Abstract

We calculate a sheaf line in $CP^3$ which is the real line supporting sheaf points on $CP^3$ of $SL(2, C)$ Yang-Mills instanton (or $SU(2)$ complex Yang-Mills instanton) sheaves for some given ADHM data we obtained previously. We found that this sheaf line is indeed a special jumping line over $S^4$ spacetime. In addition, we calculate the singularity structure of the connection $A$ and the field strength $F$ at the corresponding singular point on $S^4$ of this sheaf line. We found that the order of singularity at the singular point on $S^4$ associated with the sheaf line in $CP^3$ is higher than those of other singular points associated with normal jumping lines. We conjecture that this is a general feature for sheaf lines among jumping lines.
## I. Introduction

The discovery of classical Yang-Mills (YM) instanton began in 1975 [1–4]. In a few years, the complete instanton solutions with $8k - 3$ moduli parameters for each $k$-th homotopy class were solved by ADHM [5] in 1978 using theory in algebraic geometry. By using the

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Acknowledgments

References
monad construction combining with the Penrose-Ward transform, ADHM constructed the ADHM instanton solutions by establishing an one to one correspondence between anti-self-dual $SU(2)$-connections on $S^4$ and global holomorphic vector bundles of rank two on $CP^3$. The explicit closed forms of the complete $SU(2)$ instanton solutions with $k \leq 3$ were calculated by physicists in [6, 7]. There have been tremendous applications of YM instanton in quantum field theory [8, 9] and geometry [10] for the last few decades. For references, see some review works in [11].

In a series of recent papers [12–14], instead of quaternion calculation for the $SU(2)$ YM instanton, the present authors developed the biquaternion method with biconjugation operation [15] to construct $SL(2, C)$ [16] YM instanton (or $SU(2)$ complex YM instanton) solutions with $16k - 6$ moduli parameters for each $k$-th homotopy class. These new $SL(2, C)$ instanton solutions contain previous $SL(2, C)$ $(M, N)$ instanton solutions as a subset constructed in 1984 [17]. The number of parameters constructed in [12] is consistent with the conjecture made by Frenkel and Jardim in [18] and was proved recently in [19] from the mathematical point of view [20–23].

Moreover, for the first time, in addition to the holomorphic vector bundles on $CP^3$ in the ADHM construction which have been well studied in the $SU(2)$ instantons, the authors in [13, 14] discovered and explicitly constructed the so-called YM instanton sheaves on $CP^3$. In constrast to the smooth vector bundle on $CP^3$ induced by $SU(2)$ instanton on $S^4$, the vector bundle structure breaks down and the dimension of vector space attached on $CP^3$ may vary from point to point for $SL(2, C)$ YM instanton sheaves. In a previous publication [13], the authors calculated explicitly examples of sheaf points on $CP^3$ where the dimension of the attached vector space changes.

Since there is a fibration of $CP^3$ down to $S^4$ with fiber being $CP^1$ line, one important follow-up issue related to these sheaf points is to study how to identify the corresponding points on $S^4$ spacetime. A related issue is to calculate the sheaf line, or the real line in $CP^3$ which connecting the sheaf point on $CP^3$ and the corresponding singular point on $S^4$. We will introduce the Plücker coordinate to describe these sheaf lines in $CP^3$ in this paper.

Moreover, one would like to calculate the singularity structures of the connection $A$ and the field strength $F$ on these singular points of $S^4$. We will show that the order of singularity at the singular points on $S^4$ associated with sheaf line in $CP^3$ is higher than those of other singular points associated with normal jumping lines. We conjecture that this is a general
One unexpected result we obtained in our search of the sheaf lines was the great simplification of the calculation of $v$ in Eq. (4.85) and the corresponding connection $A$ and the field strength $F$ associated with the sheaf ADHM data. In fact, we will see that for this sheaf ADHM data the explicit form of $SL(2, C)$ YM 2-instanton field strength without removable singularities can be exactly calculated!

This paper is organized as following. In section two, we briefly review the construction of YM instanton sheaves calculated in [13]. In section three, we introduce Plücker coordinate to calculate jumping lines and sheaf lines of YM 2-instanton sheaves calculated in [13]. A duality symmetry among YM instanton sheaf solutions was pointed out with application to the known sheaf solutions. In section four, we calculate the singularity structure of connection and field strength on $S^4$ spacetime associated with jumping lines and sheaf lines of YM instanton sheaf. An explicit $SL(2, C)$ YM 2-instanton field strength will be exactly calculated. The calculable exact 2-instanton field strength is believed to be related to the 2-instanton sheaf structure. A conclusion is presented in section five.

II. THE $SL(2, C)$ YANG-MILLS TWO INSTANTON SHEAVES

In this section, we briefly review the biquaternion construction of $SL(2, C)$ ADHM instantons [12, 13]. We will pay attention to the existence of jumping lines and sheaf structures of YM 2-instanton sheaves [12–14].

A. The Biquaternion construction of $SL(2, C)$ ADHM instantons

In the biquaternion construction of $SL(2, C)$ ADHM instanton, the quadratic condition on the biquaternion matrix $\Delta(x)$ of $SL(2, C)$ instantons reads

$$\Delta(x) \circledast \Delta(x) = f^{-1} = \text{symmetric, non-singular } k \times k \text{ matrix for } x \notin J \quad (2.1)$$

where for $x \in J$,

$$\det \Delta(x) \circledast \Delta(x) = 0. \quad (2.2)$$
The set $J$ is called singular locus or "jumping lines". There are no singular locus for $SU(2)$ instantons on $S^4$. The biconjugation of a biquaternion

$$ z = z_\mu e_\mu, \quad z_\mu \in \mathbb{C}, \tag{2.3} $$

is defined to be

$$ z^\oplus = z_\mu e^\dagger_\mu = z_0 e_0 - z_1 e_1 - z_2 e_2 - z_3 e_3 = x^\dagger + y^\dagger i. \tag{2.4} $$

Occasionally the unit quaternions can be expressed as Pauli matrices

$$ e_0 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_i \rightarrow -i\sigma_i; \quad i = 1, 2, 3. \tag{2.5} $$

The norm square of a biquaternion is defined to be

$$ |z|^2_c = z^\oplus z = (z_0)^2 + (z_1)^2 + (z_2)^2 + (z_3)^2, \tag{2.6} $$

which is a complex number in general.

As a simple example, for the case of $SL(2,\mathbb{C})$ diagonal CFTW 2-instanton

$$ \Delta(x) = \begin{bmatrix} \lambda_1 & \lambda_2 \\ x - y_1 & 0 \\ 0 & x - y_2 \end{bmatrix}, \tag{2.7} $$

$$ \Delta^\oplus(x) = \begin{bmatrix} \lambda_1^\oplus x^\oplus - y_1^\oplus \\ \lambda_2^\oplus \end{bmatrix} \begin{bmatrix} 0 \\ x^\oplus - y_2^\oplus \end{bmatrix}, \tag{2.8} $$

where in the ADHM data $\lambda_j$ a complex number, and $y_j$ a biquaternion.

One can calculate the gauge potential as

$$ A_\mu = v^\oplus \partial_\mu v = \frac{1}{4} [e^\dagger_\mu e_\nu - e^\dagger_\nu e_\mu] \partial_\nu \ln(1 + \frac{\lambda_1^2}{|x - y_1|^2_c} + \frac{\lambda_2^2}{|x - y_2|^2_c} ) \\
= \frac{1}{4} [e^\dagger_\mu e_\nu - e^\dagger_\nu e_\mu] \partial_\nu \ln(\phi) \tag{2.9} $$

where

$$ v = \frac{1}{\sqrt{\phi}} \begin{bmatrix} 1 \\ -\frac{\lambda_1(x_\mu - y_1_\mu) e_\mu}{|x - y_1|^2} \\ -\frac{\lambda_2(x_\mu - y_2_\mu) e_\mu}{|x - y_2|^2} \end{bmatrix} \tag{2.10} $$
and
\[
\phi = 1 + \frac{\lambda_1^2}{|x - y_1|^2_c} + \frac{\lambda_2^2}{|x - y_2|^2_c}. \tag{2.11}
\]

To get the non-removable singularities or jumping lines, it turned out that one needs to calculate zeros of the normalization factor \(\phi\) or
\[
|x - y_1|^2_c|x - y_2|^2_c\phi = |x - y_1|^2_c|x - y_2|^2_c + |\lambda_2|^2_c|x - y_1|^2_c + |\lambda_1|^2_c|x - y_2|^2_c
\]
\[= P_4(x) + iP_3(x) = 0. \tag{2.12}\]

For the \(SL(2, C)\) CFTW general \(k\)-instanton case, one encounters intersections of zeros of \(P_{2k}(x)\) and \(P_{2k-1}(x)\) polynomials with degrees \(2k\) and \(2k - 1\) respectively
\[
P_{2k}(x) = 0, \quad P_{2k-1}(x) = 0. \tag{2.13}\]

One notes that Eq.(2.12) can be written as
\[
\det \Delta(x)^\circ \Delta(x) = |x - y_1|^2_c|x - y_2|^2_c + |\lambda_2|^2_c|x - y_1|^2_c + |\lambda_1|^2_c|x - y_2|^2_c = 0 \tag{2.14}\]
which gives the jumping lines of the \(SL(2, C)\) diagonal CFTW 2-instanton. It was shown that there is no sheaf line structure for the \(SL(2, C)\) diagonal CFTW \(k\)-instanton. The complete jumping lines of ADHM 2-instanton and 3-instanton of Eq.(2.2) were calculated in [12]. However, the existence of sheaf lines was not known and not calculated there. We will calculate and identify some sheaf lines of the \(SL(2, C)\) ADHM 2-instanton in the next section.

