Casimir effect of strongly interacting scalar fields

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Abstract

Non-trivial $\phi^4$-theory is studied in a renormalisation group invariant approach inside a box consisting of rectangular plates and where the scalar modes satisfy periodic boundary conditions at the plates. It is found that the Casimir energy exponentially approaches the infinite volume limit, the decay rate given by the scalar condensate. It therefore essentially differs from the power law of a free theory. This might provide experimental access to properties of the non-trivial vacuum. At small interplate distances the system can no longer tolerate a scalar condensate, and a first order phase transition to the perturbative phase occurs. The dependence of the vacuum energy density and the scalar condensate on the box dimensions are presented.

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1 Introduction

The Casimir effect \[1, 2, 3, 4\] in quantum field theory is the change of the vacuum energy density due to constraints on the quantum field induced by boundary conditions in space-time. The contribution to the energy density by the quantum fluctuation of the electromagnetic field was experimentally observed by Sparnaay \[5\] in 1958, thus verifying its quantum nature. Following this observation the Casimir effect was extensively studied, the renormalisation procedure that must be used in order to extract physical numbers out of divergent mode sums, being of particular interest. This procedure is most elegantly formulated in a path-integral approach \[6\], and leads to a full understanding of the Casimir effect for non-interacting quantum fields. Perturbative corrections to the free Casimir arising from a weak interaction of the fluctuating fields can also be obtained \[7\]. It was shown that the net effect of the boundaries is to produce a topological mass for the fluctuating modes \[8\]. In the recent past there has been a renaissance of the Casimir effect due to its broad span of applications, which range from gravity models \[9\] to QCD bag models \[10\] to non-linear meson-theories describing baryons as solitons \[11\]. A closely related subject is Quantum Field Theory at finite temperature since it can be described in the path-integral formalism by implementing periodic boundary conditions in Euclidean time direction \[12\]. Despite these many different applications, it is possible to understand the basic features of the Casimir effect by investigating a scalar theory. It is also of general interest to study $\phi^4$-theory due to its important applications, e.g. in the Weinberg-Salam model of weak interactions (see e.g. \[13\]) and in solid state physics \[14\]. Many different approaches \[15, 16\] to strongly interacting $\phi^4$-theory were designed to understand its non-trivial vacuum structure. Even though the Casimir effect of free quantum fields is well understood, there is not yet an understanding of the Casimir effect for strongly interacting fields. This is simply due to the lack of knowledge of the true vacuum of an interacting quantum theory. Recently, a non-perturbative path integral approach to $\phi^4$-theory has yielded some insight into the vacuum structure of the strongly interacting scalar theory \[17, 18\]. In particular, it was found that its perturbative phase is unstable (at zero temperature) because a second phase with non-vanishing scalar condensate has lower vacuum energy density \[17\]. In this phase, the connection between the scalar condensate and the vacuum energy density, which is provided by the scale anomaly, has been verified by an explicit calculation \[18\]. The structure of this new phase, describing strongly interacting scalar modes, was also investigated at finite temperature \[18\]. It was found that at a critical temperature the energy densities of the non-trivial and perturbative phases are equal, and the non-trivial phase undergoes a first order phase transition to the perturbative one. Using these results it is possible to study the Casimir effect of strongly interacting scalar fields. Since the non-trivial phase provides an intrinsic energy scale (i.e. the
magnitude of the scalar condensate at zero temperature), one expects deviations of the Casimir force from the free field law. This presumably provides access to non-perturbative vacuum properties.

In this paper, we investigate the non-trivial phase of four-dimensional \( \phi^4 \)-theory in a rectangular box consisting of \( p \) pairs of oppositely layered plates separated by a distance \( a_p \), with the scalar modes satisfying periodic boundary conditions at the plates. We shall find at large interplate distances, that the Casimir energy decays exponentially with increasing distance, the decay rate given by the magnitude of the scalar condensate. At small distances, the field theory no longer tolerates a scalar condensate, and the perturbative phase is adopted.

The paper is organised as follows: in the second section we briefly review the Casimir effect of a free theory and the recently proposed non-perturbative approach \cite{17, 18} to \( \phi^4 \)-theory. The renormalisation procedure is discussed and renormalisation group invariance is shown. In the subsequent section results are presented. The Casimir energy as a function of large (compared with the scalar condensate) interplate distance is obtained analytically, and the deviations from the energy in a free field theory are discussed. The phase-transition from the non-trivial vacuum to the perturbative phase at small interplate distances is studied and the vacuum energy density and the scalar condensate is calculated as function of the plate distances. Discussions and concluding remarks are given in the final section.

