We study the quantum matrix algebra $R_{21}x_1x_2 = x_2x_1R$ and for the standard $2 \times 2$ case propose it for the co-ordinates of $q$-deformed Euclidean space. The algebra in this simplest case is isomorphic to the usual quantum matrices $M_q(2)$ but in a form which is naturally covariant under the Euclidean rotations $SU_q(2) \otimes SU_q(2)$. We also introduce a quantum Wick rotation that twists this system precisely into the approach to $q$-Minkowski space based on braided-matrices and their associated spinorial $q$-Lorentz group.

1 Introduction

The problem of constructing a $q$-deformed or non-commutative analogue of spacetime co-ordinates has been extensively studied in recent years. One established approach is based on the idea that $q$-Minkowski space should be modelled by $2 \times 2$ braided matrices\cite{1,2}, which approach turned out to be compatible also with the approaches of \cite{3,4,5} based on other ideas of spinor decomposition and with \cite{8} based on the idea that the $q$-Lorentz group should be the quantum double. The braided matrix approach has been developed in detail by U. Meyer\cite{7} who introduced a (braided) coaddition law for $q$-Minkowski space 4-vectors as well as clarifying covariance and other properties. The connection with the quantum double point of view was established in \cite{8, Sec.4} while the connection with the spinorial approach was established in \cite{4}. The identification of the corresponding $q$-Lorentz group quantum enveloping algebra as a ‘twisted square’\cite{10} was introduced by the author in \cite{8, Sec. 4}.

In this paper we supplement this line of development with a fully compatible proposal for $q$-Euclidean space. It is obvious from the above context that $SU_q(2) \otimes SU_q(2)$ should...
be the corresponding rotation quantum group in the Euclidean case within a $2 \times 2$ matrix approach, which will be our point of view also. Not known however, is what should play the role of $q$-Euclidean space itself. We need once again a matrix algebra, but neither the braided-matrices (which are naturally hermitian) nor $2 \times 2$ quantum matrices in their usual form (which have the right $\ast$-structure but are not properly covariant under the obvious matrix action) will do. Rather, we give a general construction in which we start with a copy of the usual quantum matrices $A(R)$ with generators $t$ and define on their linear space a new product. The new algebra $\bar{A}(R)$ has generators $x = t$ but different products according to

$$x_1 \cdot x_2 = \lambda_0 R t_1 t_2$$

giving the commutation relations in the abstract. This is given in Section 3 of the paper.

Our general strategy here is to introduce a new technique for ‘covariantisation’ by twisting of module algebras and comodule algebras. It is well-known that a 2-cocycle on a quantum group allows one to ‘twist’ the quantum group\[[11]\] to a new one by conjugation. Less well known is that this same cocycle can also be used to twist anything on which the quantum group acts. This general theory is recalled first in Section 2.

In Section 4, we study the properties of this algebra $\bar{A}(R)$ in the standard $2 \times 2$ case. Thus we apply our construction to the usual quantum matrices $M_q(2)$ of \[[12]\]\[[13]\] and obtain our algebra $\bar{M}_q(2)$ which we call $\mathbb{R}^4_q$ or $q$-Euclidean space. It turns out to be isomorphic as an algebra to the usual $M_q(2)$, but not as a $\ast$-algebra or coalgebra.

Finally, in Section 5 we show that our twisting theory for algebras on which quantum groups act, can be used once more to twist our $q$-Euclidean algebra into exactly the above-mentioned $q$-Minkowski space algebra. The twisting cocycle also takes $SU_q(2) \otimes SU_q(2)$ into the $q$-Lorentz group in the correct way. In this way we have a precise notion of quantum Wick-rotation.

Note that in our original work\[[1]\], we obtained the braided-matrices algebra by a novel procedure of ‘transmutation’ from the usual quantum matrices. Thus too involves a new algebra $B(R)$ built on $A(R)$ with generators $u = t$ but a new ‘transmuted’ product\[[14]\]. At a mathematical level, our new result is to express such transmutation of a quantum group into a braided one as a two-step twisting. The first twisting gives $q$-Euclidean space and the second twisting gives $q$-Minkowski space.
Preliminaries: Twisting of Comodule Algebras

We begin with some general constructions for quantum groups, needed in Section 3 and again in Section 5. By quantum group we mean a quasitriangular Hopf algebra $H$ of enveloping algebra type (with universal R-matrix $R$ [12]). Later on we will cover also the case of a dual-quasitriangular Hopf algebra $A$ (of function algebra type) where this time $R : A \otimes A \rightarrow \mathbb{C}$ is the ‘universal R-matrix functional’ [15].