**B. The \(SL(2, C)\) complex ADHM equations**

The second method to construct \(SL(2, C)\) ADHM data is to solve the complex ADHM equations
\[
[B_{11}, B_{12}] + I_1 J_1 = 0, \tag{2.15a}
\]
\[
[B_{21}, B_{22}] + I_2 J_2 = 0, \tag{2.15b}
\]
\[
[B_{11}, B_{22}] + [B_{21}, B_{12}] + I_1 J_2 + I_2 J_1 = 0. \tag{2.15c}
\]

In this approach, one key step is to use the explicit matrix representation (EMR) of the biquaternion and do the rearrangement rule to explicitly identify the complex ADHM
data \((B_{lm}, I_m, J_m)\) with \(l, m = 1, 2\) from the \(\Delta(x)\) matrix in Eq.\((2.1)\).

As an explicit example and for illustration, we calculate the \(SL(2,C)\) CFTW 2-instanton case. In the EMR, a biquaternion can be written as a \(2 \times 2\) complex matrix

\[
\begin{align*}
z &= z^0 e_0 + z^1 e_1 + z^2 e_2 + z^3 e_3 \\
&= \left[ (a^0 + b^3) + i (b^0 - a^3) \right] (-a^2 + b^1) + i (-b^2 - a^1) \\
&\quad + \left[ (a^2 + b^1) + i (b^2 - a^1) \right] (a^0 - b^3) + i (b^0 + a^3)
\end{align*}
\]

where \(a^\mu\) and \(b^\mu\) are real and imaginary parts of \(z^\mu\) respectively. For the CFTW 2-instanton case

\[
a = \begin{bmatrix}
\lambda_1 & \lambda_2 \\
y_{11} & 0 \\
0 & y_{22}
\end{bmatrix} \rightarrow 
\begin{bmatrix}
p_1 + i q_1 & p_2 + i q_2 & 0 \\
0 & p_1 + i q_1 & 0 & p_2 + i q_2 \\
y_{11}^0 - i y_{11}^3 & 0 & - (y_{11}^2 + i y_{11}^1) & 0 \\
0 & y_{22}^0 - i y_{22}^3 & 0 & - (y_{22}^2 + i y_{22}^1) \\
y_{11}^2 - i y_{11}^1 & 0 & y_{11}^0 + i y_{11}^3 & 0 \\
0 & y_{22}^2 - i y_{22}^1 & 0 & y_{22}^0 + i y_{22}^3
\end{bmatrix} = 
\begin{bmatrix}
J_1 & J_2 \\
B_{11} & B_{21} \\
B_{12} & B_{22}
\end{bmatrix}
\]

where in Eq.\((2.18)\) we have done the \textit{rearrangement rule} for an element \(z_{ij}\) in \(a\)

\[
\begin{align*}
z_{2n-1,2m-1} &\rightarrow z_{n,m} \\
z_{2n-1,2m} &\rightarrow z_{n,k+m} \\
z_{2n,2m-1} &\rightarrow z_{k,n,m} \\
z_{2n,2m} &\rightarrow z_{k+n,k+m}
\end{align*}
\]

The EMR and the rearrangement rule for \(a^\circ\) can be similarly performed.
For the $SU(2)$ ADHM instantons, one imposes the conditions
\begin{align*}
I_1 &= J_1^†, I_2 = -I, J_1 = I_1^†, J_2 = J, \\
B_{11} &= B_2^†, B_{12} = B_1^†, B_{21} = -B_1, B_{22} = B_2
\end{align*}
(2.20a)
to recover the real ADHM equations
\begin{align*}
[B_1, B_2] + IJ &= 0, \quad (2.21a) \\
\left[B_1, B_1^†\right] + \left[B_2, B_2^†\right] + II^† - J^†J &= 0. \quad (2.21b)
\end{align*}

C. The monad construction and YM 2-instanton sheaves

The third method to construct $SL(2, C)$ ADHM instanton is the monad construction. This method is particular suitable for constructing instanton sheaves. One introduces the $\alpha$ and $\beta$ matrices as functions of homogeneous coordinates $z, w, x, y$ of $CP^3$ and defines
\begin{align*}
\alpha &= \begin{bmatrix}
    zB_{11} + wB_{21} + x \\
    zB_{12} + wB_{22} + y \\
    zJ_1 + wJ_2
\end{bmatrix}, \quad (2.22a) \\
\beta &= \begin{bmatrix}
    -zB_{12} - wB_{22} - y & zB_{11} + wB_{21} + x & zI_1 + wI_2
\end{bmatrix}. \quad (2.22b)
\end{align*}

It can be shown that the condition
\begin{equation}
\beta \alpha = 0 \quad (2.23)
\end{equation}
is satisfied if and only if the complex ADHM equations in Eq.(2.15a) to Eq.(2.15c) holds.

In the monad construction of the holomorphic vector bundles, either $\beta$ is not surjective or $\alpha$ is not injective at some points of $CP^3$ for some ADHM data, the dimension of $(\text{Ker } \beta/ \text{Im } \alpha)$ varies from point to point on $CP^3$, and one encounters "instanton sheaves" on $CP^3$ [18]. In our previous publication [13], we discovered that for some ADHM data at some sheaf points on $CP^3$, there exists common eigenvector $u$ in the costable condition $\alpha u = 0$ or [18]
\begin{align*}
(zB_{11} + wB_{21}) u &= -xu, \quad (2.24a) \\
(zB_{12} + wB_{22}) u &= -yu, \quad (2.24b) \\
(zJ_1 + wJ_2) u &= 0. \quad (2.24c)
\end{align*}
So $\alpha$ is not injective there and the dimension of $(\text{Ker } \beta/ \text{Im } \alpha)$ is not a constant over $CP^3$.

The first example of YM instanton sheaf discovered in [13] was the 2-instanton sheaf. For points $[x : y : z : w] = [0 : 0 : 1 : \pm 1]$ on $CP^3$ with the ADHM data

$$
\begin{bmatrix}
\lambda_1 & \lambda_2 \\
y_{11} & y_{12} \\
y_{12} & y_{22}
\end{bmatrix} = 
\begin{bmatrix}
a & 0 & 0 & ia \\
0 & a & ia & 0 \\
\frac{-i}{\sqrt{2}}a & 0 & 0 & \frac{a}{\sqrt{2}} \\
0 & \frac{-i}{\sqrt{2}}a & \frac{a}{\sqrt{2}} & 0 \\
0 & \frac{a}{\sqrt{2}} & \frac{i}{\sqrt{2}}a & 0 \\
\frac{a}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}}a
\end{bmatrix}, \ a \in C, \ a \neq 0,
$$

(2.25)

$\alpha$ is not injective. The second example of YM 2-instanton sheaf discovered [13] was for points $[x : y : z : w] = [0 : 0 : 1 : \pm i]$ on $CP^3$ with the ADHM data

$$
\begin{bmatrix}
\lambda_1 & \lambda_2 \\
y_{11} & y_{12} \\
y_{12} & y_{22}
\end{bmatrix} = 
\begin{bmatrix}
a & 0 & 0 & a \\
0 & a & -a & 0 \\
\frac{-i}{\sqrt{2}}a & 0 & 0 & \frac{-ia}{\sqrt{2}} \\
0 & \frac{i}{\sqrt{2}}a & \frac{i}{\sqrt{2}}a & 0 \\
0 & \frac{ia}{\sqrt{2}} & \frac{i}{\sqrt{2}}a & 0 \\
\frac{ia}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}}a
\end{bmatrix}, \ a \in C, \ a \neq 0,
$$

(2.26)

$\alpha$ is not injective.

### III. JUMPING LINES AND SHEAF LINES OF INSTANTON SHEAVES

In the previous section, we have obtained sheaf points on $CP^3$ with some examples of given ADHM data. One natural issue arisen then is to study how to identify the corresponding points on $S^4$ and calculate the singularity structure of the connection $A$ and the field strength $F$ on these points. The latter issue will be studied in the next section. In this section, we first define and calculate the sheaf line, or the real line which connecting the sheaf point on $CP^3$ and the corresponding singular point on $S^4$ and see whether the sheaf line is a jumping line or not.

In our previous publication [13], we have shown that there is no sheaf line structure for the $SL(2, C)$ diagonal CFTW $k$-instanton. On the other hand, the complete jumping lines
of ADHM 2-instanton and 3-instanton of Eq.(2.2) were calculated in section IV D. of [12].
However, the existence of sheaf lines was not known and not calculated there. In this section,
we will calculate and identify some sheaf line of the $SL(2, C)$ ADHM 2-instanton.

A. Real lines in $CP^3$

It is well known that there is a fibration from $CP^3$ to $S^4$ with fibers being $CP^1$. In the
Plücker coordinate representation of a (complex) line $CP^1$ in $CP^3$, one uses six homogeneous
coordinates to represent each line. More specifically, given two points $[a : b : c : d]$ and
$[x : y : z : w]$ on $CP^3$, the Plücker coordinates $z_{ij}$ of the line $L$ connecting the two points
are defined as

\[
\begin{align*}
  z_{12} &= ay - bx, \\
  z_{13} &= az - cx, \\
  z_{14} &= aw - dx, \\
  z_{23} &= bz - cy, \\
  z_{24} &= bw - dy, \\
  z_{34} &= cw - dz, 
\end{align*}
\]

or in short

\[
[z_{12} : z_{13} : z_{14} : z_{23} : z_{24} : z_{34}] = [a : b : c : d] \wedge [x : y : z : w].
\]

(3.27)

Note that the Plücker coordinates defined above are uniquely determined by $L$ up to a
common nonzero factor and not all six components can be zero. Thus $z_{ij}$ can be thought of
as homogeneous coordinates of a point in $CP^5$. However, not all points in $CP^5$ correspond
to lines in $CP^3$. The Plücker coordinates of a line satisfy the quadratic relations

\[
z_{12}z_{34} + z_{13}z_{42} + z_{14}z_{23} = 0,
\]

(3.29)
as can be easily verified from the definition in Eq.(3.27). So the set of lines in $CP^3$ constitutes
a manifold of complex dimension 4 rather than 5.

