SCHAUDER BASES AND OPERATOR THEORY

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Abstract. In this paper, we firstly give a matrix approach to the bases of a separable Hilbert space and then correct a mistake appearing in both review and the English translation of the Olevskii’s paper. After this, we show that even a diagonal compact operator may map an orthonormal basis into a conditional basis.

1. Introduction and preliminaries

In operator theory, an invertible operator on an infinite dimensional complex Hilbert space $\mathcal{H}$ means the bounded operator which has a bounded inverse operator, and it is well-known that, for an $n \times n$ matrix $M_n$ (seen as an operator on finite dimensional Hilbert space $\mathbb{C}^n$), $M_n$ is invertible if and only if its column vectors are linearly independent in $\mathbb{C}^n$. In other words, the column vectors of $M_n$ comprise a basis of $\mathbb{C}^n$. From this point of view, we could generalize the "invertibility" of $\omega \times \omega$ matrix $M$ (the representation of a bounded operator on an orthonormal basis of $\mathcal{H}$) in the following manner: all column vectors of $M$ form some kind of basis of $\mathcal{H}$. Actually, the invertible operator do have a natural understanding in the ‘basis’ language. That is, the column (or row) vectors of the matrix of an invertible operator always comprise a ‘Riesz basis’ (it is a direct corollary of theorem 2, paper [1], although the authors do not state it in this way). From the above observation, it suggests us to consider the $\omega \times \omega$ matrix whose column vectors form more general kind of bases.

Naturally we consider the $\omega \times \omega$ matrix whose column vectors comprise a Schauder basis. We shall call them the Schauder matrix therefrom. An operator which has a Schauder matrix representation under some orthonormal basis (ONB) will be called a Schauder operator. An easy fact is that an operator is a Schauder operator if and only if it maps some ONB into a Schauder basis. Many scholars have studied some kind of these operators. A. M. Olevskii gave a surprising result on the bounded operators which map some ONB into a conditional quasinormal basis ([5], theorem 1, p479); Stephane Jaffard and Robert M. Young proved that a Schauder basis always can be given by an one-to-one positive transformation ([1], theorem 1, p554). I. Singer gave lots of examples of bases of $\mathcal{H}$ which can be rewritten into a matrix form (see, [6], p429, p497). Besides these results, as for a joint research both on operator theory and the basis theory but not in this direction, the paper [24, 26] by Gowers, the paper [13] by Kwapien, S. and Pelczynski, A. and the elegant book [2] by M. Young are remarkable examples.
Nevertheless, there is still a gap between the researches in the field of basis theory and operator theory. There are few joint works on both basis of Hilbert space and the operators on the Hilbert space. The reason reflects on two aspects. One is the different terminology systems and the other one is that there are scanty common objects to study with. The main purpose of this paper is to show that the Schauder matrix is a candidate to fill this gap. As basic and traditional tools, the matrix representation of operators plays an important role in the study of the operators on the Hilbert space $H$. So the matrix approach to the basis theory is a good beginning to the joint research on the bases of the Hilbert space $H$ and the operators on it.

In this paper, the matrix representation of operators and bases will be the bridge between basis theory and operator theory. We firstly give a matrix approach to the bases of a separable Hilbert space and then correct a mistake appearing in both review and the English translation of the Olevskii’s paper. After this, we follow the Olevskii’s result to consider the operators which can map some ONB into a conditional Schauder basis. We shall call them conditional operators therefrom. In matrix language, it is equivalent to study the operator $T$ which has a matrix representation $M$ under some ONB such that the column vector sequence of $M$ comprise a conditional Schauder basis.

2. An Operator Theory Description of Schauder basis

2.1. Suppose that $\{e_k\}_{k=1}^{\infty}$ is an ONB of $H$. An $\omega \times \omega$ matrix $M = (m_{ij})$ automatically represents an operator under this ONB. In more details, for a vector $x \in H$ there is an unique $l^2-$sequence $\{x_n\}_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} x_n e_n$ in which the series converges in the norm of $H$. Let $y_n = \sum_{k=1}^{\infty} m_{ik} x_k, y = \sum_{n=1}^{\infty} y_n e_n$, then the operator $T_M$ defined by $T_M x = y$ is just the corresponding operator represented by $M$. In general, $T$ is not a bounded operator. We shall identify the $\omega \times \omega$ matrix $M$ and the operator $T_M$, and denote them by the same notation $M$ if we have fixed an ONB and there is no confusion.

Recall that a sequence $\psi = \{f_n\}_{n=1}^{\infty}$ is called a Schauder basis of the Hilbert space $H$ if and only if for every vector $x \in H$ there exists an unique sequence $\{\alpha_n\}_{n=1}^{\infty}$ of complex numbers such that the partial sum sequence $x_k = \sum_{n=1}^{k} \alpha_n f_n$ converges to $x$ in norm.

Denote by $P_k$ the the diagonal operator with the first $k-$th entries on diagonal line equal to 1 and 0 for others. Then as an operator $P_k$ represents the orthogonal projection from $H$ to the subspace $H^{(k)} = \text{span}\{e_1,e_2,\cdots,e_k\}$.

**Lemma 2.1.** Assume that $\{e_k\}_{k=1}^{\infty}$ is a fixed ONB of $H$. Suppose that an $\omega \times \omega$ matrix $F = (f_{ij})$ satisfies the following properties:
1. Each column of the matrix $F$ is a $l^2-$sequence;
2. $F$ has an unique left inverse matrix $G^* = (g_{ik})$ such that each row of $G^*$ is also a $l^2-$sequence;
3. Operators $Q_k$ defined by the matrix $Q_k = FP_kG^*$ are well-defined projections on $H$ and converges to the unit operator $I$ in the strong operator topology.
Then the sequence $\{f_k\}_{k=1}^{\infty}, f_k = \sum_{n=1}^{\infty} f_{ik} e_i$ must be a Schauder basis.

Here we use the term “left reverse” in the classical means, that is, the series $\sum_{j=1}^{\infty} g_{kj} f_{jn}$ converges absolutely to $\delta_{kn}$ for $k,n = 1,2,\cdots.$ $G^*$ does not mean the adjoint of $G$, it is just a notation.
Proof. Property 1 just ensure that series \( \{f_k = \sum_{j=1}^{\infty} f_{jk} e_j\}_{k=0}^{\infty} \) converges to a well-defined vector \( f_k \) in \( \mathcal{H} \) by norm. Property 2 implies that \( \text{span}\{f_n; n = 1, 2, \cdots\} = \mathcal{H} \) by the uniqueness of the left inverse. Moreover, the \( k \)-th row of the matrix \( G^* \) is just the vector \( g_k^* \) such that \( (g_k^*, f_n) = \delta_{kn} \). Therefore the vector sequence \( \{f_n\}_{n=1}^{\infty} \) must be minimal by the Hahn-Banach theorem(cf, [7] corollary 6.8, p82) and the Riesz representation theorem(see, [7], theorem 3.4, p12).

Now for each vector \( x = (x_1, x_2, \cdots) \) denote by \( \alpha_k^x = (g_k^*, x) \), it is easy to check that \( Q_k^x = Q_k \) and

\[
Q_k x = F P_k G^* x = \sum_{n=1}^{\infty} \alpha_k^x f_n.
\]

By property 3, we have \( Q_k x \rightarrow x \) since \( Q_k \) converges to \( I \) in strong operator topology(SOT). That is, series \( \sum_{n=1}^{\infty} \alpha_k^x f_n \) converges to the vector \( x \) in norm. So we have proved that each vector \( x \) in \( \mathcal{H} \) can be represented by the sequence \( \{f_n\}_{n=1}^{\infty} \) with coefficients \( \{\alpha_k^x\}_{n=1}^{\infty} \).

