CARLEMAN INEQUALITIES AND UNIQUE CONTINUATION FOR THE POLYHARMONIC OPERATORS

EUNHEE JEONG, YEHYUN KWON, AND SANGHYUK LEE

Abstract. We obtain a complete characterization of $L^p - L^q$ Carleman estimates with weight $e^{v\cdot x}$ for the polyharmonic operators. Our result extends the Carleman inequalities for the Laplacian due to Kenig–Ruiz–Sogge. Consequently, we obtain new unique continuation properties of higher order Schrödinger equations relaxing the integrability assumption on the solution spaces.

1. Introduction

Let $d$ and $k$ be positive integers and $\Omega$ a non-empty connected open set in $\mathbb{R}^d$. We denote by $V$ a complex-valued function defined on $\Omega$ of which precise description to be given below. We say that the differential inequality

$$|\Delta^k u| \leq |Vu| \text{ in } \Omega$$

has the unique continuation property (UCP) in the Sobolev space $W^{2k,p}_\text{loc}(\Omega)$ if every $u \in W^{2k,p}_\text{loc}(\Omega)$ satisfying (1.1) and vanishing on a non-empty open subset of $\Omega$ is identically zero in $\Omega$.

When $k = 1$ and $d \geq 3$, the inequality (1.1) includes the Schrödinger equation $(-\Delta + V)u = 0$ as a special case, of which UCP has been extensively studied over the past several decades by numerous authors. We refer the reader to the survey articles [15, 27, 17] and references therein. For $V \in L^{d/2}_\text{loc}(\Omega)$, Kenig, Ruiz, and Sogge [16] proved that the inequality

$$|\Delta u| \leq |Vu| \text{ in } \Omega$$

has the UCP in $W^{2,p}_\text{loc}(\Omega)$ whenever $p > \frac{2d}{d+3}$. The main ingredient in their argument is the Carleman inequality of the form

$$\|e^{v\cdot x}u\|_{L^q(\mathbb{R}^d)} \leq C\|e^{v\cdot x}\Delta u\|_{L^p(\mathbb{R}^d)}$$

where $C$ is a constant independent of $v \in \mathbb{R}^d$ and $u \in C_0^\infty(\mathbb{R}^d)$. For $p, q$ satisfying

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{d} \quad \text{and} \quad \frac{d+1}{2d} < \frac{1}{p} < \frac{d+3}{2d},$$

they deduced (1.3) from the more general uniform Sobolev inequality

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C\|(-\Delta + a \cdot \nabla + b)u\|_{L^p(\mathbb{R}^d)}, \quad \forall (a, b) \in \mathbb{C}^d \times \mathbb{C},$$

which holds if and only if (1.4) is satisfied.

In [13] the authors characterized the full range of $(p, q)$ on which the Carleman inequality (1.3) holds. More precisely, under the assumption $1 < p, q < \infty$, it was proved that (1.3) holds if and only if

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{d} \quad \text{and} \quad \frac{d^2-4}{2(d-1)} \leq \frac{1}{p} \leq \frac{d^2+2}{2(d-1)}.$$  

This range is larger than that of (1.4) whenever $d \geq 4$. Consequently, the integrability assumption on the solution space in which the differential inequality (1.2)
Theorem 1.1. Carleman inequality (1.6) holds with a constant $k$ the case order elliptic operators and have abundant applications in various physical contexts (1.6). In this paper, we aim to study the UCP of (1.1), which is a natural higher order analogue of (1.2). The UCP of (1.1) is closely tied to the Carleman inequality (1.6) described by the thick line segments.

Let $1 < p, q < \infty$. The inequality (1.6) holds and if only if

$$
\frac{1}{p} - \frac{1}{q} = \frac{2k}{d'} \quad \text{and} \quad \frac{(d+2k)(d-2)}{2d(d-1)} \leq \frac{1}{p} \leq \frac{d+2k}{2(d-1)}.
$$

If $\frac{d-2}{2} \leq k < \frac{d}{2}$, then the second condition in (1.7) is vacuously true provided that $1 < p, q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{2k}{d'}$. Hence, in this case, the inequality (1.6) holds for all $1 < p, q < \infty$ satisfying the gap condition $\frac{1}{p} - \frac{1}{q} = \frac{2k}{d'}$. For $k < \frac{d-2}{2}$, however, the second condition in (1.7) is nontrivial. For $k \geq \frac{d}{2}$, (1.6) is not true because the set (1.7) is empty.

By a standard density argument the Carleman inequality (1.6) is also valid for all $u \in W^{2k,p}_0(R^d)$ for every $p$ given as in Theorem 1.1. We shall use this fact in Section 4 to deduce a unique continuation result (Theorem 1.2) from Theorem 1.1.

In order to prove Theorem 1.1 we basically follow the strategy of the authors’ previous work [13]. This reduces the Carleman inequality (1.6) to obtaining $L^p - L^q$ boundedness of a Fourier multiplier operator of which singularity is on the $d - 2$ dimensional unit sphere $S^{d-2}$ embedded in $R^d$. We dyadically decompose the multiplier near the sphere $S^{d-2}$, and obtain sharp bounds for the dyadic pieces. In this approach, the Fourier restriction-extension operator (see Section 2) defined by $S^{d-2}$ naturally arises if we write the Fourier multiplier in the cylindrical coordinates in $R^d$.

However, direct use of the bounds on the restriction-extension operator as in [13] does not yield any sharp estimate when $k \geq 2$. In such cases, the associated multiplier $m$ can be regarded to have singularity of degree $k$ on $S^{d-2} \times \{0\}$. So, the bounds on the Fourier restriction-extension operator, whose multiplier is of order $-1$, is not enough to handle the operator defined by $m$. Our novelty in this paper is in overcoming the difficulty by making use of the Bochner–Riesz operators of indices less than $-1$ (Lemma 2.2). However, the Bochner–Riesz operators become more singular. This necessitates manipulation of the associated distribution by raising and reducing the order of operators. See Lemmas 2.1 and 3.3 and Proof of Proposition 7.2.

If we make use of Theorem 1.1 and adjust the argument in [16], then we obtain the following unique continuation result.

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1See, for example, Figures 1 and 2 in Section 3 where the necessary and sufficient condition for (1.6) is described by the thick line segments.
Theorem 1.2. Let $1 \leq k < \frac{d}{2}$, $\Omega$ a non-empty connected open set in $\mathbb{R}^d$, and let

$$X = X_{k,d}(\Omega) := \begin{cases} \bigcup_{p>1} W^{2k,p}_{\text{loc}}(\Omega) & \text{if } k \geq \frac{d-2}{2}, \\ W^{2k, \frac{d-2}{d+2}}_{\text{loc}}(\Omega) & \text{if } k < \frac{d-2}{2}. \end{cases}$$

Then, for every $V \in L^{d/2k}_{\text{loc}}(\Omega)$ the differential inequality (1.1) has the UCP in $X$.

When $k = 1$, Koch and Tataru [18] proved that the potential class $L^{d/2}_{\text{loc}}$ is critical in the scale of Lebesgue spaces for (1.2) to have the UCP; for every $r < d/2$ they constructed a nontrivial smooth function $u \in C^\infty_0(\mathbb{R}^d)$ such that $\Delta u/u \in L^r(\mathbb{R}^d)$. By following their construction, it is possible to show that for every $k \in [2, d/2)$ the $L^{d/2k}_{\text{loc}}$-potentials are critical for (1.1) to have the UCP in $C^\infty_0(\mathbb{R}^d)$ (12).

On the other hand, the problem of finding the largest possible solution space where the UCP holds is an interesting problem. That is, given a $V \in L^{d/2k}_{\text{loc}}(\mathbb{R}^d)$, one can ask whether or not the integrability of the solution space $X$ in Theorem 1.2 is optimal for the UCP of (1.1). Although we obtain Theorem 1.2 by means of the Carleman inequality (1.6), which is completely characterized in Theorem 1.1, the condition (1.7) itself does not give the sharpness of the integrability assumption of $X$ in Theorem 1.2. Even when $k = 1$, it is an open problem to find the smallest $p^*$ such that (1.2) with $V \in L^{d/2}_{\text{loc}}(\Omega)$ has the UCP in $W^{2,p}_{\text{loc}}(\Omega)$ if $p > p^*$. Also, it seems an interesting open problem to consider the UCP of (1.1) when $k \geq d/2$.

Organization. In Section 2 we present some tools from harmonic analysis, namely, the estimates for the Bochner–Riesz operators of negative indices and their consequences. These are crucial in our argument obtaining the complete set of the Lebesgue exponents for which (1.6) holds. In Section 3 we prove Theorem 1.1 and in Section 4 we prove Theorem 1.2. As a rigorous proof of an estimate which yields the necessity part in Theorem 1.1 is somewhat involved, we postpone the proof until the last section.

Notations. For a set $A \subset \mathbb{R}^n$ we denote by $C^\infty_0(A)$ the class of smooth functions each of which can be extended to a smooth function defined on some open set $U$ containing $A$ and supported in $A$. Denoting by $\alpha \in \mathbb{N}^n_0$ a multi-index we use the standard notation

$$W^{m,p}(\Omega) = \{ f \in L^p(\Omega) : \| \partial^\alpha f \|_{L^p(\Omega)} < \infty, |\alpha| \leq m \}$$

for the usual Sobolev space. Moreover, $W^{m,p}_0(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in $W^{m,p}(\Omega)$. For $1 \leq p, q \leq \infty$, we define

$$\| T \|_{p \rightarrow q} = \sup \{ \| Tf \|_{L^q(\mathbb{R}^d)} : f \in \mathcal{S}(\mathbb{R}^d), \| f \|_{L^p(\mathbb{R}^d)} = 1 \}$$

for a linear operator $T$ acting on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. For the Fourier transform we follow the convention $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$, so the inverse Fourier transform is defined by $f(x) = \mathcal{F}^{-1} f(x) = (2\pi)^{-d} \mathcal{F}f(-x)$. For a bounded function $m$ we denote by $m(D)$ the associated Fourier multiplier operator, that is, $m(D)f = \mathcal{F}^{-1}(m\mathcal{F}f)$. We slightly abuse notation so that the dimension of the Fourier transform and its inversion may vary depending on context.

2. The Bochner–Riesz operator of negative index

Let us recall from [11] Chapter III the analytic family of distributions $\chi_+^a \in \mathcal{D}'(\mathbb{R})$ defined by

$$\chi_+^a = \frac{x^a}{\Gamma(a+1)}, \quad \text{Re} \, a > -1.$$

$\chi_+^a$ can be continued analytically to all $a \in \mathbb{C}$ so that $d\chi_+^a/dx = \chi_+^{a-1}$. Since $\chi_0^a$ is the Heaviside function it follows that $\chi_+^{a-k} = \delta_0^{(k-1)}$ for $k \in \mathbb{N}$. 

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Let us denote by $\lambda$ the smooth function from $\mathbb{R}^n \setminus \{0\}$, $n \geq 2$, defined by $\lambda(\theta) = 1 - |\theta|^2$. Then the Bochner–Riesz operator $T_\alpha$ on $\mathbb{R}^n$ of (negative) index $-\alpha \in [-\frac{n+1}{2}, 0)$ is the Fourier multiplier operator defined by

$$T_\alpha f = \mathcal{F}^{-1}((\lambda^* \chi_+^{-\alpha}) \mathcal{F} f),$$

where $\lambda^* \chi_+^{-\alpha} \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ is the pullback of $\chi_+^{-\alpha}$ via the function $\lambda$ (see [11, Chapter VI]). In particular, denoting by $d\sigma_{S^{n-1}}$ the surface measure on the unit sphere $S^{n-1}$ we have

$$\lambda^* \chi_+^{-1} = \lambda^* \delta_0 = c \, d\sigma_{S^{n-1}}$$

for some constant $c$ depending on $n$. Hence $T_1$ is the Fourier restriction-extension operator on the sphere $S^{n-1}$.

The $L^p - L^q$ boundedness of $T_\alpha$ with $\alpha \in (0, \frac{n+1}{2}]$ was studied by several authors ([4, 5, 6, 2, 10, 6, 20]). We will make use of it to prove the Carleman inequality (1.0) for all admissible $p, q$ in the following section. Here, let us for the moment digress and introduce some convenient notations to describe the set of $p, q$ for which $\|T_\alpha\|_{p \to q}$ is bounded.

We set $Q = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, and for $(x, y) \in Q$ we define $(x, y)' = (1 - y, 1 - x)$. Similarly, for $R \subset Q$ we denote $R' = \{ (x, y) \in Q : (x, y)' \in R \}$. For $X_1, \ldots, X_m \in Q$, we use the notation $[X_1, \ldots, X_m]$ to denote the convex hull of the points $X_1, \ldots, X_m$. In particular, if $X, Y \in Q$, $[X, Y]$ is the closed line segment connecting $X$ and $Y$ in $Q$. By $(X, Y)$ and $[X, Y]$ we denote the open interval $[X, Y] \setminus \{X, Y\}$ and the half-open interval $[X, Y] \setminus \{Y\}$, respectively. Furthermore, for $d \geq 3$ and $0 < \alpha < d/2$, let us set

$$B^d_\alpha = \left( \frac{d-2+2\alpha}{2(d-1)}, \frac{d-2+2\alpha}{2d(d-1)} \right), \quad D^d_\alpha = \left( \frac{d-2+2\alpha}{2(d-1)}, 0 \right), \quad H = (1, 0).$$

See Figures [1] and [2].

We resume our discussion of the operator $T_\alpha$. It was shown by Börjeson [4] that $\|T_\alpha\|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} < \infty$ only if $\left( \frac{1}{p}, \frac{1}{q} \right)$ lies in the set

$$P^{n+1}_\alpha := [B^{n+1}_\alpha, (B^{n+1}_\alpha)', (D^{n+1}_\alpha, (D^{n+1}_\alpha)', H)] \setminus \{ [B^{n+1}_\alpha, D^{n+1}_\alpha], (B^{n+1}_\alpha, D^{n+1}_\alpha)' \}$$

$$= \{ (x, y) \in Q : x - y > \frac{2\alpha}{n+1}, x > \frac{n+2-2\alpha}{2n}, y < \frac{n+2-2\alpha}{2n} \}.$$ 

When $n = 2$ the sufficiency was proved by Bak [2] for $0 < \alpha \leq 3/2$. In higher dimensions, when $0 < \alpha < \alpha^*$ for some $\alpha^* < 1/2$, the complete characterization of $L^p - L^q$ boundedness of $T_\alpha$ still remains open ([3, 6, 20]). For more on this problem we refer the reader to [20], where the currently widest range of $\alpha$ can be found. We also mention the recent result [23] in which new Bochner–Riesz estimates with negative index associated to non-elliptic surfaces were proven.

