Microcanonical D-branes and Back Reaction

Esko Keski-Vakkuri\textsuperscript{1} and Per Kraus\textsuperscript{2}

\textit{California Institute of Technology}
\textit{Pasadena CA 91125, USA}
\textit{e-mail: esko, perkraus@theory.caltech.edu}

Abstract:

We compare the emission rates from excited D-branes in the microcanonical ensemble with back reaction corrected emission rates from black holes in field theory. In both cases, the rates in the high energy tail of the spectrum differ markedly from what a canonical ensemble or free field theory approach would yield. Instead of being proportional to a Bose-Einstein distribution function, the rates in the high energy tail are proportional to $e^{-\Delta S_{BH}}$, where $\Delta S_{BH}$ is the difference in black hole entropies before and after emission. After including the new effects, we find agreement, at leading order, between the D-brane and field theory rates over the entire range of the spectrum.

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1 Introduction

In the past year remarkable progress has been made in obtaining a microscopic description of certain black holes in string theory [1]. The most striking results have been found in the case of black holes which can be understood, in particular regions of moduli space, as being weakly coupled bound states of D-branes [2, 3, 4, 5, 6, 7]. Various quantities such as the extremal and near-extremal entropies, and emission and absorption rates, have been computed in this regime as an expansion in small parameters, and then extrapolated to the strong coupling regime where a field theory description of the black hole is valid [8, 9, 10, 11, 12]. Although there would appear to be no simple justification for such an extrapolation, the results so obtained agree precisely with the predictions of field theory. The agreement is especially provocative considering the very different natures of the D-brane and field theory calculations.

In computing the emission rate a number of approximations are made, on both the D-brane and field theory sides. In this paper we will consider processes for which one such approximation becomes invalid, and show how including some previously neglected effects changes the emission rates and allows the D-brane and field theory predictions to match in a non-trivial manner. In particular, we consider the emission of relatively high energy quanta, those with energies comparable to the total excitation energy of the black hole above extremality. On the D-brane side, the approximation which becomes invalid is the use of the canonical ensemble to compute the distribution functions for open string excitations. In the context of the canonical ensemble the distribution functions are given by the Bose-Einstein or Fermi-Dirac distributions; however, the D-brane is properly in the microcanonical ensemble — fixed energy — and the distribution functions which follow give entirely different results for the high energy tail of the spectrum. Given that this is the case, in order for the agreement to persist there must be a corresponding effect on the field theory side. We show that the effect is the gravitational self-interaction of the emitted quanta, and that properly taking this into account leads to a modified emission rate which agrees with the D-brane result in the high energy tail.

A method for including self-interaction effects was developed in [13]. (See also [14].) If attention is restricted to the s-wave, then the gravitational field by itself has no dynamical degrees of freedom, and it becomes sensible to integrate it out. This can be done by working in the Hamiltonian formalism and solving constraints. What is left is an effective action for the remaining matter degrees of freedom, which can then be used to obtain a corrected field
equation. In contrast to the standard approach of quantizing a field on a fixed background, this approach allows the geometry to change in response to the matter, and enforces energy conservation for the complete gravity plus matter system. This distinction is directly analogous to that between the canonical and microcanonical ensembles, in that the energy is only held fixed in the latter case.

Although our calculations will go through more generally, for definiteness we consider the specific D-brane configuration studied in [3] and [4]. The black hole is realized as a bound state of $Q_1$ D1-branes, $Q_5$ D5-branes, and total left moving momentum number $N$, all wrapped around $S^1 \times T^4$. The size of the $S_1$ is $L = 2\pi R$, and that of $T^4$ is $V$. In addition, we adopt the model of [9] in which the open strings attached to the D-brane effectively behave as though they were moving on a single D1-brane wrapped $Q_1Q_5$ times around $S_1$, with a total length $L' = 2\pi Q_1Q_5 R$. In this picture, the total left moving momentum is given in terms of $N'$ as $P = N/R = N'/(Q_1Q_5 R)$. If only left moving momentum is present the black hole is extremal and BPS saturated. The near-extremal hole is obtained by adding left and right moving momentum, while keeping $N' = N'_L - N'_R$ fixed. The entropy of the black hole is given by

$$S_{BH} = 2\pi (\sqrt{N'_L} + \sqrt{N'_R}).$$

We will only be considering the dilute gas regime [12], so that the contribution to the entropy from anti-branes is suppressed. The momentum is carried by massless excitations of open strings connecting the D1-brane to the D5-brane. These excitations correspond to a one dimensional gas of four species of massless bosons and fermions confined to a box of length $L'$. The average number of bosonic quanta of a particular species and with energy $\omega_k$ is given by the distribution functions $\rho_L(\omega_k)$, $\rho_R(\omega_k)$.

