Optimal first-order error estimates of a fully segregation scheme for the Navier-Stokes equations*

F. Guillén-González† M.V. Redondo-Neble‡

November 27, 2014

Abstract

A first-order linear fully discrete scheme is studied for the incompressible time-dependent Navier-Stokes equations in three-dimensional domains. This scheme, based on an incremental pressure projection method, decouples each component of the velocity and the pressure, solving in each time step, a linear convection-diffusion problem for each component of the velocity and a Poisson-Neumann problem for the pressure.

Using first-order \textit{inf-sup} stable $C^0$-finite elements, optimal error estimates of order $O(k + h)$ are deduced without imposing constraints on $h$ and $k$, the mesh size and the time step, respectively.

Finally, some numerical results are presented according the theoretical analysis, and also comparing to other current first-order segregated schemes.

Subject Classification. 35Q30, 65N15, 76D05.

Keywords: Navier-Stokes Equations, incremental pressure projection schemes, segregated scheme, error estimates, finite elements.

---

*The authors have been partially supported by MINECO (Spain), Grant MTM2012–32325 and the second author is also partially supported by the research group FQM-315 of Junta de Andalucía.

†Departamento de Ecuaciones Diferenciales y Análisis Numérico and IMUS. Universidad de Sevilla. Aptdo 1160, 41080 Sevilla (Spain), email: guillen@us.es, fax: ++ 34 5 4552898, phone: ++ 34 5 4559907.

‡Departamento de Matemáticas. Universidad de Cádiz. C.A.S.E.M. Polígono Río San Pedro S/N, 11510 Puerto Real. Cádiz (Spain), email: victoria.redondo@uca.es, phone: ++ 34 5 6016058.
Introduction

Let us consider the Navier-Stokes system, associated to the dynamics of viscous and incompressible fluids filling a bounded domain $\Omega \subset \mathbb{R}^3$ in a time interval $(0,T)$:

\[
\begin{aligned}
&P \\
&\begin{cases}
&u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f \quad \text{in } \Omega \times (0,T), \\
&\nabla \cdot u = 0 \quad \text{in } \Omega \times (0,T), \\
&u = 0 \quad \text{on } \partial \Omega \times (0,T), \\
&u|_{t=0} = u_0 \quad \text{in } \Omega.
\end{cases}
\end{aligned}
\]

where the unknowns are $u : (x,t) \in \Omega \times (0,T) \rightarrow \mathbb{R}^3$ the velocity field and $p : (x,t) \in \Omega \times (0,T) \rightarrow \mathbb{R}$ the pressure, and data are $\nu > 0$ the viscosity coefficient (which is assumed constant for simplicity) and $f : \Omega \times (0,T) \rightarrow \mathbb{R}^3$ the external forces. We denote by $\nabla$ the gradient operator and $\Delta$ the Laplace operator.

We consider a (uniform) partition of $[0,T]$ related to a fixed time step $k = T/M$: $t_0 = 0, t_1 = k, \ldots, t_m = mk, \ldots, t_M = T$. If $u = (u^m)_{m=0}^M$ is a given vector with $u^m \in X$ (a Banach space), let us to introduce the following notation for discrete in time norms:

\[
\begin{align*}
\|u\|_{l^2(X)} &= \left( k \sum_{m=0}^M \|u^m\|^2_X \right)^{1/2} \quad \text{and} \quad \|u\|_{l^\infty(X)} = \max_{m=0,\ldots,M} \|u^m\|_X,
\end{align*}
\]

For simplicity, we will denote $H^1 = H^1(\Omega)$ etc., $L^2(H^1) = L^2(0,T;H^1)$ etc., and $H^1 = H^1(\Omega)^3$ etc. We will denote by $C > 0$ different constants, always independent of discrete parameters $k$ and $h$.

The numerical analysis for the Navier-Stokes problem $(P)$ has received much attention in the last decades and many numerical schemes are now available. The main (numerical) difficulties are: the coupling between the pressure term $\nabla p$ and the incompressibility condition $\nabla \cdot u = 0$ and the nonlinearity given by the convective terms $(u \cdot \nabla)u$.

Fractional-step projection methods are becoming widely used, splitting the different operators appearing in the problem. The origin of these methods is generally credited to the works of Chorin [4] and Temam [30]. They developed the well known *Chorin-Temam projection* method, which is a two-step scheme, computing firstly an intermediate velocity via a convection-diffusion problem and secondly a velocity-pressure pair via a divergence-free $L^2(\Omega)$-projection problem.

Afterwards, a modified projection scheme (called *incremental-pressure* or *Van-Kan scheme*) was developed [23], adding an explicit pressure term in the first step and a pressure correction term in the projection step. The main drawbacks of projection methods are that the end-of-step velocity does not satisfy the exact boundary conditions and the discrete pressure satisfies an “artificial” Neumann boundary condition.

Some current variants of projection methods are: rotational pressure-correction schemes ([33], [14], [15]), velocity-correction schemes ([11], [12]), consistent-splitting schemes ([13],[15],...
and penalty pressure-projection schemes ([1], [2], [7]). Other variants can be seen in [24] and [25].

The convergence of the Chorin-Temam projection method was proved first in [31] for the time discrete scheme and afterwards in [5] for a fully discrete finite element (FE) scheme.

On the other hand, error estimates for projection methods were obtained (see [27], [28] for time discrete schemes and [10] for a fully discrete FE scheme). Basically, the Chorin-Temam scheme has order $O(k^{1/2})$ in $l^\infty(L^2) \cap l^2(H^1)$ and $O(k)$ in $l^2(L^2)$ for the velocity, and $O(k^{1/2})$ in $l^2(L^2)$ for the pressure. For the incremental-pressure scheme, these error estimates are improved in [27] and [28] to order $O(k)$ in $l^\infty(L^2) \cap l^2(H^1)$ for the velocity and $O(k)$ in $l^2(L^2)$ for the pressure (although this last estimate is proved only for the linear problem). In fact, these optimal error estimates are extended in [10] to a fully discrete FE-stable scheme (see (21) below) under the constraint $k^2 \leq C h$ in 3D domains or $k^2 \leq \alpha (1 + \log(h^{-1}))$ in 2D ones. The argument done in [10] is based on the direct comparison between an appropriate spatial interpolation of the exact solution and the fully discrete scheme.

By the contrary, in this paper, we will obtain optimal error estimates without imposing restrictions on $h$ and $k$ for a FE decoupled scheme different from scheme studied in [10] (which was not decoupled because the projection step is solved by means of a mixed velocity-pressure formulation). The argument used now is also different from [10], because the corresponding time discrete scheme will be introduced as an intermediate problem. This argument has already been used in [16, 17, 18] for a different splitting scheme (with decomposition of viscosity) applied to Navier-Stokes equations.

The particular property that some projection methods (without and with incremental pressure) can be rewritten as segregated methods (decoupling velocity and pressure), was observed in [26, 27]. For a segregated fully discrete FE scheme based on the non-incremental projection method, the convergence and sub-optimal error estimates $O(k^{1/2} + h)$ for the pressure have been obtained in [3], without imposing inf-sup condition, but under the double constraint $\alpha h^2 \leq k \leq \beta h^2$.

In this paper, we obtain optimal order $O(k + h)$ for the velocity and pressure, without imposing constraints on $h$ and $k$, for a time segregated scheme with first-order inf-sup stable FE spaces. Up to our knowledge, optimal first order for the pressure of a fully segregated scheme for the Navier-Stokes problem have not been proved before.

Ideas of this paper are being used to design a segregated second order in time scheme ([19]).

This paper is organized as follows:

In Section 1, we study the time discrete scheme (see Algorithm 1 below). Firstly, the stability of this scheme is deduced, and we introduce the discrete in time problems satisfied by errors and the regularity hypotheses that must be imposed on the exact solution. Afterwards, we obtain $O(k)$ accuracy for the velocity in $l^\infty(L^2) \cap l^2(H^1)$. As a consequence, the velocity is bounded in
$l^\infty(H^1)$. Then, we deduce $O(k)$ for the discrete in time derivative of velocities in $l^\infty(L^2) \cap l^2(H^1)$. Finally, $O(k)$ for the velocity in $l^\infty(H^1)$ and for the pressure in $l^\infty(L^2)$ hold.

Section 2 is devoted to study the fully discrete FE scheme (see Algorithm 2 below). We present the FE-stable spaces and their approximation properties, the fully discrete segregated scheme and the problems satisfied by the errors (comparing the time discrete Algorithm 1 with the fully discrete Algorithm 2). With respect to the spatial error estimates, firstly we obtain $O(h)$ for the velocity in $l^\infty(L^2) \cap l^2(H^1)$. Then, the velocity is bounded in $l^\infty(H^1)$. Afterwards, by using some additional estimates for the time discrete scheme, $O(h)$ for the discrete in time derivative of velocity in $l^\infty(L^2) \cap l^2(H^1)$ is obtained. Finally, $O(h)$ for the velocity in $l^\infty(H^1)$ and for the pressure in $l^\infty(L^2)$ are deduced.

In Section 3, some numerical simulations are presented, showing first order accuracy in time for velocity and pressure. These simulations are also compared with the segregated versions of the rotational, consistent and penalty-projection schemes.

Finally, some conclusions are given in Section 4.

In this paper, the following discrete Gronwall’s lemma will be used ([22, p. 369]):

**Lemma 1 (Discrete Gronwall inequality)** Let $k$, $B$ and $a_m$, $b_m$, $c_m$, $\gamma_m$ be nonnegative numbers. If we assume

$$a_{r+1} + k \sum_{m=0}^{r} b_m \leq k \sum_{m=0}^{r} \gamma_m a_m + k \sum_{m=0}^{r} c_m + B \quad \forall r \geq 0,$$

then, one has

$$a_{r+1} + k \sum_{m=0}^{r} b_m \leq \exp \left( k \sum_{m=0}^{r} \gamma_m \right) \left\{ k \sum_{m=0}^{r} c_m + B \right\} \quad \forall r \geq 0.$$

### 1 Time discrete scheme (Algorithm 1)

The norm and inner product in $L^2(\Omega)$ will be denoted by $\| \cdot \|$ and $(\cdot, \cdot)$, whereas the norm in $H_0^1(\Omega)$ of the gradient in $L^2(\Omega)$ will be denoted by $\| \cdot \|$. Any other norm in a space $X$ will be denoted by $\| \cdot \|_X$.

Let us to introduce the standard Hilbert spaces in the Navier-Stokes framework:

$$H = \{ v \in L^2(\Omega) : \nabla \cdot v = 0 \text{ in } \Omega, \ v \cdot n_{\partial \Omega} = 0 \},$$

$$V = \{ v \in H_0^1(\Omega) : \nabla \cdot v = 0 \text{ in } \Omega \},$$

where $n_{\partial \Omega}$ denotes the normal outwards vector to $\partial \Omega$.

In the sequel, the following standard skew-symmetric form of the convective term will be used:

$$C(u, v) = (u \cdot \nabla) v + \frac{1}{2} (\nabla \cdot u) v \quad \forall u \in H_0^1, \ v \in H^1,$$
and the corresponding trilinear form
\[ c(u, v, w) = \int_{\Omega} C(u, v) \cdot w = \int_{\Omega} \left\{ (u \cdot \nabla) v \cdot w + \frac{1}{2}(\nabla \cdot u) v \cdot w \right\}, \quad \forall u \in H^1_0, \ v \in H^1, \ w \in H^1 \]
or equivalently
\[ c(u, v, w) = \frac{1}{2} \int_{\Omega} \left\{ (u \cdot \nabla) v \cdot w - (u \cdot \nabla) w \cdot v \right\} = -\int_{\Omega} \left\{ (u \cdot \nabla) w \cdot v + \frac{1}{2}(\nabla \cdot u) v \cdot w \right\}. \]

Previous equalities hold even in the fully discrete case, hence we can use, in the sequel, any of these three possibilities.

The trilinear form \( c(\cdot, \cdot, \cdot) \) satisfies
\[ c(u, v, v) = 0, \quad \forall u \in H^1_0, \ v \in H^1, \]
\[ c(u, v, w) \leq C \left\{ \|u\| \|v\| \|w\| \right\} \]
where the role of \( u, v, w \) can be interchanged, using the appropriate expression of \( c(\cdot, \cdot, \cdot) \).

1.1 Description of the time scheme (Algorithm 1)

Given \( f^m = f(t_m) \), we define an approximation \((u^m, p^m)_{m=1}^M\) of the solution \((u, p)\) of \((P)\) at time \( t = t_m \), by means of an incremental pressure projection scheme of Van-Kan type [23], splitting the nonlinearity \((u \cdot \nabla)u\) and the diffusion term \(-\Delta u\) to the incompressibility condition \(\nabla \cdot u = 0\). Moreover, an explicit pressure term is introduced in the convection-diffusion problem for the velocity (Sub-step 1), with a pressure-correction in the divergence-free projection step (Sub-step 2). See Algorithm 1 for a description of the time scheme.

Notice that the convection term has been taken in \((S_1)^{m+1}\) in the semi-implicit linear form \(C(\tilde{u}^m, \tilde{u}^{m+1})\). On the other hand, adding \((S_1)^{m+1}\) and \((S_2)^{m+1}\), we arrive at
\[ (S_3)^{m+1} \begin{cases} \frac{1}{k}(u^{m+1} - u^m) + C(\tilde{u}^m, \tilde{u}^{m+1}) - \Delta \tilde{u}^{m+1} + \nabla p^{m+1} = f^{m+1} & \text{in } \Omega, \\ \tilde{u}^{m+1}|_{\partial \Omega} = 0, \quad \nabla \cdot u^{m+1} = 0 & \text{in } \Omega. \end{cases} \]
In fact \((S_3)^{m+1}\) can be viewed as consistence relations, because if \(\tilde{u}^{m+1}\) and \(u^{m+1}\) converge to the same limit velocity \(u\) as \(k\) goes to zero, then taking limits in \((S_3)^{m+1}\), one has at least formally that \(u\) will be a solution of the exact problem \((P)\).

Now, some remarks about Sub-step 2 are in order:

- Sub-step 2 can be viewed as a projection step. In fact, \(u^{m+1} = P_H \tilde{u}^{m+1}\) where \(P_H\) is the \(L^2(\Omega)\)-projector onto \(H\), because \((S_2)^{m+1}\) implies in particular
\[ (u^{m+1} - \tilde{u}^{m+1}, u) = 0 \quad \forall u \in H. \]
Algorithm 1 Time discrete algorithm

Initialization: Let $p^0$ be given and to take $u^0 = \tilde{u}^0 = u(0) (= u_0)$.

Step of time $m + 1$: Let $u^m$, $\tilde{u}^m$ and $p^m$ be given.