In this paper, we particularly make use of the following estimates for $T_k$ with $k \in [1, \frac{n+1}{2}) \cap \mathbb{N}$, which are due to Gutiérrez [10], Bak [2], and Cho–Kim–Lee–Shim [6]:

$$\|T_k f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for} \quad \left( \frac{1}{p}, \frac{1}{q} \right) \in P^{n+1}_k,$$

$$\|T_k f\|_{L^{q'}(\mathbb{R}^n)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^n)} \quad \text{for} \quad \left( \frac{1}{p}, \frac{1}{q} \right) \in (B^{n+1}_k, D^{n+1}_k],$$

$$\|T_k f\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^n)} \quad \text{for} \quad \left( \frac{1}{p}, \frac{1}{q} \right) = B^{n+1}_k.$$

In order to prove the Carleman inequality (1.0) for all $p, q$ described in Theorem 1.1 we make use of the estimates (2.3)–(2.5); see Lemma 2.2. To this end, we relate the multiplier $\lambda^* \chi_+^{-k}$ to $\lambda^* \chi_+^{-1}$. We use the standard notation $\mathcal{E}'(\mathbb{R}^n \setminus \{0\})$ denoting the set of distributions compactly supported in $\mathbb{R}^n \setminus \{0\}$. It is clear that $\lambda^* \chi_+^{-k}$ is supported on $S^{n-1}$. 


Lemma 2.1. Let \( n \geq 2 \) and \( k \geq 1 \) be integers, and let \( \langle \cdot, \cdot \rangle \) be the duality pairing on \( C^0_c(\mathbb{R}^n \setminus \{0\}) \times C^\infty(\mathbb{R}^n \setminus \{0\}) \). Then, for \( \phi \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) we have

\[
\langle \lambda^*_\chi^-_+, \phi \rangle = \langle \lambda^*\chi^-_+L^{k-1} \phi \rangle,
\]

where \( L \) is a differential operator defined by

\[
L\phi(\theta) = \frac{1}{2|\theta|^2}(n - 2 + \theta \cdot \nabla)\phi(\theta), \quad \theta \in \mathbb{R}^n \setminus \{0\}.
\]

Proof. It is enough to prove (2.6) for \( k \geq 2 \). We denote by \( \{\partial_j = \frac{\partial}{\partial \theta_j} : 1 \leq j \leq n\} \) the standard orthonormal frame on \( \mathbb{R}^n \). The chain rule (see, e.g., [11, p. 135]) gives

\[
\sum_{j=1}^n \theta_j \partial_j \lambda^* \delta_0^{(k-2)} = \lambda^* \delta_0^{(k-1)} \sum_{j=1}^n \theta_j \partial_j \lambda = -2|\theta|^2 \lambda^* \delta_0^{(k-1)}.
\]

Hence,

\[
\langle \lambda^*_\chi^-_+, \phi \rangle = \langle \lambda^* \delta_0^{(k-1)}, \phi \rangle = -\frac{1}{2} \sum_{j=1}^n \langle \frac{\partial}{|\theta|^2} \partial_j \lambda^* \delta_0^{(k-2)}, \phi \rangle = \langle \lambda^* \chi^-_+^{-(k-1)}, L\phi \rangle.
\]

Repeating this argument we get the identity (2.6). \( \square \)

For \( \rho > 0 \) let \( \lambda_\rho(\theta) := \rho^2 - |\theta|^2 \). Since \( \partial_j \lambda_\rho = \partial_j \lambda \) the proof of Lemma 2.1 gives

\[
\langle \lambda_\rho^* \chi^-_+, \phi \rangle = \langle \lambda_\rho^* \chi^-_+L^{k-1} \phi \rangle.
\]

We denote by \( T_{k,\rho} \) the Bochner–Riesz operator of which multiplier is the pullback of \( \chi^-_+ \) via \( \lambda_\rho \) instead of \( \lambda \), that is,

\[
T_{k,\rho}f = F^{-1}(\langle \lambda_\rho^* \chi^-_+ \rangle Ff).
\]

The estimates (2.3)–(2.5) also hold for \( T_{k,\rho} \) with bounds \( O(\rho^{\frac{n}{q} - \frac{n}{q} - 2k}) \).

When a test function \( \phi(\theta, \tau, y) \) depends on several parameters including the variable \( \theta \), we write \( \langle T, \phi \rangle = \langle T, \phi \rangle_\theta \) to clarify that a distribution \( T \) acts on the function \( \phi(\theta, \tau, y) \).

Lemma 2.2. Let \( d \) and \( k \) be positive integers such that \( 1 \leq k < d/2 \) and let \( \phi \in C^\infty_c([-2, 2]) \). If \( (\frac{1}{p}, \frac{1}{q}) \in P^d_k \cup \{B^d_k, D^d_k\} \), then

\[
\left\| \int \phi(\tau) e^{it\tau} \lambda^*_\chi^-_+, \hat{\phi}(\theta, \tau) e^{iy\theta}\right\|_{L^{p, \infty}(\mathbb{R}^d)} \lesssim \|\phi\|_{C^2} \|f\|_{L^p(\mathbb{R}^d)}
\]

uniformly in \( \rho \sim 1 \). Here \( x = (y,t) \) and \( (\theta, \tau) \) denote the spatial and frequency variables, respectively, in \( \mathbb{R}^{d-1} \times \mathbb{R}^d \). In fact, if \( (\frac{1}{p}, \frac{1}{q}) \in (B^d_k, D^d_k) \), then the \( L^{q, \infty} \) in (2.8) can be replaced with \( L^q \). Furthermore, if \( (\frac{1}{p}, \frac{1}{q}) \in P^d_k \), then the stronger \( L^p - L^q \) estimate holds.

Proof. We only prove the restricted weak type estimate (2.8). The stronger estimates for \( (\frac{1}{p}, \frac{1}{q}) \in P^d_k \cup \{B^d_k, D^d_k\} \) follow similarly by using the estimates (2.3) or (2.4) in place of (2.5).

We note

\[
\int \phi(\tau) e^{it\tau} \langle \lambda^*_\chi^-_+, \hat{\phi}(\theta, \tau) e^{iy\theta} \rangle \, d\tau = 2\pi \int \phi'(t-s) \langle \lambda^*_\chi^-_+, \mathcal{F}(f(\cdot, s)) e^{iy\theta} \rangle \, ds,
\]

where \( \mathcal{F} \) denotes the \((d-1)\)-dimensional Fourier transform. Hence

\[
\int \phi(\tau) e^{it\tau} \langle \lambda^*_\chi^-_+, \hat{\phi}(\theta, \tau) e^{iy\theta} \rangle \, d\tau = (2\pi)^d \int \phi'(t-s) T_{k,\rho}(f(\cdot, s))(y) \, ds,
\]

(2.9)
where $T_{k,ρ}$ is the $d - 1$ dimensional Bochner–Riesz operator of index $−k$. We also use the following simple inequalities (see, e.g., [13, p. 780])

\[(2.10) \quad \|f\|_{L^q_{t=0}L^∞_x} \leq \|f\|_{L^q_{t=0}L^q_x} \quad \text{and} \quad \|f\|_{L^q_{t=0}L^{q,1}_x} \leq \|f\|_{L^q_{t=0}L^{q,1}_x}.
\]

Making use of the equality (2.9), the first inequality in (2.10), and Minkowski’s inequality,

\[\left\| \int_R \phi(\tau)e^{i\tau \langle \lambda_{ρ}^k, \hat{f}(\theta, \tau) e^{iy_0} \rangle} d\tau \right\|_{L^q_{t=0}L^∞_x} \lesssim \left\| \int_R \phi(t-s)\| T_{k,ρ}(f(\cdot, s))\|_{L^q_{t=0}L^∞_x} ds \right\|_{L^q_{t=0}L^∞_x}.
\]

Applying Young’s inequality, the estimate (2.5) with $n = d − 1$, and the second inequality in (2.10), we see that

\[\left\| \int_R \phi(t-s)\| T_{k,ρ}(f(\cdot, s))\|_{L^q_{t=0}L^∞_x} ds \right\|_{L^q_{t=0}L^∞_x} \lesssim \|\phi\|_{L^r}\|f\|_{L^p_{t=0}L^{q,1}_x} \leq \|\phi\|_{L^r}\|f\|_{L^p_{t=0}L^{q,1}_x}.
\]

for $r \in [1, ∞]$ satisfying $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{q}$, since $\text{supp } \phi \subset [-2, 2]$, we have $\|\phi\|_r \lesssim \|\phi\|_{C^2}$ for every $r \in [1, ∞]$. Therefore, we obtain (2.8).

\[\square\]

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Let us start with three simple observations. First, if $v = 0$ then (1.6) is the Hardy–Littlewood–Sobolev inequality, which is valid if and only if $1 < p, q < ∞$ satisfy the gap condition

\[(3.1) \quad \frac{1}{p} - \frac{1}{q} = \frac{2k}{d}.
\]

Hence it is enough to consider $v \neq 0$. Secondly, since the estimate (1.6) is invariant under rotations we may assume $v = |v|e_d$. Thirdly, by scaling it is easy to see that the gap condition (3.1) is necessary for (1.6) to hold.

Under the condition (3.1), by rescaling the Carleman inequality (1.6) (with nonzero $v$) is equivalent to the following estimate for a single Fourier multiplier operator:

\[(3.2) \quad \left\| F^{-1}(\frac{\hat{f}(\xi)}{(|\xi|^2 + 2iε_d - 1)^k}) \right\|_{L^q(R^d)} \leq C\|f\|_{L^p(R^d)}.
\]

We first make a decomposition in $ε_d$. Let $ψ \in C_0^∞([-2, −1/2] ∪ [1/2, 2])$ be a nonnegative even function such that $\sum_{j∈Z} ψ(2^{-j}t) = 1$ for $t \neq 0$, and let $ψ_0(t) = 1 - \sum_{j≥1} ψ(2^{-j}t)$. We write $ξ = (η, τ) ∈ R^{d-1} × R$ and, fixing a small positive dyadic number $ε_0 ≤ 2^{-5}$, set $D := \{2^ν ∈ (0, ε_0]: ν ∈ Z\}$. Then, we break

\[(3.3) \quad \frac{1}{(|η|^2 - 1 + τ^2 + 2iτ)^k} = m_L(η, τ) + m_G(η, τ)
\]

where

\[(3.4) \quad m_L(η, τ) = \sum_{ε ∈ D} m_ε(η, τ) := \sum_{ε ∈ D} ψ_0(ε_0 − 1 + |η|^2)ψ(ε_1 τ) \quad \text{and} \quad \frac{1}{(|η|^2 - 1 + τ^2 + 2iτ)^k}.
\]

Since $m_G$ vanishes in a neighborhood of the set $S^{d-2} × \{0\}$ where the original multiplier is singular, it is clear that

\[|∂^α ξ|^{2k}m_G(ξ)| \lesssim \|ξ\|^{-|α|}\]

for every $α ∈ N_0^d$. Thus, the Mikhlin multiplier theorem and the Hardy–Littlewood–Sobolev inequality yield the inequalities

\[\|m_G(D)f\|_{L^q(R^d)} \lesssim \|F^{-1}(\cdot |^{−2k}f)\|_{L^q(R^d)} \lesssim \|f\|_{L^p(R^d)}\]

for all $p, q$ satisfying $1 < p ≤ q < ∞$ and (3.1). Thus, the remaining task is to clarify $L^p - L^q$ boundedness of $m_L(D)$. 
3.1. Estimates for localized frequencies. In the summation (3.4), if we rescale each term \( m_\varepsilon \) by \( \tau \to \varepsilon \tau \), then
\[
\| m_\varepsilon(D) \|_{p\to q} \leq \varepsilon^{\frac{1}{p} - \frac{1}{q}} \| m_\varepsilon(D) \|_{p\to q},
\]
where
\[
\tilde{m}_\varepsilon(\eta, \tau) = m_\varepsilon(\eta, \varepsilon \tau) = \frac{\psi_0(\varepsilon^{-1}(1 - |\eta|^2))\psi(\tau)}{(|\eta|^2 - 1 + \varepsilon^2\tau^2 + 2\varepsilon \tau)^k}.
\]
Hence, it is enough to study \( L^p - L^q \) boundedness of \( \tilde{m}_\varepsilon(D) \) instead of \( m_\varepsilon(D) \), which is the main result in this section (see Proposition 3.1).

Let us set \( C = (\frac{1}{2}, 0) \) and
\[
A^d = (\frac{1}{2}, \frac{d-2}{2d}) \quad \text{for} \quad d \geq 3.
\]
For a positive integer \( k \) such that \( k < d/2 \), we define
\[
\mathcal{T}_k^d = [A^d, B_k^d; C, D_k^d] \setminus \{B_k^d, D_k^d\} = \{(x, y) \in \mathbb{Q} : \frac{1}{2} < x < \frac{d-2+2k}{2(d-1)}, \quad 0 \leq y < \frac{d-2}{d}(1-x)\}.
\]
See Figures 1 and 2.

**Proposition 3.1.** Let \( \varepsilon \in \mathbb{D} \). If \( (\frac{1}{p}, \frac{1}{q}) \in \mathcal{T}_k^d \), then
\[
\| \tilde{m}_\varepsilon(D) \|_{p\to q} \lesssim \varepsilon^{\frac{d-1}{p} - \frac{d-2+2k}{2}}.
\]

The estimate (3.7), combined with the identity (3.5), yields the following.

**Corollary 3.2.** Let \( \varepsilon \in \mathbb{D} \). If \( (\frac{1}{p}, \frac{1}{q}) \in \mathcal{T}_k^d \), then
\[
\| m_\varepsilon(D) \|_{p\to q} \lesssim \varepsilon^{\frac{d-k}{p} - \frac{d-2+2k}{2}}.
\]
In particular, if \( (\frac{1}{p}, \frac{1}{q}) \in [A^d, B_k^d] \), then
\[
\| m_\varepsilon(D) \|_{p\to q} \lesssim \varepsilon^{\frac{d-k}{p} - k}.
\]

The second estimate (3.9) follows by (3.8) since \( [A^d, B_k^d] \subset \{(x, y) : y = \frac{d-2}{d}(1-x)\} \). In fact, all the estimates (3.7), (3.8), and (3.9) are sharp, which we prove in the next section (Propositions 3.7 and 3.8).

