Estimation of Bandlimited Signals in Additive Gaussian Noise: a “Precision Indifference” Principle

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Abstract

The sampling, quantization, and estimation of a bounded dynamic-range bandlimited signal affected by additive independent Gaussian noise is studied in this work. For bandlimited signals, the distortion due to additive independent Gaussian noise can be reduced by oversampling (statistical diversity). The pointwise expected mean-squared error is used as a distortion metric for signal estimate in this work. Two extreme scenarios of quantizer precision are considered: (i) infinite precision (real scalars); and (ii) one-bit quantization (sign information). If \( N \) is the oversampling ratio with respect to the Nyquist rate, then the optimal law for distortion is \( O(1/N) \). We show that a distortion of \( O(1/N) \) can be achieved irrespective of the quantizer precision by considering the above-mentioned two extreme scenarios of quantization. Thus, a quantization precision indifference principle is discovered, where the reconstruction distortion law, up to a proportionality constant, is unaffected by quantizer’s accuracy.

Index Terms

bandlimited signals, sampling, estimation, quantization

I. INTRODUCTION

Consider a bounded-dynamic signal (or field) quantization problem, where the samples are affected by additive independent and identically distributed (i.i.d.) Gaussian noise. For example, a spatial signal affected by additive i.i.d. Gaussian noise has to be sampled using an array of sensors. In a distributed setup, where filtering before sampling is not possible, noise in bandlimited signals can be reduced by statistical averaging of independent noisy samples. In addition, quantization error can be reduced by oversampling as well as by increasing the analog-to-digital converter (ADC) or quantizer precision. The fundamental tradeoff between oversampling, quantizer precision, and (statistical) average distortion is of interest.

The tradeoffs between all subsets of these three quantities has been studied in the literature. Tradeoffs for average distortion and oversampling, additive Gaussian noise and average distortion with unquantized samples, and oversampling and quantization have a flurry of work (e.g., see \([1], [2], [3], [4], [5]\)). In this work, the tradeoff between oversampling, quantizer precision, and the average distortion is of interest.

If extremely high-precision ADCs are used, then the sample distortion is noise limited. On the other hand, if lowest precision single-bit ADCs are used, then the sample distortion is limited by quantization. At a high-level, it is expected that the distortion optimal ADC precision should be in between these two extreme cases, in the sense that it should be able to resolve the signal up to the noise level. Contrary to this intuition, in this work it is shown that a distortion inversely proportional to the oversampling above the Nyquist rate is achievable with single-bit quantizers. With unquantized (infinite precision) samples, the optimal distortion is speculated to be inversely proportional to the oversampling above the Nyquist rate in the presence of independent Gaussian noise. Accordingly, the focus of this work is on the quantization of an additive independent Gaussian noise affected bandlimited signal using single-bit ADCs and oversampling. The key result of this paper is the uncovering of a quantization precision indifference principle, which is stated next.

Precision indifference principle: Consider a bounded-dynamic range bandlimited signal with samples affected by additive independent Gaussian noise and observed through quantizers. If \( N \) is the oversampling ratio, with respect to the Nyquist rate, then the optimal law for maximum pointwise mean-squared error is \( O(1/N) \), irrespective of the quantizer precision. In other words, for large \( N \), the quantizer precision only affects the proportionality constant of the distortion.

Prior art: Averaging and other properties of independent random variables are well studied in statistics \([6]\). Quantization error can be reduced by oversampling as well as by increasing the ADC (quantizer) precision (see \([7], [8], [9], [10]\) for the entire range of results). Estimation of square-integrable signals in the presence of Gaussian noise was studied by Pinsker \([2]\); however, quantization is not addressed in his work. Signal quantization with additive noise as a dither has been studied by Masry \([8]\), however signal was not assumed to be bandlimited. Masry’s results give a decay of \( O(1/N^{2/3}) \) for bandlimited signals, where \( N \) is the oversampling above the Nyquist rate; this decay is slower than an \( O(1/N) \) decay that we are after. The sampling of signals defined on a finite support, while using single-bit quantizers in the presence of ambient noise, has been also studied \([9], [10]\).
Notation: The set of bounded signals and the set of finite energy signals will be denoted by \( L^\infty(\mathbb{R}) \) and \( L^2(\mathbb{R}) \), respectively. The signal of interest will be denoted by \( g(t) \). For a signal \( s(t) \) in \( L^2(\mathbb{R}) \), the Fourier transform will be denoted by \( \hat{s}(\omega) \). The Fourier transform and its inverse are defined as,

\[
\hat{s}(\omega) = \int_{\mathbb{R}} s(t) \exp(-j\omega t) dt; \quad s(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{s}(\omega) \exp(j\omega t) d\omega.
\]

The indicator function of a set \( A \) is denoted by \( \mathbb{1}(x \in A) \). Random variables or processes will be denoted by uppercase letters. The additive independent Gaussian noise is denoted by \( W(t) \). The set of reals and integers will be denoted by \( \mathbb{R} \) and \( \mathbb{Z} \), respectively. The cumulative distribution function (cdf) of a Gaussian random variable with mean zero and variance \( \sigma^2 \) will be denoted by \( F(x) \), \( x \in \mathbb{R} \). The convolution and expectation operation will be denoted by \(*\) and \( \mathbb{E} \), respectively. It is assumed that all probability models have an underlying sample space, sigma-field, and probability measure such that (weighted) averages and indicator functions are measurable.

Organization: The mathematical formulation of our sampling problem is discussed in Sec. [II]. Short review of stable interpolation kernels and smoothness properties of associated signals are discussed in Sec. [III]. The discussion on precision indifference principle appears in Sec. [IV]. Estimation with perfect samples and single-bit quantized samples are discussed in Sec. [III]. The discussion on precision indifference principle appears in Sec. [IV]. The discussion on precision indifference principle appears in Sec. [IV]. Conclusions are presented in Sec. [V]. To maintain the flow of the paper, long proofs appear in the Appendix.

II. Problem Formulation

The discussion begins with a quick review of a stable bandlimited kernel, which is essential for stable interpolation in \( L^\infty(\mathbb{R}) \) and for defining bandlimited signals. For \( \lambda > 1 \) and \( a = (\lambda - 1)/2 \), consider the kernel \( \phi(t) \) that is given by

\[
\phi(t) = \frac{1}{\pi at^2} \sin((\pi + a)t) \sin(at); \quad \phi(0) = 1 + \frac{a}{\pi}.
\]  
(1)

The kernel decreases sufficiently fast (approximately as \( 1/t^2 \)) and therefore it is absolutely and square integrable. Its Fourier transform is illustrated in Fig. [1]. This kernel can be used to define the set of bounded bandlimited signals, which is a subset of the Zakai class of bandlimited signals [11]. Consider

\[
BL_{\text{int}} := \{ g(t) : |g(t)| \leq 1 \text{ and } g(t) * \phi(t) = g(t) \quad \forall t \in \mathbb{R} \}.
\]  
(2)

The above definition ensures that \( g(t) \) is continuous everywhere. It is easy to verify that the set of bounded bandlimited signals in \( L^2(\mathbb{R}) \) with Fourier spectrum zero outside \([-\pi, \pi]\) also belongs to the set \( BL_{\text{int}} \). The set \( BL_{\text{int}} \) also includes (almost-surely) any sample path of a bounded-dynamic range bandlimited wide-sense stationary process [12]. The quantization of bandlimited signals from the set \( BL_{\text{int}} \) in the presence of additive independent Gaussian noise is studied in this work. The derived results are applicable to finite energy bounded bandlimited signals as well as (almost surely) to any sample path of a bounded wide-sense stationary bandlimited process.

