Abstract

A set of vertices $S$ is a determining set of a graph $G$ if every automorphism of $G$ is uniquely determined by its action on $S$. The determining number of $G$ is the minimum cardinality of a determining set of $G$. This paper studies determining sets of Kneser graphs from a hypergraph perspective. This new technique lets us compute the determining number of a wide range of Kneser graphs, concretely $K_{n,k}$ with $n \geq \frac{k(k+1)}{2} + 1$. We also show its usefulness by giving shorter proofs of the characterization of all Kneser graphs with fixed determining number 2, 3 or 4, going even further to fixed determining number 5. We finally establish for which Kneser graphs $K_{n,k}$ the determining number is equal to $n - k$, answering a question posed by Boutin.

Keywords: Determining set, determining number, Kneser graph, hypergraph.

1 Introduction

The determining number of a graph $G$ is the minimum cardinality of a set $S \subset V(G)$ such that the automorphism group of the graph obtained from $G$ by fixing every vertex in $S$ is trivial. The set $S$ is called a determining set of $G$. Although they were first defined as fixing sets by Harary [15], we follow the terminology of [6] (see also [1]) since the author develops a study on Kneser graphs. Specifically, tight bounds for their determining numbers are obtained and all Kneser graphs with determining number 2, 3 or 4 are provided. The main tools used in [6] to find a determining set or to bound a determining number of a Kneser graph are based on characteristic matrices and vertex orbits.

The notion of determining set has its origin in the idea of distinguishing the vertices in a graph, particularly in the concept of symmetry breaking which was introduced by Albertson and Collins [2] and, independently, by Harary [14] [15]. Symmetry breaking has several applications; among them those related to the problem of programming a robot to manipulate objects [18]. Determining sets have been since then widely studied. There exists by now an extensive literature on this topic. Besides the above-mentioned references, see for instance [1] [7] [11] [13].
On the other hand, Kneser’s conjecture states that the graph with all $k$–element subsets of \{1, \ldots, n\} as vertices and with edges connecting disjoint sets has chromatic number $n - 2k + 2$. Kneser proposed this conjecture in [16] in connection with a study of quadratic forms. Its first proof by Lovasz [17] was the beginning of topological combinatorics as a field of research. Kneser graphs arose then as an interesting family of graphs to explore, and several topological proofs of the Kneser’s conjecture have been published; among them those of Bárány [3] and Sarkaria [20]. In 2004, it appeared the first combinatorial proof of this conjecture, due to Matoušek [19]. Besides, in [12] extremal problems concerning these graphs are investigated.

This paper addresses a general study of determining sets of Kneser graphs. Our main contribution is to introduce hypergraphs as a tool for finding determining sets which is done in Section 3. Indeed, we prove that every subset of vertices $S$ of a Kneser graph $K_n:k$ has an associated hypergraph $H_S$. The set is determining whenever the hypergraph is $k$-regular and has either $n$ or $n-1$ edges. We also show that every $k$–regular simple hypergraph with either $n$ or $n - 1$ edges and $n \geq 2k + 1$ has an associated determining set of $K_n:k$. This characterization lets us compute the determining number of all Kneser graphs $K_n:k$ verifying that $n \geq \frac{k(k+1)}{2} + 1$, which is a significant advance since the only exact values obtained previously are for $n = 2^r - 1$ and $k = 2^{r-1} - 1$ (see [6] for details).

In Section 4, we list all Kneser graphs with fixed determining number 2, 3, 4 or 5. In the first three cases, we provide shorter proofs of those developed in [6] in order to show that hypergraphs play an important role in the study of the determining number of Kneser graphs. Indeed, our technique lets us go further to fixed determining number equal to 5.

Section 5 concerns the question of whether there exists an infinite family of Kneser graphs $K_n:k$ with $k \geq 2$ and determining number $n - k$, which was posed by Boutin in [6]. The answer to this question is given by Theorem 5.2 to reach it we use as a main tool our approach of determining sets by hypergraphs.

We conclude the paper with some remarks and open problems.

2 Preliminaries

Graphs in this paper are finite and undirected. The vertex set and the edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. A hypergraph is a generalization of a graph, where edges can connect any number of vertices. Formally, a hypergraph $H$ is a pair $(V(H),E(H))$ where $V(H)$ is the set of vertices, and $E(H)$ is a set of non-empty subsets of $V(H)$ called hyperedges or simply edges. When edges appear only once, the hypergraph is called simple. The order of a hypergraph is the number of its vertices, written as $|V(H)|$, and the size is the number of its edges $|E(H)|$. A hyperedge containing $m$ vertices is said to be an edge of size $m$. Thus, given a hypergraph with $n$ vertices there are edges of size ranging over the set \{1, \ldots, n\}. The degree of a vertex $v$ is the number of hyperedges containing $v$. A hypergraph is called $k$-regular if every vertex has degree $k$. For more terminology we follow [3] and [21]. In all the figures in this paper, the hyperedges of size greater than 2 are illustrated as shadowed regions.
2.1 Determining Sets

An automorphism of $G$, $f : V(G) \rightarrow V(G)$, is a bijective mapping such that $f(u)f(v) \in E(G) \iff uv \in E(G)$. As usual $\text{Aut}(G)$ denotes the automorphism group of $G$.

A subset $S \subseteq V(G)$ is said to be a determining set of $G$ if whenever $g, h \in \text{Aut}(G)$ so that $g(s) = h(s)$ for all $s \in S$, then $g(v) = h(v)$ for all $v \in V(G)$. The minimum cardinality of a determining set of $G$, denoted by $\text{Det}(G)$, is called the determining number of $G$.

Note that every graph has a determining set. It suffices to consider any set containing all but one vertex. Thus, $\text{Det}(G) \leq |V(G)| - 1$. The only connected graphs with $\text{Det}(G) = |V(G)| - 1$ are the complete graphs. A graph $G$ with $\text{Det}(G) = 0$ is called asymmetric or rigid graph (see for instance [2]). In [4], it is proved that almost all graphs are rigid.

