Abstract

It was recently shown that non-Abelian vortex strings supported in a version of four-dimensional $\mathcal{N} = 2$ supersymmetric QCD (SQCD) become critical superstrings. In addition to four translational moduli, non-Abelian strings under consideration have six orientational and size moduli. Together they form a ten-dimensional target space required for a superstring to be critical, namely, the product of the flat four-dimensional space and conifold – a non-compact Calabi-Yau threefold. In this paper we report on further studies of low-lying closed string states which emerge in four dimensions and identify them as hadrons of our four-dimensional $\mathcal{N} = 2$ SQCD. We use the approach based on “little string theory,” describing critical string on the conifold as a non-critical $c = 1$ string with the Liouville field and a compact scalar at the self-dual radius. In addition to massless hypermultiplet found earlier we observe several massive vector multiplets and a massive spin-2 multiplet, all belonging to the long (non-BPS) representations of $\mathcal{N} = 2$ supersymmetry in four dimensions. All the above states are interpreted as baryons formed by a closed string with confined monopoles attached. Our construction presents an example of a “reverse holography.”
In this paper we continue studying the spectrum of four-dimensional “hadrons” formed by the closed critical string \[1\] which in turn can be obtained from a solitonic vortex string under an appropriate choice of the coupling constant \[2\]. One of our main tasks is to analyze the structure of the 4D supermultiplets emerging from quantization of the closed string mentioned above. We will start though from a brief review of the setup.

The problem of understanding confining gauge theories splits into two different equally fundamental tasks. The first one is to understand the physical nature of confinement and describe the formation of confining strings. There was a great progress in this direction in supersymmetric gauge theories due to the breakthrough papers by Seiberg and Witten \[3, 4\] in which the monopole condensation was shown to occur in the monopole vacua of $\mathcal{N} = 2$ supersymmetric QCD (SQCD). This leads to the formation of Abelian Abrikosov-Nielsen-Olesen (ANO) vortices \[5\] which confine color electric charges. Attempts to find a non-Abelian generalization of this mechanism led to the discovery of the so called “instead-of-confinement” phase which occurs in the quark vacua of $\mathcal{N} = 2$ SQCD, see \[6\] for a review. In this phase the (s)quarks condense while the monopoles are confined.

Once the nature of the confining string is understood the second task is to quantize this string in four-dimensional (4D) theory outside the critical dimension to study the hadron spectrum. Most solitonic strings, such as the ANO strings, have a finite thickness manifesting itself in the presence of an infinite series of unknown higher-derivative corrections in the effective sigma model on the string world sheet. This makes the task of quantizing such a string virtually impossible.

Recent advances in this direction \[2\] demonstrated that the non-Abelian solitonic vortex in a particular version of 4D $\mathcal{N} = 2$ SQCD becomes a critical superstring. This particular 4D SQCD has the U(2) gauge group, four quark flavors and the Fayet-Iliopoulos (FI) \[7\] parameter $\xi$.

Non-Abelian vortices were first discovered in $\mathcal{N} = 2$ SQCD with the U($N$) gauge group and $N_f \geq N$ flavors of quark hypermultiplets \[8, 9, 10, 11\]. In addition to four translational moduli characteristic of the ANO strings \[5\], the non-Abelian strings carry orientational moduli, as well as the size moduli if $N_f > N$ \[8, 9, 10, 11\] (see \[12, 13, 14, 15\] for reviews). If $N_f > N$ their dynamics are described by effective two-dimensional sigma model on the
string world sheet with the target space

\[ \mathcal{O}(-1)_{\text{CP}^1}^{(N_f-N)}, \quad (1.1) \]

to which we will refer to as the weighted CP model \((\text{WCP}(N, N_f-N))\).

For \(N_f = 2N\) the model becomes conformal. Moreover, for \(N = 2\) the dimension of the orientational/size moduli space is six and they can be combined with four translational moduli to form a ten-dimensional space required for superstring to become critical.\(^1\)

In this case the target space of the world sheet 2D theory on the non-Abelian vortex string is \(\mathbb{R}^4 \times Y_6\), where \(Y_6\) is a non-compact six-dimensional Calabi-Yau manifold, the so-called resolved conifold \([16, 17]\).

Since non-Abelian vortex string on the conifold is critical it has a perfectly good UV behavior. This opens the possibility that it can become thin in a certain regime \([2]\). The string transverse size is given by \(1/m\), where \(m\) is a typical mass scale of the four-dimensional fields forming the string. The string cannot be thin in a weakly coupled 4D theory because at weak coupling \(m \sim g\sqrt{T}\) and is always small in the units of \(\sqrt{T}\) where \(T\) is the tension. Here \(g\) is the gauge coupling constant of the 4D \(\mathcal{N} = 2\) QCD and \(T\) is the string tension.

A conjecture was put forward in \([2]\) that at strong coupling in the vicinity of a critical value of \(g_c^2 \sim 1\) the non-Abelian string on the conifold becomes thin, and higher-derivative corrections in the action can be ignored. It is expected that the thin string produces linear Regge trajectories even for small spins \([2]\). The above conjecture implies\(^2\) that \(m(g^2) \rightarrow \infty\) at \(g^2 \rightarrow g_c^2\).

A version of the string-gauge duality for 4D SQCD was proposed \([2]\): at weak coupling this theory is in the Higgs phase and can be described in terms of \((s)\)quarks and Higgsed gauge bosons, while at strong coupling hadrons of this theory can be understood as string states formed by the non-Abelian vortex string.

The vortices in the \(\text{U}(N)\) theories under consideration are topologically stable and cannot be broken. Therefore the finite-length strings are closed.

\(^1\)The non-Abelian vortex string is 1/2 BPS saturated and, therefore, has \(\mathcal{N} = (2,2)\) supersymmetry on its world sheet. Thus, we actually deal with a superstring in the case at hand.

\(^2\)At \(N_f = 2N\) the beta function of the 4D \(\mathcal{N} = 2\) QCD is zero, so the gauge coupling \(g^2\) does not run. Note, however, that conformal invariance in the 4D theory is broken by the FI parameter \(\xi\) which does not run either.
Thus, we focus on the closed strings. The goal is to identify the closed string states with the hadrons of 4D $\mathcal{N} = 2$ SQCD.

The first step of this program, namely, identifying massless string states was carried out in [18, 19] using supergravity formalism. In particular, a single matter hypermultiplet associated with the deformation of the complex structure of the conifold was found as the only 4D massless mode of the string. Other states arising from the massless ten-dimensional graviton are not dynamical in four dimensions. In particular, the 4D graviton and unwanted vector multiplet associated with deformations of the Kähler form of the conifold are absent. This is due to non-compactness of the Calabi-Yau manifold we deal with and non-normalizability of the corresponding modes over six-dimensional space $Y_6$.

The next step was done in [1] where a number of massive states of the closed non-Abelian vortex string was found. This step required a change of strategy. The point is that the coupling constant $1/\beta$ of the world sheet WCP(2,2) is not small. Moreover $\beta$ tends to zero once the 4D coupling $g^2$ approaches the critical value $g_2^c$ we are interested in. At $\beta \to 0$ the resolved conifold develops a conical singularity. The supergravity approximation does not work for massive states.

To analyze the massive states the little string theories (LST) approach (see [23] for a review) was used in [1]. Namely, we used the equivalence between the critical string on the conifold and non-critical $c = 1$ string which contains the Liouville field and a compact scalar at the self-dual radius [24, 25]. The latter theory (in the mirror Wess-Zumino-Novikov-Witten (WZNW) formulation) can be analyzed by virtue of algebraic methods. This leads to identification of towers of massive states with spin zero and spin two [1].

In this paper we focus on the 4D multiplet structure of the states found earlier in [19, 1]. In addition to the massless BPS hypermultiplet associated with deformations of the complex structure of the conifold we identify several massive vector multiplets and a massive spin-2 multiplet, all belonging to long non-BPS representations of $\mathcal{N} = 2$ supersymmetry in four dimensions. We interpret all states we found as baryons formed by a closed string with confined monopoles attached.

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3This is in contradistinction to the massless states. For the latter, we can perform computations at large $\beta$ where the supergravity approximation is valid and then extrapolate to strong coupling. In the sigma-model language massless states corresponds to chiral primary operators. They are protected by $\mathcal{N} = (2, 2)$ world-sheet supersymmetry and their masses are not lifted by quantum corrections.
The paper is organized as follows. In Sec. 2 we review the description of non-Abelian vortex as a critical superstring on a conifold and identify massless string state. In Sec. 3 we review LST approach in terms of non-critical $c = 1$ string and the spectrum of massive states. In Sec. 4 we introduce 4D supercharges and construct massless BPS hypermultiplet. In Sec. 5 we consider the lowest massive string excitations and show that they forms a long vector supermultiplet. Section 6 deals with the construction of $\mathcal{N} = 2$ spin-2 stringy supermultiplet. In Sec. 7 we discuss linear Regge trajectories, while Section 8 summarizes our conclusions. In Appendix A we describe the BRST operator and transitions between different pictures. In Appendix B we review long $\mathcal{N} = 2$ supermultiplets in 4D.

2 Non-Abelian vortex string

2.1 Four-dimensional $\mathcal{N} = 2$ SQCD

As was already mentioned non-Abelian vortex-strings were first found in 4D $\mathcal{N} = 2$ SQCD with the gauge group $U(N)$ and $N_f \geq N$ flavors (i.e. the quark hypermultiplets) supplemented by the FI term $\xi$ [8, 9, 10, 11], see for example [14] for a detailed review of this theory. Here we just mention that at weak coupling, $g^2 \ll 1$, this theory is in the Higgs phase in which the scalar components of the quark multiplets (squarks) develop vacuum expectation values (VEVs). These VEVs breaks the $U(N)$ gauge group Higgsing all gauge bosons. The Higgsed gauge bosons combine with the screened quarks to form long $\mathcal{N} = 2$ multiplets with mass $m \sim g \sqrt{\xi}$.

The global flavor $SU(N_f)$ is broken down to the so called color-flavor locked group. The resulting global symmetry is

$$SU(N)_{C+F} \times SU(N_f - N) \times U(1)_B,$$  \hspace{1cm} (2.1)

see [14] for more details.

The unbroken global $U(1)_B$ factor above is identified with a baryonic symmetry. Note that what is usually identified as the baryonic $U(1)$ charge is a part of our 4D theory gauge group. “Our” $U(1)_B$ is an unbroken by squark VEVs combination of two $U(1)$ symmetries: the first is a subgroup of the flavor $SU(N_f)$ and the second is the global $U(1)$ subgroup of $U(N)$ gauge symmetry.
Figure 1: Examples of the monopole “necklace” baryons: Open circles denote monopoles.

