Novel scale-free small-world networks from Koch curves

Zhongzhi Zhang1,2†, Jihong Guan3‡, Lichao Chen1,2, Ming Yin1,2, and Shuigeng Zhou1,2∗

1School of Computer Science, Fudan University, Shanghai 200433, China
2Shanghai Key Lab of Intelligent Information Processing, Fudan University, Shanghai 200433, China
3Department of Computer Science and Technology, Tongji University, 4800 Cao’an Road, Shanghai 201804, China
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The class of Koch fractals is one of the most interesting families of fractals, and the study of complex networks is a central issue in the scientific community. In this paper, inspired by the famous Koch fractals, we propose a mapping technique converting Koch fractals into a family of deterministic networks, called Koch networks. This novel class of networks incorporates some key properties characterizing a majority of real-life networked systems—a power-law distribution with exponent in the range between 2 and 3, a high clustering coefficient, small average path length, and degree correlations. All these features are obtained exactly according to the proposed generation algorithm of the networks considered. The network representation approach could be used to investigate the complexity of some real-world systems from the perspective of complex networks, and our networks may describe various real-life networked systems.

I. INTRODUCTION

The past decade has witnessed a great deal of activity devoted to complex networks by the scientific community, since many systems in the real world can be described and characterized by complex networks [1, 2, 3, 4]. Prompted by the computerization of data acquisition and the increased computing power of computers, researchers have done a lot of empirical studies on diverse real networked systems, unveiling the presence of some generic properties of various natural and manmade networks: power-law degree distribution $P(k) \sim k^{-\gamma}$ with characteristic exponent $\gamma$ in the range between 2 and 3 [2], small-world effect including large clustering coefficient and small average path length (APL) [3], and degree correlations [7, 8].

The empirical studies have inspired researchers to construct network models with the aim to reproduce or explain the striking common features of real-life systems [1, 2]. In addition to the seminal Watts-Strogatz’s (WS) small-world network model [6] and Barabási-Albert’s (BA) scale-free network model [3], a considerable number of models and mechanisms have been developed to mimic real-world systems, including initial attractiveness [3], aging and cost [10], fitness model [11], duplication [12], weight or traffic driven evolution [13, 14], geographical constraint [15], accelerating growth [16], coevolution [17], visibility graph [18], to name but a few. Although significant progress has been made in the field of network modeling and has led to a significant improvement in our understanding of complex systems, it is still a fundamental task and of current interest to construct models mimicking real networks and reproducing their generic properties from different angles.

In this present article, inspired by the famous class of Koch fractals, we propose a family of deterministic mathematical networks, called Koch networks, which integrates the observed properties of real networks in a single framework. We derive analytically exact scaling laws for the degree distribution, the clustering coefficient, the average path length, and the degree correlations. The obtained precise results show that Koch networks have rich topological features: they obey power-law degree distribution with exponent lies between 2 and 3; they have a large clustering coefficient, their average path length grows logarithmically with the total number of nodes; and they may be either disassortative or uncorrelated.

This work unfolds an alternative perspective in the study of complex networks. Instead of searching generation mechanisms for real networks, we explore deterministic mathematical networks that exhibit some typical properties of real-world systems. As the classical Koch fractals are important for the understanding of geometrical fractals in real systems [19], we believe that Koch networks could provide valuable insights into real-world systems.

II. NETWORK CONSTRUCTION

In order to define the networks, we first introduce a classical fractal—Koch curve, which was proposed by Helge von Koch [20]. The Koch curve, denoted as $S_k(t)$ after $t$ generations, can be constructed in a recursive way. To produce this well-known fractal, we begin with an equilateral triangle and let this initial configuration be $S_1(0)$. In the first generation, we perform the following operations: firstly, we trisect each side of the initial equilateral triangle; secondly, on the middle segment of each side, we construct new equilateral triangles whose
interiors lie external to the region exclosed by the base triangle; thirdly, we remove the three middle segments of the base triangle, upon which new triangles were established. Thus, we get $S_1(1)$. In the second generation, for each line segment in $S_1(1)$, repeat above procedure of three operations to obtain $S_1(2)$. This process is then repeated for successive generations. As $t$ tends to infinite, the Koch curve is obtained, and its Hausdorff dimension is $d_f = \frac{\ln 4m+1}{\ln(2m+1)}$. Figure 1 depicts the structure of $S_1(2)$.

