Degeneracies when $T=0$ Two Body Matrix Elements are Set Equal to Zero and Regge’s 6j Symmetry Relations

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Abstract

The effects of setting all $T=0$ two body interaction matrix elements equal to a constant (or zero) in shell model calculations (designated as $< T = 0 >= 0$) are investigated. Despite the apparent severity of such a procedure, one gets fairly reasonable spectra. We find that using $< T = 0 >= 0$ in single $j$ shell calculations degeneracies appear e.g. the $I = \frac{1}{2}^-$ and $\frac{13}{2}^-$ states in $^{43}$Sc are at the same excitation energies; likewise the $I=3_{2}^{+},7_{2}^{+},9_{1}^{+}$ and $10_{1}^{+}$ states in $^{44}$Ti. The above degeneracies involve the vanishing of certain 6j and 9j symbols. The symmetry relations of Regge are used to explain why these vanishings are not accidental. Thus for these states the actual deviation from degeneracy are good indicators of the effects of the $T=0$ matrix elements. A further indicator of the effects of the $T=0$ interaction in an even - even nucleus is to compare the energies of states with odd angular momentum with those that are even.
I. INTRODUCTION

In the early 1960’s single j shell calculations in the f_{7/2} region were performed by McCullen, Bayman, and Zamick (MBZ) [1,2] and Ginocchio and French[3]. In these calculations the two body matrix elements were taken from experiment. However the T=0 neutron-proton spectrum in ^{42}Sc was not well determined. Calculations with correct T=0 matrix elements were later performed by Kutschera, Brown, and Ogawa[4].

In order to see how neutron-proton two body matrix elements with isospin T=0 affect the low lying spectra of nuclei, we have set them to a constant in a single j shell calculation in the f_{7/2} region. We can then write \( V^{T=0} = c(1/4 - t_1 \cdot t_2) \) where c is a constant. Hence \( \sum_{i<j} V_{ij}^{T=0} = c/8(n(n-1)+6) - c/2T(T+1) \). This means that the spectrum of states of a given isospin e.g. T=0 in ^{44}Ti is independent of what the constant is, it might as well be zero. What the constant is will affect only the energy splittings of states with different isospin. We shall denote this matrix element input as \( <T=0> = 0 \).

Although setting all T=0 matrix elements to a constant may seem like a severe approximation, it will be seen that one gets a fairly good representation of the spectrum. When the T=0 matrix elements are reintroduced, there is some fine tuning which improves the spectrum.

While the problem of T=1 pairing is better understood and studied, there exists a very extensive literature on the possibility of T=0 pairing, both pro and con. We here include some of the relevant references.[5-13]

In a shell model calculation the effects of both T=0 and T=1 pairing are automatically included. The problem then is to sort out as much as possible the individual effects.

In the next sections we will consider calculations in the f_{7/2} shell and in the full fp space.

II. RESULTS OF SINGLE J SHELL CALCULATIONS

In the following tables we show T=T_{min} calculated yrast spectra for ^{43}Ti(Table I) and ^{44}Ti(Table II) where we use 2 different sets of matrix elements. In the first two columns we show \( <T=0> = 0 \) for the ^{42}Sc matrix elements. The last two columns consists of matrix elements from ^{42}Sc with the T=0 matrix elements now included. Also to gain some insight into how configuration mixing affects our results, we present full fp space results for ^{43}Ti and ^{44}Ti in Tables III and IV respectively.

In the single j shell calculation for which the matrix elements were taken from the spectrum of ^{42}Sc the values of these matrix elements for J =0 to J=7 were 0.000 MeV, 0.6110
MeV, 1.5863 MeV, 1.4904 MeV, 2.8153 MeV, 1.5101 MeV, 3.242 MeV, and 0.6163 MeV respectively. The yrast spectrum is also shown in Fig 1. Note that with a \( j^2 \) configuration the even \( J \) states have \( T \) equal to one and the odd \( J \) state equal to zero. This is also true experimentally for these levels. Note that the \( J=1^+ \) and \( 7^+ \) are nearly degenerate near 0.6 MeV and the \( J=3^+ \) and \( 5^+ \) are nearly degenerate near 1.5 MeV. Thus the act of setting \( T=0 \) matrix elements to a constant is equivalent to moving the \( J=1^+ \) and \( 7^+ \) together up about 0.9 MeV. Or putting it another way, the act of removing the degeneracy is to lower the energies of the \( J=1^+ \) and \( 7^+ \) by about the same amount. This is in contrast to most studies in which only the effects of lowering the \( J=1^+ \) state are studied.

We will point out several features to be found in the tables. We observe many levels that were considerably separated in the 'normal' interaction become degenerate when we go to \(< T = 0 \>= 0 \). We explore this further in the next section. We find that in general with few exceptions that the odd \( I \) levels of \(^{44}\)Ti are at a lower excitation energy when we go to the \(< T = 0 \>= 0 \) version of the interactions and that the \(^{43}\)Ti spectra is lowered in total.