The non-extremal configuration will decay due to the collision of left and right moving excitations. The resulting rate for the emission of a single species of neutral scalars was computed in [10]:

$$\Gamma_D(\omega_k) = \frac{\kappa_5^2 L}{4} \omega_k \rho_L(\omega_k/2)\rho_R(\omega_k/2) \frac{d^4k}{(2\pi)^4}$$

On the other hand, the black hole decay rate is

$$\Gamma_H(\omega_k) = \sigma_{abs}(\omega_k) \rho_H(\omega_k) \frac{d^4k}{(2\pi)^4}$$

1We will be taking the term ‘black hole entropy’ and the symbol $S_{BH}$ to mean $A/(4G_N)$. The true entropy of the black hole is equal to this plus corrections.
where $\sigma_{\text{abs}}(\omega_k)$ is the grey body factor, equal to the classical absorption cross section, and $\rho_H(\omega_k)$ is the Hawking amplitude for particle creation by a black hole. If we follow [10] and restrict ourselves to the near-extremal region and also take $N'_L \gg N'_R \gg 1$, then these rates reduce to:

$$\Gamma_D(\omega_k) = A_H \rho_R(\omega_k/2) \frac{d^4k}{(2\pi)^4}; \quad \Gamma_H(\omega_k) = A_H \rho_H(\omega_k) \frac{d^4k}{(2\pi)^4}$$

(4)

The two rates thus agree provided $\rho_R(\omega_k/2) = \rho_H(\omega_k)$. For sufficiently small $\omega_k$ the canonical ensemble can be used to compute $\rho_R(\omega_k/2)$, and self-interaction effects can be ignored in computing $\rho_H(\omega_k)$. The distribution functions then each take the Bose-Einstein form, with the temperature given by the Hawking temperature, and the rates agree. However, these approximations fail for larger $\omega_k$ and additional analysis is required to demonstrate agreement.

The remainder of this paper is organized as follows. In section 2 we study distribution functions in the microcanonical ensemble. For the extreme high energy tail, i.e. for quanta with energies equal to the total energy of the system, simple reasoning leads to the result that the average number of quanta present is $e^{-S}$, where $S$ is the microcanonical entropy of the system. This is so because there is only one state, out of $e^S$ total states, for which all of the energy is concentrated in a single quantum. For lower energies we perform a saddle point calculation to derive the leading corrections to the canonical distribution function. In section 3 we turn to the field theory calculation of $\rho_H(\omega_k)$ in the presence of self-interaction effects, using a refined version of the techniques developed in [13]. The calculation is particularly clean in the case where the black hole decays to the extremal state by emitting only a single quantum, for then multi-particle interaction effects can be ignored. Remarkably, the amplitude turns out to be simply $e^{-\Delta S_{\text{BH}}}$, where $\Delta S_{\text{BH}}$ is the difference in entropies of the black hole before and after emission. This form holds for a general spherically symmetric black hole. We therefore find agreement at leading order between the D-brane and field theory emission rates in the high energy tail. In section four we further discuss our results, and consider how agreement beyond lowest order might be reached by a more detailed field theory analysis.

\footnote{We are required to be sufficiently near extremality that the wavelengths of emitted quanta are much greater than the Schwarzschild radius, $R_s$. Since the energy of an emitted quantum can be at most $4\pi N'_R L'$, we will require that $N'_R \ll L'/R_s$.}
2 Boltzmann Factors from Microcanonical Ensemble

In the ‘fat’ black hole model of [9], the rightmoving excitations on the multiply wound D1-brane consist of 4 species of massless bosonic and fermionic open string excitations. In what follows, we shall consider an arbitrary number \( f \) of species; in the end one can set \( f = 4 \). Half of the excess energy above extremality is distributed among the rightmoving quanta, which have momenta quantized in units of \( p_R = 2\pi/L' \). The total momentum of the rightmovers must add up to \( P_R = 2\pi N'_R/L' \). Thus the counting problem consists roughly of adding up positive integers to get a total of \( N'_R \).