Sub-step 1: Find $\tilde{u}^{m+1} : \Omega \to \mathbb{R}^3$ solving

$$
(S_1)^{m+1} \begin{cases}
\frac{1}{k}(\tilde{u}^{m+1} - u^m) + C(\tilde{u}^m, \tilde{u}^{m+1}) - \Delta \tilde{u}^{m+1} + \nabla p^m = f^{m+1} & \text{in } \Omega, \\
\tilde{u}^{m+1}|_{\partial \Omega} = 0 & \text{on } \partial \Omega.
\end{cases}
$$

Sub-step 2: Find $u^{m+1} : \Omega \to \mathbb{R}^3$ and $p^{m+1} : \Omega \to \mathbb{R}$ solution of

$$
(S_2)^{m+1} \begin{cases}
\frac{1}{k}(u^{m+1} - \tilde{u}^{m+1}) + \nabla (p^{m+1} - p^m) = 0 & \text{in } \Omega, \\
\nabla \cdot u^{m+1} = 0 & \text{in } \Omega, \\
\nabla \cdot u^{m+1} = 0 & \text{in } \Omega, \\
u^{m+1} \cdot n|_{\partial \Omega} = 0.
\end{cases}
$$

• By using $\nabla \cdot u^{m+1} = 0$ in $\Omega$ and $u^{m+1} \cdot n|_{\partial \Omega} = 0$, one has the orthogonality property

$$
(u^{m+1}, \nabla q) = 0 \quad \forall \ q \in H^1(\Omega).
$$

• It is well known that Sub-step 2 is equivalent to the following two (decoupled) problems:

1. Find $p^{m+1} : \Omega \to \mathbb{R}$ such that

$$
(S_2)^{m+1}_a \begin{cases}
k \Delta (p^{m+1} - p^m) = \nabla \cdot \tilde{u}^{m+1} & \text{in } \Omega, \\
k \nabla (p^{m+1} - p^m) \cdot n|_{\partial \Omega} = 0 & \text{on } \partial \Omega.
\end{cases}
$$

2. Find $u^{m+1} : \Omega \to \mathbb{R}^3$ as

$$
(S_2)^{m+1}_b \quad u^{m+1} = \tilde{u}^{m+1} - k \nabla (p^{m+1} - p^m) \quad \text{in } \Omega.
$$

1.2 Unconditional stability and convergence of Algorithm 1

Lemma 2 (Continuous dependence of the projection step).

a) (Continuous dependence with respect to $L^2$) If $\tilde{u}^{m+1}$ and $u^m \in L^2(\Omega)$, then there exists an unique $u^{m+1} \in H$ solution of $(S_2)^{m+1}$. Moreover,

$$
|\tilde{u}^{m+1}|^2 = |u^{m+1}|^2 + |k \nabla (p^{m+1} - p^m)|^2
$$

$$
|u^{m+1} - \tilde{u}^{m+1}| \leq |\tilde{u}^{m+1} - u^m|.
$$

b) (Continuous dependence with respect to $H^1$) If $\tilde{u}^{m+1} \in H^1_0(\Omega)$ then $u^{m+1} \in H^1(\Omega) \cap H$. Moreover,

$$
\|u^{m+1}\| \leq C \|\tilde{u}^{m+1}\|.
$$
Proof.

a) Since \( u^{m+1} = P_H \tilde{u}^{m+1} \), one has (3). Moreover, estimate (4) can be obtained directly from the best approximation property of the \( L^2 \)-projection:

\[
|u^{m+1} - \tilde{u}^{m+1}| = \min_{u \in H} |u - \tilde{u}^{m+1}|.
\]

b) By applying the \( H^2(\Omega) \)-regularity of problem \((S_2)^{m+1}_a\), there exists a unique \( p^{m+1} - p^m \in H^2 \cap L^2_0 \) satisfying

\[
k \| \nabla (p^{m+1} - p^m) \|_{H^1} \leq C \| \tilde{u}^{m+1} \|.
\]

Therefore, \( u^{m+1} \in H^1(\Omega) \) and

\[
\| u^{m+1} \| \leq C \left\{ \| \tilde{u}^{m+1} \| + k \| \nabla (p^{m+1} - p^m) \|_{H^1} \right\} \leq C \| \tilde{u}^{m+1} \|.
\]

This estimate can be understood as the \( H^1 \)-stability of the \( L^2 \)-projector onto \( H \).

Lemma 3 (Stability of Algorithm 1) Let \( f \in L^2(0,T;\mathcal{H}^{-1}(\Omega)) \) (\( \mathcal{H}^{-1}(\Omega) \) being the dual space of \( \mathcal{H}_0^1(\Omega) \)) and \( u_0 \in \mathcal{H} \). Assuming the following constraint on the initial discrete pressure \( k |\nabla p^0| \leq C_0 \), then there exists a constant \( C = C(C_0,u_0,f,\Omega) > 0 \) such that,

\[
|\tilde{u}^{r+1}|^2 + |u^{r+1}|^2 + |k \nabla p^{r+1}|^2 \leq C, \quad \forall r = 0, \ldots, M - 1,
\]

\[
k \sum_{m=0}^{M-1} \left\{ \| \tilde{u}^{m+1} \|^2 + \| u^{m+1} \|^2 \right\} \leq C.
\]

Proof. We only give here an outline of the proof, which follows the same lines given in the proof of Theorem 7 below. By making

\[
2k \left( (S_1)^{m+1}_{\tilde{u}} \tilde{u}^{m+1} \right) + k \left( (S_2)^{m+1}_{\tilde{u}} \tilde{u}^{m+1} + u^{m+1} + k (\nabla p^{m+1} + \nabla p^m) \right),
\]

and using orthogonality property (2):

\[
|u^{m+1}|^2 + |k \nabla p^{m+1}|^2 - |u^{m+1}|^2 - |k \nabla p^m|^2 + |\tilde{u}^{m+1} - u^{m+1}|^2 + 2k \| \tilde{u}^{m+1} \|^2 = 2k (f^{m+1}, \tilde{u}^{m+1}),
\]

hence, by using the discrete Gronwall’s Lemma (Lemma 1):

\[
\| u^{m+1} \|_{L^\infty(L^2)} + k \| \nabla p^{m+1} \|_{L^\infty(L^2)} + \| \tilde{u}^{m+1} \|_{L^2(\mathcal{H}^1)} \leq C \quad \text{and} \quad \sum_{m=0} \| \tilde{u}^{m+1} - u^{m+1} \|^2 \leq C.
\]

Now, accounting Lemma 2, the following supplementary stability estimates hold:

\[
\| \tilde{u}^{m+1} \|_{L^\infty(L^2)} \leq C \quad \text{and} \quad \| u^{m+1} \|_{L^2(\mathcal{H}^1)} \leq C.
\]

Starting from the previous stability estimates and taking limits as \( k \downarrow 0 \) in \((S_3)^{m+1}_a\), the convergence of the velocity approximations have already been established (for instance, see [32]). Concretely, defining \( u_k : (0,T] \rightarrow H \cap H^1(\Omega) \) as the piecewise constant functions taking the value \( u^{m+1} \) in \( (t_m, t_{m+1}] \), the following result holds:
Proposition 4 (Convergence of Algorithm 1) Under conditions of Lemma 3, there exists a subsequence \((k')\) of \((k)\), and a weak solution \(u \in L^\infty(0,T; H) \cap L^2(0,T; V)\) of \((P)\) in \((0,T)\), such that: \(u_{k'} \to u\) weakly-* in \(L^\infty(0,T; H)\), weakly in \(L^2(0,T; H^1(\Omega) \cap H)\) and strongly in \(L^2(0,T; H)\), as \(k' \downarrow 0\).

1.3 Differential problems satisfied by the errors

We will obtain error estimates (for velocity and pressure) with respect to a sufficiently regular (and unique) solution \((u, p)\) of \((P)\). For this, we introduce the following notations for the errors in \(t = t_{m+1}\):

\[
\tilde{e}^{m+1} := u(t_{m+1}) - \tilde{u}^{m+1}, \quad e^{m+1} := u(t_{m+1}) - u^{m+1}, \quad e_p^{m+1} := p(t_{m+1}) - p^{m+1},
\]

and for the discrete in time derivative of errors

\[
\delta_t e^{m+1} := \frac{e^{m+1} - e^m}{k}, \quad \delta_t \tilde{e}^{m+1} := \frac{\tilde{e}^{m+1} - \tilde{e}^m}{k}.
\]

Subtracting \((S_1)^{m+1}\) with the momentum system of \((P)\) at \(t = t_{m+1}\), using the integral rest and manipulating the convective terms, one has:

\[
\begin{aligned}
(E_1)^{m+1} \quad \left\{ \begin{array}{l}
\frac{1}{k}(\tilde{e}^{m+1} - e^m) - \Delta \tilde{e}^{m+1} + \nabla (e_p^{m+1} + k \delta_t p(t_{m+1})) = e^{m+1} + \text{NL}^{m+1} \quad \text{in } \Omega, \\
e^{m+1} \big|_{\partial \Omega} = 0,
\end{array} \right.
\end{aligned}
\]

where

\[
\mathcal{E}^{m+1} = -\frac{1}{k} \int_{t_m}^{t_{m+1}} (t - t_m) u_{tt}(t) \, dt - \left( \int_{t_m}^{t_{m+1}} u_t \cdot \nabla \right) u(t_{m+1}) := e_1^{m+1} + e_2^{m+1}
\]

is the consistency error, and

\[
\text{NL}^{m+1} = -C\left(\tilde{e}^m, u(t_{m+1})\right) - C\left(\tilde{u}^m, e^{m+1}\right)
\]

are terms depending of the convective terms.

On the other hand, adding and subtracting the term \(u(t_{m+1})\) in \((S_2)^{m+1}\),

\[
(E_2)^{m+1} \quad \left\{ \begin{array}{l}
\frac{1}{k}(e^{m+1} - e^m) + \nabla (e_p^{m+1} - e_p^m - k \delta_t p(t_{m+1})) = 0 \quad \text{in } \Omega, \\
\nabla \cdot e^{m+1} = 0 \quad \text{in } \Omega, \quad e^{m+1} \cdot n|_{\partial \Omega} = 0.
\end{array} \right.
\]

Finally, adding \((E_1)^{m+1}\) and \((E_2)^{m+1}\), we arrive at:

\[
(E_3)^{m+1} \quad \left\{ \begin{array}{l}
\frac{1}{k}(e^{m+1} - e^m) - \Delta e^{m+1} + \nabla e_p^{m+1} = e^{m+1} + \text{NL}^{m+1} \quad \text{in } \Omega, \\
\nabla \cdot e^{m+1} = 0 \quad \text{in } \Omega, \quad e^{m+1} \cdot n|_{\partial \Omega} = 0.
\end{array} \right.
\]
Lemma 5 (Continuous dependence of the projection errors) The following inequalities hold
\[ |\tilde{e}^{m+1}|^2 = |e^{m+1}|^2 + k \nabla (e^{m+1} - e^m - k \delta_p(t_{m+1}))^2, \]
\[ |e^{m+1} - \tilde{e}^{m+1}| \leq |e^{m+1} - e^m| \]
\[ \|e^{m+1}\| \leq C \|\tilde{e}^{m+1}\|. \]

Proof. The proof is similar to Lemma 2, by using that \( e^{m+1} = P_H \tilde{e}^{m+1} \).

1.4 Regularity hypotheses.

We will assume the following regularity hypothesis on \( \Omega \):

(H0) \( \Omega \subset \mathbb{R}^3 \) such that Poisson problems in \( \Omega \) have \( H^2(\Omega) \)-regularity.

In order to obtain the different error estimates, the following regularity hypotheses for the (unique) solution \((u, p)\) of \((P)\) will be appearing:

(H1) \( u \in L^\infty(\mathbb{H}^2 \cap \nabla), \quad p_t \in L^2(H^1), \quad u_t \in L^2(L^2), \quad u_{tt} \in L^2(H^{-1}) \)

(H2) \( p_{tt} \in L^2(H^1), \quad u_t \in L^\infty(L^3) \cap L^2(H^1), \quad u_{tt} \in L^2(L^2), \quad u_{ttt} \in L^2(H^{-1}) \)

(H3) \( u_{tt} \in L^\infty(H^{-1}) \)

Remark 6 Unfortunately, to obtain hypotheses (H1)-(H3) is necessary to assume that \( u_t(0) \in H^1 \), which implies a non-local compatibility condition for the data \( u_0 \) and \( f(0) \). In particular, it is proved in [21] that (H1)-(H3) is satisfied (at least locally in time), if there exists \( p_0 \in H^1(\Omega) \) (the initial pressure) solution of the following overdetermined Neumann problem

\[
\begin{cases} 
\Delta p_0 = \nabla \cdot \left( f(0) - (u_0 \cdot \nabla)u_0 \right) & \text{in} \ \Omega, \\
\nabla p_0|_{\partial\Omega} = \left( \Delta u_0 + f(0) - (u_0 \cdot \nabla)u_0 \right)|_{\partial\Omega}. 
\end{cases}
\]

which in practice is hard to fulfill (see [21]).

In [24], error estimates for the (non-incremental) Chorin-Temam projection scheme are deduced without requiring this non-local compatibility condition, arriving at the optimal order \( O(k) \) in \( l^\infty(L^2) \) for the velocity and in \( l^\infty(H^{-1}) \) for the pressure, where a weight at the initial time steps must be included to deduce the optimal order for the pressure (only possible in a negative norm).

Nevertheless, for the incremental scheme Algorithm 1 it is not clear how to avoid this compatibility on the data using adequate weights at the initial time steps.
1.5 $O(k)$-error estimates for both velocities

**Theorem 7** Under conditions of Lemma 3, (H1) and the bound for the initial error pressure $|\nabla p_0| \leq C$, the following error estimates hold:

\[
\|e^m_{m+1}\|_{L^\infty(L^2)\cap L^2(H^1)} + \|e^{m+1}_{m+1}\|_{L^\infty(L^2)\cap L^2(H^1)} \leq C k, \quad \|e_{p_{m+1}}\|_{L^\infty(H^1)} \leq C, \\
\|e^m - e^m_{m+1}\|_{L^2(L^2)} + \|e^{m+1} - e^m_{m+1}\|_{L^2(L^2)} \leq C k^{3/2}.
\]

**Proof.** The proof follows similar lines of [10] and [27].

By multiplying $(E_1)_{m+1}$ by $2k e^{m+1}$ and integrating in $\Omega$, one has:

\[
\|e^{m+1}\|^2 - \|e^m\|^2 + |k \nabla e^{m+1}|^2 - |k \nabla e^m|^2 - 2k \|e^{m+1}\|^2 + 2k \left( \nabla e^m_p, e^{m+1} \right) \\
= 2k \left( e^{m+1} + NL^{m+1}, e^{m+1} \right) - 2k^2 \left( \nabla \delta p(t_{m+1}), e^{m+1} \right). 
\]

On the other hand, multiplying $(E_2)^{m+1}$ by $k(e^{m+1} + e^{m+1}) + k^2 \left( \nabla e^{m+1}_p + \nabla e^m_p \right)$ and using that $(e^{m+1}, \nabla e^m_{p,m+1}) = 0 = (e^{m+1}, \nabla e^m_{p,m+1}) = (e^{m+1}, \nabla p(t_{m+1}))$ (see (2)), we obtain

\[
|e^{m+1}|^2 - |e^m|^2 + |k \nabla e^{m+1}|^2 - |k \nabla e^m|^2 - 2k \left( e^{m+1}, \nabla e^m_p \right) \\
= k^2 \left( e^{m+1}, \nabla \delta p(t_{m+1}) \right) + k^3 \left( \nabla \delta p(t_{m+1}), e^{m+1} \right). 
\]

By adding (8) and (9), the term $2k \left( e^{m+1}, \nabla e^m_p \right)$ vanish, obtaining

\[
|e^{m+1}|^2 - |e^m|^2 + |k \nabla e^{m+1}|^2 - |k \nabla e^m|^2 - 2k \|e^{m+1}\|^2 + 2k \|e^m\|^2 \\
\leq 2k \left( e^{m+1} + NL^{m+1}, e^{m+1} \right) - k^2 \left( \nabla \delta p(t_{m+1}), e^{m+1} \right) \\
+ k^3 \left( \nabla \delta p(t_{m+1}), e^{m+1} \right). 
\]

