THE SPACES OF GEODESIC TRIANGULATIONS ON SURFACES

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Abstract. In this paper, we study the topology of the space of geodesic triangulations on a surface. We give a new proof of the contractibility of the space of geodesic triangulations for the case of a convex polygon. We also show that the space of geodesic triangulations on a flat torus is homotopy equivalent to a torus. Finally, we give a constructive method to generate geodesic triangulations for star-shaped polygons by minimizing the weighted length energy.

1. Introduction

A triangulation of a fixed combinatorial type of $T$ on a surface with a Riemannian metric $(S,g)$ is a geodesic triangulation if each edge in $T$ is embedded as a geodesic arc in $S$. We study the space of geodesic triangulations with a fixed combinatorial type on certain surfaces, including a polygonal region in the Euclidean plane and a flat torus. We focus on the following two problems.

1. The embeddability problem: Given a surface $(S,g)$ with a triangulation $T$, can we construct a geodesic triangulation with the combinatorial type of $T$? In particular, if $S$ is a 2-disk with a triangulation $T$ and we specify the positions of the boundary vertices of $T$ in the plane so that they form a polygon, can we find positions of the interior vertices in the plane to construct a geodesic triangulation of $S$ with the combinatorial type of $T$?

2. The contractibility problem: If the space of geodesic triangulations on $(S,g)$ with a fixed combinatorial type of $T$ is not empty, what is the topology of this space? In particular, is it a contractible space?

In this paper, we first give a new proof of the contractibility of the space of geodesic triangulations of a fixed combinatorial type of $T$ for the case of a convex polygon $\Omega$ in $\mathbb{R}^2$. We construct a homotopy equivalence from this space to an affine subspace in Euclidean space using the idea of Tutte’s theorem[28], significantly simplifying the previous argument in [4]. We then give a constructive method to produce geodesic triangulations with a fixed combinatorial type for a star-shaped polygon under a mild assumption on the triangulation. This problem has been studied by Hong and Nagamochi [21]. The construction of geodesic triangulations of general polygon has been solved by Xu et al. [29]. Finally, we show that the idea of Tutte’s theorem can be generalized to determine the homotopy type of the space of geodesic triangulations of flat tori.

Key words and phrases. geodesic triangulations, Tutte’s embedding.

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These results can be regarded as discrete versions of classical results by Smale [25] and Earle and Eells [12] about surface diffeomorphisms. The group of diffeomorphisms of the 2-disk fixing the boundary, denoted by $D_0(D^2)$, is contractible. Similarly, the group of diffeomorphisms of a torus isotopic to the identity $D_0(T^2)$ is homeomorphic to $T^2 \times D_0(T^2_0, x_0)$, where $D_0(T^2_0, x_0)$ is a contractible space containing the space of all the diffeomorphisms in $D_0(T^2)$ fixing $x_0$.

These two problems have been studied in [3, 4, 6, 20], partly because they are closely related to the problem of determining the existence and uniqueness of differentiable structures on a triangulated manifolds [8]. They are also used to produce effective algorithms to solve graph morphing problems in [9, 15, 26, 27].

In the general setting, we can consider a finite $n$-dimensional simplicial complex $T$, whose polyhedron $|T|$ is homeomorphic to the $n$-dimensional disk $D^n$. A geodesic triangulation of $D^n$ with the combinatorial type of $T$ is determined by the positions of vertices of $T$ in $\mathbb{R}^n$. The space of all such geodesic triangulations is denoted by $\mathcal{GT}(D^n, T)$.

We can also interpret this space in terms of homeomorphisms. First assume there exists an initial geodesic triangulation of $D^n$. Then all the other geodesic triangulations are the images of the initial triangulation under simplexwise linear homeomorphisms fixing the boundary vertices of $T$, determined by the images of the interior vertices of $T$ in $\mathbb{R}^n$. The space of all such simplexwise linear homeomorphisms is denoted by $L(D^n, T)$. Ho showed in [20] that it was homeomorphic to $\mathcal{GT}(D^n, T)$.

When we restrict to the 2-dimensional case, Cairns [5, 6] initiated an investigation of the topology of the space of geodesic triangulations of a geometric triangle in the Euclidean plane and the round 2-sphere.

**Theorem 1.1.** If $\Omega$ is a geometric triangle with a triangulation $T$ in the plane, then $\mathcal{GT}(\Omega, T)$ is path-connected.

Ho [20] proved that this space was simply-connected.

**Theorem 1.2.** If $\Omega$ is a geometric triangle with a triangulation $T$ in the plane, then $\mathcal{GT}(\Omega, T)$ is simply-connected.

A *dividing edge* in a triangulation $T$ is an interior edge connecting two boundary vertices. Using an induction argument, Bing and Starbird [3] considered the general case of star-shaped polygons.

**Figure 1.** Three examples of geodesic triangulations of the 2-disk with fixed boundary vertices.
Theorem 1.3. If $\Omega$ is a star-shaped polygon with a triangulation $T$ in the plane, and $T$ does not contain any dividing edge, then $\mathcal{GT}(\Omega, T)$ is non-empty and path-connected.

Bing and Starbird\cite{3} showed that $\mathcal{GT}(\Omega, T)$ was not necessarily path-connected if we didn’t assume star-shaped boundary. Bloch, Connelly, and Henderson\cite{4} proved the contractibility of the space of simplexwise linear homeomorphisms of a convex 2-disk. In a very recent paper, Cerf \cite{7} improved the original argument in \cite{4} to give a new proof of the Bloch-Connelly-Henderson theorem.

Theorem 1.4 (Bloch-Connelly-Henderson). If $\Omega$ is a convex polygon with a triangulation $T$ in the plane, and $T$ does not contain any dividing edge, then $\mathcal{GT}(\Omega, T)$ is homeomorphic to $\mathbb{R}^{2k}$, where $k$ is the number of interior vertices of $T$.

This paper is organized as follows. In Section 2, we recall Tutte’s theorem and its generalizations. In Section 3, we give a new proof of the contractibility of $\mathcal{GT}(\Omega, T)$ if $\Omega$ is a convex polygon using Tutte’s method. In Section 4, we give an explicit construction of a geodesic triangulation in $\mathcal{GT}(\Omega, T)$ if $\Omega$ is a strictly star-shaped polygon, assuming the triangulation does not contain any dividing edge. In Section 5, we give a characterization of a special class of geodesic triangulations corresponding to the minimizers of weighted length energies. In Section 6, we show that $\mathcal{GT}(\mathbb{T}^2, T)$ has the homotopy type of the torus. In Section 7, we discuss some open problems about the space of geodesic triangulations for other surfaces.

2. Tutte’s embedding and its generalization

2.1. Tutte’s embedding for the disk. Given a triangulation $T = (V, E, F)$ of the 2-disk with the sets of vertices $V$, edges $E$ and faces $F$, the 1-skeleton of $T$ is a planar graph. There is no canonical method to embed this graph in the plane. Tutte\cite{28} provided an efficient method to construct a straight-line embedding of a 3-vertex-connected planar graph by specifying the coordinates of vertices of one face as a convex polygon and solving for the coordinates of other vertices with a linear system of equations. Using a discrete maximal principle, Floater\cite{13} proved the same result for triangulations of the 2-disk. Gortler, Gotsman, and Thurston\cite{17} reproved Tutte’s theorem with discrete one forms and generalized this results to the case of multiple-connected polygonal regions with appropriate assumptions on the boundaries. Since we are dealing with triangulations, we use the formulation given by Floater\cite{13}.

![Figure 2. Tutte’s embedding](image-url)
Theorem 2.1. Assume $T = (V, E, F)$ is a triangulation of a convex polygon $\Omega$, and $\phi$ is a simplexwise linear homeomorphism from $T$ to $\mathbb{R}^2$. If $\phi$ maps every interior vertex in $T$ into the convex hull of the images of its neighbors, and maps the cyclically ordered boundary vertices of $T$ to the cyclically ordered boundary vertices of $\Omega$, then $\phi$ is one to one.

As Floater pointed out, this theorem gave a discrete version of the Rado-Kneser-Choquet theorem about harmonic maps from the disk to a convex polygon. Moreover, it gives a constructive method to produce geodesic triangulations of a convex polygon with the combinatorial type of $T$ as follows.

