PARTIAL REGULARITY OF SOLUTIONS OF FULLY NONLINEAR
UNIFORMLY ELLIPTIC EQUATIONS

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Abstract. We prove that a viscosity solution of a uniformly elliptic, fully nonlinear equation is $C^{2,\alpha}$ on the complement of a closed set of Hausdorff dimension at most $\varepsilon$ less than the dimension. The equation is assumed to be $C^1$, and the constant $\varepsilon > 0$ depends only on the dimension and the ellipticity constants. The argument combines the $W^{2,\varepsilon}$ estimates of Lin with a result of Savin on the $C^{2,\alpha}$ regularity of viscosity solutions which are close to quadratic polynomials.

1. Introduction

In this paper, we prove a partial regularity result for viscosity solutions of the uniformly elliptic equation

$$F(D^2u) = 0.$$ 

The operator $F$ is assumed to be uniformly elliptic and to have uniformly continuous first derivatives (these hypotheses are precisely stated in the next section).

If $F$ is concave or convex, then solutions of (1.1) in a domain $\Omega \subseteq \mathbb{R}^n$ are known to belong to $C^{2,\alpha}(\Omega)$ for some small $\alpha > 0$, according to the famous theorem of Evans [5] and Krylov [6] (see also [3] for a simple proof). Viscosity solutions of (1.1) have also been shown to be classical solutions for certain classes of nonconvex operators by Yuan [11] as well as Cabrér and Caffarelli [1]. The latter result applies, for example, to an $F$ which is the minimum of a convex and a concave operator. However, $C^2$ estimates for solutions of (1.1) are unavailable for general $F$, as attested by the recent counterexamples of Nadirashvili and Viñard [8, 9]. In fact, a counterexample to $C^{1,1}$ regularity was presented in [9], and therefore the best available regularity for solutions of (1.1) is $C^{1,\alpha}$.

In this paper, we study the singular set of a solution $u$ of (1.1), consisting of those points $x$ for which $u$ of (1.1) fails to be $C^{2,\alpha}$ in any neighborhood of $x$. Our result asserts that the singular set has Hausdorff dimension at most $n - \varepsilon$, where the constant $\varepsilon > 0$ depends only on the ellipticity of $F$ and the dimension of the ambient space. The hypotheses (F1) and (F2) are stated in Section 2.

Theorem 1. Assume that $F$ satisfies (F1) and (F2). Let $u \in C(\Omega)$ be a viscosity solution of (1.1) in a domain $\Omega \subseteq \mathbb{R}^n$. Then there is a constant $\varepsilon > 0$, depending only on $n$, $\lambda$, and $\Lambda$, and a closed subset $\Sigma \subseteq \Omega$ of Hausdorff dimension at most $n - \varepsilon$, such that $u \in C^{2,\alpha}(\Omega \setminus \Sigma)$ for every $0 < \alpha < 1$.

As far as we know, Theorem 1 is the first result which provides an estimate on the smallness of the singular set of a solution of a general uniformly elliptic, fully nonlinear...
equation. The constant $\varepsilon > 0$ which appears in the statement of the theorem is the same $\varepsilon$ as in the $W^{2,\varepsilon}$ estimate of Lin [7]; see Remark 5.4.

Let us describe the idea of the proof of Theorem 1. By differentiating (1.1) and applying $W^{2,\varepsilon}$ estimates to $Du$, we effectively obtain a $W^{3,\varepsilon}$ estimate for the solution $u$. Precisely formulated, this implies that $u$ has a global second-order Taylor expansion almost everywhere and that the constant in front of the cubic error term lies in $L^\varepsilon$. Near points possessing quadratic expansions, we apply a generalization of a result of Savin [10], which asserts that any viscosity solution of (1.1) that is sufficiently close to a quadratic polynomial must be $C^{2,\alpha}$. The $L^\varepsilon$ integrability of the modulus of the quadratic expansion then restricts the Hausdorff dimension of the singular set.

While $\varepsilon$ does not depend on the modulus $\omega$ in (F2), the assumption that $F$ is $C^1$ with a uniformly continuous first derivative is crucial to our method of proof. In particular, Theorem 1 does not apply to Bellman-Isaacs equations, which have the form

\[ F(D^2u) := \inf_{\alpha \in I} \sup_{\beta \in J} (-\text{tr}(A_{\alpha\beta}D^2u)) = 0. \]

Such operators $F$ are positively homogeneous, and obviously any function which is positively homogeneous and differentiable at the origin is linear. Therefore, the assumption (F2) is incompatible with nonlinearity if $F$ is positively homogeneous. We do not know whether such a partial regularity result is true for equations of the form (1.2).

In the next section, we state our notation and some preliminary results needed in the proof of Theorem 1. In Sections 3 and 4, we give complete arguments for the $W^{2,\varepsilon}$ estimate and the $C^{2,\alpha}$ regularity for flat solutions of (1.1). The proof of Theorem 1 is presented in Section 5.

2. Preliminaries

In this section we state our hypotheses and collect some standard ingredients needed in the proof of Theorem 1.

**Notation and hypotheses.** Let $\mathcal{M}_n$ denote the set of real $n$-by-$n$ matrices, and $\mathcal{S}_n \subseteq \mathcal{M}_n$ the set of symmetric matrices. Recall that the Pucci extremal operators are defined for constants $0 < \lambda \leq \Lambda$ and $M \in \mathcal{S}_n$ by

\[ \mathcal{P}^+_{\lambda,\Lambda}(M) := \sup_{\lambda I_n \leq A \leq \Lambda I_n} -\text{tr}(AM) \quad \text{and} \quad \mathcal{P}^-_{\lambda,\Lambda}(M) := \inf_{\lambda I_n \leq A \leq \Lambda I_n} -\text{tr}(AM). \]

Throughout this paper, we assume the nonlinear operator $F : \mathcal{S}_n \to \mathbb{R}$ satisfies the following:

