Elementary derivation of the expressions of momentum and energy in special relativity

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Abstract. The expressions of momentum and energy of a particle in special relativity are often derived in a quite unconvincing manner in elementary text, by resorting either to electrodynamic or quantum considerations, or via the introduction of the less-than-elementary concept of a four-vector. It is instead possible, by exploiting considerations introduced by P. Epstein and A. Einstein and exploited later by Feynman, to obtain a fully elementary derivation of these expressions and of the $E = mc^2$ formula exploiting only Lorentz transformations and the postulate of the conservation of quantities defined for point-like particles which reduce to the Newtonian expressions of momentum and energy in the classical limit.

PACS numbers: 03.30+p Special Relativity, 01.40.gb Teaching methods
1. Introduction

Several texts provide an elementary derivation of the kinematics of special relativity, eventually based on the first part of Einstein’s fundamental paper of 1905 \cite{1}. Starting from the two postulates of the total equivalence of inertial reference systems, and of the constancy of the speed of light in all such systems, it is in fact easy to obtain the expression of the Lorentz transformation of space-time coordinates. A particularly simple and appealing derivation is obtained by exploiting Bondi’s so-called \( k \)-calculus \cite{2}, itself based on the Doppler effect. Nevertheless going from kinematics (the Lorentz transformation) to dynamics, and in particular to the relativistic expressions of momentum and energy, is often achieved by resorting to more sophisticated concepts, like that of a four-vector, or by the use of quantum considerations. Einstein himself had originally derived the mass-energy equivalence via an explicit use of electrodynamics, and not just by the kind of kinematic considerations which he had developed in the first part of his 1905 paper, which are only based on the constancy of the speed of light, but do not otherwise depend on Maxwell’s equations.

Einstein had remarked this problem, and he proposed in 1935 an elementary derivation of the mass-energy relation, independent of his 1905 argument, motivating it with the following words \cite{4}:

The special theory of relativity grew out of the Maxwell electromagnetic equations. So it came about that even in the derivation of the mechanical concepts and their relations the consideration of those of the electromagnetic field has played an essential role. The question as to the independence of those relations is a natural one because the Lorentz transformation, the real basis of the special relativity theory, in itself has nothing to do with the Maxwell theory and because we do not know the extent to which the energy concepts of the Maxwell theory can be maintained in the face of the data of molecular physics. In the following considerations, except for the Lorentz transformation, we will depend only on the assumption of the conservation principles for impulse and energy.

Einstein’s considerations exploit a conceptual experiment introduced by G. N. Lewis and R. C. Tolman \cite{5} and further discussed by P. S. Epstein \cite{6}, where one considers collisions between pairs of particles in different inertial reference frames, and one looks for the expressions of momentum and energy by postulating their conservation. It is interesting to point out that Lewis and Tolman’s paper, as well as Epstein’s one, only consider elastic collision and derive the relativistic expression of momentum, while they provide a doubtful argument for the mass-energy equivalence by the consideration of the change of the “relativistic mass” with speed. Einstein derives instead the equivalence by simply extending the argument to inelastic collisions. The advantage of this approach for

\footnote{One example of this approach is the “elementary derivation” suggested by F. Rohrlich \cite{3}, which exploits the expressions of the momentum and energy of a photon of frequency \( \nu \).}
introducing the basic concepts of special relativity has been well remarked by Feynman who, in his Lectures [7, Vol. I. Secs. 16–4, 16–5], derives the relativistic expressions of momentum and energy in a way that closely resembles Einstein’s one. Einstein’s argument has been more recently discussed by F. Flores [8], who identifies three closely related but different claims within the mass-energy equivalence concept, and compares Einstein’s 1935 argument with his original 1905 derivation [11] and with M. Friedman’s 1983 derivation [9, p. 142ff], which rests upon the consideration of Newton’s equations in special relativity.

In this note, I present this line of thought in the hope that it may be found useful for the presentation of these fundamental concepts of special relativity in introductory courses for students of physics and mathematics. While the derivation of the relativistic expression of momentum 3 is close to Epstein’s and Feynman’s arguments, the discussion of the expression of the kinetic energy and of the mass-energy equivalence is closer to Einstein’s one.

2. Lorentz transformations and dynamic postulates

Following Einstein’s 1905 paper [1, § 2], the kinematic concepts of special relativity rest on the following postulates:

(i) The laws that govern the transformations of the state of physical systems take the same form in reference frames animated by uniform translational motion one with respect to the other.

