Color-Singlet Three-Quark States in the $su(4)$-Algebraic Many-Quark Model

An Example of the $su(4) \otimes su(4)$-Model

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As announced in last paper by the present authors, the color-singlet three-quark states are investigated for the case of the spherical $j-j$ coupling and $L-S$ coupling shell models. The latter case automatically leads to the $su(4) \otimes su(4)$-model. For a certain type of Hamiltonian, the states of individual “nucleons” are characterized.

The $su(4)$-algebraic quark model, which is an example of a many-fermion model, may be quite attractive. With the aid of this model, we can learn various aspects of a many-quark system, for example, the quark-triplet formation and the pairing correlation. The first aspect is related to describing the structure of the “nucleon” and the second one to describing color-superconductivity. In our last paper,1 which will be referred to as (I), we presented a possible systematic treatment of this model. In §6 of (I), we derived two forms which result in the “nucleon”. Both are based on the $j-j$ coupling and the $L-S$ coupling shell model and the explicit forms of the former and the latter are shown in the relations (I·6·3) and (I·6·6), respectively. The form (I·6·3) is of the same form as that presented by Petry et al.2 and the form (I·6·6) is derived under our idea. Main aim of this note is to give our own interpretation of the form (I·6·3) in connection with the form (I·6·6). We will not present a reinterpretation of the notation appearing in (I).

Let us start by rewriting the first quark-triplet operator in (I·6·6) in spherical tensor representation. For this aim, we use the quantum number $l_s$ explicitly, but omit 1/2 for the spin and the isospin. The first quark-triplet operator in (I·6·6) can be expressed as follows:

$$\tilde{B}_{l_s \mu \sigma \tau}^* = \sum_{k=1}^{3} \tilde{S}^k(l_s) \tilde{c}_{kl_s \mu \sigma \tau}^*, \quad (1)$$
$$\tilde{S}^k(l_s) = \sum_{\mu \sigma \tau} (-)^k \tilde{c}_{l_s \mu \sigma \tau}^* \tilde{c}_{3l_s \mu \sigma \tau} . \quad (2)$$

The cases $k = 2$ and 3 are obtained by cyclic permutation of the color $i = 1$, 2 and 3. By considering the addition of the orbital and the spin angular momentum, we
can define $\tilde{c}_{kl,j,\pm m}\sigma$ in the form
\[ \tilde{c}_{kl,j,\pm m}\sigma = \sum_{\mu\sigma} (l_s\mu 1/2\sigma|j\pm m)c_{kl,\mu\sigma}^* . \] (3)

Here, $\langle l_s\mu 1/2\sigma|j\pm m \rangle$ denotes the Clebsch-Gordan coefficient and $j\pm$ are given as
\[ j_+ = l_s + 1/2, \quad j_- = l_s - 1/2. \quad (l_s \geq 1) \] (4)

Since $\tilde{S}^k(l_s)$ is scalar, $\tilde{B}_{l_s,j,\pm m}\sigma$ can be defined as
\[ \tilde{B}_{l_s,j,\pm m}\sigma = \sum_{\mu\sigma} (l_s\mu 1/2\sigma|j\pm m)c_{l_s,\mu\sigma}^* . \] (5)

Then, we can prove the following relation:
\[ \tilde{B}_{l_s,j,\pm m}\sigma = \sum_{k=1}^{3} \left( \tilde{S}^k(l_s,j^+) + \tilde{S}^k(l_s,j^-) \right) \tilde{c}_{kl,j,\pm m}\sigma . \] (6)

Here, $\tilde{S}^i(l_s,j^\pm)$ is defined as
\[ \tilde{S}^i(l_s,j^\pm) = \sum_{m\tau} (-)^{j\pm-m}(-)^{3\over 2-\tau}c_{2l_s,j,\pm m}\sigma \tilde{c}_{3l_s,j,\pm m}\tau . \] (7)

The cases $k = 2$ and $3$ are obtained by cyclic permutation of the color $i = 1, 2$ and $3$. Since we are treating two single-particle levels specified by $(l_s,j^+)$ and $(l_s,j^-)$, the present $su(4)$-algebra is an $su(4) \otimes su(4)$-algebra:
\[ \tilde{S}^i(l_s) = \tilde{S}^i(l_s,j^+) + \tilde{S}^i(l_s,j^-) , \] (8a)
\[ \tilde{S}^i(l_s) = \tilde{S}^i(l_s,j^+) + \tilde{S}^i(l_s,j^-) . \] (8b)

The levels specified by $(l_s,j^\pm)$ will be called the spin-orbit partner levels.

