On massive spin-2 in the Fradkin-Vasiliev formalism. 
II. General massive case.

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Abstract

In this work we apply the Fradkin-Vasiliev formalism based on the frame-like gauge invariant description of the massive and massless spin 2 to the construction of the cubic interactions vertices for massive spin 2 self-interaction as well as its gravitational interaction. In the first case we show that the vertex can be reduced (by field redefinitions) to the set of the trivially gauge invariant terms. There are four such terms which are not equivalent on-shell and do not contain more than four derivatives. Moreover, one their particular combination reproduces the minimal (with no more than two derivatives) vertex. As for the gravitational vertex, we show that due to the presence of the massless spin 2 there exist two abelian vertices (besides the three trivially gauge invariant ones) which are not equivalent to any trivially gauge invariant terms and can not be removed by field redefinitions. Moreover, their existence appears to be crucial for the possibility to reproduce the minimal two derivatives vertex.
1 Introduction

Gauge invariance serves as a main guiding principle for the investigation of the consistent interactions for the massless higher spin fields. It severely restricts possible interactions and provides their complete classification. The similar problem for the massive higher spin fields appears to be much more difficult. The well known description of the free massive higher spins [1,2] does not have any gauge symmetry and requires an introduction of a lot of auxiliary fields. The free Lagrangian is constructed in such a way that all constrains, which are necessary to exclude all auxiliary and nonphysical degrees of freedom, follow from the Lagrangian equations. In principle, one can try to construct an interacting Lagrangian such that all the required constrains still follow, but even for the massive spin 2 case such approach appears to be rather complicated (see e.g. [3–5]). Till now the most general classification of the cubic interaction vertices for arbitrary spins massless and/or massive fields has been developed in the so-called light-cone formalism [6,7]. As a Lorentz covariant analogue of the light-cone formalism one can use a so-called TT-approach [8–10], where one works with the fields which already satisfy the transversality and tracelessness conditions. For the massive case (where there is no any gauge symmetry) any vertex, which can be constructed is considered to be the correct one. In this, one assumes that it may be possible to relax the TT-conditions and restore all necessary auxiliary fields, but, as far as we know, it has never been shown.

Taking into account a crucial role, that gauge invariance plays for the massless higher spins, it seems natural to extend this notion to the massive higher spins as well. It is indeed appears to be possible due to the introduction of the so-called Stueckelberg fields. This has been shown in a number of different approaches such as metric-like [11,12], BRST approach [13–17] (see [18] for review), the quartet unconstrained formalism [19] and the frame-like gauge invariant one [20–25]. Some applications of such approach to the construction of the interaction vertices have already appeared, e.g. in the metric-like [26–28], and the frame-like [29–35] formalisms. The most close to the light-cone classification results were obtained in the BRST-BV formalism [36] in terms of the reducible sets of fields (see, however, [37]).

In any investigation of the interaction vertices one has to take into account possible field redefinitions. In many such cases one often restrict oneself with the redefinitions which do not raise the number of derivatives in the vertex. What happens if one relax all such restrictions, working with the Stueckelberg description for the massive fields, has been investigated recently in [28]. At first, it has been shown that there always exist enough field redefinitions to bring the vertex into an abelian form. Note, that we call the vertex abelian if its gauge invariance requires some non-trivial corrections to the gauge transformations but they are such that the commutator of the gauge transformations is zero and the algebra remains to be abelian. Moreover, by using further (even higher derivative) field redefinitions, any vertex can be rewritten in the trivially gauge invariant form, i.e. in terms of the gauge invariant objects of the free theory.

Recently [35] we considered this problem in the frame-like gauge invariant formalism using a gravitational interaction for massive spin 3/2 as an example. From one hand, we have shown that in this case it is also possible to convert the vertex into the abelian form by

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1Note that the classic results of [1,2] can be reproduced by gauge fixing; moreover, it is this procedure that allowed to generalize these Lagrangians into (anti) de Sitter background.
the appropriate field redefinitions. From the other hand, it appeared that due to the presence of massless graviton there exists a couple of abelian vertices which are not equivalent to any trivially gauge invariant ones and can not be removed by field redefinitions. Moreover, the existence of these abelian vertices was crucial for the possibility to reproduce the minimal (with no more than one derivative) gravitational vertex.

In this paper working in the Fradkin-Vasiliev formalism [38–40] based on the frame-like description of massive and massless spin 2 [20,24], we consider cubic vertices for massive spin 2 self-interactions as well as its gravitational interaction. Let us briefly collect our findings here.

**Self-interaction**

- We considered the most general non-abelian ansatz for such vertex and showed that it indeed can be converted into the abelian form by field redefinitions.

- We have shown that all the abelian vertices are equivalent to some trivially gauge invariant ones and/or can be removed by field redefinitions.

- There are four on-shell non-equivalent trivially gauge invariant vertices having no more than four derivatives. Moreover, one their particular combination reproduces the minimal (having no more than two derivatives) one, constructed for the first time in [26].

- Our results are in complete agreement with those of [28] obtained in the metric-like formalism. We also compared our results with the ones obtained recently in the stringy context [41].

**Gravitational interaction**

- In this case also there exist enough field redefinitions to bring the vertex into an abelian form.

- There exist three on-shell non-equivalent trivially gauge invariant vertices.

- Besides, there are two abelian vertices, which are not equivalent to the trivially gauge invariant ones and can not be removed by field redefinitions. Moreover, their existence is crucial for the possibility to reproduce the minimal (with no more than two derivatives) gravitational vertex.

Our paper is organized as follows. In Section 2 we provide all necessary kinematical information on the frame-like gauge invariant description of massive spin 2. Section 3 describes massive spin 2 self-interaction, while Section 4 devoted to the gravitational vertex. A couple of appendices provides the minimal (with no more than two derivatives) vertices obtained by direct constructive approach.

**Notations and conventions** We work in the (anti) de Sitter space with the background frame $e^a$ and its inverse $\hat{e}_a$. We heavily use short-hand notations for their wedge products:

$$E^{a[k]} = e^{a_1} \wedge e^{a_2} \wedge \ldots \wedge e^{a_k},$$

$$\hat{E}_{a[k]} = \hat{e}_{a_1} \wedge \hat{e}_{a_2} \wedge \ldots \wedge \hat{e}_{a_k}.$$
Here and in what follows square brackets denote antisymmetrization. A couple of useful relations:

\[
\hat{E}_a[k] \wedge e^b = \delta^b_a \hat{E}_a[k-1],
\]

\[
\hat{E}_a[k] \wedge e^a = (d - k + 1)\hat{E}_a[k-1].
\]

An \((A)dS\) covariant derivative \(D\) is defined so that

\[
D \wedge D \xi^a = -\kappa E^a_{\ b} \xi^b.
\]

In the main text we systematically omit the wedge product sign \(\wedge\).

## 2 Kinematics

In this section we provide all necessary kinematical information on the frame-like gauge invariant description of massive spin 2 in \((A)dS_d\) space with \(d \geq 4\) \([20,24,25]\).

### 2.1 General formalism

In general, to construct a gauge invariant description of massive spin \(s\) one uses a set of fields necessary for description of massless spin \(s, s - 1, \ldots\). So for the frame-like (first order) gauge invariant formulation of massive spin 2 we introduce one-forms \(\Omega^{a[2]}, f^a\) for massless spin 2, one form \(A\) and zero-form \(B^{a[2]}\) for spin 1 and zero-forms \(\pi^a, \varphi\) for spin 0. Then the Lagrangian, describing massive spin 2 in \((A)dS_d\) space, has the form:

\[
\mathcal{L}_0 = \frac{1}{2} \hat{E}_{a[2]} \Omega^{a c} \Omega^c - \frac{1}{2} \hat{E}_{a[3]} \Omega^{a[2]} Df^a + \frac{1}{2} B_{ab} B^{ab} - \hat{E}_{a[2]} B^{a[2]} DA - \frac{(d - 1)(d - 2)}{2} \pi_a \pi^a + (d - 1)(d - 2) \hat{e}_a \pi^a D \varphi
\]

\[
+ m[\hat{E}_{a[2]} \Omega^{a[2]} A + \hat{e}_a B^{ab} f_b] - 2(d - 1) M \hat{e}_a \pi^a A_\mu
\]

\[
+ \frac{M^2}{2} \hat{E}_{a[2]} f^a f^a - (d - 1) m M \hat{e}_a f^a \varphi + \frac{d(d - 1)}{2} m^2 \varphi^2,
\]

where

\[
M^2 = m^2 - (d - 2) \kappa.
\]

The structure of the Lagrangian is common to the gauge invariant approach, namely, the first two lines are just the sum of the kinetic terms for massless spin 2, 1 and 0 (to simplify the formulas we use non-canonical normalization for spin 0); the fourth line contains all possible mass-like terms, with the mixing terms in the third one. The main requirement here, that completely fixes the whole construction, is that the Lagrangian must still be invariant under all (appropriately modified) gauge transformations of initial massless components. Indeed, it is not hard to check the the Lagrangian is invariant under the following gauge transformations:

\[
\delta_0 f^a = D \xi^a + e_b \eta^{ba} + \frac{2m}{(d - 2)} e^a \xi,
\]
\[\delta_0 \Omega^{[2]} = D\eta^{[2]} - \frac{M^2}{(d-2)} e^a \xi^a,\]
\[\delta_0 A = D\xi + \frac{m}{2} \xi, \quad \delta_0 B^{[2]} = -m\eta^{[2]},\]
\[\delta_0 \varphi = \frac{2M}{(d-2)} \xi, \quad \delta_0 \pi^a = -\frac{Mm}{(d-2)} \xi^a.\]  

One of the nice features of the frame-like formalism is that for all fields (physical or auxiliary) one can construct gauge invariant field strength:

\[F^{[2]}_a = D\Omega^{[2]}_a - \frac{m}{(d-2)} e^a B^a - \frac{M^2}{(d-2)} e^a f^a + \frac{2mM}{(d-2)} E^{[2]}_a \varphi,\]
\[T^a = D f^a + e_b \Omega^{ba} + \frac{2m}{(d-2)} e^a A,\]
\[B^{[2]}_a = DB^{[2]}_a + m\Omega^{[2]}_a - Me^a \pi^a,\]
\[A = DA - \frac{1}{2} E^{[2]}_a B^{[2]}_a + \frac{m}{2} e^a f^b,\]
\[\Pi^a = D\pi^a + \frac{M}{(d-2)} e_b B^{ba} + \frac{Mm}{(d-2)} f^a - \frac{m^2}{(d-2)} e^a \varphi,\]
\[\Phi = D\varphi - e_b \pi^b - \frac{2M}{(d-2)} A.\]  

Note that \(F^{[2]}_a, T^a\) and \(A\) are two-forms, while \(B^{[2]}_a, \Pi^a\) and \(\Phi\) are one-forms. In what follows, we will collectively call all of them as curvatures.

