Rough Path Analysis Via Fractional Calculus

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Abstract
Using fractional calculus we define integrals of the form $\int_a^b f(x_t)dy_t$, where $x$ and $y$ are vector-valued Hölder continuous functions of order $\beta \in (\frac{1}{3}, \frac{1}{2})$ and $f$ is a continuously differentiable function such that $f'$ is $\lambda$-Hölder continuous for some $\lambda > \frac{1}{2} - \frac{1}{\beta}$. Under some further smooth conditions on $f$ the integral is a continuous functional of $x$, $y$, and the tensor product $x \otimes y$ with respect to the Hölder norms. We derive some estimates for these integrals and we solve differential equations driven by the function $y$. We discuss some applications to stochastic integrals and stochastic differential equations.

1 Introduction
The theory of rough path analysis has been developed from the seminal paper by Lyons [4]. The purpose of this theory is to analyze dynamical systems $dx_t = f(x_t)dy_t$, where the control function $y$ is not differentiable. If the rough control $y$ has finite $p$-variation on bounded intervals, where $p \geq 2$, then the dynamical system is a continuous function, in the $p$-variation norm, of $y$ and the associated multiplicative functionals $y \otimes \cdots \otimes y$, with $k = 2, \ldots, [p]$. In the case $1 \leq p < 2$, the dynamical system can be formulated using Riemann-Stieltjes integrals and applying the results of Young [8]. In this case, $x_t$ is a continuous function of $y$ in the $p$-variation norm (see Lyons [3]).

Suppose that $f$ and $g$ are Hölder continuous functions on the interval $[a, b]$, of order $\lambda$ and $\mu$, respectively, with $\lambda + \mu > 1$. Then, the Riemann-Stieltjes integral $\int_a^b f dg$ can be expressed as a Lebesgue integral using fractional derivatives (see Zähle [9] and Proposition 2.1 below). This fact has been exploited by Nualart and Răşcanu in [6] to analyze dynamical systems driven by a control function.

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y which is Hölder continuous of order $\beta > \frac{1}{2}$. In this case further results are obtained in [2] along the line of the present paper.

The purpose of this paper is to analyze dynamical systems $dx_t = f(x_t)dy_t$, where the control function $y$ is Hölder continuous of order $\beta \in (\frac{1}{3}, \frac{1}{2})$, using the techniques of the classical fractional calculus, and following an approach inspired by the work of Nualart and Răşcanu [6] in the case $\beta > \frac{1}{2}$. In order to achieve this objective, we first provide in Section 3 an explicit formula for integrals of the form $\int_a^b f(x_t)dy_t$, where $x$ and $y$ are Hölder continuous of order $\beta \in (\frac{1}{3}, \frac{1}{2})$. This formula, given in Theorem 3.1, is based on the fractional integration by parts formula, and it involves the functions $x$, $y$, and the quadratic multiplicative functional $x \otimes y$. Notice that this explicit formula does not depend on any approximation scheme. As a consequence, we derive estimates in the Hölder norm for the indefinite integral.

Section 4 is devoted to establish the existence and uniqueness of a solution for the dynamical system $dx_t = f(x_t)dy_t$. The main ingredient in the proof of these results is to transform this equation into a system of integral equations for $x$ and $x \otimes y$ that can be solved by a standard application of a fixed point argument. We show how the solution depends continuously on the Hölder norm of $y$ and $y \otimes y$. We also prove some stability results for the differential equations which are interesting, new and may be difficult to obtain by other approaches. Remark that to derive our results we do not make use of the theory of rough paths, and we obtain explicit formulas that do not depend on any approximation argument.

These results can be applied to implement a path-wise approach to define stochastic integrals and to solve stochastic differential equations driven by a multidimensional Brownian motion. As an application of the deterministic results obtained for dynamical systems we derive a sharp rate of almost sure convergence of the Wong-Zakai approximation for multidimensional diffusion processes. We couldn’t find this kind of estimates elsewhere. Similar results hold in the case of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$. The approximation of the solutions of stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$ is more involved and it will be treated in a forthcoming paper.

2 Fractional Integrals and Derivatives

Let $a, b \in \mathbb{R}$ with $a < b$. Denote by $L^p(a, b)$, $p \geq 1$, the space of Lebesgue measurable functions $f : [a, b] \rightarrow \mathbb{R}$ for which $\|f\|_{L^p(a, b)} < \infty$, where

$$\|f\|_{L^p(a, b)} = \begin{cases} \left( \int_a^b |f(t)|^p \, dt \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \text{ess sup} \{|f(t)| : t \in [a, b]\}, & \text{if } p = \infty. \end{cases}$$

Let $f \in L^1(a, b)$ and $\alpha > 0$. The left-sided and right-sided fractional Riemann-
Liouville integrals of $f$ of order $\alpha$ are defined for almost all $x \in (a, b)$ by

$$I^\alpha_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds$$

and

$$I^\alpha_{b-} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) \, ds,$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} \, dr$ is the Euler gamma function. Let $I^\alpha_{a+}(L^p)$ (resp. $I^\alpha_{b-}(L^p)$) be the image of $L^p(a, b)$ by the operator $I^\alpha_{a+}$ (resp. $I^\alpha_{b-}$). If $f \in I^\alpha_{a+}(L^p)$ (resp. $f \in I^\alpha_{b-}(L^p)$) and $0 < \alpha < 1$ then the Weyl derivatives are defined as

$$D^\alpha_{a+} f(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} \, ds \right) \quad (2.1)$$

and

$$D^\alpha_{b-} f(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left( \frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} \, ds \right) \quad (2.2)$$

where $a \leq t \leq b$ (the convergence of the integrals at the singularity $s = t$ holds point-wise for almost all $t \in (a, b)$ if $p = 1$ and moreover in $L^p$-sense if $1 < p < \infty$).

For any $\lambda \in (0, 1)$, we denote by $C^\lambda(a, b)$ the space of $\lambda$-Hölder continuous functions on the interval $[a, b]$. Recall from [7] that we have:

- If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1-\alpha p}$ then
  $$I^\alpha_{a+}(L^p) = I^\alpha_{b-}(L^p) \subset L^q(a, b).$$

- If $\alpha > \frac{1}{p}$ then
  $$I^\alpha_{a+}(L^p) \cup I^\alpha_{b-}(L^p) \subset C^{\alpha - \frac{1}{p}}(a, b).$$

The following inversion formulas hold:

$$I^\alpha_{a+}(D^\alpha_{a+} f) = f, \quad \forall f \in I^\alpha_{a+}(L^p) \quad (2.3)$$

and

$$D^\alpha_{a+}(I^\alpha_{a+} f) = f, \quad \forall f \in I^\alpha_{a+}(L^p) \quad (2.4)$$

On the other hand, for any $f, g \in L^1(a, b)$ we have

$$\int_a^b I^\alpha_{a+} f(t) g(t) \, dt = (-1)^{\alpha} \int_a^b f(t) I^\alpha_{b-} g(t) \, dt, \quad (2.6)$$

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and for $f \in \mathcal{I}_{a+}^\alpha (L^p)$ and $g \in \mathcal{I}_{a-}^\alpha (L^p)$ we have
\[
\int_a^b D_{a+}^\alpha f(t) g(t) dt = (-1)^\alpha \int_a^b f(t) D_{b-}^\alpha g(t) dt.
\] (2.7)

Suppose that $f \in \mathcal{C}^\lambda (a,b)$ and $g \in \mathcal{C}^\mu (a,b)$ with $\lambda + \mu > 1$. Then, from the classical paper by Young [8], the Riemann-Stieltjes integral $\int_a^b f dg$ exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral $\int_a^b f dg$ in terms of fractional derivatives (see [9]).

**Proposition 2.1** Suppose that $f \in \mathcal{C}^\lambda (a,b)$ and $g \in \mathcal{C}^\mu (a,b)$ with $\lambda + \mu > 1$. Then the Riemann-Stieltjes integral $\int_a^b f dg$ exists and it can be expressed as
\[
\int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f(t) D_{b-}^{1-\alpha} g_b(t) dt,
\] (2.8)
where $g_b(t) = g(t) - g(b)$.

We will make use of the following two-variable fractional integration by parts formula, whose proof is given in the Appendix.

**Lemma 2.2** Let $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ be two functions of class $C^2$ defined on $a \leq \xi \leq \eta \leq b$. Suppose $\psi(\xi,\eta)$ vanishes on the diagonal. The following fractional integration by parts formula holds for any $0 < \alpha < 1$.
\[
\int_a^b \frac{d\xi}{\xi} \int_a^b \frac{d\eta}{\eta} \frac{\partial^2 \psi}{\partial \xi \partial \eta}(\xi,\eta) d\xi d\eta = - \int_a^b \frac{d\eta}{\eta} \int_a^\eta D_{a+}^{2-\alpha} D_{b-}^{\alpha} \varphi(\xi,\eta) \Gamma^\alpha \psi(\xi,\eta) d\xi,
\] (2.9)
where $D_{a+}^{2-\alpha}$ denotes the fractional derivative on variable $\xi$, $D_{b-}^{\alpha}$ denotes the fractional derivative on the variable $\eta$, and the operator $\Gamma^\alpha$ is defined by
\[
\Gamma^\alpha \psi(\xi,\eta) = D_{a-}^{1-\alpha} D_{b+}^{1-\alpha} \psi(\xi,\eta).
\] (2.10)

### 3 Integration of Rough Functions

Fix $\frac{1}{2} < \beta < \frac{3}{4}$. Suppose that $x : [0,T] \to \mathbb{R}^m$ and $y : [0,T] \to \mathbb{R}^d$ are $\beta$-Hölder continuous functions. Following [4] we assume that $x \otimes y$ is well-defined and it is a continuous function defined on $\Delta := \{(s,t) : 0 \leq s \leq t \leq T\}$ with values on $\mathbb{R}^m \otimes \mathbb{R}^d$ verifying the following properties:

**i)** For all $s \leq u \leq t$ we have (multiplicative property)
\[
(x \otimes y)_{s,u} + (x \otimes y)_{u,t} + (x_u - x_s) \otimes (y_t - y_u) = (x \otimes y)_{s,t}.
\] (3.1)
For all \((s, t) \in \Delta\)

\[
(x \otimes y)_{s, t} \leq k|t - s|^{2\beta}.
\]

That is, \((x, y, x \otimes y)\) constitutes a multiplicative functional in the sense of the rough paths analysis theory. We will say that \((x, y, x \otimes y)\) is a \(\beta\)-Hölder continuous multiplicative functional on \(\mathbb{R}^m \otimes \mathbb{R}^d\).

If \(x\) and \(y\) are smooth functions, then

\[
(x \otimes y)_{s, t}^{i, j} = \int_{s < \xi < \eta < t} dx^i_{\xi} dy^j_{\eta}
\]

(3.3) clearly defines a \(\beta\)-Hölder continuous multiplicative functional.

Let \(f : \mathbb{R}^m \to \mathbb{R}^d\) be a continuously differentiable function such that \(f'\) is \(\lambda\)-Hölder continuous, where \(\lambda > \frac{1}{\beta} - 2\). Our aim is to define the integral

\[
\int_a^b f(x_r) dy_r = \sum_{i=1}^d \int_a^b f_i(x_r) dy^i_r
\]

(3.4) using fractional calculus.

Fix a number \(\alpha\) such that \(1 - \beta < \alpha < 2\beta\) and \(\alpha < \frac{\lambda\beta + 1}{2}\). This is possible because \(3\beta > 1\) and \(\frac{\lambda\beta + 1}{2} > 1 - \beta\).

Notice first that the fractional integration by parts formula (2.8) cannot be used to define the integral (3.4) because the fractional derivative \(D^\alpha_{a+} f(x)\) is not well-defined under our hypotheses. For this reason we introduce the following compensated fractional derivative:

\[
\tilde{D}^\alpha_{a+} f(x)(r) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(x_r)}{(r - a)^\alpha} \right) + \alpha \int_a^r \frac{f(x_r) - f(x_\theta) - f'(x_\theta)(x_r - x_\theta)}{(r - \theta)^{\alpha + 1}} d\theta.
\]

(3.5)

This derivative is well-defined under our hypotheses because

\[
\frac{|f(x_r) - f(x_\theta) - f'(x_\theta)(x_r - x_\theta)|}{(r - \theta)^{\alpha + 1}} \leq K|r - \theta|^{(1 + \lambda)\beta - \alpha - 1},
\]

where \(K = \|f'\|_\lambda \|x\|_{\beta}^{1 + \lambda}\) and \((1 + \lambda)\beta - \alpha > 0\) since \(\alpha < \frac{\lambda\beta + 1}{2} < (1 + \lambda)\beta\).

For \(\theta < \xi < \eta\) introduce the kernel

\[
G(\theta, \xi, \eta) := \frac{1}{\alpha \Gamma(\alpha) \Gamma(2\alpha - 1)} (\xi - \theta)^{\alpha - 1} (\eta - \xi)^{\alpha - 1}
\]

\[
\times \int_0^1 q^{2\alpha - 2} (1 - q)^{\alpha - 1} \left( 1 + (1 - q) \frac{\xi - \theta}{\eta - \xi} \right)^{-1} dq.
\]

(3.6)
Define for $\varepsilon < \alpha + \beta - 1$ and $\theta < \xi < \eta < b$

$$K_{\theta,b}(\xi,\eta) = -D_{\theta+}^{1,\alpha-\varepsilon}D_{b-}^{2,\alpha-\varepsilon} [G_{b-}(\theta,\xi,\eta)] . \quad (3.7)$$

In Lemma 6.2 we will show that this kernel satisfies

$$\sup_{0 \leq s \leq t} \int_{s < \xi < \eta < t} |K_{s,t}(\xi,\eta)| \, d\xi \, d\eta < \infty.$$ 

Finally, we denote

$$A^b_a(x \otimes y) := \int_a^b \int_a^\eta K_{a,b}(\xi,\eta) \Gamma^{\alpha-\varepsilon} (x \otimes y)_{\xi,\eta} \, d\xi \, d\eta. \quad (3.8)$$

We are ready now to define the integral $\int_a^b f(x_r) \, dy_r$.

**Definition 3.1** Let $(x, y, x \otimes y)$ be a $\beta$-Hölder continuous multiplicative functional on $\mathbb{R}^m \otimes \mathbb{R}^d$. Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^d$ be a continuously differentiable function such that $f'$ is $\lambda$-Hölder continuous, where $\lambda > \frac{1}{\beta} - 2$. Fix $\alpha > 0$ and $\varepsilon > 0$ such that $1 - \beta < \alpha < 2\beta$, $\alpha < \frac{\lambda + 1}{2}$ and $\varepsilon < \alpha + \beta - 1$. Then, for any $0 \leq a < b \leq T$ we define

$$\int_a^b f(x_r) \, dy_r = (-1)^\alpha \sum_{i=1}^d \int_a^b \bar{D}^\alpha_{a+} f_i(x) (r) D_{b-}^{1-\alpha} y_{b-}(r) \, dr + \sum_{i=1}^m \sum_{j=1}^d \int_a^b D^{2\alpha-1}_{a+} \partial_i f_j(x) (r) A^b_a(x^i \otimes y^j) \, dr. \quad (3.9)$$

Notice that if $y$ is $\beta$-Hölder continuous, the fractional derivative $D^{1-\alpha}_{b-} y_{b-}(r)$ is well-defined because

$$\frac{|y_\sigma - y_r|}{(\sigma - r)^{2-\alpha}} \leq \|y\|_\beta |\sigma - r|^{\beta + \alpha - 2}$$

and $\beta + \alpha - 2 > -1$. The following theorem asserts that this definition is coherent with the classical notion of integral and will allow us to deduce estimates in the Hölder norm.

