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A new multifidelity method is developed for nonlinear orbit uncertainty propagation. This approach guarantees improved computational efficiency and limited accuracy losses compared with fully high-fidelity counterparts. The initial uncertainty is modeled as a weighted sum of Gaussian distributions whose number is adapted online to satisfy the required accuracy. As needed, univariate splitting libraries are used to split the mixture components along the direction of maximum nonlinearity. Differential Algebraic techniques are used to propagate these Gaussian kernels and compute a measure of nonlinearity required for the split decision and direction identification. Taylor expansions of the flow of the dynamics are computed using a low-fidelity dynamical model to maximize computational efficiency and corrected with selected high-fidelity samples to minimize accuracy losses. The effectiveness of the proposed method is demonstrated for different dynamical regimes combining SGP4 theory and numerical propagation as low- and high-fidelity models respectively.

I. Introduction

Accurate propagation of uncertainties is key for Space Situational Awareness (SSA) applications. Ranging from Space Surveillance and Tracking (SST) to Collision Avoidance Maneuver (CAM) planning, these activities require the estimation of the spacecraft state at a future epoch given the Initial Conditions (ICs) at an earlier time. These ICs are the result of ground-based observations or on-board Orbit Determination (OD) techniques, which are stochastic in nature. The propagation of a nominal state is thus not sufficient, but it is rather required to propagate the associated initial Probability Density Function (PDF) resulting from the estimation procedure. Given the intrinsic nonlinearity of the dynamics, no analytical solution exists to extrapolate the aforementioned PDF. On the contrary, several methods exist to approximate this quantity with increasing level of accuracy, usually at the expenses of higher computational efforts. Instead, this paper focuses on a novel multifidelity approach to tackle the Uncertainty Propagation (UP) problem in orbital dynamics which guarantees a similar accuracy of high-fidelity techniques with a reduced computational load.

Previous approaches tackled the UP problem in different ways, the most widely used being linear methods and sample-based techniques [1]. Linear methods build on the assumptions that a linearized dynamical model is sufficient to capture the dynamics of neighboring trajectories and that the uncertainty can be completely characterized by a multivariate Gaussian distribution. Under these assumptions, only the mean and the covariance matrix need to be
propagated, and the problem reduces to the integration of the State Transition Matrix (STM) known as Linear Covariance (LinCov) propagation. However, if the dynamics is highly nonlinear, this approach fails to accurately describe the time evolution of the state PDF. An extension of the LinCov method is that of State Transition Tensors (STTs), which are higher-order Taylor expansions of the dynamics about the nominal trajectory [2]. However, this method is not ideal for complex dynamics for high orders since it requires the derivation of analytical expression for the variational equations and their numerical integration along with the reference trajectory.

On the other side, Monte Carlo (MC) simulations are widely used in operational scenarios since they provide highly accurate estimation of the state uncertainty and are easy to implement. Yet their accuracy comes with a major computational cost, which makes MC techniques not applicable for real-time applications or maintenance of very large space objects catalogs. Sample based methods also include Unscented Transform (UT) and Conjugate Unscented Transform (CUT) [3–5], which are deterministic in nature as opposed to stochastic MC simulations. These methods are based on the idea that it is easier to approximate a state PDF than an arbitrary nonlinear transformation. Assuming a Gaussian distribution for the initial state uncertainty, they enforce the corresponding Moment Constraint Equations (MCEs) up to a given order to solve for a deterministic set of samples, denoted as sigma points, whose weighted statistics correctly capture the first statistical moments of the state PDF. To reduce the search space, these points are constrained to lie on carefully selected sets of axes, and a fully symmetric set is chosen to automatically satisfy the MCEs for any odd moment of the Gaussian distribution. If the UT can only capture the statistical moments up to third order, CUT sigma points sets have been obtained to match these moments up to ninth order. However, this approach suffers from a rapid increase in the number of samples needed to satisfy the MCEs for increasing orders and state dimension.

Polynomial Chaos Expansion (PCE) is another samples-based technique developed for UP [6]. This method aims at obtaining a functional representation of the propagated uncertainty with respect to the input random variables thus providing accurate information on higher-order statistical moments of the state PDF. Inputs and outputs of the considered transformation are represented by series approximations of standard random variables rewritten as series expansions of orthogonal polynomials for which the coefficients are sought. This technique requires a number of PCE terms which varies exponentially with both the order of the polynomial basis and the dimensions of the input random variables, leading to the course of dimensionality for large problems.

A new multifidelity technique based on Taylor polynomials is developed in this work to address the need for accurate and less computationally expensive methods for nonlinear Uncertainty Propagation in orbital dynamics. In this context, multifidelity techniques combine different fidelity force models to accurately propagate the state PDF at a reduced cost [6, 7]. With sample-based methods such as PCE, this approach requires the propagation of most samples in low-fidelity dynamics and the evaluation of the high-fidelity model on few carefully chosen points to correct the initial approximate solution. On the contrary, the proposed method relies on a Taylor expansion of the low-fidelity dynamics for which only the constant part needs to be propagated with the high-fidelity model.

Taylor algebra, or Differential Algebra (DA), provides the tools to compute the arbitrary-order derivatives of a function within a computer environment [8]. In the context of UP, DA is similar to STTs in the sense that it provides an expansion of the flow of the dynamics around the nominal state in the form of a Taylor polynomial of order $n$. This information has been successfully employed in previous work to map the initial state uncertainties through the nonlinear orbital dynamics [9]. However, if the transformation under consideration is strongly nonlinear, a single Taylor polynomial might fail to accurately describe the ODE flow in the domain of interest around the nominal solution. Automatic Domain Splitting (ADS) techniques were developed to overcome this issue [10]. The idea is to continuously monitor the accuracy of the Taylor expansion by estimating the magnitude of the truncated terms (terms of order $n+1$) and react consequently when the imposed error threshold is violated. When it happens, the single polynomial is split into two new expansions, each of them covering only half of the initial domain, so as to meet the specified accuracy. This algorithm requires an expansion order $\geq 3$ and its accuracy increases for increasing orders. In this work, a novel splitting algorithm specifically developed for second-order Taylor expansions and named LOADS (Low-Order Automatic Domain Splitting) is employed instead [11].