A line in $CP^3$ is called a real line if it is a fiber on $S^4$. To characterize a real line in $CP^3$,
one introduces the $\sigma$ map which preserves a real line $L$

\[
\sigma(L) = L \text{ if and only if } L = \text{ real line}.
\]

(3.30)
The $\sigma$ map can be defined as following. Let $\pi$ be the projection of the fibration from $CP^3$ to $S^4$

$$\pi : CP^3 \rightarrow S^4 \cong HP^1$$

(3.31)

where $HP^1$ is the quaternion projective space. We can parametrize the projection $\pi$ as

$$\pi : [z_1 : z_2 : z_3 : z_4] \rightarrow [z_1 + z_2j : z_3 + z_4j]$$

(3.32)

where $j \equiv e_2$ is a unit quaternion defined in Eq.(2.4). The $\sigma$ map in Eq.(3.30) can then be written as

$$\sigma : [z_1 : z_2 : z_3 : z_4] \rightarrow [\bar{z}_2 : -\bar{z}_1 : \bar{z}_4 : -\bar{z}_3].$$

(3.33)

It can be shown that the $\sigma$ map preserves real lines as illustrated in Eq.(3.30) or

$$\pi \circ \sigma = \pi.$$  

(3.34)

In fact (we use the notation $(1, i, j, k) = (e_0, e_1, e_2, e_3)$)

$$\pi([x : y : z : w]) = [x + ye_2 : z + we_2]$$

$$= [x^0e_0 + x^1e_1 + (y^0e_0 + y^1e_1)e_2 : z^0e_0 + z^1e_1 + (w^0e_0 + w^1e_1)e_2]$$

$$= [x^0e_0 + x^1e_1 + y^0e_2 + y^1e_3 : z^0e_0 + z^1e_1 + w^0e_2 + w^1e_3].$$

(3.35)

In Eq.(3.35), $x^0$ and $x^1$ are the real part and imaginary part of the complex number $x = x^0e_0 + x^1e_1 = x^0 + x^1\sqrt{-1}$, etc. On the other hand

$$\pi \circ \sigma[x : y : z : w] = \pi([\bar{y} : -\bar{x} : \bar{w} : -\bar{z}])$$

$$= \pi([y^0e_0 - y^1e_1 : -x^0e_0 + x^1e_1 : w^0e_0 - w^1e_1 : -z^0e_0 + z^1e_1])$$

$$= [y^0e_0 - y^1e_1 + (-x^0e_0 + x^1e_1)e_2 : w^0e_0 - w^1e_1 + (-z^0e_0 + z^1e_1)e_2]$$

$$= [y^0e_0 - y^1e_1 - x^0e_2 + x^1e_3 : w^0e_0 - w^1e_1 - z^0e_2 + z^1e_3]$$

$$\cong e_2[y^0e_0 - y^1e_1 - x^0e_2 + x^1e_3 : w^0e_0 - w^1e_1 - z^0e_2 + z^1e_3]$$

$$= [x^0e_0 + x^1e_1 + y^0e_2 + y^1e_3 : z^0e_0 + z^1e_1 + w^0e_2 + w^1e_3]$$

$$= \pi([x : y : z : w]),$$

(3.36)

which proves Eq.(3.34).
B. A Duality symmetry

With the $\sigma$ map introduced in the previous subsection, we can show an important duality symmetry [25] among instanton sheaf solutions. In [18], it was noted that given a set of ADHM data, one can generate a new set of ADHM data through the map

$$\Sigma : (B_{11}, B_{12}, B_{21}, B_{22}, I_1, I_2, J_1, J_2) \rightarrow (B_{22}^+, -B_{21}^+, -B_{12}^+, B_{11}^+, J_2^+, -J_1^+, -I_2^+, I_1^+) \quad (3.37)$$

Recall that in the monad construction of instanton bundle, if either $\alpha$ is not injective or $\beta$ is not surjective, then the dimension of $(\text{Ker } \beta / \text{Im } \alpha)$ may vary from point to point on $CP^3$, and one is led to use sheaf description for YM instantons or "instanton sheaves" on $CP^3$. The costable conditions in Eq.(2.24a) to Eq.(2.24c) or $\alpha u = 0 \quad (3.38)$

imply $\alpha$ is not injective. another choice is $\beta^+ u = 0 \quad (3.39)$

or the stable condition [18]

$$\begin{align*}
(zB_{11}^+ + \bar{w}B_{21}^+) u &= -\bar{x}u, \\
(zB_{12}^+ + \bar{w}B_{22}^+) u &= -\bar{y}u, \\
(zI_1^+ + \bar{w}I_2^+) u &= 0
\end{align*} \quad (3.40a - 3.40c)$$

which imply $\beta$ is not surjective. One notes that by applying the $\Sigma$ map on the ADHM data and the $\sigma$ map on the point $(x, y, z, w)$ on $CP^3$, Eq.(2.24a) to Eq.(2.24c) are transformed to Eq.(3.40a) to Eq.(3.40c).

To be more precisely, with a sheaf solution of Eq.(2.24a) to Eq.(2.24c), one can define a set of new ADHM data

$$(B_{11}', B_{12}', B_{21}', B_{22}', I_1', I_2', J_1', J_2') = (B_{22}^+, -B_{21}^+, -B_{12}^+, B_{11}^+, J_2^+, -J_1^+, -I_2^+, I_1^+) \quad (3.41)$$

and at the new point

$$[x', y', z', w'] = [\bar{y} : -\bar{x} : \bar{w} : -\bar{z}] \quad (3.42)$$

on $CP^3$. One can verify that Eq.(3.41) together with Eq.(3.42) constitute a new sheaf solution of Eq.(3.40a) to Eq.(3.40c). In fact, if one plugs Eq.(3.41) and Eq.(3.42) into
Eq. (3.40a) to Eq. (3.40c), one ends up with precisely Eq. (2.24a) to Eq. (2.24c). That is, \( \alpha \) is not injective for the old sheaf solution and \( \beta' \) is not surjective for the new sheaf solution.

As an example of the dual symmetry discussed above, we use the old sheaf point \([x : y : z : w] = [0 : 0 : 1 : 1]\) with the old ADHM data in Eq. (2.25)

\[
B_{11} = \begin{pmatrix} \frac{-ia}{\sqrt{2}} & 0 \\ 0 & \frac{ia}{\sqrt{2}} \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 0 & \frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} & 0 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 0 & \frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} & 0 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} \frac{-ia}{\sqrt{2}} & 0 \\ 0 & \frac{ia}{\sqrt{2}} \end{pmatrix},
\]

\[
J_1 = \begin{pmatrix} a & 0 \\ 0 & ia \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & ia \\ a & 0 \end{pmatrix}, \quad I_1 = \begin{pmatrix} 0 & a \\ -ia & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} -a & 0 \\ 0 & ia \end{pmatrix}, \quad \tag{3.43}
\]

which give \( \alpha \) not injective, then we can calculate the new sheaf point \([x' : y' : z' : w'] = [\bar{y} : -\bar{x} : \bar{w} : -\bar{z}] = [0 : 0 : 1 : -1]\) with the new ADHM data

\[
B'_{11} = \begin{pmatrix} \frac{ia}{\sqrt{2}} & 0 \\ 0 & \frac{-ia}{\sqrt{2}} \end{pmatrix}, \quad B'_{21} = \begin{pmatrix} 0 & -\frac{a}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} & 0 \end{pmatrix}, \quad B'_{12} = \begin{pmatrix} 0 & -\frac{a}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} & 0 \end{pmatrix}, \quad B'_{22} = \begin{pmatrix} \frac{i\bar{a}}{\sqrt{2}} & 0 \\ 0 & \frac{-i\bar{a}}{\sqrt{2}} \end{pmatrix},
\]

\[
J'_1 = \begin{pmatrix} \frac{\bar{a}}{\sqrt{2}} & 0 \\ 0 & \frac{i\bar{a}}{\sqrt{2}} \end{pmatrix}, \quad J'_2 = \begin{pmatrix} 0 & i\bar{a} \\ \frac{\bar{a}}{\sqrt{2}} & 0 \end{pmatrix}, \quad I'_1 = \begin{pmatrix} 0 & \bar{a} \\ -i\bar{a} & 0 \end{pmatrix}, \quad I'_2 = \begin{pmatrix} -\bar{a} & 0 \\ 0 & i\bar{a} \end{pmatrix}, \quad \tag{3.44}
\]

which give \( \beta' \) not surjective. Note that at the point \([0 : 0 : 1 : -1]\) with the ADHM data in Eq. (2.25), \( \alpha \) is not injective. It’s important to see that the ADHM data in Eq. (3.44) can not be obtained from the ADHM data in Eq. (2.25) by re-naming the parameter \( a \).

C. Jumping lines

In constrast to the \( SU(2) \) ADHM construction, the \( SL(2, C) \) ADHM instanton construction in Eq. (2.21) contains a set of jumping lines \( J \) for the instanton bundle \( E \). For those spacetime points \( x \in J \subset S^4 \), \( \det \Delta(x)^{40} \Delta(x) = 0 \), there are singular points on the connections \( A \) and the field strength \( F \). On the other hand, the real lines which connect points \([a : b : c : d]\) and \([x : y : z : w]\) on \( CP^3 \) are jumping lines of the instanton bundle \( E \) if \( \det(\beta_{[a:b:c:d]}(x:y:z:w)) = 0 \). It turns out that there is an one to one correspondence between jumping lines of the instanton bundle \( E \) and singular points of \( A \) and \( F \) on \( S^4 \) spacetime.

