Harmonic functions which vanish on coaxial cylinders

Stephen J. Gardiner and Hermann Render

Abstract

It was recently established that a function which is harmonic on an infinite cylinder and vanishes on the boundary necessarily extends to an entire harmonic function. This paper considers harmonic functions on an annular cylinder which vanish on both the inner and outer cylindrical boundary components. Such functions are shown to extend harmonically to the whole of space apart from the common axis of symmetry. One of the ingredients in the proof is a new estimate for the zeros of cross product Bessel functions.

1 Introduction

The Schwarz reflection principle is a beautiful and important result concerning the extension of a harmonic function \( h \) on a domain \( \Omega \subset \mathbb{R}^N \) through a relatively open subset \( E \) of \( \partial \Omega \) on which \( h \) vanishes. The extension is defined by a simple formula, and the domain of extension is independent of the choice of \( h \). When \( N = 2 \) such a reflection principle holds whenever \( E \) is contained in an analytic arc (see Chapter 9 of [13]). When \( N \geq 3 \) and \( N \) is odd, Ebenfelt and Khavinson [5] (see also Chapter 10 of [13]) have shown that a point-to-point reflection law can only hold when the containing real analytic surface is either a hyperplane or a sphere. Thus, for other surfaces in higher dimensions, more elaborate arguments are required to investigate whether such harmonic extension is still possible.

An important particular case concerns cylindrical surfaces, since a cylinder is the Cartesian product of a line and a sphere, each of which separately admits Schwarz reflection. Indeed, prior to the results of [5], the existence of a point-to-point reflection law for cylinders in \( \mathbb{R}^3 \) had already been investigated and disproved by Khavinson and Shapiro [14]. Nevertheless, Khavinson asked whether, using \( B' \) to denote the open unit ball in \( \mathbb{R}^{N-1} \), a harmonic function on the cylinder \( B' \times \mathbb{R} \) which vanishes on \( \partial B' \times \mathbb{R} \) must automatically have a harmonic extension to the whole of \( \mathbb{R}^N \).

2010 Mathematics Subject Classification 31B05, 33C10.
Keywords: harmonic continuation, Green function, cylindrical harmonics, cross product Bessel functions
This was verified in a recent paper of the authors [6]. More generally, for any $a > 0$, it was shown there that a harmonic function on a finite cylinder $B' \times (-a, a)$ which vanishes on $\partial B' \times (-a, a)$ has a harmonic extension to the strip $\mathbb{R}^{N-1} \times (-a, a)$. The proof relied on a study of the Green function $G_{\Omega}(\cdot, y)$ for the infinite cylinder $\Omega = B' \times \mathbb{R}$ with pole at $y \in \Omega$. It is a classical fact that, in three dimensions, $G_{\Omega}(\cdot, y)$ can be represented as a double series involving Bessel functions $J_n$ of the first kind of order $n$ and their zeros, and Chebychev polynomials. In [6] such a representation was established for all dimensions (ultraspherical polynomials take the place of Chebychev polynomials when $N \geq 4$), and a rigorous analysis of its convergence properties outside $\Omega$ revealed that $G_{\Omega}(\cdot, y)$ possesses a harmonic extension to $\mathbb{R}^{N-1} \times (\mathbb{R} \setminus \{ y_N \})$.

In this paper we turn our attention to the corresponding problem for annular cylinders. Let $(x', x_N)$ denote a typical point of $\mathbb{R}^{N-1} \times \mathbb{R}$ and $\|x'\| = \text{Euclidean norm of } x'$. We define

$$\Omega_b = A'_b \times \mathbb{R}, \text{ where } A'_b = \{ x' : 1 < \|x'\| < b \} \quad (b > 1).$$

Any harmonic function $h$ on $\Omega_b$ that vanishes on the outer cylindrical boundary component was shown in [7] to have a harmonic extension to the set $\{ x' : 1 < \|x'\| < 2b - 1 \} \times \mathbb{R}$. We will now establish that considerably more can be said when $h$ also vanishes on the inner cylindrical boundary component.

**Theorem 1** If $h$ is a harmonic function on $\Omega_b$ that vanishes on $\partial\Omega_b$, then $h$ has a harmonic extension to $\left( \mathbb{R}^{N-1} \setminus \{ 0 \} \right) \times \mathbb{R}$.

The proof again depends on an analysis of the Green function, but this turns out to be more challenging for the annular cylinder. Instead of $J_\nu$, the double series expansions now involve factors of the form $J_\nu (\rho t) Y_\nu (\rho b) - J_\nu (\rho b) Y_\nu (\rho t)$, where $Y_\nu$ is the Bessel function of the second kind, and the sequence $(\rho_{\nu,m})_{m \geq 1}$ of positive $\rho$-zeros of this expression when $t = 1$. Known asymptotic estimates for $\rho_{\nu,m}$ for fixed $\nu$ are insufficient for our purposes, so we are led to establish a universal lower bound. We use this to show that a harmonic function on $A'_b \times (-a, a)$ which vanishes on $\partial A'_b \times (-a, a)$ must extend harmonically to all of $\left( \mathbb{R}^{N-1} \setminus \overline{B} \right) \times (-a, a)$. It also extends to a specified part of $\overline{B} \times (-a, a)$, which increases with $a$. Theorem 1 then follows on letting $a \to \infty$.

The proof of Theorem 1 will be developed in Sections 2 - 5, subject to verification of the estimates for $\rho_{\nu,m}$. These estimates are then established in the final two sections of the paper.

From now on we will assume that $N \geq 3$. 

2
2 Zeros of cross product Bessel functions

We refer to Watson \cite{21} for the definition of $J_\nu$ and $Y_\nu$, the usual Bessel functions of order $\nu \geq 0$ of the first and second kinds, respectively, and define $N_\nu = J_\nu^2 + Y_\nu^2$. Further, let $C_\nu$ denote any cylinder function of order $\nu$, that is, $C_\nu = \alpha J_\nu + \beta Y_\nu$ for some $\alpha, \beta \in \mathbb{R}$. We collect below some properties of these functions for later use.

**Lemma 2**

(i) \( \frac{d}{dz} z^\nu C_\nu(z) = z^\nu C_{\nu-1}(z) \) and \( \frac{d}{dz} \frac{C_\nu(z)}{z^\nu} = -\frac{C_{\nu+1}(z)}{z^\nu} \).

(ii) \( C_{\nu-1}(z) + C_{\nu+1}(z) = \frac{2\nu}{z} C_\nu(z) \) and \( C_{\nu-1}(z) - C_{\nu+1}(z) = 2C'_\nu(z) \).

(iii) \( J_\nu(t)Y'_\nu(t) - Y_\nu(t)J'_\nu(t) = \frac{2\pi t}{t>0} \).

(iv) If $\nu \geq \frac{1}{2}$, then the function $t \mapsto tN_\nu(t)$ is decreasing on $(0, \infty)$ and

\[
\frac{2}{\pi t} \leq N_\nu(t) < \frac{2}{\pi} \frac{1}{\sqrt{t^2 - \nu^2}} \quad (t > \nu).
\]

If $0 \leq \nu < \frac{1}{2}$, then the function $t \mapsto tN_\nu(t)$ is increasing on $(0, \infty)$ and tends to $2/\pi$ as $t \to \infty$.

(v) The function $N_\nu$ is strictly decreasing on $(0, \infty)$.

(vi) If $y(t)$ denotes $\sqrt{\kappa}C_\nu(\kappa t)$, where $\kappa$ is a non-zero constant, then

\[
\frac{d^2 y}{dt^2} + \left( \kappa^2 + \frac{1}{t^2} - \nu^2 \right) y = 0 \quad (t > 0).
\]

**Proof.** (i) and (ii) See pp.45, 66 of Watson \cite{21}.

(iii) See p.76, (1) of \cite{21}.

(iv) and (v) See Section 13.74 of \cite{21}.

(vi) See p.17, (1.8.9) of Szegö \cite{19}.

We now fix $b > 1$ and define

\[
U_\nu(\rho, t) = J_\nu(\rho t)Y'_\nu(\rho b) - J'_\nu(\rho b)Y_\nu(\rho t) \quad (\rho > 0, t > 0).
\]

It is known \cite{3} (cf. Theorem X of Chapter VII in \cite{8}, and the paragraph following the proof of Lemma 20 below) that the zeros of the function $\rho \mapsto U_\nu(\rho, 1)$ are all real and simple. We denote by $(\rho_{\nu, m})_{m \geq 1}$ the infinite sequence formed by the positive zeros of this function arranged in increasing order. Clearly,

\[
\text{the function } x' \mapsto U_\nu(\rho_{\nu, m}, \|x'\|) \text{ vanishes on } \partial A'_b \quad (\nu \geq 0, m \geq 1).
\]

Although the sequence $(\rho_{\nu, m})_{m \geq 1}$ has been studied over many years (as illustrated by \cite{16, 17}), the following important tool in the proof of Theorem \cite{4} appears to be new. We defer its proof until Section \cite{6}.

3
Theorem 3 If \( \nu \geq \frac{1}{2} \), then
\[
\rho_{\nu,m+1} - \rho_{\nu,m} > \frac{\pi}{2b-1} \quad (m \geq 2).
\]

Some further facts about \( U_\nu \) and \( (\rho_{\nu,m}) \) are assembled below.