### III. THE DEGENERACIES THAT OCCUR IN \(< T = 0 \>= 0 \) AND EXPLANATIONS

As can be seen from Tables I and II some energy levels are degenerate when the \( T=0 \) matrix elements are set equal to a constant. The degenerate pairs \((I_1, I_2)\) include

- \(^{43}\)Ti \((\frac{1}{2}^−, \frac{13}{2}^−)(\frac{17}{2}^−, \frac{19}{2}−)\)
- \(^{44}\)Ti \((9^+, 10^+)\)

The wavefunctions for the Titanium isotopes are written as

\[
\psi = \sum D^I_{\alpha}(J_p, J_n)[(j^2)_{J_p}(j^n)_{J_n}]^I_{\alpha}
\]

where \( D^\pm(J_p, J_n) \) is the probability amplitude that in a state of total angular momentum \( I \) the protons couple to \( J_p \) and the neutrons to \( J_n \). The elements \( D^I(J_p, J_n) \) form a column vector.

Let us first consider \((\frac{1}{2}^−, \frac{13}{2}−)\) in \(^{43}\)Ti. The basis states can be written as \([J_p, J_n]^I\) where \( J_p \) is the angular momentum of the two protons. The interaction matrix element 

\[
< [J_p, J_n]^I V[J_p, J_n]^I > = \delta_{J_p, J_p} E_{J_p} + 2\Sigma J U(j j I j, J_p J) U(j j I j, J_p) E_J
\]

where \( E_J \) is the two particle matrix element \(< [jj]^I V[jj]^I > \). For even \( J \), \( T \) is equal to one while for odd \( J \), \( T \) is equal to zero.

We next consider \(^{44}\)Ti The interaction matrix element 

\[
< [J_p' J_n']^I V[J_p J_n]^I >
\]

is given by 

\[
E_{J_p} \delta_{J_p, J_p'} \delta_{J_n, J_n'} + E_{J_n} \delta_{J_p, J_p'} \delta_{J_n, J_n'} + 4\Sigma J J_A < (jj)^I (jj)^J_A > ^I<
\]
\[(jj)^{J_p}(jj)^{J_n}|(j)^I(jj)^{I_A} \rangle \implies E_I\text{ where the unitary recouping coefficients are related to the Wigner 9j symbols}

\[
< (ab)^c(de)^f|(ad)^x(be)^y >^I = \sqrt{(2c + 1)(2f + 1)(2x + 1)(2y + 1)} \begin{vmatrix}
 a & b & c \\
 d & e & f \\
 x & y & I 
\end{vmatrix} .
\] (2)

For symmetry relations the 9j symbols are more convenient than the unitary coefficients.

It is instructive to look at the energies and wavefunctions (ie column vectors) for the \( I = \frac{1}{2}^- \) and \( I = \frac{13}{2}^- \) states that appear in the NYO Technical reports (which included T=0 matrix elements.)

| Energy(MeV) | \( I = \frac{1}{2} \) | \( I = \frac{13}{2} \) |
|------------|----------------|----------------|
| \( J_p \)  | \( J_n \) | Energy | Energy |
| 5.4809    | 3.8477         | 5.8122 |
| 4 7/2     | 1.000          | 0.9942 -0.1076 |
| 6 7/2     | 0.000          | 0.1076 0.9942 |

In the \( f_{7/2} \) model the \( I = \frac{1}{2}^- \) configuration is unique \([J_p = 4, J_n = \frac{7}{2}]\). There are two configurations for the \( I = \frac{13}{2}^- \) state \([4 \frac{7}{2}]\) and \([6 \frac{7}{2}]\).

When we go to \(< T = 0 > 0 \) what basically happens is that the eigenvalues and eigenfunctions become

\[
\begin{array}{ccc}
I = \frac{1}{2} & I = \frac{13}{2} \\
J_p & J_n & E_1 & E_1 & E_2 \\
4 7/2 & 1.000 & 1.000 & 0.000 & \\
6 7/2 & 0.000 & 0.000 & 1.000 & 
\end{array}
\]

In order for this to happen the matrix element \(< [J_p = 4, J_n = \frac{7}{2}]^{I=\frac{13}{2}} \rangle V[J_p = 6, J_n = \frac{7}{2}]^{I=\frac{44}{2}} V^{-1} >\) must vanish. This vanishing is carried by the Racah coefficients \(U(\frac{7}{2} \frac{7}{2} \frac{13}{2}; 4J)U(\frac{7}{2} \frac{13}{2}; 6J)\) where J is the angular momentum of a neutron-proton pair.

