QUASIARITHMETIC-TYPE INVARIANT MEANS ON PROBABILITY SPACE

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ABSTRACT. For a family \((A_x)_{x \in (0, 1)}\) of integral quasiarithmetic means satisfying certain measurability-type assumptions we search for an integral mean \(K\) such that \(K((A_x(P))_{x \in (0, 1)}) = K(P)\) for every compactly supported probabilistic Borel measure \(P\).

Also some results concerning the uniqueness of invariant means will be given.

1. Introduction

For a continuous, strictly monotone function \(f: I \to \mathbb{R}\) (\(I\) is an interval) define a (discrete) quasiarithmetic mean \(A[f]: \bigcup_{k=1}^{\infty} I^k \to I\) by

\[
A[f](x_1, \ldots, x_k) := f^{-1} \left( \frac{f(x_1) + \cdots + f(x_k)}{n} \right),
\]

where \(k \in \mathbb{N}\) and \(x_1, \ldots, x_k \in I\). This notion was introduced in 1930s by Aumann, Knopp [21] and Jessen independently and then characterized by Kolmogorov [22], Nagumo [28] and de Finetti [14]. For the detail concerning the early history of this family we refer the reader to the book of Hardy-Littlewood-Pólya [17]. From now on, a family of all continuous, strictly monotone functions on the interval \(I\) will be denoted by \(\mathcal{CM}(I)\).

It is well known that for \(\pi_p: \mathbb{R}_+ \to \mathbb{R}\) given by \(\pi_p(x) := x^p\) if \(p \neq 0\) and \(\pi_0(x) := \ln x\), the quasiarithmetic mean \(A[\pi_p]\) is a \(p\)-th power mean \(P_p\). Remarkably, the mean \(P_1\) is the arithmetic mean.

For a vector \(f = (f_1, \ldots, f_k)\) of functions in \(\mathcal{CM}(I)\) one can define a selfmapping \(A[f]: I^k \to I^k\) by

\[
A[f](x_1, \ldots, x_k) := \left( A[f_1](x_1, \ldots, x_k), \ldots, A[f_k](x_1, \ldots, x_k) \right).
\]

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Based on a classical result by Borwein-Borwein [5, Theorem 8.8] it is known that there exists exactly one \( A^f \)-invariant mean, that is a mean \( K: I^k \to I \) (a function satisfying the inequality \( \min(x) \leq K(x) \leq \max(x) \) for all \( x \in I^k \)) such that \( K \circ A^f = K \). Furthermore the sequence of iterations of \( A^f \) tends to \((K, \ldots, K)\) pointwise.

Invariant means in a family of quasi-arithmetic means were studied by many authors, for example Burai [7], Daróczy-Páles [11], J. Jarczyk [18], J. Jarczyk and Matkowski [20]. In fact invariant means were extensively studied during recent years, see for example the papers by Baják–Páles [1, 2, 3, 4], by Daróczy–Páles [10, 12, 13], by Głązowska [15, 16], by Matkowski [23, 24, 25], by Matkowski–Páles [27], by Pasteczka [29, 32, 30] and Matkowski–Pasteczka [26]. For details we refer the reader to the recent paper of J. Jarczyk and W. Jarczyk [19].

In (nearly) all of this paper authors referred to some counterpart of a result by Borwein-Borwein which guarantees that the invariant mean is uniquely determined. Regretfully such consideration cannot be generalized to the integral setting. Therefore our paper is based on a recent result by Matkowski–Pasteczka [26] and Pasteczka [32] for noncontinuous means.

1.1. Integral means. Hereafter \( I \) stands for the arbitrary subinterval of \( I \), \( \mathcal{B}(I) \) and \( \mathcal{L}(I) \) denote the Borel and the Lebesgue \( \sigma \)-algebra on \( I \), respectively. Furthermore, let \( \mathcal{P}(I) \) be a family of all compactly supported probabilistic measures on \( \mathcal{B}(I) \). An (integral) mean on \( I \) is a function \( M: \mathcal{P}(I) \to I \) such that

\[
M(\mathcal{P}) \in [\inf \text{supp } \mathcal{P}, \sup \text{supp } \mathcal{P}] \quad \text{for all } \mathcal{P} \in \mathcal{P}(I).
\]

Using the notion \( \gamma(\mathcal{P}) := [\inf \text{supp } \mathcal{P}, \sup \text{supp } \mathcal{P}] \) we can rewrite it briefly as \( M(\mathcal{P}) \in \gamma(\mathcal{P}) \).

