The evolution of tensor perturbations in scalar-tensor theories of gravity

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The evolution equations for tensor perturbations in a generic scalar tensor theory of gravity are presented. Exact solution are given for a specific class of theories and Friedmann-Lemaître-Robertson-Walker backgrounds. In these cases it is shown that, although the evolution of tensor models depends on the choice of parameters of the theory, no amplification is possible if the gravitational interaction is attractive.

I. INTRODUCTION

In spite of the enormous technological problems connected with their detection, gravitational waves (GW) could become a very important future source of data in cosmology. Cosmological GW are produced at very early stages of the evolution of the universe and decouple from the cosmic fluid at very early times. Consequently they carry information about the conditions of the early universe that are not accessible via the electromagnetic spectrum. One of the most important applications of a future detection of primordial GW is the possibility of constraining models of inflation \[1\].

Even if GW are decoupled from the cosmic fluid their presence still influences some features of the observable universe. In particular, a GW background will produce a signature that can be found in the anisotropies \[2\] and polarization \[3\] of the Cosmic Microwaves Background (CMB). This, together with the extraordinary improvement in the sensitivity of CMB measurements, opens the possibility of obtaining important information about GW in an indirect way, even though conventional GW detectors are not yet fully operational.

The study of cosmological GW via the CMB is also very important in investigations of alternative theories of gravity. In fact, it is well known \[4\] that the features of GW in GR are rather peculiar and the detection of any deviation from this behavior would be a genuine proof of the break down of standard GR.

For this reason it is important to develop a theory of gravitational waves (or equivalently of tensor perturbations) for alternative theories of gravity. These problems have been widely studied using standard metric-based approaches \[5\], but as yet a fully covariant and gauge-invariant approach, based on the 1+3 formalism \[6, 7\] has not yet been developed. The aim of this paper is to show that this method leads to a clear, mathematically well defined description of the evolution of tensor perturbations that, unlike metric based methods is easily generalized to non–standard gravity. In order to illustrate the power of this approach, we will apply it to one of the most studied extensions of the Einstein theory: Scalar Tensor Gravity. This study extends a previous work on the covariant gauge invariant scalar perturbations in this framework \[8\].

The paper is organized as follows. In section II we will briefly review the 1+3 covariant approach in a general setting. In section III we derive equations that govern the evolution of tensor perturbations for a general imperfect fluid. In section IV we adapt these equations the study of scalar tensor gravity and apply them to a specific example. Finally, conclusions are given in section V.

Unless otherwise specified, natural units \(\hbar = c = k_B = 8\pi G = 1\) will be used throughout this paper, Latin indices run from 0 to 3. The symbol \(\nabla\) represents the usual covariant derivative and \(\partial\) corresponds to partial differentiation. We use the \(-, +, +, +\) signature and the Riemann tensor is defined by

\[
R_{\;\;\;\;\;\;bced}^{a} = W_{\;\;\;\;\;\;bde,c}^{a} - W_{\;\;\;\;\;\;bcd,d}^{a} + W_{\;\;\;\;\;\;bd}^{e}W_{\;\;\;\;\;\;ce}^{a} - W_{\;\;\;\;\;\;bc}^{f}W_{\;\;\;\;\;\;df}^{a} ,
\]

where the \(W_{\;\;\;\;\;\;bd}^{e}\) is the usual Christoffel symbol (i.e. symmetric in the lower indices), defined by

\[
W_{\;\;\;\;\;\;bd}^{e} = \frac{1}{2}g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}) .
\]

The Ricci tensor is obtained by contracting the first and the third indices

\[
R_{ab} = g^{cd}R_{acbd} .
\]

Finally the Hilbert–Einstein action in presence of matter is defined by

\[
\mathcal{A} = \int dx^4\sqrt{-g} \left[ \frac{1}{2}R + L_m \right] .
\]
II. 1+3 COVARIANT APPROACH TO COSMOLOGY

The starting point (and the cornerstone) of our analysis is the 1+3 covariant approach to cosmology [7]. This approach consists of deriving a set of first order differential equations and constraints for some suitable, geometrically well defined quantities (the 1+3 equations) that are completely equivalent to the Einstein field equations. This has the advantage of simplifying the analysis of general spacetimes which can be foliated as a set of three dimensional (spacelike) surfaces. In the following we give a very brief introduction to the parts of this formalism used in this paper.

A. Kinematics

In order to derive the 1+3 equations we have to choose a set of observers i.e. a 4-velocity field \( u^a \). This choice depends strictly on the theory of gravity that we are treating. In this section we give the set of equations for a general velocity field and then specialize them to the case of scalar-tensor gravity in section IV.

