Characterizing electron entanglement in multiterminal mesoscopic conductors

Vittorio Giovannetti¹, Diego Frustaglia¹,², Fabio Taddei¹, and Rosario Fazio³,¹
¹NEST-CNR-IFM and Scuola Normale Superiore, I-56126 Pisa, Italy.
²Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain.
³International School for Advanced Studies (SISSA), I-34014 Trieste, Italy

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We show that current correlations at the exit ports of a beam splitter can be used to detect electronic entanglement for a fairly general input state. This includes the situation where electron pairs can enter the beam splitter from the same port or be separated due to backscattering. The proposed scheme allows to discriminate between occupation-number and degree-of-freedom entanglement.

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Generation, manipulation and detection of entangled electrons is central for realizing integrated solid-state quantum computers. Among the several possibilities, a lot of attention has been devoted to the study of entanglement in multiterminal mesoscopic conductors (see Refs. [1, 2] for a review). In this context, it was shown that entanglement between spatially separated electrons can be detected by means of a beam splitter (BS) [3]. Indeed the BS, allowing the incoming (and possibly entangled) electrons to be interchanged, gives rise to two-particle interference effects. As a result, the symmetry of the incoming state influences the current-noise correlations at the exit ports. Bunching (enhanced) and antibunching (suppressed) behavior in the shot noise were predicted for spin singlet and triplet entangled states, respectively [3]. The role of entanglement was later analyzed in the whole probability distribution of the current fluctuations (being the noise power only its second moment) [4]. Further analysis were subsequently performed in the presence of spin-orbit coupling [5], and for states generated in an Andreev double-dot entangler [6]. More recently, Burkard and Loss [7] found a bound for the entanglement of arbitrary mixed spin states through shot-noise measurements by applying a reduction mapping into Werner states [8]. We generalized this result to multi-mode input states by introducing an electronic Hong-Ou-Mandel interferometer [9].

All the previous analysis rely on the assumption that only one electron per port enters the analyzer. Most of the electronic entangler devices proposed so far [1, 2], however, generate states having a finite probability amplitude that two electrons enter the analyzer at the same input port [10]. This gives rise to two distinct forms of entanglement: occupation-number and electronic degree-of-freedom entanglement [11, 12]. Under this generalized initial condition the analysis of the entanglement is complicated by super-selection rules (SSR) induced by particle number conservation [1, 11, 12, 13].

In this paper we present a detection strategy which addresses this more general situation. As in Ref. [9], it is based on the study of current correlations at the exit ports of a BS as a function of controllable phase shifts.

FIG. 1: Sketch of the analyzer. An external entangler prepares electron pairs in the input state of Eq. (1) and injects them into ports 1 and 2. The electrons are allowed to enter the analyzer from the same port (cases |Φ11⟩ and |Φ22⟩), from different ports (case |Φ12⟩), or in any superposition of the previous cases. Electrons propagating along lead 2 undergo an additional (orbital/spin-dependent) controllable phase shift (white circle in the figure) before impinging on a 50% beam splitter BS. Current correlations are measured at 3 and 4.

We shall show that, for the whole class of two-particle input states, simple data processing of the measured current cross-correlators can be used to address separately the various entanglement components. Moreover, we also account for the case in which less than two electrons can enter the interferometer due to backscattering. Our scheme of detection is particularly suitable when the entanglement is generated through superconducting-normal metal structures or multichannel non-interacting structures [14, 15].

The proposed setup is described in Fig. 1. We start by considering a pair of electrons prepared by an external device (the “entangler” we want to monitor) at a given energy E above the Fermi sea and injected into the interferometer at the input ports 1 and 2. A generic input state has the form

$$|\Psi\rangle = \sin \theta (\cos \phi |\Phi_{11}\rangle + \sin \phi |\Phi_{22}\rangle) + \cos \theta |\Phi_{12}\rangle,$$

