Preparation Uncertainty Implies Measurement Uncertainty in a Class of Generalized Probabilistic Theories

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Abstract

In quantum theory, it is known for a pair of noncommutative observables that there is no state on which they take simultaneously definite values, and that there is no joint measurement of them. They are called preparation uncertainty and measurement uncertainty respectively, and research has unveiled that they are not independent from but related with each other in a quantitative way. This study aims to reveal whether similar relations to quantum ones hold also in generalized probabilistic theories (GPTs). In particular, a certain class of GPTs is considered which can be characterized by transitivity and self-duality and regarded as extensions of quantum theory. It is proved that there are close connections expressed quantitatively between two types of uncertainty on a pair observables also in those theories: if preparation uncertainty exists, then measurement uncertainty also exists, and they are described by similar inequalities. Our results manifest that their correspondences are not specific to quantum theory but more universal ones.

1 Introduction

Since it was propounded by Heisenberg [1], the existence of uncertainty relations, which is not observed in classical theory, has been regarded as one of the most significant features of quantum theory. The importance of uncertainty relations lies not only in their conceptual aspects but also in practical use such as the security proof of quantum key distribution [2].

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There have been researches to capture and formulate the notion of “uncertainty” in several ways. One of the most outstanding works was given by Robertson [3]. There was shown an uncertainty relation in terms of standard derivation which stated that the probability distributions obtained by the measurements of a pair of noncommutative observables cannot be simultaneously sharp. While this type of uncertainty (called preparation uncertainty) has been studied also in a more direct way [4, 5, 6] or the entropic way [7, 8, 9], another type of uncertainty called measurement uncertainty is known to exist in quantum theory [10]. It describes that when we consider measuring jointly a pair of noncommutative observables, there must exist measurement error for the joint measurement, that is, we can only conduct their approximate joint measurement. There have been researches on measurement uncertainty with measurement error formulated in terms of standard derivation [11, 12, 13] or entropy [14]. Their measurement uncertainty relations were proved mathematically by using preparation uncertainty relations. It implies that there may be a close connection between those two kinds of uncertainty. From this point of view, one of us [15] proved simple inequalities which demonstrate in a more explicit way than other previous studies that preparation uncertainty indicates measurement uncertainty and the bound derived from the former also bounds the latter. The main results of [15] were obtained with preparation uncertainty quantified by overall widths and minimum localization error, and measurement uncertainty by error bar widths, Werner’s measure, and $L_\infty$ distance [16, 17, 18, 19]. On the other hand, researches on generalized probabilistic theories (GPTs) [20, 21, 22, 23, 24, 25, 26], which are the most general theories of physics, have revealed that phenomena such as no-cloning and teleportation used to be regarded as peculiar to quantum theory are indeed possible in a broader class of theories [27, 28]. However, concerning about uncertainty, although both preparation and measurement uncertainty can be formulated naturally also in GPTs, little is known about how two types of generalized uncertainty are related with each other.

In this paper, we study the relations between two kinds of uncertainty in GPTs. We focus on a class of GPTs which are transitive and self-dual including finite dimensional classical and quantum theories, and demonstrate similar results to [15] in the GPTs: preparation uncertainty relations indicate measurement uncertainty relations. More precisely, it is proved in a certain class of GPTs that if a preparation uncertainty relation gives some bound, then it is also a bound on the corresponding measurement uncertainty relation with the quantifications of uncertainty in [15] generalized to GPTs. Our results manifest that the close connections between two kinds of uncertainty exhibited in quantum theory are more universal ones.

This paper is organized as follows. In section 2, we give a brief review of GPTs. There are introduced fundamental descriptions of GPTs and several mathematical assumptions imposed in order to derive our main the-
orems. Some examples of GPTs such as classical, quantum, and regular polygon theories are also explained. In section 3, we introduce measures which quantify the width of a probability distribution. These measures are used for considering whether it is possible to localize jointly two probability distributions obtained by two kinds of measurement on one certain state, that is, they are used for describing preparation uncertainty. We also introduce measures quantifying measurement error by means of which we can formulate measurement uncertainty resulting from approximate joint measurements of two incompatible measurements. After the introductions of those quantifications, we present our main theorems and their proofs. In section 4, we conclude this paper with several discussions.

2 GPTs

In this section, we give a brief review of the mathematical formulation of GPTs. Our mathematical formulation and terms are in accord with [29, 30, 31, 32], where more detailed descriptions are given. Note that in the remaining of this paper we restrict ourselves to theories embedded in finite dimensional vector spaces, i.e. finite dimensional GPTs.

2.1 States, effects, and measurements

A physical experiment is described by three procedures: to prepare an object system, to perform a measurement, and to obtain a probability distribution onto the outcome values of the measurement [21]. Each theory of GPTs gives an intuitive description of physical experiments.

In each theory of GPTs, preparation procedures are called states. The set of all states is represented by a nonempty compact convex set \( \Omega \), which we call the state space, in some locally convex Hausdorff topological vector space \( V \) on \( \mathbb{R} \). For simplicity, we assume in this paper that \( V \) is finite dimensional. Let us denote the affine hull of \( \Omega \) by \( \text{aff}(\Omega) := \{ \sum_{i=1}^{k} \theta_i \omega_i \mid k \in \mathbb{N}, \omega_i \in \Omega, \theta_i \in \mathbb{R}, \sum_{i=1}^{k} \theta_i = 1 \} \), and assume in the remaining of this paper that \( \text{aff}(\Omega) \) is a \( N \)-dimensional \( (N < \infty) \) affine space, i.e. \( \text{dim}_{\text{aff}}(\Omega) = N^1 \).

We remark that the convex structure of \( \Omega \) is derived from the notion of probability mixtures of states: if \( \omega_1, \omega_2 \in \Omega \), then \( \omega := p \omega_1 + (1 - p) \omega_2 \in \Omega \) for \( p \in [0, 1] \), where \( \omega \) means the state obtained by the mixture of \( \omega_1 \) and \( \omega_2 \) with probability weights \( \{p, 1 - p\} \). Since \( \Omega \) is a compact convex set, thanks to the Krein-Milman theorem [33], there exist extreme points of \( \Omega \) which generate the whole set. We denote the set of all extreme points of \( \Omega \) by \( \Omega^\text{ext} := \{ \omega_i^\text{ext} \} \) (\( \neq \emptyset \)), and call its elements pure states (the other states are called mixed states).

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1The affine dimension of an affine set \( X \) is defined by the dimension of the set \( X - x_0 \) \( (x_0 \in X) \) as a vector space.
In GPTs, measurements are defined through the notion of *effects*. Let us consider a GPT whose state space is \( \Omega \), and let \( \mathcal{A}(\Omega) \) be the set of all affine functions on \( \Omega \), that is, \( \mathcal{A}(\Omega) = \{ h: \Omega \to \mathbb{R} \mid h(p\omega_1 + (1 - p)\omega_2) = ph(\omega_1) + (1 - p)h(\omega_2) \text{ for all } p \in [0, 1], \; \omega_1, \omega_2 \in \Omega \} \). An affine function \( e \in \mathcal{A}(\Omega) \) is called an *effect* if \( 0 \leq e(\omega) \leq 1 \) for all \( \omega \in \Omega \), where \( e(\omega) \) represents the probability of obtaining a specific outcome when \( \omega \) is prepared. We call the set of all effects \( \mathcal{E}(\Omega) = \{ e \in \mathcal{A}(\Omega) \mid 0 \leq e(\omega) \leq 1 \text{ for all } \omega \in \Omega \} \) the *effect space* of the theory. Note that in this paper we assume the *no-restriction hypothesis* that all effects are allowed physically [34]. The *unit effect* \( u \in \mathcal{E}(\Omega) \) is defined as the effect satisfying \( u(\omega) = 1 \) for all \( \omega \in \Omega \). It can be easily shown that \( \mathcal{E}(\Omega) \) is convex (the extreme elements are called *pure effects*), the unit effect \( u \) is pure, and \( u - e \in \mathcal{E}(\Omega) \) whenever \( e \in \mathcal{E}(\Omega) \).

An effect \( e \) is called *indecomposable* if \( e \neq 0 \) and a decomposition \( e = e_1 + e_2 \), where \( e_1, e_2 \in \mathcal{E}(\Omega) \), implies that both \( e_1 \) and \( e_2 \) are scalar multiples of \( e \). It can be seen that there exist pure and indecomposable effects in \( \mathcal{E}(\Omega) \), and we denote the set of all pure and indecomposable effects by \( \mathcal{E}^{\text{ext}}(\Omega) = \{ e_i^{\text{ext}} \} \).

In quantum theory, they correspond to rank-1 projections (see Example 2.3.4 in subsection 2.3). A *measurement* or *observable* (with \( n \) outcomes) is defined by an \( n \)-tuples \( \{ e_i \}_{i=1}^n \) of effects such that \( \sum_{i=1}^n e_i = u \), where \( e_i(\omega) \) represents the probability of observing the \( i \)th outcome of the measurement when a state \( \omega \) is prepared. The condition \( \sum_{i=1}^n e_i = u \) ensures that the total probability is 1. We describe a measurement \( E \) also as \( E = \{ e_a \}_{a \in A} \) satisfying \( \sum_{a \in A} e_a = u \) in this paper, where \( A \) is a sample space, namely the set of outcomes possible to be observed when \( E \) is measured. In this case, \( e_a(\omega) \) means the probability of observing the value \( a \in A \) in the measurement of \( E \) on a state \( \omega \). We assume in this paper that all measurements are with finite outcomes and composed of nonzero effects, and do not consider the trivial measurement \( \{ u \} \).

It is possible to represent a GPT in another way. GPTs with state spaces \( \Omega_1 \) and \( \Omega_2 \) are called equivalent if there exists an affine bijection (affine isomorphism) \( \psi \) such that \( \psi(\Omega_1) = \Omega_2 \). It is easy to show that if \( \Omega_1 \) and \( \Omega_2 \) are equivalent with an affine isomorphism \( \psi \), then \( \mathcal{E}(\Omega_2) = \mathcal{E}(\Omega_1) \circ \psi^{-1} \), and thus physical predictions are covariant (equivalent) in those GPTs. This allows us to assume that in a GPT with its state space \( \Omega \subset V \) the affine hull of \( \Omega \) does not include the origin \( O \) of \( V \), that is, \( O \notin \text{aff}(\Omega) \). We also assume for mathematical convenience that the dimension of the embedding vector space \( V \) satisfies \( \dim V = N + 1 \) (remember that \( N \) is the dimension of \( \text{aff}(\Omega) \)), and thus we can set \( V = \mathbb{R}^{N+1} \) with the standard Euclidean inner product \( (\cdot, \cdot)_E \) because any finite dimensional Hausdorff topological vector space is isomorphic linearly and topologically to the Euclidean space with the same dimension [35]. Note that by virtue of letting \( O \notin \text{aff}(\Omega) \) and \( \dim V = N + 1 \), any affine function on \( \Omega \) can be extended uniquely to a linear function on \( V \), so \( \mathcal{E}(\Omega) \subset V^* \cong \mathbb{R}^{N+1} \), where \( V^* \) is the dual space of \( V \).
For a state space $\Omega$, we define the set $V_+ \subset V$ as $V_+ := \{ x \in V \mid x = \lambda \omega, \omega \in \Omega, \lambda \geq 0 \}$ and call $V_+$ the positive cone generated by $\Omega$. Physically, $V_+$ represents the set of all “unnormalized” states, which are not necessarily mapped to 1 by $u$. We also define the cone $V_+^*$ dual to $V_+$ as $V_+^* := \{ y \in V^* \mid y(x) \geq 0 \text{ for all } x \in V_+ \}$. It is obvious that $\mathcal{E}(\Omega) = V_+^* \cap (u - V_+^*)$, and an effect $e \in \mathcal{E}(\Omega)$ is indecomposable if and only if $e$ is on an extremal ray of $V_+^*$.2

In the remaining of this paper, we follow mainly those assumptions and notations described above.

2.2 Physical equivalence of pure states

It is known that in quantum theory all pure states are physically equivalent via unitary (and antiunitary) transformations [10]. Similar notion to this physical equivalence of pure states can be introduced also in GPTs.

Let $\Omega$ be a state space. A map $T : \Omega \to \Omega$ is called a state automorphism on $\Omega$ if $T$ is an affine bijection. We denote the set of all state automorphisms on $\Omega$ by $GL(\Omega)$, and say that a state $\omega_1 \in \Omega$ is physically equivalent to a state $\omega_2 \in \Omega$ if there exists a $T \in GL(\Omega)$ such that $T \omega_1 = \omega_2$. It is shown in [29] that the physical equivalence of $\omega_1, \omega_2 \in \Omega$ is equal to the existence of some unit-preserving affine bijection $T' : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ satisfying $e(\omega_1) = T'(e)(\omega_2)$ for all $e \in \mathcal{E}(\Omega)$, which means $\omega_1$ and $\omega_2$ have the same physical contents on measurements. Because any affine map on $\Omega$ can be extended uniquely to a linear map on $V$, it holds that $GL(\Omega) = \{ T : V \to V \mid T : \text{linear, bijective, } T(\Omega) = \Omega \}$. It is clear that $GL(\Omega)$ forms a group, and we can represent the notion of physical equivalence of pure states by means of the transitive action of $GL(\Omega)$ on $\Omega^\text{ext}$.

Definition 2.2.1 (Transitive state space)

A state space $\Omega$ is called transitive if $GL(\Omega)$ acts transitively on $\Omega^\text{ext}$, that is, for any pair of pure states $\omega_i^\text{ext}, \omega_j^\text{ext} \in \Omega^\text{ext}$ there exists an affine bijection $T_{ji} \in GL(\Omega)$ such that $\omega_j^\text{ext} = T_{ji} \omega_i^\text{ext}$.

We remark that the equivalence of pure states does not depend on how the theory is expressed. In fact, when $\Omega$ is a transitive state space and $\Omega' := \psi(\Omega_1)$ is equivalent to $\Omega$ with a linear bijection $\psi$, it is easy to check that $GL(\Omega') = \psi \circ GL(\Omega) \circ \psi^{-1}$ and $\Omega'$ is also transitive.

In the remaining of this subsection, we let $\Omega$ be a transitive state space. In a transitive state space, we can introduce successfully the maximally mixed state as a unique invariant state with respect to every state automorphism.

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2A ray $E \subset V_+^*$ is called an extremal ray of $V_+^*$ if $x \in E$ and $x = y + z$ with $y, z \in V_+^*$ imply $y, z \in E$. 

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Proposition 2.2.2 ([36])
For a transitive state space \( \Omega \), there exists a unique state \( \omega_M \in \Omega \) (which we call the maximally mixed state) such that \( T \omega_M = \omega_M \) for all \( T \in GL(\Omega) \).

The unique maximally mixed state \( \omega_M \) is given by

\[
\omega_M = \int_{GL(\Omega)} T \omega^{\text{ext}} d\mu(T),
\]

where \( \omega^{\text{ext}} \) is an arbitrary pure state and \( \mu \) is the normalized two-sided invariant Haar measure on \( GL(\Omega) \).

