A New Hilbert-Type Inequality with Positive Homogeneous Kernel and Its Equivalent Forms

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Abstract: We establish a new inequality of Hilbert-type containing positive homogeneous kernel \((\min\{m,n\})^t\) and derive its equivalent forms. Based on the obtained Hilbert-type inequality, we discuss its equivalent forms and give the operator expressions in some particular cases.

Keywords: Hilbert-type inequality; homogeneous kernel; equivalent statement; operator expression; Euler–Maclaurin summation formula

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1. Introduction

If \(a_m \geq 0, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty\) and \(0 < \sum_{n=1}^{\infty} b_n^q < \infty\), then

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left( \sum_{m=1}^{\infty} a_m^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}},
\]

(1)

where the constant factor \(\pi\) is the best possible. Inequality (1) is the celebrated Hilbert’s inequality (see [1]). Inequality (1) was generalized by Hardy as follows:

If \(p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty\) and \(0 < \sum_{n=1}^{\infty} b_n^q < \infty\), then

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left( \sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}},
\]

(2)

where the constant factor \(\frac{\pi}{\sin(\pi/p)}\) is the best possible. Inequality (2) is called Hardy–Hilbert’s inequality (c.f. [1], Theorem 315).

The following analogue of Hardy–Hilbert’s inequality

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < pq \left( \sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}
\]

(3)

is known in the literature as Hardy–Littlewood–Polya’s inequality, and the constant factor \(pq\) in (3) is the best possible (c.f. [1], Theorem 341).
In 2006, Krnić and Pečarić [2] presented an extension of inequality (1) by introducing parameters \( \lambda_1 \) and \( \lambda_2 \) as follows:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_mb_n}{(m+n)^{\lambda}} < B(\lambda_1, \lambda_2) \left[ \sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q \right]^{\frac{1}{q}}
\]

where \( \lambda_i \in (0,2] \ (i = 1, 2) \), \( \lambda_1 + \lambda_2 = \lambda \in (0, 4] \),

\[
B(u,v) = \int_0^\infty \frac{t^{u-1}}{(1 + t)^{u+v}} dt \ (u,v > 0)
\]

is the beta function, in (4) the constant factor \( B(\lambda_1, \lambda_2) \) is the best possible.

For \( \lambda = 1 \), \( \lambda_1 = \frac{1}{p} \), \( \lambda_2 = \frac{1}{q} \), inequality (4) reduces to inequality (2); for \( p = q = 2 \), \( \lambda_1 = \lambda_2 = \frac{1}{2} \), inequality (4) reduces to Yang’s inequality given in [3]. It is well known that inequalities (1–3) and their integral analogues play an important role in analysis and its applications (see [4–14]).

Recently, by applying inequality (3), Adiyasuren, Batbold and Azar [15] gave a new Hilbert-type inequality with the kernel \( \frac{1}{(m+n)^{r}} \) and partial sums.

In 2016, Hong and Wen [16] studied the equivalent statements of the extended inequalities (1) and (2), and estimated the best possible constant factor for several parameters.

The results proposed in [2,15,16] have greatly attracted our interest. In 2019, Yang, Wu and Wang [17] established the following Hardy–Hilbert-type inequality and discussed its equivalent forms:

\[
\int_0^\infty \left[ \sum_{n=1}^{\infty} \frac{f(n)a_n}{(n+x)^{\lambda}} \right] dx > B^\frac{1}{p}(\sigma, \lambda - \sigma)B^\frac{1}{q}(\mu, \lambda - \mu)
\]

\[
\times \left\{ \int_0^\infty (1 - \rho_{\sigma}(x))x^{\frac{1}{p}(1-(\frac{\lambda}{p} + \frac{\mu}{q})) - 1} f^{\frac{1}{p}}(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(\frac{1}{q} - \frac{\lambda}{p} - \frac{\mu}{q} - 1)} a_n \right\}^{\frac{1}{q}},
\]

where \( 0 < p < 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( \lambda \in (0,5] \), \( \sigma \in (0,2] \cap (0,\lambda) \), \( \mu \in (0,\lambda) \),

\[
\rho_{\sigma}(x) = \frac{(1 + \theta_x)^{-\lambda}}{\sigma B(\sigma, \lambda - \sigma) x^\sigma} = O\left(\frac{1}{x^{\sigma}}\right) \in (0, 1) \ (\theta_x \in (0, \frac{1}{x}) ; x > 0),
\]

\[
f(x) \geq 0, \ x \in (0, \infty), \ a_n \geq 0
\]

In a recent paper [18], Yang, Wu and Liao gave an extension of Hardy–Hilbert’s inequality for \( k_L(\lambda_i) = \frac{n}{\lambda \sin(n \pi / L)} \ (i = 1, 2) \), as follows:

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_mb_n}{m^n + n^\lambda} < k^\frac{1}{p}(\lambda_2)k^\frac{1}{q}(\lambda_1)
\]

\[
\times \left\{ \sum_{m=1}^{\infty} m^{p(1-(\frac{1}{p} + \frac{1}{q})) - 1} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-(\frac{1}{p} + \frac{1}{q})) - 1} b_n^q \right\}^{\frac{1}{q}},
\]

where \( p > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( \lambda \in (0, \frac{5}{2}] \), \( \lambda_i \in (0, \frac{5}{2}) \cap (0,\lambda) \) \((i = 1, 2) \ a_m, b_n \geq 0 \).

For more results related to the extensions of inequalities (1) and (2) and their equivalent statements, we refer the reader to [19–24] and references cited therein.

