Existence of a positive hyperbolic Reeb orbit in three spheres with finite free group actions

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Abstract

Let $(Y, \lambda)$ be a non-degenerate contact three manifold. D. Cristofaro-Gardiner, M. Hutchings and D. Pomerleano showed that if $c_1(\xi = \text{Ker} \lambda)$ is torsion, then the Reeb vector field of $(Y, \lambda)$ has infinity many Reeb orbits otherwise $(Y, \lambda)$ is a lens space or three sphere with exactly two simple elliptic orbits. In the same paper, they also showed that if $b_1(Y) > 0$, $(Y, \lambda)$ has a simple positive hyperbolic orbit directly from the isomorphism between Seiberg-Witten Floer homology and Embedded contact homology. In addition to this, they asked whether $(Y, \lambda)$ with infinity many simple orbits also has a positive hyperbolic orbit under $b_1(Y) = 0$. In the present paper, we answer this question under $Y \cong S^3$ with nontrivial finite free group actions. In particular, it gives a positive answer in the case of a lens space $(L(p, q), \lambda)$ with odd $p$.

1 Introduction

Let $(Y, \lambda)$ be a contact three manifold and $X_\lambda$ be the Reeb vector field of it. That is, $X_\lambda$ is the unique vector field satisfying $i_{X_\lambda} \lambda = 1$ and $d\lambda(X_\lambda) = 0$. A smooth map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ is called a Reeb orbit with periodic $T$ if $\dot{\gamma} = X_\lambda(\gamma)$ and simple if $\gamma$ is an embedding map. In this paper, two Reeb orbits are considered equivalent if they differ by reparametrization.

The three-dimensional Weinstein conjecture which states that every contact closed three manifold $(Y, \lambda)$ has at least one simple periodic orbit was shown by C. H. Taubes by using Seiberg-Witten Floer (co)homology [T1]. After that, D. Cristofaro-Gardiner and M. Hutchings showed that every contact closed three manifold $(Y, \lambda)$ has at least two simple periodic orbit by using Embedded contact homology (ECH) in [CH]. ECH was introduced by M. Hutchings in several papers (for example, it is briefly explained in [H2]).

Let $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ be a Reeb orbit with periodic $T$. If its return map $d\phi^T |_{Ker \lambda=\xi} : \xi(0) \to \xi(0)$ has no eigenvalue 1, we call it a non-degenerate Reeb

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orbit and we call a contact manifold \((Y, \lambda)\) non-degenerate if all Reeb orbits are non-degenerate.

According to the eigenvalues of their return maps, non-degenerate periodic orbits are classified into three types. A periodic orbit is negative hyperbolic if \(d\phi^T|_\xi\) has eigenvalues \(h, h^{-1} < 0\), positive hyperbolic if \(d\phi^T|_\xi\) has eigenvalues \(h, h^{-1} > 0\) and elliptic if \(d\phi^T|_\xi\) has eigenvalues \(e^{\pm 2\pi i \theta}\) for some \(\theta \in \mathbb{R}\setminus \mathbb{Q}\).

For a non-degenerate contact three manifold \((Y, \lambda)\), the following theorems were proved using ECH in an essential way.

**Theorem 1.1** ([HT3]). Let \((Y, \lambda)\) be a closed contact non-degenerate three manifold. Assume that there exists exactly two simple Reeb orbit, then both of them are elliptic and \(Y\) is a lens space (possibly \(S^3\)).

**Theorem 1.2** ([CHP]). Let \((Y, \lambda)\) be a non-degenerate contact three manifold. Let \(\text{Ker} \lambda = \xi\). Then

1. if \(c_1(\xi)\) is torsion, there exists “infinitely many periodic orbits, or there exists exactly two elliptic simple periodic orbits and \(Y\) is diffeomorphic to a lens space (that is, it reduces to [HT3]).
2. if \(c_1(\xi)\) is not torsion, there exists at least four periodic orbit.

**Theorem 1.3** ([CHP]). If \(b_1(Y) > 0\), there exists at least one positive hyperbolic orbit.

In general, ECH splits into two parts \(\text{ECH}_{\text{even}}\) and \(\text{ECH}_{\text{odd}}\). In particular, \(\text{ECH}_{\text{odd}}\) is the part which detects the existence of a positive hyperbolic orbit and moreover if \(b_1(Y) > 0\), we can see directly from the isomorphism between Seiberg-Witten Floer homology and ECH (Theorem 2.7) that \(\text{ECH}_{\text{odd}}\) does not vanish. Theorem 1.3 was proved by using these facts.

In contrast to the case \(b_1(Y) > 0\), if \(b_1(Y) = 0\), \(\text{ECH}_{\text{odd}}\) may vanish, so this method doesn’t work. As a generalization of this phenomena, D. Cristofaro-Gardiner, M. Hutchings and D. Pomerleano asked the next question in the same paper.

**Question 1.4** ([CHP]). Let \(Y\) be a closed connected three-manifold which is not \(S^3\) or a lens space, and let \(\lambda\) be a nondegenerate contact form on \(Y\). Does \(\lambda\) have a positive hyperbolic simple Reeb orbit?

The reason why the cases \(S^3\) and lens spaces are excluded in Question 1.4 is that they admit contact forms with exactly two simple elliptic orbits as stated in Theorem 1.3 (for example, see [HT3]). So in general, we can change the assumption of Question 1.4 to the one that \((Y, \lambda)\) is not a lens space or \(S^3\) with exactly two elliptic orbits (this is generic condition, see [H]).

For this purpose, the author proved the next theorem in [S].

**Theorem 1.5** ([S]). Let \((Y, \lambda)\) be a nondegenerate contact three manifold with \(b_1(Y) = 0\). Suppose that \((Y, \lambda)\) has infinity many simple periodic orbits (that is, \((Y, \lambda)\) is not a lens space with exactly two simple Reeb orbits) and has at least one elliptic orbit. Then, there exists at least one simple positive hyperbolic orbit.
By Theorem 1.5 for answering Question 1.4, we can see that it is enough to consider the next problem.