The signal affected by additive noise, \( g(t) + W(t) \), is available for sampling. It is assumed that \( W(t) \sim \mathcal{N}(0, \sigma^2) \) for all \( t \in \mathbb{R} \). Independence of noise implies that \( W(t_1), W(t_2), \ldots, W(t_n) \) for distinct \( t_1, t_2, \ldots, t_n \in \mathbb{R} \) are i.i.d. with \( \mathcal{N}(0, \sigma^2) \) distribution. The Nyquist rate at which \( g(t) \) should be sampled for perfect reconstruction is one sample/second. In the noise-free regime, when \( \sigma = 0 \), it is sufficient to sample \( g(t) \) at the Nyquist rate for convergence in \( L^\infty(\mathbb{R}) \). In the noise-limited regime, when \( \sigma > 0 \), the reconstruction based on samples of \( g(t) \) will have distortion (statistical mean-squared error). This distortion can be reduced by oversampling. Let \( N \), a positive integer, be the oversampling rate. For any statistical estimate \( \hat{G}_{\text{rec}}(t) \) of the signal \( g(t) \), the maximum pointwise mean-squared error \( D_{\text{rec}} \) is defined as the distortion, i.e.,

\[
D_{\text{rec}} := \sup_{t \in \mathbb{R}} D_{\text{rec}}(t) = \sup_{t \in \mathbb{R}} \mathbb{E} \left| \hat{G}_{\text{rec}}(t) - g(t) \right|^2.
\]  
(3)

For a pointwise-consistent reconstruction, the distortion in (3) should decrease to zero as the oversampling rate \( N \) increases to infinity [6]. Consistent reconstruction of smooth signals with a random dither, in the presence of single-bit quantizers,
has been obtained in the past [8]: therefore, the asymptotic rate of decrease in \( D_{\text{rec}} \) with \( N \) is of interest to us. Due to finite precision limitations (ADC operation) during acquisition, the signal samples are quantized. Since quantization is a lossy operation [13], \( D_{\text{rec}} \) is expected to depend upon the ADC precision employed. As mentioned in Sec. I it will be shown that \( D_{\text{rec}} \) decreases as \( O(1/N) \), irrespective of the sensor precision. Thus, the ADC precision only manifests in the proportionality constant (independent of the oversampling factor \( N \)) in the optimal asymptotic reconstruction distortion.

To show the proposed precision indifference principle, two extreme cases of quantizer precision will be analyzed and their distortion will be compared: (i) signal distortion with perfect samples; and (ii) signal distortion with samples quantized using single-bit ADCs. The sampling setup for these two cases are illustrated in Fig. 2. In Fig. 2(a), the estimator works with infinite precision (unquantized) samples while in Fig. 2(b), the estimator works with poorest precision (one-bit) noisy samples. The role of extra dither noise \( W_d(t) \) will be explained later in Sec. [IV-B] The estimator \( \hat{G}_{1\text{-bit}}(t) \) will be designed and its distortion performance will be analyzed in this work.

![Fig. 2. Two extreme scenarios of quantization: In both the scenarios the signal \( g(t) \) is observed with additive independent Gaussian noise \( W(t) \). In (a), the estimator works with infinite precision (unquantized) samples \( \{Y(n\tau), n \in \mathbb{Z}\} \). In (b), the estimator works with poorest precision (one-bit) samples \( \{X(n), n \in \mathbb{Z}\} \) where \( X(n\tau) = 1(Y(n\tau) \geq 0) \).

Before we move on to the next section, it should be noted that the kernel \( \phi(t) \) and its derivative \( \phi'(t) \) are absolutely integrable. This absolute integrability and square integrability of \( \phi(t) \) can be translated into the following observations, which will be useful in Sec. [IV-B] during distortion analysis:

\[
C_{\phi} := \int_{t \in \mathbb{R}} |\phi(t)| dt < \infty, \tag{4}
\]

\[
C_{\phi}' := \sup_{\{t_k: t_k \in [k/\lambda, (k+1)/\lambda], k \in \mathbb{Z}\}} \sum_{k \in \mathbb{Z}} |\phi'(t_k)| < \infty. \tag{5}
\]

\[\text{and } C_{\phi}'' := \sup_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} \left| \phi\left( t - \frac{k}{\lambda} \right) \right|^2 < \infty. \tag{6}\]

The next section will review pertinent mathematical results which will be used in the later sections.

### III. BACKGROUND

The stable interpolation formula for Zakai sense bandlimited signals is discussed first. The necessity of \( W_d \) and associated variance-conditions on Gaussian noise (see Fig. 2b) are discussed. If the pointwise error in interpolation is bounded with bounded perturbation of samples, it is called as a stable interpolation. The properties of stable interpolation and its implications on filtering of bounded signals are given at the end of this section.

\[\text{From the interpolation formula for Zakai sense bandlimited signals, the signal of interest } g(t) \text{ can be perfectly reconstructed from its samples taken at the Nyquist rate. For } g(t) \in BL_{\text{int}}, \text{ the interpolation formula is given by [5] Lemma 3.1}, \]

\[
g(t) = \lambda \sum_{n \in \mathbb{Z}} g\left( \frac{n}{\lambda} \right) \phi\left( t - \frac{n}{\lambda} \right). \tag{7}\]

where the equality holds absolutely, pointwise, and in \( L^\infty(\mathbb{R}) \). Thus, in the absence of noise, it is sufficient to sample \( g(t) \) at a rate of \( \lambda \) sample per second (or per meter in the context of spatial fields). In the presence of quantization, the reconstruction in (7) is stable in \( L^\infty(\mathbb{R}) \).

The role of \( W_d(t) \) in Fig. 2b will now be highlighted. If the noise variance \( \sigma \) is very small compared to the dynamic range of the signal \( g(t) \), i.e., \( |\sigma| \ll 1 \), then the samples \( \mathbb{1}\{g(t) + W(t) \geq 0\} \) will not capture small scale local variation in \( g(t) \). Due to quantization, estimators such as maximum likelihood are expected to be non-linear and their analysis is too complex. To
alleviate this issue, if \( \text{var}(W(t)) = \sigma^2 \) is very small, then an extra additive independent Gaussian dither \( W_d(t) \) can be added to ensure that \( \mathbb{I}(g(t) + W(t) + W_d(t) \geq 0) \) is sufficiently random. It is assumed that \( W_d(t) \) and \( W(t) \) are independent. Such dithering allows us to use an analytically tractable reconstruction procedure, which has order-optimal distortion. The block-diagram for sampling with one-bit ADCs is illustrated in Fig. 2(b). The technical condition on \( \sigma^2 = \text{var}(W(t)) + \text{var}(W_d(t)) \) is stated using the cdf of \( W + W_d \). Let \( F : \mathbb{R} \to [0, 1] \) be the cdf of \( W + W_d \). Let \( f(x) \) be the associated probability density function with \( f(\pm C_\phi) = \delta \) and \( f(0) = \Delta \). Observe that \( \Delta > \delta \), since \( f(x) = \frac{\pi}{\sqrt{2\pi}} \exp(-x^2/2\sigma^2) \). It is required that there is a parameter \( \mu > 0 \) such that
\[
\left( 1 - \frac{1}{\sqrt{2C_\phi^2}} \right) \frac{1}{\delta} \leq \mu < \frac{1}{\Delta},
\]
where \( C_\phi \) is the constant in (4). First fix a \( \lambda > 1 \). Then, \( C_\phi = \int_{x \in \mathbb{R}} |\phi(t)| dt > \int_{t \in \mathbb{R}} \phi(t) dt = \hat{\phi}(0) = 1 \). That is, \( C_\phi^2 \sqrt{2} > \sqrt{2} > 1 \). Therefore, the lower bound on \( \mu \) in (9) is positive. Next, observe that if \( \sigma \) is large but fixed, then \( \delta = f(C_\phi) = f(0) = \Delta \). Then \( \delta \) and \( \Delta \) are close enough and the inequality in (9) can be satisfied. In other words, for a fixed \( \lambda \) and hence \( C_\phi \), there is a finite number \( \sigma_0 \) for which (9) is satisfied for all \( \sigma > \sigma_0 \). If \( \text{var}(W(t)) < \sigma_0^2 \), then \( \text{var}(W_d(t)) > \sigma_0^2 - \text{var}(W(t)) \) will ensure that \( \text{var}(W + W_d) > \sigma_0^2 \). If \( \text{var}(W(t)) \geq \sigma_0^2 \), then the extra dither is not needed. This condition will be used in the distortion analysis in Sec. IV-B.

