A Differential Private Method for Distributed Optimization in Directed Networks via State Decomposition

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Abstract—In this paper, we study the problem of consensus-based distributed optimization, where a network of agents, abstracted as a directed graph, aims to minimize the sum of all agents’ cost functions collaboratively. In existing distributed optimization approaches (Push-Pull/AB) for directed graphs, all agents exchange their states with neighbors to achieve the optimal solution with a constant stepsize, which may lead to the disclosure of sensitive and private information. For privacy preservation, we propose a novel state-decomposition based gradient tracking approach (SD-Push-Pull) for distributed optimization over directed networks that preserves differential privacy, which is a strong notion that protects agents’ privacy against an adversary with arbitrary auxiliary information. The main idea of the proposed approach is to decompose the gradient state of each agent into two sub-states. Only one substate is exchanged by the agent with its neighbours over time, and the other one is not shared. That is to say, only one substate is visible to an adversary, protecting the sensitive information from being leaked. It is proved that under certain decomposition principles, a bound for the sub-optimality of the proposed algorithm can be derived and the differential privacy is achieved simultaneously. Moreover, the trade-off between differential privacy and the optimization accuracy is also characterized. Finally, a numerical simulation is provided to illustrate the effectiveness of the proposed approach.

Index Terms—distributed optimization, directed graph, decomposition, differential private.

I. INTRODUCTION

With the rapid development in networking technologies, distributed optimization over multi-agent networks has been a heated research topic during the last decade, where agents aim to collaboratively minimize the sum of local functions possessed by each agent through local communication. Compared with centralized ones, distributed algorithms allow more flexibility and scalability due to its capability of breaking large-scale problems into sequences of smaller ones. In view of this, distributed algorithms are inherently robust to environment uncertainties and communication failures and are widely adopted in power grids [1], sensor networks [2] and vehicular networks [3].

The most commonly used algorithms for distributed optimization is the Decentralized Gradient Descent (DGD), requiring diminishing step-sizes to ensure optimality [4]. To overcome this challenge, Xu et al. [5] replaced the local gradient with an estimated global gradient based on the dynamic average consensus [6] and then proposed a gradient tracking method for distributed optimization problem. Recently, Pu et al. [7] and Xin and Khan [8] devised a modified gradient-tracking algorithm called Push-Pull/AB algorithm for consensus-based distributed optimization, which can be applied to a general directed graph including undirected graph as a special case.

The above conventional distributed algorithms require each agent to exchange their state information with the neighbouring agent, which is not desirable if the participating agents have sensitive and private information, as the transmitted information is at risk of being intercepted by adversaries. By hacking into communication links, an adversary may have access to all conveyed messages, and potentially obtain the private information of each agent by adopting an attack algorithm. The theoretical analysis of privacy disclosure in distributed optimization is presented by Mandal [9], where the parameters of cost functions and generation power can be correctly inferred by an eavesdropper in the economic dispatch problem. As the number of privacy leakage events is increasing, there is an urgent need to preserve privacy of each agent in distributed systems.

For the privacy preservation in distributed optimization, there have been several research results. Wang [10] proposed a privacy-preserving average consensus in which the state of an agent is decomposed into two substates. Zhang et al. [11] and Lu et al. [12] combined existing distributed optimization approaches with the partially homomorphic cryptography. However, these approaches suffer from high computation complexity and communication cost which may be inapplicable for systems with limited resources. As an appealing alternative, differential privacy has attracted much attention in light of its rigorous mathematical framework, proven security properties, and easy implementation [13]. The main idea of differential private approaches is noise perturbation, leading to a tradeoff between privacy and accuracy. Huang et al. [14] devised a differential private distributed optimization algorithm by adding Laplacian noise on transmitted message with a decaying stepsize, resulting in a low convergence rate. A constant stepsize is achieved by Ding et al. [15], [16] where linear convergence is enjoyed by gradient tracking method and differential privacy is achieved by perturbing states.
None of the aforementioned approaches, however, is suitable for directed graphs with weak topological restrictions, which is more practical in real applications. In practice, the information flows among sensors may not be bidirectional due to the different communication ranges, e.g., the coordinated vehicle control problem [17] and the economic dispatch problem [18]. To address privacy leakage in distributed optimization for agents interacting over an unbalanced graphs, Mao et al. [19] designed a privacy-preserving algorithm based on the push-gradient method with a decaying stepsize, which is implemented via a case study to the economic dispatch problem. Nevertheless, the algorithm in [19] lacked a formal privacy notion and it cannot achieve differential privacy.

All the above motivates us to further develop a differential private distributed optimization algorithm over directed graphs. Inspired by [10], a novel differential private distributed optimization approach based on state decomposition is proposed for agents communicating over directed networks. Under the proposed state decomposition mechanism, a Laplacian noise is perturbed on the gradient state and the global gradient is still tracked after state decomposition. The main contributions of this paper are summarized as follows:

1) We propose a state-decomposition based gradient tracking approach (SD-Push-Pull) for distributed optimization over unbalanced directed networks, where the gradient state of each agent is decomposed into two substates to maintain the privacy of all agents. Specifically, one sub-state replacing the role of the original state is communicated with neighboring agents while the other substate is not shared. Compared to the privacy-preserving approaches in [14] and [15], our proposed approach can be applied to more general and practical networks.

2) By carefully designing the state decomposition mechanism, we only need to add noise to one substate of directions instead of perturbing both states and directions [16]. Moreover, our proposed SD-Push-Pull algorithm does not require the stepsize or noise variance to be diminishing, which ensures a linear convergence to a neighborhood of the optimal solution in expectation exponentially fast under a constant stepsize policy (Theorem 1).

3) Different from the privacy notion in [10] and [19], we adopt the definition of differential privacy, which ensures the privacy of agents regardless of any auxiliary information that an adversary may have and enjoys a rigorous formulation. In addition, we prove that the proposed SD-Push-Pull algorithm can achieve \( \epsilon \)-differential privacy (Theorem 2).

**Notations:** In this paper, \( \mathbb{N} \) and \( \mathbb{R} \) represent the sets whose components are natural numbers and real numbers. \( \mathbb{R}^{++} \) denotes the set of all positive real numbers. \( \mathbb{R}^n \) and \( \mathbb{R}^{n \times p} \) represent the set of \( n \)-dimensional vectors and \( n \times p \)-dimensional matrices. We let \( \mathbb{R}^{n \times p} \) denote the space of vector-valued sequences in \( \mathbb{R}^{n \times p} \). \( 1_n \in \mathbb{R}^n \) and \( I_n \in \mathbb{R}^{n \times n} \) represent the vector of ones and the identity matrix, respectively. The spectral radius of matrix \( A \) is denoted by \( \rho(A) \). \( [x]_r \) denotes the \( r \)-th element of the vector \( x \). For a given constant \( \theta > 0 \), \( \text{Lap}(\theta) \) is the Laplace distribution with probability function \( f_L(x, \theta) = \frac{1}{2\theta} e^{-\frac{|x|}{\theta}} \). In addition, \( \mathbb{E}(x) \) and \( P(x) \) denote the expectation and probability distribution of a random variable \( x \), respectively.