A characterization of determining set is provided by Boutin in [6] by using the pointwise stabilizer of $S$, that is, for any $S \subseteq V(G)$

$$\text{Stab}(S) = \{g \in \text{Aut}(G) \mid g(v) = v, \forall v \in S\} = \bigcap_{v \in S} \text{Stab}(v).$$

**Proposition 2.1.** [6] $S \subseteq V(G)$ is a determining set of $G$ if and only if $\text{Stab}(S) = \{id\}$.

There are many graphs whose determining number can be easily computed. Among them the path $P_n$, the cycle $C_n$ and the complete graph $K_n$. An extreme is a minimum determining set of $P_n$ and so $\text{Det}(P_n) = 1$. Any pair of non-antipodal vertices is a determining set of a cycle, thus $\text{Det}(C_n) = 2$. A minimum determining set of $K_n$ is any set containing all but one vertex, and hence $\text{Det}(K_n) = n - 1$. Example 1 of [6] shows that the determining number of the Petersen graph is equal to three. Nevertheless, computing the determining number can require intricate arguments (see for instance [6, 7, 8]). Figure 1 shows minimum determining sets of some Cartesian products of graphs.

![Figure 1](image)

**Figure 1:** (a) The squared vertices form a minimum determining set of $C_5 \square C_3$. (b) The squared vertices form a minimum determining set of $K_7 \square P_3$. For the sake of clarity, we omit some edges of the complete graph $K_7$.

2.2 Kneser Graphs

The Kneser graph $K_{n,k}$ has vertices associated with the $k$–subsets of the $n$–set $[n] = \{1, \ldots, n\}$ and edges connecting disjoint sets (see Figure 2). This family of graphs is usually
considered for \( n \geq 2k \) since for \( n < 2k \) we obtain \( \binom{n}{k} \) isolated vertices. Moreover, the case \( n = 2k \) gives a set of disconnected edges which is not an interesting case for computing the determining number that is equal to \( \frac{1}{2} \binom{2k}{k} \), that is, half the number of vertices. Therefore, throughout this paper we shall assume that \( n > 2k \) and vertices will simultaneously be considered as \( k \)-sets and as vertices. Thus, the complementary of a vertex \( W \in V(K_{n:k}) \), written as \( W^c \), is the \((n-k)\)-subset \([n] \setminus W\).

Figure 2: Kneser graph \( K_{7:3} \). The neighbours of \( \{1, 2, 3\} \) are \( \{4, 5, 7\}, \{4, 5, 6\}, \{5, 6, 7\} \) and \( \{4, 6, 7\} \), since they have no elements in common.

Boutin in [6] provides a first characterization of determining sets of Kneser graphs which is a key tool also in this paper.

**Lemma 2.2.** [6] The set \( \{V_1, \ldots, V_r\} \) is a determining set of \( K_{n:k} \) if and only if there exists no pair of distinct elements \( a, b \in [n] \) so that for each \( i \) either \( \{a, b\} \subseteq V_i \) or \( \{a, b\} \subseteq V_i^c \).

Observe that the above lemma implies that every determining set of \( K_{n:k} \) has to contain all the elements of \([n]\) but at most one.

Lemma 2.2 is used in [6] to obtain tight upper and lower bounds for \( \text{Det}(K_{n:k}) \). Concretely, the author shows that \( \log_2(n + 1) \leq \text{Det}(K_{n:k}) \leq n - k \) and provides the exact value \( \text{Det}(K_{2^r-1:2^r-1}) = r \).

### 3 Computing the determining number of Kneser graphs

In this section, we characterize determining sets of Kneser graphs in terms of hypergraphs. This approach is our key tool to compute the determining number of a wide range of Kneser graphs.

For any set of vertices \( S \subseteq V(K_{n:k}) \) denote by \( \mathcal{H}_S \) the \( k \)-regular hypergraph obtained as follows. The vertex set \( V(\mathcal{H}_S) \) is equal to \( S \), and two vertices belong to the same hyperedge whenever they contain a common element. When an element of \([n]\) appears only once in the vertices of \( S \), we have a loop in the corresponding vertex of \( \mathcal{H}_S \). Figure 3 shows an instance of hypergraph associated to a set \( S \subseteq V(K_{6:3}) \).
Figure 3: Hypergraph associated to $S = \{\{1, 2, 3\}, \{3, 4, 5\}, \{1, 4, 5\}, \{4, 5, 6\}, \{3, 5, 6\}\}$. Number 2 appears only once in $S$ so we have a loop attached to the vertex $\{1, 2, 3\}$. There are two edges of size 2 determined by numbers 1 and 6, two edges of size 3 since numbers 3 and 4 appear in three vertices of $S$, and one edge of size 4 determined by number 5.

The two following results state that the condition of being determining set can be captured from the structure of the associated hypergraph.

**Lemma 3.1.** A vertex set $S$ is a determining set of $K_{n,k}$ with $n \geq 2k + 1$ if and only if the associated $k$-regular hypergraph $H_S$ is simple and has either $n$ or $n - 1$ edges.

**Proof.** ($\Rightarrow$) Consider a determining set $S$ of $K_{n,k}$ and the associated $k$-regular hypergraph $H_S$. By Lemma 2.2, there exists no pair of distinct elements $a, b \in [n]$ so that for each vertex $V \in S$ either $\{a, b\} \subseteq V$ or $\{a, b\} \subseteq V^c$. Hence, $H_S$ is simple. Indeed, having multiple edges in $H_S$ is equivalent to have at least two elements of $[n]$ in exactly the same vertices of $S$, what implies that they are not distinguishable by any vertex of $S$.

It remains to prove that $H_S$ has either $n$ or $n - 1$ edges. By Lemma 2.2, the vertices of every determining set $S$ have to contain all the elements of $[n]$ but at most one. The result follows since elements of $[n]$ in $S$ correspond to edges in $H_S$.

($\Leftarrow$) Suppose that $H_S$ is simple and has either $n$ or $n - 1$ edges. Then, for every $a, b \in [n]$ the corresponding edges in $H_S$ are different and at most one element of $[n]$ is contained in no vertex of $S$. By Lemma 2.2 yields the result.

**Lemma 3.2.** For any $k$–regular simple hypergraph $H$ with either $n$ or $n - 1$ edges and $n \geq 2k + 1$, there exists a determining set $S$ of $K_{n,k}$ such that $H \cong H_S$.

**Proof.** We first label every edge of $H$ with the elements of either $[n]$ or $[n - 1]$ depending on the number of edges. The vertices of $H$ are labeled with the labels of their incident edges, giving rise to $|V(H)|$ different $k$–subsets of $[n]$. Take $S$ as the set formed by these $k$–subsets. Clearly, $H \cong H_S$ and by Lemma 3.1 the result follows.

