New algebraic methods for calculating the heat kernel and the effective action in quantum gravity and gauge theories

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Abstract

An overview about recent progress in the calculation of the heat kernel and the one-loop effective action in quantum gravity and gauge theories is given. We analyse the general structure of the standard Schwinger-De Witt asymptotic expansion and discuss the applicability of that to the case of strongly curved manifolds and strong background fields. We argue that the low-energy limit in gauge theories and quantum gravity, when formulated in a covariant way, should be related to background fields with covariantly constant curvature, gauge field strength and potential term. It is shown that the condition of the covariant constancy of the background curvatures brings into existence some Lie algebra. The heat kernel operator for the Laplace operator is presented then as an average over the corresponding Lie group with some nontrivial Gaussian measure. Using this representation the heat kernel diagonal is obtained. The result is expressed purely in terms of curvature invariants and is explicitly covariant. Related topics concerning the structure of symmetric spaces and the calculation of the effective action are discussed.

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1. Introduction

The heat kernel proved to be a very powerful tool both in quantum field theory and in mathematical physics. It has been the subject of much investigation in recent years in physical as well as in mathematical literature [1-22]. The study of the heat kernel is motivated, in particular, by the fact that it gives a general framework of covariant methods for investigating the effective action in quantum field theories with local gauge symmetries, such as quantum gravity and gauge theories, due to special advantages achieved by using geometric methods.

Let us consider a set of quantized (bosonic) fields \( \varphi = \{\varphi^A(x)\} \) on a \( d \)-dimensional Riemannian manifold \( M \) without boundary (\( \partial M = \emptyset \)) of metric \( g_{\mu\nu} \) with Euclidean signature. The one-loop contribution of the field \( \varphi \) to the effective action is expressible in terms of the functional determinant of some elliptic differential operator \( F \) that can best be presented using the \( \zeta \)-function regularization [1]

\[
\Gamma(1) = \frac{1}{2} \log \det \frac{F}{\mu^2} = \frac{1}{2} \text{Tr} \log \frac{F}{\mu^2} = -\frac{1}{2} \zeta'(0),
\]

where \( \mu \) is a renorm parameter introduced to preserve dimensions. The \( \zeta \)-function is defined by

\[
\zeta(p) = \mu^{2p} \text{Tr} F^{-p} = \frac{\mu^{2p}}{\Gamma(p)} \int_0^\infty dt \, t^{p-1} \text{Tr} U(t),
\]

where \( \text{Tr} \) means the functional trace

\[
\text{Tr} U(t) = \int_M dx g^{1/2} \text{tr}[U(t)],
\]

\( \text{tr} \) is the usual matrix trace, \( U(t) = \{U^A_B(t|x,x')\} \) is the corresponding heat kernel

\[
U(t|x,x') = \exp(-tF)P(x,x')g^{-1/2}\delta(x,x'),
\]

where \( P(x,x') = \{P^A_B(x,x')\} \) is the parallel displacement operator of the field \( \varphi = \{\varphi^A\} \) from the point \( x \) to the point \( x' \) along the geodesic, and \( [U(t)] \) is the heat kernel diagonal

\[
[U(t)] = U(t|x,x).
\]

For a very wide range of models in quantum field theory it is sufficient to consider the second order operators of Laplace type

\[
F = -\Box + Q + m^2,
\]

where

\[
\Box = g^\mu\nu \nabla_\mu \nabla_\nu = g^{-1/2}(\partial_\mu + A_\mu)g^{1/2}g^{\mu\nu}(\partial_\nu + A_\nu)
\]
is the Laplacian (or Dalambertian in hyperbolic case), $A_\mu = \{A^A_{B\mu}\}$ being an arbitrary linear connection, $Q(x) = \{Q^A_B(x)\}$ an arbitrary matrix-valued potential term and $m$ a mass parameter. The connection $A_\mu$ includes both the Levi-Civita connection, the appropriate spin one and the vector gauge connection and is determined by the commutator of covariant derivatives

$$[\nabla_\mu, \nabla_\nu]\varphi = R_{\mu\nu}\varphi. \quad (1.8)$$

The Riemann curvature tensor $R_{\mu\nu\alpha\beta}$, the curvature of background connection $\mathcal{R}_{\mu\nu} = \{\mathcal{R}^A_{B\mu\nu}\}$ and the potential term completely describe the background metric and connection, at least locally. In the following we will call these quantities the background curvatures or simply curvatures and denote them symbolic by $\mathcal{R} = \{R_{\mu\nu\alpha\beta}, \mathcal{R}_{\mu\nu}, Q\}$.

Let us make some remarks on the subject.

- First of all, it is obviously impossible to evaluate the heat kernel and the effective action exactly, even at the one-loop order. There are, of course, some simple special cases of background fields and geometries that allow the exact and even explicit calculation of the heat kernel or the effective action. However, the effective action is an action, i.e. a functional of background fields that should be varied to get the Green functions, the vacuum expectation values of various fields observables, such as energy-momentum tensor and Yang-Mills currents etc. That is why one needs the effective action for the general background and, therefore, one has to develop consistent approximate methods for its calculation.

- Second, in quantum gravity and gauge theories the effective action is a covariant functional, i.e. invariant under diffeomorphisms and local gauge transformations. That is why the approximations for calculating the effective action have to preserve the general covariance at each order. The flat space perturbation theory is an example of bad approximation because it is not covariant.

### 2. Schwinger - De Witt asymptotic expansion

First of all, let us mention the very important so called Schwinger - De Witt asymptotic expansion of the heat kernel at $t \to 0$ [1-5]

$$\text{Tr} U(t) = (4\pi t)^{-d/2} \exp(-tm^2) \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} B_k, \quad (2.1)$$

$$B_k = \int_M dx g^{1/2} \text{tr} b_k. \quad (2.2)$$

This expansion is purely local and does not depend, in fact, on the global structure of the manifold. Its famous coefficients $b_k$ (we call them Hadamard - Minakshisundaram - De Witt - Seely (HMDS) coefficients) are local invariants built from the curvature, the potential term and their covariant derivatives [1,5,6,14]. They play a very important role both in physics and mathematics and are closely connected with various sections of mathematical physics such as spectral geometry, index theorem, Korteweg - de Vries hierarchy etc.. [14,22]. Therefore, the calculation of HMDS-coefficients in general case
of arbitrary background is in itself of great importance. However, it offers a complicated technical problem. Various methods were used for calculating these coefficients. The pioneering De Witt’s method [1] is quite simple but gets very cumbersome at higher orders. By means of it only three first coefficients $b_0, b_1, b_2$ were calculated. The approach of mathematicians [5-9, 12, 14] differs considerably from that of physicists. It is very general but also very complicated and seems not to be well adopted to physical problems. It allowed to compute in addition the next coefficient $b_3$ [9].