In general J can be 4,5,6, or 7. However in \(< T = 0 > 0 \), only the even J’s contribute i.e. J=4 or J=6. In either case one of the Racah coefficients will be \(U(\frac{7}{2} \frac{13}{2}; 46)\). This Racah coefficient is zero. This guarantees a decoupling of \([4 \frac{7}{2}]\) from \([6 \frac{7}{2}]\) but does not in itself lead to a degeneracy of the \( I = \frac{1}{2}^- \) and \( I = \frac{13}{2}^- \) states. That happens because of this additional condition

\[
U(\frac{7}{2} \frac{13}{2} \frac{44}{2}; 44) = U(\frac{7}{2} \frac{13}{2}; 44) = \frac{1}{2}
\] (3)
We next consider the degeneracy of $I = 9^+$ and $10^+$ in $^{44}$Ti in $< T = 0 > = 0$. It is again instructive to write down the eigenfunctions as they appear in the NYO report:

\[ \begin{array}{c|cccc}
\text{Energy} & 8.7799 & 8.8590 & 11.5951 & 7.8429 & 9.8814 & 10.5110 \\
\text{Isospin} & T=1 & T=1 & T=1 \\
\end{array} \]

\[
\begin{pmatrix}
J_p & j_n \\
4 & 6 & -0.7071 & 0.5636 & -0.4270 & 0.7037 & -0.0696 & 0.7071 \\
6 & 4 & 0.7071 & 0.5636 & -0.4270 & 0.7037 & -0.0696 & -0.7071 \\
6 & 6 & 0.0000 & 0.6039 & 0.7971 & 0.0984 & 0.9951 & 0.0000
\end{pmatrix}
\]

Before proceeding, we remind the reader of a general rule that can clearly be seen in the wave functions above. For even total angular momentum $I$ the wave functions of even $T$ states of $N=Z$ nuclei do not change sign under the interchange of neutrons and protons but the $T=1$ wavefunctions do change sign. For odd $I$ it is the opposite. This can be summarized by $D_{IT}(J_p, J_n) = (-1)^{I+T} D_{IT}(J_n, J_p)$.

We focus on the $T=0$ states. This makes the life much simpler. Instead of three states each we need only worry about one $I=9^+$ and two $I=10^+$ states. Note that for $I=9^+$ $T=0$ the state was the simple wavefunction

\[
\begin{pmatrix}
-\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
0
\end{pmatrix}
\]

What clearly happens for $I=10^+$ in $< T = 0 > = 0$ is that there is a decoupling of $[6,4]$ and $[4,6]$ from $[6,6]$ So that the wavefunctions of the two $T=0$ states become

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

and the eigenvalues of the first one becomes the same as that of the unique $I=9$ state.

We further note that aside from the yrast degeneracies there are other degeneracies. For example, the $7_2^+$ and $3_2^+$ are degenerate with the $I=9_1^+, 10_1^+$ pair in $^{44}$Ti. At first this is puzzling because the dimensions are different. There are seven basis states for $I=3^+$ and six for $I=7^+$, whereas for $I=9^+$ and $10^+$ there are only three basis states. However, of the seven $I=3^+$ states, five have isospin one, and only two have isospin zero. Of the six $I=7^+$ states, four have isospin one and only two have isospin zero. Since we are focusing on $T=0$ we only show only these wavefunctions in Table V. When the $T=0$ two particle matrix elements are set equal to zero the wave functions simplify as shown in the table.

We now begin to see a connection between $I = 3_2^+, 7_2^+, 9_1^+$, and $10_1^+$. For the $9_1^+$ and $10_1^+$ the only non-zero components of the wave function in the $< T = 0 > = 0$ are $D(4,6)$
and $D(6,4)$ both having magnitude $\frac{1}{\sqrt{2}}$. The $3^+_1$ state has nonzero components $D(2,4)$ and $D(4,2)$. There is no connection with the $9^+_1$ and $10^+_1$ states. However for the $3^+_2$ state the only non-vanishing matrix elements are $D(4,6)$ and $D(6,4)$ each with magnitude $\frac{1}{\sqrt{2}}$. This is the same as what occurs for the $9^+_1$ and $10^+_1$ states.

A similar story is written by $I=7^+$. The non vanishing components for the $7^+_1$ state in the $<T=0>=0$ case are $D(2,6)$ and $D(6,2)$ however for the $7^+_2$ state they are $D(4,6)$ and $D(6,4)$ each with magnitude $\frac{1}{\sqrt{2}}$. Thus a common theme emerges for $I=3^+_2, 7^+_2, 9^+_1, 10^+_1$ (all T=0) in that for the $<T=0>=0$ case the only non-vanishing components of the wave functions are $D(4,6)$ and $D(6,4)$. Visually, the column vectors look the same. And it is precisely these states that are degenerate.

Let us now show why in the case of $<T=0>=0$ the matrix element $\langle J^p = 4 | J_A = 6 | J^p = 6 | J_N = 6 | J_A = 10 \rangle$ vanishes. This is a necessary condition for the wave functions to have the simple form discussed in this section.

