Sine-Squared Pulse Approximation for Matched Filter Design Using Generalized Bessel Polynomials and Particle Swarm Optimization

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Abstract. This paper presents the study of mathematical characteristics of generalized Bessel polynomial that can be applied to approximate a sine-squared pulse for designing matched filters in communication systems. The proposed pulse can be designed by using the transfer function, in which the numerator is the five pairs of a transmissions zero pairs, and the generalized Bessel polynomial is used as the denominator. A parameters of generalized Bessel polynomials can be adjusted by particle swarm optimization to find the best parameter value. From the simulation results can be found that a parameter can be adjusted. A proposed pulse is close to the ideal response in mainlobe, and one sidelobe to four sidelobes with stability, which outperformed previous research.

1. Introduction

In the communication system, a matching of signal transmission between transmitter and receiver through a band-limit channel is very important. A matched filter [1] is one of a solution to solve this problem. A sine-squared pulse [2] is a selected pulse to design matched filters and needs to derive in form of a polynomial equation that can be applied to an electrical circuit. From the previous study, the Bessel polynomial, applied from the Bessel filter [3-5]. This polynomial has an advantage and good characteristic to be created the analog filter circuit, such as the maximally flat delay, linear phase. Especially, a mathematical characteristics of Bessel polynomials is similar to a mathematical characteristic of sine-squared pulse. Bessel polynomials can write in form of generalized Bessel polynomials [6] that is used for applying in this paper. In the literature review of this paper, S. Mneina, et al. [7] presented the linear phase matched filter using Bessel polynomials. However, this work is still far from an ideal response and illustrates only main lobe and one sidelobe. V. Pirajnanchai, et al. [8] applied the generalized Bessel transformation to design a data transmission filter. Unfortunately, the response of magnitude and attenuation is still incomplete. Moreover, it has high-order polynomials of the transfer function and high complexity when used to synthesize an electrical circuit. R. Singthongchai, [9] et al. applied q-Bessel polynomials with the fifth-order to the seventh-order with four sidelobes responses. However, an order of a denominator polynomial of the transfer function is more than a nominator polynomial. It can cause not stability in the operation. The previous works in [10-11]
can be seen that the use of generalized Bessel polynomial with parameters tuned to achieve a frequency response closest to the ideal pulse. However, the adjustment experiment was also a trial-and-error adjustment of the pulse parameters. The side lobes in the second lobe, the third lobe, and the fourth lobe have different values from the ideal value. The trial-and-error approach to adjust parameters to the closest to ideal value would be a huge waste of time. Therefore, there must be a way to optimize the pulse to find the most suitable parameter with the least error value. In this paper, we will present the techniques for finding the most suitable values.

The particle swarm optimization (PSO) [13] technique was used to determine the optimal parameters for simulating the closest ideal squared sine wave signal. This paper proposes an optimization method to approximate sine-squared pulse for designing matched filters by using generalized Bessel polynomials with particle swarm optimization. Generalized Bessel polynomial is used as a dominator in the transfer function and the transmission zero pairs are used as a nominator of a transfer function. And then simulation of this transfer function to close with an ideal sine-squared pulse signal. This paper has contained four sections as follows: section two is about preliminaries of mathematical modelling such as sine-squared pulse signal, generalized Bessel polynomials, and particle swarm optimization. Section 3 presents the simulation results. The final section is the conclusion.

2. Mathematical Modelling

2.1. Sine-squared pulse signal

The Sine-squared pulse is used for applying to design matched filters [10] in baseband communication. The ideal of the sine-squared pulse is given in Eq. (1)

\[ h(t) = \sin^2 \left( \frac{\pi t}{2\tau} \right); \quad 0 \leq t \leq 2\tau \]  

(1)

From Eq. (1), using the Laplace’s Transform and then can be written as

\[ h(s) = \left( \frac{4}{s^2 + 4} \right) \left( \frac{1}{1 + \coth \left( \frac{s\tau}{2} \right)} \right) \]  

(2)

However, the sine-squared pulse is derived in form of a mathematical hyperbolic function. It is not suitable to design matched filter circuits. By equation, sine-squared pulses represent ideal equations. Therefore, it requires specific features of sine-squared pulse that are similar to a special function or special polynomials [14]. From Storch’s approach [2], The Bessel polynomials have a mathematical characteristic that can show in a transfer function as an equation

\[ h(s) = e^{-s} \]  

(3)

From equation (3), fractional this equation into nominator and dominator. And then can be derived as a form of a hyperbolic function as follows

\[ h(s) = \frac{\frac{1}{\sinh \left( \frac{s\tau}{2} \right)}}{\sinh \left( \frac{s\tau}{2} \right) + \cosh \left( \frac{s\tau}{2} \right)} = \frac{1}{\sinh \left( \frac{s\tau}{2} \right)} \]  

(4)
From equation (4), hyperbolic equation of the dominator of transfer function has a formation similar to equation (2). It confirms that the Bessel polynomial can apply to approximate a sine-squared signal for matched filter design.

