Chaos in ocular aberration dynamics of the human eye

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Abstract: Since the characterization of the eye’s monochromatic aberration fluctuations in 2001, the power spectrum has remained the most widely used method for analyzing their dynamics. However, the power spectrum does not capture the complexities of the fluctuations. We measured the monochromatic aberration dynamics of six subjects using a Shack-Hartmann sensor sampling at 21 Hz. We characterized the dynamics using techniques from chaos theory. We found that the attractor embedding dimension for all aberrations, for all subjects, was equal to three. The embedding lag averaged across aberrations and subjects was 0.31 ± 0.07 s. The Lyapunov exponent of the rms wavefront error was positive for each subject, with an average value of 0.44 ± 0.15 µm/s. This indicates that the aberration dynamics are chaotic. Implications for future modeling are discussed.

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1. Introduction

The dynamics of the monochromatic aberrations of the human eye, during steady-state fixation, were first characterized just over a decade ago [1]. In this study, power spectrum analysis revealed that the fluctuations can be described by an inverse power law. Since this time, power spectrum based measures, in particular the slope of the power spectrum, have remained the most widely used methods for characterizing monochromatic aberration dynamics; see for example [2–9]. However, characterizing the dynamics in this way is likely to be an oversimplification. For example, it has recently been demonstrated that the dynamics are multifractal and characterized by a range of statistically distributed exponents [10]. This has implications for modeling aberration dynamics, and determining differences both between subjects, and for the same subject with differing experimental conditions.

In the investigations of other physiological signals, such as the heartbeat, more advanced analytical methods are commonplace. One such technique is chaos theory [11]. Chaos theory characterizes nonlinear dynamical systems that are sensitive to initial conditions [12]. Unlike the usual meaning of the word chaos, chaotic systems have underlying laws that determine their behavior. The reason they appear to be ‘chaotic’ is because of their sensitivity to initial conditions—the so called Butterfly Effect. Figure 1 shows an example of a chaotic time series. Examples of chaotic signals in physiology are the heartbeat [13], brain waves [14], pupillary unrest [15], and blood pressure variations [16].

Fig. 1. Example of a chaotic time series. The generating equation is the so-called logistic equation: \( x_{i+1} = k x_i (1-x_i) \), which describes the relative size of a population over time, \( t \). In the graphs the growth rate, \( k \), is 3.8. A change in the initial relative population, \( x_0 \), by only 0.1% results in a divergence of the plots.

The aim of this work was to determine if chaos is present in the fluctuations of the eye’s monochromatic aberrations.

2. Method

2.1. Subjects

The aberrations of the right eye of six subjects were measured in this study. Aberration measurements were performed over the natural pupil size of each subject. All subjects gave...
informed consent to take part in the study. Their demographics are shown in Table 1. Those who required spectacles wore them during the experiment.

| Subject | Age (yrs) | Pupil (mm) | Refraction (RE) |
|---------|-----------|------------|-----------------|
| KH      | 29        | 5.5        | −1.75/-0.50 × 90 |
| EM      | 34        | 5.0        | −6.00/-0.50 × 90 |
| JC      | 32        | 6.0        | Plano           |
| YP      | 24        | 5.0        | Plano           |
| CS      | 25        | 4.5        | −6.00/-0.50 × 60 |
| CV      | 28        | 4.0        | −4.25/-1.25 × 5 |

2.2. Aberration measurements

The aberrations were measured over a 23 s time period using an open-view Shack-Hartmann sensor sampling at 21 Hz. The aberrations of subjects KH and CV were measured over 11.5 s as they found it difficult to maintain fixation on the stimulus for 23 s. The subjects were stabilized using a bite bar. A simplified schematic of the sensor is shown in Fig. 2. For further details of the system see [17]. The stimulus was a Maltese cross displayed on an LCD monitor at an accommodative demand of 0.4 D. It subtended 11.32 minutes of arc at the eye and had a luminance of 255cd/m². The left eye of each subject remained open and fixated on the stimulus. For each subject the time evolution of each aberration (up to and including 5th radial order excluding tip and tilt), and the rms wavefront error were calculated. Blinks were removed using a cubic spline function [5]. The analysis was carried out using custom written software in Matlab.

