A unified formulation for entanglement of distinguishable and indistinguishable particles

Swapnamay Mondal

Harish-Chandra Research Institute
Chhatnag Road, Jhunsi, Allahabad 211019, India
E-mail: swapno@hri.res.in

Abstract

Entanglement is a well understood concept only for distinguishable particles. However fundamental particles being inherently indistinguishable, it is desirable to have a single formulation for entanglement of distinguishable and indistinguishable particles. We take such a unified notion to be defined in connection with a measurement (a) whose outcome gives Rényi entropies when two subsystems are perfectly distinguishable and (b) which remains well defined even when two subsystems can not be distinguished perfectly. An explicit formula for outcome of such a measurement, which we call “exact Rényi entropy”, is conjectured in terms of “normalized” Rényi entropies of spatial regions in field theory. Same idea is used to define “exact Von Neumann entropy” as well. Our formula works both for bosons and fermions. To leading order it reproduces Rényi/ Von Neumann entropies of distinguishable particles. Corrections due to indistinguishability are finite in number. They depend on spatial overlap of the wave functions and are not invariant under local unitaries and are also sensitive to dynamics of the system. We argue that in a generic situation no finite dimensional density matrix can account for all these Rényi entropies. These results provide some insight into how quantum mechanics emerges from quantum field theory in the context of entanglement. We point out some qualitative features of our formula that are relatively easy to verify experimentally.
1 Introduction

What makes quantum mechanics mathematically nice is linearity. On the other hand what makes quantum mechanics physically somewhat counterintuitive is also linearity. The best manifestation of this is entanglement, non triviality of which has been noted as early as 1935 [1]. Entanglement is the situation when description of the whole system is not same as individual descriptions of its subsystems. Clearly, this requires a clear distinction between the subsystems considered, one can only talk about entanglement of different subsystems. In the simplest possible case, each subsystem consists of a single particle. But if this is a fundamental particle like electron, then we have 2 indistinguishable subsystems, which seems to be a problem. Hence it is of fundamental importance to have an understanding of entanglement of indistinguishable particles.

Many attempts have been made in recent years to generalize the notion of entanglement for indistinguishable particles [2], [3], [4], [5], [6], [7]. Look at [8], [9] for recent review and references there in. These approaches mostly explore two directions, namely tensor product...
structure of the Hilbert space and occupation number entanglement between various modes (which is indeed well suited for many systems considered in laboratory).

In this paper, first we try to make the question sharper. In real life we do not face the issue of indistinguishability because usually electrons or other fundamental particles can be effectively distinguished by their physical properties, i.e. we can use physical properties (e.g. position) of the particles to “label” them. Such a distinction is not possible when their wave functions start overlapping and now indistinguishability is expected to play a role. But now the notion of 2 different particles is itself a bit vague. In the extreme case when the wave functions completely coincide (this can happen for fermions as well because we are not talking about the full state) any notion of distinguishability is absolutely lost. So first we should answer the question - “entanglement of what and what ?” and then proceed to quantify that. A good way to do this would be to think of some measurement apparatus that (a) produces usual entanglement measures (say Von Neumann or Rényi entropies) as outcome of measurement, when subsystems are perfectly distinguishable and (b) when wave functions overlap, the measurement remains well defined. The measurement outcome of such an apparatus can consistently be defined as entanglement measure of indistinguishable particles. Now it is a well defined theoretical problem to predict the outcome of such a measurement. This amounts to having a unified formulation for entanglement of indistinguishable as well as distinguishable particles.

One may justly worry that such a measurement is clear only for Rényi entropies with \( n > 1 \), i.e. not for Von Neumann entropy, which one usually is most interested in. Nevertheless it is worth pursuing this even for Rényi entropies because if one has a formula for Rényi entropies for \( n > 1 \) with a parametric dependence on \( n \), from that a formula for Von Neumann entropy can easily be extracted by taking \( n \to 1 \) limit. Apart from that Rényi entropies are interesting in their own right\(^1\).

We are not aware of the existence of any such unified formulation and this is what we attempt to achieve in this paper. We address this question in the case when two subsystems are effectively distinguished by their positions. Distinguishing objects by their positions is not only most common to our perception, but also relevant for real life situations such as electrons/atoms/molecules at different lattice sites, quantum dots etc. The spirit is still the same as entanglement of modes but now that we are distinguishing by position, the modes take value in a continuum and subtleties of continuum field theory are expected to appear.

\(^1\)E.g. it has been noted \([10]\) in the context of non-Abelian fractional quantum hall effect that they contain much more information than Von Neumann entropy alone.
Here we summarize the main results of our paper. Given a quantum mechanical state, one first writes down a field theory state that represents the same situation and then computes the Rényi entropies of the region over which the particle (or the subsystem considered) is localized (more precisely the region probed by the apparatus), in this state. But such quantities are well known to be divergent. This difficulty is overcome simply by subtracting the vacuum contribution and we arrive at the formula 3.5. In this formula we conjecture a precise expression, which we call $S_{n}^{\text{exact}}$, for $n$-th Rényi entropy (for arbitrary positive integer $n$), as measured by previously mentioned apparatus. Von Neumann entropy being a special case, namely $n = 1$ all our results go though for Von Neumann entropy as well. Along with being simple and natural, $S_{n}^{\text{exact}}$ reduces to Rényi entropies of distinguishable particles smoothly, when considered subsystems are localized in disjoint regions of space. But when this is not the case corrections due indistinguishability creep in. These corrections are finitely many in number and primarily depend on overlap of wave functions. Interestingly, they turn out to be sensitive to the dynamics of the system as well. They exhibit interesting properties such as not being invariant under local unitaries. To account for all the $\{S_{n}^{\text{exact}}\}$ one needs an infinite dimensional density matrix. This density matrix can be thought as normalized density matrix for that region. But as the quantities $S_{n}^{\text{exact}}$ are same as quantum mechanical Rényi entropies to leading order, this normalized density matrix must contain the naive quantum mechanical density matrix of the corresponding subsystem as a block and all other eigenvalues should be very small. In particular this means any attempt to explain entanglement of indistinguishable particles within finite dimensional Hilbert space is bound to fail. Our results also provide some insight regarding how quantum mechanics emerges from quantum field theory, in the context of entanglement. One can take the field theory to be Schrödinger field theory as a toy model or a relativistic one for more accurate description. Our results are valid both for bosons and fermions.