## II. PRELIMINARIES AND PROBLEM FORMULATION

### A. Network Model

We consider a group of agents which communicate with each other over a directed graph. The directed graph is denoted as a pair \( G = (\mathcal{N}, \mathcal{E}) \), where \( \mathcal{N} \) denotes the agents set and \( \mathcal{E} \subset \mathcal{N} \times \mathcal{N} \) denotes the edge set, respectively. A communication link from agent \( i \) to agent \( j \) is denoted by \( (j, i) \in \mathcal{E} \), indicating that agent \( i \) can send messages to agent \( j \). Given a nonnegative matrix \( M = [m_{ij}] \in \mathbb{R}^{n \times n} \), the directed graph induced by \( M \) is denoted by \( \mathcal{G}_M = (\mathcal{N}, \mathcal{E}_M) \), where \( \mathcal{N} = \{1, 2, \ldots, n\} \) and \( (j, i) \in \mathcal{E}_M \) if and only if \( m_{ij} > 0 \). The agents can directly send messages to agent \( i \) are represented as in-neighbours of agent \( i \) and the set of these agents is denoted as \( \mathcal{N}_{\text{in}}^{\text{ir}} \), the set of these agents is denoted as \( \mathcal{N}_{\text{out}}^{\text{ir}} \). Similarly, the agents who can directly receive messages from agent \( i \) are represented as out-neighbours of agent \( i \) and the set of these agents is denoted as \( \mathcal{N}_{\text{out}}^{\text{ir}} \) for \( i \in \mathcal{N} \).

### B. Differential Privacy

Differential privacy serves as a mathematical notion which quantify the degree of the involved individuals’ privacy guarantee in a statistical database. We give the following definitions for preliminaries of differential privacy in distributed optimization.

**Definition 1:** (Adjacency [20]) Two function sets \( S^{(1)} = \{f^{(1)}_i\}_{i=1}^n \) and \( S^{(2)} = \{f^{(2)}_i\}_{i=1}^n \) are said to be adjacent if there exists some \( i_0 \in \{1, 2, \ldots, n\} \) such that \( f^{(1)}_i = f^{(2)}_j \), \( \forall i \neq i_0 \) and \( f^{(1)}_{i_0} \neq f^{(2)}_{i_0} \).

**Definition 1** implies that two function sets are adjacent only if one agent changes its objective function.

**Definition 2:** (Differential privacy [21]) Given \( \epsilon > 0 \), for any pair of adjacent function sets \( S^{(1)} \) and \( S^{(2)} \) and any observation \( \mathcal{O} \subseteq \text{Range}(\mathcal{A}) \), a randomized algorithm \( \mathcal{A} \) keeps \( \epsilon \)-differentially private if

\[
P\{\mathcal{A}(S^{(1)}) \in \mathcal{O}\} \leq e^\epsilon P\{\mathcal{A}(S^{(2)}) \in \mathcal{O}\}
\]

where \( \text{Range}(\mathcal{A}) \) denotes the output codomain of \( \mathcal{A} \).

**Definition 2** illustrates that a random mechanism is differentially private if its outputs are nearly statistically identical over two similar inputs which only differ in one element. Hence, an eavesdropper cannot distinguish between two function sets with high probability based on the output of the mechanism. Here, a smaller \( \epsilon \) represents a higher level of privacy since the eavesdropper has less chance to distinguish sensitive information of each agent from the observations. Nevertheless, a high privacy level will sacrifice the accuracy of the optimization algorithm. Hence, the constant \( \epsilon \) determines a tradeoff between the privacy level and the accuracy.
C. Problem Formulation

Consider an optimization problem in a multi-agent system of $n$ agents. Each agent has a private cost function $f_i$, which is only known to agent $i$ itself. All the participating agents aim to minimize a global objective function

$$\min_{x \in \mathbb{R}^p} \sum_{i=1}^{n} f_i(x)$$

where $x$ is the global decision variable.

To solve Problem 1, assume each agent $i$ maintains a local copy of $x_i \in \mathbb{R}^p$ of the decision variable and an auxiliary variable $y_i \in \mathbb{R}^p$ tracking the average gradients. Then we can rewrite Problem (1) into local optimization problem of each agent with an added consensus constraint as follows

$$\min_{x_i \in \mathbb{R}^p} \sum_{i=1}^{n} f_i(x_i) \quad \text{s.t.} \quad x_i = x_j, \quad \forall i, j,$$

where $x_i$ is the local decision variable of the agent $i$.

Let $x := [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^{n \times p}$, $y := [y_1, y_2, \ldots, y_n]^\top \in \mathbb{R}^{n \times p}$.

Denote $F(x)$ as an aggregate objective function of the local variables, i.e., $F(x) = \sum_{i=1}^{n} f_i(x_i)$.

With respect to the objective function in Problem 1, we assume the following strong convexity and smoothness conditions.

**Assumption 1:** Each objective function $f_i$ is $\mu$-strongly convex with $L$-Lipschitz continuous gradients, i.e., for any $x, y \in \mathbb{R}^p$,

$$\langle \nabla f_i(x) - \nabla f_i(y), x - y \rangle \geq \mu \|x - y\|^2,$$

$$||\nabla f_i(x) - \nabla f_i(y)|| \leq L \|x - y\|.$$

Under Assumption 1, Problem 1 has a unique optimal solution $x^* \in \mathbb{R}^p$.

III. PRIVATE GRADIENT TRACKING ALGORITHM VIA STATE DECOMPOSITION

In this section, we propose a state-decomposition based gradient tracking method (SD-Push-Pull) for distributed optimization over directed graphs, which is described in Algorithm 1. The main idea is to let each agent decompose its gradient state $y_i$ into two substates $y_i^\alpha$ and $y_i^\beta$. The substate $y_i^\alpha$ is used in the communication with other agents while $y_i^\beta$ is never shared with other agents except for agent $i$ itself, so the substate $y_i^\beta$ is imperceptible to the neighbouring agents of agent $i$.

**Algorithm 1** SD-Push-Pull

**Step 1. Initialization:**

1) Agent $i \in \mathcal{N}$ chooses in-bound mixing/pulling weights $R_{ij} \geq 0$ for all $j \in \mathcal{N}_{R,i}^n$, out-bound pushing weights $C_{il} \geq 0$ for all $l \in \mathcal{N}_{C,i}^n$, and the two sub-state weights $\alpha_i, \beta_i \in (0, 1)$.

2) Agent $i \in \mathcal{N}$ picks any $x_{i,0}, y_{i,0}^\alpha, y_{i,0}^\beta \in \mathbb{R}^p$, $\theta_i \in \mathbb{R}_+$. The step size $\eta > 0$ is known to each agent.

**Step 2.** At iteration $k = 0, 1, 2, \ldots, K$

1) Agent $i \in \mathcal{N}$ pushes $\bar{C}_{il} y_{l,k}^\alpha$ to each $l \in \mathcal{N}_{C,i}^n$ to agent $i$.

2) Agent $i \in \mathcal{N}$ draws a random vector $\xi_{i,k}$ consisting of $p$ Laplacian noise independently drawn from $\text{Lap}(\delta_i)$ and updates $y_{i,k+1}^\beta$ as follows:

$$y_{i,k+1}^\beta = \sum_{j \in \mathcal{N}_{R,i}^n \cup \{i\}} \bar{C}_{ij} y_{j,k}^\beta + (1 - \beta_i) y_{i,k}^\beta + \xi_{i,k}$$

3) Agent $i \in \mathcal{N}$ pulls $[x_{j,k} - \eta (y_{j,k+1}^\alpha - y_{j,k}^\alpha)]$ from each $j \in \mathcal{N}_{R,i}^n$.

