The $\tau$-function of the Ablowitz-Segur family of solutions to Painlevé II as a Widom constant

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Abstract

$\tau$-functions of certain Painlevé equations (PVI,PV,PIII) can be expressed as a Fredholm determinant. Further, the minor expansion of these determinants provide an interesting connection to Random partitions. This paper is a step towards understanding whether the $\tau$-function of Painlevé II has a Fredholm determinant representation. The Ablowitz-Segur family of solutions are special one parameter solutions of Painlevé II and the corresponding $\tau$-function is known to be the Fredholm determinant of the Airy Kernel. We develop a formalism for open contour in parallel to the one formulated in [11] in terms of the Widom constant and verify that the Widom constant for Ablowitz-Segur family of solutions is indeed the determinant of the Airy Kernel. Finally, we construct a suitable basis and obtain the minor expansion of the Ablowitz-Segur $\tau$-function.

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1 Introduction

Painlevé equations are nonlinear second order ODEs whose solutions are widely recognized as important special functions with a broad range of applications. The integrability property of these equations was obtained by representing them as an isomonodromic system of ordinary differential equations. The Riemann-Hilbert (RH) method then proved to be a powerful technique to study solutions and their properties. An important object related to the solutions is the so called $\tau$-function.

In the theory of isomonodromic deformations, the $\tau$-function ($\tau_{JMU}$) was introduced by the Kyoto school and it is constructed starting from a certain 1-form $\omega_{JMU}$ on the space of the deformation parameters [18]. If the parameters are of isomonodromic type, then...
the form $\omega_{JMU}$ is closed with respect to differentiation with respect to the parameters. The corresponding $\tau_{JMU}$ function is defined locally as

$$d \log \tau_{JMU} = \omega_{JMU} \quad (1.1)$$

where $d$ denotes total differentiation with respect to the parameters. A notable example of $\tau$-function of the Painlevé II equation is the Tracy-Widom distribution [22]. Such $\tau$-function has the property of being expressed as a Fredholm determinant of the Airy kernel. Many relevant solutions of the Painlevé equations that appear in various branches of mathematics turn out to be expressed as a Fredholm determinant of some integrable operator, as defined by Its, Izergin, Korepin, and Slavnov [17]. For example the gap probability distribution in random matrices is Fredholm determinant with the sine kernel (Painlevé V) [22], the correlation function of stochastic point processes on a one-dimensional lattice originated from representations of the infinite symmetric group is a Fredholm determinant with hypergeometric kernel (Painlevé VI) [6], [7].

It is natural to inquire whether all solutions of Painlevé equations can be expressed as a Fredholm determinant of some integrable operator. In a series of recent papers Cafasso, Gavrylenko, Lisovsky [13], [11] showed that the generic $\tau$-function of the PVI, PV, PIII equation can be expressed as a Fredholm determinant. The key feature of this construction is to reduce the Riemann-Hilbert problem (RHP) associated to the isomonodromic system to a RHP on the circle for a jump matrix $G$. Then one can define a Toeplitz operator $T_G = \Pi_+ G$ where $G$ is the jump (called symbol in the literature of Toeplitz determinants) of the RHP and $\Pi_+$ the projection operator to analytic functions in the interior of the circle. It has been shown in [13], [11] that the Fredholm determinant

$$\tau[G] = \det (T_{G^{-1}} \circ T_G) \quad (1.2)$$

coincides up to a factor with the isomonodromic $\tau$ function [1,1]. The above $\tau$ function is also called Widom constant since such quantity was obtained by Widom in the description of the asymptotic behaviour of Toeplitz determinants [23], [24], when the size of the matrix tends to infinity, as a refinement of the strong Szegő theorem.

This approach cannot be directly implemented to the cases where the RHP is formulated on a contour that is not a circle as is the case for the Painlevé equations PI, PII and PIV. There are several examples of $\tau$-functions expressed as a Fredholm determinant like the Tracy-Widom distribution related to Painlevé II [22], or like the example obtained in [3] related to Painlevé IV. However the generic $\tau$-function of the Painlevé I, II and IV equations does not seem to have a Fredholm determinant representation. The main obstacle to develop the procedure implemented in [13] and [11] is the impossibility to reduce the RHP problem of the Painlevé I, II and IV equations to a RHP on the circle. However it is expected that the generic RHP for these equations could be reduced to a RHP on the line for a jump matrix $G$. Then one considers the projection operator $\Pi_+$ to holomorphic functions on the semi-plane and define the operator $T_G = \Pi_+ G$. For the case considered in this manuscript, this operator is trace class and therefore the Fredholm determinant (1.2) is well defined. In this manuscript we develop this machinery for the Painlevé II equation by considering the Ablowitz-Segur family of solutions for PII as a toy model. This example serves as a starting point to obtain the generic Painlevé II $\tau$-function.

This paper is structured as follows. We will first setup the machinery to extend the formalism in [13], [11] to a line contour and show in [1] that the $\tau$-function can be written as a Widom constant. Next we show in [3] that the Widom constant coincides with the isomonodromic $\tau$-function [1,1]. Finally in [4] we construct an appropriate basis and study the minor expansion of the Widom constant.
2 Setup

Let $J(z,t) : i\mathbb{R} \to SL(2,\mathbb{C})$ be a smooth matrix function of $z$ depending analytically on the complex parameter $t$ in some domain. We assume that $||J(z) - 1|| = O(|z|^{-1})$ as $z \to \pm i\infty$. We shall refer to $J$ as the jump matrix. In association with the data of the contour $(i\mathbb{R})$ and jump matrix $J$ one can introduce two Riemann Hilbert problems, also known as factorization problems. They consist of two $2 \times 2$ matrices $\Theta(z,t)$ and $\Psi(z,t)$ such that:

- $\Theta(z,t)$, $\Psi(z,t)$ are analytic in $z \in \mathbb{C}/i\mathbb{R}$ and admit continuous boundary values from the left (+) and right (−) sides of $i\mathbb{R}$.

- The boundary values $\Theta\pm$, $\Psi\pm$ satisfy the jump conditions

  $$J(z,t) = \Theta^{-1}(z,t)\Theta^+(z,t) = \Psi^{-1}(z,t)\Psi^-(z,t) ; \quad z \in i\mathbb{R}$$  

  \hspace{1cm} (2.1)

- The functions $\Theta(z,t)$ and $\Psi(z,t)$ are normalized at infinity.

  $$\lim_{z \to \infty} \Theta(z,t) = I ; \quad \lim_{z \to \infty} \Psi(z,t) = I$$  

  \hspace{1cm} (2.2)

where the limit is intended as limit in any proper subsector of the left/right half-planes.

