A Bollobás–type theorem for affine subspaces

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Abstract

Let $W$ denote the $n$-dimensional affine space over the finite field $\mathbb{F}_q$. We prove here a Bollobás–type upper bound in the case of the set of affine subspaces. We give a construction of a pair of families of affine subspaces, which shows that our result is almost sharp.

1 Introduction

First we introduce some notation.

In the following let $q = r^\alpha$ be a fixed prime power, $n \geq 1$ be a nonnegative integer. Let $W$ denote the $n$-dimensional affine space over the finite field $\mathbb{F}_q$.

B. Bollobás proved in [2] the following famous result.

Theorem 1.1 Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be two families of sets such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Then

$$\sum_{i=1}^{m} \frac{1}{(|A_i|+|B_i|)} \leq 1.$$  

In particular if $|A_i| = r$ and $|B_i| = s$ for each $1 \leq i \leq m$, then

$$m \leq \binom{r+s}{r}.$$  

The following strengthening of the uniform version of Bollobás’s theorem was proved by L. Lovász in [4] using tensor product methods.
Theorem 1.2 If $F = \{A_1, \ldots, A_m\}$ is an $r$-uniform family and $G = \{B_1, \ldots, B_m\}$ is an $s$-uniform family such that

(a) $A_i \cap B_i = \emptyset$

for each $1 \leq i \leq m$ and

(b) $A_i \cap B_j \neq \emptyset$

if $i < j$ ($1 \leq i, j \leq m$), then

$$m \leq \binom{r + s}{r}.$$

L. Lovász also proved the following generalization of Bollobás’ theorem for subspaces of a vector space in [5]:

**Theorem 1.3** Let $F$ be an arbitrary field and $V$ be an $n$-dimensional vector space over the field $F$.

Let $U_1, \ldots, U_m$ denote $r$-dimensional subspaces of $V$ and $V_1, \ldots, V_m$ denote $s$-dimensional subspaces of the vector space $V$. Assume that

(a) $U_i \cap V_i = \emptyset$

for each $1 \leq i \leq m$ and

(b) $U_i \cap V_j \neq \emptyset$

whenever $i < j$ ($1 \leq i, j \leq m$). Then

$$m \leq \binom{r + s}{r}.$$

In the following we give an affine version of Theorem 1.3.

We say that a pair of families of affine subspaces $(A_i, B_i)_{1 \leq i \leq m}$ of $W$ is cross–intersecting if

1. $A_i \cap B_i = \emptyset$,

for each $1 \leq i \leq m$ and

2. $A_i \cap B_j \neq \emptyset$

whenever $i < j$, ($1 \leq i, j \leq m$).

Let $m(n, q)$ denote the maximal size of a cross–intersecting pair of families of affine subspaces $(A_i, B_i)_{1 \leq i \leq n}$.

Our main result is the following modification of Lovász’ Theorem 1.3.
Theorem 1.4 Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be affine subspaces of an $n$-dimensional affine space $W$ over the finite field $\mathbb{F}_q$, where $q \neq 2$. Assume that $(A_i, B_i)_{1 \leq i \leq m}$ is cross-intersecting. Then

$$m \leq q^n + 1,$$

Remark. Theorem 1.4 means that

$$m(n, q) \leq q^n + 1.$$

Remark. Our result is a strengthening of Theorem 1.2 in the case of affine hyperplanes.

In Section 2 we prove Theorem 1.4. In the proof we use the polynomial subspace method (see [1]).

In Section 3 we give a simple construction, which shows that $m(n, q) \geq \frac{q^n - 1}{q - 1}$.

Finally in Section 4 we collect some open problems.

2 The proof of the main result

We use the following obvious observation in our proof.

Proposition 2.1 The intersection of a family of affine subspaces is either empty or equal to a translate of the intersection of their corresponding vector subspaces.

Recall that our main result was the following:

Theorem 2.2 Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be affine subspaces of an $n$-dimensional affine space $W$ over the finite field $\mathbb{F}_q$, where $q \neq 2$. Assume that $(A_i, B_i)_{1 \leq i \leq m}$ is cross-intersecting. Then

$$m \leq q^n + 1,$$
Proof:
Let \( p \) be an arbitrary, but fixed prime divisor of \( q - 1 \). Since \( q \neq 2 \), hence \( p > 1 \). We can assign for each subset \( F \subseteq \mathbb{F}_q^n \) its characteristic vector \( v_F \in \{0, 1\}^n \subseteq \mathbb{F}_p^n \) such that \( v_F(s) = 1 \) iff \( s \in F \). Here \( v_F(s) \) denotes the \( s^{th} \) coordinate of the vector \( v_F \).

Let \( 1 \leq j \leq m \) be fixed. Let \( v_j = (v_j(1), \ldots, v_j(q^n)) \) denote the characteristic vector of the affine subspace \( A_j \) and let \( w_j = (w_j(1), \ldots, w_j(q^n)) \) denote the characteristic vector of the affine subspace \( B_j \). Here \( v_j(i) \) denotes the \( i^{th} \) coordinate of the vector \( v_j \). Similarly, \( w_j(i) \) denotes the \( i^{th} \) coordinate of the vector \( w_j \).

Consider the polynomials
\[
P_i(x_1, \ldots, x_{q^n}) := 1 - \left( \sum_{k=1}^{q^n} v_i(k)x_k \right) \in \mathbb{F}_p[x_1, \ldots, x_{q^n}]
\]
for each \( 1 \leq i \leq m \).