Following the notion of Hardy-Littlewood-Pólya [17] for all \( f \in \mathcal{CM}(I) \) we can define the (integral) quasiarithmetic mean \( A^f: \mathcal{P}(I) \to I \) by

\[
A^f(\mathcal{P}) := f^{-1}\left( \int f(x) \, d\mathcal{P}(x) \right).
\]

We slightly abuse the notion of quasiarithmetic mean as \( A^f \) is both discrete and integral quasiarithmetic means. However it do not cause misunderstandings as they are defined of disjoint domains. Moreover for \( k \in \mathbb{N} \) and a vector \( (x_1, \ldots, x_k) \in I^k \) we have

\[
A^f\left( \frac{1}{k}(\delta_{x_1} + \cdots + \delta_{x_k}) \right) = A^f(x_1, \ldots, x_n),
\]
where \( \delta_x \) stands for the Dirac delta. Thus this definition generalizes the discrete one. Similarly to the discrete setting we define a \( p \)-th power mean by \( P_p := A^{[\pi_p]} \).

The aim of this paper is to generalize the notion of invariant means to infinite families of integral quasiarithmetic means.

2. Auxiliary results

Let us first prove a simple result concerning the properties of a distance between two quasiarithmetic means.

**Proposition 1.** Let \( I \subset \mathbb{R} \) be a compact interval and \( f, g \in \text{CM}(I) \). Define \( d_{f,g}: (0, |I|] \to [0, |I|] \) by
\[
d_{f,g}(t) := \sup_{P: |\gamma(P)| \leq t} |A^{[f]}(P) - A^{[g]}(P)|.
\]
Then \( d_{f,g} \) is nondecreasing and continuous. Moreover \( d_{f,g}(t) < t \) for all \( t \in (0, |I|] \).

**Proof.** Denote briefly \( d \equiv d_{f,g} \). For \( t \in (0, |I|] \) define
\[
S_t := \{(x, y, \theta) \in I \times I \times [0, 1]: |x - y| \leq t\}.
\]
and \( m: I^2 \times [0, 1] \to \mathbb{R} \) by
\[
m(x, y, \theta) := |A^{[f]}(\theta \delta_x + (1 - \theta)\delta_y) - A^{[g]}(\theta \delta_x + (1 - \theta)\delta_y)|.
\]
Then \( m \) is continuous and \( m(x, y, \theta) < |x - y| \) unless \( x = y \).

On the other hand by [8] we have
\[
d(t) = \sup_{(x,y,\theta) \in S_t} m(x, y, \theta) = \sup_{S_t} m.
\]
Since \( S_t \) is compact we have \( d(t) < t \) for all \( t \in (0, |I|] \).
Moreover for all \( t_1 \leq t_2 \) we have \( S_{t_1} \subseteq S_{t_2} \), thus
\[
d(t_1) = \sup_{S_{t_1}} m \leq \sup_{S_{t_2}} m = d(t_2)
\]
which implies that \( d \) is nondecreasing.

Now we prove that \( d \) is continuous. Fix \( t_0 \in U =: U_0 \) and consider a monotone sequence \( (t_n)_{n=1}^{\infty} \), \( \lim_{n \to \infty} t_n = t_0 \). Due to the monotonicity of \( d \) we obtain that \( (d(t_n))_{n=1}^{\infty} \) is convergent.