Note in Proposition 2.2.2 that the transitivity of \( \Omega \) guarantees the independence of \( \omega_M \) on the choice of \( \omega^{\text{ext}} \). When \( \Omega^{\text{ext}} \) is finite and \( \Omega^{\text{ext}} = \{ \omega^{\text{ext}}_i \}_{i=1}^n \), \( \omega_M \) has a simpler form

\[
\omega_M = \frac{1}{n} \sum_{i=1}^n \omega^{\text{ext}}_i.
\]

We should recall that the action of the linear bijection \( \frac{1}{\|\omega_M\|_E} \mathbb{1}_V \) on \( \Omega \) does not change the theory, where \( \|\omega_M\|_E = (\omega_M, \omega_M)_E^{1/2} \) and \( \mathbb{1}_V \) is the identity map on \( V \). We can see that this rescaling makes \( GL(\Omega) \) invariant, and the unique maximally mixed state of the rescaled state space is \( \frac{1}{\|\omega_M\|_E} \omega_M \). In the remaining of this paper, when a transitive state space is discussed, we apply this rescaling and assume that \( \|\omega_M\|_E = 1 \) holds.

The Haar measure \( \mu \) on \( GL(\Omega) \) makes it possible for us to construct a convenient representation of the theory. First of all, we define an inner product \( \langle \cdot, \cdot \rangle_{GL(\Omega)} \) on \( V \) as

\[
\langle x, y \rangle_{GL(\Omega)} := \int_{GL(\Omega)} (Tx, Ty)_E \ d\mu(T) \quad (x, y \in V).
\]

Remark that in this paper we adopt \( \langle \cdot, \cdot \rangle_E \) as the reference inner product of \( \langle \cdot, \cdot \rangle_{GL(\Omega)} \) although the following discussion still holds even if it is not \( \langle \cdot, \cdot \rangle_E \).

Thanks to the properties of the Haar measure \( \mu \), it holds that

\[
\langle Tx, Ty \rangle_{GL(\Omega)} = \langle x, y \rangle_{GL(\Omega)} \quad \forall T \in GL(\Omega),
\]

which proves any \( T \in GL(\Omega) \) to be an orthogonal transformation on \( V \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{GL(\Omega)} \). Therefore, together with the transitivity of \( \Omega \), we can see that all pure states of \( \Omega \) are of equal norm, that is,

\[
\|\omega_i^{\text{ext}}\|_{GL(\Omega)} = \langle \omega_i^{\text{ext}}, \omega_i^{\text{ext}} \rangle_{GL(\Omega)}^{1/2} = \langle T_0 \omega_0^{\text{ext}}, T_0 \omega_0^{\text{ext}} \rangle_{GL(\Omega)}^{1/2} = \langle \omega_0^{\text{ext}}, \omega_0^{\text{ext}} \rangle_{GL(\Omega)}^{1/2} \quad \text{(2.2.1)}
\]

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holds for all $\omega^\text{ext}_i \in \Omega^\text{ext}$, where $\omega^\text{ext}_0$ is an arbitrary reference pure state. We remark that when $\|\omega_M\|_E = 1$, we can obtain from the invariance of $\omega_M$ for $GL(\Omega)$

\[
\|\omega_M\|^2_{GL(\Omega)} = \int_{GL(\Omega)} (T\omega_M, T\omega_M)_E \, d\mu(T) \\
= \int_{GL(\Omega)} (\omega_M, \omega_M)_E \, d\mu(T) \\
= \|\omega_M\|^2_E \int_{GL(\Omega)} \, d\mu(T) \\
= \|\omega_M\|^2_E,
\]

and thus $\|\omega_M\|_{GL(\Omega)} = 1$. The next proposition allows us to give a useful representation of the theory (the proof is given in Appendix A).

**Proposition 2.2.3**

For a transitive state space $\Omega$, there exists a basis $\{v_l\}_{l=1}^{N+1}$ of $V$ orthonormal with respect to the inner product $(\cdot, \cdot)_{GL(\Omega)}$ such that $v_{N+1} = \omega_M$ and

\[
x \in \text{aff}(\Omega) \iff x = \sum_{l=1}^{N} a_l v_l + v_{N+1} = \sum_{l=1}^{N} a_l v_l + \omega_M \quad (a_1, \cdots, a_N \in \mathbb{R}).
\]

By employing the representation shown in Proposition 2.2.3, an arbitrary $x \in \text{aff}(\Omega)$ can be written as a vector form that

\[
x = \begin{pmatrix} x \\ 1 \end{pmatrix} \quad \text{with} \quad \omega_M = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

(2.2.2)

where the vector $x$ is sometimes called the **Bloch vector** [37, 32] corresponding to $x$.

### 2.3 Self-duality

In this part, we introduce the notion of self-duality, which plays an important role in our work. We also describe some examples of GPTs with relevant structures to transitivity or self-duality.

Let $V_+$ be the positive cone generated by a state space $\Omega$. We define the **internal dual cone** of $V_+$ relative to an inner product $(\cdot, \cdot)$ on $V$ as $V^\text{int}_+ := \{y \in V \mid (x, y) \geq 0, \forall x \in V_+\}$, which is isomorphic to the dual cone $V^*_+$ because of the Riesz representation theorem [33]. The self-duality of $V_+$ can be defined as follows.

**Definition 2.3.1 (Self-duality)**

$V_+$ is called **self-dual** if there exists an inner product $(\cdot, \cdot)$ on $V$ such that $V_+ = V^\text{int}_+$. 

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We remark similarly to Definition 2.2.1 that if $V_+$ generated by a state space $Ω$ is self-dual, then the cone $V'_+$ generated by $Ω' := ψ(Ω)$ with a linear bijection $ψ$ (i.e. $V'_+ = ψ(V_+)$) is also self-dual. In fact, we can confirm that if $V_+ = V'^{\text{int}}_{+}$ holds for some inner product $\langle \cdot, \cdot \rangle$, then $V'_+ = V'^{\text{int}}_{+}$ holds, where the inner product $\langle \cdot, \cdot \rangle'$ is defined as $(x, y)' = (ψ^{-1}x, ψ^{-1}y)$ $(x, y ∈ V)$.

Let us consider the case when $Ω$ is transitive and $V_+$ is self-dual with respect to the inner product $\langle \cdot, \cdot \rangle_{GL(Ω)}$. Since $V_+ = V'^{\text{int}}_{+}$, we can regard $V_+$ also as the set of unnormalized effects. In particular, every pure state $ω^\text{ext}_i ∈ Ω^\text{ext}$ can be considered as an unnormalized effect, and if we define

$$e_i := \frac{ω^\text{ext}_i}{\|ω^\text{ext}_i\|_{GL(Ω)}^2} = \frac{ω^\text{ext}_0}{\|ω^\text{ext}_0\|_{GL(Ω)}^2}, \quad (2.3.1)$$

then from Cauchy-Schwarz inequality

$$\langle e_i, ω^\text{ext}_k \rangle_{GL(Ω)} ≤ \|e_i\|_{GL(Ω)} \|ω^\text{ext}_k\|_{GL(Ω)} = 1$$

holds for any pure state $ω^\text{ext}_k ∈ Ω^\text{ext}$ (thus $e_i$ is indeed an effect). The equality holds if and only if $ω^\text{ext}_k$ is parallel to $e_i$, i.e. $ω^\text{ext}_k = ω^\text{ext}_0$, and we can also conclude that an effect is pure and indecomposable if and only if it is of the form defined as (2.3.1) together with the fact that effects on the extremal rays of $V'^{\text{int}}_{+}$ are indecomposable (for more details see [30]). Therefore, we can rewrite (2.3.1) as

$$e_i^\text{ext} := \frac{ω^\text{ext}_i}{\|ω^\text{ext}_i\|_{GL(Ω)}^2} = \frac{ω^\text{ext}_0}{\|ω^\text{ext}_0\|_{GL(Ω)}^2} \cdot (2.3.2)$$

When $|Ω^\text{ext}| < ∞$, it is sufficient for the discussion above that $Ω$ is transitive and self-dual with respect to an arbitrary inner product.

**Proposition 2.3.2**

Let $Ω$ be transitive with $|Ω^\text{ext}| < ∞$ and $V_+$ be self-dual with respect to some inner product. There exists a linear bijection $Ξ: V → V$ such that $Ω' := ΞΩ$ is transitive and the generating positive cone $V'_+$ is self-dual with respect to $\langle \cdot, \cdot \rangle_{GL(Ω')}$, i.e. $V'_+ = V'^{\text{int}}_{+}$.

The proof is given in Appendix B. Proposition 2.3.2 reveals that if a theory with finite pure states is transitive and self-dual, then the theory can be expressed in the way it is self-dual with respect to $\langle \cdot, \cdot \rangle_{GL(Ω)}$.

In the following, we present some examples of GPTs with transitivity or self-duality.

**Example 2.3.3 (Finite dimensional classical theories)**

Let us denote by $Ω_{\text{CT}}$ the state space of a finite dimensional classical system. $Ω_{\text{CT}}$ can be represented by means of some finite $N ∈ \mathbb{N}$ as the set of all
probability distributions (probability vectors) \( \{ p = (p_1, \cdots, p_{N+1}) \} \subset V = \mathbb{R}^{N+1} \) on some sample space \( \{ a_1, \cdots, a_{N+1} \} \), i.e. \( \Omega_{\text{CT}} \) is the \( N \)-dimensional standard simplex. It is easy to justify that the set of all pure states \( \Omega_{\text{CT}}^\text{ext} \) is given by \( \Omega_{\text{CT}}^\text{ext} = \{ p^\text{ext}_{i} \}_{i=1}^{N+1} \), where \( p^\text{ext}_{i} \) is the probability distribution satisfying \( (p^\text{ext}_{i})_{j} = \delta_{ij} \), and the positive cone \( V_+ \) by \( V_+ = \{ \sigma = (\sigma_1, \cdots, \sigma_{N+1}) \in V \mid \sigma_i \geq 0, \forall i \} \). Remark that the set \( \{ p^\text{ext}_{i} \}_{i=1}^{N+1} = \{(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, 0, \cdots, 1)\} \) forms a standard orthonormal basis of \( V \), and thus any linear map on \( V \) is determined completely by its action on \( \Omega_{\text{CT}}^\text{ext} \). Since any state automorphism maps pure states to pure states, it can be seen that the set \( GL(\Omega_{\text{CT}}) \) of all state automorphisms on \( \Omega_{\text{CT}} \) is exactly the set of all permutation matrices with respect to the orthonormal basis \( \{ p^\text{ext}_{i} \}_{i=1}^{N+1} \) of \( V \). Therefore, \( \Omega_{\text{CT}} \) is a transitive state space, and any \( T \in GL(\Omega_{\text{CT}}) \) is orthogonal, which results in

\[
\langle x, y \rangle_{GL(\Omega_{\text{CT}})} = \int_{GL(\Omega_{\text{CT}})} (Tx, Ty) E \, d\mu(T) \\
= \int_{GL(\Omega_{\text{CT}})} (x, y) E \, d\mu(T) \\
= (x, y) E \int_{GL(\Omega_{\text{CT}})} d\mu(T) \\
= (x, y) E. \tag{2.3.3}
\]

The set of all positive linear functions on \( \Omega_{\text{CT}} \) can be identified with the internal dual cone \( V_+^{*\text{int} \subset+_(\cdot)E} \), and any \( h \in V_+^{*\text{int}} \) can be represented as \( h = (h(p^\text{ext}_1), \cdots, h(p^\text{ext}_{N+1})) \) with all entries nonnegative since

\[
h(p^\text{ext}_i) = (h, p^\text{ext}_i) = (h)_i \geq 0
\]

holds for all \( i \). Therefore, we can conclude together with (2.3.3) \( V_+ = V_+^{*\text{int} \subset+_(\cdot)E} = V_+^{*\text{int} \subset+_{GL(\Omega_{\text{CT}})}} \). Note that we can find the representation (2.2.2) to be valid for this situation by taking a proper basis of \( V = \mathbb{R}^{N+1} \) and normalization.

**Example 2.3.4 (Finite dimensional quantum theories)**

The state space of a finite dimensional quantum system denoted by \( \Omega_{\text{QT}} \) is the set of all density operators on \( N(< \infty) \) dimensional Hilbert space \( \mathcal{H} \), that is, \( \Omega_{\text{QT}} := \{ \rho \in \mathcal{L}(\mathcal{H}) \mid \rho \geq 0, \text{Tr}[\rho] = 1 \} \), where \( \mathcal{L}(\mathcal{H}) \) is the set of all self-adjoint operators on \( \mathcal{H} \). The set of all pure states \( \Omega_{\text{QT}}^\text{ext} \) is given by the rank-1 projections: \( \Omega_{\text{QT}}^\text{ext} = \{ |\psi\rangle \langle \psi | \mid |\psi\rangle \in \mathcal{H}, \langle \psi | \psi \rangle = 1 \} \). It has been demonstrated in [38] that with the identity operator \( \mathbb{1}_N \) on \( \mathcal{H} \) and the generators \( \{ \sigma_i \}_{i=1}^{N^2-1} \) of \( SU(N) \) satisfying

\[
\sigma_i \in \mathcal{L}(\mathcal{H}), \quad \text{Tr}[\sigma_i] = 0, \quad \text{Tr}[\sigma_i \sigma_j] = 2\delta_{ij}, \tag{2.3.4}
\]

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any $A \in \mathcal{L}_S(\mathcal{H})$ can be represented as

$$A = c_0 \mathbb{I}_N + \sum_{i=1}^{N^2 - 1} c_i \sigma_i \quad (c_0, c_1, \cdots, c_{N^2 - 1} \in \mathbb{R})$$

(2.3.5)

and any $B \in \text{aff}(\Omega_{QT})$ as

$$B = \frac{1}{N} \mathbb{I}_N + \sum_{i=1}^{N^2 - 1} c_i \sigma_i \quad (c_1, \cdots, c_{N^2 - 1} \in \mathbb{R}).$$

(2.3.6)

Since (2.3.4) implies that $\{\mathbb{I}_N, \sigma_1, \cdots, \sigma_{N^2 - 1}\}$ forms an orthogonal basis of $\mathcal{L}_S(\mathcal{H})$ with respect to the Hilbert-Schmidt inner product $\langle \cdot, \cdot \rangle_{HS}$ defined by

$$\langle X, Y \rangle_{HS} = \text{Tr}[X^\dagger Y],$$

and (2.3.5) and (2.3.6) prove $\dim(\mathcal{L}_S(\mathcal{H})) = \dim(\text{aff}(\Omega_{QT})) + 1$, it seems natural to consider $\Omega_{QT}$ to be embedded in $V = \mathcal{L}_S(\mathcal{H})$ equipped with $\langle \cdot, \cdot \rangle_{HS}$. Because it holds that

$$\mathcal{E}(\Omega_{QT}) = \{ E \in \mathcal{L}_S(\mathcal{H}) \mid 0 \leq \text{Tr}[E \rho] \leq 1, \forall \rho \in \Omega_{QT}\}$$

$$= \{ E \in \mathcal{L}_S(\mathcal{H}) \mid 0 \leq E \leq \mathbb{I}_N \},$$

we can see $V_+ = V_{+\langle \cdot, \cdot \rangle_{HS}}^{\text{int}} = \{ A \in \mathcal{L}_S(\mathcal{H}) \mid A \geq 0 \}$, and rank-1 projections mean pure and indecomposable effects in quantum theories.