The two solutions (if they exist) are the two Birkhoff factorizations; we stipulate to call $\Theta(z,t)$ the direct Riemann-Hilbert problem and $\Psi(z,t)$ the dual. The contour $i\mathbb{R}$ divides the complex plane into the right half (negative side) and the left half (positive side). The space $L^2(i\mathbb{R}, |dz|) \otimes \mathbb{C}^2$ can be split as the direct sum of two closed subspaces (Hardy spaces):

$$\mathcal{H} = L^2(i\mathbb{R}, \mathbb{C}^2) = \mathcal{H}_+ \oplus \mathcal{H}_-$$

the functions on $\mathcal{H}$ are all column vectors. The two subspaces consist of (vector valued) functions in $L^2(i\mathbb{R})$ that are boundary values from the left(+)/right(−) of analytic functions that tend to 0 at infinity. Notice that this splitting is orthogonal. On these spaces, one can define projection operators $\Pi_{\pm}$ such that

$$\Pi_+ : \mathcal{H} \to \mathcal{H}_+ ; \quad \Pi_- : \mathcal{H} \to \mathcal{H}_-$$

explicitly, $\Pi_{\pm}$ are just the Cauchy transforms

$$\Pi_+(f)(z) = \int_{i\mathbb{R}} \frac{dw}{2\pi i} \frac{f(w)}{w - z} \quad \mathbb{R}z < 0$$  

$$\Pi_-(f)(z) = -\int_{i\mathbb{R}} \frac{dw}{2\pi i} \frac{f(w)}{w - z} \quad \mathbb{R}z > 0$$  

\hspace{1cm} (2.3)

with the equality $\Pi_+ + \Pi_- \equiv \mathbb{I}$. To define the $\tau$-function, we first define the operator $T_{J^{-1}} : \mathcal{H} \to \mathcal{H}_+$ for the symbol $J^{-1}$ by multiplication followed by projection

$$T_{J^{-1}}(f) = \Pi_+(J^{-1} f)$$  

\hspace{1cm} (2.5)

$T_J$ is similarly defined

$$T_J(f) = \Pi_+(J f)$$  

\hspace{1cm} (2.6)

With the operators $T_J$ and $T_{J^{-1}}$, we define the $\tau$-function along the same lines as the Widom constant.
Definition 1. We define the Widom constant with respect to the operators $T_J$, $T_{J^{-1}}$

$$\tau[J] = \det (T_{J^{-1}} \circ T_J)$$ (2.7)

It is a Fredholm determinant for the case of Ablowitz-Segur solutions as will be shown later.

Proposition 1. $\tau[J]$ as defined in (2.7) admits an equivalent representation as the determinant

$$\tau[J] = \det \mathcal{H}[1 + U]$$ (2.8)

where $1$ denotes the identity operator on $\mathcal{H}$, $U : \mathcal{H} \to \mathcal{H}$ is an operator represented in the splitting $\mathcal{H}_{\pm}$ as $U = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ and $a : \mathcal{H}_- \to \mathcal{H}_+$; $b : \mathcal{H}_+ \to \mathcal{H}_-$ are given by,

$$a = \Theta_+ \Pi_+ \Theta_+^{-1} - \Pi_+; \quad b = \Pi_- - \Theta_- \Pi_- \Theta_-^{-1}.$$ (2.11)

Proof. Substituting (2.5) in (2.7) and manipulating the terms gives the familiar form of the determinant representation of $\tau$-function in [11].

$$\tau[J] = \det \mathcal{H}_+ [T_{J^{-1}} \circ T_J] = \det \mathcal{H}_+ [\Pi_+ J^{-1} \Pi_+] = \det \mathcal{H}_+ [\Pi_+ \Theta_+^{-1} \Theta_- \Pi_+ \Theta_-^{-1} \Theta_+] = \det \mathcal{H}_+ [\Theta_+ \Pi_+ \Theta_+^{-1} \Theta_- \Pi_+ \Theta_-^{-1}] = \det \mathcal{H}_+ [\Theta_+ \Pi_+ \Theta_+^{-1} \Theta_- (1 - \Pi_- \Theta_-^{-1})] = \det \mathcal{H}_+ [1 - (\Theta_+ \Pi_+ \Theta_+^{-1} - \Pi_+) (\Pi_- - \Theta_- \Pi_- \Theta_-^{-1})] = \det \mathcal{H}_+ [1 + U]$$ (2.9)

where

$$U = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$ (2.10)

and

$$a = \Theta_+ \Pi_+ \Theta_+^{-1} - \Pi_+; \quad b = \Pi_- - \Theta_- \Pi_- \Theta_-^{-1}.$$ (2.11)

Notice that

$$a : \mathcal{H}_- \to \mathcal{H}_+; \quad b : \mathcal{H}_+ \to \mathcal{H}_-.$$

Now, one can repeat the same computation as above in terms of the dual RHP $\Psi_\pm$ and get the following.

$$a = \Psi_+ \Pi_+ \Psi_+^{-1} - \Pi_+; \quad b = \Pi_- - \Psi_- \Pi_- \Psi_-^{-1}.$$ (2.12)

3 Toy model: Ablowitz-Segur solution

The Ablowitz-Segur family [20] of solutions of the Painlevé II equation

$$\partial_s^2 u = su + 2u^3$$ (3.1)

are specified uniquely by the boundary condition

$$u(s) \simeq \kappa \text{Ai}(s); \quad s \to +\infty, \quad \kappa \in \mathbb{C}.$$ (3.2)
It is well known that the solution of a Painlevé equation can be obtained by solving a Riemann-Hilbert problem for a matrix valued function \( \Gamma(z, s) \) analytic in the complex \( z \) domain \( \mathbb{C} \) minus some contours.

\[
G_1 = \begin{pmatrix} 1 & 0 \\ \kappa e^{-\nu(z)} & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} \kappa e^{\nu(z)} & 0 \\ 0 & 1 \end{pmatrix}
\]

\[\Theta_1 \quad \Theta_2\]

\[\gamma_1 \quad i\mathbb{R} \quad \gamma_2\]

Figure 1: Contour

For the Ablowitz-Segur family of solutions, the contours are shown in figure 1, and the Riemann-Hilbert problem satisfied by \( \Gamma(z, s) \) is as follows

- \( \Gamma(z, s) \) is analytic in \( \mathbb{C} \setminus \Sigma \) with \( \Sigma = \gamma_1 \cup \gamma_2 \);
- the boundary values \( \Gamma_{\pm}(z, s) \) on the oriented contours \( \gamma_1 \) and \( \gamma_2 \) satisfying the following jump conditions
  \[
  \Gamma_{+}(z, s) = G_1(z, s)\Gamma_{-}(z, s) \quad z \in \gamma_1 \tag{3.3}
  \]
  \[
  \Gamma_{+}(z, s) = G_2(z, s)\Gamma_{-}(z, s) \quad z \in \gamma_2 \tag{3.4}
  \]
- the asymptotic behaviour at infinity is specified by
  \[
  \Gamma(z, s) = 1 + \frac{\Gamma^{(1)}(s)}{z} + O(z^{-2}), \quad \text{as} \quad |z| \to \infty. \tag{3.5}
  \]

The solution \( \Gamma(z, s) \) of the above Riemann Hilbert problem (when it exists) determines the Painlevé transcendent via the relation

\[
\Gamma^{(1)}(s) = 2\Gamma_{12}^{(1)}(s). \tag{3.6}
\]

From the above data, the isomonodromic \( \tau \)-function is defined by

\[
\partial_s \log \tau_{JMUV}(s) = -\Re e_{z=\infty} \Tr \left[ (z\sigma_3)^{\nu(z, s)} \Gamma^{-1}(z, s) \Gamma(z, s) \right] \tag{3.7}
\]
and we have the relation
\[ u^2(s) = -\frac{d^2}{ds^2} \log \tau_{JMU}(s). \]

