Universal instability of hairy black holes in Lovelock-Galileon theories in $D$ dimensions

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Abstract

We analyze spherically symmetric black hole solutions with time-dependent scalar hair in a class of Lovelock-Galileon theories, which are the scalar-tensor theories with second-order field equations in arbitrary dimensions. We first show that known black hole solutions in five dimensions are always plagued by the ghost/gradient instability in the vicinity of the horizon. We then generalize such black hole solutions to higher dimensions and show that the same instability found in five dimensions appears universally in any number of dimensions.

1 Introduction

In four-dimensional spacetime, an arbitrary linear combination of the Ricci scalar and a constant defines the Lagrangian density for the unique covariant metric theory for which field equations for the metric are of second-order differential equations [1]. When we extend this theory by including a scalar field and still require the second-order nature of the field equations both for the metric and the scalar field, we obtain the theory space spanned by the four arbitrary functions of the kinetic term of the scalar field and the scalar field itself [2, 3, 4, 5]. This space is now known as the Horndeski or generalized Galileon class.

One of the interesting topics to address is to find the black hole (BH) solutions in the Horndeski class and to check if scalar hair can exist or not. From this perspective, there is an interesting subclass in the Horndeski class for which the scalar field appears in the Lagrangian density only through the contraction of the rank-2 tensor $\nabla_\mu \phi \nabla_\nu \phi$ with the metric and the Einstein tensor. This Lagrangian has a manifest shift symmetry $\phi \to \phi + \text{const}$. A no-hair theorem for general scalar-tensor theories possessing the shift symmetry has been established in [6], although some assumptions made in proving the theorem could be violated [7]. Actually, for the above mentioned theories having coupling between $\nabla_\mu \phi \nabla_\nu \phi$ and the Einstein tensor, static and spherically symmetric BH solutions with nontrivial scalar profile have been derived in [8] [9] [10] [11] [12].
Electrically charged BHs have also been found in [13]. In particular, in [9], it was found that the theory allows the scalar field to depend linearly on time with nontrivial radial profile in the static and spherically symmetric BH spacetime. Thus, a BH can have time-dependent scalar hair. Such a linearly time-dependent scalar field was shown to be allowed for a wider class of theories in [14]. Although the scalar field diverges at the horizon for a fixed time coordinate, this divergent part and the term linear in time can be combined together into the ingoing (or outgoing) Eddington-Finkelstein coordinate. Thus, when the scalar field is expressed in terms of the Eddington-Finkelstein and the radial coordinate, its radial part remains finite even at the horizon. This means that any freely infalling observer records a finite value of the scalar field even at the horizon and nothing singular happens there. In this sense, the scalar field is regular on the horizon. In addition to this interesting property, this theory has the following intriguing mathematical property: the derivative of the scalar field $\nabla_\mu \phi \nabla_\nu \phi$ couples to the tensors derived from the variation of the constant and the Ricci scalar, which are the only quantities in the Lagrangian density giving second-order field equations for the metric. In other words, there is a well-defined one-to-one correspondence between the Lagrangian density involving only the metric which results in second order field equations and the rank-2 tensor to which $\nabla_\mu \phi \nabla_\nu \phi$ couples. In four dimensions, there are only two such pairs. The first pair is a constant and $g_{\mu \nu}$ and the second one is the Ricci scalar $R$ and the Einstein tensor $G_{\mu \nu}$.

This correspondence in four-dimensional spacetime can be straightforwardly extended to higher dimensions. It is known that the most general Lagrangian density involving only the metric giving rise to second-order differential equations for the metric is given by an arbitrary linear combination of the Lovelock invariants [15]. The number $N$ of the independent Lovelock invariants depends on the dimension of spacetime [see Eq. (4)]. Then, from the correspondence mentioned above it is natural to consider the coupling between $\nabla_\mu \phi \nabla_\nu \phi$ and the rank-2 tensors obtained from variation of the Lovelock invariants with respect to the metric. The Lagrangian density of the theory constructed in this way consists of $N$ Lovelock invariants and their $N$ counterparts. Therefore, the theory is specified by the $2N$ free parameters. In the rest of the paper, we call such a theory the Lovelock-Galileon theory. It turns out that the field equations both for the metric and the scalar field are of second order [16].

Recently, BH solutions with scalar hair for the Lovelock-Galileon theory have been found in five-dimensional spacetime where the Gauss-Bonnet combination constitutes the third Lovelock invariant [16]. Similarly to the case in four dimensions, the BH allows the scalar field to depend linearly on time.