To show that \( \{f_n\}_{n=1}^{\infty} \) is a Schauder basis, we just need to show that this representation is unique. Suppose that \( \{\alpha_n\}_{n=1}^{\infty} \) is a sequence such that the series \( \sum_{n=1}^{\infty} \alpha_n f_n \) converges to 0 in the norm of the Hilbert space \( \mathcal{H} \). Assume that the integer \( n_0 \) is the first number satisfying \( \alpha_{n_0} \neq 0 \). Then we have

\[
f_{n_0} = -\frac{1}{\alpha_{n_0}} \sum_{n=n_0+1}^{\infty} \alpha_n f_n
\]

in which the series also converges in the norm topology. It counter to the fact that the sequence \( \{f_n\}_{n=1}^{\infty} \) is a minimal sequence. □

Conversely, suppose that \( \psi = \{f_n\}_{n=1}^{\infty} \) is a basis of \( \mathcal{H} \). For a fixed ONB \( \{e_n\}_{n=1}^{\infty} \), each vector \( f_n \) has a representation \( f_n = \sum_{k=1}^{\infty} f_{kn} e_k \). Denote \( F_\psi = (f_{kn}) \). We shall call \( F_\psi \) the Schauder matrix corresponding to the basis \( \psi \). The following lemma is the inverse of the above lemma.

**Lemma 2.2.** Assume that \( \psi = \{f_n\}_{n=1}^{\infty} \) is a Schauder basis. Then the corresponding Schauder matrix \( F_\psi \) satisfies the following properties:

1. Each column of the matrix \( F_\psi \) is a \( l^2 \)-sequence;
2. \( F_\psi \) has an unique left inverse matrix \( G^*_\psi = (g_{kl}) \) such that each row of \( G^*_\psi \) is also a \( l^2 \)-sequence;
3. Operators \( Q_k \) defined by the matrix \( Q_k = F_\psi P_k G^*_\psi \) are well-defined projections on \( \mathcal{H} \) and converges to the unit operator \( I \) in the strong operator topology.

**Proof.** Property 1 comes from the fact that \( f_n \) is a vector in \( \mathcal{H} \).

If \( \{f_k\}_{k=1}^{\infty} \) is a Schauder basis, then the subspace \( \mathcal{H}_k = \text{span}\{f_n; n \neq k\} \) for each \( k \) satisfying \( f_k \notin \mathcal{H}_k \)(cf. [6], p50-51). So we must have a unique linear functional \( \varphi_k \) such that \( \varphi_k(f_n) = \delta_{kn} \). Then by the Riesz representation theorem, there is a unique vector \( g_k^* \) such that \( \sum_{j=1}^{\infty} g_{kj} f_{jn} = \delta_{kn} \) in which \( \{g_{kl}\}_{l=1}^{\infty} \) is a \( l^2 \)-sequence. The uniqueness holds because the sequence \( \{f_k\}_{k=1}^{\infty} \) spans the Hilbert space. Hence a Schauder matrix must have a unique left inverse matrix whose rows are \( l^2 \)-sequence. Then we have proved the property 2.

Property 3 is just a direct corollary of the definition of Schauder basis. Denote by \( G = (G^* G) \) the adjoint matrix of \( G^* \), then we have \( g_{kn} = g_{nk}^* \). Moreover, denote by \( y_n \) the \( n \)-th column vector and for a vector \( x = \sum_{n=1}^{\infty} x_n e_n \) denote by \( y_n = \sum_{k=1}^{\infty} g_{nk} x_k \). Then trivially we have \( y_n = (x, g_n) \) and \( (f_k, y_n) = \delta_{kn} \). Suppose
that \( x = \sum_{k=1}^{\infty} \alpha_k f_k \) is the representation of the vector \( x \) under the basis \( \psi \). Then we must have \( \alpha_n = y_n \) since

\[
y_n = (x, g_n) = \left( \sum_{k=1}^{\infty} \alpha_k f_k, g_n \right) = \alpha_n.
\]

Therefore we have \( Q_k x = \sum_{n=1}^{\infty} \alpha_n f_n \). Clearly we have \( Q_k x \to x \) in the norm topology. In other words, \( ||Q_k x - x|| \to 0 \) when \( k \to \infty \) which implies \( Q_k \to I \) in SOT (cf. [4], proposition 1.3, p262).

The matrix \( G^* \) is unique and decided completely by \( F_\psi \). In fact the matrix \( G^* \) is also the “right inverse” of the matrix \( F \) in the classical sense. For more details, let \( F = (f_{kn})_{\omega \times \omega} \), \( G^* = (g_{mk})_{\omega \times \omega} \), \( f_n = (f_{kn})_{k=1}^{\infty} \) and \( g_m^* = (g_{mk})_{m=1}^{\infty} \). Moreover, denote their adjoint matrices by \( F^* = (f_{kn}^*)_{\omega \times \omega} = (f_{kn})_{k=1}^{\infty} \), \( G^* = (g_{mk}^*)_{\omega \times \omega} = (g_{mk})_{m=1}^{\infty} \). Then both \( \psi = \{f_n\}_{n=1}^{\infty} \) and \( \psi^* = \{g_m\}_{m=1}^{\infty} \) are biorthogonal basis to each other. That is, \( \psi \) and \( \psi^* \) are bases and we have \( (f_n, g_m) = \delta_{nm} \) for all \( n, m \in \mathbb{N} \). Now we show that the series \( \sum_{k=1}^{\infty} f_{nk} g_{km} \) converges to \( \delta_{nm} \) as \( k \to \infty \) for all \( n, m \in \mathbb{N} \). Let \( \{e_l\}_{l=1}^{\infty} \) be the corresponding ONB. We write \( e_n, e_m \) into the linearly combinations of basis vector in \( \psi \) and \( \psi^* \) as follows:

\[
e_n = \sum_{k=1}^{\infty} \alpha_{nk} f_k, e_m = \sum_{k=1}^{\infty} \beta_{mk} g_k^*.
\]

Then we have \( \alpha_{nk} = g_{kn}^* \) and \( \beta_{mk} = f_{km}^* = f_{mk} \). Hence for any integer \( N \)

\[
\sum_{k=1}^{N} f_{nk} g_{km} = (\sum_{k=1}^{N} \alpha_{nk} f_k, \sum_{k=1}^{N} \beta_{mk} g_k^*) = (e_n - \sum_{k=1}^{\infty} \alpha_{nk} f_k, e_m - \sum_{k=1}^{\infty} \beta_{mk} g_k^*) = (e_n - \sum_{k=N}^{\infty} \alpha_{nk} f_k, e_m - \sum_{k=N}^{\infty} \beta_{mk} g_k^*) = \frac{1}{2} ||e_n - \sum_{k=N}^{\infty} \alpha_{nk} f_k|| + \frac{1}{2} ||e_m - \sum_{k=N}^{\infty} \beta_{mk} g_k^*|| < \frac{\epsilon}{2},
\]

\( ||e_n - \sum_{k=N}^{\infty} \alpha_{nk} f_k|| < \frac{\epsilon}{2}, ||e_m - \sum_{k=N}^{\infty} \beta_{mk} g_k^*|| < \frac{\epsilon}{2} \) hold. Then we have

\[
\begin{align*}
&= \left| \sum_{k=1}^{N} f_{nk} g_{km}^* - (e_n, e_m) \right| \\
&\leq \left| \sum_{k=N}^{\infty} \alpha_{nk} f_k, e_m - \sum_{k=N}^{\infty} \beta_{mk} g_k^* \right| + \left| \sum_{k=N}^{\infty} \alpha_{nk} f_k, \sum_{k=N}^{\infty} \beta_{mk} g_k^* \right| \\
&\leq \epsilon (1 + \frac{\epsilon}{N} + \frac{\epsilon}{4}).
\end{align*}
\]

For this reason, we have the following definition.

**Definition 2.3.** For a Schauder matrix \( F_\psi \), the corresponding matrix \( G^*_\psi \) is called the inverse matrix of \( F_\psi \).