**Question 1.6.** Let $Y$ be a closed connected three manifold with $b_1(Y) = 0$. Does $Y$ admit a non-degenerate contact form $\lambda$ such that all simple orbits are negative hyperbolic?

The next theorems is the main theorem of this paper.

**Theorem 1.7.** Let $L(p,q)$ ($p \neq \pm1$) be a lens space with odd $p$. Then $L(p,q)$ cannot admit a non-degenerate contact form $\lambda$ all of whose simple periodic orbits are negative hyperbolic.

Immediately, we have the next corollary.

**Corollary 1.8.** Let $(L(p,q), \lambda)$ ($p \neq \pm1$) be a lens space with a non-degenerate contact form $\lambda$. Suppose that $p$ is odd and there are infinity many simple Reeb orbits. Then, $(L(p,q), \lambda)$ has a simple positive hyperbolic orbit.

In addition to Theorem 1.7, the next Theorem 1.9 and Corollary 1.10 hold. The proof of Theorem 1.9 is shorter than the one of Theorem 1.7 and Corollary 1.10 follows from Theorem 1.7 and Theorem 1.9 immediately. We prove them at the end of this paper.

**Theorem 1.9.** Let $(S^3, \lambda)$ be a non-degenerate contact three sphere with a free $\mathbb{Z}/2\mathbb{Z}$ action. Suppose that $(S^3, \lambda)$ has infinitely many simple periodic orbits. Then $(S^3, \lambda)$ has a simple positive hyperbolic orbit.

**Corollary 1.10.** Let $(S^3, \lambda)$ be a non-degenerate contact three sphere with a nontrivial finite free group action. Suppose that $(S^3, \lambda)$ has infinitely many simple periodic orbits. Then $(S^3, \lambda)$ has a simple positive hyperbolic orbit.

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## 2 Preliminaries

For a non-degenerate contact three manifold $(Y, \lambda)$ and $\Gamma \in H_1(Y; \mathbb{Z})$, Embedded contact homology $\text{ECH}(Y, \lambda, \Gamma)$ is defined. At first, we define the chain complex $(\text{ECC}(Y, \lambda, \Gamma), \partial)$. In this paper, we consider $\text{ECH}$ over $\mathbb{Z}/2\mathbb{Z} = \mathbb{F}$.

**Definition 2.1 ([HI, Definition 1.1]).** An orbit set $\alpha = \{(\alpha_i, m_i)\}$ is a finite pair of distinct simple periodic orbit $\alpha_i$ with positive integer $m_i$. If $m_i = 1$ whenever $\alpha_i$ is hyperbolic orbit, then $\alpha = \{(\alpha_i, m_i)\}$ is called an admissible orbit set.
Set \( [\alpha] = \sum m_i[\alpha_i] \in H_1(Y) \). For two orbit set \( \alpha = \{(\alpha_i, m_i)\} \) and \( \beta = \{(\beta_j, n_j)\} \) with \( [\alpha] = [\beta] \), we define \( H_2(Y, \alpha, \beta) \) to be the set of relative homology classes of 2-chains \( Z \) in \( Y \) with \( \partial Z = \sum m_i\alpha_i - \sum_j n_j\beta_j \). This is an affine space over \( H_2(Y) \).

**Definition 2.2** ([H1] Definition 2.2). Let \( Z \in H_2(Y; \alpha, \beta) \). A representative of \( Z \) is an immersed oriented compact surface \( S \) in \([0, 1] \times Y \) such that:

1. \( \partial S \) consists of positively oriented (resp. negatively oriented) covers of \( \{1\} \times \alpha \) (resp. \( \{0\} \times \beta_j \)) whose total multiplicity is \( m_i \) (resp. \( n_j \)),
2. \( [\pi(S)] = Z \), where \( \pi : [0, 1] \times Y \to Y \) denotes the projection.
3. \( S \) is embedded in \((0, 1) \times Y \), and \( S \) is transverse to \( \{0, 1\} \times Y \).

From now on, we fix a trivialization \( \xi \) of some simple orbit \( \gamma \) and write it by \( \tau \).

For a non-degenerate Reeb orbit \( \gamma \), \( \mu_r(\gamma) \) denotes its Conley-Zehnder index with respect to a trivialization \( \tau \) in this paper. If \( \gamma \) is hyperbolic (that is, not elliptic), then \( \mu_r(\gamma^p) = p\mu_r(\gamma) \) for all positive integer \( p \) where \( \gamma^p \) denotes the \( p \) times covering orbit of \( \gamma \) (for example, see [H1 Proposition 2.1]).

**Definition 2.3** ([H1] §8.2). Let \( \alpha_1, \beta_1, \alpha_2 \) and \( \beta_2 \) be orbit sets with \([\alpha_1] = [\beta_1] \) and \([\alpha_2] = [\beta_2] \). For a fixed trivialization \( \tau \), we can define

\[
Q_\tau : H_2(Y; \alpha_1, \beta_1) \times H_2(Y; \alpha_2, \beta_2) \to \mathbb{Z}. \tag{1}
\]

by \( Q_\tau(Z_1, Z_2) = -I_r(S_1, S_2) + \#(S_1 \cap S_2) \) where \( S_1, S_2 \) are representatives of \( Z_1, Z_2 \) for \( Z_1 \in H_2(Y; \alpha_1, \beta_1) \), \( Z_2 \in H_2(Y; \alpha_2, \beta_2) \) respectively, \( \#(S_1 \cap S_2) \) is their algebraic intersection number and \( I_r \) is a kind of crossing number (see [H1] §8.3 for details).

