General Analytical Solution to Consolidation of Unsaturated Soil with Vertical Drain considering Drain Resistance

Aifang Qin*, Weifang Xu, Tianyi Li and Lianghua Jiang

Department of Civil Engineering, Shanghai University, Shanghai, 200072, China
E-mail: qinaifang@shu.edu.cn

Abstract. This paper presented a general analytical solution to consolidation of unsaturated soil with vertical drain considering drain resistance. Based on Fredlund’s one-dimensional consolidation theory and the equal-strain assumption, the coupled governing equations of excess pore-water and pore-air pressures are first decoupled by introducing two new variables. The method of eigenfunction and undetermined coefficients are then used to obtain the general solution under the drain resistance condition. Moreover, the good agreement between the equal-strain results and the typical free-strain results confirms the validity of the proposed solution. The consolidation behaviours, concluding average excess pore-air and pore-water pressures, are investigated against the drain resistance and the ratio of influence radius to vertical drain radius.

1. Introduction

In dry climate areas, it is possible that vertical drains installed to a great depth of soil may encounter some unsaturated layers in the proximity of the ground surface. The consolidation behaviours of these unsaturated layers, which have considerable differences with saturated soils, should be analysed.

On the Fredlund’s consolidation theory of unsaturated soil[1], Qin et al.[2-3] obtained semi-analytical solution to consolidation of unsaturated soils with the free drainage well by using Laplace transform. Wang et al.[4] proposed a semi-analytical solution with symmetric semi-permeable drainage boundary by applying Laplace transform. Ho et al.[5] presented an analytical solution for the axisymmetric consolidation with linearly depth-dependent initial conditions. Shan and Zhou[6] predicted the axisymmetric consolidation behaviour by introducing the differential quadrature method (DQM). All abovementioned studies are based on the free-strain consolidation theory, Ho et al.[7] introduced the equal-strain theory and derived the analytical solution to unsaturated soil stratum considering the smear effects. Zhou et al.[8] derived an equal-strain analytical solution by using the method of eigenfunction. The equal-strain theory simplified the calculation process and captured more realistic behaviours of vertical drain foundation.

This paper obtains a general analytical solution to predict the unsaturated consolidation in vertical drain foundation considering drain resistance. Firstly, the governing equations considering the drain resistance condition are decoupled into equivalent homogeneous PDEs. Then, the method of eigenfunction and undetermined coefficients are used to derive the final solution. Additionally, effects of the drain resistance factor and the ratio of influence radius to vertical drain radius on the dissipation of average excess pore pressures are highlighted.

2. Mathematical model

Figure 1 illustrates a typical foundation of unsaturated soil with vertical drain, which includes a finite depth \( H \), the radius of influence zone \( r_c \), the radius of vertical drain \( r_d \). Respectively, the coefficients \( k_w \),
**2.1 Basic assumptions**

The basic assumptions for the consolidation model of unsaturated soil are the same as in Ref. [2], and the other assumptions are made as follows:

- The change of radial excess pore pressures in vertical drain can be neglected.
- The vertical drain foundation has no lateral strain, and the vertical strains at the same depth are equal (equal-strain assumption).
- The coefficients of permeability related to the air and water phases and the volume changes for the soil are constant.

The permeability change is nonlinear properties in the consolidation of unsaturated soil. However, it is acceptable that these coefficients remain to be constant at a very small loading increment [5].