On the other hand, it is known that in quantum theory any state automorphism is either a unitary or antiunitary transformation [10], and for any pair of pure states one can find a unitary operator which links them. Thus, $\Omega_{QT}$ is transitive, and any state automorphism is of the form

$$\rho \mapsto U \rho U^\dagger \quad \forall \rho \in \Omega_{QT},$$

where $U$ is unitary or antiunitary. Considering that

$$\langle UXU^\dagger, UYYU^\dagger \rangle_{HS} = \text{Tr}[UXU^\dagger UYYU^\dagger]$$

$$= \text{Tr}[X^\dagger Y]$$

$$= \langle X, Y \rangle_{HS}$$

holds for any unitary or antiunitary operator $U$, we can obtain in a similar way to (2.3.3)

$$\langle X, Y \rangle_{\text{GL}(\Omega_{QT})} = \langle X, Y \rangle_{HS}.$$  

(2.3.7)

Therefore, we can conclude $V_+ = V_{+\langle \cdot, \cdot \rangle_{HS}}^{\text{int}} = V_{+\langle \cdot, \cdot \rangle_{\text{GL}(\Omega_{QT})}}^{\text{int}}$. We remark similarly to the classical cases that we may rewrite (2.3.6) as (2.2.2) by taking a suitable normalization and considering that $\omega_M = \mathbb{I}_N/N$. 

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Example 2.3.5 (Regular polygon theories)

If the state space of a GPT is in the shape of a regular polygon with \( n \) sides, then we call it a regular polygon theory and denote the state space by \( \Omega_n \). We set \( V = \mathbb{R}^3 \) when considering regular polygon theories, and it can be seen in [39] that the pure states of \( \Omega_n \) are described as

\[
\Omega_n^{\text{ext}} = \{ \omega_n^{\text{ext}}(i) \}_{i=0}^{n-1}
\]

with

\[
\omega_n^{\text{ext}}(i) = \left( \begin{array}{c} r_n \cos\left( \frac{2\pi i}{n} \right) \\ r_n \sin\left( \frac{2\pi i}{n} \right) \\ 1 \end{array} \right), \\
 r_n = \sqrt{\frac{1}{\cos\left( \frac{\pi}{n} \right)}}
\]

(2.3.8)

when \( n \) is finite, and when \( n = \infty \) (the state space \( \Omega_\infty \) is a disc),

\[
\Omega_\infty^{\text{ext}} = \{ \omega_\infty^{\text{ext}}(\theta) \}_{\theta \in [0,2\pi]}
\]

with

\[
\omega_\infty^{\text{ext}}(\theta) = \left( \begin{array}{c} \cos \theta \\ \sin \theta \\ 1 \end{array} \right)
\]

(2.3.9)

The state space \( \Omega_3 \) represents a classical trit system (the 2-dimensional standard simplex), while \( \Omega_\infty \) represents a qubit system with real coefficients since the unit disc can be considered to be an equatorial plane of the Bloch ball. Regular polygon theories can be regarded as intermediate theories of those theories [40].

The state space of the regular polygon theory with \( n \) sides (including \( n = \infty \)) defines its positive cone \( V_+ \), and it is also shown in [39] that the corresponding internal dual cone \( V_+^{\text{int}} \subset \mathbb{R}^3 \) is given by the conic hull\(^3\) of the following extreme effects (in fact, those effects are also indecomposable)

\[
e_n^{\text{ext}}(i) = \frac{1}{2} \left( \begin{array}{c} r_n \cos\left( \frac{(2i-1)\pi}{n} \right) \\ r_n \sin\left( \frac{(2i-1)\pi}{n} \right) \\ 1 \end{array} \right), \\
i = 0, 1, \ldots, n - 1 \ (n : \text{even}) ;
\]

\[
e_n^{\text{ext}}(i) = \frac{1}{1 + r_n^2} \left( \begin{array}{c} r_n \cos\left( \frac{2\pi i}{n} \right) \\ r_n \sin\left( \frac{2\pi i}{n} \right) \\ 1 \end{array} \right), \\
i = 0, 1, \ldots, n - 1 \ (n : \text{odd}) ;
\]

\[
e_\infty^{\text{ext}}(\theta) = \frac{1}{2} \left( \begin{array}{c} \cos \theta \\ \sin \theta \\ 1 \end{array} \right), \ \ \theta \in [0,2\pi) \ (n = \infty).
\]

(2.3.10)

\(^3\)The conic hull of a set \( X \) is defined by \( \text{cone}(X) := \{ \sum_{k=1}^{N} \theta_k x_k : \theta_k \in \mathbb{R}, x_k \in X, \theta_k \geq 0 \} \).
Moreover, for finite \( n \), we can see that the group \( GL(\Omega_n) \) (named the dihedral group) is composed of orthogonal transformations with respect to \( \langle \cdot, \cdot \rangle_E \) \cite{41}, which also holds for \( n = \infty \). Similar calculations to (2.3.3) or (2.3.7) demonstrate \( \langle \cdot, \cdot \rangle_E = \langle \cdot, \cdot \rangle_{GL(\Omega)} \) for \( n = 3, 4, \cdots, \infty \). Therefore, from (2.3.8) - (2.3.10), we can conclude that \( V_+ \) is self-dual, i.e. \( V_+ = V^\text{int}_{+\langle \cdot, \cdot \rangle_E} = V^\text{int}_{+\langle \cdot, \cdot \rangle_{GL(\Omega_n)}} \), when \( n \) is odd or \( \infty \), while \( V_+ \) is not identical but only isomorphic to \( V^\text{int}_{+\langle \cdot, \cdot \rangle_{GL(\Omega_n)}} \) when \( n \) is even (in that case, \( V_+ \) is called weakly self-dual \cite{28, 39}).

### 3 Preparation Uncertainty and Measurement Uncertainty in a Class of GPTs

In this section, our main results on the relations between preparation uncertainty and measurement uncertainty are given in GPTs with transitivity and self-duality with respect to \( \langle \cdot, \cdot \rangle_{GL(\Omega)} \). Measures quantifying the width of a probability distribution or measurement error are also given to formulate those results. Throughout this section, we consider measurements whose sample spaces are finite metric spaces.

#### 3.1 Widths of probability distributions

In this subsection, we give two kinds of measure to quantify how concentrated a probability distribution is.

Let \( A \) be a finite metric space equipped with a metric function \( d_A \), and \( O_{d_A}(a; w) \) be the ball defined by \( O_{d_A}(a; w) := \{ x \in A : d_A(x, a) \leq w/2 \} \). For \( \epsilon \in [0, 1] \) and a probability distribution \( p \) on \( A \), we define the overall width (at confidence level \( 1 - \epsilon \)) \cite{15, 16} as

\[
W_\epsilon(p) := \inf \{ w > 0 : \exists a \in A : p(O_{d_A}(a; w)) \geq 1 - \epsilon \}.
\]

We can give another formulation for the width of \( p \). We define the minimum localization error \cite{15} of \( p \) as

\[
LE(p) := 1 - \max_{a \in A} p(a).
\]

Both (3.1.1) and (3.1.2) can be applied to probability distributions observed in physical experiments. Let us consider a GPT with \( \Omega \) being its state space. For a state \( \omega \in \Omega \) and a measurement \( F = \{ f_a \}_{a \in A} \) on \( A \), we denote by \( \omega^F \) the probability distribution obtained by the measurements of \( F \) on \( \omega \), i.e.

\[
\omega^F := \{ f_a(\omega) \}_{a \in A}.
\]

The overall width and minimum localization error for \( \omega^F \) can be defined as

\[
W_\epsilon(\omega^F) := \inf \{ w > 0 : \exists a \in A : \sum_{a' \in O_{d_A}(a; w)} f_{a'}(\omega) \geq 1 - \epsilon \}.
\]
and

\[ LE(\omega^F) := 1 - \max_{a \in A} f_a(\omega) \] (3.1.4)

respectively. Note that as in [15, 16], overall widths can be defined properly even if the sample spaces of probability distributions are infinite. For example, overall widths are considered in [16] for probability measures on \( \mathbb{R} \) derived from the measurement of position or momentum of a particle.

Those two measures above are used for the mathematical description of preparation uncertainty relations (PURs). As a simple example, we consider a qubit system with Hilbert space \( \mathcal{H} = \mathbb{C}^2 \). For two PVMs \( Z = \{ |0\rangle\langle 0|, |1\rangle\langle 1| \} \) and \( X = \{ |+\rangle\langle +|, |\times\rangle\langle \times| \} \), where \( \{|0\rangle, |1\rangle\} \) and \( \{|+\rangle, |\times\rangle\} = \{ \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \} \) are the z-basis and x-basis of \( \mathcal{H} \) respectively, it holds from [5, 8] that

\[ LE(\rho^Z) + LE(\rho^X) \geq 1 - \frac{1}{\sqrt{2}} > 0 \] (3.1.5)

for any state \( \rho \). The inequality (3.1.5) shows that there is no state \( \rho \) which makes both \( LE(\rho^Z) \) and \( LE(\rho^X) \) zero, that is, \( \rho^Z \) and \( \rho^X \) cannot be localized simultaneously even if the measurements are ideal ones (PVMs). PURs in terms of overall widths were also discussed in [16] for the position and momentum observables.

### 3.2 Measurement error

In this subsection, we introduce the concept of measurement error in GPTs, which derives from joint measurement problems, and describe how to quantify it.

Let us consider a GPT with its state space \( \Omega \), and two measurements \( F = \{ f_a \}_{a \in A} \) and \( G = \{ g_b \}_{b \in B} \) on \( \Omega \). We call \( F \) and \( G \) are jointly measurable (compatible) if there exists a joint measurement \( M^{FG} = \{ m_{ab}^{FG} \}_{(a,b) \in A \times B} \) of \( F \) and \( G \) satisfying

\[
\sum_{b \in B} m_{ab}^{FG} = f_a \quad \text{for all } a \in A
\]
\[
\sum_{a \in A} m_{ab}^{FG} = g_b \quad \text{for all } b \in B,
\]

and if \( F \) and \( G \) are not jointly measurable, then they are called incompatible [42, 43]. It was shown in [44] that all measurements are jointly measurable if and only if the theory is a simplex, i.e. a classical theory. Thus, in most GPTs, there exist pairs of measurements which are incompatible, but we can nevertheless conduct their approximate joint measurements allowing measurement error. Assume that \( F \) and \( G \) are incompatible. It is known that one way to compose their approximate joint measurement is adding some trivial noise to them. To see this, we consider as a simple
example the incompatible pair of measurements $Z = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ and $X = \{|+\rangle\langle +|, |\rangle\langle -|\}$ in a qubit system described in the last subsection. It was demonstrated in [45] that the measurements

$$
\tilde{Z}^\lambda := \lambda Z + (1 - \lambda)I
$$

$$
= \left\{ \lambda |0\rangle\langle 0| + \frac{1 - \lambda}{2} I_2, \lambda |1\rangle\langle 1| + \frac{1 - \lambda}{2} I_2 \right\}
$$

$$
\tilde{X}^\lambda := \lambda X + (1 - \lambda)I
$$

$$
= \left\{ \lambda |+\rangle\langle +| + \frac{1 - \lambda}{2} I_2, \lambda |\rangle\langle -| + \frac{1 - \lambda}{2} I_2 \right\}
$$

are jointly measurable for $0 \leq \lambda \leq \frac{1}{\sqrt{2}}$, where $I := \{I_2/2, I_2/2\}$ is a trivial measurement. The joint measurability of (3.2.1) implies that the addition of trivial noise described by a trivial observable makes incompatible measurements compatible in an approximate way. In fact, it is observed also in GPTs that adding trivial noise results in approximate joint measurements of incompatible measurements [43, 45, 46].

Because the notion of measurement error derives from the difference between ideal and approximate measurements as discussed above, we have to define ideal measurements in GPTs in order to quantify measurement error. In this paper, they are defined in an analogical way with the ones in finite dimensional quantum theories, where PVMs are considered to be ideal [10]. If we denote a PVM by $E = \{P_a\}_a$, then each effect is of the form

$$
P_a = \sum_{\tilde{t}(a)} |\psi_{(a)}\rangle \langle \psi_{(a)}|.
$$

In particular, every effect is a sum of pure and indecomposable effects, and we call in a similar way a measurement $F = \{f_a\}_a \in A$ on $\Omega$ ideal if each effect $f_a$ satisfies

$$
f_a = \sum_{\tilde{t}(a)} e^\text{ext}_{(a)}, \quad \text{or} \quad f_a = u - \sum_{\tilde{t}(a)} e^\text{ext}_{(a)}, \quad (3.2.2)
$$

where we should recall that the set of all pure and indecomposable effects is denoted by $\{e^\text{ext}_i \}_{i}$ and we do not consider the trivial measurement $F = \{u\}$. It is easy to see that measurements defined as (3.2.2) result in PVMs in finite dimensional quantum theories. This type of measurement was considered also in [47].

The introduction of ideal measurements makes it possible for us to quantify measurement error. Consider an ideal measurement $F = \{f_a\}_a$ and a general measurement $\tilde{F} = \{\tilde{f}_a\}_a$, and suppose similarly to the previous subsection that $A$ is a finite metric space with a metric $d_A$. $F$ and $\tilde{F}$ may be understood as measurements to be measured ideally and measured actually respectively. Taking into consideration the fact that for each nonzero pure
effect there exists at least one state which is mapped to 1 (an “eigenstate” [30]), we can define for \( \epsilon \in [0, 1] \) the error bar width of \( \hat{F} \) relative to \( F \) [15, 16] as

\[
W_{\epsilon}(\hat{F}, F) = \inf \{ w > 0 \mid \forall a \in A, \forall \omega \in \Omega : \sum_{a' \in O_{d_A}(a; w)} \hat{f}_{a'}(\omega) \geq 1 - \epsilon \}. \tag{3.2.3}
\]

\( W_{\epsilon}(\hat{F}, F) \) represents the spread of probabilities around the “eigenvalues” of \( F \) observed when the corresponding “eigenstates” of \( F \) are measured by \( \hat{F} \), and thus it can be thought to be one of the quantifications of measurement error. Note that although error bar widths in general (not necessarily finite) metric spaces were defined in [16], we consider only finite metric spaces in this paper, so we employ their convenient forms (3.2.3) in finite metric spaces shown in [15]. Another measure is the one given by Werner [19] as the difference of expectation values of “slowly varying functions” on the probability distributions obtained when \( F \) and \( \hat{F} \) are measured. It is defined as

\[
D_{\hat{W}}(\hat{F}, F) := \sup_{\omega \in \Omega} \sup_{h \in \Lambda} \left| (\hat{F}[h])(\omega) - (F[h])(\omega) \right|, \tag{3.2.4}
\]

where

\[
\Lambda := \{ h : A \to \mathbb{R} \mid |h(a_1) - h(a_2)| \leq d_A(a_1, a_2), \forall a_1, a_2 \in A \}
\]

is the set of all “slowly varying functions” (called the Lipshitz ball of \( (A, d_A) \)) and

\[
F[h] := \sum_{a \in A} h(a)f_a
\]

is a map which gives the expectation value of \( h \in \Lambda \) when \( F \) is measured on a state \( \omega \) (similarly for \( \hat{F}[h] \)). There is known a simple relation between (3.2.3) and (3.2.4).