By straightforward calculations one can show that each curvature satisfies a corresponding differential identity:

\[DF^{[2]}_a = -\frac{m}{(d-2)} E^a B^{ba} + \frac{M^2}{(d-2)} e^a T^a + \frac{2mM}{(d-2)} E^{[2]}_a \Phi,\]
\[DT^a = -e_b F^{ba} - \frac{2m}{(d-2)} e^a A,\]
\[DB^{[2]}_a = mF^{[2]}_a + Me^a \Pi^a,\]
\[DA = -\frac{1}{2} E^{[2]}_a B^{[2]}_a - \frac{m}{2} e_b T^b,\]
\[D\Pi^a = -\frac{M}{(d-2)} e_b B^{ba} + \frac{Mm}{(d-2)} T^a + \frac{m^2}{(d-2)} e^a \Phi,\]
\[D\Phi = e_b \Pi^b - \frac{2M}{(d-2)} A.\]  

In what follows ”on-shell” means ”on auxiliary fields equations”, i.e.:

\[T^a \approx 0, \quad A \approx 0, \quad \Phi \approx 0.\]  

In this case for the remaining three curvatures we obtain both algebraic

\[e_b F^{ba} \approx 0, \quad E^{[2]}_a B^{[2]}_a \approx 0, \quad e_b \Pi^b \approx 0,\]
as well as differential identities:

\[ D \mathcal{F}^a[2] \approx -\frac{m}{(d-2)} E^a_b B^{ba}, \]
\[ DB^a[2] \approx m \mathcal{F}^a[2] + M e^a \Pi^a, \]
\[ D \Pi^a \approx -\frac{M}{(d-2)} e_b B^{ba}. \]

Naturally, the Lagrangian equations, being gauge invariant, can be expressed in terms of these curvatures. Indeed, a variation of the Lagrangian under the arbitrary variations of all fields (physical and auxiliary) has the form:

\[ \delta \mathcal{L}_0 = -\frac{1}{4} \hat{E}_{a[3]} \mathcal{F}^a[2] \delta f^a - \frac{1}{4} \hat{E}_{a[3]} T^a \delta \Omega^a[2] \]
\[ + \hat{E}_{a[2]} B^a[2] \delta \Lambda - \frac{1}{2} \hat{E}_{a[2]} A \delta B^a[2] \]
\[ - (d-1)(d-2) \hat{e}_a \Pi^a \delta \varphi + (d-1)(d-2) \hat{e}_a \Phi \delta \pi^a. \] (9)

One more nice feature of the frame-like formalism is that using these curvatures the Lagrangian can also be rewritten in the manifestly gauge invariant form. The most general ansatz looks like:

\[ \mathcal{L}_0 = a_1 \hat{E}_{a[4]} \mathcal{F}^a[2] \mathcal{F}^a[2] + a_2 \hat{E}_{a[2]} B^a b + a_3 \hat{E}_{a[2]} \Pi^a \]
\[ + a_4 \hat{E}_{a[3]} B^a[2] \Theta^a + a_5 \hat{E}_{a[3]}^{[2]} \Pi^a + a_6 \hat{E}_{a[2]} B^a[2] \Phi. \] (10)

It appears that the general solution for the coefficients \( a_1-6 \) has two arbitrary parameters. This ambiguity is related with the two identities (here we use the fact that the Lagrangian is determined only up to the total derivative):

\[ 0 = \hat{E}_{a[4]} D [ \mathcal{F}^a[2] B^a[2] ] \]
\[ = \frac{m}{2} \hat{E}_{a[4]} \mathcal{F}^a[2] \mathcal{F}^a[2] + \frac{8(d-3)m}{(d-2)} \hat{E}_{a[2]} B^a b + 2(d-3) M \hat{E}_{a[3]} \mathcal{F}^a[2] \Pi^a \]
\[ + \frac{2(d-3) M^2}{(d-2)} \hat{E}_{a[3]} B^a[2] \Theta^a + 2(d-3) M m \hat{E}_{a[2]} B^a[2] \Phi, \] (11)

\[ 0 = \hat{E}_{a[3]} D [ B^a[2] \Pi^a ] \]
\[ = -\frac{2M}{(d-2)} \hat{E}_{a[2]} B^a b + 2(d-2) M \hat{E}_{a[2]} \Pi^a \Pi^a + \frac{m}{2} \hat{E}_{a[3]} \mathcal{F}^a[2] \Pi^a \]
\[ - \frac{M m}{(d-2)} \hat{E}_{a[3]} B^a[2] \Theta^a + m^2 \hat{E}_{a[2]} B^a[2] \Phi. \] (12)

Thus the Lagrangian is determined up to the shifts:

\[ \mathcal{L}_0 \Rightarrow \mathcal{L}_0 + \rho_1 \hat{E}_{a[4]} D [ \mathcal{F}^a[2] B^a[2] ] + \rho_2 \hat{E}_{a[3]} D [ B^a[2] \Pi^a ] \] (13)

Using this freedom we choose (the reason for such choice will become clear later) \( a_2 = a_5 = 0 \). The most straightforward way to find explicit solution for the coefficient is to consider variations of the Lagrangian and compare them with (9). For example:

\[ \delta f \mathcal{L}_0 = \frac{M m a_4 - 8(d-3) M^2 a_1}{(d-2)} \hat{E}_{a[3]} \mathcal{F}^a[2] \delta f^a + \frac{2M m a_3}{(d-2)} - 4M^2 a_4 \hat{E}_{a[2]} \Pi^a \delta f^a, \] (14)
and this gives us the equations:

\[
\frac{Ma_4 - 8(d - 3)M^2a_1}{(d - 2)} = -\frac{1}{4} \quad \frac{ma_3}{(d - 2)} = 2Ma_4.
\] (15)

Taking into account variations for all other fields we find:

\[
a_1 = -\frac{1}{32(d - 3)\kappa}, \quad a_3 = -\frac{(d - 2)}{\kappa}, \quad a_4 = -\frac{m}{4M\kappa}, \quad a_6 = -\frac{(d - 2)}{2M}.
\] (16)

Note the the coefficient \(a_1\) is the same as in the purely massless spin 2 case.

### 2.2 Partial gauge fixing

To simplify the presentation we make a partial gauge fixing, namely we set scalar field \(\varphi\) to zero and solve its equation \(\Phi \approx 0\):

\[
\varphi = 0 \quad \Rightarrow \quad A = -\frac{(d - 2)}{2M} e_a \pi^a.
\] (17)

Note that this is possible only for \(M \neq 0\) thus excluding the so-called partially massless case, which has been considered in [32]. The Lagrangian takes the form (we make rescaling \(\pi^a \Rightarrow \frac{M}{(d - 2)} \pi^a\) reflecting the fact that \(\pi^a\) plays the role of physical field now):

\[
\mathcal{L}_0 = \frac{1}{2} \hat{E}_{a[2]} \Omega^{a} \Omega_{b}^a - \frac{1}{2} \hat{E}_{a[3]} \Omega^{a} \Omega_{b}^{a[2]} Df^a + \frac{1}{2} B_{a[2]} B_{b}^{a[2]} - \hat{e}_a DB_{a}^{b} \pi_{b}^a \\
- m \hat{e}_a \Omega_{a}^b \pi_{b}^a + m \hat{e}_a B_{a}^{b} f_{b}^a + \frac{M^2}{2} \hat{E}_{a[2]} f^a f^a + \frac{(d - 1)M^2}{2(d - 2)} \pi_{a}^a \pi_{a}^a.
\] (18)

Such Lagrangian is still invariant under the remaining gauge transformations:

\[
\delta \Omega^{a[2]} = D\eta^a - \frac{M^2}{(d - 2)} e^a \xi^a, \quad \delta f^a = D\xi^a + e_b \eta_{ba}, \quad \delta B^a_{a[2]} = -m\eta^a_{[2]}, \quad \delta \pi^a = -m\xi^a.
\] (19)

Moreover, for each field we still have a corresponding gauge invariant curvature:

\[
\mathcal{F}^{a[2]} = D\Omega^{a[2]} - \frac{m}{(d - 2)} E_{a}^{b} B_{a}^{b} - \frac{M^2}{(d - 2)} e^a f^a, \\
\mathcal{T}^{a[2]} = Df^a + e_b \Omega_{ba} - \frac{m}{(d - 2)} E_{b}^{a} \pi_{b}, \\
\mathcal{B}^{a[2]} = DB^{a[2]} + m\Omega^{a[2]} - \frac{M^2}{(d - 2)} e^a \pi^a, \\
\Pi^a = D\pi^a + e_b B_{b}^{a} + mf^a.
\] (20)

\[^2\text{We have explicitly checked that all our main results remain to be the same.}\]
Let us stress that on-shell only $T^a \approx 0$, while three others are non-zero and satisfy (7) as well as:

\[ D\tilde{F}^a[2] \approx -\frac{m}{(d-2)}E^a_bB^{ba}, \]
\[ DB^a[2] \approx m\tilde{F}^a[2] + \frac{M^2}{(d-2)}e^a\Pi^a, \]
\[ D\Pi^a \approx -e_bB^{ba}. \]

As before, all variations of the Lagrangian can be expressed in terms of curvatures:

\[ \delta L_0 = -\frac{1}{4} \hat{E}_{a[3]}\tilde{F}^a[2] \delta f^a - \frac{1}{4} \hat{E}_{a[3]} T^a \delta \Omega^a[2] \]
\[ -\hat{e}_a B^{ab} \delta \pi_b + \hat{e}_a \Pi_b \delta B^{ab}. \]  (22)

The most general ansatz for the Lagrangian written in terms of curvatures now:

\[ L_0 = a_1 \hat{E}_{a[4]} \tilde{F}^a[2] \tilde{F}^a[2] + a_2 \hat{E}_{a[2]} B^{ab} B^{ba} + a_3 \hat{E}_{a[2]} \Pi^a \Pi^a \]
\[ + a_4 \hat{E}_{a[3]} \tilde{F}^a[2] \Pi^a + a_5 \hat{E}_{a[3]} B^{a[2]} \mathcal{T}^a. \]  (23)

We again use the freedom to choose $a_2 = a_5 = 0$ and obtain:

\[ a_1 = -\frac{1}{32(d-3)\kappa}, \quad a_3 = \frac{16(d-3)M^2a_1}{(d-2)}, \quad a_4 = \frac{8(d-3)ma_1}{(d-2)}. \]  (24)