(1)

where $\theta, \phi \in [0, \pi/2]$ and $|\Phi_{ij}\rangle$ describe two electrons of energy E entering the BS from the $i$-th and $j$-th port respectively ($i, j = 1, 2$). The latter can be written as $|\Phi_{ij}\rangle = \sum_{\alpha, \beta} a_{i, \alpha}^\dagger(E) a_{j, \beta}^\dagger(E)|0\rangle$, with $a_{i, \alpha}^\dagger(E)$ being...
the creation operator of an incoming electron in lead $j$ with orbital/spin degree-of-freedom index $\alpha$ and energy $E$, while $\Phi^{(12)}_{\alpha,\beta}$ and $\Phi^{(j)}_{\alpha,\beta} = -\Phi^{(j)}_{\beta,\alpha}$ satisfy the normalization condition $\sum_{\alpha,\beta} |\Phi^{(12)}_{\alpha,\beta}|^2 = 1$ and $\sum_{\alpha,\beta} |\Phi^{(j)}_{\alpha,\beta}|^2 = 1/2$. In this notation $|0\rangle \equiv |0\rangle_1 \otimes |0\rangle_2$ is the vacuum representing the filled Fermi state of the two leads.

The analysis of the entanglement contained in the state $|\Psi\rangle$, can be done naturally by bipartitioning the system with respect to the port labels 1 and 2. For $\theta = 0$, the state (1) reduces to the previously studied situation [3,7,9]. There are however many cases (e.g. the Anderson entangler [8,15]) where the incoming state instead has the form (11) with $\theta \neq 0$. Our approach allows to analyze this situation as well. Following Refs. [11,12], the task is to identify both occupation-number and electronic degree-of-freedom entanglement. The first one coincides with the “variance-EP” entanglement of Ref. [11] and with the “fluffy bunny” entanglement of Ref. [12]. It is present whenever we have a non trivial superposition among terms where each lead supports a different number of incoming electrons, i.e. $|\Phi^{(1)}\rangle$, $|\Phi^{(2)}\rangle$, and $|\Phi^{(12)}\rangle$. The second one, instead, coincides with the “entanglement-EP” of Ref. [11] and originates from the component $|\Phi^{(12)}\rangle$ of Eq. (1). It is present when the leads 1 and 2 possess one electron each and are entangled through the electronic orbital/spin modes $\alpha$. Under the constraints imposed by the particle number SSR it can be shown [11,12] that occupation-number and electronic degree-of-freedom entanglement are both quantum information resources which are crucial to solve specific tasks (examples are classical and quantum data hiding and standard quantum teleportation, respectively).

As shown in Fig. 1, electrons propagating along lead 2 undergo an additional (channel/spin-dependent) controllable phase shift before impinging on a 50% BS [16]. For simplicity we assume that the BS is symmetric and does not suffer from backscattering. The BS is described by a scattering matrix $\hat{S}$ defined through the relation $b_{j,\alpha}(E) = \sum_{j'=1}^{2} s^{(\alpha)}_{j,j'} a_{j',\alpha}(E)$, with $j = 3,4$ and $b_{j,\alpha}(E)$ the annihilation operator of an outgoing electron in lead $j$ and channel $\alpha$ at energy $E$. The scattering matrix is defined as $s^{(\alpha)} = \sqrt{1/2} \begin{pmatrix} 1 & e^{i\varphi_{\alpha}} \\ 1 & -e^{i\varphi_{\alpha}} \end{pmatrix}$. For definiteness we have imposed that no channel mixing occurs in the scattering process at the BS. In the case of spin entanglement, this can be easily implemented by introducing a BS conserving spin. For orbital entanglement, a spatial separation of the orbital channels may be necessary. Finally, the tunable phases $\varphi_{\alpha}$ are introduced at the input port 2 by some externally controlled parameters [9].

The (dimensionless) current correlator at the output ports 3 and 4 is

$$C_{34}(t) = \frac{\hbar^2 e^2}{2\epsilon^2} \lim_{T \to \infty} \int_0^T dt_1 dt_2 \frac{J_3(t_1) J_4(t_2)}{T^2},$$

where [17,18],

$$I_j(t) = \frac{e}{h\nu} \sum_{E,\omega,\alpha} e^{-i\omega t} \langle b_{j,\alpha}(E) b_{j,\alpha}(E + h\omega) - a_{j,\alpha}^\dagger(E) a_{j,\alpha}(E + h\omega) \rangle$$

