Abstract

A new model of self-organized criticality is proposed. An algebra of operators is introduced which is similar to that used for the Abelian sandpile model. The structure of the configurational space is determined and the number of recurrent states is found.

Since the introduction of self-organized criticality (SOC) [1], there has been considerable interest in the study of cellular automata which demonstrate how power-law correlations emerge during the evolution of extended dissipative systems. Of them, the Abelian sandpile model [2] expresses most clearly the idea of dissipative dynamics, where a small disturbance exceeding a threshold grows and propagates through the system as an avalanche.

Avalanches seem to be crucial for SOC, while the significance of a threshold is not completely clear. Indeed, some of the characteristics of sandpiles are purely diffusive even though one might expect more complicated behaviour due to the presence of thresholds of stability. Thresholds in the forest fire model [3] are hidden in two parameters which are the probabilities of growth and ignitions. In the Bak-Sneppen model [4], a threshold value
appears as a result of infinitely long evolution.

In this Letter, I propose a cellular automaton model exhibiting SOC and containing no threshold parameters. However, the model has a common algebraic structure with the Abelian sandpile and, therefore, a similar structure of recurrent configurations corresponding to the critical state. The parallel consideration of these models should elucidate the critical dynamics of both.

Consider a two dimensional square lattice \( L \) of size \( L \times L \). Each site \( i \) of \( L \) is characterized by the radius-vector \( \mathbf{r}_i \) with integer Cartesian coordinates \((x_i, y_i)\) and by one of the unit vectors \( e_i(1), e_i(2), e_i(3), e_i(4) \) directed up, right, down or left from \( i \). At each discrete moment of time, one drops a particle on a site of \( L \) chosen at random and allows it to walk by the following rules:

(i) at each step, the particle coming to a site \( j \in L \) turns the vector \( e_j(\nu) \) clockwise by the right angle:

\[
e_j(\nu) \rightarrow e_j(\mu)
\]

where \( \mu = \nu + 1 (mod 4) \).

(ii) performs the unit step in the direction \( e_j(\mu) \) to the neighbour site \( j' \):

\[
\mathbf{r}_{j'} = \mathbf{r}_j + e_j(\mu)
\]

(iii) if the walk reaches a boundary site \( j \) and the new position \( j' \) is outside the lattice, the particle leaves the system.

The walk is assumed to be quick enough to be completed by the next discrete moment of time. As a result of the walk, a configuration of vectors \( C \) specified by a unit vector \( e_i(\nu) \) on each lattice site \( i \in L \) transforms into a new configuration \( C' \) which can generally differ from \( C \) by directions of vectors on sites visited by the particle.

To describe the transformation resulting from dropping a particle on the site \( i \), we define the operator \( a_i \) acting on \( C \) and producing \( C' \):

\[
a_i C = C'
\]

\textbf{Theorem 1} The operator \( a_i \) exists, that is, for any \( C \) the walk started at \( i \) never enters a non-trivial cycle.
Proof: If a cycle contains a boundary site, the walk visits this site more than four times. Since one of these visits has as a consequence a step outside the lattice, no cycle can contain a boundary site. Next, consider a site one step away from the boundary. It cannot be visited in a cycle as, in this case, a particle must hit one of the boundary sites. By induction, there can be no cycle containing an arbitrary site of the lattice and, therefore, the operator $a_i$ exists.

The operators $a_i$ all commute. This property enables one to construct an Abelian group quite similar to that defined by Dhar [2] for the sandpile cellular automata.

**Theorem 2** For arbitrary sites $i$ and $j$ and for any configuration of vectors $C$

$$a_i a_j C = a_j a_i C \quad (2)$$

Proof: Consider the updating procedure $a_i C = C'$ as a sequence of elementary steps $\alpha_{j_1} \cdots \alpha_{j_2} \alpha_{j_1}$ with $j_1 = i$. The operator $\alpha_j$ corresponds to the rotation of the vector $e_j(\nu) \rightarrow e_j(\mu)$, $\nu = \nu + 1 \pmod{4}$ and a consequent single step in the direction $e_j(\mu)$. Then $a_i a_j C$ can be written in the form

$$(\prod \alpha^{(2)}_j)(\prod \alpha^{(1)}_k)C$$

where subscripts (1) and (2) refer to different particles. If $j \neq j'$, the operators $\alpha^{(1)}_j$ and $\alpha^{(2)}_{j'}$ commute. If $j = j'$, they also commute due to identity of particles. Therefore, $a_i$ and $a_j$ commute.

