BOREL COMPLEXITY OF SETS OF POINTS WITH
PRESCRIBED BIRKHOFF AVERAGES IN POLISH DYNAMICAL
SYSTEMS WITH A SPECIFICATION PROPERTY

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Abstract. We study the descriptive complexity of sets of points defined by placing restrictions on statistical behaviour of their orbits in dynamical systems on Polish spaces. A particular examples of such sets are the set of generic points of a T-invariant Borel probability measure, but we also consider much more general sets (for example, \( \alpha \)-Birkhoff regular sets and the irregular set appearing in multifractal analysis of ergodic averages of a continuous real-valued function). We show that many of these sets are Borel. In fact, all these sets are Borel when we assume that our space is compact. We provide examples of these sets being non-Borel, properly placed at the first level of the projective hierarchy (they are complete analytic or co-analytic). This proves that the compactness assumption is in some cases necessary to obtain Borelness. When these sets are Borel, we use the Borel hierarchy to measure their descriptive complexity. We show that the sets of interest are located at most at the third level of the hierarchy. We also use a modified version of the specification property to show that for many dynamical systems these sets are properly located at the third level. To demonstrate that the specification property is a sufficient, but not necessary condition for maximal descriptive complexity of a set of generic points, we provide an example of a compact minimal system with an invariant measure whose set of generic points is \( \Pi^0_3 \)-complete.

1. Introduction

Every Borel subset of an uncountable Polish metric space \( X \) is a result of the following inductive procedure, whose steps are enumerated by countable ordinals: for \( \alpha = 1 \) we take all open subsets of \( X \) and their complements (closed subsets of \( X \)). We continue by transfinite induction: if \( \beta < \omega_1 \) is a countable ordinal and we completed our procedure for all ordinals \( 0 < \alpha < \beta \), then at the step \( \beta \) we add all possible countable unions (or intersections) of sets obtained so far and their complements. This completes level \( \beta \) of the procedure. So at the step \( \alpha = 2 \) we add all \( G_\delta \) and \( F_\sigma \) subsets, which are neither open nor closed, and in the next step we obtain all sets which are countable intersections of \( F_\sigma \) sets and all countable unions of \( G_\delta \) sets, which are neither \( G_\delta \) nor \( F_\sigma \) themselves. This introduces a natural hierarchy of Borel subsets of \( X \) which reflects their descriptive complexity. It is known that for an uncountable Polish space these levels do not collapse: at each level there appear new sets (sets that do not belong at any lower level of the hierarchy). Determining the level where a “naturally arising” or “non-ad hoc” Borel set appears in the hierarchy becomes a challenging problem. Only a few concrete examples of Borel sets are known to be located above the third level.
Here, we study the Borel complexity of sets appearing naturally in ergodic theory, dynamics, number theory, and fractal geometry. We develop tools which allow us to determine the exact position of many Borel sets examined in the literature. We show that without compactness some sets studied in topological dynamics become completely analytic (completely analytic sets are, in some sense, the simplest instances of non-Borel sets).

We state our main result in the framework of a dynamical system given by iteration of a continuous map \( T \) on a Polish metric space \( X \). In this setting the sets of interest consist of points whose orbits have prescribed statistical behaviour with respect to a bounded continuous observable \( \varphi: X \to \mathbb{R} \). Here by “statistical behaviour of an orbit of \( x \in X \)” we mean the asymptotic behaviour of the sequence of Birkhoff averages of \( \varphi \) along the orbit of \( x \). In the classical topological dynamics, one usually assumes that the space \( X \) is compact, but some expansions, most notably, continued fraction expansions come from dynamical systems acting on non-compact Polish spaces, so we work in this greater generality. We refer the reader to [14, 15, 16, 29] for related problems for dynamical systems (linear operators, countable Markov chains) on non-compact phase spaces. We study the following questions:

**Problem.** Given a continuous map \( T \) from a Polish space \( X \) to itself and a \( T \)-invariant measure \( \mu \), what is the complexity of the set \( G_\mu \) of generic points for the system \((G_\mu \) is defined precisely below)? More generally, given a nonempty, closed, and connected subset \( V \) of the space \( \mathcal{M}_T(X) \) of all \( T \)-invariant Borel probability measures on \( X \) endowed with the weak* topology, what is the Borel complexity of the set \( G(V) \) of all points \( x \) in \( X \) whose orbits generate \( V \), that is, the set of limit points of a sequence \( \mathcal{E}(x, N) \) equals \( V \), where \( \mathcal{E}(x, N) \) denotes the normalized counting measure concentrated on \( x, T(x), \ldots, T^{N-1}(x) \). Finally, given a bounded continuous function \( \varphi: X \to \mathbb{R} \), we study Borel complexity of sets of those \( x \in X \) that the sequence of Birkhoff averages

\[
A_k \varphi(x) = \frac{1}{k} \sum_{j=0}^{k-1} \varphi(T^j(x))
\]

converge as \( k \to \infty \) to a given value \( \alpha \in \mathbb{R} \). We also examine the set of points where this sequence does not converge for some or all bounded continuous function(s) (the irregular set).

Our initial interest in these questions comes from a result of H. Ki and T. Linton, who answered a question posed by A. Kechris and showed in [19] that for any integer \( r \geq 2 \), the set of numbers that are normal in base \( r \) is a \( \Pi^0_3 \)-complete set (a \( \Pi^0_3 \) set is a countable intersection of \( F_\sigma \) sets, the general definitions are given in [2]). Recall that we say that a real number \( \xi \) is *normal in base \( r \) if in its \( r \)-adic expansion every block of digits of length \( k \) appears with asymptotic frequency \( 1/r^k \). After [19] many authors have studied the Borel complexity of various sets related to normal numbers, and have extended Ki and Linton’s result in different directions. V. Becher, P. A. Heiber, and T. A. Slaman [4] settled a conjecture of A. S. Kechris by showing that the set of absolutely normal numbers (a real number is *absolutely normal* if it is normal to all bases \( r \geq 2 \)) is \( \Pi^0_4 \)-complete. Furthermore, V. Becher and T. A. Slaman [5] proved that the set of numbers normal in at least one base is \( \Sigma^0_4 \)-complete. See also [1], [6]. Also, sets of interest in dynamics are sometimes
Borel complexity of sets of orbits

It is well-known that a real number $\xi$ being normal in base $r$ is equivalent to the sequence $(r^k \xi)_k$ being uniformly distributed mod 1 (cf. [20]). The latter statement can be rephrased as $\xi \mod 1$ is a generic point for the Lebesgue measure on the unit circle $T = \mathbb{R}/\mathbb{Z}$ under the action of the map $T \to T$ given by $x \mapsto rx \mod 1$. Thanks to this dynamical point of view, in [2] we were able to study the Borel complexity of sets of normal numbers in several numeration systems. We employed a unified treatment for $r$-ary expansions, continued fraction expansions, $\beta$-expansions, and generalized GLS-expansions. In fact, we considered a dynamical system given by subshifts generated by all possible expansions of numbers in $[0, 1]$ in full-shifts over an at most countable set of digits (alphabet). In that setting normal numbers with respect to a given expansion method correspond to generic points of an appropriately selected shift-invariant measure on a subshift containing all possible expansions. It turns out that for these subshifts the set of generic points for any shift-invariant probability measure is precisely at the third level of the Borel hierarchy (it is a $\Pi_3^0$-complete set). The crucial dynamical feature we used was a feeble form of specification. All expansions named above generate subshifts with this specification property.

Our aim is to extend the results of [2]. We generalise [2] in two directions: First, we consider arbitrary (in particular, non-symbolic) dynamical systems on Polish spaces. In [2] only symbolic systems were considered. Second, we determine the descriptive complexity of sets of points defined by placing restrictions on statistical distribution of their orbits with respect to all bounded continuous observables. By statistical distribution of the orbits we understand the asymptotic behaviour of a sequence of Birkhoff averages along an orbit. In [2] only sets of generic points were considered. Our level of generality still allows us to study dynamical systems leading to various expansions considered in [2], but it also contains many more examples: we are now able to study the descriptive complexity of sets of numbers defined in terms of the asymptotic behaviour of the frequencies of the digits in their expansions. So far, only of the Hausdorff dimension of these sets was examined (see, for example, L. M. Barreira, B. Saussol and J. Schmeling, [4], A. H. Fan, D. J. Feng and J. Wu, [12], or L. O. R. Olsen, [26]). To apply our results in other branches of mathematics it remains to explain how to reduce the set we want to examine to the relevant dynamical form.

There are very simple examples of compact dynamical systems with ergodic invariant measures whose generic points are Borel sets of low complexity: If $X$ is the unit circle and $T$ is an irrational rotation, then every point is generic for the Lebesgue measure $\lambda$ so $G_\lambda$ is clopen. On the other hand, if $X = [0, 1]$ and $T(x) = x^2$ for $0 \leq x \leq 1$, then $T$ has two ergodic invariant measures, namely Dirac-point measures $\delta_0$ and $\delta_1$ concentrated at fixed points 0 and 1. We clearly have $G_{\delta_0} = [0, 1)$ is open but not closed, and $G_{\delta_1} = \{1\}$ is closed but not open. Hence, similarly as in [2] some assumptions about the dynamical system are necessary to achieve the maximal complexity of sets with prescribed Birkhoff averages for some (or all) observables. It turns out that the right assumption to impose on the system is a form of the specification property. The specification property was introduced by R. Bowen in his paper on Axiom A diffeomorphisms [5]. However, for
the main results of our paper only a certain weak form of the specification property (coined the *strong approximate product structure*) is needed. It implies the feeble specification property used in [2] for symbolic systems. We refer the reader to [21] for a discussion of the specification property and its many variants as well as their significance in dynamics.

We study at which Borel hierarchy levels the sets \( G_\mu \) (or more generally, sets like \( G(V) \), see §4 for the full list) may appear in the above situation and present a theorem saying that under fairly general assumptions these sets always have the maximal possible complexity.

Organization of the paper. After reviewing the notation and terminology in §2, we discuss the specification property and its weakening in §3. In §4 we define various sets described by the statistical behaviour of Birkhoff averages, and in §5 we prove that all these sets are (co-)analytic, some are always Borel, and all are always Borel when the space \( X \) is compact. In other words, in §5 we gather upper bounds for the descriptive complexity of our sets of interest. These computations hold in general, that is, they do not require specification hypotheses on the system. In §7 we state and prove our main results regarding lower bounds assuming our variant of the specification property (the strong approximate product structure). Whenever these bounds match the upper bounds obtained in the previous section, we obtain a completeness result. In particular, this is the case for all these sets when \( X \) is compact. For example, we see that the set of generic points \( G_\mu \) is \( \Pi^0_3 \)-complete. In §8 we present a (non-compact) dynamical system for which \( G(V) \) (here \( V \) may consist of a single measure \( \mu \)) is \( \Pi^1_1 \)-complete. Thus, we see a dichotomy in the complexity of \( G(V) \) depending on whether \( X \) is compact. Finally, in §9 we give an example of a minimal compact symbolic system (subshift) \((X, T)\) with an invariant ergodic measure \( \mu \) such that \( G_\mu \) is \( \Pi^0_0 \)-complete. This example shows that the specification property is sufficient for maximal complexity, but by no means necessary.

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2. Vocabulary/definitions/notation

2.1. General. Throughout this paper \( \omega = \{0, 1, 2, \ldots \} \), \( \mathbb{N} \) stands for the set of positive integers, the variables \( i, j, k, \ell, m, n \) are integers, and \( X \) always denotes a Polish topological space. Note that we are not assuming compactness of \( X \) in general; any compactness assumptions on \( X \) will be stated explicitly as needed (our main result does not require any compactness assumptions).

By a *Polish dynamical system*, which we also just call a *dynamical system*, we mean a pair \((X, T)\) where \( T : X \to X \) is a continuous map. We stress that whenever we consider a Polish dynamical system \((X, T)\) we tacitly assume that \( X \) is equipped with a fixed compatible complete metric \( \rho \). If \( X \) is non-compact, then various properties we will be considering, such as the specification property and its variants, depend on \( \rho \) in a sense that changing the metric to an equivalent one may affect these properties. For an example where the specification property is lost with a change of metric, see [23, Proposition 7.1] (in other words, for non-compact spaces
the specification property is no longer an invariant for topological conjugacy). For compact spaces the choice of metric is irrelevant.

2.2. Borel, analytic, and co-analytic sets. We recall some basic notions from descriptive set theory which we use to measure the complexity of sets in Polish spaces. The collection of Borel sets $\mathcal{B}(X)$, for $X$ a topological space, is the smallest $\sigma$-algebra containing the open sets.