Proposition 4

(i) If \( \nu \geq \frac{1}{2} \), then
\[
\rho_{\nu,m} > \frac{1}{b} \left( \nu + \frac{m}{4} \right) \quad (m \geq 1).
\] (2)

Also, for any \( \nu \geq 0 \),
\[
\frac{\rho_{\nu,m}}{m} \to \frac{\pi}{b-1} \quad (m \to \infty). \quad (3)
\]

(ii) If \( U_\nu(\rho_{\nu,m},a) = 0 \), where \( 0 < a \leq b \), then
\[
\left\{ \frac{\partial U_\nu(\rho_{\nu,m},a)}{\partial t} \right\}^2 = \frac{4}{\pi^2 a^2} \frac{N_\nu(\rho_{\nu,m}b)}{N_\nu(\rho_{\nu,m}a)} \leq \frac{4}{\pi^2 a^2}.
\]

(iii) If \( U_\nu(\rho_{\nu,m},a) = 0 \), where \( 0 < a < b \), then
\[
\rho_{\nu,m}^2 \int_a^b \{ U_\nu(\rho_{\nu,m},t) \}^2 t \, dt = \frac{2}{\pi^2} \left( 1 - \frac{N_\nu(\rho_{\nu,m}b)}{N_\nu(\rho_{\nu,m}a)} \right) \leq \frac{2}{\pi^2}.
\]

(iv) If \( \nu \geq \frac{1}{2} \), then
\[
\rho_{\nu,m}^2 \int_1^b \{ U_\nu(\rho_{\nu,m},t) \}^2 t \, dt \geq \frac{2}{\pi^2} \frac{b-1}{b} \quad (m \geq 1).
\] (4)

Also,
\[
\rho_{0,m}^2 \int_1^b \{ U_0(\rho_{0,m},t) \}^2 t \, dt \geq \frac{2}{\pi^2} \left( 1 - \frac{N_0(\rho_{0,1}b)}{N_0(\rho_{0,1})} \right) \quad (m \geq 1). \quad (5)
\]

(v) If \( a \) is the least positive zero of \( U_\nu(\rho_{\nu,m}, \cdot) \), where \( \nu > 0 \), then
\[
|U_\nu(\rho_{\nu,m},t)| \leq \frac{t-\nu}{\pi \nu} \quad (0 < t < a, m \geq 1).
\]

Proof. (i) Let \( j_{\nu,1}' \) denote the first positive zero of \( J_{\nu}' \). Then \( J_{\nu} \) is strictly increasing on \((0, j_{\nu,1}')\), so \( J_{\nu}(\rho)/J_{\nu}(\rho b) < 1 \) when \( \rho \in (0, j_{\nu,1}'/b) \), because \( b > 1 \). Since \( N_\nu \) is decreasing, by Lemma 2(v), we see that
\[
\left\{ \frac{J_{\nu}(\rho)}{J_{\nu}(\rho b)} \right\}^2 < \frac{N_\nu(\rho)}{N_\nu(\rho b)} \quad (0 < \rho \leq \frac{j_{\nu,1}'}{b}),
\]
Thus \( ρ \) for any cylinder function \( C \) by Lemma 2(iii). We know from p.486, (3) of Watson \[21\] that \( j_{ν,1}′ \) is a zero of \( N_{ν,1}(b) \leq 2(α^2 + β^2)/π \), and so (2) holds when \( m = 1, 2 \). The general case now follows from Theorem 3. The limit (3) is contained in asymptotic estimates of McMahon \[16\] (cf. Cochran \[3\]).

(ii) Let \( C_ν = αJ_ν + βY_ν \) and \( D_ν = −βJ_ν + αY_ν \), where \( α^2 + β^2 ≠ 0 \). Then

\[
\{C_ν\}^2 + \{D_ν\}^2 = (α^2 + β^2) \left( \{J_ν\}^2 + \{Y_ν\}^2 \right) = (α^2 + β^2)N_ν,
\]

and

\[
(C_νD_ν′ - C_ν′D_ν)(t) = (α^2 + β^2)(J_νY_ν′ - J_ν′Y_ν)(t) = \frac{2(α^2 + β^2)}{πt}
\]

by Lemma \[2\]iii). If \( ρ \) is a zero of \( C_ν \), we thus see that \( ρC_ν′(ρ)D_ν(ρ) = -2(α^2 + β^2)/π \), and so

\[
\{ρC_ν′(ρ)\}^2 = \frac{4}{π^2} \frac{(α^2 + β^2)^2}{\{C_ν(ρ)\}^2 + \{D_ν(ρ)\}^2} = \frac{4}{π^2} \frac{α^2 + β^2}{N_ν(ρ)}.
\]

We will now apply this formula to the cylinder function \( C_ν(t) \) defined by \( C_ν(ρ_{ν,m}t) = U_ν(ρ_{ν,m}, t) \). Thus \( α = Y_ν(ρ_{ν,m}b), \ β = −J_ν(ρ_{ν,m}b) \) and so \( α^2 + β^2 = N_ν(ρ_{ν,m}b) \). By putting \( ρ = ρ_{ν,m}a \), and noting that

\[
\frac{∂U_ν}{∂t}(ρ_{ν,m}, t) = ρ_{ν,m}C_ν′(ρ_{ν,m}t), \quad (6)
\]

we obtain the stated equality, and the subsequent inequality follows from Lemma \[2\]v).

(iii) We know from p.135, (11) of \[21\] that

\[
\int_a^b \{C_ν(pt)\}^2 t \, dt = \left[ \frac{t^2}{2} \left\{ \left( 1 - \frac{p^2}{ρ^2t^2} \right) \{C_ν(ρt)\}^2 + \{C_ν′(ρt)\}^2 \right\} \right]_a^b
\]

for any cylinder function \( C_ν \). When \( C_ν(ρ_{ν,m}t) = U_ν(ρ_{ν,m}, t) \), we can use (5), and then part (ii), to see that

\[
ρ_{ν,m}^2 \int_a^b \{U_ν(ρ_{ν,m}, t)\}^2 t \, dt = \frac{b^2}{2} \left\{ \frac{∂U_ν}{∂t}(ρ_{ν,m}, b) \right\}^2 - \frac{a^2}{2} \left\{ \frac{∂U_ν}{∂t}(ρ_{ν,m}, a) \right\}^2
\]

\[
= \frac{2}{π^2} \left( 1 - \frac{N_ν(ρ_{ν,m}b)}{N_ν(ρ_{ν,m}a)} \right) \leq \frac{2}{π^2}.
\]
(iv) If $\nu \geq \frac{1}{2}$, then we know from Lemma 2(iv) that $bN_\nu(\rho_{\nu,m}) \leq N_\nu(\rho_{\nu,m})$, so (4) follows from part (iii), with $a = 1$. Next, we note from Section 4.1 of Landau [15] and the Nicholson integral formula for $N_0$ (see p.444, (1) of [21]) that the function $t \mapsto -tN_0'(t)/N_0(t)$ is strictly increasing on $(0, \infty)$, whence

$$-\frac{btN_0'(bt)}{N_0(bt)} > \frac{tN_0'(t)}{N_0(t)},$$

or $bN_0(t)N_0'(bt) - N_0(bt)N_0'(t) < 0 \ (t > 0)$.

It follows that the function $t \mapsto N_0(bt)/N_0(t)$ is decreasing, so

$$1 - \frac{N_0(\rho_{\nu,m}b)}{N_0(\rho_{\nu,m})} \geq 1 - \frac{N_0(\rho_{0,1}b)}{N_0(\rho_{0,1})},$$

and (5) now follows from part (iii).

(v) Let $y(t) = \sqrt{t}U_\nu(\rho_{\nu,m}, t)$, where $\nu > 0$. Then

$$\frac{d}{dt} \left( t^{\nu - 1/2} y \right) = t^{1/2 - \nu} y'' - (\nu^2 - 1/4)t^{-3/2 - \nu} y = -\rho_{\nu,m} t^{1/2 - \nu} y,'$$

by Lemma 2(vi). Thus the left hand side of the above equation has the opposite sign to $y$ on $(0, a)$. Let

$$c = t^{1-2\nu} \frac{d}{dt} \left( t^{\nu - 1/2} y \right) \big|_{t=a} = a^{1/2 - \nu} y'(a) = a^{1-\nu} \frac{\partial U_\nu}{\partial t}(\rho_{\nu,m}, a). \quad (7)$$

If $y < 0$ on $(0, a)$, then $c > 0$ and $t^{1-2\nu} \frac{d}{dt} \left( t^{\nu - 1/2} y \right) < c$ on $(0, a)$. These last two inequalities are reversed if $y > 0$ on $(0, a)$. In either case, since $y(a) = 0$, we see that

$$\left| t^{\nu - 1/2} y(t) \right| \leq |c| \int_t^a \tau^{2\nu - 1} d\tau \leq |c| \frac{a^{2\nu}}{2\nu} \quad (0 < t < a),$$

whence

$$\left| U_\nu(\rho_{\nu,m}, t) \right| = \left| t^{-1/2} y(t) \right| \leq |c| \frac{a^{2\nu}}{2\nu} \leq \frac{a^{\nu} t^{-\nu}}{\nu} \leq \frac{t^{-\nu}}{\pi \nu} \quad (0 < t < a),$$

by (7), part (ii) and the fact that $a \leq 1$. □

3 Some integrals and inequalities

It will be convenient to define

$$\psi_\nu(t) = t^{\nu} - t^{-\nu} \quad (t > 0, \nu > 0).$$

6
Proposition 5 Let $0 < a < s < b$.

(i) If $I_{\nu}(s) = \int_{a}^{b} f_{\nu,s} (t) C_{\nu} (pt) t \, dt$, where $\nu > 0$, $\rho > 0$ and

$$f_{\nu,s} (t) = \begin{cases} \frac{\psi_{\nu}(t/a)\psi_{\nu}(b/s)}{\psi_{\nu}(b/a)} & (a \leq t \leq s) \\ \frac{\psi_{\nu}(s/a)\psi_{\nu}(b/t)}{\psi_{\nu}(b/a)} & (s < t \leq b) \end{cases},$$

then

$$\frac{\rho^2}{2\nu} I_{\nu}(s) = C_{\nu}(\rho s) - C_{\nu}(\rho a) \psi_{\nu}(b/s) + C_{\nu}(\rho b) \psi_{\nu}(s/a).$$

(ii) If $I_{0}(s) = \int_{a}^{b} f_{0,s} (t) C_{0} (pt) t \, dt$, where $\rho > 0$ and

$$f_{0,s} (t) = \begin{cases} \frac{\log(t/a)\log(b/s)}{\log(b/a)} & (a \leq t \leq s) \\ \frac{\log(s/a)\log(b/t)}{\log(b/a)} & (s < t \leq b) \end{cases},$$

then

$$\rho^2 I_{0}(s) = C_{0}(\rho s) - C_{0}(\rho a) \log(b/s) + C_{0}(\rho b) \log(s/a).$$