The commutativity rule enables us to consider several walks simultaneously updating them concurrently.

As usual, in the theory of Markov chains, we divide the set of all configurations $\{C\}$ into two subsets, recurrent and transient. The first subset denoted by $\{R\}$ includes those configurations which can be obtained from an arbitrary configuration by a sequential action by operators $a_i$. It follows from the definition that the subset $\{R\}$ is closed under multiple action by operators $a_i$. Once the system gets into $\{R\}$, it never gets out under subsequent evolution. All nonrecurrent configurations are called transient and form the subset $\{T\}$ which is the complement of the set $\{R\}$.

Similarly to the avalanche operators of the sandpile model, the operator $a_i$ has a unique inverse. The proof of this statement can be carried out in a way very close to the sandpile construction [3].
**Theorem 3** For any recurrent configuration $C \in \{R\}$, there exists a unique

$$(a_i^{-1}C) \in \{R\}$$

such that

$$a_i(a_i^{-1}C) = C$$

(3)

Proof: Consider a standard recurrent configuration $C^*$ which can be chosen as the set of parallel vectors $e_i(1)$ on all $i \in L$.

Construct an identity operator $EC^* = C^*$. To this end, we take the product of boundary operators $a_i (i \in B)$ multiplied by the product of corner operators $a_i (i \in A)$

$$
\prod_{i \in B} \prod_{i \in A} a_i
$$

(4)

where $B$ is the set of all boundary sites and $A$ is the set of corner sites. Evidently, the operator (4) does not change the standard configuration since it is nothing but a sequence of four successive rotations of arrows in all rows and columns of the lattice. Besides (4), we can find another representation of $E$, having a nonzero degree of $a_i$ at any $i \in L$.

For instance, let us construct an identity operator $E$ having operators $a_i$ in the next neighbours of the boundary sites. Replace $a_i$ by $\alpha_i$ and consider the product

$$
\left( \prod_{i \in B} \prod_{i \in A} \alpha_i \right)^4
$$

(5)

Let $n_{i,i} \in L$ be the occupation numbers of particles resulting from $C^*$ after the action by the operator (4). Each site a step away from the boundary receives at least one particle. Thus, the operator

$$
\prod_{i} a_i^{n_i}
$$

(6)

is the identity operator for $C^*$ having $n_i > 0$ at each next neighbour of edges. Repeating this procedure, we can construct the identity operator $E(n_i)$ having $n_i > 0$ at an arbitrary chosen site $i \in L$.

Now, drop a particle on the configuration $C^*$ to obtain a new recursive configuration $a_i C^*$. The operator

$$
P_1 = E(n_i - 1)
$$

(7)
being combined with $a_i$ gives the identity operator again:

$$P_1a_iC^* = C^*$$

(8)

By definition, any recurrent configuration $C$ can be obtained from $C^*$:

$$C = P_2C^*$$

(9)

where $P_2$ is a product of the $a_i$. Using the commutativity property, we have

$$C = P_2P_1a_iC^* = a_iP_2P_1C^*$$

(10)

The intermediate result $P_2P_1C^*$ is the seeking configuration $(a_i^{-1}C)$.

The proof of uniqueness doesn’t differ from that for sandpiles [5]. Repeating the construction (10), we can find a sequence $C_n \in \{R\}$ such that

$$(a_i)^nC_n = C$$

(11)

Since the total number of configurations doesn’t exceed $4L^2$, this sequence must enter a loop of length $m > 1$. This loop must contain $C$ as $C \in \{R\}$ is attainable from an arbitrary point of the loop. We have $a_i^nC = C$. Then $a_i^{m-1}C = a_i^{-1}C$ is the unique inverse.