Here $H$ is assumed to have a coproduct $\Delta$, counit $\epsilon$, and an antipode $S$ etc. A 2-cocycle in $H$ is an invertible element $\chi \in H \otimes H$ such that

$$\chi_{23}(id \otimes \Delta)\chi = \chi_{12}(\Delta \otimes id)\chi, \quad (\epsilon \otimes id)\chi = 1.$$  \hspace{1cm} (1)

In this case an easy application of ideas of Drinfeld in [11] tells us that the same algebra with

$$\Delta_\chi = \chi(\Delta)\chi^{-1}, \quad R_\chi = \chi_2 R \chi^{-1}, \quad S_\chi = U(S) U^{-1}$$ \hspace{1cm} (2)

remains a quasitriangular Hopf algebra, which we denote $H_\chi$. Here $U = \chi^{(1)}(S\chi^{(2)})$. This is well-known by now. One of the first places where it was explicitly studied in this Hopf algebra setting is [14].

To this theory, we need to add the following less well-known observation: If $B$ is an algebra on which $H$ acts covariantly in the sense

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \quad h \triangleright 1 = \epsilon(h)1$$ \hspace{1cm} (3)

then it easy to see from the 2-cocycle condition that

$$a \cdot_\chi b = \cdot \chi^{-1} \triangleright (a \otimes b)$$ \hspace{1cm} (4)

defines a new associative algebra which we call $B_\chi$, and that this is covariant in the sense of (3) under $H_\chi$. So the covariant system consisting of $(H, B)$ is completely ‘gauge transformed’ to $(H_\chi, B_\chi)$. To see the covariance of the new system, one needs only to check

$$h \triangleright (a \cdot_\chi b) = h \triangleright \cdot \chi^{-1} \triangleright (a \otimes b) = \cdot ((\Delta h)\chi \triangleright (a \otimes b)) = \cdot (\chi^{-1}(\Delta h) \triangleright (a \otimes b)) = \cdot \chi ((\Delta h) \triangleright (a \otimes b))$$

as required.
In practice, it is convenient for us not to work with quantum enveloping algebras but with quantum function algebras. In this case we need the dual theory to that above. Thus a 2-cocycle on $A$ is $\chi : A \otimes A \to C$ such that
\[
\chi(b_{(1)} \otimes c_{(1)})\chi(a \otimes b_{(2)}c_{(2)}) = \chi(a_{(1)} \otimes b_{(1)})\chi(a_{(2)}b_{(2)} \otimes c)
\]
for all $a, b, c$. This exactly generalises a group 2-cocycle. We assume $\chi$ is convolution-invertible in the sense that there is $\chi^{-1}$ which is inverse relative to $\chi$ to $\Delta$ in a standard way. We also assume that $\chi(a \otimes 1) = \chi(1 \otimes a) = \epsilon(a)$. Then the dual of Drinfeld’s twisting is that
\[
\begin{align*}
    a \cdot \chi b &= \sum \chi(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)}\chi^{-1}(a_{(3)} \otimes b_{(3)}) \\
    R_{\chi}(a \otimes b) &= \sum \chi(b_{(1)} \otimes a_{(1)})R(a_{(2)} \otimes b_{(2)})\chi^{-1}(a_{(3)} \otimes b_{(3)}) \\
    S_{\chi}(a) &= \sum U(a_{(1)})Sa_{(2)}U^{-1}(a_{(3)}), \quad U(a) = \sum \chi(a_{(1)} \otimes Sa_{(2)}).
\end{align*}
\]
is also a dual-quasitriangular Hopf algebra $A_{\chi}$ say. Likewise, if $B$ is an algebra coacted upon by $A$ in a covariant way (a comodule-algebra) then
\[
    c \cdot \chi d = \chi^{-1}(c_{(1)} \otimes d_{(1)})c_{(2)}d_{(2)}
\]
is a new algebra $B_{\chi}$ coacted upon covariantly by $A_{\chi}$. The covariance condition is that the coaction $\beta$ should be an algebra homomorphism. Thus the covariant system $(A, B)$ is completely ‘gauge transformed’ to the new system $(A_{\chi}, B_{\chi})$ with the same coaction $\beta$.

An introduction to the abstract terminology here is in [17], while a recent paper on the theory of twisting is in [18].

