Analysis on some powered integral inequalities with retarded argument and application

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ABSTRACT
We investigate a certain class of nonlinear Gronwall–Bellman–Pachpatte type of integral inequalities involving retarded term and nonlinear integrals. Our results derive some new inequalities as well as give an upper bound estimation of the unknown function. The significance of these inequalities originate from the truth that it is far applicable in specific situations in which different available inequalities do not apply legitimately. Our consequences additionally can be used as powerful tools in the analysis of certain classes of differential equations, integral equations, and evolution equations. An example is also illustrated to show the validity of our established theorems.

1. Introduction
Integral inequalities possess an exceptionally advantageous position in the improvement of the hypothesis of ordinary differential and integral equations. In this account, fundamental integral inequality by Gronwall [1] and Bellman [2] contributes a significant part in examining the stability and asymptotic behaviour in order to find the solutions of integral equations. In the ongoing years, Gronwall–Bellman type inequalities have been enormously improved by the affirmation of their importance and inherent value in numerous branches of the applied sciences. These inequalities with their linear and nonlinear generalizations have earned much consideration in the works of Abdeldaim et al. [3], Bainov et al. [4], Bihari [5], El-Owaidy et al. [6], Jiang et al. [7], Kim [8], Liu et al. [9], Ngoc et al. [10], Oguntuase [11], Pachpatte [12], Pachpatte [13], Wang et al. [14], Zhou et al. [15].

Furthermore, El-Owaidy et al. [16] introduced a new essential inequality of the type
\[
r(u) \leq r_0 + \int_0^u l(s) \left[ r^\varrho(s) + \int_0^s t(\varsigma) r(\varsigma) d\varsigma \right] ds, \quad u \in [0, \infty),
\]

in which the integrand function incorporates the power and makes it tougher to estimate the unknown function where \( 0 \leq \varrho < 1 \). Nevertheless, many real life issues that have been formulated for differential equations in the past were sometimes based on initial value problems. As a matter of fact, include a critical memory impact that can be interpreted in a more sophisticated version of differential equation that integrates delayed or retarded arguments. In these circumstances, we need to speak about some retarded nonlinear integral inequalities. In this record, Agarwal et al. [17] discovered the integral inequality as
\[
r(u) \leq x(u) + \sum_{i=1}^n \int_{u_0}^{u_{i+1}} l_i(u, s) \rho_i(r(s)) ds, \quad u_0 \leq u < u_1,
\]

where \( \rho_i, i = 1, \ldots, n \) are positive, continuous and non-decreasing functions on \([0, \infty)\) and \( \rho_1 \propto \rho_2 \cdots \propto \rho_n \) with \( n \) terms on the right integral of inequality. The concept of recursion is expected to get the estimation. Very recently, Abdeldaim et al. [18] proved the retarded inequality of another kind
\[
r(u) \leq r_0 + \int_0^u \left[ l(s) r(s) + q(s) \right] ds + \int_0^\omega(u) l(s) \left( \int_0^s e(\varsigma) r(\phi) d\varsigma \right) ds, \quad u \in [0, \infty),
\]
such that \( \omega(u) \leq u \) and \( \omega(0) = 0 \) and multiplication of two functions exist on the right of the third term, that’s a unique form of inequality (2). Moreover, El-Deeb et al.
[19] investigated the retarded Volterra–Fredholm type integral inequality
\[ r^0(u) \leq c(u) + \int_{a}^{\omega(u)} l(s) r(s) \, ds \]
\[ + \int_{a}^{b} e(s) r^0(s) \, ds, \quad u \in [a, b], \]  
(4)

where \( \omega(u) \leq u, \omega(0) = a \) and \( \varphi \geq 1 \) is a constant. Other than the results referenced over, a number of examiners have observed many beneficial integral inequalities, mainly stimulated by their applications in exclusive parts of the differential equations (see Boudeliou [20], Ferreira et al. [21], Khan et al. [22], Lipovan [23], Belmor et al. [24], Lipovan [25], Wang et al. [26], Ul-Haq et al. [27], Xu et al. [28]).

However, in some circumstances, various classes of powered delay integral and differential equations, it is expected to inspect new delay inequalities so as to survive and accomplish the ideal objectives. Therefore, in this article, we explore certain new generalizations of retarded integral inequalities with power for previous well-known results, which can be utilized as helpful apparatuses to demonstrate the classifications of integral inequalities and integral equations. Lastly, we give one example to describe the benefit of our results.