When a determining set $S$ is minimum, Lemma 3.2 guarantees that it does not exist a $k$–regular, simple hypergraph $H$ with either $n$ or $n - 1$ edges and $|V(H)| < |S|$. Therefore, $H_S$ is a hypergraph of minimum order. More generally, we say that a $k$–regular, simple hypergraph $H$ with either $n$ or $n - 1$ edges has minimum order if it does not exist a $k$–regular, simple hypergraph $H'$ with either $n$ or $n - 1$ edges and $|V(H')| < |V(H)|$. Thus, the following lemma is straightforward.
Lemma 3.3. A vertex set $S$ is a minimum determining set of $K_{n,k}$ with $n \geq 2k + 1$ if and only if the hypergraph $H_S$ is simple, has either $n$ or $n - 1$ edges and minimum order.

Remark. The characterization provided by Lemma 3.3 ensures us that it does not exist an infinite family of Kneser graphs $K_{n,k}$ with fixed determining number, say $d$. Indeed, the hypergraph associated to any minimum determining set must have a fixed number of vertices $d$ which implies that neither $k$ nor $n$ can take infinite values.

Lemmas 3.1 and 3.2 are the main tools in order to compute the determining number of all Kneser graphs $K_{n,k}$ with $n \geq k \left(\frac{k+1}{2}\right) + 1$, what is done in Theorems 3.4 and 3.5 below.

Theorem 3.4. Let $k$ and $d$ be two positive integers such that $k \leq d$ and $d > 2$. Then,

$$\text{Det} \left( K_{\left\lfloor \frac{d(k+1)}{2} \right\rfloor + 1 : k} \right) = d$$

Proof. We first show that $\text{Det} \left( K_{\left\lfloor \frac{d(k+1)}{2} \right\rfloor + 1 : k} \right) \leq d$. By Lemma 3.2, it suffices to prove that there exists a $k$-regular simple hypergraph, say $H_{k,d}$, with order $d$ and either $\left\lfloor \frac{d(k+1)}{2} \right\rfloor$ or $\left\lfloor \frac{d(k+1)}{2} \right\rfloor + 1$ edges. We shall use the fact that every complete graph $K_d$ has $d - 1$ pairwise disjoint perfect matchings whenever $d$ is even, and $\frac{d-1}{2}$ pairwise disjoint hamiltonian cycles whenever $d$ is odd (see for example [9]). We distinguish three cases according to the parity of $k$ and $d$.

Case 1. $d$ even: Consider $d$ vertices, a loop attached at each vertex, and the edges of $k - 1$ pairwise disjoint perfect matchings of the complete graph $K_d$ (see Figure 4(a)). Clearly, this hypergraph $H_{k,d}$ is $k$-regular and has $\frac{d(k+1)}{2}$ edges. Note that its construction does not depend on the parity of $k$.

Case 2. $d$ odd and $k$ odd: $H_{k,d}$ is the hypergraph formed by $d$ vertices with loops attached at each vertex, and $\frac{k-1}{2}$ pairwise disjoint hamiltonian cycles of $K_d$ (see Figure 4(b)). It is easy to check that $H_{k,d}$ is a $k$-regular hypergraph with $\frac{d(k+1)}{2}$ edges.

Case 3. $d$ odd and $k$ even: Consider the hypergraph $H_{k+1,d}$ obtained from case 2. Take any hamiltonian cycle $C$ with edges, say $\{e_1, e_2, \ldots, e_d\}$. Now, delete the edges with even
index and the loop attached at the vertex in which \( e_1 \) and \( e_d \) are incident. This construction gives rise to a \( k \)-regular hypergraph \( \mathcal{H}_{k,d} \) with \( \frac{d(k+1)-1}{2} = \left\lceil \frac{d(k+1)}{2} \right\rceil \) edges.

To complete the proof, it remains to show that \( \det(K_{\left\lfloor \frac{d(k+1)}{2} \right\rfloor+1,k}) \) is exactly equal to \( d \). By Lemma \[3.2\] it suffices to prove that every \( k \)-regular hypergraph \( \mathcal{H} \) with \( \left\lceil \frac{d(k+1)}{2} \right\rceil \) or \( \left\lfloor \frac{d(k+1)}{2} \right\rfloor \) + 1 edges has at least \( d \) vertices. Suppose on the contrary that there exists a \( k \)-regular hypergraph \( \mathcal{H} \) with \( \left\lceil \frac{d(k+1)}{2} \right\rceil \) edges (analogous for \( \left\lfloor \frac{d(k+1)}{2} \right\rfloor \) + 1 edges) and \( d' < d \) vertices. By Theorem 2.8 of [10] it follows that \( d' = |V(\mathcal{H})| \geq \left\lceil \frac{2|E(\mathcal{H})|}{k+1} \right\rceil \). Hence,

\[ d' \geq \left\lceil \frac{2}{k+1} \frac{d(k+1)}{2} \right\rceil = d \]

which is a contradiction.

Our next aim is to extend the result of Theorem \[3.4\] to Kneser graphs \( K_{n,k} \) verifying that \( d \geq k, d > 2 \) and \( \left\lfloor \frac{(d-1)(k+1)}{2} \right\rfloor < n - 1 < \left\lceil \frac{d(k+1)}{2} \right\rceil \). For our purpose, we first need to introduce an operation on the edge set of any hypergraph \( \mathcal{H} \).

Let \( e_1 = \{v_1, v_2, \ldots, v_s\} \) and \( e_2 = \{w_1, w_2, \ldots, w_r\} \) be two edges of \( \mathcal{H} \) of sizes \( s, r \geq 1 \) with possibly common vertices. We say that \( e_1 \) and \( e_2 \) are merged obtaining a new hypergraph \( \mathcal{H}' \) if \( V(\mathcal{H}') = V(\mathcal{H}) \) and \( E(\mathcal{H}') = (E(\mathcal{H}) \setminus \{e_1, e_2\}) \cup \{v_1, \ldots, v_s, w_1, \ldots, w_r\} \). Denote by \( e_1 \cup e_2 \) the set \( \{v_1, \ldots, v_s, w_1, \ldots, w_r\} \) in which obviously the possible common vertices are considered only once. Now, we can extend this operation to merge a finite set of edges, say \( \{e_1, e_2, \ldots, e_t\} \) obtaining the hypergraph \( \mathcal{H}' \) with \( E(\mathcal{H}') = (E(\mathcal{H}) \setminus \{e_1, e_2, \ldots, e_t\}) \cup \{e_1 \cup e_2 \cup \ldots e_t\} \). Note that \( |E(\mathcal{H}')| = |E(\mathcal{H})| - t + 1 \). We shall apply the operation of merging edges in regular hypergraphs and for \( e_i \neq e_j \) whenever \( i \neq j \). Observe that if \( \mathcal{H} \) is \( k \)-regular and \( e_1, e_2, \ldots, e_t \) are pairwise disjoint edges, that is, they have no vertex in common, then \( \mathcal{H}' \) is \( k \)-regular (see Figure \[5\]).