For a system of \( f \) superconformal fields, the number of states consistent with a total level number \( n \) is given by (see e.g. [15]):

\[
d_n \equiv e^S \approx n^{-(f+3)/4} \exp\{\pi\sqrt{fn}\}.
\]

(5)

We would now like to compute the microcanonical Boltzmann factor, \( P(n_r = k) \), which is the probability to find \( k \) bosonic quanta of a given species in a state with energy \( 2\pi r/L' \). \( P(n_r = k) \) is proportional to the number, \( d(n_r = k) \), of such states. If there are \( k \) quanta at level \( r \), they contribute \( kr \) to the total level number, leaving \( n - kr \) units of momentum to be distributed among the remaining levels. \( d(n_r = k) \) is thus equal to the number of states with total level number \( n - kr \) for a system of \( f \) species of bosons and fermions, with one bosonic level \( r \) removed. Then, the Boltzmann factor is given by

\[
P(n_r = k) = \frac{1}{\mathcal{N}} d(n_r = k)
\]

(6)

where \( 1/\mathcal{N} \) is a normalization factor such that \( \sum_k P(n_r = k) = 1 \). The distribution function is defined to be the average number of quanta of a given bosonic species in level \( r \):

\[
\rho(r) = \sum_k P(n_r = k) k.
\]

(7)

The simplest case to consider is \( r = n \). There is clearly only one such state, so \( d(n_n = 1) = 1 \). Since \( d(n_n = 0) = d_n - 1 \), we find

\[
\rho(r = n) = 1/d_n.
\]

(8)

For \( r < n \) it is not possible to proceed so simply, but over most of the spectrum we can compute perturbatively and find the leading corrections to the values given by the canonical
ensemble. The counting is done in the usual manner, by using the partition function as a

generating function for the level degeneracies and then projecting out the individual factors

\[ G_r(w) = (1 - w^r) \prod_{s=1}^{\infty} \left( \frac{1 + w^s}{1 - w^s} \right)^f. \]  

(9)

Asymptotically, as \( w \to 1 \),

\[ G_r(w) \approx (1 - w^r) \left( \frac{\ln w}{-2\pi} \right)^{f/2} \exp \left( \frac{-f \pi^2}{4 \ln w} \right). \]  

(10)

The degeneracy is given by

\[ d(n_r = k) = \frac{1}{2\pi i} \oint \frac{dw}{w^{n+1-rk}} G_r(w). \]  

(11)

The integral will be approximated by the saddle point method, using the asymptotic formula (10) for \( G_r \). For this to be accurate we require that

\[ n - rk \gg 1. \]

If we are interested in finding the microcanonical Boltzmann factor up to leading corrections

away from its canonical form, it is appropriate to identify the relevant small parameters. It will

be convenient to perform expansions in

\[ \frac{1}{\sqrt{n}} \text{ and } \frac{r}{n}, \]

We skip the tedious details and simply state the result:

\[ d(n_r = k) \sim \exp \left\{ -\frac{\pi \sqrt{f} kr}{2 n^{1/2}} - \frac{\pi \sqrt{f} (kr)^2}{8 n^{3/2}} + \frac{(f + 3) kr}{4 n} + \frac{\pi}{4 e^{\pi r \sqrt{f/4n}} k r^2}{n^{3/2}} + \ldots \right\}. \]  

(12)

We have omitted writing the \( k \) independent factors, as these will just drop out when we compute

\( P(n_r = k) \). The first term in the exponent is the usual thermal canonical ensemble result, and

the next three terms give the leading microcanonical corrections. These corrections have
different origins in the saddle point calculation. The first correction term has the same origin
as the canonical term. The second correction term comes from two different contributions: The
term with the coefficient \( f/4 \) can be traced back to the term

\[ \left( \frac{\ln w}{-2\pi} \right)^{f/2} \]
in the partition function (10), and the term with coefficient $3/4$ comes from the square root prefactor in the saddle point evaluation. The last correction term represents the effect of the removed bosonic level $r$ in the partition function. The omitted terms $+ \ldots$ in the exponent are subleading corrections.

It is useful to distinguish the following parameter regions:

1. In the low energy part of the spectrum, $r \ll n$, the degeneracy reduces to the canonical result, but with a corrected temperature coming from the third and fourth terms.

2. If $r \sim \sqrt{n}$, the two expansion parameters are of the same order of magnitude, and the different microcanonical corrections contribute roughly equally. This region corresponds to the energy of the level being of the order of the temperature.