Proof. (i) By Lemma 2(i)

$$\rho \int_{s}^{a} t^{\nu+1} C_{\nu} (pt) \, dt = \left[ t^{\nu+1} C_{\nu+1} (pt) \right]_{a}^{s},$$

$$\rho \int_{s}^{a} t^{1-\nu} C_{\nu} (pt) \, dt = \left[ t^{1-\nu} C_{\nu-1} (pt) \right]_{a}^{s},$$

so

$$\rho \int_{a}^{s} t^{1-\nu} a^{-\nu} C_{\nu+1} (ps) + s^{1-\nu} a^{\nu} C_{\nu-1} (ps) - \frac{2\nu}{\rho} C_{\nu}(\rho a),$$

by Lemma 2(ii). Similarly,

$$\rho \int_{s}^{b} t^{1-\nu} (b/t) C_{\nu} (pt) \, dt = \rho b^{-\nu} \int_{s}^{b} t^{1-\nu} C_{\nu} (pt) \, dt - \rho b^{-\nu} \int_{s}^{b} t^{\nu+1} C_{\nu} (pt) \, dt$$

$$= -b^{-\nu} \left[ t^{1-\nu} C_{\nu-1} (pt) \right]_{s}^{b} - b^{1-\nu} \left[ t^{\nu+1} C_{\nu+1} (pt) \right]_{s}^{b}$$

$$= b^{1-\nu} C_{\nu-1} (ps) + b^{1-\nu} s^{\nu+1} C_{\nu+1} (ps) - \frac{2\nu}{\rho} C_{\nu}(\rho b).$$
Hence
\[
\psi_\nu(b/a)\rho I_\nu(s) = \psi_\nu(b/s) \rho \int_a^s t \psi_\nu(t/a) C_\nu(\rho t) \, dt + \psi_\nu(s/a) \rho \int_s^b t \psi_\nu(t/b) C_\nu(\rho t) \, dt
\]
\[
= \psi_\nu(b/s) \left( s^{\nu+1} a^{-\nu} C_{\nu+1}(\rho s) + s^{-\nu} a^{\nu} C_{\nu-1}(\rho s) - \frac{2\nu}{\rho} C_\nu(\rho a) \right) + \psi_\nu(s/a) \left( b^\nu s^{1-\nu} C_{\nu-1}(\rho s) + b^{-\nu} s^{\nu+1} C_{\nu+1}(\rho s) - \frac{2\nu}{\rho} C_\nu(\rho b) \right).
\]
The coefficients of the cylinder functions $C_{\nu+1}, C_{\nu-1}$ in the above expression are, respectively,
\[
a^{-\nu} \left( b^\nu s - b^{-\nu} s^{2\nu+1} \right) + b^{-\nu} \left( a^{-\nu} s^{2\nu+1} - a^\nu s \right) = s \psi_\nu(b/a),
\]
\[
a^\nu \left( b^\nu s^{1-2\nu} - b^{-\nu} s \right) + b^\nu \left( a^{-\nu} s - a^\nu s^{1-2\nu} \right) = s \psi_\nu(b/a).
\]
Thus we can again use Lemma 2(ii) to see that
\[
\psi_\nu(b/a)\rho I_\nu(s) = s \psi_\nu(b/a) \frac{2\nu}{\rho s} C_\nu(\rho s) - \frac{2\nu}{\rho} C_\nu(\rho a) \psi_\nu(b/s) - \frac{2\nu}{\rho} C_\nu(\rho b) \psi_\nu(s/a),
\]
as claimed.

(ii) By Lemma 2(i)
\[
\rho \int_a^s \log(t/a) C_0(\rho t) \, dt = \left[ t C_1(\rho t) \log(t/a) \right]_a^s - \int_a^s C_1(\rho t) \, dt
\]
\[
= s C_1(\rho s) \log(s/a) + \rho^{-1} C_0(\rho t) |_a^s,
\]
\[
\rho \int_s^b \log(b/t) C_0(\rho t) \, dt = \left[ t C_1(\rho t) \log(b/t) \right]_s^b + \int_s^b C_1(\rho t) \, dt
\]
\[
= -s C_1(\rho s) \log(b/s) - \rho^{-1} C_0(\rho t) |_s^b.
\]
Hence
\[
\rho^2 \log(b/a) I_0(s) = \rho^2 \log(b/s) \int_a^s \log(t/a) C_0(\rho t) \, dt + \rho^2 \log(s/a) \int_s^b \log(b/t) C_0(\rho t) \, dt
\]
\[
= \log(b/s) \left( \rho s C_1(\rho s) \log(s/a) + C_0(\rho s) - \rho C_0(\rho a) \right)
\]
\[
+ \log(s/a) \left( -\rho s C_1(\rho s) \log(b/s) - C_0(\rho b) + \rho C_0(\rho s) \right)
\]
\[
= \log(b/a) C_0(\rho s) - \log(b/s) C_0(\rho a) - \log(s/a) C_0(\rho b),
\]
as required.

**Proposition 6** If $0 < a \leq s \leq b$ and $U_\nu(\rho_{\nu,m}, a) = 0$, then
\[
|U_\nu(\rho_{\nu,m}, s)| \leq \frac{\rho_{\nu,m} b}{2\pi \nu} \quad (\nu > 0), \quad |U_0(\rho_{0,m}, s)| \leq \frac{\rho_{0,m} b}{4\pi} \log \frac{b}{a} \quad (8)
\]
and
\[
\left| \frac{\partial U_\nu}{\partial t}(\rho_{\nu,m}, s) \right| \leq \frac{\rho_{\nu,m} b}{\pi} \quad (\nu \geq 0).
\]

---

8
Proof. We may assume that \( s \in (a, b) \), since (8) trivially holds when \( s \in \{a, b\} \) and (9) extends by continuity to the endpoints. Let

\[
I_\nu(s) = \int_a^b f_{\nu, s}(t) U_\nu(\rho_{\nu, m}, t) \, t \, dt,
\]

where \( \nu \geq 0 \) and \( f_{\nu, s} \) is defined as in the previous proposition. It is easy to see that

\[
\max_{t \in [a, b]} f_{\nu, s}(t) = f_{\nu, s}(s) \quad \text{and} \quad \max_{s \in [a, b]} f_{\nu, s}(s) = f_{\nu, \sqrt{ab}}(\sqrt{ab}).
\]

Further,

\[
f_{\nu, \sqrt{ab}}(\sqrt{ab}) = \left\{ \frac{\psi_\nu(\sqrt{b/a})}{\psi_\nu(b/a)} \right\}^2 = \frac{1 - (a/b)^\nu}{1 + (a/b)^\nu} \leq 1 \quad (\nu > 0)
\]

and

\[
f_{0, \sqrt{ab}}(\sqrt{ab}) = \left\{ \frac{\log(\sqrt{b/a})}{\log b/a} \right\}^2 = \frac{\log(b/a)}{4}.
\]

Thus, by the Cauchy-Schwarz inequality and Proposition 4(iii),

\[
|I_\nu(s)| \leq \left\{ \int_a^b \{|f_{\nu, s}(t)|^2\} t \, dt \right\}^{1/2} \left\{ \int_a^b \{|U_\nu(\rho_{\nu, m}, t)|^2\} t \, dt \right\}^{1/2}
\]

\[
\leq \left\{ \frac{b^2 - a^2}{2} \right\}^{1/2} \left\{ \frac{2}{\pi^2 \rho_{\nu, m}^2} \right\}^{1/2} \leq \frac{b}{\pi \rho_{\nu, m}} \quad (\nu > 0), \tag{10}
\]

and similarly

\[
|I_0(s)| \leq \frac{b \log(b/a)}{4\pi \rho_{0, m}}. \tag{11}
\]

Next, we observe that

\[
I'_\nu(s) = \int_a^b g_{\nu, s}(t) U_\nu(\rho_{\nu, m}, t) \, t \, dt \quad (a < s < b),
\]

where

\[
g_{\nu, s}(t) = \frac{d}{ds} f_{\nu, s}(t) = \begin{cases} \frac{-\nu \psi_\nu(t/a)}{s \psi_\nu(b/a)} \left( \left( \frac{b}{s} \right)^\nu + \left( \frac{b}{s} \right)^{-\nu} \right) & (a \leq t < s) \\ \frac{\nu \psi_\nu(b/t)}{s \psi_\nu(b/a)} \left( \left( \frac{s}{a} \right)^\nu + \left( \frac{s}{a} \right)^{-\nu} \right) & (s < t \leq b) \end{cases}
\]

and

\[
\int_0^\infty U_\nu(\rho_{\nu, m}, t) \, t \, dt = \frac{\pi \rho_{\nu, m}}{\sqrt{\nu}}.
\]
when \( \nu > 0 \), and

\[
g_{0,s}(t) = \frac{d}{ds} f_{0,s}(t) = \begin{cases} 
-\frac{1}{s} \log(t/a) & (a \leq t < s) \\
\frac{1}{s} \log(b/t) & (s < t \leq b)
\end{cases}.
\]

Since \( \psi_{\nu}(t/a) \leq \psi_{\nu}(s/a) \) \((a \leq t < s)\) and \( \psi_{\nu}(b/t) \leq \psi_{\nu}(b/s) \) \((s < t \leq b)\), and

\[
\psi_{\nu} \left( \frac{s}{a} \right) \left( \left( \frac{b}{s} \right)^{\nu} + \left( \frac{s}{b} \right)^{-\nu} \right) = \psi_{\nu} \left( \frac{b}{a} \right) + \psi_{\nu} \left( \frac{a}{s} \frac{2}{ab} \right) \leq 2 \psi_{\nu} \left( \frac{b}{a} \right) ,
\]

\[
\psi_{\nu} \left( \frac{b}{s} \right) \left( \left( \frac{s}{a} \right)^{\nu} + \left( \frac{s}{b} \right)^{-\nu} \right) = \psi_{\nu} \left( \frac{b}{a} \right) + \psi_{\nu} \left( \frac{a}{s} \frac{2}{b} \right) \leq 2 \psi_{\nu} \left( \frac{b}{a} \right) ,
\]

we see that \( |g_{\nu,s}(t)| \leq 2\nu/s \leq 2\nu/a \) when \( \nu > 0 \). Thus, by the Cauchy-Schwarz inequality,

\[
|I_{\nu}'(s)| \leq \frac{2\nu}{a} \left\{ \frac{b^2 - a^2}{2} \right\}^{1/2} \left\{ \frac{2}{\pi^2 \rho_{\nu,m}} \right\}^{1/2} \leq \frac{2\nu}{\pi \rho_{\nu,m} a} \quad (\nu > 0). \tag{12}
\]

Similarly, since clearly \( |g_{0,s}(t)| \leq 1/s \leq 1/a \), we have

\[
|I_{0}'(s)| \leq \frac{1}{a} \left\{ \frac{b^2 - a^2}{2} \right\}^{1/2} \left\{ \frac{2}{\pi^2 \rho_{0,m}} \right\}^{1/2} \leq \frac{1}{\pi \rho_{0,m} a}. \tag{13}
\]

The inequalities (8) and (9) follow from (10) - (13), since we can put \( C_{\nu}(pt) = U_{\nu}(p,t) \) in Proposition 5 to see that

\[
U_{\nu}(\rho_{\nu,m},s) = \frac{\rho_{\nu,m}^2}{2\nu} I_{\nu}(s) \quad (\nu > 0) \quad \text{and} \quad U_{0}(\rho_{0,m},s) = \rho_{0,m}^2 I_{0}(s).
\]

\[\blacksquare\]

### 4 Intermediate series expansions

We recall the following result from Section 1.11 of Titchmarsh [20]. (We have reformulated it using equation (1.6.4) there and Proposition 4(iii) above.)