As \( m \) is continuous for all \( n \geq 0 \) the set \( S_{t_n} \) is compact, and we have
\[
d(t_n) = m(s_n) \quad \text{for some } s_n \in S_{t_n} \subset I^2 \times [0, 1], \quad n \in \{0, 1, \ldots\}.
\]
As \( I^2 \times [0, 1] \) is compact, there exists a subsequence \( (s_{n_k})_{k=1}^{\infty} \) convergent to some element \( \bar{s} \). Then \( \bar{s} \) belongs to a topological limit of \( S_{t_{n_k}} \), i.e. \( \bar{s} \in S_{t_{n_0}} \).
Therefore
$$\lim_{n \to \infty} d(t_n) = \lim_{k \to \infty} m(s_{n_k}) = m(\lim_{k \to \infty} s_{n_k}) = m(\bar{s}) \leq d(t_0).$$

To prove the converse inequality take a sequence \((x_n)_{n=1}^{\infty}\) convergent to \(s_0\) such that \(x_n \in S_n\) for all \(n \in \mathbb{N}\). Then
$$d(t_0) = m(s_0) = m(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} m(x_n) \leq \liminf_{n \to \infty} m(s_n) = \lim_{n \to \infty} d(t_n).$$

Therefore \(d(t_0) = \lim_{n \to \infty} d(t_n)\). \(\square\)

At the end of this section let us recall a folk result for discrete dynamical systems with a trivial attractor.

**Lemma 1.** Let \(I\) be an interval with \(\inf I = 0\) and \(d : I \to I\) be a continuous function such that \(d(x) < x\) for all \(x \in I \setminus \{0\}\). Then the sequence of iterates \((d^n(x))\) converges to zero for all \(x \in I\).

### 3. Invariance of Quasiarithmetic Means

In this section we study the invariance of infinite family of quasiarithmetic means. First, we need to define a selfmapping which is a counterpart of (1.1). Contrary to the discrete case where such mapping is well-defined for every tuple we need some additional restrictions.

Family \(F := (f_x)_{x \in [0,1]}\) of functions \(f_x : I \to \mathbb{R}\) is called *admissible* if

1. each \(f_x\) is continuous and strictly monotone,
2. a bivariate function \(I \times [0,1] \ni (t, x) \mapsto f_x(t)\) is measurable with respect to the product \(\sigma\)-algebra \(\mathcal{B}(I) \times \mathcal{L}[0,1]\).

For an admissible family \(F := (f_x)_{x \in [0,1]}\) and \(\mathbb{P} \in \mathcal{P}(I)\) define a measure \(A_F(\mathbb{P})\) on \(\mathbb{R}\) by

\[
A_F(\mathbb{P}) : S \mapsto \left| \{x \in [0,1] : A^{[f_x]}(\mathbb{P}) \in S\} \right|
\]

Now we are in the position to prove one of the most important results in this note.

**Lemma 2.** Let \(F := (f_x)_{x \in [0,1]}\) be an admissible family and \(\mathbb{P} \in \mathcal{P}(I)\). Then each \(A^{[f_x]}(\mathbb{P})\) is well-defined and, moreover, \(A_F(\mathbb{P}) \in \mathcal{P}(I)\).

**Proof.** Let \(h(x) := \int_I f_x \, d\mathbb{P}\) for \(x \in [0,1]\). Since \(\mathbb{P}\) is probabilistic measure with support contained in \(I\) and \(f_x\) is continuous, strictly monotone function, we have \(h(x) \in f_x(I)\) and thus \(A^{[f_x]}(\mathbb{P}) = f_x^{-1}(h(x))\) is well defined.

Moreover by the measurability of the map \(I \times [0,1] \ni (t, x) \mapsto f_x(t)\) and Fubini-Tonelli theorem, we get that \(h\) is Lebesgue measurable. Let \(S \in \mathcal{B}\). Because \(f_x\) is continuous, injective mapping defined on an
Theorem 1. Let $\gamma$ be a $\mathcal{A}_F$-invariant measure on $I$. Then both $L$ and $\gamma$ which proves that the upper-invariant mean $M$ is measurable in the sense of Lebesgue. Therefore, $A_F(\mathcal{P})$ is well-defined Borel measure on $I$. Obviously, $A_F(\mathcal{P})(I) = 1$, which concludes the proof.