As was already noted, we consider $\mathcal{N} = 2$ SQCD in the Higgs phase: $N$ squarks condense. Therefore, non-Abelian vortex strings confine monopoles. In the $\mathcal{N} = 2$ 4D theory these strings are 1/2 BPS-saturated; hence, their tension is determined exactly by the FI parameter,

$$T = 2\pi \xi.$$  \hfill (2.2)

However, the monopoles cannot be attached to the string endpoints. In fact, in the $U(N)$ theories confined monopoles are junctions of two distinct elementary non-Abelian strings \cite{26,10,11} (see \cite{14} for a review). As a result, in four-dimensional $\mathcal{N} = 2$ SQCD we have monopole-antimonopole mesons in which the monopole and antimonopole are connected by two confining strings. In addition, in the $U(N)$ gauge theory we can have baryons appearing as a closed “necklace” configurations of $N\times$(integer) monopoles \cite{14}. For the $U(2)$ gauge group the lightest baryon presented by such a “necklace” configuration consists of two monopoles, see Fig. 1.

Both stringy monopole-antimonopole mesons and monopole baryons with spins $J \sim 1$ have mass determined by the string tension, $\sim \sqrt{\xi}$ and are heavier at weak coupling than perturbative states, which have mass $m \sim g\sqrt{\xi}$. However, according to our conjecture, at strong coupling near the critical point $g_5^2$ $m \to \infty$, see \cite{2} and Sec. 2.3 below. In this regime perturbative states decouple and we are left with hadrons formed by the closed string states. All hadrons identified as closed string states in this paper turn out to be baryons and look like monopole “necklaces,” see Fig. 1.

\footnote{There are also massless bifundamental quarks, charged with respect to both non-Abelian factors in \cite{21}. These are associated with the Higgs branch present in 4D QCD, see \cite{14,19} for details.}
2.2 World sheet sigma model

The presence of color-flavor locked group $\text{SU}(N)_{C+F}$ is the reason for the formation of the non-Abelian vortex strings [8, 9, 10, 11] in our 4D SQCD. The most important feature of these vortices is the presence of the so-called orientational zero modes.

Let us briefly review the model emerging on the world sheet of the non-Abelian critical string [2, 18, 19]. If $N_f = N$ the dynamics of the orientational zero modes of the non-Abelian vortex, which become orientational moduli fields on the world sheet, is described by two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric CP$(N-1)$ model [14].

If one adds extra quark flavors, non-Abelian vortices become semilocal. They acquire size moduli [27]. In particular, for the non-Abelian semilocal vortex at hand, in addition to the orientational zero modes $n^P$ ($P = 1, 2$), there are the so-called size moduli $\rho^K$ ($K = 1, 2$) [27, 8, 11, 28, 29, 30].

The target space of the WCP(2, 2) sigma model on the string world sheet is defined by the $D$-term condition

$$|n^P|^2 - |\rho^K|^2 = \beta, \quad (2.3)$$

and a U(1) phase is gauged away.

The total number of real bosonic degrees of freedom in this model is six, where we take into account the constraint $(2.3)$ and the fact that one U(1) phase is gauged away. As was already mentioned, these six internal degrees of freedom are combined with four translational moduli to form a ten-dimensional space needed for superstring to be critical.

At weak coupling the world sheet coupling constant $\beta$ in $(2.3)$ is related to the 4D SU(2) gauge coupling as follows: $g^2$

$$\beta \approx \frac{4\pi}{g^2}, \quad (2.4)$$

see [14]. Note that the first (and the only) coefficient is the same for the 4D SQCD and the world-sheet model $\beta$ functions. Both vanish at $N_f = 2N$. This ensures that our world-sheet theory is conformal.

Since non-Abelian vortex string is 1/2 BPS it preserves $\mathcal{N} = (2, 2)$ in the world sheet sigma model which is necessary to have $\mathcal{N} = 2$ space-time supersymmetry [31, 32]. Moreover, as was shown in [19], the string theory of the non-Abelian critical vortex is type IIA.
The global symmetry of the world-sheet sigma model is
\[ \text{SU}(2) \times \text{SU}(2) \times \text{U}(1), \] (2.5)
i.e. exactly the same as the unbroken global group in the 4D theory, cf. (2.1), at \( N = 2 \) and \( N_f = 4 \). The fields \( n \) and \( \rho \) transform in the following representations:
\[ n : \ (2, \ 0, \ 0), \quad \rho : \ (0, \ 2, \ 1). \] (2.6)

### 2.3 Thin string regime

The coupling constant of 4D SQCD can be complexified
\[ \tau \equiv i \frac{4\pi}{g^2} + \frac{\theta_{4D}}{2\pi}, \] (2.7)
where \( \theta_{4D} \) is the four-dimensional \( \theta \) angle. Note that SU(\( N \)) version of the four-dimensional SQCD at hand possesses a strong-weak coupling duality, namely, \( \tau \to -\frac{1}{\tau} \) [20, 21]. This suggests that the self-dual point \( g^2 = 4\pi \) would be a natural candidate for a critical value \( g^2_c \), where our non-Abelian vortex string becomes thin\(^5\). However, as was shown recently in [22], \( S \)-duality maps our U(\( N \)) theory to a theory in which a different U(1) subgroup of the flavor group is gauged. In particular, in our U(\( N \)) theory all quark flavors have equal charges with respect to the U(1) subgroup of the U(2) gauge group, while in the \( S \)-dual version only one flavor is charged with respect to the U(1) gauge group. As a result, the \( S \)-dual version supports a different type of non-Abelian strings [22].

This means that \( S \)-duality is broken in our 4D theory by the choice of the U(1) subgroup which is gauged\(^6\). We do not consider \( S \)-duality and its consequences here.

The two-dimensional coupling constant \( \beta \) can be naturally complexified too if we include the two-dimensional \( \theta \) term,
\[ \beta \to \beta + i \frac{\theta_{2D}}{2\pi}. \] (2.8)

\(^5\)We suggested this earlier in [18, 19].

\(^6\)We are grateful to E. Gerchkovitz and A. Karasik for pointing out to us this circumstance.
The exact relation between the complexified 4D and 2D couplings is as follows:

\[ \exp(-2\pi \beta) = -h(\tau)[h(\tau) + 2], \]  

(2.9)

where the function \( h(\tau) \) is a special modular function of \( \tau \) defined in terms of the \( \theta \)-functions,

\[ h(\tau) = \theta_4^4/(\theta_2^4 - \theta_1^4). \]

This function enters the Seiberg-Witten curve in our 4D theory [20, 21]. Equation (2.9) generalizes the quasiclassical relation (2.4). It can be derived using 2D-4D correspondence, namely, the match of the BPS spectra of the 4D theory at \( \xi = 0 \) and the world-sheet theory on the non-Abelian string [33, 34, 10, 11]. Details of this derivation will be presented elsewhere. Note that our result (2.9) differs from that obtained in [22] using the localization technique.

According to the hypothesis formulated in [2], our critical non-Abelian string becomes infinitely thin at strong coupling at the critical point \( \tau_c \) (or \( g_{\text{c}}^2 \)). Moreover, in [19] we conjectured that \( \tau_c \) corresponds to \( \beta = 0 \) in the world-sheet theory via relation (2.9). Thus, we assume that \( m \to \infty \) at \( \beta = 0 \), which corresponds to \( g^2 = g_{\text{c}}^2 \) in 4D SQCD.

At the point \( \beta = 0 \) the non-Abelian string becomes infinitely thin, higher derivative terms can be neglected and the theory of the non-Abelian string reduces to the WCP(2,2) model. The point \( \beta = 0 \) is a natural choice because at this point we have a regime change in the 2D sigma model. This is the point where the resolved conifold defined by the \( D \) term constraint (2.3) develops a conical singularity [17].

### 2.4 Massless 4D baryon as deformation of the conifold complex structure

In this section we will briefly review the only 4D massless state associated with the deformation of the conifold complex structure. It was found in [19]. As was already mentioned, all other modes arising from the massless 10D graviton have non-normalizable wave functions over the conifold. In particular, the 4D graviton is absent [19]. This result matches our expectations since from the very beginning we started from \( \mathcal{N} = 2 \) SQCD in the flat four-dimensional space without gravity.

The target space of the world sheet WCP(2,2) model is defined by the \( D \)-term condition (2.3). We can construct the U(1) gauge-invariant “mesonic”
variables

\[ w^{PK} = n^P \rho^K. \]  

(2.10)

These variables are subject to the constraint \( \det w^{PK} = 0 \), or

\[ \sum_{\alpha=1}^{4} w_\alpha^2 = 0, \]  

(2.11)

where

\[ w^{PK} \equiv \sigma^P_{\alpha^K} w_\alpha, \]

and the \( \sigma \) matrices above are \((1, -i\tau^a)\), \( a = 1, 2, 3 \). Equation (2.11) defines the conifold \( Y_6 \). It has the Kähler Ricci-flat metric and represents a non-compact Calabi-Yau manifold [16, 35, 17]. It is a cone which can be parametrized by the non-compact radial coordinate

\[ \tilde{r}^2 = \sum_{\alpha=1}^{4} |w_\alpha|^2 \]  

(2.12)

and five angles, see [16]. Its section at fixed \( \tilde{r} \) is \( S_2 \times S_3 \).

At \( \beta = 0 \) the conifold develops a conical singularity, so both \( S_2 \) and \( S_3 \) can shrink to zero. The conifold singularity can be smoothed out in two distinct ways: by deforming the Kähler form or by deforming the complex structure. The first option is called the resolved conifold and amounts to introducing a non-zero \( \beta \) in (2.3). This resolution preserves the Kähler structure and Ricci-flatness of the metric. If we put \( \rho^K = 0 \) in (2.3) we get the \( \mathbb{CP}(1) \) model with the \( S_2 \) target space (with the radius \( \sqrt{\beta} \)). The resolved conifold has no normalizable zero modes. In particular, the modulus \( \beta \) which becomes a scalar field in four dimensions has non-normalizable wave function over the \( Y_6 \) manifold [19].

As explained in [36, 19], non-normalizable 4D modes can be interpreted as (frozen) coupling constants in the 4D theory. The \( \beta \) field is the most straightforward example of this, since the 2D coupling \( \beta \) is related to the 4D coupling, see Eq. (2.9).