First assign a positive weight $c_{ij}$ to a directed edge $(i, j) \in \bar{E}$, where $\bar{E}$ is the set of directed edges of $T$. We normalize the weights by $w_{ij} = \frac{c_{ij}}{\sum_{j \in N(v_i)} c_{ij}}$ where the set $N(v_i)$ consists of all the vertices that are neighbors of $v_i$, so that $\sum_{j \in N(v_i)} w_{ij} = 1$ for all $i = 1, 2, ..., N_I$. Notice that we don’t impose symmetry condition $w_{ij} = w_{ji}$. We are given the coordinates $\{(b^x_i, b^y_i)\}_{i=N_I+1}^{|V|}$ for all the boundary vertices such that they form a convex polygon $\Omega$ in $\mathbb{R}^2$. Then we can solve the following linear system

\[
\sum_{j \in N(v_i)} w_{ij} x_j = x_i, \quad i = 1, 2, ..., N_I;
\]
\[
\sum_{j \in N(v_i)} w_{ij} y_j = y_i, \quad i = 1, 2, ..., N_I;
\]
\[
x_i = b^x_i, \quad i = N_I + 1, N_I + 2, ..., N_I + N_B = |V|;
\]
\[
y_i = b^y_i, \quad i = N_I + 1, N_I + 2, ..., N_I + N_B = |V|
\]

where $N_I = |V_I|$ is the size of the set of interior vertices $V_I$, and $N_B = |V_B|$ is the size of the set of boundary vertices $V_B$. The solution to this linear system produces the coordinates of all the interior vertices in $\mathbb{R}^2$. We put the vertices in the positions given by their coordinates, and connect the vertices based on the combinatorics of the triangulation $T$. Tutte’s theorem claims that the result is a geodesic triangulation of $\Omega$ with the combinatorial type of $T$.

The linear system above implies that the $x$-coordinate(or $y$-coordinate) of one interior vertex is a convex combination of the $x$-coordinates(or $y$-coordinates) of its neighbors. Notice that the coefficient matrix of this system is not necessarily symmetric but it is diagonally dominant, so the solution exists uniquely.

Tutte’s theorem solves the embeddability problem for a triangulation of a convex polygon. We can vary the coefficients $w_{ij}$ to construct families of geodesic triangulations of a convex polygon. We will see that this idea will lead to a simple proof of the contractibility of the space of geodesic triangulations.

2.2. Tutte’s embedding for flat tori. In the case of a flat torus $(\mathbb{T}^2, g)$ with a triangulation $T$, the situation is similar to the disk case, because we can lift a geodesic triangulation of $(\mathbb{T}^2, g)$ to the universal covering $\mathbb{R}^2$. Using the method in Gu and Yau[13] and Gortler, Gotsman, and Thurston[17], we can compute the harmonic one form to produce geodesic triangulations on $\mathbb{T}^2$ with a fixed combinatorial type of $T$. 

Specifically, we first assign a positive weight \( c_{ij} \) to each directed edge in \( T \) and normalize the weights as in the case of the 2-disk to produce positive weights \( w_{ij} \) satisfying \( \sum_{j \in N(v_i)} w_{ij} = 1 \) for all \( i = 1, 2, ..., N_I \). Instead of computing the coordinates for vertices in \( T \) directly, we compute the harmonic one forms \( \Delta z : \bar{E} \to \mathbb{R} \) by solving the following system of equations

\[
\Delta z_{ij} = -\Delta z_{ji} \quad \text{for all directed edges } (i, j) \in \bar{E};
\]

\[
\sum_{v_j \in N(v_i)} w_{ij} \Delta z_{ij} = 0 \quad \text{for all vertices } v_i \in V;
\]

\[
\Delta z_{ij} + \Delta z_{jk} + \Delta z_{ki} = 0 \quad \text{for all faces } f_{ijk} \in F.
\]

(2.1)

Gortler, Gotsman, and Thurston[17] showed that this linear system had exactly two independent solutions, denoted by \( \Delta x \) and \( \Delta y \). Then we can assign a vertex \( v_0 \) to the origin in \( \mathbb{R}^2 \) and compute the coordinates for other vertices \( v \) by summing the entries of the discrete one forms along a path \( p \) consisting a sequence of directed edges in \( T \) from \( v_0 \) to \( v \).

(2.2) \((x_0, y_0) = (0, 0) \) and \((x_i, y_i) = (\sum_{(i,j) \in p} \Delta x_{ij}, \sum_{(i,j) \in p} \Delta y_{ij}) \) for other vertices.

Since the discrete form is closed, the coordinates for \((x_i, y_i)\) are independent of the choice of the paths.

**Theorem 2.2** ([17]). Given a triangulation \( T \) of \((T^2, g)\) whose 1-skeleton is a 3-vertex-connected graph, the two linearly independent solutions of the system above produce embeddings of any sub-triangulations \( T' \) of \( T \) with the topology of a disk.

Gortler, Gotsman, and Thurston[17] pointed out that this statement of local injectivity produced a globally injective map from the universal cover of the torus to the Euclidean plane. We can generate families of equivariant geodesic triangulations in \( \mathbb{R}^2 \) projecting to geodesic triangulations on \((T^2, g)\) by varying the weights \( w_{ij} \) in the linear system. If we choose a different pair of harmonic one forms \( \Delta x' \) and \( \Delta y' \), then the resulting geodesic triangulation in \( \mathbb{R}^2 \) is the image of the original geodesic triangulation under an affine transformation. This method was extended by Aigerman and Lipman[1] to Euclidean orbifolds with spherical topology.

### 3. Geodesic Triangulations of the 2-Disk with Convex Boundary

In this section, we define the space of geodesic triangulations for the disk, and give a new proof of the contractibility of \( \mathcal{GT}(\Omega, T) \) if \( \Omega \) is a convex polygon.

**Definition 3.1.** Given a triangulation \( T = (V, E, F) \) of the 2-disk, fix the boundary vertices \( \{v_i\}_{i=1}^{V_I} \) of \( T \) in \( \mathbb{R}^2 \) with coordinates \( (b_i^x, b_i^y) \) and connect them based on \( T \) such that they form a convex polygon \( \Omega \) in \( \mathbb{R}^2 \). The space of geodesic triangulations \( \mathcal{GT}(\Omega, T) \) is defined as the set of all the geodesic triangulations of \( \Omega \) with the combinatorial type of \( T \) whose boundary vertices \( \{v_i\}_{i=1}^{V_I} \) have the corresponding coordinates \( (b_i^x, b_i^y) \).

Every geodesic triangulation is uniquely determined by the positions of the interior vertices in \( V_I \), so its topology is the subspace topology induced by \( \Omega \). Notice that this space could be empty if the boundary is complicated. For
instance, if the polygon is not star-shaped, then there doesn’t exist any geodesic embedding of a triangulation with only one interior vertex. Nevertheless, Tutte’s theorem shows that this space is not empty if the polygonal region \( \Omega \) is convex.

Let us consider the topology of the space \( \mathcal{GT}(\Omega, T) \) where \( \Omega \) is a fixed convex polygon in \( \mathbb{R}^2 \). Let \( E_I \) be the set of interior edges in \( T \) and \( E_B \) be the set of boundary edges in \( T \).

**Definition 3.2.** Given a triangulation \( T \) of \( \Omega \) with coordinates of the boundary vertices \( \{(b^x_i, b^y_i) \}_{i=1}^{\left| V \right|} \), define \( W \) to be the space of positive weights \( (w_{ij}) \in \mathbb{R}^{\left| E_I \right|} \) on the set of directed edges of \( T \) satisfying the normalization condition \( \sum_{j \in N(v_i)} w_{ij} = 1 \) for all \( v_i \in V_I \). The Tutte map \( \Psi \) sends the weights in \( W \) to the solution to the linear system in Tutte’s theorem with coefficients \( (w_{ij}) \) and \( \{(b^x_i, b^y_i)\} \).

The weight space \( W \) is a \( 2 \left| E_I \right| - \left| V_I \right| \) dimensional affine manifold in \( \mathbb{R}^{\left| E_I \right|} \). The image \( \mathcal{GT}(\Omega, T) \) is a \( 2 \left| V_I \right| \) dimensional manifold. By Euler characteristic \( \chi(\Omega) = \left| V \right| - \left| E \right| + \left| F \right| = 1 \) and the requirement of simplicial complex \( 3 \left| F \right| = 2 \left| E_I \right| + \left| E_B \right| \), we can deduce that \( \left| E_I \right| - 3 \left| V_I \right| = \left| E_B \right| - 3 \). Hence the dimension of the space of weights \( W \) is not lower than the dimension of \( \mathcal{GT}(\Omega, T) \).

**Lemma 3.3.** The Tutte map \( \Psi \) is continuous and surjective from the space of weights \( W \) to \( \mathcal{GT}(\Omega, T) \).

**Proof.** By Tutte’s theorem, for any \( (w_{ij}) \in W \), the solution to the linear system generates a geodesic triangulation of \( T \). The continuity follows from the continuous dependence of the solutions on the coefficients in the linear system. To show surjectivity, given a geodesic triangulation \( \tau \), any interior vertex \( v_i \) in \( \tau \) is in the convex hull of its neighbors. Then we can construct the weights \( (w_{ij}) \) for a geodesic triangulation \( \tau \) using the mean value coordinates defined in [14] below.