We let $\Sigma^0_1$ be the collection of open sets and $\Pi^0_1 = \Sigma^0_1 \setminus \{X \setminus A : A \in \Sigma^0_1\}$ be the collection of closed sets. In general, for $\alpha < \omega_1$ we let $\Sigma^0_\alpha$ be the collection of countable unions $A = \bigcup_n A_n$ where each $A_n \in \Pi^0_\alpha$, for some $\alpha_n < \alpha$. We also let $\Pi^0_\alpha = \Sigma^0_\alpha = \{X \setminus A : A \in \Sigma^0_\alpha\}$. Equivalently, $A \in \Pi^0_\alpha$ if $A = \bigcap_n A_n$ where $A_n \in \Sigma^0_\alpha$, and each $\alpha_n < \alpha$. We also set $\Delta^0_\alpha = \Sigma^0_\alpha \cap \Pi^0_\alpha$. In particular, $\Delta^0_1$ is the collection of clopen sets. In classical terminology, $\Sigma^0_\alpha$ is the collection of $F_\sigma$ sets and $\Pi^0_\alpha$ is the collection of $G_\delta$ sets. For any topological space, $\mathcal{B}(X) = \bigcup_{\alpha < \omega_1} \Sigma^0_\alpha = \bigcup_{\alpha < \omega_1} \Pi^0_\alpha$. A basic fact (see [13]) is that for any uncountable Polish space $X$, all of the classes $\Delta^0_\alpha, \Sigma^0_\alpha, \Pi^0_\alpha$, for $\alpha < \omega_1$, are distinct. We say a set $A \subseteq X$ is $\Sigma^0_\alpha$ (resp. $\Pi^0_\alpha$) hard if $A \notin \Pi^0_\alpha$ (resp. $A \notin \Sigma^0_\alpha$). This says that $A$ is “no simpler” than a $\Sigma^0_\alpha$ set. We say that $A$ is $\Sigma^0_\alpha$-complete if $A \in \Sigma^0_\alpha \setminus \Pi^0_\alpha$, that is, $A \in \Sigma^0_\alpha$ and $A$ is $\Sigma^0_\alpha$ hard. This says that $A$ is exactly at the complexity level $\Sigma^0_\alpha$. Likewise, $A$ is $\Pi^0_\alpha$-complete if $A \in \Pi^0_\alpha \setminus \Sigma^0_\alpha$.

The Borel sets lie at the base of another hierarchy, the projective hierarchy. The $\Sigma^1_1$ (or analytic) sets are the images of Borel sets (via continuous functions from a Polish space to a Polish space). The same collection results if one uses Borel images instead of continuous images and one can also take just images of closed sets. The $\Pi^1_1$ (or co-analytic) sets are the complements of the analytic sets.

We define $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$. The pointclasses $\Sigma^1_1$ and $\Pi^1_1$ are closed under countable unions and intersections and so contain all the Borel sets. A fundamental result of Suslin (see [13]) says that in any Polish space $\mathcal{B}(X) = \Delta^1_1$. One can continue to define the $\Sigma^1_n$ and $\Pi^1_n$ sets for $n \geq 2$, forming the projective hierarchy, but we will not consider any classes beyond $\Pi^1_3$ and $\Sigma^1_3$ in this paper.

Also, all of the collections $\Delta^0_\alpha, \Sigma^0_\alpha, \Pi^0_\alpha, \Pi^1_1$, and $\Sigma^1_1$ are pointclasses, that is, they are closed under inverse images of continuous functions.

The levels of the Borel (and projective) hierarchy can be used to calibrate the descriptive complexity of a set. Determining an upper bound on the complexity of the set $A$ generally involves writing a condition defining $A$ which shows that it is in the appropriate place in the hierarchy. Establishing lower bounds is generally more difficult. One technique is the method of “Wadge reduction” whereby a set $C \subseteq Y$ (Y a Polish space) which is known to be hard for a certain level $\Gamma$ of the hierarchy is “reduced” to the set $A$. By this we mean we have a continuous function $f : Y \to X$ such that $C = f^{-1}(A)$. Since $\Gamma$ is a pointclass, this shows that $A$ is $\Gamma$-hard.

2.3. Invariant measures, generic points. We denote the set of all Borel probability measures on $X$ by $\mathcal{M}(X)$ and equip it with the weak* topology, which in our setting is known to be Polish [12], §17.E. Recall that $\mu_n$ converges to $\mu$ in the weak* topology if and only if for every $f \in C_b(X)$ (i.e., $f$ is a bounded, continuous, real-valued function on $X$) we have $\int f \, d\mu_n \to \int f \, d\mu$. If $X$ is compact, then $\mathcal{M}(X)$ is also compact [12, Theorem 17.22]. Although our main results are of a
topological nature, it is convenient for us to fix once and for all a particular metric $D$ for $\mathcal{M}(X)$. We choose the Prohorov metric associated to our fixed metric $\rho$ on $X$ and given by

$$D(\mu, \nu) = \inf \{ \varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ for all Borel sets } A \subseteq X \},$$

where $\mu, \nu \in \mathcal{M}(X)$ and $A^\varepsilon$ denotes the $\varepsilon$-neighborhood of $A$, that is,

$$A^\varepsilon = \{ y \in X : \rho(x, y) < \varepsilon \text{ for some } x \in A \}.$$  

Although (1) does not seem to yield a symmetric function, we do obtain a metric since we consider only Borel probability measures.

Given a Polish dynamical system $(X, T)$, we say that $\mu \in \mathcal{M}(X)$ is $T$-invariant if $\mu(T^{-1}(A)) = \mu(A)$ for every Borel set $A \subseteq X$. We let $\mathcal{M}_T(X)$ be the set of $T$-invariant Borel probability measures on $X$. If $X$ is compact, then $\mathcal{M}_T(X)$ must be nonempty, but this is no longer true if $X$ is just Polish, even if $T$ exhibits non-trivial recurrence (see [14, p. 374]). A measure $\mu \in \mathcal{M}_T(X)$ is ergodic if for every Borel set $A \subseteq X$ satisfying $T^{-1}(A) \subseteq A$ it holds that $\mu(A)$ is either 0 or 1. We write $\mathcal{M}_T^e(X)$ for the set of $T$-invariant ergodic measures. The set $\mathcal{M}_T(X)$ is always convex, so $\mathcal{M}_T^e(X)$ is always non-empty if $\mathcal{M}_T(X) \neq \emptyset$.

**Definition 1.** We say that $x \in X$ is a generic point for $\mu \in \mathcal{M}_T(X)$ if the sequence of empirical measures, whose $n$-th entry is given by

$$\mathcal{E}(x, n) = \frac{1}{n} (\delta_{T^0(x)} + \delta_{T^1(x)} + \cdots + \delta_{T^{n-1}(x)})$$

converge in the weak* topology on $\mathcal{M}(X)$ to $\mu$ as $n \to \infty$. Here $\delta_x$ is the Dirac measure concentrated at $x \in X$. We let $G_\mu = G_\mu(X, T)$ denote the set of generic points for $\mu$.

If $\mu$ is ergodic, then it follows from the ergodic theorem that $\mu$-almost every $x \in X$ is a generic point for $\mu$. We will not, however, be assuming $\mu$ is necessarily ergodic in our results, and so the existence of generic points is not automatic.

The notion of generic point can be viewed as generalizing the notion of a normal number. If we take $X = \mathbb{R}^\omega$, $T$ the shift map on $\mathbb{R}^\omega$, and $\mu$ the standard Bernoulli $(\frac{1}{2}, \ldots, \frac{1}{2})$ measure on $\mathbb{R}^\omega$, then $x \in \mathbb{R}^\omega$ is a generic point if and only if $x$ is the base $r$ expansion of a normal number.

### 3. Specification property and its weakening

In order to present our modification of the specification property we develop a terminology allowing us to compare various specification-like notions. It is inspired by [21]. Given a Polish dynamical system $(X, T)$ and $x \in X$, we define the orbit of $x$ to be the set $O(x) = \{ T^n(x) : n \in \omega \}$. Although the orbit is a set, it is customary to consider it as a sequence $\langle T^n(x) : n \in \omega \rangle$. Using this interpretation, and given $a, b \in \omega$ with $a \leq b$, we define an orbit segment of $x$ from $a$ to $b$ (synonymously, over $[a, b]$) to be the sequence $T^{[a,b]}(x) = \langle T^a(x), \ldots, T^b(x) \rangle$. Every orbit segment (up to the choice of $a$ and $b$) is determined by its initial point $y = T^a(x)$ and length $n = b - a + 1 > 0$, and hence without loss of generality an orbit segment may be identified with a pair $\langle y, n \rangle$ in $X \times \omega \setminus \{0\}$. We will use these two formulations concurrently.

**Definition 2.** A specification of rank $k > 0$ is a finite sequence $\xi$ of $k$ orbit segments, that is, $\xi \in (X \times \omega \setminus \{0\})^k$. Let $\varepsilon > 0$. We say that a point $y \in X$
Definition 3. A Polish dynamical system $(X, T)$ is \( \varepsilon \)-close shadows (or traces or follows) a specification \( \xi = (x_1 + n_1)_{k=1}^j \in (X \times \omega \setminus \{0\})^k \) if there are integers (called gaps) \( s_1, \ldots, s_{k-1} \in \omega \) such that for every \( 1 \leq j \leq k \) we have
\[
\rho(T^{(\sum_{i=1}^{j-1} (n_i + s_i) + t(y)}, T^n(x_j))) < \varepsilon \quad \text{for } 0 \leq t < n_j.
\]

The number \( L = n_k + \sum_{i=1}^{k-1} (n_i + s_i) \) is the number of iterates of \( y \) required to trace \( \xi \). We may also say that the orbit segment \( T^i(x) \) traces \( \varepsilon \)-close the specification \( \xi \).

Definition 4. We say that \( T \) has the specification property if for every \( \varepsilon > 0 \) there is a constant \( N_\varepsilon \in \omega \) such that for each \( k > 0 \) and every specification \( \xi \in (X \times \omega \setminus \{0\})^k \) there is a point \( y \) which traces it \( \varepsilon \)-close with all gaps of size \( \leq N_\varepsilon \).

To state our weakening of the specification property we will need the following notation: By \( \rho_\Lambda(x, y) \) we denote the Bowen distance between \( x, y \in X \) along a finite set \( \Lambda \subseteq \omega \) given by
\[
\rho_\Lambda(x, y) = \max_{j \in \Lambda} \rho(T^i(x), T^j(y)).
\]

The Bowen ball of radius \( \varepsilon > 0 \), along \( \Lambda \) and centered at \( x \in X \) is defined as
\[
B_\Lambda(x, \varepsilon) = \{ y \in X : \rho_\Lambda(x, y) < \varepsilon \}.
\]

When \( \Lambda \subseteq \{0, 1, \ldots, n-1\} \) for some \( n \in \mathbb{N} \) and \( |\Lambda|/n \) is close to 1, then the Bowen ball \( B_\Lambda(x, \varepsilon) \) can be seen as a set of those points \( y \in X \) whose orbits almost \( \varepsilon \)-trace the orbit segment \( (x, n) \). Here ‘almost’ means that the number of ‘errors’ that is, those \( 0 \leq j < n \) for which \( \rho(T^i(x), T^j(y)) \geq \varepsilon \) is small. If \( \Lambda = \{0, 1, \ldots, n-1\} \) for some \( n \in \mathbb{N} \) then no errors are allowed and we denote a Bowen ball \( B_\Lambda(x, \varepsilon) \) as \( B_n(x, \varepsilon) \).

We will now introduce a weakening of the specification property for a dynamical system. Although a huge variety of generalizations of the specification property are available in the literature [21], we find it convenient to introduce yet another one. The newly defined property suffices to prove our main results.

Definition 4. We say that a point \( y \in X \) \( (\varepsilon, \delta_1, \delta_2) \)-approximately-traces a specification \( \xi = (x_j, n_j)_{j=1}^k \) if
\begin{itemize}
  \item[(i)] \( y \in B_{n_j}(x_1, \varepsilon) \),
  \item[(ii)] for each \( 2 \leq j \leq k \) there is a set \( \Lambda_j \subseteq \{0, \ldots, n_j - 1\} \) with \( |\Lambda_j| \geq (1 - \delta_1)n_j \) and an integer \( 0 \leq s_{j-1} \leq \delta_2 n_j \) such that
    \[ T^{(\sum_{i=1}^{j-1} s_i + n_i)}(y) \in B_{\Lambda_j}(x_j, \varepsilon). \]
\end{itemize}

Definition 5. A Polish dynamical system \( (X, T) \) has the strong approximate product structure if the following holds: Given \( \varepsilon, \delta_1, \delta_2 > 0 \), there exists a nonegative integer \( N = N(\varepsilon, \delta_1, \delta_2) \) such that for any specification \( \xi = (x_j, n_j)_{j=1}^k \) with \( n_j \geq N \) for \( 1 \leq j \leq k \), there exists a point \( y \in X \) that \( (\varepsilon, \delta_1, \delta_2) \)-approximately-traces \( \xi \).