(ii) In each such reference frame the speed of light assumes the same value $c$, independently of the state of motion of its source.

We shall choose from now on units in which $c = 1$. Based on these postulates one can easily derive Lorentz transformations in the following form. Let us consider two reference frames, $K$ and $K'$, such that $K'$ is in uniform translational motion in the $x$ direction and with speed $V$ with respect to $K$. Then the event of coordinates $(t', x', y', z')$ in $K'$ has in $K$ the coordinates $(t, x, y, z)$, where

\[
\begin{align*}
t &= \gamma(V)(t' + Vx'); \\
x &= \gamma(V)(x' + Vt'); \\
y &= y'; \\
z &= z',
\end{align*}
\]

and we have defined

\[
\gamma(V) = \frac{1}{(1 - V^2)^{1/2}}.
\]
The same relation holds for the differentials $dt$, $dx$, etc. Dividing by $dt$ we obtain the rules for the transformation of velocities:

\[
\begin{align*}
    u_x &= \frac{dx}{dt} = \frac{u'_x + V}{1 + u'_x V}, \\
    u_y &= \frac{dy}{dt} = \frac{u'_y}{\gamma(V)(1 + u'_x V)}, \\
    u_z &= \frac{dz}{dt} = \frac{u'_z}{\gamma(V)(1 + u'_x V)}.
\end{align*}
\]

(3)

To introduce dynamical concepts we obviously need supplementary postulates. We shall therefore postulate the following:

(iii) The momentum $P$ and the energy $E$ of a particle possessing the velocity $u$ in the reference frame $K$ have respectively the expressions

\[
P = m u F(u); \quad E = E_0 + m G(u),
\]

(4)

where $E_0$ is a constant, that can be called the rest energy, $m$ is a positive constant (which does not change as long as the particle’s state changes only by a change of its velocity) and which we shall call its rest mass, and $F(u)$ and $G(u)$ are monotonically increasing universal functions of $u = |u|$.

(iv) For $u \ll 1$ these expressions reduce to the well-known classical ones. One has in particular

\[
F(u) = 1 + O(u^2); \quad G(u) = \frac{1}{2} u^2 + o \left( u^2 \right).
\]

(5)

(v) The total momentum $P^{\text{tot}}$ and the total energy $E^{\text{tot}}$ of a system of several particles are respectively given by the sum of $P$ and of $E$ running over all the particles of the system.

(vi) Conservation of momentum and energy: Let us assume that (elastic or inelastic) collisions occur in a system of particles. Then $P^{\text{tot}}$ and $E^{\text{tot}}$ maintain the same values before and after each collision.

3. Elastic collisions and relativistic momentum

Let us now consider a particle pair, i.e., a system made of two identical particles (possessing the same value of $m$). Let us assume that in a reference frame $K$ they have opposite velocities $u_+, u_- = -u_+$, where $u_+ = (V, v, 0)$, with $|v| \ll V$, $V > 0$. Thus $|u^\pm| \approx V$. Let us moreover assume that the particles undergo an elastic collision, and assume respectively the velocities $w_+ = (W, w, 0)$ and $w_- = (W', w', 0)$ after it. By the conservation of momentum one must have $W' = -W$ and $w' = -w$, independently of the form of the function $F(u)$. Indeed, one has $P_{\text{in}}^{\text{tot}} = 0$ before the collision. By the conservation law one must have

\[
P_{\text{out}}^{\text{tot}} = 0 = m w_+ F(w_+) + m w_- F(w_-).
\]

(6)
Thus the vectors $w_+ e w_-$ are parallel, and one has
\[ \frac{|w_+|}{|w_-|} = \frac{F(w_-)}{F(w_+)} = \text{const}. \tag{7} \]
Since the function $F(u)$ is monotonically increasing, this equation can only be satisfied if $|w_+| = |w_-|$, and we have therefore $w_+ = -w_-$. Now energy conservation imposes $w^\pm = u^\pm$. Indeed, the total kinetic energy before the collision is given by $2mG(u)$, and after it is given by $2mG(w)$. Since $G(u)$ is a monotonically increasing function of $u$, this condition can only be satisfied if $w^\pm = u^\pm$.