**Theorem 3.2** Suppose $y : [0, T] \rightarrow \mathbb{R}^d$ is a continuously differentiable function. Let $x : [0, T] \rightarrow \mathbb{R}^m$ be a $\beta$-Hölder continuous function and let $x \otimes y$ be defined by $(x \otimes y)^{i,j}_{\alpha+} = \int_0^t \left( x^i_\xi - x^i_s \right) (y^j)_\xi \, d\xi$. Assume that $f$ satisfies the assumptions of Definition 3.1. Then, the integral $\int_a^b f(x_r) \, dy_r$ introduced in (3.9) coincides with $\sum_{i=1}^d \int_a^b f_i(x_r) (y^i)^\prime \, dr$.

**Proof.** To simplify the proof we take $m = d = 1$. From (3.8) and (3.9) we get

$$\int_a^b f(x_r) y_r^\prime \, dr = (-1)^\alpha \int_a^b D^\alpha_{a+} f(x) (r) D_{b-}^{1-\alpha} y_{b-}(r) \, dr$$

$$= (-1)^\alpha \int_a^b \bar{D}^\alpha_{a+} f(x) (r) D_{b-}^{1-\alpha} y_{b-}(r) \, dr + A_2,$$
Thus where

\[ A_2 = \frac{\alpha(-1)^\alpha}{\Gamma(1 - \alpha)} \int_a^b \int_a^r \frac{f'(x_\theta)(x_r - x_\theta)}{(r - \theta)^\alpha + 1} D_{b-}^{1-\alpha} y_b-(r) dr d\theta \]

\[ = \frac{\alpha(-1)^\alpha}{\Gamma(1 - \alpha)} \int_a^b f'(x_\theta) \left( \int_\theta^r \frac{x_r - x_\theta}{(r - \theta)^\alpha + 1} D_{b-}^{1-\alpha} y_b-(r) dr \right) d\theta. \tag{3.10} \]

So, it suffices to show that

\[ A_2 = \int_a^b D_{a+}^{2\alpha-1} f'(x)(r) \Lambda_\theta^b (x \otimes y) dr. \tag{3.11} \]

Formula \( \text{(3.11)} \) should be first proved for \( x \) of class \( C^1 \) and then extended to a general \( \beta \)-Hölder continuous function. Applying \( 2.3, 2.0 \), and \( 2.5 \) we obtain

\[ A_2 = \frac{\alpha(-1)^\alpha}{\Gamma(1 - \alpha)} \int_a^b D_{b-}^{1-\alpha} y_b-(r) \int_a^r f'(x_\theta)(x_r - x_\theta) \frac{d\theta}{(r - \theta)^\alpha + 1} dr \]

\[ = \frac{\alpha(-1)^{3\alpha-1}}{\Gamma(1 - \alpha)} \int_a^b D_{b-}^{1-\alpha} y_b-(r) \int_a^r D_{a+}^{2\alpha-1} f'(x)(\theta) \frac{1}{(r - \theta)^\alpha + 1} \left( \frac{x_r - x_\theta}{(r - \theta)^\alpha + 1} \right) d\theta dr \]

\[ = \frac{\alpha(-1)^{3\alpha-1}}{\Gamma(1 - \alpha)} \int_a^b D_{a+}^{2\alpha-1} f'(x)(\theta) \int_\theta^r \frac{1}{(r - \theta)^\alpha + 1} \left( \frac{x_r - x_\theta}{(r - \theta)^\alpha + 1} \right) D_{b-}^{1-\alpha} y_b-(r) dr d\theta \]

where \( f^{\alpha, \beta} \) denotes the fractional integral applied to a function of \( \theta \) and a similar notation is used for fractional derivatives. Now

\[ f^{\alpha, \beta} f^{2\alpha-1, \beta} \]

\[ = f^{\alpha, \beta} \left[ \frac{1}{\Gamma(2\alpha - 1)} \int_\Theta \left( \theta' - \Theta \right)^{2\alpha - 2} \frac{x_r - x_\theta}{(r - \theta')^{\alpha + 1}} d\theta' \right] \]

\[ = \frac{(-1)^{2\alpha - 2}}{\Gamma(2\alpha - 1) \Gamma(\alpha)} \int_\Theta \int_\Sigma \int_{\Theta < \Sigma < r < r'} \left( r - r' \right)^{\alpha - 1} \left( \theta' - \Theta \right)^{2\alpha - 2} (r' - \theta')^{-\alpha - 1} dx_\xi dy_\eta dr'. \]

Thus

\[ A_2 = \frac{1}{\Gamma(2\alpha - 1) \Gamma(\alpha)} \int_a^b D_{a+}^{2\alpha-1} f'(x)(\theta) \]

\[ \times \int_\Theta \left( \int_{\Theta < \Sigma < r < r'} \left( \eta - r' \right)^{\alpha - 1} \left( \theta' - \Theta \right)^{2\alpha - 2} (r' - \theta')^{-\alpha - 1} d\theta' dr' \right) dx_\xi dy_\eta d\theta. \tag{3.12} \]
Making the change of variable \( \frac{r - \xi}{\eta - \xi} = w \) and using formula 3.196 in Gradshteyn and Ryzhik \([1]\) we obtain

\[
\begin{align*}
\int_\xi^\eta (\eta - r')^{\alpha - 1} (r' - \theta')^{-\alpha - 1} \, dr' \\
= (\eta - \xi)^{-1} \int_0^1 (1 - w)^{\alpha - 1} (w + \frac{\xi - \theta'}{\eta - \xi})^{-\alpha - 1} \, dw \\
= \frac{1}{\alpha} (\eta - \xi)^{-1} \left( \frac{\xi - \theta'}{\eta - \xi} \right)^{-\alpha - 1} F(1, \alpha + 1, \alpha + 1, \frac{\eta - \xi}{\xi - \theta'}) \\
= \frac{1}{\alpha} (\eta - \xi)^{-1} \left( \frac{\xi - \theta'}{\eta - \xi} \right)^{-\alpha - 1} (1 + \frac{\eta - \xi}{\xi - \theta'})^{-1} \\
= \frac{1}{\alpha} (\eta - \xi)^{\alpha} (\xi - \theta')^{-\alpha} (\eta - \theta')^{-1},
\end{align*}
\]

and substituting this expression into (3.12) yields

\[
A_2 = \frac{1}{\Gamma(2\alpha - 1)\Gamma(\alpha)\alpha} \int_a^b D_{\alpha + 1}^{2\alpha - 1} f'(x)(\theta) \\
\times \int_{\theta < \xi < \eta < b} (\eta - \xi)^{\alpha} \left( \int_{\theta}^{\xi} (\xi - \theta')^{-\alpha} (\eta - \theta')^{-1} (\theta' - \theta)^{2\alpha - 2} \, d\theta' \right) \, dx \, dy \, d\theta.
\]

Using (3.9) we get

\[
(\eta - \xi)^\alpha \int_\theta^\xi (\xi - \theta')^{-\alpha} (\eta - \theta')^{-1} (\theta' - \theta)^{2\alpha - 2} \, d\theta' \\
= (\eta - \xi)^{\alpha - 1}(\xi - \theta)^{\alpha - 1} \int_0^1 (1 - q)^{-\alpha} q^{2\alpha - 2} \left( 1 + (1 - q)\frac{\xi - \theta}{\eta - \xi} \right)^{-1} \, dq \\
= G(\theta, \xi, \eta) \Gamma(2\alpha - 1)\Gamma(\alpha)\alpha.
\]

Hence,

\[
A_2 = \int_a^b D_{\alpha + 1}^{2\alpha - 1} f'(x)(\theta) \int_{\theta < \xi < \eta < b} G(\theta, \xi, \eta) \, dx \, dy \, d\theta.
\]

Applying the two-dimensional fractional integration by parts formula (2.9) to \( \varphi(\xi, \eta) = G(\theta, \xi, \eta) \) and \( \psi(\xi, \eta) = (x \otimes y)\xi,\eta \) and using (5.7) we obtain

\[
\begin{align*}
\int_{\theta < \xi < \eta < b} G(\theta, \xi, \eta) \, dx \, dy \, d\eta \\
= -\int_{\theta}^{\beta} \int_{\theta}^{\eta} D_{\alpha + 1}^{\alpha - \varepsilon, \theta} D_{\beta + 1}^{\alpha - \varepsilon, \theta} G_{\beta + 1}(\theta, \xi, \eta) \\
\times \Gamma^{\alpha - \varepsilon} (x \otimes y)\xi,\eta \, d\xi \, d\eta \\
= \int_{\theta}^{\beta} \int_{\theta}^{\eta} K_{\theta, \beta}(\xi, \eta) \Gamma^{\alpha - \varepsilon} (x \otimes y)\xi,\eta \, d\xi \, d\eta.
\end{align*}
\]

Thus

\[
A_2 = \int_a^b D_{\alpha + 1}^{2\alpha - 1} f'(x)(r) \int_r^\eta K_{r, \beta}(\xi, \eta) \Gamma^{\alpha - \varepsilon} (x \otimes y)\xi,\eta \, d\xi \, dy \, dr.
\]
This proves the theorem. 

For any \((s, t) \in \Delta\), and given a \(\beta\)-H"older continuous multiplicative functional \((x, y, x \otimes y)\), we define

\[
\|x\|_{s,t,\beta} = \sup_{s \leq \theta < r \leq t} \frac{|x_r - x_\theta|}{|r - \theta|^{\beta}},
\]

\[
\|x \otimes y\|_{s,t,\beta} = \sup_{s \leq \theta < r \leq t} \frac{|(x \otimes y)_{\theta, r}|}{|r - \theta|^{2\beta}}.
\]

(3.13)

(3.14)

We also set \(\|x\|_\beta = \|x\|_{0,T,\beta}\), and \(\|x \otimes y\|_\beta = \|x \otimes y\|_{0,T,\beta}\). Also, \(\|\cdot\|_{s,t,\infty}\) will denote the supremum norm in the interval \([s, t]\). In the sequel, \(k\) will denote a constant that may depend on the parameters \(\beta, \alpha, \lambda, \varepsilon\) and \(T\).

The following estimate is useful.

**Proposition 3.3** Under the hypotheses of Definition 3.1 we have, if \(b - a \leq 1\)

\[
\left\| \int f(x_r) dy_r \right\|_{a,b,\beta} \leq k \|f\|_\infty \|y\|_{a,b,\beta} + k \left( \|x \otimes y\|_{a,b,\beta} + \|x\|_{a,b,\beta} \|y\|_{a,b,\beta} \right) \\
\times \left( \|f'\|_\infty + \|f''\|_{\lambda} \|x\|_{a,b,\beta} (b - a)^{\lambda} \right) (b - a)^{3 - 2\varepsilon}. \tag{3.15}
\]

Moreover, if the second derivative \(f''\) is \(\lambda\)-H"older continuous and bounded, and \((\tilde{x}, y, \tilde{x} \otimes y)\) is also a \(\beta\)-H"older continuous multiplicative functional on \(\mathbb{R}^m \otimes \mathbb{R}^d\), then

\[
\left\| \int f(x_r) dy_r - \int f(\tilde{x}_r) dy_r \right\|_{a,b,\beta} \leq kH_1 \|x - \tilde{x}\|_{a,b,\infty} + kH_2 \|x - \tilde{x}\|_{a,b,\beta} + kH_3 \|x - \tilde{x}\| \otimes y\|_{a,b,\beta}, \tag{3.16}
\]

where

\[
H_1 = \|y\|_{a,b,\beta} \left( \|f'\|_\infty + \|f''\|_\lambda \|x\|_{a,b,\beta} (b - a)^{\beta(1 + \lambda)} \right) \left( \|x\|_{a,b,\beta} + \|\tilde{x}\|_{a,b,\beta} \right) \left( \|x\|_{a,b,\beta} + \|\tilde{x}\|_{a,b,\beta} \right) (b - a)^{3 - 2\varepsilon},
\]

\[
H_2 = \|f''\|_\infty \|y\|_{a,b,\beta} \left( \|x\|_{a,b,\beta} + \|\tilde{x}\|_{a,b,\beta} \right) (b - a)^{\beta(1 + \lambda)} \left( \|x\|_{a,b,\beta} + \|\tilde{x}\|_{a,b,\beta} \right) \left( \|x\|_{a,b,\beta} + \|\tilde{x}\|_{a,b,\beta} \right) (b - a)^{3 - 2\varepsilon},
\]

\[
H_3 = (\|f\|_\infty + \|f\|_\lambda \|\tilde{x}\|_{a,b,\beta} (b - a)^{\lambda\beta}) (b - a)^{3 - 2\varepsilon}.
\]

**Remark:** In (3.15) we can replace \(\|f\|_\infty\) and \(\|f'\|_\infty\) by \(\|f(x)\|_{a,b,\infty}\) and \(\|f'(x)\|_{a,b,\infty}\), respectively.
Proof. First we have, for any \( r \in [a, b] \)
\[
|D_a^\alpha f(x)(r)| \leq k \left( |f(x_r)| (r-a)^{-\alpha} + \|f\|_\lambda \|x\|_{a,r,\beta}^\lambda (r-a)^{\lambda\beta-\alpha} \right),
\] (3.17)
\[
|\tilde{D}_a^\alpha f(x)(r)| \leq k \left( |f(x_r)| (r-a)^{-\alpha} + \|f'\|_\lambda \|x\|_{a,r,\beta}^{1+\lambda} (r-a)^{(\lambda+1)\beta-\alpha} \right),
\] (3.18)
and
\[
|D_{b-}^\alpha y(b-r)| \leq k \|y\|_{r,b,\beta} (b-r)^{\alpha+\beta-1}. \tag{3.19}
\]
The expression (6.1) of \( \Gamma \) yields
\[
|\Gamma^\alpha \varepsilon (x \otimes y)_{a,b}| \leq k \left( \|x \otimes y\|_{a,b,\beta} + \|x\|_{a,b,\beta} \|y\|_{a,b,\beta} \right) (b-a)^{2\beta+2\alpha-2\varepsilon}. \tag{3.20}
\]
Consequently, from (3.8) Lemma 6.2 we obtain the estimate
\[
|\Lambda^b_a (x \otimes y)| \leq k \left( \|x \otimes y\|_{a,b,\beta} + \|x\|_{a,b,\beta} \|y\|_{a,b,\beta} \right) (b-a)^{2\beta+2\alpha-2\varepsilon}. \tag{3.21}
\]
Thus
\[
\left| \int_a^b f(x_r)dy_r \right| \leq k \|y\|_{a,b,\beta} \left( \int_a^b |f(x_r)| (r-a)^{-\alpha} (b-r)^{\alpha+\beta-1}dr \right.
\]
\[
+ \|f'\|_\lambda \|x\|_{a,b,\beta}^{1+\lambda} \int_a^b (r-a)^{(\lambda+1)\beta-\alpha} (b-r)^{\alpha+\beta-1}dr \\
+ k \int_a^b \left( \frac{\|f'\|_\infty}{(r-a)^{2\alpha-1}} + \|f'\|_\lambda \|x\|_{a,b,\beta}^{\lambda} (r-a)^{\lambda\beta-2\alpha+1} \right)
\]
\[
\times \left( \|x \otimes y\|_{a,b,\beta} + \|x\|_{a,b,\beta} \|y\|_{a,b,\beta} \right) (b-r)^{2\beta+2\alpha-2\varepsilon}dr.
\]
Therefore, we obtain
\[
\left| \int_a^b f(x_r)dy_r \right| \leq k \|f\|_\infty \|y\|_{a,b,\beta} (b-a)^{\beta}
\]
\[
+ k \left( \|x \otimes y\|_{a,b,\beta} + \|x\|_{a,b,\beta} \|y\|_{a,b,\beta} \right) \|f'\|_\infty (b-a)^{2\beta-2\varepsilon}
\]
\[
+ k \|f'\|_\lambda \|x\|_{a,b,\beta}^{\lambda} \|y\|_{a,b,\beta} \|x \otimes y\|_{a,b,\beta} (b-a)^{\lambda+2\beta-2\varepsilon}
\]
and this implies (3.16) easily.

