A STOCHASTIC CONTROL PROBLEM AND RELATED FREE BOUNDARIES IN FINANCE

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Abstract. In this paper, we investigate an optimal stopping problem (mixed with stochastic controls) for a manager whose utility is nonsmooth and nonconcave over a finite time horizon. The paper aims to develop a new methodology, which is significantly different from those of mixed dynamic optimal control and stopping problems in the existing literature, so as to figure out the manager’s best strategies. The problem is first reformulated into a free boundary problem with a fully nonlinear operator. Then, by means of a dual transformation, it is further converted into a free boundary problem with a linear operator, which can be consequently tackled by the classical method. Finally, using the inverse transformation, we obtain the properties of the optimal trading strategy and the optimal stopping time for the original problem.

1. Introduction. Optimal stopping problems have important applications in many fields such as science, engineering, economics and, particularly, finance. The theory in this area has been well developed for stochastic dynamic systems over the past decades. In the field of financial investment, however, an investor frequently runs into investment decisions where investors stop investing in risky assets so as to maximize their expected utilities with respect to their wealth over a finite time horizon. These optimal stopping problems depend on underlying dynamic systems as well as investors’ optimization decisions (controls). This naturally results in a mixed

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optimal control and stopping problem, and [3] is one of the typical representatives along this line of research. In the general formulation of such models including a pair of control and stopping time, the theory has also been studied in [1, 6, 7, 26] and applied in finance in [5, 11, 12, 19, 20, 25].

In the finance field, finding an optimal stopping time point has been extensively studied for pricing American-style options, which allow option holders to exercise the options before or at the maturity. Typical examples that are applicable include, but do not limited to, those presented in [4, 5]. In the mathematical finance literature, choosing an optimal stopping time point is often related to a free boundary problem for a class of diffusions (see [7, 22]). In many applied areas, especially in more extensive investment problems, however, one often encounters more general controlled diffusion processes. In real financial markets, the situation is even more complicated when investors expect to choose as little time as possible to stop portfolio selection over a given investment horizon so as to maximize their profits (see [3, 10], [13]-[16], [18]-[20] and [24]).

The initial motivation of this paper comes from the recent studies on choosing an optimal point at which an investor stops investing and/or sells all his risky assets (see [2, 11]). The objective is to find an optimization process and a stopping time so as to meet certain investment criteria, such as, the maximum of an expected nonsmooth and nonconcave utility value before or at the maturity. This is a typical yet important problem in the area of financial investment. In our model, the corresponding HJB equation of the problem is formulated as a variational inequality with a fully nonlinear operator. We make a dual transformation for the problem and obtain a new free boundary problem with a linear operator. Tackling this new free boundary problem, we characterize the properties of the free boundary and optimal trading and stopping strategies for the original problem.

In our previous works, the closest one to this paper is [12], where the utility function is smooth and concave. The value function is continuous up to the terminal time $T$; moreover, the exercise region is connected. It turns out that the problem considered in this paper is more difficult than the above one (see following innovations) due to the nonconcave utility of the manager.

The main innovations of this paper, by contrast with [12], include that: First, we consider a nonsmooth yet nonconcave utility function $g(x)$, and rigorously prove the limit of the value function is its concave hull $\varphi(x)$ when the time approaches the terminal time $T$ (see Theorem 2.1). Second, we prove the equivalence between the linear problem (15) and the original problem (12). Third, we develop a new method to study the free boundary while the exercise region is not connected (see (24)-(26) and Lemma 4.2) so that we can shed light on the monotonicity and differentiability of the free boundaries (see Figures 4.1-4.4) under any cases of parameters. It is very difficult in analysis of the free boundaries.

The remainder of the paper is organized as follows. In Section 2, the mathematical formulation of the model is presented, and the corresponding HJB equation with certain boundary and terminal conditions are posed. In particular, we show that the value function $V(x, t)$ is not continuous at $t = T$, i.e.,

$$\lim_{t \uparrow T} V(x, t) \neq V(x, T).$$

In Section 3, we make a dual transformation to convert the free boundary problem with a fully nonlinear operator (12) to a new free boundary problem with a linear operator (15). Section 4 devotes to the study of the free boundary problem (15).
in different cases of parameters. In Section 5, using the corresponding inverse dual transformation, we construct the solution of the original problem (12) and present the properties (including the monotonicity and differentiability) of its free boundaries in different cases. In Section 6, we develop a verification theorem to show that the solution of the problem (12) is the value function defined in (3). Section 7 concludes the paper.

2. Model formulation.

2.1. The manager’s problem. A manager operates in a complete, arbitrage-free, continuous-time financial market consisting of a risk-free asset with instantaneous interest rate $r$ and $n$ risky assets. The risky asset prices $S_i$ are governed by the stochastic differential equations

$$
\frac{dS_{i,t}}{S_{i,t}} = (r + \mu_i)dt + \sum_{j=1}^n \sigma_{ij}dW_j^t, \quad \text{for } i = 1, 2, \ldots, n,
$$

where the interest rate $r$, the excess appreciation rates $\mu_i$, and the volatility coefficients $\sigma_{ij}$ are constants, $W_t$ is a standard $n$-dimensional Brownian motion. In addition, the matrix $\sigma'\sigma$ is strongly nondegenerate.

A trading strategy for the manager is an $n$-dimensional process $\pi_t$, whose $i$-th component $\pi_{i,t}$ is the amount invested in the $i$-th risky asset in the portfolio at time $t$. An admissible trading strategy $\pi_t$ must be progressively measurable with respect to the filtration $\{F_t\}$ such that $X_t \geq 0$. Note that $X_t = \pi_{0,t} + \sum_{i=1}^n \pi_{i,t}$, where $\pi_{0,t}$ is the amount invested in the risk-free asset. Hence, the value of the portfolio $X_t$ evolves according to

$$
\frac{dX_t}{X_t} = (rX_t + \mu^t \pi_t)dt + \pi^t \sigma dW_t.
$$

Now, we begin with any fixed time $t$ and suppose the value at the time $t$ is $x$, then

$$
\frac{dX_s}{X_s} = (rX_s + \mu^s \pi_s)ds + \pi^s \sigma dW_s, \quad s \geq t,
$$

$$
X_t = x,
$$

where $\pi_s$, $s \in [t, T]$ belongs to the set of admissible trading strategies

$$
\Pi_t := \{\pi_s \in L^2_F([t, T]; \mathbb{R}) | X_s \geq 0, \ t \leq s \leq T\}.
$$

The manager’s dynamic problem is to choose an admissible trading strategy $\pi_t$ and a stopping time $\tau$ ($t \leq \tau \leq T$) to maximize his expected utility at the terminal time $T$:

$$
V(x, t) = \sup_{\pi, \tau} E_{t,x}[e^{-\beta(\tau-t)}g(X_\tau)] := \sup_{\pi, \tau} E[\pi, e^{-\beta(\tau-t)}g(X_\tau)|X_t = x],
$$

where $g(x)$ is the utility of the manager and $\beta > 0$ is the discount factor.

Suppose the manager’s wealth at the terminal time $T$ consists of a call option on the portfolio’s wealth and a constant $K > 0$. If the strike price is $b > 0$, then his wealth at $T$ is

$$
W_T = (X_T - b)^+ + K.
$$

His utility function $U(\cdot)$ of wealth is the risk averse, the strictly increasing, strictly concave function

$$
U(W) = \frac{1}{\gamma} W^\gamma
$$
with $0 < \gamma < 1$. So in this model,
\[
g(x) := U((x-b)^+ + K) = \frac{1}{\gamma}((x-b)^+ + K)^\gamma.
\]

If $X_t = 0$, in order to keep $X_s \geq 0$, the only choice of $\pi_s$ is 0 and thus $X_s \equiv 0$, $t \leq s \leq T$. Hence
\[
V(0,t) = \sup_{\pi,\tau} E[e^{-\beta(\tau-t)}g(0)] = g(0) = \frac{1}{\gamma}K^\gamma.
\]

In this case the optimal stopping time $\tau$ is the initial time $t$.

2.2. **Discontinuity of value function at the terminal time $T$.** From the definition (3), we can see that $V(x,T) = g(x) = \frac{1}{\gamma}[(x-b)^+ + K]^\gamma$. Since the portfolio $\pi_t$ is unrestricted, $V(x,t)$ may be discontinuous at the terminal time $T$. Therefore, we should pay attention to $V(x,T-) := \lim_{t \uparrow T} V(x,t)$.