2.2. Generalized Bessel Polynomials

The explicit form of the generalized Bessel polynomials [6],[11] order \( n \) is introduced as

\[
B_n(s, \alpha, \beta) = \sum_{k=0}^{n} \left( \frac{(\alpha + n - 1)(-n)_k}{k!} \right) (-\frac{s}{\beta})^k
\]

where \((a)_k = \prod_{i=0}^{k-1} (a + i)\) and \( \alpha, \beta \) are parameters of generalized Bessel polynomials, generalized Bessel polynomials have stability with all poles in left-half of the \( s \)-complex plane and have mathematical characteristics similar to sine-squared pulse equation. The sine-squared pulse of the matched filter is required to have the same shape as the transmit. However, it is reversed in time and shifts with constant time delay \( t_d \), the magnitude and attenuation can be derived as

\[
P(f) = t_d S_a(2\pi ft_d) + \frac{t_d}{2} S_a(2\pi ft_d + \pi) + \frac{t_d}{2} S_a(2\pi ft_d - \pi)
\]

where \( S_a = \sin(x)/x \) for \( t_d = 1 \). The ideal magnitude response in a time domain and attenuation response of the sine-squared pulse are shown in Figure 1 and Figure 2, respectively.

![Figure 1. Ideal sine-squared pulse magnitude response](image-url)
2.3. Transfer function

To design the matched filter, transfer function can be written as factional polynomial for sine-squared pulse test signal generating. The transfer function has a numerator with transmission zero pairs $\pm jk\pi, k = 2, 3, 4, \ldots, m$. The overall system has minimum stopband attenuation. Therefore, the transfer function $H(s)$ can be written as

$$H(s) = \frac{P_m(s)}{Q_n(s)} = \frac{\text{Transmission Zeros}}{\text{Generalized Bessel Polynomials}} = \prod_{k=1}^{n/2} (s^2 + \alpha_k^2) / \left( \prod_{k=1}^{n} (s - p_k) \right)$$

where $m$ is the degree of the numerator of polynomial, $n$ is the degree of the denominator polynomial or generalized Bessel polynomials, $\alpha_k$ is the transmission zero pairs $\pm jk\pi$ and $p_k$ is the poles in the left half of $s$-plane.

The numerator of the transfer function are used five pairs of transmissions zero at $\pm j2\pi$, $\pm j3\pi$, $\pm j4\pi$, $\pm j5\pi$ and $\pm j6\pi$. In addition, the dominator of a transfer function is used the generalized Bessel polynomials. Therefore, this paper proposes a method to reduce the attenuation by using the GPB as dominator of the transfer function and varies the parameters alpha and beta in the function. The proposed transfer function is given by

$$H(s) = \prod_{k=1}^{n/2} (s^2 + \alpha_k^2) / \left( \sum_{k=0}^{n} \frac{(\alpha + n - 1)(-n)_k}{k!} \right) (-\frac{s}{\beta})$$

alpha and beta parameters can adjust values to control the response that is close to the ideal response. In this paper, we consider the main lobe and the first sidelobe to the fourth sidelobe.

2.4. Particle swarm optimization

Particle swarm optimization is one of the optimization algorithms [13], which is a population-based optimization algorithm inspired by the social movement of a swarm of animals. This algorithm starts by randomly positioning every particle in the swarm within the search space and then calculates the fitness from the particles and determines the GBest value, which is equal to the best PBest value in the herd in a single iteration. Particles are adjusted based on their PBest and GBest positions. The algorithm
operates to find the best parameters. An equation of the particle $P$ in each $t$ in a swarm can be explained as

$$X_{p,t} = (X_{p1}, X_{p2}, ..., X_{pt})$$

(9)