In order to prove Proposition 3.1 we use induction on \( k \). For this, it is convenient to slightly generalize the definition of \( \tilde{m}_\varepsilon \) as follows. For \( \zeta \in C_0^\infty([-2, 2]) \) and \( 0 < \delta < 1/2 \), let us define
\[
\tilde{m}_\varepsilon^\delta(\zeta, \delta)(\eta, \tau) = \frac{\zeta(\delta^{-1}(1 - |\eta|^2))\psi(\tau)}{(|\eta|^2 - 1 + \varepsilon^2\tau^2 + 2\varepsilon \tau)^k}, \quad (\eta, \tau) \in \mathbb{R}^{d-1} \times \mathbb{R}.
\]
When \( \delta = \varepsilon_0 \), we simply write \( \tilde{m}_\varepsilon^\delta(\zeta, \varepsilon_0) = \tilde{m}_\varepsilon(\zeta) \). Also, notice that the multiplier \( \tilde{m}_\varepsilon \) defined in (3.10) can be expressed as \( \tilde{m}_\varepsilon = \tilde{m}_\varepsilon(\psi_0) = \tilde{m}_\varepsilon(\psi_0, \varepsilon_0) \). In the next lemma, we obtain identities that are important for our induction arguments in the proof of Proposition 3.2.

**Lemma 3.3.** Let \( k \) be a positive integer and \( L \) the differential operator defined in Lemma 2.7 with \( n \) replaced by \( d - 1 \). That is,
\[
Lh(\eta) = \frac{1}{2|\eta|^2}(d - 3 + \eta \cdot \nabla)h(\eta), \quad \eta \in \mathbb{R}^{d-1}.
\]
For \( h \in C^\infty(\mathbb{R}^{d-1}) \) supported away from the origin, we have
\[
\langle \tilde{m}_\varepsilon^k(\zeta)(\cdot, \tau), h \rangle = \sum_{\ell=0}^{k-1} (-1)^\ell \varepsilon_0^{k-\ell}(k-1-\ell)! \langle \tilde{m}_\varepsilon^\ell(\zeta^{(\ell)})(\cdot, \tau), L^{k-\ell}h \rangle,
\]
\[
\langle \tilde{m}_\varepsilon^1(\zeta, \delta)(\cdot, \tau), L^{k-1}h \rangle = \sum_{\ell=0}^{k-1} \frac{(k-1)!}{\delta^{\ell}} \langle \tilde{m}_\varepsilon^{k-\ell}(\zeta^{(\ell)})(\cdot, \tau), h \rangle.
\]
Proof. First, we prove the identity (3.11). It is trivial if \( k = 1 \), so we assume \( k \geq 2 \).

Direct differentiation gives
\[
\tilde{m}_\varepsilon^k[\zeta](\eta, \tau) = -\frac{1}{k - 1} \left( \frac{1}{\varepsilon_0} \tilde{m}_\varepsilon^{k-1}[\zeta'](\eta, \tau) + \frac{\eta}{2|h|^2} \cdot \nabla_{\eta} \tilde{m}_\varepsilon^{k-1}[\zeta](\eta, \tau) \right).
\]

Hence, integrating by parts we have
\[
\langle \tilde{m}_\varepsilon^k[\zeta](\cdot, \tau), h \rangle = -\frac{1}{k - 1} \left( \frac{1}{\varepsilon_0} \langle \tilde{m}_\varepsilon^{k-1}[\zeta'](\cdot, \tau), h \rangle - \sum_{j=1}^{d-1} \langle \tilde{m}_\varepsilon^{k-1}[\zeta](\cdot, \tau), \frac{\partial}{\partial \eta_j} \left( \frac{\eta}{2|h|^2} h \right) \rangle \right)
\]
\[
= -\frac{1}{(k - 1)\varepsilon_0} \langle \tilde{m}_\varepsilon^{k-1}[\zeta'](\cdot, \tau), h \rangle + \frac{1}{k - 1} \langle \tilde{m}_\varepsilon^{k-1}[\zeta](\cdot, \tau), Lh \rangle.
\]

By this identity and the induction hypothesis we obtain
\[
\langle \tilde{m}_\varepsilon^k[\zeta](\cdot, \tau), h \rangle = -\frac{1}{(k - 1)\varepsilon_0} \sum_{\ell=0}^{k-2} \varepsilon_0^\ell (k - 2 - \ell)! \langle \tilde{m}_\varepsilon^{1}[\zeta^{(\ell+1)}](\cdot, \tau), L^{k-2-\ell} h \rangle
\]
\[
+ \frac{1}{k - 1} \sum_{\ell=0}^{k-2} \varepsilon_0^\ell (k - 2 - \ell)! \langle \tilde{m}_\varepsilon^{1}[\zeta^{(\ell)}](\cdot, \tau), L^{k-1-\ell} h \rangle.
\]

Rearranging the summands gives (3.11).

Since \( L^j = -\frac{\eta}{2|h|^2} \cdot \nabla \) we see that
\[
\langle \tilde{m}_\varepsilon^j[\zeta, \delta](\cdot, \tau), Lh \rangle = \langle L^j \tilde{m}_\varepsilon^j[\zeta, \delta](\cdot, \tau), h \rangle
\]
\[
= \frac{1}{\delta} \langle \tilde{m}_\varepsilon^j[\zeta', \delta](\cdot, \tau), h \rangle + j \langle \tilde{m}_\varepsilon^{j+1}[\zeta, \delta](\cdot, \tau), h \rangle
\]
(3.13)

for \( j \in \mathbb{N} \). Now, we prove the second identity (3.12). It is clear when \( k = 1 \), so let us assume \( k \geq 2 \). By the induction hypothesis,
\[
\langle \tilde{m}_\varepsilon^k[\zeta, \delta](\cdot, \tau), L^{k-1} h \rangle = \sum_{\ell=0}^{k-2} \frac{(k - 2)!}{\delta^\ell \ell!} \langle \tilde{m}_\varepsilon^{k-1-\ell}[\zeta^{(\ell)}], \delta(\cdot, \tau), h \rangle.
\]

By (3.13) this is equal to
\[
\sum_{\ell=0}^{k-2} \frac{(k - 2)!}{\delta^\ell \ell!} \left( \frac{1}{\delta} \langle \tilde{m}_\varepsilon^{k-1-\ell}[\zeta^{(\ell+1)}], \delta(\cdot, \tau), h \rangle + (k - 1 - \ell) \langle \tilde{m}_\varepsilon^{k-\ell}[\zeta^{(\ell)}], \delta(\cdot, \tau), h \rangle \right),
\]
from which (3.12) follows. \( \square \)

In the following, making use of Tomas–Stein’s restriction estimate (20, 24), we prove sharp \( L^2 - L^{2d} \) estimate for the multiplier operators given by \( \tilde{m}_\varepsilon^k[\zeta, 2^j \varepsilon] \).

Lemma 3.4. Let \( 1 \leq k < \frac{d}{2} \) and \( \frac{2d}{d-k} \leq q \leq \infty \). For \( \zeta \in C_0^\infty([-2, 2] \setminus [-\frac{1}{2}, \frac{1}{2}]) \), \( \varepsilon \in \mathbb{D} \), and \( j = 0, 1, \ldots \) satisfying \( 2^j \leq \frac{1}{4\varepsilon} \), we have
\[
\| \tilde{m}_\varepsilon^k[\zeta, 2^j \varepsilon](D) \|_{2 \to q} \lesssim (2^j \varepsilon)^{\frac{d-k}{2}} \| \zeta \|_{L^\infty(\mathbb{R})}.
\]
When \( j = 0 \) the estimate also holds with \( \zeta \in C_0^\infty([-2, 2]) \).

Proof. For brevity’s sake, let us set
\[
M_{\varepsilon, j}^k = \tilde{m}_\varepsilon^k[\zeta, 2^j \varepsilon].
\]

By interpolation it is enough to prove that
\[
\| M_{\varepsilon, j}^k(D) f \|_{L^\infty(\mathbb{R}^d)} \lesssim (2^j \varepsilon)^{\frac{d-k}{2}} \| \zeta \|_{L^\infty(\mathbb{R})} \| f \|_{L^2(\mathbb{R}^d)},
\]
(3.14)
\[
\| M_{\varepsilon, j}^k(D) f \|_{L_{\mathbb{R}^d}^{2d}} \lesssim (2^j \varepsilon)^{\frac{d-k}{2}} \| \zeta \|_{L^\infty(\mathbb{R})} \| f \|_{L^2(\mathbb{R}^d)}.
\]
(3.15)
Obviously,
\[ |M_{ε,j}^k(D)f(x)| \lesssim \|M_{ε,j}^k\|_{L^2(\mathbb{R}^d)} \|\hat{f}\|_{L^2(\mathbb{R}^d)} \lesssim |\text{supp } M_{ε,j}^k| \frac{1}{2} \|M_{ε,j}^k\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}. \]
If \( ζ \in C_0^\infty([-2, 2] \setminus [-\frac{1}{2}, \frac{1}{2}]) \), then \( M_{ε,j}^k \) is supported in
\[ \{(η, τ) \in \mathbb{R}^{d-1} \times \mathbb{R} : \sqrt{1 - 2τ^2} \leq |η| \leq \sqrt{1 + 2τ^2}, \frac{1}{2} \leq |τ| \leq 2\} \cup \{(η, τ) \in \mathbb{R}^{d-1} \times \mathbb{R} : \sqrt{1 + 2τ^2} \leq |η| \leq \sqrt{1 + 2τ^2}, \frac{1}{2} \leq |τ| \leq 2\}. \]
(3.16)
Hence it is clear that \( |\text{supp } M_{ε,j}^k| \lesssim 2^j ε \). On the support of \( M_{ε,j}^k \) we have
\[ |||2^j ε - 1 + ε^2 τ^2 + 2iετ| \geq |||2^j ε - 1 - ε^2 τ^2 \geq 2^j ε - 4ε^2 ε, \]
which is greater than \( 2^j ε^2 \) since \( 4ε^2 \leq 2 - 3 < 2^2 \). Hence
\[ \|M_{ε,j}^k\|_{L^\infty(\mathbb{R}^d)} \lesssim (2^j ε)^{-k}|||L^\infty(\mathbb{R}), \]
and we obtain the estimate (3.17).
If \( \text{supp } ζ \subset [-\frac{1}{2}, \frac{1}{2}] \), then \( \text{supp } M_{ε,0}^k \) is contained in the set
\[ \{(η, τ) \in \mathbb{R}^{d-1} \times \mathbb{R} : \sqrt{1 - 2ε} \leq |η| \leq \sqrt{1 + 2ε}, \frac{1}{2} \leq |τ| \leq 2\} \]
on which \( |||2ε - 1 + ε^2 τ^2 + 2iετ| \geq |||2ε - 1 - ε^2 τ^2 \geq 2ε - 4ε^2 ε, \]
thus, a similar argument gives (3.14) with \( j = 0 \) and \( ζ \in C_0^\infty([-2, 2]). \)
To prove (3.15), let us choose \( \tilde{ψ} \in C_0^\infty([-3, 3]) \) such that \( \tilde{ψ}ζ = ζ \). Writing
\( x = (y, t) \in \mathbb{R}^{d-1} \times \mathbb{R} \) and using the spherical coordinates we note that
\[ M_{ε,j}^k(D)f(x) = \frac{1}{(2π)^d} \int_0^\infty \tilde{ψ} \left( \frac{1 - ρ^2}{2ε} \right) \int_\mathbb{R} \int_{S^{d-2}} \hat{f}(ρ, τ)dσ(θ)dτρ^{d-2}dρ. \]
Thus, by the Minkowski inequality \( \|M_{ε,j}^k(D)f\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)} \) is bounded by a constant times
\[ \int_0^\infty \tilde{ψ} \left( \frac{1 - ρ^2}{2ε} \right) \| \int_\mathbb{R} \int_{S^{d-2}} e^{i(ρy + τt)}(M_{ε,j}^k\hat{f})(ρ, τ)dσ(θ)dτ\|_{L^{\frac{2d}{d+2}}(\mathbb{R})} ρ^{d-2}dρ. \]
(3.19)
Applying the Hausdorff-Young and Minkowski inequalities successively gives
\[ \| \int_\mathbb{R} \int_{S^{d-2}} e^{i(ρy + τt)}(M_{ε,j}^k\hat{f})(ρ, τ)dσ(θ)dτ\|_{L^{\frac{2d}{d+2}}(\mathbb{R})} \lesssim \| \int_{S^{d-2}} e^{iyθ}(M_{ε,j}^k\hat{f})(ρ, τ)dσ(θ)\|_{L^{\frac{2d}{d+2}}(\mathbb{R})} \]
\[ \lesssim \| \int_{S^{d-2}} e^{iyθ}(M_{ε,j}^k\hat{f})(ρ, τ)dσ(θ)\|_{L^{\frac{2d}{d+2}}(\mathbb{R})} \]
\[ \lesssim \| \int_{S^{d-2}} e^{iyθ}(M_{ε,j}^k\hat{f})(ρ, τ)dσ(θ)\|_{L^{\frac{2d}{d+2}}(\mathbb{R})} \]
Since \( 2^j ε \leq \frac{1}{2} \) we see from (3.16) (or (3.18)) that \( M_{ε,j}^k(ρ, τ) \neq 0 \) only if \( ρ \sim 1 \).
Making use of the adjoint restriction estimate due to Tomas [26] and Stein [25] and the Hölder inequality, the last is
\[ \lesssim \|(M_{ε,j}^k\hat{f})(ρ, τ)\|_{L^2(S^{d-2} \times \mathbb{R})} \|L^{\frac{2d}{d+2}}(\mathbb{R}) \lesssim \|(M_{ε,j}^k\hat{f})(ρ, τ)\|_{L^2(\mathbb{R})}. \]
Since the \( ρ \)-support of the integrand in (3.19) is contained in an interval of length \( \lesssim 2^j ε \), the Cauchy-Schwarz inequality, (3.11), and the Plancherel theorem yield
\[ \|M_{ε,j}^k(D)f\|_{L^{\frac{2d}{d+2}}(\mathbb{R})} \leq \sqrt{2ε} \left( \int_0^\infty \tilde{ψ} \left( \frac{1 - ρ^2}{2ε} \right)^2 \|(M_{ε,j}^k\hat{f})(ρ, τ)\|_{L^2(\mathbb{R})}^2 \rho^{d-2}dρ \right)^{\frac{1}{2}} \]
\[ = \sqrt{2ε} \|(M_{ε,j}^k\hat{f})(η, τ)\|_{L^{2}(\mathbb{R})} \]
\[ \lesssim (2^j ε)^{-\frac{1}{2} - k} |||L^\infty(\mathbb{R})|||f|||L^2(\mathbb{R}) \]
This gives the estimate (3.15). \( \square \)
Now, we prove Proposition 3.1.

