The kernel of chromatic quasisymmetric functions on graphs and nestohedra

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Abstract. We study canonical Hopf algebra morphisms from generalised permutahedra and from graphs to $\text{QSym}$ by finding generators for their kernel and image.

Keywords: chromatic symmetric function, quasisymmetric functions, Hopf algebras, generalized permutahedra

This is an extended abstract, of which the full version [11] is yet to be published.

1 Introduction

Chromatic function on graphs

For a graph $G$ with vertex set $V(G)$, a colouring $f$ of the graph $G$ is a map $f : V(G) \to \mathbb{N}$. A colouring is proper if no edge is monochromatic. Stanley defines in [15] the chromatic symmetric function of $G$ in commuting variables $\{x_i\}_{i \geq 1}$ as

$$\Psi_G(G) = \sum_f x_f,$$

where we write $x_f = \prod_{v \in V(G)} x_{f(v)}$, and the sum runs over proper colourings of the graph $G$. Note that $\Psi_G(G)$ is in the ring $\text{Sym}$ of symmetric functions. The ring $\text{Sym}$ is a Hopf subalgebra of $\text{QSym}$, the ring of quasisymmetric functions. A long standing conjecture in this subject, commonly referred to as the tree conjecture, is that if two trees $T_1, T_2$ are not isomorphic, then $\Psi_{G}(T_1) \neq \Psi_{G}(T_2)$.

When $V(G) = [n]$, the natural ordering on the vertices allows us to consider a non-commutative analogue of $\Psi_G$, as done by Gebhard and Sagan in [6]. They define the chromatic symmetric function on non-commutative variables $\{a_i\}_{i \geq 1}$ as

$$\Psi_G(G) = \sum_f a_f,$$

where we write $a_f = \prod_{v=1}^{n} a_{f(v)}$, and we sum over the proper colourings $f$ of $G$.

Note that $\Psi_G(G)$ is also symmetric in the variables $\{a_i\}_{i \geq 1}$. Such functions are word symmetric function. The ring of word symmetric functions, $\text{WSym}$ for short, was
introduced in [13], and is sometimes called the ring of symmetric functions in non-commutative variables.

We consider graphs whose vertex sets are of the form \([n]\) for some \(n \geq 0\), and write \(G\) for the free linear space generated by such graphs. This can be endowed with a Hopf algebra structure, as described by Schmitt in [14].

In this paper we describe generators for \(\ker \Psi_{G}\) and \(\ker \Psi_{C}\). A similar problem was already considered for posets. In [5], Féray studies \(\Psi_{\text{Pos}}\), the Gessel quasisymmetric function defined on the poset Hopf algebra, and describes a set of generators of its kernel.

Some elements of the kernel of \(\Psi_{G}\) have already been constructed in [8] by Guay-Paquet and independently in [10] by Orellana and Scott. These relations, called modular relations, extend naturally to the non-commutative case. We introduce them now.

Given a graph \(G\) and an edge set \(E\) that is disjoint from \(E(G)\), let \(G \cup E\) denote the graph \(G\) with the edges in \(E\) added. In [8] and [10], it was observed that for a graph \(G\), if we have edges \(e_3 \in G\) and \(e_1, e_2 \notin G\) such that \(\{e_1, e_2, e_3\}\) forms a triangle, then

\[
\Psi_{G}(G) - \Psi_{G}(G \cup \{e_1\}) - \Psi_{G}(G \cup \{e_2\}) + \Psi_{G}(G \cup \{e_1, e_2\}) = 0. \tag{1.1}
\]

For such a graph \(G\), we call the formal sum \(G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}\) in \(G\) a modular relation on graphs. An example is given in Figure 1. Our goal is to show that these modular relations span the kernel of the chromatic symmetric function.

Theorem 1 (Kernel and image of \(\Psi_{G} : G \rightarrow WSym\)). The modular relations span \(\ker \Psi_{G}\). The image of \(\Psi_{G}\) is \(WSym\).

Two graphs \(G_1, G_2\) are said to be isomorphic if there is a bijection between the vertices that preserves edges. For the commutative version of the symmetric function, if two isomorphic graphs \(G_1, G_2\) are given, we know that \(\Psi_{G}(G_1)\) and \(\Psi_{G}(G_2)\) are the same. The formal sum in \(G\) given by \(G_1 - G_2\) is called an isomorphism relation on graphs.

Theorem 2 (Kernel and image of \(\Psi_{G} : G \rightarrow Sym\)). The modular relations and the isomorphism relations generate the kernel of the commutative chromatic symmetric function \(\Psi_{G}\). The image of \(\Psi_{G}\) is \(Sym\).

The second part of this theorem follows from previous work. For instance, in [4], several bases of \(Sym\) are constructed that are of the form \(\{\Psi_{G}(G_{\lambda})|\lambda \vdash n\}\).