Let us now consider the special case in which $W = V$, i.e., where the velocity change is parallel to the $y$ axis. Let the particles’ velocities be given by $u_{\pm,0} = (\pm V_0, \pm v_0, 0)$ (cf. fig. 1) in the $K_0$ reference frame. Let us now look at the same collision in a reference frame $K$ moving with a velocity $V_0$ in the $x$ direction with respect to $K_0$. In this reference frame, the velocities $u_\pm$ of the particles are respectively given by
\begin{align*}
u_{+} &= (0, v, 0), \\
u_{-} &= (-V, -w, 0),
\end{align*}
where
\begin{align*}V &= \frac{2V_0}{1 + V_0^2}, \\
v &= \frac{v}{\gamma(V_0)(1 + V_0^2)}, \\
w &= \frac{v}{\gamma(V_0)(1 - V_0^2)}. \tag{9}\end{align*}
One can easily check that
\[ \gamma(V) = \frac{1}{\sqrt{1 - V^2}} = \frac{1 + V_0^2}{1 - V_0^2}, \tag{10}\]
and therefore that
\[ w = \frac{v}{\gamma(V)}. \tag{11}\]
One can also obtain this result by applying equation (3) to the transformation from the $K$ reference frame to a frame $K'$ animated, with respect to $K$, by a uniform translational
motion with velocity $-V$ parallel to the $x$ axis. In this frame the $x$ component of the velocity $u_-$ vanishes, that of the $u_+$ velocity is equal to $V$, and thus the speeds of the two particles are interchanged.

Let us now consider the conservation of momentum. The change $\delta P_+$ of the momentum of the “$+$” particle is given by

$$\delta P_+ = -2mv F(v) e_y,$$

where $e_y$ is the $y$ axis versor. The corresponding quantity for the “$-$” particle is given by

$$\delta P_- = 2mw F(u_-) e_y,$$

where $u_- = \sqrt{V^2 + w^2}$. Let us assume $v, w \ll 1$: then $F(v) \simeq 1$ and $u_- \simeq V$. From momentum conservation we obtain $\delta P_+ + \delta P_- = 0$, which implies

$$mv = mw F(V).$$

Since $w = v/\gamma(V)$, we obtain

$$F(V) = \gamma(V) = \frac{1}{\sqrt{1 - \frac{v^2}{1 - \frac{V^2}{1 - \frac{V^2}{1 - \frac{V^2}}}}}}.$$

Having obtained this result for $v, w \ll 1$, it is easy to see that it also holds for larger values of $v$ and $w$, by substituting $V$ with the speed of the corresponding particle. We have in fact

$$mv F(v) = mw F(u) = \frac{mv}{\gamma(V)} F(u),$$

from which it follows

$$F(u) = F(v) \gamma(V),$$

namely

$$\frac{1}{\sqrt{1 - u^2}} = \frac{1}{\sqrt{1 - v^2}} \frac{1}{\sqrt{1 - \frac{V^2}{1 - \frac{V^2}{1 - \frac{V^2}}}}}.$$

an identity which is easy to check directly.

4. Kinetic energy conservation

Let us consider a particle having the velocity $u' = (u', 0, 0)$, parallel to the $x$ axis, in the $K'$ reference frame. Its velocity $u$ in the $K$ frame, in uniform translational motion with respect to $K'$ with a velocity $-V$ parallel to the $x$ axis, is given by $u = (u, 0, 0)$, with

$$u = \frac{u' + V}{1 + u'V}.$$  

One can easily see that

$$\gamma(u) = (1 + u'V) \gamma(u') \gamma(V).$$
If \( \mathbf{u}' \) is not parallel to the \( x \) axis, but one has instead \( \mathbf{u}' = (u'_{x}, u'_y, u'_z) \), one has the more general relation

\[
\gamma(u) = (1 + u'_{x}V)\gamma(u')\gamma(V),
\]
which can be obtained with a little algebra. It is also easy to check that

\[
\begin{align*}
    u_{x}\gamma(u) &= (u'_{x} + V)\gamma(u')\gamma(V); \\
    u_{y}\gamma(u) &= u'_{y}\gamma(u')\gamma(V); \\
    u_{z}\gamma(u) &= u'_{z}\gamma(u')\gamma(V).
\end{align*}
\]

Let us now consider a particle pair, which have opposite velocities \( \mathbf{u}'_{+}, \mathbf{u}'_{-} = -\mathbf{u}'_{+} \) in the \( K' \) frame. Let us denote by \( \mathbf{u}_{+} \) and \( \mathbf{u}_{-} \) the corresponding velocities in the \( K \) frame. We obtain

\[
\gamma(u_{+}) + \gamma(u_{-}) = 2\gamma(u')\gamma(V).
\]

