SOME CONJECTURES IN ELEMENTARY NUMBER THEORY

ANGELO B. MINGARELLI

Abstract. We announce a number of conjectures associated with and arising from a study of primes and irrationals in \( \mathbb{R} \). All are supported by numerical verification to the extent possible.

The Conjectures

Bhargava factorials. For definitions and basic results dealing with Bhargava’s factorial functions we refer to \[3\], \[4\], \[5\] and \[8\]. Briefly, let \( X \subseteq \mathbb{Z} \) be a finite or infinite set of integers. Following \[5\], one can define the notion of a \( p \)-ordering on \( X \) and use it to define a set of generalized factorials of the set \( X \) inductively. By definition \( 0!_X = 1 \). Whenever \( p \) a prime, we fix an element \( a_0 \in X \) and, for \( k \geq 1 \), we select \( a_k \) such that the highest power of \( p \) dividing \( \prod_{i=0}^{k-1} (a_k - a_i) \) is minimized. The resulting sequence of \( a_i \) is then called a \( p \)-ordering of \( X \). As one can gather from the definition, \( p \)-orderings are not unique, as one can vary \( a_0 \). On the other hand, associated with such a \( p \)-ordering of \( X \) we define an associated \( p \)-sequence \( \{\nu_k(X,p)\}_{k=1}^{\infty} \) by

\[
\nu_k(X,p) = w_p(\prod_{i=0}^{k-1} (a_k - a_i)),
\]

where \( w_p(a) \) is, by definition, the highest power of \( p \) dividing \( a \) (e.g., \( w_2(80) = 16 \)). One can show that although the \( p \)-ordering is not unique the associated \( p \)-sequence is independent of the \( p \)-ordering used. Since this quantity is an invariant it can be used to define generalized factorials of \( X \) by setting

\[
k!_X = \prod_p \nu_k(X,p),
\]

where the (necessarily finite) product extends over all primes \( p \).

Definition 1. \[12\]. An abstract (or generalized) factorial is a function \( !_a : \mathbb{N} \rightarrow \mathbb{Z}^+ \) that satisfies the following conditions:

\[
(1) \quad 0!_a = 1,
\]
For every non-negative integers $n, k$, $0 \leq k \leq n$ the generalized binomial coefficients

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} \in \mathbb{Z}^+,$$

(2) For every positive integer $n$, $n!$ divides $n!_a$.

It is easy to see that the collection of all abstract factorials forms a commutative semigroup under ordinary pointwise multiplication. In fact, it is easy to see that Bhargava’s factorial function is an abstract factorial. (Indeed, Hypothesis 1 of Definition 1 is clear by definition of the factorial in question. Hypothesis 2 of Definition 1 follows by the results in [5].)

The context of these first three conjectures is the construction in [5] as applied to the ring of integers. In this case, the factorial function for the set of rational primes

$$\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}$$

is given by [5]

$$n!_p = \prod_p p^{\sum_{m=1}^{\infty} \frac{n^{m-1}}{(p-1)m}}. \tag{2}$$

We call this simply the B-factorial for the set under consideration. In the sequel, the statement “For every $n \geq 1$” means “for every integer $n \geq 1$ for which the factorials are defined”.

Let $\mathbb{P}_2 \subset \mathbb{P}$ denote the subset of all twin primes, i.e., those primes of the form $p, p+2$ as usual. Let $n!_{\mathbb{P}_2}$ denote the B-factorial of the set $\mathbb{P}_2$. In the following conjectures the notation $w_p(n)$ is used to identify the highest power of $p$ that divides $n$. So, for example, if $n$ has the representation $n = 2^a \alpha$ and $(\alpha, 2) = 1$, then $w_2(n) = 2^a$.

**Conjecture 1.** For every $n \geq 1$,

$$\frac{n!_{\mathbb{P}_2}}{n!_p} = 2 w_2(n).$$

In analogy with the preceding we let $\mathbb{P}_3 \subset \mathbb{P}$ denote that subset of all prime triplets of the form $p, p+2, p+6. Let n!_{\mathbb{P}_3}$ denote the B-factorial of the set $\mathbb{P}_3$.

**Conjecture 2.** For every $n \geq 1$,

$$\frac{n!_{\mathbb{P}_3}}{n!_p} = \begin{cases} 3! w_2(n) w_3(n), & \text{if } n \text{ is even}, \\ 2, & \text{if } n \text{ is odd}. \end{cases}$$

Next, let $\mathbb{P}_4 \subset \mathbb{P}$ denote that subset of all prime quadruplets written in the form $p, p+2, p+6, p+8$. Since $p, p+2$ and $p+6, p+8$ are both twin primes we can view $\mathbb{P}_4 \subset \mathbb{P}_2$, and so we must have $n!_{\mathbb{P}_2} n!_{\mathbb{P}_4}$, by [5], Lemma 13. In fact, we claim that,

**Conjecture 3.** For every $n \geq 1$,

$$\frac{n!_{\mathbb{P}_4}}{n!_{\mathbb{P}_2}} = \begin{cases} 3 w_3(n), & \text{if } n \text{ is even}, \\ 1, & \text{if } n \text{ is odd}. \end{cases}$$

These three conjectures have been verified using Crabbe’s algorithm [10] to the limits available by the hardware. For motivation see [12].
Prime number inequalities. Now let \( p_n \) denote the \( n \)-th prime. Then, see [12],

Conjecture 4.

\[
p_n \geq p_k + p_{n-k-1}, \quad 1 \leq k \leq n - 1,
\]

and all \( n \geq 2 \).

