Forces and Torques Near to Impact in the Golf Swing

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Motivated by MacKenzie’s observation of a negative force couple near to impact [1, 2], this paper explores a model for how the golf club moves near to impact. It assumes the golf club is moving as the distal arm of a double pendulum. At impact the club head is moving straight down the target line, at its maximum speed, on a path with a specified radius of curvature. From this model the forces and torques required to move the club near to impact are calculated. The results are shown to be quantitatively consistent with data reported by MacKenzie to within a few percent. The negative couple near to impact is a robust feature of this model, balancing the torque associated with the force that drives the center of mass of the golf club. The negative couple allows the golfer to maintain a larger radius of curvature of the path of the club head as it moves through impact. Because the negative couple can also serve to reduce the rotational speed of the club, the presence of a negative couple at impact in the golf swing manifests a trade between distance and direction.

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II. SUMMARY FOR GOLFERS

This section summarizes several of the salient points discussed in this paper which may be of general interest to golfers. They are presented in this summary without the mathematical detail provided in the paper.

• This paper explores the forces and torques that move the club at impact. It assumes the club is moving as the distal arm of a double pendulum, as depicted in Fig. 2. At impact the club head is moving straight down the target line at maximum speed.

• There is a geometry particular to the double pendulum which allows the club head to access points along the target line, as is shown in Fig. 3. The length of the target line that is accessible depends on how far the golfer stands from the ball, but is typically 8-16 inches long, covering the distance from the middle of the stance to the forward foot. In this geometry the hands are always ahead of the club head. The path of the hands through this region is up and in. This geometry is the organizing principle for the model explored in this paper.

• In practice it is not possible to keep the club head moving on a straight line for an extended distance as it moves through impact at speed. Rather, the club head moves on an arc, as depicted in Fig. 10.
It is possible to make the radius of curvature of the club head path sufficiently large that the deviation from a straight line is negligible for several inches before and after impact, Fig. [11]. This allows some margin for error in the golf swing.

- At impact the rotational speed of the proximal arm of the double pendulum (i.e. the shoulders, arms, and hands) is decreasing while the rotational speed of the distal arm (i.e. the club) is increasing, as can be inferred from Fig 10. This happens in a balanced way so as to allow the club head to move at maximum speed in a direction straight down the target line at impact. The deceleration of the proximal arm in vicinity of impact is consistent with previous studies of the kinematic sequence [3].

- As is shown in Fig. [12] the force applied to the club by the golfer at impact is oriented in the general direction of the hub (i.e. the fixed pivot about which the proximal arm of the double pendulum rotates, which corresponds roughly to the middle of the sternum). It is is dominated by the centripetal force needed to keep the center of mass of the club moving on an arc. Both the magnitude and orientation of the force are consistent with the inverse dynamics measurements of MacKenzie [2]. The direction of the applied force at impact is an important result, and could be an organizing theme around which a golfer’s biomechanics at impact are optimized.

- This force applied by the golfer at impact results in a torque applied to the club which serves to increase the rotational speed of the club. However, this torque also serves to decrease the radius of curvature of the path of the club head. To compensate for this, an additional torque is applied to the club so as to moderate the total torque without applying any additional net force. The details of the balancing of these two torques are shown in Fig. [14]. This additional torque takes the form of a force couple [4], which can be though of as two forces, equal in magnitude, opposite direction, separated through a distance. A force couple generates a torque, but does not accelerate the center of mass.

This force couple has been measured by MacKenzie [1, 2] throughout the entire swing. It is negative within a few tens of milliseconds of impact, where it also acquires its largest magnitude. This large, negative force couple in the vicinity of impact is ubiquitous among the golfers that have been measured. It is surprising because a negative couple would reduce the rotational speed of the club, which seems contrary to the goals of most golfers.

This paper shows that this negative couple in the vicinity of impact is a robust feature of the double pendulum model of the golf swing. It serves to reduce the total torque applied to the club, allowing the club head path to maintain a larger radius of curvature through the ball. As such, the negative couple is a manifestation of the trade between distance and direction.

- It remains the subject of future work to explain exactly how this negative force couple is generated. Given that it occurs over an imperceptibly short period of time near to impact, and that nobody was aware of it before MacKenzie’s experiments, this negative couple is possibly an involuntary feature of the body when the hands/wrists are rotating at very high speed. If so, it suggests golfers have learned to incorporate this natural negative couple into their golf swings in a way which allows them to hit the ball straighter. Indeed, when training golfers it may be better to simply focus on the path of the club through the ball rather than trying to measure the force couple at impact.

- Golfers are going to ask how this information can be used to improve their golf swing. This question is best addressed by professional golf instructors. However, it is interesting to point out that the deceleration of the hands and the orientation of the force at impact highlighted in this paper is reminiscent of an approach to training the golf swing named the ‘Rotor Method’ that was pioneered by Nichols in the 1970s [3] and recently demonstrated in a video by Malaska [6]. Quoting from [1], the downswing was characterized by ‘the explosive movement of the ... right side against the resistance of the left’. This serves to enhance the deceleration of the torso/arms/hands at impact. At impact Nichols stressed ‘ ... the weight of the club head must go down the line until just after impact and then upward’. Pulling the club upward just after impact serves to help the golfer orient the forces at impact towards the hub. When done correctly, this style of ‘swing produces a very shallow arc resulting in long, thin divots’. This is suggestive of the club head paths of Figs. [10] and [11]. Perhaps this training methodology from the 1970s can be adapted to the modern golf swing as a means of training the deceleration of the body and the hub-centric orientation of the applied force near to impact.

III. INTRODUCTION

This paper is motivated by the results of MacKenzie [1, 2, 7–9], Kwon [10] and Nesbit [11–13], who have used 3-d motion analysis of the golf club to infer the forces and torques necessary to move the club throughout the swing. A goal of this paper is to understand the role of the negative couple in the immediate vicinity of impact, as reported by MacKenzie [2].
The golf swing has long been modeled as a double pendulum \[\text{[14–16]}\]. This paper makes use of this model in the immediate vicinity of impact. There has been much discussion about the general applicability of the double pendulum to the entire golf swing. For instance, it is known the hub (i.e. the fixed pivot about which the proximal arm of the double pendulum rotates) is not rigorously fixed throughout the entire swing \[\text{[10]}\], and there are claims the length of the proximal arm can change significantly during the swing \[\text{[12]}\]. This paper is focused on the dynamics in the immediate vicinity of impact. An explicit assumption of this paper is that near to impact the hub is reasonably fixed and the proximal arm is of constant length. Under these conditions the double pendulum is a good approximation.