Next, we consider the first quark-triplet operator in $(I^3_6)$. In the shell model, the quantum number $j^\pm$ is obtained by adding the orbital and the spin angular momentum, $l_s$ and $1/2$. In the case of the $L$-$S$ coupling scheme, the spin-orbit partner levels are expected to be close to each other and they are strongly correlated. The relation [3] tells that $\tilde{B}_{l_s,j,\pm m}\sigma$ depends on the two forms $\tilde{S}^k(l_s,j^+)$ and $\tilde{S}^k(l_s,j^-)$ in an equal weight. But, in the case of the $j$-$j$ coupling scheme, the two levels are expected to be separated and they are weakly correlated. Then, we can treat $\tilde{B}_{l_s,j,\pm m}\sigma$ independently of $\tilde{S}^k(l_s,j^-)$ and vice versa. The above consideration allows to rewrite the first quark-triplet operator in $(I^3_6)$ in the following form:
\[ \tilde{B}_{l_s,j,\pm m}\sigma = \sum_{k=1}^{3} \tilde{S}^k(l_s,j^\pm)\tilde{c}_{kl,j,\pm m}\sigma . \] (9)

As was already mentioned, the main aim of this note is to discuss the form [3] in connection with the form [6].
Let us define the following four operators:

\[
\tilde{B}^{*}_{lsj+m\tau}(+) = \sum_{k} \tilde{S}^{k}(lsj+) \tilde{c}^{*}_{klmj}, \\
\tilde{B}^{*}_{lsj+m\tau}(-) = \sum_{k} \tilde{S}^{k}(lsj-) \tilde{c}^{*}_{klmj}, \\
\tilde{B}^{*}_{lsj-m\tau}(+) = \sum_{k} \tilde{S}^{k}(lsj+) \tilde{c}^{*}_{klmj}, \\
\tilde{B}^{*}_{lsj-m\tau}(-) = \sum_{k} \tilde{S}^{k}(lsj-) \tilde{c}^{*}_{klmj}.
\]

The above four operators commute with \(\tilde{S}^{k}(lsj)\) \((i \neq j)\) defined by the relation \((8b)\) and, thus, they are color-singlet. The relations \((6)\) and \((9)\) can be expressed in the form

\[
\tilde{B}^{*}_{lsj+m\tau}(A) = \tilde{B}^{*}_{lsj+m\tau}(+) + \tilde{B}^{*}_{lsj+m\tau}(-), \\
\tilde{B}^{*}_{lsj-m\tau}(A) = \tilde{B}^{*}_{lsj-m\tau}(-) + \tilde{B}^{*}_{lsj-m\tau}(+), \\
\tilde{B}^{*}_{lsj+m\tau}(B) = \tilde{B}^{*}_{lsj+m\tau}(+), \\
\tilde{B}^{*}_{lsj-m\tau}(B) = \tilde{B}^{*}_{lsj-m\tau}(-).
\]

In order to discriminate between both forms, we used the superscripts \((A)\) and \((B)\). It may be interesting to search under which condition the forms \((12)\) and \((13)\) are realized for a given Hamiltonian.

