Riemannian Hyperbolization

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Classical flat geometry has formed part of basic human knowledge since ancient times. It is characterized by the almost universally known condition that the sum of the internal angles of a triangle $\triangle$ is equal to $\pi$. We write $\Sigma(\triangle) = \pi$. Other fundamental geometries are defined by replacing the equality $\Sigma(\triangle) = \pi$ by inequalities; thus positively curved geometries and negatively curved geometries are determined by the inequalities $\Sigma(\triangle) > \pi$ and $\Sigma(\triangle) < \pi$, respectively, where $\triangle$ runs over all small non-degenerate triangles in a space. It is natural then to try to find spaces that admit such geometries, and this task has been a driving force in Riemannian Geometry for many decades. But surprisingly there are not too many examples of smooth closed manifolds that support either a positively curved or a negatively curved metric. For instance, besides spheres, in dimensions $\geq 17$ (and $\neq 24$) the only positively curved simply connected known examples are complex and quaternionic projective spaces. In negative curvature the situation is arguably more striking because negative curvature has been studied extensively in many different areas in mathematics. Indeed, from the ergodicity of their geodesic flow in Dynamical Systems to their topological rigidity in Geometric Topology; from the existence of harmonic maps in Geometric Analysis to the well-studied and greatly generalized algebraic properties of their fundamental groups, negatively curved Riemannian manifolds are the main object in many important and well-known results in mathematics. Yet the fact remains that very few examples of closed negatively curved Riemannian manifolds are known. Besides the hyperbolic ones ($\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$), the other known examples are the Mostow-Siu examples (complex dimension 2) which are local branched covers of complex hyperbolic space (1980, [23]), the Gromov-Thurston examples (1987, [18]) which are branched covers of real hyperbolic ones, the exotic Farrell-Jones examples (1989, [12]) which are homeomorphic but not diffeomorphic to real hyperbolic manifolds (and there are other examples of exotic type), and the three examples of Deraux (2005, [10]) which are of the Mostow-Siu type in complex dimension 3. Hence, excluding the Mostow-Siu and Deraux examples (in dimensions 4 and 6, respectively), all known examples of closed negatively curved Riemannian manifolds are homeomorphic to either a hyperbolic one or a branched cover of a hyperbolic one.

Remark. The Mostow-Siu and Deraux examples have covers that have maps to complex hyperbolic space that look locally like branched covers. It is not known whether all these examples are global branched covers.

This lack of examples in negative curvature changes dramatically if we allow singularities, and a very rich and abundant class of negatively curved spaces (in the geodesic sense) exists due to the strict hyperbolization process of Charney and Davis [6]. The hyperbolization process was originally introduced by M. Gromov [14], and later studied by Davis and Januszkiewicz [9], and Charney-Davis strict hyperbolization is built on these previous versions. The hyperbolization process is conceptually (but not technically) quite simple since it has a lego type flavor: in the same way as simplicial complexes

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and cubical complexes are built from a basic set of pieces, basic “hyperbolization pieces” are chosen, and anything that can be built or assembled with these pieces will be negatively curved. This conceptual simplicity could be in some sense a bit deceptive because hyperbolization produces an enormous class of examples with a very fertile set of properties. But the richness and complexity of the hyperbolized objects are matched by the richness and complexity of the singularities obtained, and hyperbolized smooth manifolds are very far from being Riemannian. Interestingly one can relax and lose even more regularity and consider negative curvature from the algebraic point of view, that is consider Gromov’s hyperbolic groups, and it can be argued [26] that “almost every group” is hyperbolic. So, negative curvature is in some weak sense generic, but Riemannian negative curvature seems very scarce. It is natural then to inquire about the difference between the class of manifolds with negatively curved metrics with singularities and its subclass of more regular Riemannian counterparts. More specifically we can ask whether the strict hyperbolization process can be brought into the Riemannian universe. In this paper we give a positive answer to this question, and we do this by proving that all singularities of the Charney-Davis strict hyperbolization of a closed smooth manifold can be smoothed, provided the “hyperbolization piece” is large enough (which can always be done). Moreover we prove that we can do this process in a $\epsilon$-pinched way. Here is the statement of our Main Theorem.

**Main Theorem.** Let $M^n$ be a closed smooth manifold and let $\epsilon > 0$. Then there is a closed Riemannian manifold $N^n$ and a smooth map $f : N \to M$ such that

(i) The Riemannian manifold $N$ has sectional curvatures in the interval $[-1 - \epsilon, -1]$.

(ii) The induced map $f_* : H_\ast(N, R) \to H_\ast(M, R)$ is surjective, for every (untwisted) $R$.

(iii) If $M$ is $R$-orientable then $N$ is $R$-orientable. In this case (ii) implies that $f$ has degree one and $f^* : H^\ast(M, R) \to H^\ast(N, R)$ is injective.

(iv) The map $f^*$ sends the rational Pontryagin classes of $M$ to the rational Pontryagin classes of $N$.

**Addendum to Main Theorem.** The manifold $N$ is the Charney-Davis strict hyperbolization of $M$ but with a different smooth structure. The hyperbolization is done with a sufficiently “large” hyperbolization piece $X$.

By “large” above we mean that the width of the normal neighborhoods of the faces of $X$ are very large. These large pieces always exist (see 9.1). Corollaries 1, 2 and 3 below are the $\epsilon$-pinched Riemannian versions of classical applications of hyperbolization.

**Corollary 1.** Every closed smooth manifold is smoothly cobordant to a closed Riemannian manifold with sectional curvatures in the interval $[-1 - \epsilon, -1]$, for every $\epsilon > 0$.

**Corollary 2.** The cohomology ring of any finite CW-complex embeds in the cohomology ring of closed Riemannian manifold with sectional curvatures in the interval $[-1 - \epsilon, -1]$, for every $\epsilon > 0$.

**Proof.** Let $X$ be a finite CW-complex. Embed $X$ in some $\mathbb{R}^n$ and let $P$ be a compact neighborhood of $X$ that retracts to $X$. Let $M$ be the double of $P$. Then there is a retraction $M \to X$, and Corollary 2 follows from (iii) in the Main Theorem.

Since degree one maps between closed orientable manifolds are $\pi_1$- surjective we obtain the following result.
Corollary 3. For every finite CW-complex $X$ there is a closed Riemannian manifold $N$ and a map $f : N \to X$ such that: (i) $N$ has sectional curvatures in the interval $[-1 - \epsilon, -1]$, (ii) $f$ is $\pi_1$-surjective, (iii) $f$ is homology surjective.

All known examples of closed negatively curved Riemannian manifolds with less than $\frac{1}{4}$-pinched curvature have zero rational Pontryagin classes (for the Gromov-Thurston branched cover examples this was proved by S. Ardanza [1]). The next corollary gives examples of such manifolds with nonzero rational Pontryagin classes.

Corollary 4. For every $\epsilon > 0$ and $n \geq 4$ there is a closed Riemannian $n$-manifold with sectional curvatures in the interval $[-1 - \epsilon, -1]$ and nonzero rational Pontryagin classes.

Proof. Take $M$ in the Main Theorem orientable with nonzero Pontryagin classes. All manifolds given in Corollary 4 are new examples of closed negatively curved manifolds. This follows from Novikov’s topological invariance of the rational Pontryagin classes [25], and the $\frac{1}{2}$-pinched rigidity results given in (or implied by) the work of Hernández [20], Yau and Zheng [37], Corlette [7], Gromov [17] and Mok-Siu-Yeung [22]. We state this in the next corollary.

Corollary 5. For any $\epsilon > 0$ and $n \geq 4$ there are closed Riemannian $n$-manifolds with sectional curvatures in the interval $[-1 - \epsilon, -1]$ that are neither homeomorphic to a hyperbolic manifold $(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$ nor homeomorphic to the Gromov-Thurston branched cover of a real hyperbolic one, nor homeomorphic to one of the Mostow-Siu or Deraux examples.

The next application was suggested to us by Stratos Prassidis some time ago and deals with cusps of negatively curved manifolds. Recall that if $M$ is a complete finite volume noncompact real hyperbolic manifold then $M$ has finitely many cusps isometric to manifolds of the form $Q \times [b, \infty)$ with metric $e^{-2t}h + dt^2$, where $(Q, h)$ is a closed flat manifold and $b \in \mathbb{R}$. If $M$ has exactly one cusp diffeomorphic to $Q \times [b, \infty)$ we say that the manifold $Q$ bounds geometrically a hyperbolic manifold.

More generally, in 1978 M. Gromov defined almost flat manifolds in [16] and similar facts hold for them replacing hyperbolic manifolds by pinched negatively curved manifolds. That is, let $M$ be a complete finite volume noncompact manifold with pinched negative curvature (i.e all sectional curvatures lie in a fixed interval $[-a, -b]$, $0 < b \leq a < \infty$). Then $M$ has finitely many cusps diffeomorphic to manifolds of the form $Q \times [b, \infty)$, where $Q$ is an almost flat manifold. If $M$ has exactly one cusp diffeomorphic to $Q \times [b, \infty)$ we say that the manifold $Q$ bounds geometrically a negatively curved manifold. Of course a necessary condition for $Q$ to bound geometrically as above is to smoothly bound a compact manifold.

Remark. Here we do not assume $Q$ to be connected. Hence if $Q$ bounds geometrically then the number of connected components of $Q$ is the same as the number of connected cusps.

It was proved by Hamrick and Royster [19] that every closed flat manifold bounds smoothly. This together with the work of Gromov in [15], [16] motivated Tom Farrell and Smilka Zdravkovska to make the following well-known conjectures in [13] 30 years ago.

Conjecture 1. Every closed almost flat manifold bounds smoothly. This conjecture was also proposed, independently, by S.-T. Yau in [36].

Conjecture 2. Every closed flat manifold bounds geometrically a hyperbolic manifold.
Conjecture 3. Every closed almost flat manifold bounds geometrically a negatively curved manifold.

It was showed by Long and Reid [21] that Conjecture 2 is false by giving examples of three dimensional flat manifolds that do not bound. The following result says Conjecture 1 implies Conjecture 3.

**Theorem A.** Let $Q$ be a closed almost flat manifold. Assume that $Q$ bounds smoothly. Then $Q$ bounds geometrically a negatively curved manifold $M$.

Conjecture 1 has generated a lot of research in the last 30 years and it is known to be true for an almost flat manifold in the following cases. Let $Q$ be almost flat. Then $Q$ is covered by a nilmanifold, that is, the quotient of a simply connected nilpotent Lie group $L$ by a uniform lattice. Denote by $G$ the holonomy of $Q$.

(a) The manifold $Q$ is a nilmanifold.

(b) The holonomy $G$ has order $k$ or $2k$, where $k$ is odd, due to Farrell-Zdravkovska [13].

(c) The holonomy $G$ of $Q$ acts effectively on the center of $L$, also due to Farrell-Zdravkovska [13].

(d) The holonomy $G$ is cyclic, due to J. Davis and F. Fang [8]. Also Upadhyay [35] had proved that Conjecture 1 is true when the following conditions hold: $G$ is cyclic, $G$ acts trivially on the center of $L$, and $L$ is 2-step nilpotent.

Hence in all of the above cases $Q$ bounds geometrically a pinched negatively curved manifold. Note that for any closed $Q$ we have $\partial (Q \times I) = Q \amalg Q$. Thus we get the following corollary of Theorem A.

**Corollary 6.** Let $Q$ be a closed almost flat manifold. Then $Q \amalg Q$ bounds geometrically a pinched negatively curved manifold.

In other words, for every closed connected almost flat manifold there is a complete finite volume pinched negatively curved manifold with exactly two connected cusps, each diffeomorphic to $Q \times [b, \infty)$. A complete pinched negatively curved metric $g$ on $Q \times \mathbb{R}$ is called a (pinched negatively curved) cusp metric if the $g$-volume of $Q \times [0, \infty)$ is finite. And we say that a cusp metric $g$ on $Q \times \mathbb{R}$ is an eventually warped cusp metric if $g = e^{-2t}h + dt^2$, for $t < c$, for some $c \in \mathbb{R}$ and a metric $h$ on $Q$. I. Belegradek and V. Kapovitch [3] (see also [2]) show, based on earlier work by Z.M. Shen [34], that if $Q$ is almost flat then $Q \times \mathbb{R}$ admits an eventually warped cusp metric.

**Addendum to Theorem A.** Let $g$ be an eventually warped cusp metric on $Q \times \mathbb{R}$. If the sectional curvatures of $g$ lie in $(a, b)$, with $a < -1 < b$, then we can take $M$ in Theorem A with sectional curvatures also in $(a, b)$. Moreover the sectional curvatures of $M$ away from a cusp can be taken in $[-\epsilon - 1, -1]$, for any $\epsilon > 0$.

Even though a flat manifold may not necessarily bound geometrically a hyperbolic manifold the next corollary says it does bound geometrically an $\epsilon$-pinched to -1 manifold, for any $\epsilon > 0$. It follows from the Hamrick and Royster result [19], Theorem A and its addendum.

**Corollary 7.** Every closed flat manifold bounds geometrically a manifold with sectional curvatures in $[-\epsilon - 1, -1]$, for any $\epsilon > 0$.

Here is a brief description of the paper. In Section 1 we introduce some notation and basic concepts, including the definition of $\epsilon$-close to hyperbolic metrics. This is a slightly technical but important
concept. The idea is to try to measure how close a metric is to being hyperbolic; we do this in a chart by chart fashion. In Section 2 we define and study the “hyperbolic extension” of a metric (or space), which is a key geometric construction. In this section there are no proofs and we essentially collect the main results of [31]. In Section 3 we describe another key geometric construction, hyperbolic forcing; it is the composition of two deformations: warp forcing and the two-variable deformation, which are studied with more detail in [30] and [29], respectively. Section 4 is a family version of Section 3. Again, in sections 3 and 4 there are essentially no proofs and we mostly collect the main results of [30], [29], and [32]. In Section 5 we study neighborhoods of simplices of all-right spherical complexes. In this section we introduce a technical device that we called sets of widths. These are sets of positive real numbers that are used as widths for normal neighborhoods of simplices of all-right spherical complexes. We prove that there are sets of widths, independent of the complex, that satisfy very useful properties. These are fundamental objects that make all matching processes work. Section 6 is a sort of a “cone version” of Section 5; in it we study (all-right) piecewise hyperbolic cone complexes, which are just cones over all-right spherical complexes with metric warped by sinh. In Section 7 we deal with the smoothing issue for cubical and all-right spherical complexes; here we collect the main concepts and results of [27]. We put everything together in Section 8 to smooth hyperbolic cones. Section 9 is dedicated to the Charney-Davis strict hyperbolization process; in this section we collect the results in [28], in particular we mention that strictly hyperbolized smooth manifolds have nice “normal differentiable structures”. Finally we prove the Main Theorem in Section 10 and Theorem A in Section 11. Subsections have been added at the end of sections 7, 8, and 9, that deal with generalizations to the case of manifolds with codimension zero singularities. These subsections are used in Section 11.

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Section 1. Some Notation, Definitions, and Metrics ε-Close to Hyperbolic.

In this paper ρ will denote a fixed smooth function ρ : R → [0, 1] such that: (i) ρ|_{[−∞,0+δ]} ≡ 0, and (ii) ρ|_{[1−δ,∞]} ≡ 1, where δ > 0 is small.

The standard flat metric on R^l will be denoted by σ_{g^l}. Similarly, σ_{g^l} and σ_{g^{l−1}} will denote the standard hyperbolic and round metrics on H^l and S^{l−1}, respectively.

Let B = B^{l−1} ⊂ R^{l−1} be the unit ball, with metric σ_{g^{l−1}}. Write I_ξ = (−(1 + ξ), 1 + ξ) ⊂ R, ξ > 0. Our basic models are T_ξ = T_ξ = B × I_ξ ⊂ R^l, with hyperbolic metric σ = e^{2t}σ_{g^{l−1}} + dt^2. In what follows we may sometimes suppress the subindex ξ, if the context is clear. The number ξ is the excess of T_ξ.

Remarks.
1. One of the reasons to introduce the excess is that the process of hyperbolic extension (see Section 2) decreases the excess of the charts, as shown in the statement of Theorem 2.7.
2. In the applications we may actually need warped metrics with warping functions that are multiples of hyperbolic functions. All these functions are close to the exponential e^t (for t large), so instead of introducing one model for each hyperbolic function we introduced only the exponential model.

Let |.| denote the uniform C^2-norm of R^a-valued functions on T_ξ = B × I_ξ ⊂ R^l. Given a metric g on T, |g| is computed considering g as the R^l^2-valued function (x,t) → (g_{ij}(x,t)), (x,t) ∈ T, where, as usual, g_{ij} = g(e_i,e_j), and the e_i’s are the canonical vectors in R^l. Let ε > 0. We will say that a metric g on T is ε-close hyperbolic if |g − σ| < ε.
Let $\epsilon > 0$. A Riemannian manifold $(M^l, g)$ is $\epsilon$-close hyperbolic if there is $\xi > 0$ such that for every $p \in M$ there is an $\epsilon$-close to hyperbolic chart with center $p$, that is, there is a chart $\phi : T_\xi \to M$, $\phi(0, 0) = p$, such that $\phi^* g$ is $\epsilon$-close to hyperbolic. Note that all charts are defined on the same model space $T_\xi$. The number $\xi$ is called the excess of the charts (which is fixed). More generally, a subset $S \subset M$ is $\epsilon$-close to hyperbolic if every $p \in S$ is the center of an $\epsilon$-close to hyperbolic chart in $M$ with fixed excess $\xi$.

Let $M^n$ be a complete Riemannian manifold. We say that a point $o \in M$ is a center of $M$ if the exponential map $\exp_o : T_o M \to M$ is a diffeomorphism. In particular $M$ is diffeomorphic to $\mathbb{R}^n$. For instance if $M$ is Hadamard manifold every point is a center point. In this paper we will always use the same symbol “$o$” to denote a center of a Riemannian manifold, unless it is necessary to specify the manifold $M$, in which case we will write $o_M$. Using the diffeomorphism $\exp_o$ onto $M$ and an identification of $T_o M$ with $\mathbb{R}^n$ via some fixed choice of an orthonormal basis in $T_o M$, we can identify $M$ with $\mathbb{R}^n$ and $M - \{o\}$ with $S^{n-1} \times \mathbb{R}^+$. Therefore the metric of $M$, restricted to $M - \{o\}$, can be written as $h_t + dt^2$ on $S^{n-1} \times \mathbb{R}^+$, where $\{h_t\}_{t>0}$ is a one-parameter family of metrics on $S^{n-1}$. The set of rays $t \mapsto (x, t) \in S^{n-1} \times \mathbb{R}^+$ are geodesics on $M - \{o\}$, for every $t > 0$, and we call this set the ray structure of $M$ with respect to $o$.

If $N^l$ has center $o$ we say that $S \subset N$ is radially $\epsilon$-close to hyperbolic (with respect to $o$) if, in addition, the $\epsilon$-close to hyperbolic charts $\phi$ respect the product structure of $T$ and $N - \{o\} = S^{l-1} \times \mathbb{R}^+$, that is $\phi(., t) = (\phi_1(., t) + a_1, t + a_2)$, for some $a$ depending on the $\phi$. The term “radially” in the definition above refers to the decomposition of the manifold $N - \{o\}$ as a product $S^{l-1} \times \mathbb{R}^+$.

Of course a radially $\epsilon$-close to hyperbolic manifold is $\epsilon$-close to hyperbolic.

Remarks 1.1.

1. The definition of radially $\epsilon$-close to hyperbolic metrics is well suited to studying metrics of the form $g_t + dt^2$ for $t$ large, but for small $t$ this definition is not useful because: (1) we need some space to fit the charts, and (2) the form of our specific fixed model $T$ (note that the warping function used in the metric of $T$ is the exponential). An undesired consequence is that even punctured hyperbolic space $H^n - \{o\} = S^{n-1} \times \mathbb{R}^+$ (with warp metric $\sinh^2(t) \sigma_{S^{n-1}} + dt^2$) is not radially $\epsilon$-close to hyperbolic for $t$ small. In fact there is $a = a(n, \epsilon)$ such that hyperbolic $n$-space is $\epsilon$-close to hyperbolic for $t > a$ (and not for all $t \leq a$), see 3.9 [29].

2. For every $n$ there is a function $\epsilon' = \epsilon'(\epsilon, \xi, n)$ such that: if a Riemannian metric $g$ on a manifold $M^n$ is $\epsilon'$-close to hyperbolic, with charts of excess $\xi$, then the sectional curvatures of $g$ all lie $\epsilon'$-close to -1. This choice is possible, and depends only on $n$ and $\xi$, because the curvature depends only of the derivatives up to order 2 of $\phi^* g$ on $T_\xi$, where $\phi$ is an $\epsilon'$-close to hyperbolic chart with excess $\xi$.

Lemma 1.2. Let $\phi : T_\xi \to M$ be a warped $\epsilon$-hyperbolic chart centered at $p \in M$. Then, for every $q \in T_\xi$ we have $d_M(\phi(q), p) \leq (2 + \xi) + n^2 \epsilon$.

Proof. Write $q = (x_0, t_0) \in \mathbb{R} \times I_\xi$. Consider the path $\alpha(t) = (tx_0, 0)$, $t \in [0, 1]$, $\beta(t) = (x_0, t_0)$, $t \in [0, 1]$, and $\gamma = \alpha \ast \beta$. Write $g' = \phi^* g$ and we have $g' = \sigma + h$, with $|h| < \epsilon$. Then the $g'$-length $\ell_{g'}(\phi \circ \gamma)$ of $\phi \circ \gamma$ is $\ell_{g'}(\gamma) = \ell_{g'}(\alpha) + \ell_{g'}(\beta) \leq \ell_{\phi}(\alpha) + \ell_{h}(\alpha) + (1 + \xi) \leq 1 + n\epsilon^2 + (1 + \xi)$. Hence $d_M(\phi(q), p) \leq \ell_{g'}(\phi \circ \gamma) \leq (2 + \xi) + n^2 \epsilon$. This proves the lemma.
Let $M^n$ have center $o$ and let $B_a = B_a(o)$ be the ball in $M$ of radius $a$ centered at $o$. We say that a metric $h$ on $M$ is $(B_a, \epsilon)$-close to hyperbolic, with charts of excess $\xi$, if

1. On $B_a - \{o\} = \mathbb{S}^{n-1} \times (0, a)$ we have $h = \sinh^2(t)\sigma_{\mathbb{S}^{n-1}} + dt^2$. Hence $h$ is hyperbolic on $B_a$.

2. The metric $h$ is radially $\epsilon$-close to hyperbolic outside $B_{a-1-\xi}$, with charts of excess $\xi$.

Remarks.

1. We have dropped the word “radially” to simplify the notation. But it does appear in condition (2), where “radially” refers to the center of $B_a$.

2. We will always assume $a > a + 1$, where $a$ is as in 1.1 (1). Therefore conditions (1), (2) and Remark 1.1 (1) imply a stronger version of (2) above:

$(2')$ the metric $h$ is radially $\epsilon$-close to hyperbolic outside $B_a$, with charts of excess $\xi$.

This is the reason why we demanded radius $a - 1 - \xi$ in (2), instead of just $a$.

Let $c > 1$. A metric $g$ on a compact manifold $M$ is $c$-bounded if $|g| < c$ and $|\det g|_{C^0} > 1/c$. A set of metrics $\{g_\lambda\}$ on the compact manifold $M$ is $c$-bounded if every $g_\lambda$ is $c$-bounded.

Remarks.

1. Here the uniform $C^2$-norm $|.|$ is taken with respect to a fixed finite atlas $\mathcal{A}$.

2. We will assume that the finite atlas $\mathcal{A}$ is “nice”, that is, it has “extendable” charts, i.e. charts that can be extended to the (compact) closure of their domains.

Section 2. Hyperbolic Extensions

Recall that hyperbolic $n$-space $\mathbb{H}^n$ is isometric to $\mathbb{H}^k \times \mathbb{H}^{n-k}$ with warp metric $(\cosh^2 r) \sigma_{\mathbb{H}^k} + \sigma_{\mathbb{H}^{n-k}}$, where $\sigma_{\mathbb{H}^l}$ denotes the hyperbolic metric of $\mathbb{H}^l$, and $r : \mathbb{H}^{n-k} \to [0, \infty)$ is the distance to a fixed point in $\mathbb{H}^{n-k}$. For instance, in the case $n = 2$, since $\mathbb{H}^1 = \mathbb{R}^1$ we have that $\mathbb{H}^2$ is isometric to $\mathbb{R}^2 = \{(u, v)\}$ with warped metric $\cosh^2 v\, du^2 + dv^2$. In the following paragraph we give a generalization of this construction.

Let $(M^n, h)$ be a complete Riemannian manifold with center $o = o_M \in M$. The warp metric
\[ g = (\cosh^2 r) \sigma_{\mathbb{H}^k} + h \]
on $\mathbb{H}^k \times M$ is the hyperbolic extension (of dimension $k$) of the metric $h$. Here $r$ is the distance-to-o function on $M$. We write $\mathcal{E}_k(M, h) = (\mathbb{H}^k \times M, g)$, and $g = \mathcal{E}_k(h)$. We also say that $\mathcal{E}_k(M) = \mathcal{E}_k(M, h)$ is the hyperbolic extension (of dimension $k$) of $(M, h)$ (or just of $M$). Hence, for instance, we have $\mathcal{E}_k(\mathbb{H}^l) = \mathbb{H}^{k+l}$. For $S \subset M$ and $A \subset \mathbb{H}^k$ we write $\mathcal{E}_A(S) = A \times S \subset \mathcal{E}_k(M)$. Also write $\mathbb{H}^k = \mathbb{H}^k \times \{o_M\} \subset \mathcal{E}_k(M)$ and we have that any $p \in \mathbb{H}^k$ is a center of $\mathcal{E}_k(M)$ (see [31] or 2.3 below).

Note that $\mathbb{H}^k$ and every $\{y\} \times M$ are convex in $\mathcal{E}_k(M)$ (see [4], p.23). Let $\eta$ be a complete geodesic line in $M$ passing though $o$ and let $\eta^+$ be one of its two geodesic rays (beginning at $o$). Then $\eta$ is a totally geodesic subspace of $M$ and $\eta^+$ is convex (see [31]). Also, let $\gamma$ be a complete geodesic line in $\mathbb{H}^k$. The following two results are proved in [31].

**Lemma 2.1.** We have that $\gamma \times \eta^+$ is a convex subspace of $\mathcal{E}_k(M)$ and $\gamma \times \eta$ is totally geodesic in $\mathcal{E}_k(M)$. 

7
Corollary 2.2. We have that $\mathbb{H}^k \times \eta^+$ and $\gamma \times M$ are convex in $\mathcal{E}_k(M)$. Also $\mathbb{H}^k \times \eta$ is totally geodesic in $\mathcal{E}_k(M)$.

Remarks 2.3.
1. Note that $\mathbb{H}^k \times \eta$ (with metric induced by $\mathcal{E}_k(M)$) is isometric to $\mathbb{H}^k \times \mathbb{R}$ with warped metric $\cosh^2 v \sigma_{ik} + dv^2$, which is just hyperbolic ($k + 1$)-space $\mathbb{H}^{k+1}$. Also $\gamma \times \eta$ is isometric to $\mathbb{R} \times \mathbb{R}$ with warped metric $\cosh^2 v du^2 + dv^2$, which is just hyperbolic 2-space $\mathbb{H}^2$. In particular every point in $\mathbb{H}^k = \mathbb{H}^k \times \{0\} \subset \mathcal{E}_k(M)$ is a center point.

2. It follows from Lemma 2.1 and Remark 2.3(1) that the ray structure of $\mathcal{E}_k(h)$ with respect to any center $o_{ik} \in \mathbb{H}^k \subset \mathcal{E}_k(M)$ only depends on the ray structure of $M$ and the center $o_{ik}$.

3. Denote by $\mathbb{R}_r(M)$ the ball of radius $r$ of $M$. Note that if $h$ and $h'$ on $M$ have the same ray structures then the balls $\mathbb{R}_r(M)$ coincide.

4. Recall that $\mathbb{H}^k$ is convex in $\mathcal{E}_k(M)$. Moreover, for $l \leq k$, we also have $\mathbb{H}^l \subset \mathbb{H}^k \subset \mathcal{E}_k(M)$ is convex. If $h$ and $h'$ on $M$ have the same ray structures then the $r$-neighborhoods (with respect to $h$ and $h'$) of the convex subset $\mathbb{H}^l$ coincide.