In the last section of this paper we introduce a new graph invariant \(\tilde{\Psi}(G)\). That modular relations on graphs are in the kernel of \(\tilde{\Psi}\) is easy to see. It will follow from
Theorem 2 that \( \text{ker } \Psi_G \subseteq \text{ker } \tilde{\Psi} \). This reduces the tree conjecture in \( \Psi_G \) to this new invariant \( \tilde{\Psi}_G \).

The maps \( \Psi_G \) and \( \Psi_G \) arise as a more general construction in Hopf algebras. For a Hopf algebra \( H \), a character \( \eta \) of \( H \) is a linear map \( \eta : H \to K \) that preserves the multiplicative structure and the unit of \( H \). In [2], Aguiar, Bergeron, and Sotille define a combinatorial Hopf algebra as a pair \((H, \eta)\) where \( H \) is a Hopf algebra and \( \eta : H \to K \) a character of \( H \). For any combinatorial Hopf algebra \((H, \eta)\), a canonical Hopf algebra morphism to \( QSym \) is constructed in [2]. The maps \( \Psi_G : G \to \text{Sym} \) and \( \Psi_G : G \to \text{WSym} \) are Hopf algebra morphisms that can be obtained in such a manner: If we take the character \( \eta(G) = 1 \) [\( G \) has no edges], the canonical Hopf algebra morphism for \((G, \eta)\) is exactly the map \( \Psi_G \). The map \( \Psi_G \) arises from a parallel result in Hopf monoids, as presented in [11]. The Gessel quasisymmetric function \( \Psi_{\text{Pos}} \) on posets arises similarly.

We establish similar results to Theorem 1 and Theorem 2 in the combinatorial Hopf algebra of nestohedra, which is a Hopf subalgebra of generalised permutahedra.

**Generalised Permutahedra**

Generalised permutahedra are particular polytopes that include permutahedra, associahedra and graph zonotopes. The reader can see some results in the topic in [12].

The Minkowsky sum of two polytopes \( a, b \) is set as \( a +_M b = \{ a + b | a \in a, b \in b \} \). The Minkowsky difference \( a -_M b \) is defined as the unique polytope \( c \) that satisfies \( b +_M c = a \), if it exists. We denote the Minkowsky sum of several polytopes as \( \bigoplus a_i \). If we let \( \{ e_i | i \in I \} \) be the canonical basis of \( \mathbb{R}^I \), a simplex is a polytope of the form \( s_J = \text{conv}\{ e_j | j \in J \} \) for non-empty \( J \subseteq I \), and a generalised permutahedron in \( \mathbb{R}^I \) is a polytope of the form

\[
q = \left( \bigoplus_{J \neq \emptyset, a_j > 0} a_J s_J \right) -_M \left( \bigoplus_{J \neq \emptyset, a_j < 0} |a_j| s_J \right),
\]

for reals \( L(q) = \{ a_J | \emptyset \neq J \subseteq I \} \) that can be either positive, negative or zero. We identify a generalised permutahedron \( q \) with the list \( L(q) \). Note that not every list of reals will give us a generalised permutahedron, since the Minkowsky difference is not always defined.

A nestohedron is a generalised permutahedron where the coefficients \( a_J \) are in \( \{0, 1\} \). We identify a nestohedron \( q \) with the family \( \mathcal{F}(q) \subseteq 2^I \setminus \emptyset \) corresponding to those \( J \) such that \( a_J = 1 \). Finally, for a set \( A \subseteq 2^I \setminus \emptyset \), we write \( \mathcal{F}^{-1}(A) \) for the unique nestohedron \( q \) that satisfies \( \mathcal{F}(q) = A \). Note that in this case, every subset \( A \subseteq 2^I \setminus \emptyset \) will give rise to a nestohedron.

In [1], Aguiar and Ardila define \( \text{GP} \), a Hopf algebra structure on the linear space generated by generalised permutahedra in \( \mathbb{R}^n \) for \( n \geq 0 \). The Hopf subalgebra \( \text{Nesto} \)
is the linear space generated by nestohedra. In [7], Grujić introduced a quasisymmetric map in generalised permutahedra $\Psi_{GP}: GP \to QSym$ that we will recall now.

For a polytope $q \subseteq \mathbb{R}^I$, Grujić defines a function $f : I \to \mathbb{N}$ as $q$-generic if the face of $q$ that minimises $\sum_{i \in I} f(i)x_i$, denoted $q_f$, is a point. Equivalently, $f$ is $q$-generic if it lies in the interior of the normal cone of some vertex.

Then Grujić defines for $\{x_i\}_{i \geq 1}$ commutative variables, the quasisymmetric function:

$$\Psi_{GP}(q) = \sum_{f \text{ is } q\text{-generic}} x_f. \quad (1.2)$$

If we consider the character $\eta(q) = 1[q \text{ is a point}]$, then $\Psi_{GP}$ is the canonical Hopf algebra morphism associated with the combinatorial Hopf algebra $(GP, \eta)$.

