COUNTING BUNDLES

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1 Zetas for \((G, P)/\mathbb{F}_q(X)\)

Let \(X\) be an irreducible reduced regular projective curve defined over \(\mathbb{F}_q\) with \(F\) the field of rational functions. Let \(G\) be a reductive group of rank \(r\) with \(B\) a fixed Borel subgroup, both defined over \(F\). As usual, \(\Delta\) stands for the corresponding collection of simple roots; \(W\) the associated Weyl group; for a root \(\alpha\), \(\alpha^\vee\) the corresponding coroot; and \(\rho := \frac{1}{2} \sum_{\alpha > 0} \alpha\), the Weyl vector . . .

Definition 1 The period for \(G\) over \(F\) is defined by, for \(\text{Re} \lambda \in \mathbb{C}^+\),

\[
\omega_F^G(\lambda) := \sum_{w \in W} \left( \prod_{\alpha \in \Delta} \frac{1}{1 - q^{-\langle w\lambda - \rho, \alpha^\vee \rangle}} \cdot \prod_{\alpha > 0, w \alpha < 0} \widehat{\zeta}_F(\langle \lambda, \alpha^\vee \rangle) \right) \prod_{\alpha > 0, w \alpha < 0} \left( \frac{\widehat{\zeta}_F(\langle \lambda, \alpha^\vee \rangle + 1)}{\widehat{\zeta}_F(\langle \lambda, \alpha^\vee \rangle)} \right)
\]

where \(\mathbb{C}^+\) denotes the so-called positive chamber of \(\mathfrak{a}_0\), the space of characters associated to \((G, B)\), and \(\widehat{\zeta}_F(s)\) the complete Artin zeta function of \(X/\mathbb{F}_q\).

Corresponding to a fixed maximal standard parabolic subgroup \(P\) is a simple root \(\alpha_P \in \Delta\). Write \(\Delta \setminus \{\alpha_P\} =: \{\beta_{1,P}, \beta_{2,P}, \ldots, \beta_{r-1,P}\}\).

Definition 2 The period for \((G, P)\) over \(F\) is defined by, for \(\text{Re} \lambda_p \gg 0\),

\[
\omega_F^{G/P}(\lambda_P) := \text{Res}_{\langle \lambda - \rho, \beta_{1(P)(G)-1,P}^\vee \rangle = 0} \cdots \text{Res}_{\langle \lambda - \rho, \beta_{2,P}^\vee \rangle = 0} \text{Res}_{\langle \lambda - \rho, \beta_{1,P}^\vee \rangle = 0} (\omega_F^G(\lambda))
\]

Here, starting from \(r\)-variable \(\lambda \in \mathfrak{a}_{0, \mathbb{C}}^*\), after taking residues along with \((r-1)\) (independent) singular hyperplanes

\[
\langle \lambda - \rho, \beta_{1,P}^\vee \rangle = 0, \langle \lambda - \rho, \beta_{2,P}^\vee \rangle = 0, \ldots, \langle \lambda - \rho, \beta_{r-1(P)(G)-1,P}^\vee \rangle = 0,
\]

we are left with only one variable, which we call \(\lambda_P\).

Clearly, there is a minimal integer \(I(G/P)\) and finitely many factors (depending on the choice of \(\lambda_P\)),

\[
\widehat{\zeta}_F\left( a_1^{G/P} \lambda_P + b_1^{G/P} \right), \widehat{\zeta}_F\left( a_2^{G/P} \lambda_P + b_2^{G/P} \right), \ldots, \widehat{\zeta}_F\left( a_{I(G/P)}^{G/P} \lambda_P + b_{I(G/P)}^{G/P} \right),
\]

such that no \(\widehat{\zeta}_F(a\lambda_P + b)\) factors appear in the denominators of (all terms of) the product \(\prod_{i=1}^{I(G/P)} \widehat{\zeta}_F\left( a_i^{G/P} \lambda_P + b_i^{G/P} \right) \cdot \omega_F^{G/P}(\lambda_P)\).
Definition 3 (i) The zeta function $\hat{\zeta}_{G/P}^F$ for $(G, P)$ over $F$ is defined by

$$\hat{\zeta}_{G/P}^F(s) := \prod_{i=1}^{I(G/P)} \zeta_F\left(a_i\frac{G/P}{s + b_i\frac{G/P}}\right) \cdot \omega_F^G \left(\frac{G/P}{s}\right), \quad \text{Re } s \gg 0$$

Functional Equation There exists a constant $c_{G/P} \in \mathbb{Q}$ such that

$$\hat{\zeta}_{G/P}^F\left(-s + c_{G/P}\right) = \hat{\zeta}_{G/P}^F\left(s\right).$$