### 2.3 New variables

From the explicit forms of the gauge transformations (3) or (19) one can see that the one-form fields play the double role being gauge fields and (in some sense) the Stueckelberg ones simultaneously. As a last refinement, we make a field redefinition so that clearly separate their roles for all fields (this, in-particular, simplifies the comparison of massless and massive cases):

\[ \tilde{\Omega}^a[2] = \Omega^a[2] - \frac{m}{(d-2)}e^a\pi^a \Rightarrow \delta \tilde{\Omega}^a[2] = D\eta^a[2] + \kappa e^a\xi^a. \]  (25)

Then the new curvatures take the form:

\[ \tilde{F}^a[2] = D\tilde{\Omega}^a[2] + \kappa e^a f^a, \]
\[ T^a = Df^a + e_b\tilde{\Omega}^{ba}, \]
\[ B^a[2] = DB^a[2] + m\tilde{\Omega}^a[2] + \kappa e^a\pi^a, \]
\[ \Pi^a = D\pi^a + e_bB^{ba} + mf^a, \]  (26)

so that the first two lines are exactly the same as in the massless case. On-shell identities now:

\[ D\tilde{F}^a[2] \approx 0, \quad DB^a[2] \approx m\tilde{F}^a[2] - \kappa e^a\Pi^a, \quad D\Pi^a \approx -e_bB^{ba}. \]  (27)

At last, the Lagrangian now has a very simple form:

\[ \mathcal{L}_0 = a_1 \hat{E}_{a[4]} \tilde{F}^a[2] \tilde{F}^a[2] + \frac{1}{2} \hat{E}_{a[2]} \Pi^a \Pi^a, \]  (28)

where the first term is exactly the same as in the massless case, while the second one is just the mass term appropriately dressed with the Stueckelberg fields.
3 Self-interaction

In this section we elaborate on the cubic vertex for the massive spin 2 self-interaction following the general pattern of the Fradkin-Vasiliev formalism (see comparison of the massless and massive cases in [33]).

3.1 Deformations

The first step of the Fradkin-Vasiliev formalism is to consider the most general consistent quadratic deformations for all gauge invariant curvatures. In our case the most general ansatz looks like (from now on we omit tildes on $\Omega$):

$$\Delta F^a[2] = b_1 \Omega^{ab} \Omega_b^a + b_2 f^a f^a + b_3 \Omega^{a[2]} e_b \pi^b - b_4 e_b \Omega^{ba} \pi^a + b_5 e^a \Omega^{ab} \pi_b$$

$$+ b_6 e_b B^{ba} - b_7 e_b f^b B^{a[2]} + b_8 e^a B^{ab} f_b + b_9 E_{b[2]} B^{ba} B^{ba} + \frac{b_{10}}{2} B_{b[2]} E_{b[2]} B^{b[2]}$$

$$+ b_{11} E^a e_b B^{ab} \pi_b + b_{12} E^a e_b \pi^a \pi^b,$$

$$\Delta T^a = b_{13} \Omega^{ab} f_b + b_{14} \Omega^{ab} e_c B^c_b - b_{15} B^{ab} e_c \Omega^{cb} + b_{16} e^a \Omega^{b[2]} B_{b[2]} + b_{17} f^a e_b \pi^b$$

$$- b_{18} e_b f^b \pi^a + b_{19} e^a f_b \pi^b + b_{20} E^b B^b_a \pi^a + \frac{b_{21}}{2} E_{b[2]} B^{b[2]} \pi^a + b_{22} E^a B_{bc} \pi^c,$$

$$\Delta B^a[2] = b_{23} \Omega^{ab} C^a c + b_{24} f^a \pi^a + b_{25} e_b B^{ba} \pi^b + b_{26} B^{a[2]} e_b \pi^b + b_{27} e^a B^{ac} \pi^c,$$

$$\Delta \Pi^a = b_{28} \Omega^{ab} \pi_b + b_{29} B^{ab} f_b + b_{30} B^{ab} e^c B_{cb} + b_{31} e^a B^{b[2]} B_{b[2]} + b_{32} \pi^a e_b \pi^b + b_{33} e^a \pi^b \pi_b.$$

Here consistency means that the deformed curvatures ($\hat{F}^a[2] = F^a[2] + \Delta F^a[2]$ and so on) transform covariantly under all gauge transformations. For example, from the explicit form of the deformation $\Delta F^a[2]$ one can directly read out the corresponding corrections to the gauge transformations:

$$\delta_1 \Omega^{a[2]} = 2 b_1 \eta^a \Omega^e_c - b_3 \eta^{a[2]} e_b \pi^b - b_4 e_b \Omega^{ba} \pi^a + b_5 e^a \Omega^{ac} \pi_c$$

$$+ 2 b_2 f^a \pi^a + b_6 e_b B^{ba} \pi^a - b_7 B^{a[2]} e_b \pi^b + b_{18} e_b \pi^a \pi^b + b_{19} e^a f_b \pi^b + b_{20} E^b B^b_a \pi^a$$

$$+ \frac{b_{21}}{2} E_{b[2]} B^{b[2]} \pi^a + b_{22} E^a B_{bc} \pi^c,$$  \hspace{1cm} (30)

This, in turn, means that the deformed curvature $\hat{F}^a[2] = F^a[2] + \Delta F^a[2]$ must transform as follows:

$$\delta \hat{F}^a[2] = 2 b_1 \eta^a \hat{F}^e_c + b_{13} \eta^{a[2]} e_b \Pi^b - b_4 e_b \hat{F}^{ba} \Pi^a - b_5 e^a \hat{F}^{ac} \Pi_c$$

$$+ 2 b_2 \hat{T}^a \xi^a - b_6 e_b B^{ba} \xi^a - b_7 B^{a[2]} e_b \xi^b + b_{18} e_b \xi^a \xi^b + b_{19} e^a f_b \xi^b + b_{20} E^b B^b_a \xi^a$$

$$+ \frac{b_{21}}{2} E_{b[2]} B^{b[2]} \xi^a + b_{22} E^a B_{bc} \xi^c,$$ \hspace{1cm} (31)

Such procedure nicely work in the massless case, but in the massive, one due to the presence of one-form Stueckelberg fields, we have a lot of possible field redefinitions to take into account:

$$\Omega^{a[2]} \Rightarrow \Omega^{a[2]} + \kappa_1 B^{ba} \Omega^a_b + \kappa_2 f^a f^a + \kappa_3 B^{a[2]} e_b \pi^b + \kappa_4 e_b B^{ba} \pi^a + \kappa_5 e^a B^{ac} \pi_c,$$

$$f^a \Rightarrow f^a + \kappa_6 B^{ab} f_b + \kappa_7 \Omega^{ab} \pi_b + \kappa_8 B^{ab} e^c B_{cb} + \kappa_9 e^a B^{b[2]} B_{b[2]} + \kappa_{10} e_b \pi^b \pi^a + \kappa_{11} e^a \pi^b \pi_b,$$

$$\pi^a \Rightarrow \pi^a + \kappa_{12} B^{ab} \pi_b.$$ \hspace{1cm} (32)
To see the effect of such redefinitions, let us consider the one with the parameter $\kappa_1$ as an illustration. It produces the following corrections to the curvature:

$$
\frac{1}{\kappa_1} \Delta F^{a[2]} = B^{ca} \Omega^a_c + B^{ca} F^a_c - m \Omega^a_c \Omega^b_c - \kappa e_c \Omega^a c \pi^b + \kappa e_b B^{ba} f^a - \kappa e^a B^{ab} f_b.
$$

(33)

This is equivalent to the shifts of the deformation parameters:

$$
b_1 \Rightarrow b_1 - m \kappa_1
$$

$$
b_4 \Rightarrow b_4 + \kappa \kappa_1
$$

$$
b_5 \Rightarrow b_5 + \kappa \kappa_1
$$

$$
b_6 \Rightarrow b_6 - \kappa \kappa_1
$$

$$
b_8 \Rightarrow b_8 - \kappa \kappa_1
$$

(34)

and introduction of the abelian deformations

$$
\Delta F^{a[2]} \sim B^{ca} \Omega^a c + B^{ca} F^a_c
$$

(35)

as well as the corresponding abelian corrections to the gauge transformations:

$$
\delta_1 \Omega^a b \sim B^{ca} \eta^a c
$$

(36)

By straightforward but rather lengthy calculations we have obtained all the equations on the deformation parameters $b_1-33$ which follows from the consistency conditions. Moreover, we have shown that all these equations are invariant under all shifts generated by the field re-definitions $\kappa_1-12$ which serves as a quite non-trivial check for our calculations. We have found that the number of free parameters in the general solution for the obtained equations is equal to the number of possible field redefinitions. Moreover, by these redefinitions all the parameters of the non-abelian deformations can be set to zero. This leaves us with the abelian deformations only in agreement with the general statement of [28].

### 3.2 Abelian vertices

Recall that by abelian vertices we mean the cubic vertices which contain two gauge invariant curvatures and one explicit field which can be one-form $\Omega$, $f$ or zero-form $B$, $\pi$. Any cubic vertex with one-form by making substitutions

$$
\Omega^{a[2]} \Rightarrow \frac{1}{m} [B^{a[2]} - DB^{a[2]} - \kappa e^a \pi^a],
$$

$$
f^a \Rightarrow \frac{1}{m} [\Pi^a - D \pi^a - e_b B^{ba}].
$$

(37)

integrating by parts and using differential identities for curvatures, can be reduced to the combinations of trivially gauge invariant vertices and some abelian ones containing zero-forms. As for the later, any such gauge invariant vertex can be removed by field redefinitions. Let us consider zero-form $B^{a[2]}$ as an example. Due to the two on-shell identities:

$$
0 \approx \hat{E}_{a[4]} F^{a[2]} B^{a[2]} e_b \Pi^b = -2 \hat{E}_{a[3]} [F^{a[2]} B^{ab} \Pi_b + F^{ab} B^{a[2]} \Pi_b],
$$

$$
0 \approx \hat{E}_{a[4]} e_b F^{ba} B^{a[2]} \Pi^a = \hat{E}_{a[3]} \delta_{ad} [B^{a[2]} \Pi_d - 2 B^{a} \Pi^a],
$$

(38)
there is only one independent vertex:

$$\mathcal{L}_a \sim \hat{E}_{a[3]} F^{a[2]} B^{ab} \Pi_b.$$  \hfill (39)

Its variation gives:

$$\delta \mathcal{L}_a \sim m \hat{E}_{a[3]} F^{a[2]} \eta^{ab} \Pi_b$$  \hfill (40)

and can be compensated by the correction to the gauge transformations (showing its abelian nature):

$$\delta f^a \sim \eta^{ab} \Pi_b.$$  \hfill (41)

At the same time, this vertex can be removed by the field redefinition

$$f^a \Rightarrow f^a + B^{ab} \Pi_b.$$  \hfill (42)

The situation with the abelian vertices with $\pi^a$ is similar. Net result is that all abelian vertices are equivalent to the trivially gauge invariant ones and/or can be removed by field redefinitions again in agreement with [28].

