Macroscopic Reality from Quantum Complexity

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Abstract
Beginning with the Everett–DeWitt many-worlds interpretation of quantum mechanics, there have been a series of proposals for how the state vector of a quantum system might split at any instant into orthogonal branches, each of which exhibits approximately classical behavior. Here we propose a decomposition of a state vector into branches by finding the minimum of a measure of the mean squared quantum complexity of the branches in the branch decomposition. In a non-relativistic formulation of this proposal, branching occurs repeatedly over time, with each branch splitting successively into further sub-branches among which the branch followed by the real world is chosen randomly according to the Born rule. In a Lorentz covariant version, the real world is a single random draw from the set of branches at asymptotically late time, restored to finite time by sequentially retracing the set of branching events implied by the late time choice. The complexity measure depends on a parameter $b$ with units of volume which sets the boundary between quantum and classical behavior. The value of $b$ is, in principle, accessible to experiment.

1 Introduction

Microscopic particles have wave functions spread over all possible positions. Macroscopic objects simply have positions, or at least center-of-mass positions. How to apply the mathematics of quantum mechanics to extract predictions registered in the macroscopic world of positions from experiments on microscopic systems having wave functions but not definite positions is well understood for all practical purposes. But less well understood, or at least not a subject on which there is a clear consensus, is how in principle the definite positions of the macroscopic world emerge from the microscopic matter of which it is composed, which has only wave functions but not definite positions. There is a long list of proposals. In the present article we add another.
We begin in Sect. 2 with a brief reminder of “the problem of measurement” which arises for an experiment in which a microscopic system interacts with a macroscopic measuring device with both systems assumed governed by quantum mechanics. Among the proposals which address this problem are the many-worlds interpretation [1, 2] and environmentally-induced decoherence [3–8]. Shared by these is the hypothesis that the quantum state of the universe, as time goes along, naturally splits into a set of orthogonal branch states each of which displays a distinct configuration of macroscopic reality. We will argue, however, that the rules according to which these proposals are to be applied to the world are intrinsically uncertain and can be made precise only by the arbitrary choice of auxiliary parameters. The uncertainty is not simply the approximate nature of the macroscopic description of an underlying microscopic system, but rather that the branching process of the microscopic system itself, in each of these proposals, occurs according to uncertain rules. And as a consequence, it seems to me implausible that the corresponding branches are, by themselves, macroscopic reality. In addition, missing from these proposals is a mathematical structure that allows even the process of choosing the auxiliary parameters to be stated precisely. These various limitations we will try to address in a sequence of several steps.

A main feature of the proposal we present here is that branch formation does not follow from unitary time evolution by itself nor does it entail a modification of unitary time evolution. Instead, branch formation consists of an additional layer of the world that sits on top of unitary time evolution [9].

In Sect. 3, modifying ideas from [10], for a lattice approximation to a non-relativistic field theory of fermions and spinless bosons in 3-dimensional space, we define a version of quantum complexity designed to measure, at any instant of time, the spatial structure of entanglement in a state vector. For a system evolving according to a local Hamiltonian through a sequence of states with complexity much less than the system’s maximum possible, the conjectured second law of quantum complexity of [11] yields an approximation to the time evolution of complexity. In Sect. 4 we introduce a family of entangled multi-fermion states with, for simplicity, particles’ wave functions constant across corresponding cubic regions and then show in Appendices 2 and 3 that the complexity of each of these states is bounded both from below and from above by quantities proportional to the square root of the volume on which the particles’ wave functions differ from zero. In Sect. 5 we then propose finding a branch decomposition of any state by minimizing the decomposition’s net complexity, which we define to be a linear combination of the average squared complexity of the branches and the classical entropy of the ensemble of branch weights. The coefficient of the classical entropy term in the net complexity is a parameter with units of volume, the branching threshold \( b \), which turns out to set the boundary between quantum and classical behavior.

For the non-relativistic theory, the evolving state vector of the world can be decomposed into an evolving set of optimally chosen branches. The optimal set of branches is a piecewise continuous function of time. For sufficiently large \( b \), the continuous evolution process will consist almost entirely of Hamiltonian evolution of each branch. In Sect. 6, we propose the hypothesis that the discontinuous part of branch evolution, for a local Hamiltonian and a sufficiently large value of \( b \), will
consist of a sequence of events in which some single branch splits, with high probability permanently, into a corresponding pair of sub-branches. We then argue that if \( b \) is sufficiently large, this conjecture is satisfied by a system with a large number of degrees of freedom which follows the estimate of the time evolution of complexity in Sect. 3. The real world we then propose follows through time a single thread of the resulting tree’s branches and sub-branches, with a sub-branch at each splitting chosen randomly according to the Born rule.

In Sect. 7 we look at a model of an experiment in which the result of scattering by a microscopic system with small complexity is recorded by a macroscopic measuring device with large complexity. For a measuring device with sufficient complexity, the entanglement of the final state arising from this recording process yields an increase in net complexity of the combined system which triggers branching, with each branch carrying a different final configuration of the microscopic system.

In Sect. 8, we consider the time evolution of an isolated 2-fermion system with a smooth static internal wave-function with compact support and center-of-mass position wave function spreading according to free time evolution. After an amount of time determined by the initial center-of-mass wave function and by the value of \( b \), the wave function of the center-of-mass position will undergo branching.

In Sect. 9, we consider branching for two different examples of entangled multiparticle states. In Sect. 10, based on the examples in Sect. 9, we propose a structure for the residual entanglement left in a state not immediately subject to further branching.

In Sect. 11 we consider a class of experiments to measure the value of \( b \). Although the branching process which \( b \) governs may be viewed as a kind of wave function collapse, since branch formation as proposed here rides on top of exact unitary time evolution, \( b \) can not be determined by experiments which search for forms of collapse which violate unitary time evolution. Such experiments we believe will yield null results. Instead, the evidence for the existence of branches is solely human registration of macroscopic reality. Correspondingly we consider possible determination of the value of \( b \) by experiments in which a human observer registers either the presence or absence of branching. While branching in general, according to the proposal presented here, is a physical processes which occurs with or without the presence of a human observer, the registration of a branching event we will assume occurs when a sufficient collection of the degrees of freedom of which the observer is composed participate in the event. A possible experimental test of this account of branching is to see if different members of the class of experiments to measure \( b \) yield the same result.

In Sects. 12–18 we redo Sects. 3–5 for a relativistic field theory of fermions and spinless bosons in 3+1-dimensional space. To obtain a lattice approximation to covariance with respect to Lorentz boosts, in place of the non-relativistic definition of complexity at fixed coordinate time, complexity for the relativistic theory is defined on a random lattice on a finite volume chunk of a hyperboloid of fixed proper time. Branching based on complexity defined at fixed proper time, however, loses translational covariance. We then argue that a lattice approximation to translational covariance is restored in the limit of branching at asymptotically late proper time. Full Poincaré covariance should then result if infinite volume and zero lattice
spacing limits of branching exist. The loss of translational covariance for branching at fixed proper time is a version of the problem exposed by the EPR experiment. A discussion of this issue in a different setting and a solution related to the one we consider appear in [12–14].

The macroscopic real world, we propose, consists of a single random choice among the asymptotic set of late time branches according to a measure based on the Born rule [15]. In the case of the non-relativistic theory, we conjectured that nearly all branching events yield permanent results. A random choice among late time branches is then nearly equivalent to the continuing branch choice in the non-relativistic theory, but with the bookkeeping for the choice process performed all at once rather than sequentially over time. The real world at any particular finite time can then be recovered from the asymptotic late time choice by sequentially retracing the set of branching events the late time choice implies.

We conclude in Sect. 25 with a summary of the conjectures on which the present proposal rests which could be tested by numerical experiment and an additional comment on the relationship between branching and thought.

**Problems**

Let $S$ be a microscopic system to be measured, with corresponding state space $\mathcal{H}_S$, for which a basis is $\{|s_i\rangle\}, i > 0$. Let $\mathcal{M}$ be a macroscopic measuring device with corresponding state space $\mathcal{H}_\mathcal{M}$ containing the set of vectors $\{|m_i\rangle\}, i \geq 0$. For each different value of $i > 0$ the state $|m_i\rangle$ is a macroscopically distinct meter reading. Let $|m_0\rangle$ be an initial state showing no reading. In the combined system-meter product state space $\mathcal{H}_S \otimes \mathcal{H}_\mathcal{M}$ a measurement of $S$ by $\mathcal{M}$ over some time interval takes each possible initial state $|s_i\rangle|m_0\rangle$ into the corresponding final state $|s_i\rangle|m_i\rangle$ with the measuring device displaying the measured value of the microscopic system’s variable $|s_i\rangle|m_0\rangle \rightarrow |s_i\rangle|m_i\rangle$. (1)

By linearity of quantum mechanical time evolution, however, it then follows that a measurement with a linear superposition in the initial state will yield a final state also with a superposition

$$(\alpha|s_1\rangle + \beta|s_2\rangle|m_0\rangle \rightarrow \alpha|s_1\rangle|m_1\rangle + \beta|s_2\rangle|m_2\rangle).$$ (2)

In the measured final state, the meter no longer has a single value but a combination of two values which cannot, by itself, be connected to a recognizable configuration of a macroscopic object. The absence of a recognizable configuration for the macroscopic device is the “problem of measurement”.

The resolution of this problem proposed by the many-worlds interpretation of quantum mechanics [1, 2] is that the states $|s_1\rangle|m_1\rangle$ and $|s_2\rangle|m_2\rangle$ actually represent two different worlds. In each world the meter has a definite position but with different positions in the two different worlds. For an interaction between two systems, the splitting into separate worlds is done in the Schmidt basis, in which the density matrix of the measured system is diagonalized. Among the problems of the
many-worlds interpretation, however, is that in general, for plausible models of a measurement process, the individual worlds given by the Schmidt basis do not have sufficiently narrow coordinate dispersions to count as classical reality [16]. In addition, it is unclear under what circumstances and according to what basis a system larger than just a micro system and a measuring device should be split into separate worlds.

A resolution to the first of these problems, the absence of classical behavior in the split branches, is proposed to occur through environmentally-induced decoherence [3–6]. According to this proposal, the system-meter combination should not be considered in isolation but instead an account is required of the rest of the macroscopic environment with which the meter can interact. When the value of a macroscopic meter is changed by recording the value of a microscopic coordinate, the meter’s new state rapidly becomes entangled with a large number of other degrees of freedom in the environment

\[
\langle \alpha | s_1 \rangle | m_1 \rangle + \beta | s_2 \rangle | m_2 \rangle | e_0 \rangle \rightarrow \alpha | s_1 \rangle | m_1 \rangle | e_1 \rangle + \beta | s_2 \rangle | m_2 \rangle | e_2 \rangle. \tag{3}
\]

For a particular choice of bases for system, meter and environment, determined by the combined system’s dynamics, entanglement of the meter with the environment proceeds as quickly as possible, \(| e_1 \rangle\) and \(| e_2 \rangle\) almost do not mix in the course of further time development, and \(| e_1 \rangle\) and \(| e_2 \rangle\) include many redundant copies of the information in \(| s_1 \rangle | m_1 \rangle\) and \(| s_2 \rangle | m_2 \rangle\), respectively. Based on these various considerations it is argued that entangled environmental states \(| e_1 \rangle\) and \(| e_2 \rangle\) behave essentially as permanent classical records of the experimental results. Correspondingly, for many-worlds augmented with decoherence [7], the circumstance under which a system splits into distinct worlds is when a superposition has been produced mixing distinct values of one of these effectively classical degrees of freedom. Each distinct value of the coordinate in such a superposition goes off into a distinct world.

A step toward resolving the second problem, the absence of a criterion for branching for the universe as a whole rather than simply for some system-apparatus pair, takes the form of a theorem [8] according to which, for a system as a whole, if a particular spatial pattern of redundant records happens to occur, then there is a unique corresponding decomposition of a state vector into effectively classical branches.

A residual problem of [3–8], however, is that the rules governing their application to the world are intrinsically uncertain. In particular, the record production needed for environmentally-induced decoherence occurs over some nonzero intervals of time and space, and perhaps is entirely completed only asymptotically in long time and large distance limits. What fraction of the initial state in Eq. (3) must become entangled with the environment for splitting into classical branches to occur? Could there be micro systems which become entangled with their environment but not sufficiently to split into classical branches? When exactly over the time interval of decoherence does the splitting of the world in parts occur? And since the process extends over space, this timing will differ in different frames related by a Lorentz boost. Which is the correct choice? These various questions may be of no practical consequence in treating the meter readings as nearly classical degrees of freedom after entanglement and using
the resulting values to formulate observable predictions. But what seems to me to be clear is that something is missing from the theory. From outside the theory, something additional and arbitrary needs to be supplied by hand to resolve each of these issues. Moreover, no mathematical machinery is present in any of these proposal which allows the process of filling in what is missing to be stated precisely. As a consequence of all of which it appears to me to be implausible that these accounts provide, by themselves, a complete account of the mechanism giving rise to macroscopic reality.

A discussion of issues concerning environmentally induced decoherence and its combination with the many-worlds interpretation of quantum mechanics appears in [17].

The goal of the remainder of this paper is to construct a possible candidate for the missing mathematical machinery, first for non-relativistic many particle quantum mechanics and then for a relativistic quantum field theory.

3 Complexity

Modifying ideas from [10], we now construct a complexity measure at a single instant of time for a 3-dimensional, non-relativistic field theory of fermions and, for simplicity, spinless bosons.

3.1 Non-Relativistic Hilbert Space

Let \( L \) be a cubic lattice with coordinates \( a\hat{x}^1, a\hat{x}^2, a\hat{x}^3 \), integer \( \hat{x}^i \), lattice spacing \( a \), spanning the region \(-aB \leq a\hat{x}^i < aB \). Let \( \Psi(x, s) \) and \( \Phi(x) \) be, respectively, fermion and boson lattice field operators for lattice site \( x \), spin \( s \) and time \( t \), which we omit as an explicit argument. These operators are normalized to have anticommutators and commutators

\[
\{\Psi(x, s), \Psi^\dagger(x', s')\} = \delta_{xx'} \delta_{ss'},
\]

\[
[\Phi(x), \Phi^\dagger(x')] = \delta_{xx'}.
\]

Let \( \mathcal{H} \) be the Hilbert space spanned by all polynomials in the \( \Psi^\dagger(x, s) \) and \( \Phi^\dagger(x) \) for any \( x \) and \( s \) acting on the physical vacuum \( |\Omega\rangle \). We will assume the vacuum expectation of \( \Phi^\dagger(x) \) vanishes. Let \( \mathcal{H}_x \) be the Hilbert space spanned by polynomials in the \( \Psi^\dagger(x, s) \) and \( \Phi^\dagger(x) \) for a fixed \( x \) and any \( s \) acting on the local vacuum at point \( x \), \( |\Omega\rangle_x \). The space \( \mathcal{H} \) is then isomorphic to an ordered version of the tensor product

\[
\mathcal{H} = \bigotimes_x \mathcal{H}_x,
\]

and the vacuum \( |\Omega\rangle \) given by the product

\[
|\Omega\rangle = \bigotimes_x |\Omega\rangle_x,
\]
for which we will use the conventional unordered tensor product symbol $\otimes$. For any particular ordering of the points of $L$ and any collection of operators $O_x$ indexed by $x \in L$

$$\bigotimes_x (O_x | \Omega) = \left( \prod_x O_x \right) | \Omega\rangle,$$  \hspace{1cm} (7)

where the products over $x$ on the left and right sides of Eq. (7) are ordered identically. We will also use $\otimes$ elsewhere in this paper to represent other versions of ordered tensor products, the details of which will generally be clear from context and not spelled out explicitly.

We define in $H$ a set of product states. For a non-zero complex-valued fermion wave function $p(x, s)$ and boson wave function $q(x)$, define the fermion and boson creation operators $d_f^\dagger(p)$ and $d_b^\dagger(q)$

$$d_f^\dagger(p) = \sum_{xs} p(x, s) \Psi^\dagger(x, s), \hspace{1cm} \text{(8a)}$$

$$d_b^\dagger(q) = \sum_x q(x) \Phi^\dagger(x). \hspace{1cm} \text{(8b)}$$

From a sequence of $n$ fermion wave functions and $m$ boson wave functions define an $n$ fermion, $m$ boson product state to be

$$d_f^\dagger(p_{n-1}) \cdots d_f^\dagger(p_0) d_b^\dagger(q_{m-1}) \cdots d_b^\dagger(q_0) | \Omega\rangle.$$  \hspace{1cm} (9)

Let $P$ be the set of all product states.

It is perhaps useful to point out that the definition of product state here is distinct from the definition sometimes used elsewhere as states of the form

$$| \psi \rangle = \bigotimes_x | \psi_x \rangle,$$  \hspace{1cm} (10a)

$$| \psi_x \rangle \in H_x.$$  \hspace{1cm} (10b)

The time evolution of states in $H$ we will assume governed by a Hamiltonian given by a Hermitian polynomial in the $\Psi(x, s), \Psi^\dagger(x', s'), \Phi(z)$ and $\Phi^\dagger(z')$ which conserves fermion number and couples only $x, x', z$ and $z'$ either identical or nearest neighbors. Beyond these general requirements, we will leave the system’s Hamiltonian unspecified.

The space $H$ is defined to include bosons so that the class of permitted local Hamiltonians includes potentially interesting interacting theories. The proof in Appendix 2 of a lower bound on the complexity of the entangled fermion states considered in Sect. 4 turns out to be a bit more difficult than the corresponding proof would have been in a pure fermion theory.
### 3.2 Hermitian Operator Hilbert Space

We now define a Hilbert space over the reals of Hermitian operators acting on $\mathcal{H}$. For each $x$ let $N_x$ be the fermion number operator on $\mathcal{H}_x$, for nearest neighbor $\{x, y\}$ let $N_{xy}$ be $N_x + N_y$ and let $N$ be the total of $N_x$ over all $x$. We assume $N$ is conserved in time. For any pair of nearest neighbor sites $\{x, y\}$, let $\mathcal{F}_{xy}$ be the set of Hermitian operators $f_{xy}$ acting on $\mathcal{H}_x \otimes \mathcal{H}_y$ which conserve $N_{xy}$, have a finite norm defined to be

$$\|f_{xy}\|^2 = \text{Tr}_{xy}(f_{xy}^2),$$

where $\text{Tr}_{xy}$ is the trace on $\mathcal{H}_x \otimes \mathcal{H}_y$, and for which the partial traces $\text{Tr}_x$ and $\text{Tr}_y$ over $\mathcal{H}_x$ and $\mathcal{H}_y$, respectively, both vanish

$$\text{Tr}_xf_{xy} = 0,$$

$$\text{Tr}_yf_{xy} = 0.$$

The vector space $\mathcal{F}_{xy}$ can be made into a Hilbert space with inner product

$$\langle f_{xy}, f'_{xy} \rangle = \text{Tr}_{xy}(f_{xy}f'_{xy}).$$

Any $f_{xy}$ in some $\mathcal{F}_{xy}$ can be made into an operator $\hat{f}_{xy}$ on $\mathcal{H}$ by

$$\hat{f}_{xy} = f_{xy} \bigotimes_{q \neq x,y} I_q,$$

where $I_q$ is the identity operator on $\mathcal{H}_q$. We now drop the hat and use the same symbol for an operator acting on $\mathcal{H}_x \otimes \mathcal{H}_y$ and the corresponding operator on $\mathcal{H}$.

The total boson number we do not assume conserved in time and place no boson number constraint on $f_{xy} \in \mathcal{F}_{xy}$.

Let $K$ be the vector space over the reals of Hermitian linear operators $k$ on $\mathcal{H}$ given by sums of the form

$$k = \sum_{xy} f_{xy},$$

for any collection of $f_{xy} \in \mathcal{F}_{xy}$ for a set of nearest neighbor pairs $\{x, y\}$. We define an inner product on $K$ by

$$\langle k, k' \rangle = \sum_{xy} \langle f_{xy}, f'_{xy} \rangle.$$  

The difference between Eqs. (12a), (12b) and (16) and corresponding parts of the operator Hilbert space in [10] is a consequence of the infinite dimensionality of each $\mathcal{H}_x$. In Appendix 1 we begin from a starting point closer to the Hilbert space in [10] with the number of bosons allowed at any site $x$ restricted to be less than some finite $n$ reducing each $\mathcal{H}_x$ to finite dimension, then arrive at $K$ and Eqs. (12a), (12b) and (16) by taking the limit $n \to \infty$. As a consequence of taking $n \to \infty$, the norm of operators which act on a single site also goes to $\infty$ thereby removing such operators.
from $K$. To reintroduced single site operators by hand would require introducing also an arbitrary finite normalization constant in place of $\infty$.

### 3.3 Complexity from Unitary Trajectories

From this machinery, for any pair of states $|\omega\rangle, |\psi\rangle \in \mathcal{H}$ with equal norm and fermion number we define the complexity $C(|\psi\rangle, |\omega\rangle)$ of $|\psi\rangle$ with respect to $|\omega\rangle$. For $0 \leq \nu \leq 1$, let $k(\nu) \in K$ be a piecewise continuous trajectory of operators. Let the unitary operator $U_k(\nu)$ on $\mathcal{H}$ be the solution to the differential equation and boundary condition

$$\frac{dU_k(\nu)}{d\nu} = -ik(\nu)U_k(\nu), \quad (17a)$$

$$U_k(0) = I. \quad (17b)$$

We show in Appendix 4 that the topological closure of the group $G$ of all $U_k(1)$ realizable as solutions to Eqs. (17a) and (17b) has a subgroup which is the direct product

$$\tilde{G} = \times_n G_n, \quad (18)$$

where $G_n$ is the special unitary group on the subspace of $\mathcal{H}$ with eigenvalue $n$ of the total fermion number operator $N$. In particular $G_{03}$ acts on the subspace of $\mathcal{H}$ of pure boson states and $G_{16B^1}$ acts on the isomorphic subspace with all sites occupied by two fermions.

Thus for any pair of $|\psi\rangle, |\omega\rangle \in \mathcal{H}$ with equal fermion number, there exists a sequence of trajectories $k_i(\nu)$ and phases $\xi_i$ such that for the corresponding $U_k(1)$ we have

$$\lim_{i \to \infty} \xi_i U_k(1) |\omega\rangle = |\psi\rangle. \quad (19)$$

The complexity $C(|\psi\rangle, |\omega\rangle)$ is defined to be the minimum over all such sequences of $k_i(\nu)$ of the limit of the integral

$$C(|\psi\rangle, |\omega\rangle) = \min \lim_{i \to \infty} \int_0^1 d\nu \| k_i(\nu) \|. \quad (20)$$

Finally, any product state in $\mathcal{P}$ we assign 0 complexity. The complexity $C(|\psi\rangle)$ of any state $|\psi\rangle$ not in $\mathcal{P}$ is defined to be the distance to the nearest product state

$$C(|\psi\rangle) = \min_{|\omega\rangle \in \mathcal{P}} C(|\psi\rangle, |\omega\rangle). \quad (21)$$

Since every product state in $\mathcal{P}$ is an eigenvector of $N$, and since all operators in $K$ preserve $N$, $|\psi\rangle$ will be reachable by a sequence of unitary trajectories in Eq. (19) from a product state $|\omega\rangle$ only if $|\psi\rangle$ itself is an eigenvector of $N$. For states $|\psi\rangle$ which
are not eigenvectors of $N$, the minimum in Eq. (21) and thus the value of $C(|\psi\rangle)$ is, in effect, $\infty$.

For any $|\psi\rangle$, $|\omega\rangle$, $|\phi\rangle \in \mathcal{H}$, $C(|\psi\rangle, |\omega\rangle)$ is symmetric, 0 only if $|\psi\rangle = |\omega\rangle$, and satisfies the triangle inequality

$$C(|\psi\rangle, |\omega\rangle) \leq C(|\psi\rangle, |\phi\rangle) + C(|\phi\rangle, |\omega\rangle).$$

Thus $C(|\psi\rangle, |\omega\rangle)$ can be used to define a metric on the unit sphere in the subspace of $\mathcal{H}$ with any particular fixed eigenvalue of $N$. The identity map from the unit sphere with the topology given by $C(|\psi\rangle, |\omega\rangle)$ to the unit sphere with the topology given by the inner product on $\mathcal{H}$ is continuous. However, the identity map from the unit sphere with the topology given by the inner product on $\mathcal{H}$ to the unit sphere with the topology given by $C(|\psi\rangle, |\omega\rangle)$ is not continuous. For any $|\psi\rangle$ with eigenvalue $n$ of $N$ and any $\delta$ and $\epsilon$, it is possible to find a $|\phi\rangle$ with eigenvalue $n$ of $N$ such that

$$\langle \phi | \phi \rangle < \epsilon,$$

but in addition

$$C(|\psi\rangle, |\psi\rangle + |\phi\rangle) > \delta.$$ 

This can be proved by an adaptation of the proof of the lower bound in Appendix 2. 

The triangle inequality combined with Eq. (21) implies that for any pair of states $|\psi\rangle$ and $|\phi\rangle$

$$C(|\psi\rangle) \leq C(|\phi\rangle) + C(|\phi\rangle, |\psi\rangle),$$

$$C(|\phi\rangle) \leq C(|\psi\rangle) + C(|\phi\rangle, |\psi\rangle),$$

and therefore

$$|C(|\psi\rangle) - C(|\phi\rangle)| \leq C(|\phi\rangle, |\psi\rangle).$$

Equation (26) is, of course, trivial except for $|\phi\rangle$ and $|\psi\rangle$ with equal norms and eigenvalues of $N$.

The restriction in Eq. (15) to nearest neighbor Hamiltonians is a departure from [10] which allows Hamiltonians coupling pairs of sites at all distances. We introduce this restriction here in order to obtain a version of complexity which measures the spatial distribution of entanglement.

It seems plausible that the real world begins in a state of zero or low complexity and that the complexity the real world acquires over time occurs only as the result of time evolution by a Hamiltonian which itself carries only local interactions. If so, the admission of all possible nearest neighbor local interaction trajectories to the scope of the minimization in Eq. (20) should result in a finite value for the complexity $C(|\psi\rangle)$ of the state of the real world at any finite time.

Complexity as defined by Eq. (20) is not in general a continuous function with respect to the metric on $\mathcal{H}$ so that small deviations from any $|\psi\rangle$ can lead to large changes in complexity. Applications of [10] typically handle this issue by minimizing Eq. (20) over an $\epsilon$ neighborhood of any $|\psi\rangle$. A further departure from [10] is that
Eq. (20) will be applied without the use of an $\epsilon$ neighborhood, the role of which will be subsumed by the parameter $b$ to be introduced in the definition of branching in Sect. 5.

Although not continuous with respect to the metric on $\mathcal{H}$, the complexity $C[|\psi(t)\rangle]$ of a state $|\psi(t)\rangle$ evolving in time according to a local Hamiltonian is a continuous function of time. At any instant $t$, there will be a discrete set of trajectories $k(v)$ each of which yields a $U_k(v)$ that connects $|\psi(t)\rangle$ to some product state and is a local minimum of

$$C_k[|\psi(t)\rangle] = \int_0^1 dv \parallel k(v) \parallel.$$  \hfill (27)

The complexity $C[|\psi(t)\rangle]$ is the global minimum of this set of local minima. But since each trajectory $k(v)$ is chosen from the space $K$ of all possible local interactions, as $|\psi(t)\rangle$ evolves in $t$ according to a local Hamiltonian, the corresponding $k(v)$ at each $t$ will be part of a $t$ dependent family of $k(v)$ that varies continuously with $t$. Thus $C[|\psi(t)\rangle]$ is the minimum over a set of continuous functions of $t$ and therefore itself a continuous function of $t$.

A consequence of the restriction in Eq. (15) to nearest neighbor Hamiltonians is that state vectors $|\psi\rangle$ which carry entanglement over large volumes require $k(v)$ with many steps and thus are assigned high complexity. In Sect. 4 we define a class of multi-fermion entangled states, and then in Appendices 2 and 3 derive lower and upper bounds on the complexity of these states.

Eqs. (20) and (21) immediately yield a formula for the complexity of the tensor product $|\chi\rangle \otimes |\phi\rangle$ of a pair of states localized on regions $R_\chi$ and $R_\phi$ sufficiently distant from each other. For this case we have

$$C(|\chi\rangle \otimes |\phi\rangle)^2 = C(|\chi\rangle \otimes |\Omega_\phi\rangle)^2 + C(|\Omega_\chi\rangle \otimes |\phi\rangle)^2,$$  \hfill (28)

where $|\Omega_\chi\rangle$ and $|\Omega_\phi\rangle$ are the vacuum states on regions $R_\chi$ and $R_\phi$, respectively. For sufficiently distant $R_\chi$ and $R_\phi$, the optimal trajectories $k_{i\chi}(v)$ and $k_{i\phi}(v)$ in Eq. (20) for $|\chi\rangle \otimes |\Omega_\phi\rangle$ and $|\Omega_\chi\rangle \otimes |\phi\rangle$ will commute. The optimal product state in Eq. (21) for $|\chi\rangle \otimes |\phi\rangle$ will be the product $|\chi\rangle_0 \otimes |\phi\rangle_0$, where $|\chi\rangle_0$ and $|\phi\rangle_0$ are the optimal product states for $|\chi\rangle \otimes |\Omega_\phi\rangle$ and $|\Omega_\chi\rangle \otimes |\phi\rangle$ respectively, and $k_{i\chi}(v) + k_{i\phi}(v)$ will give an optimal trajectory in Eq. (20) for $|\chi\rangle \otimes |\phi\rangle$ if the time parametrization of $k_{i\chi}(v)$ and $k_{i\phi}(v)$ are chosen to fulfill

$$\parallel k_{i\chi}(v) \parallel = \lambda \parallel k_{i\phi}(v) \parallel$$  \hfill (29)

for some $\lambda$ independent of $t$. Eq. (28) then follows.

### 3.4 Second Law of Quantum Complexity

An estimate of the change in complexity over time of a system evolving according to a local Hamiltonian through a sequence of states each with much less than the system’s maximum possible complexity follows from the conjectured second law of quantum complexity of [11]. Let $|\phi(t)\rangle$ for $t \geq t_0$ be the trajectory given by a local
Hamiltonian $H$ of a state starting from some $|\phi(t_0)\rangle$. For a closely spaced pair of times $t, t + \delta$, the hypothesis that $H$ is local implies there is at least one operator $k(t)$ in the operator space $K$ of Sect. 3.2 and a phase factor $\xi(t)$ such that

$$|\phi(t + \delta)\rangle = \xi(t) \exp[-i\delta k(t)]|\phi(t)\rangle.$$  \hfill (30)

The incremental complexity $C(|\phi(t + \delta)\rangle, |\phi(t)\rangle)$ is then given by

$$C(|\phi(t + \delta)\rangle, |\phi(t)\rangle) = \delta \| k(t) \|,$$  \hfill (31)

for the $k(t)$ which fulfills Eq. (30) and minimizes $\| k(t) \|$. For any $t \geq t_0$ it then follows that

$$C(|\phi(t)\rangle, |\phi(t_0)\rangle) \leq \int_{t_0}^{t} dt \| k(t) \|. \hfill (32)$$

Let $\mathcal{H}(c)$ be the region of state space $\mathcal{H}$ with complexity bounded by $c$

$$\mathcal{H}(c) = \{|\phi\rangle \in \mathcal{H}|C(|\phi\rangle) \leq c\}.$$  \hfill (33)

According to the conjectured second law of quantum complexity the size of $\mathcal{H}(c)$ rises extremely rapidly as a function of $c$, sufficiently rapidly that the overwhelming majority of $|\phi\rangle \in \mathcal{H}(c)$ have complexity $C(|\phi\rangle)$ nearly equal to $c$. In particular, it is conjectured that a sequence of evolving states each with much less than the system’s maximum possible complexity, at each time step very probably increase their complexity to the maximum available on the region of state space accessible by one step of Hamiltonian time evolution. Eqs. (32) then implies that with high probability

$$C(|\phi(t)\rangle, |\phi(t_0)\rangle) = \int_{t_0}^{t} dt \| k(t) \| - \epsilon$$  \hfill (34)

for some very small $\epsilon > 0$.

### 4 Complexity of Entangled Multi-Fermion States

We introduce a family of entangled multi-fermion states, then in Appendices 2 and 3 prove lower and upper bounds for the complexity of these states. For simplicity, the states will be built up from single fermion wave functions which are constant across cubic regions. The complexity bounds will depend both on the size and on the distance between entangled regions. At the cost of additional detail, the results can be extended to more general entangled states.

For indices $0 \leq i < m$, $0 \leq j < n$, let $\{D_{ij}\}$ be a set of cubic regions each with volume $V$ in lattice units and let $\{s_{ij}\}$ be a set of spins either 1 or -1. Pairs of regions with opposite spin may overlap. Suppose in addition, there is a set of surfaces $\{S_{\ell}\}, 0 \leq \ell < q$, each of which divides the lattice $L$ into a pair of disjoint pieces, with each point of each $S_{\ell}$ at least one nearest neighbor step from each point of each distinct $S_{\ell}$, and from each point of each $D_{ij}$, and such that, for a fixed pair of
nonzero integers \( n_0, n_1 \) which sum to \( n \), for every \( 0 \leq i < m \), each \( S_\ell \) divides the set \( \{D_{ij}\}, 0 \leq j < n \), into a pair of disjoint subsets of size \( n_0 \) and \( n_1 \). Thus the set of surfaces \( \{S_\ell\}, 0 \leq \ell < q \), mark a collection of gaps within the set of \( D_{ij} \) the aggregate width of which is at least \( q \) units of lattice spacing. The particular features we require of the \( S_\ell \) are a way of specifying gaps in the \( D_{ij} \) sufficient to permit the derivation of a contribution to the complexity lower bound determined by gap width. For later convenience, we will assume \( m \) and \( V \) are both multiples of 4. No restrictions are placed on \( q \), however. In particular, \( q \) can be 0 so that the set \( \{S_\ell\} \) is empty.

From the \( \{D_{ij}\} \) and \( \{s_{ij}\} \), define a set of \( n \)-fermion product states

\[
|p_i⟩ = V^{-\frac{n}{2}} \prod_{0 \leq j < n} \left[ \sum_{x \in D_{ij}} \Psi^\dagger(x, s_{ij}) \right] |Ω⟩.
\]  

The entangled states we consider are then

\[
|ψ⟩ = m^{-\frac{1}{2}} \sum_{0 \leq i < m} \zeta_i |p_i⟩
\]  

for complex \( \zeta_i \) with \( |\zeta_i| = 1 \).

For \( n \)-fermion entangled states of the form in Eq. (36) with \( m > 4 \), \( n > 1 \), we prove in Appendix 2 a lower bound on complexity

\[
C(|ψ⟩) \geq c_0 \sqrt{mV} + \frac{c_1 q}{\sqrt{n}}
\]  

with \( c_0, c_1 \) independent of \( q, m, n \) and \( V \).

In Appendix 3 we prove in addition

\[
C(|ψ⟩) \leq c_2 \sqrt{mnV} + c_3 mn + c_4 \sqrt{mn}r,
\]  

where \( c_2, c_3 \) and \( c_4 \) are independent of \( q, m, n \) and \( V \). The distance \( r \) is given by

\[
r = \min_{x_{00}} \max_{ij} r_{ij}
\]  

where \( r_{ij} \) is the number of nearest neighbor steps in the shortest path between lattice points \( x_{ij} \) and \( y_{ij} \) such that no pair of paths for distinct \( \{i, j\} \) intersect, \( y_{ij} \) is the center point of \( D_{ij} \) and \( x_{ij} \) is an \( m \times n \) rectangular grid of nearest neighbors in the positive \( x^1 \) and \( x^2 \) directions with base point \( x_{00} \).

Equations (37) and (38) constrain the behavior of a possible continuum limit of the lattice definition of complexity. Assume a limit as \( a \to 0 \) exists for a multiplicatively renormalized version of \( C(|ψ⟩) \) evaluated on a \( |ψ⟩ \) which is held fixed in scaled units. One such state is the \( |ψ⟩ \) of Eq. (36) with the regions \( D_{ij} \) kept fixed in scaled units and therefore \( V \) of the form

\[
V = a^{-3} \hat{V},
\]
for $\hat{V}$ fixed as $a \to 0$. Since both the lower bound of Eq. (37) and and the upper bound of Eq. (38) are proportional to $\sqrt{V}$, $C(|\psi\rangle)$ will have to be related to renormalized complexity $\hat{C}(|\psi\rangle)$ by

$$C(|\psi\rangle) = a^{-\frac{3}{2}} \hat{C}(|\psi\rangle).$$

(41)

For renormalized complexity, in the limit $a \to 0$, the terms in Eqs. (37) and (38) proportional to $c_1$, $c_3$ and $c_4$ will vanish.

For multi-boson states similar to the fermion states of Eq. (36) the proof of the upper bound of Appendix 3 goes through with only minor adjustments. The proof of the lower bound of Appendix 2, however, depends on the conservation of fermion number and does not carry over to entangled states which consist purely of bosons.

5 Branching

Using the complexity measure of Sect. 3 we now define a decomposition of a state vector into a set of branches which minimizes a measure of the aggregate complexity of the branch decomposition.

The state vector of the real world, we will propose, follows through time a single continuously evolving branch in the optimal decomposition. Then at various instants the branch followed in the optimal decomposition will split into two sub-branches. Each time a split occurs, the real world, we assume, randomly chooses one of the resulting sub-branches according to the Born rule.