For single-bit estimation, the signal \( F(g(t)) - 1/2 \) will be encountered, where \( F : \mathbb{R} \to [0, 1] \) is the cumulative distribution function of the stationary noise random variable \( W(t) + W_d(t) \). Since \( g(t) \in [-1, 1] \), and \( F(x) \) has a wider support than the dynamic range of signal (i.e., \([-1, 1]\)), therefore \( F'(x) \) is finite and non-zero for \( x \in [-1, 1] \). Since \( F(0) = 1/2 \) by symmetry, therefore, \( F(g(t)) - 1/2 \) is more convenient than \( F(g(t)) \) to work with. For simplicity of notation, let \( l(t) = F(g(t)) - 1/2 \). Then \( |l(t)| \leq |F(1)| - 1/2 \), i.e., \( l(t) \) is bounded. The bound depends only on the noise distribution and the dynamic range of \( g(t) \). Finally \( |l'(t)| = |F'(g(t))g'(t)| \leq |F'(0)|2\pi^2 \) since \( F'(0) \) maximizes \( F'(x) \) in \([-1, 1]\) and \( |g'(t)| \leq 2\pi^2 \) (see [5] Proposition 3.1).

The definition of \( BL_{\text{lin}} \) involves convolution with a stable kernel and convolution will often appear in the context of error analysis. The following short lemma will be quite useful later on.

**Lemma 3.1:** Let \( p(t) \) be a signal such that \( ||p||_\infty \) is finite and \( P(t) \) be any random process such that \( P(t) \) is bounded (i.e., \( \sup_{t \in \mathbb{R}} \mathbb{E}(P^2(t)) \) is finite). Then,
\[
||p \ast \phi||_\infty \leq C_\phi ||p||_\infty, \quad (9)
\]
and
\[
\mathbb{E}([|P(t)| \ast |\phi(t)|]^2) \leq C_\phi^2 \sup_{t \in \mathbb{R}} \mathbb{E}(P^2(t)), \quad (10)
\]
where the convolutions are well defined since \( \phi(t) \) is absolutely integrable.

**Proof:** The proof follows by the definition of convolution and the triangle inequality. We have
\[
|p(t) \ast \phi(t)| = \left| \int_{u \in \mathbb{R}} p(u)\phi(t-u)du \right|,
\]
\[
\leq \int_{u \in \mathbb{R}} |p(u)||\phi(t-u)|du,
\]
\[
\leq ||p||_\infty \int_{u \in \mathbb{R}} |\phi(t-u)|du,
\]
\[
= C_\phi ||p||_\infty.
\]

For the second moment bound, note that
\[
\mathbb{E}([|P(t)| \ast |\phi(t)|]^2)
\]
\[
= \mathbb{E} \left( \int_{u, v \in \mathbb{R}} |P(u)||P(v)||\phi(t-u)||\phi(t-v)|dudv \right),
\]
\[
= \int_{u, v \in \mathbb{R}} \mathbb{E}(|P(u)||P(v)||\phi(t-u)||\phi(t-v)|)dudv,
\]
\[
\leq \sup_{t \in \mathbb{R}} \mathbb{E}(P^2(t)) \int_{u, v \in \mathbb{R}} |\phi(t-u)||\phi(t-v)|dudv,
\]
\[
= C_\phi^2 \sup_{t \in \mathbb{R}} \mathbb{E}(P^2(t)),
\]
where \( (a) \) follows by \( \mathbb{E}(2|P(u)||P(v)|) \leq \mathbb{E}(P^2(u) + P^2(v)) \leq 2 \sup_{t \in \mathbb{R}} \mathbb{E}(P^2(t)) \). Thus the proof is complete.

The two extreme scenarios of quantization as depicted in Fig. 2 and their distortions will now be analyzed in the next section.
IV. ESTIMATION OF BANDLIMITED SIGNAL

Interpolation of bandlimited signals with perfect samples is a well known topic \cite{1}. Loosely speaking, a bandlimited signal of duration $T$ and bandwidth $\pi$ has $2\pi T$ degrees of freedom \cite{14}. With $NT$ noisy samples of the field $g(t) + W(t)$ in duration $T$, the optimal distortion is expected to be $O(1/N)$ \cite{6}. With this note, sampling schemes with oversampling rate $N$ are designed to achieve a distortion of $O(1/N)$ for sampling $g(t)$.

A. Estimation with perfect samples

A brief review of estimation with perfect samples will be highlighted first. Optimal minimum mean-squared method can be found in the work of Pinsker \cite{2}. For illustration and to get a distortion proportional to $O(1/N)$, it suffices to use the frame expansion. Let the integer-valued oversampling ratio (above the Nyquist rate) be $N$, and $\tau = 1/(\lambda N)$. Then, the samples $\{Y(n\tau), n \in \mathbb{Z}\}$ are available for the reconstruction of $g(t)$. Using frame expansion or the shift-invariance of bandlimited signals,

$$ g(t) = \frac{1}{N} \sum_{n \in \mathbb{Z}} \lambda g(n\tau) \phi(t - n\tau), $$

where the equality holds pointwise and in $L^\infty(\mathbb{R})$. It must be noted that the basic operation in \cite{11} is that of averaging; hence, the noise is expected to average out while the signal will be retained. This intuition motivates the following estimator for $g(t)$ from noisy data (see Fig. 2(a)). Define

$$ \hat{G}_{fr}(t) := \frac{1}{N} \sum_{n \in \mathbb{Z}} \lambda Y(n\tau) \phi(t - n\tau), $$

$$ = \frac{1}{N} \sum_{i=0}^{N-1} \sum_{k \in \mathbb{Z}} \lambda g \left( \frac{k}{\lambda} + i\tau \right) \phi \left( t - \frac{k}{\lambda} - \frac{i}{N\lambda} \right). $$

The distortion of $\hat{G}_{fr}(t)$ is given by the following proposition.

Proposition 4.1 (Frame estimate with $O(1/N)$ distortion): Let $\hat{G}_{fr}(t)$ in \cite{12} be an estimate for the bandlimited field $g(t)$ corrupted by additive independent Gaussian noise. Let $D_{fr}(t) := \mathbb{E}[|\hat{G}_{fr}(t) - g(t)|^2]$. Then,

$$ \sup_{t \in \mathbb{R}} D_{fr}(t) \leq \frac{C_{fr}^2 \lambda^2 \sigma^2}{N} $$

where the constants $\sigma^2$ and $C_{fr}^2$ from \cite{6} do not depend on $N$.

Proof: See Appendix VI-A.

The signal term in \cite{12} converges in $L^\infty(\mathbb{R})$ to $g(t)$. The noise term results in an independent sum of zero-mean random variables at every $t \in \mathbb{R}$. This sum of random variables has a variance that decreases as $1/N$ due to the finite energy of the interpolation kernel $\phi(t)$. The constant $C_{fr}^2$ depends on the properties of the kernel $\phi(t)$. The estimation with single-bit quantizers and associated distortion analysis will be presented next.

B. Estimation with single-bit quantized samples

This section will present the key result of this work. Consider the system illustrated in Fig. 4(b). In this section, a $\hat{G}_{1-bit}(t)$ will be obtained such that $D_{1-bit}$ scales as $O(1/N)$. This is non-trivial to achieve because the non-linear quantization operation is coupled with the statistical estimation procedure. The result will be established in two parts: (i) it will be shown that suitable interpolation of one-bit samples converges to a non-linear one-to-one function of $g(t)$ with an error term having a pointwise variance of $O(1/N)$; and (ii) the obtained non-linear function of $g(t)$ can be inverted in a stable manner using recursive computation based on contraction-mapping. It will be assumed that $\var(W_d(t)) = (\sigma^2 - \var(W(t)))_+$, where $\sigma$ is such that \cite{5} is satisfied.