2. Results and discussion

Given notations will be followed throughout this text for the ease of reading: \( \mathbb{R}_+ = [0, \infty) \), \( \mathbb{I} \) denotes the set of real numbers, \( C(N, S) \) be the class of all continuous functions defined on \( N \) with range in the set \( S \) and

(H1) The function \( r(u) \) is a nonnegative and continuous on \([u_0, \infty)\).

(H2) \( \omega(u) \) be a continuous, differentiable and increasing function on \([u_0, 0)\) with \( \omega(u) \leq u, \omega(u_0) = u_0 \).

(H3) Let \( m(u), j(u), l(u), n(u) \in C([u_0, \infty), \mathbb{R}_+) \) and \( m(u) \) be a non-decreasing function.

(H4) \( \mu, \phi \in C(\mathbb{R}_+, \mathbb{R}_+) \) be non-decreasing function and \( \mu(r) > 0, \phi(r) > 0 \) for \( r > 0 \).

Presently, a principle lemma which will be utilized later in verification of the exhibited paper is given below:

**Lemma 2.1** ([29]): Let \( z \geq 0, \varphi \geq \eta \geq 0 \) and \( \varphi \neq 0 \), then
\[ z^\varphi \leq \eta^\eta \varphi + \frac{\varphi - \eta}{\varphi} \]  
(5)

**Proof:** inequality (5) is valid for \( \eta = 0 \), but if \( \eta > 0 \) and \( \delta = \eta / \varphi \), then \( \delta = 1 \) by [29], we have
\[ z^\varphi \leq \frac{\eta}{\varphi} L^{(\eta-\varphi)/\varphi} z + \frac{\varphi - \eta}{\varphi} L^{\eta/\varphi}, \]
for any \( L > 0 \). Let \( L = 1 \), we get (5).

**Theorem 2.2:** If the conditions (H1)–(H3) and
\[ r(u) \leq m(u) + \int_{u_0}^{\omega(u)} j(s) r(s) \, ds + \int_{u_0}^{\omega(u)} l(s) \]
\[ \times \left( r^2(s) + \int_{u_0}^{\omega(u)} n(s) r^2(s) \, ds \right), \quad \forall u \in \mathbb{R}_+, \]  
(6)
hold. Then
\[ r(u) \leq \frac{1}{\exp(-m(u)) - \int_{u_0}^{\omega(u)} l(s) \left( 1 + \int_{u_0}^{\omega(u)} n(s) r^2(s) \, ds \right)} \]  
(7)
provided with
\[ m(u) = \ln(m(u)) + \int_{u_0}^{\omega(u)} j(s) \, ds. \]  
(8)

**Proof:** Denote
\[ P(u) = m(u) + \int_{u_0}^{\omega(u)} j(s) r(s) \, ds \]
\[ + \int_{u_0}^{\omega(u)} l(s) \left( r^2(s) + \int_{u_0}^{\omega(u)} n(s) r^2(s) \, ds \right), \]  
(9)

\( P(u) \) is non-decreasing function, so (6) restates as
\[ r(u) \leq P(u), \quad P(u_0) = m(u_0), \]  
(10)
differentiating (9) and employ (10), we get
\[ P'(u) = m'(u) + j(u) \omega'(u) r(u) \omega(u) \]
\[ + l(u) \omega'(u) \left( r^2(u) + \int_{u_0}^{u} n(s) r^2(s) \, ds \right) \]
\[ \leq m'(u) + j(u) \omega'(u) P(u) \omega(u) \]
\[ + l(u) \omega'(u) P^2(u) + \int_{u_0}^{u} n(s) P^2(s) \, ds, \]  
(11)