The paper is divided into six sections. The first section introduces a geometry particular to the double pendulum in which the club can access points along the target line. The length of the target line that is accessible depends on how far the golfer stands from the ball, but generally extends from the middle of the stance out towards the forward foot.

The second section uses this geometry to constrain the dynamics of the double pendulum so as to limit consideration to golf swings where the club head reaches maximum speed as it moves down the target line at impact on a path with a specified radius of curvature.

The third section begins with a derivation of the double pendulum Lagrangian, done in the coordinate system used in this paper. The Lagrange equations of motion are used to calculate the external torques required to drive the system at impact, given the constraints in the second section. It is in this section that the negative couple reported by MacKenzie is found to be a robust feature of the model.

In the fourth section the equations of motion are used to simulate the motion of the club in a region near to impact. The external applied torques are assumed constant throughout this region, equal to the values required at impact. Using inverse dynamics, similar to the approach of MacKenzie \[\text{[2]}\], Kwon \[\text{[10]}\] and Nesbit \[\text{[11]}\], the simulated motion is used to recover the forces and torques that move the club.

This fourth section provides the opportunity to look at the problem from various different frames of reference, both inertial and non-inertial. This exercise serves to emphasize that the answer does not depend on the frame of reference in which the problem is solved. Hopefully, the discussions in this section can help to make clear some of the issues associated with working in different frames of reference \[\text{[17]}\].

The fifth section of the paper performs a search over the parameters of the model to find the best fit to the forces and torques at impact as reported by MacKenzie \[\text{[2]}\] for one particular golfer. It is demonstrated solutions to the model can be found which agree quantitatively with MacKenzie’s measurements to within a few percent.

The final section of the paper speculates about various mechanisms by which the negative force couple can be generated.

### IV. GEOMETRY OF THE DOUBLE PENDULUM NEAR TO IMPACT

The coordinate system is shown in Fig. 1. The x-axis is perpendicular to the target line, while the y-axis is oriented parallel to the target line. The double pendulum consists of two arms, a proximal arm of length \(R_1\) and a distal arm of length \(R_2\). The angles \(\theta\) and \(\phi\) describe the angle of the proximal and distal arms relative to the x-axis. The stationary end of the proximal arm (i.e. the hub) is attached to the origin, but is free to rotate about the origin. The proximal arm is an approximation to the shoulders/arms/hands. The hinge between the proximal arm and the distal arm is where the hands attach to the handle of the club. The distal arm is the golf club.

The position \((x_1, y_1)\) of the far end of the proximal arm (i.e. the hinge between the hands on the club handle) is

\[
\begin{align*}
    x_1 &= R_1 \cos \theta \\
    y_1 &= R_1 \sin \theta
\end{align*}
\]

Similarly, the position \((x_2, y_2)\) of the end of the distal arm (i.e. the club head) is

\[
\begin{align*}
    x_2 &= R_1 \cos \theta + R_2 \cos \phi \\
    y_2 &= R_1 \sin \theta + R_2 \sin \phi
\end{align*}
\]

Assume the club is constrained to move straight down the target line, parallel to the y-axis, a distance \(x_0 = R_1 + R_2 - \delta\) from the origin, where \(\delta > 0\). This is shown on the left side of Fig. 2 That \(x_0 < R_1 + R_2\) allows the club head to access a family of points straight down target line. This family is defined by the constraint

\[
R_1 + R_2 - \delta = R_1 \cos \theta + R_2 \cos \phi
\]

An additional constraint is that the hands should be slightly cocked \((\theta - \phi) > 0\). Note that Jorgensen \[\text{[16]}\] refers to the angle \(\theta - \phi\) as \(\beta\). Subsequently, Nesbit \[\text{[11]}\] popularized the use of the Euler angle naming convention \(\alpha, \beta, \gamma\) to describe rotation in the swing plane, out of the swing plane, and about the axis of the shaft, respectively. This convention has become popular in golf teaching circles, and thus the convention \(\alpha = \theta - \phi\) is adopted in this paper.

The final constraint is that the hands uncock as the club moves towards impact

\[
\frac{\delta \alpha}{\delta y} < 0
\]

These constraints yield a set of points along the target line, starting at \((x, y) = (x_0, y_{\text{min}})\) through \((x, y) = \)
FIG. 1. Geometry of the double pendulum, defining the angles θ and φ. The hub of the proximal arm is attached at the origin, indicated by the green circle, and is free to rotate about the origin. The proximal arm is meant to approximate the shoulders/arms/hands of the golfer. The distal arm is the club head. The hands attach to the golf club at the hinge of the club head on the target line.

The hands attach to the golf club at the hinge indicated by the red circle. The blue circle at the far end of the distal arm is the club head.

Changing parameters again, this time to a = B² + C², b = −2AC and c = A² − B², and solving for sin θ,

\[ \sin \theta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]  (10)

Then use \( y_0 = R_1 \sin \theta + R_2 \sin \phi \) to solve for sin φ.

The resulting family of orientations of the double pendulum for which the club head can access the target line is shown in Fig. 3. The length of the distal arm (i.e. the club) \( R_2 = 1.092 \text{ m} \), consistent with the value used in [12]. The length of the proximal arm \( R_1 = 0.7R_2 \) for no particular reason other than the aspect ratio looks about correct. \( R_1 + R_2 = 1.856 \text{ m} \approx 73 \text{ in} \). Finally, δ is chosen to be 7.84 cm ≈ 3 in, which makes \( R_1 + R_2 - \delta = 1.778 \text{ m} = 70.0 \text{ in} \). The length \( \sqrt{2R_1^2} \approx 13.3 \text{ in} \), which spans the distance from the center of the stance out towards the middle of the stance out towards the forward foot. Note that \( \phi \leq 0 \) at all accessible points along the target line.

Eliminate φ from these coupled equations by squaring and adding together,

\[ (R_2 \cos \phi)^2 = (x_0 - R_1 \cos \theta)^2 \]  (6a)
\[ (R_2 \sin \phi)^2 = (y_0 - R_1 \sin \theta)^2 \]  (6b)

yielding

\[ 2x_0 R_1 \cos \theta + 2y_0 R_1 \sin \theta = x_0^2 + y_0^2 + R_1^2 - R_2^2 \]  (7)

Simplify by defining the parameters \( A = x_0^2 + y_0^2 + R_1^2 - R_2^2 \), \( B = 2x_0 R_1 \), and \( C = 2y_0 R_1 \). Reduce to terms only involving sin θ by again taking the square,

\[ (B \cos \theta)^2 = B^2 (1 - \sin^2 \theta) = (A - C \sin \theta)^2 \]  (8)

which yields a quadratic equation in sin θ

\[ (B^2 + C^2) \sin^2 \theta - 2AC \sin \theta + (A^2 - B^2) = 0 \]  (9)

One can solve for θ and φ at all points where the club can access the target line subject to these constraints, as follows. The parameters \( x_0 \) and \( y_0 \) describe the position of the club head on the target line,

\[ x_0 = R_1 \cos \theta + R_2 \cos \phi \]  (5a)
\[ y_0 = R_1 \sin \theta + R_2 \sin \phi \]  (5b)

(\( x_0, y_{\text{max}} \)), where \( y_{\text{min}} = 0 \) and \( y_{\text{max}} = \sqrt{2R_1 \delta - \delta^2} \approx \sqrt{2R_1 \delta} \). Note that at \( y_{\text{max}} \), \( \phi = 0 \). For all other points along the line, \( \phi < 0 \). Similarly, \( \theta > 0 \) at all points along the target line.