**Theorem 2.1.** The value function $V$ defined in (3) is not continuous at the terminal time $T$ and satisfies
\[
\lim_{t \uparrow T} V(x,t) = \varphi(x),
\]
where
\[
\varphi(x) = \begin{cases} 
kx + \frac{1}{\gamma}K^\gamma, & 0 < x < \hat{x}, \\
\frac{1}{\gamma}(x-b+K)^\gamma, & x \geq \hat{x},
\end{cases}
\]
is the concave hull of $\frac{1}{\gamma}[(x-b)^+ + K]^\gamma$ (see Fig. 2.1); here $k$ and $\hat{x}$ satisfy
\[
\begin{cases} 
k\hat{x} + \frac{1}{\gamma}K^\gamma = \frac{1}{\gamma}(\hat{x} - b + K)^\gamma, \\
k = (\hat{x} - b + K)^{\gamma-1}.
\end{cases}
\]

![Utility](image)

**Fig. 2.1.** $\varphi(x)$

**Proof.** We first prove
\[
\limsup_{t \uparrow T} V(x,t) \leq \varphi(x).
\]

Define
\[
\zeta_t = e^{-(\gamma + \frac{1}{\gamma}(\sigma \sigma^{-1})^\gamma)^2 t - \mu \sigma^{-1} W_t},
\]
then
\[
d\zeta_t = \zeta_t \{ -\mu dt - \mu \sigma^{-1} dW_t \}. 
\]
Hence, therefore, by (6) and (7),

and

\[
d(\zeta_t X_t) = \zeta_t dX_t + X_t d\zeta_t + d\zeta_t dX_t
\]

\[
= \zeta_t ((rX_t + \mu' \pi_t) dt + \pi'_t \sigma dW_t - rX_t dt - \mu' \sigma^{-1} X_t dW_t - (\mu' \sigma^{-1}) (\pi'_t \sigma)' dt)
\]

\[
= \zeta_t |\pi'_t \sigma - \mu' \sigma^{-1} X_t| dW_t.
\]  

(5)

Thus, \(\zeta_t X_t\) is a martingale. Hence, for any admissible strategy \(\pi\) and stopping time \(\tau\) \((t \leq \tau \leq T)\), by Jensen’s inequality,

\[
E_{t,x} \varphi \left( \frac{\zeta_t}{\zeta_t} X_\tau \right) \leq \varphi \left( E_{t,x} \left( \frac{\zeta_t}{\zeta_t} X_\tau \right) \right) = \varphi(x).
\]

Thus

\[
\limsup_{t \uparrow T} \sup_{\tau, \pi} E_{t,x} \varphi \left( \frac{\zeta_t}{\zeta_t} X_\tau \right) \leq \varphi(x).
\]

(6)

We now come to prove

\[
\limsup_{t \uparrow T} \sup_{\tau, \pi} E_{t,x} \left| \varphi(X_\tau) - \varphi \left( \frac{\zeta_t}{\zeta_t} X_\tau \right) \right| = 0.
\]

(7)

Indeed, owing to \(\varphi(x)\) is differentiable and for all \(x, y \geq \hat{x}\),

\[
| |(x - b + K)^\gamma - (y - b + K)^\gamma| \leq |x - y|^\gamma,
\]

so there exits a constant \(C > 0\) such that for all \(x, y > 0\),

\[
|\varphi(x) - \varphi(y)| \leq C |x - y|^\gamma.
\]

Thus, for any admissible strategy \(\pi\) and stopping time \(t \leq \tau \leq T\),

\[
E_{t,x} \left| \varphi(X_\tau) - \varphi \left( \frac{\zeta_t}{\zeta_t} X_\tau \right) \right| \leq CE_{t,x} \left( \left( \frac{\zeta_t}{\zeta_t} X_\tau \right)^\gamma \left| \frac{\zeta_t}{\zeta_t} - 1 \right| \right).
\]

Using Hölder’s inequality, we obtain

\[
E_{t,x} \left| \varphi(X_\tau) - \varphi \left( \frac{\zeta_t}{\zeta_t} X_\tau \right) \right| \leq C \left( E_{t,x} \left( \frac{\zeta_t}{\zeta_t} X_\tau \right)^\gamma \right) \left( E_{t,x} \left| \frac{\zeta_t}{\zeta_t} - 1 \right| \right)^{1-\gamma}
\]

\[
\leq C x^\gamma \left( E_{t,x} \sup_{t \leq s \leq T} \left| \frac{\zeta_t}{\zeta_s} - 1 \right| \right)^{1-\gamma}.
\]

Hence,

\[
\limsup_{t \uparrow T} \sup_{\tau, \pi} E_{t,x} \left| \varphi(X_\tau) - \varphi \left( \frac{\zeta_t}{\zeta_t} X_\tau \right) \right| \leq C x^\gamma \lim_{t \uparrow T} \left( E_{t,x} \sup_{t \leq s \leq T} \left| \frac{\zeta_t}{\zeta_s} - 1 \right| \right)^{1-\gamma} = 0.
\]

Therefore, by (6) and (7),

\[
\limsup_{t \uparrow T} V(x, t) = \limsup_{t \uparrow T} E_{t,x} \left( e^{-\beta(t-s)} g(X_\tau) \right) \leq \limsup_{t \uparrow T} E_{t,x} \varphi(X_\tau) \leq \limsup_{t \uparrow T} E_{t,x} \varphi \left( \frac{\zeta_t}{\zeta_t} X_\tau \right) + \limsup_{t \uparrow T} E_{t,x} \left| \varphi(X_\tau) - \varphi \left( \frac{\zeta_t}{\zeta_t} X_\tau \right) \right| \leq \varphi(x).
\]

Next, we further prove

\[
\liminf_{t \uparrow T} V(x, t) \geq \varphi(x).
\]  

(8)
For a fixed \( t < T \), if \( x \geq \hat{x} \) or \( x = 0 \), we have
\[
V(x, t) \geq g(x) = \varphi(x),
\]
which implies that (8) holds true.

If \( 0 < x < \hat{x} \), choose \( \tau = T \) and choose \( \pi_s \) such that
\[
\left[ \frac{\xi_s}{\zeta_t} [\pi'_t - \mu' \sigma^{-1} X_t] \right] = (s_n^N)^' := N \chi_{\{0 < \frac{\xi_s}{\zeta_t} < \hat{x}\}} I_n, \quad \forall N > 0,
\]
where \( I_n \) is an \( n \)-dimensional unit column vector. Let \( X_s^N = \frac{\xi_s}{\zeta_t} X_s \). Then (5) implies
\[
dX_s^N = (\pi_s^N)'dW_s, \quad t \leq s \leq T.
\]
It is not hard to obtain
\[
0 \leq X_s^N \leq \hat{x}, \quad t \leq s \leq T,
\]
and since
\[
\{0 < X_s^N < \hat{x}\} = \{0 < X_s^N = x + NI'_n(W_s - W_t) < \hat{x}, t \leq s \leq T\}
\subset \{0 < x + NI'_n(W_T - W_t) < \hat{x}\},
\]
we have
\[
P(0 < X_s^N < \hat{x}) \leq P(0 < x + NI'_n(W_T - W_t) < \hat{x}) \to 0, \quad N \to \infty.
\]
Note that
\[
\hat{\varphi}P(X_T^N = \hat{x}) \leq EX_T^N \leq \hat{\varphi}P(X_T^N = \hat{x}) + \hat{\varphi}P(0 < X_T^N < \hat{x}).
\]
Therefore,
\[
\lim_{N \to \infty} P(X_T^N = \hat{x}) = \frac{EX_T^N}{\hat{x}} = \frac{x}{\hat{x}}, \quad \lim_{N \to \infty} P(X_T^N = 0) = 1 - \frac{x}{\hat{x}}.
\]
As a result,
\[
\lim_{N \to \infty} E\varphi(X_T^N) = \frac{x}{\hat{x}} g(\hat{x}) + \left(1 - \frac{x}{\hat{x}}\right) g(0) = \frac{x}{\hat{x}} \left[ k\hat{x} + \frac{1}{\gamma} K_y \right] + \left(1 - \frac{x}{\hat{x}}\right) \frac{1}{\gamma} K_y = kx + \frac{1}{\gamma} K_y = \varphi(x).
\]
Thus
\[
\sup_{\tau, \pi} E_{t, x} \left( e^{-\beta(\tau - t)} g\left( \frac{\xi_T}{\zeta_t} X_T \right) \right) \geq e^{-\beta(T - t)} \lim_{N \to \infty} E\varphi(X_T^N) = e^{-\beta(T - t)} \varphi(x).
\]
Meanwhile, similar to (7), we have
\[
\lim_{t \to T, \tau, \pi} E_{t, x} \left| g(X_T) - g\left( \frac{\xi_T}{\zeta_t} X_T \right) \right| = 0.
\]
Therefore,
\[
\lim_{t \to T} V(x, t) = \lim_{t \to T} \sup_{\tau, \pi} E_{t, x} \left( e^{-\beta(\tau - t)} g(X_T) \right) \geq \lim_{t \to T} \sup_{\tau, \pi} E_{t, x} \left( e^{-\beta(\tau - t)} g\left( \frac{\xi_T}{\zeta_t} X_T \right) \right) - \lim_{t \to T, \tau, \pi} E_{t, x} \left| g(X_T) - g\left( \frac{\xi_T}{\zeta_t} X_T \right) \right| \geq \varphi(x).
\]
The proof is complete.
Since the value function is not continuous at the terminal time \( T \), we introduce its corresponding HJB equation with the terminal condition \( V(x, T^-) = \varphi(x) \), \( x > 0 \) in the next subsection.