(11) becomes
\[ \frac{P'(u)}{P(u)} \leq \frac{m'(u)}{m(u)} + \frac{j(u) \omega'(u) P(u)}{P(\omega(u))} \]
\[ + \frac{l(u) \omega'(u) P^2(u)}{P(\omega(u))} \left( 1 + \int_{u_0}^{\omega(u)} n(s) \, ds \right) \]
\[ \leq m'(u) + \frac{j(u) \omega'(u) P(u)}{P(\omega(u))} \]
\[ + \frac{l(u) \omega'(u) P^2(u)}{P(\omega(u))} \left( 1 + \int_{u_0}^{\omega(u)} n(s) \, ds \right) \]
\[ = \frac{m'(u)}{m(u)} + j(u) \omega'(u) \]
\[ + l(u) \omega'(u) \left( 1 + \int_{u_0}^{\omega(u)} n(s) \, ds \right) P(u). \]  
(12)
Integrate the both sides of (12) from $u_0$ to $u$, we have
\[
\ln(P(u)) \leq m_1(u) + \int_{u_0}^{u} l(s) \left(1 + \int_{u_0}^{s} n(\zeta) \, d\zeta\right) P(s) \, ds,
\]
where $m_1(u)$ be as defined in (8). Consider
\[
\ln(P(u)) \leq L(u), \quad P(u) \leq \exp(L(u)),
\]
and
\[
L(u) = m_1(u) + \int_{u_0}^{u} l(s) \left(1 + \int_{u_0}^{s} n(\zeta) \, d\zeta\right) P(s) \, ds.
\]
Clearly $m_1(u) > 0$ in (15), so $L(u) > 0$ and $m(u) > 1$ from (8), taking derivative (15) and from (14), we deduce
\[
L'(u) = m_1'(u) + l(\omega(u)) \left(1 + \int_{u_0}^{u} n(s) \, ds\right) \omega'(u) P(\omega(u)) \\
\leq m_1'(u) + l(\omega(u)) \\
\times \left(1 + \int_{u_0}^{u} n(s) \, ds\right) \omega'(u) \exp(L(\omega(u))),
\]
or, equivalently,
\[
\frac{L'(u)}{\exp(L(u))} \leq \frac{m_1'(u)}{\exp(m_1(u))} \\
\times \frac{l(\omega(u)) \left(1 + \int_{u_0}^{u} n(s) \, ds\right) \omega'(u) \exp(L(\omega(u)))}{\exp(L(\omega(u)))} \\
= \frac{m_1'(u) + l(\omega(u))}{\exp(m_1(u))} \\
\times \left(1 + \int_{u_0}^{u} n(s) \, ds\right) \omega'(u),
\]
by integrating (17) from $u_0$ to $u$, it is noticed that
\[
L(u) \leq \ln \left(\frac{1}{\exp(-m_1(u)) - \int_{u_0}^{u} l(s) \left(1 + \int_{u_0}^{s} n(\zeta) \, d\zeta\right) \, ds}\right),
\]
the required estimation in (7) can be obtained by combining (10), (14) and (18).

**Theorem 2.3:** Suppose that (H1), (H2), (H3) be satisfied. Moreover
\[
r(u) \leq m(u) + \int_{u_0}^{u} j(s) r(s) \left(1 + \int_{u_0}^{s} l(\zeta) r(\zeta) \, d\zeta\right)^{\varrho} \, ds
\]
\[
+ \int_{u_0}^{u} j(s) r(s) \left(1 + \int_{u_0}^{s} n(\zeta) r(\zeta) \, d\zeta\right) \, ds,
\]
for all $u \in \mathbb{R}_+$ such that $\varrho \in (0, 1]$ is a positive constant. Then
\[
r(u) \leq m(u) + \frac{m_2(u) \exp \left(\int_{u_0}^{u} \chi(s) \, ds\right)}{1 - m_2(u) \int_{u_0}^{u} \chi(s) \, ds} \left(\int_{u_0}^{u} \chi(s) \, ds\right) \, ds,
\]
where
\[
m_2(u) = \int_{u_0}^{u} j(s) m(s) \left(\varrho \left(m(s) + \int_{u_0}^{s} l(\zeta) m(\zeta) \, d\zeta\right) - \varrho \left(m(s) + \int_{u_0}^{s} l(\zeta) m(\zeta) \, d\zeta\right) + \int_{u_0}^{s} n(\zeta) m(\zeta) \, d\zeta\right) \, ds.
\]