3 Construction of quantum algebras $\bar{A}(R)$

Every Hopf algebra $A$ coacts on itself by the coproduct $\Delta$ both from the left and from the right. If we are interested only in coactions from the right, we can convert the left coaction to a right one as follows. Firstly
\[
    \beta(a) = a_{(2)} \otimes Sa_{(1)}
\]
is a right coaction, but it is not covariant (a comodule algebra) under $A$ but rather under $A^{\text{op}}$ because the antipode $S$ is an anti-homomorphism.
Proposition 3.1 Consider $A$ as a comodule algebra under $A^{\text{op}}$ by (11). Consider $\chi = R^{-1}$ as a 2-cocycle on $A^{\text{op}}$. Then the twisted covariant system to $(A^{\text{op}}, A)$ is $(A, \bar{A})$ where $\bar{A}$ is $A$ equipped with the new product

$$c \cdot d = R(c_{(1)} \otimes d_{(1)})c_{(2)}d_{(2)}$$

Proof Here $R^{-1}$ is a dual-quasitriangular structure on $A^{\text{op}}$ because $R$ is one on $A$. Twisting by it gives the product of $A^{\text{op}}$ from (6) as $a \cdot \chi b = \sum R^{-1}(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)}R(a_{(3)} \otimes b_{(3)}) = ab$ since a dual-quasitriangular Hopf algebra is commutative up to conjugation by $R$ (it obeys axioms dual to those of Drinfeld in [12]), cf [19] for twisting a quantum enveloping algebra to its opposite coproduct. On the other hand, the copy of $A$ on which $A^{\text{op}}$ acted is also twisted and from (9) its becomes $\bar{A}$ as stated. The theory in the preliminaries ensures that $\bar{A}$ is an associative algebra and that $A$ coacts on it by (10). $\square$

Note that it is not really necessary here for the $A$ that is acted upon to be a quantum group in that the formulae for $\bar{A}$ works also at the bialgebra level. The acting copy needs to be a quantum group for (10) to be defined. We consider now the simplest matrix quantum group case. Thus consider the bialgebras $A(R)$ as in [13] [12] for $R$ a solution of the QYBE (Quantum Yang-Baxter Equations). We suppose that $A(R)$ also has a quotient $A$ which is a dual-quasitriangular Hopf algebra with antipode. The dual quasitriangular structure is the one introduced by the author in an equivalent form in [17] and is defined by $R(t_1 \otimes t_2) = \lambda_0 R$ on the generators, extended uniquely to the whole algebra. Here $\lambda_0$ is a normalisation parameter and can be set equal to 1 if $R$ is given in the quantum group normalisation as explained in [14].

Proposition 3.2 Let $A(R)$ with matrix generator $z$ be coacted from the right by $A^{\text{op}}$ as

$$z^i_j \mapsto z^a_j \otimes S t^a_i, \text{ i.e. } z \mapsto \bar{t}^{-1}z$$

where $\bar{t}$ is the matrix generator of $A^{\text{op}}$. Then this covariant system $(A^{\text{op}}, A(R))$ twists under $\chi = R^{-1}$ to the covariant system $(A, \bar{A}(R))$ where $\bar{A}(R)$ has generators $x$ and relations and covariance

$$R_{21} x_1 x_2 = x_2 x_1 R, \quad x \mapsto t^{-1}x$$

under the usual matrix generator $t$ of $A$. 

5
Proof We are in the setting of Section 2 with the initial system being the quantum group $A^\text{op}$ coacting on $B = A(R)$ as stated. We take twisting cocycle $\chi = R^{-1}$ and twisting by this, the acting quantum group $A^\text{op}$ and the acted-upon $A(R)$ have new products

$$\tilde{t}_1 \cdot \chi \tilde{t}_2 = R^{-1} \tilde{t}_1 \tilde{t}_2 R = \tilde{t}_2 \tilde{t}_1 = R_{21} \tilde{t}_2 \cdot \chi \tilde{t}_1 R_{21}^{-1}$$

$$z_1 \cdot \chi z_2 = \lambda_0 R z_1 z_2 = z_2 z_1 \lambda_0 R = R_{21}^{-1} z_2 \cdot \chi z_1 R.$$

We see that $A^\text{op}$ with its generators $\tilde{t}$ becomes twisted back to the usual $A$ with matrix generators $t$ say. Meanwhile, the relations for $\tilde{A}(R)$ come out as stated with $x$ denoting $z$ with the new product. □

Before studying this new algebra $\tilde{A}(R)$ in detail, we consider now in the same spirit our general Hopf algebra $A$ as an $A^\text{op} \otimes A$-comodule algebra, where we view the left coaction as a right coaction of $A^\text{op}$ as in (10) and at the same time make a usual right coaction via $\Delta$. So the combined coaction is

$$\beta(a) = a_{(2)} \otimes S a_{(1)} \otimes a_{(3)}$$

(11)

We take on the Hopf algebra $A^\text{op} \otimes A$ the 2-cocycle

$$\chi((a \otimes b) \otimes (c \otimes d)) = R^{-1}(a \otimes c)e(b)e(d)$$

(12)

which is $R^{-1}$ on $A^\text{op}$ and trivial on $A$.