3. Chaos theory analysis

A feature of chaos is sensitivity to initial conditions. This manifests itself as the exponential divergence (separation) of neighboring trajectories in phase space [12]. Phase space is a plot in which each axis is effectively a variable needed to specify the 'state' of the system over time [12]. The plotted points map out the so-called attractor. Figure 3(a) shows a schematic of the phase space plot for the logistic equation in Fig. 1. Figure 3(b) shows a schematic of the change in separation of neighboring trajectories versus time for such a chaotic time series. For a chaotic series there is an initial increase in the rate of divergence for a limited period of time. This is the limit of predictability [12]. Following this, the divergence no longer increases as the trajectories have diverged so much they are effectively moving randomly with respect to each other [12]. The slope of the linear region is the Lyapunov exponent. For a stochastic time series the divergence plot is flat and the Lyapunov exponent is zero, and for a non-chaotic time series in which trajectories converge, the exponent is negative. The aim of the analysis was to reconstruct the attractor of the aberration fluctuations for each subject and determine the Lyapunov exponents.
Fig. 3. (a) Schematic of the phase space plot for the logistic equation in Fig. 1. The divergence of neighboring trajectories is also shown. (b) A schematic of the change in the separation of neighboring trajectories versus time for such a chaotic time series.

3.1. Phase space reconstruction

The first step in chaos theory analysis is reconstruction of the phase space. To reconstruct the phase space from the one dimensional time series, time-delay embedding was used [18]. In this method, a point in $d$-dimensional phase space is given by

$$ p(t) = [x(t), x(t+\tau), \ldots, x(t+(d-1)\tau)], $$

where the embedding lag $\tau = t_0 + i\Delta t$, with $\Delta t$ being the time between frames [18]. The embedding dimension $d$ is equal to the number of axes and number of ‘sub-series’. The number of data points, $n$, per sub-series is given by

$$ n = N - (d-1)\times i, $$

where $N$ is the total number of data points, and $i$ is the embedding lag in units of data points.

3.1.1. Embedding lag

The lag, $\tau$, must be large enough so that the data points in each sub-series are not correlated, otherwise there is redundancy, as one sub-series can predict another. In this case the data points will accumulate along a diagonal line [18]. However, the lag must not be so large that the data points in each sub-series are completely unrelated [18]. In this case, the phase space plot will look like random scatter [12]. The optimum lag was determined by using the first minimum of the mutual information [11]. For two time series $x$ and $y$, the mutual information is given by

$$ I_{xy} = \sum_{x_i,y_j} P_{xy}(x_i,y_j) \log_2 \left( \frac{P_{xy}(x_i,y_j)}{P_x(x_i)P_y(y_j)} \right), $$

where $P_{xy}$ is the joint probability, and $P_x$ and $P_y$ are the marginal probabilities [12]. For the purpose of determining $\tau$, the two time series are the original series and the lagged series. Hence
\[ I(\tau) = \sum_{x(t),x(t+\tau)} P(x(t),x(t+\tau)) \log_2 \left( \frac{P(x(t),x(t+\tau))}{P(x(t))P(x(t+\tau))} \right). \] (4)

A high value indicates that the two series contain the same information, i.e. the data points in one series can predict that of the other. The mutual information method is preferred over using an autocorrelation-based method as autocorrelation relies on a linear basis [12].

For each subject, for each Zernike aberration, \( I(\tau) \) was calculated for a lag of \( \tau = 1 \) to 30 data points. Prior to calculating \( I(\tau) \), the time course signals were also detrended as trends cause unwanted correlations [12].

3.1.2. Embedding dimension

The embedding dimension, \( d \), was calculated using a false nearest neighbors (FNN) approach [19]. This involves measuring how the distances between neighboring points in phase space vary with increasing dimension. Suppose a data point located in \( d \)-dimensional space is given by

\[ p(t) = [x(t),x(t+\tau),\ldots,x(t+(d-1)\tau)], \] (5)

and its nearest neighbor is given by

\[ p_{NN}(t_{NN}) = [x(t_{NN}),x(t_{NN}+\tau),\ldots,x(t_{NN}+(d-1)\tau)]. \] (6)

The distance between the points is

\[ R^2_d(p,p_{NN}) = \sum_{k=1}^{d} [x(t+(k-1)\tau) - x(t_{NN}+(k-1)\tau)]^2. \] (7)