**Definition 2.4** ([H1] Definition 1.5). For \( Z \in H_2(Y, \alpha, \beta) \), we define ECH index by

\[
I(\alpha, \beta, Z) := c_1(\xi)_{|Z, \tau} + Q_\tau(Z) + \sum_i m_i \sum_{k=1}^{m_i} \mu_r(\alpha_i^k) - \sum_j n_j \sum_{k=1}^{n_j} \mu_r(\beta_j^k). \tag{2}
\]

Here, \( c_1(\xi)_{|Z, \tau} \) is a relative Chern number and, \( Q_\tau(Z) = Q_\tau(Z, Z) \). Moreover this is independent of \( \tau \) (see [H1] for more details).

**Proposition 2.5** ([H1] Proposition 1.6). The ECH index \( I \) has the following properties.

1. For orbit sets \( \alpha, \beta, \gamma \) with \([\alpha] = [\beta] = [\gamma] = \Gamma \in H_1(Y) \) and \( Z \in H_2(Y, \alpha, \beta) \), \( Z' \in H_2(Y, \beta, \gamma) \),

\[
I(\alpha, \beta, Z) + I(\beta, \gamma, Z') = I(\alpha, \gamma, Z + Z'). \tag{3}
\]

2. For \( Z, Z' \in H_2(Y, \alpha, \beta) \),

\[
I(\alpha, \beta, Z) - I(\alpha, \beta, Z') = c_1(\xi) + 2\text{PD}(\Gamma), Z - Z' >. \tag{4}
\]
3. If $\alpha$ and $\beta$ are admissible orbit sets,

$$I(\alpha, \beta, Z) = \epsilon(\alpha) - \epsilon(\beta) \mod 2.$$  \hspace{1cm} (5)

Here, $\epsilon(\alpha), \epsilon(\beta)$ are the numbers of positive hyperbolic orbits in $\alpha, \beta$ respectively.

For $\Gamma \in H_1(Y)$, we define $\text{ECC}(Y, \lambda, \Gamma)$ as freely generated module over $\mathbb{Z}/2$ by admissible orbit sets $\alpha$ such that $[\alpha] = \Gamma$. That is,

$$\text{ECC}(Y, \lambda, \Gamma) := \bigoplus_{\alpha \text{admissible orbit set with } [\alpha] = \Gamma} \mathbb{Z}_2(\alpha).$$  \hspace{1cm} (6)

To define the differential $\partial : \text{ECC}(Y, \lambda, \Gamma) \to \text{ECC}(Y, \lambda, \Gamma)$, we pick a generic $\mathbb{R}$-invariant almost complex structure $J$ on $\mathbb{R} \times Y$ which satisfies $J(\frac{\partial}{\partial s}) = X_\lambda$ and $J\xi = \xi$.

We consider $J$-holomorphic curves $u : (\Sigma, j) \to (\mathbb{R} \times Y, J)$ where the domain $(\Sigma, j)$ is a punctured compact Riemann surface. Here the domain $\Sigma$ is not necessarily connected. Let $\gamma$ be a (not necessarily simple) Reeb orbit. If a puncture of $u$ is asymptotic to $\mathbb{R} \times \gamma$ as $s \to \infty$, we call it a positive end of $u$ at $\gamma$ and if a puncture of $u$ is asymptotic to $\mathbb{R} \times \gamma$ as $s \to -\infty$, we call it a negative end of $u$ at $\gamma$ (For more details $[\text{H1}]$).

Let $u : (\Sigma, j) \to (\mathbb{R} \times Y, J)$ and $u' : (\Sigma', j') \to (\mathbb{R} \times Y, J)$ be two $J$-holomorphic curves. If there is a biholomorphic map $\phi : (\Sigma, j) \to (\Sigma', j')$ with $u' \circ \phi = u$, we regard $u$ and $u'$ as equivalent.

Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_i, n_i)\}$ be orbit sets. Let $\mathcal{M}^J(\alpha, \beta)$ denote the set of $J$-holomorphic curves with positive ends at covers of $\alpha_i$ with total covering multiplicity $m_i$, negative ends at covers of $\beta_j$ with total covering multiplicity $n_j$, and no other punctures. Moreover, in $\mathcal{M}^J(\alpha, \beta)$, we consider two $J$-holomorphic curves to be equivalent if they represent the same current in $\mathbb{R} \times Y$. For $u \in \mathcal{M}^J(\alpha, \beta)$, we naturally have $[u] \in H_2(Y; \alpha, \beta)$ and we set $I(u) = I(\alpha, \beta, [u])$. Moreover we define

$$\mathcal{M}_1^J(\alpha, \beta) := \{u \in \mathcal{M}^J(\alpha, \beta) \mid I(u) = k\}$$  \hspace{1cm} (7)

In this notations, we can define $\partial_J : \text{ECC}(Y, \lambda, \Gamma) \to \text{ECC}(Y, \lambda, \Gamma)$ as follows.

For admissible orbit set $\alpha$ with $[\alpha] = \Gamma$, we define

$$\partial_J(\alpha) = \sum_{\beta \text{admissible orbit set with } [\beta] = \Gamma} \#(\mathcal{M}_1^J(\alpha, \beta)/\mathbb{R}) \cdot \langle \beta \rangle.$$  \hspace{1cm} (8)

Note that the above counting $\#$ is well-defined since $\mathcal{M}_1^J(\alpha, \beta)/\mathbb{R}$ is a set of finite points (this follows from Proposition $2.6$ and the compactness of the moduli space $([\text{H1}])$). Moreover $\partial_J \circ \partial_J = 0$ (see $[\text{HT1}, \text{HT2}]$) and the homology defined by $\partial_J$ does not depend on $J$ (see Theorem $2.7$ or see $[\text{H1}]$).