From the expression for the neutron-proton interaction previously given the above matrix element is ($j=\frac{7}{2}$)

$$\langle J^p = 4 | J_A = 6 | J^p = 6 | J_N = 6 | J_A = 10 \rangle = \sum_{J_A} (c)(13)(9) \left\{ \begin{array}{c} j \ j \ 4 \\ j \ j \ 6 \\ 4 \ 6 \ 10 \end{array} \right\} \left\{ \begin{array}{c} j \ j \ 4 \\ j \ j \ 6 \\ 4 \ 6 \ 10 \end{array} \right\} E^4 + \sum_{J_A} (c)(13) \left\{ \begin{array}{c} j \ j \ 4 \\ j \ j \ 6 \\ 6 \ J_A \ 10 \end{array} \right\} \left\{ \begin{array}{c} j \ j \ 6 \\ j \ j \ 6 \\ 6 \ J_A \ 10 \end{array} \right\} E^6$$

(4)

where the proportionality constant $c$ is $156\sqrt{13}$. (Note that $E^5$ and $E^7$ are equal to zero because all odd J have T=0) Because the last 9j above has two rows identical it is necessary for $J_A$ to be even ie $J_A=4$ or 6. Thus the coefficient of $E^6$ is

$$\langle J^p = 4 | J_A = 6 | J^p = 6 | J_N = 6 | J_A = 10 \rangle = \sum_{J_A} (c)(13)(9) \left\{ \begin{array}{c} j \ j \ 4 \\ j \ j \ 6 \\ 6 \ 4 \ 10 \end{array} \right\} \left\{ \begin{array}{c} j \ j \ 4 \\ j \ j \ 6 \\ 6 \ 4 \ 10 \end{array} \right\} + \sum_{J_A} (c)(13)(13) \left\{ \begin{array}{c} j \ j \ 4 \\ j \ j \ 6 \\ 6 \ 6 \ 10 \end{array} \right\} \left\{ \begin{array}{c} j \ j \ 6 \\ j \ j \ 6 \\ 6 \ 6 \ 10 \end{array} \right\}$$

(5)

Using symmetry properties of 9j symbols we note that every term in the above expression (both for $E^4$ and $E^6$) contains the 9j symbol \[\left\{ \begin{array}{c} j \ j \ 6 \\ j \ j \ 6 \\ 6 \ 4 \ 10 \end{array} \right\} \]. This 9j symbol is zero and hence we have shown why the above neutron-proton matrix element vanishes. It is by no means obvious why this 9j vanishes. There will be considerable discussion in the next section of why some of the 6j’s and 9j’s we encounter vanish.
Although in Table V we have only shown T=0 wave functions there are several T=1 states interspersed amongst the T=0 states. For example, in the Technical Report NYO-9891 [2] for I=3+ the lowest state calculated to be at 6.2357 MeV has T=1. The calculated energy for this state is about 300 keV lower than the lowest T=0 state shown in Table V. Other T=1 states are calculated to be at 9.2334, 10.0321 and 10.9022 MeV. For I=7+ the lowest T=1 state is calculated to be at 6.7094 MeV, just above the other the lowest T=0 state shown in Table V. The other T=1 states for I=7+ are calculated to be at 9.0744, 9.5141 and 12.1535 MeV. The closeness of T=0 and T=1 states was previously discussed by Goode and Zamick [15].

IV. WHY SOME RACAH COEFFICIENTS VANISH - REGGE SYMMETRIES

Thus far we have explained how degeneracies arise by matrices that certain Racah or 9j symbols vanish. In this section we look for a deeper meaning. We were aided in this by many insightful articles collected in Biedenharn and Van Dam [16].

For convenience we shall switch from unitary Racah coefficients to Wigner 6j symbols

\[ U(abcd;ef) = (-1)^{a+b+c+d} \sqrt{(2e+1)(2f+1)} \binom{a}{b}{c}^{d}{e}{f} \]

In the previous section we noted that the 6j symbol \( \binom{7/2}{7/2}{4}{7/2}{6} \) vanished. We note that this is a particular case of a wider class of 6j’s that vanish. All 6j’s of the form \( \binom{j}{j}{(2j-3)}{j}{(3j-4)}{(2j-1)} \) vanish for all j, both half integer and integer. Besides the six j above other examples are \( \binom{5/2}{5/2}{2}{5/2}{4} \), \( \binom{9/2}{9/2}{6}{9/2}{8} \), and \( \binom{4}{4}{5} \).