![Figure 3. The mean value coordinate at \( v_0 \)](image)

The mean value coordinates on the directed edges of a geodesic triangulation are given by

\[
w_{ij} = \frac{c_{ij}}{\sum_{j \in N(v_i)} c_{ij}} \quad \text{and} \quad c_{ij} = \frac{\tan(\alpha_{i-1}^j/2) + \tan(\alpha_i^j/2)}{||v_i - v_j||}
\]

where the two angles \( \alpha_{i-1}^j \) and \( \alpha_i^j \) at \( v_i \) sharing the edge \((i, j) \in E_I\) in the Figure 3. The mean value coordinates provide a smooth map from \( \mathcal{GT}(\Omega, T) \) to \( W \). \( \square \)
There are various ways to construct the weights from a given geodesic triangulation other than the mean value coordinates. Floater proposed another construction by taking the average of barycentric coordinates\(^{15}\). An alternative method to construct weights from a geodesic triangulation \(\tau\) is to take the center of mass of the space of weights \((w_{ij})\in W\) such that \(\Psi((w_{ij})) = \tau\). This subspace is a convex subspace of \(W\) and the center of mass is well-defined. All three methods agree with the barycentric coordinates of a vertex when the star of this vertex is a triangle.

**Definition 3.4.** The map \(\sigma : \mathcal{G}T(\Omega, T) \rightarrow W\) sends a geodesic triangulation \(\tau\) to weights \((w_{ij})\) in \(W\) determined by the mean value coordinates.

**Theorem 3.5.** If \(\Omega\) is a convex polygon in \(\mathbb{R}^2\) with a triangulation \(T\), the space of geodesic triangulations \(\mathcal{G}T(\Omega, T)\) is contractible.

**Proof.** The map \(\sigma\) is continuous. By Tutte’s theorem, \(\Psi(\sigma(\tau)) = \tau\) for any \(\tau \in \mathcal{G}T(\mathbb{D}^2, T)\), so the map \(\sigma\) is a global section of \(\Psi\) from \(\mathcal{G}T(\Omega, T)\) to \(W\). We need to show \(\sigma \circ \Psi\) is homotopic to the identity map on \(W\). From the previous discussion, we know that \(W\) is an affine manifold in \(\mathbb{R}^{|E|}\), so we can use the isotopy \((1-t)\sigma \circ \Psi + t1\) where \(1\) is the identity map on \(W\). Since \(W\) is a contractible space, \(\mathcal{G}T(\Omega, T)\) is contractible by this homotopy equivalence. \(\square\)

Although we mainly consider triangulations in this paper, this argument can be generalized to the case of the convex geodesic embedding of a 3-vertex-connected graph \(G\), which is defined to be a geodesic embedding of \(G\) in the plane such that all its faces are convex. Then using the same idea of Tutte’s theorem, we can show the contractibility of the space of convex geodesic triangulations of \(G\) with the prescribed convex boundary \(\Omega\).

We can extend this result to convex polygons in other geometries of constant curvature. More precisely, if we have a convex polygon in the hyperbolic plane or a convex polygon in the round 2-sphere contained in a hemisphere, we can reduce it to the case of convex polygon in the Euclidean plane.

For a hyperbolic convex polygon \(\Omega_H\), we embed it in the Klein model of the hyperbolic plane so that all the edges of \(\Omega_H\) are straight arcs in the Euclidean metric, inducing a convex polygon \(\Omega\) in the Euclidean plane. Given a triangulation \(T\) of \(\Omega_H\), there is a bijection between the space of all hyperbolic geodesic triangulations of \(\Omega_H\) represented in the Klein model and \(\mathcal{G}T(\Omega_H, T)\), induced by the identity map on \(\Omega^{|V|}\). Hence the space of hyperbolic geodesic triangulations \(\mathcal{G}T(\Omega_H, T)\) is also contractible.

Similarly, if \(\Omega_S\) is a spherical convex polygon contained in a hemisphere with a triangulation \(T\), we can apply the gnomonic transformation from the center of the 2-sphere to the plane \(P\) tangent to the center of the hemisphere containing \(\Omega_S\). Then \(\Omega_S\) is mapped to a convex polygon \(\Omega\) in the plane \(P\) under the gnomonic transformation. This projective transformation keeps the incidence and maps geodesic arcs in hemisphere to the straight arcs in \(P\). Hence it induces a bijection between the space of spherical geodesic triangulations of \(\Omega_S\) with combinatorial type of \(T\) and \(\mathcal{G}T(\Omega, T)\) in \(P\).

**Corollary 3.6.** Assume \(\Omega\) is a hyperbolic convex polygon, or a spherical convex polygon contained in a hemisphere, and \(T\) is a triangulation of \(\Omega\). Then the space of geodesic triangulations \(\mathcal{G}T(\Omega, T)\) is contractible.
4. Geodesic Triangulations of the 2-Disk with star-shaped Boundary

In this section, we consider a star-shaped subset \( \Omega \) of \( \mathbb{R}^2 \). An eye of a star-shaped region \( \Omega \) is a point \( p \) in \( \Omega \) such that for any other point \( q \) in \( \Omega \) the line segment \( l(t) = tp + (1-t)q \) lies inside \( \Omega \). The set of eyes of \( \Omega \) is called the kernel of \( \Omega \). A set is called strictly star-shaped if the interior of the kernel is not empty.

In the case of polygons in \( \mathbb{R}^2 \), the kernel is the intersection of a family of closed half-spaces, each defined by the line passing one boundary edge of \( \Omega \). Every closed half space contains a half disk in \( \Omega \) centered at one point on its corresponding boundary edge. If the star-shaped polygon is strict, the intersection of the open half-spaces is not empty. This means that we can pick an eye \( e \) with a neighborhood \( U \) of \( e \) such that if \( q \in U \), then \( q \) is also an eye of \( \Omega \).

The first question to address is how to construct a geodesic triangulation of a strictly star-shaped polygon \( \Omega \) with a combinatorial type of \( T \). As Bing and Starbird\[3\] pointed out, it was not always possible if there was a dividing edge. Assuming there was no dividing edge in \( T \), they proved that such geodesic triangulations existed by induction.

We give an explicit method to produce a geodesic triangulation for a strictly star-shaped polygon. We can regard all the edges \( e_{ij} \) in \( T \) as ideal springs with Hook constants \( w_{ij} \). Fixing the boundary vertices, the equilibrium state corresponds to the critical point of the weighted length energy defined as

\[
E = \frac{1}{2} \sum_{e_{ij} \in E} w_{ij} L_{ij}^2
\]

where \( L_{ij} \) is the length of the edge connecting \( v_i \) and \( v_j \). This energy can be regarded as a discrete version of the Dirichlet energy \[10\] \[19\], and it has a unique minimizer corresponding to the equilibrium state. Tutte’s theorem guarantees that the equilibrium state is a geodesic embedding of \( T \) if the boundary is a convex polygon.

Given a triangulation \( T \) of a fixed strictly star-shaped polygon \( \Omega \), assume that the weighted length energy \( E \) satisfies \( \sum_{e_{ij} \in E} w_{ij} = 1 \). Notice that if the polygon is star-shaped but not convex, we can’t choose arbitrary weights to generate a geodesic embedding of \( T \). Hence we need to assign weights carefully to avoid singularities such as intersections of edges and degenerate triangles.

The idea is to distribute more and more weights to the interior edges connecting two interior vertices. As the weights for interior edges connecting two interior vertices tend to 1, all the interior vertices will concentrate at a certain point. If we can choose this point to be an eye of the polygon, we will produce an geodesic embedding of \( T \) of \( \Omega \).

Fix a polygon \( \Omega \) with a triangulation \( T \) and the coordinates \( \{(b_x^i, b_y^i)\}_{i=1}^{N_I+1} \) for its boundary vertices. Given a set of coordinates in \( \mathbb{R}^2 \) for all the interior vertices \( \{(x_i, y_i)\}_{i=1}^{N_I} \), we define a family of weighted length energies with a parameter \( 0 < \epsilon < 1 \) as

\[
E(\epsilon) = \frac{1-\epsilon}{2M_I} \sum_{e_{ij} \in E_I^I} L_{ij}^2 + \frac{\epsilon}{2M_B} \sum_{e_{ij} \in E_I^B} L_{ij}^2
\]

where \( E_I^B \) is the set of all the interior edges connecting an interior vertex to a boundary vertex and \( E_I^I \) is the set of all the interior edges connecting two interior vertices. Let \( M_B = |E_I^B| \) and \( M_I = |E_I^I| \). The edge lengths \( L_{ij} \) are determined by
Proof. The minimizer of the energy of the vertices

\[ L_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2. \]

As \( \epsilon \to 0 \), most weights are assigned to interior edges in \( E_I \), forcing all the interior vertices of the minimizer of \( \mathcal{E}(\epsilon) \) to concentrate to one point.