![Figure 5](image)

Figure 5: The edge \{\(a, b, c, d, e, f\)\} of the hypergraph in (b) is the result of merging three edges of the hypergraph in (a): \{\(a\), \(b, c\)\} and \{\(d, e, f\)\}. Both hypergraphs are 5-regular.

**Theorem 3.5.** Let \( k \) and \( d \) be two positives integers verifying that \( 3 \leq k + 1 \leq d \). For every \( n \in \mathbb{N} \) such that \( \left\lfloor \frac{(d-1)(k+1)}{2} \right\rfloor < n < \left\lceil \frac{d(k+1)}{2} \right\rceil \) it holds \( \det(K_{n+1,k}) = d \).
Proof. Since \( \left\lfloor \frac{(d-1)(k+1)}{2} \right\rfloor < n < \left\lceil \frac{d(k+1)}{2} \right\rceil \) then there exists \( r \in \mathbb{N} \) such that \( n = \left\lfloor \frac{d(k+1)}{2} \right\rfloor - r \) with \( r \leq \left\lfloor \frac{k-1}{2} \right\rfloor \) whenever \( d \) is odd or \( d \) is even and \( k \) is odd, and \( r \leq \frac{k}{2} \) whenever \( k \) is even and \( d \) is even. We first prove that \( \text{Det}(K_{n+1,k}) \leq d \) by distinguishing four cases according to the parity of \( d \) and \( k \). By Lemma 3.2 it suffices to show that there exists a \( k \)-regular simple hypergraph with \( d \) vertices and \( n \) edges.

Case 1. \( d \) even and \( k \) even: Consider the hypergraph \( H_{k,d} \) constructed as in case 1 of the proof of Theorem 3.4. Since \( k + 1 \leq d \) then \( k \leq d - 2 \) and so \( r \leq \frac{k}{2} \leq \frac{d-2}{2} < \frac{d}{2} \). Hence we can take \( r + 1 \leq \frac{d}{2} \) edges of any perfect matching on the vertices of \( H_{k,d} \) and merge them obtaining the hypergraph \( H'_{k,d} \). Since the edges of the perfect matching are disjoint then \( H'_{k,d} \) is \( k \)-regular. Moreover, by construction \( d = |V(H_{k,d})| = |V(H'_{k,d})| \) and

\[
|E(H'_{k,d})| = |E(H_{k,d})| - r = \frac{d(k+1)}{2} - r = n
\]

Case 2. \( d \) even and \( k \) odd: Analogous to the previous case but considering,

\[ r \leq \left\lfloor \frac{k-1}{2} \right\rfloor \leq \left\lfloor \frac{d-2}{2} \right\rfloor < \frac{d}{2} \]

and so \( r + 1 \leq \frac{d}{2} \).

Case 3. \( d \) odd and \( k \) odd: \( H_{k,d} \) is constructed by considering \( d \) vertices with loops attached at each vertex, and \( \frac{d-1}{2} \) pairwise disjoint hamiltonian cycles (see case 2 of the proof of Theorem 3.4). Each cycle has \( d \) edges and \( r \leq \frac{d-1}{2} \leq \frac{d-3}{2} \), then we can merge \( r \) disjoint edges of any hamiltonian cycle \( C \). Denote by \( \{e_1, \ldots, e_d\} \) the edge set of \( C \). Merge those edges with even index plus the loop attached to the vertex in which \( e_1 \) and \( e_d \) are incident, \( r + 1 \) disjoint edges in total. Thus, we obtain a \( k \)-regular simple hypergraph \( H'_{k,d} \) with \( d \) vertices and \( n = \left\lfloor \frac{d(k+1)}{2} \right\rfloor - r \) edges.

Case 4. \( d \) odd and \( k \) even: \( H_{k,d} \) is \( k \)-regular, has \( d \) vertices and \( \frac{d(k+1)-1}{2} \) edges (see case 3 of the proof of Theorem 3.4). Note that we can merge \( r + 1 \) disjoint edges of a hamiltonian cycle \( C \) of order \( d \) since \( r \leq \left\lfloor \frac{d-1}{2} \right\rfloor = \frac{d-2}{2} \) and so \( r + 1 \leq \frac{k}{2} \leq \frac{d-1}{2} < \frac{d}{2} \). It suffices to consider \( r + 1 \) pairwise disjoint edges among the odd labeled edges of \( C \). The resulting hypergraph \( H'_{k,d} \) is simple, \( k \)-regular, has \( d \) vertices and \( n = \left\lfloor \frac{d(k+1)}{2} \right\rfloor - r \) edges.

It remains to prove that \( \text{Det}(K_{n+1,k}) = d \). Suppose on the contrary that \( \text{Det}(K_{n+1,k}) \leq d - 1 \), then by Lemma 3.1 there exists a \( k \)-regular simple hypergraph \( \mathcal{H} \) with \( d - 1 \) vertices and either \( n + 1 \) or \( n \) edges.

Assume first that \( \mathcal{H} \) has \( n + 1 \) edges. The size-edge-sequence \( r_1 \geq \ldots \geq r_{n+1} \), where \( r_i \) is the size of the edge \( e_i \), satisfies (see [10])

\[
\sum_{i=1}^{n+1} r_i = \sum_{v \in V(\mathcal{H})} \delta(v) = k(d-1).
\]

Note that the number of loops in \( \mathcal{H} \) is at most \( d - 1 \), so the other \( n + 1 - (d - 1) = n - d + 2 \) edges have size at least 2. Hence, we obtain the following inequalities about the sum on the edge sizes:
\[
\begin{align*}
\sum_{i=1}^{n+1} r_i &\geq d - 1 \\
\sum_{i=n+1-(d-2)}^{n+1-(d-2)-1} r_i &\geq 2(n - d + 2)
\end{align*}
\]

\[
\Rightarrow k(d - 1) = \sum_{i=1}^{n+1} r_i \geq d - 1 + 2(n - d + 2) = 2n - d + 3
\]

Therefore, \( n \leq \frac{(d-1)(k+1)}{2} - 1 \) which is a contradiction since \( \lfloor \frac{(d-1)(k+1)}{2} \rfloor < n \).