If \( \beta = 0 \) another option exists, namely a deformation of the complex structure [17]. It preserves the Kähler structure and Ricci-flatness of the conifold and is usually referred to as the deformed conifold. It is defined by deformation of Eq. (2.11), namely,

\[ \sum_{\alpha=1}^{4} w_\alpha^2 = b, \]  

(2.13)
where \( b \) is a complex number. Now the \( S_3 \) can not shrink to zero, its minimal size is determined by \( b \).

The modulus \( b \) becomes a 4D complex scalar field. The effective action for this field was calculated in \[19\] using the explicit metric on the deformed conifold \[16, 37, 38\],

\[
S(b) = T \int d^4x |\partial_\mu b|^2 \log \frac{T^2 L^4}{|b|},
\]  

(2.14)

where \( L \) is the size of \( \mathbb{R}^4 \) introduced as an infrared regularization of logarithmically divergent \( b \) field norm\[7\]

We see that the norm of the \( b \) modulus turns out to be logarithmically divergent in the infrared. The modes with the logarithmically divergent norm are at the borderline between normalizable and non-normalizable modes. Usually such states are considered as “localized” on the string. We follow this rule. We can relate this logarithmic behavior to the marginal stability of the \( b \) state, see \[19\].

The field \( b \), being massless, can develop a VEV. Thus, we have a new Higgs branch in 4D \( \mathcal{N} = 2 \) SQCD which is developed only for the critical value of the coupling constant \( g^2 \).

The logarithmic metric in (2.14) in principle can receive both perturbative and non-perturbative quantum corrections in \( 1/\beta \), the sigma model coupling. However, in the \( \mathcal{N} = 2 \) theory the non-renormalization theorem of \[21\] forbids the dependence of the Higgs branch metric on the 4D coupling constant \( g^2 \). Since the 2D coupling \( \beta \) is related to \( g^2 \) we expect that the logarithmic metric in (2.14) will stay intact. This expectation is confirmed in \[1\].

In \[19\] the massless state \( b \) was interpreted as a baryon of 4D \( \mathcal{N} = 2 \) SQCD. Let us explain this. From Eq. (2.13) we see that the complex parameter \( b \) (which is promoted to a 4D scalar field) is singlet with respect to both SU(2) factors in (2.5), i.e. the global world-sheet group\[8\]. What about its baryonic charge?

Since

\[
w_\alpha = \frac{1}{2} \text{Tr} \left[ (\bar{\sigma}_\alpha)_{KP} n^P \rho^K \right]
\]  

(2.15)

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\[7\]The infrared regularization on the conifold \( \tilde{r}_{\text{max}} \) translates into the size \( L \) of the 4D space because the variables \( \rho \) in (2.12) have an interpretation of the vortex string sizes, \( \tilde{r}_{\text{max}} \sim TL^2 \).

\[8\]Which is isomorphic to the 4D global group (2.1) at \( N = 2, N_f = 4 \).
we see that the $b$ state transforms as

$$(1, 1, 2),$$

where we used (2.6) and (2.13). Three numbers above refer to the representations of (2.5). In particular it has the baryon charge $Q_B(b) = 2$.

To conclude this section let us note that in our case of type IIA superstring the complex scalar associated with deformations of the complex structure of the Calabi-Yau space enters as a component of a massless 4D $\mathcal{N} = 2$ hypermultiplet, see [39] for a review. Instead, for type IIB superstring it would be a component of a vector BPS multiplet. Non-vanishing baryonic charge of the $b$ state confirms our conclusion that the string under consideration is a type IIA.

3 Massive states from non-critical $c = 1$ string

As was explained in Sec. 1, the critical string theory on the conifold is hard to use for calculating the spectrum of massive string modes because the supergravity approximation does not work. In this section we review the results obtained in [1] based on the little string theory (LST) approach. Namely, in [1] we used the equivalent formulation of our theory as a non-critical $c = 1$ string theory with the Liouville field and a compact scalar at the self-dual radius [24, 25]. We intend to use the same formulation in this paper to analyze the 4D hypermultiplet structure of the massive states.

3.1 Non-critical $c = 1$ string theory

Non-critical $c = 1$ string theory is formulated on the target space

$$\mathbb{R}^4 \times \mathbb{R}_\phi \times S^1,$$

where $\mathbb{R}_\phi$ is a real line associated with the Liouville field $\phi$ and the theory has a linear in $\phi$ dilaton, such that string coupling is given by

$$g_s = e^{-\frac{Q}{2} \phi}.$$  

(3.2)

We will determine $Q$ in Eq. (3.1).

Genericity the above equivalence is formulated between the critical string on non-compact Calabi-Yau spaces with an isolated singularity on the one
hand, and non-critical $c = 1$ string with the additional Ginzburg-Landau
$\mathcal{N} = 2$ superconformal system \[24\], on the other hand. In the conifold case
this extra Ginzburg-Landau factor in (3.1) is absent \[40\].

In \[41, 24, 40\] it was argued that non-critical string theories with the
string coupling exponentially falling off at $\phi \to \infty$ are holographic. The
string coupling goes to zero in the bulk of the space-time and non-trivial
dynamics (LST)\[9\] is localized on the “boundary.” In our case the “boundary”
is the four-dimensional space in which $\mathcal{N} = 2$ SQCD is defined. (It is worth
emphasizing that in our case the boundary 4D dynamics is the starting point
while the extra six dimensions represent an auxiliary mathematical construct.
Perhaps, it can be referred to as a “reverse holography.”)

In other words, holography for our non-Abelian vortex string theory is
most welcome and expected. We start with $\mathcal{N} = 2$ SQCD in 4D space and
study solitonic vortex strings. In our approach 10D space formed by 4D
“actual” space and six internal moduli of the string is an artificial construction
needed to formulate the string theory of a special non-Abelian vortex. Clearly we expect that all non-trivial “actual” physics should be localized exclusively on the 4D “boundary.” In other words, we expect that LST in
our case is nothing but 4D $\mathcal{N} = 2$ SQCD at the critical value of the gauge
coupling $g_c^2$ (in the hadronic description).

The linear dilaton in (3.2) implies that the bosonic stress tensor of $c = 1$
matter coupled to 2D gravity is

$$T_{-\cdot} = -\frac{1}{2} \left[ (\partial_\phi)^2 + Q \partial_\phi^2 + (\partial_Y)^2 \right], \tag{3.3}$$

where $\partial_- = \partial_z$. The compact scalar $Y$ represents $c = 1$ matter and satisfies
the following condition:

$$Y \sim Y + 2\pi Q. \tag{3.4}$$

Here we normalize the scalar fields in such a way that their propagators are

$$\langle \phi(z), \phi(0) \rangle = -\log z\bar{z}, \quad \langle Y(z), Y(0) \rangle = -\log z\bar{z}. \tag{3.5}$$

The central charge of the supersymmetrized $c = 1$ theory above is

$$c_{SUSY}^{\phi+Y} = 3 + 3Q^2. \tag{3.6}$$

\[9\]The main example of this behavior is non-gravitational LST in the flat six-dimensional
space formed by the world volume of parallel NS5 branes.
The criticality condition for the string on (3.1) implies that this central charge should be equal to 9. This gives
\[ Q = \sqrt{2}, \] (3.7)
to be used in Eq. (3.2).

Deformation of the conifold (2.13) translates into adding the Liouville interaction to the world-sheet sigma model [24],
\[ \delta L = b \int d^2 \theta e^{-\phi} \frac{\phi + Y}{Q}. \] (3.8)

The conifold singularity at \( b = 0 \) corresponds to the string coupling constant becoming infinitely large at \( \phi \to -\infty \), see (3.2). At \( b \neq 0 \) the Liouville interaction regularize the behavior of the string coupling preventing the string from propagating to the region of large negative \( \phi \).

In fact the \( c = 1 \) non-critical string theory can also be described in terms of two-dimensional black hole [42], which is the \( \text{SL}(2, R)/U(1) \) coset WZNW theory [43, 25, 44, 24] at level
\[ k = \frac{2}{Q^2}. \] (3.9)

In [45] it was shown that \( \mathcal{N} = (2, 2) \) \( \text{SL}(2, R)/U(1) \) coset is a mirror description of the \( c = 1 \) Liouville theory. The relation above implies in the case of the conifold \( (Q = \sqrt{2}) \) that
\[ k = 1, \] (3.10)
where \( k \) is the total level of the Kač-Moody algebra in the supersymmetric version (the level of the bosonic part of the algebra is then \( k_b = k + 2 = 3 \)).
The target space of this theory has the form of a semi-infinite cigar; the field \( \phi \) associated with the motion along the cigar cannot take large negative values due to semi-infinite geometry. In this description the string coupling constant at the tip of the cigar is \( g_s \sim 1/b \).

### 3.2 Vertex operators

Vertex operators for the string theory on (3.1) are constructed in [24], see also [43, 40]. Primaries of the \( c = 1 \) part for large positive \( \phi \) (where the target space becomes a cylinder \( \mathbb{R}_\phi \times S^1 \)) take the form
\[ V_{j,m_L}^L \times V_{j,m_R}^R \approx \exp \left( \sqrt{2} j \phi + i \sqrt{2} (m_L Y_L + m_R Y_R) \right), \] (3.11)
where we split $\phi$ and $Y$ into left and right-moving parts, say $\phi = \phi_L + \phi_R$. For the self-dual radius (3.7) (or $k = 1$) the parameter $2m$ in Eq. (3.11) is integer. For the left-moving sector $2m_L \equiv 2m$ is the total momentum plus the winding number along the compact dimension $Y$. For the right-moving sector we introduce $2m_R$ which is the winding number minus momentum. We will see below that for our case type IIA string $m_R = -m$, while for type IIB string $m_R = m$.

The primary operator (3.11) is related to the wave function over “extra dimensions” as follows:

$$V_{j,m} = g_s \Psi_{j,m}(\phi, Y).$$

The string coupling (3.2) depends on $\phi$. Thus,

$$\Psi_{j,m}(\phi, Y) \sim e^{\sqrt{2}(j+\frac{1}{2})\phi + i\sqrt{2}mY}. \quad (3.12)$$

We will look for string states with normalizable wave functions over the “extra dimensions” which we will interpret as hadrons in 4D $\mathcal{N} = 2$ SQCD. The condition for the string states to have normalizable wave functions reduces to

$$j \leq -\frac{1}{2}. \quad (3.13)$$

The scaling dimension of the primary operator (3.11) is

$$\Delta_{j,m} = m^2 - j(j + 1). \quad (3.14)$$

Unitarity implies that it should be positive,

$$\Delta_{j,m} > 0. \quad (3.15)$$

Moreover, to ensure that conformal dimensions of left and right-moving parts of the vertex operator (3.11) are the same we impose that $m_R = \pm m_L$.

