Abstract: The commuting graph of a finite non-abelian group $G$ with center $Z(G)$, denoted by $\Gamma_c(G)$, is a simple undirected graph whose vertex set is $G \setminus Z(G)$, and two distinct vertices $x$ and $y$ are adjacent if and only if $xy = yx$. Alwardi et al. (Bulletin, 2011, 36, 49-59) defined the common neighborhood matrix $CN(G)$ and the common neighborhood energy $E_{cn}(G)$ of a simple graph $G$. A graph $G$ is called CN-hyperenergetic if $E_{cn}(G) > E_{cn}(K_n)$, where $n = |V(G)|$ and $K_n$ denotes the complete graph on $n$ vertices. Two graphs $G$ and $H$ with equal number of vertices are called CN-equieenergetic if $E_{cn}(G) = E_{cn}(H)$. In this paper we compute the common neighborhood energy of $\Gamma_c(G)$ for several classes of finite non-abelian groups, including the class of groups such that the central quotient is isomorphic to group of symmetries of a regular polygon, and conclude that these graphs are not CN-hyperenergetic. We shall also obtain some pairs of finite non-abelian groups such that their commuting graphs are CN-equieenergetic.

Keywords: commuting graph; CN-energy; finite group

MSC: 20D99; 05C50; 15A18; 05C25

1. Introduction

Let $G$ be a simple graph whose vertex set is $V(G) = \{v_1, v_2, \ldots, v_n\}$. The common neighborhood of two distinct vertices $v_i$ and $v_j$, denoted by $C(v_i, v_j)$, is the set of vertices adjacent to both $v_i$ and $v_j$ other than $v_i$ and $v_j$. The common neighborhood matrix of $G$, denoted by $CN(G)$, is a matrix of size $n \times n$ whose $(i,j)$th entry is 0 or $|C(v_i, v_j)|$ according as $i = j$ or $i \neq j$. The common neighborhood matrix is a symmetric matrix, hence all its eigenvalues are real. The common neighborhood eigenvalues are symmetric with respect to the origin for some special class of graphs. There is a nice relation between $CN(G)$ and $A(G)$, the adjacency matrix of $G$. More precisely, if $i \neq j$ then the $(i,j)$th entry of $CN(G)$ is same as the $(i,j)$th entry of $A(G)^2$, which is the number of 2-walks between the vertices $v_i$ and $v_j$. Further, the $(i,i)$th entry of $CN(G)$ is equal to the degree of $v_i$. Hence, $CN(G) = A(G)^2 - D(G)$, where $D(G)$ is the degree matrix of $G$. Let $CN-spec(G)$ be the spectrum of $CN(G)$. Then $CN-spec(G)$ is the set of all the eigenvalues of $CN(G)$ with multiplicities. If $a_1, a_2, \ldots, a_k$ are the distinct eigenvalues of $CN(G)$ with multiplicities $a_1, a_2, \ldots, a_k$, respectively, then we write $CN-spec(G) = \{a_1^{m_1}, a_2^{m_2}, \ldots, a_k^{m_k}\}$. The common neighborhood energy (abbreviated as CN-energy) of the graph $G$ is given by

$$E_{cn}(G) = \sum_{i=1}^{k} a_i |a_i|.$$
The study of CN-energy of graphs was introduced by Alwardi et al. in [1]. Various properties of CN-energy of a graph can also be found in [1,2]. The motivation of studying $E_{cn}(G)$ comes from the study of $E(G)$, which is well-known as energy of $G$, a notion introduced by Gutman [3]. Many results on $E(G)$, including some bounds and chemical applications, can be found in [4–15]. It is worth recalling that $E(G)$ is the sum of the absolute values of the eigenvalues of the adjacency matrix of $G$. It is also interesting to note that $E(G)$ can be obtained if $E_{cn}(G)$ is known for some classes of graphs. For instance, $E(K_n) = E_{cn}(K_n)/(n-2)$ and $E(K_{m,n}) = \sqrt{E_{cn}(K_{m,n}) + 2(n + m)}$, where $K_n$ is the complete graph on $n$ vertices and $K_{m,n}$ is the complete bipartite graph on $(m + n)$ vertices. A graph $G$ is called CN-hyperenergetic if $E_{cn}(G) > E_{cn}(K_n)$, where $n = |V(G)|$. It is still an open problem to produce a CN-hyperenergetic graph or to prove the non-existence of such graph (see [1] (Open problem 1)). In this paper we give an attempt to answer this problem by considering commuting graphs of finite groups.

The commuting graph of a finite non-abelian group $G$ with center $Z(G)$ is a simple undirected graph whose vertex set is $G \setminus Z(G)$ and two vertices $x$ and $y$ are adjacent if and only if $xy = yx$. We write $\Gamma_c(G)$ to denote this graph. In [16–23], various aspects of $\Gamma_c(G)$ are studied. In Section 2 of this paper, we derive an expression for computing CN-energy of a particular class of graphs and list a few already known results. In Section 3, we compute CN-energy of commuting graph of certain metacyclic group, dihedral group (which is the group of symmetries of a regular polygon), quasidihedral group, generalized quaternion group, Hanaki group etc. We also consider some generalizations of dihedral group and generalized quaternion group. Two graphs $G$ and $H$ with equal number of vertices are called CN-equiequigeretic if $E_{cn}(G) = E_{cn}(H)$. In Section 3, we shall also obtain some pairs of finite non-abelian groups such that their commuting graphs are CN-equiequigeretic. As consequences of our results, in Section 4, we show that $\Gamma_c(G)$ for all $G$ considered in Section 3 are not CN-hyperenergetic. We also identify some positive integers $n$ such that $\Gamma_c(G)$ is not CN-hyperenergetic if $G$ is an $n$-centralizer group. It is worth mentioning that CN-spectrums of $\Gamma_c(G)$ for certain classes of finite groups have been computed in [24] recently. However, the method adopted here, in computing CN-energy of $\Gamma_c(G)$ for various families of finite groups, is independent of CN-spec($\Gamma_c(G)$).