The stability property of kernel $\phi(t)$ has been discussed Sec. III. For this section, fix $\tau = 1/(N\lambda)$, where $\lambda > 1$ is an arbitrary stability constant. Analogous to \cite{11}, consider the random process obtained from the single-bit samples $X(n\tau), n \in \mathbb{Z},$

$$ H_N(t) = \tau \sum_{n \in \mathbb{Z}} (X(n\tau) - 1/2) \phi(t - n\tau). $$

Then, the following proposition establishes the convergence of $H_N(t)$ to a function of the signal of interest $g(t)$.

\^A single bounded constant in additive independent Gaussian noise with $N$ independent readings can be estimated up to a distortion of $O(1/N)$ \cite{6}.\[^1\]
Proposition 4.2 (Convergence of single-bit interpolation): Let \( l(t) = (F(g(t)) - 1/2) \) and \( H_N(t) \) be as defined in (14). Then

\[
\sup_{t \in \mathbb{R}} \mathbb{E}(H_N(t) - l(t) * \phi(t))^2 \leq \frac{C_2}{N} + \frac{C_3}{N^2},
\]

where \( C_2 > 0 \) and \( C_3 > 0 \) are constants independent of \( N \).

Proof: See Appendix [VT-8].

The factor \( \tau = 1/(\lambda N) \) provides the normalization for averaging in (14), while the terms \( (X(n\tau) - 1/2)\phi(t - n\tau) \) are weighted independent one-bit samples. The average in (15) converges in mean-square to a convolution. The signal \( l(t) \in L^\infty(\mathbb{R}) \) and the limit \( l(t) * \phi(t) \) is a lowpass version of \( l(t) \). The dependence of \( l(t) * \phi(t) \) on \( g(t) \) is non-linear due to quantization, which results in the \( F(g(t)) \) term. The original signal \( g(t) \) is Zakai sense bandlimited and it has one degree of freedom per unit time. The degree of freedom per unit time of \( l(t) * \phi(t) \) can be up to one as well, and \( F(x) \) has ‘nice’ properties as a function. Thus, it is not unreasonable to expect that there might be a class of \( F(x) \) such that \( F(g(t)) - 1/2) * \phi(t) \) can be inverted to find \( g(t) \), even though this equation is nonlinear.

Consider compandors defined by Landau and Miranker [15].

Definition 4.1: [15 pg 100] A compandor is a monotonic function \( Q(x) \) which has the property that \( Q(m(t)) \in L^2(\mathbb{R}) \) if \( m(t) \in L^2(\mathbb{R}) \).

Landau and Miranker have shown that if \( g(t) \in L^2(\mathbb{R}) \) and \( g(\omega) \) is zero outside \([-\pi, \pi]\), and if \( Q : [-1, 1] \to \mathbb{R} \) is a compandor with non-zero slope, then there is one to one correspondence between \( g(t) \) and \( Q(g(t)) \). Further, given any signal \( m(t) \in L^2(\mathbb{R}) \), then exists a unique \( g_m(t) \) which converges to \( g(t) \) as well. This procedure of Landau and Miranker does not extend directly to bandlimited signals in \( L^\infty(\mathbb{R}) \), especially in the presence of statistical perturbations. Suitable modifications of their approach will be used to obtain the results for our problem.

The dependence between \( g(t) \) and \( l(t) * \phi(t) \) is quite non-linear. There is no clear or obvious equation by which \( g(t) \) can be obtained from \( l(t) * \phi(t) \). Therefore, this inversion problem is casted into a recursive setup, where Banach’s fixed-point theorem can be leveraged along with contraction mapping [16, Ch. 5]. This approach is inspired from the work of Landau and Miranker. Their recursive setup is related to channel model with finite variance which is not in \( L^2(\mathbb{R}) \). Therefore, our recursive procedure to obtain an estimate of \( g(t) \) from \( H_N(t) \) (see (15) and its analysis is non-trivial and it will be presented in detail.

In summary, an estimate for \( g(t) \) is required. Due to quantization and noise, which is a non-linear operation, an approximation \( H_N(t) \) of \( (F(g(t)) - 1/2) * \phi(t) \) is available. The estimate \( H_N(t) \), which converges to \( (F(g(t)) - 1/2) * \phi(t) \) with sample density \( N \uparrow \infty \), will be inverted to obtain an estimate \( \hat{G}_{1-bit}(t) \) for the signal \( g(t) \). To establish the precision indifference principle, we wish to show that the mean-square error \( \sup_{t \in \mathbb{R}} \mathbb{E}(\hat{G}_{1-bit}(t) - g(t))^2 \) decreases as \( O(1/N) \). The details are presented next.

A ‘clip to one’ function Clip\([x]\) is defined first.

\[
\text{Clip}[x] = x \quad \text{if } |x| \leq 1
\]

\[
= \text{sgn}(x) \quad \text{otherwise.}
\]

Since \( g(t) \) has a dynamic range bounded by one, by assumption, it will be unaffected by clipping. Note that under the \( L^\infty \) norm, this transformation reduces the distance between any two scalars \( x_1 \) and \( x_2 \), i.e., \( |\text{Clip}[x_1] - \text{Clip}[x_2]| \leq |x_1 - x_2| \). This can be verified on a case by case basis. For example, if \( x_1 > 1 \) and \( x_2 \in [-1, 1] \), then \( |\text{Clip}[x_1] - \text{Clip}[x_2]| = 1 - x_2 \leq |x_1 - x_2| \). Other cases can be similarly enumerated. This clipping procedure is non-linear and complicates some of the presented analysis; however, we feel that its presence is essential for analysis.

Let \( \psi(t) = \phi(\lambda t) \). Then \( \hat{\psi}(\omega) = \phi(\lambda/\lambda) \). Thus, \( \hat{\psi}(\omega) \) is flat in \([-\lambda \pi, \lambda \pi]\) and in \([\pm \lambda \pi, \lambda \pi]\) decreases linearly to zero. Consider the set of bandlimited signals defined by

\[
\mathcal{S}_{BL,bdd} = \{ m(t) : |m(t)| \leq C_\phi \text{ and } m(t) * \psi(t) = m(t) \}.
\]

Then, \( \mathcal{S}_{BL,bdd} \) is a complete subset of the Banach space \( L^\infty(\mathbb{R}) \).

Lemma 4.1 (\( \mathcal{S}_{BL,bdd} \) is a complete metric space): Let \( \mathcal{S}_{BL,bdd} \) be as defined in (17). Then \( (\mathcal{S}_{BL,bdd}, \| \cdot \|_\infty) \) is a complete subset of \( (L^\infty(\mathbb{R}), |\cdot|_\infty) \).

Proof: Define the function \( d : \mathcal{S}_{BL,bdd} \times \mathcal{S}_{BL,bdd} \to \mathbb{R}^+ \) as \( d(m_1, m_2) = |m_1 - m_2|_\infty \). It is easy to verify the axioms of distance metric [16]: (i) \( d \geq 0 \) and \( d \leq \infty \); (ii) \( d(m_1, m_2) = 0 \) if and only if \( m_1(t) = m_2(t) \); (iii) \( d(m_1 + m_2, m_3 + m_4) = d(m_1, m_3) + d(m_2, m_4) \); and (iv) \( d(m_1, m_2) \leq d(m_1, m_3) + d(m_3, m_2) \) for any \( m_1(t), m_2(t), m_3(t) \in \mathcal{S}_{BL,bdd} \).

