I. INTRODUCTION

A. Background and motivation

GW170817 [1] – the first coalescence of a binary neutron-star (NS) system detected by the gravitational-wave (GW) interferometers LIGO and Virgo – is a milestone in GW astronomy. With more NS-NS coalescence signals expected in the near future, it will be possible to constrain the equation of state of the NS core [1,2], to test gravity in the highly relativistic/strong-curvature/supranuclear-density regime [1], and to detect coincident electromagnetic signals emitted by these sources in various bands [3,4].

A major challenge in the parameter estimation of NS binaries is the modeling of the GW signal during the late inspiral, merger, and postmerger phases [10]. This is typically achieved by using GW templates obtained either phenomenologically or using the effective-one-body approach [11,14], fitted to numerical-relativity waveforms [13,15]. A ubiquitous ingredient of these templates is an accurate description of the early-inspiral phase as described by the post-Newtonian (PN) formalism [10,17,18] (i.e., a weak-field/slow-velocity expansion of Einstein’s equations), where the dynamics of the binary is driven by energy and angular momentum loss, and the two bodies are modeled as two point particles endowed with a series of multipole moments and with finite-size tidal corrections [19–21]. The latter are encoded in the way a NS responds when acted upon by the external gravitational field of its companion – through the tidal Love numbers (TLNs) (see, e.g., [22] and references therein).

To the leading order in the tidal field, the TLNs are proportional to the induced multipole moments. As such, they can be divided into two categories: electric (or even parity) TLNs, which are related to the induced mass multipole moments, and magnetic (or odd parity) TLNs, which are related to the induced current multipole moments and do not have an analog in Newtonian theory. Tidal deformability introduces a 5PN correction to the GW phase relative to the leading-order GW term [23,24], this correction being proportional to the quadrupolar electric TLN. The next-to-leading order correction from quadrupolar electric TLNs was computed in Ref. [20] and enters at 6PN order, which is also the leading-order correction from quadrupolar magnetic TLNs [25,26]. Moreover, the leading tail contribution from quadrupolar electric TLNs, appearing at 6.5PN order, has been computed in Ref. [21].

So far, the tidal corrections to the GW phase have been computed only for nonspinning objects, i.e., neglecting the coupling between the angular momentum of one body and the tidal field produced by its companion. In this paper, we make an important step forward in the PN modeling of the GW signal from spinning NS binaries, by computing the leading-order tidal interaction of spinning bodies in a binary to leading order in the spin, and the corresponding corrections to the GW phase. Although the dimensionless spin of NSs in coalescing binaries is expected to be small [27,28], neglecting the spin-tidal coupling might introduce systematics in the parameter estimation, especially when using uniform priors that extend to high values of the spin [1,5]. Furthermore, spin-tidal corrections might be important to improve current tests of the nature of compact objects using the tidal effects in the inspiral [29,30].

In recent years, there has been remarkable progress in studying the tidal deformability of spinning compact ob-

1 We warn the reader that the quadrupolar magnetic Love numbers affect the GW phase at 6PN order also for equal-mass binaries, see erratum in Ref. [24] and Ref. [21]. This point will be important for the following discussion.
jects. Tidal deformations of slowly-spinning black holes were studied in Refs. [36, 37], which confirmed that the TLNs of a black hole are precisely zero [38, 41] also in the spinning case, at least to quadratic order in the spin in the axisymmetric case and to linear order in the spin in general. Furthermore, the coupling between the tidal fields and the angular momentum introduces new families of TLNs, which were dubbed rotational tidal Love numbers (RTLNs) [42–46]. While also the RTLNs of a black hole are precisely zero [38–41] also in the axisymmetric case and to linear order in the spin.

Finally, a different choice of assumptions on the dynamics of the fluid within the star (e.g., whether the fluid is rotational or static) can provide a more realistic configuration for a stationary tidally distorted object in a binary system, and affect the magnetic TLNs and the RTLNs [13, 40, 47].

B. Notation and conventions

We denote the speed of light in vacuum by \( c \) and set the gravitational constant \( G = 1 \) throughout the paper. Latin indices \( i, j, k \), etc. run over three-dimensional spatial coordinates and are contracted with the Euclidean flat metric \( \delta^{ij} \). Since there is not distinction between upper and lower spatial indices, we will use only the upper indices throughout the paper. The complete antisymmetric Levi-Civita symbol is denoted by \( \epsilon^{ijk} \). Following the STF notation [48], we use capital letters in the middle of the alphabet \( L, K \), etc. as shorthand for multi-indices \( a_1 \ldots a_l, b_1 \ldots b_k \), etc. Round ( ), square \([ \, ]\), and angular \( \langle \, \rangle \) brackets in the indices indicate symmetrization, antisymmetrization and trace-free symmetrization, respectively. For instance,

\[
T^{(ab)} = T^{(ab)} - \frac{1}{3} \delta^{ab} T^{cc} = \frac{1}{2} (T^{ab} + T^{ba}) - \frac{1}{3} \delta^{ab} T^{cc}. \tag{1}
\]

We call symmetric trace-free (STF) those tensors \( T^{i_1 \ldots i_l} \) that are symmetric on all indices and whose contraction of any two indices vanishes

\[
T^{(i_1 \ldots i_l)} = T^{(i_2 \ldots i_l)},
\]

\[
T^{i_1 \ldots i_l} = 0,
\]

\[
T^{(i_1 \ldots i_l)} = T^{(i_1 \ldots i_l)}. \tag{2}
\]

The contraction of a STF tensor \( T^L \) with a generic tensor \( U^L \) is \( T^L U^L = T^L U^L \). For a generic vector \( u^a \) we define \( u^{i_1 \ldots i_l} = u^a u^b \ldots u^k \) and \( u^k \). Derivatives with respect to the coordinate time \( t \) are expressed by over-dots.

For a generic body \( A \), the mass and current multipole moments are denoted by \( M_A^L \) and \( J_A^L \), respectively. We indicate the electric and magnetic tidal moments, which affect the body \( A \), respectively, by \( G_A^L \) and \( H_A^L \). All of them are STF tensors on all indices.

Restricted to a two-body system, \( A = 1, 2 \), we define the mass ratios \( \eta_A = m_A / M \), where \( M = m_1 + m_2 \) is the total mass and \( m_A \) is the mass monopole \( M_A \) in the Newtonian limit. The symmetric mass ratio is \( \nu = \eta_1 \eta_2 \) and the reduced mass is \( \mu = \nu M \). We define the dimensionless spin parameters \( \chi_A = c J_A / (\eta_A M)^2 \), where \( J_A = \sqrt{J_A^2 / A} \) is the absolute value of the current dipole moment. The body position, velocity and acceleration vectors are denoted by \( \mathbf{z}_A, \mathbf{v}_A, \mathbf{a}_A \) and \( \mathbf{a}_A = \mathbf{z}_A \), respectively. We define the two-body relative position, velocity and acceleration vectors by \( \mathbf{z} = \mathbf{z}_2 - \mathbf{z}_1, \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 \) and \( \mathbf{a} = \mathbf{a}_2 - \mathbf{a}_1 \), respectively. We also define the relative unit vector \( n^i = \mathbf{z}^i / r \), where \( r = \sqrt{z^2 + z^2} \) is the orbital distance. We define the derivatives with respect to the spatial coordinates \( z^i \) as \( \partial_1 = \partial_{i_1} \ldots \partial_{i_l} \). In particular, we denote the derivatives with respect to \( z^j \) by \( \mathbf{z}^j \).

We shall also make use of the following identity

\[
\partial_L \frac{1}{r} = \partial_L (2) \frac{1}{r} = (-1)^l \partial_L (1) \frac{1}{r} = (-1)^l (2l - 1)!! \eta^{(L)}_L . \tag{3}
\]

We shall denote \( \lambda_i (\sigma_i) \) the electric (magnetic) TLN of multipolar order \( l \), whereas \( \lambda_{l \nu} \) and \( \sigma_{l \nu} \) are the RTLNs. As discussed below, for our computation it is sufficient to consider that the multipole moments higher than the dipole are induced only on the second body by the tidal field produced by its companion. For this reason, to avoid burdening the notation, we define the quadrupolar and octupolar moments as \( Q^{abc} = M_2^{abc} \), \( Q^{abc} = M_2^{abc} \), \( S^{abc} = M_2^{abc} \), \( S^{abc} = M_2^{abc} \). To our order of approximation, the moments induced on object 1 due to the tidal field produced by object 2 can be included a posteriori by inverting the indices in the final formulas. We do so only when presenting the final GW phase, Eq. (15).

Finally, for a binary system in circular orbit we define the PN expansion parameter \( x = (\omega M)^{2/3} / c^2 \), where \( \omega \) is the orbital angular velocity. Note that \( x = v^2 / c^2 + O(c^{-4}) \).

C. Tidal deformations of rotating stars

Finite-size effects due to the deformability of compact objects enter the GW phase through the TLNs. Loosely speaking, the TLNs can be defined as the multipole moments induced on an object by an external tidal field per unit of the external field itself [22]. Within linear perturbation theory, the TLNs do not depend on the source of the tidal field but only on the internal properties of the central object.

To linear order in the spin, and assuming small and slowly varying external tidal fields, the TLNs relevant for

\[\footnote{We note that the recent analysis in Ref. [48] found disagreement with the RTLNs previously computed by some of us [42], especially for low-compactness stars. The source of such disagreement is under investigation but is irrelevant for the analysis of this work.} \]
this paper can be defined through the following relations:

\[
Q^{ab} = \lambda_2 G^{ab} + \frac{\lambda_3}{c^2} J^c H^{abc}
\]

\[
Q^{abc} = \sigma_3 G^{abc} + \frac{\sigma_2}{c^2} J^c H^{abc}
\]

\[
S^{ab} = \frac{\sigma_2}{c^2} H^{ab} + \sigma_3 J^c G^{abc}
\]

\[
S^{abc} = \frac{\sigma_3}{c^2} H^{abc} + \sigma_2 J^c G^{abc}
\]

(4)

These are called *adiabatic relations* because the TLNs are assumed to be constant, neglecting the oscillatory response of the star to a variation of the tidal field (however, see Ref. [42]).

