Abstract. We study the $C^*$-algebra $A_{n,\theta}$ generated by the Anzai flow on the $n$-dimensional torus $\mathbb{T}^n$. It is proved that this algebra is a simple quotient of the group $C^*$-algebra of a lattice subgroup $\mathbb{D}_n$ of a $(n + 2)$-dimensional connected simply connected nilpotent Lie group $F_n$ whose corresponding Lie algebra is the generic filiform Lie algebra $f_n$. Other simple infinite dimensional quotients of $C^*(\mathbb{D}_n)$ are also characterized and represented as matrix algebras over simple affine Furstenberg transformation group $C^*$-algebras of the lower dimensional tori. The $K$-groups of the $A_{n,\theta}$ and other simple quotients of $C^*(\mathbb{D}_n)$ are studied, the Pimsner-Voiculescu 6-term exact sequence being a useful tool. The rank of the $K$-groups of $A_{n,\theta}$ is studied as explicitly as possible, and is proved to be the same as for more general transformation group $C^*$-algebras of $\mathbb{T}^n$ including the Furstenberg transformation group $C^*$-algebras $A_{F,f,\theta}$.

An error (about these $K$-groups) in the literature is addressed.

1. Introduction

In 3 dimensions there is a unique (up to isomorphism) connected, simply connected, nilpotent Lie group, which we call $G_3$ (following Nielsen [21]); $G_3 (= \mathbb{R}^3$ as a set) is the Heisenberg group with multiplication

$$(k, m, n)(k', m', n') = (k + k' + nm', m + m', n + n').$$

The faithful irreducible representations of the lattice subgroup $H_3 (= \mathbb{Z}^3$ as a set) of $G_3$ generate the irrational rotation algebras $A_\theta$. In 4 dimensions there is also a unique such connected group $G_4$, in 5 dimensions there are 6 such groups $G_{5,i}$ for $1 \leq i \leq 6$, and in 6 dimensions there are 24 such groups $G_{6,j}$ for $1 \leq j \leq 24$. The main thrust in [17] [18] [19] was to find cocompact subgroups $H_4 \subset G_4$, $H_{5,i} \subset G_{5,i}$, and $H_{6,10} \subset G_{6,10}$, that would be analogous to $H_3 \subset G_3$, and then for these $H$’s to identify the infinite dimensional simple quotients of $C^*(H)$, both the faithful ones (generated by a faithful representation of $H$) and the non-faithful ones, and also to give matrix representations over lower dimensional algebras for as many of the non-faithful quotients as possible.

The most attractive concrete representations on $L^2(\mathbb{T})$ of the irrational rotation algebras $A^3_\theta$ (for irrational $\theta$), the infinite dimensional simple quotients of the group $C^*$-algebra $C^*(H_3)$, use the rotation flow $(\mathbb{Z}, \mathbb{T}), v \mapsto \lambda v,$
on the circle $\mathbb{T}$ (with $\lambda = e^{2\pi i \theta}$). The analogous 2, 3, and 4 dimensional (Anzai) flows, generated by $(w, v) \mapsto (\lambda w, uv)$ on $\mathbb{T}^2$, $(x, w, v) \mapsto (\lambda x, xw, uv)$ on $\mathbb{T}^3$, and $(y, x, w, v) \mapsto (\lambda y, yx, xw, uv)$ on $\mathbb{T}^4$ give the analogous concrete representations of the algebras $A^3_\theta$, $A^5_\theta$, and $A^{6,10}_\theta$ on $L^2(\mathbb{T}^2)$, $L^2(\mathbb{T}^3)$, and $L^2(\mathbb{T}^4)$ of simple quotients of $C^*(H_4)$ \cite{17}, $C^*(H_{5,5})$ \cite{18}, and $C^*(H_{6,10})$ \cite{19}.

The $K$-groups of $A^3_\theta = A_\theta$ were computed in \cite{24} and \cite{26}, and those of the Heisenberg $C^*$-algebra $A^4_\theta$ were studied by J. Packer in \cite{23} (where it is referred to as class 2). Since $A^{5,5}_\theta$ is isomorphic to a crossed product $\mathcal{C}(\mathbb{T}^3) \rtimes_\sigma \mathbb{Z}$ as in \cite{18}, S. Walters in \cite{30} used the Pimsner-Voiculescu six term exact sequence \cite{24} to compute its $K$-groups, and in order to calculate the action of the underlying automorphism $\sigma$ of the crossed product on $K_*(\mathcal{C}(\mathbb{T}^3))$ he made use of Connes’ non-commutative geometry involving cyclic cocycles and the Connes-Chern character. Application of this method to compute the $K$-groups of $A^{6,10}_\theta$ and its higher dimensional analogues is much more difficult, and no computations for these cases seem to have been attempted. In Section 6 this problem will be discussed in a general form via another approach (looking at $K^*_*(\mathcal{C}(\mathbb{T}^n))$ as an exterior algebra over $\mathbb{Z}^n$); we have calculated the $K$-groups of $A_{n,\theta}$ for $1 \leq n \leq 11$ (see Table 1).

In the present paper, we consider the $n$-dimensional Anzai flow $\mathcal{F} = (\mathbb{Z}, \mathbb{T}^n)$ generated by the Anzai transformation

$$\sigma : (v_1, v_2, \ldots, v_n) \mapsto (\lambda v_1, v_1 v_2, \ldots, v_{n-1} v_n)$$

with $\lambda = e^{2\pi i \theta}$ for an irrational $\theta$, and the associated operator equations

\[(\text{CR})_n \quad [U, V_1] = \lambda, \quad [U, V_2] = V_1, \quad \ldots, \quad [U, V_n] = V_{n-1}.\]

We show that the group $\mathcal{D}_n$ ($\mathcal{D}_1 = H_3$, $\mathcal{D}_2 = H_4$, $\mathcal{D}_3 = H_{5,5}$, and $\mathcal{D}_4 = H_{6,10}$), to which these are related, is a cocompact subgroup of a connected $(n + 2)$-dimensional group $F_n$ ($F_1 = G_3$, $F_2 = G_4$, $F_3 = G_{5,5}$, and $F_4 = G_{6,10}$). In Theorems 3.1 and 4.2 the infinite dimensional simple quotients of $C^*(\mathcal{D}_n)$ are identified and displayed as $C^*$-crossed products generated by minimal actions. The faithful ones using the flow $\mathcal{F}$ are called $A_{n,\theta}$ ($A_{1,\theta} = A^3_\theta$, $A_{2,\theta} = A^4_\theta$, $A_{3,\theta} = A^{5,5}_\theta$, and $A_{4,\theta} = A^{6,10}_\theta$), and the others using suitable modifications of $\mathcal{F}$ are called $A^{(n)}_1, A^{(n)}_2, \ldots, A^{(n)}_{n-1}$. The non-faithful ones $A^{(n)}_i$ are displayed also as matrix algebras over simple $C^*$-algebras from groups of lower dimension (Theorem 5.8); these groups are certain subgroups of $\mathcal{D}_{n-i}$ for $i = 1, 2, \ldots, n-1$. These $C^*$-algebras are completely classified in Corollary 6.20 by $K$-theoretic invariants, namely the ranks of the $K$-groups and the range of the unique tracial state. In Section 6, we make use of the algebraic properties of $K^*_*(\mathcal{C}(\mathbb{T}^n))$. More precisely, $K^*_*(\mathcal{C}(\mathbb{T}^n))$ is an exterior algebra over $\mathbb{Z}^n$ with a certain natural basis, and the induced automorphism $\sigma_*$ is in fact a ring automorphism, which makes computations much easier. In fact, it is shown in Proposition 6.1 that the problem of finding the
$K$-groups of any transformation group $C^\ast$-algebras of the tori is completely computable in the sense that one only needs to calculate the kernels and cokernels of a finite number of integer matrices.

We have learned that a similar method was used by R. Ji in his Ph.D. thesis [12, unpublished] to study the $K$-groups of the $C^\ast$-algebras $A_{F_{f, \theta}}$ associated with the descending Furstenberg transformations $F_{f, \theta}$ on tori. He has computed a 2-dimensional case [12, Corollary 2.20] (which includes $A^4_\theta$), and has made a claim [12, Proposition 2.17] about the form of the torsion subgroup of $K_\ast(A_{F_{f, \theta}})$ that is not true in general (see Remark 1.6 below), and does not deal with the rank of the $K$-groups. Thus the $K$-groups of $A_{F_{f, \theta}}$ (including $A_{n, \theta}$) have not yet been calculated in general. In Corollary 6.2, the $K$-groups of any transformation group $C^\ast$-algebra of an $n$-torus are proved to be finitely generated with the same rank. In the case of $A_{n, \theta}$, this common rank is called $a_n$, and is studied in detail in Section 6, and given as explicitly as possible via generating functions (Theorem 6.35). It is proved in Theorem 6.16 that $a_n$ is the common rank of the $K$-groups of many other transformation group $C^\ast$-algebra of $T^n$ as well, including the $C^\ast$-algebras from Furstenberg transformations on $T^n$. An explicit formula for the torsion parts of the $K$-groups of these algebras seems much more challenging to find.

To present the results and proofs of the paper, especially for Section 6, we need some definitions about transformations on the tori and the corresponding $C^\ast$-crossed products.

Let $T^n$ denote the $n$-dimensional torus with coordinates $(v_1, v_2, \ldots, v_n)$. We consider homotopy classes of continuous functions from $T^n$ to $T$. It is known that in each class there is a unique “linear” function $f(v_1, \ldots, v_n) = v_1^{b_1} v_2^{b_2} \ldots v_n^{b_n}$, $b_1, b_2, \ldots, b_n \in \mathbb{Z}$. Following [4, p. 35], we denote the exponent $b_i$, which is uniquely determined by the homotopy class of $f$, as $b_i = A_i[f]$.

**Definition 1.1.** An affine transformation on $T^n$ is given by

$$
\alpha(v_1, v_2, \ldots, v_n) = 
(e^{2\pi it_1} v_1^{b_1} \ldots v_n^{b_n}, e^{2\pi it_2} v_1^{b_1} \ldots v_n^{b_n}, \ldots, e^{2\pi it_n} v_1^{b_1} \ldots v_n^{b_n}),
$$

where $t := (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$ and $A := [b_{ij}]_{n \times n} \in \text{GL}(n, \mathbb{Z})$. We identify the pair $(t, A)$ with $\alpha$.

Note that any automorphism of $T^n$ followed by a rotation can be expressed in such a fashion. The set of affine transformations on $T^n$ form a group $\text{Aff}(T^n)$, which can be identified with the semidirect product $\mathbb{R}^n \rtimes \text{GL}(n, \mathbb{Z})$. More precisely, for two affine transformations $\alpha = (t, A)$ and $\alpha' = (t', A')$ on $T^n$, we have

$$
\alpha \circ \alpha' = (t + At', AA') \quad \text{and} \quad \alpha^{-1} = (-A^{-1}t, A^{-1}).
$$
(In the expression $At$, $t$ is a column vector, but for convenience we write it as a row vector.)

We have the following definitions in accordance with [12]. These transformations of the tori appear in applications of ergodic theory to number theory [4], and sometimes are called skew product transformations of the tori.

**Definition 1.2.** (a) A **Furstenberg transformation** $F_{f,\theta}$ on $\mathbb{T}^n$ is given by

$$F_{f,\theta}(v_1, v_2, \ldots, v_n) = (e^{2\pi i \theta} v_1, f_1(v_1) v_2, f_2(v_1, v_2) v_3, \ldots, f_{n-1}(v_1, \ldots, v_{n-1}) v_n),$$

where $\theta$ is a real number and each $f_i : \mathbb{T}^i \to \mathbb{T}$ is a continuous function with $A_i[f_i] \neq 0$ for $i = 1, \ldots, n-1$.

(b) An **affine Furstenberg transformation** on $\mathbb{T}^n$ is given by

$$\alpha(v_1, v_2, \ldots, v_n) = (e^{2\pi i \theta} v_1, v_1^{b_{12}} v_2, v_1^{b_{13}} v_2 v_3, \ldots, v_1^{b_{1n}} v_2^{b_{2n}} \cdots v_n^{b_{n-1,n}} v_n),$$

where $\theta$ is a real number and the exponents $b_{ij}$ are integers and $b_{i,i+1} \neq 0$ for $i = 1, \ldots, n-1$.

(c) An **ascending Furstenberg transformation** on $\mathbb{T}^n$ is given by

$$\alpha(v_1, v_2, \ldots, v_n) = (e^{2\pi i \theta} v_1, v_1^{k_1} v_2, v_2^{k_2} v_3, \ldots, v_{n-1}^{k_{n-1}} v_n),$$

where $\theta$ is a real number and the exponents $k_i$ are nonzero integers and $k_i | k_{i+1}$ for $i = 1, \ldots, n-2$.

(d) In (c), if $k_i = 1$ for $i = 1, \ldots, n-1$, the transformation is called the **Anzai transformation** on $\mathbb{T}^n$. Thus it is given by

$$\sigma(v_1, v_2, \ldots, v_n) = (e^{2\pi i \theta} v_1, v_1 v_2, \ldots, v_{n-1} v_n),$$

where $\theta$ is a real number.

Note that one can easily verify that $F_{f,\theta}$ is a homeomorphism. Also, in the above definition, we have converted “descending”, which is used in [12, Definition 2.16], to “ascending” since the order of coordinates there is opposite to ours. For certain Furstenberg transformations on $\mathbb{T}^n$, we have the following theorem.

**Theorem 1.3.** ([5], 2.3) If $\theta$ is irrational, then $F_{f,\theta}$ defines a minimal dynamical system on $\mathbb{T}^n$. If in addition, each $f_i$ satisfies a uniform Lipschitz condition in $v_i$ for $i = 1, \ldots, n-1$, then $F_{f,\theta}$ is a uniquely ergodic transformation and the unique invariant measure is the normalized Lebesgue measure on $\mathbb{T}^n$. In particular, every affine Furstenberg transformation defines a minimal and uniquely ergodic dynamical system if $\theta$ is irrational.

As a conclusion, we have the following result for the Furstenberg transformation group $C^*$-algebra $A_{F_{f,\theta}} := C(\mathbb{T}^n) \rtimes_{F_{f,\theta}} \mathbb{Z}$ as introduced in [12].
Corollary 1.4. \( A_{f,\theta} = C(T^n) \times_{F_{f,\theta}} \mathbb{Z} \) is a simple \( C^* \)-algebra for irrational \( \theta \). If in addition, each \( f_i \) satisfies a uniform Lipschitz condition in \( v_i \) for \( i = 1, \ldots, n-1 \), then \( A_{F_{f,\theta}} \) has a unique tracial state.

Proof. For the first part, the minimality of the action as stated in the preceding theorem implies the simplicity of \( A_{F_{f,\theta}} \) \([3, 29]\). For the second part, one can easily check that since \( \theta \) is irrational, the action of \( \mathbb{Z} \) on \( T^n \) generated by \( F_{f,\theta} \) is free. So, there are no periodic points in \( T^n \). This and the unique ergodicity of \( F_{f,\theta} \) yield the result \([29\) Corollary 3.3.10, p. 91]. \( \square \)

Remark 1.5. Using the preceding corollary and much like the proof of Theorem 3.1, one can prove that for irrational \( \theta \), \( A_{F_{f,\theta}} \) is in fact the unique \( C^* \)-algebra generated by unitaries \( U, V_1, \ldots, V_n \) satisfying the commutator relations

\[
(CR)_f \quad [U, V_1] = e^{2\pi i \theta}, [U, V_2] = f_1(V_1), \ldots, [U, V_n] = f_{n-1}(V_1, \ldots, V_{n-1})
\]

(where \( [a, b] := aba^{-1}b^{-1} \) and all other pairs of operators from \( U, V_1, \ldots, V_n \) commute).

Remark 1.6. In \([12\) Proposition 2.17], R. Ji claims to have proved

\((*)\) If \( f_{\theta} \) is an ascending Furstenberg transformation on \( T^n \) with the ascending sequence \( \{k_1, k_2, \ldots, k_{n-1}\} \), then the torsion subgroup of \( K_*(A_{F,\theta}) \) is isomorphic to \( \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2}^{(m_2)} \oplus \cdots \oplus \mathbb{Z}_{k_{n-1}}^{(m_{n-1})} \), where the group \( \mathbb{Z}_{k_i}^{(m_i)} \) is the direct product of \( m_i \) copies of the cyclic group \( \mathbb{Z}_{k_i} = \mathbb{Z}/k_i\mathbb{Z} \).

From this claim one can immediately deduce that the \( K \)-groups of the \( C^* \)-algebra \( A_{n,\theta} := C(T^n) \times_{\sigma} \mathbb{Z} \) generated by the Anzai transformation \( \sigma \) on \( T^n \) are torsion-free. We will show in the last section that this is not true in general. As the first counterexample, we will see that \( K_1(A_{6,\theta}) \cong \mathbb{Z}^{13} \oplus \mathbb{Z}_2 \) (Example 6.2). In fact, the error in the proof of \((*)\) is in \([12\) p. 29, l. 2]; there it is “clearly” assumed that using a matrix \( S \) in \( GL(2^n, \mathbb{Z}) \), one can delete all entries denoted by \( * \)'s in \( K_5 \), where \( K_5 \) is the \( 2^n \times 2^n \) integer matrix corresponding to \( A_{F,\theta} \) that acts on \( K_5(C(T^n)) = \Lambda^* \mathbb{Z}^n \) with respect to a certain ordered basis. This error arose originally from the general form of the matrix \( K_5 \) in \([12\) p. 27], which is not correct. R. Ji went on to use the torsion subgroup in \((*)\) as an invariant to classify the \( C^* \)-algebras generated by ascending transformations and matrix algebras over them \([12\) Theorem 3.6]. We do not know whether that classification holds.

2. The Anzai Flow \( \mathcal{F} = (\mathbb{Z}, T^n) \) and the Group \( \mathfrak{D}_n \)

Let \( \lambda := e^{2\pi i \theta} \) for an irrational number \( \theta \), and consider the Anzai transformation

\[
\sigma : (v_1, v_2, \ldots, v_n) \mapsto (\lambda v_1 v_2, v_3, \ldots, v_n v_{n-1} v_n)
\]
on $\mathbb{T}^n$, which generates (by iteration) the Anzai flow $F = (\mathbb{Z}, \mathbb{T}^n)$

$$(v_1, \ldots, v_n) \mapsto \sigma^m(v_1, \ldots, v_n) = (\lambda^n v_1, \lambda^{(n)} v_1 v_2, \lambda^{(n)} v_1^2 v_3, \ldots, \lambda^{(n)} v_1^{m-1} \ldots v_{n-1} v_n).$$

With $v_1, \ldots, v_n$ denoting also the functions in $\mathcal{C}(\mathbb{T}^n)$,

$$(v_1, \ldots, v_n) \mapsto v_1, \ldots, v_n,$$

we then get unitaries on $L^2(\mathbb{T}^n)$,

$$Uf = f \circ \sigma, \quad V_1 f = v_1 f, \quad \ldots, \quad V_n f = v_n f.$$ 

These unitaries satisfy the commutator equations

$$(\text{CR})_n \quad [U, V_1] = \lambda, \quad [U, V_2] = V_1, \quad \ldots, \quad [U, V_n] = V_{n-1},$$

(where $[a, b] := aba^{-1}b^{-1}$ and all other pairs of operators from $U, V_1, \ldots, V_n$ commute). A “discrete group construction” [17] shows how to construct a group from unitaries like this; use $(\text{CR})_n$ to collect terms in the product

$$(\lambda^k V_1^k V_2^k \ldots V_n^{k+1} U^k)(\lambda^{k'} V_1^{k'} V_2^{k'} \ldots V_n^{k'+1} U^{k'})$$

then, the exponents give the multiplication for a group $\mathcal{D}_n (= \mathbb{Z}^{n+1} \times \mathbb{Z},$ as a set). In fact $\mathcal{D}_n = \mathbb{Z}^{n+1} \rtimes \eta \mathbb{Z}$ is a semidirect product for which $\eta : \mathbb{Z} \to \text{GL}(n + 1, \mathbb{Z})$ is given by $\eta(k) = \eta_k = M_n^k$, where $M_n$ is a $(n + 1) \times (n + 1)$ matrix defined as

$$M_n = \begin{bmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & 0 & 1 & 1 \\
0 & \cdots & 0 & 0 & 1
\end{bmatrix}_{(n+1) \times (n+1)},$$

and (by induction) one can show that $M_n^k = [m_{ij}^{(k)}]$, where

$$m_{ij}^{(k)} = \begin{pmatrix} k \\ j - i \end{pmatrix}$$

with the following notation.