Note that for any \( a \leq \theta \leq r \leq b \) we can write
\[
f(x_r) - f(x_\theta) - f'(x_\theta)(x_r-x_\theta) - [f(\bar{x}_r) - f(\bar{x}_\theta) - f'(\bar{x}_\theta)(\bar{x}_r-\bar{x}_\theta)]
\]
\[
= \int_0^1 \left[ f'(x_\theta + z(x_r-x_\theta)) - f'(x_\theta) \right] (x_r-x_\theta - \bar{x}_r + \bar{x}_\theta)dz
\]
\[
+ \int_0^1 \left[ f'(x_\theta + z(x_r-x_\theta)) - f'(\bar{x}_\theta + z(\bar{x}_r-\bar{x}_\theta)) - f'(x_\theta) + f'(\bar{x}_\theta) \right] (\bar{x}_r-\bar{x}_\theta)dz
\]
\[
= a_1 - a_2.
\]
We have
\[ |a_1| \leq \|f''\|_\infty \|x\|_{a,b,\beta} \|x - \tilde{x}\|_{a,b,\beta} (r - \theta)^{2\beta}. \]

For the term \(a_2\) we make the decomposition
\[
a_2 = (\bar{x}_r - \tilde{x}_\theta) \int_0^1 \int_0^1 f''(x_\theta + z(x_r - x_\theta) + t(\tilde{x}_\theta - x_\theta) + tz(\bar{x}_r - \tilde{x}_\theta - x_r + x_\theta)) dt dz
\]
\[
- (\bar{x}_r - \tilde{x}_\theta)(\tilde{x}_\theta - x_\theta) \int_0^1 f''(x_\theta + t(\tilde{x}_\theta - x_\theta)) dt
\]
\[
- (\bar{x}_r - \tilde{x}_\theta)(\tilde{x}_\theta - x_\theta) \int_0^1 f''(x_\theta + z(x_r - x_\theta) + t(\tilde{x}_\theta - x_\theta) + tz(\bar{x}_r - \tilde{x}_\theta - x_r + x_\theta)) dt dz
\]
\[
- f''(x_\theta + t(\tilde{x}_\theta - x_\theta)) dt dz.
\]

Thus,
\[
|a_2| \leq \|f''\|_\infty \|\bar{x}\|_{a,b,\beta} \|x - \tilde{x}\|_{a,b,\beta} (r - \theta)^{2\beta}
\]
\[
+ \|f''\|_\lambda \|\tilde{x}\|_{a,b,\beta} \|x - \tilde{x}\|_{a,b,\infty} (\|x\|_{a,b,\beta} + \|x - \tilde{x}\|_{a,b,\beta}) \|r - \theta\|^{\beta(1 + \lambda)}.
\]

As a consequence,
\[
|f(x_r) - f(x_\theta) - f'(x_\theta)(x_r - x_\theta) - [f(\tilde{x}_r) - f'(\tilde{x}_\theta)(\bar{x}_r - \tilde{x}_\theta)]|
\]
\[
\leq kI_1 (r - \theta)^{\beta(1 + \lambda)},
\]
\[(3.22)\]

where
\[
I_1 = \|f''\|_\infty \{\|x\|_{a,b,\beta} + \|\tilde{x}\|_{a,b,\beta}\} \|x - \tilde{x}\|_{a,b,\beta}
\]
\[
+ \|f''\|_\lambda \|\tilde{x}\|_{a,b,\beta} (\|x\|_{a,b,\beta} + \|\tilde{x}\|_{a,b,\beta}) \|x - \tilde{x}\|_{a,b,\infty}.
\]

On the other hand, we have
\[
D_{s+1}^{2\alpha - 1} f'(x)(r) - D_{s+1}^{2\alpha - 1} f'(\tilde{x})(r)
\]
\[
= \frac{1}{\Gamma(2 - 2\alpha)} \left\{ f'(x_r) - f'(\tilde{x}_r)
\right.
\]
\[
+ (2\alpha - 1) \int_s^r \left[ f'(x_r) - f'(\tilde{x}_r) - f'(x_\theta) + f'(\tilde{x}_\theta) \right] \frac{d\theta}{(r - \theta)^{2\alpha - 1}} \right\}.
\]

Using the decomposition
\[
f'(x_r) - f'(\tilde{x}_r) - f'(x_\theta) + f'(\tilde{x}_\theta)
\]
\[
= \int_0^1 f''(x_r + t(\tilde{x}_r - x_r))(\bar{x}_r - \bar{x}_\theta) dt - \int_0^1 f''(x_\theta + t(\tilde{x}_\theta - x_\theta))(\tilde{x}_\theta - \tilde{x}_\theta) dt,
\]
we obtain

\[
|D_{a+}^{2\alpha} f'(r) - D_{a+}^{2\alpha} f'(\tilde{x}(r))| \\
\leq k(r-s)^{1-2\alpha} \|f''\|_\infty \|x-\tilde{x}\|_{s,r,\infty} + k(r-s)^{\beta-2\alpha+1} \|f''\|_\infty \|x-\tilde{x}\|_{s,r,\beta} \\
+ k(r-s)^{\beta-2\alpha+1} \|f''\|_\lambda \left(\|x\|_{s,r,\beta} + \|\tilde{x}\|_{s,r,\beta}\right) \|x-\tilde{x}\|_{s,r,\infty} \\
= kI_2(r-s)^{1-2\alpha}.
\]

(3.23)

where

\[
I_2 = (\|f''\|_\infty + \|f''\|_\lambda \left(\|x\|_{a,b,\beta} + \|\tilde{x}\|_{a,b,\beta}\right) (b-a)^{\beta\lambda}) \|x-\tilde{x}\|_{a,b,\infty} \\
+ \|f''\|_\infty \|x-\tilde{x}\|_{a,b,\beta} (b-a)^{\beta}.
\]

Now using (3.19), (3.22), (3.23) we obtain

\[
\left| \int_a^b [f(x_r) - f(\tilde{x}_r)] dy_r \right| \leq k \|y\|_{a,b,\beta} \\
\times \left( \int_a^b |f(x_r) - f(\tilde{x}_r)| (r-a)^{-\alpha} (b-r)^{\alpha+\beta-1} dr \\
+ I_1 \int_a^b (r-a)^{\beta(1+\lambda)-\alpha} (b-r)^{\alpha+\beta-1} dr \\
+ kI_2 \int_a^b (b-r)^{1-2\alpha} |\Lambda^b_{a}(x \otimes y)| dr \\
+ \int_a^b |D_{a+}^{2\alpha} f'(\tilde{x}(r))| \left|\Lambda^b_{a}(x - \tilde{x}) \otimes y\right| dr.
\]

Finally, using (3.21) we get

\[
\left| \int_a^b [f(x_r) - f(\tilde{x}_r)] dy_r \right| \leq k \|y\|_{a,b,\beta} \\
\times \left( \|f''\|_\infty (b-a)^{\beta} \|x-\tilde{x}\|_{a,b,\infty} + I_1 (b-a)^{\beta(2+\lambda)} \right) \\
+ kI_2 \left( \|x \otimes y\|_{a,b,\beta} + \|x\|_{a,b,\beta} \|y\|_{a,b,\beta}\right) (b-a)^{2\beta-2\varepsilon} \\
+ \left[ \|f''\|_\infty + \|f''\|_\lambda \left|\tilde{x}\right|_{a,b,\beta} (b-a)^{\lambda \beta}\right] \\
\times \left( \|x - \tilde{x}\|_{a,b,\beta} + \|x - \tilde{x}\|_{a,b,\beta} \|y\|_{a,b,\beta}\right) (b-a)^{2\beta-2\varepsilon}.
\]

(3.24)

This implies (3.16). \[\blacksquare\]

The following corollary is the direct consequence of the proposition.

**Corollary 3.4** Assume $b - a \leq 1$. Under the hypotheses of Definition 3.1, if $(x, y, x \otimes y)$ is also a $\beta$-Hölder continuous multiplicative functional on $\mathbb{R}^m \otimes \mathbb{R}^d$, 

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we have
\[
\left\| \int f(x_r)dy_r - \int \tilde{f}(x_r)dy_r \right\|_{a,b,\beta} \\
\leq k \|f\|_\infty \|y - \tilde{y}\|_{a,b,\beta} + k \left( \|x \otimes (y - \tilde{y})\|_{a,b,\beta} + \|x\|_{a,b,\beta} \|y - \tilde{y}\|_{a,b,\beta} \right) \\
\times \left( \|f'\|_\infty + \|f''\|_{a,b,\beta} \lambda (b-a)^\lambda \right) (b-a)^{\beta - 2\varepsilon}. \tag{3.25}
\]

On the other hand, if the derivative \(f''\) is \(\lambda\)-Hölder continuous and bounded, \((\tilde{x}, y, \tilde{x} \otimes y)\) is another \(\beta\)-Hölder continuous multiplicative functional on \(\mathbb{R}^m \otimes \mathbb{R}^d\), and \(\tilde{f}\) is another function satisfying the hypotheses of Definition 3.4, then
\[
\left\| \int f(x_r)dy_r - \int \tilde{f}(x_r)dy_r \right\|_{a,b,\beta} \\
\leq k H_1^f \|x - \tilde{x}\|_{a,b,\infty} + k H_2^f \|x - \tilde{x}\|_{a,b,\beta} + k H_3^f (x - \tilde{x}) \otimes y \|y\|_{a,b,\beta} \\
+ k \|f - \tilde{f}\|_\infty \|y\|_{a,b,\beta} + k \left( \|x \otimes y\|_{a,b,\beta} + \|x\|_{a,b,\beta} \|y\|_{a,b,\beta} \right) \\
\times \left( \|f' - \tilde{f}'\|_\infty + \|f'' - \tilde{f}''\|_{a,b,\beta} \lambda (b-a)^\lambda \right) (b-a)^{\beta - 2\varepsilon}. \tag{3.26}
\]

The estimate (3.26) implies that for a fixed \(x\), the mapping \((y, x \otimes y) \rightarrow \int f(x_r)dy_r\) is continuous with respect to the \(\beta\)-norm. As a consequence, if \(y^n\) is a sequence of continuously differentiable functions (or Lipschitz functions) such that
\[
\|y - y^n\|_\beta \rightarrow 0, \\
\|x \otimes y - x \otimes y^n\|_\beta \rightarrow 0
\]
as \(n\) tends to infinity, then
\[
\left\| \int f(x_r)dy_r - \int f(x_r)dy^n_r \right\|_\beta \rightarrow 0. \tag{3.27}
\]

Hence, the integral \(\int f(x_r)dy_r\) introduced in Definition 3.4 does not depend on the parameters \(\alpha\) and \(\varepsilon\), and it coincides with the classical integral \(\int f(x_r)y^r_sdr\) when \(y\) is continuously differentiable.

Set \(t^n_i = \frac{iT}{n}\) for \(i = 0, 1, \ldots, n\). If \(y\) is \(\beta\)-Hölder continuous, the sequence of functions
\[
y^n_i = y_0 1_{(0)}(t) + \sum_{i=1}^{n} 1_{(t_{i-1}^{n}, t_i^{n})}(t) \left[ y_{t_i^{n-1}} + \frac{n}{T} \left( t - t_{i-1}^{n} \right) \left( y_{t_i^{n}} - y_{t_{i-1}^{n}} \right) \right]
\]
converge to \(y\) in the \(\beta'\)-norm for any \(\beta' < \beta\). Assume that the multiplicative functional \(\int f(x_r - x_s)dy^n_r\) converges in the \(\beta'\)-norm as \(n\) tends to infinity to \((x \otimes y)_{s,t}\). Then (3.27) holds with \(\beta = \beta'\). In particular, this means that
\[
\int_0^T f(x_r)dy_r = \lim_{n \rightarrow \infty} \sum_{i=1}^{n} \frac{n}{T} \left( \int_{t_{i-1}^{n}}^{t_i^{n}} f(x_s)ds \right) (y_{t_i^{n}} - y_{t_{i-1}^{n}}). \tag{3.28}
\]
For any $p \geq 1$, the $p$ variation of a function $x : [0,T] \to \mathbb{R}$ is defined as
\[
\text{Var}_p(x) = \sup_\pi \left( \sum_{i=1}^n |x(t^n_i) - x(t^n_{i-1})|^p \right)^{1/p},
\]
where $\pi = \{0 = t_0 < \cdots < t_n = T\}$ runs over all partitions of $[0,T]$. Notice that
\[
\text{Var}_1/\beta(x) \leq \|x\|_{\beta},
\]
Then, for any $\beta$-Hölder continuous multiplicative functional $(x,y,x \otimes y)$ on $\mathbb{R}^m \otimes \mathbb{R}^d$ and any function $f$ satisfying the hypotheses of Definition 3.1, the integral $\int_0^T f(x_r)dy_r$ coincides with the integral defined using the $1/\beta$ variation norm (see [5]). This implies that $\int_0^T f(x_s)dy_s$ is given by the limit of the Riemann sums of the form
\[
\int_0^T f(x_s)dy_s = \lim_{|\pi| \to 0} \sum_{i=1}^n f(x_{t_{i-1}})(y_{t_i} - y_{t_{i-1}}) + f'(x_{t_{i-1}})(x \otimes y)_{t_{i-1},t_i},
\]
where $\pi = \{0 = t_0 < \cdots < t_n = T\}$ runs over all partitions of $[0,T]$.

In order to handle differential equations we need to introduce the tensor product of two multiplicative functionals:

**Definition 3.5** Suppose that $(x,y,x \otimes y)$ and $(y,z,y \otimes z)$ are $\beta$-Hölder continuous real valued multiplicative functionals. Then, for all $a \leq b \leq c$, we define
\[
(x \otimes (y \otimes z)_{a,c})_{a,b} = \int_a^b \Lambda_{a,r,b,c}(x,z)D^1_{r-a}y_b -(r)dr,
\]
\[
\frac{1}{\Gamma(2-2\alpha)} \int_{a<r<\xi<\eta<b} K_{r,b}(\xi,\eta) 
\times \frac{\Gamma^\alpha(\alpha+\varepsilon) (x \otimes y)_{\xi,a} (z_{\xi} - z_{\eta}) - (x_{r} - x_{a}) \Gamma^{\alpha-\varepsilon} (y \otimes z)_{\xi,a}}{(r-a)^{2\alpha-1}}d\xi d\eta,
\]
where
\[
\Lambda_{a,r,b,c}(x,z) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left( \frac{(x_{r} - x_{a})(z_{\xi} - z_{r})}{(r-a)^{\alpha}} + \alpha \int_a^r \frac{(z_{r} - z_{\theta})(x_{\theta} - x_{r})}{(r-\theta)^{\alpha+1}}d\theta \right).
\]

We have the following result.