Given the velocity \( u^a \), we can define the projection tensor into the tangent 3-spaces orthogonal to the flow vector:

\[ h_{ab} \equiv g_{ab} + u_a u_b \Rightarrow h_{ab} h^b c = h^a c, \ h_{ab} u^b = 0 \] (5)

and the kinematical quantities can be obtained by splitting the covariant derivative of \( u_a \) into its irreducible parts:

\[ \nabla_b u_a = \tilde{\nabla}_b u_a - A_b u_a, \ \tilde{\nabla}_b u_a = \frac{1}{3} \Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \] (6)

where \( \tilde{\nabla}_a \) is the spatially totally projected covariant derivative operator orthogonal to \( u^a \), \( A_a = \dot{u}_a \) is the acceleration \( (a_b u^b = 0) \), \( \Theta \) is the expansion parameter, \( \sigma_{ab} \) the shear \( (\sigma_{ab} = \sigma_{(ab)}, \ \sigma^c_{ab} = \sigma_{ab} u^c = 0) \) and \( \omega_{ab} \) is the vorticity \( (\omega_{ab} = \omega_{[ab]}, \ \omega_{ab} u^b = 0) \) [24].

In the \( u^a \) frame, the Weyl or conformal curvature tensor \( C_{abcd} \) can be split into its electric \( (E_{ab}) \) and Magnetic \( (H_{ab}) \) respectively:

\[ E_{ab} = C_{abcd} u^c u^d \Rightarrow E^a_{a} = 0, \ E_{ab} = E_{(ab)}, \ E_{ab} u^b = 0, \] (7)

\[ H_{ab} = \frac{1}{2} q_{ade} C^{cde}_{bc} u^c \Rightarrow H^a_{a} = 0, \ H_{ab} = H_{(ab)}, \ H_{ab} u^b = 0. \] (8)

In what follows we will use orthogonal projections of vectors and orthogonally projected symmetric trace-free part of tensors. They are defined as follows:

\[ v^{(a)} = h^a_{b} v^b, \quad T^{(ab)} = \left[ h^{(a}_{c} h^{b)}_{d} - \frac{1}{3} h^{ab} h_{cd} \right] T^{cd}. \] (9)

Angle brackets may also be used to denote orthogonal projections of covariant time derivatives along \( u^a \):

\[ \dot{v}^{(a)} = h^a_{b} \dot{v}^b, \quad \dot{T}^{(ab)} = \left[ h^{(a}_{c} h^{b)}_{d} - \frac{1}{3} h^{ab} h_{cd} \right] \dot{T}^{cd}. \] (10)

B. Thermodynamics

The velocity field \( u^a \) and the projection tensor \( h_{ab} \) allow one to split a generic stress-energy tensor \( T_{ab} \) as follows:

\[ T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \] (11)

where \( \mu \) is the energy density and \( p \) is the pressure of the fluid, \( q_a \) represents the energy flux, \( \pi_{ab} \) is the anisotropic pressure and

\[ q_a u^a = 0, \quad \pi^a_{a} = 0, \quad \pi_{ab} = \pi_{(ab)}, \quad \pi_{ab} u^b = 0. \]

In a general fluid the pressure, energy density and entropy are related to each other by an equation of state \( p = p(\mu, s) \). A fluid is considered perfect if \( q_a \) and \( \pi_{ab} \) vanish, and barotropic if the entropy is a constant i.e. the equation of state reduces to \( p = p(\mu) \).
C. Evolution Equations

Writing the Ricci and the Bianchi identities in terms of the 1+3 variables defined above, we obtain a set of evolution equations\(^{[23]}\):

\[
\text{leq: ray} \dot{\Theta} - \tilde{\nabla}_a \dot{\Theta}^a = - \frac{1}{2} \Theta^2 + (\ddot{\Theta}_a \dot{\Theta}^a) - 2 \sigma^2 + 2 \omega^2 - \frac{1}{3} (\mu + 3p) + \Lambda ,
\]

\[
\text{leq: omdot} \dot{\omega}^{(a)} - \frac{1}{2} \eta^{abc} \tilde{\nabla}_b \dot{\omega}^c = - \frac{2}{3} \Theta \omega^a + \sigma^a \omega^b ; ,
\]

\[
\text{leq: sigdot} \dot{\omega}^{(ab)} - \tilde{\nabla}^{(a} \dot{\omega}^{b)} = - \frac{1}{2} \Theta \sigma^{ab} + \dot{u}^{(a} \dot{\omega}^{b)} - \sigma^{(a} \sigma^{b)c} - \omega^{(a} \omega^{b)} - (\varepsilon^{ab} - \frac{1}{2} \pi^{ab}) ,
\]