**Notation 2.1.**

$$\begin{align*}
{k \choose r} &= \begin{cases}
\frac{k(k-1)\cdots(k-r+1)}{r!}, & \text{if } 0 \leq r < k \text{ or } (k < 0 \text{ and } r > 0); \\
1, & \text{if } r = (k + |k|)/2; \\
0, & \text{otherwise.}
\end{cases}
\end{align*}$$
One can easily check that \( D \) with \( x,y \) and \( y \) is a unitary representation of \( n \) such that \( x,y \) and \( y \) are given by

\[
((k_1, \ldots, k_{n+1}), k) = ((k'_1, \ldots, k'_{n+1}), k') = (k_1, \ldots, k_{n+1}) + \eta_k(k_1, \ldots, k_{n+1}, k + k').
\]

One can easily check that \( D \) is the discrete group generated by \( x, y, \ldots, y_n \) such that \( xy_0 = y_0x \) and \( y_i y_j = y_j y_i \) for \( 0 \leq i, j \leq n \) and

\[
[x, y_1] = y_0, \quad [x, y_2] = y_1, \quad \ldots, \quad [x, y_n] = y_{n-1}
\]

The group \( D \) is discrete, nilpotent, finitely generated and torsion-free. So by a result of Malcev \cite{16}, it follows that \( D \) is isomorphic to a discrete cocompact subgroup of a connected \((n+2)\)-dimensional nilpotent Lie group \( F_n \). More precisely, one can verify that the corresponding Lie algebra is the so called \textbf{generic filiform Lie algebra} \cite{16}, \cite{17} \( F_n \), which is spanned by \( X, Y, Y_1, \ldots, Y_n \) with non-zero brackets

\[
[X, Y_1] = Y_0, \quad [X, Y_2] = Y_1, \quad \ldots, \quad [X, Y_n] = Y_{n-1}.
\]

Let \( \mathcal{A}_{n, \theta} \) denote the \( C^* \)-algebra generated by the unitaries \( U, V_1, \ldots, V_n \). Note that \( U, V_n \) generate \( \mathcal{A}_{n, \theta} \); the unitaries \( V_1, \ldots, V_n \) have been introduced only to control the notation. An obvious property of the construction is that

\[
\pi : ((k_1, \ldots, k_{n+1}), k) \mapsto \lambda^{k_1} V_1^{k_2} V_2^{k_3} \cdots V_n^{k_{n+1}} U^k
\]

is a unitary representation of \( D_n \) on \( L^2(\mathbb{T}^n) \) that generates \( \mathcal{A}_{n, \theta} \).

3. The faithful simple quotients \( \mathcal{A}_{n, \theta} \) of \( C^*(D_n) \)

Let \( \lambda := e^{2\pi i \theta} \) for an irrational number \( \theta \). Since \( \mathcal{A}_{n, \theta} \) is generated by a representation of \( D_n \), it is a quotient of \( C^*(D_n) \).

**Theorem 3.1.** Let \( \lambda := e^{2\pi i \theta} \) for an irrational number \( \theta \).

(a) There is a unique (up to isomorphism) \( C^* \)-algebra \( \mathcal{A}_{n, \theta} \) generated by unitaries \( U, V_1, \ldots, V_n \) satisfying (CR)\(n\); \( \mathcal{A}_{n, \theta} \) is simple and is universal for the equations (CR)\( n \). Let \( \sigma \) be the homeomorphism used in the definition of \( \mathcal{A}_{n, \theta} \). Then

\[
\mathcal{A}_{n, \theta} \cong C(\mathbb{T}^n) \rtimes_{\sigma} \mathbb{Z}.
\]

(b) Let \( \pi' \) be a representation of \( D_n \) such that \( \pi = \pi' \) (as scalars) on the center \((\mathbb{Z}, 0, \ldots, 0)\) of \( D_n \), and let \( \mathcal{A} \) be a \( C^* \)-algebra generated by \( \pi' \). Then \( \mathcal{A} \cong \mathcal{A}_{n, \theta} \) via a unique isomorphism \( \omega : \mathcal{A}_{n, \theta} \rightarrow \mathcal{A} \) such that \( \omega \circ \pi = \pi' \).

(c) The \( C^* \)-algebra \( \mathcal{A}_{n, \theta} \) has a unique tracial state.

(d) There is an automorphism \( \alpha \) of \( \mathcal{A}_{n-1, \theta} \) such that \( \mathcal{A}_{n, \theta} \cong \mathcal{A}_{n-1, \theta} \rtimes_{\alpha} \mathbb{Z} \).
Proof. (a) The flow $\mathcal{F}$ used in the definition of $A_{n,\theta}$ is minimal \[5\], so the $C^*$-crossed product $\mathbb{C}(T^n) \rtimes_{\sigma} Z$ is simple $\[3, 25\]$. On the other hand, $A_{n,\theta}$ provides a covariant representation of the dynamical system $(\mathbb{C}(T^n), Z, \sigma)$. More precisely, let $\varphi : k \mapsto U^k; \varphi(k)(g) = g \circ \sigma^k$ be the unitary representation of $Z$ on $L^2(T^n)$ and $\nu : f \mapsto M_f; M_f(g) = fg$ be the $*$-representation of $\mathbb{C}(T^n)$ on $L^2(T^n)$ ($g \in L^2(T^n)$). Then it is easy to see that $(\nu, \varphi)$ is a covariant pair for $(\mathbb{C}(T^n), Z, \sigma)$. Hence by the universal property of $\mathbb{C}(T^n) \rtimes_{\sigma} Z$, there is a $*$-homomorphism $\rho : \mathbb{C}(T^n) \rtimes_{\sigma} Z \to \mathbb{C}(\mathbb{C}(T^n), \nu(\mathbb{C}(T^n)), \varphi(Z))$ obtained by setting $\rho(\sum_k f_k u^k) = \sum_k M_f u^k$ ($u$ is the unitary implementing the automorphism $\sigma$) on the dense $*$-subalgebra $\mathbb{C}(T^n)Z$ and extending by continuity on $\mathbb{C}(T^n) \rtimes_{\sigma} Z$. $\rho$ is surjective since $\mathbb{C}(T^n)$ is unital and $\mathbb{C}(\mathbb{C}(T^n), \nu(\mathbb{C}(T^n)), \varphi(Z))$ is the $C^*$-subalgebra of $\mathcal{B}(L^2(T^n))$ generated by $U$ and the multiplication operators $M_f$ coming from $\mathbb{C}(T^n)$. But, as $\mathbb{C}(T^n)$ is generated by the coordinate functions $v_1, \ldots, v_n, \nu(\mathbb{C}(T^n))$ is generated by special unitary operators $\nu(v_i) = M_{v_i} = V_i$ for $i = 1, \ldots, n$. So, $\mathcal{C}(\nu(\mathbb{C}(T^n)), u(Z)) = A_{n,\theta}$. We now have a surjective $*$-homomorphism $\rho : \mathbb{C}(T^n) \rtimes_{\sigma} Z \to A_{n,\theta}$ with $\mathbb{C}(T^n) \rtimes_{\sigma} Z$ simple. Therefore $\rho$ is an isomorphism and $A_{n,\theta}$ is simple too.

Now, let $A'$ be another $C^*$-algebra generated by unitaries $U', V_1', \ldots, V_n'$ satisfying (CR)$_n$. Since $V_1', \ldots, V_n'$ commute, there is a $*$-homomorphism $\mu : \mathbb{C}(T^n) \to A'$ such that $\mu(v_i) = V_i'$ for $i = 1, \ldots, n$. In fact $\mu(f) = f(V_1', \ldots, V_n')$. Let $\tilde{\omega} : Z \to A'$ be the unitary representation $\tilde{\omega}(k) = U^k$. Noting that $\mu(f \circ \sigma^k) = \tilde{\omega}(k) \mu(f) \tilde{\omega}(k)^*$ holds for $f = v_1, \ldots, v_n$ and hence for all $f \in \mathbb{C}(T^n)$; by the universal property of $\mathbb{C}(T^n) \rtimes_{\sigma} Z$, the covariant pair $(\mu, \tilde{\omega})$ yields a homomorphism of $\mathbb{C}(T^n) \rtimes_{\sigma} Z$ onto $A'$ mapping $v_i$ to $V_i'$ for $i = 1, \ldots, n$, and $u$ to $U'$. So, $A_{n,\theta}$ is universal for equations (CR)$_n$.

(b) The hypothesis imply that (CR)$_n$ is satisfied by the unitaries $U', V_1', \ldots, V_n'$ given by

$$\pi'((k_1, \ldots, k_{n+1}), k) = \lambda^{k_1} V_1^{k_2} V_2^{k_3} \ldots V_n^{k_{n+1}} U^k.$$ 

Part (a) and its proof now yields the result.

(c) This flow is minimal and uniquely ergodic with respect to (normalized) Haar measure $\lambda$ on $T^n$ \[5\]. So, $A_{n,\theta}$ has a unique tracial state $\tau$ given by $\tau(\sum_k f_k u^k) = f \, f^* \, \lambda$ $\[3, 25\]$ Corollary VIII.3.8, p. 91.

(d) Let $A_{n-1,\theta}$ be generated by operators $U', V_1', \ldots, V_{n-1}'$ satisfying (CR)$_{n-1}$. Define $\alpha$ as $\alpha(U') = V_{n-1}' U', \alpha(V_i') = V_i'$ for $i = 1, \ldots, n - 1$. Since $\alpha(U')$, $\alpha(V_1')$, $\ldots, \alpha(V_{n-1}')$ also satisfy (CR)$_{n-1}$, $\alpha$ can be extended to an automorphism of $A_{n-1,\theta}$. Define a $*$-homomorphism $p : A_{n-1,\theta} \to A_{n,\theta}$ by $p(U') = U$ and $p(V_i') = V_i$ for $i = 1, \ldots, n - 1$, and a unitary representation $\psi : Z \to A_{n,\theta}$ by $\psi(k) = V_n^k$. $(p, \psi)$ is a covariant pair for $(A_{n-1,\theta}, Z, \alpha)$ since the equality $p(\alpha^{(k)}(a)) = V_n^k \pi(a) V_n^{-k}$ holds for $a = U', V_1', \ldots, V_{n-1}'$; hence for all $a \in A_{n-1,\theta}$. So, there is a $*$-homomorphism from $A_{n-1,\theta} \rtimes_{\alpha} Z$ onto $A_{n,\theta}$ mapping $U'$ to $U$ and $V_i'$ to $V_i$ for $i = 1, \ldots, n - 1$, and $V_n'$ (the unitary implementing the automorphism $\alpha$) to $V_n$. 


Conversely, $A_{n-1,\theta} \rtimes_\alpha \mathbb{Z}$ is generated by the unitaries $U', V'_1, \ldots, V'_{n-1}$ and $V'_n$ satisfying (CR)$_n$. So, by universality of $A_{n,\theta}$, there is a $*$-homomorphism from $A_{n,\theta}$ onto $A_{n-1,\theta} \rtimes_\alpha \mathbb{Z}$ mapping $U'$ and $V_i$ to $U_i'$ for $i = 1, \ldots, n$. Clearly, these two $*$-homomorphisms are inverses of each other. \hfill \Box

4. Non-faithful simple quotients of $C^*(\mathfrak{D}_n)$

When $\lambda = e^{2\pi i \theta}$ for an irrational number $\theta$, $A_{n,\theta}$ is a simple quotient of $C^*(\mathfrak{D}_n)$ and the representation

$$\pi : ((k_1, \ldots, k_{n+1}), k) \mapsto \lambda^{k_1} V_1^{k_2} V_2^{k_3} \cdots V_{n+1} U^k; \quad \mathfrak{D}_n \rightarrow A_{n,\theta}$$

is faithful. But there are other infinite dimensional simple quotients of $C^*(\mathfrak{D}_n)$; for them $\pi$ is not faithful.

Suppose that $\lambda$ is a primitive $q_1$-th root of unity and that $A$ is a simple quotient of $C^*(\mathfrak{D}_n)$ that is irreducibly represented and generated by unitaries $U, V_1, \ldots, V_n$ satisfying (CR)$_n$. Then $V_i^{q_1}$ commutes with $U$ and $V_i$ for $i = 1, \ldots, n$ and so by irreducibility equals $\tilde{\mu}_1 I$, a multiple of identity. Take $V_1 = \mu_1 V_1'$ for $\mu_1^q = \tilde{\mu}_1$, so that $V_1^{q_1} I = 1$, and substitute $V_1 = \mu_1 V_1'$ in (CR)$_n$ to get

$$(\text{CR})_{n,1} \begin{cases} [U, V_1'] = \lambda, \quad [U, V_2] = \mu_1 V_1', \\ [U, V_3] = V_2, \ldots, \quad [U, V_n] = V_{n-1}, \\ V_1^{q_1} = I \end{cases}$$

1. If $\mu_1$ is not a root of unity, then we can modify the presentation $C(T^n) \rtimes_\sigma \mathbb{Z}$ for $A_{n,\theta}$ in Theorem [5, 2.1] and present the operators $U, V_1', V_2, \ldots, V_n$, and their generated algebra $A_1^{(n)}$, with the flow $F_1 = (\mathbb{Z}, \mathbb{Z}_{q_1} \times T^{n-1})$ generated by the homeomorphism $\phi_1$ of $X_1 := \mathbb{Z}_{q_1} \times T^{n-1}$,

$$\phi_1(v_1, v_2, \ldots, v_n) = (\lambda v_1, \mu_1 v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n).$$

To see that $F_1$ is minimal, we use a lemma on minimality of skew products of dynamical systems with $T$ [5, 2.1, 2.3]. In fact, let $(X, \phi)$ be a dynamical system, where $X$ is a compact metric space and $\phi$ is a homeomorphism of $X$. Let $g : X \rightarrow \mathbb{T}$ be a continuous function and consider the dynamical system $(X \times \mathbb{T}, \Phi)$ defined by $\Phi(x, \zeta) = (\phi(x), g(x) \zeta)$, which is called a skew product of $(X, \phi)$ with $T$. Then we have the following lemma.

**Lemma 4.1.** ([5, 2.1, 2.3]) Let $(X, \phi)$ be a minimal dynamical system and $g : X \rightarrow \mathbb{T}$ be a continuous function. Consider the skew product dynamical system $(X \times \mathbb{T}, \Phi)$ as above. Then $(X \times \mathbb{T}, \Phi)$ is minimal if, and only if, for any non-zero integer $k$, the functional equation

$$(\triangle) \quad g^k = \frac{R \circ \phi}{R}$$

has no continuous solution $R : X \rightarrow \mathbb{T}$. 


Using the preceding lemma, we can prove the minimality of $\mathcal{F}_1$ as follows by induction on $n$. For $n = 2$, we use the preceding lemma, although [17, Theorem 3] yields the result. In this case, $X = \mathbb{Z}_{q_1}$, $\phi(v_1) = \lambda v_1$ and $g(v_1) = \mu_1 v_1$ for $v_1 \in \mathbb{Z}_{q_1}$. $\phi$ is clearly minimal and suppose that $(\Delta)$ has a solution $R : \mathbb{Z}_{q_1} \to \mathbb{T}$ for some $0 \neq k \in \mathbb{Z}$. Then one can easily obtain $R(\lambda^k) = R(1)\mu_1^k \lambda^k$, which yields the contradiction $\mu_1^k \lambda^k = 1$ since $\mu_1$ is not a root of unity but $\lambda$ is a root of unity. Thus $\mathcal{F}_1$ is minimal. Now, suppose that $\mathcal{F}_1$ is minimal for $n = m - 1$. In this case $X = \mathbb{Z}_{q_1} \times \mathbb{T}^{m-2}$, $\phi(v_1, v_2, \ldots, v_{m-1}) = (\lambda v_1, \mu_1 v_1 v_2, v_2 v_3, \ldots, v_{m-2} v_{m-1})$ and $g(v_1, \ldots, v_{m-1}) = v_{m-1}$. If $(\Delta)$ has a solution $R : X \to \mathbb{T}$ for some $0 \neq k \in \mathbb{Z}$, then we have

$$v_{m-1}^k = R(\lambda v_1, \mu_1 v_1 v_2, v_2 v_3, \ldots, v_{m-2} v_{m-1})/R(v_1, v_2, \ldots, v_{m-1})$$

for all $(v_1, v_2, \ldots, v_{m-1}) \in X$. But this equality is impossible, for $R(v_1, \ldots, v_{m-1})$ would have a certain degree in $v_{m-1}$ and one verifies that $R(\lambda v_1, \mu_1 v_1 v_2, v_2 v_3, \ldots, v_{m-2} v_{m-1})$ has the same degree in $v_{m-1}$, hence the right side of the above equality would have degree 0 but the left side has degree $k \neq 0$. So the skew product system $(X \times \mathbb{T}, \Phi) = (\mathbb{Z}_{q_1} \times \mathbb{T}^{m-1}, \phi_1)$ is minimal.

Now take $Y_1 = \mathbb{Z}_{q_1}$. So the $C^*$-crossed product $\mathcal{C}(Y_1 \times \mathbb{T}^{n-1}) \rtimes_{\phi_1} Z$ is simple and isomorphic to $A_1^{(n)}$.

2. Suppose that $\mu_1$ is also a root of unity, say a primitive $p_1$-th root of unity, and let $q_2 = \text{lcm}\{q_1, p_1\}$, the least common multiple of $q_1$ and $p_1$. Then $V_{q_2} = \tilde{\mu}_2 I$, a multiple of the identity. If $\tilde{\mu}_2$ is not a root of unity, substitute $V_2 = \mu_2 V_2'$ (as well as $V_1 = \mu_1 V_1'$) in $(\mathbf{CR})_n$, where $\mu_2^{q_2} = \tilde{\mu}_2$, and get

$$(\mathbf{CR})_{n,2} \begin{cases} [U, V_1'] = \lambda, & [U, V_2'] = \mu_1 V_1', & [U, V_3] = \mu_2 V_2', \\
[U, V_4'] = V_3, & \ldots, & [U, V_n] = V_{n-1}, \\
V_1'^{q_1} = V_2'^{q_2} = I \end{cases}$$

Then we can present the generated algebra $A_2^{(n)}$ using the homeomorphism $\phi_2$ on $X_2 := \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \mathbb{T}^{n-2}$,

$$\phi_2(v_1, v_2, \ldots, v_n) = (\lambda v_1, \mu_1 v_1 v_2, \mu_2 v_2 v_3, v_3 v_4, \ldots, v_{n-1} v_n).$$

The flow $(Z, X_2)$ that $\phi_2$ generates is usually not minimal, but we can restrict $\phi_2$ to $Y_2 \times \mathbb{T}^{n-2} \subset X_2$, where $Y_2 \subset \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2}$ is the finite set

$$Y_2 = \{(v_1, v_2) \mid (v_1, v_2, 1, 1, \ldots, 1) \in \phi_2^r(1, 1, \mathbb{T}^{n-2}) \text{ for some } r \in \mathbb{N}\} = \{ (\lambda^r, \lambda^{\ell(r)} \mu_1^r) \mid r \in \mathbb{N} \}.$$ 

Then the flow $\mathcal{F}_2 = (Z, Y_2 \times \mathbb{T}^{n-2})$ is minimal; the proof of this is similar to the minimality proof in case 1 above. So $\mathcal{C}(Y_2 \times \mathbb{T}^{n-2}) \rtimes_{\phi_2} Z$ is simple.
and isomorphic to $A_2^{(n)}$.