FIG. 2. Geometry of the double pendulum, defining the angles θ and φ, near to impact. The club head, indicated by the blue circle, is fixed to the target line, which is a distance \( R_1 + R_2 - \delta \) from the y-axis. As is described in the text, the club head can access the target line from \( y = 0 \) thru \( y = \sqrt{2(R_1 \delta)} \). In practice, this spans club positions from the middle of the stance out towards the forward foot. Note that \( \phi \leq 0 \) at all accessible points along the target line.
The club is constrained to move straight down the line at impact, so $\dot{x}_0 = 0$ and $\dot{y}_0 = v$.

$$\begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} -\sin \theta & -\sin \phi \\ \cos \theta & \cos \phi \end{bmatrix} \begin{bmatrix} R_1 \dot{\theta} \\ R_2 \dot{\phi} \end{bmatrix}$$ (13)

Solving for $\dot{\theta}$ and $\dot{\phi}$ requires inverting the matrix

$$\begin{bmatrix} -\sin \theta & -\sin \phi \\ \cos \theta & \cos \phi \end{bmatrix}^{-1} = \frac{1}{\sin \alpha} \begin{bmatrix} -\cos \phi & -\sin \phi \\ \cos \theta & \sin \theta \end{bmatrix}$$ (14)

where $\alpha = \theta - \phi$ and $\sin \alpha = \sin \theta \cos \phi - \sin \phi \cos \theta$. Thus

$$\begin{bmatrix} R_1 \dot{\theta} \\ R_2 \dot{\phi} \end{bmatrix} = \frac{1}{\sin \alpha} \begin{bmatrix} -\cos \phi & -\sin \phi \\ \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} = \frac{v}{\sin \alpha} \begin{bmatrix} -\sin \phi \\ \sin \theta \end{bmatrix}$$ (15)

Through this entire region $\phi < 0$, and $\alpha > 0$. Therefore, both $\dot{\theta} > 0$ and $\dot{\phi} > 0$.

The club is constrained to move straight down the line at impact, so $\dot{x}_0 = 0$ and $\dot{y}_0 = v$.

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where $\alpha = \theta - \phi$ and $\sin \alpha = \sin \theta \cos \phi - \sin \phi \cos \theta$. Thus

$$\begin{bmatrix} R_1 \dot{\theta} \\ R_2 \dot{\phi} \end{bmatrix} = \frac{1}{\sin \alpha} \begin{bmatrix} -\cos \phi & -\sin \phi \\ \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix}$$ (15)

Through this entire region $\phi < 0$, and $\alpha > 0$. Therefore, both $\dot{\theta} > 0$ and $\dot{\phi} > 0$. 

FIG. 4. Angular speeds $\dot{\phi}$ and $\dot{\theta}$ for points along the target line from $y = 0$ thru $y = y_{\text{max}}$. The abscissa is setup to align with the image in Fig. 3. These angular speeds are calculated for the case of a club speed of 44.7 m/s.
that $\dot{\theta} = \phi$, and thus the proximal and distal arms move together. Out near

$$y = y_{\text{max}}$$

at the end of the accessible target line, $\dot{\theta} = 0$ and all motion of the club head is related to $\phi$.

The acceleration is constrained such that the club comes to its maximum speed at impact, $\dot{y}_0 = 0$. Additionally, the club travels from inside the line to inside the line, so at impact $\dot{x}_0 < 0$. In principle, the magnitude $|\dot{x}_0|$ should be as small as possible so that the club head travels a reasonably straight path down the target line. In practice, it requires larger forces and torques as the golfer makes $|\dot{x}_0|$ smaller, and it becomes impractical to get the club head to travel perfectly straight down the target line for an extended distance at speed. But, as will be shown below, the resulting radius of curvature of the club path can be sufficiently large that the club path is reasonably approximated as a straight line near to impact.

The radius of curvature of the club head path at impact is given by the expression $R_c = \dot{y}_0^2/\dot{x}_0$ \[\text{18}\]. As will be shown, it is useful to parameterize $R_c$ in terms of the distance of the hub from the target line, $R_1 + R_2 - \delta$. In particular, define $R_c$ in terms of the parameter $\xi$ such that $\dot{y}_0^2/|\dot{x}_0| = \xi(R_1 + R_2 - \delta)$. The condition $\xi = 1$ corresponds to the case when the path is approximated by the perimeter of a circle of radius $(R_1 + R_2 - \delta)$. Expressing $\dot{x}_0$ in terms of $\xi$,

$$\dot{x}_0 = -\frac{\dot{y}_0^2}{\xi(R_1 + R_2 - \delta)} \quad \text{(16)}$$

The second derivatives $\ddot{x}$ and $\ddot{y}$ are given by the expression

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} -\sin \theta & -\sin \phi \\ \cos \theta & \cos \phi \end{bmatrix} \begin{bmatrix} R_1 \ddot{\theta} \\ R_2 \ddot{\phi} \end{bmatrix} - \begin{bmatrix} \cos \theta & \cos \phi \\ \sin \theta & \sin \phi \end{bmatrix} \begin{bmatrix} R_1 \dot{\theta}^2 \\ R_2 \dot{\phi}^2 \end{bmatrix} \quad \text{(17)}$$

where we impose the condition.

$$\begin{bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \end{bmatrix} = \begin{bmatrix} -\frac{\ddot{y}_0^2}{\xi(R_1 + R_2 - \delta)} \\ 0 \end{bmatrix} \quad \text{(18)}$$

Solving for $R_1 \dot{\theta}$ and $R_2 \dot{\phi}$ using the same matrix inversion from above

$$\begin{bmatrix} R_1 \dot{\theta} \\ R_2 \dot{\phi} \end{bmatrix} = \frac{1}{\sin \alpha} \begin{bmatrix} -\cos \phi & -\sin \phi \\ \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \end{bmatrix} + \frac{1}{\sin \alpha} \begin{bmatrix} -\cos \alpha & -1 \\ 1 & \cos \alpha \end{bmatrix} \begin{bmatrix} R_1 \dot{\theta}^2 \\ R_2 \dot{\phi}^2 \end{bmatrix} \quad \text{(19)}$$

With these equations, $(\theta, \dot{\theta}, \ddot{\theta})$, and $(\phi, \dot{\phi}, \ddot{\phi})$ are fully specified at impact.