Gaussian Mixture Models (GMMs) can approximate an arbitrary PDF using a weighted sum of Gaussian distributions, and this approximation converges to the true PDF as the number of components approaches infinity. In orbit UP, algorithms have been developed to adapt online the number of GMM components required to correctly capture the entire state PDF as the assumption of Gaussianity becomes less accurate for longer propagation times or stronger nonlinearities [12, 13]. In this scenario, the initial uncertainty is described using a single Gaussian kernel and a metric is used throughout the propagation step to continuously monitor the error between linearized and fully nonlinear models. When the specified threshold is violated, the propagation is halted to split the single kernel into components and then resumed on the newly obtained kernels. Hybrid methods have been proposed that combine adaptive GMMs with Extended
Kalman Filter (EKF), Unscented Kalman Filter (UKF) [12] and PCEs [13], but no attempt has been made to split the GMM components using the information coded in a Taylor expansion of the dynamics around the nominal solution.

In this context, a novel multifidelity (MF) method that combines DA-based propagation and splitting of GMMs in low-fidelity (LF) dynamics with point-wise (PW) propagation of the polynomials centers in high-fidelity (HF) one is presented. In the LF step, a second-order Taylor expansion of the flow is used to compute the nonlinearity index $\nu$. When its magnitude crosses a given threshold, the polynomial is split into three new expansions such that the assumption of a quasi-linear transformation still holds in each subdomain. Assuming a Gaussian distribution for the initial state, the IC is constructed such that the former distribution is correctly captured within the domain of validity of its Taylor polynomial. As the propagation steps forward, more domains are generated by the LOADS algorithm, each of them corresponding to a new GMM component such that their weighted sum correctly captures the true non-Gaussian state PDF. Once the DA-based propagation has ended, evaluation of the resulting polynomials on UT sigma-points is used to map the statistics of the initial components to final time. The HF step is then simply a PW propagation of the means of these Gaussian kernels for the same time span. The resulting final states will constitute the polynomials centers and components means of the updated MF solution.

The paper is organized as follows. After a short description of the DA formalism in Section II, the LOADS algorithm is described in Section II.A. GMMs are then introduced in Section II.B. Section II.B.1 describes the univariate splitting libraries for standard normal distributions while Section II.B.2 discusses their extension to the multivariate case. Section II.B.3 presents the combined LOADS-GMM algorithm for UP that constitutes the LF step of the proposed method. The MF approach is then described in Section II.C. Three numerical applications to the problem of orbit UP are then included in Section III and the accuracy of the MF method compared against reference MC simulations. The importance of the HF correction and the computational efficiency of the overall method are then assessed in Sections III.D and III.E respectively. Conclusions are finally drawn in Section IV.

**II. Uncertainty Propagation**

Consider the generic nonlinear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$

$$y = f(x)$$ (1)

with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ multivariate random variables with PDFs $p(x)$ and $p(y)$ respectively. The problem of nonlinear UP consists in estimating $p(y)$ from the only knowledge of $p(x)$ and $f$. In DA framework, $x$ is expressed as a function of its nominal value and associated uncertainty as

$$[x] = \bar{x} + \beta \cdot \delta x$$ (2)

with $\bar{x}$ nominal value of $x$, $\beta$ Confidence Interval (CI) semi-amplitude and $\delta x$ first-order variations around $\bar{x}$. Evaluating $f$ in the DA framework results in the Taylor expansion of $y$ in terms of $\delta x$ expressed as

$$[y] = f([y]) = T_y(\delta x)$$ (3)

The Taylor series in Eq. (3) can then be used to efficiently compute the propagated statistics as described in Section II.B.

**A. Low-Order Automatic Domain Splitting**

The Low-Order Automatic Domain Splitting (LOADS) algorithm is a DA-based method that aims at solving the UP problem by splitting the uncertainty set into a manifold of second-order expansions [11]. The splitting decision and direction are determined by a proper measure of nonlinearity that is derived by the analysis of the Taylor expansion of the Jacobian of the transformation. This nonlinearity index, here defined $\nu$, represents a reformulation in DA sense of the parameter introduced in [14] as a measure of nonlinearity.

Following the description given in Losacco et al. [11], consider the generic transformation $f$ and a second order expansion for $[x]$. If the transformation is linear, the Taylor expansion of the Jacobian of the transformation shall coincide with its constant part, i.e. $[J] = \bar{\mathbf{J}}$. On the contrary, if the transformation is nonlinear, each term of the Jacobian $[J_{ij}]$ will be a first order Taylor expansion in the deviations $\delta x$, i.e.

$$[J_{ij}] = \bar{J}_{ij} + \delta J_{ij} = T_{\bar{J}_{ij}}(\delta x)$$ (4)
Consider now the non-constant part of the Jacobian matrix elements, \( \delta I_{ij} \). Each element can be expressed as a linear combination of the variables \( \delta x_j \), i.e.

\[
\delta I_{ij} = T_{\delta I_{ij}} = \sum_{k=1}^{n} a_{ij,k} \delta x_k
\]  

as a result, a rigorous estimation of the variation range of each term \( \delta I_{ij} \) can be obtained as

\[
\begin{align*}
 b_{\delta I_{ij}} &= \sum_{k=1}^{n} |a_{ij,k}| \\
 \delta I_{ij} &\in [-b_{\delta I_{ij}}, b_{\delta I_{ij}}]
\end{align*}
\]

The idea of the LOADS algorithm is to use these bounds as a measure of nonlinearity of the transformation. More specifically, the nonlinearity index is expressed as

\[
\nu = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} b_{\delta I_{ij}}}{\sum_{i=1}^{m} \sum_{j=1}^{n} J_{ij}} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij,k}|}{\sum_{i=1}^{m} \sum_{j=1}^{n} J_{ij}}
\]  

Therefore, the index \( \nu \) is obtained as the ratio between the entry-wise 1-norm of the matrix of the bounds and the entry-wise 1-norm of the constant part of the Jacobian. In case of linear transformation, all elements \( b_{\delta I_{ij}} \) are identically equal to zero, so \( \nu = 0 \). As the transformation departs from linearity, the numerator increases, thus a hint on the nonlinear nature of the investigated problem is obtained.