Suppose now that \( \mathcal{H} \) has \( n \) edges. Then,

\[
\begin{align*}
\sum_{i=n-(d-2)}^{n} r_i &\geq d - 1 \\
\sum_{i=1}^{n-(d-2)-1} r_i &\geq 2(n - d + 1)
\end{align*}
\]

\[
\Rightarrow k(d - 1) = \sum_{i=1}^{n} r_i \geq d - 1 + 2(n - d + 1) = 2n - d + 1
\]

Hence, \( n \leq \frac{(d-1)(k+1)}{2} \) which contradicts \( \lfloor \frac{(d-1)(k+1)}{2} \rfloor < n \). \( \square \)

As it was said before, Theorems 3.4 and 3.5 provide the determining number for all Kneser graphs \( K_{n,k} \) with \( n \geq \frac{k(k+1)}{2} + 1 \). We want to stress the usefulness of our technique by illustrating in Figure 6 the values of \( n \) and \( k \) for which we have computed \( \det(K_{n,k}) \), those obtained in [6], the trivial cases, and the values of \( n \) and \( k \) for which \( \det(K_{n,k}) \) remains to compute.

![Figure 6](image)

Figure 6: The shadow area corresponds to the values of \( n \) and \( k \) for which the determining number is provided by Theorems 3.4 and 3.5; the values obtained in [6], the trivial cases, and the values that remain to compute are those on the line \( n = 2k + 1 \) with \( n \neq 2^r - 1 \) and all the values in between the line \( n = 2k + 1 \) and the curve \( n = \frac{k(k+1)}{2} + 1 \).

4 Kneser graphs with fixed determining number

In [6] Boutin characterizes all Kneser graphs with determining numbers 2, 3 or 4, for which she has to assemble a heavy machinery. Our technique allows us to prove the same results.
but with shorter proofs. This is done in Proposition 4.2 below. Moreover, the approach of determining sets by hypergraphs lets us go further, obtaining all Kneser graphs with determining number 5. We first need a technical lemma.

**Lemma 4.1.** Let \( \mathcal{H} \) be a \( k \)-regular simple hypergraph with \( d \) vertices and \( m \) edges. Then the following statements hold:

(a) \( k \leq 2^{d-1} \) and \( m \leq 2^d - 1 \).

(b) If \( m > d + \binom{d}{2} \) then \( kd \geq 3m - 2d - \binom{d}{2} \).

(c) If \( m > d + \binom{d}{2} + \binom{d}{3} \) then \( kd \geq 4m - 3d - 2\binom{d}{2} - \binom{d}{3} \).

**Proof.** Statement (a) follows from the fact that the cardinality of the power set \( \mathcal{P}(V(\mathcal{H})) \) is \( 2^d \), and a hyperedge is a non-empty subset of vertices.

To prove statement (b), assume that \( m > d + \binom{d}{2} \) and consider the size-edge sequence of \( \mathcal{H} \), say \( r_1 \geq r_2 \geq \ldots \geq r_{m-1} \geq r_m \). Then,

\[
k d = \sum_{i=1}^{m} r_i = \sum_{i=m-d+1}^{m} r_i + \sum_{i=m-\left\lceil \frac{d}{2} \right\rceil +1}^{m-d} r_i + \sum_{i=1}^{m-\left\lceil \frac{d}{2} \right\rceil} r_i \geq d + 2\binom{d}{2} + 3 \left( m - d - \binom{d}{2} \right) = 3m - 2d - \binom{d}{2}.
\]

Suppose now that \( m > d + \binom{d}{2} + \binom{d}{3} \). We have,

\[
k d = \sum_{i=1}^{m} r_i = \sum_{i=m-d+1}^{m} r_i + \sum_{i=m-\left\lceil \frac{d}{2} \right\rceil +1}^{m-d} r_i + \sum_{i=m-\left\lceil \frac{d}{2} \right\rceil +1}^{m-\left\lceil \frac{d}{2} \right\rceil + \binom{d}{3}} r_i + \sum_{i=1}^{m-\left\lceil \frac{d}{2} \right\rceil - \binom{d}{3}} r_i \geq d + 2\binom{d}{2} + 3\binom{d}{3} + 4 \left( m - d - \binom{d}{2} - \binom{d}{3} \right) = 4m - 3d - 2\binom{d}{2} - \binom{d}{3}.
\]

Hence, statement (c) holds. \( \square \)

The following result comprises Propositions 12, 13 and 14 of [6]. The statements are the same but we provide shorter proofs by using hypergraphs.

**Proposition 4.2.**

(a) The only Kneser graph with determining number 2 is \( K_{3,1} \).

(b) The only Kneser graphs with determining number 3 are \( K_{4,1}, K_{5,2} \) and \( K_{7,3} \).

(c) The only Kneser graphs with determining number 4 are \( K_{5,1}, K_{6,2}, K_{7,2}, K_{8,3}, K_{9,3}, K_{9,4}, K_{10,4}, K_{11,4}, K_{11,5}, K_{12,5}, K_{13,6} \) and \( K_{15,7} \).

**Proof.** (a) Consider the Kneser graph \( K_{n+1,k} \) and assume that \( \text{Det}(K_{n+1,k}) = 2 \). By Lemma 3.3 there exists a \( k \)-regular simple hypergraph with minimum order 2 and either \( n \) or \( n + 1 \) edges. There are only three simple, regular hypergraphs with two vertices: a pair of loops, an edge of size 2 and an edge of size 2 with a loop attached at each vertex. However, only the first one is an associated hypergraph to the non-trivial Kneser graph \( K_{3,1} \).
(b) Since every Kneser graph $K_{n+1;1}$ is isomorphic to the complete graph $K_{n+1}$, then only $K_{4;1}$ can have determining number 3. Consider now $K_{n+1;k}$ with $k \geq 2$ and suppose that $\text{Det}(K_{n+1;k}) = 3$. By Lemma 3.3 there exists a $k$-regular simple hypergraph with minimum order 3 and either $n$ or $n+1$ edges. Then, Lemma 4.1 implies that $n \leq 7$ and $k \leq 4$. Since $n+1 \geq 2k+1$, we obtain the following candidates: $K_{5;2}$, $K_{6;2}$, $K_{7;2}$, $K_{8;2}$, $K_{7;3}$ and $K_{8;3}$. By Theorems 3.4 and 3.5 it is easy to check that the only Kneser graphs with $\text{Det}(K_{n+1;k}) = 3$ are $K_{5;2}$ and $K_{7;3}$. For instance, the graph $K_{6;2}$ has determining number 4 since $n = 5$ and $n < \left\lfloor \frac{3d-1}{2} \right\rfloor < n < \left\lfloor \frac{3d}{2} \right\rfloor$ for $d = 4$.