Proposition 3.3 The algebra $\tilde{A}$ of Proposition 3.1 is covariant under $A \otimes A$ coacting as in (11).

Proof The twisting of the covariant system $(A^\text{op} \otimes A, A)$ coacting as in (11) by the 2-cocycle in (12) is $(A \otimes A, \tilde{A})$ where the computation is strictly as in the proof of Proposition 3.1. The cocycle is trivial in the $A$ part of $A^\text{op} \otimes A$ so no further change is caused by this additional part of the coaction (11). Thus $\tilde{A}$ is as before. Also in the same way, twisting changes $A^\text{op} \otimes A$ to $A \otimes A$. □

In the quantum matrix example, we learn that $\tilde{A}(R)$ is covariant under

$$x^i_j \mapsto x^a_b \otimes (S s^i) t^b_j, \quad \text{i.e.} \quad x \mapsto s^{-1} xt, \quad [s, t] = 0$$

(13)
where $A \otimes A$ has matrix generators $s, t$ for its two factors.

Another general feature of interest to us concerns $\ast$-structures on our Hopf algebra or bialgebra $A$. A $\ast$-bialgebra means that the algebra is a $\ast$-algebra and $(\ast \otimes \ast) \circ \Delta = \Delta \circ \ast$. We say that a dual-quasitriangular structure on it is of real-type if $R(a \otimes b) = R(b^* \otimes a^*)$. We have

**Proposition 3.4** If $A$ is a $\ast$-bialgebra and $R$ real then $\bar{A}$ is a $\ast$-algebra too with the same $\ast$-operation

**Proof** We have that $\bar{A}$ is a $\ast$-algebra as

$$(a \bar{\cdot} b)^* = \overline{R(a(1) \otimes b(1))b^*(2)a^*(2)} = b^* \bar{\cdot} a^*$$

precisely because $R$ is of real type. □

Finally, we remark that there is an operator $\psi : \bar{A} \otimes \bar{A} \to \bar{A} \otimes \bar{A}$ defined by

$$\psi(a \otimes b) = b(1) \otimes R^{-1}(a(1) \otimes b(2))a(2)$$

such that

$$(\bar{\cdot} \otimes \text{id}) \circ \psi_{23} \circ \psi_{12} = \psi \circ (\text{id} \otimes \bar{\cdot}), \quad (\text{id} \otimes \bar{\cdot}) \circ \psi_{12} \circ \psi_{23} = \psi \circ (\bar{\cdot} \otimes \text{id})$$

where we apply $\psi$ as shown and multiply its output from the left by $a(1)$ and from the right by $b(2)$, using the product of $\bar{A}$. This is an elementary computation from the definitions above and the axioms of a dual-quasitriangular structure. It means that $(\bar{A}, \Delta, \psi)$ (the new product and the original coproduct) is something like a braided-group where the tensor products of $\bar{A}$ must be treated with non-commuting statistics according to

$$(a \otimes b)(c \otimes d) = a \bar{\cdot} \psi(b \otimes c) \bar{\cdot} d.$$  

Such a product law for tensor products of $\bar{A}$ is characteristic of a braided tensor product but does not in fact require $\psi$ to be a braiding. We need only (13) in order for this to give an associative algebra on $\bar{A} \otimes \bar{A}$. Then (13) says that $\Delta$ is an algebra homomorphism with respect to this product. In general, $\psi$ in (13) does not obey the QYBE nor commute.
in the required way with $\Delta$ for a true braided-group (or braided-Hopf algebra) so we have something slightly weaker.

In the quantum matrix example it means that we have an operator

$$\psi(x_1 \otimes x_2) = x_2 \otimes \lambda_0^{-1} R^{-1} x_1, \quad \text{i.e.} \quad x'_1 x_2 = x_2 \lambda_0^{-1} R^{-1} x'_1$$  \hspace{1cm} (18)$$

which extends to products via (15) and defines the tensor product algebra for two copies $x, x'$ say with relations as stated. The coproduct $\Delta x = x \otimes x$ is an algebra homomorphism to this algebra, i.e., $x'' = xx'$ obeys the relations of $\bar{A}(R)$ if $x, x'$ do and have the cross relations shown. On the other hand, $\psi$ does not in general obey the QYBE (and moreover is not covariant under (13)). This weaker structure can be viewed as falling in between the usual quantum matrices bialgebra $A(R)$ with coproduct $\Delta t = t \otimes t$ and the braided matrices $B(R)$ with coproduct $\Delta u = u \otimes u$ into which it will become under the q-Wick rotation in Section 5.

