Superstring Vertex Operators in an $AdS_5 \times S^5$ Background

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A quantizable action has recently been proposed for the superstring in an $AdS_5 \times S^5$ background with Ramond-Ramond flux. In this paper, we construct physical vertex operators corresponding to on-shell fluctuations around the $AdS_5 \times S^5$ background. The structure of these $AdS_5 \times S^5$ vertex operators closely resembles the structure of the massless vertex operators in a flat background.

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1. Introduction

A quantizable action has recently been proposed for the superstring in an $AdS_5 \times S^5$ background with Ramond-Ramond flux [1]. This action is closely related to the $AdS_2 \times S^2$ and $AdS_3 \times S^3$ actions of [2] [3] [4], and differs from the Green-Schwarz action of Metsaev and Tseytlin [5] in that quantization is straightforward using a BRST charge constructed out of the fermionic constraints and a pure spinor. Physical vertex operators can be described in a manifestly spacetime-supersymmetric manner as states in the cohomology of this BRST charge [1] [6]. In an $AdS_3 \times S^3$ background with Ramond-Ramond flux, physical vertex operators corresponding to on-shell fluctuations of the $AdS_3 \times S^3$ background were explicitly constructed by Dolan and Witten in [7]. In this paper, we shall generalize their construction to the $AdS_5 \times S^5$ background. The resulting $AdS_5 \times S^5$ vertex operators will closely resemble the massless Type IIB vertex operators in a flat ten-dimensional background.

In a flat background, the massless Type IIB vertex operators in the pure spinor formalism for the superstring are constructed using a bispinor superfield, $A_{\beta \hat{\gamma}}(x, \theta, \hat{\theta})$, satisfying the equations of motion and gauge invariances

$$\gamma^{\alpha \beta}_{mpqr} D_\alpha A_{\beta \hat{\gamma}} = \tilde{\gamma}^{\alpha \hat{\gamma}}_{mpqr} D_{\hat{\alpha}} A_{\beta \gamma} = 0,$$

(1.1)

$$\delta A_{\beta \hat{\gamma}} = D_\beta \hat{\Omega}_{\hat{\gamma}} + \tilde{D}_{\hat{\beta}} \Omega_{\gamma} \quad \text{with} \quad \gamma^{\alpha \beta}_{mpqr} D_\alpha \Omega_\beta = \gamma^{\alpha \hat{\gamma}}_{mpqr} D_{\hat{\alpha}} \hat{\Omega}_{\hat{\gamma}} = 0,$$

where $D_\alpha = \partial + (\gamma^m \theta)_\alpha \partial_m$ and $D_{\hat{\alpha}} = \partial + (\gamma^m \hat{\theta})_{\hat{\alpha}} \partial_m$ are the supersymmetric derivatives of flat $N = 2 D = 10$ superspace and $\gamma^m_{\alpha \beta}$ are the symmetric $16 \times 16$ SO(9,1) Pauli matrices. As will be reviewed in section 2, the components of $A_{\alpha \hat{\beta}}$ which satisfy (1.1) describe the linearized on-shell Type IIB supergravity fields.

As will be shown in section 3, the vertex operators for on-shell fluctuations around the $AdS_5 \times S^5$ background can also be constructed using a bispinor superfield, $A_{\beta \hat{\gamma}}(x, \theta, \hat{\theta})$, which now must satisfy the equations of motion and gauge invariances

$$\gamma^{\alpha \beta}_{mpqr} \nabla_\alpha A_{\beta \hat{\gamma}} = \tilde{\gamma}^{\alpha \hat{\gamma}}_{mpqr} \nabla_{\hat{\alpha}} A_{\beta \gamma} = 0,$$

(1.2)

3 Throughout this paper, we shall use the notation $\alpha$ and $\hat{\alpha}$ to represent the two real sixteen-component Majorana-Weyl spinors of $N = 2 D = 10$ superspace. Although they transform as the same SO(9,1) representation for the Type IIB superstring, it will be convenient to use two separate labels for the two different spinors.
\[ \delta A_{\beta \gamma} = \nabla_\beta \tilde{\Omega}_\gamma + \nabla_\gamma \tilde{\Omega}_\beta \quad \text{with} \quad \gamma_{\alpha \beta}^{\alpha \beta} \nabla_\alpha \Omega_\beta = \gamma_{\alpha \beta}^{\alpha \beta} \nabla_\alpha \tilde{\Omega}_\gamma = 0, \]

where \( \nabla_\alpha \) and \( \nabla_\alpha \) are the covariant supersymmetric derivatives in the \( AdS_5 \times S^5 \) superspace background. Note that in a generic Type IIB supergravity background, the equations of motion and gauge invariances of (1.2) would be inconsistent because \( \{ \nabla_\alpha, \nabla_\beta \} \) is non-vanishing. Although this anti-commutator is also non-vanishing in the \( AdS_5 \times S^5 \) background, the symmetrical structure of \( \{ \nabla_\alpha, \nabla_\beta \} \) in the \( AdS_5 \times S^5 \) background is enough to make (1.2) consistent.

It is sometimes convenient to choose a gauge in which the vertex operator is a worldsheet primary field. In a flat background, the bispinor superfield \( A_{\beta \gamma} \) satisfies in this gauge

\[ \partial_n \partial^n A_{\alpha \beta} = 0, \quad \partial^n A_{n \beta} = \partial^n A_{\alpha n} = 0, \quad (1.3) \]

\[ \delta A_{\beta \gamma} = \nabla_\beta \tilde{\Omega}_\gamma + \nabla_\gamma \tilde{\Omega}_\beta \quad \text{with} \quad \partial_n \partial^n \Omega_\beta = \partial_n \partial^n \tilde{\Omega}_\gamma = \partial^n \Omega_\beta = \partial^n \tilde{\Omega}_\gamma = 0 \]

where \( A_{m \gamma} = \frac{1}{16} \gamma^{\alpha \beta} D_\alpha A_{\alpha \beta}, \quad A_{\alpha n} = \frac{1}{16} \gamma^{\alpha \beta} D_\alpha \tilde{\Omega}_\beta, \quad \Omega_m = \frac{1}{16} \gamma^{\alpha \beta} D_\alpha \Omega_\beta, \quad \tilde{\Omega}_m = \frac{1}{16} \gamma^{\alpha \beta} D_\alpha \tilde{\Omega}_\beta. \) As will be shown in section 4, there exists an analogous gauge choice in the \( AdS_5 \times S^5 \) background in which the bispinor superfield \( A_{\beta \gamma} \) satisfies

\[ \nabla_B \nabla^B A_{\alpha \beta} = 0, \quad \nabla^B A_{B \beta} = \nabla^B A_{\alpha B} = 0, \quad (1.4) \]

\[ \delta A_{\beta \gamma} = \nabla_\beta \tilde{\Omega}_\gamma + \nabla_\gamma \tilde{\Omega}_\beta \quad \text{with} \quad \nabla_B \nabla^B \Omega_\beta = \nabla_B \nabla^B \tilde{\Omega}_\gamma = \nabla^B \Omega_\beta = \nabla^B \tilde{\Omega}_\gamma = 0 \]

where \( B \) ranges over all tangent-space indices of the super Lie-algebra of \( PSU(2,2|4) \). The equations of (1.4) are manifestly invariant under the AdS isometries and reduce in the flat limit to the equations of (1.3).