We present an example illustrative of the idea how to treat the simplest case of the quark-triplet formation as a color-singlet, namely, one “nucleon” problem. First, we introduce the following orthonormalized color-singlet states consisting of three quarks:

\[
|lsj+m\tau; +) = (\sqrt{6(ls+2)})^{-1} \tilde{B}^{*}_{lsj+m\tau}(+) \mid 0 \rangle (\mid j_+; +\rangle), \\
|lsj+m\tau; -) = (\sqrt{6ls})^{-1} \tilde{B}^{*}_{lsj+m\tau}(-) \mid 0 \rangle (\mid j_+; -\rangle), \\
|lsj-m\tau; -) = (\sqrt{6(ls+1)})^{-1} \tilde{B}^{*}_{lsj-m\tau}(-) \mid 0 \rangle (\mid j_-; -\rangle), \\
|lsj-m\tau; +) = (\sqrt{6(ls+1)})^{-1} \tilde{B}^{*}_{lsj-m\tau}(+) \mid 0 \rangle (\mid j_-; +\rangle).
\]

Fig. 1. Configurations of three quarks in one “nucleon” in the states \((14)\) and \((15)\) are shown illustratively.
The above four states can be shown pictorially in Fig. 1. In the frame of the above orthonormalized states, we investigate the following Hamiltonian:

$$\tilde{H} = \tilde{H}_0 + \tilde{H}_1,$$

$$\tilde{H}_0 = g_0(l_s + 1)\tilde{N}_{l_0}(-) - g_0l_s\tilde{N}_{l_0}(+) , \quad (g_0 \geq 0) \tag{17a}$$

$$\tilde{H}_1 = -g_1 \sum_{i=1}^{3} \vec{S}_i(l_s)\vec{S}_i(l_s) . \quad (g_1 \geq 0) \tag{17b}$$

Here, $\tilde{N}_{l_0}(-)$ and $\tilde{N}_{l_0}(+)$ denote quark-number operators of the states specified by $(-)$ and $(+)$, respectively, and $g_0$ and $g_1$ denote parameters. The first part $\tilde{H}_0$ comes from the spin-orbit interaction for the quark, $-2g_0 \vec{l} \cdot \vec{s}$ and it may be natural for our two-level model to adopt this interaction. The second part $\tilde{H}_1$ characterizes the quark-triplet formation. We will search the eigenstates of the Hamiltonian (16). As is clear from the states (14a) and (14b), the Hamiltonian (16) has two eigenstates $(\pm)$ for a given $j_+$ and the same happens in the case $j_-$. We pay an attention to the state with the lower energy and require that this state is one “nucleon” state. The operation of $\tilde{H}$ on the above four states gives us the following results:

$$\tilde{H}(j_+;+) = -[3g_0l_s + 2g_1(l_s + 2)]|j_+;+\rangle - 2g_1\sqrt{l_s(l_s + 2)}|j_+;+\rangle , \quad (18a)$$

$$\tilde{H}(j_+;-) = -2g_1\sqrt{l_s(l_s + 2)}|j_+;+\rangle + [g_0(l_s + 2) - 2g_1l_s]|j_+;+\rangle , \quad (18b)$$

$$\tilde{H}(j_-;-) = [3g_0(l_s + 1) - 2g_1(l_s + 1)]|j_-;+\rangle - 2g_1(l_s + 1)|j_-;+\rangle , \quad (19a)$$

$$\tilde{H}(j_-;+) = -2g_1(l_s + 1)|j_-;+\rangle - [g_0(l_s - 1) + 2g_1(l_s + 1)]|j_-;+\rangle . \quad (19b)$$

Let $|j_+\rangle$ and $|j_-\rangle$ be the eigenstates of $\tilde{H}$:

$$\tilde{H}|j_+\rangle = E_+|j_+\rangle , \quad |j_+\rangle = \psi_+|j_+;+\rangle + \psi_-|j_+;-\rangle , \quad (20)$$

$$\tilde{H}|j_-\rangle = E_-|j_-\rangle , \quad |j_-\rangle = \phi_-|j_-;-\rangle + \phi_+|j_-;+\rangle . \quad (21)$$

The relations (18a)–(21) give the linear equations for the eigenvalue equations:

(i) for $j_+$-states:

$$-3g_0l_s + 2g_1(l_s + 2)]\psi_+ - 2g_1\sqrt{l_s(l_s + 2)}\psi_+ = E_+\psi_+ , \quad (22a)$$

$$-2g_1\sqrt{l_s(l_s + 2)}\psi_+ + [g_0(l_s + 2) - 2g_1l_s]\psi_+ = E_+\psi_- . \quad (22b)$$