Continue this process assuming $\mu_2$ is also a root of unity and so on down to the following last cases

$(n - 1)$. When $\mu_{n-2}$ is also a root of unity, say a primitive $p_{n-2}$-th root of unity, let

$$q_{n-1} = \text{lcm}\{q_{n-2}, p_{n-2}\} = \text{lcm}\{q_1, p_1, \ldots, p_{n-2}\}$$

Then $V_{n-1}^{q_{n-1}} = \tilde{\mu}_{n-1} I$, a multiple of the identity. If $\tilde{\mu}_{n-1}$ is not a root of unity, substitute $V_{n-1} = \mu_{n-1} V_{n-1}'$ (as well as $V_i = \mu_i V_i'$ for $i = 1, \ldots, n - 2$) in (CR)$_n$, where $\mu_{n-1}^q = \tilde{\mu}_{n-1}$, and get

$$(CR)_{n,n-1}$$

$$\begin{cases}
[U, V'_i] = \lambda, & \lbrack U, V'_{i+1} \rbrack = \mu_i V'_i, \\
[U, V_n] = \mu_{n-1} V'_{n-1}, \\
V_1^{q_1} = V_2^{q_2} = \ldots = V_{n-1}^{q_{n-1}} = I
\end{cases}$$

Then we can present the generated algebra $A_{n-1}^{(n)}$ using the homeomorphism $\phi_{n-1}$ on $X_{n-1} := \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \ldots \times \mathbb{Z}_{q_{n-1}} \times \mathbb{T}$,

$$\phi_{n-1}((v_1, v_2, \ldots, v_{n-1})) = (\lambda v_1, \mu_1 v_1 v_2, \mu_2 v_2 v_3, \ldots, \mu_{n-1} v_{n-1} v_n).$$

The flow $(\mathbb{Z}, X_{n-1})$ that $\phi_{n-1}$ generates is usually not minimal, but we can restrict $\phi_{n-1}$ to $Y_{n-1} \times \mathbb{T} \subset X_{n-1}$, where $Y_{n-1} \subset \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \ldots \times \mathbb{Z}_{n-1}$ is the finite set

$$Y_{n-1} = \{(v_1, v_2, \ldots, v_{n-1}) \mid (v_1, v_2, \ldots, v_{n-1}, 1) \in \phi_{n-1}((1, 1, \ldots, 1, 1)), \text{ for some } r \in \mathbb{N}\}$$

$$= \{(\lambda^r, \lambda^r \mu_1^r, \lambda^r \mu_1^r \mu_2^r, \ldots, \lambda^r \mu_1^r \mu_2^r \ldots \mu_{n-2}^r) \mid r \in \mathbb{N}\}.$$

Then the flow $\mathcal{F}_{n-1} = (\mathbb{Z}, Y_{n-1} \times \mathbb{T})$ is minimal; the proof of this is similar to the minimality proof in case 1 above. So $\mathcal{C}(Y_{n-1} \times \mathbb{T}) \rtimes \phi_{n-1} \mathbb{Z}$ is simple and isomorphic to $A_{n-1}^{(n)}$.

$n$. When $\mu_{n-1}$ is a root of unity (as well as $\mu_i$ for $i = 1, \ldots, n - 2$), all the unitaries are of finite order, so the generated $C^*$-algebra $A$ is finite dimensional.

The preceding comments are summarized in the next theorem.

**Theorem 4.2.** A $C^*$-algebra $A$ is isomorphic to a simple infinite dimensional quotient of $C^*(\mathcal{D}_n)\text{ if, and only if, } A$ is isomorphic to $\mathcal{A}_{n,\theta}$ for some irrational number $\theta$, or to an $A_i^{(n)} = \mathcal{C}(Y_i \times \mathbb{T}^{n-i}) \rtimes \phi_i \mathbb{Z}$ for a suitable finite set $Y_i$ as in cases $i = 1, 2, \ldots, n - 1$ above.

A result that has been used implicitly above, and should be stated explicitly, is the analogue of Theorem 3.1 that holds for $A_i^{(n)}$ for $i = 1, \ldots, n - 1$. 
Theorem 4.3. For $i = 1, 2, \ldots, n-1$, $A_i^{(n)}$ is the unique (up to isomorphism) $C^*$-algebra generated by unitaries $U, V_1, V_2, \ldots, V_n$ satisfying $(\text{CR})_{n,i}$; $A_i^{(n)}$ is simple and is universal for the equations $(\text{CR})_{n,i}$.

As for Theorem 3.1, the result is a consequence of the minimality of the flow involved.

Remark 4.4. There are concrete representations for the $A_i^{(n)}$’s that are analogous to the concrete representation on $L^2(\mathbb{T}^n)$ used in the definition of $A_n, \theta$ (and indeed the $A_i^{(n)}$’s could have been defined in terms of these concrete representations). For $i = 1, 2, \ldots, n-1$, the representation for $A_i^{(n)}$ uses the flow $\mathcal{F}_i$ and is on $L^2(Y_i \times \mathbb{T}^{n-i})$.

5. Matrix representations for non-faithful quotients of $C^*(\mathcal{D}_n)$

The algebras $A_i^{(n)}$ above have representations as matrix algebras with entries in simple $C^*$-algebras from faithful representations of groups of lower dimension. More precisely, let $B_i^{(n)}$ be the universal $C^*$-algebra generated by unitaries $\tilde{U}, \tilde{V}_1, \ldots, \tilde{V}_{n-i}$ satisfying the following commutator relations

$$(\text{CR})_{n,i}$$

$$\begin{align*}
[\tilde{U}, \tilde{V}_1] &= \zeta_i \\
[\tilde{U}, \tilde{V}_2] &= \tilde{V}_1^{C_i} \\
[\tilde{U}, \tilde{V}_3] &= \tilde{V}_1^{(C_i)} \tilde{V}_2^{C_i} \\
& \vdots \\
[\tilde{U}, \tilde{V}_{n-i}] &= \tilde{V}_1^{(C_{i-1})} \tilde{V}_2^{(C_{i-2})} \ldots \tilde{V}_{n-i-1}^{C_i} \\
[\tilde{V}_r, \tilde{V}_s] &= 1 \quad (1 \leq r, s \leq n-i),
\end{align*}$$

where $\zeta_i$ is not a root of unity. Then we will see in Theorem 5.8 below that $A_i^{(n)} \cong M_{\zeta_i}(B_i^{(n)})$. Note that according to Remark 1.5, $B_i^{(n)}$ is a simple (affine) Furstenberg transformation group $C^*$-algebra of $\mathbb{T}^{n-i}$ since $C_i \neq 0$ (note that $b_{n,i} = (C_i \zeta_i)$). The group generated by $(\text{CR})_{n,i}$ (as $\mathcal{D}_n$ was generated by $(\text{CR})_{n}$) is $\mathcal{D}_{n,i} = (\mathbb{Z}^{n-i+1} \rtimes \mathbb{Z})$. We will see in Corollary 5.5 below that $\mathcal{D}_{n,i}$ is isomorphic to a subgroup of $\mathcal{D}_{n-i}$.

More generally, let $\alpha : \mathbb{T}^n \to \mathbb{T}^n$ be an affine Furstenberg transformation given by

$$\alpha(v_1, v_2, \ldots, v_n) = (e^{2\pi i \zeta} v_1, v_1^{b_1} v_2, v_1^{b_1} v_2^{b_2} v_3, \ldots, v_1^{b_1} v_2^{b_2} \cdots v_n^{b_{n-1,n}}),$$

where $\zeta$ is an irrational number and $b_{j,i+1} \neq 0$ for $i = 1, \ldots, n - 1$. Let $A := C(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$. Then $A$ is a simple $C^*$-algebra, which is the unique $C^*$-algebra generated by unitaries $U, V_1, \ldots, V_n$ satisfying the commutator relations

$$(\text{CR})_{\alpha} \quad [U, V_1] = e^{2\pi i \zeta} \quad [U, V_2] = V_1^{b_1} \quad \ldots \quad [U, V_n] = V_1^{b_1} \ldots v_n^{b_{n-1,n}}.$$
Since such that all other pairs of operators from the group generated by (CR)\(_k\). Then \(\Gamma_\alpha = \mathbb{Z}^{n+1} \rtimes_{\gamma} \mathbb{Z}\), in which \(\gamma : \mathbb{Z} \to \text{GL}(n+1, \mathbb{Z})\) is given by \(\gamma(k) = \gamma_k = G_k\), where \(G_k\) is a matrix defined as

\[
G_\alpha = \begin{bmatrix}
1 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & b_{12} & b_{13} & \cdots & b_{1n} \\
0 & 0 & 1 & b_{23} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & b_{n-2,n} \\
0 & \cdots & \cdots & \cdots & 0 & b_{n-1,n} \\
0 & \cdots & \cdots & \cdots & 0 & 1
\end{bmatrix}_{(n+1) \times (n+1)}
\]

One can check that \(\Gamma_\alpha\) is the discrete group generated by \(x', y'_0, y'_1, \ldots, y'_n\) such that \(x'y'_0 = y'_0 x'\) and \(y'_i y'_j = y'_j y'_i\) for \(0 \leq i, j \leq n\) and

\[
[x', y'_i] = y'_0, \quad [x', y'_j] = y'_{b_{12}}^{b_{13}}, \ldots, \quad [x', y'_n] = y'_{b_{1n}} \cdots y'_{b_{n-1,n}}.
\]

Since \(A\) is generated by a representation of \(\Gamma_\alpha\), it is a quotient of \(C^*(\Gamma_\alpha)\).

**Lemma 5.1.** Let \(\Gamma_\alpha\) denote the group generated by (CR)\(_\alpha\) as above. Then there is a monomorphism \(\iota : \Gamma_\alpha \to \mathfrak{D}_n\).

**Proof.** We construct \(\iota\) recursively. Let \(x, y_0, \ldots, y_n\) be the generators of \(\mathfrak{D}_n\) as in Section 1 satisfying (1) and \(x', y'_0, y'_1, \ldots, y'_n\) be the generators of \(\Gamma_\alpha\) satisfying (4), as above. Take \(\iota(x') = x\) and \(\iota(y'_j) = y_j\) for \(j = 0, 1\). Now to define \(\iota(y'_k)\), note that we should have \(\iota([x', y'_k]) = [\iota(x'), \iota(y'_k)] = [x, \iota(y'_k)] = \iota(y'_1 b_{1k} \cdots y_{k-1} b_{k-1,k}) = \iota(y'_1) b_{1k} \cdots \iota(y'_{k-1}) b_{k-1,k}\). For example, \([x, \iota(y'_2)] = \iota(y'_1) b_{12} = y_{b_{12}}\), so \(\iota(y'_2)\) must be defined as \(y_{b_{12}}\) according to equations (1). Similarly, \(\iota(y'_3) = y_{b_{12}} b_{23} b_{34}\) and \(\iota(y'_4) = y_{b_{12}} b_{23} b_{34} b_{45}\) and so on. Therefore using induction, one can show that \(\iota(y'_k)\) is of the form \(y_{c_{1k}} y_{c_{2k}} \cdots y_{c_{kk}}\) for \(k > 2\) and \(c_{kk} = \prod_{i=1}^{k-1} b_{i,i+1} \neq 0\). This will also guarantee the injectivity of \(\iota\) for the matrix \((c_{ij})\) for \(2 \leq i, j \leq n; c_{ij} = 0\) for \(i > j\) is upper triangular with non-zero diagonal entries.

**Remark 5.2.** In a similar way, one can see that there is also a monomorphism \(\iota' : \mathfrak{D}_n \to \Gamma_\alpha\).

**Lemma 5.3.** Let \(\Gamma := \mathbb{Z}^m \rtimes_{\gamma} \mathbb{Z}\), where \(\gamma(k) = G^k\) for some \(G \in \text{GL}(m, \mathbb{Z})\). Then \([\Gamma, \Gamma] = (G - 1)\mathbb{Z}^m \rtimes \{0\}\) and

\[
\frac{\Gamma}{[\Gamma, \Gamma]} \cong \frac{\mathbb{Z}^m}{(G - 1)\mathbb{Z}^m} \cong \mathbb{Z} = \text{coker}(G - 1) \oplus \mathbb{Z}.
\]

**Proof.** For arbitrary \(x_i \in \mathbb{Z}^m\) and \(k_i \in \mathbb{Z}\) \((i = 1, 2)\), one has

\[
[(x_1, k_1), (x_2, k_2)] = (x_1, k_1)(x_2, k_2)(x_1, k_1)^{-1}(x_2, k_2)^{-1}
\]

\[
= (x_1, k_1)(x_2, k_2)(G^{-k_1}(-x_1), -k_1)(G^{-k_2}(-x_2), -k_2)
\]

\[
= (x_1 + G^{k_1}(x_2) + G^{k_2}(-x_1) - x_2, 0)
\]

\[
= ((G^{k_1} - 1)x_2 - (G^{k_2} - 1)x_1, 0).
\]
So it is easily seen that

$$\{\Gamma, \Gamma\} = \{((G_1 - I)x, 0) \mid x \in \mathbb{Z}^m\} = (G_1 - I)^m \times \gamma \{0\}.$$  

For the next part, one can check that \(\varphi : \Gamma \to \text{coker}(G_1 - I) \oplus \mathbb{Z}\) defined by \(\varphi(x, k) = (x + (G_1 - I)^m, k)\) is an epimorphism with \(\ker \varphi = ((G_1 - I)^m, 0) = [\Gamma, \Gamma]\).

**Corollary 5.4.** If \(|b_{i,i+1}| \neq 1\) for some \(i \in \{1, \ldots, n - 1\}\), then \(\Gamma_\alpha\) is not isomorphic to \(\mathcal{D}_n\).

**Proof.** Using the preceding lemma, \(\mathcal{D}_n / [\mathcal{D}_n, \mathcal{D}_n] \cong \text{coker}(M_n - I) \oplus \mathbb{Z} \cong \mathbb{Z}^2\) where the matrix \(M_n\) was defined in Section 2, and \(\Gamma_\alpha / [\Gamma_\alpha, \Gamma_\alpha] \cong \text{coker}(G_\alpha - I) \oplus \mathbb{Z}\) and \(G_\alpha\) is the matrix defined in (3). But it is easy to see that \(\text{coker}(G_\alpha - I) = \text{coker}(B) \oplus \mathbb{Z}\), where

\[
B = \begin{bmatrix}
b_{12} & b_{13} & \cdots & b_{1n} \\
0 & b_{23} & \cdots & \vdots \\
\vdots & \ddots & \ddots & b_{n-2,n} \\
0 & \cdots & 0 & b_{n-1,n}
\end{bmatrix}_{(n-1) \times (n-1)}
\]

Thus \(\Gamma_\alpha / [\Gamma_\alpha, \Gamma_\alpha] \cong \text{coker}(B) \oplus \mathbb{Z}^2\). So a necessary condition for \(\Gamma_\alpha \cong \mathcal{D}_n\) is that \(\text{coker}(B) = 0\) i.e. \(|\det B| = \prod_{i=1}^{n-1} |b_{i,i+1}| = 1\), which ends the proof.

**Corollary 5.5.** \(\mathcal{D}_{n,i}'\) is isomorphic to a subgroup of \(\mathcal{D}_{n-i}\), and is not isomorphic to \(\mathcal{D}_{n-i}\) unless \(C_i = 1\).

**Proof.** Note that in this special case, \(b_{rs} = (C_i)_{s-r}\). Now use the preceding corollary and Lemma 5.1.

**Lemma 5.6.** Let \(C_i = |Y_i|\) be the cardinality of \(Y_i\) for \(i = 1, \ldots, n - 1\). Then the flow \((\mathbb{Z}, Y_i \times \mathbb{T}^{n-i})\) generated by \(\phi_i\) is topologically conjugate to a flow \((\mathbb{Z}, \mathbb{Z}_{C_i} \times \mathbb{T}^{n-i})\) generated by

\[
\psi_i : (v_i, v_{i+1}, \ldots, v_n) \mapsto (\lambda_i v_i, \eta_i v_{i+1}, v_{i+1} v_{i+2}, \ldots, v_{n-1} v_n),
\]

where \(\lambda_i\) is a primitive \(C_i\)-th root of unity and \(\eta_i \in \mathbb{T}\) is chosen appropriately.

**Proof.** We construct a homeomorphism \(\tau_i : Y_i \times \mathbb{T}^{n-i} \to Z_{C_i} \times \mathbb{T}^{n-i}\) that commutes with the actions of \(\mathbb{Z}\) i.e. \(\tau_i \circ \phi_i = \psi_i \circ \tau_i\). For the moment, fix \(\nu_{i+1}, \nu_{i+2}, \ldots, v_{n-1} \in \mathbb{T}\), and define \(\tau_i\) as follows for \(v_{i+1}, \ldots, v_n \in \mathbb{T}\):

\[
\tau_i(1, \ldots, v_{i+1}, v_{i+2}, \ldots, v_{n-1} v_{n-1}, v_n) = (1, v_{i+1} v_{i+1}, v_{i+2} v_{i+2}, \ldots, v_{n-1} v_{n-1}, v_n)
\]
step (1):

\[
\tau_i \circ \phi_i(1, \ldots, 1, v_{i+1}, \ldots, v_n) = \tau_i(\lambda, \mu, \ldots, v_{i+1}, v_{i+2}, \ldots, v_{n-1}v_n)
\]

\[
= \psi_i \circ \tau_i(1, \ldots, 1, v_{i+1}, \ldots, v_n)
\]

\[
= \psi_i(1, v_{i+1}, v_{i+2}, \ldots, v_{n-1}v_{n-1}, v_{n-1}v_{n-1})
\]

\[
= (\lambda, \eta_1 \nu_1 v_{i+1,1}, \nu_1 v_{i+1,2} v_{i+2} v_{i+3}, \ldots, v_{n-2} v_{n-1} v_{n-1}, v_{n-1} v_{n-1})
\]

step (2):

\[
\tau_i \circ \phi_i^2(1, \ldots, 1, v_{i+1}, \ldots, v_n) = \tau_i(\lambda^2, \lambda \mu_i^2 \mu_i^2, \ldots, v_{i-2} \mu_i^2 v_{i-1}, v_{i-1} v_{i+1}^2, v_{i+2} v_{i+3} v_{i+4}, \ldots, v_{n-3} v_{n-1} v_{n-2} v_{n-1}, v_{n-2} v_{n-1} v_{n-1})
\]

\[
= \psi_i(\lambda, \eta_1 \nu_1 v_{i+1,1}, \nu_1 v_{i+1,2} v_{i+2} v_{i+3}, \ldots, v_{n-2} v_{n-1} v_{n-1}, v_{n-1} v_{n-1})
\]

and so on down to step \((C_i)\):

\[
\tau_i \circ \phi_i^C_i(1, \ldots, 1, v_{i+1}, \ldots, v_n)
\]

\[
= \tau_i(\lambda^{C_i}, \lambda^{(C_i)^2} \mu_1^{C_i}, \lambda^{(C_i)^3} \mu_2^{C_i}, \ldots, \lambda^{(C_i)^{C_i-1}} \mu_2^{C_i} \ldots \mu_2^{C_i-1}, \ldots, \mu_2^{C_i-1}) \mu_2^{C_i-1}, \ldots, \mu_2^{C_i-1})
\]

\[
\end{align*}

\[
= \psi_i^{C_i}(1, \ldots, 1, v_{i+1}, \ldots, v_n) = \psi_i^{C_i}(1, v_{i+1} v_{i+1}, v_{i+2}, \ldots, v_n)
\]
The definition of $\tau_i$ on $(1, \ldots, 1, \mathbb{T}^{n-i})$ at this last step must coincide with the definition at the first step, so $\eta_i$ and $\nu_{i+1}, \nu_{i+2}, \ldots, \nu_{n-1}$ are chosen to satisfy the equations

$$\lambda_i(C_i) \mu_1^{(C_i)} \mu_2^{(C_i)} \ldots \mu_i^{(C_i)} = \lambda_i^{(C_i)} \eta_i^{(C_i)}$$

$$\lambda_i^{C_i+1} \mu_1^{(C_i)} \mu_2^{(C_i)} \ldots \mu_i^{(C_i)} \eta_i^{(C_i)} = (-1)^{C_i+1} \lambda_i^{(C_i)} \mu_1^{(C_i)} \mu_2^{(C_i)} \ldots \mu_i^{(C_i)}$$

The above equations are true since $\lambda_i$ is a primitive $C_i$-th root of unity, and then choose $\nu_{i+1}$ to be a $C_i$-th root of

$$(\eta_{i+1}^{(C_i)})$$

and then choose $\nu_{i+2}$ to be a $C_i$-th root of

$$(\eta_{i+2}^{(C_i)})$$

and so on choose $\nu_{i+3}, \ldots, \nu_{n-1}$ recursively to satisfy the above equations.