**Proposition 3.6** If the function $y$ is continuously differentiable and for all $a \leq b$
\[
(y \otimes z)_{a,b} = \int_a^b (z_{b} - z_{r}) y'_r dr,
\]
\[
(x \otimes y)_{a,b} = \int_a^b (x_{r} - x_{a})y'_r dr,
\]

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then
\[ (x \otimes (y \otimes z),c)_{a,b} = \int_a^b (x_r - x_a)(z_c - z_r)y'_r dr. \]

**Proof.** We are going to use formula (3.9) with \( m = 2, d = 1, f(x, z) = xz \) and the functions \( x_t - x_a \) and \( z_c - z_t \). In this way we obtain

\[
\int_a^b (x_\theta - x_a)(z_c - z_\theta)dy_\theta = \frac{(-1)^{\alpha}}{\Gamma(1 - \alpha)} \int_a^b \left( \frac{(x_r - x_a)(z_c - z_r)}{(r - a)^{\alpha}} + \alpha \int_a^r \frac{(z_r - z_\theta)(x_\theta - x_r)}{(r - \theta)^{\alpha + 1}} d\theta \right) \\
\times D_{a - \alpha}^{1 - \alpha} y_b - (r) dr
\]

\[
+ \frac{1}{\Gamma(2 - 2\alpha)} \int_a^b \left[ \frac{z_c - z_r}{(r - a)^{2\alpha - 1}} + (2\alpha - 1) \int_a^r \frac{z_r - z_\theta}{(r - \theta)^{2\alpha}} d\theta \right] \\
\times \int_r^b \int_r^\eta K_{r,b}(\xi, \eta) \Gamma_{-\varepsilon}(x \otimes y)_{\xi, \eta} d\xi d\eta dr
\]

\[
- \frac{1}{\Gamma(2 - 2\alpha)} \int_a^b \left[ \frac{x_r - x_a}{(r - a)^{2\alpha - 1}} + (2\alpha - 1) \int_a^r \frac{x_r - x_\theta}{(r - \theta)^{2\alpha}} d\theta \right] \\
\times \int_r^b \int_r^\eta K_{r,b}(\xi, \eta) \Gamma_{-\varepsilon}(y \otimes z)_{\xi, \eta} d\xi d\eta dr,
\]

and this completes the proof. ■

It is easy to obtain the following estimate

**Proposition 3.7** Suppose that \( (x, y, x \otimes y) \) and \( (y, z, y \otimes z) \) are \( \beta \)-Hölder continuous real valued multiplicative functionals. Then, for any \( a \leq b \leq c \), we have

\[
\left| (x \otimes (y \otimes z),c)_{a,b} \right| \leq k \left( \|y\|_{a,b,\beta} \|x\|_{a,b,\beta} \|z\|_{a,b,\beta} \right.
\]

\[
+ \|z\|_{a,b,\beta} \|y \otimes x\|_{a,b,\beta} + \|x\|_{a,b,\beta} \|y \otimes z\|_{a,b,\beta} ) (b - a)^{3\beta}
\]

\[
+ k \|z\|_{a,c,\beta} \left( \|x\|_{a,b,\beta} \|x\|_{a,b,\beta} + \|y \otimes x\|_{a,b,\beta} \right) (b - a)^{2\beta}(c - a)\beta.
\]

If \( b = c \) we write \( (x \otimes (y \otimes z),c)_{a,b} = (x \otimes y \otimes z)_{a,b} \). If the functions \( x, y \) and \( z \) are continuously differentiable, then

\[
(x \otimes y \otimes z)_{a,b} = \int_{a < r < \theta < \sigma < b} x'_r y'_\theta z'_\sigma dr d\theta d\sigma.
\]

Define

\[
\|x \otimes y \otimes z\|_{a,b,\beta} = \sup_{a \leq b < r < \sigma \leq b} \frac{|(x \otimes y \otimes z)_{c,c}|}{[r - \theta]^{3\beta}}.
\]
Then, Proposition 3.7 implies that
\[
\| x \otimes y \otimes z \|_{a,b,\beta} \leq \kappa \left( \| y \|_{a,b,\beta} \| x \|_{a,b,\beta} \| z \|_{a,b,\beta} + \| z \|_{a,b,\beta} \| y \otimes x \|_{a,b,\beta}
\right.
\]
\[
+ \| x \|_{a,b,\beta} \| y \otimes z \|_{a,b,\beta} \right) .
\]
(3.29)

Proposition 3.7 also implies that \( (x, (y \otimes z)_{-c}, (x \otimes (y \otimes z))_{-c}) \) is a \( \beta \)-Hölder continuous functional on the interval \([0,c]\). As a consequence, if \( f \) satisfies the assumptions of Definition 3.1, we can define the integral \( \int_{a}^{b} f(x) \, d_{r} \, (y \otimes z)_{r,c} \), for all \( a \leq b \leq c \). The following estimate for this integral will be needed to solve differential equations.

**Proposition 3.8** Suppose that \( (x,y,x \otimes y) \) and \( (y,z,y \otimes z) \) are \( \beta \)-Hölder continuous multiplicative functionals on \( \mathbb{R}^{m} \otimes \mathbb{R}^{d} \). Let \( f : \mathbb{R}^{m} \to \mathbb{R}^{d} \) be a continuously differentiable function such that \( f' \) is \( \lambda \)-Hölder continuous, where \( \lambda > \frac{1}{\beta} - 2 \).

Fix \( \alpha > 0 \) and \( \epsilon > 0 \) such that \( 1 - \beta < \alpha < 2 \beta \), \( \alpha < \frac{\lambda \beta + 1}{2} \) and \( \epsilon < \alpha + \beta - 1 \). Then the following estimate holds:
\[
\sup_{a \leq \xi \leq \eta \leq b} \frac{1}{(\eta - \xi)^{2 \beta}} \left| \int_{\xi}^{\eta} f(x_{r}) \, d_{r} \, (y \otimes z)_{r,c} \right| \leq k \left[ A_{a,b}
\right.
\]
\[
+ B_{a,b} \| x \otimes y \|_{a,b,\beta} (b - a)^{\beta - 2 \epsilon} \right] ,
\]
(3.30)
where
\[
A_{a,b} = \left( \| y \otimes z \|_{a,b,\beta} + \| y \|_{a,b,\beta} \| z \|_{a,b,\beta} \right) \times \left[ \| f \|_{\infty} + \left( \| x \|_{a,b,\beta} \| f' \|_{\infty} + \| f' \|_{\lambda} \| x \|_{a,b,\beta}^{1+\lambda} (b - a)^{\lambda \beta} \right) (b - a)^{\beta - 2 \epsilon} \right] ,
\]
and
\[
B_{a,b} = \| z \|_{a,b,\beta} \left( \| f' \|_{\infty} + \| f' \|_{\lambda} \| x \|_{a,b,\beta}^{\lambda} (b - a)^{\lambda \beta} \right) .
\]
(3.31)

**Proof.** To simplify the proof we will assume \( d = m = 1 \). From (3.1) it is easy to see that
\[
\left( x \otimes y \right)_{-b} \leq \left( \| x \|_{a,b,\beta} + \| y \|_{a,b,\beta} \right) (b - a)^{\beta} ,
\]
(3.33)
and from Proposition 3.7 we have
\[
\| x \otimes (y \otimes z)_{-b} \|_{a,b,\beta} \leq k \left( \| x \|_{a,b,\beta} \| y \|_{a,b,\beta} \| z \|_{a,b,\beta} 
\right.
\]
\[
+ \| x \|_{a,b,\beta} \| y \otimes z \|_{a,b,\beta} + \| z \|_{a,b,\beta} \| x \otimes y \|_{a,b,\beta} \right) (b - a)^{\beta} .
\]
(3.34)
From (26), (3.33), and (3.34) we obtain

\[
\left| \int_{a}^{b} f(x) dr (y \otimes z)_{r,t} \right| \\
\leq k \|f\|_{\infty} \| (y \otimes z)_{.b} \|_{a,b,\beta} (b-a)^{\beta} \\
+ k \left( \|x \otimes (y \otimes z)_{.b} \|_{a,b,\beta} + \|x\|_{a,b,\beta} \right) \| (y \otimes z)_{.b} \|_{a,b,\beta} \\
\times \left( \|f'\|_{\infty} + \|f'\| \|x\|_{a,b,\beta} (b-a)^{\lambda \beta} \right) (b-a)^{2\beta-2\varepsilon}
\]

\[
\leq k \|f\|_{\infty} \left( \|y \otimes z\|_{a,b,\beta} + \|y\|_{a,b,\beta} \|z\|_{a,b,\beta} \right) (b-a)^{2\beta} \\
+ k \left( \|x\|_{a,b,\beta} \|y\|_{a,b,\beta} \|z\|_{a,b,\beta} + \|x\|_{a,b,\beta} \|y \otimes z\|_{a,b,\beta} + \|z\|_{a,b,\beta} \|x \otimes y\|_{a,b,\beta} \right) \\
\times \left( \|f'\|_{\infty} + \|f'\| \|x\|_{a,b,\beta} \|x \otimes y\|_{a,b,\beta} \right) (b-a)^{3\beta-2\varepsilon},
\]

which implies the desired result. ■

4 Differential Equations Driven by Rough Paths

Let \( y : [0, 1] \rightarrow \mathbb{R}^{d} \) be a \( \beta \)-Hölder continuous function. Suppose that \((y^i, y^j, y^i \otimes y^j)\) is a \( \beta \)-Hölder continuous multiplicative function, for each \( i, j = 1, \ldots, d \). We aim to solve the differential equation

\[
x_t = x_0 + \int_{0}^{t} f(x_r) dy_r,
\]

where \( f = \mathbb{R}^{m} \rightarrow \mathbb{R}^{md} \).

Formula (3.9) and Definition 3.5 allow us to transform this equation into the following system of integral equations:

\[
x_t = x_0 + (-1)^{\alpha} \int_{0}^{t} \hat{D}_{0+}^{\alpha} f(x) (s) D_{t-s}^{1-\alpha} y_t(s) ds \\
+ \int_{0}^{t} \hat{D}_{0+}^{2\alpha-1} f(x) (s) \int_{s}^{t} \int_{s}^{\eta} K_{0,s} (\xi, \eta) \Gamma^{\alpha-\varepsilon} (x \otimes y)_{\xi,\eta} d\xi d\eta ds,
\]

\[
(x \otimes y)_{s,t} = (-1)^{\alpha} \int_{s}^{t} \hat{D}_{s+}^{\alpha} f(x) (r) D_{t-r}^{1-\alpha} (y \otimes y)_{t-r} (r) dr \\
+ \int_{s}^{t} \hat{D}_{s+}^{2\alpha-1} f(x) (r) \\
\times \int_{r}^{t} \int_{r}^{\eta} K_{s,r} (\xi, \eta) \Gamma^{\alpha-\varepsilon} (x \otimes (y \otimes y))_{\xi,\eta} d\xi d\eta dr.
\]

**Theorem 4.1** Let \( y : [0, 1] \rightarrow \mathbb{R}^{d} \) be a \( \beta \)-Hölder continuous function. Suppose that \((y^i, y^j, y^i \otimes y^j)\) is a real valued \( \beta \)-Hölder continuous multiplicative function, for each \( i, j = 1, \ldots, d \). Let \( f : \mathbb{R}^{m} \rightarrow \mathbb{R}^{md} \) be a continuously differentiable
function such that $f'$ is $\lambda$-H"older continuous, where $\lambda > \frac{1}{\beta} - 2$, and $f$ and $f'$ are bounded. Set

$$\rho_f := ||f||_\infty + ||f'||_\infty + ||f'||_\lambda.$$ 

Then there is a solution to Equations (4.2)-(4.3), such that $(x, y, x \otimes y)$ is a $\beta$-H"older continuous multiplicative functional. Moreover, for any $\gamma > \frac{1}{\beta}$ the function $x$ satisfies the estimate

$$\sup_{0 \leq t \leq T} |x_t| \leq |x_0| + T \left\{ 2k\rho_f \left[ ||y||_\beta + \frac{||y \otimes y||_\beta}{||y||_\beta} \right] \lor 1 \right\}^{\gamma}, \tag{4.4}$$

where $k$ is a universal constant depending only on $\beta$ and $\gamma$.

**Proof.** To simplify the proof we will assume $d = m = 1$. The proof will be done in several steps.

**Step 1.** Fix $\alpha > 0$ and $\varepsilon > 0$ such that $1 - \beta < \alpha < 2\beta$, $\alpha < \frac{\lambda(\beta + 1)}{2}$, $\varepsilon < \alpha + \beta - 1$, $\varepsilon < \frac{\beta}{2}$, and $(1 - 2\varepsilon)/(\beta - 2\varepsilon) < \gamma$.

We will write the Equations (4.2) and (4.3) in the compact form

$$x = \Phi_1(x, y, x \otimes y),$$

$$x \otimes y = \Phi_2(x, y, y \otimes y, x \otimes y).$$

Consider the mapping $J : (x, x \otimes y) \to (J_1 x, J_2 (x \otimes y))$ defined by

$$J_1 x = \Phi_1(x, y, x \otimes y),$$

$$J_2 (x \otimes y) = \Phi_2(x, y, y \otimes y, x \otimes y).$$

We need some a priori estimates of the H"older norms of $J_1 x$ and $J_2 (x \otimes y)$ in terms of the H"older norms of $x$ and $x \otimes y$. From (3.15) it follows that

$$||J_1 x||_{s,t,\beta} \leq k ||f||_\infty ||y||_{s,t,\beta} + \left( ||x \otimes y||_{s,t,\beta} + ||x||_{s,t,\beta} ||y||_{s,t,\beta} \right)$$

$$\times \left( ||f'||_\infty + ||f'||_\lambda ||x||_{s,t,\beta} (t - s)^{\lambda \beta} (t - s)^{\beta - 2\varepsilon} \right). \tag{4.5}$$

On the other hand, Proposition 3.3 implies that

$$||J_2 (x \otimes y)||_{s,t,\beta} \leq k \left[ A_{s,t} + B_{s,t} ||x \otimes y||_{s,t,\beta} (t - s)^{\beta - 2\varepsilon} \right], \tag{4.6}$$

where $A_{s,t}$ and $B_{s,t}$ are defined by (3.31) and (3.32), respectively.

**Step 2.** Set

$$\alpha(y) := \left( 2k\rho_f \left[ ||y||_\beta + \frac{||y \otimes y||_\beta}{||y||_\beta} \right] \lor 1 \right)^{1/(\beta - 2\varepsilon)},$$

where $k$ is the constant appearing in formulas (4.5) and (4.6). Suppose that

$$0 < t - s \leq \frac{1}{\alpha(y)}. \tag{4.7}$$
Then, the inequalities
\[ \|x\|_{s,t,\beta} \leq 2k\rho_f \|y\|_\beta \] (4.8)
\[ (t-s)^{\beta-2\epsilon} \|x \otimes y\|_{s,t,\beta} \leq \|y\|_\beta \] (4.9)

imply that
\[ \|J_1 x\|_{s,t,\beta} \leq 2k\rho_f \|y\|_\beta \] (4.10)
\[ (t-s)^{\beta-2\epsilon} \|J_2 (x \otimes y)\|_{s,t,\beta} \leq \|y\|_\beta \] (4.11)

In fact, from the definition of \( \alpha(y) \) and (4.8) we deduce
\[ (t-s)^{\beta-2\epsilon} \|x\|_{s,t,\beta} \leq 1. \] (4.12)

By the definition of \( B_{s,t} \) and \( A_{s,t} \) we have
\[ B_{s,t} \leq (\|f\|_\infty + \|f'\|_\lambda) \|y\|_\beta \leq \rho_f \|y\|_\beta \leq \frac{(t-s)^{-(\beta-2\epsilon)}}{2k}, \] (4.13)
and
\[ A_{s,t} \leq (\|f\|_\infty + \|f'\|_\infty + \|f'\|_\lambda) \left( \|y \otimes y\|_\beta + \|y\|_\beta^2 \right) \]
\[ \leq \frac{(t-s)^{-(\beta-2\epsilon)}}{2k} \|y\|_\beta. \] (4.14)

Therefore, substituting (4.13) and (4.14) into (4.6) we obtain (4.11). Finally, from (4.13) we get (4.11).