Proof of Proposition 3.1 Let $h_{y,\tau}(\eta) = \hat{f}(\eta,\tau)e^{i\eta y}$. We recall the notation $\tilde{m}_\varepsilon = m_k^b[\psi_0]$, and write

$$\tilde{m}_\varepsilon(D)f(y, t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}} e^{it\tau} \langle \tilde{m}_\varepsilon^1[\psi_0] \rangle(\cdot, \tau), h_{y,\tau} d\tau.$$  

As mentioned before, we need to dyadically decompose the inner integral near $S^{d-2}$, and obtain sharp estimates for each dyadic piece. More precisely, for $(\frac{1}{p}, \frac{1}{q}) \in [B_k^d, D_k^d]$, we show the $\varepsilon$-uniform $L^{k,1} - L^{q,\infty}$ estimates for the dyadic operators. When $k \geq 2$, however, this is not possible as can be seen by a simple example. Prior to dyadic decomposition, we must relax the order $k$ by using the identity (3.11). This gives, for $\varepsilon \in \mathbb{D}$,

$$\tilde{m}_\varepsilon(D)f(y, t) = \sum_{\ell=0}^{k-1} c_{k,\ell,\varepsilon} \int_{\mathbb{R}} e^{it\tau} \langle \tilde{m}_\varepsilon^1[\psi_0] \rangle(\cdot, \tau), L^{k-1-\ell}h_{y,\tau} d\tau,$$

where $c_{k,\ell,\varepsilon} = \frac{(-1)^\ell}{(2\pi)^d \varepsilon_{\varepsilon}(k-\ell)!}$.

Let us define

$$I_{\ell_j}(y, t) = \int_{\mathbb{R}} e^{it\tau} \langle \tilde{m}_\varepsilon^1[\psi_0] \rangle(\cdot, \tau), L^{k-1-\ell}h_{y,\tau} d\tau.$$

For $0 \leq \ell \leq k-1$ and $(\frac{1}{p}, \frac{1}{q}) \in T_k^d$, we need only to prove

$$\|I_{\ell_j}\|_{L^q(\mathbb{R}^d)} \lesssim \varepsilon^{\frac{d-1}{p} - \frac{d-2+2k}{2}} \|f\|_{L^p(\mathbb{R}^d)}.$$

We dyadically decompose $\tilde{m}_\varepsilon^1[\psi_0](\cdot, \tau)$ away from $S^{d-2}$ in the $\varepsilon$ scale. Recalling the smooth cutoff function $\psi$ introduced at the beginning of Section 3, we define $\psi_j(t) = \psi(2^{-j}t)$ for $j \geq 1$. Then $\sum_{j \geq 0} \psi_j = 1$ on $\mathbb{R}$, so we can write $I_{\ell}(y, t) = \sum_{j \geq 0} I_{\ell, j}(y, t)$ where

$$I_{\ell, j}(y, t) := \int_{\mathbb{R}} e^{it\tau} \langle \psi_j(\varepsilon^{-1}(1 - |\cdot|^2)) \tilde{m}_\varepsilon^1[\psi_0] \rangle(\cdot, \tau), L^{k-1-\ell}h_{y,\tau} d\tau.$$

In fact, the set of summation indices $j$ is finite since $\psi_j(\varepsilon^{-1}(1 - |\cdot|^2)) \tilde{m}_\varepsilon^1[\psi_0] \rangle(\cdot, \tau) \neq 0$ only if $2^{-j} \varepsilon \leq 2\varepsilon_0$.

We aim to prove that

$$\|I_{\ell, j}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \text{ if } (\frac{1}{p}, \frac{1}{q}) \in [B_k^d, D_k^d],$$

$$\|I_{\ell, j}\|_{L^q(\mathbb{R}^d)} \lesssim (2^j \varepsilon)^{\frac{d-1}{p} - \frac{d-2+2k}{2}} \|f\|_{L^p(\mathbb{R}^d)} \text{ if } \frac{2d}{d-2} \leq q \leq \infty.$$  

If we assume these estimates for the moment, then we have by real interpolation

$$\|I_{\ell, j}\|_{L^q(\mathbb{R}^d)} \lesssim (2^j \varepsilon)^{\frac{d-1}{p} - \frac{d-2+2k}{2}} \|f\|_{L^p(\mathbb{R}^d)}$$

for $(\frac{1}{p}, \frac{1}{q}) \in T_k^d$. Since $\frac{1}{p} < \frac{2d-2+2k}{2d-2}$ summing over $j$ we obtain the desired estimate (3.20). It remains to prove (3.22) and (3.23).

First, we prove (3.22). Using the spherical coordinates, the identity (2.1), and scaling, we have

$$I_{\ell, j}(y, t) = \int_{0}^{\infty} \psi_j \left(\frac{1 - \rho^2}{\varepsilon}\right) \int_{\mathbb{R}} e^{it\tau} \phi_{\varepsilon, \ell}(\rho, \tau) \langle \lambda_\rho \chi_+^{k-1}, L^{k-1-\ell}h_{y,\tau} \rangle d\tau d\rho,$$

where $\lambda_\rho(\theta) = \rho^2 - |\theta|^2$ and

$$\phi_{\varepsilon, \ell}(\rho, \tau) = \frac{\psi(\varepsilon^{-1}(1 - \rho^2))\psi(\tau)}{\rho^2 - 1 + \varepsilon^2 \tau^2 + 2i\varepsilon \tau}.$$
Here we omitted harmless constant multiplication depending only on $d$. The identity (3.12) gives
\begin{equation}
I_{\ell,j}(y,t) = \int \mathcal{F}_j \left( \frac{1 - \rho^2}{\varepsilon} \right) \int e^{i \langle \xi, \xi \rangle} \mathcal{F}_j(\lambda_\rho \chi_{(k-\ell)}(\theta,\tau)) f(\xi,\theta,\tau) e^{i \eta \cdot y} d\theta d\rho.
\end{equation}
Let us denote by $\mathcal{I}_j$ the support of the function $\rho \mapsto \mathcal{F}_j((1 - \rho^2)/\varepsilon)$, that is,
\begin{equation}
\mathcal{I}_j := \left\{ \left[ \sqrt{1 - 2j^{-1} \varepsilon}, \sqrt{1 - 2j^{-1} \varepsilon} \right] \cup \left[ \sqrt{1 + 2j^{-1} \varepsilon}, \sqrt{1 + 2j^{-1} \varepsilon} \right] \right\} \quad \text{if } j \geq 1,
\end{equation}
\begin{equation}
\left[ \sqrt{1 - 2j^{-1} \varepsilon}, \sqrt{1 + 2j^{-1} \varepsilon} \right] \quad \text{if } j = 0.
\end{equation}
From the Minkowski inequality and Lemma 2.2 it follows that
\begin{equation}
\|I_{\ell,j}\|_{L^q, \infty(\mathbb{R}^d)} \leq \int_0^\infty \mathcal{F}_j \left( \frac{1 - \rho^2}{\varepsilon} \right) \int e^{i \langle \xi, \xi \rangle} \mathcal{F}_j(\lambda_\rho \chi_{(k-\ell)}(\theta,\tau)) f(\xi,\theta,\tau) e^{i \eta \cdot y} d\theta d\rho.
\end{equation}
\begin{equation}
\leq 2^j \varepsilon \sup_{\rho \in \mathcal{I}_j} \int e^{i \langle \xi, \xi \rangle} \mathcal{F}_j(\lambda_\rho \chi_{(k-\ell)}(\theta,\tau)) f(\xi,\theta,\tau) e^{i \eta \cdot y} d\theta d\rho.
\end{equation}
\begin{equation}
\leq 2^j \varepsilon \left[ \sup_{\rho \in \mathcal{I}_j} \|\mathcal{F}_j(\theta,\tau)\|_{C^2} \right] \|f\|_{L^p,1(\mathbb{R}^d)}
\end{equation}
for $(\frac{1}{p}, \frac{1}{q}) \in [B^d_\infty, D^d_\infty]$. Elementary calculation shows that $\sup_{\rho \in \mathcal{I}_j} \|\mathcal{F}_j(\theta,\tau)\|_{C^2} \lesssim (2^j \varepsilon)^{-1}$ (see Lemma 3.3 below). Hence we obtain (3.22).

Now, we turn to prove (3.23). We cannot follow the strategy of the proof of (3.22) which relies on (3.24) and boundedness of the Bochner–Riesz operator $T_{k-\ell}$ of order $-(k-\ell)$, since $T_{k-\ell}$ is unbounded from $L^2$ to $L^q$ for any $1 \leq q \leq \infty$ and $0 \leq \ell \leq k - 1$. To get over the issue, using the identity (3.12), we integrate by parts again (in the definition (3.21) of $I_{\ell,j}$) to remove $L^{k-1-\ell}$ and then apply Lemma 3.4.

Let us define $\zeta_0 \in C_0^\infty([-2, 2])$ and $\zeta_j \in C_0^\infty([-2, 2] \setminus [-1/2, 1/2])$, $j \geq 1$, by setting
\begin{equation}
\zeta_j(t) = \begin{cases} 
\psi(t / \varepsilon) \zeta_0(t) & \text{if } j = 0, \\
\psi(t / \varepsilon) \zeta_0(t) & \text{if } j \geq 1.
\end{cases}
\end{equation}
Then
\begin{equation}
\psi((1 - |\eta|^2)) \hat{m} \hat{m}_j(\psi(t / \varepsilon)) \zeta_j(\eta, \tau) = \hat{m} \hat{m}_j(\zeta_j, 2^j \varepsilon)(\eta, \tau), \quad j \geq 0,
\end{equation}
so
\begin{equation}
I_{\ell,j}(y,t) = \int e^{i \langle \xi, \xi \rangle} \hat{m} \hat{m}_j(\zeta_j, 2^j \varepsilon)(\eta, \tau, \theta, \tau, h_y, \tau) d\tau.
\end{equation}
Applying the identity (3.12) we have
\begin{equation}
I_{\ell,j}(y,t) = \sum_{r=0}^{k-1-\ell} \binom{k - 1 - \ell}{2^j \varepsilon}^{r} \int e^{i \langle \xi, \xi \rangle} \hat{m} \hat{m}_j(\zeta_j, 2^j \varepsilon)(\eta, \tau, h_y, \tau) d\tau.
\end{equation}
We notice that $\sup \zeta_j^{(r)}(\eta) \in [-2, 2] \setminus [-1/2, 1/2]$ if $j \geq 1$ and $\sup \zeta_0^{(r)}(\eta) \in [-2, 2]$. Since $2^j \varepsilon \leq 4 \varepsilon$, we also observe that $\|\zeta_j^{(r)}\|_{L^\infty(\mathbb{R})}$ is bounded uniformly in $j$ and $\varepsilon$. Thus, by Lemma 3.4 we get
\begin{equation}
\|I_{\ell,j}\|_{L^q(\mathbb{R}^d)} \lesssim \sum_{r=0}^{k-1-\ell} (2^j \varepsilon)^{-r} (2^j \varepsilon)^{r-k-1-\ell} \|f\|_{L^p(\mathbb{R}^d)} \lesssim (2^j \varepsilon)^{k-1-\ell} \|f\|_{L^2(\mathbb{R}^d)}
\end{equation}
for $\frac{2d}{q} \leq q \leq \infty$. Since $2^j \varepsilon \lesssim 1$ this estimate gives (3.23).
Lemma 3.5. Let $I_j$ and $\phi_{\varepsilon,\ell}$ be as in the Proof of Proposition 3.7. We have
\[
\sup_{\rho \in I_j} \|\phi_{\varepsilon,\ell}(\rho, \cdot)\|_{C^2(\mathbb{R})} \lesssim (2^j \varepsilon)^{-1}.
\]

Proof. If $\rho \in I_j$, $j \geq 1$, and $\tau \in \text{supp} \psi$, then
\[
|\rho^2 - 1 + \varepsilon^2 \tau^2 + 2i\varepsilon \tau| \geq |\rho^2 - 1| - |\varepsilon^2 \tau^2| \geq (2^{j-1} - 4\varepsilon) \varepsilon \gtrsim 2^j \varepsilon.
\]
The last inequality holds since $0 < \varepsilon \leq \varepsilon_0 \leq 2^{-5}$. On the other hand, if $\rho \in I_0$ and $\tau \in \text{supp} \psi$, then
\[
|\rho^2 - 1 + \varepsilon^2 \tau^2 + 2i\varepsilon \tau| \geq 2\varepsilon \tau \geq \varepsilon.
\]
Therefore, direct calculation shows, for $\rho \in I_j$ and $r = 0, 1, 2$, $|\partial^r_x \phi_{\varepsilon,\ell}(\rho, \tau)| \lesssim (2^j \varepsilon)^{-1}$. \hfill $\square$

3.2. Sharpness of the local estimates. The estimate (3.8) is sharp in the sense that the exponent of $\varepsilon$ in (3.8) cannot be made larger. To show this, we begin with a general fact regarding $L^p - L^q$ norm of Fourier multipliers.

Lemma 3.6. Let $1 \leq p, q \leq \infty$. For $a \in L^\infty(\mathbb{R}^d)$,
\[(3.25) \quad \|a(D)\|_{p \to q} = \|\Pi(D)\|_{p \to q}.
\]
Consequently,
\[(3.26) \quad \|\text{Re} a(D)\|_{p \to q} \leq \|a(D)\|_{p \to q} \quad \text{and} \quad \|\text{Im} a(D)\|_{p \to q} \leq \|a(D)\|_{p \to q}.
\]

Proof. By the definition, $\Pi(D) f(x) = \widetilde{a}(D) h(-x)$ where $h$ is defined by $\hat{h}(\xi) = f(\xi)$ so that $h(x) = \overline{f(-x)}$. Hence
\[
\|\Pi(D) f\|_{L^p} = \|a(D) h\|_{L^p} \leq \|a(D)\|_{p \to q} \|h\|_{L^q} = \|a(D)\|_{p \to q} \|f\|_{L^p},
\]
which shows $\|\Pi(D)\|_{p \to q} \leq \|a(D)\|_{p \to q}$. Similarly, $\|a(D)\|_{p \to q} \leq \|\Pi(D)\|_{p \to q}$, and this gives the identity (3.25). The inequalities in (3.26) are clear. \hfill $\square$

Sharpness of the estimate (3.25) can be proved by a Knapp type example adapted to the cylinder $S^{d-2} \times [1/2, 2] \subset \mathbb{R}^d$.