2 The Casimir effect of scalar fields

\( \phi^4 \)-theory is described by the Euclidean generating functional for Green’s functions, i.e.

\[
Z[j] = \int \mathcal{D}\phi \exp\left\{ -\int d^4x \left( \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{m^2}{2} \phi^2 + \frac{\lambda}{24} \phi^4 - j(x)\phi^2(x) \right) \right\}, \quad (1)
\]

where \( m \) denotes the bare mass of the scalar field and \( \lambda \) the bare coupling strength of the \( \phi^4 \)-interaction. \( j(x) \) is an external source for \( \phi^2(x) \) which is introduced so that we can derive the effective potential \cite {19, 20} of the composite field \( \phi^2 \) later on. It was observed in \cite{18} that it is more convenient to use the effective potential of \( \phi^2 \) to study the phase structure of the theory. In particular, its minimum value is the vacuum energy density and thus provides access to the Casimir effect, if it is calculated by imposing adequate boundary conditions to the scalar modes. For these initial investigations we adopt the simplest geometry and consider a rectangular box consisting of \( p \) pairs of oppositely layered plates separated by distances \( a_p \). We expect that the Casimir energy will not be sensitive to the detailed shape of the finite volume as is known in a free theory \cite{21}. The integration over the field \( \phi \) in (1),
only extends over configurations which satisfy periodic boundary conditions at the plates. In the case of Dirichlet or Neumann boundary conditions, surface counter terms must be added to \( \bar{Z} \). In that case the results would sensitively depend on the physical structure of the surface, and such effects are beyond the scope of this paper.

The effective action is defined by a Legendre transformation of the generating functional \( \bar{Z}[j] \), i.e.

\[
\Gamma[\phi^2_c] := -\ln Z[j] + \int d^4x \phi^2_c(x) j(x) , \quad \phi^2_c(x) := \frac{\delta \ln Z[j]}{\delta j(x)}. \tag{2}
\]

From here the effective potential \( U(\phi^2_c) \) is obtained by restricting \( \phi_c \) to constant classical fields \( (\Gamma[\phi^2_c = \text{const.}] = \int d^4x U(\phi^2_c)) \), which are obtained for a constant external source \( j \). The minimum value of the effective potential \( U_{min} \) is the vacuum energy density and is obtained from (2) at zero external source, i.e.

\[
\frac{dU}{d\phi^2_c} \bigg|_{\phi^2_c = \phi^2_{c\,0}} = j = 0. \tag{3}
\]

The minimum classical configuration \( \phi^2_{c\,0} \) represents the scalar condensate.

2.1 Equivalence of effective action and sum of zero-point energies

In this subsection we review the Casimir effect of a free scalar theory \( (\lambda = 0) \) using Schwinger’s proper-time regularisation. We demonstrate that the minimum of the effective potential \( U \) coincides with the mode sum usually considered when studying the Casimir effect [1, 2, 3, 4]. This equivalence was also obtained by using another regularisation scheme [2], and previously observed with proper-time regularisation in the context of chiral solitons [22].

The minimum of the effective potential of a free scalar theory is

\[
U_{min} = \frac{1}{2TV_3} \text{Tr} \ln(-\partial^2 + m^2), \tag{4}
\]

where \( T \) is the Euclidean time interval, and \( V_3 \) is the space volume. The trace in (4), extending over all modes satisfying periodic boundary conditions, is a divergent object and needs regularisation. For definiteness we use Schwinger’s proper-time regularisation, but note, however, that the specific choice of the regularisation prescription has no influence on the renormalised (finite) result (e.g., compare [20] and [23]). In proper-time regularisation, the vacuum energy density becomes

\[
U_{min} = -\frac{1}{2TV_3} \text{Tr} \int_{1/\Lambda^2}^\infty ds \frac{e^{-s(-\partial^2 + m^2)}}{s}, \tag{5}
\]
where \( \Lambda \) is the ultraviolet cutoff.

The trace over the temporal degree of freedom can be easily performed, i.e.,

\[
U_{\text{min}} = -\frac{1}{2V_3} \int \frac{dk_0}{2\pi} \text{Tr}_V \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-s(k_0^2 + E)},
\]

(6)

The trace \( \text{Tr}_V \) extends over the spatial degrees of freedom and \( E \) is the energy obtained from the eigenvalue equation

\[
(-\nabla^2 + m^2)\phi(x) = E^2\phi(x),
\]

(7)

where the eigenfunction \( \phi \) satisfies periodic boundary conditions. If the \( k_0 \)-integration in (6) is performed, a partial integration in the \( s \)-integral yields

\[
V_3U_{\text{min}} = \frac{1}{2\sqrt{\pi}} \text{Tr}_V \left\{ E \Gamma\left(\frac{1}{2}, \frac{E^2}{\Lambda^2}\right) \right\} - \frac{\Lambda}{2\sqrt{\pi}} \text{Tr}_V \exp\left\{ -\frac{E^2}{\Lambda^2} \right\},
\]

(8)

where \( \Gamma\left(\frac{1}{2}, x\right) \) is the incomplete \( \Gamma \)-function. The first term of (8) is precisely the mode sum \( \frac{1}{2} \text{Tr}_V E \) in cutoff regularisation, with the particular cutoff function \( \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \frac{E^2}{\Lambda^2}\right) \) provided by Schwinger’s proper-time regularisation. In the limit of large \( \Lambda \), the second term only contributes a constant to the action which is subtracted by demanding that the Casimir energy approaches zero for large interplate distances.