**Proposition 7** Let \( f : [1,b] \to \mathbb{R} \) be a continuous function of bounded variation and let

\[
a_m = \frac{1}{J_{1}^{b} \left\{ U_{\nu}(\rho_{\nu,m},\tau) \right\}^{2}} \int_{1}^{b} f(\tau) U_{\nu}(\rho_{\nu,m},\tau) \tau \, d\tau.
\]

Then the series \( \sum_{m=1}^{\infty} a_m U_{\nu}(\rho_{\nu,m},t) \) converges pointwise to \( f(t) \) on \((1,b)\).
Formula (14) below is stated without proof by Carslaw [2].

**Proposition 8** Let $1 < s < b$.

(a) If $\nu > 0$, then

\[
2\nu \sum_{m=1}^{\infty} \frac{U_{\nu}(\rho_{\nu,m}, s)U_{\nu}(\rho_{\nu,m}, t)}{\rho_{\nu,m}^2 \int_1^b \{U_{\nu}(\rho_{\nu,m}, \tau)\}^2 \tau \, d\tau} = \begin{cases} 
\frac{\psi_{\nu}(t)\psi_{\nu}(b/s)}{\psi_{\nu}(b)} & (1 \leq t \leq s) \\
\frac{\psi_{\nu}(s)\psi_{\nu}(b/t)}{\psi_{\nu}(b)} & (s < t \leq b)
\end{cases}
\]

and the series converges uniformly for $t \in [1, b]$.

(b) In the case where $\nu = 0$,

\[
\sum_{m=1}^{\infty} \frac{U_0(\rho_{0,m}, s)U_0(\rho_{0,m}, t)}{\rho_{0,m}^2 \int_1^b \{U_0(\rho_{0,m}, \tau)\}^2 \tau \, d\tau} = \begin{cases} 
\frac{(\log t) \log(b/s)}{\log b} & (1 < t \leq s) \\
\frac{(\log s) \log(b/t)}{\log b} & (s < t < b)
\end{cases}
\]

and the series converges uniformly for $t \in [1, b]$.

**Proof.** We know from p.199 of [21] that any cylinder function $C_\nu$ satisfies $C_\nu(t) = O(t^{-1/2})$ as $t \to \infty$. Applying this estimate separately to each factor in the definition of $U_{\nu}(\rho_{\nu,m}, t)$, we see that $|U_{\nu}(\rho_{\nu,m}, \cdot)| \leq C(b, \nu)/\rho_{\nu,m}$ on $[1, b]$. Thus, by parts (i) and (iv) of Proposition 4, the series in (14) converges uniformly on $[1, b]$. Part (a) now follows from Proposition 7 and the fact that

\[
\int_1^b f_{\nu,s}(\tau)U_{\nu}(\rho_{\nu,m}, \tau) \tau \, d\tau = \frac{2\nu}{\rho_{\nu,m}^2} U_{\nu}(\rho_{\nu,m}, s),
\]

by Proposition 5(i), where $f_{\nu,s}(t)$ denotes the right hand side of (14).

Part (b) follows in similar fashion from Proposition 5(ii). $\blacksquare$

If $\lambda > 0$, let $P_n^{(\lambda)}$ be the usual ultraspherical (Gegenbauer) polynomial defined by the expansion

\[
(1 - 2tu + u^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(t)u^n \quad (|t| \leq 1, |u| < 1).
\]

(See Section 4.7 of Szegö [19], or Chapter IV of Stein and Weiss [18].) We note for future reference that

\[
|P_n^{(\lambda)}(t)| \leq P_n^{(\lambda)}(1) = \left(\frac{n + 2\lambda - 1}{n}\right) \quad (|t| \leq 1) \quad (16)
\]
(see Lemma 6(i) of [8]). Also, let $T_n(t)$ be the Chebyshev polynomial given by $\cos(n \cos^{-1}(t))$ when $|t| \leq 1$, and let

$$\nu_n = n + \frac{N - 3}{2} \quad (n \geq 0).$$

We will need the following known expansions for the Green function $G_{A_n}'(\cdot, \cdot)$ of the annular region $A_n'$ in $\mathbb{R}^N$ ($N \geq 3$).

**Proposition 9** Suppose that $y' \in A_n'$.

(i) Let $1 < \|x\| < \|y\|$, then

$$G_{A_n}'(x', y') = \left(\frac{3}{2}\right)^{N} \sum_{n=0}^{\infty} \frac{N-3}{n} \left(\frac{\langle x', y' \rangle}{\|y\|^2}\right) \left(\frac{\psi_{\nu_n}(\|x\|) \psi_{\nu_n}(b/\|y\|)}{\psi_{\nu_n}(b)}\right);$$

(17)

and, if $\|y\| < \|x\| < b$, then

$$G_{A_n}'(x', y') = \left(\frac{3}{2}\right)^{N} \sum_{n=0}^{\infty} \frac{N-3}{n} \left(\frac{\langle x', y' \rangle}{\|y\|^2}\right) \left(\frac{\psi_{\nu_n}(\|y\|) \psi_{\nu_n}(b/\|x\|)}{\psi_{\nu_n}(b)}\right).$$

(18)

(ii) Let $N = 3$. If $1 < \|x\| < \|y\|$, then

$$G_{A_n}'(x', y') = \frac{\log(b/\|y\|)}{\log b} \log \|x\| + \sum_{n=1}^{\infty} \frac{1}{n} T_n \left(\frac{\langle x', y' \rangle}{\|x\| \|y\|}\right) \left(\frac{\psi_{\nu_n}(\|x\|) \psi_{\nu_n}(b/\|y\|)}{\psi_{\nu_n}(b)}\right);$$

(19)

and, if $\|y\| < \|x\| < b$, then

$$G_{A_n}'(x', y') = \frac{\log(b/\|x\|)}{\log b} \log \|y\| + \sum_{n=1}^{\infty} \frac{1}{n} T_n \left(\frac{\langle x', y' \rangle}{\|x\| \|y\|}\right) \left(\frac{\psi_{\nu_n}(\|y\|) \psi_{\nu_n}(b/\|x\|)}{\psi_{\nu_n}(b)}\right).$$

(20)

**Proof.** (i) This follows by dilation from Corollary 1.1 of Grossi and Vujadinovic [10].

(ii) This follows by combining Proposition 2.1 of Grossi and Takahashi [9] (cf. Hickey [11]) with the expansions

$$-\log \left(\frac{x'}{\|y\|} - \frac{y'}{\|y\|}\right) = \sum_{n=1}^{\infty} \frac{1}{n} T_n \left(\frac{\langle x', y' \rangle}{\|x\| \|y\|}\right) \left(\frac{\|x\|}{\|y\|}\right)^n \quad (\|x\| < \|y\|),$$

$$-\log \left(\frac{x'}{\|x\|} - \frac{y'}{\|x\|}\right) = \sum_{n=1}^{\infty} \frac{1}{n} T_n \left(\frac{\langle x', y' \rangle}{\|x\| \|y\|}\right) \left(\frac{\|y\|}{\|x\|}\right)^n \quad (\|y\| < \|x\|).$$
Let \( y' \in A'_b \) and \( \delta \in (0, 1) \), and let \( S_{y'} \) be the sphere in \( \mathbb{R}^{N-1} \) centred at \( 0' \) that contains \( y' \). We define \( \mu_{y', \delta} \) to be the probability measure on \( S_{y'} \) that has density with respect to surface area measure proportional to

\[
\exp \left( -\frac{1}{2} \frac{\|z'-y'\|^2}{\delta^2 \|y'\|^2} \right)^{-1} \quad \text{when } \|z'-y'\| < \delta \|y'\|, \text{ and 0 otherwise.}
\]

We further define the Green potential \( G_{A'_b, \mu_{y', \delta}}(x') = \int G_{A'_b}(x', z') \, d\mu_{y', \delta}(z') \quad (x' \in A'_b) \),

and the function

\[
P_n^{(N-2)} \mu_{y', \delta}(x') = \int P_n^{(N-2)} \left( \frac{\langle x', z' \rangle}{\|x'\| \|z'\|} \right) \, d\mu_{y', \delta}(z') \quad (x' \in \mathbb{R}^{N-1} \setminus \{0'\})
\]

when \( N \geq 4 \). When \( N = 3 \) the function \( T_n \mu_{y', \delta} \) is defined from \( T_n \) analogously.

We recall the following result (see [12]).

**Proposition 10** Let \( f \in C^\infty(\partial B') \) and let \( c_{i,j} \) be the Fourier coefficients of \( f \) with respect to an orthonormal basis \( \{H_{i,j} : j = 1, ..., M(i)\} \) of the spherical harmonics of degree \( i \) in \( \mathbb{R}^{N-1} \). Then the series \( \sum_{i=0}^\infty \sum_{j=1}^{M(i)} c_{i,j} H_{i,j} \) converges uniformly on \( \partial B' \) to \( f \), and so the series

\[
\sum_{i=0}^\infty \|x'\|^i \sum_{j=1}^{M(i)} c_{i,j} H_{i,j} \left( \frac{x'}{\|x'\|} \right)
\]

converges uniformly on \( B' \setminus \{0'\} \) to the Poisson integral of \( f \) in \( B' \).