In the above equations, \(Q^L\) and \(S^L\) are, respectively, the mass and current multipole moments of order \(l\) induced on the spinning object (with spin vector \(J^c\)), whereas \(G^L\) and \(H^L\) are the external electric and magnetic tidal moments of order \(l\) evaluated at the location of the object. The constants \(\lambda_l\) and \(\sigma_l\) are the ordinary electric and magnetic TLNs, whereas \(\lambda_{3l}\) and \(\sigma_{3l}\) are the RTLNs [42–44]. The powers of \(c\) in the above equations guarantee that, at Newtonian order, a magnetic tidal field does not induce any multipole moment. The magnetic tidal moments source the mass multipole moments only starting at 1PN order, in agreement with the discussion in Ref. [46]. On the other hand, an electric tidal field can induce also current multipole moments at Newtonian order, but these moments affect the metric only at higher PN order, as discussed below.

The above relations generalize to spinning objects the standard proportionality relations among the quadrupole moments \(Q^{ab}\) of a nonspinning object and the external quadrupolar tidal moments \(G^{ab}\) [22–24]. In particular, the structure of Eq. (4) corresponds to the spin-tidal couplings introduced in Ref. [42] :

(i) in the nonspinning case, Eq. (4) reduces to \(Q^L = \lambda_l G^L\) and \(S^L = \frac{\sigma_l}{c} H^L\). In other words, an \(l\)-pole tidal moment can induce only an \(l\)-pole multipole moment with the same parity [4]. For example, a quadrupolar electric (respectively, magnetic) tidal moment \(G^{ab}\) (respectively, \(H^{ab}\)) induces a mass (respectively, current) quadrupole moment, \(Q^{ab}\) (respectively, \(S^{ab}\)). For \(l = 2\) and \(l = 3\), the induced multipole moments depend on four independent TLNs, namely \(\lambda_2, \lambda_3, \sigma_2,\) and \(\sigma_3\). It is well known that the dominant correction to the GW phase depends on \(\lambda_2\) through a 5PN term [23, 24];

(ii) the spin of the binary components couples tidal moments and multipole moments with different \(l\) order and opposite parity [42, 51]. In particular, a *magnetic* quadrupolar (respectively, octupolar) tidal moment can induce a *mass* octupole (respectively, quadrupole) moment through a term proportional to the spin and to the rotational Love number \(\lambda_{32}\) (respectively, \(\lambda_{23}\)). Likewise, an *electric* quadrupolar (respectively, octupolar) tidal moment can induce a *current* octupole (respectively, quadrupole) moment through a term proportional to the spin and to the rotational Love number \(\sigma_{32}\) (respectively, \(\sigma_{23}\)).

D. Summary of results

For the busy reader, we summarize here the main results of our work, which are derived in detail in the rest of the paper. We follow the notation described in Sec. 1B. Our main result is the GW phase with all tidal corrections included up to 6.5PN order and to linear order in the spin, see Eq. (13) below.

1. Lagrangian

The Lagrangian describing the two-body interaction can be written as

\[
\mathcal{L} = \mathcal{L}_{\text{orb}} + \mathcal{L}_{\text{int}}.
\]  

(5)

Here, \(\mathcal{L}_{\text{orb}}\) describes the orbital motion of the bodies

\[
\mathcal{L}_{\text{orb}} = \mathcal{L}_M + \mathcal{L}_J + \mathcal{L}_{Q2} + \mathcal{L}_{Q3} + \mathcal{L}_{S2} + \mathcal{L}_{S3},
\]  

(6)

where \(\mathcal{L}_M\) and \(\mathcal{L}_J\) are the contributions that depend only on the masses of the two bodies and on their spin vectors (to linear order in the spin), respectively,

\[
\mathcal{L}_M = \frac{\mu v^2}{2} + \frac{\mu M}{r} + \frac{\mu}{c^2} \left( 1 - \frac{3\nu}{8} v^4 + \frac{M}{r^2} \left[ 3 + (3 + \nu) v^2 - \frac{M}{r} \right] \right) + O \left( e^{-4} \right),
\]  

(7)

\[
\mathcal{L}_J = \frac{\epsilon_{abv}}{c^2} v^b \left( \eta_2 J_1^a + \eta_1 J_2^a \right) \frac{2M}{r^2} v^c + \frac{\epsilon_{abv}}{2} v^b \left( \eta_2 J_1^a + \eta_1 J_2^a \right) \frac{a_c}{2} + O \left( e^{-4} \right).
\]  

(8)

The mass quadrupole term reads, to next-to-leading PN order, respectively. Likewise, mass and current multipole moments have even and odd parity, respectively.

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3 We remind that the index \(L\) represents \(l\) indices \(i_1, \ldots, i_l\) running from one to three (see Sec. 1B).

4 Electric and magnetic tidal moments have even and odd parity,
where $E_{\text{int}}^{\text{int}}$ is the internal energy of body 2 which, to this level of approximation, can be expressed through its Newtonian value given in Eq. (82) below. At Newtonian order, $L_{Q2}$ is simply a coupling between the mass quadrupole moment and the quadrupolar electric tidal moment, $L_{Q2} = \frac{1}{2} G_2^{ab} Q_{ab} + O(c^{-2})$.

Likewise, the mass octupole term reads, to leading order,

$$L_{Q3} = \frac{1}{6} G_2^{abc} Q_{abc} \equiv -\frac{5\eta_1 M}{2r^4} Q_{abc} R_{abc} + O(c^{-2}),$$

whereas the current quadrupole and octupole terms, respectively, read (again to leading order)

$$L_{S2} = \frac{1}{3c^2} H_2^{ab} S_{ab} = \frac{4\eta_1 M}{c^2 r^3} \delta^{abc} R_{abc} + \frac{8}{c^4 r^4} J_1^d S_{abc} (5n_{abc} - 2\delta^{abc} n^a) + O(c^{-4}),$$

and

$$L_{S3} = \frac{1}{8c^2} H_2^{abc} S_{abc} = \frac{15\eta_1 M}{2c^2 r^4} \delta^{abc} R_{abc} + \frac{45}{4c^2 r^4} J_1^d S_{abc} (\delta^{dabc} - \frac{7}{3} n_{abcd}) + O(c^{-4}).$$

The other term appearing in Eq. (5), $L_{\text{int}}^{\text{int}}$, takes into account the internal dynamics of the body 2 (which we remind is the only one tidally deformed at this stage),

$$L_{\text{int}}^{\text{int}} = -\frac{1}{4\lambda_2} Q_{ab} Q^{ab} - \frac{1}{12\lambda_3} Q_{ab} Q^{ab} - \frac{1}{6\sigma_2} S_{ab} S^{ab} - \frac{1}{16\sigma_3} S_{ab} S^{ab} + \alpha J_2^d S_{abc} Q^{abc} + \beta J_2^d S_{abc} Q^{abc},$$

where $\alpha$ and $\beta$ are related to the RTLNs in a way to be defined shortly. Those above are all possible couplings among the multipole moments included in our model, and by the requirement that $L_{\text{int}}^{\text{int}}$ is scalar, parity invariant, at most linear in the spin, and quadratic in the higher multipole moments.

The Lagrangian [13] provides the correct equations of motion to linear order in the spin, up to 1PN order in the electric quadrupolar TLN, and to leading order in the other tidal deformations. As discussed below, this is sufficient to completely describe the tidal contribution to the phase up to 6.5PN order. Furthermore, the interaction term [13] guarantees that Euler-Lagrange equations for the multipole moments yield the adiabatic relations [4] with the following identification:

$$\lambda_{23} = 2\lambda_2 \sigma_2 \alpha \quad \lambda_{32} = 6\lambda_3 \sigma_2 \beta \quad \sigma_{23} = 3\lambda_2 \sigma_3 \beta \quad \sigma_{32} = 8\lambda_2 \sigma_3 \alpha.$$

We note that, with the above definition, the RTLNs are proportional to those defined in Ref. [12] in the axisymmetric case. The explicit relations between them are given in Appendix [4].

Crucially, the Lagrangian formulation enforces Eq. (14). Therefore, only two out of four RTLNs are independent. In particular, $\lambda_{23}$ is proportional to $\sigma_{32}$ and $\lambda_{32}$ is proportional to $\sigma_{23}$. This proportionality does not emerge from the perturbative computation performed in Ref. [12]. We discuss this issue in more detail in Sec. [13].

2. GW phase

Finally, from the above Lagrangian one can compute the GW phase to linear order in the spin. For circular orbits, up to 1.5PN order in the point-particle terms and up to 6.5PN order in the tidal-deformability terms, the GW phase for the TaylorF2 approximant [10, 17, 18] reads

$$\psi(x) = \frac{3}{128\nu x^{5/2}} \left[ 1 + \frac{3715}{756} + \frac{55}{9} \nu \right] x + \left( \frac{113}{3} \times \left( 38 \frac{\nu (\chi_1 + \chi_2)}{3} - 16\pi x^{1.5} + O(x^2) + \Delta x^5 + (\delta \Lambda + \Sigma) x^6 + (\bar{\Lambda} + \bar{\Sigma} + \bar{\Gamma} - \pi \Lambda) x^{6.5} + O(x^7) \right) \right],$$

where $x = \frac{1}{2}(\dot{\nu} \omega)^{2/3}$, $\omega$ is the orbital angular velocity, and $\chi_i$ are the dimensionless spin parameters introduced in Sec. [13].

The first two lines in the above equation denote the point-particle contribution to the GW phase, whereas the
other terms are due to the tidal deformability. The 5PN term is the usual leading-order tidal contribution where

\[
\Lambda = \left( 264 - \frac{288}{\eta_1} \right) c^{10} \frac{\lambda_2^{(1)}}{M^3} + (1 \leftrightarrow 2),
\]

and \(\lambda_2^{(1)}\) is the (quadrupolar, electric) TLN of the \(A\)-th body. The 6PN term contains the next-to-leading order contribution of the previous term,

\[
\delta \Lambda = \left( \frac{4595}{28} - \frac{15895}{28\eta_1} + \frac{5715\eta_1}{14} - \frac{325\eta_1^2}{7} \right) c^{10} \frac{\lambda_2^{(1)}}{M^3} + (1 \leftrightarrow 2),
\]

and the leading-order contribution from the quadrupolar, magnetic TLN,

\[
\Sigma = \left( \frac{6920}{21\eta_1} - \frac{20740}{21\eta_1} \right) c^8 \frac{\sigma_2^{(1)}}{M^3} + (1 \leftrightarrow 2).
\]

Note that the magnetic term corrects some errors in the first version of Ref. [22], and agrees with that recently derived in Ref. [20]. In particular, \(\Sigma\) affects the phase at 6PN order for equal-mass binaries and is therefore degenerate with \(\delta \Lambda\).\(^5\) The leading-order tail-tidal term, proportional to the quadrupolar electric TLN \(\lambda_2\), enters in the GW phase at 6.5PN order with the same combination \(\Lambda\) as in the 5PN term [21].