\[
(\dot{E}^{(ab)} + \frac{1}{2} \pi^{(ab)}) - (\dot{H})^{ab} + \frac{1}{2} \tilde{\nabla}^{(a} q^{b)} = - \frac{1}{2} (\mu + p) \sigma^{ab} - \Theta (\varepsilon^{ab} + \frac{1}{6} \pi^{ab})
+ 3 \sigma^{(a} \varepsilon^{b)c} - \frac{1}{3} \pi^{c)} - \dot{u}^{(a} q^{b)}
+ \eta^{c} \sigma^{d} [ 2 \dot{\omega}^{a} H^{b} d + \omega^{c} (E^{b} d + \frac{1}{2} \pi^{b) d} ] ,
\]

\[
\dot{H}^{(ab)} + (E)^{ab} - \frac{1}{2} (\pi)^{ab} = - \Theta H^{ab} + 3 \sigma^{(a} \varepsilon^{b) c} + \frac{1}{2} \omega^{(a} q^{b)}
- \eta^{c} \sigma^{d} [ 2 \dot{\omega}^{a} E^{b} d - \frac{1}{2} \sigma^{b} q^{d} - \omega^{c} H^{b) d} ] ,
\]

\[
\text{leq: cons1} \dot{\mu} + \tilde{\nabla}_{a} q^{a} = - \Theta (\mu + p) - 2 (\dot{\omega}^{a} q^{a}) - (\sigma^{a} \pi^{b} ) ,
\]

\[
\text{leq: cons2} \dot{q}^{(a)} + \tilde{\nabla}^{a} p + \tilde{\nabla}_{b} \pi^{ab} = - \frac{1}{3} \Theta q^{a} - \sigma^{a} q^{b} - (\mu + p) \dot{u}^{a} - \dot{u}^{b} \pi^{ab} - \eta^{abc} \omega_{c q} ,
\]

and a set of constraints

\[
\text{leq: div} E \tilde{\nabla}_{b} E^{(ab)} + \frac{1}{2} \pi^{(ab)} - \frac{1}{2} \tilde{\nabla}^{a} \mu + \frac{1}{3} \Theta q^{a} - \frac{1}{3} \sigma^{a} q^{b} - 3 \omega_{a} H^{b} - \eta^{abc} [ \sigma_{d} H^{a b} - \frac{2}{3} \omega_{b c} ] = 0 ,
\]

\[
\text{leq: div} E \tilde{\nabla}_{b} H^{ab} + (\mu + p) \omega^{a} + 3 \omega_{a} (E^{ab} - \frac{1}{6} \pi^{ab}) + \eta^{abc} [ \frac{1}{2} \tilde{\nabla} q^{c} + \sigma_{b d} (E^{c} d + \frac{1}{2} \pi^{d}) ] = 0 ,
\]

\[
\text{leq: onu} \tilde{\nabla}_{a} \sigma^{a b} - \frac{2}{3} \tilde{\nabla}^{a} \Theta + \eta^{abc} [ \tilde{\nabla} q_{c} + 2 \dot{u} \omega_{c} ] + q^{a} = 0 ,
\]

\[
\tilde{\nabla}_{a} \omega^{a} - (\dot{u} a \omega^{a}) = 0 ,
\]

\[
lhconstr H^{ab} + 2 \dot{u} \psi^{(a} H^{b) c} + \tilde{\nabla}^{(a} \omega^{b)} - (\sigma)^{ab} = 0 ,
\]

that are completely equivalent to the Einstein equations. It is from these equations that we derive the general evolution equations for the tensor perturbations in scalar-tensor gravity.
III. PERTURBATIONS EQUATIONS

A. The Background

The equations presented in the previous section hold in any spacetime we wish to analyze. However, in what follows we will focus on the class of spacetimes that do not differ too much from a Friedmann-Lemaître-Robertson-Walker (FLRW) model. The reason is that current observations suggest that the universe appears to deviate only slightly from homogeneity and isotropy. We can define a FLRW spacetimes in terms of the variables above. Homogeneity and isotropy imply:

\[ \sigma = \omega = 0 , \quad \tilde{\nabla}_a f = 0 , \]

where \( f \) is any scalar quantity; in particular

\[ \tilde{\nabla}_a \mu = \tilde{\nabla}_a p = 0 \quad \Rightarrow \quad \dot{u}_a = 0 . \]

It follows that the governing equations for this background are

\[ \dot{\Theta} + \frac{1}{3} \Theta^2 + \frac{1}{2} (\mu + 3p) = 0 , \]

\[ \tilde{R} = 2 \left[ \frac{1}{2} \Theta^2 + \mu \right] , \]

\[ \mu_{\phi} + \Theta (\mu + p) = 0 . \]

Now in order to describe small deviations from a FLRW spacetime we simply take all the quantities that are zero in the background as being first order, and retain in the equations (24-32) only the terms that are linear in these quantities, i.e. we drop all second order terms. This procedure corresponds to the linearization in the 1+3 covariant approach and it greatly simplifies the system (24-32). In particular, the scalar vector and tensor parts of the perturbations are decoupled, so that we are able to treat them separately. In what follows we will focus only on the tensor perturbations.