**Proposition 3.2.1 ([15, 16])**

For \( \epsilon \in (0, 1) \),

\[
W_{\epsilon}(\hat{F}, F) \leq \frac{2}{\epsilon} D_{\hat{W}}(\hat{F}, F)
\]

holds for any pair of an ideal measurement \( F \) and a general measurement \( \hat{F} \).

On the other hand, there can be introduced a more intuitive quantification of measurement error called \( l_{\infty} \) distance [18]:

\[
D_{\infty}(\hat{F}, F) := \sup_{\omega \in \Omega} \max_{a \in A} \left| \hat{f}_a(\omega) - f_a(\omega) \right|. \tag{3.2.5}
\]

By means of those quantifications of measurement error above, we can formulate measurement uncertainty relations (MURs). As an illustration,
we again consider the joint measurement problem of incompatible measurements $Z$ and $X$ in a qubit system. Suppose that $\tilde{M}^{ZX}$ is an approximate joint measurement of $Z$ and $X$, and $\tilde{M}^Z$ and $\tilde{M}^X$ are its marginal measurements corresponding to $Z$ and $X$ respectively. It was proved in [18] that

\[ D_\infty(\tilde{M}^Z, Z) + D_\infty(\tilde{M}^X, X) \geq 1 - \frac{1}{\sqrt{2}} > 0. \]  

(3.2.6) gives a quantitative representation of the incompatibility of $Z$ and $X$ that $D_\infty(\tilde{M}^Z, Z)$ and $D_\infty(\tilde{M}^X, X)$ cannot be simultaneously zero, that is, measurement error must occur when conducting any approximate joint measurement of $Z$ and $X$ (see [17] for another inequality). MURs for the position and momentum observables were given in [16] and [19] in terms of (3.2.3) and (3.2.4) respectively.

### 3.3 Main results: relations between preparation uncertainty and measurement uncertainty

In the previous subsections, we have introduced several measures to review two kinds of uncertainty, preparation uncertainty and measurement uncertainty. In this part, we shall manifest as our main results how they are related with each other in GPTs, which is a generalization of the quantum ones in [15].

Before demonstrating our main theorems, we confirm the physical settings and mathematical assumptions to state them. In the following, we focus on a GPT with $\Omega$ being its state space, and suppose that $\Omega$ is transitive and its positive cone $V_+$ is self-dual with respect to $\langle \cdot, \cdot \rangle_{GL(\Omega)}$. In addition, we consider ideal measurements $F = \{f_a\}_{a \in A}$ and $G = \{g_b\}_{b \in B}$ on $\Omega$, whose sample spaces are finite metric spaces $(A, d_A)$ and $(B, d_B)$ respectively, and consider a measurement $\tilde{M}^{FG} := \{\tilde{m}_{ab}^{FG}\}_{(a,b) \in A \times B}$ as an approximate joint measurement of $F$ and $G$, whose marginal measurements are given by

\[
\tilde{M}^F := \{\tilde{m}_a^F\}_a, \quad \tilde{m}_a^F := \sum_{b \in B} \tilde{m}_{ab}^{FG}; \\
\tilde{M}^G := \{\tilde{m}_b^G\}_b, \quad \tilde{m}_b^G := \sum_{a \in A} \tilde{m}_{ab}^{FG}.
\]

Remember that as shown in Subsection 3.2 the ideal measurement $F = \{f_a\}_a$ satisfies

\[
f_a = \sum_{i(a)} e_{i(a)}^{\text{ext}}, \quad \text{or} \quad f_a = u - \sum_{i(a)} e_{i(a)}^{\text{ext}} \]  

(3.3.1) in terms of the pure and indecomposable effects $\{e_{i(a)}^{\text{ext}}\}_i$ shown in (2.3.2) (similarly for $G = \{g_b\}_b$). The following lemma is needed to prove our main results.
Lemma 3.3.1
If \( \Omega \) is transitive, then the unit effect \( u \in V_{\text{fin}}^\ast \subset V \) is identical to the maximally mixed state \( \omega_M \), i.e. \( u = \omega_M \).

Lemma 3.3.1 is an easy consequence of Proposition 2.2.3. In fact, (2.2.2) gives
\[
u = \omega_M = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Now, we can state our main theorems connecting PURs and MURs. Note that one of us [15] proved similar results to ours for finite dimensional quantum theories. Because GPTs shown above include those theories, our theorems can be considered to demonstrate that the relations between PURs and MURs introduced in [15] are more general ones.

Theorem 3.3.2
Let \( \Omega \) be a transitive state space and its positive cone \( V_+ \) be self-dual with respect to \( \langle \cdot, \cdot \rangle_{GL(\Omega)} \), and let \( (F, G) \) be a pair of ideal measurements on \( \Omega \). For an arbitrary approximate joint measurement \( \tilde{M}^{FG} \) of \( (F, G) \) and \( \epsilon_1, \epsilon_2 \in [0, 1] \) satisfying \( \epsilon_1 + \epsilon_2 \leq 1 \), there exists a state \( \omega \in \Omega \) such that
\[
W_{\epsilon_1}(\tilde{M}^F, F) \geq W_{\epsilon_1+\epsilon_2}(\omega^F) \\
W_{\epsilon_2}(\tilde{M}^G, G) \geq W_{\epsilon_1+\epsilon_2}(\omega^G).
\]

Theorem 3.3.2 manifests that if one cannot make both \( W_{\epsilon_1+\epsilon_2}(\omega^F) \) and \( W_{\epsilon_1+\epsilon_2}(\omega^G) \) vanish, then one also cannot make both \( W_{\epsilon_1}(\tilde{M}^F, F) \) and \( W_{\epsilon_2}(\tilde{M}^G, G) \) vanish. That is, if there exists a PUR, then there also exists a MUR. Moreover, Theorem 3.3.2 also demonstrates that bounds for MURs in terms of error bar widths can be given by ones for PURs described by overall widths.

Proof (Proof of Theorem 3.3.2)
In this proof, we denote the inner product \( \langle \cdot, \cdot \rangle_{GL(\Omega)} \) and the norm \( \| \cdot \|_{GL(\Omega)} \) simply by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) respectively.

Since we assume that \( f_a \) in (3.3.1) is an effect (thus \( u - f_a \) is also an effect), \( \sum_{i(a)} \epsilon^\text{ext}_{i(a)} \) is an effect and it satisfies \( 0 \leq \langle \sum_{i(a)} \epsilon^\text{ext}_{i(a)}, \omega \rangle \leq 1 \) for any state \( \omega \in \Omega \). However, if we act \( \sum_{i(a)} \epsilon^\text{ext}_{i(a)} \) on the pure state \( \omega^\text{ext}_{j(a)} \), then (2.3.2) shows that \( \langle \epsilon^\text{ext}_{i(a)}, \omega^\text{ext}_{j(a)} \rangle = 1 \), and thus we have
\[
\langle \epsilon^\text{ext}_{i(a)}, \omega^\text{ext}_{j(a)} \rangle = 0 \quad \text{for} \quad i(a) \neq j(a),
\]
that is,
\[
\langle \epsilon^\text{ext}_{i(a)}, \epsilon^\text{ext}_{j(a)} \rangle = 0 \quad \text{for} \quad i(a) \neq j(a). \tag{3.3.2}
\]

Because
\[
\langle \epsilon^\text{ext}_{i(a)}, \epsilon^\text{ext}_{i(a)} \rangle = \frac{1}{\|\omega^\text{ext}_{0}\|^2} \quad \text{and} \quad \langle u, \epsilon^\text{ext}_{i(a)} \rangle = \frac{1}{\|\omega^\text{ext}_{0}\|},
\]

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hold from (2.3.2), we obtain
\[
\left< \sum_{i(a)} e^\text{ext}_{i(a)}, \sum_{i(a)} e^\text{ext}_{i(a)} \right> = \frac{\#i(a)}{\omega_0^\text{ext}}
\]
\[
\left< u, \sum_{i(a)} e^\text{ext}_{i(a)} \right> = \frac{\#i(a)}{\omega_0^\text{ext}}
\]
\[
\left< u - \sum_{i(a)} e^\text{ext}_{i(a)}, u - \sum_{i(a)} e^\text{ext}_{i(a)} \right> = 1 - \frac{\#i(a)}{\omega_0^\text{ext}}
\]
\[
\left< u, u - \sum_{i(a)} e^\text{ext}_{i(a)} \right> = 1 - \frac{\#i(a)}{\omega_0^\text{ext}}
\]  
(3.3.3)

where \((\#i(a))\) is the number of elements of the index set \(\{i(a)\}\) and we use \(\left< u, u \right> = \left< u, \omega_M \right> = 1\).

On the other hand, for any element \(e \in V^\text{int} + \langle \cdot, \cdot \rangle\), \(e/\langle u, e \rangle\) defines a state because \(V_+ = V^\text{int} \). In particular, for the effect \(f_a = \sum_{i(a)} e^\text{ext}_{i(a)}\) or \(u - \sum_{i(a)} e^\text{ext}_{i(a)}\), we can see from (3.3.3) that
\[
\left< f_a, \frac{f_a}{\langle u, f_a \rangle} \right> = 1 \quad (3.3.4)
\]

holds, that is, the state \(f_a/\langle u, f_a \rangle\) is an eigenstate of \(f_a\) (similarly for \(g_b\)). Therefore, considering that any \(a\) satisfies (3.3.4), we have for any \(w_1 \geq W_{\epsilon_1}(\bar{M}_F, F)\)
\[
\sum_{a' \in O_{dA}(a; w_1)} \left< \tilde{m}_{d'}^{a'}, \frac{f_a}{\langle u, f_a \rangle} \right> \geq 1 - \epsilon_1,
\]
equivalently,
\[
\sum_{b \in B} \sum_{a' \in O_{dA}(a; w_1)} \left< \tilde{m}_{d'}^{FG}, \frac{f_a}{\langle u, f_a \rangle} \right> \geq 1 - \epsilon_1
\]
for all \(a \in A\) because of the definition of \(W_{\epsilon_1}(\bar{M}_F, F)\) (3.2.3). Multiplying both sides by \(\langle u, f_a \rangle = \langle \omega_M, f_a \rangle(> 0)\) and taking the summation over \(a\) yield
\[
\sum_{a \in A} \sum_{b \in B} \sum_{a' \in O_{dA}(a; w_1)} \left< \tilde{m}_{d'}^{FG}, f_a \right> \geq 1 - \epsilon_1,
\]
or
\[
\sum_{a' \in A} \sum_{b \in B} \sum_{a \in O_{dA}(a'; w_1)} \left< \tilde{m}_{d'}^{FG}, f_a \right> \geq 1 - \epsilon_1,
\]
where we use the relations that \(\sum_{a \in A} \langle u, f_a \rangle = \langle u, u \rangle = 1\) and
\[
\sum_{a \in A} \sum_{a' \in O_{dA}(a; w_1)} = \sum_{a' \in A} \sum_{a \in O_{dA}(a'; w_1)}.
\]
Overall, we obtain

$$\sum_{a' \in A} \sum_{b \in B} \sum_{a \in O_{d,A}(a'; w_1)} \langle u, \hat{m}_{a'b}' \rangle \left( \frac{f_a}{\langle u, \hat{m}_{a'b}' \rangle} \right) \geq 1 - \epsilon_1.$$  \hfill (3.3.5)

Similar calculations show that for any $w_2 \geq W_{\epsilon_2}(\hat{M}^G, G)$

$$\sum_{a' \in A} \sum_{b \in B} \sum_{b \in O_{d,B}(b'; w_2)} \langle u, \hat{m}_{a'b}' \rangle \left( \frac{g_b}{\langle u, \hat{m}_{a'b}' \rangle} \right) \geq 1 - \epsilon_2$$  \hfill (3.3.6)

holds. We obtain from (3.3.5) and (3.3.6)

$$\sum_{a' \in A} \sum_{b \in B} \langle u, \hat{m}_{a'b}' \rangle \left[ \left( \sum_{a \in O_{d,A}(a'; w_1)} \frac{\hat{m}_{a'b}'}{\langle u, \hat{m}_{a'b}' \rangle} \right) \right] \geq 2 - \epsilon_1 - \epsilon_2,$$

which implies that there exists a $(a_0', b_0') \in A \times B$ such that

$$\left( \sum_{a \in O_{d,A}(a_0'; w_1)} \frac{\hat{m}_{a'b}'}{\langle u, \hat{m}_{a'b}' \rangle} \right) \geq 2 - \epsilon_1 - \epsilon_2$$  \hfill (3.3.7)

since $\sum_{a' \in A} \sum_{b' \in B} \langle u, \hat{m}_{a'b}' \rangle = \langle u, u \rangle = 1$ and $0 \leq \langle u, \hat{m}_{a'b}' \rangle \leq 1$ for all $(a', b') \in A \times B$. We can see from (3.3.7) that

$$\sum_{a \in O_{d,A}(a_0'; w_1)} \frac{\hat{m}_{a'b}'}{\langle u, \hat{m}_{a'b}' \rangle} \geq 1 - \epsilon_1 - \epsilon_2 + \left( 1 - \sum_{b \in O_{d,B}(b_0'; w_2)} \frac{\hat{m}_{a'b}'}{\langle u, \hat{m}_{a'b}' \rangle} \right) \geq 1 - \epsilon_1 - \epsilon_2$$  \hfill (3.3.8)

holds for an arbitrary $w_1 \geq W_{\epsilon_1}(\hat{M}^F, F)$, where we use

$$\sum_{b \in O_{d,B}(b_0'; w_2)} \frac{\hat{m}_{a'b}'}{\langle u, \hat{m}_{a'b}' \rangle} \leq \sum_{b \in B} \frac{\hat{m}_{a'b}'}{\langle u, \hat{m}_{a'b}' \rangle} = 1,$$

and similarly

$$\sum_{b \in O_{d,B}(b_0'; w_2)} \frac{\hat{m}_{a'b}'}{\langle u, \hat{m}_{a'b}' \rangle} \geq 1 - \epsilon_1 - \epsilon_2$$  \hfill (3.3.9)
holds for an arbitrary \( w_2 \geq W_2(\tilde{M}^G, G) \). Therefore, because

\[
\omega':= \frac{\tilde{m}_{ab}^F}{\langle u, \tilde{m}_a^F \rangle}
\]

defines a state, (3.3.8) and (3.3.9) together with the definition of the overall width (3.1.3) result in

\[
W_{\epsilon_1}(\tilde{M}^F, F) \geq W_{\epsilon_1+\epsilon_2}(\omega_F^F)
\]

\[
W_{\epsilon_2}(\tilde{M}^G, G) \geq W_{\epsilon_1+\epsilon_2}(\omega_G^G),
\]

which proves the theorem.

\[\blacksquare\]

The next corollary results immediately from Proposition 3.2.1. It describes a similar content to Theorem 3.3.2 in terms of another measure.