Finally, all spin-tidal terms enter at 6.5PN order in the GW phase, through three different (albeit degenerate) terms. The first two are due to the coupling between the spin and the ordinary quadrupolar (electric and magnetic) TLNs:

\[
\tilde{\Lambda} = \left[ \frac{593}{4} - \frac{1105}{8\eta_1} + \frac{567\eta_1}{8} - 81\eta_1^2 \right] \chi_2 + \left( -\frac{6607}{8} + \frac{6639\eta_1}{8} - 81\eta_1^2 \right) \chi_1 c^{10} \frac{\lambda_2^{(1)}}{M^3} + (1 \leftrightarrow 2),
\]

\[
\tilde{\Sigma} = \left[ \frac{-9865}{3} + \frac{4933}{3\eta_1} + 1644\eta_1 \right] \chi_2 - \chi_1 c^8 \frac{\sigma_2^{(1)}}{M^3} + (1 \leftrightarrow 2),
\]

whereas the third 6.5PN term in Eq. (15) is proportional to the RTLNs,

\[
\tilde{\Gamma} = \frac{c^{10}}{M^4} \left[ (856\eta_1 - 816\eta_1^2) \lambda_{23}^{(1)} \right. \\
- \frac{833\eta_1}{3} - 278\eta_1^2 \sigma_{23}^{(1)} \\
- \nu \left( 272\lambda_{32}^{(1)} - 204\sigma_{32}^{(1)} \right) \left] + (1 \leftrightarrow 2). \right.
\]

The fact that these terms all enter at 6.5PN order can be understood as follows. Let us first focus on the terms proportional to the RTLNs, i.e., on \(\tilde{\Gamma}\). The mass quadrupole moment \(Q_{ab}\) enters at 2PN order in the phase \([19, 53, 54]\). From the adiabatic relations \([4]\), \(Q_{ab}\) acquires a contribution proportional to the spin and to \(H_{abc} \sim v/r^4 \sim 4.5\) PN, so that overall these corrections enter the GW phase at \(2 + 4.5 = 6.5\) PN order. Likewise, \(Q_{abc}\) affects the phase at 3PN order and its spin-tidal coupling is proportional to \(H_{abc} \sim v/r^4 \sim 3.5\) PN so that also this term enters at \(3 + 3.5 = 6.5\) PN order. Similar arguments can be made for the terms proportional to \(S_{ab}\) and \(S_{abc}\), since the latter enter the GW phase at 2.5PN and 3.5PN order, respectively, and they are coupled to \(G_{abc} \sim 4\) PN and \(G_{ab} \sim 3\) PN, respectively.

We generalize this argument in Sec. III D, showing that the spin-tidal terms arising from \(l\)-pole RTLNs to linear order in the spin enter the GW phase at \((2l + 1/2 + 2\eta_1)\) PN order. Therefore, for any \(l \geq 3\), this contribution enters at lower PN order relative to the standard electric TLNs of order \(l\) (the latter entering at \((2l + 1)\) PN order).

Corrections proportional to the spin and to the ordinary \(l = 2\) TLNs, i.e. the \(\tilde{\Lambda}\) and \(\tilde{\Sigma}\) terms, also enter at 6.5PN order. This is due to the fact that, as discussed below, the leading-order spin terms in \(G_{ab}\) and \(H_{abc}\) enter, respectively, at 4.5PN and at 4PN order and, since they enter the GW phase, respectively, through the induced \(Q_{ab}\) and \(S_{ab}\) (at 2PN and 2.5PN order, respectively), their overall contribution is again 6.5PN. We generalize this argument in Sec. III D, showing that the spin-tidal terms arising from \(l\)-pole TLNs (both electric and magnetic) to linear order in the spin enter the GW phase at \((2l + 5/2)\) PN order. It is also worth noting that these terms effectively couple higher-order point-particle terms (the spins) to the tidal terms (the ordinary TLNs), thus breaking the “decoupling” that exists between the point-particle phase and the tidal phase at the leading order [55].

The tidal terms entering the GW phase at leading order in the spin are summarized in Table I. For completeness, in Appendix I we also provide higher-order terms entering the GW phase [15] which are proportional to the TLNs and are computed as a by-product of our analysis.

We stress that – within our Lagrangian approach – only two out of \(\lambda_{32}, \lambda_{23}, \sigma_{32},\) and \(\sigma_{23}\) are independent, these four quantities being related to \(\alpha\) and \(\beta\) [the only two extra parameters entering our interaction Lagrangian (13)] through Eq. (13). While this is an unsolved issue (see discussion in Sec. III D) for the sake of generality we will consider these terms as independent. In any case, these terms enter the GW phase only through the combination \(\tilde{\Gamma}\).
TABLE I. Schematic representation of the PN contributions of the TLNs and of the RTLNs to the GW phase of a binary system to linear order in the spin. “LO,” “NLO,” and “NNLO” stand for Leading Order, Next-to-Leading Order, etc. The entries in boldface are the new 6.5PN terms computed in this work (we omit the leading-order tail effect entering at 6.5PN order derived in [23]). They are all proportional to the spins of the binary components and would be zero in the nonspinning case. For generic l-poles, the contribution from TLNs enters at (2l + 1/2)PN order, whereas the spin-tidal contribution from the ordinary TLNs enters at (2l + 5/2)PN order (see Sec. III B). For comparison, in the nonspinning case the electric and magnetic TLNs enter at (2l + 1)PN and (2l + 2)PN order, respectively.

| PN order | \( \lambda_2 \) | \( \sigma_2 \) | \( \lambda_{23,32} \), \( \sigma_{23,32} \) | \( \lambda_3 \) | \( \sigma_3 \) |
|----------|----------------|----------------|----------------|----------------|----------------|
| 5        | LO \( \propto \Lambda \) |
| 6        | NLO \( \propto \delta \Lambda \), \( \propto \Sigma \) |
| 6.5      | NNLO \( \propto \tilde{\Lambda} \), NLO \( \propto \tilde{\Sigma} \), LO \( \propto \tilde{\Gamma} \) |
| 7        | ... | ... | ... | LO |
| 8        | ... | ... | ... | ... |

II. PN TIDAL INTERACTIONS OF INTERACTING, STRUCTURED BODIES

In this section, we summarize the PN theory of tidal interactions in binary systems, which has been mainly developed in Refs. [20, 56, 58].

A. Coordinate frames and multipole expansions of the PN potentials

Let us consider \( N \) interacting, arbitrarily structured bodies immersed in a strong-field environment. It is possible to define a harmonically and conformally Cartesian coordinate system \((t, \mathbf{x}^i)\), which we call “global frame”, covering the entire spacetime except the strong-field region near each body. In this frame, the spacetime metric, in 1PN approximation [i.e., including terms up to \( O(c^{-2}) \)] has the form

\[
ds^2 = - \left(1 + \frac{2\Phi_g}{c^2} + \frac{2\Phi^2_g}{c^4} \right) c^2 dt^2 + \frac{2\Phi^i}{c^3} c dt dx^i + \left(1 - \frac{2\Phi_g}{c^2} \right) \delta^{ij} dx^i dx^j + O(c^{-4}) .
\]

The scalar field \( \Phi_g(t, \mathbf{x}) \) can be decomposed as \( \Phi_g = \phi_g + c^{-2} \psi_g \), where \( \phi_g \) is the Newtonian (0PN) potential and \( \psi_g \) is its 1PN correction; \( \zeta_j^i(t, \mathbf{x}) \) is the gravito-magnetic vector potential, at 1PN order.

For each body \( A (A = 1, \ldots, N) \) we assume the existence of a local coordinate system \((s_A, y_A^i)\), which we call “body frame” or “local frame”, covering the body, including the strong-field worldtube \( W_A \) defined as the product of the ball \( |y_A| < r_A^- \) with an open interval of time. Moreover, for each body \( A \), there exists a buffer region \( B_A \) defined as the product of \( r_A^- < |y_A| < r_A^+ \) with an open time interval, which is covered by both the global frame \((t, \mathbf{x}^i)\) and the local frame \((s_A, y_A^i)\).

In the buffer region \( B_A \), the gravitational field is weak and the local coordinates \( s_A, y_A^i \) are harmonic and conformally Cartesian, therefore the metric can be written in the 1PN form shown in Eq. (22), in terms of potentials \( \Phi_A(s_A, y_A), \zeta_A^i(s_A, y_A) \). In the buffer region, the coordinate transformation between the global frame and the local frame has the form

\[
x^i(s_A, y_A) = y_A^i + z^i(s_A) + c^{-2} [\text{1PN terms}] ,
\]

where the vector \( z^i \) describes a time-dependent spatial translation between the two frames. We do not explicitly write the 1PN terms in Eq. (22) for brevity; they depend on a set of freely specifiable functions encoding the residual gauge freedom.

