Mathematical model and computational experiments for the calculation of three-layer plates of complex configuration

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Abstract. Mathematical model and computational experiments to solve the problems of bending of three-layer plates of complex configuration are considered in the paper. Mathematical model and computational algorithm given in the paper are developed on the basis of the Hamilton-Ostrogradsky’s variation principle, the Bubnov-Galerkin methods and the Rvachev R-functions. Numerical results of the problem are presented. Results are given in tabular and graphical form. Numerical studies of the results obtained are analyzed using the R-function method in combination with the Bubnov-Galerkin variational method to solve the problem of calculating the vibration of three-layer plates of complex configuration.

1. Introduction
Currently, three-layer structures are widely used in various fields of modern structures. They are important structural elements in modern construction, mechanical engineering, etc. Layered structures are widely used in modern technology. Everyday practice is constantly putting forward new tasks in the area of design of such structures and imposes ever-increasing demands on the reliability of calculations, thus stimulating the development of the theory of three-layer structures.

Wide use of elastic elements of three-layer structures in modern technology requires the development of mathematical models for the study of their statics and dynamics. The problems of element oscillations of multilayer structures under static and dynamic loads are well studied for the structures of round or rectangular configuration. However, an elastic behavior of three-layer structures of complex configuration has been studied insufficiently.

Based on these judgments, a mathematical model, calculation algorithms for three-layer plate oscillations have been considered in the paper; computational experiments to solve the problems of bending of three-layer plates of complex configuration have been conducted.

In recent years, a significant contribution to the study of theoretical and applied issues of mathematical modeling of multilayer structures has been done by the following scientists:
A.Ya.Aleksandrov, N.A.Alfutov, S.A.Ambartsumyan, A.N. Andreev, A.E.Bogdanovich, K.Z.Galimov, V.V. Bolotin, Kh.M.Mushtari, A.T. Vasilenko, V.V. Vasilyev, V.E. Verijenko, K.Z. Galimov, E.I.Grigolyuk, Ya.M.Grigorenko, G.M.Kulikov, A.K.Malmeister, V.L.Narusberg, Yu.V.Nemirovsky, Yu.N.Novichkov, I.F.Obraztsov, V.V.Pikut, A.O.Raskazov, R.Kristansen, L. Librescu, J. Reddy, E. Reisner, S.P.Timoshenko, L.V.Kurpa, A.V. Bogdanov and many others, including the scientists of the Republic of Uzbekistan: V.K.Kabulov, T.Sh.Shirinkulov, Sh.A.Nazirov, F.B. Badalov and others. Studies of other scientists are devoted to various aspects of informatization and software development. However, investigations in elastic behavior of three-layer structures of complex configuration have not been sufficiently studied.

The method considered in this paper consists in constructing a special coordinate sequence of functions that allows one to precisely satisfy different boundary conditions with practically arbitrary geometry of sections.

L.V. Kurpa has studied the theory of R-functions for the solution of linear problems of bending and oscillations of single-layer and multilayer plates of arbitrary configuration in plan and various ways of fixing. Mathematical statement of the problems of bending and free vibrations of multilayer plates is considered within the framework of the classical theory and the refined theories of the first and second orders taking into account shear strains. In [1], various variation statements of problem are considered as well as special applications of the theory of R-functions to this class of problems and numerical solutions of the problems of bending of composite plates.

To solve the problems of physical-mathematical fields of complex configuration, V.L.Rvachev [2, 7] has developed a new method - the R-function method, which allows the construction of coordinate sequences for the fields of practically arbitrary configuration and boundary conditions of complex form. The use of the theory of R-functions apparatus allows one to obtain a solution in the form of a formula, called a decision structure, containing an explicit dependence on geometric and physical parameters. The obtained theoretical results have been used to create a modern programming technology in mathematical physics for the implementation of problem-oriented languages created under the guidance of V.L.Rvachev. The R-function method has also been successfully applied to calculate the plates of various configuration by Sh.A.Nazirov [8, 9], F.M.Nuraliev [18, 20], Sh.A.Anarova [21, 23], O.T.Amanov [24] and their followers.

In [18], numerical-analytical methods and algorithms for solving partial differential equations with initial-boundary conditions were developed that describe the effect of electromagnetic fields on the strain state of thin electrically conductive bodies of complex configuration at combined using the Bubnov-Galerkin variational method and the structural method of R-functions; resolving equations are obtained. Solution structures are formed by the R-function method for the basic conditions of problems of magnetoelastic plates and shells having a complex configuration; normalized equations for complex sections of thin bodies are constructed using card operations of algebraic theories of R-functions. Numerical-analytical methods and the reliability of calculation results of the magnetoelasticity of thin plates for sections of classical configuration have been developed by comparing exact and approximate solutions by the R-function method; the plates with rigidly clamped and hinged-supported boundary conditions were considered. The convergence of a computational algorithm for calculating magnetoelastic thin shells and plates with a complex structural configuration with respect to the number of coordinate functions in solution structure constructed by the R-function method was investigated.

In [24], theoretical statements of flexible three-layer reinforced concrete elements with a middle layer from wood concrete (arbolite) of up to 1 MPa strength were developed using mathematical modeling. A new mathematical model of three-layer reinforced concrete structures based on the Hamilton-Ostrogradsky’s variation principle has been created, a computational calculation algorithm has been developed at combined application of the Bubnov-Galerkin variation method and the R-function structural method (RFM), as well as a software package
that automates the process of studying three-layer structures. And in work [25] some problems of mathematical modeling of multidimensional problems of nonlinear heat conduction are considered.

2. Problem statement

Basic equations of the theory of elasticity of multilayer plates are derived on the basis of the Kirchhoff-Love theory and the Hamilton-Ostrogradsky’s variation principle [10, 11]. When constructing specific models, the Cauchy geometric relations and physical relations are used, in the inverse form of the Hooke law, as well as the law of change in displacements [12].

