Tomographic reconstruction of binary fields

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Abstract. A novel algorithm is proposed for reconstructing binary images from their projection along a set of different orientations. Based on a nonlinear transformation of the projection data, classical back-projection procedures can be used iteratively to converge to the sought image. A multiscale implementation allows for a faster convergence. The algorithm is tested on images up to 1 Mb definition, and an error free reconstruction is achieved with a very limited number of projection data, saving a factor of about 100 on the number of projections required for classical reconstruction algorithms.

1. Introduction

Binary reconstruction is a classical tomographic problem with the additional information that the reconstructed image is binary (i.e. 0-1 valued fields). The latter condition is a severe constraint that can help reducing drastically the number of projections needed for reconstruction without loss of information [1].

In the early 2000’s, various algorithms have been proposed showing the feasibility of the reconstruction on modest definition images (typically $64^2$ pixels). However, the proposed methods are generally based on sophisticated optimization strategies of multiple minima problems, e.g. simulated annealing, genetic programming [2]. Hence, the very slow convergence (typically hours) and poor success rate (few percent remaining errors) tended to confine such techniques to specific cases where projection data acquisition is limited.

Besides, in a seminal contribution, Candés et al. [3] have shown that images having a few number of gray levels could be reconstructed exactly from a very limited number of projections. The proposed strategy was to use a regularization based on the total variation of the image. Independently, Batenburg [4] proposed a specific algebraic algorithm to reconstruct binary images that turns out to be extremely efficient as compared to alternative published algorithms, allowing for larger images to be considered. In the line of the latter algorithm, a new approach is proposed herein to deal with large definition images requiring minutes on a standard PC, thereby allowing for a much larger class of applications. For example, one-megapixel images are reconstructed error-free from a few projections (the number depends on the “complexity” of the image pattern, but 5-20 is a typical order of magnitude) in a few minutes on a standard PC without sophisticated implementation.

2. Proposed algorithm

The binary image is denoted as $f(x)$, such that at a pixel located in $x$, the gray level value $f$ is either 0 or 1. A projection in the direction $n_\theta$ is defined as the sum of $f(x)$ for all pixels $x$ located along the same
ray of direction \( \mathbf{n}_\theta \). More precisely, introducing the unit vector \( \mathbf{t}_\theta \) normal to \( \mathbf{n}_\theta \), the projection \( S(y, \theta) \) is written as

\[
S(y, \theta) = \int f(x) \delta(x \cdot \mathbf{t}_\theta - y) \, dx
\]

\[
\equiv \int P_\theta(y, x)f(x) \, dx
\]

(1)

where \( P_\theta \) is the projection operator. For each direction \( \theta \), the detected signal is discretized with a bin size equal to the pixel size. In the following, for the sake of simplicity a pixel \( x \) is considered to project on exactly one detector site \( y \) (the closest to its projected position \( x \cdot \mathbf{t}_\theta \)) so that the sparse matrix \( P_\theta \) is itself binary-valued. Moreover, non-zero values of \( f \) are assumed to lie with a known domain, \( \mathcal{D} \), the indicator function of which is characterized by its projection \( S_0(y, \theta) = P_\theta(y, x)1_{\mathcal{D}}(x) \). The problem consists in finding \( f(x) \) from a collection of projection vectors \( S(y, \theta) \), for a few evenly distributed \( \theta \) angles.

2.1. Initialization

The algorithm consists in an initialization step and an iterative correction loop. The initialization step constructs the probabilities \( \pi_i \), that a given pixel \( i \) be 1, as can be estimated from the different projections. Each projection \( \theta_j \) passing through the site independently provides such a probability \( p_j = S(y, \theta_j)/S_0(y, \theta_j) \). It can be shown that when two projections are available for the same site, the probability for the site to be valued 1 is

\[
\pi = \frac{p_1 p_2}{p_1 p_2 + (1 - p_1)(1 - p_2)}
\]

(2)

These probabilities are very conveniently represented by their nonlinear transforms

\[
\Phi(p) \equiv \log \left( \frac{p}{1 - p} \right)
\]

(3)

since the combination of probabilities, Eq. (2), is equivalent to

\[
\Phi(\pi) = \Phi(p_1) + \Phi(p_2)
\]

(4)

This argument holds for an arbitrary number of projections, so that the probability \( \pi_i \) is obtained from

\[
\pi = \Phi^{-1} \left( \sum_{\text{projection } i} \Phi(p_i) \right)
\]

(5)

Thus the initial probability field is obtained from a mere back-projection algorithm of the transformed projections, \( \Phi(p) \), and the final probability field is retrieved from the \( \Phi^{-1} \) nonlinear transform.

