ARAK INEQUALITIES FOR CONCENTRATION FUNCTIONS AND THE LITTLEWOOD–OFFORD PROBLEM: A SHORTENED VERSION

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Abstract. Let $X, X_1, \ldots, X_n$ be independent identically distributed random variables. In this paper we study the behavior of concentration functions of weighted sums $\sum_{k=1}^n X_k a_k$ with respect to the arithmetic structure of coefficients $a_k$ in the context of the Littlewood–Offord problem. Concentration results of this type received renewed interest in connection with distributions of singular values of random matrices. Recently, Tao and Vu proposed an Inverse Principle in the Littlewood–Offord problem. We discuss the relations between the Inverse Principle of Tao and Vu as well as that of Nguyen and Vu and a similar principle formulated for sums of arbitrary independent random variables in the work of Arak from the 1980’s.

This paper is a shortened and edited version of the preprint [8]. Here we present the results without proofs.

At the beginning of 1980’s, Arak [1, 2] has published new bounds for the concentration functions of sums of independent random variables. These bounds were formulated in terms of the arithmetic structure of supports of distributions of summands. Using these results, he has obtained the final solution of an old problem posed by Kolmogorov [11]. In this paper, we apply Arak’s results to the Littlewood–Offord problem which was intensively investigated in the last years. We compare the consequences of Arak’s results with recent results of Nguyen, Tao and Vu [13], [14] and [18].

Let $X, X_1, \ldots, X_n$ be independent identically distributed (i.i.d.) $\mathbb{R}$-valued random variables. The concentration function of a $\mathbb{R}^d$-dimensional random vector $Y$ with distribution $F = \mathcal{L}(Y)$ is defined by the equality

$$Q(F, \lambda) = \sup_{x \in \mathbb{R}^d} \mathbb{P}(Y \in x + \lambda B), \quad \lambda \geq 0,$$

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where \( B = \{ x \in \mathbb{R}^d : \| x \| \leq 1/2 \} \) is the centered Euclidean ball of radius 1/2. Let \( a = (a_1, \ldots, a_n) \neq 0 \), where \( a_k = (a_{k1}, \ldots, a_{kd}) \in \mathbb{R}^d, \ k = 1, \ldots, n \). Starting with seminal papers of Littlewood and Offord [12] and Erdős [9], the behavior of the concentration functions of the weighted sums \( S_a = \sum_{k=1}^n X_k a_k \) is studied intensively. In the sequel, let \( F_a \) denote the distribution of the sum \( S_a \).

In the last ten years, refined concentration results for the weighted sums \( S_a \) play an important role in the study of singular values of random matrices (see, for instance, Nguyen and Vu [13], Rudelson and Vershynin [15, 16], Tao and Vu [18, 19], Vershynin [21], and the authors of the present paper [4], [5], and [7]). These results reflect the dependence of the bounds on the arithmetic structure of coefficients \( a_k \) under various conditions on the vector \( a \in (\mathbb{R}^d)^n \) and on the distribution \( L(X) \).

Several years ago, Tao and Vu [18] and Nguyen and Vu [13] proposed the so-called Inverse Principles in the Littlewood–Offord problem. In the present paper, we discuss the relations between these Inverse Principles and similar principles formulated by Arak (see [1] and [2]) in his work form the 1980’s.

Apparently the authors of the publications mentioned above were not aware of the principles introduced in Arak [1] and [2]. Although not named as such, his Inverse Principle is related to general bounds for concentration functions of distributions of sums of independent one-dimensional random variables. The results were used for the estimation of the rate of approximation of \( n \)-fold convolutions of probability distributions by infinitely divisible ones. Later, the methods based on Arak’s Inverse Principle admitted to prove a number of other important results concerning infinitely divisible approximation of convolutions of probability measures. In 1986, Arak and Zaitsev have published a monograph [3] containing the aforementioned results and a discussion of the underlying Inverse Principle.