**Theorem 4.1.** Let \( \Omega \) be a polygonal region with a triangulation \( T \) of \( \Omega \). Let \( v_j^B = (x_j^B, y_j^B) = (b_j^x, b_j^y) \) for \( j = 1, \ldots, N_B \) be the coordinates of the boundary vertices of \( \Omega \) and \( v_i^I(\epsilon) = (x_i^I(\epsilon), y_i^I(\epsilon)) \) for \( i = 1, \ldots, N_I \) be the coordinates of the interior vertices of the minimizer of the energy \( \mathcal{E}(\epsilon) \). Then for all \( i = 1, 2, \ldots, N_I \),

\[
\lim_{\epsilon \to 0} v_i^I = (x_i^I(\epsilon), y_i^I(\epsilon)) = (x_0, y_0) = v_0
\]

where

\[
v_0 = \sum_{j=1}^{N_B} \lambda_j v_j^B \quad \text{and} \quad \lambda_j = \frac{\deg(v_j^B) - 2}{\sum_{j=1}^{N_B} \deg(v_j^B) - 2} \frac{M_B}{\deg(v_j^B)},
\]

assuming \( \deg(v) \) is the degree of the vertex \( v \) in \( T \).

Proof. The minimizer of \( \mathcal{E}(\epsilon) \) satisfies the following linear system formed by taking derivatives with respect to \( x_i \) and \( y_i \) for all \( i = 1, 2, \ldots, N_I \)

\[
\frac{1 - \epsilon}{M_I} \sum_{i \in N(v_i^I)} (v_i^I - v_i^B) + \frac{\epsilon}{M_B} \sum_{j \in N(v_i^I)} (v_j^I - v_j^B) = 0 \quad \text{for } k = 1, 2, \ldots, N_I.
\]

Notice that we separate the interior vertices \( v_i^I \in V_I \) and the boundary vertices \( v_j^B \in V_B \) in the summation. This system can be represented as

\[
M(\epsilon)x = b_x \quad M(\epsilon)y = b_y
\]

where the variables are

\[
x = (x_1^I, x_2^I, \ldots, x_{N_I}^I, x_1^B, \ldots, x_{N_B}^B)^T
\]

and

\[
y = (y_1^I, y_2^I, \ldots, y_{N_I}^I, y_1^B, \ldots, y_{N_B}^B)^T.
\]

The boundary conditions are

\[
b_x = (0, 0, \ldots, 0, x_1^B, \ldots, x_{N_B}^B)^T
\]

and

\[
b_y = (0, 0, \ldots, 0, y_1^B, \ldots, y_{N_B}^B)^T.
\]

The coefficient matrix \( M(\epsilon) \) is an \((N_I + N_B) \times (N_I + N_B)\) matrix, and it can be decomposed as

\[
M(\epsilon) = \begin{pmatrix} S(\epsilon) & -\epsilon W \\ 0 & Id \end{pmatrix}
\]

where \( W \) is an \( N_I \times N_B \) matrix, \( S(\epsilon) \) is a square matrix of size \( N_I \), and \( Id \) is the identity matrix of size \( N_B \). The matrix \( W \) is defined as

\[
W(i, j) = \begin{cases} \frac{1}{M_B} & \text{if } v_i^I \text{ is connected to } v_j^B; \\ 0 & \text{if } v_i^I \text{ is not connected to } v_j^B. \end{cases}
\]
The matrix $S$ is defined as

$$
S(i,j)(\epsilon) = \begin{cases} 
- \sum_{i \neq k} S(i,k) + \epsilon \sum_{k=1}^{N_B} W(i,k) & \text{if } i = j; \\
- \frac{\epsilon^2}{M_T} & \text{if } v^j_i \text{ is connected to } v^j_i; \\
0 & \text{if } v^j_i \text{ is not connected to } v^j_i.
\end{cases}
$$

Notice that for the first $N_I$ rows in $M(\epsilon)$, the sums of their respective entries are zero, and all the off-diagonal terms are non-positive. The matrix $W$ represents the relations of the boundary vertices with the interior vertices, and the sum of all its entries equals one. The matrix $S(\epsilon)$ is symmetric, strictly diagonally-dominant, and the sum of all its entries equals $\epsilon$.

To show the limiting behavior of the solution to the system as $\epsilon \to 0$, we need the lemma below.

**Lemma 4.2.** Given the notations above, we have

$$\lim_{\epsilon \to 0} \epsilon S(\epsilon)^{-1} = 1$$

where the matrix $1$ is the $N_I \times N_I$ matrix with all entries equal to 1.

**Proof.** Notice that $S(\epsilon)$ is symmetric and strictly diagonally dominant, so it is invertible. Let $S = S(0)$ and $M = M(0)$, then $S$ has an eigenvalue $\lambda = 0$ with the normalized eigenvector $v = (1/\sqrt{N_I}, 1/\sqrt{N_I}, ..., 1/\sqrt{N_I})^T$.

First, we show that $\lambda = 0$ is a simple eigenvalue for $S$. If $S$ has another eigenvector $u = (u_1, u_2, ..., u_{N_I})^T$ corresponding to $\lambda = 0$ not parallel to $v$, then it is orthogonal to $v$ so $\sum_{i=1}^{N_I} u_i = 0$. Without loss of generality, we assume that $u_1 > 0$ achieves the maximal absolute value among $u_i$. Then we have

$$Su = 0 \implies \sum_{i=1}^{N_I} S(i,1)u_i = 0 \implies S(1,1)u_1 = -\sum_{i=2}^{N_I} S(1,i)u_i.$$

Notice that $S$ is weakly diagonally dominant, $S(1,1) > 0$, and $S(1,i) \leq 0$, so we can deduce that

$$S(1,1)u_1 \geq -\sum_{i=2}^{N_I} S(1,i)u_1 \implies -\sum_{i=2}^{N_I} S(1,i)(u_i - u_1) \geq 0.$$

By our assumption, $u_i - u_1 \leq 0$ for all $i = 1, ..., N_I$, so the only possibility is $u_i = u_1$ for all $i$, which contradicts the fact that $u$ is orthogonal to $v$. Hence all the other eigenvalues of $S$ are positive by Gershgorin circle theorem. (See, e.g. [10])

Second, we show that the eigenvalue $\lambda(\epsilon)$ of $S(\epsilon)$ approaching to 0 satisfies

$$\lim_{\epsilon \to 0} \frac{\lambda(\epsilon)}{\epsilon} = \frac{1}{N_I}.$$ 

This means that the derivative $(d\lambda/d\epsilon)(0) = 1/N_I$. To compute the derivative, notice that the sum of all the entries of $S(\epsilon)$ is $\epsilon$, hence we have

$$v^T S(\epsilon) v = \frac{1}{N_I} (1, 1, ..., 1) S(\epsilon) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{\epsilon}{N_I}. $$
The derivative of a simple eigenvalue of a symmetric matrix is given in [23] by

\[
\frac{d\lambda}{d\epsilon}(0) = \frac{d(v^T S(\epsilon)v)}{d\epsilon} = \frac{d(\epsilon/N_I)}{\epsilon} = \frac{1}{N_I}.
\]

Finally, we are ready to prove the lemma. Since \(S(\epsilon)\) is symmetric, we have the diagonalization with an orthonormal matrix \(P(\epsilon)\)

\[
\epsilon S^{-1}(\epsilon) = P(\epsilon) \begin{pmatrix}
\epsilon \lambda_1^{-1}(\epsilon) & & \\
& \epsilon \lambda_2^{-1}(\epsilon) & \\
& & \ddots \\
& & & \epsilon \lambda_N^{-1}(\epsilon)
\end{pmatrix} P^T(\epsilon).
\]

Without loss of generality, we assume the first eigenvalue \(\lim_{\epsilon \to 0} \lambda_1(\epsilon) = 0\). Given any \(0 < \delta < 1\), we can choose small \(\epsilon > 0\) such that the following three inequality holds

\[
\lambda_i(\epsilon) > C > 0 \text{ for } i = 2, 3, ..., N_I;
\]

\[
\|P(\epsilon) \begin{pmatrix}
\epsilon \lambda_1^{-1}(\epsilon) & & \\
& \epsilon \lambda_2^{-1}(\epsilon) & \\
& & \ddots \\
& & & \epsilon \lambda_N^{-1}(\epsilon)
\end{pmatrix} P^T(\epsilon) - P(\epsilon)
\begin{pmatrix}
N_I \\
& 0 \\
& & \ddots \\
& & 0
\end{pmatrix} P^T(\epsilon)\|_2 < \delta;
\]

and the eigenvector \(v_1(\epsilon)\) of \(S(\epsilon)\) corresponding to the eigenvector \(\lambda_1(\epsilon)\) satisfies

\[
\|v_1(\epsilon) - \frac{1}{\sqrt{N_I}} \begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix}\|_\infty < \delta.
\]