**Proof:** Let
\[
Y(u) = \int_{u_0}^{u} j(s) r(s) \left(\varrho \left(m(s) + \int_{u_0}^{s} l(\zeta) r(\zeta) \, d\zeta\right) \, ds
\]
\[
+ \int_{u_0}^{u} j(s) r(s) \left(\int_{u_0}^{s} n(\zeta) r(\zeta) \, d\zeta\right) \, ds,
\]
therefore, (19) reaches to
\[
r(u) \leq m(u) + Y(u), \quad Y(u_0) = 0,
\]
by applying Lemma 2.1 and (25) into (24), it follows that
\[
Y(u) \leq \left[\int_{u_0}^{u} j(s) r(s) \left(\varrho \left(m(s) + Y(s) \right) + \int_{u_0}^{s} n(\zeta) r(\zeta) \, d\zeta\right) \, ds
\]
\[
+ \int_{u_0}^{u} j(s) r(s) \left(\int_{u_0}^{s} n(\zeta) r(\zeta) \, d\zeta\right) \, ds\right] \, ds
\]
\[
\leq \left[\int_{u_0}^{u} j(s) \left[m(s) + Y(s) \right] \, ds
\]
\[
+ \int_{u_0}^{u} j(s) \left[m(s) + Y(s) \right] \, ds\right] \, ds
\]
\[
\leq \left[\int_{u_0}^{u} j(s) m(s) \left(\varrho \left(m(s) + \int_{u_0}^{s} l(\zeta) m(\zeta) \, d\zeta\right) \, ds
\]
\[
+ (1 - \varrho) \int_{u_0}^{s} n(\zeta) m(\zeta) \, d\zeta\right) \, ds
\]
\[
+ (1 - \varrho) j(s) m(s)\right] \, ds\right] \, ds.
\]
+ j(s)m(s) \left( \int_{u_0}^{s} n(\zeta) m(\zeta) \, d\zeta \right) \, ds + \int_{u_0}^{\omega(u)} \left[ j(s) \left( 2 q m(s) \left( 1 + \int_{u_0}^{s} l(\zeta) \, d\zeta \right) \right) + \int_{u_0}^{\omega(u)} \left[ j(s) \left( 1 + \int_{u_0}^{s} l(\zeta) \, d\zeta \right) \right] \right] \chi(s) \exp \left( \int_{u_0}^{\omega(u)} \gamma(\zeta) \, d\zeta \right) \, ds \right],

(26)

where \( m_2(u) \), \( \gamma(u) \) and \( \chi(u) \) are quoted in (21), (22) and (23) respectively. For an arbitrary \( u \in [u_0, U^*] \), fixed \( U^* \in [u_0, \infty) \) and taking \( Y_1(u) \) by the right-hand side of (26), we deduce

\[
Y_1(u) = m_2(U^*) \int_{u_0}^{\omega(u)} \gamma(s) Y(s) \, ds + \int_{u_0}^{\omega(u)} \chi(s) Y^2(s) \, ds,
\]

(27)

(26) can be carried out as

\[
Y(u) \leq Y_1(u), \quad Y(\omega(u)) \leq Y_1(\omega(u)) \leq Y_1(U(u)) = m_2(U^*),
\]

(28)

since \( Y_1(u) \) is non-decreasing, hence we can write (27) by using (28) as the following

\[
Y'_1(u) \leq \gamma(\omega(u)) \omega'(u) Y_1(u) + \chi(\omega(u)) \omega'(u) Y'_1(u),
\]

(29)

take \( Z(u) = Y_1^{-1}(u) \), then \( Z'(u) = -Y_1^{-2}(u) Y'_1(u) \), therefore (29) yields

\[
Z'(u) + \omega(u) \omega'(u) Z(u) \geq -\chi(\omega(u)) \omega'(u).
\]

(30)

Multiplying both sides of (30) by \( \exp(\int_{u_0}^{\omega(u)} \gamma(s) \, ds) \), integrating the resultant inequality and applying \( Z(u_0) = m_2^{-1}(U^*) \), we attain

\[
Z(u) \geq \frac{1 - m_2(U^*) \int_{u_0}^{\omega(u)} \chi(\omega(s)) \omega'(s) \, ds}{m_2(U^*) \exp \left( \int_{u_0}^{\omega(u)} \gamma(s) \, ds \right)},
\]

(31)

substituting \( Z(u) = Y_1^{-1}(u) \) in (31) and from (28), we get

\[
Y(u) \leq \frac{m_2(U^*) \exp \left( \int_{u_0}^{\omega(u)} \gamma(s) \, ds \right) \chi(u) \exp \left( \int_{u_0}^{s} \gamma(\zeta) \, d\zeta \right) \, ds}{1 - m_2(U^*) \int_{u_0}^{\omega(u)} \chi(s) \exp \left( \int_{u_0}^{s} \gamma(\zeta) \, d\zeta \right) \, ds},
\]