Motivated by the ideas of [2] and [16], in the present paper we deal with a new Hilbert-type inequality containing positive homogeneous kernel \( \min(m,n)^{\lambda} \) and deduce its equivalent forms. Furthermore, we discuss the equivalent statements relating to the best possible constant factor, based on the obtained Hilbert-type inequality.
2. Some Lemmas

In what follows, we suppose that \( p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \lambda \in (0, \frac{34}{11} [\lambda_i \in (0, \frac{11}{8}) \cap (0, \lambda) \ (i = 1, 2), \ a_m, b_n \geq 0 \ (m, n \in \mathbb{N} = \{1, 2, \ldots \}) \) such that

\[
0 < \sum_{m=1}^{\infty} m^{(1 + \frac{1}{p} + \frac{\lambda - 1}{p}) - 1} a_m < \infty \quad \text{and} \quad 0 < \sum_{n=1}^{\infty} n^{q(1 + \frac{1}{p} + \frac{\lambda - 1}{p}) - 1} b_n < \infty.
\]

**Lemma 1.** Define the weight coefficient with positive homogeneous kernel

\[
\omega_{\lambda}(\lambda_1, m) := \frac{1}{m!} \sum_{n=1}^{\infty} \frac{(\min|m, n|)^{\lambda}}{m^{\lambda - \lambda_1 + 1}} \quad (m \in \mathbb{N})
\]

Then, we have the following inequalities

\[
k_1(1 - \frac{\lambda - \lambda_1}{\lambda m^{\lambda_1}}) < \omega_{\lambda}(\lambda_1, m) < k_1(1 - \frac{\lambda - \lambda_1}{\lambda m^{\lambda_1}}) \quad (m \in \mathbb{N}) \quad (7)
\]

**Proof.** For fixed \( m \in \mathbb{N}, \) we define a function \( g_m(t) := \frac{(\min|m, t|)^{\lambda}}{t^{\lambda - \lambda_1 + 1}} \ (t > 0), \) and obtain

\[
g_m(t) = \begin{cases}
\frac{1}{\lambda_1^{\lambda_1 - 1}}, & 0 < t < m, \\
\frac{1}{m^{\lambda_1}} t^{\lambda_1 - 1}, & t \geq m
\end{cases}, \quad g_m'(t) = \begin{cases}
(\lambda_1 - 1) t^{\lambda_1 - 2}, & 0 < t < m, \\
-(\lambda - \lambda_1 + 1) \frac{m^{\lambda_1}}{t^{\lambda_1 - 2}}, & t > m
\end{cases}
\]

\( g_m(1) = 1, \) and \( \int_0^1 g_m(t) \, dt = \frac{1}{\lambda_1}. \)

To prove the inequalities in (7), we consider two cases below:

(i) For \( \lambda_1 \in (0, 1] \cap (0, \lambda), \) it is easy to observe that \( g_m(t) \) is decreasing in \((0, \infty), \) and strictly decreasing in \([m, \infty). \) By following the decreasing property of the series, we find

\[
\omega_{\lambda}(\lambda_1, m) < \frac{1}{m!} \int_0^{\infty} \frac{(\min|m, t|)^{\lambda}}{t^{\lambda - \lambda_1 + 1}} \, dt = \frac{1}{m!} \int_0^{m} \frac{t^{\lambda_1 - 1}}{t^{\lambda - \lambda_1 + 1}} + \int_m^{\infty} m^{\lambda_1} \frac{t^{\lambda_1 - 1}}{t^{\lambda - \lambda_1 + 1}} = k_1(\lambda_1),
\]

\[
\omega_{\lambda}(\lambda_1, m) > \frac{1}{m!} \int_0^{\infty} \frac{(\min|m, t|)^{\lambda}}{t^{\lambda - \lambda_1 + 1}} \, dt = \frac{1}{m!} \int_0^{m} \frac{(\min|m, t|)^{\lambda}}{t^{\lambda - \lambda_1 + 1}} - \frac{1}{m!} \int_0^{m} \frac{t^{\lambda_1 - 1}}{t^{\lambda - \lambda_1 + 1}} = k_1(\lambda_1) - \frac{1}{\lambda m^{\lambda_1}} = k_1(\lambda_1)(1 - \frac{\lambda - \lambda_1}{\lambda m^{\lambda_1}}),
\]

which implies the required inequalities in (7).

(ii) For \( \lambda_1 \in (1, \frac{1}{2}] \cap (0, \lambda), \) by using the Euler–Maclaurin summation formula (c.f. [2,3]) with the Bernoulli function of 1-order \( \rho(t) := t - \lfloor t \rfloor - \frac{t}{2}, \) we obtain

\[
\sum_{n=2}^{m} g_m(n) = \int_1^{m} g_m(t) \, dt + \frac{1}{2} g_m(t)_1^{\infty} + \int_1^{m} \rho(t) g'_m(t) \, dt
\]

\[
= \int_1^{m} g_m(t) \, dt + \frac{1}{2} g_m(t)_1^{\infty} + (\lambda_1 - 1) \int_1^{m} \rho(t) t^{\lambda_1 - 2} \, dt
\]

\[
= \int_1^{m} g_m(t) \, dt + \frac{1}{2} g_m(t)_1^{\infty} + (\lambda_1 - 1) \left( \frac{1}{12} t_1^{\lambda_1 - 2} \right)
\]

\[
\leq \int_1^{m} g_m(t) \, dt + \frac{1}{2} g_m(t)_1^{\infty} (1 < \lambda_1 < 2, 0 < \varepsilon_1 < 1),
\]