If we do not ask that each row of \( G^* \) is a \( l^2 \)-sequence, an \( \omega \times \omega \) matrix may have a “left inverse” in the classical sense.

**Example 2.4.** Let \( F \) be the matrix

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & \cdots \\
0 & -1 & 1 & 0 & \cdots \\
0 & 0 & -1 & 1 & \cdots \\
0 & 0 & 0 & -1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
and \(G^*\) be the matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & \\
0 & -1 & -1 & -1 & \\
0 & 0 & -1 & -1 & \\
0 & 0 & 0 & -1 & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

It is trivial to check \(G^*F = FG^* = I\). Then by above lemma 2.1, we know \(F\) is not a Schauder matrix since the rows of its inverse matrix are not \(l^2\)-sequence. Moreover, if we denote by \(g_n\) the \(n\)-th column vector, then the sequence \(\xi = \{g_n\}_{n=1}^\infty\) is a complete minimal sequence (see [5], p24 and p50 for definitions). It is easy to check \(\xi\) is complete since the \(l^2\)-sequence \(h_n = \{h_n(j)\}_{j=1}^\infty\), \(h_n(j) = \delta_{nj}\) is in its range: On the other hand, the row vector sequence \(\{f_k\}_{k=1}^\infty\) satisfies \((g_n, f_k) = \delta_{kn}\) which implies \(g_n \notin \bigvee_{m \neq n} g_m\) (or in notations of Singer, we have \(g_n \notin \{g_1, \ldots, g_{n-1}, g_{n+1}, \ldots\}\)) by the fact \(\bigvee_{m \neq n} g_m = \ker \varphi_k\) in which \(\varphi_k(x) = (x, f_k)\) is a bounded functional by Riesz’s theorem. Therefore \(\xi\) is an example which is complete and minimal sequence but not a basis sequence.

By above lemmas 2.1 and 2.2, we have

**Theorem 2.5.** An \(\omega \times \omega\) matrix \(F\) is a Schauder matrix if and only if it satisfies property 1, 2 and 3.

For a Schauder matrix \(F\), the column vector sequence \(\{g_n\}_{n=1}^\infty\) of \(G\) defined in above lemmas is also a Schauder basis which is called the biorthogonal basis \(\{\delta_{m,n}\}_{m,n=1}^\infty\) (cf. [2], pp23-29, [3] pp23-25).

The projection \(FP_n G^*\) is just the \(n\)-th “natural projection” so called in [3] (p354). It is also the \(n\)-th partial sum operator so called in [6] (definition 4.4, p25). Now we can translate theorem 4.1.15 and corollary 4.1.17 in [6] into the following

**Proposition 2.6.** If \(F\) is a Schauder matrix, then \(M = \sup_n \{\|FP_n G^*\|\}\) is a finite const.

The const \(M\) is called the const for the basis \(\{f_n\}_{n=1}^\infty\).

Assume that \(\psi = \{f_n\}_{n=1}^\infty\) is a basis. For a subset \(\Delta\) of \(\mathbb{N}\), denote by \(P_\Delta\) the diagonal matrix defined as \(P_\Delta(nn) = 1\) for \(n \in \Delta\) and \(P_\Delta(nn) = 0\) for \(n \notin \Delta\). The projection \(Q_\Delta = F_\psi P_\Delta G_{\psi}^*\) defined in above lemmas is called a natural projection (see, definition 4.2.24, [4], p378). In fact for a vector \(x = \sum_{n=1}^\infty x_n f_n\), it is trivial to check \(Q_\Delta x = \sum_{n\in \Delta} x_n f_n\). Then we have a same result for the unconditional basis const (cf. definition 4.2.28, [4], p379):

**Proposition 2.7.** If \(F_\psi\) is a Schauder matrix, then the unconditional basis const of the basis \(\psi\) is \(M_{ub} = \sup_{\Delta \subseteq \mathbb{N}} \{\|F_\psi P_\Delta G_{\psi}^*\|\}\).

In virtue of the proposition 4.2.29 and theorem 4.2.32 in the book [4], we have

**Proposition 2.8.** For a Schauder basis \(\psi\), it is an unconditional basis if and only if \(\sup_{\Delta \subseteq \mathbb{N}} \{\|F_\psi P_\Delta G_{\psi}^*\|\} < \infty\).

Following the notations in lemma 2.1 as a direct corollary of lemma 2.1 and theorem 6 in [2] (p28), we have

**Proposition 2.9.** \(F\) is a Schauder matrix if and only if the adjoint matrix (conjugate transpose) \(G\) of its left inverse \(G^*\) is a Schauder matrix.


As well known that a sequence of operators $T_n$ converges to an operator $T$ in SOT dose not imply $T_n^*$ converging to $T$ in SOT, so the above proposition is not trivial.

**Corollary 2.10.** $M = \sup_n \{||FP_nG^*||\} < \infty$ if and only if $M' = \sup_n \{||GP_nF^*||\} < \infty$.

### 2.2. From the definition of the Schauder matrix $F_\psi$, basic properties of Schauder matrix have natural relations to the Schauder basis $\psi$. This understanding lead us to the following definition.

**Definition 2.11.** A matrix $F$ is called an unconditional, conditional, Riesz, normalized or quasinormal respectively if and only if the sequence of its column vectors comprise an unconditional, conditional, Riesz, normalized or quasinormal basis.

Two Schauder matrices $F_\psi, F_\phi$ are called equivalent if and only if the corresponding bases $\psi$ and $\phi$ are equivalent.

Here we use the term quasinormal instead of “bounded” to avoid ambiguity(cf [5] p476, [6] p21). Arso uses the word “similar” in the same meaning as the word “equivalent”(cf, [10] p19, [4]p387).

Denote by $\pi_\infty$ the set of all permutations of $\mathbb{N}$ (see [6], p361). Denote by $U_\pi$ both the unitary operator which maps $e_{\pi(n)}$ to $e_n$ and the corresponding matrix under the ONB $\{e_n\}_{n=1}^\infty$.

**Theorem 2.12.** Assume that $F$ is a Schauder matrix and $G^*$ is its inverse matrix. We have

1. For each invertible matrix $X$, $XF$ is also a Schauder matrix. Moreover, $XF$ is unconditional(conditional) if and only if $F$ is unconditional(conditional);
2. For each diagonal matrix $D = \text{diag}(\alpha_1, \alpha_2, \cdots)$ in which each diagonal element $\alpha_k$ is nonzero, $FD$ is also a Schauder matrix. Moreover, $FD$ is unconditional(conditional) if and only if $F$ is unconditional(conditional);
3. For a unconditional matrix $F$, $FU$ is also a unconditional matrix for $U \in \pi_\infty$;
4. Two Schauder matrix $F$ and $F'$ are equivalent if and only if there is a invertible matrix $X$ such that $XF = F'$.

**Proof.** Property 1, 2, 3 and 4 are basic facts about basis just in a matrix language. Their counterparts are proposition 4.1.8, 4.2.14, 4.1.5, 4.2.12, and corollary 4.2.34 in [4], Theorem 1 in [10]. Some of those facts are easy to check by our lemma 2.1. As an example, we shall prove property 1. Let $F' = XF$, then clearly $G^{*'} = GX^{-1}$ is its inverse matrix. Both properties 1 and 2 in lemma 2.1 hold immediately. To verify property 3, we know that $FP_nG^*$ converges to $I$ in SOT if and only if $XFP_nG^*X^{-1}$ converges to $I$ in SOT. Also we have

$$||XFPG^*X^{-1}|| \leq ||X|| \cdot ||X^{-1}|| \cdot ||FG^*||$$

for any natural projection $P$, which implies the last part of property 1(cf, [3] theorem 4.2.32).