Applying the above lemma we can introduce the notion of invariance in the spirit of Matkowski. Namely for an admissible family $\mathcal{F} := (f_x : I \to \mathbb{R})_{x \in [0,1]}$ a mean $M : \mathcal{P}(I) \to I$ is called $A_F$-invariant provided $M = M \circ A_F$. We are going to study properties of $A_F$-invariant means. Adapting some general results from [32] define the lower- and the upper-invariant mean $\mathcal{L}_F, \mathcal{U}_F : \mathcal{P}(I) \to I$ by

$$
\mathcal{L}_F(\mathcal{P}) := \lim_{n \to \infty} \left( \inf \gamma(A^n_F(\mathcal{P})) \right),
$$

$$
\mathcal{U}_F(\mathcal{P}) := \lim_{n \to \infty} \left( \sup \gamma(A^n_F(\mathcal{P})) \right).
$$

Now we show that these means are $A_F$-invariant. Moreover, similarly to the discrete case, $\mathcal{L}_F$ and $\mathcal{U}_F$ are the smallest and the biggest $A_F$-invariant means, respectively.

**Theorem 1.** Let $\mathcal{F} := (f_x : I \to \mathbb{R})_{x \in [0,1]}$ be an admissible family. Then both $\mathcal{L}_F$ and $\mathcal{U}_F$ are $A_F$-invariant means. Moreover for every $A_F$-invariant mean $M : \mathcal{P}(I) \to I$ the inequality $\mathcal{L}_F \leq M \leq \mathcal{U}_F$ holds.

**Proof.** Take $\mathcal{P} \in \mathcal{P}(I)$ arbitrarily. By virtue of Lemma 2 we obtain that $A^n_F(\mathcal{P}) \in \mathcal{P}(I)$ for all $n \in \mathbb{N}$.

Moreover as $A^n_F(\mathcal{P}) \in \gamma(A^n_F(\mathcal{P}))$ for all $x \in [0,1]$ we obtain $\gamma(A^{n+1}_F(\mathcal{P})) \subseteq \gamma(A^n_F(\mathcal{P}))$. In particular for every $\mathcal{P} \in \mathcal{P}(I)$ we have

$$
\mathcal{L}_F(\mathcal{P}) = \lim_{n \to \infty} \left( \inf \gamma(A^n_F(\mathcal{P})) \right) \subseteq \gamma(A^0_F(\mathcal{P})) = \gamma(\mathcal{P}),
$$

which proves that $\mathcal{L}_F$ is a mean. Moreover

$$
\mathcal{L}_F(\mathcal{P}) = \lim_{n \to \infty} \left( \inf \gamma(A^n_F(\mathcal{P})) \right) = \lim_{n \to \infty} \left( \inf \gamma(A^{n+1}_F(\mathcal{P})) \right)
$$

$$
= \lim_{n \to \infty} \left( \inf \gamma(A^n_F(\mathcal{P})) \right) = \mathcal{L}_F(\mathcal{P}),
$$

which shows that $\mathcal{L}_F$ is $A_F$-invariant. Similarly $\mathcal{U}_F$ is an $A_F$-invariant mean.

Now let $M : \mathcal{P}(I) \to I$ be an arbitrary $A_F$-invariant mean. Then, applying the definition of $A_F$-invariance iteratively, we obtain

$$
M(\mathcal{P}) = M \circ A^n_F(\mathcal{P}) \text{ for all } \mathcal{P} \in \mathcal{P}(I) \text{ and } n \in \mathbb{N}.
$$
By mean property it follows that for all $P \in \mathcal{P}(I)$ we have
$$M(P) \in \gamma(A^n_F(P)) \quad (n \in \mathbb{N})$$
and therefore, as $\gamma(A^n_F(P)) \subseteq \gamma(A^{n-1}_F(P))$, we obtain
$$M(P) \in \bigcap_{n=1}^{\infty} \gamma(A^n_F(P)) = [\mathcal{L}_F(P), \mathcal{U}_F(P)].$$
The latter inequality can be rewritten as $\mathcal{L}_F \leq M \leq \mathcal{U}_F$. \quad \square