\[
\sum_{n=m+1}^{\infty} g_m(n) = \int_m^{\infty} g_m(t) \, dt + \frac{1}{2} g_m(t)_m^{\infty} + \int_m^{\infty} \rho(t) g'_m(t) \, dt
\]

\[
= \int_m^{\infty} g_m(t) \, dt + \frac{1}{2} g_m(t)_m^{\infty} + \frac{\lambda_1 - \lambda_1 - 1}{12} m^{\lambda_1 - 2} \epsilon_1 t_1^{\lambda_1 - 2} m^{\lambda_1 - 2}
\]

\[
< \int_m^{\infty} g_m(t) \, dt + \frac{1}{2} g_m(t)_m^{\infty} + \frac{\lambda_1 - \lambda_1 + 1}{12 m^{\lambda_1 - 1}} (\lambda_1 < \lambda, 0 < \varepsilon_1 < 1),
\]
and then one has
\[
\sum_{n=1}^{\infty} g_m(n) < \int_0^{\infty} g_m(t) dt + \frac{1}{2} g_m(1) + \frac{(\lambda-\lambda_1+1)}{12m^{2-\lambda_1}}
\]
\[
= \int_0^{\infty} g_m(t) dt - h_m(\lambda, \lambda_1),
\]
where \(h(\lambda_1) := 12 - (7 + \lambda) \lambda_1 + \lambda_1^2\) and
\[
h_m(\lambda, \lambda_1) := \int_0^{1} g_m(t) dt - \frac{1}{2} g_m(1) - \frac{(\lambda-\lambda_1+1)}{12m^{2-\lambda_1}} - \lambda_1 - \frac{1}{2} - \frac{(\lambda-\lambda_1+1)}{12}
\]
\[
= \frac{h(\lambda_1)}{12 \lambda_1}.
\]
Since \(h'(\lambda_1) = -(7 + \lambda) + 2 \lambda_1 < 0\) \((\lambda_1 \in (1, \frac{11}{8}], \lambda \in (0, \frac{34}{11}])\), it follows that
\[
h_m(\lambda, \lambda_1) \geq \frac{12 - (7 + \lambda) \times (\frac{11}{8}) + (\frac{11}{8})^2}{12 \lambda_1} \geq \frac{273 - 88 \lambda}{768 \lambda_1} > 0 \hspace{1em} (\lambda \in (0, \frac{34}{11}])
\]
Thus, we get
\[
\omega_{\lambda}(\lambda_1, m) = \frac{1}{m^{\lambda_1}} \sum_{n=1}^{\infty} g_m(n) < \frac{1}{m^{\lambda_1}} \int_0^{\infty} g_m(t) dt = k_\lambda(\lambda_1) = \frac{\lambda}{\lambda_1(\lambda-\lambda_1)}.
\]
On the other hand, we have
\[
\sum_{n=2}^{m} g_m(n) = \int_0^{m} g_m(t) dt + \frac{1}{2} g_m(1) \frac{(\lambda_1-1)}{12m^{2-\lambda_1}} + \frac{(\lambda_1-1)}{12m^{2-\lambda_1}} (m^{\lambda_1-2} - 1),
\]
\[
\sum_{n=m+1}^{\infty} g_m(n) = \int_0^{\infty} g_m(t) dt + \frac{1}{2} g_m(1) \frac{(\lambda_1-1)}{12} + \frac{(\lambda_1-1)}{12} (m^{\lambda_1-2} - 1)
\]
\[
< \int_0^{\infty} g_m(t) dt + \frac{1}{2} g_m(1) \frac{(\lambda_1-1)}{12} m^{\lambda_1-2}
\]
and then by \(\frac{1}{2} - \frac{\lambda_1-1}{12} > \frac{1}{2} - \frac{1}{12} > 0\) \((\lambda_1 < 2)\), we find
\[
\sum_{n=1}^{\infty} g_m(n) \geq \frac{1}{2} g_m(1) + \frac{(\lambda_1-1)}{12} (m^{\lambda_1-2} - 1)
\]
\[
> \frac{1}{2} g_m(1) + \frac{(\lambda_1-1)}{12} \int_0^{\infty} g_m(t) dt - \int_0^{1} g_m(t) dt.
\]
Hence, from the expression \(g_m(t)\) we deduce the inequalities in (7). The proof of Lemma 1 is thus complete. \(\Box\)

Next, we shall establish a new inequality of Hilbert type for positive homogeneous kernel.

**Lemma 2.** The following Hilbert-type inequality holds true:
\[
I = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min(m, n))^{\lambda} a_m b_n < k^2_{\lambda} (\lambda_1) k^2_{\lambda} (\lambda_2)
\]
\[
\times \left[ \sum_{n=1}^{\infty} n^{p(1+\frac{\lambda_1}{\lambda} + \frac{\lambda_1-1}{\lambda})-1} d_m \right]^{\frac{1}{p}} \int \left[ \sum_{n=1}^{\infty} n^{p(1+\frac{\lambda_2}{\lambda} + \frac{\lambda_2-1}{\lambda})-1} d_n \right]^{\frac{1}{p}}
\]
\(\lambda_1, \lambda_2 \in (1, \frac{11}{8}], \lambda \in (0, \frac{34}{11}]\).
\(\frac{34}{11}\).
Proof. Following the pattern in which the proof of Lemma 1 was obtained, for \( n \in \mathbb{N}, \lambda \in (0, \frac{11}{4}], \lambda_2 \in (0, \frac{11}{8}] \cap (0, \lambda) \), we have the following inequality:

\[
k_1(\lambda_2) \left( 1 - \frac{\lambda - \lambda_2}{\lambda \lambda_2} \right) \leq \omega(\lambda_2, n) := \frac{1}{n^{\lambda_2}} \sum_{m=1}^{\infty} \frac{(\min(m, n))^{\lambda}}{m^{1-\lambda_2+1}} < k_1(\lambda_2).
\]