The consistency error can be bounded as follows:

\[
2k \left( e^{m+1}_1, e^{m+1}_1 \right) \leq \frac{k}{3} \|e^{m+1}\|^2 + C k^2 \iint_{t_m}^{t_{m+1}} \|u_t\|^2_{H^{-1}} dt, \\
2k \left( e^{m+1}_2, e^{m+1}_2 \right) \leq 2k \left( \int_{t_m}^{t_{m+1}} \|u_t\|_{L^3}, \|e^{m+1}_1\|_{L^3} \right) \|e^{m+1}_1\|_{L^6} \leq \frac{k}{3} \|e^{m+1}\|^2 + C k^2 \iint_{t_m}^{t_{m+1}} \|u_t\|^2.
\]

By using the antisymmetry property $c(u^m, e^{m+1}, e^{m+1}) = 0$ and equality (5), we bound the convective terms as follows:

\[
2k \left( NL^{m+1}_1, e^{m+1}_1 \right) = 2k \left( e^m, u(t_{m+1}), e^{m+1} \right) \leq \frac{k}{3} \|e^{m+1}\|^2 + C k \|u(t_{m+1})\|^2_{L^\infty W^{1,3}} \|e^m\|^2 \\
\leq \frac{k}{3} \|e^{m+1}\|^2 + C k \|e^m\|^2 + C k \left( |k \nabla e^m|^2 + |k \nabla e^{m+1}|^2 + k^2 |\nabla \delta p(t_{m+1})|^2 \right)
\]

Now, by using that $(e^m, \nabla \delta p(t_{m+1})) = 0$, we bound the third term at RHS of (10):

\[
-k^2 \left( e^{m+1}, \nabla \delta p(t_{m+1}) \right) = -k^2 \left( e^{m+1} - e^m, \nabla \delta p(t_{m+1}) \right) \leq \frac{1}{4} \|e^{m+1} - e^m\|^2 + C k^4 \|\nabla \delta p(t_{m+1})\|^2.
\]
Finally, we bound the last term at RHS of (10):
\[ k^3 \left( \nabla \delta_t p(t_{m+1}), \nabla e_{p}^{m+1} + \nabla e_{p}^{m} \right) = k^3 \left( \nabla \delta_t p(t_{m+1}), \nabla e_{p}^{m+1} - \nabla e_{p}^{m} \right) + k^3 \left( \nabla \delta_t p(t_{m+1}), 2 \nabla e_{p}^{m} \right) = I_1 + I_2 \]

By using \((E_2)^{m+1}\), the \(I_1\)-term can be rewritten as
\[
I_1 = k^2 \left( \nabla \delta_t p(t_{m+1}), \tilde{e}_{m+1} - e_{m} \right) + k^2 \left| \nabla \delta_t p(t_{m+1}) \right|^2 
\leq \frac{1}{4} \| \tilde{e}_{m+1} - e_{m} \|^2 + C k^3 \int_{t_m}^{t_{m+1}} \| \nabla p_t \|^2 
\]

We bound \(I_2\) as:
\[
I_2 \leq C k \|k \nabla e_{p}^{m}\|^2 + C k^3 \| \nabla \delta_t p(t_{m+1}) \|^2 \leq C k \|k \nabla e_{p}^{m}\|^2 + C k^2 \int_{t_m}^{t_{m+1}} \| \nabla p_t \|^2 
\]

By applying these bounds in (10),
\[
|e_{m+1}^3| - |e_{m}|^2 + k \nabla e_{p}^{m+1} - |k \nabla e_{p}^{m}|^2 + \frac{1}{2} \| \tilde{e}_{m+1} - e_{m} \|^2 + k \| \tilde{e}_{m+1} \|^2 \leq C k \|e_{m}\|^2 
+ C k^2 \int_{t_m}^{t_{m+1}} \left( \| u_t \|^2_{H-1} + |u_t|^2 + |\nabla p_t|^2 \right) dt + C k \left( |k \nabla e_{p}^{m}|^2 + |k \nabla e_{p}^{m-1}|^2 \right). 
\]

Adding up from \(m = 1\) to \(r\), and applying the discrete Gronwall inequality, we arrive at:
\[
\|e_{m+1}\|_{L^2} + \|e_{m+1}\|_{L^2(H^1)} \leq C k, \quad \|e_{m+1} - e_{m}\|_{L^2} \leq C k^{3/2} \quad \text{and} \quad \|e_{p}^{m+1}\|_{L^\infty(H^1)} \leq C. 
\]

Finally, by applying Lemma 5, estimates \(\|e_{m+1}\|_{L^\infty(L^2)} \leq C k\) and \(\|e_{m+1}\|_{L^2(H^1)} \leq C k\) hold. 

Notice that the error estimate \(\|e_{m}\|_{L^\infty(H^1)} \leq C k\) implies in particular the uniform estimates
\[
\|e_{m}\|_{H^1} \leq C \quad \text{and} \quad \|u_{m}\|_{H^1} \leq C \quad \forall m. 
\]

1.6 \(O(k)\)-error estimates for the pressure

First, we are going to obtain error estimates for the discrete time derivative of velocity, and then the optimal order \(O(k)\) for the pressure.

Lemma 8 (Continuous dependence of discrete derivatives for the projection step) It holds
\[
|\delta_t \tilde{e}_{m+1}^1| = |\delta_t e_{m+1}^1| + |k \nabla \delta_t (e_{p}^{m+1} - e_{p}^{m} - k \delta_t p(t_{m+1}))|^2, \tag{11}
\]
\[
|\delta_t e_{m+1}^1 - \delta_t \tilde{e}_{m+1}^1| \leq |\delta_t e_{m+1}^1 - \delta_t e_{m}^1|,
\]
\[
|\delta_t e_{m+1}^1| \leq C \|\delta_t \tilde{e}_{m+1}^1\|. 
\]

Proof. The proof is similar to Lemma 2 and Lemma 5, using that \(\delta_t e_{m+1}^1 = P_{H}(\delta_t \tilde{e}_{m+1}^1)\).
Theorem 9 Assuming hypotheses of Theorem 7, (H2) and the following constraints on the first-step approximation

\[ |\delta_t \mathbf{e}^1| + |k \nabla \delta_t \mathbf{e}_p^1| \leq C k \quad \text{and} \quad |\delta_t \mathbf{e}^1| \leq C \sqrt{k}, \]

one has

\[ \|\delta_t \mathbf{e}^{m+1}\|_{L^\infty(L^2) \cap L^2(H^1)} + \|\delta_t \mathbf{e}^{m+1}\|_{L^\infty(L^2) \cap L^2(H^1)} \leq C k \quad \text{and} \quad \|\delta_t \mathbf{e}_p^{m+1}\|_{L^\infty(H^1)} \leq C. \]

Proof. By making \( \delta_t (E_1)^{m+1} \) and \( \delta_t (E_2)^{m+1} \):

\[ (D_1)^{m+1} \quad \frac{\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^m}{k} - \Delta \delta_t \mathbf{e}^{m+1} + \nabla (\delta_t \mathbf{e}_p^m + k \delta_t \mathbf{p}(t_{m+1})) = \delta_t \mathbf{e}^{m+1} + \delta_t \mathbf{NL}^{m+1} \]

where \( \delta_t \mathbf{p}(t_{m+1}) = \frac{1}{k} (\delta_t \mathbf{p}(t_{m+1}) - \delta_t \mathbf{p}(t_m)) \), and

\[ (D_2)^{m+1} \quad \frac{\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^m}{k} + \nabla (\delta_t \mathbf{e}_p^{m+1} - \delta_t \mathbf{e}_p^m - k \delta_t \mathbf{p}(t_{m+1})) = 0. \]

The proof follows similar lines of Theorem 7. Multiplying \((D_1)^{m+1}\) by \(2 k \delta_t \mathbf{e}^{m+1}\), we get:

\[ |\delta_t \mathbf{e}^{m+1}|^2 - |\delta_t \mathbf{e}^m|^2 + |\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^m|^2 + 2 k |\delta_t \mathbf{e}^{m+1}|^2 + 2 k \left( \nabla \delta_t \mathbf{e}_p^m, \delta_t \mathbf{e}^{m+1} \right) \]

\[ = 2 k \left( \delta_t \mathbf{e}^{m+1} + \delta_t \mathbf{NL}^{m+1}, \delta_t \mathbf{e}^{m+1} \right) - 2 k^2 \left( \nabla \delta_t \mathbf{p}(t_{m+1}), \delta_t \mathbf{e}^{m+1} \right). \]  \hspace{1cm} (12)

On the other hand, multiplying \((D_2)^{m+1}\) by \(k (\delta_t \mathbf{e}^{m+1} + \delta_t \mathbf{e}^{m+1}) + k^2 (\nabla \delta_t \mathbf{e}_p^{m+1} + \nabla \delta_t \mathbf{e}_p^m)\),

\[ |\delta_t \mathbf{e}^{m+1}|^2 - |\delta_t \mathbf{e}^m|^2 + |k \nabla \delta_t \mathbf{e}_p^{m+1}|^2 - |k \nabla \delta_t \mathbf{e}_p^m|^2 - 2 k \left( \delta_t \mathbf{e}^{m+1}, \nabla \delta_t \mathbf{e}_p^m \right) \]

\[ = k^2 \left( \nabla \delta_t \mathbf{p}(t_{m+1}), \delta_t \mathbf{e}^{m+1} \right) + k^3 \left( \nabla \delta_t \mathbf{p}(t_{m+1}), \nabla \delta_t \mathbf{e}_p^{m+1} + \nabla \delta_t \mathbf{e}_p^m \right). \]  \hspace{1cm} (13)

By adding \((12)\) and \((13)\), the term \(2 k \left( \delta_t \mathbf{e}^{m+1}, \nabla \delta_t \mathbf{e}_p^m \right)\) cancels, arriving at

\[ |\delta_t \mathbf{e}^{m+1}|^2 - |\delta_t \mathbf{e}^m|^2 + |\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^m|^2 + |k \nabla \delta_t \mathbf{e}_p^{m+1}|^2 - |k \nabla \delta_t \mathbf{e}_p^m|^2 + 2 k |\delta_t \mathbf{e}^{m+1}|^2 \]

\[ = 2 k \left( \delta_t \mathbf{e}^{m+1} + \delta_t \mathbf{NL}^{m+1}, \delta_t \mathbf{e}^{m+1} \right) - k^2 \left( \nabla \delta_t \mathbf{p}(t_{m+1}), \delta_t \mathbf{e}^{m+1} \right) + k^3 \left( \nabla \delta_t \mathbf{p}(t_{m+1}), \nabla \delta_t \mathbf{e}_p^{m+1} + \nabla \delta_t \mathbf{e}_p^m \right). \]  \hspace{1cm} (14)

We bound the RHS of \((14)\) as follows:

\[ 2 k \left( \delta_t \mathbf{e}_1^{m+1}, \delta_t \mathbf{e}^{m+1} \right) \leq \varepsilon k |\delta_t \mathbf{e}^{m+1}|^2 + C k^2 \int_{t_{m-1}}^{t_{m+1}} \|u_{tt}\|^2_{H^{-1}} \]

\[ 2 k \left( \delta_t \mathbf{e}_2^{m+1}, \delta_t \mathbf{e}^{m+1} \right) = 2 k \left( \delta_t u(t_{m+1}) \cdot \nabla (u(t_{m+1}) - u(t_m)), \delta_t \mathbf{e}^{m+1} \right) \]

\[ + 2 k \left( \delta_t (u(t_{m+1}) - \delta_t u(t_m)) \cdot \nabla u(t_m), \delta_t \mathbf{e}^{m+1} \right) := I_1 + I_2 \]

\[ I_1 \leq \varepsilon k |\delta_t \mathbf{e}^{m+1}|^2 + C k \|\delta_t u(t_{m+1})\|_{L^2}^2 \int_{t_m}^{t_{m+1}} \|\delta_t u\|^2_{H^1} \leq \varepsilon k |\delta_t \mathbf{e}^{m+1}|^2 + C k^2 \|u_t\|_{L^\infty(L^2)} \int_{t_m}^{t_{m+1}} \|u_t\|^2_{H^1}. \]
\[ I_2 \leq \varepsilon k \| \delta_t \tilde{e}^{m+1} \|^2 + C k \| \nabla u(t_m) \|_{L^3}^2 \| \delta_t \left( \int_{t_m}^{t_{m+1}} u_t \right) \|^2 \leq \varepsilon k \| \delta_t \tilde{e}^{m+1} \|^2 + C k^2 \int_{t_{m-1}}^{t_{m+1}} \| u_{tt} \|^2 \]

(in the above inequality we have used estimates obtained in [28]).

Now, we bound the non-linear terms:

\[ 2k \left( \delta_t N L^{m+1}, \delta_t \tilde{e}^{m+1} \right) = 2k \left( \delta_t \tilde{e}^m, u(t_{m+1}), \delta_t \tilde{e}^{m+1} \right) + 2k \left( \delta_t \bar{u}^m, \tilde{e}^{m+1}, \delta_t \tilde{e}^{m+1} \right) \]

\[ + 2k \left( \bar{e}^{m-1}, \delta_t u(t_{m+1}), \delta_t \tilde{e}^{m+1} \right) + 2k \left( \bar{u}^{m-1}, \delta_t \bar{e}^{m+1}, \delta_t \tilde{e}^{m+1} \right) := \sum_{i=1}^4 L_i \]

\[ L_1 \leq k \| \delta_t \tilde{e}^m \| \| u(t_{m+1}) \|_{L^\infty W^{1,3}} \| \delta_t \tilde{e}^{m+1} \| \leq \varepsilon k \| \delta_t \tilde{e}^{m+1} \|^2 + C k \| \delta_t \tilde{e}^m \|^2 \]

\[ \leq \varepsilon k \| \delta_t \tilde{e}^{m+1} \|^2 + C k \| \delta_t \tilde{e}^m \|^2 + C k \left( \| k \nabla \delta_t e_p^m \|^2 + \| k \nabla \delta_t e_p^{m-1} \|^2 + C k^2 \| \nabla \delta_t \delta P(t_m) \|^2 \right) \]

(here (11) is used),

\[ L_2 = 2k \left( \delta_t \bar{e}^m, \tilde{e}^{m+1}, \delta_t \tilde{e}^{m+1} \right) + 2k \left( \delta_t u(t_m), \bar{e}^{m+1}, \delta_t \tilde{e}^{m+1} \right) := L_{21} + L_{22} \]

\[ L_{21} \leq 2k \| \bar{e}^{m+1} \| \| \delta_t \tilde{e}^m \| \| \delta_t \tilde{e}^{m+1} \| \leq \varepsilon k \left( \| \delta_t \tilde{e}^{m+1} \|^2 + \| \delta_t \tilde{e}^m \|^2 \right) + C k \| \delta_t \tilde{e}^m \|^2 \]

where we have used that \( \| \tilde{e}^{m+1} \| \leq C \),

\[ L_{22} \leq k \| \delta_t u(t_m) \|_{L^3} \| \delta_t \tilde{e}^{m+1} \| \| \tilde{e}^{m+1} \| \leq \varepsilon k \| \delta_t \tilde{e}^{m+1} \|^2 + C k \| \tilde{e}^{m+1} \|^2 \]

(in the above estimate we have used the regularity \( u_t \in L^\infty(L^3) \)), and from a similar way,

\[ L_3 \leq \varepsilon k \| \delta_t \tilde{e}^{m+1} \|^2 + C k \| \tilde{e}^{m+1} \|^2. \]

Finally,

\[ L_4 = 0. \]