As before (see Section 1) we use $h$ to identify $M - \{o\}$ with $\mathbb{S}^{n-1} \times \mathbb{R}^+$. Sometimes we will denote a point $v = (u, r) \in \mathbb{S}^{n-1} \times \mathbb{R}^+ = M - \{o\}$ by $v = ru$. Fix a center $o \in \mathbb{H}^k \subset \mathcal{E}_k(M)$. Then, for $y \in \mathbb{H}^k - \{o\}$ we can also write $y = tu, (w, t) \in \mathbb{S}^{k-1} \times \mathbb{R}^+$. Similarly, using the exponential map we can identify $\mathcal{E}_k(M) - \{o\}$ with $\mathbb{S}^{k+n-1} \times \mathbb{R}^+$, and for $p \in \mathcal{E}_k(M) - \{o\}$ we can write $p = sx, (x, s) \in \mathbb{S}^{k+n-1} \times \mathbb{R}^+$. We denote the metric on $\mathcal{E}_k(M)$ by $f$ and we can write $f = f_o + ds^2$. Since $\mathbb{H}^k$ is convex in $\mathcal{E}_k(M)$ we can write $\mathbb{H}^k - \{o\} = \mathbb{S}^{k-1} \times \mathbb{R}^+ \subset \mathbb{S}^{k+n-1} \times \mathbb{R}^+ \subset \mathbb{S}^{k+n-1}$.

A point $p \in \mathcal{E}_k(M) - \mathbb{H}^k$ has two sets of coordinates: the polar coordinates $(x, s) = (x(p), s(p)) \in \mathbb{S}^{k+n-1} \times \mathbb{R}^+$ and the hyperbolic extension coordinates $(y, v) = (y(p), v(p)) \in \mathbb{H}^k \times M$. Write $M_o = \{o\} \times M$. Therefore we have the following functions:

| Function Type | $p$-Function | $o$-Function |
|--------------|--------------|--------------|
| the distance to $o$ function | $s : \mathcal{E}_k(M) \rightarrow [0, \infty)$ | $s(p) = d_{\mathcal{E}_k(M)}(p, o)$ |
| the direction to $p$ function | $x : \mathcal{E}_k(M) - \{o\} \rightarrow \mathbb{S}^{k+n-1}$ | $p = s(p)x(p)$ |
| the distance to $\mathbb{H}^k$ function | $r : \mathcal{E}_k(M) \rightarrow [0, \infty)$ | $r(p) = d_{\mathcal{E}_k(M)}(p, \mathbb{H}^k)$ |
| the projection on $\mathbb{H}^k$ function | $y : \mathcal{E}_k(M) \rightarrow \mathbb{H}^k$ | |
| the projection on $M$ function | $v : \mathcal{E}_k(M) \rightarrow M$ | |
| the projection on $\mathbb{S}^{n-1}$ function | $u : \mathcal{E}_k(M) - \mathbb{H}^k \rightarrow \mathbb{S}^{n-1}$ | $v(p) = r(p)u(p)$ |
| the length of $y$ function | $t : \mathcal{E}_k(M) \rightarrow [0, \infty)$ | $t(w) = d_{\mathcal{E}_k(M)}(y, o)$ |
| the direction of $y$ function | $w : \mathcal{E}_k(M) - M_o \rightarrow \mathbb{S}^{k+n-1}$ | $y(p) = t(p)w(p)$ |

Note that $r = d_M(v, o)$. Note also that, by 2.1, the functions $w$ and $u$ are constant on geodesics emanating from $o \in \mathcal{E}_k(M)$, that is $w(sx) = w(x)$ and $u(sx) = u(x)$.

Let $\partial_r$ and $\partial_s$ be the gradient vector fields of $r$ and $s$, respectively. Since the $M$-fibers $M_y = \{y\} \times M$ are convex the vectors $\partial_r$ are the velocity vectors of the speed one geodesics of the form $a \mapsto (y, au), u \in \mathbb{S}^{n-1} \subset M$. These geodesics emanate from (and orthogonally to) $\mathbb{H}^k \subset \mathcal{E}_k(M)$. Also the vectors $\partial_s$ are the velocity vectors of the speed one geodesics emanating from $o \in \mathcal{E}_k(M)$. For $p \in \mathcal{E}_k(M)$, denote by $\triangle = \triangle(p)$ the right triangle with vertices $o, y = y(p), p$ and sides the geodesic segments $[o, p] \in \mathcal{E}_k(M), [o, y] \in \mathbb{H}^k, [p, y] \in \{y\} \times M \subset \mathcal{E}_k(M)$. (These geodesic segments are unique and well-defined because: (1) $\mathbb{H}^k$ is convex in $\mathcal{E}_k(M)$, (2) $(y, o) = o_{[y] \times M}$ and $o$ are centers in $\{y\} \times M$ and $\mathbb{H}^k \subset \mathcal{E}_k(M)$, respectively.)
Let \( \alpha : \mathcal{E}_k(M) - \mathbb{H}^k \to \mathbb{R} \) be the angle between \( \partial_s \) and \( \partial_r \) (in that order), thus \( \cos \alpha = f(\partial_r, \partial_s) \), \( \alpha \in [0, \pi] \). Then \( \alpha = \alpha(p) \) is the interior angle, at \( p = (y, v) \), of the right triangle \( \Delta = \Delta(p) \). We call \( \beta(p) \) the interior angle of this triangle at \( o \), that is \( \beta(p) = \beta(x) \) is the spherical distance between \( x \in \mathbb{S}^{k+n-1} \) and the totally geodesic sub-sphere \( \mathbb{S}^{k-1} \). Alternatively, \( \beta \) is the angle between the geodesic segment \([o, p] \subset \mathcal{E}_k(M)\) and the convex submanifold \( \mathbb{H}^k \). Therefore \( \beta \) is constant on geodesics emanating from \( o \in \mathcal{E}_k(M) \), that is \( \beta(sx) = \beta(x) \). The following corollary follows from 2.1 (see 2.1 in [31]).

**Corollary 2.4.** Let \( \eta^+ \) (or \( \eta \)) be a geodesic ray (line) in \( M \) through \( o \) containing \( v = v(p) \) and \( \gamma \) a geodesic line in \( \mathbb{H}^k \) through \( o \) containing \( y = y(p) \). Then \( \Delta(p) \subset \gamma \times \eta^+ \subset \gamma \times \eta \).

Note that the right geodesic triangle \( \Delta(p) \) has sides of length \( r = r(p) \), \( t = t(p) \) and \( s = s(p) \). By Lemma 2.1 and Remark 2.3 we can consider \( \Delta \) as contained in hyperbolic 2-space. Hence using hyperbolic trigonometric identities we can find relations between \( r, t, s, \alpha \) and \( \beta \). For instance, using the hyperbolic law of cosines we get:

\[
\cosh(s) = \cosh(r) \cosh(t)
\]

Note that this implies \( t \leq s \). Here is an application of this equation.

**Proposition 2.6 (Iterated hyperbolic extensions)** We have that

\[
\mathcal{E}_l(\mathcal{E}_k(M)) = \mathcal{E}_{l+k}(M)
\]

where we are identifying \( \mathbb{H}^{l+k} \) with \( \mathbb{H}^l \times \mathbb{H}^k \) with warped metric \( (\cosh^2(t)) \sigma_{\mathbb{H}^l} + \sigma_{\mathbb{H}^k} \).

This proposition is proved in [31].

**Remarks.**
1. Note that the identification of \( \mathbb{H}^{l+k} \) with \( \mathbb{H}^l \times \mathbb{H}^k \) (with warp metric) depends on the order of \( l \) and \( k \), that is, on the order in which the hyperbolic extensions are taken.
2. As before, here the function \( t : \mathbb{H}^k \to [0, \infty) \) is the distance in \( \mathbb{H}^k \) to the point \( o \in \mathbb{H}^k \).

We next explore the relationship between hyperbolic extensions and metrics \( \epsilon \)-close to hyperbolic. Since \( \mathcal{E}_k(\mathbb{H}^l) = \mathbb{H}^{k+l} \) one would expect that if \( M \) is “close” to \( \mathbb{H}^l \), then \( \mathcal{E}_k(M) \) would be close to \( \mathbb{H}^{k+l} \). This motivates the following question.

**Question.** What can we say about the hyperbolic extension of a \((B_a, \epsilon)\)-close to hyperbolic metric?

(Recall that metrics \((B_a, \epsilon)\)-close to hyperbolic are metrics that are already hyperbolic on the ball \( B_a = B_a(0) \) of radius \( a \), and are radially \( \epsilon \)-close to hyperbolic outside \( B_a' \), see Section 1.)

The next result answers this question; it is Theorem B in [31].

**Theorem 2.7.** Let \( M^n \) have center \( o \). Assume \( M \) is \((B_a, \epsilon)\)-close to hyperbolic, with charts of excess \( \xi > 0 \). Then \( \mathcal{E}_k(M) \) is \((B_a, C\epsilon)\)-close to hyperbolic, with charts of excess \( \xi' \), provided \( a \) is sufficiently large. Explicitly we want

\[
a \geq R = R(\epsilon, k, \xi)
\]

Here \( C = C(n, k, \xi) \), and \( \xi' = \xi - e^{-a/2} > 0 \).

This theorem is proved in [31]. Explicit formulas for \( C \) and \( R \) are given in [31] (the constant \( C \) here is called \( C_2 \) in [31]). Note that the excess of the charts decreases. This is one of the main reasons to
introduce the excess. In Section 3 (see also [30]) we describe another geometric process, warp forcing, which also reduces the excess of the charts.

Section 3. Deformations of Metrics

The goal of this section is to describe the “hyperbolic forcing” method. It has as input a metric on $\mathbb{R}^n$ of the form $g = g_r + dr^2$ (or, more generally a metric on a manifold with center) and output a metric still of the form $g_r' + dr^2$, but which is hyperbolic on a ball centered at the origin.

Hyberbolic forcing is defined as the composition of two other metric deformations: the two-variable deformation and warp forcing. We present these first.

3.1. The Two Variable Warping Deformation.

Let $g'$ be a metric on the $(n - 1)$-sphere $\mathbb{S}^{n-1}$ and consider the warped metric $g = \sinh^2 t g' + dt^2$ on $\mathbb{S}^{n-1} \times \mathbb{R}^+$. Recall that $\rho : \mathbb{R} \to [0, 1]$ is a fixed smooth function with $\rho(t) = 0$ for $t \leq 0$ and $\rho(t) = 1$ for $t \geq 1$. Given positive numbers $a$ and $d$ define $\rho_{a,d}(t) = \rho(2t - a/d)$. Also fix an atlas $\mathcal{A}_{\mathbb{S}^{n}}$ on $\mathbb{S}^{n-1}$ as before (see remarks at the end of Section 1). All norms and boundedness constants will be taken with respect to this atlas. Recall that $\sigma_{\mathbb{S}^{n-1}}$ is the round metric on $\mathbb{S}^{n-1}$. Write $g_t = \left(1 - \rho_{a,d}(t)\right) \sigma_{\mathbb{S}^{n-1}} + \rho_{a,d}(t) g'$ and define the metric

$$\mathcal{T}_{a,d} g = \sinh^2 t g_t + dt^2$$

We call the correspondence $g \mapsto \mathcal{T}_{a,d} g$ the two variable warping deformation. By construction we have that $\mathcal{T}_{a,d} g$ satisfies the following property:

$$\mathcal{T}_{a,d} g = \begin{cases} \sinh^2 (r) \sigma_{\mathbb{S}^{n-1}} + dr^2 & \text{on } B_a \\ g & \text{outside } B_a + \frac{d}{2} \end{cases}$$

Hence, the two variable warping deformation changes a warp metric $h$ inside the ball $B_a + \frac{d}{2}$ making it (radially) hyperbolic on the smaller ball $B_a$. The warp metric $h$ does not change outside $B_a + \frac{d}{2}$.

Remarks. 3.1.1.
1. Note that if we choose $g$ to be the warped-by-sinh hyperbolic metric, that is, $g = \sinh^2 t \sigma_{\mathbb{S}^{n-1}} + dt^2$, then $\mathcal{T}_{a,d} g = g$.
2. To be able to define $\mathcal{T}_{a,d} g$ the metric $g$ does not need to be a warp metric everywhere. It only needs to be a warp metric in the ball $B_a + \frac{d}{2}$.

3.2. Warp Forcing.

Let $(M^n, g)$ be a complete Riemannian manifold with center $o \in M$. Recall that we can write the metric on $M - \{o\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$ as $g = g_r + dr^2$. We denote by $\mathbb{S}_r = \mathbb{S}_r(M) = \mathbb{S}^{n-1} \times \{r\}$ the sphere of radius $r$. For a fixed $r_o > 0$ we can think of the metric $g_{r_o}$ as being obtained from $g = g_r + dr^2$ by “cutting” $g$ along the sphere of radius $r_o$, so we call $g_{r_o}$ the warped spherical cut of $g$ at $r_o$. In the same vein, we call the metric
\[ \tilde{g}_{r_0} = \left( \frac{1}{\sinh^2(r_0)} \right) g_{r_0} \]

the \textit{(unwarped by sinh) spherical cut of} \( g \) at \( r_0 \). Note that in the particular case where \( g = g_r + dr^2 \) is already a warped-by-sinh metric (that is, \( g_r = \sinh^2(r)g' \) for some fixed \( g' \) independent of \( r \)) we have that the warped spherical cut of \( g = \sinh^2(r)g' + dt^2 \) at \( r_0 \) is \( \sinh^2(r_0)g' \), and the the spherical cut at \( r_0 \) is \( \tilde{g}_{r_0} = g' \). Hence the terms “warped” and “unwarped” (usually we will omit the term “unwarped”).

Fix \( r_0 > 0 \). We define the warped-by-sinh metric \( \tilde{g}_{r_0} \) by:

\[ \tilde{g}_{r_0} = \sinh^2(t)\tilde{g}_{r_0} + dr^2 = \sinh^2(t)\left( \frac{1}{\sinh^2(r_0)} \right)g_{r_0} + dr^2 \]

We now force the metric \( g \) to be equal to \( \tilde{g}_{r_0} \) on \( B_{r_0} = B_{r_0}(M) \) and stay equal to \( g \) outside \( B_{r_0 + \frac{1}{2}} \). For this we define the warped forced (on \( B_{r_0} \)) metric as:

\[ W_{r_0} g = (1 - \rho_{r_0})\tilde{g}_{r_0} + \rho_{r_0} g \]

where \( \rho_{r_0}(t) = \rho(2t - 2r_0) \), and \( \rho : \mathbb{R} \to [0,1] \) is as before (see Section 1). Hence we have

\[ W_{r_0} g = \begin{cases} \tilde{g}_{r_0} & \text{on } B_{r_0} \\ g & \text{outside } B_{r_0 + \frac{1}{2}} \end{cases} \]

Hence warp forcing changes the metric only on \( B_{r_0 + \frac{1}{2}} \), making it a warped metric inside \( B_{r_0} \). The metric \( g \) does not change outside \( B_{r_0 + \frac{1}{2}} \). We call the process \( g \mapsto W_{r_0} g \) \textit{warp forcing}.

**Remarks 3.2.1.**

1. Notice that to define \( W_{r_0} g \) we only need \( g_r \) to be defined for \( r \geq r_0 \).
2. Note that if we choose \( g \) to be the warped-by-sinh hyperbolic metric, that is, \( g = \sinh^2 t \sigma_{\mathbb{S}^{n-1}} + dt^2 \), then \( W_{r_0} g = g \)

### 3.3. Hyperbolic Forcing.

Let \( (M^n, g) \) have center \( o \). As before we write \( g = g_r + dr^2 \). Let \( r_0 > d > 0 \). We define the metric \( H_{r_0,d}(g) \) in the following way. First warp-force the metric \( g \), i.e take \( W_{r_0} g \). Recall \( W_{r_0} g \) is a warped metric on \( B_{r_0} \) (see also remarks 3.1.1 and 3.2.1). Hence we can use two variable warping deformation (see 3.1) and define

\[ H_{r_0,d} g = T_{r_0,d}(W_{r_0} g) \]

The process \( g \mapsto H_{r_0,d} g \) is called \textit{hyperbolic forcing}. Write \( h = H_{r_0,d} g \). Note that \( h \) also has the form \( h = h_r + dr^2 \). In the next results we explicitly describe \( h_r \) and give some properties of the metric \( h = H_{r_0,d} g \). These results are proved in \([32]\).

**Proposition 3.3.2.** We have

\[ h_r = \begin{cases} g_r & \text{if } r_0 + \frac{1}{2} \leq r \\ (1 - \rho_{r_0}(r)) \sinh^2(r)\tilde{g}_{r_0} + \rho_{r_0}(r) g_r & \text{if } r_0 \leq r \leq r_0 + \frac{1}{2} \\ \sinh^2(r)\left( (1 - \rho_{r_0-d})(r) \right) \sigma_{\mathbb{S}^{n-1}} + \rho_{r_0-d}(r)\tilde{g}_{r_0} & \text{if } r_0 - d \leq r \leq r_0 \\ \sinh^2(r) \sigma_{\mathbb{S}^{n-1}} & \text{if } r \leq r_0 - d \end{cases} \]
where the gluing functions $\rho_{r_0}$ and $\rho_{(r_0-d),d}$ are defined in 3.2 and 3.1, respectively.

**Proposition 3.3.3.** The metric $h = H_{r_0} g$ has the following properties.

(i) The metric $h$ is canonically hyperbolic on $B_{r_0-d}$, i.e equal to $\sinh^2(r)\sigma g + dr^2$ on $B_{r_0-d}$.

(ii) We have that $g = h$ outside $B_{r_0 + \frac{d}{2}}$.

(iii) The metric $h$ coincides with $W_{r_0}(g_{r_0})$ outside $B_{r_0 - \frac{d}{2}}$.

(iv) The metric $h$ coincides with $T_{(r_0-d),d} g_{r_0}$ on $B_{r_0}$.

(v) All the $g$-geodesic rays $r \mapsto ru$, $u \in S^d$, emanating from the center are geodesics of $(M,h)$. Hence, the space $(M,h)$ has center $o$. Moreover the function $r$ (distance to the center $o$) is the same on the spaces $(M,g)$ and $(M,h)$. In other words, the spaces $(M,g)$ and $(M,h)$ have the same ray structures.

Next we discuss the following question:

*Is the hyperbolically forced metric $h = H_{r_0} g$ close to hyperbolic, when $g$ is close to hyperbolic?*

Notice that from 3.1.1 and 3.2.1 it follows that if we choose $g$ to be the warped-by-sinh hyperbolic metric, that is, $g = \sinh^2(r)\sigma g + dr^2$, then $H_{r_0} g = g$. Therefore one would expect that the answer to the previous question is “yes”. So, it is better ask a more quantitative question:

*To what extent is the hyperbolically forced metric $h = H_{r_0} g$ close to hyperbolic, when $g$ is close to hyperbolic?*

The next theorem deals with this question. This theorem is proved in [32].

**Theorem 3.3.4.** Let $M^n$ have center $o$ and metric $g = g_r + dr^2$. Assume the spherical cut $\hat{g}_{r_0}$ is $c$-bounded. If the metric $g$ is radially $\epsilon$-close to hyperbolic outside $B_{r_0-(1+\xi)}$ with charts of excess $\xi > 1$, then the metric $H_{r_0} g$ is $(B_{r_0-d},\eta)$-close to hyperbolic with charts of excess $\xi - 1$, provided

$$\eta \geq C_1 \left( \frac{1}{d} + e^{-(r_0-d)} \right) + C_2 \epsilon$$

Here $C_1$ is a constant depending only on $n$, $\xi$, $c$, and $C_2$ depends only on $\xi$.

**Remarks 3.3.5.**

1. An important point here is that by taking $r_0$ and $d$ large we can make $H_{r_0} g$ $2C_2\epsilon$-close to hyperbolic. How large we have to take $d$ and $r_0$ depends on $c$, which is a $C^2$ bound for the the $\hat{g}_{r_0}$, the spherical cut of $g$ at $r_0$ (see 3.2).

2. Note that the excess of the charts decreases by 1. This is because of warp forcing (see [30]).

**Section 4. Deformations of Families of Metrics**

In this section we give a one-parameter version of the concepts and results presented in Section 3. Let $(M^n,g)$ be a complete Riemannian manifold with center $o \in M$. Recall that we can write the metric on $M - \{o\} = \mathbb{R}^n - \{0\} = S^{n-1} \times \mathbb{R}^+$ as $g = g_r + dr^2$, where $r$ is the distance to $o$.

Fix $\xi > 0$, and let $\lambda_0 > 1 + \xi$. We say that the collection $\{g_\lambda\}_{\lambda \geq \lambda_0}$ is a $\odot$-family of metrics on $M$ if each $g_\lambda$ is a metric of the form $g_\lambda = (g_\lambda)_r + dr^2$ defined (at least) for $r > \lambda - (1 + \xi)$.
Remark. We will always assume that the family of metrics \( \{g_\lambda\} \) is smooth, that is, the map \((x, \lambda) \mapsto g_\lambda(x)\) is smooth, \(x \in M\), \(\lambda \geq \lambda_0\).

We say that the \( \{g_\lambda\} \) has (spherical) cut limit at \( b \) if there is a \( C^2 \) metric \( \hat{g}_{\infty+b} \) on \( S^{n-1} \) such that

\[
(4.1) \quad |(g_\lambda)_{\lambda+b} - \hat{g}_{\infty+b}| \to 0 \quad \text{as} \quad \lambda \to \infty
\]

Remarks 4.2.
1. Recall that the metric \((g_\lambda)_{\lambda+b}\) is the spherical cut of \( g_\lambda \) at \( \lambda + b \). See Section 3.2.
2. The arrow above means convergence in the \( C^2 \)-norm on the space of \( C^2 \) metrics on \( S^{n-1} \). See remarks at the end of Section 1.
3. The definition above implies that \((g_\lambda)_{\lambda+b}\) is defined for large \( \lambda \), even if \( b < -(1 + \xi) \).
4. Note that the concept of cut limit at \( b \) depends strongly on the indexation of the family.
5. In [32] it is proved the following. If an \( \circ \)-family \( \{g_\lambda\} \) has cut limits at \( b \), then the family \( \{(g_\lambda)_{\lambda+b}\}_\lambda \) is \( c \)-bounded, for some \( c \).

Consider the \( \circ \)-family \( \{g_\lambda\} \) and let \( d > 0 \). Apply hyperbolic forcing to get

\[
h_\lambda = H_{\lambda,d}g_\lambda
\]

We say that the family \( \{h_\lambda\} \) is the hyperbolically forced family corresponding to the \( \circ \)-family \( \{g_\lambda\} \). Note that we can write \( h_\lambda = (h_\lambda)_r + dr^2 \). Using Proposition 3.3.2 we can explicitly describe \( (h_\lambda)_r \): (see [32] for more details)

\[
(4.3) \quad (h_\lambda)_r = \begin{cases} 
(g_\lambda)_r, & \lambda + \frac{1}{2} \leq r \\
(1 - \rho_\lambda(r)) \sinh^2(r) (g_\lambda)_{\lambda} + \rho_\lambda(r) (g_\lambda)_r, & \lambda \leq r \leq \lambda + \frac{1}{2} \\
\sinh^2(r) (1 - \rho_{(\lambda-d),d}(r)) \sigma_{\infty} + \rho_{(\lambda-d),d}(r) (g_\lambda)_{\lambda}, & \lambda - d \leq r \leq \lambda \\
\sinh^2(r) \sigma_{\infty-1}, & r \leq \lambda - d
\end{cases}
\]

The next proposition is a one-parameter version of 3.3.3. It is proved in [32].

Proposition 4.4. The metrics \( h_\lambda \) have the following properties.
(i) The metrics \( h_\lambda \) are canonically hyperbolic on \( B_{\lambda-d} \), i.e. equal to \( \sinh^2(r) \sigma_{\infty} + dr^2 \) on \( B_{\lambda-d} \), provided \( \lambda > d \).
(ii) We have that \( g_\lambda = h_\lambda \) outside \( B_{\lambda+\frac{1}{2}}^\lambda \).
(iii) The metric \( h \) coincides with \( W_\lambda(g_\lambda) \) outside \( B_{\lambda-\frac{1}{2}}^\lambda \).
(iv) The metric \( h \) coincides with \( H_{(\lambda-d),d}(\hat{g}_\infty)_{\lambda} \) on \( B_\lambda \).
(v) If the \( \circ \)-family \( \{g_\lambda\} \) has cut limits for \( b = 0 \) then \( \{h_\lambda\} \) has cut limits on \((-\infty, 0]\). In fact we have

\[
\hat{h}_{\infty+b} = \begin{cases} 
\hat{g}_\infty, & b = 0 \\
(1 - \rho(2 + \frac{2b}{d})) \sigma_{\infty} + \rho(2 + \frac{2b}{d}) \hat{g}_\infty, & -d \leq b \leq 0 \\
\sigma_{\infty}, & b \leq -d
\end{cases}
\]

where \( \rho \) is as in Section 2.
(vi) If we additionally assume that \( \{g_\lambda\} \) has cut limits on \([0, \frac{1}{2}]\), then \( \{h_\lambda\} \) has also cut limits on \([0, \frac{1}{2}]\). In fact, for \( b \in [0, \frac{1}{2}] \) we have

\[
h_{\infty+b} = (1 - \rho(b)) \hat{g}_\infty + \rho(b) \hat{g}_{\infty+b}
\]
where $\rho$ is as in Section 3. Of course if $\{g_\lambda\}$ has a cut limit at $b > \frac{1}{2}$ then $\{h_\lambda\}$ has the same cut limit at $b$ (see item 2).

(vii) All the rays $r \leftrightarrow ru$, $u \in \mathbb{S}^n$, emanating from the origin are geodesics of $(M, h_\lambda)$. Hence, all spaces $(M, h_\lambda)$ have center $o \in M$ and have the same geodesic rays emanating from the common center $o$. Moreover the function $r$ (distance to $o \in M$) is the same on all spaces $(M, h_\lambda)$.

We now state one of our most important results. It is used in an essential way in smoothing Charney-Davis strict hyperbolizations. It is proved in \cite{32} using 3.3.4. Before, we need a definition. We say that an $\odot$-family $\{g_\lambda\}$ is radially $\epsilon$-close to hyperbolic, with charts of excess $\xi$, if each $g_\lambda$ is radially $\epsilon$-close to hyperbolic outside $B_{\lambda - (1+\xi)}$, with charts of excess $\xi$.

**Theorem 4.5.** Let $M$ have center $o$, $\{g_\lambda\}$ an $\odot$-family on $M$, and $\epsilon' > 0$. Assume that $\{g_\lambda\}$ has cut limits at $b = 0$. If $\{g_\lambda\}$ is radially $\epsilon$-close to hyperbolic, with charts of excess $\xi > 1$, then $H_{\lambda, d} g_\lambda$ is $(B_{\lambda - d, \epsilon' + C_2 \epsilon})$-close to hyperbolic, with charts of excess $\xi - 1$, provided

(i) $\lambda - d > \ln(2C_1)$

(ii) $d \geq \frac{2C_2}{\epsilon'}$

Here $C_1$ and $C_2$ are as in Theorem 3.3.4.

**Remarks 4.6.**

1. Note that we can take $\epsilon'$ as small as we want hence $\epsilon' + C_2 \epsilon$ as close as $C_2 \epsilon$ as we desire, provided we choose $d$ and $\lambda$ sufficiently large. How large depending on $\epsilon'$ and $c$.

2. The constant $C_1(c, n, \xi)$ in Theorem 3.3.4 depends on $c$, which is a $c$ bound for the limit metric $\hat{g}_{\infty + b}$ (see 4.1). This $c$ exists (see 4.2 (5)).

4.7 Cuts Limits and Hyperbolic Extensions.

At the beginning of this section we gave the definition of cut limit (see 4.1). More generally, let $I \subset \mathbb{R}$ be an interval (compact or noncompact). We say the $\odot$-family $\{g_\lambda\}$ has cut limits on $I$ if the convergence in (4.1) is uniform in $b \in I$. Explicitly this means: for every $b \in I$ and $\epsilon > 0$, there are $\lambda_*$ and neighborhood $U$ of $b$ in $I$ such that $| (g_\lambda)_{\lambda + b'} - \hat{g}_{\infty + b'} | < \epsilon$, for $\lambda > \lambda_*$ and $b' \in U$. In particular $\{g_\lambda\}$ has a cut limit at $b$, for every $b \in I$.

If the $\odot$-family $\{g_\lambda\}$ has cut limits on $\mathbb{R}$ we will just say that $\{g_\lambda\}$ has cut limits. Here is a natural question:

*If $\{h_\lambda\}_b$ has a cut limits, does $\{E_k(h_\lambda)\}_\lambda$ have cut limits?*

**Remark.** More generally we can ask whether $\{E_k(h_\lambda)\}_{\lambda'}$ has cut limits, where $\lambda = \lambda(\lambda')$. Of course the answer would depend on the change of variables $\lambda = \lambda(\lambda')$.

The next result gives an affirmative answer to this question provided the family $\{h_\lambda\}$ is, in some sense, nice near the origin. Explicitly, we say that $\{h_\lambda\}$ is hyperbolic around the origin if there is a $B \in \mathbb{R}$ such that

$$\overline{(h_\lambda)_{\lambda + b}} = \sigma_{\mathbb{S}^{n-1}}$$

for every $b \leq B$ and every (sufficiently large) $\lambda$. Note that this implies that each $h_\lambda$ is canonically hyperbolic on the ball of radius $\lambda + B$. Examples of $\odot$-families that are hyperbolic around the origin are families obtained using hyperbolic forcing, as above.
As mentioned before the next result answers affirmatively the question above. Moreover it also says that some reparametrized families \( \{E_k(h_\lambda)\}_\lambda \), for certain change of variables \( \lambda = \lambda(\lambda') \), have cut limits as well. Write \( \lambda = \lambda(\lambda', \theta) = \sinh^{-1}(\sinh(\lambda') \sin \theta) \), for fixed \( \theta \). We say that \( \{E_k(h_\lambda)\}_\lambda \) is the \( \theta \)-reparametrization of \( \{E_k(h_\lambda)\}_\lambda \). Note that if we consider an hyperbolic right triangle with one angle equal to \( \theta \) and side (opposite to \( \theta \)) of length \( \lambda \), then \( \lambda' \) is the length of the hypothenuse of the triangle. All \( \theta \)-reparametrizations, in the limit \( \lambda' \to \infty \), differ just by translations. The next proposition is proved in [33].