Using the Hölder’s inequality (see [25]), we obtain

\[
I = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\min(m, n))^\lambda \frac{m^{(\lambda-\lambda_2+1)/q}}{n^{(\lambda-\lambda_1+1)/p}} a_m \frac{n^{(\lambda-\lambda_1+1)/p}}{m^{(\lambda-\lambda_2+1)/q}} b_n
\]

\[
\leq \left[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\min(m, n))^\lambda n^{(\lambda-\lambda_2+1)(q-1)} a_m \right] \left[ \sum_{n=1}^{\infty} \omega(\lambda_2, n) n^{(1+\frac{1}{q})-1} b_n \right]^{\frac{1}{q}}.
\]

Hence, by using the inequalities in (7) and (9), we derive inequality (8). This completes the proof of Lemma 2. \( \square \)

As a consequence of Lemma 2, we can deduce the following Hilbert-type inequality for the positive homogeneous kernel.

Remark 1. By inequality (8), for \( \lambda_1 + \lambda_2 = \lambda \in (0, \frac{11}{4}] (\subset (0, \frac{13}{8})), \lambda_i \in (0, \frac{11}{8}] \cap (0, \lambda) \) \((i = 1, 2)\) we obtain

\[
0 < \sum_{m=1}^{\infty} m^{p(1+\lambda_1)-1} a_m^p < \infty, \quad 0 < \sum_{n=1}^{\infty} n^{q(1+\lambda_2)-1} b_n^q < \infty
\]

and the following inequality:

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min(m, n))^\lambda a_m b_n < \frac{\lambda}{\lambda_1 \lambda_2} \left[ \sum_{m=1}^{\infty} m^{p(1+\lambda_1)-1} a_m^p \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{q(1+\lambda_2)-1} b_n^q \right]^{\frac{1}{q}}.
\]

In Lemma 3 below, we show that the constant factor given in (10) is the best possible.

Lemma 3. For \( \lambda_1 + \lambda_2 = \lambda \in (0, \frac{11}{4}], \lambda_i \subset (0, \frac{11}{8}] \cap (0, \lambda) \) \((i = 1, 2)\), the constant factor \( \frac{\lambda}{\lambda_1 \lambda_2} \) in (10) is the best possible.

Proof. For any \( 0 < \varepsilon < q \lambda_1 \), we set

\[
\overline{a}_m := m^{-\lambda_1 - \frac{\varepsilon}{q}-1}, \quad \overline{b}_n := n^{-\lambda_2 - \frac{\varepsilon}{q}-1} \quad (m, n \in \mathbb{N})
\]

If there exists a constant \( M \leq \frac{\lambda}{\lambda_1 \lambda_2} \) such that (10) is valid when replacing \( \frac{\lambda}{\lambda_1 \lambda_2} \) by \( M \), then in particular, by substitution of \( a_m = \overline{a}_m \) and \( b_n = \overline{b}_n \) in (10), we have

\[
\overline{I} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min(m, n))^\lambda \overline{a}_m \overline{b}_n < M \left[ \sum_{m=1}^{\infty} m^{p(1+\lambda_1)-1} \overline{a}_m^p \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{q(1+\lambda_2)-1} \overline{b}_n^q \right]^{\frac{1}{q}}.
\]
In the following, we shall prove that $M \geq \frac{1}{\lambda_1 \lambda_2}$, which would reveal that $M = \frac{1}{\lambda_1 \lambda_2}$ is the best possible constant factor in (10).

By inequality (11) and the decreasing property of the series, we obtain

$$\widetilde{I} < M \left( \sum_{m=1}^{\infty} m^p(1+\lambda_1)^{-1} m^{-p\lambda_1 - \varepsilon - p} \right)^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^q(1+\lambda_2)^{-1} n^{-q\lambda_2 - \varepsilon - q} \right]^{\frac{1}{q}}$$

$$= M \left( 1 + \sum_{m=2}^{\infty} m^{-\varepsilon - 1} \right)^{\frac{1}{p}} \left( 1 + \sum_{n=2}^{\infty} n^{-\varepsilon - 1} \right)^{\frac{1}{q}}$$

$$< M \left( 1 + \int_1^{\infty} x^{-\varepsilon - 1} dx \right)^{\frac{1}{p}} \left( 1 + \int_1^{\infty} y^{-\varepsilon - 1} dy \right)^{\frac{1}{q}} = M (\varepsilon + 1).$$

By inequalities in (9) and setting

$$\lambda_1 = \lambda_1 - \frac{\varepsilon}{q} \in (0, \frac{11}{8}) \cap (0, \Lambda) \quad (0 < \lambda_2 = \lambda_2 + \frac{\varepsilon}{q} = \lambda - \lambda_1 < \lambda),$$

we obtain

$$\widetilde{I} = \sum_{m=1}^{\infty} \left[ m^p(1+\lambda_1 - \frac{\varepsilon}{q} \frac{\varepsilon}{q}) \sum_{n=1}^{\infty} (\min(m, n))^{\lambda_1 - \frac{\varepsilon}{q}} n^{-\lambda_2 + \frac{\varepsilon}{q} - 1} \right] m^{-\varepsilon - 1}$$

$$= \frac{1}{\lambda_1 \lambda_2} \sum_{m=1}^{\infty} \frac{1}{\lambda_1} \sum_{n=1}^{\infty} \frac{1}{m^{1+\frac{\varepsilon}{q}+1}} \left( \frac{1}{\lambda_1} \right)^{\lambda_1} \left( \frac{1}{\lambda_2} \right)^{\lambda_2} \left( \int_1^{\infty} x^{-\varepsilon - 1} dx - O(1) \right)$$

$$= \frac{1}{\varepsilon \lambda_1 \lambda_2} (1 - \varepsilon O(1)).$$

Then, we have

$$\frac{\lambda}{(\lambda_1 - \frac{\varepsilon}{q})(\lambda_2 + \frac{\varepsilon}{q})} (1 - \varepsilon O(1)) < M (\varepsilon + 1).$$