(ii) for $j_-$-states:

$$[3g_0(l_s + 1) - 2g_1(l_s + 1)]\phi_- - 2g_1(l_s + 1)\phi_+ = E_-\phi_- , \quad (23a)$$

$$-2g_1(l_s + 1)\phi_- - [g_0(l_s - 1) + 2g_1(l_s + 1)]\phi_+ = E_-\phi_+ . \quad (23b)$$

The normalization is performed by

$$\psi_+^2 + \psi_-^2 = 1 , \quad \phi_-^2 + \phi_+^2 = 1 . \quad (24)$$
Let us start by searching solutions of the eigenvalue equations (22) and (23). We specify two solutions by the indices 1 and 2. First, we consider the case \( g_0 = 0 \). In this case, we have the following solutions for the eigenvalue equation:

\[
\begin{align*}
(1a) \quad \mathcal{E}_{j_+}^{(1)} &= -4g_1(l_s + 1), \\
|j_+; 1\rangle &= \frac{1}{\sqrt{2(l_s + 1)}} \left( \tilde{B}_{l_s j_+ m \tau}^{(+)} + \tilde{B}_{l_s j_+ m \tau}^{(-)} \right) |0\rangle \\
&= \frac{1}{2\sqrt{3(l_s + 1)}} \left( \tilde{B}_{l_s j_+ m \tau}^{(A)} \right) |0\rangle.
\end{align*}
\]

Here, we used the relation (12a).

\[
(1b) \quad \mathcal{E}_{j_+}^{(2)} &= 0, \\
|j_+; 2\rangle &= \frac{1}{\sqrt{2(l_s + 1)}} \left( \tilde{B}_{l_s j_+ m \tau}^{(+)} - \tilde{B}_{l_s j_+ m \tau}^{(-)} \right) |0\rangle \\
&= \frac{1}{2\sqrt{3(l_s + 1)}} \tilde{B}_{l_s j_+ m \tau}^{(A)} |0\rangle.
\]

For the eigenvalue equation (23), we have the following solutions:

\[
\begin{align*}
(2a) \quad \mathcal{E}_{j_-}^{(1)} &= -4g_1(l_s + 1), \\
|j_-; 1\rangle &= \frac{1}{\sqrt{2}} |j_-; -\rangle + \frac{1}{\sqrt{2}} |j_-; +\rangle \\
&= \frac{1}{2\sqrt{3(l_s + 1)}} \left( \tilde{B}_{l_s j_- m \tau}^{(-)} + \tilde{B}_{l_s j_- m \tau}^{(+)} \right) |0\rangle \\
&= \frac{1}{2\sqrt{3(l_s + 1)}} \tilde{B}_{l_s j_- m \tau}^{(A)} |0\rangle.
\end{align*}
\]

Here, we used the relation (12b).

\[
\begin{align*}
(2b) \quad \mathcal{E}_{j_-}^{(2)} &= 0, \\
|j_-; 2\rangle &= \frac{1}{\sqrt{2}} |j_-; -\rangle - \frac{1}{\sqrt{2}} |j_-; +\rangle \\
&= \frac{1}{2\sqrt{3(l_s + 1)}} \left( \tilde{B}_{l_s j_- m \tau}^{(-)} - \tilde{B}_{l_s j_- m \tau}^{(+)} \right) |0\rangle.
\end{align*}
\]