5.1 Net Complexity of a Branch Decomposition

For any $|\psi\rangle \in \mathcal{H}$ let $|\psi\rangle = \sum_i |\psi_i\rangle$ be a candidate orthogonal decomposition into branches. We define the net complexity $Q(\{|\psi_i\rangle\})$ of this decomposition to be

$$Q(\{|\psi_i\rangle\}) = \sum_i \langle \psi_i | \psi_i \rangle [C(|\psi_i\rangle)]^2 - b \sum_i \langle \psi_i | \psi_i \rangle \ln(\langle \psi_i | \psi_i \rangle),$$

(42)

with branching threshold $b > 0$. For any choice of $b$, the branch decomposition of $|\psi\rangle$ is defined to be the $\{|\psi_i\rangle\}$ which minimizes $Q(\{|\psi_i\rangle\})$. The first term in Eq. (42) is the mean squared complexity of the branches split off from $|\psi\rangle$. But each branch can also be thought of as specifying, approximately, some macroscopic classical configuration of the world. The second term represents the entropy of this random ensemble of classical configurations.

Since the complexity of any state which is not an eigenvector of particle number $N$ is $\infty$, each branch in a decomposition $\{|\psi_i\rangle\}$ which minimizes $Q(\{|\psi_i\rangle\})$ will be an eigenvector of $N$. The requirement that each branch be an eigenvector of $N$ becomes a superselection rule.

The quantity $Q(\{|\psi_i\rangle\})$ is nonnegative and, with nonzero lattice spacing, there is at least one choice of orthogonal decomposition for which $Q(\{|\psi_i\rangle\})$ is bounded.
from above. Any $|\psi\rangle$ with fermion number $n$ can be expressed as a linear combination of a finite set of product states of the form

$$|\{x_j, s_j\}, \{y_k\} \rangle = \prod_{0 \leq j < n} \Psi^\dagger(x_j, s_j) \prod_{0 \leq k < m} \Phi^\dagger(y_k) |\Omega\rangle.$$  \hspace{1cm} (43)

For this decomposition all $C(|\psi_i\rangle)$ are 0 and the second term in Eq. (42)

$$- \sum_{x_j, x_j', y_k} \left[ \langle \{x_j, s_j\}, \{y_k\} |\psi\rangle \langle \psi | \{x_j, s_j\}, \{y_k\} \rangle \right]$$

$$\times \ln(\langle \{x_j, s_j\}, \{y_k\} |\psi\rangle \langle \psi | \{x_j, s_j\}, \{y_k\} \rangle),$$  \hspace{1cm} (44)

is finite. Since $Q(|\psi_i\rangle)$ is nonnegative, it follows that $Q(|\psi_i\rangle)$ has a finite minimum. We will assume without proof that this minimum is unique and realized by some decomposition $|\psi_i\rangle$, except possibly for $|\psi\rangle$ in a lower dimensional submanifold of the unit sphere in $\mathcal{H}$.

For a $|\psi\rangle$ with multi-particle wave function that is $C^\infty$ and has compact support, a finite maximum of $Q(|\psi_i\rangle)$ also persists in the continuum limit $a \to 0$ with lattice dimension $2aB$ held fixed. Returning to scaled positions $ax$, an orthonormal basis for the $n$-fermion, $m$-boson subspace of $\mathcal{H}$ consists of the set of plane-wave states

$$|\{p_j, s_j\}, \{q_k\} \rangle = (8B)^{-3n+m}$$

$$\times \prod_{0 \leq j < n, 0 \leq k < m} \left\{ \sum_{x_j, y_k} \exp[ip_j \cdot (ax_j) + iq_k \cdot (ay_k)] \right\} \Psi^\dagger(ax_j, s_j) \Phi^\dagger(ay_k) |\Omega\rangle,$$  \hspace{1cm} (45)

for momenta $p_j, q_k$ each component of which is an integer multiple of $\frac{a}{\sqrt{B}}$. Each of the plane-waves in Eq. (45) is a product state and therefore has complexity 0. Thus the first term in Eq. (42) is 0. Since the wave function of $|\psi\rangle$ is $C^\infty$ and has compact support, however, the expansion coefficients $\langle \psi | \{p_j, s_j\}, \{q_k\} \rangle$ fall off at large $|p_j|$ and $|q_k|$ faster than any power. In addition, for small $z$ and any small positive $\epsilon$ we have

$$- \ln(z) < e^{-1}z^{-\epsilon}.$$  \hspace{1cm} (46)

Thus the second term in Eq. (42) is bounded

$$- \sum_i \langle \psi_i |\psi_i\rangle \ln(\langle \psi_i |\psi_i\rangle) \leq \sum_{p_j, q_k} e^{-1}[\langle \{k_j, s_j\}, \{q_k\} |\psi\rangle \langle \psi | \{k_j, s_j\}, \{q_k\} \rangle]^{-\epsilon}.$$  \hspace{1cm} (47)

As a result of the rapid fall off of $\langle \psi | \{p_j, s_j\}, \{q_k\} \rangle$ at large $|p_j|$ and $|q_k|$, the sum in Eq. (47) and therefore $Q(|\psi_i\rangle)$ has a finite limit as $a \to 0$.

For $b$ either extremely small or extremely large, the branches which follow from Eq. (42) will look nothing like the macro reality we see. For small enough $b$, the minimum of $Q(|\psi_i\rangle)$ will be dominated by the complexity term. It follows from the discussion of Sect. 3 that the minimum of the complexity term will occur for a
set of branches each of which is nearly a pure, unentangled multi-particle product state. Thus bound states will be sliced up into unrecognizable fragments. On the other hand, for very large \( b \), the minimum of \( Q(|\psi_i\rangle) \) will be dominated by the entropy term, leading to only a single branch consisting of the entire coherent quantum state. Again, unlike the world we see.

The result of all of which is that for the branches given by minimizing \( Q(|\psi_i\rangle) \) of Eq. (42) to have any chance of matching the macro world, \( b \) has to be some finite number. Experiments to measure \( b \) will be discussed in more detail in Sect. 11.

In Sect. 4 we argued that the results of Appendices 2 and 3 imply that if a continuum limit exists for the lattice definition of complexity, the multiplicatively renormalized continuum complexity \( \hat{C}(\langle \varphi \rangle) \) will be related to lattice complexity \( C(|\psi\rangle) \) by Eq. (41). For net complexity of Eq. (42) to have a renormalized continuum version, \( b \) will therefore have to be given by

\[
\hat{b} = a^{-3} \hat{b},
\]

for renormalized continuum \( \hat{b} \), which will then have units of volume.

5.2 Net Complexity of a Tensor Product

The choice of \( [C(|\psi_i\rangle)]^2 \) in Eq. (42) defining \( Q(|\psi_i\rangle) \) rather than some other power of \( C(|\psi_i\rangle) \) is dictated by the plausible requirement that branching occur independently for remote, unentangled factors of a tensor product state.

Consider a state \( |\psi\rangle \) given by the tensor product \( |\chi\rangle \otimes |\phi\rangle \) of a pair of states localized on regions \( R_\chi \) and \( R_\phi \) sufficiently distant from each other. A candidate branch decomposition then becomes

\[
|\psi\rangle = \sum |\chi_i\rangle \otimes |\phi_j\rangle.
\]

Eqs. (28) and (42) then imply

\[
Q(|\chi_i\rangle \otimes |\phi_j\rangle) = Q(|\chi_i\rangle \otimes |\Omega_\phi\rangle) + Q(|\Omega_\chi\rangle \otimes |\phi_j\rangle)).
\]

Thus branching of each of the remote states will occur independently unaffected by branching of the other.

5.3 Time Evolution of Optimal Branch Decomposition

Suppose \( Q(|\psi_i\rangle) \) is minimized at each \( t \) for some evolving \( |\psi(t)\rangle \). The set of possible branch decompositions over which \( Q(|\psi_i\rangle) \) is minimized can be viewed as a topological space with topology given by the product of the Hilbert space topology on each \( |\psi_i\rangle \). At any time \( t \), the net complexity function \( Q(|\psi_i\rangle) \) will then have some set of local minima, each an absolute minimum on a corresponding open set of branch decompositions. The optimal decomposition will be the global minimum over this set of local minima. For time evolution by a local Hamiltonian, the complexity \( C[|\psi(t)\rangle] \) of \( |\psi(t)\rangle \) and the complexity \( C[|\psi_i(t)\rangle] \) of any branch \( |\psi_i(t)\rangle \) will be
continuous functions of time. Thus the local minima of \( Q(\{ |\psi_i\rangle \}) \) will themselves track continuously in time. But at a set of isolated points in time, which of the competing local minima is the overall global minimum can potentially change. At such instants, the optimal decomposition will jump discontinuously. Thus the optimal decomposition is a piecewise continuous function of \( t \).

Continuous Hamiltonian time evolution of each branch leaves the classical entropy term in Eq. (42) unchanged, while the quantum complexity term in Eq. (42) potentially changes during Hamiltonian time evolution, thereby causing a continuous drift in the optimal branch configuration. For a sufficiently large \( b \), however, the classical entropy term in Eq. (42) can be made arbitrarily more important than the quantum term. Thus for large enough \( b \), the continuous part of time evolution will consist almost entirely of Hamiltonian time evolution of each branch.

For the discontinuous part of branch evolution, the requirement that the \( \{ |\psi_i\rangle \} \) be an orthogonal decomposition of \( |\psi(t)\rangle \) implies that a single \( |\psi_i\rangle \) cannot jump by itself. The simplest possible discontinuity allowed by the requirement that the \( \{ |\psi_i\rangle \} \) be orthogonal is for some single branch \( \phi_0 \rangle \) to split into two pieces

\[
|\phi\rangle = |\phi_0\rangle + |\phi_1\rangle.
\]  

(51)

The terms in \( Q(\{ |\psi_i\rangle \}) \) arising from \( |\phi\rangle \) before the split are

\[
\langle \phi|\phi\rangle \left[ (C(|\phi\rangle))^2 - b \ln(\langle \phi|\phi\rangle) \right].
\]  

(52)

The terms from \( |\phi_0\rangle, |\phi_1\rangle \) after the split can be written in the form

\[
\langle \phi|\phi\rangle \left\{ \rho (C(|\phi_0\rangle))^2 + (1 - \rho) (C(|\phi_1\rangle))^2 - b \rho \ln(\rho) - b(1 - \rho) \ln(1 - \rho) - b \ln(\langle \phi|\phi\rangle) \right\},
\]  

(53)

for \( \rho \) defined by

\[
\langle \phi_0|\phi_0\rangle = \rho \langle \phi|\phi\rangle.
\]  

(54)

Thus a split will occur if

\[
[C(|\phi\rangle)]^2 - \rho [C(|\phi_0\rangle)]^2 - (1 - \rho) [C(|\phi_1\rangle)]^2 > -b \rho \ln(\rho) - b(1 - \rho) \ln(1 - \rho).
\]  

(55)

The condition for a split is a saving in average squared complexity by an amount linear in \( b \). Splitting occurs as soon as the required threshold saving in average squared complexity is crossed.

A split could also reverse itself if as a result of time evolution the complexity of \( |\phi\rangle, |\phi_0\rangle \) or \( |\phi_1\rangle \) changes sufficiently to reverse the inequality in Eq. (55). In Sect. 6, however, we will present an argument for the hypothesis that such reversals almost never occur, and that, in addition, for a system evolving through a sequence of states each with much less than the system’s maximum possible complexity, a permanent split of a single branch into two pieces according to Eq. (53) accounts for nearly all of the discontinuities in the time evolution of the optimal branch decomposition.
6 Pair Splits Persist, Other Discontinuities Absent

In Sect. 3.4 based on the conjectured second law of quantum complexity [11], we derived the estimate

\[ C(|\phi(t), \phi(t_0)\rangle) = \int_{t_0}^{t} dt \| k(t) \| - \epsilon \]  

(56)

for the time evolution of complexity of a system governed by a local Hamiltonian, evolving through a sequence of states each with much less than the system’s maximum possible complexity. Based on Eq. (56), we now present an argument for the hypothesis that a pair of branches \(|\phi_0\rangle\) and \(|\phi_1\rangle\) produced according to Eqs. (51) and (55) by a split at some \(t_0\) of a branch \(|\phi\rangle\) with much less than maximal complexity, for a system with a large number of degrees of freedom and \(b\) sufficiently large, with high probability will not merge back into \(|\phi\rangle\) at \(t > t_0\). If Eq. (55) holds at \(t_0\), with high probability it will continue to hold at all \(t > t_0\).

In addition, other possible events merging two branches into a single result we will argue are similarly improbable. Rearrangements at a single instant of \(n\) branches into a new configuration of \(n'\) branches with \(n, n' \geq 2\) are also highly improbable. Splits with \(n = 1\) and \(n' \geq 3\) at a single instant we believe occur with zero probability and are realized instead as a sequence of events at distinct times each with \(n = 1\) and \(n' = 2\).

The time evolution of the set of optimal branches then yields a tree structure of branches each eventually splitting into a pair of sub-branches. The state vector of the real world we propose follows through the tree a single sequence of branches and sub-branches, with the sub-branch at each splitting event chosen randomly according to the Born rule.

A key element in the arguments we give for the persistence and dominance of pair splits is the hypothesis that the system considered has not yet reached a configuration of maximal complexity. This hypothesis could eventually fail for a sufficiently isolated bound subsystem of the universe. Thus branches associated with this subsystem potentially might recombine. As a consequence of isolation from the rest of the universe, however, such recombination processes would necessarily result in no external record. In principle, events simultaneously producing or recombining more than two branches might also occur. These, however, are something like simultaneous \(n\)-body collisions, \(n > 2\), in a random gas and we assume do not occur even for a system which has reached maximal complexity. In the relativistic formulation of branching, the real world is assumed to consist of a random choice among the final set of branching records left at asymptotically late \(\tau\). The real world at any particular finite time is then recovered by retracing the set of branching events the late time choice implies. For this purpose, it will be technically convenient to treat branches which recombine in the evolving optimal branch configuration as remaining distinct through the recombination event. But again, since recombination events occur only in subsystems sufficiently isolated from the rest of the universe, it is plausible that their treatment in relativistic branching is without observable consequences.
6.1 Complexity After a Split

The first half of the argument for the persistence of pair splits is a bound on the change in complexity, following a branching event, of either child branch by the change in complexity of the parent. At some time \(t_0\), assume a particular \(|\phi\rangle\) of an optimal branch decomposition \(\{|\psi_j\rangle\}\) splits into sub-branches \(|\phi_0\rangle\) and \(|\phi_1\rangle\) according to Eqs. (51) and (55). According to the discussion of Sect. 3.4, there is an operator \(k(t)\) which satisfies

\[
|\phi(t + \delta)) = \xi(t) \exp[-i\delta k(t)]|\phi(t)\rangle,
\]

with minimal \(\|k(t)\|\) yielding

\[
C(|\phi(t)\rangle, |\phi(t_0)\rangle) = \int_{t_0}^t dt \|k(t)\| - \epsilon.
\]

For the branches \(|\phi_0\rangle\) and \(|\phi_1\rangle\) we can then define \(k_i(t)\) to accomplish

\[
|\phi_i(t + \delta)) = \xi_i(t) \exp[-i\delta k_i(t)]|\phi_i(t)\rangle,
\]

with minimal \(\|k_i(t)\|\). The argument leading to Eq. (58) then implies that with high probability

\[
C(|\phi_i(t)\rangle, |\phi_i(t_0)\rangle) = \int_{t_0}^t dt \|k_i(t)\| - \epsilon_i
\]

for some very small \(\epsilon_i \geq 0\).

For sufficiently large \(b\), for \(|\phi\rangle\) the state of a system with a large number of degrees of freedom, we can obtain bounds on the \(\|k_i(t)\|\).

For some nearest neighbor pair \(\{x, y\}\), define

\[
Q = H_x \otimes H_y,
\]

\[
R = \bigotimes_{z \neq x, y} H_z,
\]

\[
H = Q \otimes R.
\]

Let the corresponding Schmidt decompositions of \(|\phi_0\rangle, |\phi_1\rangle\) be

\[
|\phi_i\rangle = \sum_j |\psi_{ij}\rangle \otimes |\chi_{ij}\rangle.
\]

For sufficiently large \(b\), for a system with a large number of degrees of freedom, the states \(|\phi_0\rangle\) and \(|\phi_1\rangle\) on reaching the branching threshold and after will have wandered off into high dimensional spaces. We therefore expect the burden of orthogonality between \(|\phi_0\rangle\) and \(|\phi_1\rangle\) to be spread over many lattice spacings. The reduced
states produced by averaging $|\phi_0\rangle$ and $|\phi_1\rangle$ over the 2 site Hilbert space $Q$ should then still be orthogonal. If so, we have

$$\langle X_{0j} | X'_{1\ell'} \rangle = 0,$$

for all $j, \ell'$.

We now temporarily approximate $Q$ by the corresponding space defined in Appendix 1 with the number of bosons at each site $x$ and $y$ restricted to some large but finite $n$.

Let $h_{xy}$ be the piece of the Hamiltonian $H$ acting on $Q$. For $i$ of 0 and 1, let $P_{ixy}$ be the projection onto the subspace of $Q$ spanned by the set of $|\psi_{ij}\rangle$ and $h_{xy}|\psi_{ij}\rangle$ for all $j$. Let $P_{xy}$ be the projection onto the subspace spanned by the set of $|\psi_{ij}\rangle$ and $h_{xy}|\psi_{ij}\rangle$ for all $i$ and $j$. Equation (59) combined with Eqs. (62) and (63) imply the minimal norm $k_{ixy}$, the part of each $k_i$ acting on $Q$, is given by

$$\hat{k}_{ixy} = P_{ixy} h_{xy} P_{ixy},$$

$$k_{ixy} = \hat{k}_{ixy} - \frac{\text{Tr}_{xy}(\hat{k}_{ixy})}{\text{Tr}_{xy}(I_{xy})} I_{xy},$$

where $I_{xy}$ is the identity operator on $Q$. Similarly Eqs. (57), (62) and (63) imply the minimal norm $k_{xy}$ is given by

$$\hat{k}_{xy} = P_{xy} h_{xy} P_{xy},$$

$$k_{xy} = \hat{k}_{xy} - \frac{\text{Tr}_{xy}(\hat{k}_{xy})}{\text{Tr}_{xy}(I_{xy})} I_{xy},$$

However,

$$P_{ixy} = P_{ixy} P_{xy},$$

and therefore

$$\text{Tr}_{xy}(\hat{k}_{ixy}^2) = \text{Tr}_{xy}(P_{ixy} h_{xy} P_{ixy} h_{xy} P_{ixy})$$

$$\leq \text{Tr}_{xy}(P_{xy} h_{xy} P_{xy} h_{xy} P_{xy})$$

$$= \text{Tr}_{xy}(\hat{k}_{xy}^2).$$

In addition, as the limit on the number of bosons $n \to \infty$

$$\text{Tr}_{xy}(k_{ixy}^2) \to \text{Tr}_{xy}(\hat{k}_{ixy}^2),$$

$$\text{Tr}_{xy}(k_{xy}^2) \to \text{Tr}_{xy}(\hat{k}_{xy}^2).$$

Equations (67)–(68b) combined across all nearest neighbor pairs $\{x, y\}$ then imply
\[ \| k_i(t) \| \leq || k(t) ||, \tag{69} \]

for \( i = 0, 1. \)

Combining Eq. (69) with Eqs. (58) and (60) implies
\[ C(|\phi_i(t)\rangle, |\phi_i(t_0)\rangle) < C(|\phi(t)\rangle, |\phi(t_0)\rangle) + \epsilon, \tag{70} \]

A further iteration of the argument leading to Eq. (58) then yields
\[ C(|\phi_i(t)\rangle) - C(|\phi_i(t_0)\rangle) < C(|\phi(t)\rangle) - C(|\phi(t_0)\rangle) + \epsilon. \tag{71} \]

### 6.2 Net Complexity After a Split

We now show that combined with Eq. (53) at \( t_0, \) Eq. (71) leads to Eq. (53) for all \( t > t_0. \)

At \( t > t_0, \) the left hand side of Eq. (55) is given by \( p(t) \)
\[ p(t) = \left[ D(t) + C(|\phi(t_0)\rangle) \right]^2 - \rho \left[ D_0(t) + C(|\phi_0(t_0)\rangle) \right]^2 \]
\[ - (1 - \rho)\left[ D_1(t) + C(|\phi_1(t_0)\rangle) \right]^2, \tag{72} \]

with the definitions
\[ D(t) = C(|\phi(t)\rangle) - C(|\phi(t_0)\rangle), \tag{73a} \]
\[ D_0(t) = C(|\phi_0(t)\rangle) - C(|\phi_0(t_0)\rangle), \tag{73b} \]
\[ D_1(t) = C(|\phi_1(t)\rangle) - C(|\phi_1(t_0)\rangle). \tag{73c} \]

We can then expand \( p(t) \) as a sum of three terms
\[ q(t) = D(t)^2 - \rho D_0(t)^2 - (1 - \rho) D_1(t)^2, \tag{74} \]
\[ r(t) = 2D(t)C(|\phi(t_0)\rangle) - 2\rho D_0(t)C(|\phi_0(t_0)\rangle) - 2(1 - \rho)D_1(t)C(|\phi_1(t_0)\rangle), \tag{75} \]
\[ s = C(|\phi(t_0)\rangle)^2 - \rho C(|\phi_0(t_0)\rangle)^2 - (1 - \rho)C(|\phi_1(t_0)\rangle)^2. \tag{76} \]

Equations (71) and (73a)–(73c) imply \( q(t) \) is greater than some \(-\epsilon. \) Also \( s \) is the left hand side of Eq. (55) so strictly greater than the right hand side, \( D(t) \) greater than \(-\epsilon \) by a further application of the second law of quantum complexity, and \( C(|\phi(t_0)\rangle) \) is nonnegative by the definition of complexity. The Cauchy-Schwartz inequality

\[ \sum_{i=0}^{\infty} \langle \phi_i | k_i(t) \rangle \leq \sum_{i=0}^{\infty} \langle \phi_i | k(t) \rangle, \tag{69} \]
combined with the bounds on \(q(t)\) and \(s\), then implies that \(r(t)\) is greater than some \(\frac{-\epsilon}{\omega}\). It follows that

\[
 p(t) > s - \epsilon > -b\rho \ln(\rho) - b(1-\rho) \ln(1-\rho) - \epsilon.
\]  

Thus Eq. (55) is highly likely satisfied for all \(t > t_0\). A split which first occurs at some time \(t_0\) with high probability persists for all \(t > t_0\).

### 6.3 Other Mergers of Pairs Similarly Improbable

The argument supporting the hypothesis that splits persist can equally well be applied to show that any pair of branches \(|\phi_0\rangle\) and \(|\phi_1\rangle\) which exists at some time \(t_0\), whether or not they were born from the split of a single shared parent branch, are highly unlikely to merge into a single branch at \(t > t_0\).

Let the sum of the branches \(|\phi_0\rangle\) and \(|\phi_1\rangle\) at \(t_0\) be given again by

\[
|\phi\rangle = |\phi_0\rangle + |\phi_1\rangle.
\]

Then since the optimal branch decomposition \(\{\psi_i\}\) at \(t_0\) includes \(|\phi_0\rangle\) and \(|\phi_1\rangle\), rather than their replacement by \(|\phi\rangle\), the inequality

\[
[C(|\phi\rangle)]^2 - \rho[C(|\phi_0\rangle)]^2 - (1-\rho)[C(|\phi_1\rangle)]^2
\]

\[
> -b\rho \ln(\rho) - b(1-\rho) \ln(1-\rho).
\]  

must again hold at \(t_0\). The discussion of Sects. 6.1 and 6.2 then supports the hypothesis that Eq. (80) continues to hold for all \(t > t_0\).

### 6.4 No Other Discontinuities

The remaining class of possible discontinuities in branch time evolution are events rearranging \(n\) branches at a single instant into a new configuration of \(n'\) branches with \(n + n' > 3\). A further extension of the argument showing splits persist shows that rearrangements with \(n, n' \geq 2\) are highly improbable.

Consider the case of \(n\) and \(n'\) both \(2\). Suppose at time \(t_0\) the optimal branch configuration includes a pair of branches \(|\phi_0\rangle, |\phi_1\rangle\). For a system evolving through a sequence of states with much less than the system’s maximum possible complexity, we will show at \(t > t_0\) it is highly improbable for \(|\phi_0\rangle, |\phi_1\rangle\) to jump to a distinct pair \(|\phi'_0\rangle, |\phi'_1\rangle\) with

\[
|\phi_0\rangle + |\phi_1\rangle = |\phi'_0\rangle + |\phi'_1\rangle,
\]

while all other branches vary continuously with time at \(t_0\).
Since all branches in the optimal decomposition \{ |\psi_i\rangle \} aside from \{|\phi_0\rangle \} and \{|\phi_1\rangle \}

\[
|\phi'_i\rangle = \sum_j z_{ij} |\phi_j\rangle. 
\]  

(82)

By applying the argument leading to Eq. (71) first with \(|\phi'_0\rangle\) in the place of \(|\phi\rangle\),

\[
z_{00}|\phi_0\rangle \text{ in place of } |\phi_0\rangle \text{ and } z_{01}|\phi_1\rangle \text{ in place of } |\phi_1\rangle,
\]

then with \(|\phi'_1\rangle\) in the place of \(|\phi\rangle\),

\[
z_{10}|\phi_0\rangle \text{ in place of } |\phi_0\rangle \text{ and } z_{11}|\phi_1\rangle \text{ in place of } |\phi_1\rangle,
\]

we obtain for \(t > t_0\) and \(i\) and \(j\) and combination of 0 and 1

\[
C(|\phi_j(t)\rangle) - C(|\phi_i(t_0)\rangle) < C(|\phi'_j(t)\rangle) - C(|\phi'_i(t_0)\rangle) + \epsilon,
\]

(83)

since for any complex number \(z\) and vector \(|\phi\rangle\)

\[
C(z|\phi\rangle) = C(|\phi\rangle). 
\]  

(84)

Consider now Eq. (83) with \(t\) and \(t_0\) exchanged and \(t\) run back to some early time \(t_1\).

The discussion in Sect. 3.4 yielding

\[
C(|\phi(t)\rangle, |\phi(t_0)\rangle) = \int_{t_0}^t dt \| k(t) \| - \epsilon 
\]

(85)

implies the complexity of almost all states will increase monotonically from \(t_1\) to \(t_0\).

As already mentioned, we assume the system begins at \(t_1\) in a state with small or 0 complexity and arrives at \(t_0\) in a state with complexity much larger. Eq. (83) then yields the result

\[
C(|\phi_i(t_0)\rangle) < C(|\phi'_i(t_0)\rangle) + \epsilon,
\]

(86)

for some small \(\epsilon\) much smaller than \(C(|\phi_i(t_0)\rangle)\) or \(C(|\phi'_i(t_0)\rangle)\) and \(i, j\) any combination of 0 and 1.

An adaptation of the discussion of Sect. 6.2 combined with Eqs. (83) and (86) implies \(|\phi_0\rangle, |\phi_1\rangle\) can not jump discontinuously to \(|\phi'_0\rangle, |\phi'_1\rangle\) at \(t_0\).

Let \(t_2\) be some time preceding \(t_0\) at which the optimal branch configuration still includes \(|\phi_0\rangle, |\phi_1\rangle\) and not \(|\phi'_0\rangle, |\phi'_1\rangle\). Then at \(t_2\) the complexities satisfy

\[
\rho'[C(|\phi'_0(t_2)\rangle)^2 + (1 - \rho')[C(|\phi'_1(t_2)\rangle)^2] - \rho[C(|\phi_0(t_2)\rangle)^2] \]

\[> b\rho' \ln(\rho') + b(1 - \rho') \ln(1 - \rho') - b\rho \ln(\rho) - b(1 - \rho) \ln(1 - \rho),\]

(87)

where \(\rho, \rho'\), both time independent, are defined by

\[
\langle \phi_0(t_2)|\phi_0(t_2)\rangle = \rho(\langle \phi_0(t_2)|\phi_0(t_2)\rangle + \langle \phi_1(t_2)|\phi_1(t_2)\rangle),
\]

(88)
\[ \langle \phi_0(t_2) \vert \phi_0(t_2) \rangle = \rho' \langle (\phi_0(t_2) \vert \phi_0(t_2) \rangle + \langle \phi_1(t_2) \vert \phi_1(t_2) \rangle . \]  

(89)

For \( \vert \phi_0 \rangle, \vert \phi_1 \rangle \) to jump to \( \vert \phi'_0 \rangle, \vert \phi'_1 \rangle \) at \( t_0 > t_2 \) the inequality in Eq. (87) would have to reverse. At \( t_0 \) the left hand side of Eq. (87) is given by

\[
p(t_0) = \rho' [D'_0(t_0) + C(\vert \phi'_0(t_0) \rangle] + (1 - \rho') [D'_1(t_0) + C(\vert \phi'_1(t_0) \rangle)]
- \rho [D_0(t_0) + C(\vert \phi_0(t_0) \rangle)] - (1 - \rho) [D_1(t_0) + C(\vert \phi_1(t_0) \rangle)],
\]

(90)

with the definitions

\[
D'_0(t_0) = C(\vert \phi'_0(t_0) \rangle) - C(\vert \phi'_0(t_2) \rangle),
\]

(91a)

\[
D'_1(t_0) = C(\vert \phi'_1(t_0) \rangle) - C(\vert \phi'_1(t_2) \rangle),
\]

(91b)

\[
D_0(t_0) = C(\vert \phi_0(t_0) \rangle) - C(\vert \phi_0(t_2) \rangle),
\]

(91c)

\[
D_1(t_0) = C(\vert \phi_1(t_0) \rangle) - C(\vert \phi_1(t_2) \rangle).
\]

(91d)

We can then expand \( p(t) \) as a sum of three terms

\[
q(t_0) = \rho' D'_0(t_0) + (1 - \rho') D'_1(t_0) - \rho D_0(t_0) - (1 - \rho) D_1(t_0),
\]

(92)

\[
r(t_0) = 2 \rho D'_0(t_0) C(\vert \phi'_0(t_0) \rangle) + 2 (1 - \rho) D'_1(t_0) C(\vert \phi'_1(t_0) \rangle)
- 2 \rho D_0(t_0) C(\vert \phi_0(t_0) \rangle) - 2 (1 - \rho) D_1(t_0) C(\vert \phi_1(t_0) \rangle),
\]

(93)

\[
s = \rho' C(\vert \phi'_0(t_2) \rangle) + (1 - \rho') C(\vert \phi'_1(t_2) \rangle) - \rho C(\vert \phi_0(t_2) \rangle) - (1 - \rho) C(\vert \phi_1(t_2) \rangle).
\]

(94)

By Eq. (83), \( q(t_0) \) is greater than some small \( - \epsilon \), by Eqs. (83) and (86), \( r(t_0) \) is greater than some other small \( - \epsilon \) and \( s \) is the left hand side of Eq. (87) and therefore strictly greater than the right hand side. Thus \( p(t_0) \) is highly probably greater than the right hand side of Eq. (87) and a jump from \( \vert \phi_0 \rangle, \vert \phi_1 \rangle \) to \( \vert \phi'_0 \rangle, \vert \phi'_1 \rangle \) at any \( t_0 > t_2 \) is highly unlikely to occur.

By a similar argument, for a system evolving through a sequence of states with much less than the system’s maximum possible complexity, rearrangements of \( n \) branches at a single instant into a new configuration of \( n' \) branches for any other \( n, n' \geq 2 \) can be shown also to be highly unlikely to occur.

The end result of all of which is support for the hypothesis that for a system evolving through a sequence of states with much less than the system’s maximum possible complexity, the discontinuities in branch time evolution are highly probably only splits of a single branch into a pair of sub-branches.
7 Scattering Experiment

We will apply the branching proposal of Sect. 5 to a model of an experiment in which a microscopic system scatters and produces a final state recorded by a macroscopic measuring device.

Let $\mathcal{H}$ be the product

$$\mathcal{H} = Q \otimes R,$$  \hspace{1cm} (95)

where $Q$ is the space of states of the macroscopic measuring device and $R$ is the space of states of the microscopic system which undergoes scattering.

Assume an unentangled initial state

$$|\psi^{\text{in}}\rangle = |\psi^0_Q\rangle \otimes |\psi^0_R\rangle,$$  \hspace{1cm} (96)

for which the complexity measure $Q(|\psi^{\text{in}}\rangle)$ is already at a minimum and cannot be reduced by splitting $|\psi^{\text{in}}\rangle$ into orthogonal parts. For the microscopic system, this can be accomplished by a microscopic state $|\psi^0_R\rangle$ with probability concentrated on a scale smaller than the branching volume $b$. The macroscopic state $|\psi^0_Q\rangle$ we assume spread on a scale much larger than $b$ but without entanglement on a scale larger than $b$. We assume also that the number of fermions $n_Q$ in the macroscopic system is much larger than the number of fermions $n_R$ in the microscopic system.

A macroscopic state satisfying these assumptions is the $n$-fermion product states of Sect. 4,

$$|\psi^0_Q\rangle = V^{-\frac{n}{2}} \sum_{x_j \in D_{ij}} \prod_j \Psi^i(x_j, s_{ij}) \Omega,$$  \hspace{1cm} (97)

where, as before, the spins $s_{ij}$ are either 1 or -1 and the $D_{ij}$ are a set of regions each of volume $V$ any pair of which may overlap if their spins are opposite.

The microscopic system then undergoes scattering which produces a final state which is a superposition of $|\psi^i_R\rangle$, we assume for simplicity equally weighted, which is then detected by the macroscopic device eventually leading to the entangled result

$$|\psi^{\text{out}}\rangle = m^{-\frac{1}{2}} \sum_{0 \leq i < m} |\psi^i_Q\rangle \otimes |\psi^i_R\rangle.$$  \hspace{1cm} (98)

As was the case for the initial state, the macroscopic factor $|\psi^i_Q\rangle$ in each of these terms we assume spread on a scale $V$ large with respect to $b$, but without additional entanglement beyond the entanglement explicit in Eq. (98), and the microscopic factor $|\psi^i_R\rangle$ we assume spread on a scale small with respect to $b$. We assume also the macroscopic factors for distinct $i$ are orthogonal. Macroscopic final states which accomplish this are the rest of the $n$-fermion product states of Sect. 4,

$$|\psi^i_Q\rangle = V^{-\frac{n}{2}} \sum_{x_j \in D_{ij}} \prod_j \Psi^i(x_j, s_{ij}) \Omega.$$  \hspace{1cm} (99)
With Eq. (99) for the macroscopic factors $|\psi_Q^i\rangle$, Appendix 2 can be adapted to provide an estimate of the effect on $C(|\psi^{out}\rangle)$ of the microscopic factors $|\psi_R^i\rangle$ in Eq. (98). For $|\psi^{out}\rangle$, consider a version of Eqs. (35) and (36) with fermion number $n$ now replaced by $n_Q + n_R$. For the additional values of $n_Q \leq j < n_Q + n_R$ assume the regions $D_{ij}$ extend only over volumes $V_R$ much smaller than $V$. Construct the regions $E_{ij}$ without change. As a consequence of the small size of the $D_{ij}$ of the microscopic state, fermions in these regions will make almost no contribution to the final Schmidt vectors $\{\lambda_j(1)\}$ which will remain unchanged from the discussion in Appendix 2. But an estimate of $C(|\psi^{out}\rangle)$ now requires a trajectory $k(v)$ from a product state $|\omega\rangle$ with a total of $n_Q + n_R$ fermions. As a result, the bound on Schmidt vector rotation angles of Eq. (B15) becomes

$$\int_0^1 |\theta_k(v)|dv \geq \arcsin \left(\sqrt{\frac{n_Q-n_R}{mV}}\right).$$

Similarly, as a consequence of the fermion number of $C(|\psi^{out}\rangle)$, the bound in Eq. (B56) becomes

$$\sum_{x\in D^f, y\in D^o} \langle \psi(v) | (I-z^0(x,y)) | \psi(v) \rangle \leq 6(n_Q + n_R).$$

The net result of these two changes is that the bound of Eq. (37) becomes

$$C(|\psi^{out}\rangle) \geq c_0 \sqrt{\frac{mV(n_Q-n_R)}{n_Q + n_R}}.$$ (102)

For $n_Q$ large with respect to $n_R$, the lower bound on $C(|\psi^{out}\rangle)$ is almost unchanged from Eq. (37).

For the net complexity of $|\psi^{out}\rangle$ as a single branch we obtain

$$Q(|\psi^{out}\rangle) \geq (c_0)^2 m V.$$ (103)

On the other hand a decomposition of $|\psi^{out}\rangle$ taking each of the $m$ terms in the sum in Eq. (99) as a branch and assuming low complexity for each of the microscopic $|\psi_R^i\rangle$ gives

$$Q((m^{-1}|\psi_Q^i\rangle \otimes |\psi_R^i\rangle) = b \ln(2m),$$ (104)

which will be smaller than $Q(|\psi^{out}\rangle)$ since $V$ is assumed much larger than $b$. Thus the final state will split into $m$ separate branches, one of which, chosen randomly according to the Born rule, becomes the real world. For a more detailed model filling in the evolution from $|\psi^{in}\rangle$ to $|\psi^{out}\rangle$ the branching process would occur not in a single step but sequentially over some short time as the entangled form of Eq. (98) emerges.
8 2-Fermion System

We consider an isolated 2-fermion system with smooth static internal wave function with compact support and center-of-mass position wave-function spreading according to free time evolution. The center-of-mass wave function we will show eventually undergo branching. The proof that branching will occur does not by itself show what branches the state will split into, only that it will split. For the wave function of a single fermion system, no amount of spreading would result in branching since all states of a single fermion system are product states.