In case nonlinearities are detected, the LOADS algorithm proceeds by splitting the uncertainty set along a determined direction \( k \). The identification of the splitting direction is done with the same procedure. The only element that changes is the processed Jacobian. More specifically, given an investigated splitting direction \( e \) and the associated variable \( \delta x_e \), defining as \( \delta x_e \) the vector

\[
\delta x_e = \{0, \ldots, 0, \delta x_e, 0, \ldots, 0\}^T \quad e \in [1, \ldots, n]
\]

a directional Jacobian can be built as the composition of \([J(\delta x)]\) with \( \delta x_e \), i.e.

\[
[J|_e] = T_{\delta x_e} \circ \delta x_e = T_{J|_e}(\delta x_e)
\]

The directional nonlinearity index can be then computed as

\[
\nu_e = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} b_{\delta I_{ij}|_e}}{\sum_{i=1}^{m} \sum_{j=1}^{n} J_{ij}}
\]

The splitting direction \( k \) is then selected as the one corresponding to the maximum directional index, i.e.

\[
k : \nu_k = \max_e \{\nu_e\}
\]

Once the splitting direction has been determined, the current set \([x]\) can be split according to a selected splitting library as described in Section II.B generating a list of subsets \([x^{(p)}]\) whose size is determined by the library itself.

The described approach is repeated taking into account some imposed constraints, i.e. a maximum number of splits \( N_{\text{max}} \) and a maximum nonlinearity level \( e\). The algorithm is initialized by considering the initial domain \([x]\) and adding it to a list of sets \( S \) to be processed. The algorithm creates the so-called manifold of sets \( M \) that satisfy the imposed constraints. The steps are the following

1) Consider the first set \([x^{(p)}]\) of \( S \) and remove it from the list.
2) Apply the transformation \([x^{(p)}] = f([x^{(p)}])\).
3) Compute the nonlinearity index $\psi(p)$: if $\psi(p) < \varepsilon$, store $[y^{(p)}]$ into $M$ and go to last step, otherwise proceed.
4) Compute the splitting direction $k$.
5) Check if a split can be performed: if $N = N_{\text{max}}$ store $[y^{(p)}]$ into $M$ and go to last step, otherwise proceed.
6) Split $[x^{(p)}]$ thus generating $[x^{(i)}]$ for $i = 1, \ldots, N_{\text{split}}$ and store them into $S$.
7) Check $S$: if not empty, go back to step 1 and proceed, otherwise stop.

A more exhaustive analysis of the described algorithm can be found in [11], where the LOADS algorithm is applied to the investigated case of orbit uncertainty propagation, and a detailed analysis of the role played by the different control parameters is offered.

B. Gaussian Mixture Model

Consider a multivariate random variable $x \in \mathbb{R}^n$ with PDF $p(x)$. The probability density function $p(x)$ has no general closed-form solution and might be approximated using different techniques. However, if $x$ is Gaussian with mean $\mu \in \mathbb{R}^n$ and covariance $P \in \mathbb{R}^{n \times n}$, its PDF has the well-known analytical expression

$$p(x; \mu, P) = \frac{1}{\sqrt{2\pi|P|}} \cdot \exp\left(-\frac{1}{2}(x - \mu)^T P^{-1} (x - \mu)\right) \tag{12}$$

where $| \cdot |$ represents the matrix determinant.

GMMS leverage this property to obtain analytic approximations of arbitrary PDFs as weighted sums of Gaussian distributions. For a generic random variable $x$, its PDF is thus expressed as

$$p(x) \approx \sum_{i=1}^{L} \alpha_i p(x; \mu_i, P_i) \tag{13}$$

with $L$ total number of components and $\alpha_i$ mixing proportions or weights subject to

$$\alpha_i \geq 0 \quad \forall i \in \{1, \ldots, L\} \tag{14a}$$

$$\sum_{i=1}^{L} \alpha_i = 1 \tag{14b}$$

If $x$ is subject to a nonlinear transformation such as Eq. (1), the number of components $L$ that guarantees an accurate representation of the input PDF $p(x)$ might fail to correctly capture the shape of the transformed PDF $p(y)$. A splitting algorithm is thus employed to dynamically adapt this number so that $p(y)$ is approximated with the desired accuracy.

1. Univariate Splitting Library

Gaussian distributions remain Gaussian under linear transformations. Moreover, if $x \sim \mathcal{N}(\mu, P)$ then $p(x) > 0 \forall x \in \mathbb{R}^n$. In practice $p(x)$ can however be considered zero outside a CI around its mean $\mu$. For univariate PDFs, a typical value for its semi-amplitude is that of $3\sigma$ which accounts for $\approx 99.7\%$ of all realizations. In this case $p(x)$ is given by Eq. (12) for $x \in [\mu - 3\sigma, \mu + 3\sigma]$ and equal to zero otherwise. In this sense, the splitting procedure increases the number of components in Eq. (13) and reduces their covariance matrices $P_i$ so that the assumption of a linear transformation holds in the neighborhood of $\mu_i$ in which $p(x; \mu_i, P_i) > 0$. This condition must be verified separately for each component resulting in the recursive algorithm described below.

Given a single Gaussian kernel $p(x; \mu_i, P_i)$ with weight $\alpha_i$ for which a split is deemed necessary, a univariate splitting library is used to perform this operation in the direction defined by the $k^{th}$ eigenvector of $P_i$ [12]. In this work, the splitting direction $k$ is determined by the LOADS algorithm described in Section II.A. Moreover, the univariate splitting library is computed offline and reused as many time as necessary during the splitting procedure.