3.1. Conjugacy of means. Following the idea contained in Bullen [6] and Chudziak-Páles-Pasteczka [9], let us introduce the notion of conjugacy of means. For a continuous and strictly monotone function $u: J \to I$ and a mean $M: \mathcal{P}(I) \to I$ define a the conjugancy $M[u]: \mathcal{P}(J) \to J$ by
$$M[u](P) = u^{-1}\left(M\left(u(x) \, d\mathbb{P}(x)\right)\right).$$
It is easy to see that $(M[u])[u^{-1}] = M$. Moreover for every $f \in \mathcal{CM}(I)$ the quasiarithmetic mean $A[f]$ is a $f$-conjugant of the arithmetic mean (which coincides with the expected value).

The following lemma is easy to see

Lemma 3. Let $\mathcal{F} := (f_x: I \to \mathbb{R})_{x \in [0,1]}$ be an admissible family, $u: J \to I$, and $\mathcal{G} := (g_x = f_x \circ u)_{x \in [0,1]}$. Then $M: \mathcal{P}(I) \to I$ is a $A_\mathcal{F}$-invariant mean if and only if $M[u]$ is a $A_\mathcal{G}$-invariant mean.

3.2. Uniqueness of invariant means. In what follows we show few sufficient condition in order to guarantee the uniqueness of $A_\mathcal{F}$-invariant mean. First observe that Theorem 1 has the following corollary

Corollary 1. Let $\mathcal{F} := (f_x: I \to \mathbb{R})_{x \in [0,1]}$ be an admissible family. Then $\mathcal{L}_F = \mathcal{U}_F$ if and only is there exists exactly one $A_\mathcal{F}$-invariant mean.

The main disadvantage of this result is that it is very difficult to verify this condition in practice. In the next result we show that whenever $\mathcal{F}$ is bounded from one side then the invariant mean is uniquely determined in a weak sense.

Theorem 2. Let $\mathcal{F} := (f_x: I \to \mathbb{R})_{x \in [0,1]}$ be an admissible, upper (lower) bounded family. Then there exists a (uniquely determined) $A_\mathcal{F}$-invariant mean $K_F: \mathcal{P}(I) \to I$ such that
$$\lim_{n \to \infty} A^{[k]} \circ A^n_F(P) = K_F(P) \quad \text{for all } k \in \mathcal{CM}(I) \text{ and } P \in \mathcal{P}(I).$$
Proof. Assume that $A^{[f_x]} \leq A^{[u]}$ for some $u : I \to \mathbb{R}$ and define
$$G := (g_x := f_x \circ u^{-1})_{x \in [0,1]}.$$ 
As $A^{[f_x]} \leq A^{[u]}$ we get $A^{[g_x]} \leq A$ for all $x \in [0, 1]$.