Koch curve can be easily generalized to other dimensions by introducing a parameter $m$. Denoted by $S_m(t)$ the generalization after $t$ generations, which is constructed as follows: Start with an equilateral triangle as the initial configuration $S_m(0)$. In the first generation, we perform the following operations similar to those described in last paragraph: partition each side of the initial triangle into $2m+1$ ($m$ is a positive integer) segments, which are consecutively numbered $1, 2, \cdots, 2m, 2m+1$ from one endpoint of the side to the other; construct a new small equilateral triangle on each even-numbered segment so that the interiors of the new triangles lie in the exterior of the base triangle; remove the segments upon which triangles were constructed. In this way we obtain $S_m(1)$. Analogously, we can get $S_m(t)$ from $S_m(t-1)$ by repeating recursively the procedure of above three operations for each existing line segments in generation $t-1$. In the infinite $t$ limit, the Hausdorff dimension of the generalized Koch curves $d_f = \frac{\ln(4m+1)}{\ln(2m+1)}$. Figure 2 shows the structure of $S_2(2)$.

The generalized Koch curves can be used as a basis of a new class of networks: sides (excluding those deleted) of the triangles constructed at arbitrary generations are mapped to nodes, which are connected if the corresponding sides are in contact. For uniformity, the three sides of the initial equilateral triangle of $S_m(0)$ also correspond to three different nodes. We shall call the resultant networks Koch networks. Note that after the birth of each side of a triangle constructed at a given generation, although some segments of it will be removed at subsequent steps, we look on its remain segments as a whole and map it to only one node. Figures 3 and 4 show two networks corresponding to $S_1(2)$ and $S_2(2)$, respectively.

FIG. 1: (Color online) The first two generations of the construction for Koch curve.

FIG. 2: (Color online) The first two generations of the construction for generalized Koch curve in the case of $m = 2$.

FIG. 3: (Color online) The network derived from $S_1(2)$.

FIG. 4: (Color online) The network derived from $S_2(2)$.

III. GENERATION ALGORITHM

According to the construction process of the generalized Koch curves and the proposed method of mapping from Koch curves to Koch networks, we can introduce with ease an iterative algorithm to create Koch networks, denoted by $K_{m,t}$ after $t$ generation evolutions. The al-
algorithm is as follows: Initially \((t = 0)\), \(K_{m,0}\) consists of three nodes forming a triangle. For \(t \geq 1\), \(K_{m,t}\) is obtained from \(K_{m,t-1}\) by adding \(m\) groups of nodes for each of the three nodes of every existing triangles in \(K_{m,t-1}\). Each node group has two nodes. These two new nodes and its “mother” node are linked to one another shaping a new triangle. In other word, to obtain \(K_{m,t}\) from \(K_{m,t-1}\), we replace each of the existing triangles of \(K_{m,t-1}\) by the connected clusters on the right hand of Fig. 5. Figures 3 and 4 illustrate the growing process of the networks for two particular cases of \(m = 1\) and \(m = 2\), respectively.

Let us compute the order and size (number of all edges) of Koch networks \(K_{m,t}\). To this end, we first consider the total number of triangles \(L_\Delta(t)\) that exist at step \(t\). By construction (see Fig. 5), this quantity increases by a factor of \(3m + 1\), i.e., \(L_\Delta(t) = (3m + 1)L_\Delta(t-1)\). Considering \(L_\Delta(0) = 1\), we have \(L_\Delta(t) = (3m + 1)^t\). Denote \(L_v(t)\) and \(L_e(t)\) as the numbers of nodes and edges created at step \(t\), respectively. Note that each triangle in \(K_{m,t-1}\) will give birth to \(6m\) new nodes and \(9m\) new edges at step \(t\), then one can easily obtain: \(L_v(t) = 6mL_\Delta(t-1) = 6m(3m + 1)^{t-1}\) and \(L_e(t) = 9mL_\Delta(t-1) = 9m(3m + 1)^{t-1}\), both of which hold for arbitrary \(t > 0\). Then, the total number of nodes \(N_t\) and edges \(E_t\) present at step \(t\) is

\[
N_t = \sum_{t_i=0}^{t} L_v(t_i) = 2(3m + 1)^t + 1
\]

and

\[
E_t = \sum_{t_i=0}^{t} L_e(t_i) = 3(3m + 1)^t,
\]

respectively. Thus, the average degree is

\[
\langle k \rangle = \frac{2E_t}{N_t} = \frac{6(3m + 1)^t}{2(3m + 1)^t + 1},
\]

which is approximately 3 for large \(t\), showing that Koch networks are sparse as most real-life networks.