Consider the univariate standard Gaussian distribution $x \sim \mathcal{N}(0, 1)$ with PDF

$$p(x) = p(x; 0, 1) \tag{15}$$

for the purpose of computing a univariate splitting library, it is desired to approximate Eq. (15) with a GMM such that

$$p(x) \approx \tilde{p}(x) = \sum_{i=0}^{L-1} \tilde{\alpha}_i p(x; \mu_i, \sigma_i^2) \tag{16}$$
with \( L \) number of components and \( \tilde{\alpha}_i, \tilde{\mu}_i, \tilde{\sigma}^2_i \) mixture weights, means and variances respectively. After selecting the number of components \( L \), a minimization problem is formulated to determine the remaining \( 3L \) parameters. The number of free variables can be then further reduced constraining the GMM to be homoscedastic, i.e.

\[
\tilde{\alpha}_i^2 = \tilde{\sigma}^2 \quad \forall i \in \{0, \ldots, L - 1\} \tag{17}
\]

and taking advantage of the symmetric shape of \( p(x) \) about the mean. Equation (16) is thus rewritten as

\[
\tilde{p}(x) = \tilde{\alpha}_0 \cdot p(x; 0, \tilde{\sigma}^2) + \sum_{i=1}^{L} \tilde{\alpha}_i \cdot \left[p(x; -\tilde{\mu}_i, \tilde{\sigma}^2) + p(x; \tilde{\mu}_i, \tilde{\sigma}^2) \right] \tag{18}
\]

with \( l = \lfloor L/2 \rfloor \) and \( \tilde{\alpha}_0 = 0 \) if \( L \) is even so that only \( L + 1 \) parameters remain to be determined.

To solve for the optimal \( \tilde{\alpha}_i, \tilde{\mu}_i \) and \( \tilde{\sigma}^2 \) a performance index \( J \) is developed starting from the \( L_2 \) distance between PDFs defined as

\[
L_2(p_1, p_2) = \int_{\mathbb{R}^n} (p_1(x) - p_2(x))^2 \, dx \tag{19}
\]

The \( L_2 \) distance is preferred here instead of other measures since an analytical expression for Eq. (19) exists if both \( p_1, p_2 \) are approximated by GMMs as

\[
p_1(x) = \sum_{i=1}^{L_1} \alpha_{1,i} \cdot p(x; \mu_{1,i}, \Sigma_{1,i}) \tag{20a}
\]

\[
p_2(x) = \sum_{j=1}^{L_2} \alpha_{2,j} \cdot p(x; \mu_{2,j}, \Sigma_{2,j}) \tag{20b}
\]

substituting Eq. (20) into Eq. (19) leads in fact to the following expression for \( L_2 \)

\[
L_2(p_1, p_2) = d_{1,1} + d_{2,2} - 2 \cdot d_{1,2} \tag{21}
\]

with

\[
d_{m,n} = \sum_{i=1}^{L_m} \sum_{j=1}^{L_n} \alpha_{m,i} \alpha_{n,j} K(\mu_{m,i}, \mu_{n,j}, \Sigma_{m,i} + \Sigma_{n,j}) \quad m, n = 1, 2 \tag{22}
\]

and \( K \) defined as

\[
K(\mu_1, \mu_2, \Sigma_1 + \Sigma_2) = \frac{1}{\sqrt{2\pi (\Sigma_1 + \Sigma_2)}} \cdot \exp \left\{-\frac{1}{2} (\mu_1 - \mu_2)^T (\Sigma_1 + \Sigma_2)^{-1} (\mu_1 - \mu_2) \right\} \tag{23}
\]

If \( p_1 = p(x; 0, 1) \) and \( p_2 = \tilde{p}(x) \) as given by Eq. (18), Eq. (22) further simplifies into

\[
L_2(p, \tilde{p}) = \frac{1}{2\sqrt{\pi}} - \frac{2\tilde{\alpha}_0}{\sqrt{2\pi(1 + \tilde{\sigma}^2)}} + \frac{1}{2\sqrt{\pi\tilde{\sigma}^2}} \left[ \tilde{\alpha}_0^2 + 4\tilde{\alpha}_0 \sum_{i=1}^{l} \tilde{\alpha}_i \exp \left\{-\frac{\tilde{\mu}_i^2}{4\tilde{\sigma}^2} \right\} \right] \tag{24}
\]
The nonlinear optimization problem is finally stated as follows.

Minimize

\[ J = L_2(p, \bar{p}) + \lambda \tilde{\sigma}^2 \]

subject to the single equality constraint

\[ 0 = 1 - \tilde{\alpha}_0 - 2 \sum_{i=1}^{l} \tilde{\alpha}_i \]

and inequality constraints

\[ 0 < \tilde{\alpha}_i < \tilde{\alpha}_{i-1} \quad i = 2, \ldots, l \]  \hspace{1cm} (27a)
\[ 0 < \tilde{\mu}_{i-1} < \tilde{\mu}_i \quad i = 2, \ldots, l \]  \hspace{1cm} (27b)

where \( \lambda > 0 \) is a penalty factor that scales the importance of minimizing \( \tilde{\sigma}^2 \) with respect to \( L_2(p, \bar{p}) \). The resulting Nonlinear Programming Problem (NLP) is then solved numerically using the interior point optimizer IPOPT [15] compiled against the HSL Mathematical Software Library for the solution of sparse linear systems [16]. The optimal solution for \( L = 3 \) and \( \lambda = 10^{-3} \) is given in Table 1.

| \( i \) | \( \tilde{\alpha}_i \) | \( \tilde{\mu}_i \) | \( \tilde{\sigma} \)  |
|-------|----------------|----------------|-----------------|
| 1     | 0.2252246852539708 | -1.0575150485760967 | 0.6715664864669252 |
| 2     | 0.5495506294920584 | 0.0              | 0.6715664864669252 |
| 3     | 0.2252246852539708 | 1.0575150485760967  | 0.6715664864669252 |

Table 1  Univariate splitting library for \( L = 3 \) and \( \lambda = 10^{-3} \)