4) Agent $i \in \mathcal{N}$ updates $x_{i,k+1}$ through

$$x_{i,k+1} = \sum_{j \in \mathcal{N}_{R,i}^n \cup \{i\}} R_{ij} [x_{j,k} - \eta (y_{j,k+1}^\alpha - y_{j,k}^\alpha)].$$

Denote $R := [R_{ij}], \quad \Lambda_\alpha := \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n), \quad \bar{C} := [\bar{C}_{ij}], \quad \Lambda_\beta := \text{diag}(\beta_1, \beta_2, \ldots, \beta_n), \quad \bar{\xi}_k := [\bar{\xi}_{i,k}, \bar{\xi}_{2,k}, \ldots, \bar{\xi}_{n,k}]^\top, \quad y_k := [y_{1,k}^\alpha, \ldots, y_{n,k}^\alpha(k), y_{1,k}^\beta, \ldots, y_{n,k}^\beta(k)]^\top, \quad \nabla F(x_k) = [\nabla f_1(x_k), \ldots, \nabla f_n(x_k)]^\top, \quad \nabla \bar{F}(x_k) = [\bar{\xi}_k^\top, \nabla \bar{F}(x_k)]^\top, \quad C := \begin{bmatrix} \bar{C} & \Lambda_\alpha \\ \Lambda_\alpha^\top & \Lambda_\beta \end{bmatrix}, \quad \Gamma := [I_n \quad 0_n].$

Algorithm 1 can be rewritten in a matrix form as follows:

$$y_{k+1} = \bar{C} y_k + \nabla \bar{F}(x_k),$$

$$x_{k+1} = R [x_k - \eta \Gamma (y_{k+1} - y_k)],$$

where $x_0$ and $y_0$ are arbitrary.

**Remark 1:** In SD-Push-Pull, we design a state decomposition mechanism for the direction state (shown in (3)), where sensitive information $\nabla f_i(x_{i,k})$ is contained in one substate of direction, $y_{i,k}^\beta$. Since the substate $y_{i,k}^\beta$ is not shared in communication link, the private information $\nabla f_i(x_{i,k})$ of agent $i$ is protected from being leaked. Although $y_{ij,k}^\beta$ will be shared through $y_{i,k}^\alpha$, the noise $\xi_{i,k}$ added in the updated of $y_{i,k}^\alpha$ (shown in (3a)) helps to avoid privacy breaches of $f_i$. The above discussion intuitively illustrates how state decomposition mechanism achieves differential privacy of each agent. Rigorous theoretical analysis will be provided in Section V.

**Assumption 2:** The matrix $R \in \mathbb{R}^{n \times n}$ is a nonnegative row-stochastic matrix and $C \in \mathbb{R}^{2n \times 2n}$ is a nonnegative column-stochastic matrix, i.e., $R_{11} = 1_n$ and $1_{2n}^\top C = 1_{2n}$. Moreover, the diagonal entries of $R$ and $C$ are positive, i.e., $R_{ii} > 0, C_{ii} > 0, \forall i \in \mathcal{N}$.

Assumption 2 can be satisfied by properly designing the weights in $R$ and $C$ by each agent locally. For instance, each agent may choose $R_{ij} = \frac{1}{|N_{R,i}| + c_R}$ for some constant $c_R > 0$ for all $j \in N_{R,i}^n$ and let $R_{ii} = 1 - \sum_{j \in N_{R,i}^n} R_{ij}$. Similarly,
agent $i$ may choose $\alpha_i = \zeta$ and $\tilde{C}_{ii} = \frac{1 - \zeta}{N_{C,i}^0 + c_{C}}$ for some constant $0 < \zeta < 1$, $c_{C} > 0$ for all $l \in N_{C,i}^0$, and let $\tilde{C}_{ii} = 1 - \zeta - \sum_{l \in N_{C,i}^0} \tilde{C}_{ii}$. Such a choice of weights renders $\mathbf{R}$ row-stochastic and $\mathbf{C}$ column-stochastic, thus satisfying Assumption 2.

Assumption 3: The graphs $\mathcal{G}_R$ and $\mathcal{G}_{C^T}$ induced by matrices $\mathbf{R}$ and $\mathbf{C}$ contain at least one spanning tree. In addition, there exists at least one agent that is a root of spanning trees for both $\mathcal{G}_R$ and $\mathcal{G}_{C^T}$.

Assumption 3 is weaker than assumptions in most previous works (e.g., [8], [23], [24]), where graphs $\mathcal{G}_R$ and $\mathcal{G}_{C^T}$ are assumed to be strongly connected. The relaxed assumption about graph topology enables us to design graphs $\mathcal{G}_R$ and $\mathcal{G}_{C^T}$ more flexibly. Similar assumption are adopted in [22], [25].

Lemma 1 ([26]): Under Assumption 2, the matrix $\mathbf{R}$ has a unique nonnegative left eigenvector $u^\top$ (w.r.t. eigenvalue) with $u^\top 1_n = n$, and matrix $\mathbf{C}$ has a unique nonnegative right eigenvector $v$ (w.r.t. eigenvalue) with $1_n^\top v = n$.

To make connections with the traditional push-pull algorithm, we let $\tilde{y}_k := y_{k+1} - y_k$. Then one can have

$$x_{k+1} = \mathbf{R}(x_k - \eta \mathbf{T}\tilde{y}_k),$$

$$\tilde{y}_{k+1} = \mathbf{C}\tilde{y}_k + \nabla\tilde{F}(x_{k+1}) - \nabla\tilde{F}(x_k).$$

Since $\mathbf{C}$ is column stochastic,

$$1_{2n}^\top \tilde{y}_{k+1} = 1_{2n}^\top \tilde{y}_k + 1_{2n}^\top \nabla\tilde{F}(x_{k+1}) - 1_{2n}^\top \nabla\tilde{F}(x_k).$$

From $1_{2n}^\top \tilde{y}_0 = 1_{2n}^\top y_1 - 1_{2n}^\top y_0 = 1_{2n}^\top \nabla\tilde{F}(x_0)$, we have by induction that

$$1_{2n}^\top \tilde{y}_k = 1_{2n}^\top \nabla\tilde{F}(x_k) = \frac{1}{n} 1_n^\top \nabla\tilde{F}(x_k) + \frac{1}{n} 1_n^\top \xi_k. \quad (7)$$

IV. Convergence Analysis

In this section, we analyze the convergence performance of the proposed private push-pull algorithm. For the sake of analysis, we define the following variables:

$$\bar{x}_k := \frac{1}{n} u^\top x_k, \quad \bar{y}_k = \frac{1}{n} v^\top \tilde{y}_k.$$

The main idea of our strategy is to bound $\mathbb{E}[||\tilde{x}_{k+1} - x^*||^2]/2, \mathbb{E}[||x_{k+1} - 1_n^\top \bar{x}_{k+1}||^2], \mathbb{E}[||\tilde{y}_{k+1} - v\bar{y}_k||^2]$ on the basis of the linear combinations of their previous values, where $|| \cdot ||_R$ and $|| \cdot ||_C$ are specific norms to be defined later. By establishing a linear system of inequalities, we can derive the convergence result.

Definition 3: Given an arbitrary vector norm $|| \cdot ||$, for any $x \in \mathbb{R}^{n \times p}$, we define a matrix norm

$$||x|| := \left(\sum_{i=1}^p ||(x^{(1)}||, ||x^{(2)}||, \ldots, ||x^{(p)}||)^2 \right)^{1/2},$$

where $x^{(1)}, x^{(2)}, \ldots, x^{(p)} \in \mathbb{R}^n$ are columns of $x$.