Reasoning as in Theorem 7, taking into account the above estimates and choosing \( \varepsilon \) small enough, we arrive at

\[ \| \delta_t e^{m+1} \|^2 - \| \bar{\delta} e^m \|^2 + \frac{1}{2} \| \delta_t \bar{e}^{m+1} - \delta_t e^m \|^2 + \| k \nabla \delta_t e_p^m \|^2 \]

\[ \leq C k \| \delta_t e^m \|^2 + C k^2 \int_{t_{m-1}}^{t_{m+1}} \left( \| u_{tt} \|^2_{H^{-1}} + \| u_{tt} \|^2 \right) + C k^2 \int_{t_{m-1}}^{t_{m+1}} \| u_t \|^2 + \frac{k}{2} \| \delta_t \tilde{e}^m \|^2 \]

\[ + C k \left( \| \bar{e}^{m-1} \|^2 + \| \bar{e}^{m+1} \|^2 \right) + C k \left( \| k \nabla \delta_t e_p^m \|^2 + \| k \nabla \delta_t e_p^{m-1} \|^2 \right) + C k^2 \int_{t_{m-1}}^{t_{m+1}} | \nabla p_{tt} |^2. \]

Now, by adding from \( m = 1 \) to \( r \) and using error estimates of Theorem 7, we arrive at

\[ \| \delta_t e^{r+1} \|^2 + \| k \nabla \delta_t e_p^{r+1} \|^2 + \frac{1}{2} \sum_{m=1}^r \| \delta_t \bar{e}^{m+1} - \delta_t e^m \|^2 + \frac{k}{2} \sum_{m=1}^r \| \delta_t \tilde{e}^{m+1} \|^2 \]

\[ \leq \| \delta_t e^1 \|^2 + \| k \nabla \delta_t e_p^1 \|^2 + \frac{k}{2} \| \delta_t \tilde{e}^1 \|^2 + C k \sum_{m=1}^r \left( \| \delta_t e^m \|^2 + \| k \nabla \delta_t e_p^m \|^2 + \| k \nabla \delta_t e_p^{m-1} \|^2 \right) + C k^2. \]
Then, applying the discrete Gronwall Lemma, we obtain the estimates
\[ \| \delta t e^{m+1} \|_{L^\infty(L^2)} \leq C k, \quad \sum_{m=1}^{r} |\delta_t e^{m+1} - \delta_t e^{m}|^2 \leq C k^2, \quad \| \delta_t e^{m+1} \|_{L^2(H^1)} \leq C k \]
and
\[ \| \delta_t e_p^{m+1} \|_{L^\infty(H^1)} \leq C. \]
After that, taking into account Lemma 8,
\[ \| \delta_t e^{m+1} \|_{L^2(H^1)} \leq C k \sum_{m=1}^{r} |\delta_t e^{m+1} - \delta_t e^{m}| \leq C k^2 \quad \text{and} \quad \| \delta_t e^{m+1} \|_{L^\infty(L^2)} \leq C k \]
hence the proof is finished.

**Theorem 10** Under hypothesis of Theorem 9 and (H3), the following error estimates hold
\[ \| \bar{e}^{m+1} \|_{L^\infty(H^1)} + \| e_p^{m+1} \|_{L^\infty(L^2)} \leq C k. \]  
(15)

**Proof.**

**Step 1.** To prove
\[ \| e_p^{m+1} \|_{L^2(L^2)} \leq C k. \]  
(16)

We are going to deduce the estimate (16) from Theorem 9 and the continuous inf-sup condition applied to \((E_3)^{m+1}\). Indeed, rewritten \((E_3)^{m+1}\) as
\[ -\nabla e_p^{m+1} = \delta_t e^{m+1} - \Delta \bar{e}^{m+1} - E^{m+1} - NL^{m+1}, \quad e_p^{m+1} \in L_0^2(\Omega), \]
then, applying the continuous inf-sup condition
\[
\| e_p^{m+1} \|_{L^2} \leq C \left\{ \| \delta_t e^{m+1} \|_{H^{-1}} + \| \bar{e}^{m+1} \| + \| e^{m+1} \| + \| NL^{m+1} \|_{H^{-1}} \right\} \\
\leq C \left\{ \| \delta_t e^{m+1} \|_{H^{-1}} + \| \bar{e}^{m+1} \| + \| e^{m} \| + k \left\| u \right\|_{L\infty(0,T;H^{-1})} + k \left\| u \right\|_{L\infty(0,T;L^3)} \right\},
\]
where we have used the estimate
\[ \| NL^{m+1} \|_{H^{-1}} \leq C(\| e^{m} \| + \| u(t_{m+1}) \|_{L^3} + \| e^{m+1} \| + \| \bar{e}^{m} \|_{L^3}) \leq C(\| e^{m} \| + \| e^{m+1} \|). \]

By taking into account that \( \| \bar{e}^{m+1} \|_{L^2(H^1)} \leq C k \) and \( \| \delta_t e^{m+1} \|_{L^\infty(L^2)} \leq C k \) and hypothesis (H2) and (H3), we arrive at (16).

**Step 2.** To prove (15) for \( \bar{e}^{m+1} \).

From \((E_3)^{m+1}\) we have
\[ -\Delta \bar{e}^{m+1} = -\delta_t e^{m+1} - \nabla e_p^{m+1} + E^{m+1} + NL^{m+1}, \quad \bar{e}^{m+1} |_{\partial\Omega} = 0. \]
Multiplying by \( 2k \delta_t \bar{e}^{m+1} \), we obtain
\[
|\nabla \bar{e}^{m+1}|^2 - |\nabla e^{m+1}|^2 + |\nabla \bar{e}^{m+1} - \nabla e^{m+1}|^2 \leq 2k \left( -\nabla e_p^{m+1} - \delta_t e^{m+1} + E^{m+1} + NL^{m+1}, \delta_t \bar{e}^{m+1} \right) \\
\leq C k |e_p^{m+1}|^2 + C k |\nabla \delta_t \bar{e}^{m+1}|^2 + C k |\delta_t e^{m+1}|^2 + C k |E^{m+1}|^2_{H^{-1}} + C k |NL^{m+1}|^2_{H^{-1}} \\
\leq C k |e_p^{m+1}|^2 + C k |\nabla \delta_t \bar{e}^{m+1}|^2 + C k |\delta_t e^{m+1}|^2 + C k^3 + C k \left( \| e^{m} \|^2 + \| e^{m+1} \|^2 \right)
\]
(18)
where we have bounded the two last terms at RHS of (18) as in (17). Adding (18) from \( m = 0 \) to \( r \) and applying the estimates of Theorems 7 and 9 and (16), we arrive at (15) for \( \tilde{e}_m^{m+1} \).

**Step 3.** To prove (15) for \( e_{p}^{m+1} \).

By using the inequality (17) and taking into account that

\[
\| \delta_t e_{m+1} \|_{H^{-1}} \leq C \| \delta_t e_m \| \leq C k \quad \text{and} \quad \| \tilde{e}_m^{m+1} \| \leq C k,
\]

we arrive at (15) for \( e_{p}^{m+1} \).

### 1.7 Additional estimates

Now, we are going to obtain some \( H^2 \) stability estimates which will be necessary in next Section to get optimal error estimates in space.

**Lemma 11** Under hypotheses of Theorem 7 and \((H0)\), one has

\[
\| e_{m+1} \|_{H^2} \leq C, \quad \forall m.
\]

**Proof.** From the \( H^2 \)-regularity of the Poisson problem \((E_1)^{m+1}\), one has

\[
\| \tilde{e}_{m+1} \|_{H^2} \leq C \left( \left( \frac{e_{m+1} - e_m}{k} \right)^2 + \| \nabla e_{p}^m \|^2 + k^2 \| \nabla \delta_t p(t_{m+1}) \|^2 + \| \tilde{e}_{m+1} \|^2 + |\text{NL}^{m+1}|^2 \right). \tag{19}
\]

The first and second term of the RHS of (19) are bounded using that \( |\tilde{e}_{m+1} - e_m| \leq C k \) from (7) and \( \| e_{p}^{m+1} \|_{\infty(H^1)} \leq C \) from (6). It is easy to bound the third and the forth term of the RHS of (19). Finally, we bound the nonlinear term as follows

\[
|\text{NL}^{m+1}|^2 \leq C \left( \| e^m \|_{L^\infty(W_{1,3})} \| u(t_{m+1}) \|^2 + \| \tilde{u}^m \|^2 \| \tilde{e}_{m+1} \|_{L^\infty(W_{1,3})}^2 \right) \leq C \left( \| e^m \| \| \tilde{e}_{m+1} \|_{H^2} + \| e_{m+1} \| \| \tilde{e}_{m+1} \|_{H^2} \right) \leq \varepsilon \left( \| e^m \|_{H^2}^2 + \| e_{m+1} \|_{H^2}^2 + C \right).
\]

Then, by applying these estimates in (19) and taking a small enough \( \varepsilon \), there exists \( \alpha < 1 \) such that

\[
\| e_{m+1} \|_{H^2}^2 \leq \alpha \| e^m \|_{H^2}^2 + C,
\]

hence, by an induction process,

\[
\| e_{m+1} \|_{H^2}^2 \leq \alpha^m \| e^0 \|_{H^2}^2 + C (\alpha^m + \cdots + \alpha + 1) \leq C
\]

and the proof is concluded.

**Remark 12** As a consequence of the \( l^\infty \) in time estimates \( \| \tilde{e}_{m+1} \|_{H^2} \leq C \) and \( \| e_{p}^{m+1} \| \leq C, \forall m \), one also has

\[
\| \tilde{u}_{m+1} \|_{H^2} \leq C \quad \text{and} \quad \| p_{m+1} \| \leq C \quad \forall m.
\]
On the other hand, as a direct consequence of Theorem 9, one has
\[ \|\delta_t \delta_t e^{m+1}\|_{L^\infty(L^2)} + \|\delta_t \delta_t \tilde{e}^{m+1}\|_{L^\infty(L^2)} \leq C. \]
In particular, using that \( u_{tt} \in L^2(L^2) \) (see (H2)), this estimate can be extended to the scheme as
\[ \|\delta_t \delta_t \tilde{u}^{m+1}\|_{L^2(L^2)} \leq C. \]

Lemma 13 \textit{Under hypotheses of Theorem 9 and (H0), one has}
\[ \|\delta_t \tilde{e}^{m+1}\|_{L^2(H^2)} \leq C. \]
In particular \( \|\delta_t \tilde{u}^{m+1}\|_{L^2(H^2)} \leq C. \)

\textbf{Proof.} The idea is to argue as in Lemma 11, using the \( H^2 \)-regularity of the Poisson problem \((D_1)^{m+1}\) and applying Theorem 9. \hfill \blacksquare

\section{Fully discrete scheme (Algorithm 2)}

In this section, we will denote by \( C \) different constants, always independent of \( k \) and \( h \).

\subsection{Finite element approximation and fully discrete scheme}

We consider a segregated FE approximation of the time discrete Algorithm 1. We restrict ourselves to the case where \( \Omega \) is a 2D polygon or a 3D polyhedron satisfying the regularity hypothesis \((\text{H0})\). We consider two FE spaces \( Y_h \subset H^1_0(\Omega) \) and \( Q_h \subset H^1(\Omega) \cap L^2_0(\Omega) \) associated to a regular family of triangulations \( T_h \) of the domain \( \Omega \) of mesh size \( h \) (regular in the Ciarlet’s sense \([6]\)). For simplicity, we restrict \( Y_h \) and \( Q_h \) to globally continuous functions and locally polynomials of degree at least 1. Finally, we will assume:

1. The inverse inequality \( \|u_h\| \leq C h^{-1}|u_h| \) for each \( u_h \in Y_h \) holds.
2. The stable “\( \inf-sup \)” condition \(([9])\) for \((Y_h,Q_h)\): There exists \( \beta > 0 \) independent of \( h \) such that,
\[ \inf_{q_h \in Q_h \setminus \{0\}} \left( \sup_{v_h \in Y_h \setminus \{0\}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\| |q_h|} \right) \geq \beta. \]
3. There exists some interpolation operators with the following properties:
   \( I_h : L^2 \to Y_h \) such as
   \[ (u - I_h u, \nabla q_h) = 0, \ \forall q_h \in Q_h \]
satisfying the approximation properties:
\[ \| u - I_h u \|_{H^{-1}} \leq C h \| u \|_{L^2} \quad \forall u \in L^2(\Omega), \]  \hspace{1cm} \text{(23)}
\[ \| u - I_h u \|_{L^2} \leq C h \| u \|_{H^1} \quad \forall u \in H_0^1(\Omega), \]
\[ \| u - I_h u \|_{H^1} \leq C h \| u \|_{H^2} \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega), \]
and the stability property:
\[ \| I_h u \|_{H^1} \leq C \| u \|_{H^1} \quad \forall u \in H_0^1(\Omega). \]

(b) \( J_h : H^1(\Omega) \cap L^2_0(\Omega) \rightarrow Q_h \) defined by
\[ \left( \nabla (J_h p - p), \nabla q_h \right) = 0 \quad \forall q_h \in Q_h, \]
satisfied the approximation property
\[ \| p - J_h p \|_{L^2} \leq C h \| p \|_{H^1} \quad \forall p \in H^1(\Omega) \cap L^2_0(\Omega). \]

Remark 14 (Choice of \( I_h \)) For instance, if we consider the \( \mathbb{P}_1\)-bubble \( \times \mathbb{P}_1 \) approximation to construct the space \( Y_h \times Q_h \), then a possible manner to choose \( I_h \) is as follows: Let \( \tilde{I}_h \) be a regularization interpolation operator (of Clément or Scott-Zhang type) onto the globally continuous and locally \( \mathbb{P}_1 \) FE space, that is \( \tilde{I}_h u \in C^0(\overline{\Omega}) \) and \( \tilde{I}_h u|_T \in \mathbb{P}_1 \) for each \( T \in T_h \). Then, \( \tilde{I}_h \) satisfies
\[ |\tilde{I}_h u - u| \leq C h \| u \|, \quad |\tilde{I}_h u - u| \leq C h \| u \|_{H^2}. \]  \hspace{1cm} \text{(24)}
We define \( I_h u = \tilde{I}_h u + R_h u \), where \( R_h u = \sum_T \mathcal{B}_T \alpha_T(u) \) with \( \mathcal{B}_T \) a bubble function and \( \alpha_T \in \mathbb{R}^3 \) such as
\[ \int_T (u - I_h u) = 0, \quad \forall T \in T_h, \]  \hspace{1cm} \text{(25)}
that is
\[ \alpha_T(u) = \frac{\int_T (u - \tilde{I}_h u)}{\int_T \mathcal{B}_T} \quad \forall T \in T_h. \]
Then, (22) can be deduced from (25). Moreover, by using again (25), it is known by means of a duality argument ([9]) that
\[ | u - I_h u |_{H^{-1}} \leq C h | u - I_h u |. \]

Now, in order to obtain estimate (24) but changing \( \tilde{I}_h \) by \( I_h \) it suffices to prove
\[ | R_h(u) | \leq C h \| u \| \quad \text{and} \quad | \nabla R_h u | \leq C h \| u \|_{H^2}. \]
Indeed, by using orthogonality of the bubble functions,
\[ \| R_h(u) \|_{L^2}^2 = \sum_T \| \alpha_T(u) \|^2 \| \mathcal{B}_T \|^2_{L^2} = \sum_T \left( \int_T u - \tilde{I}_h u \right) \left( \int_T |u - \tilde{I}_h u|^2 \right) \]
\[ \leq \left( \sum_T |T| \left( \int_T |u - \tilde{I}_h u|^2 \right) \right) \frac{|T|^2}{|T|^2} \leq C \| u - \tilde{I}_h u \|_{L^2}^2. \]
hence $|R_h(u)| \leq C h \|u\|$, owing to the approximation property $|u - \tilde{I}_h u| \leq C h \|u\|$. 

