Refinements of Nash equilibrium in a Pentagonal Fuzzy Bimatrix Game

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ABSTRACT—In this paper, we introduce the concept of symmetric pentagonal fuzzy bimatrix and constant pentagonal fuzzy matrix for which we define the existence of Nash equilibrium in both pure and mixed strategies along with some numerical illustrations. By applying Ranking to the pay-offs, we convert the fuzzy valued game problem to crisp valued game problem. Therefore, Solution concept of perfect equilibrium for normal form game is reviewed and a new concept of proper equilibrium of pentagonal fuzzy bimatrix game is discussed.

Keywords—Bimatrix game, Nash equilibrium Approximation, fuzzy numbers, pentagonal fuzzy number, expected payoff, symmetric pentagonal fuzzy bimatrix, constant pentagonal fuzzy matrix, Defuzzification.

1. Introduction
In real world, people were overcoming lot of uncertainty in the day to day life. Fuzzy environment has a potential to solve such kind of uncertainty. Fuzzy set theory was introduced by L.A.Zadeh in the year 1965\(^[13]\) which plays an vital role in predicting the solution for the problems, it involves in many fields namely medicine, engineering etc.

Depending on the nature of the impreciseness and problems in various applications, we are using different fuzzy numbers and interval valued fuzzy number. Pentagonal fuzzy number for the first time was used by Raj Kumar and T.Pathinathan \(^[7]\).Also they developed the generalised concepts of pentagonal fuzzy number in 2015 along with the set theoretic operations. The concept of Equilibrium, as defined by John Nash \(^[5]\) is one of the most important and elegant idea in game theory. Unfortunately a game can have many Nash Equilibria and some of the Equilibria may be inconsistent with our intuitive notions about what should be the outcome of a game.

Neumann and Morgenstern \(^[12]\) invented the mathematical theory of games. Bimatrix games with PFN as payoffs have been discussed in many articles. Ranking of fuzzy numbers is one of the fundamental problems of fuzzy arithmetic and fuzzy decision making .we use ranking functions to defuzzified the fuzzy numbers.

In this paper, we consider the bimatrix game under pentagonal fuzzy environment and we focus on the solution method of Equilibria for the bimatrix games with PFN as payoffs and then the result achieved.

2. Preliminaries
A. Bimatrix game
A two person finite game in a strategic form which is defined as the matrix of ordered pairs is called a Bimatrix game. A Bimatrix game is a 2 player regular game where

Player 1 with a finite set of strategy \(S = \{s_1, s_2, \ldots, s_m\}\)
Player 2 with a finite set of strategy $T = \{t_1, t_2, \ldots, t_n\}$

When the pair of strategies $(s_i, t_j)$ is chosen, the first player’s payoff is $a_{ij} = u_1(s_i, t_j)$ and the second player’s payoff is $b_{ij} = u_2(s_i, t_j)$ such that $u_1, u_2$ are called the payoff functions.

The outcomes of payoff values can be represented by a Bimatrix

$\begin{array}{|c|c|c|c|}
\hline
\text{Strategy} & t_1 & t_2 & \ldots & t_n \\
\hline
s_1 & (a_{11}, b_{11}) & (a_{12}, b_{12}) & \ldots & (a_{1n}, b_{1n}) \\
\hline
s_2 & (a_{21}, b_{21}) & (a_{22}, b_{22}) & \ldots & (a_{2n}, b_{2n}) \\
\hline
\ldots & \ldots & \ldots & \ldots & \ldots \\
\hline
s_m & (a_{m1}, b_{m1}) & (a_{m2}, b_{m2}) & \ldots & (a_{mn}, b_{mn}) \\
\hline
\end{array}$

The payoff matrix of Player 1 and Player 2 is

$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$

Player 1’s payoff representation is the first component of the ordered pairs and the player 2’s payoff representation is the second component of the ordered pairs.

**B. Nash Equilibrium**

A Nash equilibrium also called strategic equilibrium, is a list of strategies one for each player, which has the property that no player can unilaterally change his strategy to get a better payoff.

A Nash equilibrium for a game $\gamma = (x, y)$ is a Nash equilibrium for a Bimatrix game $\gamma = (A, B)$ if

(i) For every mixed strategy $x$ of the row player $x^T Ay \leq x^T A\tilde{y}$ and

(ii) For every mixed strategy $y$ of the column player $x^T By \leq \tilde{x}^T B y$

**C. 2.3 Expected payoff**

For a mixed strategy Nash equilibrium the expected payoff for that player is given by multiplying each probability in each cell by his/her respective payoff in that cell.