The spectrum of the allowed values of $j$ and $m$ in (3.11) was exactly determined by using the Kač-Moody algebra for the coset $\mathrm{SL}(2, R)/\mathrm{U}(1)$ in [46, 47, 48, 49, 43], see [50] for a review. Both discrete and continuous representations were found. Parameters $j$ and $m$ determine the global quadratic Casimir operator and the projection of the spin on the third axis,

$$J^2 |j, m\rangle = -j(j+1) |j, m\rangle, \quad J^3 |j, m\rangle = m |j, m\rangle \quad (3.16)$$

We include the case $j = -\frac{1}{2}$ which is at the borderline between normalizable and non-normalizable states. In [1] it is shown that $j = -\frac{1}{2}$ corresponds to the norm logarithmically divergent in the infrared in much the same way as the norm of the $b$ state, see (2.14)
where $J^a$ ($a = 1, 2, 3$) are the global SL(2,R) currents.

We will focus on **discrete representations** with

\[
j = \pm \frac{1}{2}, -1, -\frac{3}{2}, ..., \quad m = \pm \{j, j-1, j-2, ...\}. \tag{3.17}
\]

Discrete representations include the normalizable states localized near the tip of the cigar (see (3.13)), while the continuous representations contain non-normalizable states.

Discrete representations contain states with negative norm. To exclude these ghost states a restriction for spin $j$ is imposed [46, 47, 48, 49, 50]

\[-\frac{k+2}{2} < j < 0. \tag{3.18}\]

Thus, for our value $k = 1$ we are left with only two allowed values of $j$,

\[
j = \pm \frac{1}{2}, \quad m = \pm \left\{ \frac{1}{2}, \frac{3}{2}, ... \right\} \tag{3.19}
\]

and

\[
j = -1, \quad m = \pm \{1, 2, ...\}. \tag{3.20}
\]

### 3.3 Scalar and spin-2 states

Four-dimensional spin-0 and spin-2 states were found in [1] using vertex operators ((3.11)). The 4D scalar vertices $V^S$ in the $(-1, -1)$ picture have the form [24]

\[
V_{j,m}^{S,L} \times V_{j,-m}^{S,R} = e^{-\varphi_L - \varphi_R} e^{ip_\mu x^\mu} V_{j,m}^{L} \times V_{j,-m}^{R}, \tag{3.21}
\]

where superscript $S$ stands for scalar, $\varphi_{L,R}$ represents bosonized ghost in the left and right-moving sectors, while $p_\mu$ is the 4D momentum of the string state.

The condition for the state (3.21) to be physical is

\[
\frac{1}{2} + \frac{p_\mu p^{\mu}}{8\pi T} + m^2 - j(j+1) = 1, \tag{3.22}
\]

where $1/2$ comes from the ghost and we used (3.14). We note that the conformal dimension of the ghost operator $e^{(q\varphi)}$ is equal to $-(q + q^2/2)$, where $q$ is the picture number.
The GSO projection restricts the integer $2m$ for the operator in (3.21) to be odd \[51, 24\],

$$m = \frac{1}{2} + \mathbb{Z}.$$ \hspace{1cm} (3.23)

For half-integer $m$ we have only one possibility $j = -\frac{1}{2}$, see (3.19). This determines the masses of the 4D scalars,

$$\frac{(M^S_m)^2}{8\pi T} = -\frac{p_\mu p^\mu}{8\pi T} = m^2 - \frac{1}{4},$$ \hspace{1cm} (3.24)

where the Minkowski 4D metric with the diagonal entries $(-1, 1, 1, 1)$ is used.

In particular, the state with $m = \pm 1/2$ is the massless baryon $b$, associated with deformations of the conifold complex structure [1], while states with $m = \pm (3/2, 5/2, ...)$ are massive 4D scalars.

At the next level we consider 4D spin-2 states. The corresponding vertex operators are given by

$$(V^L_{j,m} \times V^R_{j,-m}(p_\mu))^\text{spin-2} = \xi_{\mu\nu} \psi^\mu_L \psi^\nu_R e^{-\varphi_L - \varphi_R} e^{ip_\mu x^\mu} V^L_{j,m} \times V^R_{j,-m},$$ \hspace{1cm} (3.25)

where $\psi^\mu_{L,R}$ are the world-sheet superpartners to 4D coordinates $x^\mu$, while $\xi_{\mu\nu}$ is the polarization tensor.

The condition for these states to be physical takes the form

$$\frac{p_\mu p^\mu}{8\pi T} + m^2 - j(j + 1) = 0.$$ \hspace{1cm} (3.26)

The GSO projection selects now $2m$ to be even, $|m| = 0, 1, 2, ...$ [24], thus we are left with only one allowed value of $j$, $j = -1$ in (3.20). Moreover, the value $m = 0$ is excluded. This leads to the following expression for the masses of spin-2 states:

$$(M^{\text{spin-2}}_m)^2 = 8\pi T m^2, \hspace{1cm} |m| = 1, 2, ....$$ \hspace{1cm} (3.27)

We see that all spin-2 states are massive. This confirms the result in [19] that no massless 4D graviton appears in our theory. It also matches the fact that our “boundary” theory, 4D $\mathcal{N} = 2$ QCD, is defined in flat space without gravity.

To determine baryonic charge of these states we note that $U(1)_B$ transformation of $b$ in the Liouville interaction (3.8) is compensated by a shift of

\[\text{We will demonstrate this in the next section.}\]
Figure 2: Spectrum of spin-0 and spin-2 states as a function of the baryonic charge. Closed and open circles denote spin-0 and spin-2 states, respectively.

The baryonic charge of $b$ is two, see (2.16). Below we use the following convention: upon splitting $Y$ into left and right-moving parts $Y = Y_L + Y_R$ we define that only $Y_L$ is shifted under $U(1)_B$ transformation,

$$ b \rightarrow e^{2i \theta} b, \quad Y_L \rightarrow Y_L + 2\sqrt{2} \theta, \quad Y_R \rightarrow Y_R. \quad (3.28) $$

This gives for the baryon charge of the vertex operator (3.11)

$$ Q_B = 4m. \quad (3.29) $$

We see that the momentum $m$ in the compact $Y$ direction is in fact the baryon charge of a string state. All states we found above are baryons. Their masses as a function of the baryon charge are shown in Fig. 2.

The momentum $m$ in the compact dimension is also related to the $R$-charge. On the world sheet we can introduce the left and right $R$-charges separately. Normalizing charge of $\theta^+$, namely, $R^{(2)}_L(\theta^+) = 1$, we see that $Y$ should be shifted under the $R^{(2)}_L$ symmetry to make invariant the Liouville interaction (3.8).

This gives

$$ R^{(2)}_L(V^L_{j,m}) = -2m \quad (3.30) $$
for the $R_L^{(2)}$ charge of the vertex (3.11), which is the bottom component of the world sheet supermultiplet. The $R_R^{(2)}$ charge in the right-moving sector is defined similarly. Here superscript (2) denotes the world sheet $R$-charge.

As was discussed above, the massless baryon $b$ corresponds to $j = -1/2$, $m = \pm 1/2$. Thus, the associated vertex $V_{j,m}$ has $R_L^{(2)} = \pm 1$ and conformal dimension $\Delta = 1/2$, see (3.14). Therefore it satisfies the relation

$$\Delta = \frac{|R_L^{(2)}|}{2}$$

as expected for the bottom component of a chiral primary operator, which defines the short representation of supersymmetry algebra (and similar relation in the right-moving sector). In 4D theory $b$ is a component of a short $\mathcal{N} = 2$ BPS multiplet, namely hypermultiplet.

## 4 Massless hypermultiplet

The remainder of this paper is devoted to the study the supermultiplet structure of the 4D string states described in the previous sections. Our strategy is as follows: we explicitly construct 4D supercharges and use them to generate all components of a given multiplet starting from a scalar or spin-2 representative shown in (3.21) or (3.25). We will generate supermultiplets originating from the lowest states with $j = -1/2$, $m = \pm (1/2, 3/2)$ and $j = -1$, $m = \pm 1$. In this section we will start with the massless baryon $b$.

### 4.1 4D supercharges

First we bosonize world sheet fermions $\psi_{\mu}$, $\psi_\phi$ and $\psi_Y$, the superpartners of $x_\mu$, the Liouville field $\phi$ and the compact scalar $Y$, respectively. Following the standard rule we divide them into pairs

$$\psi_k = \frac{1}{\sqrt{2}}(\psi_{2k-1} - i\psi_{2k}), \quad \bar{\psi}_k = \frac{1}{\sqrt{2}}(\bar{\psi}_{2k-1} + i\bar{\psi}_{2k}), \quad k = 1, 2,$$

$$\psi = \frac{1}{\sqrt{2}}(\psi_\phi - i\psi_Y), \quad \bar{\psi} = \frac{1}{\sqrt{2}}(\bar{\psi}_\phi + i\bar{\psi}_Y),$$

and define

$$\psi_k \bar{\psi}_k = i\partial H_k \quad \text{(no summation)}, \quad \psi \bar{\psi} = i\partial H,$$
where the bosons $H_k$ and $H$ have the standard propagators

$$\langle H_k(z), H_l(0) \rangle = -\delta_{kl} \log z, \quad \langle H(z), H(0) \rangle = -\log z \quad (4.4)$$

and

$$\psi_k \sim e^{iH_k}, \quad \psi \sim e^{iH}. \quad (4.5)$$

The above formulas are written for the left-moving sector. In the right-moving sector bosonization is similar with the replacement $z \rightarrow \bar{z}$ and $\partial_z \rightarrow \partial_{\bar{z}}$.

As usual, we define spinors in terms of scalars $H$. Namely,

$$S_\alpha = e^{\sum_k i s_k H_k}, \quad \bar{S}_\dot{\alpha} = e^{\sum_k i \bar{s}_k H_k} \quad (4.6)$$

are 4D spinors, $\alpha = 1, 2, \dot{\alpha} = 1, 2$. Moreover,

$$S = e^{i \frac{H}{2}}, \quad \bar{S} = e^{-i \frac{H}{2}} \quad (4.7)$$

are spinors associated with "extra" dimensions $\phi$ and $Y$. Here $s_k = \pm \frac{1}{2}$, $k = 1, 2$ and the choices of the allowed values of $s_k$ are restricted by the GSO projection, see below.