is the current operator of the $j$-th port ($j = 3,4$). In these expressions the average $\langle \cdots \rangle$ is taken over the incoming electronic state, $T$ is the measurement time, and $\nu$ is the density of states of the leads (a discrete spectrum has been considered to ensure a proper regularization of the current correlations). $C_{34}$ is a measurable quantity which can be written in terms of the shot noise $S_{34}$ and the average current $i_{3,4} = \int dt (I_{3,4}(t))/T$ through the relation $C_{34} = \frac{\nu T}{2\sqrt{\nu}} [S_{34} + \hbar \nu i_{3,4}]$. Differently from $S_{34}$, the $C_{34}$ is linear in the input state of the system (10). Therefore, for any given density matrix $\rho = \sum_{\alpha,\beta} p_{\alpha,\beta} |\Psi_{\alpha,\beta}\rangle\langle \Psi_{\alpha,\beta}|$ as in Eq. (1) we obtain $C_{34}(\rho) = \sum_{\alpha,\beta} p_{\alpha,\beta} C_{34}(\psi_{\alpha,\beta})$, where the correlators $C_{34}(\psi_{\alpha,\beta})$ take the form

$$C_{34}(\psi_{\alpha,\beta}) = \left[ 1 + w \cos^2 \theta + v \sin^2 \theta \sin(2\phi) \right]/4.$$  \hspace{1cm} (4)

Here, $w$ and $v$ are real quantities satisfying $|w|, |v| \leq 1$. They depend upon the interferometer phases $\varphi_{\alpha}$ and the input state parameters as

$$w = \sum_{\alpha,\beta} \left( \Phi^{(12)}_{\alpha,\beta} \right)^* \Phi^{(12)}_{\beta,\alpha} e^{i(\varphi_{\alpha} - \varphi_{\beta})}$$

$$v = 2 |\text{Re} \left[ \sum_{\alpha,\beta} \left( \Phi^{(11)}_{\alpha,\beta} \right)^* \Phi^{(22)}_{\beta,\alpha} e^{i(\varphi_{\alpha} + \varphi_{\beta})} \right] |.$$  \hspace{1cm} (5)

Consider first the case where there is just one incoming electron per port, i.e., $\theta = 0$ and $|\Psi\rangle = |\Phi^{12}\rangle$. In this situation one has $i_{3,4} = e/(h\nu)$ and Eq. (4) gives the same Fano factor $F_{34} = S_{34}/(2e\sqrt{i_{3,4}}) = (w - 1)/4$ of Ref. [8], where it was shown that all separable mixtures of states $|\Phi^{12}\rangle$ must exhibit non-negative values of $w$, i.e. non-negative values of $C_{34}(\rho) - 1/4$.

In the case where $\theta$ is generic, the same result can be obtained by relating the correlator $C_{34}$ to the entanglement of formation $F_I(\rho)$ [20] of the input state. This is done by applying generalized twirling transformations to map all density matrices for ports 1 and 2 into generalized Werner states [21]. The latter are constructed as in Ref. [8] by introducing a joint orthonormal basis for ports 1 and 2 formed by the states $|\chi_k\rangle_1 \otimes |\chi_k\rangle_2$ and $|\Psi^{(1)}_{kk}\rangle = (|\chi_k\rangle_1 \otimes |\chi_k\rangle_2 \pm |\chi_{k'}\rangle_1 \otimes |\chi_{k'}\rangle_2)/\sqrt{2}$ with $k < k'$. Here, the label $k$ enumerates all configurations with two or less particles in each port described, respectively, by vectors $|0\rangle_1$, $a_{1,\alpha}^\dagger(E)|0\rangle_1$, $a_{1,\alpha}^\dagger(E)a_{2,\alpha}^\dagger(E)|0\rangle_1$ and $|0\rangle_2$, $e^{-i\varphi_{\alpha}}a_{2,\alpha}^\dagger(E)|0\rangle_2$, $e^{-i(\varphi_{\alpha} + \varphi_{\beta})}a_{2,\alpha}^\dagger(E)a_{2,\beta}^\dagger(E)|0\rangle_2$. This, indeed, generalizes the approach of Refs. [3,18], since we have introduced a larger basis that accounts for ports occupied by a different number of electrons. Following the derivation of Refs. [2,18], it is easy to show
that $E_f(\rho)$ can be lower bounded by the quantity

$$W(\rho) = \sum_{kk'} (\Psi_{kk'}^{(\pm)})_\rho (\Psi_{kk'}^{(\pm)}) / 2$$

through the inequality

$$E_f(\rho) \geq E(W(\rho)),$$  

where $E(x) = H(\frac{1}{2} + x) - x \log_2 x - (1 - x) \log_2 (1 - x)$ for $x \in [1/2, 1]$ and null otherwise [here $H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$]. In our case, Eq. (6) yields the identity

$$W(\rho) = 1 - 2C_{34}(\rho).$$  

It follows that $C_{34}(\rho) < 1/4$ implies $W(\rho) > 1/2$ and hence, from Eq. (7), $E_f(\rho) > 0$. Therefore, we can conclude that also in the general case in which the two electrons can enter the same port, values of $C_{34}$ smaller than 1/4 are direct evidence of entanglement in the input state. 