Proposition 3.7. Let $d$ and $k$ be positive integers and let $1 \leq p, q \leq \infty$. Then, if $0 < \varepsilon \leq \varepsilon_0$ for some small $\varepsilon_0 > 0$,
\[
|m_\varepsilon(D)\|_{p \to q} \gtrsim \varepsilon^{d/2} (\frac{1}{p} - \frac{1}{q}) - k.
\]

Proof. By (3.5) it is enough to prove
\[
\|m_\varepsilon(D)\|_{p \to q} \gtrsim \varepsilon^{d/2} (\frac{1}{p} - \frac{1}{q}) - k,
\]
and furthermore, by the second inequality in (3.20), we need only show
\[(3.27) \quad \|\text{Im} m_\varepsilon(D)\|_{p \to q} \gtrsim \varepsilon^{d/2} (\frac{1}{p} - \frac{1}{q}) - k.
\]
We note that $\text{Im} m_\varepsilon(\eta, \tau)$ is equal to
\[(3.28) \quad \psi_0(\varepsilon^{-1} (1 - |\eta|^2)) \psi(\tau) \frac{\sum_{l=1}^{k+1} (-1)^l \binom{k}{2l-1} (\frac{1}{2l-1})^k |\eta|^2 - 1 + \varepsilon^2 \tau^2)^{k-2l+1} (2\varepsilon \tau)^{2l-1}}{|(|\eta|^2 - 1 + \varepsilon^2 \tau^2)^2 + 4\varepsilon^2 \tau^2|^2}.
\]
Let us choose a nonnegative smooth function $\phi$ on $\mathbb{R}$ such that $\text{supp} \phi \subset [1/2, 2]$ and $\phi = 1$ on $[1, 3/2]$, and define
\[
\hat{f}_\varepsilon(\eta, \tau) = \phi(\tau) \phi \left( \frac{\eta_{d-1} - 1}{\delta_0 \varepsilon} \right) \prod_{j=1}^{d-2} \phi \left( \frac{\eta_j}{\sqrt{\delta_0 \varepsilon}} \right)
\]
for a small constant $\delta_0 > 0$ and $0 < \varepsilon \leq \delta_0$. It is clear that
\[(3.29) \quad \|f_\varepsilon\|_{L^p} \sim (\delta_0 \varepsilon)^{\frac{d}{2} - \frac{d}{p}}.
\]
If \((\eta, \tau) \in \text{supp} \hat{f}_\varepsilon\), we have \(\tau \sim 1\) and \(|\eta|^2 - 1 + \varepsilon^2 \tau^2 \sim \delta_0 \varepsilon\). This yields

\[
|\text{Im} \tilde{m}_\varepsilon(\eta, \tau)| \sim \frac{1}{\varepsilon^{2k}} \left( \delta_0 \varepsilon^k - \sum_{l=2}^{k+1} \delta_0^{2l-1} \varepsilon^k \right) \sim \delta_0 \varepsilon^{-k}
\]

if

\[
(\eta, \tau) \in Q_\varepsilon := \left[ \sqrt{\delta_0 \varepsilon}, \frac{3}{2} \sqrt{\delta_0 \varepsilon} \right]^{d-2} \times \left[ 1 + \delta_0 \varepsilon, 1 + \frac{3}{2} \delta_0 \varepsilon \right] \times [1, \frac{3}{2}]
\]

whenever \(\delta_0\) is small enough. Clearly, \(\text{Im} \tilde{m}_\varepsilon\) is either negative or positive on \(Q_\varepsilon\).

Hence, if \(x\) lies in the set

\[
S_\varepsilon := \{ x \in \mathbb{R}^d : |x_d| \leq 10^{-3}, |x_{d-1}| \leq \varepsilon^{-1}, |x_j| \leq \varepsilon^{-\frac{d}{q}} \text{ for } 1 \leq j \leq d-2 \}
\]

for \(0 < \varepsilon \leq \delta_0\) and \(\delta_0\) is small enough, then

\[
|\text{Im} \tilde{m}_\varepsilon(D)f_\varepsilon(x)| \geq \left| \int \int \cos (x_d \tau + x_{d-1}(\eta_{d-1} - 1) + \sum_{j=1}^{d-2} x_j \eta_j) \text{Im} \tilde{m}_\varepsilon(\eta, \tau) \hat{f}_\varepsilon(\eta, \tau) d\eta d\tau \right| \sim |Q_\varepsilon| \delta_0 \varepsilon^{-k} \sim (\delta_0 \varepsilon) \frac{d}{q} \delta_0 \varepsilon^{-k}.
\]

Taking a sufficiently small \(\delta_\varepsilon\), we have

\[
\| \text{Im} \tilde{m}_\varepsilon(D)f_\varepsilon \|_{L^q(\mathbb{R}^d)} \geq \| \text{Im} \tilde{m}_\varepsilon(D)f_\varepsilon \|_{L^q(S_\varepsilon)} \gtrsim \varepsilon^{-\frac{d}{q}} \varepsilon^{-\frac{d}{q} - k}
\]

for \(0 < \varepsilon \leq \delta_\varepsilon\). Combined with (3.29), this implies

\[
\| \text{Im} \tilde{m}_\varepsilon(D) \|_{p \rightarrow q} \gtrsim \frac{\| \text{Im} \tilde{m}_\varepsilon(D)f_\varepsilon \|_{L^q(\mathbb{R}^d)}}{\| f_\varepsilon \|_{L^p(\mathbb{R}^d)}} \gtrsim \varepsilon^{\frac{d}{q} \left( \frac{1}{q} - \frac{1}{p} \right) - k},
\]

which gives the desired estimate (3.30).

We can also prove that the estimate (3.3) is sharp. In particular, the estimate (3.30) is sharp when \(\frac{1}{q} = \frac{d-2}{d} \left( 1 - \frac{1}{p} \right)\).

**Proposition 3.8.** Let \(d, k\) be positive integers, \(1 \leq p, q \leq \infty\), and \(0 < \varepsilon \ll 1\). Then we have

\[
|\text{Im} \tilde{m}_\varepsilon(D)f_\varepsilon| \gtrsim \varepsilon^{-\frac{d}{q} \left( \frac{1}{q} - \frac{1}{p} \right) - k}. \tag{3.30}
\]

By (3.15) and duality, the estimate (3.30) follows from

\[
|\text{Im} \tilde{m}_\varepsilon(D)| \gtrsim \varepsilon^{-\frac{d}{q} \left( \frac{1}{q} - \frac{1}{p} \right) - k}. \tag{3.31}
\]

When \(k = 1\), it is relatively simple to obtain the lower bound (3.31) by analyzing \(\text{Im} \tilde{m}_\varepsilon\) and using (3.26). Indeed, this was done in [13]. For larger \(k\), however, \(\text{Im} \tilde{m}_\varepsilon\) is given by a summation of \(\sim k/2\) terms with alternating signs (see (3.28)). Furthermore, it can be shown that each of those yields a Fourier multiplier whose \(L^p - L^q\) norm is \(\gtrsim \varepsilon^{-\frac{d}{q} \left( \frac{1}{q} - \frac{1}{p} \right) - k}\). In other words, there is no ‘leading term’ in the alternating sum (3.25). This makes it difficult to determine the lower bound for \(\| \text{Im} \tilde{m}_\varepsilon(D) \|_{p \rightarrow q}\).

We get around this problem and prove (3.31) replacing the order of denominator of \(\tilde{m}_\varepsilon\) by integration by parts. As the proof is rather involved, we shall postpone it until the last section; see Section 5.
Proof of Theorem 1.1. As mentioned before, the condition (3.1) is necessary for the Carleman inequality (1.6). In the preliminary decomposition (3.3), the global part \(m_G(D)\) is bounded from \(L^p(\mathbb{R}^d)\) to \(L^q(\mathbb{R}^d)\) if \(p, q \in (1, \infty)\) satisfy (3.1). The other condition in (1.7) is determined by the local part \(m_L(D)\).

Proposition 3.9. Let \(k\) be a positive integer such that \(k < d/2\), and let \(1 < p, q < \infty\). Then

\[
\|m_L(D)f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}
\]

if and only if

\[
\frac{1}{p} - \frac{1}{q} \geq \frac{2k}{d+2}, \quad \frac{d}{p} - \frac{1}{q} \geq \frac{d-2+2k}{2}, \quad \text{and} \quad \frac{d}{q} - \frac{1}{p} \leq \frac{d-2k}{2}.
\]

Before proving the proposition we additionally define two points \(E_k^d\) and \(F_k^d\) in \(Q\). For every \((d, k) \in \mathbb{N} \times \mathbb{N}\) satisfying \(1 \leq k < d/2\), we define

\[
E_k^d = \left(\frac{d^2+2kd-4}{2(d+2)(d-1)}, \frac{(d-2)(d+2)}{2(d^2+2d)(d-1)}\right), \quad F_k^d = \left(\frac{d-2+2k}{d}, 0\right).
\]

We note that \(E_k^d\) and \(F_k^d\) are on the line \(\{(x, y) \in Q: dx - y = \frac{d-2+2k}{d}\}\), while \(E_k^d\) and \(E_k^d'\) are on the line \(\{(x, y): x - y = \frac{2k}{d-2}\}\). See Figures 1 and 2. The pairs \((\frac{1}{p}, \frac{1}{q})\) satisfying the conditions in (3.33) are contained in the closed pentagon \([E_k^d, F_k^d, (E_k^d)', (F_k^d)', H]\).

The line segments \([E_k^d, F_k^d]\) and \(L_k^d := \{(x, y) \in \text{int } Q: x - y = \frac{2k}{d-2}\}\) intersect in the interior of \(Q\) if and only if \(k < \frac{d-2}{2}\). In this case, we denote the intersection point by \(G_k^d\), that is,

\[
G_k^d = \left(\frac{(d+2k)(d-2)}{2(d^2-1)}, \frac{d-2k-2}{2(d-1)}\right).
\]

Combining the conditions (3.1) and (3.33), we conclude that the Carleman inequality (1.6) holds if and only if \((\frac{1}{p}, \frac{1}{q})\) lies on the line

\[
\|m_d^L (\mathbb{R}^d) f \|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}
\]

which is \([G_k^d, (G_k^d)']\) if \(k < \frac{d-2}{2}\) and \(L_k^d\) if \(\frac{d-2}{2} \leq k < \frac{d}{2}\). Therefore, the proof of Theorem 1.1 is completed.
Proof of Proposition 3.9. We first prove the sufficiency part. By duality the estimate (3.32) is equivalent to \( \|m_L(D)\|_{q' \to p'} \lesssim 1 \). By Lemma 3.6 this is equivalent to
\[
\|m_L(D)\|_{q' \to p'} \lesssim 1.
\]
Thus, if \((\frac{1}{p}, \frac{1}{q}) \in [E_k^d, F_k^d], (E_k^d)' \setminus \{F_k^d, (F_k^d)\}' \) by duality and interpolation. Moreover, since \( m_L \) is supported in the ball of radius 2 centered at the origin, it is easy to see \( \|m_L(D)\|_{1 \to \infty} \lesssim 1 \). Indeed, for a cutoff function \( \phi \in C_0^\infty(\mathbb{R}^d) \) such that \( \phi = 1 \) on \( \text{supp} \ m_L \), we have \( m_L(D) = (\phi^2 m_L)(D) \). So, from the Young inequality and (3.32) with \((\frac{1}{p}, \frac{1}{q}) \in [E_k^d, F_k^d] \) it follows that
\[
\|m_L(D)f\|_{L^\infty} \lesssim \|\phi\|_{L^{q'}} \left\| (\phi m_L)(D)f \right\|_{L^q} \lesssim \|\phi\|_{L^p} \|f\|_{L^1} \lesssim \|f\|_{L^1}.
\]
Again, interpolation gives (3.32) for all \( 1 < p, q < \infty \) satisfying (3.33). Hence, it is enough to prove (3.32) for \((\frac{1}{p}, \frac{1}{q}) \in [E_k^d, F_k^d] \).

Let \( \beta \in C_0^\infty([-4, -1/4] \cup [1/4, 4]) \) be such that \( \beta \psi = \psi \). Since \( q \geq 2 \), using (3.4), by the Littlewood–Paley inequality and the Minkowski inequality we have
\[
\|m_L(D)f\|_{L^q(\mathbb{R}^d)} \approx \left( \sum_{z \in \mathbb{Z}} |\beta \left( \frac{Dz}{\varepsilon} \right) m_{\varepsilon}(D)f(z)|^2 \right)^{\frac{1}{2}} \|m_L(D)f\|_{L^q(\mathbb{R}^d)} \leq \left( \sum_{z \in \mathbb{Z}} |\beta \left( \frac{Dz}{\varepsilon} \right) m_{\varepsilon}(D)f(z)|^2 \right)^{\frac{1}{2}}.
\]
Combined with Corollary 3.2 this gives
\[
\|m_L(D)f\|_{L^q(\mathbb{R}^d)} \lesssim \left( \sum_{z \in \mathbb{Z}} 2^{2(d-\frac{1}{q}-\frac{d-2+2k}{2})} \|\beta \left( \frac{Dz}{\varepsilon} \right) f(z)|_{L^p(\mathbb{R}^d)}^2 \right)^{\frac{1}{2}},
\]
if \((\frac{1}{p}, \frac{1}{q}) \in \mathcal{T}_k^d \). We notice that \([E_k^d, F_k^d] \subset \mathcal{T}_k^d \) and \( \frac{4}{p} - \frac{1}{q} - \frac{d-2+2k}{2} = 0 \) if \((\frac{1}{p}, \frac{1}{q}) \in [E_k^d, F_k^d] \). Since \( p \leq 2 \) the Minkowski inequality and the Littlewood–Paley inequality give
\[
\|m_L(D)f\|_{L^q(\mathbb{R}^d)} \lesssim \left( \sum_{z \in \mathbb{Z}} |\beta \left( \frac{Dz}{\varepsilon} \right) f(z)|^2 \right)^{\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^d)} \sim \|f\|_{L^p(\mathbb{R}^d)}
\]
whenever \((\frac{1}{p}, \frac{1}{q}) \in [E_k^d, F_k^d] \).
Now, we prove the necessity part. We need only show that the inequality (3.32) implies the first and the second inequalities in (3.33) since the other in (3.33) follows from the second via duality. From the assumption (3.32) it follows \(|m_{\varepsilon}(D)|_{p \to q} \lesssim 1\) uniformly in \(\varepsilon \in \mathbb{D}\). Hence, by Propositions 3.7 and 3.8 we have

\[
\varepsilon \frac{d+2s}{d} \left( \frac{2}{d} \right) - k \lesssim 1, \quad \varepsilon \frac{d}{d} \frac{d+2s}{2} \lesssim 1
\]

for \(\varepsilon \ll 1\). Thus, considering the limiting case \(\varepsilon \to 0\) gives the first and the second inequalities in (3.33).

\[Q.E.D. \]

4. Unique continuation (proof of Theorem 1.2)

For the differential inequality \(|\Delta u| \leq |Vu|\), the argument deducing the UCP from the Carleman inequality (1.3) is well-known (16). One can prove Theorem 1.2 combining Theorem 1.1 and an argument in [16, Corollary 3.2]. We need only replace the Kelvin transform by an analogous point inversion transform preserving the polyharmonicity.