Consider a multilayer plate shown in Figure 1. Here we will assume that some layers of a multilayer plate can be interpreted as isotropic elastic plates obeying the Kirchhoff-Love hypothesis. The remaining layers are considered transversally soft; their contribution to the potential energy of strain of the stress layers $\sigma_{11}, \sigma_{22}$ and $\sigma_{12}$ is negligible compared to the contribution of the stresses $\tau_{13}, \tau_{23}$ and $\sigma_{33}$, so, the density of potential energy of strain is determined by formula [13]

$$u = \frac{1}{2} \left( \tau_{13} \gamma_{13} + \tau_{23} \gamma_{23} + \sigma_{33} \varepsilon_{33} \right) + \bar{E} \varepsilon^{2} O (\delta)$$

The case of soft layers is easily obtained by limit transition. For brevity, we will call transversally soft layers simply soft layers.

Combine the adjacent rigid layers: the resulting layers are also rigid, but due to the variability of elastic characteristics they are inhomogeneous in thickness. The Kirchhoff-Love hypothesis remains valid for combined rigid layers. Similarly, combine the adjacent soft layers.

Hereinafter, rigid and soft layers are understood to be precisely these combined inhomogeneous layers. If the extreme layer is soft, then its strain energy is, as a rule, negligible compared to the strain energy of the adjacent rigid layer. Therefore, without any particular loss of generality, the extreme layers can be considered to be rigid ones.

Let $n$ be the number of rigid layers. Number the rigid layers, starting from one of the outer planes of the plate: $k = 1, 2, \ldots, n$. Similarly, number the soft layers located between them: $k = 1, 2, \ldots, n-1$. The indices indicating the number of a rigid layer are enclosed in parentheses, the indices indicating the number of a soft layer in square brackets. This makes it easy to distinguish between parameters and values related to the layers of different rigidity. The thickness of rigid layers is denoted by $h_{(k)}$, the elastic modulus by $E_{(k)}$, the Poissons ratios by $\nu_{(k)}$. Assume that $E_{(k)} = E_{(k)}(z)$. Assume that the Poissons ratio $\nu_{(k)}$ for each rigid layer is constant; the middle surface of each rigid layer is chosen from the condition [13]
where $z^{(k)}$ — is the distance measured by a normal to the middle surface of the layer. Here the integral is taken over the entire thickness of the layer.

The stress-strain state of a rigid layer is fully defined if the normal $w^{(k)}(x_1, x_2)$ and tangential $v_1^{(k)}(x_1, x_2)$ and $v_2^{(k)}(x_1, x_2)$ displacements of the points belonging to the middle surface are known. Based on the Kirchhoff-Love hypothesis, the displacements $v_j^{(k)}(x_1, x_2, z)$ ($j = 1, 2, 3$) of arbitrary points of this layer are determined by formulas:

$$ u_3^{(k)} = v_3^{(k)} = w^{(k)}. \quad (2) $$

The presence of a free index $\alpha$ means that the first formula replaces the two corresponding formulas at $\alpha = 1, 2$. Using these formulas, the strain components of a rigid layer in its middle surface is obtained [13]:

$$ \varepsilon^{(k)}_{\alpha\beta} = \frac{1}{2} \left( \frac{\partial v_1^{(k)}}{\partial x_\beta} + \frac{\partial v_2^{(k)}}{\partial x_\alpha} \right) - z^{(k)} \frac{\partial^2 w^{(k)}}{\partial x_\alpha \partial x_\beta}. \quad (3) $$

This formula replaces the three corresponding formulas at $\alpha = 1, 2, \beta = 1, 2$.

Taking into account formulas (3), it is easy to obtain the expressions for the corresponding components of the stress tensor

$$ \sigma_{11}^{(k)} = \frac{E^{(k)}}{1 - \nu^2} \left( \frac{\partial v_1^{(k)}}{\partial x_1} + v_2^{(k)} \frac{\partial v_2^{(k)}}{\partial x_2} \right) - \frac{E^{(k)} z^{(k)}}{1 - \nu^2} \left( \frac{\partial w_1^{(k)}}{\partial x_1} + v_1^{(k)} \frac{\partial w_2^{(k)}}{\partial x_2} \right), \quad (4) $$

where (1 $\leftrightarrow$ 2)

$$ \tau_{12}^{(k)} = \frac{E^{(k)} z^{(k)}}{2(1 + \nu^{(k)})} \left( \frac{\partial v_1^{(k)}}{\partial x_2} + \frac{\partial v_2^{(k)}}{\partial x_1} \right) - \frac{E^{(k)} z^{(k)} \partial^2 w^{(k)}}{1 + \nu^{(k)}}, \quad (5) $$

The expression 1 $\leftrightarrow$ 2 means that the first of the formulas (5) corresponds to two formulas: the second is obtained from the first by circular replacement of indices 1 and 2.

Consider the assumptions for soft layers. The simplest of these is that for all components of the displacement vector, the linear law of variation in thickness is taken. Then the displacements at an arbitrary point of the soft layer are [12]:

$$ u_3^{[k]} = u_3^{[k]} + z^{[k]} \xi_3^{[k]}; \quad u_3^{[k]} = w^{[k]} + z^{[k]} \xi_3^{[k]} \quad (6) $$

Here $v_1^{[k]}, v_2^{[k]}$ and $w^{[k]}$ — are the displacements of the middle surface of soft layer; $\xi_1^{[k]}, \xi_2^{[k]}$ and $\xi_3^{[k]}$ — are normal components of the gradient from these displacements.

Taking into account formulas (2), expressions of strains in the soft layer are obtained:

$$ \varepsilon_{\alpha\beta}^{[k]} = \frac{1}{h^{[k]}} \left( v_3^{[k+1]} - v_3^{[k]} + c' k \frac{\partial w^{[k]}}{\partial x_\alpha} + c'' k \frac{\partial w^{[k+1]}}{\partial x_\alpha} \right); \quad \varepsilon_{33}^{[k]} = \frac{1}{h^{[k]}} \left( w^{[k+1]} - w^{[k]} \right), \quad (7) $$

where $c'_k$ and $c''_k$ are the distances from the middle plane of the k-th soft layer to the middle planes of the k-th and (k+1)-th rigid layers, respectively (Figure 2):
\[ c'_k = \frac{1}{2}(h_{(k)} + h_{[k]}); \quad c''_k = \frac{1}{2}(h_{(k+1)} + h_{[k]}). \] (8)

It follows from formulas (7) that the strains of the soft layer are completely determined by the displacements of the rigid layers bounding this layer.