Remark 6. We will also say that a point \( y \) \( \varepsilon \)-approximately-traces a specification \( \xi \), if it \( (\varepsilon, \varepsilon, \varepsilon) \)-approximately-traces \( \xi \) as in Definition 4. We define \( N(\varepsilon) = N(\varepsilon, \varepsilon, \varepsilon) \), where \( N(\cdot, \cdot, \cdot) \) is as in Definition 5.

Thus, strong approximate product structure implies that, for any \( \varepsilon, \delta_1, \delta_2 > 0 \) and any specification \( \xi = (x_1, n_1), \ldots, (x_n, n_k) \) with all \( n_i \) sufficiently large, we can find a point \( y \) which \( \varepsilon \)-traces the first orbit segment \( (x_1, n_1) \), and \( \varepsilon \)-traces the segments \( (x_i, n_i) \) for \( i \geq 2 \) with at most \( \delta_2 n_i \) errors, allowing gaps of sizes at most \( \delta_2 n_i \) between the segments \( (x_{i-1}, n_{i-1}) \) and \( (x_i, n_i) \).
The strong approximate product structure is similar to and implies the approximate product structure of Pfister and Sullivan [28]. The difference between these two notions is that in Definition 4, $\varepsilon$-approximate-tracing allows no errors in the first orbit segment. Below we picture the implications between the strong approximate product structure (SAPS) and specification properties investigated by other authors: the specification property (SP), the weak specification property (WSP), and approximate product structure (APS), see [21] for more details:

$$\text{SP} \Rightarrow \text{WSP} \Rightarrow \text{SAPS} \Rightarrow \text{APS}$$

Although definition of the specification property looks prohibitively restrictive, there are many classes of systems exhibiting a variant of that property. Familiar examples of symbolic systems (subshifts) that have the specification property are mixing shifts of finite type and mixing sofic shifts. There are many other symbolic systems with the specification property. All topologically mixing graph maps (in particular, interval maps) have the specification property. The specification property is closely related with hyperbolicity: every transitive uniformly hyperbolic diffeomorphism of a compact connected manifold has the specification property. Specification or its variants also holds for systems of algebraic origin. Following Lind [24], we say that a toral automorphism is quasi-hyperbolic if its associated matrix $A$ has no roots of unity as eigenvalues. Lind [24, Theorem (ii)-(iii)] and Marcus [25] proved that all quasi-hyperbolic toral automorphisms must satisfy weak specification. As the strong approximate product structure implies the feeble specification property used in [2], our results also apply to the systems considered in [2], most notably to $\beta$-shifts.

4. Sets defined by conditions on statistical behaviour of their orbits

Generalizing the notion of a generic point, we define $V(x)$ as the set of all limit points in $M(X)$ of the sequence $(E(x,n))_{n \geq 0}$. Thus if $x \in G_\mu$ then $V(x) = \{\mu\}$, and the converse is true if $X$ is compact. It is well known that $V(x)$ is a closed connected subset of $M_T(X)$, which is necessarily nonempty provided that $X$ is compact.

Given a set $V \subseteq M_T(X)$, we can relate it to $V(x)$ and define sets of points, whose orbits exhibit particular statistical behaviour.

**Definition 7.** If $V \subseteq M(X)$, we let $G^V = G^V(X,T) = \{x \in X : V \subseteq V(x)\}$. Similarly, let $G_V = G_V(X,T) = \{x \in X : V(x) \subseteq V\}$. We set $G(V) = G^V \cap G_V$. Finally, we let $VG = \{x \in X : V(x) \cap V \neq \emptyset\}$.

We refer to the set $G(V)$ as the set of generic points for $V$. We caution the reader that in the non-compact case it is not necessarily true that $G_\mu = G(\{\mu\})$ (see Remark 34).

**Definition 8.** Given $\varphi \in C_b(X)$, $k > 0$ and $y \in X$ we define $k$-th ergodic average of $\varphi$ along the orbit of $y$ by

$$A_k \varphi(y) = \frac{1}{k} \sum_{j=0}^{k-1} \varphi(T_j(y)) = \int \varphi dE(y,k).$$

**Definition 9.** Given $\varphi \in C_b(X)$ we say that a point $x \in X$ is Birkhoff regular for $\varphi$ if the sequence of ergodic averages $A_k \varphi(x)$ converges as $k \to \infty$. For $\alpha \in \mathbb{R}$ we
define the \( \alpha \)-level sets for Birkhoff averages (\( \alpha \)-Birkhoff regular points) of \( \varphi \) by
\[
B_\varphi(\alpha) = \{ x \in X : \lim_{k \to \infty} A_k \varphi(x) = \alpha \}.
\]
The set of all Birkhoff regular points for \( \varphi \) is denoted
\[
B_\varphi(X, T) = \bigcup_{\alpha \in \mathbb{R}} B_\varphi(\alpha).
\]

**Definition 10.** The irregular set for \( \varphi \) is the set \( I_\varphi(X, T) \) consisting of all points \( y \in X \) such that the sequence of ergodic averages \( (A_k \varphi(y))_{k \in \mathbb{N}} \) does not converge. The irregular set \( I(X, T) \) of the Polish dynamical system \((X, T)\) is the union of \( I_\varphi(X, T) \) when \( \varphi \) runs over all bounded continuous real-valued functions on \( X \).

Thus, for each \( \varphi \in C_b(X) \) we have a partition
\[
X = I_\varphi(X, T) \cup \bigcup_{\alpha \in \mathbb{R}} B_\varphi(\alpha).
\]

By the Birkhoff ergodic theorem, the irregular set is universally null: \( \mu(I_\varphi(X, T)) = \mu(I(X, T)) = 0 \) for every \( T \)-invariant Borel probability measure \( \mu \) on \( X \) and \( \varphi \in C_b(X) \). The complement of the irregular set, denoted \( Q(X, T) \), is called the quasi-regular set of \((X, T)\). If \( X \) is compact, then it is easy to see that every quasi-regular point must be also generic for some \( T \)-invariant measure (cf. \cite{27}). It seems to be a little known fact that each quasi-regular point must be generic for some \( T \)-invariant measure also in the general situation, that is, for every Polish metric space. For example, Oxtoby in [27, Section 7] (following Fomin) adds an additional condition to the definition of quasi-regular point for a Borel automorphism to get a similar result.

**Fact 11.** We have \( Q(X, T) = \bigcup_{\mu \in \mathcal{M}_T(X)} G_\mu. \)

*Proof.* Note that if \( x \) is generic for \( \mu \), then the definition of weak* convergence and \cite{3} imply that \( A_k \varphi(y) \to \int \varphi \, d\mu \) as \( k \to \infty \), so every generic point is \( \alpha(\varphi) \)-Birkhoff regular for every \( \varphi \in C_b(X) \) with \( \alpha(\varphi) = \int \varphi \, d\mu \). In particular, each generic point is quasi-regular. On the other hand, the space of Borel measures on \( X \) is weakly sequentially complete by \cite[Theorem 8.7.1]{7}. Here, weak sequential completeness means that every fundamental sequence converges (cf. \cite[Section 8.1]{7}). A sequence \( (\mu_n)_{n \geq 1} \) of Borel probability measures is fundamental if for every bounded continuous function \( f \), the sequence \( \int f \, d\mu_n \) is Cauchy. Therefore if \( x \) is quasi-regular, then the sequence \( (\mathcal{E}(x, n))_{n \geq 1} \) is fundamental and so converges to a Borel probability measure \( \mu \) on \( X \), which is easily seen to be \( T \)-invariant. \( \square \)

Note that for all Polish dynamical systems \((X, T)\), for each \( \mu \in \mathcal{M}_T(X) \), and for each \( \varphi \in C_b(X) \) we have
\[
G_\mu \subseteq Q(X, T) \subseteq B_\varphi(X, T) \quad \text{and} \quad I_\varphi(X, T) \subseteq I(X, T).
\]

**5. Upper bounds**

We will study which sets defined by placing conditions on the statistical behaviour of orbits are Borel (assuming that \( X \) is compact if necessary). Recall that \((X, T)\) is a Polish dynamical system and \((f_n)\) is a sequence of bounded continuous real-valued functions on \( X \) which generate the weak* topology on \( \mathcal{M}(X) \) (cf. Theorem 17.19 of \cite{18}). That is, \( \mu_k \to \mu \) as \( k \to \infty \) if and only if for each \( n \) we have
\[ \int f_n \, d\mu_k \to \int f_n \, d\mu \text{ as } k \to \infty. \] Recall also that \( A_k \varphi \) stands for the \( k \)-th ergodic average of \( \varphi \) along the orbit of \( y \), see (3).

**Fact 12.** If \( \varphi \in C_b(X) \) and \( \alpha \in \mathbb{R} \), then the set \( B_{\varphi}(\alpha) = \{ x \in X : \lim_{k \to \infty} A_k \varphi(x) = \alpha \} \) of \( \alpha \)-Birkhoff regular points is a \( \Pi_3^0 \) set.

*Proof.* It is easy to see that
\[
B_{\varphi}(\alpha) = \bigcap_{m=1}^{\infty} \bigcap_{K=1}^{\infty} \bigcap_{k=K}^{\infty} \bigcap_{\ell=K}^{\infty} \{ y \in X : |A_k \varphi(y) - \alpha| \leq 1/m \}.
\]

\[ \square \]

**Fact 13.** If \( \mu \in \mathcal{M}_T(X) \), then the set \( G_\mu \) of \( \mu \)-generic points is a \( \Pi_3^0 \) set.

*Proof.* A point \( y \in X \) belongs to \( G_\mu \) if and only if for every \( n \) the sequence \( A_k f_n(y) \) converges to \( c_n = \int f_n \, d\mu \) as \( k \to \infty \). Thus
\[
G_\mu = \bigcap_{n=0}^{\infty} B_{f_n}(c_n).
\]

By Fact 12, \( G_\mu \) is a \( \Pi_3^0 \) set. \[ \square \]

**Fact 14.** If \( \varphi \in C_b(X) \), then the irregular set \( I_\varphi(X,T) \) is a \( \Sigma_3^0 \) set and \( B_{\varphi}(X,T) = X \setminus I_\varphi(X,T) \) is a \( \Pi_3^0 \) set.

*Proof.* A point \( x \in X \) is irregular for \( \varphi \) if and only if
\[
\liminf_{k \to \infty} A_k \varphi(x) < \limsup_{k \to \infty} A_k \varphi(x).
\]

Therefore
\[
I_\varphi(X,T) = \bigcup_{N=1}^{\infty} \bigcap_{K=1}^{\infty} \bigcup_{k=K}^{\infty} \bigcup_{\ell=K}^{\infty} \{ x \in X : A_k \varphi(x) + 1/N < A_\ell \varphi(x) \}.
\]

By continuity, the set \( \{ x \in X : A_k \varphi(x) + 1/N < A_\ell \varphi(x) \} \) is open in \( X \) for any choice of \( \varphi \) and \( k, \ell, m \). It follows that \( I_\varphi(X,T) \) is a \( \Sigma_3^0 \) set. \[ \square \]

**Fact 15.** The irregular set \( I(X,T) \) is a \( \Sigma_3^0 \) set and the quasi-regular set \( Q(X,T) = X \setminus I(X,T) \) is a \( \Pi_3^0 \) set.