**Step 3.** We can now proceed with the proof of the existence. Let \( N \) be a natural number such that \( \frac{T}{N} = \delta \leq \frac{1}{\alpha(y)} \). We partition the interval \([0,T]\) in \( N \) subintervals of the same length and set \( t_i = \frac{iT}{N}, \) \( i = 0, 1, \ldots, N-1 \). We will make use of the notation \( \|x\|_i = \|x\|_{t_{i-1},t_i,\beta} \) and \( \|x \otimes y\|_i = \|x \otimes y\|_{t_{i-1},t_i,\beta} \), for \( i = 1, \ldots, N-1 \). From Step 2 we know that if that \( x \) and \( x \otimes y \) satisfy
\[ \|x\|_i \leq 2k\rho_f \|y\|_\beta \]
\[ \|x \otimes y\|_i \leq \|y\|_\beta \delta^{-(\beta-2\epsilon)}, \]

for any \( i = 1, \ldots, N-1 \), then the same inequalities hold for \( Jx \) and \( Jx \otimes y \), that is
\[ \|J_1 x\|_i \leq 2k\rho_f \|y\|_\beta \]
\[ \|J_2 (x \otimes y)\|_i \leq \|y\|_\beta \delta^{-(\beta-2\epsilon)}. \]

Consequently, there is a constant \( C_1 \) such that
\[ \|J_1^n x\|_\beta + \|J_2^n (x \otimes y)\|_\beta \leq C_1. \]

This implies that the sequence of functions \( J_1^n x \) is equicontinuous and bounded in \( C^{\beta} \). Therefore, there exists a subsequence which converges in the \( \beta'-\text{Hölder} \)
norm if $\beta' < \beta$. In the same way, there is a subsequence of $J^n_2(x \otimes y')$ which converges in the $\beta'$-Hölder norm. The limit $(x, x \otimes y)$ defines a $\beta$-Hölder continuous multiplicative functional $(x, y, x \otimes y)$. Using the continuity of the solution in this norm it is not difficult to show that the limit is a solution. This implies the existence of a solution, which satisfies (4.8) and (4.9).

**Step 4.** Let us now prove the estimate (4.14). By step 2, the solution we have constructed satisfies the estimates (4.8) and (4.9) if (4.7) holds. Then it follows that for any $r \in [s, t]$

$$\sup_{r \in [s, t]} |x_r| \leq |x_s| + (t - s)^\beta \|x\|_{s,t,\beta} \leq |x_s| + (t - s)^{2\varepsilon}.$$ 

Since the interval $[0, T]$ can be divided into $[T/\tau]$ intervals of length $\tau = \frac{1}{\alpha(y)}$, the inequality (4.14) follows.

**Theorem 4.2** Let $y : [0, 1] \to \mathbb{R}^d$ be a $\beta$-Hölder continuous function. Suppose that $(y_i, y_j, y_i \otimes y_j)$ is a real valued $\beta$-Hölder continuous multiplicative function, for each $i, j = 1, \ldots, d$. Let $f : \mathbb{R}^m \to \mathbb{R}^{md}$ be a twice continuously differentiable function such that $f''$ is $\lambda$-Hölder continuous, where $\lambda > \frac{1}{\beta} - 2$, and $f$, $f'$ and $f''$ are bounded. Then there is a unique solution to Equations (4.12) - (4.13) such that $(x, y, x \otimes y)$ is a $\beta$-Hölder continuous multiplicative functional.

Moreover, if $\tilde{x}$ satisfies $\tilde{x}_t = \tilde{x}_0 + \int_0^t f(\tilde{x}_r)dy_r \epsilon$ and $\tilde{y}$ verifies the same hypotheses as $y$, then

$$\sup_{0 \leq t \leq T} |x_t - \tilde{x}_t| \leq C \|y\|_\beta \|y \otimes y\|_\beta + (y \otimes (y - \tilde{y}))_\beta,$$

(4.15)

where $C$ depends on $\|y\|_\beta$, $\|y \otimes y\|_\beta$, $\beta$, $\lambda$, and $\hat{p}_f$, and where

$$\hat{p}_f = \|f\|_\infty + \|f'\|_\lambda + \|f''\|_\infty + \|f'\|_\lambda + \|f''\|_\lambda + \|f''\|_\lambda.$$  

**Proof.** To simplify the proof we will assume $d = m = 1$. Notice that uniqueness follows from the estimate (4.14). So it suffices to show this inequality. We fix $s < t$ such that $t - s \leq \frac{1}{\alpha(y)}$, where $\beta$ is defined as follows

$$\beta(y) = \left(2k\hat{p}_f \left[\|y\|_\beta + \|y\|_\beta + \|y \otimes y\|_\beta + \frac{\|y \otimes y\|_\beta}{\|y\|_\beta}\right] \vee 1\right)^{1/(\beta \lambda)}.$$  

(4.16)

The constant $k$ appearing in the definition of $\beta$ will be chosen later. We choose $\alpha > 0$ and $\varepsilon > 0$ such that $1 - \beta < \alpha < 2\beta$, $\alpha < \frac{\lambda \beta + 1}{\beta}$, and $\varepsilon < \alpha + \beta - 1$, $\varepsilon < \frac{2}{\beta}$. We also assume that the solutions $x$ and $\tilde{x}$ satisfy the following inequalities:

$$(t - s)^{\beta - 2\varepsilon} \|x\|_\beta \leq 1,$$  

(4.17)

$$(t - s)^{\beta - 2\varepsilon} \|\tilde{x}\|_\beta \leq 1,$$  

(4.18)

$$(t - s)^{\beta - 2\varepsilon} \|x \otimes y\|_\beta \leq \|y\|_\beta.$$  

(4.19)
Our first purpose is to estimate the Hölder norm \( \| x - \bar{x} \|_{s,t,\beta} \). We can write

\[
\| x - \bar{x} \|_{s,t,\beta} \leq \left\| \int [f(x_s) - f(\bar{x}_s)] \, dy_s \right\|_{s,t,\beta} + \left\| \int f(\bar{x}_s) d(y_s - \bar{y}_s) \right\|_{s,t,\beta} = I_{1,s,t} + I_{2,s,t}.
\]

The term \( I_{1,s,t} \) can be estimated using (3.16) and we obtain

\[
I_{1,s,t} \leq k[H_1 \| x - \bar{x} \|_{s,t,\infty} + H_2 \| x - \bar{x} \|_{s,t,\beta} + H_3 \| (x - \bar{x}) \otimes y \|_{s,t,\beta}],
\]

where

\[
H_1 = \| y \|_{s,t,\beta} \left( \| f' \|_{\infty} + \| f'' \|_{\lambda} \| \bar{x} \|_{s,t,\beta} \right) (t - s)^{\beta(1 + \lambda)} + \left( \| f'' \|_{\infty} + \| f'' \|_{\lambda} \right) \| \bar{x} \|_{s,t,\beta} (t - s)^{\beta}.
\]

\[
H_2 = \| f'' \|_{\infty} \| y \|_{s,t,\beta} \left( \| x \|_{s,t,\beta} + \| \bar{x} \|_{s,t,\beta} \right) (t - s)^{\beta(1 + \lambda)} + \| f'' \|_{\infty} \left( \| x \|_{s,t,\beta} + \| \bar{x} \|_{s,t,\beta} \| y \|_{s,t,\beta} \right) (t - s)^{2\beta - 2\varepsilon} + \left( \| f'' \|_{\infty} + \| f'' \|_{\lambda} \right) \| \bar{x} \|_{s,t,\beta} (t - s)^{\beta - 2\varepsilon}.
\]

\[
H_3 = \left( \| f'' \|_{\infty} + \| f'' \|_{\lambda} \right) \| \bar{x} \|_{s,t,\beta} (t - s)^{\beta - 2\varepsilon}.
\]

Then, using the inequalities (4.17), (4.18), and (4.19) we get the following estimates

\[
H_1 \leq \| y \|_{\beta} \left( \| f'' \|_{\infty} + 2 \| f'' \|_{\infty} + 6 \| f'' \|_{\lambda} \right),
\]

\[
H_2 \leq \| y \|_{\beta} \left( 3 \| f'' \|_{\infty} + \| f'' \|_{\lambda} \right) (t - s)^{\beta},
\]

\[
H_3 \leq \left( \| f'' \|_{\infty} + \| f'' \|_{\lambda} \right).
\]

It remains to handle the term \( \| (x - \bar{x}) \otimes y \|_{s,t,\beta} \) in (4.20). To get estimates for this term we apply again the inequality (3.16) and we have

\[
\| (x - \bar{x}) \otimes y \|_{s,t,\beta} = \left\| \int_s^t [f(x_r) - f(\bar{x}_r)] \, d_r(y \otimes y)_{r,t} \right\|_{s,t,\beta}
\leq k(t - s)^{\beta} \left( \tilde{H}_1 \| x - \bar{x} \|_{s,t,\infty} + \tilde{H}_2 \| x - \bar{x} \|_{s,t,\beta} + \tilde{H}_3 \| (x - \bar{x}) \otimes (y \otimes y) \|_{s,t,\beta} \right),
\]

where

\[
\tilde{H}_1 = \| y \|_{\beta} \cdot \| x \|_{s,t,\beta} \left( \| f'' \|_{\infty} + \| f'' \|_{\lambda} \| \bar{x} \|_{s,t,\beta} \right) (t - s)^{\beta(1 + \lambda)} + \left( \| f'' \|_{\infty} + \| f'' \|_{\lambda} \right) \| \bar{x} \|_{s,t,\beta} (t - s)^{\beta}.
\]

\[
\tilde{H}_2 = \| y \|_{\beta} \cdot \| x \|_{s,t,\beta} \| \bar{x} \|_{s,t,\beta} (t - s)^{\beta - 2\varepsilon}.
\]
\[ H_2 = \| f'' \|_{\infty} \|(y \otimes y) \cdot t, \xi \|_{s, t, \beta} (\|x\|_{s, t, \beta} + \|\bar{x}\|_{s, t, \beta}) (t-s)^{\beta(1+\lambda)} \]
\[ + \| f'' \|_{\infty} (\|x \otimes (y \otimes y) \cdot t, \xi \|_{s, t, \beta} + \|x\|_{s, t, \beta} \|(y \otimes y) \cdot t, \xi \|_{s, t, \beta}) (t-s)^{2\beta-2\varepsilon} \]
\[ + (\|f\|_{\infty} + \|f\|_{\lambda} \|\bar{x}\|_{s, t, \beta}(t-s)^{\lambda\beta}) \|(y \otimes y) \cdot t, \xi \|_{s, t, \beta} (t-s)^{\beta-2\varepsilon}, \]
and
\[ H_3 = H_3. \]

Using \ref{333}, \ref{334}, \ref{335}, \ref{336} and \ref{139} we get the following estimates
\[ \tilde{H}_1 = \left(\|y \otimes y\|_{s, t, \beta} + \|y\|_{s, t, \beta}^2\right) \]
\[ \times \left(\|f\|_{\infty} + \|f''\|_{\lambda} \|\bar{x}\|_{s, t, \beta} (\|x\|_{s, t, \beta} + \|\bar{x}\|_{s, t, \beta})\right) (t-s)^{3(2+\lambda)} \]
\[ + k \left(\|f''\|_{\infty} + \|f''\|_{\lambda} (\|x\|_{s, t, \beta} + \|\bar{x}\|_{s, t, \beta}) (t-s)^{3\beta-2\varepsilon}\right) \]
\[ \times \left(\|x\|_{s, t, \beta} \|y\|_{s, t, \beta} + \|x\|_{s, t, \beta} \|y\|_{s, t, \beta} + \|y\|_{s, t, \beta} \|x \otimes y\|_{s, t, \beta}\right) (t-s)^{3\beta-2\varepsilon} \]
\[ \leq k \left(\|y \otimes y\|_{\beta} + \|y\|_{\beta}^2\right) \left(\|f''\|_{\infty} + \|f''\|_{\lambda} (t-s)^{\beta}\right), \tag{4.25} \]
and
\[ \tilde{H}_2 = \|f''\|_{\infty} \left(\|y \otimes y\|_{s, t, \beta} + \|y\|_{s, t, \beta}^2\right) \left(\|x\|_{s, t, \beta} + \|\bar{x}\|_{s, t, \beta}\right) (t-s)^{3(2+\lambda)} \]
\[ + k \left(\|f''\|_{\infty} \left(\|y\|_{s, t, \beta} \|x \otimes y\|_{s, t, \beta}\right) \right) \]
\[ + \|x\|_{s, t, \beta} \|y \otimes y\|_{s, t, \beta} + \|x\|_{s, t, \beta} \|y\|_{s, t, \beta}^2 \right) (t-s)^{3\beta-2\varepsilon} \]
\[ + k \left(\|f\|_{\infty} + \|f\|_{\lambda} \|\bar{x}\|_{s, t, \beta}(t-s)^{\lambda\beta}\right) \left(\|y \otimes y\|_{s, t, \beta} + \|y\|_{s, t, \beta}^2\right) (t-s)^{3\beta-2\varepsilon} \]
\[ \leq k \left(\|f\|_{\infty} + \|f\|_{\lambda} + \|f''\|_{\infty}\right) \left(\|y \otimes y\|_{\beta} + \|y\|_{\beta}^2\right) (t-s)^{\beta}. \tag{4.27} \]

On the other hand, from \ref{334} we get
\[ \|x - \bar{x}\|_{s, t, \beta} (y \otimes y)_{s, t, \beta} \leq k \left(\|x - \bar{x}\|_{s, t, \beta} \|y\|_{\beta}^2\right) \]
\[ + \|x - \bar{x}\|_{s, t, \beta} \|y \otimes y\|_{\beta} + \|y\|_{\beta} \|x \otimes y\|_{s, t, \beta}\right) (t-s)^{\beta}. \tag{4.28} \]

Thus, substituting \ref{328}, \ref{329}, \ref{328} and \ref{328} into \ref{328} yields
\[ \|(x - \bar{x}) \otimes y\|_{s, t, \beta} \leq k(t-s)^{\beta} \left(\|y \otimes y\|_{\beta} + \|y\|_{\beta}^2\right) \]
\[ \times \left(\|f''\|_{\infty} + \|f''\|_{\lambda} + \|f''\|_{\infty}\right) \|(x - \bar{x})\|_{s, t, \infty} \]
\[ + \left(\|f\|_{\infty} + \|f\|_{\lambda} + \|f''\|_{\infty}\right) \left(\|y \otimes y\|_{\beta} + \|y\|_{\beta}^2\right) \left\|x - \bar{x}\right\|_{s, t, \beta} \]
\[ + \left(\|f\|_{\infty} + \|f\|_{\lambda}\right) \left(\|x - \bar{x}\|_{s, t, \beta} \|y\|_{\beta} + \|x - \bar{x}\|_{s, t, \beta} \|y\|_{\beta} \|x \otimes y\|_{s, t, \beta}\right) \]
\[ \leq k(t-s)^{\beta} \rho_f \left(\|y \otimes y\|_{\beta} + \|y\|_{\beta}^2\right) \left(\|x - \bar{x}\|_{s, t, \infty} + \|x - \bar{x}\|_{s, t, \beta}\right) \]
\[ + k(t-s)^{\beta} \left(\|f\|_{\infty} + \|f\|_{\lambda}\right) \|y\|_{\beta} \|(x - \bar{x}) \otimes y\|_{s, t, \beta}. \]
The condition $t - s \leq 1/\beta(y)$, if the constant in $\beta(y)$ is chosen in an appropriate way, implies that

$$k(t - s)^\beta (\|f\|_\infty + \|f\|_\lambda) \|y\|_\beta \leq \frac{1}{2}.$$ 

Hence,

$$\|(x - \tilde{x}) \otimes y\|_{s,t,\beta} \leq k(t-s)^\beta \tilde{\rho}_f \left( \|y\|_\beta (\|y\|_\beta + \|y\|_\beta^2) (\|x - \tilde{x}\|_{s,t,\infty} + \|x - \tilde{x}\|_{s,t,\beta}) \right).$$