IV. TOPOLOGICAL PROPERTIES

Now we study some relevant characteristics of the Koch networks \(K_{m,t}\), focusing on degree distribution, clustering coefficient, average path length, and degree correlations.

A. Degree distribution

Let \(k_i(t)\) be the degree of a node \(i\) at time \(t\). When node \(i\) enters the network at step \(t_i\) \((t_i \geq 0)\), it has a degree of 2, viz. \(k_i(t_i) = 2\). To determine \(k_i(t)\), we first consider the number of triangles involving node \(i\) at step \(t\) that is denoted by \(L_\Delta(i, t)\). These triangles will give rise to new nodes linked to the node \(i\) at step \(t+1\). Then at step \(t_i\), \(L_\Delta(i, t_i) = 1\). By construction, for any triangle involving node \(i\) at a given step, it will lead to \(m\) new triangles passing by node \(i\) at next step. Thus, \(L_\Delta(i, t) = (m + 1)L_\Delta(i, t-1)\). Considering the initial condition \(L_\Delta(i, t_i) = 1\), we have \(L_\Delta(i, t) = (m + 1)^{t-t_i}\). On the other hand, every triangle passing by node \(i\) contains two links connected to \(i\), therefore we have \(k_i(t) = 2L_\Delta(i, t)\). Then we obtain

\[
k_i(t) = 2L_\Delta(i, t) = 2(m + 1)^{t-t_i}.
\]

In this way, at time \(t\) the degree of arbitrary node \(i\) of Koch networks has been computed explicitly. From Eq. (4), it is easy to see that at each step the degree of a node increases \(m\) times, i.e.,

\[
k_i(t) = (m + 1)k_i(t-1).
\]

Equation (4) shows that the degree spectrum of Koch networks is discrete. Thus, we can get the degree distribution \(P(k)\) of the Koch networks via the cumulative
degree distribution \[ P_{\text{cum}}(k) = \frac{1}{N_t} \sum_{r \leq t_i} L_r(\tau) = \frac{2 \times (3m + 1)^t + 1}{2 \times (3m + 1)^t + 1} \] (6)

Substituting for \( t_i \) in this expression using \( t_i = t - \frac{\ln(\frac{k_i}{\gamma})}{\ln(m + 1)} \)
gives

\[ P_{\text{cum}}(k) = \frac{2 \times (3m + 1)^t \times \left( \frac{k_i}{\gamma} \right)^{\frac{\ln(3m + 1)}{\ln(m + 1)}} + 1}{2 \times (3m + 1)^t + 1} \] (7)

In the infinite \( t \) limit, we obtain

\[ P_{\text{cum}}(k) = 2 \frac{\ln(3m + 1)}{\ln(m + 1)} \times k - \frac{\ln(3m + 1)}{\ln(m + 1)} \] (8)

So the degree distribution follows a power-law form \( P(k) \sim k^{-\gamma} \) with the exponent \( \gamma = 1 + \frac{\ln(3m + 1)}{\ln(m + 1)} \) belonging to the interval [2, 3]. When \( m \) increases from 1 to infinite, \( \gamma \) decreases from 3 to 2. It should be stressed that the exponent of degree distribution of most real scale-free networks also lies in the same range between 2 and 3.

B. Clustering coefficient

By definition, the clustering coefficient \[ C \] of a node \( i \) with degree \( k_i \) is the ratio between the number of triangles \( e_i \) that actually exist among the \( k_i \) neighbors of node \( i \) and the maximum possible number of triangles involving \( i \), \( k_i(k_i - 1)/2 \), namely, \( C_i = 2e_i/k_i(k_i - 1) \).

For Koch networks, we can obtain the exact expression of clustering coefficient \( C(k) \) for a single node with degree \( k \). By construction, for any given node having a degree \( k \), there are just \( e = \frac{k}{2} \) triangles connected this node, see also Eq. \[ 3 \]. Hence there is a one-to-one corresponding relation between the clustering coefficient of a node and its degree: for a node of degree \( k \),

\[ C(k) = \frac{1}{k - 1} \] (9)

which shows a power-law scaling \( C(k) \sim k^{-1} \) in the large limit of \( k \), in agreement with the behavior observed in a variety of real-life systems \[ 22 \].