As shown in Ref. [57], under these assumptions the potentials \( \Phi_A, \zeta_A^i \) for each body can be written in terms of a set of multipole moments. The internal degrees of freedom of the body are described by its mass multipole moments \( M_A^l(s_A) \) (with \( l \geq 0 \)) and its current multipole moments \( J_A^l(s_A) \) (with \( l \geq 1 \)) [23, 56, 61] (for a recent account in the context of tests of the black-hole no-hair theorem, see also [61]). The tidal field due to the bodies \( B \neq A \) is described by the electric tidal moments \( G_A^{EL}(s_A) \) and the magnetic tidal moments \( H_A^{ML}(s_A) \) (defined for \( l \geq 0 \) and \( l \geq 1 \), respectively). Both the body and the tidal moments are STF tensors on all indices. The explicit expansion of the PN potentials in the body frame is

\[
\Phi_A(s_A, y_A) = - \sum_{l=0}^{\infty} \frac{1}{l!} \left\{ (-1)^l M_A^l(s_A) \partial_L \frac{1}{|y_A|} + G_A^{EL}(s_A) y_A^L \right. \\
+ \frac{1}{c^2} \left\{ (-1)^l (2l + 1) \frac{\mu_A^l(s_A)}{(l + 1)(2l + 3)} \partial_L \frac{1}{|y_A|} + \frac{(-1)^l}{2} \bar{J}^A_L(s_A) \partial_L \frac{1}{|y_A|} - \bar{V}_A^L(s_A) y_A^L \right. \\
+ \left. \frac{1}{2(2l + 3)} \tilde{G}^A_L(s_A) y_A^{2L} \right\} + O(c^{-4}) ,
\]

\[
\zeta_A^i(s_A, y_A) = - \sum_{l=0}^{\infty} \frac{1}{l!} \left\{ (-1)^l Z_A^i_L(s_A) \partial_L \frac{1}{|y_A|} + Y_A^{il}(s_A) y_A^L \right. \\
+ \left. \frac{(-1)^l}{2} \bar{J}^A_L(s_A) \partial_L \frac{1}{|y_A|} - \bar{V}_A^L(s_A) y_A^L \right\} + O(c^{-4}) ,
\]

\footnote{The harmonic gauge condition is \( \partial_\nu (\sqrt{-g} g^{\mu \nu}) = 0 \), which implies \( 4\Phi + \partial_\mu \zeta^\mu = O(c^{-2}) \).}

\footnote{Conformally Cartesian coordinates are a special case of isotropic coordinates and require \( g_{00} y_{ij} = -\delta_{ij} + O(c^{-4}) \) [24].}
where

\[ Z_A^{(l)}(s_A) = \frac{M_A^{(l)}}{l+1} s_A + \frac{4l}{l+1} \frac{\mu}{l+1} s_A^{(l-1)} + O(c^{-2}), \]

\[ Y_A^{(l)}(s_A) = Y_A^{(l)} + \frac{1}{l+1} \frac{\nu}{l+1} s_A + \frac{2l-1}{2l+1} \frac{\nu}{l+1} s_A^{(l-1)} + O(c^{-2}). \]

Mass and electric moments are defined up to 1PN order, while current and magnetic moments are defined just to Newtonian level. The quantities \( \mu_A^l, \nu_A^l \) (defined for \( l \geq 0 \) and \( l \geq 1 \), respectively, and not to be confused with symmetric mass ratio \( \nu \) and reduced mass \( \mu \)) are called internal and external gauge moments, respectively, because they do not contain gauge-invariant information. As we shall see below, they will be set to zero by choosing the body-frame coordinate system.

In Eq. (24), the separation between the interior and exterior degrees of freedom is clear and unique: the terms with negative powers of \( |y_A| \) depend on the body multipole moments; the terms with positive powers of \( |y_A| \) depend on the tidal moments. This expression is defined in the buffer region where \( r_A < |y_A| < r_A + \alpha \); the body multipole moments encode the structure of the strong-field region \( |y_A| < r_A + \alpha \), while the tidal moments encode the gravitational fields generated by external \( |y_A| > r_A + \alpha \) sources and the inertial effects due to the motion of the local asymptotic rest frame with respect to the global frame.

Using the residual gauge freedom in the coordinate transformation (23), we choose the body-adapted gauge for the local frame, by setting \( M_A^0 = 0 \) and \( G_A^0 = 0 \); the former ensures that the center of mass-energy of body \( A \) is at \( y_A^0 = 0 \), the latter that replacing the body by a freely falling observer at \( y_A^0 = 0 \), the proper time is measured by the coordinate \( s_A \). Moreover, we set to zero the internal and external gauge moments, \( \mu_A^0 = \nu_A^0 = 0 \), and we choose the orientation of the body-frame spatial axes to coincide with those of the global frame (see 57, 58).

In the body-adapted gauge, the coordinate transformation (23) yields a function \( z_A^l(t) \) such that the equation \( x^l = z_A^l(t) \) describes the position of the body \( A \) in the global frame. This is called the “center-of-mass worldline” of the body \( A \), but in general it does not parametrize an actual worldline in spacetime (the global frame \( t, x^l \) is not defined in the strong-field region of the body, and thus it is not defined in its center of mass). This function parametrizes the location of the local frame of the body \( A \) in the global coordinate system. The same procedure gives the functions \( s_A(t) \) relating the proper and coordinate times of each body.

The PN potentials of the global frame (22) can be expressed in terms of the global multipole moments of the different bodies \( M_{gA}^l(t), Z_{gA}^{(l)}(t) \) (defined for \( l \geq 0 \) and STF tensors on all (the last) \( l \)-indices):

\[ \Phi_g(t, x) = -\sum_{A=1}^N \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \left\{ \frac{1}{l!} |M_{gA}^l(t)| \partial_L \frac{1}{|x - z_A(t)|} \right\} \]

\[ + \sum_{A=1}^N \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} |Z_{gA}^{(l)}(t)| \partial_L \frac{1}{|x - z_A(t)|} + O(c^{-4}) \]

\[ + \sum_{A=1}^N \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} |Y_{gA}^{(l)}(t)| \partial_L \frac{1}{|x - z_A(t)|} + O(c^{-2}). \]
where the multipole moments of the bodies by the relations \[ M^{l}_{sys} = \sum_{A=1}^{N} \sum_{k=0}^{l} \left( \frac{l}{k} \right) M^{(l-K, K)}_{g,A} \] and
\[ Z^{l}_{sys} = \sum_{A=1}^{N} \sum_{k=0}^{l} \left( \frac{l}{k} \right) Z^{(l-K, K)}_{g,A} \] + \mathcal{O}(e^{-4}) .

\[ J^{l}_{sys} = \frac{1}{4} \sum_{A=1}^{N} \sum_{k=0}^{l} \left( \frac{l}{k} \right) J^{(l-K, K)}_{g,A} \] + \mathcal{O}(e^{-4}) .

The equations of motion of the body \( A \) are the orbital equation of motion [i.e., a differential equation for \( z^{i}_{A}(t) \)], and the multipole equations of motion (i.e., a set of differential equations for the lowest-order body-frame multipole moments).

The multipole equations of motion have been derived in Refs. \[ 56, 57 \], and have the form:
\[ \dot{M}_{A} = - \frac{1}{c^{2}} \sum_{l=0}^{\infty} \frac{1}{l !} \left[ (l+1)M^{l}_{A} G^{l}_{A} + l \dot{M}^{l}_{A} G^{l}_{A} \right] + \mathcal{O}(e^{-4}) , \]
\[ \dot{M}^{l}_{A} = \sum_{l=0}^{\infty} \frac{1}{l !} \left[ M^{l}_{A} G^{l}_{A} + \frac{1}{c^{2}} \left( \frac{1}{l+2} \epsilon^{ijk} M^{l}_{A} H^{k}_{B} \right) \right] + \mathcal{O}(e^{-4}) , \]
\[ \dot{J}^{l}_{A} = \sum_{l=0}^{\infty} \frac{1}{l !} \left[ \epsilon^{ijk} M^{l}_{A} J^{k}_{B} \right] + \mathcal{O}(e^{-4}) , \]

where \( M^{l}_{A}(t) = M^{l}_{A}(s_{A}(t)) \) and the same holds for the other body-frame moments. These equations have to be supplemented by the equations for the multipole moments with \( l \geq 2 \), which depend on the internal dynamics of the bodies. In the adiabatic approximation, they are given by Eq. (31).

The orbital equation of motion,
\[ z^{i}_{A} = \mathcal{F}^{i}_{A}[z_{B} - z_{B}^{i}] / r_{B} , \]
\[ \dot{z}^{i}_{A} = \mathcal{F}^{i}_{A} z_{B} - \dot{z}^{i}_{B} , \]
\[ \mathcal{F}^{i}_{A} = \frac{1}{2} \left[ M^{l}_{A} \dot{z}^{i}_{A} \right] \]
\[ \mathcal{F}^{i}_{A} = \frac{1}{2} \left[ M^{l}_{A} \dot{z}^{i}_{A} \right] \]

can be obtained from the condition \( \dot{M}_{A} = 0 \), which follows from the gauge condition \( \dot{M}_{A} = 0 \). Using the relations between multipoles in different frames [Appendix A, Eqs. \[ A9 \sim A15 \]], Eq. (31) yields the explicit form of Eq. (36).

In the case of a binary system (\( N = 2 \)), the dynamics in the center-of-mass (COM) frame is described by the orbital separation \( z^{i} = z^{i}_{2} - z^{i}_{1} \). If we denote the two velocities \( v^{i}_{A} = \dot{z}^{i}_{A} \), the relative velocity is \( v^{i} = v^{i}_{2} - v^{i}_{1} \). We also define the radial separation \( r = |z| = \sqrt{\delta^{ij} z_{i} z_{j}} \), and the unit vector \( n^{i} = z_{i} / r \). The radial velocity is \( \dot{r} = v^{i} n^{i} \). The equation of motion of the orbital separation has the form
\[ \ddot{z}^{i} = z^{i}_{2} - z^{i}_{1} = \mathcal{F}^{i}_{2} - \mathcal{F}^{i}_{1} . \]

If the orbit is circular (as expected in the late inspiral) one gets \( \dot{r} = O(e^{-4}) \). In this case, Eq. (37) yields the radius-frequency relation \( r(\omega) \), where \( \omega / (2\pi) \) is the orbital frequency.
C. Lagrangian formulation and gravitational waveform

The equations of motion (37), together with the equations for the multipole moments in the adiabatic approximation, Eq. (41), can be derived from a generalized action principle, in terms of a Lagrangian function $L(z, \dot{z}, \dot{\omega}, M_{r1}^A, M_{r2}^A, J_{s1}^A)$. Therefore, given the explicit expression of the equations of motion, it is possible to derive the corresponding Lagrangian. The binding energy of the two-body system can be then obtained using the standard techniques of Lagrangian and Hamiltonian mechanics.

For a circular orbit, at 1PN order the energy is a function of the radial distance $r$ only. Replacing the radius-frequency relation $r(\omega)$, it is possible to express the energy as a function of the orbital angular velocity $\omega$.