### 2.3. Now we turn to study the basic properties of Schauder operators. Recall that a Schauder operator $T$ is an operator mapping some ONB into a Schauder basis. In his paper [5], Olevskii call an operator to be generating if and only if it maps some ONB into a quasinormal conditional basis. Hence our definition of Schauder operator is a generalization of Olevskii’s one.
Theorem 2.13. Following conditions are equivalent:
1. T is a Schauder operator;
2. T maps some ONB \( \{e_n\}_{n=1}^\infty \) into a basis;
3. T has a polar decomposition \( T = UA \) in which A is a Schauder operator;
4. Assume that T has a matrix representation F under a fixed ONB \( \{e_n\}_{n=1}^\infty \). There is some unitary matrix U such that \( FU \) is a Schauder matrix.

Proof. 2 \( \Rightarrow \) 1. The \( k \)-th column of the matrix of T under the ONB \( \{e_n\}_{n=1}^\infty \) is just the \( l^2 \)-coefficients of \( Te_k \).

1 \( \Rightarrow \) 3. Assume that \( \{f_n\}_{n=1}^\infty \) is a basis in which \( f_n \) is the \( n \)-th column of the matrix \( F \) of \( T \) under some ONB. Then if we denote the matrix of \( U \) and A also by the same notations, we have \( UA = F \). Property 1 of lemma 2.12 tell us \( U^*F = A \) is also a Schauder matrix.

3 \( \Rightarrow \) 4. Assume that \( \{g_n\}_{n=1}^\infty \) is an ONB such that the matrix of A under it is a Schauder matrix. Then the operator \( U \) defined as \( Ue_n = g_n \) is a unitary operator and the \( n \)-th column of its matrix under the ONB \( \{e_n\}_{n=1}^\infty \) is just the \( l^2 \)-coefficients of \( g_n \). Hence we have \( AU \) is a Schauder matrix.

4 \( \Rightarrow \) 1. The column vector sequence of the unitary matrix \( U \) is an ONB. The matrix of \( T \) under this ONB is just \( U^*FU \). Property 1 of lemma 2.12 shows that \( U^*FU \) is a Schauder matrix since \( FU \) is a Schauder matrix itself. \( \square \)

The equivalence 1 \( \Leftrightarrow \) 3 had been used in proof of the theorem 1’ of [5], although Olevskii had not given an explanation.

Proposition 2.14. A Schauder operator T must be injective and has a dense range in \( \mathcal{H} \).

Proof. T must be injective since the representation of 0 is unique. For a basis \( \{f_n\}_{n=1}^\infty \), the finite linear combination of \( \{f_n\}_{n=1}^\infty \) is dense in the Hilbert space \( \mathcal{H} \). Therefore the range of \( T \) must be dense in \( \mathcal{H} \). \( \square \)

2.4. If T is a Schauder operator, does for each ONB sequence \( \{e_n\}_{n=1}^\infty \) the vector sequence \( \{Te_n\}_{n=1}^\infty \) always be a basis? In this subsection, we shall show that the answer is negative in general and it is true only in the case that \( T \) is an invertible operator.

Lemma 2.15. Assume that A is a positive operator satisfying \( \sigma(A) \subseteq [\lambda_1, \lambda_2] \) and \( \lambda_1, \lambda_2 \in \sigma(A) \) for some \( \lambda_1 > 0 \). Then for any const \( \varepsilon > 0 \) small enough, there is a rank 1 projection \( P \) such that \( \frac{1}{\sqrt{2}} \frac{\lambda_2 - \lambda_1}{\lambda_1} \varepsilon < ||APA^{-1}|| \).

Proof. Let \( e_1, e_2 \) be two normalized vectors in \( \mathcal{H} \) such that
\[
e_1 \in E_{[\lambda_1, \lambda_1 + \delta]} , e_2 \in E_{[\lambda_2 - \delta, \lambda_2]},
\]
in which \( E_{[\lambda_1, \lambda_1 + \delta]} \) and \( E_{[\lambda_2 - \delta, \lambda_2]} \) is the spectral projection of \( A \) on the interval \([\lambda_1, \lambda_1 + \delta]\) and \([\lambda_2 - \delta, \lambda_2]\) respectively(cf. [7], pp269-272). Then for \( \delta < \frac{\lambda_2 - \lambda_1}{\lambda_2} \), we have \( \langle e_1, e_2 \rangle = 0 \) and
\[
\lambda_1 \leq ||Ae_1|| \leq \lambda_1 + \delta, \lambda_2 - \delta \leq ||Ae_2|| \leq \lambda_2.
\]
Consider the vector \( e = \frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_2 \) and the operator \( P = e \otimes e \) defined as:
\[
Px = (x, e)e.\]
It is trivial to check that \( P \) is a rank 1 orthogonal projection. Now we
then it maps each ONB into an unconditional basis. By virtue of theorem 2.13, we can assume that
\[ 0 \in \{ \cdot \} \]
we have \( 0 \) sequence \( \psi \Delta = \{ \cdot \} \)
it is enough to show that it is not an unconditional basis, which can be verified by
its unconditional const. Assume that the claim is not true, that is, \( \text{Ran}E \)
we have \( 0 \) sequence \( \Delta ) = (0, 0, 0, \ldots) \) \( \text{and} \)
and
\[ \|Ae\|^2 \geq \frac{1}{2} (\lambda_1 + \lambda_2)^2. \]
Therefore the following inequality holds:
\[
\|APA^{-1}e\| \geq \frac{1}{2} (\lambda_1 + \lambda_2)^2 \sqrt{\frac{1}{2} \lambda_1^2 + \frac{1}{2} (\lambda_2 - \delta)^2}
\]
Let \( \varepsilon \) be a const satisfying \( \varepsilon < \frac{1}{2\sqrt{2}} \). Hence for the positive number \( \delta < \frac{2\varepsilon^2 \lambda_1^2}{|1 - 2\sqrt{2} \lambda_1 \lambda_2|} \)
the required inequality holds.

**Theorem 2.16.** If an operator \( A \) maps every ONB sequence into a basis, then \( A \)
must be an invertible operator.

**Proof.** A direct result of 2.12 is that if an operator \( A \) maps every ONB into a basis then it maps each ONB into a unconditional basis. By virtue of theorem 2.13 we can assume that \( T \) is a positive operator. We need to show that \( 0 \notin \sigma_p(A) \).
Firstly, we have \( 0 \notin \sigma_p(A) \) by above proposition 2.13 since \( A \) is a Schauder operator. If \( 0 \in \sigma(T) \) then 0 must be an accumulation point of \( \sigma(T) \). Hence we can choose a sequence \( \{ \lambda_k \}_{k=1}^\infty \) such that:
1. \( \{ \lambda_k \}_{k=1}^\infty \subseteq \sigma(A) \); and
2. \( \lambda_{k+1} < \lambda_k \) and \( \lambda_{\frac{1}{2n}} \leq \frac{1}{n+1} \).
Denote by \( I_0 = \sigma(A) - \cup_{n=1}^\infty [\lambda_{2n}, \lambda_{2n-1}] \) and \( A_0 \) = \( \text{Ran}E_{I_0} \). Let \( A_n = AE_{[\lambda_{2n}, \lambda_{2n-1}]} \), then we have \( A = A_0 \oplus A_1 \oplus A_2 \oplus A_3 \cdots \). And each operator \( A_n \) is an invertible positive operator for \( n \geq 1 \). Now by above lemma 2.15 we can choose a vector \( e^{(n)}_1 \in \text{Ran}E_{[\lambda_{2n}, \lambda_{2n-1}]} \) such that the projection \( P^{(n)}_1 = e^{(n)}_1 \otimes e^{(n)}_1 \) satisfying
\[
AP^{(n)}_1 A^{-1} = A_n P^{(n)}_1 A_n > n
\]
for each \( n \). Here we use the fact
\[
E_{[\lambda_{2n}, \lambda_{2n-1}]} P_1^{(n)} = P_1^{(n)} E_{[\lambda_{2n}, \lambda_{2n-1}]} = P_1^{(n)}.
\]
Now for each subspace \( \text{Ran}E_{[\lambda_{2n}, \lambda_{2n-1}]} \) we choose an ONB \( \{ f^{(n)}_k \}_{k=1}^{\alpha_k} \) such that \( e^{(n)}_1 = f^{(n)}_1 \). Moreover, choose an ONB \( \{ e^{(0)}_k \}_{k=1}^{\alpha_0} \) of the subspace \( \text{Ran}E_{I_0} \). Here \( \alpha_k \) is a finite number or the countable cardinal which is equal to the dimension of the subspace \( \text{Ran}E_{[\lambda_{2n}, \lambda_{2n-1}]} \) and \( \text{Ran}E_{I_0} \) respectively. Clearly the set \( \{ f^{(n)}_k, n = 0, 1, 2, \cdots \text{ and } k = 1, 2, \cdots, \alpha_k \} \) is an ONB for \( \mathcal{H} \) itself. It is a countable set and each its arrangement \( \psi \) give an ONB sequence of \( \mathcal{H} \). In more details, denote by \( \Delta = \{(n, k); n = 0, 1, 2, \cdots \text{ and } k = 1, 2, \cdots, \alpha_k \} \). For any bijection \( \sigma : \Delta \to \mathbb{N} \), define \( g_n = f^{(n)}_{\sigma^{-1}(n)} \). Then \( \psi_{\sigma} = \{ g_n \}_{n=1}^\infty \) is an ONB sequence.