It is the general manifestly covariant technique for calculation of the HMDS - coefficients $b_k$ that was developed in our papers [10,4]. It proved to be very effective and allowed to calculated the next coefficient $b_4$. Moreover, this technique allows to analyze the general structure of all HMDS-coefficients and to calculate them in some approximation, e.g. as expansion in curvature etc. In the case of scalar operators the coefficient $b_4$ is also calculated in [11]. Analytic approach was developed in [7], where a general expression in closed form for these coefficients was obtained. Very good reviews of the calculation of the HMDS-coefficients are given in recent papers [14].

The Schwinger - De Witt expansion is good for small $t$, viz.\[ t \mathcal{R} \ll 1, \quad (2.3) \]

and thereby in the case of massive quantized fields in weak background fields when \[ \mathcal{R} \ll m^2, \quad (2.4) \]
i.e. in the case when the Compton wave length of the massive quantum field is much smaller than the characteristic length scale of the background fields \[ \frac{\hbar}{mc} \ll L. \quad (2.5) \]

In this case the local HMDS-coefficients $b_k$ are much smaller than the corresponding power of the mass parameter \[ b_k \ll m^{2k}. \quad (2.6) \]

Therefore, one can simply integrate over $t$ to get the $1/m^2$ asymptotic expansion of the one-loop effective action: for odd $d$

\[
\Gamma_{(1)} = \frac{1}{2} (4\pi)^{-d/2} \pi (-1)^{d+1} \sum_{k=0}^{\infty} \frac{m^{d-2k}}{k! \Gamma \left( \frac{d}{2} - k + 1 \right)} B_k, \quad (2.7)
\]

and for even dimension

\[
\Gamma_{(1)} = \frac{1}{2} (4\pi)^{-d/2} \left\{ (-1)^{d/2} \sum_{k=0}^{d/2} \frac{m^{d-2k}}{k! \Gamma \left( \frac{d}{2} + 1 - k \right)} \right\} \\
\times B_k \left[ \ln \frac{m^2}{\mu^2} - \Psi \left( \frac{d}{2} - k + 1 \right) - C \right] + \sum_{k=\frac{d}{2}+1}^{\infty} \frac{\Gamma \left( k - \frac{d}{2} \right) (-1)^k}{k! m^{2k-d}} B_k \right\}. \quad (2.8)
\]
where $\Psi(q) = (d/dq) \log \Gamma(q)$, $C = -\Psi(1)$.

This approximation describes well the physical effect of the vacuum polarization effect of massive quantized field in weak background fields.

3. General structure of the asymptotic expansion and partial summation

However the Schwinger - De Witt approximation is of very limited applicability. It is absolutely inadequate for large $t$ ($tR \gg 1$), in strongly curved manifolds and strong background fields ($R \gg m^2$) and becomes meaningless in massless theories. Therefore, this approximation can not describe essentially nonperturbative effects such as particle creation and the vacuum polarization by strong background fields, which are, in fact, nonlocal and nonanalytical.

In fact, the effective action is a nonlocal and nonanalytical functional and possesses a sensible massless limit. But its calculation requires quite different methods. One of such methods which would exceed the limits of the Schwinger - De Witt asymptotic expansion is the partial summation procedure [2]. It is based on the analysis of the general structure of the HMDS-coefficients. The HMDS-coefficients are the local polynomial invariants built from the curvature and its covariant derivatives. The first two HMDS-coefficients have the well known form

$$B_0 = \int dx g^{1/2} tr 1,$$

$$B_1 = \int dx g^{1/2} tr \left\{ Q - \frac{1}{6} R \right\}$$

(3.1)

One can classify all the terms in the higher order coefficients $B_k$, ($k \geq 2$), according to the number of curvatures and their derivatives. The terms with leading derivatives can be shown to have the following structure $R \Box^{k-2} R$. Then it follows the class of terms cubic in the curvatures etc.. The last class of terms does not contain any covariant derivatives at all but only the powers of the curvature

$$B_k = \int dx g^{1/2} tr \left\{ R \Box^{k-2} R + \sum_{0 \leq i \leq 2k-6} R \nabla i R \nabla^{2k-6-i} R \right.$$

$$+ \cdots + \sum_{0 \leq i \leq k-3} R^i (\nabla R) R^{k-i-3} (\nabla R) + R^k \right\}$$

(3.2)

Now one can try to sum up each class of terms separately to get the corresponding expansion of the heat kernel

$$\text{Tr} U(t) = \int dx g^{1/2}(4\pi t)^{-d/2} \exp(-tm^2) tr \left\{ 1 - t \left( Q - \frac{1}{6} R \right) \right. $$

$$+ t^2 R \chi(t \Box) R + \cdots + t^3 \nabla R \Psi(t R) \nabla R + \Phi(t R) \right\}$$

(3.3)
and that of the effective action
\[
\Gamma_{(1)} = \int dx g^{1/2} tr \left\{ \mathcal{R} F(\square) \mathcal{R} + \cdots + \nabla \mathcal{R} Z(\mathcal{R}) \nabla \mathcal{R} + V(\mathcal{R}) \right\}
\] (3.4)

The idea of partial summation consists in comparing the values of all terms in HMDS-coefficients \( B_k \) (3.2), picking up the main (the largest in some approximation) terms and summing up the corresponding partial sum. There is always a lack of uniqueness concerned with the global structure of the manifold, when doing so. But, hopefully, fixing the topology, e.g. the trivial one, one can obtain a unique, well defined, expression that would reproduce the Schwinger-De Witt expansion, being expanded in curvature.

One should mention that these expansions are asymptotic ones and do not converge, in general. So, one has to use some methods of summation of the divergent asymptotic series. This can be done by using an integral transform and analytic continuation [4].

Actually, the effective action is a covariant functional of the metric and depends on the geometry of the manifold as a whole, i.e. it depends on both local characteristics of the geometry like invariants of the curvature tensor and its global topological structure. However, we are not going to investigate the influence of the topology but concentrate our attention, as a rule, on the local effects. Then the possible approximations for evaluating the effective action can be based on the assumptions about the local behavior of the background fields, dealing with the real physical gauge invariant variations of the local geometry, i.e. with the curvature invariants, but not with the behavior of the metric and the connection which is not invariant. Comparing the value of the curvature with the values of its covariant derivatives one comes to two possible approximations: i) the short-wave (or high-energy) approximation characterized by
\[
\nabla \nabla \mathcal{R} \gg \mathcal{R} \mathcal{R}
\] (3.5)

and ii) the long-wave (or low-energy) one
\[
\nabla \nabla \mathcal{R} \ll \mathcal{R} \mathcal{R}.
\] (3.6)

Such a formulation is manifestly covariant and, therefore, more suitable for calculations in quantum gravity and gauge theories than the usual flat space perturbation theory.