In order to illustrate the equivalence of the mode sum approach and the approach provided by the effective potential, we calculate the Casimir energy for a massless scalar particle in a box consisting of \( p \) pairs of rectangular plates and in \( d \) space-time dimensions. In this case we have

\[
-\ln Z = -\frac{L^{d-p}}{2} \int \frac{d^{d-p}k}{(2\pi)^{d-p}} \sum_{\{n_i\} = -\infty}^{\infty} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp\left\{ -s[k^2 + \sum_{i=1}^{p} n_i^2 (\frac{2\pi}{a_i})^2] \right\},
\]

(9)

where \( a_i, i = 1 \ldots p \) is the distance of the hyperplanes in \( i \)th direction and \( L \gg a_i \) is the length of the box of the unconstrained modes. The integration over the continuous degrees of freedom can be performed in a straightforward manner, i.e.,

\[
-\ln Z = -\frac{L^{d-p}}{2} \left(\frac{1}{d-p}\right)^{d-p} \sum_{\{n_i\} = -\infty}^{\infty} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^{1+\frac{d-p}{2}}} e^{-s\sum n_i^2 (\frac{2\pi}{a_i})^2},
\]

(10)

In order to extract the ultra-violet divergences, we apply Poisson’s formula

\[
\sum_{n = -\infty}^{\infty} f(n) = \sum_{\nu = -\infty}^{\infty} c(\nu), \quad \text{with} \quad c(\nu) = \int_{-\infty}^{\infty} dn \ f(n) e^{i2\pi \nu n}.
\]

(11)

Rewriting

\[
\sum_{n = -\infty}^{\infty} e^{-sn^2 (\frac{2\pi}{a})^2} = \frac{a}{2\sqrt{\pi} s} \sum_{\nu = -\infty}^{\infty} e^{-\nu^2 a^2 \frac{2\pi}{4\nu}}
\]

(12)
equation (10) becomes
\begin{equation}
- \ln Z = - \frac{L^{d-p}}{2} \frac{1}{(2\sqrt{\pi})^{d-p}} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^{1+d-p}} \prod_{i=1}^{p} \left( \frac{a_i}{2\sqrt{\pi} s} \right) \sum_{\nu_i=-\infty}^{\infty} e^{-\nu_i^2 s^2}.
\end{equation}

Note that the ultra-violet behaviour is dominated by the integrand at small \( s \) and the only divergences come from the term with all \( \nu_i \) are zero. The divergent term
\begin{equation}
- \ln Z_{\text{div}} = - \frac{L^{d-p}}{2} \frac{1}{(2\sqrt{\pi})^{d-p}} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^{1+d-p}} \prod_{i=1}^{p} \left[ \frac{a_i}{2\sqrt{\pi} s} \right],
\end{equation}
is proportional to the \( d \)-dimensional volume \( V \) and a pure constant, which can be absorbed by a redefinition of the action. After the substitution \( s \rightarrow 1/s \), the \( s \)-integral can be performed in (13), yielding for the finite part in the limit \( \Lambda \rightarrow \infty \)
\begin{equation}
- \frac{1}{V} \ln Z = U_{\text{min}} = - \frac{1}{2 \pi^{d/2}} \Gamma \left( \frac{d}{2} \right) Z(a_1 \ldots a_p, d),
\end{equation}
where
\begin{equation}
Z(a_1 \ldots a_p, d) = \sum_{\{\nu_i\}=-\infty}^{\infty} \frac{1}{(a_1^2 \nu_1^2 + \ldots + a_p^2 \nu_p^2)^{d/2}},
\end{equation}
is the Epstein Zeta-function (the prime indicates that the contribution with all \( \nu_i \) = 0 is excluded from the sum). For \( p = 1 \) and four space-time dimensions \( (d = 4) \) one obtains the analytic result for the vacuum energy density and the Casimir energy \( E_c \), respectively, i.e.,
\begin{equation}
U_{\text{min}} = - \frac{1}{\pi^2 a^4} \Gamma(2) \zeta(4), \quad E_c = V_3 U_{\text{min}} = - \frac{\pi^2 L^2}{90 a^3},
\end{equation}
where \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) is Riemann's \( \zeta \)-function. This is precisely the result usually obtained by evaluating the mode sum of zero point energies \[3\].