Our paper is organized as follows. In 2 we argue Rényi entropies are in principle measurable quantities, so that the validity of our formula can in principle be tested experimentally. We also discuss how would such an apparatus experience a “particle”. In 3 we present our conjectured formula for Von Neumann and Rényi entropies for indistinguishable particles, as perceived by

\footnote{Note that this normalized density matrix is no good for analyzing vacuum entanglement.}

\footnote{The case for anyons could be interesting. The author is not aware of any anyonic field theory, hence straightforward application of the formula 3.5 may not work. But one can think of anyons as normal bosons or fermions coupled to a Chern Simons theory and appropriate flux attached to the particles. So one would expect that in 3.5 taking $|\Psi\rangle$ to be such a state may work, but we do not attempt this problem in this paper.}
the earlier mentioned apparatus. We show that to leading order this reproduces Von Neumann / Rényi entropies for a distinguishable quantum particle. Some novel features of the corrections are pointed out. We go on to mention some of the qualitative features of this formula, that are easy to test experimentally. We conclude the paper in §4.

2 Measurement of Rényi entropies

2.1 In theory

In order to argue that Rényi entropies are in principle measurable quantities, we need to find the corresponding Hermitian operator and then express Rényi entropy as expectation value of that operator. Now $n$-th Rényi entropy (we will take $n$ to be integer and $n \geq 1$) reads

$$S_n(\rho) = \frac{1}{1-n} \text{log } \text{Tr} \rho^n. \quad (2.1)$$

In the limit $n \to 1$, this boils down to Von Neumann entropy (also referred as entanglement entropy). We concentrate on the simpler quantity $R_n(\rho) := \text{Tr} \rho^n$. Clearly it is enough to argue that $R_n$ is a measurable quantity. Now one can express this as

$$R_n(\rho) = \text{Tr}_n \left[ \rho^\otimes_n \tilde{E}^{(n)} \right]. \quad (2.2)$$

where $\text{Tr}_n$ denotes trace over $\mathcal{H}^\otimes_n$, $n$-th tensor product of single particle state space $\mathcal{H}$ and

$$\tilde{E}^{(n)}_{j_1...j_n;i_1...i_n} = \frac{1}{2} \left[ \prod_{k=1}^n \delta_{j_k,i_{k+1}} + \prod_{k=1}^n \delta_{j_k,i_{k-1}} \right]. \quad (2.3)$$

This operator is easily checked to be Hermitian. We will call this operator “entangler”. This establishes Rényi entropies are indeed measurable quantities, but not on single particle Hilbert space, but on $n$-th tensor product of single particle Hilbert space.

In fact recently, the problem of measurement of Rényi entropies has been addressed in [11], [12].

Note that here $\mathcal{H}$ need not be single particle Hilbert space, it is simply the Hilbert space of the subsystem considered. In particular the subsystem can be some region of space as well. In that case the entangler corresponding to some spatial region $\Omega$ reads

$$E^{(n)}_{\Omega} = \frac{1}{2} \int \prod_{j=1}^n \prod_{a \in \Omega} d_{a}(j) \prod_{b \in \Omega} d_{b}(j') \langle \{q_b^{(n')}, \ldots q_a^{(n)} \} | \{q_b^{(1')}, \ldots q_a^{(n)} \} \rangle \langle \{q_b^{(1')}, \ldots q_a^{(n)} \} | \{q_b^{(1')}, \ldots q_a^{(n)} \} \rangle \left[ \delta \left( q_b^{(1')} - q_a^{(2)} \right) \ldots \delta \left( q_b^{(n')} - q_a^{(1)} \right) + \delta \left( q_b^{(1')} - q_a^{(n')} \right) \ldots \delta \left( q_b^{(n')} - q_a^{(n-1)} \right) \right]. \quad (2.4)$$
Here \( q_a^{(k)} \) denotes the local field variable at the point \( a \) on the \( k^{th} \) copy of space. This is just to formally show that Rényi entropies are observables in field theory as well.

### 2.2 In practice

The above analysis is for distinguishable particles. Let us take them to be distinguished by their positions, i.e. one particle has wave-function sharply peaked around \( x_1 \) and another around \( x_2 \), with \( x_1 \) and \( x_2 \) far enough to ensure that their wave functions do not overlap. Let their Hilbert spaces be \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) respectively. Now let us try to measure \( n \)-th Rényi entropy for the 1\(^{st} \) particle, i.e. the one at position \( x_1 \). So one would need \( n \) copies of the system. The apparatus would be sensitive to \( \mathcal{H}_1^{\otimes n} \). Hence one should make measurement on all the copies of 1\(^{st} \) particle simultaneously, i.e. around position \( x_1 \) over some region \( A \), where the wave function of the 1\(^{st} \) particle has support (and of course on all the \( n \) copies together, but now on we will take this to be understood). So whatever result the measurement gives would be interpreted as properties of the “1\(^{st} \) particle”. This is unambiguous as we have taken the 2\(^{nd} \) particle far enough to ensure it has no support over \( A \).