We will first find a lower bound on the complexity arising from spreading of the system’s center-of-mass wave function. A version of Eq. (47) then yields an upper bound on the smallest net complexity of a branch decomposition of the state. At some instant at or before the time at which the two limits cross, the wave function of the center-of-mass position will split into branches.

8.1 Lower Bound on State Complexity

Let $|\psi\rangle$ be the state for a 2-fermion system with Gaussian wave function for the center-of-mass position

$$|\psi\rangle = \sum_X g(X)|\psi(X)\rangle,$$

(105a)

$$g(X) = \frac{1}{R^3 \pi^{3/2}} \exp \left( -\frac{|X|^2}{2R^2} \right),$$

(105b)

$$|\psi(X)\rangle = \sum_{x_0 + x_1 = 2X, s_0, s_1} [\phi(x_0 - x_1, s_0, s_1)$$

$$\times \Psi^\dagger(x_0, s_0)\Psi^\dagger(x_1, s_1)]|\Omega\rangle,$$

(105c)

where $R$ is some large constant, the wave function $\phi(x, s_0, s_1)$ is antisymmetric

$$\phi(x, s_0, s_1) = -\phi(-x, s_1, s_0),$$

(106)

vanishes beyond some distance $r$ which is large in lattice units but still much smaller than $R$

$$\phi(x, s_0, s_1) = 0, |x| > r,$$

(107)

and is a sufficiently smooth function of $x$ that sums over lattice points are nearly given by integrals over corresponding continuous variables. The state $|\psi(X)\rangle$ is normalized

$$\langle \psi(X)|\psi(X)\rangle = 1.$$  

(108)

Dropping a term which is small by a factor of $r^{-3}$, Eq. (108) implies
A lower bound on the complexity of $|\psi\rangle$ follows from Appendix 2 with a change in the choice of regions $E_{\epsilon}$. Divide the lattice $L$ into $L'$ and $L''$ as before. All of the points in $L'$ we group into disjoint sets $E_{\epsilon}$ with 4 points in each set. Each $E_{\epsilon}$ we require chosen in such a way that any pair $x, y \in E_{\epsilon}$ have $R > |x - y| > r$.

For each $E_{\epsilon}$ there is a corresponding tensor product decomposition of $H$ according to Eqs. (B7a)–(B8) and Schmidt decomposition of $|\psi\rangle$ following Eqs. (B9)–(B11)

$$|\psi\rangle = \sum_{j} \lambda_{j\epsilon} |\phi_{j\epsilon}\rangle |x_{j\epsilon}\rangle.$$ (110)

Consider the subspace of $Q_{\epsilon}$ with $N[Q_{\epsilon}]$ of 1. Because no pair of $x, y \in E_{\epsilon}$ is within the range of the same $|\psi(X)\rangle$ in the superposition in Eq. (105a), the subspace of $Q_{\epsilon}$ with $N[Q_{\epsilon}]$ of 1 is a direct sum of terms, each with dimension 2, one such term for each $x \in E_{\epsilon}$ formed from the copies of $|\psi(X)\rangle$ it intersects. For some particular $x$, let the corresponding terms in the Schmidt decomposition of Eq. (110) be

$$|\omega_{x}\rangle = \sum_{j=1,2} \lambda_{jx} |\phi_{jx}\rangle |x_{jx}\rangle.$$ (111)

Equations (105a), (105c) and (106) then imply

$$|\omega_{x}\rangle = 2g(x) \sum_{y,s_0,s_1} |\phi(x - y, s_0, s_1)\rangle \times$$

$$\Psi^{\dagger}(x, s_0) \Psi^{\dagger}(y, s_1) |\Omega\rangle,$$ (112)

and therefore

$$\langle \omega_{x} | \omega_{x} \rangle = 4[g(x)^2 \sum_{x,s_1,s_2} |\phi(x, s_1, s_2)|^2].$$ (113)

Equation (109) then implies

$$\langle \omega_{x} | \omega_{x} \rangle = 16[g(x)^2].$$ (114)

In place of Eq. (B15), for each $E_{\epsilon}$ we wind up with

$$\int_{0}^{1} |\theta_{\epsilon}(v)| dv \geq \arcsin[4\sqrt{2}g(x)],$$ (115)

for some arbitrarily chosen single $x \in E_{\epsilon}$. Summed over all $E_{\epsilon}$, the replacement for Eq. (B17) becomes

$$\sum_{\epsilon} \int_{0}^{1} |\theta_{\epsilon}(v)| dv \geq \frac{4R^2}{\pi^{\frac{3}{2}}}. $$ (116)
The final bound on \( C(|\psi\rangle) \) is then

\[
C(|\psi\rangle) \geq \frac{2^{\frac{1}{3}} R^3}{3^{\frac{2}{3}} \pi^{\frac{1}{3}}}.
\]  

(117)

If the wave function \( g(X) \) of the center-of-mass position of \(|\psi\rangle\) evolves according to the free Schrödinger equation for total mass \( M \), the value of \( R \) in Eq. (117) at any time \( t \) will grow according to

\[
R^2 = R_0^2 + \frac{t^2}{16M^2R_0^2},
\]

(118)

where \( R_0 \) is the value of \( R \) at time 0. If \( R \) evolves with \( t \) according to Eq. (118), \( C(|\psi\rangle) \) according to Eq. (117) will eventually grow arbitrarily large.

### 8.2 Upper Bound on Minimal Net Complexity of Branches

The orthonormal set of plane waves of Eq. (45) gives a possible branch decomposition of the state \(|\psi\rangle\) of Eq. (105a). We now show that if \( R \) in Eq. (105a) and therefore in Eq. (117) grows with \( t \) according to Eq. (118), \( Q(\{|\psi_i\rangle\}) \) of a possible branch decomposition of \(|\psi\rangle\) into plane wave nonetheless remains bounded.

Since the plane waves of Eq. (45) are product states, the complexity of each vanishes. Thus to bound \( Q(\{|\psi_i\rangle\}) \) it is sufficient to bound the second term on the right hand side of Eq. (42). For a 2-fermion plane-wave \(|p_0, s_0, p_1, s_1\rangle\) given by Eq. (45), the matrix element \( \langle p_0, s_0, p_1, s_1 |\psi\rangle \), for \( \phi(x, s_0, s_1) \) smooth on the lattice scale, has the factored form

\[
\langle p_0, s_0, p_1, s_1 |\psi\rangle = 2\hat{g}(p_0 + p_1)\hat{\phi}\left(\frac{p_0 - p_1}{2}, s_0, s_1, \right),
\]

(119a)

\[
\hat{g}(p) = (8B)^{-\frac{2}{3}} \sum_x \exp(-ip \cdot x)g(x).
\]

(119b)

\[
\hat{\phi}(p, s_0, s_1) = (8B)^{-\frac{2}{3}} \sum_x \exp(-ip \cdot x)\phi(x, s_0, s_1).
\]

(119c)

The second term on the right hand side of Eq. (42) becomes

\[
-\sum_{p_0, s_0, p_1, s_1} \{\langle p_0, s_0, p_1, s_1 |p_0, s_0, p_1, s_1\rangle \ln[\langle p_0, s_0, p_1, s_1 |p_0, s_0, p_1, s_1\rangle]\}
\]

(120)

\[
= S_0 + S_1,
\]

where according to the normalizations of Eqs. (105b) and (109)

\[
S_0 = 4R^{-3} \pi^{-\frac{3}{2}} \int d^3x \frac{x^2}{R^2} \exp[-\frac{x^2}{R^2}]
\]

(121a)
The term $S_0$ is equal to 6. The term $S_1$, as a result of the assumption that $\phi(x, s_1, s_2)$ is a smooth function with compact support, following the discussion of Eq. (47) of Sect. 5, is some other finite number independent of $R$.

Thus while according to Eqs. (117) and (118), the net complexity $Q(|\psi\rangle)$ will rise linearly with large $t$, the net complexity of the decomposition of $|\psi\rangle$ into plane waves will remain bounded. Therefore the minimum net complexity will not be at the original unbranched state. It will occur instead for branching into plane waves or, much more likely, into still some other configuration with net complexity still smaller than that of plane waves.

8.3 Branching by an Isolated Proton

A extension of the branching result shown here presumably implies that the center-of-mass wave function of a single isolated proton would also eventually undergo branching as a result of its 3 quark constituents. Branching based on truly elementary fields is thus not expected be entirely equivalent to branching based on composites built from these fields even if the elementary constituents are permanently bound within the composites. As discussed in Sect. 11, however, branching by itself will not automatically produce observable consequences. In particular, the evolving wave function of a proton put through a 2-slit experiment would show interference fringes unchanged by the presence of a branching process. Observable consequences occur only if the branching event is coupled to an observer in such a way that its occurrence is registered. For branching driven by spreading of a center-of-mass wave function, the coupling mechanism itself potentially introduces sufficient complexity to cause branching thereby obscuring the branching process of the underlying proton in isolation. It is not obvious how to set up an experiment to detect center-of-mass branching driven by a proton’s composite structure. However if such an experiment could be found, it might provide a way to determine the value of $b$.

9 Multi-Particle Systems

For the entangled multi-fermion states of Sect. 4, we will show that branching occurs if the volume occupied by the entangled states or the squared gap width exceed a threshold proportional to $b$. For an entangled superposition of identical copies of a general multi-particle state, we then show the resulting optimal branches each have complexity squared equal to $b$.

9.1 Multi-Fermion System with Large Volume

According to Eq. (38), if $|\psi\rangle$ of Eq. (36) is split into $r$ branches $|\psi_i\rangle$ each the sum of $m/r$ distinct $|p_i\rangle$ of Eq. (35), the net complexity $Q(\{|\psi_i\rangle\})$ will be bounded by
where, for simplicity, we assume \( mV \) sufficiently large that the \( c_2 \) and \( c_3 \) terms in Eq. (38) can be dropped. The minimum of the bound in Eq. (122) occurs at

\[
\frac{c_1^2mnV}{r} + b \ln r,
\]

for which value Eq. (122) becomes

\[
Q(|\psi_i\rangle) \leq b + b \ln(b)\left(\frac{c_1^2mnV}{b}\right).
\]

On the other hand, according to Eq. (37), if \(|\psi\rangle\) is not split into branches

\[
Q(|\psi\rangle) \geq c_0^2 mV.
\]

Eqs. (124) and (125) imply the branch configuration \(|\psi_i\rangle\) for \( r \) of Eq. (123) will have lower net complexity than \(|\psi\rangle\) left unsplit if

\[
mV \geq sb,
\]

where \( s \) is the solution to

\[
c_0^2 s = 1 + \ln(c_0^2 ns).
\]

There may be some set of branches with complexity still lower than \(|\psi_i\rangle\), but \(|\psi\rangle\) left unsplit will not be the minimum.

9.2 Multi-Fermion System with Large Gap Width

Now suppose the term in Eq. (37) proportional to the gap width \( q \) is much larger than the term proportional to \( V \). The term proportional to \( q \) is independent of \( m \) and holds for any \( m \geq 2 \). For the entangled \(|\psi\rangle\) of Eq. (36) left as a single branch we have

\[
Q(|\psi\rangle) \geq \frac{c_1^2 q^2}{n}.
\]

The upper bound of Eq. (38) remains above Eq. (37) for all \( m \geq 2 \). If the entangled \(|\psi\rangle\) of Eq. (36) is split into \( m \) branches each consisting of one of the product states \(|p_i\rangle\) of Eq. (35), each of which has complexity 0, the result is

\[
Q(|p_i\rangle) = b \ln m.
\]

Branching will therefore occur if

\[
c_1 q \geq \sqrt{nb \ln m}.
\]
Although Eq. (130) is a lower bound on the length scale of entanglement sufficient to cause branching for lattice spacing $a > 0$, it probably does not do so in the limit $a \to 0$. Based on the discussion of Sect. 4, it is likely that $q$ and $b$ will be related to renormalized continuum $\hat{b}$ and $\hat{q}$ by

$$\hat{q} = aq, \quad \hat{b} = a^3 b.$$  \hfill (131a)

so that for renormalized quantities Eq. (130) becomes

$$c_1 \hat{q} \geq \sqrt{\frac{n\hat{b} \ln m}{a}}. \quad \hfill (132)$$

In the limit $a \to 0$, Eq. (130) requires branching only for $\hat{q} \to \infty$. There is, however, a separate mechanism which will potentially cause branching in place of entanglement over arbitrary distance. For a system which begins in the distant past with small complexity and in some bounded region, for entanglement eventually to develop over large distances will require particle wave functions to spread out over large distances. This process will plausibly lead to branching as a result of the volume thresholds derived for a 2-fermion system in Sect. 8 and for $n$-fermions in Sect. 9.1. The argument which says the constraint of Eq. (130) goes away in the limit $a \to 0$, if adapted to the constraints of Sect. 8 and Sect. 9.1 suggests these both survive.

### 9.3 Displaced Copies of a Multi-Particle System

Let $|\psi(t)\rangle$ be the state of a multi-particle system evolving in infinite volume from a configuration with small complexity at $t_0$. For a system in infinite volume, the example considered in Sect. 8 supports the assumption that the range of possible complexity of states accessible to this system at $t > t_0$ is not bounded from above. The discussion of Sect. 3.4, based on the conjectured second law of quantum complexity, then suggests the complexity of $|\psi(t)\rangle$ will be a monotonically increasing function of $t$ which grows without bound.

Assume, for simplicity, there is a limiting $k(\nu, t) \in K$ in Eq. (19) which satisfies Eqs. (17a) and (17b) connecting $|\psi(t)\rangle$ to some product state and optimizing Eq. (20). Let $|\psi_i(t)\rangle$, $0 \leq i < m$, be copies of $|\psi(t)\rangle$ sufficiently displaced from each other that the corresponding $k_i(\nu, t)$ have disjoint support. Define $|\phi(t)\rangle$ to be the sum

$$|\phi(t)\rangle = \frac{1}{\sqrt{m}} \sum_i |\psi_i(t)\rangle. \quad \hfill (133)$$

The complexity of $|\phi(t)\rangle$ will then be given by

$$C(|\phi(t)\rangle) = \int d\nu \parallel \sum_i k_i(\nu, t) \parallel + c(t). \quad \hfill (134a)$$
\[ \begin{align*}
&= \sqrt{m} \int d\nu \|k(v, t)\| + c(t), \\
&= \sqrt{md(t)} + c(t),
\end{align*} \]

where \(d(t)\) grows monotonically in \(t\) without bound and \(c(t)\) remains bounded.

If \(|\phi(t)\rangle\) is split into \(p\) branches \(|\phi_j(t)\rangle, 0 \leq j < p\), each the sum of \(\frac{m}{p}\) distinct \(|\psi_i(t)\rangle\), the net complexity \(\Phi(|\{\phi_j(t)\}\rangle)\) will become for large \(t\)

\[ \Phi(|\{\phi_j(t)\}\rangle) = \frac{md(t)^2}{p} + b \ln p. \]

The minimum of Eq. (135) occurs at

\[ p = \frac{md(t)^2}{b}, \]

for which

\[ C(|\phi_j(t)\rangle) = \sqrt{b}, \]

for each \(0 \leq j < p\).

### 10 Residual Entanglement

According to Sect. 5.2, each factor of an unentangled tensor product will go through branching independently. Sect. 9 on the other hand supports the hypothesis that entanglement will extend only over a finite range in a branch state which is not itself subject to further branching. Combining these pieces leads to the hypothesis that the most general form of a branch state not immediately subject to further branching will consist approximately of a tensor product of a set of factors each entangled only over a finite range.

We will assume that the limit \(2B \rightarrow \infty\) has been taken of the number of lattice steps in the edge of the cubic lattice \(L\) of Sect. 3.1, or alternatively, that \(2B\) is much larger than any of the lengths, in lattice units, that occur in the following.

Let \(|\psi\rangle\) be a branch left after a branching event and not itself immediately subject to further branching. Let \(S\) be a sphere with volume \(V\). Define the spaces \(Q\) and \(R\) to be

\[ Q = \bigotimes_{x \in S} \mathcal{H}_x, \]

\[ R = \bigotimes_{x \notin S} \mathcal{H}_x, \]

so that the full Hilbert space \(\mathcal{H}\) is then
Define the Schmidt decomposition of $|\psi\rangle$ to be

$$|\psi\rangle = \sum_i \lambda_i |\phi_i\rangle \otimes |\chi_i\rangle,$$

$$(140a)$$

$$|\phi_i\rangle \in \mathcal{Q},$$

$$(140b)$$

$$|\chi_i\rangle \in \mathcal{R},$$

$$(140c)$$

$$\langle \phi_i|\phi_j\rangle = \delta_{ij},$$

$$(140d)$$

$$\langle \chi_i|\chi_j\rangle = \delta_{ij}.$$  

$$(140e)$$

Let $C_\mathcal{Q}$ be

$$C_\mathcal{Q} = \sup_i C(|\phi_i\rangle).$$

$$(141)$$

Let $\lambda_0$ be the largest of the $\lambda_i$. The hypothesis we propose is that if $V$ is made sufficiently large that $C_\mathcal{Q} \gg b$, then entanglement in $|\psi\rangle$ across the boundary of $S$ is small and the sum in Eq. (140a) nearly reduces to a single term

$$|\psi\rangle \approx \lambda_0 |\phi_0\rangle \otimes |\chi_0\rangle.$$  

$$(142)$$

The error in Eq. (142), we propose can be made progressively smaller by making $S$ progressively larger. For any particular $|\psi\rangle$ and $S$, however, there will remain some small residual error in Eq. (142) as long as $S$ is small enough not to encompass the entire region on which $|\psi\rangle$ differs from the vacuum.

11 Determination of $b$

Each of the branches which results from the scattering experiment in Sect. 7 looks like what would be left behind if some observer standing outside the normal universe made an observation of the scattering results and thereby caused the reduction of the final state according to the projection postulate. Since the process of forming these branches depends on $b$, it may seem that at least in principle there should be some further measurement revealing the loss of coherence in the final branch configuration which would thereby provide a determination of the value of $b$. The obstacle to finding such a measurement is that branch formation is solely an extra layer of the world sitting on top of exact unmodified unitary Hamiltonian time evolution. No process governed by the underlying Hamiltonian dynamics depends in any way on $b$. So no such process can be used to find $b$. In particular, the time evolution of a state vector is entirely unaffected by the occurrence of a branching event.

So what gives? Are branches simply fictions of some kind?
We believe they are not. But their status is at the least peculiar. The world as seen by human observers we believe incorporates elements that can not be identified simply with state vectors. The hypothesis that human observers encounter such additional elements of reality we believe is also present, at least implicitly, in the various proposals for environmentally-induced decoherence [3–8]. A determination of $b$ consists of finding a value which yields branches which agree with the macroscopic world seen by a human observer. A class of experiments to determine $b$ is as follows.

Suppose a macroscopic system, including now a human observer, is designed to register one of two different possible outcomes of an observation of a microscopic system. Let $|\psi(t)\rangle$ be the state of the total system and let $|\psi_0(t)\rangle$ and $|\psi_1(t)\rangle$ be the branches of the possible outcomes so that, as usual,

$$|\psi(t)\rangle = |\psi_0(t)\rangle + |\psi_1(t)\rangle. \quad (143)$$

The experiment begins at $t_0$ with state $|\psi(t_0)\rangle$ as optimal branch configuration. At some $t_1 > t_0$ the inequality in Eq. (55) becomes first an equality

$$[C(|\phi(t_1)\rangle)]^2 - \rho[C(|\phi_0(t_1)\rangle)]^2 - (1 - \rho)[C(|\phi_1(t_1)\rangle)]^2$$

$$= -b\rho \ln(\rho) - b(1 - \rho) \ln(1 - \rho), \quad (144)$$

where $\rho$ is given by

$$\langle \phi_0 | \phi_0 \rangle = \rho \langle \phi | \phi \rangle. \quad (145)$$

Then on the time interval $t > t_1$, Eq. (55) holds and branching occurs. Eq. (144) implies $b$ is given by

$$b = \sigma^{-1} \{ [C(|\phi(t_1)\rangle)]^2$$

$$- \rho[C(|\phi_0(t_1)\rangle)]^2 - (1 - \rho)[C(|\phi_1(t_1)\rangle)]^2 \} \quad (146)$$

where $\sigma$ is

$$\sigma = -\rho \ln(\rho) - (1 - \rho) \ln(1 - \rho). \quad (147)$$

Then at some time $t_2 \geq t_1$ the result of the experiment is registered by a human observer who recognizes that branching has occurred. The discussion of Sect. 6.2 implies that for $t_2 > t_1$ the left hand side of Eq. (144) almost certainly increases from $t_1$ to $t_2$ and thus $b$ is bounded by

$$b \leq \sigma^{-1} \{ [C(|\phi(t_2)\rangle)]^2$$

$$- \rho[C(|\phi_0(t_2)\rangle)]^2 - (1 - \rho)[C(|\phi_1(t_2)\rangle)]^2 \}. \quad (148)$$

While the values of $t_0$ and $t_2$ are known, the value of $t_1$ is not known. To go from the inequality of Eq. (148) to the equality of Eq. (146) would require determination of $t_1$. But no amount of information about the set up of the experiment leading to Eq. (148) and no amount of calculation of the dependence of $C(|\phi(t_2)\rangle)$, $C_0(|\phi(t_2)\rangle)$ and $C_1(|\phi(t_2)\rangle)$ on $t_2$ based on such information would be sufficient to find where $t_1$ lies in the interval between $t_0$ and $t_2$. The position of $t_1$ in the range between $t_0$ and $t_2$ is
determined by the value of $b$, which the experiment is intended to measure and is otherwise unknown.

However, now imagine repeating the experiment leading to Eq. (148) with $|\phi\rangle$, $|\phi_0\rangle$ and $|\phi_1\rangle$ successively modified to yield a progressively smaller bound on $b$. At some point in this sequence, the inequality in Eq. (148) will turn into the equality of Eq. (146) and $t_1$ will become $t_2$. For still smaller values of the right hand side of Eq. (148) branching will no longer occur at any $t_1$ in the range $t_0 < t_1 \leq t_2$, and therefore no longer be registered by a human observer at $t_2$.

The critical value of the bound in Eq. (148) at which branching goes away is the value of $b$.

A primary issue in the design of any such experiment is finding a possible branching process which a human observer might potentially register without automatically causing branching as a consequence of the complexity of the machinery by which the branching event is rendered observable. In particular, it could turn out that the minimum complexity change in the state of the degrees of freedom of an observer corresponding to registration of an event is by itself always sufficient to cause branching. If so, a measurement of $b$ would depend on information about the complexity of states of matter corresponding to states of human thought. These difficulty may, perhaps, offer some explanation for why a value of $b$ has not so far been coincidentally produced as a by-product of some otherwise unrelated experiment.

An experimental test of the overall account of branching which we propose would be to see if different versions of the experiment just described yield a single value for $b$.

How the experiment just described might be realized in practice is a subject we hope to return to elsewhere.

12 Lattice Approximation to Lorentz Covariant Branching

The definitions of complexity and branching in Sects. 3 and 5 were for a non-relativistic field theory. We now propose an extension of these definitions to a relativistic field theory of fermions and spinless bosons.

An immediate problem with potential Lorentz covariance of the branches found by minimizing $Q(|\psi_t\rangle)$ in Eq. (42) is that the underlying definition of complexity is based on hyperplanes of fixed $t$, which are themselves not Lorentz invariant. We will therefore replace the constant $t$ hyperplanes with boost invariant hyperboloids of constant proper time $\tau$. Hyperboloids of constant $\tau$, however, are not translationally invariant.

The loss of translational invariance shows itself as a variant of the problem exposed by the EPR experiment. This difficulty in only slightly different clothing we already briefly mentioned in Sect. 2 and is a general problem for any formulation of branches as the substance of reality [3–8].

Consider some branch viewed in two different frames related by a translation. For some period of proper time assume the branch’s representation in each frame remain related by a translation. But then in a pair of disjoint regions with spacelike separation, suppose processes occur each of which, by itself, is sufficient to cause splitting of the
branch the two processes share. Assume in addition, that in one frame one of these events occurs at smaller \( \tau \) but in the other frame, as a consequence of the of the regions’ spacelike separation, the other event occurs at smaller \( \tau \). The result will be that in the proper time interval between the events the branch structure seen by the two different frames will be different. But our goal is to be able to interpret branch state vectors as the underlying substance of reality. That interpretation fails if branch structure is different according to different reference frames.

For any pair of distinct frames, however, for any pair of spacelike separated events each capable of causing a branch to split, there is some proper time sufficiently late that splitting will have been completed in both frames. Correspondingly, we will argue that the definition to be introduced for branching on a hyperboloid of fixed \( \tau \) should approach translational covariance as \( \tau \to \infty \). A related proposal, in a somewhat different setting, is considered in [12–14].

We will therefore assume macroscopic reality is a single random choice among the set of branches at asymptotically late \( \tau \) according to a measure based on the Born rule. If the branches which make up macroscopic reality are permanent once formed, a random choice among the accumulated set of branches at late \( \tau \) is equivalent to the continuing branching choice in the non-relativistic theory, but with the bookkeeping for the choice process performed all at once rather than in sequential steps. The real world at some finite \( \tau \) in any particular frame would be recovered from the asymptotic late \( \tau \) choice by tracing back through proper time the branching tree according to that frame.

The limiting branching tree found as \( \tau \to \infty \), we propose as the underlying real object. The indirect relation between branches found as \( \tau \to \infty \) and branches found at finite time is then qualitatively similar to the indirect relation, in a Lorentz covariant quantum field theory, between a final out scattering state and a Shroedinger representation state at some finite time.

Details of this proposal we now fill in. As first step, we will reformulate the definitions of complexity and branching in Sects. 3 and 5 with the regular lattice at fixed time of Sect. 3.1 replaced by a finite random lattice chosen according to a Lorentz invariant density on a hyperboloid with fixed proper time.

### 13 Hyperbolic Random Lattice

Let \( L(\tau) \) be the the spacelike hyperboloid with fixed proper time \( \tau \)

\[
(x^0)^2 - \sum_i (x^i)^2 = \tau^2
\]

(149)

and let \( L(\tau, \sigma) \) be the intersection of \( L(\tau) \) with the ball \( B(\sigma \tau) \) of radius \( \sigma \tau \)

\[
\sum_i (x^i)^2 < \sigma^2 \tau^2.
\]

(150)

From \( L(\tau_0, \sigma) \) for some initial \( \tau_0 \), we construct a finite random set of points \( L(\tau_0, \sigma, \rho) \) chosen according to the Lorentz invariant density on \( L(\tau_0, \sigma) \), where \( \rho \) is a proper distance which will set the lattice spacing. Then for \( \tau > \tau_0 \), a piecewise continuous
\( L(\tau, \sigma, \rho) \) will be obtained by an iterative sequence of transformations applied to \( L(\tau_0, \sigma, \rho) \).

Choose an initial point \( x_0 \) from \( L(\tau_0, \sigma) \) randomly according to the Lorentz invariant volume measure on \( L(\tau_0, \sigma) \). Then iteratively choose \( x_{i+1} \) according to the invariant measure on \( L(\tau_0, \sigma) \) but from the subset of \( L(\tau_0, \sigma) \) with proper distance from each \( x_j, 0 \leq j \leq i, \) greater than \( \rho \). Stop this process at the smallest \( n \) such that for each \( x \in L(\tau, \sigma) \) there is at least one \( x_j, 0 \leq j < n, \) with proper distance from \( x \) less than or equal to \( \rho \). Let \( L(\tau_0, \sigma, \rho) \) be the set of all such \( x_j \). For each \( x \in L(\tau_0, \sigma, \rho) \), let the \( c(x) \) be the Voronoi cell centered on \( x \), the set of points in \( L(\tau_0, \sigma, \rho) \) closer to \( x \) than to any other \( y \in L(\tau_0, \sigma, \rho) \). For every \( y \), the set of sites closer to \( x \) is convex. Since \( c(x) \) is the intersection of all such sets, it is also convex. Every point in \( c(x) \) is at most a proper distance of \( \rho \) from \( x \). Every point with distance from \( x \) less than \( \frac{\xi}{2} \) is contained in \( c(x) \). Thus for \( \rho \) much smaller than \( \tau \), the proper volume of each \( c(x) \) is less than \( \frac{4\pi\rho^3}{3} \) and greater than \( \frac{2\rho^3}{6} \). Pairs of points \( \{x, y\} \) will be considered nearest neighbors if \( c(x) \) and \( c(y) \) share a 2-dimensional boundary surface.

The set of points \( L(\tau, \sigma, \rho) \) for \( \tau > \tau_0 \), we obtain from \( L(\tau - \delta, \sigma, \rho) \) for some small value of \( \delta \). Let \( L(\tau, \sigma, \rho) \) consist of the points of \( L(\tau - \delta, \sigma, \rho) \), each rescaled by a factor of \( 1 + \frac{\delta}{\rho} \), with any resulting hole in \( L(\tau, \sigma) \) which is a proper distance greater than \( \rho \) from the rescaled points of \( L(\tau - \delta, \sigma, \rho) \) filled by an additional point chosen randomly according to the invariant measure on \( L(\tau, \sigma) \). For small enough \( \delta \), at most one such region will be found and adding a single point will leave no such region remaining. The resulting set is \( L(\tau, \sigma, \rho) \).

Unlike the field operators for the non-relativistic theory of Sect. 3, which were assumed to be taken from a lattice field theory, for the relativistic theory it is technically more convenient to assume field operators at any \( x \) obtained from averages over \( c(x) \) of corresponding field operators of either a continuum field theory or, alternatively, a Minkowski space lattice field theory with lattice spacing much smaller than \( \rho \). For simplicity we will assume a continuum field theory. Let \( \Psi(x, s) \) be a continuum field operator averaged over the cell \( c(x) \). Let \( \Phi(x) \) and \( \Pi(x) \) be, respectively, Hermitian boson field and conjugate momentum operators for \( x \) also in the rest frame at \( x \) averaged over \( c(x) \). We will assume the vacuum expectation values of \( \Phi(x) \) and \( \Pi(x) \) vanish. Since \( L(\tau, \sigma) \) is spacelike, we can assume the \( \Psi(x, s), \Psi^\dagger(x, s), \Phi(x), \) and \( \Pi(x) \) are normalized to obey the anticommutation and commutation relations

\[
\{\Psi(x, s), \Psi(x', s')\} = 0, \tag{151a}
\]

\[
\{\Psi^\dagger(x, s), \Psi^\dagger(x', s')\} = 0, \tag{151b}
\]

\[
[\Phi(x), \Phi(x')] = 0, \tag{151c}
\]

\[
[\Pi(x), \Pi(x')] = 0, \tag{151d}
\]

\[
\{\Psi(x, s), \Psi^\dagger(x', s')\} = \delta_{x,x'}\delta_{s,s'}, \tag{151e}
\]
Eqs. (151e) and (151f) satisfy lattice approximations to Lorentz covariance. Let $\mathcal{H}$ be the subspace of the full relativistic Hilbert space, $\mathcal{H}^R$, spanned by all polynomials in the $\Psi(x, s), \Psi^\dagger(x', s'), \Phi(y)$ and $\Pi(y')$ for any $x, x', y, y'$ and $s, s'$ acting on the physical vacuum $|\Omega\rangle$ but restricted to order at most $n_b$ in any single $\Phi(x)$ or $\Pi(x)$. The resulting $\mathcal{H}$ is finite dimensional. By redefining the field operators $\Phi(x)$ and $\Pi(x)$ to be sandwiched between projection operators onto the space generated by the restricted set of polynomials, we can enforce the subsidiary relations

$$\Phi(x)^{n_b} = 0, \quad (152a)$$

$$\Pi(x)^{n_b} = 0. \quad (152b)$$

14 Auxiliary Field Theory

For the non-relativistic theory of Section 3.1, the Hilbert space $\mathcal{H}$ is isomorphic to a tensor product of the local Hilbert spaces $\mathcal{H}_x$. For the relativistic theory we have just defined, as a remnant of the the Reeh-Schlieder theorem for the underlying continuum field theory, a similar tensor product form does not hold. The tensor product form of $\mathcal{H}$, however, is a key ingredient in the construction of the non-relativistic complexity measure of Sect. 3.3.

Rather than viewing the non-relativistic complexity measure as acting on states, however, it can also be viewed as acting on the algebra of fields. This perspective suggests an extension to the relativistic case.

From any element of the algebra $A$ of polynomials in the $\Psi(x, s), \Psi^\dagger(x, s), \Phi(x), \Pi(x)$, we define a linear map $f$ to an isomorphic algebra $B$ of polynomials in the auxiliary fields $\Sigma_i(x, s)$ and $\Upsilon_i(x)$, $0 \leq i \leq 1$, which obey the anticommutation and commutation relations

$$\{\Sigma_i(x, s), \Sigma_j(x', s')\} = 0, \quad (153a)$$

$$[\Upsilon_i(x), \Upsilon_j(x')] = 0, \quad (153b)$$

$$\{\Sigma_0(x, s), \Sigma_1(x', s')\} = \delta_{xx'}\delta_{ss'}, \quad (153c)$$

$$[\Upsilon_0(x), \Upsilon_1(x')] = i\delta_{xx'}, \quad (153d)$$

along with the boson cutoff

$$\Upsilon_i^{n_b} = 0. \quad (154)$$

The map $f$ is defined by
along with the requirement

\[ f(a \cdot a') = f(a) \cdot f(a'), \quad (156) \]

for all \( a, a' \in A \).

To obtain a complexity measure on \( B \), we will turn \( \Sigma_i(x, s) \) and \( \Upsilon_i(x) \) into field operators on an auxiliary Hilbert space \( \mathcal{H}^B \) generated by all elements of \( B \) acting on an auxiliary vacuum \( |\Omega^B\rangle \). But unlike the fields \( \Psi(x, s), \Psi^*(x, s), \Phi(x), \Pi(x) \), which are a mix of creation and annihilation operators, the \( \Sigma_i(x, s) \) and \( \Upsilon_i(x) \) will be purely creation operators.

Realizations of Eqs. (153a)–(154) by creation operators acting on \( |\Omega^B\rangle \) are discussed in Appendix 5. The space \( \mathcal{H}^B \) is generated by all polynomials in \( B \) acting on \( |\Omega^B\rangle \) as specified in Appendix 5. For each lattice point \( x \), the set of all polynomials in \( \Sigma_i(x, s), \Upsilon_i(x) \), acting on the local vacuum, \( |\Omega^B\rangle \) generates a Hilbert space \( \mathcal{H}^B_x \). The space \( \mathcal{H}^B \) is then isomorphic to the ordered tensor product

\[ \mathcal{H}^B = \bigotimes_x \mathcal{H}^B_x. \quad (157) \]

The cost of realizing Eqs. (153a)–(154) purely with creation operators, however, is that while the energy spectrum of \( h \), the continuum Hamiltonian on \( \mathcal{H}^R \) projected into \( \mathcal{H} \), is bounded from below by 0 so that any \( a \in A \) which according to \( h \) carries a negative increment of energy annihilates \( |\Omega\rangle \), the spectrum of \( f(h) \) is not bounded from below and \( f(a) \in B \) does not in general annihilate \( |\Omega^B\rangle \). Approximate Lorentz covariance of Eqs. (153a)–(156) combined with Eq. (157), according to Reeh-Schlieder, makes the presence of negative energy states in \( \mathcal{H}^B \) pretty much unavoidable. But \( \mathcal{H}^B \) will be used only to define complexity, not for time evolution. The time evolution of physical states in \( \mathcal{H} \) remains governed by the Hamiltonian of \( \mathcal{H}^R \), the spectrum of which is bounded from below by 0.

The fields \( \Sigma_i(x, s) \) and \( \Upsilon_i(x) \) become versions of the non-relativistic \( \Psi^i(x, s) \) and \( \Phi^i(x) \) of Sect. 3.1. But with the hyperplane of constant \( t \) of the non-relativistic theory now replaced by the hyperboloid \( L(\tau) \) of constant proper time \( \tau \), and the translational invariance of the hyperplane of constant \( t \) replaced by the Lorentz boost invariance of \( L(\tau) \).
15 Hermitian Operator Hilbert Space Again

Equation (157) makes possible for \( \mathcal{H}_B^\mathcal{H} \) a relativistic version of the non-relativistic complexity measure of Sect. 3. Then from the map \( f : A \to B \) we will retrieve a definition of complexity on \( \mathcal{H} \).