The sign of $C_{34} - 1/4$, however, is not sufficient to discriminate between occupation-number and electronic degree-of-freedom entanglement. In order to do so we use the symmetries $w(\{\varphi_\alpha + \pi/2\}) = w(\{\varphi_\alpha\})$ and $v(\{\varphi_\alpha + \pi/2\}) = -v(\{\varphi_\alpha\})$ to define $C_{34}^{(\pm)}(\rho) = |C_{34}(\rho; \{\varphi_\alpha\}) \pm \rho_{34}(\{\varphi_\alpha + \pi/2\})| / 2$. Equation (1) shows that $C_{34}^{(\pm)}$ depends only on the superposition among $|\Phi_{11}\rangle$ and $|\Phi_{22}\rangle$, i.e.,

$$C_{34}^{(\pm)}(\Psi) = \frac{1}{4} \left[ 1 + w \cos^2 \theta \right],$$

where $C_{34}^{(\pm)}(\Psi) = \frac{v}{4} \sin^2 \theta \sin(2\phi)$. From the discussion of the $\theta = 0$ case it follows that the presence of electronic degree-of-freedom entanglement can be detected by finding values of $\varphi_\alpha$ such that $C_{34}^{(\pm)} < 1/4$. Vice-versa, we observe that any value of $C_{34}^{(\pm)}$ different from zero is indicative of occupation-number entanglement in the system. Indeed, $C_{34}^{(\pm)} \neq 0$ is possible only for $\theta \neq 0$ and $\phi \neq 0, \pi/2$ which correspond to a non-trivial superposition between terms with different local occupation number. Moreover, one can relate $C_{34}^{(\pm)}$ to the superselection induced variance (SIV), $V(\Psi) = \frac{4}{2} \left( |N_1^2| |\Psi\rangle \langle \Psi| - |N_1| |\Psi_1| |\Psi_2| \right)^2$. This is a measure of occupation number entanglement, where $N_1 = \sum_{i} a_{1,i}^\dagger a_{1,i}$ is the total number operator of particles in port 1. It is easy to show that $V(\Psi) \geq (4C_{34}^{(\pm)})^2$.

We finally discuss the case in which the entangler suffers from backscattering, a situation often encountered in proposed entanglers. In this case, due to conservation of the total number of particles 1, 2, the most general input state in ports 1 and 2 is described by a convex combination $R$ of density matrices $\rho''$, $\rho'$ and $|0\rangle \langle 0|$ having, respectively, two, one or zero impinging electrons. It reads $R = \rho'' + \rho' + (1 - \rho'' - \rho') |0\rangle \langle 0|$, where $\rho''$ is the probability that both electrons enter the ports (no backscattering occurred) and $\rho'$ is the probability that only one of them enters the ports. The entanglement of formation of the state $R$ can be bounded as before. In this case, however, Eqs. (6) and (8) give

$$W(R) = q'' W(\rho'') + q' W(\rho') \geq q'' W(\rho'') + q' [1 - 2C_{34}(\rho'')] = q'' - 2C_{34}(R),$$  

where we used the fact that $W(\rho'' | 0\rangle \langle 0|) = 0$ and the linearity of $C_{34}$, yielding $C_{34}(R) = q'' C_{34}(\rho'')$ (notice that since $\rho'$ and $|0\rangle$ have less than two electrons they do not contribute to the current cross-correlator). From the monotonicity of the function $E(x)$ we finally obtain