**Claim 2.17.** For each ONB sequence \( \psi_{\sigma}, \{ Ag_n \}_{n=1}^\infty \) is not a basis.

We have shown that if \( \{ Ag_n \}_{n=1}^\infty \) is a basis it must be an unconditional one. So it is enough to show that it is not an unconditional basis, which can be verified by its unconditional const. Assume that the claim is not true, that is, \( \{ Ag_n \}_{n=1}^\infty \) is a
basis. It is trivial to check that \( A_n P_1^{(n)} A_n^{-1} \) is a natural projection corresponding to the basis \( \{ A_n \} \). In fact, we have
\[
A_n P_1^{(n)} A_n^{-1} = P_{\sigma(n,1)} - P_{\sigma(n,1)-1}.
\]
Here we denote by \( P_n \) the \( n \)-th partial sum operator so called in the book [6]. But now we have \( \|A_n P_1^{(n)} A_n^{-1}\| \to \infty \) which counters to the fact that a unconditional basis must have a finite unconditional const (cf, [4], corollary 4.2.26).

**Corollary 2.18.** If an operator \( T \) is not invertible, then there is some ONB \( \{ e_n \}_{n=1}^{\infty} \) such that the sequence \( \{ T e_n \}_{n=1}^{\infty} \) is not a basis.

By the theorem 1 of [5], a generating operator never be invertible. Hence we have

**Corollary 2.19.** For a generating operator \( T \), there is some ONB \( \{ e_n \}_{n=1}^{\infty} \) such that the sequence \( \{ T e_n \}_{n=1}^{\infty} \) is not a basis.

Both the English translation and the review (MR0318848) of the paper [5] by A. M. Oleskii make a pity clerical mistake:

Review (MR0318848): “The author obtains a spectral characterization for the linear operators that transform every complete orthonormal system into a conditional basis in a Hilbert space.”

The English translation: “Definition. A bounded noninvertible linear operator \( T : \mathcal{H} \to \mathcal{H} \) is said to be generating if it maps every orthonormal basis \( \varphi \) into a quasinormed basis \( \psi \).”

The word “every” should be “some” in both of them. Note that in the proof of the theorem 1 ([5]), Olevskii had shown that an operator never can maps every ONB into a conditional basis. Even the theorem 1 of [5] itself shows it, but need a little operator theory discussion.

Since in the Hilbert space \( \mathcal{H} \) all quasinormal unconditional bases are equivalent (cf, Theorem 18.1, [6], p529) and in addition with theorem 2.12, we have

**Proposition 2.20.** An \( \omega \times \omega \) matrix \( F \) is a Riesz matrix if and only if it represents an invertible operator.

Above result also can be obtained directly from theorem 2 of the paper [1].

**Corollary 2.21.** An operator \( T \) is invertible if and only if there is some ONB such that the matrix \( F \) under this ONB of \( T \) is a Riesz matrix.

**Corollary 2.22.** For an invertible operator \( T \), its matrix always be a Riesz matrix under any ONB.

2.5. Conditional and unconditional bases have very different behaviors. On the other side, properties of operators given by Schauder matrices are strongly dependent on the related bases. Both the theorem 1 of the paper [5] and the behaviors of Riesz matrix (cf, proposition 2.20) support this observation. In this subsection, we give a same classification of operators dependent on their matrix representation (or equivalently, on their actions on ONBs). And then we give some more remarks on Olevskii’s paper.

**Definition 2.23.** A Schauder operator \( T \) will be called a conditional operator if and only if there is some ONB \( \{ e_n \}_{n=1}^{\infty} \) such the column vector sequence of its matrix representation \( F \) of \( T \) under the ONB comprise a conditional basis. Otherwise, \( T \) will be called a unconditional operator.
By the theorem 2.13, we have

**Corollary 2.24.** A Schauder operator $T$ is conditional if and only if it maps some ONB $\{e_n\}_{n=1}^{\infty}$ into a conditional basis $\{Te_n\}_{n=1}^{\infty}$.

For convenience, we correct the error appearing in the translation and rewrite Olevskii’s definition as follows:

**Definition 2.25.** A bounded operator $T \in \mathcal{L}(\mathcal{H})$ is said to be generating if and only if it maps some ONB into a quasinormal conditional basis.

Above definition modifies slightly from the original form on the Olevskii’s paper. We write down the original one to compare them in details:

**Definition 2.26.** ([5], p476) A bounded non-invertible operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is said to be generating if and only if it maps some ONB into a quasinormal basis.

**Proposition 2.27.** Above two definitions are equivalent.

**Proof.** If a bounded operator $T \in \mathcal{L}(\mathcal{H})$ maps some ONB into a quasinormal conditional basis, then it must be non-invertible since an invertible operator maps each ONB into a Riesz basis (hence a unconditional basis) by proposition 2.20. On the other side, if a bounded non-invertible operator $T : \mathcal{H} \rightarrow \mathcal{H}$ maps some ONB into a quasinormal basis. Then the quasinormal basis must be a conditional one otherwise $T$ must be invertible again by proposition 2.20. □

**Corollary 2.28.** A generating operator is a conditional operator; An invertible operator is a unconditional operator.

### 3. A Criterion for Operators to be Conditional

#### 3.1. Question:
Is $K = \text{diag}\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ a conditional operator?

From Olevskii’s result, we can not obtain the confirm answer. In this section, we will improve the Olevskii’s technology and gain a confirm answer.

First, let us recall some notations in the line of Olevskii.