Let \( \mathcal{P}_B^\mathcal{P} \) be the set of all product states in \( \mathcal{H}_B^\mathcal{H} \) defined by adapting Eqs. (8a)–(9). For fermion wave function \( p(x, s) \), anti-fermion wave function \( q(x, s) \), and boson wave function \( r(x, i) \), define the fermion, anti-fermion and boson creation operators, \( d_f(p) \), \( \overline{d}_f(q) \), and \( d_b(r) \), respectively, to be

\[
\begin{align*}
  d_f(p) &= \sum_{xs} p(x, s) \Sigma_1(x, s), \\
  \overline{d}_f(q) &= \sum_{xs} q(x, s) \Sigma_0(x, s), \\
  d_b(r) &= \sum_{xi} r(x, i) \Upsilon_i(x).
\end{align*}
\]  

(158a)  
(158b)  
(158c)

Then for a sequence of \( k \) fermion, \( \ell \) anti-fermion and \( m \) boson creation operators, the corresponding product state is

\[
\begin{align*}
  d_f(p_{k-1}) \ldots d_f(p_0) \overline{d}_f(q_{\ell-1}) \ldots \overline{d}_f(q_0) \\
  \times d_b(r_m) \ldots d_b(r_0) |\Omega_B^B\rangle.
\end{align*}
\]  

(159)

As a consequence of Eq. (154), the dimension of the Hilbert space \( \mathcal{H}_B^\mathcal{H} \) is finite. A Hilbert space of Hermitian operators to be used to define complexity can therefore be constructed following, with some minor changes, the version in Appendix 1. For any site \( x \), let \( \mathcal{F}_x^\mathcal{H} \) be the set of Hermitian operators on \( \mathcal{H}_x^\mathcal{H} \) which have finite norm

\[
\|f_x\|^2 = \text{Tr}_x(f_x)^2,
\]  

(160)

vanishing trace

\[
\text{Tr}_x f_x = 0,
\]  

(161)

and conserve \( N_B^\mathcal{H} \), a copy on \( \mathcal{H}_x^\mathcal{H} \) of the fermion number of the underlying field \( \Psi(x, s) \). \( N_B^\mathcal{H} = 0 \) on \( |\Omega_B^B\rangle \), is raised by 1 by \( \Sigma_1(x, s) \) and lowered by 1 by \( \Sigma_0(x, s) \). For any pair of nearest neighbors \( \{x, y\} \), let \( \mathcal{F}_{xy}^\mathcal{H} \) be the set of Hermitian operators on \( \mathcal{H}_x^\mathcal{H} \otimes \mathcal{H}_y^\mathcal{H} \) which have finite norm

\[
\|f_{xy}\|^2 = \text{Tr}_{xy}(f_{xy})^2,
\]  

(162)

vanishing partial traces

\[
\text{Tr}_{xy} f_{xy} = 0,
\]  

(163a)
\[ \text{Tr}_x f_{xy} = 0, \quad (163b) \]

and conserve \( N^B \).

Inner products on \( \mathcal{F}_x^B \) and \( \mathcal{F}_{xy}^B \) are defined by

\[ \langle f_x, f'_x \rangle = \text{Tr}_x (f_x f'_x), \quad (164a) \]

\[ \langle f_{xy}, f'_{xy} \rangle = \text{Tr}_{xy} (f_{xy} f'_{xy}). \quad (164b) \]

Operators \( f_x \in \mathcal{F}_x^B \) and \( f_{xy} \in \mathcal{F}_{xy}^B \) can be made into operators on \( \mathcal{H}^B \) by

\[ \hat{f}_x = f_x \bigotimes_{q \neq x} I_q, \quad (165a) \]

\[ \hat{f}_{xy} = f_{xy} \bigotimes_{q \neq x, y} I_q, \quad (165b) \]

where \( I_q \) is the identity operator on \( \mathcal{H}_q^B \). As usual, we now drop the hat and use the same symbol for operators on \( \mathcal{H}_x^B, \mathcal{H}_x^B \otimes \mathcal{H}_y^B \), and the corresponding operators on \( \mathcal{H}^B \). Let \( K^B \) be the vector space over the reals of linear operators \( k \) on \( \mathcal{H}^B \) given by

\[ k = \sum_{xy} f_{xy} + \frac{1}{\sqrt{d_q}} \sum_x f_x \quad (166) \]

for any collection of \( f_{xy} \in \mathcal{F}_{xy}^B \) for a set of nearest neighbor pairs \( \{x, y\} \), any collection of \( f_x \in \mathcal{F}_x^B \) for a set of sites \( x \), and \( d_q \) the dimension of \( \mathcal{H}_q^B \) for some site \( q \). The inner product on \( K^B \) is given by

\[ \langle k, k' \rangle = \sum_{xy} \langle f_{xy}, f'_{xy} \rangle + \sum_x \langle f_x, f'_x \rangle. \quad (167) \]

### 16 Complexity from Auxiliary States

Adapting Sect. 3.3 yields from the operator space \( K^B \) a definition of complexity \( C^B(|\psi\rangle) \) on states \( |\psi\rangle \in \mathcal{H}^B \). For \( 0 \leq \nu \leq 1 \), let \( k(\nu) \in K^B \) be a piecewise continuous trajectory of operators. Let the unitary operator \( U_k(\nu) \) on \( \mathcal{H}^B \) be the solution to the differential equation and boundary condition

\[ \frac{dU_k(\nu)}{d\nu} = -ik(\nu)U_k(\nu), \quad (168a) \]

\[ U_k(0) = I. \quad (168b) \]

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For any pair of $|\psi\rangle, |\omega\rangle \in \mathcal{H}^B$ with equal norm and fermion number, there exists a sequence of trajectories $k_i(v)$ and phases $\xi_i$ such that for the corresponding $U_{k_i}(1)$ we have

$$\lim_{i \to \infty} \xi_i U_{k_i}(1)|\omega\rangle = |\psi\rangle. \quad (169)$$

The complexity $C^B(|\psi\rangle, |\omega\rangle)$ is defined to be the minimum over all such sequences of $k_i(v)$ of the limit of the integral

$$C^B(|\psi\rangle, |\omega\rangle) = \min \lim_{i \to \infty} \int_0^1 dv \, \| k_i(v) \|. \quad (170)$$

As before, any product state in $\mathcal{P}^B$ we assign 0 complexity. The complexity $C^B(|\psi\rangle)$ of any state $|\psi\rangle$ not in $\mathcal{P}^B$ is defined to be the distance to the nearest product state

$$C^B(|\psi\rangle) = \min_{|\omega\rangle \in \mathcal{P}^B} C^B(|\psi\rangle, |\omega\rangle). \quad (171)$$

Since every product state in $\mathcal{P}^B$ is an eigenvector of $N^B$, and since all operators in $K^B$ preserve $N^B$, $|\psi\rangle$ will be reachable by a sequence of unitary trajectories in Eq. (169) from a product state $|\omega\rangle$ only if $|\psi\rangle$ itself is an eigenvector of $N^B$. For states $|\psi\rangle$ which are not eigenvectors of $N^B$, the minimum in Eq. (171) and thus the value of $C^B(|\psi\rangle)$ is, in effect, $\infty$.

The complexity of any $a|\Omega\rangle$ for $a \in A$ is then defined to be

$$C(a|\Omega\rangle) = C^B[f(a)|\Omega^B\rangle]. \quad (172)$$

An immediate consequence of Eq. (172) is that since $C^B[f(a)|\Omega^B\rangle]$ is finite only if $f(a)|\Omega^B\rangle$ is an eigenvector of $N^B$, $C(|\psi\rangle)$ is finite only if $|\psi\rangle$ is an eigenvector of $N$.

Since $A$ and $\mathcal{H}$ are both finite dimensional, the set of $a|\Omega\rangle, a \in A$, is closed and every $|\psi\rangle \in \mathcal{H}$ is given by some $a|\Omega\rangle, a \in A$. In addition, each $|\psi\rangle \in \mathcal{H}$ is given by only a single $a|\Omega\rangle$. There are no nonzero $a \in A$ which annihilate $|\Omega\rangle$. Although the full infinite dimensional algebra of field operators on the underlying continuum relativistic Hilbert space $\mathcal{H}^R$ does contain operators which annihilate $|\Omega\rangle$, none of these can make their way into $A$ since it is generated by a finite set of continuum field averages each taken over a bounded region.

On the other hand, there are potentially $a \in A$ extremely close to annihilation operators for which

$$\| a|\Omega\rangle \| \ll 1, \quad (173a)$$

$$\| f(a)|\Omega^B\rangle \| = O(1), \quad (173b)$$

$$C^B[f(a)|\Omega\rangle] \gg 1. \quad (173c)$$

For some otherwise ordinary $b \in A$, we might then have
\[ b|\Omega\rangle \approx (a + b)|\Omega\rangle, \]  
\[ C[(a + b)|\Omega\rangle] \gg C[b|\Omega\rangle]. \]  

In this case, according Sect. 18, if \((a + b)|\Omega\rangle\) is split into branches, \(a|\Omega\rangle\) will land in a branch of its own with extremely small weight and therefore not much overall effect. The closer \(a\) is to a true annihilation operator, the smaller the weight of the \(a|\Omega\rangle\) branch and the more negligible the effect of the presence of \(a\) in the state \((a + b)|\Omega\rangle\).

The utility of Eq. (172) as a definition of complexity is dependent, as was the case for the non-relativistic complexity of Eq. (21), on the distinction between states created by field operators acting in a region \(V\) and field operators acting in a distant region \(V'\). For the underlying continuum field theory, the Reeh-Schlieder theorem implies that any state created by operators in \(V\) can be expressed as the limit, with respect to the topology of \(\mathcal{H}^R\), of a sequence of states created by operators in \(V\). In particular, for the continuum field theory, an entangled combination of states created by field operators acting in \(V\) and field operators acting in a distant \(V'\) can be arbitrarily well approximated by an entangled combination of states created by field operators acting purely in \(V\). But for the lattice field theory, the complexity of these various states is determined only after all are turned into states in \(\mathcal{H}^B\), and in the topology of \(\mathcal{H}^B\) an approximating sequence of states created by field operators acting purely in \(V\) will not, in general, approximate the entangled combination of states created by field operators acting in \(V\) and in a distant \(V'\). Moreover, as shown for the non-relativistic version of complexity in Sect. 3.3, \(C(|\psi\rangle)\) is not in general continuous with respect to the Hilbert space topology on \(|\psi\rangle\). As a consequence, the complexity of the approximating sequence of states created by operators purely in \(V\) will not, in general, converge to the complexity of the entangled combination created by operators in \(V\) and in \(V'\). In Section 17 we will consider an example of a state in the relativistic \(\mathcal{H}\) which has large complexity as a result of entanglement extended over a large volume.

### 17 Complexity of Entangled States Again

We assume now the lattice spacing parameter \(\rho\) is much smaller than the proper time \(\tau\) of hyperboloid \(L(\tau, \sigma)\). Entangled multi-fermion relativistic states in \(\mathcal{H}\) analogous to the non-relativistic states of Sect. 4, for large values of the volume \(V\), we will show have complexity satisfying upper and lower bounds analogous to Eqs. (37) and (38).

For indices \(0 \leq i < m, 0 \leq j < n\), let \(\{D_{ij}\}\) be a set of disjoint, nearly cubic regions each centered on a corresponding point \(y_{ij}\). The region \(D_{ij}\) is the set of all center points \(x\) of all cells \(c(x)\) crossed by starting at \(y_{ij}\) and traveling along a geodesic in \(L(\tau, \sigma)\) a proper distance \(\leq d\) in the positive or negative \(x^1\)-direction, then traveling along a geodesic a proper distance \(\leq d\) in the positive or negative \(x^2\)-direction, then traveling along a geodesic a proper distance \(\leq d\) in the positive or negative \(x^3\)
-direction. We assume $\tau$ much larger than $d$ and $d$ much larger than $\rho$. For $d$ large, the mean number of points in each such $D_{ij}$ will approach some limit $V$ with small relative dispersion. Since the proper volume of each $D_{ij}$ is $8d^3$ and the proper volume of each $c(x)$ is between $\frac{4\pi\rho^3}{3}$ and $\frac{\pi\rho^3}{6}$, $V$ will be between $\frac{48d^3}{\pi\rho^3}$ and $\frac{6d^3}{\pi\rho^3}$.

Let $u^k(x)$ and, for later use, $v^k(x)$ be orthogonal spinor wave functions obtained by boosting from the origin of $L(\tau, \sigma)$ to point $x$ a pair of orthogonal spinors for a free fermion at rest in the rest frame at the origin of $L(\tau, \sigma)$.

From the $\{D_{ij}\}$ define a set of $n$-fermion monomials

$$p_i = V^{-\frac{n}{2}} \prod_{0\leq j<n} \left[ \sum_{x \in D_{ij}, k} u^k(x)\Psi^k(x, k) \right],$$

(175)

and an entangled polynomial $q$ and corresponding state $|\psi\rangle \in \mathcal{H}$

$$q = z^{-\frac{1}{2}} \sum_{0\leq i<m} \zeta_i p_i,$$

(176a)

$$|\psi\rangle = q|\Omega\rangle,$$

(176b)

for complex $\zeta_i$ with $|\zeta_i| = 1$ and $z$ chosen to normalize $|\psi\rangle$ to 1. The $V^{-\frac{n}{2}}$ normalization of $p_i$ insures that $z$ has a finite limit for large $d$.

The corresponding monomials in $B$

$$p^B_i = V^{-\frac{n}{2}} \prod_{0\leq j<n} \left[ \sum_{x \in D_{ij}, k} u^k(x)\Sigma_1(x, k) \right],$$

(177)

and entangled polynomial $q^B$ and state $|\psi^B\rangle \in \mathcal{H}^B$ become

$$q^B = z^{-\frac{1}{2}} m^{-\frac{1}{2}} \sum_{0\leq i<m} \zeta_i p^B_i,$$

(178a)

$$|\psi^B\rangle = q^B|\Omega^B\rangle.$$ (178b)

According to Eq. (172)

$$C(|\psi\rangle) = C^B(|\psi^B\rangle).$$

(179)

For $|\psi^B\rangle$ of Eq. (178b) with $m > 4$, $n > 1$, we prove in Appendix 6a lower bound on complexity of the same form as the non-relativistic lower bound of Eq. (37)

$$C^B(|\psi^B\rangle) \geq c_0 \sqrt{mV},$$

(180)

with $c_0$ independent of $m$, $n$ and $V$.

In Appendix 7 we prove an upper bound of almost the same form as the non-relativistic upper bound of Eq. (38)
where $c_1, c_2, c_3$ and $c_4$ are independent of $m, n$ and $V$. The distance $r$ is given by
\[ r = \min_{x_{00}} \max_{ij} r_{ij} \] (182)
where $r_{ij}$ is the number of nearest neighbor steps in the shortest path between lattice points $x_{ij}$ and $y_{ij}$ such that no pair of paths for distinct $\{i,j\}$ intersect, $y_{ij}$ is the center point of $D_{ij}$ and $x_{ij}$ is an $m \times n$ rectangular grid consisting of the center points of the cells crossed by a geodesic starting at $x_{00}$ of $m$ steps of $4/\rho^1$ each in the $x^1$ direction each point of which then forms the base for $n$ nearest neighbor steps along a geodesic in the $x^2$ direction.

As was the case for the non-relativistic upper bound, if $C_B(|\psi^B\rangle)$ is scaled with a factor of $\rho^3/2$, and the limit $\rho \to 0$ taken with the regions $D_{ij}$ kept fixed in scaled units, the bounds of Eqs. (180) and (181) have continuum limits, from which the term proportional to $c_2, c_3$ and $c_4$ vanish.

18 Branching Again

Let $P(\tau, \sigma, \rho)$ be the projection operator from the Schroedinger-like representation of the continuum relativistic Hilbert space $\mathcal{H}^R$ on $L(\tau)$ to its finite dimensional lattice subspace $\mathcal{H}$ based on $L(\tau, \sigma, \rho)$. Define the complexity of any continuum relativistic state $|\psi\rangle \in \mathcal{H}^R$ on the hyperboloid $L(\tau)$ to be
\[ C(\tau, \sigma, \rho, |\psi\rangle) = C[P(\tau, \sigma, \rho)|\psi\rangle], \] (183)
for the lattice complexity $C[P(\tau, \sigma, \rho)|\psi\rangle]$ of Eq. (172). For $|\psi\rangle \in \mathcal{H}^R$ define the net complexity $Q(\tau, \sigma, \rho, \{|\psi_i\rangle\})$ of a branch decomposition $\{|\psi_i\rangle\}$ as before by Eq. (42), with $C(|\psi_i\rangle)$ replaced by $C(\tau, \sigma, \rho, |\psi_i\rangle)$. The optimal branch decomposition as before is found by minimizing $Q(\tau, \sigma, \rho, \{|\psi_i\rangle\})$. The resulting branch decomposition has a finite volume, lattice approximation to Lorentz covariance.

Rather than defining the lattice fields on $L(\tau, \sigma, \rho)$ to be averages of continuum fields on $L(\tau)$, an alternative starting point for relativistic branching would have been to assume, as in the non-relativistic case, a pure lattice field theory with time development in $\tau$ governed by some corresponding hamiltonian $h$ consisting of nearest-neighbor polynomials in the lattice fields. From that stating point, we could have then used $h$ translated into $f(h)$ acting on $\mathcal{H}^B$ for a version of the argument of Sect. 6 to support the hypothesis that the branching predicted by $Q(\tau, \rho, \{|\psi_i\rangle\})$ for evolution in $\tau$ also consists purely of irreversible splits of some parent branch into a pair of orthogonal sub-branches. We will assume this hypothesis holds.
19 \sigma Large

Now suppose that \( Q(\tau, \sigma, \rho, \{|\psi_i\}\}) \) either has a limit \( Q(\tau, \rho, \{|\psi_i\}\}) \) as \( \sigma \to \infty \) or alternatively that \( \sigma \) has been made large enough that nothing in the following ever comes close to bumping into the boundary of \( L(\tau, \sigma, \rho) \). Whether the limit of branching as \( \sigma \to \infty \) actually exist is beyond the scope of the present discussion. In any case, for notational simplicity, we will now drop \( \sigma \) as an argument.

If a limiting \( Q(\tau, \rho, \{|\psi_i\}\}) \) does exit, the underlying Hilbert space \( \mathcal{H} \) will be defined on the lattice \( L(\tau, \rho) \) consisting of a set of points \( \{x_i\} \subset L(\tau) \) chosen randomly according to the invariant measure on \( L(\tau) \) but subject to the requirements that the proper distance between any pair of distinct \( x_i \) is no less than \( \rho \) and that no point in \( L(\tau) \) is a proper distance greater than \( \rho \) from all \( x_j \). In addition, for small \( \delta \), the set of point \( L(\tau, \rho) \) will consist of the points of \( L(\tau - \delta, \rho) \), each rescaled by a factor of \( 1 + \frac{\delta}{\tau} \) with any resulting hole in \( L(\tau) \) more than a proper distance of \( \rho \) from the rescaled points of \( L(\tau - \delta, \rho) \) filled by an additional point chosen randomly according to the invariant measure on \( L(\tau) \).

20 Residual Entanglement Again

Let \( |\psi(\tau)\rangle \in \mathcal{H} \) be a branch not immediately subject to further branching. Let \( Z \) be some large spatial region and \( W \) the intersection of \( Z \) with \( L(\tau, \rho) \). Let \( X \) be the intersection of the complement of \( Z \) with \( L(\tau, \rho) \). Let \( \mathcal{Q} \) be the subspace of \( \mathcal{H} \) spanned by the set of field operators with support in \( W \) acting on the vacuum and \( \mathcal{R} \) the subspace of \( \mathcal{H} \) spanned by the set of field operators with support in \( X \) acting on the vacuum. Although for reasons already briefly mentioned in Sect. 14, \( \mathcal{H} \) will not simply be isomorphic to a tensor product of \( \mathcal{Q} \) and \( \mathcal{R} \), the non-vacuum sectors of these spaces remain linearly independent. Let \( \{|\psi_Qi\rangle\} \) and \( \{|\psi_Ri\rangle\} \) be orthonormal bases for \( \mathcal{Q} \) and \( \mathcal{R} \), respectively. Let \( S_i \) and \( T_i \) be operators with support in \( W \) and in \( X \), respectively, such that

\[
|\psi_Qi\rangle = S_i |\Omega\rangle, \tag{184a}
\]

\[
|\psi_Ri\rangle = T_i |\Omega\rangle. \tag{184b}
\]

Then any branch \( |\psi(\tau)\rangle \) has a unique expansion of the form

\[
|\psi(\tau)\rangle = \sum_{ij} \alpha_{ij} S_i T_j |\Omega\rangle, \tag{185}
\]

A polar decomposition of the matrix \( \alpha_{ij} \) then yields a Schmidt decomposition for \( |\psi(\tau)\rangle \)

\[
|\psi(\tau)\rangle = \sum_i \lambda_i U_i V_i |\Omega\rangle, \tag{186a}
\]

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where $U_i$ and $V_i$ are operators with support in $W$ and $X$, respectively.

According to a version of the hypothesis in Sect. 10 copied over to relativistic complexity and branching, the state $|\psi(\tau)\rangle$ will be entangled only over bounded regions in $L(\tau, \rho)$, so that for sufficiently large $Z$, the sum in Eq. (186a) nearly reduces to a single term

$$|\psi(\tau)\rangle \approx \lambda(\tau) U(\tau)V(\tau)|\Omega\rangle.$$  (187)

Now define $Q^\perp$ to be the projection of $Q$ orthogonal to $R$. Since the union of $W$ and $X$ is all of $L(\tau, \rho)$, $Q^\perp$ will hold the degrees of freedom present in $\mathcal{H}$ but missing from $R$. We therefore expect

$$\mathcal{H} = Q^\perp \otimes R.$$  (188)

For sufficiently large $Z$, an alternative version of Eq. (191) becomes

$$|\psi(\tau)\rangle \approx \lambda(\tau)|\psi_{Q^\perp}\rangle \otimes |\psi_R\rangle,$$  (189)

where

$$|\psi_{Q^\perp}\rangle \in Q^\perp,$$  (190a)

$$|\psi_R\rangle \in R.$$  (190b)

The difference between $|\psi_{Q^\perp}\rangle$ and $U(\tau)|\Omega\rangle$ and between $|\psi_R\rangle$ and $V(\tau)|\Omega\rangle$, however, should be only near the boundary of $Z$. For sufficiently large $Z$, Eq. (189) then becomes

$$|\psi(\tau)\rangle \approx \lambda(\tau)[U(\tau)|\Omega\rangle] \otimes [V(\tau)|\Omega\rangle].$$  (191)

### 21 $\rho$ Small, $\tau \to \infty$

Now assume that $\rho$ has been made much smaller than any of the length scales occurring in the following. For notational simplicity we will drop $\rho$ as an argument of the various functions in which it appears. The example of Sect. 8 shows that as $\tau \to \infty$, for any system not confined to a bounded region, branch splitting will continue without stop. To frame a plausible hypothesis for the behavior of branching as $\tau \to \infty$ which takes continued splitting into account, we define a branch labelling scheme which then facilitates the definition of a set of summed branches.

For a system beginning in some initial state $|\psi\rangle$ with complexity close to 0 at proper time $\tau_0$, consider the set of branch states which result from minimizing $Q(\tau, \{|\psi_i\rangle\})$ for $\tau \geq \tau_0$. Let $E$ be the set of all branching events. As discussed in

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Sect. 6 on non-relativistic branching, we will assume $E$ consists almost entirely of splits of branches into pairs of sub-branches. In addition, however, $E$ may also include rare events for isolated subsystems of the universe for which associated branches split and then later recombine, leaving no record in the rest of the universe. It will turn out to be technically convenient to adopt a rule according to which recombining branches remain distinct so that all splits effectively become permanent. A corresponding rule will then be needed to handle subsequent splits of the remaining distinct branches.

The hypothesis that all splitting events in $E$ consist of some branch splitting permanently into a pair of sub-branches yields a labeling scheme for branches. Each branch state $|\psi(s, \tau)\rangle$ can be labelled with a set of pairs

$$s = \{(e_0, \ell_0), \ldots (e_{n-1}, \ell_{n-1})\}, \ n > 0,$$

(192)

giving a corresponding history of splitting events $e_i \in E$ and branch indices $\ell_i \in \{0, 1\}$. For a splitting event $e \in E$ at time $\tau$ of a state $|\psi(w, \tau)\rangle$ with history

$$w = \{w_0, \ldots w_{n-1}\},$$

(193)

the resulting branch states $|\psi(u, \tau)\rangle, |\psi(v, \tau)\rangle$ have

$$u = \{w_0, \ldots w_{n-1}, (e, 0)\},$$

(194a)

$$v = \{w_0, \ldots w_{n-1}, (e, 1)\}. $$

(194b)

As a consequence of allowing branch states to pass through recombination events unaffected, it follows that each branch in the optimal set $\{|\psi_i\rangle\}$ at $\tau$ will be a sum of some corresponding set $S_i$ of orthogonal branch states

$$|\psi_i\rangle = \sum_{w \in S_i} |\psi(w, \tau)\rangle.$$

(195)

Suppose $|\psi_i\rangle$ is a branch state at $\tau$ with $w \in S_i$ and suppose that the branching event $e$ at $\tau$ splits $|\psi_i\rangle$ into branches $|\phi_0\rangle$ and $|\phi_1\rangle$. The rule needed to compensate for having ignored events in which branches can rejoin is then

$$|\psi(u, \tau)\rangle = \langle \psi(w, \tau) | \phi_0\rangle |\phi_0\rangle,$$

(196a)

$$|\psi(v, \tau)\rangle = \langle \psi(w, \tau) | \phi_1\rangle |\phi_1\rangle,$$

(196b)

The initial state we assign branch index 0 of an initial null branching event $\emptyset \in E$ at $\tau_0$. Thus $|\psi\rangle$ at $\tau_0$ becomes $|\psi(\{(\emptyset, 0)\}, \tau_0)\rangle$. Each $s$ can also be viewed as a map from some subset of $E$ into $\{0, 1\}$. For $s$ of Eq. (192), define $|s|$ to be $n$.

For $\tau_0$ sufficiently large, $Q(\tau, \{|\psi_i\rangle\})$ will be nearly translationally invariant over the spatial region contributing to any branching event. Since $Q(\tau, \{|\psi_i\rangle\})$ is also Lorentz invariant, it seems reasonable to assume at the least that for each branching event and resulting branch in any Poincaré frame there will be
For any \( \tau \geq \tau_0 \), let \( S(\tau) \) be the set of \( s \) corresponding to the set of branches which minimize \( Q(\tau, \{ |\psi_i| \}) \). Each \( S(\tau) \) can be viewed as a set of maps, each map in the set taking a subset of \( E \) into \( \{0, 1\} \). Define \( S \) to be the set of all such maps, each taking some subset of \( E \) into \( \{0, 1\} \). Appending, for the moment, a reference frame label \( f \) to \( S_f(\tau) \), the set \( S \) then contains at least
\[
S \supseteq U_f \cup_r S_f(\tau). \tag{197}
\]
For any \( s \in S \), and any \( \tau \), define \( |\chi(s, \tau)\rangle \) to be the sum of all the \( \tau \) branches with histories containing \( s \)
\[
|\chi(s, \tau)\rangle = \sum_{s'\in S(\tau), s'\supseteq s} |\psi(s', \tau)\rangle. \tag{198}
\]
For any \( \tau \), there will be a corresponding \( n_\tau \), such that
\[
|\chi(s, \tau)\rangle = 0, |s| > n_\tau. \tag{199}
\]
On the other hand, for every \( s \in S \) there is a \( \tau_s \) such that
\[
|\chi(s, \tau)\rangle \neq 0, \tau > \tau_s. \tag{200}
\]
For any \( \tau_0 \leq \tau_1 \leq \tau \), selecting the \( s' \in S(\tau_1) \) which are descendents of some \( s \in S(\tau_0) \) yields
\[
|\chi(s, \tau)\rangle = \sum_{s'\in S(\tau_1), s'\supseteq s} |\chi(s', \tau)\rangle. \tag{201}
\]
For any pair of distinct \( s, s' \in S(\tau) \), for any \( \tau' \geq \tau \), the states \( |\chi(s, \tau')\rangle \) and \( |\chi(s', \tau')\rangle \) are orthogonal. For any \( s \in S(\tau) \), the only \( s' \in S(\tau) \) which satisfies \( s' \supseteq s \) is \( s' = s \) itself, in which case
\[
|\chi(s, \tau)\rangle = |\psi(s, \tau)\rangle. \tag{202}
\]
Let \( U(\tau) \) be the unitary operator on the full relativistic Hilbert space \( \mathcal{H}^R \) which takes the Schroedinger-like representation of a state on the hyperboloid \( L(\tau) \) to the representation of that state on the hyperplane with \( x^0 = 0 \). Define \( |\hat{\chi}(s, \tau)\rangle \) to be the \( x^0 = 0 \) representation of \( |\chi(s, \tau)\rangle \)
\[
|\hat{\chi}(s, \tau)\rangle = U(\tau)|\chi(s, \tau)\rangle. \tag{203}
\]
For any \( \tau_0 \leq \tau_1 \leq \tau \), Eq. \((201)\) implies
\[
|\hat{\chi}(s, \tau)\rangle = \sum_{s'\in S(\tau_1), s'\supseteq s} |\hat{\chi}(s', \tau)\rangle. \tag{204}
\]
For any \( \tau \geq \tau_0 \)
\[
|\hat{\chi}([(\emptyset, 0)], \tau)\rangle = U(\tau_0)|\psi\rangle. \tag{205}
\]
The example of Sect. 8 shows that as \( \tau \to \infty \), for any system not confined to a bounded region, branch splitting will continue without stop and the values of \(|s|\) for \( s \in S(\tau) \) will grow without bound. Thus there is no fixed \( s \in S \), for which \(|\psi(s, \tau)|\) remains defined as \( \tau \to \infty \). For every \( s \in S \), however, the summed branch \(|\hat{\mathcal{X}}(s, \tau)|\) remains defined and potentially has a limit as \( \tau \to \infty \).

The discussion in Sect. 5.3 of the evolution with increasing \( t \) of the optimal branch decomposition arising from the non-relativistic \( Q(|\psi_i|) \), now applied to \( Q(\tau, \{ \psi_i \}) \), implies the evolution with increasing \( \tau \) of the optimal relativistic branch decomposition will be piecewise continuous. The discontinuous piece, according to Sects. 6 and 19, will consist almost entirely of permanent splitting of some branch into a pair of sub-branches. The continuous piece, according to Sect. 5.3, for sufficiently large \( b \) will consist almost entirely of continuous unitary evolution with \( \tau \) of the branches which do not split. If branch splitting is entirely permanent splitting into pairs of sub-branches and branches which don’t split change purely by unitary evolution in \( \tau \), then each \(|\hat{\mathcal{X}}(s, \tau)|\) of Eq. (203), for any \( s \in S(\tau') \) for any \( \tau \geq \tau' \), will be constant in \( \tau \). Thus the existence, for any \( s \in S \), of the limit

\[
\lim_{\tau \to \infty} |\hat{\mathcal{X}}(s, \tau)| = |\hat{\mathcal{X}}(s)|
\]  

appears to be a plausible hypothesis.

### 22 Translational Covariance

Let \(|\psi\rangle \in \mathcal{H}^R \) be the representation of a state on the \( x^0 = 0 \) hyperplane and \( \{ |\hat{\mathcal{X}}(s)| \} \) the corresponding set of \( \tau \to \infty \) branches. Let \( P^\mu, 0 \leq \mu < 4 \), be the momentum operator on \( \mathcal{H}^R \). Let \(|\psi_{z}\rangle \) be a copy of \(|\psi\rangle \) translated by \( z_\mu \)

\[
|\psi_{z}\rangle = \exp(-iz_\mu P^\mu)|\psi\rangle.
\]  

(207)

Let \( \{ |\hat{\mathcal{X}}_{z}(s)| \} \) be the set of \( \tau \to \infty \) branches arising from \(|\psi_{z}\rangle \). We now give an argument in support of the hypothesis that for every \( s \in S \)

\[
|\hat{\mathcal{X}}_{z}(s)| = \exp(-iz_\mu P^\mu)|\hat{\mathcal{X}}(s)|.
\]  

(208)

If Eq. (208) holds for \( z_\mu \) in a neighborhood of 0, it holds for all \( z_\mu \). We will assume \( z_\mu \) small in the following.

Let \(|\psi(\tau)\rangle \) be \(|\psi\rangle \) represented on \( L(\tau) \)

\[
|\psi(\tau)\rangle = U^{\dagger}(\tau)|\psi\rangle,
\]  

(209)

and let \(|\psi(s, \tau)\rangle \) be the corresponding branch decomposition

\[
|\psi(\tau)\rangle = \sum_s |\psi(s, \tau)\rangle.
\]  

(210)

Let \( \{ Y_j \} \) be a partition of space into disjoint regions, \( W_j \) the intersection of \( Y_j \) with \( L(\tau) \).
Let $\mathcal{Q}_j$ be the subspace of $\mathcal{H}^R$ spanned by the operators with support in $W_j$ acting on the vacuum. Assume the $\{W_j\}$ are all sufficiently large that Eq. (191) yields for each $|\psi(s, \tau)\rangle$

$$|\psi(s, \tau)\rangle \approx \lambda(s, \tau) \bigotimes_j |\psi_j(s, \tau)\rangle,$$ (211a)

$$|\psi_j(s, \tau)\rangle \in \mathcal{Q}_j,$$ (211b)

so that

$$|\psi(\tau)\rangle \approx \sum_s \lambda(s, \tau) \bigotimes_j |\psi_j(s, \tau)\rangle.$$ (212)

Let $E_j(\tau)$ be the set of events in the causal past of $W_j$. Any $s \in S(\tau)$ is a union of the overlapping sets

$$s = \bigcup_j s_j$$ (213a)

$$s_j = s \cap \left( E_j(\tau) \times \{0, 1\} \right).$$ (213b)

Let $S_j(\tau)$ be the set of $s_j$ arising from $s \in S(\tau)$. For the $W_j$ sufficiently large in comparison to the size of the entangled region driving any branching event, the set of events with ambiguous classification according to these definitions is a small fraction of the size of each set. We will assume the vector $|\psi_j(s_j, \tau)\rangle$ actually has the form $|\psi_j(s_j, \tau)\rangle$ since it is unchanged by parts of $s$ outside $s_j$. If each of the $W_j$ is sufficiently large, $|\psi_j(s_j, \tau)\rangle$ and $|\psi_{j'}(s_{j'}, \tau)\rangle, j' \neq j$, will be orthogonal. The orthogonality of distinct branches implies in addition

$$\langle \psi_j(s_j, \tau) | \psi_j(s_{j'}, \tau) \rangle = \delta_{s_j s_{j'}}.$$ (214)

Let $|\psi_\zeta(\tau)\rangle$ be

$$|\psi_\zeta(\tau)\rangle = U^\dagger(\tau) \exp(-i\bar{\zeta}_\mu P^\mu) |\psi\rangle,$$ (215a)

$$= U^\dagger(\tau) \exp(-i\bar{\zeta}_\mu P^\mu) U(\tau) |\psi(\tau)\rangle.$$ (215b)

If $\tau$ is sufficiently large that each of the regions $W_j$ is nearly flat, we will now argue that Eqs. (212) and (215b) should give

$$|\psi_\zeta(\tau)\rangle \approx \sum_s \lambda(s, \tau) \bigotimes_j |\psi_{\zeta j}(s_j, \tau)\rangle,$$ (216a)

$$|\psi_{\zeta j}(s_j, \tau)\rangle = \exp(-i\bar{\zeta}_{\mu j} P^\mu) |\psi_j(s_j, \tau)\rangle,$$ (216b)
\[ z_{\mu j} = a_{\mu j}^0 z_0, \]  

where \( a_{\mu j}^0 \) is the Lorentz boost which takes points in the hyperplane with \( x^0 = x_j^0 \) to points in the hyperplane holding \( W_j \), where \( x_j^0 \) is the time component of the center point of \( W_j \).

An argument in support of Eqs. (216a)–(216c) is as follows. Suppose first that only a single \( |\psi_j(s_j, \tau)\rangle \) differs from the vacuum and that the corresponding \( W_j \) is entirely flat. Then the effect of \( U(\tau) \) on \( |\psi_j(s_j, \tau)\rangle \) should consist of a boost which takes states represented in the the hyperplane holding \( W_j \) to states represented in the hyperplane with \( x^0 = x_j^0 \) followed by a time development operator taking states represented in the \( x^0 = x_j^0 \) hyperplane to states in the \( x^0 = 0 \) hyperplane. But since time development itself is assumed translationally covariant, only the boost components of \( U(\tau) \) and \( U^j(\tau) \) in Eq. (215b) will have an effect on \( \exp(i z_{\mu j} P^\mu) \). The time development parts of \( U(\tau) \) and \( U^j(\tau) \) will commute through. The effect of the boost components of \( U(\tau) \) and \( U^j(\tau) \) on the translation operator is then given Eqs. (216a)–(216c). If each of the \( W_j \) is sufficiently large, since \( |\psi_j(s_j, \tau)\rangle \) and \( |\psi_{j'}(s_j, \tau)\rangle, j' \neq j \), will be orthogonal, \( U(\tau) \) should act nearly independently on each \( |\psi_j(s_j, \tau)\rangle \). The result is Eqs. (216b)–(216c) for the full expansion of \( |\psi(\tau)\rangle \) in Eq. (216a).

