Bubble with Global Monopole

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Abstract

The first-order phase transition of $O(3)$ symmetric models is considered in the limit of high temperature. It is shown that this model supports a new bubble solution where the global monopole is formed at the center of the bubble in addition to the ordinary $O(3)$ bubble. Though the free energy of it is larger than that of normal bubble, the production rate can considerably be large at high temperatures.
Since a general theory of the decay of the metastable phase was developed [1, 2], the study of first-order phase transitions have attracted the attention due to their possible relevance to the physics of the early universe. The semiclassical expression of the tunneling rate from false vacuum to true vacuum is given by the bubble solution of lowest (Euclidean) action both for the first-order phase transitions at zero temperature field theories [2] and those at finite temperature [3, 4]. However, the above case does not include the possibility that there exist various decay modes between two given classical vacua. In this note, we explore such problem by examining a scalar model of internal $O(3)$ symmetry at finite temperature and examine a new bubble solution with a global monopole.

Suppose that the model of our interest contains a series of stationary points which support bubble solutions, a decay probability per unit time per unit volume is given by

$$\Gamma/V = \sum_n A_ne^{-B_n/h}, \quad (1)$$

where $n$ represents the $n$-th local minimum. For each $n$, $B_n$ is given by a value of Euclidean action for $n$-th bubble solution and $A_n$ is estimated by integrating out the fluctuations around a given $n$-th configuration.

We choose the $O(3)$-symmetric scalar models described in terms of an isovector field $\phi^a (a = 1, 2, 3)$ as the first possibility for sample calculations. At finite temperature the Euclidean action is

$$S_E = \int^\beta_0 dt_E \int d^3x \left\{ \frac{1}{2} \left( \frac{\partial\phi^a}{\partial t_E} \right)^2 + \frac{1}{2} (\nabla\phi^a)^2 + V(\phi^a) \right\}, \quad (2)$$

where $\beta = \hbar/k_B T$. Although our argument followed is general and does scarcely depend on the detailed form of scalar potential if it has the vacuum structure with both a true and a false vacuum, we will consider a specific model, i.e. a $\phi^6$-potential

$$V(\phi) = \frac{\lambda}{v^4}(\phi^2 + \alpha v^2)(\phi^2 - v^2)^2 + V_0 \quad (3)$$
where $\phi$ denotes the amplitude of scalar fields $\phi^a$ defined by $\phi = \sqrt{\phi^a\phi^a}$. Here we consider the transition from the symmetric vacuum to the broken vacuum, i.e. $0 < \alpha < 1/2$, and choose $V_0$ as $-\lambda \alpha v^4$ in order to make the value of $S_E$ coincide with $B_\alpha$ in Eq. (1). This does not lose generality since the case of the transition from the broken vacuum to the symmetric one ($-1 < \alpha < 0$) also possesses the same type of bubble solutions if we replace $\phi$ to $v - \phi$.

It has been known that the rate of vacuum transitions at finite temperature has relevant amount of contributions from $O(3)$-symmetric sphaleron-type bubbles \cite{3, 4}. In high temperature limit this contribution dominates and then we can neglect the dependence along $t_E$-axis. Here we are interested in the regime where thermal fluctuations are much larger than quantum fluctuations and address the problem for different kind of finite temperature bubbles which connect global $O(3)$ internal symmetry to that of spatial rotation. We now ask for a solution of the field equations that is time-independent and spherically symmetric, apart from the angle dependence due to the mapping between the $(\theta, \varphi)$ angles in space and those of isovector space $\hat{\phi}_n^a(\equiv \phi^a/\phi)$ such as

$$\hat{\phi}_n^a = (\sin n\theta \cos n\varphi, \sin n\theta \sin n\varphi, \cos n\theta),$$

where the allowed $n$ is 0 or 1 in order to render the scalar amplitude $\phi$ a function of $r$ only. Under this assumption, the Euler-Lagrange equation becomes

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} - \delta_{n1}\frac{2}{r^2}\phi = \frac{dV}{d\phi},$$

where $\delta_{n1}$ in the third term of Eq. (5) denotes Kronecker delta for $n = 0$ or 1.

The condition that the theory should be in false vacuum at spatial infinity fixes the boundary value of field, i.e. $\lim_{r \to \infty} \phi \to 0$. To be nonsingular solution at the origin of coordinates, the boundary condition is

$$\begin{cases}
\left. \frac{d\phi}{dr} \right|_{r=0} = 0 & \text{if } n = 0 \\
\phi(0) = 0 & \text{if } n = 1.
\end{cases}$$
When \( n = 0 \), it is well-known bubble solution at finite temperature [3]. We will analyze \( n = 1 \) case and show that there always be \( n = 1 \) solution if the equation contains \( n = 0 \) solution. A brief argument of the existence of \( n = 1 \) solution is as follows.