A. Preliminary Analysis

From Eqs. (6a) and Lemma 1, we can obtain

$$\bar{x}_{k+1} = \frac{1}{n} u^\top \mathbf{R}(x_k - \eta \mathbf{T}\tilde{y}_k) = \bar{x}_k - \frac{\eta}{n} u^\top \mathbf{T}\tilde{y}_k. \quad (8)$$

Furthermore, let us define

$$g_k := \frac{1}{n} \sum_{i=1}^n \nabla f_i(x_k), \quad h_k := \frac{1}{n} \sum_{i=1}^n \nabla f_i(x_{i,k}),$$

which leads to $\bar{y}_k = h_k + \frac{1}{n} 1_n^\top \xi_k$. Then, from equation (8)

$$\bar{x}_{k+1} = \bar{x}_k - \frac{\eta}{n} u^\top \mathbf{T}(\bar{y}_k - v\bar{y}_k + v\bar{y}_k)$$

$$= \bar{x}_k - \frac{\eta}{n} u^\top \mathbf{T}v\bar{y}_k - \frac{\eta}{n} u^\top \mathbf{T}(\bar{y}_k - v\bar{y}_k)$$

$$= \bar{x}_k - \eta'(h_k - g_k) - \frac{\eta}{n} u^\top \mathbf{T}(\bar{y}_k - v\bar{y}_k) - \frac{\eta'}{n} 1_n^\top \xi_k,$$

$$\eta' := \frac{2}{n} u^\top \mathbf{T}v.$$ Based on Lemma 1 and equation (8), we obtain

$$\bar{x}_{k+1} - 1_n^\top \bar{x}_{k+1} = \left(\mathbf{R} - \frac{1}{n} u^\top \mathbf{R} \right) \bar{x}_k - \frac{\eta}{n} u^\top \mathbf{T}v \bar{y}_k.$$

Similarly, we have

$$\bar{y}_{k+1} = \frac{v}{n} \bar{y}_k + \frac{v}{n} \bar{y}_k + \frac{v}{n} \bar{y}_k$$

$$= \mathbf{C}\tilde{y}_k + \nabla\tilde{F}(x_{k+1}) - \nabla\tilde{F}(x_k) - (v(y_{k+1} - y_k) - v\bar{y}_{k+1})$$

$$= (C - \frac{v^2}{n} 1_n^\top \mathbf{C})(\bar{y}_{k+1} - v\bar{y}_k)$$

$$+ (v - \frac{v^2}{n} 1_n^\top \mathbf{C})(\nabla\tilde{F}(x_{k+1}) - \nabla\tilde{F}(x_k)).$$

Denote $\mathcal{F}_k$ as the $\sigma$-algebra generated by $\{\xi_0, \ldots, \xi_{k-1}\}$, and define $\mathbb{E}[\cdot | \mathcal{F}_k]$ as the conditional expectation given $\mathcal{F}_k$.

B. Supporting lemmas

We next prepare a few useful supporting lemmas for further convergence analysis.

Lemma 2: Under Assumption 1, there holds

$$||h_k - g_k||^2 \leq \frac{L}{\sqrt{n}} ||x_k - 1_n^\top \bar{x}_{k+1}||^2,$$

$$||g_k||^2 \leq \frac{L}{2} ||\bar{x}_k - x^*||^2,$$

$$\mathbb{E}[||\tilde{y}_k - h_k||^2 | \mathcal{F}_k] \leq \frac{2p^2}{n},$$

where $\bar{\theta} = \max_i \theta_i$.

Proof: In view of Assumption 1

$$||h_k - g_k||^2 = \frac{1}{n} ||1_n^\top \nabla F(x_k) - 1_n^\top \nabla F(1_n^\top \bar{x}_{k+1})||^2$$

$$\leq \frac{L}{n} \sum_{i=1}^n ||x_{i,k} - \bar{x}_k||^2 \leq \frac{L}{\sqrt{n}} ||x_k - 1_n^\top \bar{x}_{k+1}||^2.$$
Suppose Assumptions 2 and 3 hold. There exist vector norms $\| \cdot \|_R$ and $\| \cdot \|_C$ such that

$$\| C \|_R \leq \| R \|_C < 1 \quad \text{and} \quad \rho := \| C \|_R / \| R \|_C < 1,$$

and $\rho$ is satisfied:

$$\mathbb{E}[|| \tilde{x}_k - x^* \|_2^2 | F_k] = \frac{1}{n^2} \mathbb{E}[|| \tilde{x}_k ||_2^2 | F_k] \leq \frac{2 \rho \theta^2}{n},$$

where $\theta = \max \{ \| I - \mu \theta \|, 1 - L \theta \}.$

Lemma 3: (Adapted from Lemma 10 in [27]) Under Assumption 1, for any $x \in \mathbb{R}^p$ and $0 < \theta < 2/\mu$, we have

$$\| x - \theta F(x) - x^* \|_2 \leq \tau \| x - x^* \|_2,$$

where $\tau = \max (1 - \mu \theta, 1 - L \theta)$. 

Lemma 4: (Adapted from Lemma 3 and Lemma 4 in [7]) Suppose Assumptions 2 and 3 hold. There exist vector norms $\| \cdot \|_R$ and $\| \cdot \|_C$, such that

$$\| R \|_C \leq \rho < 1, \quad \| C \|_R \leq \frac{1}{\rho}, \quad \text{and} \quad \| \cdot \|_R \leq \| \cdot \|_C,$$

and $\rho$ is arbitrarily close to the spectral radius $\rho := \| C \|_R / \| R \|_C < 1$, and $\tau_R$ and $\tau_C$ are defined in (23). Then, $\sup_{k \geq 2} \mathbb{E}[|\hat{x}_k - x^*|_2]$ converges to

$$\mathbb{E}[|x_k - x^*|_2] \leq \min \left\{ \sqrt{\frac{1 - \sigma_R^2}{6 \sigma_I}}, \sqrt{\frac{1 - \sigma_C^2}{6 \sigma_C}} \right\}$$

where $\sigma_I = \max \{ \sqrt{1 - \mu \theta}, 1 - L \theta \}$.

Proof: See Appendix VIII-A.

The following theorem shows the convergence properties for the SD-Push-Pull algorithm in (6).

**Theorem 1:** Suppose Assumptions 3 holds and the stepsize $\eta$ satisfies

$$\eta \leq \min \left\{ \sqrt{\frac{1 - \sigma_R^2}{6 \sigma_I}}, \sqrt{\frac{1 - \sigma_C^2}{6 \sigma_C}} \right\}$$

where $d_1, d_2, d_3$ are defined in (23). Then $\sup_{k \geq 2} \mathbb{E}[|\hat{x}_k - x^*|_2]$ converges to

$$\min \left\{ \sqrt{\frac{1 - \sigma_R^2}{6 \sigma_I}}, \sqrt{\frac{1 - \sigma_C^2}{6 \sigma_C}} \right\}$$

where $((I - A)^{-1}B)_{ii}$ denotes the $i$th element of the vector $(I - A)^{-1}B$. Their specific forms are given in (25) and (26), respectively.

Proof: In terms of Lemma 9 by induction we have

$$\mathbb{E}[|\hat{x}_k - x^*|_2] \leq \mathbb{E}[|x_k - x^*|_2] \leq 1 \mathbb{E}[|x_k - x^*|_2]$$

$$\leq 1 \mathbb{E}[|x_{k+1} - x^*|_2]$$

$$\leq \sum_{i=0}^{k-1} 1 \mathbb{E}[|x_{i+1} - x^*|_2]$$

$$\leq \sum_{i=0}^{k-1} 1 \mathbb{E}[|x_{i+1} - x^*|_2]$$

From equation (17), we can see that if $\rho (A) < 1$, then $\sup_{k \geq 2} \mathbb{E}[|\hat{x}_k - x^*|_2]$ converges to a neighborhood of $0$ at the linear rate $O(\rho(A)^k)$.