### 2.2 Non-trivial \( \phi^4 \)-theory with boundary conditions

In this subsection we describe the non-perturbative approach to \( \phi^4 \)-theory provided by the modified loop expansion \[17\], taking into account the constraints on the scalar field imposed by boundary conditions. We demonstrate that the renormalisation procedure is not affected by the presence of a rectangular box, implying that renormalisation group invariance is preserved as it is in the infinite volume limit \( (a_i \rightarrow \infty) \).
The modified loop expansion [17] is based on a linearisation of the $\phi^4$-interaction in the path-integral (1) by means of an auxiliary field $\chi(x)$

$$Z[j] = \int \mathcal{D}\phi \mathcal{D}\chi \exp\{-\int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{6}{\lambda} \chi^2(x) + \left( \frac{m^2}{2} - i \chi(x) \right) \phi^2(x) - j(x) \phi^2(x) \right] \} .$$

This linearisation was first proposed in [24]. The integral over the fundamental field $\phi$ is then easily performed, yielding

$$Z[j] = \int \mathcal{D}\chi \exp\{-S[\chi, j] \} ,$$

$$S[\chi, j] = \frac{6}{\lambda} \int d^4x \chi^2 + \frac{1}{2} \text{Tr}_{(R)} \ln \mathcal{D}^{-1}[\chi, j] ,$$

$$\mathcal{D}^{-1}[\chi, j]_{xy} = (-\partial^2 + m^2 - 2i \chi(x) - 2j(x)) \delta_{xy} .$$

The trace $\text{Tr}_{(R)}$ extends over all eigenmodes of the operator $\mathcal{D}^{-1}[\chi, j]$ which satisfy the periodic boundary conditions and the subscript $(R)$ indicates that a regularisation prescription is required. Note that the boundary conditions to the field $\phi$ do not give rise to any constraint for the auxiliary field $\chi$. The approach of [17] is defined by a modified expansion with respect to the field $\chi$ around its mean field value $\chi_0$ defined by

$$\frac{\delta S[\chi, j]}{\delta \chi(x)}|_{\chi=\chi_0} = 0 .$$

(22)

The modified loop expansion of [17, 18] coincides with an $1/N$-expansion of $O(N)$ symmetric $\phi^4$-theory [24] for $N = 1$, implying that the convergence of the expansion is doubtful. However, it was seen in zero dimensions that the effective potential of this approximation rapidly converges to the exact one obtained numerically. Recent results show that the same is true for four dimensional $\phi^4$-theory [26]. Reasonable results are obtained even at mean-field level. At this level we obtain from [18]

$$-\ln Z[j](a_1 \ldots a_p) =$$

$$\int d^4x \left\{ -\frac{3}{2\lambda} (M - m^2 + 2j)^2 \right\} + \frac{1}{2} \text{Tr}_{(R)} \ln(-\partial^2 + M) ,$$

where $M$ is related to the mean field value $\chi_0$ by $\chi_0 = i(M - m^2 + 2j)/2$. The mean field equation for $\chi_0$ (22) can be recast into an equation for $M$, i.e.,

$$\frac{\delta \ln Z[j]}{\delta M} = 0 .$$

(24)

For a constant external source $j$, this equation is satisfied for constant $M$. 
The only effect of the rectangular box is contained in the loop contribution, which is, in Schwinger’s proper-time regularisation \((d = 4)\)

\[
L = \frac{1}{2} \text{Tr}(R) \ln(-\partial^2 + M)
\]

\[
= -\frac{L^{d-p}}{2} \int \frac{d^{d-p}k}{(2\pi)^{d-p}} \sum_{\{n_i\}} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp\{-s[\sum_{i=1}^{d-p} k_i^2 + M + \sum_i \left(\frac{2\pi}{a_i}\right)^2 n_i^2]\},
\]

with \(i = 1 \ldots p\) and \(L\) the linear extension in the unconstrained directions. The sum over the unconstrained modes (\(k\)-integral) can be easily performed. Applying Poisson’s formula \(\text{(11)}\) to extract the divergent terms as we did for the free theory (section \(\text{[I]}\)) we obtain

\[
L = -\frac{V}{32\pi^2} M^2 \Gamma(-2, \frac{M}{\Lambda^2}) - \frac{V}{32\pi^2} \sum' \int_{1/\Lambda^2}^{\infty} ds \frac{e^{-sM}}{s^3} \exp\{-\sum_i \frac{a_i^2}{4s} \nu_i^2\},
\]

where \(\Gamma\) is the incomplete \(\Gamma\)-function, \(V\) the space-time volume and the prime indicates that the contribution with all \(\nu_i = 0\) is excluded. This implies that the second term of the right hand side of \(\text{(26)}\) is ultraviolet finite, so we can remove the regulator in this term \((\Lambda \to \infty)\). Using the asymptotic expression of the incomplete \(\Gamma\)-function, we find