2. Massless Vertex Operators in a Flat Background

In this section, we review the construction of massless vertex operators for the Type IIB superstring in a flat background using the pure spinor formalism. We will first review the manifestly super-Poincaré invariant worldsheet action and will then discuss the physical massless vertex operators.
2.1. Worldsheet action in a flat background

In a flat background, the worldsheet action in conformal gauge in this formalism is

\[ S = \int d^2z \left[ \frac{1}{2} \partial x^m \partial x_m + p_\alpha \partial \theta^\alpha + \tilde{p}_\alpha \partial \tilde{\theta}^\alpha \right] + S_\lambda + S_{\tilde{\lambda}}. \]  

(2.1)

where \( m = 0 \) to 9, \( \alpha \) and \( \tilde{\alpha} = 1 \) to 16, and \( S_\lambda \) and \( S_{\tilde{\lambda}} \) are the worldsheet free-field actions for an independent left and right-moving spinor variable, \( \lambda^\alpha \) and \( \tilde{\lambda}^{\tilde{\alpha}} \), satisfying the pure spinor conditions

\[ \lambda \gamma^m \lambda = 0 \quad \text{and} \quad \tilde{\lambda} \gamma^m \tilde{\lambda} = 0 \quad \text{for} \quad m = 0 \) to 9. \]  

(2.2)

Note that all unhatted worldsheet variables (except for \( x^m \)) in (2.1) are left-moving and all hatted worldsheet variables are right-moving.

The free-field action of (2.1) is manifestly invariant under \( N = 2 \) \( D = 10 \) spacetime-supersymmetry and it is convenient to define the following spacetime-supersymmetric combinations of the worldsheet fields,

\[ d_\alpha = p_\alpha - (\Pi^m - \frac{1}{2} \theta \gamma^m \partial \theta)(\gamma_m \theta)_\alpha, \quad \Pi^m = \partial x^m + \theta \gamma^m \partial \theta, \]  

\[ \hat{d}_\alpha = \hat{p}_\alpha - (\hat{\Pi}^m - \frac{1}{2} \hat{\theta} \hat{\gamma}^m \partial \hat{\theta})(\hat{\gamma}_m \hat{\theta})_{\alpha}, \quad \hat{\Pi}^m = \overline{\partial} x^m + \hat{\theta} \hat{\gamma}^m \overline{\partial} \hat{\theta}, \]  

(2.3)

which satisfy the OPE’s

\[ d_\alpha (y) d_\beta (z) \rightarrow -2 \gamma^m _{\alpha \beta} (y - z)^{-1} \Pi_m, \quad \hat{d}^\alpha (\mathbf{y}) \hat{d}^\beta (\mathbf{z}) \rightarrow -2 \gamma^m _{\alpha \beta} (\mathbf{y} - \mathbf{z})^{-1} \hat{\Pi}_m. \]  

(2.4)

2.2. Physical massless vertex operators

Physical states in the pure spinor formalism are defined as vertex operators of ghost-number (1, 1) in the cohomology of the nilpotent BRST-like charges

\[ Q = \oint d\zeta \lambda^\alpha d_\alpha \quad \text{and} \quad \overline{Q} = \oint d\overline{\zeta} \tilde{\lambda}^{\tilde{\alpha}} \hat{d}_{\tilde{\alpha}}, \]  

(2.5)

where \( \lambda^\alpha \) and \( \tilde{\lambda}^{\tilde{\alpha}} \) carry ghost-number (1, 0) and (0, 1) respectively. The massless states are constructed from zero modes only and are therefore represented by the vertex operator

\[ \text{[1]} \]

Although an explicit construction of \( S_\lambda \) and \( S_{\tilde{\lambda}} \) requires breaking SO(9,1) to a subgroup, the OPE’s of \( \lambda^\alpha \) and \( \tilde{\lambda}^{\tilde{\alpha}} \) with their Lorentz currents \( N^{mn} \) and \( \hat{N}^{mn} \) are manifestly SO(9,1) covariant. This allows all vertex operator and scattering amplitude computations to be manifestly SO(9,1) super-Poincaré covariant\[1\].
\[ U = \lambda^{\beta} \hat{\lambda}^{\gamma} A_{\beta \gamma}(x, \theta, \hat{\theta}) \] where \( A_{\beta \gamma}(x, \theta, \hat{\theta}) \) is a bispinor \( N = 2 \ D = 10 \) superfield which depends only on the worldsheet zero modes of \( x^m, \theta^\alpha \) and \( \hat{\theta}^{\hat{\alpha}} \).

The cohomology condition for a physical vertex operator \( U \) is that it satisfies the equations and gauge invariances:

\[ QU = \overline{Q} U = 0, \quad \delta U = QA + \overline{Q} \hat{\Lambda} \quad \text{with} \quad \overline{Q} A = Q \hat{\Lambda} = 0. \]

Applying these conditions to \( U = \lambda^{\beta} \hat{\lambda}^{\gamma} A_{\beta \gamma} \) where \( \Lambda = \hat{\lambda}^{\alpha} \hat{\Omega}_\alpha \) and \( \hat{\Lambda} = \lambda^\alpha \Omega_\alpha \), and using that pure spinors satisfy

\[ \lambda^\alpha \lambda^\beta = \frac{1}{1920} (\lambda mnpqr \gamma)_{\gamma}^{\alpha \beta} mnpqr \quad \text{and} \quad \hat{\lambda}^{\alpha} \hat{\lambda}^{\beta} = \frac{1}{1920} (\hat{\lambda} mnpqr \hat{\gamma})^{\alpha \beta} mnpqr, \]

one finds that \( A_{\beta \gamma} \) must satisfy the conditions

\[ \gamma_{mnpqr}^{\alpha \beta} D_\alpha A_{\beta \gamma} = \gamma_{mnpqr}^{\alpha \beta} D_\beta A_{\gamma \alpha} = 0, \]