B. Covariant decomposition of the 1+3 variables

Even if the geometrical nature of the kinematical 1+3 variables are known from their definition, the same cannot be said for the thermodynamic variables and the terms that contain derivatives. In the first case, this is because they depend strictly on the details of the energy momentum tensor of the fluid we are dealing with and in the second case, the index carried by the derivative might, in principle, change the geometrical nature of the quantity. For this reason, we have to define a general procedure that allows us to recognize the nature of these new quantities or split them in their irreducible parts. Giving a completely general procedure is a highly non-trivial task. Fortunately, however, for vectors and trace-free symmetric tensor, which describe the key quantities in the covariant approach, this task is relatively easy and can be summarized as follows.

In the case of a vector field the decomposition procedure is based on the Helmholtz’s theorem [10]. This theorem says that a continuous vector field \( V_a \) that vanishes (together with its first derivative) on the boundary of a manifold \( M \) can be decomposed, modulus a constant, into an irrotational field \( \tilde{V}_a \) and a solenoidal field \( \bar{V}_a \) [26]:

\[ V_a = \tilde{V}_a + \bar{V}_a = \eta^{abc} \tilde{\nabla}_b \bar{V}_c + \tilde{\nabla}^a \bar{V}_a ; , \]

where

\[ \tilde{\nabla}^a \tilde{V}_a = 0 , \quad \eta^{abc} \tilde{\nabla}_b \bar{V}_c = 0 . \]

With a little more work we can generalize this decomposition to any symmetric tensor that vanish on the boundary of \( M \) [11] obtaining:

\[ W_{ab} = \tilde{W}_{ab} + \bar{W}_{ab} + W_{ab}^* = \tilde{W}_{ab} + \tilde{\nabla}_a \bar{W}_b + \tilde{\nabla}_a \tilde{\nabla}_b W^* , \]

where

\[ \tilde{\nabla}^a \tilde{W}_{ab} = 0 , \quad (\tilde{W})_{ab} = 0 , \quad (W^*)_{ab} = 0 \]
and both of these decompositions are unique. Now we can define scalars, vectors or tensors as quantities that transform like scalars, solenoidal vectors or symmetric tensors, or are obtained from them using the $h_{ab}$ or $\nabla_a$ operators.

In what follows we will consider only purely tensor perturbations which are characterized by the fact that vector and scalar contributions to the 1+3 variables and equations vanish, i.e.

\[ f, \, \tilde{\nabla}^a f, \, \tilde{\nabla}^a \tilde{\nabla}^b f = 0, \quad \forall \, f \quad \text{(scalar)}, \]

and

\[ \tilde{V}_a, \, \tilde{\nabla}_a \tilde{V}_b = 0, \quad \forall \, \tilde{V}_a \quad \text{(solenoidal vector)}. \]

C. Tensor perturbation equations

Focusing only on tensor perturbations, i.e. applying (33) and (34), the evolution equations (??)-(??) reduce to

\[ \ddot{\sigma}_{ab} + \frac{2}{3} \Theta \sigma_{ab} + E_{ab} - \frac{1}{2} \pi_{ab} = 0, \]

\[ \dot{H}_{ab} + H_{ab} \Theta + (E)_{ab} - \frac{1}{2} (\pi)_{ab} = 0, \]

\[ \dot{E}_{ab} + E_{ab} \Theta - (H)_{ab} + \frac{1}{2} (p + \mu) \sigma_{ab} + \frac{1}{6} \Theta \pi_{ab} + \frac{1}{2} \tilde{\pi}_{ab} = 0, \]

together with the conditions

\[ \tilde{\nabla}_b H^{ab} = 0, \quad \tilde{\nabla}_b E^{ab} = 0, \quad H_{ab} = (\sigma)_{ab}. \]

Taking the time derivative of the above equations we obtain

\[ \dddot{\sigma}_{ab} - \tilde{\nabla}^2 \sigma + \frac{5}{3} \Theta \dot{\sigma}_{ab} + \left( \frac{1}{9} \Theta^2 + \frac{1}{6} (1 - 9 w) \mu \right) \sigma_{ab} = - \left( \dot{\pi}_{ab} + \frac{2}{3} \Theta \pi_{ab} \right), \]

\[ \ddot{H}_{ab} - \tilde{\nabla}^2 H_{ab} + \frac{7}{3} \Theta \dot{H}_{ab} + \frac{2}{3} (\Theta^2 - 3 w \mu) H_{ab} = (\pi)_{ab} + \frac{2}{3} \Theta (\pi)_{ab}, \]