The GW flux emitted by the system is due to the presence of time-varying multipole moments, and is given (up to next-to-leading order in the PN expansion) by [19]

$$F = \frac{1}{5c^2} M_{sys} \omega \cdot \dot{M}_{sys} + \frac{1}{189c^2} \nu^{ij} \nu^{ij} + \frac{16}{45c^2} \dot{\nu}^{ij} \cdot \dot{\nu}^{ij} + F_{\text{tail}} + O(c^{-3}),$$

where

$$F_{\text{tail}} = \frac{2}{5c^2} \dot{A}_{sys}^{ij} \dot{U}_{tail}^{ij}$$

is the tail contribution to the flux, entering at 1.5PN order beyond the leading term (see, e.g., [19, 63]). The explicit expression of $U_{\text{tail}}^{ij}$ for the two-body system is given in Eq. (A3). Using Eqs. (30)–(32) and the equations in Appendix A it is then possible to compute the flux in terms of the mass and current moments of the bodies of the system, $M_{r1}^A$ and $J_{s1}^A$.

In the adiabatic approximation, the orbital and internal energy are related to the GW flux through the energy balance relation

$$\dot{E} = -F.$$  

Assuming Eq. (40) and the stationary phase approximation, the Fourier transform of the gravitational waveform can be written as $h = A e^{i\psi}$ where the phase $\psi(\omega)$ is given in terms of the flux and the energy $E(\omega)$ by the differential relation

$$\frac{d^2 \psi}{d\omega^2} = -2 \frac{dE}{F} \frac{d\omega}{d\omega},$$

(see, e.g., [20] and references therein).

III. TIDAL INTERACTIONS OF A SPINNING BINARY SYSTEM

In [20, 58], the approach described in the previous section has been applied to the so-called “$M_1 - M_2 - J_2 - Q_2$” truncation. This is a system composed by two bodies, the first characterized by its monopole mass moment $M_1$ only, the second characterized by its monopole mass moment $M_2$, its dipole current moment (i.e., its spin) $J_2$, and its quadrupole mass moment $M_2^Q$, which we call $Q_2$; all other multipole moments identically vanish. Assuming the adiabatic approximation [19] (which in this truncation reduces to $Q_{ab} = \lambda_2 G^{ab}_2$) and neglecting PN orders larger than one (i.e., neglecting $O(c^{-1})$ terms), the $M_1 - M_2 - J_2 - Q_2$ truncation describes a binary system of two nonspinning bodies with masses $M_1$ and $M_2$ and (electric, quadrupolar) TLN $\lambda_2$. Indeed, in this approximation it turns out that the spin $J_2$ is constant and can be consistently set to zero. Moreover, the quadrupole moment $M_2^Q$ tidally induced by body 2 (which is set to zero in the $M_1 - M_2 - J_2 - Q_2$ truncation) can be easily derived a posteriori by exchanging the indices 1 $\leftrightarrow$ 2, as explained below.

The PN waveform derived in [20, 58] includes the tidal contribution to the phase up to next-to-leading order, i.e. to overall 6PN order. The quadrupolar magnetic contribution to the waveform of a nonspinning, tidally interacting binary system, which also appears at 6PN order, has been derived in Refs. [25, 26].

In this section, we extend the results of [20, 25, 58] by including the effects of spin. To this aim, we apply the approach described in Sec. III to a larger truncation in which body 1 is characterized by its mass $M_1$ and spin $J_1$, while body 2 is characterized by its mass $M_2$, its spin $J_2$, its mass quadrupole moment $Q_2$, its current quadrupole moment $S_2 = J_2^Q$, its mass octupole moment $Q_{ijk} = M_2^{ij}$, its current octupole moment $S_{ijk} = J_2^{ijk}$, and its current octupole moment $S_{ijk} = J_2^{ijk}$. Moreover, we neglect the terms quadratic in the spin. We remark that while the (mass and current) multipole moments with $l \geq 2$ are assumed to be induced by tidal interactions only [19], the masses and spins $M_2^Q$, $J_2^Q$ are a priori features of the system. Therefore, in our derivation we set to zero the $l \geq 2$ moments of body 1, but include its mass and spin. At the end of the computation, the tidally induced ($l \geq 2$) moments of body 1 will be easily obtained by simply exchanging the indices $A = 1, 2$ of the two bodies, as in [20, 58].

9 We remark that, although the current PN waveforms of compact binaries only include up to 3.5PN-order terms in the point-particle phase, the tidal interaction – which appears at 5PN order – cannot be neglected (and indeed is detectable [1, 6, 7]). This is due to the fact that a new dimensionful scale – the NS radius $R$ – appears in the tidal interaction, and the tidal terms in the GW phase are magnified by a factor $\sim (c^2 R/M)^5$ [64].

10 The spin-induced quadrupole moment, anyway, would only enter at quadratic order in the spin.
A. Equations of motion

With our truncation, the equations of motion of the mass monopole (i.e., the mass), the mass dipole and the current dipole (i.e., the spin), namely Eqs. [33], [34], and [35], respectively, reduce to

\[ \begin{align*}
\ddot{M}_1^i &= M_1 G_i^1 + \frac{1}{c^2} \left[ \frac{1}{2} J_i^1 H_{ij}^1 - \epsilon^{ijk} J_i^1 \dot{G}_j^k - 2 \epsilon^{ijk} J_i^1 G_j^k \right] + O(e^{-4}) , \\
\ddot{M}_2^i &= M_2 G_i^2 + \frac{1}{2} Q^{ij} G_i^j + \frac{1}{6} Q^{jka} G_i^{jka} + \frac{1}{c^2} \left[ \frac{1}{3} \epsilon^{ijk} Q^j a H_{ka}^1 + \frac{1}{2} \epsilon^{ijk} Q^j a H_{ka}^2 + \frac{1}{4} \epsilon^{ijk} Q^j a H_{ka}^2 + \frac{1}{3} \epsilon^{ijk} \dot{Q}^{j a} H_{ka}^2 \right] , \\
\ddot{J}^i_1 &= O(e^{-2}) , \\
\ddot{J}^i_2 &= \epsilon^{ijk} Q^{j a} G_i^{ka} + \epsilon^{ijk} Q^{j a} G_i^{ka} + O(e^{-2}) .
\end{align*} \]

The \( l \geq 2 \) multipole moments are given, in the adiabatic approximation, by the algebraic relations [see Eq. [4]]

\[ \begin{align*}
Q^{ab} &= \lambda_2 G_2^{ab} + \frac{\lambda_{23}}{c^2} J_2 H_{2}^{ab} , \\
Q^{abc} &= \lambda_3 G_2^{abc} + \frac{\lambda_{23}}{c^2} J_2 H_{2}^{abc} , \\
S^{ab} &= \frac{\sigma_2}{c^2} H_{2}^{ab} + \sigma_{23} J_2 \frac{G_2^{ab}}{c^2} , \\
S^{abc} &= \frac{\sigma_2}{c^2} H_{2}^{abc} + \sigma_{23} J_2 \frac{G_2^{abc}}{c^2} .
\end{align*} \]

As discussed in Sec. [11] the orbital equations of motion can be obtained by replacing Eq. [13] in the condition \( \dot{N}_A = 0 \), which is a consequence of the gauge condition \( M_1 = 0 \).

At 0PN (i.e., Newtonian) order, the mass monopoles are conserved and, setting to zero the right-hand side of Eq. [13], we get

\[ \begin{align*}
M_1 G_i^1 &= O(e^{-2}) , \\
M_2 G_i^2 &= \frac{1}{2} Q^{ij} G_i^j + \frac{1}{6} Q^{jka} G_i^{jka} = O(e^{-2}) ,
\end{align*} \]

where \( G_i^A = G_{g,A}^i - \tilde{z}_A^i \) [Eq. [33]], and the global electric tidal moments are [see Eq. [34]]

\[ \begin{align*}
G_i^{Lg,A} &= -\partial_i^{(A)} \phi^{ext}_A + O(e^{-2}) ,
\end{align*} \]

where \( \phi^{ext}_A \) is the Newtonian potential on the body \( A \) due to the other bodies,

\[ \phi^{ext}_A = \sum_{B \neq A} \phi_B^{int} = - \sum_{B \neq A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} M_B^{(A)} \frac{1}{|z_A - z_B|} . \]

With our truncation, \( \phi^{ext}_i = -M_2/r - Q^{ij} \phi_B^{(1)}(1/r) + Q^{ijk} \phi_B^{(1)}(1/r) \), \( \phi_B^{ext} = -M_1/r \), and

\[ \begin{align*}
M_1 \tilde{z}_1 &= -M_1 \phi_B^{(1)}(1) \\
&= \frac{M_1 M_2}{r^2} n^i + \frac{15 M_1 M_5}{2 r^4} Q^{jk \nu(ijk)} \\
&- \frac{35 M_1}{2 r^5} Q^{jkm \nu(ijk)} + O(e^{-2}) ,
\end{align*} \]

while \( M_2 \tilde{z}_2 = -M_2 \tilde{z}_1 + O(e^{-2}) \). In the COM frame, \( z^i = z^i_1 - z^i_2 \) (note that \( z^2 = \eta z^1 \), \( z^1 = \eta z^1 \)).