Therefore, The Expected payoffs are defined by the relations

Player 1: $\pi_1(p, q) = \Sigma \Sigma p_i q_j a_{ij}$ where $i = 1 \text{ to } m$ and $j = 1 \text{ to } n$

Player 2: $\pi_2(p, q) = \Sigma \Sigma p_i q_j b_{ij}$ where $i = 1 \text{ to } m$ and $j = 1 \text{ to } n$

**D. Strictly dominating strategy**
A strategy $a_i \in A_i$ is strictly dominated by $a_i \in A_i$ if $U_i(a_i, a_{-i}) < U_i(a_i, a_{-i}) \forall a_{-i} \in A_{-i}$ set

E. Fuzzy set(13)

A fuzzy set $\tilde{A}$ is defined by $\mu_{\tilde{A}}(x): R \rightarrow [0,1] \tilde{A} = \{(x, \mu_{\tilde{A}}(x)): x \in A, \mu_{\tilde{A}}(x) \in [0,1]\}$

In the pair $(x, \mu_{\tilde{A}}(x))$, the first element $x$ belong to the classical set $A$, the second element $\mu_{\tilde{A}}(x)$, belong to the interval $[0,1]$ called Membership function.

F. Fuzzy number(13)

A fuzzy subset $\tilde{A}$ defined on $R$, is said to be a fuzzy number if its membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions

- There exist at least one $x_0 \in R$, $\mu_{\tilde{A}}(x_0) = 1$
- $\mu_{\tilde{A}}(x)$ is piecewise continuous
- $\tilde{A}$ must be normal and convex

G. Symmetric bimatrix games (12)

A 2-player Strategic game is said to be symmetric, if the pure strategy of the players are the same and the Player’s payoff functions $U_1$ and $U_2$ are such that $U_1(S_1, S_2) = U_2(S_2, S_1)$ (i.e.) a symmetric game does not change when the players change roles.

Using the notation of bimatrix games, an $m \times n$ Bimatrix game $\gamma = (A, B)$ is symmetric if $m = n$ and $a_{ij} = b_{ji}$ for all $j \in \{1, 2, \ldots, n\}$ or equivalently $B = A^T$

3. Pentagonal fuzzy number (1,7)

A Pentagonal Fuzzy number (PFN) of a fuzzy set $A$ is defined as $A_P = \{a, b, c, d, e\}$, and its membership function is given by,

$$
\mu_{A_P}(x) = \begin{cases} 
0 & \text{for } x < a, \\
\frac{(x-a)}{(b-a)} & \text{for } a \leq x \leq b \\
\frac{(x-b)}{(c-b)} & \text{for } b \leq x \leq c \\
1 & \text{for } x = c \\
\frac{(d-x)}{(d-c)} & \text{for } c \leq x \leq d \\
\frac{(e-x)}{(e-d)} & \text{for } d \leq x \leq e \\
0 & \text{for } x > e
\end{cases}
$$
Fig. 1: Graphical representation of Pentagonal Fuzzy Number (PFN)

A. Constant pentagonal fuzzy matrix (CPFM)

A square PFM $A = (a_{ij})$ of order $n \times n$ is called a constant PFM if all the rows are equal to each other, i.e.,

$$(a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}, a_{5ij}) = (a_{1rj}, a_{2rj}, a_{3rj}, a_{4rj}, a_{5rj}) \ \forall \ i, r, j$$

4. Defuzzification

Defuzzification is an operation of translating an output fuzzy variable into a unique number (crisp)

Ranking function of a pentagonal number $A (a, b, c, d, e)$ be a PFN then,

$$R (A) = \frac{2a + 3b + 2c + 3d + 2e}{4}$$

5. Numerical example: 1

Let us consider two students Suvetha and Anu participated in the dance competition. They are the top both contestants of this competition. Choreographers will judge based on the criteria listed below and will receive judges brief comments immediately following their performance.