(c) Reasoning as in the previous case, the only Kneser graph $K_{n+1;1}$ with determining number 4 is $K_{5;1}$. Take $K_{n+1;k}$ with $k \geq 2$ and assume that $\text{Det}(K_{n+1;k}) = 4$. By Lemma 3.3 there exists a $k$-regular simple hypergraph with minimum order 4 and either $n$ or $n+1$ edges. Hence, Lemma 4.1 implies that $2k \leq n \leq 2k + 1$. Since $n+1 \geq 2k+1$, we obtain the following candidates: $K_{5;2}$, $K_{6;2}$, $K_{7;2}$, $K_{8;2}$, $K_{7;3}$ and $K_{8;3}$. By Theorems 3.4 and 3.5 it is easy to check that the only Kneser graphs with $\text{Det}(K_{n+1;k}) = 3$ are $K_{5;2}$ and $K_{7;3}$. For instance, the graph $K_{6;2}$ has determining number 4 since $n = 5$ and $\left\lfloor \frac{3d-1}{2} \right\rfloor < n < \left\lfloor \frac{3d}{2} \right\rfloor$ for $d = 4$.

Suppose now that $2k \leq n < \frac{k(k+1)}{2}$. Then, $4 \leq k \leq 8$. When $k = 4$, the possible values of $n$ are 8 or 9. Figure 7 shows 4-regular hypergraphs with order 4 and having 8 and 9 edges respectively. By Lemma 3.2, we have $\text{Det}(K_{9;4}) \leq 4$ and $\text{Det}(K_{10;4}) \leq 4$. Obviously, $\text{Det}(K_{9;4}) = \text{Det}(K_{10;4}) = 4$ since they are not in the above list of graphs with determining number either 2 or 3. When $k = 5$, then $10 \leq n \leq 14$. Also, Lemma 4.1 gives $n \leq 11$. Figure 7 shows that $\text{Det}(K_{11;5}) = \text{Det}(K_{12;5}) = 4$. Similarly, when $k$ is 6 or 7, the only Kneser graphs with determining number 4 are $K_{13;6}$ and $K_{15;7}$, and Figure 7 shows their associated hypergraphs. Finally, when $k = 8$ there are not suitable values of $n$, and that completes the proof.

![Figure 7: Hypergraphs associated to Kneser graphs with determining number 4.](image-url)

Our technic lets us go further obtaining all Kneser graphs with determining number 5.
Proposition 4.3. The Kneser graphs with determining number 5 are $K_{6:1}$, $K_{8:2}$, $K_{10:3}$, $K_{11:3}$, $K_{12:4}$, $K_{13:4}$, $K_{13:5}$, $K_{14:5}$, $K_{15:5}$, $K_{16:5}$, $K_{14:6}$, $K_{15:6}$, $K_{16:6}$, $K_{17:6}$, $K_{17:7}$, $K_{18:7}$, $K_{19:7}$, $K_{17:8}$, $K_{18:8}$, $K_{19:8}$, $K_{20:8}$, $K_{21:8}$, $K_{19:9}$, $K_{20:9}$, $K_{21:9}$, $K_{22:9}$, $K_{21:10}$, $K_{22:10}$, $K_{23:10}$, $K_{24:10}$, $K_{23:11}$, $K_{24:11}$, $K_{25:11}$, $K_{26:11}$, $K_{25:12}$, $K_{26:12}$, $K_{27:12}$, $K_{27:13}$, $K_{28:13}$, $K_{29:14}$, and $K_{31:15}$.

Proof. As it was said before, the Kneser graph $K_{n+1:1}$ is isomorphic to the complete graph $K_{n+1}$ then only $K_{6:1}$ has determining number 5. Consider the graph $K_{n+1:k}$ with $k \geq 2$ and suppose that $\text{Det}(K_{n+1:k}) = 5$. By Lemma 3.3 there exists a $k$-regular simple hypergraph $H$ with minimum order 5 and either $n$ or $n+1$ edges. By Lemma 4.1 it follows that $2k \leq n \leq 31$ and $2 \leq k \leq 16$. Thus, the list of candidate graphs increases now to 196 Kneser graphs. When $n \geq \frac{k(k+1)}{2}$ (it happens for 157 graphs among the 196) we can apply Theorems 3.4 and 3.5 obtaining that only $K_{8:2}$, $K_{10:3}$, $K_{11:3}$, $K_{12:4}$, $K_{13:4}$ and $K_{16:5}$ have determining number 5.

Assume now that $2k \leq n < \frac{k(k+1)}{2}$ what implies that $4 \leq k \leq 16$. When $k = 4$, the possible values of $n$ are 8 or 9 but they correspond to Kneser graphs with determining number 4 (see Proposition 4.2). When $k = 5$, Lemma 4.1 gives $10 \leq n \leq 14$. However for $n$ equal to either 10 or 11, we obtain Kneser graphs already studied in Proposition 4.2 whose determining numbers are equal to 4. The remaining values correspond to the Kneser graphs $K_{13:3}$, $K_{14:5}$ and $K_{15:5}$ whose associated hypergraphs with 5 vertices are illustrated in Figure 8. Table 1 shows the rest of the values of $n$ and $k$ for which $\text{Det}(K_{n+1:k}) = 5$. In all the cases, it is easy to construct the associated hypergraph. On the other hand, note that for $k = 6$ and $n = 12$ or $k = 7$ and $n = 14$, the corresponding Kneser graph has determining number 4. For the remaining available values of $n$ and $k$, we use conditions (b) and (c) of Lemma 4.1 in order to show that the hypergraph $H$ does not exist and hence the determining in those cases cannot be equal to 5. For instance, if $k = 6$ and $n = 18$ then it is straightforward to check that condition (b) of Lemma 4.1 does not hold when $d = 5$.

Figure 8: Hypergraphs associated to Kneser graphs with determining number 5.