In [1], Aguiar and Ardila define the graph zonotope $Z : G \to GP$, a Hopf algebra morphism that is injective and maps $\Psi_G$ to $\Psi_{GP}$. They also define other maps from other combinatorial Hopf algebras, like matroids, to $GP$, that preserve the canonical Hopf algebra morphisms. If we are able to describe $\ker \Psi_{GP}$, then such maps $Z : H \to GP$ give us some information on $\ker \Psi_H$ using that $Z(\ker \Psi_H) = \ker \Psi_{GP} \cap Z(H)$.

We discuss now a non-commutative version of $\Psi_{GP}$, where we will establish an analogue of Theorem 1 to nestohedra. Consider the Hopf algebra of word quasisymmetric functions $WQSym$, a version of $QSym$ in non-commutative variables introduced in [9].

For a generalised permutahedron $q$ and non-commutative variables $\{a_i\}_{i \geq 1}$, we set

$$\Psi_{GP}(q) = \sum_{f \text{ is } q\text{-generic}} a_f.$$

It is easily seen (and shown in [11]) that $\Psi_{GP}(q)$ is a word quasisymmetric function. This defines a Hopf algebra morphism between $GP$ and $WQSym$. Let us call $\Psi_{Nesto}$ and $\Psi_{Nesto}^\prime$ to the restrictions of $\Psi_{GP}$ and $\Psi_{GP}$ to $Nesto$, respectively.

Our next theorems describe the kernel of the maps $\Psi_{Nesto}$ and $\Psi_{Nesto}^\prime$, using some modular relations that we will present later, in Theorem 14. In fact, these modular relations generalise the ones for graphs, presented in [8] and mentioned earlier in (1.1), in the sense that the graph zonotope embedding $Z : G \to GP$, presented in [1], maps modular relation on graphs to modular relations on nestohedra.

**Theorem 3** (*Kernel and image of of $\Psi_{Nesto} : Nesto \to WQSym$*). The space $\ker \Psi_{Nesto}$ is generated by the modular relations in nestohedra. The image of $\Psi_{Nesto}$ is $SC$, a proper subspace of $WQSym$ introduced in Definition 9 below.

Let us denote by $WQSym_n$ the linear space of homogeneous word quasisymmetric functions of degree $n$, and let $SC_n = SC \cap WQSym_n$. A monomial basis for $SC$ is presented in Definition 9. The dimension of $SC_n$ is computed in [11], where in particular it is shown that it is exponentially smaller than the dimension of $WQSym_n$. 

Two generalised permutahedra $q_1, q_2$ are isomorphic if one can be obtained from the other by permuting the coordinates. If $q_1, q_2$ are isomorphic, the commutative chromatic quasisymmetric functions $\Psi_{GP}(q_1)$ and $\Psi_{GP}(q_2)$ are the same. We call to $q_1 - q_2$ an isomorphism relation on nestohedra.

**Theorem 4** (Kernel and image of $\Psi_{Nesto} : Nesto \to \mathcal{QSym}$). The space $\ker \Psi_{Nesto}$ is generated by the modular relations and the isomorphism relations. The image of $\Psi_{Nesto}$ is $\mathcal{QSym}$.

A description of $\ker \Psi_{Nesto}$ is less general than a description of $\ker \Psi_{GP}$. Nevertheless, most of the combinatorial objects embedded in $GP$ are also in $Nesto$, such as graphs and matroids, so the result in the $Nesto$ Hopf subalgebra can already be used to help us on other kernel problems.

We will use boldface for non-commutative Hopf algebras, their elements, and the associated combinatorial objects, like word symmetric functions, for sake of clarity.

## 2 Preliminaries

For an equivalence relation $\sim$ on a set $A$, we call $[x]_\sim$ to the equivalence class of $x$ in $\sim$, and we write $[x]$ when $\sim$ is clear from context. We write both $\mathcal{E}(\sim)$ and $A/\sim$ for the set of equivalence classes of $\sim$.

### 2.1 Linear algebra preliminaries

The following easy linear algebra lemmas will be useful to compute generators of the kernels and the images of $\Psi$ and $\Psi$. These lemmas describe a sufficient condition for a set $B$ to span the kernel of a linear map $\phi : V \to W$. The proofs of these lemmas are basic linear algebra and can be found in [11].

**Lemma 5.** Let $V$ be a finite dimensional vector space with a basis $\{a_i|i \in I\}$ indexed by $I = [m]$, $\phi : V \to W$ be a linear map, and $B = \{b_j|j \in J\} \subseteq \ker \phi$ be a family of relations.