After \( t \) step growth, the average clustering coefficient \( C_t \) of the whole network \( K_{m,t} \), defined as the mean of \( C_i \)'s over all nodes in the network, is given by

\[ C_t = \frac{1}{N_t} \sum_{r=0}^{t} \left[ \frac{1}{G_r - 1} \times L_r(\tau) \right] \] (10)

where the sum runs over all the nodes of all generations and \( G_r \) is the degree of those nodes created at step \( r \), which is given by Eq. \[ 4 \]. In the limit of large \( N_t \), Equation \[ 10 \] converges to a nonzero value \( C \), as reported in Fig. \[ 5 \]. For \( m = 1, 2, \) and 3, \( C \) are 0.82008, 0.88271, and 0.91316, respectively. As \( m \) approaches infinite, \( C \) converges to 1. Thus, \( C \) increases with \( m \); when \( m \) grows from 1 to infinite, \( C \) increases form 0.82008 to 1. Therefore, for the full range of \( m \), the the average clustering coefficient of Koch networks is very high.

C. Average path length

Using a method similar to but different from those in literature \[ 23, 24, 25 \], we now study analytically the average path length (APL) \( d_i \) of Koch networks \( K_{m,t} \). It follows that

\[ d_t = \frac{D_t}{N_t(N_t - 1)/2} \] (11)

where \( D_t \) is the total distance between all couples of nodes, i.e.,

\[ D_t = \sum_{i \in K_{m,t}, j \in K_{m,t}, i \neq j} d_{ij} \] (12)

in which \( d_{ij} \) is the shortest distance between node \( i \) and \( j \).

Notice that Koch networks have a self-similar structure, which allows us to address \( d_t \) analytically. This
self-similar structure is obvious from an equivalent network construction method: to obtain \( K_{m,t} \), one can make \( 3m + 1 \) copies of \( K_{m,t-1} \) and join them in the hubs (namely nodes with largest degree). As shown in Fig. 7, network \( K_{m,t+1} \) may be obtained by the juxtaposition of \( 3m + 1 \) copies of \( K_{m,t} \), which are labeled as \( K_{m,t}^1, K_{m,t}^2, \cdots, K_{m,t}^{3m} \) and \( K_{m,t}^{3m+1} \), respectively.

We continue by exhibiting the procedure of the determination of the total distance and present the recurrence formula, which allows us to obtain \( D_{t+1} \) of the \( t + 1 \) generation from \( D_t \) of the \( t \) generation. From the obvious self-similar structure of Koch networks, it is easy to see that the total distance \( D_{t+1} \) satisfies the recursion relation

\[
D_{t+1} = (3m + 1) D_t + \Omega_t,
\]

where \( \Omega_t \) is the sum over all shortest paths whose endpoints are not in the same \( K_{m,t}^\alpha \) branch. The solution of Eq. (13) is

\[
D_t = (3m + 1)^{t-1} D_1 + \sum_{\tau=1}^{t-1} (3m + 1)^{t-\tau-1} \Omega_{\tau}.
\]

The paths contributing to \( \Omega_t \) must all go through at least one of the three edge nodes (i.e., grey nodes \( X, Y \) and \( Z \) in Fig. 7) at which the different \( K_{m,t}^\alpha \) branches are connected. The analytical expression for \( \Omega_t \), called the length of crossing paths, is found below.

Let \( \Omega_t^{\alpha,\beta} \) be the sum of the length of all shortest paths with endpoints in \( K_{m,t}^\alpha \) and \( K_{m,t}^\beta \). According to whether or not two branches are adjacent, we sort the crossing path length \( \Omega_t^{\alpha,\beta} \) into two classes: If \( K_{m,t}^\alpha \) and \( K_{m,t}^\beta \) meet at an edge node, \( \Omega_t^{\alpha,\beta} \) rules out the paths where either endpoint is that shared edge node. For example, each path contributed to \( \Omega_t^{1,2} \) should not end at node \( X \). If \( K_{m,t}^\alpha \) and \( K_{m,t}^\beta \) do not meet, \( \Omega_t^{\alpha,\beta} \) excludes the paths where either endpoint is any edge node. For instance, each path contributed to \( \Omega_t^{2,m+2} \) should not end at nodes \( X \) or \( Y \). We can easily compute that the numbers of the two types of crossing paths are \( 3m^2 + 3m \) and \( 3m^2 \), respectively. On the other hand, any two crossing paths belonging to the same class have identical length. Thus, the total sum \( \Omega_t \) is given by

\[
\Omega_t = \frac{3m^2 + 3m}{2} \Omega_t^{1,2} + 3m^2 \Omega_t^{2,m+2}.
\]