Let $A_k = (a_{ij}) \in M_{2^k}(\mathbb{C})$ (where $1 \leq i, j \leq 2^k$) be defined as follows: $a_{i1} = 2^{-\frac{k}{2}}, 1 \leq i \leq 2^k$; and if $j = 2^s + v(1 \leq v \leq 2^s)$, then

$$a_{ij} = \begin{cases} 2^{\frac{j-k}{2}}, & (v-1)2^{k-s} < i \leq (2v-1)2^{k-s-1}, \\ -2^{\frac{j-k}{2}}, & (2v-1)2^{k-s-1} < i \leq v2^{k-s}. \end{cases}$$

For $\alpha, \frac{1}{\sqrt{2}} < \alpha < 1$, let $T_{(k, \alpha)} \in M_{2^k}(\mathbb{C})$ be defined as follows:

$$T_{(k, \alpha)} = \begin{bmatrix} \alpha^k & \alpha & \cdots & \alpha^{k-1} \\ \alpha^k & \alpha & \cdots & \alpha^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{k-1} & \cdots & \cdots & \alpha^{k-1} \end{bmatrix}.$$

For $\alpha, \frac{1}{\sqrt{2}} < \alpha < 1$, let $T_{(k, \alpha)} \in M_{2^k}(\mathbb{C})$ be defined as follows:

$$T_{(k, \alpha)} = \begin{bmatrix} \alpha^k & \alpha & \cdots & \alpha^{k-1} \\ \alpha^k & \alpha & \cdots & \alpha^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{k-1} & \cdots & \cdots & \alpha^{k-1} \end{bmatrix}.$$

For $\alpha, \frac{1}{\sqrt{2}} < \alpha < 1$, let $T_{(k, \alpha)} \in M_{2^k}(\mathbb{C})$ be defined as follows:

$$T_{(k, \alpha)} = \begin{bmatrix} \alpha^k & \alpha & \cdots & \alpha^{k-1} \\ \alpha^k & \alpha & \cdots & \alpha^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{k-1} & \cdots & \cdots & \alpha^{k-1} \end{bmatrix}.$$
In this section, we will show that if the positive operator $T$ does not admit the eigenvalue zero and $\sigma(T)$ has a decreasing sequence $\{\lambda_n, n = 1, 2, \ldots\}$ which converges to zero and
\[
\lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1,
\]
then $T$ must be a conditional operator. Thus the compact operator $K = \text{diag}\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is a conditional operator.

3.2. Now, we give a key lemma.

**Lemma 3.1.** Let $T$ be a diagonal operator with entries $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ under the ONB $\{e_k\}_{k=1}^\infty$, where $\lambda_n > 0$. Given $\alpha, \frac{1}{\sqrt{2}} < \alpha < 1$. If for each $k \geq 1$, there exist positive numbers $c_k \leq d_k$, such that

a) $\sup_k \frac{c_k}{d_k} < \infty$,

b) there exists subset $\Delta_k = \{n_1^k, n_2^k, \ldots, n_{2^k}^k\}$ of $\mathbb{N}$ such that $c_k \leq \frac{\alpha}{\lambda_{n_{2^k}-1}^k}, \lambda_{n_{2^k}}^k \leq d_k$, and $c_k \leq \frac{n_{j}^k}{\sqrt{2}} \leq d_k$ when $1 \leq j \leq k-1, 2^k(1 - \frac{1}{2^k}) + 1 \leq i \leq 2^k(1 - \frac{1}{2^k})$,

c) $\max \Delta_k < \min \Delta_{k'}$ when $k < k'$,

then $T$ is a conditional operator.

**Proof.** In this proof, we shall identify the operators and the $\omega \times \omega$ matrix representation of the operators under ONB $\{e_k\}_{k=1}^\infty$.

We rearrange $n_1^k, n_2^k, \ldots, n_{2^k}^k$ into a increasing sequence and denote it by $m_1^k, m_2^k, \ldots, m_{2^k+1}^k (m_1^k < m_2^k < \cdots < m_{2^k}^k)$. Let $m_0^k = 1$ and $\mathcal{H}_k = \text{span}\{e_{m_1^k}, e_{m_1^k+1}, \ldots, e_{m_{2^k+1}-1}\}$ for $k \geq 0$, then since $\max \Delta_k < \min \Delta_{k'}$ when $k < k'$, we know $\mathcal{H}_k \cap \mathcal{H}_{k'} = \{0\}$ when $k \neq k'$ and $\oplus_{k \geq 0} \mathcal{H}_k = \mathcal{H}$. Moreover, $\{\lambda_{m_1^k}, \lambda_{m_1^k+1}, \ldots, \lambda_{m_{2^k+1}-1}\}$ is a conditional operator.

Let $T_k \in \mathcal{L}(\mathcal{H}_k)$ the k-th block of $T$ on $\mathcal{H}_k$, i.e.

\[
T_k = \begin{bmatrix}
\lambda_{m_1^k} & \lambda_{m_1^k+1} & \cdots & e_{m_1^k} \\
\lambda_{m_1^k+1} & \cdots & \lambda_{m_2^k+1-1} & e_{m_1^k+1} \\
\cdots & \cdots & \cdots & \cdots \\
\lambda_{m_{2^k+1}-1} & e_{m_1^k+2^k-1} & \cdots & \cdots
\end{bmatrix}
\]

then $\bigoplus_{k \geq 0} T_k = T$. Denote $T_0 = T_0$. For $k \geq 1$, let

\[
\tilde{T}_k = \begin{bmatrix}
\lambda_{m_1^k} & \cdots & e_{m_1^k} \\
\lambda_{m_1^k+1} & \cdots & e_{m_1^k+1} \\
\cdots & \cdots & \cdots \\
\lambda_{m_{2^k+1}-1} & e_{m_1^k+2^k-1} & \cdots & \cdots
\end{bmatrix}
\]

where $\tilde{\mathcal{H}}_k = \bigvee \{e_{m_1^k+2^k}, \ldots, e_{m_{2^k+1}-1}\}$ and $S_k$ is a diagonal operator with entries $\{\lambda_{m_1^k}, \lambda_{m_1^k+1}, \ldots, \lambda_{m_{2^k+1}-1}\}\backslash (\lambda_{m_1^k}, \lambda_{m_1^k+1}, \ldots, \lambda_{m_{2^k+1}-1})$. It is easy to see that the entries of $\tilde{T}_k$ are just a rearrangement of entries of $T_k$ for $k \geq 1$.

We will prove $\tilde{T} \triangleq \bigoplus_{k \geq 0} \tilde{T}_k$ is a conditional operator and then show $T$ is a conditional operator.
Let $X_0 = I \in \mathcal{L}(\mathcal{H}_0)$. For $k \geq 1$, let

$$X_k = \begin{bmatrix}
\frac{\lambda_{n_k}^k}{\alpha^k} & \frac{\lambda_{n_k}^k}{\alpha^k} & \cdots & \frac{\lambda_{n_k}^k}{\alpha^k} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{\lambda_{n_k}^1}{\alpha^1} & \cdots & \frac{\lambda_{n_k}^1}{\alpha^1} & \frac{\lambda_{n_k}^k}{\alpha^k} \\
\end{bmatrix} \in \mathcal{L}(\mathcal{H}_k),
$$

since

$$\sup_k \max \left\{ \frac{\lambda_{n_k}^k}{\alpha^k}, \ldots, \frac{\lambda_{n_k}^1}{\alpha^1}, \frac{\alpha^1}{\lambda_{n_k}^1}, \ldots, \frac{\alpha^k}{\lambda_{n_k}^k} \right\} \leq \sup_k \max \left\{ \frac{d_k}{c_k} \right\} < \infty,$$

we have $X \triangleq \oplus_{k \geq 0} X_k$ is an invertible operator.

Moreover for $k \geq 1$,

$$\tilde{T}_k = X_k \cdot \begin{bmatrix} T_{(k,\alpha)}c_k^{-1} \\ S_k \end{bmatrix},$$

so

$$\tilde{T} = \oplus_{k \geq 0} \tilde{T}_k = X \cdot \oplus_{k \geq 0} \begin{bmatrix} T_{(k,\alpha)}c_k^{-1} \\ S_k \end{bmatrix},$$

where we denote $\begin{bmatrix} T_{(k,\alpha)}c_k^{-1} \\ S_k \end{bmatrix}$ by $\tilde{T}_0$ when $k = 0$.