Note that a bundle \( E \) on \( CP^3 \) can descend down to a bundle over \( S^4 \) if and only if no fiber of the twistor fibration is a jumping line for \( E \). This is the case for \( SU(2) \) instanton and thus there are no jumping lines on \( E \) and no singular points on \( S^4 \) spacetime.
To see the correspondence, similar to Eq.(2.22a) and Eq.(2.22b), we define $\alpha$ and $\beta$ matrices at different points $[x : y : z : w]$ and $[a : b : c : d]$ on $CP^3$ as

$$
\alpha_{[x;y;z:w]} = \begin{pmatrix}
I_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} & I_{2 \times 2}
\end{pmatrix}
x + \begin{pmatrix}
0_{2 \times 2} & I_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2}
\end{pmatrix}
y + \begin{pmatrix}
B_{11} & B_{12} & J_1
\end{pmatrix}
z + \begin{pmatrix}
B_{21} & B_{22} & J_2
\end{pmatrix}
w, \quad (3.45)
$$

$$
\beta_{[a;b;c;d]} = \begin{pmatrix}
0_{2 \times 2} & I_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2}
\end{pmatrix}
a + \begin{pmatrix}
0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} & I_{2 \times 2}
\end{pmatrix}
b + \begin{pmatrix}
-B_{12} & B_{11} & I_1
\end{pmatrix}
c + \begin{pmatrix}
-B_{22} & B_{21} & I_2
\end{pmatrix}
d. \quad (3.46)
$$

It is straightforward to calculate the product map

$$
\beta_{[a;b;c;d]} \alpha_{[x;y;z:w]} = (ay - bx) + B_{12} (az - cx) + B_{22} (aw - dx) + B_{11} (cy - bz) + B_{21} (dy - bw)
$$

$$
+ (-B_{12}B_{11} + B_{11}B_{12} + \imath_1 j_1)cz + (-B_{12}B_{21} + B_{11}B_{22} + \imath_1 j_2)cw
$$

$$
+ (-B_{22}B_{11} + B_{21}B_{12} + \imath_2 j_1)dz + (-B_{22}B_{21} + B_{21}B_{22} + \imath_2 j_2)dw
$$

$$
= z_{12} + B_{12}z_{13} + B_{22}z_{14} - B_{11}z_{23} - B_{21}z_{24} + (-B_{12}B_{21} + B_{11}B_{22} + \imath_1 j_2)z_{34} \quad (3.47)
$$

where we have applied the complex ADHM equations in Eq.(2.15a) to Eq.(2.15c). We have also used the Plücker coordinate representation in Eq.(3.27) to reduce the above result.

As an example, for the sheaf point $[0 : 0 : 1 : 1]$ on $CP^3$ obtained in the previous section, we can calculate

$$
\beta_{[0;0;1;1]} \alpha_{[0;0;1;1]} = \beta_{[0;0;1;1]} \alpha_{[0;0;1;1]} = \begin{pmatrix}
0 & 0
\end{pmatrix} \quad (3.48)
$$

On the other hand, we can also calculate $\Delta \otimes \Delta$ in Eq.(2.1) on $S^4$. To do the calculation, we introduce the coordinates for $x$ ($x_0$ and $x_1$ in Eq.(3.49) are not to be confused with $x^0$ and $x^1$ in Eq.(3.35))

$$
x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3
$$

$$
= \begin{pmatrix}
x_0 - \imath x_3 & - (x_2 + \imath x_1)
x_2 - \imath x_1 & x_0 + \imath x_3
\end{pmatrix}
$$

$$
= \begin{pmatrix}
x_{11} & x_{21}
x_{12} & x_{22}
\end{pmatrix} \quad (3.49)
$$
The result is

\[
\Delta^\ast \Delta
\]

\[
= \begin{pmatrix}
-I_2 & x_{22} + B_{22} & -x_{21} - B_{21} \\
I_1 & -x_{12} - B_{12} & x_{11} + B_{11}
\end{pmatrix}
\begin{pmatrix}
J_1 & J_2 \\
x_{11} + B_{11} & x_{21} + B_{21} \\
x_{12} + B_{12} & x_{22} + B_{22}
\end{pmatrix}
\]

\[
= (x_{11}x_{22} - x_{12}x_{21} + x_{11}B_{22} - x_{12}B_{21} - x_{21}B_{12} + x_{22}B_{11} + I_1J_2 + B_{11}B_{22} - B_{12}B_{21})
\]

where again the complex ADHM equations have been used to reduce the calculation above.

Finally if we use the identification \([18]\) for Eq.(3.27) and Eq.(3.49)

\[D' = z_{12} = ay - bx,
\]

\[-x_{21} = z_{13} = az - cx,
\]

\[x_{11} = z_{14} = aw - dx,
\]

\[-x_{22} = z_{23} = bz - cy,
\]

\[x_{12} = z_{24} = bw - dy,
\]

\[1 = z_{34} = cw - dz
\]

where

\[
D' = x_{11}x_{22} - x_{12}x_{21}
\]

is fixed by the quadratic relations in Eq.(3.29), and restrict \([z_{12} : z_{13} : z_{14} : z_{23} : z_{24} : z_{34}]\) to be a real line, we end up with the correspondence

\[
\Delta^\ast \Delta = \beta_{[a:b:c:d]}^\ast \alpha_{[x:y:z:w]}.
\]

So the jumping line connecting two points \([a : b : c : d]\) and \([x : y : z : w]\) on \(CP^3\) can be calculated from the jumping line equation

\[
\det \beta_{[a:b:c:d]}^\ast \alpha_{[x:y:z:w]} = 0.
\]

On the other hand, the corresponding singular point on \(S^4\) associated with jumping line can be calculated from Eq.(2.2).
Before moving to the next subsection, let’s look at the identification in Eq. (3.51) in more details. Note that the four complex number ($x_{11}, x_{12}, x_{21}, x_{22}$) in Eq. (3.51) represent a line in $CP^3$. If we choose $[a : b : c : d] = \sigma[x : y : z : w] = [\bar{y} : -\bar{x} : \bar{w} : -\bar{z}]$ in Eq. (3.51) and Eq. (3.52), we get

\[
D' = z_{12} = \bar{y}y - (-\bar{x})x = \bar{y}y + \bar{x}x,
-x_{21} = z_{13} = \bar{y}z - \bar{w}x = \bar{y}z - \bar{w}x,
-x_{22} = z_{23} = -\bar{x}z - \bar{w}y = -(\bar{x}z + \bar{w}y),
-x_{12} = z_{24} = -\bar{x}w - (-\bar{z})y = -\bar{x}w + \bar{z}y,
1 = z_{34} = \bar{w}w - (-\bar{z})z = \bar{w}w + \bar{z}z,
\]

(3.54)

and

\[
\Delta^\oplus \Delta = \beta_{\sigma[x:y:z:w]} \sigma_{3:1:2} \lambda_{x:y:z:w}.
\]

(3.55)

One can easily see that

\[
x_{11} = \bar{x}_{22},
-x_{12} = -\bar{x}_{21},
\]

(3.56)

which constrain ($x_{11}, x_{12}, x_{21}, x_{22}$) to contain only four real parameters to represent a real line over $S^4$. This real line is in an one to one correspondence with a point $x$ with four real coordinates on $S^4$. To be more specific, with the identification in Eq. (3.49), the four real coordinates in $x_{\mu} = (x_0, x_1, x_2, x_3)$ represents a point on $S^4$, while ($x_{11}, x_{12}, x_{21}, x_{22}$) in Eq. (3.54) represents the corresponding real line in $CP^3$ over $S^4$. On the other hand, Eq. (3.55) gives an exact relation between coordinates of the sheaf point $[x : y : z : w]$ on $CP^3$ and coordinates of the corresponding singular point ($x_0, x_1, x_2, x_3$) on $S^4$.

To compare the parametrization used in Eq. (3.35), we note that Eq. (3.35) can be further
calculated to be

\[ \pi([x : y : z : w]) = [x + ye_2 : z + we_2] \]

\[ = [x^0e_0 + x^1e_1 + y^0e_2 + y^1e_3 : z^0e_0 + z^1e_1 + w^0e_2 + w^1e_3] \]

\[ \simeq [(z^0e_0 - z^1e_1 - w^0e_2 - w^1e_3)(x^0e_0 + x^1e_1 + y^0e_2 + y^1e_3)] \]

\[ : (z^0e_0 - z^1e_1 - w^0e_2 - w^1e_3)(z^0e_0 + z^1e_1 + w^0e_2 + w^1e_3) \]

\[ \simeq \frac{(z^0x^0 + z^1x^1 + w^0y^0 + w^1y^1)e_0 + (z^0x^1 - z^1x^0 - w^0y^1 + w^1y^0)e_1}{(z^0)^2 + (z^1)^2 + (w^0)^2 + (w^1)^2} : e_0 \]

\[ = [x_0e_0 - x_1e_3 + x_2e_2 - x_3e_3 : e_0] \quad (3.57) \]

where in the last step of the above calculation, we have used the identifications in Eq. (3.49) and Eq. (3.54). The quaternion \((x_0e_0 - x_1e_3 + x_2e_2 - x_3e_3)\) in the above equation represents a point in \(S^4\) with parametrization used in Eq. (3.35) which is different from parametrization used in Eq. (3.49).