\[ \delta A_{\beta \gamma} = D_\beta \hat{\Omega}_\gamma + D_\gamma \Omega_\beta \quad \text{with} \quad \gamma_{mnpqr}^{\alpha \beta} D_\alpha \Omega_\beta = \gamma_{mnpqr}^{\alpha \beta} D_\beta \hat{\Omega}_\gamma = 0, \]

where \( D_\alpha = \frac{\partial}{\partial \theta^\alpha} + (\gamma m \theta)_\alpha \partial_m \) and \( D_\beta = \frac{\partial}{\partial \hat{\theta}^{\hat{\alpha}}} + (\gamma m \hat{\theta})^{\hat{\alpha}} \partial_m \) are the supersymmetric derivatives of flat \( N = 2 \ D = 10 \) superspace. Note that \( Q \Phi = \lambda^\alpha D_\alpha \Phi \) and \( \overline{Q} \Phi = \hat{\lambda}^{\alpha} D_\alpha \Phi \) for any superfield \( \Phi(\lambda, \hat{\lambda}, x, \theta, \hat{\theta}) \).

It will now be shown that the conditions of (2.8) correctly reproduce the Type IIB supergravity spectrum. The easiest way to check this is to use the fact that closed superstring vertex operators can be understood as the left-right product of open superstring vertex operators. The massless open superstring vertex operator is described by an \( N = 1 \ D = 10 \) spinor superfield \( A_\beta(x, \theta) \) satisfying the conditions

\[ \gamma_{mn \alpha \beta}^{\alpha \beta} D_\alpha A_\beta(x, \theta) = 0, \quad \delta A_\beta(x, \theta) = D_\beta \Omega(x, \theta). \]

The conditions of (2.9) imply the super-Maxwell spectrum since there exists a gauge choice such that

\[ A_\beta(x, \theta) = (\gamma m \theta)_\beta a_m(x) + (\theta \gamma mnp \theta)(\gamma_{mnp} \xi(x))_\beta + ... \]

\[ \text{For the ordinary closed bosonic string with the standard definitions of} \ Q \ \text{and} \ \overline{Q}; \ \text{these cohomology conditions reproduce the usual physical spectrum for states with non-zero momentum. For zero momentum states, there are additional subtleties associated with the} \ b_0 - \overline{b}_0 \ \text{condition which will be ignored in the discussion of this paper.} \]
where $a_m(x)$ and $\xi^\alpha(x)$ satisfy the super-Maxwell equations $\partial^m(\partial_m a_n - \partial_n a_m) = \partial_m(\gamma^m \xi)_\alpha = 0$ and where ... contains only auxiliary fields.

Similarly, the equations of (2.8) imply that there exists a gauge choice such that

$$A_{\beta\gamma}(x, \theta, \hat{\theta}) = (\gamma^m \theta)_\beta(\gamma^n \theta)^{\gamma}_{\gamma} h_{mn}(x) + (\gamma^m \theta)_\beta(\hat{\theta} \gamma^{pq\hat{r}} \hat{\theta})(\gamma_{pq\hat{r}} \hat{\psi}_m(x))_{\gamma}$$  \hspace{1cm} (2.11)

$$+ (\theta \gamma^{pq\hat{r}} \hat{\theta})(\gamma^m \theta)^{\gamma}_{\gamma} (\gamma_{pq\hat{r}} \psi_m(x))_\beta + (\theta \gamma^{mnp} \theta)_{\gamma} \gamma_{pq pqr} \gamma_{\gamma}^{qrs} F^{\alpha \hat{\alpha}}(x) + ...$$

where ... contains only auxiliary fields and where $h_{mn}(x), \psi^\alpha_m(x), \hat{\psi}^\alpha_m(x)$ and $F^{\alpha \hat{\alpha}}(x)$ satisfy the equations

$$\partial^m(\partial_m h_{pn} - \partial_p h_{mn}) = \partial^m(\partial_m h_{pm} - \partial_n h_{pm}) = 0,$$  \hspace{1cm} (2.12)

$$\partial^m(\partial_m \psi^\alpha_n - \partial_n \psi^\alpha_m) = \partial^m(\partial_m \hat{\psi}^\alpha_n - \partial_n \hat{\psi}^\alpha_m) = 0, \quad \partial_n(\gamma^n \psi_m)_\alpha = \partial_n(\gamma^n \hat{\psi}_m)_\alpha = 0,$$

$$\gamma^m_{\alpha \beta} \partial_n F^{\gamma \hat{\gamma}} = \gamma^m_{\alpha \hat{\beta}} \partial_n F^{\alpha \hat{\alpha}} = 0.$$  

The equations of (2.12) are those of linearized Type IIB supergravity where $h_{mn}$ describes the dilaton, graviton and anti-symmetric two-form, $\psi^\alpha_m$ and $\hat{\psi}^\alpha_m$ describe the two gravitini and dilatini, and $F^{\alpha \hat{\alpha}}$ describes the Ramond-Ramond field strengths.

3. **Vertex Operator in $AdS_5 \times S^5$ Background**

In this section, it will be shown how to generalize the equations of (2.8) in an $AdS_5 \times S^5$ background. We shall first discuss the superstring action in an $AdS_5 \times S^5$ background and will then discuss the physical vertex operators which describe on-shell fluctuations of this background.

3.1. **Action in $AdS_5 \times S^5$ background**

In an $AdS_5 \times S^5$ background with Ramond-Ramond flux, the worldsheet action for the superstring in the pure spinor formalism is

$$S = \int d^2 z \left[ \frac{1}{2}(\eta_{cd} J^c \bar{J}^d + \eta_{c'd'} J^{c'} \bar{J}^{d'}) + \delta_{\alpha \beta}(3 J^c \bar{J}^\alpha - J^\alpha \bar{J}^c) \right]$$  \hspace{1cm} (3.1)

$$+ N_{cd} J^{[cd]} + N_{c'd'} J^{[c'd']} + \hat{N}_{cd} J^{[cd]} + \hat{N}_{c'd'} J^{[c'd']} + \frac{1}{2}(N_{cd} \hat{N}_{cd} - N_{c'd'} \hat{N}_{c'd'}) + S_\lambda + S_{\hat{\lambda}}$$

