GEODESIC MULTIPLICATION AND QUANTUM KINEMATICS IN A NEWTONIAN SPACETIME

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Abstract

We consider a quantum test particle in the background of a Newtonian gravitational field in the framework of Cartan’s formulation of nonrelativistic spacetime. We have proposed a novel quantization of a point particle which amounts to introducing its position operators as multiplication with the corresponding Riemann normal coordinates and momentum operators as infinitesimal right translation operators determined by geodesic multiplication of points of the spacetime. We present detailed calculations for the simplest model of a two-dimensional Newtonian spacetime.

1. Introduction. Let us consider a 4-dimensional differentiable manifold $M$ with a symmetric affine connection (a spacetime with vanishing torsion). Every reasonable neighbourhood $M_e \in M$ can be equipped with a nonassociative binary operation called the geodesic multiplication [1, 2, 3, 4].

Its infinitesimal left and right translations can be used to define the (geodesic) momentum operators of a quantum test particle in the curved spacetime [5, 6]. The corresponding commutation relations can be taken as the quantum kinematic algebra. It coincides with the usual canonical Poisson algebra (Weyl’s kinematics) only in the case of the flat spacetime. In the
present paper, detailed calculations are performed for the spacetime which

describes a nonrelativistic Newtonian gravitational field.

2. Geodesic multiplication and translations. The local geodesic

multiplication of points \( x, y \in M_e \) is defined [1, 2] by

\[
xy \equiv Ly \equiv R_y x = (\exp_y \circ \tau^e_y \circ \exp^{-1}_e)x.
\]

Here we have used the exponential mapping \( T_e M \rightarrow M_e \)

and the parallel transport mapping \( \tau^e_y : T_e M \rightarrow T_y M \)

along a geodesic \( y(s) \) emerging from \( e \in M_e \). The local geodesic multiplication can be constructed in every

neighbourhood \( M_e \) where all required exponential mappings and parallel transport operations are well defined. \( M_e \)

is a finite region of \( M \), where geodesics emerging from \( e \) do not intersect and which does not contain singular points.

By introducing the local geodesic multiplication, \( M_e \) can be seen to be a binary algebraic system called the

local geodesic loop [1, 2, 3]. The point \( e \in M \) is the unit element of the loop \( M_e \) and the (left) inverse \( x^{-1}_L \)

of \( x \in M_e \) is defined by \( x^{-1}_L x = e \).

In general, the geodesic multiplication need not be commutative and associative [2].

The local geodesic multiplication ([1]) determines the following left (L) and right (R) infinitesimal translation matrices:

\[
(xy)^\mu = y^\mu + L^\mu_\nu(y)x^\nu + \ldots, \quad L^\mu_\nu(y) \equiv \frac{\partial(xy)^\mu}{\partial x^\nu} \bigg|_{x=e}, \quad (2)
\]

\[
(xy)^\mu = x^\mu + R^\mu_\nu(x)y^\nu + \ldots, \quad R^\mu_\nu(x) \equiv \frac{\partial(xy)^\mu}{\partial y^\nu} \bigg|_{y=e}. \quad (3)
\]

Note that the lower indices of matrices \( L^\mu_\nu \), \( R^\mu_\nu \) in fact belong to the tangent space \( T_e M \). In the following let us denote tangent space (flat) indices with letters from the beginning of the alphabets. Now two local frame fields can be introduced

\[
L_\alpha(x) \equiv L^\mu_\alpha(x)\partial_\mu, \quad R_\alpha(x) \equiv R^\mu_\alpha(x)\partial_\mu. \quad (4)
\]

It is well known that for two vector fields their commutator is again a vector field. We know that \( L_\alpha(x) \) and \( R_\alpha(x) \) are frame fields, so it is quite natural to define the structure functions \( \lambda^\gamma_{\alpha\beta}(x) \) and \( \rho^\gamma_{\alpha\beta}(x) \) by

\[
[L_\alpha(x), L_\beta(x)] = -\lambda^\gamma_{\alpha\beta}(x)L_\gamma(x), \quad (5)
\]
\[ [R_\alpha(x), R_\beta(x)] = +\rho^\gamma_{\alpha\beta}(x) R_\gamma(x). \] (6)

In general, the structure functions \( \lambda^\gamma_{\alpha\beta}(x) \), \( \rho^\gamma_{\alpha\beta}(x) \) do not coincide.