Substituting (4.29), (4.21), (4.22) and (4.23) into (4.20) yields

$$I_{1,s,t} \leq k \tilde{\rho}_f \sup_{k} \left( \|y\|_\beta \|x - \tilde{x}\|_{s,t,\infty} + \|y\|_\beta \|x - \tilde{x}\|_{s,t,\beta} (t-s)^\beta \right) \leq k \tilde{\rho}_f \|y\|_\beta + k \|x \otimes (y - \tilde{y})\|_{s,t,\beta} (\|f\|_\infty + \|f\|_\lambda) (t-s)^{\beta-2\varepsilon}.$$ 

Again, condition $t - s \leq 1/\beta(y)$, if the constant in $\beta(y)$ is chosen in an appropriate way, implies that

$$I_{1,s,t} \leq k \tilde{\rho}_f \|y\|_\beta \|x - \tilde{x}\|_{s,t,\infty} + \frac{1}{2} \|x - \tilde{x}\|_{s,t,\beta}.$$ (4.30)

For the term $I_{2,s,t}$ we have the following estimates, using (3.25) and (3.26)

$$I_{2,s,t} \leq k \|f\|_\infty \|y - \tilde{y}\|_{s,t,\beta} + k \left( \|x \otimes (y - \tilde{y})\|_{s,t,\beta} + \|x\|_{s,t,\beta} \|y - \tilde{y}\|_{s,t,\beta} \right) \times \left( \|f\|_\infty + (\|f\|_\lambda \|x\|_{s,t,\beta} (t-s)^{\beta-2\varepsilon} \right) \leq k \tilde{\rho}_f \|y - \tilde{y}\|_\beta + k \|x \otimes (y - \tilde{y})\|_{s,t,\beta} (\|f\|_\infty + \|f\|_\lambda) (t-s)^{\beta-2\varepsilon}.$$ (4.31)

In order to estimate $\|x \otimes (y - \tilde{y})\|_{s,t,\beta}$ we make use of Proposition 3.8 and we obtain

$$\|x \otimes (y - \tilde{y})\|_{s,t,\beta} = \sup_{\xi \leq \xi \leq \eta \leq t} \left\{ \frac{1}{(\eta - \xi)^{2\beta}} \left| \int_{\xi}^{\eta} f(x_r) d_r (y \otimes (y - \tilde{y}))_{r,\eta} \right| \right\} \leq k \left[ A_{s,t} + B_{s,t} \|x \otimes y\|_{s,t,\beta} (t-s)^{\beta-2\varepsilon} \right],$$ (4.32)

where

$$A_{s,t} = \left( \|y \otimes (y - \tilde{y})\|_{s,t,\beta} + \|y\|_{s,t,\beta} \|y - \tilde{y}\|_{s,t,\beta} \right) \times \left( \|f\|_\infty + \left( \|x\|_{s,t,\beta} \|f\|_\infty + \|f\|_\lambda \|x\|_{s,t,\beta} (t-s)^{\beta-2\varepsilon} \right) \right) \leq \rho_f \left( \|y \otimes (y - \tilde{y})\|_{s,t,\beta} + \|y\|_{s,t,\beta} \|y - \tilde{y}\|_{s,t,\beta} \right),$$ (4.33)

and

$$B_{s,t} = \|y - \tilde{y}\|_{s,t,\beta} \left( \|f\|_\infty + \|f\|_\lambda \|x\|_{s,t,\beta} (t-s)^{\beta} \right) \leq \|y - \tilde{y}\|_{s,t,\beta} \left( \|f\|_\infty + \|f\|_\lambda \right).$$ (4.34)
Substituting (4.33) and (4.34) into (4.32) yields

\[ \|x \otimes (y - \tilde{y})\|_{s,t,\beta} \leq k \rho_f \left( \|y\|_\beta \|x - \tilde{x}\|_{s,t,\infty} + \frac{1}{2} \|x - \tilde{x}\|_{s,t,\beta} \right) + k \rho_f \|y - \tilde{y}\|_\beta + k \rho_f^2 \|y \otimes (y - \tilde{y})\|_{s,t,\beta}(t - s)^{\beta - 2\varepsilon}. \]  

(4.35)

Finally, from (4.35) and (4.38) we obtain

\[ I_{2,s,t} \leq k \rho_f \|y - \tilde{y}\|_\beta + k \rho_f^2 \left( \|y \otimes (y - \tilde{y})\|_{s,t,\beta} + \|y\|_{s,t,\beta}(t - s)^{\beta - 2\varepsilon}. \right) \]

(4.36)

Now from (4.30) and (4.36) we get

\[ \|x - \tilde{x}\|_{s,t,\beta} \leq k \rho_f \|y\|_\beta \|x - \tilde{x}\|_{s,t,\infty} + \frac{1}{2} \|x - \tilde{x}\|_{s,t,\beta} + k \rho_f \|y - \tilde{y}\|_\beta + k \rho_f^2 \|y \otimes (y - \tilde{y})\|_{s,t,\beta}(t - s)^{\beta - 2\varepsilon}. \]  

(4.37)

Notice that

\[ \|x - \tilde{x}\|_{s,t,\infty} \leq |x_s - \tilde{x}_s| + (t - s)^{\beta} \|x - \tilde{x}\|_{s,t,\beta}. \]  

(4.38)

Hence,

\[ \|x - \tilde{x}\|_{s,t,\beta} \leq k \rho_f \|y\|_\beta \|x_s - \tilde{x}_s| + (t - s)^{\beta} \|x - \tilde{x}\|_{s,t,\beta} + k \rho_f \|y - \tilde{y}\|_\beta + k \rho_f^2 \|y \otimes (y - \tilde{y})\|_{s,t,\beta}(t - s)^{\beta - 2\varepsilon}. \]  

And consequently,

\[ \|x - \tilde{x}\|_{s,t,\beta} \leq k \rho_f \|y\|_\beta |x_s - \tilde{x}_s| + k \rho_f \|y - \tilde{y}\|_\beta + k \rho_f^2 \|y \otimes (y - \tilde{y})\|_{s,t,\beta}(t - s)^{\beta - 2\varepsilon}. \]  

(4.39)

Substituting (4.39) into (4.38) yields

\[ \|x - \tilde{x}\|_{s,t,\infty} \leq |x_s - \tilde{x}_s| + (t - s)^{\beta} \left( k \rho_f \|y\|_\beta |x_s - \tilde{x}_s| + k \rho_f \|y - \tilde{y}\|_\beta + k \rho_f^2 \|y \otimes (y - \tilde{y})\|_{s,t,\beta}(t - s)^{\beta - 2\varepsilon}. \right) \]  

(4.40)

Suppose that \( y = \tilde{y} \). Then, Equation (4.40) implies that \( x = \tilde{x} \) in a small interval \([0, \delta]\), and by a recursive argument, the uniqueness follows.

Denote \( \kappa = \frac{1}{\beta(y)} \) and \( t_n = n \kappa \). Set

\[ Z_n = \sup_{0 \leq s \leq t_n} |x_s - \tilde{x}_s| \]

Then inequality (4.40) states that

\[ Z_{n+1} \leq (1 + k \rho_f \kappa^2)Z_n + k \rho_f \kappa^2 \|y - \tilde{y}\|_\beta + k \rho_f^2 \kappa^{2\beta - 2\varepsilon} \|y \otimes (y - \tilde{y})\|_{s,t,\beta} \]  

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Therefore
\[ Z_T \leq k(1 + k\rho f^\beta)^{T/\kappa}|x_0 - \tilde{x}_0| \]
\[ + k \sum_{l=0}^{T/\kappa} (1 + k\rho f^\beta)^l \left[ \rho f^\beta \|y - \tilde{y}\|_\beta + \rho f^2 k^{2\beta - 2\epsilon} \|y \otimes (y - \tilde{y})\|_\beta \right] . \]

This implies the desired estimate. ■

The following corollary is direct consequence of (4.37) and (4.15).

**Corollary 4.3** If \( f \) is twice continuously differentiable and \( f'' \) is Lipschitz continuous and if \( x \) and \( \tilde{x} \) satisfy
\[
x_t = x_0 + \int_0^t f(x_s)ds \quad \text{and} \quad \tilde{x}_t = \tilde{x}_0 + \int_0^t f(\tilde{x}_s)ds ,
\]
then
\[
\|x_t - \tilde{x}_t\|_\beta \leq C \{|x_0 - \tilde{x}_0| + \|y - \tilde{y}\|_\beta + \|y \otimes (y - \tilde{y})\|_\beta \} , \tag{4.41}
\]
where we use the notation of Theorem 4.1.

## 5 Stochastic Differential Equations

Suppose that \( B = \{B_t = (B^1_t, B^2_t, \ldots, B^d_t)\} \) is a \( d \)-dimensional Brownian motion. Fix a time interval \([0, T]\). Define
\[
(B \otimes B)_{s,t} = \int_s^t (B_r - B_s) d \circ B_r ,
\]
where the stochastic integral is a Stratonovich integral. That is,
\[
(B \otimes B)^{i,j}_{s,t} = \begin{cases} \frac{1}{2}(B^i_r - B^i_s)^2 & \text{if} \ i = j \\ \int_s^t (B^i_r - B^i_s) dB^j_r & \text{if} \ i \neq j \end{cases} ,
\]
where the stochastic integral is an Itô integral. It is not difficult to show that we can choose a version of \((B \otimes B)_{s,t}\) in such a way that \((B, B, B \otimes B)\) constitutes a \(\beta\)-Hölder continuous multiplicative functional, for a fixed \(\beta \in (1/3, 1/2)\).

As a first application of Theorem 4.2 and 3.25 we deduce that the Stratonovich stochastic integral \( \int_0^T f(B_r) d \circ B_r \) has the following path-wise expression
\[
\int_0^T f(B_r) d \circ B_r = (-1)^\alpha \sum_{i=1}^d \int_0^T \tilde{D}^\alpha_{i,i} f_i(B)_r (D^1_{T-r} - B^i_{T-r})_r dr
\]
\[
+ \sum_{i=1}^m \sum_{j=1}^d \int_0^T \tilde{D}^{2\alpha-1}_{0+} \partial_i f_j(B)_r \Lambda^T(B^i \otimes T^j) dr . \tag{5.1}
\]
We can apply Theorem 4.1 and deduce the existence of a solution for the stochastic differential equation in \( \mathbb{R}^m \)
\[
X_t = X_0 + \int_0^t f(X_s) dB_s,
\]
where the initial condition \( X_0 \) is an arbitrary random variable, and the function \( f : \mathbb{R}^m \to \mathbb{R}^{md} \) is a continuously differentiable function such that \( f' \) is \( \lambda \)-Hölder continuous, where \( \lambda > \frac{\beta}{2} - 2 \), and \( f \) and \( f' \) are bounded. By Theorem 4.2 the solution is unique if \( f \) is twice continuously differentiable with bounded derivatives and \( f'' \) is \( \lambda \)-Hölder continuous, where \( \lambda > \frac{1}{\beta} - 2 \). The stochastic integral here is a path-wise integral which depends on \( B \) and \( B \otimes B \).

We have also the stability type results 4.15 and 4.41. In particular, if \( B^\varepsilon \) is a piece-wise smooth approximation of \( B \) such that
\[
\|B^\varepsilon - B\|_\beta \quad \text{and} \quad \|B \otimes (B^\varepsilon - B)\|_\beta
\]
converge to zero with a certain rate, then according to Corollary 4.3 \( \|X - X^\varepsilon\|_\beta \) will also converge to 0 with the same rate, where
\[
X^\varepsilon_t = X_0 + \int_0^t f(X^\varepsilon_s) dB^\varepsilon_s.
\]
In particular, this implies that the stochastic process solution of (5.2) coincides with the solution of the Stratonovich stochastic differential equation
\[
X_t = X_0 + \int_0^t f(X_s) d\theta B_s^H.
\]

In this section we will apply these results in order to obtain the almost sure rate of convergence of the Wong-Zakai approximation to the stochastic differential equation (5.2). That is, we will consider the rate of convergence in Hölder norm when we approximate the Brownian motion by a polygonal line.

In order to get a precise rate for these approximations we will make use of the following exact modulus of continuity of the Brownian motion. There exists a random variable \( G \) such that almost surely for any \( s, t \in [0, T] \) we have
\[
|B_t - B_s| \leq G|t - s|^{1/2} \sqrt{\log \left( \frac{|t - s|^{-1}}{2} \right)} \quad \text{(5.4)}
\]

Let \( \pi = \{0 = t_0 < t_1 < \cdots < t_n = T\} \) be the uniform partition of the interval \( [0, T] \). That is \( t_k = \frac{kT}{n} \), \( k = 0, \ldots, n \). We denote by \( B^\varepsilon \) the polygonal approximation of the Brownian motion defined by
\[
B^\varepsilon_t = \sum_{k=0}^{n-1} \left( B_{t_k} + \frac{n}{T} (t - t_k) \left( B_{t_{k+1}} - B_{t_k} \right) \right) 1_{(t_k, t_{k+1})}(t).
\]

We have the following result
Lemma 5.1 There exist a random variable $C_{T,\beta}$ such that
\[
\|B - B^n\|_\beta \leq C_{T,\beta} n^{\beta - 1/2} \sqrt{\log n} \quad (5.5)
\]
\[
\|B \otimes (B - B^n)\|_\beta \leq C_{T,\beta} n^{\beta - 1/2} \sqrt{\log n}. \quad (5.6)
\]

Proof. Fix $0 < s < t < T$ and assume that $s \in [t_1, t_{t+1}]$ and $t \in [t_k, t_{k+1}]$.

Let us first estimate
\[
h_1(s, t) = \frac{1}{(t-s)^\beta} |B_t^n - B_t - (B_s^n - B_s)|.
\]
If $t - s \geq \frac{T}{n}$, then using (5.4) we obtain
\[
|h_1(s, t)| \leq T^{-\beta} n^\beta \left[ \left| B_{t_k} - B_s + \frac{n}{T} (t-t_k) (B_{t_{k+1}} - B_{t_k}) \right|
+ \left| B_{t_1} - B_s + \frac{n}{T} (s-t_1) (B_{t_{t+1}} - B_{t_1}) \right| \right]
\leq 4G T^{-\beta + 1/2} n^{-1/2 + \beta} \sqrt{\log (n/T)}.
\]
If $t - s < \frac{T}{n}$, then there are two cases. Suppose first that $s, t \in [t_k, t_{k+1}]$. In this case, if $n$ is large enough ($n > T e^{2/(1 - 2\beta)}$) we obtain using (5.4)
\[
|h_1(s, t)| \leq \frac{|B_t - B_s|}{(t-s)^\beta} + \frac{n}{T} |B_{t_{k+1}} - B_{t_k}| (t-s)
\]
\[
\leq G |t-s|^{2-\beta} \sqrt{\log |t-s|^{-1}} + GT^{-1/2} \sqrt{\log (n/T)} n^{1-1/2} (t-s)^{1-\beta}
\leq GT^{-\beta + 1/2} n^{-1/2 + \beta} \sqrt{\log (n/T)}.
\]
On the other hand, if $s \in [t_{k-1}, t_k]$ and $t \in [t_k, t_{k+1}]$ we have, again if $n$ is large enough
\[
|h_1(s, t)| \leq \frac{1}{(t-s)^\beta} \left| B_{t_k} - B_t + \frac{n}{T} (t-t_k) (B_{t_{k+1}} - B_{t_k})
- \left( B_{t_k} - B_s - \frac{n}{T} (t_k - s) (B_{t_{k-1}} - B_{t_k}) \right) \right|
\leq \frac{1}{(t-s)^\beta} \left| B_t - B_s + \frac{n}{T} (t-s) (|B_{t_k} - B_{t_{k-1}}| + |B_{t_{k+1}} - B_{t_k}|) \right|
\leq G \left| t-s \right|^{1/2} \sqrt{\log |t-s|^{-1}} + 2(t-s) \left( \frac{n}{T} \right)^{1/2} \sqrt{\log (n/T)}
\leq 3G T^{-\beta + 1/2} n^{-1/2 + \beta} \sqrt{\log (n/T)}.
\]
This proves (5.6).