If we regard the radius \( r \) as time and the scalar amplitude \( \phi(r) \) as the coordinate of a particle, Eq.(5) describes a one-dimensional motion of a unit-mass hypothetical particle under the conserved force due to the potential \(-V(\phi)\) and two nonconservative forces, i.e. one is the friction of time-dependent coefficient \(-\frac{2}{r} \frac{d \phi}{d r}\) and the other is time-dependent repulsion \( \frac{2}{r^2} \phi \). Hence, in the terminology of Newton equation, \( n = 1 \) solution in Fig. 1 is interpreted as the motion of a particle that starts at time zero at the origin (\( \phi(0) = 0 \)), turns at an appropriate nonzero position at a certain time (\( \phi(t_{\text{turn}}) = \phi_{\text{turn}} \)), and stops at the origin at infinite time (\( \phi(\infty) = 0 \)).

At first, if one considers a hypothetical particle at the origin at time zero, it is accelerated by the time-dependent repulsion of which coefficient is divergently large for small \( r \). Since one initial condition is fixed by the starting point (\( \phi(0) = 0 \)), such motions near the origin are characterized by another initial condition \( C \)

\[
\phi(r) \approx C \left\{ r + \mathcal{O}(r^3) + \cdots \right\},
\]

where \( C \) is the initial velocity of a particle which should be tuned by the proper boundary condition at infinite time (\( \phi(\infty) = 0 \)). From now on let us consider a set of solutions specified by a real parameter \( C \) and show that there always exists the unique motion of \( \phi(\infty) = 0 \) for an appropriate \( C \). When \( C \) is sufficiently large, the acceleration near the origin due to time-dependent repulsion and conservative force is too strong and then the particle overshoots the top of potential \(-V(\phi = v)\) and goes to infinity (\( \phi(\infty) = \infty \)), despite the deceleration due to the friction. Since the solution of our interest is the motion which includes a return, we will look at the motions of which \( C \) is smaller than a critical value \( C_{\text{top}} \). Here \( C_{\text{top}} \) gives the motion that the particle stops at the hilltop of the potential \(-V\) at infinite time \( \phi(\infty) = v \).

Suppose that \( C \) be too small, i.e. \( C \) is smaller than another critical value \( C_0 \) (\( C_0 < C_{\text{top}} \)),

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particle turns at a point smaller than $\phi_0$ ($0 < \phi_0 < v$) where $-V(\phi = \phi_0) = 0$ and thereby it can not return to the origin. Moreover, the particle which turns at a point too close to $v$ arrives at the origin at a finite time, since the time-dependent friction and repulsion during returning do not play a role due to the vanishing of their coefficients $1/r$ and $1/r^2$ while the particle stays near $v$ for sufficiently long time. By continuity for an appropriate $C (C_0 < C < C_{top})$ there is a solution which describes the motion that starts at time zero at the origin turns at a position $\phi_{\text{turn}}$ between $\phi_0$ and $v$, and then stops at the origin at infinite time. Here for the existence proof of $n = 1$ solutions, we have used the properties of scalar potential $V(\phi)$ that it has a local minimum at $\phi = 0$, a local maximum at $\phi_{\text{bottom}}$ ($0 < \phi_{\text{bottom}} < \phi_0$) and an absolute minimum at $v$. It is exactly the same condition as that for the existence of $n = 0$ solution. Hence we completes our argument that the equation always contains $n = 1$ solutions when it has $n = 0$ solution. A rigorous proof for the existence of $n = 1$ solution is demonstrated in ref.[5]. We have no analytic form of solutions, so numerical solutions for $n = 0$ and $n = 1$ bubbles are given in Fig. 1.

Suppose that each classical solution describes a critical bubble corresponding to a different decay channel, from the profile of energy density $T_0^0$ (see Fig. 2) we can read the following characteristics of $n = 1$ bubbles. First, the boundary value of scalar field at the origin always has that of false vacuum, i.e. $\phi(0) = 0$, so this implies that there remains a false-vacuum core inside the true-vacuum region of the bubble due to the winding between $O(3)$ internal symmetry and spatial rotation. Second, the value of energy density has a local minimum at the origin, i.e. $3C^2/2 + V(0)$ in which $C$ is a constant appeared in Eq.(7), increases to the maximum at $R_m$ in Fig. 2 and decays below $V(0)$. This implies that a matter droplet is created inside the bubble and is surrounded by inner bubble wall with size of order $R_m \sim 1/m_{\text{Higgs}} = \sqrt{4(3 + 2\alpha)\lambda v}$. Third, if we read the expression of energy density when the scalar amplitude has maximum value $\phi = \phi_{\text{turn}}$, its derivative term vanishes and
then it becomes

$$T_0^0 = \frac{\phi_{\text{turn}}^2}{r^2} + V(\phi_{\text{turn}}).$$  \hfill (8)