Consider the \( |\chi_j(s, \tau)\rangle \) found from the branches of \( |\psi_j(\tau)\rangle \). Suppose again to begin that in Eq. (216a) for \( |\psi_j(\tau)\rangle \) only a single \( |\psi_j(s_j, \tau)\rangle \) differs from the vacuum, suppose that \( W_j \) is entirely flat, and ignore for the moment the \( z_{\mu j} \) component of \( z_{\mu j} \). For \( W_j \) entirely flat, the net complexity function \( Q(\tau, \{ |\psi_j\rangle \}) \) is translationally covariant. It follows that the \( W_j \) factor of any branch arising from \( |\psi_j(\tau)\rangle \) will be the translation the \( W_j \) factor of a corresponding branch of \( |\psi(\tau)\rangle \), and similarly for the summed branches:

\[
|\chi_j(s_j, \tau)\rangle = \exp(-i \sum_{\mu > 0} z_{\mu j} P^\mu) |\chi_j(s_j, \tau)\rangle. \tag{217}
\]

Now ignore the \( z_{\mu j}, \mu > 0 \), and assume \( z_{0 j} > 0 \). Then \( |\psi_j(s_j, \tau)\rangle \) will be \( |\psi_j(s_j, \tau)\rangle \) developed forward to \( \tau + z_0 \) and thus potentially subject to additional branching. According to Sect. 5.2, each factor of a tensor product of states branches independently, and therefore even if more than one \( |\psi_j(s_j, \tau)\rangle \) differs from the vacuum, the branches of \( |\psi_j(s_j, \tau + z_0)\rangle \) will be of the form \( \mu(s'_j)|\psi_j(s'_j, \tau + z_0)\rangle \) for \( s'_j \geq s_j \) and some set of nonnegative real \( \mu(s'_j) \). We therefore have

\[
|\chi_j(s_j, \tau)\rangle = \sum_{s'_j \geq s_j} \mu(s'_j)|\psi_j(s'_j, \tau + z_0)\rangle, \tag{218a}
\]

\[
= \exp(-i z_{0 j} P^\mu)|\psi_j(s_j, \tau)\rangle, \tag{218b}
\]

so that by Eq. (202) for \( |\chi_j(s_j, \tau)\rangle \)

\[
|\chi_j(s_j, \tau)\rangle = \exp(-i z_{0 j} P^\mu)|\chi_j(s_j, \tau)\rangle. \tag{219}
\]

Combining Eqs. (217) and (219) gives
\[ |\chi_j(s_j, \tau)\rangle = \exp(-iz_{\mu j} P^\mu) |\chi_j(s_j, \tau)\rangle, \]  
(220)

for \( z_{0j} > 0, s_j \in S_j(\tau) \).

On the other hand, for \( z_{0j} < 0 \), the derivation of Eq. (219) can be repeated but with the roles of \(|\psi_j(\tau)\rangle\) and \(|\psi(\tau)\rangle\) reversed, so that

\[ |\chi_j(s_{ij}, \tau)\rangle = \exp(i z_{\mu j} P^\mu) |\chi_j(s_{ij}, \tau)\rangle, \]  
(221)

and therefore

\[ |\chi_j(s_{ij}, \tau)\rangle = \exp(-i z_{\mu j} P^\mu) |\chi_j(s_{ij}, \tau)\rangle, \]  
(222)

for \( s_{ij} \in S_{ij}(\tau) \).

The argument for Eq. (218a) implies for both positive and negative \( z_{0j} \)

\[ S_{ij}(\tau) = S_j(\tau + z_{0j}). \]  
(223)

Thus the sets of possible \( s_{ij} \) for Eq. (220) and for Eq. (222) differ. This difference can be removed by choosing some third \( \tau' < \tau - |z_{0j}| \) and then using Eq. (201) to obtain Eqs. (220) and (222) both for \( s_{ij} \in S_{ij}(\tau') \).

If the \( W_j \) are all sufficiently large, which is possible for \( \tau \) sufficiently large, if all of the \(|\psi_j(s_j, \tau)\rangle\) and \(|\psi_j(s_j, \tau)\rangle\) differ from the vaccum, Eqs. (220) and (222) should still apply to each independently. We then have for an arbitrary \(|\psi\rangle\) and displaced \(|\exp(-i z_\mu P^\mu)|\psi\rangle\) the summed branches

\[ |\chi(s, \tau)\rangle \approx \hat{\lambda}(s) \bigotimes_j |\chi_j(s_j, \tau)\rangle, \]  
(224a)

\[ |\chi'(s, \tau)\rangle \approx \hat{\lambda}(s) \bigotimes_j |\chi'_j(s_j, \tau)\rangle, \]  
(224b)

remain related by Eqs. (220) and (222). The derivation of Eqs. (215a)–(216c) then implies

\[ U(\tau) |\chi'(s, \tau)\rangle \approx \exp(-iz_\mu P^\mu) U(\tau) |\chi(s, \tau)\rangle, \]  
(225)

and therefore

\[ |\dot{\chi}(s, \tau)\rangle \approx \exp(-iz_\mu P^\mu) |\dot{\chi}(s, \tau)\rangle. \]  
(226)

If the limit in Eq. (206) exists as assumed, the \( \tau \rightarrow \infty \) limit of Eq. (226) is then Eq. (208).

23 Born Rule As an Invariant Measure on Branching Histories

To begin, assume a particular Poincaré frame, \( f \). Consider an infinite sequence \( s_i \) such that

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s_i \in \cup_t S_t(\tau), \quad (227a)

|s_i| = i, \quad (227b)

s_i \subset s_{i+1}. \quad (227c)

A version of the Born rule based on asymptotic late time branches says the probability a state with history which begins as $s_i$ at the next branching event lands in $s_{i+1}$ is

$$P(s_{i+1}|s_i) = \frac{\langle \hat{\chi}(s_{i+1})|\hat{\chi}(s_i) \rangle}{\langle \hat{\chi}(s_{i})|\hat{\chi}(s_i) \rangle}. \quad (228)$$

The Born rule we now formulate as a measure on the set of branching histories, each extending over all time, beginning from some initial state $|\psi\rangle$. An all-time branching history $\hat{s}$ is an infinite set of pairs which assigns each event $e \in E$ to a corresponding branch index $i \in \{0, 1\}$. 

$$\hat{s} = \{(e_0, i_0), (e_1, i_1), \ldots\}. \quad (229)$$

Let $\hat{S}$ be the set of all such all-time histories $\hat{s}$. For every $s \in S$, let $v(s) \subset \hat{S}$ be the collection of $\hat{s} \in \hat{S}$ which are supersets of $s$,

$$v(s) = \{\hat{s} \in \hat{S}|\hat{s} \supset s\}. \quad (230)$$

For every such $v(s)$ define the function $\mu[v(s)]$ to be

$$\mu[v(s)] = \langle \hat{\chi}(s)|\hat{\chi}(s) \rangle. \quad (231)$$

Let $\Sigma$ be the $\sigma$-algebra of sets in $\hat{S}$ generated by all $v(s)$ for $s \in S$. The complement of any $v(s)$ is given by the finite union

$$v(s)^c = \cup_{s' \in c(s)} v(s'), \quad (232)$$

where $c(s)$ is the set of $s'$ each consisting of exactly one of the events in $s$ but with branch index reversed

$$c[\{(e_0, i_0), \ldots, (e_{n-1}, i_{n-1})\}] = \left\{\{(e_0, \neg i_0), \ldots, (e_{n-1}, \neg i_{n-1})\}\right\}. \quad (233)$$

In addition, for any $s, s' \in S$,

$$v(s) \cap v(s') = v(s \cup s'). \quad (234)$$

It follows that every element of $\Sigma$ is given by a union of a countable collection of pairwise disjoint $v(s)$. For every countable collection of pairwise disjoint sets $\{v(s_i)\}$, define
\[ \mu[\bigcup_i v(s_i)] = \sum_i \mu[v(s_i)]. \] (235)

Equation (235) turns \( \mu \) into a probability measure on \( \Sigma \).

Equation (228) follows from Eq. (231). Since the \( |\hat{\chi}(s)\rangle \) are Poincaré covariant and the algebra \( \Sigma \) is frame independent, the measure \( \mu \) is Poincaré invariant. The Born rule can then be formulated as the hypothesis that world’s history of branching events is an \( \hat{s} \in \hat{S} \) chosen randomly according to the measure \( \mu \).

### 24 Time Dependent View of Branching History

The Poincaré covariant set of \( \tau \to \infty \) branches \( |\hat{\chi}(s)\rangle \) and corresponding branching history \( \hat{s} \) chosen according to the Born measure of Sect. 23 we take to be the physical objects underlying macroscopic reality. From these, a view of branching history unfolding in time in any particular Poincaré frame can be constructed.

In any particular frame, for any all-time history of events \( \hat{s} \), there is a corresponding sequence of partial branch histories \( s_n \in S, n \geq 1 \), with

\begin{align*}
|s_n| &= n, \quad \text{(236a)} \\
 s_n &\subset s_{n+1}, \quad \text{(236b)} \\
 \bigcup_n s_n &= \hat{s}, \quad \text{(236c)}
\end{align*}

ordered in such a way that for every \( n \) the last event in \( s_n \) occurs after the last event in \( s_{n-1} \). Let \( |\hat{\chi}(s_n)\rangle \) be the corresponding sequence of states represented on the \( x^0 = 0 \) hyperplane. From these define \( |\psi_n(\tau)\rangle \) to be

\[ |\psi_n(\tau)\rangle = U^\dagger(\tau) |\hat{\chi}(s_n)\rangle, \] (237)

where \( U^\dagger(\tau) \) is the unitary operator taking states represented on the \( x^0 = 0 \) hyperplane to their representation on the \( L(\tau) \) hyperboloid. The system begins at \( \tau_0 \) evolution from the initial state \( |\psi\rangle \)

\[ |\psi(\tau_0)\rangle = |\psi\rangle, \] (238)

then at a sequence of proper times \( \tau_n, n \geq 1 \), successively branches from \( |\psi_n(\tau_n)\rangle \) to \( |\psi_{n+1}(\tau_n)\rangle \).

The \( \tau_n \) can be found as follows. Define \( |\psi_n(\tau)\rangle \) and \( \rho_n \) to be

\begin{align*}
|\phi_n(\tau)\rangle &= |\psi_n(\tau)\rangle - |\psi_{n+1}(\tau)\rangle \quad \text{(239a)} \\
\rho_n &= \frac{\langle \psi_{n+1}(\tau) | \psi_{n+1}(\tau) \rangle}{\langle \psi_n(\tau) | \psi_n(\tau) \rangle}. \quad \text{(239b)}
\end{align*}

From these define
Each $\tau_n$ will then be the smallest $\tau$ such that
\[ \Delta_n(\tau) \geq 0. \] (241)

By choice of the $s_n$, the sequence of $\tau_n$ is guaranteed to be increasing.

### 25 Conclusion

In Sect. 2 we argued that the branching which follows from environmentally induced decoherence by itself looks like it’s missing something. The present article consists of a series of conjectures which propose to fill in what’s missing.

What are the odds these various guesses might be right? With the exception of the experiments proposed in Sect. 11, the various conjectures can all, at least in principle, be tested by numerical experiments. Among the hypotheses which could be checked numerically are the proposal in Sect. 6 that branching is almost always a permanent split of a single branch into a pair of sub-branches, the proposal in Sect. 10 that branches on the large scale are nearly tensor products each factor of which is entangled only over limited distance, and the conjecture in Sect. 21 that the infinite proper time limit exists for the $|\hat{\chi}(s, \tau)\rangle$. On the other hand, the complexities needed for the experiments in Sect. 11 might also, at least in principle, be filled in numerically leading to a realizable attempt to estimate $b$.

The non-relativistic version of the proposal here is something like the Von Neumann-Wigner interpretation [18–20] turned on its head. Instead of conscious observation causing branching events, branching events occur with or without an observer but those which include a sufficient set of an observer’s degrees of freedom register as a transition in thought. A somewhat related possibility is that distinct mental states might in all cases be associated with distinct branches because the complexity arising from the superposition of distinct mental states is itself sufficient to cause branching. The sequence of mental transitions associated with a trajectory of thought might then be a sequence of successive branching events.

### Appendix 1: Truncated Hermitian Operator Hilbert Space

Let $\mathcal{H}_x^n$ be the subspace of $\mathcal{H}_x$ with less than $n$ bosons. The dimension $d_n$ of each $\mathcal{H}_x^n$ is finite. Let $\mathcal{H}^n$ be the product over $x$ of all $\mathcal{H}_x^n$

\[ \mathcal{H}^n = \bigotimes_x \mathcal{H}_x^n. \] (A1)

For any site $x$, let $\mathcal{F}_x^n$ consist of all Hermitian $f_x$ on $\mathcal{H}_x^n$ with finite
\[ \| f_x \|^2 = \text{Tr}_x(f_x)^2, \quad (A2) \]

and vanishing trace
\[ \text{Tr}_x f_x = 0. \quad (A3) \]

For any pair of nearest neighbor sites \( \{x, y\} \), let \( \mathcal{F}_{xy} \) consist of all Hermitian \( f_{xy} \) on \( \mathcal{H}_x \otimes \mathcal{H}_y \) with finite
\[ \| f_{xy} \|^2 = \text{Tr}_{xy}(f_{xy})^2, \quad (A4) \]

and vanishing traces
\[ \text{Tr}_x f_{xy} = 0, \quad (A5a) \]
\[ \text{Tr}_y f_{xy} = 0. \quad (A5b) \]

Inner products on \( \mathcal{F}_x \) and \( \mathcal{F}_{xy} \) are
\[ \langle f_x, f'_x \rangle = \text{Tr}_x(f_x f'_x), \quad (A6a) \]
\[ \langle f_{xy}, f'_{xy} \rangle = \text{Tr}_{xy}(f_{xy} f'_{xy}). \quad (A6b) \]

Operators \( f_x \in \mathcal{F}_x \) and \( f_{xy} \in \mathcal{F}_{xy} \) can be made into operators on \( \mathcal{H}^n \) by
\[ \hat{f}_x = f_x \bigotimes_{q \neq x} I_q, \quad (A7a) \]
\[ \hat{f}_{xy} = f_{xy} \bigotimes_{q \neq x, y} I_q, \quad (A7b) \]

where \( I_q \) is the identity operator on \( \mathcal{H}_q \). As usual, we now drop the hat and use the same symbol for operators on \( \mathcal{H}_x, \mathcal{H}_y \otimes \mathcal{H}_y \), and the corresponding operators on \( \mathcal{H}^n \).

Let \( \mathcal{K}^n \) be the vector space over the reals of linear operators \( k \) on \( \mathcal{H}_x \) given by sums of the form
\[ k = \sum_{xy} f_{xy} + \frac{1}{\sqrt{d_n}} \sum_x f_x \quad (A8) \]

for any collection of \( f_{xy} \in \mathcal{F}_{xy} \) for a set of nearest neighbor pairs \( \{x, y\} \) and any collection of \( f_x \in \mathcal{F}_x \) in a set of sites \( x \). The inner product on \( \mathcal{K}^n \) is
\[ \langle k, k' \rangle = \sum_{xy} \langle f_{xy}, f'_{xy} \rangle + \sum_x \langle f_x, f'_x \rangle. \quad (A9) \]

An equivalent inner product on \( \mathcal{K}^n \), which is a version of the inner product on operator Hilbert space in [10], is
\[ \langle k, k' \rangle = \frac{\text{Tr}(kk')}{d_n^{n_L - 2}}, \quad (A10) \]

where \( \text{Tr} \) is the trace on all of \( \mathcal{H}^{\text{det}} \) and \( n_L \) is the number of sites in the lattice \( L \). As a result of the factor of \( \frac{1}{\sqrt{d_n}} \) in Eq. (A8), if \( d_n \) is made large, matrix elements of \( k \) given by Eq. (A8) will approach those of \( k \) given by Eq. (15) and \( K^\alpha \) will become equivalent to the operator space \( K \) of Sect. 3.2.

### Appendix 2: Lower Bound on the Complexity of Entangled States

The proof of Eq. (37) proceeds as follows. The trajectories \( k(\nu) \in K \) and \( U_k(\nu) \) which determine any \( C(\ket{\psi}, \ket{\omega}) \), according to Eqs. (17a)–(20), we characterize by a corresponding set of trajectories of Schmidt spectrum vectors. We then find the rotation matrices which govern the motion of these vectors as \( \nu \) changes. A bound on the time integral of the angles which occur in these matrices by a time integral of \( \| k(\nu) \| \) yields Eq. (37).

#### 2.1 Schmidt Spectra

Consider some entangled \( n \)-fermion \( \ket{\psi} \) of form Eq. (36). For a trajectory \( k(\nu) \in K \), let \( U_k(\nu) \) be the solution to Eqs. (17a) and (17b). Define \( \ket{\omega(\nu)} \) to be

\[ \ket{\omega(\nu)} = U_k(\nu)\ket{\omega}, \quad (B1) \]

for some product state \( \ket{\omega} \) and assume that \( k(\nu) \) has been chosen to give

\[ \ket{\omega(1)} = \xi \ket{\psi}, \quad (B2) \]

for a phase factor \( \xi \). Since all \( k(\nu) \) conserve fermion number, \( \ket{\omega} \) according to Eq. (9) must have the form

\[ \ket{\omega} = d^\dagger_f(p_{n-1}) \ldots d^\dagger_f(p_0) \times d^\dagger_b(q_{m-1}) \ldots d^\dagger_b(q_0) \ket{\Omega}, \quad (B3) \]

for some number of bosons \( m \).

We now divide the lattice \( L \) into a collection of disjoint regions and define a corresponding collection of Schmidt decompositions of the trajectory of states which determine any \( C(\ket{\psi}, \ket{\omega}) \). Divide \( L \) into subsets \( L^e, L^o \), with, respectively, even or odd values of the sums of components \( \hat{x}_i \). The sites in each subset have nearest neighbors only in the other. Let \( D^e_{ij}, D^o_{ij}, D^e, D^o \) be

\[ D^e_{ij} = L^e \cap D_{ij}, \quad (B4a) \]

\[ D^o_{ij} = L^o \cap D_{ij}, \quad (B4b) \]
\[ D^e = \bigcup_{ij} D^e_{ij}, \quad \text{(B4c)} \]
\[ D^o = \bigcup_{ij} D^o_{ij}, \quad \text{(B4d)} \]

Between \( D^e \) and \( D^o \) choose the larger, or either if they are equal. Assume the set chosen is \( D^e \). Among the \( nm \) spins \( s_{ij} \), at least \( \frac{nm}{2} \) will have the same value and therefore correspond to \( D_{ij} \) which do not intersect. The corresponding collection of \( D^e_{ij} \) will then include at least \( \frac{nmV}{4} \) points.

From this set of \( D^e_{ij} \) construct a set of subsets \( E_\ell \) each consisting of \( 2n \) distinct points chosen from \( 2n \) distinct \( D^e_{ij} \). The total number of \( E_\ell \) will then be at least \( \frac{mV}{8} \). We will consider only the first \( \frac{mV}{8} \) of these.

The Hilbert space \( \mathcal{H} \) is given by a tensor product
\[ \mathcal{H} = \mathcal{H}_f \otimes \mathcal{H}_b, \quad \text{(B5)} \]
of a fermion space \( \mathcal{H}_f \) and a boson space \( \mathcal{H}_b \). Similarly the space \( \mathcal{H}_x \) at each \( x \) is given by a tensor product
\[ \mathcal{H}_x = \mathcal{H}_x^f \otimes \mathcal{H}_x^b, \quad \text{(B6)} \]
of a fermion space \( \mathcal{H}_x^f \) and a boson space \( \mathcal{H}_x^b \). The dimensions of \( \mathcal{H}_x^f \) and \( \mathcal{H}_x^b \) are, respectively, 4 and \( \infty \).

For each set \( E_\ell \) form the tensor product spaces
\[ Q_\ell = \bigotimes_{x \in E_\ell} \mathcal{H}_x^f, \quad \text{(B7a)} \]
\[ R_\ell = \bigotimes_{q \not\in E_\ell} \mathcal{H}_q^f \quad \text{(B7b)} \]

It follows that \( Q_\ell \) has dimension \( 4^{2n} \) and
\[ \mathcal{H} = Q_\ell \otimes R_\ell. \quad \text{(B8)} \]

A Schmidt decomposition of \( |\omega(v)\rangle \) according to Eq. (B8) then becomes
\[ |\omega(v)\rangle = \sum_j \lambda_{j\ell}(v)|\phi_{j\ell}(v)\rangle|\chi_{j\ell}(v)\rangle, \quad \text{(B9)} \]
where
\[ |\phi_{j\ell}(v)\rangle \in Q_\ell \quad \text{(B10a)} \]
\[ |\chi_{j\ell}(v)\rangle \in R_\ell, \quad \text{(B10b)} \]
for \( 0 \leq j < 4^{2n} \) and real non-negative \( \lambda_{j\ell}(v) \) which fulfill the normalization condition.
\[ \sum_{j} |\lambda_{je}(\nu)|^2 = 1. \]  

(B11)

Each \(|\phi_{je}(\nu)\rangle\) is a pure fermion state while the \(|\chi_{je}(\nu)\rangle\) can include both fermions and bosons.

The fermion number operators \(N[Q_\ell]\) and \(N[R_\ell]\) commute and \(|\omega(\nu)\rangle\) is an eigenvector of the sum with eigenvalue \(n\). It follows that the decomposition of Eq. (B9) can be done with \(|\phi_{je}(\nu)\rangle\) and \(|\chi_{je}(\nu)\rangle\) eigenvectors of \(N[Q_\ell]\) and \(N[R_\ell]\), respectively, with eigenvalues summing to \(n\). Let \(|\phi_{0e}\rangle\) be \(|\Omega_\ell\rangle\), the vacuum state of \(Q_\ell\), and let \(|\phi_{ie}(\nu)\rangle, 1 \leq i \leq 4n,\) span the \(4n\)-dimensional subspace of \(Q_\ell\) with \(N[Q_\ell]\) of \(1\). We assume the corresponding \(\lambda_{ie}(\nu), 1 \leq i \leq 4n,\) are in nonincreasing order. Consider Eq. (B9) for \(\nu = 1\). By Eq. (B2), for any choice of \(\ell\) there will be a set of \(2n\) nonzero orthogonal \(|\phi_{1\ell}(1)\rangle, \ldots, |\phi_{2n\ell}(1)\rangle\) with

\[ \lambda_{je}(1) = \sqrt{\frac{1}{mV}}, \]  

(B12)

for \(1 \leq j \leq 2n\).

On the other hand, for \(\nu = 0\), Eq. (B9) becomes a decomposition of the product state \(|\omega(0)\rangle\). The boson part of \(|\omega(0)\rangle\) will occur as the same overall tensor factor in each \(|\chi_{1\ell}(0)\rangle, \ldots, |\chi_{n\ell}(0)\rangle\). The fermion part of \(|\omega(0)\rangle\) is a product of \(n\) independent single fermion states, the space spanned by the projection of these into some \(Q_\ell\) is at most \(n\) dimensional, and as a result at most \(n\) orthogonal \(|\phi_{1\ell}(0)\rangle, \ldots, |\phi_{n\ell}(0)\rangle\) can occur. Therefore at \(\nu = 0\), there will be at most \(n\) nonzero \(\lambda_{1\ell}(0), \ldots, \lambda_{n\ell}(0)\). For \(n < j \leq 2n\), we have

\[ \lambda_{je}(0) = 0. \]  

(B13)

But according to Eq. (B11), for each fixed value of \(\ell\) the set of components \(\{\lambda_{je}(\nu)\}\) indexed by \(j\) is a unit vector. Eqs. (B13) and (B12) then imply that as \(\nu\) goes from 0 to 1, \(\{\lambda_{je}(\nu)\}\) must rotate through a total angle of at least \(\arcsin(\sqrt{n/mV})\).

For the small interval from \(\nu\) to \(\nu + \delta\nu\) let \(\mu_{je}(\nu)\) and \(\theta_{\ell}(\nu)\) be

\[ \lambda_{je}(\nu + \delta\nu) = \lambda_{je}(\nu) + \delta\nu \mu_{je}(\nu), \]  

\[ \theta_{\ell}(\nu)^2 = \sum_{j} |\mu_{je}(\nu)|^2. \]  

(B14a)

(B14b)

We then have

\[ \int_{0}^{1} |\theta_{\ell}(\nu)| d\nu \geq \arcsin \left( \sqrt{\frac{n}{mV}} \right). \]  

(B15)

Summed over the \(\frac{mV}{8}\) values of \(\ell\), Eq. (B15) becomes
\[ \sum_{\ell} \int_{0}^{1} |\theta_{\ell}(v)| dv \geq \frac{mV}{8} \arcsin \left( \sqrt{\frac{n}{mV}} \right), \quad (B16) \]

and therefore

\[ \sum_{\ell} \int_{0}^{1} |\theta_{\ell}(v)| dv \geq \frac{1}{4\pi} \sqrt{mnV}. \quad (B17) \]

### 2.2 More Schmidt Spectra

Replacing the subsets \(E_\ell\) defined in Appendix B1, with subsets of \(L\) obtained from the \(S_\ell\) of Sect. 4 leads to an additional bound similar to Eq. (B17).

For each \(0 \leq \ell < q\), of the two subsets of \(L\) defined by \(S_\ell\), let \(T_\ell\) be the subset which, for each \(0 \leq i < m\), holds \(n_0\) of the sets \(D_{ij}, 0 \leq j < n\). Redefine \(Q_\ell, R_\ell\) of Eqs. (B7a) and (B7b), to be

\[ Q_\ell^T = \bigotimes_{x \in T_\ell} \mathcal{H}_x^l, \quad (B18a) \]

\[ R_\ell^T = \mathcal{H}_q^b \bigotimes_{q \neq T_\ell} \mathcal{H}_q^l. \quad (B18b) \]

For each \(0 \leq \ell < q\) there is again a corresponding Schmidt decomposition of \(|\omega(v)\rangle\) of Eqs. (B1) and (B2)

\[ |\omega(v)\rangle = \sum_j \lambda_{j\ell}^T(v)|\phi_{j\ell}^T(v)\rangle|\chi_{j\ell}^T(v)\rangle, \quad (B19) \]

where

\[ |\phi_{j\ell}^T(v)\rangle \in Q_\ell^T, \quad (B20a) \]

\[ |\chi_{j\ell}^T(v)\rangle \in R_\ell^T. \quad (B20b) \]

Each \(|\phi_{j\ell}^T(v)\rangle\) is a pure fermion state while the \(|\chi_{j\ell}^T(v)\rangle\) can include both fermions and bosons. For \(v = 1\), for every \(0 \leq \ell < q\), the sum over \(j\) in Eq. (B19) has \(m\) nonzero entries each with

\[ \lambda_{j\ell}^T(1) = \frac{1}{\sqrt{m}}, \quad (B21) \]

with \(|\phi_{j\ell}^T(1)\rangle\) carrying fermion number \(n_0\) and \(|\chi_{j\ell}^T(1)\rangle\) carrying fermion number \(n_1\).

Duplicating the discussion of Appendix B1, a trajectory of angles \(\theta_{\ell}^T(v)\) can be defined which rotates the unit vector \([\lambda_{j\ell}^T(0)]\) arising from the product state \(|\omega(0)\rangle\) into the unit vector \([\lambda_{j\ell}^T(1)]\) of Eq. (B21). For each \(0 \leq \ell < q\), a version of the lower bound

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of Eq. (B15) can be obtained by finding the product state $|\omega(0)\rangle$ which gives $[\hat{\lambda}^T_{j\epsilon}(0)]$ closest to $[\hat{\lambda}^T_{j\epsilon}(1)]$ for the set of $0 \leq j < m$ corresponding to $|\phi^T_{j\epsilon}(0)\rangle$ and $|\chi^T_{j\epsilon}(0)\rangle$ with fermion numbers $n_0$ and $n_1$, respectively.

According to Eq. (9), the product state $|\omega(0)\rangle$ includes $n$ fermion creation operators $d_{j\epsilon}^\dagger(p_j)$ given by Eq. (8a). Since $|\omega(1)\rangle$ and therefore $|\omega(0)\rangle$ are normalized to 1, we can require the $p_j(x,s)$ to be orthonormal. The simplest way to insure $n_0$ and $n_1$, respectively, for $|\phi^T_{0\epsilon}(0)\rangle$ and $|\chi^T_{0\epsilon}(0)\rangle$ is for the support of $p_j(x,s)$ to be entirely within $T_\epsilon$ for $0 \leq i < n_0$ and entirely outside $T_\epsilon$ for $n_0 \leq i < n$. The Schmidt decomposition of Eq. (B19) then yields a vector $[\hat{\lambda}^T_{j\epsilon}(0)]$ with only a single nonzero entry and therefore

$$\sum_j \hat{\lambda}^T_{j\epsilon}(0)\hat{\lambda}^T_{j\epsilon}(1) = \frac{1}{\sqrt{m}}.$$  \hfill (B22)

A larger value of the sum in Eq. (B22) is possible only if an even number of $p_j(x,s)$ have support both within $T_\epsilon$ and outside $T_\epsilon$. For some $r \leq n_0, n_1$, define $z$ to be the set

$$z = \{i|0 \leq i < r\} \cup \{i|n_0 \leq i < n_0 + r\}.$$  \hfill (B23)

Then for $i \in z$, suppose

$$p_i(x,s) = p_i^0(x,s) + p_i^1(x,s),$$  \hfill (B24)

where the $p_i^0(x,s)$ have support entirely within $T_\epsilon$ and the $p_i^1(x,s)$ have support entirely outside $T_\epsilon$. Since $p_i(x,s)$ is normalized and the support of $p_i^0(x,s)$ is disjoint from the support of $p_i^1(x,s)$, we have

$$\parallel p_i^0 \parallel^2 + \parallel p_i^1 \parallel^2 = 1.$$  \hfill (B25)

The piece $|\tilde{\omega}(0)\rangle$ of $|\omega(0)\rangle$ with fermion number $n_0$ on $T_\epsilon$ and $n_1$ outside $T_\epsilon$ is given by

$$|\tilde{\omega}(0)\rangle = \sum_u \left( \bigotimes_{i \in u} |p_i^0\rangle \bigotimes_{j \in z - u} |p_j^1\rangle \right),$$  \hfill (B26)

where the sum is over all $r$ element subsets $u \subset z$.

The vector $[\hat{\lambda}^T_{j\epsilon}(0)]$ corresponding to $|\tilde{\omega}(0)\rangle$ will have at most $\frac{(2r)!}{(r)!^2}$ nonzero entries, one for each of the sets $u$ in the sum in Eq. (B26). The Cauchy-Schwarz inequality then yields

$$\sum_j \hat{\lambda}^T_{j\epsilon}(0)\hat{\lambda}^T_{j\epsilon}(1) \leq \sqrt{\frac{(2r)!}{m(r)!^2}} \parallel |\tilde{\omega}(0)\rangle \parallel.$$  \hfill (B27)

Let $|\hat{p}_j^0\rangle$ be the projection of $|p_j^0\rangle$ orthogonal to all $|\hat{p}_j^0\rangle, j < i$, and let $|\hat{p}_i^1\rangle$ be the projection of $|p_i^1\rangle$ orthogonal to all $|\hat{p}_j^1\rangle, j < i$. Substituting $\{ |\hat{p}_j^0\rangle \}$ and $\{ |\hat{p}_i^1\rangle \}$ for $\{ |p_j^0\rangle \}$
and \(\{|p_i\rangle\}\), respectively, in Eq. (B26) leaves \(|\tilde{\omega}(0)\rangle\) unchanged. The value of \(\| |\tilde{\omega}(0)\rangle\|\) will then be maximized if the resulting \(\| |\tilde{\rho}_i^0\rangle\|\) and \(\| |\tilde{\rho}_i^1\rangle\|\) are increased as needed to satisfy Eq. (B25).

Suppose now that \(\| |\tilde{\omega}(0)\rangle\|\) has been maximized with respect to \(\| |\tilde{\rho}_0^0\rangle\|\) and \(\| |\tilde{\rho}_1^0\rangle\|\) for all \(0 \leq i < r\), except some pair of values \(j, k\). The remaining dependence on \(\| |\tilde{\rho}_0^0\rangle\|\) and \(\| |\tilde{\rho}_1^0\rangle\|\) for \(i = j, k\), is maximized at

\[
\| |\tilde{\rho}_j^0\rangle\| = \| |\tilde{\rho}_k^0\rangle\|, \quad (B28a)
\]

\[
\| |\tilde{\rho}_j^1\rangle\| = \| |\tilde{\rho}_k^1\rangle\|, \quad (B28b)
\]

If \(\| |\tilde{\omega}(0)\rangle\|\) is then maximized with respect to the remaining \(i\) independent \(\| |\tilde{\rho}_0^0\rangle\|\) and \(\| |\tilde{\rho}_1^0\rangle\|\), Eq. (B27) becomes

\[
\sum_j \lambda_{je}^T(0)\lambda_{je}^T(1) \leq \sqrt{\frac{(2r)!}{m2^r(r!)^2}}, \quad (B29)
\]

Suppose \(m\) has the form \(\frac{(2r)!}{(r!)^2}\). For any \(r' \leq r\) Eq. (B29) becomes

\[
\sum_j \lambda_{je}^T(0)\lambda_{je}^T(1) \leq \frac{(2r')!}{\sqrt{m2^r(r')^2}}. \quad (B30)
\]

An induction argument then shows that Eq. (B30) is an increasing function of \(r'\). For \(r' > r\) on the other hand, the sum in Eq. (B30) becomes

\[
\sum_j \lambda_{je}^T(0)\lambda_{je}^T(1) \leq \frac{\sqrt{m}}{2^{r'}}, \quad (B31)
\]

which is a decreasing function of \(r'\). The maximum of Eq. (B30) will therefore be at \(r' = r\).

Now suppose \(m\) lies between \(\frac{(2r)!}{(r!)^2}\) and \(\frac{(2r+2)!}{(r+1)!^2}\). For \(r' \leq r\) the maximum Eq. (B30) will still be at \(r' = r\) and given by

\[
\sum_j \lambda_{je}^T(0)\lambda_{je}^T(1) \leq \frac{(2r)!}{\sqrt{m2^r(r!)^2}}, \quad (B32a)
\]

\[
< \sqrt{\frac{(2r)!}{2^{2r}(r!)^2}}, \quad (B32b)
\]

For \(r' = r + 1\), Eq. (B30) becomes

\[
\sum_j \lambda_{je}^T(0)\lambda_{je}^T(1) \leq \frac{\sqrt{m}}{2^{r+1}}, \quad (B33a)
\]
\begin{equation}
\sqrt{\frac{(2r + 2)!}{2^{2r+2}[(r+1)!]^2}}.
\tag{B33b}
\end{equation}

A further induction argument shows that Eqs. (B32b) and (B33b) are decreasing functions of \( r \). Thus for \( m \geq 2 \), we have

\begin{equation}
\sum_j \lambda^T_{j'}(0) \lambda^T_{j'}(1) \leq \frac{1}{\sqrt{2}}.
\tag{B34}
\end{equation}

A duplicate of the argument leading to Eq. (B17) then yields

\begin{equation}
\sum_{j} \int_{0}^{1} |\theta^T_{j'}(\nu)| d\nu \geq \frac{\pi q}{4}.
\tag{B35}
\end{equation}

\subsection*{2.3 Schmidt Rotation Matrix}

A lower bound on \( C(\ket{\psi}) \) follows from Eqs. (B17) and (B35). Appendices 2.3 and 2.4 derive the consequence of Eq. (B17). A derivation of the additional terms in the bound on \( C(\ket{\psi}) \) which follow from Eq. (B35) is briefly summarized in Appendix 2.5.

The rotation of \( \lambda_{j'}(\nu) \) during the interval from \( \nu \) to \( \nu + \delta \nu \) will be determined by \( k(\nu) \). For each \( f_{xy} \) in Eq. (15) for \( k(\nu) \) which can contribute to a nonzero value of \( \theta_{j'}(\nu) \), the nearest neighbor pair \{\( x, y \)\} has one point, say \( x \) in \( E_\nu \). Since \( E_\nu \subset D^\nu \) and the nearest neighbors of all points in \( D^\nu \) are in \( D^\nu \), \( y \) can not be in \( E_\nu \). Let \( g_{j'}(\nu) \) be the sum of all such \( f_{xy} \). The effect of all other terms in Eq. (15) on the Schmidt decomposition of Eq. (B9) will be a unitary transformation on \( R_\nu \) and identity on \( Q_\nu \). All other terms will therefore leave \( \lambda_{j'}(\nu) \) unchanged.