Assume that there exists $\mathcal{I} \subseteq I$ such that:

- the elements $(\phi(a_i))_{i \in \mathcal{I}}$ form a linearly independent family in $W$,
- for $i \in I \setminus \mathcal{I}$ we have $a_i = b + \sum_{k=i+1}^{m} \lambda_k a_k$ for some $b \in B$ and some scalars $\lambda_k$;

Then $B$ spans $\ker \phi$. Additionally, we have that $(\phi(a_i))_{i \in \mathcal{I}}$ is a basis of the image of $\phi$.

The following lemma will help us dealing with the composition $\Psi = \text{comu} \circ \Psi$: we give a sufficient condition for a natural enlargement of the set $B$ to generate $\ker \Psi$. 

Lemma 6. We will use the same notation as in Lemma 5. Let \( \phi_1 : W \to W' \) be a linear map and call \( \phi' = \phi_1 \circ \phi \). Take an equivalence relation \( \sim \) in \( \{a_i\}_{i \in I} \) that satisfies \( \phi'(a_i) = \phi'(a_j) \) whenever \( a_i \sim a_j \). Define \( C = \{a_i - a_j | a_i \sim a_j\} \) and write \( \phi'([a_i]) = \phi'(a_i) \) with no ambiguity.

Assume the hypothesis in Lemma 5 and, additionally, suppose that \( (\phi'([a_i]))_{[a_i] \in E(\sim)} \) is linearly independent.

Then \( \ker \phi' \) is generated by \( B \cup C \). Furthermore, \( (\phi'([a_i]))_{[a_i] \in E(\sim)} \) is a basis of \( \operatorname{im} \phi' \).

2.2 Hopf algebras and associated combinatorial objects

In the following, all the Hopf algebras \( H \) have a grading, denoted as \( H = \oplus_{n \geq 0} H_n \).

An integer composition, or simply a composition, of \( n \), is a list \( \alpha = (a_1, \cdots, a_k) \) of positive integers which sum is \( n \). We write \( \alpha \models n \). We denote \( l(\alpha) \) for the length of the list and we denote as \( C_n \) the set of compositions of size \( n \).

An integer partition, or simply a partition, of \( n \) is a non-increasing list \( \lambda = (\lambda_1, \cdots, \lambda_k) \) of positive integers which sum is \( n \). We denote \( \lambda \vdash n \). We write \( l(\lambda) \) for the length of the list and we denote as \( P_n \) the set of partitions of size \( n \). By disregarding the order of the parts on a composition \( \alpha \) we obtain a partition denoted \( \lambda(\alpha) \).

A set partition \( \pi = \{\pi_1, \cdots, \pi_k\} \) of a set \( I \) is a collection of non-empty disjoint subsets of \( I \), called blocks, that cover \( I \). We write \( \pi \models I \). We denote \( l(\pi) \) for the size of the set partition. We write \( \mathbf{P}_I \) for the family of set partitions of \( I \), or simply \( \mathbf{P}_n \) if \( I = [n] \). By counting the elements on each block we obtain an integer partition denoted \( \lambda(\pi) \models \#I \). We identify a set partition \( \pi \in \mathbf{P}_I \) with an equivalence relation \( \sim_\pi \) on \( I \), where \( x \sim_\pi y \) if \( x, y \in I \) are on the same block of \( \pi \).

A set composition \( \vec{\pi} = S_1 \cdots S_j \) of \( I \) is a list of non-empty disjoint subsets of \( I \) that cover \( I \). We write \( \vec{\pi} \models S \). We denote \( l(\vec{\pi}) \) for the size of the set composition. We call \( \mathbf{C}_I \) to the family of set compositions of \( I \), or simply \( \mathbf{C}_n \) if \( I = [n] \). By disregarding the order of a set composition \( \vec{\pi} \), we obtain a set partition \( \lambda(\vec{\pi}) \models I \). By counting the elements on each block we obtain a composition denoted \( \alpha(\vec{\pi}) \models \#I \). A set composition is naturally identified with a total preorder \( P_{\vec{\pi}} \) on \( I \), where \( xP_{\vec{\pi}} y \) if \( x \in S_i, y \in S_j \) for \( i \leq j \).

A colouring of the set \( I \) is a function \( f : I \to \mathbb{N} \). The set composition type \( \vec{\pi}(f) \) of a colouring \( f : I \to \mathbb{N} \) is the set composition obtained after deleting the empty sets of \( f^{-1}(1) | f^{-1}(2) | \cdots \).

We recall that in partitions and in set partitions, it is defined a classical coarsening order \( \leq \), where we say that \( \lambda \leq \tau (\pi \leq \tau, \text{ resp.}) \) if \( \tau \) is obtained from \( \pi \) by adding some parts (resp. if \( \tau \) is obtained from \( \pi \) by merging some blocks).