3. Geodesic translations in the Riemann normal coordinates.

Let us introduce the Riemann normal coordinates \( y^\alpha \) for which \( \Gamma^\alpha_{\beta\gamma}(e) = 0 \). In these coordinates, expansions of \( R^\alpha_{\beta}(y) \) and \( L^\alpha_{\beta}(y) \) can be found using Eqs. (2), (3) and the geodesic multiplication formula of Akivis [2]

\[(zy)^\alpha = z^\alpha + y^\alpha - \frac{1}{2} \Gamma^\alpha_{\beta\gamma}(e) z^\beta z^\gamma y^\delta - \frac{1}{2} \Gamma^\alpha_{\beta(\gamma,\delta)}(e) z^\beta y^\gamma y^\delta + \ldots \] (7)

We get

\[ R^\alpha_{\beta}(y) = \delta^\alpha_{\beta} - \frac{1}{2} \Gamma^\alpha_{\beta\gamma}(e) y^\delta y^\gamma + \ldots \] (8)

\[ L^\alpha_{\beta}(y) = \delta^\alpha_{\beta} - \frac{1}{2} \Gamma^\alpha_{\beta(\gamma,\delta)}(e) y^\gamma y^\delta + \ldots \] (9)

Now we can introduce local frame fields \( L_\alpha(y) \) and \( R_\alpha(y) \). Their structure functions \( \lambda^\gamma_{\alpha\beta}(y) \) and \( \rho^\gamma_{\alpha\beta}(y) \) have expansions [3, 7]

\[ \lambda^\gamma_{\alpha\beta}(y) = -R^\gamma_{[\alpha\beta\delta]}(e) y^\delta + \ldots \] (10)

\[ \rho^\gamma_{\alpha\beta}(y) = R^\gamma_{\delta[\alpha\beta]}(e) y^\delta + \ldots \] (11)

The commutator of two frame fields can also be calculated [3, 7]

\[ [L_\alpha(x), R_\beta(x)] = \frac{1}{2} R^\kappa_{\alpha\delta\beta}(e) y^\delta \partial_\kappa + \ldots \] (12)

Here \( R^\gamma_{[\alpha\beta\delta]}(e) \) denote components of the curvature tensor at the unit element \( e \). We use definitions and sign conventions given in [3].

4. Structure of a nonrelativistic spacetime. Let us consider a four–dimensional differentiable manifold \( V \) with an affine connection and a vanishing torsion. Following [9], let us introduce the absolute time by imposing two conditions.

1. There exists a function \( t(x^\mu) \) (the absolute time) and therefore 1–form \( dt \) that is exact,

\[ dt \equiv t_\mu dx^\mu = t_\mu dx^\mu, \quad d \wedge dt = 0. \]

2. 1–form \( dt \) is covariantly constant,

\[ \nabla_u dt = \nabla_u (t_\mu dx^\mu) \equiv u^\nu (t_\mu,\nu - \Gamma^\nu_{\mu\sigma} t_\sigma) dx^\mu = 0 \]
for all vectors \( u \in TV \).

For introducing the absolute space we have proposed [10, 11] to impose the condition of vanishing curvature of three-hyperplanes \( t = \text{const} \) with coordinates \( x^i (i = 1, 2, 3) \), \( R^a_{bcd}(x^i) = 0 \). It is slightly less restrictive than the corresponding conditions \( R^a_{b\mu\nu} = 0, R^a_{\beta mn} = 0 \) given in [8].