If the soft layer is non-uniform in thickness, then significant deviations from assumptions (6) can be observed. Then, instead of these assumptions, it is better to introduce a hypothesis of a uniform distribution of tangential stresses \( \tau_{13} \) and \( \tau_{23} \) along the thickness of \( h_{[k]} \). The error of this assumption, obviously, is less the larger the layer and the lower its elastic characteristics compared to the elastic characteristics of adjacent rigid layers. Under this assumption, the stresses in the soft layer can be defined as

\[
\tau^{[k]}_{\alpha3} = B_k \bar{\gamma}^{[k]}_{\alpha3}, \quad text{t} = \sigma^{[k]}_{33} = C_k \bar{\varepsilon}^{[k]}_{33},
\] (9)

where \( \bar{\gamma}^{[k]}_{\alpha3} \) and \( \bar{\varepsilon}^{[k]}_{33} \) — are the thickness-averaged strains (Figure 3); \( B_k \) and \( C_k \) are the reduced rigidities determined by the following conditions [13]:

\[
\int_{h_{[k]}} \tau^{[k]}_{\alpha3} dz = \bar{\gamma}^{[k]}_{\alpha3} h_{[k]}; \quad \int_{h_{[k]}} \sigma^{[k]}_{33} dz = \bar{\varepsilon}^{[k]}_{33} h_{[k]}
\] (10)

Substituting the average strains obtained by formulas (7) into expressions (10), and taking into account (9), we obtain the formulas for the stresses

\[
\tau^{[k]}_{\alpha3} = B_k \left( v_{(k+1)}^{(k)} - v_{(k)}^{(k)} + c'_k \frac{\partial w^{(k)}}{\partial x_\alpha} + c''_k \frac{\partial w^{(k+1)}}{\partial x_\alpha} \right); \quad \sigma^{[k]}_{33} = C_k (w^{(k+1)} - w^{(k)}).
\] (11)

For the given rigidities the following relationships are obtained

\[
\frac{1}{B_k} = \int_{h_{[k]}} \frac{dz}{G(z)}; \quad \frac{1}{C_k} = \int_{h_{[k]}} \frac{dz}{E(z)}
\] (12)

Note that the transversal modulus in the general case does not coincide with the elastic modulus of material of the soft layer. If a layer can be considered transversally soft, conditions \( \varepsilon_{11} \ll \varepsilon_{33}, \quad \varepsilon_{22} \ll \varepsilon_{33} \), are met, the soft layer compression occurs almost without its expansion.
obtained in the form $[12, 14, 15]$:

$$E_z = \frac{(1 - v)E}{(1 + v)(1 - 2v)}.$$  

Obviously, for a layer to be considered transversely soft, the material must possess sufficient compressibility.

3. Mathematical model

Three-layer plates are a special case of multilayer plates, their mathematical models are obtained according to the Hamilton-Ostrogradky's variation principle given in $[9, 12, 14]$:

$$\delta \int_t (K - \Pi + A) \, dt = 0, \quad (13)$$

where $K$ is the kinetic energy of the system, $\Pi$ is the potential energy of the system, and $A$ is the potential of external forces, $\delta$ is the operation of variation.

Consider the case when

$$K = \int \int \int \rho \left[ \left( \frac{\partial u^{(k)}_1}{\partial t} \right)^2 + \left( \frac{\partial u^{(k)}_2}{\partial t} \right)^2 + \left( \frac{\partial u^{(k)}_3}{\partial t} \right)^2 + \left( \frac{\partial u^{(k)}_{12}}{\partial t} \right)^2 \right] \, dx \, dy \, dz,$$

$$\Pi = \int \int \left[ \left( \sigma^{(k)}_{11} \varepsilon^{(k)}_{11} + \sigma^{(k)}_{22} \varepsilon^{(k)}_{22} + \tau^{(k)}_{12} \varepsilon^{(k)}_{12} \right) + \left( \varepsilon^{(k)}_{13} \gamma^{(k)}_{13} + \varepsilon^{(k)}_{23} \gamma^{(k)}_{23} + \varepsilon^{(k)}_{33} \gamma^{(k)}_{33} \right) \right] \, dV, \quad (14)$$

$$A = -\sum_{k=1}^{n} \int \left( q_1^{(k)} \nu_n^{(k)} + q_2^{(k)} \nu_l^{(k)} + q_3^{(k)} w^{(k)} \right) \, dV,$$

$$-\sum_{k=1}^{n} \int \left[ N_n^{(k)} \nu_n^{(k)} + N_l^{(k)} \nu_l^{(k)} - Q^{(k)} w^{(k)} + M_n^{(k)} \frac{\partial w^{(k)}}{\partial n} + M_l^{(k)} \frac{\partial w^{(k)}}{\partial l} \right] \, d.$$

Here, the displacements $u_i^{(k)} (i = 1,3)$ correspond to the rigid layers; $u_i^{[k]} (i = 1,3)$ to the soft layers; $\sigma_{ij}^{(k)}$ is the stress component of the rigid layer; $\varepsilon_{ij}^{(k)}$ is the strain component of the rigid layer; $\tau_{ij}^{[k]}$ is the stress component of the soft layer; $\gamma_{ij}^{[k]}$ is the strain component of the soft layer; $k$ is the number of layers; $q_i^{(k)} (i = 1,3)$ is the external load; $N_n^{(k)}, N_l^{(k)}$ are the acting external transverse force; $M_n^{(k)}, M_l^{(k)}$ are the external bending and torques; $\Gamma$ is the boundary field.