Once a solution is obtained, it is quite natural to ask whether the solution in the Lovelock-Galileon theory is stable or not. As for the case in four dimensions, this issue has been addressed recently in [17]. Interestingly, it was found that irrespective of the choice of the parameters in the Lagrangian, BH spacetime with linearly time-dependent scalar hair suffers from ghost or gradient instabilities in the vicinity of the event horizon. In this paper, we will show that the solution in five-dimensional spacetime given in [16] has the same type of instabilities when the scalar field depends linearly on time. To see whether this instability is universally inherent independently of the dimensionality of spacetime or a special feature limited only to four and five dimensions, we further consider BH solutions in generic Lovelock-Galileon theories in higher dimensions. We show that, in any number of dimensions, as is the case in four and five dimensions, the Lovelock-Galileon theory admits solutions for which the metric is static and spherically symmetric but the scalar field has a term linear in time in addition to the piece dependent only on the radial
coordinate. However, it turns out that our generalized solutions also have the aforementioned instability. This means that the instability is not particular to four and five dimensions, but arises regardless of the dimensionality of spacetime in the Lovelock-Galileon class of scalar-tensor theories.

This paper is organized as follows. In Sec. 2 we introduce Lovelock-Galileon theory and see that the field equations are of second order. In Sec. 3 we review the BH solution given in [16] and show its instability. Then, we generalize these results to general Lovelock-Galileon theories in arbitrary dimensions in Sec. 4. Finally, we draw our conclusions in Sec. 5.

2 Lovelock-Galileon theory

Lovelock theory [15] is one of the natural extensions of general relativity in arbitrary dimensions, whose action is written as

\[ S_L = \int d^D x \sqrt{-g} \sum_{n=0}^{M} a_n \mathcal{R}^{(n)}, \]

where \( D \geq 4 \) is the spacetime dimension, \( a_n \)'s are constants, and

\[ \mathcal{R}^{(n)} = \frac{1}{2^n} \delta^{\alpha_1 \alpha_2 \cdots \alpha_{2n-1} \alpha_{2n}}_{\beta_1 \beta_2 \cdots \beta_{2n-1} \beta_{2n}} R^{\beta_1 \alpha_2 \cdots \beta_{2n-1} \alpha_{2n}}_{\alpha_1 \beta_2 \cdots \beta_{2n-1} \beta_{2n}} \]

is the Lovelock invariant of \( n \)th order. Here \( \delta^{\alpha_1 \alpha_2 \cdots \alpha_p}_{\beta_1 \beta_2 \cdots \beta_p} \) denotes the generalized Kronecker delta:

\[ \delta^{\alpha_1 \alpha_2 \cdots \alpha_p}_{\beta_1 \beta_2 \cdots \beta_p} \equiv p! \delta^{\alpha_1 \alpha_2 \cdots \alpha_p}_{[\beta_1 \beta_2 \cdots \beta_p]} \]

One can easily find that \( \mathcal{R}^{(1)} \) is nothing but the Ricci scalar. For \( n = 0 \), it is natural to define \( \mathcal{R}^{(0)} \equiv 1 \), so that it serves as a cosmological constant. For \( 2n > D \), the Lovelock invariants are identical to zero since there exists at least one spacetime coordinate that appears more than once in the indices of the generalized delta. Note also that, for \( 2n = D \), \( \mathcal{R}^{(n)} \) is a topological invariant and so it does not contribute to the equation of motion. We therefore take

\[ M = \left\lfloor \frac{D - 1}{2} \right\rfloor, \]

where \( \lfloor \cdot \rfloor \) is the floor function, and thus there is only a finite number \( N = M + 1 \) of the independent Lovelock invariants. So, it follows that Lovelock theory reduces to general relativity with a cosmological constant in four dimensions. Varying the action (1) with respect to the metric gives us the following equation of motion:

\[ \sum_{n=0}^{M} a_n H^{(n)}_{\mu \nu} = 0, \]

where \( H^{(n)}_{\mu \nu} \) is the Lovelock tensor of \( n \)th order:

\[ H^{(n)}_{\mu \nu} \equiv \frac{1}{2^{n+1}} \delta^{\alpha_1 \alpha_2 \cdots \alpha_{2n-1} \alpha_{2n}}_{\beta_1 \beta_2 \cdots \beta_{2n-1} \beta_{2n}} R^{\beta_1 \beta_2 \cdots \beta_{2n-1} \beta_{2n}}_{\alpha_1 \alpha_2 \cdots \alpha_{2n-1} \beta_{2n}}. \]
n obviously the field equations in Lovelock-Galileon theory are also of second order. Where $c$ is a constant. The action is invariant under the shift of the scalar field: $\phi \rightarrow \phi + c$, where $c$ is a constant. The equations of motion are written as follows:

$$\sum_{n=0}^{M} \left( a_n H_{\mu\nu}^{(n)} + b_n E_{\mu\nu}^{(n)} \right) = 0,$$