Recall that an $n$-centralizer group $G$ is a group such that $|\text{Cent}(G)| = n$, where $\text{Cent}(G) = \{C_G(w) : w \in G\}$ and $C_G(w) = \{v \in G : vw = vw\}$ is the centralizer of $w$ (see [25,26]). We also identify some $r \in \mathbb{Q}^{2,0}$ such that $\Gamma_c(G)$ is not CN-hyperenergetic if $\text{Pr}(G) = r$. Also recall that the commutativity degree of $G$, denoted by $\text{Pr}(G)$, is the probability that a randomly chosen pair of elements of $G$ commute.

Readers may review [27–32] for the background and various results regarding this notion. Further, we show that $\Gamma_c(G)$ is not CN-hyperenergetic if $\Gamma_c(G)$ is not planar or toroidal. Note that a graph is planar or toroidal according as its genus is zero or one respectively. Finally, we conclude the paper with a few conjectures.

2. A Useful Formula and Prerequisites

We write $G = G_1 \sqcup G_2$ to denote that $G$ has two components namely $G_1$ and $G_2$. Also, $IK_m$ denotes the disjoint union of $l$ copies of the complete graph $K_m$ on $m$ vertices. We begin this section with the following two key results of Alwardi et al. [1].

**Theorem 1** ([1] Proposition 2.4). If $G = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_m$ then $E_{cn}(G) = \sum_{i=1}^{m} E_{cn}(G_i)$.

**Lemma 1** ([1] Example 2.1). If $K_n$ denotes the complete graph on $n$ vertices then $E_{cn}(K_n) = 2(n-1)(n-2)$.

Now we derive a formula for CN-energy of graphs which are disjoint unions of some complete graphs. The following theorem is very useful in order to compute CN-energy of commuting graphs of finite groups.
Theorem 2. Let $G = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \cdots \sqcup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of $l_i$ copies of the complete graphs $K_{m_i}$ on $m_i$ vertices for $1 \leq i \leq k$. Then

$$E_{cn}(G) = 2 \sum_{i=1}^{k} l_i(m_i - 1)(m_i - 2).$$

Proof. By Theorem 1 we have

$$E_{cn}(G) = \sum_{i=1}^{k} l_iE_{cn}(K_{m_i}).$$

Therefore, the result follows from Lemma 1. \hfill \Box

We conclude this section with the following useful results from [17,18].

Lemma 2. Let $G$ be a finite group with center $Z(G)$. If $\frac{G}{Z(G)}$ is isomorphic to

1. The Suzuki group $Sz(2)$, presented by $\langle u, v : u^5 = v^4 = 1, v^{-1}uv = u^2 \rangle$, then $\Gamma_c(G) = 5K_{|Z(G)|} \sqcup K_{|Z(G)|}$.
2. $\mathbb{Z}_p \times \mathbb{Z}_p$, for any prime $p$, then $\Gamma_c(G) = (p + 1)K_{(p-1)|Z(G)|}$.
3. The dihedral group $D_{2n}$ $(m \geq 2)$, presented by $\langle u, v : u^n = v^2 = 1, uvv^{-1} = u^{-1} \rangle$, then $\Gamma_c(G) = K_{(m-1)|Z(G)|} \sqcup mK_{|Z(G)|}$.

Lemma 3. Let $G$ be a non-abelian group. If $G$ is isomorphic to

1. A group of order $pq$, where $p$ and $q$ are primes with $p \mid (q - 1)$, then $\Gamma_c(G) = K_{q-1} \sqcup qK_{p-1}$.
2. The quasidihedral group $QD_{2n}$ $(n \geq 4)$, presented by $\langle u, v : u^{2^{n-1}} = v^2 = 1, uvv^{-1} = u^{2^{n-2} - 1} \rangle$, then $\Gamma_c(G) = K_{2^{n-1}-1} \sqcup 2^{n-2}K_2$.
3. $PSL(2, 2^k)$, the projective special linear group for $k \geq 2$, then $\Gamma_c(G) = 2^k-1(2^k - 1)K_{2^k} \sqcup (2^k + 1)K_{2^k-1} \sqcup 2^{k-1}(2^k + 1)K_{2^k-2}$.
4. $GL(2, q)$, the general linear group where $q = p^n > 2$ and $p$ is a prime, then $\Gamma_c(G) = \frac{q(q-1)}{2}K_q \sqcup \frac{q(q+1)}{2}K_{q^2-3q+2} \sqcup (q + 1)K_{q^2-2q+1}$.

Lemma 4. Let $G$ be a non-abelian group. If $G$ is isomorphic to

1. The Hanaki group $A(n, \sigma)$ $(n \geq 2)$ of order $2^{2n}$ given by

$$\left\{ U(x, y) = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & \sigma(x) & 1 \end{bmatrix} : x, y \in F \right\},$$

under matrix multiplication where $F = GF(2^n)$ and $\sigma \in \text{Aut}(F)$ given by $\sigma(u) = u^2$, then $\Gamma_c(G) = (2^n - 1)K_{2^n}$.
2. The Hanaki group $A(n, p)$ of order $p^{3n}$ given by

$$\left\{ V(x, y, z) = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} : x, y, z \in F \right\},$$

under matrix multiplication where $F = GF(p^n)$ and $p$ is a prime, then $\Gamma_c(G) = (p^n + 1)K_{p^{2n-1}p^n}$.

3. CN-Energy of Commuting Graphs

In this section, we compute $E_{cn}(\Gamma_c(G))$ for several classes of finite non-abelian groups.