For $u \in \mathcal{M}^J(\alpha, \beta)$, the its (Fredholm) index is defined by

$$\text{ind}(u) := -\chi(u) + 2c_1(\xi|_{\alpha}], \tau) + \sum_k \mu_\tau(\gamma_k^+) - \sum_l \mu_\tau(\gamma_l^-).$$  \hspace{1cm} (9)
Here \( \{ \gamma_k^+ \} \) is the set consisting of (not necessarily simple) all positive ends of \( u \) and \( \{ \gamma_k^- \} \) is that one of all negative ends. Note that for generic \( J \), if \( u \) is connected and somewhere injective, then the moduli space of \( J \)-holomorphic curves near \( u \) is a manifold of dimension \( \text{ind}(u) \) (see \cite{HT1} Definition 1.3).

Let \( \alpha = \{ (\alpha_i, m_i) \} \) and \( \beta = \{ (\beta_i, n_i) \} \). For \( u \in \mathcal{M}^J(\alpha, \beta) \), it can be uniquely written as \( u = u_0 \cup u_1 \) where \( u_0 \) are unions of all components which maps to \( \mathbb{R} \)-invariant cylinders in \( u \) and \( u_1 \) is the rest of \( u \).

**Proposition 2.6** (\cite{HT1} Proposition 7.15). Suppose that \( J \) is generic and \( u = u_0 \cup u_1 \in \mathcal{M}^J(\alpha, \beta) \). Then

1. \( I(u) \geq 0 \)
2. If \( I(u) = 0 \), then \( u_1 = \emptyset \)
3. If \( I(u) = 1 \), then \( u_1 \) is embedded and \( u_0 \cap u_1 = \emptyset \). Moreover \( \text{ind}(u_1) = 1 \).
4. If \( I(u) = 2 \) and \( \alpha \) and \( \beta \) are admissible, then \( u_1 \) is embedded and \( u_0 \cap u_1 = \emptyset \). Moreover \( \text{ind}(u_1) = 2 \).

If \( c_1(\xi) + 2\text{PD}(\Gamma) \) is torsion, there exists the relative \( \mathbb{Z} \)-grading:

\[
\text{ECH}(Y, \lambda, \Gamma) := \bigoplus_{* : \mathbb{Z}-\text{grading}} \text{ECH}_*(Y, \lambda, \Gamma).
\]

(10)

Let \( Y \) be connected. Then there is degree \(-2\) map \( U \).

\[
U : \text{ECH}_*(Y, \lambda, \Gamma) \to \text{ECH}_{*-2}(Y, \lambda, \Gamma).
\]

(11)

To define this, choose a base point \( z \in Y \) which is not on the image of any Reeb orbit and let \( J \) be generic. Then define a map

\[
U_{J,z} : \text{ECC}_*(Y, \lambda, \Gamma) \to \text{ECC}_{*-2}(Y, \lambda, \Gamma)
\]

(12)

by

\[
U_{J,z}(\alpha) = \sum_{\beta : \text{admissible orbit set with } [\beta] = \Gamma} \# \{ u \in \mathcal{M}^J_2(\alpha, \beta) / \mathbb{R} \mid (0, z) \in u \} \cdot \langle \beta \rangle.
\]

(13)

The above map \( U_{J,z} \) is a chain map, and we define the \( U \) map as the induced map on homology. Under the assumption, this map is independent of \( z \) (for a generic \( J \)). See \cite{HT3} §2.5 for more details. Moreover, in the same reason as \( \partial \), \( U_{J,z} \) does not depend on \( J \) (see Theorem 2.7 and see \cite{IT1}). In this paper, we choose a suitable generic \( J \) as necessary.

The next isomorphism is important.

**Theorem 2.7** (\cite{IT1}). For each \( \Gamma \in H_1(Y) \), there is an isomorphism

\[
\text{ECH}_*(Y, \lambda, \Gamma) \cong \text{HM}_*(\mathbb{R}Y, \mathfrak{s}(\xi) + \text{PD}(\Gamma))
\]

(14)

of relatively \( \mathbb{Z}/d\mathbb{Z} \)-graded abelian groups. Here \( d \) is the divisibility of \( c_1(\xi) + 2\text{PD}(\Gamma) \) in \( H_1(Y) \) mod torsion and \( \mathfrak{s}(\xi) \) is the spin-\( c \) structure associated to
the oriented 2–plane field as in [KM]. Moreover, the above isomorphism inter-
changes the map \( U \) in (14) with the map

\[
U_1 : \hat{HM}_s(-Y, s(\xi) + \text{PD}(\Gamma)) \longrightarrow \hat{HM}_{s-2}(-Y, s(\xi) + \text{PD}(\Gamma))
\]
defined in [KM].

Here \( \hat{HM}_s(-Y, s(\xi) + 2\text{PD}(\Gamma)) \) is a version of Seiberg-Witten Floer homology
with \( \mathbb{Z}/2\mathbb{Z} \) coefficients defined by Kronheimer-Mrowka [KM].

The action of an orbit set \( \alpha = \{(\alpha_i, m_i)\} \) is defined by

\[
A(\alpha) = \sum m_i A(\alpha_i) = \sum m_i \int_{\alpha_i} \lambda.
\]

Note that if two admissible orbit sets \( \alpha = \{(\alpha_i, m_i)\} \) and \( \beta = \{(\beta_i, n_i)\} \) have
\( A(\alpha) \leq A(\beta) \), then the coefficient of \( \beta \) in \( d\alpha \) is 0 because of the positivity of \( J \)
holomorphic curves over \( d\lambda \) and the fact that \( A(\alpha) - A(\beta) \) is equivalent to the
integral value of \( d\lambda \) over \( J \)-holomorphic punctured curves which is asymptotic
to \( \alpha \) at \( +\infty \), \( \beta \) at \( -\infty \).