Instead of solving the Riemann Hilbert problem for \( \Gamma \) we factorize it into two separate Riemann-Hilbert problems, one for the function \( \Theta_1(z, s) \) analytic in \( z \in \mathbb{C} \setminus \gamma_1 \) and one for the function \( \Theta_2(z, s) \) analytic in \( z \in \mathbb{C} \setminus \gamma_2 \) with boundary values

\[
\Theta_1(z, s) = G_1(z, s)\Theta_1(z, s) \quad z \in \gamma_1 \tag{3.8}
\]
\[
\Theta_2(z, s) = G_2(z, s)\Theta_2(z, s) \quad z \in \gamma_2 \tag{3.9}
\]

and
\[
\Theta_1(z) = 1 + \mathcal{O}(z^{-1}), \quad \Theta_2(z, s) = 1 + \mathcal{O}(z^{-1}) \quad \text{as} \quad z \to \infty. \tag{3.10}
\]

It is straightforward to solve the Riemann Hilbert problems (3.8), (3.9). The solutions are given by the Cauchy transforms of the respective jumps \( G_1, G_2 \)

\[
\Theta_1(z, s) = \begin{bmatrix} 1 & 0 \\ \kappa \int_{\gamma_1} e^{\frac{\nu(\lambda, s)}{\lambda - z}} d\lambda / 2\pi i & 1 \end{bmatrix} \tag{3.11}
\]
\[
\Theta_2(z, s) = \begin{bmatrix} 1 & \kappa \int_{\gamma_2} e^{\nu(\lambda, s)} d\lambda / 2\pi i \\ 0 & 1 \end{bmatrix}. \tag{3.12}
\]

Next we define the matrix valued function \( \Theta(z, s) \) such that

\[
\Theta(z, s) = \begin{cases} 
\Theta_2(z, s) & \Re z < 0 \\
\Theta_1(z, s) & \Re z \geq 0
\end{cases} \tag{3.13}
\]

Clearly the matrix function \( \Theta(z, s) \) is analytic in \( \mathbb{C} \setminus i\mathbb{R} \) and the boundary values \( \Theta_{\pm} \) on \( i\mathbb{R} \) satisfy the jump condition

\[
\Theta_{-}(z, s)^{-1}\Theta_{+}(z, s) = J(z, s) = \begin{bmatrix} 1 & \kappa \int_{\gamma_1} e^{\nu(\lambda, s)} d\lambda / 2\pi i \\ -\kappa \int_{\gamma_1} e^{-\nu(\lambda, s)} d\lambda / 2\pi i & 1 \end{bmatrix}. \tag{3.14}
\]

and
\[
\lim_{z \to \infty} \Theta(z, s) = 1.
\]

### 3.1 Computing the \( \tau \)-function

In this section we want to make sense of the quantity \( \det (T_{j^{-1}} \circ T_j) \) introduced in (2.7) when the matrix \( J \) is as in (3.14). In Proposition 1 we show that

\[
\tau[J] = \det (T_{j^{-1}} \circ T_j) = \det \mathcal{H} [1 + U] \tag{3.15}
\]

where \( 1 \) denotes the identity operator on \( \mathcal{H} \), and \( U = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \) with \( a : \mathcal{H}_- \to \mathcal{H}_+; \)
\( b : \mathcal{H}_+ \to \mathcal{H}_- \) given by,

\[
a = \Theta_+ \Pi_+ \Theta_+^{-1} - \Pi_+ ; \quad b = \Pi_- - \Theta_- \Pi_- \Theta_-^{-1}
\]

with \( \Theta_{\pm} \) the boundary values of the matrix \( \Theta \) defined in (3.13).

We want to show that the quantity (3.15) is a Fredholm determinant and coincides with the \( \tau \)-function defined in [1].
We remind the reader of the result in [1] where the \( \tau \)-function of the Ablowitz-Segur family of solutions for Painlevé II is given by the following Fredholm determinant:

\[
\tau(s) = \det \left[ \text{Id}_{L^2(\gamma_+ \cup \gamma_-)} - \kappa \begin{bmatrix} 0 & \mathcal{F} \\ \mathcal{G} & 0 \end{bmatrix} \right] = \det \left[ \text{Id}_{L^2(\gamma_+)} - \kappa^2 \mathcal{F} \circ \mathcal{G} \right]
\]

with

\[
\mathcal{F} : L^2(\gamma_-) \rightarrow L^2(\gamma_+) \quad \mathcal{G} : L^2(\gamma_+) \rightarrow L^2(\gamma_-)
\]

and

\[
(\mathcal{F}g)(z) = e^{-\frac{1}{2} \nu(z,s)} \int_{\mathbb{R}^{-i\epsilon}} \frac{dw}{2\pi i} e^{\frac{1}{2} \nu(w,s)} g(w)
\]

\[
(\mathcal{G}g)(z) = e^{\frac{1}{2} \nu(z,s)} \int_{\mathbb{R}^{+i\epsilon}} \frac{dw}{2\pi i} e^{-\frac{1}{2} \nu(w,s)} g(w)
\]

**Theorem 1.** The \( \tau \)-function (3.16) of the Ablowitz-Segur family of solutions of the Painlevé II equation is the Widom constant defined in (3.15).

**Proof.** The Widom constant can be obtained from (3.15) by computing the operators \( a \), \( b \) explicitly. Let \( f(z) \in \mathcal{H}_- \), \( h(z) \in \mathcal{H}_+ \) be vector valued functions

\[
f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},
\]

then

\[
a f(z) = \int_{\mathbb{R}} \frac{dw}{2\pi i} \Theta_2(z) \Theta_2^{-1}(w) - 1 f(w)
\]

\[
h h(z) = \int_{\mathbb{R}} \frac{dw}{2\pi i} \Theta_1(z) \Theta_1^{-1}(w) h(w),
\]

with \( \Theta_1 \) and \( \Theta_2 \) are as in (3.12). We begin by computing (3.20)

\[
\Theta_2(z) \Theta_2^{-1}(w) - 1 = \begin{bmatrix} 0 & \kappa \int_{\gamma_2} \left( \frac{e^{\nu(\lambda,s)}}{\lambda-z} - \frac{e^{\nu(\lambda,s)}}{\lambda-w} \right) \frac{d\lambda}{2\pi i} \\ 0 & 0 \end{bmatrix}
\]

substituting (3.22) in (3.21) and focusing on the only non-zero entry \( a_{12} \),

\[
a_{12} f_2(z) = -\kappa \int_{\mathbb{R}} \frac{dw}{2\pi i} \int_{\gamma_2} \frac{d\lambda}{2\pi i} \frac{e^{\nu(\lambda,s)}}{(\lambda-z)(\lambda-w)} f_2(w)
\]

integrating over \( \lambda \),

\[
a_{12} f_2(z) = -\kappa \int_{\mathbb{R}-\epsilon} \frac{dw}{2\pi i} \frac{e^{\nu(w,s)}}{w-z} f_2(w)
\]