Supercharges for non-critical string are defined in [51]. In our case four 4D $\mathcal{N} = 1$ supercharges

$$Q_\alpha = \frac{1}{2\pi i} \frac{\bar{b}}{|b|} \int dz e^{-\frac{i}{2} S} S_\alpha S \exp \left( \frac{1}{\sqrt{2}} Y \right),$$

$$\bar{Q}_{\dot{\alpha}} = \frac{1}{2\pi i} \frac{b}{|b|} \int dz e^{-\frac{i}{2} \bar{S}_{\dot{\alpha}}} \bar{S} \exp \left( -\frac{i}{\sqrt{2}} Y \right) \quad (4.8)$$

act in the left-moving sector, where we used the $\left( -\frac{1}{2} \right)$ picture. We have to multiply these supercharges in the left-moving sector by the phase factors $\bar{b}/|b|$ and $b/|b|$ to make them neutral with respect to baryonic $U(1)_B$. Other four supercharges of $\mathcal{N} = 2$ 4D supersymmetry are given by similar formulas and act in the right-moving sector. The action of the supercharge on a vertex is understood as an integral around the location of the vertex on the world sheet.

Supercharges (4.8) satisfy 4D space-time supersymmetry algebra

$$\{ Q_\alpha, \bar{Q}_{\dot{\alpha}} \} = 2P_\mu \sigma^\mu, \quad (4.9)$$

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while all other anti-commutators vanish. Note that $P_\mu$ is the 4D momentum operator, the anti-commutator (4.9) does not produce translation in the Liouville direction.

The GSO projection is the requirement of locality of a given vertex operator with respect to the supercharges (4.8).

Let us start with $Q_\alpha$ with $s_k^{(0)} = (1/2, 1/2)$. Then mutual locality of the supercharges (4.8) selects polarizations

$$s_k = \pm \left( \frac{1}{2}, \frac{1}{2} \right), \quad \bar{s}_k = \pm \left( \frac{1}{2}, -\frac{1}{2} \right)$$

(4.10)

associated with four supercharges $Q_\alpha$ and $\bar{Q}_\dot{\alpha}$.

As an example, let us check the GSO selection rule (3.23) for 10D "tachyon" vertices (3.21). We have

$$\langle Q_\alpha, V^S_L(w) \rangle \sim \int dz \left\{ (z - w)^{-(\frac{1}{2} - m)} + \ldots \right\}$$

(4.11)

where dots stand for less singular OPE terms and 1/2 comes from the ghost $\varphi$. We see that locality requirement selects half-integer $m$ as shown in (3.23). Note that an important feature of the supercharges (4.8) is the dependence on momentum $m$ in the compact direction $Y$. Without this dependence all 10D "tachyon" vertices (3.21) would be projected out as it happens for critical strings. Note also that none of the states (3.21) are tachyonic in 4D.

Now we can introduce 4D space-time $R$-charges. We normalize them as follows:

$$R^{(4)} = R_L^{(4)} + R_R^{(4)}, \quad R_L^{(4)} (Q_\alpha) = -1, \quad R_L^{(4)} (\bar{Q}_\dot{\alpha}) = 1,$$

(4.12)

and use the same normalizations for $R_R^{(4)}$. This definition ensures that for a given vertex operator we have

$$R_L^{(4)} = -2m_L, \quad R_R^{(4)} = -2m_R.$$

(4.13)

Note that the scalars $H$ are not shifted upon $R^{(4)}$ rotations, so the worldsheet fermions $\psi_k, \bar{\psi}$ do not have $R^{(4)}$ charges. This is in contrast with the action of the world sheet $R^{(2)}$ symmetry.
4.2 Fermion vertex

To generate fermion vertex for the $b$ state we apply supercharges (4.8) to the left-moving part of the vertex (3.21) with $j = -1/2$ and $m = \pm 1/2$. To get the fermion vertex in the standard ($-1/2$) picture we have to convert the vertex (3.21) from the ($-1$) to (0) picture. This is done in Appendix A using the BRST operator. The left-moving part of the scalar vertex (3.21) in the (0) picture has the form

$$V_{j,m}^{(0)}(p_{\mu}) = \left[ \sqrt{2}(j\psi_\phi + im\psi_Y) + \frac{i}{\sqrt{4\pi T}} p_{\mu} \psi^\mu \right] e^{ip_{\mu}x^\mu + \sqrt{2}j\phi + i\sqrt{2}mY}, \quad (4.14)$$

where we skip the subscripts $L$.

Let us start with $j = -1/2$ and $m = 1/2$. The vertex (4.14) reduces to

$$V_{-1/2,1/2}^{(0)}(p_{\mu}) = \left[ -\psi + \frac{i}{\sqrt{4\pi T}} p_{\mu} \psi^\mu \right] e^{ip_{\mu}x^\mu - \phi + iY}. \quad (4.15)$$

Applying the supercharge $Q_\alpha$ we find that correlation function does not contain pole contribution and hence gives zero. On the other hand $\Bar{Q}_\dot{\alpha}$ produces the following fermion vertex

$$\Bar{V}_{\dot{\alpha}}^{(-1/2)} = \langle \Bar{Q}_{\dot{\alpha}}, V_{-1/2,1/2}^{(0)}(p_{\mu}) \rangle$$

$$\sim e^{-\phi/2} \left[ -\Bar{S}_{\dot{\alpha}} S + \frac{ip_{\mu}}{\sqrt{4\pi T}} (\Bar{\sigma}_{\mu})_{\dot{\alpha}\alpha} S^\alpha \Bar{S} \right] e^{ip_{\mu}x^\mu - \phi/2}. \quad (4.16)$$

where we used

$$\langle \psi(z), \Bar{S}(w) \rangle \sim \frac{1}{\sqrt{(z-w)}} S,$$

$$\langle e^{-iY(z)}, e^{-iY(w)} \rangle \sim \frac{1}{\sqrt{(z-w)}},$$

$$\langle \psi_{\mu}(z), \Bar{S}(w)_{\dot{\alpha}} \rangle \sim \frac{1}{\sqrt{(z-w)}} (\Bar{\sigma}_{\mu})_{\dot{\alpha}\alpha} S^\alpha. \quad (4.17)$$

Note that the momentum $m$ along the compact direction is zero for the fermion vertex (4.16).
As a check we can calculate the conformal dimension of the vertex (4.16). The condition for this vertex to be physical is

\[
\frac{3}{8} + \frac{3}{8} + \frac{p_\mu p^\mu}{8\pi T} - j(j + 1) = 1,
\]

(4.18)

where the first and the second contributions come from the ghost $\phi$ and the scalars $H_k$ and $H$, respectively. We see that for $j = -1/2$ this state is massless, as expected.

By the same token, for $m = -1/2$ we consider the action of the supercharges on the vertex in (4.15) with $\psi \to \bar{\psi}$ and $m = -1/2$. Only the action of $Q_\alpha$ gives non-trivial fermion vertex. We get

\[
V^{\alpha,-\frac{1}{4}} = \langle Q_\alpha, V^{(0)}_{-\frac{1}{2}, m=-\frac{1}{2}}(p_\mu) \rangle
\]

\[
\sim e^{-\frac{\bar{\psi}}{2}} \left[ -S^\alpha \bar{S} + \frac{ip_\mu}{\sqrt{4\pi T}} (\sigma_\mu)^{\alpha\dot{\alpha}} \bar{S} \right] e^{ip_\mu x_\mu - \frac{\psi}{\sqrt{2}}}.
\]

(4.19)

To conclude this subsection we note that if we apply supercharges to the fermion vertices (4.16) and (4.19) we do not generate new states. For example, acting on (4.16) with $Q_\alpha$ gives (the left-moving part of) the scalar vertex (3.21),

\[
\langle Q_\alpha, \bar{V}^{(-\frac{1}{2})}_{\dot{\alpha}} \rangle \sim \bar{\psi}_{\dot{\alpha}} V^{S,L}_{\frac{1}{4}, m=\frac{1}{2}}
\]

in the picture ($-1$). This result is in full accord with supersymmetry algebra (4.9). Acting with $Q_{\dot{\alpha}}$ produces the scalar vertex (3.21) with $m = -1/2$,

\[
\langle Q_{\dot{\alpha}}, \bar{V}^{(-\frac{1}{2})}_{\dot{\beta}} \rangle \sim \varepsilon_{\dot{\alpha}\dot{\beta}} V^{S,L}_{-\frac{1}{4}, m=-\frac{1}{2}}.
\]

(4.21)

### 4.3 Building the hypermultiplet

In this section we will use the bosonic and fermionic vertices obtained above to construct hypermultiplet of the massless $b$ states. For simplicity in this section and below we will consider only bosonic components of supermultiplets. As was already mentioned, in the case of type IIA superstring we should consider the states with $m_R = -m_L \equiv -m$. We will prove this statement below, in this and the subsequent subsections.

In the NS-NS sector we have one complex (or two real) scalars (3.21),

\[
b = V^{S,L}_{j=-\frac{1}{2}, m} \times V^{S,R}_{j=-\frac{1}{2}, -m}
\]

(4.22)
associated with $m = \pm 1/2$.

Since for the scalar states the momentum $m$ is opposite in the left- and right-moving sectors, for the R-R states we get the product of fermion vertices (4.16) and (4.19), namely,

$$V_{\dot{\alpha}\alpha} = \bar{V}_L^\dot{\alpha} \times V_R^\alpha, \quad \bar{V}_{\dot{\alpha}\dot{\alpha}} = V_L^\dot{\alpha} \times \bar{V}_R^\dot{\alpha}.$$  \hspace{1cm} (4.23)

The vertices above define a complex vector $C^\mu$ via

$$V_{\dot{\alpha}\alpha} = (\dot{\sigma}_\mu)_{\dot{\alpha}\alpha} C^\mu.$$  \hspace{1cm} (4.24)

However, as is usual for the massless R-R string states, the number of physical degrees of freedom reduces because the fermion vertices (4.16) and (4.19) satisfy the massless Dirac equations which translate into the Bianchi identity for the associated form. For 1-form (vector) we have

$$\partial_\mu C_\nu - \partial_\nu C_\mu = 0,$$ \hspace{1cm} (4.25)

which ensures that the complex vector reduces to a complex scalar,

$$C_\mu = \partial_\mu \tilde{b}.$$ \hspace{1cm} (4.26)

Altogether we have two complex scalars, $b$ and $\tilde{b}$, which form the bosonic part of the hypermultiplet. As was already mentioned, deformations of the complex structure of a Calabi-Yau manifold gives a massless hypermultiplet for type IIA theory and massless vector multiplet for type IIB theory. The derivation above shows that our choice $m_R = -m_L$ corresponds to type IIA string.