Taking the $L^2$-norm of the gradient,

$$\|\nabla R_h(u)\|_{L^2}^2 = \sum_T \left( \int_T u - \tilde{I}_h u \right)^2 \frac{1}{|T|} \leq C \sum_T \int_T |u - \tilde{I}_h u|^2 \leq \frac{C}{h^2} \|u - \tilde{I}_h u\|_{L^2}^2$$

hence $\|\nabla R_h(u)\|_{L^2}^2 \leq C h^2 \|u\|_{H^2}^2$, owing to the approximation property $\|u - \tilde{I}_h u\|_{L^2} \leq C h^2 \|u\|_{H^2}$.

Now, following the equality $u^{m+1} = \tilde{u}^{m+1} - k \nabla (p^{m+1} - p^m)$, we define:

$$K_{h,k}u^{m+1} := I_h \tilde{u}^{m+1} - k \nabla J_h (p^{m+1} - p^m). \quad (26)$$

Note that $K_{h,k}u^{m+1} \in Y_h + \nabla Q_h$. By comparing (26) with the time discrete Algorithm 1:

$$u^{m+1} - K_{h,k}u^{m+1} = \tilde{u}^{m+1} - I_h \tilde{u}^{m+1} - k \nabla \left( (p^{m+1} - J_h p^{m+1}) - (p^m - J_h p^m) \right),$$

hence, using the $L^2$ approximation property for $I_h$, the $H^1$-stability for $J_h$ and the $H^2 \times H^1$ estimates for $(\tilde{u}^{m+1}, p^{m+1})$:

$$|u^{m+1} - K_{h,k}u^{m+1}| \leq C \left( h^2 \|\tilde{u}^{m+1}\|_{H^2} + k \|p^{m+1} - p^m\| \right) \leq C(k + h^2) \quad \forall m.$$  

The fully discrete scheme is described in Algorithm 2.

**Algorithm 2** Fully discrete algorithm

**Initialization:** Let $(\tilde{u}^0_h, p^0_h) \in Y_h \times Q_h$ be an approximation of $(u^0, p^0)$. Put $u^0_h = \tilde{u}^0_h$.

**Step of time $m+1$:** Let $(\tilde{u}_h^m, p_h^m) \in Y_h \times Q_h$ and $u_h^{m+1} \in Y_h + \nabla Q_h$ be given.

**Sub-step 1:** Find $\tilde{u}_h^{m+1} \in Y_h$ such that,

$$(S_1)_{a,h}^{m+1} \left( \frac{\tilde{u}_h^{m+1} - u_h^m}{k}, v_h \right) + c \left( \tilde{u}_h^{m+1}, \tilde{u}_h^{m+1}, v_h \right) + \left( \nabla \tilde{u}_h^{m+1}, \nabla v_h \right) + \left( \nabla p_h^m, v_h \right) = \left( f^{m+1}, v_h \right).$$

**Sub-step 2:** Find $p_h^{m+1} \in Q_h$ such that

$$(S_2)_{a,h}^{m+1} \left( k \nabla (p_h^{m+1} - p_h^m), \nabla q_h \right) = \left( \tilde{u}_h^{m+1}, \nabla q_h \right) \quad \forall q_h \in Q_h.$$ 

Now, we define $u_h^{m+1} \in Y_h + \nabla Q_h$ by

$$(S_2)_{b,h}^{m+1} u_h^{m+1} = \tilde{u}_h^{m+1} - k \nabla (p_h^{m+1} - p_h^m).$$

Notice that, adding both sub-steps of Algorithm 2, we obtain:

$$(S_3)_{h}^{m+1} \left( \frac{u_h^{m+1} - u_h^m}{k}, v_h \right) + c \left( u_h^{m+1}, u_h^{m+1}, v_h \right) + \left( \nabla u_h^{m+1}, \nabla v_h \right) + \left( \nabla p_h^{m+1}, v_h \right) = \left( f^{m+1}, v_h \right).$$
From \((S_2)^{m+1}_{b,h}\), one has the orthogonality property
\[
\left( \mathbf{u}^{m+1}_h, \nabla q_h \right) = 0 \quad \forall q_h \in Q_h. \tag{27}
\]

**Remark 15 (Segregated version of Algorithm 2)** We introduce the end-of-step velocity \(\mathbf{u}^m_h\) only for doing the numerical analysis. For practical implementations, this velocity \(\mathbf{u}^m_h\) can be eliminated, rewriting Algorithm 2 as follows:

Let \((p_h^{m-1}, p_h^m, \mathbf{u}^m_h) \in Q_h \times Q_h \times \mathbf{Y}_h\) be given.

(a) Find \(\mathbf{u}^{m+1}_h \in \mathbf{Y}_h\) such that, \(\forall \mathbf{v}_h \in \mathbf{Y}_h:\)
\[
\left( \frac{\mathbf{u}^{m+1}_h - \mathbf{u}^m_h}{k}, \mathbf{v}_h \right) + c \left( \mathbf{u}^m_h, \mathbf{u}^{m+1}_h, \mathbf{v}_h \right) + \left( \nabla \mathbf{u}^{m+1}_h, \nabla \mathbf{v}_h \right) + \left( \nabla (2p_h^m - p_h^{m-1}), \mathbf{v}_h \right) = \left( \mathbf{f}^{m+1}, \mathbf{v}_h \right).
\]

(b) Find \(p_h^{m+1} \in Q_h\) such that, \(\forall q_h \in Q_h:\)
\[
\left( k \nabla (p_h^{m+1} - p_h^m), \nabla q_h \right) = \left( \mathbf{u}^{m+1}_h, \nabla q_h \right).
\]

Then, computations for pressure \(p_h^{m+1}\) and velocity \(\mathbf{u}^{m+1}_h\) are decoupled. In fact, (a) is a linear convection-diffusion-Dirichlet problem for \(\mathbf{u}^{m+1}_h\) (where each component of \(\mathbf{u}^{m+1}_h\) is also decoupled from the other ones) and (b) is a Poisson-Neumann problem for \(p_h^{m+1}\). Therefore, Algorithm 2 can be rewritten as a fully decoupled scheme.

Note that, in order to initialize the scheme we have to start with a pressure \(p_h^{-1}\) which has no sense. We can avoid it starting from an auxiliary initial step given by either one-step scheme or by the scheme written as Algorithm 2, i.e., given \(\mathbf{u}^0_h, p^0_h\) and \(\mathbf{u}^0_h = \mathbf{u}^0_h\), we compute first \(\mathbf{u}^1_h\) from \((S_1)^1_{h}\) and after \(p^1_h\) from \((S_2)^1_{a,h}\).

### 2.2 Stability and convergence of Algorithm 2

It is easy to extend the results given in the previous Section about the continuous dependence of the projection step of Algorithm 1 to the fully discrete Algorithm 2. Indeed, from \((S_2)^{m+1}_{b,h}\) and the orthogonality property (27), we have
\[
|\mathbf{u}^{m+1}_h|^2 = |\mathbf{u}^{m+1}_h|^2 + |k \nabla (p_h^{m+1} - p_h^m)|^2 \tag{28}
\]
hence, in particular, \(|\mathbf{u}^{m+1}_h| \leq |\mathbf{u}^{m+1}_h|\). From \((S_2)^{m+1}_{a,h}\)
\[
|k \nabla (p_h^{m+1} - p_h^m)|^2 = \left( \mathbf{u}^{m+1}_h, k \nabla (p_h^{m+1} - p_h^m) \right) = \left( \mathbf{u}^{m+1}_h - \mathbf{u}_h^m, k \nabla (p_h^{m+1} - p_h^m) \right),
\]
hence
\[
|\mathbf{u}^{m+1}_h - \mathbf{u}^{m+1}_h| \leq |\mathbf{u}^{m+1}_h - \mathbf{u}^m_h|.
\]
Moreover, using the antisymmetric property $c(\tilde{u}_h^m, \tilde{u}_{h+1}^m, \tilde{u}_{h+1}^m) = 0$ (see (1)), one can extend the stability and convergence results of Algorithm 1 to the fully discrete Algorithm 2. In particular, for any $r < N$, the following stability estimates hold:

$$\|u_{h+1}^{r+1}\|_{l^\infty(L^2)} + \|\tilde{u}_{h+1}^{r+1}\|_{l^\infty(L^2)\cap L^2(H^1)} + \|k\nabla p_{h+1}^{r+1}\|_{l^\infty(L^2)} \leq C,$$

$$\sum_{m=0}^{r} |\tilde{u}_{h+1}^{m+1} - u_{h+1}^{m}|^2 + \sum_{m=0}^{r} |u_{h+1}^{m+1} - \tilde{u}_{h+1}^{m+1}|^2 \leq C. \quad (29)$$

Indeed, by making $\left((S_1)_{h}^{m+1}, 2k \tilde{u}_{h+1}^{m+1}\right)$, using the fact that

$$2k(\nabla p_{h+1}^m, \tilde{u}_{h+1}^{m+1}) = 2(k \nabla p_{h+1}^m, k \nabla (p_{h+1}^{m+1} - p_{h+1}^m)),$$

and the equalities $(a - b)2a = a^2 - b^2 + (a - b)^2$ and $(a - b)2b = a^2 - b^2 - (a - b)^2$, we have

$$|\tilde{u}_{h+1}^{m+1}|^2 - |u_{h+1}^{m}|^2 + |\tilde{u}_{h+1}^{m} - u_{h+1}^{m}|^2 + k \|\tilde{u}_{h+1}^{m+1}\|^2 + |k \nabla p_{h+1}^{m+1}|^2 - |k \nabla p_{h+1}^m|^2$$

$$- |k \nabla (p_{h+1}^{m+1} - p_{h+1}^m)|^2 \leq k \|f_{m+1}\|^2_{H^{-1}} \quad (30)$$

Adding (28) and (30), the negative term $-|k \nabla (p_{h+1}^{m+1} - p_{h+1}^m)|^2$ of (30) cancel and we arrive at

$$|u_{h+1}^{m+1}|^2 - |u_{h+1}^{m}|^2 + |\tilde{u}_{h+1}^{m} - u_{h+1}^{m}|^2 + |k \nabla p_{h+1}^{m+1}|^2 - |k \nabla p_{h+1}^m|^2 + k \|\tilde{u}_{h+1}^{m+1}\|^2 \leq k \|f_{m+1}\|^2_{H^{-1}}$$

Now, adding from $m = 0$ to $r$ ($r < N$), we obtain the desired stability estimates (29).

### 2.3 Problems related to the spatial errors

We will present an error analysis for the fully discrete Algorithm 2 $(\tilde{u}_{h+1}^{m+1}, u_{h+1}^{m+1}, p_{h+1}^{m+1})$ as an approximation of the time discrete Algorithm 1 $(\tilde{u}^{m+1}, u^{m+1}, p^{m+1})$. Consequently, we define the following errors:

$$e_d^{m+1} = u^{m+1} - u_{h+1}^{m+1}, \quad \tilde{e}_d^{m+1} = \tilde{u}^{m+1} - \tilde{u}_{h+1}^{m+1}, \quad e_{p,d}^{m+1} = p^{m+1} - p_{h+1}^{m+1}.$$  

Splitting the discrete part and the interpolation one:

$$e_d^{m+1} = e_h^{m+1} + e_i^{m+1}, \quad \tilde{e}_d^{m+1} = \tilde{e}_h^{m+1} + \tilde{e}_i^{m+1}, \quad e_{p,d}^{m+1} = e_{p,h}^{m+1} + e_{p,i}^{m+1}$$

where $e_i$ are interpolation errors and $e_h$ space discrete errors, concretely

$$e_h^{m+1} = K_{h,k} u^{m+1} - u_{h+1}^{m+1} \quad \text{and} \quad e_i^{m+1} = u^{m+1} - K_{h,k} u^{m+1},$$

$$\tilde{e}_h^{m+1} = I_{h} u^{m+1} - \tilde{u}_{h+1}^{m+1} \quad \text{and} \quad \tilde{e}_i^{m+1} = \tilde{u}^{m+1} - I_{h} \tilde{u}^{m+1},$$

$$e_{p,h}^{m+1} = J_{h} p^{m+1} - p_{h+1}^{m+1} \quad \text{and} \quad e_{p,i}^{m+1} = p^{m+1} - J_{h} p^{m+1}.$$
Remark 16 From the equalities $u^{m+1} = \tilde{u}^{m+1} - k \nabla (p^{m+1} - p^m)$ and $K_h^k u^{m+1} = I_h \tilde{u}^{m+1} - k \nabla J_h (p^{m+1} - p^m)$, one has

$$e_i^{m+1} = \tilde{e}^{m+1}_i - k \nabla (e^{m+1}_{p,i} - e^{m}_{p,i}).$$

(31)

In particular, subtracting $\tilde{e}_i^{m+1}$ and (31) replacing $m$ for $m - 1$, we get

$$\frac{1}{k} (e_i^{m+1} - e_i^m) = e_i (\delta t \tilde{u}^{m+1}) + \nabla (e_{p,i}^m - e_{p,i}^{m-1}),$$

(32)

where $e_i (\delta t \tilde{u}^{m+1}) = (\tilde{e}_i^{m+1} - \tilde{e}_i^m)/k$. Moreover, owing to the choice of the interpolation operators $I_h$ and $J_h$, from (31)

$$(e_i^{m+1}, \nabla q_h) = (\tilde{e}_i^{m+1}, \nabla q_h) - k \left( \nabla (e_{p,i}^{m+1} - e_{p,i}^m), \nabla q_h \right) = 0, \quad \forall q_h \in Q_h. \tag{33}$$

On the other hand, since $(u_h^{m+1}, \nabla q_h) = 0$ $\forall q_h \in Q_h$ and $(u^{m+1}, \nabla q) = 0$ $\forall q \in H^1 \cap L^2_0$, then

$$(e_d^{m+1}, \nabla q_h) = 0 \quad \forall q_h \in Q_h. \tag{34}$$

Finally, from (33) and (34), we arrive at

$$(e_h^{m+1}, \nabla q_h) = 0 \quad \forall q_h \in Q_h.$$ 

By comparing $(S_1)^{m+1}, (S_2)^{m+1}$ and $(S_1)_h^{m+1}, (S_2)_h^{m+1}$, we have the following problems satisfied by the spatial errors $\tilde{e}_d^{m+1}$ and $(e_d^{m+1}, e_p^{m+1})$ respectively:

$$\frac{1}{k} (\tilde{e}_d^{m+1} - e_d^m, \nabla v_h) + (\nabla \tilde{e}_d^{m+1}, \nabla v_h) + (\nabla e_p^{m+1}, v_h) = NL_h^{m+1} (v_h), \quad \forall v_h \in Y_h,$$

and

$$e_i^{m+1} = \tilde{e}_i^{m+1} - k \nabla (e_{p,i}^{m+1} - e^{m}_{p,i}),$$

where

$$NL_h^{m+1} (v_h) = c (\tilde{u}_h^{m+1} - \tilde{u}_h^m, \nabla v_h) - c (\tilde{u}_h^{m+1} - \tilde{u}_h^m, \nabla v_h) = c (\tilde{u}_h^m, \nabla v_h) = c (\tilde{u}_h^m, \nabla v_h) = c (\tilde{u}_h^m, \nabla v_h).$$