where $J^A = (g^{-1} \partial g)^A$ and $\bar{J}^A = (g^{-1} \bar{\partial} g)^A$ are left-invariant currents constructed from the supergroup element $g(x, \theta, \hat{\theta}) \in PSU(2, 2|4)$, $[N_{cd}, N_{c'd'}]$ and $[\hat{N}_{cd}, \hat{N}_{c'd'}]$ are the
SO(4,1) × SO(5) components of the Lorentz current for \( \lambda^\alpha \) and \( \tilde{\lambda}^\alpha \), and \( S_\lambda \) and \( \tilde{S}_\lambda \) are the same as in (2.1). The indices \((c, [cd], c', [c'd'], \alpha, \tilde{\alpha})\) range over the tangent space indices of the super-Lie algebra of \( PSU(2,2|4) \) where \((c, [cd])\) describe the SO(4,2) isometries of \( AdS_5 \) with \( c = 0 \) to 4, and \((c', [c'd'])\) describe the SO(6) isometries of \( S^5 \) with \( c' = 5 \) to 9. It will be convenient to preserve the notation \( \alpha \) and \( \tilde{\alpha} \) for the two sixteen-component Majorana-Weyl spinors, but unlike the flat case, one can now contract an \( \alpha \) index with an \( \tilde{\alpha} \) index in an \( SO(4,1) \times SO(5) \) invariant manner using the matrix \( \delta^{\alpha\tilde{\alpha}} = \gamma_{\alpha01234}^{\tilde{\alpha}} \) and \( \delta_{\alpha\tilde{\alpha}} = \gamma_{\alpha01234}^{\tilde{\alpha}} \). The non-vanishing structure constants \( f^C_{AB} \) of the \( PSU(2,2|4) \) algebra are

\[
\begin{align*}
  f^e_{\alpha\beta} &= 2\gamma^e_{\alpha\beta}, & f^e_{\tilde{\alpha}\tilde{\beta}} &= 2\gamma^e_{\alpha\beta}, \\
  f^{[ef]}_{\alpha\beta} &= f^{[ef]}_{\beta\alpha} = (\gamma^{ef})_{\alpha\beta} \delta_{\gamma\tilde{\gamma}}, & f^{[e'f']}_{\alpha\beta} &= f^{[e'f']}_{\beta\alpha} = -(\gamma^{e'f'})_{\alpha\beta} \delta_{\gamma\tilde{\gamma}}, \\
  f^\beta_{\alpha\tilde{c}} &= -f^\beta_{\tilde{\alpha}\alpha} = \frac{1}{2}(\gamma_\zeta)_{\alpha\beta} \delta_{\gamma\tilde{\gamma}}, & f^\beta_{\tilde{\alpha}\alpha} &= -f^\beta_{\alpha\tilde{\alpha}} = \frac{1}{2}(\gamma_\zeta)_{\alpha\beta} \delta_{\gamma\tilde{\gamma}}, \\
  f^{[gh]}_{[cd][ef]} &= \eta_{ef} \delta^{[gh]}_{[cd]}, & f^{[ef]}_{[cd][\alpha]} &= -f^{[ef]}_{[\alpha][cd]} = \frac{1}{2}(\gamma_{\alpha\alpha\beta})_{\alpha\beta}, & f^\beta_{[cd]\tilde{\alpha}} &= -f^\beta_{\tilde{\alpha}[cd]} = \frac{1}{2}(\gamma_{\alpha\alpha\beta})_{\alpha\beta},
\end{align*}
\]
where \( \zeta \) signifies either \( c \) or \( c' \) and \([cd]\) signifies either \([cd]\) or \([c'd']\).

The action of (3.1) was derived in [1] by plugging the background values of the \( AdS_5 \times S^5 \) superfields into the sigma model action for the superstring in a generic Type II background, and integrating out the \( d_\alpha \) and \( d_\tilde{\alpha} \) worldsheet fields as in [2][3][4]. This action is invariant under \( \delta g = M g + g \Omega(x, \theta, \tilde{\theta}) \) where \( M \) is a global \( PSU(2,2|4) \) transformation and \( \Omega(x, \theta, \tilde{\theta}) \) is a local \( SO(4,1) \times SO(5) \) transformation which also rotates the pure spinors as

\[
\delta \lambda^\alpha = \frac{1}{2} \Omega_{[cd]} (\gamma_{cd} \lambda)^\alpha, \quad \delta \tilde{\lambda}^\alpha = \frac{1}{2} \Omega_{[cd]} (\gamma_{cd} \tilde{\lambda})^\alpha,
\]

so \( g(x, \theta, \tilde{\theta}) \) can be defined to take values in the coset supergroup \( PSU(2,2|4) / SO(4,1) \times SO(5) \).

6 The \( \frac{1}{2}(N^{cd}\tilde{N}_{cd} - N^{c'd'}\tilde{N}_{c'd'}) \) term in (3.1) comes from \( R^{ef}_{\alpha\beta} N^{cd} \tilde{N}^{ef}_{cd} \) where \( R^{ef}_{\alpha\beta} \) is the background Riemann tensor, and was mistakenly omitted in [1]. The need for such a term was first pointed out by Bershadsky [3] based on one-loop conformal invariance arguments. Also, we have changed our normalization of the second term in (3.1) so that our \( PSU(2,2|4) \) structure constants agree with those of [3].
The first line of (3.1) is the ten-dimensional version of the action proposed in [3] for $AdS_2 \times S^2$. As discussed in [3], it is one-loop conformally invariant and differs from the action of Metsaev and Tseytlin [5] in that $\kappa$-symmetry is replaced by the condition that the currents $\tilde{J}^\alpha$ and $\tilde{J}^\tilde{\alpha}$ are “covariantly” holomorphic and anti-holomorphic. In other words, the equations of motion from the first line imply that [3]

$$
\overline{\partial} J^\alpha = \frac{1}{2} [J^{[cd]}(\gamma_{cd} J)^\alpha + J^{[c^d]}(\gamma_{c^d} J)^\alpha] \quad \text{and} \quad \overline{\partial} \tilde{J}^{\tilde{\alpha}} = \frac{1}{2} [J^{[cd]}(\gamma_{cd} \tilde{J})^{\tilde{\alpha}} + J^{[c^d]}(\gamma_{c^d} \tilde{J})^{\tilde{\alpha}}].
$$

(3.4)

The second line of (3.1) is necessary so that the currents $\delta_{a\bar{a}}^\alpha \lambda^\alpha J^\alpha$ and $\delta_{a\bar{a}}^\tilde{\alpha} \tilde{\lambda}^\tilde{\alpha} J^\tilde{\alpha}$ satisfy

$$
\overline{\partial} (\delta_{a\bar{a}}^\alpha \lambda^\alpha J^\alpha) = 0 \quad \text{and} \quad \overline{\partial} (\delta_{a\bar{a}}^\tilde{\alpha} \tilde{\lambda}^\tilde{\alpha} J^\tilde{\alpha}) = 0,
$$

(3.5)

which will be used later for constructing the $AdS$ versions of the BRST charges $Q$ and $\overline{Q}$.