If the embedding dimension is too small the points will appear to be closer than they actually are, as illustrated in Fig. 4(a). Hence as the embedding dimension increases the measured distance will increase until the correct embedding dimension is found (Fig. 4(b)), where after the distance remains constant (Fig. 4(c)). The separation of points when the dimension increases by one is given by

\[ R^2_{d+1}(p,p_{NN}) = R^2_d(p,p_{NN}) + [x(t+d\tau) - x(t_{NN}+d\tau)]^2. \] (8)

And so the change in distance can be calculated as

\[ \sqrt{\frac{R^2_{d+1}(p,p_{NN}) - R^2_d(p,p_{NN})}{R^2_d(p,p_{NN})}} = \frac{x(t+d\tau) - x(t_{NN}+d\tau)}{R_d(p,p_{NN})}. \] (9)

A point is considered a FNN if two conditions are satisfied:

\[ \sqrt{\frac{R^2_{d+1}(p,p_{NN}) - R^2_d(p,p_{NN})}{R^2_d(p,p_{NN})}} > R_{tol} \] (10)

and

\[ \frac{R_{d+1}}{R_d} > A_{tol}, \] (11)

where \( R_d \) is the size of the attractor, which can be estimated using the standard deviation of the time series [19]. This second condition accounts for the fact that the nearest neighbor of a point may not necessarily be close to it. Without this extra condition, the number of false nearest neighbors starts to rise again as \( d \) increases beyond the appropriate dimension.
For a data series, the nearest neighbor of each point is determined for a given dimension. For each pair of points, Eqs. (10) and (11) are then evaluated. The embedding dimension is then determined from a plot of the percentage of false nearest neighbors versus dimension. For each subject, for each Zernike aberration, the percentage of false nearest neighbors was determined for \( d = 1:10 \). A value of 15 was used for \( R_{\text{Tol}} \) [20]. \( A_{\text{Tol}} \) was set to 5. This was the lowest value such that the percentage of false nearest neighbors did not start to rise again as \( d \) continued to increase.

For each aberration and the rms wavefront error for each subject, the left-hand side of Eq. (14) was evaluated for \( i = 1:60 \). The Lyapunov exponent was determined from the slope of the linear rise, if applicable, in the divergence averaged across neighbors, versus time \((i\Delta t)\) [21]. To determine the region of the linear rise, the data were fitted using two straight lines given by

\[
y_{j} = m_{j}x(1:n) + C_{j}
\]

and

\[
\ln d_{j}(i) = \ln C + L(i\Delta t).
\]

Fig. 4. Principle of the false nearest neighbors method to determine the embedding dimension. The correct embedding dimension in the illustration is 2. A point and its neighbor are separated by a distance \( R_{\text{True}} \). (a) Data points are incorrectly embedded in one-dimensional phase space. The points appear to be closer to each other than they actually are \((R_{\text{Meas}} < R_{\text{True}})\), and therefore the nearest neighbor is false. (b) Data points are correctly embedded in two-dimensional phase space and the true distance is determined. (c) Data points embedded in a higher dimension. Again the measured distance is equal to the true distance.

3.2. Lyapunov exponent

The number of Lyapunov exponents is equal to the number of axes in the phase space reconstruction [12]. However in practice, only the largest exponent is calculated as this dominates the trajectory divergence [21]. There are several algorithms which calculate the largest Lyapunov exponent from the reconstructed attractor [12]. We used the algorithm of Rosenstein et al. as it is robust to noise and suitable for small data sets [21]. The divergence of trajectories at a time \( t \) is given by

\[
d(t) = Ce^{Lt},
\]

where \( C \) is a constant and \( L \) is the largest Lyapunov exponent. For each data point in the reconstructed attractor, its nearest neighbor is found. The nearest neighbor has to be separated in time by more than the average period of the time series. This constraint allows one to consider the two points to represent two different nearby trajectories. The mean period is calculated as the inverse of the mean frequency of the power spectrum [21]. From Eq. (12), for the \( j^{th} \) pair of neighbors, and time step \( i \), the separation is

\[
d_{j}(i) = Ce^{L(i\Delta t)},
\]

where \( \Delta t \) is the time between frames. Hence

\[
\ln d_{j}(i) = \ln C + L(i\Delta t).
\]
where $n$ is the break-point time step. $n$ was determined as the value which minimizes

$$m_2 x(n) + C_2 - (m_1 x(n) + C_1).$$

Hence $m_2$ is the Lyapunov exponent, and $(n\Delta t)$ is the limit of predictability.