So far so trivial. Now let us flatten the wave functions a little and bring \( x_1 \) and \( x_2 \) a bit closer so that the wave functions start overlapping. This means while making measurements on \( A \), sometimes one would end up measuring the 2\(^{nd} \) particle! But then as the wave functions have started overlapping, the very notion of 1\(^{st} \) particle and 2\(^{nd} \) particle itself is ambiguous. What is not ambiguous is the readings of this apparatus. In real life, one would not know the precise wave functions of all the particles before performing this measurement, hence would not know whether the wave functions have overlap or not. So it is desirable to have a formula for the measurement of this apparatus, which works even when the wave functions have started overlapping. It is the outcome of this experiment, that we discuss in this paper.

### 3 Our conjecture

#### 3.1 The statement

In the above mentioned situation, it is best to think in the framework of field theory. It is the natural framework to handle indistinguishable particles and till now our understanding of nature shows fundamental particles are best described as excitations of some relativistic quantum field. In quantum field theory, we assign degrees of freedom to each point in space.
So from this point of view the above measurement is best thought as measuring some properties of the field variables in the region \( A \). So it is natural to expect that the entanglement properties of the region \( A \) can be related with those of the “1st particle”, as perceived by our field theory apparatus\(^4\).

So corresponding to the given quantum mechanical state \(|\psi\rangle\), which treats them as distinguishable particles, one first has to write down a field theory state \(|\Psi\rangle\). Then our conjecture reads

\[
S_n^{\text{exact}} = S_n (\rho_\Omega (\Psi)) - S_n (\rho_\Omega (0)) .
\]  

(3.5)

where \( \Omega \) is the region where the considered subsystem of particles has their wave functions mostly supported\(^5\), \( \rho_\Omega (\Psi) \) is the density matrix of region \( \Omega \) when the field is in the state \(|\Psi\rangle\) and \( \rho_\Omega (0) \) is the density matrix of region \( \Omega \) when the field is in vacuum state. The subtraction of vacuum contribution is just reminiscent of cancellation of bubble diagrams in usual scattering amplitude computations and should be thought as normalizing the Rényi entropies. By \( S_n^{\text{exact}} \) we mean the Rényi entropy as measured by the apparatus (actually it measures \( R_n \), but then one can compute \( S_n \) from that). When two particles can be distinguished with absolute certainty, i.e. there is no overlap between their wave functions, then \( S_n^{\text{exact}} \) is simply given by \( S_n (\rho (\psi)) \), which treats them as distinguishable particles. But otherwise \( S_n^{\text{exact}} \) is expected to deviate from \( S_n (\rho (\psi)) \) and we conjecture that it would be given by the right hand side of 3.5.

Strictly speaking the measurement is not very clear for \( n \to 1 \) case, i.e. for Von Neumann entropy. Nevertheless we use the same intuition as for \( n > 1 \) and in our opinion \( S_1^{\text{exact}} \) is what quantifies the Von Neumann entropy both for distinguishable and indistinguishable particles.

### 3.2 Gathering the tools

Now let us gather some technical tools required to show that \( S_n (\rho_\Omega (\Psi)) - S_n (\rho_\Omega (0)) \) boils down to \( S_n (\rho (\psi)) \) in suitable conditions and to compute the corrections. We would actually work with the simpler quantity

\[
\frac{R_n (\rho_\Omega (\Psi))}{R_n (\rho_\Omega (0))} .
\]

---

\( ^4\)Note when both particles have support in \( A \), then this apparatus actually senses both of them and the notion of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) breaks down. Nevertheless experimentally this measurement remains well defined and theoretically entanglement of regions remains well defined.

\( ^5\)More precisely, \( \Omega \) is the region over which the apparatus disturbs the field variables.
which is related to \( S_n (\rho_\Omega (\Psi)) - S_n (\rho_\Omega (0)) \) by a scaling and then exponentiation. Now we need to express \( R_n (\rho_\Omega (\Psi)) \) in a form, suitable to do manipulations.

First we note that we can express \( R_n \) also as

\[
R_n (\rho) = \text{Tr}_n [\rho^{\otimes n} E^{(n)}] .
\]  

with

\[
E^{(n)}_{j_1...j_n;i_1...i_n} = \prod_{k=1}^{n} \delta_{j_k,i_{k+1}} .
\]  

\( E^{(n)}_{j_1...j_n;i_1...i_n} \) is not Hermitian and hence not observable. We will just use this for technical purpose. Currently, this is defined on \( \mathcal{H}_1^{\otimes n} \). But one can easily extend its action on the full tensor product Hilbert space \( (\mathcal{H}_1 \otimes \mathcal{H}_2)^{\otimes n} \) by defining

\[
E^{(n)}_{j_1,b_1...j_n,b_n;i_1...i_n,a_n} = \prod_{k=1}^{n} \delta_{j_k,i_{k+1}} \delta_{a_k,b_k} .
\]

where \( i_k, j_k \) are the indices of \( k \)-th copy of \( \mathcal{H}_1 \) and \( a_k, b_k \) are the indices of \( k \)-th copy of \( \mathcal{H}_2 \). In fact if \( \rho_1 = \text{Tr}_2 \rho_{12} \), then

\[
R_n (\rho_1) = \text{Tr}_n [\rho_{12}^{\otimes n} E^{(n)}] .
\]

where now trace is over \( (\mathcal{H}_1 \otimes \mathcal{H}_2)^{\otimes n} \). From now on we will suppress the subscripts on trace and assume it is clear from the context.