Lemma 4.1. For \(0 < s < \frac{d}{2}\) and \(u\) supported away from the origin in \(\mathbb{R}^d\), let

\[
T_s u(x) = |x|^{-d+2s} u \circ \Phi(x), \quad x \neq 0,
\]

where \(\Phi(x) = |x|^{-2s} x\). Then, we have

\[
(-\Delta)^s T_s u(x) = |x|^{-d-2s} ((-\Delta)^s u) \circ \Phi(x), \quad x \neq 0.
\]

Lemma 4.1 is already known. For example, see [1, Proposition 2] and [7, Lemma 3]. Nevertheless, we provide a short proof different from those in [1] and [7] for the sake of completeness.

Proof of Lemma 4.1. If we set \(f = (-\Delta)^s u\), then (4.2) is equivalent to

\[
| \cdot |^{-d+2s} ((-\Delta)^{-s} f) \circ \Phi = (-\Delta)^{-s} (| \cdot |^{-d-2s} f \circ \Phi).
\]

Since

\[
(-\Delta)^{-s} f(x) = c_{d,s} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2s}} dy
\]

and \(|\det J\Phi(y)| = |y|^{-2d}\),

\[
|x|^{-d+2s} ((-\Delta)^{-s} f) \circ \Phi(x) = c_{d,s} |x|^{-d+2s} \int_{\mathbb{R}^d} \frac{f(y)}{|x|^{d-2s}} dy
\]

\[
= c_{d,s} |x|^{-d+2s} \int_{\mathbb{R}^d} \frac{f(\Phi(y)) |y|^{-2d}}{|x|^{d-2s}} dy.
\]

Since \(\frac{|x|^2}{|y|^2} = |x|^{-2} |y|^{-2} |x-y|^2\), this is equal to

\[
c_{d,s} \int_{\mathbb{R}^d} \frac{f(\Phi(y)) |y|^{-d-2s}}{|x-y|^{d-2s}} dy = (-\Delta)^{-s} (| \cdot |^{-d-2s} f \circ \Phi)(x),
\]

and we obtain (4.3).

Using (1.0), Lemma 4.1 and the argument in [16, p. 345], we obtain the following.

Lemma 4.2. Let \(d \geq 3\) and \(\Omega\) be a connected open set in \(\mathbb{R}^d\) containing \(S^{d-1}\). Let \(k, V\), and \(X\) be given as in Theorem 1.2. Suppose that \(u \in X\) satisfies (1.1). If \(u\) vanishes in the annulus \(A_+ (\delta_1) := \{ x \in \mathbb{R}^d; 1 < |x| < 1 + \delta_1 \}\) for some \(\delta_1 > 0\), then \(u\) also vanishes in \(A_- (\delta_2) := \{ x \in \mathbb{R}^d; 1 - \delta_2 < |x| < 1 \}\) for some \(0 < \delta_2 < 1\), and vice versa.
Proof. First, we assume $u = 0$ in $A_+(\delta_1)$ and show that $u$ vanishes in $A_-(\delta_2)$ for some $0 < \delta_2 < 1$. By rotational symmetry, it is enough to prove that $u = 0$ in a neighborhood of $e_1 := (1, 0, \ldots , 0) \in \mathbb{R}^d$.

Let $p > 1$ if $k \geq \frac{d+2}{2}$ and $p = \frac{2(d-1)}{d+2k}$ if $k < \frac{d+2}{2}$, and let $q \in (1, \infty)$ be such that (1.7) holds. We denote by $B(x, r)$ the open ball in $\mathbb{R}^d$ of radius $r$ centered at $x$. We choose a cutoff function $\phi \in C_0^\infty(B(0, \delta_1))$ satisfying $\phi = 1$ in $B(0, \frac{\delta_1}{2})$, and define $\tilde{u}(x) = u(x)\phi(x - e_1)$. Since $V \in L^{d/2k}_\text{loc}(\Omega)$ and $B(0, 1)$ is convex, there is a small $\rho > 0$ such that

$$\tag{4.4} C\|V\|_{L^{\frac{d}{2k}}(S_\rho \cap B(e_1, \frac{\delta_1}{2}))} \leq \frac{1}{2} \quad \text{and} \quad B(0, 1) \cap S_\rho \subset B(e_1, \frac{\delta_1}{2}),$$

where $S_\rho := \{ x \in \mathbb{R}^d : 1 - \rho < x_1 \leq 1 \}$ and $C$ is the constant in the Carleman inequality (1.6). See Figure 3.

Obviously,

$$\text{supp } \tilde{u} \subset B(e_1, \delta_1) \setminus A_+(\delta_1) \subset [S_\rho \cap (B(e_1, \delta_1) \setminus A_+(\delta_1))] \cup \{ x_1 \leq 1 - \rho \}.$$ 

So, by the inclusion in (1.3) we have

$$S_\rho \cap (B(e_1, \delta_1) \setminus A_+(\delta_1)) \subset S_\rho \cap B(0, 1) \subset S_\rho \cap B(e_1, \frac{\delta_1}{2}).$$

Therefore, the Carleman inequality (1.6) gives

$$\tag{4.5} \|e^{\lambda x_1} \tilde{u}\|_{L^q(S_\rho)} \leq C\|e^{\lambda x_1} \Delta^k \tilde{u}\|_{L^p(S_\rho \cap B(e_1, \frac{\delta_1}{2}))} + C\|e^{\lambda x_1} \Delta^k \tilde{u}\|_{L^p(\{ x_1 \leq 1 - \rho \})}$$

for all $\lambda > 0$. Since $\tilde{u} = u$ in $B(e_1, \frac{\delta_1}{2})$, by (1.1), Hölder’s inequality, and (4.4)

$$C\|e^{\lambda x_1} \Delta^k \tilde{u}\|_{L^p(S_\rho \cap B(e_1, \frac{\delta_1}{2}))} \leq C\|e^{\lambda x_1} V u\|_{L^p(S_\rho \cap B(e_1, \frac{\delta_1}{2}))} \leq C\|V\|_{L^{\frac{d}{2k}}(S_\rho \cap B(e_1, \frac{\delta_1}{2}))}\|e^{\lambda x_1} u\|_{L^q(S_\rho \cap B(e_1, \frac{\delta_1}{2}))} \leq \frac{1}{2}\|e^{\lambda x_1} V u\|_{L^q(S_\rho)}.$$

Since $\tilde{u} \in W^{2k,p}(\mathbb{R}^d)$, it is clear that $\|e^{\lambda x_1} \tilde{u}\|_{L^q(S_\rho)} < \infty$ by the Sobolev embedding.

Therefore, combining the estimates (4.5) and (4.6), we have

$$\|e^{\lambda x_1} \tilde{u}\|_{L^q(S_\rho)} \leq 2C\|e^{\lambda x_1} \Delta^k \tilde{u}\|_{L^p(\{ x_1 < 1 - \rho \})} \leq 2C\|e^{\lambda (1 - \rho)}\|_{L^p(\{ x_1 \leq 1 - \rho \})},$$

Figure 3. Continuation of $u = 0$ from $A_+(\delta_1)$ to $S_\rho \cap B(e_1, \frac{\delta_1}{2})$ in the proof of Lemma 4.2.
equivalently,
\[ \|e^{λ(x_1-1+ρ)}u\|_{L^p(S_ρ \cap B(ε_1, 1/ρ))} \leq 2C\|Δ^k u\|_{L^p(1 \leq x < 1-ρ)} \]
uniformly in λ. The right hand side is clearly finite, so the inequality leads to a contradiction as λ → ∞ unless \( u \) were identically zero in \( S_ρ \).

We next prove the converse assertion, that is, we show \( u = 0 \) in a neighborhood of \( e_1 \) assuming \( u = 0 \) in \( A_-(δ_2) \).

Making use of the Kelvin transform (4.1) we notice that \( T_k u = 0 \) in \( A_+(δ_1) \) with \( δ_1 = δ_2 / T_δ \). If we set \( V^*(x) = |x|^{-d-2k} V \circ Φ(x) \), then for every \( K \subset \subset Φ(Ω) \)
\[ ||V^*||_{L^p(Φ^{-1}(K))} = ||V||_{L^p(Φ^{-1}(K))} < ∞, \]
so \( V^* \in L^{d/2k}(Ω) \). It follows from (4.2) and (1.1) that
\[ |Δ^k T_k u(x)| = |x|^{-d-2k}|Δ^k u| Φ(x)| \leq |V^*(x) T_k u(x)|. \]

Now, we may use the preceding argument proving continuation of \( u = 0 \) from the outside of \( Ω^{d-1} \) to the inside. Thus we see \( T_k u \) vanishes near \( e_1 \). Consequently, \( u \) also vanishes in a neighborhood of \( e_1 \). □

By the argument in [16] p. 344 Theorem 1.2 is rather a straightforward consequence of the above lemma.

**Proof of Theorem 1.2** Let \( u ∈ X \) vanish in a nonempty open subset of \( Ω \), and we denote by \( Ω_0 \) the maximal open set in which \( u = 0 \). We aim to prove \( Ω_0 = Ω \).

Suppose to the contrary that \( Ω_0 \neq Ω \). Then there exists an \( x_0 ∈ Ω \cap ∂Ω_0 \) and an open ball \( B ⊂ Ω_0 \) such that \( ∂B \cap ∂Ω_0 = \{ x_0 \} \). By translation and scaling, we may assume \( B = B(0,1) \). Since \( u = 0 \) in \( B \), Lemma 1.2 shows that \( u = 0 \) in a neighborhood of \( x_0 \). This contradicts to the maximality of \( Ω_0 \). Therefore, \( Ω_0 = Ω \). □

**5. Proof of Proposition 3.8**

In this section, we prove (3.31) by constructing examples. In the first place, it would be convenient to define notations for quantities relevant to the example and obtain estimates for those quantities.

For \( 1/4 ≤ τ ≤ 2, ε ∈ R^{d-1}, 0 < ε ≤ 1 \), and a nonnegative \( ϕ ∈ C_0^∞(R) \) supported near 1, let us define
\[
I_1(τ, ε; y) := \int_R \frac{2ετ}{(ρ^2 - 1 + ε^2 τ^2)^2} + 4ε^2 τ^2 ϕ(ρ) cos((ρ - 1) |y|) dρ,
I_2(τ, ε; y) := \int_R (ρ^2 - 1 + ε^2 τ^2)^2 + 4ε^2 τ^2 ϕ(ρ) cos((ρ - 1) |y|) dρ,
I_3(τ, ε; y) := \int_R \frac{2ετ}{(ρ^2 - 1 + ε^2 τ^2)^2} + 4ε^2 τ^2 ϕ(ρ) sin((ρ - 1) |y|) dρ,
I_4(τ, ε; y) := \int_R (ρ^2 - 1 + ε^2 τ^2)^2 + 4ε^2 τ^2 ϕ(ρ) sin((ρ - 1) |y|) dρ.
\]

We also set
\[
\bar{I}_1(τ; ε) := \int_R \frac{2ετ}{(ρ^2 - 1 + ε^2 τ^2)^2} + 4ε^2 τ^2 ϕ(ρ) dρ,
\bar{I}_2(τ; ε) := \int_R (ρ^2 - 1 + ε^2 τ^2)^2 + 4ε^2 τ^2 ϕ(ρ) dρ.
\]

By a simple calculation we get the following.
Lemma 5.1. For a fixed \( \delta_0 \in (0, \frac{1}{4}) \) let \( \varphi \in C_0^\infty([1 - 2\delta_0, 1 + 2\delta_0]) \). Then there is a constant \( C \) independent of \( 0 < \varepsilon \ll 1 \) such that
\[
|\mathcal{I}_1(\tau; \varepsilon)| \leq C \quad \text{and} \quad |\mathcal{I}_2(\tau; \varepsilon)| \leq C \log \varepsilon^{-1}.
\]
Consequently,
\[
|\mathcal{I}_1(\tau, y; \varepsilon)|, |\mathcal{I}_3(\tau, y; \varepsilon)| \leq C,
\]
\[
|\mathcal{I}_2(\tau, y; \varepsilon)|, |\mathcal{I}_4(\tau, y; \varepsilon)| \leq C \log \varepsilon^{-1}.
\]
Proof. Since the function \( \rho \mapsto \rho^2 - 1 + \varepsilon^2 \tau^2 \) is invertible on \([1 - 2\delta_0, 1 + 2\delta_0]\), by changing variables, it is easy to see
\[
|\mathcal{I}_1(\tau; \varepsilon)| \leq \|\varphi\|_{L^\infty} \int_{-\infty}^{\infty} \frac{2\varepsilon \tau}{\tau^2 + 4\varepsilon^2 \tau^2} \, dt \lesssim \int_{0}^{\infty} \frac{1}{t^2 + 1} \, dt \lesssim 1.
\]
Similarly, we get
\[
|\mathcal{I}_2(\tau; \varepsilon)| \leq \|\varphi\|_{L^\infty} \int_{-\infty}^{\infty} \frac{t}{\tau^2 + 4\varepsilon^2 \tau^2} \, dt \lesssim \int_{0}^{1/\varepsilon} \frac{t}{t^2 + 1} \, dt \lesssim \log \varepsilon^{-1}.
\]
The estimates for \( \mathcal{I}_3(\tau, y; \varepsilon) \) follow in a similar manner. \( \square \)

Fortunately, under additional assumptions on \( \varphi \) we can improve the estimates for \( \mathcal{I}_2 \) and \( \mathcal{I}_4 \). We also obtain a proper lower bound for \( \mathcal{I}_1 \).