The velocity of each particle can be defined as

$$V_{p,t} = (V_{p1}, V_{p2}, ..., V_{pt})$$

(10)

defines PBest is the best value parameter of particle $p$ and GBest is the best value parameter of particle swarm the current iteration and $t$ is number of iteration in positive number and $Obj_{p,t}$ is the objective function of each particle $p$ at position $X_p$

$$Obj_{p,t} = f(X_{p,t})$$

(11)

Therefore, an updated position of each particle can be derived as

$$X_{p,t+1} = X_{p,t} + V_{p,t+1}$$

(12)

The velocity of each particle can be explained in this equation

$$V_{p,t} = wV_{p,t-1} + n_1r_1(P\text{Best}_{p}, p - X_{p,t}) + n_2r_2(G\text{Best} - X_{p,t})$$

(13)

where $w$ is weight factor, $n_1, n_2$ is acceleration factor, and $r_1, r_2$ is distributed randomly in range (0,1).

Applying the PSO algorithm with the sine-squared pulse optimization problem can be explained in this below pseudo code.

**Generate Initial Solution**

Setting $\alpha = \beta = 2$ As initial velocity
Setting order $n$
Setting Iteration = 30
Particle = 50
Objective Function = min(Error)

**Do**

Find Pbest of $\alpha$ and $\beta$
Find Gbest of $\alpha$ and $\beta$
Calculate Velocity
Update position of $\alpha$ and $\beta$ by using velocity
**If** error < maximum possible value
    Update Pbest of $\alpha$ and $\beta$
    Maximum possible value = error

**End**

Gbest = min(Maximum Possible Value)
Updating Velocity for all particles
**Repeat Until** completed of all iteration and particles
**Show Gbest of $\alpha$ and $\beta$**
We apply this algorithm to find the best parameter of $\alpha$ and $\beta$ in each generalized Bessel polynomial order of transfer function with a minimum mean squared error that will discuss in the next section.

3. Simulation Results
In this section, we propose a simulation result from a transfer function and optimization method from the previous section. MATLAB software is used for simulation. To simulate an optimized pulse with PSO, we define 30 iterations with 50 swarms. A comparing with main lobe and sidelobes from th order 10 to order 14 to find an optimal parameters and mean squared error value of each lobes. For main lobe is $\omega_0 \leq \omega \leq \omega_1$, first sidelobe is $\omega_1 \leq \omega \leq \omega_2$, the second sidelobe is $\omega_2 \leq \omega \leq \omega_3$, the third sidelobe is $\omega_3 \leq \omega \leq \omega_4$, the fourth sidelobe is $\omega_4 \leq \omega \leq \omega_5$. From Table 1, substitutes a parameters into transfer function equation (8).

**Table 1.** Simulation results using PSO with five transmission zero pairs or four sidelobes

| Order | $\alpha$ | $\beta$ | MSE | MSE 1 | MSE 2 | MSE 3 | MSE 4 | MSE 5 |
|-------|---------|--------|-----|-------|-------|-------|-------|-------|
| 10    | 4.8544  | 1.6570 | 3.880-05 | 7.908e-05 | 2.001e-04 | 8.578e-05 | 3.348e-04 |
| 11    | 5.3242  | 1.7510 | 1.456e-05 | 1.148e-04 | 6.817e-05 | 1.984e-04 |
| 12    | 5.6999  | 1.8361 | 1.226e-05 | 6.534e-05 | 5.140e-05 | 1.187e-04 |
| 13    | 5.9743  | 1.9020 | 1.194e-05 | 3.701e-05 | 3.712e-05 | 7.129e-05 |
| 14    | 6.1424  | 1.9833 | 2.048e-05 | 2.566e-05 | 2.425e-05 | 2.425e-05 |