**Corollary 3.3.3**

Let \( \Omega \) be a transitive state space and its positive cone \( V_+ \) be self-dual with respect to \( \langle \cdot, \cdot \rangle_{GL(\Omega)} \), and let \( (F, G) \) be a pair of ideal measurements on \( \Omega \).

For an arbitrary approximate joint measurement \( \tilde{M}^{FG} \) of \((F, G)\) and \( \epsilon_1, \epsilon_2 \in (0,1] \) satisfying \( \epsilon_1 + \epsilon_2 \leq 1 \), there exists a state \( \omega \in \Omega \) such that

\[
D_8(\tilde{M}^F, F) \geq \epsilon_1 2 W_{\epsilon_1+\epsilon_2}(\omega_F^F)
\]

\[
D_8(\tilde{M}^G, G) \geq \epsilon_2 2 W_{\epsilon_1+\epsilon_2}(\omega_G^G).
\]

There is also another formulation by means of minimum localization error and \( l_8 \) distance.

**Theorem 3.3.4**

Let \( \Omega \) be a transitive state space and its positive cone \( V_+ \) be self-dual with respect to \( \langle \cdot, \cdot \rangle_{GL(\Omega)} \), and let \( (F, G) \) be a pair of ideal measurements on \( \Omega \).

For an arbitrary approximate joint measurement \( \tilde{M}^{FG} \) of \((F, G)\), there exists a state \( \omega \in \Omega \) such that

\[
D_{L_8}(\tilde{M}^F, F) + D_{L_8}(\tilde{M}^G, G) \geq LE(\omega_F^F) + LE(\omega_G^G).
\]

**Proof**

Because \( \{ f_a/\langle u, f_a \rangle \}_{a} \) are states, we can see from (3.3.4) and the definition of the \( l_{\infty} \) distance (3.2.5) that

\[
\left| \left\langle f_a, \frac{f_a}{\langle u, f_a \rangle} \right\rangle - \left\langle \tilde{m}_a^F, \frac{f_a}{\langle u, f_a \rangle} \right\rangle \right| \leq D_{L_8}(\tilde{M}^F, F)
\]

holds for all \( a \in A \), which can be rewritten by means of (3.3.4) as

\[
1 - \sum_{b \in B} \left\langle \tilde{m}_{ab}^{FG}, \frac{f_a}{\langle u, f_a \rangle} \right\rangle \leq D_{L_8}(\tilde{M}^F, F),
\]
for all $a \in A$. Multiplying both sides by $\langle f_a, u \rangle$ and taking the summation over $a$, we have

$$1 - \sum_{a \in A} \sum_{b \in B} \langle \tilde{m}_{ab}^{FG}, f_a \rangle \leq D_\varepsilon(\tilde{M}^F, F),$$

namely

$$1 - \sum_{a' \in A} \sum_{b' \in B} \langle u, \tilde{m}_{a'b'}^{FG} \rangle \left( f_{a'}, \frac{\tilde{m}_{a'b'}^{FG}}{\langle u, \tilde{m}_{a'b'}^{FG} \rangle} \right) \leq D_\varepsilon(\tilde{M}^F, F) \quad (3.3.10)$$

In a similar way, we also have

$$1 - \sum_{a' \in A} \sum_{b' \in B} \langle u, \tilde{m}_{a'b'}^{FG} \rangle \left( g_{b'}, \frac{\tilde{m}_{a'b'}^{FG}}{\langle u, \tilde{m}_{a'b'}^{FG} \rangle} \right) \leq D_\varepsilon(\tilde{M}^G, G). \quad (3.3.11)$$

Since $\sum_{a' \in A} \sum_{b' \in B} \langle u, \tilde{m}_{a'b'}^{FG} \rangle = 1$, (3.3.10) and (3.3.11) give

$$\sum_{a' \in A} \sum_{b' \in B} \langle u, \tilde{m}_{a'b'}^{FG} \rangle \left[ \left( 1 - \langle f_{a'}, \frac{\tilde{m}_{a'b'}^{FG}}{\langle u, \tilde{m}_{a'b'}^{FG} \rangle} \rangle \right) + \left( 1 - \langle g_{b'}, \frac{\tilde{m}_{a'b'}^{FG}}{\langle u, \tilde{m}_{a'b'}^{FG} \rangle} \rangle \right) \right] \leq D_\varepsilon(\tilde{M}^F, F) + D_\varepsilon(\tilde{M}^G, G),$$

which indicates that there exists a $(a'_0, b'_0) \in A \times B$ satisfying

$$\left( 1 - \langle f_{a'_0}, \frac{\tilde{m}_{a'_0b'_0}^{FG}}{\langle u, \tilde{m}_{a'_0b'_0}^{FG} \rangle} \rangle \right) + \left( 1 - \langle g_{b'_0}, \frac{\tilde{m}_{a'_0b'_0}^{FG}}{\langle u, \tilde{m}_{a'_0b'_0}^{FG} \rangle} \rangle \right) \leq D_\varepsilon(\tilde{M}^F, F) + D_\varepsilon(\tilde{M}^G, G). \quad (3.3.12)$$

Because

$$\omega'_0 := \frac{\tilde{m}_{a'_0b'_0}^{FG}}{\langle u, \tilde{m}_{a'_0b'_0}^{FG} \rangle}$$

is a state, we can conclude from (3.3.12) and the definition of the minimum localization error (3.1.4) that

$$LE(\omega'_0^F) + LE(\omega'_0^G) \leq D_\varepsilon(\tilde{M}^F, F) + D_\varepsilon(\tilde{M}^G, G),$$

which proves the theorem.

\[ \square \]

Our theorems above have been proved only for restricted theories such as finite dimensional classical and quantum theories, and regular polygon theories with odd sides (see Example 2.3.3 - 2.3.5). What is essential to the proofs of the theorems is that we can see effects as states (the self-duality), and that every effect of an ideal measurement is an “eigenstate” of itself (see (3.3.4)). In fact, taking those points into consideration, although it may be a minor generalization, we can demonstrate similar theorems for even-sided regular polygon theories.
Theorem 3.3.5
Let $n$ be an even integer. Theorem 3.3.2, Corollary 3.3.3, and Theorem 3.3.4 hold also for the $n$-sided regular polygon theory in Example 2.3.5.

Proof
We again denote the inner product $\langle \cdot, \cdot \rangle_{GL(\Omega_n)}$ by $\langle \cdot, \cdot \rangle$ in this proof.

In the $n$-sided regular polygon theory with even $n$, if $F = \{f_a\}_a$ is an ideal measurement, then it is of the form

$$F = \{f_0, f_1\}$$  \hspace{1cm} (3.3.13)

with

$$f_0 = e_n^{ext}(i) \quad \text{and} \quad f_1 = u - e_n^{ext}(i) = e_n^{ext}(i + \frac{n}{2})$$  \hspace{1cm} (3.3.14)

for some $i$ (remember that we do not consider the trivial measurement $F = \{u\}$). Let us introduce an affine bijection

$$\psi := \begin{pmatrix} r_n & 0 & 0 \\ 0 & r_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$  \hspace{1cm} (3.3.15)

on $\mathbb{R}^3$. Because $(e, \omega)_E = (\psi^{-1}(e), \psi(\omega))_E$ holds for any $\omega \in \Omega_n$ and $e \in \mathcal{E}(\Omega_n)$, we can consider an equivalent expression of the theory with $\psi(\Omega_n) =: \hat{\Omega}_n$ and $\psi^{-1}(\mathcal{E}(\Omega_n))$ being its state and effect space respectively. The pure states (2.3.8) and the extreme effects (2.3.10) are modified as

$$\omega_n^{ext}(i) \rightarrow \hat{\omega}_n^{ext}(i) := \psi(\omega_n^{ext}(i)) = \begin{pmatrix} \frac{r_n^2 \cos(\frac{2\pi i}{n})}{r_n^2 \sin(\frac{2\pi i}{n})} \\ 0 \\ 1 \end{pmatrix}$$  \hspace{1cm} (3.3.16)

and

$$e_n^{ext}(i) \rightarrow \hat{e}_n^{ext}(i) := \psi^{-1}(e_n^{ext}(i)) = \frac{1}{2} \begin{pmatrix} \cos(\frac{(2i-1)\pi}{n}) \\ \sin(\frac{(2i-1)\pi}{n}) \\ 1 \end{pmatrix}$$  \hspace{1cm} (3.3.17)

respectively, and their conic hull (the positive cone and the internal dual cone) as

$$V_+ \rightarrow \hat{V}_+ := \psi(V_+)$$

$$V^{* \text{int}}_{+\langle \cdot, \cdot \rangle} \rightarrow \hat{V}^{* \text{int}}_{+\langle \cdot, \cdot \rangle} := \psi^{-1}(V^{* \text{int}}_{+\langle \cdot, \cdot \rangle})$$

respectively. Note in the equations above that $GL(\Omega_n) = GL(\hat{\Omega}_n)$ and $\langle \cdot, \cdot \rangle_E = \langle \cdot, \cdot \rangle_{GL(\Omega_n)} = \langle \cdot, \cdot \rangle_{GL(\hat{\Omega}_n)} = \langle \cdot, \cdot \rangle$ hold, and $\omega_M = u = ^t(0, 0, 1)$ is invariant for $\psi$ (and $\psi^{-1}$). We can also find that a measurement $E = \{e_a\}_a$ in the original expression is rewritten as $\tilde{E} := \{\tilde{e}_a\}_a$ with $\tilde{e}_a := \psi^{-1}(e_a)$, and that an ideal measurement $F$ in (3.3.13) and (3.3.14) gives

$$\tilde{F} = \{\tilde{f}_0, \tilde{f}_1\}$$  \hspace{1cm} (3.3.18)
with
\[ \tilde{f}_0 = \tilde{\epsilon}_n^{\text{ext}}(i) \quad \text{and} \quad \tilde{f}_1 = u - \tilde{\epsilon}_n^{\text{ext}}(i) = \tilde{\epsilon}_n^{\text{ext}}(i) + \frac{r}{2}, \]
(3.3.19)
which is also ideal in the rewritten theory.

It is important to confirm that all of our measures (3.1.3), (3.1.4), (3.2.3), (3.2.4), and (3.2.5) depend only on probabilities, and thus they are invariant for the modification above. For example, for a pair of measurement \( M = \{m_a\}_a \) and \( F = \{f_a\}_a \) on the original state space \( \Omega_n \), we can see easily from (3.1.4) and (3.2.5) that
\[
LE(p^F) = 1 - \max_{a \in A} f_a(\omega) \\
= 1 - \max_{a \in A} \tilde{f}_a(\omega) \\
= LE(\hat{\omega}^{\tilde{F}})
\]
and
\[
D_{\tilde{X}}(M, F) = \sup_{\omega \in \Omega_n} \max_{a \in A} |m_a(\omega) - f_a(\omega)| \\
= \sup_{\tilde{\omega} \in \Omega_n} \max_{a \in A} |\tilde{m}_a(\tilde{\omega}) - \tilde{f}_a(\tilde{\omega})| \\
= D_{\tilde{X}}(\tilde{M}, \tilde{F})
\]
respectively. It results in that if Theorem 3.3.4 holds in the modified theory, then it holds also in the original theory. Similar considerations for the other measures reveal that for the successful proof of Theorem 3.3.5 we need to prove the claim of Theorem 3.3.5 to be true in the rewritten expression.

It can be easily seen from (3.3.16) and (3.3.17) that \( \hat{V}_+ \) generated by (3.3.16) includes \( \hat{V}_+^{\text{sint}} \) generated by (3.3.17), i.e. \( \hat{V}_+^{\text{sint}} \subset \hat{V}_+ \), which proves that
\[
\frac{\hat{\epsilon}}{\langle u, \hat{\epsilon} \rangle} \in \hat{\Omega}_n
\]
(3.3.20)
for any effect \( \hat{\epsilon} \in \hat{V}_+^{\text{sint}} \). Considering that
\[
\langle \tilde{\epsilon}_n^{\text{ext}}(i), \frac{\tilde{\epsilon}_n^{\text{ext}}(i)}{\langle u, \tilde{\epsilon}_n^{\text{ext}}(i) \rangle} \rangle = 1
\]
(3.3.21)
holds for any \( i \), we can see together with (3.3.18) and (3.3.19) that (3.3.4) also holds in this case. Therefore, we can repeat by means of (3.3.20) and (3.3.21) the same calculations as in Theorem 3.3.2 - Theorem 3.3.4, and obtain similar results to them in the modified theory.

\[ \square \]

Theorem 3.3.4 (and Theorem 3.3.5) has an application to evaluate the degree of incompatibility \([43, 45, 46]\) of a GPT.
Example 3.3.6 (Evaluation of degree of incompatibility)

Suppose that $\Omega$ is an arbitrary state space, and $F$ and $G$ are two-outcome measurements on $\Omega$, namely $F = \{f_0, f_1\}$ and $G = \{g_0, g_1\}$, and consider similarly to (3.2.1) their “fuzzy” versions

$$
\tilde{F}^\lambda := \lambda F + (1 - \lambda) \left( \frac{u}{2}, \frac{u}{2} \right) = \left\{ \lambda f_0 + \frac{1 - \lambda}{2} u, \lambda f_1 + \frac{1 - \lambda}{2} u \right\}
$$

$$
\tilde{G}^\lambda := \lambda G + (1 - \lambda) \left( \frac{u}{2}, \frac{u}{2} \right) = \left\{ \lambda g_0 + \frac{1 - \lambda}{2} u, \lambda g_1 + \frac{1 - \lambda}{2} u \right\}
$$

(3.3.22)

for $\lambda \in [0, 1]$. It is known that we can find a $\lambda_{F,G} \geq \frac{1}{2}$ such that the distorted measurements $\tilde{F}^\lambda$ and $\tilde{G}^\lambda$ in (3.3.22) are jointly measurable for any $\lambda \in [0, \lambda_{F,G}]$, and $\lambda_{opt} := \inf_{F,G} \lambda_{F,G}$ can be thought describing the degree of incompatibility of the theory. $\lambda_{opt}$ has been calculated in various theories: for example, $\lambda_{opt} = \frac{1}{\sqrt{2}}$ in finite dimensional quantum theories [45], and $\lambda_{opt} = \frac{1}{2}$ in the square theory (a regular polygon theory with $n = 4$) [48].