In view of Lemma 7 it suffices to ensure $a_{11}, a_{22}, a_{33} < 1$ and $\det(1 - A) > 0$, or

$$\det(1 - A) = (1 - a_{11})(1 - a_{22})(1 - a_{33}) - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} - (1 - a_{22})(1 - a_{33})(1 - a_{11})$$

which is equivalent to

$$\frac{1}{2}(1 - a_{11})(1 - a_{22})(1 - a_{33}) - c_1c_6c_9\eta^3$$

$$- c_2c_4c_9\eta^3 - (1 - a_{22})c_2c_8\eta^3 - (1 - a_{11})c_6c_9\eta^2$$

and

$$b = [c_3 \eta^2, c_7 \eta^2, c_{11}]^T \theta^2$$

respectively, where constants $c_i$'s and $b_1$ are defined in (39), (41) and (42).

Proof: See Appendix VIII-A.
requiring
\[ \eta \leq \min \left\{ \sqrt{\frac{1 - \sigma^2_R}{6c_5}}, \sqrt{\frac{1 - \sigma^2_C}{6c_{10}}} \right\}. \] (21)

Second, in view of relation (20), \( a_{22} > \frac{1 + \sigma^2_R}{2} \), and \( a_{33} > \frac{1 + \sigma^2_C}{2} \), we have
\[
\begin{align*}
\frac{1}{2} (1-a_{11})(1-a_{22})(1-a_{33}) & - c_1 c_6 c_8 \eta^5 - c_2 c_4 \eta^3 \\
(1-a_{22}) c_6 \eta^3 & - (1-a_{11}) c_6 c_8 \eta^2 - (1-a_{33}) c_1 c_4 \eta^3 \\
> \frac{\eta \mu}{2} & - \frac{\sigma^2_R}{3} - \frac{\sigma^2_C}{3} - c_1 c_6 c_8 \eta^5 - c_2 c_4 \eta^3 - \eta \mu c_6 c_8 \eta^2 \\
- \frac{1}{2} \sigma^2_R c_2 c_4 \eta^3 & - \frac{1}{2} \sigma^2_C c_1 c_4 \eta^3.
\end{align*}
\] (22)

Then, relation (19) is equivalent to
\[ d_1 \eta^4 + d_2 \eta^2 - d_3 < 0, \]
where
\[
\begin{align*}
d_1 & := c_1 c_6 c_8, \\
d_2 & := c_2 c_4 c_9 + \frac{1 - \sigma^2_R}{2} c_2 c_8 + \frac{1 - \sigma^2_C}{2} c_1 c_4 + \frac{u^T T_{v\mu} c_6 c_8}{n}, \\
d_3 & := \frac{u^T T_{v\mu}}{18n} (1 - \sigma^2_R)(1 - \sigma^2_C).
\end{align*}
\] (23)

Hence, a sufficient condition for \( \det(I - A) > 0 \) is
\[ \eta \leq \sqrt{\frac{2d_3}{d_2 + \sqrt{d_2^2 + 4d_1 d_3}}}. \] (24)

Relation (21) and (24) yield the final bound on the stepsize \( \eta \).

Moreover, in light of (18) and (20), we can obtain from (17) that
\[
\begin{align*}
[(I - A)^{-1} B]_i & = \left[ ((1 - a_{11})(1-a_{33}) - a_{22} a_{32}) c_6 \eta^2 \right] \\
& + (a_{13} a_{32} + a_{12} (1 - a_{33})) c_7 \eta^2 \\
& + (a_{12} a_{23} + a_{13} (1 - a_{22})) c_1 \eta^2 \det(I - A) \\
& \leq \frac{18 \sigma^2_R}{\eta \mu} \left[ ((1 - \sigma^2_R)(1 - \sigma^2_C) - c_6 c_8 \eta^2) c_3 \eta^2 \\
& + \left( c_2 c_9 + \frac{c_1 (1 - \sigma^2_C)}{2} \right) c_7 \eta^3 + \left( c_1 c_6 \eta^3 + \frac{c_2 c_4 (1 - \sigma^2_R)}{2} \right) \right] c_1, \\
\end{align*}
\] (25)

and
\[
\begin{align*}
[(I - A)^{-1} B]_i & = \left[ (a_{23} a_{31} + a_{21} (1 - a_{33})) c_3 \eta^2 \right] \\
& + ((1 - a_{11})(1-a_{33}) - a_{13} a_{32}) c_7 \eta^2 \\
& + (a_{13} a_{21} + a_{23} (1 - a_{11})) c_1 \eta^2 \det(I - A) \\
& \leq \frac{18 \sigma^2_R}{\eta \mu} \left[ (c_6 c_8 \eta^4 + \frac{c_4 \eta^2 (1 - \sigma^2_C)}{2}) c_3 \eta^2 \\
& + \left( \frac{\eta \mu (1 - \sigma^2_C)}{2} c_2 c_8 \eta^3 \right) c_7 \eta^2 + \left( c_2 c_9 \eta^3 + \frac{c_6 c_8 \eta^2 \mu}{2} \right) \right] c_1. \\
\end{align*}
\] (26)

Remark 2: When \( \eta \) is sufficiently small, it can be shown that the linear rate indicator \( \rho(A) \approx 1 - \eta^4 \mu. \) From Theorem 1 it is worth noting that the upper bounds in (25) and (26) are functions of \( \eta, \mu \), and other problem parameters, and they are decreasing in terms of \( \mu. \) Fixing the system parameter and the privacy level \( \min_i \epsilon_i \), the optimization accuracy has the order of \( d \sim O(1/ \min_i \epsilon_i) \) for small \( \epsilon_i \). As \( \epsilon_i \) converges to 0, that is, for complete privacy for each agents, the accuracy becomes arbitrarily low.

V. DIFFERENTIAL PRIVACY ANALYSIS

In this section, we analyze the differential privacy property of SD-Push-Pull. The observation \( O_{\delta} \) denotes the message transmitted between agents, where \( O_{\delta} = \{x_{i,k} - \eta (y_{i,k+1}^n - y_{i,k}^n), c_0 y_{i,k}^n \forall i, j \in N \}. \) Considering the differential privacy of agent \( i_0 \)'s objective function \( f_{i_0} \), we assume there exists a type of passive adversary called eavesdropper defined as follows, who is interested in the agent \( i_0 \).

Definition 4: An eavesdropper is an external adversary who has access to all transmitted data \( O_{\delta}, \forall k \) by eavesdropping on the communications among the agents. Moreover, he knows all agents’ functions \( \{f_i\}, i \neq i_0 \), the network topology \( \{R, C\} \) and the initial value of all agents \( \{x_0, y_0\} \).

Before analyzing the differential privacy, we need the following assumption to bound the gradient of the objective function \( f_i \) in the adjacency definition in Definition 1

Assumption 4: Given a finite number of iterations \( k \leq K \), the gradients of all local objective functions \( \nabla f_i(x_{i,k}) \forall i \in N, \forall k \leq K \), are bounded, i.e., there exists a positive constant \( C \) such that for all \( k \leq K, \|\nabla f_i(x_{i,k})\| \leq C, \forall i \in N \).

Next, we derive condition on the noise variance under which SD-Push-Pull satisfies \( \epsilon \)-differential privacy.