**2.2 Governing equations**

Governing equations of air and water phases for consolidation in unsaturated soil with vertical drains under equal-strain conditions are as follows [2]:

\[
\frac{\partial \bar{u}_a}{\partial t} = -C_a \frac{\partial \bar{u}_w}{\partial t} - C_v \left( \frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r} \frac{\partial u_a}{\partial r} \right) + C_q \frac{\partial q}{\partial t} \tag{1a}
\]

\[
\frac{\partial \bar{u}_w}{\partial t} = -C_a \frac{\partial \bar{u}_w}{\partial t} - C_v \left( \frac{\partial^2 u_w}{\partial r^2} + \frac{1}{r} \frac{\partial u_w}{\partial r} \right) + C_q \frac{\partial q}{\partial t} \tag{1b}
\]

where, \( \bar{u}_a = \int_{r_0}^{r} u_a 2\pi rdr \left[ \pi (r_a^2 - r^2) \right] \), \( \bar{u}_w = \int_{r_0}^{r} u_w 2\pi rdr \left[ \pi (r_a^2 - r^2) \right] \) \( \bar{u}_a, \bar{u}_w \)

\[
C_a = m_a^t \left( m_{ik}^w - m_{ik}^a - u_{im} n_s (1 - S_{sw}) \right) \left( \bar{u}_w - \bar{u}_a \right) \right)^{1/3}; C_v = k_v RT \left( \frac{m_{ik}^w - m_{ik}^v}{\gamma_v m_w^v} \right) \left( \bar{u}_w - \bar{u}_a \right) \right)^{1/3};
\]

\[
C_q = m_q^t \left( m_{ik}^w - m_{ik}^a - u_{im} n_s (1 - S_{sw}) \right) \left( \bar{u}_w - \bar{u}_a \right) \right)^{1/3}; C_{w} = m_{ik}^w - m_{ik}^v / \gamma_w m_w^v; C_{q} = m_{ik}^w / m_w^w .
\]

where \( u_{im}, u_{im}, u_{im}, u_{im} \) are the excess pore-air and pore-water pressures in the influence zone and in vertical drain (kPa), respectively, \( \bar{u}_a \) and \( \bar{u}_w \) are the average excess pore-air and excess pore-water pressure in the entire influence zone (kPa). \( R \) is the universal air constant (\(-8.314 \text{ J(mol·K)}^{-1}\)); \( T \) (ie, \( T^2 + 273 \)) is
the absolute temperature (K); \( u_a^0 \) and \( u_w^0 \) are the initial excess pore-air and pore-water pressure (kPa); \( \bar{u}_a^0 \) (ie, \( u_a^0 + u_{\text{atm}} \)) is the absolute initial excess pore-air pressure (kPa); \( u_{\text{atm}} \) is the atmospheric pressure (~100kPa); M is the air mass molecular (~0.029 kg/mol); \( n_0 \) and \( S_0 \) are the initial porosity and saturation; \( m_k^d, m_d^d, m_k^w \) and \( m_d^w \), respectively, are the coefficients related to the variations of net normal stress and suction for the air and water phases (kPa\(^{-1}\)); \( \gamma_w \) is the unit weight of water (~9.8 kN/m\(^3\)).

2.3 Initial and Boundary Conditions
The initial condition shows in figure 1:

\[ u_a(r,z,0) = u_a^0, \quad u_w(r,z,0) = u_w^0 \quad (3) \]

The boundary conditions are:

\[ u_a(r_a, z, t) = u_{a0}(z, t), \quad u_w(r_a, z, t) = u_{w0}(z, t) \quad (4) \]

\[ \frac{\partial u_a(r, z, t)}{\partial r} - \frac{u_a}{r} = 0 \quad \frac{\partial u_w(r, z, t)}{\partial z} = 0 \quad (5) \]

\[ u_{w0}(0, t) = u_{w0}(0, t) = 0 \quad (6) \]

The drain resistance conditions are expressed as at \( r = r_u \):

\[ \frac{\partial u_a(r_u, z, t)}{\partial r} = -\frac{r_{su}}{k_u} \frac{\partial^2 u_a}{\partial z^2}, \quad \frac{\partial u_w(r_u, z, t)}{\partial r} = -\frac{r_{sw}}{k_w} \frac{\partial^2 u_w}{\partial z^2} \quad (7) \]