*Proof.* Note that by Fact 11 we have \( x \in Q(X,T) \) if and only if \( x \) is generic for some \( \mu \in \mathcal{M}_T(X) \). The latter is equivalent to the sequence \( (E(x,n))_{n \geq 1} \) being a Cauchy sequence in the Prohorov metric \( D \) (starting with the complete metric \( \rho \) on \( X \) we obtain a complete metric \( D \) on \( \mathcal{M}(X) \), see [13, Cor. 18.6.3]). Hence
\[
Q(X,T) = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \bigcup_{\ell=N}^{\infty} \{ x \in X : D(E(x,k),E(x,\ell)) \leq 1/m \}.
\]

Note that for any \( n \geq 1 \), the map \( x \mapsto E(x,n) \) is continuous. Thus the set \( \{ x \in X : D(E(x,k),E(x,\ell)) \leq 1/m \} \) is closed for any choice of the parameters \( k, \ell \) and \( m \). Therefore \( Q(X,T) \) is a \( \Pi_3^0 \) set. \[ \square \]

**Fact 16.** If \( V \subseteq \mathcal{M}_T(X) \) is closed, then \( G^V = \{ x \in X : V \subseteq V(x) \} \) is a \( \Pi_2^0 \) set.
Proof. Let $V_0$ be a countable dense subset of $V$. Then
\[
G^V = \bigcap_{\nu \in V_0} G^{(\nu)} = \bigcap_{\nu \in V_0} \bigcap_{N=1}^\infty \bigcap_{m=1}^\infty \bigcup_{n=N}^\infty \{x \in X : D(\mathcal{E}(x,n),\nu) < 1/m\}.
\]
Note that for any $n \geq 1$, the map $x \mapsto \mathcal{E}(x,n)$ is continuous. Thus the set \(\{x \in X : D(\mathcal{E}(x,n),\nu) < 1/m\}\) is open for any choice of the parameters $n$ and $\nu$. Thus $G^V$ is a $\Pi^0_2$ set.

Recall that a point $x \in X$ is said to have maximal oscillation, if $V(x) = \mathcal{M}_T(X)$ [10, Def. 21.17]. Taking $V = \mathcal{M}_T(X)$ in Fact 16 we immediately obtain the following corollary:

Fact 17. The set of points with maximal oscillation is a $\Pi^0_2$ set.

Fact 18. If $U \subseteq \mathcal{M}_T(X)$ is open, then the set $U^G$ is analytic (is $\Sigma^0_1$).

Proof. Let $R = \{(x,\mu) \in X \times \mathcal{M}(X) : \mu \in V(x)\} \subseteq X \times \mathcal{M}_T(X)$. We have
\[
R = \bigcap_{m=1}^\infty \bigcap_{N=1}^\infty \bigcup_{n=N}^\infty \{(x,\mu) \in X \times \mathcal{M}_T(X) : D(\mu,\mathcal{E}(x,n)) < 1/m\}.
\]
Clearly, the set \(\{(x,\mu) \in X \times \mathcal{M}_T(X) : D(\mu,\mathcal{E}(x,n)) < 1/m\}\) is open in $X \times \mathcal{M}_T(X)$ for every $n,m > 0$, so \(\{(x,\mu) \in X \times \mathcal{M}(X) : \mu \in V(x)\} \subseteq X \times \mathcal{M}_T(X)\) is Borel. Since $U^G$ is a projection of the set $R \cap (X \times U)$ onto the first coordinate, we get that $U^G$ is $\Sigma^0_1$ (see [18, p. 86]).

As an immediate corollary we have:

Fact 19. For every closed $V \subseteq \mathcal{M}_T(X)$ the set $G_V$ is co-analytic (is $\Pi^0_1$).

Proof. Recall that $X^V G = X \setminus G_V$ and apply Fact 18.

Fact 20. Assume $X$ is compact. If $V \subseteq \mathcal{M}_T(X)$ is closed, then $G_V = \{x \in X : V(x) \subseteq V\}$ is a $\Pi^0_3$ set. If $U \subseteq \mathcal{M}_T(X)$ is open, then $U^G = \{x \in X : V(x) \cap U \neq \emptyset\}$ is a $\Sigma^0_3$ set.

Proof. Observe that
\[
V(x) \subseteq V \iff D(\mathcal{E}(x,n),V) \to 0 \text{ as } n \to \infty
\iff x \in \bigcap_{m=1}^\infty \bigcup_{N=1}^\infty \bigcup_{n=N}^\infty \{x \in X : D(\mathcal{E}(x,n),V) \leq 1/m\}.
\]
Here we employed the standard notation $D(\mathcal{E}(x,n),V) = \inf_{\nu \in V} D(\mathcal{E}(x,n),\nu)$. We use the compactness of $X$ in the first equivalence; without compactness, only the implication $\Leftarrow$ holds. Since the map $x \mapsto \mathcal{E}(x,n)$ is continuous for every $n \geq 1$, the sets $\{x \in X : D(\mathcal{E}(x,n),V) \leq 1/m\}$ are closed. Thus $G_V$ is a $\Pi^0_3$ set. For the second part of the fact, note that $U^G = X \setminus G_{X \setminus U}$ for every open set $U \subseteq X$.

Fact 21. If $V \subseteq \mathcal{M}_T(X)$ is closed, then the set $G(V)$ is co-analytic (is $\Pi^0_1$). If, in addition, $X$ is compact, then $G(V)$ is a $\Pi^0_3$ set.

Proof. Use the equality $G(V) = G^V \cap G_V$, Fact 19 and, respectively, Fact 19 or Fact 20.

The following question is related to those considered in this paper.
**Question.** Given a dynamical system \((X, T), \varphi \in \mathcal{C}_b(X)\), and \(A \subseteq \mathbb{R}\) what is the exact complexity of the set

\[
B(\varphi, A) = \bigcup_{\alpha \in A} B_\varphi(\alpha).
\]

6. Auxiliary results

The next three results record some basic estimates which we will use in the proof of our main result. To formulate these results we have to extend our terminology.

The **empirical measure** \(E(\bar{x})\) of a sequence \(\bar{x} = \langle x_1, \ldots, x_k \rangle \in X^k\), where \(k > 0\) is the normalized sum of the Dirac (point-mass) measures along points in the sequence, that is

\[
E(\bar{x}) = \frac{1}{k} \sum_{p=1}^{k} \delta_{x_p}.
\]

Given a specification \(\xi = ((x_j, n_j))_{j=1}^{k} \in (X \times \omega \setminus \{0\})^k\) we define the **empirical measure** of \(\xi\) denoted by \(E(\xi)\) as the empirical measure of the sequence \(\text{seq}(\xi) \in X_{\sum_{j=1}^{k} n_j}\) obtained by concatenation of the orbits segments in \(\xi\), that is, \(E(\xi) = E(\text{seq}(\xi))\) where

\[
\text{seq}(\xi) = \langle x_1, T(x_1), \ldots, T^{n_1-1}(x_1), \ldots, x_k, T(x_k), \ldots, T^{n_k-1}(x_k) \rangle \in X_{\sum_{j=1}^{k} n_j}.
\]

Note that if \(\xi = \langle (x, n) \rangle\) is a single orbit segment for some \(n > 0\), then we have

\[
E(\xi) = E(\langle x, T(x), \ldots, T^{n-1}(x) \rangle) = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)} = E(x, n),
\]

where \(E(x, n)\) is the empirical measure defined by (2).

**Lemma 22.** Let \(\gamma, \delta, \varepsilon > 0\). Assume that \(\bar{x} = \langle x_1, \ldots, x_k \rangle \in X^k\) and \(\bar{y} = \langle y_1, \ldots, y_\ell \rangle \in X^\ell\), where \(0 < k \leq \ell \leq (1 + \delta)k\). If there exists \(j\) with \((1-\gamma)k \leq j \leq k\) and subsequences \(\bar{x}' = \langle x_{a_1}, \ldots, x_{a_j} \rangle \in X^j\) and \(\bar{y}' = \langle y_{b_1}, \ldots, y_{b_\ell} \rangle \in X^\ell\) with \(1 \leq a_1 < \ldots < a_j \leq k\) and \(1 \leq b_1 < \ldots < b_\ell \leq \ell\) such that \(\rho(x_{a_i}, y_{b_j}) < \varepsilon\) for \(1 \leq i \leq j\), then

\[
D(E(\bar{x}), E(\bar{y})) \leq 2\gamma + \delta + \varepsilon.
\]

**Proof.** By the triangle inequality,

(4) \[
D(E(\bar{x}), E(\bar{y})) \leq D(E(\bar{x}, E(\bar{x}')) + D(E(\bar{x}'), E(\bar{y}')) + D(E(\bar{y}'), E(\bar{y})).
\]

For each \(1 \leq i \leq j\) we have from the definition of \(D\) that \(D(\delta_{x_{a_i}}, \delta_{y_{b_j}}) < \varepsilon\). It follows that \(D(E(\bar{x}'), E(\bar{y}')) < \varepsilon\).

Next, observe that

\[
E(\bar{x}')(A) \leq \frac{k}{j} E(\bar{x})(A) \leq E(\bar{x})(A) + \frac{k-j}{k} \leq E(\bar{x})(A) + \gamma,
\]

which means that \(D(E(\bar{x}), E(\bar{x}')) \leq \gamma\). By our assumptions \((1-\gamma-\delta)k \leq \ell \leq k\); this implies \(D(E(\bar{y}), E(\bar{y}')) \leq \gamma + \delta\) by a similar reasoning. Plugging all the mentioned inequalities into (4) finishes the proof.

We have the following immediate corollary.
Corollary 23. Let $k > 0$. If a point $y$ $\varepsilon$-approximately-traces a specification $\xi = ((x_j, n_j))_{j=1}^k \in (X \times \omega \setminus \{0\})^k$ with gaps $s_1, \ldots, s_{k-1} \in \omega$ bounded by $\delta_2 n_j$ and errors bounded by $\delta_1 n_j$ and

$$L = n_k + \sum_{i=1}^{k-1} (s_i + n_i),$$

then $D(\mathcal{E}(\xi), \mathcal{E}(y, L)) < 2\gamma + \delta + \varepsilon$.

Lemma 24. If $(X, T)$ is a Polish dynamical system with the strong approximate product structure and $\mu$ is a $T$-invariant measure on $X$, then for each $\varepsilon > 0$ there exists a positive integer $Q$ such that for every $n > 0$ there is $x \in X$ satisfying $D(\mu, \mathcal{E}(x, nQ)) < \varepsilon$.

Proof. Assume that $\mu$ is a $T$-invariant (possibly non-ergodic) probability measure. It is well-known that $\mu$ can be approximated by affine combinations of finite collections of ergodic measures with rational coefficients, that is, for every $\varepsilon > 0$ there is a measure $\nu \in \mathcal{M}_T(X)$ such that

$$D(\mu, \nu) < \varepsilon/4 \quad \text{and} \quad \nu = \sum_{i=1}^k q_i \nu_i,$$

where $k > 0$ and $\nu_1, \ldots, \nu_k \in \mathcal{M}_T(X)$ are ergodic, $q_1, \ldots, q_k \in \mathbb{Q}$ are positive and $q_1 + \cdots + q_k = 1$. Let $Q$ be a positive integer such that for some $p_1, \ldots, p_k$ we have $q_i = p_i/\varepsilon$ for $i = 1, \ldots, k$. We choose $z_1, \ldots, z_k$ such that $z_i$ is generic for $\nu_i$. Next, let $K \geq N(\varepsilon/8)$ (here $N(\cdot)$ is as in Remark 6) be large enough so that

$$D(\mathcal{E}(z_i, K), \nu_i) < \varepsilon/8 \quad \text{for} \quad i = 1, \ldots, k.$$

For each $n > 0$ we define a specification $\xi$ by

$$\xi = ((x'_j, K))_{j=1}^{nQ} \in (X \times \omega \setminus \{0\})^{nQ},$$

where each $z_i$ appears $n p_i$ times on the list $x'_1, \ldots, x'_nQ$, that is,

$$(x'_j)_{j=1}^{nQ} = (z_{1, \text{np}_1 \text{ times}}, z_1, \ldots, z_{2, \text{np}_2 \text{ times}}, \ldots, z_{k, \text{np}_k \text{ times}}).$$

By definition of $\xi$ we have

$$\mathcal{E}(\xi) = \sum_{i=1}^k q_i \mathcal{E}(z_i, K).$$

(5)

Let $x$ be a point which $\varepsilon/8$-approximately-traces the specification $\xi$ and let $L$ be the number of iterates required to do so. Then $D(\mathcal{E}(\xi), \mathcal{E}(x, L)) < \varepsilon/4$, hence $D(\mu, \mathcal{E}(x, L)) < 5\varepsilon/8$ because of (5).

Since $nQ \leq L < (1+\varepsilon/4)nQ$ we use Lemma 22 to get $D(\mathcal{E}(x, nQ), \mathcal{E}(x, L)) < \varepsilon/4$ which ends the proof.