\[
L = \frac{V}{32\pi^2} \{M \Lambda^2 + \frac{1}{2} M^2 (\ln \frac{M}{\Lambda^2} - \frac{3}{2} + \gamma)\} - \frac{V}{32\pi^2} F_3(M, a_1 \ldots a_p),
\]

where \(\gamma = 0.577\ldots\) is Euler’s constant and the function \(F_3\) is defined by

\[
F_\epsilon(M, a_1 \ldots a_p) = \sum' \int_{0}^{\infty} ds \frac{e^{-sM}}{s^\epsilon} e^{-\sum_i \frac{a_i^2}{s} \nu_i^2}
\]

with \(\epsilon = 3\). The first term on the right hand side of \(\text{(27)}\) is precisely the effective potential in the infinite volume limit since the function \(F_3\) vanishes for \(a_i \to \infty\). The second term in \(\text{(27)}\) is thus the modification of the effective potential due to the presence of the plates. Note that this term is finite, implying that the boundaries do not affect the renormalisation procedure. This is the desired result.

Following the renormalisation scheme given in \(\text{[18]}\), we absorb the divergences in the bare parameters \(\lambda, m, j\) by setting

\[
\frac{6}{\lambda} = \frac{6}{\lambda_R} - \frac{1}{16\pi^2} (\ln \frac{\Lambda^2}{\mu^2} - \gamma + 1),
\]

\[
\frac{6}{\lambda} - \frac{m^2}{\lambda} = \frac{6}{\lambda_R} j_R - \frac{3m^2_R}{\lambda_R},
\]

\[
j - m^2 = 0,
\]
where \( \mu \) is an arbitrary renormalisation point and a subscript \( R \) refers to the renormalised quantities. Later we will check that physical quantities do not depend on \( \mu \). In the following, we consider the massless case \( m_R = 0 \). The coupling strength renormalisation in (29) was earlier used by Coleman et al. [24] and, as pointed out by Stevenson, it implies that the bare coupling becomes (infinitesimally) negative, if the regulator \( \Lambda \) is taken to infinity [16]. It was shown that this behaviour of the bare coupling strength is hidden in the standard perturbation theory [17]. In fact, we have from (29)

\[
\lambda = \frac{\lambda_R}{1 - \beta_0 \lambda_R (\ln \frac{\Lambda^2}{\mu^2} - \gamma + 1)} , \quad \beta_0 = \frac{1}{96\pi^2} \tag{32}
\]

implying \( \lambda \to 0^- \) for \( \Lambda \to \infty \), whereas in contrast an expansion of (32) with respect to the renormalised coupling strength, i.e.

\[
\lambda = \lambda_R(\mu) [1 + \beta_0 \lambda_R (\ln \frac{\Lambda^2}{\mu^2} - \gamma + 1) + O(\lambda_R^2)] \tag{33}
\]

suggests that \( \lambda \to +\infty \), if \( \Lambda \to \infty \).

Inserting (29-31) and \( L \) from (27) in (23) one obtains

\[
- \frac{1}{V} \ln Z[j](a_1 \ldots a_p) = - \frac{3}{2\lambda_R} M^2 - \frac{6}{\lambda_R} M j_R + \frac{\alpha}{2} M^2 (\ln \frac{M}{\mu^2} - \frac{1}{2}) - \alpha F_3(M, a_1 \ldots a_p) \tag{34}
\]

where \( \alpha = 1/32\pi^2 \), and \( M \) is defined by the mean field equation (24), i.e.

\[
- \frac{3}{\lambda_R} M - \frac{6}{\lambda_R} M j_R + \alpha M (\ln \frac{M}{\mu^2} - \frac{1}{2}) + \alpha F_2(M, a_1 \ldots a_p) = 0 \tag{35}
\]

It is now straightforward to perform the Legendre transformation (2). The final result for the effective potential is

\[
U(\phi^2_c) = \frac{\alpha}{2} M^2 (\ln \frac{M}{\mu^2} - \frac{1}{2}) - \frac{3}{2\lambda_R} M^2 - \alpha F_3(M, a_1 \ldots a_p) , \tag{36}
\]

where

\[
\phi^2_c = \frac{1}{V} \frac{\delta \ln Z[j_R]}{\delta j_R} = \frac{6}{\lambda_R} M . \tag{37}
\]

The effective potential is renormalisation group invariant, since a change in the renormalisation point \( \mu \) can be absorbed by a change of the renormalised coupling strength [17, 18]. Note that due to field renormalisation (31), \( M = \lambda_R \phi^2_c/6 \) is renormalisation group invariant rather than \( \phi^2_c \). Thus \( M \) is a physical quantity and is referred to as scalar condensate. In the infinite volume limit \((a_i \to \infty)\) the
effective potential has a global minimum for $M = M_0 \neq 0$, implying that the ground state has a non-vanishing scalar condensate [18]. Furthermore, the minimum value of the effective potential (vacuum energy density) is related to the scalar condensate by [18]

$$U_0 = -\frac{\alpha}{144}\lambda_R^2(\phi_c^2)^2 \to -\frac{1}{4}\beta(\lambda_R)\langle \phi^4 \rangle,$$

which yields the correct scale anomaly at this level of approximation.