Suppose that \( b_1(Y) = 0 \). In this situation, for any orbit sets \( \alpha \) and \( \beta \) with
\( [\alpha] = [\beta] \), \( H_2(Y, \alpha, \beta) \) consists of only one component since \( H_2(Y) = 0 \). So we
may omit the homology component from the notation of ECH index \( I \), that is,
\( I(\alpha, \beta) \) just denotes the ECH index. Furthermore, for an orbit set \( \alpha \) with \( [\alpha] = 0 \),
we set \( I(\alpha) := I(\alpha, \emptyset) \).

**Proposition 2.8.** Let \( (Y, \lambda) \) be a non-degenerate connected contact three manifold
with \( b_1(Y) = 0 \). Let \( \rho : Y \rightarrow Y \) be a \( p \)-fold cover with \( b_1(Y) = 0 \). Let \( (\tilde{Y}, \tilde{\lambda}) \)
be a non-degenerate contact three manifold induced by the covering map. Suppose that \( \alpha \) and \( \beta \) be admissible orbit sets in \( (Y, \lambda) \) consisting of only hyperbolic
orbits. Then

\[
I(\rho^* \alpha, \rho^* \beta) = p I(\alpha, \beta)
\]

Here \( \rho^* \alpha \) and \( \rho^* \beta \) are inverse images of \( \alpha \), \( \beta \) and so admissible orbit sets in
\( (\tilde{Y}, \tilde{\lambda}) \).

**Proof of Proposition 2.8.** Let \( \tau \) be a fixed trivialization of \( \xi = \text{Ker}\lambda \) defined
over every simple orbit in \( (Y, \lambda) \) and \( \tilde{\tau} \) be its induced trivialization of \( \xi = \text{Ker}\tilde{\lambda} \)
defined over every simple orbit in \( (\tilde{Y}, \tilde{\lambda}) \). More precisely, it is defined as follows.
Let \( \tilde{\gamma} : \mathbb{R}/T\mathbb{Z} \rightarrow \tilde{Y} \) be a simple orbits. Since \( \rho : Y \rightarrow Y \) is a \( p \)-fold cover, there
are a positive integer \( r \) and simple orbit \( \gamma : \mathbb{R}/r\mathbb{Z} \rightarrow Y \) with \( \gamma \circ \pi = \rho \circ \tilde{\gamma} \)
where \( \pi : \mathbb{R}/T\mathbb{Z} \rightarrow \mathbb{R}/r\mathbb{Z} \) is the natural projection and \( p \) is divisible by \( r \).
Let \( \pi_* : \gamma^* \xi \rightarrow \gamma^* \xi \) be the induced bundle map. Then for a trivialization
\( \tau : \gamma^* \xi \rightarrow \mathbb{R}/r\mathbb{Z} \times \mathbb{C} \), we define the induced trivialization \( \tilde{\tau} : \tilde{\gamma}^* \xi \rightarrow \mathbb{R}/T\mathbb{Z} \times \mathbb{C} \)
so that \((\pi \times \text{id}) \circ \tilde{\tau} = \tau \circ \pi_* \).

Let \( \{Z\} = H_2(Y; \alpha, \beta) \) and \( \{	ilde{Z}\} = H_2(\tilde{Y}; \rho^* \alpha, \rho^* \beta) \). It is sufficient to prove
that each term in \( I(\rho^* \alpha, \rho^* \beta) \) defined by \( \tilde{\tau} \) and \( \tilde{Z} \) is \( p \) times of corresponding
term in \( I(\alpha, \beta) \) defined by \( \tau \) and \( Z \).

At first, we consider the term of Conley-Zehnder indices of orbits. As just
before Definition 2.3, for every hyperbolic orbit \( \gamma \) in \( (Y, \lambda) \), we have \( \mu_\tau(\gamma^\rho) = \)
$p\mu_\tau(\gamma)$ and so for a simple hyperbolic orbit $\gamma$ in $(Y, \lambda)$, we have $p\mu_\tau(\gamma) = \mu_\tau(\rho^*\gamma)$ (in the right hand side, if several orbits appear in $\rho^*\gamma$, we add their Conley-Zehnder indices all together).

Next, consider the terms $c_1(\xi|_\tau, \tau)$. Recall that this is defined as follows. Let $S$ be a surface with boundary and $f : S \to Y$ be a map representing $Z$. Take a generic section $\psi : S \to f^*\xi$ such that $\psi|_{\partial S}$ is a nonvanishing $\tau$-trivial section and $\psi$ is transverse to the zero section (see [H1] for the definition of $\tau$-trivial). Then, there is a sequence of admissible orbit sets $\{\alpha_i\}_{i=0,1,2,...}$ satisfying the following conditions.

Proposition 3.1. Suppose that all simple orbits in $(S^3, \lambda)$ are negative hyperbolic. Then, there is a sequence of admissible orbit sets $\{\alpha_i\}_{i=0,1,2,...}$ satisfying the following conditions.

Note that without the isomorphisms to Seiberg-Witten Floer homology, we can compute $\text{ECH}$ of $S^3$ and $L(p, q)$ themselves under ellipsoids (see [H3, Section 3.7]) and M. Hutchings gives a sketch of the proof of (18) and (19) including the structure of $U$-map in a special case (see [H3, Section 4.1]).

This completes the proof of Proposition 2.8.

3 Proof of the results

There are isomorphisms as follows.

Proposition 3.1.

$$\text{ECH}(S^3, \lambda, 0) = F[U^{-1}, U]/UF[U]$$

and for any $\Gamma \in H_1(L(p, q))$,

$$\text{ECH}(L(p, q), \lambda, \Gamma) = F[U^{-1}, U]/UF[U].$$

Proof of Proposition 3.1. These come from the isomorphisms between $\text{ECH}$, Seiberg-Witten Floer homology and Heegaard Floer homology. See Theorem 2.7 [KM, T2, OZ] and [KLT].