### 2.3. HJB equation.

Applying the dynamic programming principle, one can obtain the following HJB equation

\[
\begin{cases}
\min \left\{ -V_t - \max_{\pi} \left[ \frac{1}{2} (\pi' \sigma' \pi)V_{xx} + \mu' \pi V_x \right] - r x V_x + \beta V, \right. \\
V - \frac{1}{\gamma} [(x - b)^+ + K]^\gamma = 0, & x > 0, \ 0 < t < T, \\
V(0, t) = \frac{1}{\gamma} K^\gamma, & 0 < t < T, \\
V(x, T^-) = \varphi(x), & x > 0, \\
\end{cases}
\]

(9)

From the definition (3), we easily see that \( V \) is increasing in \( x \), so we need to find an increasing solution of (9).

Intuitively speaking, the Hamiltonian operator

\[
\max_{\pi} \left\{ \frac{1}{2} (\pi' \sigma' \pi)V_{xx} + \mu' \pi V_x \right\} - r x V_x + r V
\]

is singular if \( V_{xx} > 0 \); or \( V_{xx} = 0 \), \( V_x > 0 \). Thus, \( V_{xx} \leq 0 \). Moreover, if \( V_x = 0 \) holds on \((x_0, t_0)\), then for any \( x \geq x_0 \), \( V_x(x, t_0) = 0 \), which contradicts to \( V(x, t) \geq \frac{1}{\gamma} [(x - b)^+ + K]^\gamma \rightarrow +\infty, \ x \rightarrow +\infty \).

The above thinking motives us to find a solution of (9) such that

\[
V_x > 0, V_{xx} < 0, \ x > 0, \ 0 < t < T.
\]

(11)

Note that the gradient of \( \pi' \sigma' \pi \) with respect to \( \pi \) is

\[
\nabla_\pi (\pi' \sigma' \pi) = 2 \pi \sigma' \pi.
\]

Hence, the optimal trading strategy is

\[
\pi^* = - (\sigma \sigma')^{-1} \mu \frac{V_x(x, t)}{V_{xx}(x, t)}.
\]

Applying \( V_{xx} < 0 \), we have

\[
V - \frac{1}{\gamma} [(x - b)^+ + K]^\gamma \geq 0 \quad \text{if and only if} \quad V - \varphi(x) \geq 0.
\]

Define \( a^2 = \mu' (\sigma \sigma')^{-1} \mu \), then the variational inequality (9) reduces to

\[
\min \left\{ -V_t + a^2 \frac{V_x^2}{V_{xx}} - r x V_x + \beta V, \ V - \varphi(x) \right\} = 0,
\]

Hence, we formulate our problem as the following variational inequality

\[
\begin{cases}
\min \left\{ -V_t + a^2 \frac{V_x^2}{V_{xx}} - r x V_x + \beta V, \ V - \varphi(x) \right\} = 0, \ x > 0, \ 0 < t < T, \\
V(0, t) = \frac{1}{\gamma} K^\gamma, & 0 < t < T, \\
V(x, T^-) = \varphi(x), & x > 0. \\
\end{cases}
\]

(12)

We will show that the problem (12) has a (unique) solution which satisfies (11). Also, in Section 6 we will present a verification theorem which ensures this solution is just \( V \) defined in (3).
3. Dual problem. In this section, we formulate the dual problem (namely, (15) below) of the problem (12). After this section, we will study the dual problem first, and then use the inverse dual transformation to construct a desired solution to the problem (12).

Define the dual transformation of $\varphi(x)$ as

$$
\psi(y) = \max_{x \geq 0} (\varphi(x) - xy), \quad y > 0,
$$

then the optimal $x$ for a fixed $y$, denoted by $x_\varphi(y)$, is

$$
x_\varphi(y) = \begin{cases} 
y \frac{1}{\gamma} - (K - b), & \text{for } 0 < y < k, \\
\in [0, \bar{x}], & \text{for } y = k, \\
0, & \text{for } y > k,
\end{cases}
$$

and

$$
\psi(y) = \varphi(x_\varphi(y)) - x_\varphi(y)y
$$

$$
= \begin{cases} 
\frac{1 - \gamma}{\gamma}y \frac{1}{\gamma} + (K - b)y, & \text{for } 0 < y < k, \\
\frac{1}{\gamma}K, & \text{for } y \geq k,
\end{cases}
$$

(13)

(see Fig. 3.1).

Now define the dual transformation of $V(x, t)$ (see Pham [23]) as

$$
v(y, t) = \max_{x \geq 0} (V(x, t) - xy), \quad y > 0, \quad 0 < t < T.
$$

(14)

From the variational inequality problem (12), we deduce the variational inequality for $v$ as following

$$
\begin{cases}
\min \{-v_t - \frac{a^2}{2}y^2 v_{yy} - (\beta - r)yv_y + \beta v, \ v - \psi\} = 0, & y > 0, \ 0 < t < T, \\
v(y, T-) = \psi(y), & y > 0.
\end{cases}
$$

(15)

Remark 1. The equation in (15) is degenerate on the boundary $y = 0$. According to Fichera’s theorem (see Olečnik and Radkevič [21]), we must not put boundary condition on $y = 0$.

Remark 2. Pay attention to the exceptional case $b = K = 0$ in (15), accordingly, $k = +\infty$ in (4); and (13) becomes

$$
\psi(y) = \frac{1 - \gamma}{\gamma}y \frac{1}{\gamma}, \quad \text{for } y > 0.
$$
Note that $\psi(y) = -y^{1-\gamma}, \psi_{yy}(y) = \frac{1}{1-\gamma}y^{\gamma-2}$, thus

$$-\psi_t - \frac{a^2}{2}y^2\psi_{yy} - (\beta - r)y\psi_y + \beta\psi$$

$$= y^{1-\gamma} \left[ -\frac{a^2}{2} \frac{1}{1-\gamma} + (\beta - r) + \frac{\beta}{1-\gamma} \right]$$

$$= \frac{1}{\gamma} y^{1-\gamma} \left( \beta - \frac{a^2}{2} \frac{\gamma}{1-\gamma} - r\gamma \right).$$

So if

$$\kappa := \beta - \frac{a^2}{2} \frac{\gamma}{1-\gamma} - r\gamma \geq 0,$$

then all the domain $\{y > 0, 0 < t < T\}$ is the optimal stopping region; and

$$v(y, t) = \psi(y) = \frac{1 - \gamma}{\gamma} y^{1-\gamma}$$

and

$$V(x, t) = \phi(x) = \frac{1}{\gamma} x^{\gamma}.$$

On the other hand, if

$$\kappa < 0,$$

then the optimal stopping domain is empty and the solution $v(y, t)$ of (15) satisfies

$$\begin{cases}
-\psi_t - \frac{a^2}{2}y^2\psi_{yy} - (\beta - r)y\psi_y + \beta\psi = 0, & y > 0, 0 < t < T; \\
v(y, T-) = \psi(y), & y > 0.
\end{cases}$$

which we can easily find an explicit solution

$$v(y, t) = \frac{1 - \gamma}{\gamma} e^{\frac{\kappa}{\gamma} (t-T)} y^{1-\gamma}$$

and

$$V(x, t) = \frac{1}{\gamma} e^{\kappa(t-T)} x^{\gamma}.$$

This case is just like American call option without dividend: American call option is the same as European call option.