We have seen that an elastic collision in the \( K \) frame cannot change the common value \( u' \) of the particles' speed. Thus the right-hand side of this equation cannot change in the collision. But then neither can its left-hand side. If we denote by \( \mathbf{w}_{+} \) and \( \mathbf{w}_{-} \) the particles' speeds in the \( K \) frame after the collision, we obtain

\[
\gamma(u_{+}) + \gamma(u_{-}) = \gamma(w_{+}) + \gamma(w_{-}).
\]

As Einstein [4, p. 227] points out, these equations have the form of conservation laws. One can thus interpret \( m(\gamma(u) - 1) \) as the kinetic energy of a particle with rest mass \( m \) animated by a velocity \( \mathbf{u} \). This quantity vanishes for \( u \to 0 \), and for small values of \( u \) is given by

\[
m(\gamma(u) - 1) \simeq \frac{1}{2}mu^{2},
\]
in agreement with the classical limit. We can thus set

\[
G(u) = \gamma(u) - 1.
\]

Let us remark moreover that, by applying the equations (22) to a particle pair, we obtain

\[
\mathbf{u}_{+}\gamma(u_{+}) + \mathbf{u}_{-}\gamma(u_{-}) = 2V\gamma(u')\gamma(V).
\]

We can thus derive the following relation:

\[
\mathbf{u}_{+}\gamma(u_{+}) + \mathbf{u}_{-}\gamma(u_{-}) = \mathbf{w}_{+}\gamma(w_{+}) + \mathbf{w}_{-}\gamma(w_{-}),
\]

which can be interpreted as the conservation law for the momentum. We thus recover the relativistic expression of the momentum derived in sec. [3].

\[\text{§ Equations (24) imply the conservation of } \gamma(u) + \text{const. for a particle pair. One must set const. } = -1 \text{ to ensure that } G(u) \text{ vanishes as } u \to 0.\]
5. Mass-energy equivalence

Let us now consider a totally inelastic collision in a particle pair. In the center of mass reference frame, the total kinetic energy before the collision is given by

\[ T' = 2m (\gamma(u') - 1). \] (29)

The total kinetic energy after the collision vanishes, but the energy of the resulting particle has increased by \( T' \). By momentum conservation the resulting particle is at rest in the center of mass frame \( K' \) and has thus velocity \( V \) parallel to the \( x \) axis in the \( K \) frame. Let us denote by \( M \) its rest mass. In the \( K \) frame the total momentum before the collision is given by

\[ P = m (u_+ \gamma(u_+) + u_- \gamma(u_-)) = 2mV \gamma(u') \gamma(V), \] (30)

while after the collision it has the value

\[ P = MV \gamma(V). \] (31)

We obtain therefore

\[ M = 2m \gamma(u'). \] (32)

Thus the total rest mass of the system has increased by a quantity that is exactly equal, in our units, to the kinetic energy “dissipated” in the collision, i.e., transformed in other energy forms. One can thus introduce a “natural” choice of the energy at rest \( E_0 \), by equating it (in our units) to the rest mass of the particle, also because, “from the nature of the concept, [that] is determined only to within an additive constant, one can stipulate that \( E_0 \) should vanish together with \( m \)” [4, p. 229]. We can thus consider \( m \gamma(u) \) as the expression of the total energy of a particle, and we can associate with the change \( \delta E \) of energy from the kinetic to a different form with a change \( \delta m = \delta E \) of the rest mass of the particle.

One should point out that this line of thought does not rest on Maxwell’s equations (keeping only the constancy of the speed of light) and neither does on other mechanical concepts, in particular on the concept of force, that is difficult to justify in special relativity. Einstein criticizes the use of the concept of force in the derivation of the relativistic expression of momentum, contained in the book by G. D. Birkhoff and R. E. Langer, Relativity and Modern Physics, [10] exactly for this reason, as made clear by the closing paragraphs of his 1935 paper [4]:

Thus, in the book just mentioned, essential use is made of the concept of force, which in the relativity theory has no such direct significance as it has in classical mechanics. This is connected with the fact that, in the latter, the force is to be considered as a given function of the coordinates of all the particles, which is obviously not possible in the relativity theory. Therefore I have avoided introducing the force concept.

Furthermore, I was concerned with avoiding making any assumption concerning the transformation character of impulse and energy with respect to a Lorentz transformation.
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