Before proceeding with the derivation of the equations of motion at 1PN order, we note that the Newtonian equations of motion can also be obtained from a Lagrangian function

\[ \begin{align*}
\mathcal{L} &= \sum_{A=1}^{N} \left( \frac{1}{2} M_A \tilde{z}_A^2 + \frac{1}{2} \sum_{l=0}^{\infty} \frac{1}{l!} M_A^{(l)} \frac{G_{g,A}^i}{c^2} + \mathcal{L}_A^{int} \right) \\
&+ O(e^{-2}) ,
\end{align*} \]

\[ \quad \]
which in our truncation becomes

\[ L = \frac{1}{2} M \dot{z}_{1,2}^2 + \frac{1}{2} M \dot{z}_{2,2}^2 + \frac{M_1 M_2}{r} + \frac{1}{2} Q^{ij} G_{g,2}^{ij} \]

\[ + \frac{1}{6} Q^{ijk} G_{g,2}^{ijk} + L_{2}^{int} + O(c^{-2}) \]

\[ = \frac{\mu z_{2,2}^2}{2} + \mu \frac{M}{r} - U_{Q2} - U_{Q3} + L_{2}^{int} + O(c^{-2}), \quad (52) \]

where [see Eqs. (B1) and (B2)]

\[ U_{Q2} = -\frac{1}{2} Q^{ij} G_{g,2}^{ij} \]

\[ U_{Q3} = -\frac{1}{6} Q^{ijk} G_{g,2}^{ijk} \]

are the quadrupolar and octupolar Newtonian gravitational potential energy, and \( L_{2}^{int} \) describes the internal dynamics and depends on some internal degrees of freedom of body 2, which we call \( q_{2}^{a} \). The internal energy of body 2 is

\[ E_{2}^{int} = q_{2}^{a} \frac{\partial L_{2}^{int}}{\partial \dot{q}_{2}^{a}} - L_{2}^{int}. \quad (55) \]

Remarkably, without any assumption on the dependence of \( L_{2}^{int} \) on the variables \( q_{2}^{a} \), it is possible to derive an equation for the internal energy. Indeed, the Euler-Lagrange equations for the Lagrangian (52) give

\[ \frac{d}{dt} \frac{\partial L_{2}^{int}}{\partial \dot{q}_{2}^{a}} = \frac{\partial L_{2}^{int}}{\partial q_{2}^{a}} + \frac{1}{2} G_{g,2}^{ij} \frac{\partial Q^{ij}}{\partial q_{2}^{a}} + \frac{1}{6} G_{g,2}^{ijk} \frac{\partial Q^{ijk}}{\partial q_{2}^{a}}, \quad (56) \]

and replacing in the time derivative of Eq. (55) yields

\[ \dot{E}_{2}^{int} = \left( \frac{1}{2} G_{g,2}^{ij} \frac{\partial Q^{ij}}{\partial q_{2}^{a}} + \frac{1}{6} G_{g,2}^{ijk} \frac{\partial Q^{ijk}}{\partial q_{2}^{a}} \right) \dot{q}_{2}^{a} + O(c^{-2}) \]

\[ = \frac{1}{2} G_{g,2}^{ij} \dot{q}_{2}^{a} + \frac{1}{6} G_{g,2}^{ijk} \dot{q}_{2}^{a} + O(c^{-2}). \quad (57) \]

Equation (57) represents the energy transferred by the tidal field to body 2 (tidal heating).

At 1PN order, the mass monopole of the body 2 is not conserved anymore. Its evolution equation, Eq. (42), can be written as

\[ \dot{M}_{2} = \frac{1}{c^2} \left( \dot{E}_{2}^{int} + 3 \dot{U}_{Q2} + 4 \dot{U}_{Q3} \right) + O(c^{-4}), \quad (58) \]

where \( \dot{E}_{2}^{int} \) is given in Eq. (57). The above equation provides a way to partition the mass \( M_{2} \),

\[ M_{2} = \frac{n M_{2}}{c^4} \left( E_{2}^{int} + 3 U_{Q2} + 4 U_{Q3} \right) + O(c^{-4}), \quad (59) \]

where \( n M_{2} \) is the conserved Newtonian mass of body 2. As we shall discuss below, this partitioning of \( M_{2} \) is useful to find an action principle for the system.

In order to derive the expressions of \( z_{1}(t) \) and \( z_{2}(t) \) at 1PN order, the right-hand side of Eq. (13) has to be expressed in terms of the body-frame multipole moments up to the same order. To this aim, the expressions of \( G_{l}^{i} \) and \( G_{l}^{F} \) with \( l = 1, \ldots, 4 \) are needed up to 1PN order, while those of \( H_{l}^{ij} \), \( H_{l}^{F} \) with \( l = 2, \ldots, 4 \) are needed up to 0PN order. These expressions are given, for the general case of \( N \) tidally interacting bodies, in Appendix A [Eqs. (A9)–(A15)]. The computation of \( G_{l}^{F} \), \( H_{l}^{F} \) with \( l = 2, 3 \) for the truncated system is explicitly shown in Appendix B (the derivation of the other tidal moments is similar). With our truncation, replacing the evolution equations for masses and spins, Eqs. (42) and (44), we find the orbital equations of motion in the form

\[ M_{1} \dot{z}_{1}^{a} = F_{1}^{i} + F_{1,j}^{i} + F_{1,Q2}, \quad (60) \]

\[ M_{2} \dot{z}_{2}^{a} = F_{2}^{i} + F_{2,j}^{i} + F_{2,Q2}, \quad (61) \]

where the explicit expressions of the terms \( F_{1}^{i} \), \( F_{1,j}^{i} \), \( F_{1,Q2} \), \( F_{2}^{i} \), \( F_{2,j}^{i} \), \( F_{2,Q2} \) are defined in terms of the Newtonian masses \( n M_{A} \), see Sec. II.B:

\[ a^{i} = \dot{z}^{i} - \dot{z}^{i}_{1} - \dot{z}^{i}_{2} = a^{i}_{M} + a^{i}_{Q2} + a^{i}_{Q3} + a^{i}_{S2} + a^{i}_{S3}. \quad (62) \]

The mass and spin contributions are

\[ a^{i}_{M} = -\frac{M}{r^{3}} \eta^{i} - \frac{1}{c^{2} r^{2}} \left( \eta^{i} \left[ (1 + 3 \nu) \dot{r}^{2} - \frac{3 \nu}{2} \dot{r}^{2} - 2(2 + \nu) \frac{M}{r} \right] - 2(2 - \nu) r \dot{v}^{i} \right) + O(c^{-4}), \quad (63) \]

\[ a^{i}_{S} = \frac{\epsilon_{abc} F^{a}_{i} J^{b}_{2}}{c^{2} \eta^{c} r^{3}} \left[ (3 + \eta_{2}) v^{a} \dot{r}^{b} - 3(1 + \eta_{2}) v^{a} \dot{r}^{b} + 6 n^{a} v^{b} \right] \]

\[ + \frac{\epsilon_{abc} F^{a}_{i} J^{b}_{3}}{c^{2} \eta^{c} r^{3}} \left[ (3 + \eta_{1}) v^{a} \dot{r}^{b} - 3(1 + \eta_{1}) v^{a} \dot{r}^{b} + 6 n^{a} v^{b} \right] + O(c^{-4}). \quad (64) \]

The mass quadrupole contribution is
The current quadrupole contribution is
\[
a^i_{Q2} = -\frac{3Q^{ab}}{2r^4} [5n^{abi} - 2n^a \delta^b i] + \frac{1 \{ Q^{ab} \}}{r^4} n^{abi} \left( -\frac{15}{2\eta_2} (1 + 3\nu) v^2 + \frac{105\eta_1 v^2}{4} + \frac{12}{\eta_2} (5 - 2\eta_2^2) M \right) + n^a \delta^b i \left( \frac{3}{\eta_2} (2 + 2\eta_2 - 3\eta_2^2) v^2 - \frac{15\eta_1}{2\eta_2} (2 - \eta_2 - 3\eta_2^2) v^2 - \frac{3}{\eta_2} (8 - \eta_2 - 3\eta_2^2) M \right) + \frac{15}{\eta_2} (2 - \eta_2) \nu n^a v_i
\]
\[
= \frac{3}{2\eta_2} (7 - 2\eta_2 + 3\eta_2^2) n^a v^i - \frac{15\eta_1}{2\eta_2} \nu n^a v_i + \frac{2}{\eta_2} (5 - 4\eta_2 - 2\eta_2^2) \nu v^i
\]
\[
+ \frac{\hat{Q}^{ab}}{r^3} \left( \frac{3}{2\eta_2} (4 - \eta_2) n^a \delta^b i - \frac{15\eta_1}{2\eta_2} \nu n^a v_i + \frac{6}{\eta_2} n^a v_i - \frac{3}{\eta_2} \nu n^a v_i + \frac{3}{\eta_2} (1 - 2\eta_2 - \eta_2^2) \nu n^a v_i \right)
\]
\[
+ \frac{\hat{Q}^{ab}}{r^2} \left[ \frac{3}{4} n^a \delta^b i - \frac{2}{3} n^a \delta^b \right] - \frac{E^{int}}{r^2} n^i \right) \}
\]
\[
+ \frac{3n^i}{c^2 Mn_\eta} \left( \frac{Q^{ab}}{r^5} \left[ \frac{5}{2} n^a \nu(7n^d - v^d) + (\delta^{ad} - 5n^{ad}) v^b + 5\nu \delta^{ad} n^b \right] + \frac{\hat{Q}^{ab}}{r^4} (\delta^{ad} - \frac{5}{2} n^{ad}) n^b \right)
\]
\[
+ \frac{3}{c^2 Mn_\eta} \left( \frac{Q^{ab}}{r^5} \left[ \frac{5}{2} n^a \nu(7n^d - v^d) + (\delta^{ad} - 5n^{ad}) v^b + 5\nu \delta^{ad} n^b \right] + \frac{\hat{Q}^{ab}}{r^4} (\delta^{ad} - \frac{5}{2} n^{ad}) n^b \right)
\]
\[
+ \frac{3c^4 \nu J_{\nu}^a}{c^2 Mn_\nu} \left( \frac{Q^{bd}}{r^5} \left[ n^a (\delta^{ic} v^c - \delta^{iv} v^c) + 5n^{ac} (\delta^{ib} v^c - \delta^{ic} v^b) + 5n^{bc} (\delta^{ia} v^c - \delta^{ic} v^a) + 35n^{abc} (\nu \delta^{ic} n^i) \right)
\]
\[
+ \delta^{ab} (\delta^{ic} v^c - \delta^{ic} v^c) + \delta^{ac} (\delta^{ib} v^c - \delta^{ib} v^c) + 5 (\delta^{ab} n^c + \delta^{ac} n^b) n^i - \frac{3}{\tau^2} \delta^{ic} (5n^{abc} - \delta^{ab} n^c - \delta^{ac} n^b) \right) \}
\]
\[
+ O(c^{-4}).
\]  
(65) 

The mass octupole contribution is
\[
a^i_{Q3} = \frac{5Q^{abc}}{2\eta_2 r^5} (7n^{abc} - 3\delta^{ic} n^{ab}) + O(c^{-2}).
\]  
(66) 