(F1) $F$ is uniformly elliptic and Lipschitz; precisely, we assume that there exist constants $0 < \lambda \leq \Lambda$ such that, for every $M, N \in \mathcal{S}_n$,

\[ \mathcal{P}^-_{\lambda,\Lambda}(M - N) \leq F(M) - F(N) \leq \mathcal{P}^+_{\lambda,\Lambda}(M - N). \]

(F2) $F$ is $C^1$ and its derivative $DF$ is uniformly continuous, that is, there exists an increasing continuous function $\omega : [0, \infty) \to [0, \infty)$ such that $\omega(0) = 0$, and for every $M, N \in \mathcal{S}_n$,

\[ |DF(M) - DF(N)| \leq \omega(|M - N|). \]

We call a constant *universal* if it depends only on the dimension $n$, the ellipticity constants $\lambda$ and $\Lambda$, and the modulus $\omega$. Throughout, $c$ and $C$ denote positive universal constants which may vary from line to line. We denote by $Q_{x,r}$ the cube centered at $x$ and of side
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length 2r. That is, we define $Q_{x,r} := \{ y \in \mathbb{R}^n : |y_i - x_i| \leq r \}$ and $Q_r := Q_{0,r}$. Balls are written $B(x,r) := \{ y \in \mathbb{R}^n : |y - x| < r \}$ and $B_r := B(0,r)$.

Recall that the Hausdorff dimension of a set $E \subseteq \mathbb{R}^n$ is defined by

$$\mathcal{H}_{d_{\text{im}}}(E) := \inf \left\{ 0 \leq s < \infty : \text{for all } \delta > 0, \text{ there exists a collection } \{B(x_j, r_j)\} \text{ of balls such that } E \subseteq \bigcup_{j=1}^{\infty} B(x_j, r_j) \text{ and } \sum_{j=1}^{\infty} r_j^s < \delta \right\}.$$  

**Standard results.** In this subsection, we state three results needed below. Proofs of all three of these results can be found in [2]. We first recall the statement of the Alexandroff-Bakelman-Pucci (ABP) inequality. We use the notation $u^+ = \max\{0, u\}$ and $u^- := -\min\{0, u\}$.

**Proposition 2.1 (ABP inequality).** Assume that $B_R \subseteq \mathbb{R}^n$ and $f \in C(B_R) \cap L^\infty(B_R)$. Suppose that $u \in C(B_R)$ satisfies inequality

$$\mathcal{D}^+_{\lambda,\Lambda}(D^2 u) \geq f \text{ in } B_R,$$

$$u \geq 0 \text{ on } \partial B_R.$$  

Then

$$\sup_{B_R} u^- \leq CR \|f^+\|_{L^\infty(\{\Gamma_u = u\})},$$  

where $C$ is a universal constant and $\Gamma_u$ is the convex envelope in $B_{2r}$ of $-u^-$, where we have extended $u \equiv 0$ outside $B_R$.

We next recall an interior $C^{1,\alpha}$ regularity result for solutions of (1.1).

**Proposition 2.2.** If $u$ is a viscosity solution of (1.1) in $B_1$, then $u \in C^{1,\alpha}(B_{1/2})$ for some universal $0 < \alpha < 1$.

Finally, we recall a consequence of the Calderón-Zygmund cube decomposition. This appeared in a slightly different form as [2, Lemma 4.2].

**Proposition 2.3.** Suppose that $D \subseteq E \subseteq Q_1$ are measurable and $0 < \delta < 1$ is such that:

- $|D| \leq \delta|Q_1|$: and
- if $x \in \mathbb{R}^n$ and $r > 0$ such that $Q_{x,3r} \subseteq Q_1$ and $|D \cap Q_{x,r}| \geq \delta|Q_r|$, then $Q_{x,3r} \subseteq E$.

Then $|D| \leq \delta|E|$.

3. $W^{2,\varepsilon}$ estimate

An integral estimate for the second derivatives of strong solutions of linear, uniformly elliptic equations in nondivergence form with only measurable coefficients was first obtained by Lin [3]. It was later extended to viscosity solutions of fully nonlinear equations in [2].

To state the estimate, we require some notation. Given a domain $\Omega \subseteq \mathbb{R}^n$ and a function $u \in C(\Omega)$, define the quantities

$$\Theta(x) = \Theta(u, \Omega)(x) := \inf \left\{ A \geq 0 : \text{there exists } p \in \mathbb{R}^n \text{ such that for all } y \in \Omega, \right.$$  

$$u(y) \geq u(x) + p \cdot (x - y) - \frac{1}{2} A |x - y|^2 \right\},$$
\[ \Theta(x) = \Theta(u, \Omega)(x) := \inf \{ A \geq 0 : \text{there exists } p \in \mathbb{R}^n \text{ such that for all } y \in \Omega, \]
\[ u(y) \leq u(x) + p \cdot (x - y) + \frac{1}{2} A |x - y|^2 \}, \]
and
\[ \Theta(x) := \Theta(u, \Omega)(x) = \max \{ \Theta(u, \Omega)(x), \Theta(u, \Omega)(x) \} . \]
The quantity \( \Theta(u, \Omega)(x) \) is the minimum curvature of any paraboloid that touches \( u \) from below at \( x \). If \( u \) cannot be touched from below at \( x \) by any paraboloid, then \( \Theta(u, \Omega)(x) = +\infty \). A similar statement holds for \( \Theta(x) \), where we touch from above instead.

The form of the \( W^{2,\varepsilon} \) estimates we need is contained in the following proposition.

**Proposition 3.1.** If \( u \in C(B_1) \) satisfies the inequality
\[ P_{\lambda, \Lambda}^{+}(D^2 u) \geq 0 \quad \text{in } B_1, \]
then for all \( t > t_0 \) \( \sup_{B_1} |u| \),
\[ \{ x \in B_{1/2} : \Theta(u, B_1)(x) > t \} \leq C t^{-\varepsilon}, \]
where the constants \( C, t_0, \varepsilon > 0 \) are universal.