3. For the higher levels, $n \gg r \gg \sqrt{n}$, the second term gives the leading correction.

4. When $r \sim n$, the saddle point method is no longer valid.

Let us focus on region 3. In this region, up to the accuracy considered, we can write

$$d(n_r = k) \approx \exp \left[ \pi \sqrt{fn} - \pi \sqrt{f(n - kr)} \right]. \quad (13)$$

Furthermore, in computing $P(n_r = k)$ it is accurate to replace $\sum_k$ by the $k = 0$ term, and for $\rho(r)$ by the $k = 1$ term. Therefore, we find

$$\rho(r) \approx \exp \left[ \pi \sqrt{f(n - r)} - \pi \sqrt{fn} \right]. \quad (14)$$

To translate this back into language appropriate for the black hole, we use $\omega_k/2 = 2\pi r/L'$ and the expression for the black hole entropy (1), and set $f = 4$, $n = N_R'$. In region 3, our result then takes the form

$$\rho_R(\omega_k/2) \approx \exp [S_{BH}(M - \omega_k) - S_{BH}(M)], \quad (15)$$

where $M$ is the mass of the black hole. This formula also gives the leading result in the extreme high energy tail, when $r = n$. However, there we also know the more accurate result,

$$\rho_R(\omega_k/2) \approx \left( \frac{S_{BH}(M) - S_{BH}(M - \omega_k)}{2\pi} \right)^{7/2} \exp [S_{BH}(M - \omega_k) - S_{BH}(M)], \quad (16)$$

---

3 The momentum of the left movers, $N_L'$, can be thought of as a constant, as the change only gives a subleading correction.
where we used (8), (3) and (1). Note that (16) contains all terms of (12) except those arising from removing the rth bosonic level in the partition function.

We have found that the correct microcanonical description of the gas of open string excitations on the D-string leads to a prediction: there are corrections to the exactly thermal behavior, and presumably these should correspond to some new effects on the field theory side. Since the corrections were seen to have different mathematical origins, it is possible that the corresponding field theory corrections are associated with a variety of different physical effects. Furthermore, the result is partially dependent on the specific D-brane model. For example, in (3) the prefactor contains an explicit dependence on the number of species $f$ of superconformal fields — $f$ does not appear in the combination $fn$ as in the exponential term — and this dependence shows up as the $7/2$ power in (16). These subleading terms would be different in the model appropriate to the black string limit, where there are $Q_1$ ($Q_5$) singly wound D1 (D5) -branes, and the number of species is $f = 4Q_1Q_5$ instead of 4. However, the same leading exponential terms in (3), (16) would still appear. In the next section we shall show that the leading term in the high energy tail region can be computed in field theory, provided gravitational self-interaction effects are properly included.

### 3 Emission Rate From Field Theory

#### 3.1 Metric

In this section we will be using a slightly non-standard set of coordinates for the black hole metric [16]:

$$ds^2 = - [N_t(r) \ dt]^2 + [dr + N_r(r) \ dt]^2 + r^2 \ d\Omega_{d-2}^2.$$  \hspace{1cm} (17)

The metric can always be put in this form, provided that the geometry is spherically symmetric and has a Killing vector which is timelike outside the horizon. The advantage of these coordinates is that they are well behaved at the horizon. For example, for a four dimensional Reissner-Nordström solution: $N_t = 1$, $N_r = \sqrt{2M/r - Q^2/r^2}$.

Since the equation for null geodesics is $\dot{r} = \pm N_t - N_r$, with the plus(minus) sign applying for outgoing(ingoing) trajectories, we see that the horizon, $r = R$, is determined from the condition $N_t(R) - N_r(R) = 0$. Near the horizon $N_t - N_r$ behaves as

$$N_t(r) - N_r(r) \sim (r - R)\kappa + O((r - R)^2)$$  \hspace{1cm} (18)
where $\kappa$ can easily be seen to be the surface gravity of the black hole. For comparison with other forms of the metric, it is useful to note that a change of time coordinate brings the metric into the form

$$
 ds^2 = -\left[ N_t^2(r) - N_r^2(r) \right] dt^2 + \frac{N_t^2(r)}{N_t^2(r) - N_r^2(r)} dr^2 + r^2 d\Omega_{d-2}^2. \tag{19}
$$

$N_t, N_r$ are generally functions of various charges as well as the mass of the black hole; in what follows we only show the mass dependence explicitly. We thus write: $N_t(r; M), N_r(r; M), R(M), \kappa(M)$.

### 3.2 Black hole radiance including self-interaction

We now turn to the computation of emission probabilities in field theory, and include effects due to self-interaction. Only the s-wave modes are considered, as these dominate the emission process at low energies. Black hole radiance results from the mismatch between the two natural vacuum states which arise in the quantization of a field propagating on a black hole spacetime. The first vacuum state to consider is the one most naturally employed by an asymptotic observer. Such an observer would expand the field operator in terms of mode solutions which have the time dependence $\exp(-i\omega t)$, where $t$ is the Killing time:

$$
 \phi(t, r) = \int dk \left[ a_k u_k(r) e^{-i\omega t} + a_k^\dagger u_k^*(r) e^{i\omega t} \right]. \tag{20}
$$