To prove (3.5), first note that the equations of motion for $\lambda^\alpha$ and $\tilde{\lambda}^{\tilde{\alpha}}$ are

$$
\overline{\partial} \lambda^\alpha = \left( \frac{1}{2} J^{[cd]} + \frac{1}{4} \tilde{N}^{cd} \right) (\gamma_{cd} \lambda)^\alpha + \left( \frac{1}{2} J^{[c^d]} - \frac{1}{4} \tilde{N}^{c^d} \right) (\gamma_{c^d} \lambda)^\alpha,
$$

(3.6)

$$
\partial \tilde{\lambda}^{\tilde{\alpha}} = \left( \frac{1}{2} J^{[cd]} + \frac{1}{4} N^{cd} \right) (\gamma_{cd} \tilde{\lambda})^{\tilde{\alpha}} + \left( \frac{1}{2} J^{[c^d]} - \frac{1}{4} N^{c^d} \right) (\gamma_{c^d} \tilde{\lambda})^{\tilde{\alpha}}.
$$

If one includes the contribution from the second line, the equations of motion of (3.4) for $\tilde{J}^\alpha$ and $\tilde{J}^\tilde{\alpha}$ get modified to

$$
\overline{\partial} J^\tilde{\alpha} = \frac{1}{2} [J^{[cd]}(\gamma_{cd} J)^\tilde{\alpha} + J^{[c^d]}(\gamma_{c^d} J)^\tilde{\alpha}]
$$

(3.7)

$$
+ \frac{1}{4} [\tilde{N}^{cd} (\gamma_{cd} J)^\tilde{\alpha} - \tilde{N}^{c^d} (\gamma_{c^d} J)^\tilde{\alpha} + N^{cd} (\gamma_{cd} \tilde{J})^{\tilde{\alpha}} - N^{c^d} (\gamma_{c^d} \tilde{J})^{\tilde{\alpha}}],
$$

$$
\overline{\partial} \tilde{J}^{\tilde{\alpha}} = \frac{1}{2} [J^{[cd]}(\gamma_{cd} \tilde{J})^{\tilde{\alpha}} + J^{[c^d]}(\gamma_{c^d} \tilde{J})^{\tilde{\alpha}}]
$$

$$
+ \frac{1}{4} [N^{cd} (\gamma_{cd} \tilde{J})^{\tilde{\alpha}} - N^{c^d} (\gamma_{c^d} \tilde{J})^{\tilde{\alpha}} + \tilde{N}^{cd} (\gamma_{cd} J)^\alpha - \tilde{N}^{c^d} (\gamma_{c^d} J)^\alpha].
$$

Putting (3.6) and (3.7) together, one finds

$$
\overline{\partial} (\delta_{a\bar{a}}^\alpha \lambda^\alpha J^\tilde{\alpha}) = - \frac{1}{4} (N^{cd} (\gamma_{cd} \lambda)^\alpha - N^{c^d} (\gamma_{c^d} \lambda)^\alpha) \delta_{a\bar{a}}^\alpha \tilde{J}^{\tilde{\alpha}},
$$

(3.8)

$$
\partial (\delta_{a\bar{a}}^\tilde{\alpha} \tilde{\lambda}^{\tilde{\alpha}} J^\alpha) = - \frac{1}{4} (\tilde{N}^{cd} (\gamma_{cd} \tilde{\lambda})^{\tilde{\alpha}} - \tilde{N}^{c^d} (\gamma_{c^d} \tilde{\lambda})^{\tilde{\alpha}}) \delta_{a\bar{a}}^\tilde{\alpha} J^\alpha.
$$

It will now be argued that the right-hand sides of these two equations vanish. As was explained in [4], one can write the Lorentz currents as $N^{cd} = \frac{1}{2} (\lambda \gamma^{cd} w)$ and $\tilde{N}^{cd} = \frac{1}{2} (\tilde{\lambda} \gamma^{cd} \tilde{w})$.
\[ \frac{1}{2}(\lambda \gamma^{cd}\w) \text{ where } \omega_{\alpha} \text{ and } \bar{\omega}_{\dot{\alpha}} \text{ are the conjugate momenta to } \lambda^{\alpha} \text{ and } \bar{\lambda}^{\dot{\alpha}}. \] And since \( \lambda^{\alpha} \lambda^{\beta} = \frac{1}{1920}(\lambda \gamma^{mn\rhoqr}\lambda) \gamma^{\alpha\beta}_{mn\rhoqr} \) and \( \bar{\lambda}^{\dot{\alpha}} \lambda^{\dot{\beta}} = \frac{1}{1920}(\lambda \gamma^{mn\rhoqr}\lambda) \gamma^{\dot{\alpha}\dot{\beta}}_{mn\rhoqr} \), the right-hand sides of (3.8) are proportional to

\[ \gamma^{cd}_m \gamma^{\cd}_n \gamma^c d^e_1 - \gamma^{c'd'}_e \gamma^{mnqr} \gamma^{e'd'}_n \] (3.9)

But one can easily check that (3.9) vanishes for any choice of the five-form direction \( mn\rhoqr \), implying that (3.5) is satisfied. The vanishing of (3.9) will also later be used to argue that the BRST charges \( Q \) and \( \bar{Q} \) anti-commute.

### 3.2. Physical vertex operators

Since \( \delta_{\alpha\dot{\alpha}} \lambda^{\alpha} \bar{\lambda}^\dot{\alpha} \) and \( \delta_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \lambda^{\alpha} \) are holomorphic and anti-holomorphic, one can construct the conserved charges

\[ Q = \oint d\bar{z} \delta_{\alpha\dot{\alpha}} \lambda^{\alpha} \bar{\lambda}^\dot{\alpha}, \] and \[ \bar{Q} = \oint d\bar{z} \delta_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \lambda^{\alpha} \] (3.10)

which will be used to define physical vertex operators in the \( AdS_5 \times S^5 \) background as was done for the flat background in section 2. On-shell fluctuations around the \( AdS_5 \times S^5 \) background are therefore described by vertex operators of the form \( U = \lambda^{\alpha} \bar{\lambda}^\dot{\alpha} A_{\alpha\dot{\alpha}}(x, \theta, \bar{\theta}) \) satisfying (2.6) where \( Q \) and \( \bar{Q} \) are defined in (3.10).