Theorem 2: Given a finite number of iterations \( k \leq K \), under Assumption 4, SD-Push-Pull preserves \( \epsilon \)-differential privacy for a given \( \epsilon_i > 0 \) of each agent \( i \)'s cost function if the noise parameter in (35) satisfies \( \theta_i > 2\sqrt{\B_K / \epsilon_i}. \)

Proof: Since the eavesdropper is assumed to know the initial states of the algorithm, we have \( x_0^{(1)} = x_0^{(2)} \) and \( y_0^{(1)} = y_0^{(2)}. \) From Algorithm 1, it can be seen that given initial state \( \{x_0, y_0\} \), the network topology \( \{R, C\} \) and the function set \( S \), the observation sequence \( z = \{x_{0,k}\}_{k=0} \) is uniquely determined by the noise sequence \( \xi = \{\xi_k\}_{k=0} \). Hence, we use function \( Z_{\mathcal{F}} : (\mathbb{R}^{n \times p})^K \rightarrow (\mathbb{R}^{n \times p})^K \) to represent the relation, where \( \mathcal{F} = \{x_0, y_0, R, C, S\} \), i.e., \( z = Z_{\mathcal{F}}(\xi) \). From Definition 2 keeping \( \epsilon \)-differential privacy is equivalent to guarantee that for any pair of adjacent function sets \( S^{(1)} \) and \( S^{(2)} \) and any observation \( O \subseteq \text{Range}(\mathcal{F}), \)

\[ P\{x \in \Omega | Z_{\mathcal{F}^{(1)}}(x) \in O\} \leq e^\epsilon P\{x \in \Omega | Z_{\mathcal{F}^{(2)}}(x) \in O\}, \]

where \( \mathcal{F}^{(i)} = \{x_0, y_0, R, C, S^{(i)}\}, i = 1, 2 \) and \( \Omega = (\mathbb{R}^{n \times p})^K \) denotes the sample space.

Hence, we need to consider that \( Z_{\mathcal{F}^{(1)}}(x^{(1)}) = Z_{\mathcal{F}^{(2)}}(x^{(2)}). \) It is indispensable to guarantee that \( \forall k \leq K, \forall i \in N, y_{i,k+1}^{(1)} = y_{i,k+1}^{(2)} \) and \( x_{i,k}^{(1)} = x_{i,k}^{(2)} \). From the update rule of Algorithm 1 we can obtain that \( y_{i,k+1}^{(1)} = y_{i,k+1}^{(2)}, \forall k \leq K, \forall i \in N \) can be inferred.
lead to $x_{i,k}^{(1)} = x_{i,k}^{(2)}, \forall k \leq K, \forall i \in N$. Thus, we only need to guarantee that $y_{i,k+1}^{(1)} = y_{i,k+1}^{(2)}, \forall k \leq K, \forall i \in N$.

For any $i \neq i_0$, to ensure $y_{i_0,1}^{(1)} = y_{i_0,1}^{(2)}$, the noise should satisfy $\xi_{i_0}^{(1)} = \xi_{i_0}^{(2)}$ from (3a). It then follows $y_{i_1,1}^{(1)} = y_{i_1,1}^{(2)}$ from (3b) since $f_{i_1}^{(1)} = f_{i_1}^{(2)}$. Based on the above analysis, we can finally obtain

$$\xi_{i,k}^{(1)} = \xi_{i,k}^{(2)}, \quad \forall k \leq K, \forall i \neq i_0. \quad (27)$$

Similarly, for agent $i_0$, at iteration $k = 0$, we have

$$\xi_{i_0,0}^{(1)} = \xi_{i_0,0}^{(2)} \quad (28)$$

For $k \geq 1$, since $f_{i_0}^{(1)} \neq f_{i_0}^{(2)}$, the noise should satisfy

$$\Delta \xi_{i_0,k} = -(1 - \beta_{i_0})\Delta y_{i_0,k}, 1 \leq k \leq K, \quad (29)$$

where $\Delta \xi_{i_0,k} = \xi_{i_0,k}^{(1)} - \xi_{i_0,k}^{(2)}$ and $\Delta y_{i_0,k} = y_{i_0,k}^{(1)} - y_{i_0,k}^{(2)}$. In light of equation (3b), we have

$$\Delta y_{i_0,k} = \beta_{i_0} \Delta y_{i_0,k-1} + \Delta f_{i_0,k-1}, \forall k \geq 1, \quad (30)$$

where $\Delta f_{i_0,k-1} = \nabla f_{i_0}^{(1)}(x_{i_0,k}) - \nabla f_{i_0}^{(2)}(x_{i_0,k})$.

Given $\Delta y_{i_0,0} = 0$, by induction we have the following relationship, for any $k \geq 1$,

$$\Delta y_{i_0,k} = \beta_{i_0}^{k-1} \Delta f_{i_0,0} + \beta_{i_0}^{k-2} \Delta f_{i_0,1} + \cdots + \beta_{i_0} \Delta f_{i_0,k-1}. \quad (31)$$

Combining (29) and (31), we have

$$\sum_{k=0}^{K} ||\Delta \xi_{i_0,k}||_1 \leq (1 - \beta_{i_0}) \sum_{k=0}^{K} ||\Delta y_{i_0,k}||_1$$

$$\leq (1 - \beta_{i_0}) \sum_{k=0}^{K} \sum_{t=0}^{k-1} \beta_{i_0}^{k-1-t} ||\Delta f_{i_0,t}||_1$$

$$\leq (1 - \beta_{i_0}) \sum_{k=0}^{K} \sum_{t=0}^{k-1} \beta_{i_0}^{k-1-t} 2\sqrt{pC}$$

$$< (1 - \beta_{i_0}) \sum_{k=1}^{K} (\beta_{i_0}^{k-1}) 2K\sqrt{pC}$$

$$= \frac{2(1 - \beta_{i_0})\sqrt{pC}K(1 - \beta_{i_0}^K)}{1 - \beta_{i_0}}$$

$$< 2\sqrt{pCK},$$

where the second inequality holds based on Assumption (4).

Denote $\mathcal{R}^{(i)} = \{\xi^{(i)} | \mathcal{Z}_{\mathcal{F}^{(i)}}(\xi^{(i)}) \in \mathcal{O}, l = 1, 2\}$, then

$$P\{\xi \in \mathcal{O} | \mathcal{Z}_{\mathcal{F}^{(i)}}(\xi) \in \mathcal{O}\} = P\{\xi^{(i)} \in \mathcal{R}^{(i)}\} = \int_{\mathcal{R}^{(i)}} f_\xi(\xi^{(i)}) d\xi^{(i)}, \quad (33)$$

where $f_\xi(\xi^{(i)}) = \prod_{i=0}^{K} \prod_{j=1}^{n_i} \prod_{r=1}^{n_i} f_{L_r}(\xi^{(i)}_{j,r}, \theta_i)$.