By splitting the error in the discrete and the interpolation parts and using (31) and (32),

$$(E_1)_h^{m+1} \begin{cases} \frac{1}{k} (\tilde{e}_h^{m+1} - e_h^m, \nabla v_h) + (\nabla \tilde{e}_h^{m+1}, \nabla v_h) + (\nabla e_{p,h}^{m+1}, v_h) = NL_h^{m+1} (v_h) \\ - e_i (\delta t \tilde{u}^{m+1}), v_h) - (\nabla \tilde{e}_i^{m+1}, \nabla v_h) - (\nabla (e_{p,i}^{m+1} - e_{p,i}^{m-1}), v_h), \quad \forall v_h \in Y_h \end{cases}$$

$$(E_2)_h^{m+1} \begin{cases} e_h^{m+1} = \tilde{e}_h^{m+1} - k \nabla (e_{p,h}^{m+1} - e_{p,h}^m). \end{cases}$$

Finally, adding $(E_1)_h^{m+1}$ and $(E_2)_h^{m+1}$,

$$(E_3)_h^{m+1} \begin{cases} \frac{1}{k} (e_h^{m+1} - e_h^m, \nabla v_h) + (\nabla e_h^{m+1}, \nabla v_h) + (\nabla e_{p,h}^{m+1}, v_h) = NL_h^{m+1} (v_h) \\ - e_i (\delta t \tilde{u}^{m+1}), v_h) - (\nabla e_i^{m+1}, \nabla v_h) - (\nabla (e_{p,i}^{m+1} - e_{p,i}^{m-1}), v_h), \quad \forall v_h \in Y_h. \end{cases}$$
2.4 \(O(h)\) error estimates for \(\tilde{e}_h^{m+1}\) in \(l^\infty(L^2) \cap l^2(H^1)\) and for \(e_h^{m+1}\) in \(l^\infty(L^2)\)

Theorem 17 We assume hypotheses of Theorem 7 and the initial approximation
\[
|e^0_h| + |k \nabla e^0_{p,h}| \leq C h.
\]

Then, the following error estimates hold
\[
\|\tilde{e}_h^{m+1}\|_{l^\infty(L^2) \cap l^2(H^1)}^2 + \|e_h^{m+1}\|_{l^\infty(L^2)}^2 + \|k \nabla e_{p,h}^{m+1}\|_{l^\infty(L^2)}^2 \leq C h^2,
\]
(35)
\[
\|\tilde{e}_h^{m+1} - e_h^{m}\|_{l^2(L^2)}^2 \leq C k h^2.
\]
(36)

Remark 18 By using the \(O(k)\) accuracy for the time discrete Algorithm 1, we arrive at the following optimal order for the total error of the velocity:
\[
\|u(t_{m+1}) - \tilde{u}_h^{m+1}\|_{l^\infty(L^2) \cap l^2(H^1)} \leq C (k + h).
\]

Proof: By making \(\langle E_1 \rangle_h^{m+1}, 2k \tilde{e}_h^{m+1}\) and using the equalities
\[
\left(\nabla e_{p,h}^m, e_h^{m+1}\right) = 0,
\]
\[
2k \left(\nabla e_{p,h}^m, \tilde{e}_h^{m+1}\right) = 2k \left(\nabla e_{p,h}^m, \nabla (e_{m+1}^m - e_p^m)\right) = |k \nabla e_{p,h}^m|^2 - |k \nabla (e_{m+1}^m - e_p^m)|^2,
\]
and the \(L^2\)-orthogonality property
\[
|\tilde{e}_h^{m+1}|^2 = |e_h^{m+1}|^2 + |k \nabla (e_{p,h}^{m+1} - e_{p,h}^m)|^2,
\]
(37)
we arrive at
\[
|e_h^{m+1}|^2 - |e_h^m|^2 + |e_h^{m+1} - e_h^m|^2 + 2k \|e_h^{m+1}\|^2 + |k \nabla e_{p,h}^{m+1}|^2 - |k \nabla e_{p,h}^m|^2
\]
\[
= -2k \left(e_p^m \partial t_1 \tilde{u}_h^{m+1}, e_h^{m+1}\right) - 2k \left(\nabla e_{h}^{m+1}, \nabla \tilde{e}_h^{m+1}\right) - 2k \left(\nabla (2e_{p,i}^m - e_{p,i}^{m-1}), e_h^{m+1}\right)
\]
\[
+2k c(e_h^m, \tilde{u}_h^{m+1}, \tilde{e}_h^{m+1}) + 2k c(e_h^m, \tilde{u}_h^{m+1}, e_h^{m+1}) - 2k c(\tilde{u}_h^m, e_h^{m+1}, \tilde{e}_h^{m+1})
\]
(38)
\[
-2k c(\tilde{u}_h^m, \tilde{e}_h^{m+1}, \tilde{e}_h^{m+1}) := \sum_{i=1}^7 I_i
\]

We bound the RHS of (38) as follows (using Remark 12):
\[
I_1 \leq \varepsilon k \|e_h^{m+1}\|^2 + C k e_p^m \|e_h^{m+1}\|^2 \leq \varepsilon k \|e_h^{m+1}\|^2 + C k^2 \|e_{p,h}^{m+1}\|^2
\]
\[
I_2 \leq \varepsilon k \|e_h^{m+1}\|^2 + C h^2 k \|\tilde{u}_h^{m+1}\|_{L^2}^2 \leq \varepsilon k \|e_h^{m+1}\|^2 + C k^2 h^2
\]
\[
I_3 = 2k \left(2e_{p,i}^m - e_{p,i}^{m-1}, \nabla \tilde{e}_h^{m+1}\right) \leq \varepsilon k \|e_h^{m+1}\|^2 + C k h^2 \|p_{m+1} - p_{m-1}\|_{L^2}^2 \leq \varepsilon k \|e_h^{m+1}\|^2 + C k h^2
\]

With respect to the nonlinear terms,
\[
I_4 = 2k c(e_h^m, \tilde{u}_h^{m+1}, \tilde{e}_h^{m+1}) \leq C k \|e_h^m\| \|\tilde{u}_h^{m+1}\|_{W^{1,3} \cap L^\infty} \|\tilde{e}_h^{m+1}\| \leq \varepsilon k \|e_h^{m+1}\|^2 + C k \|e_h^m\|^2
\]

22
\[ \leq \varepsilon k \| \tilde{e}^{m+1} \|^2 + C k \left( \| e_h^{m} \|^2 + 2 |k \nabla e_{p,h}^{m}|^2 + 2 |k \nabla e_{p,h}^{m-1}|^2 \right) \]

(here, (37) has been used),

\[ I_5 = 2 k c \left( \tilde{e}_i^{m}, \tilde{u}_h^{m+1}, \tilde{e}_h^{m+1} \right) \leq \varepsilon k \| \tilde{e}^{m+1} \|^2 + C k \| e_i^{m} \|^2 \leq \varepsilon k \| \tilde{e}^{m+1} \|^2 + C h^4 k \| \tilde{u}^m \|^2_{H^2} \]

\[ \leq \varepsilon k \| \tilde{e}^{m+1} \|^2 + C k h^4, \]

\[ I_6 = 2 k c \left( \tilde{u}_h^{m}, \tilde{e}_h^{m+1}, \tilde{e}_h^{m+1} \right) = 0, \]

\[ I_7 = 2 k c \left( \tilde{u}_h^{m}, \tilde{e}_h^{m+1}, \tilde{e}_h^{m+1} \right) \leq C k \| \tilde{u}_h^{m} \|^2 \| e_i^{m+1} \|^2_{L^2} + \varepsilon k \| \tilde{e}_h^{m+1} \|^2 \]

\[ \leq C k \| \tilde{u}_h^{m} \|^2 \| e_i^{m+1} \| \| e_i^{m+1} \| + \varepsilon k \| \tilde{e}_h^{m+1} \|^2 \]

\[ \leq C k h^3 \| \tilde{u}_h^{m} \|^2 \| \tilde{u}_h^{m+1} \|^2_{H^2} + \varepsilon k \| \tilde{e}_h^{m+1} \|^2 \]

\[ \leq C k h^3 \| \tilde{u}_h^{m} \|^2 + \varepsilon k \| \tilde{e}_h^{m+1} \|^2 \]

Then, using these bounds in (38) we obtain

\[ |e_h^{m+1}|^2 - |e_h^{m}|^2 + |k \nabla e_{p,h}^{m+1}|^2 - |k \nabla e_{p,h}^{m}|^2 + |\bar{e}_h^{m+1} - e_h^{m}|^2 + k \| e_{h}^{m+1} \|^2 \leq C k \left( |e_h^{0}|^2 + |k \nabla e_{p,h}^{0}|^2 + h^2 \right) \]

(39)

Finally, by adding (39) from \( m = 0 \) to \( r \) (with any \( r < M \)), and using that \( k \sum \| \tilde{u}_h^m \|^2 \leq C \) and Theorem 9, the discrete Gronwall’s Lemma yields to

\[ |e_h^{r+1}|^2 + |k \nabla e_{p,h}^{r+1}|^2 + \sum_{m=0}^{r} |\bar{e}_h^{m+1} - e_h^{m}|^2 + k \sum_{m=0}^{r} \| e_{h}^{m+1} \|^2 \leq C \left( |e_h^{0}|^2 + |k \nabla e_{p,h}^{0}|^2 + h^2 \right) \]

hence the estimates (35)-(36) hold.

Theorem 17 and the inverse inequality \( \| u_h \| \leq C h^{-1} |u_h| \) for each \( u_h \in Y_h \), imply the uniform estimate

\[ \| \tilde{e}_h^{m+1} \| \leq \frac{C}{h} \| \tilde{e}_h^{m+1} \| \leq C. \] (40)

2.5 \( O(h) \) for \( \delta_t e_{h}^{m+1} \) in \( L^\infty(L^2) \), \( \delta_t \bar{e}_{h}^{m+1} \) in \( L^\infty(L^2) \cap L^2(\mathbf{H}^1) \) and \( (\bar{e}_{h}^{m+1}, e_{p,d}^{m+1}) \) in \( L^\infty(\mathbf{H}^1 \times L^2) \)

By making \( \delta_t(E_1)_h^{m+1} \) and \( \delta_t(E_2)_h^{m+1} \), one arrives at (\( \forall m \geq 1 \):

\[ \frac{1}{k} \left( \delta_t \bar{e}_d^{m+1} - \delta_t \bar{e}_d^{m}, v_h \right) + \left( \nabla \delta_t \bar{e}_d^{m+1}, \nabla v_h \right) - \left( \delta_t \nabla e_{p,d}^{m}, v_h \right) = \delta_t \mathbf{N} l_h^{m+1}(v_h) \quad \forall v_h \in Y_h \]

and

\[ \delta_t \bar{e}_d^{m+1} = \delta_t \bar{e}_d^{m+1} - k \nabla (\delta_t e_{p,d}^{m+1} - \delta_t e_{p,d}) \]
where
\[ \delta_t N L_h^{m+1}(v_h) = c(\delta_t \tilde{e}_d^m, \tilde{u}^{m+1}, v_h) + c(\delta_t \tilde{u}_h^m, \tilde{e}_d^{m+1}, v_h) + c(\tilde{e}_d^{m+1}, \delta_t \tilde{u}_h^{m+1}, v_h) + c(\tilde{u}_h^{m-1}, \delta_t \tilde{e}_d^{m+1}, v_h). \]

On the other hand, the following \( L^2 \)-orthogonality property holds:
\[ (\delta_t e_d^{m+1}, \nabla q_h) = 0, \quad \forall q_h \in Q_h. \]
Consequently, for each \( v_h \in Y_h \), one has
\[ (D_1)_h^{m+1} = \left\{ \frac{1}{k} (\delta_t \tilde{e}_d^{m+1} - \delta_t e_d^m, v_h) + (\nabla \delta_t e_d^{m+1}, \nabla v_h) + (\nabla \delta_t e_p^{m+1} - e_l(\delta_t \tilde{u}^{m+1}), v_h) - (\nabla \delta_t e_t^{m+1}, \nabla v_h) - (\nabla (2 \delta_t e_p^{m+1} - \delta_t e_{p,h}^{m+1}), v_h) \right\}, \]
\[ (D_2)_h^{m+1} = \delta_t e_d^{m+1} - k \nabla (\delta_t e_d^{m+1} - \delta_t e_{p,h}^{m+1}), \]
and the following discrete \( L^2 \)-orthogonality property:
\[ (\delta_t e_d^{m+1}, \nabla q_h) = 0, \quad \forall q_h \in Q_h. \tag{41} \]

In the last two equalities, some properties of the interpolation operators have been used.

**Theorem 19** Under the hypotheses of Theorems 9 and 17, assuming the following approximation for the first step of Algorithm 2
\[ |\delta_t e_t^1| + |k \nabla \delta_t e_{p,h}^1| \leq C h, \quad k \|\delta_t e_t^1\|^2 \leq C h^2 \tag{42} \]
then
\[ \|\delta_t e_d^{m+1}\|_{\infty(L^2)} + \|\delta_t e_m^{m+1}\|_{\infty(L^2) \cap (H^1)} + \|k \delta_t \nabla e_{p,h}^{m+1}\|_{\infty(L^2)} \leq C h. \tag{43} \]

**Proof:** Since the initial estimate \( |\delta_t e_t^1| + |k \nabla \delta_t e_{p,h}^1| \leq C h \) is assumed, it suffices to prove (43) for each \( m \geq 1 \).