Mathematical models of n-layered anisotropic plates have been studied in $[13]$; if to limit the number of rigid layers by $n = 2$, then the equations of motion for three-layer plates are obtained.

Consider a special case when material of the rigid layers is orthotropic, and the main directions coincide with the direction of the axes of coordinates. The equations for three-layer plates are obtained in the form $[12, 14, 15]$: 

in the plane of the layer. In this case, the transversal modulus $E_z$ is related to the modulus of elasticity $E$ and the Poisson’s ratio $v$ by dependence
\[
\begin{align*}
\begin{aligned}
A_1^{(1)} \frac{\partial^2 v_{1}^{(1)}}{\partial y^2} + A_2^{(1)} \frac{\partial^2 v_{1}^{(1)}}{\partial x^2} + A_3^{(1)} \frac{\partial^2 v_{2}^{(1)}}{\partial x \partial y} + B_1 \left( v_{1}^{(2)} - v_{1}^{(1)} \right) + B_1 c \frac{\partial w}{\partial x} - q_1^{(1)} &= 0, \\
A_1^{(2)} \frac{\partial^2 v_{1}^{(2)}}{\partial y^2} + A_2^{(2)} \frac{\partial^2 v_{1}^{(2)}}{\partial x^2} + A_3^{(2)} \frac{\partial^2 v_{2}^{(2)}}{\partial x \partial y} - B_1 \left( v_{1}^{(2)} - v_{1}^{(1)} \right) - B_1 c \frac{\partial w}{\partial x} - q_1^{(2)} &= 0, \\
A_2^{(2)} \frac{\partial^2 v_{1}^{(2)}}{\partial y^2} + A_3^{(2)} \frac{\partial^2 v_{2}^{(2)}}{\partial x^2} + A_1^{(2)} \frac{\partial^2 v_{1}^{(2)}}{\partial x \partial y} + B_1 \left( v_{1}^{(2)} - v_{1}^{(1)} \right) + B_1 c \frac{\partial w}{\partial y} - q_2^{(2)} &= 0,
\end{aligned}
\end{align*}
\]

Here \( D_0 = \frac{E h^3}{12(1-v^2)} \); \( D_k = \frac{G h^3}{12} \); \( D_3 = 2D_k + \nu D_1 \); \( A_k^{(1)} = \frac{E h^3}{12(1-v^2)} \); \( A_k^{(k)} = G h \); \( A_3^{(k)} = A_k^{(k)} + \nu A_1^{(k)} \); \( B = \frac{G(z)}{h} \); \( \nu = \frac{h_k}{h} \) \( (k = 1, 2, \alpha = 1, 2) \),

where \( w(x, y) \) is the normal displacement, \( v_{1}^{(k)}(x, y) \) and \( v_{2}^{(k)}(x, y) (k = 1, 2) \) are the tangential displacements, \( h \) is the thickness of the plate; \( E \) is the modulus of elasticity; \( \nu \) - is the Poisson's ratio; \( G \) is the shear modulus for the main directions; \( q_1^{(1)}, q_1^{(2)}, q_2^{(1)}, q_2^{(2)}, q_3 \) are the external loads.

First, in the equilibrium equations (15) we use the dimensionless coordinates, considering \( \bar{x} = ax, \bar{y} = by, \bar{h} = h_0 h, \bar{v}_1^{(1)} = hv_1^{(1)}, \bar{v}_1^{(2)} = hv_1^{(2)}, \bar{v}_2^{(1)} = hv_2^{(1)}, \bar{v}_2^{(2)} = hv_2^{(2)}, \bar{w} = hw \)
Equilibrium equation (16) in vector - matrix form is written as

$$SU = Q$$

where \(U(\varphi_1^{(1)}, \varphi_1^{(2)}, \varphi_2^{(1)}, \varphi_2^{(2)}; w)\) are the displacements, \(S\) is the matrix-operator of the form

\[ S = S_1 \frac{\partial^4}{\partial x^4} + S_2 \frac{\partial^4}{\partial x^2 \partial y^2} + S_3 \frac{\partial^4}{\partial y^4} + S_4 \frac{\partial^2}{\partial x^2} + S_5 \frac{\partial^2}{\partial x \partial y} + S_6 \frac{\partial^2}{\partial y^2} + S_7 \frac{\partial}{\partial x} + S_8 \frac{\partial}{\partial y} + S_9 u. \]

Below the basic boundary conditions for some ways of fixing the edges are given.

1. Rigidly fixed edge:

$$u^{(k)}_1 \bigg|_\Gamma = u^{(k)}_2 \bigg|_\Gamma = v^{(k)}_1 \bigg|_\Gamma = v^{(k)}_2 \bigg|_\Gamma = w^{(k)}_1 \bigg|_\Gamma = w^{(k)}_2 \bigg|_\Gamma = 0, \quad k = \Gamma_1, 2$$

$$\frac{\partial u^{(k)}_1}{\partial n} \bigg|_\Gamma = \frac{\partial u^{(k)}_2}{\partial n} \bigg|_\Gamma = 0, \quad k = \Gamma_1, 2$$

2. Sliding fixing:

$$w^{(k)}_1 \bigg|_\Gamma = w^{(k)}_2 \bigg|_\Gamma = v^{(k)}_1 \bigg|_\Gamma = v^{(k)}_2 \bigg|_\Gamma = w^{(k)}_1 \bigg|_\Gamma = w^{(k)}_2 \bigg|_\Gamma = 0, \quad k = \Gamma_1, 2;$$

$$\sigma^{(k)}_n \bigg|_\Gamma = \tau^{(k)}_n \bigg|_\Gamma = 0, \quad k = \Gamma_1, 2.$$  

3. Sliding hinge:

$$w^{(k)}_1 \bigg|_\Gamma = w^{(k)}_2 \bigg|_\Gamma = 0, \quad k = \Gamma_1, 2;$$

$$M_n \bigg|_\Gamma = 0; \quad \sigma_n \bigg|_\Gamma = \tau_n \bigg|_\Gamma = 0.$$

