and

$$\gamma_2 = k(t)^{1/(n+1)}, \quad c \neq 1,$$

$$t^2k''(t) - (n + 1)tk'(t) + (n + 1)ck(t) = 0. \quad (4.12)$$

In particular, if $k(t)$ is a power function of $t$, we can solve this equation to obtain that

$$\gamma_2(t) = \begin{cases} c_1 t^{\frac{n+2}{n+1}}, & \text{if } c = \frac{(n+2)^2}{4(n+1)}, \\ (c_2 t^{\tau_1} + c_3 t^{\tau_2}) \frac{1}{\sqrt{n+1}}, & \text{if } c < \frac{(n+2)^2}{4(n+1)}, \\ 0, & \text{if } c > \frac{(n+2)^2}{4(n+1)}, \end{cases} \quad (4.13)$$

where $c_1$ is a positive constant, the constants $c_2, c_3$ are chosen such that $\gamma_2 > 0$, and $\tau_1, \tau_2$ are the solutions of the quadric equation $\tau^2 - (n + 2)\tau + (n + 1)c = 0$. We obtain the immersion (3) if $c > 0$, and the immersion (4) if $c < 0$.

Case 2 It was shown in [1] that if

$$\mu_2^2 + (\alpha - \lambda_2)^2 \neq 0, \quad \mu_2 - \lambda_2^2 + \alpha^2 = 0, \quad (4.14)$$

then $M^n$ is affinely equivalent to the immersion

$$(\gamma_1(t)x_1, \cdots, \gamma_1(t)x_{n-1}, \gamma_1(t)g(x_1, \cdots, x_{n-1}) + \gamma_2(t), \gamma_1(t)).$$
where \( g(x_1, \cdots, x_{n-1}) \) is a convex function whose graph immersion is a parabolic affine hypersphere with the difference tensor \( L^2 \) defined by (4.5). It follows from the first equation of (4.2), (4.7), (4.8) and (4.14) that \( L^2 = 0, f = t, c = 0 \), and that this parabolic affine hypersphere has a flat affine metric, and thus \( g(x_1, \cdots, x_{n-1}) = \frac{1}{2}(x_1^2 + \cdots + x_{n-1}^2) \).

Moreover, we recall from the computations of the hypersurfaces on page 294 of [3] that \( \gamma_1, \gamma_2 \) satisfy that

\[
(\gamma_1'' - \gamma_1') f^2 = \gamma_1' \quad f = |\gamma_1'|^{1/(n+1)}. \tag{4.15}
\]

As \( f = t \), we have that \( (\gamma_1')' = \epsilon(n+1)t^{n+1}, \epsilon \in \{-1, 1\} \), which gives that \( \gamma_1^n = \frac{n+1}{n+2}t^{n+2} + c_1 \). By applying an affine reflection, we may assume that \( \gamma_1 > 0 \), then put \( \epsilon = 1 \) and \( \gamma_1 = (\frac{n+1}{n+2}t^{n+2} + c_1)^{1/(n+1)} \). By (4.15), we get that

\[
\left( \frac{\gamma_2}{\gamma_1} \right)' = t^{-2}\gamma_1' = \frac{n+1}{n+2}t^{-1} + c_1t^{-n-3},
\]

which yields that

\[
\gamma_2/\gamma_1 = \frac{n+1}{n+2}\ln t - \frac{c_1}{(n+2)t^{n+2}} + c_2.
\]

Then, since \( \gamma_1^n = t^{n+1} \) and \( c_1 = \gamma_1^n - \frac{n+1}{n+2}t^{n+2} \), we have that

\[
\gamma_2' = \frac{n+1}{n+2}\gamma_1' \ln t - \frac{\gamma_1}{(n+2)t} + \left( \frac{n+1}{(n+2)^2} + c_2 \right) \gamma_1',
\]

and

\[
\gamma_2 = \int \left( \frac{n+1}{n+2}\gamma_1' \ln t - \frac{\gamma_1}{(n+2)t} + \left( \frac{n+1}{(n+2)^2} + c_2 \right) \gamma_1 \right) dt + \left( \frac{n+1}{(n+2)^2} + c_2 \right) \gamma_1 + c_3.
\]

Here, by applying equiaffine transformations, we may set that \( c_2 = -(n+1)/(n+2)^2 \) and \( c_3 = 0 \). We have the immersion (5).

**Case 3** It was shown in [1] that if

\[
\mu_2^2 + (\alpha - \lambda_2)^2 = 0, \tag{4.16}
\]

then \( M^n \) is affinely equivalent to the immersion \( (x_1, \cdots, x_{n-1}, g(x_1, \cdots, x_{n-1}) + \gamma_1(t), \gamma_2(t)) \), where \( g(x_1, \cdots, x_{n-1}) \) is a convex function whose graph immersion is a parabolic affine hypersphere with the difference tensor \( L^2 \) defined by (4.5). It follows from the analysis above, \( \lambda_2 \neq 0 \) in (4.2), (4.7), (4.8) and (4.16), that \( L^2 = 0, \mu_2 = 0, \lambda_2 = \alpha = -\frac{1}{t}, f = t \) and \( c = 0 \). As before, \( g(x_1, \cdots, x_{n-1}) = \frac{1}{2}(x_1^2 + \cdots + x_{n-1}^2) \).

Moreover, we see from the computations of the hypersurfaces on page 295 of [3] that \( \gamma_1, \gamma_2 \) satisfy that

\[
\gamma_2'^3 = (\gamma_1'' \gamma_2 - \gamma_1' \gamma_2') f^{2(n+2)}, \quad f = |\gamma_2'|^{1/(n+1)}. \tag{4.17}
\]

Then, taking \( f = t \) into this equation, we deduce that

\[
\gamma_2' = t^{n+1}, \quad t^2 \gamma_1'' - (n+1)t \gamma_1' - 1 = 0, \tag{4.18}
\]

where \( \epsilon \in \{-1, 1\} \). Then, we can directly solve these equations to obtain that

\[
\gamma_2(t) = \frac{\epsilon}{n+2}t^{n+2} + c_1, \quad \gamma_1(t) = -\frac{\ln t}{n+2} + c_2 t^{n+2} + c_3.
\]

By applying a translation and a reflection in \( \mathbb{R}^{n+1} \), we may assume that \( c_1 = c_3 = 0 \) and \( \gamma_2 > 0 \), i.e., \( \epsilon = 1 \). Also, by possibly applying an equiaffine transformation, we may set that \( c_2 = 0 \). Hence, we obtain the immersion (6).

\[\square\]

**Conflict of Interest** The authors declare no conflict of interest.
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