### 3.3 Trivially invariant terms

Now let us turn to the last remaining possibility, namely, to the trivially gauge invariant vertices. We have three curvatures $\mathcal{F}$, $\mathcal{B}$ and $\Pi$ which do not vanish on-shell, so the most general ansatz is:

$$\mathcal{L}_1 = h_1 \hat{E}_{a[6]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} + h_2 \hat{E}_{a[5]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \Pi^a + h_3 \hat{E}_{a[4]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \Pi^a + h_4 \hat{E}_{a[3]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \Pi^a + \hat{E}_{a[4]} [h_5 \mathcal{F}^{a[2]} \mathcal{B}^{ab} \mathcal{B}^{a} b + h_6 \mathcal{F}^{ab} \mathcal{B}^{a} b \mathcal{B}^{a} b] + \hat{E}_{a[3]} [h_7 \mathcal{B}^{ab} \mathcal{B}^{a} b \Pi^a + h_8 \mathcal{B}^{a[2]} \mathcal{B}^{ab} \Pi_b].$$  \hfill (43)

But these terms are not independent. Once again using the fact that the Lagrangian is defined only up to total derivative and also differential identities for the curvatures we obtain:

$$0 = \hat{E}_{a[6]} D[\mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \mathcal{B}^{a[2]}]$$

$$0 \approx \frac{m}{2} \hat{E}_{a[6]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} - (d - 5) \kappa \hat{E}_{a[5]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \Pi^a,$$  \hfill (44)

$$0 = \hat{E}_{a[5]} D[\mathcal{F}^{a[2]} \mathcal{B}^{a[2]} \Pi^a]$$

$$0 \approx \frac{m}{2} \hat{E}_{a[5]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \Pi^a - 2(d - 4) \kappa \hat{E}_{a[4]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \Pi^a \Pi^a$$

$$-2 \hat{E}_{a[4]} [\mathcal{F}^{a[2]} \mathcal{B}^{ab} \mathcal{B}^{a} b + \mathcal{F}^{ab} \mathcal{B}^{a} b \mathcal{B}^{a} b],$$  \hfill (45)

$$0 = \hat{E}_{a[4]} D[\mathcal{B}^{a[2]} \Pi^a]$$

$$0 \approx \frac{m}{2} \hat{E}_{a[4]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \Pi^a - 2(d - 3) \kappa \hat{E}_{a[3]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} \Pi^a \Pi^a$$

$$-2 \hat{E}_{a[3]} [2 \mathcal{B}^{ab} \mathcal{B}^{a} b \Pi^a + 2 \mathcal{B}^{a[2]} \mathcal{B}^{ab} \Pi_b],$$  \hfill (46)

$$0 = \hat{E}_{a[3]} D[\mathcal{B}^{a[2]} \mathcal{B}^{ab} \mathcal{B}^{a} b]$$

$$0 \approx \frac{m}{2} \hat{E}_{a[3]} \mathcal{F}^{a[2]} \mathcal{B}^{ab} \mathcal{B}^{a} b + 2 \mathcal{F}^{ab} \mathcal{B}^{a} b \mathcal{B}^{a} b]$$

$$-2 \kappa \hat{E}_{a[3]} [(d - 1) \mathcal{B}^{ab} \mathcal{B}^{a} b \Pi^a - (d - 4) \mathcal{B}^{a[2]} \mathcal{B}^{ab} \Pi_b].$$  \hfill (47)
Thus we have only four independent terms and we choose:
\[
\mathcal{L}_1 = h_5 \hat{E}_a[4] \mathcal{F}^{a[2]} \mathcal{B}^{ab} \mathcal{B}_b + \hat{E}_a[3] [h_7 \mathcal{B}^{ab} \mathcal{B}_b \Pi^a + h_8 \mathcal{B}^{a[2]} \mathcal{B}^{ab} \Pi_b + h_4 \Pi^a \Pi^a].
\] (48)

To understand the physical meaning of these results, it is convenient to consider them in the unitary gauge \( B^{a[2]} = 0, \pi^a = 0 \), where
\[
\mathcal{B}^{a[2]} = m \Omega^a[2], \quad \Pi^a = m f^a.
\] (49)

Let us stress that all the possible field redefinitions we discussed above necessarily contain Stueckelberg fields and so do not change this part of the Lagrangian. We obtain:
\[
\mathcal{L}_1 \approx m^2 h_5 \hat{E}_a[4] \mathcal{F}^{a[2]} \Omega^{ab} \Omega^b
\]
\[
+ m^3 \hat{E}_a[3] [h_7 \Omega^{ab} \Omega^b f^a + h_8 \Omega^{a[2]} \Omega^{ab} f_b + h_4 f^a f^a f^a].
\] (50)

It is instructive to compare these expression with the minimal two derivative vertex (see appendix A.1) which in the unitary gauge has the form:
\[
\mathcal{L}_1 \sim \hat{E}_a[3] [\Omega^{ab} \Omega^b f^a + \Omega^{a[2]} \Omega^{ab} f_b - M^2 f^a f^a f^a].
\] (51)

Our results are in a complete agreement with the results of [28]. Indeed, in the metric-like formalism there exists only one main gauge invariant object (here we follow the notations of the original paper):
\[
H_{\mu
u} = h_{\mu
u} + \frac{1}{m} \nabla_{(\mu} B_{\nu)} - \frac{1}{\mu m} \nabla_\mu \nabla_\nu \varphi - \frac{2m}{\mu(D-2)} g_{\mu\nu} \varphi
\]
(39)

(compare with one-form \( \Pi^a \)), but there exist also two gauge invariant objects which do not contain more than two derivatives of the physical field:
\[
H_{\mu
u\rho} = \nabla_\mu H_{\nu\rho} - \nabla_\nu H_{\mu\rho}
\]
(40)

(\( H^{\underline{\mu\nu}} \))
(\( H_{\mu\nu\rho\sigma} = \nabla_\mu \nabla_{[\mu} H_{\nu]\rho\sigma} - \nabla_\nu \nabla_{[\nu} H_{\mu]\rho\sigma} + \nabla_\rho \nabla_{[\rho} H_{\mu\nu]\sigma} - \nabla_\sigma \nabla_{[\sigma} H_{\mu\nu]\rho} \))
(39)

(compare with one form \( \mathcal{B}^{a[2]} \) and two-form \( \mathcal{F}^{a[2]} \)). The authors of [28] have shown that besides the minimal two derivative vertex there exist just three vertices which do not contain more than four derivatives (here we provide slightly different but equivalent form just to stress the similarity with our frame-like results):
\[
\tilde{a}^{(dRGT)}_0 \sim \left\{ \frac{\mu_1 \nu_2 \mu_3 \nu_1 \nu_2 \nu_3}{\nu_2 \nu_3} \right\} H_{\mu_1}^{\nu_1} H_{\mu_2}^{\nu_2} H_{\mu_3}^{\nu_3},
\]
\[
\tilde{a}^{(PL1)}_0 \sim \left\{ \frac{\mu_1 \mu_2 \mu_3 \mu_4}{\nu_2 \nu_3 \nu_4} \right\} H^{\underline{\mu_1}}_{\mu_1} H^{\underline{\mu_2}}_{\mu_2} H^{\underline{\mu_3}}_{\mu_3} H^{\underline{\mu_4}}_{\mu_4},
\]
\[
\tilde{a}^{(PL2)}_0 \sim \left\{ \frac{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5}{\nu_2 \nu_3 \nu_4 \nu_5} \right\} H^{\underline{\mu_1}}_{\mu_1} H^{\underline{\mu_2}}_{\mu_2} H^{\underline{\mu_3}}_{\mu_3} H^{\underline{\mu_4}}_{\mu_4} H^{\underline{\mu_5}}_{\mu_5}.
\]
(40)

The first one is related with the so-called dRGT gravity [12,43], while the other two are related with the so-called pseudo-linear terms [44,45]. The authors also noted that the vertices \( \tilde{a}^{(PL1,2)}_0 \) can be rewritten in terms of \( H^{\underline{\mu\nu}} \) instead of \( H^{\underline{\mu\nu}} \), so that there exist differential identities for these objects similar to the ones given at the beginning of this subsection.
Recently, an interesting paper [41] appeared, where the cubic vertices for massive spin 2 self-interaction as well as its interaction with massless graviton were elaborated in the stringy context. The massive spin 2 comes from the first massive level of open bosonic or superstring, while massless graviton comes from the closed (super)string. The aim of the paper was to compare the stringy results with those of the bigravity theory [46]. To this purpose, the authors calculated three point amplitudes and then converted them into a so-called transverse-traceless parts of the cubic vertices. They obtained (here we also follow the notations of the original paper; in-particular, symmetric tensor $M_{\mu\nu}$ corresponds to the massive spin 2):

- bigravity

$$L_{M^3} = \frac{(-\beta_1 + \beta_2)(1 + \alpha^2)^{3/2}}{6\alpha} m_g [M^3] + \frac{(1 - \alpha^2)}{m_g \alpha \sqrt{1 + \alpha^2}} M^{\mu\nu} (\partial_\mu M_{\rho\sigma} \partial_\nu M^{\rho\sigma} - 2 \partial_\nu M_{\rho\sigma} \partial^\rho M_\mu^\sigma),$$

- superstring

$$L_{M^3}^{\text{eff}} = \frac{g_0}{\alpha'} \left\{ [M^3] + 2\alpha' M^{\mu\nu} [\partial_\mu M_{\rho\sigma} \partial_\nu M^{\rho\sigma} - 3\partial_\nu M_{\rho\sigma} \partial^\rho M_\mu^\sigma] + 4\alpha'^2 \partial^\mu \partial^\nu M_{\rho\sigma} \partial^\rho M_\mu^\sigma \partial^\kappa M_{\rho\sigma} \partial^\lambda M_{\rho\sigma} \partial^\kappa M_{\rho\sigma} \partial^\lambda M_{\rho\sigma} \right\},$$

- bosonic string

$$L_{M^3}^{\text{eff,bos}} = \frac{g_0}{\alpha'} \left\{ 2[M^3] + 3\alpha' M^{\mu\nu} [\partial_\mu M_{\rho\sigma} \partial_\nu M^{\rho\sigma} - 4\partial_\nu M_{\rho\sigma} \partial^\rho M_\mu^\sigma] + 2\alpha'^2 \partial^\mu \partial^\nu M_{\rho\sigma} \partial^\rho M_\mu^\sigma \partial^\kappa M_{\rho\sigma} \partial^\lambda M_{\rho\sigma} \partial^\kappa M_{\rho\sigma} \partial^\lambda M_{\rho\sigma} \right\}.$$

Thus the general structure is indeed the same (one term without derivatives, two terms with two derivatives and so on), while the relative coefficients are different in all three cases.