To evaluate the implications of (2.6), one has to know the OPE’s of \( Q \) and \( \bar{Q} \). Instead of trying to directly compute these OPE’s using the action of (3.1), it will be simpler to deduce them from the requirement that they preserve the AdS isometries and that they reduce correctly in the flat limit. As will be argued below, when acting on a ghost-number \((M, N)\) vertex operator \( \Phi = \lambda^{\alpha_1}...\lambda^{\alpha_M} \bar{\lambda}^{\dot{\beta}_1}...\bar{\lambda}^{\dot{\beta}_N} A_{\alpha_1...\alpha_M\dot{\beta}_1...\dot{\beta}_N}(x, \theta, \bar{\theta}) \),

\[ Q \Phi = \lambda^{\kappa} \lambda^{\alpha_1}...\lambda^{\alpha_M} \bar{\lambda}^{\dot{\beta}_1}...\bar{\lambda}^{\dot{\beta}_N} \nabla_\kappa A_{\alpha_1...\alpha_M\dot{\beta}_1...\dot{\beta}_N} \] and

\[ \bar{Q} \Phi = \lambda^{\alpha_1}...\lambda^{\alpha_M} \bar{\lambda}^{\dot{\beta}_1}...\bar{\lambda}^{\dot{\beta}_N} \nabla_\bar{\kappa} A_{\alpha_1...\alpha_M\dot{\beta}_1...\dot{\beta}_N} \] (3.11)

where \( \nabla_\alpha = E^M_\alpha (\partial_M + \omega_M) \) and \( \nabla_{\dot{\alpha}} = E^M_{\dot{\alpha}} (\partial_M + \omega_M) \) are the covariant supersymmetric derivatives in the \( AdS_5 \times S^5 \) background, and \( E^M_B \) and \( \omega_M \) are the super-vierbein and spin connection in the \( AdS_5 \times S^5 \) background with \( B \) ranging over the tangent-superspace indices \((c, \alpha, \bar{\alpha})\) and \( M \) ranging over the curved superspace indices \((m, \mu, \bar{\mu})\). One can express \( E^M_B \) and \( \omega_M \) in terms of the coset elements \( g(x, \theta, \bar{\theta}) \) by defining \( E^M_B = (E^M_M)^{-1} \) and \( \omega_{\alpha}^{[cd]} = E^M_{M}^{[cd]} \) where \((g^{-1}dg)^B = E^B_M dX^M \) for \( X^M = (x^m, \theta^\mu, \bar{\theta}^{\bar{\mu}}) \).
To justify (3.11), one can check that it is the unique definition which preserves all AdS isometries and reduces to \( Q\Phi = \lambda^\alpha D_\alpha \Phi \) and \( \overline{Q}\Phi = \lambda^\alpha D_\alpha \Phi \) in the flat limit. Furthermore, (3.11) is consistent with the nilpotency conditions \( Q^2 = \overline{Q}^2 = \{ Q, \overline{Q} \} = 0 \). The conditions \( Q^2 \Phi = \overline{Q}^2 \Phi = 0 \) follow from the fact that \( \gamma_{\alpha\beta\gamma}^{\alpha\beta\gamma} \{ \nabla_\alpha, \nabla_\beta \} = \gamma_{\alpha\beta\gamma}^{\alpha\beta\gamma} (\nabla_\alpha, \nabla_\beta) = 0 \). Although \( \{ \nabla_\alpha, \nabla_\beta \} \) is non-vanishing, its symmetrical structure allows \( \{ Q, \overline{Q} \} \Phi \) to vanish since

\[
\{ Q, \overline{Q} \} \Phi = \lambda^\alpha \hat{\lambda}^\alpha_1 ... \lambda^{\alpha M} \hat{\lambda}^\alpha_1 ... \hat{\lambda}^\beta_1 ... \hat{\lambda}^\delta N \{ \nabla_\kappa, \nabla_\tau \} A_{\alpha_1 ... \alpha M \beta_1 ... \beta N} (3.12)
\]

\[
= \lambda^\alpha \hat{\lambda}^\alpha_1 ... \lambda^{\alpha M} \hat{\lambda}^\alpha_1 ... \hat{\lambda}^\beta_1 ... \hat{\lambda}^\beta N \left( (\gamma_{\alpha \beta}^{\alpha \beta})_\kappa \delta^\sigma_{\tau} \nabla_{[cd]} - (\gamma_{\epsilon \delta}^{\epsilon \delta})_\kappa \delta^\sigma_{\tau} \nabla_{[cd]} \right) A_{\alpha_1 ... \alpha M \beta_1 ... \beta N}, \tag{3.13}
\]

where \( \nabla_{[cd]} \) acts as a Lorentz rotation in the \( [cd] \) direction on the \( M + N \) spinor indices of \( A_{\alpha_1 ... \alpha M \beta_1 ... \beta N} \), i.e.

\[
\nabla_{[cd]} A_{\alpha_1 ... \alpha M \beta_1 ... \beta N} = \frac{1}{2} \left( (\gamma_{cd})_{\alpha_1 \gamma} A_{\gamma \alpha_2 ... \beta N} + (\gamma_{cd})_{\alpha_2 \gamma} A_{\alpha_1 \gamma \alpha_3 ... \beta N} + ... + (\gamma_{cd})_{\beta N \gamma} A_{\alpha_1 ... \alpha M \beta_1 ... \beta N-1 \gamma} \right). \tag{3.14}
\]

But since all indices of \( A_{\alpha_1 ... \alpha M \beta_1 ... \beta N} \) are contracted with either \( \lambda^\alpha \) or \( \hat{\lambda}^\alpha \), one can use (2.7) to argue that all terms in (3.13) are proportional to (3.9) which identically vanishes. So \( \{ Q, \overline{Q} \} \Phi = 0 \) as desired.

Using (3.11), (2.6) implies that the bispinor superfield \( A_{\alpha \beta} (x, \theta, \hat{\theta}) \) in the physical vertex operator \( U = \lambda^\alpha \hat{\lambda}^\beta A_{\alpha \beta} \) must satisfy the equations of motion and gauge invariances

\[
\gamma_{\alpha \beta \gamma}^{\alpha \beta \gamma} \nabla_\alpha A_{\beta \gamma} = \gamma_{\alpha \beta \gamma}^{\alpha \beta \gamma} \nabla_\alpha A_{\beta \gamma} = 0, \tag{3.15}
\]

\[
\delta A_{\beta \gamma} = \nabla_\beta \hat{\Omega}_\gamma + \nabla_\gamma \Omega_\beta \quad \text{with} \quad \gamma_{\alpha \beta \gamma}^{\alpha \beta \gamma} \nabla_\alpha \Omega_\beta = \gamma_{\alpha \beta \gamma}^{\alpha \beta \gamma} \nabla_\alpha \hat{\Omega}_\gamma = 0.
\]

Although one could do a component analysis to check that (3.13) correctly describes the on-shell fluctuations around the AdS5 \( \times \) S5 background, this is guaranteed to work since (3.13) are the unique equations of motion and gauge invariances which are invariant under the AdS isometries and which reduce to the massless Type IIB supergravity equations of (2.8) in the flat limit.