According to the above relation (27), (29), we can obtain for any $\xi^{(1)}$, there exists a $\xi^{(2)}$ such that $\mathcal{Z}_{\mathcal{F}^{(1)}}(\xi^{(1)}) = \mathcal{Z}_{\mathcal{F}^{(2)}}(\xi^{(2)})$. As the converse argument is also true, the above defines a bijection. Hence, for any $\xi^{(2)}$, there exists a unique $(\xi^{(1)}, \Delta \xi)$ such that $\xi^{(2)} = \xi^{(1)} + \Delta \xi$. Since $\Delta \xi$ is fixed and is not dependent on $\xi^{(2)}$, we can use a change of variables to obtain

$$P\{\xi^{(2)} \in \mathcal{R}^{(2)}\} = \int_{\mathcal{R}^{(1)}} f_\xi(\xi^{(1)} + \Delta \xi) d\xi^{(1)}. \quad (34)$$

Hence, we have

$$P\{\xi \in \mathcal{O} | \mathcal{Z}_{\mathcal{F}^{(i)}}(\xi) \in \mathcal{O}\} = \frac{P\{\xi^{(1)} \in \mathcal{R}^{(1)}\}}{P\{\xi^{(2)} \in \mathcal{R}^{(2)}\}} = \frac{\int_{\mathcal{R}^{(1)}} f_\xi(\xi^{(1)} + \Delta \xi) d\xi^{(1)}}{\int_{\mathcal{R}^{(1)}} f_\xi(\xi^{(1)}) d\xi^{(1)}}. \quad (35)$$

Since

$$\frac{f_\xi(\xi^{(1)} + \Delta \xi) d\xi^{(1)}}{f_\xi(\xi^{(1)}) d\xi^{(1)}} = \prod_{k=0}^{K} \prod_{i=1}^{n} \prod_{r=1}^{n_i} f_{L_r}(\xi^{(1)}_{j,r}, \theta_i) \prod_{k=0}^{K} \prod_{i=1}^{n} \prod_{r=1}^{n_i} \exp \left( \left| \Delta \xi_{j,r} \right| \right), \quad \theta_i$$

$$\leq \prod_{k=0}^{K} \prod_{i=1}^{n} \prod_{r=1}^{n_i} \exp \left( \frac{||\Delta \xi_{i,k}||_1}{\theta_i} \right)$$

$$= \exp \left( \sum_{k=0}^{K} \frac{||\Delta \xi_{i,k}||_1}{\theta_i} \right) \leq e^{\epsilon_{i_0} \delta}.$$

Then, integrating both sides over $\mathcal{R}^{(1)}$, we have

$$P\{\xi \in \mathcal{O} | \mathcal{Z}_{\mathcal{F}^{(i)}}(\xi) \in \mathcal{O}\} \leq e^{\epsilon_{i_0} \delta} P\{\xi \in \mathcal{O} | \mathcal{Z}_{\mathcal{F}^{(i)}}(\xi)\}, \quad (37)$$

which establishes the $\epsilon_{i_0}$-differential privacy of agent $i_0$. The fact that $i_0$ can be arbitrary agent without loss of generality guarantees $\epsilon_i$-differential privacy of each agent $i$.

VI. SIMULATIONS

In this section, we illustrate the effectiveness of SD-Push-Pull.

Consider a network containing $N = 5$ agents, shown in Fig. 1. The optimization problem is considered as the ridge regression problem, i.e.,

$$\min_{x \in \mathbb{R}^p} f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) = \frac{1}{2} \sum_{i=1}^{n} \left( (u_i^T x - v_i)^2 + \rho ||x||_2^2 \right), \quad (38)$$

where $\rho > 0$ is a penalty parameter. Each agent $i$ has its private sample $(u_i, v_i)$ where $u_i \in \mathbb{R}^p$ denotes the features and $v_i \in \mathbb{R}$ denotes the observed outputs. The vector $u_i \in [-1, 1]^p$ is drawn from the uniform distribution. Then the observed outputs $v_i$ is generated according to $v_i = u_i^T \tilde{x}_i + \gamma_i$, where $\tilde{x}_i$ is evenly located in $[0, 10]^p$ and $\gamma_i \sim \mathcal{N}(0, 5)$. In terms of the above parameters, problem (1) has a unique solution $x^* = \left( \sum_{i=1}^{n} u_i u_i^T + n\rho I \right)^{-1} \sum_{i=1}^{n} u_i u_i^T \tilde{x}_i$.

**Fig. 1.** A digraph of 5 agents.

The weight between two states, $\alpha_i$ and $\beta_i$, are set to be 0.01 and 0.5 for each agent $i \in \{1, 2, 3, 4, 5\}$. The matrix $R$
and \( C \) are designed as follows: for any agent \( i \), \( R_{ij} = \frac{1}{|N_{R,j}^i|+1} \) for \( j \in N_{R,j}^i \) and \( R_{ij} = 1 - \sum_{j \in N_{R,j}^i} R_{ij} \); for any agent \( i \), \( C_{li} = \frac{1-\alpha_i}{|N_{C,i}^l|+1} \) for all \( l \in N_{C,i}^l \) and \( C_{li} = 1 - \alpha_i - \sum_{l \in N_{C,i}^l} C_{li} \).

Assume \( \epsilon_i = \epsilon, \forall i \in \{1, 2, 3, 4, 5\} \) and \( \delta = 10 \). To investigate the dependence of the algorithm accuracy with differential privacy level, we compare the performance SD-Push-Pull for three cases: \( \epsilon = 1, \epsilon = 5 \) and \( \epsilon = 10 \), in terms of the normalized residual \( \frac{1}{2}E \left[ \sum_{i=1}^{5} \left\| x_{i,k} - x^* \right\|^2 \right] \). The results are depicted in Fig. 2, which reflect that SD-Push-Pull becomes suboptimal to guarantee differential privacy, and the constant \( \epsilon \) determines a tradeoff between the privacy level and the optimization accuracy.

**First inequality:** By Lemma 2, Lemma 3, Lemma 6 and Lemma 8 we can obtain from (9) that
\[
\mathbb{E}[[x_{k+1} - x^*]_F^2] = \mathbb{E}[|x_k - \eta y_k - x^* - \eta (h_k - g_k)| - \eta n_2 T(\tilde{y} - \eta y_k)]_F^2 \\
+ \mathbb{E}[|x_k - \eta y_k - x^* - \eta (h_k - g_k)| - \frac{n}{n} \eta y_k]_F^2, \\
\leq \tau_1 \mathbb{E}[|x_k - \eta y_k - x^* - \eta (h_k - g_k)|_F^2] + \frac{2\tau_1}{\tau_1 - 1} \mathbb{E}[|\eta y_k - h_k|_F^2] \\
+ \frac{2\tau_1}{\tau_1 - 1} \mathbb{E}[|\eta y_k - h_k|_F^2] + \frac{2\eta y_k^2}{n} \bar{g}^2, \\
\leq \tau_1 (1 - \eta y_k^2) \mathbb{E}[|x_k - x^*|_F^2] + \frac{2\tau_1}{\tau_1 - 1} \mathbb{E}[|\eta y_k - h_k|_F^2] \\
+ \frac{2\tau_1}{\tau_1 - 1} \mathbb{E}[|\eta y_k - h_k|_F^2] + \frac{2\eta y_k^2}{n} \bar{g}^2.
\]
Taking \( \tau_1 = \frac{1}{1 - \eta y_k^2} \), we have
\[
\mathbb{E}[|x_{k+1} - x^*|_F^2] \\
\leq (1 - \eta y_k^2) \mathbb{E}[|x_k - x^*|_F^2] + c_1 \mathbb{E}[|x_k - 1n x_k|_F^2] \\
+ c_2 \eta \mathbb{E}[|y_k - y_k|_F^2] + c_3 \eta y_k^2 \bar{g}^2,
\]
where
\[
c_1 = \frac{2u^T T v L^2}{\mu n^2}, \quad c_2 = \frac{2|u^T T v L^2}{\mu n^2}, \quad c_3 = \frac{2p(u^T T v L^2)}{n^3}.
\]
**Second inequality:** By relation (10), Lemma 3, Lemma 6 and Lemma 8, we can obtain
\[
\mathbb{E}[|x_{k+1} - 1n x_k|_F^2] \\
\leq \tau_2 (1 - \eta y_k^2) \mathbb{E}[|x_k - 1n x_k|_F^2] + \frac{2\tau_2}{\tau_2 - 1} \mathbb{E}[|\eta y_k - 1n x_k|_F^2] \\
+ \frac{2\tau_2}{\tau_2 - 1} \mathbb{E}[|\eta y_k - 1n x_k|_F^2] + \frac{2\eta y_k^2}{n} \bar{g}^2.
\]
In view of Lemma 2,
\[
\mathbb{E}[|x_{k+1} - 1n x_k|_F^2] \\
\leq \frac{2\eta y_k^2}{n} + \frac{2\eta y_k^2}{n} \mathbb{E}[|x_k - 1n x_k|_F^2] \\
+ 2L^2 \mathbb{E}[|x_k - x^*|_F^2].
\]
Taking \( \tau_2 = \frac{1 + \eta y_k^2}{2\eta y_k^2} \), given \( 1 + \sigma_R^2 < 2 \), we can obtain
\[
\mathbb{E}[|x_{k+1} - 1n x_k|_F^2] \\
\leq \frac{1 + \sigma_R^2}{2} \mathbb{E}[|x_k - 1n x_k|_F^2] \\
+ \frac{4\sigma_R^2 \eta y_k^2 \delta_R^2}{1 - \sigma_R^2} \mathbb{E}[|T v|_F^2] \\
+ \frac{4\sigma_R^2 \eta y_k^2 \delta_R^2}{1 - \sigma_R^2} \mathbb{E}[|T v|_F^2] \\
\leq c_4 \eta y_k^2 \mathbb{E}[|x_k - x^*|_F^2] + (1 + \frac{\sigma_R^2}{2} + c_5 \eta y_k^2) \\
\times \mathbb{E}[|x_k - 1n x_k|_F^2] + c_6 \eta y_k^2 \mathbb{E}[|\tilde{y}_k - y_k|_F^2] + c_7 \eta y_k^2 \bar{g}^2,
\]
where
\[
c_4 = \frac{8\sigma_R^2 L^2 \delta_R^2}{1 - \sigma_R^2}, \quad c_5 = \frac{8\sigma_R^2 L^2 \delta_R^2}{1 - \sigma_R^2 n}, \quad c_6 = \frac{4\sigma_R^2 \delta_R^2}{1 - \sigma_R^2}, \quad c_7 = \frac{8\sigma_R^2 \delta_R^2}{1 - \sigma_R^2 n}.
\]