(32)

so (25) leads to

\[
r(U^*) \leq m(U^*) \exp \left( \int_{u_0}^{\omega(U^*)} \gamma(s) \, ds \right),
\]

(33)

the arbitrariness of \( U^* \) in (33) gives the desired bound in (20). This completes the proof. \( \blacksquare \)

Remark 2.1: If \( m(u) = c, \omega(u) \leq u, n(u) = 0 \), then Theorem 2.3 converts to Theorem 3.2 due to Abdeldaim et al. [3].

Remark 2.2: By taking \( m(u) = u_0, \omega = 1, r(u) \leq q(u(t)), m(u) = g(t) \) and \( g(u) = f(t) \), then Theorem 2.3 will be a slight variation of Theorem 2.2 with \( q(t) = 0 \) given in Abdeldaim et al. [18].

Another interesting and slightly different version of the nonlinear integral inequality may be stated as follows.

Corollary 2.4: If (H1), (H2), (H5) \( m(u), j(u), l(u) \in C([u_0, \infty), \mathbb{R}_+) \) be fulfilled and

\[
r^2(u) \leq m(u) + \int_{u_0}^{\omega(u)} r(s) \, ds \leq \frac{1}{\exp(-m_4(u)) - \int_{u_0}^{\omega(u)} \gamma_1(s) \, ds},
\]

(34)

with \( \varphi > 0 \). Then

\[
r(u) \leq \frac{1}{m_2(u)} \left( 1 + \frac{m^{-1}(u)}{m(u)} \right),
\]

(35)

where \( m_4(u) = \ln(m_3(u)) + \int_{u_0}^{\omega(u)} \gamma_1(s) \, ds \), \( m_3(u) = j(u)(m(u) + \varphi - 1)/\varphi \), \( j(u)(\int_{u_0}^{\omega(u)} m(s) \, ds) \), \( \gamma_1(u) = (1/\varphi) j(u)(\int_{u_0}^{\omega(u)} l(s) \, ds) \) and \( \gamma_1(u) = (1/\varphi) j(u)(2m(u) + \varphi - 1) \).

Proof: The proof of Corollary 2.4 proceeds in much a comparable way as in the evidence of Theorem 2.2 with a few alterations. We leave the information to the reader. \( \blacksquare \)

Theorem 2.5: The assumptions (H1), (H2), (H4), (H5) be satisfied. Also

\[
r(u) \leq m(u) + \int_{u_0}^{\omega(u)} r(s) \, ds \leq \frac{m^{-1}(u)}{m(u)} \left( 1 + \frac{m(u)}{m(u)} \right),
\]

(36)
with $\varrho > 0$, $\lambda \geq 1$ are constants and $\varrho + \lambda > 1$. Then

$$r(u) \leq \xi^{-1}\left(\xi(m(u)) + \int_{0}^{\omega(u)} j(s)Q(s)\,ds\right),$$  
(37)

$\forall u \in [u_0, U^*]$ provided with

$$Q(u) = \left[\Omega^{-1}\left[\Lambda^{-1}\left(\Lambda(\Omega(\varrho^2(u))) + \int_{u_0}^{u} l(s)\,ds + \lambda \int_{u_0}^{\omega(u)} j(s)\,ds\right)\right]\right]^\varrho,$$  
(38)

$$\Omega(z) = \int_{1}^{z} \frac{\,ds}{\phi(s)}, \quad z > 0,$$  
(39)

$$\Lambda(z) = \int_{1}^{z} \frac{\phi(\Omega^{-1}(s))\,ds}{(\mu(\Omega^{-1}(s))\Omega^{-1}(s))^{(\varrho+\lambda-1)}}, \quad z > 0,$$  
(40)

$$\xi(z) = \int_{1}^{z} \frac{\,ds}{\phi(s)}, \quad z > 0,$$  
(41)