To see that $\tau_i$ commutes with the actions of $\mathbb{Z}$, take a point $P \in Y_i \times T^{n-i}$. Then $P = \phi_i^r(1, \ldots, 1, v_{i+1}, \ldots, v_n)$ for some $0 \leq r < C_i$ and $v_{i+1}, \ldots, v_n \in \mathbb{T}$.
and
\[
\tau_i \circ \phi_i(P) = \tau_i \circ \phi_i^{r+1}(1, \ldots, 1, v_{i+1}, \ldots, v_n)
\]
\[
= \psi_i^{r+1} \circ \tau_i(1, \ldots, 1, v_{i+1}, \ldots, v_n)
\]
\[
= \psi_i \circ \tau_i \circ \phi_{i+1}^{r}(1, \ldots, 1, v_{i+1}, \ldots, v_n)
\]
\[
= \psi_i \circ \tau_i(P),
\]
as required. \hfill \square

**Corollary 5.7.** \(A_i^{(n)} \cong \mathcal{C}(\mathbb{Z}_C \times \mathbb{T}^{n-i}) \rtimes \psi_i \mathbb{Z}, \) where \(\psi_i\) was introduced in the preceding lemma. In particular, the algebra \(A_i^{(n)}\) is also a simple infinite dimensional quotient of \(C^*(\mathfrak{D}_{n-i+1})\).

**Proof.** The first part is clear, in view of the preceding lemma. For the second part, note that \(\mathcal{C}(\mathbb{Z}_C \times \mathbb{T}^{n-i}) \rtimes \psi_i \mathbb{Z}\) is generated by unitaries satisfying (CR) \((n, i, 1)\) for \(i = 1, \ldots, n-1\). \(\square\)

**Theorem 5.8.** The algebra \(A_i^{(n)} = \mathcal{C}(\mathbb{Y}_i \times \mathbb{T}^{n-i}) \rtimes \psi_i \mathbb{Z}\) above is isomorphic to \(M_{C_i}(B_i^{(n)}\), where \(B_i^{(n)}\) is the \(C^*\)-algebra generated by \((\mathfrak{CR})_{n-i, 1}\) for \(i = 1, \ldots, n-1\).

**Proof.** By Lemma 5.6, \(A_i^{(n)} \cong \mathcal{C}(\mathbb{Z}_C \times \mathbb{T}^{n-i}) \rtimes \psi_i \mathbb{Z},\) hence this crossed product is simple too. For convenience, put \(q := C_i\) and \(m := n - i\) and let \(D = \mathcal{C}(\mathbb{Z}_q \times \mathbb{T}^{m}) \rtimes \psi \mathbb{Z}\), where \(\psi(v, v_1, v_2, \ldots, v_m) = (\lambda v, \eta v_1 v_2, \ldots, v_{m-1} v_m)\), in which \(\lambda\) is a primitive \(q\)-th root of unity and \(\eta\) not a root of unity. Since \(D\) is simple, it is the unique \(C^*\)-algebra generated by unitaries \(U, V, V_1, \ldots, V_m\) such that \(V^q = I\) and

\[(5) \quad [U, V] = \lambda, [U, V_1] = \eta V, [U, V_2] = V_1, \ldots, [U, V_m] = V_{m-1}\]

(all other pairs of unitaries from \(U, V, V_1, \ldots, V_m\) commute).

Let \(\zeta := (-1)^{q+1} \eta^q\) and \(B\) be the unique (simple) \(C^*\)-algebra generated by unitaries \(\check{U}, V_1, \ldots, \check{V}_m\) such that

\[(6) \quad [\check{U}, \check{V}_1] = \zeta, [\check{U}, \check{V}_2] = \check{V}_1^q, \ldots, [\check{U}, \check{V}_m] = \check{V}_{m-1}^{(m-1)} \check{V}_{m-2}^{(m-2)} \cdots \check{V}_1^q\]

(all other pairs of unitaries from \(\check{U}, \check{V}_1, \ldots, \check{V}_m\) commute). We prove that \(D \cong M_{q}(B)\). Define unitaries \(U', V', V_1', \ldots, V_m'\) in \(M_{q}(B)\) as follows (all unspecified entries being 0).

\(U'\) has \(\check{U}\) in the upper right-hand corner and 1’s on the subdiagonal.

\(V' = \text{diag}(1, \lambda, \lambda^2, \ldots, \lambda^{q-1}).\)

\(V_1' = \text{diag}(b\check{V}_1, b\eta \lambda \check{V}_1, b\eta^2 \lambda^2 \check{V}_1, \ldots, b\eta^{q-1} \lambda^{(q/2)} \check{V}_1).\)

\(V_2' = \text{diag}(c_2 \check{V}_2, c_22 \check{V}_1^{-1} \check{V}_2, c_22 \check{V}_1^{-2} \check{V}_2, \ldots, c_2 q \check{V}_1^{-(q-1)} \check{V}_2).\)
$$V'_3 = \text{diag}(c_{31}V_3, c_{32}V_1V_2^{-1}V_3, c_{33}V_1^2V_2^{-2}V_3, \ldots, c_{3q}V_1^{(q)}V_2^{-(q-1)}V_3).$$

$$\vdots$$

$$V'_m = \text{diag}(c_{m1}V_m, c_{m2}V_1^{(-1)^{m-1}}V_2^{(-1)^{m-2}}\ldots V_m, \ldots, c_{mq}V_1^{(-1)^{m-1}(q+m-3)}V_2^{(-1)^{m-2}(q+m-4)}\ldots V_{m-1}^{-(q-1)}V_m).$$

Thus the $j$-th entry of $V'_i$ equals $c_{ij} \prod_{r=1}^{i} \tilde{V}_r^{(-1)^{j-r}(1+r-2)}$, and the constants $b_i$, $c_{ij} \in \mathbb{T}$ must be chosen so that the unitaries $U', V', V'_1, \ldots, V'_m$ satisfy (4). Note that $V'^q = I_q$ and $U'$ has the property that

$$[U', \text{diag}(\tilde{W}_1, \ldots, \tilde{W}_q)] = \text{diag}(\tilde{U}\tilde{W}_q\tilde{U}^{-1}\tilde{W}_1^{-1}, \tilde{W}_1\tilde{W}_2^{-1}, \ldots, \tilde{W}_{q-1}\tilde{W}_q^{-1})$$

for arbitrary invertibles $\tilde{W}_1, \ldots, \tilde{W}_q \in B$. Therefore

$$[U', V'] = \text{diag}(\tilde{\lambda}^{q-1}, \lambda, \ldots, \lambda) = \lambda I_q.$$

$$[U', V'_1] = \text{diag}(\tilde{U}(b\eta^{-1}\lambda^{(q)}_1)\tilde{V}_1)\tilde{U}^{-1}(b\tilde{V}_1)^{-1}, (b\tilde{V}_1)(b\eta\lambda\tilde{V}_1)^{-1}, \ldots,$$

$$(b\eta^{-2}\lambda^{(q')_1})\tilde{V}_1)(b\eta^{-1}\lambda^{(q)}_1)\tilde{V}_1)^{-1}$$

$$= \text{diag}(\eta^{-1}\lambda^{(q)}_1)[U, \tilde{V}_1], \eta\lambda, \ldots, \eta\lambda^{(q')_1})$$

$$= \text{diag}(\eta^{-1}\lambda^{(q)}_1\zeta, \eta\lambda, \ldots, \eta\lambda^{(q')_1})$$

$$= \text{diag}(\eta^{-1}\lambda^{(q)}_1, \eta\lambda, \ldots, \eta\lambda^{(q')_1})$$

$$= \eta V'$$

(note that since $\lambda$ is a primitive $q$-th root of unity, one can easily see that $\lambda^{(q)}_1 = (-1)^{q+1}$).

$$[U', V'_2] = \text{diag}(\tilde{U}(c_{2q}V_1^{-(q-1)}\tilde{V}_2)\tilde{U}^{-1}(c_{21}\tilde{V}_2)^{-1}, (c_{21}\tilde{V}_2)(c_{22}\tilde{V}_1V_2^{-1}\tilde{V}_2)^{-1}, \ldots,$$

$$(c_{2,q-1}\tilde{V}_1^{-(q-2)}\tilde{V}_2)(c_{2q}\tilde{V}_1^{-(q-1)}\tilde{V}_2)^{-1})$$

$$= \text{diag}(\frac{c_{2q}}{c_{21}}(\tilde{U}\tilde{V}_1^{-(q-1)}\tilde{U}^{-1})(\tilde{U}\tilde{V}_2\tilde{U}^{-1}\tilde{V}_2)^{-1}, \frac{c_{21}}{c_{22}}\tilde{V}_1, \ldots, \frac{c_{2,q-1}}{c_{2q}}\tilde{V}_1)$$

$$= \text{diag}(\frac{c_{2q}}{c_{21}}(\zeta^{-(q-1)}\tilde{V}_1^{-(q-1)})(\tilde{V}_1\tilde{V}_2\tilde{V}_2)^{-1}, \frac{c_{21}}{c_{22}}\tilde{V}_1, \ldots, \frac{c_{2,q-1}}{c_{2q}}\tilde{V}_1)$$

$$= \text{diag}(\frac{c_{2q}}{c_{21}}\zeta^{-(q-1)}\tilde{V}_1, \frac{c_{21}}{c_{22}}\tilde{V}_1, \ldots, \frac{c_{2,q-1}}{c_{2q}}\tilde{V}_1)$$

$$= V'_1.$$
By multiplying these equations, one obtains $b^q \tilde{\eta}^{(q)} \lambda^{(q+1)} = \tilde{\zeta}^{-1}$, so one must choose $b$ to be a $q$-th root of
\[
\tilde{\lambda}^{(q+1)} \tilde{\eta}^{(q)} \tilde{\zeta}^{-1}.
\]

\[
[U', V'_q] = \text{diag}(\tilde{U}(c_{3q})(\tilde{V}_1)^{(q)}) V_2^{-(q-1)} V_3 (c_{3q}^{-1})(c_{3q}^{-1}) V_2^{-(q-1)} V_3^{-1},
\]
\[
\ldots, (c_{3q}^{-1} V_1^{(q)} V_2^{-(q-2)} V_3 (c_{3q}^{-1}) V_2^{-(q-1)} V_3^{-1})
\]
\[
= \text{diag}(c_{3q} \tilde{U} V_1^{(q)} V_2^{-(q-1)} V_3^{-(q-1)}), c_{3q}^{-1} V_2^{-(q-1)} V_3^{-(q-1)}),
\]
\[
= \text{diag}(c_{3q} \tilde{U} V_1^{(q)} V_2^{-(q-1)} V_3^{-(q-1)}), c_{3q}^{-1} V_2^{-(q-1)} V_3^{-(q-1)}),
\]
\[
= c_{3q}^{-1} V_2^{-(q-1)} V_3^{-(q-1)}),
\]
\[
= V'_q.
\]

The last equality holds if, and only if,
\[
(8) \quad \frac{c_{3q}}{c_{31}} \tilde{\zeta}^{(q)} = c_{21}, \quad \frac{c_{31}}{c_{32}} = c_{22}, \quad \ldots, \quad \frac{c_{3q}^{-1}}{c_{3q}} = c_{2q}.
\]

By multiplying the above equalities, one obtains $c_{21} \ldots c_{2q} = \zeta^{(q)}$ and combining with (6) one must choose $c_{21}$ to be a $q$-th root of
\[
\lambda^{(q+1)} \tilde{\eta}^{(q)} b^{(q)} \zeta^{(q)}
\]
and then
\[
c_{2j} = c_{21} \lambda^{(q+1)} \tilde{\eta}^{(q)} b^{(q)} \zeta^{(q)}
\]

One can continue this procedure down to
\[
[U', V'_m] = \text{diag}(\tilde{U}(c_{mq} V_1^{(q)}) V_2^{-(q-1)} V_3 (c_{mq}^{-1})(c_{mq}^{-1}) V_2^{-(q-1)} V_3^{-1},
\]
\[
\ldots, (c_{m,q}^{-1} V_1^{(q)} V_2^{-(q-2)} V_3 (c_{m,q}^{-1}) V_2^{-(q-1)} V_3^{-1})
\]
\[
= \text{diag}(c_{m,q} \tilde{U} V_1^{(q)} V_2^{-(q-1)} V_3^{-(q-1)}), c_{m,q}^{-1} V_2^{-(q-1)} V_3^{-(q-1)}),
\]
\[
= \text{diag}(c_{m,q} \tilde{U} V_1^{(q)} V_2^{-(q-1)} V_3^{-(q-1)}), c_{m,q}^{-1} V_2^{-(q-1)} V_3^{-(q-1)}),
\]
\[
= c_{m,q}^{-1} V_2^{-(q-1)} V_3^{-(q-1)}),
\]
\[
= V'_m.
\]
Using the next lemma and the paragraph following it, we have

\[
\begin{align*}
\frac{c_{mq}}{c_{m1}} \tilde{V}_1^{(-1)^{m-1} (q+m-3)_{m-1}} \cdots \tilde{V}_m^{-(q-1)} \tilde{V}_m \tilde{V}_m^{-1},
\frac{c_{m1}}{c_{m2}} \tilde{V}_2^{(-1)^{m-2}} \cdots \tilde{V}_m^{-1},
\frac{c_{mq}}{c_{m1}} \tilde{V}_1^{(-1)^{m-2} (q+m-4)_{m-2}} \cdots \tilde{V}_m^{-(q-1)} \tilde{V}_m^{-1}
\end{align*}
\]

\[
= \text{diag} \left( \frac{c_{mq}}{c_{m1}} \left( \tilde{V}_1^{(-1)^{m-1} (q+m-3)_{m-1}} \cdots \tilde{V}_m^{-(q-1)} \tilde{V}_m \tilde{V}_m^{-1} \right) \right)
\]

\[
= \text{diag} \left( \frac{c_{mq}}{c_{m1}} \left( \tilde{V}_1^{(-1)^{m-2}} \cdots \tilde{V}_m^{-1} \right) \right)
\]

\[
= \text{diag} \left( \frac{c_{mq}}{c_{m1}} \left( \tilde{V}_1^{(-1)^{m-2} (q+m-4)_{m-2}} \cdots \tilde{V}_m^{-1} \right) \right)
\]

\[
= \text{diag} \left( \frac{c_{mq}}{c_{m1}} \left( \tilde{V}_1^{(-1)^{m-2} (q+m-4)_{m-2}} \cdots \tilde{V}_m^{-1} \right) \right)
\]

\[
= \text{diag} \left( \frac{c_{mq}}{c_{m1}} \left( \tilde{V}_1^{(-1)^{m-2} (q+m-4)_{m-2}} \cdots \tilde{V}_m^{-1} \right) \right)
\]

\[
= \text{diag} \left( \frac{c_{mq}}{c_{m1}} \left( \tilde{V}_1^{(-1)^{m-2} (q+m-4)_{m-2}} \cdots \tilde{V}_m^{-1} \right) \right)
\]

\[
= V_{m-1}.
\]

Using the next lemma and the paragraph following it, we have

\[
\sum_{r=1}^{m} (-1)^{m-r} \binom{q + m - r - 2}{m - r} \binom{q}{r - s} = \delta_{s,m-1} \quad (1 \leq s \leq m - 1)
\]

where \( \delta \) denotes the delta function. Therefore the last equality holds if, and only if,

\[
\frac{c_{mq}}{c_{m1}} \zeta^{(-1)^{m-1} (q+m-3)_{m-1}} = c_{m-1,1}, \quad \frac{c_{m1}}{c_{m2}} = c_{m-1,2}, \ldots, \quad \frac{c_{mq}}{c_{m2}} = c_{m-1,q}.
\]

By multiplying the above equalities, one obtains

\[
\prod_{j=1}^{q} c_{m-1,j} = \zeta^{(-1)^{m-1} (q+m-3)_{m-1}}.
\]
One can see that $c_{m-1,1}$ must be chosen as a $q$-th root of

$$
\zeta(-1)^{m-1} \binom{q+m-2}{m-1} \prod_{k=-1}^{m-2} c_{k1}^{(-1)^{m-k-1} \binom{q+m-k-3}{m-k}}
$$

where $c_{-1,1} := \lambda$, $c_{01} := \eta$ and $c_{11} := b$. Then

$$
c_{m-1,j} = \prod_{k=-1}^{m-1} c_{k1}^{(-1)^{m-k-1} \binom{q+m-k-3}{m-k}}
$$

Then one can show that

$$
c_{m,j} := \prod_{k=-1}^{m} c_{k1}^{(-1)^{m-k-1} \binom{q+m-k-2}{m-k}}
$$

is a solution for (8), where $c_{k1}$ was chosen recursively in the previous steps for $k = 1, \ldots, m - 1$ and $c_{m1} \in \mathbb{T}$ is chosen arbitrary. In accordance with the previous steps, we let $c_{m1}$ be a $q$-th root of

$$
\zeta(-1)^{m} \binom{q+m-2}{m-2} \prod_{k=-1}^{m-1} c_{k1}^{(-1)^{m-k+1} \binom{q+m-k-1}{m-k}}
$$

So the unitaries $U', V', V'_1, \ldots, V'_m$ satisfy (5). It just remains to prove that they generate $M_q(B) = M_q(\mathbb{C}) \otimes B$. Let $E_{ij}$ denote the $q \times q$ matrix with all zero entries except for a 1 in the $i,j$ entry ($1 \leq i, j \leq q$). These form a set of matrix units for $M_q(\mathbb{C})$. We first show that all $E_{ij}$’s are generated by $U', V'$. Take $E = \frac{1}{q} \sum_{k=0}^{q-1} V'^k$. Then $E = \text{diag}(1, 0, 0, \ldots, 0) = E_{11}$ and one can check that $E_{ij} = U'^{(i-1)}E^{V'_{j-1}}$. Now that we have a copy of $M_q(\mathbb{C})$ generated by $U'$ and $V'$, it is enough to prove that the elements $E_{11} \otimes \tilde{U}$ and $E_{11} \otimes \tilde{V}_i$ (for $i = 1, \ldots, m$) belong to $C^*(U', V', V'_1, \ldots, V'_m)$ (for these elements generate $E_{11} \otimes B$ and moving this around with the matrix units will generate $M_q(\mathbb{C}) \otimes B$). But according to the definition of $U', V', V'_1, \ldots, V'_m$, one can easily verify that $E_{11} \otimes \tilde{U} = U'E_{q1}$ and $E_{11} \otimes \tilde{V}_i = \tilde{c}_{i1} E_{11} V'_i$ (for $i = 1, \ldots, m$), in which the constants $c_{i1} \in \mathbb{T}$ for $i > 1$ were introduced above and $c_{11} := b$. This completes the proof. \qed

The following combinatorial lemma presents a generalized form of the identity $\binom{m-1}{k} = \binom{m}{k} - \binom{m-1}{k-1}$.