Shown in Fig.6 are $\ddot{\phi}$ and $\ddot{\theta}$ as a function of the distance along the target line for various values of $\xi \geq 1$. Again,
the abscissa is setup to align with the image in Fig. 3. Note that at \( y/y_{\text{max}} = 0 \) and for \( \xi = 1.0 \), \( \dot{\theta} = \phi \approx 0 \). For larger values of \( y/y_{\text{max}} \) the magnitude of the required angular acceleration increases, with \( \phi \) accelerating and \( \theta \) decelerating. The deceleration of \( \theta \) and the acceleration of \( \phi \) near to impact is consistent with previous studies of the kinematic sequence \[3\].

Fig. 6 details \( \phi \) and \( \theta \) as a function of the distance along the target line for various values of \( \xi \). Here the curves have a dashed region and a solid region. The dashed regions occur when either \( \dot{\phi} < 0 \) (i.e. deceleration of rotation of the club) or \( \dot{\theta} > 0 \) (i.e. acceleration of the rotation of the arms/hands). Neither solution is likely to be realized in practice. The solid regions correspond to \( \dot{\phi} \geq 0 \) and \( \dot{\theta} \leq 0 \), and are the solutions which have a better chance of matching what is realized in actual golf swings. Note that for the case of the solid lines, the magnitude of the acceleration gets smaller at smaller values of \( \xi \), which corresponds to the club path through impact being more curved.

VI. THE LAGRANGIAN OF THE DOUBLE PENDULUM

The double pendulum was originally used as a model for the golf swing by Cochran and Stobbs \[14\]. The Lagrangian of the double pendulum and its application to the dynamics of the golf swing was subsequently pioneered by Jorgensen \[15\]. In this section, the Lagrangian is re-derived using the coordinate system of this paper.

A. Kinetic Energy

The Lagrangian of a rigid body can be calculated as the difference between the kinetic energy and potential energies \[19\]. Thus, the first step is to define the kinetic energy of the moving parts in the double pendulum.

Start by considering the proximal arm. Let \( r_1 \) denote the distance along the arm. The velocity of a point along the proximal arm is

\[
\begin{align*}
\dot{x}_1 &= -r_1 \dot{\theta} \sin \theta \\
\dot{y}_1 &= r_1 \dot{\theta} \cos \theta
\end{align*}
\]

The square of the velocity is

\[
v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = r_1^2 \dot{\theta}^2
\]

The kinetic energy is calculated by integrating the local kinetic energy over the entire proximal arm.

\[
KE_1 = \frac{1}{2} \int dm_1 v_1^2
\]

Defining the linear mass density \( \rho_1(r_1) \) such that \( dm = dr_1 \rho_1(r_1) \), the integral becomes

\[
KE_1 = \frac{1}{2} \int_0^{R_1} dr_1 \rho_1 r_1^2 \dot{\theta}^2
\]

The integral of the linear mass density is just the mass,

\[
M_1 = \int_0^{R_1} dr_1 \rho_1
\]

As such, \( \rho_1/M_1 \) is a probability density,

\[
1 = \int_0^{R_1} dr_1 \frac{\rho_1}{M_1} r_1^2
\]

With this interpretation, the integral over \( r_1^2 \) is the second moment,

\[
< R_1^2 > = \int_0^{R_1} dr_1 \frac{\rho_1}{M_1} r_1^2
\]

The kinetic energy can then be parameterized as

\[
KE_1 = \frac{1}{2} M_1 < \dot{R}_1^2 > \dot{\theta}^2
\]

Now consider the distal arm. Define \( r_2 \) to be the distance along the distal arm. The velocity of a point along the distal arm is

\[
\begin{align*}
\dot{x}_2 &= -R_1 \dot{\theta} \sin \theta - r_2 \dot{\phi} \sin \phi \\
\dot{y}_2 &= R_1 \dot{\theta} \cos \theta + r_2 \dot{\phi} \cos \phi
\end{align*}
\]

The square of the velocity is

\[
v_2^2 = \dot{x}_2^2 + \dot{y}_2^2
\]

which simplifies to

\[
v_2^2 = R_1^2 \dot{\theta}^2 + r_2^2 \dot{\phi}^2 + 2 R_1 r_2 \dot{\theta} \dot{\phi} \cos (\theta - \phi)
\]

Once again, define the kinetic energy of the distal arm as an integral of the local kinetic energy over the entire distal arm

\[
KE_2 = \frac{1}{2} \int dm_2 v_2^2
\]

Defining the linear mass density of the distal arm, \( \rho_2 \), and using the definitions of the first and second moments of the distal arm,
Similarly, another couple, $K_\alpha$ is applied at the hinge between the two arms. It is applied such that it increases the angle $\phi$ relative to $\theta$, and thus decreases $\theta - \phi$. The potential energy associated with a constant version of this couple is then

$$PE_2 = K_\alpha (\theta - \phi)$$

As will be shown below, $K_\alpha$ corresponds to the couple reported by MacKenzie [2].

The total potential energy is

$$PE = -K_\theta \theta + K_\alpha (\theta - \phi)$$

The resulting Lagrangian $L$ is

$$L = \frac{1}{2} A \dot{\theta}^2 + \frac{1}{2} B \dot{\phi}^2 + C \dot{\phi} \cos \alpha + K_\theta \theta - K_\alpha (\theta - \phi)$$ (42)

### C. Equations of Motion

The Lagrangian is of the form $L(x_i, \dot{x}_i)$, where $i$ ranges over the independent coordinates, in this case $\theta$ and $\phi$. The associated equation of motion for each coordinate is given by (19),

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{x}_i} - \frac{\delta L}{\delta x_i} = 0$$ (43)

The equation of motion associated with $\theta$ is

$$A \ddot{\theta} + C \dot{\phi} \cos (\theta - \phi) + C \dot{\phi}^2 \sin (\theta - \phi) = K_\theta - K_\alpha$$ (44)

Similarly, the equation of motion associated with $\phi$ is

$$B \ddot{\phi} + C \dot{\theta} \cos (\theta - \phi) - C \dot{\theta}^2 \sin (\theta - \phi) = K_\alpha$$ (45)

These are the two equations of motion which govern the motion of the double pendulum subject to couples $K_\theta$ and $K_\alpha$. Given initial conditions ($\theta_0$, $\dot{\theta}_0$) and ($\phi_0$, $\dot{\phi}_0$), and the couples $K_\theta$ and $K_\alpha$, the equations of motion can be solved for $\theta(t)$ and $\phi(t)$.