The effect of \( g_{j'}(\nu) \) on \( \lambda_{j'}(\nu) \) over the interval from \( \nu \) to \( \nu + \delta \nu \) can be determined from the simplification

\begin{equation}
\ket{\omega(\nu + \delta \nu)} = \exp[i\delta v g_{j'}(\nu)] \ket{\omega(\nu)}.
\tag{B36}
\end{equation}

From \([\omega(\nu + \delta \nu)\langle\omega(\nu + \delta \nu)]\) of Eq. (B36), construct the density matrix \( \rho(\nu + \delta \nu) \) by a partial trace over \( R_\nu \), using the basis for \( R_\nu \) from the Schmidt decomposition in Eq. (B9) of \([\omega(\nu)]\) at \( t \)

\begin{equation}
\rho(\nu + \delta \nu) = \sum_j [\langle \chi_j(\nu) | \omega(\nu + \delta \nu) \rangle \times \langle \omega(\nu + \delta \nu) | \chi_j(\nu) \rangle].
\tag{B37}
\end{equation}

An eigenvector decomposition of \( \rho(\nu + \delta \nu) \) exposes the \( \lambda_{j'}(\nu + \delta \nu) \)

\begin{equation}
\rho(\nu + \delta \nu) = \sum_j [\lambda_{j'}(\nu + \delta \nu)^2 \times |\phi_{j'}(\nu + \delta \nu)\rangle \langle \phi_{j'}(\nu + \delta \nu) |].
\tag{B38}
\end{equation}

A power series expansion through first order in \( \delta \nu \) applied to Eqs. (B36), (B37) and (B38) then gives for \( \mu_{j'}(\nu) \) of Eq. (B14a)
\[\mu_{j\epsilon}(v) = \sum_k r_{jk\epsilon}(v) \lambda_{k\epsilon}(v),\] (B39)

for the rotation matrix \(r_{jk\epsilon}(v)\)

\[r_{jk\epsilon}(v) = -\text{Im}[\langle \phi_{j\epsilon}(v) | \langle \chi_{j\epsilon}(v) | g_{\epsilon}(v) | \phi_{k\epsilon}(v) \rangle | \chi_{k\epsilon}(v) \rangle].\] (B40)

### 2.4 Rotation Angle Bounds

Since the \(f_{xy}\) contributing to \(g_{\epsilon}(v)\) conserve total fermion number \(N\), \(g_{\epsilon}(v)\) can be expanded as

\[g_{\epsilon}(v) = \sum_{xy} g_{\epsilon}(x, y, v),\] (B41a)

\[g_{\epsilon}(x, y, v) = \sum_{i=0,1} a^i(x, y, v) z^i(x, y, v)\] (B41b)

where \(z^0(x, y, v)\) acts only on states with \(N(H_{x} \otimes H_{y}) = 0\), \(z^1(x, y, v)\) acts only on states with \(N(H_{x} \otimes H_{y})\) strictly greater than 0, and the \(z^i(x, y, v)\) are normalized by

\[\| z^i(x, y, v) \| = 1.\] (B42)

The operator \(z^0(x, y, v)\) will be

\[z^0(x, y, v) = z_{0f}(x, y) \otimes g^b(x, y, v),\] (B43a)

\[z_{0f}(x, y, v) = P_f(x, y) \bigotimes_{q \neq x, y} I_q,\] (B43b)

where \(P_f(x, y)\) projects onto the vacuum state of \(H_{x}^f \otimes H_{y}^f\) and \(g^b(x, y, v)\) is a normalized Hermitian operator acting on \(H_{x}^b \otimes H_{y}^b\).

Combining Eqs. (B14b),(B39) - (B41b) gives

\[|\theta_{\epsilon}(v)| \leq \sum_{xyl} |\theta^i_{\epsilon}(x, y, v)|\] (B44a)

\[|\theta^i_{\epsilon}(x, y, v)|^2 = \sum_j [\mu^i_{j\epsilon}(x, y, v)]^2,\] (B44b)

with the definitions

\[\mu^i_{j\epsilon}(x, y, v) = -a^i(x, y, v) \sum_k \text{Im}[\langle \phi_{j\epsilon}(v) | \langle \chi_{j\epsilon}(v) | g_{\epsilon}(v) | \phi_{k\epsilon}(v) \rangle | \chi_{k\epsilon}(v) \rangle \lambda_{k\epsilon}(v)],\] (B45)
Since the $|\phi_{je}(v)\rangle$ are orthonormal, $g^b(x,y,v)$ is Hermitian and the $\lambda_{ke}(v)$ are real, we have

$$\text{Im}\{\langle \phi_{je}(v)|\phi_{ke}(v)\rangle\langle \chi_{je}(v)|g^b(x,y,v)|\chi_{ke}(v)\rangle\lambda_{ke}(v)\} = 0.$$  \hfill (B46)

Equation (B45) for $i = 0$ can then be turned into

$$\mu^0_{je}(x,y,v) = a^0(x,y,v) \sum_k \text{Im}\{\langle \phi_{je}(v)|\langle \chi_{je}(v)|[I - z^0f(x,y)]

\times g^b(x,y,v)|\phi_{ke}(v)\rangle|\chi_{ke}(v)\rangle\lambda_{ke}(v)\}.$$ \hfill (B47)

But in addition

$$|\omega(v)\rangle = \sum_k |\phi_{ke}(v)\rangle|\chi_{ke}(v)\rangle\lambda_{ke}(v).$$ \hfill (B48)

Also $I - z^0f(x,y)$ is a projection operator so that

$$[I - z^0f(x,y)]^2 = I - z^0f(x,y).$$ \hfill (B49)

The normalization condition on $z^0(x,y,v)$ implies $[g^b(x,y,v)]^2$ has trace 1 as an operator on $\mathcal{H}_x \otimes \mathcal{H}_y$ and therefore all eigenvalues bounded by 1. Eqs. (B43a), (B44b), (B47), (B48), and (B49) then give

$$[\theta^0_{je}(x,y,v)]^2 \leq [a^0(x,y,v)]^2 \langle \omega(v)|[I - z^0f(x,y)]|\omega(v)\rangle.$$ \hfill (B50)

For $\mu^1_{je}(x,y,v)$, since $z^1(x,y,v)$ is nonzero only on the subspace with $N(\mathcal{H}_x \otimes \mathcal{H}_y)$ nonzero, we have

$$\mu^1_{je}(x,y,v) = -a^1(x,y,v)\text{Im}\{\langle \phi_{je}(v)|\langle \chi_{je}(v)|

\times z^1(x,y,v)[I - z^0f(x,y)]|\omega(v)\rangle\}.$$ \hfill (B51)

Equations (B44b) and (B51) give

$$[\theta^1_{je}(x,y,v)]^2 \leq [a^1(x,y,v)]^2 \langle \omega(v)|[I - z^0f(x,y)]

\times [z^1(x,y,v)]^2[I - z^0f(x,y)]|\omega(v)\rangle.$$ \hfill (B52)

But by Eq. (B42), $[z^1(x,y,v)]^2$ as an operator on $\mathcal{H}_x \otimes \mathcal{H}_y$, has trace 1 and therefore all eigenvalues bounded by 1. Thus Eq. (B52) implies

$$[\theta^1_{je}(x,y,v)]^2 \leq [a^1(x,y,v)]^2 \langle \omega(v)|[I - z^0f(x,y)]|\omega(v)\rangle.$$ \hfill (B53)

By construction of $D^r$, each nearest neighbor pair $\{x,y\}$ with $x \in D^r$ must have $y \in D^r$. Also any $x \in D^r$ is contained in at most a single $E_f$. As a result Eqs. (B44a), (B50) and (B53) imply
\[
\sum_{\ell} |\theta_{\ell}(v)| \leq \sum_{x \in D^e, y \in D^o} \{ |a^0(x, y, v)| + |a^1(x, y, v)| \} \times \sqrt{\langle \omega(v) | I - z^0_{\ell}(x, y) | \omega(v) \rangle}.
\]  
(B54)

The Cauchy–Schwartz inequality then gives
\[
[\sum_{\ell} |\theta_{\ell}(v)|]^2 \leq \sum_{x \in D^e, y \in D^o} \{ |a^0(x, y, v)| + |a^1(x, y, v)| \}^2 \times \sum_{x \in D^e, y \in D^o} \langle \omega(v) | I - z^0_{\ell}(x, y) | \omega(v) \rangle.
\]  
(B55)

The state \(|\omega(v)\rangle\) can be expanded as a linear combination of orthogonal states each with \(n\) fermions each at a single position. A state with fermions at \(n\) positions will survive the projection \(I - z^0_{\ell}(x, y)\) only if at least one of the fermions is either at \(x\) or \(y\). Each \(x \in D^e\) can be the member of only a single such pair of nearest neighbor \(\{x, y\}\). A \(y \in D^o\) can be in \(6x, y\) pairs for an \(x \in D^e\). Thus a term with \(n\) fermion positions in the expansion of \(|\omega(v)\rangle\) will pass \(I - z^0_{\ell}(x, y)\) for at most \(6n\) pairs of \(x\) and \(y\). Therefore
\[
\sum_{x \in D^e, y \in D^o} \langle \omega(v) | I - z^0_{\ell}(x, y) | \omega(v) \rangle \leq 6n.
\]  
(B56)

By Eq. (16)
\[
\| k(v) \|^2 \geq \sum_{\ell, x \in D^e, y \in D^o} \| g_{\ell}(x, y, v) \|^2
\]  
(B57)

In addition, \(z^0(x, y, v)\) is orthogonal to \(z^1(x, y, v)\). It follows that
\[
\| k(v) \|^2 \geq \sum_{x \in D^e, y \in D^o} \{ |a^0(x, y, v)|^2 + |a^1(x, y, v)|^2 \}.
\]  
(B58)

Assembling Eqs. (B55), (B56) and (B58) gives
\[
\| k(v) \|^2 \geq \frac{1}{2} \sum_{x \in D^e, y \in D^o} \{ |a^0(x, y, v)| + |a^1(x, y, v)| \}^2
\]  
\[
\geq \frac{1}{12n} \sum_{\ell} |\theta_{\ell}(v)|^2
\]  
(B59)

Eq. (B17) then implies
\[
\int_0^1 \| k(v) \| \geq \frac{1}{\pi} \sqrt{\frac{mV}{192}},
\]  
(B60)

and therefore
\[ C(\langle \psi \rangle, \langle \omega \rangle) \geq \frac{1}{\pi} \sqrt{\frac{mV}{192}}. \]  
(B61)

Since Eq. (B61) holds for all product \( \langle \omega \rangle \) we obtain
\[ C(\langle \psi \rangle) \geq \frac{1}{\pi} \sqrt{\frac{mV}{192}}. \]  
(B62)

### 2.5 Additional Terms

The nearest neighbor \( \{x, y\} \) which contribute to each \( \theta^T(\nu) \) in Eq. (B35) are all distinct from the pairs which contribute to \( \theta(\nu) \) in Eq. (B17). A repeat of the steps leading to Eq. (B59) yields\[
\| k(\nu) \| \geq \frac{1}{12n} \left[ \sum_{\ell} |\theta(\nu)| + \sum_{\ell} |\theta^T(\nu)| \right]^2.
\]  
(B63)

Eq. (B62) becomes
\[ C(\langle \psi \rangle) \geq \frac{1}{\pi} \sqrt{\frac{mV}{192}} + \frac{\pi q}{\sqrt{192}}. \]  
(B64)

### Appendix 3: Upper Bound on the Complexity of Entangled States

An upper bound on \( C(\langle \psi \rangle) \) of the \( n \)-particle entangled state of Eq. (36) is given by \( C(\langle \psi \rangle, \langle \omega \rangle) \) for any \( n \)-particle product state \( \langle \omega \rangle \), for which in turn an upper bound is given by
\[ C(\langle \psi \rangle, \langle \omega \rangle) \leq \int_0^1 dt \| k(\nu) \|, \]  
(C1)

for any trajectory \( k(\nu) \in K \) fulfilling Eqs. (B1) and (B2). Beginning with an \( \langle \omega \rangle \) consisting of \( n \) particles each at one of a corresponding set of \( n \) single points, we construct a sufficient \( k(\nu) \) in three stages. First, \( \langle \omega \rangle \) is split into a sum of \( m \) orthogonal product states, each again consisting of \( n \) particles one at each of a corresponding set of \( n \) single points. Then the position of each of the particles in the product states is moved to the center of the wave function of one of the single particle states of Eq. (35). Finally, by approximately \( \ln(V)/\ln(8) \) iterations of a fan-out tree, the \( mn \) wave functions concentrated at points are spread over the \( mn \) cubes \( D_{ij} \).

### 3.1 Product State to Entangled State

Define the set of positions \( x_{ij} \) to be
for \(0 \leq i < m, 0 \leq j < n\) and arbitrary base point \(x_{00}\). Let the set of \(n\)-particle product states \(|\omega_i\rangle\) be

\[
|\omega_i\rangle = \prod_{0 \leq j < n} \Psi^i(x_{ij}, 1)|\Omega\rangle.
\]

(C3)

The entangle \(n\)-particle state \(|\chi\rangle\)

\[
|\chi\rangle = \sqrt{\frac{1}{m}} \sum_i |\omega_i\rangle
\]

(C4)

we generate from a sequence of unitary transforms acting on \(|\omega\rangle = |\omega_0\rangle\).

Let \(k_0\) acting on \(\mathcal{H}_{x_{00}} \otimes \mathcal{H}_{x_{01}}\) have matrix elements

\[
\langle \Omega | \Psi(x_{00}, -1)\Psi(x_{01}, -1)k_0 \Psi^i(x_{00}, 1)\Psi^i(x_{01}, 1)|\Omega\rangle = -i,
\]

(C5)

and extend \(k_0\) to \(\mathcal{H}\) by Eq. (14). We then have

\[
\exp(i\theta_0 k_0)|\omega_0\rangle = \sqrt{\frac{1}{m}} |\omega_0\rangle + \sqrt{\frac{m-1}{m}} \prod_{0 \leq j < n} \Psi^i(x_{ij}, s_{ij})|\Omega\rangle.
\]

(C7)

where

\[
\theta_0 = \arcsin(\sqrt{\frac{m-1}{m}}).
\]

(C8)

and the set of spin indices \(s_{ij}, 0 \leq i, j < n\) is

\[
s_{ij} = -1, j \leq i,
\]

(C9a)

\[
s_{ij} = 1, j > i.
\]

(C9b)

Now let \(k_1\) acting on \(\mathcal{H}_{x_{01}} \otimes \mathcal{H}_{x_{02}}\) have matrix elements
and extend $k_1$ to $\mathcal{H}$ by Eq. (14). We then have

$$\exp(i\theta_j k_1) \exp(i\theta_0 k_0) |\omega_0\rangle = \sqrt{\frac{1}{m}} |\omega_0\rangle + \sqrt{\frac{m-1}{m}} \prod_{0 \leq j < n} \Psi^j(x_{0j}, s_{2j}) |\Omega\rangle,$$

for $\theta_j$ given by $\frac{\pi}{2}.$

Continuing in analogy to Eqs. (C5)–(C12), for a sequence of operators $k_j, 0 \leq j < n-1$, acting on $\mathcal{H}_{x_{nj}} \otimes \mathcal{H}_{x_{n+1}}$, and corresponding $\theta_j$ we obtain

$$\exp(i\theta_{n-2} k_{n-2}) \ldots \exp(i\theta_0 k_0) |\omega_0\rangle = \sqrt{\frac{1}{m}} |\omega_0\rangle + \sqrt{\frac{m-1}{m}} \prod_{0 \leq j < n} \Psi^j(x_{0j}, -1) |\Omega\rangle,$$

Let $k_{n-1}$ acting on $\mathcal{H}_{x_{n0}} \otimes \mathcal{H}_{x_{10}}$ have matrix elements

$$\langle \Omega | \Psi(x_{10}, 1) k_{n-1} \Psi^j(x_{00}, -1) |\Omega\rangle = -i,$$

(C14a)

$$\langle \Omega | \Psi(x_{00}, -1) k_{n-1} \Psi^j(x_{10}, 1) |\Omega\rangle = i,$$

(C14b)

extend $k_{n-1}$ to $\mathcal{H}$ by Eq. (14), and let $\theta_{n-1}$ be $\frac{\pi}{2}.$ Applying $\exp(i\theta_{n-1} k_{n-1})$ to Eq. (C13), followed by a similar sequence of $\exp(i\theta_j k_j), n \leq j < 2n-1$ acting on $\mathcal{H}_{x_{nj}} \otimes \mathcal{H}_{x_{nj+1}}$ gives

$$\exp(i\theta_{2n-2} k_{2n-2}) \ldots \exp(i\theta_0 k_0) |\omega_0\rangle = \sqrt{\frac{1}{m}} |\omega_0\rangle + \sqrt{\frac{m-1}{m}} |\omega_1\rangle,$$

(C15)

Multiplying Eq. (C15) by $\exp(i\theta_j k_j), 2n-1 \leq j < 3n-2$ on $\mathcal{H}_{x_{1j}} \otimes \mathcal{H}_{x_{1j-2n+2}},$ and then $\exp(i\theta_j k_j), 3n-2 \leq j < 4n-2$ on $\mathcal{H}_{x_{1j-3n+2}} \otimes \mathcal{H}_{x_{1j-3n+2}}$ gives

$$\exp(i\theta_{4n-3} k_{4n-3}) \ldots \exp(i\theta_0 k_0) |\omega_0\rangle = \sqrt{\frac{1}{m}} |\omega_0\rangle + \sqrt{\frac{1}{m}} |\omega_1\rangle + \sqrt{\frac{m-2}{m}} |\omega_2\rangle.$$

(C16)

The end result of a sequence of $2mn - m$ such steps is $|\chi\rangle$ of Eq. (C4)

$$\exp(i\theta_{2mn-m-1} k_{2mn-m-1}) \ldots \exp(i\theta_0 k_0) |\omega_0\rangle = \sqrt{\frac{1}{m} \sum_i |\omega_i\rangle}.$$

(C17)
The $k_i$ and $\theta_i$ of Eq. (C17) have

$$\| k_i \| = \sqrt{2},$$  \hspace{1cm} (C18a)$$

$$| \theta_i | \leq \frac{\pi}{2}. \hspace{1cm} \text{ (C18b)}$$

Thus Eq. (C17) implies

$$C(\langle \mathcal{H} \rangle, \langle \omega \rangle) \leq \sqrt{2} \pi m \left( n - \frac{1}{2} \right). \hspace{1cm} \text{ (C19)}$$

### 3.2 Entangled State Repositioned

Let $y_{ij}$ be the center of cube $D_{ij}$ of Eq. (35), $s_{ij}$ the spins of Eq. (35) and $\zeta_i$ the phases of Eq. (36). Define the entangled $n$-particle state $|\phi\rangle$ be

$$|\phi\rangle = \sum_i \zeta_i \prod_j \Psi^i(y_{ij}, s_{ij})|\Omega\rangle. \hspace{1cm} \text{ (C20)}$$

For each $0 \leq i < m, 0 \leq j < n$, let $z_{ij}^0, z_{ij}^1, \ldots z_{ij}^r$ be the shortest sequence of nearest neighbor sites such that

$$z_{ij}^0 = x_{ij}, \hspace{1cm} \text{ (C21a)}$$

$$z_{ij}^r = y_{ij}, \hspace{1cm} \text{ (C21b)}$$

for the $x_{ij}$ in Eqs. (C2a)–(C4) and such that all $z_{ij}^{\ell'}$ for distinct $\ell', i, j$, are themselves distinct. For each $0 \leq \ell < r_{ij} - 1$, for nearest neighbor pair $z_{ij}^{\ell}, z_{ij}^{\ell+1}$, let $k_{ij}^{\ell}$ acting on $\mathcal{H}_{z_{ij}^{\ell}} \otimes \mathcal{H}_{z_{ij}^{\ell+1}}$ have matrix elements

$$\langle \Omega | \Psi(z_{ij}^{\ell+1}, 1)k_{ij}^{\ell} \Psi^\dagger(z_{ij}^{\ell}, 1)|\Omega\rangle = -i, \hspace{1cm} \text{ (C22a)}$$

$$\langle \Omega | \Psi(z_{ij}^{\ell}, 1)k_{ij}^{\ell} \Psi^\dagger(z_{ij}^{\ell+1}, 1)|\Omega\rangle = i, \hspace{1cm} \text{ (C22b)}$$

and extend $k_{ij}^{\ell}$ to $\mathcal{H}$ by Eq. (14). For each $i, j$ pair with $j < n - 1$, for the final nearest neighbor step $\exp(ik_{ij}^{r_{ij} - 1})$, $\ell' = r_{ij} - 1$, Eqs. (C22a) and (C22b) are modified to produce spin orientation $s_{ij}$ at $y_{ij}$

$$\langle \Omega | \Psi(z_{ij}^{r_{ij} - 1}, s_{ij})k_{ij}^{r_{ij} - 1} \Psi^\dagger(z_{ij}^{r_{ij} - 1}, 1)|\Omega\rangle = -i, \hspace{1cm} \text{ (C23a)}$$

$$\langle \Omega | \Psi(z_{ij}^{r_{ij} - 1}, 1)k_{ij}^{r_{ij} - 1} \Psi^\dagger(z_{ij}^{r_{ij} - 1}, s_{ij})|\Omega\rangle = i, \hspace{1cm} \text{ (C23b)}$$
and for \( j = n - 1 \) for the final \( \exp(ik_{jn-1}) \), \( \ell' = r_{in-1} - 1 \), Eqs. (C22a) and (C22b) are modified in addition to generate the phase \( \zeta_i \)

\[
\langle \Omega | \Psi(z_{ij}^{\ell+1}, s_{ij})k_{ij}^\ell \Psi^\dagger(z_{ij}^{\ell}, 1)|\Omega \rangle = -i\zeta_i,
\]

(C24a)

\[
\langle \Omega | \Psi(z_{ij}^{\ell+1}, 1)k_{ij}^\ell \Psi^\dagger(z_{ij}^{\ell+1}, s_{ij})|\Omega \rangle = i\zeta_i^*.
\]

(C24b)

Define \( r \) to be

\[
r = \max_{ij} r_{ij},
\]

(C25)

and for each \( i, j \) pair define

\[
k_{ij}^\ell = 0, \quad r_{ij} \leq \ell < r.
\]

(C26)

Let \( k_{ij}^\ell \) be

\[
k_{ij}^\ell = \sum_{j} k_{ij}^\ell.
\]

(C27)

Then we have

\[
\prod_{ij}[\exp(i\frac{\pi}{2}k_{ij}^{s-1}) \ldots \exp(i\frac{\pi}{2}k_{ij}^{0})]|\chi\rangle = |\phi\rangle,
\]

(C28)

for \( |\chi\rangle \) of Eq. (C4).

The \( k_{ij}^\ell \) of Eqs. (C27), (C22a) - (C24b) have

\[
\| k_{ij}^\ell \| \leq \sqrt{2mn}.
\]

(C29)

Thus Eq. (C28) implies

\[
C(|\phi\rangle, |\chi\rangle) \leq \frac{\pi\sqrt{mnr}}{\sqrt{2}}.
\]

(C30)

We now minimize \( r \) over the base point \( x_{00} \)

\[
\hat{r} = \min_{x_{00}} r,
\]

(C31)

with the result

\[
C(|\phi\rangle, |\chi\rangle) \leq \frac{\pi\sqrt{mnr}}{\sqrt{2}},
\]

(C32)

where we have dropped the hat on \( r \).
3.3 Fan-Out

The state $|\psi\rangle$ with particles at the centers of the cubes $D_{ij}$ we now fan-out to the state $|\psi\rangle$ with particle wave functions spread uniformly over the cubes $D_{ij}$. For sufficiently small lattice spacing $a$ nearly all of the complexity of the bound on $C(|\psi\rangle)$ is generated in this step.

Let $d$ be the length of the edge of the $D_{ij}$. Each edge of $D_{ij}$ then consists of $d + 1$ sites. The volume $V$ is then $d^3$. We begin with case

$$d = 2^p,$$  \hfill (C33)

for some integer $p$. For simplicity we present the fan-out applied to a prototype single particle state $|v_0\rangle$ on prototype cube $G$ with edge length $d$, and center at some point $y$

$$|v_0\rangle = \Psi^\dagger(y, 1)|\Omega\rangle.$$  \hfill (C34)

The first stage of the fan-out process consists of splitting $|v_0\rangle$ into a pair of components displaced from each other in lattice direction 1. For integer $-2^p - 2 \leq i \leq 2^p - 2$ define $y(i)$ to be $y$ incremented by $i$ nearest neighbor steps in lattice direction 1. For $1 \leq j \leq 2^p - 2$ define $k_j$ on $\mathcal{H}_{y(j-1)} \otimes \mathcal{H}_{y(j)}$ to have matrix elements

$$\langle \Omega | \Psi[y(j), 1]k_j \Psi^\dagger[y(j-1), 1] | \Omega \rangle = -i,$$  \hfill (C35a)

$$\langle \Omega | \Psi[y(j-1), 1]k_j \Psi^\dagger[y(j), 1] | \Omega \rangle = i.$$  \hfill (C35b)

For $-2^p - 2 \leq j \leq -1$ define $k_j$ by Eqs. (C35a) and (C35b) but with $j + 1$ in place of $j - 1$. Then define $\tilde{k}_j$ by

$$\tilde{k}_1 = \frac{1}{\sqrt{2}}(k_1 + k_{-1}),$$  \hfill (C36a)

$$\tilde{k}_j = k_j + k_{-j}, 2 \leq j \leq 2^p - 2.$$  \hfill (C36b)

With these definitions it then follows that

$$|v_1\rangle = \exp(i \frac{\pi}{2} \tilde{k}_m) \cdots \exp(i \frac{\pi}{2} \tilde{k}_1) |v_0\rangle,$$  \hfill (C37)

for $m = 2^p - 2$, is given by

$$|v_1\rangle = \frac{1}{\sqrt{2}} \sum_{i=-2^p-2}^{2^p-2} \Psi^\dagger[y(i), 1] |\Omega\rangle.$$  \hfill (C38)

Equations (C36a) and (C36b) imply

$$\| \tilde{k}_1 \| = \sqrt{2},$$  \hfill (C39a)
It then follows that
\[ C(|v_1\rangle, |v_0\rangle) < 2^{p-2} \pi, \]  
(C40)

where for simplicity we have used an overestimate for \( \| \bar{k}_1 \| \).

The next stage of the fan-out consists of splitting each of the 2 components of \(|v_1\rangle\) but now in lattice direction 2. For \( \bar{k}_j, 2^{p-2} < j \leq 2^{p-1} \), defined by adapting Eqs. (C35a)–(C36b), we have
\[ |v_2\rangle = \exp\left(i \frac{\pi}{2} \bar{k}_m\right) \cdots \exp\left(i \frac{\pi}{2} \bar{k}_1\right) |v_0\rangle, \]  
(C41)

with \( m = 2^{p-1} \), given by
\[ |v_2\rangle = \frac{1}{2} \sum_{i=-2^{p-2}}^{2^{p-1}} \sum_{j=-2^{p-2}}^{2^{p-2}} \Psi^\dagger[y(i,j), 1]|\Omega\rangle, \]  
(C42)

for \( y(i,j) \) defined to be \( y(i) \) displaced \( j \) steps in lattice direction 2. Eqs. (C36a) and (C36b) adapted to the fan-out in direction 2 give \( \bar{k}_j, 2^{p-2} < j \leq 2^{p-1} \) each acting on twice as many sites as was the case for the direction 1 fan-out and therefore
\[ \| \bar{k}_{2^{p-1}+1} \| = 2, \]  
(C43a)
\[ \| \bar{k}_j \| = 2 \sqrt{2}, 2^{p-2} + 2 \leq j \leq 2^{p-1}. \]  
(C43b)

It then follows that
\[ C(|v_2\rangle, |v_1\rangle) < 2^{p-2} \sqrt{2} \pi. \]  
(C44)

Splitting yet again, now in lattice direction 3, yields
\[ |v_3\rangle = \exp\left(i \frac{\pi}{2} \bar{k}_m\right) \cdots \exp\left(i \frac{\pi}{2} \bar{k}_1\right) |v_0\rangle, \]  
(C45)

for \( m = 2^{p-1} + 2^{p-2} \), given by
\[ |v_3\rangle = \frac{1}{\sqrt{8}} \sum_{i=-2^{p-2}}^{2^{p-1}} \sum_{j=-2^{p-2}}^{2^{p-2}} \sum_{j'=-2^{p-2}}^{2^{p-2}} \Psi^\dagger[y(i,j, j'), 1]|\Omega\rangle, \]  
(C46)

for \( y(i,j, j') \) defined to be \( y(i, j) \) displaced \( j' \) steps in lattice direction 3.

Eqs. (C36a) and (C36b) adapted to the fan-out in direction 3 give \( \bar{k}_j, 2^{p-1} < j \leq 2^{p-1} + 2^{p-2} \), each acting on twice as many sites as was the case for the direction 2 fan-out and therefore
\[ \| \bar{k}_{2^{p-1}+1} \| = 2 \sqrt{2}, \]  
(C47a)
It then follows that

\[ \| \tilde{k}_j \| = 4, 2^{p-1} + 2 \leq j \leq 2^{p-1} + 2^{p-2}. \]  \hspace{1cm} (C47b)

The weight originally concentrated in \( |\psi_0\rangle \) at the center point \( y \) of \( G \), with edge length \( d \), in \( |\psi_3\rangle \) is distributed equally over the center points of \( 8 \) sub-cubes of \( G \) each with edge length \( \frac{d}{2} \). Combining Eqs. (C40), (C44) and (C48) gives

\[ C(|\psi_3\rangle, |\psi_0\rangle) < 2^{p-1} \pi. \]  \hspace{1cm} (C48)

The fan-out process of Eqs. (C37)–(C49) we now repeat a total of \( p - 1 \) iterations arriving at a state \( |\psi_{3p-3}\rangle \) with weight equally distributed over the center points of \( 2^{3p-3} \) cubes each with edge length \( 2 \). Eqs. (C49) rescaled for iteration \( \ell' \) give

\[ C(|\psi_{3\ell'}\rangle, |\psi_{3\ell'-3}\rangle) < (3 + \sqrt{2})2^{p-\ell'-1}2^{\frac{3\ell'-1}{2}} \pi. \]  \hspace{1cm} (C50)

The term \( 2^{p-\ell'-1} \) counts the decreasing number of lattice steps between cube centers as the fan-out process is iterated, while the term \( 2^{\frac{3\ell'-1}{2}} \) counts the growing number of cubes and therefore of sites which each subsequent operator \( \tilde{k}(i) \) acts on simultaneously.

To complete the fan-out process, the weight at the center of each of the cubes with edge length \( 2 \) needs to be distributed to the \( 26 \) points forming its boundary. This process can be carried out in \( 3 \) additional steps thereby defining \( |\psi_{3p-2}\rangle, |\psi_{3p-1}\rangle \) and \( |\psi_{3p}\rangle \).

To obtain \( |\psi_{3p-2}\rangle \) from \( |\psi_{3p-3}\rangle \), the weight at the center of each edge length \( 2 \) cube is distributed simultaneously and equally to the points at the centers of the \( 6 \) edge length \( 2 \) squares forming the cube’s boundary. This process itself is done simultaneously across all \( 2^{3p-3} \) cubes. The result is

\[ C(|\psi_{3p-2}\rangle, |\psi_{3p-3}\rangle) \leq \frac{\pi2^{\frac{3\ell'-1}{2}}}{2^{\frac{3\ell'-1}{2}}}. \]  \hspace{1cm} (C51)

To obtain \( |\psi_{3p-1}\rangle \) from \( |\psi_{3p-2}\rangle \), the weight at the center of each edge length \( 2 \) square is distributed simultaneously and equally to the center point of the \( 4 \) length \( 2 \) lines forming the boundary of that square. This process itself is done simultaneously across all faces of all \( 2^{3p-3} \) cubes. The result is

\[ C(|\psi_{3p-1}\rangle, |\psi_{3p-2}\rangle) \leq \frac{\sqrt{3}\pi2^{\frac{3\ell'-1}{2}}}{2\sqrt{2}}. \]  \hspace{1cm} (C52)

To obtain \( |\psi_{3p}\rangle \) from \( |\psi_{3p-1}\rangle \), the weight at the center of each length \( 2 \) line is distributed simultaneously and equally to that line’s pair of end points. This process itself is done simultaneously across all lines forming the boundaries of the faces of all \( 2^{3p-3} \) cubes. The result is

\[ C(|\psi_{3p}\rangle, |\psi_{3p-1}\rangle) \leq \frac{\sqrt{3}\pi2^{\frac{3\ell'-1}{2}}}{2\sqrt{2}}. \]
The bound on \( C(\ket{v_{3p}}, \ket{v_{3p-1}}) \) obtained by summing Eqs. (C51)–(C53) turns out to be less than the bound in Eq. (C50) for \( \ell' = p \). We therefore sum Eq. (C50) from \( \ell' \) of 1 to \( p \) and obtain

\[
C(\ket{v_{3p}}, \ket{v_0}) < \frac{(3 + \sqrt{2})(2 + \sqrt{2})}{4\sqrt{2}} \pi 2^{\frac{3p}{2}}.
\]

(C54)

Substituting \( V \) for \( 2^{3p} \), we then have

\[
C(\ket{v_{3p}}, \ket{v_0}) < \frac{(3 + \sqrt{2})(2 + \sqrt{2})}{4\sqrt{2}} \pi \sqrt{V}.
\]

(C55)

The bound of Eq. (C54) is derived assuming Eq. (C33) giving the edge \( d \) of cube \( G \) as an even power of 2. Consider now the case

\[
2^{p-1} < d < 2^p.
\]

(C56)

Assume again that at each iteration \( \ell' \) of the fan-out process, each edge length of each parent cube is split as evenly as possible into halves to produce 8 child cubes with all edges nearly equal. Suppose \( d \) is \( 2^p - 1 \). After iteration \( \ell' \) has been completed, the total number of cubes will still be \( 2^{3\ell'} \). Orthogonal to each direction, the cubes can be grouped into \( 2^{\ell'} \) planes, each holding \( 2^{2\ell'} \) cubes. But for each direction one of these orthogonal planes will have an edge in that direction which is one lattice unit shorter than the corresponding edge of the other \( 2^{\ell'} \) planes. It follows that the update process in each direction can proceed with \( 2^{p-\ell'-1} - 1 \) steps occurring simultaneously across all cubes, and one final update skipped for the cubes with a single edge in that direction one lattice unit shorter. The bound of Eq. (C50) will hold without modification. For \( d \) given by \( 2^p - 2 \), after iteration \( \ell' \), for each direction, there will be two planes of \( 2^{2\ell'} \) cubes each with the edge in that direction one lattice unit shorter. The bound of Eq. (C50) will continue to hold. Similarly for \( d \) given by \( 2^p - q \) for any \( q < 2^{p-1} \).

For \( d \) of Eq. (C56), when \( \ell' \) reaches \( p - 1 \) the resulting cubes (no longer exactly cubes) will have a mix of edges of length 2 and of length 1. The argument leading to Eqs. (C51)–(C53) can be adapted to show they continue to hold for the final pass with \( \ell' \) of \( p \). The bound of Eq. (C54) remains in place for the net result of the entire fan-out process. By assumption, according to Eq. (C56) we have

\[
2d > 2^p.
\]

(C57)

Then since \( V \) is \( d^3 \), Eq. (C54) gives

\[
C(\ket{v_{3p-1}}, \ket{v_0}) < \frac{(3 + \sqrt{2})(2 + \sqrt{2})}{2} \pi \sqrt{V},
\]

(C58)
which is weaker than Eq. (C55) and therefore holds whether or not $d$ is an even power of 2.

The bound of Eq. (C55) applies to a fan-out process on a single prototype state on cube $G$. Assume the process repeated in parallel on the $mn$ cubes $D_{ij}$, thereby generating $|\psi\rangle$ of Eq. (36). For $|\phi\rangle$ of Eq. (C20) we then have

$$C(|\psi\rangle, |\phi\rangle) \leq \frac{(3 + \sqrt{2})(2 + \sqrt{2})}{2} \pi \sqrt{mnV}. \quad (C59)$$

From Eqs. (C19) and (C32), it follows that for a product state $|\omega\rangle$ we have

$$C(|\psi\rangle, |\omega\rangle) \leq c_1 \sqrt{mnV} + c_2 mn + c_3 \sqrt{mn}, \quad (C60)$$

where

$$c_1 = \frac{(3 + \sqrt{2})(2 + \sqrt{2})}{2} \pi, \quad (C61a)$$

$$c_2 = \sqrt{2} \pi, \quad (C61b)$$

$$c_3 = \frac{\pi}{\sqrt{2}}, \quad (C61c)$$

for $r$ of Eq. (C32). Eq. (38) then follows.

### Appendix 4: Complexity Group

We now show that the topological closure of the group $G$ of all $U_k(1)$ realizable as solutions to Eqs. (17a) and (17b) has as a subgroup the direct product

$$\hat{G} = \times_n SU(d_n), \quad (D1)$$

where $SU(d_n)$ acts on the subspace of $\mathcal{H}$ with eigenvalue $n$ of the fermion number operator $N$, $d_n$ is the dimension of this subspace, and the product is over the range $0 \leq n \leq 16B^3$.

#### 4.1 Lie Algebras

The $8B^3$ sites of the lattice $L$ we reorder as a 1-dimensional array of distinct sites, successive pairs of which are nearest neighbors with respect to the original lattice $L$. The new array of sites we label with an integer valued index $z$ ranging from 0 to $8B^3 - 1$.

For any pair of nearest neighbor $\{z, z'\}$, let $\mathcal{F}_{zz'}$ be the set of operators of the form
\[ f_{z'z} = g_{z'z} \bigotimes_{q \neq z,z'} I_q, \]  

where \( I_q \) is the identity operator on \( \mathcal{H}_q \) and \( g_{z'z} \) is a traceless Hermitian operator acting on \( \mathcal{H}_z \otimes \mathcal{H}_{z'} \) which commutes with \( N_{z'z} \), the fermion number operator on \( \mathcal{H}_z \otimes \mathcal{H}_{z'} \). Let \( K_p \) be the vector space over the reals of operators of the form

\[ k = \sum_{z'z} f_{z'z}, \]  

for any collection of \( f_{z'z} \in \mathcal{F}_{z'z} \) for \( z, z' \leq p \).