More important for physics is not the multiplication law, but the addition law. Here we do have a braided-group whenever $R$ is Hecke with eigenvalues $q, -q^{-1}$ say for the associated braiding operator. The additive braiding and braid-statistics are then

$$\Psi(x_1 \otimes x_2) = R(x_2 \otimes x_1) R, \quad \text{i.e.} \quad x'_1 x_2 = R x_2 x'_1 R$$  \hspace{1cm} (19)$$

extending to products as a braiding and corresponding to the additive braid statistics shown. Thus $x'' = x + x'$ obeys the relations of $\bar{A}(R)$ if $x, x'$ do and have the mutual relations in (19). This is proven following exactly the same steps as in [8][20] for the braided-addition law of $A(R)$, to which we refer the reader for details.

**Proposition 3.5** The addition law on $\bar{A}(R)$ with the additive braiding (14) is covariant under the coaction (13).

**Proof** We verify covariance as

$$(s_1^{-1} x'_1 t_1)(s_2^{-1} x_2 t_2) = s_1^{-1} s_2^{-1} x'_1 x_2 t_1 t_2 = s_1^{-1} s_2^{-1} R x_2 x'_1 R t_1 t_2 = R s_2^{-1} s_1^{-1} x_2 x'_1 t_2 t_1 R = R(s_2^{-1} x_2 t_2)(s_1^{-1} x'_1 t_1) R$$

using the relations in each algebra and $[s, t] = 0$. Another way to see this covariance is to write a multi-index R-matrix $R_{i+}^{j-} k^L = R_{i_0}^{j_0, i_0 k_0} R_{j_1, k_1}^{i_1, x'_1 t_1}$ so that the braiding is the canonical form for a braided covector space with $x_I = x'^{0} i_1$ in the framework introduced in [3]. This is similar to the strategy used in [20] for addition of usual quantum matrices. One can easily see that $A(R_+)$ with matrix generator $\Lambda_{i,j}^I$ maps onto $A \otimes A$ by $\Lambda_{i,j}^I = s^{-1} x'^{0} i_1$ and its matrix right coaction
\[ x_J \mapsto x_J A^I J \] (with respect to which the braiding is necessarily covariant) maps onto (13). \( \Box \)

Armed with this covariance, one has at once a \( q \)-Poincaré group according to the general construction in [8, Theorem 6]. This is generated by \( \mathbf{p} \) (a copy of \( \mathbf{x} \) but regarded as momentum) and \( \Lambda \) with cross relations from \( \mathbb{R}_+ \). There is a dilaton element \( \varsigma \) for \( \lambda_0 \neq 1 \). The spinorial version follows the same lines with the rotations generated now by \( \mathbf{s}, \mathbf{t} \) and cross relations from \( \mathcal{R} \). The key step is to note that \( A \otimes A \) has a tensor product dual-quasitriangular structure consisting of \( \mathcal{R} \) on each factor. The coaction (13) becomes a right action of \( A \otimes A \) induced by evaluation against this as

\[
\mathbf{p}_1 \triangleleft \mathbf{t}_2 = \mathbf{p}_1 \lambda_0 \mathbf{R}, \quad \mathbf{p}_1 \triangleleft \mathbf{s}_2 = \lambda_0^{-1} \mathbf{R}^{-1} \mathbf{p}_1.
\]

(20)

The semidirect product \( (A \tilde{\otimes} A) \triangleleft A(\mathcal{R}) \) therefore has the corresponding cross relations

\[
\mathbf{p}_1 \mathbf{t}_2 = \lambda_0 \mathbf{t}_2 \mathbf{p}_1 \mathbf{R}, \quad \mathbf{p}_1 \mathbf{s}_2 = \mathbf{s}_2 \lambda_0^{-1} \mathbf{R}^{-1} \mathbf{p}_1, \quad \mathbf{p}_\varsigma = \lambda_0^{-2} \mathbf{p}.
\]

(21)

This computation follows the same steps as made recently for the \( q \)-Minkowski case in [9, Sec. 4] but we see that the relations are significantly simpler than there. The coproduct is the same, namely the standard cross coproduct by the coaction of the form (13). As in [9] this comes out as \( \Delta \mathbf{p} = \mathbf{p} \otimes \mathbf{s}^{-1}(\ ) \varsigma \mathbf{p} + \mathbf{p} \otimes \mathbf{s} \) where the indices of \( \mathbf{p} \) have to be inserted into the space.