The potential term $V(\phi_{\text{turn}})$ can be neglected in thin-wall limit, so the object inside the bubble has a long-range hair which penetrates the inner bubble wall. Since the central region of this matter is in false symmetric vacuum due to the hedgehog ansatz of scalar field in Eq.(4) and for large $r$ the scalar field goes to the true broken vacuum but the long-range tail of energy proportional to $r$ has a cutoff at the outer bubble wall at $R_{n=1}$ in Fig. 2, we can interpret the matter aggregate inside the $n = 1$ bubble as a global monopole of size $R_m$ [6].

Even though there is a no-go theorem for static scalar objects in spacetime dimensions more than two [7], the global monopole of this case can be supported inside the Euclidean bubble configuration as a smooth finite-energy configuration due to a natural cutoff introduced by the outer bubble wall. Fig. 2 shows that the radius of $n = 1$ bubble is larger than that of $n = 0$ bubble. It can be easily understood by the conservation of energy, i.e. the additional energy used to make a matter aggregate is equal to the loss of energy due to the increase of the radius of bubble.

Though we do not have any analytic solution, we can estimate $B_n$ in Eq.(11) by use of the obtained bubble configurations through numerical analysis for given parameters of theory. It has already been proved that $n = 0$ solution describes the nontrivial solution of lowest action [8] and from Fig. 3 we read that the value of action for $n = 1$ solution, $vB'_1/T$, is larger than that of $n = 0$, $vB'_0/T$, irrespective of the shape of scalar potential. Fig. 3 also shows that the ratio $B'_1/B'_0$ becomes small while the difference $B'_1 - B'_0$ increases as the size of bubble becomes large in comparison with the mass scale of theory, i.e. thin-wall limit. This can be understood as follows; the amount of energy consumed to support a global monopole at the center of bubble is proportional to the radius of bubble due to its long-range tail (see Eq.(8)), however the free energy to support the bubble itself, $vB'_0$ or $vB'_1$, is proportional to the cubic of bubble radius in thin-wall limit.
The next task is to estimate the pre-exponential factor $A_n$ in Eq.(1). The scheme of computing $A_0$ for $n = 0$ bubble solution which is the lowest action solution with one negative mode was given in the second paper of Ref.[2]. Here, let us attempt to calculate the pre-exponential factor $A_1$ for $n = 1$ bubble. Considering the small fluctuation around $n = 1$ bubble $\delta \phi_{n=1}^a = \sum_k c_k^a \psi_k^a$, we obtain a Schrödinger-type equation for three particles in three dimensions

$$(-\nabla^2 \delta^{ab} + \hat{r}^a \hat{r}^b \frac{d^2 V}{d\phi_{n=1}^a} + (\delta^{ab} - \hat{r}^a \hat{r}^b) \frac{1}{\phi} \frac{dV}{d\phi} \bigg|_{\phi_{n=1}^a}) \psi_k^b = \lambda_k \psi_k^a.$$  \hspace{1cm} (9)

It looks too difficult to solve this equation since it includes $\theta$ and $\varphi$ dependent off-diagonal terms in its Hamiltonian and the potential form is only determined numerically, however the eigenfunctions of six zero modes due to three translations and three rotations are explicitly given in a form

$$\psi_{k,i}^a = N_t \nabla_i \phi_{n=1}^a,$$

$$= N_t \left\{ \hat{r}_i \hat{r}^a \frac{d\phi_{n=1}^a}{dr} + (\delta_i^a - \hat{r}_i \hat{r}^a) \phi_{n=1}^a \right\}, \hspace{1cm} (10)$$

and

$$\psi_{k,i}^a = N_r \epsilon_{ijk} x^j \nabla^k \phi_{n=1}^a = N_r \epsilon_{ijk} \hat{r}^j \phi_{n=1}, \hspace{1cm} (11)$$

where $N_t$ and $N_r$ are normalization constants. Here we give few comments on the number of negative modes. First, every component of translational zero-mode eigenfunction in Eq.(10) has $(\theta, \varphi)$-dependence where its radial part, $\frac{d\phi_{n=1}^a}{dr} - \frac{2n-1}{r}$, has single node at the origin, and each $a = i$ component includes an additional $(\theta, \varphi)$-independent part, $\phi_{n=1}$, which has no node. Every $a \neq i$ component of rotational ones, $\phi_{n=1}$, has one node at the origin. For the sake of simplicity, let us examine the problem for the perturbation with specific direction; (i) one for amplitude ($\delta \phi_{n=1}^a = \hat{r}^a \delta \phi_{n=1}$) and (ii) two for transverse ($\delta \phi_{n=1}^a = (\delta^{ab} - \hat{r}^a \hat{r}^b) \delta \phi_{n=1}^b$).