Let \( G_p \) be the group on \( \mathcal{H} \) of all \( U_k(1) \) realizable as solutions to Eq. (17a) for \( k(v) \in K_p \). The topological closure of the group \( G_p \) consists of all operators of the form \( \exp(\text{i}h) \) for \( h \in L_p \), where \( L_p \) is the Lie algebra generated by \( K_p \) [21]. Said differently, \( L_p \) is the smallest set of operators such that \( K_p \subseteq L_p \) and, in addition, for any \( h_0, h_1 \in L_p \), and any real \( r_0, r_1 \), there are \( h_2, h_3 \in L_p \) given by

\[ h_2 = r_0 h_0 + r_1 h_1, \]  

\[ h_3 = \text{i} [h_0, h_1]. \]  

The requirement that \( L_p \) be closed under sums in Eq. (D4a) follows from the Trotter product formula applied to the large \( t \) limit

\[ \exp(\text{i}r_0 h_0 + \text{i}r_1 h_1) = \lim_{t \to \infty} [\exp(\text{i}t^{-1}r_0 h_0) \exp(\text{i}t^{-1}r_1 h_1)]'. \]  

The requirement that \( L_p \) be closed under commutation in Eq. (D4b) follows from the Baker–Campbell–Hausdorff formula applied to the large \( t \) limit

\[ \exp([h_0, h_1]) = \lim_{t \to \infty} [\exp(\text{i}t^{-1/2}h_0) \exp(-\text{i}t^{-1/2}h_1) \times \exp(-\text{i}t^{-1/2}h_0) \exp(\text{i}t^{-1/2}h_1)]'. \]  

The requirement of taking a topological closure of the group generated by \( U_k(1) \) in order to generate \( L_p \) is a consequence of the appearance of limits in Eqs. (D5) and (D6).

4.2 Induction

For any integer \( 0 < p \leq 8B^3 - 1 \), divide \( \mathcal{H} \) into the product

\[ Q_p = \bigotimes_{q \leq p} \mathcal{H}_q, \]  

\[ R_p = \bigotimes_{q > p} \mathcal{H}_q, \]
\[ \mathcal{H} = Q_p \otimes R_p. \]  

(D7c)

By induction on \( p \), we will show that the closure of \( G_p \) includes the subgroup \( \hat{G}_p \)

\[ \hat{G}_p = \times_n \hat{G}_{pn}, \]  

(D8a)

\[ \hat{G}_{pn} = SU(d_{pn}) \bigotimes_{z \geq p} I_z, \]  

(D8b)

where \( SU(d_{pn}) \) acts on the subspace \( Q_{pn} \) of \( Q_p \), with eigenvalue \( n \) of the total number operator \( N \), and \( d_{pn} \) is the dimension of \( Q_{pn} \). The product in Eq. (D8a) is over \( 0 \leq n \leq 2p + 2 \). Equations (D8a) and (D8b) for the case \( p = 8B^3 - 1 \) become Eq. (D1).

The set of \( g_{zz'} \) in Eq. (D2) is a subset of the set of \( f_{xy} \) in Eq. (14) of Sect. 3. Thus \( \hat{G}_p \) for \( p = 8B^3 - 1 \) is a subgroup of the group \( G \) of Sect. 3. Proof of Eq. (D8a) therefore implies Eq. (18) of Sect. 3.

For \( p = 1 \), Eqs. (D8a) and (D8b) follow immediately from the definition of \( K_p \). Assuming Eqs. (D8a) and (D8b) for some \( p - 1 \), we will prove them for \( p \).

Let \( S_{pn} \) be an orthonormal basis for \( Q_{pn} \) consisting of all \( n \)-fermion, \( m \)-boson, \( m \leq b_{\text{max}}(p + 1) \), vectors of the form

\[ |\psi\rangle = \prod_{i \leq n} \Psi^i(z^f_i, s_i) \prod_{j \leq m} \Phi^j(z^b_j) |\Omega\rangle \]  

(D9a)

\[ s_i \in \{-1, 1\}, \]  

(D9b)

\[ z^f_i, z^b_j \leq p, \]  

(D9c)

for any list of \( n \) distinct pairs of \( (z^f_i, s_i) \) and any list of \( m \) integers \( z^b_j \) such that each \( z^b_j \) coincides with at most \( b_{\text{max}} - 1 \) other \( z^b_{j'} \). For any pair of distinct \( |\psi_0\rangle, |\psi_1\rangle \in S_{pn} \), and \( 2 \times 2 \) traceless Hermitian \( h \), define

\[ H(|\psi_0\rangle, |\psi_1\rangle, h) = \sum_{ij} |\psi_i\rangle\langle\psi_j| h_{ij}, \]  

(D10a)

\[ H_p(|\psi_0\rangle, |\psi_1\rangle, h) \]  

\[ = H(|\psi_0\rangle, |\psi_1\rangle, h) \bigotimes_{z \geq p} I_z. \]  

(D10b)

The set of all such \( H_p(|\psi_0\rangle, |\psi_1\rangle, h) \) is a linear basis for the Lie algebra \( L_{pn} \) of the group \( \hat{G}_{pn} \) of Eq. (D8b).

Thus to prove Eqs. (D8b) and (D8a) for \( p \) it is sufficient to show that any \( H_p(|\psi_0\rangle, |\psi_1\rangle, h) \) for some \( |\psi_0\rangle, |\psi_1\rangle \in S_{pn} \) and \( 2 \times 2 \) traceless Hermitian \( h \), given the induction hypothesis, is contained in the Lie algebra generated by \( L_{p-1n} \) for some \( n' \) and \( F_{p-1p'} \).
4.3 Without Bosons

We consider first $|\psi_0\rangle$ and $|\psi_1\rangle$ both with $m$ of 0 in Eqs. (D9a)–(D9c). We will work backwards starting from some $H_p(|\psi_0\rangle, |\psi_1\rangle, h)$ for $|\psi_0\rangle, |\psi_1\rangle \in S_{pn}$. Since $|\psi_0\rangle$ and $|\psi_1\rangle$ have the same value of total $N$ on the region $z \leq p$, it follows that a $U_0$ can be found in $\hat{G}_{p-1}$ such that

$$|\psi_2\rangle = U_0 |\psi_0\rangle,$$

$$|\psi_3\rangle = U_0 |\psi_1\rangle$$

are orthogonal vectors in $S_{pn}$, their restrictions to the region $p - 1 \leq z \leq p$ are also orthogonal but have equal total particle counts on $p - 1 \leq z \leq p$. The particle count difference between $|\psi_0\rangle$ and $|\psi_1\rangle$ at point $p$ is at most 2, and equal and opposite to the difference between the corresponding totals on the region $z \leq p - 1$. This compensating difference can be moved by $U_0$ to the point $p - 1$.

A $k$ in $\mathcal{F}_{p-1p}$ can then be found such that

$$|\psi_4\rangle = \exp(ik) |\psi_2\rangle,$$

$$|\psi_5\rangle = \exp(ik) |\psi_3\rangle,$$

$$|\psi_4\rangle = |\psi_6\rangle \otimes |\nu\rangle,$$

$$|\psi_5\rangle = |\psi_7\rangle \otimes |\nu\rangle,$$

for some $|\nu\rangle \in \mathcal{H}_p$, with particle number $n_\nu$ and $|\psi_6\rangle$ and $|\psi_7\rangle$ orthogonal vectors in $S_{(p-1)m}$ with $m = n - n_\nu$.

It is then possible to find a $U_2$ in $\hat{G}_{p-1}$ such that

$$|\psi_8\rangle = U_2 |\psi_4\rangle,$$

$$|\psi_9\rangle = U_2 |\psi_5\rangle,$$

$$|\psi_8\rangle = |\chi\rangle \otimes |\phi_0\rangle \otimes |\nu\rangle,$$

$$|\psi_9\rangle = |\chi\rangle \otimes |\phi_1\rangle \otimes |\nu\rangle,$$

$$|\phi_0\rangle = \Psi^\dagger(p - 1, -1)|\Omega\rangle,$$

$$|\phi_1\rangle = \Psi^\dagger(p - 1, 1)|\Omega\rangle,$$

for some $|\chi\rangle$ in $S_{(p-2)(m-1)}$.
Combining Eqs. (D11a)–(D13f), the induction hypothesis implies the existence of $U_0, U_2 \in \hat{G}_{p-1}$ and $k \in F_{(p-1)p}$ such that

$$U_2 \exp(ik) U_0 H_p (|\psi_0\rangle, |\psi_1\rangle, h) U_0^\dagger \exp(-ik) U_2^\dagger = |\chi\rangle \langle \chi | \otimes \sum_{ij} |\phi_i\rangle \langle \phi_j | h_{ij} \otimes |v\rangle \langle v |.$$  \hspace{1cm} (D14)

The expression on the right-hand side of Eq. (D14) can then be obtained from a commutator between an operator $k \in F_{(p-1)p}$ and an operator $g \in L_{(p-1)m}$ for $m = n - n_v$. For $2 \times 2$ traceless Hermitian $k_{ij}$, define

$$k = \sum_{ij} |\phi_i\rangle \langle \phi_j | k_{ij} \otimes |v\rangle \langle v | \otimes I_q,$$  \hspace{1cm} (D15)

and for a $2 \times 2$ traceless Hermitian $g_{ij}$, define

$$g = |\chi\rangle \langle \chi | \otimes \sum_{ij} |\phi_i\rangle \langle \phi_j | g_{ij} \otimes \otimes I_q.$$  \hspace{1cm} (D16)

For any traceless, Hermitian $2 \times 2 h_{ij}$, there are $k_{ij}$ and $g_{ij}$ such that

$$h = i[k, g].$$  \hspace{1cm} (D17)

Combining Eqs. (D14), (D15), (D16) and (D17) then gives

$$H_p (|\psi_0\rangle, |\psi_1\rangle, h) = U_0^\dagger \exp(-ik) U_2^\dagger i[k, g] U_2 \exp(ik) U_0,$$  \hspace{1cm} (D18)

which completes the induction step and for $|\psi_0\rangle$ and $|\psi_1\rangle$ with $m = 0$ in Eqs. (D9a) - (D9c).

### 4.4 With Bosons

We consider next $|\psi_0\rangle$ and $|\psi_1\rangle$ both with nonzero $n$ and $m$ in Eqs. (D9a)–(D9c).

Suppose $0 < n < 2p + 2$.

If the boson factors in $|\psi_0\rangle$ and $|\psi_1\rangle$ are identical, then by a combination of a rotation by a $U_0$ in $\hat{G}_{p-1}$ and by a $U_1$ in the group generated by $k \in F_{(p-1)p}$ the boson factors can both be turned into the case $m = 0$, already covered in Appendix 4.3.

Suppose the boson factors in $|\psi_0\rangle$ and $|\psi_1\rangle$ are not identical but the fermion factors in $|\psi_0\rangle$ and $|\psi_1\rangle$ are identical. Then again, but a combination of a rotation by a $U_0$ in $\hat{G}_{p-1}$ and by a $U_1$ in the group generated by $k \in F_{(p-1)p}$ the boson factors can both be turned into the case $m = 0$ but with orthogonal fermion factors in $|\psi_0\rangle$ and $|\psi_1\rangle$. Thus back to the case covered in Appendix 4.3.

Suppose both the fermion factors and the boson factors in $|\psi_0\rangle$ and $|\psi_1\rangle$ are not identical. The induction step of Appendix 4.3 shows that the action of $\hat{G}_p$ is available at least on the fermion factors in $|\psi_0\rangle$ and $|\psi_1\rangle$. A $U_0$ in $\hat{G}_p$ can therefore be found which makes the fermion factors in $|\psi_0\rangle$ and $|\psi_1\rangle$ distinct both on the region $p - 1 \leq z \leq p$ and on
the region $0 \leq z \leq p - 1$. It follows that a $U_1$ in $\hat{G}_{p-1}$ and a $U_2$ in the group generated by $k \in \mathcal{F}_{(p-1)p}$ can then be found which take $|\psi_0\rangle$ and $|\psi_1\rangle$ back to $m$ of 0.

Suppose finally either $n$ is 0 and $|\psi_0\rangle$ and $|\psi_1\rangle$ have only fermions or $n$ is $2p + 2$ and all sites are filled with fermions. In either case, $\hat{G}_{p-1}$ and $\hat{G}_p$ act purely on boson states. The induction step to show that the Lie algebra of $\hat{G}_p$ is generated by the Lie algebra of $L_{(p-1)p'}$, either for $n'$ of 0 or $n'$ of $2p$, and $\mathcal{F}_{p-1p}$ becomes nearly a direct translation of the induction step in Appendix 4.3 from fermion states to boson states. We omit the details.

Appendix 5: Auxiliary Field Algebra

We will construct a Hilbert space $H^B$ generated by the algebra $B$ of polynomials in the $\Sigma_i(x, s)$ and $\Upsilon_i(x)$ acting purely as creation operators on $|\Omega^B\rangle$ and satisfying Eqs. (153a)–(153d).

Let $B^\Sigma$ be the algebra generated by the set of all $\Sigma_i(x, s)$, for any $x$, $s$ and $i$, and let $B^\Upsilon$ be the algebra generated by the set of all $\Upsilon_i(x)$, for any $x$ and $i$. Since every $a \in B^\Sigma$ commutes with every $b \in B^\Upsilon$, the algebra $B$ is the tensor product

$$B = B^\Sigma \otimes B^\Upsilon.$$  \hspace{1cm} (E1)

For every $x$, let $B^\Sigma_x$ be the algebra generated by the set of $\Sigma_i(x, s)$, for any $s$ and $i$, and let $B^\Upsilon_x$ be the algebra generated by the set of $\Upsilon_i(x)$ for any $i$. Then for every $x \neq y$, every $a_x \in B^\Sigma_x$ commutes or anticommutates with every $a_y \in B^\Sigma_y$, and every $a_x \in B^\Upsilon_x$ commutes with every $a_y \in B^\Upsilon_y$. Therefore the algebras $B^\Sigma$ and $B^\Upsilon$ are the products

$$B^\Sigma = \bigotimes_x B^\Sigma_x;$$  \hspace{1cm} (E2a)

$$B^\Upsilon = \bigotimes_x B^\Upsilon_x.$$  \hspace{1cm} (E2b)

Now let $\eta_\alpha$ be a boost that takes the point $x$ to the point $(\tau, 0, 0, 0)$. For Eqs. (155a) and (155b) to be covariant, $\Sigma_0(x, s)$ has to transform under boosts like $\Psi(x, s)$ and $\Sigma_1(x, s)$ has to transform under boosts like $\Psi^T(x, s)$. Let $S^x_{ss'}$ and $\bar{S}^x_{ss'}$ be the spin transformation matrices corresponding to $\eta_\alpha$ and define $\hat{\Sigma}_0(x, s)$ and $\hat{\Sigma}_1(x, s)$ to be

$$\hat{\Sigma}_0(x, s) = \sum_{s'} S^x_{ss'} \Sigma_0(x, s'),$$ \hspace{1cm} (E3a)

$$\hat{\Sigma}_1(x, s) = \sum_{s'} \bar{S}^x_{ss'} \Sigma_1(x, s').$$ \hspace{1cm} (E3b)

For each $x$ and $s$, let $B^\Sigma_{xs}$ be the algebra generated by $\hat{\Sigma}_0(x, s)$ and $\hat{\Sigma}_1(x, s)$. Then for $s \neq s'$, every $a_{xs} \in B^\Sigma_{xs}$ either commutes or anticommutates with every $a_{xs'} \in B^\Sigma_{xs'}$. Therefore the algebra $B^\Sigma_x$ is the product

\[\text{Springer}\]
\[ B_{x}^{\Sigma} = \bigotimes_{s} B_{xs}^{\Sigma}. \] (E4)

Equation (E1) implies \( \mathcal{H}^{B} \) is a tensor product
\[ \mathcal{H}^{B} = \mathcal{H}^{\Sigma} \otimes \mathcal{H}^{\Upsilon}, \] (E5)
of a space generated by \( B^{\Sigma} \) acting on \( |\Omega^{B}\rangle \) and a space generated by \( B^{\Upsilon} \) acting on \( |\Omega^{B}\rangle \) and Eqs. (E2a) and (E2b) imply \( \mathcal{H}^{\Sigma} \) and \( \mathcal{H}^{\Upsilon} \) are themselves products of spaces \( \mathcal{H}_{x}^{\Sigma} \) and \( \mathcal{H}_{x}^{\Upsilon} \) generated, respectively, by \( B_{x}^{\Sigma} \) and \( B_{x}^{\Upsilon} \) acting on \( |\Omega^{B}\rangle \)
\[ \mathcal{H}_{x}^{\Sigma} = \bigotimes_{x} \mathcal{H}_{x}^{\Sigma}, \] (E6a)
\[ \mathcal{H}_{x}^{\Upsilon} = \bigotimes_{x} \mathcal{H}_{x}^{\Upsilon}. \] (E6b)

Similarly, Eq. (E4) implies \( \mathcal{H}_{xs}^{\Sigma} \) is a product of \( \mathcal{H}_{xs}^{\Sigma} \) generated by \( B_{xs}^{\Sigma} \) acting on \( |\Omega^{B}\rangle \)
\[ \mathcal{H}_{xs}^{\Sigma} = \bigotimes_{s} \mathcal{H}_{xs}^{\Sigma}. \] (E7)

For the pair of operators \( \hat{\Sigma}_{0}(x, s) \) and \( \hat{\Sigma}_{1}(x, s) \) which generate \( B_{xs}^{\Sigma} \), Eqs. (153a) and (153c) become
\[ [\hat{\Sigma}_{0}(x, s)]^{2} = 0, \] (E8a)
\[ [\hat{\Sigma}_{1}(x, s)]^{2} = 0, \] (E8b)
\[ \{\hat{\Sigma}_{0}(x, s), \hat{\Sigma}_{0}(x, s)\} = \gamma_{ss}^{0}. \] (E8c)

Equations (E8a)–(E8c) combined with approximate Lorentz and charge conjugation invariance of the complexity of states in \( \mathcal{H} \) imply that for the field polynomials \( P_{i}[\hat{\Sigma}_{0}(x, s), \hat{\Sigma}_{1}(x, s)] \)
\[ P_{0}[\hat{\Sigma}_{0}(x, s), \hat{\Sigma}_{1}(x, s)] = 1, \] (E9a)
\[ P_{1}[\hat{\Sigma}_{0}(x, s), \hat{\Sigma}_{1}(x, s)] = \mu \hat{\Sigma}_{0}(x, s), \] (E9b)
\[ P_{2}[\hat{\Sigma}_{0}(x, s), \hat{\Sigma}_{1}(x, s)] = \mu \hat{\Sigma}_{1}(x, s), \] (E9c)
\[ P_{3}[\hat{\Sigma}_{0}(x, s), \hat{\Sigma}_{1}(x, s)] = \nu \{\hat{\Sigma}_{0}(x, s), \hat{\Sigma}_{1}(x, s)\}, \] (E9d)
where \( \mu \) and \( \nu \) are normalization constants independant of \( x \) and \( s \), an orthonormal basis for \( \mathcal{H}_{xs}^{\Sigma} \) must have the form
\[ |x, s, i\rangle = P_i [\hat{\Sigma}_0(x, s), \hat{\Sigma}_1(x, s)]|\Omega^B\rangle, \]  

(E10)

up to an overall unitary rotation of the basis. Equations (E8a)–(E8c) imply the result of any other polynomial in \( \hat{\Sigma}_0(x, s) \) and \( \hat{\Sigma}_1(x, s) \) acting on \( |\Omega^B\rangle \) is equal to some corresponding linear combination of \( |x, s, i\rangle \) of Eq. (E10). The complexity of a state in \( \mathcal{H}_x^B \) is independent of overall normalization, however, so \( u \) can be arbitrarily set to 1. The remaining constant \( v \) determines the contribution to complexity arising from sites occupied by more than a single boson. In the continuum limit of complexity, if a continuum limit exists, the weight of multiply occupied sites in any state will go to 0. The continuum limit should therefore be independent of \( v \).

For the pair of operators \( Y_0(x) \) and \( Y_1(x) \) which generate \( B_x^1 \), Eq. (153d) becomes

\[ [Y_0(x), Y_1(x)] = i. \]  

(E11)

Equation (E11) combined with approximate Lorentz and charge conjugation invariance of the complexity of states in \( \mathcal{H} \) imply that, up to an overall unitary rotation of the basis, an orthonormal basis for \( \mathcal{H}_x^1 \) will consist of a family of states \( \{|x, n_0, n_1\rangle \} \) labeled by a pair of nonnegative integers \( n_0, n_1 \). For each \( n_0, n_1 \) pair \( P_{n_0 n_1} [Y_0(x), Y_1(x)] \) is an ordered product, independent of \( x \), of \( n_0 \) copies of \( Y_0(x) \) and \( n_1 \) copies of \( Y_1(x) \) subject to the requirement

\[ P_{n_0 n_1} [Y_0(x), Y_1(x)] = P_{n_1 n_0} [Y_1(x), Y_0(x)]. \]  

(E12)

The \( \{|x, n_0, n_1\rangle \} \) are given by

\[ |x, n_0, n_1\rangle = u_{n_0 n_1} P_{n_0 n_1} [Y_0(x), Y_1(x)]|\Omega^B\rangle, \]  

(E13)

where the \( u_{n_0 n_1} \) are normalization constants independent of \( x \) and symmetric in the indices \( n_0, n_1 \). Equation (E11) implies the result of any other polynomial in \( Y_0(x) \) and \( Y_1(x) \) acting on \( |\Omega^B\rangle \) is equal to some corresponding linear combination of the \( |x, n_0, n_1\rangle \) of Eq. (E13). To be consistent with the normalization choice for fermions, \( u_{00}, u_{01} \) and \( u_{10} \) will be set to 1. The remaining \( u_{n_0 n_1} \) determine the contribution to complexity arising from sites occupied by more than a single boson and should have no effect on the continuum limit of complexity, if a continuum limit exists.

Equation (154) implies the \( P_{n_0 n_1} [Y_0(x), Y_1(x)] \) identically vanish for \( n_0 \geq n \) or \( n_1 \geq n \).

The end result of Eqs. (E5)–(E7) is an \( \mathcal{H}_x^B \) generated by the algebra \( B \) acting on \( |\Omega^B\rangle \) which is an ordered tensor product

\[ \mathcal{H}_x^B = \bigotimes_x \mathcal{H}_x^B, \]  

(E14)

on which, according to Eqs. (E8a)–(E13), the \( \Sigma_i(x, s), \Sigma_i(x), \) satisfy Eqs. (153a)–(153d).

It is convenient to define at this point an orthonormal basis \( P \) for \( B \). In particular, no linear combination of elements of \( P \) is 0 as a result of the anticommutation and commutation relations of Eqs. (153a)–(153d). Each \( p \in P \) consists of a product of a \( p^\Sigma \in P^\Sigma \) and a \( p^\Upsilon \in P^\Upsilon \), where \( P^\Sigma \) and \( P^\Upsilon \) are orthonormal bases for the fermion
field algebra $B^\Sigma$ and the boson field algebra $B^\Upsilon$, respectively. Each $p^\Sigma$ is defined to be a product over all distinct $x$ and $s$ of one of the fermion field combinations in Eqs. (E9a)–(E9d). Each $p^\Upsilon$ is defined to be a product over all distinct $x$ of one of the normalized boson field combinations $u_{n_0,n_1} P_{n_0,n_1} [Y_0(x), Y_1(x)]$.

**Appendix 6: Lower Bound on the Complexity of Entangled Relativistic States**

The proof of Eq. (180) bounding from below the complexity of the entangled relativistic state $|\psi^B\rangle$ of Eq. (178b) is a version of the proof in Appendix 2 of a lower bound on the complexity of the entangled non-relativistic state of Eq. (36), but with the regular lattice of Sect. 3.1 replaced by the random lattice of Sect. 13 and with the inclusion in $\mathcal{H}^B$ of anti-fermion states. The proof in Appendix 2 can be adapted to the presence of anti-fermion states in $\mathcal{H}^B$ by treating fermion-anti-fermion pairs in $\mathcal{H}^B$ following the treatment of bosons in Appendix 2. To do this we convert the complexity calculation in $\mathcal{H}^B$ into an equivalent complexity calculation in yet another auxiliary Hilbert space.

### 6.1 More Auxiliary Hilbert Spaces

For a trajectory $k^B(v) \in K^B$, let $U_{k^B}(v)$ be the solution to

$$
\frac{dU_{k^B}(v)}{dv} = -ik^B(v)U_{k^B}(v),
$$

\[ \text{F1a} \]

$$
U_{k^B}(0) = I.
$$

\[ \text{F1b} \]

Define $|\omega(v)^B\rangle$ to be

$$
|\omega^B(v)\rangle = U_{k^B}(v)|\omega^B\rangle.
$$

\[ \text{F2} \]

for a product state $|\omega^B(0)\rangle \in \mathcal{H}^B$

$$
|\omega^B(0)\rangle = d_j(p_{j-1}) \ldots d_j(p_0)d_j(q_{k-1}) \ldots d_j(q_0)
\times
\prod db(r_{\ell-1}) \ldots db(r_0)|\Omega^B\rangle,
$$

\[ \text{F3} \]

with $j$ fermions, $k$ anti-fermions, and $\ell$ bosons. Assume that $|\omega^B(0)\rangle$ and $k^B(v)$ have been chosen to give

$$
|\omega(1)^B\rangle = \xi|\psi^B\rangle,
$$

\[ \text{F4} \]

for a phase factor $\xi$. Fermion number conservation by $k^B(v)$ implies $j - k$ must equal the fermion number $n$ of $|\psi^B\rangle$.

To deal with the presence of anti-fermions in $\mathcal{H}^B$, we will make use of yet one more auxiliary Hilbert space, $\mathcal{H}^C$, which consists purely of fermion states generated by all
polynomials in an auxiliary field $\Sigma^C_1(x, s)$ acting on an auxiliary vacuum $|\Omega^C\rangle$. The tensor product $\mathcal{H}^C \otimes \mathcal{H}^B$ we name $\mathcal{H}^D$.

There is a natural map $M$ from $\mathcal{H}^D$ to $\mathcal{H}^B$ defined by

$$M[P(\Sigma^C_1)|\Omega^C\rangle \otimes |\psi^B\rangle] = P(\Sigma_1)|\psi^B\rangle,$$

where $P(\Sigma^C_1)$ is a polynomial in the field $\Sigma^C_1(x, s)$, $P(\Sigma_1)$ is the corresponding polynomial but in the field $\Sigma_1(x, s)$ and $|\psi^B\rangle$ is any state in $\mathcal{H}^B$. The map $M$ takes a subspace of $\mathcal{H}^D$ to the null vector in $\mathcal{H}^B$ and thus does not have an inverse.

Corresponding to the decomposition of $\mathcal{H}^D$ and $\mathcal{H}^B$ as tensor products over all sites

$$\mathcal{H}^D = \bigotimes_x \mathcal{H}^D_x,$$

$$\mathcal{H}^B = \bigotimes_x \mathcal{H}^B_x,$$

the map $M$ is given by the product

$$M = \prod_x M_x,$$

where each $M_x$ maps $\mathcal{H}^D_x$ to $\mathcal{H}^B_x$. The maps $M_x$ and $M_y$ for distinct $x$ and $y$ commute.

Let $K^D$ be the Hilbert space of Hermitian operators of Sect. 15 for $\mathcal{H}^D$ in place of $\mathcal{H}^B$ and with the additional requirement that $k^D \in K^D$ separately conserve both the fermion number $N^B$ of $\mathcal{H}^B$ and the fermion number $N^C$ of $\mathcal{H}^C$.

We now convert $k^B(\cdot) \in K^B$, $|\omega^B(\cdot)\rangle \in \mathcal{H}^B$ connecting

$$|\omega^B(1)\rangle = \xi|\psi^B\rangle,$$

for a phase factor $\xi$, to the product state $|\omega^B(0)\rangle$ into corresponding $k^D(\cdot) \in K^D$, $|\omega^D(\cdot)\rangle \in \mathcal{H}^D$ connecting some $|\omega^D(1)\rangle$ to a product state $|\omega^D(0)\rangle$ along a path such that for $0 \leq \nu \leq 1$

$$M|\omega^D(\nu)\rangle = |\omega^B(\nu)\rangle,$$

$$\| k^D(\cdot) \| \leq 9 \| k^B(\cdot) \| .$$

In addition, while $|\omega^B(\nu)\rangle$ is an eigenvector of $N^B$ with eigenvalue $n$, $|\omega^D(\nu)\rangle$ is an eigenvector of $N^B$ with eigenvalue 0 and of $N^C$ with eigenvalue $n$. Equation (F9b) implies

$$C^D[|\omega^D(1)\rangle, |\omega^D(0)\rangle] \leq 9C^B[|\omega^B(1)\rangle, |\omega^B(0)\rangle].$$

Thus a lower bound on $C^D[|\omega^D(1)\rangle, |\omega^D(0)\rangle]$ will give a lower bound on $C^B[|\omega^B(1)\rangle, |\omega^B(0)\rangle]$.

Let the product state $|\omega^D(0)\rangle$ be $|\omega^C\rangle \otimes |\omega^B\rangle$ where
\[ |\omega^C \rangle = d_f^C (p_{n+m-1}) \ldots d_f^C (p_m) |\Omega^C \rangle \] (F11a)

\[ |\omega^B \rangle = d_f (p_{m-1}) \ldots d_f (p_0) d_f (q_{m-1}) \ldots d_f (q_0) \times d_b (r_{\nu-1}) \ldots d_b (r_0) |\Omega^B \rangle. \] (F11b)

for \( d_f (p_i), d_f (q_i) \) and \( d_b (r_i) \) from Eq. (F3), and \( d_f^C (p_i) \) constructed from \( d_f (p_i) \) of Eq. (F3) by substituting \( \Sigma^C_1 (x, s) \) for \( \Sigma_1 (x, s) \).

Equations (F8) and (F9a) imply the state \(|\omega^D (1) \rangle\) will satisfy

\[ M |\omega^D (1) \rangle = \xi |\psi^B \rangle. \] (F12)

In addition, since the trajectory \( k^D (v) \) conserves \( N^B \) and \( N^C \) and \(|\omega^D (0) \rangle\), by Eqs. (F11a) and (F11b), has \( N^B \) of 0 and \( N^C \) of \( n \), \(|\omega^D (1) \rangle\) must have these same eigenvalues. Also, since \( M \) acts only on the \( \Sigma^C_1 (x, s) \) fermion content of \(|\omega^D (1) \rangle\) and \(|\psi^B \rangle\), by Eqs. (178a) and (178b), has no boson content and no \( \Sigma^B_0 (x, s) \) anti-fermion content, \(|\omega^D (1) \rangle\) must have no boson content, no \( \Sigma^B_0 (x, s) \) and \( \Sigma^B_1 (x, s) \) content and be given instead by

\[ |\omega^D (1) \rangle = \xi |\psi^C \rangle \otimes |\Omega^B \rangle, \] (F13)

where \(|\psi^C \rangle\) is

\[ q^C = z^{-1} m^{-\frac{1}{2}} \sum_{0 \leq i < m} \xi_i p_i^C, \] (F14a)

\[ |\psi^C \rangle = q^C |\Omega^C \rangle, \] (F14b)

for \( p_i^C \) given by

\[ p_i^C = V^{-\frac{z}{2}} \prod_{0 \leq i < m} \left[ \sum_{x \in D_{ij}, k} u^k (x) \Sigma^C_1 (x, k) \right]. \] (F15)

for the same \( \xi_i, u^k (x) \) and \( D_{ij} \) in Eqs. (175)–(178b) for \(|\psi^B \rangle\).

For both the nonrelativistic version of complexity in Sect. 3.3 and the relativistic version in Sect. 16, \( C(|\psi \rangle) \) is actually independent of the normalization of \(|\psi \rangle\). We can therefore safely set \( z \) to 1 in Eq. (F14a). The result is that \(|\omega^D (1) \rangle\) in Eq. (F13) is normalized to 1, which for consistency we now assume also for \(|\omega^D (0) \rangle\).

Now approximate Eqs. (17a), (17b), (B1) and (B2) for \(|\omega^B (v) \rangle\) by a series of discrete steps

\[ |\omega^B (v + \delta) \rangle = [1 - i \delta k^B (v)] |\omega^B (v) \rangle. \] (F16)

We will prove Eqs. (F9a) and (F9b) by induction in \( v \). Eqs. (F5), (F3), (F11a) and (F11b) give Eq. (F9a) and (F9b) for \( v = 0 \).

Now assume \( k^B (v) \) satisfying Eq. (F9b) has been found for \( v < v' \) such that \(|\omega^D (v) \rangle\) given by
satisfies Eq. (F9a) for $\nu \leq \nu'$. We will show that a $k^D(\nu')$ exists satisfying Eq. (F9b) and extending Eq. (F9a) to $\nu' + \delta$.

According to Eq. (A8), $k^B$ in Eq. (F16) consists of a sum of operators of the form

$$\hat{f}^B_{xy} = f^B_{xy} \otimes I_q,$$

where $f^B_{xy}$ is a Hermitian operator on $\mathcal{H}_x^B \otimes \mathcal{H}_y^B$ for a pair of nearest neighbor sites $\{x, y\}$ which conserves $N^B$ and has vanishing partial traces for both $\mathcal{H}_x^B$ and $\mathcal{H}_y^B$. We assume the dimension $d_\xi$ of $\mathcal{H}_x^D$, and the corresponding slightly smaller dimension of $\mathcal{H}_x^B$, are large enough that the contribution to $k^B$ of single site operators of the form given in Eq. (A7a) can be neglected.

Then the required $k^D(\nu')$ can be found if for every allowed $\hat{f}^B_{xy}$ there is a $\hat{f}^D_{xy}$ of the form

$$\hat{f}^D_{xy} = f^D_{xy} \otimes f^D_q,$$

where $f^D_{xy}$ is a Hermitian operator on $\mathcal{H}_x^D \otimes \mathcal{H}_y^D$ which has vanishing partial traces for both $\mathcal{H}_x^D$ and $\mathcal{H}_y^D$, conserves $N^B$ and $N^C$ and for which

$$M^D_{xy} |\omega^D(\nu')\rangle = \hat{f}^D_{xy} |\omega^B(\nu')\rangle,$$

$$\| f^D_{xy} \| \leq 9 \| f^B_{xy} \|. \quad (F20b)$$

To find the required $f^D_{xy}$, decompose $\mathcal{H}_x^D \otimes \mathcal{H}_y^D$ into a direct sum of subspaces

$$\mathcal{H}_x^D \otimes \mathcal{H}_y^D = \bigoplus_{mn} \mathcal{H}^D_{mn}, \quad (F21)$$

with eigenvalues $m$ and $n$ of $N^B$ and $N^C$, respectively. Similarly, decompose $\mathcal{H}_x^B \otimes \mathcal{H}_y^B$ into a direct sum of subspaces

$$\mathcal{H}_x^B \otimes \mathcal{H}_y^B = \bigoplus_m \mathcal{H}^B_m, \quad (F22)$$

with eigenvalue $m$ of $N^B$.

Let $P^D_{mn}$ be the projection operator onto $\mathcal{H}^D_{mn}$. Define $M_{mn}$ to be

$$M_{mn} = M_x M_y P^D_{mn}, \quad (F23)$$

for $M_x$ and $M_y$ from Eq. (F7). Then $M_{mn}$ maps $\mathcal{H}^D_{mn}$ onto $\mathcal{H}^B_{m+n}$. Let $\mathcal{H}^D_{mn} \perp$ be the orthogonal complement of the subspace of $\mathcal{H}^D_{mn}$ which is mapped to 0 by $M_{mn}$. For each $|\psi^B\rangle \in \mathcal{H}^B_{m+n}$ there is a unique $|\psi^D\rangle \in \mathcal{H}^D_{mn}$ such that

$$M_x M_y |\psi^D\rangle = |\psi^B\rangle. \quad (F24)$$

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For each such \( \psi^B \), define \( M_{mn}^{-1} \)
\[
M_{mn}^{-1}\psi^B = \psi^D, \tag{F25}
\]
and for any \( \psi^B \in \mathcal{H}_\ell^B \) with \( \ell \) other than \( m + n \)
\[
M_{mn}^{-1}\psi^B = 0. \tag{F26}
\]
Equations (F24)–(F26) imply
\[
M_x M_y M_{mn}^{-1} = P_{m+n}^B, \tag{F27}
\]
where \( P_{m+n}^B \) is the projection operator onto \( \mathcal{H}_{m+n}^B \). Define \( g_{xy}^D \) to be
\[
g_{xy}^D = \sum_{mn} M_{mn}^{-1} f_{xy}^B M_{mn}. \tag{F28}
\]
By Eq. (F23), \( g_{xy}^D \) maps each \( \mathcal{H}_{mn}^D \) into itself and therefore conserves both \( N_C \) and \( N_B \). We then have
\[
M_x M_y g_{xy}^D = \sum_{mn} P_{m+n}^B f_{xy}^B M_{mn},
\]
\[
= f_{xy}^B \sum_{mn} M_{mn},
\]
\[
= f_{xy}^B M_x M_y, \tag{F29}
\]
where the first line follows from by Eq. (F27) and the second follows because \( M_{mn}^{-1} \) maps onto \( \mathcal{H}_{m+n}^B \) and \( f_{xy}^B \) conserves \( N_B \). Equations (F7), (F29) and the induction hypothesis, Eq. (F9a) for \( \nu' \), give
\[
M_{xy}^D \omega^D (\nu') = f_{xy}^B M_\nu \omega^D (\nu'),
\]
\[
= f_{xy}^B |\omega^B (\nu')\rangle, \tag{F30}
\]
which is Eq. (F20a).