4. **The solution and the free boundary of (15).** Recall the function space [17]

$$W^{2,1}_p(\Omega) = \{ u(y, t) \in L^p(\Omega) \mid u_{yy}, u_t \in L^p(\Omega) \}$$

where $\Omega$ is a domain in $\mathbb{R}^2$, and

$$W^{2,1}_{p, loc}(\Omega) = \{ u \in W^{2,1}_p(\Omega') \mid \text{for any } \Omega' \subset \subset \Omega \}.$$ 

Moreover, denote

$$C(\Omega) = \{ u \text{ is continuous in } \Omega \}.$$

Now we find the solution of (15) which is under liner growth condition with respect to $y$. 


Theorem 4.1. The problem (15) has a unique solution \( v(y,t) \in W_p^{2,1}(Q_y \setminus B_e(k,T)) \cap C(Q_y) \) for any \( p > 2 \) and small \( \varepsilon > 0 \). Moreover,

\[
\begin{align*}
\psi(y) & \leq v \leq Ae^{B(T-t)}y^{\frac{\gamma}{\gamma-1}} + 1, \\
v_t & \leq 0, \\
v_y & \leq 0, \\
v_{yy} & \geq 0,
\end{align*}
\]

where \( Q_y = (0, +\infty) \times (0, T) \), \( B_e(k,T) \) is the disk with center \((k,T)\) and radius \( \varepsilon \), \( A = \max\{\frac{1-\gamma}{\gamma}, \frac{1}{\gamma}K^\gamma, |K-b|k\} \), \( B = \frac{a^2}{2} - \frac{\gamma}{\gamma-1} + \frac{\beta-r}{\gamma-1} \).

Proof. The solution of system (15) can be proved by a standard penalty method [9]. The frame of the proof is as following. The first step is to define a penalty function, for \( \varepsilon > 0 \),

\[
\alpha_\varepsilon(x) = \begin{cases} 
0, & x > \varepsilon, \\
-1, & x = 0,
\end{cases}
\]

and \( \alpha_\varepsilon(x) \) is increasing in \((-\infty, +\infty)\); moreover

\[
\lim_{\varepsilon \to 0} \alpha_\varepsilon(x) = \begin{cases} 
0, & x > 0, \\
[0, -\infty], & x = 0.
\end{cases}
\]

The second step is to consider the penalty problem

\[
\begin{align*}
-v_t - \frac{a^2}{2} y^2 v_{yy} - (\beta - r)yv_y + \beta v + \alpha_\varepsilon(v - \psi) &= 0, & y > 0, & 0 < t < T, \\
v(y, T-) &= \psi(y), & y > 0.
\end{align*}
\]

According to the results of existence and uniqueness of \( W_p^{2,1} \) in [17], the above problem has a unique solution \( v_\varepsilon \in W_p^{2,1}(Q_y \setminus B_e(k,T)) \cap C(Q_y) \). The last step is to define

\[
v(y,t) = \lim_{\varepsilon \to 0} v_\varepsilon(y,t)
\]

and prove that it is the solution of the problem (15) indeed. Furthermore, by Sobolev embedding theorem [8], we have

\[
v \in C((Q_y \cup \{t = 0, T\}), \quad v_y \in C((Q_y \cup \{t = 0, T\}) \setminus (k,T)).
\]

The first inequality in (16) follows from (15) directly, we now prove the second inequality in (16). Denote

\[
w(y,t) = Ae^{B(T-t)}y^{\frac{\gamma}{\gamma-1}} + 1.
\]

Then

\[
\begin{align*}
-w_t - \frac{a^2}{2} y^2 w_{yy} - (\beta - r)yw_y + \beta w &= Ae^{B(T-t)}y^{\frac{\gamma}{\gamma-1}} \left( B + \frac{a^2}{2} \frac{\gamma}{\gamma-1} - (\beta - r)\left(\frac{\gamma}{\gamma-1}\right) + \beta A \right) \\
&\geq 0,
\end{align*}
\]

and \( w(y,t) \geq w(y,T) \geq \psi(y) \). Using the comparison principle of variational inequality (see Friedman [8]), we know that \( w \) is a super solution of (15).
Next we prove (17). Let \( v(y, t) = v(y, t - \delta) \) for small \( \delta > 0 \), then \( \tilde{v} \) satisfies
\[
\begin{cases}
\min\{-\tilde{v} - \frac{\alpha^2}{2} y^2 \tilde{v}_{yy} - (\beta - r) y \tilde{v}_y + \beta \tilde{v} - \psi(y)\} = 0, & y > 0, \quad \delta < t < T; \\
\tilde{v}(y, T) \geq \psi(y), & y > 0.
\end{cases}
\]
Hence, by the comparison principle, we have \( \tilde{v} \geq v \), i.e. \( v_t \leq 0 \).

Define
\[
\begin{align*}
\mathcal{ER}_y &= \{(y, t) \in Q_y | v = \psi\}, & & \text{exercise region}, \\
\mathcal{CR}_y &= \{(y, t) \in Q_y | v > \psi\}, & & \text{continuation region}.
\end{align*}
\]

Note that \( k \) is the only discontinuity point of \( \psi'(y) \) and \( \psi''(y) \). Now, we claim \((k, t) \) could not be contained in \( \mathcal{ER}_y \) for all \( t \in (0, T) \). Otherwise, if \((k, t_0) \in \mathcal{ER}_y \) for some \( t_0 < T \), then it belongs to the minimum points of \( v - \psi(y) \), thus \( v_y(k-, t_0) \leq \psi'(k-) < \psi'(k+) \leq v_y(k+, t_0) \), which implies \( v_y \) is not continues at the point \((k, t_0) \), contradicting (21).

Here, we present the proofs of (18) and (19). Recall that
\[
\begin{align*}
\psi'(y) &< 0, & & \psi''(y) > 0, & y \in (0, k), \\
\psi'(y) &= 0, & & \psi''(y) = 0, & y \in (k, +\infty).
\end{align*}
\]
Since \( v \) gets its minimum value in \( \mathcal{ER}_y \), we have \( v_y = \psi' \leq 0 \) in \( \mathcal{ER}_y \). Moreover, \( v_y(y, T) = \psi' \leq 0 \). Taking the derivative for the following equation
\[
-v_t - \frac{\alpha^2}{2} y^2 v_{yy} - (\beta - r) y v_y + \beta v = 0 \quad \text{in} \quad \mathcal{CR}_y
\]
with respect to \( y \) leads to
\[
-\partial_t v_y - \frac{\alpha^2}{2} y^2 \partial_{yy} v_y - (\alpha^2 + \beta - r) y \partial_y v_y + rv_y = 0 \quad \text{in} \quad \mathcal{CR}_y.
\] (22)
Note that \( v_y = \psi' \leq 0 \) on \( \partial(\mathcal{CR}_y) \), where \( \partial(\mathcal{CR}_y) \) is the boundary of \( \mathcal{CR}_y \) in the interior of \( Q_y \). Using the maximum principle, we obtain (18).

In addition, \( v \geq \psi \), together with that \( v = \psi, \quad v_y = \psi' \) in \( \mathcal{ER}_y \), yields \( v_{yy} = \psi'' \geq 0 \) in \( \mathcal{ER}_y \). It is not hard to prove that
\[
\lim_{\mathcal{CR}_y \ni y \to \partial(\mathcal{CR}_y)} v_{yy}(y, t) \geq 0,
\]
and \( v_{yy}(y, T) = \psi'' \geq 0 \). Taking the derivative for the equation (22) with respect to \( y \), we obtain
\[
-\partial_{yy} v_{yy} - \frac{\alpha^2}{2} y^2 \partial_{yy} v_{yy} - 2a^2 y \partial_y v_{yy} + (r - a^2) v_{yy} = 0 \quad \text{in} \quad \mathcal{CR}_y.
\]
Using the maximum principle, we obtain
\[
v_{yy} \geq 0 \quad \text{in} \quad \mathcal{CR}_y.
\] (23)
The proof is complete.