As all recurrent configurations can be obtained from an arbitrary one by the successive acting by operators $a_i$, we can represent any $C \in \{R\}$ in the form

$$C = \prod_{i \in L} (a_i)^n C^*$$

(12)

The $L^2$-dimensional vector $\mathbf{n}$ labels the recurrent configurations. We can note that the operator $a_i^4$ returns the vector $\mathbf{e}_i(\nu)$ to the former position and initiates a motion of four particles at neighbouring sites of $i$. Therefore, the operator $a_i^4a_{j_1}^{-1}a_{j_2}^{-1}a_{j_3}^{-1}a_{j_4}^{-1}$, where $j_1, j_2, j_3, j_4$ are the nearest neighbours of $i$, doesn’t change the initial configuration and is actually the identity operator $E$. Using the Laplacian matrix $\Delta$ with elements

$$\Delta_{i,j} = \begin{cases} 4 & i = j \\ -1 & i, j \text{ nearest neighbours} \\ 0 & \text{otherwise} \end{cases}$$

one can write down the identity operator in the form

$$E = \prod_j a_j^{\Delta_{ij}}$$

(13)
Eq. (13) shows that two vectors \( \mathbf{n} \) and \( \mathbf{n}' \) label the same configuration of arrows if the difference between them is \( \sum_j m_j \Delta_{ij} \) where \( m_j \) are integers. The \( L^2 \)-dimensional space \( \{ \mathbf{n} \} \) has a periodic structure with the elementary cell of the form of a hyper-parallelepiped with the base edges \( \Delta_{ij}, j = 1, 2, ..., L^2 \). Thus, the number of non-equivalent recurrent configurations is

\[
N = \det \Delta
\]

(14)

which is Kirhhoff’s formula for spanning trees [3] and Dhar’s formula for sandpiles [2].

The similarity to trees is not accidental. By construction, vectors involved into rotation don’t form closed cycles if the motion is over. Due to finiteness of the lattice, each vector takes part in the motion some time or other. Each time the updating procedure is over, the collection of vectors reproduces the set of bonds of a tree. Thus, starting from an arbitrary noncorrelated set of vectors, we come to the set of spanning trees that are characterized by power-law correlations between different sites.

The correspondence with sandpiles is not surprising as well. The algebra of the operators \( a_i \) completely coincides with that of avalanche operators of the Abelian sandpile model [2]. Moreover, the identity operator (13) has the same form for both the models. This is the reason why the numbers of recurrent configurations coincide.

Continuing the analogy between self-directing walks and sandpiles one can find the expected number \( G_{ij} \) of full rotations of the vector at site \( j \), due to the particle dropped at \( i \) [2]. During the walk, the expected number of steps outside \( j \) is \( \Delta_{jj} G_{ij} \) whereas \( -\sum_{k \neq j} G_{ik} \Delta_{kj} \) is the average flux into \( j \). Equating both fluxes one gets

\[
\sum_k G_{ik} \Delta_{kj} = \delta_{ij}
\]

(15)

or

\[
G_{ij} = [\Delta^{-1}]_{ij}
\]

(16)

The close analogy with sandpiles calls for a definition of avalanches in our model. The first step after landing of the particle generally leads to emergence of a cyclic configuration of arrows. As a result, the system leaves the recurrent set. It is natural to define the avalanche as a process of restoration of the recurrent state. It corresponds to successive rotations from the beginning of
the motion up to the moment when an acyclic configuration is restored for
the first time and the structure of the spanning tree is reconstructed. The
number of steps $n$ is the duration of an avalanche, the number of different
visited sites $s$ is its size. If the first step doesn’t lead to a cyclic configuration,
we put $n = 1, s = 1$.

When a closed loop appears on the given lattice, a branch of the dual
tree gets disconnected on the dual lattice. The probability distribution $P(s)$
of disconnected clusters follows the power law

$$P(s) \sim \frac{1}{s^{11/8}}$$

(17)

where $s$ is the number of lattice sites belonging to a cluster. Since the number
of steps $n$ which are neccessary to restore the tree is proportional to $s$, one
can expect the similar power law for the avalanche distribution $P(n)$.

The proposed model admits a natural generalization on an arbitrary
graph and arbitrary order of numeration of unit vectors directed to nearest
neighbours of a given site. The main result (14) remains unchanged where
$\Delta$ should be defined as the Laplacian matrix of the given graph.

References

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