\[ \ddot{E}_{ab} - \tilde{\nabla}^2 E_{ab} + \frac{7}{3} \Theta \dot{E}_{ab} + \frac{2}{3} (\Theta^2 - 3 w \mu) E_{ab} - \frac{1}{6} \Theta \mu (1 + w) (1 + 3 c_s^2) \sigma_{ab} \]

\[ = - \left[ \frac{1}{2} \ddot{\pi}_{ab} - \frac{1}{2} \tilde{\nabla}^2 \pi_{ab} + \frac{5}{6} \Theta \dot{\pi}_{ab} + \frac{1}{3} (\Theta^2 - \mu) \pi_{ab} \right], \]

where $c_s^2$ is the sound speed of the fluid and we have used the Raychaudhuri equation (??), the energy conservation equation (28) and the commutator identity

\[ (\dot{X})_{ab} = (X)_{ab} + \frac{1}{3} (X) \Theta. \]

These equations generalize the tensor perturbation equations for an imperfect fluid that were derived in [12] (note the different signature!).

Once the form of the anisotropic pressure has been determined the (39)-(41) can be solved to give the evolution of tensor perturbations. As already noted in [13] the presence of a term that contains the shear in the (41) makes this equation effectively third order, so that it is not possible to write down a closed wave equation for $E_{ab}$. If $\pi_{ab} = 0$, it is easy to show that for consistency, the solution for this field must also satisfy a wave equation because the shear is a solution of a wave equation and equation (35) holds. This will also be the case here because in our case $\pi_{ab} \propto \sigma_{ab}$ and so the anisotropic pressure will also satisfy a wave equation.
D. Harmonic analysis

Following standard harmonic analysis, equations (39) and (40) may be reduced to ordinary differential equations. It is standard [14] to use trace-free symmetric tensor eigenfunctions of the spatial the Laplace-Beltrami operator defined by:

\[ \tilde{\nabla}^2 Q_{ab} = -\frac{k^2}{a^2} Q_{ab}, \]  

where \( k = 2\pi a/\lambda \) is the wavenumber and \( \tilde{Q}_{ab} = 0 \). Developing \( \sigma_{ab} \) and \( H_{ab} \) in terms of the \( Q_{ab} \), (39) and (40) reduce to:

\[ \dot{\varphi}^{(k)} + \frac{5}{3} \Theta \dot{\varphi}^{(k)} + \left( \frac{1}{9} \Theta^2 + \frac{1}{6} (1 - 9w) \mu - \frac{k^2}{a^2} \right) \sigma_{ab}^{(k)} = \dot{\pi}_{ab}^{(k)} + \frac{2}{3} \Theta \pi^{(k)} , \] 

\[ \ddot{H}^{(k)} + \frac{7}{3} \Theta \dot{H}^{(k)} + \frac{2}{3} \left( \Theta^2 - 3w \mu - \frac{k^2}{a^2} \right) H^{(k)} = (\dot{\pi})^{(k)} + \frac{2}{3} \Theta (\pi)^{(k)} , \] 

and equation (47) reads:

\[ E^{(k)} = -\dot{\varphi}^{(k)} - \frac{2}{3} \Theta \sigma^{(k)} + \frac{1}{2} \pi^{(k)} . \] 

IV. TENSOR PERTURBATION IN SCALAR TENSOR GRAVITY

A. Scalar Tensor gravity

In Scalar-Tensor theories of gravity the inclusion of Mach’s principle implies the introduction of a scalar field \( \phi \) non-minimally coupled to the geometry [16]. In particular, this field is associated with the gravitational “constant” \( G \) and leads to it varying in time. Like many of the extensions of GR, scalar-tensor theories appear in many fundamental schemes and have been proposed as models for dark energy because their cosmology naturally leads to the phenomenon of cosmic acceleration, which is the characteristic footprint of dark energy [17, 18].

The most general action for Scalar-Tensor Theories of gravity is given by (conventions as in Wald [19])

\[ \mathcal{A} = \int dx^4 \sqrt{-g} \left[ \frac{1}{2} F(\phi) R - \frac{1}{2} \nabla_a \phi \nabla^a \phi - V(\phi) + \mathcal{L}_m \right] , \]  

where \( V(\phi) \) is a general (effective) potential expressing the self interaction of the scalar field and \( \mathcal{L}_m \) represents the matter contribution.

Varying the action with respect to the metric gives the gravitational field equations:

\[ F(\phi) G_{ab} = F(\phi) \left( R_{ab} - \frac{1}{2} g_{ab} R \right) = T^m_{ab} + \nabla_a \phi \nabla_b \phi - g_{ab} \left( \frac{1}{2} \nabla_c \phi \nabla^c \phi + V(\phi) \right) + \left( \nabla_b \nabla_a - g_{ab} \nabla_c \nabla^c \right) F(\phi) , \]  

while the variation with respect to the field \( \phi \) gives the curved spacetime version of the Klein-Gordon equation

\[ \nabla_a \nabla^a \phi + \frac{1}{2} F'(\phi) R - V'(\phi) = 0 \]  

where the prime indicates a derivative with respect to \( \phi \). Both these equations reduce to the standard equations for GR and a minimally coupled scalar field when \( F(\phi) = 1 \).