Let

$$U = \oplus_{k \geq 0} \begin{bmatrix} A_k^* \\ I \end{bmatrix},$$

where we denote $\begin{bmatrix} A_k^* \\ I \end{bmatrix} = I \in \mathcal{L}(\mathcal{H}_0)$ when $k = 0$, then it is an unitary operator and

$$\tilde{T} U = X \cdot \oplus_{k \geq 0} \begin{bmatrix} T_{(k,\alpha)}c_k^{-1} \\ S_k \end{bmatrix} \cdot \oplus_{k \geq 0} \begin{bmatrix} A_k^* \\ I \end{bmatrix}$$

$$= X \cdot \oplus_{k \geq 0} \begin{bmatrix} T_{(k,\alpha)}A_k^*c_k^{-1} \\ S_k \end{bmatrix}$$

$$= X \cdot \oplus_{k \geq 0} \begin{bmatrix} T_{(k,\alpha)}A_k^* \\ S_k \end{bmatrix} \cdot \oplus_{k \geq 0} \begin{bmatrix} c_k^{-1}I \\ I \end{bmatrix}.$$

To show $\tilde{T}$ is conditional, from theorem 2.12 it suffices to show that

$$F \triangleq \oplus_{k \geq 0} \begin{bmatrix} T_{(k,\alpha)}A_k^* \\ S_k \end{bmatrix}$$

is a conditional matrix.

We will deal with it by theorem 2.3 and proposition 2.8. First, one can easily see that $F$ has an unique left inverse matrix

$$G^* = \oplus_{k \geq 0} \begin{bmatrix} A_kT_{(k,\alpha)}^{-1} \\ S_k^{-1} \end{bmatrix},$$

where each row is a $l^2$–sequence.
Second, $Q_n = FP_nG^*$ are obviously projections. Let
\[
A_1 = \{m_1^k, m_1^k + 1, \ldots, m_1^k + 2^k - 1; \ k \geq 1\} \subseteq \mathbb{N},
\]
\[
A_2 = \{m_1^k + 2^k, m_1^k + 2^k + 1, \ldots, m_1^{k+1} - 1; \ k \geq 1\} \subseteq \mathbb{N}.
\]
For any $x \in \mathcal{H}$, we have
\[
x = \sum_{j=1}^{\infty} x_j e_j = \sum_{j \in \Lambda_1} x_j e_j + \sum_{j \in \Lambda_2} x_j e_j,
\]
and
\[
FP_nG^*(x) = FP_nG^*(\sum_{j \in \Lambda_1} x_j e_j + \sum_{j \in \Lambda_2} x_j e_j) = (\oplus_{k \geq 0} T(k,\alpha) A_k^*) P_n^{(1)} (\oplus_{k \geq 0} A_k T_{(k,\alpha)}^{-1})(\sum_{j \in \Lambda_1} x_j e_j) + P_n^{(2)}(\sum_{j \in \Lambda_2} x_j e_j),
\]
where $\oplus_{k \geq 0} T(k,\alpha) A_k^*$ and $P_n^{(1)}$ are the operators on $\mathcal{H}^{(1)} = \bigvee_{j \in \Lambda_1} \{e_j\}$, $P_n^{(1)}$ converges to $I$ in the strong operator topology; $P_n^{(2)}$ is the operator on $\mathcal{H}^{(2)} = \bigvee_{j \in \Lambda_2} \{e_j\}$ and also converges to $I$ in the strong operator topology.

It follows from the result of Olevskii that $\oplus_{k \geq 0} T(k,\alpha) A_k^*$ is quasinormal conditional matrix. Then from theorem 2.8, we have
\[
\lim_{n \to \infty} (\oplus_{k \geq 0} T(k,\alpha) A_k^*) P_n^{(1)} (\oplus_{k \geq 0} A_k T_{(k,\alpha)}^{-1})(\sum_{j \in \Lambda_1} x_j e_j) = \sum_{j \in \Lambda_1} x_j e_j.
\]
Thus $FP_nG^*(x)$ converges to $x$ as $n \to \infty$ and $F$ is a Schauder matrix.

Moreover, since the unconditional basis const of $\oplus_{k \geq 0} T(k,\alpha) A_k$ is smaller than the unconditional basis const of $F$ and the unconditional basis const of $\oplus_{k \geq 0} T(k,\alpha) A_k$ is infinity, we have that the unconditional basis const of $F$ is infinity. Thus from proposition 2.8, we know that $F$ is a conditional matrix and $\overline{TU}$ is a conditional matrix.

Since the entries of $\overline{T}$ is just a rearrangement of $T$, one can easily find an unitary matrix (operator) $\overline{U}$ such that $\overline{UU}^* = T$, it follows that $\overline{U}^* \overline{T} \overline{U}$ is a conditional matrix. Again from theorem 2.12 $\overline{T} \overline{U}$ is a conditional matrix. Thus $T$ is a conditional operator, since it maps orthonormal basis $\{(\overline{U})e_1, \ldots, (\overline{U})e_n, \ldots\}$ into a conditional basis.

Now, we come to the main results.

**Theorem 3.2.** Let $T \in \mathcal{L} (\mathcal{H})$ which does not admit the eigenvalue zero. If there exists a constant $\delta > 1$ such that
\[
\lim_{t \to 0^+} \text{Card} \left( \left\{ \frac{t}{\delta} \mid \sigma(T) \right\} \right) = \infty,
\]
then $T$ is a conditional operator.

**Proof.** First step, we choose a sequence $\{\lambda_n\} \subseteq \sigma(T)$ satisfying the conditions of lemma 3.1. We will find it by induction.

For $k = 1$, $\Delta_1 = \{\lambda_1, \lambda_2\} \subseteq \sigma(T)$ and $c_1, d_1$ can be easily chosen such that
\[
\frac{d_1}{c_1} \leq \delta \text{ and } c_1 \leq \frac{\alpha}{\lambda_1}, \frac{\alpha}{\lambda_2} \leq d_1.
\]
Suppose we have found $\Delta_{k-1} = \{\lambda_{2^{k-1}-1}, \lambda_{2^{k-1}}, \lambda_{2^{k-1}+1}, \ldots, \lambda_{2^k-2}\} \subseteq \sigma(T)$ which satisfies
\[
\Delta_{k-1} \cap \bigcup_{1 \leq j \leq k-2} \Delta_j = \emptyset,
\]
and $c_{k-1}, d_{k-1}$ such that the first two conditions of lemma 3.1 are satisfied. Since
\[
\lim_{t \to 0^+} \text{Card}\{[\frac{t}{\delta}, t] \cap \sigma(T)\} = \infty,
\]
we can find $t_0 < \min\{\lambda; \lambda \in \bigcup_{1 \leq j \leq k-1} \Delta_j\}$ such that $t \leq t_0$,
\[
\text{Card}\{[\frac{t}{\delta}, t] \cap \sigma(T)\} \geq 2^k.
\]
Choose arbitrary two elements $\{\lambda_{2^k+1-3}, \lambda_{2^k+1-2}\} \subseteq \sigma(T) \cap [\frac{t_0\alpha^k}{\delta}, t_0\alpha^k]$, then choose one after one as follows,
\[
\{\lambda_{2k+1-5}, \lambda_{2k+1-4}\} \subseteq \{\sigma(T) \cap [\frac{t_0\alpha^k}{\delta}, t_0\alpha^k]\}\{\lambda_{2^k+1-3}, \lambda_{2^k+1-2}\}
\]
\[
\vdots
\]
\[
\{\lambda_{(2j-1)2^k-j+1-1}, \lambda_{(2j-1)2^k-j+1}, \lambda_{(2j-1)2^k-j+1+1}, \ldots, \lambda_{(2j+1-1)2^k-j-2}\} \subseteq \{\sigma(T) \cap [\frac{t_0\alpha^k}{\delta}, t_0\alpha^k]\}\{\lambda_{(2j+1-1)2^k-j-1}, \lambda_{(2j+1-1)2^k-j}, \lambda_{(2j+1-1)2^k-j+1}, \ldots, \lambda_{2^k+1-2}\}
\]
\[
\vdots
\]
\[
\{\lambda_{2^k-1}, \lambda_{2^k}, \ldots, \lambda_{3,2^k-1-2}\} \subseteq \{\sigma(T) \cap [\frac{t_0\alpha^k}{\delta}, t_0\alpha^k]\}\{\lambda_{3,2^k-1-1}, \lambda_{3,2^k-1}, \ldots, \lambda_{2^k+1-2}\}.
\]
Since $\text{Card}\{[\frac{t_0\alpha^j}{\delta}, t_0\alpha^j] \cap \sigma(T)\}$ is more than $2^k$, the above process is reasonable.
Denote $c_k = t_0^{-1}, d_k = \delta t_0^{-1}$, then obviously
\[
c_k \leq \frac{\alpha}{\lambda_{2^k-1}}, \ldots, \frac{\alpha}{\lambda_{3,2^k-1-1}}, \frac{\alpha^2}{\lambda_{3,2^k-1-2}}, \ldots, \frac{\alpha^2}{\lambda_{7,2^k-2-2}},
\]
\[
\ldots, \frac{\alpha^{k-1}}{\lambda_{2^k+1-5}}, \frac{\alpha^{k-1}}{\lambda_{2^k+1-4}}, \frac{\alpha^{k}}{\lambda_{2^k+1-3}}, \frac{\alpha^{k}}{\lambda_{2^k+1-2}} \leq d_k.
\]
Thus we have found a sequence $\{\lambda_n\} \subseteq \sigma(T)$ satisfying the conditions of lemma 3.1. Obviously, $\lambda_n$ converges to zero as $n \to \infty$.