We can see that the states \( |j_\pm; 1\rangle \) are lower than \( |j_\pm; 2\rangle \) in energy and the lower states \( |j_\pm; 1\rangle \) are expressed in the form \( \tilde{B}_{l_s j_\pm m \tau}^{(A)} |0\rangle \). Since the above is the case in which the spin-orbit interaction is switched off, the above solution is quite natural and the Hamiltonian (16) reproduces the above feature. In addition, we must mention that in any case with \( g_0 \neq 0 \), we cannot derive the forms (25a) and (26a).
Next, we investigate the case $g_1 = 0$. As is clear from the eigenvalue equations (22) and (23), in this case, the two levels specified by $+$ and $-$ do not couple with each other. The eigenvalue equation (22) gives the following solution:

\[
\begin{align*}
E_{j+}^{(1)} &= -3g_0 l_s , \\
|j+;1\rangle &= |j+;+\rangle = \frac{1}{\sqrt{6(l_s + 2)}} \tilde{B}_{ts,j,mr}^{*}(+) |0\rangle \\
&= \frac{1}{\sqrt{6(l_s + 2)}} \tilde{B}_{ts,j,mr}^{*(B)} |0\rangle . \quad (27a)
\end{align*}
\]

Here, we used the relation (13a).

\[
\begin{align*}
E_{j+}^{(2)} &= g_0 (l_s + 2) , \\
|j+;2\rangle &= |j+;+\rangle = \frac{1}{\sqrt{6l_s}} \tilde{B}_{ts,j,mr}^{*}(-) |0\rangle . \quad (27b)
\end{align*}
\]

Since $E_{j+}^{(1)} < E_{j+}^{(2)}$, the state (27a) is nothing but the state we are looking for. The eigenvalue equation (23) leads to the following solution:

\[
\begin{align*}
E_{j-}^{(1)} &= 3g_0 (l_s + 1) , \\
|j-;1\rangle &= |j-;+\rangle = \frac{1}{\sqrt{6(l_s + 1)}} \tilde{B}_{ts,j,mr}^{*}(-) |0\rangle \\
&= \frac{1}{\sqrt{6(l_s + 1)}} \tilde{B}_{ts,j,mr}^{*(B)} |0\rangle . \quad (28a)
\end{align*}
\]

Here, we used the relation (13b).

\[
\begin{align*}
E_{j-}^{(2)} &= -g_0 (l_s - 1) , \\
|j-;2\rangle &= |j-;+\rangle = \frac{1}{\sqrt{6(l_s + 1)}} \tilde{B}_{ts,j,mr}^{*}(+) |0\rangle . \quad (28b)
\end{align*}
\]

Different from the previous three cases, the present gives us the relation $E_{j-}^{(1)} > E_{j-}^{(2)}$. Therefore, we have to conclude that certainly we have the state related to the form (13b), but, we find a color-singlet state which is lower than the state we intend to search.

The above analysis suggests us the following: The Hamiltonian (16) can produce the color-singlet three-quark states and in energy, one of them is lower than the state generated by the operators (12) and (13). In this sense, the Hamiltonian (16) is unsatisfactory for our purpose. By adding the third part to $\tilde{H}$, we adopt the following Hamiltonian:

\[
\tilde{H} = \tilde{\mathcal{H}} + \tilde{H}_2 , \quad (29)
\]

\[
\tilde{H}_2 = -g_2 \left( \tilde{N}(+) - \tilde{N}(-) \right) \left( \sum_{i=1}^{3} \tilde{S}_{i}^{(l_s)} \tilde{S}_{i}(l_s) \right) \left( \tilde{N}(+) - \tilde{N}(-) \right) . \quad (30)
\]
With the use of the relations (20) and (21), in which \( \tilde{H} \) and result:

\[
\begin{align*}
\text{(31a)} & \quad \tilde{H}|j_+;+\rangle = -[3g_0l_s + 2(g_1 + g_2)(l_s + 2)]|j_+;+\rangle \\
& \quad -2(g_1 - 3g_2)\sqrt{l_s(l_s + 2)}|j_+;+\rangle , \\
\tilde{H}|j_+;\rangle = -2(g_1 - 3g_2)\sqrt{l_s(l_s + 2)}|j_+;\rangle + [g_0(l_s + 2) - 2(g_1 + g_2)l_s]|j_+;\rangle , \tag{31b}
\end{align*}
\]

\[
\begin{align*}
\tilde{H}|j_-;\rangle &= [3g_0 - 2(g_1 + g_2)](l_s + 1)|j_-;\rangle - 2(g_1 - 3g_2)(l_s + 1)|j_-;\rangle , \\
\tilde{H}|j_-;+\rangle &= -2(g_1 - 3g_2)(l_s + 1)|j_-;+\rangle - [g_0(l_s - 1) + 2(g_1 + g_2)(l_s + 1)]|j_-;+\rangle . \tag{32b}
\end{align*}
\]