### 3.4 Non-abelian version

In the kinematical section we have seen that the free Lagrangian for massive spin 2 can be written as the massless one plus mass term appropriately dressed with the Stueckelberg fields. At the same time, the cubic vertex in its trivially gauge invariant form drastically differs from that of the massless case being non-abelian in nature. In this subsection we reconstruct our vertex in the form as close to the massless one as possible. For this purpose, we start with the following ansatz for the deformation of the massive spin 2 curvatures:

$$\Delta F^{a[2]} = b_1 \Omega^{b a} \Omega^b_a + b_2 f^a f^a,$$

$$\Delta T^a = b_3 \Omega^{a b} f_b,$$

$$\Delta B^{a[2]} = b_4 B^{b a} \Omega^b_a + b_5 f^a \pi^a,$$

$$\Delta \Pi^a = b_6 \Omega^{a b} \pi_b + b_7 B^{a b} f_b$$

(52)
and require that the deformed curvatures transform covariantly.

**η**

-transformation**s** In this case corrections to the gauge transformations have the form:

\[ \delta \Omega^{a[2]} = -b_1 \eta^{ba} \Omega^a_b, \quad \delta f^a = -b_3 \eta^{ab} f_b, \quad \delta B^{ab} = b_4 \eta^{ba} B^a_b, \quad \delta \pi^a = -b_6 \eta^{ab} \pi_b, \quad (53) \]

while the variations of the deformed curvatures are:

\[ \delta \hat{\mathcal{F}}^{ab} = -b_1 \eta^{ba} D \Omega^a_b + b_2 f^a e_b \eta^{ba} - \kappa b b \eta^{a} f_b, \]
\[ \delta \hat{T}^a = -b_3 \eta^{ab} D f_b + (b_3 - b_1) \Omega^a_b e_c \eta^{cb} - b_4 \eta^{ab} e_c \Omega_b, \]
\[ \delta \hat{\mathcal{B}}^{a[2]} = b_4 \eta^{ba} D B^a_b - m(b_1 + b_4) \Omega^a_b e_c \eta^{cb} - b_5 e_b e_c \pi^a - \kappa b_b e_c \eta^{ab} \pi_b, \]
\[ \delta \hat{\Pi}^a = -b_6 \eta^{ab} D \pi_b + (b_7 + b_4) e_c \eta^{cb} - m(b_3 + b_7) \eta^{ab} f_b + b_4 \eta^{ab} e_c B^c_b. \]

For consistency this must coincide with

\[ -b_1 \eta^{ba} \mathcal{F}^a_b = -b_1 \eta^{ba} D \Omega^a_b + \kappa b_1 f^a e_b \eta^{ba} - \kappa b_1 e^a \eta^{ab} f_b, \]
\[ -b_3 \eta^{ab} \mathcal{T}_b = -b_3 \eta^{ab} [D f_b + e \Omega^a_b], \]
\[ b_4 \eta^{ba} \mathcal{B}^a_b = b_4 \eta^{ba} D B^a_b + m b_4 \eta^{ba} \Omega^a_b - \kappa b_4 \eta^{ab} \pi^a + \kappa b_4 e^a \eta^{ab} \pi_b, \]
\[ -b_6 \eta^{ab} \pi_b = -b_6 \eta^{ab} [D \pi_b + e^c B^c_b + m f_b], \]

which gives us:

\[ b_2 = \kappa b_1, \quad b_3 = b_1, \quad b_4 = -\frac{b_1}{2}, \quad b_5 = -\kappa b_4, \quad b_6 = -b_4, \quad b_7 = b_4. \quad (56) \]

**ξ**

-transformation**s** Here the corrections look like:

\[ \delta \Omega^{a[2]} = b_2 f^a \xi^a, \quad \delta f^a = b_3 \Omega^{ab} \xi_b, \quad \delta B^{a[2]} = b_5 \pi^a \xi^a, \quad \delta \pi^a = -b_7 B^{ab} \xi_b, \quad (57) \]

while the variations have the form:

\[ \delta \mathcal{F}^{a[2]} = b_2 D f^a \xi^a + \kappa b_1 e_b \Omega^{ba} \xi^a + \kappa (b_3 - b_1) e^a \Omega^{ab} \xi_b, \]
\[ \delta \hat{T}^a = b_3 D \Omega^{ab} \xi_b + \kappa b_3 e^a f^b \xi_b - (\kappa b_3 - b_2) e_b f^b \xi_a + b_2 f^a e_b \xi_b, \]
\[ \delta \hat{\mathcal{B}}^{a[2]} = b_5 D \pi^a \xi^a + \kappa (b_4 - b_7) B^a_b e^c \xi_b - \kappa b_4 e_b B^{ba} \xi^a + m(b_2 - b_5) f^a \xi^a, \]
\[ \delta \hat{\Pi}^a = -b_7 D B^{ab} \xi_b + m(b_3 - b_5) \Omega^{ab} \xi_b + \kappa b_6 e^a \pi^b \xi_b + (b_5 - \kappa b_6) e_b \pi^b \xi^a - b_5 \pi^a e_b \xi^b. \]

Comparing them with

\[ b_2 \mathcal{T}^a \xi^a = b_2 [D f^a + e_b \Omega^{ba} \xi^a], \]
\[ b_3 \mathcal{R}^{ab} \xi_b = b_3 D \Omega^{ab} \xi_b + \kappa b_3 e^a f^b \xi_b + \kappa b_3 f^a e_b \xi^b, \]
\[ b_5 \Pi^a \xi^a = b_5 [D \pi^a + e_b B^{ba} + m f^a] \xi^a, \]
\[ -b_7 B^{ab} \xi_b = -b_7 D B^{ab} \xi_b - m b_7 \Omega^{ab} \xi_b - \kappa b_7 e^a \pi^b \xi_b + \kappa b_7 \pi^a e_b \xi^b, \quad (59) \]

we obtain the same solution for the coefficients \( b_{1-7} \). Thus we obtain (we change \( b_1 \to 2b_0 \)):

\[ \Delta \mathcal{F}^{a[2]} = 2b_0 \Omega^{ba} \Omega^a_b + 2 \kappa b_0 f^a f^a, \]
\[ \Delta \mathcal{T}^a = 2b_0 \Omega^{ab} f_b, \]
\[ \Delta \mathcal{B}^{a[2]} = -b_0 B^{ba} \Omega^a_b + \kappa b_0 f^a \pi^a, \]
\[ \Delta \Pi^a = b_0 \Omega^{ab} \pi_b - b_0 B^{ab} \hat{f}_b. \quad (60) \]
In this, the variations of the deformed curvatures which do not vanish on-shell have the form:

\[
\delta \hat{F}^a_{[2]} \approx 2b_0 F^{ba} \eta^a_b, \\
\delta \hat{B}^a_{[2]} \approx -b_0 \eta^a_b B^{a}_{b} + \kappa b_0 \Pi^a \xi^a, \\
\delta \hat{\Pi}^a \approx -b_0 \eta^{ab} \Pi^b + b_0 B^{ab} \xi^b.
\] (61)

Now as in the massless cases we take the free Lagrangian but with initial curvatures replaced by the deformed ones and add possible abelian terms having no more than four derivatives:

\[
L = a_1 \hat{E}^{[4]} \hat{F}^{[2]} \hat{F}^{[2]} + d_1 \hat{E}^{[5]} \hat{F}^{[2]} \hat{F}^{[2]} f^a + \\
\frac{1}{2} \hat{E}^{[3]} \hat{\Pi}^a \hat{\Pi}^a + d_2 \hat{E}^{[3]} \hat{\Pi}^a \hat{\Pi}^a f^a.
\] (62)

Note that the term with the coefficient \(d_1\) (which exists in \(d > 4\) only) is gauge invariant by itself and the first line is the same as in the massless spin 2 case. As for the second line, its variation looks like

\[
\delta L_1 = (2d_2 - b_0) \hat{E}^{[3]} B^a \eta^{ab} \Pi^b + (b_0 - 2d_2) \hat{E}^{[5]} B^{ab} \Pi^a \xi^b + 2d_2 \hat{E}^{[4]} B^{ab} \Pi^b \xi^a.
\] (63)

Thus we have to put \(d_2 = \frac{b_0}{2}\), while with the help of identity

\[
0 \approx \hat{E}^{[3]} B^{a[2]} e_b \Pi^b \xi^a = -\hat{E}^{[4]} [B^{a[2]} \Pi^b \xi^c - 2B^{ab} \Pi^b \xi^a]
\]

we find that the last term vanish on-shell.

4 Gravitational interaction

In this section we consider gravitational interaction of massive spin 2 in the same framework. At first, we provide the kinematics of massless spin 2 in our notations.