We stress again that this massless hypermultiplet is a short BPS representation of $\mathcal{N} = 2$ supersymmetry algebra in 4D and is characterized by the non-zero baryonic charge $Q_B(b) = \pm 2$.

Let us also note that the four-dimensional space-time $R^{(4)}$ charge of the vertex operator (4.22) vanishes due to cancellation between left and right-moving sectors, see (4.13). For the vertex (4.23) it is also zero since both $m_L$ and $m_R$ are zero. Thus we conclude that $b$ and $\tilde{b}$ have the vanishing $R^{(4)}$ charge, as expected for the scalar components of a hypermultiplet.

### 4.4 What would we get for type IIB superstring?

Our superstring is of type IIA. This is fixed by derivation of our string theory as a description of non-Abelian vortex in 4D $\mathcal{N} = 2$ SQCD, see [19]. In this
subsection we “forget” for a short while about this and consider superstring theory on the manifold (3.1) on its own right. Then, as usual in string theory, we have two options for a closed string: type IIA and type IIB. We will show below that type IIB option corresponds to the choice $m_R = m_L$.

For this choice the massless state with $j = -1/2$ is described as follows. In the NS-NS sector we have one complex scalar,

$$a = V^{S,L}_{j=-\frac{1}{2},m} \times V^{S,R}_{j=-\frac{1}{2},m},$$  \hspace{1cm} (4.27)

associated with $m = \pm 1/2$. In the R-R sector we now obtain

$$V_{\alpha\beta} = V^L_{\alpha} \times V^R_{\beta}, \quad \bar{V}_{\dot{\alpha}\dot{\beta}} = \bar{V}^L_{\dot{\alpha}} \times \bar{V}^R_{\dot{\beta}}. \hspace{1cm} (4.28)$$

Expanding the complex vertex $V_{\alpha\beta}$ in the basis of $\sigma$ matrices

$$V^\alpha_{\beta} = F \delta^\alpha_{\beta} + (\sigma_\mu \bar{\sigma}_\nu)^\alpha_{\beta} C_{\mu\nu}$$  \hspace{1cm} (4.29)

we get a complex scalar $F$ and a complex 2-form $C_{\mu\nu}$ which can be expressed in terms of a real 2-form, $C_{\mu\nu} = F_{\mu\nu} - i F^*_{\mu\nu}$, where $F_{\mu\nu}$ is real and $F^*_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} F^{\rho\lambda}$. The Dirac equations for the fermion vertices (4.16) and (4.19) imply that $F$ is a constant, while $F_{\mu\nu}$ satisfies the Bianchi identity. This ensures that $F_{\mu\nu}$ can be constructed in terms of a real vector potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  \hspace{1cm} (4.30)

We see that we get a massless $\mathcal{N} = 2$ BPS vector multiplet with the bosonic components given by the complex scalar $a$ and the gauge potential $A_\mu$. This is what we expect from deformation of the complex structure of a Calabi-Yau manifold for type IIB string.

Let us note that $R$ charges also match since the $R^{(4)}$ charge of $a$ in (4.27) is $R^{(4)} = \pm 2$ (see (4.13)) while the $R^{(4)}$ charge of (4.28) and $A_\mu$ are zero as expected.

However, if we try to interpret this $\mathcal{N} = 2$ vector multiplet as a state of the non-Abelian vortex in $\mathcal{N} = 2$ SQCD we will get an inconsistency. To see this one can observe that our state has non-zero baryonic charge which cannot be associated with a gauge multiplet. This confirms our conclusion that the string theory for our non-Abelian vortex-string is of IIA type.
5 Exited state with \( j = -1/2 \)

Below we consider the supermultiplet structure of the lowest massive states given by the vertex operators (2.21) and (2.25). In this section we start with the first excited state of the scalar vertex (2.21) with \( j = -1/2 \) and \( m = \pm 3/2 \). The mass of this state is

\[
\left( \frac{M_{j=-\frac{1}{2}, m=\pm\frac{3}{2}}}{8\pi T} \right)^2 = 2, \quad (5.1)
\]

see (3.24).

5.1 Action of supercharges

The left-moving part of the vertex operator in the \((0)\) picture is given by (4.14). For \( m = 3/2 \) we obtain

\[
V_{-\frac{3}{2}, \frac{3}{2}}(p_\mu) = \left[ -\left(2\psi - \bar{\psi}\right) + \frac{i}{\sqrt{4\pi T}} p_\mu \psi^\mu \right] e^{i\sigma_\mu x^\mu - \frac{\phi}{2} + i\frac{Y}{\sqrt{2}}} \cdot (5.2)
\]

In much the same way as for the \( b \) state, the supercharge \( Q \) acting on the vertex above gives zero while the supercharge \( \bar{Q} \) produces the following fermion vertex in the picture \((-\frac{1}{2})\):

\[
\bar{V}_{\frac{1}{2}} = \langle \bar{Q}_{\dot{\alpha}}, V_{-\frac{1}{2}, m=\frac{3}{2}}(p_\mu) \rangle \sim e^{-\frac{x}{2}} \left[ -2\bar{S}_\dot{\alpha} S \right. \\
+ \left. \frac{ip_\mu}{\sqrt{4\pi T}} (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} \bar{S}_\alpha S \right] (\partial_- Y + \psi \psi Y) e^{i\sigma_\mu x^\mu - \frac{\phi}{2} + i\frac{Y}{\sqrt{2}}} \cdot (5.3)
\]

Note that the momentum \( m \) along the compact dimension is \( m = 1 \) for this vertex. It is easy to check that the mass of this fermion is given by (5.1).

In a similar manner, for \( m = -3/2 \) we use the bosonic vertex (5.2) with \( \psi \to \bar{\psi} \) and \( m = -3/2 \). Action of supercharge \( Q \) gives the following fermion vertex:

\[
V_{\alpha,-\frac{1}{2}} = \langle Q^\alpha, V_{-\frac{1}{2}, m=-\frac{3}{2}}(p_\mu) \rangle \sim e^{-\frac{x}{2}} \left[ -2S^\alpha \bar{S} \right. \\
+ \left. \frac{ip_\mu}{\sqrt{4\pi T}} (\sigma_\mu)^{\alpha\dot{\alpha}} \bar{S}_\dot{\alpha} S \right] (\partial_- Y + \psi \psi Y) e^{i\sigma_\mu x^\mu - \frac{\phi}{2} - i\frac{\sqrt{2}Y}{2}} \cdot (5.4)
\]
with \( m = -1 \).

Now let us apply the supercharges to the fermion vertices (5.3) and (5.4). Action of \( Q \) on (5.3) does not produce new states, while \( \bar{Q} \) gives

\[
\langle \bar{Q}_\alpha, \bar{V}^{(-\frac{1}{2})}_\beta \rangle \sim \varepsilon_{\alpha\beta} V^{S,\text{excited}}_{m=\frac{1}{2}},
\]

where the new excited scalar vertex in the picture \((-1)\) with \( m = 1/2 \) has the form

\[
V^{S,\text{excited}}_{m=\frac{1}{2}} = \left[ -2\partial_2^2 Y + \frac{i\gamma^\mu}{\sqrt{\pi}} \psi^\mu \bar{\psi} \partial_1 Y \right] e^{-\phi} e^{i\gamma^\mu x^\mu - \frac{\phi}{\sqrt{2}} + i\gamma^5}.
\]

The mass of this state is still given by (5.1). Action of supercharge \( Q \) on the fermion vertex (5.4) produces the conjugated scalar with \( m = -1/2 \).

### 5.2 Building massive vector supermultiplet

Now we can use the vertices obtained in the previous subsection to construct supermultiplets at the level (5.1). We have two scalar vertices with \( m = \pm 3/2 \) and \( m = \pm 1/2 \), see left-hand side of (5.7) and (5.6). Using these vertices we can construct the scalar states in the NS-NS sector. Namely, we have one complex scalar

\[
V^{S,L}_{j=-\frac{1}{2}, m=\pm \frac{3}{2}} \times V^{S,R}_{j=-\frac{1}{2}, m=\mp \frac{3}{2}}
\]

formed by the \( m = \pm 3/2 \) vertices and one complex scalar

\[
V^{S,\text{excited},L}_{j=-\frac{1}{2}, m=\pm \frac{3}{2}} \times V^{S,\text{excited},R}_{j=-\frac{1}{2}, m=\mp \frac{3}{2}}
\]

formed by the \( m = \pm 1/2 \) vertices (5.6).

Moreover, we have also another two complex scalars,

\[
V^{S,L}_{j=-\frac{1}{2}, m=\pm \frac{3}{2}} \times V^{S,\text{excited},R}_{j=-\frac{1}{2}, m=\mp \frac{3}{2}}
\]

and

\[
V^{S,\text{excited},L}_{j=-\frac{1}{2}, m=\pm \frac{3}{2}} \times V^{S,R}_{j=-\frac{1}{2}, m=\mp \frac{3}{2}}
\]

formed by products of two different vertices. Altogether in the NS-NS sector we observe four complex scalars.

In the R-R sector we have

\[
V^{\text{excited}}_{\dot{\alpha}\dot{\alpha}} = \bar{V}^{L}_{\dot{\alpha}} \times V^{R}_{\dot{\alpha}}, \quad \bar{V}^{\text{excited}}_{\alpha\dot{\alpha}} = V^{L}_{\alpha} \times \bar{V}^{R}_{\dot{\alpha}},
\]

(5.11)
where now the fermion vertices are given by (5.3) and (5.4). Expanding these vertices in the basis of $\sigma$ matrices

\[ V_{\aa}^{\text{excited}} = (\bar{\sigma}_\mu)_{\aa\dot{\alpha}} B^\mu + (\bar{\sigma}_\mu \sigma_\nu \bar{\sigma}_\rho)_{\aa\dot{\alpha}} B^{\mu\nu\rho} \]  

(5.12)

we arrive at the complex vector field $B^\mu$ and the complex 3-form $B^{\mu\nu\rho}$.

In four dimensions the massive 3-form is dual to a massive scalar [52]. Generically the rules of dualizing can be summarized as follows [52]. In $D$ dimensions massless $p$-forms have

\[ c_{D-2}^p = \frac{(D-2)!}{p!(D-2-p)!} \]  

(5.13)

physical degrees of freedom. Therefore, the rule of dualizing of the massless $p$-form is

\[ p \rightarrow (D-2-p). \]  

(5.14)

In particular, 3-form in 4D has no degrees of freedom.