Taking $\varepsilon \to 0^+$, we deduce that $\frac{1}{\lambda_1 \lambda_2} \leq M$. Hence, $M = \frac{1}{\lambda_1 \lambda_2}$ is the best possible constant factor in (10). Lemma 3 is thus proven. \( \Box \)

Setting $\lambda_0 = \frac{\lambda}{p} + \frac{\lambda - \lambda_2}{q}$, $\lambda_2 := \frac{\lambda_2}{\lambda} + \frac{\lambda - \lambda_1}{p}$, we find $\lambda_0 + \lambda_2 = \lambda$, and then we can reduce inequality (8) to the following:

$$I = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min(m, n))^{\lambda_0} a_m b_n < k_0^p(\lambda_0) k_1^q(\lambda_1)$$

$$\times \left[ \sum_{m=1}^{\infty} m^{p\lambda_1 - 1} b_m^p \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{q\lambda_2 - 1} a_n^q \right]^{\frac{1}{q}} \quad (12)$$

It is worth noting that inequality (12) is an analogue of the Hilbert-type inequality (8). In the following lemma, we present a relation between the parameters $\lambda$, $\lambda_1$ and $\lambda_2$ on the best possible constant factor in inequality (12).

**Lemma 4.** If inequality (12) has the best possible constant factor $k_0^p(\lambda_0) k_1^q(\lambda_1)$ for various parameters, then $\lambda = \lambda_0 + \lambda_2$.

**Proof.** From the assumption conditions of inequality (12), it follows that

$$\lambda_0 = \frac{\lambda_1}{p} + \frac{\lambda - \lambda_2}{q} > 0, \quad \lambda_0 < \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda, \quad 0 < \lambda_2 = \lambda - \lambda_1 < \lambda.$$

Hence, we have

$$k_0(\lambda_0) = \frac{\lambda}{\lambda_1} \lambda_2 \in \mathbb{R}_+ = (0, \infty)$$

$$\lambda = \lambda_0 + \lambda_2.$$
If the constant factor $k_\lambda^1 (\lambda_1) k_\lambda^1 (\lambda_2)$ in (12) is the best possible, then in view of inequality (10), we have

$$k_\lambda^1 (\lambda_1) k_\lambda^1 (\lambda_2) \leq k_\lambda (\lambda_1)$$

By Hölder’s inequality with weight, we find

$$k_\lambda (\lambda_1) = k_\lambda \left( \frac{\lambda_1}{p} + \frac{\lambda_2 - \lambda_1}{q} \right)$$

$$= \int_0^\infty \left( \min(1, u)^\lambda u^{-\lambda_1 + \frac{\lambda_1}{p} + \frac{1-\lambda_1}{q}} \right) du = \int_0^\infty \left( \min(1, u)^\lambda u^{-\lambda_1 + \frac{1-\lambda_1}{q}} \right) du$$

$$\leq \int_0^\infty \left( \min(1, u)^\lambda u^{-\lambda_1 - 1} du \right)^{\frac{1}{2}} \left[ \int_0^\infty \left( \min(1, u)^\lambda u^{-\lambda_1 - 1} du \right)^{\frac{1}{2}} \right]$$

$$= k_\lambda^1 (\lambda_1) k_\lambda^1 (\lambda_2).$$

It follows that $k_\lambda^1 (\lambda_1) k_\lambda^1 (\lambda_2) = k_\lambda (\lambda_1)$, and thus (13) keeps the form of equality.

It is easy to see that (13) keeps the form of equality if, and only if, there exist constants $A$ and $B$ (not all zero) such that (c.f. [25])

$$Au^{\lambda - \lambda_1 - 1} = Bu^{-(\lambda - \lambda_2) - 1} \quad \text{a.e. in } R_+.$$

Assuming that $A \neq 0$, we have $u^{\lambda - \lambda_2 - \lambda_1} = \frac{B}{A} \quad \text{a.e. in } R_+$, and this yields $\lambda - \lambda_2 - \lambda_1 = 0$, hence $\lambda = \lambda_1 + \lambda_2$. The proof of Lemma 4 is thus complete. 

3. Main Results and Some Particular Cases

**Theorem 1.** Inequality (8) is equivalent to the following inequality:

$$J : = \left\{ \sum_{n=1}^\infty \frac{n^{-p(\frac{1}{p} + \frac{1-\lambda_1}{q}) - 1}}{n^{-p(\frac{1}{p} + \frac{1-\lambda_1}{q}) - 1}} \left[ \sum_{m=1}^\infty (\min(m, n))^\lambda a_m \right]^p \right\}^{\frac{1}{2}}$$

$$< k_\lambda^1 (\lambda_1) k_\lambda^1 (\lambda_2) \times \sum_{m=1}^\infty \frac{1}{n^{p(1 + \frac{1}{p} + \frac{1-\lambda_1}{q}) - 1}} b_n^p.$$ (14)

If the constant factor in (8) is the best possible, then so is the constant factor in (14).