Equation (48) can be recast in the form:

\[ G_{ab} = T_{ab}^{TOT} = \frac{T^m_{ab}}{F(\phi)} + T_{ab}^{(eff)} , \]  

where \( T_{ab}^{(eff)} \) is defined as

\[ T_{ab}^{(eff)} = T_{ab}^{\phi} = \frac{1}{F(\phi)} \left[ \nabla_a \phi \nabla_b \phi - g_{ab} \left( \frac{1}{2} \nabla_c \phi \nabla^c \phi + V(\phi) \right) + \nabla_b \nabla_a F(\phi) - g_{ab} \nabla_c \nabla^c F(\phi) \right] , \] 

and an effective gravitational coupling \( G_{eff} = F(\phi)^{-1} \) appears. Provided that \( \phi, \alpha \neq 0 \), equation (49) also follows from the conservation equations

\[ \nabla^b T_{ab}^{\phi} = 0 . \]  

(51)
B. Perturbation equations in Scalar-Tensor gravity

At this point we are ready to derive the equations for the evolution of tensor perturbations in scalar-tensor gravity. The key point we will use here is that, as we have seen, the field equations of these models can be recast in the form (50), where together with matter, an effective fluid with energy momentum tensor \( T_{\phi}^{\text{eff}} \) appears. Following the prescription given in section II, it is then relatively easy to generalize (39-41) to this case.

A crucial point in this procedure is the choice of frame: since (50) describes a two fluids system, it is not immediately obvious which choice of \( u^a \) is the more convenient. It turns out that if we assume standard matter to be a perfect fluid, the best choice of frame is one comoving with the effective fluid. In this frame, standard matter is in general imperfect, i.e. it acquires a frame-induced heat flux and anisotropic pressure. However, it can be shown [20], that for a perfect fluid only the frame–induced heat flux is first order, i.e. standard matter does not produce any frame–induced first order tensor contributions. Consequently, in the effective fluid frame we avoid any frame–induced contributions to both the effective fluid thermodynamic quantities and standard matter. We therefore obtain the simplest possible form for the total anisotropic pressure appearing in the R.H.S. of equations (39-41).

In order to define the effective fluid frame, we assume that the momentum density \( \nabla_a \phi \) is timelike [8, 15, 21]:

\[
\nabla_a \phi \nabla^a \phi < 0 ,
\]

in an open region \( A \) of spacetime and we choose the 4-velocity to be

\[
u^a \equiv - \nabla^a \phi \quad , \quad u^a = -1 , \quad \dot{\psi} \equiv \dot{\phi} = u^a \nabla_a \phi = (- \nabla_a \phi \nabla^a \phi)^{1/2} .
\]

This implies

\[
\nabla_a \phi = 0 ,
\]

and

\[
\omega_{ab} = - h_a^c h_b^d \nabla_d \left( \frac{1}{\psi} \nabla_c \phi \right) = 0 ,
\]

i.e. our foliation selects vorticity free spacelike hypersurfaces in which \( \phi = \text{const} \).

Using the choice (53), the tensor \( T_{ab}^{\text{TOT}} \) can be decomposed as follows [8]:

\[
\mu_{\text{TOT}} = \mu_m + \frac{1}{F(\phi)} \left[ \frac{1}{2} \psi^2 + V(\phi) - \Theta \dot{F}(\phi) \right] ,
\]

\[
p_{\text{TOT}} = p_m + \frac{1}{F(\phi)} \left[ \frac{1}{2} \psi^2 - V(\phi) + \left( \dot{F}(\phi) + \frac{2}{3} \Theta \dot{F}(\phi) \right) \right] ,
\]

\[
q_{aT}^{\text{TOT}} = q_a^m + \dot{F}(\phi) \frac{\dot{F}(\phi)}{F(\phi)} \dot{u}_a ,
\]

\[
\pi_{ab}^{\text{TOT}} = \pi_{ab}^m - \frac{\dot{F}(\phi)}{F(\phi)} \sigma_{ab} .
\]