For large $r$, $u_k(r) \rightarrow e^{ikr}/r^{(d-2)/2}$. $a_k$ and $a_k^\dagger$ destroy and create quanta of definite energy, and the corresponding vacuum state $\ket{0_u}$ is defined by $a_k \ket{0_u} = 0$. However, the modes $u_k(r)$ become singular at the horizon, and the state $\ket{0_u}$ yields an infinite result for the energy-momentum density measured by a freely falling observer crossing the horizon. Thus if physics is to be well behaved at the horizon, the state of the field resulting from black hole formation by collapsing matter cannot be $\ket{0_u}$. A state which *is* well behaved at the horizon can be obtained by expanding the field in terms of modes $v_k(t, r)$ which are nonsingular there,

$$
 \phi(t, r) = \int dk \left[ b_k v_k(t, r) + b_k^\dagger v_k^*(t, r) \right]. \tag{21}
$$

Then, the state $\ket{0_v}$ determined by $b_k \ket{0_v} = 0$ results in a finite energy-momentum density at the horizon, and so is a viable candidate.

The difference between the states $\ket{0_u}$ and $\ket{0_v}$ is characterized by the fact that the modes
\(v_k(t, r)\) contain both positive and negative frequency components. Defining

\[
\alpha_{kk'} = \int_{-\infty}^{\infty} dt e^{i\omega_k t} v_{k'}(t, r_f) \quad ; \quad \beta_{kk'} = \int_{-\infty}^{\infty} dt e^{-i\omega_k t} v_{k'}(t, r_f),
\]

(22)

the standard result is that in the state \(|0_\nu\rangle\), the probability per unit time to emit particles with energy in the range \(\omega_k\) to \(\omega_k + d\omega_k\) is controlled by \(|\beta_{kk'}/\alpha_{kk'}|^2\). If the emissions are uncorrelated, as is the case in free field theory, then the total flux of outgoing particles takes the form of (3) with

\[
\rho_H(\omega_k) = \frac{|\beta_{kk'}/\alpha_{kk'}|^2}{1 - |\beta_{kk'}/\alpha_{kk'}|^2}. \tag{23}
\]

On the other hand, if we consider sufficiently large \(\omega_k\) such that at most one particle can be emitted, then

\[
\rho_H(\omega_k) = \frac{|\beta_{kk'}/\alpha_{kk'}|^2}{1 + |\beta_{kk'}/\alpha_{kk'}|^2} \approx |\beta_{kk'}/\alpha_{kk'}|^2. \tag{24}
\]

In the latter case the emission probability is low, \(|\beta_{kk'}/\alpha_{kk'}|^2 \ll 1\). As an example, for a free field propagating on the four dimensional Schwarzschild metric, it is conventional to take \(v_{k'}(t, r) = e^{ik'U}\), where \(U\) is the Kruskal coordinate. Such a mode has \(t\) dependence \(\exp(-ik'e^{-t/4M})\) leading to \(|\beta_{kk'}/\alpha_{kk'}|^2 = e^{-8\pi M \omega_k}\). From (23), the outgoing flux is then that of a thermal body (with grey-body factor) at the Hawking temperature \(T_H = 1/(8\pi M)\).

In the free field approximation the field obeys \(\Box \phi = 0\); now we would like to consider how the field equation is altered due to gravitational self-interaction. The strategy will be to obtain a corrected field equation and then to use the corresponding mode solutions to calculate \(|\beta/\alpha|\).

To proceed, we will quantize a massless, gravitating, spherical shell surrounding the black hole. Such an object has a position coordinate \(r\), canonical momentum \(p\), and Hamiltonian \(H(r, p)\). By deriving the explicit form for \(H(r, p)\), and turning classical quantities into quantum mechanical operators, we can derive the modified field equation. In [13], this procedure was carried out systematically starting from the full action for the gravity plus shell system. The reduced Hamiltonian for the shell variables alone was obtained by solving the gravitational constraints and inserting the solutions back into the action. Here we will use a shortcut to arrive at the same result. The key point, which emerges from the analysis in [13], is that for a

\[\text{The precise location of } r_f \text{ in the integrals does not matter, provided it is outside the horizon where } u_k(r) \text{ breaks down. Also, note that our definitions are not quite identical to the standard Bogoliubov coefficients, due to the absence of } u_k(r_f) \text{ factors; these factors will just cancel out in the ratio } |\beta/\alpha| \text{ and so have been omitted from the start.}\]

\[\text{This is true provided } \beta_{kk'}/\alpha_{kk'} \text{ is independent of } k', \text{ as will seen to be the case throughout this paper.}\]
black hole of mass $M$ and shell energy $H$, the classical trajectory of the shell is a null geodesic in the metric
\[ ds^2 = - [N_t(r; M + H) \ dt]^2 + [dr + N_r(r; M + H) \ dt]^2, \]
that is,
\[ \dot{r}(r) = N_t(r; M + H) - N_r(r; M + H). \]