(9)

$$\sum_{n=0}^{M} b_n J^{(n)}_{\alpha} : \alpha = 0,$$

(10)

where

$$E_{\mu\nu}^{(n)} \equiv -\frac{1}{2} g_{\mu\nu} H^{(n)\alpha\beta} \phi;_{\alpha} \phi;_{\beta} + H^{(n)}_{\alpha} (\mu ; \nu) \phi;_{\alpha}$$

$$- \frac{n}{2^{n+1}} g_{\mu \nu} (\mu ; \nu) \delta^{a_2 a_2 \ldots a_2 \ldots a_2}_{b_2 b_2 \ldots b_2 \ldots b_2} R^{\lambda \beta_2} R^{\beta_3 \beta_4} \ldots R^{\beta_{2n-1} \beta_{2n}} \phi;_{\alpha} \phi;_{\beta},$$

$$- \frac{n}{2^{n+1}} g_{\mu \nu} (\mu ; \nu) \delta^{a_2 a_2 \ldots a_2 \ldots a_2}_{b_2 b_2 \ldots b_2 \ldots b_2} R^{\beta_3 \beta_4} \ldots R^{\beta_{2n-1} \beta_{2n}} \phi;_{\alpha} \phi;_{\beta} \phi;_{\lambda},$$

$$- \frac{n}{2^{n+1}} g_{\mu \nu} (\mu ; \nu) \delta^{a_2 a_2 \ldots a_2 \ldots a_2}_{b_2 b_2 \ldots b_2 \ldots b_2} R^{\beta_3 \beta_4} \ldots R^{\beta_{2n-1} \beta_{2n}} \phi;_{\alpha} \phi;_{\beta},$$

(11)

$$J^{(n)}_{\alpha} \equiv -H^{(n)\alpha\beta} \phi;_{\alpha \beta},$$

(12)

Obviously, the field equations in Lovelock-Galileon theory are also of second order. This fact again follows from the Bianchi identity.

### 3 Instability of the Einstein-Gauss-Bonnet-Galileon black holes

A static and spherically symmetric BH solution for the action (7) in five dimensions was obtained in [16]. Here we rederive the solution and discuss its stability under a tensor perturbation.

Since we have three independent Lovelock invariants for $D = 5$, we consider the following Einstein-Gauss-Bonnet-Galileon action:

$$S = \int d^{5}x \sqrt{-g} \left[ a_0 + a_1 R + a_2 R^{(2)} \right] - \frac{b_0}{2} \phi;_{\alpha} \phi;^{\alpha} + b_1 G^{\alpha\beta} \phi;_{\alpha} \phi;_{\beta} + b_2 H^{(2)\alpha\beta} \phi;_{\alpha} \phi;_{\beta},$$

(13)

This second-order nature of the field equations is not violated even if the coefficients $a_n$ and $b_n$ are functions of $\phi$. 

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where
\[ R^{(2)} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}, \]
\[ H^{(2)}_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R^{(2)} + 2RR_{\mu\nu} - 4R_{\mu\alpha}R^\alpha_{\nu} + 4R_{\mu\alpha\beta\gamma}R^{\alpha\beta\gamma} + 2R_{\mu\alpha\beta\gamma}R^{\alpha\beta\gamma}. \]

In [16], the form of the metric was chosen as
\[ ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2\bar{\gamma}_{ij}dx^i dx^j, \]
where \( \bar{\gamma}_{ij} \) represents the metric of a three-dimensional maximally symmetric space with spatial curvature \( \kappa \). It was also assumed that \( \phi \) has the form of
\[ \phi(t, r) = qt + \psi(r), \]
with \( q \neq 0 \) being a constant. Nonvanishing \( q \) is essential for BH solutions with regular scalar hair. Although the scalar field is time dependent, the metric can be static thanks to the shift symmetry of the theory.\(^2\)

Let us write down the equations of motion. The scalar equation reads
\[ -\frac{b_0}{6} + b_1 \left( \frac{ff'}{2rh} - F \right) + 2b_2 \frac{fh'}{rh} F = 0, \]  
and the metric equation yields
\[ -\frac{1}{6} \left( a_0 + \frac{b_0q^2}{2h} \right) + \left( a_1 - \frac{b_1q^2}{2h} \right) \frac{fh'}{2rh} - \left( a_1 + \frac{b_1q^2}{2h} \right) F \\
+ \left( a_2 - \frac{b_2q^2}{2h} \right) \frac{2fh'}{r^2} F + f^2\psi'^2 \left[ \frac{b_1}{2} \left( \frac{h'}{rh} + \frac{2}{r^2} \right) + 2b_2 \left( \frac{h'}{rh} F - \frac{fh'}{r^3} \right) \right] = 0, \]
\[ -\frac{1}{6} \left( a_0 - \frac{b_0q^2}{2h} \right) + \left( a_1 - \frac{b_1q^2}{2h} \right) \left( \frac{f'}{2r} - F \right) + \left( a_2 - \frac{b_2q^2}{2h} \right) \frac{2f'}{r} F \\
+ (b_1 + 4b_2F) \frac{f^2}{2r} (\psi'^2) + f^2\psi'^2 \left[ (b_1 + 4b_2F) \left( \frac{3f'}{4rf} + \frac{1}{4} \frac{h'}{rh} \right) + \frac{b_1}{r^3} - \frac{2b_2}{r^3} f' \right] = 0. \]

Here we have defined
\[ F(r) \equiv \frac{\kappa - f(r)}{r^2}. \]

Since Eq. (15) is a quadratic equation with respect to \( f \), one can easily express \( f \) by \( h \) and \( h' \) as follows:
\[ f = \frac{6b_1rh + 3(b_1r^2 + 4\kappa b_2)h'}{24b_2h'} \sqrt{9(2b_1rh + (b_1r^2 + 4\kappa b_2)h')^2 - 48b_2r(b_0r^2 + 6\kappa b_1)hh'}. \]

With this relation, the square of \( \psi' \) can be obtained as a functional of \( h \) and \( h' \) from Eq. [19]. Then Eq. (20) becomes an ordinary differential equation with respect to \( h \) and can be solved in principle. For more details about the solution, see [16].