Take $P_0 \in \mathcal{P}(u(I))$ arbitrarily and let $P_n := A_G(P_{n-1})$ for all $n \in \mathbb{N}_+$. Then we know that
$$P_1(P_{n+1}) \leq P_2(P_{n+1}) \leq \sup \gamma(P_{n+1}) \leq A(P_n) = P_1(P_n).$$
This implies that all intervals $[P_1(P_n), P_2(P_n)]$ are disjoint. In particular
$$\lim_{n \to \infty} P_2(P_n) = \lim_{n \to \infty} P_1(P_n) =: m(P).$$
Thus we obtain
$$\lim_{n \to \infty} \text{Var}(P_n) = \lim_{n \to \infty} \left( P_2(P_n) - (P_1(P_n))^2 \right) = 0.$$ 
In view of Chebyshev’s inequality we have
$$\Pr\left( \|P_n - m(P)\| \geq \xi \right) \leq \Pr\left( \|P_n - P_1(P_n)\| \geq \xi - |P_1(P_n) - m(P)| \right) \leq \frac{\text{Var}(P_n)}{(\xi - |P_1(P_n) - m(P)|)^2} \text{ for all } \xi > 0.$$ 
Whence in view of (3.2) and (3.3) we obtain
$$\lim_{n \to \infty} \Pr\left( \|P_n - m(P)\| \geq \xi \right) = 0 \text{ for all } \xi > 0$$
which shows that $P_n \to \delta_{m(P)}$ in a (Lebesgue) measure. As each $P_n$ is compactly supported we obtain that
$$\lim_{n \to \infty} A_k^n(P_n) = m(P) \text{ for all } k \in \mathcal{CM}(u(I)).$$
Consequently, as $A^{[f_x]} = A^{[g_x \circ u]} = (A^{[g_x]})^{[u]}$ for all $x \in [0,1]$ we have
$$\lim_{n \to \infty} A_k^n(P) = m^n(P) \text{ for all } k \in \mathcal{CM}(I) \text{ and } P \in \mathcal{P}(I)$$
which yields (3.1) with $K_F := m[u].$ \hfill \Box

Now we show a result in a case when the family $\mathcal{F}$ satisfy some sort of boundedness. It is important to emphasize that even a finite family of quasi-arithmetic means can be unbounded (in the family of quasiarithmetic means with a pointwise ordering), see [31] for details.

**Theorem 3.** Let $\mathcal{F} := (f_x : I \to \mathbb{R})_{x \in [0,1]}$ be an admissible family and $\mathcal{T}$ be a finite subset of $\mathcal{CM}(I)$. Assume that for every $x \in [0, 1]$ there exists $l_x, u_x \in \mathcal{T}$ such that $A^{[l_x]} \leq A^{[f_x]} \leq A^{[u_x]}$. Then there exists a uniquely determined $A_{\mathcal{F}}$-invariant mean $K_{\mathcal{F}} : \mathcal{P}(I) \to I$. 
Proof. Define $d : (0, |I|] \to [0, |I|]$ by

$$d(t) := \max_{l,u \in T} \sup_{P : |\gamma(P)| \leq t} |A[l](P) - A[u](P)| = \max_{l,u \in T} d_{l,u}(x).$$

Then by Proposition 1 we obtain that $d$ is continuous and $d(x) < x$ for all $x \in (0, |I|]$. Therefore by Lemma 1 we obtain that the sequence of iterations $(d^n)_{n=1}^\infty$ tends to zero pointwise.

On the other hand for every mean $P \in \mathcal{P}(I)$ and $x \in [0, 1]$ we obtain

$$\min_{l \in T} A[l](P) \leq A[f_{x}](P) \leq \max_{u \in T} A[u](P).$$

Therefore

$$\sup_{x \in [0, 1]} A[f_{x}](P) - \inf_{x \in [0, 1]} A[f_{x}](P) \leq \max_{u \in T} A[u](P) - \min_{l \in T} A[l](P) \leq d(|\gamma(P)|).$$

Thus $|\gamma(A_{F}(P))| \leq d(|\gamma(P)|)$ for every $P \in \mathcal{P}(I)$. Therefore

$$\mathcal{U}_{F}(P) - \mathcal{L}_{F}(P) = \lim_{n \to \infty} |\gamma(A_{F}^{n}(P))| \leq \lim_{n \to \infty} d^n(|\gamma(P)|) = 0,$$

which proves $\mathcal{U}_{F}(P) = \mathcal{L}_{F}(P)$. As $P$ was taken arbitrarily we obtain $K_{F} := \mathcal{U}_{F} = \mathcal{L}_{F}$, which by Corollary 1 implies that $A_{F}$-invariant mean is uniquely determined. \[\square\]

Applying this theorem we can easily show the finite counterpart of this result

**Corollary 2.** Let $F := (f_{x} : I \to \mathbb{R})_{x \in [0, 1]}$ be an admissible family which contains finitely many distinct functions. Then there exists a uniquely determined $A_{F}$-invariant mean $K_{F} : \mathcal{P}(I) \to I$.

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