$\Omega^{-1}, \Lambda^{-1}, \xi^{-1}$ are the inverses of $\Omega, \Lambda, \xi$ and $U^*$ is chosen such that

$$\Lambda(\Omega(\varrho^2(u))) + \int_{u_0}^{u} l(s)\,ds + \lambda \int_{u_0}^{\omega(u)} j(s)\,ds$$  
$$\leq \int_{1}^{z} \frac{\phi(\Omega^{-1}(s))\,ds}{(\mu(\Omega^{-1}(s))\Omega^{-1}(s))^{(\varrho+\lambda-1)}},$$  
(42)

$$\xi(m(u)) + \int_{u_0}^{\omega(u)} j(s)Q(s)\,ds \leq \frac{\,ds}{\phi(s)},$$  
(43)

$$\Lambda^{-1}\left(\Lambda(\Omega(\varrho^2(u))) + \int_{u_0}^{u} l(s)\,ds + \lambda \int_{u_0}^{\omega(u)} j(s)\,ds\right) \leq \frac{\,ds}{\phi(s)},$$  
(44)

Proof: Fixing $U^* \in [u_0, \infty)$ for an arbitrary $u \in [u_0, U^*]$ and letting

$$V(u) = m(U^*) + \int_{u_0}^{\omega(u)} j(s)\mu(r(s))$$  
$$\times \left((r^2(s) + \int_{u_0}^{s} l(\xi)\phi(r(\xi))\,d\xi\right)^\varrho, ds,$$  
(45)

(36), (45) imply that

$$r(u) \leq V(u), \quad V(u_0) = m(U^*),$$  
(46)

since $V(u)$ is non-decreasing, then (45) equals to

$$V'(u) = j(\omega(u))\omega'(u)\mu(r(\omega(u)))$$  
$$\times \left((r^2(\omega(u)) + \int_{u_0}^{\omega(u)} l(s)\phi(r(s))\,ds\right)^\varrho$$  
$$\leq j(\omega(u))\omega'(u)\mu(V(\omega(u)))$$  
$$\times \left(V^2(\omega(u)) + \int_{u_0}^{\omega(u)} l(s)\phi(V(s))\,ds\right)^\varrho$$  
$$\leq j(\omega(u))\omega'(u)\mu(V(\omega(u)))W^\varrho(\omega(u)), $$  
(47)

where

$$W(\omega(u)) = V^2(\omega(u)) + \int_{0}^{\omega(u)} l(s)\phi(V(s))\,ds,$$  
(48)

and

$$W(\omega(u_0)) = m^2(U^*), \quad V(\omega(u)) \leq W(\omega(u)),$$  
(49)

by the definition of $W(\omega(u))$, utilizing (47), (49) and $W(\omega(u)) > 0$, we have

$$W'(\omega(u))\omega'(u)$$  
$$= \lambda V^\varrho-1(\omega(u))\omega'(u) + l(\omega(u))\omega'(u)\phi(V(\omega(u)))$$  
(50)

so that

$$W'(\omega(u)) \leq \lambda j(\omega(u))\omega'(u)\mu(W(\omega(u)))W^{(\varrho+\lambda-1)}(\omega(u))$$  
$$+ l(\omega(u))\phi(W(\omega(u)))$$  
(51)

$$W'(\omega(u)) \leq \frac{\lambda j(\omega(u))\omega'(u)\mu(\omega(u))}{\phi(W(\omega(u)))} + l(\omega(u)),$$  
(52)

integrate (52), using (39) and (49), we get

$$\Omega(\omega(u))$$  
$$\leq \Omega(m^2(U^*)) + \int_{u_0}^{u} l(\omega(s))\,ds$$  
$$+ \int_{u_0}^{u} \frac{\lambda j(\omega(s))\omega'(s)\mu(W(\omega(s)))W^{(\varrho+\lambda-1)}(\omega(s))}{\phi(W(\omega(s)))}\,ds$$  
$$\leq \Omega(m^2(U^*)) + \int_{u_0}^{U^*} l(\omega(s))\,ds$$  
$$\geq \int_{u_0}^{U^*} \frac{\lambda j(\omega(s))\omega'(s)\mu(W(\omega(s)))W^{(\varrho+\lambda-1)}(\omega(s))}{\phi(W(\omega(s)))}\,ds,$$  
(53)

for $u < U^*$. Denoting

$$V_1(u) = \int_{u_0}^{u} \frac{\lambda j(\omega(s))\omega'(s)\mu(W(\omega(s)))W^{(\varrho+\lambda-1)}(\omega(s))}{\phi(W(\omega(s)))}\,ds,$$  
(54)