Notice that the columns of \(P(\epsilon) = (v_1, v_2, ..., v_{N_I})\) form a set of the orthonormal basis formed by eigenvectors \(v_i\), where the first eigenvector \(v_1(\epsilon)\) approaches \(v = (1/\sqrt{N_I}, ..., 1/\sqrt{N_I})\). Then we have

\[
\|\epsilon S^{-1}(\epsilon) - I\|_2 \leq \|P(\epsilon) \begin{pmatrix}
\epsilon \lambda_1^{-1}(\epsilon) & & \\
& \epsilon \lambda_2^{-1}(\epsilon) & \\
& & \ddots \\
& & & \epsilon \lambda_N^{-1}(\epsilon)
\end{pmatrix} P^T(\epsilon) - P(\epsilon)
\begin{pmatrix}
N_I \\
& 0 \\
& & \ddots \\
& & 0
\end{pmatrix} P^T(\epsilon)\|_2
\]

\[
+ \|P(\epsilon) \begin{pmatrix}
N_I \\
& 0 \\
& & \ddots \\
& & 0
\end{pmatrix} P^T(\epsilon) - I\|_2 \leq \delta + \|N_I v_1^T(\epsilon)v_1(\epsilon) - I\|_2.
\]

Notice that

\[
\|N_I v_1^T(\epsilon)v_1(\epsilon) - I\|_2 \leq 2N_I^2 \delta.
\]

Hence

\[
\|\epsilon S^{-1}(\epsilon) - I\|_2 \leq (1 + 2N_I^2)\delta.
\]

\(\square\)
The inverse of the matrix $M(\epsilon)$ can be represented as

$$M^{-1}(\epsilon) = \begin{pmatrix} S^{-1}(\epsilon) & \epsilon S^{-1}(\epsilon) W \\ 0 & I \end{pmatrix}.$$  

Then the solution of the linear system $M(\epsilon)x = b$ is $x = M^{-1}(\epsilon)b$, whose first $N_I$ entries are given by

$$\begin{pmatrix} x_1^I(\epsilon) \\ x_2^I(\epsilon) \\ \vdots \\ x_{N_I}^I(\epsilon) \end{pmatrix} = \epsilon S^{-1}(\epsilon) W \begin{pmatrix} x_1^B \\ x_2^B \\ \vdots \\ x_{N_B}^B \end{pmatrix}.$$  

As $\epsilon \to 0$, the solution approaches $Wx^B$. All the $x_i^I$ approach the same point

$$\lim_{\epsilon \to 0} x_i^I = (1, \ldots, 1)\, \text{W}$$

A similar result holds for $y$-coordinates of the interior vertices. Hence we conclude the limit of the solutions $\lim_{\epsilon \to 0} v_i^I = v_0$.  

Notice that the matrix $W$ can be replaced with more general matrices. The original energy $E(\epsilon)$ distributes $\epsilon$ percentage of weights evenly to all the edges in $E^B_I$. We can define new energies by redistributing the weights

$$E^W(\epsilon) = \frac{1 - \epsilon}{2M_I} \sum_{e_{ij} \in E^I_i} L^2_{ij} + \frac{\epsilon}{2} \sum_{e_{ij} \in E^B_i} w_{ij} L^2_{ij}$$

with $w_{ij} > 0$ and $\sum_{(i,j) \in E^B} w_{ij} = 1$. The matrix $W$ is defined as

$$W(i,j) = \begin{cases} w_{ij} & \text{if } v_i^I \text{ is connected to } v_j^B \\
0 & \text{if } v_i^I \text{ is not connected to } v_j^B \end{cases}.$$  

The limit of the solution is

$$v_0 = \sum_{j=1}^{N_B} \lambda_j v_j^B \quad \text{where} \quad \lambda_j = \sum_{i=1}^{N_I} w_{ij}.$$  

To construct a geodesic triangulation, pick an eye $e$ of $\Omega$ such that $e = \sum_{i=1}^{N_I} \lambda_i v_i^B$ where $\lambda_i > 0$ and $\sum_{i=1}^{N_I} \lambda_i = 1$, then define

$$W(i,j) = \begin{cases} w_{ij} \frac{\lambda_j}{\deg(v_j^B) - 2} & \text{if } v_i^I \text{ is connected to } v_j^B \\
0 & \text{if } v_i^I \text{ is not connected to } v_j^B \end{cases}.$$  

and the corresponding energy $E^W(\epsilon)$. The remaining task is to show that the critical point of $E^W(\epsilon)$ is a geodesic embedding of $T$ for small $\epsilon$.

If $\Omega$ is not convex, there exists a reflex vertex, defined as a boundary vertex of $\Omega$ where the turning angle is negative. We use the result by Gortler, Gotsman and Thurston[17] to show that the minimizer of $E^W(\epsilon)$ constructed above is an embedding for some $\epsilon > 0$. 
Theorem 4.3 ([17]). Given a strictly star-shaped polygon $\Omega$ with a triangulation $T$ without dividing edges, if the reflex vertices of $\Omega$ are in the convex hull of their respective neighbors, then the solution to the linear system generates a straight-line embedding of $T$.

Theorem 4.4. Given a strictly star-shaped polygon $\Omega$ with a triangulation $T$ without dividing edges, and an eye $e$ in $\Omega$ with coefficients $W$, there exists an $\epsilon > 0$ such that the critical point of the energy $E^W(\epsilon)$ generates a geodesic embedding of $T$.

Proof. Theorem 4.3 implies that we only need to check that the reflex vertices $v_r$ are in the convex hulls of their respective neighbors.

Choose an $\epsilon$ small enough such that the vertices of the critical point of $E^W(\epsilon)$ defined above are eyes of $\Omega$. Assume $v_r$ is a reflexive point on the boundary of $\Omega$. Let $v$ be an interior vertex of the geodesic triangulation in the star of $v_r$, and let $v_1$ and $v_2$ be the two boundary vertices connecting to $v_r$. Since there is no dividing edge in $T$, $v_1$ and $v_2$ are the only boundary vertices connecting to $v_r$. We want to show that $v_r$ is in the convex hull of its neighbors.

Assume the opposite, then all the edges connecting to $v_r$ lie in a closed half plane, so the inner product of any pair of three vectors $\overrightarrow{v_r v_1}$, $\overrightarrow{v_r v_2}$ and $\overrightarrow{v_r v}$ is non-negative. But the inner angle at $v_r$ is larger than $\pi$, then either angle $\angle v_1 v_r v$ or $\angle v_2 v_r v$ is strictly larger than $\frac{\pi}{2}$, which means one inner product is negative. This leads to a contradiction. $\square$

This result solves the embeddability problem for strictly star-shaped polygons $\Omega$ with a triangulation $T$. We can construct a geodesic triangulation of $\Omega$ as follows. Pick an eye $e$ of $\Omega$ with the coefficients $W$ defined above. Then choose $\epsilon = 1/2$ and solve the linear system corresponding to the critical point of $E^W(1/2)$. If the solution is not an embedding, replace $\epsilon$ by $\epsilon/2$ and continue.

We conjecture that the space of geodesic triangulations for strictly star-shaped polygon with a fixed combinatorial type is contractible.

5. A Characterization of Geodesic Triangulations From Energies

We use the weighted length energy to generate families of geodesic triangulations for both convex polygons and strictly star-shaped polygons in the previous sections. One interesting question is whether we can realize any given geodesic triangulation in $\mathcal{GT}(\Omega, T)$ as the critical point of certain weighted length energy by choosing appropriate weights. Unfortunately, this is not the case, given the example in Eades, Healy, and Nikolov [11].

Example 5.1.

We have two equilateral triangles with different sizes determined by the vertices below

\[
\begin{align*}
v_1 &= \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -\sin \epsilon \\ \cos \epsilon \end{bmatrix}, \\
v_5 &= \begin{bmatrix} -\frac{\sqrt{3}}{2} \cos \epsilon + \frac{1}{2} \sin \epsilon \\ -\frac{\sqrt{3}}{2} \sin \epsilon - \frac{1}{2} \cos \epsilon \end{bmatrix}, \text{ and } v_6 = \begin{bmatrix} \frac{\sqrt{3}}{2} \cos \epsilon + \frac{1}{2} \sin \epsilon \\ \frac{\sqrt{3}}{2} \sin \epsilon - \frac{1}{2} \cos \epsilon \end{bmatrix},
\end{align*}
\]
and the triangulation given in Figure 4. The weighted length energy is given by
\[
E(\epsilon) = 3((2 - \cos \epsilon)^2 + \sin^2 \epsilon + (2 + \frac{\sqrt{3}}{2} \sin \epsilon + \frac{1}{2} \cos \epsilon)^2) + (-\frac{\sqrt{3}}{2} \cos \epsilon + \frac{1}{2} \sin \epsilon)^2)
= 30 - 6 \cos \epsilon + 6\sqrt{3} \sin \epsilon.
\]

Notice that when \(\epsilon\) is close to zero, \(E(\epsilon)\) is a monotonic increasing function with respect to \(\epsilon\). Moreover, the length of every interior edge decreases or at least stays with the same length when \(\epsilon \to 0^+\). Then it can’t be a critical point of any energy in the form of \(E = \frac{1}{2} \sum w_{ij} L_{ij}^2\).