We find we can relate the above 6j symbol to a simpler one using one of the six remarkable relations discovered by Regge in 1959 [17] We follow the notation of Rotenberg et. al. [18]

\[ \begin{pmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{pmatrix} = \begin{pmatrix} A & B & C \\ D & E & F \end{pmatrix} \]

\( A = \frac{1}{2}(j_1 + j_2 + l_1 - l_2) \) \( B = \frac{1}{2}(j_2 + j_3 + l_2 - l_3) \)
\( C = \frac{1}{2}(j_1 + j_3 - l_1 + l_3) \) \( D = \frac{1}{2}(j_1 - j_2 + l_1 + l_2) \)
\( E = \frac{1}{2}(j_2 - j_3 + l_2 + l_3) \) \( F = \frac{1}{2}(-j_1 + j_3 + l_1 + l_3) \)
From this Regge symmetry relation we find that

\[
\begin{align*}
\{ j & j (2j - 3) \} = \{ 2 & (2j - 3) (2j - 2) \} = \{ (2j - 2) & (2j - 3) 2 \} \\
\{ j & (3j - 4) (2j - 1) \} = \{ (2j - 2) & (2j - 1) (2j - 2) \} = \{ (2j - 2) & (2j - 1) (2j - 2) \}
\end{align*}
\] (6)

We note that 6j symbols with a “two” in them have been worked out by Biedenharn, Blatt, and Rose [19]. Using their notation we find from their results that \( \{ l_1 & J_1 2 \} \) for \( l_2 = J_1 + 1 \) and \( l_1 = J_1 + 1 \) is proportional to \( X \) where

\[
X = [(J_1 + 1)(J_1 - J_2) - L(L + 1) + J_2(J_2 + 2)]
\] (7)

We have \( L = 2j - 2, J_1 = 2j - 3, l_1 = 2j - 2, J_2 = 2j - 2, \) and \( l_2 = 2j - 1 \). With these values we see that \( X \) vanishes.

In Regge’s paper [17] he states “although no direct connection has been established between these wider symmetries it seems very probably that it will be found in the future.” He also states “We see therefore that there are 144 identical Racah’s coefficients.... It should be pointed out that this wider 144-group is isomorphic to the direct product of the permutation group of 3 and 4 objects.”

Following Regge’s work Bargmann presented, amongst other things, his derivation of the Regge symmetries [20]. He there stated “While the following analysis does not lead to a deeper understanding of the Regge symmetries it yields, at least a fairly transparent derivation of the symmetries.”

In section III we pointed out that a certain 9j symbol “unexpectedly” vanished. Perhaps there are some symmetries involving the 9j symbols as well. The only comment by Bargmann on this [20] is “Schwinger has computed the generating function for the 9j symbol. This does not reveal any new symmetries - at least none to be obtained by a permutation of the relevant quantities k_{\alpha\beta}.”

Nevertheless the Regge symmetries for 6j symbols do have some implications for 9j’s. The 9j mentioned in the previous section \( \{ \frac{7}{2} & \frac{7}{2} 6 \} \) is part of a wider class of identically zero 9j symbols. These are of the form \( \{ j & j (2j - 1) \} \) (Other examples \( \{ (2j - 1) & (2j - 3) (4j - 4) \} \).
Following the notation of Rotenberg et al. [18] we first write down the well known expression for a 9j as a sum over three 6j symbols.

\[
\begin{pmatrix}
  j & j & (2j - 1) \\
  j & j & (2j - 1) \\
  (2j - 1) & (2j - 3) & (4j - 4)
\end{pmatrix} = \Sigma_\beta (-1)^{2\beta}(2\beta + 1)
\]

The parameter \( \beta \) is constrained by triangle relations in each of the 6j symbols. In particular first 6j symbol constrains \( \beta \) as follows

\[
\beta \geq (3j - 4) \tag{9}
\]
\[
\beta \leq (3j - 3). \tag{10}
\]

From these constraints \( \beta = (3j-3) \) or \( (3j-4) \) or the first 6j symbol is zero. If \( \beta = (3j-4) \) the second 6j symbol becomes the one previously discussed above in equation 6 and was there shown to be zero. This leaves \( \beta = (3j-3) \).

In this case the last 6j symbol becomes \( \begin{pmatrix} (2j - 1) & (2j - 1) & (4j - 4) \\ (3j - 3) & j & j \end{pmatrix} \) which we now show vanishes.

We will use the Regge symmetry [18]

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  l_1 & l_2 & l_3
\end{pmatrix} = \begin{pmatrix}
  A & B & C \\
  D & E & F
\end{pmatrix}
\]

\[
A = j_1, \quad B = \frac{1}{2}(j_2 + j_3 - l_2 + l_3), \quad C = \frac{1}{2}(j_2 + j_3 + l_2 - l_3),
\]
\[
E = \frac{1}{2}(-j_2 + j_3 + l_2 + l_3), \quad F = \frac{1}{2}(j_2 - j_3 + l_2 + l_3)
\]

so that we can now write

\[
\begin{pmatrix}
  (2j - 1) & (2j - 1) & (4j - 4) \\
  (3j - 3) & j & j
\end{pmatrix} = \begin{pmatrix}
  (2j - 1) & (3j - \frac{5}{2}) & (3j - \frac{5}{2}) \\
  (3j - 3) & (2j - \frac{3}{2}) & \frac{3}{2}
\end{pmatrix} = \begin{pmatrix}
  (3j - \frac{5}{2}) & (3j - \frac{5}{2}) & (2j - 1) \\
  \frac{3}{2} & (2j - \frac{3}{2}) & (3j - 3)
\end{pmatrix} \tag{11}
\]