4. Fixed hinge:

$$u^{(k)}_1 = u^{(k)}_2 = v^{(k)}_1 = v^{(k)}_2 = 0, \quad k = \Gamma_1, 2; \quad M_n \bigg|_\Gamma = 0.$$

5. Free edge:

$$M_n \bigg|_\Gamma = 0; \quad (Q_n + \frac{\partial}{\partial x} M_{n\tau}) \bigg|_\Gamma = 0,$$

where \(\sigma_n = \sigma_1 l_1^2 + \sigma_2 l_2^2 + 2\sigma_1 l_1 l_2, \quad \tau_n = \tau_1 (l_1^2 - l_2^2) + (\sigma_2 - \sigma_1) l_1 l_2, \quad M_n = (M_1 l_1^2 + 2M_2 l_1 l_2 + M_3 l_2^2), \quad M_{n\tau} = M_1 (l_1^2 - l_2^2) + (M_2 - M_1) l_1 l_2, \quad Q_n = Q_1 l_1 + Q_2 l_2. \)

In practice, there are combinations of the above boundary conditions (3) - (7), when the edges of the plate are fixed in different ways.

4. **Computational algorithm**

Three-layer plates are a special case of multilayer plates, their mathematical models are obtained according to the Hamilton-Ostrogradsky’s variation principle given in [9,12,14]:

$$\delta \int_t (K - \Pi + A) \, dt = 0, \quad (17)$$

where \(K\) is the kinetic energy of the system, \(\Pi\) is the potential energy of the system, \(A\) is the potential of external forces, \(\delta\) is the operation of variation.

Consider the case when
\[ K = \iiint_{x'y'z'} \left[ \frac{\partial u^{(k)}_1}{\partial t} \right]^2 + \left[ \frac{\partial u^{(k)}_2}{\partial t} \right]^2 + \left[ \frac{\partial u^{(k)}_3}{\partial t} \right]^2 \right] \, dx \, dy \, dz, \]

\[ \Pi = \int_{V} \left[ \left( \sigma^{(k)}_{11} \varepsilon^{(k)}_{11} + \sigma^{(k)}_{22} \varepsilon^{(k)}_{22} + \tau^{(k)}_{12} \varepsilon^{(k)}_{12} \right) + \left( \frac{[k]}{\gamma^{(k)}_{13} + [k]} + \frac{[k]}{\gamma^{(k)}_{23} + [k]} \right) \right] dV, \]

Here, the displacements \( u^{(k)}_i \) correspond to the rigid layers; \( u^{[k]}_i \) to the soft layers; \( \sigma^{(k)}_{ij} \) is the stress component of the rigid layer; \( \varepsilon^{(k)}_{ij} \) is the strain component of the rigid layer; \( \tau^{(k)}_{ij} \) is the stress component of the soft layer; \( \gamma^{(k)}_{ij} \) is the strain component of the soft layer; \( k \) is the number of layers; \( q^{(k)}_i \) is the external load; \( N^{(k)}_n \), \( N^{(k)}_t \) are the acting external forces; \( Q^{(k)} \) is the external transverse force; \( M^{(k)}_n \), \( M^{(k)}_t \) are the external bending and torques; \( \Gamma \) is the boundary field.

Mathematical models of n-layered anisotropic plates have been studied in [13]; if to limit the number of rigid layers by \( n = 2 \), then the equations of motion for three-layer plates are obtained.

Consider a special case when material of the rigid layers is orthotropic, and the main directions coincide with the direction of the axes of coordinates. The equations for three-layer plates are obtained in the form [12, 14, 15]:

\[ \begin{align*}
A^{(1)}_1 \frac{\partial^2 v^{(1)}_1}{\partial x^2} + A^{(2)}_2 \frac{\partial^2 v^{(2)}_1}{\partial y^2} + A^{(3)}_3 \frac{\partial^2 v^{(3)}_1}{\partial x \partial y} + B_1 \left( v^{(2)}_1 - v^{(1)}_1 \right) + B_1 c \frac{\partial w}{\partial x} - q^{(1)}_1 &= 0, \\
A^{(2)}_1 \frac{\partial^2 v^{(2)}_1}{\partial x^2} + A^{(3)}_2 \frac{\partial^2 v^{(3)}_1}{\partial y^2} + A^{(3)}_3 \frac{\partial^2 v^{(3)}_1}{\partial x \partial y} - B_1 \left( v^{(2)}_1 - v^{(1)}_1 \right) - B_1 c \frac{\partial w}{\partial x} - q^{(2)}_1 &= 0, \\
A^{(3)}_1 \frac{\partial^2 v^{(3)}_1}{\partial y^2} + A^{(2)}_2 \frac{\partial^2 v^{(2)}_1}{\partial y^2} + A^{(3)}_3 \frac{\partial^2 v^{(3)}_1}{\partial x \partial y} + B_1 \left( v^{(2)}_1 - v^{(1)}_1 \right) + B_1 c \frac{\partial w}{\partial y} - q^{(2)}_3 &= 0, \\
2D_1 \frac{\partial^4 w}{\partial x^4} + 2D_2 \frac{\partial^4 w}{\partial y^4} + 4D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} - B c \left[ \frac{\partial}{\partial x} \left( v^{(2)}_1 - v^{(1)}_1 \right) + \frac{\partial}{\partial y} \left( v^{(2)}_2 - v^{(1)}_2 \right) \right] + \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - q_3 &= 0.
\end{align*} \]

Here \( D_k = \frac{E_k h^3}{12(1-\nu^2)}; \ D_k = \frac{G_k h^3}{12}; \ D_3 = 2D_k + \nu D_1; \ A^{(k)}_n = \frac{E^{(k)}_n h}{1-\nu^2}; \ A^{(k)}_k = G^{(k)} h^{(k)}; \ A^{(k)}_3 = A^{(k)}_k + \nu A^{(k)}_1; \ B = \frac{G}{h}; \ c = \frac{1}{2} \left( h^{(k)} - h^{(k)} \right); \ (k = 1, 2; \ \alpha = 1, 2),

where \( w(x, y) \) is the normal displacement, \( v^{(k)}_1(x, y) \) and \( v^{(k)}_2(x, y) \) are the tangential displacements, \( h \) is the thickness of the plate; \( E \) is the modulus of elasticity; \( \nu \) - is the Poissons
ratio; \( G \) is the shear modulus for the main directions; \( q_1^{(1)}, q_1^{(2)}, q_2^{(1)}, q_2^{(2)}, q_3 \) are the external loads.