Note that without the isomorphisms to Seiberg-Witten Floer homology, we can compute $\text{ECH}$ of $S^3$ and $L(p, q)$ themselves under ellipsoids (see [H3, Section 3.7]) and M. Hutchings gives a sketch of the proof of (18) and (19) including the structure of $U$-map in a special case (see [H3, Section 4.1]).

Lemma 3.2. Suppose that all simple orbits in $(S^3, \lambda)$ are negative hyperbolic. Then, there is a sequence of admissible orbit sets $\{\alpha_i\}_{i=0,1,2,...}$ satisfying the following conditions.
1. For any admissible orbit set \( \alpha, \alpha \) is in \( \{\alpha_i\}_{i=0,1,2,...} \).
2. \( A(\alpha_i) < A(\alpha_j) \) if and only if \( i < j \).
3. \( I(\alpha_i, \alpha_j) = 2(i - j) \) for any \( i, j \).

**Proof of Lemma 3.2.** The assumption that there is no simple positive hyperbolic orbit means \( \partial = 0 \) because of the fourth statement in Proposition 2.5. So the ECH is isomorphic to a free module generated by all admissible orbit sets over \( \mathbb{F} \). Moreover, from (18), we can see that for every two admissible orbit sets \( \alpha \) and \( \beta \) with \( A(\alpha) > A(\beta) \), \( U^k(\alpha) = \langle \beta \rangle \) for some \( k > 0 \). So for every non negative even number \( 2i \), there is exactly one admissible orbit set \( \alpha_i \), whose ECH index relative to \( \emptyset \) is equal to \( 2i \). By considering these arguments, we obtain Lemma 3.2. \( \square \)

In the same way as before, we also obtain the next Lemma.

**Lemma 3.3.** Suppose that all simple orbits in \((L(p,q), \lambda)\) are negative hyperbolic. Then, for any \( \Gamma \in H_1(L(p,q)) \), there is a sequence of admissible orbit sets \( \{\alpha_i^\Gamma\}_{i=0,1,2,...} \) satisfying the following conditions.

1. For any \( i = 0, 1, 2, ... \), \( [\alpha_i^\Gamma] = 0 \) in \( H_1(L(p,q)) \).
2. For any admissible orbit set \( \alpha \) with \( [\alpha] = \Gamma \), \( \alpha \) is in \( \{\alpha_i^\Gamma\}_{i=0,1,2,...} \).
3. \( A(\alpha_i^\Gamma) < A(\alpha_j^\Gamma) \) if and only if \( i < j \).
4. \( I(\alpha_i^\Gamma, \alpha_j^\Gamma) = 2(i - j) \) for any \( i, j \).

**Lemma 3.4.** Suppose that all simple orbits in \((L(p,q), \lambda)\) are negative hyperbolic. Then there is no contractible simple orbit.

**Proof of Lemma 3.4.** Let \( \rho : (S^3, \hat{\lambda}) \rightarrow (L(p,q), \lambda) \) be the covering map where \( \hat{\lambda} \) is the induced contact form of \( \lambda \) by \( \rho \). Suppose that there is a contractible simple orbit \( \gamma \) in \((L(p,q), \lambda)\). Then the inverse image of \( \gamma \) by \( \rho \) consists of \( p \) simple negative hyperbolic orbits. By symmetry, they have the same ECH index relative to \( \emptyset \). This contradicts the results in Lemma 3.2. \( \square \)

Recall the covering map \( \rho : (S^3, \hat{\lambda}) \rightarrow (L(p,q), \lambda) \).

By Lemma 3.4, we can see that under the assumptions, if \( p \) is prime, there is an one-to-one correspondence between periodic orbits in \((S^3, \hat{\lambda})\) and ones in \((L(p,q), \lambda)\). That is, for each orbit \( \gamma \) in \((L(p,q), \lambda)\), there is an unique orbit \( \hat{\gamma} \) in \((S^3, \hat{\lambda})\) satisfying \( \rho^* \gamma = \hat{\gamma} \) and for each orbit \( \hat{\gamma} \) in \((S^3, \hat{\lambda})\), there is an unique orbit \( \gamma \) in \((L(p,q), \lambda)\) satisfying \( \rho^* \gamma = \hat{\gamma} \). As above, we distinguish orbits in \((S^3, \hat{\lambda})\) from ones in \((L(p,q), \lambda)\) by adding tilde. Furthermore, we also do the same way in orbit sets. That is, for each orbit set \( \alpha = \{(\alpha_i, m_i)\} \) over \((L(p,q), \lambda)\), we set \( \hat{\alpha} = \{\hat{\alpha}_i, m_i\}\).

**Lemma 3.5.** Under the assumptions and notations in Lemma 3.3, there is a labelling \( \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{p-1}\} = H_1(L(p,q)) \) satisfying the following conditions.

1. \( \Gamma_0 = 0 \) in \( H_1(L(p,q)) \).
2. If \( A(\alpha_i^\Gamma') < A(\alpha_j^\Gamma') \), then \( i < i' \) or \( j < j' \).

3. For any \( i = 0, 1, 2, \ldots \) and \( \Gamma_j \in \{ \Gamma_0, \Gamma_1, \ldots, \Gamma_{p-1} \} \), \( \frac{1}{2} I(\alpha_i^\Gamma) = j \in \mathbb{Z}/p\mathbb{Z} \).

**Proof of Claim 3.7.** By Proposition 2.8 for each \( \Gamma \in H_1(L(p,q)) \), \( \frac{1}{2} I(\alpha_i^\Gamma) \) in \( \mathbb{Z}/p\mathbb{Z} \) is independent of \( i \). Moreover, by Lemma 3.4, every admissible orbit set of \((S^3, \lambda)\) comes from some of \((L(p,q), \lambda)\) and so in the notation of Lemma 3.2 and Lemma 3.3, the set \( \{\alpha_i^\Gamma\}_{i=0,1,...,r \in H_1(L(p,q))} \) is exactly equivalent to \( \{\alpha_i\}_{i} \). These arguments imply Lemma 3.5 (see the below diagram).