The results of 6j symbols with a “\( \frac{3}{2} \)” are found in Varshalovich, Moskalev, and Khersonski
\[ [21] \begin{pmatrix} a & b & c \\ \frac{3}{2} & e & f \end{pmatrix} \] for \( e=c-\frac{1}{2} \) and \( f=b-\frac{1}{2} \), as we have here, is proportional to

\[ 3(-a(a+1)+b(b+1)+c(c+1))-2(b+1)(c+1) \] (12)

which for \( a=(3j-\frac{5}{2}) \), \( b=(3j-\frac{5}{2}) \), and \( c=(2j-1) \) is zero. Thus in the lone remaining case of \( \beta=(3j-3) \) the final 6j symbol in the sum is zero. So for any allowed value of \( \beta \) one of the 6j symbols is zero implying that the 9j symbol

\[
\begin{pmatrix} j & j & \frac{2j-1}{2} \\ j & j & \frac{2j-1}{2} \\ \frac{2j-1}{2} & \frac{2j-1}{2} & \frac{4j-4}{2} \end{pmatrix}
\]
is zero.

\section{V. FULL F-P CALCULATION FOR \textit{43}TI AND \textit{44}TI}

We have performed full fp calculations for \textit{44}Ti and \textit{43}Ti with the FPD6 interaction. We shall show these and also compare the \textit{44}Ti calculations with single j results using the spectrum of \textit{42}Sc as input. The later is shown in Fig 1.

We first discuss \( T=0 \) states in the even-even nucleus \textit{44}Ti. In Table II and Fig 2 we show the single j results. The first two columns show the results when the \( T=0 \) two body matrix elements are set to zero i.e. \(< T=0 >= 0 \). In Fig 2 we show the even I states of \textit{44}Ti in the first column and the odd I in the second column. Note that the \( I=9^+ \) and \( I=10^+ \) states are degenerate as has been previously discussed.

In the last two columns we have the single j shell results when the full spectrum of \textit{42}Sc is introduced including the \( T=0 \) matrix elements. We note that there is much more change in the odd I spectrum than in the even I. The odd I spectrum raises considerably. The even I spectrum gets spread out a bit but this is tame in comparison to the alteration in the odd I spectrum.

In Table IV and Fig 3 we show results for a full f-p calculation using FPD6. We use the same format as for Table II. When the two body \( T=0 \) matrix elements are set equal to zero (first two columns), we find surprisingly that there is not much difference with the single j shell calculation shown in Table II and Fig 2. The \( I=9^+ \) and \( 10^+ \) state which were exactly degenerate in the single j shell calculation are still nearly degenerate in the full fp calculation. The overall spectra do not look very different (see first two columns in Tables II and IV and Fig 2 and 3).

There is one difference however the appearance in Table IV and Fig 3 of \( I=1^+ \) and \( 11^+ \) \( T=0 \) states. In a single j shell calculation the \( I=1^+ \) and \( 11^+ \) states all have isospin \( T=1 \).
We now come to the full f-p calculation in which all the two body matrix elements of the FPD6 interaction are in play- both T=0 and T=1. Now we see major differences for both the even I and odd I states of \(^{44}\text{Ti}\). (See Table IV and Fig 3 right hand columns).

If we look at the low spin states, \(I = 0^+, 2^+, \text{ and } 4^+\) they are largely unaffected when the T=0 two body matrix elements are put back in. The main difference comes from the higher spin states. With the full FPD6 the spectrum of the even I gets spread out more looking somewhat rotational. For example the I=10\(^+\) state increases in energy from 6.476 MeV to 7.790 MeV. In the corresponding single j shell calculation there was hardly any change in the I=10\(^+\) energy. Likewise the I=12\(^+\) energy goes up from 7.192 MeV to 8.574 MeV when the T=0 two body matrix elements are put back into FPD6.

The odd I states experience a substantial upward shift in the spectrum. Now the I=9\(^+\) state is considerably higher than the \(I = 10^+\) state (9.030 vs 7.790 MeV).

In the single j shell calculation with matrix elements from \(^{42}\text{Sc}\) the even I columns corresponding to \(< T = 0 > = 0\) and full spectrum (the first and third columns of energy levels) are not that different. It appears that the reintroduction of the T=0 two body matrix elements does not make much difference. In Fig 3 however the third column, again even I, gets more spread out relative to the first column going a bit in the direction of giving a more rotational spectrum. Thus is would appear that for even I the T=0 two body matrix elements will affect the spectrum in a significant way only when configuration mixing is present.