3. Analytical Solution
Eqs. (1a) and (1b) can also be transformed as follows:

\[ \frac{\partial \bar{u}_a}{\partial t} = A_1 \left( \frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r} \frac{\partial u_a}{\partial r} \right) + A_2 \left( \frac{\partial^2 u_w}{\partial r^2} + \frac{1}{r} \frac{\partial u_w}{\partial r} \right) + A_3 \frac{dq}{dt} \quad (8a) \]

\[ \frac{\partial \bar{u}_w}{\partial t} = W_1 \left( \frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r} \frac{\partial u_a}{\partial r} \right) + W_2 \left( \frac{\partial^2 u_w}{\partial r^2} + \frac{1}{r} \frac{\partial u_w}{\partial r} \right) + W_3 \frac{dq}{dt} \quad (8b) \]

where, \( A_1 = -C_q / (1 - C_a C_w) \); \( A_2 = C_q w / (1 - C_a C_w) \); \( A_3 = C_q w - C_q w / (1 - C_a C_w) \);
\( W_1 = C_q w / (1 - C_a C_w) \); \( W_2 = -C_q w / (1 - C_a C_w) \); \( W_3 = C_q w - C_q w / (1 - C_a C_w) \).

Eqs. (8a) and (8b) can be converted into equivalent PDEs by introducing variables of \( \Phi_1 \) and \( \Phi_2 \):

\[ \frac{\partial}{\partial r} \left( r \frac{\partial \Phi_1}{\partial r} \right) = r \frac{\partial \Phi_1}{\partial t} - \beta_1 \frac{dq}{dt} \quad (9a) \]

\[ \frac{\partial}{\partial r} \left( r \frac{\partial \Phi_2}{\partial r} \right) = r \frac{\partial \Phi_2}{\partial t} - \beta_2 \frac{dq}{dt} \quad (9b) \]

where, \( \Phi_1 = \bar{u}_a + q_u \bar{u}_a \), \( \Phi_2 = q_i \bar{u}_a + \bar{u}_w \)
\( \beta_1 \) (ie, \( A_1 q_u W_q \)) and \( \beta_2 \) (ie, \( A_2 q_u + W_q \)), the derivation process and relevant parameters are presented in Appendix A. Based on the boundary conditions provided in Eq. (4) and (5), integrating Eqs. (9a) and (9b) against the radius yields:

\[ \Phi_1 = -\frac{1}{2Q_1} \left[ r^2 \ln \left( \frac{r}{r_u} \right) - \frac{r^2 - r_u^2}{2} \right] \left( \frac{\partial \Phi_1}{\partial t} - \beta_1 \frac{dq}{dt} \right) + \Phi_{1d} \quad (11a) \]

\[ \Phi_2 = -\frac{1}{2Q_2} \left[ r^2 \ln \left( \frac{r}{r_u} \right) - \frac{r^2 - r_u^2}{2} \right] \left( \frac{\partial \Phi_2}{\partial t} - \beta_2 \frac{dq}{dt} \right) + \Phi_{2d} \quad (11b) \]
where, \( \Phi_{1d} = u_{a_1} + u_{a_2}q_{21} \), \( \Phi_{2d} = u_{a_2}q_{12} + u_{a_1} \).

Substituting Eqs. (11a) and (11b) into Eq. (10) gives:

\[
\Phi_1 = \eta_1 \left( \frac{\partial \Phi_1}{\partial t} - \beta_1 \frac{dq_1}{dt} \right) + \Phi_{1d} \quad (12a)
\]

\[
\Phi_2 = \eta_2 \left( \frac{\partial \Phi_2}{\partial t} - \beta_2 \frac{dq_2}{dt} \right) + \Phi_{2d} \quad (12b)
\]

where,

\[
\eta_1 = \left[ e^2 / 8Q_N (N^2 - 1) \right] (-4N^4 \ln N + 3N^4 - 4N^2 + 1) ;
\]

\[
\eta_2 = \left[ e^2 / 8Q_N (N^2 - 1) \right] (-4N^4 \ln N + 3N^4 - 4N^2 + 1) ;
\]

\( N \) (ie, \( r_H/R_w \)) is the ratio of influence radius to vertical drain radius.