In order to make renormalisation group invariance obvious, we remove the renormalisation point dependence in (36) by subtracting $U_0$ from the effective potential $U(M)$ (36). Both the renormalisation point $\mu$ and the renormalised coupling $\lambda_R$ drop out, and we obtain

$$U(M) = \frac{\alpha}{2}M^2(\ln \frac{M}{M_0} - \frac{1}{2}) - \alpha F_3(M, a_1 \ldots a_p).$$

The effective potential $U$ as a function of the scalar condensate $M$ for one pair of plates ($p = 1$) is shown in figure 1. In this case, the results are equivalent to that of a finite temperature field theory if the inverse distance $1/a$ between the plates is identified with the temperature (in units of Boltzmann’s constant) [12]. Further results of finite temperature $\phi^4$-theory are given in [18]. For a large interplate distance $a$ (zero temperature), the continuum effective potential is obtained, and the effective potential has a minimum at a nonvanishing value of the scalar condensate. At finite $a$, a second minimum at zero condensate $M$ develops, which is referred to as the perturbative phase. At large $a$, this trivial phase is unstable, because the non-perturbative minimum has lower vacuum energy density. Decreasing $a$ (increasing temperature), lowers the difference in the energy density between the perturbative and non-perturbative phases. At a critical distance $a_c$ the non-trivial phase becomes degenerate with the perturbative one (at $M = 0$). If $a$ is decreased further, the non-trivial phase becomes meta-stable and a first order phase transition to the trivial phase at $M = 0$ can occur, either by quantum or statistical fluctuations.

3 Results

In a free massless field theory (with $p = 1$) there is no intrinsic energy scale in competition with that of the interplate distance. This implies that the vacuum energy density $U_0$ scales as $1/a^4$ with the interplate distance from dimensional arguments. This scaling law was experimentally observed by Sparnaay in the case of QED [3]. In the case of non-trivial $\phi^4$-theory an intrinsic energy scale is provided by the scalar condensate. Thus one expects deviations from the $1/a^4$ scaling law of the free theory. Such deviations might provide experimental access to properties of the non-trivial phase.
3.1 Near the infinite volume limit

The scalar condensate \( M_v \) at the minimum of the effective potential is given by the gap equation

\[
\frac{dU}{dM}\big|_{M=M_v} = M_v \ln \frac{M_v}{M_0} + F_2(M_v, a_1\ldots a_p) = 0.
\]  

The vacuum energy density \( U_v \) is obtained by inserting \( M_v \) back into (39). For large interplate separations \( a_i^2 \gg 1/M_v \) the function \( F_2(M_v, a_1\ldots a_p) \) can be analytically estimated by noting that only terms with a single \( \nu_i \neq 0 \) and all others \( \nu_j \neq i \) contribute to the sum (28), i.e.

\[
F_2(M_v, a_1\ldots a_p) \approx \sum_i \int_0^\infty \frac{ds}{s^\epsilon} e^{-sM_v^2/a_i^2}.
\]  

A simple rescaling yields

\[
F_2(M, a_1\ldots a_p) = \sum_i \left( \frac{4}{a_i^4} \right) \epsilon^{-1} f_\epsilon(M, a_i^2/4), \quad f_\epsilon(x) = \int_0^\infty \frac{ds}{s^\epsilon} e^{-sx} e^{-\frac{x}{4s}}.
\]  

After some technical manipulations, the functions \( f_\epsilon(x) \) can be related to the modified Bessel functions of the second kind, i.e.,

\[
f_\epsilon(x) = 2 x^{\frac{\epsilon-1}{2}} K_{\epsilon-1}(2\sqrt{x}) \approx \sqrt{\pi} x^{2\epsilon-3} e^{-2\sqrt{x}},
\]  

where the last approximate expression is just the asymptotic form of the Bessel function for \( x \to \infty \). Thus we have near the infinite volume limit

\[
F_2 \approx M^{1/4} \sqrt{\pi} \sum_i (a_i^2/4)^{-3/4} e^{-\sqrt{M}a_i}, \quad F_3 \approx M^{3/4} \sqrt{\pi} \sum_i (a_i^2/4)^{-5/4} e^{-\sqrt{M}a_i}.
\]  

Solving (40) for the scalar condensate \( M_v \) in perturbation theory around \( M_0 \) we obtain the change in the condensate due to the presence of the plates, i.e.

\[
M_v = M_0 \left[ 1 - \sqrt{\pi} \sum_{i=1}^p \left( \frac{M_0 a_i^2}{4} \right)^{-3/4} e^{-\sqrt{M_0}a_i} + \ldots \right].
\]  