First, in the equilibrium equations (15) we use the dimensionless coordinates, considering \( \vec{x} = ax, \vec{y} = by, \vec{h} = h_0h, \vec{v}_1^{(1)} = hv_1^{(1)}, \vec{v}_1^{(2)} = hv_1^{(2)}, \vec{v}_2^{(1)} = hv_2^{(1)}, \vec{v}_2^{(2)} = hv_2^{(2)}, \vec{w} = hw \)

\[
\begin{align*}
\frac{\partial^2 v_1^{(1)}}{\partial y^2} + \frac{A_{(1)}^{(1)} b^2 \partial^2 v_1^{(1)}}{A_{(2)}^{(1)} a^2 \partial x^2} + \frac{A_{(1)}^{(1)} b \partial^2 v_1^{(1)}}{A_{(2)}^{(1)} a \partial x \partial y} &+ \frac{B_1 b^2}{A_{(2)}^{(1)}} (v_1^{(2)} - v_1^{(1)}) + \frac{B_1 b^2 \partial w}{A_{(2)}^{(1)} a \partial x} = \frac{q_1^{(1)} b^2}{A_{(2)}^{(1)} h}, \\
\frac{\partial^2 v_1^{(2)}}{\partial y^2} + \frac{A_{(2)}^{(1)} b^2 \partial^2 v_1^{(2)}}{A_{(2)}^{(2)} a^2 \partial x^2} + \frac{A_{(2)}^{(1)} b \partial^2 v_1^{(2)}}{A_{(2)}^{(2)} a \partial x \partial y} &+ \frac{B_1 b^2}{A_{(2)}^{(2)} a} (v_1^{(2)} - v_1^{(1)}) - \frac{B_1 b^2 \partial w}{A_{(2)}^{(2)} a \partial x} = \frac{q_1^{(2)} b^2}{A_{(2)}^{(2)} h}, \\
\frac{\partial^2 v_2^{(1)}}{\partial y^2} + \frac{A_{(1)}^{(1)} b^2 \partial^2 v_2^{(1)}}{A_{(2)}^{(1)} a^2 \partial x^2} + \frac{A_{(1)}^{(1)} b \partial^2 v_2^{(1)}}{A_{(2)}^{(1)} a \partial x \partial y} &+ \frac{B_1 b^2}{A_{(2)}^{(1)} a} (v_2^{(2)} - v_2^{(1)}) + \frac{B_1 c b \partial w}{A_{(2)}^{(1)} a \partial y} = \frac{q_2^{(1)} b^2}{A_{(2)}^{(1)} h}, \\
\frac{\partial^2 v_2^{(2)}}{\partial y^2} + \frac{A_{(2)}^{(1)} b^2 \partial^2 v_2^{(2)}}{A_{(2)}^{(2)} a^2 \partial x^2} + \frac{A_{(2)}^{(1)} b \partial^2 v_2^{(2)}}{A_{(2)}^{(2)} a \partial x \partial y} &+ \frac{B_1 h}{A_{(2)}^{(2)} a} (v_2^{(2)} - v_2^{(1)}) - \frac{B_1 c b \partial w}{A_{(2)}^{(2)} a \partial y} = \frac{q_2^{(2)} b^2}{A_{(2)}^{(2)} h}, \\
\frac{\partial^4 w}{\partial y^4} + \frac{D_1 b^4 \partial^4 w}{D_2 a^4 \partial x^4} + \frac{2D_3 b^2}{D_2 a^2} \frac{\partial^2 w}{\partial x^2 \partial y^2} &- \frac{B_c}{2D_2} \left[ \frac{b^4}{a} \frac{\partial}{\partial x} \left( v_1^{(2)} - v_1^{(1)} \right) + \frac{b^3}{a} \frac{\partial}{\partial y} \left( v_2^{(2)} - v_2^{(1)} \right) + \frac{b^4 \partial^2 w}{a^2 \partial x^2} + b^2 \frac{\partial^2 w}{\partial y^2} \right] = \frac{q_4 b^4}{2D_2 h}.
\end{align*}
\]

Equilibrium equation (16) in vector - matrix form is written as

\[ SU = Q \]

where \( U(v_1^{(1)}, v_2^{(1)}, v_2^{(1)}, v_2^{(2)}, w) \) are the displacements, \( S \) is the matrix-operator of the form

\[ S = S_1 \frac{\partial^4}{\partial x^4} + S_2 \frac{\partial^4}{\partial x^2 \partial y^2} + S_3 \frac{\partial^2}{\partial x^2} + S_4 \frac{\partial^2}{\partial x \partial y} + S_5 \frac{\partial^2}{\partial y^2} + S_6 \frac{\partial^2}{\partial x^2 \partial y^2} + S_7 \frac{\partial}{\partial x} + S_8 \frac{\partial}{\partial y} + S_9 u. \]

Below the basic boundary conditions for some ways of fixing the edges are given.