**Proposition 4.7.1.** Let \( M \) have center \( o \). Let \( \{h_\lambda\}_\lambda \) be \( \circ \)-family of metrics on \( M \). Assume \( \{h_\lambda\}_\lambda \) is hyperbolic around the origin. If \( \{h_\lambda\} \) has cut limits, then the \( \theta \)-reparametrization \( \{E_k(h_\lambda)\}_\lambda' \) has cut limits as well. Here \( \theta \in (0, \pi/2] \).

### Section 5. Normal Neighborhoods on All-Right Spherical Complexes

In this section we define and and give some properties of neighborhoods of simplices in all-right spherical complexes. The goal is to define “natural normal neighborhoods” of simplices in all-right spherical complexes, and give some of its properties.

We use the definition and properties of a spherical complex given in Section 1 of [6]. Recall that a spherical complex is an all-right spherical complex if all of its edge lengths are equal to \( \pi/2 \). Given an all-right spherical complex \( P \) we will use the same symbol \( P \) for the complex itself (the collection of all simplices), and its realization (the union of all its simplices). In this paper we shall assume that all spherical complexes satisfy the “intersection condition” of simplicial complexes: every two simplices intersect in at most one common face.

**Remark 5.0.1.** Let \( P \) be an all-right spherical complex and \( \Delta \in P \). The symbol \( \hat{\Delta} \) denotes the interior of \( \Delta \). In this paper we will use the three definitions of link \( \text{Link}(\Delta, P) \) of \( \Delta \) in \( P \). The geometric link \( \text{Link}(\Delta, P) \) is the union of the end points of geodesic segments of small length \( \beta > 0 \) emanating perpendicularly (to \( \Delta \)) from some point \( x \in \hat{\Delta} \). If we want to specify \( \beta \) and \( x \) we say that \( \text{Link}_\beta(\Delta, P) \) is the \( \beta \)-link based at \( x \). The geometric star \( \text{Star}(\sigma, K) \) is the union of the corresponding segments. The simplicial link is the subcomplex of \( P \) formed by all simplices \( \Delta' \) such that (1) \( \Delta' \) is disjoint from \( \Delta \), (2) \( \Delta' \) and \( \Delta \) span a simplex (this simplex is the join \( \Delta * \Delta' \in P \), and \( \Delta' \) is the opposite face of \( \Delta \) in \( \Delta * \Delta' \)). Note that if we continue a geodesic \( [x, u] \), with \( u \) in the geometric \( \beta \)-link at \( x \), we will hit a unique point in \( \Delta' \). This radial geodesic projection gives a relationship between geometric links and simplicial links. The simplicial star is the subcomplex of \( P \) formed by all simplices \( \Delta' \) that contain \( \Delta \). For \( x \in \hat{\Delta}^k \) the direction link of \( \Delta \) in \( P \) at \( x \) is the set of all vectors at \( x \) perpendicular to \( \hat{\Delta}^k \). Using geodesics emanating from \( x \) perpendicularly to \( \Delta \) we also get a relationship between geometric links and the direction links. These different definitions of link all come with natural all-right spherical metrics: the geometric link with the rescaled induced metric, the simplicial link with the induced metric and the direction link with the angle metric. The relationships between the different definitions of link mentioned above all respect the metrics.

#### 5.1 Sets of Widths of Normal Neighborhoods on the Sphere \( S^m \).

We consider the \( m \)-sphere \( S^m \subset \mathbb{R}^{m+1} = \{x = (x_1, \ldots, x_{m+1})\} \) with the canonical all-right spherical structure whose \( m \)-simplices are \( S^m \cap \{(-1)^s x_i \geq 0\} \), for any choice \( s_i \in \{0, 1\}, i = 1, \ldots, m + 1 \). Let
\[ \beta \in (0, \pi/2) \text{ and } \Delta \in S^m. \] The closed normal neighborhood of \( \Delta \) in \( S^m \) of width \( \beta \) is the union of (images of) geodesics of length \( \beta \) emanating perpendicularly from \( \Delta \). It will be denoted by \( N_\beta(\Delta, S^m) \). For the special case \( \text{dim} \Delta = m \) we will take \( N_\beta(\Delta^m, S^m) = \Delta^m \), for any \( \beta \).

Let \( B = \{ \beta_k \}_{k=0,1,2,\ldots} \) be an indexed set of real numbers with \( \beta_k \in (0, \pi/2) \) and \( \beta_{k+1} < \beta_k \). We write \( B(m) = \{ \beta_0, \ldots, \beta_{m-1} \} \). The set \( B \) determines the set of spherical \( B \)-neighborhoods \( N_B(S^m) = N_{B(m)}(S^m) = \{ N_{\beta_k}(\Delta^k, S^m) \}_{\Delta^k \in S^m, k < m} \), for any sphere \( S^m \) (of any dimension). Note that the normal neighborhoods of all \( k \)-simplices \( \Delta^k \) have the same width \( \beta_k \). The set \( B \) is called a set of widths of spherical normal neighborhoods or simply a set of widths. The set \( B(m) \) is a finite set of widths of length \( m \). The definitions above still make sense if we replace \( S^m \) by \( S^m_\mu \) (for small enough \( \beta_k \)'s).

We are interested in pairs of sets of widths \( (B, A) \), \( B = \{ \beta_k \} \) and \( A = \{ \alpha_j \} \), having the following Disjoint Neighborhood Property:

\[ \text{(5.1.1) DNP: For every } k \text{ and } m \text{ the following sets are disjoint} \]
\[
\left\{ N_{\beta_k}(\Delta^k, S^m) - \bigcup_{j<k} N_{\alpha_j}(\Delta^j, S^m) \right\}_{\Delta^k \in S^m}
\]

The disjoint neighborhood property obtained by fixing \( k \) and \( m \) above will be denoted by \( \text{DNP}(k, m) \). In this case we allow the sets of widths to be finite of length at least \( k+1 \). Note that the ordering of the pair \( (B, A) \) is important. It is straightforward to verify that \( \text{DNP}(k, m) \) (and \( \text{DNP} \)) is equivalent to the following property. For fixed (any) \( k \) and \( m \) we have: for different \( k \)-simplices \( \Delta^k_1 \) and \( \Delta^k_2 \) we have

\[ (5.1.1)' \quad N_{\beta_k}(\Delta^k_1, S^m) \cap N_{\beta_k}(\Delta^k_2, S^m) \subset \bigcup_{j<k} N_{\alpha_j}(\Delta^j, S^m) \]

That is, the \( B \)-neighborhoods of different \( k \)-simplices intersect only inside the \( A \)-neighborhood of the \((k-1)\)-skeleton (which is equal to \( \bigcup_{j<k} N_{\alpha_j}(\Delta^j, S^m) \)).

Proposition 5.1.2. The pair of (finite or infinite) sets of widths \( (B, A) \) satisfy \( \text{DNP}(k, m) \) if and only if

\[ \frac{\sin \beta_k}{\sin \alpha_k} \leq \frac{\sqrt{2}}{2}. \]

Note that the inequality condition is independent of \( m \). The proposition follows directly from lemmas 5.1.3 (taking \( k = l \) and \( \beta = \gamma \)) and 5.1.4 given below, and the fact that \( \{ \alpha_k \} \) is decreasing.

Lemma 5.1.3. Let \( \Delta^k, \Delta^l \in S^m \) and \( \Delta^j = \Delta^k \cap \Delta^l \). Let \( \alpha, \beta, \gamma \in (0, \pi/2) \) such that

\[ \frac{\sin \beta}{\sin \alpha}, \frac{\sin \gamma}{\sin \alpha} \leq \frac{\sqrt{2}}{2}. \]

Then

\[ N_{\beta}(\Delta^k, S^m) \cap N_{\gamma}(\Delta^l, S^m) \subset N_{\alpha}(\Delta^j, S^m) \]

Proof. In this proof \( \text{Link}(\Delta^j, S^m) \) shall denote the simplicial link and \( \text{Star}(\Delta^j, S^m) \) the simplicial star (see 5.0.1). Note that \( N_{\beta}(\Delta, S^m) \subset \text{Star}(\Delta, S^m) \), for every \( \Delta \in S^m \). Write \( S = \text{Link}(\Delta, S^m) \), \( \Delta'_1 = S \cap \Delta^k \) and \( \Delta'_2 = S \cap \Delta^l \). Then \( \Delta'_1 \) is a simplex in the all-right triangulation of \( S \). Also \( \Delta'_1 \) and \( \Delta'_2 \) are disjoint. Hence their distance in \( S \) is at least \( \pi/2 \).

Suppose there is \( q \in N_{\beta}(\Delta^k, S^m) \cap N_{\gamma}(\Delta^l, S^m) \). Since both of these neighborhoods lie in \( \text{Star}(\Delta^j, S^m) \) there is geodesic segment \([p, q]\) in \( \text{Star}(\Delta^j, S^m) \) with \( p \in \Delta^j \) and \([p, q]\) perpendicular to \( \Delta^j \) (note that \( p \) may lie in \( \partial \Delta^j \) and that the geodesic may not be unique if \( q \in S \)). Write \( \alpha' = d_{S^m}(p, q) \). We have to prove \( \alpha' \leq \alpha \). We assume \( \alpha' > \alpha \) and get a contradiction. Let \( q_1 \) be the closest point in \( \Delta^k \) to \( q \) and \( q_2 \) be the closest point in \( \Delta^l \) to \( q \). We have \( a_1 = d_{S^m}(q_1, q) \leq \beta \) and \( a_2 = d_{S^m}(q_2, q) \leq \gamma \). We get a right (at \( q_i \)) spherical triangle with one side equal to \( a_i \) and hypotenuse equal to \( \alpha' \). Let \( \theta_i \) be the angle at \( p \),
Lemma 5.1.4. Let $\alpha, \beta$ that verify the lemma for $S$, that is, the angle opposite to the side of length $q$. Analogously let $q'$ be the intersection of $S$ with the ray at $p$ with direction $q$. Let $z_i$ be the intersection of $S$ with the ray at $p$ with direction $q$. Note that $z_i \in \Delta_i$. Also note that $d_S(q, z_i)$ is equal to the length of the arc $qz_i$ in $S$. This together with the fact that $S$ is a $(m-j-1)$-sphere of radius one imply that $d_S(q, z_i) = \theta_i$. Therefore

$$\frac{\pi}{2} \leq d_S(\Delta_i, \Delta_j' \leq d_S(z_1, z_2) \leq d_S(z_1, q) + d_S(q', z_2) \leq (\theta_1 + \theta_2)$$

Hence $\frac{\pi}{2} \leq \theta_1 + \theta_2 < \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$ which is a contradiction. This proves the lemma.

Lemma 5.1.5. Suppose the pair of sets of widths $(B, \mathcal{A})$ satisfies DNP. Let $\Delta^j = \Delta^k \cap \Delta^l$, $j < k \leq l$. Then

$$\mathcal{N}_{1\beta_j}(\Delta^k, S^m) \cap \mathcal{N}_{1\beta_l}(\Delta^l, S^m) \subset \bigcup_{j<k} \mathcal{N}_{1\beta_i}(\Delta^i, S^m)$$

Remark. Note that the condition $\Delta^j = \Delta^k \cap \Delta^l$, $j < k, \ell$, is equivalent to $\Delta^k \not\subset \Delta^l$ and $\Delta^l \not\subset \Delta^k$.

Proof of Lemma 5.1.5. From Proposition 5.1.2 we have $\frac{\sin \beta_j}{\sin \alpha_j} \leq \frac{\sin \beta_k}{\sin \alpha_k} < \frac{\sqrt{2}}{2}$. The lemma now follows from Lemma 5.1.3. This proves the lemma.

5.2. Natural Neighborhoods on the Sphere $S^m$.

Let $\Delta \in S^m$. In this section the $\beta$-geometric link at the barycenter of $\Delta$ will be called the linked sphere of $\Delta$ of radius $\beta$, and will be denoted by $S_\Delta$. Rescaling gives an identification between $S_\Delta$ and $S^m_{\Delta}$, thus we will consider $S_\Delta$ as an all-right spherical complex (alternatively we can consider $S_{\Delta}$ with the angle metric). In this case the simplices of $S_\Delta$ are $S_\Delta \cap \Delta'$, for all $\Delta' \supset \Delta$.

Let $B = \{\beta_k\}$ be a set of widths. Let $\Delta = \Delta^k$ be the linked sphere of $\Delta$ of radius $\beta_k$. By intersecting $S_\Delta$ with each element of the set $N_B(S^m)$ we get the set $N(S_\Delta, B) = \{S_\Delta \cap N_{\beta_j}(\Delta^j, S^m)\}_{j \in S^m}$. It is straightforward to verify that for simplices $\Delta^j$, with $\Delta \subset \Delta^j$, there are decreasing $\beta_j' > 0$ such that

$$S_\Delta \cap N_{\beta_j}(\Delta^j, S^m) = N_{\beta'_j}(\Delta \cap \Delta^j, S_\Delta)$$

where the last term is the $\beta_j'$-normal neighborhood of the simplex $S_\Delta \cap \Delta^j$ in $S_\Delta$ (recall that we are identifying $S_\Delta$ with $S^m_{\Delta}$, or, using the angle metric). Note that, since the $\beta_j$'s are decreasing, we have $\beta_j' < \pi/2$. Hence we can write $N(S_\Delta, B) = N_{B'}(m-k-1)(S_\Delta)$ where $B' = \{\beta_0, \ldots, \beta_{m-k-2}\}$ and we also say that $N_{B'}(m-k-1)(S_\Delta)$ is the set of $B'(m-k-1)$-neighborhoods of $S_\Delta$. Note that $B'(m-k-1)$ depends only on $B$ and the dimension $k$ of $\Delta^k$. The next lemma gives this relation explicitly.
Lemma 5.2.1. For \( l = 0, \ldots, m - k - 2 \) we have 
\[
\sin(\beta_i^l) = \frac{\sin(\beta_{k+1})}{\sin(\beta_k^l)}.
\]

Proof. Let \( p \in S_\Delta \cap (N_{s_j}(\Delta^j, S^m)) \), where \( \Delta = \Delta^k \subset \Delta^j \). Then there is a \( q \in \Delta^j \) such that 
\[
d = d_{sm}(p,q) = d_{sm}(p,\Delta^j) \leq \beta_j.
\]
We are interested in the case when \( d \) is maximum, so we assume 
\( d = \beta_j \). Let \( o \) be the barycenter of \( \Delta \). Since \( d_{sm}(o,p) = \beta_k \) we get a right (at \( q \)) spherical triangle with one side equal to \( \beta_j \) and hypotenuse equal to \( \beta_k \). The angle opposite to the side of length \( \beta_j \) is \( \beta_j^l - k - 1 \).

Then, by the spherical law of sines we get 
\[
\sin(\beta_j^l - k - 1) = \frac{\sin(\beta_j^l)}{\sin(\beta_k^l)}.
\]
This proves the lemma.

Therefore the set of widths \( B \) on \( S^m \) induces the set \( B'(m - k - 1) \) on \( S^{m-k-1} \), by considering \( S^{m-k-1} \) as the (rescaled) link of the \( k \) simplices in the all-right triangulation of \( S^m \). Lemma 5.2.1 gives a relationship between \( B(m) \) and \( B'(m - k - 1) \).

Let \( B = \{\beta_i\}_{i=0,1,\ldots} \) be a set of widths. We say that \( B \) is a natural set of neighborhood widths for all spheres if \( B(m - k - 1) = B'(m - k - 1) \) for all \( m \) and \( k \) with \( m > k \).

Corollary 5.2.2. The set of widths \( B = \{\beta_i\} \) is natural if and only if \( \sin(\beta_i) = \sin(i+1)(\beta_0) \) and \( \beta_0 < \pi/4 \).

Proof. It follows from 5.2.1 with \( l = 0 \) that \( \sin(\beta_{k+1}) = \sin(\beta_k) \sin(\beta_0) \). This proves the corollary.

Given \( c \in (0,1) \) we define \( B(c) = \{\beta_i\} \) by \( \beta_i = \sin^{-1}(c^{i+1}) \). Hence the corollary says that \( B \) is natural if and only if \( B = B(c) \), for some \( c \in (0,1) \). In fact, in this case we have \( c = \sin(\beta_0) \).

Let \( c \in (0,1) \) and \( c > 1 \). We denote by \( B(c;\mathcal{D}) = \{\gamma_i\} \) the set defined by \( \gamma_i = \sin^{-1}(c^{i+1}) \). Note that \( B(c;\mathcal{D}) \) is a set of widths provided \( c < 1 \). Proposition 5.1.2 implies the next corollary.

Corollary 5.2.3. The pair of sets of widths \( (B(s;\mathcal{D}), B(c;\mathcal{D})) \) satisfy DNP provided \( \frac{s}{c} \leq \frac{\sqrt{s}}{2} \).

Note that we can take \( c = c' = 1 \) which implies that a natural set of widths \( B(s;\mathcal{D}) \) satisfy DNP with \( A = B = B(s;\mathcal{D}) \).

5.3. Neighborhoods in Piecewise Spherical complexes.

This subsection is essentially a version of 5.1 in which we replace \( S^m \) by an all-right spherical complex. Let \( P \) be an all-right spherical complex and \( \Delta^k \in P \). We can write \( \text{Link}(\Delta^k, P) = \bigcup_{\Delta^j \subset \Delta^i \in P} \text{Link}(\Delta^k, \Delta^j) \) as sets and complexes. Hence the set \( \{\Delta^j\}_{\Delta^k \subset \Delta^i \in P} \) is in one-to-one correspondence with the set of spherical simplices of \( \text{Link}(\Delta^k, P) \), that is \( \Delta^j \) corresponds to \( \text{Link}(\Delta^k, \Delta^j) \), which is an all-right spherical simplex of dimension \( j - k - 1 \) in \( \text{Link}(\Delta^k, P) \). The all-right spherical metric on \( \text{Link}(\Delta^k, P) \) will be denoted by \( \sigma_{\text{Link}(\Delta^k, P)} \).

Remark 5.3.1. In the paragraph above we did not specify the type of link we were using (see 5.0.1). Any of these will lead to corresponding definition of \( \text{Link}(\Delta, P) \), but they are all equivalent as metric complexes (note that we are assuming \( P \) has the “intersection condition”). We will use any of the definitions depending on the situation.

Lemma 5.3.2. We have that 
\[
\text{Link}(\text{Link}(\Delta^k, \Delta^j), \text{Link}(\Delta^k, P)) = \text{Link}(\Delta^j, P)
\]
provided \( \Delta^k \subset \Delta^j \). (This is an equality of all-right spherical metric complexes.)
Remark. If we use the simplicial definition of links this equality is actually an equality of sets.

Proof. Let $\Delta^k \subset \Delta^j$. Let $\Delta^i$ be the opposite face of $\Delta^k$ in $\Delta^j$. The statement in the lemma written using the simplicial definition of link is: \( \text{Link}(\Delta^i, \text{Link}(\Delta^k, P)) = \text{Link}(\Delta^j, P) \). We prove this. We have that $\Delta^i \in \text{Link}(\Delta^j, \text{Link}(\Delta^k, P))$ is equivalent to (recall $\Delta^k \cap \Delta^j = \emptyset$) (1) $\Delta^i \cap \Delta^k = \emptyset$, (2) $\Delta^i \cup \Delta^k$ is contained in a simplex, (3) $\Delta^j \cap \Delta^i = \emptyset$, (4) $\Delta^i \cup \Delta^j \cup \Delta^k$ is contained in a simplex. On the other hand $\Delta^i \in \text{Link}(\Delta^j, P)$ is equivalent to (a) $\Delta^i \cap \Delta^j = \emptyset$, (b) $\Delta^i \cup \Delta^j$ is contained in a simplex. Since $\Delta^i$ is opposite to $\Delta^k$ in $\Delta^j$ we have that (a)+(b) if and only if (1)-(4). This proves the lemma.

Define the closed normal neighborhood of $\Delta^k$ in $\Delta^j$ of width $\beta$ as $N_\beta(\Delta^k, \Delta^j) = N_\beta(\Delta^k, S^m) \cap \Delta^j$. If $\Delta^k$ is a simplex in the all-right spherical complex $P$, we define the closed normal neighborhood of $\Delta^k$ in $P$ of width $\beta$ as

\[
N_\beta(\Delta^k, P) = \bigcup_{\Delta^i \subset \Delta^j \in P} N_\beta(\Delta^k, \Delta^j)
\]

Let $B = \{\beta_k\}$ be a set of widths. Then, for any all-right spherical complex $P$ the set $B$ induces the set of neighborhoods $N_B(P) = \{N_{\beta_k}(\Delta^k, P)\}_{\Delta^k \in P}$. The next lemma is the spherical complex version of Lemma 5.1.3.

Corollary 5.3.4. Let $\Delta^k, \Delta^l \in P$ and $\Delta^j = \Delta^k \cap \Delta^l$. Let $\alpha, \beta, \gamma \in (0, \pi/2)$ such that $\frac{\sin \beta}{\sin \alpha}, \frac{\sin \gamma}{\sin \alpha} \leq \frac{\sqrt{2}}{2}$. Then $N_\beta(\Delta^k, P) \cap N_\gamma(\Delta^l, P) \subset N_\alpha(\Delta^j, P)$.

Proof. The proof is the same as the proof of Lemma 5.1.5 with minor obvious changes. Just recall that we are assuming $P$ to have the intersection condition. Also note that if $p, q, q_i$ are as in the proof of 5.1.5 then they all three lie in an all-right simplex in $P$. This proves the corollary.

As in Section 5.1, the next two results follow directly from corollary 5.3.4. The first is a version of DNP (see 5.1.1) for $P$, obtained by replacing $S^m$ by $P$.

Corollary 5.3.5. Let the pair of sets of widths $(B, A)$ satisfy DNP. Then for any all-right spherical complex $P$ and $k$ that the following sets are disjoint

\[
\left\{ N_{\beta_k}(\Delta^k, P) - \bigcup_{j < k} N_{\beta_j}(\Delta^j, P) \right\}_{\Delta^k \in P}
\]

The next is a version of Lemma 5.1.5 for general $P$.

Corollary 5.3.6. Let the pair of sets of widths $(B, A)$ satisfy DNP. Then for any all-right spherical complex $P$ and $\Delta^j = \Delta^k \cap \Delta^l$, $j < k \leq l$, simplices in $P$, we have that

\[
N_{\beta_k}(\Delta^k, P) \cap N_{\beta_l}(\Delta^l, P) \subset \bigcup_{i < k} N_{\alpha_i}(\Delta^i, P)
\]

Section 6. Normal Neighborhoods on Hyperbolic Cones

In this section we define and give some properties of neighborhoods of faces in hyperbolic cones. Hyperbolic cones are cones over all-right spherical complexes; they admit a canonical piecewise hyperbolic metric. To define the neighborhoods on hyperbolic cones we will use the objects and results of Section 5.

6.1. Neighborhoods in Piecewise Hyperbolic cones.

We write $\mathbb{R}^{k+1}_+ = (0, \infty)^{k+1}$, $\overline{\mathbb{R}}^{k+1}_+ = \overline{[0, \infty)^{k+1}}$, and $\mathbb{B}^{k+1}_H = \mathbb{B}^{k+1}_H \cap \overline{\mathbb{R}}^{k+1}_+$, where $\mathbb{B}^{k+1}_H$ is the disc model of $\mathbb{H}^{k+1}$. The canonical all-right spherical $k$-simplex is $\Delta_{sk} = S^k \cap \overline{\mathbb{R}}^{k+1}_+$. We denote the origin of
\(\mathbb{H}^{k+1}\) by \(o = o_{ik+1}\). We can identify \(\mathbb{H}^{k+1}_+ - \{o\}\) with \(\Delta_g \times \mathbb{R}^+\) with metric \(\sinh^2(s) \sigma_{\Delta_g} + ds^2\), where \(s\) is the distance to the "vertex" \(o\). We say that \(\mathbb{H}^{k+1}_+\) is the infinite hyperbolic cone of \(\Delta_g\) and write \(C \Delta_g = \mathbb{H}^{k+1}_+\).

Let \(P\) be an all-right spherical complex. The piecewise spherical metric on \(P\) will be denoted by \(\sigma_P\). Recall that \(P\) is constructed by gluing the all-right spherical simplices \(\Delta \in P\), where each \(\Delta = \Delta^k\) is a copy of \(\Delta_g\). The infinite piecewise hyperbolic cone of \(P\) is the space \(CP\) obtained by gluing the hyperbolic cones \(C \Delta, \Delta \in P\) using the same rules used for obtaining \(P\). Note that all vertex points of the \(C \Delta\) get glued to a unique vertex \(o = o_{CP}\). The cones \(C \Delta, \Delta \in P\), are the cone simplices of \(CP\) and the faces of the cone simplex \(C \Delta\) are the \(C \Delta', \Delta' \subset \Delta\). The set of all cone simplices will also be denoted by \(CP\). The complex \(CP\) (i.e. \(CP\) together with its cone faces) is an all-right hyperbolic cone complex.

The piecewise hyperbolic metric on \(CP\) shall be denoted by \(\sigma_{CP}\) and its corresponding geodesic metric by \(d_{CP}\). Note that the metric \(\sigma_{CP}\) can be deduced from the hyperbolic cone structure. All (constant speed) rays emanating from \(o\) are length minimizing geodesics defined on \([0, \infty)\). Then we can identify \(CP - \{o\}\) with \(P \times \mathbb{R}^+\) with warped metric \(\sinh^2(r) \sigma_P + dr^2\), where \(r\) is the distance to the vertex \(o\). Even though \(\sigma_{CP}\) is not (generally) smooth, the set of rays emanating from the vertex \(o_{CP}\) gives a well defined ray structure as in Section 1.

For \(s \geq 0\) we denote the open ball of radius \(s\) of \(CP\) centered at \(o\) by \(B_s(CP)\). Note that this ball is the “finite open cone” \(P \times (0, s) \cup \{o\}\), where we are using the identification above. The closed ball will be denoted by \(B_s(CP)\) and the sphere of radius \(s\), \(s > 0\), will be denoted by \(S_s(CP)\), which we shall sometimes identify with \(P \times \{s\}\) or simply with \(P\).

Let \(\Delta \in P\). In this section \(\text{Star}(\Delta, P)\) will denote the simplicial star of \(\Delta\) in \(P\). Since \(\text{Star}(\Delta, P)\) is an all-right spherical complex then \(C(\text{Star}(\Delta, P))\) is a well defined all-right hyperbolic cone complex, which we could interpret as the the simplicial star of \(C \Delta\) in \(CP\). To save parentheses we will write \(C \text{Star}(\Delta, P)\) instead of \(C(\text{Star}(\Delta, P))\). Note that \(C \text{Star}(\Delta, P) \subset CP\).

We will use the following three identifications, given in 6.1.1, 6.1.2, and 6.1.3 below.

**6.1.1.** Let \(\Delta^j \subset \Delta^k\) and let \(\Delta^l\) be the opposite face of \(\Delta^j\) in \(\Delta^k\). We have \(l = k - j - 1\). Since \(C \Delta^j = \mathbb{H}^{j+1}_+ \subset \mathbb{H}^{j+1}\), \(C \Delta^l = \mathbb{H}^{k+1}_+ \subset \mathbb{H}^{k+1}\) and \(C \Delta^k = \mathbb{H}^{k+1}_+ \subset \mathbb{H}^{j+1}_+ \subset \mathbb{H}^{k+1}\) we can write

\[
C \Delta^k = C \Delta^j \times C \Delta^l \subset \mathbb{H}^{j+1}_+ \times \mathbb{H}^{k+1}_+ = \mathbb{E}_{k+1}(\mathbb{H}^{l+1}_+)
\]

with warped metric \(\cosh^2(r) \sigma_{\mathbb{H}^{l+1}_+} + \sigma_{\mathbb{H}^{j+1}_+}\), where \(r\) is the distance in \(\mathbb{H}^{l+1}_+\) to \(o\). Thus we can write \(C \Delta^k = \mathcal{E}_{C \Delta^j}(C \Delta^l)\). Note that the order of the decomposition here is important (see 2.6). The identification above can be done explicitly in the following way. Let \(p \in C \Delta^k \subset \mathbb{H}^{k+1} = \mathcal{E}_{\mathbb{H}^{l+1}_+}(\mathbb{H}^{l+1}_+)\). We use the functions (or coordinates) given in Section 2: \(s, r, t, y, v, x, u, w\). Then \(p = sx \in C \Delta^k, (s, x) \in \mathbb{R}^+ \times \Delta^k, \) corresponds to \((y, v) = (tw, ru) \in C \Delta^j \times C \Delta^l, (t, w) \in \mathbb{R}^+ \times \Delta^j, (r, u) \in \mathbb{R}^+ \times \Delta^l.\) Note that \(x = \lceil w, u \rceil(\beta)\), where \(\beta\) is as in Section 2, i.e. it is the angle between \(w\) and \(x\).