Theorem 3. Let $G$ be a finite group with center $Z(G)$. If $\frac{G}{Z(G)}$ is isomorphic to
1. The Suzuki group $Sz(2)$, then
   \[ E_{cn}(\Gamma_c(G)) = 2(61|Z(G)|^2 - 57|Z(G)| + 12). \]

2. $\mathbb{Z}_p \times \mathbb{Z}_p$, then
   \[ E_{cn}(\Gamma_c(G)) = 2(p + 1)((p - 1)|Z(G)| - 1)((p - 1)|Z(G)| - 2). \]

3. The dihedral group $D_{2m}$ ($m \geq 2$), then
   \[ E_{cn}(\Gamma_c(G)) = 2((m^2 - m + 1)|Z(G)|^2 - (6m - 3)|Z(G)| + 2m + 2). \]

**Proof.** By Lemma 2 and Theorem 2 we have

\[
E_{cn}(\Gamma_c(G)) = \begin{cases} 
2(4|Z(G)| - 1)(4|Z(G)| - 2) + 10(3|Z(G)| - 1)(3|Z(G)| - 2), & \text{if } \frac{G}{Z(G)} \cong Sz(2) \\
2(p + 1)((p - 1)|Z(G)| - 1)((p - 1)|Z(G)| - 2), & \text{if } \frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p \\
2((m - 1)|Z(G)| - 1)((m - 1)|Z(G)| - 2) + 2m(|Z(G)| - 1)(|Z(G)| - 2), & \text{if } \frac{G}{Z(G)} \cong D_{2m}.
\end{cases}
\]

Hence, the result follows on simplification. \hfill \Box

We have the following two corollaries of Theorem 3.

**Corollary 1.** Let $G$ be isomorphic to one of the following groups

1. $\mathbb{Z}_2 \times Q_8$,
2. $\mathbb{Z}_2 \times D_8$,
3. $\mathbb{Z}_4 \times \mathbb{Z}_4 = \langle u, v : u^4 = v^4 = 1, uvv^{-1} = u^{-1} \rangle$,
4. $M_{16} = \langle u, v : u^8 = v^2 = 1, uvv^{-1} = u^{-1} \rangle$,
5. $SG(16, 3) = \langle u, v : u^4 = v^4 = 1, uv = v^{-1}u^{-1}, uvv^{-1} = vv^{-1} \rangle$,
6. $D_8 \times \mathbb{Z}_4 = \langle u, v, w : u^4 = v^2 = w^2 = 1, uv = vu, uw = wu, vw = uvv^{-1} \rangle$.

Then $E_{cn}(\Gamma_c(G)) = 36$.

**Proof.** If $G$ is isomorphic to one of the above listed group then it is of order 16. Therefore, $|Z(G)| = 4$ and so $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, putting $p = 2$ in Theorem 3 (2) we get the required result. \hfill \Box

**Corollary 2.** Let $G$ be a non-abelian group.

1. If $G$ is of order $p^3$, for any prime $p$, then
   \[ E_{cn}(\Gamma_c(G)) = 2(p + 1)(p^2 - p - 1)(p^2 - p - 2). \]

2. If $G$ is the metacyclic group $M_{2mn}$ ($m \geq 3$), presented by $\langle u, v : u^m = v^{2m} = 1, uvv^{-1} = u^{-1} \rangle$, then
   \[ E_{cn}(\Gamma_c(G)) = \begin{cases} 
2((m^2 - m + 1)n^2 - (6m - 3)n + 2m + 2), & \text{if } m \text{ is odd} \\
2((m^2 - 2m + 4)n^2 - (6m - 6)n + m + 2), & \text{if } m \text{ is even}.
\end{cases} \]

3. If $G$ is the dihedral group $D_{2m}$ ($m \geq 3$), then
   \[ E_{cn}(\Gamma_c(G)) = \begin{cases} 
2(m - 2)(m - 3), & \text{if } m \text{ is odd} \\
2(m - 3)(m - 4), & \text{if } m \text{ is even}.
\end{cases} \]
4. If \( G \) is the generalized quaternion group \( Q_{4n} \) \((n \geq 2)\), presented by \( \langle u, v : v^{2n} = 1, u^2 = v^n, uvu^{-1} = v^{-1} \rangle \), then
\[
E_{cn}(\Gamma_c(G)) = 2(2n - 3)(2n - 4). 
\]

**Proof.** (1) If \( G \) is of order \( p^3 \) then \( |Z(G)| = p \) and \( \frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p \). Hence the result follows from Theorem 3 (2).

(2) We have
\[
|Z(M_{2mn})| = \begin{cases} 
n, & \text{if } m \text{ is odd} \\
2n, & \text{if } m \text{ is even} 
\end{cases}
\]
and
\[
\frac{M_{2mn}}{Z(M_{2mn})} \cong \begin{cases} 
D_{2m}, & \text{if } m \text{ is odd} \\
D_{nr}, & \text{if } m \text{ is even}. 
\end{cases}
\]
Hence, the result follows from Theorem 3 (3).

(3) Follows from part (2), considering \( n = 1 \).

(4) Follows from Theorem 3 (3), since \( |Z(Q_{4n})| = 2 \) and \( \frac{Q_{4n}}{Z(Q_{4n})} \cong D_{2n} \). \( \square \)

In the following theorems we compute \( E_{cn}(\Gamma_c(G)) \) for more families of groups.