### D. Solving for Couples

Consider the situation at impact. The values ($\theta_0$, $\dot{\theta}_0$, $\dot{\theta}_0$) and ($\phi_0$, $\dot{\phi}_0$, $\dot{\phi}_0$) are known from the constraint that at impact the club moves down the target line at peak speed on a path with a specified radius of curvature. In this section the equations of motion are inverted to solve for the values of $K_\theta$ and $K_\alpha$ that are consistent with this condition.

Start with the equations of motion above, now written in matrix notation

$$
< R_2 > = \int_0^{R_2} dv_2 \frac{\rho_2}{M_2} v_2^2
$$

$$
< R_2^2 > = \int_0^{R_2} dv_2 \frac{\rho_2}{M_2} v_2^2
$$

the following expression for the kinetic energy of the distal arm is obtained,

$$KE = \frac{1}{2} M_2 \left( R_2^2 \dot{\theta}^2 + < R_2 > \dot{\phi}^2 + 2 R_1 < R_2 > \dot{\theta} \dot{\phi} \cos (\theta - \phi) \right)$$ (35)

$$KE = \frac{1}{2} \left( M_1 < R_1^2 > + M_2 R_1^2 \right) \dot{\theta}^2 + \frac{1}{2} M_2 R_1 < R_2 > \dot{\phi}^2
+ M_2 R_1 < R_2 > \dot{\phi} \dot{\theta} \cos (\theta - \phi)$$ (36)

It is useful to define the following parameters

$$A = M_1 < R_1^2 > + M_2 R_1^2$$ (37a)
$$B = M_2 < R_2 >$$ (37b)
$$C = M_2 R_1 < R_2 >$$ (37c)
$$\alpha = \theta - \phi$$ (37d)

With these definitions, the kinetic energy simplifies to

$$KE = \frac{1}{2} A \dot{\theta}^2 + \frac{1}{2} B \dot{\phi}^2 + C \dot{\phi} \dot{\theta} \cos \alpha$$ (38)

The values of the parameters in $A$, $B$, and $C$ which are used in subsequent calculations in this paper are listed in Appendix A.

### B. Potential Energy

We will want to apply some external torques to the system. These torques are better described as force couples [20], where a force couple $K$ can be thought of as the torque obtained by two forces, equal in magnitude $F$ but opposite in direction, acting at two different points separated by a distance $d$. The net force is zero, so the couple does not accelerate the center of mass. The net torque is $K = Fd$, and results in rotation about the center of mass.

Assume a couple of constant magnitude $K_\theta$ is applied at the hub and has the orientation such that it increases the angle $\theta$. The potential energy for this couple is

$$PE_1 = -K_\theta \theta$$ (39)
Solve for $K$. Multiplying out the matrix equations,

\[
\begin{bmatrix}
A & C \cos \alpha \\
C \cos \alpha & B
\end{bmatrix}
\begin{bmatrix}
\dot{\theta} \\
\dot{\phi}
\end{bmatrix}
+ \begin{bmatrix}
0 & C \sin \alpha \\
-\dot{C} \sin \alpha & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta} \\
\ddot{\phi}
\end{bmatrix}
= \begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
K_\theta \\
K_\alpha
\end{bmatrix}
\]

\tag{46}

Invert the matrix in front of $K_\theta$ and $K_\alpha$,

\[
\begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix}
^{-1}
= \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\]

\tag{47}

Solve for $K_\theta$ and $K_\alpha$,

\[
\begin{bmatrix}
K_\theta \\
K_\alpha
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
A & C \cos \alpha \\
C \cos \alpha & B
\end{bmatrix}
\begin{bmatrix}
\dot{\theta} \\
\dot{\phi}
\end{bmatrix}
+ \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & C \sin \alpha \\
-\dot{C} \sin \alpha & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta} \\
\ddot{\phi}
\end{bmatrix}
\]

\tag{48}

Multiplying out the matrix equations,

\[
K_\theta = (A + C \cos \alpha) \ddot{\theta} + (B + C \cos \alpha) \ddot{\phi}
+ C \sin \alpha (-\ddot{\theta} + \ddot{\phi})
\]

\tag{49}

\[
K_\alpha = C \cos \alpha \ddot{\theta} + B \ddot{\phi} - C \sin \alpha \ddot{\theta}^2
\]

\tag{50}

Thus, given $(\theta_0, \dot{\theta}_0, \ddot{\theta}_0)$ and $(\phi_0, \dot{\phi}_0, \ddot{\phi}_0)$, the couples $K_\theta$ and $K_\alpha$ are determined.

This formalism has been used to calculate the required couples at points along the target line for various values of $\xi$. Shown in Fig. 7 are results for $\xi = (1.0, 1.1, 1.2, 1.3, 1.4)$. The top graphics shows $K_\alpha$ while the lower graphic shows $K_\theta$. Shown in Fig. 8 are results for $\xi = (0.6, 0.7, 0.8, 0.9, 1.0)$. The solid lines indicate the regions where $\ddot{\theta} \leq 0$ and $\ddot{\phi} \geq 0$. The dashed lines extend beyond this range and are shown for completeness; however, it is unlikely one would want to implement a solution in these regions.

$K_\theta$ is the primary couple driving $\theta$, which is decelerating into impact. Thus, it is not surprising $K_\theta < 0$. The absolute scale of $K_\theta$ depends linearly on our choice of the inertial moment of the proximal arm of the pendulum. In these numerical experiments, that value was chosen by fiat and is not based on a biomechanical model. Thus, the absolute scale of $K_\theta$ is not meaningful.

There is likely some surprise that $K_\alpha$ is negative, as it was discussed above that $\ddot{\phi} > 0$. The reason $K_\alpha < 0$ is described in detail in the next section. The magnitude of $K_\alpha$ in these calculations should be close to what is observed in experiment, as the inertial properties of the distal arm of the double pendulum are based on those of a golf club. Note that in all realizable cases $K_\alpha$ is negative with magnitude of order tens of N.m. This is consistent with the experiments of MacKenzie \[2\], and is the central point of this paper.

---

**FIG. 7.** Couples $K_\alpha$ and $K_\theta$ for points along the target line with $\xi \geq 1$. $K_\alpha$ is robustly negative, of magnitude -50 N.m. As the radius of curvature decreases, the magnitude of negative force couple $K_\alpha$ gets smaller.