Recall that the homogeneous component \( QSym_n \) (resp. \( Sym_n, WSym_n, \) \( WQSym_n \)) of the Hopf algebra \( QSym \) (resp. \( Sym, WSym, WQSym \)) has a monomial basis indexed by compositions (resp. partitions, set partitions, set compositions). We will denote this basis by \( \{M_{\vec{\pi}}\}_{\vec{\pi} \in C_n} \) (resp. \( \{m_{\lambda}\}_{\lambda \in P_n}, \{m_{\pi}\}_{\pi \in P_n}, \{M_{\vec{\pi}}\}_{\vec{\pi} \in C_n} \)).
2.3 Monomial basis and graph Hopf algebra

We now discuss the monomial expansion of chromatic symmetric function on graphs:

**Lemma 7** ([6, Proposition 3.2]). For a graph $G$ we say that a set partition $\tau$ of $V(G)$ is proper if no block of $\tau$ contains an edge. Then have that $\Psi_G(G) = \sum_{\tau} m_{\tau}$, where the sum runs over all proper set partitions of $V(G)$.

For a set partition $\pi$, we define the graph $K_{\pi}$ where $\{i, j\} \in E(K_{\pi})$ if $i \sim_{\pi} j$. A set partition $\tau$ is proper in $K_{\pi}$ if and only if $\tau \leq \pi$. Hence, as a consequence of Lemma 7,$$
\Psi_G(K_{\pi}) = \sum_{\tau \leq \pi} m_{\tau}. \quad (2.1)
$$

2.4 Monomial basis and nestohedra Hopf algebra

We define, for a non-empty set $A \subseteq [n]$, the set $A_{\pi} = \{\text{minima of } A \text{ in } P_{\pi}\}$, where we recall that $P_{\pi}$ is a total preorder on $[n]$. We say that $A_{\pi}$ is a singleton.

The following lemma is well known in the folklore of generalised permutahedra and is shown, for instance, in [11].

**Lemma 8** (Vertex normal cone characterization). Let $q$ be a nestohedron. A colouring $f$ is $q$-generic if and only if $A_{\pi}(f) = \text{pt}$ for every $A \in F(q)$. Furthermore, the face $q_f$ that minimizes $\sum_i f(i)x_i$ only depends on the set composition $\pi(f)$. We write $q_{\pi}$ for the face $q_f$ for any $f$ of set composition type $\pi$, without ambiguity.

For $\pi \in C_n$, we define the fundamental nestohedron as $p_{\pi} = \mathcal{F}^{-1}\{A \subseteq [n] | A_{\pi} = \text{pt}\}$.

On set compositions, we write that $\pi_1 \preceq \pi_2$ whenever $\mathcal{F}(p_{\pi_1}) \subseteq \mathcal{F}(p_{\pi_2})$. Equivalently, $\pi_1 \preceq \pi_2$ if for any non-empty $A \subseteq [n]$ we have $A_{\pi_1} = \text{pt} \Rightarrow A_{\pi_2} = \text{pt}$. With this, $\preceq$ is a preorder, called singleton commuting preorder or SC preorder.

Additionally, we define the equivalence relation $\sim$ in $C_n$ as $\pi \sim \tau$ if $p_{\pi} = p_{\tau}$. A combinatorial interpretation of this equivalence relation can be found below in **Proposition 10**, which also motivates the name of the preorder defined above.

It is natural to consider $N_{[\pi]} = \sum_{\pi \sim \tau} M_{\tau}$, which forms a linear independent family.

**Definition 9.** We define the singleton commuting Hopf algebra, or SC for short, as the graded vector subspace $\bigoplus_{n \geq 0} SC_n$ of $WQSym$, where $\{N_{[\pi]} : [\pi] \in C_n / \sim\}$ is a basis of each $SC_n$. As a consequence of **Theorem 3**, SC is a Hopf algebra.

The following proposition will not be used in the proof of the main theorems, but gives us a way to describe the equivalence classes of $\sim$. In particular, in [11], it allows us to compute the dimensions of $SC_n$. The proof of **Proposition 10** can be found in [11].

**Proposition 10.** For $\pi, \tau \in C_I$, the following are equivalent.
We have that $p_{\bar{\pi}} = p_{\bar{\tau}}$.

We have $\lambda(\bar{\pi}) = \lambda(\bar{\tau})$ and each $a, b \in I$ that satisfies both $a P_{\bar{\pi}} b$ and $b P_{\bar{\tau}} a$ are either singletons or in the same block in $\lambda(\bar{\pi})$.

From the definition of $\preceq$, we have the following consequence of Lemma 8.

$$\Psi_{GP}(p_{\bar{\pi}}) = \sum_{\bar{\pi} \preceq \bar{\tau}} M_{\bar{\tau}}.$$  \hspace{1cm} (2.2)

As presented, (2.2) seems to show that $(\Psi_{GP}(p_{\bar{\pi}}))_{\bar{\pi} \in C_n}$ writes triangularly with respect to the monomial basis. Since $\preceq$ is not an order, that is not the case, but we obtain a related result with this reasoning:

**Lemma 11.** The family $(\Psi((p_{|\bar{\pi}|}))_{|\bar{\pi}| \in C_n/\sim}$ forms a basis of $SC$.

The following lemma is helpful to show Theorem 4 and is shown in [11].