This can be straightforwardly generalized to the case of quantum field theory. Now the analog of “1st particle” would be some spatial region \( \Omega \) and that of “2nd particle” would be the rest of space, which we denote as \( \Omega^c \). So we will have

\[
\text{Tr} \rho_\Omega^n = \text{Tr} \left( \rho^{\otimes n} E_\Omega^{(n)} \right) .
\]  

where \( \rho \) is the state of the whole system (which may be mixed as well) and \( \rho_\Omega := \text{Tr}_{\Omega^c} \rho \) is the reduced density matrix for the region \( \Omega \). The operator \( E_\Omega^{(n)} \) has been studied in great details in a recent paper [13] and we will heavily borrow their results. If we denote local field variable

\[\text{Note that field degrees of freedom at different points are perfectly distinguishable and hence there is no confusion due to distinguishability in the field theory set up. Indistinguishability of field excitations stem from our interpretation of harmonic oscillator states as particles.}\]
at point $a$ as $q_a$, then $E^{(n)}_\Omega$ reads
\[
E^{(n)}_\Omega = \int \prod_{j=1}^n \prod_{a \in \Omega} dq_a^{(j)} \prod_{b \in \Omega} dq_b^{(j')} \{q_b^{(1')} \cdots q_b^{(n')}}\} \langle \{q_a^{1} \cdots q_a^{n}\} | \delta \left(q_b^{(1')} - q_a^{(1)}\right) \cdots \delta \left(q_b^{(n')} - q_a^{(n)}\right) \rangle \times I_\Omega^n. \tag{3.9}
\]
This operator has many nice properties, as explored in [13]. The one particularly useful for us is
\[
\langle \psi_1 \cdots \psi_n | E^{(n)}_\Omega | \phi_1 \cdots \phi_n \rangle = \langle \phi_1 \cdots \phi_n | E^{(n)}_\Omega | \psi_n \cdots \psi_{n-1} \rangle^*. \tag{3.10}
\]
A derivation of this property is given in A.

It should also be noted, that we have reached field theory expression simply by assigning degrees of freedom to each point in space, but no further assumption regarding the nature of degrees of freedom has been made. So this may both be a bosonic field or a fermionic field. No information of dynamics is being used, so this goes through for interacting field theories as well.

Here we are interested in cases where quantum mechanics has internal degrees of freedom, like spin so that they can be entangled through that. So the field theory also needs to have internal degrees of freedom.

### 3.3 The results

Now we explicitly compute $R_n (\rho_\Omega (\Psi))$. Let us consider the following quantum mechanical state of two distinguishable particles
\[
|\psi\rangle = \frac{1}{\sqrt{|a|^2 + |b|^2}} \left\{ a |\phi_1, i\rangle |\phi_2, j\rangle + b |\phi_1, k\rangle |\phi_2, l\rangle \right\}. \tag{3.11}
\]
here $|\phi_1, i\rangle$ is a state centered around the point $x_1$, supported mostly in a region $A$ and with spin index $i$. Similarly $|\phi_2, j\rangle$ denotes a state centered around the point $x_2$, supported mostly in a region $B$, with $A \cap B = \emptyset$. If we label the particles by their spatial wave function and concentrate on spin part, then we have
\[
|\psi_{\text{spin}}\rangle = \frac{1}{\sqrt{|a|^2 + |b|^2}} \left\{ a |i\rangle |j\rangle + b |k\rangle |l\rangle \right\}. \tag{3.12}
\]

\footnote{In gauge theories it is more natural to think of the field variables being associated to links rather than points, if thought on a lattice. So there may be some subtlety in that case. Nevertheless we expect the qualitative results should go through.}
We assume \( i \neq k, j \neq l \), as otherwise the state is not entangled at all.

This situation can be mimicked by the following field theory state

\[
\ket{\Psi} = N \left(a O_{x_1,i} O_{x_2,j} + b O_{x_1,k} O_{x_2,l}\right) \ket{0}.
\]

here e.g. \( O_{x_1,i} \) denotes creation operator for a mode corresponding to the spatial wave function \( \phi_1(x) \) (which is concentrated around the point \( x_1 \)) and internal index \( i \). Note as \( O\text{-s}^8 \) contain only creation operators, they (anti)commute among themselves. We choose the following normalization for \( O\text{-s} \)

\[
\langle 0 | O^\dagger_{x_p,i_p} O_{x_p,j_p} | 0 \rangle = \delta_{i_p j_p}.
\]

Then the normalization constant \( N \) is equal to \( \frac{1}{\sqrt{|a|^2 + |b|^2}} \) to leading order but will receives corrections if the modes which are excited are not truly orthogonal. Now we explicitly calculate \( \langle \Psi^{\otimes n} | E_A^{(n)} | \Psi^{\otimes n} \rangle \). We take the field variables to be anti-commuting. The case for commuting field variables is even easier.

There would be \( 4^n \) terms in \( \langle \Psi^{\otimes n} | E_A^{(n)} | \Psi^{\otimes n} \rangle \), each of the following form

\[
\langle 0 | \left( O^{(n)}_{x_2,j_2} \right)^\dagger \left( O^{(n)}_{x_1,i_1} \right)^\dagger \cdots \left( O^{(1)}_{x_2,j_1} \right)^\dagger \left( O^{(1)}_{x_1,i_1} \right)^\dagger E_A^{(n)} O^{(1)}_{x_1,p_1} O^{(1)}_{x_2,q_1} \cdots O^{(n)}_{x_1,p_n} O^{(n)}_{x_2,q_n} | 0 \rangle
\]

\[
= (-1)^{n^2} \langle 0 | \left\{ \left( O^{(n)}_{x_1,i_1} \right)^\dagger \cdots \left( O^{(1)}_{x_1,i_1} \right)^\dagger \right\} \left\{ \left( O^{(n)}_{x_2,j_1} \right)^\dagger \cdots \left( O^{(1)}_{x_2,j_1} \right)^\dagger \right\} \right| 0 \rangle.
\]