**6.1.2.** The following identification is an important one; it will be used many times. Let \(\Delta = \Delta^k \in P\). We have that \(C \text{Star}(\Delta, P)\) can be identified with \(C \Delta \times C(\text{Link}(\Delta, P))\) with metric \(\cosh^2(r) \sigma_{\mathbb{H}^{k+1}_+} + \sigma_{C \text{Link}(\Delta, P)}\), where \(r\) is the distance in \(C(\text{Link}(\Delta, P))\) to the vertex \(o \in C(\text{Link}(\Delta, P))\). Note that the vertex of \(C \text{Star}(\Delta, P)\) is identified with \((o', o'')\), where \(o', o''\) are the vertices of \(C \Delta\) and \(C(\text{Link}(\Delta, P))\), respectively. The identification here is an identification of all-right hyperbolic cone complexes and it
is obtained using 6.1.1 simplexwise. Explicitly using the coordinates $s, r, t, y, v, x, u, w$ given in Section 2 we see that an element $p = sx \in C \text{Star}(\Delta^k, P)$ can be written as $(tw, ru) \in C \Delta^k \times C \Delta^l \subset C \Delta^k \times C \text{Link}(\Delta^k, P)$, where $\Delta^k \ast \Delta^l$ is a simplex in the (simplicial) star $\text{Star}(\Delta^k, P)$; that is, $\Delta^l$ is a simplex in $\text{Link}(\Delta^k, P)$. Since we can write $x = [w, u](\beta)$, $\beta$ is the angle between $w$ and $x$, the identification is given by $s[w, u](\beta) = (t w, r u)$.

6.1.3. As mentioned above, even though $\sigma_{\text{C Link}(\Delta^k, P)}$ is not in general smooth it has a well defined ray structure, where we are taking $o_{\Delta^k} = o_{\text{C Link}(\Delta^k, P)}$ as the center of $\text{C Link}(\Delta^k, P)$. Hence it makes sense to consider, as in Section 2, the hyperbolic extension $\mathcal{E}_k(C \text{Link}(\Delta^k, P)) = C \Delta^k \times C (\text{Link}(\Delta^k, P))$ with metric $\cosh^2(r) \sigma_{g_{\Delta^k}} + \sigma_{\text{C Link}(\Delta^k, P)}$. Therefore, using 6.1.2, we can write

$$C \text{Star}(\Delta^k, P) = \mathcal{E}_{C \Delta^k} \left( C \text{Link}(\Delta^k, P) \right) \subset \mathcal{E}_k \left( C \text{Link}(\Delta^k, P) \right)$$

where we consider $C \text{Star}(\Delta^k, P) \subset C P$ with metric $\sigma_{\Delta^k}$ and $C \left( \text{Link}(\Delta^k, P) \right)$ with metric $\sigma_{\text{C Link}(\Delta^k, P)}$.

For a cone simplex $C \Delta \subset C P$, we define its closed normal neighborhood of width $s$ by

$$N_s(C \Delta, C P) = C \Delta \times \overline{B}_s(C \text{Link}(\Delta^k, P)) \subset C \text{Star}(\Delta^k, P)$$

where we are using the identification given in 6.1.2. Hence $N_s(C \Delta, C P)$ is the union of (the images of) all geodesics of length $s$ emanating perpendicularly from $C \Delta$. The open normal neighborhood of width $s$ will be denoted by $\tilde{N}_s(C \Delta, C P)$.

**Lemma 6.1.5.** Let $\Delta^j \subset \Delta^k \subset P$. Then

$$C \text{Star}(\Delta^k, P) = C \Delta^k \times C \text{Link}(\Delta^k, P) = C \Delta^j \times C \text{Star}(\Delta^j, \text{Link}(\Delta^j, P)) \subset C \Delta^j \times C \text{Link}(\Delta^j, P)$$

where $\Delta^j = \text{Link}(\Delta^j, \Delta^k)$ is the opposite face of $\Delta^j$ in $\Delta^k = \Delta^j \ast \Delta^l$.

**Remark.** The first equality is given in 6.1.2 above. The last inclusion follows from the inclusion $\text{Star}(\Delta^j, \text{Link}(\Delta^j, P)) \subset \text{Link}(\Delta^j, P)$. The middle equality in the statement of the lemma is an equality of hyperbolic cone complexes.

**Proof.** We have

$$C \Delta^k \times C \text{Link}(\Delta^k, P) = C \Delta^j \times \left( C \Delta^l \times C \text{Link}(\Delta^k, P) \right) = C \Delta^j \times C \text{Star}(\Delta^j, \text{Link}(\Delta^j, P))$$

where the first equality follows from 6.1.1 and the second equality from 5.3.2 and 6.1.2 above. This proves the lemma.

Here is a metric version of Lemma 6.1.5. Let $\Delta^j, \Delta^k,$ and $\Delta^l$ be as in Lemma 6.1.5. Let $h : S^{m-k-1} \to \text{Link}(\Delta^k, P)$ be a homeomorphism and consider the cone of $h$, $C h : \mathbb{R}^{m-k} \to C \text{Link}(\Delta^k, P)$. Let $f'$ be a metric on $\mathbb{R}^{m-k}$ of the form $f' = f'_r + dr^2$. Thus $f'$ and $\sigma_{g_{\mathbb{R}^{m-k}}}$ have the same ray structure. The metric $f = h_*f'$ is a metric on $C \text{Link}(\Delta^k, P)$, and it has the same ray structure as $\sigma_{C \text{Link}(\Delta^k, P)}$. We can consider the (restriction of the) metric $\mathcal{E}_k(f)$ defined on $\mathcal{E}_k(C \text{Link}(\Delta^k, P))$ to $\mathcal{E}_{C \Delta^k}(C \text{Link}(\Delta^k, P)) = C \Delta^k \times C \text{Link}(\Delta^k, P)$. And, since we have $\text{Link}(\Delta^k, P) = \text{Link}(\Delta^l, C \text{Link}(\Delta^k, P))$ (see 5.3.2) the metric $f$ is also a metric on $C \text{Link}(\Delta^l, \text{Link}(\Delta^j, P))$, and we can consider the metric $\mathcal{E}_j(\mathcal{E}_l(f))$ on $\mathcal{E}_{C \Delta^j}(\mathcal{E}_{C \Delta^l}(C \text{Link}(\Delta^l, \text{Link}(\Delta^j, P)))) = C \Delta^j \times C \Delta^l \times C \text{Link}(\Delta^l, \text{Link}(\Delta^j, P))$.

**Corollary 6.1.6.** Using the identification in 6.1.5 we get $\mathcal{E}_k(f) = \mathcal{E}_j(\mathcal{E}_l(f))$

**Proof.** The proof follows from Proposition 2.6 and the proof of Lemma 6.1.5. This proves the corollary.
We can allow $f$ above to be non-smooth, e.g. we can take $f = \sigma_{\text{Link}(\Delta^k, P)}$, and we obtain the following corollary. It follows from 6.1.3 and 6.1.6.

**Corollary 6.1.7.** We have
\[
\text{CStar}(\Delta, P) = \mathcal{E}_{\text{C}}(C(\text{Link}(\Delta, P))) = \mathcal{E}_{\text{C}}(\mathcal{E}_{\text{C}}(C(\text{Link}(\Delta^i, \text{Link}(\Delta, P))))
\]
where we consider $\text{CStar}(\Delta, P) \subset CP$ with metric $\sigma_{\text{C}, P}$, $C(\text{Link}(\Delta, P))$ with metric $\sigma_{\text{C}, \text{Link}(\Delta, P)}$, and $C(\text{Link}(\Delta^i, \text{Link}(\Delta, P)))$ with metric $\sigma_{\text{C}, \text{Link}(\Delta^i, \text{Link}(\Delta, P))}$.

The next two results will be needed in 6.2.

**Lemma 6.1.8.** Let $\Delta^j \subset \Delta^k \in P$. Then
\[
N_s(C \Delta^k, C P) = C \Delta^j \times N_s(C \Delta^i, \text{Link}(\Delta^j, P))
\]
where $\Delta^i = \text{Link}(\Delta^j, \Delta^k)$. A similar statement holds if we replace $N$ by $\bar{N}$.

**Remark.** Note that the left-hand side of the equality, $N_s(C \Delta^k, C P)$, is a subset of $\text{CStar}(\Delta^k, P)$. The right-hand side is a subset of $C \Delta^i \times \text{Link}(\Delta^j, P)$. By Lemma 6.1.5 we can write $\text{CStar}(\Delta^k, P) \subset C \Delta^j \times \text{Link}(\Delta^j, P)$. Lemma 6.1.8 says that under this inclusion $N_s(C \Delta^k, C P)$ corresponds to $C \Delta^j \times N_s(C \Delta^i, \text{Link}(\Delta^j, P))$.

**Proof.** We have
\[
N_s(C \Delta^k, C P) = C \Delta^j \times \left( C \Delta^i \times \overline{B}_s(\text{Link}(\Delta^k, P)) \right) = C \Delta^j \times N_s(C \Delta^i, \text{Link}(\Delta^j, P))
\]
where the first equality follows from 6.1.1 and 6.1.4 and the last from 5.3.2 and 6.1.4. This proves the lemma.

**Lemma 6.1.9.** Let $s > 0$, $\beta \in (0, \pi/2)$ and $\Delta \in P$. Then
\[
N_{s_{\beta}}(C \Delta, C P) \cap \mathcal{S}_{s}(C P) = N_{\beta}(\Delta, P) \times \{s\}
\]
where $s_{\beta} = \sinh^{-1}\left(\sinh(s) \sin(\beta)\right)$ and we are identifying $\mathcal{S}_{s}(C P)$ with $P \times \{s\}$ (thus $N_{\beta}(\Delta, P) \times \{s\} \subset P \times \{s\} = \mathcal{S}_{s}(C P)$).

**Proof.** Denote the vertex of $C(\text{Link}(\Delta, P))$ by $o'$. Note that both sides of the equality above are contained in $\mathcal{S}_{s}(C P)$. From 6.1.4 and $\beta < \pi/2$ we also get that both sides are contained in $\text{CStar}(\Delta, P)$. Let $p \in \mathcal{S}_{s}(C P)$, then $d_{\text{C}, P}(o, p) = s$. From 6.1.4 we can write $p = (x, y) \in C \Delta \times \overline{B}_s(C \text{Link}(\Delta, P))$, for some $s'$. Consider the geodesic segments $a = [o, p]$, $b = [(x, o')] \subset \{x\} \times \text{Link}(\Delta, P)$ and $c = [o, (x, o')] \subset C \Delta \times \{o'\}$. The length of $a$ is $s$. Since each $\{x\} \times \text{Link}(\Delta, P)$ is totally geodesic (see 6.1.2) we get that the length of $b$ is $s'$. Also since $p \in \text{CStar}(\Delta, P)$ we have that $p \in C \Delta^i$ for some $\Delta^j \subset \Delta$ containing $\Delta$. But $C \Delta^j = \mathbb{H}^{j+1}_{++} \subset \mathbb{H}^{j+1}$ is totally geodesic in $C P$ hence all three segments $a, b, c$ are contained in $C \Delta^j$. Therefore we get a hyperbolic geodesic triangle with sides $a, b, c$, whose angle at $(x, o')$ is $\pi/2$ (because $C \Delta \times \{o'\}$ and $\{x\} \times \text{Link}(\Delta, P)$ are perpendicular, see 6.1.2). Let $\beta'$ be the angle at $o$. Then $p \in N_{s_{\beta}}(C \Delta, C P)$ if and only if $s' \leq s_{\beta}$. Also $p \in N_{\beta}(\Delta, P)$ if and only if $\beta' \leq \beta$. But $s_{\beta} = \sinh^{-1}\left(\sinh(s) \sin(\beta)\right)$ and by the hyperbolic law of sines we also get that $s' = \sinh^{-1}\left(\sinh(s) \sin(\beta')\right)$. Consequently $s' \leq s_{\beta}$ is equivalent to $\beta' \leq \beta$. This proves the lemma.

### 6.2. Construction of the Fundamental Neighborhoods in Hyperbolic Cones.

In this section we construct the fundamental sets $\mathcal{Y}$ and $\mathcal{X}$ on the cone of a given all-right spherical...
complex $P$. These sets depend on a number of pre-fixed variables. This subsection is a bit involved and technical, but the sets $Y$, $X$ are key objects which will be used in Section 8 to smooth the metric $\sigma_{cP}$ on $CP$. The results that will be used in Section 8 are propositions 6.2.1, 6.2.3, and 6.2.4.

Let $\xi > 0$, $\varsigma > 0$ and $c > 1$ with $c\varsigma < e^{-4}$. Write $B = B(\varsigma; c) = \{\beta_i\}$ and $A = B(\varsigma) = \{\alpha_i\}$ be set of widths as in 5.2. We have $\sin \beta_i = c\varsigma^{-i+1}$, $\sin \alpha_i = \varsigma^{-i+1}$. Since $e^{-4} < \sqrt{2}$, the condition $c\varsigma < e^{-4}$ together with corollary 5.2.3 imply that $(B, A)$ and $(B, B)$ satisfy DNP in Section 5.1.

Given a number $r > 0$ and an integer $k \geq 0$ we define $r_k = r_k(DNP) = sinh^{-1}\left(\frac{\sinh(r)}{\sin(\alpha_k)}\right)$. By convention we also set $r_{-1} = r$. (Alternatively we could declare that every set of widths $\{\alpha_k\}$ has a (-1) term $\alpha_{-1}$ always equal to $\pi/2$.) Let $k$ and $m$ be integers with $m \geq 2$ and $0 \leq k \leq m - 2$. Define $s_{m,k} = sinh^{-1}\left(\frac{\sinh(r)\sin(\beta_k)}{\sin(\alpha_{m-2})}\right) = sinh^{-1}\left(\sinh(r_{m-2})\sin(\beta_k)\right)$. We write $r_{m,k} = r_{m-k-3}$. Note that $r_{m,k} < s_{m,k}$.

Let $P = P^m$ be an all-right spherical complex with $m \leq \xi$, and let $r > (4 + \xi)$. For every $\Delta^k \in P$, $0 \leq k \leq m - 2$, define the following subsets of $CP$:

$$Y(P, \Delta^k, r, \xi, (c, \varsigma)) = \hat{N}_{\Delta^k, m,k}(C \Delta^k, CP) \subset \bigcup_{j < k} N_{\Delta^j, r, m,j}(C \Delta^j, CP) - B_{r_{m-2}-(2+\xi)}(CP)$$

$$Y(P, r, \xi, (c, \varsigma)) = CP - \bigcup_{j < m-1} N_{\Delta^j, r, m,j}(C \Delta^j, CP) - B_{r_{m-2}-(2+\xi)}(CP)$$

Since $\xi$, $c$ and $\varsigma$ will remain constant, in the rest of this section we will write $Y(P, \Delta^k, r)$ instead of $Y(P, \Delta^m, r, \xi, (c, \varsigma))$.

**Proposition 6.2.1.** For $r > (4 + \xi)$ we have the following properties

(i) $Y(P, \Delta^k, r) \subset \hat{N}_{\Delta^k, m,k}(C \Delta^k, CP) \subset \operatorname{int} \operatorname{Star} \Delta^k, CP)$

(ii) $Y(P, \Delta^k, r) \cap N_{\Delta^j, r, m,j}(C \Delta^j, CP) = \emptyset$, for $j < k$.

(iii) $Y(P, \Delta^j, r) \cap B_{r_{m-2}-(2+\xi)}(CP) = \emptyset$,

(iv) $CP - B_{r_{m-2}-(2+\xi)}(CP) = Y(P, r) \cup \bigcup_{\Delta^k \in P, k \leq m-2} Y(P, \Delta^k, r)$

(v) $\Delta^j \cap \Delta^k = \emptyset$ implies $N_{\Delta^j, r, m,j}(C \Delta^j, CP) \cap N_{\Delta^k, r, m,k}(C \Delta^k, CP) = \emptyset$

(vi) $\Delta^j \cap \Delta^k = \emptyset$ implies $Y(P, \Delta^j, r) \cap Y(P, \Delta^k, r) = \emptyset$

(vii) $\Delta^k = \Delta^i \cap \Delta^j$, with $k < i, j$, implies $Y(P, \Delta^i, r) \cap Y(P, \Delta^j, r) = \emptyset$

(viii) for any two different $k$-simplices $\Delta^i_k, \Delta^j_k$ we have $Y(P, \Delta^i_k, r) \cap Y(P, \Delta^j_k, r) = \emptyset$

(ix) $Y(P, r) \cap N_{\Delta^j, r, m,j}(C \Delta^j, CP) = \emptyset$, for $j < m - 1$.

**Proof.** The statements (ii), (iii), and (ix) follow from the definition of $Y$. We prove (i). The second contention holds because $\hat{N}_{\Delta^k, m,k}(C \Delta^k, CP)$ is open. We prove the first contention. By definition we
have \( \mathcal{Y}(P, \Delta^k, r) \subset N_{s_m,k} (C \Delta^k, C P) \). If a point \( p \in \hat{N}_{s_m,k} (C \Delta^k, C P) - \hat{N}_{s_m,k} (C \Delta^k, C P) \) then its distance to \( C \partial \Delta^k \) is \( < s_{m,k} \). Hence \( p \in \hat{N}_{s_m,k} (C \Delta^j, C P) \) for some \( \Delta^j \subset \partial \Delta^k \); thus \( j < k \). But it can be checked that \( r_{m,j} > s_{m,k}, j < k \) (this follows from \( c_\xi < e^{-4} < 1 \)). Therefore \( p \in \hat{N}_{s_m,j} (C \Delta^j, C P) \), which implies \( p \not\in \mathcal{Y}(P, \Delta^k, r) \). This proves (i). Item (ix) follows from (i). Next we prove (iv). Using \( r_{m,j} < s_{m,j} \) and the definition of \( \mathcal{Y}(P, r) \) we have

\[
CP - B_{r_{m-2} - (2 + \xi)}(C P) = \mathcal{Y}(P, r) \cup \bigcup_{j \leq m - 2} N_{r_{m,j}} (C \Delta^j, C P) \subset \mathcal{Y}(P, r) \cup \bigcup_{j \leq m - 2} N_{s_{m,j}} (C \Delta^j, C P)
\]

This together with (iii) imply that we can prove (iv) by showing, by induction on \( k \), that \( U = \bigcup_{l \leq m - 2} \mathcal{Y}(P, \Delta^l, r) \) contains \( \hat{N}_{s_m,j} (C \Delta^j, C P) - B_{r_{m-2} - (2 + \xi)}(C P) \) for every \( k \)-simplex of \( P \), \( k \leq m - 2 \).

For \( k = 0 \) this statement holds because \( \mathcal{Y}(\Delta^0, P) = \hat{N}_{s_m,0} (C \Delta^0, C P) - B_{r_{m-2} - (2 + \xi)}(C P) \). Assume \( U \) contains every \( \hat{N}_{s_m,j} (C \Delta^j, C P) - B_{r_{m-2} - (2 + \xi)}(C P) \), for all \( j < k \). By the definition of \( \mathcal{Y}(\Delta^k, P) \) we have that \( \hat{N}_{s_m,k} (C \Delta^k, C P) - B_{r_{m-2} - (2 + \xi)}(C P) \) is contained in

\[
\left[ \mathcal{Y}(\Delta^k, P) \cup \bigcup_{j < k} N_{r_{m,j}} (C \Delta^j, C P) \right] - B_{r_{m-2} - (2 + \xi)}(C P)
\]

This together with the fact that \( s_{m,k} > r_{m,k} \) and the inductive hypothesis imply that \( \hat{N}_{s_m,k} (C \Delta^k, C P) - B_{r_{m-2} - (2 + \xi)}(C P) \subset U \). This proves (iv). To prove the other two statements we need a lemma.

**Lemma 6.2.2.** For \( t \geq r_{m-2} - (2 + \xi) \) and \( r > (4 + \xi) \) we have (see Lemma 6.1.9)

\[
N_{r_{m,k}} (C \Delta, C P) \cap S_t(C P) = N_{\theta_{m,k}(t)}(\Delta, P) \times \{ t \}
\]

\[
N_{s_{m,k}} (C \Delta, C P) \cap S_t(C P) = N_{\phi_{m,k}(t)}(\Delta, P) \times \{ t \}
\]

where \( \theta_{m,k}(t) \) and \( \phi_{m,k}(t) \) are defined by the equations

\[
\sin(\theta_{m,k}(t)) = c'' \sin(\alpha_k), \quad \sin(\phi_{m,k}(t)) = c'' \sin(\beta_k),
\]

with \( c'' = \frac{\sinh(r_{m-2})}{\sinh(t)} < 2e^2 \). Moreover \( \theta_{m,k}(t) \) and \( \phi_{m,k}(t) \) are well defined and less than \( \pi/4 \).

**Proof.** From Lemma 6.1.9 and the definitions of \( \alpha_k \) and \( \beta_k \) we have

\[
\sin(\theta_{m,k}(t)) = \frac{\sinh(r_{m,k})}{\sinh(t)} = \frac{\sinh(r_{m-2})}{\sinh(t)} \frac{\sinh(r_{m,k})}{\sinh(r_{m-2})} = c'' \sin(\alpha_k)
\]

and

\[
\sin(\phi_{m,k}(t)) = \frac{\sinh(s_{m,k})}{\sinh(t)} = \frac{\sinh(r_{m-2})}{\sinh(t)} \frac{\sinh(s_{m,k})}{\sinh(r_{m-2})} = c'' \sin(\beta_k)
\]

Since \( \xi > 0 \), simple calculation shows that \( c'' < 2e^2 \), provided \( t \geq r_{m-2} - (2 + \xi), r > 4 + \xi \) (thus \( r_{m-2} > 4 + \xi \)). Hence the definitions of \( \alpha_k \) and \( \beta_k \) at the beginning of this section imply \( c'' \sin(\alpha_k) = c'' \zeta^{k+1} < \frac{\alpha}{2^2} \) and \( c'' \sin(\beta_k) = c'' \zeta^{k+1} < \frac{\beta}{2^2} \). This proves the lemma.

We now finish the proof of Proposition 6.2.1. Statement (v) follows from Lemma 6.2.2 and the fact that \( \beta \)-neighborhoods, \( \beta < \pi/4 \), of disjoint simplices in an all-right spherical complex are disjoint. Statement (vi) follows from (v). We prove (vii). Note that \( c'' = c''(m, t) \). Using items (i), (ii), and lemmas 6.2.2 and 5.15 it is enough to prove that, for fixed \( t \) and \( m \), the pair of sets of widths \( \{ \phi_{m,k}(t) \}, \{ \theta_{m,k}(t) \} \) satisfies DNP. But from the definitions we have \( \{ \phi_{m,k}(t) \} = B(\zeta, c'' \alpha) \) and \( \{ \theta_{m,k}(t) \} = B(\zeta, c'' \beta) \). Therefore Lemma 5.2.3 and the condition \( c \zeta < e^{-4} \) imply \( \{ \phi_{m,k}(t) \}, \{ \theta_{m,k}(t) \} \) satisfies DNP. This proves (vii). Statement (viii) follows from (vii) by taking \( i = j \). This proves Proposition 6.2.1.
Define the sets
\[ \mathcal{X}(P^m, \Delta^k, r) = \mathcal{Y}(P^m, \Delta^k, r) - B_{r_{m-2}}(C P^m) \]
\[ \mathcal{X}(P^m, r) = \mathcal{Y}(P^m, r) - B_{r_{m-2}}(C P^m) \]

Alternatively, we can define \( \mathcal{X}(P^m, \Delta^k, r) \) by the same formula that defines \( \mathcal{Y}(P^m, \Delta^k, r) \) with just one small change: in the last term replace the radius \( r_{m-2} - (2 + \xi) \) by \( r_{m-2} \). Similarly for \( \mathcal{X}(P^m, r) \).

**Proposition 6.2.3.** For \( \Delta^j \subset \Delta^k \subset P \) we have
\[ \mathcal{Y}(P, \Delta^k, r) \subset C \Delta^j \times \mathcal{X}(\text{Link}(\Delta^j, P), \Delta^l, r) \]
where \( \Delta^l = \Delta^k \cap \text{Link}(\Delta^j, P) \) is opposite to \( \Delta^j \) in \( \Delta^k \).

**Remark.** The left term in the proposition is a subset of \( C \text{Star}(\Delta^k, P) \), thus also a subset of \( C \text{Star}(\Delta^j, P) \). The right term is a subset \( C \Delta^j \times C \text{Link}(\Delta^j, P) \) and, by 6.1.2, we can write \( C \Delta^j \times C \text{Link}(\Delta^j, P) = C \text{Star}(\Delta^j, P) \). Proposition 6.2.3 says that \( \mathcal{Y}(P, \Delta^k, r) \) is a subset of \( C \Delta^j \times \mathcal{X}(\text{Link}(\Delta^j, P), \Delta^l, r) \) under this identification.

**Proof.** By the (alternative) definition of \( \mathcal{X} \), it is enough to prove the following three statements

1. \[ \mathcal{Y}(P, \Delta^k, r) \subset C \Delta^j \times N_{s_{m-j-1,l}} \left( C \Delta^j, C \text{Link}(\Delta^j, P) \right) \]
2. For \( \Delta^j \in \text{Link}(\Delta^j, P) \), \( i < l = k - j - 1 \), we have
   \[ \mathcal{Y}(P, \Delta^k, r) \cap \left[ C \Delta^j \times N_{r_{m-j-1,i}} \left( C \Delta^j, C \text{Link}(\Delta^j, P) \right) \right] = \emptyset \]
3. \[ \mathcal{Y}(P, \Delta^k, r) \cap \left[ C \Delta^j \times B_{r_{m-j-3}} \left( C \text{Link}(\Delta^j, P) \right) \right] = \emptyset \]

Statement (1) follows from (i) of Proposition 6.2.1, Lemma 6.1.8 and the equalities \( s_{m,k} = s_{m-j-1,k-j-1} \) and \( l = k - j - 1 \). Statement (2) follows from (ii) of Proposition 6.2.1, Lemma 6.1.8 and the statements \( r_{m,i+j+1} = r_{m-j-1,i}, i + j + 1 < k \). For (3) note that (6.1.4) and the definition of \( r_{m,j} \) imply
\[ C \Delta^j \times B_{r_{m-j-3}} \left( C \text{Link}(\Delta^j, P) \right) = N_{r_{m,j}} \left( C \Delta^j, C P \right) \]
This together with (ii) of Proposition 6.2.1 imply (3). This proves the proposition.

**Proposition 6.2.4.** For \( \Delta^k \subset P \), \( k \leq m - 2 \), we have
\[ \mathcal{Y}(P, r) \cap \mathcal{Y}(P, \Delta^k, r) \subset C \Delta^k \times \mathcal{X}(\text{Link}(\Delta^k, P), r) \]

**Proof.** Using the definition of \( \mathcal{X}(\text{Link}(\Delta^k, P), r) \), it is enough to prove the following three statements

1. \[ \mathcal{Y}(P, \Delta^k, r) \subset C \Delta^k \times C \text{Link}(\Delta^k, P) \]
2. For \( \Delta^j \in P \), \( \Delta^k \subset \Delta^j \), \( l \leq m - k - 3 \), and \( \Delta^l \) opposite to \( \Delta^k \) in \( \Delta^j \), we have
   \[ \mathcal{Y}(P, r) \cap \left[ C \Delta^k \times N_{r_{m-k-1,l}} \left( C \Delta^l, C \text{Link}(\Delta^k, P) \right) \right] = \emptyset \]
3. \[ \mathcal{Y}(P, r) \cap \left[ C \Delta^k \times B_{r_{m-k-3}} \left( C \text{Link}(\Delta^k, P) \right) \right] = \emptyset \]

Statement (1) follows from (i) of Proposition 6.2.1, and 6.1.2. Statement (2) follows 6.2.1 (ix), Lemma 6.1.8, the identities \( r_{m-k-1,j,k-1} = r_{m,j} \), \( k + l + 1 = j \), the fact that \( l \leq m - k - 3 \) if and only if \( j \leq m - 2 \), and the definition of \( \mathcal{Y}(P, r) \). Finally (3) follows from 6.2.1 (ix), the definition of \( r_{m,k} \) and the definition of \( \mathcal{Y}(P, r) \). This proves the lemma.
6.3. Radial Stability of the Sets $\mathcal{Y}(P, \Delta^k, r)$.

In Section 8 we will need a sort of a stable property for the sets $\mathcal{Y}$. We use the objects and notation used in Section 6.2. Recall that $\text{Star}(\Delta, P)$ is the simplicial star of $\Delta$ in $P$, and that an element in $C \Delta$ can be written as $sx$, $s \in [0, \infty)$, $x \in P$. Let $\theta \in (0, \pi/2)$, and write $a(s) = a(s) = \sinh^{-1}(\sinh(s) \sin \theta)$.