The text on p. 10 of [3] is an analogue of descriptions of the Inverse Principles in the papers of Nguyen, Tao and Vu [13, 14] and [18]. A difference being that they restrict themselves to the classical Littlewood–Offord problem while discussing the arithmetic structure of the coefficients \( a_1, \ldots, a_n \) only. This means that one deals with distributions of sums of non-identically distributed random vectors of a special type. A further difference is that, in [13] and [14], the multivariate case is studied as well.

Nevertheless, there are some consequences of Arak’s results which provide some Inverse Principles for the Littlewood–Offord problem too. Some of them have a non-empty intersection with the results of Nguyen, Tao and Vu [13, 14, 18, 20] (see Theorem [2]). Moreover, in [3], there are some structural results which would be apparently new in the Littlewood–Offord problem (see Theorems [3] and [4]) and have no analogues in the literature. We would like to emphasize that there are of course also some results from [13, 14, 18, 20] which do not follow from the results of Arak.

We denote by \([B]_\tau\) the closed \( \tau \)-neighborhood of a set \( B \in \mathbb{R} \). For a finite set \( K \), we denote by \( |K| \) the number of elements \( x \in K \). The symbol \( \times \) is used for the direct product of sets.
In the sequel we use the notation $G = \mathcal{L}(X_1 - X_2)$. For $\delta \geq 0$, we denote
\[ p(\delta) = G \{ \{ z : |z| > \delta \} \}. \tag{1} \]
Below we will use the condition
\[ G \{ \{ x \in \mathbb{R} : C_1 < |x| < C_2 \} \} \geq C_3, \tag{2} \]
where the values of $C_1, C_2, C_3$ will be specified in the formulations below.

For $\lambda \geq 0$, introduce the distribution $H^\lambda$, with the characteristic function
\[ \hat{H}^\lambda(t) = \exp \left( -\frac{\lambda}{2} \sum_{k=1}^{n} (1 - \cos \langle t, a_k \rangle) \right), \quad t \in \mathbb{R}^d. \tag{3} \]
We should note that $H^\lambda$, is a symmetric infinitely divisible distribution with the Lévy spectral measure $M^\lambda = \lambda^4 M^\ast$, where $M^\ast = \sum_{k=1}^{n} (E_{a_k} + E_{-a_k})$, and $E_b$ is the distribution concentrated at a point $b \in \mathbb{R}^d$.

Lemma 1 below represents an obvious bridge between the Littlewood–Offord problem and general bounds for concentration functions, in particular Arak’s results.

**Lemma 1.** For any $\kappa, \delta > 0$, $\tau \geq 0$, we have
\[ Q(F_a, \tau) \leq c_1(d) (1 + [\kappa/\delta])^d Q(H^{p(\tau/\kappa)}, \delta), \tag{4} \]
where $c_1(d)$ depends on $d$ only, and $[x]$ is the largest integer $k$ that satisfies the inequality $k < x$.

In a recent paper of Eliseeva and Zaitsev [6], a more general statement than Lemma 1 is obtained. It gives useful bounds if $p(\tau/\kappa)$ is small, even if $p(\tau/\kappa) = 0$. Letting $\delta \to 0$, we see that (4) implies that
\[ Q(F_a, 0) \leq c_1(d) Q(H^{p(0)}, 0) = c_1(d) H^{p(0)} \{ \{0\} \}, \tag{5} \]
see Zaitsev [22] for details.

In the monograph [3], it is also shown that if the concentration function of a one-dimensional infinitely divisible distribution is large enough, then the corresponding Lévy spectral measure is concentrated approximately on a set with a special arithmetic structure up to a difference of small measure (see Theorems 3.3 and 4.3 of Chapter II in [3]). Coupled with Lemma 1, these results provide bounds in the Littlewood–Offord problem, see Theorems 1–4.

A set $K \subset \mathbb{R}^d$ is a symmetric Generalized Arithmetic Progression (GAP) of rank $r$ if it can be expressed in the form
\[ K = \{ m_1 g_1 + \cdots + m_r g_r : -L_j \leq m_j \leq L_j, \ m_j \in \mathbb{Z} \text{ for all } 1 \leq j \leq r \} \]
for some $g_1, \ldots, g_r \in \mathbb{R}^d$; $L_1, \ldots, L_r > 0$. The numbers $g_j$ are the generators of $K$, and $\text{Vol}(K) = \prod_{j=1}^{r} (2[L_j] + 1)$ is the volume of $K$ (see [13], [14] and [18]).