In addition, since \( M_{mn}^{-1} \) maps into \( \mathcal{H}_{mn}^D \), Eqs. (F23) and (F27) imply
\[
M_{m'n'} M_{mn}^{-1} = \delta_{m'm} \delta_{n'n} P_{m+n}^B. \tag{F31}
\]
We then have
\[
Tr_{xy}^D \left( s_{xy}^D \right)^2 = \sum_{mn} Tr_{xy}^B \left[ P_{m+n}^B f_{xy}^B \left( f_{xy}^B \right)^2 \right]
\]
\[
= \sum_{mn} Tr_{xy}^B \left[ P_{m+n}^B (f_{xy}^B)^2 \right], \tag{F32}
\]
where the first line follows from Eqs. (F28) and (F31) and the second holds because \( f_{xy}^B \) conserves \( N_B \). Since the index \( s \) of \( \Sigma_s (x, s) \) is in the range \( 0 \leq s < 4 \), the maximum possible value of \( N_B \) for \( x \) and \( y \) together is 8. As a result there are at most 9
different combinations of \(m\) and \(n\) giving any value of \(m + n\). Equation (F32) then implies

\[
\text{Tr}_D^D (g_{xy}^D)^2 \leq 9 \text{Tr}_B^B (f_{xy}^B)^2,
\]

which is Eq. (F20b).

Finally, \(g_{xy}^D\) can be split into

\[
g_{xy}^D = f_{xy}^D + \frac{1}{\sqrt{d_H}} I_x \otimes f_{xy}^D + \frac{1}{\sqrt{d_H}} I_Y \otimes I_y + \frac{1}{d_H} f_{xy} I_x \otimes I_y,
\]

where

\[
\text{Tr}_x f_{xy}^D = 0,
\]

\[
\text{Tr}_y f_{xy}^D = 0,
\]

\[
\text{Tr}_x f_{xy}^D = 0,
\]

\[
\text{Tr}_y f_{xy}^D = 0,
\]

and \(d_H\) is the dimension of each \(\mathcal{H}_x^D\). Eqs. (F34)–(F35d) imply

\[
\text{Tr}_x (g_{xy}^D)^2 = \text{Tr}_x (f_{xy}^D)^2 + \text{Tr}_y (f_{xy}^D)^2 + \text{Tr}_x (f_{xy}^D)^2 + (f_{xy}^D)^2.
\]

Equations (F36) and (F33) imply \(f_{xy}^D\) satisfies Eq. (F20b). For \(d_H\) large enough, Eqs. (F34) and (F30) imply \(f_{xy}^D\) satisfies Eq. (F20a).

Which completes the induction step of the proof of Eqs. (F9a) and (F9b). Equation (F10) follows. To obtain a lower bound on \(C^D[|\omega^D(1)|, |\omega^D(0)|]\) we now derive a lower bound on \(C^D[|\omega^D(1)|, |\omega^D(0)|]\).

### 6.2 Schmidt Spectra Again

From each region \(D_{ij}\), extract a subset \(\tilde{D}_{ij}\), consisting of the center points \(x\) of all cells \(c(x)\) reached by starting at \(y_{ij}\) and traveling along a geodesic in \(L(\tau, \sigma)\) in the positive or negative \(x_1\) direction a number \(\geq \frac{d}{2\rho}\) of discrete steps each of proper length \(2\rho\), then traveling along a geodesic in the positive or negative \(x_2\) direction a number \(\geq \frac{d}{2\rho}\) of discrete steps each of proper length \(2\rho\), then traveling along a geodesic in the positive or negative \(x_3\) direction a number \(\geq \frac{d}{2\rho}\) of discrete steps each of proper length \(2\rho\). Since each \(c(x)\) is contained within a sphere of radius \(\rho\) around its center point, none of the points in \(\tilde{D}_{ij}\) will be nearest neighbors and for large \(d\), the total number of points in each \(\tilde{D}_{ij}\) will be nearly \(\frac{d^3}{\rho^3}\). Since \(V\) is between
$\frac{48d^3}{\pi \rho^3}$ and $\frac{6d^3}{\pi \rho^3}$, the number of points in each $\hat{D}_{ij}$ is $zV$ for $z$ between $\frac{z}{6}$ and $\frac{z}{48}$. We will assume $V$ is large enough that we can ignore the statistical uncertainty in the number of points in each $\hat{D}_{ij}$.

From this set of $\hat{D}_{ij}$ construct a set of subsets $E_\ell$ each consisting of $2n$ distinct points chosen from $2n$ distinct $\hat{D}_{ij}$. Since there are $mn$ sets $\hat{D}_{ij}$, there will be $\frac{zmV}{2}$ sets $E_\ell$.

For each $E_\ell$ form the tensor product spaces

$$Q_\ell = \bigotimes_{x \in E_\ell} \mathcal{H}_x^C,$$  \hspace{1cm} (F37a)

$$R_\ell = \bigotimes_{x \notin E_\ell} \mathcal{H}_x^C \otimes \mathcal{H}_B.$$  \hspace{1cm} (F37b)

The space $Q_\ell$ has dimension $16^{2n}$ and $\mathcal{H}^D$ becomes

$$\mathcal{H}^D = Q_\ell \otimes R_\ell.$$  \hspace{1cm} (F38)

A Schmidt decomposition of $|\omega^D(v)\rangle$ according to Eq. (F38) then becomes

$$|\omega^D(v)\rangle = \sum_j \lambda_{j\ell}(v)|\phi_{j\ell}(v)\rangle|\chi_{j\ell}(v)\rangle,$$  \hspace{1cm} (F39)

where

$$|\phi_{j\ell}(v)\rangle \in Q_\ell,$$  \hspace{1cm} (F40a)

$$|\chi_{j\ell}(v)\rangle \in R_\ell,$$  \hspace{1cm} (F40b)

for $0 \leq j < 16^{2n}$ and real non-negative $\lambda_{j\ell}(v)$ which fulfill the normalization condition

$$\sum_j [\lambda_{j\ell}(v)]^2 = 1.$$  \hspace{1cm} (F41)

Each $|\phi_{j\ell}(v)\rangle$ is a pure fermion state while the $|\chi_{j\ell}(v)\rangle$ can include fermions, antifermions and bosons.

The fermion number operators $N^C[Q_\ell]$ and $N^C[R_\ell]$ commute and $|\omega^D(v)\rangle$ is an eigenvector of the sum with eigenvalue $n$. It follows that the decomposition of Eq. (F39) can be done with $|\phi_{j\ell}(v)\rangle$ and $|\chi_{j\ell}(v)\rangle$ eigenvectors of $N^C[Q_\ell]$ and $N^C[R_\ell]$, respectively, with eigenvalues summing to $n$. Let $|\phi_{0\ell}\rangle$ be $|\Omega_\ell\rangle$, the vacuum state of $Q_\ell$, and let $|\phi_{i\ell}(v)\rangle$, $1 \leq i \leq 8n$, span the $8n$-dimensional subspace of $Q_\ell$ with $N^C[Q_\ell]$ of $1$. We assume the corresponding $\lambda_{i\ell}(v)$, $1 \leq i \leq 8n$, are in nonincreasing order. Consider Eqs. (F13)–(F15) for $|\omega^D(1)\rangle$. For any choice of $\ell$ there will be a set of $2n$ nonzero orthogonal $|\phi_{1\ell}(1)\rangle, \ldots, |\phi_{2n\ell}(1)\rangle$ with
\[ \lambda_{j\ell}(1) = \sqrt{\frac{1}{mV}}. \]  

(F42)

On the other hand, for the product state \(|\psi^D(0)\rangle\) in Eqs. (F11a) and (F11b), the \(|\phi_{j\ell}(\nu)\rangle\) come entirely from \(|\omega^C\rangle\), which is a product of \(n\) independent single fermion states. The space spanned by the projection of these into some \(Q_\ell\) is at most \(n\) dimensional, and as a result at most \(n\) orthogonal \(|\phi_{1\ell}(0)\rangle, \ldots |\phi_{n\ell}(0)\rangle\) can occur. Therefore at \(\nu = 0\), there will be at most \(n\) nonzero \(\lambda_{1\ell}(0), \ldots \lambda_{n\ell}(0)\). For \(n < j \leq 8n\), we have

\[ \lambda_{j\ell}(0) = 0. \]  

(F43)

Since \(\{\lambda_{j\ell}(\nu)\}\) is a unit vector, Eqs. (F42) and (F43) imply that as \(\nu\) goes from 0 to 1, \(\{\lambda_{j\ell}(\nu)\}\) must rotate through a total angle of at least \(\arcsin(\sqrt{\frac{n}{mV}})\).

For the small interval from \(\nu\) to \(\nu + \delta\nu\) let \(\mu_{j\ell}(\nu)\) and \(\theta_{\ell}(\nu)\) be

\[ \lambda_{j\ell}(\nu + \delta\nu) = \lambda_{j\ell}(\nu) + \delta\nu \mu_{j\ell}(\nu), \]  

(F44a)

\[ \theta_{\ell}(\nu)^2 = \sum_j |\mu_{j\ell}(\nu)|^2. \]  

(F44b)

We then have

\[ \int_0^1 |\theta_{\ell}(\nu)| d\nu \geq \arcsin(\sqrt{\frac{n}{mV}}). \]  

(F45)

Summed over the \(\frac{zmV}{2}\) values of \(\ell\), Eq. (F45) becomes

\[ \sum_\ell \int_0^1 |\theta_{\ell}(\nu)| d\nu \geq \frac{zmV}{2} \arcsin \left(\sqrt{\frac{n}{mV}}\right), \]  

(F46)

and therefore

\[ \sum_\ell \int_0^1 |\theta_{\ell}(\nu)| d\nu \geq \frac{r}{\pi} \sqrt{mnV}, \]  

(F47a)

\[ \geq \frac{1}{48} \sqrt{mnV}, \]  

(F47b)

since \(z\) is greater than \(\frac{z}{48}\).

### 6.3 Rotation Matrix and Rotation Angle Bounds

The rest of the proof of the lower bound on relativistic complexity, Eq. (180), is a copy of Sections 2.3 and 2.4 of the proof in Appendix 2 of the non-relativistic lower bound,
Eq. (37), but with the non-relativistic fermion charge \( N \) and Hilbert spaces \( \mathcal{H}^{a} \) and \( \mathcal{H}^{b} \) replaced, respectively, by \( N^{c}, \mathcal{H}^{c} \) and \( \mathcal{H}^{b} \).

As in Appendix 2.3, the rotation of \( \lambda_{j\epsilon}(v) \) during the interval from \( v \) to \( v + \delta v \) will be determined by the sum \( g^{D}_{\epsilon}(v) \) of all contributions to \( k^{D}(v) \) of Eq. (F17) arising from \( f^{D}_{xy} \) for nearest neighbor pairs \( \{x, y\} \) with one point, say \( x \), in \( E_{\epsilon} \). By construction of the \( E_{\epsilon} \), if \( x \) is in \( E_{\epsilon} \), \( y \) can not be in \( E_{\epsilon} \) or any distinct \( E_{\epsilon'} \). A repeat of the derivation of Eqs. (B36)–(B38) then leads to

\[
\mu_{j\epsilon}(v) = \sum_{k} r_{jk\epsilon}(v)\lambda_{k\epsilon}(v), \tag{F48}
\]

for the rotation matrix \( r_{jk\epsilon}(v) \)

\[
r_{jk\epsilon}(v) = -\text{Im}[\langle \phi_{j\epsilon}(v)|\langle \chi_{k\epsilon}(v)|g^{D}_{\epsilon}(v)|\phi_{k\epsilon}(v)|\chi_{k\epsilon}(v)\rangle], \tag{F49}
\]

for \( |\phi_{k\epsilon}(v)\rangle \) and \( |\chi_{k\epsilon}(v)\rangle \) of Eq. (F39) and \( \mu_{j\epsilon}(v) \) of Eq. (F44a).

Since the \( f^{D}_{xy} \) contributing to \( g^{D}_{\epsilon}(v) \) conserve total fermion number \( N^{c} \), \( g^{D}_{\epsilon}(v) \) can be expanded as

\[
g^{D}_{\epsilon}(v) = \sum_{x\in E_{\epsilon}, y\not\in E_{\epsilon}} g^{D}_{\epsilon}(x, y, v), \tag{F50a}
\]

\[
g^{D}_{\epsilon}(x, y, v) = \sum_{i=0,1} a^{i}(x, y, v)z^{i}(x, y, v) \tag{F50b}
\]

where \( z^{0}(x, y, v) \) acts only on states with \( N^{c}(\mathcal{H}^{D}_{x} \otimes \mathcal{H}^{D}_{y}) \) of 0, \( z^{1}(x, y, v) \) acts only on states with \( N^{c}(\mathcal{H}^{D}_{x} \otimes \mathcal{H}^{D}_{y}) \) strictly greater than 0, and the \( z^{i}(x, y, v) \) are normalized by

\[
\| z^{i}(x, y, v) \| = 1. \tag{F51}
\]

The operator \( z^{0}(x, y, v) \) will be

\[
z^{0}(x, y, v) = z^{0c}(x, y) \otimes g^{B}(x, y, v), \tag{F52a}
\]

\[
z^{0c}(x, y, v) = P^{C}(x, y) \bigotimes_{q \neq x, y} I_{q}, \tag{F52b}
\]

where \( P^{C}(x, y) \) projects onto the vacuum state of \( \mathcal{H}^{C}_{x} \otimes \mathcal{H}^{C}_{y} \) and \( g^{B}(x, y, v) \) is a normalized Hermitian operator acting on \( \mathcal{H}^{B}_{x} \otimes \mathcal{H}^{B}_{y} \).

Combining Eqs. (F44b),(F48) - (F50b) gives

\[
|\theta_{\epsilon}(v)| \leq \sum_{x\in E_{\epsilon}, y\not\in E_{\epsilon}} |\theta^{j}_{\epsilon}(x, y, v)| \tag{F53a}
\]

\[
|\theta^{j}_{\epsilon}(x, y, v)|^2 = \sum_{j} [\mu^{j}_{\epsilon}(x, y, v)]^2, \tag{F53b}
\]
with the definitions
\[
\mu_{j\ell}^i(x, y, \nu) = -a^i(x, y, \nu) \sum_k \text{Im}\{\langle \phi_{j\ell}(v)|\chi_{j\ell}(v)\rangle \times z_j^i(x, y, \nu)|\phi_{k\ell}(v)|\chi_{k\ell}(v)\}.
\] (F54)

A duplicate of the proof of Eqs. (B46)–(B52) then yields
\[
[\theta_0^i(x, y, \nu)]^2 \leq [a^0(x, y, \nu)]^2 \langle \omega(v)|[I - z^0(x, y)]|\omega(v)\rangle,
\] (F55a)
\[
[\theta_1^i(x, y, \nu)]^2 \leq [a^1(x, y, \nu)]^2 \langle \omega(v)|[I - z^0(x, y)]|\omega(v)\rangle,
\] (F55b)
which combined with Eq. (F53a) imply
\[
\sum_\ell |\theta_\ell(v)| \leq \sum_{x \in E, y \in E} \left| [|a^0(x, y, \nu)| + |a^1(x, y, \nu)|] \times \sqrt{\langle \omega(v)|[I - z^0(x, y)]|\omega(v)\rangle} \right|.
\] (F56)

where
\[
E = \bigcup_\ell E_\ell.
\] (F57)

The Cauchy–Schwarz inequality then gives
\[
[\sum_\ell |\theta_\ell(v)|]^2 \leq \sum_{x \in E, y \in E} \left[ [|a^0(x, y, \nu)| + |a^1(x, y, \nu)|]^2 \times \sum_{x \in E, y \in E} \langle \omega(v)|[I - z^0(x, y)]|\omega(v)\rangle \right].
\] (F58)

A repeat of the argument leading to Eq. (B56) implies
\[
\sum_{x \in E, y \in E} \langle \omega(v)|[I - z^0(x, y)]|\omega(v)\rangle \leq Mn,
\] (F59)

where \( M \) is the maximum number of nearest neighbors of any lattice point \( x \). An upper bound on \( M \) can be found as follows. Recall any \( c(x) \) is contained in a sphere with center \( x \) and radius \( \rho \) and contains a sphere with center \( x \) and radius \( \frac{\xi}{2} \). It follows that \( M \) is less than or equal to the number of disjoint spheres of radius \( \frac{\xi}{2} \) that can be placed with centers on a sphere with center \( x \) and radius \( 2\rho \). For each of the \( \frac{\xi}{2} \) spheres, a slice through its center orthogonal to the line from its center to \( x \) will be contained in a sphere with center \( x \) and radius \( \frac{\sqrt{17}}{2} \). The area of each of these slices is \( \frac{\pi\rho^2}{4} \), the area of the radius \( \frac{\sqrt{17}}{2} \) sphere is \( \frac{68\pi\rho^2}{4} \), and therefore
\[
M \leq 68.
\] (F60)

By Eq. (16)
\[ \| k^D(v) \| ^2 \geq \sum_{\ell, x \in E, y \not\in E} \| g^D_\ell(x, y, v) \| ^2 \]  \hspace{1cm} (F61)

In addition, \( z^0(x, y, v) \) is orthogonal to \( z^1(x, y, v) \). It follows that
\[ \| k^D(v) \| ^2 \geq \sum_{x \in E, y \not\in E} [\| a^0(x, y, v) \| ^2 + \| a^1(x, y, v) \| ^2]. \]  \hspace{1cm} (F62)

Assembling Eqs. (F58), (F59), (F60) and (F62) gives
\[ \| k^D(v) \| ^2 \geq \frac{1}{2} \sum_{x \in E, y \not\in E} \| a^0(x, y, v) \| + \| a^1(x, y, v) \| ^2 \]
\[ \geq \frac{1}{136n} \| \theta_\ell(v) \| ^2 \]  \hspace{1cm} (F63)

Eq. (F47b) then implies
\[ \int_0^1 \| k(v) \| \geq \frac{1}{2348} \sqrt{mV} , \]
and therefore
\[ C^D(|\omega^D(1)\rangle, |\omega^D(0)\rangle) \geq \frac{1}{2348} \sqrt{mV} , \]  \hspace{1cm} (F65)

which by Eqs. (F10) and (F8) yields
\[ C^B(|\psi^B\rangle, |\omega^B(0)\rangle) \geq \frac{1}{21132} \sqrt{mV} . \]  \hspace{1cm} (F66)

Since Eq. (F66) holds for all product \( |\omega^B(0)\rangle \) we finally obtain
\[ C^B(|\psi^B\rangle) \geq \frac{1}{21132} \sqrt{mV} . \]  \hspace{1cm} (F67)

**Appendix 7: Upper Bound on the Complexity of Entangled Relativistic States**

The proof of Eq. (181) bounding from above the complexity of the entangled relativistic state \( |\psi^B\rangle \) of Eq. (178b) follows the proof in Appendix 3 of an upper bound on the complexity of the entangled non-relativistic state of Eq. (36), but with the regular lattice of Sect. 3.1 replaced by the random lattice of Sect. 13 and \( \mathcal{H} \) replaced by \( \mathcal{H}^B \).

An upper bound on \( C^B(|\psi^B\rangle) \) is given by \( C^B(|\psi^B\rangle, |\omega^B\rangle) \) for any product state \( |\omega^B\rangle \), for which in turn an upper bound is given by

\[ \sum_{\ell, x \in E, y \not\in E} \| g^D_\ell(x, y, v) \| ^2 \geq \frac{1}{21132} \sqrt{mV} . \]
\[ C^B(|\psi^B\rangle, |\omega^B\rangle) \leq \int_0^1 dt \ |k^B(v)|, \tag{G1} \]

for any trajectory \( k^B(v) \in K^B \) fulfilling

\[ \frac{d\omega^B(v)}{dv} = -ik^B(v)\omega^B(v), \tag{G2a} \]

\[ \omega^B(0) = |\omega^B\rangle, \tag{G2b} \]

\[ \omega^B(1) = \xi |\psi^B\rangle, \tag{G2c} \]

for a phase factor \( \xi \).

As in Appendix 3, construction of a sufficient \( k^B(v) \) begins with an \( |\omega^B\rangle \) consisting of \( n \) fermions each at one of a corresponding set of \( n \) single points. Then \( |\omega^B\rangle \) is split into a sum of \( m \) orthogonal product states, each again consisting of \( n \) fermions one at each of a corresponding set of \( n \) single points. Then the position of each of the fermions in the product states is moved to the center of one of the monomials of Eq. (177). Finally, by approximately \( \ln(V)/\ln(8) \) iterations of a fan-out tree, the \( mn \) wave functions concentrated at points are spread over the \( mn \) cubes \( D_{ij} \).

### 7.1 Cell Count Bound

The bound on \( C^B(|\psi^B\rangle) \) relies on a bound we will derive first on the number of distinct cells \( c(x), x \in L(\tau, \sigma, \rho) \), which intersect a geodesic \( g(\lambda) \in L(\tau, \sigma), 0 \leq \lambda \leq \lambda_{\text{max}}, \) of length \( \lambda_{\text{max}} \).

Let \( \bar{g} \) be the set of all points within a proper distance \( 2\rho \) of any point on \( g(\lambda) \). Since every \( c(x) \) is contained in a sphere with center \( x \) and radius \( \rho \), it follows that \( \bar{g} \) contains all \( c(x) \) which intersect \( g(\lambda) \). On the other hand, each \( c(x) \) within \( \bar{g} \) contains a sphere with center \( x \) and radius \( \frac{\rho}{2} \) which is disjoint from all other \( c(x') \) contained in \( \bar{g} \). The total volume occupied by the collection of disjoint radius \( \frac{\rho}{2} \) spheres has to be less than the total volume of \( \bar{g} \). The number \( p(\lambda_{\text{max}}) \) of \( c(x) \) which intersect \( g(\lambda) \) is therefore bounded by

\[ p(\lambda_{\text{max}}) \leq 24 \frac{\lambda_{\text{max}}}{\rho} + 64. \tag{G3} \]

### 7.2 Product State to Entangled State

For each value of \( 0 \leq i < m \), let \( x_{i0} \) be the center point of the cell found by traveling from an arbitrarily chosen starting point, \( x_{00} \), along a geodesic in the \( x^1 \) direction a proper distance of \( 4i\rho \). Then from each \( x_{i0} \) travel along a geodesic in the \( x^2 \) direction. For each \( 0 < j < n \), let \( x_{ij} \) be the center point of the cell the geodesic beginning at \( x_{i0} \) enters after leaving the cell with center point \( x_{ij-1} \). All points on the geodesics...
beginning at \(x_{i_0}\) and at \(x_{i_0}\) for \(i \neq i'\) will be at least a distance of \(4\rho\) apart. As a result each \(x_{ij}\) will be both distinct from and not a nearest neighbor of each \(x_{i'j'}\) with \(i \neq i'\). The gap between \(x_{ij}\) and \(x_{i'j'}\) accomplishes the goal of making it possible, despite the random placement of cells, to insure that \(x^B_{ij}\) and \(x^B_{ij+1}\) are nearest neighbors as will turn out to be required.

Let the set of \(n\)-particle product states \(|\omega_i^B\rangle\) be

\[
|\omega_i^B\rangle = \prod_{0 \leq j < n} \frac{1}{m} \sum_k u^k(x_{ij}) \Sigma_1(x_{ij}, k) |\Omega^B\rangle.
\]

The entangle \(n\)-particle state \(|\chi^B\rangle\)

\[
|\chi^B\rangle = \sqrt{\frac{1}{m}} \sum_i |\omega_i^B\rangle
\]

we generate from a sequence of unitary transforms acting on \(|\omega^B\rangle = |\omega_0^B\rangle\).

The sequence of \(k^B\) which convert the product state \(|\omega_i^B\rangle\) into the entangled state \(|\chi^B\rangle\) follows the sequence of \(k\) mapping the product state \(|\omega_i^B\rangle\) to the entangled state \(|\chi^B\rangle\) in Appendix 3.1, with the non-relativistic fermion operator \(\Psi^T(x, s)\) replaced by the relativistic \(\hat{\Sigma}(x, s)\).

From \(k^B_0, \ldots, k^B_{n-2}\) in place of \(k_0, \ldots, k_{n-2}\) of Eqs. (C5)–(C13) we obtain

\[
\exp(i\theta^B_{n-2} k^B_{n-2}) \ldots \exp(i\theta^B_0 k^B_0) |\omega_0^B\rangle = \sqrt{\frac{1}{m}} |\omega_0^B\rangle + \sqrt{\frac{m-1}{m}} \prod_{0 \leq j < n} (\sum_k \nu^k(x_{ij}) \Sigma_1(x_{ij}, k)) |\Omega^B\rangle,
\]

with

\[
|| k^B_i || = \sqrt{2}, \quad \theta^B_i \leq \frac{\pi}{2}.
\]

as in Eqs. (C18a) and (C18b) and therefore total cost

\[
\sum_{0 \leq j \leq n-2} |\theta^B_j| \cdot || k^B_j || \leq \frac{\pi(n-1)}{\sqrt{2}}.
\]

The spinor \(\nu^k(x)\) in Eq. (G6), as defined in Section 17, is orthogonal to \(u^k(x)\) of Eq. (G4) and obtained, as is \(u^k(x)\), by boosting from the origin of \(L(\tau, \sigma)\) to \(x\) a spin state of a free fermion at rest at the origin of \(L(\tau, \sigma)\).

Then from \(k^B_{n-1}, \ldots, k^B_{n-1+p}, 3n - 2 \geq p < 48n^2 + 159n,\) in place of \(k_{n-1}, \ldots, k_{2n-2}\) of Eqs. (C14a)–(C15) we obtain

\[
\exp(i\theta^B_{n-1+p} k^B_{n-1+p}) \ldots \exp(i\theta^B_0 k^B_0) |\omega_0^B\rangle = \sqrt{\frac{1}{m}} |\omega_0^B\rangle + \sqrt{\frac{m-1}{m}} |\omega_1^B\rangle,
\]

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with $\| k^B_i \|, |\theta^B_i|$ satisfying Eqs. (G7a) and (G7b) and incremental cost

$$\sum_{n-1 \leq j \leq n-1+p} |\theta^B_j| \| k^B_j \| < 24\sqrt{2\pi n^2} + \frac{159\pi}{\sqrt{2}} n.$$  \hfill (G10)

The count of additional $k^B_i$ required for Eq. (G9) arises as follows. A geodesic between $x^B_{ij}$ and $x^B_{i+1j}$ has proper length $\lambda$ in the range $2\rho \leq \lambda < (2n + 4)\rho$ and therefore, according to Eq. (G3), can pass through a total of between 3 and $48n + 160$ cells, and thus requires between 2 and $48n + 159$ nearest neighbor steps. The sequence of $k^B_{n-1}, \ldots, k^B_{n-1+p}$ for the map of Eq. (G9) can be required to complete between 2 and $48n + 159$ such steps from $x^B_{ij}$ and $x^B_{i+1j}$ for each $0 \geq j < n$, hence $3n - 2 \geq p < 48n^2 + 159n$.

Following Eqs. (C16) and (C17), we now apply copies of the maps of Eqs. (G6) and (G9) along the $x^2$ direction geodesics at $x_{10}, \ldots, x_{n0}$ with end result

$$\exp(iq^B_{k^B}k^B_q) \ldots \exp(iq^B_0k^B_0)|\phi^B_0\rangle = |\phi^B_i\rangle |\phi^B_j\rangle \ldots |\phi^B_n\rangle,$$

where all $k^B_i$ satisfy Eq. (G7a), $\theta^B_i$ satisfy Eq. (G7b) and

$$q < 48mn^2 + 160mn.$$  \hfill (G12)

The cost of the transition from $|\omega^B\rangle$ to $|\chi^B\rangle$ is then bounded by

$$C^B(|\chi^B\rangle, |\omega^B\rangle) \leq 24\sqrt{2\pi mn^2} + 80\sqrt{2\pi mn}.$$  \hfill (G13)

### 7.3 Entangled State Repositioned

Let the entangled $n$-particle state $|\phi^B\rangle$ be

$$|\phi^B\rangle = \sum_i \xi_i \prod_j \left[ \sum_k u^k(y_{ij}) \Sigma_1(y_{ij}, k) \right] |\Omega\rangle,$$

where, as defined in Section 17, $y_{ij}$ is the center of cube $D_{ij}$ in Eq. (175) and $\xi_i$ is the phase factor of monomial $p_i$ in Eq. (176a).

Equations (C21a)–(C32) translate directly from the non-relativistic field theory to the relativistic case, with the result

$$C^B(|\phi^B\rangle, |\chi^B\rangle) \leq \frac{\pi \sqrt{mn r}}{\sqrt{2}}.$$  \hfill (G15)

The distance $r$ is given by

$$r = \min_{\delta_{ij}} \max_{\delta_{ij}} r_{\delta_{ij}}$$  \hfill (G16)
where \( r_{ij} \) is the number of nearest neighbor steps in the shortest path between lattice points \( x_{ij} \) and \( y_{ij} \) such that no pair of paths for distinct \( \{i,j\} \) intersect, \( y_{ij} \) is the center point of \( D_{ij} \) and \( x_{ij} \) is the \( mxn \) grid of points of Appendix 7.2.

### 7.4 Fan-Out

Following Appendix 3.3 of the proof of the non-relativistic upper bound in Appendix 3, the state \( |\phi^B\rangle \) with particles at the centers of the cubes \( D_{ij} \) we now fan-out to the state \( |\psi^B\rangle \) with particle wave functions spread uniformly over the cubes \( D_{ij} \). For sufficiently small \( \rho \) nearly all of the complexity in the bound on \( C^B(|\psi^B\rangle) \) is generated in this step.

We will construct a fan-out initially for \( D_{00} \), which will then be duplicated on the remaining \( D_{ij} \). Recall the \( x \in D_{00} \) are the centers of all cells crossed by starting at \( y_{00} \) and traveling along a geodesic in the positive or negative \( x^1 \) direction a proper distance of less than \( d \), then in the positive or negative \( x^2 \) direction a proper distance less than \( d \), then in the positive or negative \( x^3 \) direction a proper distance less than \( d \).

The set of \( x \in D_{00} \) we will arrange as the endpoints of a tree constructed in a sequence of stages most of which increase the number of endpoints of the tree by a factor of 8. Starting at \( y_{00} \), travel along a geodesic in the positive or negative \( x^1 \) directions a proper distance of \( \frac{d}{2} \). Define this set of 2 points to be \( s(1) \). From each of the points of \( s(1) \), travel along a geodesic in the positive or negative \( x^2 \) direction a proper distance of \( \frac{d}{2} \). Let this set of 4 points be \( s(2) \). From each of the points of \( s(2) \), travel along a geodesic in the positive or negative \( x^3 \) direction a proper distance of \( \frac{d}{2} \). The resulting set of 8 points is \( s(3) \). Repeating this sequence of 3 steps a total of \( p \) times yields a set \( s(3p) \) of \( 8^p \) endpoints, each a distance of \( \frac{d}{2^{p-1}} \) from its nearest neighbor. For each \( y \in s(3p) \) let \( s(y) \) be the set of 8 \( y' \in s(3p+3) \) reached by a sequence of geodesic segments originating at \( y \).

Now choose \( q \) such that

\[
\rho < \frac{d}{2^q} \leq 2\rho. \tag{G17}
\]

Each pair of distinct points in \( s(3q) \) will be separated by a distance of at least \( 2\rho \). Since every cell \( c(x) \) is contained in a sphere of radius \( \rho \) around \( x \), each \( y \in s(3q) \) will lie in a distinct cell. Similarly, for all \( r < q \), each \( y \in s(3r) \) will lie in a distinct cell. For each \( y \in s(3r), r \leq q \), let \( x(y) \) be the center point of the cell containing \( y \).

At the outset of Sect. 17 we assumed \( \rho \) is much smaller than the proper time \( \tau \) of the hyperboloid \( L(\tau, \sigma) \). The region in \( L(\tau, \sigma) \) occupied by a collection of nearby \( y \in s(3q) \) will therefore be nearly flat and can be divided up into disjoint cubes each with edge length \( \frac{d}{2r-1} \) centered on a corresponding \( y \in s(3q) \). Let the cube for \( y \in s(3q) \) be \( t(y) \). The union of all \( t(y) \) covers \( D_{00} \). Let \( w(y) \) be

\[
w(y) = t(y) \cap D_{00}. \tag{G18}
\]

Define \( n(y) \) to be the number of points in \( w(y) \). Working backwards iteratively from \( s(3q) \), define \( n(y) \) for \( y \in s(3p), p < q \), by
Carried back to \( n(y_{00}) \) the result is \( V \), the total number of points in \( D_{00} \).

For any \( r \leq q \), define the state \( \nu_{3r}^B \) to be

\[
|\nu_{3r}^B\rangle = \sum_{y \in \mathcal{S}(3r), k} \frac{1}{\sqrt{V}} u^k(y) \sum_{x(y), k} |x(y), k\rangle |\Omega^B\rangle.
\] (G20)

Equations (C35a)–(C49) of the non-relativistic fan-out process in Appendix 3.3 can then be adapted to generate a sequence of \( \exp(i \frac{\pi}{2} k_B^2) \) which maps \( |\nu_{3r-3}^B\rangle \) into \( |\nu_{3r}^B\rangle \). For the non-relativistic fan-out process, each step in which a state is split yields a pair of equally weighted pieces. For the splitting process corresponding to the states of Eq. (G20) the resulting pair will not in general be weighted equally, but this difference by itself does not affect the complexity bound. In the course of the map taking \( |\nu_{3r-3}^B\rangle \) into \( |\nu_{3r}^B\rangle \), each of the 3 geodesic segments by which any point in \( s(3r) \) is reached from its parent point in \( s(3r - 3) \) will be of length \( \frac{d}{2} \). Equation (G3) implies that the number of nearest neighbor steps to traverse a segment of length \( \frac{d}{2} \) is bounded by \( 24 \frac{d}{\rho^2} + 63 \). A repetition of the derivation of Eq. (C50) then yields

\[
C^B(|\nu_{3r}^B\rangle, |\nu_{3r-3}^B\rangle) < (3 + \sqrt{2})(24 \frac{d}{\rho^2} + 63)2^{\frac{r-3}{2}} \pi.
\] (G21)

The last step in the fan-out process consists of splitting the piece of \( |\nu_{3q}^B\rangle \) at each \( x(y) \) into \( n(y) \) equally weighted components, then distributing these across the cube \( t(y) \) to produce the state

\[
|\nu_{3q+1}^B\rangle = \sum_{x \in D_{00}, k} \frac{1}{\sqrt{V}} u^k(x) \sum_{x(y), k} |x(y), k\rangle |\Omega^B\rangle.
\] (G22)

The complexity of the map taking \( |\nu_{3q}^B\rangle \) to \( |\nu_{3q+1}^B\rangle \) can be bounded as follows. For each \( y \in s(3q) \) the length of the shortest line connecting the cell holding \( y \) to the cell holding any \( x \in w(y) \) is bounded by \( \frac{\sqrt{3}d}{2q} \), the distance from \( y \) to a corner of \( t(y) \), which according to Eq. (G17) is bounded by \( 2\sqrt{3}\rho \). Equation (G3) implies that the number of nearest neighbor steps to traverse a segment of length \( 2\sqrt{3}\rho \) is bounded by \( 48\sqrt{3} + 63 \). For any \( x \in w(y) \), at each \( z \in w(y) \) along the path from \( x \) to \( y \), the remaining path from \( z \) to \( y \) is the shortest nearest neighbor route to \( y \). It follows that if the paths from some \( x \in w(y) \) to \( y \) and from a distinct \( x' \in w(y) \) to \( y \) coincide at \( z \) the remaining segments from \( z \) to \( y \) will also coincide. The collection of shortest paths from all \( x \in w(y) \) to \( y \) must therefore form a tree, each branch of which consists of at most \( 48\sqrt{3} + 63 \) nearest neighbor steps. By adapting the derivation of Eq. (C50) the cost of all such paths executed in parallel for all \( x \in D_{00} \), the total count of which is \( V \), can then be bounded to give
Summing Eq. (G21) over $r$ from 1 to $q$, adding Eq. (G23) and using Eq. (G17) gives

$$C^B(|v^B_{3q+1}>, |v^B_{3q+1}>) \leq (48 \sqrt{3} + 63) \frac{\pi}{\sqrt{2}} \sqrt{V}.$$  \hfill (G23)

The bound of Eq. (G24) applies to a fan-out process on a single cube $D_{00}$. Assume the process repeated in parallel on the $mn$ cubes $D_{ij}$, thereby generating $|\psi^B\rangle$ of Eq. (178b). For $|\phi^B\rangle$ of Eq. (G14) we then have

$$C^B(|\psi^B\rangle, |\phi^B\rangle) \leq c_1 \sqrt{mnV}.$$  \hfill (G26)

From Eqs. (G13) and (G15), it follows that for a product state $|\omega^B\rangle$ we have

$$C^B(|\psi^B\rangle, |\omega^B\rangle) \leq c_1 \sqrt{mnV} + c_2 mn^2 + c_3 mn + c_4 \sqrt{mn},$$  \hfill (G27)

for $c_1$ of Eq. (G25), $r$ of Eq. (G16) and

$$c_2 = 24 \sqrt{2} \pi,$$  \hfill (G28a)

$$c_3 = 80 \sqrt{2} \pi,$$  \hfill (G28b)

$$c_4 = \frac{\pi}{\sqrt{2}}.$$  \hfill (G28c)

Equation (181) then follows.

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