The Bimatrix represent the scores in the form of PFN.

| S.No | Criteria                                         | Possible points |
|------|--------------------------------------------------|-----------------|
| 1.   | Presentation skills (Body expression, Eye contact) | 10              |
| 2.   | Audience response to performance (Are people kidding or ignoring the performance) | 10              |
Table 2: Anu

| Suvetha | $E_1$ | $E_2$ |
|---------|-------|-------|
| $M_1$   | $P((1,3,6,7,9)(3,6,7,8,10))$ | $P((4,5,7,8,9)(5,6,7,8,9))$ |
| $M_2$   | $P((3,5,6,8,10)(2,3,7,8,8))$ | $P((1,2,6,6,7)(2,4,5,7,8))$ |

After defuzzification

| Suvetha | $E_1$ | $E_2$ |
|---------|-------|-------|
| $M_1$   | $(14,20.5)$ | $(19.75,21)$ |
| $M_2$   | $(19,25,16.7)$ | $(13,15.75)$ |

By best reply response method,

Best reply of Player 1 to the strategic $T$ of Player 2 is defined as the set

$$R_1 (t) = \{ s^* \in S; u_1(s^*, t) \geq u_1(s, t), \forall s \in S \}$$

Similarly Best reply of Player 2 to the strategy $S$ of Player 1 is defined as

$$R_2 (S) = \{ t^* \in T; u_2(s, t^*) \geq u_2(s, t), \forall t \in T \}$$

$(M_1, E_2)$ and $(M_2, E_1)$ are the two pure Nash equilibrium points by best reply response method for the given pentagonal fuzzy bimatrix game.

Numerical Example: 2

Nowadays there are numerous of online marketing houses, namely Amazon, flipkart, snap deal, stub borne etc...Among these we consider two companies whose targeted aims are to increase their market shares under increasing demands of product in market. They are following the two categories strategy 1: COD/ CASH ON DELIVERY, Strategy 2: Net banking payment or Debit card payment. We are considering two companies as player 1 and 2 respectively. Here we use PFN as payoffs to represent such ambiguity of the data character.

Table 3: Company 2

| Company 1 | $T_1$ | $T_2$ |
|-----------|-------|-------|
| $S_1$     | $P((24,6,8,10),(1,3,6,9,11))$ | $(0.5,0.3)$ |
| $S_2$     | $P((1,3,5,7,9)(0,2,5,8,10))$ | $(0.6,0.1)$ |

Table 4: Solution

| Company 1 | $T_1$ | $T_2$ |
|-----------|-------|-------|
| $S_1$     | $(18,18)$ | $(0.5,0.3)$ |
Therefore we have (18, 18) as the dominate strategy pure Nash equilibrium point.

**Numerical Example: 3** Consider a pentagonal fuzzy bimatrix game and computing the equilibrium solution strategies.

| Table 5: Player 2 |
|-------------------|
| **Player 1**      | **Player 2** |
| $b_1$             | $b_2$       |
| $a_1$             | P((1,2,3,4,5)(0,1,5,3,4,5,5,5)) | (0.7,0.2) |
| $a_2$             | P((1,2,2,8,4,5)(0,1,5,2,8,4,5,5,5)) | (0.7,0.2) |

After defuzzification, we obtain the bimatrix as

| Player 1 | Player 2 |
|----------|----------|
| $b_1$    | $b_2$    |
| $a_1$    | (9,7.625) | (0.7,0.2) |
| $a_2$    | (8.9,8.65) | (0.7,0.2) |

By Expected payoff method, Expected payoff of $a_1$ and $a_2$ given by

$$E(a_1) = 9q_1 + 0.7q_2$$
$$E(a_2) = 8.9q_1 + 0.7q_2$$

$$E(a_1) = E(a_2)$$

On simplification,

$q_1 = 0$ Such that $q_1 + q_2 = 1$

$\Rightarrow q_2 = 1$

Therefore (0,1) is a mixed Nash equilibrium point.

Similarly, (9, 7.625) is the only pure Nash equilibrium point.