5 Kneser graphs $K_{n:k}$ with determining number $n - k$

The characterization of determining sets in terms of hypergraphs provided in Section 3 drives us to answer the question posed by Boutin in [6]: We know that $\text{Det}(K_{n:k}) = n - k$ for $K_{n:1}$ for any $n$, $K_{5:2}$ and $K_{6:2}$. Is there an infinite family of Kneser graphs with $k \geq 2$ for which $\text{Det}(K_{n:k}) = n - k$?
Table 1: Values of \( n \) and \( k \) with \( k \geq 6 \) for which \( \text{Det}(K_{n+1: k}) = 5 \).

| \( k \) | \( n \) |
|-------|-------|
| 6     | 13,14,15,16 |
| 7     | 15,16,17,18 |
| 8     | 16,17,18,19,20 |
| 9     | 18,19,20,21 |
| 10    | 20,21,22,23 |
| 11    | 22,23,24,25 |
| 12    | 24,25,26 |
| 13    | 26,27 |
| 14    | 28 |
| 15    | 30 |

Lemma 5.1. Let \( k \) and \( n \) be two positive integers such that \( 2k \leq n < \frac{k(k+1)}{2} \). Then, \( \text{Det}(K_{n+1: k}) \leq k \).

Proof. By Lemma 3.2, it suffices to prove that there exists a \( k \)-regular simple hypergraph \( H \) with \( k \) vertices and \( n \) edges. Consider the \( k \)-regular simple hypergraph \( H_{k,d} \) constructed in the proof of Theorem 3.4. Recall that, independently of the parity of \( k \) and \( d \), the hypergraph \( H_{k,d} \) is \( k \)-regular, has \( d \) vertices and \( \lfloor \frac{d(k+1)}{2} \rfloor \) edges. Assume that \( k = d \), then we have a hypergraph \( H_{k,k} \) with \( k \) vertices and \( k \frac{(k+1)}{2} \) edges.

Let \( \{v_0, v_1, ..., v_{k-1}\} \) be the vertex set of \( H_{k,k} \). We distinguish two cases according to the parity of \( k \). Note that all the indices below are taken modulo \( k \).

Case 1. \( k \) odd: \( H_{k,k} \) is the hypergraph formed by \( k \) vertices with loops attached at each vertex, and \( \frac{k-1}{2} \) pairwise disjoint hamiltonian cycles of \( K_k \) (see case 2 of Theorem 3.4). We assign the following set of \( \frac{k-1}{2} \) edges to each vertex \( v_i \in V(H_{k,k}) \) (see Figure 9):

\[
E_i = \{\{v_{i-1}, v_{i+1}\}, \{v_{i-2}, v_{i+2}\}, ..., \{v_{i-\frac{k-1}{2}}, v_{i+\frac{k-1}{2}}\}\}
\]

Note that the edges of \( E_i \) are disjoint and two sets \( E_i, E_j \) have no edges in common whenever \( i \neq j \). These facts guarantee that in the process of merging that we are going to describe next, the \( k \)-regularity is preserved. We consider again two cases.

1.1. If \( \frac{k(k+1)}{2} - \frac{k-1}{2} + 1 < n < \frac{k(k+1)}{2} - \frac{k-1}{2} \) then merge a subset of \( \frac{k(k+1)}{2} - \frac{k-1}{2} \) edges of \( E_0 \), obtaining a \( k \)-regular hypergraph \( H'_{k,k} \) with \( \frac{k(k+1)}{2} - \frac{k-1}{2} - n + 1 = n \) edges.

1.2. If \( 2k \leq n \leq \frac{k(k+1)}{2} - \frac{k-1}{2} + 1 \) then we can merge the edges of at least one set \( E_i \), obtaining a \( k \)-regular hypergraph which has a number of edges bigger or equal to \( n \). If the number is \( n \) then the process is concluded. Otherwise, suppose that we can merge the edges of \( s \) subsets with \( 0 \leq s \leq k-1 \), say \( E_0, E_2, ..., E_{s-1} \), obtaining a \( k \)-regular hypergraph \( H'_{k,k} \) with \( \frac{k(k+1)}{2} - \frac{s(k-1)}{2} + s \) edges and verifying that

\[
\frac{k(k+1)}{2} - \frac{(s-1)(k-1)}{2} + (s-1) < n < \frac{k(k+1)}{2} - \frac{s(k-1)}{2} + s
\]
Then the edges of $E_s$ cannot be merged since if so the resulting hypergraph would have a number of edges smaller than $n$. Hence, we proceed as in case 1.1 merging $\frac{k(k+1)}{2} - s(k-1) + s - n + 1$ edges of $E_s$. This process leads to a $k$-regular simple hypergraph $H$ with $k$ vertices and $n$ edges.

Figure 9: Hypergraph $H_{7,7}$. The selected edges form the set $E_0$.

Case 2. $k$ even: $H_{k,k}$ is a hypergraph with $k$ vertices, a loop attached at each vertex, and the edges of $k - 1$ pairwise disjoint perfect matchings of the complete graph $K_k$ (see case 1 of Theorem 3.4). We distinguish three cases.

2.1. If $\frac{k(k+1)}{2} - \frac{k(k-1)}{2} + \frac{k}{2} \leq n < \frac{k(k+1)}{2}$ then we can follow an analogous process of merging than in case 1 preserving also the $k$-regularity, but instead of assigning the sets $E_i$ to each vertex $v_i$, we now assign the following set of edges to $v_i$ for $i = 0, \ldots, \frac{k}{2} - 1$ (see Figure 10(a)):

$$F_i = \{\{v_{i-1}, v_{i+1}\}, \{v_{i-2}, v_{i+2}\}, \ldots, \{v_{i-\frac{k}{2}+1}, v_{i+\frac{k}{2}-1}\}\}$$

Note first that the assignment is done to half of the vertices since $F_i = F_{i+\frac{k}{2}}$ because of the parity of $k$. Observe also that the edges of $F_i$ are disjoint and two sets $F_i$, $F_j$ have no edges in common whenever $i \neq j$.

The process described in case 1 leads to a $k$-regular simple hypergraph $H'_{k,k}$ which is the result of merging at most the edges of all sets $F_i$ in $H_{k,k}$, that is, merging at most $\frac{k}{2}(\frac{k}{2} - 1)$ edges and obtaining in such case a hypergraph with $\frac{k(k+1)}{2} - \frac{k}{2}(\frac{k-1}{2}) + \frac{k}{2} \leq n$ with $\frac{k}{2}$ edges. Note that the edges obtained by this procedure are all of size $k - 2$ but at most one of smaller size.