**FIG. 8.** Couples $K_\alpha$ and $K_\theta$ for points along the target line with $\xi \leq 1$. As the radius of curvature decreases, the magnitude of negative force couple $K_\alpha$ gets smaller.
\[ B\ddot{\phi} = -C\dot{\theta} \cos \alpha + C\dot{\theta}^2 \sin \alpha + K_\alpha \]  
\( \text{(51)} \)

This is the equation of motion for the rotation of the club in the non-inertial frame of the handle of the club (i.e. at the hinge between the proximal and distal arms of the double pendulum). The parameter \( B \) is the moment of inertia of the club about the handle. What follows on the right hand side are the various torques which drive rotation about the handle, in the frame of reference of the handle. Because \( \dot{\phi} > 0 \), the total torque on the club is positive.

The first two terms on the left hand side are fictitious forces due to the fact the position of the handle defines the origin of a non-inertial reference frame. The first term is the torque due to the Euler force associated with the linear acceleration of the handle, acting through the center of mass of the golf club. The second term is the centrifugal force associated with the rotation of the handle, acting through the center of mass of the golf club. The final term is the couple \( K_\alpha \).

The four terms in this equation of motion are shown in Fig. 9 for the case \( \xi = 1 \), for points along the target line. The solid black line is \( B\ddot{\phi} \), the solid red line is \(-C\dot{\theta} \cos \alpha \), the solid green line is \( C\dot{\theta}^2 \sin \alpha \), the solid blue line is \( K_\alpha \), and the black open circles are the sum of the terms \( C\dot{\theta} \cos \alpha, C\dot{\theta}^2 \sin \alpha, \) and \( K_\alpha \). Near to \( y = 0 \), the fictitious centrifugal torque dominates the release (i.e. the green curve), while closer to \( y = y_{\text{max}} \) the torque is dominated by the fictitious Euler force (i.e. the red curve). In all cases the sum of the Euler and centrifugal torques are larger than \( B\ddot{\phi} \) (i.e. the black curve). Thus, to achieve the requisite motion of the club one must include the couple \( K_\alpha < 0 \) (i.e. the blue curve). The sum of the red, green, and blue curves (i.e. the right hand side of the equation of motion) is represented as the black open circles, verifying that they equal the black line.

**VII. CALCULATING THE CLUB PATH USING LAGRANGIAN DYNAMICS**

In this section the equations of motion for the double pendulum are used to calculate the motion of the double pendulum near to impact. The initial conditions are obtained from the considerations of the previous sections with \( \xi = 1 \) and \( y_0 = 0.5 y_{\text{max}} \). The couples \( K_\theta \) and \( K_\alpha \) are assumed constant over the range of motion, and set equal to the values required at impact from the considerations above. The value of \( K_\alpha \) is 43.4 Nm.

Shown in Fig. 10 is the calculated path of the double pendulum as it moves through impact. There is no actual impact with a golf ball in this calculation, so the club moves unimpeded through impact. The gray circles mark the center of mass of the golf club. The hinge between the proximal and distal arms is marked as small black circles. The point of impact is marked as a larger black circle. The gray circles indicate the positions of the center of mass of the golf club. From these points the inverse dynamics problem can be solved for the linear forces that act on the club in the frame of reference of the handle of the club, for the case \( \xi = 1 \). The various curves in graphic correspond to terms defined in Eq. (51). The black curve is the torque required to keep the club head moving on the specified radius of curvature. The red and green curves indicate the torques associated with the fictitious Euler and centrifugal forces due to the acceleration of the non-inertial reference frame. The sum of these two torques is always larger than the that of the black curve. To moderate these two forces, a negative couple is applied. This is shown as the blue curve. The sum of the red, blue, and green curves is shown as the black open circles, and is equal to the black line.

**Fig. 9.** Detail of the terms in the equation of motion for the rotation of the club in the frame of reference of the handle of the club, for the case \( \xi = 1 \). The various curves in graphic correspond to terms defined in Eq. (51). The black curve is the torque required to keep the club head moving on the specified radius of curvature. The red and green curves indicate the torques associated with the fictitious Euler and centrifugal forces due to the acceleration of the non-inertial reference frame. The sum of these two torques is always larger than the that of the black curve. To moderate these two forces, a negative couple is applied. This is shown as the blue curve. The sum of the red, blue, and green curves is shown as the black open circles, and is equal to the black line.
move the center of mass. This inverse dynamics calculation is meant to enable comparison with the inverse dynamics analysis of golf club motion, as implemented by MacKenzie [2], Kwon [10] and Nesbit [11]. The forces are depicted in Fig. 12, shown as arrows acting at the handle. Note that they all point in the general direction of the hub, which is consistent with the measurements of MacKenzie [2]. The scale is not indicated in the figure, but is of order 260 N.

The forces obtained using inverse dynamics can be compared with theory. The double pendulum imposes constraints on the motion of the proximal and distal arms, such as the fixed pivot around which the proximal arm rotates and the hinged connection between two arms. These constraints result in forces that constrain the motion of the system, but are not explicit in the Lagrangian. The implicit force due to constraints acting on the club can be described as the sum of four terms. The first two terms originate from the dynamics of the proximal arm. They look as if the center of mass of the distal arm were located at the hinge,

\[ F_{\theta} = ( -\sin \theta \dot{x} + \cos \theta \dot{y} ) M_{2} R_{1} \ddot{\theta} \]

\[ F_{\phi} = ( -\cos \phi \dot{x} - \sin \phi \dot{y} ) M_{2} R_{2} \dot{\phi} \]

The second two terms involve the dynamics of the distal arm

\[ F_{\phi} = ( -\sin \phi \dot{x} + \cos \phi \dot{y} ) M_{2} R_{2} \ddot{\phi} \]

\[ F_{\phi} = ( -\cos \phi \dot{x} - \sin \phi \dot{y} ) M_{2} R_{2} \phi^{2} \]

The sum of the x-components and y-components of these forces are shown in Fig. 13 in comparison with the forces obtained from inverse dynamics. The solid lines are calculated from the theoretical expressions, above. The open circles are obtained from the inverse dynamics. Indeed, the inverse dynamics recover the theoretical answer.

A. Whence art thou, \( \alpha < 0 \) (II)

The results of the previous section allow us to calculate the torques on the club about the center of mass of the club in the reference frame of the center of mass of the club. While the position of the center of mass defines the origin of a non-inertial reference frame, the fictitious forces associated with the acceleration of this reference frame act through the center of mass and thus yield no torque because the moment arm is zero. This is why the center of mass reference frame is always a
FIG. 12. The forces that move the center of mass of the golf club. The forces are obtained from the inverse dynamics analysis. The forces are shown as arrows being applied at the handle of the club (i.e. at the hinge between the proximal and distal arms of the double pendulum). As is shown, they are all oriented in the general direction of the hub. The magnitude of the force at impact is 260 N.