**VII. CONCLUSION AND FUTURE WORK**

In this paper, we considered a distributed optimization problem with differential privacy in the scenario where a network is abstracted as an unbalanced directed graph. We proposed a state-decomposition-based differentially private distributed optimization algorithm (SD-Push-Pull). In particular, the state decomposition mechanism was adopted to guarantee the differential privacy of individuals’ sensitive information. In addition, we proved that each agent reach a neighborhood of the optimum in expectation exponentially fast under a constant stepsize policy. Moreover, we showed that the constants \( (\epsilon, \delta) \) determine a tradeoff between the privacy level and the optimization accuracy. Finally, a numerical example was provided that demonstrates the effectiveness of SD-Push-Pull. Future work includes improving the accuracy of the optimization and considering the optimization problem with constraints.

**VIII. APPENDIX**

**A. Proof of Lemma 2**

The three inequalities embedded in (13) come from (9), (10) and (11), respectively.
where

\[ \eta < \frac{1}{L}. \]

Third inequality: It follows from (11), Lemma 5, Lemma 6 and Lemma 8 that

\[
E[||\tilde{y}_{k+1} - v\tilde{y}_{k+1}||^2_f | F_k] \\
\leq \frac{1 + \sigma_\ell^2}{2}E[||y_k - v\tilde{y}_k||^2_f | F_k] \\
+ \frac{(1 - \sigma_\ell^2)^2 \delta_\ell^2}{(1 - \sigma_\ell^2)} ||I_{2_n} - \frac{v1^T}{n}||^2 E[||\nabla \tilde{F}(x_{k+1}) - \nabla \tilde{F}(x_k)||^2_f | F_k].
\]

Next, we bound \(E[||\nabla \tilde{F}(x_{k+1}) - \nabla \tilde{F}(x_k)||^2_f | F_k] \)

\[
= E[||((\xi_k + 1 - \xi_k)^T, (\nabla F(x_{k+1}) - \nabla F(x_k))^T)|^2_f | F_k] \\
= E[||\xi_k - \xi_k||^2_f | F_k] + E[||\nabla F(x_{k+1}) - \nabla F(x_k)||^2_f | F_k] \\
\leq 4p\theta^2 + E[||\nabla F(x_{k+1}) - \nabla F(x_k)||^2_f | F_k]
\]

Then, From Assumption 1 and equation (40), we have

\[ E[||\nabla F(x_{k+1}) - \nabla F(x_k)||^2_f | F_k] \]

\[ \leq L^2 E[||x_{k+1} - x_k||^2_f | F_k] \\
\leq L^2 E[||R - I||^2_f ||x_k - 1_n \tilde{x}_k||^2_f | F_k] \\
- \eta^2 E[||\tilde{y}_k - v\tilde{y}_k||^2_f | F_k] \\
\leq 3L^2 ||R - I||^2_E E[||x_k - 1_n \tilde{x}_k||^2_f | F_k] + 3L^2 \eta^2 E[||R||^2_f | F_k] \\
+ 6L^2 \eta^2 E[||R||^2_f ||v||^2_f | F_k] \\
+ 3L^2 \eta^2 E[||R||^2_f ||v||^2_f | F_k]
\]

In light of inequality (40), we further have

\[ E[||\nabla F(x_{k+1}) - \nabla F(x_k)||^2_f | F_k] \]

\[ \leq (3L^2 ||R - I||^2 + \frac{6L^2 \eta^2}{n} E[||R||^2_f ||v||^2_f | F_k]) E[||x_k - 1_n \tilde{x}_k||^2_f | F_k] \\
+ 3L^2 \eta^2 E[||R||^2_f ||v||^2_f | F_k] + 6L^2 \eta^2 E[||R||^2_f ||v||^2_f | F_k]
\]

\[
E[||\tilde{y}_{k+1} - v\tilde{y}_{k+1}||^2_f | F_k] \\
\leq c_8 \eta^2 E[||x_k - x^* - v||^2_f | F_k] + c_9 E[||x_k - 1_n \tilde{x}_k||^2_f | F_k] \\
+ \frac{1 + \sigma_\ell^2}{2} E[||\tilde{y}_k - v\tilde{y}_k||^2_f | F_k] + c_{11} \theta^2,
\]

where

\[
c_8 = \frac{12 \delta_\ell^2 + L^2}{(1 - \sigma_\ell^2)} ||I_{2_n} - \frac{v1^T}{n}||^2_f (3L^2 ||R - I||^2 + \frac{6L^2 \eta^2}{n} E[||R||^2_f ||v||^2_f | F_k]),
\]

\[
c_9 = \frac{2 \delta_\ell^2}{(1 - \sigma_\ell^2)} ||I_{2_n} - \frac{v1^T}{n}||^2_f \left( \frac{3L^2}{n} ||R - I||^2 + \frac{6L^2 \eta^2}{n} E[||R||^2_f ||v||^2_f | F_k] \right),
\]

\[
c_{10} = \frac{6L^2 \eta^2}{n} \left( \frac{1 + \sigma_\ell^2}{2} \right) ||I_{2_n} - \frac{v1^T}{n}||^2_f ||R||^2_f ||v||^2_f,
\]

\[
c_{11} = \frac{12 \delta_\ell^2 \rho}{(1 - \sigma_\ell^2)n} ||I_{2_n} - \frac{v1^T}{n}||^2_f ||R||^2_f ||v||^2_f.
\]
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