We now consider the odd-even spectrum \(^{43}\text{Ti}(^{43}\text{Sc})\). The results are shown in Tables I and III and in Fig 4. In the figure we only show a full calculations with FPD6 and compare results when the T=0 two body matrix elements are set equal to zero (first column) with those where the full FPD6 interaction is included (second column).

The results at first look a bit complicated but a careful examination shows systematic behavior.

For I less than \(\frac{7}{2}^-\) the states come down in energy (relative to the I=\(\frac{7}{2}^-\) ground state). For I greater than \(\frac{7}{2}^-\) there is another systematic. When the T=0 two body matrix elements are set to zero there are nearly degenerate doublets \((\frac{9}{2}^-, \frac{11}{2}^-)\) \((\frac{13}{2}^-, \frac{15}{2}^-)\) and \((\frac{17}{2}^-, \frac{19}{2}^-)\). The effect of putting T=0 two body matrix elements back in is to cause the lower spin member of each doublet to rise in energy by a substantial amount, while the higher spin member lowers itself a small amount, ie I= \(\frac{9}{2}^-, \frac{11}{2}^-\), and \(\frac{17}{2}^-\) rise noticeably but I= \(\frac{11}{2}^-, \frac{13}{2}^-\), and \(\frac{19}{2}^-\) drop slightly. This spectral staggering should be good evidence of the importance of T=0 two body matrix elements.

Work on the effect of L=0, T=1 and L=1, T=0 pairing in the f-p shell has already been
performed by Poves and Martinez-Pinedo.[22] They start with a realistic interaction, KB3, and study the effects of removing the T=1 pairing from the T=0 S=1 pairing. They focused on binding energies and on the even spin states of $^{48}$Cr. Relative to their work, whose conclusions we certainly agree with, we have made a more severe approximation of setting all T=0 matrix elements equal to zero. The payoff for us is that certain degeneracies appear between states, the deviation of which in the physical spectrum can largely be attributed to T=0 two body matrix elements. Also, we focussed on odd I excited states. The deviation in the physical spectrum of the energies of odd I states from even I is also a good indication of the effects of T=0 matrix elements.

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TABLES

TABLE I. Spectra of $^{43}\text{Ti}$

$^{42}\text{Sc} < T = 0 >= 0$ interaction

| I   | E(MeV) | $^{42}\text{Sc}$ interaction |
|-----|--------|-------------------------------|
| 7/2 | 0.000  |                               |
| 9/2 | 1.640  |                               |
| 3/2 | 1.831  |                               |
| 11/2| 2.061  |                               |
| 5/2 | 2.832  |                               |
| 1/2 | 3.279  |                               |
| 13/2| 3.279  |                               |
| 15/2| 3.425  |                               |
| 17/2| 3.919  |                               |
| 19/2| 3.919  |                               |

TABLE II. Spectra of $^{44}\text{Ti}$

$^{42}\text{Sc} < T = 0 >= 0$ interaction

| I   | E(MeV) | $^{42}\text{Sc}$ interaction |
|-----|--------|-------------------------------|
| 0   | 0.000  |                               |
| 2   | 1.303  |                               |
| 4   | 2.741  |                               |
| 6   | 3.500  |                               |
| 3   | 4.716  |                               |
| 5   | 4.998  |                               |
| 7   | 5.356  |                               |
| 8   | 5.656  |                               |
| 9   | 7.200  |                               |
| 10  | 7.200  |                               |
| 12  | 7.840  |                               |
### TABLE III. $^{43}$Ti full fp calculation

| I   | E(MeV)  | I   | E(MeV)  |
|-----|---------|-----|---------|
| $7/2$ | 0.000   | $7/2$ | 0.000   |
| $3/2$ | 1.668   | $3/2$ | 0.871   |
| $9/2$ | 1.970   | $1/2$ | 1.805   |
| $11/2$ | 2.000  | $11/2$ | 1.889   |
| $5/2$ | 2.638   | $5/2$ | 2.305   |
| $1/2$ | 2.940   | $9/2$ | 2.633   |
| $15/2$ | 3.065  | $15/2$ | 2.948   |
| $13/2$ | 3.070  | $19/2$ | 3.401   |
| $17/2$ | 3.325  | $13/2$ | 3.718   |
| $19/2$ | 3.417  | $17/2$ | 4.429   |