Now we turn to the estimate of the term
\[
h_2(s, t) = \frac{1}{(t-s)^{2\beta}} \left| \int_s^t (B_u^n - B_u^t) dB_u^n - \int_s^t (B_u^t - B_u^t) dB_u \right|
\]

for $i \neq j$ (the case $i = j$ is obvious from (5.5)). We claim that there exists a random variable $Z$ such that, almost surely, for all $s, t \in [0, T]$ we have

$$\left| \int_s^t (B_u^i - B_s^i) dB_u^j \right| \leq Z |t - s| \log |t - s|^{-1}. \quad (5.7)$$

In fact, it suffices to show this inequality almost surely for all $s$ and $t$ rational numbers. If we fix $s$, the process $\{M_t, t \in [s, T]\}$

$$M_t = \int_s^t (B_u^i - B_s^i) dB_u^j$$

is a continuous martingale and it can be represented as a time-changed Brownian motion:

$$M_t = W_{\int_s^t (B_u^i - B_s^i)^2 du}.$$

As a consequence, applying (5.4) there exists a random variable $G$ such that

$$|M_t| = |W_{\int_s^t (B_u^i - B_s^i)^2 du}| \leq G \left( \int_s^t (B_u^i - B_s^i)^2 du \right)^{1/2} \sqrt{\log \left( \int_s^t (B_u^i - B_s^i)^2 du \right)}^{-1}$$

and again (5.4), applied to $B_u^i - B_s^i$, yields

$$|M_t| \leq G_1 G_2 \left( \int_s^t (u - s) \log |u - s|^{-1} du \right)^{1/2} \sqrt{\log \left( G_2 \int_s^t (u - s) \log |u - s|^{-1} du \right)}^{-1},$$

for some random variable $G_2$. We have for $|t - s| \leq 1$

$$\int_s^t (u - s) \log |u - s|^{-1} du = (t - s)^2 \left( \frac{1}{4} + \frac{1}{2} \log |t - s|^{-1} \right),$$

and this implies easily the estimate (5.7).

Suppose first that $t - s \geq \frac{T}{n}$. Then

$$h_2(t, s) = \frac{1}{(t - s)^{2\beta}} \left| \int_s^t (B_u^i - B_s^i) dB_u^j \right|$$

$$= \frac{1}{(t - s)^{2\beta}} \left| (B_t^i - B_s^i)(B_t^j - B_s^j) - \int_s^t (B_u^j - B_s^j) dB_u^i \right|$$

$$\leq \frac{1}{(t - s)^{2\beta}} \left| (B_t^i - B_s^i)(B_t^j - B_s^j) \right| + \frac{1}{(t - s)^{2\beta}} \left| \int_s^t [B_u^j - B_s^j] dB_u^i \right|$$

$$= A_1 + A_2.$$
Finally, if \( s \leq t \), and we obtain \( (5.7) \) yields

\[
B_{s}^{T,\beta}(t,s) \leq C_{T,\beta}n^{\beta-1/2} \sqrt{\log n}.
\]

For the term \( A_{2} \) we proceed as in the proof of the estimate \( (5.5) \). We have

\[
\int_{s}^{t} [B_{u}^{i,\beta} - B_{u}^{i}] dB_{u}^{i} = W_{f_{s}^{i}}(B_{u}^{i,\beta} - B_{u}^{i})^{2} du
\]

where \( W \) is a Brownian motion. As a consequence, using that

\[
\|B - B^{T}\|_{\infty} \leq C_{T,\beta}n^{-1/2} \sqrt{\log (n/T)}
\]

(this estimate is proved as \( (5.8) \)) we get

\[
A_{2} \leq \frac{G}{(t-s)^{2\beta}} \left( \int_{s}^{t} (B_{u}^{i,\beta} - B_{u}^{i})^{2} du \right)^{1/2} \sqrt{\log \left( \int_{s}^{t} (B_{u}^{i,\beta} - B_{u}^{i})^{2} du \right)^{-1}}
\]

\[
\leq C_{T,\beta}(t-s)^{1/2-2\beta}n^{-1/2} \sqrt{\log n} \sqrt{\log [(t-s)^{-1}n(\log n)^{-1}]}
\]

\[= C_{T,\beta}n^{\beta-1/2} \sqrt{\log n}.
\]

Suppose now that \( t-s < \frac{T}{n} \). We make the decomposition

\[
h_{2}(s,t) = \frac{1}{(t-s)^{2\beta}} \left( \int_{s}^{t} (B_{u}^{i} - B_{u}^{i}) dB_{u}^{i} + \int_{s}^{t} (B_{u}^{i} - B_{u}^{i}) dB_{u}^{i} \right) = B_{1} + B_{2}.
\]

Then \( (5.7) \) yields

\[
B_{2} \leq Z(t-s)^{-2\beta} \log |t-s|^{-1} \leq C_{T,\beta}n^{2\beta-1} \log n.
\]

In order to handle the term \( B_{1} \), assume first that \( s, t \in [t_{k}, t_{k+1}] \). Then

\[
\int_{s}^{t} (B_{u}^{i} - B_{u}^{i}) dB_{u}^{i} = \frac{n}{t} \left( B_{t}^{i} - B_{t_{k}}^{i} \right) \int_{s}^{t} (B_{u}^{i} - B_{u}^{i}) du
\]

and we obtain

\[
B_{1} \leq C_{T,\beta}(t-s)^{1-2\beta} \log n \leq C_{T,\beta}n^{1-2\beta} \log n.
\]

Finally, if \( s \in [t_{k-1}, t_{k}] \) and \( t \in [t_{k}, t_{k+1}] \) we have

\[
B_{1} \leq (t-s)^{-2\beta} \frac{n}{T} \left( B_{t}^{i} - B_{t_{k+1}}^{i} \right) \int_{s}^{t_{k}} (B_{u}^{i} - B_{u}^{i}) du + \left( B_{t_{k+1}}^{i} - B_{t_{k}}^{i} \right) \int_{t_{k}}^{t} (B_{u}^{i} - B_{u}^{i}) du
\]

\[\leq C_{T,\beta}n^{1-2\beta} \log n.
\]

The proof is now complete.  \( \blacksquare \)

As a consequence, we can establish the following result.
Theorem 5.2 Let $f : \mathbb{R}^m \to \mathbb{R}^{md}$ be continuously differentiable with bounded derivative up to fourth order and let $X$ satisfy

$$X_t = X_0 + \int_0^t f(X_s)dB_s$$

If $X^\pi_t$ satisfies the following ordinary differential equation

$$X^\pi_t = X_0 + \int_0^t f(X^\pi_s)dB^\pi_s,$$

then for any $\beta \in (1/3, 1/2)$, there is a random constant $C_{T,\beta} \in (0, \infty)$ such that

$$\|X - X^\pi\|_\beta \leq C_{T,\beta}n^{\beta - 1/2}\sqrt{\log n}. \quad (5.11)$$

**Proof.** The result is a straightforward consequence of Lemma 5.1 and Theorem 4.2. ■

6 Appendix

Proof of Lemma 2.4 The fractional integration by parts formula (2.8) yields

$$\int_a^b d\xi \int_\xi^b \varphi(\xi, \eta) \frac{\partial^2 \psi}{\partial \xi \partial \eta}(\xi, \eta) d\eta$$

$$= (-1)^{1-\alpha} \int_a^b d\xi \int_\xi^b D^{\alpha-1}_b \varphi_b - (\xi, \cdot)(\eta) D^{\alpha-1}_\xi \frac{\partial \psi}{\partial \xi} (\xi, \cdot)(\eta) d\eta.$$

The operators $D^{1-\alpha}_\xi$ and $\frac{\partial \psi}{\partial \xi}$ commute, as it follows from the following computations:

$$D^{1-\alpha}_\xi \frac{\partial \psi}{\partial \xi}(\xi, \eta)$$

$$= \frac{1}{\Gamma(\alpha)} \left( \frac{(\eta - \xi)^{\alpha-1} \partial \psi}{\partial \xi}(\xi, \eta) + (1 - \alpha) \int_\xi^\eta \frac{\partial \psi}{\partial \xi}(\xi, \eta) - \frac{\partial \psi}{\partial \xi}(\xi, \eta') d\eta' \right)$$

$$= \frac{1}{\Gamma(\alpha)} \left\{ \frac{\partial}{\partial \xi} \left[ (\eta - \xi)^{\alpha-1} \psi(\xi, \eta) \right] + (\alpha - 1)(\eta - \xi)^{\alpha-2} \psi(\xi, \eta) \right.$$

$$+ (1 - \alpha) \frac{\partial}{\partial \xi} \int_\xi^\eta \frac{\psi(\xi, \eta) - \psi(\xi, \eta')}{(\eta - \eta')^{2-\alpha}} d\eta' + (1 - \alpha) \frac{\psi(\xi, \eta)}{(\eta - \xi)^{2-\alpha}} \right\}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial \xi} \left\{ (\eta - \xi)^{\alpha-1} \psi(\xi, \eta) + (1 - \alpha) \int_\xi^\eta \frac{\psi(\xi, \eta) - \psi(\xi, \eta')}{(\eta - \eta')^{2-\alpha}} d\eta' \right\}$$

$$= \frac{\partial}{\partial \xi} D^{1-\alpha}_\xi \psi(\xi, \eta).$$
Hence, applying again (2.8) we obtain (2.9) with \( \Gamma_{\alpha} \).

Remarks:

1. Formula (2.8) holds if \( \psi \) is of class \( C^2 \) in \( a < \xi < \eta < b \) and

\[
\int_a^{b} d\xi \int_a^{b} d\eta \left| D_{\alpha+}^{\alpha, \xi} D_{\beta-}^{\alpha, \eta} \varphi_{\psi - \psi}(\xi, \eta) \right| \leq \infty.
\]

2. Under the conditions of the above lemma, we also have \( \Gamma_{\alpha} \psi(\xi, \eta) = D_{\xi+}^{1-\alpha, \eta} D_{\eta-}^{1-\alpha, \xi} \psi(\xi, \eta) \).

3. The operator \( \Gamma_{\alpha} \) can also be expressed as follows.

\[
\Gamma_{\alpha} \psi(\xi, \eta) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)^2} \left\{ (\eta - \xi)^{\alpha - 1} \left( (\eta - \xi)^{\alpha - 1} \psi(\xi, \eta) + (1 - \alpha) \int_\xi^{\eta} \frac{\psi(\xi, \eta) - \psi(\xi, \eta')}{(\eta - \eta')^{2 - \alpha}} d\eta' \right) \\
+ (1 - \alpha) \int_\xi^{\eta} (\xi' - \xi)^{\alpha - 2} \frac{(\eta - \xi)^{\alpha - 1} \psi(\xi, \eta) - (\eta - \xi')^{\alpha - 1} \psi(\xi', \eta)}{(\eta - \eta')^{2 - \alpha}} d\xi' \\
+ (1 - \alpha) \left\{ \int_\xi^{\eta} \frac{\psi(\xi, \eta) - \psi(\xi', \eta)}{(\eta - \eta')^{2 - \alpha}} d\xi' - \int_\xi^{\eta} \frac{\psi(\xi', \eta) - \psi(\xi', \eta')}{(\eta - \eta')^{2 - \alpha}} d\eta' \right\} \right\}.
\]

Exchanging the integration order, we see the last double integral equals to

\[
\int_\xi^{\eta} \frac{\psi(\xi, \eta) - \psi(\xi, \eta')}{(\eta - \eta')^{2 - \alpha}} d\xi' \int_{\eta'}^{\eta} \frac{1}{(\xi' - \xi)^{2 - \alpha}} d\xi'.
\]
This leads to the following expression for \( \Gamma^\alpha \)

\[
\Gamma^\alpha \psi(\xi, \eta) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)^2} \left\{ (\eta - \xi)^{2\alpha-2} \psi(\xi, \eta) 
+ (1 - \alpha) \int_{\xi}^{\eta} (\eta - \xi)^{\alpha-1} \psi(\xi, \eta) - (\eta - \xi')^{\alpha-1} \psi(\xi', \eta) d\xi' 
+ (1 - \alpha)^2 \int_{\xi}^{\eta} \int_{\xi'}^{\eta} \frac{\psi(\xi, \eta) - \psi(\xi, \eta') - \psi(\xi', \eta) + \psi(\xi', \eta')}{(\xi' - \xi)^{2-\alpha} (\eta - \eta')^{2-\alpha}} d\eta' d\xi' 
+ (\alpha - 1) \int_{\xi}^{\eta} \frac{\phi(\xi, \eta) - \phi(\xi, \eta')}{(\eta - \eta')^{2-\alpha}} (\eta' - \xi)^{\alpha-1} d\eta' \right\}.
\]  

(6.1)

Consider the kernel \( K_{s,t}(\xi, \eta) \) defined in (3.7), that is,

\[
K_{s,t}(\xi, \eta) = D_{1+\alpha-\epsilon} D_{\alpha-\epsilon}^{2-\alpha} G_{t-s}(s, \xi, \eta),
\]

where

\[
G(s, \xi, \eta) = C_\alpha (\xi - s)^{\alpha-1} (\eta - \xi)^{\alpha-1} \int_0^1 q^{2\alpha-2}(1-q)^{-\alpha}(1+(1-q)\frac{\xi - s}{\eta - \xi})^{-1} dq.
\]

(6.2)

\[
\phi(z) = C_\alpha \int_0^1 q^{2\alpha-2}(1-q)^{-\alpha}(1+(1-q)z)^{-1} dq = C_\alpha \int_0^1 (1-q)^{2\alpha-2} q^{-\alpha}(1+qz)^{-1} dq,
\]

(6.3)

and \( C_\alpha \) is given as the coefficient in (3.6).

Lemma 6.1 Let \( 1/2 < \alpha < 1 \). The function \( \phi(z) \) defined in (6.3) satisfies \( \phi(0) < \infty \), \( \phi \) is decreases to zero as \( z \) tends to infinity. If \( \beta < 1 - \alpha \), then

\[
\phi(z) \leq cz^{-\beta}
\]

(6.4)

Moreover, if \( \beta < 2 - \alpha \),

\[
|\phi'(z)| \leq cz^{-\beta}.
\]

(6.5)

Lemma 6.2 The kernel \( K_{s,t}(\xi, \eta) \) satisfies

\[
\sup_{0 \leq s < t \leq T} \int_{s < \xi < \eta < t} |K_{s,t}(\xi, \eta)| d\xi d\eta < \infty.
\]

(6.6)

Proof. To simplify the notation we omit the dependence on the variable \( s \) in \( G(s, \xi, \eta) \). Also, \( c \) will denote a generic constant depending on \( \alpha \) and \( \epsilon \). We
have

\[ D_1^1 \alpha \varepsilon D_2^2 \alpha \varepsilon G_{t-}(\xi, \eta) \]

\[ = c D_1^1 \alpha \varepsilon \left( \frac{G(\xi, \eta) - G(\xi, t)}{(t-\eta)\alpha-\varepsilon} \right) + (\alpha - \varepsilon) \int_\eta^t \frac{G(\xi, \eta) - G(\xi, \eta')}{(\eta' - \eta)\alpha - \varepsilon + 1} d\eta' \]

\[ + (\alpha - \varepsilon^2) \int_\eta^t \int_\eta^\xi \frac{G(\xi, \eta) - G(\xi, \eta') - G(\xi', \eta) + G(\xi', \eta')}{(\eta' - \eta)\alpha - \varepsilon + 1 (\xi - \xi')\alpha - \varepsilon + 1} d\eta' d\xi' \].