The current quadrupole contribution is
\[
a^i_{S2} = \frac{4e^{bcd}}{c^2 \eta_2} \left[ \frac{S^{ad}}{r^4} \left[ n^a (\delta^{ib} v^c - \delta^{ic} v^b) + n^b (\delta^{ia} v^c - \delta^{ic} v^a) + 5n^{ab} (\nu \delta^{ic} n^i) \right] - \frac{\hat{S}^{ad}}{r^3} \delta^{ic} n^{ab} \right)
\]
\[
+ \frac{\nu i}{c^2 M_\nu} \frac{S^{ab}}{r^5} \left[ 4\delta^{ib} (5n^{ia} - \delta^{ia}) + 10 (\delta^{ia} n^{bc} + \delta^{ib} n^{ac} + \delta^{ic} n^{ab}) - 70n^{iabc} \right] + O(c^{-4}).
\]  
(67) 

Finally, the current octupole contribution is
\[
a^i_{S3} = \frac{15}{c^2 \eta_2} \left[ \frac{S^{bcd}}{r^5} n^{ab} v^c \left( \frac{7}{2} \epsilon^{iabc} n^c - \epsilon^{carn} n^d \right) - (\epsilon^{iad} \delta^{ce} + \epsilon^{icd} \delta^{ae} + \epsilon^{aeb} \delta^{ic}) \left( \frac{\hat{S}^{bcd}}{2r^4} \right) \right]
\]
\[
+ \frac{45\nu^i}{c^2 M_\nu} \frac{S^{abc}}{r^6} \left[ \delta^{cd} (\delta^{ia} n^b + \delta^{ib} n^a - 7n^{ib}) - \frac{7}{3} (\delta^{ia} n^{bcd} + \delta^{ib} n^{acd} + \delta^{ic} n^{abd} + \delta^{id} n^{abc} - 9n^{iabcd}) \right]
\]
\[
+ O(c^{-4}).
\]  
(68) 

In the above equations, all the contributions to the orbital acceleration are given up to 1PN order, with the exception of the mass octupole contribution [65], which is given to 0PN order only. This is sufficient to determine the GW phase up to 6.5PN order.

As we shall discuss in Sec. [1112] in a circular, compact binary system, up to first order in the spins (parallel to the orbital angular momentum), the tidally induced mass and current l-pole moments contribute to the GW waveforms to order (2l + 5/2)PN through the ordinary TLNs and to order (2l + 1/2 + 2\delta_2)PN through the RTLNs, respectively.

**B. Lagrangian**

The orbital equation of motion in the COM frame, 
\[
\ddot{a}^i = a^i_M + a^i_j + a^i_{Q2} + a^i_{Q3} + a^i_{S2} + a^i_{S3} [Eq. (62)],
\] can
be derived from an action principle. One first writes the most general Lagrangian consistent with the truncation and at most linear in the spin, which will depend on a set of free coefficients. Then, applying the Euler-Lagrange equations (Eq. 70, see below) to the Lagrangian, replacing the evolution equations for $J_1$, $J_2$ and $E_2^a$, Eqs. 44 and 57, and comparing with the orbital equations of motion [Eq. 1], it is possible to find the values of the coefficients, which will only depend on the masses of the two bodies. Following this approach we find that the Lagrangian is

$$L_{\text{orb}}(z, v, a) = L_M + L_J + L_{Q2} + L_{Q3} + L_{S2} + L_{S3}. \tag{69}$$

The explicit expressions of the different terms in Eq. 69 are given in Sec. 1D in Eqs. 70–12. Note that Eq. 69 is a generalized Lagrangian, since it depends on the (relative) acceleration $a'$, together with the (relative) position and velocity; the action is stationary if the generalized Euler-Lagrange equations are satisfied,

$$\left( \frac{\partial}{\partial z^2} - \frac{d}{dt} \frac{\partial}{\partial v^i} + \frac{d^2}{dt^2} \frac{\partial}{\partial a^j} \right) L_{\text{orb}} = 0. \tag{70}$$

As discussed in 58, a generalized Lagrangian is needed in order to obtain the spin contribution of the orbital equation of motion, $a'_j$, from an action principle. We remark that the mass monopole contribution to the acceleration $a'_M$, Eq. 63, is only due to the monopole term of the Lagrangian, $L_M$. The spin contribution to the acceleration $a'_J$, Eq. 64, is only due to the spin term of the Lagrangian, $L_J$. The mass quadrupole contribution to the acceleration $a'_{Q2}$, Eq. 65, arises from terms in $L_M$, $L_J$ and $L_{Q2}$. Finally, the current quadrupole and the (mass and current) octupole contributions to the acceleration $a'_{Q2}$, $a'_{Q3}$, $a'_{S2}$, Eqs. 60–65, arise from $L_{S2}$, $L_{Q3}$ and $L_{S3}$, respectively.

It is possible to extend the Lagrangian $L_{\text{orb}}$ in order to also describe the adiabatic evolution of the mass and current quadrupole and octupole moment ($Q^{ij}, S^{ij}, Q^{ijk}, S^{ijk}$), i.e., to enforce the adiabatic relations in Eq. 44. In this derivation, we shall use the explicit expressions of the $l = 2, 3$ tidal moments of body 2, which have been derived in Appendix 13 [see Eqs. 66–110].

At 0PN order, the mass multipole contributions to the Lagrangian are

$$L_{Q2} = \frac{1}{2} G_2^{ab} Q^{ab} + O(c^{-2}), \tag{71}$$
$$L_{Q3} = \frac{1}{6} G_2^{abc} Q^{abc} + O(c^{-2}), \tag{72}$$

while $L_{S2} \sim L_{S3} \sim O(c^{-2})$.\footnote{For a generic mass multipole moment of order $l$, the contribution to the Newtonian Lagrangian is $L_{Ql} = \frac{1}{l!} G_2^{abc} Q^{abc} + O(c^{-2})$. In the case of current multipole moments, the structure is akin to the Newtonian one, but at order 1PN, $L_{Ql} = \frac{1}{l!} \frac{1}{2} \frac{1}{2} H_2^{abc} S^{abc} + O(c^{-2})$.}

Up to 1PN order, the mass quadrupole contribution $L_{Q2}$ can be written as

$$L_{Q2} = U^{ab} Q^{ab} + V^{ab} \dot{Q}^{ab} + W E_2^{int} + O(c^{-4}), \tag{73}$$

where $U^{ab}(z, v)$, $V^{ab}(z, v)$, $W(z, v)$ are the coefficients appearing in Eq. 49, i.e.,

$$U^{ab} = \frac{3 \eta_1 M}{2 r^3} n^{ab} + \frac{1}{c^2} M \left[ n^{ab} \left( \frac{3 \eta_1}{4} (3 + \nu) v^2 \right) + \frac{15 \nu \eta_1}{r^2} - \frac{3 \eta_1}{2} (1 + 3 \eta_1) \frac{M}{r} \right] + \frac{3 \eta_1^2}{2} v^{ab} - \frac{3 \eta_1^2}{2} (3 + \eta_2) \hat{r} n^{a} v^{b} \right], \tag{74}$$
$$V^{ab} = \frac{1}{c^2 r^2} \eta_1 \left( 5 n^{abcd} - \delta^{ab} n^{c} - \delta^{cd} n^{b} \right) v^c, \tag{75}$$
$$W = \frac{1}{c^2} \left[ \frac{\eta_1^2}{2} v^2 + \eta_1 M \frac{1}{r} \right]. \tag{76}$$

As discussed in Sec. III A we do not explicitly compute the $O(c^{-2})$ corrections in the mass octupole contribution because they do not affect the tidal contribution to the GW phase to 6.5PN order. The current quadrupole and octupole contributions to the Lagrangian, up to 1PN order, can be written as:

$$L_{S2} = \frac{1}{3 c^2} H_2^{ab} S^{ab} + O(c^{-4}), \tag{77}$$
$$L_{S3} = \frac{1}{8 c^2} H_2^{abc} S^{abc} + O(c^{-4}). \tag{78}$$

Comparing Eqs. 74–76 with the expression of $G_2^{ab}$ at 1PN order [Eq. 110], we find that

$$G_2^{ab} = 2 (1 + W) (U^{ab} - \dot{V}^{ab}) + O(c^{-4}), \tag{79}$$

an expression which will be useful below.

Let us now define the Lagrangian

$$L(z, v, a, Q^L, \dot{Q}^L, S^L) = L_{\text{orb}}(z, v, a, Q^L, \dot{Q}^L, S^L) + L_{\text{int}}^2(Q^L, S^L). \tag{80}$$

The internal Lagrangian $L_{\text{int}}^2$ only depends on the internal degrees of freedom $Q^{ab}, Q^{abc}, S^{ab}, S^{abc}$, and describes the internal dynamics, while the orbital Lagrangian depends both on the orbital degrees of freedom and on the momenta $Q^L$, $S^L$. Note that $L_{Q2}$ also depends on
\[ E_2^{\text{int}} / c^2 \] see Eqs. \([73], (70)\); in order to write the Euler-Lagrange equations, we need to know the explicit form of \( E_2^{\text{int}}(Q^L, S^L) \), at 0PN order. To this aim, note that, replacing the adiabatic relations \[^{14}\] in Eq. \((44)\) we find, at leading order,

\[
\dot{E}_2^{\text{int}} = \frac{1}{2} G_2^{ab} \dot{Q}^{ab} + \frac{1}{6} G_2^{abc} \dot{Q}^{abc} + O(c^{-2})
\]

\[
= \frac{1}{4\lambda_2} \frac{d}{dt} (Q^{ab} \dot{Q}^{ab}) + \frac{1}{12\lambda_3} \frac{d}{dt} (Q^{abc} \dot{Q}^{abc}) + O(c^{-2}).
\]  

(81)

Therefore, the internal energy (at Newtonian level) has the form

\[
E_2^{\text{int}} = \frac{1}{4\lambda_2} Q^{ab} \dot{Q}^{ab} + \frac{1}{12\lambda_3} Q^{abc} \dot{Q}^{abc} + O(c^{-2}).
\]  

(82)

The corresponding internal Lagrangian is \( L_2^{\text{int}} = -E_2^{\text{int}} \), and the Euler-Lagrange equations

\[
\left( \frac{\partial}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial}{\partial q} \right) \mathcal{L} = 0
\]  

(83)

(where \( q = \{ Q^{ab}, Q^{abc}, S^{ab}, S^{abc} \} \)) give the Newtonian adiabatic relations, \( Q^{ab} = \lambda_2 G_2^{ab} + O(c^{-2}) \), \( Q^{abc} = \lambda_3 G_2^{abc} + O(c^{-2}) \).