(53), (54) gives

$$W(\omega(u)) \leq \Omega^{-1}(V_1(u)),$$  
(55)

$$V_1(u_0) = \Omega(m^2(U^*)) + \int_{u_0}^{U^*} l(\omega(s))\,ds,$$  
(56)

differentiate (54) and applying (55), we observe that

$$V_1'(u) = \frac{\lambda j(\omega(u))\omega'(u)\mu(W(\omega(u)))W^{(\varrho+\lambda-1)}(\omega(u))}{\phi(W(\omega(u)))}$$  
$$\leq \frac{\lambda j(\omega(u))\omega'(u)\mu(V_1^{-1}(V_1(U)))\Omega^{-1}}{\phi(V_1^{-1}(V_1(U)))},$$  
(57)

implies

$$\frac{\phi(V_1^{-1}(V_1(U)))V_1'(u)}{\mu(V_1^{-1}(V_1(U)))\Omega^{-1}(V_1^{-1}(V_1(U)))} \leq \lambda j(\omega(u))\omega'(u),$$  
(58)
inequality (57) by integrating and using (40), (55) yields the estimate

\[
\Lambda(V_1(u)) \leq \Lambda(\Omega(m^2(U^*))) + \int_{u_0}^{U^*} l(\omega(s)) \, ds \\
+ \lambda \int_{u_0}^{U^*} j(\omega(s)) \omega'(s) \, ds,
\]

(58)

\[
V_1(t) \leq \Lambda^{-1} \left( \Lambda(\Omega(m^2(U^*))) + \int_{u_0}^{U^*} l(\omega(s)) \, ds + \lambda \int_{u_0}^{\omega(u)} j(s) \, ds \right),
\]

(59)

we conclude from (55) and (59) that

\[
W^\omega(\omega(u)) \leq \left( \Lambda^{-1} \left( \Lambda(\Omega(m^2(U^*))) + \int_{u_0}^{U^*} l(\omega(s)) \, ds + \lambda \int_{u_0}^{\omega(u)} j(s) \, ds \right) \right)^\Theta \\
= Q(u),
\]

(60)

substitute (60) in (47), take integral from \( u_0 \) to \( u \) in the resulting inequality, use (41) and let \( u = U^* \), we obtain

\[
V(U^*) \leq \xi^{-1} \left( \xi(m(U^*)) + \int_{u_0}^{\omega(u)} j(s)Q(s) \, ds \right),
\]

(61)

therefore from (46)

\[
r(U^*) \leq \xi^{-1} \left( \xi(m(U^*)) + \int_{u_0}^{\omega(u)} j(s)Q(s) \, ds \right),
\]

(62)

the arbitrary nature of \( U^* \) in (62) produces the acquired inequality in (37).

3. Application

This segment is about to indicate a prompt use of Theorem 2.3 for analysing the boundedness of the following retarded integral equation of Volterra type is indicated

\[
y^2(u) - \int_{u_0}^{\omega(u)} t(s)y(s) \left( y(s) + \int_{u_0}^{S} z(v)y(v) \, dv \right) \frac{1}{2} \, ds \\
+ \int_{u_0}^{\omega(u)} t(s)y(s) \left( \int_{u_0}^{S} n(v)y(v) \, dv \right) \, ds = c(u),
\]

(63)

Example 3.1: Let \( y(u), r(u), z(u), n(u) \) and \( c(u) \) be continuous functions on \( [0, \infty) \), \( \omega(u) \) be as defined in (H2). If \( y(u) \) satisfies (63), then

\[
|y(u)| \leq \left[ |c(u)| + \frac{K(u) \exp \left( \frac{1}{6} \int_{u_0}^{\omega(u)} \Theta(s) \, ds \right)}{1 - \frac{1}{6} K(u) \int_{u_0}^{\omega(u)} N(s) \, ds} \right]^{\frac{1}{2}},
\]

(64)

provided with

\[
K(u) = \int_{u_0}^{\omega(u)} \frac{1}{6} |t(s)| \left[ \frac{1}{3} |c(s)|^2 + \frac{7}{3} |c(s)| \right] \exp \left( \frac{1}{6} \int_{u_0}^{\omega(u)} \Theta(v) \, dv \right) \, ds.
\]