Obviously (3.2) implies that for any \( 0 < \hat{\varepsilon} < \varepsilon \),
\[ \int_{B_{1/2}} (\Theta(u, B_1)(x))^{\hat{\varepsilon}} \, dx \leq C \sup_{B_1} |u|^{\hat{\varepsilon}}, \]
where the constant \( C \) depends additionally on a lower bound for \( \varepsilon - \hat{\varepsilon} \).

Proposition 3.1 is stated differently than the corresponding estimate in [2]. We emphasize here that \( \Theta(u, \Omega)(x) \) is defined in terms of quadratic polynomials which touch \( u \) at \( x \) and stay below \( u \) in the full domain \( \Omega \). That Proposition 3.1 is stated in terms of such a quantity is crucial to its application in the proof of Theorem 1. Indeed, if instead of Proposition 3.1 we had the weaker statement that \( u \) is twice differentiable at almost every point, and \( |D^2 u| \in L^\varepsilon \), then this would be insufficient to prove the partial regularity result.

For completeness and because of our alternative formulation, we give a simplified proof of Proposition 3.1 following the along the lines of the argument in [2]. The heart of the proof is following consequence of the ABP inequality. We recall that \( Q_1 \subset Q_3 \) and \( Q_3 \subset B_{6\sqrt{n}} \).

**Lemma 3.2.** Assume that \( \Omega \subset \mathbb{R}^n \) is open and \( B_{6\sqrt{n}} \subset \Omega \). Suppose \( u \in C(\Omega) \) satisfies
\[ P_{\lambda, \Lambda}^{+}(D^2 u) \geq 0 \quad \text{in } \Omega, \]
such that for some \( t > 0 \),
\[ \{ \Theta(u, \Omega) \leq t \} \cap Q_3 \neq \emptyset. \]
Then there are universal constants \( M > 1 \) and \( \sigma > 0 \) such that
\[ \{(\Theta(u, \Omega) \leq M t) \} \cap Q_1 \geq \sigma > 0. \]

**Proof.** Since the operator \( P_{\lambda, \Lambda}^{+} \) and the quantity \( \Theta \) are positively homogeneous, we may assume that \( t = 1 \). By adding an affine function to \( u \), we may suppose that the paraboloid
\[ P(x) := \frac{1}{2} (36n - |x|^2) \]
touches \( u \) from below at some point \( x_0 \in Q_3 \), that is,
\[ \inf_{\Omega} (u - P) = u(x_0) - P(x_0) = 0. \]
In particular,
\[ u \geq P \geq 0 \quad \text{in } B_{6\sqrt{n}} \quad \text{and} \quad u(x_0) = P(x_0) \leq \sup_{Q_3} P = 18n. \]

According to [2, Lemma 4.1], there exist smooth functions \( \varphi \) and \( \xi \) on \( \mathbb{R}^n \) and universal constants \( C \) and \( K > 1 \) such that
\[
\begin{aligned}
\mathcal{P}^+_{\lambda, \sigma}(D^2\varphi) &\geq -C\xi \quad \text{in } \mathbb{R}^n, \\
0 \leq \xi \leq 1, \quad \xi &\equiv 0 \quad \text{on } \mathbb{R}^n \setminus Q_1, \\
\varphi &\geq -K \quad \text{in } \mathbb{R}^n, \quad \varphi \geq 0 \quad \text{in } \mathbb{R}^n \setminus B_{6\sqrt{n}}, \quad \text{and} \quad \varphi \leq -1 \quad \text{in } Q_3.
\end{aligned}
\]

Define \( w := u + A\varphi \), with \( A > 0 \) selected below. It is easy to check that \( w \) satisfies
\[
\begin{aligned}
\mathcal{P}^+_{\lambda, \sigma}(D^2w) &\geq -CA\xi \quad \text{in } B_{6\sqrt{n}}, \\
w &\geq 0 \quad \text{on } \partial B_{6\sqrt{n}}.
\end{aligned}
\]

Let \( \Gamma_w \) denote the convex envelope of \(-w^{-}\chi_{6\sqrt{n}}\) in \( B_{12\sqrt{n}} \). According to the ABP inequality (Proposition [2, Lemma 4.1] above),
\[ -18n + A \leq -u(x_0) - \varphi(x_0) \leq \sup_{B_{6\sqrt{n}}} w^{-} \leq CA |\{ x \in Q_1 : \Gamma_{\omega}(x) = w(x)\}|. \]

Choosing \( A = 19n \) yields
\[ |\{ x \in Q_1 : \Gamma_{\omega}(x) = w(x)\}| \geq \sigma, \]
for a universal constant \( \sigma > 0 \). We also have that \( u^{-} \leq C \) in \( B_{6\sqrt{n}} \). We finish the proof by showing that, for some universal constant \( M > 0 \),
\[
\{ x \in Q_1 : \Gamma_{\omega}(x) = w(x)\} \subseteq \{ x \in Q_1 : \omega(u, \Omega)(x) \leq M \}.
\]

If \( x \in Q_1 \) is such that \( \Gamma_{\omega}(x) = w(x) \), then since \( \Gamma_{\omega} \) is convex and negative in \( B_{12\sqrt{n}} \), there exists an affine function \( L \) that touches \( \Gamma_{\omega} \), and hence \( w \), from below at \( x \). It follows that \( L \leq 0 \) in \( B_{12\sqrt{n}} \). We have
\[ L - A\varphi \leq u \quad \text{in } B_{6\sqrt{n}} \]
with equality holding at \( x \). Since \( L(x) = u(x) + A\varphi(x) \geq -KA \) and \( L \geq 0 \) in \( B_{12\sqrt{n}} \), we deduce that \( DL \leq KA/(6\sqrt{n}) \). Since \( |D^2\varphi| \) is bounded by a universal constant, we can find a concave paraboloid \( \tilde{P} \) with opening \( M \), with \( M \) universal, such that \( \tilde{P} \leq u \) in \( B_{6\sqrt{n}} \) and equality holding at \( x \). Since dist\( (Q_1, \mathbb{R}^n \setminus B_{6\sqrt{n}}) \geq 5\sqrt{n} \) and \( |DL| \leq C \), by making \( M \) larger if necessary, we may assume that \( \tilde{P} \leq P \) on the set \( \mathbb{R}^n \setminus B_{9\sqrt{n}} \). Hence \( \tilde{P} \leq P \leq u \) on \( \Omega \). Therefore \( \tilde{P} \leq u \) on \( \Omega \) with equality holding at \( x \). This completes the proof of (3.7).