With the use of the relations (20) and (21), in which \( \tilde{H} \) and \( E_{j\pm} \) are replaced with \( \tilde{H} \) and \( E_{j\pm} \), we have the following eigenvalue equations:

(i)’ for \( j_+ \)-states:

\[
\begin{align*}
\text{(33a)} & \quad -[3g_0l_s + 2(g_1 + g_2)(l_s + 2)]\psi_+ - 2(g_1 - 3g_2)\sqrt{l_s(l_s + 2)}\psi_- = E_{j_+}\psi_+ , \\
& \quad -2(g_1 - 3g_2)\sqrt{l_s(l_s + 2)}\psi_+ + [g_0(l_s + 2) - 2(g_1 + g_2)l_s]\psi_- = E_{j_+}\psi_- , \tag{33b}
\end{align*}
\]

(ii)’ for \( j_- \)-states:

\[
\begin{align*}
\text{(34a)} & \quad [3g_0 - 2(g_1 + g_2)](l_s + 1)\phi_- - 2(g_1 - 3g_2)(l_s + 1)\phi_+ = E_{j_-}\phi_- , \\
& \quad -2(g_1 - 3g_2)(l_s + 1)\phi_- - [g_0(l_s - 1) + 2(g_1 + g_2)(l_s + 1)]\phi_+ = E_{j_-}\phi_+ . \tag{34b}
\end{align*}
\]

Of course, if \( g_2 = 0 \), the relations (33) and (34) are reduced to the relations (22) and (23), respectively and the condition (24) is also used in the present case.

Let us investigate the eigenvalue equations (33) and (34). First, we search the condition which leads to the same solution as that shown by the relation (25a). By substituting \( \psi_+ = \sqrt{l_s + 2}/2(l_s + 1) \) and \( \psi_- = \sqrt{l_s}/2(l_s + 1) \) into Eq. (33), we have the following condition:

\[
g_0(2l_s + 1) + 8(l_s + 3)g_2 = 0 . \tag{35}
\]

Further, by substituting \( \phi_+ = 1/\sqrt{2} \) and \( \phi_- = 1/\sqrt{2} \) into Eq. (34), we have

\[
g_0(2l_s + 1) - 12g_2(l_s + 1) = 0 . \tag{36}
\]

Therefore, the common solution to the conditions (35) and (36) is given by

\[
g_0 = 0 , \quad g_2 = 0 . \tag{37}
\]
The solution (37) gives the results which are the same as those presented in (1a), (1b), (2a) and (2b) in Eqs.(25) and (26). For the case $g_0 = g_2 = 0$, we can summarize the result as follows:

(1a) : (1a) ,
(1b) : (1b) ,
(2a) : (2a) ,
(2b) : (2b) .

Next, we investigate the case where the spin-orbit partner levels are uncoupled with each other. As is clear from the eigenvalue equation (33), two levels become uncoupled with each other under the condition

$$g_1 - 3g_2 = 0 , \quad \text{i.e.,} \quad g_1 = 3g_2 . \quad (g_2 \geq 0) \quad (40)$$

In this case, we have

$$E_{j_+}^{(1)} = -3g_0l_s - 24g_2(l_s + 2) , \quad (41a)$$
$$E_{j_+}^{(2)} = g_0(l_s + 2) - 8g_2l_s . \quad (41b)$$

Since $E_{j_+}^{(2)} - E_{j_+}^{(1)} = 2g_0(2l_s + 1) + 16g_2(l_s + 3) > 0$, the state with $E_{j_+}^{(1)}$ is nothing but the state we are looking for. For the states with $j_-$, the relation (40) is also valid. In this case, the eigenvalue equation (33) gives us

$$E_{j_-}^{(1)} = (3g_0 - 24g_2)(l_s + 1) , \quad (42a)$$
$$E_{j_-}^{(2)} = -g_0(l_s - 1) - 8g_2(l_s + 1) . \quad (42b)$$