4. Results

Figure 5 shows the mutual information for each subject averaged across Zernike coefficients. The mean lag across all subjects was $0.31 \pm 0.07$ s. The average percentage of false nearest neighbors averaged across all aberrations for each subject is shown in Fig. 6. The embedding dimension was taken as the lag at which the number of false nearest neighbors was $\leq 5\%$. This gave an embedding dimension of 3 for each subject. The embedding dimension for each individual aberration for each subject was also 3. Figure 7 shows the phase space reconstruction for each subject for the rms wavefront error.

Fig. 5. The mutual information averaged across Zernike coefficients for each subject. The lag is in units of data points. * Indicates the location of the first minimum, and hence the embedding lag.
Fig. 6. Graph of the percentage of FNN averaged across all Zernike coefficients for each subject. * Indicates the embedding dimension, which was taken as the point where the lag was ≤5%.

Fig. 7. Reconstructed attractors for the rms wavefront error for each subject. Units are in µm.
Figure 8 shows the average divergence across the attractor for the rms wavefront error for each subject. Only the first 40 data points, corresponding to a time span of up to 2 s, have been included for clarity. The linear rise is also shown. For comparison purposes the divergence of the rms wavefront error, randomly reordered in time, is also plotted. This so-called random-shuffle surrogate data preserves the statistical properties of the data and is used to confirm that the data is not random but chaotic [14]. As can be seen from the plots, reordering the data eliminates the linear rise, and the divergence plot is characteristic of stochastic data. The mean limit of predictability for the rms wavefront error was $0.99 \pm 0.20$ s, and the mean Lyapunov exponent was $0.44 \pm 0.15 \mu m/s$. The lag, limit of predictability, and Lyapunov exponent for each subject for the rms wavefront error are shown in Table 2.

| Subject | Lag (s)  | $P_{rms}$ (s) | $L_{rms}$ (µm/s) |
|---------|---------|--------------|-----------------|
| KH      | 0.24    | 0.82         | 0.51            |
| EM      | 0.34    | 1.02         | 0.27            |
| JC      | 0.34    | 1.16         | 0.35            |
| YP      | 0.39    | 1.21         | 0.52            |
| CS      | 0.34    | 1.02         | 0.33            |
| CV      | 0.19    | 0.68         | 0.66            |
| Mean    | 0.31 ± 0.07 | 0.99 ± 0.20 | 0.44 ± 0.15     |

Figure 9 shows the limit of predictability and Lyapunov exponent for each individual Zernike aberration coefficient averaged across subjects. The lags used in the phase space reconstruction for each subject were obtained from Fig. 5, the first minimum of the average mutual information, and the embedding dimension was 3. The attractors for individual Zernike aberrations for each subject were similar to those shown in Fig. 7.
5. Discussion

In this study we measured the monochromatic aberration dynamics of 6 subjects during steady-state fixation. We used techniques from chaos theory to reconstruct the attractors and determine the Lyapunov exponents.

5.1. Chaotic nature of aberration dynamics and implications for modeling

There are currently very few models of ocular aberration dynamics [22]. Modeling of aberration dynamics is important, for example, in developing algorithms to better control closed-loop adaptive optics systems [23]. From Fig. 8 it can be seen that in comparison to the randomly shuffled data, all subjects show an initial rise in the divergence of neighboring trajectories for the rms wavefront error. The mean Lyapunov exponent was 0.44 ± 0.15 µm/s. All individual Zernike aberration coefficients also had a positive Lyapunov exponent. This indicates that like other physiological signals, the fluctuations in the aberrations are chaotic [11,13–16]. Consequently, the aberration fluctuations are not stochastic and there are underlying equations that determine their behavior. The phase space portraits from the rms wavefront error shown in Fig. 7 are similar to those of other physiological systems and are characteristic of a so-called strange attractor [12].

The mean lag averaged across aberrations and subjects for the rms wavefront error was 0.31 ± 0.07 s. This value indicates the memory of the system generating the fluctuations, as beyond this time the data are reasonably uncorrelated. The average limit of predictability for the rms wavefront error was 0.99 ± 0.20 s. Hence after this time period it is difficult to quantify the separation of trajectories in phase space. The limit of predictability was similar across all individual aberration coefficients. For all subjects, the embedding dimension for all Zernike aberration coefficients was 3. This indicates that there are ~3 variables controlling the fluctuations [20]. Hence, when developing models, at least three variables should be included.