It is straightforward to see that \( \mathcal{S}_{BL,bdd} \subset L^\infty(\mathbb{R}) \) since \( |m|_\infty \leq C_\phi \) for every \( m(t) \in \mathcal{S}_{BL,bdd} \). To show that the subset is complete, consider any Cauchy sequence \( m_n(t) \in \mathcal{S}_{BL,bdd} \). Since \( L^\infty(\mathbb{R}) \) is complete, therefore \( m_n(t) \to s(t) \), where \( s(t) \in L^\infty(\mathbb{R}) \). It remains to show that \( s(t) \) belongs to \( \mathcal{S}_{BL,bdd} \).
For any $\epsilon > 0$, there is an $n_0$ such that $||m_n - s||_\infty < \epsilon$ for all $n > n_0$. Since $\int_{\mathbb{R}} |\psi(t)|dt = C_0/\lambda$, therefore, $||m_n * \psi - s * \psi||_\infty \leq ||m_n - s||_\infty ||C_0/\lambda|| = C\epsilon/\lambda$ for all $n > n_0$ (see Lemma 3.1). Thus, $m_n(t) * \psi(t) \rightarrow s(t) * \psi(t)$. However, $m_n(t) * \psi(t) \equiv m_n(t)$ since $m_n(t) \in S_{BL,bdd}$. Therefore, it follows that $s(t) = s(t) * \psi(t)$, or $s(t) \in S_{BL,bdd}$. Thus, $S_{BL,bdd}$ is complete.

A map $T : S_{BL,bdd} \rightarrow S_{BL,bdd}$ will be defined next. This map will result in a recursive procedure to obtain $g(t)$ from $h(t) := l(t) * \phi(t)$. Define

$$T[m(t)] = \text{Clip} \left[ \mu h(t) + [m(t) - \mu(F(m(t)) - 1/2)] * \phi(t) \right] * \phi(t).$$

(18)

It will be shown that $T$ is a contraction on $(S_{BL,bdd}, ||.||_\infty)$.

**Lemma 4.2 (T is a contraction):** Let $(S_{BL,bdd}, ||.||_\infty)$ be the metric space as defined in (17). Let $T : S_{BL,bdd} \rightarrow S_{BL,bdd}$ be a map as defined in (18). If the condition in (8) is satisfied, then there is a choice of $\mu$ such that $T$ is a contraction, i.e.,

$$||T[m_1] - T[m_2]||_\infty \leq \alpha ||m_1 - m_2||_\infty,$$

(19)

for some $0 < \alpha < 1$ and any $m_1(t), m_2(t) \in S_{BL,bdd}$. The parameter $\alpha$ does not depend on the choice of $m_1$ and $m_2$.

**Proof:** See Appendix VI-C.

Now the key recursive equation will be stated. Let $l(t) = (F(g(t)) - 1/2)$ and $h(t) = l(t) * \phi(t)$ be available for obtaining $g(t)$. Then,

$$g_{k+1}(t) := T[g_k(t)] = \text{Clip} \left[ \mu h(t) + [g_k(t) - \mu(F(g_k(t)) - 1/2)] * \phi(t) \right] * \phi(t),$$

(20)

where $k \geq 0, k \in \mathbb{Z}$ and $\mu > 0$ is a constant that will be chosen according to Lemma 4.2. Set $g_0(t) \equiv 0$. The original signal $g(t)$ is a fixed point of this equation and it can be verified by substitution. The following proposition shows that $g(t)$ is the only fixed point of the equation in (20). The proof hinges on Banach’s fixed point theorem or contraction theorem [16, Ch. 5].

**Proposition 4.3 (Signal of interest is the fixed point of $T$):** Let $g(t) \in BL_{int} \subset S_{BL,bdd}$ be a continuous bounded bandlimited signal. Let $h(t) = l(t) * \phi(t)$, where $l(t) = F(g(t)) - 1/2$. Consider the recursion $g_k(t) = T[g_{k-1}(t)]$, where $T$ is as defined in (18). Set $g_0(t) \equiv 0$. If $\mu$ is selected as in (8), then

$$\lim_{k \rightarrow \infty} ||g_k - g||_\infty = 0.$$

(21)

**Proof:** The proof is straightforward with Lemma 4.1 and Lemma 4.2 in place. Define $d(m_1, m_2) = ||m_1 - m_2||_\infty$ for any $m_1(t), m_2(t) \in S_{BL,bdd}$. From Lemma 4.1 note that $(S_{BL,bdd}, d)$ is a complete metric space. The signal $g(t)$ is in $S_{BL,bdd}$ and it satisfies $g(t) = T[g(t)]$, i.e., it is a fixed point for $T$ defined in (18).

Pick $\mu$ as in (8). Then $T$ is a contraction on $(S_{BL,bdd}, d)$. Thus, by Banach’s fixed point theorem (contraction theorem) [16, Ch. 5], there is exactly one fixed point in $S_{BL,bdd}$ for the equation $g(t) = T[g(t)]$. Since $g_k(t)$ converges to a fixed point, it must converge to $g(t)$ in the distance metric $d$. Thus the proof is complete.

Proposition 4.3 holds with perfect information about $l(t) * \phi(t)$. The estimation of signal from $H_N(t)$, the statistical approximation of $l(t) * \phi(t)$, will be discussed now. Let $G_k(t)$ be the sequence of random waveforms generated from $H_N(t)$ when it is applied to the recursion in (20). That is, fix $G_0(t) \equiv 0$ and define

$$G_{k+1}(t) := T[G_k(t)] = \text{Clip} \left[ \mu H_N(t) + [G_k(t) - \mu(F(G_k(t)) - 1/2)] * \phi(t) \right] * \phi(t).$$

(22)

Let $\tilde{G}_{1,\text{bit}}(t) = \lim_{k \rightarrow \infty} G_k(t)^2$. For the same choice of $\mu$ which ensures that $T$ is a contraction on $(S_{BL,bdd}, ||.||_\infty)$, the distortion of $|\tilde{G}_{1,\text{bit}}(t) - g(t)|$ has to be established. To this end, the following proposition is noted.

**Proposition 4.4 (1-bit estimation has distortion $O(1/N)$):** Let $H_N(t)$ be the estimate of $l(t)$ as described in (14) and $\mu$ be selected as in (8). With $G_0(t) \equiv 0$, let $G_k(t)$ be the sequence of random waveforms as defined in (22). Define $\lim_{k \rightarrow \infty} G_k(t) = \tilde{G}_{1,\text{bit}}(t)$. Then,

$$D_{1,\text{bit}} := \sup_{t \in \mathbb{R}} \mathbb{E}(\tilde{G}_{1,\text{bit}}(t) - g(t))^2 = O(1/N),$$

i.e., the distortion $D_{1,\text{bit}}$ decreases as $O(1/N)$.

**Proof:** See Appendix VI-D.

The results of Proposition 4.1 and Proposition 4.4 can be summarized into the following theorem.

**Theorem 4.1 (Precision indifference principle):** Let $g(t)$ be a bounded dynamic-range bandlimited-signal as defined in (2). Assume that $g(t) + W(t)$ is available for sampling, where $W(t)$ an additive independent Gaussian random process with finite

This limit exists since it can be shown that $||G_k - G_{k-1}||_\infty \leq \alpha ||G_{k-1} - G_{k-2}||_\infty$ for some $0 < \alpha < 1$ by using an analogous procedure as in Lemma 4.2.
variance. Fix an oversampling factor of $N$, where $N$ is large for statistical averaging. There exists an estimate $\hat{G}_{1\text{-bit}}(t)$ obtained from single-bit samples of $g(t) + W(t)$ such that

$$\sup_{t \in \mathbb{R}} |\hat{G}_{1\text{-bit}}(t) - g(t)|^2 = O(1/N).$$

This distortion is proportional to the best possible distortion of $O(1/N)$ that can be obtained with unquantized or perfect samples.

A few remarks highlighting the importance of the results obtained will conclude this section.

C. Remarks on the results obtained

1) Comparison with the bit-conservation principle: The bit-conservation principle is somewhat in contrast to the precision indifference principle. Loosely speaking, bit-conservation principle states that for sampling a bandlimited signal in a noiseless setting, the oversampling density can be traded-off against ADC precision while maintaining a fixed bit-rate per Nyquist interval and an order-optimal pointwise distortion. In the presence of additive independent Gaussian noise, this tradeoff between ADC precision and oversampling is absent while studying pointwise mean-distorted signal. In the noisy setup, the distortion is proportional to $1/N$, where $N$ is the oversampling density irrespective of the ADC precision. The presence of noise shifts the role of ADC precision towards only the proportionality constant in distortion!