As an application of the above calculation, we can calculate for example the real line corresponding to the sheaf point \([0 : 0 : 1 : 1]\) or the sheaf line in short obtained in the previous section to be

\[ [0 : 0 : 1 : 1] \land \sigma[0 : 0 : 1 : 1] = [0 : 0 : 1 : 1] \land [0 : 0 : 1 : -1] \]

\[ = [0 : 0 : 0 : 0 : 0 : -2] \sim [0 : 0 : 0 : 0 : 0 : 1]. \quad (3.58) \]

For the sheaf point \([0 : 0 : 1 : i]\), similar calculation gives

\[ [0 : 0 : 1 : i] \land \sigma[0 : 0 : 1 : i] = [0 : 0 : 1 : i] \land [0 : 0 : 1 : -i] \]

\[ = [0 : 0 : 0 : 0 : 0 : -2i] \sim [0 : 0 : 0 : 0 : 0 : 1]. \quad (3.59) \]

So all four sheaf points calculated in the previous section lie on the same sheaf line. To calculate the projection of the sheaf point \([0 : 0 : 1 : 1]\) on \(CP^3\) down to \(S^4\), we note from Eq. (3.54) and Eq. (3.58) that

\[ (x_{11}, x_{12}, x_{21}, x_{22}) = (0, 0, 0, 0) \quad (3.60) \]
which means
\[ x_\mu = (0, 0, 0, 0) \quad (3.61) \]
by Eq.\((3.49)\). The projection of all other three sheaf points on \(CP^3\) down to \(S^4\) is \(x_\mu = (0, 0, 0, 0)\) too. Here we note that \(S^4\) contains two parts
\[ S^4 = R^4 \cup \{\infty\}, \quad (3.62) \]
or, in the language of quaternion projective space in Eq.\((3.31)\),
\[ S^4 \cong HP^1 = [R^4 : 1] \cup [1 : 0]. \quad (3.63) \]

D. Properties of the Sheaf line as Jumping line

For the YM 2-instanton data obtained in the last section
\[
J_1 = \begin{pmatrix} a & 0 \\ 0 & ia \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & ia \\ a & 0 \end{pmatrix},
\]
\[
B_{11} = \begin{pmatrix} \frac{ia}{\sqrt{2}} & 0 \\ 0 & \frac{ia}{\sqrt{2}} \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 0 & \frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} & 0 \end{pmatrix},
\]
\[
B_{12} = \begin{pmatrix} 0 & \frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} & 0 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} -\frac{ia}{\sqrt{2}} & 0 \\ 0 & \frac{ia}{\sqrt{2}} \end{pmatrix},
\]
\[
I_1 = \begin{pmatrix} 0 & a \\ -ia & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} -a & 0 \\ 0 & ia \end{pmatrix}, \quad (3.64)
\]
we can calculate the singular points on \(S^4\) associated with the jumping line. The two delta matrices can be written as
\[
\Delta = \begin{pmatrix} ae_0 & -ae_1 \\ x + \frac{i}{\sqrt{2}}ae_0 & \frac{i}{\sqrt{2}}e_1 \end{pmatrix}, \quad \Delta^\oplus = \begin{pmatrix} ae_0 & x^\dagger + \frac{i}{\sqrt{2}}ae_0 & -\frac{i}{\sqrt{2}}e_1 \\ ae_1 & -\frac{i}{\sqrt{2}}e_1 & x^\dagger + \frac{i}{\sqrt{2}}ae_0 \end{pmatrix}, \quad (3.65)
\]
and their product can be calculated to be

\[ \Delta \otimes \Delta = \begin{pmatrix} ae_0 & x^\dagger - \frac{i}{\sqrt{2}} ae_0 \\ ae_1 & -\frac{i}{\sqrt{2}} e_1 \end{pmatrix} \begin{pmatrix} ae_0 & -ae_1 \\ x + \frac{i}{\sqrt{2}} ae_0 & \frac{i}{\sqrt{2}} e_1 \end{pmatrix} \]

\[ = \begin{pmatrix} |x|^2 - \sqrt{2}iax_0 & \sqrt{2}iax_1 \\ \sqrt{2}iax_1 & |x|^2 + \sqrt{2}iax_0 \end{pmatrix}, \quad (3.66) \]

which gives

\[ \det \Delta \otimes \Delta = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^2 + 2a^2 (x_0^2 + x_1^2). \quad (3.67) \]

We conclude that

\[ (x_0^2 + x_1^2 + x_2^2 + x_3^2)^2 + 2a^2 (x_0^2 + x_1^2) = 0 \quad (3.68) \]

gives the singular locus on \( S^4 \). One important observation is that for the special singular point

\[ \mu = (0, 0, 0, 0) \quad (3.69) \]

associated with sheaf line calculated in Eq.(3.61), Eq.(3.67) gives

\[ \det \Delta \otimes \Delta_{\text{sheaf}} = 0. \quad (3.70) \]

So this sheaf line is indeed a jumping line. This is a general statement. Indeed, for the case of sheaf lines, either \( \alpha \) is not injective or \( \beta \) is not surjective. If \( \alpha \) is not injective, then \( \beta \alpha \) is not injective, which implies \( \det \beta \alpha = 0 \) or Eq.(3.70). If \( \beta \) is not surjective, then \( \beta \alpha \) is not surjective, which again implies \( \det \beta \alpha = 0 \) or Eq.(3.70). This completes the proof that sheaf lines are special jumping lines. We thus have seen that the following equation holds

\[ \{\text{lines in } CP^3\} \supset \{\text{real lines over } S^4\} \supset \{\text{jumping lines over } S^4\} \supset \{\text{sheaf lines over } S^4\}. \quad (3.71) \]

To identify sheaf lines among jumping lines, in the next section, we will see that the order of singularity of the connection \( A \) and the field strength \( F \) at the singular point on \( S^4 \) associated with sheaf line in \( CP^3 \) is higher than those of other singular points associated with normal jumping lines.

Another interesting observation is that the location of the sheaf point \( x_\mu = (0, 0, 0, 0) \) seems reasonable since it is exactly the geometrical center of "positions" \( y_{11} \) and \( y_{22} \) of the
2-instantons in the ADHM data \[13\]

\[y_{11} = -de_0 = \begin{bmatrix} -d & 0 \\ 0 & -d \end{bmatrix}, \quad y_{22} = de_0 = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix},\]

(3.72)

which we have chosen to obtain Eq. (2.25) and Eq. (2.26).

For the case of diagonal CFTW $SL(2, C)$ 2-instanton solutions, there are no sheaf lines although the jumping lines do exist. The jumping lines or singular locus calculated in Eq. (2.12) are

\[P_4(x) = 0, \quad P_3(x) = 0.\]

(3.73)

The result of Eq. (3.68) can also be obtained by calculating the determinant of $\beta\alpha$ in Eq. (3.53)

\[\det \beta\alpha = \left( x_0^2 + x_1^2 + x_2^2 + x_3^2 \right)^2 + 2a^2 \left( x_0^2 + x_1^2 \right),\]

(3.74)

which is consistent with Eq. (3.67). In this calculation, we have used the identifications in Eq. (3.54) and Eq. (3.49).

Finally, to understand the change of dimensionality of vector bundles at the sheaf points, we can calculate the ranks of $\alpha$ and $\beta$ for a given ADHM data at the sheaf points. For the ADHM data in Eq. (3.64),

\[\alpha_{[x:y:z:w]} = \begin{pmatrix} I_{2 \times 2} \\ 0_{2 \times 2} \\ 0_{2 \times 2} \end{pmatrix} x + \begin{pmatrix} 0_{2 \times 2} \\ I_{2 \times 2} \\ 0_{2 \times 2} \end{pmatrix} y + \begin{pmatrix} B_{11} \\ B_{12} \\ J_1 \end{pmatrix} z + \begin{pmatrix} B_{21} \\ B_{22} \\ J_2 \end{pmatrix} w\]

\[= \begin{pmatrix} I_{2 \times 2} \\ 0_{2 \times 2} \\ 0_{2 \times 2} \end{pmatrix} x + \begin{pmatrix} 0_{2 \times 2} \\ I_{2 \times 2} \\ 0_{2 \times 2} \end{pmatrix} y + \begin{pmatrix} -ia \sqrt{2} & 0 \\ 0 & ia \sqrt{2} \\ 0 & a \sqrt{2} \end{pmatrix} z + \begin{pmatrix} 0 & a \sqrt{2} \\ a \sqrt{2} & 0 \\ 0 & ia \sqrt{2} \end{pmatrix} w,\]

(3.75)
and
\[
\beta_{[x:y:z:w]} = \begin{pmatrix} 0_{2 \times 2} & I_{2 \times 2} & 0_{2 \times 2} \end{pmatrix} x + \begin{pmatrix} -I_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} \end{pmatrix} y + \begin{pmatrix} -B_{12} & B_{11} & I_{1} \end{pmatrix} z + \begin{pmatrix} -B_{22} & B_{21} & I_{2} \end{pmatrix} w
\]
\[
= \begin{pmatrix} 0_{2 \times 2} & I_{2 \times 2} & 0_{2 \times 2} \end{pmatrix} x + \begin{pmatrix} -I_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} \end{pmatrix} y
\]
\[
+ \begin{pmatrix} 0 & -\frac{a}{\sqrt{2}} - \frac{ia}{\sqrt{2}} & 0 & 0 & a \\ -\frac{a}{\sqrt{2}} & 0 & 0 & \frac{ia}{\sqrt{2}} & -ia \\ 0 & -\frac{a}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & ia & \end{pmatrix} z + \begin{pmatrix} \frac{ia}{\sqrt{2}} & 0 & 0 & \frac{a}{\sqrt{2}} & -a \\ 0 & -\frac{ia}{\sqrt{2}} & 0 & 0 & ia \end{pmatrix} w, \quad (3.76)
\]
we can calculate \( \alpha \) and \( \beta \) at the sheaf point \([x:y:z:w] = [0:0:1:1]\) to be \((a \neq 0)\)
\[
\alpha_{[0:0:1:1]} = \begin{pmatrix} -\frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} & \frac{ia}{\sqrt{2}} \\ \frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} \\ a & ia \end{pmatrix}, \beta_{[0:0:1:1]} = \begin{pmatrix} \frac{ia}{\sqrt{2}} & -\frac{a}{\sqrt{2}} & -\frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} & -a & a \\ -\frac{a}{\sqrt{2}} & -\frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} & \frac{ia}{\sqrt{2}} & -ia & ia \end{pmatrix}, \quad (3.77)
\]
which are both of rank 1. So the dimensions of \( \text{Ker} \alpha_{[0:0:1:1]} \) and \( \text{Ker} \beta_{[0:0:1:1]} \) are 1 and 5 respectively, which imply the dimension of the quotient space
\[
\dim(\text{Ker} \beta_{[0:0:1:1]}/\text{Im} \alpha_{[0:0:1:1]}) = 5 - 1 = 4. \quad (3.78)
\]
Note that for points other than sheaf points \( \dim(\text{Ker} \beta/\text{Im} \alpha) = 4 - 2 = 2. \)