**Lemma 3.6.** Suppose that all simple orbits in \((L(p,q), \lambda)\) are negative hyperbolic and \( p \) is prime. For \( \Gamma \in H_1(L(p,q)) \), we set \( f(\Gamma) = \frac{1}{2} I(\alpha_i^\Gamma) = j \in \mathbb{Z}/p\mathbb{Z} \) for some \( i \geq 0 \). Then this map has to be isomorphism as cyclic groups. Here we note that by Lemma 3.5, this map has to be well-defined and a bijective map from \( H_1(L(p,q)) \) to \( \mathbb{Z}/p\mathbb{Z} \).

**Proof of Lemma 3.6.** Since under the assumption there are infinitely many simple orbits and \( |H_1(L(p,q))| < \infty \), we can pick up \( p \) different simple periodic orbits \( \{\gamma_1, \gamma_2, \ldots, \gamma_p\} \) in \((L(p,q), \lambda)\) with \( |\gamma_1| = |\gamma_2| = \cdots = |\gamma_p| = \Gamma \) for some \( \Gamma \in H_1(L(p,q)) \). Since there is no contractible simple orbit (Lemma 3.4), \( \Gamma \neq 0 \) and so \( f(\Gamma) \neq 0 \). For \( i = 1, 2, \ldots, p \), let \( \tilde{\gamma}_i \) be the corresponding orbit in \((S^3, \lambda)\) of \( \gamma_i \) and \( C_{\tilde{\gamma}_i} \) be a representative of \( Z_{\tilde{\gamma}_i} \) where \( \{Z_{\tilde{\gamma}_i}\} = H_2(S^3; \tilde{\gamma}_i, \emptyset) \) (see Definition 2.2).

**Claim 3.7.** Suppose that \( 1 \leq i, j \leq p \) and \( i \neq j \). Then the intersection number \#([0,1] \times \gamma_i \cap C_{\tilde{\gamma}_i}) \) in \( \mathbb{Z}/p\mathbb{Z} \) does not depend on the choice of \( i, j \) where \( \#([0,1] \times \gamma_i \cap C_{\tilde{\gamma}_i}) \) is the algebraic intersection number in \([0,1] \times Y \) (see Definition 2.3).

**Proof of Claim 3.7.** By the definition, we have

\[
\frac{1}{2} I(\gamma_i \cup \gamma_j, \gamma_i) = \frac{1}{2} I(\tilde{\gamma}_j) + \#([0,1] \times \gamma_i \cap C_{\tilde{\gamma}_i})
\] (21)

So in \( \mathbb{Z}/p\mathbb{Z} \),

\[
\#([0,1] \times \gamma_i \cap C_{\tilde{\gamma}_i}) = \frac{1}{2} I(\gamma_i \cup \gamma_j, \gamma_i) - \frac{1}{2} I(\gamma_j)
\]
\[
= \frac{1}{2} I(\gamma_i \cup \gamma_j) - \frac{1}{2} I(\gamma_i) - \frac{1}{2} I(\gamma_j) = f(2\Gamma) - 2f(\Gamma)
\] (22)
This implies that the value $\#([0,1] \times \tilde{\gamma}_i \cap C_{\tilde{\gamma}_i})$ in $\mathbb{Z}/p\mathbb{Z}$ depends only on $f(2\Gamma)$ and $f(\Gamma)$. This complete the proof of Claim 3.7. \hfill \Box

Return to the proof of Lemma 3.6 we set $l := \#([0,1] \times \tilde{\gamma}_i \cap C_{\tilde{\gamma}_i}) \in \mathbb{Z}/p\mathbb{Z}$ for $i \neq j$.

In the same way as Claim 3.7 for $1 \leq n \leq p$, we have

$$\frac{1}{2} I(\bigcup_{1 \leq i \leq n} \tilde{\gamma}_i, \bigcup_{1 \leq i \leq n-1} \tilde{\gamma}_i) = \frac{1}{2} f(\tilde{\gamma}_n) + \sum_{1 \leq i \leq n-1} \#([0,1] \times \tilde{\gamma}_i \cap C_{\tilde{\gamma}_n})$$

and so

$$f(n\Gamma) - f((n-1)\Gamma) = f(\Gamma) + (n-1)l \in \mathbb{Z}/p\mathbb{Z}. \quad (24)$$

Suppose that $l \neq 0$. Since $p$ is prime, there is $1 \leq k \leq p$ such that $f(\Gamma) + (k-1)l = 0$. This implies that $f(k\Gamma) - f((k-1)\Gamma) = 0$. But this contradicts the bijectivity of $f$. So $l = 0$ and therefore $f(n\Gamma) = nf(\Gamma)$. Since $f(\Gamma) \neq 0$, we have that $f$ is isomorphism. \hfill \Box

**Lemma 3.8.** Suppose that all simple orbits in $(L(p,q), \lambda)$ are negative hyperbolic. Let $\gamma_{\min}$ and $\gamma_{\sec}$ be orbits with smallest and second smallest actions in $(L(p,q), \lambda)$ respectively. Then,

$$I(\tilde{\gamma}_{\min}) = I(\tilde{\gamma}_{\sec}, \tilde{\gamma}_{\min}) = 2 \quad (25)$$

and moreover,

$$6 < I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\sec}) \leq 2p. \quad (26)$$

**Proof of Lemma 3.8.** Consider the diagram (20) and Lemma 3.2 As admissible orbit sets, $\gamma_{\min}$ and $\gamma_{\sec}$ correspond to $\tilde{\alpha}_1^{T_0}$ and $\tilde{\alpha}_2^{T_0}$ respectively. This implies (25).