2. Splitting Multivariate Distributions

In the multivariate case, the univariate splitting library is applied for each component in the direction identified by the \( k^{th} \) eigenvector of its covariance matrix \( P \). Without loss of generality, consider a single kernel of Eq. (13) with weight, mean and covariance \( \alpha, \mu, P \) respectively. The spectral factorization of \( P \) is then obtained as

\[ P = V \Lambda V^T \]  \hspace{1cm} (28)

where \( V \) is an orthogonal matrix of right eigenvectors of \( P \) and \( \Lambda \) a diagonal matrix of corresponding eigenvalues. Performing the split along the \( k^{th} \) axis of the spectral factorization results in the following weights, means and covariance for the \( L \) new GMM components

\[ \alpha_i = \tilde{\alpha}_i \alpha \]  \hspace{1cm} (29a)
\[ \mu_i = \mu + \sqrt{\lambda_k} \tilde{\mu}_i v_k \]  \hspace{1cm} (29b)
\[ P_i = V \Lambda_i V^T \]  \hspace{1cm} (29c)

with \( v_k \) \( k^{th} \) column of \( V \), \( \lambda_k \) corresponding eigenvalue and \( \Lambda_i \) defined as

\[ \Lambda_i = \begin{bmatrix} \lambda_1 \\
\vdots 
\tilde{\sigma}^2 \lambda_k \\
\vdots 
\lambda_n \end{bmatrix} \]  \hspace{1cm} (30)
3. Combined LOADS-GMM algorithm

In this work, a univariate splitting library such as the one in Table 1 is embedded within the LOADS framework and the nonlinearity indexes \( \nu, \nu_e \) defined in Eqs. (7) and (10) are used to identify the need for new splits and the corresponding directions. Moreover, an additional constraint is taken into account to bound the components weights, i.e. \( \alpha^{(p)} \geq \alpha_{\min} \forall p \). Given the nonlinear transformation in Eq. (1) the developed algorithm is as follows.

Firstly, the initial uncertainty on the function input \( x \) is modeled with a multivariate Gaussian distribution with known mean \( \mu_0 \) and covariance \( P_0 \). A CI is then chosen and the input DA state is initialized as

\[
[x] = \mu_0 + cV_0\sqrt{L_0} \cdot \delta x = \bar{x}_0 + \beta_0 \cdot \delta x
\]  

(31)

with \( \lambda, V \) eigenvalues and eigenvectors of \( P_0 \) such that \( P_0 = V_0 \lambda_0 V_0^T \). A single GMM component with \( \alpha = 1, \mu_0, P_0 \) is then associated to \( [x] \).

Secondly, a univariate splitting library is selected and the steps of the iterative LOADS algorithm described in Section II.A reformulated as follows

1) Initialize the list of sets \( S \) to be processed with \( [x^{(1)}] \) given by Eq. (31).
2) Consider the first set \( [x^{(p)}] \) of \( S \) and remove it from the list.
3) Apply the nonlinear transformation \( [y^{(p)}] = f([x^{(p)}]) \).
4) Check if a split can be performed: if \( \alpha^{(p)} < \alpha_{\min} \) store \( [y^{(p)}] \) into \( M \) and go to last step, otherwise proceed.
5) Compute the nonlinearity index \( \nu \) as in Eq. (7) with the Jacobian of the transformation \( [J^{(p)}] \) obtained as

\[
[J^{(p)}] = \frac{\partial f_i}{c \Lambda^{(p)}(i) \delta x_j} \quad i = 1, \ldots, m \quad j = 1, \ldots, n
\]  

(32)

if \( y^{(p)} < \nu_{\chi} \) store \( [y^{(p)}] \) into \( M \) and go to last step, otherwise proceed.
6) Compute the splitting direction \( k \) with the directional Jacobian obtained as

\[
[J^{(p)}] = [J^{(p)}] \circ \delta x_e
\]  

(33)

with \( [J^{(p)}], \delta x_e \) given by Eqs. (8) and (32).
7) Check if a split can be performed: if \( N = N_{\text{max}} \) store \( [y^{(p)}] \) into \( M \) and go to last step, otherwise proceed.
8) Split \( [x^{(p)}] \) and the associated GMM component according to the selected library

\[
[x^{(i)}] = [x^{(p)}] \circ \left\{ \bar{x}_0 + \bar{\sigma} \delta x_k, \ldots, 0 \right\}^T
\]  

(34)

\[
\alpha^{(i)} = \tilde{\alpha}_i \alpha^{(p)}
\]  

(35a)

\[
\mu^{(i)} = \mu^{(p)} + \sqrt{\Lambda_k^{(p)} \bar{\mu}_k^{(p)}}
\]  

(35b)

\[
P^{(i)} = \sqrt{\Lambda_k^{(p)} \bar{V}_k^{(p)}}
\]  

(35c)

with \( \bar{\Lambda}_k^{(p)} \) \( k \)th eigenvector of \( P^{(p)} \), \( \bar{\Lambda}_k^{(p)} \) corresponding eigenvalue and \( \Lambda_k^{(p)} \) as in Eq. (30).
9) Store the resulting \( [x^{(i)}] \) for \( i = 1, \ldots, L \) in \( S \).
10) Check \( S \), if not empty, go back to step 2, otherwise stop.

Once the iterative procedure has ended, the initial Gaussian PDF \( p(x) \sim N(\mu_0, P_0) \) is approximated by a multivariate GMM whose components weights, means and covariances are the output of the LOADS algorithm, namely

\[
p(x) \approx \sum_{p=1}^{P} \alpha^{(p)} p \left(x; \mu^{(p)}, P^{(p)}\right)
\]  

(36)

with \( P \) dimension of the output manifold \( M \). The transformed PDF \( p(y) \) is then estimated as follows.
Firstly, a sampling scheme is chosen to draw samples from each component of the initial distribution in Eq. (36). Four schemes, one stochastic and three deterministic have been tested with similar results [3, 5]

1) Monte Carlo (MC) sampling
2) Unscented Transform (UT)
3) Fourth-Order Conjugate Unscented Transform (CUT4)
4) Sixth-Order Conjugate Unscented Transform (CUT6)

Secondly, each GMM component is sampled according to its mean $\mu^{(p)}$ and covariance $P^{(p)}$. The drawn samples are then mapped through $f$ evaluating the corresponding Taylor expansion of the output $[y^{(p)}] = f([x^{(p)}])$ stored in $M$. The first two moments of the transformed PDF are then obtained from the mapped samples as

$$
\mu^{(p)}_y = \frac{1}{N_s} \sum_{s=0}^{N_s-1} y^{(p)}_s
$$

(37a)

$$
P^{(p)}_{y,ij} = \frac{1}{N_s-1} \sum_{s=0}^{N_s-1} \left( [y^{(p)}_s - \mu^{(p)}_i] [y^{(p)}_s - \mu^{(p)}_j] \right) 
\quad i, j = 1, \ldots, m
$$

(37b)

for MC samples and

$$
\mu^{(p)}_y = \sum_{s=0}^{N_s-1} w_s y^{(p)}_s
$$

(38a)

$$
P^{(p)}_y = \sum_{s=0}^{N_s-1} w_s \left[ y^{(p)}_s - \mu^{(p)} \right] \left[ y^{(p)}_s - \mu^{(p)} \right]^T
$$

(38b)

for UT and CUT samples. Here $N_s$ is the total number of samples and $w_i$ the associated weights.