Then substituting Eq. (7) into the first integration of (9a) and (9b) at domain \( r \in (r_w, r) \) yields:

\[
\frac{\partial^2 \Phi_{1d}}{\partial z^2} = \rho_1 \left( \frac{\partial \Phi_1}{\partial t} - \beta_1 \frac{dq_1}{dt} \right) \quad (13a)
\]

\[
\frac{\partial^2 \Phi_{2d}}{\partial z^2} = \rho_2 \left( \frac{\partial \Phi_2}{\partial t} - \beta_2 \frac{dq_2}{dt} \right) \quad (13b)
\]

where, \( \alpha_q = k_z/k_{x_1} = k_z/k_{x_2} ; \rho_1 = \alpha_q (N^2 - 1)/Q ; \rho_2 = \alpha_q (N^2 - 1)/Q_z \).

Combining Equation (12a), (12b), (13a), and (13b), as follows:

\[
\frac{\partial^3 \Phi_{1d}}{\partial z^2 \partial t} - \frac{1}{\eta_1} \frac{\partial^2 \Phi_{1d}}{\partial z^2} = \frac{\beta_1}{\eta_1} \frac{dq_1}{dt} \quad (14a)
\]

\[
\frac{\partial^3 \Phi_{2d}}{\partial z^2 \partial t} - \frac{1}{\eta_2} \frac{\partial^2 \Phi_{2d}}{\partial z^2} = \frac{\beta_2}{\eta_2} \frac{dq_2}{dt} \quad (14b)
\]

Based on the homogeneous boundary conditions given in Eqs. (5) and (6) and the eigenfunction method, \( \Phi_{1d} \) and \( \Phi_{2d} \) can be expanded that:

\[
\Phi_{1d}(z,t) = \sum_{i=0}^{\infty} \phi_i(t) \sin \left( \frac{Mz}{H} \right) , \quad \Phi_{2d}(z,t) = \sum_{i=0}^{\infty} \phi_i(t) \sin \left( \frac{Mz}{H} \right) \quad (15)
\]

where \( \phi_i(t) \) is the undetermined function with respect to \( t \), and \( M = (2m + 1)\pi/2, (m = 0,1,2,\cdots) \).

Based on the orthogonality of the sine function (ie, \( \sin (Mz/H) \)), substituted Eq (15) into Eqs. (14a) and (14b) and the differential equations with respect to \( t \) will be converted:

\[
\frac{d\phi_i(t)}{dt} + \delta_i \phi_i(t) = \xi_i \frac{dq_i(t)}{dt} \quad (16a)
\]

\[
\frac{d\phi_i(t)}{dt} + \delta_i \phi_i(t) = \xi_i \frac{dq_i(t)}{dt} \quad (16b)
\]

where, \( \delta_i = M^2 / \rho_1 H^2 - M^2 \eta_1 \quad \xi_i = \frac{Q_{1d}}{d_e^2} \left( \frac{1}{M^2} \frac{N^2 - 1}{N^2} G - \frac{Q_{1d}}{d_e^2} \right)^i ; \xi_1 = \frac{2 \rho_1 \beta_1 \delta_i H^2}{M^3} ; \xi_2 = \frac{2 \rho_1 \beta_1 \delta_i H^2}{M^3} ; \)

\[
G \) (ie, \( \alpha_d (H/d_e)^2 \)) is the drain resistance factor.