Suppose that \( |y(u)| = r(u) \), hence (68) takes the form

\[
r(u)^3 \leq |c(u)| + \int_{u_0}^{\omega(u)} |t(s)|r(s) \\
\times \left( r(s) + \int_{u_0}^{S} |z(v)r(v) \, dv \right)^{1/2} \\
+ \int_{u_0}^{\omega(u)} |t(s)| |y(s)| \left( \int_{u_0}^{S} |n(v)| |y(v) \, dv \right) \, ds,
\]

(68)

Using Lemma 2.1 on the right hand side of the inequality (69), we get

\[
r(u)^3 \leq |c(u)| + \int_{u_0}^{\omega(u)} |g(s)|r(s) \\
\times \left( \frac{1}{2} r(s) + \frac{1}{2} \int_{u_0}^{S} |z(v)r(v) \, dv \right)^{1/2} \\
+ \int_{u_0}^{\omega(u)} |t(s)|r(s) \left( \int_{u_0}^{S} |n(v)|r(v) \, dv \right) \, ds.
\]

(69)

put

\[
r_1(u) = \int_{u_0}^{\omega(u)} |t(s)|r(s)
\]

(70)
\[ r(u) \leq [(c(u)) + r_1(u)]^2 \leq \frac{1}{3}[(c(u)) + r_1(u)] + \frac{2}{3}, \quad (72) \]

substituting (72) in (71), we gather
\[ r_1(u) \leq \int_{u_0}^{\theta(u)} |t(s)| \left( \frac{1}{3} |c(s)| + r_1(s) \right) \] 
\[ \times \left( \frac{1}{6} \int_{u_0}^{\theta(u)} |t(s)| \left( \frac{1}{3} |c(s)| + \frac{1}{3} \right) \right) ds \] 
\[ + \int_{u_0}^{\theta(u)} |t(s)| \left( \frac{1}{3} |c(s)| + r_1(s) \right) \] 
\[ \times \left( \frac{1}{6} \int_{u_0}^{\theta(u)} |z(v)| \left( \frac{1}{3} |c(v)| + r_1(v) \right) dv \right) ds \] 
\[ + \int_{u_0}^{\theta(u)} |t(s)| \left( \frac{1}{3} |c(s)| + r_1(s) \right) \] 
\[ \times \left( \frac{1}{6} \int_{u_0}^{\theta(u)} |z(v)| \left( \frac{1}{3} |c(v)| \right) dv \right) ds \] 
\[ + \int_{u_0}^{\theta(u)} |t(s)| \left( \frac{1}{3} |c(s)| \right) ds + \int_{u_0}^{\theta(u)} |t(s)| \] 
\[ \times \left( \frac{1}{3} |c(s)| + r_1(s) \right) \left( \int_{u_0}^{\theta(u)} |n(v)| dv \right) ds \] 
\[ + \int_{u_0}^{\theta(u)} |t(s)| \left( \frac{1}{3} + \frac{1}{3} |c(s)| + r_1(s) \right) \] 
\[ \times \left( \frac{1}{3} \int_{u_0}^{\theta(u)} |n(v)| dv \right) \] 
\[ \times \left( \frac{1}{3} |c(s)| + r_1(s) \right) + 2 + \int_{u_0}^{\theta(u)} |z(v)| |c(v)| dv \] 
\[ ds \] 
\[ \leq K(u) + \frac{1}{6} \int_{u_0}^{\theta(u)} \Theta(s)r_1(s) ds \] 
\[ + \frac{1}{6} \int_{u_0}^{\theta(u)} N(s)r_1^2(s) ds. \quad (73) \]

where \( K(u), \Theta(u), N(u) \) are as mentioned in (65), (66) and (67), respectively. The required bound (64) can be attained by the suitable application of Theorem 2.3 in (73) with some alterations, therefore we omit the details.

\[ \blacksquare \]

4. Conclusion

The intent of this paper is to formulate some new non-linear Gronwall–Bellman–Pachpatte type of inequalities of one independent variable. Unlike some former results in the literature, the integral inequalities taken into consideration right here include the retarded term, which brings about challenges in the estimation of unknown functions on the upper bounds. It can be
accompanied that the received inequalities generalize and expand some current consequences and permit us to contemplate the stability, boundedness and asymptotic behavior of the solutions of a class of more general retarded nonlinear differential, integral and integro-differential equations. Using our method, the integral inequality can be further studied with more dimensions.

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