4.1 Graviton

We describe massless graviton with the one-forms \(h^a\) and \(\omega^{a[2]}\) and the free Lagrangian:

\[
\mathcal{L}_0 = \frac{1}{2} \hat{E}^{[4]} \omega^{ab} \omega_a^b - \frac{1}{2} \hat{E}^{[3]} \omega^{a[2]} Dh^a - \frac{(d - 2) \kappa}{2} \hat{E}^{[2]} h^a h_a.
\] (64)

This Lagrangian is invariant under the following gauge transformations:

\[
\delta_0 \omega^{a[2]} = D \eta^{a[2]} + \kappa e^a \xi^a, \quad \delta_0 h^a = D \xi^a + e_b \eta^{ba}.
\] (65)

Two main gauge invariant objects (curvature and torsion) are:

\[
R^{a[2]} = D \omega^{a[2]} + \kappa e^a h^a, \quad T^a = Dh^a + e_b \omega^{ba}.
\] (66)

They satisfy the usual differential identities:

\[
DR^{a[2]} = -\kappa e^a T^a, \quad DT^a = -e_b R^{ba},
\] (67)
while on-shell we have:

\[ T^a \approx 0 \quad \Rightarrow \quad e_b R^{ba} \approx 0, \quad DR^{a[2]} \approx 0. \] 

(68)

At last, the free Lagrangian can be rewritten as

\[ \mathcal{L}_0 = a_0 \hat{E}_a [4] R^{a[2]} R^{a[2]}, \quad a_0 = -\frac{1}{32(d - 3)\kappa}. \] 

(69)

### 4.2 Deformations

We begin with the deformations for the massless spin 2. The most general ansatz has the form:

\[ \Delta R^{a[2]} = c_1 \Omega^{ba} \Omega^a_b + c_2 f^a f^b + c_3 \Omega^{[a[2]} e_b \pi^b - c_4 e_b \Omega^{ba} \pi^a + c_5 e^a \Omega^{ab} \pi_b \]

\[ + c_6 f^a e_b B^{ba} - c_7 e_b f^b B^{[a[2]} + c_8 e^a B^{ab} f_b + c_9 E_b [2] B^{ba} B^b + \frac{c_{10}}{2} B^{a[2]} B^{[a[2]} B^b + \]

\[ + c_{11} E^{ab} B^{ac} B_{bc} + c_{12} E^{ab} \pi^a \pi_b, \] 

(70)

\[ \Delta T^a = c_{13} \Omega^{ab} f_b + c_{14} \Omega^{ab} e_c B^{cb} - c_{15} B^{ab} \pi^b + c_{16} e^a \Omega^{[b[2]} B_{a[2]} + c_{17} f^a e_b \pi^b \]

\[ - c_{18} e_b f^b \pi^a + c_{19} e^a f^b \pi_b + c_{20} E_b [2] B^{ba} \pi^c + \frac{c_{21}}{2} E_b [2] B^{b[2]} \pi^a + c_{22} E^{ab} B_{bc} \pi^c. \]

Here we also have a lot of possible field redefinitions:

\[ \omega^{a[2]} \quad \Rightarrow \quad \omega^{a[2]} + \kappa_1 B^{ba} \Omega^a_b + \kappa_2 f^a f^b + \kappa_3 B^{[a[2]} e_b \pi^b - \kappa_4 e_b \Omega^{ba} \pi^a + \kappa_5 e^a \Omega^{ab} \pi_b, \]

\[ h^a \quad \Rightarrow \quad h^a + \kappa_6 B^{ab} f_b + \kappa_7 \Omega^{ab} \pi_b - \kappa_8 B^{ab} e^c B_b ^{bc} + \kappa_9 e^a \Omega^{[b[2]} B_{b[2]} + \kappa_{10} e_b \pi^b \pi^a + \kappa_{11} e^a \pi^b \pi_b. \] 

(71)

The most general ansatz for the massive curvatures deformations is:

\[ \Delta \Phi^{a[2]} = b_1 \Omega^{ba} \omega^b_a + b_2 h^a f^b + b_3 \omega^{a[2]} e_b \pi^b - b_4 e_b \omega^{ba} \pi^a + b_5 e^a \omega^{ac} \pi_c \]

\[ + b_6 h^a e_b B^{ba} - b_7 e_b h^b B^{[a[2]} + b_8 e^a B^{ab} h_b, \]

\[ \Delta T^a = b_9 \Omega^{ab} h_b + b_{10} \omega^{ab} f_b + b_{11} \omega^{ab} e^c B_b ^{cb} - b_{12} B^{ab} e^c \omega^{cb} + b_{13} e^a \omega^{[b[2]} B_{b[2]} \]

\[ + b_{14} h^a e_b \pi^b - b_{15} e_b h^b \pi^a + b_{16} e^a h^b \pi_b, \]

(72)

\[ \Delta B^{a[2]} = b_{17} \omega^{ba} B_b ^{a}, \quad \Delta \Pi^a = b_{18} \omega^{ab} \pi_b + b_{19} B^{ab} h_b, \]

with the possible fields redefinitions:

\[ \Omega^{a[1]} \quad \Rightarrow \quad \Omega^{a[1]} + \rho_1 B^{ba} \omega^a_b + \rho_2 h^a \pi^a, \]

\[ f^a \quad \Rightarrow \quad f^a + \rho_3 B^{ab} h_b + \rho_4 \omega^{ab} \pi_b. \]

(73)

In both cases we have obtained all the equations on the deformation parameters which follow from the consistency requirement and checked that all of them are invariant under the shifts generated by all field redefinitions. In both cases the number of the free parameters in the general solutions are the same as the number of the fields redefinitions so that all the non-abelian deformations can be set to zero.
4.3 Trivially invariant terms

In this case it is convenient to proceed with the trivially invariant terms and then consider the abelian ones. The most general ansatz:

\[ \mathcal{L} = h_1 \hat{E}_{a[6]} R^a R^b [ \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} ] + h_2 \hat{E}_{a[5]} R^{a[2]} \mathcal{F}^{a[2]} \Pi^a + h_3 \hat{E}_{a[4]} R^{a[2]} \Pi^a \Pi^a + \hat{E}_{a[4]}[h_4 R^{a[2]} B^{ab} B^a_b + h_5 R^{ab} B^a_b B^{a[2]}]. \] (74)

Here we also have a couple of identities:

\[
0 = \frac{m}{2} \hat{E}_{a[6]} D[R^{a[2]} \mathcal{F}^{a[2]} B^a_b] \\
\approx \frac{m}{2} \hat{E}_{a[6]} R^{a[2]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} - 2(d - 5)\kappa \hat{E}_{a[5]} R^{a[2]} \mathcal{F}^{a[2]} \Pi^a, \tag{75}
\]

\[
0 = \hat{E}_{a[5]} D[R^{a[2]} B^a_b] \\
\approx \frac{m}{2} \hat{E}_{a[5]} R^{a[2]} \mathcal{F}^{a[2]} \Pi^a + 2(d - 4)\kappa \hat{E}_{a[4]} R^{a[2]} \Pi^a \Pi^a \\
- 2\hat{E}_{a[4]} [R^{a[2]} B^{ab} B^a_b - R^{a[2]} B^{ab} B^{a[2]}], \tag{76}
\]

so we choose:

\[ \mathcal{L}_t = \hat{E}_{a[4]} [h_4 R^{a[2]} B^{ab} B^a_b + h_5 R^{ab} B^a_b B^{a[2]} + h_3 R^{a[2]} \Pi^a \Pi^a]. \] (77)

Note that in \( d = 4 \) we have one additional identity:

\[ 0 = -\hat{E}_{a[5]} R^{a[2]} e_b B^{ab} B^a_b = 2\hat{E}_{a[4]} [R^{a[2]} B^{ab} B^a_b - R^{ab} B^a_b B^{a[2]}]. \] (78)

Substituting the explicit expression for \( R^{a[2]} \), integrating by parts and using differential identities for massive spin 2 curvatures, one can show that our trivially invariant terms are equivalent to some combinations of the abelian terms:

\[
V_3 = 2\hat{E}_{a[4]} \Pi^a \Pi^a [D\omega^{a[2]} + 2\kappa e^a h^a] \\
\approx 4\hat{E}_{a[3]} B^a_b [2g^{ab} \Pi^a \omega^{ab} + \Pi^b \omega^{a[2]}] + 4(d - 3)\kappa \hat{E}_{a[3]} \Pi^a \Pi^a h^a, \tag{79}
\]

\[
V_4 = 2\hat{E}_{a[4]} B^{ab} B^a_b [D\omega^{a[2]} + 2\kappa e^a h^a] \\
\approx -2m\hat{E}_{a[4]} \mathcal{F}^{ab} B^a_b \omega^{a[2]} + 4\kappa \hat{E}_{a[3]} B^a_b [2\Pi^a \omega^{ab} - (d - 4)\Pi^b \omega^{a[2]}] \\
+ 4(d - 3)\kappa \hat{E}_{a[3]} B^{ab} B^a_b h^a, \tag{80}
\]

\[
V_5 = 2\hat{E}_{a[4]} B^a_b B^{a[2]} [D\omega^{ab} + \kappa (e^a h^b - e^b h^a)] \\
\approx -m\hat{E}_{a[4]} [\mathcal{F}^{ab} B^{a[2]} - \mathcal{F}^{a[2]} B^{ab}] \omega^{a[2]} \\
- 2\kappa \hat{E}_{a[3]}[(d - 4)B^{ab} B^{a[2]} h^b + 2B^{ab} B^a_b h^a]. \tag{81}
\]

4.4 Abelian vertices

There are two possible types of abelian vertices: those with the massive spin 2 components \( \Omega, f, B, \pi \) and with the massless spin 2 \( \omega, h \) ones. Exactly as in the massive spin 2
self-interaction case all gauge invariant abelian vertices of the first type are equivalent to some combinations of the trivially gauge invariant vertices and/or can be removed by field redefinitions. So we consider here only second type.

Taking into account on-shell identities:

\[
0 \approx \hat{E}_{a[5]} e_b \mathcal{F}^{ab} \mathcal{B}^{a[2]} \omega^{a[2]} = -2 \hat{E}_{a[4]} \mathcal{F}^{ab} [\mathcal{B}^{ab} \omega^{a[2]} + \mathcal{B}^{a[2]} \omega^{ab}],
\]

\[
0 \approx -\hat{E}_{a[4]} \mathcal{B}^{a[2]} e_c \Gamma^c \omega^{a[2]} = -2 \hat{E}_{a[3]} [\mathcal{B}^{a[2]} \Pi_b \omega^{ab} + \mathcal{B}^{ab} \Pi_b \omega^{a[2]}],
\]

(82)

the most general ansatz for such abelian vertices has the form (for terms with five and four derivatives respectively):

\[
\mathcal{L}_{a5} = \hat{E}_{a[4]} [g_1 \mathcal{F}^{a[2]} \mathcal{B}^{ab} + g_2 \mathcal{F}^{ab} \mathcal{B}^{a[2]}] \omega^a_b
\]

\[
+ \hat{E}_{a[3]} [g_3 \mathcal{B}^{a[2]} \Pi^b + g_4 \mathcal{B}^{ab} \Pi^a] \omega^a_b,
\]

(83)

\[
\mathcal{L}_{a4} = \hat{d}_1 \hat{E}_{a[5]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} h^a + \hat{d}_2 \hat{E}_{a[4]} \mathcal{F}^{a[2]} \Pi^a h^a
\]

\[
+ \hat{E}_{a[3]} [d_3 \mathcal{P}^a \Pi^a h^a + d_4 \mathcal{B}^{a} b \mathcal{B}^{ab} h^a + d_5 \mathcal{B}^{a[2]} \mathcal{B}^{a} b h^a].
\]

(84)