5.2 Possible inputs to the chaotic dynamics

5.2.1. The heartbeat

We speculate that one of the major inputs into the chaotic dynamics of the eyes’ aberrations is the effect of the heartbeat. A number of studies have shown that the aberration dynamics show some correlation with the heartbeat [4,8,24], and there is much evidence that the heartbeat is...
itself a chaotic signal, see for example [13,20]. The heartbeat has a wide-spread impact on the eye, and its effect on aberrations is likely to mediate itself via a number of mechanisms. These include fundus pulsation [25], corneal pulsation [26], and the effect on the crystalline lens owing to blood flow through the ciliary body [24]. Furthermore, the heartbeat is correlated with fixational eye movements [27,28], changes in intraocular pressure [29,30], and fluctuations in pupil size [31]. These events are likely to alter the shape of the eye and affect the aberrations. Although these events show some correlation with the heartbeat, we note that their dynamic properties unrelated to the heartbeat may also affect the chaos observed in this study. Chaos has been observed in fixational eye movements [32], blood pressure control [16], and pupillary hippus [15].

The average values of the Lyapunov exponents found here are similar to that of the heartbeat [20]. Future work will involve measuring simultaneously the heartbeat signal alongside the ocular aberrations, and processing both signals using chaos theory. The likely impact of the heartbeat on the chaotic nature of the aberration dynamics raises the interesting possibility that changes in the chaos in the heartbeat, for example due to ill health [33,34], may reveal itself as changes in the chaotic properties of aberration dynamics.

5.2.2. Accommodative microfluctuations

Another plausible major input into the chaotic dynamics is the microfluctuations in accommodation [35]. Aside from the impact of the heartbeat [36], unlike other aberrations, there is evidence that components of accommodation fluctuations are actively changed by the accommodation control system, see for example [37]. Specifically, the magnitude of the fluctuations increases in response to a decrease in blur sensitivity, and so are affected by retinal image quality [38–41]. Several authors have proposed that the accommodation control system monitors the rate of change of image contrast with microfluctuations in defocus, to help the eye stay in focus on a stationary target, and also to determine the response to changes in object distance [42].

In this study we also analyzed the microfluctuations in accommodation using the defocus and spherical aberration Zernike coefficients [43]. We found the attractor to be similar to that of the rms wavefront error. The mean limit of predictability across subjects was $1.16 \pm 0.35$ s, and the mean Lyapunov exponent was $0.36 \pm 0.26$ D/s. As the other aberrations in the eye show some correlation with accommodation microfluctuations, this may partly explain the chaos observed in their fluctuations [44,45]. From Fig. 9(b) the Lyapunov exponent of Zernike defocus is notably larger than that of the other aberrations. Owing to the variability between subjects, this difference was not statistically significant however. The larger value may be due to the active input of the accommodation control system to this aberration. Future work will involve altering retinal image quality, with adaptive optics for example [38], and determining how the chaotic properties of microfluctuations in accommodation and other aberrations change.

5.2.3. Tear film fluctuations

Changes in tear film are known to have some impact on the aberration dynamics, see for example [46]. To examine the effect of tear film fluctuations we fitted subject EM with a scleral contact lens [47]. The front of the lens was dried once the lens was in place on the eye. This lens preserves the tear film between the eye and the lens. Figure 10(a) shows the fluctuations in the rms wavefront error with and without the lens in place. For this subject, there were no blinks during the measurement without and with the scleral lens. Hence we were comparing the effect of the tear film breaking up. Figure 10(b) shows the corresponding divergence plots. The scleral lens did not affect the limit of predictability. The Lyapunov exponent was $0.33 \mu m/s$ for the scleral lens case, in comparison to a value of $0.27 \mu m/s$ for the no lens case. The increase of the Lyapunov exponent for the scleral lens case indicates that the fluctuations in tear film may be a source of noise that ‘dilutes’ the chaos, as a flat
Fig. 10. (a) Time course of the rms wavefront error for subject EM with and without a scleral lens. Both signals have been detrended and the plot for the no lens case has been displaced by an amount of +0.15 µm for clarity. (b) Corresponding divergence plots. Again the no lens case has been displaced for clarity (+0.55 µm/s).

divergence plot indicates noise. Hence the Lyapunov exponents obtained in this study should be regarded as a lower limit.