FIG. 13. The components $F_x$ and $F_y$ of the forces moving the center of mass of the golf club. The solid lines are calculated from theory, as described in the text. The open circles are obtained from the inverse dynamics analysis. The forces obtained from the inverse dynamics analysis are shown to recover the forces calculated from theory.

particularly convenient frame of reference from which to calculate torques [21].

There are only two torques which are relevant. The first is the torque generated by the linear force that move the center of mass, detailed in Eqs. (52) - (55). MacKenzie refers to this torque as the moment of force, and is indicated here as $M_\alpha$. The other torque is the couple $K_\alpha$. Combined, these two torques must equal the total torque which rotates the club, $I_{cm}\ddot{\phi} = T_\alpha = M_\alpha + K_\alpha$.

These torques are shown in Fig. 14 as the club moves through impact, from the simulation above. The solid red line is $M_\alpha$. The blue line is $K_\alpha = 43.4$ N m. The solid black line is $T_\alpha = I_{cm}\ddot{\phi}$. The open black circles are calculated as the sum $M_\alpha + K_\alpha$. This analysis confirms $T_\alpha = M_\alpha + K_\alpha$.

Once again, we see that while the total torque on the club $T_\alpha > 0$, the couple $K_\alpha$ has to be negative because the other torque in the problem $M_\alpha$ would otherwise provide more torque than what is required to move the club head on the path defined by the radius of curvature.

It is interesting to point out that the value for $K_\alpha$ was set by balancing torques in the non-inertial frame of reference of the handle of the club. In this section the analysis was done in the non-inertial frame of reference of the center of mass of the club. In both cases, the couple $K_\alpha$ has the same value. This serves to emphasize that if you solve for the forces and torques which move a rigid body in multiple reference frames, even non-inertial reference frames, you should always recover the same answer.

VIII. SEEKING A MATCH TO MACKENZIE’S DATA

The majority of MacKenzie’s video ‘In-Plane Couple and Moment of Force During the Golf Swing’ [2] highlights the golf swing of a single golfer. For this golfer in the last frame before impact, the club head is moving at 116.5 mph and the measured values of force and torques are force $F_0 = 456$ N, moment of force $M_0 = 55.8$ N m, and couple $K_0 = -59.1$ N m. In this section the double pendulum model is solved at impact over a grid of parameter values $\delta, \xi$, and $y_0$, in an attempt to find the best fit to $F_0, M_0$ and $K_0$. All other parameters in the problem, such as the length of arms of the double pendulum and the inertial properties of the golf club, are as defined in Appendix A. As such they are just approximations to what may have been used in the experiments of MacKenzie.

The result of this search is summarized in the charts of Fig. 15. The different panels correspond to different values of $\delta$, ranging from 2 to 5 inches. The abscissa corresponds to different values of $y_0$, ranging from 0 to
This difference is calculated as the sum-of-squares average and the result obtained from the model \( F \). The required torque of the center of mass of the golf club. The solid red line is \( M_\alpha \). The blue line is \( K_\alpha = -43.4 \text{ N-m} \). The solid black line is \( T_\alpha = I_{cm} \phi \). The open black circles are calculated as the sum \( M_\alpha + K_\alpha \). This analysis confirms \( T_\alpha = M_\alpha + K_\alpha \). This result is the central point to the paper: \( M_\alpha \) by itself is larger than the required torque \( T_\alpha \). To compensate for this, the torque \( K_\alpha < 0 \) must be applied so as to keep the club moving on the path defined by the radius of curvature.

FIG. 14. Accounting of the torques in the frame of reference of the center of mass of the golf club. The solid red line is \( M_\alpha \). The blue line is \( K_\alpha = -43.4 \text{ N-m} \). The solid black line is \( T_\alpha = I_{cm} \phi \). The open black circles are calculated as the sum \( M_\alpha + K_\alpha \). This analysis confirms \( T_\alpha = M_\alpha + K_\alpha \). This result is the central point to the paper: \( M_\alpha \) by itself is larger than the required torque \( T_\alpha \). To compensate for this, the torque \( K_\alpha < 0 \) must be applied so as to keep the club moving on the path defined by the radius of curvature.

These data show that one can use the same set of forces and torques \( F, M, \) and \( K \) to hit the ball standing different distances from the ball, \( \delta \), and from different positions in the stance, \( y_0 \). As the ball is moved further forward in the stance, the golfer must stand closer to the ball and the radius of curvature of the club head path becomes smaller.

IX. SPECULATION ABOUT HOW \( K_\alpha \) IS GENERATED

The scale of \( K_\alpha \) is of order 50 N-m. What can generate a couple of this magnitude? This section explores three possibilities.

It is important to remember this particular torque is a force couple. It can be thought of as being generated by two linear force vectors, equal in magnitude \( F_K \) but opposite in direction, separated through a distance \( d \). Because the linear sum of the forces is zero, there is no net force on the center of mass of the object due to the two force vectors. However, because they are separated through the distance \( d \), they yield a torque of magnitude \( d F_K \) perpendicular to the plane defined by the two force vectors, and thus generate rotation.

A. The Hands

Suppose this couple is generated by forces applied by the hands. This could be either because the hands are actively applying force, or because the hands can not keep up with the linear and/or rotational speeds at impact.

For a right handed golfer, imagine the left hand applying a force in the direction of motion of the club, and the right hand applying a force of equal magnitude in the opposite direction (i.e. opposing the motion of the club). Suppose the distance from the pinky finger of the left hand to the forefinger of the right hand when a right handed golfer grips the club is 1/6 meter (i.e. 6-7 inches) and is the distance through which the couple is applied. Then to generate a couple applying 50 N m of torque, each hand would have to be applying 300 N of force in opposite directions. This is in addition to the hundreds of Newtons of linear force already discussed above, which is presumably split between the two hands. 300 N of force amounts to 70 lbs force. That seems like a lot of force for each hand to be applying. For this reason, it would seem that this explanation alone is insufficient to provide all of \( K_\alpha \).

However, it is important to note that this negative couple is applied only 10-20 ms before impact. Thus, the resulting impulse (i.e. torque multiplied by time) is not particularly large. If this torque were due to the fact the hands can not keep up with the release of the club, it might be difficult for the golfer to perceive this applied torque. It would be quite spectacular if golfers have learned to harness this natural drag to help them to hit the ball straighter.
FIG. 15. The results of a search over a grid of parameters in an attempt to match to force $F$ and torques, $M$ and $K$, reported by MacKenzie. The the color scale encodes $\log_{10}(E)$, where $E$ is the average fractional error, as described in the text. The value of $\delta$ is given in the top left corner of each panel. The minimum error is indicated in the top right corner of each panel. The values $F$, $M$, and $K$ at the minimum are listed at the top of each panel. The values to which they are being fit are $F_0 = 456$ N, $M_0 = 55.8$ N m, and $K_0 = -59.1$ N m. It is a primary result of this paper that the double pendulum model of the golf swing is able to obtain the results reported by MacKenzie to within a few percent.