$$E_f(R) \geq E(q'' - 2C_{34}(R)).$$

Once the transmission probability $q''$ of the setup is known, Eq. (12) provides an estimate of the entanglement of formation of the system. Notice that, differently from the case discussed in the first part of the paper, the state $R$ is a mixture of terms with different total number of particles. In this case the standard entanglement of formation $E_f(R)$ measures the amount of entanglement needed to create the state $R$ without taking into account the constraints posed by particle-number conservation (i.e., considering possible decompositions of $R$ including coherent superpositions of states with different particle number). For this reason Schuch, Verstraete and Cirac have proposed to replace the quantity $E_f(R)$ with the SSR-entanglement of formation. This is defined as $E_f^{(SSR)}(R) = \min_{\rho, \psi} \sum_i p_i E_f(\psi_i)$, where the minimization is performed over all decomposition of $R$ with $|\psi_i\rangle$ being eigenstates of the particle-number operator. By construction, $E_f^{(SSR)}(R)$ is always greater or equal to $E_f(R)$. In our case it reads

$$E_f^{(SSR)}(R) = q'' E_f(\rho'') + q' E_f(\rho') \geq q'' E_f(\rho''),$$

where we used $E_f(|0\rangle \langle 0|) = 0$. The rhs term of this expression can now be lower bounded by noticing that $\rho''$ represents a density matrix formed by vectors of the form $|\psi_i\rangle$. Employing Eqs. (7) and (8) we find

$$E_f^{(SSR)}(R) \geq q'' E(1 - 2C_{34}(R)/q''),$$

where we used, again, the relation $C_{34}(R) = q'' C_{34}(\rho'')$ by linearity of $C_{34}$. As in the case of Eq. (12), once the probability $q''$ is known, Eq. (14) provides an estimate of the SSR-entanglement of formation of the system.

In the remaining part of the paper we make a connection between the results obtained so far and some proposed electronic entanglers. The situation we consider applies naturally to the case of superconductor-based entanglers. It has been shown [4] that Cooper pairs can be injected into normal metal ports 1 and
2 through a (double) tunnel barrier in the form of a wave packet \(|\Psi\rangle = \sum E |\Psi_E\rangle\). Here, the states \(|\Psi_E\rangle\) have the form given in Eq. (1), with the difference that the two electrons do not possess the same energy. In fact, they have opposite sign with respect to the chemical potential of the superconductor, instead, such that \(\Phi_{ij} = \sum_{a,b} \Phi^*_{a,b} a^\dagger_\alpha a^\dagger_\beta (E_a - E_b)\). When calculating the cross-correlator \(C_{34}\) of Eq. (2) one notices that, actually, the energy labels \(E - E^\prime\) can be effectively incorporated into the mode labels \(\alpha\) and \(\beta\) in \(|\Phi_{ij}\rangle\). This is possible thanks to the limit \(T \to \infty\) in Eq. (2), which allows only zero-frequency contributions \((\omega = 0)\) to the current operators \(\mathcal{J}\).

The state \(|\Psi\rangle\) describes \(N = eV/\delta E\) incoming electron pairs. In the case \(N > 1\), the cross-correlator \(C_{34}\) exhibits extra terms scaling with \(N(N - 1)\) which do not allow to separate the occupation-number entanglement from the electronic degree-of-freedom one details shall be given elsewhere. However, the problem may be solved by introducing small measurement times instead, such that \(eV \ll \delta E\) and \(N \to 1\). It can be shown that a calculation of the the cross-correlator \(C_{34}\) in these conditions leads to equivalent results.

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with the SSR-entanglement of formation of Ref. [11] discussed at the end of the paper. For the pure input state of Eq. (1) one obtains $E_{N-SSR}(\Psi) = \cos^2 \theta E_f(\Phi_{12})$, where we used the fact that $E_f(\Phi_{jj}) = 0$ for $j = 1, 2$. Thus, $E_{N-SSR}(\Psi)$ is proportional to the entanglement of formation of $|\Psi_{12}\rangle$ and can be used to quantify the electronic degree-of-freedom entanglement component of the system. A lower bound can be obtained by replacing $\rho$ with $|\Phi_{12}\rangle$ in Eqs. (7) and (8), and by noticing that $C_3^+(\Psi) = [4C_{34}(\Phi_{12}) \cos^2 \theta + \sin^2 \theta]/4$. This yields $E_{N-SSR}(\Psi) \geq \cos^2 \theta \mathcal{E}\left(\frac{1}{2} + \frac{1}{4C_3^+(\Psi)}\right)$, which gives $E_{N-SSR}(\Psi) > 0$ for $C_3^+(\Psi) < 1/4$. 