**Remark 11** By Proposition 9, we obtain formulae for \( G_{A'_b, \mu_{y', \delta}}(x') \) if we replace \( P_n^{(N-2)} \left( \frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \) by \( P_n^{(N-2)} \mu_{y', \delta}(x') \) in (17) and (18), and \( T_n \left( \frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \) by \( T_n \mu_{y', \delta}(x') \) in (19) and (20). Further, the series in (17) and (19) would now converge uniformly on \( \{x' : 1 < \|x'\| \leq \|y'\|\} \). When \( N \geq 4 \) this follows from the proof of Corollary 1.1 in [10] with the additional ingredient that the restriction of the Newtonian potential

\[
x' \mapsto \int \|x' - z'\|^{3-N} \, d\mu_{y', \delta}(z') \quad (x' \in \mathbb{R}^{N-1})
\]

to \( S_{y'} \) is \( C^\infty \) (cf. Theorem 3.3.3 of [11]), and so we can appeal to the preceding proposition. The case where \( N = 3 \) follows similarly from [9]. Further, inversion can be used to show that the series in (18) and (20) would converge uniformly on \( \{x' : \|y'\| \leq \|x'\| < b\} \).
5 Proofs of main results

Let \( \eta_t \) denote the unit measure concentrated at \( t \in \mathbb{R} \).

**Lemma 12** For any \( n \geq 0, m \geq 1 \) and \( y \in \Omega_b \), let \( u_{n,m,y} \) be the function on \((\mathbb{R}^{N-1}\setminus\{0\}') \times \mathbb{R} \) defined by

\[
x \mapsto \|x'\|^\frac{n-N}{2} P_n(\frac{x'}{\|x'\|}) \left( \frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \frac{U_{\nu_n}(\rho_{\nu_n,m}, \|x'\|) U_{\nu_n}(\rho_{\nu_n,m}, \|y'\|)}{\rho_{\nu_n,m} \int_1^b U_{\nu_n}(\rho_{\nu_n,m}, t) \, dt} e^{-\rho_{\nu_n,m} |x_N - y_N|} \quad (N \geq 4),
\]

\[
x \mapsto T_n \left( \frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \frac{U_{\nu_n}(\rho_{\nu_n,m}, \|x'\|) U_{\nu_n}(\rho_{\nu_n,m}, \|y'\|)}{\rho_{\nu_n,m} \int_1^b U_{\nu_n}(\rho_{\nu_n,m}, t) \, dt} e^{-\rho_{\nu_n,m} |x_N - y_N|} \quad (N = 3).
\]

Then \( u_{n,m,y} \)

(i) is harmonic on \((\mathbb{R}^{N-1}\setminus\{0\}') \times (\mathbb{R}\setminus\{y_N\}));

(ii) continuously vanishes on \( \partial A'_b \times \mathbb{R} \);

(iii) has distributional Laplacian on \((\mathbb{R}^{N-1}\setminus\{0\}') \times \mathbb{R} \) given by

\[-2\rho_{\nu_n,m} u_{n,m,y}(x', y_N) dx' dy_N .
\]

**Proof.** Parts (i) and (iii) are proved by the same arguments as were used to establish parts (i) and (iv) of Lemma 11 in [6]. Part (ii) follows from [1].

The binomial coefficient \( \binom{n+N-4}{n} \), which appears in several estimates below, should be interpreted as 1 when \( N = 3 \). We denote the distance from \( y' \) to \( \mathbb{R}^{N-1}\setminus A_b' \) by

\[ d(y') = \min \{\|y'\| - 1, b - \|y'\|\} \quad (y' \in \overline{A_b'}). \]

Also, we will write \( C(\alpha, \beta, ...) \) for a positive constant depending at most on \( \alpha, \beta, ... \), not necessarily the same on any two occurrences.

**Lemma 13** Let \( n \geq 0, m \geq 1, y \in \Omega_b \), and let \( u_{n,m,y} \) be as in Lemma 12.

Then

(i) \( |u_{n,m,y}(x)| \leq C(b) \binom{n+N-4}{n} \rho_{\nu_n,m}^3 d(y') e^{-\rho_{\nu_n,m} |x_N - y_N|} \quad (1 \leq \|x'\| \leq b) ;
\]

(ii) \( |u_{n,m,y}(x)| \leq C(b) \binom{n+N-4}{n} \rho_{\nu_n,m}^2 m d(y') e^{-\rho_{\nu_n,m} |x_N - y_N|} \quad (b < \|x'\|) ;
\]

(iii) \( |u_{n,m,y}(x)| \leq C(b) \binom{n+N-4}{n} \rho_{\nu_n,m}^{\frac{3}{2}} \|x'\|^{(N-1)/2} d(y') e^{-\rho_{\nu_n,m} |x_N - y_N|} \quad (0 < \|x'\| < 1) .
\]
Proof. Since either \( \nu_n \geq \frac{1}{2} \) or \( \nu_n = 0 \), we see from Proposition 4(iv) that

\[
\frac{1}{\rho_{\nu_n,m}} \int_1^b \{ U_{\nu_n}(\rho_{\nu_n,m}, t) \}^2 \, dt \leq C(b)\rho_{\nu_n,m}.
\]

Further, by (9), (11) and the mean value theorem,

\[
|U_{\nu_n}(\rho_{\nu_n,m}, \|y'\|)| \leq \frac{\rho_{\nu_n,m}b}{\pi} d(y').
\]

In view of (16) it only remains to establish appropriate estimates for \( U_{\nu_n}(\rho_{\nu_n,m}, \|x'\|) \) in each of the three stated regions.

Proposition 6 (with \( a = 1 \)) shows that

\[
|U_{\nu_n}(\rho_{\nu_n,m}, \|x'\|)| \leq C(b)\rho_{\nu_n,m} \quad (1 \leq \|x'\| \leq b),
\]

so part (i) is established.

When \( \|x'\| > b \), we use the arithmetic-geometric means inequality and then Lemma 2(iv) to see that

\[
|U_{\nu_n}(\rho_{\nu_n,m}, \|x'\|)| \leq \frac{N_{\nu_n}(\rho_{\nu_n,m} \|x'\|) + N_{\nu_n}(\rho_{\nu_n,m}b)}{2} \leq N_{\nu_n}(\rho_{\nu_n,m}b),
\]

and hence that

\[
|U_{\nu_n}(\rho_{\nu_n,m}, \|x'\|)| \leq \frac{C}{m},
\]

by Proposition 3(i) and Lemma 2(iv). Part (ii) now follows.

To prove part (iii), let \( a \) denote the least positive zero of \( U_{\nu_n}(\rho_{\nu_n,m}, \cdot) \). Thus \( a \in (0,1] \). If \( \nu_n \geq \frac{1}{2} \), then we see from Proposition 6 that

\[
|U_{\nu_n}(\rho_{\nu_n,m}, \|x'\|)| \leq \frac{\rho_{\nu_n,m}b}{\pi} \quad (a \leq \|x'\| \leq 1),
\]

and from Proposition 3(iv) that

\[
|U_{\nu_n}(\rho_{\nu_n,m}, \|x'\|)| \leq \frac{2\|x'\|^{-\nu_n}}{\pi} \quad (0 < \|x'\| < a),
\]

whence

\[
|U_{\nu_n}(\rho_{\nu_n,m}, \|x'\|)| \leq C(b)\rho_{\nu_n,m} \|x'\|^{-\nu_n} \quad (0 < \|x'\| < 1).
\]

If \( \nu_n = 0 \), we instead observe that

\[
|U_{0}(\rho_{0,m}, \|x'\|)| \leq \frac{N_0(\rho_{0,m} \|x'\|) + N_0(\rho_{0,m}b)}{2} \leq N_0(\rho_{0,m} \|x'\|)
\]

\[
\leq \frac{2\|x'\|}{\pi \rho_{0,m} \|x'\|} \quad (0 < \|x'\| < 1),
\]

15
by Lemma 2(iv). ■

Let \( n \geq 0, \, \delta \in (0, 1) \) and \( y' \in A'_1 \). We define

\[
\begin{aligned}
h_{n,y'}^{\delta} (x') &= \begin{cases} \\
\frac{||x'||^{{\frac{3-N}{2}}} P_n (\frac{x'}{||x'||}) \mu_{y',\delta} (x') \frac{\psi_{\nu_n} (||x'||) \psi_{\nu_n} (b/ ||y'||)}{\psi_{\nu_n} (b)} }{\nu_n} & (1 < ||x'|| \leq ||y'||) \\
\frac{||x'||^{{\frac{3-N}{2}}} P_n (\frac{x'}{||y'||}) \mu_{y',\delta} (x') \frac{\psi_{\nu_n} (||y'||) \psi_{\nu_n} (b/ ||x'||)}{\psi_{\nu_n} (b)}}{\nu_n} & (||y'|| < ||x'|| < b)
\end{cases}
\end{aligned}
\]

if \( N \geq 4 \),

\[
\begin{aligned}
h_{n,y'}^{\delta} (x') &= \begin{cases} \\
\frac{1}{n} T_n \mu_{y',\delta} (x') \frac{\psi_{\nu_n} (||x'||) \psi_{\nu_n} (b/ ||y'||)}{\psi_{\nu_n} (b)} & (1 < ||x'|| \leq ||y'||) \\
\frac{1}{n} T_n \mu_{y',\delta} (x') \frac{\psi_{\nu_n} (||y'||) \psi_{\nu_n} (b/ ||x'||)}{\psi_{\nu_n} (b)} & (||y'|| < ||x'|| < b)
\end{cases}
\end{aligned}
\]

if \( N = 3 \) and \( n \geq 1 \), and when \( N = 3 \) and \( n = 0 \) we write

\[
\begin{aligned}
h_{0,y'}^{\delta} (x') &= \begin{cases} \\
2 \frac{\log (b/ ||y'||)}{\log b} \log ||x'|| & (1 < ||x'|| \leq ||y'||) \\
2 \frac{\log (b/ ||x'||)}{\log b} \log ||y'|| & (||y'|| < ||x'|| < b)
\end{cases}
\end{aligned}
\]

Further, let \( u_{n,m,y}^{\delta} \) have the same definition as \( u_{n,m,y}^{\delta} \), except that we use

\[
P_n (\frac{x'}{||y'||}) \mu_{y',\delta} (x') \) and \( T_n \mu_{y',\delta} (x') \) in place of \( P_n (\frac{x'}{||x'||}) \) and \( T_n (\frac{x'}{||x'|| ||y'||}) \), respectively.

**Remark 14** Lemmas 12 and 13 clearly remain true if we replace \( u_{n,m,y}^{\delta} \) by \( u_{n,m,y}^{\delta} \) throughout.