For any positive integers $r, m \in \mathbb{N}$ we define $\mathcal{K}^{(d)}_{r,m}$ as the collection of all symmetric GAPs of rank $\leq r$ and of volume $\leq m$. 


We have to mention that, instead of $\mathcal{K}^{(1)}_{r,m}$, Arak [2] has considered one-dimensional projections of sets of integer points of cardinality $\leq m$ contained in symmetric convex subsets of $\mathbb{R}^d$. However, it may be shown that each of these projections of rank $r$ and of volume $\leq c_2(r)$ may be imbedded into a one-dimensional symmetric GAP of rank $r$ and of volume $\leq c_2(r)m$. Therefore, Arak’s results may be easily reformulated in terms of $\mathcal{K}^{(1)}_{r,m}$.

For any Borel measure $W$ on $\mathbb{R}$ and $\tau \geq 0$ we define $\beta_{r,m}(W, \tau)$ by

$$\beta_{r,m}(W, \tau) = \inf_{K \in \mathcal{K}^{(1)}_{r,m}} W\{x \in [K] \cap \mathbb{R} \setminus \mathbb{R} \}.$$ (6)

**Theorem 1.** Let $\kappa, \delta > 0$, $\tau \geq 0$, and let $X$ be a real random variable satisfying condition (2) with $C_1 = \tau/\kappa$, $C_2 = \infty$ and $C_3 = p(\tau/\kappa) > 0$. Let $d = 1$, $r, m \in \mathbb{N}$. Then

$$Q(F_a, \tau) \leq c_3(r)(1 + [\tau/\delta]) \left( \frac{1}{m \sqrt{\beta_{r,m}(M, \delta)}} + \frac{1}{(\beta_{r,m}(M, \delta))^{|r+1|/2}} \right),$$ (7)

where $M = \frac{p(\tau/\kappa)}{4} M^*$, $M^* = \sum_{k=1}^{n} (E_{a_k} + E_{-a_k})$ and where $c_3(r)$ depends on $r$ only.

In order to prove Theorem 1 it suffices to apply Lemma 1 and Theorem 4.3 of Chapter II in [3].

Theorem 2 follows from Theorem 1. The conditions of this theorem are weakened conditions of those used in the results of Nguyen, Tao and Vu [13], [14] and [18].

**Theorem 2.** Let $X$ be a real random variable satisfying condition (2) with $C_1 = 1$, $C_2 = \infty$ and $C_3 = p(1) > 0$. Let $d \geq 1$, $0 < \varepsilon \leq 1$, $0 < \theta \leq 1$, $A > 0$, $B > 0$ be constants and $	au = \tau_n \geq 0$ be a parameter that may depend on $n$. Suppose that $a = (a_1, \ldots, a_n) \in (\mathbb{R}^d)^n$ is a multi-subset of $\mathbb{R}^d$ such that $q_j = Q(F^{(j)}_a, \tau) \geq n^{-A}$, $j = 1, \ldots, d$, where $F^{(j)}_a$ are distributions of coordinates of the vector $S_n$. Let $\rho_n$ denote a non-random sequence satisfying $n^{-B} \leq \rho_n \leq 1$. Then, for any number $n'$ between $\varepsilon n^\varepsilon$ and $n$, there exists a symmetric GAP $K$ such that

1. At least $n - dn'$ elements of $a$ are $\tau \rho_n$-close to $K$ in the norm $|x| = \max_j |x_j|$ (this means that for these elements $a_k$ there exist $y_k \in K$ such that $|a_k - y_k| \leq \tau \rho_n$).

2. $K$ has small rank $R = O(1)$, and small cardinality

$$|K| \leq \prod_{j=1}^{d} \max \left\{ O\left( q_j^{-1} \rho_n^{-1} (n')^{-1/2} \right), 1 \right\}. $$ (8)

Here we write $O(\cdot)$ if the involved constants depend on the parameters named “constants” in the formulation, but not on $n$.