**Proof.** Suppose that inequality (14) is valid. By Hölder’s inequality (c.f. [25]), we have

$$I = \int_{\min(m, n)^{\lambda}}^\infty [n^{p(\frac{1}{p} + \frac{1-\lambda_1}{q}) - 1}] \sum_{m=1}^\infty (\min(m, n))^\lambda a_m \left[ \frac{1}{n^{\frac{1}{p} + \frac{1-\lambda_1}{q}}} + \frac{1}{n^{\frac{1}{p} + \frac{1-\lambda_1}{q}}} b_n \right] \leq \int_{\min(m, n)^{\lambda}}^\infty n^{p(1 + \frac{1}{p} + \frac{1-\lambda_1}{q}) - 1} b_n.$$ (15)

Then, by using inequality (14), we obtain inequality (8). On the other hand, assuming that inequality (8) is valid, we set

$$b_n := n^{-p(\frac{1}{p} + \frac{1-\lambda_1}{q}) - 1} \left[ \sum_{m=1}^\infty (\min(m, n))^\lambda a_m \right]^{p-1}, \quad n \in \mathbb{N}.$$
If \( J = 0 \), then inequality (14) is naturally valid; if \( J = \infty \), then it is impossible to make inequality (14) valid, which implies \( J < \infty \). Suppose that \( 0 < J < \infty \). By inequality (8), we have

\[
\sum_{n=1}^{\infty} \eta^{\left(1+\frac{\lambda_{2}}{p} + \frac{1-\lambda_{1}}{q}\right)-1} b_{n}^{p} = J^{p} = I < k_{\lambda_{1}}^{\frac{1}{\lambda_{1}}} (\lambda_{1}) k_{\lambda_{2}}^{\frac{1}{\lambda_{2}}} (\lambda_{2}) \times \left[ \sum_{m=1}^{\infty} m^{p \left(1+\frac{1}{p} + \frac{1-\lambda_{2}}{q}\right)-1} a_{m}^{p} \right]^{\frac{1}{p}},
\]

\[
I = \left[ \sum_{n=1}^{\infty} \eta^{\left(1+\frac{\lambda_{2}}{p} + \frac{1-\lambda_{1}}{q}\right)-1} b_{n}^{p} \right]^{\frac{1}{p}} < k_{\lambda_{1}}^{\frac{1}{\lambda_{1}}} (\lambda_{1}) k_{\lambda_{2}}^{\frac{1}{\lambda_{2}}} (\lambda_{2}) \left[ \sum_{m=1}^{\infty} m^{p \left(1+\frac{1}{p} + \frac{1-\lambda_{2}}{q}\right)-1} a_{m}^{p} \right]^{\frac{1}{p}}.
\]

Thus, inequality (14) follows, and we conclude that inequality (8) is equivalent to inequality (14).

Furthermore, we show that if the constant factor in (8) is the best possible, then the constant factor in (14) is also the best possible. Otherwise, from inequality (15) we would reach a contradiction, namely that the constant factor in (8) is not the best possible. The proof of Theorem 1 is thus completed. \( \square \)

In the following theorem, we give some equivalent statements of the best possible constant factor related to several parameters.

**Theorem 2.** The statements (i), (ii), (iii) and (iv) below are equivalent:

(i) \( k_{\lambda_{1}}^{\frac{1}{\lambda_{1}}} (\lambda_{1}) k_{\lambda_{2}}^{\frac{1}{\lambda_{2}}} (\lambda_{2}) \) is independent of \( p, q \);

(ii) \( k_{\lambda_{1}}^{\frac{1}{\lambda_{1}}} (\lambda_{1}) k_{\lambda_{2}}^{\frac{1}{\lambda_{2}}} (\lambda_{2}) \) is expressible as a single integral

\[
k_{\lambda_{1}} (\lambda_{1}) = \int_{0}^{\infty} (\min(1, u))^{\lambda_{1}} u^{-\lambda_{1}-1} du \quad (0 < \lambda_{1} < \lambda)
\]

(iii) \( k_{\lambda_{1}}^{\frac{1}{\lambda_{1}}} (\lambda_{1}) k_{\lambda_{2}}^{\frac{1}{\lambda_{2}}} (\lambda_{2}) \) in (8) is the best possible constant factor;

(iv) \( \lambda = \lambda_{1} + \lambda_{2} \quad (\in (0, \frac{11}{4}) \)\).

If the statement (iv) is valid, namely, \( \lambda = \lambda_{1} + \lambda_{2} \in (0, \frac{11}{4}) \), then we have inequality (10) and the following equivalent inequality with the best possible constant factor \( \lambda_{1}\lambda_{2} \):

\[
\left\{ \sum_{n=1}^{\infty} \frac{1}{n^{p\lambda_{1}+1}} \left[ \sum_{m=1}^{\infty} \left(\min(m, n)\right)^{\lambda_{1}} a_{m} \right]^{p} \right\}^{\frac{1}{p}} < \frac{\lambda}{\lambda_{1}\lambda_{2}} \left[ \sum_{m=1}^{\infty} m^{p\left(1+\lambda_{1}\right)-1} a_{m}^{p} \right]^{\frac{1}{p}}
\]