Let us introduce basis 1–forms \( \omega^\alpha \). 1–form \( \omega^0 \) can be chosen in the direction of \( dt \), \( \omega^0 \sim dt \). The scalar product between basis 1–forms is defined as

\[
\omega^\alpha \cdot \omega^\beta = e^{\alpha\beta} \equiv \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\] (13)

Now the relation \( \rho \cdot dt = 0 \) holds for every 1–form \( \rho \). Singular metric \( e^{\alpha\beta} \) is supposed to be covariantly constant, \( \nabla_\mu e^{\alpha\beta} = 0 \).

Let us introduce also connection 1–forms \( \omega^\alpha_\beta \equiv \Gamma^\alpha_\beta_\delta \omega^\delta \). By definition the following relation holds:

\[
\nabla_\beta \omega^\alpha = -\Gamma^\alpha_\beta_\delta \omega^\delta.
\]

Applying the covariant differentiation operator \( \nabla_\beta \) to condition (13) we get the following properties of the connection 1–forms:

\[
\omega^a_d \delta^{bd} + \omega^b_d \delta^{ad} = 0,
\]

\[
\omega^0_a = 0.
\]

We don’t get any conditions for connection 1–forms \( \omega^0_0 \) because of the special form of singular metric \( e^{\alpha\beta} \).

We can introduce also a dual basis consisting of vectors \( e_\alpha = e_\alpha^\mu \frac{\partial}{\partial x^\mu} \), \( < \omega^\alpha, e_\beta > = \delta^\alpha_\beta \). But because of the singularity of metric \( e^{\alpha\beta} \) one cannot introduce the corresponding scalar product between vectors \( e_\alpha \).

5. **Nonrelativistic spacetime with a Newtonian gravitational field.** In a nonrelativistic spacetime where all the abovementioned conditions hold there exists a preferred coordinate system (Galilean coordinates \( x \)) where the basis 1–forms are simply differentials of holonomic coordinates \( \omega^\mu = dx^\mu \). In particular, \( \omega^0 = dx^0 = dt \).

Let us consider a nonrelativistic spacetime specified by a Newtonian gravitational potential \( \Phi(x) \). Let us identify trajectories of freely falling test particles

\[
\frac{d^2 x^j}{dt^2} + \frac{\partial \Phi}{\partial x^j} = 0
\] (14)

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with geodesic lines of a curved spacetime

\[ \frac{d^2x^j}{dt^2} + \Gamma^j_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0. \] (15)

As a result, the Newtonian spacetime has a nontrivial affine connection and a nonvanishing curvature tensor \[8, 9\]

\[
\Gamma^j_{00} = \frac{\partial \Phi}{\partial x^j}, \quad R^j_{0i0k} = -R^j_{00k} = \frac{\partial^2 \Phi}{\partial x^j \partial x^k},
\] (16)

all other components vanish.

Let us introduce the Riemann normal coordinates \( \{t_R, y^i\} \) with the origin of coordinates \( e^\mu = 0 \). In the linear approximation in gravitational potential \( \Phi \) the transformation formulas read

\[
t = t_R, \\
x^j = y^j - \frac{1}{2} \frac{\partial \Phi(0)}{\partial x^j} t^2_R - \frac{1}{6} \frac{\partial^2 \Phi(0)}{\partial x^j \partial x^k} t^2_R y^k + \ldots
\] (17)

(18)

The Galilean absolute time \( t \) does not transform and in the following we need not specify the coordinates to which it belongs.

Geodesic lines emerging from \( e = 0 \) – trajectories of freely falling test particles – are now given by equations

\[
y^j(t) = v^i(0)t
\] (19)

where \( v^i(0) = \text{const} \) are initial velocities, \( v^i(0) = \dot{y}^i(0) = \dot{x}^i(0) \). Information about the gravitational field is encoded in the transformation (18) from Galilean to Riemannian coordinates: according to Eq. (14), \( -\partial \Phi(x)/\partial x^j \) is the gravitational force acting upon a test particle with a unit mass.