4. Primary Vertex Operators

It is sometimes convenient to choose a gauge for physical vertex operators such that they are dimension zero worldsheet primary fields, i.e. they have no double poles with the stress tensor. For the ordinary closed bosonic string, this is the gauge where \( b_0 U = \overline{b}_0 U = 0 \). In the pure spinor formalism, there is no natural candidate for the background, so one needs to analyze the stress-tensor to find this gauge-fixing condition. This will be done first in a flat background and then in the AdS5 \( \times \) S5 background.
4.1. Primary vertex operators in a flat background

In a flat background, the left and right-moving stress tensors are

\[ T = \frac{1}{2} \partial x^m \partial x_m + p_\alpha \partial \theta^\alpha + T_\lambda \quad \text{and} \quad \overline{T} = \frac{1}{2} \overline{\partial x^m} \overline{\partial x_m} + \overline{p}_\alpha \overline{\partial \theta^\alpha} + \overline{T}_\lambda, \]  

(4.1)

where \( T_\lambda \) and \( \overline{T}_\lambda \) are the \( c = 22 \) stress-tensors constructed from the pure spinor variables \( \lambda^\alpha \) and \( \hat{\lambda}^{\dot{\alpha}} \) \([1]\). When acting on the massless vertex operator \( U = \lambda^\alpha \hat{\lambda}^{\dot{\alpha}} A_{\alpha\dot{\beta}}(x, \theta, \dot{\theta}) \), the condition of no double poles with \( T \) or \( \overline{T} \) implies that \( \partial_m \partial^m A_{\alpha\dot{\beta}} = 0 \). Furthermore, the on-shell conditions of (1.1) imply that \( \partial \lambda \in \delta A \) on-shell conditions of (1.1) imply that

\[ \partial \lambda \Rightarrow \delta A \in \hat{\lambda}^{\dot{\beta}} \]  

and

\[ \lambda \Rightarrow \delta A \in (\lambda, \dot{\lambda}) \]  

(4.2)

where \( A_{m\gamma} = \frac{1}{16} \gamma^m \beta_{A\beta\gamma} \) and \( A_{nm} = \frac{1}{16} \gamma^m \beta_{A\beta\gamma} \). So the gauge-fixed equations for \( A_{\alpha\dot{\beta}} \) are

\[ \partial^m \partial_m A_{\alpha\dot{\beta}} = 0, \quad \partial^m A_{m\beta} = \partial^m A_{m\dot{\beta}} = 0. \]  

(4.3)

Under the gauge transformation \( \delta A_{m\beta} = D_\beta \hat{\omega}_m - \hat{D} \hat{\omega}_m, \delta A_{m\dot{\beta}} = \delta A_{m\beta} \) and \( \delta A_{nm} = \delta A_{m\dot{\beta}} - D_\beta \hat{\omega}_m \) where \( \hat{\omega} = \frac{1}{16} \gamma^m D_\beta \hat{\omega}_m \) and \( \hat{\omega} = \frac{1}{16} \gamma^m D_\beta \hat{\omega}_m \). So the residual gauge transformations which leave (1.2) invariant are

\[ \delta A_{m\beta} = D_\beta \hat{\omega}_m + D_\beta \hat{\omega}_m \]  

with

\[ \partial_m \partial^m \hat{\omega}_m = \partial_m \partial^m \hat{\omega}_m = \partial^m \hat{\omega}_m = \partial^m \hat{\omega}_m = 0. \]  

(4.4)

The conditions of (1.2) and (1.3) will now be generalized in an \( AdS_5 \times S^5 \) background.

4.2. Primary vertex operators in an \( AdS_5 \times S^5 \) background

In the \( AdS_5 \times S^5 \) background, the left and right-moving stress tensors associated with the action of (1.1) are

\[ T = \frac{1}{2} (\eta_{cd} J_c J^d + \eta_{c'd'} J_{c'} J^{d'}) - 4 \delta_{\alpha\dot{\beta}} J^{\alpha\dot{\beta}} + N_{cd} J^{[cd]} + N_{c'd'} J^{[c'd']} + T_\lambda, \]  

(4.5)

\[ \overline{T} = \frac{1}{2} (\eta_{cd} \overline{J}_c \overline{J}^d + \eta_{c'd'} \overline{J}_{c'} \overline{J}^{d'}) - 4 \delta_{\alpha\dot{\beta}} \overline{J}^{\alpha\dot{\beta}} + \overline{N}_{cd} \overline{J}^{[cd]} + \overline{N}_{c'd'} \overline{J}^{[c'd']} + \overline{T}_\lambda \]

where \( T_\lambda \) and \( \overline{T}_\lambda \) are defined as in (1.1). As was done earlier with \( Q \) and \( \overline{Q} \), instead of directly computing the OPE’s of \( T \) and \( \overline{T} \), it will be simpler to deduce them from the requirements that they preserve the \( AdS \) isometries and reduce correctly in the flat limit.

When acting on the physical vertex operator \( U = \lambda^\alpha \hat{\lambda}^{\dot{\alpha}} A_{\alpha\dot{\beta}}(x, \theta, \dot{\theta}) \), the condition of no double poles with \( T \) or \( \overline{T} \) implies that

\[ \nabla_B \nabla^B A_{\alpha\dot{\beta}}(x, \theta, \dot{\theta}) = 0. \]  

(4.6)
where $\nabla_B \nabla^B = \eta^{BC} \nabla_B \nabla_C$, $B = (c, [cd], \alpha, \hat{\alpha})$ ranges over all tangent space indices of $PSU(2,2|4)$, $\nabla_B = E_B^M (\partial_M + \omega_M)$ are the covariant derivatives in the $AdS_5 \times S^5$ background when $B = (c, c', \alpha, \hat{\alpha})$, and $\nabla_{[cd]}$ acts as a Lorentz rotation in the $cd$ direction on all tangent space indices. The only non-zero components of $\eta^{BC}$ are

$$\eta^{\hat{c}\hat{d}} = -\eta^c_d = -\frac{1}{4} \delta^{\hat{c}\hat{d}}, \quad \eta^{[cd][ef]} = -\frac{1}{4} \eta^{\hat{c} [e \eta] d}, \quad \eta^{[c'd'] [e' f']} = \frac{1}{4} \eta^{e' [e' \eta] f' d'}, \quad (4.6)$$

and $\eta^{cd}$ is the $SO(1,9)$ Minkowski metric. Note that $[\nabla_A, \nabla_B] = f^{AB}_C \nabla_C$ where $f^{AB}_C$ are the $PSU(2,2|4)$ structure constants defined in (3.2), and $[\nabla_B \nabla^B, \nabla_C] = 0$ for all $C$. In the flat limit, $\nabla_B \nabla^B$ reduces to $\partial_\gamma \partial^n$ since $\eta^{cd}$ is the only surviving component of $\eta^{BC}$.