Hence,

\[
\phi_i(t) = e^{-\delta_i t} \left[ C_i + \int_0^t e^{\delta_i \tau} d\tau \right] e^{\xi_i \frac{dq_i(t)}{dt} d\tau} \quad (17a)
\]
\[
\phi_s(t) = e^{-Bt} \left( C_1 + \int_0^t \left[ e^{B(t-\tau)} \frac{dq(\tau)}{d\tau} \right] d\tau \right)
\]  
(17b)

Substituting Equation (17a) and (17b) into Equation (15):

\[
\Phi_{1d}(z, t) = \sum_{n=0}^{\infty} e^{-Bz} \left( C_1 + \int_0^t \left[ e^{B(t-\tau)} \frac{dq(\tau)}{d\tau} \right] \sin \left( \frac{M}{H} \right) \right) 
\]  
(18a)

\[
\Phi_{2d}(z, t) = \sum_{n=0}^{\infty} e^{-Bz} \left( C_2 + \int_0^t \left[ e^{B(t-\tau)} \frac{dq(\tau)}{d\tau} \right] \sin \left( \frac{M}{H} \right) \right) 
\]  
(18b)

Eqs (18a) and (18b) can now be substituted back into Eqs. (13) and (14), and then the final solution obtains by combining Eq. (10):

\[
\bar{u}_i = \frac{1}{q_{12}q_{21}} \left[ q_{12} \Phi_{1} - \Phi_{2} \right] 
\]  
(19a)

\[
\bar{u}_w = \frac{1}{q_{21}q_{31}} \left[ q_{21} \Phi_{2} - \Phi_{1} \right] 
\]  
(19b)

where, \( \Phi = \sum_{n=0}^{\infty} \frac{2}{M} \left[ \left( u_a^n + q_{12}u_w^n \right) e^{-Bz} + \beta_1 \int_0^t \left[ e^{B(t-\tau)} \frac{dq(\tau)}{d\tau} \right] \sin \left( \frac{M}{H} \right) \right] \)

\[
\Phi = \sum_{n=0}^{\infty} \frac{2}{M} \left[ \left( q_{12}u_a^n + u_w^n \right) e^{-Bz} + \beta_2 \int_0^t \left[ e^{B(t-\tau)} \frac{dq(\tau)}{d\tau} \right] \sin \left( \frac{M}{H} \right) \right] 
\]  

4. Verification and Examples

A typical example is used to validate the general analytical solution and to illustrate the consolidation behaviours of vertical drain foundation, as shown in figure 1. This study mainly examines the effects of drainage resistance factor \( G \) (ie, \( \alpha_0/(H/d_w)^2 \)) and ratio of influence radius to drain radius \( N \) (ie, \( N=r_f/r_w \)) on the consolidation behaviours of vertical drain foundation subjected to arbitrary loads. Following the Qin et al.[2], the example in this section adopts the material parameters as follows:

\( q_0 = 100 \text{kPa} \) (\( u_0 = 20 \text{kPa}, u_w = 40 \text{kPa} \)), \( r_w = 0.2 \text{m}, r_e = 1.8 \text{m} \) (\( N=9 \)), \( H = 10 \text{m}, k_s = 1\times10^{-8} \text{m/s}, k_w = 1\times10^{-10} \text{m/s}, S_i = 80\%, n_0 = 50\%, m_{a1} = -5\times10^{-5} \text{kPa}^{-1}, m_{a2} = -2\times10^{-4} \text{kPa}^{-1}, m_{w1} = 1\times10^{-4} \text{kPa}^{-1}, m_{w2} = 2.4\times10^{-9} \text{kPa}^{-1} \).

4.1 Verification

Based on the special consolidation example without considering drain resistance (ie, \( G=0 \)) and subjected to instantaneous load (ie, \( q(t)=q_0 \)), the validity of the equal-strain model is verified by comparing with the free-strain model proposed by Qin et al.[2]. Figures 2 demonstrates the dissipation curve of the average excess pore-air and pore-water pressures with different \( k_s/k_w \) values under the equal-strain and free-strain conditions. It can be observed that the solution obtained from these two different models are close to each other. The slight difference mainly appears at the earlier stage of the consolidation process, but the difference has negligible influence on the average excess pore pressures, which was also mentioned by Zhou et al.[7]. It shows that the equal-strain model is simple and efficient in solving the unsaturated soil consolidation problem. Moreover, there are two stages (ie, before and after the "plateau period") of dissipation curve of average excess pore-water pressures in figure 2B. It is predicted that the single curve converged at second stage are mainly due to the average excess pore-air pressure almost completely dissipates. With an increase of \( k_s/k_w \), the dissipation rate of average excess pore-air and pore-water pressures increase gradually.
4.2 Influence of Drain Resistance.