6. A two-dimensional model. For simplicity, let us consider a two-dimensional Newtonian spacetime with an absolute time and one space coordinate. In the Riemann normal coordinates \( \{t, y\} \), nonvanishing connection coefficients have the following expansions in \( \{t, y\} \) and in \( \Phi \):

\[
\Gamma^y_{00} = \frac{2}{3} Cy + \ldots, \quad \Gamma^y_{y0} = -\frac{1}{3} Ct + \ldots
\] (20)
where we have introduced a notation $\frac{\partial^2 \Phi(0)}{\partial x \partial x} \equiv C$. From Eqs. (8), (9) the right and the left translation matrices can be calculated. Expansions of the right and the left local frame fields (4) read:

$$
R_0 = R_0^s \partial_s = \partial_t + \left( \frac{1}{3} C ty + \ldots \right) \partial_y,
$$

$$
R_y = R_y^s \partial_s = \left( 1 - \frac{1}{3} C t^2 + \ldots \right) \partial_y;
$$

$$
L_0 = L_0^s \partial_s = \partial_t - \left( \frac{1}{6} C t^2 + \ldots \right) \partial_y,
$$

$$
L_y = L_y^s \partial_s = \left( 1 + \frac{1}{6} C t^2 + \ldots \right) \partial_y.
$$

The corresponding structure functions are

$$
\rho^y_0 = -C t + \ldots \quad \lambda^y_0 = -\frac{1}{2} C t + \ldots
$$

and

$$
[L_0, R_0] = \left( \frac{1}{2} C y + \ldots \right) \partial_y,
$$

$$
[L_0, R_y] = \left( -\frac{1}{2} C t + \ldots \right) \partial_y,
$$

$$
[L_y, R_0] \approx 0, \quad [L_y, R_y] \approx 0.
$$

Note that in the lowest approximation, $\rho$ and $\lambda$ depend only on time $t$.

7. **A quantum test particle in a curved spacetime.** Let us consider a quantum test particle moving in a curved spacetime [5, 6]. Let us introduce an action (on scalar valued functions) of the position operators $y^\alpha$ as multiplication with the Riemann normal coordinates $y^\alpha$. Then we have

$$
[y^\alpha, y^\beta] = 0.
$$

We have proposed to define (geodesic) momentum operators $p_\alpha$ via infinitesimal right geodesic translations $p_\alpha(y) = -i\hbar R^\beta_\alpha(y) \partial_\beta$. Then,

$$
[y^\alpha, p_\beta(y)] = i\hbar R^\alpha_\beta(y)
$$

and

$$
[p_\alpha, p_\beta] = -i\hbar \rho^\delta_\alpha_\beta p_\delta.
$$
Commutators (29)–(31) constitute a modification of canonical quantum commutation relations.

8. Modified canonical formalism for a test particle in a two-dimensional Newtonian spacetime. To clarify the physical and geometrical meaning of the quantum commutation relations (29)–(31), let us consider the corresponding modified canonical formalism in the classical mechanics for a test particle moving in a two-dimensional Newtonian spacetime. Let us replace quantum commutators with classical brackets as follows:

\[
\frac{[y^\alpha, p_\beta]}{\hbar} \to [y^\alpha, p_\beta], \quad \frac{p_\beta}{\hbar} \to p_\beta. \tag{32}
\]

Using results of Sec. 6, nonvanishing brackets in the lowest approximation are

\[
[y, p_y] = 1 - \frac{1}{3}Ct^2, \tag{33}
\]
\[
[t, p_0] = 1, \tag{34}
\]
\[
[y, p_0] = \frac{1}{3}Cty \tag{35}
\]

and

\[
[p_y, p_0] = -Ctp_y. \tag{36}
\]

Let us introduce the canonical momentum vector \(p_{k^{\text{can}}}\) with brackets

\[
[y^i, p_{k^{\text{can}}}] = \delta^i_k. \tag{37}
\]

Then brackets (33)–(35) can be considered as defining the components of geodesic momentum vector \(p_k\) in terms of canonical momentum \(p_{k^{\text{can}}}\) as follows:

\[
p_y = p_{y^{\text{can}}}(1 - \frac{1}{3}Ct^2), \tag{38}
\]
\[
p_0 = p_{0^{\text{can}}} + \frac{1}{3}Cty p_{y^{\text{can}}}. \tag{39}
\]

Note that the canonical momentum of a test particle is the main part of its geodesic momentum.