We claim that the polynomials \( \{P_i : 1 \leq i \leq m\} \) are linearly independent functions over \( \mathbb{F}_p \). Namely
\[
P_i(w_i) = 1 - \sum_{k=1}^{q^n} v_i(k)w_i(k) = 1 - |A_i \cap B_i| = 1
\]
and
\[
P_i(w_j) = 1 - \sum_{k=1}^{q^n} v_j(k)w_j(k) = 1 - |A_i \cap B_j| = 1 - q^t,
\]
where \( t \geq 0 \), because \( (A_i, B_i)_{1 \leq i \leq m} \) is a cross–intersecting pair of families of affine subspaces and hence we can apply Proposition 2.1. Since
\[
q \equiv 1 \pmod{p},
\]
thus
\[
1 - q^t \equiv 0 \pmod{p}.
\]
Consider a linear combination
\[
\sum_{r=1}^{m} \lambda_r P_r = 0,
\]
where $\lambda_r \in \mathbb{F}_p$. It is easy to prove that $\lambda_r = 0$ for each $1 \leq r \leq m$. Namely for contradiction, suppose that there exists a nontrivial linear relation

$$
\sum_{s=1}^{m} \lambda_s P_s = 0. \quad (3)
$$

Let $s_0$ be the smallest $s$ such that $\lambda_s \neq 0$. Substitute $w_{s_0}$ for the variable of each side of (3). Then by equations (1) and (2), all but the $s_0^{th}$ term vanish, and what remains is

$$
\lambda_{s_0} P_{s_0}(w_{s_0}) = 0.
$$

But $P_{s_0}(w_{s_0}) \neq 0$ implies that $\lambda_{s_0} = 0$, a contradiction. Hence the polynomials $P_1, \ldots, P_m$ are linearly independent functions over $\mathbb{F}_p$.

We infer that the linearly independent polynomials $\{P_1, \ldots, P_m\}$ are in the $\mathbb{F}_p$-space spanned by the monomials

$$
\{x^u \in \mathbb{F}_p[x_1, \ldots, x_q^n] : \deg(x^u) \leq 1\}.
$$

Clearly

$$
|\{x^u : \deg(x^u) \leq 1\}| \leq q^n + 1,
$$

hence

$$
m \leq q^n + 1,
$$

which was to be proved. \hfill \Box

### 3 A simple construction

We use in our construction the following simple proposition.

**Proposition 3.1** Let $F_j$ be arbitrary affine subspaces for each $1 \leq j \leq m$. Let $G_j := \alpha_j + F_j$, where $\alpha_j \notin F_j$. Then $F_i \cap G_j \neq \emptyset$ iff $\alpha_j \in F_i - F_j$.

**Proof.**

First suppose that $\alpha_j \in F_i - F_j$. Then we can write $\alpha_j$ into the form

$$
\alpha_j = f_i - f_j,
$$

where $f_i \in F_i$ and $f_j \in F_j$. Hence $f_i = \alpha_j + f_j \in \alpha_j + F_j = G_j$. 

On the other hand, suppose that \( F_i \cap G_j \neq \emptyset \). Let \( v \in F_i \cap G_j \), i.e., \( v \in F_i \) and \( v \in \alpha_j + F_j \). Then there exists \( f_j \in F_j \) such that \( v = \alpha_j + f_j \) by definition. Hence \( \alpha_j = v - f_j \in F_i - F_j \).

**Proposition 3.2** Let \( n \geq 1 \) and \( q \) be an arbitrary prime power. Then \( m(n, q) \geq \frac{q^n - 1}{q - 1} \).

**Proof.** Let \( m = \frac{q^n - 1}{q - 1} \). We give a concrete cross-intersecting pair of families of affine subspaces \( \{A_1, \ldots, A_m\} \) and \( \{B_1, \ldots, B_m\} \) of an \( n \)-dimensional affine space \( W \) over the finite field \( \mathbb{F}_q \). Let

\( H = \{H_1, \ldots, H_m\} \)

denote an enumeration of the set of hyperplanes of the vector space \( \mathbb{F}_q^n \). It is easy to see that \( m = \frac{q^n - 1}{q - 1} \). For each \( 1 \leq i \leq m \) we fix a vector \( \beta_i \in \mathbb{F}_q^n \setminus H_i \). Define

\[ A_i := H_i, \]

and

\[ B_i := H_i + \beta_i. \]

Clearly \( A_i, B_i \) are affine subspaces of \( W \) for each \( 1 \leq i \leq m \).

Since \( \beta_i \notin H_i \) for each \( 1 \leq i \leq m \), hence \( A_i \cap B_i = \emptyset \) by the definition of \( A_i \) and \( B_i \).

On the other hand, since \( \beta_i \in H_i - H_j = \mathbb{F}_q^n \), hence it follows from Proposition 3.1 that \( A_i \cap B_j \neq \emptyset \) for each \( 1 \leq i < j \leq m \).

**4 Open problems**

Here we collect some interesting open problems.

-open problem 1: What can we say about \( m(n, 2) \)?

-open problem 2: What is the precise value of \( m(n, q) \), if \( q > 2 \)?

Finally we conjecture the following projective version of Theorem 1.4:
Conjecture 1 Let $\mathbb{F}$ be an arbitrary field. Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be projective subspaces of an $n$-dimensional projective space $W$ over the field $\mathbb{F}$. Assume that $(A_i, B_i)_{1 \leq i \leq m}$ is cross-intersecting (i.e. $A_i \cap B_i = \emptyset$ for each $1 \leq i \leq m$ and $A_i \cap B_j \neq \emptyset$ whenever $1 \leq i < j \leq m$). Then

$$m \leq 2^{n+1} - 2.$$ 

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