2) Interpretation of precision-indifference principle: First, it can be argued that the precision indifference principle holds while estimating a constant signal (one degree of freedom) in additive independent Gaussian noise. Assume that a constant $c \in [-1, 1]$ has to be estimated based on $N$ noisy readings $Y_i = c + W_i, 1 \leq i \leq N$, where $\{W_i, 1 \leq i \leq N\}$ are i.i.d. $N(0, \sigma^2)$. In the absence of quantization, $\hat{C}_N = (\sum_{i=1}^N Y_i)/N$ converges to $c$ in the mean-square sense, and $\mathbb{E}(\hat{C}_N - c)^2 = \sigma^2/N$. This is the optimal distortion if (perfect (unquantized) samples) are available. Now consider the case where single-bit readings $B_i = 1(c + W_i \geq 0), 1 \leq i \leq N$ are available. The random variables $\{B_i, 1 \leq i \leq N\}$ are i.i.d. Ber$(q)$ where $q = \mathbb{P}(W \geq -c) = \mathbb{P}(W \leq c) = F(c)$. Assume $\hat{B}_N = (\sum_{i=1}^N B_i)/N$. It can be shown that $\mathbb{E}(\hat{B}_N - F(c))^2 \leq 1/(4N)$ since each $\var(B_i) \leq F(c)(1 - F(c)) \leq 1/4$. Define $\hat{C}_{1\text{-bit}} = F^{-1}(\hat{B}_N)$ if $\hat{B}_N \in [F(-1), F(1)]$ and $\hat{C}_{1\text{-bit}} = \pm 1$ otherwise. Since $F(x)$ is invertible and $dF^{-1}(x)/dx$ is bounded for $x \in [F(-1), F(1)]$, therefore, using the delta method, $\hat{C}_{1\text{-bit}}$ obtained from $\hat{B}_N$ has a mean-squared error which decreases as $(1/N)$.

Next, it should be noted that bandlimited signals have one degree of freedom in every Nyquist interval. An oversampling factor of $N$ means that there are $N$ samples to observe each degree of freedom on an average. Finally, observing the Nyquist samples of a bandlimited signals with a distortion of $O(1/N)$ results, by stable interpolation with kernel $\phi(t)$, in a pointwise distortion of $O(1/N)$ for the signal estimate at any point.

3) Precision-indifference for a larger class of noise: Consider the model where each sample $Y(n\tau) = g(n\tau) + W(n\tau)$ is affected by some non-Gaussian noise. Focus on the case where $V(n\tau)$ can be written as $V(n\tau) = W(n\tau) + U(n\tau)$, where $W(n\tau)$ and $U(n\tau)$ are i.i.d. for all $n \in \mathbb{Z}, \tau \in \mathbb{R}$. If $W(n\tau)$ is Gaussian, $\mathbb{V}(V(n\tau)) = \sigma^2 < \infty$, and $F_V(x)$ satisfies (8), then the precision indifference principle will hold. The extension of existing proofs is simple and only its key steps will be mentioned here. In the perfect sample case (see Fig. [2.3]), the Gaussian part of $V(n\tau)$ will limit the best possible (optimal) distortion to $O(1/N)$; this is because even if the values of $U(n\tau)$ are (magically) known the residual $W(n\tau)$ will limit the distortion. With single-bit quantization, note that all the proofs in Sec. IV-B only depend upon the existence of a $\delta$ and $\Delta$ such that (8) is satisfied, monotonicity of $F_V(x)$ such that its derivative is bounded away from zero, and $F_V(0) = 1/2$. The recursive procedure in (20), however, requires the knowledge of $F_V(x)$.

V. CONCLUSIONS AND FUTURE WORK

The sampling, quantization, and estimation of a bounded dynamic-range bandlimited signal affected by additive independent Gaussian noise was studied. Such setup naturally arises in distributed sampling or where the sampling device itself is noisy. For bandlimited signals, the distortion due to additive independent Gaussian noise can be reduced by oversampling (statistical diversity). The maximum pointwise expected mean-squared error (statistical $L^2$ error) was used as a distortion metric. Using two extreme scenarios of quantizer precision, namely infinite precision and single-bit precision, a quantizer precision indifference principle was illustrated. It was shown that the optimal law for distortion is $O(1/N)$, where $N$ is the oversampling ratio with respect to the Nyquist rate. This scaling of distortion is unaffected by the quantizer precision, which is the key message of the precision indifference principle. In other words, the reconstruction distortion law, up to a proportionality constant, is unaffected by quantizer precision.

Extensions of the precision indifference principle to other classes of parametric or non-parametric signals is of immediate interest. Further, this work assumed sufficient dithering by noise because the estimators were linear. It is of interest to look towards estimation techniques which do not require extra dithering.

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A. Unquantized samples result in a distortion of $O(1/N)$

By using (11) and the definition of $\hat{G}_{fr}(t)$, first note that,

$$\hat{G}_{fr}(t) - g(t) = \frac{1}{N} \sum_{i=0}^{N-1} \sum_{k \in \mathbb{Z}} W \left( k \lambda + i \tau \right) \phi \left( t - k \lambda - i \lambda \right).$$

Since $W(k + i\tau)$ are i.i.d. with variance $\sigma^2$, therefore,

$$\mathbb{E}[|\hat{G}_{fr}(t) - g(t)|^2] = \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{k \in \mathbb{Z}} \lambda^2 \sigma^2 \left| \phi \left( t - k \lambda - i \lambda \right) \right|^2.$$ 

From (6), $\sum_{k \in \mathbb{Z}} |\phi(t - k/\lambda)|^2 \leq C''_{\phi}$. Thus,

$$\mathbb{E}[|\hat{G}_{fr}(t) - g(t)|^2] \leq \frac{C''_{\phi} \lambda^2 \sigma^2}{N}. \quad (23)$$

The proof is now complete.

B. Estimation of non-linear function of the signal $g(t)$

By the linearity of expectation, it is easy to see that $\mathbb{E}(H_N(t)) = \tau \sum_{n \in \mathbb{Z}} l(n\tau) \phi(t - n\tau)$. For variance calculations, note that each $X(n\tau)$ is an indicator random variable. Thus, $\text{var}(X(n\tau)) \leq (1/4)$. Using the independence of $\{X(n\tau), n \in \mathbb{Z}\}$, we get

$$\text{var}(H_N(t)) = \tau^2 \sum_{n \in \mathbb{Z}} \text{var}(X(n\tau)) |\phi(t - n\tau)|^2,$$

$$\leq \frac{\tau^2}{4} \sum_{n \in \mathbb{Z}} \left| \phi(t - \frac{1}{\lambda} - i\tau) \right|^2,$$

$$\leq \frac{\tau^2}{4} NC''_{\phi} = \frac{C''_{\phi}}{4\lambda^2 N} = \frac{C_2}{N},$$

where $C''_{\phi}$ is given by (6), and is finite since $\phi(t)$ decays rapidly and is in $C^2(\mathbb{R})$. The constant $C_2 := C''_{\phi}/(4\lambda^2)$.