The transformed PDF is finally estimated as

$$
p(y) \approx \sum_{p=1}^{P} \alpha^{(p)} \rho \left( y; \mu^{(p)}_y, P^{(p)}_y \right)
$$

(39)

with $\mu^{(p)}_y, P^{(p)}_y$ given by Eq. (37) or Eq. (38).

4. Likelihood Agreement Measure between Distributions

The Likelihood Agreement Measure (LAM) is introduced as a measure for the accuracy of the GMM representation of the transformed PDF. Given two multivariate PDFs, the LAM is defined as [12]

$$
\mathcal{L} (p, q) = \int_{\mathbb{R}^n} p(x) q(x) dx
$$

(40)

and describes the amount of overlap between the two distributions, being larger for $p, q$ in greater agreement between each other.

If the agreement of a GMM with respect to samples resulting from a MC simulation is of interest, Eq. (40) can be reformulated treating the MC samples as a Dirac Mixture Model (DMM) as

$$
\mathcal{L} (p, q) = \sum_{i=1}^{N_s} \sum_{j=1}^{L} \gamma_i \alpha_j \rho (x_i; \mu_j, P_j)
$$

(41)

with $N_s, L$ number of MC samples and Gaussian kernels respectively. Moreover, $\alpha_j, \mu_j, P_j$ are the GMM components weights, means and covariances while $x_i, \gamma_i$ with $\gamma_i = 1/N_s \forall i$ are the propagated random samples and corresponding weights.
C. Multifidelity Orbit Uncertainty Propagation

The combined LOADS-GMM algorithm presented in Section II.B can be applied to any nonlinear transformation $f$. In the followings, its application within a multifidelity (MF) method for UP in orbital dynamics is presented. The MF method developed in this work comprises two steps, a low-fidelity (LF) propagation carried out in DA framework and an high-fidelity (HF) one executed point-wise (PW) for a posteriori correction of the LF step. The Simplified General Perturbations 4 (SGP4) model is chosen for the LF dynamics since its analytical formulation allows a straightforward integration with the LOADS-GMM algorithm. An accurate numerical propagator is then used for the HF correction. The MF orbit UP thus comprises the following steps.

1) Model the initial uncertainty on the orbit state as a multivariate Gaussian distribution

$$ p(x(t_0)) \sim N(\mu(t_0), P(t_0)) \tag{42} $$

2) Run the LOADS algorithm with $[x_L^{(1)}(t_0)]$ obtained from $\mu(t_0), P(t_0)$ as in Eq. (31) with $f$ given by SGP4

$$ [x_L^{(p)}(t)] = f([x_L^{(p)}(t_0)]) \quad \forall p = 1, \ldots, P \tag{43} $$

3) Approximate the initial PDF as in Eq. (36)

$$ p(x(t_0)) \approx \sum_{p=1}^{P} a^{(p)} p \left( x; \mu_L^{(p)}(t_0), P_L^{(p)}(t_0) \right) \tag{44} $$

4) Propagate the initial means $\mu_L^{(p)}(t_0)$ in HF to obtain an accurate representation of the nominal trajectories

$$ \mu_L^{(p)}(t_0) \mapsto \mu_H^{(p)}(t) \tag{45} $$

5) Separate the constant part and nilpotent part of the Taylor expansions $[x_L^{(p)}(t)]$

$$ [x_L^{(p)}(t)] = \bar{x}_L^{(p)}(t) + \mathcal{T}_{\delta x_L^{(p)}(t)}(t_0, \delta x_0) \tag{46} $$

6) Substitute $\bar{x}_L^{(p)}(t)$ in Eq. (46) with $\mu_H^{(p)}(t)$

$$ [x_M^{(p)}(t)] = \mu_H^{(p)}(t) + \mathcal{T}_{\delta x_L^{(p)}(t)}(t_0, \delta x_0) \tag{47} $$

7) Estimate the transformed PDF from $[x_M^{(p)}(t)]$ as in Eqs. (36) and (39)

$$ p(x(t)) \approx \sum_{p=1}^{P} a^{(p)} p \left( x; \mu_M^{(p)}(t), P_M^{(p)}(t) \right) \tag{48} $$

Three numerical applications of the above method are included in Section III.

III. Numerical Results

Three applications of the multifidelity method for orbit Uncertainty Propagation are presented in this section. The analytical SGP4 model is used for the LF dynamics while the HF propagation is carried out numerically taking into account the following force models

1) Earth non-uniform gravity field with $8 \times 8$ spherical harmonics potential
2) Sun and Moon third-body forces with bodies positions from JPL ephemerides
3) Isotropic drag force with modified Harris-Priester static atmosphere model
4) Isotropic Solar Radiation Pressure (SRP) including Earth’s umbra and penumbra transitions

The propagators and force models used in this work are provided by the open-source library Orekit [17].