Remark 25. Lemma 24 shows, assuming $(X, T)$ has the strong approximate product structure, that the set of measures of the form $\mathcal{E}(x, n)$ for $x \in X$ and $n > 0$ is weak* dense in $\mathcal{M}_T(X)$ in the sense that every $\mu \in \mathcal{M}_T(X)$ is the weak* limit of a sequence $\mu_k = \mathcal{E}(x_k, n_k)$ (in general, the measures $\mathcal{E}(x, n)$ are not $T$-invariant).
7. Main Results

In this section we prove our main results about the complexity of the set of generic points for systems satisfying the strong approximate product structure. If the system $(X,T)$ is compact, the set $\mathcal{M}_T(X)$ is nonempty (a theorem of Bogolyubov and Krylov). If $X$ is compact and $\mathcal{M}_T(X)$ consists of a single element $\mu$, then $X = G_\mu$. Thus for compact $X$, the statistical behavior of the system is trivial if there exist less than two $T$-invariant measures. For this reason, in this section we mostly work under the assumption that $|\mathcal{M}_T(X)| \geq 2$. It should be kept in mind, however, that for general Polish $X$, the set $\mathcal{M}_T(X)$ may be empty.

Example 26. Let $\hat{X}$ be a compact metric space and $\hat{T}: \hat{X} \to \hat{X}$ be a continuous map with the specification property. For example, we can take the compact unit interval, $\hat{X} = [0,1]$ and $\hat{T}$ to be the tent map $\hat{T}(x) = 1 - |2x|$. Let $X$ be the set of points with the maximal oscillation. It was noted by Sigmund that $X$ is a dense $G_\delta$ subset of $\hat{X}$, and $X$ is a $T$-invariant set. Hence restricting $\hat{T}$ to $X$ we obtain a Polish dynamical system which does not have any invariant measures, since $I(X,T) = X$. It is easy to see that $(X,T)$ has the specification property. It shows that we need to assume that $|\mathcal{M}_T(X)| \geq 2$

In Lemma 27, we show (assuming that $(X,T)$ has the strong approximate product structure) that given a sequence of $T$-invariant measures $(\mu_n)_{n \geq 1}$, satisfying $D(\mu_n, \mu_{n+1}) \to 0$, one can find a point $y \in X$ and piecewise constant non-decreasing function $j \mapsto n(j)$ such that the sequence of empirical measures $E(y,j)$, is asymptotic to the sequence $\mu_{n(j)}$ (the distance $D(E(y,j), \mu_{n(j)})$ converges to 0 as $j \to \infty$).

Part 1 of Theorem 29 gives the exact complexity of the set of generic points $G_\mu$ for a Polish dynamical system $(X,T)$ with the strong approximate product structure satisfying $|\mathcal{M}_T(X)| \geq 2$. More generally, Part 1 of Theorem 29 uses the same proof method, to obtain lower bounds for the complexity of the set $G(V)$, where $V \subseteq \mathcal{M}_T(X)$ is a non-trivial, closed, connected subset of $\mathcal{M}_T(X)$. If $X$ is compact then $G_\mu = G(\{\mu\})$ and so in this case 1 follows directly from the statement of 1 of Theorem 29. In this section we also prove lower bounds on the complexity of the sets $G^V$, $G_V$ and other sets introduced in Section 5.

Lemma 27. Let $(X,T)$ be a Polish dynamical system with the strong approximate product structure.

1. Given a sequence $(\mu_n)_{n \geq 1}$ in $\mathcal{M}_T(X)$ such that $D(\mu_n, \mu_{n+1}) \to 0$, there exists a point $y \in X$ and a sequence $L_n \nearrow \infty$ such that

\begin{equation}
D(E(y,j), \mu_{n(j)}) \to 0 \quad \text{as } j \to \infty,
\end{equation}

where $n(j)$ denotes the unique integer satisfying $L_{n(j)} - 1 \leq j < L_{n(j)}$. In particular, this implies $V(y) = \{ \text{limit points of the sequence } (\mu_k)_{k \geq 1} \}$.

2. Given any $y_0 \in X$ and $\varepsilon > 0$, we can define $y$ in (1) so that $y \in B(y_0, \varepsilon)$.

3. If $y^1$ and $y^2$ are points obtained as in (1) for sequences of measures $(\mu^1_n)_{n \geq 1}$ and $(\mu^2_n)_{n \geq 1}$ such that for some $N \geq 1$ we have that $\mu^1_i = \mu^2_i$ for $1 \leq i \leq N$, then $\rho(y^1, y^2) \leq 1/2^N$.

\[\text{Constant on finite sets of consecutive } j\text{'s.}\]
Proof: Fix $0 < \varepsilon < 1$. We will define sequences of integers $(K_n)_{n \geq 0}$ and points $(x_n)_{n \geq 0} \subseteq X$, which will be parameters for our construction. For each $n \geq 0$ we use Lemma 24 to pick $x_n \in X$ and $K_n$ so that the following hold:

\begin{align}
(7) & \quad K_n \geq N(\varepsilon/2^{n+1}), \\
(8) & \quad D(\mathcal{E}(x_n, K_n), \mu_n) \leq \varepsilon/2^{n+1}.
\end{align}

Here, $N(\varepsilon/2^{n+1})$ is defined as in Remark 6.

Next, we will define $(\ell_n)_{n \geq 1}, (L_n)_{n \geq 0}$ and $(y_n)_{n \geq 0} \subseteq X$. We begin with $L_0 = K_1$ and $y_0$ an arbitrary point in $X$. For all $n \geq 0$ we will have

\begin{align}
(9) & \quad L_{n-1} \geq nK_n.
\end{align}

We proceed inductively, for $n \geq 1$:

- pick $\ell_n$ such that $\ell_nK_n \geq (n+1)K_{n+1}$ and $\ell_nK_n \geq nL_{n-1}$,
- define the specification $\xi_n$ to consist of the pair $(y_{n-1}, L_{n-1})$ repeated $\ell_n$ times,
- define $y_n$ to be a point that $\varepsilon/2^{n+1}$-approximately-traces the specification $\xi_n$,
- define $L_n$ to be the length of orbit segment required for $y_n$ to trace $\xi_n$. Note that condition (9) is satisfied, as $L_n \geq \ell_nK_n \geq nK_{n+1}$.

It follows from this definition that

\begin{align}
(10) & \quad \rho(T^i(y_{n-1}), T^i(y_n)) \leq \varepsilon/2^{n+1} \quad \text{for } 0 \leq i < L_{n-1}.
\end{align}

In particular, the sequence $(y_n)$ must converge to a limit $y$, and we have

\begin{align}
(11) & \quad \rho(T^i(y_{n-1}), T^i(y)) \leq \varepsilon/2^n \quad \text{for } 0 \leq i < L_{n-1}.
\end{align}

We will now examine the empirical measures $\mathcal{E}(y, j)$ for $j > 0$. First, consider $\mathcal{E}(y, L_n)$ for some integer $n$. By (11), we have

\begin{align}
(12) & \quad D(\mathcal{E}(y, L_n), \mathcal{E}(y_n, L_n)) \leq \varepsilon/2^n.
\end{align}

Recall that we have chosen $y_n$ so that it $\varepsilon/2^{n+1}$-approximately-traces $\xi_n$ along the orbit segment of length $L_n$. Together with Lemma 22 this gives $D(\mathcal{E}(y_n, L_n), \mathcal{E}(\xi_n)) \leq 4\varepsilon/2^{n+1} = \varepsilon/2^{n+1}$.

Define $\xi_n$ to be the specification consisting of $\ell_n$ repetitions of pair $(x_n, K_n)$. Thus $\xi_n$ only differs from $\xi_n$ by the lack of initial $(y_{n-1}, L_{n-1})$ segment. We defined $\ell_n$ so that $\ell_nK_n \geq nL_{n-1}$, thus an application of Lemma 22 gives

\begin{align}
(13) & \quad D(\mathcal{E}(\xi_n), \mathcal{E}(\xi_n)) \leq 1/n.
\end{align}

Finally, notice that

\begin{align}
(14) & \quad D(\mathcal{E}(\xi_n), \mu_n) = D(\mathcal{E}(x_n, K_n), \mu_n) \leq \varepsilon/2^{n+1}.
\end{align}

Combining (12), (13) and (14) gives

\begin{align}
(15) & \quad D(\mathcal{E}(y, L_n), \mu_n) \leq 1/n + 7\varepsilon/2^{n+1} \to 0 \quad \text{as } n \to \infty.
\end{align}

We are now prepared to estimate $\mathcal{E}(y, j)$ for arbitrary $j > 0$. There exists a unique integer $n = n(j)$ such that $L_{n-1} \leq j < L_n$. Recall the definition of $y_n$ as a point that $\varepsilon/2^{n+1}$-approximately-traces $\xi_n$. Let $s_1, \ldots, s_{\ell_n}$ be the lengths of gaps in this tracing, as in Definition 4. The orbit segment $(y_n, L_n)$ is made up of orbit segments corresponding to the elements of $\xi_n$ (interspersed by gaps of length...
Thus, the initial subsegment \((y, j)\) fully contains some number \(b + 1\) of those segments. Formally, \(b\) is the unique integer \(0 \leq b < \ell_n\) such that

\[
B := L_{n-1} + bK_n + \sum_{i < b} s_i \leq j < L_{n-1} + (b + 1)K_n + \sum_{i \leq b} s_i.
\]

We have \(j - B \leq K_n + s_b \leq (1 + \varepsilon/2^{n+1})K_n\). We also have

\[
\frac{j - B}{B} \leq \frac{(1 + \varepsilon/2^{n+1})K_n}{B} \leq \frac{(1 + \varepsilon/2^{n+1})K_n}{L_{n-1}} \leq \frac{1 + \varepsilon/2^{n+1}}{n},
\]

where the last inequality follows from (19). Next, note that \(y_n\), over the orbit segment of length \(B\), \(\varepsilon/2^{n+1}\)-approximately-traces the specification

\[
\zeta = \langle (y_{n-1}, L_{n-1}), (x_n, K_n), \ldots (x_n, K_n) \rangle.
\]

By the triangle inequality, we obtain

\[
D(\mathcal{E}(\zeta, \mu_n)) \leq D(\mathcal{E}(\zeta, \mu_n)) + D(\mathcal{E}(\zeta, \mu_n)).
\]

Let us examine the terms on the right hand side of \(\leq\) in (17):

- \(D(\mathcal{E}(y, j), \mathcal{E}(y, B)) \leq (1 + \varepsilon/2^{n+1})/n\) by (16),
- \(D(\mathcal{E}(y, B), \mathcal{E}(y, B)) \leq \varepsilon/2^{n+1}\) by (11),
- \(D(\mathcal{E}(y, B), \mathcal{E}(\zeta)) \leq 4\varepsilon/2^{n+1}\) by \(\varepsilon/2^{n+1}\)-approximate-tracing and Lemma 22,
- \(D(\mathcal{E}(\zeta, \mu_n)) \leq \max\{D(\mathcal{E}(y_{n-1}, L_{n-1}), \mu_n), D(\mathcal{E}(x_n, K_n), \mu_n)\}\). Now,
  - \(D(\mathcal{E}(y_{n-1}, L_{n-1}), \mu_n) \to 0\) as \(n \to \infty\), as shown by (15), and by our assumption \(D(\mu_n, \mu_{n-1}) \to 0\),
  - \(D(\mathcal{E}(x_n, K_n), \mu_n) \leq \varepsilon/2^{n+1} \to 0\) as \(n \to \infty\).

All of this, applied to the right hand side of \(\leq\) in (17), shows that

\[
D(\mathcal{E}(y, j), \mu_n(j)) \to 0 \quad \text{as} \quad j \to \infty.
\]

It follows easily that \(V(y) \subseteq \{\text{limit points of the sequence } (\mu_k)_{k \geq 1}\}\). Indeed, if \(\mathcal{E}(y, j_k) \to \bar{\mu}\) for some subsequence \((j_k)\), then also \(\mu_{n(j_k)} \to \bar{\mu}\). Conversely, if \(\mu_{n_k} \to \bar{\mu}\) for some \(\bar{\mu}\), then \(\mathcal{E}(y, L_{n_k-1}) \to \bar{\mu}\).

It remains to address the “moreover” part of the lemma statement:

(i) Regardless of the sequence \((\mu_n)\), by (11) we have \(\rho(y, y_0) \leq \varepsilon/2\).