**Proof.** (i) \( \Rightarrow \) (ii). By (i), we have

\[
k_{\lambda_{1}}^{\frac{1}{\lambda_{1}}} (\lambda_{1}) k_{\lambda_{2}}^{\frac{1}{\lambda_{2}}} (\lambda_{2}) = \lim_{p \to 1+} k_{\lambda_{1}}^{\frac{1}{\lambda_{1}}} (\lambda_{1}) k_{\lambda_{2}}^{\frac{1}{\lambda_{2}}} (\lambda_{2}) = k_{\lambda_{1}} (\lambda_{1}).
\]

Namely, \( k_{\lambda_{1}}^{\frac{1}{\lambda_{1}}} (\lambda_{1}) k_{\lambda_{2}}^{\frac{1}{\lambda_{2}}} (\lambda_{2}) \) is expressible as a single integral

\[
k_{\lambda_{1}} (\lambda_{1}) = \int_{0}^{\infty} (\min(1, u))^{\lambda_{1}} u^{-\lambda_{1}-1} du \quad (0 < \lambda_{1} < \lambda)
\]

(ii) \( \Rightarrow \) (iv). If \( k_{\lambda_{1}}^{\frac{1}{\lambda_{1}}} (\lambda_{1}) k_{\lambda_{2}}^{\frac{1}{\lambda_{2}}} (\lambda_{2}) \) is expressible as a single integral

\[
k_{\lambda_{1}} (\lambda_{1}) = \int_{0}^{\infty} (\min(1, u))^{\lambda_{1}} u^{-\lambda_{1}-1} du \quad (0 < \lambda_{1} < \lambda)
\]
then for $\tilde{\lambda}_1 = \Lambda_1 \ (\in (0, \lambda))$, (13) keeps the form of equality. In view of the proof of Lemma 4, it follows that $\lambda = \lambda_1 + \lambda_2$.

(iv) $\Rightarrow$ (i). If $\lambda = \lambda_1 + \lambda_2$, then $k_{\lambda}^{\frac{1}{2}}(\lambda_1)k_{\lambda}^{\frac{1}{2}}(\lambda_2) = k_{\lambda}(\lambda_1)$, which is independent of $p, q$. Thus, we deduce that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv).

(iii) $\Rightarrow$ (iv). By Lemma 4, we get $\lambda = \lambda_1 + \lambda_2$.

(iv) $\Rightarrow$ (iii). By Lemma 3, for $\lambda = \lambda_1 + \lambda_2, k_{\lambda}^{\frac{1}{2}}(\lambda_1)k_{\lambda}^{\frac{1}{2}}(\lambda_2) = \frac{1}{\lambda_1, \lambda_2}$ is the best possible constant factor in (8). It follows that (iii) $\Leftrightarrow$ (iv).

Therefore, we assert that the statements (i), (ii), (iii) and (iv) are equivalent. This completes the proof of Theorem 2. $\Box$

Now, we discuss some particular cases of the inequalities obtained above, from which we will derive some interesting inequalities.

**Remark 2.** (i) Putting $\lambda = 1, \lambda_1 = \frac{1}{3}, \lambda_2 = \frac{1}{3}$ in (10) and (16), we obtain the following equivalent inequalities with the best possible constant factor $pq$:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \min(m, n)a_mb_n < pq \left( \sum_{m=1}^{\infty} m^{2(p-1)}a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} n^{2(q-1)}b_n^q \right)^{\frac{1}{q}}$$

(17)

$$\left[ \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \min(m, n)a_m \right)^p \right]^{\frac{1}{p}} < pq \left[ \sum_{m=1}^{\infty} m^{2(p-1)}a_m^p \right]$$

(18)

(ii) Putting $\lambda = 1, \lambda_1 = \frac{1}{3}, \lambda_2 = \frac{1}{3}$ in (10) and (16), we get the following equivalent inequalities with the best possible constant factor $pq$:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \min(m, n)a_mb_n < pq \left( \sum_{m=1}^{\infty} (ma_m) \right)^{\frac{1}{\lambda}} \left( \sum_{n=1}^{\infty} (nb_n) \right)^{\frac{1}{q}}$$

(19)

$$\left[ \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{m=1}^{\infty} \min(m, n)a_m \right)^p \right]^{\frac{1}{p}} < pq \left[ \sum_{m=1}^{\infty} (ma_m) \right]$$

(20)

(iii) Setting $p = q = 2$, both (17) and (19) reduce to the inequality:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \min(m, n)a_mb_n < 4 \left( \sum_{m=1}^{\infty} (ma_m) \right)^{2} \left( \sum_{n=1}^{\infty} (nb_n)^2 \right)^{\frac{1}{2}}$$

(21)

Furthermore, both (18) and (20) reduce to the equivalent form of (21) as follows:

$$\left[ \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{m=1}^{\infty} \min(m, n)a_m \right)^2 \right]^{\frac{1}{2}} < 4 \left[ \sum_{m=1}^{\infty} (ma_m)^2 \right]^{\frac{1}{2}}$$

(22)

(iv) Putting $\lambda = 2, \lambda_1 = \lambda_2 = 1$ in (10) and (16), we have the following equivalent inequalities with the best possible constant factor 2:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \min(m, n) \cdot a_mb_n < 2 \left( \sum_{m=1}^{\infty} m^{2(p-1)}a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} n^{2(q-1)}b_n^q \right)^{\frac{1}{q}}$$

(23)
\begin{align}
\left\{ \sum_{n=1}^{\infty} \frac{1}{q^p+1} \left( \sum_{m=1}^{\infty} (\min(m,n))^2 a_m \right)^{\frac{p}{2}} \right\}^\frac{1}{p} < 2 \left( \sum_{m=1}^{\infty} m^{2p-1} d_m^p \right)^{\frac{1}{p}}
\tag{24}
\end{align}