### TABLE IV. $^{44}$Ti full fp calculation

| I   | E(MeV)  | I   | E(MeV)  |
|-----|---------|-----|---------|
| 0   | 0.000   | 0   | 0.000   |
| 2   | 1.515   | 2   | 1.317   |
| 4   | 2.587   | 4   | 2.536   |
| 6   | 3.223   | 6   | 3.843   |
| 3   | 4.717   | 3   | 6.241   |
| 5   | 4.932   | 8   | 6.383   |
| 8   | 5.292   | 5   | 7.579   |
| 7   | 5.391   | 10  | 7.790   |
| 10  | 6.476   | 7   | 7.921   |
| 9   | 6.574   | 12  | 8.574   |
| 1   | 7.070   | 9   | 9.030   |
| 12  | 7.192   | 1   | 9.681   |
| 11  | 9.914   | 11  | 11.028  |
TABLE V. Comparison of wave functions of MBZ\textsuperscript{a} with those for which $< T = 0 >= 0$ matrix elements are set equal to zero.

| $I=3$ | MBZ | $< T = 0 >= 0$ | MBZ | $< T = 0 >= 0$ |
|-------|-----|----------------|-----|----------------|
| Energy(MeV) | 6.533 | 10.493 |
| $J_P$ | $J_N$ | $J_P$ | $J_N$ | $J_P$ | $J_N$ |
| 2 | 2 | 0.0000 | 0 | 0.0000 | 0 |
| 2 | 4 | 0.6968 | $\frac{1}{\sqrt{2}}$ | -0.1202 | 0 |
| 4 | 2 | -0.6968 | $\frac{1}{\sqrt{2}}$ | 0.1202 | 0 |
| 4 | 4 | 0.0000 | 0 | 0.0000 | 0 |
| 4 | 6 | 0.1202 | 0 | 0.6968 | $\frac{1}{\sqrt{2}}$ |
| 6 | 4 | -0.1202 | 0 | -0.6968 | $\frac{1}{\sqrt{2}}$ |
| 6 | 6 | 0.0000 | 0 | 0.0000 | 0 |
| $I=7$ | MBZ | $< T = 0 >= 0$ | MBZ | $< T = 0 >= 0$ |
| Energy(MeV) | 6.5723 | 9.6570 |
| $J_P$ | $J_N$ | $J_P$ | $J_N$ | $J_P$ | $J_N$ |
| 2 | 6 | 0.6965 | $\frac{1}{\sqrt{2}}$ | 0.1220 | 0 |
| 4 | 4 | 0.0000 | 0 | 0.0000 | 0 |
| 4 | 6 | 0.1220 | 0 | -0.6965 | $\frac{1}{\sqrt{2}}$ |
| 6 | 2 | -0.6965 | $\frac{1}{\sqrt{2}}$ | -0.1220 | 0 |
| 6 | 4 | -0.1220 | 0 | 0.6965 | $\frac{1}{\sqrt{2}}$ |
| 6 | 6 | 0.0000 | 0 | 0.0000 | 0 |
| $I=9$ | MBZ | $< T = 0 >= 0$ | MBZ | $< T = 0 >= 0$ |
| Energy(MeV) | 8.7799 |
| $J_P$ | $J_N$ | $J_P$ | $J_N$ |
| 4 | 6 | -0.7071 | $\frac{1}{\sqrt{2}}$ |
| 6 | 4 | 0.7071 | $\frac{1}{\sqrt{2}}$ |
| 6 | 6 | 0.0000 | 0 |
| $I=10$ | MBZ | $< T = 0 >= 0$ | MBZ | $< T = 0 >= 0$ |
| Energy(MeV) | 7.8429 | 9.8814 |
| $J_P$ | $J_N$ | $J_P$ | $J_N$ |
| 4 | 6 | 0.7037 | $\frac{1}{\sqrt{2}}$ | -0.0696 | 0 |
| 6 | 4 | 0.7037 | $\frac{1}{\sqrt{2}}$ | -0.0696 | 0 |
| 6 | 6 | 0.0084 | 0 | 0.9951 | 1 |

\textsuperscript{a} From Technical Report NYO 9801 [2]
FIG. 1. Spectrum of \( ^{42}\text{Sc} \)
| MeV | $\langle T=0 \rangle = 0$ | Full Interaction |
|-----|--------------------------|-----------------|
| 10  |                          |                 |
| 9   |                          |                 |
| 8   |                         12 |                 |
| 10  |                         9 |                 |
| 7   |                          |                 |
| 6   |                          |                 |
| 8   |                          |                 |
| 5   |                          |                 |
| 4   |                          |                 |
| 6   |                          |                 |
| 3   |                          |                 |
| 4   |                          |                 |
| 2   |                          |                 |
| 2   |                          |                 |
| 1   |                          |                 |
| 2   |                          |                 |
| 0   |                          |                 |
| 0   |                          |                 |

FIG. 2. Single $j \ T=0$ $^{44}$Ti with matrix elements from $^{42}$Sc
FIG. 3. Full f-p T=0 $^{44}$Ti with FPD6 interaction
FIG. 4. Full f-p $^{43}$Ti ($^{43}$Sc) with FPD6 interaction
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