Here we have ignored the normalization factor \( N^{2n} \). To shorten the expressions, we introduce some notations

\[
\left( O^{(n)}_{x_1,i_1} \right)^\dagger \cdots \left( O^{(1)}_{x_1,i_1} \right)^\dagger =: O^\dagger_{1,i}
\]

\[
\left( O^{(n)}_{x_2,j_1} \right)^\dagger \cdots \left( O^{(1)}_{x_2,j_1} \right)^\dagger =: O^\dagger_{2,j}
\]

\[
O^{(1)}_{x_1,p_1} \cdots O^{(n)}_{x_1,p_n} =: O_{1,p}
\]

\[
O^{(1)}_{x_2,q_1} \cdots O^{(n)}_{x_2,q_n} =: O_{2,q}.
\]

So we have

\[
(-1)^n \langle 0 | O^\dagger_{1,i} O^\dagger_{2,j} E_A^{(n)} O_{1,p} O_{2,q} | 0 \rangle.
\]

\(^8\)e.g. if we think of a real scalar theory with internal indices, then for a Gaussian state (both in position and momentum space) \( O_{x,1} \sim \int \frac{dp}{\sqrt{2E_p}} e^{-i p \cdot x} e^{-\frac{p^2}{2m^2}} a^\dagger_{1,p,} \).
Remember, in present notation all the $O, O^\dagger$-s are clusters of $n$ creation/annihilation operators.

Before we proceed further, let us note that for even $n$,

$$[A, B_1 \ldots B_n] = \sum_{p=0}^{n} (-1)^{p-1} B_1 \ldots B_{p-1} \{A, B_p\} B_{p+1} \ldots B_n. \quad (3.14)$$

and for odd $n$

$$\{A, B_1 \ldots B_n\} = \sum_{p=0}^{n} (-1)^{p-1} B_1 \ldots B_{p-1} \{A, B_p\} B_{p+1} \ldots B_n. \quad (3.15)$$

We treat the case of even and odd $n$ separately.

**even n** For even $n$, we write

$$(-1)^n \langle 0| O_{1,i}^{\dagger} O_{2,j}^{\dagger} E_A^{(n)} O_{1,p} O_{2,q} |0\rangle = \langle 0| O_{1,i}^{\dagger} E_A^{(n)} O_{1,p} O_{2,j}^{\dagger} O_{2,q} |0\rangle + \langle 0| O_{1,i}^{\dagger} [O_{2,j}^{\dagger}, E_A^{(n)} O_{1,p}] O_{2,q} |0\rangle. \quad (3.16)$$

1st piece in 3.16 can be further simplified as

$$\langle 0| O_{1,i}^{\dagger} E_A^{(n)} O_{1,p} O_{2,j}^{\dagger} O_{2,q} |0\rangle = \langle 0| E_A^{(n)} |0\rangle \langle 0|(P(O_{1,i}))^{\dagger} O_{1,p} |0\rangle \langle 0|O_{2,j}^{\dagger} O_{2,q} |0\rangle + \langle 0| [O_{1,i}^{\dagger}, E_A^{(n)}] P(O_{1,i}) |0\rangle^* \langle 0|O_{2,j}^{\dagger} O_{2,q} |0\rangle. \quad (3.17)$$

by using 3.10 and inserting complete basis repeatedly. Among these complete basis states, contributions other than vacuum vanish due to unequal number creation or annihilation operators. Here $P$ denotes the permutation that rotates the internal indices by one step, i.e.

$$P(O_{1,i}) = P \left( O_{x_1,i_1}^{(1)} \ldots O_{x_1,i_n}^{(n)} \right) = O_{x_1,i_n}^{(n)} \ldots O_{x_1,i_{n-1}}^{(n-1)}.$$

We decompose the 2nd terms in 3.16 as,

$$\langle 0| [O_{1,i}^{\dagger}, E_A^{(n)} O_{1,p}] O_{2,q} |0\rangle = \langle 0| [O_{1,i}^{\dagger}, E_A^{(n)}] O_{1,p} O_{2,q} |0\rangle + \langle 0| O_{1,i}^{\dagger} E_A^{(n)} [O_{2,j}^{\dagger}, O_{1,p}] O_{2,q} |0\rangle. \quad (3.18)$$

By 3.14 both these commutators decompose in terms of anti-commutators (as $n$ is even). Using 3.12, we have

$$1^{st} piece of 3.17 = \langle 0| E_A^{(n)} |0\rangle \prod_{k=1}^{n} \delta_{i_k p_{k+1}} \delta_{j_k q_k}. \quad (3.19)$$
This is the leading piece. When this piece is added for all the $4^n$ terms, one ends up with

$$N^{2n} \langle 0 | E_A^{(n)} | 0 \rangle (|a|^{2n} + |b|^{2n}) = \langle 0 | E_A^{(n)} | 0 \rangle \left( N \sqrt{|a^2| + |b^2|} \right)^{2n} R_n(\rho^{QM}_1).$$

As $N \sim (|a^2| + |b^2|)^{-1/2}$, this shows

$$S_n (\rho_A(\Psi)) - S_n (\rho_A(0)) = S_n \left( \rho^{QM}_1(\psi) \right). \quad (3.20)$$

to leading order and hence the expression in 3.5 has the correct quantum mechanics limit.

3.18 and 2\textsuperscript{nd} piece of 3.17 and deviation of $N$ from $(|a^2| + |b^2|)^{-1/2}$ constitute corrections to this. In particular, we note 2\textsuperscript{nd} piece of 3.18 would have a factor of $\prod_k j_k p_k$ coming from the commutator. This will give non-zero contribution if $i = l$ or $j = k$ or $i = j$ or $k = l$ in 3.11.