We now prove Proposition [3.1] by applying Proposition [2.3] to the contrapositive of Lemma [3.2].

**Proof of Proposition [3.7]** According to the previous lemma, there are universal constants \( M, \sigma > 0 \), such that, for all \( t > 0 \) and \( Q_{x,3r} \subseteq Q_{1/6\sqrt{n}} \), we have
\[ |\Theta(u, B_1) > M \} | \cap Q_{x,r} > (1 - \sigma)|Q_r| \quad \text{implies that } \quad \Theta > t \quad \text{in } Q_{x,3r}. \]

Since \( u \) is bounded, we can touch it from below in \( Q_{1/2\sqrt{n}} \) by a paraboloid with an opening proportional to \( \sup_{B_1}|u| \), and hence for some \( x \in Q_{1/2\sqrt{n}} \),
\[ \Theta(u, B_1)(x) \leq C \sup_{B_1}|u|. \]
Then according to Lemma 3.2, there exists a universal $t_0$ such that, for all $t > t_0 \sup_{B_1} |u|$,  
\[
\{ \Theta(u, B_1) > Mt \} \cap Q_{1/6\sqrt{n}} \leq (1 - \sigma)|Q_{1/6\sqrt{n}}|.
\]
It follows from Lemma 2.3 that, for every $t > t_0 \sup_{B_1} |u|$,  
\[
(3.8) \quad \{ \Theta(u, B_1) > Mt \} \cap Q_{1/6\sqrt{n}} \leq (1 - \sigma)|\{ \Theta(u, B_1) > t \} \cap Q_{1/6\sqrt{n}}|.
\]
By iterating (3.8), we obtain a universal constants $C, \varepsilon > 0$ such that for all $t > t_0 \sup_{B_1} |u|$,  
\[
\{ \Theta(u, B_1) > t \} \cap Q_{1/6\sqrt{n}} \leq Ct^{-\varepsilon}.
\]
The proposition now follows from an easy covering argument. \hfill\Box

**Remark 3.3.** It is natural to wonder what, if anything, can be said about the exponent $\varepsilon$ in Proposition 3.1. By constructing an explicit example, we will show that $\varepsilon \to 0$ as the ellipticity $\Lambda/\lambda \to \infty$.

Fix $\alpha, R > 0$, and define the function $u$ in $\mathbb{R}^2 \setminus \{0\}$ by 
\[
(3.9) \quad u(x) := \begin{cases} 
R^{\alpha + 2}|x|^{-\alpha} + \frac{2}{3}|x|^2 - (1 + \frac{2}{3})R^2 & \text{if } 0 < |x| < R, \\
0 & \text{if } |x| \geq R.
\end{cases}
\]
Observe that $u \in C^1(\mathbb{R}^2 \setminus \{0\})$. An easy computation confirms that for $0 < |x| < R$, 
\[
D^2 u(x) = \alpha (\alpha + 2) R^{\alpha + 2}|x|^{-\alpha - 4} x \otimes x - \alpha |x|^{-\alpha - 2}(R^{\alpha + 2} - |x|^{\alpha + 2})I_0, 
\]
from which we can see that, in the punctured ball $0 < |x| < R$, the eigenvalues of $D^2 u(x)$ are $-\alpha |x|^{-\alpha - 2}(R^{\alpha + 2} - |x|^{\alpha + 2})$ and $\alpha |x|^{-\alpha - 2}(|x|^{\alpha + 2} + (\alpha + 1)R^{\alpha + 2})$. Therefore,  
\[
(3.10) \quad \mathcal{P}_{\lambda, \alpha}^+(D^2 u) = \Lambda \alpha |x|^{-\alpha - 2}(R^{\alpha + 2} - |x|^{\alpha + 2}) - \lambda \alpha |x|^{-\alpha - 2}(|x|^{\alpha + 2} + (\alpha + 1)R^{\alpha + 2}) 
\]
\[
\geq - (\Lambda + \lambda) \alpha \geq -2\Lambda \alpha \quad \text{in } B_R \setminus \{0\},
\]
provided that $0 < \alpha \leq \Lambda/\lambda - 1$.

Since $u \equiv 0$ in $\mathbb{R}^2 \setminus B_R$, the inequality (3.10) also holds in $\mathbb{R}^2 \setminus B_R$. Using that $u \in C^1(\mathbb{R}^2 \setminus \{0\})$, it follows that the inequality (3.10) holds in the viscosity sense in $\mathbb{R}^2 \setminus \{0\}$.

For any neighborhood $N$ of $x \in B_R \setminus \{0\}$, we have  
\[
(3.11) \quad \Theta(u, N)(x) \geq \alpha |x|^{-\alpha - 2}(R^{\alpha + 2} - |x|^{\alpha + 2}).
\]
This is easily deduced from the fact that if $\varphi$ is a smooth function touching $u$ from below at $x$, then $D^2 \varphi(x) \leq D^2 u(x)$, and the latter has an eigenvalue of $-\alpha |x|^{-\alpha - 2}(R^{\alpha + 2} - |x|^{\alpha + 2})$. It follows that  
\[
\Theta(u, N)(x) \geq c\alpha R^{\alpha + 2}|x|^{-\alpha - 2} \quad \text{in } B_{R/2} \setminus \{0\}
\]
where $c$ depends only on $\alpha$.