\(^2\)If the coefficients \( a_\alpha \) or \( b_\alpha \) depend on \( \phi \), the time dependence of the scalar field spoils the staticity of the metric.
Now we argue that this solution is actually unstable under the tensor perturbation of the following form:

$$\delta g_{ab} = \delta g_{ai} = 0, \quad \delta g_{ij} = r^2 \chi(t, r) \bar{h}_{ij}(x^k),$$

where $a, b = (t, r)$. Here $\chi$ represents the dynamical degree of freedom of the perturbation and $\bar{h}_{ij}$ are symmetric tensor spherical harmonics which satisfy [18]

$$\bar{\nabla}^k \bar{\nabla}_k \bar{h}_{ij} = -\gamma_t \bar{h}_{ij}, \quad \bar{\nabla}^i \bar{h}_{ij} = 0, \quad \bar{h}_{ii} = 0,$$

where $\bar{\nabla}_i$ denotes a covariant derivative with respect to $\bar{\gamma}_{ij}$, and the eigenvalue $\gamma_t$ takes continuous positive numbers for $\kappa = 0, -1$ or discrete values $\gamma_t = \ell(\ell + D - 3) - 2$ ($\ell = 2, 3, \cdots$) in $D$-dimensional spacetime for $\kappa = 1$. The scalar field $\phi$ does not have the tensor perturbation and hence we set $\delta \phi = 0$.

Plugging the decomposition (23) into the original field equations and expanding them to first order in perturbation, we obtain the linear differential equation for $\chi$. From the derived perturbation equation, we can construct the corresponding Lagrangian up to an irrelevant constant factor. This overall factor can be fixed by the comparison with the second-order action in the simple case for which direct computation of the second-order action is relatively simple. Since a series of these procedures is straightforward, we omit the intermediate computations and give the final form of the second-order action:

$$S^{(2)} = \int d^5 x \sqrt{-\bar{g}} \left( \frac{\lambda_0}{2} \chi^2 - \frac{\lambda_1}{2} \chi'^2 + \frac{\lambda_2}{2} \chi^2 \frac{h'}{r} - \lambda_3 \right) \bar{h}^{kl} \bar{h}_{kl},$$

where $\bar{g}$ denotes the determinant of the background metric and $\lambda_0, \cdots, \lambda_3$ are the background-dependent coefficients. As we will see below, among these coefficients, $\lambda_0$ and $\lambda_1$ determine the presence of ghost/gradient instability. Their explicit forms are given by

$$\lambda_0 = \frac{a_1}{2h} - \frac{a_2 f'}{r h} + \frac{q^2}{h} \frac{b_1 f'}{r^2} - \frac{b_2 f'}{r} + X \left( \frac{b_1}{2} + \frac{b_2 f'}{r} \right) + 2b_2 f X',$$

$$\lambda_1 = \frac{a_1}{2} f - \frac{a_2 f'^2}{r h} + r^2 f \left[ \frac{q^2}{h} \left( \frac{b_1}{2} - \frac{b_2 f'}{r} \right) - X \left( \frac{b_1}{2} - 3b_2 f' \right) \right],$$

where $X$ is the canonical kinetic term of the scalar field:

$$X \equiv -\frac{1}{2} \phi \phi' = \frac{q^2}{2h} - \frac{f \phi'^2}{2}.$$

To construct the Hamiltonian, let us introduce the canonical momentum conjugate to $\chi$ as

$$\pi = \sqrt{-\bar{g}} \left( \lambda_0 \dot{\chi} + \frac{\lambda_2}{2} \chi' \right) \bar{h}^{kl} \bar{h}_{kl}.$$
For this Hamiltonian to be bounded below, the coefficients \(\lambda_0\), \(\lambda_1\), and \(\lambda_3\) must be positive. Let us check the stability in the vicinity of the horizon where \(h \simeq 0\). Near the horizon, one can approximate \(\lambda_0\) and \(\lambda_1\) as

\[
\lambda_0 \approx -\frac{q^2 r^2}{h^2} \left( \frac{b_1}{2} - b_2 \frac{f h'}{r h} \right),
\]

\[
\lambda_1 \approx \frac{q^2 r^2 f}{h} \left( \frac{b_1}{2} - b_2 \frac{f h'}{r h} \right).
\]

Therefore, when the scalar velocity charge \(q\) is nonzero, we get

\[
\lambda_0 \lambda_1 \approx -\frac{q^4 r^4 f}{h^3} \left( \frac{b_1}{2} - b_2 \frac{f h'}{r h} \right)^2 < 0.
\]

This means that either \(\lambda_0\) or \(\lambda_1\) is negative. Thus, the solution given in [16] is always plagued either by the ghost or gradient instability. This instability is akin to that found in [17] for four-dimensional BHs with time-dependent scalar hair in shift-symmetric scalar-tensor theories.