Define free boundaries
\[
\begin{align*}
h(t) &= \inf\{y \in [0, k]| v(y, t) = \psi(y)\}, & & 0 < t < T, \\
g(t) &= \sup\{y \in [0, k]| v(y, t) = \psi(y)\}, & & 0 < t < T, \\
f(t) &= \inf\{y \in [k, +\infty)| v(y, t) = \psi(y)\}, & & 0 < t < T.
\end{align*}
\] (24) (25) (26)
Owing to \( \partial_t (v(y, t) - \psi(y)) = v_t \leq 0 \), the functions \( h(t) \) and \( f(t) \) are decreasing and \( g(t) \) is increasing in \( t \).
Substituting the first expression of (13) into the equation (15) yields
\[
-\partial_t \psi - \frac{a^2}{2} y^2 \partial_{yy} \psi - (\beta - r) y \partial_y \psi + \beta \psi = \frac{a^2}{2} \left( \frac{\gamma}{\gamma - 1} - 1 \right) y \frac{\partial}{\partial t} y - (\beta - r) y \left[ - \frac{\partial}{\partial t} y + (K - b) \right] + \beta \left[ \frac{1 - \gamma}{\gamma} y \frac{\partial}{\partial t} y + (K - b) y \right]
\]
\[
= \left( \frac{\beta - r \gamma}{\gamma} - \frac{a^2}{2} \frac{1}{1 - \gamma} \right) y \frac{\partial}{\partial t} y + r(K - b) y, \quad y < k,
\]
and note that
\[
-\partial_t \psi - \frac{a^2}{2} y^2 \partial_{yy} \psi - (\beta - r) y \partial_y \psi + \beta \psi = \frac{\beta}{\gamma} K > 0, \quad y > k.
\]
Denote the right hand side of (27) by \( \Psi(y) \). It is not hard to see that
\[
\mathcal{E} \mathcal{R}_y \subset \{ \Psi(y) \geq 0, \ y < k \} \cup \{k, +\infty\} \times (0, T).
\]

**Lemma 4.2.** We have
\[
\mathcal{E} \mathcal{R}_y = \{ (y, t) \in Q_y | h(t) \leq y \leq g(t) \} \cup \{ (y, t) \in Q_y | y \geq f(t) \}.
\]

**Proof.** By the definitions of \( h(t) \), \( g(t) \) and \( f(t) \), we get
\[
\mathcal{E} \mathcal{R}_y \subset \{ (y, t) \in Q_y | h(t) \leq y \leq g(t) \} \cup \{ (y, t) \in Q_y | y \geq f(t) \}.
\]

Now, we prove
\[
\Omega = \{ (y, t) \in Q_y | h(t) \leq y \leq g(t) \} \subset \mathcal{E} \mathcal{R}_y.
\]
Since \( \{ (h(t), t) \in Q_y \cap \mathcal{E} \mathcal{R}_y \cap \{ y < k \} \subset \{ \Psi \geq 0 \} \) and \( \{ \Psi \geq 0 \} \) is a connected region, we have
\[
\Omega \subset \{ \Psi \geq 0 \}.
\]
Assume that (30) is false. Since \( \mathcal{C} \mathcal{R}_y \) is an open set, there exists an open subset \( \mathcal{N} \) such that \( \mathcal{N} \subset \Omega \) and \( \partial_p \mathcal{N} \subset \mathcal{E} \mathcal{R}_y \), where \( \partial_p \mathcal{N} \) is the parabolic boundary of \( \mathcal{N} \). Thus,
\[
\begin{align*}
-\nu_t - \frac{a^2}{2} y^2 \nu_{yy} - (\beta - r) y \nu_y + \beta \nu &= 0 \quad \text{in } \mathcal{N}, \\
-\psi_t - \frac{a^2}{2} y^2 \psi_{yy} - (\beta - r) y \psi_y + \beta \psi &\geq 0 \quad \text{in } \mathcal{N}, \\
v &= \psi \quad \text{on } \partial_p \mathcal{N}.
\end{align*}
\]
By the comparison principle, \( v \leq \psi \) in \( \mathcal{N} \), which implies \( \mathcal{N} = \emptyset \).

Similar proof yields
\[
\{ (y, t) \in Q_y | y \geq f(t) \} \subset \mathcal{E} \mathcal{R}_y.
\]

Therefore, the desired result (29) holds.

Thanks to Lemma 4.2, \( h(t) \), \( g(t) \) and \( f(t) \) are three free boundaries of (15).

**Theorem 4.3.** The free boundaries \( h(t) \), \( g(t) \) and \( f(t) \) are in \( C^\infty(0, T) \). Moreover, we have the following.

**Case I.** \( \beta \geq \frac{a^2}{2} \frac{1}{\gamma - 1} + r \gamma \), \( \Psi(k) \geq 0 \), (Fig. 4.1)
\[
h(t) \equiv 0 \leq g(t) \leq g(T-) = k = f(T-) \leq f(t).
\]
Case II. $\beta \geq \frac{a^2}{2} \frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) < 0$, (Fig 4.2)

If $\beta > \frac{a^2}{2} \frac{\gamma}{1-\gamma} + r\gamma$,

$$h(t) \equiv 0 \leq g(t) \leq g(T-) = \left(\frac{-r(K-b)}{\frac{1-r\gamma}{\gamma} - \frac{a^2}{2} \frac{1}{1-\gamma}}\right)^{\gamma-1} < k = f(T-) \leq f(t).$$

If $\beta = \frac{a^2}{2} \frac{\gamma}{1-\gamma} + r\gamma$, (Fig 4.4)

$$\mathcal{E}R_y \cap ((0,k) \times (0,T)) = \emptyset, \quad k = f(T-) \leq f(t).$$

Case III. $\beta < \frac{a^2}{2} \frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) > 0$, (Fig 4.3)

$$\left(\frac{-r(K-b)}{\frac{1-r\gamma}{\gamma} - \frac{a^2}{2} \frac{1}{1-\gamma}}\right)^{\gamma-1} = h(T-) \leq h(t) \leq g(t) \leq g(T-) = k = f(T-) \leq f(t).$$

Case IV. $\beta < \frac{a^2}{2} \frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) \leq 0$, (Fig 4.4)

$$\mathcal{E}R_y \cap (0,k) \times (0,T) = \emptyset, \quad k = f(T-) \leq f(t).$$

Proof. By the method of [8], one can prove $h(t)$, $g(t)$, $f(t)$ are in $C^\infty(0,T)$; we omit the details.

Here, we only prove the results in Case II, the remaining situations are similar.

If $\beta > \frac{a^2}{2} \frac{\gamma}{1-\gamma} + r\gamma$ and $\Psi(k) < 0$, then $K < b$, Denote $y_T = \left(\frac{-r(K-b)}{\frac{1-r\gamma}{\gamma} - \frac{a^2}{2} \frac{1}{1-\gamma}}\right)^{\gamma-1}$, then $\Psi(k) < 0$ implies $y_T < k$. By (28) and $\{\Psi \geq 0\} = (0,y_T)$,

$$\mathcal{E}R_y \subset (0,y_T] \cup (k,\infty) \times (0,T),$$
thus \[0 \leq h(t) \leq g(t) \leq y_T < k \leq f(t).\]

Now, we prove \(h(t) \equiv 0\). Set \(\mathcal{N} = \{(y, t) | 0 < y \leq h(t), 0 < t < T\}\). It follows from (31) that \(v \leq \psi\) in \(\mathcal{N}\). By the definition of \(h(t)\), we have \(\mathcal{N} = \emptyset\) as well as \(h(t) \equiv 0\).

Here, we aim to prove \(f(T-) = \lim_{t \uparrow T} f(t) = k\). Otherwise, if \(f(T-) > k\), then there exists a contradiction that
\[
0 = -v_t - \frac{a^2}{2} y^2 v_{yy} - (\beta - r) y v_y + \beta v
= -v_t - \frac{a^2}{2} y^2 \psi_{yy} - (\beta - r) y \psi_y + \beta \psi
= -\partial_t v + \frac{1}{\gamma} K > 0, \quad k < y < f(T-), \quad t = T.
\]
So \(f(T-) = k\). The proof of \(g(T-) = y_T\) is similar that if \(g(T-) > y_T\), there exists contradiction
\[
0 = -v_t - \frac{a^2}{2} y^2 v_{yy} - (\beta - r) y v_y + \beta v
= -v_t - \frac{a^2}{2} y^2 \psi_{yy} - (\beta - r) y \psi_y + \beta \psi
= -\partial_t v + \Psi(y) > 0, \quad g(T-) < y < y_T, \quad t = T.
\]
If \(\beta = \gamma - r\) and \(\Psi(k) < 0\), then \(K < b\) as well as \(\Psi(y) < 0\) for all \(0 < y < k\), thus \((0, k] \times (0, T) \subset CR_y\), so \(h(t), \ g(t)\) do not exist.