We build the example by making $R > 0$ small and replicating the function $u$. 
\[
v(x) := -|x|^2 + \sum_{y \in \mathbb{Z}^2} \min \left(1, \frac{\lambda}{\Lambda \alpha} u(x - 2Ry)\right)
\]
Note that for $R$ small, the minimum inside the summation takes the second value when $|x - 2Ry| > cR^{(\alpha + 2)/\alpha}$.

It is routine to check that  
\[
|v| \leq 1 \quad \text{in } B_1 \quad \text{and} \quad \mathcal{P}_{\lambda, \alpha}^+(D^2 v) \geq 0 \quad \text{in } \mathbb{R}^2.
\]
Theorem 1.3 in [10]. For completeness, and because we need the result under slightly different hypotheses on $F$ depending also on $\alpha$, we present a refinement of a result of Savin [10], which states that a viscosity solution of a classical solution.

Moreover, the following estimate holds

$$C/\lambda$$ arranged if the ellipticity satisfies $$(\Lambda/\lambda + 1)\varepsilon > 2.$$ Now fix $y \in \mathbb{Z}^2$ with $B(2Ry, R) \subseteq B_{1/2}$,

$$\int_{B(2Ry, R)} (\Theta(v, B_1)(x))^\varepsilon \, dx = \int_{B_R} (\Theta(v, B_1)(x))^\varepsilon \, dx \geq \int_{B_{R/2}\setminus B_{cR(2+\alpha)/\alpha}} (\Theta(\frac{1}{\lambda\varepsilon}u, B_{1/2})(x - 2Ry))^\varepsilon \, dx \geq \int_{B_{R/2}\setminus B_{cR(2+\alpha)/\alpha}} (c(\alpha, \lambda) R^{\alpha+2}|x|^{-\alpha-2})^\varepsilon \, dx = 2\pi c(\alpha, \lambda)\varepsilon R^{\alpha+2} \int_{cR(\alpha+2)/\alpha}^{R/2} t^{-(\alpha+2)\varepsilon + 1} \, dt \geq c(\lambda, \alpha)\varepsilon R^{2(\alpha+2)(1-\varepsilon)/\alpha}$$

There exist $c/R^2$ disjoint balls of the form $B(2Ry, R)$, with $y \in \mathbb{Z}^2$, inside $B_{1/2}$. Therefore,

$$\int_{B_{1/2}} (\Theta(v, B_1)(x))^\varepsilon \, dx \geq c(\alpha, \lambda)\varepsilon R^{2(\alpha+2)(1-\varepsilon)/\alpha - 2} = c(\lambda, \alpha)\varepsilon R^{2(2-\alpha+2(1-\varepsilon)/\alpha}.$$

Observe that the exponent $2(2-\alpha+2\varepsilon)/\alpha < 0$. Thus $\|\Theta(v, B_1)\|_{L^\infty(B_{1/2})} \to +\infty$ as $R \to 0$, keeping $\lambda$, $\Lambda$, $\alpha$, and $\varepsilon$ fixed.

This demonstrates that the $W^{2,\varepsilon}$ estimate as stated in Proposition 3.1 is false in dimension $n = 2$ if we have $(\Lambda/\lambda + 1)\varepsilon > 2$. It is false in all dimensions $n \geq 2$, for the same range of $\varepsilon$ and $\Lambda/\lambda$, since we may add dummy variables to our example at no cost. In particular, the exponent $\varepsilon$ in Proposition 3.1 is never greater than 1.

Conjecture 3.4. The optimal exponent in Proposition 3.1 is $\varepsilon = 2(\Lambda/\lambda + 1)^{-1}$.

It is not difficult to show that Conjecture 3.4 is true in the case that $\Lambda = \lambda$.

4. $C^{2,\alpha}$ Regularity for Flat Solutions

We present a refinement of a result of Savin [10], which states that a viscosity solution of a uniformly elliptic equation that is sufficiently close to a quadratic polynomial is, in fact, a classical solution.

Proposition 4.1. Assume in addition that $F(0) = 0$. Suppose that $0 < \alpha < 1$ and $u \in C(B_1)$ is a solution of (3.10) in $B_1$. Then there exists a universal constant $\delta_0 = \delta_0(\alpha) > 0$, depending also on $\alpha$, such that

$$\|u\|_{L^\infty(B_1)} \leq \delta_0 \quad \text{implies that} \quad u \in C^{2,\alpha}(B_{1/2}).$$

Moreover, the following estimate holds

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty}.$$

In the case that $F \in C^2$ and $|D^2F|$ is bounded, Proposition 4.1 is a special case of Theorem 1.3 in [10]. For completeness, and because we need the result under slightly different hypotheses on $F$, we give a proof of Proposition 4.1 here, following the argument of [10].

The key step in the proof of Proposition 4.1 is given by the following Lemma.
Lemma 4.2. Suppose in addition that \( F(0) = 0 \) and fix \( 0 < \alpha < 1 \). Then there exist universal constants \( \delta_0 > 0 \) and \( 0 < \eta < 1 \), depending also on \( \alpha \), such that, if \( u \in C(B_1) \) is a solution of (4.1) in \( B_1 \), and
\[
\sup_{B_1} |u| \leq \delta,
\]
then there is a quadratic polynomial \( P \) satisfying \( F(D^2P) = 0 \) and
\[
\sup_{B_0} |u - P| \leq \eta^{2+\alpha} \sup_{B_1} |u|.
\]

Proof. We argue by compactness. With \( \eta > 0 \) to be chosen below, assume on the contrary that there exist sequences \( \{F_k\} \) and \( \{u_k\} \), such that:

- \( F_k : \mathcal{S}_n \to \mathbb{R} \) satisfies (F1) and (F2) with the same \( \lambda, \Lambda, \omega, \) and \( F_k(0) = 0 \);
- \( u_k \in C(B_1) \) satisfies \( F_k(D^2u_k) = 0 \) in \( B_1 \);
- \( \delta_k := \sup_{B_1} |u_k| \to 0 \) as \( k \to \infty \); and
- there is no quadratic polynomial \( P \) satisfying (4.1) for \( u = u_k \).