4. High-energy approximation

The idea of partial summation was realized in short-wave approximation for investigating the nonlocal aspects of the effective action (in other words the high-energy limit of that) in [15,4]. The leading terms with higher derivatives in HMDS-coefficients (quadratic in curvatures) have the following form
\[
B_k = \int dx g^{1/2} tr \left\{ f_k^{(1)} Q \square^{k-2} Q + 2 f_k^{(2)} R_{\alpha\mu} \nabla^\alpha \square^{k-3} \nabla_\nu R^\nu_{\mu} \\
- 2 f_k^{(3)} Q \square^{k-2} R + f_k^{(4)} R_{\mu\nu} \square^{k-2} R^\mu_{\nu} + f_k^{(5)} R \square^{k-2} R + O(R^3) \right\}
\] (4.1)
The coefficients $f_k^{(i)}$ were calculated explicitly in [15,4] and completely independent in [16].

Moreover, the local Schwinger - De Witt expansion can now be summed up to get the nonlocal heat kernel

$$
\text{Tr} \, U(t) = \int dx \, g^{1/2} (4\pi t)^{-d/2} \exp(-tm^2) \text{tr} \left\{ 1 - t \left( Q - \frac{1}{6} R \right) \right. \\
+ \frac{t^2}{2} \left[ Q\gamma^{(1)}(t \Box)Q + 2R_{\alpha\mu} \nabla_\alpha \frac{1}{\Box} \gamma^{(2)}(t \Box) \nabla_\nu R^\nu\mu - 2Q\gamma^{(3)}(t \Box)R \right. \\
+ \left. R_{\mu\nu} \gamma^{(4)}(t \Box)R^{\mu\nu} + R\gamma^{(5)}(t \Box)R \right] + O(R^3) \right\}.
$$

(4.2)

and the corresponding high-energy effective action

$$
\Gamma_{(1)} = \Gamma_{(1)\text{loc}} + \Gamma_{(1)\text{nonloc}}
$$

(4.3)

where the essential nonlocal part has the form

$$
\Gamma_{(1)\text{nonloc}} = \frac{1}{2} (4\pi)^{-d/2} \int dx \, g^{1/2} \text{tr} \left\{ Q\beta^{(1)}(\Box)Q + 2R_{\alpha\mu} \nabla_\alpha \frac{1}{\Box} \beta^{(2)}(\Box) \nabla_\nu R^\nu\mu \\
- 2Q\beta^{(3)}(\Box)R + R_{\mu\nu} \beta^{(4)}(\Box)R^{\mu\nu} + R\beta^{(5)}(\Box)R + O(R^3) \right\}
$$

(4.4)

The explicit form of the formfactors $\gamma^{(i)}(t \Box)$ and $\beta^{(i)}(\Box)$ is given in [15,4].

Another approach to study the high-energy limit of the effective action, so called covariant perturbation theory, is developed in [17]. By means of it the next third order in curvatures of the nonlocal high-energy limit is investigated.

### 5. Low-energy approximation

The low-energy effective action, in other words, the effective potential, presents a very natural tool for investigating the vacuum of the theory, its stability and the phase structure [23]. In the case of gauge theories and quantum gravity it is much more complicated problem as the high-energy limit. Here only partial success is achieved and various approaches to this problem are only outlined (see, e.g. our recent papers [20,21]).

The long-wave (or low-energy) approximation is determined, as it was already stressed above, by strong slowly varying background fields. This means that the derivatives of all invariants are much smaller than the products of the invariants themselves. The zeroth order of this approximation corresponds to covariantly constant background curvatures

$$
\nabla_\mu R_{\alpha\beta\gamma\delta} = 0, \quad \nabla_\mu R_{\alpha\beta} = 0, \quad \nabla_\mu Q = 0.
$$

(5.1)

In this case the HMDS-coefficients are simply polynomials in curvature invariants and potential term of dimension $\Re^k$ up to terms with one or more covariant derivatives of the
background curvatures \( O(\nabla \mathcal{R}) \)

\[
\begin{align*}
b_k &= \sum_{n=0}^{k} \binom{k}{n} Q^{k-n} a_n + O(\nabla \mathcal{R}), \\
a_k &= b_k \bigg|_{Q=\nabla \mathcal{R}=0} = \sum \mathcal{R}^k.
\end{align*}
\]

(5.2)  

(5.3)

Mention that the commutators \([Q, \mathcal{R}_{\mu \nu}]\) are of order \( O(\nabla \nabla \mathcal{R}) \) and, therefore are neglected here.

Then after summing the Schwinger-De Witt expansion we obtain for the heat kernel, the \( \zeta \)-function and the effective action

\[
\begin{align*}
\operatorname{Tr} U(t) &= \int_M dx \, g^{1/2}(4\pi t)^{-d/2} \left\{ \exp \left(-t(m^2 + Q)\right) (\Omega(t) + O(\nabla \mathcal{R})) \right\}, \\
\zeta(p) &= \int_M dx \, g^{1/2}(4\pi)^{-d/2} \frac{\mu^{2p}}{\Gamma(p)} \int_0^\infty dt \, t^{p-d/2-1} \\
&\quad \times \operatorname{tr} \left\{ \exp \left(-t(m^2 + Q)\right) (\Omega(t) + O(\nabla \mathcal{R})) \right\}, \\
\Gamma(1) &= \int_M dx \, g^{1/2} \{ V(\mathcal{R}) + O(\nabla \mathcal{R}) \},
\end{align*}
\]

(5.4)  

(5.5)  

(5.6)

with

\[
V(\mathcal{R}) = \frac{1}{2} (4\pi)^{-d/2} \frac{1}{\Gamma\left(\frac{d}{2} + 1\right)} \int_0^\infty dt \left( \log(\mu^2 t) + \psi \left( \frac{d}{2} + 1 \right) \right) \\
\times \left( \frac{\partial}{\partial t} \right)^{\frac{d}{2}+1} \operatorname{tr} \left\{ \exp(-t(m^2 + Q))\Omega(t) \right\}
\]

for even \( d \) and

\[
V(\mathcal{R}) = \frac{1}{2} (4\pi)^{-d/2} \frac{1}{\Gamma\left(\frac{d}{2} + 1\right)} \\
\times \int_0^\infty dt \, t^{-1/2} \left( \frac{\partial}{\partial t} \right)^{\frac{d+1}{2}} \operatorname{tr} \left\{ \exp(-t(m^2 + Q))\Omega(t) \right\}
\]

(5.7)  

(5.8)

for odd \( d \), where

\[
\Omega(t) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} a_k,
\]

(5.9)

is a function of local invariants of the curvatures (but not of the potential).