5. The solution and the free boundary of (12).

Lemma 5.1. We have
\[v_{yy} > 0, \quad 0 < y < f(t), \quad 0 < t < T.\] \hfill (32)

Proof. Apply the strong maximum principle,
\[v_{yy} > 0 \text{ in } CR_y,\]



together with
\[v_{yy} = \psi'' > 0 \text{ in } ER_y \cap \left((0, k) \times (0, T)\right),\]

then (32) is true. \hfill \Box

Lemma 5.2. We have
\[\lim_{y \to 0^+} v_y(y, t) = -\infty, \quad 0 < t < T,\] \hfill (33)
\[\lim_{y \to f(t)} v_y(y, t) = 0, \quad 0 < t < T.\] \hfill (34)

Proof. For any \(t \in (0, T)\), it is not hard to see that \(\lim_{y \to 0^+} v(y, t) \geq \lim_{y \to 0^+} \psi(y) = +\infty\).

And by \(v_{yy} \geq 0\), for a fixed \(y_0 > 0\),
\[v_y(y, t) \leq \frac{v(y_0, t) - v(y, t)}{y_0 - y} \to -\infty, \quad y \to 0^+.\]

And (34) is due to \(v_y\) is continuously through the free boundary \(y = f(t)\). \hfill \Box
By Lemma 5.1 and Lemma 5.2, we can define a transformation
\[ y = J(x, t) = \begin{cases} (v_y(\cdot, t))^{-1}(-x), & \text{for } x > 0; \\ f(t), & \text{for } x = 0, \end{cases} \]
then \( J(x, t) \in C([0, +\infty) \times (0, T]) \) and is decreasing to \( x \).

**Lemma 5.3.** We have
\[ \lim_{x \to 0^+} J(x, t) = f(t), \quad 0 < t < T, \]  \( (35) \)
\[ \lim_{x \to +\infty} J(x, t) = 0, \quad 0 < t < T, \]  \( (36) \)
\[ \lim_{t \uparrow T} J(x, t) = \varphi'(x), \quad x \geq 0. \]  \( (37) \)

**Proof.** Clearly (35) and (36) are the results of Lemma 5.2. Now we prove (37).

**The case of** \( x > \hat{x} \). Owing to the regularity of \( v_y \) on \( t = T \),
\[ \lim_{t \uparrow T} v_y(y, t) = \psi'(y), \quad 0 < y < k. \]
Notice that \( \psi'(y) \) maps onto \(( -\infty, -\hat{x})\) for \( 0 < y < k \), and \( \psi''(y) > 0, \ 0 < y < k \), thus
\[ \lim_{t \uparrow T} J(x, t) = \lim_{t \uparrow T} (v_y(\cdot, t))^{-1}(-x) = (\psi'(\cdot))^{-1}(-x) = \varphi'(x), \quad x > \hat{x}. \]  \( (38) \)

**The case of** \( 0 \leq x \leq \hat{x} \). Due to \((v_y(\cdot, t))^{-1}(-x)\) is decreasing to \( x \) for all \( t \in (0, T) \),
\[ (v_y(\cdot, t))^{-1}(-x) \leq (v_y(\cdot, t))^{-1}(0) = f(t). \]  \( (39) \)
If \( y < k \), then \( \lim_{t \uparrow T} v_y(y, t) = \psi'(y) < -\hat{x} \leq -x \), so when \( t \) is sufficiently close to \( T \),
\[ v_y(y, t) < -x, \]
thus
\[ (v_y(\cdot, t))^{-1}(-x) > y, \]
hence
\[ \lim\inf_{t \uparrow T} (v_y(\cdot, t))^{-1}(-x) \geq y. \]
Since \( y < k \) is arbitrary, we see that
\[ \lim\inf_{t \uparrow T} (v_y(\cdot, t))^{-1}(-x) \geq k. \]
Together with (39),
\[ k \leq \lim\inf_{t \uparrow T} (v_y(\cdot, t))^{-1}(-x) \leq \lim\sup_{t \uparrow T} (v_y(\cdot, t))^{-1}(-x) \leq \lim_{t \uparrow T} f(t) = k, \]
thus
\[ \lim_{t \uparrow T} J(x, t) = \lim_{t \uparrow T} (v_y(\cdot, t))^{-1}(-x) = k = \varphi'(x), \quad 0 \leq x \leq \hat{x}. \]  \( (40) \)
Now (37) follows from (38) and (40).

We set
\[ \tilde{V}(x, t) = \min_{y > 0} (v(y, t) + xy). \]  \( (41) \)
Theorem 5.4. The function \( \hat{V} \) is a strong solution of (12) and satisfies the following
\[
\hat{V}, \hat{V}_x, \hat{V}_t \in C(Q_x),
\]
(42)
\[
\hat{V}_{xx} \in C(Q_x \setminus \partial(CR_x)).
\]
(43)
Moreover,
\[
\hat{V}_t \leq 0, \quad \hat{V}_x > 0, \quad \hat{V}_{xx} < 0, \quad (x, t) \in Q_x,
\]
(44)
\[
\lim_{x \to +\infty} \hat{V}(x, t) = +\infty, \quad \lim_{x \to +\infty} \hat{V}_x(x, t) = 0, \quad \forall t \in (0, T).
\]
(45)
Proof. From Lemma 5.1 and Lemma 5.2, it is easily seen that \( J(x, t) \in \arg \min_{y \geq 0} (v(y, t) + xy) \) for all \((x, t) \in Q_x\), thus
\[
\hat{V}(x, t) = v(J(x, t), t) + xJ(x, t), \quad (x, t) \in Q_x.
\]
In addition,
\[
\hat{V}_x(x, t) = v_y(J(x, t), t)J_x(x, t) + xJ_x(x, t) + J(x, t) = J(x, t) \geq 0,
\]
(46)
\[
\hat{V}_{xx}(x, t) = J_x(x, t) = \partial_x[(v_y(\cdot), t)^{-1}(x)] = -\frac{1}{v_{yy}(J(x, t), t)} < 0,
\]
(47)
\[
\hat{V}_t(x, t) = v_y(J(x, t), t)J_t(x, t) + v_t(J(x, t), t) + xJ_t(x, t) = v_t(J(x, t), t) \leq 0,
\]
(48)
Using \( J(x, t) \in C(Q_x) \), one can prove \( \hat{v}_t(y, t) \in C(Q_y), \hat{v}_{yy}(y, t) \in C(Q_y \setminus \partial(CR_y)) \) (see [9]), so (42) and (43) are true. Moreover,
\[
\lim_{x \to +\infty} \hat{V}(x, t) = \lim_{x \to +\infty} [v(J(x, t), t) + xJ(x, t)]
\geq \lim_{x \to +\infty} v(J(x, t), t) = \lim_{y \to 0^+} v(y, t) = +\infty, \quad 0 < t < T,
\]
\[
\lim_{x \to +\infty} \hat{V}_x(x, t) = \lim_{x \to +\infty} J(x, t) = 0, \quad 0 < t < T.
\]
Here, we verify \( \hat{V} \) is the strong solution of (12). First,
\[
\lim_{x \to 0^+} \hat{V}(x, t) = \lim_{x \to 0^+} [v(J(x, t), t) + xJ(x, t)] = v(f(t), t) = \frac{1}{\gamma}K^\gamma, \quad t \in (0, T),
\]
\[
\lim_{t \to T} \hat{V}(x, t) = \lim_{t \to T}[v(J(x, t), t) + xJ(x, t)]
\geq v(\lim_{t \to T} J(x, t), T) + x \lim_{t \to T} J(x, t)
\geq \psi(\varphi'(x)) + x\varphi'(x)
\geq \psi(x), \quad x \geq 0,
\]
so \( \hat{V} \) meets the boundary and terminal conditions in (12).
Second, due to (46), (47) and (48),
\[
\left( -\hat{V}_t + \frac{a^2}{2} \hat{V}_{xx}^2 - r_x\hat{V}_x + \beta \hat{V} \right)(x, t)
= \left( -v_t - \frac{a^2}{2} y^2 v_{yy} - (\beta - r) y v_y + \beta v \right)(J(x, t), t) \geq 0.
\]
(49)
On the other hand,
\[ v(y, t) \geq \psi(y), \; \forall \; y > 0 \]
\[ \Rightarrow \min_{y > 0} (v(y, t) + xy) \geq \min_{y > 0} (\psi(y) + xy), \; \forall \; x \geq 0 \]
\[ \Rightarrow \hat{V}(x, t) \geq \varphi(x), \; \forall \; x \geq 0. \]

