On a Problem of Hajdu and Tengely

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Abstract. We prove a result that finishes the study of primitive arithmetic progressions consisting of squares and fifth powers that was carried out by Hajdu and Tengely in a recent paper: The only arithmetic progression in coprime integers of the form \((a^2, b^2, c^5, d^5)\) is \((1,1,1,1)\). For the proof, we first reduce the problem to that of determining the sets of rational points on three specific hyperelliptic curves of genus 4. A 2-cover descent computation shows that there are no rational points on two of these curves. We find generators for a subgroup of finite index of the Mordell-Weil group of the last curve. Applying Chabauty’s method, we prove that the only rational points on this curve are the obvious ones.

1 Introduction

Euler ([9, pages 440 and 635]) proved Fermat’s claim that four distinct squares cannot form an arithmetic progression. Powers in arithmetic progressions are still a subject of current interest. For example, Darmon and Merel [8] proved that the only solutions in coprime integers to the Diophantine equation \(x^n + y^n = 2z^n\) with \(n \geq 3\) satisfy \(xyz = 0\) or \(\pm 1\). This shows that there are no non-trivial three term arithmetic progressions consisting of \(n\)-th powers with \(n \geq 3\). The result of Darmon and Merel is far from elementary; it needs all the tools used in Wiles’ proof of Fermat’s Last Theorem and more.

An arithmetic progression \((x_1, x_2, \ldots, x_k)\) of integers is said to be primitive if the terms are coprime, i.e., if \(\gcd(x_1, x_2) = 1\). Let \(S\) be a finite subset of integers \(\geq 2\). Hajdu [11] showed that if

\[(a_1^{\ell_1}, \ldots, a_k^{\ell_k})\]  \hspace{1cm} (1)

is a non-constant primitive arithmetic progression with \(\ell_i \in S\), then \(k\) is bounded by some (inexplicit) constant \(C(S)\). Bruin, Győry, Hajdu and Tengely [2] showed that for any \(k \geq 4\) and any \(S\), there are only finitely many primitive arithmetic progressions of the form (1), with \(\ell_i \in S\). Moreover, for \(S = \{2, 3\}\) and \(k \geq 4\), they showed that \(a_i = \pm 1\) for \(i = 1, \ldots, k\).

A recent paper of Hajdu and Tengely [12] studies primitive arithmetic progressions (1) with exponents belonging to \(S = \{2, n\}\) and \(\{3, n\}\). In particular, they...
show that any primitive non-constant arithmetic progression \((\ell_i)\) with exponents \(\ell_i \in \{2, 5\}\) has \(k \leq 4\). Moreover, for \(k = 4\) they show that
\[
(\ell_1, \ell_2, \ell_3, \ell_4) = (2, 2, 2, 5) \quad \text{or} \quad (5, 2, 2, 2). \tag{2}
\]
Note that if \((a_{\ell_i} : i = 1, \ldots, k)\) is an arithmetic progression, then so is the reverse progression \((a_{\ell_i} : i = k, k-1, \ldots, 1)\). Thus there is really only one case left open by Hajdu and Tengely, with exponents \((\ell_1, \ell_2, \ell_3, \ell_4) = (2, 2, 2, 5)\). This is also mentioned as Problem 11 in a list of 22 open problems recently compiled by Evertse and Tijdeman [10]. In this paper we deal with this case.

**Theorem 1.** The only arithmetic progression in coprime integers of the form
\[
(a^2, b^2, c^2, d^5)
\]
is \((1, 1, 1, 1)\).

This together with the above-mentioned results of Hajdu and Tengely completes the proof of the following theorem.

**Theorem 2.** There are no non-constant primitive arithmetic progressions of the form \((\ell_i)\) with \(\ell_i \in \{2, 5\}\) and \(k \geq 4\).

The primitivity condition is crucial, since otherwise solutions abound. Let for example \((a^2, b^2, c^2, d)\) be any arithmetic progression whose first three terms are squares — there are infinitely many of these; one can take \(a = r^2 - 2rs - s^2\), \(b = r^2 + s^2\), \(c = r^2 + 2rs - s^2\) — then \(((ad^2)^2, (bd^2)^2, (cd^2)^2, d^5)\) is an arithmetic progression whose first three terms are squares and whose last term is a fifth power.

For the proof of Thm. 1 we first reduce the problem to that of determining the sets of rational points on three specific hyperelliptic curves of genus 4. A 2-cover descent computation (following Bruin and Stoll [3]) shows that there are no rational points on two of these curves. We find generators for a subgroup of finite index of the Mordell-Weil group of the last curve. Applying Chabauty’s method, we prove that the only rational points on this curve are the obvious ones. All our computations are performed using the computer package MAGMA [1].

The result we prove here may perhaps not be of compelling interest in itself. Rather, the purpose of this paper is to demonstrate how we can solve problems of this kind with the available machinery. We review the relevant part of this machinery in Sect. 3 after we have constructed the curves pertaining to our problem in Sect. 2. Then, in Sect. 4 we apply the machinery to these curves. The proofs are mostly computational. We have tried to make it clear what steps need to be done, and to give enough information to make it possible to reproduce the computations (which have been performed independently by both authors as a consistency check).