Using interior Hölder estimates and taking a subsequence, if necessary, we may suppose that there is an operator \( F_0 \) and a function \( u_0 \in C(B_1) \) such that, as \( k \to \infty \), we have the limits:

- \( F_k \to F_0 \) locally uniformly on \( \mathcal{S}_n \);
- \( DF_k \to DF_0 \) locally uniformly on \( \mathcal{S}_n \); and
- \( m_k^{-1}u_k \to u_0 \) locally uniformly in \( B_1 \).

We claim that \( u_0 \) is a solution of the constant coefficient linear equation
\[
(4.2) \quad DF_0(0) \cdot D^2u_0 = 0 \quad \text{in} \quad B_1
\]
To verify (4.2), select a smooth test function \( \varphi \) and a point \( x_0 \in B_1 \) such that
\[
x \mapsto (u_0 - \varphi)(x) \quad \text{has a strict local maximum at} \quad x = x_0.
\]
Then we can find a sequence \( x_k \in B_1 \) such that \( x_k \to x_0 \) as \( k \to \infty \), and
\[
x \mapsto (u_k - \delta_k \varphi)(x) \quad \text{has a local maximum at} \quad x = x_k.
\]
Therefore,
\[
F_k \left( \delta_k D^2 \varphi(x_k) \right) \leq 0.
\]
Observe that
\[
F_k \left( \delta_k D^2 \varphi(x_k) \right) = \frac{d}{dt} \int_0^{\delta_k} F_k(tD^2 \varphi(x_k)) dt = \int_0^{\delta_k} DF_k(tD^2 \varphi(x_k)) \cdot D^2 \varphi(x_k) dt \\
\geq \delta_k DF(0) \cdot D^2 \varphi(x_k) - \delta_k |D \varphi(x_k)| \omega(\delta_k |D^2 \varphi(x_k)|).
\]
Combining the last two inequalities, dividing by \( \delta_k \), and letting \( k \to \infty \) yields
\[
DF(0) \cdot D^2 \varphi(x_0) \leq 0.
\]
We have shown that \( u_0 \) is a subsolution of (4.2), and checking that it is a supersolution is done by a similar argument.

Up to a change of coordinates, equation (4.2) is Laplace’s equation. Since \( \|u_0\|_{L^\infty(B_1)} \leq 1 \), standard estimates imply that \( u_0 \in C^\infty(B_1) \), and that the quadratic polynomial \( P(x) := u_0(0) + x \cdot Du_0(0) + x \cdot D^2u_0(0)x \) satisfies
\[
(4.3) \quad \sup_{B_0} |u_0 - P| \leq C \eta^3 \leq \frac{1}{2} \eta^{2+\alpha}.
\]
for a universal constant \( \eta = \eta_0 > 0 \), chosen sufficiently small and depending also on \( \alpha \), and we also have
\[
DF_0(0) \cdot D^2 P = 0.
\]
Therefore,
\[
|DF_k(0) \cdot D^2 P| = o(1) \quad \text{as } k \to \infty.
\]
It follows that
\[
F_k(\delta_k D^2 P) = \frac{d}{dt} \int_0^{\delta_k} F_k(t D^2 P) dt = \int_0^{\delta_k} DF_k(t D^2 P) \cdot D^2 P dt
\]
\[
\leq \delta_k DF_k(0) \cdot D^2 P + \delta_k |D^2 P| \omega(\delta_k |D^2 P|) = o(\delta_k) \quad \text{as } k \to \infty.
\]
Since \( F_k \) is uniformly elliptic, we can find a constant \( a_k \in \mathbb{R} \), of order \(|a_k| = o(\delta_k)\), such that \( P_k(x) := \delta_k P(x) + a_k|x|^2 \) satisfies \( F(D^2 P_k) = 0 \). Using this, the uniform convergence of \( u_k \) to \( u_0 \) on \( B_n \), and multiplying \( 4.3 \) by \( \delta_k \), we see that for large enough \( k \),
\[
\sup_{B_n} |u_k - P_k| \leq \delta_k \eta^{2+\alpha}.
\]
This contradiction completes the proof.

**Proof of Proposition 4.2.** By a standard covering argument, it is enough to show that the estimate holds at the origin. More precisely, we have to show that if \(|u|_{L^\infty(B_1)} = \delta < \delta_0\), then there is a quadratic polynomial \( P \) such that \( F(D^2 P) = 0 \), \(|P| \leq C\delta \) in \( B_1 \) and
\[
|u(x) - P(x)| \leq C\delta |x|^{2+\alpha} \quad \text{for all } x \in B_1.
\]
The idea of the proof is to apply Lemma 4.2 in a decreasing sequence of scales, obtaining a sequence of quadratic polynomials approximating \( u \) at zero with an appropriate error estimate.

Let \( \eta \in (0, 1) \) and \( \delta_0 > 0 \) be as in Lemma 4.2. We will construct by induction a sequence of quadratic polynomials \( \{P_k\}_{k=1}^\infty \) such that
\[
F(D^2 P_k) = 0 \quad \text{and} \quad |u - P_k|_{L^\infty(B_{\alpha k})} \leq \delta \eta^{2+\alpha k}.
\]
Moreover, we will show that this sequence is convergent and its limit as \( k \to \infty \) will be the desired polynomial \( P \) giving the second order expansion of \( u \) at the origin.