B. Aerodynamic Drag of the Club head

Another possible source of negative couple is the aerodynamic drag on the club head as it approaches impact. Imagine the size of that force is $F_d$ in the direction opposing the motion of the club head. Now imagine that the hands apply a force of equal magnitude but in the opposite direction, counter acting this drag. The separation between these two forces is the length of the club, $\epsilon = R_2$. Henrikson reports [22] the scale of the drag force to be 4.5 - 7.5 N. If we use 10 N as an upper limit, and assume a club of length 1 m, then this can yield a couple of order 10 N m. Again, this is too small to give values as large as 50 N m.

C. Inertia of the squaring of the club face

Missing from the model of the double pendulum is the fact that the club face goes from open to square to closed as the club moves through impact. This requires rotation of the club around the long axis of the shaft. It also requires the rotation of the arms and hands, which support the club. This motion is related to the $\beta$-torque and $\gamma$-torque described in Nesbit’s 2005 paper [11], which involve motion out of the swing plane and about the axis of the shaft, respectively.

For our purposes, consider that the motion caused by the $\beta$-torque and $\gamma$-torques is coupled to the release of the hands, defined in this paper as the angle $\alpha = \theta - \phi$. It is certainly the case that the club face is open when $\alpha \approx \pi/2$, it is square near to impact where $\alpha \approx 0$, and closed after impact, when $\alpha$ ends at $-\pi/2$.

Now posit that the $\beta$-torques and $\gamma$-torques causes motion that affects the moment of inertia relevant to the motion in the plane of the golf swing. This could involve the relative positions of the arms and hands, the rotation of the club around its axis, motion of mass above and below the swing plane, etc.

Further make the generalization that the kinetic energy associated with the squaring of the club manifests itself in the swing plane as $KE_s$ and that this can be parameterized in terms of the angular speed $\dot{\alpha}$ and a moment of inertia $I_s$. 
\[ KE_s = \frac{1}{2} I_s \dot{\alpha}^2 \]  \hspace{1cm} (57)

As long as we are only solving the double pendulum in the vicinity of impact, this additional term can then be included in the Lagrangian of this paper (i.e. not making generalizations beyond the immediate vicinity of impact).

With this addition, the equations of motion become

\[ A\ddot{\theta} + C\dot{\theta} \cos (\theta - \phi) + C\dot{\phi}^2 \sin (\theta - \phi) = K_\theta - K_\alpha + I_s(\dot{\phi} - \dot{\theta}) \]  \hspace{1cm} (58)

\[ B\ddot{\phi} + C\dot{\phi} \cos (\theta - \phi) - C\dot{\theta}^2 \sin (\theta - \phi) = K_\alpha - I_s(\dot{\phi} - \dot{\theta}) \]  \hspace{1cm} (59)

As has been shown above, \( \dot{\phi} > 0 \) and \( \dot{\theta} < 0 \), so the term \(-I_s(\dot{\phi} - \dot{\theta})\) functions as a negative torque.

In the exercises above, \( K_\alpha \) was assumed to provide the full negative couple required to keep the club moving straight down the line. For arguments sake, lets assume here that all of the negative couple comes from \( I_s \). Evaluating the example above at impact, \( (\dot{\phi} - \dot{\theta}) \approx 500 \text{rad/s}^{-2} \) which suggests \( I_s \approx 0.1 \text{kg m}^2 \). We can compare this with the value of the moment of inertial of the golf club about its handle, \( I_{R_2} = 0.24 \text{kg m}^2 \) used in this paper. Thus, \( I_s \) needs to be of order 40% of the size of \( I_{R_2} \), which would be a large perturbation. While this seems like a logical avenue for the biomechanics community to explore, it is possible it will not be large enough to explain all of \( K_\alpha \).

### D. Speculation Summary

This section has explored three physical processes that could generate \( K_\alpha \approx -50 \text{N m} \). Each one of them individually seems too small to provide a torque of sufficient magnitude. Thus, instead of there being one clean source of \( K_\alpha \), it seems likely the actual answer involves multiple terms, or phenomena not considered in this paper.

### X. SUMMARY

Motivated by MacKenzie’s observation of a negative couple near to impact \([1, 2]\), this paper has explored a model for how the golf club moves near to impact. It assumes the club is moving as the distal arm of a double pendulum and that at impact the club head is moving straight down the target line, at its maximum speed, on a path of defined curvature. From this model, the forces and torques required to move the club near to impact are calculated.

The results obtained from this model are shown to be quantitatively consistent with data reported by MacKenzie to within a few percent. Indeed, the negative couple near to impact is found to be a robust feature of this model. It balances torques resulting from the forces that drive the center of mass of the golf club. These torques reduce the radius of curvature of the path of the club head as it moves through impact. By applying a negative couple the golfer is able to achieve a larger radius of curvature. This reduces the difference between the path of the club head and the target line as the club head moves near to impact. Because the negative couple can also serve to reduce the rotational speed of the club, its presence in the golf swing manifests a trade between distance and direction.

### Appendix A: Model Parameters

The properties of the golf club were taken from Nesbit [12], for consistency. They are:

- \( R_2 = 1.092 \text{m} \), the length of the golf club in meters. Presumably measured from a place between the two hands to the middle of the club face.
- \( M_2 = 0.382 \text{kg} \), the mass of the golf club.
- \(< R_2 >= 0.661 \text{m} \), the first moment, which is the distance from the hands to the center of mass of the club.
- \( I_{2,CM} = M_2 < (R_2 - < R_2 >)^2 >= 0.071 \text{kg m}^2 \), the moment of inertia of the golf club measured about its center of mass.

The properties of the proximal arm of the double pendulum were picked by fiat, and are not based on any biomechanical model.

- \( R_1 = 0.7R_2 \), the length of the proximal arm of the double pendulum. This number is not based on any detailed measurement. It is meant to be a very crude approximation.
- \( M_1 < (R_1 - < R_1 >)^2 >= 3 * I_{2,CM} \), the moment of inertia of the proximal arm of the double pendulum about the fixed hub. This number is just a stab in the dark. Its only relevance is to scale the magnitude of \( K_\theta \).

The distance \( \delta \) is taken to be 7.84 cm, which is just about 3.1 inches. This was chosen so that \( R_1 + R_2 - \delta = 70 \text{ in} \). Again, there is no particular reason for this choice other than it made the length of the accessible points along the target line of order 12 in.
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