(ii) Note that construction of \(y_1 \ldots y_n\) depends only on \(\mu_1 \ldots \mu_n\). Consider another sequence \((\mu'_n)\), such that \(\mu_j = \mu'_j\) for \(j \leq n\). Let \(y'\) be the point obtained from this sequence. Then, since \(y_n = y'_n\), we deduce from (11) that

\[
\rho(y, y') \leq \rho(y, y_n) + \rho(y'_n, y') \leq \varepsilon/2^{n+1} + \varepsilon/2^{n+1} = \varepsilon/2^n.
\]

We make the following observation:

**Corollary 28.** If \((X, T)\) is a Polish dynamical system with the strong approximate product structure and \(V \subseteq M_T(X)\) is a nonempty closed connected set, then the set \(G(V)\) is dense in \(X\). For any \(T\)-invariant measure \(\mu\), the set \(G_\mu\) is dense.
Proof. For the first part, note that there exists a sequence \((\mu_n)_{n \geq 1}\) such that 
\[ D(\mu_n, \mu_{n+1}) \to 0 \] 
as \(n \to \infty\), and \(\{\text{limit points of } (\mu_n)_{n \geq 1}\} = V\). Thus an application of 
Lemma 27 shows that \(G(V)\) is dense. Similarly, application of Lemma 27 to the 
constant sequence \(\mu_n = \mu\) shows that \(G_\mu\) is dense.

\[ \square \]

**Theorem 29.** If a Polish dynamical system \((X, T)\) has the strong approximate 
product structure, then

(i) If \(|\mathcal{M}_T(X)| \geq 2\), then for any \(\mu \in \mathcal{M}_T(X)\) the set \(G_\mu\) of \(\mu\) generic points

is \(\Pi^0_3\)-complete.

Moreover, we have the following:

(ii) For every nonempty closed connected set \(V \subseteq \mathcal{M}_T(X)\) with nonempty

complement \(U = \mathcal{M}_T(X) \setminus V\), the sets \(G_V\) and \(G(V)\) are \(\Pi^0_3\)-hard (and so 
the set \(U^G\) is \(\Sigma^0_3\)-hard). Also, the set \(G^V\) is \(\Pi^0_2\)-complete. The sets \(G_V\),

\(G^V\), \(G(V)\), and \(U^G\) are all dense in \(X\).

Proof. We give the proof of (ii), and remark at the end how the proof shows (i).

Let \(V \subseteq \mathcal{M}_T(X)\) be as in (ii) and let \(U = \mathcal{M}_T(X) \setminus V\). Let \(\mathcal{C} = \{\beta \in \omega^\omega : \beta(n) \to \infty\}\). We construct a continuous reduction map \(\pi : \omega^\omega \to X\)

such that if \(\beta \in \mathcal{C}\) then \(\pi(\beta) \in G(V) \subseteq G_V\), and if \(\beta \notin \mathcal{C}\) then \(\pi(\beta) \in U^G = \mathcal{M}_T(X) \setminus G_V\). This will show that \(G_V\) and \(G(V)\) are \(\Pi^0_3\)-complete.

Take \(\nu \in U\) and let \(\mu' \in V\). Consider the set \(E = \{0 \leq t \leq 1 : t \mu' + (1-t) \nu \in V\}\).

It is easy to see that \(E \neq \emptyset\) is closed, \(1 \in E\), and \(0 \notin E\). Let \(t_0 = \min E\) and \(\bar{\mu} = t_0 \mu' + (1-t_0) \nu \in V\). Hence \(\alpha \bar{\mu} + (1-\alpha) \nu \in U\) for \(\alpha < 1\). Since \(V\) is closed and connected and \(\mathcal{M}_T(X)\) is separable there is a sequence \((\mu_n)_{n \geq 1}\) of \(T\)-invariant measures satisfying

\[ D(\mu_n, \mu_{n+1}) \to 0 \quad \text{as } n \to \infty, \]

\[ \bigcap_{N=1}^{\infty} \{\mu_n : n \geq N\} = V. \]

That is, the set of limit points of sequence \((\mu_n)_{n \geq 1}\) is equal to \(V\).

Select a sequence \((n_k)_{k \geq 1}\) such that \(\mu_{n_k} \to \bar{\mu}\) as \(k \to \infty\), and additionally 
\(n_{k+1} - n_k \to \infty\). Given a \(\beta \in \omega^\omega\), we define a function

\[ \psi^\beta(j) = \frac{j - n_k}{n_{k+1} - n_k} \frac{1}{\beta(k) + 1} + \frac{n_{k+1} - j}{n_{k+1} - n_k} \frac{1}{\beta(k+1) + 1} \]

for \(n_k \leq j < n_{k+1}\) and a sequence of measures

\[ \mu^\beta_j = \psi^\beta(j) \nu + (1 - \psi^\beta(j)) \mu_j. \]

Note that \(\psi^\beta(n_k) = 1/\beta(k)\) for all \(k\), and \(\psi^\beta\) is "linear" between \(n_k\) and \(n_{k+1}\). Thus we have

\[ |\psi^\beta(j) - \psi^\beta(j+1)| \leq \frac{1}{n_{k+1} - n_k} \to 0 \quad \text{as } j \to \infty. \]

It follows that \(D(\mu^\beta_j, \mu^\beta_{j+1}) \to 0\) as \(j \to \infty\). Thus we can apply Lemma 27 to the 
sequence \((\mu^\beta_j)\), and obtain a point \(\pi(\beta)\) such that \(V(\pi(\beta)) = \{\text{limit points of } \mu^\beta_j\}\).

Let us consider cases:

- If \(\beta(n) \in \mathcal{C}\), then \(\psi^\beta(j) \to 0\) as \(j \to \infty\). It follows that \(D(\mu^\beta_j, \mu_j) \to 0\), and consequently \(V(\pi(\beta)) = \{\text{limit points of } \mu^\beta_j\} = \{\text{limit points of } \mu_j\} = V\).
Finally, notice that \( \mu_0 \) is \( \Pi^0_1 \)-ergodic by Lemma 27(iii). This shows that \( G(V) \) and \( G_V \) are \( \Pi^0_3 \)-hard, and \( UG \) is \( \Sigma^0_3 \)-hard. Note that Lemma 27(iii) shows that all those sets are dense.

Suppose now that \( |\mathcal{M}_T(X)| \geq 2 \), and let \( \mu \in \mathcal{M}_T(X) \). We reapply the above proof for the case \( V = \{ \mu \} \). Since \( |\mathcal{M}_T(X)| \geq 2 \), \( U = \mathcal{M}_T(X) \setminus V \neq \emptyset \). We can take the measures \( (\mu_n) \) of the previous proof to be all equal to \( \mu \). We also let \( \nu \in U = \mathcal{M}_T(X) \setminus V \). For \( \beta \in \mathcal{C}_3 \), the previous argument shows in this case that not only is \( \pi(\beta) \in G(\{ \mu \}) \), but in fact \( \pi(\beta) \in G_\mu \). This follows from (6) in Lemma 27(iii). For \( \beta \notin \mathcal{C}_3 \), the previous argument gives that \( \pi(\beta) \notin G(\{ \mu \}) \), and so \( \pi(\beta) \notin G_\mu \). Thus, \( \beta \in \mathcal{C}_3 \) if \( \pi(\beta) \in G_\mu \), and so \( G_\mu \) is \( \Pi^0_3 \)-hard (and thus \( \Pi^0_3 \)-complete by Fact 14).

To prove that \( G^V \) is \( \Pi^0_2 \)-complete, first recall that it is a \( \Pi^0_2 \) set (Fact 16). Again by Lemma 27(iii), we see that \( G^V \) is dense. Given \( \nu \notin V \), we know that \( G_\nu \) is dense. Note that \( G_\nu \subseteq X \setminus G^V \), thus both \( G^V \) and its complement are dense. It follows that \( G^V \) is \( \Pi^0_2 \)-complete (indeed, suppose that \( G^V \) is \( \Sigma^0_2 \)-complete; then both \( G^V \) and its complement would be dense \( \Pi^0_2 \) sets, which is impossible by the Baire Category Theorem). This completes the proof of Theorem 29.

Combined with the upper bounds of Facts 20 and 21, Theorem 29 has the following immediate corollary.

**Corollary 30.** If \((X,T)\) is a compact dynamical system with the strong approximate product structure and \( V \subseteq \mathcal{M}_T(X) \) is a nonempty closed connected set whose complement \( U = \mathcal{M}_T(X) \setminus V \) is also nonempty, then

(i) the sets \( G_V \) and \( G(V) \) are \( \Pi^0_3 \)-complete,

(ii) the set \( UG \) is \( \Sigma^0_3 \)-complete.

Observe that for any ergodic probability measure \( \mu \), it follows from the Birkhoff ergodic theorem that for \( \mu \)-almost all \( x \) we have \( x \in B_\varphi(\int \varphi \, d\mu) \).

**Theorem 31.** Let \((X,T)\) be a Polish dynamical system with the strong approximate product structure. If \( \varphi \in C_0(X) \) and \( \alpha \in \mathbb{R} \) are such that

\[
V_\varphi(\alpha) = \{ \mu \in \mathcal{M}_T(X) : \int \varphi \, d\mu = \alpha \} \neq \emptyset
\]

and \( U = \mathcal{M}_T(X) \setminus V_\varphi(\alpha) \neq \emptyset \), then \( B_\varphi(\alpha) \) is a \( \Pi^0_3 \)-complete set and \( I_\varphi(X,T) \) is a \( \Sigma^0_3 \)-complete set.

**Proof.** Pick any measure \( \nu \in U \). As in the proof of Theorem 29, we can pick a measure \( \tilde{\mu} \in V_\varphi(\alpha) \) such that \( t\nu + (1-t)\tilde{\mu} \in U \) for all \( 0 < t \leq 1 \). We select an arbitrary sequence \( n_k \nearrow \infty \), with \( n_{k+1} - n_k \to \infty \). Given a sequence \( \beta \in \omega^\omega \), we define function \( \psi^\beta : \omega \to [0,1] \) by setting \( \psi^\beta(n_{2k}) = 1/(\beta(k) + 1) \), \( \psi^\beta(n_{2k+1}) = 0 \), and interpolating linearly on other arguments, that is,

\[
(22) \quad \psi^\beta(j) = \frac{j - n_k}{n_{k+1} - n_k} \psi^\beta(n_k) + \frac{n_{k+1} - j}{n_{k+1} - n_k} \psi^\beta(n_{k+1}). \quad \text{for } n_k < j < n_{k+1}.
\]
It is straightforward to check that

(i) \(|\psi^\beta(j) - \psi^\beta(j + 1)| \to 0\) as \(j \to \infty\),

(ii) if \(\beta \to \infty\), then \(\psi^\beta(j) \to 0\),

(iii) if not \(\beta \to \infty\), then both 0 and \(1/(\lim\inf \beta(j) + 1)\) are limit points of

sequence \(\psi^\beta(j)\).

We define measures

\[ \mu^\beta_j = \psi^\beta(j)\nu + (1 - \psi^\beta(j))\bar{\mu}. \]

From the properties of \(\psi^\beta\) listed above it follows that

(i) \(D(\mu^\beta_j, \mu^\beta_{j+1}) \to 0\) as \(j \to \infty\), thus we can invoke Lemma 27 to obtain a

point \(y^\beta\),

(ii) if \(\beta \to \infty\), then \(\mu^\beta_j \to \bar{\mu}\) and consequently \(y^\beta \in G_{\bar{\mu}} \subseteq B_\varphi(\alpha)\),

(iii) if not \(\beta \to \infty\), then \(y^\beta \in I_\varphi(X, T)\).

Note that the measures \(\mu^\beta_1, \ldots, \mu^\beta_k\) depend only on the values \(\beta(1) \ldots \beta(k)\). This, together with part (ii) of Lemma 27 implies that the map \(\pi: \omega^\omega \ni \beta \mapsto y^\beta \in X\) is continuous. By (ii) and (iii), \(\pi\) is a continuous reduction of \(C_3\) to \(B_\varphi(\alpha)\), and also a

reduction of \(\omega^\omega \setminus C_3\) to \(I_\varphi(X, T)\). Together with Fact 13 this finishes the proof. \(\Box\)

From the proof above, we also obtain the following corollary:

**Corollary 32.** If \((X, T)\) is a Polish dynamical system with the strong approximate

product structure with respect to a compatible metric \(\rho\) and there are at least two \(T\)-invariant measures, then

(i) the quasi-regular set \(Q(X, T)\) is \(\Pi_3^0\)-hard, thus \(Q(X, T)\) is \(\Pi_3^0\)-complete.

(ii) the irregular set \(I(X, T)\) is \(\Sigma_3^0\)-hard, and thus \(I(X, T)\) is \(\Pi_3^0\)-complete.