**Lemma 12.** There is an order $\preceq'$ on $C_n$ that satisfies $\bar{\pi} \preceq \bar{\tau} \Rightarrow \alpha(\bar{\pi}) \preceq' \alpha(\bar{\tau})$.

## 3 Main theorems on graphs

With Lemma 5, we will show that the kernel of $\Psi_{G}$ is spanned by the modular relations.

**Proof of Theorem 1.** Recall that $G_n$ is spanned by graphs with vertex set $[n]$. We choose an order $\succeq$ in this family of graphs in a way that the number of edges is non-decreasing.

Recall that for a set partition $\pi$ of the vertex set $[n]$, we define $K_\pi$ as the graph where $\{i, j\} \in E(K_\pi)$ if $i \sim_\pi j$. Then, from (2.1), we know that $\{\Psi_{G}(K_{c_\pi})|\pi \in P_n\}$ writes as a triangular matrix over the monomial basis of $WSym$, hence forms a linearly independent set in $WSym$.

In order to apply Lemma 5 to the set of modular relations on graphs, it suffices to show the following: if a graph $G$ is not of the form $K_{c_\pi}$, then we can find a formal sum $G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}$ that is a modular relation. Indeed, $G$ is the graph with least edges in that expression, so it is the smallest in the order $\succeq$. If the above holds, Lemma 5 implies that the modular relations generate the space $\ker \Psi_{G}$ and $\{\Psi_{G}(K_{c_\pi})|\pi \in P_n\}$ forms a basis of $\text{im} \Psi_{G}$, so, from (2.1), $\text{im} \Psi_{G} = WSym$.

To find the desired modular relation, it is enough to find a triangle $\{e_1, e_2, e_3\}$ such that $e_1, e_2 \not\in E(G)$ and $e_3 \in E(G)$. Consider $\tau$, the set partition given by the connected

![Figure 2: Choice of edges in proof of Theorem 1](image-url)
components of $G^c$. By hypothesis, $G \neq K^c_\pi$, so there are vertices $v, w$ in the same block of $\tau$ that are not neighbours in $G^c$. Without loss of generality we can take such $u, w$ that are at distance 2 in $G^c$, so they have a common neighbour $v$ in $G^c$. The edges $e_1 = \{v, u\}$, $e_2 = \{v, w\}$ and $e_3 = \{u, w\}$ form the desired triangle, concluding the proof. 

Proof of Theorem 2. Our goal is to apply Lemma 6 to the map $\Psi_G = \text{comu} \circ \Psi_G$ for the equivalence relation corresponding to graph isomorphism. First, if $\lambda(\pi) = \lambda(\tau)$ then $K^c_\pi$ and $K^c_\tau$ are isomorphic graphs. Define without ambiguity $r_{\lambda(\pi)} = \Psi_G(K^c_\pi)$.

From the proof of Theorem 1, to apply Lemma 6 it is enough to establish that the family $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ is linearly independent. Indeed, it would follow that $\ker \Psi_G$ is generated by the modular relations and the isomorphism relations, and $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ is a basis of $\text{im} \Psi_G$, concluding the proof.

The linear independence of $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ follows from the fact that it writes as an upper triangular matrix under the coarsening order in integer partitions. Indeed, from (2.1), if we let $\pi$ run over set partitions and $\sigma$ run over integer partitions, we have

$$r_{\lambda(\pi)} = \Psi_G(K^c_\pi) = \sum_{\tau \leq \pi} m_{\lambda(\tau)} = \sum_{\sigma \leq \lambda(\pi)} a_{\pi, \sigma} m_{\sigma},$$

where $a_{\pi, \sigma} = \#\{\tau \vdash [n]|\lambda(\pi) = \sigma, \tau \leq \pi\}$. Note that $a_{\pi, \lambda(\pi)} = 1$, so $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ is linearly independent. From a dimension argument, $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ spans $\text{Sym}_n$, so $\text{im} \Psi_G = \text{Sym}$. 

Remark 13. We have obtained in the proof of Theorem 2 that $(r_\lambda)_{\lambda \vdash [n]}$ is a basis for $\text{Sym}_n$. This basis is different from other “chromatic bases” proposed in [4]. The proof gives us a recursive way to compute the coefficients $\zeta_\lambda$ on the span $\Psi_G(G) = \sum_\lambda \zeta_\lambda r_\lambda$. It is then natural to ask if combinatorial properties can be obtained for these coefficients, which are isomorphic invariants.