Lemma 5.2. Let \( \delta_0, \varepsilon, \) and \( \varphi \) be given as in Lemma 5.1 and suppose further that \( 0 \leq \varphi(\rho) \leq 1 \) and \( \varphi(1 + \rho) = \varphi(1 - \rho) \) for \( \rho \in \mathbb{R} \), and
\[
\varphi(\rho) = 1 \quad \text{if} \quad \rho \in [1 - \delta_0, 1 + \delta_0].
\]
Then, we have the uniform estimates
\[
|\mathcal{I}_2(\tau, y; \varepsilon)| \lesssim 1 \quad \text{for all} \quad \tau \in [1/2, 2] \quad \text{and} \quad y \in \mathbb{R}^{d-1},
\]
\[
|\mathcal{I}_4(\tau, y; \varepsilon)| \lesssim 1 \quad \text{for all} \quad \tau \in [1/2, 2] \quad \text{and} \quad |y| \sim \varepsilon^{-1}.
\]
Furthermore,
\[
\mathcal{I}_1(\tau, y; \varepsilon) \gtrsim 1 \quad \text{for all} \quad \tau \in [1/2, 2] \quad \text{and} \quad |y| \sim \varepsilon^{-1}.
\]
Proof. In order to use the symmetry of the function \( \varphi \) we approximate \( \rho^2 - 1 + \varepsilon^2 \tau^2 \) by \( 2(\rho - 1) \) and write
\[
\frac{\rho^2 - 1 + \varepsilon^2 \tau^2}{(\rho^2 - 1 + \varepsilon^2 \tau^2)^2 + 4 \varepsilon^2 \tau^2}
\]
\[
= \frac{\rho - 1}{2(\rho - 1)^2 + 2 \varepsilon^2 \tau^2} \left( \frac{\rho^2 - 1 + \varepsilon^2 \tau^2}{(\rho^2 - 1 + \varepsilon^2 \tau^2)^2 + 4 \varepsilon^2 \tau^2} - \frac{\rho - 1}{2(\rho - 1)^2 + 2 \varepsilon^2 \tau^2} \right)
\]
\[
= \frac{\rho - 1}{2(\rho - 1)^2 + 2 \varepsilon^2 \tau^2} \left( (\rho - 1)^2(\rho + 1) + 2 \varepsilon^2 \tau^2(\rho - 1)^3 + \varepsilon^4 \tau^4(\rho - 3) \right)
\]
\[
=: I_1(\rho, \tau; \varepsilon) - I_2(\rho, \tau; \varepsilon).
\]
Since the function \( \rho \mapsto \rho \varphi(1 + \rho) \) is odd, we see that
\[
\int_0^{2\delta_0} \frac{\rho \varphi(1 + \rho)}{2(\rho^2 + \varepsilon^2 \tau^2)} \cos(\rho |y|) \, d\rho = 0,
\]
and
\[
\mathcal{I}_2(\tau, y; \varepsilon) = - \int_0^{2\delta_0} \frac{\rho \varphi(1 + \rho)}{2(\rho^2 + \varepsilon^2 \tau^2)} \cos((\rho - 1) |y|) \, d\rho.
\]
To estimate this we separately consider the three terms in the numerator in \( \mathcal{I}_2 \).
First, by translation $\rho \to \rho + 1$
\[
\left| \int \frac{2\rho}{2((\rho - 1)^2 + \varepsilon^2 \tau^2)} \frac{(\rho - 1)^4 (\rho + 1) \varphi(\rho) \cos((\rho - 1)|y|)}{(\rho^2 - 1 + \varepsilon^2 \tau^2)^2 + 4\varepsilon^2 \tau^2} \, d\rho \right| 
\leq \int_{-2\delta_N}^{2\delta_N} \rho^2 (\rho + 2) \frac{d\rho}{\rho (\rho + 2) + \varepsilon^2 \tau^2 + 4\varepsilon^2 \tau^2}.
\]

If $|\rho| \geq \varepsilon^2 \tau^2$ then $(\rho^2 + 2 + \varepsilon^2 \tau^2)^2 = \rho^2 ((\rho + 2) + \frac{\varepsilon^2 \tau^2}{\rho})^2 \geq \rho^2 (\rho + 1)^2$, hence
\[
\int_{-2\delta_N}^{2\delta_N} \rho^2 (\rho + 2) \frac{d\rho}{(\rho + 2) + \varepsilon^2 \tau^2 + 4\varepsilon^2 \tau^2} 
\leq \int_{\varepsilon^2 \tau^2 |\rho| \leq 2\delta_N} \frac{2}{\rho (\rho + 1)^2} \rho^2 (\rho + 1)^2 \, d\rho + \int_{|\rho| \leq \varepsilon^2 \tau^2} \frac{\rho^2 (\rho + 2)}{4\varepsilon^2 \tau^2} \, d\rho 
\leq 1.
\]

Secondly, similar computations show that
\[
\left| \int \frac{(\rho - 1)^3 \varphi(\rho) \cos((\rho - 1)|y|)}{(\rho^2 - 1 + \varepsilon^2 \tau^2)^2 + 4\varepsilon^2 \tau^2} \, d\rho \right| 
\leq \varepsilon \int_{-2\delta_N}^{2\delta_N} \frac{|\rho|^2}{\rho (\rho + 2) + \varepsilon^2 \tau^2 + 4\varepsilon^2 \tau^2} \, d\rho 
\leq \varepsilon \int_{\varepsilon^2 \tau^2 |\rho| \leq 2\delta_N} \frac{|\rho|^2}{\rho^2 (\rho + 1)^2} \, d\rho + \int_{|\rho| \leq \varepsilon^2 \tau^2} \frac{|\rho|^2}{4\varepsilon^2 \tau^2} \, d\rho 
\leq \varepsilon^2 (\log \varepsilon^2 + \varepsilon^2) \leq 1.
\]

Finally, we see that
\[
\left| \int \frac{\varepsilon^4 \tau^4 (\rho - 3) \varphi(\rho) \cos((\rho - 1)|y|)}{2((\rho - 1)^2 + \varepsilon^2 \tau^2)} \frac{d\rho}{(\rho^2 - 1 + \varepsilon^2 \tau^2)^2 + 4\varepsilon^2 \tau^2} \right| 
\leq \varepsilon \int_{-2\delta_N}^{2\delta_N} \frac{2}{\rho (\rho + 2) + \varepsilon^2 \tau^2 + 4\varepsilon^2 \tau^2} \, d\rho \leq 1.
\]

Combining these estimates, we obtain the uniform estimate (5.5).

Let us prove the estimate (5.4). As before we can write
\[
I_4(\tau, y; \varepsilon) = \int (I_1(\rho, \tau; \varepsilon) - I_2(\rho, \tau; \varepsilon)) \varphi(\rho) \sin((\rho - 1)|y|) \, d\rho.
\]

Using the argument identical to one in the above we have
\[
\left| \int I_2(\rho, \tau; \varepsilon) \varphi(\rho) \sin((\rho - 1)|y|) \, d\rho \right| \leq 1
\]
uniformly in $\tau \in [1/2, 2]$ and $y \in \mathbb{R}^d$. Thus it remains to prove
\[
(5.6) \quad \left| \int I_1(\rho, \tau; \varepsilon) \varphi(\rho) \sin((\rho - 1)|y|) \, d\rho \right| \leq 1
\]
uniformly in $\tau \in [1/2, 2]$ and $|y| \sim \varepsilon^{-1}$. By the assumption on $\varphi$ the integral in (5.6) is equal to
\[
\int_{-2\delta_N}^{2\delta_N} \frac{\rho \varphi(\rho + 1) \sin(\rho|y|)}{2(\rho^2 + \varepsilon^2 \tau^2)} \, d\rho = \int_{0}^{2\delta_N} \frac{\rho \varphi(\rho + 1) \sin(\rho|y|)}{\rho^2 + \varepsilon^2 \tau^2} \, d\rho.
\]

We break this into the following two terms:
\[
I_4^1(y) := \int_{0}^{2\delta_N} \frac{\sin(\rho|y|)}{\rho} \, d\rho,
\]
\[
I_4^2(\tau, y; \varepsilon) := \int_{0}^{2\delta_N} \left( \frac{\rho \varphi(\rho + 1)}{\rho^2 + \varepsilon^2 \tau^2} - \frac{1}{\rho} \right) \sin(\rho|y|) \, d\rho.
\]
Since \( \int_0^u \sin \frac{\pi \lambda}{\rho} \, dt \leq 4 \) for \( u > 0 \) we see \( |I_2(y)| \leq 4 \) for any \( y \). From (5.2) it follows that
\[
I_3^2(\tau, y; \varepsilon) = - \int_0^{\delta_\varepsilon} \frac{\varepsilon^2 \tau^2 \sin(|y|)}{\rho^2 + \varepsilon^2 \tau^2} \, d\rho + \int_{\delta_\varepsilon}^{\infty} \left( \frac{\rho \phi'}{\rho^2 + \varepsilon^2 \tau^2} - \frac{1}{\rho} \right) \sin(|y|) \, d\rho.
\]

The (absolute value of the) first integral in (5.7) is dominated by
\[
\varepsilon^2 \tau^2 |y|^2 \left| \int_0^{\delta_\varepsilon} \frac{\sin \rho}{\rho^2 + \varepsilon^2 \tau^2 |y|^2} \, d\rho \right| \leq \varepsilon \tau |y| \left( \int_0^{\delta_\varepsilon} \frac{1}{1 + \rho^2} \, d\rho \right),
\]
which is uniformly bounded provided that \( |y| \sim \varepsilon^{-1} \). On the other hand, the second integral in (5.7) is estimated by \( \int_{\delta_\varepsilon}^{2\delta_\varepsilon} \frac{\rho}{\mu} \leq 1 \). Thus we obtain (5.4), which yields the uniform bound (5.4).

Finally, we prove (5.5). Let us set
\[
b(u) = \frac{1}{|s^2 + 1|} \, ds, \quad 0 \leq u \leq \infty.
\]

Clearly, this is a continuous, monotonically increasing, and bounded function on the interval \( [0, \infty) \). Let us fix a large \( \lambda > 0 \) such that
\[
b(\lambda/4) \geq 2^4 (b(\infty) - b(\lambda/4)),
\]
and then let us take a small number \( \mu > 0 \) such that \( \lambda \mu \leq 2^{-7} \) and define
\[
A = \{ y \in \mathbb{R}^{d-1} : \frac{\mu}{4 \varepsilon} \leq |y| \leq \frac{\mu}{2 \varepsilon} \}.
\]

We now break the integral \( I_1(\tau, y; \varepsilon) \) into two parts as
\[
I_1(\tau, y; \varepsilon) = \int_{|\rho^2 - 1 + \varepsilon^2 \tau^2| \leq \lambda \varepsilon} \frac{2 \varepsilon \tau \phi(\rho \cos((\rho - 1)|y|))}{\rho^2 - 1 + \varepsilon^2 \tau^2 + (2 \varepsilon \tau)^2} \, d\rho + \int_{|\rho^2 - 1 + \varepsilon^2 \tau^2| \geq \lambda \varepsilon} \frac{2 \varepsilon \tau \phi(\rho \cos((\rho - 1)|y|))}{\rho^2 - 1 + \varepsilon^2 \tau^2 + (2 \varepsilon \tau)^2} \, d\rho
\]
\[
= : I_1^1(\tau, y; \varepsilon) + I_1^2(\tau, y; \varepsilon).
\]

Let us first estimate a lower bound for \( I_1^1(\tau, y; \varepsilon) \). Since \( \delta_\varepsilon \) and \( \varepsilon \) are small, if \( |\rho^2 - 1 + \varepsilon^2 \tau^2| \leq \lambda \varepsilon \), then it is easy to see that \( |\rho - 1| \leq 2 \lambda \varepsilon \). Hence whenever \( y \in A \)
\[
|\rho - 1| y \leq \lambda \mu \leq 2^{-7},
\]
and this yields
\[
\cos((\rho - 1)|y|) \geq 1 - \frac{(\rho - 1)^2 |y|^2}{2} \geq 1 - 2^{-15}.
\]

If \( \varepsilon \leq \frac{\delta_\varepsilon}{2 \tau} \), then \( |\rho - 1| \leq 2 \lambda \varepsilon \leq \delta_\varepsilon \), so \( \phi(\rho) = 1 \) by the assumption (5.2). Thus, for \( \varepsilon \leq \frac{\delta_\varepsilon}{2 \tau}, \tau \in [1/2, 2] \), and \( y \in A \), we have
\[
I_1^1(\tau, y; \varepsilon) \geq (1 - 2^{-15}) \int_{|\rho^2 - 1 + \varepsilon^2 \tau^2| \leq \lambda \varepsilon} \frac{2 \varepsilon \tau}{\rho^2 - 1 + \varepsilon^2 \tau^2 + (2 \varepsilon \tau)^2} \, 2 \rho d\rho
\]
\[
\geq 1 - 2^{-15} \int_{|\rho^2 - 1 + \varepsilon^2 \tau^2| \leq \lambda \varepsilon} \frac{2 \varepsilon \tau}{\rho^2 - 1 + \varepsilon^2 \tau^2 + (2 \varepsilon \tau)^2} \, dt \geq 1 - 2^{-15} \int_{|\rho| \leq \delta_\varepsilon} \frac{1}{t^2 + 1} \, dt
\]
\[
\geq 2^{-3} b(\lambda/4) \geq 2(b(\infty) - b(\lambda/4))
\]

\[\text{In fact, it is vacuously true that } \varepsilon \leq \frac{\delta_\varepsilon}{2 \tau} \text{ since we are assuming } \varepsilon \ll 1 \text{ and } \lambda \text{ is a large number. It follows, since } \rho \in \text{supp } \phi \subset [1 - 2 \delta_\varepsilon, 1 + 2 \delta_\varepsilon] \text{ and } \delta_\varepsilon \leq \frac{\delta_\varepsilon}{2}, \text{ that } |\rho - 1| \leq \frac{\delta_\varepsilon}{\rho + 1} \leq \frac{2 \lambda \varepsilon}{1 - \varepsilon} \leq \frac{\varepsilon}{4} \lambda.\]
by the choice of \( \lambda \) (see (5.8)). Meanwhile, since 
\[
\frac{\varphi(\rho)}{2\rho} \leq \frac{\varphi(\rho)}{2(1-2\varepsilon r^2)} \leq \varphi(\rho) \leq 1,
\]

\[
|I^2_1(\tau, y; \varepsilon)| \leq \int_{|\rho^2 - 1 + \varepsilon^2 r^2| \geq \lambda \varepsilon} 2\varepsilon \tau \frac{\varphi(\rho)}{2\rho} 2\rho d\rho \leq \int_{|\tau| \geq \lambda \varepsilon} \frac{t^2 + (2\varepsilon \tau)^2}{t^2 + (2\varepsilon \tau)^2} dt = \int_{|s| \geq \frac{\lambda}{2}} \frac{1}{s^2 + 1} ds \leq b(\infty) - b(\lambda/4).
\]

Combining the estimates for \( I^1_1 \) and \( I^2_1 \) we have, for all \((\tau, y) \in [1/2, 2] \times \mathbb{A} \) and \( \varepsilon \) small enough,
\[
I_1(\tau, y; \varepsilon) \geq I^1_1(\tau, y; \varepsilon) - |I^2_1(\tau, y; \varepsilon)| \geq b(\infty) - b(\lambda/4).
\]

Therefore, the proof of (5.9) is complete. □

Now we prove Proposition 3.8.