As anticipated, \( \pi_{ab}^m \) does not affect our tensor perturbations equations, so that the total anisotropic pressure in the frame \( u_a \) is proportional to the shear. This allows us to rewrite (59-61) as

\[
\dot{\sigma}^{(k)} + \left( \frac{5}{3} \Theta + \dot{F} \right) \ddot{\sigma}^{(k)} - \left( \frac{\dot{F}^2}{F^2} - \frac{2}{3} \Theta \dot{F} \frac{\dot{F}}{F} - \dot{F} - \frac{1}{9} \Theta^2 - \frac{1}{6} (1 - 9 w) \mu - \frac{k^2}{a^2} \right) \sigma^{(k)} = 0
\]

\[
\dot{H}^{(k)} + \left( \frac{7}{3} \Theta + \dot{F} \right) \dot{H}^{(k)} - \left( 2 w \mu - \frac{2}{3} \Theta^2 + \frac{\dot{F}^2}{F^2} - \Theta \frac{\dot{F}}{F} - \frac{\dot{F}}{F} - \frac{k^2}{a^2} \right) H^{(k)} = 0
\]
\[ \ddot{E}^{(k)} + \frac{7}{3} \Theta \dot{E}^{(k)} + \left( \frac{2}{3} \Theta^2 - 2 w \mu - \frac{3}{2} \frac{\dot{F}^2}{F^2} \dot{F} - \frac{k^2}{a^2} \right) E^{(k)} - \left( \frac{1}{6} \mu \Theta (1 + w) (1 + 3 \epsilon^2) - \frac{1}{12} (5 - 9 w) \mu \frac{\dot{F}}{F} + \frac{5}{18} \Theta^2 \frac{\dot{F}}{F} \right) \]
\[ - \frac{1}{6} \Theta \frac{\dot{F}^2}{F^2} + \frac{9}{4} \frac{\dot{F}^3}{F^3} - \frac{5}{2} \frac{\dot{F}}{F} \frac{\ddot{F}}{F} + \frac{1}{6} \Theta \frac{\dot{F}}{F} + \frac{1}{2} \frac{\dot{F}}{F} \frac{\ddot{F}}{F} \sigma^{(k)} = 0. \]

These equations describe the evolution of tensor perturbations for a general scalar tensor theory of gravity and reduce to the ones presented in [12, 13] in the GR limit \((F \to 1)\).

In the next subsection, we will apply these equations to a specific example of coupling, potential and background.

C. The case \(F(\phi) = -\xi \phi^2, V(\phi) = \lambda \phi^p\) in vacuum

As an example, let us consider the class of scalar tensor theories given by

\[ F(\phi) = -\xi(r) \phi^2, \quad V(\phi) = \lambda \phi^p(r), \]

where \(\xi\) and \(p\) depend on a single free parameter \(r\), via

\[ \xi(r) = \frac{(2r + 3)^2}{12(r + 1)(r + 2)}; \quad p(r) = \frac{3(1 + r)}{3 + 2r}. \]

Using the Noether symmetry approach \[23\] it can be shown that this class of theories admit the following exact solution in a vacuum:

\[ a(t) = \alpha t^n; \quad \phi(t) = \beta t^m; \quad K = 0, \]

with

\[ n(r) = \frac{2r^2 + 9r + 6}{r(r + 3)}; \quad m(r) = \frac{2r^2 + 9r + 9}{r(r + 3)}. \]

This represents a specific case of a general exact analytic solution for the background equations \[8, 22, 23\]. Depending on the value of \(r\), the \[63\] can represent power law inflation behaviour \((n(r) > 1)\) or a Friedmann-like phase \((0 < n(r) < 1)\) \[27\] and its character is essentially related to the choice of the form of the potential (via the choice of \(r\)). The parameter \(\alpha\) is free while \(\beta\) is linked to \(\lambda\) and \(r\) through

\[ \lambda = \frac{(6 + r)(3 + 2r)^2}{8r(3 + r)^2(2 + 3r + 2r)} \beta^{-\frac{2}{9 + 2r}}. \]

It turns out to be simpler to leave \(\beta\) free and derive \(\lambda\), but of course the real free parameter is this last one. Let us write down the time dependence of some important quantities. The expansion parameter is

\[ \Theta(t) = 3 \frac{n(r)}{t}. \]

The effective energy density is given by

\[ \mu(t) = \frac{3(6 + 9r + 2r^2)^2}{r^2(3 + r)^2} \frac{1}{t^2}, \]

which is a non negative function of \(r\). Finally the effective barotropic index is

\[ w = \frac{2}{3} - \frac{2 + r}{6 + r (9 + 2r)}, \]

which is constant. Substituting in the equations \[60\] and \[60\] we obtain

\[ \ddot{\sigma}^{(k)} + \frac{3 (4 + r) (1 + 2r)}{r (3 + r) t} \dot{\sigma}^{(k)} + \left[ \frac{3 (1 + r) (4 + r) (9 + 2r)}{r (3 + r)^2 t^2} + \frac{k}{t^{2n(r)}} \right] \sigma^{(k)} = 0, \]
The two equations above give the behaviour of the shear and the gravitomagnetic field. The gravitoelectric field can be obtained from the solution of the equations above and

\[ E^{(k)} = -\sigma^{(k)} - \frac{(3 + 9r + 2\bar{r})}{r(3 + r)t} \sigma^{(k)}. \]