Theorem 1 has been proved for one-dimensional situations and thus initially allows us to prove Theorem 2 for $d = 1$ only. However, it may be shown that this one-dimensional version of Theorem 2 provides sufficiently rich arithmetic properties for the set $a = (a_1, \ldots, a_n) \in \mathbb{R}$. For any Borel measure $W$ on $\mathbb{R}$ and $\tau \geq 0$ we define $\beta_{r,m}(W, \tau)$ by

$$\beta_{r,m}(W, \tau) = \inf_{K \in \mathcal{K}^{(1)}_{r,m}} W\{x \in [K] \cap \mathbb{R} \setminus \mathbb{R} \}.$$ (6)

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$$Q(F_a, \tau) \leq c_3(r)(1 + [\tau/\delta]) \left( \frac{1}{m \sqrt{\beta_{r,m}(M, \delta)}} + \frac{1}{(\beta_{r,m}(M, \delta))^{|r+1|/2}} \right),$$ (7)

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In the multivariate case as well. It suffices to apply the one-dimensional version of Theorem 2 to the distributions \( F_a^{(j)} \), \( j = 1, \ldots, d \).

Theorem 1 has non-asymptotical character, it is more general than Theorem 2 and gives information about the arithmetic structure of \( a = (a_1, \ldots, a_n) \) without assumptions like \( q_j = Q(F_a^{(j)}, \tau) \geq n^{-A}, j = 1, \ldots, d \).

Another one-dimensional result of Arak [1] allows us to formulate another Inverse Principle type result in the Littlewood–Offord problem.

For any \( r \in \mathbb{N} \) and \( u = (u_1, \ldots, u_r) \in (\mathbb{R}^d)^r \), \( u_j \in \mathbb{R}^d \), \( j = 1, \ldots, r \), we introduce the set

\[
K_1(u) = \left\{ \sum_{j=1}^{r} n_j u_j : n_j \in \{-1, 0, 1\}, \text{ for } j = 1, \ldots, r \right\}.
\]  

Define also a collection of sets

\[
K_r^{(d)} = \{ K_1(u) : u = (u_1, \ldots, u_r) \in (\mathbb{R}^d)^r \}.
\]  

It is easy to see that the set \( K_1(u) \) is the symmetric GAP of rank \( r \) and volume 3\(^r\).

The following Theorems 3 and 4 are consequences of Theorem 3.3 of Chapter II [3] which follows directly from the results of Arak [1].

It is interesting that, in the multivariate case, Theorems 3 and 4 are obtained by the application of their one-dimensional versions to the distributions of coordinates of the vector \( S_a \).

**Theorem 3.** Let \( X \) be a real random variable satisfying condition (2) with \( C_1 = 1 \), \( C_2 = \infty \) and \( C_3 = p(1) > 0 \). Let \( \tau_j \geq \delta_j \geq 0 \) and \( q_j = Q(F_a^{(j)}, \tau_j), j = 1, \ldots, d \). Then there exist an absolute constant \( c \), numbers \( r_1, \ldots, r_d \in \mathbb{N} \) and vectors \( u^{(j)} = (u_1^{(j)}, \ldots, u_{r_j}^{(j)}) \in \mathbb{R}^{r_j}, j = 1, \ldots, d \), such that

\[
R = \sum_{j=1}^{d} r_j \leq c \sum_{j=1}^{d} (|\log q_j| + \log(\tau_j/\delta_j) + 1),
\]  

and

\[
p(1) M^* \{ \mathbb{R}^d \setminus \times_{j=1}^{d} [K_1(u^{(j)})]_{\delta_j} \} \leq c \sum_{j=1}^{d} (|\log q_j| + \log(\tau_j/\delta_j) + 1)^3,
\]  

where \( K_1(u^{(j)}) \in K_r^{(1)} \) and

\[
M^* = \sum_{k=1}^{n} \left( E_{a_k} + E_{-a_k} \right).
\]

Furthermore, the set \( \times_{j=1}^{d} K_1(u^{(j)}) \) can be represented as \( K_1(u) \in K_r^{(d)} \), \( u = (u_1, \ldots, u_{R}) \in (\mathbb{R}^d)^{R} \). Moreover, the vectors \( u_s \in \mathbb{R}^d, s = 1, \ldots, R \), have only one non-zero coordinate each.