4 Lovelock-Galileon black holes in higher dimensions

4.1 Black hole solutions

In the following, we discuss BH solutions in Lovelock-Galileon theory in arbitrary dimensions and generalize the solutions given in the previous works. Let us consider the metric of the form

\[
ds^2 = -h(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \gamma_{ij} dx^i dx^j,
\]

where \(\gamma_{ij}\) represents the metric of a \((D-2)\)-dimensional maximally symmetric space with spatial curvature \(\kappa\). We also assume \(\phi\) depends linearly on time as in Eq. (17):

\[
\phi(t, r) = qt + \psi(r).
\]

Now we have three unknown functions of \(r\): \(h(r)\), \(f(r)\), and \(\psi(r)\). One can show that if we find a solution which satisfies the \(tt\)-, \(rr\)-, and \(tr\)-components of Eq. (9), then it solves all the other components of Eq. (9) and the scalar equation of motion (10). Thus, we have a necessary and sufficient number of equations to fully solve the system.

Let us write down these equations in terms of the unknown functions. The \(tr\)-component of Eq. (9) is written as

\[
0 = \sum_{n=0}^{M} b_n E^{(n)}_{tr} = q \psi' \sum_{n=0}^{M} b_n H^{(n)} r = -q \sum_{n=0}^{M} b_n J^{(n)} r.
\]

If we assume \(q \psi' \neq 0\), this simplifies to

\[
\sum_{n=0}^{M} \frac{b_n}{(D-2n-1)!} F^{n-1} \left[ \frac{f h'}{r h} - (D - 2n - 1) F \right] = 0,
\]

\[\text{Note that} \ X \text{is finite at the horizon for the regular solution.}\]

\[\text{We assume that not all} \ b_n \text{'s are zero; i.e., there exists at least one nonvanishing Lovelock-derivative coupling.}\]
where a prime denotes a derivative with respect to $r$, and $F$ is defined by Eq. (21). Also, the $rr$-component is equivalent to

$$\sum_{n=0}^{M} \frac{F^{n-2}}{(D-2n-1)!} \left\{ n \left( a_n - \frac{b_n q^2}{2h} \right) \frac{f h'}{r h} F - (D-2n-1) \left( a_n + \frac{b_n q^2}{2h} \right) F^2 
+ n b_n f^2 \psi'^2 \left[ \frac{h'}{r h} F + (D-2n-1) \frac{F}{r^2} - (n-1) \frac{f h'}{r^3 h} \right] \right\} = 0, \quad (38)$$

and the $tt$-component is given by

$$\sum_{n=0}^{M} \frac{F^{n-2}}{(D-2n-1)!} \left\{ \left( a_n - \frac{b_n q^2}{2h} \right) \left[ n \frac{f'}{r} F - (D-2n-1) F^2 \right] + n b_n \frac{f^2}{r} F (\psi'^2)' 
+ n b_n f^2 \psi'^2 \left[ \frac{3 f'}{2 r f} F + \frac{1}{2 r h} F + (D-2n-1) \frac{F}{r^2} - (n-1) \frac{f'}{r^3} \right] \right\} = 0. \quad (39)$$

Note that, in obtaining Eqs. (38) and (39), we used Eq. (36) to eliminate terms which are proportional to $\sum_{n=0}^{M} b_n H^{(n)} r$. Solving Eq. (37), which can be regarded as an algebraic equation with respect to $f$ through Eq. (21), one can express $f$ in terms of $h$ and $h'$. Substituting that relation into Eq. (38), $\psi'$ can also be related to $h$ and $h'$. Then, Eq. (39) reduces to a (nonlinear) second-order differential equation for $h(r)$. Once $h(r)$ is obtained, $f(r)$ and $\psi'(r)$ can be calculated successively. It must be noted that the roots of the algebraic equation (37) should be chosen so that $f$, $h$, and $\psi$ are real.

Since the original action is quite complicated, one cannot solve the field equations explicitly for the unknown functions in general. For this reason, we shall consider some simple cases in the following.

(i) **Lovelock-Galileon theory of $\ell$th order**

As an exactly solvable example, let us focus on the case where only the $n = \ell$ term of the Lovelock-Galileon action (7) is nonzero. Namely, we consider the following action:

$$S = \int d^D x \sqrt{-g} \left[ a_\ell R^{(\ell)} + b_\ell H^{(\ell)} \phi_{\alpha \beta} \phi_{\beta \alpha} \right]. \quad (40)$$