It is still to be argued that the corrections are indeed small, in the regime when quantum mechanics of distinguishable particles provide a good description. This is the regime when the wave functions are mostly supported in disjoint spatial regions. But then so are the operators whose (anti)commutators appear in the corrections. In a local quantum field theory, only operators at same space point (anti)commute. Hence, (anti)commutators pick contribution where both the operators (e.g. $O_{x_1}$ and $O_{x_2}$) have support. These are just the tails of the wave functions and hence small. A simpler way to see this is to use cluster decomposition principle [14]. Cluster decomposition itself gives the leading answer. Hence corrections must be small.

\textbf{odd n} Similar computation now gives

$$(-1)^n \langle 0 | O_{1,i}^{\dagger} O_{2,j}^{\dagger} E_A^{(n)} O_{1,p} O_{2,q} | 0 \rangle$$

$$= \langle 0 | E_A^{(n)} | 0 \rangle \langle 0 | (P(O_{1,i})^{\dagger} O_{1,p}) | 0 \rangle \langle 0 | O_{2,j}^{\dagger} O_{2,q} | 0 \rangle - \langle 0 | O_{1,i}^{\dagger} \left\{ E_A^{(n)} \right\} O_{1,p} P(O_{1,i}) | 0 \rangle \langle 0 | O_{2,j}^{\dagger} O_{2,q} | 0 \rangle$$

$$- \langle 0 | O_{1,i}^{\dagger} \left\{ O_{2,j}^{\dagger}, E_A^{(n)} \right\} O_{1,p} O_{2,q} | 0 \rangle + \langle 0 | O_{1,i}^{\dagger} E_A^{(n)} \left\{ O_{2,j}^{\dagger}, O_{1,p} \right\} O_{2,q} | 0 \rangle. \quad (3.21)$$

Using 3.14 each piece can be decomposed in terms of anti commutators. The leading piece is the same as for even $n$, i.e. we still have same quantum mechanical limit, but the correction terms may differ by some signs.

\textbf{Comments} Some comments are in order.

1. Our formula goes through both for bosons and fermions (interacting and non interacting) and smoothly reduces to the case for distinguishable particles.
2. Although in the above computation the subsystem is thought to comprise a single particle, the formula 3.5 holds for arbitrary subsystems. In a general case, $O$-s will contain products of superpositions of creation operators rather than just superposition of creation operators.

3. As pointed out earlier the corrections can sense if $i = l$ or $j = k$ or $i = j$ or $k = l$. This means states related by local unitaries may have different Rényi entropies! This is expected, because the usual invariance of entropies under local unitaries simply mean that one can rename spin up to be spin down for the particle in her access and that would not have any effect on entropy. But in field theory there is a global notion of spin up or spin down. One can not really locally change this.

4. What is the field theory to be used? For high enough energies one may need to use relativistic quantum field theories, but for low enough energies, it should suffice to use Schrödinger field theory.

5. Corrections are sensitive to the dynamics of the system. An easy way to see this is to note that in Schrödinger vacuum, field modes corresponding to energy eigenstates are in ground state. When translated to position space field variables, which we need to do in present situation, the information of wave functions of energy eigenstates would be required.

6. One can define a density matrix $\tilde{\rho}_A$ for region $A$, that would reproduce all the “normalized” Rényi entropies, i.e. the expression in 3.5. This would be an infinite dimensional density matrix. But the fact that Rényi entropies computed form this density matrix are very close to those from the quantum mechanical density matrix $\rho_1^{QM}$ implies that $\tilde{\rho}_A$ contains $\rho_1^{QM}$ as a block and then there are many other small eigenvalues, possibly infinite in number. However, according to our conjecture one would need $\tilde{\rho}_A$ to account for all the experimentally measured Rényi entropies. This means it is impossible for any finite dimensional density matrix to correctly reproduce all the Rényi entropies. Thus any analysis confined to finite dimensional Hilbert space is insufficient.

7. This also gives some insight into how quantum mechanics emerges from a continuum quantum field theory, in the context of entanglement. The modes corresponding to significant eigenvalues are the ones captured in quantum mechanical entanglement. However for a precise enough description of entanglement of indistinguishable particles, one needs
to take all other modes into account. In terms of modular Hamiltonian $\hat{H}$ (which is defined to by $\rho = e^{-\hat{H}}$) this has a nice interpretation. The field modes that are invisible to quantum mechanics correspond to very high energy states of the modular Hamiltonian.

8. Explicitly computing the corrections may be somewhat more complicated than a quantum mechanics computation of Rényi entropies. Nevertheless as we have a well defined object, it is a practical problem and in principle one can attempt to compute it for a given situation.

3.4 Experimental signatures

Now we point out some qualitative features, that may not be so hard to verify experimentally. Of course for a precise test of our conjecture these are not enough.

1. **Local unitaries change entropy**: As has been noted earlier, the corrections are not invariant under the action of local unitaries. So presumably one can take two such states related by local unitaries acting on internal degrees of freedom of one subsystem (spatial part of wave functions should have enough overlap to produce observable corrections) and measure various Rényi entropies for both of them. They are conjectured to differ.