**Lemma 6.3.1.** Let $b \in \mathbb{R}$, $\Delta^k \in P$, and $x \in \text{Star}(\Delta, P)$. Then $(s+b)x \in N_{a(s)}(C \Delta^k, C \Delta P)$ if and only if $\sin(\gamma) \frac{\sinh(s+b)}{\sinh(s)} \leq \sin \theta$, where $s > 0$, $\gamma = \gamma(x) = d_p(x, \Delta^k)$.

**Proof.** Note that $\gamma$ is the angle opposite to the cathetus of length $d(s) = d_{C \Delta P}((s+b)x, C \Delta^k)$ of the right hyperbolic triangle with hypotenuse $(s+b)$. We want $d(s) \leq a(s)$; equivalently $\sinh(d(s)) \leq \sinh(a(s))$. By the hyperbolic law of sines $\sinh(d(s)) = \sin(\gamma) \sinh(s+b)$, hence $\sinh(d(s)) \leq \sinh(a(s))$ is equivalent to $\sin(\gamma) \sinh(s+b) \leq \sinh(s) \sin \theta$. This proves the lemma.

Note that the lemma also holds if we replace $N$ by $\overset{0}{N}$ and $\leq$ by $<$. Write $R(s) = R_{x,b}(s) = (s+b)x$.

**Lemma 6.3.2.** Let $\Delta^k$, $P$ and $x$ as in Lemma 6.3.1. We have three mutually exclusive cases:

1. $e^b \sin(\gamma) < \sin \theta$, which implies that $R(s) \in N_{a(s)}(C \Delta, C \Delta P)$, $s \geq s_0$, for some $s_0$.
2. $e^b \sin(\gamma) > \sin \theta$, which implies that $R(s) \notin N_{a(s)}(C \Delta, C \Delta P)$, for all $s > 0$.
3. $e^b \sin(\gamma) = \sin \theta$, which implies that $R(s) \notin N_{a(s)}(C \Delta, C \Delta P)$, for all $s > 0$.

**Proof.** The lemma follows from 6.3.1 and the following two facts: (1) the function $s \mapsto \frac{\sinh(s+b)}{\sinh(s)}$ is strictly decreasing for $s > 0$, and (2) $\lim_{s \to \infty} \frac{\sinh(s+b)}{\sinh(s)} = e^b$. This proves the lemma.

From the definition of $r_k$ given at the beginning of 6.2, we have $r_{m-2} = r_{m-2}(r) = \sinh^{-1}(\frac{\sinh(r)}{\sin \alpha_{m-2}})$, hence we can write $r = r(r_{m-2}) = \sinh^{-1}(\sinh(r_{m-2}) \sin \alpha_{m-2})$. Therefore we can write $r_{m,k} = r_{m,k}(r)$ and $s_{m,k} = s_{m,k}(r)$ in terms of the new variable $r_{m-2}$, and a calculation shows that $r_{m,k} = a_{x,b}(r_{m-2})$ and $s_{m,k} = a_{x,b}(r_{m-2})$. We will use these identities in the proof of the next result.

**Proposition 6.3.3.** Fix $b \in \mathbb{R}$ and let $x \in P$. Then at least one of the following conditions hold.

1. There is $\Delta^k$, $k \leq m-2$, such that $R_{x,b}(r_{m-2}) \in \mathcal{Y}(P, \Delta^k, r(r_{m-2}))$, for all $r_{m-2} > r'$, for some $r'$.
2. We have that $R_{x,b}(r_{m-2}) \notin \mathcal{Y}(P, r(r_{m-2}))$, for all $r_{m-2} > r'$, for some $r'$.

Moreover, these two conditions are stable in the following sense. If $x'$ and $b'$ are sufficiently close to $x$ and $b$, respectively, and $R_{x,b}$ satisfies (i) then $R_{x',b'}$ also satisfies (i) (with the same $r'$). Similarly for condition (ii).

**Proof.** By induction. Suppose C1 of 6.3.2 holds for $R = R_{x,b}$ with $\theta = \alpha_0$, for some $\Delta^0$. Then, since $N_{a_0}(r_{m-2})(\Delta^0, P) = N_{a_0}(r_{m-2})(\Delta^0, P) \subset \mathcal{Y}(P, \Delta^0, r)$ we see that $R$ satisfies (1) for $\mathcal{Y}(P, \Delta^0, r)$ and we are done. Suppose C3 holds with $\theta = \alpha_0$, for some $\Delta^0$. Then $x \in \text{Star}(\Delta^0, P)$ and $e^b \sin(\gamma) = \sin(\gamma)$, where $\gamma = \gamma(x)$. Since $\alpha_k < \beta_k$, we have $e^b \sin(\gamma) < \sin(\beta_0)$, hence by 6.3.2 with $\theta = \beta_0$ we have that $R(r_{m-2}) \in N_{a_0}(r_{m-2})(\Delta^0, P)$, for large $r_{m-2}$, and follows that $R$ satisfies (1) for $\mathcal{Y}(P, \Delta^0, r)$ and we are done. Now suppose that C2 happens for all $\Delta^0$, with $\theta = \alpha_0$. As before we have three possibilities. First C1 holds for $R = R_{x,b}$ with $\theta = \alpha_1$, for some $\Delta^1$. This, together with the assumption that C2 holds for all $\Delta^0$ with $\theta = \alpha_0$, and the definition of $\mathcal{Y}(P, \Delta^1, r)$ imply that $R$ satisfies (1) for $\mathcal{Y}(P, \Delta^1, r)$ and we are done. Suppose C3 holds for $R$ and $\Delta^1$ (with $\theta = \alpha_0$), for some $\Delta^1$. Using the same argument as in the $\Delta^0$ case (when we assumed C3 some $\Delta^0$) we get that $R$ satisfies (1) for $\mathcal{Y}(P, \Delta^1, r)$ and we
are done. The third case is that $\mathbf{C}_2$ happens for $R$ and all $\Delta^1$. Proceeding in this way we obtain that either $R$ satisfies (1), for some $\Delta^k$, $k \leq m - 2$ or $\mathbf{C}_2$ holds for $R$ and all $\Delta^k$, $k \leq m - 2$ (with $\theta = \alpha_k$). Hence (2) holds for $R$. Moreover it does so stably. This proves the proposition.

7. Smooth Structures on Cube and All-Right Spherical Complexes.

For the basic definitions and results about cube complexes see for instance [5]. Given a (cube or all-right spherical) complex $K$ we use the same notation $K$ for the complex itself (the collection of all closed cubes or simplices) and its realization (the union of all cubes or simplices). For $\sigma \in K$ we denote its interior by $\hat{\sigma}$.

Let $M^n$ be a smooth manifold of dimension $n$. A smooth cubulation of $M$ is a pair $(K, f)$, where $K$ a cube complex and $f : K \to M$ a non-degenerate PD homeomorphism [24], that is, for all $\sigma \in K$ we have $f|_{\sigma}$ is a smooth embedding. Sometimes we will write $K$ instead of $(K, f)$. The smooth manifold $M$ together with a smooth cubulation is a smooth cube manifold or a smooth cube complex. A smooth all-right-spherical triangulation and a smooth all-right-spherical manifold (or complex) is defined analogously.

In this section $\text{Link}(\sigma^j, K)$ means the geometric link of an open $j$-cube or $j$-all-right simplex $\sigma^j$, defined as the union of the end points of straight (geodesic) segments of small length $\epsilon > 0$ emanating perpendicularly (to $\hat{\sigma}^j$) from some point $x \in \hat{\sigma}^j$. The star $\text{Star}(\sigma, K)$ as the union of such segments. We can identify the star with the cone of the link $\text{C Link}(\sigma, K)$ (or $\epsilon$-cone) defined as $\text{C Link}(\sigma, K) = \text{Link}(\sigma, K) \times [0, \epsilon) / \text{Link}(\sigma, K) \times \{0\}$. Thus a point $x$ in $\text{C Link}(\sigma, K)$, different from the cone point $o = o_{\text{C Link}(\sigma, K)}$, can be written as $x = tu$, $t \in (0, \epsilon)$, $u \in \text{Link}(\sigma, K)$. For $s > 0$ we get the cone homothety $x \mapsto sx = (st)u$ (partially defined if $s > 1$). If we want to make explicit the dependence of the link or the cone on $\epsilon$ we shall write $\text{Link}_\epsilon(\sigma, K)$ or $\text{C}_\epsilon \text{Link}(\sigma, K)$ respectively. As usual we shall identify the $\epsilon$-neighborhood of $\hat{\sigma}$ in $K$ with $\text{C}_\epsilon \text{Link}(\sigma, K) \times \hat{\sigma}$ (or just $\text{C Link}(\sigma, K) \times \hat{\sigma}$). Hence a cone homothety induces a neighborhood homothety obtained by crossing it with the identity $1_\sigma$.

In what follows we assume that $f : K \to M$ is a smooth cubulation (or all-right spherical triangulation) of the smooth manifold $M$. Since the PL structure on $M$ induced by $K$ is Whitehead compatible with $M$ we have that the link $\text{Link}(\sigma^i, K)$ is PL homeomorphic to $\mathbb{S}^{n-i-1}$. A link smoothing for $\hat{\sigma}^i$ (or $\sigma^i$) is just a homeomorphism $h_{\sigma^i} : \mathbb{D}^{n-i} \to \text{Link}(\sigma^i, K)$. The cone of $h_{\sigma^i}$ is the map $C h_{\sigma^i} : \mathbb{D}^{n-i} \longrightarrow \text{C Link}(\sigma^i, K)$ given by $t x = [x, t] \mapsto t h_{\sigma^i}(x) = [h_{\sigma^i}(x), t]$, where we are canonically identifying the $\epsilon$-cone of $\mathbb{S}^{n-i-1}$ with the disc $\mathbb{D}^{n-i}$. We remark that we are not assuming $h_{\sigma^i}$ to be smooth. A link smoothing $h_{\sigma^i}$ induces the following smoothing of the normal neighborhood of $\hat{\sigma}^i$:

$$h_{\sigma^i}^* = f \circ \left( C h_{\sigma^i} \times 1_{\sigma^i} \right) : \mathbb{D}^{n-i} \times \hat{\sigma}^i \longrightarrow M$$

The pair $(h_{\sigma^i}^*, \mathbb{D}^{n-i} \times \hat{\sigma}^i)$, or simply $h_{\sigma^i}^*$, is a normal chart on $M$. Note that the collection $\mathcal{A} = \{ (h_{\sigma^i}^*, \mathbb{D}^{n-i} \times \hat{\sigma}^i) \}_{\sigma^i \in K}$ is a topological atlas for $M$. Sometimes will just write $\mathcal{A} = \{ h_{\sigma^i}^* \}_{\sigma^i \in K}$. The topological atlas $\mathcal{A}$ is called a normal atlas. It depends uniquely on the complex $K$, the map $f$ and the collection of link smoothings $\{ h_{\sigma} \}_{\sigma \in K}$. To express the dependence of the atlas on the set of links smoothings we shall write $\mathcal{A} = \mathcal{A}(\{ h_{\sigma} \}_{\sigma \in K})$ (this is different from $\mathcal{A} = \{ h_{\sigma^i}^* \}_{\sigma^i \in K}$, as written above). The most important feature about these normal atlases is that they preserve the radial and sphere (link) structure given by $K$. 

27
Note that not every collection of link smoothings induce a smooth atlas. But when the induced atlas is smooth we call $\mathcal{A}$ a normal smooth atlas on $M$ with respect to $K$ and the corresponding smooth structure $S'$ a normal smooth structure on $M$ with respect to $K$. In this case we say that the set of link smoothings $\{h_\sigma\}_{\sigma \in K}$ is smooth. The following theorem is proved in [27].

**Theorem 7.1.** Let $M$ be a smooth cube or all-right spherical manifold, with smooth structure $S$. Then $M$ admits a normal smooth structure $S'$ diffeomorphic to $S$.

Hence if $M^n$ is a smooth manifold with smooth structure $S$ and $K$ is a cubulation (or all-right spherical triangulation) of $M$, then there are link smoothings $h_\sigma$, for all $\sigma \in K$, such that the atlas $\mathcal{A} = \mathcal{A}(\{h_\sigma\}_{\sigma \in K})$ is smooth (equivalently, $\{h_\sigma\}_{\sigma \in K}$ is smooth). Moreover the normal smooth structure $S'$, induced by $\mathcal{A}$, is diffeomorphic to $S$.

**7.2. Induced Link Smoothings.**

Let $K$ be a cubical or all-right spherical complex. Then the links of $\sigma \in K$ are all-right-spherical complexes. We explain here how to obtain from a given a collection of link smoothings $h_\sigma$ for a link in $K$ (and its corresponding normal atlas and structure) a collection of links smoothings for a link in $K$ (and its corresponding normal atlas and structure).

The all-right-spherical structure on $\text{Link}(\sigma, K)$ induced by $K$ has all-right-spherical simplices $\{\tau \cap \text{Link}(\sigma, K) \mid \tau \in K\}$. Note that $\tau \cap \text{Link}(\sigma, K)$ is non-empty only when $\sigma \subseteq \tau$, hence we can write

$$\text{Link}(\sigma, K) = \{\tau \cap \text{Link}(\sigma, K) \mid \sigma \subseteq \tau \in K\}$$

Since $\tau \cap \text{Link}(\sigma, K)$ is a simplex in the all-right spherical complex $\text{Link}(\sigma, K)$ we can consider its link $\text{Link}\left(\tau \cap \text{Link}(\sigma, K), \text{Link}(\sigma, K)\right)$. By definition we have:

$$\text{Link}\left(\tau \cap \text{Link}(\sigma, K), \text{Link}(\sigma, K)\right) = \text{Link}(\tau, K)$$

provided we choose the radii and bases of the links properly. In the formula above radii and bases are not specified but the radii are certainly not equal. The simple relationship between these radii is given by equation (1) in the proof of Lemma 1.2 [27] (or the corresponding one in the spherical case; see Remark 1 after the proof of Lemma 1.2 [27]). The next result is proved in [28].

**Proposition 7.2.1.** Let $\{h_\sigma\}_{\sigma \in K}$ be a set of link smoothings on $K$, and let $\sigma^k \in K$. Assume $\{h_\sigma\}_{\sigma \in K}$ is smooth; that is, the atlas $\mathcal{A} = \mathcal{A}(\{h_\sigma\}_{\sigma \in K})$ is smooth. Let $S'$ be the normal smooth structure on $K$ induced by $\mathcal{A}$. Then:

1. The set of link smoothings $\{h_{\sigma^k \cap \text{Link}(\sigma, K)}\}_{\sigma^k \subseteq \sigma^i}$ for the links of $\text{Link}(\sigma^k, K)$ is smooth; that is, the atlas $\mathcal{A}_{\sigma^k} = \mathcal{A}_{\text{Link}(\sigma^k, K)} = \{h_{\sigma^k \cap \text{Link}(\sigma^k, K)}\}_{\sigma^k \subseteq \sigma^i}$ is a smooth normal atlas on $\text{Link}(\sigma^k, K)$.

2. The link smoothing

$$h_{\sigma^k} : S^{n-k-1} \rightarrow \left(\text{Link}(\sigma^k, K), S_{\sigma^k}\right)$$

is a diffeomorphism. Here $S_{\sigma^k}$ is the smooth structure induced by the atlas $\mathcal{A}_{\sigma^k}$.

3. We have that $\text{Link}(\sigma^k, K)$ is a smooth submanifold of $(K, S')$. Moreover

$$S'|_{\text{Link}(\sigma^k, K)} = S_{\sigma^k}$$

where $S'|_{\text{Link}(\sigma^k, K)}$ denotes the restriction of $S'$ to $\text{Link}(\sigma^k, K)$.
7.3. The Case of Manifolds with Codimension Zero Singularities.

Here we treat the case of manifolds with a one point singularity. The case of manifolds with many (isolated) point singularities is similar.

Let $Q$ be a smooth manifold with a one point singularity $q$, that is $Q - \{q\}$ is a smooth manifold and there is a topological embedding $C_1N \rightarrow Q$, with $o_{C_N} \mapsto q$, that is a smooth embedding outside the vertex $o_{C_N}$. Here $N = (N, S_N)$ is a closed smooth manifold (with smooth structure $S_N$). Also $C_1N$ is the (closed) cone of width 1 and we identify $C_1N - \{o_{C_N}\}$ with $N \times (0, 1]$. We write $C_1N \subset Q$. We say that the singularity $q$ of $Q$ is modeled on $CN$.

Assume $(K, f)$ is a smooth cubulation of $Q$, that is

(i) $K$ is a cubical complex.
(ii) $f : K \rightarrow Q$ is a homeomorphism. Write $f(p) = q$ and $L = \text{Link}(p, K)$.
(iii) $f|\sigma$ is a smooth embedding for every cube $\sigma$ not containing $p$.
(iv) $f|_{\sigma - \{p\}}$ is a smooth embedding for every cube $\sigma$ containing $p$.
(v) $L$ is $PL$ homeomorphic to $(N, S_N)$.

Many of the definitions and results given before for smooth cube manifolds still hold (with minor changes) in the case of manifolds with a one point singularity:

(1) A link smoothing for $L = \text{Link}(p, K)$ (or $p$) is just a homeomorphism $h_p : N \rightarrow L$. Since all but one of the links of $K$ are spheres, sets of link smoothings for $K$ are defined, that is, they are sets of link smoothings for the sphere links plus a link smoothing for $L$.

(2) Given a set of link smoothings for $K$ we get a set of normal charts as before. For the vertex $p$ we mean the cone map $h_p^* = f \circ C h_p : CN \rightarrow Q$. We will also denote the restriction of $h_p^*$ to $CN - \{o_{C_N}\}$ by the same notation $h_p^*$. As before $\{h_p^*\}_{\sigma \in K}$ is a (topological) normal atlas on $Q$ with respect to $K$. The atlas on $Q$ is smooth if all transition functions are smooth, where for the case $h_p^* : CN - \{o_{C_N}\} \rightarrow Q - \{q\}$ we are identifying $CN - \{o_{C_N}\}$ with $N \times (0, 1]$ with the product smooth structure obtained from some smooth structure $S_N$ on $N$. A smooth normal atlas on $Q$ with respect to $K$ induces, by restriction, a smooth normal structure on $Q - \{q\}$ with respect to $K - \{p\}$ (this makes sense even though $K - \{p\}$ is not, strictly speaking, a cube complex).

(3) We say that the set $\{h_\sigma\}$ is smooth if the atlas $A = \{h_\sigma^*\}_{\sigma \in K}$ is smooth. In this case we say that the smooth atlas $A$ (or the induced smooth structure, or the set $\{h_\sigma\}$) is correct with respect to $N$ if $S_N$ and $S_N$ are diffeomorphic.

(4) Also it is straightforward to verify that Proposition 7.2.1 holds in our present case.

(5) In [28] the following version of Theorem 7.1 is proved:

**Theorem 7.3.1.** Let $Q$ be a smooth manifold with one point singularity $q$ modeled on $CN$, where $N$ is a closed smooth manifold. Let $(K, f)$ be a smooth cubulation of $Q$. Then $Q$ admits a normal smooth structure with respect to $K$, which restricted to $Q - \{q\}$ is diffeomorphic to $Q - \{q\}$. Moreover this normal smooth structure is correct with respect to $N$ if

(a) $\dim N \leq 4$.
(b) $\dim N \geq 5$ and the Whitehead group $\text{Wh}(N)$ of $N$ vanishes.
Section 8. Smoothing Hyperbolic Cones

Given an all-right spherical complex $P^m$ of dimension $m$ and a compatible smooth structure $\mathcal{S}_P$ on $P$, by Theorem 7.1 (see also remark 1 after the statement of 7.1) we can assume that $\mathcal{S}_P$ is a normal smooth structure, and $\mathcal{S}_P$ has a normal atlas $\mathcal{A}_P$. The atlas $\mathcal{A}_P$ and its induced differentiable structure $\mathcal{S}_P$ are constructed (uniquely) from a set of link smoothings $\mathcal{L} = \{h_\Delta\}_{\Delta \in \mathcal{P}}$. To express this dependence we will sometimes write $\mathcal{A}_P = \mathcal{A}_P(\mathcal{L})$ and $\mathcal{S}_P = \mathcal{S}_P(\mathcal{L})$.

Recall that the cone $CP$ has a piecewise hyperbolic metric $\sigma_{CP}$ induced by the piecewise spherical metric on $P$. We denote these metrics by $\sigma_{CP}$ and $\sigma_P$ respectively. As mentioned in Section 6, the piecewise hyperbolic metric $\sigma_{CP}$ has a well defined ray structure.

(8.0.1.) Consider the following data.

1. A positive number $\xi$.
2. A collection $d = \{d_2, d_3, \ldots\}$ of real numbers, with $d_i > (4 + \xi)$. We write $d(k) = \{d_2, d_3, \ldots, d_k\}$.
3. A positive number $r$, with $r > 2d_i$, $i = 2, \ldots, m + 1$, and as in item 4.
4. Real numbers $\zeta > 0$, $c > 1$, with $c\zeta < e^{-4}$. Hence we get sets of widths (see 5.2 and 5.3) $A = B(\zeta) = \{\alpha_i\}$ and $B = B(\zeta; c) = \{\beta_i\}$, where $\sin \alpha_i = \zeta^{i+1}$, $\sin \beta_i = c\zeta^{i+1}$.
5. An all-right spherical complex $P^m$, $\dim P = m$, with compatible smooth normal atlas $\mathcal{A}_P(\mathcal{L}_P)$, where $\mathcal{L}_P$ is a smooth set of link smoothings on $P$.
6. A diffeomorphism $\phi_p = \phi_{p,\mathcal{L}_P} : (P, \mathcal{S}_P(\mathcal{L}_P)) \to \mathbb{S}^m$ to the standard $m$-sphere. The map $\phi_p$ is called a global smoothing for $P$, with respect to $\mathcal{S}_P$ (or $\mathcal{A}_P$, or $\mathcal{L}_P$). For $m = 1$ we shall take $\phi_p$ in a canonical way (that is, depending only on $P$).

The smooth atlas $\mathcal{A}_P(\mathcal{L}_P)$ on $P$ induces, by coning, a smooth atlas on $CP - \{a_{CP}\}$, and this atlas together with the coning $C\phi_p : CP \to \mathbb{R}^{m+1}$ of the map $\phi_p$ induce a smooth atlas $\mathcal{A}_{CP} = \mathcal{A}_{CP}(\mathcal{L}_P, \phi_p)$ on $CP$. We denote the corresponding smooth structure by $\mathcal{S}_{CP} = \mathcal{S}_{CP}(\mathcal{L}_P, \phi_p)$. Note that we get a diffeomorphism $C\phi_p : (CP, \mathcal{S}_{CP}) \to \mathbb{R}^{m+1}$.

With the data given in items 1-6 above we will construct for every $P^m$, by induction on the dimension $m$, the smoothed Riemannian metric $\mathcal{G}(P, \mathcal{L}_P, \phi_p, r, \xi, d, (c, \zeta))$ on the cone $CP$ of $P$, where we consider $CP$ with smooth structure $\mathcal{S}_{CP}$.

In sections 8.1 and 8.2 we will assume $\xi, \text{d, c, zeta}$ fixed. In particular we shall assume $A, B$ fixed. So, to simplify our notation, we shall denote the smoothed metric by $\mathcal{G}(P, \mathcal{L}_P, \phi_p, r)$ or just $\mathcal{G}(P, r)$ or $\mathcal{G}(P)$. In sections 8.3 and 8.4 we need to make explicit the dependence of the smoothed metric on the other variables, and we will show that, given $\epsilon > 0$, we can choose $r$ and $d_i$, $i = 2, \ldots, m$, large so that $\mathcal{G}(P, \mathcal{L}_P, \phi_p, r, \xi, d, (c, \zeta))$ has curvatures $\epsilon$-pinched to -1, provided the variables satisfy certain conditions. Before we begin with dimension 1 we need to discuss induced structures.

Let $\Delta = \Delta^k \in P$. The restriction of $\mathcal{L}_P$ to $\text{Link}(\Delta, P)$ is the set $\mathcal{L}_P|_{\text{Link}(\Delta, P)} = \{h_\Delta\}_{\Delta \subseteq \Delta^k}$. Sometimes we will just write $\mathcal{L}_P|_{\text{Link}(\Delta, P)}$ or, more specifically, $\mathcal{L}_P|_{\text{Link}(\Delta, P)}(\mathcal{L}_P)$. The corresponding induced atlas on $\text{Link}(\Delta, P)$ is $\mathcal{A}_{\text{Link}(\Delta, P)}(\mathcal{L}_P) = \{h_\Delta^*\}_{\Delta \subseteq \Delta^k}$, and sometimes we will simply write $\mathcal{A}_{\text{Link}(\Delta, P)}$. The smooth structure on $\text{Link}(\Delta, P)$ induced by $\mathcal{A}_{\text{Link}(\Delta, P)}$ will be denoted by $\mathcal{S}_{\text{Link}(\Delta, P)}(\mathcal{L}_P)$, or simply by...
\( S_{\text{Link}(\Delta, P)} \). By Proposition 7.2.1 we have that, for \( \Delta \in P \), the link smoothing \( h_\Delta \) is a global smoothing for \( \text{Link}(\Delta, P) \) with respect to \( S_{\text{Link}(\Delta, P)} \). Write \( \phi_{\text{Link}(\Delta, P)} = \phi_{\text{Link}(\Delta, P)}(L_P) = h_\Delta \). Therefore we obtain the following restriction rule:

\[
(8.0.2.) \quad L_P \mapsto \left( L_{\text{Link}(\Delta, P)}(L_P), \phi_{\text{Link}(\Delta, P)}(L_P) \right)
\]

where \( L_P \) satisfies 5 in (8.0.1) for \( P \), and the objects \( L_\Delta, \phi_\Delta \) satisfy 5, 6 of (8.0.1) for \( \text{Link}(\Delta, P) \). The smooth structure on \( \text{Link}(\Delta, P) \) constructed from the data \( (L_\Delta, \phi_\Delta) \) will be denoted by \( S_{\text{Link}(\Delta, P)}(L_P) \), or \( S_{\text{Link}(\Delta, P)}(L_\Delta, \phi_\Delta) \), or simply by \( S_{\text{Link}(\Delta, P)} \). The next lemma says that the restriction rule (8.0.2) is transitive, that is, it respects the identity \( \text{Link}(\Delta^i, \text{Link}(\Delta^j, P)) = \text{Link}(\Delta^k, P) \), where \( \Delta^i = \text{Link}(\Delta^j, \Delta^k) \) (see 5.3.2).

**Lemma 8.0.3.** Let \( \Delta^j \subset \Delta^k \subset P \) and let \( \Delta^l = \text{Link}(\Delta^j, \Delta^k) \). Then we have

\[
L_{\text{Link}(\Delta^i, \text{Link}(\Delta^j, P))} \left( L_{\text{Link}(\Delta^j, P)}(L_P) \right) = L_{\text{Link}(\Delta^k, P)}(L_P)
\]

\[
\phi_{\text{Link}(\Delta^i, \text{Link}(\Delta^j, P))} \left( L_{\text{Link}(\Delta^j, P)}(L_P) \right) = \phi_{\text{Link}(\Delta^k, P)}(L_P)
\]

**Proof.** If we use the simplicial definition of link the identity \( \text{Link}(\Delta^i, \text{Link}(\Delta^j, P)) = \text{Link}(\Delta^k, P) \) is an equality of sets; hence the lemma follows from the definition of \( L \) and \( \phi \). This proves the lemma.

Recall that we have an identification \( C\text{Star}(\Delta, P) = C \Delta \times \text{Link}(\Delta, P) \) (see 6.1.2). The “open” version of this identification is \( \overset{\circ}{C}\text{Star}(\Delta, P) = \overset{\circ}{C} \Delta \times \text{Link}(\Delta, P) \), where \( \overset{\circ}{C}\text{Star}(\Delta, P) = C(\overset{\circ}{\text{Star}}(\Delta, P)) \). Here \( \overset{\circ}{\text{Star}}(\Delta, P) = \overset{\circ}{C}\overset{\circ}{\text{Star}}(\Delta, P) \). Note that \( C\text{Star}(\Delta, P) \) as an open subset of \( C P \) has the induced smooth structure \( S_{CP} | \overset{\circ}{C}\text{Star}(\Delta, P) \), and, for simplicity, we will just write \( S_{CP} \). On the other hand note that \( C \Delta = \mathbb{H}^{k+1} \subset \mathbb{H}^{k+1} \) has the natural smooth structure \( S_{\mathbb{H}^{k+1}} \), and \( \text{Link}(\overset{\circ}{\Delta}, P) \) has the smooth structure \( S_{\text{Link}(\overset{\circ}{\Delta}, P)} \). Therefore we can give \( C \overset{\circ}{\Delta} \times \text{Link}(\overset{\circ}{\Delta}, P) \) the “product” smooth structure \( S_\times = S_{C \overset{\circ}{\Delta} \times \text{Link}(\overset{\circ}{\Delta}, P)} \).