**Proof.** Pick any two \(T\)-invariant measures \(\bar{\mu} \neq \nu\) and a function \(\varphi \in C_\alpha(X)\) such that

\[ \alpha = \int_X \varphi\,d\bar{\mu} \neq \int_X \varphi\,d\nu. \]

In the proof of Theorem 31 we defined a continuous map \(\pi: \omega^\omega \to X\) such that \(\pi(C_3) \subseteq G_{\bar{\mu}}\) and \(\pi(\omega^\omega \setminus C_3) \subseteq I_\varphi(X, T)\). Since \(G_{\bar{\mu}} \subseteq Q(X, T)\) and \(I_\varphi(X, T) \subseteq I(X, T)\), the map \(\pi\) reduces \(C_3\) to \(Q(X, T)\), and reduces \(\omega^\omega \setminus C_3\) to \(I(X, T)\). In Fact 13 we have shown the matching upper bounds. \(\Box\)

8. Compactness and Higher level completeness

In Theorem 29 it was shown for a Polish dynamical system \((X, T)\) satisfying a weak form of specification (the strong approximate product structure), and having at least two invariant measures, that for every closed connected \(V \subseteq \mathcal{M}_T\) which is non-trivial (i.e., \(V \neq \emptyset, V \neq \mathcal{M}_T\)), we have that \(G(V)\) is \(\Pi_3^0\)-hard. When \(X\) is compact, Fact 31 shows that \(G(V)\) is a \(\Pi_3^0\) set (this holds for any closed \(V \subseteq \mathcal{M}_T(X)\)), and so \(G(V)\) is a \(\Pi_3^0\)-complete set in this case. Similar results hold for the sets \(G_V = \{x \in X: V(x) \subseteq V\}\) (where \(V \subseteq \mathcal{M}_T(X)\) is closed) and \(U G = \{x \in X: V(x) \subseteq U \neq \emptyset\}\) (where \(U \subseteq \mathcal{M}_T(X)\) is open). When \(X\) is not compact, the natural computation shows that \(G(V)\) and \(G_V\) are \(\Pi_3^1\) sets and \(U G\) is a \(\Sigma_3^1\) set (see Facts 13 and 19). Recall from Fact 19 that in this case we still have that for any closed \(V \subseteq \mathcal{M}_T\) that \(G^V = \{x \in X: V(x) \subseteq \hat{V}\}\) is \(\Pi_2^1\). A natural question is whether the compactness of \(X\) is necessary to reduce the complexity of \(G_V, G(V),\) and \(U G\) to a Borel set. The next result shows that this is the case, as in general the sets \(G(V)\) and \(G_V\) may be a \(\Pi_3^1\)-complete sets, while \(U G\) is a \(\Pi_3^1\)-complete set. Thus, we see a strong dichotomy in the complexity of these sets between the compact and non-compact cases.
**Theorem 33.** There is a Polish dynamical system $(X,T)$, so $T: X \to X$ is continuous, and a closed, connected, nonempty $V \subseteq \mathcal{M}_T(X)$ such that $G(V)$ is $\Pi^1_1$-complete. In fact, we may take $V = \{ \mu \}$ for some $\mu \in \mathcal{M}_T(X)$, and $(X,T)$ to have the specification property. Furthermore, setting $U = \{ \nu \in \mathcal{M}_T(X) : \nu \neq \mu \}$ we obtain that $G(V)$ is $\Pi^1_1$-complete and $U^*G$ is $\Sigma^1_1$-complete.

**Remark 34.** For $V = \{ \mu \}$, where $\mu \in \mathcal{M}_T(X)$, the sets $G_\mu$ and $G(V)$ may not be equal. For $x$ to be in $G_\mu$, that is, for $x$ to be a $\mu$ generic point, we must have that for all increasing sequences of positive integers $(n_i)_{i \geq 1}$ that the empirical measures $E(x,n_i)$ converge to $\mu$. For $x$ to be in $G(V)$, we need only have that for each such sequence $n_i$ that if the empirical measures $E(x,n_i)$ converge to a measure, then that measure must be $\mu$. This distinction is important, since the set $G_\mu$ is always $\Pi^1_3$ by Fact 13.

**Proof of Theorem 33.** Let $X = \omega^\omega$ with the (continuous) shift map $T$. For $n \in \omega$ write $\delta_n$ for the point-mass measure on the constant $n$ sequence denoted $\bar{n}$. Let $V = \{ \delta_n \}$.

To show that $G(V)$ is $\Pi^1_1$-complete, we construct a continuous map $\pi: Tr \to X$ reducing $Wf$ to $G(V)$, where $Wf$ is the set of all wellfounded trees on $\omega$. Recall that a tree $T$ on $\omega$ is a set $T \subseteq \omega^{<\omega}$ of finite length sequences from $\omega$ which is closed under the operation of taking initial segments. A tree is called wellfounded if it doesn’t contain an infinite branch, and illfounded if it does. Let $Tr$ denote the set of all trees on $\omega$. We fix an enumeration $t^{(1)}, t^{(2)}, \ldots$ of $\omega^{<\omega}$ such that $|t^{(i)}| \leq |t^{(j)}|$ for $i < j$. Using this enumeration we identify $\omega^{<\omega}$ with the (continuous) shift map $\bar{n}$ for the point-mass measure on the constant $\bar{n}$ sequence denoted $\bar{n}$. For $s \in \omega^{<\omega}$ we set $\tilde{\nu}(s) = \sum_{i < |s|} \nu(s(i),i)$. For $s = (s(0) \ldots s(\ell-1)) \in \omega^{<\omega}$ (so $\ell = |s|$ is the length of $s$) we define a measure $\mu_s$ as

$$\mu_s = \frac{1}{2^{\ell+1}} \delta_{s^\ell} + \sum_{i < \ell} \frac{1}{2^{i+1}} \nu(s(i),i).$$

Suppose $s_0, s_1, s_2, \ldots$ is a sequence from $\omega^{<\omega}$ with $\lim_i |s_i| = \infty$ and such that $s_i$ converges to $x = (x(0), x(1), \ldots) \in \omega^\omega$, that is, $\forall n \exists i \geq n$ $(s_i \upharpoonright m = x \upharpoonright m)$. Then the corresponding sequence of measures $\mu_{s_i}$ is easily seen to converge to the measure $\mu_x$ as $i \to \infty$, where $\mu_x$ is the measure given by

$$\mu_x = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \nu_{x(i),i} = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \nu(p(x(i),i)).$$

On the other hand assume that $s_0, s_1, s_2, \ldots$ is a sequence of distinct elements of $\omega^{<\omega}$ and the corresponding sequence of measures $\mu_{s_i}$ is Cauchy. We claim that
\[
\lim_i |s_i| = \infty \text{ and there is an } x \in \omega^\omega \text{ such that } s_i \text{ converges to } x. \text{ Let } k \text{ be the least integer, if one exists, such that either there are infinitely many } i \text{ with } s_i(k) \text{ not defined or } \{s_i(k)\} \text{ is not eventually constant. Assuming } k \text{ is defined, the first case cannot happen as then } s_i \upharpoonright k \text{ is defined and constant for all large enough } i \text{ but for some } i < j \text{ we would have } s_i = s_j. \text{ If the second case in the definition of } k \text{ occurs, then for large enough } i \text{ we have that } s_i \upharpoonright k \text{ is constant and either } s_i(k) \text{ takes infinitely many values or there are two distinct values of } s_i(k) \text{ both of which occur for infinitely many values of } i. \text{ However, we would then have } D(\mu_{s_i}, \mu_{s_j}) \geq 1/2^{k+1} \text{ for arbitrarily large } i < j, \text{ a contradiction. So, } k \text{ is not defined, that is, for all } k \text{ we have that for all large enough } i \text{ that } s_i(k) \text{ is defined and constant. This shows that the } s_i \text{ converge to some } x \in \omega^\omega.

Finally, for each \( s = (s(0) \ldots s(\ell - 1)) \in \omega^\omega \) and \( 0 \leq i < \ell \) we set \( n^* = p(n(s(i), i)) \) and define a periodic point

\[
y^* = \left( n_0^* \ldots n_0^* \ n_1^* \ldots n_1^* \ldots n_{\ell-1}^* \ldots n_{\ell-1}^* \ 1 \ldots 1 \right) \in \omega^\omega.
\]

Note that the primary period of \( y^* \) equals \( p_s = 2^{2\ell+1} - 2\ell+1 \) and \( D(\mathcal{E}(y^*, p_s), \mu_s) \) goes to 0 as \( \ell \) goes to infinity. We are ready to define \( \pi({\mathcal{T}}) \) for a tree \( \mathcal{T} \). Let \( c_n \) be the primary period of \( y^* \) for \( s = t^{(n)} \). We fix sequences \( (a_n) \) and \( (b_n) \) satisfying \( a_1 = 2, a_{2n} = c_n, a_{2n+1} = 1, b_{n+1} > b_n \) and

\[
a_n b_n > \max\{2^n a_{n+1}, 2^n (a_0 b_0 + \ldots + a_{n-1} b_{n-1})\}
\]

for all \( n \).

We set \( \ell_0 = 0 \). As stage \( n \geq 1 \) of the construction of \( x = \pi(\beta) \) we will define \( x(\ell_{n-1}, \ell_n) \) where \( \ell_n = \ell_{n-1} + a_n b_n \). For \( n \) odd we set \( x(\ell_{n-1}, \ell_n) = 0^{(2^\ell \times 1)}(\ell_{n-1}, \ell_n) = 0 \ldots 0 \) (here 0 appears \( a_n b_n \) times). For \( n \) even, we have two cases: if \( s = t^{(n/2)} \in \mathcal{T} \), then we put \( x(\ell_{n-1}, \ell_n) = (y^*)^{b_n} \). Otherwise (that is, if \( s = s^{(n/2)} \notin \mathcal{T} \)), we put \( x(\ell_{n-1}, \ell_n) = (q(n/2))^{a_n b_n} \). Note that \( \pi(\mathcal{T}) = x \) defined in this way depends continuously on \( x \) because \( x(0, \ell_n) \) is determined by \( \{t^{(1)}, \ldots, t^{(n)}\} \cap \mathcal{T} \). Furthermore, it follows from \( 23 \) that the measures \( \mathcal{E}(x, e_{2n-1}) \) converge to \( \delta_0 \) as \( n \to \infty \). Thus \( \delta_0 \in V(x) \). We show that \( \delta_0 \) is the only measure in \( V(x) \) if and only if \( \mathcal{T} \) is a wellfounded tree. Let \( \nu \in V(x) \) and \( i_k \) be a strictly increasing sequence with \( \lim_{k \to \infty} \mathcal{E}(x, i_k) = \nu \). Let \( (n_k) \) be a sequence of integers such that \( \ell_n \leq i_k < \ell_{n+1} \) for every \( k \). Without loss of generality we assume that the \( (n_k) \) is strictly increasing and \( n_k \) is even for every \( k \). The proof in the other case is similar.

Let \( e_k = (i_k - \ell_{n_k})/i_k \). By passing to a subsequence we may assume that \( e_k \to e \in [0, 1] \) as \( k \to \infty \). From \( 23 \) we have that the sequence \( \mathcal{E}(x, i_k) \) and the sequence with \( k \)-th term given by

\[
\nu_k = (1 - e) \mathcal{E}(x|_{\ell_{n_k - 1}}, \ell_{n_k}) + e \mathcal{E}(x|_{\ell_{n_k}, \ell_{n_k + 1}})
\]

has the same weak* limit in \( M_{\mathcal{T}}(X) \). If \( e = 1 \), then \( \nu = \delta_0 \), so assume \( e < 1 \). Take the sequence \( (m_k) \) so that \( n_k = 2m_k \) for every \( k \).