**Theorem 4.** Let \( G \) be a non-abelian group.

1. If \( G \) is of order \( pq \), where \( p \) and \( q \) are primes with \( p \mid (q - 1) \), then
\[
E_{cn}(\Gamma_c(G)) = 2(q^2 + p^2q - 5pq + q + 6). 
\]

2. If \( G \) is the quasidihedral group \( QD_{2^n} \) \((n \geq 4)\), then
\[
E_{cn}(\Gamma_c(G)) = 2(2^{n-1} - 3)(2^{n-1} - 4). 
\]

3. If \( G = PSL(2, 2^k) \) then
\[
E_{cn}(\Gamma_c(G)) = 2^{4k+1} - 4 \cdot 2^{3k+1} + 6 \cdot 2^{k+1} + 12. 
\]

4. If \( G = GL(2, q) \) then
\[
E_{cn}(\Gamma_c(G)) = 2q^6 - 6q^5 - 2q^4 + 10q^3 + 6q^2 + 2q. 
\]

**Proof.** (1) If \( G \) is of order \( pq \) then, by Lemma 3 (1) and Theorem 2, we have
\[
E_{cn}(\Gamma_c(G)) = 2((q - 2)(q - 3) + q(p - 2)(p - 3)). 
\]
This gives the required result on simplification.

(2) Follows from Lemma 3 (2) and Theorem 2.

(3) By Lemma 3 (3) and Theorem 2 we have
\[
\frac{E_{cn}(\Gamma_c(G))}{2} = (2^k + 1)(2^k - 2)(2^k - 3) + 2^{k-1}(2^k + 1)(2^k - 3)(2^k - 4) + 2^{k-1}(2^k - 1)(2^k - 1)(2^k - 2), 
\]
which gives the required result.

(4) By Lemma 3 (4) and Theorem 2 we have
\[
E_{cn}(\Gamma_c(G)) = q(q + 1)(q^2 - 3q + 1)(q^2 - 3q) + q(q - 1)(q^2 - q - 1)(q^2 - q - 2) + 2(q + 1)(q^2 - 2q)(q^2 - 2q - 1), 
\]
which gives the required result on simplification. \( \square \)

**Theorem 5.** Let \( G \) be a non-abelian group.
1. If $G$ is the Hanaki group $A(n, \sigma)$ then
$$E_{cn}(\Gamma_c(G)) = 2(2^n - 1)^2(2^n - 2).$$

2. If $G$ is the Hanaki group $A(n, p)$ then
$$E_{cn}(\Gamma_c(G)) = 2(p^n + 1)(p^{2n} - p^n - 1)(p^{2n} - p^n - 2).$$

**Proof.** The result follows from Lemma 4 and Theorem 2. □

Note that all the groups considered above are abelian centralizer group (in short, AC-group). Now we present a result on $E_{cn}(\Gamma_c(G))$ if $G$ is a finite AC-group.

**Theorem 6.** Consider that an AC-group $G$ has distinct centralizers $X_1, \ldots, X_n$ of non-central elements of $G$. Then
$$E_{cn}(\Gamma_c(G)) = 2 \sum_{i=1}^{n} (|X_i| - |Z(G)| - 1)(|X_i| - |Z(G)| - 2).$$

**Proof.** We have $\Gamma_c(G) = \biguplus_{i=1}^{n} K|X_i| - |Z(G)|$ by [17] (Lemma 1). Therefore, by Theorem 2, the result follows. □

**Corollary 3.** Let $K$ be a finite abelian group and $H$ be a finite non-abelian AC-group. If $G \cong H \times K$ then
$$E_{cn}(\Gamma_c(G)) = 2 \sum_{i=1}^{n} (|Y_i||K| - |Z(H)||K| - 1)(|Y_i||K| - |Z(H)||K| - 2),$$
where $\text{Cent}(H) = \{H, Y_1, \ldots, Y_n\}$.

**Proof.** Clearly $Z(H \times K) = Z(H) \times K$ and $\text{Cent}(H \times K) = \{H \times K, Y_1 \times K, Y_2 \times K, \ldots, Y_n \times K\}$. Hence, $H \times K$ is an AC-group and so, by Theorem 6, the result follows. □

We shall conclude this section by obtaining some pairs of finite non-abelian groups such that their commuting graphs are CN-equiequenergetic.

**Proposition 1.** The commuting graphs of $D_{4k}$ and $Q_{4k}$ for $k \geq 2$ are CN-equiequenergetic.

**Proof.** The result follows from parts (3) and (4) of Corollary 2. □

Using Corollary 2 (parts (3) and (4)) and Theorem 4 (2) we also have the following result.

**Proposition 2.** The commuting graphs of $D_{2k}$, $Q_{2k}$ and $QD_{2k}$ for $k \geq 4$ are pairwise CN-equiequenergetic.

4. Some Consequences

In this section we derive some consequences of the results obtained in Section 3.

**Theorem 7.** Let $G$ be a finite group with center $Z(G)$. If $\frac{G}{Z(G)}$ is isomorphic to $Sz(2)$, $\mathbb{Z}_p \times \mathbb{Z}_p$ or $D_{2m}$ (where $p$ is any prime and $m \geq 2$) then $\Gamma_c(G)$ is not CN-hyperenergetic.

**Proof.** If $\frac{G}{Z(G)} \cong Sz(2)$ then, by Theorem 3 (1), we have
$$E_{cn}(\Gamma_c(G)) = 2|Z(G)|^2 - 57|Z(G)| + 12).$$
Let $G$ be a finite group. If $G$ is isomorphic to $M_{2m}$, $D_{2m}$, or $Q_{4n}$, then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. Since $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the result follows from Theorem 7 considering $p = 2$. □

Corollary 5. Let $G$ be a non-abelian group. If $G$ is isomorphic to $M_{2mn}$, $D_{2m}$, or $Q_{4n}$ or a group of order $p^3$ then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. If $G$ is isomorphic to $M_{2mn}$, $D_{2m}$ or $Q_{4n}$ then $\frac{G}{Z(G)}$ is isomorphic to some dihedral groups. If $G$ is isomorphic to a group of order $p^3$ then $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Hence, by Theorem 7, the result follows. □

We have the following results regarding commuting graphs of finite $n$-centralizer groups.