Next it will be shown that as $N \to \infty$, the expression $\tau \sum_{n \in \mathbb{Z}} F(g(n\tau)) \phi(t - n\tau)$ converges to the right hand side of (15). The proof uses the fast decay of $\phi(t)$, which ensures absolute integrability of $\phi(t)$. Consider the expression

$$e_0 = \left[ \int_0^\tau l(u)\phi(t - u)du \right] - \tau l(t)\phi(t - \tau).$$

This summation is well defined since $l(t) = (F(g(t)) - 1/2)$ is bounded and $\phi(t)$ is integrable.
This expression represents the error in approximating convolution integral in a $\tau$ interval by the corresponding term in a Riemann sum [18]. This can be bounded as explained next,

$$|e_0| = \left| \int_0^\tau [l(u) - l(\tau)] \phi(t - u) du - l(\tau) \left[ \tau \phi(t - \tau) - \int_0^\tau \phi(t - u) du \right] \right|,$$

$$\leq ||l'||_\infty \tau \int_0^\tau |\phi(t - u)| du + ||l||_\infty \tau |\phi(t - \tau) - \phi(t - t_0)|,$$

where we use the triangle inequality, $||l(u) - l(\tau)|| \leq ||l'||_\infty |u - \tau|$, and finally $|u - \tau| \leq \tau$. Next, by Lagrange mean-value theorem, we note that $\int_0^\tau \phi(t - u) du = \tau \phi(t - \tau)$ for some $t_0 \in (0, \tau)$. Thus, from (24),

$$|e_0| \leq ||l'||_\infty \tau \int_0^\tau |\phi(t - u)| du + ||l||_\infty \tau |\phi(t - \tau) - \phi(t - t_0)|,$$

for some $u_0 \in (0, \tau)$. We note that $|\phi(t - u) - \phi(t - t_0)| = |\phi'(t - u_0)| |u - t_0| \leq$ for some $u_0 \in (t_0, \tau) \subseteq (0, \tau)$ (this is by Lagrange’s mean-value theorem). In the same fashion, for any $[n\tau, (n + 1)\tau]$ interval, the following bound can be established:

$$|e_n| \leq ||l'||_\infty \tau \int_n^{(n+1)\tau} |\phi(t - u)| du + ||l||_\infty \tau^2 |\phi'(t - u_n)|,$$

where $u_n \in (n\tau, (n + 1)\tau)$. Finally,

$$||E(H_N(t)) - l(t) \ast \phi(t)|| = \left| \sum_{n \in Z} l(n\tau) \phi(t - n\tau) - \int_{u \in \mathbb{R}} l(u) \phi(t - u) du \right|,$$

$$\leq \sum_{n \in Z} |e_n|,$$

$$\leq \sum_{n \in Z} |e_n|,$$

$$\leq ||l'||_\infty \tau \int_{u \in \mathbb{R}} |\phi(t - u)| du + ||l||_\infty \tau^2 \left| \sum_{n \in Z} |\phi(t - u_n)| \right|,$$

where the last step follows using (25). Using the boundedness of $C_\phi$ and $C'_\phi$ (see (11) and (13)), and using bounds for $l(t)$ and $l'(t)$, we get,

$$||E(H_N(t)) - l(t) \ast \phi(t)|| \leq \frac{F'(0)(2\pi)^2 C_\phi}{\lambda} + \frac{|F(1) - 1/2| C'_\phi}{\lambda^2} \frac{1}{N} = C_3 \frac{1}{N}.$$

Finally,

$$E(H_N(t) - l(t) \ast \phi(t))^2 = \text{var}(H_N(t)) + [E(H_N(t)) - l(t) \ast \phi(t)]^2 \leq \frac{C_2}{N} + \frac{C_3^2}{N^2} \to 0 \text{ as } N \to \infty.$$

The upper bound in the above inequality is independent of $t$; therefore, the desired result follows by maximizing the left hand side as a function of $t$. This completes the proof. It should be noted that the mean-squared error between $H_N(t)$ and $l(t) \ast \phi(t)$ is of the order of $(1/N)$. This result will be used to find the mean-squared error of $\hat{G}_{1-bd}(t) - g(t)$.

### C. The map $T$ is a contraction

From (18), first note

$$\left(1 - \frac{1}{C'_\phi} \right) \frac{1}{\delta} \leq \left(1 - \frac{1}{\sqrt{2} C'_\phi} \right) \frac{1}{\delta} \leq \mu \leq \frac{1}{\Delta}.$$

Since $F'(x)$, the pdf of the noise random variable, is positive and $F'(x) \in [\delta, \Delta]$ for $x \in [-C_\phi, C_\phi]$, therefore,

$$\delta \leq \frac{F(x) - F(y)}{x - y} \leq \Delta \text{ for } x \neq y \text{ and } x, y \in [-C_\phi, C_\phi].$$

(27)

Now consider the map in (18). First, it will be shown that if $m(t) \in S_{BL, bdd}$, then $T[m(t)] \in S_{BL, bdd}$. Define,

$$r(t) := \mu h(t) + [m(t) - \mu (F(m(t)) - 1/2)] \ast \phi(t).$$

(28)
Then $T[m(t)] = \text{Clip}[r(t)] \ast \phi(t)$. Since $|\text{Clip}[r(t)]| \leq 1$, Lemma 5.3 applied with $p(t) = \text{Clip}[r(t)]$ results in $|\text{Clip}[r(t)] \ast \phi(t)| \leq C_\phi$ for all $t \in \mathbb{R}$. Next, $T[m(t)] \ast \psi(t) = (\text{Clip}[r(t)] \ast \phi(t)) \ast \psi(t) = \text{Clip}[r(t)] \ast (\phi(t) \ast \psi(t)) = \text{Clip}[r(t)] \ast \phi(t) = T[m(t)]$. Note that $\hat{\phi}(\omega) \hat{\psi}(\omega) = \hat{\phi}(\omega)$ which results in $\hat{\phi}(t) \ast \hat{\psi}(t) = \hat{\phi}(t)$. That is, $T[m(t)]$ satisfies the convolution property needed to be present in the set $S_{\text{BL,bdd}}$.

The contraction property will now be established. Let $m_1(t)$ and $m_2(t)$ be any two signals in $S_{\text{BL,bdd}}$ with corresponding $r_1(t)$ and $r_2(t)$ as defined in (28). The transformed signals can be written in terms of $r_i(t)$ as $T[m_i(t)] = \text{Clip}[r_i(t)] \ast \phi(t)$ for $i = 1, 2$. The signals $m_1(t)$ and $m_2(t)$ are bounded in $[-C_\phi, C_\phi]$. The contraction property is established by the following steps:

$$
|r_1(t) - r_2(t)| = |[m_1(t) - m_2(t)] - \mu(F(m_1(t)) - F(m_2(t))) \ast \phi(t)|
= |\int_{u \in \mathbb{R}} (m_1(u) - m_2(u)) \left(1 - \frac{F(m_1(u)) - F(m_2(u))}{m_1(u) - m_2(u)}\right) \phi(t-u) du|
\leq \int_{u \in \mathbb{R}} |m_1(u) - m_2(u)| \left|1 - \frac{F(m_1(u)) - F(m_2(u))}{m_1(u) - m_2(u)}\right| |\phi(t-u)| du
\leq \int_{u \in \mathbb{R}} |m_1(u) - m_2(u)| |1 - \mu \delta| |\phi(t-u)| du
\leq |1 - \mu \delta| \|m_1 - m_2\|_\infty \int_{u \in \mathbb{R}} |\phi(t-u)| du
= C_\phi |1 - \mu \delta| \|m_1 - m_2\|_\infty,
$$

where (a) follows by the triangle inequality, (b) follows from (27) and $\mu \Delta < 1$, and (c) follows from the definition of the $L^\infty$-norm. Next, by using the distance-reduction property of the clip-to-one function, we get $|\text{Clip}[r_1(t)] - \text{Clip}[r_2(t)]| \leq |r_1(t) - r_2(t)| \leq C_\phi |1 - \mu \delta| \|m_1 - m_2\|_\infty$. Applying Lemma 5.1 with $p(t) = r_1(t) - r_2(t)$, we get,

$$
|T[m_1(t)] - T[m_2(t)]| \leq C^2_\phi |1 - \mu \delta| \|m_1 - m_2\|_\infty.
$$

By taking supremum on $t$ in the left hand side of the above equation, the desired contraction can be obtained:

$$
\|T[m_1] - T[m_2]\|_\infty \leq C^2_\phi |1 - \mu \delta| \|m_1 - m_2\|_\infty,
$$

independent of the choice of $m_1(t), m_2(t) \in S_{\text{BL,bdd}}$. The conditions in (8) ensure that the parameter $C^2_\phi |1 - \mu \delta| < 1$. Set $\alpha := C^2_\phi |1 - \mu \delta|$, where $\alpha < 1$. Thus,

$$
\|T[m_1] - T[m_2]\|_\infty \leq \alpha \|m_1 - m_2\|_\infty,
$$

for some $0 < \alpha < 1$. Since $C_\phi, \delta, \text{ and } \Delta$ do not depend on $m_1$ and $m_2$, therefore $\alpha$ is independent of the choice of $m_1$ and $m_2$. Thus the proof is complete.