A similar computation for \( b \) gives that the only non-zero entry is \( b_{21} \) that reads

\[
b_{21} h_1(z) = \kappa \int_{\mathbb{R}} \frac{e^{-\nu(\lambda,s)}}{(\lambda-z)(\lambda-w)} h_1(w) \frac{d\lambda}{2\pi i} \frac{dw}{2\pi i}
\]

integrating over \( \lambda \)

\[
b_{21} h_1(z) = \kappa \int_{\mathbb{R}+\epsilon} \frac{e^{-\nu(w,s)}}{(w-z)} h_1(w) \frac{dw}{2\pi i}
\]

\( \gamma_{\pm} \) are \( \gamma_{1,2} \) rotated by \( \pi/2 \) and this is also the source of the factor of \( i \) in the exponential in [1].
Substituting $a$ and $b$ back in (3.15), we get the following

$$
\tau(s) = \det \left[ Id_{L^2(\mathbb{R})} \otimes \mathbb{C}^2 - \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \right] = \det \left[ Id_{L^2(\mathbb{R})} \otimes \mathbb{C}^2 - \begin{bmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{21} & 0 & 0 & 0 \end{bmatrix} \right] (3.27)
$$

$$
= \det \left[ Id_{L^2(\mathbb{R})} - \begin{bmatrix} 0 & a_{12} \\ b_{21} & 0 \end{bmatrix} \right]. (3.28)
$$

Further, it is straightforward to see that the operator $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ is trace class. In other words, $a_{12}$ and $b_{21}$ are Hilbert-Schmidt.

$$
|a_{12}|^2 = -\kappa^2 \int_{\mathbb{R}^{++}} |dz| \int_{\mathbb{R}^{--}} |dw| \frac{e^{\nu(w,s)+\nu(s)}}{|w-z|^2} = -\kappa^2 \int_{\mathbb{R}^{++}} |dz| \int_{\mathbb{R}^{--}} |dw| \frac{e^{2\Re\nu(w,s)}}{|w-z|^2} < +\infty (3.29)
$$

$$
|b_{21}|^2 = -\kappa^2 \int_{\mathbb{R}^{++}} |dz| \int_{\mathbb{R}^{--}} |dw| \frac{e^{-\nu(w,s)-\nu(s)}}{|w-z|^2} = -\kappa^2 \int_{\mathbb{R}^{++}} |dz| \int_{\mathbb{R}^{--}} |dw| \frac{e^{-2\Re\nu(w,s)}}{|w-z|^2} < +\infty (3.30)
$$

(3.29) and (3.30) are clearly convergent, implying that $a_{21}$ and $b_{12}$ are Hilbert-Schmidt operators. Therefore, the determinant $\det \left[ 1 + U \right]$ is Fredholm and coincides with the $\tau$-function in (3.16). \hfill \Box

**Remark.** Further, it is shown in [2] that

$$
\det \left[ Id_{L^2([\gamma_1 \cup \gamma_2\\mathbb{R})]} + U \right] = \det \left[ Id_{L^2([s,\infty))] - \kappa^2 K_{A_1}|_{[s,\infty]} \right] (3.31)
$$

where $K_{A_1}$ is the Airy kernel, which implies the $\tau$-function (3.28) is the determinant of the Airy Kernel. It is a well known result [22] that the solution of (3.2) is related to the Airy Kernel as

$$
u(s)^2 = -\frac{d^2}{ds^2} \log \det \left[ 1 - \kappa^2 K_{A_1}|_{[s,\infty]} \right] (3.32)
$$

### 3.2 Relation to the JMU tau-function

The logarithmic derivative of the Widom constant (3.15) can be shown to coincide with the logarithmic derivative of the isomonodromic $\tau$-function (3.7). To this end, we begin by defining a matrix valued function $Y(z,s)$ as a ratio of $\Theta(z,s)$ and $\Gamma(z,s)$.

$$
Y(z,s) = \begin{cases} \Theta_2^{-1}(z,s)\Gamma(z,s) & \Re z < 0 \\ \Theta_1^{-1}(z,s)\Gamma(z,s) & \Re z \geq 0 \end{cases} (3.33)
$$

$Y(z,s)$ has a jump only on $i\mathbb{R}$. Its boundary values satisfy the following relation

$$
Y_+(z,s) = J^{-1}(z,s)Y_-(z,s) \quad z \in i\mathbb{R} (3.34)
$$

and $Y(z,s) \to 1$ as $z \to \infty$.

**Notation:** $' \equiv \frac{\partial}{\partial z}$ and $\overset{..}{'} \equiv \frac{\partial}{\partial s}$. All functions depend on $z,s$ unless stated otherwise.

**Proposition 2.** The logarithmic derivative of Widom constant in (3.15) is

$$
\partial_s \log \tau[J] = \int_{i\mathbb{R}} \frac{dz}{2\pi i} \operatorname{Tr} \left\{ J^{-1}J' \left[ -Y_+Y_+^{-1} + \Theta_2^{-1}\Theta_2' \right] \right\} . (3.35)
$$
Proof. Begin with the Fredholm determinant\footnote{This computation follows from Theorem 2.3 in [11]. The difference being the choice of factorisation.}

\[
\tau[J] = \det [T_{J^{-1}} \circ T_J] = \det [PQ]
\] (3.36)

where \( P = T_{J^{-1}} = \Pi_+ J^{-1} \) and \( Q = T_J = \Pi_+ J \). Inverses are \( P^{-1} = Y_+ \Pi_+ Y_+^{-1} \) and \( Q^{-1} = \Theta_+^{-1} \Pi_+ \Theta_+ \). Computing the logarithmic derivative

\[
\partial_s \log \det [PQ] = \text{Tr} \left[ \partial_s PP^{-1} + Q^{-1} \partial_s Q \right]
\] (3.37)

to obtain the last expression we use the fact that \( J^{-1} \partial_s J \) is a multiplication operator and only the diagonal parts of \( Y_+ \Pi_+ Y_+^{-1} \) and \( \Theta_+^{-1} \Pi_+ \Theta_+ \) contribute to the expression. \( \square \)

Using the above proposition we can identify the Widom constant with the isomonodromic \( \tau \) function.

Proposition 3. The logarithmic derivative of the Widom constant \eqref{eq:3.35} coincides exactly with the logarithmic derivative of the (isomonodromic) JMU \( \tau \)-function \eqref{eq:3.7} for Ablowitz-Segur family of solutions namely :

\[
\partial_s \log \tau[J] = \partial_s \log \tau_{JMU} = -\Re e s_{z=\infty} \text{Tr} \left[ \Gamma' \Gamma^{-1} (z\sigma_3) \right] .
\] (3.38)

Proof. We prove the statement by simplifying the expression of \( \partial_s \log \tau[J] \) in \eqref{eq:3.35}:

\[
\text{Tr} \left\{ J^{-1} J \left[ -Y_+ Y_+^{-1} + \Theta_2^{-1} \Theta_2 \right] \right\} .
\] (3.39)