To see how Theorem 3.3.4 contributes to the degree of incompatibility, we consider the situations in Theorem 3.3.4 (and Theorem 3.3.5) with the marginals $\tilde{M}^F$ and $\tilde{M}^G$ of the approximate joint measurement being $\tilde{F}^\lambda$ and $\tilde{G}^\lambda$ in (3.3.22) for $\lambda \in [0, \lambda_{F,G}]$ respectively. In this case, we can represent the measurement error $D_\omega(\tilde{F}^\lambda, F)$ in a more explicit way:

$$
D_\omega(\tilde{F}^\lambda, F) = \sup_{\omega \in \Omega} \max_{i \in \{0, 1\}} \left| \left( \lambda f_i + \frac{1 - \lambda}{2} u \right)(\omega) - f_i(\omega) \right|
$$

$$
= (1 - \lambda) \sup_{\omega \in \Omega} \max_{i \in \{0, 1\}} \left| f_i(\omega) - \frac{1}{2} \right|
$$

$$
= \frac{1 - \lambda}{2},
$$

(3.3.23)

where we use the relation

$$
\left| f_0(\omega) - \frac{1}{2} \right| = \left| (u - f_1)(\omega) - \frac{1}{2} \right| = \left| f_1(\omega) - \frac{1}{2} \right|
$$

and the fact that there is an “eigenstate” $\omega_i$ for each ideal effect $f_i$ satisfying $f_i(\omega_i) = 1$ as we have seen in (3.3.4) or (3.3.21). Therefore, we can conclude from Theorem 3.3.4 (and Theorem 3.3.5) that for any $\lambda \in [0, \lambda_{F,G}]$ and for some state $\omega_0$

$$
1 - \lambda \geq \left( 1 - \max_{i \in \{0, 1\}} f_i(\omega_0) \right) + \left( 1 - \max_{j \in \{0, 1\}} g_j(\omega_0) \right)
$$

holds, that is,

$$
\lambda_{F,G} \leq \sup_{\omega \in \Omega} \left( \max_{i \in \{0, 1\}} f_i(\omega) + \max_{j \in \{0, 1\}} g_j(\omega) \right) - 1
$$

(3.3.24)
holds, and $\lambda_{opt}$ can be evaluated by taking the infimum of both sides of (3.3.24) over all two-outcome measurements. If we restricted ourselves to regular polygon theories, our inequality (3.3.24) gives unfortunately meaningless bounds $\lambda_{opt} \leq 1$ for $n = 3, 4, 5, 6$, where we can always find a state $\omega^*$ such that

$$\max_{i \in \{0, 1\}} f_i(\omega^*) + \max_{j \in \{0, 1\}} g_j(\omega^*) = 2$$

for any pair of ideal measurements $F, G$. However, when $n \geq 7$, we can obtain nontrivial bounds in (3.3.24).

4 Conclusion and Discussion

In our study, although only theories with transitivity and self-duality with respect to a certain inner product were considered, it was revealed that similar quantitative relations between preparation and measurement uncertainty to quantum case [15] hold also in GPTs. Because GPTs considered in this paper include classical, quantum, and other theories such as regular polygon theories, our results can be considered as generalizations of the quantum ones. It is easy to see from the proofs that our theorems can be generalized to the case when three or more measurements are considered. While our assumptions may seem curious, it has been observed in [37] that those two conditions are satisfied simultaneously if the state space is bit-symmetric. There are also researches where they are derived from certain conditions possible to be interpreted physically [47, 49]. However, considering that our theorems also hold in regular polygon theories with even sides, which are not self-dual, future research is required to investigate whether we can loosen the assumptions.

What is also specific to our main theorems is that their proofs do not require the rules of determining composite systems, while the quantum results of the previous study [15] were proved by means of the maximally entangled state and its “ricochet property”. It is known that in GPTs there exist ambiguities when constituting the composite system of two systems [27, 28], but our theorems avoid successfully those difficulties. Future research should reveal the relations between the maximal entanglement and self-duality, which will be a key to generalize our theorems to infinite dimensional cases (remember that the maximally entangled states cannot be defined in infinite dimensional quantum theories such as $\mathcal{H} = L^2(\mathbb{R})$).

As seen Example 3.3.6, our theorems also can be considered as yielding via PURs a method for evaluating measurement error, which is in general hard to obtain [18], and it has been discussed that measurement error quantifies the degree of nonlocality [45, 46]. Although our method turns to be meaningless for theories such as the square theory, where there is no preparation uncertainty, our results will provide an application to understand the nonlocality in GPTs, which is also a future problem.
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Appendix A  Proof of Proposition 2.2.3

In this part, we give a proof of Proposition 2.2.3. We need the following proposition holding without the assumption of the transitivity of $\Omega$.

**Proposition A.1**

For a state space $\Omega$, define a linear map $P_M : V \to V$ by

$$P_M x = \int_{GL(\Omega)} T_x \, d\mu(T).$$

Then, $P_M$ is an orthogonal projection with respect to the inner product $\langle \cdot, \cdot \rangle_{GL(\Omega)}$, i.e.

$$P_M = P_M^2 \quad \text{and} \quad \langle P_M x, y \rangle_{GL(\Omega)} = \langle x, P_M y \rangle_{GL(\Omega)} \quad \text{for all } x, y \in V.$$

**Proof**

We denote the inner product $\langle \cdot, \cdot \rangle_{GL(\Omega)}$ simply by $\langle \cdot, \cdot \rangle$ in this proof.

Let $V_M := \{ x \in V \mid T x = x \text{ for all } T \in GL(\Omega) \}$ be the set of all fixed points with respect to $GL(\Omega)$. Then, it is easy to see that $P_M x_M = x_M$ for any $x_M \in V_M$ and $V_M = ImP_M$ (in particular $V_M$ is a subspace of $V$). Therefore,

$$P_M^2 x = P_M(P_M x) = P_M x$$

holds for any $x \in V$, and thus $P_M^2 = P_M$. On the other hand, we can observe

$$\langle P_M x, y \rangle = \int_{GL(\Omega)} d\mu(T) \langle TP_M x, Ty \rangle_E$$

$$= \int_{GL(\Omega)} d\mu(T) \langle P_M x, Ty \rangle_E$$

$$= \int_{GL(\Omega)} d\mu(T) \left( \int_{GL(\Omega)} d\mu(S) S x, Ty \right)_E. \quad (A.1)$$

Since the vector $\int_{GL(\Omega)} d\mu(S) S x \in V$ is constructed with its $i$th element

$$\left( w_i, \int_{GL(\Omega)} d\mu(S) S x \right)_E$$

in terms of the Euclidean orthonormal basis $\{w_i\}_{i=1}^{N+1}$ of $V$ given by

$$\int_{GL(\Omega)} d\mu(S) (w_i, S x)_E,$$
(for more details see [50]), it holds that
\[
\int_{GL(\Omega)} d\mu(T) \left( \int_{GL(\Omega)} d\mu(S) Sx, Ty \right)_E
= \int_{GL(\Omega)} d\mu(T) \left[ \int_{GL(\Omega)} d\mu(S) (Sx, Ty)_E \right]
= \int_{GL(\Omega)} d\mu(S) \left[ \int_{GL(\Omega)} d\mu(T) (Sx, Ty)_E \right]
= \int_{GL(\Omega)} d\mu(S) \left( Sx, \int_{GL(\Omega)} d\mu(T)Ty \right)_E,
\]
where we use Fubini’s theorem for the finite Haar measure \( \mu \) on \( GL(\Omega) \).
Therefore, we obtain together with (A.1)
\[
\langle P_Mx, y \rangle = \langle x, P_My \rangle.
\]
\( \square \)

Proposition A.1 enables us to give an orthogonal decomposition of a vector \( x \in V \) such that
\[
x = (\mathbb{1} - P_M)x + P_Mx,
\]
where \( (\mathbb{1} - P_M)x \in V_M^\perp \) and \( P_Mx \in V_M \). When the transitivity of \( \Omega \) is assumed, (A.2) is reduced to Proposition 2.2.3.

Proposition 2.2.3
For a transitive state space \( \Omega \), there exists a basis \( \{ v_l \}_{l=1}^{N+1} \) of \( V \) orthonormal with respect to the inner product \( \langle \cdot, \cdot \rangle_{GL(\Omega)} \) such that \( v_{N+1} = \omega_M \) and
\[
x \in \text{aff}(\Omega) \iff x = \sum_{l=1}^{N} a_l v_l + v_{N+1} = \sum_{l=1}^{N} a_l v_l + \omega_M \ (a_1, \cdots, a_N \in \mathbb{R}).
\]

Proof
Since we set \( \dim \text{aff}(\Omega) = N \), there exists a set of \( N \) linear independent vectors \( \{ v_l \}_{l=1}^{N} \subset \text{aff}(\Omega) - \omega_M \) which forms a basis of the \( N \)-dimensional vector subspace \( \text{aff}(\Omega) - \omega_M \subset V \), and we can assume by taking an orthonormalization that they are orthonormal with respect to the inner product \( \langle \cdot, \cdot \rangle \). Hence, \( x \in \text{aff}(\Omega) \) if and only if it is represented as
\[
x = \sum_{l=1}^{N} a_l v_l + \omega_M \ (a_1, \cdots, a_N \in \mathbb{R}).
\]
Moreover, because of the definition of \( \text{aff}(\Omega) \), for every \( v_l \in [\text{aff}(\Omega) - \omega_M] \) there exist \( k \in \mathbb{N} \), real numbers \( \{b_i\}_{i=1}^k \) satisfying \( \sum_{i=1}^k b_i = 1 \), and states \( \{\omega_i\}_{i=1}^k \) such that \( v_l = \sum_{i=1}^k b_i \omega_i - \omega_M \). By means of Proposition 2.2.2, we obtain for all \( l = 1, 2, \cdots, N \)

\[
P_M v_l = \sum_{i=1}^k b_i P_M \omega_i - P_M \omega_M
\]

\[
= \sum_{i=1}^k b_i \omega_M - \omega_M = 0. \tag{A.4}
\]

Therefore, because of Proposition A.1

\[
\langle \omega_M, v_l \rangle = \langle P_M \omega_M, v_l \rangle = \langle \omega_M, P_M v_l \rangle = 0
\]

holds for all \( l = 1, 2, \cdots, N \), and we can conclude together with the unit norm of \( \omega_M \) that \( \{v_1, \cdots, v_N, \omega_M\} \) in (A.3) forms an orthonormal basis of the \( (N+1) \)-dimensional vector space \( V \) with respect to \( \langle \cdot, \cdot \rangle \) and Proposition 2.2.3 is proved (we can also find that (A.3) corresponds to (A.2)).

\[\Box\]

**Appendix B  Proof of Proposition 2.3.2**

In this part, we prove Proposition 2.3.2. As we have so far, we let \( \Omega \) be a state space, \( V_+ \) be the positive cone generated by \( \Omega \), and \( \text{GL}(\Omega) \) be the set of all state automorphisms on \( \Omega \) in the following.

**Lemma B.1**

\( V_{+\langle \cdot, \cdot \rangle_{\text{GL}(\Omega)}} \) is a \( \text{GL}(\Omega) \)-invariant set. That is, \( TV_{+\langle \cdot, \cdot \rangle_{\text{GL}(\Omega)}}^* V_{+\langle \cdot, \cdot \rangle_{\text{GL}(\Omega)}} \) for all \( T \in \text{GL}(\Omega) \).

**Proof**

Let \( w \in V_{+\langle \cdot, \cdot \rangle_{\text{GL}(\Omega)}}^* \). It holds that \( \langle w, v \rangle_{\text{GL}(\Omega)} \geq 0 \) for all \( v \in V_+ \). Because any \( T \in \text{GL}(\Omega) \) is an orthogonal transformation with respect to \( \langle \cdot, \cdot \rangle_{\text{GL}(\Omega)} \), we obtain

\[
\langle T w, v \rangle_{\text{GL}(\Omega)} = \langle w, T^{-1} v \rangle_{\text{GL}(\Omega)} \geq 0
\]

for all \( v \in V_+ \). Therefore, \( TV_{+\langle \cdot, \cdot \rangle_{\text{GL}(\Omega)}}^* V_{+\langle \cdot, \cdot \rangle_{\text{GL}(\Omega)}} \) holds, and a similar argument for \( T^{-1} \in \text{GL}(\Omega) \) proves the lemma.

\[\Box\]

**Lemma B.2**

Let \( \langle \cdot, \cdot \rangle \) be an arbitrary inner product on \( V \). \( V_+ \) is self-dual if and only if there exists a linear map \( J : V \to V \) strictly positive with respect to \( \langle \cdot, \cdot \rangle \) such that \( J(V_+) = V_{+\langle \cdot, \cdot \rangle}^* \).
Thus, \( v \in V^\mathrm{int}_+ \) is equivalent to \( Jv \in V^\mathrm{int}_+ \). It concludes \( V^\mathrm{int}_+ \) is a strictly positive map with respect to \( \langle \cdot, \cdot \rangle \). We obtain

\[
V_+ = V^\mathrm{int}_+ = \{ v \mid \langle v, w \rangle \geq 0, \forall w \in V_+ \}
\]

\[
= \{ v \mid \langle v, Kw \rangle \geq 0, \forall w \in V_+ \}
\]

Thus, \( v \in V_+ = V^\mathrm{int}_+ \) is equivalent to \( Kv \in V^\mathrm{int}_+ \), i.e. \( K^{-1}V_+ = V^\mathrm{int}_+ \). Define \( J = K^{-1} \).

In Lemma B.2, we gave a necessary and sufficient condition for \( V_+ \) with an inner product \( \langle \cdot, \cdot \rangle \) to be self-dual. The condition was the existence of a strictly positive map \( J \) satisfying \( J(V_+) = V^\mathrm{int}_+ \). This map \( J \) may not be unique. For instance, let us consider a classical system in \( \mathbb{R}^2 \) whose extreme points are two points \( (1, -1) \) and \( (1, 1) \). The positive cone is a “forward lightcone” \( V_+ = \{(x_0, x_1) \mid x_0 \geq 0, x_0^2 - x_1^2 \geq 0 \} \). It is easy to see that \( V_+ = V^\mathrm{int}_+ \) with the standard Euclidean inner product \( \langle \cdot, \cdot \rangle_E \). However, every linear map of the form \( (x_0, x_1) \mapsto (\lambda_0 x_0, \lambda_1 x_1) \) for \( \lambda_0, \lambda_1 > 0 \) (which contains “Lorentz transformations” in \( 1 + 1 \) dimension) is strictly positive and makes \( V_+ \) invariant. Nevertheless, when \( |\Omega^\text{ext}| < \infty \), we can demonstrate that such strictly positive maps are “equivalent” to each other .

Lemma B.3
Let \( |\Omega^\text{ext}| < \infty \). If a linear map \( J : V \to V \) strictly positive with respect to an inner product \( \langle \cdot, \cdot \rangle \) satisfies \( J(V_+) = V_+ \), then for each \( \omega^\text{ext} \in \Omega^\text{ext} \) there exists \( \mu(\omega^\text{ext}) > 0 \) such that \( J(\omega^\text{ext}) = \mu(\omega^\text{ext})\omega^\text{ext} \).