Definition 3 (ii) The zeta function $\hat{\zeta}_{G/P}^F(s)$ for $(G, P)$ over $F$ is defined by

$$\hat{\zeta}_{G/P}^F(s) := \hat{\zeta}_{G/P}^F\left(s + \frac{c_{G/P} - 1}{2}\right)$$

2 Non-Abelian Zeta Functions for $\mathbb{F}_q(X)$

Motivated by [W1], and for the RH, introduce a new genuine pure non-abelian zetas for $X$ by

$$\zeta_{X,r}(t) := \sum_{m=0}^{\infty} \sum_{V \in [V] \in \mathcal{M}_{X,r}(d), d=rm} q^{h^0(X,V)} - 1 \cdot \left(q^{-s}\right)^{d(V)} \cdot \text{Re}(s) > 1$$

Here, as usual, $\mathcal{M}_{X,r}(d)$ denotes the moduli space of semi-stable $\mathbb{F}_q$-rational vector bundles of rank $r$, $[\ ]$ the Seshedri class defined using Jordan-Hölder graded bundles, and $\text{Aut}(V)$ denotes the automorphism group of $V$. Introduce also the complete zeta function for $X$ by

$$\hat{Z}_{X,r}(t) := \sum_{m=0}^{\infty} \sum_{V \in [V] \in \mathcal{M}_{X,r}(d), d=rm} q^{h^0(X,V)} - 1 \cdot \left(t^s\right)^{\chi(X,V)}$$

Zeta Facts (i) (Rationality)

$$Z_{X,r}(t) = \sum_{m=0}^{(g-1)-1} \alpha_{X,r}(mr) \cdot \left(t^{rm} + q^{r[g-1]-m} \cdot t^{r[2(g-1)-m]}\right) + \alpha_{X,r}(r(g-1)) \cdot t^{r(g-1)} + \beta_{X,r}(0) \cdot \frac{(q^r - 1)t^{rg}}{(1 - q^t t^r)(1 - t^r)}$$

with

$$\beta_{X,r}(0) := \sum_{V \in [V] \in \mathcal{M}_{X,r}(0)} \frac{1}{\#\text{Aut}(V)} \cdot \alpha_{X,r}(d) := \sum_{V \in [V] \in \mathcal{M}_{X,r}(d)} q^{h^0(X,V)} - 1 \cdot \frac{1}{\#\text{Aut}(V)}$$
(ii) (Functional Equation)

\[
\tilde{Z}_{X, r}(1/qt) = \tilde{Z}_{X, r}(t)
\]

We expect to have the following

**Riemann Hypothesis**

*All zeros of the zeta function \( \tilde{\zeta}_{X, r}(s) \) lie on the central line \( \text{Re} \ s = \frac{1}{2} \)*

### 3 Counting Bundles

Semi-stable bundles are naturally counted within the stratifications of the refined Brill-Noether loci defined using \( h^0 \) and \( \text{Aut} \). So the Riemann Hypothesis offers us intrinsic quantitative controls uniformly through \( \alpha \)'s and \( \beta \). More generally, write \( ss \) for the part corresponding to semi-stable principal bundles.

**Counting Conjectures** (i) (Parabolic Reduction, Stability & the Mass)

\[
\text{Vol} \left( K_G Z_{G(h)} \backslash G^1(h)/G(F) \right) = \text{Res}_{\lambda = \rho \omega_F} \zeta_{G/P}(\lambda) = \text{Res}_{s=1} \hat{\zeta}_{S, G/P}(s)
\]

(ii) (Uniformity) *There exist rational functions \( R_{r, q}(t) \) depending on \( q \) and \( t \) and rational numbers \( a_r, b_r \) depending only on \( r \), but independent of the curve \( X \), such that*

\[
\hat{\zeta}_{F, r}(s) = R_{r, q}(q^{-a_r s - b_r}) \cdot \hat{\zeta}_{G/P, r}^{S, \text{sl}_2/P_{r-1, 1}}(a_r s + b_r)
\]

Parallel structures for number fields are exposed in [W2, 3], [Ko], [KKS], and [W5] which contains an adelic approach to Atiyah-Bott & Witten ([AB], [Wi]) and to Kontsevich ([K]). For function fields, (i) is related to Harder-Narasimhan ([HN], [Z], [LR]), uniformity holds for \( G = \text{SL}_2 \) ([W4]), for which, the RH is established in [Y], and a proof of [Ko] style for the FE of general \( \hat{\zeta}_{G/P}^{SL_2}(s) \) can be obtained.

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