By adding \((D_1)_h^{m+1}\) multiplied by \(2k \delta_t e_d^{m+1} \in Y_h\), where the pressure term is writing as
\[ 2k \left( \nabla \delta_t e_{p,h}^m, \delta_t e_d^{m+1} \right) = 2k \left( \nabla \delta_t e_{p,h}^m, k \nabla (\delta_t e_p^{m+1} - \delta_t e_{p,h}^m) \right) \]
\[ = |k \nabla \delta_t e_{p,h}^m|^2 - (k \nabla \delta_t e_{p,h}^m)^2 - |k \nabla (\delta_t e_p^{m+1} - \delta_t e_{p,h}^m)|^2 \]
\[ \left( \text{here } \left( \nabla \delta_t e_{p,h}^m, \delta_t e_d^{m+1} \right) = 0 \text{ has been used} \right), \] and the equality
\[ |\delta_t e_d^{m+1}|^2 = |\delta_t e_{p,h}^{m+1}|^2 + |k \nabla (\delta_t e_p^{m+1} - \delta_t e_{p,h}^m)|^2 \]
(which is deduced from \((D_2)_h^{m+1}\) and the discrete \( L^2 \)-orthogonality (41)), one has
\[ |\delta_t e_d^{m+1}|^2 - |\delta_t e_d^m|^2 + |\delta_t e_d^{m+1} - \delta_t e_d^m|^2 + 2k \|\delta_t e_d^{m+1}\|^2 + |k \nabla \delta_t e_{p,h}^{m+1}|^2 \]
\[ - |k \nabla \delta_t e_{p,h}^m|^2 = -2k \left( e_l(\delta_t \tilde{u}^{m+1}), \delta_t e_d^{m+1} \right) - 2k \left( \nabla \delta_t e_d^{m+1}, \nabla \delta_t e_d^{m+1} \right) \]
\[ -2k \left( \nabla (2 \delta_t e_p^{m+1} - \delta_t e_{p,h}^{m+1}, \delta_t e_d^{m+1}) + 2 \delta_t N L_h^{m+1}(\delta_t e_d^{m+1}) = I_1 + I_2 + I_3 + I_4. \tag{44} \]
We bound the RHS of (44) as:

\[ I_1 \leq \varepsilon k \| \delta_t \hat{e}^{m+1}_h \|^2 + C k h^2 |\delta_t \hat{u}^{m+1}|^2 \]

(here the hypothesis (23) on the \(O(h)\)-approximation of \(I_h\) in the \(H^1\)-norm has been used),

\[ I_2 \leq \varepsilon k \| \delta_t \hat{e}^{m+1}_h \|^2 + C k h^2 \| \delta_t \hat{u}^{m+1} \|^2_{H^2} \]

\[ I_3 \leq \varepsilon k \| \delta_t \hat{e}^{m+1}_h \|^2 + C k h^2 (\| \delta_t p^m \|^2 + \| \delta_t p^{m-1} \|^2) . \]

The nonlinear terms, for \(m \geq 1\), are treated as follows:

\[ I_4 = 2 k c \left( \delta_t \hat{e}^m_d, \tilde{u}^{m+1} \right) + 2 k c \left( \delta_t \hat{u}_h, \hat{e}^{m+1}_d, \delta_t \hat{e}^{m+1}_h \right) + 2 k c \left( \hat{e}^{m-1}_d, \delta_t \tilde{u}^{m+1}_h, \delta_t \tilde{e}^{m+1}_h \right) + 2 k c \left( \tilde{u}^{m-1}_h, \delta_t \tilde{e}^{m+1}_h, \delta_t \tilde{e}^{m+1}_h \right) := \sum_{i=1}^4 J_i \]

We bound each \(J_i\)-term as follows:

\[ J_1 = 2 k c \left( \delta_t \hat{e}^m_h, \tilde{u}^{m+1} \right) + 2 k c \left( \delta_t \hat{e}^m_h, \tilde{u}^{m+1} \right) + 2 k c \left( \delta_t \hat{e}^m_i, \tilde{u}^{m+1}, \delta_t \hat{e}^{m+1}_i \right) := J_{11} + J_{12} \]

\[ J_{11} \leq \varepsilon k \| \delta_t \hat{e}^{m+1}_h \|^2 + C k \| \tilde{u}^{m+1} \|^2_{H^1} |\delta_t \hat{e}^{m+1}_h|^2 \]

\[ \leq \varepsilon k \| \delta_t \hat{e}^{m+1}_h \|^2 + C k \left( |\delta_t \hat{e}^{m+1}_h|^2 + 2(|k \nabla \delta_t e^{m+1}_p|^2 + |k \nabla \delta_t e^{m-1}_p|^2 + 2 |k \nabla \delta_t e^{m+1}_h|^2) \right) \]

\[ J_{12} \leq \varepsilon k \| \delta_t \hat{e}^{m+1}_h \|^2 + C k \| \tilde{u}^{m+1} \|^2 |\delta_t \hat{e}^{m+1}_h|^2 \leq \varepsilon k \| \delta_t \hat{e}^{m+1}_h \|^2 + C k h^2 |\delta_t \hat{u}^m|^2 \]

(\text{in the last inequality we use } \| \hat{e}^{m+1} \| \leq C \text{ due to (40) and } \| \tilde{u}^m \|_{H^2} \leq C),

\[ J_2 = 2 k c \left( \delta_t \hat{e}^m_h, \tilde{u}^{m+1} \right) + 2 k c \left( \delta_t \hat{e}^m_h, \tilde{u}^{m+1} \right) + 2 k c \left( \delta_t \hat{u}_h, \hat{e}^{m+1}_d, \delta_t \hat{e}^{m+1}_h \right) \]

\[ \leq \varepsilon k \left( \| \delta_t \hat{e}^{m+1}_h \|^2 + \| \delta_t \hat{e}^m_h \|^2 \right) + C k \| \tilde{u}^{m+1} \|^2 \| \delta_t \hat{e}^{m+1}_h \|^2 + C k \| \delta_t \hat{u}_h \| \| \hat{e}^{m+1}_d \| \left( \| \hat{e}^m_d \|^2 + \| \hat{e}^m_h \|^2 \right) \]

\[ \leq \varepsilon k \left( \| \delta_t \hat{e}^{m+1}_h \|^2 + \| \delta_t \hat{e}^m_h \|^2 \right) + C k \| \delta_t \hat{e}^{m+1}_h \|^2 \| \hat{e}^m_d \| \| \hat{e}^m_h \| \| \tilde{u}^{m+1} \|_{H^2} + \| \hat{e}^m_h \|^2 \]

\[ \leq \varepsilon k \left( \| \delta_t \hat{e}^{m+1}_h \|^2 + \| \delta_t \hat{e}^m_h \|^2 \right) + C k \left( |\delta_t \hat{e}^m_h|^2 + 2(|k \nabla \delta_t e^{m+1}_p|^2 + |k \nabla \delta_t e^{m-1}_p|^2) \right) + C k h^2 + C k \| \hat{e}^m_h \|^2 \]

\[ J_3 = 2 k c \left( \hat{e}^{m-1}_h, \delta_t \tilde{u}^{m+1}_h \right) + 2 k c \left( \hat{e}^{m-1}_h, \delta_t \tilde{u}^{m+1}_h \right) := J_{31} + J_{32} \]

\[ J_{31} \leq C k \| \hat{e}^{m-1}_h \| \| \delta_t \tilde{u}^{m+1}_h \|_{L^2} \| \delta_t \hat{e}^{m+1}_h \| \leq \varepsilon k \| \delta_t \hat{e}^{m+1/2}_h \|^2 + C k \| \hat{e}^{m-1}_h \|^2 \]

\[ J_{32} \leq \varepsilon k \| \delta_t \hat{e}^{m+1}_h \|^2 + C k \| \hat{e}^{m-1}_h \|^2 \leq \varepsilon k \| \delta_t \hat{e}^{m+1/2}_h \|^2 + C k \| \delta_t \tilde{u}^{m+1}_h \|_{H^2} \| \hat{e}^{m-1}_h \|^2 \]

\[ J_4 = 2 k c \left( \tilde{u}^{m-1}_h, \delta_t \hat{e}^{m+1}_h \right) + 2 k c \left( \tilde{u}^{m-1}_h, \delta_t \hat{e}^{m+1}_h \right) := J_{41} + J_{42} \]

\[ J_{41} = 0, \]
\[ J_{12} \leq C k \| \tilde{u}_h^{m-1} \|_3 \| \delta_t \tilde{e}_t^{m+1} \|_{L^1} \| \delta_t \tilde{e}_h^{m+1} \| \leq \varepsilon k \| \delta_t \tilde{e}_h^{m+1} \|_2 + C k \| \tilde{u}_h^{m-1} \|_{H^2}^2 \] 

(here, we use \( \| \delta_t \tilde{e}_i^{m+1} \|_{L^3} \leq C \| \delta_t \tilde{e}_i^{m+1} \|_{1/2} \| \tilde{e}_i^{m+1} \|_{1/2} \leq C h^{3/2} \| \tilde{u}_i^{m+1} \|_{H^2} \)).

By applying these estimates in (44) for a small enough \( \varepsilon \), we obtain
\[
|\delta_t e_h^{m+1}|^2 - |\delta_t e_h^m|^2 + |\delta_t \tilde{e}_h^{m+1} - \delta_t e_h^m|^2 + k |\delta_t \tilde{e}_h^{m+1}|^2 + k \nabla \delta_t e_p^{m+1}|^2 - |k \nabla \delta_t e_p^m|^2 \leq C k \left( |\delta_t e_h^m|^2 + 2(|k \nabla \delta_t e_p^m|^2 + |k \nabla \delta_t e_p^{m-1}|^2) \right) + C k h^2 + \frac{k}{2} |\delta_t \tilde{e}_h^m|^2
\]
\[+ C k h^2 \left( |\delta_t \tilde{u}^{m+1}|^2 + |\delta_t \tilde{u}^{m+1}|_{H^2}^2 \right) + C k \| \tilde{e}_h^m \|_2. \]

Therefore, by adding from \( m = 1 \) to \( r \) (with any \( r < M \)), taking into account (20), Lemma 13 and Theorem 17, the discrete Gronwall’s Lemma can be applied, yielding to
\[
|\delta_t e_h^{r+1}|^2 + k |\nabla \delta_t e_p^{r+1}|^2 + \sum_{m=1}^r |\delta_t \tilde{e}_h^{m+1} - \delta_t e_h^m|^2 + \frac{k}{2} \sum_{m=1}^r |\delta_t \tilde{e}_h^{m+1}|^2 \leq C \left( |\delta_t e_h^1|^2 + \frac{k}{2} |\delta_t \tilde{e}_h^1|^2 + |k \nabla \delta_t e_p^1|^2 + h^2 \right),
\]
hence (43) holds by using the hypotheses on the first step (42).

**Corollary 20** Assuming hypotheses of Theorem 19, the following error estimates hold
\[
\| \tilde{e}_h^{m+1} \|_{l_\infty(H^1)} \leq C h \quad \text{and} \quad \| e_p^{m+1} \|_{l_\infty(L^2)} \leq C h.
\]

**Proof:** We divide the proof into three steps:

**Step 1.** To obtain
\[
\| e_p^{m+1} \|_{l_2(L^2)} \leq C h. \quad (45)
\]

Arguing as in the time discrete Algorithm 1, from the discrete inf-sup condition applied to \( (E_3)_h \) and the estimates \( \| \tilde{e}_h^{m+1} \|_{l_2(H^1)} \leq C h \) and \( \| \delta_t e_h^{m+1} \|_{l_\infty(L^2)} \leq C h \), we have (45).

**Step 2.** To prove
\[
\| \tilde{e}_h^{m+1} \|_{l_\infty(H^1)} \leq C h. \quad (46)
\]

By multiplying \( (E_3)_h \) by \( 2k \delta_t \tilde{e}_h^{m+1} \):
\[
|\nabla \tilde{e}_h^{m+1}|^2 - |\nabla \tilde{e}_h^m|^2 + |\nabla e_h^{m+1} - \nabla e_h^m|^2 = -2k \left( \nabla e_p^{m+1} + \delta_t e_t e_h^{m+1} + \delta_t e_h^{m+1}, \delta_t e_h^{m+1} \right)
\]
\[+ 2k \left( \nabla e_t^{m+1}, \nabla \delta_t e_h^{m+1} \right) - 2k \left( |k (2 \nabla e_p^{m+1} - e_p^{m-1}) + \delta_t e_t e_h^{m+1} | + 2k \nabla e_p^{m+1} \right) + 2k \nabla e_p^{m+1} \delta_t e_h^{m+1} \right).
\]

Then, we obtain
\[
|\nabla \tilde{e}_h^{m+1}|^2 - |\nabla \tilde{e}_h^m|^2 + |\nabla e_h^{m+1} - \nabla e_h^m|^2 \leq 2k |e_p^{m+1}|^2
\]
\[+ C k |\nabla \delta_t e_h^{m+1}|^2 + C k |\delta_t e_h^{m+1}|^2 + C k |\delta_t e_h^{m+1}|^2
\]
\[+ 2k |e_t (\delta_t e_h^{m+1})|^2 + 2k |\tilde{e}_t^{m+1}|^2 + C k \left( |e_p^{m+1}|^2 + |e_p^{m-1}|^2 \right) + 2k \nabla e_p^{m+1} \delta_t e_h^{m+1} \right)\]
\[
\leq 2 k |e_{p,h}^{m+1}|^2 + C k |\nabla \cdot \delta_t e_h^{m+1}|^2 + C k |\delta_t e_h^{m+1}|^2 + C k |\delta t e_h^{m+1}|^2
\]
\[+ C k h^2 \|\delta u_{m+1}\|^2 + C k h^2 \|\tilde{u}_{m+1}\|^2_{H^2} + C k h^2 (\|p_m\|^2 + \|p_{m-1}\|^2) + 2 k N L_{h}^{m+1}(\delta e_h^{m+1}).\]

Taking into account (40), we bound the last term of the RHS as follows,
\[
2 k N L_{h}^{m+1}(\delta e_h^{m+1}) \leq \varepsilon k \|\delta e_h^{m+1}\|^2 + C k \|e_h^{m+1}\|^2 + C k \|e_h^{m}\|^2 + C k \|e_{p,h}^{m}\|^2 + C k h^2
\[
\leq \varepsilon k \|\delta e_h^{m+1}\|^2 + C k \|e_h^{m+1}\|^2 + C k \|e_h^{m}\|^2 + C k h^2
\]

hence, we arrive at
\[
|\nabla e_h^{m+1}|^2 - |\nabla e_h^{m+1}|^2 + |\nabla e_h^{m+1} - \nabla e_h^m| \leq k |e_{p,h}^{m+1}|^2 + k |\nabla \cdot \delta e_h^{m+1}|^2 + C k |\delta e_h^{m+1}|^2
\]
\[+ k |\nabla \delta t e_h^{m+1}|^2 + C k |\delta e_h^{m+1}|^2 + C k \|e_h^{m+1}\|^2 + C k \|e_h^{m}\|^2 + C k h^2 + C k h^2 \|\delta u_{m+1}\|^2.
\]

Adding from \(m = 0\) to \(r\),
\[
|\nabla e_h^{r+1}|^2 \leq |\nabla e_h^0|^2 + C k \sum_{m=0}^r |e_{p,h}^{m+1}|^2 + C k \sum_{m=0}^r |\nabla \delta t e_h^{m+1}|^2 + C k \sum_{m=0}^r |\delta t e_h^{m+1}|^2 + C k h^2.
\]

Then, by applying (45) and the estimates obtained in Theorems 17 and 19, we obtain (46).

**Step 3.** To obtain \(\|e_{p,h}^{m+1}\|_{L^\infty(L^2)} \leq C h\).

Finally, by using again the discrete inf-sup condition (21) and taking into account (46), one has \(\|e_{p,h}^{m+1}\|_{L^\infty(L^2)} \leq C h\) and the proof is finished.

**Remark 21** By combining Theorem 10 and Corollary 20, the following error estimate for the total error holds
\[
\|u(t_{m+1}) - \tilde{u}_h^{m+1}\|_{L^\infty(H^1)} + \|p(t_{m+1}) - \tilde{p}_h^{m+1}\|_{L^\infty(L^2)} \leq C (k + h).
\]

### 3 Numerical Simulations

We consider the FE approximation \(P_2 \times P_1\) related to a structured mesh of the domain \(\Omega = (0, 1)^2 \subset \mathbb{R}^2\).

The numerical results have been obtained using the software FreeFem++ ([8, 20]), and show first order accurate in time for velocity and pressure of the segregated version of Algorithm 2 given in Remark 15. These results are agree to Remark 21.

In fact, we present some numerical error orders in time for velocity \(u = (u_1, u_2)\) and pressure \(p\) using the following exact solution for \((P):\)
\[
u = e^{-t} \left( (\cos(2 \pi x) - 1) \sin(2 \pi y) - (\cos(2 \pi y) - 1) \sin(2 \pi x) \right) \quad \text{and} \quad p = 2 \pi e^{-t} (\sin(2 \pi x) + \sin(2 \pi y)).
\]
We take \( \nu = 1 \) and adjust the force \( \mathbf{f} \) to enforce this exact solution.

Note that \( \nabla \cdot \mathbf{u} = 0 \) in \( \Omega \), \( \mathbf{u}|_{\partial \Omega} = 0 \) and \( \int_{\Omega} p = 0 \). On the other hand, we have choice this regular exact solution such that \( \nabla p \cdot \mathbf{n} \neq 0 \) on the boundary \( \partial \Omega \), in order to measure the effect of the numerical boundary condition \( \nabla (p^{n+1} - p^n) \cdot \mathbf{n} = 0 \) on \( \partial \Omega \). We approach numerically the order in time for the segregated version of Algorithm 2 given in Remark 15, comparing to other current first order splitting schemes like, rotational pressure-correction, consistent splitting and penalty-projection schemes, also implemented in they segregated form.