Proof
Any \( \omega^\text{ext} \in \Omega^\text{ext} \) is represented as \( \omega^\text{ext} = c(\omega^\text{ext})w \) with \( c(\omega^\text{ext}) := \|\omega^\text{ext}\| = (\omega^\text{ext}, \omega^\text{ext})^{1/2} \) and \( w \) satisfying \( \|w\| = 1 \). Suppose that there exists a family

\[
\omega_j^\text{ext} \in \Omega^\text{ext}
\]

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such that there is no $\mu(\omega_j^{\text{ext}}) > 0$ for every $j = 1, 2, \cdots, Z$ satisfying $J(\omega_j^{\text{ext}}) = \mu(\omega_j^{\text{ext}})w_j^{\text{ext}}$, and define $W := \{w_j\}_{j=1}^{Z}$. Since $J$ maps each extreme ray of $V_+$ to an extreme ray of $V_+$, $J(w_j)$ with $w_j \in W$ is proportional to some $\omega^{\text{ext}} \in \Omega^{\text{ext}}$ (remember that an extreme ray of $V_+$ is the set of positive scalar multiples of an extreme point of $\Omega$). We can see that $J(w_j)$ is proportional to some $w_l \in W$ with $l \neq j$ considering that $J(J(w_j)) = \mu J(w_j)$ holds if and only if $J(w_j) = \mu w_j$ holds.

Let us diagonalize $J$. It is written as $J = \sum_{n=1}^{M} \tau_n R_n$, where $\tau_1 > \tau_2 > \cdots > \tau_M > 0$ and $\{R_n\}_{n=1}^{M}$ are orthogonal projections. We choose $w_1$ so that $0 \neq (w_1, R_1 w_1) \geq (w_j, R_1 w_j)$ for all $w_j \in W$. Although such $w_1$ may not be unique, the following argument does not depend on the choice. If it happens that $(w_j, R_1 w_j) = 0$ for all $w_j \in W$, we choose $w_1$ so that $0 \neq (w_1, R_2 w_1) \geq (w_j, R_2 w_j)$ for all $w_j \in W$. If still $(w_j, R_2 w_j) = 0$ for all $w_j \in W$, we repeat the argument for $R_3, R_4, \cdots$. For simplicity, we assume hereafter that $(w_1, R_1 w_1) \neq 0$ holds. The general cases can be treated similarly.

Let $r_1 := R_1 w_1/\|R_1 w_1\| \neq 0$, then $J$ is written as

$$J = \tau_1 |r_1 \times r_1| + \tau_j (R_1 - |r_1 \times r_1|) + \sum_{n \geq 2} \tau_n E_n = \tau_1 \hat{R}_0 + \tau_1 \hat{R}_1 + \sum_{n \geq 2} \tau_n \hat{R}_n,$$

where we define $\hat{R}_0 := |r_1 \times r_1|$, $\hat{R}_1 := R_1 - |r_1 \times r_1|$ and $\hat{R}_n := R_n$ for $n \geq 2$ satisfying $\hat{R}_a \hat{R}_b = \delta_{ab} \hat{R}_a$ for $a, b = 0, 1, \cdots, M$. Now we consider a vector

$$\frac{J(w_1)}{\|J(w_1)\|} = \frac{\tau_1 \hat{R}_0 w_1 + \tau_1 \hat{R}_1 w_1 + \sum_{n \geq 2} \tau_n \hat{R}_n w_1}{\left(\tau_1^2 (w_1, \hat{R}_0 w_1) + \tau_1^2 (w_1, \hat{R}_1 w_1) + \sum_{n \geq 2} \tau_n^2 (w_1, \hat{R}_n w_1)\right)^{1/2}},$$

which must coincide with some $w_l \in W$. Its “$\hat{R}_0$ -element” can be calculated as

$$\left(\frac{J(w_1)}{\|J(w_1)\|}, \hat{R}_0 \frac{J(w_1)}{\|J(w_1)\|}\right) = \frac{\tau_1^2 (w_1, \hat{R}_0 w_1)}{(w_1, \hat{R}_0 w_1) + (w_1, \hat{R}_1 w_1) + \sum_{n=2}^{M} \frac{\tau_n^2}{\tau_1^2} (w_1, \hat{R}_n w_1)}.$$

On the other hand, we can obtain that

$$(w_1, \hat{R}_0 w_1) + (w_1, \hat{R}_1 w_1) + \sum_{n=2}^{M} \frac{\tau_n^2}{\tau_1^2} (w_1, \hat{R}_n w_1) < (w_1, \hat{R}_0 w_1) + (w_1, \hat{R}_1 w_1) + \sum_{n=2}^{M} (w_1, \hat{R}_n w_1) = 1$$

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because there exists a $n \geq 2$ such that $(w_1, \hat{R}_n w_1) \neq 0$ (otherwise $w_1 = (\hat{R}_0 + \hat{R}_1) w_1 = R_1 w_1$ and thus $J(w_1) = \tau_1 w_1$ hold, which contradicts $w_1 \in W$). Therefore, (B.1) results in

$$ \left( \frac{J(w_1)}{\|J(w_1)\|}, \frac{\hat{R}_0 J(w_1)}{\|J(w_1)\|} \right) > (w_1, \hat{R}_0 w_1). $$

This observation concludes a contradiction to $J(w_1)/\|J(w_1)\| = w_1 \in W$ because $w_1 \geq (w_j, \hat{R}_0 w_j)$ for all $w_j \in W$. Overall, we find that every $\omega^{\text{ext}} \in \Omega^{\text{ext}}$ has some $\mu(\omega^{\text{ext}}) > 0$ such that $J(\omega^{\text{ext}}) = \mu(\omega^{\text{ext}}) \omega^{\text{ext}}$.

\[ \square \]

**Lemma B.4**

Let $|\Omega^{\text{ext}}| < \infty$, and suppose that linear maps $J$ and $K$ strictly positive with respect to an inner product $(\cdot, \cdot)$ satisfy $J(V_+) = K(V_+) = V_+^{\text{int}}_{+(\cdot)}$ (in particular, $V_+$ is self-dual). Then, there exists a $\mu(\omega^{\text{ext}}) > 0$ for each $\omega^{\text{ext}} \in \Omega^{\text{ext}}$ such that $K(\omega^{\text{ext}}) = \mu(\omega^{\text{ext}}) J(\omega^{\text{ext}})$ holds.

**Proof**

As was seen in Lemma B.2, the inner products $(\cdot, \cdot)_J := (\cdot, J \cdot)$ and $(\cdot, \cdot)_K := (\cdot, K \cdot)$ satisfy $V_+^{\text{int}}_{+(\cdot)_J} = V_+$ and $V_+^{\text{int}}_{+(\cdot)_K} = V_+$ respectively. Because $(\cdot, \cdot)_K$ is represented as $(\cdot, \cdot)_K = (\cdot, L \cdot)_J$ with some linear map $L$ strictly positive with respect to $(\cdot, \cdot)_J$, we have for arbitrary $v, w \in V$

$$(v, w)_K = (v, K w) = (v, L w)_J = (v, J L w),$$

and thus $L = J^{-1} \circ K$ holds. On the other hand, $L$ satisfies

$$V_+^{\text{int}}_{+(\cdot)_K} = \{ v \mid (v, w)_K \geq 0, \forall w \in V_+ \}
= \{ v \mid (v, L w)_J \geq 0, \forall w \in V_+ \}
= \{ v \mid (L v, w)_J \geq 0, \forall w \in V_+ \} = L^{-1}(V_+^{\text{int}}_{+(\cdot)_J}).$$

That is, $L(V_+) = V_+$ holds. Therefore, we can apply Lemma B.3 to $L$, and conclude that

$$L(\omega^{\text{ext}}) = \mu(\omega^{\text{ext}}) \omega^{\text{ext}} = J^{-1}(K(\omega^{\text{ext}})),$$

i.e. $K(\omega^{\text{ext}}) = \mu(\omega^{\text{ext}}) J(\omega^{\text{ext}})$ holds.

\[ \square \]

**Proposition 2.3.2**

Let $\Omega$ be transitive with $|\Omega^{\text{ext}}| < \infty$ and $V_+$ be self-dual with respect to some inner product. There exists a linear bijection $\Xi : V \rightarrow V'$ such that $\Omega' := \Xi \Omega$ is transitive and the generating positive cone $V_+'$ is self-dual with respect to $\langle \cdot, \cdot \rangle_{GL(\Omega')}$, i.e. $V_+ = V_+^{\text{int}}_{+(\cdot)_{GL(\Omega')}}$.
Proof
Because of the transitivity of $\Omega$, we can adopt the orthogonal coordinate system of $V$ introduced in Proposition 2.2.3. Since $V_+$ is self-dual, there exists a linear map $J: V \to V$ strictly positive with respect to $\langle \cdot, \cdot \rangle_{GL(\Omega)}$ such that $J(V_+) = V^\text{int} + \langle \cdot, \cdot \rangle_{GL(\Omega)}$ (Lemma B.2). We can assume without loss of generality that $J$ satisfies $\langle \omega_M, J\omega_M \rangle_{GL(\Omega)} = 1$. Let us introduce

$$\Omega^* := V^\text{int} + \langle \cdot, \cdot \rangle_{GL(\Omega)} \cap [z = 1] = \{ v \in V^\text{int} + \langle \cdot, \cdot \rangle_{GL(\Omega)} \mid \langle v, \omega_M \rangle_{GL(\Omega)} = 1 \},$$

where we identify the “$\omega_M$-coordinate” with “2-coordinate” in $V$ and define $[z = 1] := \{ x \in V \mid \langle x, \omega_M \rangle_{GL(\Omega)} = 1 \} = \text{aff}(\Omega)$ (see Proposition 2.2.3). Note that since both $V^\text{int} + \langle \cdot, \cdot \rangle_{GL(\Omega)}$ and $[z = 1]$ are $GL(\Omega)$-invariant, $\Omega^*$ is also $GL(\Omega)$-invariant. It is easy to demonstrate that $\Omega^*$ is convex (and compact), and we denote by $\Omega^\text{ext}$ the set of all extreme points of $\Omega^*$. We can also see that $\Omega^\text{ext}$ generates the extreme rays of $V^\text{int} + \langle \cdot, \cdot \rangle_{GL(\Omega)}$, because $J$ satisfying $J(V_+) = V^\text{int} + \langle \cdot, \cdot \rangle_{GL(\Omega)}$ is bijective and maps extreme rays of $V_+$ to extreme rays of $V^\text{int} + \langle \cdot, \cdot \rangle_{GL(\Omega)}$, it holds that $|\Omega^\text{ext}| = |\Omega^\text{ext}|$. Thus, there exists a bijection $f: \Omega^\text{ext} \to \Omega^\text{ext}$ and $\beta(\omega^\text{ext}) > 0$ for each $\omega^\text{ext} \in \Omega^\text{ext}$ satisfying $J(\omega^\text{ext}) = \beta(\omega^\text{ext}) f(\omega^\text{ext})$.

For each $T \in GL(\Omega)$, we introduce $J_T := T^{-1} \circ J \circ T$. It is easy to see that $J_T$ satisfies $J_T(V_+) = V^\text{int} + \langle \cdot, \cdot \rangle_{GL(\Omega)}$ by virtue of Lemma B.1. Furthermore, $J_T$ is shown to be strictly positive with respect to $\langle \cdot, \cdot \rangle_{GL(\Omega)}$ because $T \in GL(\Omega)$ is an orthogonal transformation with respect to $\langle \cdot, \cdot \rangle_{GL(\Omega)}$. Therefore, applying Lemma B.4 to $J$ and $J_T$, there exists $\mu_T : \Omega^\text{ext} \to \mathbb{R}_{>0}$ such that $J_T(\omega^\text{ext}) = \mu_T(\omega^\text{ext}) J(\omega^\text{ext})$ for $\omega^\text{ext} \in \Omega^\text{ext}$, that is,

$$J_T(\omega^\text{ext}) = \mu_T(\omega^\text{ext}) J(\omega^\text{ext})$$

$$= \mu_T(\omega^\text{ext}) \beta(\omega^\text{ext}) f(\omega^\text{ext})$$

$$= \beta_T(\omega^\text{ext}) f(\omega^\text{ext}),$$

where we define $\beta_T(\omega^\text{ext}) := \mu_T(\omega^\text{ext}) \beta(\omega^\text{ext})$. We calculate this $\beta_T(\omega^\text{ext})$. It holds that

$$J_T(\omega^\text{ext}) = T^{-1} \circ J(T\omega^\text{ext})$$

$$= T^{-1}(\beta(T\omega^\text{ext}) f(T\omega^\text{ext}))$$

$$= \beta(T\omega^\text{ext}) T^{-1} f(T\omega^\text{ext})$$

$$= \beta_T(\omega^\text{ext}) f(\omega^\text{ext}).$$

This relation shows that $T^{-1} f(T\omega^\text{ext})$ is proportional to $f(\omega^\text{ext})$. Considering that the $z$-coordinates of $f(T\omega^\text{ext})$ and $f(\omega^\text{ext})$ are 1 and that $T^{-1}$ preserves $z$-coordinates, we find that $T^{-1} f(T\omega^\text{ext}) = f(\omega^\text{ext})$ (equivalently, $f(T\omega^\text{ext}) = T f(\omega^\text{ext})$) holds. Consequently, we obtain

$$J_T(\omega^\text{ext}) = \beta(T\omega^\text{ext}) f(\omega^\text{ext}).$$

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Now we introduce
\[ J_{av} := \frac{1}{|GL(\Omega)|} \sum_{T \in GL(\Omega)} J_T. \]