Therefore, we have

$$E_{j_-}^{(2)} - E_{j_-}^{(1)} = -2 [g_0(2l_s + 1) - 8g_2(l_s + 1)] . \quad (43)$$

Since the state, which we are looking for, should obey the condition $E_{j_-}^{(1)} < E_{j_-}^{(2)}$, the relation (33) gives us

$$g_0 < 8g_2 \cdot \frac{l_s + 1}{2l_s + 1} . \quad (44)$$

It should be noted that the inequality $E_{j_-}^{(1)} < E_{j_-}^{(2)}$ cannot be derived under the Hamiltonian (16). Thus, for the case with $g_1 = 3g_2$ and $g_0 < 8g_2 \cdot (l_s + 1)/(2l_s + 1)$, the results are summarized as follows:

(3a) : $E_{j_+}^{(1)} = -3g_0l_s - 24g_2(l_s + 2)$ ,

$$\|j_+; 1\| = \frac{1}{\sqrt{6(l_s + 2)}} B_{l_s, j_+ + m\tau}^*(+) |0\rangle = \frac{1}{\sqrt{6(l_s + 2)}} B_{l_s, j_+ + m\tau}^*(B) |0\rangle , \quad (45a)$$

(3b) : $E_{j_+}^{(2)} = g_0(l_s + 2) - 8g_2l_s$ ,

$$\|j_+; 2\| = \frac{1}{\sqrt{6l_s}} \tilde{B}_{l_s, j_+ + m\tau}^*(-) |0\rangle , \quad (45b)$$
Since \( g \) with the \( \Delta \)-excitation. Under the condition (44), we have the inequality (51b). The above is the comparison in (I-6.6) is given by

\[
\tilde{D}^{s}_{l_{s},\mu \tau}|0\rangle = \prod_{i=1}^{3} \tilde{c}^{*}_{l_{i},\mu \tau}|0\rangle .
\]

(4a)

In the case \( g_{0} = g_{2} = 0 \), we have

\[
\tilde{H} \tilde{D}^{s}_{l_{s},\mu \tau}|0\rangle = E^{(\Delta)}_{l_{s}} \tilde{D}^{s}_{l_{s},\mu \tau}|0\rangle , \quad E^{(\Delta)}_{l_{s}} = 0 .
\]

(4b)

The state \( \tilde{D}^{s}_{l_{s},\mu \tau}|0\rangle \) is higher than \( |j_{+}; 1\rangle \) and \( |j_{-}; 1\rangle \) shown by relations (38a) and (39a). Next, we investigate the case described by the second quark-triplet operator in (I-6.3):

\[
\tilde{D}^{s}_{l_{s},j_{2}m_{\tau}}|0\rangle = \prod_{i=1}^{3} \tilde{c}^{*}_{l_{i},j_{2}m_{\tau}}|0\rangle .
\]

This case gives

\[
\tilde{H} \tilde{D}^{s}_{l_{s},j_{2}m_{\tau}}|0\rangle = E^{(\Delta)}_{j_{+}} \tilde{D}^{s}_{l_{s},j_{2}m_{\tau}}|0\rangle , \quad E^{(\Delta)}_{j_{+}} = -3g_{0}l_{s} ,
\]

(50a)

\[
\tilde{H} \tilde{D}^{s}_{l_{s},j_{2}m_{\tau}}|0\rangle = E^{(\Delta)}_{j_{-}} \tilde{D}^{s}_{l_{s},j_{2}m_{\tau}}|0\rangle , \quad E^{(\Delta)}_{j_{-}} = 3g_{0}(l_{s} + 1) .
\]

(50b)

We require that \( \tilde{D}^{s}_{l_{s},j_{2}m_{\tau}}|0\rangle \) is higher than \( |j_{+}; 1\rangle \) and \( |j_{-}; 1\rangle \) shown by the relation (45a) and (46a), respectively:

\[
E^{(\Delta)}_{j_{+}} - E^{(1)}_{j_{+}} = 24g_{2}(l_{s} + 2) > 0 ,
\]