We define

\[
a_N = \sigma_N (N - 2) \quad \text{when} \quad N \geq 3, \quad \text{and} \quad a_2 = \sigma_2,
\]

where \( \sigma_N \) denotes the surface area of the unit sphere in \( \mathbb{R}^N \).

**Lemma 15** Let \( y \in \Omega_b \) and \( n \geq 0 \).

(i) The series \( \sum_{m=1}^{\infty} \mu^{\delta}_{n,m} u_{n,m,y}^{\delta} \) converges uniformly on \( \overline{\Omega_b} \) to a function \( v_{n,y}^{\delta} \) which is harmonic on \( A'_1 \times (\mathbb{R} \setminus \{y_N\}) \) and continuously vanishes on \( \partial \Omega_b \).

(ii) \( (-\Delta v_{n,y}^{\delta})(z) = h_{n,y'}^{\delta} (z') d'z' d\eta_{y_N} \) on \( \Omega_b \), in the sense of distributions.

16
Proof. We know from Lemma 12 and the above remark that, inside $\Omega_b$, the function 
\[ \rho^{-2}_{\nu_{\nu_{n,m}}u_{n,m,y}} \] is the (Green) potential of the measure 
\[ 2a_N^{-1}\rho^{-1}_{\nu_{\nu_{n,m}}u_{n,m,y}}(z',y_N)dz'd\eta_{y_N}. \]

Since the potential 
\[ x \mapsto \int_{A_b'} G_{\Omega_b}(x,(z',y_N)) \, dz' \quad (x \in \Omega_b) \]

is bounded on $\Omega_b$, it only remains to note from Proposition 8 that the series 
\[ z' \mapsto 2a_N^{-1}\sum_{m=1}^{\infty} \rho^{-1}_{\nu_{\nu_{n,m}}u_{n,m,y}}(z',y_N) \]
converges uniformly on $A_b'$ to $a_N^{-1}h_{n,y}^\delta(z')$. \textcircled{1}

Lemma 16 Let $y \in \Omega_b$ and $\delta \in (0,1)$. Then the series 
\[ x \mapsto \|y\|^{\frac{3-N}{2}} \sum_{n=0}^{\infty} \nu_{\nu_{n,y}}^\delta(x) \quad (N \geq 4), \]
\[ x \mapsto \frac{1}{2}v_{\delta,0,y}^\delta(x) + \sum_{n=1}^{\infty} v_{\delta,n,y}^\delta(x) \quad (N = 3) \]

converges uniformly on $\Omega_b$ to a function $g_0^\delta$ which is the Green potential in $\Omega_b$ of the measure $G_{A_b'}\mu_{\nu_{\nu_{n,y}}}(z')dz'd\eta_{y_N}$.

Proof. We know from Lemma 13 that, inside $\Omega_b$, the function $v_{\delta,n,y}^\delta$ is the Green potential of the measure $a_N^{-1}h_{n,y}^\delta(z')dz'd\eta_{y_N}$. Further, by Proposition 9 and Remark 11, the series 
\[ z' \mapsto \|y\|^{\frac{3-N}{2}} \sum_{n=0}^{\infty} \nu_{\nu_{n,y}}^\delta(z') \quad (N \geq 4), \]
\[ z' \mapsto \frac{1}{2}h_{\delta,0,y}^\delta(z') + \sum_{n=1}^{\infty} h_{\delta,n,y}^\delta(z') \quad (N = 3) \]

converge uniformly on $A_b'$ to $G_{A_b'}\mu_{\nu_{\nu_{n,y}}}(z')$. This establishes the result. \textcircled{1}

Theorem 17 If $y \in \Omega_b$ and $x \in A_b' \times (\mathbb{R}\setminus\{y_N\})$, then

\[ G_{\Omega_b}(x,y) = \begin{cases} 
\frac{a_N}{a_{N-1}} \|y\|^{\frac{3-N}{2}} \sum_{n=0}^{\infty} \nu_n \sum_{m=1}^{\infty} u_{n,m,y}(x) & (N \geq 4) \\
\sum_{m=1}^{\infty} u_{0,m,y}(x) + 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{n,m,y}(x) & (N = 3) 
\end{cases} \]
Proof. Let $g^\delta_y$ be as in Lemma 16. By Lemma 13(i), Remark 14 and Proposition 4(i) we can differentiate term-by-term to see that

$$
\frac{\partial^2 g^\delta_y}{\partial x_N^2} = \begin{cases}
\|y\|^2 N^{-\frac{2N}{2}} \sum_{n=0}^{\infty} \nu_n \sum_{m=1}^{\infty} u_{n,m,y}^\delta(x) & (N \geq 4) \\
\frac{1}{2} \sum_{m=1}^{\infty} \omega_{0,m,y}(x) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{n,m,y}^\delta(x) & (N = 3)
\end{cases}
$$

(22)
on A'_b \times (\mathbb{R}\{y_N\}). The same estimates show that the above function has limit 0 on approach to $\infty$ within $\Omega_b$. By Lemma 16, we know that $\frac{\partial^2 g^\delta_y}{\partial x_N^2}$ is harmonic on $A'_b \times (\mathbb{R}\{y_N\})$ and vanishes on $\partial A'_b \times (\mathbb{R}\{y_N\})$. Further,

$$
-\Delta \frac{\partial^2 g^\delta_y}{\partial x_N^2} = (-\Delta G_{A'_b} \mu_{y'},\delta) d\eta_{y_N} = a_{N-1} d\mu_{y',\delta} d\eta_{y_N}
$$
in the sense of distributions, so

$$
a_N a_N^{-1} \frac{\partial^2 g^\delta_y}{\partial x_N^2} = G_{\Omega_b} (\mu_{y',\delta} \eta_{y_N}) \text{ on } \Omega_b.
$$

Finally, as $\delta \to 0$, we note that $G_{\Omega_b} (\mu_{y',\delta} \eta_{y_N}) \to G_{\Omega_b} (\cdot, y)$, and from (22) we see that $(a_N/a_{N-1}) \partial^2 g^\delta_y/\partial x_N^2$ converges to the right hand side of (21) locally uniformly on $A'_b \times (\mathbb{R}\{y_N\})$, since $a_3/a_2 = 2$. 

Corollary 18 Let $y \in \Omega_b$, $b' > b$, $\varepsilon > 0$ and $\delta \in (0, 1)$. Then $G_{\Omega_b} (\cdot, y)$ has a harmonic extension $\widetilde{G}_{\Omega_b} (\cdot, y)$ to the set

$$
L = \left\{(x', x_N) : x_N \neq y_N, \|x'\| > e^{-|x_N-y_N|/b'}\right\}
$$

which satisfies

$$
\left| \widetilde{G}_{\Omega_b} (x, y) \right| \leq C(N, b, b', \delta, \varepsilon) d(y') \text{ (} x \in L, |x_N - y_N| > \varepsilon, \|x'\| > \delta \).
$$

(23)

Proof. Theorem 17 and the estimates in Lemma 13(ii) and Proposition 4(i), together show that $G_{\Omega_b} (\cdot, y)$ has a harmonic extension to the set $(\mathbb{R}^{N-1} \setminus \mathcal{F}) \times (\mathbb{R}\{y_N\})$ that satisfies $\left| \widetilde{G}_{\Omega_b} (x, y) \right| \leq C(N, b, \varepsilon) d(y')$ when $|x_N - y_N| > \varepsilon$. Further, by Lemma 13(iii), we see that $G_{\Omega_b} (\cdot, y)$ has a harmonic extension to $L$ that satisfies (23). 

Theorem 1 is an immediate consequence of the following result, subject to the verification below of Theorem 3.
Theorem 19 Let \( c > 0 \) and \( h \) be a harmonic function on \( A'_b \times (-c, c) \) which continuously vanishes on \( \partial A'_b \times (-c, c) \). Then \( h \) has a harmonic extension to the set
\[
\left\{ (x', x_N) : |x_N| < c, \|x'\| > e^{((|x_N|-c)/b)} \right\}.
\]

Proof. Let \( 0 < c'' < c' < c \) and \( b' > b \). On \( A'_b \times (-c', c') \) we can write \( h \) as the difference \( h_1 - h_2 \) of two positive harmonic functions that vanish on \( \partial A'_b \times (-c', c') \). (We can write \( h \) as the difference of two Dirichlet solutions there with non-negative boundary data.) Next, let \( h^*_i \) \((i = 1, 2)\) be defined as \( h_i \) on \( A'_b \times [-c'', c''] \), as 0 on \( (A'_b \times (-\infty, -c']) \cup (A'_b \times [c', \infty)) \cup \partial \Omega_b \), and extended to \( \Omega_b \) by solving the Dirichlet problem in \( A'_b \times (-c', -c'') \) and in \( A'_b \times (c', c'') \). Then \( h_i^* \) is subharmonic on \( A'_b \times ((-\infty, -c'') \cup (c', \infty)) \) and superharmonic on \( A'_b \times (-c', c') \), and continuously vanishes on \( \partial \Omega_b \). We can write \( h^*_i \) as \( G_{\Omega_b} \mu_i \), where \( \mu_i \) is a signed measure on \( \Omega_b \times \{ \pm c', \pm c'' \} \) satisfying
\[
\int d(y')d|\mu_i|(y) < \infty.
\]
It follows from Corollary 18 that the formula
\[
\tilde{h}(x) = \int_{A'_b \times \{ \pm c', \pm c'' \}} \tilde{G}_{\Omega_b}(x, y)d(\mu_1 - \mu_2)(y)
\]
defines a harmonic extension of \( h \) from \( A'_b \times (-c', c'') \) to the set
\[
\left\{ x_N < c'', \|x'\| > e^{\left((x_N-c'')/b'\right)} \right\} \cap \left\{ x_N > -c'', \|x'\| > e^{-\left((x_N+c'')/b'\right)} \right\}.
\]
Since \( c'' \) can be arbitrarily close to \( c \), and \( b' \) can be arbitrarily close to \( b \), the result follows. \( \square \)

6 Proof of Theorem 3

Let
\[
u(x, y) = x^{1/2} J_\nu(xy) Y_\nu(xy) - x^{1/2} J_\nu(xy) Y_\nu(y) \quad (x > 0, y > 0).
\]