We first perform algebraic manipulation on \( Y_+ Y_+^{-1} \) using \eqref{eq:3.33}

\[
Y_+ Y_+^{-1} = (\Theta_2^{-1} \Gamma') (\Gamma^{-1} \Theta_2)
= (-\Theta_2^{-1} \Theta_1 \Theta_2^{-1} \Gamma + \Theta_2^{-1} \Gamma') (\Gamma^{-1} \Theta_2)
= \Theta_2^{-1} (-\Theta_1 \Theta_2^{-1} + \Gamma \Gamma^{-1}) \Theta_2
\] (3.40)

and expressing \( J \) in terms of \( \Theta_1 \) and \( \Theta_2 \) we obtain

\[
J^{-1} J = (\Theta_2^{-1} \Theta_1)(\Theta_1^{-1} \Theta_2)
= (\Theta_2^{-1} \Theta_1)(-\Theta_1^{-1} \Theta_2 \Theta_2^{-1} \Theta_1 + \Theta_1^{-1} \Theta_2)
= \Theta_2^{-1} (-\Theta_1 \Theta_2^{-1} + \Theta_2 \Theta_1^{-1}) \Theta_2
= \Theta_2^{-1} \Delta (\hat{\Theta} \Theta^{-1}) \Theta_2
\] (3.41)

where

\[
\Delta (\hat{\Theta} \Theta^{-1}) = \Theta_2 \Theta_2^{-1} - \Theta_1 \Theta_1^{-1} .
\]

Substituting \eqref{eq:3.40} and \eqref{eq:3.41} in \eqref{eq:3.39} and using cyclicity of trace,

\[
\text{Tr} \left\{ J^{-1} J Y_+ Y_+^{-1} \right\} = \text{Tr} \left\{ \Delta (\hat{\Theta} \Theta^{-1}) (\Gamma \Gamma^{-1} + \Theta_2 \Theta_2^{-1}) \right\}
\] (3.42)
We begin by computing the following term

\[- \int_{i\mathbb{R}} \frac{dz}{2\pi i} \text{Tr} \left\{ \Delta \left( \dot{\Theta} \Theta^{-1} \right) \Gamma' \Gamma^{-1} \right\} (3.43)\]

since \( \Gamma \) has no jump on \( i\mathbb{R} \), (3.43) can be further simplified

\[- \int_{i\mathbb{R}} \frac{dz}{2\pi i} \text{Tr} \left\{ \Delta \left( \dot{\Theta} \Theta^{-1} \right) \Gamma' \Gamma^{-1} \right\} = - \int_{i\mathbb{R}} \frac{dz}{2\pi i} \text{Tr} \Delta \left( \dot{\Theta} \Theta^{-1} \Gamma' \Gamma^{-1} \right) = \int_{\gamma} \frac{dz}{2\pi i} \text{Tr} \Delta \left( \dot{\Theta} \Theta^{-1} \Gamma' \Gamma^{-1} \right)\]

\[= \int_{\gamma_1} \frac{dz}{2\pi i} \text{Tr} \Delta \left( \dot{\Theta} \Theta^{-1} \Gamma' \Gamma^{-1} \right) + \int_{\gamma_2} \frac{dz}{2\pi i} \text{Tr} \Delta \left( \dot{\Theta} \Theta^{-1} \Gamma' \Gamma^{-1} \right) (3.44)\]

Let us begin by computing the integral on \( \gamma_1 \) in (3.44)

\[\text{Tr} \Delta \left( \dot{\Theta} \Theta^{-1} \Gamma' \Gamma^{-1} \right) = \text{Tr} \left\{ \Theta_{1+} \Theta_{1+}^{-1} \Gamma' \Gamma^{-1} \right\} - \text{Tr} \left\{ \Theta_{1-} \Theta_{1-}^{-1} \Gamma' \Gamma^{-1} \right\} (3.45)\]

computing (3.45) term by term by substituting (3.3) for \( \Gamma_+ \)

\[\Gamma_+ \Gamma_+^{-1} = (G_1 \Gamma_+^{-1})' (\Gamma_+^{-1} G_1^{-1})\]

\[= G_1 \left[ G_1^{-1} G_1 + \Gamma_+^{-1} \right]^{-1} G_1^{-1} (3.46)\]

and (3.8) for \( \Theta_{1+} \)

\[\dot{\Theta}_{1+} \Theta_{1+}^{-1} = (G_1 \dot{\Theta}_{1-}) \Theta_{1-}^{-1} G_1^{-1}\]

\[= G_1 \left[ G_1^{-1} \dot{G}_1 + \dot{\Theta}_{1-} \Theta_{1-}^{-1} \right] G_1^{-1}. (3.47)\]

Substituting (3.46), (3.47) in (3.44) and using cyclicity

\[\text{Tr} \left\{ (\dot{\Theta}_{1+} + \Theta_{1+}^{-1}) \Gamma_+ \Gamma_+^{-1} \right\} = \text{Tr} \left[ (G_1^{-1} \dot{G}_1 + \dot{\Theta}_{1-} \Theta_{1-}^{-1}) (G_1^{-1} G_1' + \Gamma_+ \Gamma_+^{-1}) \right] (3.48)\]

In (3.48), notice that the term \( (G_1^{-1} \dot{G}_1 + \dot{\Theta}_{1-} \Theta_{1-}^{-1}) (G_1^{-1} G_1') \) is traceless. Furthermore, we have the following identity \( 2G_1^{-1} \dot{G}_1 = -z G_1^{-1} \sigma_3 G_1 + z \sigma_3 \). The terms \( \dot{\Theta}_{1-} \Theta_{1-}^{-1} \Gamma_+ \Gamma_+^{-1} \) in (3.48) and (3.45) cancel each other out. So, all that is left to compute on the contour \( \gamma_1 \) is the following

\[\int_{\gamma_1} \frac{dz}{2\pi i} \text{Tr} \left[ G_1^{-1} \dot{G}_1 \Gamma_+ \Gamma_+^{-1} \right] = \frac{1}{2} \int_{\gamma_1} \frac{dz}{2\pi i} \text{Tr} \left[ (z G_1^{-1} \sigma_3 G_1 + z \sigma_3) \Gamma_+ \Gamma_+^{-1} \right] (3.49)\]

We begin by computing the following term

\[\text{Tr} \left( G_1^{-1} \sigma_3 G_1 \Gamma_+ \Gamma_+^{-1} \right) (3.50)\]

the term \( \Gamma_+ \Gamma_+^{-1} \) can be simplified by substituting (3.3)

\[\Gamma_+ \Gamma_+^{-1} = (G_1^{-1} \Gamma_+)' (\Gamma_+^{-1} G_1)\]

\[= G_1^{-1} (-G_1^{-1} G_1' + \Gamma_+ \Gamma_+^{-1}) G_1 (3.51)\]
substituting (3.51) in (3.50) and using the cyclic property of the trace

\[ \text{Tr} \left( G_1^{-1} \sigma_3 G_1 \Gamma_1 \right) = \text{Tr} \left[ \sigma_3 \Gamma_1 \Gamma_1 \right] - G_1 \cdot G_1 \sigma_3 \] (3.52)

note that \( G_1 \Gamma_1 \sigma_3 \) is traceless. Substituting (3.52) in (3.49) we have

\[ \frac{1}{2} \int_{\gamma_1} dz \frac{2\pi i}{\Sigma} \text{Tr} \left[ (z G_1^{-1} \sigma_3 - z \sigma_3) \Gamma_1 \Gamma_1 \right] = \frac{1}{2} \int_{\gamma_1} dz \frac{2\pi i}{\Sigma} \text{Tr} \left[ -z \sigma_3 (\Gamma_1 \Gamma_1 - \Gamma_1 \Gamma_1) \right] \] (3.53)

repeating this exercise and computing the integral on \( \gamma_2 \) in (3.42), we get exactly the same expression. Putting all together

\[ \partial_s \ln \tau[J] = \int_{\Sigma} dz \frac{2\pi i}{\Sigma} \text{Tr} \left[ -z \sigma_3 \Gamma_1 \Gamma_1 \right] \] (3.54)

4 Minor expansion

The Hilbert space \( L^2(S^1) \) admits a natural orthonormal basis of Fourier modes (i.e. the monomials \( z^n, n \in \mathbb{Z} \)). The minor expansion of the Fredholm determinant (1.2) in this particular basis gives rise to interesting combinatorics. In the case of Painlevé VI, V, III the combinatorics correspond to certain Nekrasov Partition functions of certain Gauge theories [13].