(51a)

\[
E^{(\Delta)}_{j_{-}} - E^{(1)}_{j_{-}} = -3(g_{0}(2l_{s} + 1) - 8g_{2}(l_{s} + 1)) > 0 ,
\]

(51b)

\[
E^{(\Delta)}_{j_{-}} - E^{(1)}_{j_{+}} = 3(g_{0}(2l_{s} + 1) + 8g_{2}(l_{s} + 1)) > 0 ,
\]

(52a)

\[
E^{(\Delta)}_{j_{-}} - E^{(1)}_{j_{-}} = 24g_{2}(l_{s} + 1) > 0 .
\]

(52b)

Since \( g_{2} > 0 \), the inequalities (51a), (52a) and (52b) are automatically satisfied. Under the condition (44), we have the inequality (51b). The above is the comparison with the \( \Delta \)-excitation.
The arguments presented in this note lead to the following conclusion for one “nucleon” states:

(A) The case ($g_0 = 0, g_2 = 0$):

\[ \langle j_+; 1 \rangle = \frac{1}{2\sqrt{3(l_s + 1)}} \tilde{B}_{l_s j_+,m_\tau}^{A(1)} |0 \rangle, \]
\[ E_{j_+}^{(1)} = -4g_1(l_s + 1), \tag{53a} \]
\[ \langle j_-; 1 \rangle = \frac{1}{2\sqrt{3(l_s + 1)}} \tilde{B}_{l_s j_-,m_\tau}^{A(1)} |0 \rangle, \]
\[ E_{j_-}^{(1)} = -4g_1(l_s + 1), \tag{53b} \]

(B) The case ($g_0 < 8g_2(l_s + 1)/(2l_s + 1), g_1 = 3g_2$):

\[ \langle j_+; 1 \rangle = \frac{1}{\sqrt{6(l_s + 2)}} \tilde{B}_{l_s j_+,m_\tau}^{B(1)} |0 \rangle, \]
\[ E_{j_+}^{(1)} = -3g_0l_s - 24g_2(l_s + 2), \tag{54a} \]
\[ \langle j_-; 1 \rangle = \frac{1}{\sqrt{6(l_s + 1)}} \tilde{B}_{l_s j_-,m_\tau}^{B(1)} |0 \rangle, \]
\[ E_{j_-}^{(1)} = 3g_0(l_s + 1) - 24g_2(l_s + 1). \tag{54b} \]

The energies $E_{j_+}^{(1)}$ and $E_{j_-}^{(1)}$ given by the relation (54) can be rewritten in the form

\[ E_{j_+}^{(1)} = E_{l_0} - \xi_{l_0}l_s, \quad E_{j_-}^{(1)} = E_{l_0} + \xi_{l_0}(l_s + 1), \tag{55} \]
\[ E_{l_0} = -48g_2 \frac{(l_s + 1)^2}{2l_s + 1}, \quad \xi_{l_0} = 3g_0 + 24g_2 \frac{1}{2l_s + 1}. \tag{56} \]

The energy $E_{l_0}$ tells that the origin measuring the energy changes from the point 0, which we adopted at the starting point. Of course, this change arises as a consequence of $\tilde{H}_2$. The symbol $\xi_{l_0}$ indicates the strength of the spin-orbit interaction for one “nucleon” state. It may be interesting to see that the value deviates from $3g_0$, the value of three bare quarks. It also comes from the effect of $\tilde{H}_2$.

Thus, we have the following conclusion: If the value of the parameters contained in the Hamiltonian (29) are chosen appropriately, we can derive the color-singlet three-quark states presented in Ref [2] for both of the spin-orbit partner levels. Therefore, it may be interesting to investigate the deviation from the states (54a) and (54b). It is our next task. In Ref [3], the states (54a) and (54b) were applied to the problem of nuclear magnetic moments and various discussions were performed. Since the present note does not aim at the application, for example, we does not contact with the problems related to those discussed in Ref [3].

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