**Lemma 8.0.4.** The following identification is a diffeomorphism

\[
(\overset{\circ}{C}\text{Star}(\Delta, P), S_{CP}) = (C \overset{\circ}{\Delta} \times \text{Link}(\overset{\circ}{\Delta}, P), S_\times)
\]

**Proof.** We use the variables \( s, t, r, y, v, x, w, u, \beta \) defined in Section 2 (also see 6.1.1 and 6.1.2). We assume that the image of the chart \( h_\Delta^\circ \) is \( \overset{\circ}{\Delta}(\Delta, P) \). By rescaling, and using the notation in 6.1.1 and 6.1.2 we can write

\[
h_\Delta^\circ : \mathbb{D}^{m-k}(\pi/2) \times \overset{\circ}{\Delta} \rightarrow P
\]

\[
(\beta u', w) \mapsto \left[ w, h_\Delta(u') \right](\beta)
\]

where \( \mathbb{D}^{m-k}(\pi/2) \) is the disc of radius \( \pi/2 \), and we are expressing and element \( \mathbb{D}^{m-k}(\pi/2) \) as \( \beta u' \), with \( \beta \in [0, \pi/2] \), \( u' \in S^{m-k-1} \). A chart for \( (\overset{\circ}{C}\text{Star}(\Delta, P), S_{CP}) \) is the cone of \( h_\Delta^\circ \), which we shall denote by \( h_\Delta^* \). Explicitly, from (1) we have

\[
h_\Delta^* : \mathbb{R}_+ \times \mathbb{D}^{m-k}(\pi/2) \times \overset{\circ}{\Delta} \rightarrow CP
\]

\[
(s, \beta u', w) \mapsto s \left[ w, h_\Delta(u') \right](\beta)
\]

31
And for $(C \Delta \times C \text{Link}(\Delta, P), S_\times)$ we can take the following chart

$$h_\Delta^\dagger : \mathbb{R}_+ \times \mathbb{R}^{m-k} \times \hat{\Delta} \longrightarrow P$$

$$(t, r u', w) \mapsto (t w, r h_\Delta(u'))$$

where we write an element in $\mathbb{R}^{m-k}$ as $ru'$, $r \in [0, \infty)$, $u' \in \mathbb{S}^{m-k-1}$. From (2) and (3) and 6.1.2 we get

$$(h_\Delta^\dagger)^{-1} \circ h_\Delta^*(s, \beta u', w) = (t, ru', w)$$

and recall that the relationship between the variables $s, \beta, t, r$ is the following. There is a right hyperbolic triangle with catheti of length $t, r$, hypotenuse of length $s$ and angle $\beta$ opposite to the cathetus of length $r$. Using hyperbolic trigonometry we can find an invertible transformation $(s, \beta) \to (t, r)$. In particular $r = \sinh^{-1}(\sin \beta \sinh(s))$. The variables $s$ and $t$ are never zero, but $\beta$ and $r$ could vanish. Note that $\beta = 0$ if and only if $r = 0$. To get differentiability at $\beta = 0$ note that the map $(s, \beta u') \to ru'$ can be rewritten as $(s, z) \to (\frac{r(s, \beta)}{\beta} z)$, $\beta = |z|$, which is smooth because $\frac{r(s, \beta)}{\beta}$ is a smooth even function on $\beta$. Similarly, the smoothness of the inverse of the map in (4) follows from the fact that the map $(t, r) \to \frac{\beta(r, t)}{r}$ is a smooth even function on $r$. This proves the lemma.

8.1 Dimension One.

An all-right spherical complex $P^1$ of dimension one satisfying (8.0.1) (6) is formed by a finite number $k'$ of segments of length $\pi/2$ glued successively forming a circle. Hence we shall canonically take (up to rotation) $\phi = \phi_P : P \to \mathbb{S}^1$ so that $\phi$ maps each 1-simplex to an arc of length $2\pi/k'$, and it does so with constant speed. Using $\phi$ we shall identify $P$ with a circle of length $k'\pi/2$. Therefore $P$ with metric $\sigma_P$, is isometric to $\mathbb{S}^1$ with metric $k \sigma_{S^1}$, $k = k'/4$. Consequently we identify $C P$ with $\mathbb{R}^2$, and $C P - \{o_{C P}\}$ to $\mathbb{R}^2 - \{0\}$ with hyperbolic metric $\sigma_{C P} = \sinh^2(t) k \sigma_{S^1} + dt^2$. Notice that this metric is smooth on $\mathbb{R}^2$ away from the cone point $o_{C P} = 0 \in \mathbb{R}^2$, and it does have a singularity at $0$ unless $k = 1$.

Recall we are assuming $r > d_2$, where both $r$ and $d_2$ are given. Next we give two constructions of $\mathcal{G}(P)$. The first one is very explicit (see Gromov-Thurston [IS]) and does not use (directly) any of the methods introduced previously. The second one looks more like the inductive construction in 8.2, and uses the construction given in Section 4. These two constructions are slightly different but both satisfy the two properties $P^1, P^2$ given below. Here is the first construction.

Let $\rho$ be as in Section 1. Define

$$\mu(t) = \mu_{d_2, r, k}(t) = k \rho(\frac{t}{d_2} - \frac{r - d_2}{d_2}) + (1 - \rho(\frac{t}{d_2} - \frac{r - d_2}{d_2}))$$

hence $\mu(t) = 1$, for $t \leq r - d_2$ and $\mu(t) = k$ for $t \geq r$. Define

$$\mathcal{G}(P, r) = \sinh^2(t) \mu(t) \sigma_{S^1} + dt^2$$

Since the metric $\mathcal{G}(P, r)$ is equal to the canonical hyperbolic warped metric $\sinh^2(t)\sigma_{S^1} + dt^2$ on the ball of radius $r - d_2$, we can extend $\mathcal{G}(P, r)$ to the cone point $o_{C P} = 0 \in \mathbb{R}^2$. It is straightforward to verify that $\mathcal{G}(P, r)$ satisfies the following three properties:

**P**. The metrics $\mathcal{G}(P, r)$ and $\sigma_{C P}$ have the same ray structure.

**P**. The metric $\mathcal{G}(P, r)$ coincides with $\sigma_{C P}$ outside the ball of radius $r$.
P’3. The metric $\mathcal{G}(P, r)$ coincides with $\sinh^2(t)\sigma_{s1} + dt^2$ on the ball of radius $r - d_2$.

P’4. The family of metrics $\{\mathcal{G}(P, r)\}_{r > d_2}$ has cut limits (that is, it has cut limits on $I = \mathbb{R}$). Notice that $d_2$ is fixed and $r$ is the index of the family.

Actually from the definition of cut limit we have that the cut limit of $\mathcal{G}(p, r)$ at $b$ is

$$\lim_{r \to \infty} \mu_{d_2, r, \xi}(r + b) \sigma_{s1} = \left(1 + (k - 1)\rho(1 + \frac{b}{d_2})\right) \sigma_{s1}.$$  \hspace{1cm} (8.1.1.)

Here is the second construction. Recall that we are identifying $C_P - \{o_{C_P}\}$ with $\mathbb{R}^2 - \{0\}$. Consider the constant $\circ$-family of metrics $\{g_r\}_{r - \frac{1}{2} > d_2}$ given by $g_r = k\sigma_{s1} + dt^2 = \sigma_{C_P}$. Now just define

$$\mathcal{G}(P, r) = \mathcal{G}(P, r, d) = \mathcal{H}_{r, \frac{1}{2}, d_2 - \frac{1}{2}}(g_r, \frac{1}{2})$$

In this case we also have that $\mathcal{G}(P)$ satisfies P’1, P’2, P’3 and P’4 (see Proposition 4.4).

8.2 The Inductive Step.

Recall that in this section we are assuming $\xi, c, \varsigma$ (hence A, B), and $d$ constant. With the data $\xi, A, B, r > 0$ and an all-right spherical complex $P$ we constructed in Section 6.2 the numbers $r_k = r_k(r)$ and the sets $\mathcal{Y}(P, \Delta^k, r), \mathcal{Y}(P, r), \mathcal{X}(P, \Delta^k, r), \mathcal{X}(P, r)$, where $\Delta^k \in P$. The inverse of the function $r_k = r_k(r)$ shall be denoted by $r = r(r_k)$. Recall also that in 6.1.3 we identified $C_{\text{Star}}(\Delta^k, P)$, with metric $\sigma_{C_P}^{(C_{\text{Star}}(\Delta^k, P))}$, with $\mathcal{E}_{C_{\Delta^k}}(\mathcal{C}_{\text{Link}}(\Delta^k))$, with metric $\mathcal{E}_{k}(\sigma_{C_{\text{Link}}(\Delta^k, P)})$. We will use these objects in this section.

Let $m \geq 2$ and suppose that for every triple $(P, \mathcal{L}_P, \phi_P), j = \text{dim} P \leq m - 1$, as in items 5 and 6 of (8.0.1) above, and $r > d_i, i = 2, ..., m + 1$ there are couple of Riemannian metrics: the smoothed metric $\mathcal{G}(P, \mathcal{L}_P, \phi_P, r, \xi, d, (c, \varsigma))$ (or simply $\mathcal{G}(P, \mathcal{L}_P, \phi_P, r)$, or even $\mathcal{G}(P, r)$), and the patched metric $\mathcal{O}(P, \mathcal{L}_P, r)$ (or just $\mathcal{O}(P, r)$), satisfying the following properties

P1. The smoothed metric $\mathcal{G}(P, r)$ is a Riemannian metric on $(C_P, S_{C_P})$, and it has the same ray structure as $\sigma_{C_P}$.

P2. The patch metric $\mathcal{O}(P, r)$ is a Riemannian metric defined on $C_P - \mathbb{B}_{r - 2^{-(j + 1)}}(C_P)$ (with smooth structure $S_{C_P}$), and it has the same ray structure as $\sigma_{C_P}$.

P3. On $\mathcal{Y}(P, \Delta^k, r), k \leq j - 2 = \text{dim} P - 2$, the patched metric $\mathcal{O}(P, r)$ coincides with the metric

$$\mathcal{E}_{C_{\Delta^k}}\left(\mathcal{G}(\text{Link}(\Delta^k, P), r)\right)$$

where $\mathcal{G}(\text{Link}(\Delta^k, P), r) = \mathcal{G}(\text{Link}(\Delta^k, P), \mathcal{L}_\Delta(\mathcal{L}_P), \phi_{\Delta}(\mathcal{L}_P), r)$ is defined on $(C_{\text{Star}}(\Delta), S_{C_P})$.

(Recall $\mathcal{Y}(P, \Delta^k, r) \subset C_{\text{Star}}(\Delta^k, P) = C_{\Delta^k} \times C_{\text{Link}}(\Delta^k, P)$, see 6.1.2, 6.1.3, and 8.0.4.)

P4. On $\mathcal{Y}(P, r)$ the patched metric $\mathcal{O}(P, r)$ coincides with the metric $\sigma_{C_P}$.

P5. The metrics $\mathcal{G}(P, r)$ and $\mathcal{O}(P, r)$ coincide on $C_P - \mathbb{B}_{r - 2}(C_P)$.

Note that the patched metric $\mathcal{O}(P, \mathcal{L}_P, r)$ does not depend on $\phi_P$. Properties P3, P4, P5 and the definition of the sets $\mathcal{X}(P, \Delta^k, r), \mathcal{X}(P, r)$ imply
**P6.** On \( \mathcal{X}(P, \Delta^k, r) \), \( k \leq j - 2 = \dim P - 2 \), the smoothed metric \( \mathcal{G}(P, r) \) coincides with the metric

\[
\mathcal{E}_{C^k} \left( \mathcal{G} \left( \text{Link}(\Delta^k, P), r \right) \right)
\]

where \( \mathcal{G} \left( \text{Link}(\Delta^k, P), r \right) = \mathcal{G} \left( \text{Link}(\Delta^k, P), L_{\Delta^k}(L_P), \phi_{\Delta^k}(L_P), r \right) \) is defined on \( (C \text{Star}(\Delta, P), S_{C^P}) \).

**P7.** On \( \mathcal{X}(P, r) \) the smoothed metric \( \mathcal{G}(P, r) \) coincides with the metric \( \sigma_{C^P} \).

Note that the metrics \( \mathcal{G}(\Delta^1, r) \) constructed for spherical all-right 1-complexes in 8.1, together with the choice \( \sigma(\Delta^1, r) = \sigma_{C^P} \), satisfy properties **P1-P5**. Indeed **P1** implies **P1**, **P2** implies **P5** (recall \( r_{-1} = r \), see 6.2) and **P2, P3, P4** are trivially satisfied.

Now, assume we are given the data: \( P, \dim P = m, L_P, \phi_P, r, \sigma \) as items 5 and 6 in (8.0.1). Define the patched metric \( \mathcal{G}(P, r) = \mathcal{G}(P, L_P, r) \) on \( C P - B_{r_{-1,2+(2+2)}(C P)} \) as in **P3 and P4** above. That is, we define \( \mathcal{G}(P, r) \) by demanding that:

**P**. On \( \mathcal{Y}(P, \Delta^k, r) \), \( k \leq \dim P - 2 \), \( \mathcal{G}(P, r) \) coincides with the metric \( \mathcal{E}_{C^k} \left( \mathcal{G} \left( \text{Link}(\Delta^k, P), r \right) \right) \).

**P**. On \( \mathcal{Y}(P, r) \), the patched metric \( \mathcal{G}(P, r) \) coincides with the metric \( \sigma_{C^P} \).

**Lemma 8.2.1.** The patched metric \( \mathcal{G}(P, r) \) defined by properties **P** and **P** is well defined.

**Proof.** The metric \( \mathcal{G}(P, r) \) is defined on the “patches” \( \mathcal{Y}(P, \Delta, r), \Delta \in P \), and \( \mathcal{Y}(P, r) \). We have to prove that these definitions coincide on the intersections \( \mathcal{Y}(P, \Delta^k, r) \cap \mathcal{Y}(P, \Delta^j, r), \mathcal{Y}(P, r) \cap \mathcal{Y}(P, \Delta^j, r) \). If \( \Delta^j \cap \Delta^k = \emptyset \) then (vi) of Proposition 6.2.1 implies \( \mathcal{Y}(P, \Delta^j, r) \cap \mathcal{Y}(P, \Delta^k, r) = \emptyset \). Also if \( \Delta^j \not\subset \Delta^k \) and \( \Delta^k \not\subset \Delta^j \) by (vii) of Proposition 6.2.1, we also get \( \mathcal{Y}(P, \Delta^j, r) \cap \mathcal{Y}(P, \Delta^k, r) = \emptyset \). Therefore we assume \( \Delta^j \subset \Delta^k, j < k \).

Recall that \( \mathcal{Y}(P, \Delta^j, r) \subset C \text{Star}(\Delta^j, r) \) and \( \mathcal{Y}(P, \Delta^k, r) \subset C \text{Star}(\Delta^k, r) \) (see 6.2.1 (i)). The metrics

\[
h = \mathcal{E}_{C \Delta^j} \left( \mathcal{G} \left( \text{Link}(\Delta^j, P), L_{\text{Link}(\Delta^j, P)}(L_P), \phi_{\text{Link}(\Delta^j, P)}(L_P), r \right) \right)
\]

\[
g = \mathcal{E}_{C \Delta^k} \left( \mathcal{G} \left( \text{Link}(\Delta^k, P), L_{\text{Link}(\Delta^k, P)}(L_P), \phi_{\text{Link}(\Delta^k, P)}(L_P), r \right) \right)
\]

are defined on the whole of \( C \text{Star}(\Delta^j, P) \) and \( C \text{Star}(\Delta^k, P) \), respectively. From 6.1.2 we have that \( C \text{Star}(\Delta^j, P) = C \Delta^j \times C \text{Link}(\Delta^j, P) \). And from Lemma 6.2.3 we have that \( \mathcal{Y}(P, \Delta^k, r) \subset C \Delta^j \times \mathcal{X}(\text{Link}(\Delta^j, P), \Delta^j, r) \), where \( \Delta^j = \Delta^k \cap \text{Link}(\Delta^j, P) \) (alternatively \( \Delta^j \) is opposite to \( \Delta^j \) in \( \Delta^k \), or \( \Delta^j = \text{Link}(\Delta^j, \Delta^k) \)). Hence it is enough to prove that the metrics \( h \) and \( g \) coincide on \( C \Delta^j \times \mathcal{X}(\text{Link}(\Delta^j, P), \Delta^j, r) \). But (2) and 6.1.6 (see also 6.1.5 and Proposition 2.6) imply

\[
g = \mathcal{E}_{C \Delta^j} \left[ \mathcal{E}_{C \Delta^j} \left( \mathcal{G} \left( \text{Link}(\Delta^j, P), L_{\text{Link}(\Delta^j, P)}(L_P), \phi_{\text{Link}(\Delta^j, P)}(L_P), r \right) \right) \right] \]

Note that the inductive hypothesis (that is, properties **P3, P5**, which imply **P6**) applied to the data \( \text{Link}(\Delta^j, P), \Delta^j \) gives us that on the set \( \mathcal{X}(\text{Link}(\Delta^j, P), \Delta^j, r) \) we have

\[
\mathcal{G} \left( \text{Link}(\Delta^j, P), L_{\text{Link}(\Delta^j, P)}(L_P), \phi_{\text{Link}(\Delta^j, P)}(L_P), r \right) = \mathcal{E}_{C \Delta^j}(f)
\]
where
\[
f = \mathcal{G}\left( \text{Link}(\Delta^i, \text{Link}(\Delta^j, P)), \mathcal{L}_{\text{Link}(\Delta^i, \text{Link}(\Delta^j, P))}(\mathcal{L}_{\text{Link}(\Delta^j, P)}), \phi_{\text{Link}(\Delta^i, \text{Link}(\Delta^j, P))}(\mathcal{L}_{\text{Link}(\Delta^j, P)}), r \right)
\]  
(5)

From (5) and transitivity of the restriction rule (see 8.0.3) we get
\[
f = \mathcal{G}\left( \text{Link}(\Delta^k, P), \mathcal{L}_{\text{Link}(\Delta^k, P)}(\mathcal{L}_{P}), \phi_{\text{Link}(\Delta^k, P)}(\mathcal{L}_P), r \right)
\]  
(6)

Putting together (1), (4) and (6) we obtain an equation with the same right-hand side as in (3) but with \( h \) instead of \( g \) on the left-hand side. This proves that \( g = h \) on \( \mathcal{Y}(P, \Delta^j, r) \cap \mathcal{Y}(P, \Delta^k, r) \).

The proof that the patched metric is well defined on \( \mathcal{Y}(P, \Delta^k, r) \cap \mathcal{Y}(P, r) \) uses a similar argument and it follows from 6.2.4, the inductive hypothesis applied to \( \text{Link}(\Delta^k, P) \) (that is properties \( \text{P4}, \text{P5} \) which imply \( \text{P7} \)) and 6.1.3. This proves the lemma.

Recall that \( r_{m-2} = r_{m-2}(r) \). Let \( r = r(r_{m-2}) \) be the inverse, where we consider \( r_{m-2} \) as a large real variable. For \( P = P^{m} \) using \( C\phi_P \) we get an identification between \( CP \) and \( \mathbb{R}^{m+1} \). Therefore we can consider the family of metrics \( \{\phi(P, r(r_{m-2}))\}_{r_{m-2}} \) as a \( \odot \)-family of metrics on \( \mathbb{R}^{m+1} \). Because of the unpleasant constant \( 1/2 \) in the warp forcing process we rescale this family and now consider the \( \odot \)-family of metrics.

\[ \{\phi(P, r(r_{m-2}))\}_{r_{m-2}} \quad (\text{recall that the indexation of the family tells us where we take the spherical cuts}) \]

Finally define
\[ \mathcal{G}(P, r) = \mathcal{H}_{r_{m-2}^{-\frac{1}{2}}, \frac{d_{m+1}}{2}}\left( \phi(P, r(r_{m-2})) \right) \]

Then, by construction, this metric satisfies \( \text{P3} \) and \( \text{P4} \). Properties \( \text{P1} \) and \( \text{P2} \) can be proved by induction on the dimension of \( P \) (using properties \( \text{P1-P5} \)), together with Remark 2.3. Property \( \text{P5} \) holds by construction and by (ii) of 4.4. This concludes the construction of the smoothed Riemannian metric \( \mathcal{G}(P, r) = \mathcal{G}(P, L_P, \phi_P, r, \xi, d, (c, \varsigma)) \).

Hence, by construction, we have the following properties.

\( \text{P8.} \) The smoothed metric \( \mathcal{G}(P^m, r) \) is hyperbolic on \( \mathbb{B}_{r_{m-2}^{-d_{m+1}}}(CP) \).

\( \text{P9.} \) The patched metric \( \phi(P^m, r) \) does not depend \( d_i, i > m \).

\( \text{P10.} \) The smoothed metric \( \mathcal{G}(P^m, r) \) does not depend \( d_i, i > m + 1 \).

### 8.3. On the Dependence of \( \mathcal{G}(P, r) \) on the Variables \( c \) and \( \xi \).

In this section we show that the smoothed metric \( \mathcal{G}(P, r) = \mathcal{G}(P, L_P, \phi_P, \xi, r, (c, \varsigma)) \) does not depend on the variables \( \xi \) and \( c \), provided \( \varsigma \) is fixed and \( c, \xi \) and \( \varsigma \) satisfy certain relation. In the next section we will show that, assuming \( d \) and \( r \) large, the metric \( \mathcal{G}(P, r) \) is \( \epsilon \)-hyperbolic. However the excess of the \( \epsilon \)-hyperbolic charts does depend on the variables \( c \) and \( \xi \). In the next result assume \( \varsigma \) and \( d \) fixed. We shall write \( \mathcal{G}(P, r, \xi, c) = \mathcal{G}(P, L_P, \phi_P, r, \xi, (c, \varsigma)) \).

**Proposition 8.3.1.** Let \( c' > c > 1 \) and \( \xi' > \xi > 0 \) be such that \( c'\varsigma < e^{-(4+\xi')} \) Then on \( CP - \mathbb{B}_{r_{m-2}^{-2+\xi}}(CP) \), for \( r > (1+\xi) \) we have
\[ G(P, r, \xi', c') = G(P, r, \xi, c) \]

**Proof.** Write \( A' = B(c', \varsigma) \). Denote by \( \mathcal{Y}(P, \Delta, r) = \mathcal{Y}(P, \Delta, r, \xi, (c', \varsigma)) \) the sets obtained by replacing \( c \) and \( \xi \) in the definition of \( \mathcal{Y}(P, \Delta, r) = \mathcal{Y}(P, r, \xi, (c, \varsigma)) \) (see 6.2) by \( c' \) and \( \xi' \), respectively. Similarly we obtain \( \mathcal{Y}'(P, r) \). We have

\[ \mathcal{Y}(P, \Delta, r) \subset \mathcal{Y}'(P, \Delta, r) \quad \text{and} \quad \mathcal{Y}(P, r) \subset \mathcal{Y}'(P, r) \quad (1) \]

We will prove the proposition by induction on the dimension \( m \) of \( P^m \). It can be checked from Section 8.1 that the case \( m = 1 \) does not depend on the variables \( c \) and \( \xi \). Assume \( G(P^k, r, \xi', c') = G(P^k, r, \xi, c) \), for every \( P^k, k < m \). Consider \( P^m \). First we prove that the corresponding patched metrics \( G(P, r, \xi', c') \) and \( G(P, r, \xi, c) \) coincide. But it follows from properties \( \text{P3} \) and \( \text{P4} \) applied to both metrics, the inductive hypothesis and (1) that \( G(P^m, r, \xi', c') = G(P^m, r, \xi, c) \) on \( \mathcal{Y}(P, \Delta^k, r) \), for all \( \Delta^k \in P \), \( k \leq m-2 \), and on \( \mathcal{Y}(P, r) \). Therefore, by 6.2.1 (iii), the metrics \( G(P^k, r, \xi', c') \), \( G(P^k, r, \xi, c) \) coincide on \( C P - B_{r_m-2-(2+\varsigma)}(C P) \). Finally note that the smoothed metrics \( G(P, r, \xi, c) \) are obtained from the corresponding patched metrics by using the hyperbolic forcing process of Section 4. But this process depends only on \( d \) and \( r_m-2 = sinh^{-1}(\frac{\sinh(\varsigma)}{\sin(\alpha_m-2)}) \). The former is fixed and the later, since \( \sin(\alpha_m-2) = \varsigma^{m-1} \) (see 6.2), is independent of \( c', \xi \) and \( \xi' \). This proves the proposition.

In the next section we will need the following result. We use the notation in the proof of the previous proposition. Let \( s_{m,k} = s_{m,k}(r) \) be obtained from \( s_{m,k} = s_{m,k}(r) \) by replacing \( c \) by \( c' \) (see 6.2).

**Lemma 8.3.2.** If \( c' > c \) we have \( (s_{m,k} - s_{m,k}) > ln(\frac{c}{c'}) - 1 \), provided \( r > 1 \).

**Proof.** From the definition at the beginning of 6.2 we have \( s_{m,k} = sinh^{-1}(c \frac{\sinh(\varsigma)}{\sin(\alpha_m-2)}) \) and \( s'_{m,k} = sinh^{-1}(c' \frac{\sinh(\varsigma)}{\sin(\alpha_m-2)}) \). A simple calculation shows that the function \( t \mapsto sinh^{-1}(c't) - sinh^{-1}(ct) \) is increasing. And another calculation shows that the value of this function at \( t = 1 \) has value at least \( ln(c') - ln(c) - 1 \). This proves the lemma.

**8.4. On the \( \epsilon \)-Close to Hyperbolicity of \( G(P, r) \).**

In this section we prove that the smoothed metrics on \( C P^m \) are \( \epsilon \)-close to hyperbolic, provided \( d_2, \ldots, d_{m+1} \) and \( r \) are large enough, \( m \leq \xi \) and \( \varsigma \) is small (how small depending only on \( \xi \)). First we need the following lemma. Recall that an element of \( C P \) can be written as \( sx, s \geq 0, x \in P \).

**Lemma 8.4.1.** The family of metrics \( \{ G(P^m, r(r_{m-2})) \}_{r_{m-2}} \) has cut limits on \( \mathbb{R} \).

**Proof.** We prove this by induction on the dimension \( m \) of \( P^m \). For \( m = 1 \) the lemma follows from (v), (vi) and (vii) of Proposition 4.4 (alternatively we can use \( \text{P4} \) and 8.1.1 in Section 8.1). Suppose the \( \odot \)-family of metrics \( \{ G(\text{Link}(\Delta^k, P^m), r(r_{m-k-3})) \}_{r_{m-k-3}} \) has cut limits on \( \mathbb{R} \). Then the \( \odot \)-family of metrics \( \{ \mathcal{E}_{C \Delta^k}(G(\text{Link}(\Delta^k, P), r(r_{m-2}))) \}_{r_{m-2}} \) also has cut limits on \( \mathbb{R} \).

**Proof of claim.** By construction (see 4.4 (i) or \( \text{P8} \)), the family \( \{ G(\text{Link}(\Delta^k, P), r(r_{m-k-3})) \}_{r_{m-k-3}} \) satisfies the hypothesis of Proposition 4.7.1: the family is hyperbolic around the origin. Since \( r_{m-k-3} = sinh^{-1}(\sinh(r_{m-2})\sin(\alpha_k)) \) the claim follows from Proposition 4.7.1. This proves the claim.

We continue with the proof of Lemma 8.4.1. Assume the lemma holds for \( P^k, k < m \). Let \( P = P^m \).
and \( \Delta^k \in P \). Suppose that the lemma does not hold for the family \( \mathcal{F} = \{ \mathcal{G}(P^m, r(r_{m-2})) \}_{r_{m-2}} \). We will show a contradiction. To simplify our notation write \( s = r_{m-2} \) and \( g_s = \mathcal{G}(P^m, r(s)) \). We have \( g_s = \sinh^2(r)(g_s)_t + dt^2 \), where \( t \) is the distance to \( o_{C^P_m} \). Since \( \mathcal{F} \) does not have cut limits on \( \mathbb{R} \) the is a bounded closed interval \( I \subset \mathbb{R} \) such that \( \mathcal{F} \) does not have cut limits on \( I \). For \( (x, b) \in P \times I \) write \( g_s^*(x, b) = (g_s)_{s+b}(x) \). Note that \( \sinh^2(s + b)g_s^*(x, b) + dt^2 = g_s((s + b)x) \). Since we are assuming that \( \mathcal{F} \) does not have cut limits on \( I \) we have that the sequence \( \{ g_s^* \} \) defined on \( P \times I \) does not converge in the \( C^2 \) topology. Hence there is a derivative \( \partial^J \), for some multi-index of order \( \leq 2 \), and sequences \( s_n \to \infty \), \( x_n \to x \), \( b_n \to b \) such that \( \partial^J g_{s_n}^*(x_n, b_n) \leq a \), \( \partial^J g_{s_n+1}^*(x_n, b_n) \leq \frac{a}{2} \) for some fixed \( a > 0 \), and \( n \) even. By proposition 6.3.3 we have that \( R_{x,b}(s) = (s + b)x \in \mathcal{Y}(P, \Delta^k, r(s)) \), for some \( \Delta^k \), \( k \leq m - 2 \), and \( s > s' \), for some \( s' \); or \( R_{x,b}(s) = (s + b)x \in \mathcal{Y}(P, r(s)) \), \( s > s' \), for some \( s' \). We assume the first case: \( R_{x,b}(s) = (s + b)x \in \mathcal{Y}(P, \Delta^k, r(s)) \), for some \( \Delta^k \), \( k \leq m - 2 \). The other case is similar. Since \( s \) is large, we also get that \( R_{x,b}(s) = (s + b)x \in \mathcal{X}(P, \Delta^k, r(s)) \), \( s > s' \). Moreover, also by 6.3.3, we can assume \( R_{s_n,b_n}(s) = (s + b_n)x_n \in \mathcal{X}(P, \Delta^k, r(s)) \), for \( s > s' \). But by property \textbf{P6}, on \( \mathcal{X}(P, \Delta^k, r(s)) \) the metric \( g_s \) is equal to \( \mathcal{E}_{\Delta^k}(\mathcal{G}(\text{Link}(\Delta^k, P), r(s))) \). Consequently the family of metrics \( \{ \mathcal{E}_{\Delta^k}(\mathcal{G}(\text{Link}(\Delta^k, P), r(s))) \} \) does not have cut limits on \( I \) either. But the claim, together with the inductive hypothesis, imply that this family does have cut limits, which leads to a contradiction. This proves the lemma.