Note that \( x(\ell_{n_k}, \ell_{n_k + 1}) \) is a sequence containing only zeros. Also,

\[
x(\ell_{n_k - 1}, \ell_{n_k}) = \begin{cases} q_k \cdot b_{n_k}, & \text{where } q_k = q(m_k), \text{ if } t^{(m_k)} \notin \mathcal{T}, \\ y^*|_{0,a_{n_k}} \cdot b_{n_k}, & \text{where } s = t^{(m_k)}, \text{ if } t^{(m_k)} \in \mathcal{T}. \end{cases}
\]

\[
24
\]
It follows that the sequence \( \nu_k \) converges to the same weak* limit as the sequence of measures

\[
(25) \quad w_k = \begin{cases} 
(1 - e)\delta_{x_0} + e\delta_0, & \text{if } t^{(m_k)} \notin \mathcal{T}, \\
(1 - e)\mu_{t^{(m_k)}} + e\delta_0, & \text{if } t^{(m_k)} \in \mathcal{T}.
\end{cases}
\]

Suppose \( t^{(m_k)} \notin \mathcal{T} \) for some subsequence \((k(j))_{j \geq 1}\). Then the sequence of measures \( \delta_{\bar{\omega}^{(m_j)}} \) would also converge - contradiction. Thus for sufficiently big \( k \), the second case of (25) holds. It follows that the sequence \( \{\mu_{t^{(m_k)}}\} \) converges. This, by our previous observations, gives that the \( t^{(m_k)} \) converge to some \( z \in \omega^\omega \). Since each \( t^{(m_k)} \in \mathcal{T} \), this show that \( z \) is a branch through \( \mathcal{T} \) and so \( \mathcal{T} \) is illfounded. We have shown that if \( V(\pi(\mathcal{T})) \neq \{\delta_0\} \), then \( \mathcal{T} \) is illfounded. Conversely, if \( \mathcal{T} \) is illfounded, say \( z \) is a branch through \( \mathcal{T} \), then along the sequence \((n_{2k})\) where \( t^{(k)} = z \upharpoonright k \) we have that \( E(\pi(\mathcal{T}), n_{2k}) \) converges to the measure \( \mu_z \) defined above, and \( \mu_z \neq \delta_0 \), and so we have \( V(\pi(\mathcal{T})) \neq \{\delta_0\} \). So \( V(\pi(\mathcal{T})) = \{\delta_0\} \) if and only if \( t \in \mathcal{W} \). This finishes the proof. It is easy to see that the above proof shows in fact that \( G_{\{\delta_0\}} \) is \( \Pi_1^1 \)-complete, while \( UG \), where \( U = \mathcal{M}(\mathcal{T}) \setminus \{\delta_0\} \) is \( \Sigma_1^1 \)-complete. 

\[ \square \]

9. Complexity of \( G_\mu \) in a Minimal Subshift

Recall that a dynamical system \((X, \mathcal{T})\) is called minimal, if \( \{T^n x; n \geq 0\} = X \) for every \( x \in X \). Minimality is a stronger condition than transitivity. We give an example that shows that \( G_\mu \) may be \( \Pi_3^0 \)-complete in a minimal system.

**Theorem 35.** There exists a minimal shift \((X, \mathcal{T})\) and an ergodic measure \( \mu \) such that \( G_\mu \) is \( \Pi_3^0 \)-complete.

We will examine a construction given by Oxtoby, originally used as an example of a minimal, not uniquely ergodic dynamical system; see [11](#) Example 10.3 or [27](#) §10. We will construct a subshift of \( \{0, 1\}^\omega \). The construction uses a sequence \((s_j)_{j \geq 1}\) of natural numbers such that

\[
(26) \quad \sum_{j=1}^{\infty} \frac{1}{s_j} < 1.
\]

We will now define certain strings over the alphabet \( \{0, 1, x\} \). Here \( x \) is a placeholder to be filled at a later stage of the construction. We begin with

\[
(27a) \quad W_0 = x, \quad 0_0 = 0, \quad 1_0 = 1,
\]

for odd \( n \):

\[
(27b) \quad W_n = 0_{n-1}W_n^{s_n-1}, \quad 0_n = 0_n^{s_n-1}, \quad 1_n = 0_n^{s_n-1}1_n^{s_n-1}.
\]

for even \( n \):

\[
(27c) \quad W_n = 1_{n-1}W_n^{s_n-1}, \quad 0_n = 1_{n-1}0_n^{s_n-1}, \quad 1_n = 1_{n-1}1_n^{s_n-1}.
\]

To illustrate, the sequence \((W_n)\) begins with

\[
W_0 = x, \\
W_1 = 0xx, \\
W_2 = 0110xx0xx0xx, \\
W_3 = 01100000000110xx0xx0xx0110xx0xx0xx0xx.
\]
For all $n$, the word $0_n$ can be obtained by replacing all $x$ symbols with zeros in $W_n$. Similarly, the word $1_n$ can be obtained by replacing all $x$ symbols with ones in $W_n$. The lengths of $W_n$, $0_n$ and $1_n$ are equal, we will denote this length by $l_n$. Observe that in the sequence $0_0, 1_1, 0_2, 1_3, \ldots$ each word is a prefix of the next one, and thus we can define $y \in \{0, 1\}^\omega$ to be the limit of this sequence. Finally, we define $X$ to be the closure of the orbit of $y$. The sequence $y$ is Toeplitz, and $X$ is a minimal shift; see [11, Chapter 14] for details. Note that the language of $X$ can be described as follows:

$$L = L(X) = \{ u \in \{0, 1\}^\omega : u \text{ is a subword of } 0_n \text{ or } 1_n \text{ for some } n \geq 0 \}.$$ 

For a word $u$, define $c(u) = |\{0 \leq i < |u| : u_i = 1\}|$ and $m(u) = c(u)/|u|$. From (27), it follows that $m(0_n)$ is a weakly increasing sequence, and therefore $m(0_n) \not< a$ for some real number $a \in [0, 1]$. Similarly, $m(1_n) \not< b$ for some $b \in [0, 1]$. Again from (27), we get $m(1_n) - m(0_n) = \prod_{j=1}^n (1 - 1/s_j)$. The condition (26) implies that $\prod_{j=1}^\infty (1 - 1/s_j) > 0$, and consequently $a \not< b$.

**Fact 36.** The set $\mathcal{M}_T^\omega(X)$ consists of two measures $\mu_0, \mu_1$. Moreover $\mu_0(C) = a$ and $\mu_1(C) = b$, where $C = \{ x \in X : x_0 = 1 \}$.

**Proof.** $X$ has exactly two ergodic measures [11, Chapter 14]. Note that for arbitrary continuous $f : X \to \mathbb{R}$, we have

$$\min_{\mu \in \mathcal{M}_T^\omega(X)} \int_X f \, d\mu = \lim_{k \to \infty} \min_{x \in X} \min_{u \in L_k(X)} m(u),$$

where $L_k(X) = \{ u \in L(X) : |u| = k \}$. Pick any $n \geq 0$, then any word in $L(X)$ can be divided into blocks, each equal to $0_n$ or $1_n$, where the first and last block may be incomplete. This shows that (29) is greater or equal to $m(0_n)$. Since this is true for any $n$, we get that (29) is greater or equal to $a$. On the other hand, by plugging $u = 0_n$, we find that (29) is less or equal than $a$. Thus there exists a measure $\mu_0 \in \mathcal{M}_T^\omega(X)$ such that $\mu_0(C) = a$. Similarly, applying (28) to $-\chi_C$ shows us that there exists a measure $\mu_1 \in \mathcal{M}_T^\omega(X)$ such that $\mu_1(C) = b$. 

**Fact 37.** For any $k$, suppose we have $j_0, j_1, \ldots, j_k$ such that $0 \leq j_i \leq s_{i+1} - 2$. Then the word $0_{j_0} 0_{j_1} \ldots 0_{j_k}$ is a suffix of $0_{k+1}$, in particular it belongs to the language $L$.

**Proof.** Let us write $u \subseteq v$ whenever the word $u$ is a suffix of $v$. We will proceed by induction. For $k = 0$, the statement is true. Now, suppose the statement holds for $k - 1$, then we have $0_{j_0} 0_{j_1} \ldots 0_{j_{k-1}} \subseteq 0_k$. From this, we obtain $0_{j_0} 0_{j_1} \ldots 0_{j_{k-1}} 0_{j_k} \subseteq 0_{j_0} 0_{j_1} \ldots 0_{j_{k-1}} 0_{j_k+1} \subseteq 0_k \subseteq 0_{k+1}$. Since the relation $\subseteq$ is transitive, we are done. 

From the fact above, it follows that the infinite sequence $0_{j_0} 0_{j_1} \ldots 0_{j_k} \ldots$ is an element of $X$, if $1 \leq j_i \leq s_{i+1} - 2$ for all $i$. Recall that the set $C_3 := \{ \beta \in \omega^\omega : \liminf \beta(n) = +\infty \}$ is a $\Pi_3^\omega$-complete subset of $\omega^\omega$. We define $f : \omega^\omega \to X$ by the formula

$$f(\beta) = 0_{j_0} 0_{j_1} \ldots 0_{j_{i+1}} \ldots, \quad \text{where } j_i = \min \{ \beta(i) + 1, s_{2i+2} - 2 \}.$$
We claim that $\beta \in \mathcal{C}_3 \iff f(\beta) \in G_{\mu_0}$. Fact 36 implies that $G_{\mu_0} = B_{\chi_{C}(a)}$. This will simplify our analysis considerably.

$\beta \notin \mathcal{C}_3$: Let $k = \liminf \beta(i) + 1$. Consider any $i$ such that $j_i = k$. From (26), note that $\#_{2i+1}$ is a prefix of $\#_{2i+3}$. Thus

$$w = \#_{2i+1}^k \#_{2i+1}^{j_i-1} \#_{2i+1}^k \#_{2i+1}^{j_i} \#_{2i+1}^k \#_{2i+1}^k \#_{2i+1}^k,$$

is a prefix of $f(\beta)$. The underlined part is a suffix of $0_{2i}$, and hence $|w| \leq |0_{2i}| + k|0_{2i+1}| + |1_{2i+1}|$. We have $|0_{2i}|/|0_{2i+1}| = 1/s_{2i+1}$ and $|1_{2i+1}| = |1_{2i+1}|$. We treat the underlined part as if it contained only zeros; this gives the estimate

$$m(w) \geq \frac{km(0_{2i+1}) + m(1_{2i+1})}{k + 1 + 1/s_{2i+1}} \to \frac{k}{k + 1} a + \frac{1}{k + 1} b > a,$$

as $i \to \infty$. Thus $f(\beta) \notin B_{\chi_{C}}(a)$.

$\beta \in \mathcal{C}_3$: Any prefix of $f(\beta)$ can be written as

$$w = \#_{2i+1}^k \#_{2i+1}^{j_i} \#_{2i+1}^k \#_{2i+1}^{j_i} \#_{2i+1}^k \#_{2i+1}^k \#_{2i+1}^k,$$

where $k \geq 0$ and $u$ is a prefix of $0_{2i+1}$. Recall that $0_{2i+1} = \#_{2i+1}^{j_i}$ and $0_{2i} = \#_{2i+1}^{j_i-1}$. So we further have either $u = \#_{2i}^k u'$, where $u'$ is a prefix of $\#_{2i-1}$, or $u = \#_{2i}^k \#_{2i-1}^k \#_{2i-1}^k u'$, where $u'$ is a prefix of $0_{2i-1}$. In both cases, we have $|u'| \leq l_{2i-1}$, and consequently

$$c(u') \leq 2l_{2i-1} + pc(0_{2i}) + qc(0_{2i-1}).$$

As in the previous case, the underlined part of (30) has length $\leq l_{2i-2}$. Thus we obtain

$$m(w) \leq \frac{l_{2i-2} + j_i-1 c(0_{2i-1}) + kc(0_{2i+1}) + 2l_{2i-1} + pc(0_{2i}) + qc(0_{2i-1})}{|w|} = \frac{l_{2i-2} + 2l_{2i-1}}{|w|} \frac{j_i-1 + q l_{2i-1}}{|w|} m(0_{2i-1}) + \frac{pl_{2i}}{|w|} m(0_{2i}) + \frac{kl_{2i+1}}{|w|} m(0_{2i+1}).$$

Finally, since $|w| \geq j_i-1 l_{2i-1}$ and $m(0_n) \leq a$ for all $n$, we get

$$m(w) \leq \frac{3}{j_i-1} + a.$$

Our assumption $\beta \in \mathcal{C}_3$ implies that $j_i \to \infty$, and hence $\limsup m(w) \leq a$, as the length of $w$ goes to infinity. Fact 30 implies that $\liminf m(w) \geq a$.

Since $w$ is an arbitrary prefix of $f(\beta)$, we get that $f(\beta) \in B_{\chi_C}(a) = G_{\mu_0}$.

Thus $f$ is a reduction of $\mathcal{C}_3$ to $G_{\mu_0}$, which proves that the latter set is $\Pi^0_3$-complete.

We have completed the proof of Theorem 35.}

We claim that the minimal system $X$ constructed in this section does not have SAPS. Consider any word $w \in L(X)$ of length $l_n$. It follows from the construction that $w$ is a subword of one of the words $0_n \#_{2i_n} \#_{2i_n} \#_{2i_n} \#_{2i_n} \#_{2i_n} \#_{2i_n} \#_{2i_n}$. Consequently, we obtain the bound $|L_{l_n}(X)| \leq 4l_n + 4$. Thus

$$\liminf_{n \to \infty} \frac{|L_{l_n}(X)|}{n} \leq 4.$$

By a result of Cyr and Kra [9, Theorem 1.1], this implies that there are at most 4 distinct measures in $\mathcal{M}_F(X)$ whose set of generic points is nonempty. Thus, for a suitable choice of $t \in [0,1]$, the measure $\nu = t \mu_1 + (1-t) \mu_0$ will have $G_\nu = \emptyset$. 
On the other hand, if $X$ had SAPS, by Corollary 25 the set $G_\nu$ would be dense, in particular nonempty.

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