Not all of them are completely independent as can be seen from

\[
0 = \hat{E}_{a[5]} D \left[ \mathcal{F}^{a[2]} \mathcal{B}^{a[2]} h^a \right]
\]

\[
0 \approx \frac{m}{2} \hat{E}_{a[5]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} h^a - 2 (d - 4) \kappa \hat{E}_{a[4]} \mathcal{F}^{a[2]} \Pi^a h^a
\]

\[
-2 \hat{E}_{a[4]} \mathcal{F}^{a[2]} \mathcal{B}^{a} b + \mathcal{F}^{ab} \mathcal{B}^{a[2]} \omega^{a b},
\]

\[
0 = \hat{E}_{a[4]} D \left[ \mathcal{B}^{a[2]} \Pi^a h^a \right]
\]

\[
0 \approx \frac{m}{2} \hat{E}_{a[4]} \mathcal{F}^{a[2]} \Pi^a h^a - 2 (d - 3) \kappa \hat{E}_{a[3]} \Pi^a \Pi^a h^a
\]

\[
- \hat{E}_{a[3]} [\mathcal{B}^{a[2]} \mathcal{B}^{ab} h^a + 2 \mathcal{B}^{ab} \mathcal{B}^{a} b h^a]
\]

\[
+ \hat{E}_{a[3]} [\mathcal{B}^{a[2]} \Pi^b - 2 \mathcal{B}^{ab} \Pi^a] \omega^a_b.
\]

(85)

This leads us to the following form of the abelian vertex which is not equivalent to any combination of the trivially gauge invariant ones (taking into account relations at the end of previous subsection):

\[
\mathcal{L}_a = \hat{E}_{a[3]} [g_3 \mathcal{B}^{a[2]} \Pi_b \omega^{ab} + d_3 \mathcal{P}^a \Pi^a h^a + d_4 \mathcal{B}^{ab} \mathcal{B}^{a} b h^a + d_5 \mathcal{B}^{a[2]} \mathcal{B}^{ab} h^a].
\]

(87)

Now let us require this vertex to be gauge invariant.

\(\hat{\eta}^{ab}\)-transformations We obtain:

\[
\delta \mathcal{L}_a = \frac{g_3 - d_4 + 2d_5}{2} \hat{E}_{a[2]} \mathcal{B}^{a[2]} \mathcal{B}^{b[2]} \hat{\eta}^{b[2]},
\]

(88)

but this can be compensated by the appropriate correction to the gauge transformation:

\[
\delta A \sim \mathcal{B}^{a[2]} \hat{\eta}^{b[2]}.
\]

(89)

\(\hat{\xi}^a\)-transformations These variations give us:

\[
\delta \xi \mathcal{L}_a = - \frac{md_5}{2} \hat{E}_{a[3]} \mathcal{F}^{a[2]} \mathcal{B}^{ab} \hat{\xi}_b + [(d - 2) \kappa g_3 + d_3 + (d - 3) \kappa (d_5 - d_4)] \hat{E}_{a[2]} \mathcal{B}^{a[2]} \Pi^b \hat{\xi}_b
\]

\[
+ m (d_5 - d_4) \hat{E}_{a[3]} \mathcal{F}^{ab} \mathcal{B}^{a} b \hat{\xi}_b - 2 [d_3 + \kappa d_4 - \kappa (d - 1) d_5] \hat{E}_{a[2]} \mathcal{B}^{ab} \Pi^a \hat{\xi}_b.
\]

(90)
Here all terms in the first line can be compensated by the corrections to the gauge transformations:

\[ \delta f^a \sim B^{ab} \hat{\xi}_b, \quad \delta A_\mu \sim \Pi^a \hat{\xi}_a, \]

while for the second line to vanish we must put

\[ d_5 = d_4, \quad d_3 = -M^2 d_4. \]

Thus we have three trivially gauge invariant vertices \( V_{3,4,5} \) and two independent abelian ones (with the coefficients \( g_3 \) and \( d_4 \)). The vertices \( V_{4,5} \) contain higher derivative terms, while the remaining three in the unitary gauge produce:

\[
\frac{1}{m^2} \mathcal{L}_1 = h_3 \hat{E}_{a[3]} R^{a[2]} f^a f^a - g_3 \hat{E}_{a[3]} \Omega^{a[2]} \omega^{ab} f_b \\
+ d_4 \hat{E}_{a[3]} \left[ \Omega^{ab} \omega_{ab} h^a + \Omega^{a[2]} \omega^{ab} h_b - M^2 f^a f^a h^a \right].
\]

Again it is instructive to compare these results with the minimal two derivative vertex also in the unitary gauge (see appendix A.2):

\[
\mathcal{L}_1 \sim \hat{E}_{a[3]} \left[ \Omega^{ab} \omega_{ab} h^a + \Omega^{a[2]} \omega^{ab} h_b + \Omega^{a[2]} \omega^{ab} f_b \right] \\
+ \frac{1}{4} \hat{E}_{a[3]} R^{a[2]} f^a f^a - M^2 \hat{E}_{a[3]} f^a f^a h^a.
\]

Note that for this vertex the two derivative part for bigravity [46] and from (super)string [41] coincides (here symmetric tensor \( G_{\mu\nu} \) corresponds to the massless graviton):

\[
\mathcal{L}_{GM^2} = \frac{1}{m_g \sqrt{1 + \alpha^2}} \left[ G^{\mu\nu} (\partial_\mu M_{\rho\sigma} \partial_\nu M^{\rho\sigma} - 4 \partial_\mu M_{\rho\sigma} \partial_\rho M^{\nu\sigma}) \\
+ 2 M^{\mu\nu} (\partial_\mu G_{\rho\sigma} \partial_\nu M^{\rho\sigma} - \partial_\rho G_{\mu\sigma} \partial_\nu M^{\rho\sigma}) \right].
\]

### 4.5 Comeback

In this subsection we reconstruct the same vertex in the non-abelian form so that the massive theory can be considered as a deformation of the massless one.

**Massive spin 2 deformations** We consider the following restricted ansatz

\[
\begin{align*}
\Delta \mathcal{F}^{a[2]} &= b_1 \Omega^{ba} \Omega^a_b + b_2 \kappa f^a h^a, \\
\Delta \mathcal{T}^a &= b_3 \Omega^{ab} f_b + b_4 \Omega^{ab} h_b, \\
\Delta \mathcal{B}^{a[2]} &= b_5 \Omega^{ba} B^a_b + b_6 \kappa h^a \pi^a, \\
\Delta \Pi^a &= b_7 \Omega^{ab} \pi_b + b_8 B^{ab} h_b
\end{align*}
\]

and require it to be consistent.

\( \hat{\eta}^{ab} \)-transformations Here the corrections to the gauge transformations look like:

\[
\begin{align*}
\delta \Omega^{a[2]} &= -b_1 \hat{\eta}^{ba} \Omega^a_b, \\
\delta f^a &= -b_3 \hat{\eta}^{ab} f_b, \\
\delta B^{a[2]} &= -b_5 \hat{\eta}^{ba} B^a_b, \\
\delta \pi^a &= -b_7 \hat{\eta}^{ab} \pi_b,
\end{align*}
\]
while the variations for the deformed curvatures are:

\[
\begin{align*}
\delta \tilde{F}^{a[2]} &= -b_1 \tilde{\eta}^{ba} D\Omega_b^a + b_2 \kappa f^a e_b \tilde{\eta}^{ba} - b_3 \kappa e^a \tilde{\eta}^{ab} f_b, \\
\delta \tilde{T}^a &= -b_3 \tilde{\eta}^{ab} Df_b + (b_4 - b_5) \Omega^a_{\ c} \tilde{\eta}^{cb} - b_1 \tilde{\eta}^{ab} e^c \Omega_{cb}, \\
\delta \tilde{B}^{a[2]} &= -b_5 \tilde{\eta}^{ba} DB^b + b_6 \kappa e_b \tilde{\eta}^{ba} \pi^a - mb_1 \tilde{\eta}^{ba} \Omega^a_b - b_7 \kappa e^a \tilde{\eta}^{ab} \pi_b, \\
\delta \Pi^a &= -b_7 \tilde{\eta}^{ab} D\pi_b + (b_8 + b_9) B^{ab} e^c \tilde{\eta}^{cb} - b_5 \tilde{\eta}^{ab} e^c B_{cb} - mb_3 \tilde{\eta}^{ab} f_b.
\end{align*}
\]

Comparing them with

\[
\begin{align*}
-b_1 \tilde{\eta}^{ba} F^a_b &= -b_1 \tilde{\eta}^{ba} [D\Omega^a_b + \kappa e^a f_b - \kappa e_b f^a] \\
&= -b_1 \tilde{\eta}^{ba} [DF^a_b + b_1 \kappa f^a e_b \tilde{\eta}^{ba} - b_1 \kappa e^a \tilde{\eta}^{ab} f_b], \\
-b_3 \tilde{\eta}^{ab} T_b &= -b_3 \tilde{\eta}^{ab} [Df^b + e_c \Omega^b], \\
-b_5 \tilde{\eta}^{ba} B^a_b &= -b_5 \tilde{\eta}^{ba} [DB^a_b + m\Omega^a_b + \kappa e^a \pi_b - \kappa e_b \pi^a] \\
&= -b_5 \tilde{\eta}^{ba} DB^a_b - mb_5 \tilde{\eta}^{ba} \Omega^a_b + b_5 \kappa e_b \tilde{\eta}^{ba} \pi^a - b_5 \kappa e^a \tilde{\eta}^{ab} \pi_b, \\
-b_7 \tilde{\eta}^{ab} \Pi_b &= -b_7 \tilde{\eta}^{ab} [D\pi_b + e^c B_{cb} + m f_b],
\end{align*}
\]

we obtain:

\[
b_2 = b_3 = b_4 = b_5 = b_6 = b_7 = -b_8 = b_1.
\]

All calculations for \(\hat{\xi}^a, \eta^{ab}\) and \(\zeta^a\) transformations are similar and produce the same solution. Complete variations of the deformed curvatures which are non-zero on-shell have the form:

\[
\begin{align*}
\delta \tilde{F}^{a[2]} &\approx -b_1 \tilde{\eta}^{ba} F^a_b - b_1 \eta^{ba} R^a_b, \\
\delta \tilde{B}^{a[2]} &\approx -b_1 \tilde{\eta}^{ba} B^a_b + b_1 \kappa \Pi^a \hat{\xi}^a, \\
\delta \Pi^a &\approx -b_1 \tilde{\eta}^{ab} \Pi_b + b_1 B^{ab} \hat{\xi}_b.
\end{align*}
\]

**Deformations for the graviton** are chosen to be:

\[
\Delta R^{a[2]} = c_1 \Omega^{ba} \Omega^a_b + c_2 \kappa f^a f_b, \quad \Delta T^a = c_3 \Omega^{ab} f_b.
\]