Since there are only trivial solutions for $\ell = 0$ or $(D-1)/2$, we assume $1 \leq \ell < (D-1)/2$. In this case, the following nontrivial solution is obtained:

$$h = C_0 - \frac{C_1}{r^{(D-2\ell-1)/\ell}}, \quad (41)$$
$$f = \frac{\kappa}{C_0} h, \quad (42)$$
$$\psi'^2 = \frac{q^2}{\kappa h^2} \frac{C_1}{r^{(D-2\ell-1)/\ell}}, \quad (43)$$

where $C_0$ and $C_1$ are constants of integration. For this solution we find that the kinetic term of the scalar field is constant: $X = q^2/2C_0$. Note that the above solution is independent
of $a_\ell$ and $b_\ell$. Since $\kappa = 0$ yields $f = 0$ which is irrelevant, let us consider the $\kappa = 1$ case. In this case one can rescale the time coordinate to have $C_0 = 1$ and $f = h$, leading to

$$h = f = 1 - \frac{C_1}{r^{(D-2\ell-1)/\ell}},$$

$$\psi' = \pm \frac{q}{h} \sqrt{\frac{C_1}{r^{(D-2\ell-1)/2\ell}}}. $$

(For $\psi'$ to be real, $C_1$ must be positive.) Equation (44) can be thought of as a generalization of the Schwarzschild solution in general relativity.

The solution (44) has a horizon at $r = C_1^{\ell/(D-2\ell-1)} \equiv r_h$. Clearly, $\phi = qt + \psi$ exhibits logarithmic divergence $\sim \ln |r - r_h|$ for fixed $t$. However, following [9], by replacing $t$ in $\phi$ with the ingoing Eddington-Finkelstein coordinate $u$ defined by

$$u = t + \int_r^r \frac{dr'}{f(r')},$$

we have $\phi = qu + \Psi(r)$. It can be confirmed that $\Psi(r)$ remains finite for $r \to r_h$ for the plus branch of Eq. (45). Thus, any infalling observer records a finite value of $\phi$ at the horizon $r_h$.

In the case of $\kappa = -1$, one may take $C_0 = -1$ and $C_1 < 0$, so that $h, f, \psi'^2 > 0$ for $0 < r < r_h$.

(ii) Schwarzschild-like metric

Even when there are a number of nonvanishing terms in the action, one can proceed further assuming that $h(r) = f(r)$. In this case, we can immediately integrate Eq. (37) and get the following algebraic equation for $F$:

$$W[F; b_n] \equiv \sum_{n=0}^M \frac{b_n}{(D-2n-1)!} F^n = \frac{\mu}{r^{D-1}},$$

where $\mu$ is an integration constant. The roots should be chosen so that $F$ is real for any $r > 0$. Interestingly, with the replacement $b_n \to a_n$ the structure of Eq. (47) is identical to that of Eq. (70) obtained in the course of deriving spherically symmetric solutions in Lovelock theory (see the Appendix). This means that the metric functions are of the same form as those of the BH solutions in Lovelock theory.

Next, we analyze Eqs. (38) and (39) and discuss when this type of solution is consistent and possible. Subtracting Eq. (39) from Eq. (38) yields

$$\frac{\mu f}{r^{D+1}F'} \left( f \psi'^2 - \frac{q^2}{f} \right)' = -\frac{2\mu f}{r^{D+1}F'} X' = 0.$$  

(48)

Assuming $\mu \neq 0$, we obtain $X \equiv X_0 = \text{const.}$ and hence have the following relation:

$$f^2 \psi'^2 = q^2 - 2X_0f.$$  

(49)
Substituting this into Eq. (38), we obtain

\[- \sum_{n=0}^{M} a_n - 2X_0 b_n = (r^{D-1} F^n)' + (q^2 - 2X_0 \kappa) \sum_{n=1}^{M} n b_n \frac{r^n}{(D-2n-1)!} (r^{D-3} F^{n-1})' = 0. \quad (50)\]

This expression is easily integrated to give

\[- \sum_{n=0}^{M} a_n - 2X_0 b_n F^n + \frac{q^2 - 2X_0 \kappa}{r^2} \sum_{n=1}^{M} n b_n \frac{F^n}{(D-2n-1)!} = \frac{\nu}{r^{D-1}}, \quad (51)\]

where \(\nu\) is an integration constant. Rewriting the right-hand side using Eq. (47) yields

\[\sum_{n=0}^{M} a_n - 2X_0 b_n + (\nu/\mu) b_n F^n = \frac{q^2 - 2X_0 \kappa}{r^2} \sum_{n=1}^{M} n b_n \frac{F^n}{(D-2n-1)!}. \quad (52)\]

Note that \(r\) in the right-hand side is related to \(F\) by Eq. (17). If \(r\) is not written as a rational function of \(F\), Eq. (52) is satisfied if and only if

\[q^2 - 2X_0 \kappa = 0, \quad a_n - 2X_0 b_n + (\nu/\mu) b_n = 0, \quad (0 \leq n \leq M). \quad (53, 54)\]

From Eq. (53) we see that the integration constant \(X_0\) must be fixed as \(X_0 = q^2/2\kappa\) for \(\kappa \neq 0\). If \(a_i = 0\) (respectively \(b_i = 0\)) then \(b_i = 0\) (respectively \(a_i = 0\)) is required from Eq. (53). The same equation also implies that for all nonvanishing pairs of \((a_j, b_j)\)

\[\frac{a_j}{b_j} - 2X_0 j = -\frac{\nu}{\mu} \quad (55)\]

must be satisfied. Thus, the solution is consistent provided that the parameters of the theory fulfill the above requirements. When \(\kappa \neq 0\), we obtain the following solution for \(\psi(r)\):

\[\psi' = \pm \frac{q}{f} \sqrt{1 - \frac{f}{\kappa}}. \quad (56)\]

Again, for the plus branch solution, it can be verified that \(\phi\) remains finite when it is written in terms of the ingoing Eddington-Finkelstein and radial coordinates. These results generalize those in [16], which dealt with the \(D = 5\) case.