Let us try to find a physical interpretation of the geodesic momentum. In our two-dimensional model, the spacetime can be considered as consisting of a
family of geodesic lines. Let us denote \( v(0) = v \), then \( v \) acts as a a parameter which enumerates the geodesics. The time \( t \) is a canonical parameter along a geodesic. Now \( y^i = \{ t, \ y(t, v) \} \), \( t > 0 \) can be considered as a change of coordinates where the new coordinate lines are \( g^i(t) = y^i(t, v = \text{const}) \) and \( h^i(v) = y^i(t = \text{const}, v) \). The corresponding tangent vectors are

\[
U^i = \frac{dg^i(t)}{dt} = \{ 1, v \}, \quad V^i = \frac{dh^i(v)}{dv} = \{ 0, t \}.
\]

(40)

Using the expressions for connection coefficients (20), let us calculate the following absolute derivatives \( \frac{D}{Dt} \) along lines \( y^i = g^i(t) \) – the trajectories of freely falling particles (19):

\[
\frac{DU^i}{Dt} = \frac{dU^i}{dt} + \Gamma^i_{kl} U^k \frac{dy^l}{dt} = 0,
\]

(41)

\[
\frac{DV^i}{Dt} = \{ 0, 1 - \frac{1}{3} C t^2 \},
\]

(42)

\[
\frac{D^2V^i}{Dt^2} = \{ 0, -C t \}.
\]

(43)

Eq. (11) confirms that lines \( y^i = g^i(t) \) are geodesic lines. Vector \( V^i dv \) is a vector joining a geodesic to a neighbouring geodesic. The rate of change of \( V^i \) along the geodesic \( g(t) \) is given by the geodesic deviation equation which now reads

\[
\frac{D^2V^1}{Dt^2} = -C t.
\]

(44)

In the case of a test particle with a unit mass, canonical commutation relations (37) allow us to identify \( p_y = 1 = v \) and write \( y = vt = p^\text{can}_y V^1 \). As a result we get in the lowest approximation

\[
\frac{Dy}{Dt} = p^\text{can}_y \frac{DV^1}{Dt} = p^\text{can}_y (1 - \frac{1}{3} C t^2) = p_y,
\]

(45)

\[
\frac{D^2y}{Dt^2} = p^\text{can}_y \frac{D^2V^1}{Dt^2} = -C t p^\text{can}_y = -C t p_y = [p_y, p_0].
\]

(46)

We see that the geodesic momentum \( p_y \) describes the dynamics of the gravitational field and its equation of motion – the geodesic deviation equation – is encoded in the commutator (36).
8. Discussion. The simple model of a test particle moving in a two-
dimensional Newtonian spacetime allows us to clarify the physical meaning 
of modified commutator algebra (33)–(35) in the lowest approximation in the 
Riemann coordinates \( \{ t, y \} \) and in the gravitational potential \( \Phi \).

All our model calculations can easily be generalized to the case of a four-
dimensional nonrelativistic Newtonian spacetime. However, difficulties can 
emerge if we consider a relativistic spacetime where the time coordinate is 
not absolute and where a nonsingular metric tensor \( g_{\mu \nu} \) exists.

In the standard quantum mechanics, commutation relations are entirely 
kinematical and dynamics is given by a Hamiltonian. As distinct from this, 
since the dynamics of the gravitational field has been encoded in the coordi-
nate transformation (18), modified commutation relations contain informa-
tion about kinematics as well as dynamics.

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