On the other hand, this trajectory must also follow from Hamilton’s equations applied to $H(r, p)$: $\dot{r} = \partial H / \partial p$. So $\partial H / \partial p = N_t(r; M + H) - N_r(r; M + H)$, and we thus find
\[ p(r, H) = \int_0^H \frac{dH'}{N_t(r; M + H') - N_r(r; M + H')}. \]
The choice for the lower limit of integration can be justified by comparing with the free field limit, or by comparison with [13]. In general, it is not possible to invert $p(r, H)$ to obtain $H(r, p)$; fortunately, such an expression will not be needed in what follows.

To pass to the quantum theory we would like to make the substitutions
\[ p \to -i \frac{\partial}{\partial r}; \quad H \to i \frac{\partial}{\partial t}, \]
and arrive at a differential equation for the field. However, one is met with factor ordering ambiguities, and at best one obtains a rather unwieldy non-local equation. Fortunately, the mode solutions we are interested in, the $v_k(t, r)$, are accurately described by the WKB approximation and are insensitive to these issues. This is because of the large redshift involved — the bulk of the emission from the black hole is governed by modes which have a very short wavelength at the horizon, and provided the curvature at the horizon does not blow up, the WKB approximation is valid for such modes. We therefore write
\[ v_{k'}(t, r) = e^{iS_{k'}(t, r)} \]
where $S_{k'}(t, r)$ satisfies the Hamilton-Jacobi equation,
\[ \frac{\partial S_{k'}}{\partial r} = p(r, -\partial S_{k'}/\partial t). \]
As a boundary condition, it is convenient to take
\[ S_{k'}(0, r) = k'r. \]
The solution to the Hamilton-Jacobi equation, (30), is given by the classical action. That is, if \( r(t) \) is a classical trajectory then
\[
S_{k'}(t, r(t)) = k'r_0 + \int_0^t dt \left[ p(r(t)) \dot{r} - H(r_0) \right],
\]
(32)
where \( r_0 = r(0) \), and the conserved energy \( H(r_0) \) of the trajectory should be chosen such that the boundary condition \( p(r_0) = k' \) is satisfied. One can check that \( \partial S/\partial r = p \) and \( \partial S/\partial t = -H \) as desired.

In [13] this construction was used to derive an explicit form for \( v_{k'}(t, r) \), but here we will proceed in a slightly more indirect fashion. We want to compute
\[
\alpha_{kk'} = \int_{-\infty}^{\infty} dt \ e^{i\omega_k t} e^{iS_{k'}(t, r_f)}; \quad \beta_{kk'} = \int_{-\infty}^{\infty} dt \ e^{-i\omega_k t} e^{iS_{k'}(t, r_f)}.
\]
(33)
For large \( k \) this can be done by saddle point evaluation. The saddle point is found from
\[
\pm \omega_k + \frac{\partial S_{k'}}{\partial t} = 0,
\]
(34)
the upper(lower) sign applying for \( \alpha_{kk'}(\beta_{kk'}) \). For \( \alpha_{kk'} \) the condition can be written as \( H = \omega_k \), which simply says that the saddle point trajectory has energy \( \omega_k \). For \( \beta_{kk'} \) the condition is instead \( H = -\omega_k \) so that the trajectory has negative energy. Calling these trajectories \( r_+(t) \) and \( r_-(t) \), we can use the formula for the action given in (32) to obtain expressions for \( \alpha_{kk'} \) and \( \beta_{kk'} \). We will approximate the integrals by the value of the integrand at the saddle point. Of course, there is also a prefactor coming from the second derivative at the saddle point; this prefactor will be ignored for now. Substituting in, we find
\[
|\alpha_{kk'}| = e^{-\text{Im} \int_{r_0}^{r_f} p_+(r) dr} \quad ; \quad |\beta_{kk'}| = e^{-\text{Im} \int_{r_0}^{r_f} p_-(r) dr}
\]
(35)
We have not included the \( k'r_0 \) terms because, as we will see momentarily, \( r_0 \) is always real. Note also that the \( \omega_k t \) term cancels with the \( \int H dt \) term in both cases. Now, from (27),
\[
p_{\pm}(r) = \int_0^{\pm \omega_k} \frac{dH'}{N_1(r; M + H') - N_r(r; M + H')}
\]
(36)
We can find \( r_{\pm 0} \) by satisfying the boundary condition \( p(r_0) = k' \). For large \( k' \), \( r_{\pm 0} \) must be very close to the horizon so that the redshift can convert the large value of \( k' \) into a finite value for the energy of the trajectory. Therefore,
\[
r_{\pm 0} = R(M \pm \omega_k) \pm \epsilon_{\pm}
\]
(37)
where $\epsilon_{\pm}$ are small and positive, $r_{-0}$ is found to be inside the horizon because of the fact that the trajectory $r_{-}(t)$ has negative energy. Now, interchanging orders of integration gives