Second step, we will complete the proof.
We rearrange the sequence $\{\lambda_n\} \subseteq \sigma(T)$ into a decreasing sequence $\{\mu_n\}$. Fix a constant $M > \|T\|$. For $n \geq 1$, cut each segment $[\mu_{n+1}, \mu_n]$ into smaller subsegments (many enough and we denote them by $[\nu_{m_{n+1}^j}, \nu_{m_n^j}], 1 \leq j \leq k(n) - 1, \nu_{m_n^1} = \mu_n, \nu_{m_n^k} = \mu_{n+1}$) in order that
\[
\frac{\nu_{m_n^j}}{\nu_{m_{n+1}^j}} \leq M, 1 \leq j \leq k(n) - 1, n = 1, 2, \ldots.
\]
From the spectral decompose theorem of self-adjoint operator, we have
\[
T = \bigoplus_{n \geq 0} \bigoplus_{1 \leq j \leq k(n) - 1} T_{(n,j)},
\]
where $T_{n,j}$ is the operator on the subspace $\mathcal{H}_{(n,j)}$ corresponding to $[\nu_{m_{j}+1}^{m_{n}}, \nu_{m_{n}}^{*}] \cap \sigma(T)$ for $n \geq 1$ and $T_{(0)}$ is the operator on the subspace $\mathcal{H}_{(0)}$ corresponding to $[\mu_{1}, \infty) \cap \sigma(T)$.

Denote

$$X = \oplus_{n \geq 0} \oplus_{1 \leq j \leq k(n)-1} \xi_{(n,j)}^{-1} T_{(n,j)},$$

where $\xi_{(0)} = \mu_{1}$, $\xi_{(n,j)} \in [\nu_{m_{j}+1}^{m_{n}}, \nu_{m_{n}}^{*}] \cap \sigma(T)$ and $\xi_{(n,1)} = \mu_{n}$. Then since

$$||\xi_{(n,j)}^{-1} T_{(n,j)}|| \leq M \text{ and } ||(\xi_{(n,j)}^{-1} T_{(n,j)})^{-1}|| \leq M, \ 1 \leq j \leq k(n)-1, \ n \geq 0,$$

we have $X$ is an invertible operator. Moreover,

$$S \triangleq \oplus_{n \geq 0} \oplus_{1 \leq j \leq k(n)-1} \xi_{(n,j)} T_{(n,j)} = X^{-1} T,$$

where $I_{(n,j)}$ is the identity operator on $\mathcal{H}_{(n,j)}$. Obviously, $S$ is a diagonal operator with $\{\lambda_{n}\}$ its subsequence. Thus $S$ satisfies the conditions of lemma 3.1 and hence it is a conditional operator. From theorem 2.12 we obtain that $T$ is a conditional operator.

Following is an easier criterion for an operator to be conditional.

**Theorem 3.3.** Let $T \geq 0$ belong to $\mathcal{L}(\mathcal{H})$ which does not admit the eigenvalue zero. If $\sigma(T)$ has a decreasing sequence $\{\lambda_{n}\}$ which converges to zero such that

$$\lim_{n \to \infty} \frac{\lambda_{n}}{\lambda_{n+1}} = 1,$$

then $T$ is a conditional operator.

**Proof.** It suffices to show that there exists a constant $\delta > 1$ such that

$$\lim_{\delta \to 0^{+}} \text{Card}\{|\frac{t}{\delta}, t| \cap \{\lambda_{n}, n \geq 1\}\} = \infty.$$

If not, then there exists $N > 0$, such that for any $t_{0} > 0$, there is a $t \leq t_{0}$,

$$\text{Card}\{|\frac{t}{\delta}, t| \cap \{\lambda_{n}, n \geq 1\}\} < N.$$

Thus there exist sequences $a_{k}, b_{k}$ converge to zero, such that for all $k$

$$\frac{b_{k}}{a_{k}} = \delta, \text{Card}\{[a_{k}, b_{k}] \cap \{\lambda_{n}, n \geq 1\}\} < N,$$

$$b_{k+1} < a_{k}, \text{Card}\{[b_{k+1}, a_{k}] \cap \{\lambda_{n}, n \geq 1\}\} \geq 1.$$

Choose $\lambda_{n_{1}}$ such that $\lambda_{n_{1}} = \min\{\lambda_{n}; \lambda_{n} \geq b_{1}\}$, choose $\lambda_{n_{2}}$ such that $\lambda_{n_{2}} = \max\{\lambda_{n}; \lambda_{n} \leq a_{1}\}$. Generally, choose $\lambda_{n_{2k-1}} = \min\{\lambda_{n}; \lambda_{n} \geq b_{k}\}$ and $\lambda_{n_{2k}} = \max\{\lambda_{n}; \lambda_{n} \leq a_{k}\}$. It is easy to see that $n_{2k} - n_{2k-1} \leq N$.

On the other hand, since

$$\lim_{n \to \infty} \frac{\lambda_{n}}{\lambda_{n+1}} = 1,$$

we have

$$\lim_{n \to \infty} \frac{\lambda_{n}}{\lambda_{n+j}} = 1,$$

for any $1 \leq j \leq N$ and hence

$$\lim_{k \to \infty} \frac{\lambda_{n_{2k-1}}}{\lambda_{n_{2k}}} = 1.$$
But
\[ \frac{\lambda_{n2k-1}}{\lambda_{n2k}} \geq \frac{b_k}{a_k} = \delta > 1 \]
for any \( k \), it is a contradiction.
Thus \( T \) is a conditional operator. \( \square \)

Remark 3.4. Actually, suppose the limit of \( \frac{\lambda_n}{\lambda_{n+1}} \) exists, then
\[ \lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1 \]
if and only if there exists a constant \( \delta > 1 \) such that
\[ \lim_{t \to 0} \text{Card} \{ \frac{t}{\delta}, t \} \cap \{ \lambda_n, n \geq 1 \} = \infty. \]
One can easily prove it. Thus the condition of theorem \( 3.3 \) is a little stronger than theorem \( 3.2 \).

Corollary 3.5. Let \( T \in \mathcal{L}(\mathcal{H}) \) such that \( T \) and \( T^* \) do not admit the eigenvalue zero. If \( \sigma((T^*T)^{\frac{1}{2}}) \) has a decreasing sequence \( \lambda_n \) which converges to zero such that
\[ \limsup_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1, \]
then \( T \) is a conditional operator.

Proof. From the polar decomposition theorem,
\[ T = U(T^*T)^{\frac{1}{2}}, \]
where \( U \) is a unitary operator. Thus from theorem \( 3.3 \) and theorem \( 2.12 \) we obtain the result. \( \square \)

Corollary 3.6. Compact operator \( K = \text{diag}\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \) is a conditional operator.

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