For a positive real number \( \xi \) and a positive integer write \( \xi_k = \xi - k + \frac{1}{k} \). Note that \( \xi_1 = \xi \).

**Proposition 8.4.2.** Let \( \xi > 0 \), \( c > 0 \) and \( \varsigma \leq \frac{\sqrt{2}}{2} \) be such that

(1) \( \varsigma < e^{-(15 + 3\xi)} \)

(2) \( c \geq e^5 \xi \)

Let \( (P^m, \mathcal{L}_P, \phi_P) \), and \( \epsilon > 0 \). Then we have that \( \mathcal{G}(P, \mathcal{L}_P, \phi_P, \xi, r, d, (c, \varsigma)) \) is \( (B_\alpha, \epsilon) \)-close to hyperbolic \((a = r_{m-2} - d_{m+1})\), with charts of excess \( \xi_m \), provided

(a) \( d_i \) and \( r - d_i \), \( i = 2, ..., m + 1 \), are sufficiently large.

(b) \( m + 1 \leq \xi \).

**Remarks.**

1. By “sufficiently large” in (a) we mean that there are \( r_i(P) \) and \( d_i(P) \), \( i = 2, ..., m + 1 \), such that the proposition holds whenever we choose \( r - d_i \geq r_i(P) \) and \( d_i \geq d_i(P) \).

2. How large \( r \) and the \( d_i \) have to be depends on \( \epsilon \).

3. Note that the choices of \( c, \xi \) and \( \varsigma \) do not depend on \( \epsilon \).

4. If we want the smoothed metric on a cone \( CP^m \) to be \( (B_\alpha, \epsilon) \)-close to hyperbolic we can choose \( \xi = m + 1 \), \( c = e^5 \xi \) and \( \varsigma = e^{-(15 + 3\xi)} \). With these choices the method would not work for \( P \) of dimension \( > m \).

5. The condition \( \varsigma = e^{-(15 + 3\xi)} \) is stronger than the condition \( \varsigma < e^{-4} \). The later is used to construct the smoothed metric but it is not strong enough to give us \( \epsilon \)-hyperbolicity.

**Proof.** We will only mention the relevant objects to our argument in the notation for the smoothed metrics. That is, we will write \( \mathcal{G}(P, d, r, \xi, (c, \varsigma)) \) or just \( \mathcal{G}(P, d, r) \). Our proof is by induction on the dimension \( m \) of \( P^m \), with \( m + 1 < \xi \). So, we assume \( c, \xi \) and \( \varsigma \) fixed and satisfying (i) and (ii), that is, \( \varsigma < e^{-(15 + 3\xi)} \) and \( c \geq e^{15 + 3\xi} \). We also assume \( \epsilon > 0 \) and without loss of generality we can assume

\[
\epsilon < \frac{1}{(1 + \xi)^2}
\]
For $m = 1$ we have that the proposition follows from 8.1 and Theorem 4.5 by writing $\lambda = r$, choosing $g_r = \sigma_{C,P}$, replacing $\xi$ by $\xi + 1$, and taking $\epsilon' = \epsilon$. Also, since $g_r = \sigma_{C,P}$ is $\epsilon$-close to hyperbolic, for every $\epsilon$, we can take the $\epsilon$ in 4.5 to be zero. With all these choices Theorem 4.5 implies that $G(P,d_2,r)$ is $\epsilon$-close to hyperbolic, with charts of excess $\xi = \xi_1$, provided $r - d_2$ and $d_2$ are large enough.

Let $m$ such that $m + 1 \leq \xi$. We write $a_k = r_{k-2} - d_{k+1}$, and note that $G(P^k,r,d)$ is, by construction (see $\text{P8}$), radially hyperbolic on the ball of radius $a_k$. We now assume that the proposition holds for all $k < m$. That is, given $\epsilon > 0$ and $P^k$, the smoothed metric $G(P^k,r,d)$ is $(B_{a_k},\epsilon)$-close to hyperbolic, with charts of excess $\xi_k$, provided $r - d_i$ and $d_i$, $i = 2,...,d_{k+1}$ are large enough. Note that, since we are assuming $k < m$, we get that $k + 1 < \xi$. We use the following notation

$$A_k = C(m - k,k + 1,\xi) \quad B = C_2(\xi) \quad \epsilon_k = \frac{\epsilon}{3A_k B}$$

where $C$ is as in Theorem 2.7 and $C_2$ as in Theorem 4.5. Let $P = P_m$. For $k < m$ write $L_k = \{\text{Link}(\Delta^k,P)\}_{\Delta^k \in P}$. A generic element in $L_k$ will be denoted by $Q = Q^j$, $j + k = m - 1$. By inductive hypothesis, for each $Q^j$ there are $r_i(Q^j)$ and $d_i(Q^j)$, $i = 2,...,j + 1$ such that $G(Q,r(Q^j),d(Q^j))$ is $(B_{a_j},\epsilon)$-close to hyperbolic, with charts of excess $\xi_j$, provided $r - d_i \geq r_i(Q^j)$ and $d_i \geq d_i(Q^j)$. For $i \leq m$, let $d_i(P)$ be defined by

$$d_i(P) = \max_{Q^j,1 \leq i \leq j + 1} \{d_i(Q^j)\}$$

We write $d(P) = \{d_2(P),...,d_m(P),...\}$ where $d_i(P)$, $i > m + 1$, is any positive number. This is just for notational purposes and the arguments given below will not depend the $d_i(P)$, $i > m + 1$. We do reserve the right to later choose $d_{m+1}(P)$ larger. Also write

$$r_i(P) = d_i(P) + \max_{Q^j,1 \leq i \leq j + 1} \{4\ln(m), r_i(Q^j)\}$$

Therefore, from the inductive hypothesis and property $\text{P8}$ we get that

(8.4.3.) For every $Q^j \in L_k$, the metric $G(Q^j,r,d)$ is $(B_{a_j},\epsilon)$-close to hyperbolic, with charts of excess, $\xi_j$ provided $r - d_i \geq r_i(P)$ and $d_i \geq d_i(P)$, $i = 2,...,k + 1$.

By definition we have $r_i(P) \geq 4\ln(m)$. Hence, if $r > r_i(P)$ and $0 \leq j \leq m - 1$ we get that $\xi_j - e^{-r/2} > \xi_j - \frac{1}{m^2} > \xi - j + \frac{1}{m} \geq \xi - (m - 1) + \frac{1}{m}$. This together with (8.4.3), Theorem 2.7 and the definitions given in (2) imply that

(8.4.4.) For every $\text{Link}(\Delta^k,P) \in L_k$, the metric $G_{\text{Link}(\Delta^k,P)}(r,d)$, defined on the space $E_{k+1}(\text{Link}(\Delta^k,P))$, is $(B_{a_j},\frac{\epsilon}{3B})$-close to hyperbolic, with charts of excess $\xi - (m - 1) + \frac{1}{m}$, provided $r - d_i \geq r_i(P)$ and $d_i \geq d_i(P)$, $i = 2,...,k + 1$.

**Lemma 8.4.5.** The patched metric $G(P,r,d)$ is radially $(\frac{\epsilon}{3B})$-close to hyperbolic, provided $r - d_i \geq r_i(P)$, $d_i \geq d_i(P)$, $i = 2,...,m$.

**Proof.** Before we prove the lemma we need some preliminaries. Recall that we have functions $r_{m,k} = r_{m,k}(r)$ and $s_{m,k}(r)$. For $\Delta = \Delta^k \in P$ write $Y_\Delta = Y(P,\Delta,r,\xi,(c,\varsigma))$ and $Y = Y(P,r,\xi,(c,\varsigma))$ (see 6.2). For $\Delta = \Delta^k$, $k \leq m$ define

$$N_\Delta = N_{m,k}(C \Delta ,CP) - \bigcup_{\Delta^l \in P, l < k} N_{m,k}(C \Delta^l ,CP) - B_{r_{m-2} - (2 + \xi)}(C P)$$

$$N = CP - \bigcup_{\Delta^l \in P, l \leq m - 2} N_{m,k}(C \Delta^l ,CP) - B_{r_{m-2} - (2 + \xi)}(C P)$$
For $k = m - 1$ in the definition of $N_{\Delta k}$ above take $s_{m,m-1} = \sinh^{-1}(c' \sinh(r) \varsigma)$. Also for $k = m$ take $N_{s_{m,m}} (C \Delta^m, C P) = C \Delta^m$. Write $N_k = \bigcup_{\Delta \in \mathcal{P} N_k}$. Define $c' = e^{a+c}$ and $\xi' = 6 + 2 \xi$. From hypothesis (i), that is from $c \varsigma < e^{-15 + 3 \xi}$, we get that $c' \varsigma < e^{-4 + \xi'}$, hence we can define the sets $Y'_\Delta = \mathcal{Y}(P, \Delta, r, \xi', (c', \varsigma))$ and $Y' = \mathcal{Y}(P, r, \xi', (c', \varsigma))$ (see 6.2). That is

$$Y'_\Delta = N_{s_{m,k}} (C \Delta^k, C P) \cup \bigcup_{\Delta' \in \mathcal{P}, \xi < k} N_{r_{m,k}} (C \Delta', C P) - B_{r_{m,k} - (2 + \xi')} (C P)$$

and analogously for $Y'$. Here $s'_{m,k}$ is defined by replacing $c$ by $c'$ in the definition of $s_{m,k}$. Note that if we replace $c$ by 1 in the definition of $s_{m,k}$ we obtain $r_{m,k}$. This together with hypothesis (ii), the definition of $c'$, and Lemma 8.3.2 imply

$$(s'_{m,k} - s_{m,k}) > 5 + \xi \quad (s_{m,k} - r_{m,k}) > 5 + \xi$$

(5)

It follows from the condition $c' \varsigma < e^{-4 + \xi'}$, Lemma 8.3.1 and properties P3, P5 that for each $\Delta^k \in P$, $k \leq m - 2$, we have that

(8.4.6.) the metrics $\mathcal{Q}(P, r, \xi, (c, \varsigma))$, $\mathcal{E}_{k+1} (\mathcal{G}((\Delta^k, P), r, \xi, (c, \varsigma)))$ and $\mathcal{E}_{k+1} (\mathcal{G}((\Delta^k, P), r, \xi', (c', \varsigma)))$ coincide on $Y'_\Delta$.

For $p \in C P$ denote the ball of radius $s$ centered at $p$ by $B_{s \cdot p} (C P)$, with respect to the metric $\sigma_{C P}$.

Claim 1. For $\Delta = \Delta^k$, $k \leq m - 2$, we have that $d_{\mathcal{Q}} (N_\Delta, C P - Y'_\Delta) \geq 5 + \xi$

Remark. Here $d_{\mathcal{Q}} (., .)$ denotes path distance with respect to the metric $\mathcal{Q}(P, r)$.

Proof of claim. Write $\Delta = \Delta^k$. From the definitions we have $N_\Delta \subset Y_\Delta \subset Y'_\Delta \subset C \text{Star}(\Delta, P)$. Note that $C \text{Star}(\Delta, P) \subset C P$ but we can also consider $C \text{Star}(\Delta, P) \subset E = \mathcal{E}_{k+1} (\mathcal{G}(\mathcal{L}(\Delta, P), r))$. We claim that we can work on $E$, that is, it is enough to prove that $d_{\mathcal{Q}} (N_\Delta, C - Y'_\Delta) \geq 5 + \xi$ ($E$ with metric $\mathcal{E}_{k+1} (\mathcal{G}(\mathcal{L}(\Delta, P), r))$). To see this note first that $\overline{N_\Delta} \subset int Y'_\Delta$ (this follows from $s_{m,k} < s_{m,j}$, $j < k$). Hence if there is $p \in N_\Delta$ and $q \notin Y'_\Delta$ and $\alpha$ a path joining $p$ to $q$ of $\mathcal{Q}$-length $< 5 + \xi$ then there a restriction $\beta$ of $\alpha$ such that: (1) it begins at $p$, (2) it ends at some point $q' \in \partial Y'_\Delta$. (3) it is totally contained in $\overline{Y'_\Delta}$, (4) its $\mathcal{Q}$-length is $< 5 + \xi$. By property P3 the path $\beta$ (now considered in $Y' \subset E$, with metric $\mathcal{E}_{k+1} (\mathcal{G}(\mathcal{L}(\Delta, P), r))$) has the same properties (1)-(4). This shows that we can work on $E$ with metric $\mathcal{E}_{k+1} (\mathcal{G}(\mathcal{L}(\Delta, P), r))$ instead of $C P$ with metric $\mathcal{Q}(P, r)$.

Let $o \in C \text{Star}(\Delta, P) \subset E$ correspond to $o_{C P}$. Let $p \in N_\Delta$, and $q \notin Y'_\Delta$. From (3) we have 3 cases.

Case 1. $q \notin N_{s_{m,k}} (C \Delta, C P)$. Since $N_{s_{m,k}} (C \Delta, C P) = \mathbb{H}^{k+1}_+ \times B_{s_{m,k}} (\mathcal{L}(\Delta, P))$ and $N_\Delta \subset N_{s_{m,k}} (C \Delta, C P) = \mathbb{H}^{k+1}_+ \times B_{s_{m,k}} (\mathcal{L}(\Delta, P))$, this case follows from the first inequality in (5). Note that the $s$ neighborhoods of $\mathbb{H}^{k+1}_+$ in $\mathcal{E}_{k+1} (\mathcal{G}(\mathcal{L}(\Delta, P), r))$ coincide with respect to the metrics $\mathcal{E}_k (\mathcal{G}(\mathcal{L}(\Delta, P), r))$ and $\mathcal{E}_{k+1} (\sigma_{C \mathcal{L}(\Delta, P)})$ (see Remark 2.3).

Case 2. $q \in N_{r_{m,j}} (\Delta^j, P)$, $j < k$. Since $p \in N_\Delta$ we have that $p \notin N_{s_{m,j}} (\Delta^j, P)$, this case is similar to case 1, but uses the second inequality in (5), instead of the first.
Case 3. \( q \in B_{r_{m-2}-(2+\xi')}(E) \), where the ball is centered at \( o \). Since \( p \notin B_{r_{m-2}-(2+\xi)}(E) \) this case follows from the fact that \((r_{m-2} - (2 + \xi)) - (r_{m-2} - (2 + \xi')) = \xi' - \xi = 6 + \xi > 5 + \xi \). This proves the claim.

Claim 2. We have that \( d_{C,P}(N, C \ P \ Y) > 5 + \xi \)

Proof of claim. First, we can extend the argument given in the proof of claim 1 one more step, that is for \( \Delta = \Delta^{m-1} \). The only problem here is that we have not defined the sets \( Y(P, \Delta^k, r) \) for \( k = m-1 \) (see 6.2). So just define \( s'_{m,m-1} = \text{sinh}^{-1}(c' \text{sinh}(r_\xi)) \), and \( Y'_{\Delta} \) accordingly (as in 6.2). It can be checked, using the results in Section 6, that the argument above goes through and we get \( d_{\phi}(N_{\Delta}, C \ P \ Y') \geq 5 + \xi \) (in this case the patched metric is just \( \sigma_{C,P} \)). Since \( Y''_{\Delta} \subset Y \) it remains to prove that for \( \Delta = \Delta^m \) we have \( d_{\phi}(N_{\Delta}, C \ P \ Y) \geq 5 + \xi \). To deal with this case define \( Y^* = C \ P \backslash \bigcup_{k < m} N_{r_{m,k}}(C \Delta^k, C \ P) \) and let \( Y^*_{\Delta} \) be the component of \( Y^* \) contained in \( C \Delta \). Here we are taking \( r_{m,m-1} = \text{sinh}^{-1}(\text{sinh}(r_\xi)) \).

Then \( N_{\Delta} \subset Y^*_{\Delta} \subset Y \). We want to prove that \( d_{\phi}(N_{\Delta}, C \ P \ Y^*) \geq 5 + \xi \), but this case now can be reduced to the case \( \Delta = \Delta^m \subset \mathbb{S}^m \subset \mathbb{H}^{m+1} = \mathbb{C}^m \), which can easily be dealt with. This proves the claim.

We are now ready to prove the lemma. Assume \( p \in N_k \). We prove by induction on \( k \) that the patched metric \( \phi(P, r, d) \) is radially \((\frac{\epsilon}{3m})\)-close to hyperbolic, with excess \( \xi'' = \xi - (m-1) + \frac{1}{m} \), that is, there is a radially \((\frac{\epsilon}{3m})\)-close to hyperbolic chart \( \phi : T_{\xi''} \rightarrow C \ P \) centered at \( p \). We begin with \( k = 0 \). Assume \( p \in N_0 \). Then \( p \in N_{\Delta^0} \) for some \( \Delta^0 \). From (8.4.4) there is a radially \((\frac{\epsilon}{3m})\)-close to hyperbolic chart \( \phi : T_{\xi''} \rightarrow \mathcal{E}_k(C \text{Link}(\Delta^0, P)) \). It follows from Lemma 1.2, claim 1 and (1) that the image \( \phi(T_{\xi''}) \subset Y^*_{\Delta^0} \). This together with (8.4.6) imply \( \phi \) is also chart for \( \phi(P, r) \). This proves the case \( k = 0 \). The inductive step \( k \leq m-2 \) is similar. It remains to prove the case \( p \in N \). But this case follows from a similar argument as above (in this case fitting a chart in \( Y \)) and using claim 2 and property \( \textbf{P5} \). This proves Lemma 8.4.5.

We now finish the proof of Proposition 8.4.2. Take \( \epsilon' = \frac{\epsilon}{\xi} \) and apply Theorem 4.5 to the family \( \{ (\phi(P, r(r_{m-2}), d)) \}_{r_{m-2} = \frac{1}{2} + \frac{1}{m}} \). Note that we have to use Lemma 8.4.1 to satisfy one of the hypothesis of Theorem 4.5. Since \( \epsilon' + B \frac{\epsilon}{3m} < \epsilon \) (recall \( B = C_2 \), see (2)) from Theorem 4.5 we obtain a number \( r_{m+1}(P) \) and a (possibly larger) number \( d_{m+1}(P) \) such that \( \mathcal{G}(P, r, d) \) is radially \( \epsilon \)-close to hyperbolic, provided \( r - d_i \geq r_i(P) \) and \( d_i \geq d_{i}(P) \), \( i = 2, \ldots, m+1 \). Finally note that the excess of the charts given by Theorem 4.5 is \( \xi'' = 1 = (\xi - (m-1) + \frac{1}{m}) - 1 = \xi_m \). This proves Proposition 8.4.2.

8.5. Smoothing Cones Over Manifolds.

As in the beginning of Section 8, let \( P^m \) be an all-right spherical complex and \( \mathcal{S}_P = S(L_P) \) a compatible normal smooth structure on \( P \). In the previous sections we have canonically constructed a Riemannian metric \( \mathcal{G}(P, L_P, \phi_P, r, \xi, d, (c, \varsigma)) \) on the cone \( C \ P \). An important assumption was that \((P, \mathcal{S}_P)\) was diffeomorphic (by \( \phi_P \)) to the sphere \( \mathbb{S}^m \). We cannot expect to do the same construction on a general manifold \( P \) because \( C \ P \) is not in general a manifold. But we will canonically construct a complete Riemannian metric on \( C \ P \) that has some of the previous properties.

We consider the same data as before: \( P^m, L_P, r, \xi, d, (c, \varsigma) \) satisfying (8.0.1) but with one change, replace the map \( \phi_P \) by a Riemannian metric \( h_P \) on the closed smooth manifold \((P, \mathcal{S}_P)\). Hence we begin with the following data: \( P^m, L_P, h_P, r, \xi, d, (c, \varsigma) \).
First note that, by Theorem 7.1, a compatible normal smooth structure on $P$ exists. We will assume that $P$ has either dimension $\leq 4$ or $Wh(\pi_1 P) = 0$, so that we can apply 7.3.1. Note also that the sets $\mathcal{V}(P, \Delta, r)$, $\mathcal{V}(P, r)$ are defined for general $P$ (no just for $P = S^m$) and satisfy all the properties given in Section 6.2. Now, since all the links of $P$ are spheres we can define, as in 8.1 and 8.2 the patch metric $\mathcal{V}(P, r) = \mathcal{V}(P, L_P, r, \xi, d, (c, s))$ on $C P - \mathbb{B}_{r_{m-2}}(C P)$, and this metric satisfies properties P2, P3, P4 given in section 8.2.

Recall that in 8.2 this construction is completed by applying the hyperbolic forcing to the $\circ$-family of metrics $\{\mathcal{V}(P, r(r_{m-2}))\}_{r_{m-2}^-}$. This method consists of two parts: warp forcing and then hyperbolic forcing. In our more general setting here we can still apply warp forcing, but we cannot directly apply hyperbolic forcing (at least not in the way given in Section 3) because we do not have Corollary 8.5.1. The metrics $P$ general case, to finish our construction we apply first warp forcing and then a version of hyperbolic forcing for $G$; this new version will use the metric $h_p$ instead of the canonical metric $\sigma_{cm}$ on the sphere $S^m$.

Consider now the $\circ$-family of metrics $\{\mathcal{V}(P, r(r_{m-2}))\}_{r_{m-2}^-}^\circ$ and apply warp forcing to obtain
\[
g_{r_{m-2}} = W_{r_{m-2}^-} \mathcal{V}(P, r(r_{m-2}))
\]
and we have that $g_{r_{m-2}}$ is warped on $\mathbb{B}_{r_{m-2}^-}^\circ(C P) - o_{C P}$, specifically we have $g_{r_{m-2}} = \sinh^2(t) g + dt^2$, where $g$ is a Riemannian metric on $P$ (it is the spherical cut of $\mathcal{V}(P, r(r_{m-2}))$ at $r_{m-2}^- = t$) and $t$ is the distance-to-the-vertex function on $C P$. Let $\rho_{a, d}$ be the function in 3.1. Now define the metric $\mathcal{V}(P, h, r) = \mathcal{V}(P, L_P, h, r, \xi, d, (c, s))$ by
\[
\mathcal{V}(P, h, r) = \begin{cases} 
    h + (\rho_{r_{m-2} - d_{m+1} - d_{m+1}^-} \circ t) (g - h) + dt^2 & \text{on } \mathbb{B}_{r_{m-2}^-}^\circ(C P) - \mathbb{B}_{r_{m-2} - d_{m+1}^-}^\circ(C P) \\
    (\mu \circ t)^2 h + dt^2 & \text{on } \mathbb{B}_{r_{m-2} - d_{m+1}}^\circ(C P)
\end{cases}
\]
where $\mu(t) = \frac{\epsilon t - \lambda t}{2}$, and $\lambda = \rho_{r_{m-2} - 2d_{m+1} + d_{m+1}}$. Also we are assuming $r_{m-2} - 2d_{m+1} > 0$. Note that the metric $\mathcal{V}(P, h, r) = \frac{1}{2} \epsilon t h + dt^2$ on $\mathbb{B}_{r_{m-2} - 2d_{m+1}}(C P) - o_{C P}$, that is for $0 < t \leq r_{m-2} - 2d_{m+1}$. We write $C P - o_{C P} = P \times (0, \infty)$ and extend the metric $\mathcal{V}(P, h, r)$ to $P \times \mathbb{R}$ by $\frac{1}{2} \epsilon t h + dt^2$ for $-\infty < t \leq r_{m-2} - 2d_{m+1}$.

**Corollary 8.5.1.** The metrics $\mathcal{V}(P, h, r)$ and $\mathcal{V}(P, r)$ have the following properties

(i) $\mathcal{V}(P, r)$ is a Riemannian metric on $P \times \mathbb{R}$ that has the same ray structure as $\sigma_{cm}$ (on $P \times (0, \infty)$).

(ii) Properties P3 and P4.

(iii) We have $\mathcal{V}(P, h, r) = \frac{1}{2} \epsilon t h + dt^2$ for $-\infty < t \leq r_{m-2} - 2d_{m+1}$.

(iv) Given $\epsilon > 0$ we have that the sectional curvatures of $\mathcal{V}(P, h, r)$ are $\epsilon$-pinched to -1 for $t \geq r_{m-2} - 2d_{m+1}$ provided $r - d_i < 1$, $d_i$, $i = 2, ..., m + 1$, and $r - 2d_{m+1}$ are large enough.

**Proof.** Item (i) follows the same argument used for P1 in the spherical case. Item (ii) is true by construction (see also Lemma 8.2.1). Item (iii) follows from the discussion above, and (iv) from 8.4.2 and Bishop-O’Neill warp curvature formula [4], p.27.

9. On Charney-Davis Strict Hyperbolization Process.

We use some of the notation used in [6]. In particular the canonical $n$-cube $[0, 1]^n$ will be denoted by $\square^n$. 41
(This differs with the notation used in Section 7, where an $n$-cube was denoted by $\sigma^n$.) Also $B_n$ is the isometry group of $\square^n$.

A Charney-Davis strict hyperbolization piece of dimension $n$ is a compact connected orientable hyperbolic $n$-manifold with corners satisfying the properties stated in Lemma 5.1 of [6]. The group $B_n$ acts by isometries on $X^n$ and there is a smooth map $f : X^n \to \square^n$ constructed in Section 5 of [6] with certain properties. We collect some facts from [6].

1. For any $k$-face $\square^k$ of $\square^n$ we have that $f^{-1} (\square^k)$ is totally geodesic in $X^n$. Moreover $X^n$ is a Charney-Davis hyperbolization piece of dimension $k$. The submanifold (with corners) $f^{-1} (\square^k)$ is a $k$-face of $X^n$. Note that the intersection of faces is a face and every $k$-face is the intersection of exactly $n - k$ distinct $(n - 1)$-faces.

2. The map $f$ is $B_n$-equivariant.

3. The faces of $X^n$ intersect orthogonally.

4. The map $f$ is transversal to the $k$-faces of $\square^n$, $k < n$.

The following is proved in [28].

Proposition 9.1. For every $n$ and $r > 0$ the is a Charney-Davis hyperbolization piece of dimension $n$ such that the widths of the normal neighborhoods of every $k$-face, $k = 0, \ldots, n - 1$, are larger that $r$.

For a $k$-face $X^k$ and $p \in X^k$, the set of inward normal vectors to $X^k$ at $p$ can be identified with the canonical all-right $(n - k - 1)$-simplex $\Delta_{\square^{n-k-1}}$. In this sense we consider $\Delta_{\square^{n-k-1}} \subset T_pX$. Similarly we can consider $\Delta_{\square^{n-k-1}} \subset T_q \square^n$, for $q \in \square^k$. We make the convention that the two identifications above are done with respect to an ordering of the $(n - 1)$-faces $X_{\square^{n-1}}$ of $X$ and the corresponding ordering for $\square^n$. For a proof of the following proposition see [28].

Lemma 9.2. For $p \in X^k$, we have that $Df_p$ sends non-zero normal vectors to non-zero normal vectors; thus $Df_p|_{\Delta_{\square^{n-k-1}}} : \Delta_{\square^{n-k-1}} \to \Delta_{\square^{n-k-1}}$. Moreover, $n \circ (Df_p|_{\Delta_{\square^{n-k-1}}}) : \Delta_{\square^{n-k-1}} \to \Delta_{\square^{n-k-1}}$ is the identity, where $n(x) = \frac{1}{|x|}$ is the normalization map.

The strict hyperbolization process of Charney and Davis is done by gluing copies of $X^n$ using the same pattern as the one used to obtain the cube complex $K$ from its cubes. This space is called $K_X$ in [6]. Note that we get a map $F : K_X \to K$, which restricted to each copy of $X$ is just the map $f : X^n \to \square^n$. We will write $X_{\square^k} = F^{-1} (\square^k)$, for a $k$-cube $\square^k$ of $K$.