**Proof of Proposition 3.8.** It is enough to prove (3.31). Let us fix a small enough \( \delta_0 > 0 \) and choose a nonnegative smooth function \( \phi \in C^\infty_0([1 - 2\delta_0, 1 + 2\delta_0]) \) such that

- the function \( \rho \mapsto (\rho + 1)^{\frac{d-2k}{2}} \phi(\rho + 1) \) is even;
- \( \sup \rho \frac{\rho}{\rho + 1} \phi(\rho) \leq 1; \)
- \( \rho^{\frac{d-2k}{2}} \phi(\rho) = 1 \) on \([1 - \delta_0, 1 + \delta_0]\).

We define \( f \in C^\infty_0(\mathbb{R}^d) \) by

\[
\tilde{f}(\eta, \tau) = \frac{\phi(|\eta|) \phi(\tau)}{\psi_0 \psi(\tau)} \quad (\eta, \tau) \in \mathbb{R}^{d-1} \times \mathbb{R},
\]

Then, it is clear that \( \|f\|_{L^p(\mathbb{R}^d)} \lesssim 1 \) for every \( p \), and we have by the spherical coordinates in \( \mathbb{R}^{d-1} \)
\[
\tilde{m}_\varepsilon(D)f(y, t) = \frac{1}{(2\pi)^d} \int e^{it\tau} \phi(\tau) \int \frac{\rho^{d-2} \phi(\rho) \tilde{d}\sigma_{d-2}(\rho y)}{(\rho^2 - 1 + \varepsilon^2 \tau^2 + 2\varepsilon \tau i)^k} d\rho d\tau.
\]

Integration by parts reduces the order \( k \) to 1 in the denominator. In fact, we notice that for any integer \( k \geq 2 \),
\[
\frac{1}{(\rho^2 - 1 + \varepsilon^2 \tau^2 + 2\varepsilon \tau i)^k} = -\frac{1}{2\rho(k-1)} \partial_\rho \left( \frac{1}{(\rho^2 - 1 + \varepsilon^2 \tau^2 + 2\varepsilon \tau i)^{k-1}} \right).
\]

Thus, integrating by parts \( (k-1) \)-times with respect to \( \rho \), we have
\[
(5.9) \quad \tilde{m}_\varepsilon(D)f(y, t) = c \int e^{it\tau} \phi(\tau) \int \frac{\Phi(\rho, y)}{\rho^2 - 1 + \varepsilon^2 \tau^2 + 2\varepsilon \tau i} d\rho d\tau
\]
for some constant \( c \), where
\[
\Phi(\rho, y) = T^{k-1}(\rho^{d-2} \phi(\rho) \tilde{d}\sigma_{d-2}(\rho y)) \quad \text{with} \quad Th := \partial_\rho(\rho^{-1} t).
\]

Let us recall the identity (see, for example, [3] Appendix B)
\[
\tilde{d}\sigma_{d-2}(\rho y) = c |\rho y|^{\frac{d-2}{2}} J_{d-2}(|\rho y|),
\]

where \( J_{\nu}(r) \) denotes the Bessel function of the first kind and has the following property (for \( \text{Re} \nu > -1/2 \))
\[
\partial_\nu (r^{-\nu} J_{\nu}(r)) = -r^{-\nu} J_{\nu+1}(r), \quad r > 0.
\]

\(^3\)Throughout the proof, we let \( c \) denote some positive constant which may differ at each occurrence depending only on \( k \) or \( d \).
Making use of these identities and the chain rule $\partial_p = |y|\partial_r$, $r = \rho|y|$, we have

\begin{equation}
\Phi(\rho, y) = \sum_{l=0}^{k-1} |py|^{\frac{3d}{2} - 2l - 1} J_{\frac{d-3}{2} + l}(\rho|y|)\phi_l(\rho)|y|^{2l}
\end{equation}

with $\phi_l \in C_0^\infty([1 - 2\delta_0, 1 + 2\delta_0])$. In particular $\phi_{k-1}(\rho) = c \rho^{d-2}\phi(\rho)$.

Setting

\[ J_l(y, t) = |y|^\frac{d-2k-1}{2} \int e^{it\tau} \phi(\tau) \int \frac{\rho^\frac{d}{2} \phi(\rho)}{\rho^2 - 1 + \varepsilon^2 \tau^2 + 2\varepsilon \tau i} J_{\frac{d-3}{2} + l}(|\rho y|) d\rho d\tau, \]

we may rewrite (5.9) as

\begin{equation}
\tilde{m}_\varepsilon(D) f(y, t) = c \sum_{l=0}^{k-1} J_l(y, t).
\end{equation}

In the rest of the proof, we aim to show that

\begin{equation}
|J_{k-1}(y, t)| \geq \varepsilon^4^{-k}
\end{equation}

on a subset $\{(y, t) \in \mathbb{R}^{d-1} \times \mathbb{R} : |y| \sim \varepsilon^{-1}, |t| \ll 1\}$, whereas for every $s > 0$

\begin{equation}
|J_l(y, t)| \lesssim \varepsilon^4^{-1-l-s}, \quad 0 \leq l \leq k - 2.
\end{equation}

Hence, in this set, $J_{k-1}$ would be the leading term and dominate the others in the summation (5.11).

First, let us analyze $J_{k-1}$. We shall use the asymptotic (see [9, p. 580])

\[ J_{\frac{d-3}{2} + k-1}(\rho|y|) = \sqrt{\frac{2}{\pi\rho|y|}} \cos \left( \rho|y| - \frac{\pi(d + 2k - 4)}{4} \right) + R(\rho|y|), \]

where $|R(\rho)| \lesssim \rho^{-\frac{d}{2}}$ for $\rho \geq 1$, and the formula

\[ \cos(u + v) = \cos u \cos v - \sin u \sin v \]

with $u = (\rho - 1)|y|$ and $v = |y| - \frac{\pi}{4}(d + 2k - 4)$. Using these we break $J_{k-1}$ as

\[ J_{k-1}(y, t) = \frac{2^{d-1}}{\pi} \int e^{it\tau} \phi(\tau) \int \frac{\rho^\frac{d}{2} \phi(\rho)}{\rho^2 - 1 + \varepsilon^2 \tau^2 + 2\varepsilon \tau i} J_{\frac{d-3}{2} + l}(|\rho y|) d\rho d\tau\]

where, setting $\alpha_{d,k} := \frac{\pi}{4}(d + 2k - 4)$, we define

\[ J_{k-1}^1 := \int e^{it\tau} \phi(\tau) \int \frac{\rho^\frac{d}{2} \phi(\rho) \cos((\rho - 1)|y|)}{\rho^2 - 1 + \varepsilon^2 \tau^2 + 2\varepsilon \tau i} d\rho d\tau, \]

\[ J_{k-1}^2 := \int e^{i\tau} \phi(\tau) \int \frac{\rho^\frac{d}{2} \phi(\rho) \sin((\rho - 1)|y|)}{\rho^2 - 1 + \varepsilon^2 \tau^2 + 2\varepsilon \tau i} d\rho d\tau, \]

\[ J_{k-1}^3 := \int e^{i\tau} \phi(\tau) \int \frac{\rho^\frac{d-2k+1}{2} \phi(\rho) R(\rho|y|)}{\rho^2 - 1 + \varepsilon^2 \tau^2 + 2\varepsilon \tau i} d\rho d\tau. \]

Regarding the lower bound for $|J_{k-1}^1|$, we estimate that for $|\Im J_{k-1}^1|$. Setting

\begin{equation}
\varphi(\rho) = \rho^\frac{d-2k}{2} \phi(\rho)
\end{equation}

and using the definitions in (5.1), we can write

\[ \Im J_{k-1}^1 = |y|^\frac{2k-d}{2} \cos(|y| - \alpha_{d,k}) \int \phi(\tau)(\sin(t\tau)I_2(\tau, y; \varepsilon) - \cos(t\tau)I_1(\tau, y; \varepsilon)) d\tau. \]

\footnote{See (5.18) below for a precise description.}
By the definition of $\phi$ the function $\varphi$ defined in (5.14) satisfies the assumption in Lemma 5.2. By the lemma if $\varepsilon$ is small enough, then we obtain the bounds

$$|I_1(\tau, y; \varepsilon)| \gtrsim 1 \quad \text{and} \quad |I_2(\tau, y; \varepsilon)| \lesssim 1$$

whenever $c_1 \varepsilon^{-1} \leq |y| \leq c_2 \varepsilon^{-1}$ for some constants $c_1$ and $c_2$. Thus, if $|t|$ is sufficiently small and $c_1 \varepsilon^{-1} \leq |y| \leq c_2 \varepsilon^{-1}$, then we conclude that

$$|\mathcal{J}_{k-1}(y, t)| \geq |\text{Im} \mathcal{J}_{k-1}(y, t)| \gtrsim |y|^{\frac{2d-1}{2}} \cos(|y| - \alpha_{d,k}).$$

On the other hand, using the definitions (5.1) and (5.14),

$$\mathcal{J}_{k-1}^2 = |y|^{\frac{2d-1}{2}} \sin(|y| - \alpha_{d,k}) \int e^{it\tau} \phi(\tau) (I_4(\tau, y; \varepsilon) - i I_3(\tau, y; \varepsilon)) d\tau.$$ 

Thanks to Lemmas 5.1 and 5.2 if $c_1 \varepsilon^{-1} \leq |y| \leq c_2 \varepsilon^{-1}$, then we have the estimate

$$|\mathcal{J}_{k-1}^2(y, t)| \lesssim |y|^{\frac{2d-1}{2}} \sin(|y| - \alpha_{d,k})|.$$ 

Moreover, since $|R(\rho)| \lesssim |\rho|^{-\frac{3}{2}}$ for $|\rho| \gtrsim 1/\rho \sim 1$, we see from Lemma 5.1 that

$$|\mathcal{J}_{k-1}^2(y, t)| \lesssim |y|^{\frac{2d-1}{2}} \int |\phi(\tau)||I_1(\tau; \varepsilon)| + |I_2(\tau; \varepsilon)| d\tau \lesssim |y|^{\frac{2d-1}{2}} \log \varepsilon^{-1}.$$ 

Combining the estimates (5.15), (5.16), and (5.17), if $|t| \ll 1$ and $y$ lies in the set

$$\mathcal{G} := \{ y \in \mathbb{R}^{d-1}: c_1 \varepsilon^{-1} \leq |y| \leq c_2 \varepsilon^{-1}, |y| - \alpha_{d,k} \in 2\pi \mathbb{Z} + [-c_0, c_0]\}$$

for a sufficiently small constant $c_0$, then

$$|\mathcal{J}_{k-1}(y, t)| \geq |\mathcal{J}_{k-1}^1(y, t)| - |\mathcal{J}_{k-1}^2(y, t)| - |\mathcal{J}_{k-1}^3(y, t)| \gtrsim |y|^{\frac{2d-1}{2}} \cos(|y| - \alpha_{d,k} - c|\sin(|y| - \alpha_{d,k})| - c|y|^{-1} \log \varepsilon^{-1}) \gtrsim \varepsilon^{\frac{2d}{2} - k}.$$ 

Thus the proof of the estimate (5.12) is complete.

Secondly, we prove the estimates (5.13). Since $|J_\rho(r)| \lesssim r^{-\frac{3}{2}}$ for $r \gtrsim 1$, Lemma 5.1 yields, for $s > 0$ and $y \in \mathcal{G}$,

$$|J_s(y, t)| \lesssim |y|^{\frac{2d-1}{2}} \int |\phi(\tau)||I_s(\tau; \varepsilon)| + |I_s(\tau; \varepsilon)| d\tau \lesssim |y|^{\frac{2d-1}{2}} (1 + \log \varepsilon^{-1}) \lesssim \varepsilon^{\frac{2d}{2} - 1 - s}.$$ 

To sum up, we have proved that if $|t| \ll 1$ and $y \in \mathcal{G}$ (with $c_0$ small enough), then

$$|\mathcal{J}_s(y, t)| \gtrsim \varepsilon^{\frac{2d}{2} - k}, \quad 0 < \varepsilon \ll 1.$$ 

Therefore, for some constant $\tilde{c} > 0$ small enough,

$$\|\tilde{m}_\varepsilon(D)f(y, t)\|_{L^\infty(\mathcal{G} \times [-\varepsilon, \varepsilon])} \gtrsim \varepsilon^{\frac{2d}{2} - k} |\mathcal{G}|^{\frac{1}{2}} \sim \varepsilon^{\frac{2d}{2} - k - \frac{2d}{2} - 1},$$ 

and the proof of (3.31) is completed. \qed

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(Eunhee Jeong) Department of Mathematics Education and Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju 54896, Republic of Korea
Email address: eunhee@jbnu.ac.kr

(Yehyun Kwon) School of Mathematics, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea
Email address: yhkwon@kias.re.kr

(Sanghyuk Lee) Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Republic of Korea
Email address: shklee@snu.ac.kr