1. Rigidly fixed edge:

\[ u_1^{(k)} \bigg|_\Gamma = v_1^{(k)} \bigg|_\Gamma = u_2^{(k)} \bigg|_\Gamma = v_2^{(k)} \bigg|_\Gamma = w_1^{(k)} \bigg|_\Gamma = w_2^{(k)} \bigg|_\Gamma = 0, \quad k = \overline{1, 2} \]

\[ \frac{\partial u_1^{(k)}}{\partial n} \bigg|_\Gamma = \frac{\partial v_1^{(k)}}{\partial n} \bigg|_\Gamma = \frac{\partial w_1^{(k)}}{\partial n} \bigg|_\Gamma = 0, \quad k = \overline{1, 2} \]

2. Sliding fixing:

\[ u_1^{(k)} \bigg|_\Gamma = u_2^{(k)} \bigg|_\Gamma = \frac{\partial w_1^{(k)}}{\partial n} \bigg|_\Gamma = \frac{\partial w_2^{(k)}}{\partial n} \bigg|_\Gamma = 0, \quad k = \overline{1, 2}; \]

\[ \sigma_n^{(k)} \bigg|_\Gamma = \tau_n^{(k)} \bigg|_\Gamma = 0, \quad k = \overline{1, 2}. \]
3. Sliding hinge:

\[ w^{(k)} \big|_{\Gamma} = w^{(k)} \big|_{\Gamma} = 0, \quad k = 1, 2; \]
\[ M_n \big|_{\Gamma} = 0; \quad \sigma_n \big|_{\Gamma} = \tau_n \big|_{\Gamma} = 0. \]

4. Fixed hinge:

\[ u^{(k)} = u^{(k)} = v^{(k)} = v^{(k)} = 0, \quad k = 1, 2; \]
\[ M_n \big|_{\Gamma} = 0. \]

5. Free edge:

\[ M_n \big|_{\Gamma} = 0; \quad (Q_n + \frac{\partial}{\partial s} M_n \tau) \big|_{\Gamma} = 0, \]
where \( \sigma_n = \sigma_1 l_1^2 + \sigma_2 l_2^2 + 2\sigma_{12} l_1 l_2, \quad \tau_n = \tau_{12} (l_1^2 - l_2^2) + (\sigma_2 - \sigma_1) l_1 l_2, \)
\[ M_n = (M_1 l_1^2 + 2M_{12} l_1 l_2 + M_2 l_2^2), \quad M_n \tau = M_{12} (l_1^2 - l_2^2) + (M_2 - M_1) l_1 l_2, Q_n = Q_1 l_1 + Q_2 l_2. \]

Here \( l_1 = \cos \alpha, \quad l_2 = \sin \alpha. \)

In practice, there are combinations of the above boundary conditions (3) - (7), when the edges of the plate are fixed in different ways.

5. Computational experiment

Consider the bending of a rigidly fixed and hinge-supported three-layer plate shown in Figure 4.

![Figure 4. Circle with four cuts](image)

In this case, the normalized equations of the boundary have the form:

\[ f_1 := \text{sqr}(R) - \text{sqr}(x) - \text{sqr}(y) \geq 0; \]
\[ f_2 := \text{sqr}(x - R) + \text{sqr}(y) - \text{sqr}(r_1) \geq 0; \]
\[ f_3 := \text{sqr}(x) + \text{sqr}(y - R) - \text{sqr}(r_1) \geq 0; \]
\[ f_4 := \text{sqr}(x + R) + \text{sqr}(y) - \text{sqr}(r_1) \geq 0; \]
\[ f_5 := \text{sqr}(x) + \text{sqr}(y + R) - \text{sqr}(r_1) \geq 0; \]
\[ w := ((f_1 \text{ and } f_2) \text{ and } f_3) \text{ and } f_4) \text{ and } f_5, \]
where R is the radius of the large circle, r1 is the radius of the cut circle.

To conduct a computational experiment, input data for solving this problem are: R=1m, r1=0.2 m, h=0.01 m (thickness), 1/2=25, G1/E2=0.5, G2/E2=0.2, ν=0.25 (Poisson’s ratio) [1].

Considering the values $\bar{w}(\bar{w}=10^4 w)$ on the cross section OX (x, y = 0), conduct a numerical study of the algorithm for calculating the bending of a three-layer plate with a rigidly fixed boundary condition and coordinates equal to 10 (i.e., $N = (nk + 1)(nk + 2)/2$, where $nk=3$) and constant values of Gauss nodes in calculating the double integrals (i.e. clutoch = 10, 20 and 32. Calculation results are shown in Table 1, the graph is given in Figure 5, whose graph is given in Fig. 5 respectively.

**Table 1.** Calculation results

| (x,y) | clut=10 | clut=20 | clut=32 |
|-------|---------|---------|---------|
| (-0.7;0) | 0.4183 | 0.5241 | 0.5114 |
| (-0.6;0) | 1.6409 | 1.8372 | 1.804 |
| (-0.5;0) | 3.3908 | 3.5081 | 3.4613 |
| (-0.4;0) | 5.2991 | 5.1833 | 5.1329 |
| (-0.3;0) | 7.045 | 6.6254 | 6.5785 |
| (-0.2;0) | 8.4063 | 7.7023 | 7.6616 |
| (-0.1;0) | 9.259 | 8.3581 | 8.3227 |
| (0;0) | 9.5485 | 8.5778 | 8.5443 |
| (0.1;0) | 9.2596 | 8.3592 | 8.3238 |
| (0.2;0) | 8.4052 | 7.7027 | 7.662 |
| (0.3;0) | 7.0388 | 6.6225 | 6.5758 |
| (0.4;0) | 5.2869 | 5.1758 | 5.1259 |
| (0.5;0) | 3.3756 | 3.4981 | 3.4522 |
| (0.6;0) | 1.6295 | 1.83 | 1.7976 |
| (0.7;0) | 0.415 | 0.523 | 0.5106 |

![Figure 5. The values of $\bar{w}(\bar{w}=10^4 w)$ in section (x,y=0) at nk =3](image)

Now consider a numerical study of the algorithm for calculating the bending of a three-layer plate with a rigidly-fixed boundary conditions and coordinates k=2 and 3, the constant values
of Gauss nodes when calculating double integrals (i.e. cl=20). The calculation results are shown in Table 2, a graph is given in Figure 3, respectively.