Theorem 8. If $G$ is a finite 4-centralizer group then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, by [25] (Theorem 2). Hence, using Theorem 7 for $p = 2$, the result follows. □

Theorem 9. Let $G$ be a finite $(p + 2)$-centralizer $p$-group. Then $\Gamma_c(G)$ is not CN-hyperenergetic.
Proof. We have \( \frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p \), by [33] (Lemma 2.7). Hence, by Theorem 7, the result follows. \( \square \)

**Theorem 10.** If \( G \) is a finite 5-centralizer group then \( \Gamma_c(G) \) is not CN-hyperenergetic.

**Proof.** We have \( \frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) or \( D_6 \), by [25] (Theorem 4). Hence, by Theorem 7, the result follows. \( \square \)

As a corollary to Theorems 8 and 10 we have the following result.

**Corollary 6.** Let \( G \) be a finite non-abelian group and \( \{x_1, x_2, \ldots, x_r\} \) be a set of pairwise non-commuting elements of \( G \) having maximal size. Then \( \Gamma_c(G) \) is not CN-hyperenergetic if \( r = 3, 4 \).

**Proof.** By [34] (Lemma 2.4), we have that \( G \) is a 4-centralizer or a 5-centralizer group according as \( r = 3 \) or \( 4 \). Hence the result follows from Theorems 8 and 10. \( \square \)

**Theorem 11.** Let \( G \) be a non-abelian group. If \( G \) is isomorphic to \( \text{QD}_{2^r} \), \( \text{PSL}(2, 2^k) \), \( A(n, r) \), \( GL(2, q) \), \( A(n, p) \) or a group of order \( pr \), where \( p \) and \( r \) are primes with \( p \mid (r - 1) \) and \( q = p^m > 2 \), then \( \Gamma_c(G) \) is not CN-hyperenergetic.

**Proof.** If \( G \) is isomorphic to \( \text{QD}_{2^r} \) then, by Theorem 4, we have \( E_{cn}(\Gamma_c(G)) = 2(2^{n-1} - 3)(2^{n-1} - 4) \). Since \( |V(\Gamma_c(G))| = 2^n - 2 \), by Lemma 1 we have

\[
E_{cn}(K_{2^{n-2}}) = 2(2^n - 3)(2^n - 4).
\]

Clearly, \( (2^n - 3)(2^n - 4) \) is a 4-centralizer or a 5-centralizer group

Since \( |V(\Gamma_c(G))| = 2^k(2^{2k} - 1) - 1 = 2^{3k} - 2^k - 1 \), by Lemma 1 we have

\[
E_{cn}(K_{2^{3k-2k-1}}) = 2(2^{3k} - 2^k - 1)(2^{3k} - 2^k - 3) = 2^{3k+1} - 2 \cdot 2^{3k+1} - 3 \cdot 2^{3k+1} + 2^{3k+1} + 5 \cdot 2^{3k+1} + 12.
\]

Therefore,

\[
E_{cn}(K_{2^{3k-2k-1}}) - E_{cn}(\Gamma_c(G)) = 2^{3k+1} - 3 \cdot 2^{3k+1} + 2^{3k+1} - 2^{3k+1} = 2^{3k+1}(2^k - 3) + 2^{3k+1}(2^k - 1).
\]

Since \( 2^k - 3 > 0 \) and \( 2^{2k} - 1 > 0 \) we have \( E_{cn}(K_{2^{3k-2k-1}}) - E_{cn}(\Gamma_c(G)) \) is positive. Hence, the result follows.

If \( G \) is isomorphic to \( GL(2, q) \) then, by Theorem 4 (4), we have

\[
E_{cn}(\Gamma_c(G)) = 2q^6 - 6q^5 - 2q^4 + 10q^3 + 6q^2 + 2q.
\]

Since \( |V(\Gamma_c(G))| = (q^2 - 1)(q^2 - q) - (q - 1) = q^4 - q^3 - q^2 + 1 \), by Lemma 1 we have

\[
E_{cn}(K_{q^4 - q^3 - q^2 + 1}) = 2(q^4 - q^3 - q^2)(q^4 - q^3 - q^2 - 1) = 2q^6 - 4q^7 - 2q^6 + 4q^5 + 2q^3 + 2q^2.
\]

Therefore,

\[
E_{cn}(K_{q^4 - q^3 - q^2 + 1}) - E_{cn}(\Gamma_c(G)) = 2q^6 - 4q^7 - 4q^6 + 10q^5 + 2q^4 - 8q^3 - 4q^2 - 2q = 2q^6(q^2 - q - 2) + 2q^7(5q^3 - 4q - 2) + 2q(q^3 - 2).
\]
We have $q^2 - 2q - 2 = q(q - 2) - 2 > 0$, $5q^3 - 4q - 2 = q(5q^2 - 4) - 2 > 0$ and $q^3 - 2 > 0$ since $q = p^m > 2$ for some prime $p$. Therefore, $E_{cn}(K_{g^2 \cdot q^2 - q^2 + 1}) - E_{cn}(\Gamma_c(G))$ is positive and hence the result follows.

If $G$ is isomorphic to $A(n, \sigma)$ then, by Theorem 5 (1), we have $E_{cn}(\Gamma_c(G)) = 2(2n - 1)^2(2n - 2)$. Since $|V(\Gamma_c(G))| = 2^n(2n - 1) = 2^{2n} - 2n$, by Lemma 1 we have

$$E_{cn}(K_{2^{2n} - 2n}) = 2(2^{2n} - 2n - 1)(2^{2n} - 2n - 2).$$

Clearly, $2^{2n} - 2n - 1 > 2^{2n} - 2 \cdot 2^n - 1 = (2n - 1)^2$ and $2^{2n} - 2n - 2 > 2^n - 2$. Therefore, $E_{cn}(K_{2^{2n} - 2n}) > E_{cn}(\Gamma_c(G))$.