By Lemma 9.2 we can use the derivative of the map $F : K_X \to K$ (in a piecewise fashion) to identify $\text{Link}(X^k, K_X)$ with $\text{Link}(\square^k, K)$, where in both cases we consider the “direction” definition of link, that is, the link $\text{Link}(X^k, K_X)$ (at $p \in X^k$) is the set of normal vectors to $X^k$ at $p$ and the link $\text{Link}(\square^k, K)$ (at $q \in \square^k$) is the set of normal vectors to $\square^k$ at $q$. Hence we write $\text{Link}(X^k, K_X) = \text{Link}(\square^k, K)$; thus the set of links for $K$ coincides with the set of links for $K_X$.

In what follows we assume that the width of the normal neighborhoods of all $X^k$ to be larger than $s_0$, for some $s_0$. Also let $r$ such that $s_0 > 2r$. By 9.1 we can take $s_0$ and $r$ arbitrarily large.

Let $X^k \subset X\square^n$ be a $k$-face of $K_X$, contained in the copy $X\square^n$ of $X$ over $\square^n$. For a non-zero vector $v$ normal to $X^k$ at $p \in X^k$, and pointing inside $X\square^n$, we have that $\exp_p(tu)$ is defined and contained in $X\square^n$, for $0 \leq t < s_0/|u|$. Let $h_k : S^{n-k-1} \to \text{Link}(\square^k, K) = \text{Link}(X^k, K_X)$ be a link smoothing of the link corresponding to $\square^k \in K$. We define the map.
\[ H_{\square_k} : \mathbb{D}^{n-k} \times \hat{X}_{\square_k} \to K_X \]
\[(tv, p) \mapsto H_{\square_k}(tv, p) = \exp_p (2\pi t h_{\square_k}(v)) \]

where \( v \in \mathbb{S}^{n-k-1} \) and \( t \in [0, 1) \). For \( k = n \) we have that \( H_{\square_n} \) is the inclusion \( \hat{X}_{\square_n} \subset K_X \) (or we can take this as a definition). Note that \( H_{\square_k} \) is a topological embedding because we are assuming the width of the normal neighborhood of \( X_{\square} \) to be larger than \( s_0 > 2r \). We call a chart of the form of \( H_{\square_k} \) a normal chart for the \( k\)-face \( X_{\square_k} \). A collection \( \{ H_{\square} \}_{\square \in K} \) of normal charts is a normal atlas, and if this atlas is smooth (or \( C^k \)) the induced differentiable structure is called a normal smooth (or \( C^k \)) structure. The following theorem is proved in [28].

**Theorem 9.3.** Let \( \mathcal{L} = \{ h_\square \}_{\square \in K} \) be a set of link smoothings for \( K \). If \( \mathcal{L} \) is smooth then the normal atlas \( \{ H_{\square} \}_{\square \in K} \) on \( K_X \) is smooth.

We will write \( \mathcal{A}_{KX} = \{ H_{\square} \}_{\square \in K} \). Note that the normal atlas \( \mathcal{A}_{KX} \) depends uniquely on the smooth set of link smoothings \( \mathcal{L} = \{ h_\square \}_{\square \in K} \) for \( K \) (hence for \( K_X \)). To express this dependence we will sometimes write \( \mathcal{A}_{KX} = \mathcal{A}_{KX}(\mathcal{L}) \). We will denote by \( S_{KX} = S_{KX}(\mathcal{L}) \) the smooth structure on \( K_X \) induced by the smooth atlas \( \mathcal{A}_{KX} \). The following Theorem is proved in [28].

**Theorem 9.4.** The smooth manifold \( (K_X, S_{KX}) \) smoothly embeds in \( (K, S') \times X \), with trivial normal bundle. Here \( S' \) is the normal smooth structure on \( K \) induced by \( \mathcal{L} \).

### 9.5. Hyperbolized Manifolds with Codimension Zero Singularities.

In this section we treat the case of manifolds with a one point singularity. The case of manifolds with many (isolated) point singularities is similar. We assume the setting and notation of Section 7.3. Let \( K_X \) be the Charney-Davis strict hyperbolization of \( K \). Denote also by \( p \) the singularity of \( K_X \). Many of the definitions and results given before still hold (with minor changes) in the case of manifolds with a one point singularity (see [28] for more details).

1. Given a set of link smoothings for \( K \) (hence for \( K_X \)) we also get a set of charts \( H_{\square} \). For the vertex \( p \) we mean the cone map \( H_p = C h_p : C N \to C L \subset K_X \). We will also denote the restriction of \( H_p \) to \( C N - \{ o_{CN} \} \) by the same notation \( H_p \). As in item (2) of 7.3 here we are identifying \( C N - \{ o_{CN} \} \) with \( N \times (0, 1] \) with the product smooth structure obtained from some smooth structure \( \hat{S}_N \) on \( N \). As before \( \{ H_{\square} \}_{\square \in K} \) is a normal atlas for \( K_X \) (or \( K_X - \{ p \} \)). A normal atlas for \( K - \{ p \} \) induces a normal smooth structure on \( K_X - \{ p \} \).

2. Again we say that the smooth atlas \( \{ H_{\square} \} \) (or the induced smooth structure, or the set \( \{ h_\square \} \)) is correct with respect to \( N \) if \( S_N \) is diffeomorphic to \( \hat{S}_N \).

3. Let the set \( \mathcal{L} = \{ h_\square \}_{\square \in K} \) induce a smooth structure on \( K - \{ p \} \), that is, \( \mathcal{L} \) is smooth. As in Theorem 9.3 we get that \( \{ H_{\square} \}_{\square \in K} \) is a smooth atlas on \( K_X - \{ p \} \) and it induces a normal smooth structure \( S_{KX} \) on \( K_X - \{ p \} \). Moreover, from Theorem 7.3.1 we get that \( S_{KX} \) is correct with respect to \( S_N \) when \( \dim N \leq 4 \) (always) or when \( \dim N > 4 \), provided \( Wh(N) = 0 \). Note that in this case we can take the domain \( C N - \{ o_{CN} \} = N \times (0, 1] \) of \( H_p \) with smooth product structure \( S_N \times S_{(0,1]} \).

4. It can be verified that a version of Theorem 9.4 also holds in this case: \( (K_X - \{ p \}, S_{KX}) \) smoothly embeds in \( (K - \{ p \}, S') \times X \) with trivial normal bundle.
Section 10. Proof of the Main Theorem.

In Section 2 the concept of hyperbolic extension over hyperbolic space was introduced. We extend next, in the obvious way, this concept to hyperbolic extensions over hyperbolic manifolds.

As in Section 2, let \((N, h)\) be a complete Riemannian manifold with center \(o = o_N\). Let \((P, \sigma_p)\) be a hyperbolic manifold. The hyperbolic extension of \(h\) over \(P\) is the Riemannian metric \(g = (\cosh^2 r)\sigma_p + h\) on \(P \times N\), where \(r : N \to [0, \infty)\) is the distance to \(o\) function on \(N\). We write \(g = \mathcal{E}_P(h)\) and \((P \times N, g) = \mathcal{E}_P(N, h)\) (or simply \(\mathcal{E}_P(N)\)) and we call \(\mathcal{E}_P(N)\) the hyperbolic extension of \(N\) over \(P\).

We now begin the proof of the Main Theorem. Let \(M^n\) be a closed smooth manifold. Let \(K\) be a smooth cubulation of \(M\) and \(K_X\) the Charney-Davis strict hyperbolization of \(M\), as in Section 9. We can assume that the Charney-Davis hyperbolization piece \(X\) has normal bundles with large widths (see 9.1), all larger than a large number \(s_0 > 0\) and \(s_0 >> 3r\), where \(r\) is as in 9.1. Let \(\mathcal{A}_{K_X} = \{H_{\square}\}_{\square \in K}\) be a smooth normal atlas for \(K_X\), and \(\mathcal{S}_{\sigma_K}\) the induced normal smooth structure on \(K_X\). Recall that the \(H_{\square}\) are constructed from a smooth set of link smoothings \(\mathcal{L}_K = \{h_{\square}\}_{\square \in K}\) for the links of \(K\) (or \(K_X\)).

Remark. The domains of the charts \(H_{\square}^{nk}\) are the sets \(\mathbb{D}^{n-k} \times \hat{X}_{\square}^{nk}\). But in this section, for notational purposes, we will consider the rescaling of \(H_{\square}^{nk}\) given by the composition \((u, p) \mapsto (\frac{1}{2r}u, p) \mapsto H_{\square}^{nk}(\frac{1}{2r}u, p)\), defined on \(\mathbb{D}^{n-k}(s_0/2r) \times \hat{X}_{\square}^{nk}\). We shall denote this chart also by \(H_{\square}^{nk}\). That is, in this section \(H_{\square}^{nk}\) is the chart given by \(H_{\square}^{nk}(tv, p) = \exp_{\hat{p}}(t H_{\square}^{nk}(v))\).

In what follows, to simplify our notation, we write \(\text{Link}(X_{\square}) = \text{Link}(X_{\square}, K_X)\). Recall that given \(\square \in K\), the set \(\mathcal{L}_K\) of link smoothings for the links \(\text{Link}(X_{\square})\) of \(K_X\) (and of \(K\)) induce, by restriction (see 7.2), the set of link smoothings \(\{h_{\square'} \in \mathcal{L}_K, \square' \subseteq \square\}\) for the links of \(\text{Link}(X_{\square})\). We denote this induced set of smoothings by \(\mathcal{L}_{\text{Link}(X_{\square})}\) or just \(\mathcal{L}_{\square}\).

The space \(K_X\) has a natural piecewise hyperbolic metric which we denote by \(\sigma_{K_X}\). The piecewise hyperbolic metric on the cones \(C\text{Link}(X_{\square})\) of the all-right spherical simplices \(\text{Link}(X_{\square})\) will be denoted by \(\sigma_{C\text{Link}(X_{\square})}\). The restriction of \(\sigma_{K_X}\) to the totally geodesic space \(X_{\square}\) shall be denoted by \(\sigma_{X_{\square}}\).

For \(\square \in K\), the (closed) normal neighborhood of \(\hat{X}_{\square}^{nk}\) in \(K_X\) of width \(s < s_0\) is the set \(N_{\square}(\hat{X}_{\square}^{nk}, K_X) = H_{\square}^{nk}(\mathbb{D}^{n-k}(s) \times \hat{X}_{\square}^{nk})\). That is, it is the union of the images of all geodesics of length \(\leq s\) in each copy of \(X\) containing \(\hat{X}_{\square}^{nk}\), that begin in (and are normal to) \(\hat{X}_{\square}^{nk}\). Similarly the open normal neighborhood of \(\hat{X}_{\square}^{nk}\) of width \(s < s_0\) is the set \(N_{\square}^o(\hat{X}_{\square}^{nk}, K_X) = H_{\square}^{nk}(\text{int}\mathbb{D}^{n-k}(s) \times \hat{X}_{\square}^{nk})\). Sometimes we will just write \(N_{\square}(\hat{X}_{\square}^{nk}) = N_{\square}(\hat{X}_{\square}^{nk}, K_X)\) and \(N_{\square}^o(\hat{X}_{\square}^{nk}) = N_{\square}^o(\hat{X}_{\square}^{nk}, K_X)\). Since the normal bundles of the \(X_{\square}\) are canonically trivial (see construction of \(X\) in [28], or Section 2 in [28]) we can canonically identify the neighborhood \(N_{\square}(\hat{X}_{\square})\) with \(\hat{X}_{\square}^{nk} \times C_{\text{Link}}(\text{Link}(X_{\square})^{nk})\), where \(C_{\text{Link}}(\text{Link}(X_{\square})^{nk}) = B_{\text{\square}}(C\text{Link}(X_{\square}))\) is the closed \(s\)-cone of length \(s\), that is, it is the ball of radius \(s\) on the (infinite) cone \(C\text{Link}(X_{\square})\) centered at the vertex, see 6.1. Similarly we have the identification \(\hat{X}_{\square}^{nk} = \hat{X}_{\square}^{nk} \times \hat{C}_{\text{Link}}(\text{Link}(X_{\square})^{nk})\), where \(\hat{C}_{\text{Link}}(\text{Link}(X_{\square})^{nk})\) is the open \(s\)-cone of length \(s\). Moreover these identifications are also metric identifications, where we consider \(N_{\square}(\hat{X}_{\square}, K_X) \subset K_X\) with the (restricted) piecewise hyperbolic metric \(\sigma_{K_X}\) and \(\hat{X}_{\square}^{nk} \times C_{\text{Link}}(\text{Link}(X_{\square})^{nk})\) with the hyperbolic extension metric

\[
\mathcal{E}_{\text{Link}(X_{\square})} = \cosh^2(t) \sigma_{X_{\square}} + \sigma_{C\text{Link}(X_{\square})}
\]

where \(t\) is the distance-to-the-vertex function on the cone \(C\text{Link}(X_{\square})\).
Lemma 10.1. Let $\Box^k = \Box^i \cap \Box^j$, $k \geq 0$. Let $s_1, s_2, s < s_0$ be positive real numbers such that $\frac{\sinh s}{\sinh s_1}, \frac{\sinh s}{\sinh s_2} \leq \frac{\sqrt{2}}{2}$. Then $N_{s_1}(\hat{X}^j) \cap N_{s_2}(\hat{X}^j) \subset N_s(\hat{X}^k)$.

Proof. Let $p \in N_{s_1}(\hat{X}^j) \cap N_{s_2}(\hat{X}^j)$. Let $q$ be the closest point to $p$ in the totally geodesic subspace $X^k$. Denote by $t$ the distance between $p$ and $q$. Consider the cone $C = C_{s_0}(\text{Link}(X^k))$ at $q$, that is $C$ is the union of the images of all geodesics of length $s_0$ normal to $X^k$ at $q$. Then $C$ is convex and $p \in C$. Write $A_l = C \cap X^l$, $l = i, j$, which are also convex. Let $q_l \in A_l$ be the closest point to $p$ in $A_l$ and let $\gamma_l$ be the geodesic segment between $p$ and $q_l$ with length $(\gamma_l)$ equal to the distance between $p$ and $q_l$. Hence $a_l = \text{length}(\gamma_l) < s_l$. Since $A_l$ is convex we have that $\gamma_l$ is in $A_l$ and $\gamma_l$ is perpendicular to $A_l$ at $q_l$. Hence $q_l$ is also the closest point to $p$ in $X^l$. We get right triangles with vertices $p, q, q_l$ (right at $q_l$), and hypotenuse equal to $t$. Let $\theta_l$ be the angle at $p$. Thus $\theta_l$ is opposite to the side with length $a_l$. By the hyperbolic law of sines we have $\sin \theta_l = \frac{\sinh a_l}{\sinh s l}$, and by hypothesis we get

$$\sin \theta_l = \frac{\sinh a_l}{\sinh s l} \leq \frac{\sinh s_i}{\sinh s} \leq \frac{\sinh s_j}{\sinh s} \leq \frac{\sqrt{2}}{2} \sinh s t.$$

We want to prove that $t \leq s$. Suppose $t > s$. It follows then from the inequality above that $\sin \theta_l < \sqrt{2}/2$, thus $\theta_l < \pi/4$. Let $S$ be the link of $X^k$ at $q$. The segments $[q, q_l]$ intersect $S$ in two different vertices $v_i$. Since $S$ is an all-right spherical complex and the sets $S \cap A_l$ are disjoint the (angle) distance $d_s(v_i, v_j)$ between $v_i$ and $v_j$ is at least $\pi/2$. Also the segment $[p, q]$ intersects $S$ in a vertex $u$ and we have $\theta_l = \frac{\sinh a_l}{\sinh s l} = \sinh d_s(u, v_i)$. Consequently

$$\frac{\pi}{4} \leq d_s(v_i, v_j) \leq d_s(v_i, u) + d_s(u, v_j) = \theta_i + \theta_j < \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

which is a contradiction. This proves the lemma.

Suppose $\Box^i \subset \Box^k \in K$. Then $\Box^k$ determines the all-right spherical simplex $\Delta_{\text{Link}(\Box^i, K)}(\Box^k) = \Box^k \cap \text{Link}(\Box^i, K) \subset \text{Link}(\Box^i, K) = \text{Link}(X^i)$. We will just write $\Delta(\Box^k)$ if there is no ambiguity. (Other definition previously used: $\Delta_{\text{Link}(\Box^i, K)}(\Box^k) = \text{Link}(\Box^i, \Box^k)$)

Lemma 10.2. Let $\Box^i \subset \Box^k$ and $s_1, s_2 < s_0$. Then

$$N_{s_1}(\hat{X}^i) \cap N_{s_2}(\hat{X}^k) = \hat{X}^i \times C_{\Delta(\Box^k), C_{s_1}(\text{Link}(X^i))}$$

Proof. Let $p \in N_{s_1}(\hat{X}^i)$. By the identification $N_{s}(\hat{X}^i) = \hat{X}^i \times C_{s}(\text{Link}(X^i))$ we can write $p = (u, x) \in \hat{X}^i \times C_{s_1}(\text{Link}(X^i))$. Since $\{x\} \times C_s(\text{Link}(X^i))$ is convex in $\hat{X}^i \times C_s(\text{Link}(X^i))$, we have

$$d_{C_{s_1}(\text{Link}(X^i))}(p, X^k) = d_{C_{s_1}(\text{Link}(X^i))}(u, C_{\Delta(\Box^i)})$$

where, as usual, $d_s$ denotes distance on a space $S$. Consequently the first term in the inequality above is $< s_2$ if and only if the second term is $< s_2$. This proves the lemma.

Remark. Clearly the open version of Lemma 10.2 also holds:

45
Now, let $d, r, \xi, c$ and $\varsigma$ be as in items 1, 2, 3, 4 at the beginning of Section 8, and let the numbers $s_{m,k} = s_{m,k}(r), r_{m,k} = r_{m,k}(r)$ be as in Section 6.2. For each $\square^k \in K$ define the sets

$$Z(X^{\square^k}) = \mathring{\mathbf{N}}_{s, k} \left( X^{\square^k} \right) - \bigcup_{i < k} \mathbf{N}_{r, i} \left( X^{\square^k} \right)$$

By 9.1 we can take $s_0$ as large as needed, hence we can assume that $Z(X^{\square^k}) \subset \mathring{\mathbf{N}}_{s_0} \left( X^{\square^k} \right)$.

We next use the sets $\mathcal{X}(P, \Delta, r)$ and $\mathcal{X}(P, r)$ of Section 6.2. The sets $\mathcal{X}(\text{Link}(X^{\square^k}), \Delta(\square^j), r)$ and $\mathcal{X}(\text{Link}(X^{\square^k}), r)$ are a subsets of the (infinite) cone $C \text{Link}(X^{\square^k})$.

**Lemma 10.3.** We have the following properties

(i) If $\square^i \cap \square^j = \emptyset$ then $Z(X^{\square^i}) \cap Z(X^{\square^j}) = \emptyset$.

(ii) If $\square^k = \square^i \cap \square^j, 0 \leq k < i, j$, then $\mathbf{N}_{s, i} \left( X^{\square^i} \right) \cap \mathbf{N}_{s, j} \left( X^{\square^j} \right) \subset \mathbf{N}_{s, k} \left( X^{\square^k} \right)$.

(iii) If $\square^k = \square^i \cap \square^j, 0 \leq k < i, j$, then $Z(X^{\square^i}) \cap Z(X^{\square^j}) = \emptyset$.

(iv) If $\square^i \subset \square^k$ then we have (see Remark 2 before 10.1)

$$Z(X^{\square^i}) \cap Z(X^{\square^k}) \subset \mathring{X} \times \mathcal{X}(\text{Link}(X^{\square^k}), \Delta(\square^k), r)$$

(v) For $k < n - 1$ we have $Z \cap Z(X^{\square^k}) \subset \mathring{X}^{\square^k} \times \mathcal{X}(\text{Link}(X^{\square^k}), r)$.

**Proof.** Let $\square^i \cap \square^j = \emptyset$. Then the distance in $K_X$ from $X^{\square^i}$ to $X^{\square^j}$ is at least $2s_0$. This proves (1). Statement (ii) follows from Lemma 10.1, item (4) at the beginning of Section 8, and the following calculation for $l = i, j$ (see 6.2 for the definition of $s_{n,i}$ and $r_{n,i}$)

$$\frac{\sinh s_{n,j}}{\sinh r_{n,k}} = \left( \frac{\sinh r \sin \beta}{\sinh n \sin \beta} \right) = c \varsigma^{i-k} \leq c \varsigma < e^{-4} < \frac{\sqrt{2}}{2}$$

Statement (iii) follows from (ii) and the definition of the sets $Z$. We next prove (iv). Write $Z = Z(X^{\square^i}) \cap Z(X^{\square^k})$. By the definition of the sets $Z$ we have

$$Z = \mathring{\mathbf{N}}_{r, i} \left( X^{\square^i} \right) \cap \mathring{\mathbf{N}}_{r, k} \left( X^{\square^k} \right) - \bigcup_{i < k} \mathbf{N}_{r, i} \left( X^{\square^i} \right)$$

This together with Lemma 10.2 imply $Z \subset \mathring{X}^{\square^i} \times A$ where

$$A = \mathring{\mathbf{N}}_{s, k} \left( C \Delta(\square^k), C_{s, k} \left( \text{Link}(X^{\square^k}) \right) \right) - \bigcup_{i < k} \mathbf{N}_{s, i} \left( C \Delta(\square^i), C_{s, i} \left( \text{Link}(X^{\square^i}) \right) \right) - \mathbf{B}_{r, i} \left( \text{Link}(X^{\square^i}) \right)$$

hence

$$A \subset \mathring{\mathbf{N}}_{s, k} \left( C \Delta(\square^k), C_{s, k} \left( \text{Link}(X^{\square^k}) \right) \right) - \bigcup_{i < k} \mathbf{N}_{s, i} \left( C \Delta(\square^i), C_{s, i} \left( \text{Link}(X^{\square^i}) \right) \right) - \mathbf{B}_{r, i} \left( \text{Link}(X^{\square^i}) \right)$$

But for $i > j$ we have $s_{n,i} = s_{n-j,i-j}, r_{n,i} = r_{n-j,i-j}$ and $r_{n,j} = r_{n-j-3}$ (see definitions in 6.2). Therefore $A \subset \mathcal{X}(\text{Link}(X^{\square^k}), \Delta(\square^k), r)$. This proves (iv). The proof of (v) is similar to the proof of (iv) with minor changes. This proves Lemma 10.3.
We now smooth the metric $\sigma_{K_X}$. For each $\square \in K$ using the construction in Section 8 we get a Riemannian metric $\mathcal{G}(\text{Link}(X_{\square}), \mathcal{L}_{\square}, h, r, \xi, d, (c, \varsigma))$ on $\text{Link}(X_{\square})$, which we shall simply denote by $\mathcal{G}(\text{Link}(X_{\square}))$. Define the Riemannian metric $\mathcal{G}(X_{\square})$ on $\tilde{X}_{\square}$ by

$$\mathcal{G}(X_{\square}) = \mathcal{E}_{\tilde{X}_{\square}} \left( \mathcal{G}(\text{Link}(X_{\square})) \right)$$

**Remark.** Recall that we can consider $\tilde{X}_{\square} \times C\text{Link}(X_{\square})$ to get that the metrics coincide on the intersection $Z(X_{\square}) \cap Z(X_{\square}^k)$, $i, j < n - 1$. Also the Riemannian metric $\mathcal{G}(X_{\square}^k)$ coincides with $\sigma_{K_X}$ on $Z \cap Z(X_{\square}^k)$.

**Proposition 10.4.** The Riemannian metrics $\mathcal{G}(X_{\square})$ and $\mathcal{G}(X_{\square}^k)$ coincide on the intersection $Z(X_{\square}) \cap Z(X_{\square}^k)$, $i, j < n - 1$. And this follows from applying 6.1.3 locally. This proves the proposition.

Finally define the metric $\mathcal{G}(K_X) = \mathcal{G}(K_X, \mathcal{L}, r, \xi, d, (c, \varsigma))$ to be equal to $\mathcal{G}(X_{\square}^k)$ on $Z(X_{\square}^k)$, for $\square^k \in K$, $k < n - 1$. And equal to $\sigma_{K_X}$ on $Z$. By lemmas 10.1 and 10.4 the metric $\mathcal{G}(K_X)$ is a well-defined Riemannian metric on the smooth manifold $(K_X, \mathcal{S}_{K_X})$.

**Corollary 10.5.** Let $\epsilon > 0$ and $M^n$ closed. Choose $\xi, c, \varsigma$ satisfying (i) and (ii) in 8.4.2, and $\xi \geq n$. Then the metric $\mathcal{G}(K_X)$ has all sectional curvatures $\epsilon$-pinched to -1, provided $d_i, r - d_i, i = 2, \ldots, n$, are sufficiently large.

**Proof.** Choose $\epsilon'$, as in Remark 1.1(2), so that an $\epsilon'$-close to hyperbolic manifold with charts of excess $\xi$ has sectional curvatures $\epsilon$-pinched to -1. Take $A$ and $a$ so that $A \geq C(n, k; \xi)$ (see 2.7), and $a \geq a_k$, for $k \leq n$. Since $M$ is compact we only have finitely many cubes in a cubulation $K$ of $M$. Hence the set of links of $K$ (hence of $K_X$) is finite. This together with Proposition 8.4.2 imply that all $\mathcal{G}(X_{\square}^k), \square^k \in K$, $k < n - 1$, are $(\frac{\epsilon'}{n})$-close to hyperbolic outside the balls of radius $a$, and are hyperbolic on the ball of radius $2a$. All this provided $d_i, r - d_i, i = 2, \ldots, n$, are sufficiently large. We can apply Theorem 2.7 (locally, see remark below) to get that the metrics $\mathcal{G}(X_{\square}^k)$ are $\epsilon'$-close to hyperbolic outside $Z(X_{\square}^k) \cap N_a(X_{\square}^k)$, and hyperbolic on $Z(X_{\square}^k) \cap N_{2a}(X_{\square}^k)$. This proves the corollary.

The corollary proves (i) of the main Theorem. Items (ii), (iii) follow from [6]. Item (iv) follows from Proposition 9.4. This proves the Main Theorem.

**Remark.** Note that it does not make sense to say that $\mathcal{G}(X_{\square}^k)$ is $\epsilon'$-close to hyperbolic because neither
$\hat{X}_k$ nor $\hat{X}_k \times C\text{Link}(X_k)$ have a center. What we mean by the “local application of Theorem 2.7” mentioned in the proof above is the following. Take $p \in \mathcal{Z}(X_k)$ and let $B \subset \hat{X}_k$ be an open ball centered at $p$. Note that we can also consider $B \times C\text{Link}(X_k) \subset \mathbb{H}^{n-k} \times C\text{Link}(X_k) = \mathcal{E}_k(C\text{Link}(X_k))$ and we can now apply 2.7 to $\mathcal{E}_k(C\text{Link}(X_k))$, where we are considering $p$ as the center.

Section 11. Proof of Theorem A.

Let $N$ be a closed smooth manifold that bounds a compact smooth manifold $M^m$. Denote the given smooth structure of $N$ by $S_N$. Let $Q$ be the smooth $m$-manifold with one point singularity formed by gluing the cone $C_1N$ to $M$ along $N \subset M$. Let $q$ be the singularity of $Q$ and note that it is modeled on $C N$ (see 7.3). A triangulation of $Q$ is obtained by coning a smooth triangulation of the manifold with boundary $M$, and let $f : K \rightarrow Q$ be the induced cubulation (see appendix G). Write $f^{-1}(q) = p$. Note that $(K, f)$ is a smooth cubulation of $Q$ in the sense of Section 7.3. By item (2) of 7.3 we have that $Q - \{q\}$ has a normal smooth structure $S'$ for $K$, induced by a set of links smoothings $L$.

Let $K_X$ be the Charney-Davis strict hyperbolization of $K$. Also denote by $p$ the singularity of $K_X$. By item (1) of 9.4, the space $K_X - \{p\}$ has a normal smooth atlas $\{H_k\}_{k \in K}$ and normal smooth structure $S_{K_X}$. Moreover, since we are assuming $Wh(N) = 0$ (if $\dim N > 4$) we have that we can take the domain $C N - \{o_{C_N}\} = N \times (0, 1]$ of $H_p$ with product smooth structure $S_N \times S_{[0, 1]}$ (see 9.4).

We can now proceed exactly as in Section 10 and define the sets $\mathcal{Z}(X_k)$, $Z$, and the metrics $\mathcal{G}(X_k)$ depending on $L, r, \xi, d, (c, \varsigma)$. For the special case $\Box^0 = p$ we use the results in Section 8.5. We obtain in this way a Riemannian metric $\mathcal{G}(K_X) = \mathcal{G}(K_X, L, r, \xi, d, (c, \varsigma))$ on $K_X - \{p\}$. Theorem A and its addendum now follow from 8.5.1 (iii), (iv) and the result of Belegradek and Kapovitch [54] mentioned in the introduction (before the addendum to Theorem A). To be able to apply 8.5.1 we need to satisfy the hypothesis made at the beginning of 8.5: that the Whitehead group $Wh(\pi_1 N)$ vanishes. But this follows from [17]. This proves Theorem A.

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