The triangulation in Figure 4 is not a critical point of any energy, because we can construct a vector field to move the interior vertices of the triangulation so that no edge is lengthened. We can show that this condition leads to a necessary and sufficient condition for a geodesic triangulation to be realized as the minimizer of a weighted length energy. Eades, Healy, and Nikolov [11] gave another characterization for this class of geodesic triangulations.

**Lemma 5.2.** A geodesic triangulation \(\tau\) of a polygon \(\Omega\) can be realized by the critical point of a weighted length energy if and only if any vector field at the set of interior vertices of \(\tau\) will shorten at least one edge and lengthen at least one edge.

**Proof.** Let \((x_i, y_i)\) be the coordinate for vertex \(v_i\) of a given geodesic triangulation in \(\mathbb{R}^2\). If there exists a vector field not increasing any edge length, then all the edge lengths will decrease or at least stay with the same length as we move the vertices of the geodesic triangulation along the vector field. Then it can’t be a critical point of \(E\) for any choice of \(w_{ij}\).

Conversely, assume that we are given a geodesic triangulation \(\tau\) such that any vector field at interior vertices of \(\tau\) will increase the length of some edge and decrease the length of another edge. We want to show that we can find some positive weights \(w_{ij}\) for every edge in \(E_I\) such that \(\tau\) is the critical point of the weighted length energy
\[
E = \frac{1}{2} \sum_{e_{ij} \in E_I} w_{ij} L_{ij}^2.
\]

To find these weights, consider the linear system corresponding to the critical point of weighted length energy, denoted by \(Vw = 0\),
\[
\sum_{j \in N(v_i)} v_i^T w_{ij} = 0 \quad i = 1, ..., N_I
\]
where we regard $w_{ij}$ as the unknowns for the system and $v_{ij} = -v_{ji} = (x_i - x_j, y_i - y_j)$ are determined by $\tau$. For each interior vertex $v_i$, we have two equations corresponding to the $x$ coordinate and the $y$ coordinate of $v_i$, so $V$ is a $2N_i \times |E_i|$ matrix. If $w_{ij}$ is the weight of an interior edge connecting two interior vertices, then the column $c_{ij}$ of $V$ corresponding to $w_{ij}$ is

$$(0, \ldots, 0, w_{ij}, 0, \ldots, 0, 0, \ldots, 0)^T.$$  

If $w_{ij}$ is the weight of an interior edge connecting one interior vertex $v_i$ with a boundary vertex $v_j$, then the column $c_{ij}$ of $V$ corresponding to $w_{ij}$ is

$$(0, \ldots, 0, v_{ij}, 0, \ldots, 0)^T.$$  

To show the existence of a positive solution, consider an arbitrary vector field $X$ defined on the set of interior vertices of $\tau$. It can be represented as $(\alpha_1, \alpha_2, \ldots, \alpha_{N_2})^T$ where $\alpha_i$ is a row vector in $\mathbb{R}^2$. Then consider the derivative of the length of an interior edge connecting two interior vertices under $X$

$$\frac{dL^2_{ij}}{dt} = \left. \frac{d}{dt} \right|_{t=0} (x_i + \alpha_i^x t - x_j - \alpha_j^x t)^2 + (y_i + \alpha_i^y t - y_j - \alpha_j^y t)^2 = v_{ij} \cdot (\alpha_i - \alpha_j) = 2X \cdot c_{ij}.$$  

Similarly for an interior edge connecting one interior vertex $v_i$ with one boundary vertex $v_j$, we have

$$\frac{dL^2_{ij}}{dt} = \left. \frac{d}{dt} \right|_{t=0} (x_i + \alpha_i^x t - x_j)^2 + (y_i + \alpha_i^y t - y_j)^2 = v_{ij} \cdot \alpha_i = 2X \cdot c_{ij}.$$  

By assumption, we know that $X$ shortens one edge with weight $w_{ij}$ and lengthens another with weight $w_{ij}'$. Hence the corresponding columns $c_{ij}$ and $c_{ij}'$ produce different signs, namely $X \cdot c_{ij}$ and $X \cdot c_{ij}'$ has different signs. This means that all the entries of $X^T V$ can’t have the same sign. Since $X$ is arbitrary, by Farkas’s alternative [24], $Vw = 0$ has a positive solution $(w_{ij})$.

\[ \square \]

6. Geodesic Triangulations of Flat Tori

In this section, we study the space of geodesic triangulations of a fixed combinatorial type of $T$ on a flat torus $(\mathbb{T}^2, g)$. Here we assume a triangulation is a simplicial complex. Notice that by the result of Colin de Verdiere [10], and Hass and Scott [19], the space $\mathcal{GT}(\mathbb{T}^2, g, T)$ is not empty, so the embeddability problem for flat tori is resolved. We define the space of geodesic triangulations using simplexwise linear homeomorphisms and study its topology.

**Definition 6.1.** Given a triangulation $T = (V, E, F)$ of a flat torus $(\mathbb{T}^2, g)$, let $\tau_0$ be a geodesic triangulation of the combinatorial type of $T$, then the space of geodesic triangulations is the set of images of $\tau_0$ under all simplexwise linear homeomorphisms isotopic to the identity of $\mathbb{T}^2$. This space is denoted by $\mathcal{GT}(\mathbb{T}^2, g, T)$.

The image of $\tau_0$ under a simplexwise linear homeomorphism is a geodesic triangulation $\tau$ of $(\mathbb{T}^2, g)$, because simplexwise linear homeomorphisms map triangulations to triangulations and geodesic arcs in the 1-skeleton of $T$ to geodesic arcs.

For flat tori, $\mathcal{GT}(\mathbb{T}^2, g, T)$ is not simply a submanifold of $(\mathbb{T}^2)^{|V|}$, because the positions of the vertices can’t uniquely determine a geodesic triangulation. Specifically, fixing two points $p$ and $q$ on a flat torus, there exist many geodesic arcs connecting them. These arcs are not necessarily homotopic to each other by a
homotopy fixing two endpoints. They are lifted to the straight arcs connecting a preimage of \( p \in \mathbb{R}^2 \) with different preimages of \( q \in \mathbb{R}^2 \). Even if the positions of the vertices of two geodesic triangulations coincide, the corresponding edge can be different geodesic arcs. Nevertheless, we can show that \( \mathcal{GT}(\mathbb{T}^2, g, T) \) is a topological manifold.

**Lemma 6.2.** \( \mathcal{GT}(\mathbb{T}^2, g, T) \) is a topological manifold of dimension \( 2|V| \).

*Proof.* We construct local charts to cover \( \mathcal{GT}(\mathbb{T}^2, g, T) \). The idea is that we can perturb the vertices of a geodesic triangulation to construct another.

For any given \( \tau \in \mathcal{GT}(\mathbb{T}^2, g, T) \), choose any vertex \( v \) of \( \tau \) and lift its star to the universal cover. Then the image of the star in the universal cover is a strictly star-shaped polygon \( \Omega \) in \( \mathbb{R}^2 \) and the preimage \( \tilde{v} \) of \( v \) is an eye of \( \Omega \). Since the kernel of the polygon \( \Omega \) is an open set, there exists an open neighborhood \( U \) of \( \tilde{v} \) such that \( p \in U \) is an eye of \( \Omega \). So we can connect \( p \) with boundary vertices of \( \Omega \) and project it to the flat torus to form another geodesic triangulation \( \tau' \). The simplexwise linear homeomorphism corresponding to \( \tau' \) is linearly isotopic to the simplexwise linear homeomorphism corresponding to \( \tau \) by the linear isotopy between \( p \) and \( \tilde{v} \) in \( \Omega \). We can project this linear isotopy to construct an isotopy between these two geodesic triangulations on \( \mathbb{T}^2 \).

Similarly, we can perturb other vertices \( v_i \) of \( \tau \) in their neighborhoods \( U_i \) to generate other geodesic triangulations and the corresponding simplexwise linear homeomorphisms. We choose \( \delta > 0 \) and choose one point \( p_i \) from \( B_\delta(v_i) \subset U_i \), the open disk centered at \( v_i \) with radius \( \delta \) in \( \mathbb{T}^2 \), and connect them with geodesic arcs based on the combinatorial type of \( T \). We can choose \( \delta \) small enough such that the if we pick one point from each \( B_\delta(v_i) \) and connect them based on \( T \), we produce a geodesic triangulation. Hence we construct a chart

\[
\psi_\tau : \prod_{v_i \in V} B_\delta(v_i) \rightarrow \mathcal{GT}(\mathbb{T}^2, g, T)
\]

covering the geodesic triangulation \( \tau \). We can construct charts for any element in \( \mathcal{GT}(\mathbb{T}^2, g, T) \) to cover the whole space. The product \( \prod_{v_i \in V} B_\delta(v_i) \) is homeomorphic to \( \mathbb{R}^{2|V|} \), so the dimension of the space is \( 2|V| \). \( \square \)

Notice that our definition for the space of geodesic triangulations requires a flat metric. For the polygons in the plane, the metric is the Euclidean metric. On the contrary, there are infinite flat metrics on \( \mathbb{T}^2 \) representing different elements in the Teichmüller space of flat tori with unit area. We show that the topology of \( \mathcal{GT}(\mathbb{T}^2, g, T) \) is independent of the choice of flat metrics.