There are two contributions to the variation of the vacuum energy density \( U_v(M_v(a_i), a_i) \) with the interplate distances, one from a change of the scalar condensate and one from the change of the effective potential \( U(M) \) via the function \( F_3 \). Equation (40) implies that a variation of the condensate does not change \( U_v \) in first order, and thus the leading contribution is from a change of \( F_3 \). Using the asymptotic form (44) for \( F_3 \) one obtains

\[
\frac{1}{\alpha} \Delta U_v = \frac{1}{\alpha} (U_v - U_\infty) \approx -\sqrt{\pi} M_0^3 \sum_{i=1}^p \left( \frac{M_0 a_i^2}{4} \right)^{-5/4} e^{-\sqrt{M_0}a_i},
\]
where $U_\infty$ is the vacuum energy density in the infinite volume limit. This is the desired result: equation (46) gives the change of the vacuum energy density due to boundary conditions. In free field theory (and $p = 1$) it decays by the power law $\sim 1/a^4$ (see (17)). In contrast, the energy density (46) of strongly interacting scalar modes decays exponentially (with a power law correction), the slope given by the magnitude of the scalar condensate $M_0$. This implies that at least in principle one can decide by observing the dependence of the Casimir force on the interplate distance, whether the theory is in a free or in a non-perturbative phase. In the latter case, it is also possible to extract ground state properties, e.g., the scalar condensate. Since in QED the $1/a^4$-power law was experimentally verified [5], the QED ground state is trivial and e.g., has no photon condensate, an expected result since photon self-interactions are absent.

3.2 At the phase transition at small interplate distances

As was seen in section 2.2 for one pair of plates ($p = 1$), the system undergoes a first order phase transition from the non-trivial vacuum to the perturbative vacuum, if the interplate distance becomes small enough. Numerical investigations of the effective potential (39) at various distances $a_i$ show that the same effect holds for $p > 1$: if the box is small enough, a first order phase transition to the perturbative vacuum occurs.

Equating the energy density $U_0$ of the perturbative state at $M = 0$ to that of the non-trivial phase (with non-zero condensate) at $M = M_v$ we obtain

$$
\frac{M_v^2}{2} \left( \ln \frac{M_v}{M_0} - \frac{1}{2} \right) - F_3(M_v, a_1 \ldots a_p) + F_3(0, a_1 \ldots a_p) = 0, \quad (47)
$$

where the dependence of the scalar condensate $M_v(a_1 \ldots a_p)$ on the interplate distances $a_i$ is implicitly given by (40). The set of equations (40, 47) defines a hypersurface in the space spanned by the distances $a_i$, which separates the non-trivial phase from the perturbative one. Note that this transition line is given in terms of renormalisation group invariant (and therefore physical) quantities.

For one pair of plates ($p = 1$), the formulation is equivalent to the finite-temperature $\phi^4$-theory (identifying $1/a$ with temperature), and the phase transition at small distance $a$ is of same structure as that in the finite-temperature theory at high temperature. Due to this correspondence, the numerical value for the critical distance $a_{(c)}$ can be taken from [18]

$$
M_0 a_{(c)}^2 = 10.29134 \ldots . \quad (48)
$$
The ratio of the scalar condensate $M_v$ at the transition point and the continuum (zero temperature) condensate $M_0$ is

$$M_v(a_{(c)}) / M_0 = 0.9041\ldots.$$  \hspace{1cm} (49)

Due to the first order nature of the phase transition, the scalar condensate has a discontinuity at the transition point $a_{(c)}$ and is zero for smaller distances.

For two pairs of plates the transition line between the two phases was obtained by solving (40, 47) numerically. The result is presented in figure 2. Numerical investigations (cf. figure 4) suggest that the transition line is approximately given by the equation

$$\frac{1}{a_{c1}^2} + \frac{1}{a_{c2}^2} = \frac{M_0}{9}.$$  \hspace{1cm} (50)

For $p = 3$ we have numerically checked, that the first order phase transition occurs, if the rectangular box is sufficiently small.

### 3.3 Boundary dependence of energy density and scalar condensate

For given vacuum energy density $U_v$, equation (39) i.e.,

$$\frac{M_v^2}{2}(\ln \frac{M_v}{M_0} - \frac{1}{2}) - F_3(M_v, a_1 \ldots a_p) = U_v$$  \hspace{1cm} (51)

with $M_v$ defined by (40) yields the hypersurface of constant energy density in the space spanned by $\{a_i, i = 1 \ldots p\}$. Comparing (51) with (17) it is easily seen that the phase separating surface is not a surface of constant energy density, implying that there are intersections between the two surfaces. We expect that the hypersurface of constant energy density is continuous at the intersection, but not differentiable due to the first order phase transition.