If $G \cong A(n, p)$ then, by Theorem 5 (2), we have $E_{cn}(\Gamma_c(G)) = 2(p^n + 1)(p^{2n} - p^n - 1)(p^{2n} - p^n - 2)$. Since $|V(\Gamma_c(G))| = (p^n + 1)(p^{2n} - p^n)$, by Lemma 1 we have

$$E_{cn}(K_{(p^n + 1)(p^{2n} - p^n)}) = 2((p^n + 1)(p^{2n} - p^n) - 1)((p^n + 1)(p^{2n} - p^n) - 2).$$

We have

$$(p^n + 1)(p^{2n} - p^n - 1)(p^{2n} - p^n - 2)$$

$$< (p^n + 1)(p^{2n} - p^n - 1)(p^{2n} - p^n - 2)$$

$$= ((p^n + 1)(p^{2n} - p^n) - (p^n + 1))(p^n + 1)(p^{2n} - p^n) - 2(p^n + 1))$$

$$< ((p^n + 1)(p^{2n} - p^n) - 1)((p^n + 1)(p^{2n} - p^n) - 2).$$

Hence, $E_{cn}(\Gamma_c(G)) < E_{cn}(K_{(p^n + 1)(p^{2n} - p^n)})$.

If $G$ is isomorphic to a non-abelian group of order $pr$ then, by Theorem 4 (1), we have

$$E_{cn}(\Gamma_c(G)) = 2(r^2 + p^2r - 5pr + r + 6).$$

Since $|V(\Gamma_c(G))| = pr - 1$, by Lemma 1 we have

$$E_{cn}(K_{pr-1}) = 2(pr - 2)(pr - 3) = 2(p^2r^2 - 5pr + 6).$$

Since $r + 1 < 2(r - 1) < p^2(r - 1)$ we have $r^2 + p^2r + r < p^2r^2$. Hence, $E_{cn}(K_{pr-1}) > E_{cn}(\Gamma_c(G))$. This completes the proof. □

It is already mentioned that $Pr(G)$, the commutativity degree of a group $G$, is the probability that a randomly chosen pair of elements of $G$ commute. Therefore, it measures the abelianness of a group. For any finite group $G$, its commutativity degree can be computed using the formula

$$Pr(G) = \frac{1}{|G|^2} \sum_{w \in G} |C_G(w)|$$

or $Pr(G) = \frac{k(G)}{|G|}$,

where $k(G)$ is the number of conjugacy classes in $G$. In finite group theory, it is an interesting problem to find all the rational numbers $r \in (0, 1]$ such that $Pr(G) = r$ for some finite group $G$. Over the decades, many values of such $r$ have obtained and characterized finite groups such that $Pr(G) = r$. In the following theorem we list some values of $r$ such that $\Gamma_c(G)$ is not CN-hyperenergetic if $Pr(G) = r$.

**Theorem 12.** If $Pr(G) \in \{ \frac{5}{18}, \frac{2}{9}, \frac{11}{27}, \frac{7}{18}, \frac{1}{2}, \frac{5}{8} \}$ then $\Gamma_c(G)$ is not CN-hyperenergetic.

**Proof.** If $Pr(G) \in \{ \frac{5}{18}, \frac{2}{9}, \frac{11}{27}, \frac{7}{18}, \frac{1}{2}, \frac{5}{8} \}$ then $\frac{G}{Z(G)}$ is isomorphic to the groups in $\{ D_{14}, D_{10}, D_8, D_6, Z_2 \times Z_2, Z_2 \times Z_3 \}$ (by [35] (p. 246) and [36] (p. 451)). Hence, the result follows from Theorem 7. □
Theorem 13. Let $G$ be a finite group and $\Pr(G) = \frac{p^2 + p - 1}{p^2}$, where $p$ is the smallest prime divisor of $|G|$. Then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, by [37] (Theorem 3). Hence the result follows from Theorem 7.  

Theorem 14. If $G$ is a finite non-solvable group and $\Pr(G) = \frac{1}{12}$ then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. We have $G \cong A_5 \times K$ for some abelian group $K$, by [27] (Proposition 3.3.7). It can be seen that $\Gamma_c(G) = 5K_{3|K|} \sqcup 10K_{2|K|} \sqcup 6K_{4|K|}$. Therefore, by Theorem 2, we have

$$E_{cn}(\Gamma_c(G)) = 2(5(3|K| - 1)(3|K| - 2) + 10(2|K| - 1)(2|K| - 2)) + 6(4|K| - 1)(4|K| - 2))$$

$$= 2(181|K|^2 - 177|K| + 42).$$

Additionally, by Lemma 1, we have $E_{cn}(K_{59|K|}) = 2(3481|K|^2 - 177|K| + 2)$. Therefore

$$E_{cn}(K_{59|K|}) - E_{cn}(\Gamma_c(G)) = 2(3300|K|^2 - 40) > 0.$$  

This completes the proof. 

The following three theorems show that $\Gamma_c(G)$ is not CN-hyperenergetic if $\Gamma_c(G)$ is planar/toroidal or the complement of $\Gamma_c(G)$ is planar.

Theorem 15. Let $G$ be a finite non-abelian group. If $\Gamma_c(G)$ is planar then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. If $G \cong D_{12}, D_{10}, D_8, D_6, Q_8$ or $Q_{12}$ then, by Corollary 5, we have that $\Gamma_c(G)$ is not CN-hyperenergetic.

If $G$ is isomorphic to one of the groups listed in Corollary 1 then, by Corollary 4, it follows that $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong A_4$ then it can be seen that $\Gamma_c(G) = K_3 \sqcup 4K_2$. Using Theorem 2, we have $E_{cn}(\Gamma_c(G)) = 4$. Also, by Lemma 1, we have $E_{cn}(K_{11}) = 180$. Therefore, $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong Sz(2)$ then $\frac{G}{Z(G)} \cong Sz(2)$. Therefore, by Theorem 7, it follows that $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong SL(2,3)$ then it can be seen that $\Gamma_c(G) = 3K_3 \sqcup 4K_4$. Therefore, by Theorem 2, we have $E_{cn}(\Gamma_c(G)) = 48$. Also, by Lemma 1, we have $E_{cn}(K_{22}) = 840$. Therefore, $\Gamma_c(G)$ is not CN-hyperenergetic.