For the massive $p$-forms we have

\[ c_{D-1}^p = \frac{(D-1)!}{p!(D-1-p)!} \]  

(5.15)

physical degrees of freedom. The rule of dualizing now becomes

\[ p \rightarrow (D-1-p). \]  

(5.16)

Thus the massive 3-form in 4D is dual to a massive scalar. Explicitly the duality relation can be written as [52, 53]

\[ B_{\mu\nu\rho} \sim \varepsilon_{\mu\nu\rho\lambda} \partial^\lambda c. \]  

(5.17)

We conclude in the R-R sector we obtained one complex scalar $c$ and the complex vector $B^\mu$. Altogether the bosonic part of the supermultiplet with mass (5.1) contains 5 scalars and a vector, all complex. This is exactly the bosonic content of two real $\mathcal{N} = 2$ long massive vector multiplets, each containing 5 scalars and a vector, see Appendix B,

\[ (\mathcal{N} = 2)_{\text{vector}} = 1_{\text{vector}} + 5_{\text{scalar}}. \]  

(5.18)

\footnote{We did not include 3-form in the expansion because in the massless case it contains no physical degrees of freedom, see below.}
Let us note that the $\mathcal{N} = 2$ massive vector multiplet can be realized as a result of Higgsing of a $U(1)$ massless gauge multiplet containing gauge field and a complex scalar (2 real scalars) by vacuum expectation values (VEVs) of a hypermultiplet which contains 4 real scalars. After Higgsing, one scalar is “eaten” by the Higgs mechanism, so we are left with massive vector field and 5 scalars. The number of degrees of freedom in this massive $\mathcal{N} = 2$ vector multiplet is $8 = 3 + 5$, where 3 comes from the massive vector.

Summarizing this section we present 4D $R$ charges of the vector multiplet components. Due to cancellation of the $R$ charges of the left and right-moving sectors, the $R^{(4)}$ charges of the R-R states (5.11) and two scalars (5.7), (5.8) of the NS-NS sector vanish, see (4.13). The $R$-charges of two scalars (5.9) and (5.10) are non-zero, $R^{(4)} = \pm 2$. These are exactly the $R$-charges of a massive $\mathcal{N} = 2$ vector multiplet. This can be easily understood in terms of Higgsing of the massless gauge multiplet by hypermultiplet VEVs. The gauge field and scalars from the hypermultiplet have the zero $R$ charge while the $R$ charges of two scalar superpartners of the gauge field in the massless vector multiplet are indeed characterized by $R^{(4)} = \pm 2$, cf. Sec. 4.4.

6 The lowest $j = -1$ multiplet

In this section we consider the lowest spin-2 supermultiplet produced by the vertex operator (3.25). The mass of the state with $j = -1$ and $m = \pm 1$ is

$$\frac{(M_{j=-1,m=\pm 1})^2}{8\pi T} = 1,$$

see (3.27).

We will see below that the spin-2 state (3.25) is the highest component of this supermultiplet. To simplify our discussion it is easier to start from a scalar component of this supermultiplet replacing the world-sheet fermions $\psi^{L,R}_\mu$ by $\psi^L_\phi$ and $\psi^R_Y$. Thus, in the left-moving sector we start from the scalar vertex which, in the picture $(-1)$, has the form

$$V^{(-1)}_{j=-1,m=1} = \psi e^{-\varphi} e^{ip_\mu x^\mu - \sqrt{2} \phi + i \sqrt{2} m Y}$$

where we skip the superscripts $L$, while $\psi$ is given by (4.2) and $m = 1$. For $m = -1$ we use a similar vertex with replacement $\psi \to \bar{\psi}$. The conformal dimension of this vertex is the same as that of the vertex in (3.25), so we have a scalar state with mass (6.1).
6.1 Action of supercharges

To convert this vertex operator into the picture (0) we use the BRST operator, see Appendix A. Then in the picture (0) we have

\[ V_{j=-1,m=1}^{(0)} = \left[ \frac{1}{\sqrt{2}} (\partial_\phi - i \partial_Y) + \frac{i p_\mu}{\sqrt{4 \pi T}} \psi^\mu \psi \right] e^{-\phi} e^{ip_\mu x^\mu - \sqrt{2} \phi + i \sqrt{2} m Y} \quad (6.3) \]

for \( m = 1 \) and a similar vertex with \( \psi \rightarrow \bar{\psi} \) for \( m = -1 \).

Now, let us apply the supercharges to generate the fermion vertices. \( Q \) acts trivially on (6.3), while \( \bar{Q} \) produces the following fermion vertex in the picture \((-1)/2\):

\[ \bar{V}_{\dot{\alpha}}^{(-\frac{1}{2})} = \langle \bar{Q}_{\dot{\alpha}}, V_{-1,m=1}^{(0)}(p_\mu) \rangle \sim e^{-\frac{\phi}{2}} [\bar{S}_{\dot{\alpha}} S + \bar{p}_\mu (\bar{\sigma}_\mu)_{\dot{\alpha}} S^\alpha S] (\partial_\phi + \psi_\phi \psi_Y) e^{ip_\mu x^\mu - \sqrt{2} \phi + i \sqrt{2} Y}, \quad (6.4) \]

This fermion vertex has \( m = 1/2 \).

In a similar manner applying supercharge \( Q \) to the scalar vertex \( V_{j=-1,m=-1}^{(0)} \) we get a fermion vertex with \( m = -1/2 \),

\[ (V^\alpha)^{-\frac{1}{2}} = \langle Q^\alpha, V_{-1,m=-1}(p_\mu) \rangle \sim e^{-\frac{\phi}{2}} [S^\alpha S + p_\mu (\sigma_\mu)^{\alpha \dot{\alpha}} S \bar{S}_{\dot{\alpha}}] (\partial_\phi + \psi_\phi \psi_Y) e^{ip_\mu x^\mu - \sqrt{2} \phi - i \sqrt{2} Y}. \quad (6.5) \]

In order to generate new bosonic vertex operators with the same mass (6.1) we apply supercharges to the fermion vertices above. Supercharge \( Q \) acting on (6.4) gives the following bosonic vertices in the picture \((-1)\):

\[ \langle Q^\alpha, \bar{V}_{\dot{\alpha}} \rangle \sim \sigma_\mu^{\alpha \dot{\alpha}} \left( \psi^\mu + \frac{p_\mu}{\sqrt{4 \pi T}} \psi \right) e^{-\phi} e^{ip_\mu x^\mu - \sqrt{2} \phi + i \sqrt{2} Y} \]

\[ = \sigma_\mu^{\alpha \dot{\alpha}} \left( V_{j=-1,m=1}^\mu + \frac{p_\mu}{\sqrt{4 \pi T}} V_{j=-1,m=1}^{(-1)} \right) \quad (6.6) \]

where \( V_{j=-1,m=1}^{(-1)} \) is the scalar vertex (6.2), while

\[ V_{j=-1,m=1}^\mu = \psi^\mu e^{-\phi} e^{ip_\mu x^\mu - \sqrt{2} \phi + i \sqrt{2} m Y} \quad (6.7) \]
is a new vector vertex operator with $m = 1$. We recognize it as a left-moving part of the spin-2 vertex (3.25). As was mentioned above, we obtained it by applying the supercharges to the scalar vertex (6.2). In a similar way we can generate the complex-conjugated vector $V^{\mu}_{j=-1, m=-1}$ with $m = -1$ if we apply the supercharge $\bar{Q}$ to the fermion vertex (6.5).

We can also apply the supercharge $\bar{Q}$ to the fermion vertex (6.4). This gives

$$\langle \bar{Q}^{\dot{\alpha}}, \bar{V}_{\dot{\beta}} \rangle \sim \delta^{\dot{\alpha}}_{\dot{\beta}} V_{j=-1, m=0} ,$$

(6.8)

where

$$V_{j=-1, m=0}^{(-1)} = \left( \bar{\psi} + \frac{p^\mu}{\sqrt{4\pi T}} \psi_\mu \right) \partial_\mu Y e^{-\varphi} e^{i p_\mu x^\mu - \sqrt{2} \phi} \tag{6.9}$$

is a new scalar vertex with $m = 0$ and mass (6.1). Similarly, the action of $Q$ on the fermion vertex (6.5) gives a complex-conjugated scalar vertex with the replacement $\bar{\psi} \rightarrow \psi$.

Finally, instead of the scalar vertex (6.2) we can start from another scalar vertex,

$$\bar{V}_{j=-1, m=1}^{(-1)} = \bar{\psi} e^{-\varphi} e^{i p_\mu x^\mu - \sqrt{2} \phi + i \sqrt{2} m Y} .$$

(6.10)

Note that this vertex is different from the one complex-conjugated to (6.2) because here we take $m = 1$. Conjugated to (6.10) is obtained by replacement $\bar{\psi} \rightarrow \psi$ and taking $m = -1$.

Following the same steps as above in the case of the vertex (6.2) one can show that the action of supercharges on the scalar vertex (6.10) produces the same states which we already obtained from (6.2).

Summarizing, in the bosonic left-moving sector for $j = -1$ multiplet we find a complex vector vertex (6.7) and three complex scalar vertices,

$$V_{j=-1, m=\pm 1}^{(-1)} , \quad \bar{V}_{j=-1, m=\pm 1}^{(-1)} , \quad V_{j=-1, m=0}^{(-1)} ,$$

(6.11)

given by (6.2), (6.10) and (6.9) respectively.

### 6.2 Building spin-2 multiplet

Now we will use bosonic and fermionic vertex operators from the previous subsection to construct the supermultiplet with $j = -1$ and mass (6.1). Let us start with the R-R sector. In much the same way as for the excited state
in Sec. 5.2 we arrive at

\[ V_{\alpha}^{j=-1} = V_{\alpha}^{L}(m = \frac{1}{2}) \times V_{\alpha}^{R}(m = -\frac{1}{2}), \]

\[ \bar{V}_{\alpha}^{j=-1} = V_{\alpha}^{L}(m = -\frac{1}{2}) \times \bar{V}_{\alpha}^{R}(m = \frac{1}{2}), \quad (6.12) \]

where the fermion vertices are given by (6.4) and (6.5). Expanding \( V_{\alpha}^{j=-1} \) and \( \bar{V}_{\alpha}^{j=-1} \) as in (5.12) we get a complex vector and a complex 3-form. As was discussed in Sec. 5.2, the massive 3-form dualizes into a massive scalar. Thus in the R-R sector we get one complex vector and one complex scalar.