Next, we show the inequality (26). See $\tilde{\alpha}_1^{T_0}$ in the diagram (20). By the diagram, $I(\tilde{\alpha}_1^{T_0}) = 2p$. Moreover, this comes from $\tilde{\alpha}_1^{T_0}$ with $[\tilde{\alpha}_1^{T_0}] = 0 \in H_T(L(p,q))$. By Lemma 3.4 $\tilde{\alpha}_1^{T_0}$ has to consist of at least two negative hyperbolic orbits. This implies that $A(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\sec}) \leq A(\tilde{\alpha}_1^{T_0})$ and so by Lemma 3.2 $I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\sec}) \leq I(\tilde{\alpha}_1^{T_0}) = 2p$.

Considering the above argument and Lemma 3.2 it is enough to show that $I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\sec}) \neq 6$. We prove this by contradiction. Suppose that $I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\sec}) = 6$. Since $\gamma_{\sec}$ corresponds to $\tilde{\alpha}_2^{T_0}$, we have $I(\tilde{\gamma}_{\sec}) = 4$ and so $I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\sec}, \tilde{\gamma}_{\sec}) = 2$. To consider the $U$-map, fix a generic almost complex structure $J$ on $\mathbb{R} \times S^3$.

Consider the $U$-map $U(\tilde{\alpha}_1^{T_0} = \tilde{\gamma}_{\min} = \emptyset)$. This implies that for each generic point $z \in S^3$, there is an embedded $J$-holomorphic curve $C_z \in \mathcal{M}^+(\tilde{\gamma}_{\min}, \emptyset)$ through $(0,z) \in \mathbb{R} \times S^3$. By using this $C_z$, we have

$$I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\sec}, \tilde{\gamma}_{\sec}) = I(\mathbb{R} \times \tilde{\gamma}_{\sec} \cup C_z). \quad (27)$$

Note that the right hand side of (27) is the ECH index of holomorphic curves $\mathbb{R} \times \tilde{\gamma}_{\sec} \cup C_z$ (see just before (7)). Since $I(\tilde{\gamma}_{\min} \cup \tilde{\gamma}_{\sec}) = 2$ and Proposition 2.5 we have $\mathbb{R} \times \tilde{\gamma}_{sec} \cap C_z = \emptyset$. 

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Consider a sequence of holomorphic curves $C_z$ as $z \to \tilde{\gamma}_{sec}$. By the compactness argument, this sequence has a convergent subsequence and this has a limiting holomorphic curve $C_{\infty}$ which may be splitting into more than one floor. But in this case, $C_{\infty}$ can not split because the action of the positive end of $C_{\infty}$ is smallest value $A(\tilde{\gamma}_{min})$. This implies that $\mathbb{R} \times \tilde{\gamma}_{sec} \cap C_{\infty} \neq \emptyset$. Since $I(\tilde{\gamma}_{min} \cup \tilde{\gamma}_{sec}, \tilde{\gamma}_{sec}) = I(\mathbb{R} \times \tilde{\gamma}_{sec} \cup C_{\infty}) = 2$, $\mathbb{R} \times \tilde{\gamma}_{sec} \cap C_{\infty} \neq \emptyset$ contradicts the fourth statement in Proposition 2.6. Therefore, we have $I(\tilde{\gamma}_{min} \cup \tilde{\gamma}_{sec}) \neq 6$ and thus complete the proof of Lemma 3.8.

**Proof of Theorem 1.7.** We may assume that $p$ is prime because the condition that all simple periodic orbits are negative hyperbolic does not change under taking odd-fold covering. By Lemma 3.6 and (25), we have $[\gamma_{sec}] = 2[\gamma_{min}]$ and so $[\gamma_{min} \cup \gamma_{sec}] = 3[\gamma_{min}]$ in $H_1(L(p,q))$. Since $f$ is isomorphic, we have $1/2 I(\tilde{\gamma}_{min} \cup \tilde{\gamma}_{sec}) = 3$ in $\mathbb{Z}/p\mathbb{Z}$. But this can not occur in the range of (26). This is a contradiction and we complete the proof of Theorem 1.7.

**Proof of Theorem 1.9.** We prove this by contradiction. By Theorem 1.6 we may assume that there is no elliptic orbit and so that all simple orbits are negative hyperbolic. In the same as Lemma 3.2, there is exactly one admissible orbit set $\alpha_i$ whose ECH index relative to $\emptyset$ is equal to $2i$. If there is a non-$\mathbb{Z}/2\mathbb{Z}$-invariant orbit $\gamma$, by symmetry there are two orbit with the same ECH index relative to $\emptyset$. This is a contradiction. So we may assume that all simple orbits are $\mathbb{Z}/2\mathbb{Z}$-invariant.

Let $(\mathbb{R}P^3, \lambda')$ be the non-degenerate contact three manifold obtained as the quotient space of $(S^3, \lambda)$ and $\gamma$ be a $\mathbb{Z}/2\mathbb{Z}$-invariant periodic orbit. Then, this orbit corresponds to a double covering of a non-contractible orbit $\gamma'$ in $(\mathbb{R}P^3, \lambda')$. This implies that the eigenvalues of the return map of $\gamma$ are square of the ones of $\gamma'$. This means that the eigenvalues of the return map of $\gamma$ are both positive and so $\gamma$ is positive hyperbolic. This is a contradiction and so we complete the proof of Theorem 1.9.

**Proof of Corollary 1.10.** Note that if $(L(p,q), \lambda)$ has a simple positive hyperbolic orbit, then its covering space $(S^3, \tilde{\lambda})$ also has a simple positive hyperbolic orbit. Considering a non-trivial cyclic subgroup acting on the contact three sphere together with Corollary 1.8 and Theorem 1.9 we complete the proof of Corollary 1.10.

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