Under these fluctuations Eq.(8) reduces to

$$(-\nabla^2 + U(r)) \psi_k^a = \lambda_k \psi_k^a,$$  \hspace{1cm} (12)
where $U(r) = \left. \frac{d^2V}{dr^2} \right|_{\phi_{n=1}}$ for the first case (i) and $U(r) = \left. \frac{1}{\phi} \frac{dV}{d\phi} \right|_{\phi_{n=1}}$ for the second case (ii). When $U(r) = \left. \frac{d^2V}{dr^2} \right|_{\phi_{n=0}}$, each single ‘a’ component of Eq. (12) is nothing but the fluctuation equation for $n = 0$ bubble which contains a nodless s-wave mode as the unique negative mode [2, 9]. For the fluctuation of scalar amplitude for $n = 1$ bubble, the lowest mode can not be nodless s-wave mode but $l = 1$ mode with single node at $r = 0$ since $\psi^a_k$ should have $\theta$ and $\varphi$ dependence proportional to $\hat{r}^a$ even though the operator in Eq. (12) is a scalar operator. It is analogous for the perturbation to the transversal directions. It implies that the eigenfunction of negative mode may take a somewhat complicated form which depends on angels. Second, the $n = 1$ solution which is a parity-odd bounce solution, $\phi(r = 0) = \phi(r = \infty)$, is supported by assuming the winding between three spatial rotations and those in internal space. It seems that the argument for the system of quantum mechanics with one time variable in Ref. [9] does not forbid directly the existence of $n = 1$ solution as a bubble configuration. We know very little on the counting of negative modes, and then there should be further work on this issue. Since the operator in Eq. (9) is even under parity transformation and is covariant under the rotation, i.e. $x^i \rightarrow O^{ij} x^j$ and $\psi^a_k(x') = O^{ab} \psi^b_k(x)$ where both $O^{ij}$ and $O^{ab}$ are the elements of $O(3)$ group, vector spherical harmonics is a method to investigate the modes of which the eigenfunctions can be chosen to be either even or odd in parity [10].

In order to compute the decay rate accurately, we should calculate the nonzero modes $\lambda_k$, which is extremely difficult even for $n = 0$ bubbles. However, the dimensional estimate may reach a rough result such as [3]

$$\frac{\Gamma^{(1)}}{\Gamma^{(0)}} \sim \left( \frac{B_1'}{B_0'} \right)^2 \exp \left[ -\frac{v}{T}(B_1' - B_0') \right].$$

(13)

At high temperature limit, $T \gg v$, both decay rates for $n = 0$ and 1 bubbles, of course, increase and, moreover, the relative decay rate of $n = 1$ bubble to that of $n = 0$ is also enhanced exponentially. When $v/T \sim 10^{-1}$, the order of $\Gamma^{(1)}/\Gamma^{(0)}$ is around 0.17 in a thin-wall case ($\lambda = 1$, $\alpha = 0.12$) and it is around 28 in a thick-wall case ($\lambda = 1$, $\alpha = 0.47$). Therefore, at high temperatures and in thick-wall case, this $n = 1$ bubbles can be
preferred to those of $n = 0$, and obviously the existence of another decay channel can enhance considerably the total nucleation rate of bubbles, $\Gamma = \Gamma^{(0)} + \Gamma^{(1)}$, except for the bubbles with extremely thin wall.

The main consideration of the paper was to gain an understanding as to how bubbles with global monopoles nucleated in first-order phase transitions at high temperatures and the next question will be how the high-temperature bubbles grow [11], particularly at the site of global monopole. Once we assume these bubbles in early universe where the gravity effect should be included, the bubbles with solitons may result in interesting phenomena [12] in relation with inflationary models, i.e. the inflation in the cores of topological defects [13].

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Figure Captions

FIG. 1. Plot of bubble solutions. $n = 0$ and $n = 1$ configurations are shown as dotted and solid lines, respectively. The parameters chosen in the figures are: $\lambda = 1$, $\alpha = 0.12$, and $V_0 = -0.12\nu^4$.

FIG. 2. Plot of energy density $T^0_0$. The parameters chosen in the figures are: $\lambda = 1$, $\alpha = 0.12$, and $V_0 = -0.12\nu^4$ which is the minimum of energy density.

FIG. 3. Plot of action $S_E$ (or equivalently $B_n$) as a function of $\alpha$. Another parameters are chosen as $\lambda = 1$ and $V_0 = -\lambda\alpha\nu^4$. 
Figure 1
Figure 2
Figure 3