Similarly, \( \alpha \) and \( \beta \) at point \([x:y:z:w] = [0:0:1:-1]\) are
\[
\alpha_{[0:0:1:-1]} = \begin{pmatrix} -\frac{ia}{\sqrt{2}} & -\frac{a}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} & -\frac{ia}{\sqrt{2}} \\ \frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} & -\frac{ia}{\sqrt{2}} \end{pmatrix}, \beta_{[0:0:1:-1]} = \begin{pmatrix} \frac{ia}{\sqrt{2}} & -\frac{a}{\sqrt{2}} & -\frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} & -a & a \\ -\frac{a}{\sqrt{2}} & -\frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} & \frac{ia}{\sqrt{2}} & -ia & ia \end{pmatrix}, \quad (3.79)
\]
which are both of rank 1, and the dimension of the quotient space is 4, same with Eq. (3.78).

Similarly, one can calculate \( \alpha \) and \( \beta \) with ADHM data in Eq. (2.26) at the sheaf point.
We can also calculate

\[ [x : y : z : w] = [0 : 0 : 1 : i] \]

to be

\[
\alpha_{[0:0:1:i]} = \begin{pmatrix}
\frac{-ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} \\
\frac{a}{\sqrt{2}} & \frac{ia}{\sqrt{2}} \\
\frac{ia}{\sqrt{2}} & -\frac{a}{\sqrt{2}} \\
a & ia \\
-ia & -a
\end{pmatrix}, \quad \beta_{[0:0:1:i]} = \begin{pmatrix}
-\frac{a}{\sqrt{2}} & -\frac{ia}{\sqrt{2}} & -\frac{ia}{\sqrt{2}} & -ia & a \\
-\frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} & \frac{ia}{\sqrt{2}} & a & ia
\end{pmatrix},
\]

(3.80)

and at the sheaf point \([x : y : z : w] = [0 : 0 : 1 : -i]\) to be

\[
\alpha_{[0:0:1:-i]} = \begin{pmatrix}
\frac{-ia}{\sqrt{2}} & -\frac{a}{\sqrt{2}} \\
\frac{a}{\sqrt{2}} & \frac{ia}{\sqrt{2}} \\
\frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} \\
a & -ia \\
-ia & iia
\end{pmatrix}, \quad \beta_{[0:0:1:-i]} = \begin{pmatrix}
\frac{a}{\sqrt{2}} & -\frac{ia}{\sqrt{2}} & -\frac{ia}{\sqrt{2}} & ia & a \\
-\frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} & \frac{ia}{\sqrt{2}} & a & -ia
\end{pmatrix}.
\]

(3.81)

We can also calculate \(\alpha\) and \(\beta\) with ADHM data in Eq.(3.44) at the sheaf point \([x : y : z : w] = [0 : 0 : 1 : -1]\) to be

\[
\alpha'_{[0:0:1:-1]} = \begin{pmatrix}
\frac{i\bar{a}}{\sqrt{2}} & \frac{\bar{a}}{\sqrt{2}} \\
\frac{\bar{a}}{\sqrt{2}} & -i\bar{a} \\
-\frac{\bar{a}}{\sqrt{2}} & \frac{\bar{a}}{\sqrt{2}} \\
\bar{a} & -i\bar{a} \\
-\bar{a} & i\bar{a}
\end{pmatrix}, \quad \beta'_{[0:0:1:-1]} = \begin{pmatrix}
\frac{i\bar{a}}{\sqrt{2}} & \frac{\bar{a}}{\sqrt{2}} & \frac{\bar{a}}{\sqrt{2}} & \bar{a} & \bar{a} \\
\frac{\bar{a}}{\sqrt{2}} & -i\bar{a} & \frac{\bar{a}}{\sqrt{2}} & -i\bar{a} & -i\bar{a}
\end{pmatrix}.
\]

(3.82)

In all cases of sheaf points, we find that \(\dim(\ker\beta / \text{Im}\alpha) = 5 - 1 = 4\).

**IV. SINGULARITY STRUCTURE OF A AND F ASSOCIATED WITH SHEAF LINE**

In the previous section, we showed that all sheaf lines are jumping lines. What makes sheaf lines different from the normal jumping lines on \(S^4\) spacetime? In this section, we will show that the order of singularity of the connection \(A\) and the field strength \(F\) at the singular point on \(S^4\) associated with sheaf line in \(CP^3\) is higher than those of other singular points associated with normal jumping lines.
A. Singularity structure of connection

In the explicit calculation of $SU(2)$ instanton connections, one needs to do a large gauge transformation to remove all the singularities on $S^4$. This can be easily done for the case of 1-instanton. For the case of diagonal CFTW 2-instanton, see the choice of large gauge transformation function in \[26\]. For the $SL(2, C)$ instanton connections, in addition to the removable singularities, there exist non-removable singularities \[12\] associated with jumping lines in $CP^3$. For example, for the case of $SL(2, C)$ diagonal CFTW 2-instanton, these non-removable singularities can be calculated from Eq.\[2.12\].

For the non-diagonal ADHM 2-instanton sheaves of the present case, we will use similar technique and identify only non-removable singularities which containing the singularity structure associated with the sheaf line. The explicit form of the 2-instanton connection without removable singularities will not be calculated. However, it is interesting to see that the explicit form of 2-instanton field strength without removable singularities can be exactly calculated and will be given in the next subsection. We begin with the two delta matrices with ADHM data given in Eq.\[3.64\]

\[
\Delta = \begin{pmatrix}
 ae_0 & -ae_1 \\
 x + \frac{i}{\sqrt{2}}ae_0 & \frac{ia}{\sqrt{2}}e_1 \\
 \frac{ia}{\sqrt{2}}e_1 & x + \frac{i}{\sqrt{2}}ae_0
\end{pmatrix},
\]

\[
\Delta^\oplus = \begin{pmatrix}
 ae_0 & x^\dagger - \frac{i}{\sqrt{2}}ae_0 & -\frac{ia}{\sqrt{2}}e_1 \\
 -\frac{ia}{\sqrt{2}}e_1 & x^\dagger + \frac{i}{\sqrt{2}}ae_0 & ae_1
\end{pmatrix}.
\]

To calculate the connection, we need to first identify $v$ vector which satisfies $\Delta^\oplus v = 0$ or

\[
\begin{pmatrix}
 ae_0 & x^\dagger + \frac{i}{\sqrt{2}}ae_0 & -\frac{ia}{\sqrt{2}}e_1 \\
 ae_1 & -\frac{ia}{\sqrt{2}}e_1 & x^\dagger + \frac{i}{\sqrt{2}}ae_0
\end{pmatrix}
\begin{pmatrix}
 v_1 \\
 v_2 \\
 v_3
\end{pmatrix} = 0
\]

which means

\[
\begin{align*}
 ae_0 v_1 + \left( x^\dagger + \frac{i}{\sqrt{2}}ae_0 \right) v_2 + \left( -\frac{ia}{\sqrt{2}}e_1 \right) v_3 &= 0, \\
 ae_1 v_1 + \left( \frac{-ia}{\sqrt{2}}e_1 \right) v_2 + \left( x^\dagger + \frac{i}{\sqrt{2}}ae_0 \right) v_3 &= 0,
\end{align*}
\]
from which one can solve \( v_2 \) and \( v_1 \) to be

\[
v_2 = -\frac{xe_1 x^\dagger}{|x|^2} v_3, \tag{4.88}
\]

\[
v_1 = \frac{1}{a} \left[ e_1 x^\dagger + \frac{ia}{\sqrt{2}} \left( e_1 - \frac{xe_1 x^\dagger}{|x|^2} \right) \right] v_3. \tag{4.89}
\]