Some numerical analysis results and computational simulations can be seen in [14] and [15] for the rotational pressure-correction projection scheme, in [13], [15] and [29] for the consistent splitting scheme and in [1], [2] and [7] for the penalty-projection scheme.

Concretely, let \( \mathbf{u}^m \in \mathbf{Y}_h \), \( \Pi_h(\nabla \cdot \mathbf{u}^m) \in Q_h \) and \( p^m \in Q_h \) be given, where \( \Pi_h \) is the \( L^2(\Omega) \)-projector operator onto the discrete pressure space \( Q_h \), the implemented segregated schemes are:

- **The rotational pressure-correction scheme:**

  (a) Find \( \mathbf{u}^{m+1}_h \in \mathbf{Y}_h \) such that \( \forall \mathbf{v}_h \in \mathbf{Y}_h \),

  \[
  \left( \frac{\mathbf{u}^{m+1}_h - \mathbf{u}^m_h}{k}, \mathbf{v}_h \right) + c \left( \mathbf{u}^m_h, \mathbf{u}^{m+1}_h, \mathbf{v}_h \right) + \nu \left( \nabla \mathbf{u}^{m+1}_h, \nabla \mathbf{v}_h \right) - \left( \mathbf{p}^m_h, \nabla \mathbf{v}_h \right) = \left( \mathbf{f}^{m+1}_h, \mathbf{v}_h \right),
  \]

  where

  \[
  \mathbf{q}^m_h = 2 \mathbf{p}^m_h - \mathbf{p}^{m-1}_h + \nu \Pi_h(\nabla \cdot \mathbf{u}^m_h) \in Q_h.
  \]

  (b) Compute \( \Pi_h(\nabla \cdot \mathbf{u}^{m+1}_h) \).

  (c) Find \( p^{m+1}_h \in Q_h \) such that

  \[
  k \left( \nabla (p^{m+1}_h - p^m_h + \nu \Pi_h(\nabla \cdot \mathbf{u}^{m+1}_h)), \nabla \mathbf{q}_h \right) = - \left( \nabla \cdot \mathbf{u}^{m+1}_h, \mathbf{q}_h \right) \quad \forall \mathbf{q}_h \in Q_h
  \]

- **The consistent splitting scheme:**

  (a) Find \( \mathbf{u}^{m+1}_h \in \mathbf{Y}_h \) such that \( \forall \mathbf{v}_h \in \mathbf{Y}_h \),

  \[
  \left( \frac{\mathbf{u}^{m+1}_h - \mathbf{u}^m_h}{k}, \mathbf{v}_h \right) + c \left( \mathbf{u}^m_h, \mathbf{u}^{m+1}_h, \mathbf{v}_h \right) + \nu \left( \nabla \mathbf{u}^{m+1}_h, \nabla \mathbf{v}_h \right) - \left( \mathbf{p}^m_h, \nabla \mathbf{v}_h \right) = \left( \mathbf{f}^{m+1}_h, \mathbf{v}_h \right)
  \]

  (b) Compute \( \Pi_h(\nabla \cdot \mathbf{u}^{m+1}_h) \).

  (c) Find \( p^{m+1}_h \in Q_h \) such that

  \[
  \left( \nabla (p^{m+1}_h - p^m_h + \nu \Pi_h(\nabla \cdot \mathbf{u}^{m+1}_h)), \nabla \mathbf{q}_h \right) = \left( \frac{\mathbf{u}^{m+1}_h - \mathbf{u}^m_h}{k}, \nabla \mathbf{q}_h \right) \quad \forall \mathbf{q}_h \in Q_h.
  \]

- **The penalty pressure-projection scheme:**
\[
\begin{array}{|c|c|c|c|}
\hline
k & 0.2 - 0.1 & 0.1 - 0.05 & 0.05 - 0.025 \\
\hline
\|u_1\|_{L^\infty(L^2)} & 1.077 & 1.326 & 1.582 \\
\|u_1\|_{L^\infty(H^1)} & 0.812 & 1.146 & 1.453 \\
\|u_2\|_{L^\infty(L^2)} & 1.095 & 1.352 & 1.585 \\
\|u_2\|_{L^\infty(H^1)} & 0.817 & 1.148 & 1.457 \\
\|p\|_{L^2(L^2)} & 0.877 & 1.282 & 1.535 \\
\|p\|_{L^\infty(L^2)} & 0.880 & 1.157 & 1.444 \\
\hline
\end{array}
\]

Table 1: Error orders in time for Algorithm 2

\[
\begin{array}{|c|c|c|c|}
\hline
k & 0.2 - 0.1 & 0.1 - 0.05 & 0.05 - 0.025 \\
\hline
\|u_1\|_{L^\infty(L^2)} & 1.048 & 1.278 & 1.475 \\
\|u_1\|_{L^\infty(H^1)} & 0.955 & 1.150 & 1.290 \\
\|u_2\|_{L^\infty(L^2)} & 1.105 & 1.314 & 1.511 \\
\|u_2\|_{L^\infty(H^1)} & 1.035 & 1.176 & 1.311 \\
\|p\|_{L^2(L^2)} & 1.241 & 1.436 & 1.490 \\
\|p\|_{L^\infty(L^2)} & 1.012 & 1.238 & 1.361 \\
\hline
\end{array}
\]

Table 2: Error orders in time for Rotational Scheme

(a) Find \(u^{m+1}_h \in Y_h\) such that, \(\forall v_h \in Y_h\),

\[
\left( \frac{u^{m+1}_h - u^m_h}{k}, v_h \right) + c(u^m_h, u^{m+1}_h, v_h) + \nu\left( \nabla u^{m+1}_h, \nabla v_h \right) + \nu\left( \nabla \cdot u^{m+1}_h, \nabla \cdot v_h \right) - \left( q^m_h, \nabla \cdot v_h \right) = \left( f^{m+1}, v_h \right).
\]

where \(q^m_h\) is given as in (47). Note that this scheme is not fully segregated because it couples the velocity components in the term \(\nu\left( \nabla \cdot u^{m+1}_h, \nabla \cdot v_h \right)\).

(b) Compute \(\Pi_h(\nabla \cdot u^{m+1}_h)\).

(c) Find \(p^{m+1} \in Q_h\) solving the Poisson-Neumann problem (48).

We consider the structured mesh taking 70 subintervals in \([0, 1]\) (with \(h = 0.0142857\)). In addition, \(k = 0.2, 0.1, 0.05\) and \(0.025\) are considered corresponding to 10, 20, 40 and 80 time iterations in the time interval \([0, 2]\).

The numerical results comparing the time accuracy can be seen in Tables 1, 2, 3 and 4, showing a little better accuracy in velocity and pressure for the incremental scheme Algorithm 2. Moreover, first order accurate in time for velocity and pressure is observed for all previous schemes.

With respect to the computational cost, the CPU time needed taking \(k = 0.025\) (80 time iterations) is shown in Table 5, showing a little lower cost in the incremental scheme Algorithm 2.
Table 3: Error orders in time for Consistent Scheme

| k   | 0.2 – 0.1 | 0.1 – 0.05 | 0.05- 0.025 |
|-----|-----------|------------|-------------|
| \|u_1\|_{L^\infty(L^2)} | 0.726  | 0.814  | 0.885  |
| \|u_1\|_{H^1} | 0.715 | 0.813  | 0.885  |
| \|u_2\|_{L^\infty(L^2)} | 0.764  | 0.843  | 0.905  |
| \|u_2\|_{H^1} | 0.775 | 0.841  | 0.908  |
| \|p\|_{L^2} | 0.822  | 0.906  | 0.952  |
| \|p\|_{L^\infty(L^2)} | 0.700  | 0.792  | 0.868  |

Table 4: Error orders in time for Penalty-Projection Scheme

| k   | 0.2 – 0.1 | 0.1 – 0.05 | 0.05- 0.025 |
|-----|-----------|------------|-------------|
| \|u_1\|_{L^\infty(L^2)} | 0.983  | 1.256  | 1.459  |
| \|u_1\|_{H^1} | 0.903 | 1.120  | 1.266  |
| \|u_2\|_{L^\infty(L^2)} | 1.012  | 1.265  | 1.484  |
| \|u_2\|_{H^1} | 0.942 | 1.135  | 1.282  |
| \|p\|_{L^2} | 1.161  | 1.354  | 1.429  |
| \|p\|_{L^\infty(L^2)} | 0.937  | 1.172  | 1.324  |

Note that in this scheme the problem related to the $L^2(\Omega)$-projector $\Pi_h$ has not to be computed.

4 Conclusions

The optimal error estimates of order $O(k + h)$ for the velocity and pressure are deduced for the first-order linear fully discrete segregated scheme based on an incremental pressure projection method (Algorithm 2) approaching the 3D Navier-Stokes problem. This convergence is unconditional, i.e. without imposing constraints on mesh size $h$ or time step $k$.

Moreover, some numerical computations of the segregated version of Algorithm 2 agree the previous numerical analysis are provided. These simulations are also compared with the segregated versions of the rotational, consistent and penalty-projection schemes, obtaining a little better accuracy in time and lower computational cost of Algorithm 2.

Finally, although this segregated scheme has the numerical boundary layer furnished by the

| Scheme: | Algorithm 2 | Rotational | Consistent | Penalty |
|--------|-------------|------------|------------|---------|
| CPU-time (s) | 2067.45 | 2113.22 | 2079.4 | 2147.9 |

Table 5: Computational cost
artificial boundary condition $\nabla (p^{m+1} - p^m) \cdot n$ on $\partial \Omega$, this fact does not perturb the optimal convergence in the energy norms $H^1(\Omega) \times L^2(\Omega)$ for the velocity and pressure, respectively.

References

[1] Ph. Angot, M. Jobelin, C. Lapuerta, J.-C. Latché, B. Piar. A finite element penalty-projection method for incompressible flows. Journal of Computational Physics, 217 (2006), 502-518.

[2] Ph. Angot, M. Jobelin, J.-C. Latché. Error analysis of the Penalty-Projection Method for the time Dependent Stokes Equations. International Journal on Finite Volumes 6, 1 (2009) 1-26.

[3] S. Badia, R. Codina. Convergence analysis of the FEM approximation of the first order projection method for incompressible flows with and without the inf-sup condition. Numer.Math., 107 (4) (2007), 533-557.

[4] A.J. Chorin. Numerical solution of the Navier-Stokes equations. Math. Comput., 22 (1968), 745-762.

[5] A.J. Chorin. On the convergence of discrete approximations of the Navier-Stokes equations. Math. Comput., 23 (1969), 341-353.

[6] P.G. Ciarlet. Basic error estimates for elliptic problems - Finite Element Methods, Part 1, Handbook of Numerical Analysis, P. G. Ciarlet and J. L. Lions, eds., North-Holland, Amsterdam, 1991.

[7] C. Février, J. Laminie, P. Poulet, Ph. Angot. On the penalty-projection method for the time Navier-Stokes equations with the MAC mesh. Journal of Computational and Applied Mathematics 226 (2009) 228-245.

[8] FreeFem++ Software. www.freefem.org

[9] V. Girault, P.A. Raviart. Finite Element Methods for Navier-Stokes Equations. Springer-Verlag, 1986.

[10] J.L. Guermond, L. Quartapelle On the approximation of the unsteady Navier-Stokes equations by finite elements projection methods. Numer.Math., 80 (1998), 207-238.

[11] J.L. Guermond, J. Shen. Quelques résultats nouveaux sur les méthodes de projection. C.R. Acad. Sci. Paris, Série I 333 (2002), 1111-1116.

[12] J.L. Guermond, J. Shen. Velocity-correction projection methods for incompressible flows. SIAM Journal on Numerical Analysis, 41 (2003), 112-134.

[13] J.L. Guermond, J. Shen. A New Class of Truly Consistent Splitting Schemes for Incompressible Flows. J. Comput. Phys., 192 (2003), 262-276.

[14] J.L. Guermond, J. Shen. On the error estimates for the rotational pressure-correction projection methods. Mathematical of Computation, 73 (2004), 1719-1737.

[15] J.L. Guermond, P. Minev, J. Shen. An overview of projection methods for incompressibility flows. Comp. Methods Appl. Mech. Engrg., 195 (2006), 6011-6045.

[16] F. Guillén-González, M.V. Redondo-Neble Sharp error estimates for a fractional-step method applied to the 3D Navier-Stokes equations C.R. Acad. Sci. Paris, Ser. I 345 (2007), 359-362.

[17] F. Guillén-González, M.V. Redondo-Neble. New error estimates for a viscosity-splitting scheme in time for the 3D Navier-Stokes equations. IMA J. Numer. Anal. (2011) 31 (2), 556-579.
[18] F. Guillén-González, M.V. Redondo-Neble. Spatial error estimates for a finite element viscosity-splitting scheme for the Navier-Stokes equations. Int. J. Numer. Anal. Mod. 10 (4) (2013), 826-844.

[19] F. Guillén-González, M.V. Redondo-Neble. A second order in time pressure segregation scheme for the Navier-Stokes equations. Proccedings of Cedia congress, 2013.

[20] F. Hecht. New development in FreeFem++. J. Numer. Math. 20 (2012), no. 3-4, 251-265.

[21] J.G. Heywood, R. Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second order error estimates for spacial discretization. SIAM J. Num. Anal., 19 (2) (1982), 275-311.

[22] J.G. Heywood, R. Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second order time discretization, SIAM J. Numer. Anal., 27 (1990), 353-384.

[23] J. van Kan. A second-order accurate pressure-correction scheme for viscous incompressible flow. SIAM J. Sci. Stat. Comput., 7 (39) (1986), 870-891.

[24] A. Prohl. Projection and quasi-compressibility methods for solving the incompressible Navier-Stokes equations. Advances in Numerical Mathematics. B. G. Teubner, Stuttgart, 1997.

[25] J.H. Pyo. The Gauge-Uzawa and Related Projection Finite Element Methods for the Evolution Navier-Stokes Equations. Thesis, University of Maryland, USA, 2002.

[26] R. Rannacher. On Chorin’s Projection Method for the Incompressible Navier-Stokes Equations. The Navier-Stokes equations II- Theory and Numerical Methods. Proceedings of a Conference held in Oberwolfach, Germany, 1991. (Eds.) J.G. Heywood, K. Masuda, R. Rautmann, V.A. Solonnikov, Springer-Verlag, (1992), 167-183.

[27] J. Shen. On error estimates of projection methods for Navier-Stokes equations: first-order schemes. SIAM Journal Num. Anal., 29 (1992), 57-77.

[28] J. Shen. Remarks on the pressure error estimates for the projection methods. Numer. Math., 67 (4) (1994), 513-520.

[29] J. Shen, X. Yang. Error estimates for finite element approximations of consistent splitting schemes for incompressible flows. Discrete and Continuous Dynamical Systems-Series B , Volume 8(3) (2007).

[30] R. Temam. Une méthode d’approximations de la solution des equations de Navier-Stokes. Bull. Soc. Math. France, 98 (1968), 115-152.

[31] R. Temam. Sur la stabilité et la convergence de la méthode des pas fractionnaires. Ann. Mat. Pura Appl., LXXIV (1968), 191-380.

[32] R. Temam. Navier-Stokes equations. Theory and Numerical Analysis. North-Holland, 1984.

[33] L.J.P. Timmermans, P.D. Minev, F.N. van de Vosse. An approximate projection scheme for incompressible flow using spectral elements. Int. J. Num. Meth. Fluids, 22 (1996), 673-688.