2.2. If $\frac{k(k+1)}{2} - \frac{k}{2}(\frac{k}{2} - 1) + \frac{k}{2} - \frac{k}{2}(\frac{k}{2} - 2) + \frac{k}{2} = 3k \leq n < \frac{k(k+1)}{2} - \frac{k}{2}(\frac{k-2}{2}) - 1$ then we first merge all sets of edges $F_i$, obtaining a hypergraph $H'_{k,k}$ with $\frac{k(k+1)}{2} - \frac{k}{2}(\frac{k-1}{2}) + \frac{k}{2}$ edges. We now assign the following set of edges to $v_i$ for $i = 0, \ldots, \frac{k}{2} - 1$ (see Figure 10(b)):

$$F_i' = \{\{v_{i-1}, v_{i+2}\}, \{v_{i-2}, v_{i+3}\}, \ldots, \{v_{i-\frac{k}{2}+2}, v_{i+\frac{k}{2}-1}\}\}$$

Again, we follow the procedure described in case 1 which gives a $k$-regular simple hypergraph that is the result of merging at most the edges of all sets $F_i'$ in $H'_{k,k}$, that
is, merging at most \( \frac{k(k+1)}{2} - \frac{k}{2} - 2 \) edges and obtaining in such case a hypergraph with \( \frac{k(k+1)}{2} - \frac{k}{2} (n-1) + \frac{k}{2} - \frac{k(k-2)}{2} + \frac{k}{2} \) edges. Observe that the edges obtained by this process are all of size \( k-4 \) but at most one of smaller size.

2.3. If \( 2k \leq n < 3k \) then merge the sets \( F_i \) and \( F'_{i} \), obtaining a hypergraph with \( 3k \) edges. These edges are: \( k \) loops, \( \frac{k}{2} \) edges of size \( k-2 \), \( \frac{k}{2} \) edges of size \( k-4 \) and \( k \) edges of size \( 2 \) forming the cycle \( \{v_0, ... v_{k-1}\} \). For every vertex \( v_i \), consider now the set of edges (see Figure 10(c)):

\[
F_i'' = \{\{v_i\}, \{v_{i+1}, v_{i+2}\}\}
\]

and merge the required sets \( F_i'' \) to attain a hypergraph with \( n \) edges.

![Figure 10: Hypergraph \( H_{8,8} \). The selected edges form the set: (a) \( F_0 \), (b) \( F'_0 \), (c) \( F''_0 \).](image)

Now, we can formulate our main result in this section.

**Theorem 5.2.** \( \text{Det}(K_{n+1:k}) = n+1-k \) if and only if \( K_{n+1:k} \) is isomorphic to \( K_{n+1:1} \), \( K_{5:2} \) or \( K_{6:2} \).

**Proof.** (\( \Rightarrow \)) By Lemma 5.1, it does not exist a Kneser graph \( K_{n+1:k} \) verifying that \( 2k \leq n < \frac{k(k+1)}{2} \) and \( \text{Det}(K_{n+1:k}) = n+1-k \) since \( \text{Det}(K_{n+1:k}) \leq k < n+1-k \). Then, we can assume that \( n \geq \frac{k(k+1)}{2} \) and \( k \geq 2 \). Indeed, when \( k = 1 \) the Kneser graph \( K_{n+1:1} \) is isomorphic to the complete graph \( K_{n+1} \) and so \( \text{Det}(K_{n+1:k}) = n \).

Suppose first that there exists \( d \in \mathbb{N} \) with \( d > 2 \) such that \( n = \lceil \frac{d(k+1)}{2} \rceil \). Then \( d \geq k \) and by Theorem 3.4 we have \( \text{Det}(K_{n+1:k}) = d \). Thus, it suffices to prove that \( d < \lceil \frac{d(k+1)}{2} \rceil + 1 - k \) except for \( d = 3 \) and \( k = 2 \) which is the graph \( K_{5:2} \). Suppose on the contrary that either \( d \neq 3 \) or \( k \neq 2 \) and \( d \geq \lceil \frac{d(k+1)}{2} \rceil + 1 - k \). We distinguish two cases.

**Case 1.** \( d \) even or \( k \) odd: The contradiction follows since \( (2-k-1)d \geq 2(1-k) \) and so \( d \leq 2 \).

**Case 2.** \( d \) odd and \( k \) even: We have \( 2d \geq d(k+1) - 2k + 1 \) what easily implies that \( (k-1)(d-2) \leq 1 \). Clearly, the inequality only holds for \( k = 2 \) and \( d = 3 \).

Assume now that there exists \( d \in \mathbb{N} \) with \( 3 \leq k + 1 \leq d \) such that \( \lfloor \frac{(d-1)(k+1)}{2} \rfloor < n < \lfloor \frac{d(k+1)}{2} \rfloor \). By Theorem 3.5, \( \text{Det}(K_{n+1:k}) = d \) and it suffices to show that \( d < n+1-k \) except
for $k = 2$ and $d = 4$ what leads to the Kneser graph $K_{6,2}$. The following expression holds for all positive integers $d, k$ and $n$ satisfying the above conditions except for $k = 2$ and $d = 4$:

$$d - 1 < \left\lfloor \frac{(d - 1)(k + 1)}{2} \right\rfloor + 1 - k < n + 1 - k$$

Hence, the result follows.

$(\Leftarrow)$ The determining numbers of $K_{n+1,1}$, $K_{5,2}$ and $K_{6,2}$ are $n$, 3 and 4 respectively.

\[\Box\]

6 Concluding Remarks

We have introduced hypergraphs for finding determining sets of Kneser graphs. This technique provides the determining number of all Kneser graphs $K_{n,k}$ with $n \geq \frac{k(k+1)}{2} + 1$. We also show the usefulness of this approach by providing shorter proofs (of those in [6]) of the characterization of all Kneser graphs with fixed determining number 2, 3 or 4, and establishing those with fixed determining number 5. Finally, we prove that it does not exists an infinite number of Kneser graphs $K_{n,k}$ with $k \geq 2$ and determining number $n - k$, answering a question posed by Boutin in [6].

It appears that our technique can also be applied to the values of $n$ and $k$ in between the line $n = 2k + 1$ and the curve $n = \frac{k(k+1)}{2} + 1$ for which Det($K_{n,k}$) remains to compute (see Figure 6). Nevertheless, the values on the line $n = 2k + 1$ with $n \neq 2^r - 1$ would probably require different arguments. We also believe that hypergraphs can be used in the study of the determining number of other families of graphs such as the Johnson graphs.

An interesting open problem is to find similar approaches to compute other parameters related to graphs such as the metric dimension. Perhaps hypergraphs can characterize not only determining sets but also resolving sets.

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