**Lemma 5.9.** Using Notation 2.1, we have

$$
\binom{m-q}{k} = \sum_{j \geq 0} (-1)^j \binom{m-j}{k-j} \binom{q}{j}
$$

for all $m, k \in \mathbb{Z}$ and $q \in \mathbb{N}$. 
Proof. First one checks that for \( q = 1 \), the identity \( \binom{m-1}{k} = \binom{m}{k} - \binom{m-1}{k-1} \) holds for all \( m, k \in \mathbb{Z} \). Then using induction on \( q \), one has

\[
\binom{m-q-1}{k} = \binom{m-q}{k} - \binom{m-q-1}{k-1} = \sum_{j \geq 0} (-1)^j \binom{m-j}{k-j} \binom{q}{j} - \sum_{j \geq 0} (-1)^j \binom{m-1-j}{k-1-j} \binom{q}{j} = \sum_{j \geq 0} (-1)^j \binom{m-j}{k-j} \binom{q}{j} - \sum_{j \geq 1} (-1)^{j-1} \binom{m-j}{k-j} \binom{q}{j-1} = \sum_{j \geq 0} (-1)^j \binom{m-j}{k-j} \binom{q}{j}.
\]

\( \Box \)

Now, to prove (9) in the preceding theorem, let \( j := r-s \) and \( k := m-s \). Then we have

\[
\sum_{r=1}^{m} (-1)^{m-r} \binom{q+m-r-2}{m-r} \binom{q}{r-s} = \sum_{j=1-s}^{k} (-1)^{k+j} \binom{q+k-2-j}{k-j} \binom{q}{j} = (-1)^k \sum_{j \geq 0} (-1)^j \binom{q+k-2-j}{k-j} \binom{q}{j} = (-1)^k \binom{k-2}{k} = \delta_{k1} = \delta_{m-s,1} = \delta_{s,m-1}.
\]

6. \( K \)-theory for \( A_{n,\theta} \)

In this section we study the \( K \)-theory of \( A_{n,\theta} \). First, we describe the method of computation for the \( K \)-groups of \( \mathcal{C}(\mathbb{T}^n) \rtimes_{\alpha} \mathbb{Z} \), where \( \alpha \) is an arbitrary homeomorphism of \( \mathbb{T}^n \). To do this, we will pay attention to the algebraic structure of \( K^*(\mathbb{T}^n) \). Note that it is sufficient to consider the special case of “linear” homeomorphisms since as stated in the introduction, we know that every continuous function from \( \mathbb{T}^n \) to \( \mathbb{T}^n \) is homotopic to a “linear” function \( f(v_1, \ldots, v_n) = v_1^{a_1} v_2^{a_2} \ldots v_n^{a_n} \) \((a_1, a_2, \ldots, a_n \in \mathbb{Z})\), and that the \( K \)-groups of \( \mathcal{C}(\mathbb{T}^n) \rtimes_{\alpha} \mathbb{Z} \) depend up to isomorphism only on the homotopy class of \( \alpha \).
It is well known that $K^*(\mathbb{T}^n)$ is a $\mathbb{Z}_2$-graded ring and by the Künneth formula \([1]\), it is an exterior algebra (over $\mathbb{Z}$) on $n$ generators, where the elements of even degree are in $K^0(\mathbb{T}^n)$ and those of odd degree are in $K^1(\mathbb{T}^n)$. The generators of this exterior algebra correspond to the generators of the dual group $\mathbb{Z}^n$ of $\mathbb{T}^n$ \([28\, \text{p. 185}]\). Indeed, in this case the Chern character
\[
\text{ch} : K^*(\mathbb{T}^n) \longrightarrow \tilde{H}^*(\mathbb{T}^n, \mathbb{Q})
\]
is integral and gives the Chern isomorphisms
\[
\begin{align*}
\text{ch}_0 : & K^0(\mathbb{T}^n) \longrightarrow \tilde{H}^{\text{even}}(\mathbb{T}^n, \mathbb{Z}), \\
\text{ch}_1 : & K^1(\mathbb{T}^n) \longrightarrow \tilde{H}^{\text{odd}}(\mathbb{T}^n, \mathbb{Z}),
\end{align*}
\]
where $\tilde{H}^*(\mathbb{T}^n, \mathbb{Z}) \cong \Lambda_2^*(e_1, \ldots, e_n)$ is the (Čech) cohomology ring of $\mathbb{T}^n$ under the cup product, and $\tilde{H}^k(\mathbb{T}^n, \mathbb{Z}) \cong \Lambda_2^k(e_1, \ldots, e_n)$. On the other hand, $K^*(\mathbb{T}^n) \cong K_*^{\bullet}(\mathbb{C}(\mathbb{T}^n))$. So, by introducing $e_i := [v_i]_1$ (i.e. the class in $K_1(\mathbb{C}(\mathbb{T}^n))$ of the coordinate function $v_i : \mathbb{T}^n \rightarrow \mathbb{T}$ as an element of $\mathcal{U}(\mathbb{C}(\mathbb{T}^n))$ ) for $i = 1, \ldots, n$, we have the isomorphisms $K_*^{\bullet}(\mathbb{C}(\mathbb{T}^n)) \cong \Lambda_2^*(e_1, \ldots, e_n) \cong \Lambda^{*}\mathbb{Z}^n$, and it is in a way that respects the canonical embedding of $\mathbb{Z}^n$. Moreover, such an isomorphism is unique since only the identity automorphism of the ring $\Lambda^*\mathbb{Z}^n$ fixes each element of $\mathbb{Z}^n$.

Now we use the Pimsner-Voiculescu six term exact sequence \([24]\) as the main tool for computing the $K$-groups of $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$. Let $\alpha_\ast (= K_\ast(\alpha))$ be the ring automorphism of $K_*^{\bullet}(\mathbb{C}(\mathbb{T}^n))$ induced by $\alpha$ and let $\alpha_i$ be the restriction of $\alpha_\ast$ on $K_i(\mathbb{C}(\mathbb{T}^n)); (i = 0, 1)$. Let $A := \mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$. Then we have the following exact sequence.

\[
\begin{array}{cccc}
K_0(\mathbb{C}(\mathbb{T}^n)) & \overset{\alpha_0 - \text{id}}{\longrightarrow} & K_0(\mathbb{C}(\mathbb{T}^n)) & \overset{j_0}{\longrightarrow} & K_0(A) \\
\delta_1 & & & \delta_0 & \\
\end{array}
\]

Here, $j : \mathcal{C}(\mathbb{T}^n) \rightarrow A$ is the canonical embedding of $\mathcal{C}(\mathbb{T}^n)$ in $A$, $j_0 := K_0(j)$ and $j_1 := K_1(j)$. Also from now on id denotes the identity function on each underlying set. As a result, we have the following short exact sequences

\[
\begin{align*}
0 & \longrightarrow \text{coker}(\alpha_0 - \text{id}) \longrightarrow K_0(\mathbb{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \longrightarrow \ker(\alpha_1 - \text{id}) \longrightarrow 0, \\
0 & \longrightarrow \text{coker}(\alpha_1 - \text{id}) \longrightarrow K_1(\mathbb{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \longrightarrow \ker(\alpha_0 - \text{id}) \longrightarrow 0.
\end{align*}
\]

Since all the groups involved are abelian and $\ker(\alpha_i - \text{id})$ is torsion-free $(i = 0, 1)$, these short exact sequences split and we have
\[
\begin{align*}
& (11) \quad K_0(\mathbb{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong \text{coker}(\alpha_0 - \text{id}) \oplus \ker(\alpha_1 - \text{id}), \\
& (12) \quad K_1(\mathbb{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong \text{coker}(\alpha_1 - \text{id}) \oplus \ker(\alpha_0 - \text{id}).
\end{align*}
\]

So, it suffices to determine the kernel and cokernel of $(\alpha_0 - \text{id})$ and $(\alpha_1 - \text{id})$ acting as endomorphisms on the finitely generated abelian groups $\Lambda_2^{\text{even}}(e_1, \ldots, e_n)(\cong \mathbb{Z}^{2n-1})$ and $\Lambda_2^{\text{odd}}(e_1, \ldots, e_n)(\cong \mathbb{Z}^{2n-1})$, respectively. Note
that from the isomorphisms (11) and (12), the $K$-groups of $\mathcal{C}(T^n) \times_\alpha \mathbb{Z}$ are finitely generated abelian groups. Now since $\alpha_*$ becomes a ring homomorphism, it suffices to know the action of $\alpha_*$ on $e_1, \ldots, e_n$. In fact for a general basis element $e_i \wedge e_{i+1} \wedge \cdots \wedge e_r$ of $K_*(\mathcal{C}(T^n)) \cong \Lambda^n_k(e_1, \ldots, e_n)$ we have

$$\alpha_*(e_i \wedge e_{i+1} \wedge \cdots \wedge e_r) = \alpha_*(e_i) \wedge \alpha_*(e_{i+1}) \wedge \cdots \wedge \alpha_*(e_r).$$

Thus if we consider $\{e_1, \ldots, e_n\}$ as the canonical basis of $\mathbb{Z}^n$ and take $\hat{\alpha} = \alpha_*|_{\mathbb{Z}^n}$, we have $\alpha_*=\hat{\alpha} = \oplus_{r=1}^n \wedge^r \hat{\alpha}$, $\alpha_0 = \wedge^{even} \hat{\alpha} = \oplus_{r\geq 0} \wedge^{2r} \hat{\alpha}$ and $\alpha_1 = \wedge^{odd} \hat{\alpha} = \oplus_{r\geq 0} \wedge^{2r+1} \hat{\alpha}$, where $\wedge^i \hat{\alpha}$ is the $i$-th exterior power of $\hat{\alpha}$, which acts on $\Lambda^i \mathbb{Z}^n$ ($i=0,1,\ldots,n$). Now let $\alpha = (f_1, \ldots, f_n)$ and $b_{ij} := A_i[f_j]$ or in other words, assume that $f_i$ is homotopic to $(e_1, \ldots, e_n) \mapsto e_i^1 v_2 e_i^2 \cdots e_i^n$.

So we can write

$$\alpha_*(e_i) = \alpha_*[v_i]_1 = [\alpha(v_i)]_1 = [f_i(v_1, \ldots, v_n)]_1 = [v_i^1 v_i^2 \cdots v_i^n]_1 = \sum_{j=1}^n b_{ij} [v_j]_1 = \sum_{j=1}^n b_{ij} e_j.$$

Therefore $\hat{\alpha}$ acts on $\mathbb{Z}^n$ via the corresponding integer matrix $A := [b_{ij}]_{n \times n} \in GL(n, \mathbb{Z})$ and $\alpha_*$ acts on $\Lambda^* \mathbb{Z}^n$ via $\Lambda^* A$, and we have the following isomorphisms

$$K_0(\mathcal{C}(T^n) \times_\alpha \mathbb{Z}) \cong \text{coker}(\alpha_0 - \text{id}) \oplus \ker(\alpha_1 - \text{id})$$

$$= \text{coker}(\oplus_{r\geq 0} \wedge^{2r} \hat{\alpha} - \text{id}) \oplus \ker(\oplus_{r\geq 0} \wedge^{2r+1} \hat{\alpha} - \text{id}),$$

so we can write

$$K_0(\mathcal{C}(T^n) \times_\alpha \mathbb{Z}) \cong \bigoplus_{r\geq 0} \text{coker}(\wedge^{2r} \hat{\alpha} - \text{id}) \oplus \ker(\wedge^{2r+1} \hat{\alpha} - \text{id}),$$

and similarly

$$K_1(\mathcal{C}(T^n) \times_\alpha \mathbb{Z}) \cong \bigoplus_{r\geq 0} \text{coker}(\wedge^{2r+1} \hat{\alpha} - \text{id}) \oplus \ker(\wedge^{2r} \hat{\alpha} - \text{id}).$$

Therefore, in order to compute the $K$-groups of $\mathcal{C}(T^n) \times_\alpha \mathbb{Z}$, we must find the kernel and cokernel of $\wedge^r \hat{\alpha} - \text{id}$ as an endomorphism of $\Lambda^r \mathbb{Z}^n$ ($r = 0, 1, \ldots, n$). Note that the matrix of $\wedge^r \hat{\alpha} - \text{id}$ with respect to the canonical basis $\{e_1, \ldots, e_r\}$ with lexicographic order is $A_{n,r} := \wedge^r A - I_r$, which is an integer matrix of order $\binom{n}{r}$ ($I_k$ is the identity matrix of order $k$, we often omit $k$ whenever it is clear). So by computing the kernel and cokernel of $A_{n,r}$ for $r = 0, 1, \ldots, n$ with appropriate tools (such as the Smith normal form for integer matrices [20, p. 26]), one can determine the $K$-groups of $\mathcal{C}(T^n) \times_\alpha \mathbb{Z}$. The authors have written a program in Maple 6.0 to do these computations.

We summarize the discussion above in the following proposition.
Proposition 6.1. Let \( \alpha \) be a homeomorphism of \( \mathbb{T}^n \) and \( \hat{\alpha} \in \text{Aut}(\mathbb{Z}^n) \) be the restriction of \( \alpha_* \) to \( \mathbb{Z}^n \) (as above). Then \( \alpha_* = \wedge^r \hat{\alpha} = \bigoplus_{r=1}^n \wedge^r \hat{\alpha} \) on \( K^*(\mathbb{T}^n) = \Lambda^* \mathbb{Z}^n \) and

\[
K_0(C(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong \bigoplus_{r \geq 0} [\text{coker}(\wedge^{2r} \hat{\alpha} - \text{id}) \oplus \ker(\wedge^{2r+1} \hat{\alpha} - \text{id})],
\]

\[
K_1(C(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong \bigoplus_{r \geq 0} [\text{coker}(\wedge^{2r+1} \hat{\alpha} - \text{id}) \oplus \ker(\wedge^{2r} \hat{\alpha} - \text{id})].
\]

Corollary 6.2. The \( K \)-groups of \( C(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z} \) are finitely generated abelian groups with the same rank. Moreover, this common rank equals

\[
\text{rank ker}(\wedge^* \hat{\alpha} - \text{id}) = \sum_{r=0}^n \text{rank ker}(\wedge^r \hat{\alpha} - \text{id}).
\]

Proof. Use the proposition and note that for any \( \varphi \in \text{End}(\mathbb{Z}^n) \) one has \( \text{rank ker} \varphi = \text{rank coker} \varphi \). \qed

Now for the \( K \)-groups of \( A_{n,\theta} \cong C(\mathbb{T}^n) \rtimes_{\sigma} \mathbb{Z} \), the “linearized” form of the corresponding affine homeomorphism \( \sigma \) is as follows

\[
(v_1, v_2, \ldots, v_n) \mapsto (v_1, v_1 v_2, \ldots, v_{n-1} v_n).
\]

So \( \sigma(e_i) = e_{i-1} + e_i \) for \( i = 1, \ldots, n \) (\( e_0 := 0 \)). The matrix with respect to the canonical basis \( \{e_1, \ldots, e_n\} \) of \( \mathbb{Z}^n \) that corresponds to \( \sigma \) is

\[
S_n := \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{n \times n}
\]

Notation 6.3. We let \( a_n := \text{rank } K_0(A_{n,\theta}) = \text{rank } K_1(A_{n,\theta}) \) and \( a_{n,r} := \text{rank ker}(\wedge^r S_n - I) \) for \( r = 0, 1, \ldots, n \). From the preceding corollary we have

\[
a_n = \text{rank ker}(\wedge^* S_n - I) = \sum_{r=0}^n a_{n,r}.
\]

Corollary 6.4. \( K_i(C^*(\mathcal{D}_n)) \cong K_i(A_{n+1,\theta}) \) for \( i = 0, 1 \). In particular

\[
\text{rank } K_0(C^*(\mathcal{D}_n)) = \text{rank } K_1(C^*(\mathcal{D}_n)) = a_{n+1}.
\]

Proof. Since \( \mathcal{D}_n \cong \mathbb{Z}^{n+1} \rtimes_{\eta} \mathbb{Z} \), so \( C^*(\mathcal{D}_n) \cong C^*(\mathbb{Z}^{n+1}) \rtimes_{\tilde{\eta}} \mathbb{Z} \cong C(\mathbb{T}^{n+1}) \rtimes_{\tilde{\eta}} \mathbb{Z} \) and the integer matrix corresponding to \( \tilde{\eta} \) is the \( (n+1) \times (n+1) \) matrix \( M_n \) introduced in Section 1, which is precisely \( S_{n+1} \). The rest of proof follows from the last proposition. \( \square \)

Some examples will illustrate the methods described.
Example 6.5. We compute the \( K \)-groups of \( A_5^{5,5} \), which have been computed in \cite{30} by another method. In fact, the Chern character and non-commutative geometry were used in \cite{30} to compute the kernel and cokernel of \( \sigma_i - \text{id} \) (\( i=0,1 \)). As \( A_5^{5,5} = A_{3,3} \), we compute the kernel and cokernel of \( S_{3,r} := \wedge^r S_3 - l_1 \) for \( r = 0, 1, 2, 3 \), in which

\[
S_3 = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

- \( r = 0 \), \( S_{3,0} = \wedge^0 S_3 - l_1 = [0] \). So, \( \ker S_{3,0} = \mathbb{Z} \) and \( \coker S_{3,0} = \mathbb{Z}/(0) \cong \mathbb{Z} \).
- \( r = 1 \), \( S_{3,1} = \wedge^1 S_3 - l_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \). So,
  \[
  \ker S_{3,1} = \{(x, y, z) \in \mathbb{Z}^3 | \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\} = (\mathbb{Z}, 0, 0) \cong \mathbb{Z}
  \]
  \[
  \coker S_{3,1} = \mathbb{Z}^3/\ker S_{3,1} = \mathbb{Z}^3/\langle e_1, e_2 \rangle \cong \mathbb{Z}.
  \]
- \( r = 2 \), \( S_{3,2} = \wedge^2 S_3 - l_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \).
  \[
  \ker S_{3,2} = \{(x, y, z) \in \mathbb{Z}^3 | x + y = 0 \} = (\mathbb{Z}, 0, 0) \cong \mathbb{Z}
  \]
  \[
  \coker S_{3,2} = \mathbb{Z}^3/\ker S_{3,2} = \mathbb{Z}^3/\langle e_1, e_2 \rangle \cong \mathbb{Z}.
  \]
- \( r = 3 \), \( S_{3,3} = \wedge^3 S_3 - l_1 = [0] \). So, \( \ker S_{3,3} = \mathbb{Z} \) and \( \coker S_{3,3} = \mathbb{Z}/(0) \cong \mathbb{Z} \).