$$\text{Im} \int_{r_{+0}}^{r_{f}} p_{+}(r) \, dr = \text{Im} \int_{0}^{\omega k} dH' \int_{r_{+0}}^{r_{f}} \frac{dr}{N_{t}(r; M + H') - N_{r}(r; M + H')}.$$  (38)

The imaginary part vanishes, since in the region considered $r$ is always outside the horizon: $N_{t} - N_{r} > 0$. Thus $|\alpha_{kk'}| = 1$.

The situation is different for $\beta_{kk'}$,

$$\text{Im} \int_{r_{-0}}^{r_{f}} p_{-}(r) \, dr = \text{Im} \int_{0}^{-\omega k} dH' \int_{r_{-0}}^{r_{f}} \frac{dr}{N_{t}(r; M + H') - N_{r}(r; M + H')}.$$  (39)

since now the $r$ integration region crosses the horizon for all values of $H'$. The imaginary part comes from integrating over the pole where,

$$\frac{1}{N_{t}(r; M + H') - N_{r}(r; M + H')} \sim \frac{1}{\kappa(M + H')} \frac{1}{r - R(M + H')}.$$  (40)

Therefore

$$\text{Im} \int_{r_{-0}}^{r_{f}} p_{+}(r) \, dr = -\pi \int_{0}^{\omega k} dH' \frac{dH'}{\kappa(M + H')}.$$  (41)

Recalling the first law of thermodynamics, $dM = (\kappa/2\pi) dS_{BH}$, we now see that

$$\text{Im} \int_{r_{+0}}^{r_{f}} p_{+}(r) \, dr = \frac{1}{2} [S_{BH}(M) - S_{BH}(M - \omega k)].$$  (42)

So

$$|\beta_{kk'}|^{2} = \exp [S_{BH}(M - \omega k) - S_{BH}(M)].$$  (43)

For small $\omega_{k}$ the term in the exponent can accurately be expanded to first order, and it is also appropriate to use the form (23) since interaction effects are small. Thus

$$\rho_{H}(\omega_{k}) \approx \frac{1}{e^{\omega_{k}/T_{H}} - 1},$$  (44)

where $T_{H} = \kappa/(2\pi)$ is the Hawking temperature. But for large $\omega_{k}$ the exponent should not be expanded, and the form (24) should be used,

$$\rho_{H}(\omega_{k}) \approx \exp [S_{BH}(M - \omega_{k}) - S_{BH}(M)].$$  (45)

We would now like to make some comments about the preceding derivation. First, the results presented here correct a calculational error in [13]. Second, an interesting feature is

\footnote{The sign of the imaginary part is chosen by comparing with the free field limit.}
the form of $\alpha_{kk'}$, $\beta_{kk'}$ given in (35), as these expressions are the standard form for a WKB transmission coefficient in quantum mechanics. Thus, as was noted in [13], the emission is naturally interpreted as being due to particles tunneling across the horizon. Finally, it is remarkable that we never had to make reference to a specific black hole geometry, so that (43) holds for any for any spherically symmetric black hole. Although this feature proved convenient here, it would also be interesting to extend the analysis to include effects which do depend on more details of the geometry.

4 Discussion

We would now like to compare the D-brane and field theory predictions. At very low energies, both $\rho_R$ and $\rho_H$ go smoothly over to a Bose-Einstein distribution. At higher energies, the D-brane result receives corrections from the various terms in (12). The third and fourth terms in (12) correspond to corrections which are not evident in our field theory analysis; we will speculate on their possible origin momentarily. As the energy is increased further, the leading order distribution functions tend to their values in the high energy tail, $\rho_R = \rho_H = e^{-\Delta S_{BH}}$. Again, the D-brane result supplies an additional correction from the prefactor in (16). Thus, while the two pictures give the same results to leading order, the D-brane analysis supplies additional information.