We know that the cylinder function \( x \mapsto u_\nu(x, y) \) has infinitely many positive zeros which are all simple (see Sections 15.21, 15.24 of [21]). Let \( x_{\nu, k}(y) \) denote the \( k \)th zero of this function in \((1, \infty)\). By Lemma 21 vi) and Sturm’s comparison theorem [4],
\[
x_{\nu, k+1}(y) - x_{\nu, k}(y) \geq \frac{\pi}{y} \left( \nu \geq \frac{1}{2} \right). \quad (24)
\]
Further, by Sturm’s convexity theorem [4],
\[
x_{\nu, k+1}(y) - x_{\nu, k}(y) < x_{\nu, k}(y) - x_{\nu, 1}(y) \quad (k \geq 2, \nu \geq \frac{1}{2}). \quad (25)
\]

We collect together some useful facts about \( u_\nu(x, y) \) below.
Lemma 20 If \( u_\nu (x_0, y_0) = 0 \), where \( x_0 > 1 \), then

\[
\frac{\partial u_\nu}{\partial x}(x_0, y_0) \frac{\partial u_\nu}{\partial y}(x_0, y_0) = 2y_0 \int_1^{x_0} \{ u(x, y_0) \}^2 \, dx > 0, \tag{26}
\]

\[
\frac{\partial u_\nu}{\partial x}(x_0, y_0) > \frac{y_0}{x_0}, \tag{27}
\]

\[
\left( \frac{\partial u_\nu}{\partial x} \left( x \frac{\partial u_\nu}{\partial x} - y \frac{\partial u_\nu}{\partial y} \right) \right)(x_0, y_0) = \frac{4}{\pi^2}, \tag{28}
\]

\[
2x_0 \frac{\partial^2 u_\nu}{\partial y \partial x}(x_0, y_0) = 2 \frac{\partial u_\nu}{\partial y}(x_0, y_0) + y_0 \frac{\partial^2 u_\nu}{\partial y^2}(x_0, y_0). \tag{29}
\]

**Proof.** Inequality (26) is known \([3]\), but we will give a short alternative proof. We abbreviate \( u_\nu \) to \( u \), define \( q(x, y) = (\nu^2 - \frac{1}{4}) x^2 - y^2 \), and note from Lemma 2(vi) that \( u_{xx} = qu \). Hence \( u_{xxy} = qu_y + qu \), and so

\[
\frac{\partial}{\partial x} (u_{xy} u - u_{xx} u_y) = u_{xxy} u - u_{xx} u_y = (q_y u + qu_y) u - qu u_y = q_y u^2 = -2u^2.
\]

Since \( u(1, \cdot) \equiv 0 \), we can set \( y = y_0 \) and integrate the above equation with respect to \( x \) over \([1, x_0] \) to obtain (26).

We fix \( y \) and define

\[
f(x) = 2y_0^2 \int_1^x \{ u(t, y) \}^2 \, dt + xu^2 - xu x u.
\]

Using the fact that \( u_{xx} = qu \), we obtain

\[
f'(x) = u^2 \left( 2y_0^2 + 2q + xq_x \right) = u^2 \left\{ 2y_0^2 + 2 \left( \nu^2 - \frac{1}{4} \right) \frac{y^2}{x^2} - y^2 \right\} - 2 \frac{\nu^2 - \frac{1}{4}}{x^2} \right\} = 0.
\]

Since \( u(1, \cdot) \equiv 0 \), we see that \( f \equiv f(1) = -(u_x(1, y))^2 \). Further,

\[
u_x(1, y) = y \left( J_\nu(y) Y'_\nu(y) - J'_\nu(y) Y_\nu(y) \right) = \frac{2}{\pi}, \tag{30}
\]

by Lemma 2(iii). Since \( u(x_0, y_0) = 0 \), we conclude that

\[
2y_0^2 \int_1^{x_0} \{ u(t, y_0) \}^2 \, dt - x_0 \left( \frac{\partial u}{\partial x}(x_0, y_0) \right)^2 = f(x_0) = f(1) = \frac{4}{\pi^2} < 0.
\]

Thus, by (26),

\[
y_0 \frac{\partial u}{\partial x}(x_0, y_0) \frac{\partial u}{\partial y}(x_0, y_0) = 2y_0^2 \int_1^{x_0} \{ u(t, y_0) \}^2 \, dt < x_0 \left( \frac{\partial u}{\partial x}(x_0, y_0) \right)^2,
\]

and (27) follows.
Let $w = xu_x - yu_y$. Direct computation shows that $w_{xx} = qw$, since
$2q + xq_x - yq_y = 0$. For any fixed value of $y$ the expression $uw_x - u_xw$ thus has a constant value. Since $u(1, \cdot) \equiv 0$, and $w(1, y) = u_x (1, y) = 2/\pi$ by (30), we conclude that
$$
uw_x - u_xw = -\frac{4}{\pi^2},
$$
which yields (28) because $u(x_0, y_0) = 0$.

Differentiation of (31) with respect to $y$ yields
$$
0 = u_yw_x + uw_{xy} - u_{xy}w - u_xw_y
= u_y(u_x + xu_{xx} - yu_{xy}) + uw_{xy} - u_{xy}(xu_x - yu_y) - u_x(xu_{xy} - u_y - yu_{yy})
= 2uyu_x + xu_{xx}uy + uw_{xy} - u_{xy}zu_x + u_xyu_{yy}
= u(xqu_y + wy) + u_x(2uy - 2yu_x + yu_{yy}),
$$
since $u_{xx} = qu$. This simplifies to (29) because $u(x_0, y_0) = 0$ and $u_x(x_0, y_0) \neq 0$.

If $x > 1$, then we see from (29) that the function $y \mapsto u_{y}(x, y)$ has only simple zeros on $(0, \infty)$. We define $y_{\nu, k}(x) > 0$ to be the $k$th positive zero. (When $x = b$ these correspond to the zeros $\rho_{\nu, k}$ defined in Section 2.) Further, in view of (26), the implicit function theorem can be applied to the function $u_{\nu} : (1, \infty) \times (0, \infty) \to \mathbb{R}$ to see that $y_{\nu, k}$ is differentiable on $(1, \infty)$, and so we can differentiate the equation $u_{\nu}(x, y_{\nu, k}(x)) = 0$ to obtain
$$
y_{\nu, k}'(x) = -\frac{\partial u_{\nu}(x, y_{\nu, k}(x))}{\partial x} \frac{\partial u_{\nu}}{\partial y}(x, y_{\nu, k}(x)) < 0,
$$
by (26) again, whence $y_{\nu, k}$ is strictly decreasing on $(1, \infty)$. The following simple observation will help us to show that $y_{\nu, k}$ is also convex.

**Lemma 21** Suppose that $u(x, y)$ is a function such that $u_{xx} = qu$, and $y_k$ is a differentiable function such that $u(x, y_k(x)) = 0$. If $u_xu_y(2u_{xy}u_y - u_{yy}u_x) > 0$ on the zero set of $u$, then $y_k$ is convex.

**Proof.** We know that $y_k'u_y(x, y_k(x)) = -u_x(x, y_k(x))$, and $u_{xx} = qu = 0$ on the zero set of $u$, so
$$
y_k''(x) = -\frac{d}{dx} \left( \frac{u_x(x, y_k(x))}{u_y} \right) = \frac{u_{xx}u_y y_k' - u_x(u_{xy} + u_{yy}y_k')}{u_y^2}(x, y_k(x))
= \left( \frac{u_x(2u_{xy}u_y - u_{yy}u_x)}{u_y} \right)(x, y_k(x)) > 0.
$$

The following result will be proved in Section 7.
Proposition 22 Let \( \nu \geq \frac{1}{2} \). Then, for each \( x > 1 \) the cross product \( u_\nu(x, y) \) satisfies a second order differential equation, \( \tilde{P} F'' - \tilde{P}' F' + \tilde{Q} F = 0 \), where \( \tilde{P}(x, y) > 0 \) and \( \tilde{P}'(x, y) < 0 \).

We now prove a result that contains Theorem 3.

Theorem 23 If \( \nu \geq \frac{1}{2} \), then the zero curves \( y_{\nu, k}(1, \infty) \to (0, \infty) \) are convex, and

\[
y_{\nu, k}(x) - y_{\nu, k}(x) > \frac{\pi}{2x - 1} \quad (k \geq 2).
\]

Proof. On the zero set of \( u_\nu \) we have, by (29) and then (28),

\[
x \left( 2 \frac{\partial^2 u_\nu}{\partial x \partial y} \frac{\partial u_\nu}{\partial y} - \frac{\partial^2 u_\nu}{\partial y^2} \frac{\partial u_\nu}{\partial x} \right) = 2 \left( \frac{\partial u_\nu}{\partial y} \right)^2 + \frac{\partial^2 u_\nu}{\partial y^2} \left( y \frac{\partial u_\nu}{\partial y} - x \frac{\partial u_\nu}{\partial x} \right)
\]

= \[
2 \left( \frac{\partial u_\nu}{\partial y} \right)^2 - \frac{\partial^2 u_\nu}{\partial y^2} \frac{4}{\pi^2} \left\{ \frac{\partial u_\nu}{\partial x} \right\}.
\]

whence

\[
\frac{\partial u_\nu}{\partial x} \frac{\partial u_\nu}{\partial y} \left( 2 \frac{\partial^2 u_\nu}{\partial x \partial y} \frac{\partial u_\nu}{\partial y} - \frac{\partial^2 u_\nu}{\partial y^2} \frac{\partial u_\nu}{\partial x} \right) = \frac{2}{x} \frac{\partial u_\nu}{\partial y} \left( \frac{\partial u_\nu}{\partial y} \right)^3 - \frac{\partial^2 u_\nu}{\partial y^2} \frac{4}{\pi^2} \frac{\partial u_\nu}{\partial x}.
\]

The first term on the right hand side is positive, by (26), and the second is negative, by Proposition 22. Hence \( y_k \) is convex, by Lemma 21.