Therefore

\[
K_0(A_5^{5,5}) = K_0(A_{3,3}) \cong (\coker S_{3,0} \oplus \coker S_{3,2}) \oplus (\ker S_{3,1} \oplus \ker S_{3,3})
\]
\[
\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^4,
\]

\[
K_1(A_5^{5,5}) = K_1(A_{3,3}) \cong (\coker S_{3,1} \oplus \coker S_{3,3}) \oplus (\ker S_{3,0} \oplus \ker S_{3,2})
\]
\[
\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^4.
\]

Example 6.6. Using our method, we have obtained the \( K \)-groups of \( A_{n,6} \) by computer for \( 4 \leq n \leq 11 \). We find the kernels and cokernels of \( S_{n,r} := \wedge^r S_n - l_1 \) for \( r = 0, 1, \ldots, n \) using the Smith normal form theorem \cite[p. 26]{20}. The results obtained are stated in Table 1. Because of computational limitations, we do not have any results yet for \( n > 11 \), except for \( \{a_n\} \) for which we have obtained the generating functions (see Subsection 6.1).

In this table, the group \( \mathbb{Z}_k^{(m)} \) is the direct product of \( m \) copies of the cyclic group \( \mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z} \), and it seems that the \( K \)-groups of \( A_{n,6} \) generally have torsion. The first example is \( K_1(A_{6,6}) \); this is in fact because \( \coker S_{6,3} = \coker(\wedge^3 S_6 - l_{20}) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_2 \). One of the things that we are interested in studying is the behavior of the sequence \( \{a_n\} \). We will show below the importance of this sequence, namely \( a_n \) is the common rank of the \( K \)-groups.
of a certain set of $\mathcal{C}^*$-algebras including the Furstenberg transformation group $\mathcal{C}^*$-algebras $A_{F_\gamma,\theta}$ of $\mathbb{T}^n$ introduced in [12].

Note that by Proposition 6.1, the $K$-groups of $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$ are completely determined by the corresponding homomorphism $\hat{\alpha} \in \text{Aut}(\mathbb{Z}^n)$ and its exterior powers. From a computational point of view, we only need the cokernels of the maps involved, since we know that for any endomorphism $\varrho$ on $\mathbb{Z}^m$, $\ker \varrho \cong \text{coker} \varrho / (\text{coker} \varrho)_{\text{tor}}$. When $\hat{\alpha} = 1$, we don’t even need to compute all the cokernels. In other words, we have the next proposition, for which we recall a definition.

**Definition 6.7.** Let $\hat{\alpha}, \hat{\beta} \in \text{End}(\mathbb{Z}^m)$. We say that $\hat{\alpha}$ is equivalent to $\hat{\beta}$ over $\mathbb{Z}$ (and write $\hat{\alpha} \equiv \hat{\beta}$) if there exist $\hat{u}, \hat{v} \in \text{Aut}(\mathbb{Z}^m)$ such that $\hat{u} \circ \hat{\alpha} \circ \hat{v} = \hat{\beta}$. Similarly, if $A$ and $B$ are integer $m \times m$ matrices, $A$ is equivalent to $B$ if there exist $U, V \in \text{GL}(m, \mathbb{Z})$ such that $UAV = B$.

Recall that $\hat{\alpha} \equiv \hat{\beta}$ if, and only if, $\text{coker} \hat{\alpha} \cong \text{coker} \hat{\beta}$ if, and only if, $\hat{\alpha}$ and $\hat{\beta}$ have the same Smith normal form. Also, $A$ equiv $B$ if, and only if, $B$ is obtainable from $A$ by a finite number of elementary operations. An elementary operation on an integer matrix is one of the following types: interchanging two rows (or two columns), adding an integer multiple of one row (or column) to another, and multiplying a row (or column) by $-1$.

**Proposition 6.8.** Let $\hat{\alpha} \in \text{SL}(n, \mathbb{Z})$ (i.e. $\det \hat{\alpha} = 1$). Then $\wedge^n \hat{\alpha} - \text{id}$ and $\wedge^{n-r} \hat{\alpha} - \text{id}$ are equivalent as endomorphisms of $\Lambda^r \mathbb{Z}^n = \Lambda^{n-r} \mathbb{Z}^n = \mathbb{Z}^{(r)}$. Equivalently, $\text{coker}(\wedge^n \hat{\alpha} - \text{id}) \cong \text{coker}(\wedge^{n-r} \hat{\alpha} - \text{id})$ for $r = 0, 1, \ldots, n$.

**Proof.** We prove the equivalence of the endomorphisms as matrices with respect to some basis. Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ be a basis for $\mathbb{Z}^n$ and put $S = \{1, 2, \ldots, n\}$. For $I = \{i_1, \ldots, i_r\} \subset S$, where $1 \leq i_1 < \ldots < i_r \leq n$, put $e_I = e_{i_1} \wedge \ldots \wedge e_{i_r} \in \Lambda^r \mathbb{Z}^n$. Then $\mathcal{E}_r := \{e_I \mid I \subset S, \ |I| = r\}$ is a
basis for $\Lambda^r \mathbb{Z}^n$. Let $\omega := e_1 \wedge \ldots \wedge e_n$, which generates $\Lambda^n \mathbb{Z}^n$. When $r = 0$, $\wedge^0 \hat{\alpha} - \text{id} = 0$ and $\wedge^n \hat{\alpha}(\omega) = \hat{\alpha}(e_1) \wedge \ldots \wedge \hat{\alpha}(e_n) = (\det \hat{\alpha})(e_1 \wedge \ldots \wedge e_n) = \omega$, so $\wedge^n \hat{\alpha} - \text{id} = 0$. Now, fix an $r \in \{1, \ldots, n-1\}$. For an arbitrary subset $I \subset S$ with $|I| = r$, take $J = E \setminus I = \{j_1, \ldots, j_{n-r}\}$, so $|J| = n - r$. Then $e_i \wedge e_J = (\text{sgn} \mu)\omega$, in which $\mu \in S_n$ is the permutation that converts $(1, 2, \ldots, n)$ to $(i_1, \ldots, i_r, j_1, \ldots, j_{n-r})$. It is easily seen that $\mu = \mu_1 \ldots \mu_r$, where $\mu_k$ is the permutation that takes $i_k$ from its position in $(1, 2, \ldots, n)$ to its new position in $(i_1, \ldots, i_r, j_1, \ldots, j_{n-r})$. One can see that $\mu_k$ is the combination of $i_k - (r - k + 1)$ transpositions $(k = 1, \ldots, r)$. Thus
\[
\text{sgn} \mu = \prod_{k=1}^r (-1)^{i_k - (r - k + 1)} = (-1)^{\ell(I) - r(r + 1)/2},
\]
where $\ell(I) := \sum_{k=1}^r i_k$. Now take $m = \binom{n}{r} = \binom{n}{n-r}$ and let $E_r = \{e_{I_1}, \ldots, e_{I_m}\}$ be a basis for $\Lambda^r \mathbb{Z}^n$. Now write $E_{n-r} = \{e_{J_1}, \ldots, e_{J_m}\}$ as the basis for $\Lambda^{n-r} \mathbb{Z}^n$ such that $J_k = E \setminus I_k$ for $k = 1, \ldots, m$. From the above argument one can write
\[
e_{I_1} \wedge e_{J_j} = (-1)^{\ell(I_1) - r(r + 1)/2} \delta_{ij} \omega,
\]
since if $i \neq j$ then $I_i \cap J_j = \emptyset$ thus $e_{I_i} \wedge e_{J_j} = 0$. Now let $A = [a_{ij}]_{m \times m}$ and $B = [b_{ij}]_{m \times m}$ be the corresponding integer matrices of $\wedge^r \hat{\alpha}$ and $\wedge^{n-r} \hat{\alpha}$ with respect to $E_r$ and $E_{n-r}$, respectively. So $\wedge^r \hat{\alpha}(e_{I_i}) = \sum_{p=1}^m a_{pi} e_{I_p}$ and $\wedge^{n-r} \hat{\alpha}(e_{J_j}) = \sum_{q=1}^m b_{qj} e_{J_q}$. What we want to show is that $A - I$ is equivalent to $B - I$. We have
\[
\wedge^n \hat{\alpha}(e_{I_i} \wedge e_{J_j}) = (-1)^{\ell(I_1) - r(r + 1)/2} \delta_{ij} \omega = \wedge^r \hat{\alpha}(e_{I_i}) \wedge \wedge^{n-r} \hat{\alpha}(e_{J_j}) = \sum_{p=1}^m a_{pi} b_{qj} (-1)^{\ell(I_p) - r(r + 1)/2} \delta_{pq} \omega;
\]
thus one obtains
\[
\sum_{k=1}^m (-1)^{\ell(I_k) - \ell(I_1)} a_{ki} b_{kj} = \delta_{ij}.
\]
Therefore if we take $c_{ij} = (-1)^{\ell(I_1) - \ell(I_1)} a_{ji}$ and $C := [c_{ij}]_{m \times m}$, then $c_{ij} - \delta_{ij} = (-1)^{\ell(I_1) - \ell(I_1)} (a_{ji} - \delta_{ji})$; therefore $C - I$ is obtained from $A - I$ by changing rows (and columns) and occasionally multiplying some rows (and columns) by -1. So $C - I$ is equivalent to $A - I$. On the other hand, (13) means that $CB = I$. Thus $C - I = C(B - I)(-I)$, hence $B - I$ is equivalent to $C - I$. So $A - I$ is equivalent to $B - I$.

**Corollary 6.9.** If $\det \hat{\alpha} = 1$, then $\text{rank ker}(\wedge^r \hat{\alpha} - \text{id}) = \text{rank ker}(\wedge^{n-r} \hat{\alpha} - \text{id})$. In particular, $a_{n,r} = a_{n,n-r}$ for $r = 0, 1, \ldots, n$.

**Corollary 6.10.** Let $A := C(\mathbb{T}^{2m-1}) \ltimes_\alpha \mathbb{Z}$ for which the corresponding homomorphism $\hat{\alpha}$ satisfies $\det \hat{\alpha} = 1$. Then $K_0(A) \cong K_1(A)$ and the (common) rank of the $K$-groups of $A$ is an even number. In particular, for every...
Furstenberg transformation group \( C^* \)-algebra \( A_{Ff,\theta} \) of an odd-dimensional torus (including \( A_{2m-1,\theta} \)), one has \( K_0(A_{Ff,\theta}) \cong K_1(A_{Ff,\theta}) \).

**Proof.** Combining the preceding proposition and Proposition 6.1, one obtains

\[
K_0(A) \cong K_1(A) \cong \bigoplus_{k=0}^{m-1}[\text{coker}(\wedge^k \hat{\alpha} - \text{id}) \oplus \text{ker}(\wedge^k \hat{\alpha} - \text{id})].
\]

As a result, the rank of the \( K \)-groups of \( A \) is an even number since the ranks of the cokernel and kernel of an endomorphism coincide. Note that for \( A_{Ff,\theta} \) the corresponding integer matrix of \( \hat{\alpha} \) is an upper triangular matrix with 1’s on diagonal. Thus \( \det \hat{\alpha} = 1 \). \( \square \)

6.1. **The rank \( a_n \) of the \( K \)-groups of \( A_{n,\theta} \).** In this part we study some general properties of \( a_n \). We specify some \( C^* \)-algebras whose ranks of \( K \)-groups are related to the sequence \( \{a_n\} \). As an application, we characterize the rank of the \( K \)-groups of Furstenberg transformation group \( C^* \)-algebras \( A_{Ff,\theta} \) and simple infinite dimensional quotients of \( C^*(\mathcal{D}_n) \), which were studied in Sections 4 and 5. To this end, we remind the reader of some linear algebraic properties of nilpotent and unipotent matrices.

**Definition 6.11.** Let \( V \) be a (complex) vector space. An \( \hat{\epsilon} \in \text{End}_C V \) is called nilpotent (respectively, unipotent) if \( \hat{\epsilon}^k = 0 \) (respectively, \( (\hat{\epsilon} - \text{id})^k = 0 \)) for some positive integer \( k \). The minimum value of \( k \) with this property is called the degree of \( \hat{\epsilon} \), denoted \( \text{deg}(\hat{\epsilon}) \).

As an example, every upper (respectively, lower) triangular matrix with zeros on the diagonal is nilpotent. Also, \( S_n \) as defined above is a unipotent matrix of degree \( n \). Note that all eigenvalues of a nilpotent (respectively, unipotent) matrix are zero (respectively, one). In particular, every unipotent matrix is invertible. Thus every unipotent endomorphism is an automorphism.

**Corollary 6.12.** Let \( V \) be a finite dimensional complex vector space and \( \hat{\epsilon} \) be a nilpotent (respectively, unipotent) endomorphism of \( V \). Then \( \text{deg}(\hat{\epsilon}) \) is equal to the maximum order of its Jordan blocks.

**Proof.** It suffices to prove the statement for the nilpotent case. Since all the eigenvalues of \( \hat{\epsilon} \) are zero, each Jordan block is a zero matrix of order one or in the form

\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 0 & 1 \\
\end{bmatrix},
\]

which is a nilpotent matrix and its degree is the same as its order, which is greater than 1. Now the rest of proof is clear. \( \square \)
**Definition 6.13.** Let $V$ be a finite dimensional complex vector space and $\hat{e} \in \text{End}_C V$ be nilpotent (respectively, unipotent). We say that $\hat{e}$ is of maximal degree if $\deg(\hat{e}) = \dim V$.

**Corollary 6.14.** Let $V$ be an $n$-dimensional complex vector space and $\hat{e} \in \text{End}_C V$ be nilpotent (respectively, unipotent). Then $\deg(\hat{e}) \leq n$. If $\deg(\hat{e}) = n$, the Jordan normal form of $\hat{e}$ is the following matrix of order $n$

$$
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & & \\
0 & \ddots & \ddots & \ddots & 0 \\
& \ddots & \ddots & \ddots & \\
& & 0 & 0 & 1 \\
& & & 0 & \cdots \\
0 & \cdots & 0 & 0 & 0 \\
\end{bmatrix}
$$

In particular, all nilpotent (respectively, unipotent) matrices of maximal degree acting on $V$ are similar.

**Proof.** Use the preceding corollary and the Jordan normal form theorem. □

**Example 6.15.** Let $B = [b_{ij}]_{n \times n}$ be any upper triangular matrix whose diagonal elements are all zero (respectively, one) and whose entries $b_{i,i+1}$ for $i = 1, \ldots, n-1$ are all nonzero. Then $B$ is nilpotent (respectively, unipotent) of maximal degree. In fact, let $n$ be the order of $B$ and let $b := \prod_{i=1}^{n-1} b_{i,i+1}$, which is a nonzero number. Then one can easily see that $B^n = 0$ (respectively, $(B - I)^n = 0$) and $B^{n-1}$ (respectively, $(B - I)^{n-1}$) is a matrix with $b$ appearing in the northeast corner and zeros elsewhere. So $\deg(B) = n$ i.e., $B$ is of maximal degree.

As a useful conclusion, we compare the ranks of the $K$-groups of $C^*$-algebras of the form $C(T^n) \rtimes_\alpha \mathbb{Z}$ in the following theorem, which shows the importance of the rank $a_n$ of the $K$-groups of $A_{n,\theta}$.

**Theorem 6.16.** Let $A := C(T^n) \rtimes_\alpha \mathbb{Z}$, in which $\alpha$ is a homeomorphism of $T^n$ whose corresponding integer matrix $A \in \text{GL}(n, \mathbb{Z})$ is unipotent of maximal degree (i.e. $\deg(A) = n$). Then

$$
\text{rank } K_0(A) = \text{rank } K_1(A) = a_n = \text{rank } K_0(A_{n,\theta}) = \text{rank } K_1(A_{n,\theta}).
$$

In particular, the rank of the $K$-groups of any Furstenberg transformation group $C^*$-algebra $A_{F_{\hat{\alpha}},\hat{\theta}} = C(T^n) \rtimes_{F_{\hat{\alpha}},\hat{\theta}} \mathbb{Z}$ is equal to the rank of the $K$-groups of $A_{n,\theta}$, i.e. to $a_n$.

**Proof.** Let $\hat{\alpha}$ denote the restriction of $\alpha$ to $\mathbb{Z}^n$ and $A$ be the corresponding matrix of $\hat{\alpha}$ acting on $\mathbb{Z}^n$. Also let $S_n$ be the corresponding matrix for $A_{n,\theta}$. Now $A$ is assumed to be unipotent of maximal degree, and $S_n$ is unipotent of maximal degree too. So the last corollary yields that $A$ is similar to $S_n$. In fact the Jordan normal form of $A - I$ is precisely $S_n - I$. On the other hand, from Corollary 6.2 we know that the rank of the $K$-groups of $A$ is equal to $\text{rank } \ker(\wedge^n A - I)$. But the similarity of $A$ and $S_n$ yields
the similarity of $\wedge^* A - 1$ and $\wedge^* S_n - 1$ as matrices acting on $\Lambda^* C^n$. So $\dim_{C^*} \ker(\wedge^* A - 1) = \dim_{C^*} \ker(\wedge^* S_n - 1)$, which yields the result.

For the second part, note that the corresponding integer matrix of a Furstenberg transformation $F_{f, \theta}$ on $T^n$ is of the form

\[
\begin{pmatrix}
1 & b_{12} & b_{13} & \cdots & b_{1n} \\
0 & 1 & b_{23} & & \\
& \ddots & \ddots & \ddots & b_{n-2,n} \\
& & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & b_{n-1,n} \\
0 & \cdots & 0 & 0 & 1
\end{pmatrix}_{n \times n},
\]

which, following the preceding example, is unipotent of maximal degree since $b_{i,i+1} \neq 0$ for $i = 1, \ldots, n - 1$. Now the proof of the first part yields the result. \qed

Remark 6.17. In the preceding theorem, the basis for $\mathbb{Z}^n$ for the matrices involved is $\{e_1, \ldots, e_n\}$, where $e_i := [v_i]_1$ as introduced at the beginning of Section 6. But it is interesting to observe that if $\hat{\alpha}$ is an arbitrary unipotent automorphism of $\mathbb{Z}^n$, then there is a basis for $\mathbb{Z}^n$ with respect to which the integer matrix $A$ of $\hat{\alpha}$ is of the form $(\heartsuit)$ above (but not necessarily with $b_{i,i+1} \neq 0$ for $i = 1, \ldots, n - 1$, unless $\hat{\alpha}$ is of maximal degree) [8, Theorems 16, 18]. The unipotence of $\hat{\alpha}$ also has important effects on the dynamics of the generated flow. For example, if $\alpha$ is an affine transformation on $T^n$ and $\hat{\alpha}$ is unipotent, then the dynamical system $(T^n, \alpha)$ has quasi-discrete spectrum [8, Theorem 19]. More generally, let $\alpha = (t, A)$ be an affine transformation on $T^n$ and take $Z_p(A) = \ker(A^p - \text{id}) \subset \mathbb{Z}^n$ for $p \in \mathbb{N}$ and consider the following conditions

1. $Z_1(A) = Z_p(A)$, $\forall p \in \mathbb{N}$,
2. $t$ is rationally independent over $Z_1(A)$, i.e., if $k = (k_1, \ldots, k_n) \in Z_1(A)$ is such that $\langle t, k \rangle := \sum_{j=1}^n t_j k_j$ is a rational number, then $k = 0$.
3. $Z_1(A) \neq \{0\}$.
4. $A$ is unipotent.