D. Mean-squared error analysis of $\tilde{G}_{1-bd}(t) - g(t)$ using contraction

To analyze the mean-squared error, two sets of recursion will be considered. One will involve $H_N(t)$, the statistical estimate of $h(t) \ast \phi(t)$, and corresponding $\tilde{G}_k(t)$. Then $\tilde{G}_{1-bd}(t)$ is the limit of $\tilde{G}_k(t)$ as $k \to \infty$. The second recursion will involve $h(t) = l(t) \ast \phi(t)$ and the corresponding estimate $g_k(t)$ for the bandlimited signal $g(t)$. In the second recursion, $g(t)$ is the limit of $g_k(t)$. Let $r_k(t)$ and $R_k(t)$ be defined as follows:

$$
r_k(t) = \mu h(t) - [g_{k-1}(t) - \mu(F(g_{k-1}(t)) - 1/2)] \ast \phi(t),
R_k(t) = \mu H_N(t) - [G_{k-1}(t) - \mu(F(G_{k-1}(t)) - 1/2)] \ast \phi(t).
$$

Note that $g_k(t) = \text{Clip}[r_k(t)] \ast \phi(t)$ and $G_k(t) = \text{Clip}[R_k(t)] \ast \phi(t)$. By subtracting (31) from (32), the following equations are obtained:

$$
R_k(t) - r_k(t) = \mu (H_N(t) - h(t)) - \left[G_{k-1}(t) - g_{k-1}(t) - \mu(F(G_{k-1}(t)) - F(g_{k-1}(t))\right] \ast \phi(t),
= \mu (H_N(t) - h(t)) - \int_{u \in \mathbb{R}} \phi(t-u) \left[G_{k-1}(u) - g_{k-1}(u) - \mu(F(G_{k-1}(u)) - F(g_{k-1}(u))\right] du.
$$

By applying the triangle inequality twice on the above equation, the following inequalities are obtained:

$$
|R_k(t) - r_k(t)| \leq \mu |H_N(t) - h(t)| + \int_{u \in \mathbb{R}} |\phi(t-u)\left[G_{k-1}(u) - g_{k-1}(u) - \mu(F(G_{k-1}(u)) - F(g_{k-1}(u))\right] du
\leq \mu |H_N(t) - h(t)| + \int_{u \in \mathbb{R}} |\phi(t-u)| |G_{k-1}(u) - g_{k-1}(u)| \left|1 - \frac{F(G_{k-1}(u)) - F(g_{k-1}(u))}{G_{k-1}(u) - g_{k-1}(u)}\right| du
$$

(33)
The use of (8) and (27) in (33) results in
\[ |R_k(t) - r_k(t)| \leq \mu |H_N(t) - h(t)| + |1 - \mu \delta| \int_{u \in \mathbb{R}} |\phi(t - u)||G_{k-1}(u) - g_{k-1}(u)| du. \]

Since the clipping operation reduces distance, therefore
\[ |\text{Clip}[R_k(t)] - \text{Clip}[r_k(t)]| \leq \mu |H_N(t) - h(t)| + |1 - \mu \delta| \int_{u \in \mathbb{R}} |\phi(t - u)||G_{k-1}(u) - g_{k-1}(u)| du. \]
\[ = \mu |H_N(t) - h(t)| + |1 - \mu \delta| (|\phi(t)| \ast |G_{k-1}(t) - g_{k-1}(t)|). \] \quad (34)

Now the mean-squared error of \( \text{Clip}[R_k(t)] - \text{Clip}[r_k(t)] \) will be bounded using (34). First note that for any two random variables \( X \) and \( Y \), \( \mathbb{E}((X + Y)^2) \leq 2\mathbb{E}(X^2 + Y^2) \). Thus, taking second moments on both sides of (33) results in
\[ \mathbb{E}(|\text{Clip}[R_k(t)] - \text{Clip}[r_k(t)]|^2) \leq 2\mu^2 \mathbb{E}(|H_N(t) - h(t)|)^2 + 2|1 - \mu \delta|^2 \mathbb{E} \left( (|\phi(t)| \ast (|G_{k-1}(t) - g_{k-1}(t)|))^2 \right), \]
\[ \leq 2\mu^2 \mathbb{E}|H_N(t) - h(t)|^2 + 2|1 - \mu \delta|^2 \sup_t \mathbb{E}|G_{k-1}(t) - g_{k-1}(t)|^2 \] \quad (35)
where the last inequality follows from Lemma 3.1 with \( P(t) = G_{k-1}(t) - g_{k-1}(t) \). Taking supremum over \( t \) on the left side, the following recursive relationship is obtained:
\[ \sup_t \mathbb{E}(|\text{Clip}[R_k(t)] - \text{Clip}[r_k(t)]|^2) \leq 2\mu^2 \left( \frac{C_2}{4\lambda N^2} + \frac{C_3}{N^2} \right) + 2C_\phi^2 |1 - \mu \delta|^2 \sup_t \mathbb{E}(|G_{k-1}(t) - g_{k-1}(t)|)^2, \] \quad (36)
where the uniform upper bound on \( \mathbb{E}|H_N(t) - h(t)|^2 \) from (15) has been used. Since \( G_k(t) - g_k(t) = (\text{Clip}[R_k(t)] - \text{Clip}[r_k(t)]) \ast \phi(t) \), by applying Lemma 3.1 with \( P(t) = \text{Clip}[R_k(t)] - \text{Clip}[r_k(t)] \) we get,
\[ \sup_t \mathbb{E}(|G_k(t) - g_k(t)|^2) \leq 2C_\phi^2 \mu^2 \left( \frac{C_2}{4\lambda N^2} + \frac{C_3}{N^2} \right) + 2C_\phi^2 |1 - \mu \delta|^2 \sup_t \mathbb{E}(|G_{k-1}(t) - g_{k-1}(t)|)^2. \] \quad (37)

From (3), it is noted that \( \sqrt{2}C_\phi^2 |1 - \mu \delta| < 1 \). Define \( 0 < \beta := 2C_\phi^2 |1 - \mu \delta|^2 < 1 \). Using the recursion in (37), it follows that
\[ \lim_{k \to \infty} \sup_t \mathbb{E}|G_k(t) - g_k(t)|^2 \leq \frac{1}{1 - \beta} 2C_\phi^2 \mu^2 \left( \frac{C_2}{4\lambda N^2} + \frac{C_3}{N^2} \right) \] \quad (38)
Since we know that \( G_k(t) \) and \( g_k(t) \) converge in \( L^\infty(\mathbb{R}) \) to \( \tilde{G}_{1\text{-bit}}(t) \) and \( g(t) \), respectively, therefore,
\[ \sup_t \mathbb{E} |\tilde{G}_{1\text{-bit}}(t) - g(t)|^2 \leq \frac{1}{1 - \beta} 2\mu^2 \left( \frac{C_2}{4\lambda N^2} + \frac{C_3}{N^2} \right) \] \quad (39)
Lastly, \( \mu, \beta, C_2, \lambda, \) and \( C_3 \) are constants that do not depend on \( N \), therefore,
\[ \sup_t \mathbb{E} |\tilde{G}_{1\text{-bit}}(t) - g(t)|^2 = O(1/N). \]

The proportionality constant depends upon the class of signal \( S_{\text{BL,bdd}} \), the noise variance \( \sigma^2 \), the stability properties of \( \phi(t) \), and the chosen constant \( \mu \). It does not depend on the individual signal \( g(t) \). This completes the proof of the accuracy indifference principle.