Since \(|u|_{L^\infty(B_1)} = \delta \), \( P_0 = 0 \) suffices for the case \( k = 0 \). Let us suppose that we have a quadratic polynomial \( \tilde{P}_k \) for which \( 4.5 \) holds. Let \( \tilde{u} \) and \( \tilde{F} \) denote
\[
\tilde{u}(x) := \eta^{-2k} (u(x) - P_k(x)) \quad \text{and} \quad \tilde{F}(M) := F(M + D^2 P_k).
\]
Observe that \( \tilde{F}(D^2 \tilde{u}) = 0 \) in \( B_1 \) and \(|\tilde{u}| \leq \delta \eta^{k\alpha} \) in \( B_1 \). Applying Lemma 4.2 we find a quadratic polynomial \( \tilde{P}_k \) such that \(|\tilde{P}_k| \leq C\delta \eta^{k\alpha} \) in \( B_1 \) and
\[
\tilde{F}(D^2 \tilde{P}_k) = 0 \quad \text{and} \quad |\tilde{u} - \tilde{P}_k|_{B_{\alpha k}} \leq C\delta \eta^{k\alpha} \eta^{2+\alpha} = \delta \eta^{2+\alpha(k+1)\alpha}.
\]
Let \( P_{k+1} = P_k + \eta^{2k} \hat{P}_k(\eta^{-k}x) \). From the estimate above, we have
\[
F(D^2 P_{k+1}) = 0 \quad \text{and} \quad |u - P_{k+1}|_{B_{\alpha k+1}} \leq C\delta \eta^{(k+1)(2+\alpha)}.
\]
This completes the inductive construction of a sequence of polynomials satisfying \( 4.4 \).

It remains to show that the sequence \( \{P_k\} \) is convergent and that its limit \( P \) satisfies \( 4.4 \). Since \( |\hat{P}_k| \leq C\delta \eta^{\rho k} \) in \( B_1 \), its coefficients are bounded by \( C\delta \eta^{\rho k} \). More precisely, if \( P = a_k + b_k \cdot x + x \cdot C_kx \), then \(|a_k| + |b_k| + |C_k| \leq C\delta \eta^{\rho k} \). Therefore
\[
P_{k+1} - P_k = \eta^{2k} \hat{P}_k(\eta^{-k}x) = \eta^{2k} a_k + \eta^{k} b_k \cdot x + x \cdot C_kx.
\]
Since \( \eta < 1 \), all the coefficients of \( P_{k+1} - P_k \) are bounded by the geometric series \( C\eta^{\alpha k} \). Therefore the sum \( \sum_{k=1}^{\infty} (P_{k+1} - P_k) \) is telescoping and hence convergent, and we may define

\[
P := \lim_{k \to \infty} P_k = \sum_{k=1}^{\infty} (P_{k+1} - P_k).
\]

Since \( F(D^2 P_k) = 0 \) for every \( k \) and \( F \) is continuous, we also have \( F(D^2 P) = 0 \).

Writing \( P(x) = a + b \cdot x + x^t C x \), we have the following estimates for the coefficients:

\[
|a - a_k| \leq \sum_{j=k}^{\infty} |a_{j+1} - a_j| \leq \sum_{j=k}^{\infty} C\eta^{(2+\alpha)j} \delta = C\eta^{(2+\alpha)k} \delta,
\]

\[
|b - b_k| \leq \sum_{j=k}^{\infty} |b_{j+1} - b_j| \leq \sum_{j=k}^{\infty} C\eta^{(1+\alpha)j} \delta = C\eta^{(1+\alpha)k} \delta,
\]

\[
|C - C_k| \leq \sum_{j=k}^{\infty} |C_{j+1} - C_j| \leq \sum_{j=k}^{\infty} C\eta^{\alpha j} \delta = C\eta^{\alpha k} \delta.
\]

Therefore \( |P(x) - P_k(x)| \leq C\delta \eta^{(2+\alpha)k} \) if \( x \in B_{\eta^k} \). In particular, \( |P| = |P - P_k| \leq C\delta \) in \( B_1 \).

Fix \( x \in B_1 \), and let \( k \) be the integer so that \( \eta^{k+1} < |x| \leq \eta^k \). Then we estimate

\[
|u(x) - P(x)| \leq |u(x) - P_k(x)| + |P_k(x) - P(x)| \leq C\delta \eta^{(2+\alpha)k} \leq C\delta |x|^{2+\alpha},
\]

which completes the proof. \( \square \)

5. Partial regularity

In this section, we prove our main result. We first apply the \( W^{2,\varepsilon} \) estimate in Proposition \( \ref{prop:2reg} \) to the derivative of \( u \), in effect deriving a \( W^{3,\varepsilon} \) estimate, and then to use this result and a scaling argument combined with Proposition \( \ref{prop:scaling} \) to obtain the theorem.

To state the \( W^{3,\varepsilon} \) estimate, we define the quantity

\[
\Psi(u, \Omega)(x) := \inf \{ A \geq 0 : \text{there exists } p \in \mathbb{R}^n \text{ and } M \in \mathcal{M}_n \text{ such that for all } y \in \Omega,

|u(y) - u(x) + p \cdot (x-y) + (x-y) \cdot M(x-y)| \leq \frac{1}{2} A|x-y|^3 \}.
\]

The following lemma records an elementary relation between \( \Psi(u, B_1) \) and \( \Theta(u_x, B_1) \).

Lemma 5.1. Assume that \( u \in C^1(B_1) \). Then for each \( x \in B_1 \),

\[
(5.1) \quad \Psi(u, B_1)(x) \leq \left( \sum_{i=1}^{n} (\Theta(u_{x_i}, B_1)(x))^2 \right)^{1/2}.
\]

Proof. Suppose that \( x \in B_1 \) and \( A_i \geq 0 \) are such that \( \Theta(u_{x_i}, B_1)(x) \leq A_i \) for each \( i = 1, \ldots, n \). Then we can find vectors \( p^1, \ldots, p^n \in \mathbb{R}^n \) such that

\[
(5.2) \quad |u_{x_i}(y) - u_{x_i}(x) + p^i \cdot (x-y)| \leq \frac{1}{2} A_i |x-y|^2 \text{ for all } y \in B_1.
\]

Let \( M \in \mathcal{M}_n \) be the matrix with entries \( \frac{1}{2} p_j^i \). It follows that

\[
(5.3) \quad u(y) - u(x) + Du(x) \cdot (x-y) + (x-y) \cdot M(x-y)