2. **Syzygies of $R_n$**: The objects $\{R_n\}$ are just invariant polynomials of the density matrix. It is known that invariant polynomials constructed out of a set of matrices form a ring and various elements of the ring obey certain polynomial relations among them, called “syzygy”. These syzygies crucially depend on dimensionality of the matrix. For a single matrix the ring is generated by trace of various powers of the matrix, which are nothing but power sum symmetric polynomials $[15]$ of the eigenvalues of the matrix. But then these are exactly $\{R_n\}$. So $R_n$-s generate the ring of invariants and would satisfy certain syzygies. Now let the dimension of the Hilbert space of internal degrees of freedoms for the considered subsystem be $d$. Then only $d$ of the $R_n$ (or equivalently Rényi entropies) are independent and rest can be determined in terms of them. A simple way to work out these relations using symmetric polynomials has been explained in B. These relations are expected to be obeyed by $\{R_n\}$, i.e. certain polynomials constructed out of $R_n$-s should identically vanish. But according to our conjecture, the actual description is provided by an infinite dimensional matrix and hence these polynomials would not be exactly 0, but some small number. So if one has high enough precision measurement of $R_n$-s, one can check whether these polynomials are zero or not.
3. **Dependence on dynamics:** The corrections involve expectation values of certain operators in the vacuum state. Now for two systems with different dynamics vacuum itself would be different. Hence if one considers two systems which have different Hamiltonians and take entangled states which are same from a quantum mechanical point of view, high enough precision measurements are conjectured to show some difference between Rényi entropies of these two systems.

4 **Conclusion**

In this paper we have posed the problem of entanglement of indistinguishable particles in reference with outcome of a measurement, which measures trace of some positive integer power of density matrix (which amounts to measuring the Rényi entropy) when the subsystems are distinguishable. In general the outcome of this measurement is taken to be the Rényi entropy of the corresponding particles and deviations from usual expressions are interpreted as effects of indistinguishability. An explicit expression for outcome of this measurement is conjectured by relating it to properly normalized Rényi entropies of spatial regions in a suitably designed field theory state. For Von Neumann entropy, which is only a special case of Rényi entropy, the measurement is not clear. Nevertheless we move on with the intuition gathered for other Rényi entropies and conjecture similar expression, namely properly normalized Von Neumann entropy of spatial regions to be the exact Von Neumann entropy of an indistinguishable subsystem.

The corrections due to indistinguishability are shown to have novel features which are in stark contrast with features of usual quantum entanglement of distinguishable particles. Due to these in a generic case the exact expressions for Rényi entropies are not invariant under local unitaries and expected relations among Rényi entropies will also not be obeyed. Moreover these corrections seems to be sensitive to dynamics of the system as well. But all of these corrections arise only when the wave functions of two particles overlap. Hence when the overlap reduces to zero, i.e. when they are perfectly distinguishable, all these corrections vanish and one gets back usual entanglement measures for distinguishable particles. This was required for consistency of our formula. One can construct a density matrix that accounts for all these exact Rényi entropies. From field theory perspective this can be thought of as normalized density matrix of that region. In a generic case this matrix would be infinite dimensional. This implies one invariably needs an infinite dimensional density matrix to deal with entanglement of indistinguishable particles, any treatment of the subject with finite dimensional Hilbert
space is bound to fail. The fact that these exact Rényi entropies are same as those of a distinguishable particle to leading order means that this infinite dimensional density matrix contains the quantum mechanical density matrix (that one would get form the spin part by treating the particles as distinguishable) as a block and rest of it has very small eigenvalues. This shed some light on how quantum mechanics with finitely many degrees of freedom emerges from an a priori quantum field theoretic\textsuperscript{9} situation, namely the degrees of freedom that are entangled enough are captured by quantum mechanics. It is proposed that while dealing with indistinguishable particles which are effectively distinguished by their positions, one should use the formula 3.5, rather than naive Rényi / Von Neumann entropies as computed from quantum mechanics.

**Acknowledgements:** I am grateful to my friend Pinaki Banerjee for collaboration in earlier part of this work and many subsequent discussions. My special thanks to Ashoke Sen for various illuminating discussions and valuable comments on the manuscript. I would also like to thank Aditi Sen(De), Nilay Kundu, Avijit Misra and M.N. Bera for useful conversations and comments. Lastly I thank people of India for their generous support for research in theoretical physics.

\textsuperscript{9}Of course when only orthogonal modes are occupied that is same as quantum mechanics of distinguishable particles. Non trivial situation appears when the occupied modes are not exactly orthogonal.
A Derivation of \(3.10\)

\[
\langle \psi_1 \ldots \psi_n | E^{(n)}_{\Omega} | \phi_1 \ldots \phi_n \rangle = \prod_{j=1}^{n} \prod_{a_j, \alpha_j \notin \Omega} dq_{a_j}^{(j)} dq_{\alpha_j}^{(j)} \psi_1^{*} (q_{a_1}^{(1)}, q_{\alpha_1}^{(1)}) \ldots \psi_n^{*} (q_{a_n}^{(n)}, q_{\alpha_n}^{(n)}) \times \phi_1 (q_{a_1}^{(n)}, q_{\alpha_1}^{(n)}) \ldots \phi_n (q_{a_{n-1}}^{(n)}, q_{\alpha_{n-1}}^{(n)})
\]

\[
= \prod_{j=1}^{n} \prod_{a_j, \alpha_j \notin \Omega} dq_{a_j}^{(j)} dq_{\alpha_j}^{(j)} [\phi_1^{*} (q_{a_1}^{(1)}, q_{\alpha_1}^{(1)}) \ldots \phi_n^{*} (q_{a_n}^{(n)}, q_{\alpha_n}^{(n)})]
\]

\[
\times \psi_1 (q_{a_1}^{(1)}, q_{\alpha_1}^{(1)}) \ldots \psi_n (q_{a_n}^{(n)}, q_{\alpha_n}^{(n)})
\]

\[
= \prod_{j=1}^{n} \prod_{a_j, \alpha_j \notin \Omega} dq_{a_j}^{(j)} dq_{\alpha_j}^{(j)} [\phi_1^{*} (q_{a_1}^{(1)}, q_{\alpha_1}^{(1)}) \ldots \phi_n^{*} (q_{a_n}^{(n)}, q_{\alpha_n}^{(n)})]
\]

\[
\times \psi_1 (q_{a_1}^{(2)}, q_{\alpha_1}^{(1)}) \ldots \psi_n (q_{a_n}^{(n)}, q_{\alpha_n}^{(n)})
\]

\[
\times \psi_1 (q_{a_1}^{(1)}, q_{\alpha_1}^{(1)}) \ldots \psi_n (q_{a_n}^{(n)}, q_{\alpha_n}^{(n)})
\]

\[
\times \psi_1 (q_{a_1}^{(1)}, q_{\alpha_1}^{(1)}) \ldots \psi_n (q_{a_n}^{(n)}, q_{\alpha_n}^{(n)})
\]

\[
= \langle \phi_1 \ldots \phi_n | E^{(n)}_{\Omega} | \psi_n \ldots \psi_{n-1} \rangle^{*}
\]