Similarly in the non-commutative case, we obtain that $\text{WSym}_n$ is spanned by $(\Psi_G(K^c_\pi))_{\pi \vdash [n]}$, and so other coefficients arise. We can again ask for combinatorial properties of these coefficients. The same can be asked in the next section for the nestohedra case.

4 Main theorems on nestohedra

For non-empty sets $A \subseteq [n]$, we define $\text{Orth} A = \{\pi \in C_n|A_\pi = pt\}$. We have:

Theorem 14 (A modular relation for $\Psi_{\text{Nesto}}$). Let $\{A_k | k \in K\}$ and $\{B_j | j \in J\}$ be two disjoint families of non-empty subsets of $[n]$. Let us write $K = \bigcup_{k \in K} (\text{Orth} A_k)^c$, and $J = \bigcup_{j \in J} \text{Orth} B_j$. Consider the nestohedron $q = \mathcal{F}^{-1}\{A_k | k \in K\}$.

Suppose that $K \cup J = C_n$. Then,

$$\sum_{T \subseteq J} (-1)^{#T} \Psi_{\text{GP}} \left[q + M \mathcal{F}^{-1}\{B_j | j \in T\}\right] = 0.$$
The proof of this result is done combinatorially, and is presented in [11].

The sum \( \sum_{T \subseteq J} (-1)^{|T|} [q + M F^{-1} \{B_i | j \in T\}] \) is called a modular relation on nestohedra.

It can be noted that, if \( I = G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1,e_2\} \) is a modular relation on graphs, then the graph zonotope \( Z(I) \) is the modular relation on nestohedra corresponding to \( q = Z(G) \) (i.e. \( \{A_k | k \in K\} = E(G) \)), \( B_1 = e_1 \) and \( B_2 = e_2 \). In this case, the condition \( K \cup J = C_n \) follows from the fact that no proper colouring of \( G \) is monochromatic in both \( e_1 \) and \( e_2 \), which is imposed by \( e_3 \in G \).

Recall that we set \( p[\vec{r}] = F^{-1} \{A \subseteq [n] | A[\vec{r}] = pt\} \), which depends only on the SC-equivalence class of \( \vec{r} \) (by definition of \( \sim \)) and are called the fundamental nestohedra. Write, without ambiguity, \( p[\vec{r}] = p[\vec{r}] \).

We follow here roughly the same idea as in the graph case: We use the family of nestohedra \( \{p[\vec{r}] | [\vec{r}] \in C_n / \sim \} \), constructed above, whose image by \( \Psi_{GP} \) is linearly independent and is rich enough to span the image, to apply Lemma 5.

**Proof of Theorem 3.** We will apply Lemma 5 with the modular relations from Theorem 14.

First recall that \( \text{Nesto}_n \) is a linear space generated by the nestohedra in \( \mathbb{R}^n \). We choose a total order \( \geq \) on the nestohedra so that \( \# F(q) \) is non decreasing.

We have seen in Lemma 11 that \( \{\Psi_{GP}(p[\vec{r}]) | [\vec{r}] \in C_n / \sim \} \) is linearly independent. Therefore, it suffices to show that for any \( q \) that is not a fundamental nestohedron, we can write some modular relation \( b \) as \( b = q + \sum_i \lambda_i q_i \), where \( \# F(q) < \# F(q_i) \forall i \). So \( q < q_i \forall i \).

Indeed, it would follow from Lemma 5 that the modular relations on nestohedra span \( \ker \Psi_{\text{Nesto}} \). As a consequence, \( \text{im} \Psi_{\text{Nesto}} \) is spanned by the sets \( \{\Psi_{GP}(p[\vec{r}]) | [\vec{r}] \in C_n / \sim \} \) for each \( n \geq 0 \). From Lemma 11, this image is \( SC_n \).

To obtain the desired modular relation, we invoke Theorem 14 on \( \{A \in F(q)\} \) and \( \{B \notin F(q)\} \). Let us write \( K = \bigcup_{A \in F(q)} (\text{Orth} A)^c \) and \( J = \bigcup_{B \notin F(q)} \text{Orth} B \). We will first show that we have \( K \cup J = C_n \).

Take, for sake of contradiction, some \( \vec{r} \notin K \cup J \). Note that \( \vec{r} \notin K \) is equivalent to \( A[\vec{r}] = pt \) for every \( A \in F(q) \). Note as well that \( \vec{r} \notin J \) is equivalent to \( B[\vec{r}] \neq pt \) for every \( B \notin F(q) \). Therefore, if \( \vec{r} \notin K \cup J \), then \( q = p[\vec{r}] \), contradicting the assumption that \( q \) is not a fundamental nestohedron. We obtain that \( K \cup J = C_n \). Finally, note that

\[
q + \sum_{T \subseteq F(q) \neq \emptyset} (-1)^{|T|} [q + M F^{-1}(T)],
\]

is a modular relation of the desired form, concluding the hypothesis of Lemma 5. \( \square \)

For the commutative case we will apply Lemma 6. Note that we already have a generator set of \( \ker \Psi_{\text{Nesto}} \), so similarly to the proof of Theorem 2, we just need to establish some linear independence.