### 4.2 Stability analysis

In the following we discuss the stability of the solutions above. We consider the tensor perturbation of the form

\[\delta g_{ab} = \delta g_{ai} = 0, \quad \delta g_{ij} = r^2 \chi(t, r) \bar{h}_{ij}(x^k), \quad \delta \phi = 0 \quad (57)\]

If \(r^2\) is written as a rational function of \(F\), there are other possibilities.
as we did in Sec. 3. For the nature of the tensor perturbation, we only need to consider the \(ij\)-components of the field equations for the metric. Then, the perturbed equation of motion is given by

\[
\sum_{n=0}^{M} \left( a_n \delta H^{(n)ij} + b_n \delta E^{(n)ij} \right) = 0, \tag{58}
\]

where

\[
\begin{align*}
\delta H^{(n)ij} &= \frac{1}{2} (D-4)! n F^{n-2} \frac{\dot{h}^i j}{2} \\
&\times \left\{ -\frac{\dot{\chi}}{h} \left[ (n-1) \frac{f'}{r} - (D-2n-1)F \right] + f \chi'' \left[ (n-1) \frac{h'}{r} - (D-2n-1)F \right] \right\}, \tag{59}
\end{align*}
\]

\[
\begin{align*}
\delta E^{(n)ij} &= \frac{r^2}{2} (D-4)! n F^{n-2} \frac{\ddot{h}^i j}{2} \left\{ \frac{\ddot{\chi}}{h} \left[ 2(n-1) \frac{f'}{r} X' \right] \\
&+ X \left[ (D-2n-1) \left( 2(n-1) \frac{f'}{r} + F \right) + (n-1) \frac{f'}{rF} \left( F - 2(n-2) \frac{f'}{r^2} \right) \right] \\
&- \frac{q^2}{h} \left[ (D-2n-1) \left( -2(n-1) \frac{f'}{r} + F \right) - (n-1) \frac{h'}{rF} \left( h' F + (n-2) \frac{f'}{r^2} \right) \right] \right\} \right. \\
&+ f \chi'' \left\{ X \left[ (D-2n-1) \left( 2(n-1) \frac{f'}{r} - F \right) + (n-1) \frac{h'}{rF} \left( 3F - 2(n-2) \frac{f'}{r^2} \right) \right] \\
&- \frac{q^2}{h} \left[ (D-2n-1) \left( (n-1) \frac{f'}{r} - F \right) + (n-1) \frac{h'}{rF} \left( F - (n-2) \frac{f'}{r^2} \right) \right] \right\} + (\chi' and \chi terms), \tag{60}
\end{align*}
\]

with \(X\) being the canonical kinetic term of the scalar field given by Eq. (28). In Eq. (60), we omitted terms containing \(\chi'\) and \(\chi\) because we are only interested in the sign in front of \(\ddot{\chi}\) and \(\chi''\). From the linear differential equation for \(\chi\), we can construct the corresponding second-order action for \(\chi\) as

\[
S^{(2)} = \int d^D x \sqrt{-g} \left( \frac{\lambda_0}{2} \dot{\chi}^2 - \frac{\lambda_1}{2} \chi'^2 + \frac{\lambda_2}{2} \dot{\chi} \chi' - \frac{\lambda_3}{2} \chi^2 \right) \tilde{h}^{kl} \tilde{h}_{kl}. \tag{61}
\]

The coefficients of the action can be read off from the perturbed equation of motion except for an overall constant. To fix this constant, we calculate the second-order Lovelock action, which corresponds to setting \(b_n = 0\) in the full action. Integrating by parts, we obtain

\[
S_L^{(2)} = \int d^D x \sqrt{-g} \sum_{n=0}^{M} \frac{a_n (D-4)! n F^{n-2}}{4} \frac{\dot{h}^i j}{2} \tilde{h}^{kl} \tilde{h}_{kl} \\
\times \left\{ -\frac{\dot{\chi}^2}{h} \left[ (n-1) \frac{f'}{r} - (D-2n-1)F \right] + f \chi'^2 \left[ (n-1) \frac{h'}{r} - (D-2n-1)F \right] \right\}. \tag{62}
\]
As a result, the full expressions for $\lambda_0$ and $\lambda_1$ are given by