In order to determine \( \Omega_t^{1,2} \) and \( \Omega_t^{2,m+2} \), we define

\[
s_t = \sum_{i \in K_{m,t}, i \neq X} d_{iX}.
\]

Considering the self-similar network structure, we can easily know that at time \( t + 1 \), the quantity \( s_{t+1} \) evolves recursively as

\[
s_{t+1} = (m + 1) s_t + 2m [s_t + (N_t - 1)]
\]

\[
= (3m + 1) s_t + 4m (3m + 1)^t.
\]

Using \( s_0 = 2 \), we have

\[
s_t = (4mt + 6m + 2) (3m + 1)^{t-1}.
\]

Have obtained \( s_t \), the next step is to compute the quantities \( \Omega_t^{1,2} \) and \( \Omega_t^{2,m+2} \) given by

\[
\Omega_t^{1,2} = \sum_{i \in K_{m,t}^1, j \in K_{m,t}^2} d_{ij}
\]

\[
= \sum_{i \in K_{m,t}^1, j \in K_{m,t}^2} (d_{iX} + d_{jX})
\]

\[
= (N_t - 1) \sum_{i \in K_{m,t}^1} d_{iX} + (N_t - 1) \sum_{j \in K_{m,t}^2} d_{jX}
\]

\[
= 2(N_t - 1) s_t - (N_t - 1)(N_t - 1)^2.
\]

where \( d_{XY} = 1 \) has been used. Substituting Eqs. (19) and (20) into Eq. (15), we obtain

\[
\Omega_t = (9m^2 + 3m) (N_t - 1) s_t + 3m^2 (N_t - 1)^2
\]

\[
= 3m(4mt + 7m + 2)(3m + 1)^{2t}.
\]

Inserting Eqs. (21) for \( \Omega_t \), into Eq. (14), and using \( D_1 = 53m + 19 \), we have

\[
D_t = \frac{(3m + 1)^{t-1}}{3} [3m + 5 + (24t + 24m + 4)(3m + 1)^t].
\]

Inserting Eq. (22) into Eq. (11), one can obtain the analytical expression for \( d_t \):

\[
d_t = \frac{6m + 10 + 8(6t + 6m + 1)(3m + 1)^t}{3(3m + 1)[(3m + 1)^t + 1]},
\]

which approximates \( \frac{16t}{3m + 1} \) in the infinite \( t \), implying that the APL shows a logarithmic scaling with network order. Therefore, Koch network exhibits a small-world behavior. We have checked our analytic result against numerical calculations for different \( m \) and various \( t \). In all the cases we obtain a complete agreement between our theoretical formula and the results of numerical investigation, see Fig. [8].

Recently, it has been suggested that for scale-free networks with order \( N \) their average path length \( d(N) \)
is relevant to the exponent $\gamma$ [26, 27]; when $\gamma = 3$, $d(N) \sim \ln N$; when $2 \leq \gamma < 3$, $d(N) \sim \ln \ln N$. However, our above exact results for APL of Koch networks show that the scaling observed in the literature [26, 27] is not a generic feature of all scale-free networks, at least it is not valid for the Koch networks.