Forces parameters used throughout all simulations are summarized in Table 2. Moreover, the Equations of Motion are integrated in modified equinoctial elements and the initial epoch time is set on January 1st, 2021 00:00:00 UTC.
| Parameter                                      | Symbol | Value                  |
|-----------------------------------------------|--------|------------------------|
| Earth’s standard gravitational parameter      | $\mu_E$ | 398 600.4355 km$^3$ s$^{-2}$ |
| Satellite mass                                | $m$    | 500.0 kg               |
| Satellite cross section area                  | $A_R$  | 1.0 m$^2$              |
| Atmospheric drag coefficient                  | $C_D$  | 2.0                    |
| SRP reflection coefficient                    | $C_R$  | 1.5                    |

**Table 2** Force models parameters

The first two simulations are adapted from [12] and include an High Earth Orbit and a Low earth Orbit (LEO) cases. The last scenario is a Medium Earth Orbit (MEO) orbit for the Galileo constellation. Nominal Initial Conditions (ICs) are summarized in Table 3. The corresponding initial uncertainties are assumed Gaussian with $1\sigma$ deviations in cartesian parameters given in Table 4.

| a, km | e     | i, deg | $\Omega$, deg | $\omega$, deg | $M$, deg |
|-------|-------|--------|---------------|---------------|----------|
| High Earth Orbit | 35 000.0 | 0.2    | 0.0           | 0.0           | 0.0      |
| Low earth Orbit   | 6603.0   | 0.01   | 0.0           | 0.0           | 0.0      |
| Medium Earth Orbit| 29 600.135 | 0.0    | 56.0          | 0.0           | 0.0      |

**Table 3** Nominal Initial Conditions for all test cases

| $\sigma_x$, km | $\sigma_y$, km | $\sigma_z$, km | $\sigma_{v_x}$, m s$^{-1}$ | $\sigma_{v_y}$, m s$^{-1}$ | $\sigma_{v_z}$, m s$^{-1}$ |
|----------------|----------------|----------------|-----------------------------|-----------------------------|-----------------------------|
| High Earth Orbit | 1.0            | 1.0            | 0.0                         | 1.0                         | 1.0                         |
| Low earth Orbit   | 1.3            | 0.5            | 0.0                         | 2.5                         | 5.0                         |
| Medium Earth Orbit | 0.5            | 1.0            | 1.0                         | 0.5                         | 0.5                         |

**Table 4** Initial uncertainties for all test cases

The selected splitting library is that of Table 1 with $L = 3$ so that each split generates three new GMM components. The Confidence Interval is also set to $c = 3$. The maximum number of splits and minimum components weight are chosen as $N_{\text{max}} = 20$ and $\alpha_{\text{min}} = 10^{-3}$ respectively so that the imposed nonlinearity threshold $\varepsilon_r$ is always satisfied. The last is set to $\varepsilon_r = 0.01$ for the High Earth Orbit and MEO cases and to $\varepsilon_r = 0.025$ for the LEO one.

The three ICs are propagated for two revolutions of the corresponding nominal orbit and the obtained results presented in Sections III.A to III.C. Moreover, the impact of $\alpha_{\text{min}}$ on the overall accuracy of the MF method is assessed in Section III.D.

A. High Earth Orbit

Results for the first test case are displayed in Fig. 1. Contour lines for the estimated PDF and reference MC samples obtained with the HF dynamical model are plotted on the same figures to visually assess the accuracy of the proposed approach. Fig. 1a shows a projection of the propagated uncertainty in position onto the $x - y$ plane while Fig. 1b represents the uncertainty in velocity on the same plane. Values of the estimated PDF are normalized with respect to their maximum located in correspondence of the nominal solution. The final GMM includes 6561 components to accurately represent the strongly non-Gaussian PDF as demonstrated by the contour lines following the curvature of the reference MC samples.
Fig. 1  Monte Carlo samples and mixture PDF after two revolutions for High Earth Orbit test case

The higher accuracy guaranteed by the MF approach compared to the LF counterpart is assessed in Fig. 2 in which a single contour line is plotted for both solutions. The contribution of the HF step is clearly seen here as a translation of the corrected mean and estimated PDF with respect to the first LF solution.

Fig. 2  Low-fidelity and multifidelity PDF contours for High Earth Orbit test case

B. Low earth Orbit

The second scenario is the equatorial LEO orbit for which the estimated PDF is displayed in Fig. 3. Figure 3a shows a projection of the uncertainty in position onto the $x - y$ plane while Fig. 3b a projection of the corresponding uncertainty in velocity onto the same plane. As before, the effects of nonlinearity are visible in the curvature of the initially Gaussian MC samples set, closely followed by the contour lines of the estimated PDF. Despite the higher nonlinearity threshold $\epsilon_v = 0.025$ and the lower eccentricity with respect to the previous case, 2187 kernels are needed to approximate the transformed PDF mainly due to the lower orbit altitude and the consequent atmospheric drag.

A contour plot for comparison with the purely LF method is offered in Fig. 4 in which a single contour line and the weighted GMM means are plotted for both LF and MF solutions. Even if less marked than in Fig. 2, the contribution of the HF step is still recognized as a translation of the corrected PDF with respect to the LF estimation thus corroborating the results presented above.
Fig. 3 Monte Carlo samples and mixture PDF after two revolutions for Low earth Orbit test case

Fig. 4 Low-fidelity and multifidelity PDF contours for Low earth Orbit test case

C. Medium Earth Orbit
The last test case is the circular, inclined orbit of the Galileo constellation. The propagated uncertainties are shown in Fig. 5 together with the reference MC samples. Figure 5a represents the uncertainty in position projected on the \( x - z \) plane while Fig. 5b the corresponding uncertainty in velocity. In this case, 729 GMM components are needed to model the propagated PDF. The great reduction in the number of kernels with respect to the previous cases is mainly due to the lower eccentricity (compared to the High Earth Orbit case), the longer orbital period and the absence of atmospheric drag (compared to the LEO case). The lower number of components is thus a hint of the weaker nonlinearity of the dynamics in this scenario demonstrated by the shape of the contour lines in Fig. 5b. This behavior also highlights the advantages of an adaptive splitting scheme with respect to more conventional GMMs with fixed number of components: the final number of Gaussian kernels is guaranteed to be the minimum required to satisfy the imposed nonlinearity threshold.

D. Sensitivity with respect to Minimum GMM Component Weight
Since the number of GMM components output of the MF method are bounded by the maximum allowed number of splits \( N_{\text{max}} \) and the minimum kernel weight \( \alpha_{\text{min}} \), it is of interest to analyze the impact of \( \alpha_{\text{min}} \) on the accuracy of the estimated PDF. For each test case, multiple solutions have been obtained for different values of \( \alpha_{\text{min}} \in [1, 10^{-8}] \) and the corresponding LAM with respect to the same set of MC samples computed using Eq. (41). These values have been then normalized with respect to the solution with the highest allowed number of kernels, i.e.
Fig. 5  Monte Carlo samples and mixture PDF after two revolutions for Medium Earth Orbit test case

\[ \mathcal{L}_{n,MF}(\alpha_{\text{min}}) = \frac{\mathcal{L}_{MF}(\alpha_{\text{min}})}{\mathcal{L}_{MF}(\alpha_{\text{min}})_{\alpha_{\text{min}}=10^{-10}}} \]

The final number of components and normalized LAM \( \mathcal{L}_n \) are shown in Fig. 6 as a function of \( \alpha_{\text{min}} \).