All we have used is that local field variables constitute a basis for quantum states of the field. So this would go through both for bosons and fermions.

B Relations between \(R_{n}\)-s

The problem is best casted in terms of symmetric polynomials \([15]\). Two kind of symmetric polynomials would be important for us.

1. **power sum symmetric function**: \(n\)-th power sum symmetric function is defined as

   \[
   p_n = \sum x_i^n
   \] (B.22)

2. **monomial symmetric function**: Monomial symmetric function for a partition \(\lambda = (\lambda_1, \ldots, \lambda_k)\) of some positive integer \(m\) is defined as

   \[
   m_\lambda = \sum_{i_1, \ldots, i_k} x_{i_1}^{\lambda_1} \ldots x_{i_k}^{\lambda_k}
   \] (B.23)
In usual mathematics literature the sum in B.22 and B.23 are just formal, in the sense that one takes the number of \( x_i \)-s to be infinite and does not worry about convergence of the sum, it is just the symmetry properties, one is worried about. However we will take the number of variables to be finite and \( d \) in number, where \( d \) is the dimensionality of Hilbert space of the subsystem concerned. So for us

\[
p_n = \sum_{i=1}^{d} x_i^n \quad \text{(B.24)}
\]

which is nothing but \( R_n \). Now

\[
p_1(x_1, \ldots, x_d)^n = \sum_{\lambda_1 \leq \cdots \leq \lambda_d} \frac{n!}{\lambda_1! \cdots \lambda_d!} m_\lambda(x_1, \ldots, x_d) \quad \text{(B.25)}
\]

Here \( \lambda_i \)-s can be 0 as well. In left hand side \( p_1^n = 1 \), as \( p_1 \) is just trace of density matrix, which is 1. In right hand side one can express monomial symmetric functions in terms of power sum symmetric functions. Doing so leaves one with an equation involving power sum symmetric functions only. Up to \( n \leq d \) these equations would be trivial, but beyond that we would have non trivial relations.

But how does one expand \( m(\lambda) \) in terms of \( p(\rho) \)-s? For small \( d \), one can work out these by hand only, otherwise this can be easily done using computer programming, e.g. SAGE. Here is a simple example done in SAGE, that expresses \( m_{(2,1)} \) in terms of power sum polynomials. Note this operation is not sensitive to the number of variables.

```
Input
Sym = SymmetricFunctions(QQ);
p = Sym.p()
m = Sym.m()
a21 = m([2,1]);
p(a21)

Output
p[2, 1] - p[3]
```
References

[1] A. Einstein, B. Podolsky and N. Rosen ; “Can quantum mechanical description of physical reality be considered complete ?” ; Phys. Rev. 47, 777 Published 15 May 1935

[2] Paolo Zanardi1 and Xiaoguang Wang ; “Fermionic entanglement in itinerant systems” ; J. Phys. A: Math. Gen. 35 (2002) 79477959

[3] Paolo Zanardi ; “Quantum entanglement in fermionic lattices” ; PHYSICAL REVIEW A, VOLUME 65, 042101 (2002)

[4] H. M. Wiseman1 and John A. Vaccaro ; “Entanglement of Indistinguishable Particles Shared between Two Parties” ; Phys. Rev. Lett. 91, 097902 (2003)

[5] K. Eckert et al ; “Quantum Correlations in Systems of Indistinguishable Particles” ; 2002, Ann. Phys. (N.Y.) 299, 88

[6] Yu Shi ; “Quantum entanglement of identical particles” ; PHYSICAL REVIEW A 67, 024301 (2003)

[7] J. K. Korbicz and M. Lewenstein ; “Group-theoretical approach to entanglement” ; PHYSICAL REVIEW A 74, 022318 (2006)

[8] Giancarlo Ghirardi, Luca Marinatto, Tullio Weber ; “Entanglement and Properties of Composite Quantum Systems: a Conceptual and Mathematical Analysis” [arXiv:quant-ph/0109017v2]

[9] Ryszard Horodecki , Pawe Horodecki , Micha Horodecki, Karol Horodecki ; “Quantum entanglement” ; Rev. Mod. Phys., Vol. 81, No. 2, April-June 2009

[10] Hui Li and F. D. M. Haldane ; “Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States” ; PRL 101, 010504 (2008)

[11] John Cardy ; “Measuring Entanglement Using Quantum Quenches” ; PRL 106, 150404 (2011)

[12] Dmitry A. Abanin , Eugene Demler ; “Measuring Entanglement Entropy of a Generic Many-Body System with a Quantum Switch” ; PRL 109, 020504 (2012)
[13] Nobura Shiba ; “Entanglement Entropy of Disjoint Regions in Excited States : An Operator Method” [arXiv: 1408.0637]

[14] Steven Weinberg ; *The Quantum Theory of Fields: Volume I* (Cambridge University Press)

[15] Bruce E. Sagan ; *The Symmetric Group : representations, combinatorial algorithms and symmetric functions* (New York, Springer, 2001)