= (y-x) \cdot \int_0^1 Du(x + t(y-x)) - Du(x) + 2tM \cdot (y-x) \, dt.
\]
According to (5.2),
\[
\int_0^1 |u_x(x + t(y - x)) - u_x(x) + t\rho_j \cdot (y - x)| \, dt \leq \frac{1}{2}A_1t^2|x - y|^2.
\]
Denoting \( A = (A_1, \ldots, A_n) \) and using the previous inequality and (5.3), we obtain
\[
|u(y) - u(x) + Du(x) \cdot (y - x) + (x - y) \cdot M(x - y)|
\leq (y - x) \cdot \int_0^1 \frac{1}{2}A_1t^2|x - y|^2 \, dt \leq \frac{1}{6}|A||x - y|^3 \, dt.
\]
Thus \( \Psi(u, B_1) \leq |A| \), as desired. \( \square \)

In the next lemma, we formulate the \( W^{3,\epsilon} \) estimate in an appropriate way for its application in the proof of Theorem 1. A similar statement was used by Caffarelli and Souganidis to obtain an algebraic rate of convergence for monotone finite difference approximations of uniformly elliptic equations.

Lemma 5.2. Suppose \( u \in C(B_1) \) solves (1.1) in \( B_1 \) and satisfies \( \sup_{B_1} |u| \leq 1 \). There are universal constants \( C, \epsilon > 0 \) such that, if \( t > 1 \), then
\[
(5.4) \quad \left| \{ x \in B_{1/2} : \Psi(u, B_1)(x) > t \} \right| \leq Ct^{-\epsilon}.
\]

Proof. According to Proposition 2.2, we have that \( u \in C^2(B_1) \). Moreover, according to Proposition 5.5, for every unit direction \( e \in \mathbb{R}^n \), \( |e| = 1 \), the function \( u_e := e \cdot Du \) satisfies the inequalities
\[
\rho^-_{\alpha, \Lambda}(D^2u_e) \leq 0 \leq \rho^+_{\alpha, \Lambda}(D^2u_e)
\]
in the viscosity sense. According to Proposition 3.1, we have, for every \( t > 1 \),
\[
\left| \{ x \in B_{1/2} : \rho(u, B_1)(x) > t \} \right| \leq Ct^{-\epsilon},
\]
where \( C, \epsilon > 0 \) are universal constants. An application of Lemma 5.1 yields (5.4). \( \square \)

Lemma 5.3. Suppose that \( u \) satisfies the hypotheses of Lemma 5.2 and that \( 0 < \alpha < 1 \). There is a universal constant \( \delta_0 > 0 \), such that for every \( y \in B_{1/2} \) and \( 0 < r < \frac{1}{16} \),
\[
(5.5) \quad \{ \Psi(u, B_1) \leq r^{-1}\delta_0 \} \cap B(y, r) \neq \emptyset \quad \text{implies that} \quad u \in C^{2,\alpha}(B(y, r)).
\]

Proof. Suppose that \( 0 < r < \frac{1}{16} \), \( y \in B_{1/2} \), and \( z \in B(y, r) \) is such that
\[
\Psi(u, B_1)(z) \leq r^{-1}\delta.
\]
Then there exist \( p \in \mathbb{R}^n \) and \( M \in \mathcal{M}_n \) such that, for every \( x \in B_1 \),
\[
(5.6) \quad |u(x) - u(z) + p \cdot (z - x) + (z - x) \cdot M(z - x)| \leq \frac{1}{3}r^{-1}\delta |z - x|^3.
\]
Replacing \( M \) by \( \frac{1}{3}(M + Mt) \), we may assume that \( M \in S_n \). Since \( u \) is a viscosity solution of (1.1), it is clear that \( F(-M) = 0 \). Define the function
\[
v(x) := \frac{1}{16r^2} \left( u(z + 4r x) - u(z) + 4rp \cdot x + 16r^2x \cdot Mx \right), \quad x \in B_1.
\]
The inequality (5.6) implies that
\[
\sup_{B_1} |v| \leq \frac{1}{3}\delta.
\]
Define the operator \( \tilde{F}(N) := F(N - M) \), and observe \( \tilde{F} \) satisfies (F1) and (F2), with the same ellipticity constants \( \lambda, \Lambda \), and modulus \( \omega \), and \( \tilde{F}(0) = F(-M) = 0 \). It is clear that \( v \) is a solution of
\[
\tilde{F}(D^2v) = 0 \quad \text{in} \quad B_1.
\]
Let $\delta_0 > 0$ be the universal constant in Proposition 4.1, which also depends on $\alpha$. Suppose that $\delta \leq 3\delta_0$. Then Proposition 4.1 yield that $v \in C^{2,\alpha}(B(z, 2r))$, from which we deduce that $u \in C^{2,\alpha}(B(z, 2r))$. Since $B(y, r) \subseteq B(z, 2r)$, we are done. □

Proof of Theorem 1. By a standard covering argument, we may fix $0 < \alpha < 1$ and assume that $\Omega = B_1$, $u \in C(B_1)$ is bounded, and to show that $u \in C^{2,\alpha}(B_1/2)$. Since $B(y, r) \subseteq B(z, 2r)$, we are done.

Remark 5.4. An inspection of the proof reveals that the codimension $\varepsilon$ in Theorem 1 is equal to the exponent $\varepsilon$ of the $W^{2,\varepsilon}$ estimate of Proposition 3.1. In particular, it does not depend on the modulus $\omega$ of $DF$. It follows that we could further reduce the dimension of the singular set if we could improve the exponent of the $W^{2,\varepsilon}$ estimate. However, it is not possible to improve the exponent $\varepsilon$ in the $W^{2,\varepsilon}$ estimate, since as we saw in Remark 3.3, the constant $\varepsilon$ is at most $2(\Lambda/\lambda + 1)^{-1}$.

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