We note that \(|GL(\Omega)| < \infty\) when \(|\Omega^{ext}| < \infty\) because \(|GL(\Omega)| \leq |\Omega^{ext}|!\). \(J_{av}\) acts on \(\omega^{ext} \in \Omega^{ext}\) as
\[ J_{av}(\omega^{ext}) = \frac{1}{|GL(\Omega)|} \sum_{T \in GL(\Omega)} \beta(T \omega^{ext}) \cdot f(\omega^{ext}) =: C f(\omega^{ext}), \]
where \(C := \frac{1}{|GL(\Omega)|} \sum_{T \in GL(\Omega)} \beta(T \omega^{ext})\) is a positive constant which does not depend on the choice of \(\omega^{ext} \in \Omega^{ext}\) because of the transitivity of \(\Omega\). Thus, the map satisfies \(J_{av}(V_+) = V^{*\text{int}}_+\) since \(J_{av}(\Omega^{ext}) = C \Omega^{ext}\), and is strictly positive with respect to \(\langle \cdot, \cdot \rangle_{GL(\Omega)}\) since it is a summation of the strictly positive operators \(\{J_T\}_{T \in GL(\Omega)}\). Moreover, it satisfies
\[ J_{av} \circ T = T \circ J_{av} \]
for any \(T \in GL(\Omega)\). We thus find that \(J_{av} \circ P_M = P_M \circ J_{av}\) holds for the orthogonal projection \(P_M\) introduced in Proposition A.1. In fact,
\[
J_{av}(P_M x) = \frac{1}{|GL(\Omega)|} J_{av} \left( \sum_{T \in GL(\Omega)} Tx \right) = \frac{1}{|GL(\Omega)|} \sum_{T \in GL(\Omega)} T(J_{av} x) = P_M(J_{av} x)
\]
holds for all \(x \in V\). Therefore, \(J_{av}\) is decomposed into two parts as
\[ J_{av} = P_M \circ J_{av} \circ P_M + P_M^\perp \circ J_{av} \circ P_M^\perp, \]
where \(P_M^\perp = 1 - P_M\). We note that \(V_M^\perp = \text{Im} P_M^\perp = [\text{aff}(\Omega) - \omega_M] = \mathbb{R}^N\) and \(\dim V_M = \dim \text{Im} P_M = 1\) hold by virtue of Proposition 2.2.3. Therefore, the first part of (B.1) is proportional to \(1_{V_M} = 1_x = P_M\), and because we set \(\langle \omega_M, J_{\omega_M}\rangle_{GL(\Omega)} = 1\) and thus
\[
\langle \omega_M, P_M \circ J_{av} \circ P_M \omega_M\rangle_{GL(\Omega)} = \langle \omega_M, J_{av} \omega_M\rangle_{GL(\Omega)} = \langle \omega_M, P_M J_{\omega_M}\rangle_{GL(\Omega)} = \langle \omega_M, J_{\omega_M}\rangle_{GL(\Omega)} = 1 = \langle \omega_M, P_M \omega_M\rangle_{GL(\Omega)}
\]
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holds, it is proved that
\[ P_M \circ J_{av} \circ P_M = P_M. \]

Let us examine the second part. Suppose that there exists a nonzero \( x \in V_M^\perp \) such that \( Tx = x \) for all \( T \in GL(\Omega) \). Then, \( P_M x = x \neq 0 \) holds, and it contradicts to (A.4). Thus, we can find that \( GL(\Omega) \) acts irreducibly on \( V_M^\perp \), that is, only \( \{0\} \) and \( V_M^\perp = \mathbb{R}^n \) itself are invariant subspaces. It concludes that \( P_M^\perp J_{av} P_M^\perp \), which commutes with every element in \( GL(\Omega) \), is proportional to \( \mathbb{I}_{V_M^\perp} = \mathbb{I}_{\mathbb{R}^N} = P_M^\perp \) due to Schur’s lemma. Consequently, we obtain for some \( \xi > 0 \)
\[ J_{av} = P_M + \xi P_M^\perp, \]
and thus
\[ J_{av}(V_+) = (P_M + \xi P_M^\perp)(V_+) = V_{+\text{int}}^{\text{int}}_{+\langle \cdot, \cdot \rangle_{GL(\Omega)}}, \quad (B.2) \]

Let us introduce a linear bijection
\[ \Xi := \sqrt{J_{av}} = P_M + \sqrt{\xi} P_M^\perp, \]
strictly positive with respect to \( \langle \cdot, \cdot \rangle_{GL(\Omega)} \), and define \( \Omega' := \Xi \Omega \). It is easy to check that the positive cone \( V_+ \) generated by \( \Omega' \) is given by \( V'_+ = \Xi V_+ \), and \( GL(\Omega') = \Xi GL(\Omega) \Xi^{-1} = GL(\Omega) \) (moreover, the unique maximally mixed state of \( \Omega' \) is still \( \omega_M \)). In addition, we can find that
\[ V_{+\text{int}}^{\text{int}}_{+\langle \cdot, \cdot \rangle_{GL(\Omega')}} = \{ v \mid \langle v, w' \rangle_{GL(\Omega)} \geq 0, \ \forall w' \in V'_+ \} \]
\[ = \{ v \mid \langle v, \Xi w \rangle_{GL(\Omega)} \geq 0, \ \forall w \in V_+ \} \]
\[ = \Xi^{-1} V_{+\text{int}}^{\text{int}}_{+\langle \cdot, \cdot \rangle_{GL(\Omega)}}, \]
holds. Since (B.2) can be rewritten as
\[ \Xi V_+ = \Xi^{-1} V_{+\text{int}}^{\text{int}}_{+\langle \cdot, \cdot \rangle_{GL(\Omega)}}, \]
we can conclude
\[ V'_+ = V_{+\text{int}}^{\text{int}}_{+\langle \cdot, \cdot \rangle_{GL(\Omega')}}. \]

\[ \square \]

Remark
In the case of \( |\Omega^{\text{ext}}| = \infty \), there exists a counterexample of Lemma B.3. Let us consider a state space
\[ \Omega = \{ \xi (1, \mathbf{x}) = \xi (1, x_1, x_2, x_3) \in \mathbb{R}^4 \mid |\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2 \leq 1 \} \]

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(the Bloch ball). Ω defines a corresponding positive cone $V_+$ as

$$V_+ = \{ x \in \mathbb{R}^4 \mid x_0^2 - |x|^2 \geq 0, x_0 \geq 0 \},$$

which can be identified with a forward light cone of a Minkowski spacetime. We examine a pure Lorentz transformation $\Lambda$ defined for $\lambda \in \mathbb{R}$ as

$$\Lambda = \begin{bmatrix}
\cosh \lambda & \sinh \lambda & 0 & 0 \\
\sinh \lambda & \cosh \lambda & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

It is easy to prove that this $\Lambda$ is strictly positive. Since the pure Lorentz transformation preserves the Minkowski metric, it satisfies $\Lambda(V_+) = V_+$. However, $\Lambda$ transforms an extreme point $x = \{1, 0, 1, 0\}$ to

$$\Lambda(x) = \{cosh \lambda, sinh \lambda, 1, 0\},$$

which is not proportional to $x$. Investigating whether Proposition 2.3.2 still holds when $|\Omega^{\text{ext}}| = \infty$ is a future problem.

References

[1] W. Heisenberg, “Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik,” *Zeitschrift für Physik*, vol. 43, pp. 172–198, Mar 1927.

[2] M. Koashi, “Unconditional security of quantum key distribution and the uncertainty principle,” *Journal of Physics: Conference Series*, vol. 36, pp. 98–102, apr 2006.

[3] H. P. Robertson, “The uncertainty principle,” *Physical Review*, vol. 34, pp. 163–164, July 1929.

[4] J. B. M. Uffink, *Measures of uncertainty and the uncertainty principle*. PhD thesis, University of Utrecht, Utrecht, 1990.

[5] J. I. de Vicente and J. Sánchez-Ruiz, “Separability conditions from the landau-pollak uncertainty relation,” *Physical Review A*, vol. 71, p. 052325, May 2005.

[6] T. Miyadera and H. Imai, “Generalized landau-pollak uncertainty relation,” *Physical Review A*, vol. 76, p. 062108, Dec. 2007.

[7] D. Deutsch, “Uncertainty in quantum measurements,” *Physical Review Letters*, vol. 50, pp. 631–633, Feb. 1983.
[8] H. Maassen and J. B. M. Uffink, “Generalized entropic uncertainty relations,” Physical Review Letters, vol. 60, pp. 1103–1106, Mar. 1988.

[9] M. Krishna and K. R. Parthasarathy, “An entropic uncertainty principle for quantum measurements,” Sankhyā: The Indian Journal of Statistics, Series A (1961-2002), vol. 64, no. 3, pp. 842–851, 2002.

[10] P. Busch, P. J. Lahti, J.-P. Pellonpää, and K. Ylinen, Quantum Measurement. Springer International Publishing, 2016.

[11] E. Arthurs and M. S. Goodman, “Quantum correlations: A generalized heisenberg uncertainty relation,” Physical Review Letters, vol. 60, pp. 2447–2449, June 1988.

[12] E. Arthurs and J. L. Kelly Jr., “On the simultaneous measurement of a pair of conjugate observables,” Bell System Technical Journal, vol. 44, no. 4, pp. 725–729, 1965.

[13] M. Ozawa, “Universally valid reformulation of the heisenberg uncertainty principle on noise and disturbance in measurement,” Physical Review A, vol. 67, p. 042105, Apr. 2003.

[14] F. Buscemi, M. J. W. Hall, M. Ozawa, and M. M. Wilde, “Noise and disturbance in quantum measurements: An information-theoretic approach,” Physical Review Letters, vol. 112, p. 050401, Feb. 2014.

[15] T. Miyadera, “Uncertainty relations for joint localizability and joint measurability in finite-dimensional systems,” Journal of Mathematical Physics, vol. 52, no. 7, p. 072105, 2011.

[16] P. Busch and D. B. Pearson, “Universal joint-measurement uncertainty relation for error bars,” Journal of Mathematical Physics, vol. 48, no. 8, p. 082103, 2007, https://doi.org/10.1063/1.2759831.

[17] T. Miyadera and H. Imai, “Heisenberg’s uncertainty principle for simultaneous measurement of positive-operator-valued measures,” Physical Review A, vol. 78, p. 052119, Nov. 2008.

[18] P. Busch and T. Heinosaari, “Approximate joint measurements of qubit observables,” Quantum Information & Computation, vol. 8, p. 797818, Sept. 2008.

[19] R. F. Werner, “The uncertainty relation for joint measurement of position and momentum,” Quantum Information & Computation, vol. 4, p. 546562, Dec. 2004.

[20] S. P. Gudder, Stochastic Methods in Quantum Mechanics. New York: Dover, 1979.
[21] H. Araki, *Mathematical Theory of Quantum Fields*. Oxford: Oxford University Press, 1999.

[22] L. Hardy, “Quantum theory from five reasonable axioms,” 2001, quant-ph/0101012.

[23] J. Barrett, “Information processing in generalized probabilistic theories,” *Physical Review A*, vol. 75, p. 032304, Mar. 2007.

[24] G. Chiribella, G. M. D’Ariano, and P. Perinotti, “Informational derivation of quantum theory,” *Physical Review A*, vol. 84, p. 012311, July 2011.

[25] H. Barnum and A. Wilce, “Information processing in convex operational theories,” *Electronic Notes in Theoretical Computer Science*, vol. 270, no. 1, pp. 3 – 15, 2011. Proceedings of the Joint 5th International Workshop on Quantum Physics and Logic and 4th Workshop on Developments in Computational Models (QPL/DCM 2008).

[26] P. Janotta and H. Hinrichsen, “Generalized probability theories: what determines the structure of quantum theory?,” *Journal of Physics A: Mathematical and Theoretical*, vol. 47, no. 32, p. 323001, 2014.

[27] H. Barnum, J. Barrett, M. Leifer, and A. Wilce, “Generalized no-broadcasting theorem,” *Physical Review Letters*, vol. 99, p. 240501, Dec. 2007.

[28] H. Barnum, J. Barrett, M. Leifer, and A. Wilce, “Teleportation in general probabilistic theories,” in *Proceedings of Symposia in Applied Mathematics*, vol. 71, pp. 25–48, 2012.

[29] G. Kimura, K. Nuida, and H. Imai, “Physical equivalence of pure states and derivation of qubit in general probabilistic theories,” 2010, 1012.5361.

[30] G. Kimura, K. Nuida, and H. Imai, “Distinguishability measures and entropies for general probabilistic theories,” *Reports on Mathematical Physics*, vol. 66, no. 2, pp. 175 – 206, 2010.

[31] H. Barnum, J. Barrett, M. Krumm, and M. P. Müller, “Entropy, majorization and thermodynamics in general probabilistic theories,” 2015, 1508.03107.

[32] M. P. Müller, O. C. O. Dahlsten, and V. Vedral, “Unifying typical entanglement and coin tossing: on randomization in probabilistic theories,” *Communications in Mathematical Physics*, vol. 316, pp. 441–487, 2012.
[33] J. B. Conway, *A Course in Functional Analysis*. Springer-Verlag New York, 1985.

[34] G. Chiribella, G. M. D’Ariano, and P. Perinotti, “Probabilistic theories with purification,” *Physical Review A*, vol. 81, p. 062348, June 2010.

[35] H. H. Schaefer, *Topological Vector Spaces*. Springer-Verlag New York, 2nd ed., 1999.

[36] E. B. Davies, “Symmetries of compact convex sets,” *The Quarterly Journal of Mathematics*, vol. 25, pp. 323–328, Jan. 1974.

[37] M. P. Müller and C. Ududec, “Structure of reversible computation determines the self-duality of quantum theory,” *Physical Review Letters*, vol. 108, p. 130401, Mar. 2012.

[38] G. Kimura, “The bloch vector for n-level systems,” *Journal of the Physical Society of Japan*, vol. 72, no. SUPPL.C, pp. 185–188, 2003.

[39] P. Janotta, C. Gogolin, J. Barrett, and N. Brunner, “Limits on nonlocal correlations from the structure of the local state space,” *New Journal of Physics*, vol. 13, no. 6, p. 063024, 2011.

[40] R. Takakura, “Entropy of mixing exists only for classical and quantum-like theories among the regular polygon theories,” *Journal of Physics A: Mathematical and Theoretical*, vol. 52, p. 465302, Oct. 2019.

[41] D. S. Dummit and R. M. Foote, *Abstract Algebra*. Hoboken, New Jersey: John Wiley & Sons, Inc., 3rd ed., 2003.

[42] T. Heinosaari, T. Miyadera, and M. Ziman, “An invitation to quantum incompatibility,” *Journal of Physics A: Mathematical and Theoretical*, vol. 49, p. 123001, Feb. 2016.

[43] P. Busch, T. Heinosaari, J. Schultz, and N. Stevens, “Comparing the degrees of incompatibility inherent in probabilistic physical theories,” *Europhysics Letters*, vol. 103, p. 10002, July 2013.

[44] M. Plávala, “All measurements in a probabilistic theory are compatible if and only if the state space is a simplex,” *Physical Review A*, vol. 94, p. 042108, Oct. 2016.

[45] M. Banik, M. R. Gazi, S. Ghosh, and G. Kar, “Degree of complementarity determines the nonlocality in quantum mechanics,” *Physical Review A*, vol. 87, p. 052125, May 2013.

[46] N. Stevens and P. Busch, “Steering, incompatibility, and bell-inequality violations in a class of probabilistic theories,” *Physical Review A*, vol. 89, p. 022123, Feb. 2014.
[47] H. Barnum, M. P. Müller, and C. Ududec, “Higher-order interference and single-system postulates characterizing quantum theory,” *New Journal of Physics*, vol. 16, p. 123029, Dec. 2014.

[48] A. Jenčová and M. Plávala, “Conditions on the existence of maximally incompatible two-outcome measurements in general probabilistic theory,” *Physical Review A*, vol. 96, p. 022113, Aug. 2017.

[49] M. Krumm, H. Barnum, J. Barrett, and M. P. Müller, “Thermodynamics and the structure of quantum theory,” *New Journal of Physics*, vol. 19, no. 4, p. 043025, 2017.

[50] G. Kimura and K. Nuida, “On affine maps on non-compact convex sets and some characterizations of finite-dimensional solid ellipsoids,” *Journal of Geometry and Physics*, vol. 86, pp. 1 – 18, 2014.