Hence,
\[ \min \left\{ -\hat{V}_t + \frac{a^2}{2} \frac{\hat{V}^2}{V_{xx}} - rx\hat{V}_x + \beta\hat{V}, \hat{V} - \varphi \right\} \geq 0 \quad \text{in} \quad Q_x. \]

Now, we prove that
\[ \hat{V}(x, t) > \varphi(x) \Rightarrow \left( -\hat{V}_t + \frac{a^2}{2} \frac{\hat{V}^2}{V_{xx}} - rx\hat{V}_x + \beta\hat{V} \right)(x, t) = 0. \quad (50) \]

We now claim
\[ \hat{V}(x, t) > \varphi(x) \Rightarrow v(J(x, t), t) > \psi(J(x, t)). \quad (51) \]

Let \( y = J(x, t) \), if \( v(y, t) = \psi(y) \) for \( y \geq k \), then
\[ v(y, t) = \psi(y) = \frac{1}{\gamma} K^\gamma \]
\[ \Rightarrow v_y(y, t) = 0 \]
\[ \Rightarrow x = 0 \]
\[ \Rightarrow \hat{V}(x, t) = \frac{1}{\gamma} K^\gamma = \varphi(x). \]

If \( v(y, t) = \psi(y) \) for \( y < k \), then
\[ v(y, t) = \psi(y) = \frac{1 - \gamma}{\gamma} y^\gamma t + (K - b)y \]
\[ \Rightarrow x = -v_y(y, t) = y^\gamma t - (K - b) \]
\[ \Rightarrow \hat{V}(x, t) = v(y, t) + xy = \frac{1 - \gamma}{\gamma} y^\gamma t + (K - b)y + (y^\gamma t - (K - b))y = \varphi(x). \]

Hence, (51) is true.

Together with (51), (15) and (49) yield
\[ \hat{V}(x, t) > \varphi(x) \]
\[ \Rightarrow v(J(x, t), t) > \psi(J(x, t)) \]
\[ \Rightarrow \left( -\hat{V}_t + \frac{a^2}{2} \frac{\hat{V}^2}{V_{xx}} - \beta\hat{V} \right)(J(x, t), t) = 0 \]
\[ \Rightarrow \left( -\hat{V}_t + \frac{a^2}{2} \frac{\hat{V}^2}{V_{xx}} - rx\hat{V}_x + \beta\hat{V} \right)(x, t) = 0. \]

Therefore, \( \hat{V}(x, t) \) satisfies the variational inequality in (12). Thus, \( \hat{V}(x, t) \) is a strong solution of (12). \( \square \)

Now, we discuss the free boundaries of (12). Define
\[ \mathcal{ER}_x = \{ \hat{V} = \varphi \}, \; \text{exercise region}, \]
\[ \mathcal{CR}_x = \{ \hat{V} > \varphi \}, \; \text{continuation region}. \]

And let
\[ H(t) = \sup \{ x \geq 0 | \hat{V}(x, t) = \varphi(x) \}, \quad 0 < t < T, \]
\[ G(t) = \inf \{ x \geq 0 | \hat{V}(x, t) = \varphi(x) \}, \quad 0 < t < T. \]

On the two free boundaries \( y = h(t) \) and \( y = g(t) \),
\(v(y, t) = \frac{1 - \gamma}{\gamma} y^{\frac{1}{\gamma}} + (K - b)y,\)
\(v_y(y, t) = -y^{\frac{1}{\gamma}} + (K - b).\)

Note that
\(x = -v_y(y, t).\) \(\tag{52}\)

Then the corresponding two free boundaries of (12) are
\(H(t) = -v_y(h(t), t) = h(t)^{\frac{1}{\gamma}} - (K - b),\)
\(G(t) = -v_y(g(t), t) = g(t)^{\frac{1}{\gamma}} - (K - b).\)

Moreover,
\(H'(t) = \frac{1}{\gamma - 1} h(t) - h' - (K - b) \geq 0,\)
\(G'(t) = \frac{1}{\gamma - 1} g(t) - g' - (K - b) \leq 0,\)

and
\(H(T^-) = h(T^-)^{\frac{1}{\gamma}} - (K - b),\)
\(G(T^-) = g(T^-)^{\frac{1}{\gamma}} - (K - b).\)

On the other hand, by (46) and (35),
\(\hat{V}_x(0, t) = J(0, t) = (v(\cdot, t))^{-1}(0) = f(t).\)

The above analysis leads to

**Theorem 5.5.** The two free boundaries of (12) \(H(t), G(t)\) are in \(C^\infty(0, T),\) and \(H'(t) \geq 0, G'(t) \leq 0, \hat{V}_x(0, t) = f(t).\) Moreover, they have the following classifications.

**Case I.** \(\beta \geq \frac{a^2}{2} \frac{\gamma}{1 - \gamma} + r\gamma, \Psi(k) \geq 0, \) \(\) (Fig 5.1)
\(H(t) \equiv +\infty, \; k^{\frac{1}{\gamma}} - (K - b) = G(T^-) \leq G(t),\)
\(\) i.e.
\(H(t) \equiv +\infty, \; \hat{x} = G(T^-) \leq G(t).\)

![Fig 5.1](image1)

**Fig 5.1.** \(\beta \geq \frac{a^2}{2} \frac{\gamma}{1 - \gamma} + r\gamma, \Psi(k) \geq 0.\)

**Case II.** \(\beta \geq \frac{a^2}{2} \frac{\gamma}{1 - \gamma} + r\gamma, \Psi(k) < 0.\)

If \(\beta > \frac{a^2}{2} \frac{\gamma}{1 - \gamma} + r\gamma, \) \(\Psi(k) < 0.\) \(\) (Fig 5.2)
\( H(t) \equiv +\infty, \quad y_T = \left( \frac{-r(K-b)}{2 - r\gamma} - \frac{a^2}{2} \right) - (K-b) = G(T-) < G(t), \)

If \( \beta = \frac{a^2}{2} \frac{1}{1-\gamma} + r\gamma, \) (Fig 5.4) \( \mathcal{E}R_x \neq \emptyset. \)

**Case III.** \( \beta < \frac{a^2}{2} \frac{1}{1-\gamma} + r\gamma, \) \( \Psi(k) > 0, \) (Fig 5.3)

\( k^{\frac{1}{\gamma-r}} - (K-b) = G(T-) \leq G(t) \leq H(t) \leq H(T-) = \left( \frac{-r(K-b)}{2 - r\gamma} - \frac{a^2}{2} \frac{1}{1-\gamma} \right) - (K-b). \)

**Case IV.** \( \beta < \frac{a^2}{2} \frac{1}{1-\gamma} + r\gamma, \) \( \Psi(k) \leq 0, \) (Fig 5.4)

\[ \mathcal{E}R_x = \emptyset. \]

![Fig 5.3](image1)

![Fig 5.4](image2)

**6. Verification theorem.** In this section, we will prove that \( \hat{V}(x,t) \) defined by (41), which is a solution to (12), coincides with the value function \( V(x,t) \) defined by (3).

Up to now, we have proved that \( \hat{V}(x,t) \) admits the following conditions:

(i) it is a strong solution to (9) and satisfies (see (42))

\[ \hat{V}(x,t), \hat{V}_x(x,t), \hat{V}_t(x,t) \in C(Q_x), \quad \hat{V}_{xx}(x,t) \in C(Q_x \setminus \partial(CR_x)). \]

(ii) There exists a constant \( C \) such that

\[ |\hat{V}(x,t)| \leq C(1 + |x|^2), \quad \forall(x,t) \in Q_x, \]

which can be deduced from (16).