To find the $AdS_5 \times S^5$ analog of (4.2), it will be convenient to define $A_{\beta\gamma}$ by

$$A_{\bar{c}\bar{\gamma}} = \frac{1}{16} \gamma^{\alpha \beta} \nabla_{\alpha} A_{\beta \gamma}, \quad A_{\bar{\gamma} \bar{c}} = \frac{1}{4} \delta^{\hat{c}\hat{d}} \gamma_{\alpha \beta\gamma} (\nabla_{\alpha} A_{\beta \gamma} - \nabla_{\beta} A_{\alpha \gamma}) + \gamma_{\alpha \beta\gamma} (\nabla_{\gamma} A_{\beta \gamma} - \nabla_{\gamma} A_{\alpha \gamma}), \quad (4.7)$$

Under the gauge transformation $\delta A_{\alpha \beta} = \nabla_{\alpha} \hat{\Omega}_{\beta} - \nabla_{\beta} \Omega_{\alpha}$, one can check that $A_{\beta\gamma}$ transforms as

$$\delta A_{\beta\gamma} = \nabla_{\gamma} \hat{\Omega}_{\beta} - (1) (B) \nabla_{\gamma} \Omega_{B} + (1) (E) f^{D}_{\beta \gamma} \eta^{CE} \eta_{BD} \Omega_{E} \quad (4.8)$$

where $(B) = 0$ if $B$ is a bosonic index, $(B) = 1$ if $B$ is a fermionic index, and

$$\Omega_{\gamma} = \frac{1}{16} \gamma^{\alpha \beta} \nabla_{\alpha} \Omega_{\beta}, \quad \Omega_{\bar{c}} = \frac{1}{5} \delta^{\hat{c}\hat{d}} \gamma_{\alpha \beta\gamma} (\nabla_{\alpha} \Omega_{\beta} - \nabla_{\beta} \Omega_{\alpha}) + \gamma_{\alpha \beta\gamma} (\nabla_{\gamma} \Omega_{\beta} - \nabla_{\gamma} \Omega_{\alpha}), \quad (4.9)$$

One can similarly define $A_{\alpha \beta}$ as

$$A_{\bar{c}} = \frac{1}{16} \gamma^{\hat{c}\hat{d}} \nabla_{\hat{c}} A_{\hat{d}} \gamma_{\hat{b}} \hat{d}, \quad A_{\bar{c} \alpha} = \frac{1}{5} \delta^{\hat{c}\hat{d}} \gamma_{\alpha \beta\gamma} (\nabla_{\alpha} A_{\gamma \beta} - \nabla_{\beta} A_{\gamma \alpha}) + \gamma_{\alpha \beta\gamma} (\nabla_{\gamma} A_{\beta \gamma} - \nabla_{\gamma} A_{\alpha \gamma}), \quad (4.10)$$

which transforms as

$$\delta A_{\alpha \beta} = \nabla_{\beta} \Omega_{\alpha} - (1) (B) \nabla_{\alpha} \hat{\Omega}_{B} + (1) (E) f^{D}_{\alpha \beta} \eta^{CE} \eta_{BD} \hat{\Omega}_{E} \quad (4.11)$$

where

$$\hat{\Omega}_{\gamma} = \frac{1}{16} \gamma^{\hat{c}\hat{d}} \nabla_{\hat{c}} \hat{\Omega}_{\hat{d}} \gamma_{\hat{b}} \hat{d}, \quad \hat{\Omega}_{\alpha} = \frac{1}{5} \delta^{\hat{c}\hat{d}} \gamma_{\alpha \beta\gamma} (\nabla_{\alpha} \hat{\Omega}_{\beta} - \nabla_{\beta} \hat{\Omega}_{\alpha}) + \gamma_{\alpha \beta\gamma} (\nabla_{\gamma} \hat{\Omega}_{\beta} - \nabla_{\gamma} \hat{\Omega}_{\alpha}), \quad (4.12)$$
Then the unique conditions on $A_{\alpha \beta}$ which preserve the $AdS$ isometries and which reduce to (4.2) in the flat limit are

$$\nabla^B \nabla_B A_{\alpha \beta} = 0, \quad \nabla^B A_{B \beta} = \nabla^B A_{\alpha B} = 0.$$  \hspace{1cm} (4.13)

Furthermore, using the gauge transformations of (4.8) and (4.11) one can check that these conditions are invariant under the residual gauge transformations

$$\delta A_{\beta \gamma} = \nabla_\beta \tilde{\Omega}_\gamma + \nabla_\gamma \tilde{\Omega}_\beta \quad \text{with} \quad \nabla_B \nabla^B \Omega_\alpha = \nabla_B \nabla^B \tilde{\Omega}_\alpha = \nabla^B \Omega_B = \nabla^B \tilde{\Omega}_B = 0,$$  \hspace{1cm} (4.14)

which reduce to (4.13) in the flat limit. So the conditions of (4.13) and (4.14) for the primary vertex operator describing on-shell fluctuations of the $AdS_5 \times S^5$ background closely resemble the conditions of (4.2) and (4.3) for the primary vertex operator describing the massless Type IIB supergravity fields in a flat background.

5. Concluding Remarks

In this paper we have shown that the structure of physical vertex operators for on-shell fluctuations around the $AdS_5 \times S^5$ background closely resembles the structure of massless vertex operators in a flat background. This is true both for the gauge-invariant form of the vertex operators and for the gauge-fixed form of the primary vertex operators.

The next step is to compute tree-level scattering amplitudes involving the $AdS_5 \times S^5$ vertex operators. Since massless tree amplitudes are straightforward to compute in a flat background using the pure spinor formalism \cite{1}, it is quite encouraging that the physical vertex operators corresponding to on-shell fluctuations around the $AdS_5 \times S^5$ background are described by superfields which closely resemble those in a flat background. Of course, to compute superstring scattering amplitudes, one also needs to know the OPE's of the worldsheet fields. In a flat background, they are free-field OPE's, however, in an $AdS_5 \times S^5$ background, the OPE's are complicated because of non-holomorphicity of the currents. Nevertheless, it might be possible to deduce their form by requiring that they are invariant under the AdS isometries and that they reduce to the appropriate free-field OPE's in the flat limit.

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