It is naturally to call the functions \(\Omega(t)\) and \(V(\mathcal{R})\), that do not contain the covariant derivatives at all and so determine the zeroth order of the heat kernel and that of the effective action, the \textit{generating function} for covariantly constant terms in HMDS-coefficients and the \textit{effective potential} in quantum gravity respectively.
Let us mention that such a definition of the effective potential is not conventional. It differs from the definition that is often found in the literature \[24\]. What is meant usually under the notion of the effective potential is a function of the potential term only \(Q\), because it does not contain derivatives of the background field (in contrast to Riemann curvature \(R_{\alpha\beta\gamma\delta}\) that contains second derivatives of the metric and the curvature \(R_{\mu\nu}\) with first derivatives of the connection). So, e.g. in \[24\] the potential term \(Q\) is summed up exactly but an expansion is made not only in covariant derivatives but also in powers of curvatures \(R_{\mu\nu\alpha\beta}\) and \(R_{\mu\nu}\), i.e. the curvatures are treated perturbatively. Thereby the validity of this approximation for the effective action is limited to small curvatures

\[
R_{\mu\nu}, R_{\mu\nu\alpha\beta} \ll Q. \tag{5.9a}
\]

Such an expansion is called ‘expansion of the effective action in covariant derivatives’. Without the potential term \((Q = 0)\) the effective potential in such a scheme is trivial. Hence we stress here once again, that the effective potential in our definition contains, in fact, much more information than the usual effective potential does when using the ‘expansion in covariant derivatives’. As a matter of fact, what we mean is the low-energy limit of the effective action formulated in a covariant way.

The conditions of integrability of these relations lead to strong algebraic restrictions on the curvatures themselves

\[
R_{\mu\nu\lambda\sigma} R_{\beta\gamma\delta} + R_{\mu\nu\lambda\sigma} R_{\gamma\delta\alpha\beta} = 0 \tag{5.10}
\]

\[
R_{\mu\nu\lambda\sigma} R_{\alpha\beta\gamma\delta} + R_{\alpha\beta\lambda\mu} R_{\gamma\delta} = 0 \tag{5.11}
\]

\[
[ R_{\mu\nu}, R_{\alpha\beta} ] + R_{\mu\nu\lambda\sigma} R_{\alpha\beta\gamma\delta} - R_{\alpha\beta\lambda\mu} R_{\gamma\delta} = 0 \tag{5.12}
\]

\[
[ R_{\mu\nu}, Q ] = 0 \tag{5.13}
\]

Mention that the conditions (5.1), (5.10)-(5.13) are local. They determine the geometry of the locally symmetric spaces. However, the manifold is globally symmetric one only in the case when it satisfies additionally some global topological restrictions (usually it has to be connected) and the condition (5.1) is valid everywhere, i.e. at any point of the manifold \[25\].

But in our case, i.e. in physical problems, the situation is radically different. The correct setting of the problem seems to be as follows. The low-energy effective action depends, in general, also essentially on the global topological properties of the space-time manifold. But, as it was mentioned above, we do not investigate in this paper the influence of the topology. Therefore, consider a complete noncompact asymptotically flat manifold without boundary that is homeomorphic to \(\mathbb{R}^d\). Let a finite not small, in general, domain of the manifold exists that is strongly curved and quasi-homogeneous, i.e. the invariants of the curvature in this region vary very slowly. Then the geometry of this region is locally very similar to that of a symmetric space. However one should have in mind that there are always regions in the manifold where this condition is not fulfilled. This is, first of all, the asymptotic Euclidean region that has small curvature and, therefore, the opposite short-wave approximation is valid.

The general situation in correct setting of the problem is the following. From infinity with small curvature and possibly radiation, where \([17]\) \(\mathcal{R} R \ll \nabla\nabla\mathcal{R}\), we pass on to
quasi-homogeneous region where the local properties of the manifold are close to those of symmetric spaces. The size of this region can tend to zero. Then the curvature is nowhere large and the short-wave approximation is valid anywhere. If one tries to extend the limits of such region to infinity, then one has also to analyze the topological properties. The space can be compact or noncompact depending on the sign of the curvature. But first we will come across a coordinate horizon-like singularity, although no one true physical singularity really exists.

This construction can be intuitively imagined as follows. Take the flat Euclidean space $\mathbb{R}^d$, cut out from it a region $M$ with some boundary $\partial M$ and stick to it along the boundary, instead of the piece cut out, a piece of a curved symmetric space with the same boundary $\partial M$. Such a construction will be homeomorphic to the initial space and at the same time will contain a finite highly curved homogeneous region. By the way, the exact effective action for a symmetric space differs from the effective action for built construction by a purely topological contribution. This fact seems to be useful when analyzing the effects of topology.

Thus the problem is to calculate the low-energy effective action (the effective potential $V(\Re)$) (5.7), (5.8), i.e. the heat kernel for covariantly constant background. Although this quantity, generally speaking, depends essentially on the topology and other global aspects of the manifold, one can disengage oneself from these effects fixing the trivial topology. Since the asymptotic Schwinger - De Witt expansion does not depend on the topology, one can hold that we thereby sum up all the terms without covariant derivatives in it.

In other words the problem is the following. One has to obtain a local covariant function of the invariants of the curvature $\Omega(t)$ (5.9) that would describe adequately the low-energy limit of the heat kernel diagonal and that would, being expanded in curvatures, reproduce all terms without covariant derivatives in the asymptotic expansion of heat kernel, i.e. the HMDS-coefficients $a_k$ (5.3). If one finds such an expression, then one can simply determine the $\zeta$-function (5.5) and, therefore, the effective potential $V(\Re)$ (5.7), (5.8).

6. Algebraic approach

There exist a very elegant indirect possibility to construct the heat kernel without solving the heat equation but using only the commutation relations of some covariant first order differential operators. The main idea is in a generalization of the usual Fourier transform to the case of operators and consists in the following.