**Lemma 6.3.** Assume \( T \) is a triangulation of \( \mathbb{T}^2 \) and let \( g_1 \) and \( g_2 \) be two flat metrics representing two distinct elements in the Teichmüller space of flat tori with unit area. Then \( \mathcal{GT}(\mathbb{T}^2, g_1, T) \) is homeomorphic to \( \mathcal{GT}(\mathbb{T}^2, g_2, T) \).

*Proof.* We can construct a map between \( \mathcal{GT}(\mathbb{T}^2, g_1, T) \) and \( \mathcal{GT}(\mathbb{T}^2, g_2, T) \) explicitly using the universal covering \( \mathbb{R}^2 \). Since the Teichmüller space \( \mathcal{T}(\mathbb{T}^2) \) is parametrized by the upper half plane \( \{ z \in \mathbb{C} | \text{Im} z > 0 \} \), let \( z_1 \) and \( z_2 \) correspond to the metrics \( g_1 \) and \( g_2 \). This means that \( (\mathbb{T}^2, g_1) \) is isometric to the quotient manifold \( \mathbb{R}^2 / \langle \alpha, \beta \rangle \), where \( \alpha \) and \( \beta \) are two isometries of \( \mathbb{R}^2 \) given by

\[
\alpha(z) = z + 1 \quad \text{and} \quad \beta(z) = z + z_1.
\]
and \(<\alpha,\beta>\) is the group of isometries generated by \(\alpha\) and \(\beta\). Similarly, \((T^2, g_2)\) is isometric to \(\mathbb{R}^2/\langle\alpha,\gamma,\gamma\rangle\), where \(\gamma(z) = z + z_2\). Then there is an orientation preserving linear map \(\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) fixing 0 and 1, sending \(z_1\) to \(z_2\). This map induces a diffeomorphism \(f : (T^2, g_1) \rightarrow (T^2, g_2)\) because \(\tilde{f} \circ \beta = \gamma\).

Notice that the linear map sends straight lines to straight lines and keeps the incidence. Assume \(\tau_1\) is a geodesic triangulation on \((T^2, g_1)\) and we lift it to the universal covering to \(\tilde{\tau}_1\). Then \(\tilde{f}(\tilde{\tau}_1)\) is a geodesic triangulation of \(\mathbb{R}^2\) and descends to a geodesic triangulation \(f(\tau_1)\) on \((T^2, g_2)\). Hence we have a bijective map between \(\mathcal{GT}(T^2, g_1, T)\) and \(\mathcal{GT}(T^2, g_2, T)\) induced by \(\tilde{f}\) and its inverse. These two maps are continuous, hence we have a homeomorphism between the two spaces. \(\square\)

Choose \(v_0 \in V\) and define \(\mathcal{GT}(T^2, g, T, v_0, x)\) to be the space of geodesic triangulations with the location of \(v_0\) fixed at the point \(x \in T^2\). To find the homotopy type of \(\mathcal{GT}(T^2, g, T, v_0, x)\), we construct a similar Tutte map \(\Psi\) from the space of weights \(W\) to \(\mathcal{GT}(T^2, g, T, v_0, x)\) using Theorem 2.2. However, the definition of Tutte map is more complicated than the definition for the case of polygons in the plane.

Without loss of generality, assume that \((T^2, g)\) is isometric to \(\mathbb{R}^2/\langle u, v \rangle\) such that

\[
u(z) = z + 1 \quad \text{and} \quad v(z) = z + z_1 \]

where \(Imz_1 > 0\). Let \(p_1\) and \(p_2\) be two loops consisting of sequences of directed edges in \(T\) based at \(v_0\) homotopic to the meridian and the longitude of \(T^2\) respectively. If we solve the system (2.1) and choose harmonic one forms \(\Delta x\) and \(\Delta y\) randomly, the resulting geodesic triangulation produced by the formula (2.2) in the plane might not project to a geodesic triangulation on \((T^2, g)\).

To find the pair of harmonic one forms \(\Delta x'\) and \(\Delta y'\) which produce a geodesic triangulation in \((T^2, g),\) assume that

\[
(\alpha, \beta) = \left( \sum_{(i,j) \in p_1} \Delta x_{ij}, \sum_{(i,j) \in p_1} \Delta y_{ij} \right)
\]

and

\[
(\gamma, \delta) = \left( \sum_{(i,j) \in p_2} \Delta x_{ij}, \sum_{(i,j) \in p_2} \Delta y_{ij} \right).
\]

Notice that these two vectors \((\alpha, \beta)\) and \((\gamma, \delta)\) can’t be the zero vector by Theorem 2.2. Then there exists a unique orientation preserving linear transformation \(A\) in \(\mathbb{R}^2\) such that \(A\) sends \((\alpha, \beta)\) to \((1, 0)\) and \((\gamma, \delta)\) to \((Rez_1, Imz_1)\), namely,

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & \alpha \\ c & \beta \end{bmatrix} = \begin{bmatrix} 1 & Rez_1 \\ 0 & Imz_1 \end{bmatrix}.
\]

Then define \(\Delta x' = a\Delta x + b\Delta y\) and \(\Delta y' = c\Delta x + d\Delta y\), and we have

\[
(1, 0) = \left( \sum_{(i,j) \in p_1} \Delta x'_{ij}, \sum_{(i,j) \in p_1} \Delta y'_{ij} \right)
\]

and

\[
(Rez_1, Imz_1) = \left( \sum_{(i,j) \in p_2} \Delta x'_{ij}, \sum_{(i,j) \in p_2} \Delta y'_{ij} \right).
\]

This means that by the formula (2.2), \(\Delta x'\) and \(\Delta y'\) produce an equivariant geodesic triangulation in the plane, projecting to a geodesic triangulation on \((T^2, g)\).
Hence we can define the harmonic one forms $\Delta x'$ and $\Delta y'$ as follows. Combine the system of equations (2.1) with the following two equations

$$(1, 0) = \left( \sum_{(i,j) \in p_1} \Delta z_{ij}, \sum_{(i,j) \in p_1} \Delta z_{ij} \right).$$

Then we can find a unique solution $\Delta x'$ to this system by the discussion above. Similarly, combine the system of equations (2.1) with another two equations

$$(\text{Re} z_1, \text{Im} z_1) = \left( \sum_{(i,j) \in p_2} \Delta z_{ij}, \sum_{(i,j) \in p_2} \Delta z_{ij} \right),$$

and we can find a unique solution $\Delta y'$. Notice that $\Delta x'$ and $\Delta y'$ are uniquely determined by an element $(w_{ij})$ in the weight space $W$. Then we can have a well-defined Tutte map.

**Definition 6.4.** Assume that $T$ is a triangulation of $T^2$, $(T^2, g)$ is a flat torus isometric to $\mathbb{R}^2/\langle u, v \rangle$ such that

$$u(z) = z + 1 \quad \text{and} \quad v(z) = z + z_1$$

where $\text{Im} z_1 > 0$. The Tutte map $\Psi$ for $(T^2, g)$ sends the weights $(w_{ij}) \in W$ to the geodesic triangulation $\tau$ in $\mathcal{GT}(T^2, g, T, v_0, x)$ constructed by projecting the equivariant geodesic triangulation produced by $\Delta x'$ and $\Delta y'$ defined above based on the formula (2.2) to $(T^2, g)$.

Then we can compute the mean value coordinates for each directed edge and define a similar section $\sigma : \mathcal{GT}(T^2, g, T, v_0, x)$ to $W$ as the case of convex polygons. To apply theorem 2.2, we need to show that the 1-skeleton of any triangulation on a torus is 3-vertex-connected.

**Lemma 6.5.** Given a triangulation $T$ of $T^2$, the 1-skeleton of $T$ is a 3-vertex-connected graph.

**Proof.** We need to check that if we remove two vertices and all the edges and faces containing one of the two vertices, the remaining space is connected. Choose any two vertices $v_1$ and $v_2$ in $T$, then remove them and all the edges and faces containing $v_1$ or $v_2$ from $T$. Let $S$ denote the remaining space.