Denote

\[
s_0 = 0 \quad \text{and} \quad s_k = \sum_{j=1}^{k} r_j, \quad k = 1, \ldots, d.
\]

For \( s_{k-1} < s \leq s_k \), the vectors \( u_s \) are non-zero in the \( k \)-th coordinates only and these coordinates are equal to the sequence of coordinates \( u_{1}^{(k)}, \ldots, u_{r_k}^{(k)} \) of the vectors \( u^{(k)} \).
Theorem 4. Let $X$ be a real random variable satisfying condition (2) with $C_1 = 1$, $C_2 = \infty$ and $C_3 = p(1) > 0$. Let $A, B > 0$. Let $\tau_j \geq \delta_j \geq 0$, $\tau_j/\delta_j \leq n^B$ and $q_j = Q(F_a^{(j)}), \tau_j \geq n^{-A}$, for $j = 1, \ldots, d$. Then there exist an absolute constant $c$, numbers $r_1, \ldots, r_d \in \mathbb{N}$ and vectors $u^{(j)} = (u^{(j)}_1, \ldots, u^{(j)}_r) \in \mathbb{R}^r$, $j = 1, \ldots, d$, such that

$$R = \sum_{j=1}^{d} r_j \leq c d \left( (A + B) \log n + 1 \right),$$

and

$$p(1) M^*\{ \mathbb{R}^d \setminus \times_{j=1}^{d} [K_1(u^{(j)})]_{\delta_j} \} \leq c d \left( (A + B) \log n + 1 \right)^3,$$

where $K_1(u^{(j)}) \in K_a^{(j)}$ and $M^* = \sum_{k=1}^{n} (E_{a_k} + E_{-a_k})$. Moreover, the description of the set $K_1(u) = \times_{j=1}^{d} K_1(u^{(j)})$ from the end of the formulation of Theorem 3 remains true.

It is easy to see that, in conditions of Theorem 4 with $\tau_j = \delta_j n^B = \tau$, $j = 1, \ldots, d$, the set $K_1(u)$ is a symmetric GAP of rank $R = O(\log n)$, of volume $3^R = O(n^D)$ (with a constant $D$) and such that at least $n - O(\log^3 n)$ elements of $a = (a_1, \ldots, a_n) \in (\mathbb{R}^d)^n$ are $\tau/n^B$-close to $K_1(u)$. Theorem 3 provide bounds by replacing $\log n$ by $|\log q|$ without the assumption $q = Q(F_a, \tau) \geq n^{-A}$. Moreover, in (12) and (14), the dependence of constants on $C_3 = p(1)$ is stated explicitly.

Notice that if $\tau_1 = \cdots = \tau_d = \tau$, then $q = Q(F_a, \tau) \leq q_j$, and $|\log q_j| \leq |\log q|, j = 1, \ldots, d$. Moreover, there exist distributions for which the quantity $q$ may be sufficiently smaller than $\max_j q_j$. Consider, for instance, the uniform distribution on the boundary of the square $\{ x \in \mathbb{R}^2 : |x| = 1 \}$.

Moreover, if all $\tau_j$ and $\delta_j$ are equal to zero, we can use (3) instead of (4) and replace $p(1)$ by $p(0)$ in (12) and (14), see Zaitsev [22] for details. In this case we agree that $\log(\tau_j/\delta_j) = 0$.

Now we compare our Theorems 2, 3 and 4 with the results discussed in a review of Nguyen and Vu [14] (see Theorems 7.5, 9.2 and 9.3 of [14]). These results were obtained under the assumption $Q(F_a, \tau) \geq n^{-A}$. This implies that $Q(F_a^{(j)}, \tau) \geq n^{-A}, j = 1, \ldots, d$, since $Q(F_a^{(j)}, \tau) \geq Q(F_a, \tau)$.