Finally \( v \) and \( v^\circ \) can be written as

\[
v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{a} \left[ e_1 x^\dagger + \frac{ia}{\sqrt{2}} \left( e_1 - \frac{xe_1 x^\dagger}{|x|^2} \right) \right] v_3 \\ \frac{-xe_1 x^\dagger}{|x|^2} v_3 \\ v_3 \end{pmatrix} \tag{4.90}
\]

and

\[
v^\circ = \left( v_3 \frac{1}{a} \left[ -xe_1 + \frac{ia}{\sqrt{2}} \left( -e_1 + \frac{xe_1 x^\dagger}{|x|^2} \right) \right], v_3 \frac{xe_1 x^\dagger}{|x|^2}, v_3 \right) \tag{4.91}
\]

respectively. The next step is to do the normalization condition

\[
v^\circ v = 1 \tag{4.92}
\]

or

\[
v_3 \left\{ \frac{1}{a^2} \left[ -xe_1 + \frac{ia}{\sqrt{2}} \left( -e_1 + \frac{xe_1 x^\dagger}{|x|^2} \right) \right] \left[ e_1 x^\dagger + \frac{ia}{\sqrt{2}} \left( e_1 - \frac{xe_1 x^\dagger}{|x|^2} \right) \right] + \frac{x e_1 x^\dagger}{|x|^2} \left( -\frac{xe_1 x^\dagger}{|x|^2} \right) \right\} v_3 = 1 \tag{4.93}
\]

to extract the non-removable singular factor similar to Eq. (2.12). After some lengthy calculation, we end up with

\[
v_3 \left\{ \frac{1}{a^2} \left[ \left( \frac{|x|^4 + 2a^2 (x_0^2 + x_1^2)}{|x|^2} \right) \right] \right\} v_3 = 1 \tag{4.94}
\]

where \(|x|^4 = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^2\). So the normalization can be done by setting

\[
v_3 = \frac{a |x|}{\sqrt{|x|^4 + 2a^2 (x_0^2 + x_1^2)}}, \tag{4.95}
\]

and the normalized \( v \) and \( v^\circ \) vector can be written as

\[
v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \frac{a |x|}{\sqrt{|x|^4 + 2a^2 (x_0^2 + x_1^2)}} \begin{pmatrix} \frac{1}{a} \left[ e_1 x^\dagger + \frac{ia}{\sqrt{2}} \left( e_1 - \frac{xe_1 x^\dagger}{|x|^2} \right) \right] \\ \frac{-xe_1 x^\dagger}{|x|^2} \\ 1 \end{pmatrix}, \tag{4.96}
\]

\[
v^\circ = \frac{a |x|}{\sqrt{|x|^4 + 2a^2 (x_0^2 + x_1^2)}} \begin{pmatrix} \frac{1}{a} \left[ -xe_1 + \frac{ia}{\sqrt{2}} \left( -e_1 + \frac{xe_1 x^\dagger}{|x|^2} \right) \right] \\ \frac{xe_1 x^\dagger}{|x|^2} \\ 1 \end{pmatrix}. \tag{4.97}
\]
The connection $A$ can be written as

$$A_\mu = v^\# \partial_\mu v.$$  \hfill (4.98)

It turns out that in order to extract non-removable singularity structure of $A$, one needs only check the normalization factor calculated in Eq.(4.95). This is similar to the calculation in Eq.(2.12) for the case of $SL(2, C)$ CFTW 2-instanton. The factor inside the square root in Eq.(4.95) is exactly the same with $\det \Delta \otimes \Delta$ and $\det \beta \alpha$ calculated in Eq.(3.67) and Eq.(3.74) respectively. We conclude that the non-removable singularities of the connection $A$ occur at

$$|x|^4 + 2a^2 \left( x_0^2 + x_1^2 \right) = 0,$$  \hfill (4.99)

which is the same with the singular locus calculated in Eq.(3.68).

**B. Singularity structure of field strength**

In this subsection, we go one step further to calculate the singularity structure of field strength $F$. It turns out that $F$ without removable singularities is much more easier to calculate than $A$. The formula for the field strength of $SL(2, C)$ ADHM instanton calculated in [12] was

$$F_{\mu\nu} = v^\# b \left( e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger \right) f b^\# v$$  \hfill (4.100)

where $v$ and $v^\#$ were given in Eq.(4.96) and Eq.(4.97) respectively, and other factors can be calculated to be

$$\Delta \otimes \Delta = f^{-1} = \begin{pmatrix} |x|^2 - \sqrt{2}i a x_0 & \sqrt{2}i a x_1 \\ \sqrt{2}i a x_1 & |x|^2 + \sqrt{2}i a x_0 \end{pmatrix},$$  \hfill (4.101)

$$f = \frac{1}{|x|^4 + 2a^2 \left( x_0^2 + x_1^2 \right)} \begin{pmatrix} |x|^2 + \sqrt{2}i a x_0 & -\sqrt{2}i a x_1 \\ -\sqrt{2}i a x_1 & |x|^2 - \sqrt{2}i a x_0 \end{pmatrix},$$  \hfill (4.102)

$$b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b^\# = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  \hfill (4.103)
The field strength can then be calculated to be

\[ F_{\mu\nu} = \frac{a |x|}{\sqrt{|x|^4 + 2a^2 (x_0^2 + x_1^2)}} \left( \frac{1}{a} \left[ -xe_1 + \frac{ia}{\sqrt{2}} \left( -e_1 + \frac{x e_1^\dagger}{|x|^2} \right) \right], \frac{x e_1^\dagger}{|x|^2}, 1 \right) \]

\[ \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger \end{pmatrix} \begin{pmatrix} |x|^2 + \sqrt{2}iax_0 & -\sqrt{2}iax_1 \\ -\sqrt{2}iax_1 & |x|^2 - \sqrt{2}iax_0 \end{pmatrix} \]

\[ = \frac{a^2}{|x|^4 + 2a^2 (x_0^2 + x_1^2)} \left( \frac{x e_1^\dagger}{|x|^2}, |x| \right) \left( e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger \right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |x|^2 + \sqrt{2}iax_0 & -\sqrt{2}iax_1 \\ -\sqrt{2}iax_1 & |x|^2 - \sqrt{2}iax_0 \end{pmatrix} \begin{pmatrix} x e_1^\dagger \\ x \end{pmatrix}. \]

\[ (4.104) \]

It is important to see that there are non-removable singularities in the prefactor of Eq.(4.104). In addition, removable singularity shows up in \( \left( \frac{x e_1^\dagger}{|x|^2}, |x| \right) \) and \( \left( \frac{x e_1^\dagger}{|x|^2}, |x| \right) \), which surprisingly can be gauged away by preforming a large gauge transformation with simple gauge function in the quaternion form as following

\[ F'_{\mu\nu} = \frac{x}{|x|} F_{\mu\nu} \frac{x}{|x|} \]

\[ = \frac{a^2}{|x|^4 + 2a^2 (x_0^2 + x_1^2)} \left( e_1 x^\dagger, x^\dagger \right) \left( e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger \right) \begin{pmatrix} |x|^2 + \sqrt{2}iax_0 & -\sqrt{2}iax_1 \\ -\sqrt{2}iax_1 & |x|^2 - \sqrt{2}iax_0 \end{pmatrix} \begin{pmatrix} x e_1 \\ x \end{pmatrix} \]

\[ = \frac{a^2}{|x|^4 + 2a^2 (x_0^2 + x_1^2)} \left( e_1, 1 \right) x^\dagger \left( e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger \right) x \begin{pmatrix} |x|^2 + \sqrt{2}iax_0 & -\sqrt{2}iax_1 \\ -\sqrt{2}iax_1 & |x|^2 - \sqrt{2}iax_0 \end{pmatrix} \begin{pmatrix} -e_1 \\ 1 \end{pmatrix}. \]

\[ (4.105) \]

We can see that the non-removable singular points occur in \(|x|^4 + 2a^2 (x_0^2 + x_1^2) = 0\), which is consistent with all our previous calculations.

It is interesting to see that the explicit form of \( SL(2, C) \) YM 2-instanton field strength without removable singularities presented in Eq.(4.105) can be exactly calculated! To the
knowledge of the authors, it seems to be a very difficult task, though it might not be impossible, to exactly calculate $SU(2)$ YM 2-instanton field strength with all singularities removed by a suitable large gauge transformation. See the discussion for the choice of large gauge transformation function in [26] for the case of $SU(2)$ CFTW 2-instanton.

To be more precisely, if one uses the $SL(2, C)$ ADHM data calculated from the costable condition of sheaf structure in Eq.(2.24a) to Eq.(2.24c), and plugs this $SL(2, C)$ sheaf ADHM data into $\Delta^\oplus$ in Eq.(4.84), then the calculation of $\nu$ in Eq.(4.85) and thus the field strength $F$ in Eq.(4.100) will be greatly simplified. A closer look for this solvability or simplification seems worthwhile.

Presumably, the simplification for the calculation of $SL(2, C)$ YM 2-instanton field strength is also due to the existence of the sheaf line with associated one single singular point at $x = (0, 0, 0, 0)$ on $S^4$, instead of two removable singular points corresponding to two positions of $SU(2)$ YM 2-instanton before doing a large gauge transformation [26].

C. Order of Singularity at the Sheaf line

In the paragraph after Eq.(3.70), we have shown that all sheaf lines are special jumping lines. In this subsection we will first define the order of singularity of a jumping line including a sheaf line. We will then give a general prescription to calculate it. Recall that in the $SL(2, C)$ ADHM construction, the jumping lines were defined by Eq.(2.1) and Eq.(2.2) which we reproduce in the following

$$\Delta(x)^\oplus \Delta(x) = f^{-1}, \quad (4.106)$$

$$\det \Delta(x)^\oplus \Delta(x) = 0. \quad (4.107)$$

Note that there are no jumping lines for the $SU(2)$ YM instanton. For a given ADHM data, the field strength can be calculated to be (see the example given in Eq.(4.100) to Eq.(4.104))

$$F_{\mu\nu} = v^\oplus(x)b(e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu)fb^\oplus v(x). \quad (4.108)$$

In the case of $SU(2)$, $v(x)$ (but not $f$) in general contains ”removable singularities” which
can be gauged away by doing a "large gauge transformation" $g$: 

$$F'_{\mu\nu} = v'^\circ(x)b(e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger)fb'(x),$$

$$= g^\circ(x)v^\circ(x)b(e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger)fb^\circ v(x)g(x). \quad (4.109)$$

For the case of $SL(2, C)$ YM instantons, in addition to the "removable singularities" in $v(x)$, $f$ contains "non-removable singularities" which cannot be gauged away and remain [12]. We define the order of singularity of a jumping line to be the singularity in $f$

$$f = (f^{-1})^{-1} = \frac{[Cof f^{-1}]^t}{\det f^{-1}} = \frac{[Cof f^{-1}]^t}{\det \Delta(x)^\circ \Delta(x)} \quad (4.110)$$

where $Cof$ means cofactor of a matrix. In the following we review [12] some explicit calculations of $\det \Delta(x)^\circ \Delta(x)$:

1. The geometry of 1-instanton jumping lines

The complete jumping lines of the $SL(2, C)$ 1-instanton can be described by ADHM data with 10 parameters $y_\mu = p_\mu + iq_\mu$ and $\lambda$. To study these singularities, let the real part of $\lambda^2$ be $c$ and imaginary part of $\lambda^2$ be $d$, we see that [12]

$$\det \Delta(x)^\circ \Delta(x) = |x - (p + qi)|^2_c + \lambda^2 = P_2(x) + iP_1(x)$$

$$= [(x_0 - p_0)^2 + (x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2$$

$$- (q_0^2 + q_1^2 + q_2^2 + q_3^2)] + c$$

$$- 2i[(x_0 - p_0)q_0 + (x_1 - p_1)q_1 + (x_2 - p_2)q_2 + (x_3 - p_3)q_3 - \frac{d}{2}] = 0, \quad (4.111)$$

which implies

$$(x_0 - p_0)^2 + (x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2 = (q_0^2 + q_1^2 + q_2^2 + q_3^2) - c, \quad (4.112)$$

$$(x_0 - p_0)q_0 + (x_1 - p_1)q_1 + (x_2 - p_2)q_2 + (x_3 - p_3)q_3 = \frac{d}{2} \quad (4.113)$$

where $P_2(x)$ and $P_1(x)$ are polynomials of 4 variables with degree 2 and 1 respectively. The geometry of the above singular structure was discussed in details in [12]. There is no sheaf line for $SL(2, C)$ 1-instanton.
2. The complete 2-instanton and 3-instanton jumping lines

Since the complete 2-instanton and 3-instanton ADHM data were worked out in [6, 12], the explicit form of \( \det \Delta(x)^\circ \Delta(x) \) can be explicitly calculated and the corresponding jumping lines can in principal be identified [12]. Since the form of the 3-instanton case is very lengthy, we list as an example only the 2-instanton case in the following [12]

\[
\det \Delta_{2-\text{ins}}(x)^\circ \Delta_{2-\text{ins}}(x) = |x - y_1|^2_c |x - y_2|^2_c + |\lambda_2|^2_c |x - y_1|^2_c + |\lambda_1|^2_c |x - y_2|^2_c \\
+ y_{12}^\circ (x - y_1)y_{12}^\circ (x - y_2) + (x - y_2)^\circ y_{12}(x - y_1)^\circ y_{12} \\
- y_{12}^\circ (x - y_1)y_1^\circ \lambda_2 - \lambda_2^\circ \lambda_1(x - y_1)^\circ y_{12} \\
- (x - y_2)^\circ y_{12}^\circ \lambda_1^\circ \lambda_2 - \lambda_2^\circ \lambda_1 y_{12}(x - y_2) \\
+ |y_{12}|^2_c (|\lambda_2|^2_c + |\lambda_1|^2_c) + |y_{12}|^4_c.
\] (4.114)

One sees that Eq.(4.114) is a polynomial of degree 4 in \( x \). So the order of singularity in \( f \) is at most 4 for a given ADHM data. In general, the order of singularity in \( f \) is at most \( 2k \) for a given \( k \)-instanton ADHM data. Although the complete 2-instanton jumping lines have been calculated in Eq.(4.114), the existence of a special sheaf line was not known in [12]. One explicit example of a 2-instanton sheaf line with order of singularity 2 was calculated in Eq.(4.102). We will discuss this example in details later.

3. A class of \( k \)-instanton jumping lines

A class of \( SL(2, C) \) \( k \)-instanton jumping lines, the \( SL(2, C) \) CFTW or the generalized \( (M, N) \) instanton jumping lines were calculated to be zeros of the following determinant [12].

\[
\det \Delta(x)^\circ \Delta(x) = |x - y_1|^2_c |x - y_2|^2_c \cdots |x - y_k|^2_c \phi = P_{2k}(x) + iP_{2k-1}(x) 
\] (4.115)

where

\[
\phi = 1 + \frac{\lambda_1 \lambda_1^\circ}{|x - y_1|^2_c} + \cdots + \frac{\lambda_k \lambda_k^\circ}{|x - y_k|^2_c}.
\] (4.116)

In Eq.(4.115), \( P_{2k}(x) \) and \( P_{2k-1}(x) \) are polynomials with degrees \( 2k \) and \( 2k - 1 \) respectively. The case of 2-instanton jumping lines was calculated in Eq.(2.12). Unfortunately, it was shown [12] that there existed no sheaf lines for this case.
4. Order of Singularities at the Sheaf line and jumping line

In this subsection, we will show that the order of singularity calculated in the previous subsections for connection and field strength at the singular point \( x_\mu = (0, 0, 0, 0) \) on \( S^4 \) associated with the sheaf line in \( CP^3 \) is higher than those of other singular points associated with normal jumping lines. The function we want to study is in the denominator of the prefactor in Eq. (4.105)

\[
h(x_0, x_1, x_2, x_3) = |x|^4 + 2a^2 \left( x_0^2 + x_1^2 \right)
= \left( x_0^2 + x_1^2 + x_2^2 + x_3^2 \right)^2 + 2a^2 \left( x_0^2 + x_1^2 \right), \quad a \in \mathbb{C}, \quad a \neq 0.
\]

(4.117)

One can easily see that

\[
h(x_0, x_1, x_2, x_3) = 0 \quad \text{and} \quad \partial_\mu h(x_0, x_1, x_2, x_3) = 0 \quad \text{for} \quad x_\mu = (0, 0, 0, 0).
\]

(4.118)

We want to show that there is no spacetime point other than \( x_\mu = (0, 0, 0, 0) \) which shares the same property as in Eq. (4.118). This means that we are looking for non-zero solution for the following system of equations

\[
\left( x_0^2 + x_1^2 + x_2^2 + x_3^2 \right)^2 + 2a^2 \left( x_0^2 + x_1^2 \right) = 0
\]

(4.119)

and

\[
4x_0 \left( x_0^2 + x_1^2 + x_2^2 + x_3^2 \right) + 4a^2 x_0 = 0,
\]

(4.120)

\[
4x_1 \left( x_0^2 + x_1^2 + x_2^2 + x_3^2 \right) + 4a^2 x_1 = 0,
\]

(4.121)

\[
4x_2 \left( x_0^2 + x_1^2 + x_2^2 + x_3^2 \right) = 0,
\]

(4.122)

\[
4x_3 \left( x_0^2 + x_1^2 + x_2^2 + x_3^2 \right) = 0
\]

(4.123)

for \( a \in \mathbb{C}, \quad a \neq 0 \).

To see that there is no non-zero solution of the above system of equations, we first note that Eq. (4.122) and Eq. (4.123) imply \( x_2 = 0 \) and \( x_3 = 0 \) respectively. So either \( x_0 \neq 0 \) or \( x_1 \neq 0 \) which, by Eq. (4.120) and Eq. (4.121), imply \( a^2 = -(x_0^2 + x_1^2) \). But then Eq. (4.119) tells us \(-a^4 = 0 \) or \( a = 0 \), which contradicts with the sheaf condition that \( a \neq 0 \). This completes the proof.

Since \( \partial_\nu \partial_\mu h(x_0, x_1, x_2, x_3) \neq 0 \) for \( x_\mu = (0, 0, 0, 0) \), the order of singularity of the sheaf line calculated in Eq. (3.58) is 2 and is higher than those of other normal jumping lines. We note
that by using Eq. (3.55), the jumping line condition is $\det \Delta \sigma \Delta = \det \beta_{\sigma[x:y:z:w]} \alpha_{[x:y:z:w]} = 0$. On the other hand, the sheaf line is further constrained by another condition that $\alpha$ is not injective (or $\beta$ is not surjective). So it seems to be reasonable to conjecture that in general the order of singularity of a sheaf line is higher than those of other normal jumping lines.

V. CONCLUSION

In this paper, we calculate a sheaf line in $CP^3$ which is a fiber line on $S^4$ spacetime supporting sheaf points on $CP^3$ of Yang-Mills instanton sheaves for some given ADHM data we obtained previously in [13]. We found that this sheaf line is a special jumping line over $S^4$ spacetime. Incidentally, we discover a duality symmetry among YM instanton sheaf solutions with dual ADHM data.

To understand the effect of sheaf line on $S^4$ spacetime, we calculate the singularity structure of the connection $A$ and the field strength $F$ at the corresponding singular point on $S^4$ of this sheaf line. We found that the order of singularity at the singular point on $S^4$ associated with the sheaf line in $CP^3$ is higher than those of other singular points associated with normal jumping lines. We conjecture that this is a general feature for sheaf lines among jumping lines.

One unexpected benefit we obtained in our search of the sheaf line was the great simplification of the calculation of $v$ in Eq. (4.85) and the corresponding connection $A$ and the field strength $F$ in Eq. (4.100) associated with the sheaf ADHM data. In fact, we have seen that for the sheaf ADHM data the explicit form of $SL(2, C)$ YM 2-instanton (or $SU(2)$ complex YM 2-instanton) field strength without removable singularities can be exactly calculated! To understand the mechanism of this simplification of the calculation of YM instanton, more explicit examples of sheaf lines will be helpful.

It will be important to calculate more examples of sheaf lines associated with YM instanton sheaves for instanton with higher topological charges. However, it was shown that there has no sheaf line structure for the $SL(2, C)$ diagonal CFTW $k$-instanton [13]. To explicitly construct YM instanton sheaves, one needs to first work out explicitly non-diagonal ADHM $k$-instanton solutions which are in general difficult to calculate for $k > 3$.

Recently, some examples of YM instanton sheaves with topological charges 3 and 4 were discovered and explicitly constructed [14]. The sheaf lines over $S^4$ of these YM instanton
sheaves with higher topological charges and the associated singular structures of $A$ and $F$ are currently under investigation.

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