We have $PSL(2,4) \cong A_5$. Therefore, if $G \cong A_5$ then it follows that $\Gamma_c(G)$ is not CN-hyperenergetic (follows from Theorem 11).

If $G \cong S_4$ then the characteristic polynomial of $CN(\Gamma_c(G))$ is given by $x^8(x - 3)^2(x + 1)^{11}(x^2 - 5x - 30)$ and so

$$CN-spec(\Gamma_c(G)) = \left\{0^8, 3^2, (-1)^{11}, \left(\frac{5 + \sqrt{145}}{2}\right)^1, \left(\frac{5 - \sqrt{145}}{2}\right)^1 \right\}.$$  

Therefore, $E_{cn}(\Gamma_c(G)) = 17 + \sqrt{145}$. Additionally, by Lemma 1, we have $E_{cn}(K_{23}) = 924$. Therefore, $\Gamma_c(G)$ is not CN-hyperenergetic. Hence, the result follows from [38] (Theorem 2.2). 

Theorem 16. Let $G$ be a finite non-abelian group. If $\Gamma_c(G)$ is toroidal then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. If $G \cong D_{14}, D_{16}$ or $Q_{16}$ then by Corollary 5 it follows that $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong QD_{16}$ then, by Theorem 11, we have that $\Gamma_c(G)$ is not CN-
hyperenergetic. If \( G \) is isomorphic to \( \mathbb{Z}_7 \times \mathbb{Z}_3 \) then \( \Gamma_c(G) \) is not CN-hyperenergetic, follows from Theorem 11 considering \( p = 3 \) and \( r = 7 \). If \( G \cong D_6 \times \mathbb{Z}_3 \) then \( \frac{G}{Z(G)} \cong D_6 \). Therefore, by Theorem 7, \( \Gamma_c(G) \) is not CN-hyperenergetic. If \( G \cong A_4 \times \mathbb{Z}_2 \) then it can be seen that \( \Gamma_c(G) = K_6 \sqcup 4K_4 \). Therefore, by Theorem 2, we have \( E_{cn}(\Gamma_c(G)) = 2(5 \cdot 4 + 4 \cdot 3 \cdot 2) = 88 \). Also, by Lemma 1, we have \( E_{cn}(K_{22}) = 2 \cdot 21 \cdot 20 = 840 \). Hence, \( \Gamma_c(G) \) is not CN-hyperenergetic. Hence, the result follows from [39] (Theorem 6.6).

We also have the following result.

**Theorem 17.** Let \( G \) be a finite non-abelian group. If the complement of \( \Gamma_c(G) \) is planar then \( \Gamma_c(G) \) is not CN-hyperenergetic.

**Proof.** The result follows from [40] (Proposition 2.3) and Corollary 5.

In view of the above results we conclude this paper with a few conjectures.

**Conjecture 1.** A planar or toroidal graph is not CN-hyperenergetic.

**Conjecture 2.** \( \Gamma_c(G) \) is not CN-hyperenergetic.

**Conjecture 3.** If \( G = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \cdots \sqcup l_kK_{m_k} \), where \( l_iK_{m_i} \) denotes the disjoint union of \( l_i \) copies of the complete graphs \( K_{m_i} \) on \( m_i \) vertices for \( 1 \leq i \leq k \), then it is not CN-hyperenergetic.

**Author Contributions:** Conceptualization, R.K.N., W.N.T.F., K.C.D. and Y.S.; investigation, R.K.N., W.N.T.F., K.C.D. and Y.S.; writing—original draft preparation, R.K.N., W.N.T.F., K.C.D. and Y.S.; writing—review and editing, R.K.N., W.N.T.F., K.C.D. and Y.S.; funding acquisition, W.N.T.F., K.C.D. and Y.S. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was funded by National Research Foundation fund from the Korean government, Grant No. 2021R1F1A1050, and UoA Flexible Fund from Northumbria University, Grant No. 201920A1001.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** No new data is created in this paper.

**Acknowledgments:** The authors are grateful to the referees for their valuable comments and suggestions. W. N. T. Fasfous is thankful to Indian Council for Cultural Relations for the ICCR Scholarship. K. C. Das is supported by National Research Foundation funded by the Korean government (Grant No. 2021R1F1A1050). Y. Shang was supported by UoA Flexible Fund No. 201920A1001 from Northumbria University.