5.2.4. Others

Another possible input into the results is cognitive demand. It has been shown that cognitive demand affects both the heartbeat and accommodation system [48]. Future work will include analyzing the chaotic properties of aberration dynamics with different cognitive tasks.

5.3. Potential impact of instrument noise, eye movement artifacts and blink removal artifacts

The Shack-Hartmann measurements are not noise free. There are noise sources intrinsic to the instrument itself, such as read noise, and sources external to the instrument owing to movement of the eye relative to the system. The effect of noise is to shorten the length of time of the linear rising region in the divergence plots [21], and also eventually flatten the divergence plots. Hence, like the Lyapunov exponents, the limits of predictability obtained in this study should be considered as a lower bound of the noise free (true) limits of predictability.

To estimate the intrinsic noise level, measurements were taken on an artificial eye consisting of a 20 mm focal length lens and a flat matte surface acting as a retina [17]. Measurements were performed over a 5 mm pupil diameter and the laser power was lowered such that the amount of light returning from the artificial eye was the same as that returning from the human eye. Figure 11(a) shows the time course of the rms wavefront error for both the artificial eye and subject YP (whose measurements were also recorded over a 5 mm pupil). As can be seen from the plot, the fluctuations in the artificial eye are considerably smaller than that of YP. Hence the contribution of intrinsic noise to the observed results is likely to be negligible. The signal to noise, measured as the ratio of the standard deviation of the rms wavefront error for YP to that of the artificial eye is 13. Figure 11(b) shows the divergence plots for both signals, which confirms that the results for the artificial eye are not chaotic but merely stochastic.

Fixational eye movements cause an effective translation of the pupil relative to the lenslet array [9]. Sahin has found that the mean pupil translation of healthy subjects during a 50 ms time span is around 40 ± 10 µm [49]. To determine the relative effect of eye movements, for each Shack-Hartmann measurement for subject KH, we calculated the rms wavefront error with the pupil shifted by +40 µm and −40 µm. Subject KH was chosen as they had the smallest fluctuations in their rms wavefront error and so pupil movements will have had a larger relative effect. Figure 12(a) shows the rms wavefront error time course for the original time series, assuming the eye’s pupil was static, and the resulting plots for a pupil movement of +40 µm and −40 µm. As can be seen from the plot the differences owing to pupil
movement are extremely small in comparison to the measured fluctuations assuming a fixed pupil. Figure 12(b) shows the divergence plots. The limit of predictability ranged from 0.78 s to 0.97 s, and the Lyapunov exponent ranged from 0.50 µm/s to 0.51 µm/s. In comparison to the fixed pupil results ($P_{rms} = 0.82$ s and $L_{rms} = 0.51$ µm/s), we therefore estimate the impact of pupil movements to be 23% for the limit of predictability and 2% for the Lyapunov exponent. Future work will involve simultaneous measurement of the pupil position during the Shack-Hartmann recordings.

Prior to analyzing the data using chaos theory, blinks were removed and the data points were replaced using cubic spline interpolation. Typically there were two to three blinks during a 23 s time period and one to two blinks during a 11.5 s time period. This amounts to up to around only 4% of data points being replaced by a cubic spline function. Hence it is unlikely that removal of blinks had a significant impact on the results.

5.4 Other applications of chaos theory analysis to aberration dynamics

A change in the chaotic properties of other physiological signals can indicate changes in the state of the system, for example owing to pathology [33,34]. Hence chaos theory may have application in detecting subtle changes in the aberration dynamics both between subjects, and within subjects for different experimental conditions. In particular, chaos theory may have utility in the study of microfluctuations of accommodation and myopia [41]. For example, in late-onset myopes, the fluctuations in accommodation display large low frequency fluctuations [50]. Methods used to control chaos in the brain and heart could also be applied to the eye to restore normal functioning of the accommodation system in such subjects [51,52].
6. Conclusion

We used techniques from chaos theory to analyze the monochromatic aberration dynamics of six subjects during steady-state viewing. We found that the fluctuations in the aberrations are chaotic and that the dynamics can be reconstructed in phase space in 3 dimensions; this has implications for future modeling. A further application of chaos theory is in studying and manipulating the accommodation system, particularly in myopic subjects.

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