D. Degree correlations

Degree correlation is a particularly interesting subject in the field of network science [7, 8, 28, 29, 30, 31], because it can give rise to some interesting network structure effects. An interesting quantity related to degree correlations is the average degree of the nearest neighbors for nodes with degree $k$, denoted as $k_{nn}(k)$, which is a function of node degree [29, 30]. When $k_{nn}(k)$ increases with $k$, it means that nodes have a tendency to connect to nodes with a similar or larger degree. In this case the network is defined as assortative [7, 8]. In contrast, if $k_{nn}(k)$ is decreasing with $k$, which implies that nodes of large degree are likely to have near neighbors with small degree, then the network is said to be disassortative. If correlations are absent, $k_{nn}(k) = \text{const.}$

We can exactly calculate $k_{nn}(k)$ for Koch networks using Eqs. (1) and (15) to work out how many links are made at a particular step to nodes with a particular degree. By construction, we have the following expression [32, 33]

$$k_{nn}(k) = \frac{1}{L_v(t_i)k(t_i,t)} \left( \sum_{t' = t_i}^{t' = t_i - 1} m L_v(t', t_i - 1)k(t',t) \right) + \sum_{t' = t_i + 1}^{t' = t} m L_v(t_i)k(t_i, t' - 1)k(t',t) + 1$$

for $k = 2(m+1)^{t-t_i}$. Here the first sum on the right-hand side accounts for the links made to nodes with larger degree (i.e., $t'_i < t_i$) when the node was generated at $t_i$. The second sum describes the links made to the current smallest degree nodes at each step $t'_i > t_i$. The last term 1 accounts for the link connected to the simultaneously emerging node. In order to compute Eq. (24), we distinguish two cases according to parameter $m$: $m = 1$ and $m \geq 2$.

When $m = 1$, we have

$$k_{nn}(k) = t + 2.$$  (25)

Thus, in the case of $m = 1$, the networks show absence of correlations in the full range of $t$. From Eqs. (25) and (11) we easily see that for large $t$, $k_{nn}(k)$ is approximately a logarithmic function of network order $N_t$, namely, $k_{nn}(k) \sim \ln N_t$, exhibiting a similar behavior as that of the BA model [30] and the two-dimensional random Apollonian network [31].

When $m \geq 2$, Equation (24) is simplified to

$$k_{nn}(k) = \frac{3m + 1}{m - 1} \left[ \frac{(m + 1)^2}{3m + 1} \right]^{t_i} - \frac{m + 3}{m - 1} + \frac{2m}{m + 1} \left( t - t_i \right). \quad (26)$$

Thus after the initial step $k_{nn}(k)$ grows linearly with time. Writing Eq. (26) in terms of $k$, it is straightforward to obtain

$$k_{nn}(k) = \frac{3m + 1}{m - 1} \left[ \frac{(m + 1)^2}{3m + 1} \right]^{t_i} \left( \frac{k}{k_i} \right)^{\ln \left( \frac{m + 1}{m + 3} \right) - \frac{m + 3}{m - 1} + \frac{2m}{m + 1} \ln \left( \frac{k}{k_i} \right).} \quad (27)$$

Therefore, $k_{nn}(k)$ is approximately a power law function of $k$ with negative exponent, which shows that the networks are disassortative. Note that $k_{nn}(k)$ of the Internet exhibits a similar power-law dependence on the degree $k_{nn}(k) \sim k^{-\omega}$, with $\omega = 0.5$ [29].

V. CONCLUSIONS

Networks having a complicated enough structure to display nontrivial properties for real-life systems and statistical models but simple enough to reveal analytical insights are few and far between. In this paper, we have presented a mapping that converts the Koch curves into complex networks, which have many important general properties observed in real networks. Our networks are characterized by closed-form, exact formulas for various properties. The rigorous solutions show that the resulting graphs have a heavy-tailed degree distribution with general exponent $\gamma \in [2, 3]$, a logarithmical average path length with network order, large clustering coefficient, and degree correlations. Our analytical technique could guide and shed light on related studies for deterministic
network models by providing a paradigm for computing the structural features.

In addition to the analytical method and the nontrivial topological characterizations of our proposed model, another contribution of our work is mapping the Koch fractals to a family of graphs. With the the proposed mapping method, a natural bridge between the theory of network science and the classic fractals has been built. This network representation technique could find application to some real systems making it possible to explore the complexity of real-life networked systems in biological and information fields within the framework of complex network theory. Recently, a similar recipe has been adopted for investigating the navigational complexity of cities [34], which has also proven useful to the study of polymer physics [35]. Finally, it should be stressed that for the special case of $m = 1$, the network is completely uncorrelated, it is thus of potential interest as a null substrate network to corroborate the dynamical behavior of complex systems, since the analytic solution of dynamical processes is usually available only for uncorrelated networks [36–39], the generation algorithm of which is often more difficult than one would expect a priori [40].

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