In making the comparison, it is important to distinguish those results which are sensitive to the details of the particular D-brane configuration and black hole geometry, from those which only rely on more general considerations [1]. For instance, the result that the distribution functions tend, at leading order, to $e^{-\Delta S_{BH}}$ is of the latter type. As long as there exists a microscopic description of the degrees of freedom, the same counting argument that we invoked for the average occupation number will go through. An analogous situation holds on the field theory side in that we never made explicit reference to the precise form of the metric. While we are thus not testing the D-brane model in the same way that, for instance, a grey-body factor calculation does, we are testing the ability of field theory to reproduce the results of a general microscopic description. Such issues are key to understanding the issues related to the apparent information loss that occurs when the field theory description is taken literally. That is, assuming that the microscopic description really is supplying a unitary S-matrix, one wants

\footnote{We thank S. Mathur for a discussion on these issues.}
to know which features field theory is incapable of reproducing.

We have also discussed corrections coming from the D-brane side which are sensitive to the specifics of the model. For example, the prefactor in (16) is raised to power which depends on the number of species of bosons and fermions moving on the D-string. It is interesting to consider whether a more detailed field theory calculation would be able to account for these sorts of corrections. With this in mind, let us mention some of the effects which were neglected in our analysis. First, we considered only self-interactions and ignored interactions between different emitted particles. One might expect that such effects can become appreciable in the lower energy part of the spectrum, where several particles can be emitted within a short time interval. Second, there are corrections to the WKB approximation, and to the saddle point method used to compute $\beta_{kk'}/\alpha_{kk'}$. We do not know how to compute these corrections, but presumably they are under control for a large, semiclassical looking black hole. It would be interesting to examine these issues further, and to study other processes such as scattering. Finally, it is very encouraging that the study of black holes from a string theory perspective has reached the stage where predictions can easily be made which are very nontrivial to verify from a field theory point of view.

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References

[1] For reviews of microscopic black holes in string theory with references, see G. Horowitz, gr-qc/9604051 and J. M. Maldacena, Ph.D. Thesis, Princeton University, hep-th/9607235.

[2] A. Strominger and C. Vafa, Phys. Lett. B379 (1996), 99 (hep-th/9601029).

[3] C. G. Callan and J. M. Maldacena, Nucl. Phys. B472 (1996), 591 (hep-th/9602043).

[4] G. Horowitz and A. Strominger, Phys. Rev. Lett. 77 (1996), 2368 (hep-th/9602051).

[5] J. Maldacena and A. Strominger, Phys. Rev. Lett. 77 (1996), 428 (hep-th/9603060).

[6] C. Johnson, R. Khuri and R. Myers, Phys. Lett. B378 (1996), 78 (hep-th/9603061).
[7] J. Breckenridge, R. Myers, A. Peet, and C. Vafa, hep-th/9602063.

[8] J. Breckenridge, D. Lowe, R. Myers, A. Peet, A. Strominger and C. Vafa, Phys. Lett. B381 (1996), 423 (hep-th/9603078); G. T. Horowitz, J. M. Maldacena and A. Strominger, hep-th/9603109; G. Horowitz, D. A. Lowe and J. M. Maldacena, Phys. Rev. Lett. 77 (1996), 430 (hep-th/9603195); J. M. Maldacena, hep-th/9605016; A. Dhar, G. Mandal and S. R. Wadia, hep-th/9605234; For an alternative approach, based on classical hair, see F. Larsen and F. Wilczek, Phys. Lett. B375 (1996), 37 (hep-th/9511064), hep-th/9604134, hep-th/9609084; M. Cvetic and A. A. Tseytlin, Phys. Rev. D53 (1996), 5619 (hep-th/9512031).

[9] J. M. Maldacena and L. Susskind, hep-th/9604042. Excitations of multi-wound D1-branes were considered by S. R. Das and S. D. Mathur, Phys. Lett. B375 (1996) 103 (hep-th/9601152).

[10] S. R. Das and S. D. Mathur, hep-th/9606185, hep-th/9607149.

[11] S. S. Gubser and I. Klebanov, hep-th/9608108.

[12] J. M. Maldacena and A. Strominger, hep-th/9609026; S. S. Gubser and I. Klebanov, hep-th/9609076.

[13] P. Kraus and F. Wilczek, Nucl. Phys. B433 (1994), 665 (hep-th/9408003); Nucl. Phys. B437 (1994), 231 (hep-th/9411219).

[14] V. Balasubramanian and H. Verlinde, Nucl. Phys. B464 (1996), 213 (hep-th/9512148).

[15] M. Green, J. Schwarz, and E. Witten, Superstring Theory, Vol. 1 (Cambridge University Press, 1987)

[16] P. Kraus and F. Wilczek, Mod. Phys. Lett. A40 (1994), 3713.