Set

\[ A_1 = \frac{G(\xi, \eta) - G(\xi, t)}{(t-\eta)\alpha-\varepsilon (\xi-s)\alpha-\varepsilon} \]

\[ A_2 = \frac{1}{(\xi-s)\alpha-\varepsilon} \int_\eta^t \frac{G(\xi, \eta) - G(\xi, \eta')}{(\eta' - \eta)\alpha - \varepsilon + 1} d\eta' \]

\[ A_3 = \int_\eta^\xi \frac{G(\xi, \eta) - G(\xi, t) - G(\xi', \eta) + G(\xi', \eta')}{(t-\eta)\alpha-\varepsilon (\xi - \xi')\alpha - \varepsilon + 1} d\xi' \]

\[ A_4 = \int_\eta^\xi \int_\eta^t \frac{G(\xi, \eta) - G(\xi, \eta') - G(\xi', \eta) + G(\xi', \eta')}{(\eta' - \eta)\alpha - \varepsilon + 1 (\xi - \xi')\alpha - \varepsilon + 1} d\eta' d\xi'. \]

It suffices to show that

\[ \sup_{0 \leq s < t \leq T} \int_{s < \xi < \eta < t} |A_i| d\xi d\eta < \infty \] (6.7)

for \( i = 1, 2, 3, 4 \).

**Step 1** Suppose \( i = 1 \). Using the fact that the function \( \phi \) is bounded we obtain

\[ G(\xi, \eta) \leq c (\xi - s)^{\alpha - 1} (\eta - \xi)^{\alpha - 1}. \]

Hence,

\[ |A_1| = \frac{|G(\xi, \eta)| + |G(\xi, t)|}{(t-\eta)\alpha-\varepsilon (\xi-s)\alpha-\varepsilon} \leq c (\xi - s)^{\varepsilon - 1} (\eta - \xi)^{\alpha - 1} (t-\eta)^{-\alpha+\varepsilon} + c(\xi - s)^{\varepsilon - 1} (t-\eta)^{-\varepsilon}. \]

and (6.7) holds for \( i = 1 \).

**Step 2** Suppose \( i = 2 \). We have

\[ G(\xi, \eta) - G(\xi, \eta') = \int_\eta^{\eta'} \frac{\partial G}{\partial y}(\xi, y) dy. \]
and
\[
\frac{\partial G}{\partial \eta} (\xi, \eta) = (\alpha - 1)(\xi - s)^{\alpha - 1}(\eta - \xi)^{\alpha - 2}\phi \left( \frac{\xi - s}{\eta - \xi} \right)
\]
\[-(\xi - s)^{\alpha - 1}(\eta - \xi)^{\alpha - 2}\phi' \left( \frac{\xi - s}{\eta - \xi} \right) \frac{\xi - s}{(\eta - \xi)^2}
\]
\[= (\xi - s)^{\alpha - 1}(\eta - \xi)^{\alpha - 2}\chi \left( \frac{\xi - s}{\eta - \xi} \right),\]
where
\[\chi(z) = (\alpha - 1)\phi(z) - z\phi'(z).
\]
Notice that, by Lemma 6.1 the function \(\chi(z)\) is uniformly bounded. Hence,
\[
|A_2| \leq c(\xi - s)^{\alpha - 1}\int_t^\eta \int_\eta^{\eta'} (y - \xi)^{\alpha - 2}(\eta' - \eta)^{-\alpha + 1}\,dy \,d\eta'
\]
\[\leq c(\xi - s)^{\alpha - 1}(\eta - \xi)^{\alpha - 1}\int_t^\eta \int_\eta^{\eta'} (\eta' - y)^{\alpha - 1}(y - \eta)^{\alpha - 1}\,dy \,d\eta'
\]
\[\leq c(\xi - s)^{\alpha - 1}(\eta - \xi)^{\alpha - 1}\int_t^\eta (\eta' - \eta)^{\alpha - 1}\,d\eta',
\]
which implies that (6.7) holds for \(i = 2\).

**Step 3** Suppose \(i = 3\). We have
\[
G(\xi, \eta) - G(\xi', \eta) = \int_\xi^{\xi'} \frac{\partial G}{\partial x}(x, \eta)\,dx,
\]
and
\[
\frac{\partial G}{\partial \xi} (\xi, \eta) = (\alpha - 1)(\xi - s)^{\alpha - 2}(\eta - \xi)^{\alpha - 1}\phi \left( \frac{\xi - s}{\eta - \xi} \right)
\]
\[-(\alpha - 1)(\xi - s)^{\alpha - 1}(\eta - \xi)^{\alpha - 2}\phi \left( \frac{\xi - s}{\eta - \xi} \right) \frac{\xi - s}{(\eta - \xi)^2}
\]
\[+ (\xi - s)^{\alpha - 1}(\eta - \xi)^{\alpha - 1}\phi' \left( \frac{\xi - s}{\eta - \xi} \right) \frac{\eta - s}{(\eta - \xi)^2}
\]
\[= (\xi - s)^{\alpha - 2}(\eta - \xi)^{\alpha - 1}(\alpha - 1)\phi \left( \frac{\xi - s}{\eta - \xi} \right)
\]
\[+(\xi - s)^{\alpha - 1}(\eta - \xi)^{\alpha - 2}\gamma \left( \frac{\xi - s}{\eta - \xi} \right),\]
where
\[\gamma(z) = (1 - \alpha)\phi(z) + (1 + z)\phi'(z).
\]
By Lemma 6.1 the function $\gamma$ is uniformly bounded. Hence,

$$|A_3| \leq c(t-\eta)^{-\alpha+\varepsilon} \int_s^\xi \int_{\xi'}^\xi (\xi - \xi')^{-\alpha+\varepsilon-1}$$

$$\times \left[(x-s)^{\alpha-2}(\eta-x)^{\alpha-1} + (x-s)^{\alpha-1}(\eta-x)^{\alpha-2}\right] dxd\xi'$$

$$\leq c(t-\eta)^{-\alpha+\varepsilon}(\eta-\xi)^{\alpha-1} \int_s^\xi \int_{\xi'}^\xi (\xi - x)\frac{1}{\alpha-1}d\xi' (\xi - \xi')^{\frac{1}{\alpha-1}}(\xi' - s)^{\frac{1}{\alpha-1}} dxd\xi'$$

$$+ c(t-\eta)^{-\alpha+\varepsilon}(\eta-\xi)^{\alpha-1} \int_s^\xi \int_{\xi'}^\xi (\xi - x)\frac{1}{\alpha-1}d\xi' (\xi - \xi')^{\frac{1}{\alpha-1}}(\xi' - s)^{\alpha-1} dxd\xi'$$

$$= c(t-\eta)^{-\alpha+\varepsilon}(\eta-\xi)^{\alpha-1}(\xi - s)^{\varepsilon-1} + c(t-\eta)^{-\alpha+\varepsilon}(\eta-\xi)^{\frac{1}{\alpha-1}}(\xi - s)^{\frac{1}{\alpha-1}}(\xi - s)^{\frac{1}{\alpha-1}+\varepsilon-1}.$$ 

which implies that (6.7) holds for $i = 3$.

**Step 4** Suppose $i = 4$. We are going to use the following decomposition

$$G(\xi, \eta) - G(\xi', \eta') = G(\xi', \eta) + G(\xi, \eta')$$

$$= - \int_{\eta'}^{\eta} \int_{\xi'}^{\xi} \partial^2 G \frac{d^2 G}{d\xi d\eta}(x,y)dxdy$$

We need to compute the second derivative:

$$\frac{\partial^2 G}{\partial \xi \partial \eta} = (\xi - s)^{\alpha-2}(\eta - \xi)^{\alpha-2} \left( (\alpha - 1)^2 - (\alpha - 1)(\alpha - 2) \frac{\xi - s}{\eta - \xi} \right) \phi \left( \frac{\xi - s}{\eta - \xi} \right)$$

$$- (\xi - s)^{\alpha-1}(\eta - \xi)^{\alpha-1} \left[ \phi'' \left( \frac{\xi - s}{\eta - \xi} \right) \frac{(\eta - s)(\xi - s)}{(\eta - \xi)^4} \right]$$

$$+ \phi' \left( \frac{\xi - s}{\eta - \xi} \right) \frac{(\eta - s) + 2(\xi - s)}{(\eta - \xi)^3}.$$ 

Hence, we can write

$$\frac{\partial^2 G}{\partial \xi \partial \eta} = (\xi - s)^{\alpha-2}(\eta - \xi)^{\alpha-2} \psi \left( \frac{\xi - s}{\eta - \xi} \right),$$

where

$$\psi(z) = ((\alpha - 1)^2 - (\alpha - 1)(\alpha - 2)z) \phi(z)$$

$$- \phi''(z) z^2(1+z) - \phi'(z) z(1+2z).$$

We are going to use the decomposition

$$\psi(z) = \psi_1(z) + \psi_2(z),$$

where

$$\psi_1(z) = -(\alpha - 1)(\alpha - 2)z\phi(z)$$

$$\psi_2(z) = (\alpha - 1)^2\phi(z) - \phi''(z) z^2(1+z) - \phi'(z) z(1+2z).$$

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This leads to
\[
\int_{s \leq \xi \leq \eta \leq t} A_4 d\xi d\eta = B_1 + B_2,
\]
where
\[
B_i = -\int_{D} (\eta' - \eta)^{-\alpha+\varepsilon-1}(\xi - \xi')^{-\alpha+\varepsilon-1}
\times (x-s)^{\alpha-2}(y-\xi)^{\alpha-2}\psi_i \left( \frac{x-s}{y-x} \right) d\xi' dxd\xi d\eta d\eta' + \int_{D} (\eta' - \eta)^{-\alpha+\varepsilon-1}(\xi - \xi')^{-\alpha+\varepsilon-1}(x-s)^{\alpha-2}(y-\xi)^{\alpha-2}\psi_i \left( \frac{x-s}{y-x} \right) d\xi' dxd\xi d\eta d\eta',
\]
i = 1, 2, and
\[
D = \{(\xi', x, \xi, \eta, y, \eta') : s < \xi' < x < \xi < \eta < y < \eta' < t\}.
\]

Step 5 Estimation of \(B_1\). Denote
\[
D_1 = \{(\xi', x, \xi, \eta, \eta') : s < \xi' < x < \xi < \eta < \eta' < t\}.
\]
Using (3.3) with \(\beta = 1 - \alpha - \delta\) with \(\delta < \varepsilon/3\), we obtain
\[
|B_1| \leq c \int_{D} (\eta' - \eta)^{-\alpha+\varepsilon-1}(\xi - \xi')^{-\alpha+\varepsilon-1}(x-s)^{\alpha-2}(y-\xi)^{\alpha-2} \phi \left( \frac{x-s}{y-x} \right) d\xi' dxd\xi d\eta d\eta'
\]
\[
\leq c \int_{D_1} (\eta' - \eta)^{-\alpha+\varepsilon-1}(\xi - \xi')^{-\alpha+\varepsilon-1}(x-s)^{\alpha-2+\delta}(y-\xi)^{\alpha-2+\delta} d\xi' dxd\xi d\eta d\eta'
\]
\[
= c \int_{D_1} (\eta' - \eta)^{-\alpha+\varepsilon-1}(\xi - \xi')^{-\alpha+\varepsilon-1}(x-s)^{\alpha-2+\delta} d\xi' dxd\xi d\eta d\eta'
\]
\[
\times \left[ (\eta-x)^{-1-\delta} - (\eta'-x)^{-1-\delta} \right] d\xi dxd\xi d\eta d\eta'
\]
\[
\leq c \int_{D_1} (\eta' - \eta)^{-\alpha+\varepsilon-1}(\xi - \xi')^{-\alpha+\varepsilon-1}(x-s)^{\alpha-2+\delta} (\eta-x)^{-1-\delta} (\eta'-\eta)^{\alpha} d\xi' dxd\xi d\eta d\eta'
\]
\[
\leq c \int_{D_1} (\eta' - \eta)^{-\alpha+\varepsilon-1}(\xi - \xi')^{-1+\delta+\alpha}(x-\xi')^{-2\alpha+\varepsilon-3\delta}
\times (x-\xi')^{2\alpha-1}(\xi' - s)^{-1+\delta}(\eta-\xi)^{-1+\delta}(\xi-x)^{-1-\delta} d\xi' dxd\xi d\eta d\eta'
\]
\[
\leq c \int_{D_1} (\eta' - \eta)^{-\alpha+\varepsilon-1}(\xi - \xi')^{-1+\delta}(x-\xi')^{2\alpha-1}(\xi' - s)^{-1+\delta}(\eta-\xi)^{-1+\delta} d\xi' dxd\xi d\eta d\eta'
\]
\[
\leq c \int_{D_1} (\eta' - \eta)^{-\alpha+\varepsilon-1}(\xi - \xi')^{-1+\delta}(x-\xi')^{2\alpha-1}(\xi' - s)^{-1+\delta}(\eta-\xi)^{-1+\delta} d\xi' dxd\xi d\eta d\eta'.
\]

Step 6 Estimation of \(B_2\). Let us compute the function \(\psi_2(z)\):
\[
\psi_2(z) = \int_{0}^{1} (1-q)^{2\alpha-2} q^{-\alpha} \left[ (\alpha-1)^2 (1+qz)^{-1} - 2q^2 (1+qz)^{-3} z^2 (1+z) + q(1+qz)^{-2} z (1+2z) \right] dq
\]
\[
= \int_{0}^{1} (1-q)^{2\alpha-2} q^{-\alpha} (1+qz)^{-3} \left[ (\alpha-1)^2 (1+qz)^2 + qz (1+(2-q)z) \right] dq.
\]
This implies that the function $\psi_2(z)$ is uniformly bounded. As a consequence, we deduce the following estimates

\[
|B_2| \leq c \int_D (\eta' - \eta)^{-\alpha + \varepsilon - 1} d\xi' dx d\eta' dy d\eta' d\xi' dx d\eta' dy d\eta' d\xi' dx d\eta' dy d\eta' \\
\leq c \int_D (\eta' - y)^{-\frac{\alpha}{2}} (y - \eta)^{-\alpha^2 + \varepsilon - 1} (x - s)^{-\frac{\alpha^2}{2}} (y - \eta)^{-\alpha^2 - 1} d\xi' dx d\eta' dy d\eta' d\xi' dx d\eta' dy d\eta' d\xi' dx d\eta' dy d\eta' \\
\leq c \int_D (\eta' - y)^{-\frac{\alpha}{2}} (y - \eta)^{-1 + \frac{\varepsilon}{2}} (\xi - x)^{-\frac{\alpha}{2}} (x - \xi')^{-\frac{\alpha}{2}} d\xi' dx d\eta' dy d\eta' d\xi' dx d\eta' dy d\eta' d\xi' dx d\eta' dy d\eta' d\xi' dx d\eta' dy d\eta' \\
\leq c \int_D (\xi' - s)^{-1 + \frac{\varepsilon}{2}} (\eta - \xi)^{-1 + \frac{\varepsilon}{2}} d\xi' dx d\eta' dy d\eta' d\xi' dx d\eta' dy d\eta' d\xi' dx d\eta' dy d\eta' d\xi' dx d\eta' dy d\eta' \\
< \infty.
\]

\[\blacksquare\]

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