Now we pass to the NS-NS sector. The scalar vertices (6.11) give \( 3 \times 3 = 9 \) scalars of the form

\[ V_{i}^{L}(m \geq 0) \times V_{j}^{R}(m \leq 0), \quad (6.13) \]

where \( V_{i}(m), i = 1, 2, 3, \) are given by (6.2), (6.10) and (6.9), respectively. Changing the sign of \( m \) together with the replacement \( \psi \rightarrow \bar{\psi} \) gives nine complex conjugated scalars in addition to those in (6.13).

Combining the vector vertex (6.7) with three scalar vertices (6.11) provides us with six vectors of the form

\[ (V_{j=1,m=1}^{\mu})^{L} \times V_{j}^{R}(m \leq 0), \]

\[ V_{i}^{L}(m \geq 0) \times (V_{j=1,m=-1}^{\mu})^{R}, \]

\[ i = 1, 2, 3. \quad (6.14) \]

Again changing the sign of \( m \) together with the replacement \( \psi \rightarrow \bar{\psi} \) gives six complex conjugated vectors to those.

Finally we can combine two vector vertices (6.7) to produce a tensor

\[ (V_{j=1,m=1}^{\mu})^{L} \times (V_{j=1,m=-1}^{\nu})^{R}. \quad (6.15) \]

Changing the sign of \( m \) gives a complex conjugated tensor. In 4D a massive vector has \( (D - 1) = 3 \) physical degrees of freedom. Therefore for the tensor state (6.15) we get

\[ 3 \times 3 = 9 = 5 + 3 + 1 \Rightarrow 1_{\text{spin}-2} + 1_{\text{vector}} + 1_{\text{scalar}} \quad (6.16) \]

massive degrees of freedom, where we show the expansion of the massive tensor into irreducible representations of \( \text{SO}(D - 1 = 3) \). Thus, from the
complex tensor (6.15) we obtain one spin-2 state, one vector and one scalar, all of them complex.

Combining all bosonic states together we get

$$1_{\text{spin-2}} + 8_{\text{vector}} + 11_{\text{scalar}} ,$$

where we show the numbers of states with the given spin.

How they split into 4D $\mathcal{N} = 2$ supermultiplets? Long $\mathcal{N} = 2$ spin-2 multiplet contains

$$(\mathcal{N} = 2)_{\text{spin-2}} = 1_{\text{spin-2}} + 6_{\text{vector}} + 1_{\text{scalar}}$$

bosonic spin states while long $\mathcal{N} = 2$ vector multiplet has

$$(\mathcal{N} = 2)_{\text{vector}} = 1_{\text{vector}} + 5_{\text{scalar}}$$

bosonic spin states, see Appendix B and Eq. (5.18).

We conclude that $j = -1$ states with mass (6.1) form

$$(j = -1) \text{ states} = 1 \times (\mathcal{N} = 2)_{\text{spin-2}} + 2 \times (\mathcal{N} = 2)_{\text{vector}}$$

(one spin-2 and two vector) $\mathcal{N} = 2$ long (non-BPS) supermultiplets, all complex.

7 Regge trajectories

In this section we will show that all states we discussed in this paper (shown in Fig. 2) are the lowest states of the corresponding linear Regge trajectories. To construct these Regge trajectories we multiply the vertex operators (3.21) or (3.25) by derivatives of flat 4D coordinates. For example, for the scalar vertices (3.21) we construct a family of vertices

$$\prod_{i=1}^{n} \partial_{-} x_{\mu_{i}} \partial_{+} x_{\nu_{i}} e^{-\varphi_{L} - \varphi_{R}} \epsilon^{ip_{\mu} x_{\mu}} V^{S,L}_{j = -\frac{1}{2}, m} \times V^{S,R}_{j = -\frac{1}{2}, -m} ,$$

where $n$ is $n = 0, 1, 2, ...$. The hadronic states associated with these vertices have at most spin $2n$. Their mass is

$$\left( \frac{M_{j = -\frac{1}{2}, m}}{8\pi T} \right)^{2} = m^{2} - \frac{1}{4} + n , \quad n = 0, 1, 2, ...$$
We see that mass squared for these states depends linearly on the spin. This linear Regge dependence appears because we use the flat 4D part of the string \( \sigma \) model to construct the Regge trajectories.

A similar construction can be developed for vertices \((3.25)\). Masses of these states are

\[
\frac{(M_{j=-1,m})^2}{8\pi T} = m^2 + n, \quad n = 0, 1, 2, ...
\]  

(7.3)

We have the same linear dependence with the same slope.

8 Conclusions

In [2] we observed that a vortex string supported in \( \mathcal{N} = 2 \) SQCD is critical provided the following conditions are met:

(i) The gauge group of the model considered is U(2);

(ii) The number of flavor hypermultiplets is \( N_f = 2N = 4 \);

The 4D theory under consideration is not conformal because of the Fayet-Iliopoulos parameter \( \xi \neq 0 \). However, the gauge coupling \( \beta \) function vanishes; the Fayet-Iliopoulos parameter does not run either.

In addition to four translational zero modes this string exhibits three orientational and three size zero modes. Their geometry is described by a non-compact six-dimensional Calabi-Yau manifold, the so-called resolved conifold \( Y_6 \). The target space takes the form \( \mathbb{R}^4 \times Y_6 \). The emergence of six extra zero modes on the string under consideration makes the target-space model conformal, the overall Virasoro central charge (including the ghost contribution) vanishes. Thus, this string is critical. The phenomenon we observed could be called a “reverse holography.”

The next question which was natural to address was the quantization of this closed critical string and the derivation of the hadronic spectrum. The present paper completes the work started in [18, 19, 1]. We calculated the masses of the massive spin-0 and spin-2 states and constructed the 4D supermultiplets to which they belong. Our formulas match the previous result for the massless states.

The massive supermultiplets are shown to be long (non-BPS saturated). We also prove that the above states are the lowest states on the corresponding Regge trajectories which are linear and parallel.
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Appendix A. BRST operator and vertices in the picture (0)

To convert vertex operator in the picture \((-1)\) into picture \((0)\) we use the BRST operator as follows \([55]\):

\[
V^{(0)} = \langle Q_{BRST}, \zeta V^{(-1)} \rangle, \tag{A.1}
\]

where

\[
Q_{BRST} = \frac{1}{2\pi i} \int dz \left[ cT^m + \gamma G^m + \frac{1}{2}(cT^{gh} + \gamma G^{gh}) \right]. \tag{A.2}
\]

Here \(c\) and \(\gamma\) are the ghosts of fermionic \((b,c)\) and bosonic \((\beta,\gamma)\) systems, respectively, while \(T^m, G^m\) and \(T^{gh}, G^{gh}\) are the energy momentum tensor and the supercurrent for matter and ghosts. Below we will need the explicit expression for the matter supercurrent,

\[
G^m = i(\psi^\mu \partial_- x_\mu + \psi_\phi \partial_- \phi + \psi_Y \partial_- Y). \tag{A.3}
\]

The ghost system \((\beta,\gamma)\) can be expressed in terms of fermions \(\eta, \zeta\),

\[
\gamma = e^\varphi \eta, \quad \beta = e^{-\varphi} \partial_- \zeta, \tag{A.4}
\]

where the propagator of \(\eta, \zeta\) is normalized as

\[
\langle \eta(z), \zeta(0) \rangle = \frac{1}{z}. \tag{A.5}
\]

To convert the left-moving part of the scalar vertex \((3.21)\) in the picture \((-1)\) into the picture \((0)\) we use the rule \((A.1)\). We arrive at the expression \((4.14)\) with the help of \((A.3)\).

Similarly, for the \(j = -1\) vertex \((6.2)\) we again use \((A.3)\) to obtain the vertex operator \((6.3)\) in the picture \((0)\).
In this Appendix we briefly review construction of $\mathcal{N} = 2$ long massive supermultiplets in four dimensions. For massive states in the rest frame supersymmetry generators $Q^{\alpha f}$ and $\bar{Q}_{f\dot{\alpha}}$ can be viewed as annihilation and creation operators, where $f = 1, 2$ is the index of two $\mathcal{N} = 1$ supersymmetries which constitute $\mathcal{N} = 2$. Assuming that the annihilation operators $Q^{\alpha f}$ produce zero upon acting on a “ground state” $|a\rangle$ we can generate all states of a given supermultiplet applying on $|a\rangle$ the creation operators $\bar{Q}_{f\dot{\alpha}}$.

For simplicity we will consider only the bosonic states in a multiplet. Assuming that $|a\rangle$ is a bosonic state we have 6 possibilities

$$\{Q_{11}Q_{21}, Q_{11}Q_{12}, Q_{11}Q_{22}, Q_{21}Q_{12}, Q_{21}Q_{22}, Q_{12}Q_{22}\} \times |a\rangle$$ (B.1)

at level 2 and only one possibility

$$\bar{Q}_{11}\bar{Q}_{21}\bar{Q}_{12}\bar{Q}_{22} \times |a\rangle$$ (B.2)

at level 4.

First let us construct the long $\mathcal{N} = 2$ massive vector supermultiplet. In this case we choose $|a\rangle$ to be a scalar with spin $J = 0$. The construction is shown in Table 1 where $J_z$ is the $z$-projection of spin and level 0 denotes the state $|a\rangle$ itself. Here we used the fact that, say, in Eq. (B.1) the product $Q_{11}Q_{21}$ acting on $|a\rangle$ increases $J_z$ by one, four product operators of the type $Q_{f1}Q_{g2}$ ($f, g = 1, 2$) do not change $J_z$, while the product $\bar{Q}_{12}\bar{Q}_{22}$ reduces $J_z$ by one.

### Table 1: Structure of the vector multiplet. We show the numbers of states with the given $J_z$ produced by the action of supercharges at each level and their sum.

| $J_z$ | Level 0 | Level 2 | Level 4 | Sum |
|-------|---------|---------|---------|-----|
| 1     | 0       | 1       | 0       | 1   |
| 0     | 1       | 4       | 1       | 6   |
| −1    | 0       | 1       | 0       | 1   |

Appendix B. Long $\mathcal{N} = 2$ vector and spin-2 multiplets in 4D.
Table 2: Spin-2 multiplet.

Overall we observe one state with $J_z = 1$, one state with $J_z = -1$ and 6 states with $J_z = 0$. This gives the decomposition (5.18).

Now let us pass to the spin-2 supermultiplet. To this end we take $|a\rangle$ to be a vector state with spin $J = 1$. The resulting structure is shown in Table 2. Here, say, 4+1+1=6 means that at level 2 four states are generated from the state at level 0 in the same row, while two other states come from states at level 0 in the upper or the lower neighboring rows. The last column gives decomposition (6.18).
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