A few years ago Tao and Vu [18] formulated the so-called Inverse Principle, stating that

A set $a = (a_1, \ldots, a_n)$ with large $Q(F_a, 0)$ must have strong additive structure.

Theorem 7.5 of [14] was obtained by Tao and Vu [18]. This theorem is named in [14] “Weak Inverse Principle”.

Later, Tao and Vu [20] improved the result of Theorem 7.5 of [14]. Nguyen and Vu [13] have extended the Inverse Principle to the continuous case proving, in particular, Theorems 9.2 and 9.3 of [14].

Theorem 2 allows us to derive Theorem 7.5 of [14] and a one-dimensional version of the first two statements of Theorem 9.3 of [14].

Lemma 1 is interesting only if we assume that $p(\tau/\kappa) > 0$. This assumption is closely related to assumption (2) in Theorems 9.2 and 9.3 of [14] (with $C_2 < \infty$). We can w.l.o.g.
take in (2) $C_1 = 1$ and $C_3 = p(1)$. Moreover, in our results, $C_2 = \infty$. We think that using Lemma 1 one could show that $C_2$ may be taken as $C_2 = \infty$ in Theorems 9.2 and 9.3 of [14] too. Note, however, that $p(1)$ is involved in our inequalities explicitly, in contrast with Theorems 9.2 and 9.3 of [14].

The basic inequality of Theorem 9.3 of [14] gives the bound

$$|K| \leq \max \left\{ O(q^{-1}(n')^{-1/2}), 1 \right\}, \quad \text{where } q = Q(F_a, \tau). \tag{15}$$

Inequality (15) and inequality (8) of Theorem 2 (with $\rho_n = 1$) are not only of the same form, but their contents are almost the same, at least for $d = 1$. Attentive readers may notice evident differences though. In particular, the last item of Theorem 9.3 of [14] is absent in Theorem 2. On the other hand, in Theorem 2, we take $C_2 = \infty$.

Sometimes, for $d > 1$, inequality (8) (with $\rho_n = 1$) may be even stronger than inequality (15). For example, if the vector $S_a$ has independent coordinates (this may happen if each of the vectors $a_j$ has only one non-zero coordinate), then

$$c_4(d) \prod_{j=1}^d q_j \leq Q(F_a, \tau) \leq \prod_{j=1}^d q_j \tag{16}$$

with a positive $c_4(d)$. Note, however, that we could derive a multivariate analogue of Theorem 9.3 of [14] from its one-dimensional version arguing precisely as in the proof of our Theorem 2. Then we get inequality (8).

Theorem 2 can be considered as an analogue of both Theorems 9.2 and 9.3 of [14]. Comparing these theorems, we should mention that the amount of approximating points is sometimes a little bit smaller in Theorem 9.2 of [14], but, in Theorem 2 $C_2 = \infty$, and we get a variety of results by choosing various $\rho_n$, while in Theorem 9.2 of [14] $\rho_n = n^{-1/2} \log n$, and in Theorem 9.3 of [14] $\rho_n = 1$.

The assertion of Theorem 4 implies that, in conditions of Theorem 9.3 of [14], there exists a symmetric GAP $K$ of rank $R = O(\log n)$, of volume $3^R = O(n^D)$ and such that at least $n - O((\log n)^3)$ elements of $a = (a_1, \ldots, a_n) \in (\mathbb{R}^d)^n$ are $\tau/n^R$-closed to $K$. Moreover, Theorem 3 provide bounds with replacing $\log n$ by $|\log q|$ without assumption $q = Q(F_a, \tau) \geq n^{-4}$ (recall that this assumption is absent in the conditions of Theorem 1 too). Comparing with the results of [14], we see that in Theorem 4 the exceptional set has a logarithmic size (which is much better than $O(n)$ and $O(n^\theta)$, $0 < \theta \leq 1$, in Theorems 9.2 and 9.3 of [14]), but this is attained at the expense of a logarithmic growth of the rank.

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