Let us consider for a moment a trivial case of vanishing curvatures but not the potential term

$$R_{\alpha\beta\gamma\delta} = 0, \quad R_{\alpha\beta} = 0, \quad Q \neq 0.$$  \hspace{1cm} (6.1)

In this case the operators of covariant derivatives obviously commute and form together with the potential term an Abelian algebra

$$[\nabla_\mu, \nabla_\nu] = 0, \quad [\nabla_\mu, Q] = 0.$$  \hspace{1cm} (6.2)
It is easy to show that the heat kernel operator can be presented in the form

$$\exp(t \Box) = (4\pi t)^{-d/2} \int dk g^{1/2} \exp \left\{ -\frac{1}{4t} k^\mu g_{\mu\nu} k^\nu + k^\mu \nabla_\mu \right\}.$$  \hspace{1cm} (6.3)

Here, of course, it is assumed that the covariant derivatives commute also with the metric

$$\left[ \nabla_\mu, g_{\alpha\beta} \right] = 0.$$  \hspace{1cm} (6.4)

Acting with this operator on the $\delta$-function and using the obvious relation

$$\exp(k^\mu \nabla_\mu) g^{-1/2} \delta(x, x') = g^{-1/2} \delta(x + k, x')$$ \hspace{1cm} (6.5)

one can simply integrate over $k$ to obtain the heat kernel in coordinate representation

$$U(t|x, x') = (4\pi t)^{-d/2} \exp \left\{ -t(m^2 + Q) - \frac{1}{4t} (x - x')^\mu g_{\mu\nu} (x - x')^\nu \right\}.$$ \hspace{1cm} (6.6)

The heat kernel diagonal is given then by

$$\left[ U(t) \right] = (4\pi t)^{-d/2} \exp \left\{ -t(m^2 + Q) \right\},$$ \hspace{1cm} (6.7)

and the function $\Omega(t)$ (5.9) is simply

$$\Omega(t) = 1.$$ \hspace{1cm} (6.7a)

In fact, the covariant derivatives do not commute and the commutators of them are proportional to the curvatures. Thus one can try to generalize the above idea in such a way that (6.3) would be the zeroth approximation in the commutators of the covariant derivatives, i.e. in the curvatures. We are going to find a representation of the heat kernel operator in the form

$$\exp(t \Box) = \int dk \Phi(t, k) \exp \left\{ -\frac{1}{4t} k^A \Psi_{AB}(t) k^B + k^A X_A \right\}$$ \hspace{1cm} (6.8)

where $X_A = X_A^\mu \nabla_\mu$ are some first order differential operators. The commutators of them should be proportional to the curvature

$$\left[ X_A, X_B \right] = O(\Re)$$ \hspace{1cm} (6.9)

and the functions $\Psi(t)$ and $\Phi(t, k)$ should have the following property

$$\Psi(t) = \gamma_{AB} + O(\Re), \hspace{1cm} \Phi(t, k) = \gamma^{1/2}(1 + O(\Re)),$$ \hspace{1cm} (6.9)

where $\gamma_{AB}$ is some constant nondegenerate positive definite matrix, $\gamma^{1/2} = \det \gamma_{AB}$ and $O(\Re)$ means the terms linear in the curvatures.
In general, the operators $X_A$ do not form a closed finite dimensional algebra because at each stage taking more commutators there appear more and more derivatives of the curvatures. It is the case of covariantly constant curvatures that actually closes the algebra. In this case the operators $X_A$ together with the curvatures form some Lie algebra.

Using this representation one could, as above, act with $\exp(k_A X^A)$ on the $\delta$-function on $M$ to get the heat kernel. The main point of this idea is that it is much more easier to calculate the action of the exponential of the first order operator $k_A X_A$ on the $\delta$-function than that of the exponential of the second order operator $\Box$.

7. Heat kernel in flat space with nonvanishing Yang-Mills background

This idea was realized in [21] for more complicated case of vanishing Riemann curvature (flat space) but nonvanishing curvature of background connection (Yang-Mills case)

$$R_{\alpha\beta\gamma\delta} = 0, \quad R_{\alpha\beta} \neq 0, \quad Q \neq 0.$$ (7.1)

Using the condition of covariant constancy of the curvatures (5.1), (5.12)-(5.13) one can show that in this case the covariant derivatives form a nilpotent Lie algebra

$$[\nabla_\mu, \nabla_\nu] = R_{\mu\nu},$$ (7.2)

$$[\nabla_\mu, R_{\alpha\beta}] = [\nabla_\mu, Q] = 0,$$

$$[R_{\mu\nu}, R_{\alpha\beta}] = [R_{\mu\nu}, Q] = 0.$$ (7.3)

For this algebra one can prove a theorem expressing the heat kernel operator in terms of an average over the corresponding Lie group

$$\exp(t \Box) = (4\pi t)^{-d/2} \det \left( \frac{t R}{\sinh(t R)} \right)^{1/2} \int dk g^{1/2} \exp \left\{ -\frac{1}{4t} k^\mu g_{\mu\lambda}(t R \coth(t R))^\lambda_\nu k^\nu + k^\mu \nabla_\mu \right\}$$ (7.4)

where $R$ means the matrix with coordinate indices $R = \{ R^\mu_\nu = g^{\mu\lambda} R^\lambda_\nu \}$ and the determinant is taken with respect to these indices, other (bundle) indices being intact. The proof of this theorem is given in [21].

Using this theorem we express the heat kernel in coordinate representation in terms of the quantity

$$\exp(k^\mu \nabla_\mu) P(x, x') g^{-1/2} \delta(x, x')$$ (7.5)

according to the definition of the heat kernel (1.4). It is not difficult to show that

$$\exp(k^\mu \nabla_\mu) P(x, x') g^{-1/2} \delta(x, x') = P(x, x') g^{-1/2} \delta(x + k, x').$$
Subsequently, the integral over $k^\mu$ becomes trivial and one obtains immediately the heat kernel in coordinate representation

$$U(t|x,x') = (4\pi t)^{-d/2} \mathcal{P}(x,x') \det \left( \frac{t\mathcal{R}}{\sinh(t\mathcal{R})} \right)^{1/2} \times \exp \left\{ -t(m^2 + Q) - \frac{1}{4t} (x - x')^\mu g_{\mu\lambda}(t\mathcal{R} \coth(t\mathcal{R}))^\lambda_\nu (x - x')^\nu \right\} \quad (7.6)$$

The heat kernel diagonal is now easily obtained by taking the coincidence limit $x = x'$

$$[U(t)] = (4\pi t)^{-d/2} \det \left( \frac{t\mathcal{R}}{\sinh(t\mathcal{R})} \right)^{1/2} \exp \{ -t(m^2 + Q) \} . \quad (7.7)$$

The generating function $\Omega(t)$ is now

$$\Omega(t) = \det \left( \frac{t\mathcal{R}}{\sinh(t\mathcal{R})} \right)^{1/2} . \quad (7.8)$$

It defines the $\zeta$-function (5.5) and the corresponding effective potential (5.7), (5.8). Expanding it in a power series in $t$ one can find all covariantly constant terms in all HMD$-\text{ coefficients}$ $a_k$.