Then $(T^n, \alpha)$ is ergodic with respect to Haar measure if and only if $\alpha$ satisfies (1) and (2) [8]. Moreover, if $\alpha$ satisfies (1) through (4), $(T^n, \alpha)$ is minimal, uniquely ergodic with respect to Haar measure, and has quasi-discrete spectrum. Conversely, any minimal transformation of $T^n$ with topologically quasi-discrete spectrum is conjugate to an affine transformation which must satisfy (1) through (4) [9]. The $C^*$-algebras corresponding to such actions are therefore simple and have a unique tracial state.
Corollary 6.18. Let $A$ be a simple infinite dimensional quotient of $C^*(\mathfrak{D}_n)$. Then $\text{rank} K_0(A) = \text{rank} K_1(A) = a_{n-i}$ for some $i \in \{0, 1, \ldots, n-1\}$, determined by the isomorphism $A \cong C(Y_i \times \mathbb{T}^{n-i}) \rtimes \phi_i \mathbb{Z}$ in Theorem 5.8.

Proof. It was proved in Section 5 that $A$ is isomorphic to a matrix algebra over a Furstenberg transformation group $C^*$-algebra $B^{(n)}_i$ on $\mathbb{T}^{n-i}$ for some suitable $i$. So $K_j(A) \cong K_j(B^{(n)}_i)$ for $j = 0, 1$. The rest of proof is clear from the preceding theorem. \hfill \Box

We will see in Proposition 6.19 below that $\{a_n\}$ is a strictly increasing sequence. Therefore the preceding corollary is a first step towards the classification of the simple infinite dimensional quotients of $C^*(\mathfrak{D}_n)$ by means of $K$-theory. But as is seen, the rank of the $K$-groups can not distinguish the algebras in the same “level” (i.e. those algebras that are included in the same case (i) in Section 5, but with different values of the parameters). The other powerful $K$-theoretical object that helps us do this is the trace invariant, i.e. the range of the unique tracial state acting on the $K_0$-group.

Proposition 6.19. Suppose $A \cong C(Y_i \times \mathbb{T}^{n-i}) \rtimes \phi_i \mathbb{Z}$ is a simple infinite dimensional quotient of $C^*(\mathfrak{D}_n)$ as in Theorem 5.8. Then $A$ has a unique tracial state $\bar{\tau}$ and $\bar{\tau}_* K_0(A) = \frac{1}{2\pi i}(\mathbb{Z} + \mathbb{Z} \phi_i)$, where $C_i = |Y_i|$ and $e^{2\pi i \phi_i} = \zeta_i = (-1)^{C_i+1} \eta_i^{C_i}$ as in Lemma 5.6 and Theorem 5.8.

Proof. Following Theorem 5.8 $A$ is isomorphic to $MC_i(B^{(n)}_i) = MC_i(\mathbb{C}) \rtimes B^{(n)}_i$, where $B^{(n)}_i$ is the simple $C^*$-algebra generated by $(\mathbb{C} \mathbb{R})_{n,i}$ for $\zeta_i = (-1)^{C_i+1} \eta_i^{C_i}$. By Corollary 1.1 $B^{(n)}_i$ has a unique tracial state $\tau$ and $\tau_* K_0(B^{(n)}_i) = \mathbb{Z} + \mathbb{Z} \phi_i$, where $e^{2\pi i \phi_i} = \zeta_i$ [12, Theorem 2.23]. Thus $A$ has the unique tracial state $\bar{\tau} = (\frac{1}{2\pi i} \text{Tr}) \rtimes \tau$, in which $\text{Tr}$ is the usual trace on $MC_i(\mathbb{C})$, and so $\bar{\tau}_* K_0(A) = \frac{1}{C_i}(\mathbb{Z} + \mathbb{Z} \phi_i)$ [12, Lemma 3.5]. \hfill \Box

Corollary 6.20. $A_{n,\theta} \cong A_{n',\theta'}$ if, and only if, $n = n'$ and there exists an integer $k$ such that $\theta = k \pm \theta'$. More generally, let $A^{(n)}_i \cong C(Y_i \times \mathbb{T}^{n-i}) \rtimes \phi_i \mathbb{Z}$ be a simple infinite dimensional quotient of $C^*(\mathfrak{D}_n)$ with the structure constants $\lambda, \mu_1, \ldots, \mu_i$ as in case (i) of Section 5 and let $A^{(n')}_i \cong C(Y'_{i'} \times \mathbb{T}^{n'-i'}) \rtimes \phi_i' \mathbb{Z}$ be a simple infinite dimensional quotient of $C^*(\mathfrak{D}_{n'})$ with the structure constants $\lambda', \mu'_1, \ldots, \mu'_i$. Suppose that $C_i = |Y_i|$ and $C_i' = |Y'_{i'}|$. Then $A^{(n)}_i \cong A^{(n')}_i$ if, and only if, $n - i = n' - i'$, $C_i = C_i'$ and

$$\lambda^{(c_i)} \mu_1^{(c_i)} \mu_2^{(c_i)} \ldots \mu_i^{(c_i)} = \lambda'^{C_i'} \mu_1^{C_i'} \mu_2^{C_i'} \ldots \mu_i^{C_i'}$$

or

$$\lambda^{(C_i)} \mu_1^{(C_i)} \mu_2^{(C_i)} \ldots \mu_i^{(C_i)} = (\lambda'^{C_i'} \mu_1^{C_i'} \mu_2^{C_i'} \ldots \mu_i^{C_i'})^{-1}.$$
Proof. Use the proposition and the fact that \( \{a_n\} \) is a strictly increasing sequence (see Proposition 6.33 below). Note that

\[
\zeta_i = (-1)^{C_i+1} \eta_i = \lambda^{(i)} \mu_1^{(i)} \mu_2^{(i-1)} \cdots \mu_i^{C_i}.
\]

\( \square \)

Remark 6.21. Note that \( C_i = |Y_i| \) is completely determined by the structural constants \( \lambda, \mu_1, \ldots, \mu_{i-1} \) (which are roots of unity). More precisely

\[
C_i = \min \{ r \in \mathbb{N} \mid \lambda^r = \lambda^{(i)} \mu_1^r = \cdots = \lambda^{(i)} \mu_1^{(i-1)} \mu_2^{(i-2)} \cdots \mu_{i-1}^1 = 1 \}.
\]

As an example, see [18, Lemma 5.4].

6.2. Some combinatorial properties of \( a_{n,r} \). As mentioned before, our main goal is to describe \( a_n \) as the rank of the \( K \)-groups of \( A_{n,\theta} \). Since \( a_n = \sum_{r=0}^{n} a_{n,r} \), we study \( a_{n,r} \) first. So this part is devoted to some combinatorial properties of \( a_{n,r} \) as the rank of \( \ker(\wedge^r \hat{\alpha} - \text{id}) \) for \( r = 0, 1, \ldots, n \). In other words we show that \( a_{n,r} \) equals the number of partitions of \( \lfloor r(n+1)/2 \rfloor \) to \( r \) distinct positive integers not greater than \( n \). To do this, we will use properties of the irreducible representations of the semisimple Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \). First, we need some more elementary properties of nilpotent linear mappings.

Lemma 6.22. Let \( V \) be an arbitrary (complex) vector space and \( \hat{\epsilon} \in \text{End}_\mathbb{C} V \) be nilpotent of degree \( k \). Then \( \exp(\hat{\epsilon}) \) is unipotent of degree \( k \). Moreover, \( \exp(\hat{\epsilon}) - \text{id} \) is similar to \( \hat{\epsilon} \).

Proof. For the first part, we know that \( \exp(\hat{\epsilon}) - \text{id} = \hat{\epsilon} + \hat{\epsilon}^2/2! + \cdots + \hat{\epsilon}^{k-1}/(k-1)! = \hat{\omega} \), where \( \omega = \text{id} + \hat{\epsilon}^2/2! + \cdots + \hat{\epsilon}^{k-2}/(k-1)! \) commutes with \( \hat{\epsilon} \) and is invertible since it is unipotent. So, \( (\exp(\hat{\epsilon}) - \text{id})^r = (\hat{\omega})^r = \hat{\epsilon}^r \omega^r \) for each positive integer \( r \). Thus \( \exp(\hat{\epsilon}) - \text{id} \) is unipotent with the same degree of \( \hat{\epsilon} \). For the second part, using the Jordan normal form of \( \hat{\epsilon} \), it is sufficient to prove the statement for the special case when \( \hat{\epsilon} \) is a Jordan block with zeros on the diagonal. Since in this case \( \hat{\epsilon} \) is of maximal degree, by the first part, \( \exp(\hat{\epsilon}) - \text{id} \) is also of maximal degree. Therefore they are similar by Corollary 6.3. \( \square \)

Let \( V \) be an arbitrary (complex) vector space and \( \hat{\phi} : V \to V \) be a linear mapping. Then \( \hat{\phi} \) can be extended in a unique way to a homomorphism \( \wedge^* \hat{\phi} : \Lambda^* V \to \Lambda^* V \) such that \( \wedge^* \hat{\phi}(1) = 1 \), yielding

\[
\wedge^* \hat{\phi}(x_1 \wedge \cdots \wedge x_p) = \hat{\phi}(x_1) \wedge \cdots \wedge \hat{\phi}(x_p) \ (x_i \in V).
\]
Let's define $D$ nilpotent.

$\phi$ yielding deduce that know that $D$ also $\hat{\phi}$ can be extended in a unique way to a derivation $D^*\hat{\phi} : \Lambda^* V \to \Lambda^* V$, yielding

$$D^*\hat{\phi}(x_1 \wedge \ldots \wedge x_p) = \sum_{i=1}^{\infty} x_1 \wedge \ldots \wedge \hat{\phi}(x_i) \wedge \ldots \wedge x_p \ (p \geq 2, x_i \in V).$$

Let's define $\wedge^r \hat{\phi} := \wedge^r \hat{\phi}|_{\Lambda^r V}$ and $D^r \hat{\phi} := D^*\hat{\phi}|_{\Lambda^r V}$ as induced linear mappings on the $r$-th exterior power of $V$ ($r \geq 0$). Then we have

$$\wedge^r \hat{\phi} = \bigoplus_{r \geq 0} \wedge^r \hat{\phi}, \ D^*\hat{\phi} = \bigoplus_{r \geq 0} D^r \hat{\phi}.$$

Lemma 6.23. With the above notation, if $\hat{\phi} : V \to V$ is nilpotent, $\wedge^r \hat{\phi}$ and $D^r \hat{\phi}$ are also nilpotent for $r \geq 1$. If $V$ is finite dimensional, $D^*\hat{\phi}$ is nilpotent.

Proof. Assume that $\hat{\phi}^t = 0$ for some $t \in \mathbb{N}$. We have $(\wedge^r \hat{\phi})^t(x_1 \wedge \ldots \wedge x_r) = \hat{\phi}^t(x_1) \wedge \ldots \wedge \hat{\phi}^t(x_r) = 0$ so $(\wedge^r \hat{\phi})^t = 0$ and $\wedge^r \hat{\phi}$ is nilpotent. For $D^r \hat{\phi}$ we know that $D^r \hat{\phi}(x_1 \wedge \ldots \wedge x_r) = \sum_{i=1}^{\infty} x_1 \wedge \ldots \wedge \hat{\phi}(x_i) \wedge \ldots \wedge x_r$ and one can deduce that

$$D^r \hat{\phi}^p(x_1 \wedge \ldots \wedge x_r) = \sum_{i_1+\ldots+i_r=p} \frac{p!}{(i_1)!(\ldots)(i_r)!} \hat{\phi}^{i_1} x_1 \wedge \ldots \wedge \hat{\phi}^{i_r} x_r.$$ 

Now since $i_1 + \ldots + i_r = p$, there exists an $i_j$ with $i_j \geq p/r$. So if $p \geq rt$ then $\hat{\phi}^{i_j} = 0$ and $D^r \hat{\phi}^p = 0$. Thus $D^r \hat{\phi}$ is nilpotent.

For the next part, let $m := \dim V$. Since $D^*\hat{\phi} = \bigoplus_{r \geq 0} D^r \hat{\phi}$ and $\hat{\phi}_0 = 0$, from the first part we have $(D^*\hat{\phi})^m = 0$ and $D^*\hat{\phi}$ is nilpotent too. \qed

Lemma 6.24. Let $\hat{\phi} : V \to V$ be a nilpotent linear mapping. Then

$$\exp(D^*\hat{\phi}) = \wedge^* \exp(\hat{\phi})$$

on $\Lambda^* V$.

Proof. We have

$$\exp(D^*\hat{\phi})(x_1 \wedge \ldots \wedge x_r) = \sum_{p \geq 0} \frac{1}{p!} D^r \hat{\phi}^p(x_1 \wedge \ldots \wedge x_r)$$

$$= \sum_{p \geq 0} \frac{1}{p!} \left( \sum_{i_1+\ldots+i_r=p} \frac{p!}{(i_1)!(\ldots)(i_r)!} \hat{\phi}^{i_1} x_1 \wedge \ldots \wedge \hat{\phi}^{i_r} x_r \right)$$

$$= \sum_{i_j \geq 0} \frac{1}{(i_1)!(\ldots)(i_r)!} \hat{\phi}^{i_1} x_1 \wedge \ldots \wedge \hat{\phi}^{i_r} x_r$$

$$= \left( \sum_{i_1 \geq 0} \frac{1}{(i_1)!} \hat{\phi}^{i_1} x_1 \right) \wedge \ldots \wedge \left( \sum_{i_r \geq 0} \frac{1}{(i_r)!} \hat{\phi}^{i_r} x_r \right)$$
\[= (\exp(\dot{\phi})x_1) \wedge \ldots \wedge (\exp(\dot{\phi})x_r)\]
\[= \wedge^r \exp(\dot{\phi})(x_1 \wedge \ldots \wedge x_r),\]

which yields the result. Note that all sums in the above equalities are finite according to the previous lemma. \hfill \square

**Remark 6.25.** The nilpotence of \( \dot{\phi} \) is not necessary in the preceding lemma. In fact, one may use the definition of \( \exp : \mathfrak{gl}(\Lambda^*V) \to \text{GL}(\Lambda^*V) \). More precisely, define \( s : \mathbb{R} \to \text{GL}(\Lambda^*V) \) by \( s(t) = \wedge^* \exp(t\dot{\phi}) \). Then one may check that \( s \) is the 1-parameter subgroup generated by \( D^*\dot{\phi} \) (i.e. \( \dot{s}(0) = D^*\dot{\phi} \)) and \( s(1) = \wedge^* \exp(\dot{\phi}) \).

**Corollary 6.26.** Let \( \hat{\phi} : V \to V \) be a nilpotent linear mapping. Then for \( r \geq 0 \), \( \exp(D^r\hat{\phi}) = \wedge^r \exp(\hat{\phi}) \) on \( \Lambda^rV \). In particular, if \( \hat{\epsilon} := \hat{\phi} + \text{id} \), then \( \wedge^r \hat{\epsilon} - \text{id} \) is similar to \( D^r\hat{\phi} \).

**Proof.** The first part follows immediately from the lemma. For the second part, we know from Lemma 6.22 that \( \exp(\hat{\phi}) - \text{id} \) is similar to \( \hat{\phi} \), hence \( \exp(\hat{\phi}) \) is similar to \( \hat{\phi} + \text{id} = \hat{\epsilon} \). So
\[\wedge^r \hat{\epsilon} - \text{id} \sim \wedge^r \exp(\hat{\phi}) - \text{id} \]
\[\exp(D^r\hat{\phi}) - \text{id} \sim D^r\hat{\phi} \]
\hfill \square

**Proposition 6.27.** Let \( \sigma \) be the Anzai transformation on \( \mathbb{T}^n \) and \( \sigma_* \) be the corresponding induced homomorphism on \( K_*(\mathbb{C}(\mathbb{T}^n)) = \Lambda^*\mathbb{Z}^n \). Let \( \dot{\sigma} \) be the restriction of \( \sigma_* \) to \( \mathbb{Z}^n \) and consider the linear mapping \( \dot{\sigma} \otimes \text{id} \) on \( V := \mathbb{Z}^n \otimes \mathbb{C} \). Take \( \hat{\varphi} = \dot{\sigma} \otimes \text{id} \) and \( D^r\hat{\varphi} = D^r(\hat{\varphi} |_{\Lambda^rV}) \) as above. Then
\[a_{n,r} = \text{rank ker}(\wedge^r \dot{\sigma} - \text{id}) = \text{dim ker} D^r\hat{\varphi} \]

**Proof.** Using the preceding corollary one has \( \wedge^r (\dot{\sigma} \otimes 1) - \text{id} \sim D^r\hat{\varphi} \). Therefore
\[\text{rank ker}(\wedge^r \dot{\sigma} - \text{id}) = \text{dim ker}(\wedge^r (\dot{\sigma} \otimes 1) - \text{id}) = \text{dim ker} D^r\hat{\varphi} \]
\hfill \square

Now, to compute \( a_{n,r} \) as explicitly as possible, we find a connection to representation theory of \( \mathfrak{sl}_2(\mathbb{C}) \). First, we recall some definitions and properties.

Let \( \mathfrak{sl}_2(\mathbb{C}) \) denote the special linear algebra over \( \mathbb{C}^2 \)
\[\mathfrak{sl}_2(\mathbb{C}) := \{a \in M_2(\mathbb{C}) \mid tr(a) = 0\} \]
Theorem 6.28. Let \( \sigma \) be a positive integer. Notation 6.29. With the above notation, Proposition 6.30. Suppose \( n, k, r \) be positive integers. \( P(n, r, k) \) denotes the number of partitions of \( k \) to \( r \) distinct positive integers not greater than \( n \). In other words

\[
P(n, r, k) = \text{card}\{(i_1, \ldots, i_r) \mid i_1 + \ldots + i_r = k, 1 \leq i_1 < \ldots < i_r \leq n\}.
\]

We conventionally take \( P(n, 0, 0) = 1 \) and \( P(n, r, 0) = P(n, 0, k) = 0 \) for \( r, k \geq 1 \).

Proposition 6.30. With the above notation, \( \dim \ker D^r \hat{\varphi} = P(n, r, [r(n + 1)/2]) \), in which \([x]\) denotes the greatest integer not greater than \( x \).
Proof. Following Weyl’s theorem [11, p. 28], \( \pi^r \) is completely irreducible, which means that \( \Lambda^r V = \oplus_{p=1}^N W_p \), where the \( W_p \)'s are \( \pi^r \)-invariant irreducible subspaces of \( \Lambda^r V \) and \( N \) is the number of such subspaces, which following the preceding theorem is equal to \( \dim E \oplus \dim E_1 \), where \( E_j = \{ v \in \Lambda^r V \mid \pi^r(h)v = jv \} \). On the other hand, \( \ker D = \oplus_{p=1}^N \ker (D^r \hat{\varphi}|_{W_p}) \), hence \( \dim \ker D = \sum_{p=1}^N \dim \ker (D^r \hat{\varphi}|_{W_p}) = N \), since \( \dim \ker (D^r \hat{\varphi}|_{W_p}) = 1 \), so \( \dim \ker D^r \hat{\varphi} = \dim E_0 + \dim E_1 \). To compute the last term, note that from the preceding theorem we have \( \dim \ker \hat{\varphi} = (2^r - 2)(n + 1) \), for even \( r(n + 1) \), \( E_0 = \{ 0 \} \) and \( \dim E_0 = P(n, r, (n + 1)/2) \) and for odd \( r(n + 1) \), \( E_1 = \{ 0 \} \) and \( \dim E_1 = P(n, r, (n + 1)/2 - 1) \). To summarize, \( \dim \ker D^r \hat{\varphi} = N = \dim E_0 + \dim E_1 = P(n, r, [r(n + 1)/2]). \)

Therefore, as the main result of this part, we have the following theorem.