**Table 2. Calculation results**

| (x,y)  | nk=2 (6 kf) | nk=3 (10 kf) | (x,y)  |
|-------|-------------|-------------|-------|
| (-0.7;0) | 0.5248      | 0.5241      | (-0.7;0) |
| (-0.6;0) | 1.8396      | 1.8372      | (-0.6;0) |
| (-0.5;0) | 3.5124      | 3.5081      | (-0.5;0) |
| (-0.4;0) | 5.1888      | 5.1833      | (-0.4;0) |
| (-0.3;0) | 6.631       | 6.6254      | (-0.3;0) |
| (-0.2;0) | 7.7068      | 7.7023      | (-0.2;0) |
| (-0.1;0) | 8.3605      | 8.3581      | (-0.1;0) |
| (0;0) | 8.5777      | 8.5778      | (0;0) |
| (0.1;0) | 8.3566      | 8.3592      | (0.1;0) |
| (0.2;0) | 7.6981      | 7.7027      | (0.2;0) |
| (0.3;0) | 6.6168      | 6.6225      | (0.3;0) |
| (0.4;0) | 5.1702      | 5.1758      | (0.4;0) |
| (0.5;0) | 3.4938      | 3.4981      | (0.5;0) |
| (0.6;0) | 1.8276      | 1.83        | (0.6;0) |
| (0.7;0) | 0.5223      | 0.523       | (0.7;0) |

**Figure 6.** Values of $\bar{w}$ ($\bar{w}=10^4w$) in section (x,y=0) at slutoch = 20

6. **Conclusion**

A mathematical model and a computational algorithm for solving the problems of bending of three-layer plates of complex configuration developed on the basis of the Hamilton-Ostrogradky's variation principle, the Bubnov-Galerkin method and the Rvachev R-functions method (RFM) are given in the paper. Numerical results of the problem are presented. Results are shown in tabular and graphical form. Analysis of numerical study of results has shown that the use of the R-function method in combination with the Bubnov-Galerkin variation method for solving the problem of calculating the oscillations of three-layer plates of complex configuration is fully substantiated and gives good results.
References

[1] Kurpa L.V. The R-function method for solving linear problems of bending and oscillations of shallow shells. Kharkiv 2009. p 391.
[2] Rvachev V.L. Geometric applications of algebra of logic. Kiev: Technic 1967. p 212.
[3] Rvachev V.L. R-functions and some of its application. Kiev Naukova Dumka 1987. p 552.
[4] Rvachev V.L., Kurpa L.V. R-functions in problems of plate theory. Kiev Naukova Dumka 1987. p 174.
[5] Rvachev V.L., Protsenko V.S. Contact problems elasticity theory for non-classical areas. - Kiev: Naukova Dumka 1977. p 235.
[6] Rvachev V.L., Sinekop N.S. The method of R-functions in problems of the theory of elasticity and plasticity. - Kiev: Naukova Dumka, 1990 .p 216.
[7] Rvachev V.L., Sklepus N.G. Application of the R-function method to solving problems of plate bending // Applied. Mechanics. -1969. 5, No. 9. -WITH. p 124-128.
[8] Nazirov S.A. Software package for solving boundary value problems by variational methods // Algorithms: Sat. scientific tr - Tashkent, EC AN RUz, 1988. - Issue. 65.- S. p 38-48.
[9] Kabulov V.K. Algorithmization in Elasticity Theory and Deformation Theory of Plasticity. Tashkent, "Fan" 1966. p 395.
[10] Bolotin V.V. To the theory of laminated plates. - Izv. USSR Academy of Sciences. Rel. Mechanics and Mechanical Engineering, 3 (1963). p 6572.
[11] Bolotin V.V. Strength, stability and vibrations of multilayer plates. - In the book. Strength calculations. M.: Mechanical Engineering, 1965, issue 11. p 3163.
[12] Bolotin V.V. Novichkov Yu.N. Mechanics of multilayer structures. - M.: Mechanical Engineering, 1980 .p 375.
[13] Nazirov Sh.A., Zhumaev S.S. The mathematical model of multilayer plates // Computational issues. and applied. mathematics: scientific. work. - Tashkent, IMIT AN RUz, 2010.- issue. 124. - p. 64-79
[14] Vasizu K. Variational methods in the theory of elasticity and plasticity.- M. : World Dumka, Kiev, 1987.
[15] Samples I.P. Saveliev L.M. Khazanov H.S. Finite element methods and tasks of structural mechanics of aircraft. - M.: Higher School, 1985. p 392.
[16] Mikhlin S.G. Variational methods in mathematical physics. M.: Nauka, 1970. p 512.
[17] Gantmakher F.R. Matrix Theory. - M.: Nauka, 1988 . p 552.
[18] Nazirov Sh.A., Nuraliev F.M. . Mathematical modeling of processes of electro-magnetic fields' effects on thin conducting plates by complex form. The 4th International Conference on Application of Information and Communication Technologies. AICT2010. Uzbekistan, Tashkent, 12-14 October 2010. p. 125-129.
[19] Nazirov Sh.A., Nuraliev F.M. Algorithmization of the decision of classes of multidimensional problems of magneto-elasticity of thin plates and shells, Second IEEE and IFIP International Conference in Central Asia on Internet, ICI 2006 September 19-21, 2006. Tashkent, Uzbekistan.
[20] Nuraliev F.M. Mathematical modeling of processes of the electromagnetic fields effects on deformingal condition of thin electro-conductive bodies by the method of r-function. Abstract of dissertation for the degree of candidate of technical sciences. Tashkent - 2016. p 86.
[21] Anarova Sh.A., Nuraliyev F.M.,Dadenova G. Mathematical model of spatially loaded bars with account of torsion function and transverse shears. International Journal of Technical Research and Applications. India - 2016. - Vol. 4, Issue 1. p 22-32.
[22] Anarova Sh.A., Nuraliev F.M.,Sadullaeva Sh. A. Algorithm of solution of the problem of bending torsion of the rod based on R-function method. International Journal of Current Research. Vol. 8, Issue 9. p 37807-37819.
[23] Amanov O.T. Calculation of three-layer reinforced concrete elements with a middle layer of wood concrete. Abstract of dissertation for the degree of candidate of technical sciences. Tashkent - 2001. p 21.