As we have seen the contribution of the bundle curvature $\mathcal{R}_{\mu\nu}$ is not as trivial as that of the potential term. However, the algebraic approach does work in this case too. This is the generalization of the well known Schwinger result in quantum electrodynamics when the bundle curvature is just the electromagnetic field strength $\mathcal{R}_{\mu\nu} = iF_{\mu\nu}$. It is a good example how one can get the heat kernel without solving any differential equations but using only the algebraic properties of the covariant derivatives.

8. Heat kernel in symmetric spaces

Let us now try to generalize this construction to the case of the curved manifolds with covariantly constant curvature, i.e. symmetric spaces. Below we follow mainly our papers [20]. Let us list shortly some known facts about symmetric spaces presented in the form that will be convenient for further use. What are the direct consequences of the condition of covariant constancy of the curvature?

First of all, let $e^\mu_a(x,x')$ be a frame that is covariantly constant (parallel) along the geodesic between points $x$ and $x'$. The frame components of all tensors will be always understood with respect to this special frame. Let us consider the Riemann tensor in more detail. It is obvious that the frame components of the curvature tensor of a symmetric space are constant. For any Riemannian manifold they can be presented in the form

$$R_{abcd} = \beta_{ik} E^i_{ab} E^k_{cd} , \quad (8.1)$$
where $E_{ab}^i$, $(i = 1, \ldots, p; p \leq d(d - 1)/2)$, is some set of antisymmetric matrices and $\beta_{ik}$ is some symmetric nondegenerate $p \times p$ matrix. The traceless matrices $D_i = \{D_{ab}^i\}$ defined by

$$D_{ab}^i = -\beta_{ik}E_{cb}^k \epsilon^{ca} = -\delta_{bi}$$

(8.2)

are known to be the generators of the isotropy algebra $\mathcal{H}$ of dimension $\dim \mathcal{H} = p$

$$[D_i, D_k] = F_{ik}^j D_j,$$

(8.3)

with $F_{ik}^j$ being the structure constants. The structure constants $F_{ik}^j$ are completely determined by these commutation relations and satisfy Jacobi identities

$$[F_i, F_k] = F_{ik}^j F_j,$$

(8.4)

where $F_i = \{F_{ik}^j\}$ are the generators of the isotropy algebra in adjoint representation. Mention that the isotropy group $\mathcal{H}$ is always compact as it is a subgroup of the orthogonal group (in Euclidean case).

It is not difficult to show that the condition of integrability (5.10) of the equations (5.1) takes the form

$$E_{ac}^i D_{bk}^c - E_{bc}^i D_{ak}^c = E_{ab}^j F_{jk}^i.$$  

(8.5)

This equation takes place only in symmetric spaces and is the most important one. It is this equation that makes a Riemannian manifold the symmetric space. From the eqs. (8.3) and (8.4) we have, in particular,

$$\beta_{ik} F_{jm}^k + \beta_{mk} F_{ji}^k = 0,$$

(8.6)

that means that the adjoint and coadjoint representations of the isotropy group are equivalent and the matrix $\beta_{ik}$ plays the role of the metric of the isotropy algebra.

Moreover, the eq. (8.5) brings into existence a much wider algebra $\mathcal{G}$ of dimension $\dim \mathcal{G} = D = p + d$. Indeed, let us define the quantities $C_{ABC}^A = -C_{CBA}^A$, $(A = 1, \ldots, D)$ by

$$C_{ab}^i = E_{ab}^i, \quad C_{ib}^a = D_{ib}^a, \quad C_{kl}^i = F_{kl}^i, \quad C_{bc}^i = C_{ka}^i = C_{ik}^a = 0,$$

(8.7)

and the matrices $C_A = \{C_{AB}^C\} = \{C_a, C_i\}$,

$$C_a = \begin{pmatrix} 0 & D_{ai}^b \\ E_{ac}^j & 0 \end{pmatrix}, \quad C_i = \begin{pmatrix} D_{ia}^b & 0 \\ 0 & F_{ik}^j \end{pmatrix}. \quad (8.8)$$

Using the eqs. (8.3)-(8.6) one can show that they satisfy the Jacobi identities

$$[C_A, C_B] = C_{ABC}^C C_C$$

(8.9)

or, more precisely,

$$[C_a, C_b] = E_{ab}^i C_i, \quad [C_a, C_i] = D_{ai}^b C_b, \quad [C_i, C_k] = F_{ik}^j C_j,$$

(8.10)
and define, therefore, some Lie algebra $\mathcal{G}$ with the structure constants $C^A_{BC} = \{E^i_{ab}, D^b_{ia}, F^j_{ik}\}$, matrices $C_A$ being the generators in adjoint representation.

Further, introducing a symmetric nondegenerate $D \times D$ matrix

$$\gamma_{AB} = \begin{pmatrix} g_{ab} & 0 \\ 0 & \beta_{ik} \end{pmatrix},$$

(8.11)

that plays the role of the metric on the algebra $\mathcal{G}$ one can show that the structure constants satisfy also the identity

$$\gamma_{AB}C^B_C + \gamma_{DB}C^B_A = 0,$$

(8.12)

that means that the adjoint and coadjoint representations of the algebra $\mathcal{G}$ are also equivalent.

In other words, the Jacobi identities (8.9) are equivalent to the identities (5.10) that the curvature must satisfy in the symmetric space. This means that the set of the structure constants $C^A_{BC}$, satisfying the Jacobi identities, determines the curvature tensor of symmetric space $R^k_{bcd}$. Vice versa the structure of the algebra $\mathcal{G}$ is completely determined by the curvature tensor of symmetric space.