In the long wavelength limit \((k = 0)\), the above equations admit the solution:

\begin{align*}
\sigma^{(k)} &= C_1 t^{3 + \frac{9}{r + 1} + \frac{1}{\alpha}} + C_2 t^{\frac{5}{r + 3} - \frac{1}{\alpha}}, \\
H^{(k)} &= C_1 t^{-5 - \frac{9}{r + 1} + \frac{1}{\alpha}} + C_2 t^{-4 - \frac{9}{r + 3} - \frac{1}{\alpha}}, \\
E^{(k)} &= C_1 t^{1 - \frac{9}{r + 1} + \frac{1}{\alpha}} - C_2 t^{1 - \frac{9}{r + 3} - \frac{1}{\alpha}},
\end{align*}

which are plotted for \(r = -1\) in Figure 1. Note that the exponents in the above solution depend on \(r\) so that there are values of \(r\) for which the above solutions present growing modes i.e. the gravitational waves are amplified on superhorizon scale. Since also the gravitational constant depends on the parameter \(r\), we are interested in values of this parameter for which we have growing modes and an attractive gravitational interaction. Unfortunately, it can be easily shown that such modes are incompatible with an attractive gravitational interaction i.e. the values of \(r\) for which the exponents in the solutions above are positive imply a negative effective gravitational constant. This means that in this class of theories and background only cosmological models driven by a repulsive gravitational constant induce an amplification of gravitational waves.

In the general case, the solutions for (65) and (66) involve Bessel functions of first and second kind \(J\) and \(Y\). We obtain:

\begin{align*}
\sigma^{(k)} &= t^{-\frac{9}{r + 1} - \frac{1}{\alpha}} \left[ C_1 J \left( \frac{1}{2} + \frac{3}{r(r + 6)\sqrt{\alpha}} \right) + C_2 Y \left( \frac{1}{2} + \frac{3}{r(r + 6)\sqrt{\alpha}} \right) \right], \\
H^{(k)} &= t^{-\frac{9}{r + 3} - \frac{1}{\alpha}} \left[ C_1 J \left( \frac{1}{2} + \frac{3}{r(r + 6)\sqrt{\alpha}} \right) + C_2 Y \left( \frac{1}{2} + \frac{3}{r(r + 6)\sqrt{\alpha}} \right) \right], \\
E^{(k)} &= t^{-\frac{9}{r + 1} - \frac{1}{\alpha}} \left[ C_1 J \left( \frac{1}{2} + \frac{3}{r(r + 6)\sqrt{\alpha}} \right) + C_2 Y \left( \frac{1}{2} + \frac{3}{r(r + 6)\sqrt{\alpha}} \right) \right] + \frac{m}{n} t^{-\frac{9}{r + 3} - \frac{1}{\alpha}} \left[ C_1 J \left( \frac{1}{2} + \frac{3}{r(r + 6)\sqrt{\alpha}} \right) + C_2 Y \left( \frac{1}{2} + \frac{3}{r(r + 6)\sqrt{\alpha}} \right) \right].
\end{align*}

The form of these solutions are similar to the one found in GR and this was expected from the structure of the equations. A plot of these solutions for \(r = -1\) is given in Figure 2. As in the long wavelength limit here growing modes are possible in principle but not compatible with attractive gravitational interaction so that also in short wavelengths only negative \(G_{eff}\) cosmological models induce an amplification of gravitational waves.

V. CONCLUSIONS

In this paper we have derived the evolution of the tensor perturbations in a general scalar-tensor theory of gravity. These equations have been obtained using the covariant gauge invariant approach to perturbation theory. The key
element of this approach is the fact that the Scalar-Tensor field equations can be recast in a form similar to the GR field equations, where the energy-momentum is described standard matter plus an effective fluid connected to the non-minimally coupled scalar field appears. In this way, equations for the evolution of the tensorial perturbations can be derived in a straightforward way.

To illustrate the method, we analyzed the the evolution of tensor perturbations of a class of theories and backgrounds obtained by the Noether symmetry approach [23]. Our analysis reveals that, although the details of the evolution of the gravitational waves depend on the specific values of the parameters, no growing modes are compatible with an
attractive gravitational interaction.

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[24] Following the standard convection we will indicate the symmetrization over two index of a tensor with round brackets and the antisymmetrization with square ones.

[25] Here the ‘curl’ is defined as $\nabla \times \mathbf{X} = \eta^{cd}(a \nabla_a X_b)$.

[26] Helmholtz’s theorem is given in a modern and more generalized form in the Kodaira-Hodge-De Rham decomposition theorem [9].

[27] This solution can represent, of course, also a contraction, but in the following we will consider only the expanding cases.