Furthermore, we can prove that

(iii) There exists a measurable function \( \hat{\pi}(x,t) \) such that

\[ -\hat{V}_t - \mathcal{L}^{\hat{\pi}}(x,t)\hat{V}(x,t) = \hat{V}_t(x,t) - \max_{\pi} \mathcal{L}^{\pi}\hat{V}(x,t) = 0, \quad \text{if} \quad \hat{V}(x,t) > \varphi(x), \quad (53) \]

where

\[ \mathcal{L}^{\pi}\hat{V}(x,t) := \frac{1}{2} \pi'\sigma'\pi_s \hat{V}_{xx}(x,t) + (rX_s + \mu'\pi_s)\hat{V}_x(x,t) - \beta\hat{V}(x,t). \]

And the following SDE

\[
\begin{align*}
\left\{ \right. \\
\begin{array}{l}
dX_s = (rX_s + \mu'\hat{\pi}(X_s,s))ds + \hat{\pi}(X_s,s)'\sigma dW_s, \\
X_t = x
\end{array} \\
\left. \right\} \quad \text{for} \quad t \leq s \leq \tau, \\
(54)
\]

and the stopping time

\[ \tau = \inf \{ s \geq t : \hat{V}(X_s, s) = \varphi(X_s, s) \}, \quad (55) \]
have a unique strong solution \((X^*_s, \tau^*)\).

Indeed, define
\[
\hat{\pi}(x, t) := -(\sigma \sigma')^{-1} \frac{\hat{V}_x(x, t)}{\hat{V}_{xx}(x, t)}.
\]

One can prove \(v(y, t) \in C^{\infty, \infty}(\mathcal{CR}_y)\) (see [17]), so \(\hat{V}(x, t) \in C^{\infty, \infty}(\mathcal{CR}_x)\), together with \(\hat{V}_{xx} < 0\) (see (44)), we have \(\hat{\pi}(x, t) \in C^{\infty, \infty}(\mathcal{CR}_x)\). Define
\[
\mathcal{CR}^n := \{(x, t) \in \left[\frac{1}{n}, n\right] \times [0, T - \frac{1}{n}] : \hat{V}(x, t) \geq \varphi(x) + \frac{1}{n}\},
\]
then \(\hat{\pi}(x, t)\) is Lipschitz continuous w.r.t. \(x\) in \(\mathcal{CR}^n\). So (54) and the stopping time
\[
\tau = \inf\{s \geq t : (X_s, s) \notin CR^n\},
\]
have a unique strong solution \((X^n_s, \tau^n)\) (see [23]), and satisfy \(X^n_m = X^n_s\), \(s \leq \tau^n \leq \tau^m\) for \(n < m\). Let \(\tau^\ast := \lim_{n \to \infty} \tau^n\), \(X^\ast_s := X^n_s\) if \(s \leq \tau^n \leq \tau^\ast\), then we have \((X^\ast_s, \tau^\ast)\) admits (54) and (55).

Now, we formulate a verification theorem as follows.

**Theorem 6.1.** Let \(\hat{V}(x, t)\) admit Conditions (i)-(iii) in the beginning of this section. Then
\[
\hat{V}(x, t) = V(x, t), \quad \forall (x, t) \in Q_x,
\]
Furthermore, \(\pi^\ast := \hat{\pi}(X^\ast_s, s)\) and \(\tau^\ast\) are the feedback control and optimal stopping time.

In order to prove Theorem 6.1, we need the following lemmas.

**Lemma 6.2.** For the value function \(V(x, t)\) defined in (3), there exists \(C_T > 0\) depending on \(T\), such that
\[
V(x, t) \leq C_T(1 + x^\gamma)
\]

**Proof.** Note that
\[
V(x, t) = \sup_{\pi, \tau} E_{t,x}[e^{-\beta(\tau-t)}g(X_\tau)]
\]
\[
= \sup_{\pi, \tau} E_{t,x}\left[\frac{e^{-\beta(\tau-t)}}{\gamma}((X_\tau - b)^+ + K)^\gamma\right]
\]
\[
\leq \sup_{\pi, \tau} E_{t,x}\left[\frac{e^{-\beta(\tau-t)}}{\gamma}X_\tau^\gamma\right] + \frac{1}{\gamma}K^\gamma.
\]

Note that \(\sup E_{t,x}\left[e^{-\beta(\tau-t)}\frac{1}{\gamma}X_\tau^\gamma\right]\) is the value function in the situation \(b = K = 0\), by Remark 2, we complete the proof. \(\square\)

**Lemma 6.3.** We have
\[
V(x, t) \geq \sup_{\pi, \tau} E_{t,x}\left[e^{-\beta(\tau-t)}g(X_\tau) \chi_{\{\tau < T\}} + e^{-\beta(T-t)}\varphi(X_T) \chi_{\{\tau = T\}}\right].
\]

**Proof.** Here, we introduce the following so-called dynamic programming principle. For any stopping time \(\theta\) \((t \leq \theta \leq T)\),
\[
V(x, t) = \sup_{\pi, \tau} E_{t,x}\left[e^{-\beta(\tau-t)}g(X_\tau) \chi_{\{\theta > \tau\}} + e^{-\beta(\theta-t)}V(X_\theta, \theta) \chi_{\{\theta \leq \tau\}}\right]. \tag{56}
\]
Owing to (56), for any admissible \((\pi_s, \tau)\),
\[
V(x, t) \geq E_{t,x}\left[e^{-\beta(\tau-t)}g(X_\tau) \chi_{\{\theta > \tau\}} + e^{-\beta(\theta-t)}V(X_\theta, \theta) \chi_{\{\theta \leq \tau\}}\right]. \tag{57}
\]
Choosing $\theta = T - \varepsilon$ in (57), letting $\varepsilon \to 0$, and using the dominated convergence theorem, Lemma 6.2 and Lemma 2.1, we have
\[ V(x,t) \geq E_{t,x} \left[ e^{-\beta(t-s)} g(X_s) \chi_{\{T < \tau\}} + e^{-\beta(T-t)} \varphi(X_T) \chi_{\{\tau = T\}} \right]. \]
Since $(\pi, \tau)$ is arbitrary, we obtain the desired result.

We now go back to prove Theorem 6.1:

Proof. For any admissible $\pi$, and stopping time $\tau (t \leq \tau \leq T)$, by Itô’s formula (after an eventual localization for removing the stochastic integral term in the expectation),
\[
\hat{V}(x,t) = E_{t,x}[e^{-\beta(T-t)}\hat{V}(X_T, \tau)] + E_{t,x}\left[ \int_t^T e^{-\beta(s-t)}(-\hat{V}_t - \mathcal{L}\hat{V})(X_s, s)ds \right] \\
\geq E_{t,x}[e^{-\beta(T-t)}\hat{V}(X_T, \tau)] + E_{t,x}\left[ \int_t^T e^{-\beta(s-t)}(-\hat{V}_t - \sup_{\pi} \mathcal{L}\hat{V})(X_s, s)ds \right] \\
\geq E_{t,x}[e^{-\beta(T-t)}\varphi(X_T)] \\
\geq E_{t,x}[e^{-\beta(T-t)}g(X_T)].
\]
Since $(\pi, \tau)$ is arbitrary, we have $\hat{V}(x,t) \geq V(x,t)$.

On the other hand, by Itô’s formula,
\[
\hat{V}(x,t) = E_{t,x}[e^{-\beta(T-t)}\hat{V}(X_T^*, \tau^*)] + E_{t,x}\int_t^{\tau^*} e^{-\beta(s-t)}[-\hat{V}_t - \mathcal{L}\hat{V}](X_s^*, s)ds.
\]
Owing to (53), we have $\hat{V}(x,t) = E_{t,x}[e^{-\beta(T-t)}\varphi(X_T^*)]$. Note that by the result in Section 5, $\{\hat{V}(x,t) = \varphi(x)\} = \{\hat{V}(x,t) = g(x)\}$ for any $t < T$, thus
\[
\hat{V}(x,t) = E_{t,x}[e^{-\beta(T-t)}\varphi(X_T^*)\chi_{\{\tau^* < T\}} + \varphi(X_T^*)\chi_{\{\tau^* = T\}}] \\
= E_{t,x}[e^{-\beta(T-t)}g(X_T^*)\chi_{\{\tau^* < T\}} + \varphi(X_T^*)\chi_{\{\tau^* = T\}}].
\]
By Lemma 6.3, we have $\hat{V}(x,t) \leq V(x,t)$.

7. Conclusions. In this paper, we present a new method to study the free boundaries while the exercise region is not connected (see (24)-(26) and Lemma 4.2) so that we can shed light on the behaviors of the free boundaries for a fully nonlinear variational inequality without any restrictions on parameters (see Figure 5.1-5.4.). The financial meaning is that if at time $t$, the portfolio’s wealth $x$ is located in $\mathcal{CR}_x$, then the manager should go on investing; and if $x$ is located in $\mathcal{ER}_x$, then he should stop investing.

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