### 4 \( q \)-Euclidean space

In this section we specialise to the standard \( R \)-matrix

\[
R = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & q & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix}
\]

(22)

and study \( A(\mathcal{R}) \) constructed in the last section for this case. The usual \( A(\mathcal{R}) \) is the usual quantum matrices \( M_q(2) \).

The relations are computed from Proposition 3.2 for a matrix \( \mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) as

\[
ba = qab, \quad ca = q^{-1}ac, \quad da = ad, \quad bc = cb + (q - q^{-1})ad \\
db = q^{-1}bd, \quad dc = qcd
\]
which we call the algebra of $q$-Euclidean space $\mathbb{R}^4_q = \bar{M}_q(2)$. We know from the last section that this algebra is fully covariant under the action of $SU_q(2) \otimes SU_q(2)$ by (13). This therefore plays the role of the 4-dimensional rotation group in this picture. We note in passing that $\bar{M}_q(2) \cong M_q(2)$ as an algebra by the permutation of generators
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \begin{pmatrix} c & d \\ a & b \end{pmatrix}.
\] (23)

Thus our construction does not give a genuinely new algebra in this simplest case. On the other hand, our matrix coaction (13) and other of our constructions would not appear very natural when mapped over to $M_q(2)$ by this identification.

Next, it is easy to see that the standard $2 \times 2$ quantum matrices $M_q(2)$ for real $q$ are a $*$-bialgebra
\[
\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix}
\] (24)
and its dual-quasitriangular structure cf[17] is of real type. Hence from Proposition 3.4 we know that our $q$-Euclidean space is also a $*$-algebra with the operation (24).

Moreover, a short computation gives that the element $\text{det} = ad - q^{-1}bc$ is central in the algebra $\mathbb{R}^4_q$ and, moreover self-adjoint. Thus we are motivated to take it as a natural 'metric'. If we make a change of variables
\[
t = \frac{a - d}{2t}, \quad x = \frac{c - qb}{2}, \quad y = \frac{c + qb}{2}, \quad z = \frac{a + d}{2}
\] (25)
then $t^* = t, x^* = x, y^* = y, z^* = z$ are some natural self-adjoint spacetime co-ordinates and
\[
\text{det} = ad - q^{-1}bc = (\frac{1 + q^{-2}}{2})t^2 + q^{-2}x^2 + q^{-2}y^2 + (\frac{1 + q^{-2}}{2})z^2
\] (26)
so that the signature of the natural metric is the Euclidean one.

The multiplication law works with the multiplicative statistics (18) between two copies, which come out as
\[
a' a = q^{-1}aa' + (q^1 - q)bc', \quad a'b = ba', \quad a'c = q^{-1}ca' + (q^{-1} - q)dc', \quad a'd = da'
\]
\[
b' a = q^{-1}ab' + (q^1 - q)bd', \quad b' b = bb', \quad b'c = q^{-1}cb' + (q^{-1} - q)dd', \quad b'd = db'
\]
\[
c' a = ac', \quad c'b = q^{-1}bc', \quad c'c = cc', \quad c'd = q^{-1}dc'
\]
\[
d' a = ad', \quad d' b = q^{-1}bd', \quad d' c = cd', \quad d'd = q^{-1}dd'
\]

times an overall factor $\lambda_0^{-1} = q^{1/2}$ on the right hand sides, which is not relevant for the multiplicativity property itself. The structure however, is not a usual bialgebra, nor a braided one since the exchange law $\psi$ underlying these statistics does not obey the QYBE.

Finally, there is a usual braided-Hopf algebra for addition with the additive braid statistics (13). In our case this (like the algebra itself) works out isomorphic under (23) to the additive braid statistics already obtained for $M_q(2)$ in [20], so we do not repeat its listing here. It means that our $q$-Euclidean 4-vectors, like $q$-Minkowski 4-vectors [7], can be added. Moreover, Proposition 3.5 assures us that this addition is covariant under $SU_q(2) \otimes SU_q(2)$ and that we have an associated 4-dimensional Euclidean $q$-Poincaré group as a semidirect product or bosonisation $SU_q(2) \tilde{\otimes} SU_q(2) \ltimes \mathbb{R}^4_q$ from (21). The multi-index generators $\Lambda^I_J$ there provide also a vectorial picture of the Euclidean rotation group $O_q(4)$ in our framework with corresponding $q$-Poincaré group $O_q(4) \ltimes \mathbb{R}^4_q$. In both cases the scaling parameter is $\lambda_0 = q^{1/2} \neq 1$ so a dilaton is needed. In this way, we obtain the analogous structure to that found for $q$-Minkowski space in [7, 9].