(v) Putting \( \lambda = e \left( < \frac{11}{4} = 2.75 \right) \), \( \lambda_1 = \lambda_2 = \frac{3}{4} \) in (10) and (16), we have the following equivalent inequalities with the best possible constant factor \( \frac{1}{2} \):

\begin{align}
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min(m,n))^p a_m b_n < \frac{4}{e^3} \left( \sum_{m=1}^{\infty} m^{p(1+\frac{1}{4})-1} d_m^p \right) \left( \sum_{n=1}^{\infty} n^{\frac{1}{4}(1+\frac{1}{4})-1} \right)^{\frac{1}{2}}
\tag{25}
\end{align}

\begin{align}
\left\{ \sum_{n=1}^{\infty} \frac{1}{n^{\frac{p}{2}+1}} \left( \sum_{m=1}^{\infty} (\min(m,n))^p a_m \right)^{\frac{1}{2}} \right\} < \frac{4}{e^3} \left( \sum_{m=1}^{\infty} m^{p(1+\frac{1}{4})-1} d_m^p \right)^{\frac{1}{2}}
\tag{26}
\end{align}

4. Operator Expressions

We choose the functions

\[ \psi(n) := n^{\frac{1}{4}(1+\frac{1}{4})-1}, \]

where, from,

\[ \psi^{1-p}(n) = n^{-p(\frac{3}{4}+\frac{1}{4})-1} \quad (m, n \in \mathbb{N}) \]

We define the following real normed spaces:

\[ l_{p,\psi} := \{ a = (a_m)_{m=1}^{\infty}, \psi \}, \| a \|_{l_{p,\psi}} := \left( \sum_{m=1}^{\infty} \psi^p(m) |a_m|^p \right)^{\frac{1}{p}} < \infty, \]

\[ l_{q,\psi} := \{ b = (b_n)_{n=1}^{\infty}, \psi \}, \| b \|_{l_{q,\psi}} := \left( \sum_{n=1}^{\infty} \psi^q(n) |b_n|^q \right)^{\frac{1}{q}} < \infty, \]

\[ l_{p,q,1-p} := \{ c = (c_n)_{n=1}^{\infty}, \psi \}, \| c \|_{l_{p,q,1-p}} := \left( \sum_{n=1}^{\infty} \psi^{1-p}(n) |c_n|^p \right)^{\frac{1}{p}} < \infty. \]

We let \( a \in l_{p,\psi} \), and set

\[ c = (c_n)_{n=1}^{\infty}, c_n := \sum_{m=1}^{\infty} (\min(m,n))^\lambda a_m, n \in \mathbb{N}. \]

Then, we can rewrite inequality (14) as follows:

\[ \| c \|_{l_{p,q,1-p}} < k_1^\lambda (\lambda_1 k_2^\lambda (\lambda_2)) \| a \|_{l_{p,\psi}} < \infty, \text{ that is } c \in l_{p,q,1-p}. \]

**Definition 1.** Define a Hilbert-type operator \( T : l_{p,\psi} \rightarrow l_{p,q,1-p} \) as follows: For any \( a \in l_{p,\psi} \), there exists a unique representation \( c \in l_{p,q,1-p} \). Define the formal inner product of \( Ta \) and \( b \in l_{q,\psi} \) and the norm of \( T \) as follows:

\[ (Ta, b) := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min(m,n))^\lambda a_m |b_n| \]

\[ \| T \| := \sup_{a \neq 0} \frac{\| Ta \|_{l_{p,q,1-p}}}{\| a \|_{l_{p,\psi}}} \]

Then, by Theorems 1 and 2, we obtain the operator expressions of inequalities (8) and (14) as follows:
**Theorem 3.** If $a \in l_{(p,q)}$, $b \in l_{(q,p)}$, $\|a\|_{l_{p,q}}$, $\|b\|_{l_{q,p}} > 0$, then we have the following inequalities:

$$
(Ta, b) < k_\lambda^\frac{1}{q} (\lambda_1) k_\lambda^\frac{1}{p} (\lambda_2) \|a\|_{l_{p,q}} \|b\|_{l_{q,p}}
$$

(27)

$$
\|Ta\|_{l_{p,q}^{1-p}} < k_\lambda^\frac{1}{q} (\lambda_1) k_\lambda^\frac{1}{p} (\lambda_2) \|a\|_{l_{p,q}}
$$

(28)

Furthermore, $\lambda_1 + \lambda_2 = \lambda \in (0, \frac{11}{4})$ if, and only if, the constant factor $k_\lambda^\frac{1}{q} (\lambda_1) k_\lambda^\frac{1}{p} (\lambda_2)$ in (27) and (28) is the best possible, namely,

$$
\|T\| = k_\lambda (\lambda_i) = \frac{\lambda}{\lambda_1 \lambda_2} \quad (\lambda_i \in (0, \frac{11}{8}) \cap (0, \lambda), \ i = 1, 2)
$$

(29)

5. Conclusions

In this paper, we give, with Lemma 2 and Theorem 1, respectively, a new inequality of the Hilbert-type containing positive homogeneous kernel and its equivalent forms. Based on the obtained Hilbert-type inequality, we discuss in Theorem 2 the equivalent statements of the best possible constant factor related to several parameters. As applications, the operator expressions of the obtained inequalities are given in Theorem 3, and some particular cases of the obtained inequalities (10) and (16) are considered in Remark 2. It is shown that the results obtained in Theorems 1 and 2 would generate more new inequalities of Hilbert-type.

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