If $v_1$ is not in the star of $v_2$ in $T$, then $S$ is homotopic to a twice punctured torus, because we remove two disjoint open disks from the torus $T^2$. This fact follows from a computation using Euler characteristic $\chi(S)$. The open star of a vertex $v$ in $T$ has one vertex, $E$ edges and $F$ faces with $E = F$, so we remove two surfaces with Euler characteristic one, namely two disks. Then $\chi(S) = 0 - 2 = -2$.

If $S$ is disconnected, let $S_1$ and $S_2$ be two connected components. Then either one of $S_1$ and $S_2$ contains two boundary components, or each of $S_1$ and $S_2$ contains one boundary component. In either case, we will produce a disconnect surface instead of a torus when gluing the two open 2-disks back to $S$.

If $v_1$ is in the star of $v_2$ in $T$, then we remove a open disk from $T$. This is because we remove one vertex, $E_1$ edges, and $F_1$ faces from the star of $v_1$, and one vertex, $E_2$ edges, and $F_2$ faces from the star of $v_2$ with $E_1 = F_1$ and $E_2 = F_2$. Notice that the intersection of the stars of $v_1$ and $v_2$ contains one edge and two faces. Hence we remove two vertices, $E_1 + E_2 - 1$ edges and $F_1 + F_2 - 2$ faces, which combines to form a disk. Hence the remaining space is homotopic to a torus with one puncture, which is a connected surface.
In both cases, the remaining space is connected, so the 1-skeleton of $T$ is 3-vertex-connected.

**Theorem 6.6.** Let $T$ be a triangulation of a flat torus $\mathbb{T}^2$, then $\mathcal{G}(\mathbb{T}^2, g, T, v_0, x)$ is contractible and $\mathcal{G}(\mathbb{T}^2, g, T) = \mathcal{G}(\mathbb{T}^2, g, T, v_0, x) \times \mathbb{T}^2$.

**Proof.** Given a geodesic triangulation in $\mathcal{G}(\mathbb{T}^2, T, g)$, we can move the image of $v_0$ to $x$ using an isometry of the flat torus isotopic to the identity. The Tutte map $\Psi$ from $W$ to $\mathcal{G}(\mathbb{T}^2, g, T, v_0, x)$ is well-defined and continuous. We have the continuous section $\sigma$ from $\mathcal{G}(\mathbb{T}^2, g, T, v_0, x)$ to $W$ defined by the mean value coordinates. Then $\Psi(\sigma(\tau)) = \tau$ for any $\tau \in \mathcal{G}(\mathbb{T}^2, g, T, v_0, x)$, and $\sigma \circ \Psi$ is homotopic to the identity map on $W$. Hence $\Psi$ and $\sigma$ provide the homotopy equivalence so $\mathcal{G}(\mathbb{T}^2, g, T, v_0, x)$ is contractible. Since the group of isometries of a flat torus isotopic to the identity is homeomorphic to $\mathbb{T}^2$, we conclude that $\mathcal{G}(\mathbb{T}^2, T, g)$ is homeomorphic to $\mathcal{G}(\mathbb{T}^2, g, T, v_0, x) \times \mathbb{T}^2$. □

Notice that this is an analogous result for the smooth counterparts: $D_0(\mathbb{T}^2)$ is homeomorphic to $D_0(\mathbb{T}^2; x) \times \mathbb{T}^2$, where $D_0(\mathbb{T}^2; x)$ is contractible.

7. Further Work

Remaining open is the contractibility problem for the spaces of geodesic triangulations on the star-shaped polygons, the round 2-sphere and hyperbolic surfaces. The space of geodesic triangulations on the 2-sphere was studied by Awartani-Henderson [2]. The conjecture is that $\mathcal{G}(S^2, T)$ is homotopic to $SO(3)$. For hyperbolic surfaces $S$, Hass and Scott [19] showed that $\mathcal{G}(S, T)$ was contractible if $T$ is an 1-vertex triangulation. It is conjectured that $\mathcal{G}(S, T)$ is contractible.

Another direction to generalize the result was proposed by Luo [22]. Instead of the space of geodesic triangulations of a flat convex polygon with a fixed combinatorial type, we can study the space of geodesic triangulations of a convex polygon $\Omega$ with prescribed curvatures at the interior vertices of the triangulation. Then $\mathcal{G}(\Omega, T)$ is the special case when the prescribed curvatures are zero at all the interior vertices. It is conjectured that these spaces with different prescribed curvatures are also contractible.

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References

[1] Noam Aigerman and Yaron Lipman, *Orbifold tutte embeddings*, ACM Trans. Graph. 34 (2015), no. 6, 190–1.
[2] Marwan Awartani and David W Henderson, *Spaces of geodesic triangulations of the sphere*, Transactions of the American Mathematical Society 304 (1987), no. 2, 721–732.
[3] RH Bing and Michael Starbird, *Linear isotopies in $\mathbb{R}^2$*, Transactions of the American Mathematical Society 237 (1978), 205–222.
[4] Ethan D Bloch, Robert Connelly, and David W Henderson, *The space of simplexwise linear homeomorphisms of a convex 2-disk*, Topology 23 (1984), no. 2, 161–175.
[5] Steward S Cairns, *Deformations of plane rectilinear complexes*, The American Mathematical Monthly 51 (1944), no. 5, 247–252.
[6] Steward S Cairns, *Isotopic deformations of geodesic complexes on the 2-sphere and on the plane*, Annals of Mathematics (1944), 207–217.
About the bloch-connelly-henderson theorem on the simplexwise linear homeomorphisms of a convex 2-disk, arXiv preprint math/1910.00240 (2019).

Robert Connelly, David W Henderson, Chung Wu Ho, and Michael Starbird, On the problems related to linear homeomorphisms, embeddings, and isotopies, Continua, decompositions, manifolds, 1983, pp. 229–239.

Éric Colin De Verdière, Michel Pocchiola, and Gert Vegter, Tutte’s barycenter method applied to isotopies, Computational Geometry 26 (2003), no. 1, 81–97.

Y Colin de Verdiere, Comment rendre géodésique une triangulation d’une surface, L’Enseignement Mathématique 37 (1991), 201–212.

Peter Eades, Patrick Healy, and Nikola S Nikolov, The weighted barycenter drawing recognition problem, International Symposium on Graph Drawing and Network Visualization, Springer, 2018, pp. 569–575.

Clifford J Earle, James Eells, et al., A fibre bundle description of teichmüller theory, Journal of Differential Geometry 3 (1969), no. 1-2, 19–43.

Michael Floater, One-to-one piecewise linear mappings over triangulations, Mathematics of Computation 72 (2003), no. 242, 685–696.

Michael S Floater, Mean value coordinates, Computer aided geometric design 20 (2003), no. 1, 19–27.

Michael S Floater and Craig Gotsman, How to morph tilings injectively, Journal of Computational and Applied Mathematics 101 (1999), no. 1-2, 117–129.

Gene H Golub and CFV Loan, Matrix computations, forth edition, 2013.

Steven Gortler, Craig Gotsman, and Dylan Thurston, Discrete one-forms on meshes and applications to 3d mesh parameterization, Computer Aided Geometric Design (2006).

Xianfeng Gu and Shing-Tung Yau, Global conformal surface parameterization, Proceedings of the 2003 Eurographics/ACM SIGGRAPH symposium on Geometry processing, Eurographics Association, 2003, pp. 127–137.

Joel Hass and Peter Scott, Simplicial energy and simplicial harmonic maps, arXiv preprint arXiv:1206.2574 (2012).

Chung Wu Ho, On certain homotopy properties of some spaces of linear and piecewise linear homeomorphisms. i, Transactions of the American Mathematical Society 181 (1973), 213–233.

Seok-Hee Hong and Hiroshi Nagamochi, Convex drawings of graphs with non-convex boundary constraints, Discrete Applied Mathematics 156 (2008), no. 12, 2368–2380.

Feng Luo, Rigidity of polyhedral surfaces, arXiv preprint math/0612714 (2006).

Kaare Brandt Petersen, Michael Syskind Pedersen, et al., The matrix cookbook, Technical University of Denmark 7 (2008), no. 15, 510.

Stephen Roman, S Axler, and FW Gehring, Advanced linear algebra, vol. 3, Springer, 2005.

Stephen Smale, Diffeomorphisms of the 2-sphere, Proceedings of the American Mathematical Society 10 (1959), no. 4, 621–626.

Vitaly Surazhsky and Craig Gotsman, Controllable morphing of compatible planar triangulations, ACM Transactions on Graphics (TOG) 20 (2001), no. 4, 203–231.

Vitaly Surazhsky and Craig Gotsman, Intrinsic morphing of compatible triangulations, International Journal of Shape Modeling 9 (2003), no. 02, 191–201.

William Thomas Tutte, How to draw a graph, Proceedings of the London Mathematical Society 3 (1963), no. 1, 743–767.

Yin Xu, Renjie Chen, Craig Gotsman, and Ligang Liu, Embedding a triangular graph within a given boundary, Computer Aided Geometric Design 28 (2011), no. 6, 349–356.

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