**Numerical Example: 4**

Consider a $4 \times 4$ pentagonal fuzzy bimatrix game and computing Equilibria strategies.

| Player 1 | Player 2 |
|----------|----------|
| $b_1$    | $b_2$    | $b_3$ | $b_4$ |
| $a_1$    | (P(7,8,9,10,11), (1,2,3,4,5)) | (P(1,2,3,4,5)) | P(4,5,6,7,8) | (P(0,2,4,6,8)) |
| $a_2$    | (P(1,3,5,7,9)) | (P(0,2,4,6,8)) | P(0,1,2,3,4) | P(4,5,6,7,8) |
By iterative elimination of strictly dominance property,

Since \( b_3 \) is strictly dominated by \( b_2 \), therefore eliminating \( b_3 \) we have

| Player 1 | Player 2 |
|----------|----------|
| \( a_1 \) | \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) |
| (27,9)   | (9,24)   | (18,2)   | (12,9)  |
| \( a_2 \) | (15,9)   | (12,15)  | (6,12)  | (18,19.25) |
| \( a_3 \) | (6,16)   | (10.5,18)| (10.25,15) | (9,12)  |
| \( a_4 \) | (10.75,16)| (15,12) | (6,12)  | (13.25,9) |

Since \( a_3 \) is strictly dominated by \( a_4 \), therefore eliminating \( a_3 \) we have

| Player 1 | Player 2 |
|----------|----------|
| \( a_1 \) | \( b_1 \) | \( b_2 \) | \( b_4 \) |
| (27,9)   | (9,24)   | (12,9)  |
| \( a_2 \) | (15,9)   | (12,15)  | (18,19.25) |
| \( a_4 \) | (10.75,16)| (15,12) | (13.25,9) |

Since \( b_1 \) is strictly dominated by \( b_2 \), therefore eliminating \( b_1 \) we have

| Player 1 | Player 2 |
|----------|----------|
| \( a_2 \) | \( b_2 \) | \( b_4 \) |
| (15,9)   | (12,15)  | (18,19.25) |
| \( a_4 \) | (10.75,16)| (15,12) | (13.25,9) |
Since $a_1$ is strictly dominated by $a_2$, therefore eliminating $a_1$ we have

\[
\begin{array}{c|cc}
\text{Player 1} & a_2 & a_4 \\
\hline
b_2 & (12,15) & (15,12) \\
\hline
b_4 & (18,19.25) & (13.25,9) \\
\end{array}
\]

Therefore we have $(a_2, b_4)$ and $(a_4, b_2)$ are the two Nash equilibrium points for the given pentagonal fuzzy game problem.

**Numerical Example: 5**

Consider a constant symmetric pentagonal fuzzy matrix given by

\[
A = \begin{pmatrix}
(-1,0,1,2,4) & (0,1,2,4,5) & (1,2,3,4,5) \\
(-1,0,1,2,4) & (0,1,2,4,5) & (1,2,3,4,5) \\
(-1,0,1,2,4) & (0,1,2,4,5) & (1,2,3,4,5)
\end{pmatrix}
\]

After defuzzification, the matrix $A$ is given by

\[
A = \begin{pmatrix}
(3.5) & (5) & (9) \\
(3.5) & (5) & (9) \\
(3.5) & (5) & (9)
\end{pmatrix}
\]

Since it is a symmetric PFM we have $A = B^T$

Therefore, $B^T = \begin{pmatrix}
(3.5) & (5) & (9) \\
(3.5) & (5) & (9) \\
(3.5) & (5) & (9)
\end{pmatrix}$

\[
B = \begin{pmatrix}
(3.5) & (5) & (9) \\
(3.5) & (5) & (9) \\
(3.5) & (5) & (9)
\end{pmatrix}
\]

Hence the obtained bimatrix is

\[
(A, B) = \begin{pmatrix}
(3.5,3.5) & (5,3.5) & (9,3.5) \\
(3.5,5) & (5,5) & (9.5) \\
(3.5,9) & (5,9) & (9,9)
\end{pmatrix}
\]
| Player1 | Player2 | $b_1$  | $b_2$  | $b_3$  |
|---------|---------|--------|--------|--------|
| $a_1$   | (3.5,3.5) | (5,3.5) | (9,3.5) |
| $a_2$   | (3.5,5)   | (5,5)   | (9,5)   |
| $a_3$   | (3.5,9)   | (5,9)   | (9,9)   |

By best reply response method, all the points are pure Nash equilibrium points.

6. Result

For a constant symmetric pentagonal fuzzy bimatrix, all the components are only pure Nash equilibrium points, there doesn’t exist mixed Nash equilibrium point for Constant symmetric pentagonal fuzzy bimatrix game.

7. Conclusion

In this paper, we have analysed the Equilibria of the Bimatrix game in both pure and mixed strategies under the pentagonal fuzzy environment. We have used Defuzzification technique to the payoff matrix to obtain crisp values of the game. Few numerical illustrations have been solved for the proposed method.

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