Figure 3 shows the vacuum energy density for one pair of plates ($p = 1$) as a function of $16/a^2$, when $a$ is the interplate distance (or equivalently the inverse temperature). For large values of $16/a^2$ (small $a$), the perturbative phase is realised, and the $1/a^4$-scaling law is observed. For small values of $16/a^2$ (large $a$) the scalar theory is in the non-trivial phase, and the energy density exponentially approaches the continuum value (see (16)) given by the scale anomaly (8).

For two pairs of plates ($p = 2$), figure 4 shows lines of constant vacuum energy density in the $16/a_1^2 - 16/a_2^2$ plane. Also shown is the phase transition line (dashed curve). The lines of constant energy density are continuous, but have a cusp at the first order phase transition point.
We have also studied the hypersurfaces of constant scalar condensate in \( a_i \)-space. For \( p = 1 \), this is equivalent investigating the temperature dependence of the scalar condensate, and thus the results are given in [18]. For \( p = 2 \), the lines of constant condensate (in units of continuum condensate \( M_0 \)) are presented in figure 5. A line of constant condensate is discontinuous at the phase transition line, and is zero in the perturbative phase. This behaviour is again due to the first order phase transition.

4 Discussion and concluding remarks

We have shown for \( \phi^4 \)-theory constrained by a rectangular box that the non-trivial ground state undergoes a first order phase transition to the perturbative vacuum, if the extension in at least one space-time direction becomes small enough. For large boxes the finite size corrections to the infinite volume limit are exponentially small. They are negligible, if the interplate distances are large compared with intrinsic scale provided by the (continuum) scalar condensate (i.e. \( a_i \sqrt{M_0} \gg 1 \)). On the other hand finite size effects become important for \( a_i \sqrt{M_0} \approx 1 \) and induces a phase transition to a non-trivial vacuum.

We believe that these properties are a common feature of a wide class of quantum field theories. Indeed, an analogous situation is observed in lattice gauge theories. Theoretical investigations show, that for high temperatures pure SU(N) lattice gauge theory has a phase transition from a non-trivial (confining) ground state to a perturbative phase [27]. Numerical simulations of the SU(N) theory use a lattice with size \( n_t n^3 \), \( n \gg n_t \), which corresponds to a system with volume \( n^3 \) and inverse temperature \( n_t \). Such a system shows two phases, a non-trivial phase for \( \beta < \beta_c(n_t) \), and a deconfined phase for \( \beta > \beta_c(n_t) \) [28] (\( \beta = 2N/g^2 \) with \( g \) the SU(N) coupling strength). This compares well with our considerations as follows. The intrinsic scale of lattice gauge theories is provided by the string tension \( \chi \) [29] (or equivalently by the gauge field condensate [30] as in the continuum Yang-Mills theory). Our investigations suggest that a finite size phase transition occurs if

\[
n_t n^3 a^4 \chi^2 \leq 1, \tag{52}
\]

where \( a \) is the lattice spacing. The string tension in units of the lattice spacing \( a^4 \), strongly depends on the inverse coupling strength \( \beta \) dictated by the renormalisation group. Numerical simulations [28] show that for fixed \( n_t, n \), \( a^4 \chi^2 \) decreases with increasing \( \beta \), implying that (52) is satisfied for \( \beta \approx \beta_c \), the coupling strength at which the phase transition occurs.

In conclusion, we have studied \( \phi^4 \)-theory in a renormalisation group invariant approach inside a rectangular box consisting of \( p \) pairs of plates, at which the scalar
modes satisfy periodic boundary conditions. We have further investigated the
ground state properties of the non-trivial phase affected by the geometrical con-
staints. The dependence of the vacuum energy density and the scalar condensate
on the interplate distances was studied in some detail. In the non-trivial phase the
vacuum energy density exponentially approaches the infinite volume limit, the decay
rate given by the magnitude of the scalar condensate. This behaviour of the energy
density essentially differs from that of a free theory, where it scales according a $1/a^4$-
power law. This implies, that at least in principle, one can determine which phase
the system has adopted by measuring the Casimir force. At small interplate dis-
tances, the system undergoes a first order phase transition to the perturbative phase.
This phase transition is of the same nature as the transition at high temperature.

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Figure captions

**Figure 1:** The effective potential as function of the scalar condensate at various interplate distances.

**Figure 2:** The transition line separating the non-trivial phase and the perturbative phase, two pairs of plates \((p = 2)\) with distances \(a_1\) and \(a_2\), respectively.

**Figure 3:** The vacuum energy density for one pair of plates \((p = 1)\) as a function of the interplate distance (inverse temperature) in units of \(1/\sqrt{M_0}\).

**Figure 4:** The lines of constant vacuum energy density \(\frac{\mu_v}{aM_0^2}\) for two pairs of plates; \(a_i^2\) in units of the inverse scalar condensate \(1/M_0\).

**Figure 5:** The lines of constant scalar condensate \(M/M_0\) for two pairs of plates.