Figure 3 illustrates the dissipation curve under the exponential load varying with the drain resistance factor $G$ ($\sim 0$ (no drain resistance), 0.1, 0.5, 1 and 2). The exponential loads as follow:

$$q(t) = q_0 + Aq_1 (1 - e^{-bt})$$

where $A$ is load constant, and $A = 1$; $b$ is the load parameter, and $b = 5 \times 10^{-4}$ s$^{-1}$.

Obviously, the variation of drain resistance factor $G$ has a significant impact on the dissipation of the average excess pore-air and pore-water pressures, the smaller the drain resistance factor $G$ is, the faster the dissipation is. It is predicted that the drainage capacity of vertical drain will be increases as $G$ decreases, which accelerates the process of consolidation, and $G$ has a similar influence on the dissipation of the two stages (ie, before and after the "plateau period") in figure 3B. However, when $G$ is less than 0.1, the drain resistance has little effect on the dissipation, and it is suggested that drain resistance should not be considered in practical engineering.

![Figure 2. Dissipation of (A) $\bar{u}_a$ and (B) $\bar{u}_w$ with different ratios of $k_a/k_w$](image)

![Figure 3. Dissipation of (A) $\bar{u}_a$ and (B) $\bar{u}_w$ under different $G$](image)

Figure 4 illustrates the comparison curves to dissipation of $\bar{u}_a$ and $\bar{u}_w$ under exponential load with different $N$ (ie, $N = r_d/r_a$), which is used to describing the influence of influence radius on the consolidation process. In this part, $G=0.5$ and $r_d=0.2$ are applied, and the variation of $N$ value depends on $r_e$. The smaller $N$ is, the faster the dissipation is in figure 4. The drainage path reduces with the decrease of $r_e$, and that accelerates the consolidation of vertical drain foundation as shown in figure 4. Otherwise, the average excess pore pressures firstly increase gradually with the external load increasing, and then reach to the peak at about $t=10^4$ in figure 3. The reason for this phenomenon is that the increasing rate of the exponential load is faster than the dissipation rate of average excess pore pressures during the early stage.
5. Conclusion

Based on the equal-strain hypothesis, the general analytical solution to consolidation of vertical drain foundation considering drain resistance were proposed. The solution of the equal-strain model established in this study has slight difference with the free-strain model at the earlier stage of the consolidation process, but it has a negligible influence on the consolidation behaviours. Specially, the calculation process was simplified by using the equal-strain model. The influence of drain resistance (ie, G) and ratio of influence radius to vertical drain radius (ie, N) on the unsaturated consolidation of vertical drain foundation are took into account. When N decreases, the drainage path of vertical drain foundation reduces, and then the average excess pore pressures would dissipate more quickly. Moreover, the dissipation rate of average excess pore pressures will be increases as drain resistance decreases, and when G is less than 0.1, it is suggested that drain resistance should not be considered in practical engineering.

6. Appendix A.

By adopting the two arbitrary constants $q_1$ and $q_2$, Eqs. (8a) and (8b) can be summarized as follows:

$$\frac{\partial (\bar{u}_d q_1 + \bar{u}_w q_2)}{\partial t} - (A_d q_1 + W_d q_2) \left( \frac{\partial \bar{u}_d}{\partial t} + \frac{1}{r} \frac{\partial \bar{u}_d}{\partial r} \right) - (A_w q_1 + W_w q_2) \left( \frac{\partial \bar{u}_w}{\partial t} + \frac{1}{r} \frac{\partial \bar{u}_w}{\partial r} \right) = (A_d q_1 + W_d q_2) \frac{dq}{dt} \quad (A1)$$