Now consider the bundle curvature $R_{ab}$. One can show analogously that because of the integrability conditions (5.11), (5.12) it must have the form

$$R_{ab} = R_i E^i_{ab},$$

(8.13)

where $E^i_{ab}$ are the same 2-forms and $R_i$ are some matrices forming a representation of the isotropy algebra

$$[R_i, R_k] = F^j_{ik} R_j.$$

(8.14)

Finally, the potential term should commute with the curvature $R_{\mu\nu}$ and, therefore, with all matrices $R_i$

$$[R_i, Q] = 0.$$ 

(8.15)

One can show that the algebra $\mathcal{G}$ is isomorphic to the algebra of the infinitesimal isometries of symmetric space. The set of all generators of infinitesimal isometries $\mathcal{G} = \{\xi_A\}, \dim \mathcal{G} = D$, can be split in two essentially different sets: $\mathcal{M} = \{P_a\}, \dim \mathcal{M} = d$, and $\mathcal{H} = \{L_i\}, \dim \mathcal{H} = p$, according to the values of their initial parameters

$$P_a \Big|_{x=x'} \neq 0, \quad L_i \Big|_{x=x'} = 0.$$ 

(8.16)

Let us introduce a two-point matrix $K = \{K^a_b(x, x')\}$

$$K^a_b = R^a_{cbd} \sigma^c \sigma^d,$$

(8.17)

where $\sigma^a$ are the frame components of the tangent vector to the geodesic connecting the points $x$ and $x'$ at the point $x'$, $\sigma^a(x, x') = g^{ab} e^\mu_b (x', x') \nabla_\mu \sigma(x, x'), \sigma(x, x')$ being the
geodetic interval defined as one half the square of the length of this geodesic. Then the
generators of isometries can be presented in the form [20]

\[
P_a = e^\mu_b \left( \cos \sqrt{K} \right)^b_a \nabla_\mu \\
L_i = e^\mu_b \left( \frac{\sin \sqrt{K}}{\sqrt{K}} \right)^b_a D^a_{\mu \nu} \nabla_\mu.
\] (8.18, 8.19)

The functions of the matrix \( K \) should be understood here as a power series in curvature.

One can show that this operators form exactly the Lie algebra \( \mathcal{G} \) generated by the
curvature tensor of the symmetric space

\[
[\xi_A, \xi_B] = \epsilon^{C}{}_{AB} \xi_C,
\] (8.20)

or, more explicitly,

\[
[P_a, P_b] = E^i \gamma_{ab} L_i, \\
[P_a, L_i] = D^b \gamma_{ai} P_b, \\
[L_i, L_k] = E^j \gamma_{ik} L_j,
\] (8.21)

Therefore, the curvature tensor of the symmetric space completely determines the
structure of the group of isometries.

Let us consider now the scalar case with vanishing bundle curvature

\[
R_{\mu\nu} = 0.
\] (8.21a)

It is not difficult to show that in this case the Laplacian in symmetric space can be presented
in terms of generators of isometries

\[
\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu = \gamma^{AB} \xi_A \xi_B = g^{ab} P_a P_b + \beta^{ik} L_i L_k,
\] (8.22)

where \( \gamma^{AB} = (\gamma_{AB})^{-1} \) and \( \beta^{ik} = (\beta_{ik})^{-1} \).

Using this representation one can prove a theorem expressing the heat kernel operator
in terms of some average over the group of isometries \( G \)

\[
\exp(t \Box) = (4\pi t)^{-D/2} \int dk \gamma^{1/2} \det \left( \frac{\sinh(k^A C_A/2)}{k^A C_A/2} \right)^{1/2} \\
\times \exp \left\{ -\frac{1}{4t} k^A \gamma_{AB} k^B + \frac{1}{6} R_G t \right\} \exp(k^A \xi_A)
\] (8.23)

where \( \gamma = \det \gamma_{AB} \) and \( R_G \) is the scalar curvature of the group of isometries \( G \)

\[
R_G = -\frac{1}{4} \gamma^{AB} C^C_{AD} C^D_{BC}.
\] (8.24)
The proof of this theorem is given in [20].

For further use it is convenient to rewrite the integral (8.23) splitting the integration variables \( k^A = (q^a, \omega^i) \) in the form

\[
\exp(t \Box) = (4\pi t)^{-D/2} \int dq\, d\omega \eta^{1/2} \beta^{1/2} \det \left( \frac{\sinh((q^a C_a + \omega^i C_i)/2)}{(q^a C_a + \omega^i C_i)/2} \right)^{1/2} \\
\times \exp \left\{ -\frac{1}{4t} (q^a g_{ab} q^b + \omega^i \beta_{ik} \omega^k) + \left( \frac{1}{8} R + \frac{1}{6} R_H \right) t \right\} \exp \left( q^a P_a + \omega^i L_i \right),
\]

(8.25)

where \( \beta = \det \beta_{ik}, \eta = \det g_{ab} \) and \( R_H \) is the scalar curvature of the isotropy subgroup \( H \)

\[
R_H = -\frac{1}{4} \beta^{ik} F_{il}^m F_{km}.
\]

(8.26)

To get the heat kernel in coordinate representation one has to act with the heat kernel operator \( \exp(t \Box) \) on the coordinate \( \delta \)-function. Below we will calculate only the heat kernel diagonal. Therefore, it is sufficient to compute only the coincidence limit \( x = x' \). Using the explicit form of the generators of the isometries (8.18), (8.19) and solving the equations of characteristics one can obtain the action of the isometries on the \( \delta \)-function [20]

\[
\exp \left( q^a P_a + \omega^i L_i \right) g^{-1/2} \delta(x, x')\bigg|_{x=x'} = \det \left( \frac{\sinh(\omega^i D_i/2)}{\omega^i D_i/2} \right)^{-1} \eta^{-1/2} \delta(q).
\]

(8.27)

Now using (8.27) one can easily integrate over \( q \) in (8.25) to get heat kernel diagonal

\[
[U(t)] = (4\pi t)^{-D/2} \int d\omega \beta^{1/2} \det \left( \frac{\sinh(\omega^i F_i/2)}{\omega^i F_i/2} \right)^{1/2} \det \left( \frac{\sinh(\omega^i D_i/2)}{\omega^i D_i/2} \right)^{-1/2} \\
\times \exp \left\{ -\frac{1}{4t} \omega^i \beta_{ik} \omega^k - \left( m^2 + Q - \frac{1}{8} R - \frac{1}{6} R_H \right) t \right\}.
\]

(8.28)

Changing the integration variables \( \omega \to \sqrt{t} \omega \) and introducing a Gaussian averaging over \( \omega \)

\[
<f(\omega)> = (4\pi)^{-p/2} \int d\omega \beta^{1/2} \exp \left( -\frac{1}{4} \omega^i \beta_{ik} \omega^k \right) f(\omega)
\]