In this spirit, we would like to propose, at least, a reasonable expansion of the Fredholm determinant of our operator in a similar guise. In our case the underlying Hilbert space \( L^2(i\mathbb{R}) \) does not immediately suggest a natural discrete orthonormal basis. Here below we want to propose a very natural such basis: the main guiding principle is that of identifying the Hardy space \( H^+ \) with the Hardy space of the interior of the disk, and pulling back the monomial basis.

**Proposition 4.** The Fredholm determinant of the \( \tau \)-function in (3.15) can be expanded, on an appropriate basis, in terms of minors, that can be labelled by Maya diagrams \( (m_X) \)

\[ \tau[s] = \sum_{m_X \in \mathbb{M}; \ |p|=|k|} \alpha_{p_X}^{h_X} \beta_{p_X}^{k_X} \] (4.1)

where the coefficients \( \alpha_m^n, \beta_m^m \) are as follows

\[ \beta_0^0 = \alpha_0^0 = \left( 4 \frac{\partial^2}{\partial s^2} - 1 \right) \left( 1 - \frac{\partial}{\partial s} \right)^{-1} \text{Ai}(s) \] (4.2)

where \( \text{Ai}(s) \) is the Airy function, and

\[ \alpha_m^n = \frac{(-1)^{m+n}}{(m!)^2 n!(m + n + 1)!} \left( \tilde{D} \right)^{m+n} \alpha_0^0 \] (4.3)

\[ \beta_m^m = \frac{(-1)^{m+n}}{(n!)^2 m!(m + n + 1)!} \left( \tilde{D} \right)^{m+n} \beta_0^0 \] (4.4)

with \( \tilde{D} = 2 \left( \frac{\partial}{\partial s} - 1 \right)^2 \left( 4 \frac{\partial^2}{\partial s^2} - 4 \right) \)
Proof. Recall that,

\[ \tau(s) = \det \left[ \text{Id}_{L^2(i\mathbb{R})} - \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \right] = \det \left[ \text{Id}_{L^2(i\mathbb{R}) \otimes \mathbb{C}^2} - \begin{bmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{21} & 0 & 0 & 0 \end{bmatrix} \right] \]  

(4.5)

of which \( a_{12} \) and \( b_{21} \) are the only non zero entries. Therefore, the determinant of the \( 4 \times 4 \) block operator can be reduced to a determinant of a \( 2 \times 2 \) block operator. Let us denote \( a_{12} \equiv \alpha, b_{21} \equiv \beta \)

\[ \tau[s] = \det \left[ \text{Id}_{L^2(i\mathbb{R})} - \begin{bmatrix} 0 & a_{12} \\ b_{21} & 0 \end{bmatrix} \right] = \det \left[ \text{Id}_{L^2(i\mathbb{R})} - \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix} \right] \]  

(4.6)

We remind the reader here that the block decomposition is due to the splitting \( L^2(i\mathbb{R}) = \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \).

The first step to obtain the minor expansion is constructing a suitable basis to expand \( \alpha(z, w) \) and \( \beta(z, w) \),

\[ \alpha(z, w) : \mathcal{H}_- \to \mathcal{H}_+ ; \quad \beta(z, w) : \mathcal{H}_+ \to \mathcal{H}_- \]  

(4.7)

Basis construction

The spaces \( \mathcal{H}_\pm \) are Hardy spaces of functions analytic on the left and right half of the complex planes respectively. To construct the bases of \( \mathcal{H}_\pm \), we employ the Paley-Weiner theorem which identifies \( \mathcal{H}_+ \) as the image under Fourier transform of functions supported on a half-line. Specifically, let \( \mathbb{C}_+ = \{ z : x = x + iy, y > 0 \} \),

\[ H^2(\mathbb{C}_+) = \left\{ f : f \text{ is analytic in } \mathbb{C}_+ \text{ and } \sup_{0 < y < +\infty} \int_{-\infty}^{+\infty} |f(z)|^2 \, dx < \infty \right\} \]  

(4.8)

By definition, the boundary values of \( f \in H^2(\mathbb{C}_+) \) on \( \mathbb{R} \) define a function in \( L^2(\mathbb{R}) \) and we can think of \( H^2(\mathbb{C}_+) \) as a (closed) subspace of \( L^2(\mathbb{R}) \). With this understanding, the Paley–Wiener theorem can be stated as the following identity:

\[ \mathcal{F}H^2(\mathbb{C}_+) = L^2[0, \infty). \]  

(4.9)

The space \( \mathcal{H}_+ \) can be isometrically mapped to \( H^2(\mathbb{C}_+) \) by a variable change \( z \to iz \). We have that Laguerre functions \( (L^n_\lambda(z) z^\lambda e^{-z}) \) provide a basis of \( L^2(\mathbb{R}_+) \). Using the Paley–Wiener theorem, upon inverse Fourier transform, they yield a basis for \( H^2(\mathbb{C}_+) \) and an innocent change of variable \( z \to iz \) gives a basis on \( \mathcal{H}_+ \). We can comfortably restrict ourselves to \( \lambda = 0 \). Following [21], the Fourier transform \( (\hat{L}_n(z)) \) of the Laguerre functions \( L^n_\lambda(x) e^{-\frac{x}{2} x^\frac{\lambda}{2}} \) for \( \lambda = 0 \), and using the notation and \( L^n_0 \equiv L_n \), is

\[ L_n(x) e^{-\frac{x}{2}} = \frac{e^{-\frac{x}{2}}}{n!} \sum_{k=0}^{n} (-n)_k x^k \]  

(4.10)

\[ \hat{L}_n(t) = \frac{-2}{n!} \left( \frac{1 + 2it}{2it - 1} \right)^n \frac{1}{2it - 1} \]  

(4.11)

\( \hat{L}_n \) forms a complete basis on \( H^2(\mathbb{C}_+, dt) \). With the change of variable \( 2it = z \), [4.7] reads

\[ \hat{L}_n(z) = \frac{-2}{n!} \left( \frac{1 + z}{z - 1} \right)^n \frac{1}{z - 1} \]  

(4.12)
and will form a basis on $H^2(\mathcal{H}_+, \tau^2 dz)$. In conclusion

$$e^n_{\mathcal{H}_+} = \frac{i}{n!} \left( \frac{1 + z}{z - 1} \right)^n \frac{1}{z - 1}$$