In order to give a true statement to the Eq. (A1), the Q must satisfy the following condition:

$$Q_{q_1} = A_d q_1 + W_d q_2, \quad Q_{q_2} = A_w q_1 + W_w q_2 \quad (A2)$$

The roots $Q_1$ and $Q_2$ of the quadratic equation Eqs. (A2) can be expressed as:

$$Q_{q_2} = \frac{1}{2} \left[ A_1 + W_1 \pm \sqrt{(A_1 - W_1)^2 + 4 A_1 W_1} \right] \quad (A3)$$

When $Q = Q_1$, the solution of $q_1$ and $q_2$ in Eq. (A2) are $q_{11}$ and $q_{21}$, respectively. Similarly, $q_1$ and $q_2$ in Eq. (A3) are $q_{12}$ and $q_{22}$ when $Q = Q_2$, respectively.

where,

$$\frac{\partial \Phi_1}{\partial t} - Q_1 \left( \frac{\partial \Phi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_1}{\partial r} \right) = \beta_1 \frac{dq}{dt} \quad (A4)$$

$$\frac{\partial \Phi_2}{\partial t} - Q_2 \left( \frac{\partial \Phi_2}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_2}{\partial r} \right) = \beta_2 \frac{dq}{dt} \quad (A5)$$

$$\Phi_1 = \bar{u}_d + q_{12} \bar{u}_w, \quad \Phi_2 = q_{21} \bar{u}_d + \bar{u}_w \quad (A6)$$

Where, $\beta_1 = A_1 + q_{12} W_1$ ; $\beta_2 = A_2 + W_1$ ; $q_{12} = W_1 \left( Q_2 - A_2 \right) / A_w$ ; $q_{21} = W_1 / (Q_1 - A_1)$ / $A_w$.

Generally, it is reasonably assumed that $q_{11} = q_{22} = 1$, so Eqs. (7) and (8) can be obtained by substituting...
Eqs. (5) and (6) into Eqs. (A1) and (A3), respectively.

Acknowledgments
We sincerely acknowledge the National Natural Science Foundation of China (Grant No. 41372279 and Grant No. 11672172) for financial support.

References
[1] Fredlund D. G., Hasan J U. (1979) One-dimensional consolidation theory: unsaturated soils. J. Can Geotech, 17:521 -31.
[2] Qin A. F., Sun D. A., Yang L P. (2010) A semi-analytical solution to consolidation of unsaturated soils with the free drainage well. J. Computers and Geotechnics, 37(7): 867-875.
[3] Qin A. F., Sun D. A., Tan Y. W. (2010) Semi-analytical solution to one-dimensional consolidation in unsaturated soils. J. Appl Math Mech (English Edition), 31(2):215–26.
[4] Wang L., Sun D. A., Qin A. F. (2017) Semi-analytical solutions to one-dimensional consolidation for unsaturated soils with symmetric semi-permeable drainage boundary. J. Computers and Geotechnics, 38(6): 831-850.
[5] HO L., Fatahi B. (2016) Axisymmetric consolidation in unsaturated soil deposit subjected to time-dependent loadings. J. International Journal of Geomechanics, 17(2):04016046.
[6] Zhou W. H. (2013) Axisymmetric consolidation of unsaturated soils by differential quadrature method. J. Mathematical Problems in Engineering, 497161.
[7] Zhou W. H. (2014) A simple analytical solution to one-dimensional consolidation for unsaturated soils. J. International Journal for Numerical and Analytical Methods in Geomechanics, 38.
[8] HO L., Fatahi B. (2018) Analytical solution to axisymmetric consolidation of unsaturated soil stratum under equal-strain condition incorporating smear effects. J. International Journal for Numerical and Analytical Methods in Geomechanics, 42(15):1890-1913.
[9] Zhou F., Chen Z., Wang X. D. (2018) An equal-strain analytical solution for the radial consolidation of unsaturated soils by vertical drains considering drain resistance. J. Advances in Civil Engineering, 5069159