(8.29)

one obtains then for the generating function \( \Omega(t) \) we get another form of this formula

\[
\Omega(t) = \exp \left\{ \left( \frac{1}{8} R + \frac{1}{6} R_H \right) t \right\} \\
\times \left( \det \left( \frac{\sinh(\sqrt{t} \omega^i F_i/2)}{\sqrt{t} \omega^i F_i/2} \right)^{1/2} \det \left( \frac{\sinh(\sqrt{t} \omega^i D_i/2)}{\sqrt{t} \omega^i D_i/2} \right)^{-1/2} \right)
\]

(8.30)

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One can present this result also in an alternative nontrivial rather formal way. Substituting the equation

$$(4\pi t)^{-p/2} \beta^{1/2} \exp \left( -\frac{1}{4t} \omega^i \beta_{ik} \omega^k \right) = (2\pi)^{-p} \int dp \exp (ip_k \omega^k - tp_k \beta^{kn} p_n)$$  \hspace{1cm} (8.31)$$

into the integral (8.28), integrating over $\omega$ and changing the integration variables $p_k \to it^{-1/2}p_k$ we get finally an expression without any integration

$$\Omega(t) = \exp \left\{ \left( \frac{1}{8} R + \frac{1}{6} R_H \right) t \right\} \det \left( \frac{\sinh(\sqrt{t} \partial^k F_k/2)}{\sqrt{t} \partial^k F_k/2} \right)^{1/2} \times \det \left( \frac{\sinh(\sqrt{t} \partial^k D_k/2)}{\sqrt{t} \partial^k D_k/2} \right)^{-1/2} \exp (p_n \beta^{nk} p_k) \bigg|_{p=0}$$.  \hspace{1cm} (8.32)$$

where $\partial^k = \partial/\partial p_k$.

This formal solution should be understood as a power series in the derivatives $\partial^i$ that is well defined and determines the heat kernel asymptotic expansion at $t \to 0$, i.e. all HMDS-coefficients $a_k$.

Let us mention that the formulae (8.30), (8.32) obtained in this section are exact (up to possible nonanalytic topological contributions) and manifestly covariant because they are expressed in terms of the invariants of the isotropy group $H$, i.e. the invariants of the curvature tensor. They can be used now to generate all HMDS-coefficients $a_k$ for any symmetric space, i.e. for any space with covariantly constant curvature, simply by expanding it in a power series in $t$. Thereby one finds all covariantly constant terms in all HMDS-coefficients in manifestly covariant way. This gives a very nontrivial example how the heat kernel can be constructed using only the commutation relations of some differential operators, namely the generators of infinitesimal isometries of the symmetric space. We are going to obtain the explicit formulae in a further work.

We considered for simplicity the case of symmetric space of compact type, i.e. with positive sectional curvatures

$$K(u,v) = R_{abcd} u^a v^b u^c v^d = \beta_{ik} (E_{ab}^i u^a v^b)(E_{cd}^k u^c v^d),$$  \hspace{1cm} (8.33)$$
i.e. positive definite matrix $\beta_{ik}$. A simply connected symmetric space is, in general, reducible, and has the following general structure [25]

$$M = M_0 \times M_+ \times M_-$$  \hspace{1cm} (8.34)$$

where $M_0$, $M_+$ and $M_-$ are the Euclidean, compact and noncompact components.

The corresponding algebra of isometries is a direct sum of ideals

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_+ \oplus \mathcal{G}_-$$  \hspace{1cm} (8.35)$$

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where $G_0$ is an Abelian ideal and $G_+$ and $G_-$ are the semi-simple compact and noncompact ones.

There is a remarkable duality relation $\ast$ between compact and noncompact objects. For any algebra $G = \mathcal{M} + \mathcal{H} = \{P_a, L_i\}$ one defines the dual one according to $G^\ast = i\mathcal{M} + \mathcal{H} = \{iP_a, L_k\}$, the structure constants of the dual algebra being

$$\{C^A_{BC}\} = \{E^{i}_{ab}, D^c_{dk}, F^j_{lm}\} = \{-E^{i}_{ab}, D^c_{dk}, F^j_{lm}\}. \quad (8.36)$$

So, the star $\ast$ only changes the sign of $E^{i}_{ab}$ but does not act on all other structure constants. This means also that the matrix $\gamma$ (3.19) for dual algebra should have the form

$$\gamma^\ast_{AB} = \begin{pmatrix} g_{ab} & 0 \\ 0 & \beta_{ik} \end{pmatrix} = \begin{pmatrix} g_{ab} & 0 \\ 0 & -\beta_{ik} \end{pmatrix} \quad (8.37)$$

and, therefore, the curvature of the dual manifold has the opposite sign

$$R^\ast_{abcd} = -R_{abcd}. \quad (8.38)$$

We hope that it is not difficult to generalize our results to the general case using the duality relation and the analytic continuation. This means that our formulae (8.28), (8.30), (8.32) should be valid in general case of arbitrary symmetric space too. Moreover, they should also be valid for the case of pseudo-Euclidean signature of the metric $g_{\mu\nu}$.

## 9. Conclusion

In present paper we have presented a brief overview of recent results in studying the heat kernel obtained in our papers [20,21]. Here we have discussed some ideas connected with the point that was left aside in previous investigations [4,10,15-17], namely, the problem of calculating the low-energy limit of the heat kernel and the effective action in quantum gravity and gauge theories. We have analyzed in detail the status of the low-energy approximation and stressed the central role of an algebraic structure that naturally appears when generalizing consistently the low-energy limit to curved space and gauge theories. We have proposed a promising new purely algebraic approach for calculating the low-energy heat kernel and realized, thereby, the idea of partial summation of the terms without covariant derivatives in local Schwinger - De Witt asymptotic expansion for computing the effective action that was suggested in [2,4].

Of course, there are left many unsolved problems. First of all, one has to obtain explicitly the covariantly constant terms in HMDS-coefficients. This would be the opposite case to the high-derivative approximation [15-17] and can be of certain interest in mathematical physics.

Then, we still do not know how to calculate the low-energy heat kernel in general case of covariantly constant curvatures, i.e. when all background curvatures ($\mathcal{R} = \{R_{\mu\nu\alpha\beta}, R_{\mu\nu}, Q\}$) are present. If it would be possible to obtain in this general case the
formulae similar to (8.30) then it would allow to find the heat kernel in a number of important cases and would lead finally to the general solution of the effective potential problem in quantum gravity and gauge theories.

Besides, it is not perfectly clear how to do the analytical continuation of Euclidean low-energy effective action to the space of Lorentzian signature for obtaining physical results.

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