(4.13)
is a basis on $H^2(\mathcal{H}_+, dz)$. Similarly,

$$e^n_{\mathcal{H}_-} = \frac{i}{n!} \left( \frac{z - 1}{z + 1} \right)^n \frac{1}{z + 1}$$

(4.14)
is a basis on $H^2(\mathcal{H}_-, dz)$

**Minor expansion**

Expanding $\alpha(z, w)$ and $\beta(z, w)$ in the basis $e_{\mathcal{H}_+}$ and $e_{\mathcal{H}_-}$, the $\tau$-function (3.15) can be expressed as a minor expansion. Starting with $\alpha(z, w)$,

$$\alpha^n_m = \langle \alpha(z, w)e^n_{\mathcal{H}_-}, e^m_{\mathcal{H}_+}(z) \rangle$$

$$= \int \frac{dz}{2\pi i} e^m_{\mathcal{H}_+}(z) \int \frac{dw}{2\pi i} \alpha(z, w)e^n_{\mathcal{H}_-}(w)$$

$$= \frac{-\kappa}{m!n!} \int \frac{dz}{2\pi i} \left( \frac{z + 1}{z - 1} \right)^m \frac{1}{(z - 1)} \int \frac{dw}{2\pi i} e^{\nu(w, s)} \left( \frac{w - 1}{w + 1} \right)^n \frac{1}{(w + 1)}$$

$$= \frac{\kappa}{m!n!} \int \frac{dw}{2\pi i} e^{\nu(w, s)} (w - 1)^{m+n+2}$$

(4.15)

Similarly for $\beta(w, z)$

$$\beta^n_m = \langle \beta(z, w)e^n_{\mathcal{H}_-}, e^m_{\mathcal{H}_+}(z) \rangle$$

$$= \int \frac{dz}{2\pi i} e^m_{\mathcal{H}_+}(z) \int \frac{dw}{2\pi i} \beta(z, w)e^n_{\mathcal{H}_-}(w)$$

$$= \frac{\kappa}{m!n!} \int \frac{dz}{2\pi i} \left( \frac{z - 1}{z + 1} \right)^m \frac{1}{(z + 1)} \int \frac{dw}{2\pi i} e^{-\nu(w, s)} \left( \frac{w + 1}{w - 1} \right)^n \frac{1}{(w - 1)}$$

$$= \frac{-\kappa}{m!n!} \int \frac{dw}{2\pi i} e^{-\nu(w, s)} (w + 1)^{m+n+2}$$

(4.16)

**Recurrence relations**

$\alpha^n_m$, $\beta^n_m$ can be made explicit by noticing that the functions such as $\int \frac{dw}{2\pi i} e^{\nu(w, s)} (w + 1)^{m+n+2}$ can be written as some derivatives of the Airy function. Define the function $\chi_{m+n}$ as

$$\chi_{m+n} = (w - 1)^{m+n} \frac{(w + 1)^{m+n+2}}{m!n!} e^{\nu(w, s)}.$$  

(4.17)

Then $\alpha^n_m$ in terms of $\chi_{m+n}$ is simply

$$\alpha^n_m = \frac{\kappa}{m!n!} \int \frac{dw}{2\pi i} e^{\nu(w, s)} (w - 1)^{m+n+2} = \frac{\kappa}{m!n!} \int \frac{dw}{2\pi i} \chi_{m+n}$$

(4.18)

and $\beta^n_m$ in terms of $\chi_{m+n}$ is

$$\beta^n_m = \frac{-\kappa}{m!n!} \int \frac{dw}{2\pi i} e^{-\nu(w, s)} (w + 1)^{m+n+2}$$

$$= \frac{\kappa}{m!n!} \int \frac{dw}{2\pi i} e^{\nu(w, s)} (w + 1)^{m+n+2} = \frac{\kappa}{m!n!} \int \frac{dw}{2\pi i} \chi_{m+n}$$

(4.19)
The function $\chi_{m+n}$ obeys a recursion relation that can be derived as follows

$$
\int \frac{dw}{2\pi i} \chi_{m+n}(w, s) = \int \frac{dw}{2\pi i} \left( \frac{1}{w+1} \right)^{m+n+1} e^{\nu(w,s)} - \int \frac{dw}{2\pi i} \left( \frac{1}{w+1} \right) e^{\nu(w,s)}
$$

which gives the following equation

$$
\int_{\mathbb{R}} \frac{dw}{2\pi i} \left[ \chi_{m+n} + \frac{2}{(m+n+1)} \left( \frac{\partial}{\partial s} - 1 \right)^2 \left( \frac{4}{\partial s^2} - s \right) \chi_{m+n-1} \right] = 0 \tag{4.21}
$$

We define a function $Ci(t)$ as follows:

$$
Ci(t) := \int \frac{dw}{2\pi i} \left( \frac{1}{w+1} \right) e^{\nu(w)} = \left( 1 - \frac{\partial}{\partial s} \right)^{-1} Ai(t) \tag{4.22}
$$

Then the function $\chi_0$ can be computed in terms of $Ci(t)$

$$
\int_{\mathbb{R}} \frac{dw}{2\pi i} \chi_0 = \int \frac{dw}{2\pi i} \left( \frac{1}{w+1} \right) e^{\nu(w,s)} = \int \frac{dw}{2\pi i} \partial_w \left( \frac{1}{w+1} \right) e^{\nu(w,s)}
$$

$$
= \int \frac{dw}{2\pi i} \left( \frac{4}{\partial s^2} - s \right) \left( \frac{1}{w+1} \right) e^{\nu(w,s)}
$$

$$
= \left( \frac{4}{\partial s^2} - s \right) Ci(s) \tag{4.23}
$$

Since both $\alpha_m^n$ and $\beta_m^n$ are integrals of $\chi_{m+n}$, the recursion relation (4.21) and the expression for $\chi_0$ (4.22) give the explicit expressions for $\alpha_m^n$ and $\beta_m^n$. Starting with the recursion relation for $\alpha_m^n$ and $\beta_m^n$

$$
n(m+n+1)\alpha_m^n + 2 \left( \frac{\partial}{\partial s} - 1 \right)^2 \left( \frac{4}{\partial s^2} - s \right) \alpha_{m-1}^n = 0 \tag{4.24}
$$

$$
n(m+n+1)\beta_m^n + 2 \left( \frac{\partial}{\partial s} - 1 \right)^2 \left( \frac{4}{\partial s^2} - s \right) \beta_{m-1}^n = 0 \tag{4.25}
$$

with

$$
\alpha_0^n = \left( \frac{4}{\partial s^2} - 1 \right) \left( 1 - \frac{\partial}{\partial s} \right)^{-1} Ai(s) \tag{4.26}
$$