$$\lambda_0 = \sum_{n=1}^{M} \frac{M}{(D-2n-1)!} \frac{(D-4)!nF^{n-2} r^2 f'}{4h} \left\{ -\frac{2a_n f'}{r^2} \left[ (n-1) \frac{f'}{r} - (D-2n-1)F \right] + 4(n-1)b_n \frac{f}{r} X' \right. \\
+ 2b_n X \left[ (D-2n-1) \left( 2(n-1) \frac{f}{r^2} + F \right) + (n-1) \frac{f'}{rF} \left( F - (2n-2) \frac{f}{r^2} \right) \right] \\
- \left. \frac{2b_n q^2}{h} \left[ (D-2n-1) \left( 2(n-1) \frac{f}{r^2} + F \right) - (n-1) \frac{f'}{r} \left( \frac{h'}{h} F + (n-2) \frac{f'}{r^2} \right) \right] \right\}, \quad (63)$$

$$\lambda_1 = \sum_{n=1}^{M} \frac{M}{(D-2n-1)!} \frac{(D-4)!nF^{n-2} r^2 f}{4} \left\{ -\frac{2a_n f'}{r^2} \left[ (n-1) \frac{f'h'}{rh} - (D-2n-1)F \right] \\
+ 2b_n X \left[ (D-2n-1) \left( 2(n-1) \frac{f}{r^2} - F \right) + (n-1) \frac{f'h'}{rF} \left( 3F - 2(n-2) \frac{f}{r^2} \right) \right] \\
- \left. \frac{2b_n q^2}{h} \left[ (D-2n-1) \left( 2(n-1) \frac{f}{r^2} - F \right) + (n-1) \frac{f'h'}{r} \left( F - (n-2) \frac{f}{r^2} \right) \right] \right\}. \quad (64)$$

Now, as in Sec. 3 both $\lambda_0$ and $\lambda_1$ must be positive in order that neither ghost nor gradient instability appears. However, near the horizon we find

$$\lambda_0 \lambda_1 \approx -\frac{q^4 r^4 f}{4h^3} \left\{ \sum_{n=1}^{M} \frac{(D-4)!nF^{n-2}}{(D-2n-1)!} \left[ (n-1) \frac{f'h'}{rh} - (D-2n-1)F \right] \right\}^2 < 0, \quad (65)$$

for $q \neq 0$. Thus, we conclude that the instability found in Sec. 3 occurs quite generically for hairy BHs in Lovelock-Galileon theories in $D$ dimensions.

### 5 Conclusions

In the first part of this paper, we analyzed the stability of the BH solutions in five-dimensional Lovelock-Galileon theory given in [16]. In the analysis we calculated the second-order action and showed that the Hamiltonian cannot be bounded below when the scalar velocity charge is nonzero. Then we have considered the Lovelock-Galileon theory in arbitrary dimensions, which generalizes the study of [16]. Our ansatz is that the metric is static and spherically symmetric, and the scalar field is allowed to contain a term linear in time in addition to the nontrivial radial profile. We have reformulated the problem of finding the solutions into the one of solving the algebraic equation and the second-order differential equation. Quite remarkably, tensor perturbation analysis for the generalized solutions shows that the Hamiltonian for the perturbation is always unbounded in the vicinity of the horizon, exactly in the same way as in the case of five dimensions. We thus conclude that the instability of BHs with time-dependent scalar hair in the Lovelock-Galileon theory is inherent in arbitrary higher dimensions.

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Appendix: Spherically symmetric solutions in Lovelock theory

In this appendix, we summarize the basic equations for a spherically symmetric metric in Lovelock theory [19]. They can be reproduced simply by setting $b_n = 0$ in Eqs. (38) and (39). Obviously, the $tr$-equation is trivial in this case. We thus have

\[
\sum_{n=0}^{M} \frac{F^{n-1}}{(D-2n-1)!} \left[ n a_n \frac{f h'}{rh} - (D - 2n - 1)a_n F \right] = 0, \tag{66}
\]

\[
\sum_{n=0}^{M} \frac{F^{n-1}}{(D-2n-1)!} \left[ n a_n \frac{f'}{r} - (D - 2n - 1)a_n F \right] = 0. \tag{67}
\]

These two equations imply that

\[
\sum_{n=0}^{M} \frac{na_n F^{n-1}}{(D-2n-1)!} \left( \frac{f'}{f} - \frac{f h'}{h} \right) = 0 \Rightarrow h \propto f. \tag{68}
\]

One can rescale the time coordinate so that $h = f$. Equation (67) can be recast in

\[
\frac{d}{dr} \sum_{n=0}^{M} \frac{a_n r^{D-1} F^n}{(D-2n-1)!} = 0, \tag{69}
\]

and this can be integrated to give

\[
W[F; a_n] \equiv \sum_{n=0}^{M} \frac{a_n}{(D-2n-1)!} F^n = \frac{\mu}{r^{D-1}}, \tag{70}
\]

where $\mu$ is an integration constant. From Eq. (70), $F$ is determined algebraically. In the case where $n \geq 3$ terms vanish, the well-known Boulware-Deser solution [20] is recovered.

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