**Table 2.** A transfer function of simulation results using PSO with five transmission zero pairs or four sidelobes

| Order | Transfer Function |
|-------|------------------|
| 10    | $H_{10}(s) = \frac{s^{10} + (4.219e-15)s^9 + 888.3s^8 + (2.007e-12)s^7 + (2.838e05)s^6 + (2.576e-10)s^5 + (3.996e07)s^4 + (8.284e-09)s^3}{s^{10} + 83.61s^9 + 3373s^8 + 8.606e04s^7 + 1.532e06s^6 + 1.918e07s^5 + 1.878e08s^4 + 1.286e09s^3}$ |
| 11    | $H_{11}(s) = \frac{s^{11} + (4.219e-15)s^{10} + 888.3s^9 + (2.007e-12)s^8 + (2.838e05)s^7 + (2.576e-10)s^6 + (3.996e07)s^5 + (8.284e-09)s^4}{s^{11} + 96.27s^{10} + 4487s^9 + 1.332e08s^8 + 2.788e06s^7 + 4.307e07s^6 + 5.08e05s^5 + 4.349e09s^4}$ |
| 12    | $H_{12}(s) = \frac{s^{12} + (4.219e-15)s^{11} + 888.3s^{10} + (2.007e-12)s^9 + (2.838e05)s^8 + (2.576e-10)s^7 + (3.996e07)s^6 + (8.284e-09)s^5}{s^{12} + 109.1s^{11} + 5787s^{10} + 1.965e08s^9 + 4.743e06s^8 + 8.555e07s^7 + 1.18e09s^6 + 1.25e10s^5 + 2.723e11s^4 + 3.42e11s^3 + 4.497e11s^2 + 4.497e11s + 4.497e11}$ |
| 13    | $H_{13}(s) = \frac{s^{13} + (4.219e-15)s^{12} + 888.3s^{11} + (2.007e-12)s^{10} + (2.838e05)s^9 + (2.576e-10)s^8 + (3.996e07)s^7 + (8.284e-09)s^6}{s^{13} + 122.1s^{12} + 7267s^{11} + 7.623e09s^{10} + 1.576e08s^9 + 2.523e07s^8 + 3.162e06s^7 + 3.095e05s^6 + 2.334e04s^5 + 1.316e3s^4 + 5.294e3s^3 + 1.325e4s^2 + 1.597e4s + 1.597e4}$ |
| 14    | $H_{14}(s) = \frac{s^{14} + (4.219e-15)s^{13} + 888.3s^{12} + (2.007e-12)s^{11} + (2.838e05)s^{10} + (2.576e-10)s^9 + (3.996e07)s^8 + (8.284e-09)s^7}{s^{14} + 135.1s^{13} + 8920s^{12} + 3.804e09s^{11} + 1.16e07s^{10} + 2.725e08s^9 + 4.976e09s^8 + 7.21e10s^7 + 8.315e11s^6 + 7.587e12s^5 + 5.383e13s^4 + 2.876e14s^3 + 1.093e15s^2 + 2.64e15s + 3.056e15}$ |
From the Table 2, the transfer functions can be used to simulate a response and compare between ideal pulse and proposed pulse. The comparison of magnitude response, attenuation response and mean squared error ideal pulse and proposed pulse are illustrated in Figure 3, Figure 4, and Figure 5 respectively.

**Figure 3.** The comparison of magnitude response of sine-squared pulse with ideal function and optimized pulse order 10 to order 14.

**Figure 4.** The comparison of attenuation response of sine-squared pulse with ideal function and optimized pulse order 10 to order 14.
Figure 5. The comparison of mean squared error of sine-squared pulse with ideal function and optimized pulse order 10 to order 14.

From the results of all these experiments, Figure 3 and Figure 4 show a results that proposed pulse is satisfy the ideal pulse along main lobe to the fourth side lobes. The 14th order polynomial is make the closest to the ideal pulse and the 10th order polynomial meets to the ideal pulse from main lobes to the third side lobes. But, in the 4th side lobe has an slightly error. However, a lower order cause to reduce a complexity for designing to a communication circuit. it can be concluded that the generalized Bessel polynomials can be used to design a sine-squared pulse and linear phase matched filter. The parameters of the generalized Bessel polynomial were optimized using the particle swarm optimization algorithm. The results of the experiment showed the efficiency of the algorithm to define a parameter with lower error than previous work [7-11]. Moreover, this approach can apply for the simulation of matched filters to the 4th side lobes with stability.

4. Conclusions
This paper presented the sine-squared pulse approximation for matched filter design by using a mathematical characteristic of generalized Bessel polynomial that can be applied to the design of matched filter in communication systems. The study of Storch’s approach can be applied to a mathematical characteristic similar to a sine-squared pulse function. This pulse can be designed by using the transfer function in which the numerator is the transmission’s zero pairs, and the generalized Bessel polynomial is used as the denominator. The simulation results can be found that a parameter can be adjusted and close to an ideal response using particle swarm optimization. it is closest to ideal pulses, from the first sidelobe to the fourth sidelobe with a minimum value of mean-squared error. To confirm the performance of generalized Bessel polynomials with particle swarm optimization when compares with previous literature. In future works, we will apply a transfer function to synthesize a communication circuit.

5. References
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