**Theorem 6.31.** \( a_{n,r} = P(n, r, [r(n + 1)/2]) \) for \( r = 0, 1, \ldots, n \).

Proof. Use Propositions 6.4 and 6.5. □

As a result, we can prove that \( \{ a_n \} \) is a strictly increasing sequence. We need a lemma first.

**Lemma 6.32.** \( P(n + 1, r, k + s) \geq P(n, r, k) \) for \( s = 0, 1, \ldots, r \).

Proof. For \( s = 0 \), the proof is clear. Now let \( 1 \leq s \leq r \) and suppose that \((j_1, \ldots, j_r)\) is a partition of \( k \) such that \( 1 \leq j_1 < \ldots < j_r \leq n \). Now define \( i_q := j_q \) for \( 1 \leq q \leq r - s \) and \( i_q := j_q + 1 \) for \( r - s + 1 \leq q \leq r \). Then \((i_1, \ldots, i_r)\) is a partition of \( k + s \) and \( 1 \leq i_1 < \ldots < i_r \leq n + 1 \). Thus \( P(n + 1, r, k + s) \geq P(n, r, k) \). □

**Proposition 6.33.** \( \{ a_n \} \) is a strictly increasing sequence.

Proof. First note that \( a_{n,0} = a_{n,n} = P(n, 0, 0) = P(n, n, n(n + 1)/2) = 1 \) and from the preceding theorem we have \( a_n = \sum_{n=0}^n P(n, r, [r(n + 1)/2]). \) Now we prove that for every \( m \in \mathbb{N} \), \( a_{2m+1} > a_{2m} > a_{2m-1} \). Applying the lemma to the terms of the following equalities yields the result.

\[
a_{2m+1} = 1 + \sum_{r=0}^m \binom{2m+1}{2r} P(2r, 2m + 2r) + \sum_{r=0}^{m-1} P(2m + 1, 2r + 1, 2rm + 2r + m + 1),
\]

\[
a_{2m} = \sum_{r=0}^m \binom{2m}{2r} P(2m, 2r, 2rm + r) + \sum_{r=0}^{m-1} P(2m, 2r + 1, 2rm + m + r)
\]

\[
= 1 + \sum_{r=0}^{m-1} \binom{2m}{2r} P(2m, 2r, 2rm + r) + \sum_{r=0}^{m-1} P(2m, 2r + 1, 2rm + m + r),
\]
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\[
a_{2m-1} = \sum_{r=0}^{m-1} P(2m - 1, 2r, 2rm) + \sum_{r=0}^{m-1} P(2m - 1, 2r + 1, 2rm + m).
\]

\[\square\]

6.3. Generating functions for \(a_n\). In this part, we express the rank of the \(K\)-groups of \(A_{n,\theta}\) as explicitly as possible. In fact, we present them as the constant terms in the polynomial expansions of certain functions. First of all, we need the following basic lemma.

**Lemma 6.34.** Let \(P(n, r, k)\) denote the number of partitions of \(k\) to \(r\) distinct positive integers not greater than \(n\). Then \(P(n, r, k)\) is the coefficient of \(u^r t^k\) in the polynomial expansion of \(F_n(u, t) := \prod_{i=1}^{n}(1 + ut^i)\). In other words

\[
\sum_{r, k \geq 0} P(n, r, k) u^r t^k = \prod_{i=1}^{n}(1 + ut^i).
\]

**Proof.**

\[
\prod_{i=1}^{n}(1 + ut^i) = 1 + \sum_{r=1}^{n} \sum_{(i_1, \ldots, i_r) \in \mathbb{N}} (ut^{i_1} \cdots ut^{i_r}) = 1 + \sum_{r=1}^{n} \sum_{k \geq 1} P(n, r, k) u^r t^k
\]

\[
= \sum_{r, k \geq 0} P(n, r, k) u^r t^k.
\]

\[\square\]

Now, we have the following expressions for the rank \(a_n\) of the \(K\)-groups of \(A_{n,\theta}\).

**Theorem 6.35.** Let \(a_n = \text{rank} K_0(A_{n,\theta}) = \text{rank} K_1(A_{n,\theta})\). Then

- for odd \(n\), \(a_n\) is the constant term in the polynomial expansion of \(\prod_{i=1}^{n}(1 + t^{i - \frac{n+1}{2}})\)
- for even \(n\), \(a_n\) is the constant term in the polynomial expansion of \((1 + t^{\frac{1}{2}}) \prod_{i=1}^{n}(1 + t^{i - \frac{n+1}{2}})\).

**Proof.** We know that \(a_n = \sum_{r=0}^{n} a_{n, r}\) and \(a_{n, r} = P(n, r, \lfloor r(n + 1)/2 \rfloor)\). Now let \(n = 2m - 1\) be an odd number. So, \(a_n = \sum_{r=0}^{2m-1} P(2m - 1, r, rm)\). Take
\( y = ut^m \). From the preceding lemma, we have

\[
F_n(u, t) = F_{2m-1}(yt^{-m}, t) = \prod_{i=1}^{2m-1} (1 + yt^{-m}) = \sum_{r,k \geq 0} P(2m - 1, r, k)yt^{k-rm}
\]

so for \( y = 1 \) we have

\[
\prod_{i=1}^{n} (1 + t^{i-\frac{n+1}{2}}) = \prod_{i=1}^{2m-1} (1 + t^{i-m}) = \sum_{r,k \geq 0} P(2m - 1, r, k)t^{k-rm},
\]

which yields the result.

For even \( n \), say \( n = 2m \), we have

\[
a_n = a_{2m} = \sum_{r=0}^{2m} P(2m, r, [r(m + \frac{1}{2})])
\]

\[
= \sum_{r=0}^{m} P(2m, 2r, (2m + 1)) + \sum_{r=0}^{m-1} P(2m, 2r + 1, 2rm + m + r)
\]

\[
= A_m + B_m.
\]

First, let’s determine \( A_m \). Note that from the lemma we have

\[
\frac{1}{2} \left\{ \prod_{i=1}^{2m} (1 + ut^i) + \prod_{i=1}^{2m} (1 - ut^i) \right\} = \sum_{r,k \geq 0} P(2m, r, k)\left\{ \frac{1 + (-1)^r}{2} \right\} u^rt^k
\]

\[
= \sum_{r,k \geq 0} P(2m, 2r, k)u^{2r}t^k.
\]

If we define \( y := u^2t^{2m+1} \), we have the following identity

\[
\frac{1}{2} \left\{ \prod_{i=1}^{2m} (1 + y^{\frac{1}{2}}t^{i-(m+\frac{1}{2})}) + \prod_{i=1}^{2m} (1 - y^{\frac{1}{2}}t^{i-(m+\frac{1}{2})}) \right\} = \sum_{r,k \geq 0} P(2m, 2r, k)y^rt^{k-r(2m+1)},
\]

which for \( y = 1 \) yields

\[
\frac{1}{2} \left\{ \prod_{i=1}^{2m} (1 + t^{i-(m+\frac{1}{2})}) + \prod_{i=1}^{2m} (1 - t^{i-(m+\frac{1}{2})}) \right\} = \sum_{r,k \geq 0} P(2m, 2r, k)t^{k-r(2m+1)},
\]

hence \( A_m \) is the constant term in polynomial expansion of

\[
\frac{1}{2} \left\{ \prod_{i=1}^{2m} (1 + t^{i-(m+\frac{1}{2})}) + \prod_{i=1}^{2m} (1 - t^{i-(m+\frac{1}{2})}) \right\}.
\]
Similarly, for $B_m$ we have
\[
\frac{1}{2} \left( \prod_{i=1}^{2m} (1 + ut^i) - \prod_{i=1}^{2m} (1 - ut^i) \right) = \sum_{r,k \geq 0} P(2m, r, k) \left( \frac{1 - (-1)^r}{2} \right) u^r t^k
\]
\[
= \sum_{r,k \geq 0} P(2m, 2r + 1, k) u^{2r+1} t^k.
\]
If we define $y^2 := u^2 t^{2m+1}$, we have the following identities
\[
\frac{1}{2} \left( \prod_{i=1}^{2m} (1 + y^{\frac{1}{2}} t^{i-(m+\frac{1}{2})}) - \prod_{i=1}^{2m} (1 - y^{\frac{1}{2}} t^{i-(m+\frac{1}{2})}) \right)
\]
\[
= \sum_{r,k \geq 0} P(2m, 2r + 1, k) y^{2r+1} t^{k-(2rm+r+m)-\frac{1}{2}}
\]
\[
= t^{-\frac{1}{2}} \sum_{r,k \geq 0} P(2m, 2r + 1, k) y^{2r+1} t^{k-(2rm+r+m)},
\]
which for $y = 1$ yields
\[
\frac{t^{\frac{1}{2}}}{2} \left( \prod_{i=1}^{2m} (1 + t^{i-(m+\frac{1}{2})}) - \prod_{i=1}^{2m} (1 - t^{i-(m+\frac{1}{2})}) \right) = \sum_{r,k \geq 0} P(2m, 2r + 1, k) t^{k-(2rm+r+m)},
\]
hence $B_m$ is the constant term in the polynomial expansion of
\[
\frac{t^{\frac{1}{2}}}{2} \left( \prod_{i=1}^{2m} (1 + t^{i-(m+\frac{1}{2})}) - \prod_{i=1}^{2m} (1 - t^{i-(m+\frac{1}{2})}) \right).
\]
Therefore $a_n = a_{2m} = A_m + B_m$ is the constant term in the polynomial expansion of
\[
\frac{1}{2} \left( \prod_{i=1}^{2m} (1 + t^{i-(m+\frac{1}{2})}) + \prod_{i=1}^{2m} (1 - t^{i-(m+\frac{1}{2})}) \right)
\]
\[
+ \frac{t^{\frac{1}{2}}}{2} \prod_{i=1}^{2m} (1 + t^{i-(m+\frac{1}{2})}) - \frac{t^{\frac{1}{2}}}{2} \prod_{i=1}^{2m} (1 - t^{i-(m+\frac{1}{2})})
\]
or equivalently, the constant term in the polynomial expansion of
\[
\frac{1}{2} \left( (1 + z) \prod_{i=1}^{2m} (1 + z^{2i-(2m+1)}) + (1 - z) \prod_{i=1}^{2m} (1 - z^{2i-(2m+1)}) \right),
\]
or equivalently, the constant term in the polynomial expansion of
\[
(1 + z) \prod_{i=1}^{2m} (1 + z^{2i-(2m+1)}),
\]
or equivalently, the constant term in the polynomial expansion of
\[(1 + t^\frac{1}{2}) \prod_{i=1}^{2m} (1 + t^{-(m+\frac{1}{2})}) ,\]
which yields the result. \(\square\)

One can use this theorem to determine the asymptotic behavior of \(a_n\).

**Corollary 6.36.** \(a_n \sim \left(\frac{24}{\pi}\right)^{\frac{1}{2}} 2^n n^{-\frac{3}{2}}\) when \(n \to \infty\).

### 7. Concluding remarks

(1) The torsion parts of the \(K\)-groups of \(A_{n,\theta}\) seem much more difficult
to describe explicitly in terms of \(n\). Nevertheless, it is an interesting problem to find such descriptions. It is of interest to compute the \(K\)-groups of \(A_{n,\theta}\) (or more general algebras \(A_{F_0,\theta}\)), since in the class of \(C^*\)-algebras generated by uniquely ergodic minimal
diffeomorphisms on a compact manifold, \(K\)-theory is a complete invariant. More precisely, suppose that \(M\) is a connected compact smooth manifold with \(\dim(M) > 0\) and \(h : M \to M\) is a uniquely ergodic minimal diffeomorphism, and put \(A := C(M) \rtimes_k \mathbb{Z}\). Let \(\tau\) be the trace induced by the unique invariant probability measure, and assume that \(\tau_*K_0(A)\) is dense in \(\mathbb{R}\). Then the 4-tuple
\[(K_0(A), K_0(A)_+, [1_A], K_1(A))\]
is a complete algebraic invariant (called the **Elliott invariant** of \(A\)) [15]. In this case, \(A\) has stable rank one, real rank zero and tracial topological rank zero in the sense of H. Lin [14]. The order on \(K_0(A)\) is also determined by the unique trace \(\tau\).

(2) The method used in section 6 above for computing the \(K\)-groups of the transformation group \(C^*\)-algebras of the tori may be extended to more general settings. Let \(G\) be a compact connected Lie group with \(\pi_1(G)\) torsion-free. Then \(K^*(G)\) is torsion-free and can be given the structure of a \(\mathbb{Z}_2\)-graded Hopf algebra over the integers [10]. Moreover, regarded as a Hopf algebra, \(K^*(G)\) is the exterior algebra on the module of the primitive elements, which are of degree 1. The module
of the primitive elements of \(K^*(G)\) may also be described as follows. Let \(U(n)\) denote the group of unitary matrices of order \(n\) and let \(U := \cup_{n=1}^{\infty} U(n)\) be the stable unitary group. Any unitary representation \(\rho : G \to U(n)\), by composition with the inclusion \(U(n) \subset U\), defines a homotopy class \(\beta(\rho)\) in \([G, U] = K^1(G)\). The module of the primitive elements in \(K^1(G)\) is exactly the module generated by all classes \(\beta(\rho)\) of this type. If in addition, \(G\) is semisimple and simply connected of rank \(l\), there are \(l\) basic irreducible representations \(\rho_1, \ldots, \rho_l\), whose maximum weights \(\lambda_1, \ldots, \lambda_l\) form a basis for the character group \(\hat{T}\) of the maximal torus \(T\) of \(G\) and the
classes $\beta(\rho_1), \ldots, \beta(\rho_l)$ form a basis for the module of the primitive elements in $K^1(G)$ and $K^*(G) = \Lambda^*(\beta(\rho_1), \ldots, \beta(\rho_l))$. In any case, to compute $K_n(C(G) \rtimes_\alpha \mathbb{Z})$, it is sufficient to determine the homotopy classes of $\alpha \circ \rho$ for irreducible representations $\rho$ of $G$ in terms of $\beta(\rho)$'s.

(3) There is a relation between the $K$-theory of transformation group $C^*$-algebras of the tori and the topological $K$-theory of compact nilmanifolds. In fact let $\alpha = (t, A)$ be an affine transformation on $T^n$ satisfying the conditions (1) through (4) in Remark 6.17. Then it has been shown in [8] that $\alpha$ is conjugate (in the group of affine transformations of $T^n$) to the transformation $\alpha' = (t', A')$, where $A'$ has an upper triangular matrix whose bottom right $k \times k$ corner is the identity matrix $I_k$ and $t' = (0, \ldots, 0, t'_1, \ldots, t'_k)$. $\alpha'$ is called a standard form [22]. Assume that $\alpha$ is given in standard form.

J. Packer associates an induced flow $(\mathbb{R}, N/\Gamma)$ to the flow $(\mathbb{Z}, T^n)$ generated by $\alpha$, where $N$ is a simply connected nilpotent Lie group of dimension $n + 1$, $\Gamma$ is a cocompact subgroup of $N$, and the action of $\mathbb{R}$ is given by translation on the left by $\exp sX$ for $s \in \mathbb{R}$ and some $X \in \mathfrak{n}$, the Lie algebra of $N$. One of the most important facts is that the $C^*$-algebra $C(N/\Gamma) \rtimes_\beta \mathbb{R}$ corresponding to the induced flow is strongly Morita equivalent to $C(T^n) \rtimes_\alpha \mathbb{Z}$ [22, Proposition 3.1]. Consequently, one has

\begin{equation}
K_i(C(T^n) \rtimes_\alpha \mathbb{Z}) \cong K_i(C(N/\Gamma) \rtimes_\beta \mathbb{R}) \cong K^{1-i}(N/\Gamma); \ i = 0, 1.
\end{equation}

The second isomorphism here, is the Connes’ Thom isomorphism. So the $K$-theory of $C(T^n) \rtimes_\alpha \mathbb{Z}$ is converted to the topological $K$-theory of the compact nilmanifold $N/\Gamma$. Following the proof of Proposition 3.1 in [22], one can conclude that for the special case of the Anzai flows, $N = F_{n-1}$ and $\Gamma = D_{n-1}$ which were defined in Section 2. On the other hand, following [27, Theorem 3.6], one has the following isomorphism

\begin{equation}
K_i(C^*(\Gamma)) \cong K^{i+n+1}(N/\Gamma); \ i = 0, 1.
\end{equation}

Combining (14) and (15) one gets

\begin{equation}
K_i(C(T^n) \rtimes_\alpha \mathbb{Z}) \cong K^{i+1}(N/\Gamma) \cong K_{i+n}(C^*(\Gamma)); \ i = 0, 1.
\end{equation}

Using the above isomorphisms, one can relate the algebraic invariants of the involved $C^*$-algebras and topological information about the corresponding nilmanifold. For example, since $N/\Gamma$ is a classifying space for $\Gamma$, one has the following isomorphisms

\begin{equation}
H^*_{\text{dr}}(N/\Gamma) \cong H^*(\mathbb{R}, \mathbb{R}) \cong H^*(\Gamma, \mathbb{R}) \cong H^*(N, \mathbb{R}) \cong H^*(\mathfrak{n}, \mathbb{R}),
\end{equation}
where $H^*_\text{dR}(N/\Gamma)$ denotes the de Rham cohomology of the manifold $N/\Gamma$, $H^*(N/\Gamma, \mathbb{R})$ denotes the Čech cohomology of $N/\Gamma$ with coefficients in $\mathbb{R}$, $H^*(\Gamma, \mathbb{R})$ denotes the group cohomology of $\Gamma$ with coefficients in the trivial $\Gamma$-module $\mathbb{R}$, $H^*(N, \mathbb{R})$ denotes the Moore cohomology group of $N$ (as a locally compact group) with coefficients in the trivial Polish $N$-module $\mathbb{R}$, and $H^*(n, \mathbb{R})$ denotes the cohomology of the Lie algebra $n$ with coefficients in the trivial $n$-module $\mathbb{R}$. Now using the Chern isomorphisms $\text{ch}_0 : K^0(N/\Gamma) \otimes \mathbb{Q} \to \tilde{H}^{\text{even}}(N/\Gamma, \mathbb{Q})$ and $\text{ch}_1 : K^1(N/\Gamma) \otimes \mathbb{Q} \to \tilde{H}^{\text{odd}}(N/\Gamma, \mathbb{Q})$, one concludes that the even and odd cohomology groups stated in (17) are all isomorphic to $\mathbb{R}^k$, where $k$ is the (common) rank of the $K$-groups of $C(T^n) \rtimes_\alpha \mathbb{Z}$ as in Corollary 6.2. As an example, if $N = F_{n-1}$, $\Gamma = D_{n-1}$, and $n = f_{n-1}$, then the even and odd cohomology groups stated in (17) are all isomorphic to $\mathbb{R}^{a_n}$, where $a_n$ is the rank of the $K$-groups of $A_{n,\theta}$ that was studied in detail in Section 6.

Conversely, one may use the topological tools for $N/\Gamma$ to get some information about $C(T^n) \rtimes_\alpha \mathbb{Z}$ and $C^*(\Gamma)$. For example, we know that $N/\Gamma$ as a compact nilmanifold can be constructed as a principal $\mathbb{T}$-bundle over a lower dimensional compact nilmanifold. Then we can compute the topological $K$-groups of $N/\Gamma$ using the six term Gysin exact sequence. As an example, one can see that $F_{n-1}/D_{n-1}$ is a principal $\mathbb{T}$-bundle over $F_{n-2}/D_{n-2}$, and the corresponding Gysin exact sequence is in fact the topological version of the Pimsner-Voiculescu exact sequence for the crossed product $A_{n,\theta} \cong A_{n-1,\theta} \rtimes_\alpha \mathbb{Z}$ as in Theorem 3.1.

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