Further,

$$
n\alpha_{m-1} = m\alpha_{m-1}^n \quad ; \quad n\beta_{m-1} = m\beta_{m-1}^n \tag{4.27}
$$
Notice that (4.24) and (4.25) are recursive relations in \( n \). Using (4.27) similar recursion relations in \( m \) can be obtained
\[
m(m + n + 1)\alpha_m^n + 2\left(\frac{\partial}{\partial s} - 1\right)^2 \left(4 \frac{\partial^2}{\partial s^2} - s\right) \alpha_m^{n-1} = 0 \tag{4.28}
\]
\[
m(m + n + 1)\beta_m^n + 2\left(\frac{\partial}{\partial s} - 1\right)^2 \left(4 \frac{\partial^2}{\partial s^2} - s\right) \beta_m^{n-1} = 0 \tag{4.29}
\]
Define \( 2\left(\frac{\partial}{\partial s} - 1\right)^2 \left(4 \frac{\partial^2}{\partial s^2} - s\right) = \tilde{D} \). From (4.24)
\[
\alpha_m^n = (-1)^{m+n} \frac{(1 + m)}{n!(m + n + 1)!} \tilde{D}^n \alpha_m^0 \tag{4.30}
\]
now using (4.28)
\[
\alpha_m^0 = (-1)^m \frac{1}{m!(m + 1)!} \tilde{D}^m \alpha_0^0 \tag{4.31}
\]
In terms of \( \alpha_0^0, \alpha_m^n \) is explicit
\[
\alpha_m^n = \frac{(-1)^{m+n}}{(m!)^2 n!(m + n + 1)!} \tilde{D}^{m+n} \alpha_0^0 \tag{4.32}
\]
Repeating the same computation for \( \beta_m^n \)
\[
\beta_m^n = \frac{(-1)^{m+n}}{(m!)^2 m!(m + n + 1)!} \tilde{D}^{m+n} \beta_0^0 \tag{4.33}
\]
Maya diagrams

The determinant of an operator \( A \in \mathbb{C}^{m \times m} \) can be expanded in terms of its principal minors [11]. For a finite \( m \times m \) matrix \( A \), the minor expansion is given by
\[
\det (1 + A) = \sum_{n=0}^{\infty} \sum_{i_1 < \ldots < i_n} \det(A_{i_p, i_q})^n_{p, q=1} \tag{4.34}
\]
This sequence obviously terminates after \( n = m \). Now generalise the minor expansion for infinite dimensional matrix.

- Let \( K \) be a infinite dimensional matrix. Instead of labelling by \( \{1, \ldots, m\} \), \( K \) is labelled by an infinite discrete set. Define a half-integer lattice \( \mathbb{Z}' = \mathbb{Z} + \frac{1}{2} \). The set of all finite subsets of \( \mathbb{Z}' \) is given by \( \{0, 1\}^{\mathbb{Z}'} \) and \( \chi \subset \{0, 1\}^{\mathbb{Z}'} \). For \( \mathbb{Z}'_\chi = \mathbb{Z}' \setminus \chi \), define 'particles' (\( p_\chi \)) to be the positions \( p_\chi = \chi \cap \mathbb{Z}'_\chi \) and 'holes' (\( h_\chi \)) to be the positions \( h_\chi = \chi \cap \mathbb{Z}'_\chi \). (\( p_\chi, h_\chi \)) define point configurations on \( \mathbb{Z}' \). Furthermore, for a block matrix, we have two indices.
  - Expansion of the block determinant, given by the particles and holes (\( p_\chi, h_\chi \))
  - The index within the block, which is called the colour index.

- Maya diagram \( m_\chi \) is constructed by drawing filled circles at the points \( (\mathbb{Z}'_\chi \setminus p_\chi) \cup h_\chi \) and empty circles at \( p_\chi \cup (\mathbb{Z}'_\chi \setminus h_\chi) \). Set of all Maya diagrams is denoted by \( \mathcal{M} = \cup_{\chi} m_\chi \).
For $\det(1 + K)$, the minors can be labelled by the half integer lattice $\mathbb{Z}'$. Rows and columns will now be labelled by $\chi \subset \{0, 1\}^{\mathbb{Z}'}$. The minor expansion is given by

$$
\det [1 + K] = \sum_{\chi \subset \{0, 1\}^{\mathbb{Z}'}} A_{\chi} \tag{4.35}
$$

Maya diagrams can also be written as young diagrams by playing the following game. Reading the maya diagram from the left end, draw a horizontal line pointing to the right ($\rightarrow$) for every filled circle ($\bullet$) and a vertical line pointing downwards ($\downarrow$) for every empty circle ($\circ$).

Let us work out a simple example. Let $K$ be a $2 \times 2$ block matrix. $A, B$ are $2 \times 2$ matrices

$$
K = \begin{bmatrix}
0 & A \\
B & 0
\end{bmatrix} \tag{4.36}
$$

The labelling goes as shown in fig. 2. The blue and red label are the colour indices.

In fig. 2 collect the blue and red terms in terms of particles and holes. It is clear from here that minor contribution for the configurations where $|p| \neq |h|$ is 0. $p_\chi = (\frac{5}{2})$, $h_\chi = (-\frac{5}{2}, -\frac{1}{2})$; $p_\chi = (\frac{5}{2}, \frac{7}{2})$, $h_\chi = (-\frac{1}{2})$. This gives the blue and red Maya diagrams.

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$\begin{align*}
\chi_p & = (\frac{5}{2}, -\frac{5}{2}, -\frac{1}{2}) \\
\chi_h & = (\frac{5}{2}, \frac{7}{2}, -\frac{3}{2})
\end{align*}$

$\begin{align*}
\chi_p & = (\frac{1}{2}, \frac{1}{2}, 1) \\
\chi_h & = (\frac{1}{2}, \frac{7}{2}, -\frac{3}{2})
\end{align*}$

Figure 2: Minor expansion

Figure 3: Maya diagrams
and the corresponding Young tableaux are

Now for the Ablowitz-Segur $\tau$-function,

$$
\tau[s] = \det \left[ 1 - \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right] \quad (4.37)
$$

$\alpha$ and $\beta$ can be expanded on a discrete basis $e_{H_{\pm}}(z)$

$$
\alpha(z, w) = \sum_{m, n \in \mathbb{Z}_+} \alpha^m_n e_{H_+}(z)^m e_{H_-}(w)^n; \quad \beta(z, w) = \sum_{m, n \in \mathbb{Z}_+} \beta^m_n e_{H_-}(z)^m e_{H_+}(w)^n \quad (4.38)
$$

Since $\alpha^m_n$ and $\beta^m_n$ in (4.38) are not matrices themselves, the corresponding Maya diagrams are "colourless". If $a^m_n$ and $b^m_n$ were $N \times N$ matrices themselves, the corresponding entries in the expansion would be $a^m_n$ and $b^m_n$ where $\alpha, \beta = \{1, ..., N\}$ would be the colour indices. Furthermore, given the off-diagonal structure of the matrix $U$, the minors with $|p| \neq |h|$ vanish. Therefore, the minor expansion reads,

$$
\tau[s] = \sum_{m \in \mathcal{M}: |p|=|h|} h^p \beta_p \frac{x^p}{h_x}. \quad (4.39)
$$

The proof is now complete.

It would be extremely interesting to interpret the terms in this minor expansion in a similar way to the case of Painlevé VI, V, III. However, to our knowledge, in the case of the second Painlevé transcendent, there is no direct analog connection with some field theory. Nonetheless the computation proceeds in a rather natural way and may prove of use in future applications.

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