**\(\eta^{ab}\)-transformations** Here the corrections to the gauge transformations are:

\[
\delta \Omega^{ab} = -c_1 \eta^{ba} \Omega^a_b, \quad \delta h^a = -c_3 \eta^{ab} f_b
\]

while the variations for the deformed curvatures have the form:

\[
\begin{align*}
\delta \tilde{F}^{a[2]} &= -c_1 \eta^{ba} D\Omega_b^a + c_2 \kappa f^a e_b \eta^{ba} - c_3 \kappa e^a \eta^{ab} f_b, \\
\delta \tilde{T}^a &= -c_3 \eta^{ab} Df_b + (c_3 - c_1) \Omega^{ab} e^c \eta^{cb} - c_1 \eta^a_b e_c \Omega^{cb}.
\end{align*}
\]

Comparing them with:

\[
\begin{align*}
-c_1 \eta^{ba} F^a_b &= -c_1 [\eta^{ba} D\Omega^a_b - \kappa f^a e_c \eta^{ca} + \kappa e^a \eta^{ab} f_b], \\
-c_3 \eta^{ab} T_b &= -c_3 \eta^{ab} [D f_b - e^c \Omega^{cb}],
\end{align*}
\]

we obtain:

\[
c_2 = c_3 = c_1
\]
Similarly for $\xi^a$-transformations. The only non-zero on-shell variation is:

$$\delta \hat{R}^{a[2]} \approx -c_1 \eta^b \mathcal{F}^a_b. \quad (106)$$

Now we write an interacting Lagrangian containing no more than four derivatives (note that we still need some abelian terms here):

$$\mathcal{L} = a_0 \hat{E}_{a[4]} \left[ \hat{R}^{a[2]} \hat{F}^{a[2]} + \hat{F}^{a[2]} \hat{F}^{a[2]} \right] + d_1 \hat{E}_{a[5]} \mathcal{F}^{a[2]} \mathcal{F}^{a[2]} h^a + \frac{1}{2} \hat{E}_{a[2]} \hat{\Pi}^a \hat{\Pi}^a + d_2 \hat{E}_{a[3]} \hat{\Pi}^a \hat{\Pi}^a h^a. \quad (107)$$

Note that the first line is exactly the same as in the massless case [32] and is invariant by itself provided $b_1 = c_1$, while for the variation of the second line we obtain:

$$\delta \mathcal{L} = \left( b_1 - 2d_2 \right) \hat{E}_{a[2]} \mathcal{B}^{ab} \hat{\Pi}^a \hat{\Pi}^b \hat{\xi}^a - 2d_2 \hat{E}_{a[2]} \mathcal{B}^{ab} \hat{\Pi}^b \hat{\xi}^a \quad (108)$$

Thus we have to put $d_2 = \frac{b_1}{2}$, while the last term vanishes on-shell due to identity:

$$0 \approx -\hat{E}_{a[3]} \mathcal{B}^{a[2]} e_b \hat{\Pi}^b \hat{\xi}^a = \hat{E}_{a[2]} \left[ \mathcal{B}^{a[2]} \hat{\Pi}^b \hat{\xi}_b - 2 \mathcal{B}^{ab} \hat{\Pi}^b \hat{\xi}^a \right]$$

5 Conclusion

In this work we applied the Fradkin-Vasiliev formalism based on the frame-like gauge invariant description of the massive and massless spin 2 to the construction of the cubic interactions vertices for massive spin 2 self-interaction as well as its gravitational interaction. In the first case we have shown that in agreement with the general results of [2 8] the vertex can be reduced to the set of the trivially gauge invariant terms. There are four such terms which are not equivalent on-shell and do not contain more than four derivatives. Moreover, one their particular combination reproduces the minimal (with no more than two derivatives) vertex [26]. As for the gravitational vertex, we have shown that due to the presence of the massless spin 2 there exist two abelian vertices (besides the three trivially gauge invariant ones) which are not equivalent to any trivially gauge invariant terms and can not be removed by field re-definitions. Moreover, their existence appeared to be crucial for the possibility to reproduce the minimal two derivatives vertex.

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A Minimal vertices in the constructive approach

The spin 2 is the highest spin where all the components of the frame-like formalism enter the free Lagrangian so one can use a very well known constructive approach. Here we provide the
results of such approach for the minimal (i.e. with no more than two derivatives) vertices both for the self-interaction as well as for the interaction with the graviton. We use the so-called modified 1 and 1/2 order formalism (see the detailed discussion and examples of the explicit calculations in [32]). In short, it means that we consider only terms which are not equivalent on-shell and also use on-shell conditions calculating all gauge variations. As a result, we obtain corrections to the gauge transformations for the physical fields only.

A.1 Self-interaction

For the massive spin 2 self-interaction we obtained the following minimal cubic vertex:

\[ \mathcal{L}_1 = a_0 \hat{E}_{a[3]}[\Omega^{ab} \Omega^a_b f^a + \Omega^{a[2]}_b \Omega^{ab} f_b] + a_1 \hat{e}_a [B^{b[2]}_b B_{b[2]} f^a + 4B^{ab} B_{bc} f^c] \]

\[ + a_2 \hat{e}_a [\pi^b \pi^b f^a - 2\pi^a \pi^b f_b] + a_3 \phi B_{a[2]} B^{a[2]} \]

\[ + b_1 \hat{E}_{a[3]} [\Omega^{a[2]}_b A f^a + b_2 \hat{e}_a B^{ab} f_b \phi + b_3 \hat{e}_a \phi A \pi^a] \]

\[ + c_1 \hat{E}_{a[3]} f^a f^a f^a + c_2 \hat{E}_{a[2]} f^a f^a \phi + c_3 \hat{e}_a f^a \phi^2 + c_4 \phi^3, \]

where

\[ a_1 = \frac{a_0}{2}, \quad a_2 = -\frac{(d-1)(d-2)a_0}{2} + \frac{(d-2)(d-4)m^2 a_0}{4M^2}, \quad a_3 = -\frac{(d-4)ma_0}{2M}, \]

\[ b_1 = \frac{(d-4)ma_0}{(d-2)}, \quad b_2 = 2ma_3, \quad b_3 = 2ma_2, \]

\[ 6c_1 = -\frac{2(2d-5)M^2 - (d-4)m^2}{(d-2)} a_0, \quad c_2 = (d-1)Ma_0 - \frac{(d-4)m^3 a_0}{2M}, \]

\[ c_3 = -\frac{(d-1)(d+6)m^2 a_0}{4} + \frac{(3d-2)(d-4)m^4 a_0}{8M^2}. \]

In particular, these formulas show that the partially massless limit \( M \to 0 \) is possible only in \( d = 4 \). Corrections to the physical fields gauge transformations have the form:

\[ \delta_1 f^a = 2a_0 \eta^{ab} f_b - 2a_0 \Omega^{ab} \xi_b + 2b_1 f^a \xi - 2b_1 A \xi^a, \]

\[ \delta_1 A = 2a_2 B^a \xi_a + \frac{b_2}{2} \phi e_a \xi^a, \]

\[ \delta_1 \phi = \frac{1}{(d-1)(d-2)} (2a_3 (\pi \xi) - b_2 \phi \xi). \]

To obtain these particular form of the vertex we used the following field redefinitions (which do not raise the number of derivatives):

\[ f^a \Rightarrow f^a + \kappa_1 f^a \phi + \kappa_2 e^a \phi^2; \]

\[ A \Rightarrow A + \kappa_3 \phi A, \]

\[ \phi \Rightarrow \phi + \kappa_4 \phi^2. \]

In the unitary gauge this vertex has a very simple form:

\[ \mathcal{L}_1 = a_0 \hat{E}_{a[3]}[\Omega^{ab} \Omega^a_b f^a + \Omega^{a[2]}_b \Omega^{ab} f_b] + c_1 \hat{E}_{a[3]} f^a f^a f^a. \]
A.2 Gravitational interaction

In this case for the minimal cubic vertex we obtained:

\[ L_1 = a_0 \hat{E}_{a[3]}[\Omega^{ab} \Omega^a_b h^a + \Omega^{a[2]} \Omega^{ab} h_b + \Omega^{a[2]} \omega^{ab} f_b] + \frac{a_0}{4} \hat{E}_{a[4]} R^{a[2]} f^a f^a \]

\[ + a_0 \hat{e}_a ] B^{b[2]} h^a + 4 B^{ab} B^{c} h^c \right] - (d - 1)(d - 2) a_0 \hat{e}_a \left[ (\pi \pi) h^a - 2 \pi^a \pi^b h_b \right] \]

\[ + b_1 \hat{E}_{a[3]} \Omega^{a[2]} \phi h^a + b_2 \hat{E}_{a[2]} B^{a[2]} f^b h_b \]

\[ + c_1 \hat{E}_{a[3]} f^a f^a h^a + c_2 \hat{E}_{a[2]} f^a h^a \phi + c_3 \hat{e}_a h^a \phi^2 , \] (113)

where

\[ b_1 = - \frac{2ma_0}{(d - 2)}, \quad b_2 = - ma_0, \]

\[ c_1 = - M^2 a_0, \quad c_2 = (d - 1) M a_0, \quad c_3 = - d(d - 1) M^2 a_0. \]

Here the corrections to the physical fields transformations have the form:

\[ \delta_1 f^a = 2 a_0 \hat{\eta}^{ab} f_b - 2 a_0 \Omega^{ab} \hat{\xi}^a + 2 a_0 \omega^{ab} h_b - 2 a_0 \omega^{ab} \xi_b - 2 b_1 A \hat{\xi}^a + 2 b_1 h^a \xi, \]

\[ \delta_1 A = 2 a_0 e_a B^{ab} \hat{\xi}^b - b_2 f^a \hat{\xi}^a + b_2 h^a \xi_a, \quad \delta_1 \phi = - 2 a_0 (\pi \hat{\xi}), \] (114)

\[ \delta_1 h^a = 2 a_0 \eta^{ab} f_b - 2 a_0 \Omega^{ab} \xi_b + \frac{4(d - 3)ma_0}{(d - 2)} [f^a \xi - A \xi^a]. \]

In this case we also used the allowed field redefinitions:

\[ f^a \Rightarrow f^a + \kappa_1 h^a \phi, \]

\[ h^a \Rightarrow h^a + \kappa_2 f^a \phi + \kappa_3 \phi^2. \] (115)

In the unitary gauge this vertex looks as follows:

\[ L_1 = a_0 \hat{E}_{a[3]}[\Omega^{ab} \Omega^a_b h^a + \Omega^{a[2]} \Omega^{ab} h_b + \Omega^{a[2]} \omega^{ab} f_b] \]

\[ + \frac{a_0}{4} \hat{E}_{a[4]} R^{a[2]} f^a f^a - M^2 a_0 \hat{E}_{a[3]} f^a f^a h^a. \] (116)

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