Fig. 6  Number of Gaussian Mixture Model components and normalized LAM as function of \( \alpha_{\text{min}} \)

As can be seen in Fig. 6a, for the High Earth Orbit test case no more than 6561 kernels with \( \alpha_i > 10^{-5} \) \( \forall i \) are needed to satisfy the imposed nonlinearity threshold \( \varepsilon_n \). For \( \alpha_{\text{min}} < 10^{-5} \) a plateau is in fact reached in both figures, thus signifying the same PDF estimation is obtained for smaller values. On the contrary, looking at Fig. 6b from right to left a rapid decrease in \( \mathcal{L}_n \) is observed for \( \alpha_{\text{min}} \rightarrow 1 \), i.e. for a number of GMM components that is progressively reduced to a single Gaussian kernel. Similar conclusions can be drawn for the LEO and MEO cases in which the plateau is reached for \( \alpha_{\text{min}} = 10^{-4} \) corresponding to 2187 and 729 components respectively. These results demonstrate the flexibility and accuracy of the proposed MF approach which eliminates the needs of a priori estimation of the required number of kernels to guarantee a certain accuracy thus minimizing the computational effort given the nonlinearity content of the problem at hand.

To demonstrate the higher accuracy of the MF method compared to the LF counterpart, the normalized LAM of the LF solution is also included in Fig. 6b for the first two scenarios. The normalized measure is now computed as
\[ L_{n,LF}(a_{\text{min}}) = \frac{L_{LF}(a_{\text{min}})}{L_{MF}(a_{\text{min}})|_{a_{\text{min}}=10^{-8}}} \]  

with \( L_{MF}(a_{\text{min}})|_{a_{\text{min}}=10^{-8}} \) as in Eq. (49). The LF LAM results in the horizontal dashed lines at \( L_n \approx 0 \) thus proving the better agreement of the MF estimate to the MC samples with respect to the LF solution and the importance of the HF correction.

### E. Computational Load

The MF method presented in this paper has been introduced as an accurate and computationally efficient alternative to more conventional HF approaches for orbit UP. In this section, the computational load of the proposed method is compared to that of an equivalent HF solution. For this purpose, a DA version of the HF numerical propagator is used with the combined LOADS-GMM algorithm to propagate the initial uncertainty directly in HF thus avoiding the needs for a posteriori correction. Details on how numerical integrators can be embedded in the LOADS framework are provided in [11]. All numerical simulations have been run on a Dell Precision with an Intel Core i5-9400H CPU @2.50 GHz and 16GB of RAM running Ubuntu 18.04.6 LTS. For both cases, the total runtime is approximated from that of a single propagation as

\[
T_{MF} \approx L \cdot (t_{DA,LF} + t_{PW,HF}) \\
T_{HF} \approx L \cdot t_{DA,HF}
\]

where \( L \) is the number of Gaussian kernels and the subscripts \( DA \) and \( PW \) indicate whether the propagation is performed in DA or point-wise. The runtimes for single GMM components are summarized in Table 5.

|                | \( t_{DA,LF}, \) ms | \( t_{PW,HF}, \) ms | \( t_{MF}, \) ms | \( t_{HF}, \) ms | \( \frac{t_{HF}}{t_{LF}}, \) |
|----------------|---------------------|---------------------|-----------------|------------------|------------------|
| High Earth Orbit | 2.805               | 67.395              | 70.200          | 1442.065         | 20.542           |
| Low Earth Orbit  | 1.855               | 353.120             | 354.975         | 5389.915         | 15.184           |
| Medium Earth Orbit | 2.350              | 60.560              | 62.910          | 1176.670         | 18.704           |

**Table 5  Single propagation runtime for all test cases**

As seen in Table 5, the MF method is 15 to 20 times faster than the HF counterpart with the High Earth Orbit case providing the highest speedup. Moreover, within the MF procedure most of the time is spent in the correction step which must be carried out numerically for maximum accuracy. This fact is more marked for the LEO case where the low orbit altitude and the presence of atmospheric drag further slow down the numerical integration.

### IV. Conclusions

A novel multifidelity method for Uncertainty Propagation is presented. The approach leverages Differential Algebra techniques and Gaussian Mixture Models to efficiently map the initial uncertainties through arbitrary nonlinear transformations. A DA-based measure of nonlinearity is computed to detect departure from linearity of each kernel and a splitting algorithm is employed to adapt on demand the number of GMM components needed to satisfy the desired accuracy. Second-order Taylor expansions of the function output with respect to the initial uncertainty are then used to efficiently map the initial statistics using Unscented Transform sigma points. The combined LOADS-GMM algorithm is run on a LF model to minimize the associated computational load and an high-fidelity propagation of the kernels means is inserted before the UT sampling to adjust the centers of the computed expansions. The MF algorithm is then applied to the problem of orbit UP and three test cases are provided to cover the high-, medium- and low- Earth orbit regimes. The estimated PDFs are compared with reference Monte Carlo simulations and a good agreement between MC samples and PDF contour lines is obtained. The advantages of an adaptive split scheme for the determination of the optimal number of kernels are then highlighted comparing solutions obtained for increasing number of GMM components. The importance of the HF correction is then assessed comparing the agreement to MC samples of the LF and MF solutions. Savings in computational time with respect to fully HF solutions that employ the LOADS-GMM algorithm to propagate the uncertainty in HF dynamics are then estimated in the order of a 15-20 times speedup for single Gaussian kernels.
Further developments are envisaged to broaden the applicability of the proposed method with the inclusion of additional dynamical models and the extension to different orbital regimes and multi-body dynamics. Moreover, the modeling of process noise due to uncertainties in time-varying parameters is also foreseen to improve the realism of the propagated state PDF.

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