The validity of this conjecture implies that the function \( f: \mathbb{N} \to \mathbb{Z}^+ \),

\[
f(n) = \begin{cases}
1, & \text{if } n = 0, \\
1, & \text{if } n = 1, \\
p_{n-1}!, & \text{if } n \geq 2,
\end{cases}
\]

is an abstract factorial. Thus, if true, it would follow from the results in [12] that for any abstract factorial \( n!_a \), the quantity \( \sum_{n \geq 1} 1/n!_a f(n) \notin \mathbb{Q} \).

Apéry numbers. We define the Apéry numbers \( A_n, B_n \) recursively, as usual, by setting \( A_0 = 1, A_1 = 5; B_0 = 0, B_1 = 6 \) whose general terms are given by the recurrence relations

\[
A_{n+1} = (P(n)A_n - n^3A_{n-1})/(n + 1)^3,
\]

and

\[
B_{n+1} = (P(n)B_n - n^3B_{n-1})/(n + 1)^3,
\]

where \( P(n) \) is the polynomial

\[
P(n) = 34n^3 + 51n^2 + 27n + 5.
\]

In a singular argument Apéry [1] showed that \( B_n/A_n \to \zeta(3) \) as \( n \to \infty \) where \( \zeta \) is the usual Riemann zeta function. In addition, he proved that \( \zeta(3) \) is irrational (though no explicit formula akin to the one known for the values of \( \zeta \) at positive even integers was given). More explicit proofs appeared since, e.g., [2], [14], [9] among others. (See [13] for extensions of the series acceleration method found in [Fischler [11], Remarque 1.3] to integer powers of \( \zeta(3) \).)

Here we propose using an old irrationality criterion due to Brun [6] (see also [7]) in order to formulate a conjecture that, if true, would give another proof of the irrationality of \( \zeta(3) \). Let \( x_n \) be a sequence of real numbers and \( \Delta \) the forward difference operator defined by \( \Delta x_n = x_{n+1} - x_n \).

**Theorem 2.** (Brun, [6]) Let \( x_n \in \mathbb{Z}^+ \) be an increasing sequence and \( y_n \in \mathbb{Z}^+ \) be such that \( \Delta(y_n/x_n) > 0 \). If

\[
\delta_n \equiv \Delta(\Delta y_n/\Delta x_n) < 0,
\]

then \( y_n/x_n \) converges to an irrational number.

Although Brun claimed later [7] that “... this theorem is simple but unfortunately not very useful” we show that perhaps it may be used to prove the irrationality of \( \zeta(3) \).

The idea is as follows: It is known that the sequence \( A_n \) of Apéry numbers is an increasing sequence of positive integers [9] and although the \( B_n \) is not necessarily a sequence of integers, the weighted sequence \( c_nB_n \) is such a sequence where
\[ e_n = 2 \cdot (\text{lcm}\{1, 2, \ldots, n\})^3, \] [9]. In addition, the sequence \( B_n/A_n = e_nB_n/e_nA_n \) is increasing, [9] and it is easily proved that the sequence \( e_nA_n \) is increasing as well.

Thus, setting \( x_n = e_nA_n \) and \( y_n = e_nB_n \) we see that the requirements \( x_n \) is increasing and \( y_n/x_n \) increasing are met in Theorem 2 (all sequences being positive and all integers). We anticipate the following

**Conjecture 5.** There is an unbounded subsequence of positive integers \( n_k \to \infty \) such that \( \delta_{n_k} < 0 \).

Since it is known that \( y_n/x_n \) increases to \( \zeta(3) \), clearly \( y_{n_k}/x_{n_k} \) does the same for any subsequence. Hence, an affirmative answer to the previous conjecture implies the irrationality of \( \zeta(3) \) by Brun’s irrationality theorem, Theorem 2. The numerical evidence seems to point to a stronger conjecture however. Indeed, it appears as if

**Conjecture 6.** For every integer \( N \geq 2 \), there is an \( n \in \mathbb{Z}^+ \) such that all

\[ \delta_n, \delta_{n+1}, \delta_{n+2}, \ldots, \delta_{n+N} < 0. \]

Of course this result, if true, implies the previous conjecture.

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School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada, K1S 5B6

E-mail address, A. B. Mingarelli: amingare@math.carleton.ca