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**

1. Alwardi, A.; Soner, N.D.; Gutman, I. On the common-neighborhood energy of a graph. *Bulletin 2011*, 36, 49–59.
2. Alwardi, A.; Arsić, B.; Gutman, I.; Soner, N.D. The common neighborhood graph and its energy. *Iran. Math. Sci. Inf. 2012*, 7, 1–8.
3. Gutman, I. The energy of a graph. *Bol. Soc. Mat. Mex. 2013*, 19, 123–127.
4. Akbari, S.; Das, K.C.; Ghahremani, M.; Koorepazan-Mohtakhar, F.; Raoufi, E. Energy of graphs containing disjoint cycles. *MATCH Commun. Math. Comput. Chem. 2021*, 86, 543–547.
5. Bozkurt, S.B.; Gürbüz, A.D. Improved bounds for the spectral radius of digraphs. *Hacet. J. Math. Stat. 2010*, 39, 313–318.
6. Das, K.C.; Alazemi, A.; Andelic, M. On energy and Laplacian energy of chain graphs. *Discret. Appl. Math. 2020*, 284, 391–400. [CrossRef]
7. Das, K.C.; Elumalai, S. On energy of graphs. *MATCH Commun. Math. Comput. Chem. 2017*, 77, 3–8.
8. Das, K.C.; Gutman, I. Bounds for the Energy of Graphs. *Hacet. J. Math. Stat. 2016*, 45, 1–9. [CrossRef]
9. Das, K.C.; Gutman, I. Upper bounds on distance energy. *MATCH Commun. Math. Comput. Chem. 2021*, 86, 611–620.
10. Das, K.C.; Mojallal, S.A.; Gutman, I. On energy and Laplacian energy of bipartite graphs. *Appl. Math. Comput. 2016*, 273, 759–766. [CrossRef]
11. Das, K.C.; Mojallal, S.A.; Sun, S. On the sum of the $k$ largest eigenvalues of graphs and maximal energy of bipartite graphs. *Linear Algebra Appl.* 2019, 569, 175–194. [CrossRef]
12. Gutman, I. Bounds for total $\pi$-electron energy. *Chem. Phys. Lett.* 1974, 24, 283–285. [CrossRef]
13. Gutman, I. New approach to the McClelland approximation. *MATCH Commun. Math. Comput. Chem.* 1983, 14, 71–81.
14. Gutman, I.; Radenović, S.; Đorđević, S.; Milovanović, Ž.I.; Milovanović, E.I. Extending the McClelland formula for total $\pi$-electron energy. *J. Math. Chem.* 2017, 55, 1934–1940. [CrossRef]
15. Milovanović, I.; Milovanović, E.; Gutman, I. Upper bounds for some graph energies. *Appl. Math. Comput.* 2016, 289, 435–443. [CrossRef]
16. Akbari, S.; Mohammadian, A.; Radjavi, H.; Raja, P. On the diameters of commuting graphs. *Linear Algebra Appl.* 2006, 418, 161–176. [CrossRef]
17. Dutta, J.; Nath, R.K. Spectrum of commuting graphs of some classes of finite groups. *Matematika* 2017, 33, 87–95.
18. Dutta, J.; Nath, R.K. Finite groups whose commuting graphs are integral. *Mat. Vesnik* 2017, 69, 226–230.
19. Dutta, J.; Nath, R.K. Laplacian and signless Laplacian spectrum of commuting graphs of finite groups. *Khayyam J. Math.* 2018, 4, 77–87.
20. Iranmanesh, A.; Jafarzadeh, A. Characterization of finite groups by their commuting graph. *Acta Math. Acad. Paedagog. Nyhazi.* 2007, 23, 7–13.
21. Morgan, G.L.; Parker, C.W. The diameter of the commuting graph of a finite group with trivial center. *J. Algebra* 2013, 393, 41–59. [CrossRef]
22. Nath, R.K. Various spectra of commuting graphs $n$-centralizer finite groups. *Int. J. Eng. Sci. Tech.* 2018, 10, 170–172. [CrossRef]
23. Parker, C. The commuting graph of a soluble group. *Bull. Lond. Math. Soc.* 2013, 45, 839–848. [CrossRef]
24. Fasfous, W.N.T.; Sharafdini, R.; Nath, R.K. Common neighborhood spectrum of commuting graphs of finite groups. *arXiv* 2020, arXiv:2002.10146.
25. Belcastro, S.M.; Sherman, G.J. Counting centralizers in finite groups. *Math. Mag.* 1994, 67, 366–374. [CrossRef]
26. Dutta, J. A Study of Finite Groups in Terms of Their Centralizers. Master’s Thesis, North-Eastern Hill University, Meghalaya, India, 2010.
27. Castelaz, A. Commutativity Degree of Finite Groups. Master’s Thesis, Wake Forest University, Winston-Salem, NC, USA, 2010.
28. Das, A.K.; Nath, R.K.; Pournaki, M.R. A survey on the estimation of commutativity in finite groups. *Southeast Asian Bull. Math.* 2013, 37, 161–180.
29. Erdős, P.; Turán, P. On some problems of a statistical group-theory IV. *Acta. Math. Acad. Sci. Hungar.* 1968, 19, 413–435. [CrossRef]
30. Shang, Y. A note on the commutativity of prime near-rings. *Algebra Colloq.* 2015, 22, 361–366. [CrossRef]
31. Nath, R.K. Commutativity Degrees of Finite Groups—A Survey. Master’s Thesis, North-Eastern Hill University, Meghalaya, India, 2008.
32. Nath, R.K.; Das, A.K. On a lower bound of commutativity degree. *Rend. Circ. Math. Palermo* 2010, 59, 137–141. [CrossRef]
33. Ashrafi, A.R. On finite groups with a given number of centralizers. *Algebra Colloq.* 2000, 7, 139–146. [CrossRef]
34. Abdollahi, A.; Jafarain, S.M.; Hassanabadi, A.M. Groups with specific number of centralizers. *Houston J. Math.* 2007, 33, 43–57.
35. Rusin, D.J. What is the probability that two elements of a finite group commute? *Pacific J. Math.* 1979, 82, 237–247. [CrossRef]
36. Nath, R.K. Commutativity degree of a class of finite groups and consequences. *Bull. Aust. Math. Soc.* 2013, 88, 448–452. [CrossRef]
37. MacHale, D. How commutative can a non-commutative group be? *Math. Gaz.* 1974, 58, 199–202. [CrossRef]
38. Afkhami, M.; Farrokh DG, M.; Khayyaramanesh, K. Planar, toroidal, and projective commuting and non-commuting graphs. *Comm. Algebra* 2015, 43, 2964–2970. [CrossRef]
39. Das, A.K.; Nongsiang, D. On the genus of the commuting graphs of finite non-abelian groups. *Int. Electron. J. Algebra* 2016, 19, 91–109. [CrossRef]
40. Abdollahi, A.; Akbari, S.; Maimani, H.R. Non-commuting graph of a group. *J. Algebra* 2006, 298, 468–492. [CrossRef]