The resulting probability field is then filtered by a convolution with a Gaussian filter with a short characteristic scale \( \delta \) typically chosen to be of the order of a few pixels. The resulting field is thresholded setting large \( (p > 0.99) \) and low \( (p < 0.01) \) probabilities to 1 and 0, respectively. A second determination of the intermediate probabilities is made securing the previous thresholded pixels (namely, \( S \) and \( S_0 \) are corrected from the contributions of these known pixels, and only undetermined pixels are updated with a new estimate of \( \pi \)).

2.2. Correction

The second step is a correction loop where the spirit consists in maximizing the similarity with the filtered \( p \)-field while satisfying the projection constraint independently for each projection direction. After the initialization step, it is not possible to simply threshold the probability field \( \pi \) at 1/2 (or equivalently \( \Phi(\pi) \) at 0) to obtain \( f \) (since in particular the nonlinear transformation does not preserve the expectation value). The principle of the algorithm consists in correcting the \( p \) values along each ray \( (y, \theta) \) so that
a binarization of $\varpi$ would match the known ray projection $S(y, \theta)$. This nonlinear problem reduces to translating uniformly all the $\Phi(\varpi)$ values along the ray so that the $S(y, \theta)$ largest ones become positive. After sweeping through all directions, the estimate $\tilde{f}$ of $f$ is constructed from the sign of $\Phi(\varpi)$ ($\tilde{f} = 1$ for positive values of $\Phi(\varpi)$ and $\tilde{f} = 0$ for negative ones). As long as $\tilde{f}$ is not consistent with the projection information, this correction step is repeated, namely, $\varpi$ is first evaluated from the convolution of the current estimate of $\tilde{f}$ by a Gaussian, and $\Phi(\varpi)$ are corrected as indicated above. As the number of iteration increases, the Gaussian used in the convolution is scaled down to a width of 1 pixel. Typically, a few tens of such steps are required to reach an error-free image. It is emphasized that the recourse to the $\Phi$-transformed projection data (and probabilities) — rather than $\varpi$ — is a crucial point for a fast convergence.

2.3. Multiscale strategy
This algorithm can be used in a multiscale strategy to tackle images with complex texture and large sizes, although simple images are more efficiently reconstructed at the finest level of description. The spirit of the multiscale approach is to coarsen the projections gathering two consecutive $y$’s into one $S'(y, \theta) = (S(2y, \theta) + S(2y + 1, \theta))/4$, and averaging $2 \times 2$ pixels in the image $f$ into “super-pixels” in a recursive fashion. Each level of this pyramidal construction is termed a “generation,” and the finest one is labeled 0. Starting from the coarsest scale, a binary reconstruction is first performed as above presented, and the result used as a predetermination of the refined level.

3. Example
This algorithm was tested on a series of artificial domains of variable complexity. Unions of ellipses or polygons were used to generate microstructures as they were used as a benchmark proposed by Batenburg [4] and compared with alternative algorithms. In all those cases, the presented algorithm revealed to perform better in terms of time (and error when a fixed number of iterations was prescribed) as compared to all reported methods in Ref. [4].

However, the choice of the image microstructures is often quite simple as compared to what may be encountered in practical examples. For complex microstructures and large image sizes, it is essential to use this multiscale strategy. In order to test a realistic case, an example shown in Figure 1 and extracted from real tomographic data is used as an illustration. The full image as reconstructed from a Filtered Back Projection algorithm is first binarized and filtered using a phase-field approach favoring a two level distribution down to very small ones. Moreover, the domain size is quite large.

Table 1 gives the number of iterations and computation time to reconstruct the binary image shown in Figure 1. This image is identical (for all single pixel) to the binary image issued from the initial binarization although it is reconstructed from 17 projections only. Note that the original data shown in Fig. 1(left) required 1500 projections. Moreover the image size is $1025 \times 1025$ pixels, a size much larger than what is usually considered in the literature. Table 1 also reports the time needed to reach convergence (i.e. not a single pixel difference between reference and reconstruction) at each generation. The present code is a simple Matlab® program run on a standard single processor PC without much optimization.

The advantage of the multiscale procedure can be seen in the fact that the same algorithm does not converge when the number of coarse-graining step is 0 (i.e. no coarse-graining) or 1. Choosing a number of generations from 2 to 5 results in a total computation time of 259, 134, 134 and 133 s respectively. Although the benefit in time is negligible when increasing the number of generations from 3 to 5, robustness increases and hence a large number of scales is preferred. Robustness means here that the number of coarse-graining steps does not need any tuning depending on the image texture. A large number of generations is at worse useless, but never a penalty for time or performance.
4. Conclusion
A novel binary reconstruction algorithm has been presented. A direct algebraic approach, based on the probability field that pixels be 0/1-valued, is proposed. The major difference with earlier approaches is the recourse to a nonlinear transform of the probability field. Such a transform is used for the initialization step and for the correction one. This leads to a significant increase in the performances allowing for error-free high definition image reconstructions (i.e. larger then 1 Mb) in minutes without any fancy optimization of the implementation.

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