Recall that two nestohedra \( q_1 \) and \( q_2 \) are isomorphic if there is a permutation matrix \( P \) such that \( x \in q_2 \iff Px \in q_1 \). Since we are in the commutative case now, if \( \vec{r}_1 \) and
\(\bar{\pi}_2\) share the same composition type, then \(p^{\bar{\pi}_1}\) and \(p^{\bar{\pi}_2}\) are isomorphic, and so we have 
\[\Psi_{GP}(p^{\bar{\pi}_1}) = \Psi_{GP}(p^{\bar{\pi}_2})\]. Set \(R_{\alpha}(\bar{\pi}) := \Psi_{GP}(p^{\bar{\pi}})\) without ambiguity.

**Proof of Theorem 4.** We will apply Lemma 6 to the map \(\Psi_{GP} = \text{com} \circ \Psi_{GP}\) on the equivalence relation corresponding to the isomorphism of nestohedra.

From the proof of Theorem 3, to apply Lemma 6 it is enough to establish that the family \((R_x)_{x\in C_n}\) is linearly independent. It would follow that \(\ker \Psi_{GP}\) is generated by the modular relations and the isomorphism relations, and \((R_x)_{x\in C_n}\) is a basis of \(\text{im} \Psi_{G}\), concluding the proof.

To show the linear independence of \((R_x)_{x\in C_n}\), we write \(R_x\) on the monomial basis of \(QSym\), and use the order \(\preceq\) mentioned in Lemma 12.

As a consequence of (2.2), if we write \(A_{\bar{\pi},\beta} = \#\{\bar{\pi} \in C_n | \bar{\pi} \preceq \bar{\pi}, \alpha(\bar{\pi}) = \beta\}\), we have:
\[
R_{\alpha}(\bar{\pi}) = \Psi_{GP}(p^{\bar{\pi}}) = \sum_{\bar{\pi} \preceq \bar{\pi}} M_{\alpha}(\bar{\pi}) = A_{\bar{\pi},\alpha(\bar{\pi})}M_{\alpha(\bar{\pi})} + \sum_{\alpha(\bar{\pi}) < \beta} A_{\bar{\pi},\beta}M_{\beta},
\]
(4.1)

It is clear that \(A_{\bar{\pi},\alpha(\bar{\pi})} > 0\), so independence follows, which completes the proof. \(\square\)

## 5 A word on graph invariants

Consider the ring \(\mathbb{K}[q_1, q_2, \ldots]\) on countable many commuting variables, and let \(R\) be such ring modulo the relations \(q_i(q_i - 1)^2 = 0\). Let \(\text{Sym}(R)\) be the ring of symmetric functions with coefficients in \(R\).

Consider the graph invariant \(\Psi(G) = \sum_f x_f q_i^{c_G(f,i)}\) in \(\text{Sym}(R)\), where the sum runs over all colourings \(f\), and \(c_G(f,i)\) stands for the number of monochromatic edges of colour \(i\) in the colouring \(f\) (i.e. edges \(\{v_1, v_2\}\) such that \(f(v_1) = f(v_2) = i\)).

Let \(l = G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}\) be a modular relation on graphs, i.e. \(\{e_1, e_2, e_3\}\) are edges that form a triangle, and for a colouring \(f\), set
\[s_f = q_i^{c_G(f,i)} - q_i^{c_{G\cup\{e_1\}}(f,i)} - q_i^{c_{G\cup\{e_2\}}(f,i)} + q_i^{c_{G\cup\{e_1, e_2\}}(f,i)}\).

It is easy to see that \(s_f\) is always zero in \(R\), so \(\Psi(l) = \sum_f x_fs_f = 0\).

It follows that any modular relation is in \(\ker \Psi\). From Theorem 2 we have that \(\ker \Psi_G \subseteq \ker \Psi\), so we obtain the following proposition.

**Proposition 15.** For any graphs \(G_1, G_2\), we have \(\Psi_G(G_1) = \Psi_G(G_2) \Rightarrow \Psi(G_1) = \Psi(G_2)\).

If we find a graph invariant satisfying Proposition 15 that takes different values for non-isomorphic trees, we obtain a proof of the tree conjecture. This is in line with what has been done in [3], where it was shown that non-isomorphic proper caterpillars have different chromatic symmetric functions. We wish to use Theorem 2 to prove Proposition 15 for other invariants.

Now we have \(\Psi(G)|_{q_1 = 0} = \Psi_G\). So \(\ker \Psi = \ker \Psi_G\). We note that other specialisations are also allowed, like \(\Psi(G)|_{q_i = 1}\) and \(\frac{d}{dq_i} \Psi(G)|_{q_i = 1}\).
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