5 $q$-Wick Rotation and Transmutation as Double-Twisting

In this section we consider ourselves again in the general setting as in Section 3 and give a new example and application of the twisting theory of Section 2. Thus, consider $A$ a dual-quasitriangular Hopf algebra and this time begin with the covariant system $(A \otimes A, \bar{A})$ obtained in Proposition 3.3. This is our set-up for Euclidean space and its transformation group, for example.

Consider now the cocycle on $A \otimes A$ given by

$$\chi((a \otimes b) \otimes (c \otimes d)) = \epsilon(a) R^{-1}(b \otimes c) \epsilon(d). \quad (27)$$

One can easily see that it obeys the 2-cocycle condition (3). In fact, it is nothing other than the dual of the cocycle $R_{23}^{-1}$ used by [10] in their construction of a ‘twisted square’ Hopf algebra. We have given the relevant dual construction for the Hopf algebra $A \rtimes A$ in [8, Sec. 4] as an algebra built on $A \otimes A$ with the new product

$$(a \otimes b)(c \otimes d) = ac_{(2)} \otimes b_{(2)} dR_{23}^{-1}(b_{(1)} \otimes c_{(1)}) R(b_{(3)} \otimes c_{(3)}). \quad (28)$$
We also gave on it a $*$-structure $(a \otimes b)^* = b^* \otimes a^*$ in the real case and identified the spinorial $q$-Lorentz group of $[4]$ as a matrix example of this construction. This abstract formulation of the $q$-Lorentz group function algebra as the dual of the twisted square construction of $[4]$ was one of the main results of the author in $[8]$.

**Proposition 5.1** The covariant system $(A \otimes A, \bar{A})$ twists under the cocycle (27) to the covariant system $(A \bowtie A, A)$ where $A$ has the product

$$a \cdot b = a_{(2)} b_{(3)} R(a_{(3)} \otimes S b_{(1)}) R(a_{(1)} \otimes b_{(2)}).$$

Here $A \bowtie A$ is the double cross product Hopf algebra in $[8]$, Sec. 4] and $A$ is the braided group of function algebra type introduced in $[13]$.

**Proof** It is obvious that the twisted square in [1] with coproduct $\Delta_\chi = R_{23}^{-1}( ) R_{23}$ is an example of twisting with $\chi = R_{23}^{-1}$. The cocycle (27) is just the dual of this on our dual-quasitriangular Hopf algebra, and with the dual twisting it is obvious that we have $(A \otimes A)_\chi = A \bowtie A$ where $A \bowtie A$ has product (28). This can also be checked easily from (3). The new part of the proposition concerns the twisting of $\bar{A}$. We compute from (4) its product twisted by (27) as

$$a \cdot \chi b = \chi^{-1}(Sa_{(1)} \otimes a_{(3)} \otimes S b_{(1)} \otimes b_{(3)}) a_{(2)}^{-1} b_{(2)}$$

$$= R(a_{(2)} \otimes S b_{(1)}) a_{(1)}^{-1} b_{(2)} = R(a_{(3)} \otimes S b_{(1)}) R(a_{(1)} \otimes b_{(2)}) a_{(2)} b_{(3)}$$

as stated. $\Box$

The passage $A \mapsto \underline{A}$ was introduced in one step in [13] as a process of covariantisation or transmutation by means of a category-theoretical (braided) Tannaka-Kein reconstruction theorem. Our new mathematical result in this proposition is to factorise this complicated construction into two steps: the construction of $\bar{A}$ followed by its Wick rotation to $A$. Note also that $A \bowtie A$ maps onto $A$ by multiplication. In this case the coaction (11) under which $A$ is covariant becomes the standard quantum adjoint coaction.

We call this twisting ‘Wick rotation’ because for the standard R-matrix (22) the algebra $SU_q(2) \bowtie SU_q(2)$ is an established spinorial form of the $q$-Lorentz group as explained above, while the algebra $M_q(2) = BM_q(2)$ as given by transmuting the quantum matrices to braided ones [1] is an established $q$-Minkowski space [2] [3] [4] cf. [3] [4] [4]. Thus
our $q$-Euclidean space and the action of $O_q(4)$ in the form $SU_q(2) \otimes SU_q(2)$ gets precisely ‘Wick rotated’ by twisting to this established Minkowski space approach. Since the Euclidean picture is important for the rigorous construction of ordinary quantum field theory, it seems likely that our $q$-Euclidean picture will be useful too in this programme of $q$-deforming physics.