Let \( y_0 = y_{\nu, k}(x_0) \) be given, where \( x_0 > 1 \). Then \( x_0 \) is the \( k \)th zero of \( x \mapsto u_\nu(x, y_0) \) in \( (1, \infty) \), so \( x_0 = x_{\nu, k}(y_0) \). We now consider the next zero, \( x_{\nu, k+1}(y_0) \). By the convexity of \( y_{\nu, k+1} \),

\[
y_{\nu, k}(x_0) + \{ x_{\nu, k}(y_0) - x_{\nu, k+1}(y_0) \} y'_{\nu, k+1}(x_{\nu, k+1}(y_0)) \leq y_{\nu, k+1}(x_0).
\]

We use (22), (27) and (24) to deduce that

\[
y_{\nu, k+1}(x_0) - y_{\nu, k}(x_0) \geq (x_{\nu, k+1}(y_0) - x_{\nu, k}(y_0)) \frac{\partial u_\nu}{\partial x} (x_{\nu, k+1}(y_0), y_0) \frac{\partial u_\nu}{\partial y} (x_{\nu, k+1}(y_0), y_0)
\]

\[
\geq (x_{\nu, k+1}(y_0) - x_{\nu, k}(y_0)) \frac{y_0}{x_{\nu, k+1}(y_0)}
\]

\[
\geq \frac{\pi}{x_{\nu, k+1}(y_0)}.
\]

Finally, by (26),

\[
x_{\nu, k+1}(y_0) < 2x_{\nu, k}(y_0) - x_{\nu, k-1}(y_0) < 2x_{\nu, k}(y_0) - 1 = 2x_0 - 1 \quad (k \geq 2),
\]

so we arrive at (33).
7 Proof of Proposition 22

Let
\[ F(y) = a(y) f(y) + b(y) g(y), \]  
where
\[ f(y) = Y_\nu(xy), \quad g(y) = -J_\nu(xy), \quad a(y) = J_\nu(y), \quad b(y) = Y_\nu(y), \]
and \( x > 1 \) is fixed. We will show that functions of the form \( (34) \) satisfy a certain second order differential equation, and that when \( (35) \) holds the signs of the coefficients in this equation are as described in Proposition 22.

Let \( f, g, a, b \) be differentiable functions defined on \( (c, d) \), and let
\[ W = fg' - f'g, \quad N = f^2 + g^2, \quad w = ab' - a'b, \quad n = a^2 + b^2. \]

Lemma 24 If \( F = af + bg \), then
\[ F' = Af + Bg \quad \text{and} \quad F'' = Cf + Dg, \]
where
\[
A = a' + \frac{aN'}{2N} + \frac{bW}{N} \quad \text{and} \quad B = b' + \frac{bN'}{2N} - \frac{aW}{N}, \\
C = A' + \frac{AN'}{2N} + \frac{BW}{N} \quad \text{and} \quad D = B' + \frac{BN'}{2N} - \frac{AW}{N}.
\]

Proof. Since
\[ 2Nf' - N'f = 2g^2f' - 2g'gf = -2gW \quad \text{and} \quad 2Ng' - N'g = 2f^2g' - 2f'fg = 2fW, \]
we see that
\[ f' = \frac{N'}{2N}f - \frac{W}{N}g \quad \text{and} \quad g' = \frac{N'}{2N}g + \frac{W}{N}f, \]
and so
\[ F' = a'f + af' + b'g + bg' = f \left( a' + \frac{aN'}{2N} + \frac{bW}{N} \right) + g \left( b' + \frac{bN'}{2N} - \frac{aW}{N} \right). \]
Thus \( F'' = Af + Bg \). The same reasoning, applied to \( F' \), shows that \( F'' = Cf + Dg \).

Proposition 25 Let \( f, g, a, b \) be smooth functions. Then the function \( F = af + bg \) satisfies the differential equation
\[ \widetilde{P}F'' - \widetilde{P}'F' + \widetilde{Q}F = 0, \]
where \( \widetilde{P} = Wn - Nw \) and \( \widetilde{Q} = N(CB - DA) \).
Proof. We know from Lemma 24 that $F' = Af + Bg$ and $F'' = C f + Dg$. Hence

$$(Ab - Ba) F'' - (Cb - Da) F' + (CB - DA) F = 0,$$  \hspace{1cm} (37)

because (trivially)

$$(Ab - Ba) C - (Cb - Da) A + (CB - DA) a = 0,$$

$$(Ab - Ba) D - (Cb - Da) B + (CB - DA) b = 0.$$

Since

$$Ab - Ba = a'b - b'a + \frac{W}{N} (b^2 + a^2),$$  \hspace{1cm} (38)

we see that

$$N (Ab - Ba) = W (a^2 + b^2) - N (ab' - a'b) = \tilde{P},$$

and we can multiply across (37) by $N$ to get

$$\tilde{P} F'' - N (Cb - Da) F' + N (CB - DA) F = 0.$$

It remains to check that $N (Cb - Da) = \tilde{P}'$. Since

$$N (Cb - Da) = N (A'b - B'a) + \frac{N'}{2} (Ab - Ba) + W (Bb + Aa)$$

and

$$\tilde{P}' = N' (Ab - Ba) + N (A'b - B'a) + N (Ab' - Ba'),$$

we see that $N (Cb - Da) - \tilde{P}' = -\Delta_1$, where

$$\Delta_1 = \frac{N'}{2} (Ab - Ba) + N (Ab' - Ba') - W (Bb + Aa),$$

and it suffices to show that $\Delta_1 = 0$. We compute

$$Ab' - Ba' = \frac{N'}{2N} (ab' - a'b) + \frac{W}{N} (a'a + b'b),$$

$$Bb + Aa = b'b + a'a + \frac{N'}{2N} (b^2 + a^2),$$

and use these identities along with (38) to obtain

$$\Delta_1 = \frac{N'}{2} (a'b - b'a) + \frac{N'}{2N} (a^2 + b^2) + \frac{N'}{2} (ab' - a'b) + W (a'a + b'b)$$

$$- W \left( b'b + a'a + \frac{N'}{2N} (b^2 + a^2) \right) = 0.$$
Proof of Proposition 22. We apply the preceding proposition to the case where (35) holds. Then $N(y) = N_{\nu}(xy)$, $n(y) = N_{\nu}(y)$,
\[ W(y) = xJ_{\nu}(xy)Y'_{\nu}(xy) - xJ'_{\nu}(xy)Y_{\nu}(xy) = x\frac{2}{\pi xy} = \frac{2}{\pi y} \]
by Lemma 2(iii), and similarly $w(y) = 2/(\pi y)$. Further,
\[ \tilde{P}(x, y) = Wn - wN = \frac{2}{\pi y} (N_{\nu}(y) - N_{\nu}(xy)) > 0 \]
by Lemma 2(v), and
\[ \tilde{P}'(x, y) = -\frac{\tilde{P}(x, y)}{y} + \frac{2}{\pi y^2} (yN'_{\nu}(y) - yxN'_{\nu}(xy)) < 0, \]
since $y \mapsto yN'_{\nu}(y)$ is increasing. (It is clear from p.446 of [21] that $(d/dt)(tN_{\nu}(t))$ is increasing when $\nu \geq \frac{1}{2}$, and we also know that $N_{\nu}$ is decreasing.) Proposition 22 is now established, because $u_{\nu}(x, y) = \sqrt{xF}(y)$. ■

References

[1] D. H. Armitage and S. J. Gardiner, *Classical potential theory*, Springer, London, 2001.

[2] H. S. Carslaw, *Integral equations and the determination of Green’s functions in the theory of potential*, Proc. Edinb. Math. Soc. 31 (1913) 71–89.

[3] J.A. Cochran, *Remarks on the zeros of cross-product Bessel functions*, J. Soc. Indust. Appl. Math. 12 (1964), 580–587.

[4] P. Duren, *Invitation to classical analysis*, Amer. Math. Soc., Providence, RI, 2012.

[5] P. Ebenfelt and D. Khavinson, *On point to point reflection of harmonic functions across real-analytic hypersurfaces in $\mathbb{R}^n$*, J. Anal. Math. 68 (1996) 145–182.

[6] S. J. Gardiner and H. Render, *Harmonic functions which vanish on a cylindrical surface*, J. Math. Anal. Appl. 433 (2016), 1870-1882.

[7] S. J. Gardiner and H. Render, *A reflection result for harmonic functions which vanish on a cylindrical surface*, J. Math. Anal. Appl. 443 (2016), 81–91.
[8] A. Gray, G.B. Mathews and T.M. MacRobert, *A Treatise on Bessel Functions and their Applications to Physics*. 2nd Ed. MacMillan, London, 1922.

[9] M. Grossi and F. Takahashi, *On the location of two blowup points on an annulus for the mean field equation*, C. R. Math. Acad. Sci. Paris 352 (2014), no. 7-8, 615–619.

[10] M. Grossi and D. Vujadinovic, *On the Green function of the annulus*, Anal. Theory Appl. 32 (2016), 52-64.

[11] D. M. Hickey, *The equilibrium point of Green’s function for an annular region*, Ann. of Math. (2) 30 (1928/29), 373–383.

[12] H. Kalf, *On the expansion of a function in terms of spherical harmonics in arbitrary dimensions*, Bull. Belg. Math. Soc. Simon Stevin 2 (1995) 361–380.

[13] D. Khavinson, *Holomorphic partial differential equations and classical potential theory*, Universidad de La Laguna, Departamento de Análisis Matemático, La Laguna, 1996.

[14] D. Khavinson and H. S. Shapiro, *Remarks on the reflection principle for harmonic functions*, J. Analyse Math. 54 (1990), 60–76.

[15] L. J. Landau, *Bessel functions: monotonicity and bounds*, J. London Math. Soc. (2) 61 (2000) 197–215.

[16] J. McMahon, *On the roots of the Bessel and certain related functions*, Ann. of Math. 9 (1895), 23–30.

[17] E. Sorolla, J. R. Mosig and M. Mattes, *Algorithm to calculate a large number of roots of the cross-product of Bessel functions*, IEEE Trans. on Antennas and Propagation, 61 (2013), 2180–2187.

[18] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean spaces*, Princeton University Press, Princeton, N.J., 1971.

[19] G. Szegő, *Orthogonal Polynomials*, 4th Edition, Amer. Math. Soc., Providence, RI, 1975.

[20] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Part I. 2nd Ed. Clarendon Press, Oxford, 1962.

[21] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, 1922.
Stephen J. Gardiner  
School of Mathematics and Statistics,  
University College Dublin,  
Belfield, Dublin 4, Ireland.  
email: stephen.gardiner@ucd.ie

Hermann Render  
School of Mathematics and Statistics,  
University College Dublin,  
Belfield, Dublin 4, Ireland.  
email: hermann.render@ucd.ie