Finally, we note that there is an enveloping algebra version of all these results based on quasitriangular Hopf algebras $H$. This time we use a cocycle as in (1) to twist a coalgebra on which $H$ acts to a new coproduct

$$\Delta_\chi(c) = \chi^\rho \Delta c.$$  \hspace{1cm} (29)

The cocycle condition gives at once that this is coassociative. The twisted quantum group $H_\chi$ acts on it covariantly in the sense that the action is a coalgebra map (a module-coalgebra). In our specific application, we can take $H$ as a coalgebra acted upon by $H \otimes H^\text{cop}$ according to

$$(h \otimes g)\cdot c = hcSg$$  \hspace{1cm} (30)

which it the analogue of (11). Here $H^\text{cop}$ means with the opposite coalgebra. We take cocycle $\chi = R_{42}$ so that $(H \otimes H^\text{cop})_\chi = H \otimes H$. According to the general theory then, this acts covariantly on the new coalgebra $\bar{H}$ with coproduct

$$\bar{\Delta}c = R_{42}^\rho \Delta c = (\Delta c)(S \otimes S)R_{21} = (\Delta c)R_{21}$$  \hspace{1cm} (31)

as $R$ is invariant under $(S \otimes S)$. So far only the action of $H^\text{cop}$ was significant and we could obtain $\bar{H}$ as acted upon by $H$ equally well, by starting with $H^\text{cop}$ acting by $g\cdot c = cSg$ and cocycle $\chi = R_{21}$, as the analogue of Proposition 3.1. Next we take this covariant system $(H \otimes H, \bar{H})$ and twist again with cocycle $\chi = R_{32}$. This gives a new coalgebra $\tilde{H}$ with coproduct

$$\tilde{\Delta}c = R_{32}^\rho \tilde{\Delta}c = c_{(1)}R^{(2)}S\mathcal{R}^{(2)}(c_{(2)}R^{(1)} = c_{(1)}S\mathcal{R}^{(2)} \otimes \text{Ad} \mathcal{R}^{(1)}(c_{(2)})$$  \hspace{1cm} (32)

where Ad is the quantum adjoint action. Here $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ denotes the decomposition of $\mathcal{R}$ as an element of $H \otimes H$ and $\mathcal{R}'$ denotes a second copy of it. We obtain exactly the transmuted coproduct $\tilde{H}$ introduced in [21] and on it an action (30) of the quantum group $(H \otimes H)_\chi$ which is a version of the twisted square of [19] with twisting by $R_{32}$. Its
subalgebra $H$ also acts, by the quantum adjoint action. This is of course nothing other than our results above in a dualised form and a convenient left-right reversal. In principle, it is not necessary to repeat the proofs again since there are diagrammatic methods to make such dualisations (by turning the diagrams up-side-down).

In addition, there is nothing stopping the reader partly dualising our above results, namely working with $\bar{A}$ or $q$-Euclidean space (not dualised) and $H \otimes H$ or $U_q(su_2) \otimes U_q(su_2)$ acting on it. This dualisation is the usual one and could be constructed using (4). The action from (11) is $(h \otimes g) \triangleright a = \langle Sh, a_{(1)} \rangle a_{(2)} \langle g, a_{(3)} \rangle$ where $\langle , \rangle$ is the duality pairing. In the concrete matrix setting the action is

$$l_1^+ \triangleright x_2 = \lambda_0^{-1} R_{21}^{-1} x_2, \quad l_1^+ \triangleright x_2 = \lambda_0 R x_2, \quad m_2^\pm \triangleright x_1 = x_1 \lambda_0 R, \quad m_2^\pm \triangleright x_1 = x_1 \lambda_0^{-1} R_{21}^{-1} \quad (33)$$

where $l^\pm$ and $m^\pm$ denotes the FRT generators[13] of the two copies of $U_q(su_2) \otimes U_q(su_2)$ or other quantum enveloping algebra. This time the quantum Wick rotation proceeds by twisting further with cocycle $R_{23}^{-1}$ and turns the $q$-Euclidean space $\bar{M}_q(2)$ to the $q$-Minkowski space of braided-matrices and turns $U_q(su_2) \otimes U_q(su_2)$ to the twisted square $U_q(su_2) \bowtie U_q(su_2)$ as the usual dual of the spinorial $q$-Lorentz group function algebra $SU_q(2) \rhd SU_q(2)$. This formulation of the $q$-Lorentz group enveloping algebra as a twisted square was explained in [8, Sec. 4] and subsequently reiterated by other authors also. Moreover, since the $q$-Euclidean system is covariant, our quantum Wick rotation ensures that $q$-Minkowski space is likewise covariant as a module algebra under the $q$-Lorentz group enveloping algebra, acting as in (33) with $x$ replaced by $u$.

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