At 1PN order we cannot use Eq. \((83)\) to derive the expression of the internal energy, because that equation is only given up to Newtonian order. We instead look for an expression which reduces to \((82)\) at 0PN order, and which yields the correct adiabatic relations \[^{14}\] at 1PN order. We find that \[^{14}\]

\[
L_2^{\text{int}} = -E_2^{\text{int}} = -\frac{1}{4\lambda_2} Q^{ab} \dot{Q}^{ab} - \frac{1}{12\lambda_3} Q^{abc} \dot{Q}^{abc}
\]

\[
- \frac{1}{6\lambda_2} S^{ab} \dot{S}^{ab} - \frac{1}{12\lambda_3} S^{abc} \dot{S}^{abc}
\]

\[
+ \alpha J_2^{ab} Q^{ab} S^{abc} + \beta J_2^{abc} S^{abc} Q^{abc},
\]  

(84)

the Euler-Lagrange equations \[^{33}\] give

\[
Q^{ab} = \lambda_2 G_2^{ab} + 2\lambda_2 \alpha J_2^{ab} S^{abc} + O(c^{-4}),
\]

\[
Q^{abc} = \lambda_3 \left( G_2^{abc} + O(c^{-2}) \right) + 6\lambda_3 \beta J_2^{(c) ab} S^{abc} + O(c^{-4}),
\]

\[
S^{ab} = \frac{\sigma_2}{c^2} H_2^{ab} + 3\sigma_2 \beta J_2^{ab} Q^{abc} + O(c^{-4}),
\]

\[
S^{abc} = \frac{\sigma_3}{c^2} H_2^{abc} + 8\sigma_3 \beta J_2^{(c) ab} Q^{abc} + O(c^{-4}),
\]  

(85)

where \( G_2^{ab} \) in the above expression is precisely given by Eq. \((79)\) to this PN order.

In order to simplify the above expressions, it is useful to substitute the adiabatic expressions for \( Q^L \) and \( S^L \) to lowest order in the spin, and truncate the result to linear order in the spin. This is clearly consistent with our perturbative scheme which neglects quadratic and higher-order spin terms. This substitution yields

\[
Q^{ab} = \lambda_2 G_2^{ab} + \frac{2\lambda_2 \sigma_3 \alpha}{c^2} J_2^{ab} H_2^{abc} + O(c^{-4}),
\]

\[
Q^{abc} = \lambda_3 \left( G_2^{abc} + O(c^{-2}) \right) + \frac{6\lambda_3 \sigma_2 \beta}{c^2} J_2^{(c) ab} H_2^{abc} + O(c^{-4}),
\]

\[
S^{ab} = \frac{\sigma_2}{c^2} H_2^{ab} + 3\lambda_2 \sigma_2 \beta J_2^{ab} G_2^{abc} + O(c^{-4}),
\]

\[
S^{abc} = \frac{\sigma_3}{c^2} H_2^{abc} + 8\lambda_2 \sigma_3 \beta J_2^{(c) ab} C^{ab} + O(c^{-4}),
\]  

(86)

which coincide with the adiabatic relations \[^{14}\] [with the replacement \([16]\)] at 1PN order, to leading order in the tidal moments, and to linear order in the spin. We remark that Eq. \((84)\) has been obtained under the assumption that the multipole moments are tidally induced and neglecting the contributions quadratic in the spin; therefore, in the Newtonian limit Eq. \((83)\) reduces to Eq. \((82)\).

Finally, we note that, replacing the adiabatic relations Eqs. \([14]\) in Eq. \((44)\), it follows that the spins of the two bodies are constant.

### C. Gravitational waveform

In order to derive the gravitational waveform, we need the radius-frequency relation (to 1PN order) for a circular binary. In this case, \( n^i = (\cos(\omega t), \sin(\omega t), 0) \), \( v^i = r \omega \phi^i \) where \( \phi^i = (\sin(\omega t), \cos(\omega t), 0) \). We assume the spins of the two bodies to be parallel to the orbital angular momentum. Therefore, we can write \( J_A^i = J_A s^i \) \((A = 1, 2)\), where \( s^i = e^{ijk} \phi^j \phi^k = (0, 0, 1) \), and define the dimensionless spin variables \( \chi_A = cJ_A / (\eta_A M)^2 \). Replacing these expressions in the orbital equations of motion, Eq. \((50)\), using the Newtonian orbital acceleration Eq. \((50)\) to simplify the expressions of higher PN order, and imposing the adiabatic relations \[^{14}\], we find

\[ n^i \]

Note that this is the most general Lagrangian function, which can be built from the multipole moments of our truncation, and which is at most quadratic in the internal degrees of freedom and linear in the spin.
\[ r = M^{1/3} \left\{ \frac{1}{\omega^{2/3}} \left[ x + \left( \frac{\eta_1 - 3\eta_2}{3} \chi_1 + \frac{(\eta_2 - 3)\eta_2}{3} \chi_2 \right) x^{1.5} + O(x^2) + \frac{3\eta_1}{\eta_2} \frac{c^{10}}{M^5} \lambda_2 x^5 \right] + \left[ \frac{-\eta_1}{2\eta_2} (6 - 2\eta_2 + \eta_2^2) \frac{c^{10}}{M^5} \lambda_2 + \frac{16\eta_1}{\eta_2^2} \frac{c^8}{M^5} \sigma_2 \right] x^6 + \left( \frac{2\eta_1 (9 - 2\eta_2) \chi_2 - \frac{4\eta_1^3}{\eta_2} \chi_1}{\eta_2^2} \right) \frac{c^{10}}{M^5} \lambda_2 - \frac{48\eta_1^2}{\eta_2} \frac{c^8}{M^5} \sigma_2 \right] \right\} x^6.5 + O(x^7), \] (87)

where the first line (up to order \( O(x^2) \)) refers to the point-particle terms, and the others refer to the tidal terms to linear order in the spin. We recall that \( x = (\omega M)^{2/3}/c^2 = v^2/c^2 + O(c^{-4}) \).

Replacing the adiabatic relations (84) in the Lagrangian (63) yields the reduced Lagrangian

\[ \mathcal{L}(z, v, a) = \mu \frac{v^2}{2} + \frac{\mu M}{r} \left( 1 + \frac{3\eta_1 \lambda_2}{2\eta_2 r^3} + \frac{15\eta_1 \lambda_3}{2\eta_2 r^7} \right) + \mu \frac{c^2}{r^3} \left\{ \frac{1 - 3\nu}{8} v^4 + \frac{M}{r} \left[ v^2 \left( \frac{3 + \nu}{2} + \frac{3\eta_1^2 (5 + \eta_2) \lambda_2}{4\eta_2} \right) \right] \right\} \]

\[ + \frac{\epsilon^{abc}}{c^2} v^b \left[ \frac{2\eta_1 J_1^a + \eta_1 J_2^a}{\eta_2} \frac{M}{r^7} \right] + \frac{\lambda_2}{c^2} \frac{9\eta_1 M}{r^7} \epsilon^{abc} n^a J_1^b v^c \]

\[ + \frac{\sigma_2}{c^4} \left( \frac{12\eta_1^2 M^2}{r^6} \left( v^2 - \dot{v}^2 \right) + \frac{24\eta_1 M}{r^7} \epsilon^{abc} n^a \dot{J}_1^b \dot{v}^c \right) + \frac{\sigma_3}{c^4} \left( \frac{60\eta_1^2 M^2}{r^8} \left( v^2 - \dot{v}^2 \right) + \frac{180\eta_1 M}{r^9} \epsilon^{abc} n^a \dot{J}_1^b \dot{v}^c \right) \]

\[ + \frac{\eta_1^2 M^2}{c^4 r^7} (48\lambda_2 \sigma_3 \alpha - 36 \lambda_3 \sigma_2 \beta) \epsilon^{abc} n^a \dot{J}_1^b \dot{v}^c. \] (88)

Note that the contributions from the RTLNs in the above equation only enter through the terms proportional to \( \alpha \) and \( \beta \). However, these terms are linear in the velocity, and therefore they do not contribute to the conserved energy below and to the GW waveform. As we show below, the RTLNs enter in the GW waveform through the radius-frequency relation and through the GW flux.

From the above reduced Lagrangian, the conserved energy of our truncation then reads (63)

\[ E = v^i \left( \frac{\partial \mathcal{L}}{\partial \dot{\omega}^i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \omega^i} \right) + a^i \frac{\partial \mathcal{L}}{\partial a^i} - \mathcal{L} \]

\[ = \mu \frac{v^2}{2} - \frac{\mu M}{r} \left( 1 + \frac{3\eta_1 \lambda_2}{2\eta_2 r^3} + \frac{15\eta_1 \lambda_3}{2\eta_2 r^7} \right) + \mu \frac{c^2}{r^3} \left\{ \frac{1 - 3\nu}{8} v^4 + \frac{M}{r} \left[ v^2 \left( \frac{3 + \nu}{2} + \frac{3\eta_1^2 (5 + \eta_2) \lambda_2}{4\eta_2} \right) \right] \right\} \]

\[ + \frac{\epsilon^{abc}}{c^2} v^b \left[ \frac{2\eta_1 J_1^a + \eta_1 J_2^a}{\eta_2} \frac{M}{r^7} \right] + \frac{\lambda_2}{c^2} \frac{9\eta_1 M}{r^7} \epsilon^{abc} n^a J_1^b v^c \]

\[ + \frac{\sigma_2}{c^4} \left( \frac{12\eta_1^2 M^2}{r^6} \left( v^2 - \dot{v}^2 \right) + \frac{24\eta_1 M}{r^7} \epsilon^{abc} n^a \dot{J}_1^b \dot{v}^c \right) + \frac{\sigma_3}{c^4} \left( \frac{60\eta_1^2 M^2}{r^8} \left( v^2 - \dot{v}^2 \right) + \frac{180\eta_1 M}{r^9} \epsilon^{abc} n^a \dot{J}_1^b \dot{v}^c \right) \]

\[ + \frac{\eta_1^2 M^2}{c^4 r^7} (48\lambda_2 \sigma_3 \alpha - 36 \lambda_3 \sigma_2 \beta) \epsilon^{abc} n^a \dot{J}_1^b \dot{v}^c. \] (89)

For a circular orbit, replacing the radius-frequency relation (87), the conserved energy can be written as

\[ E = -\frac{\mu}{M} (M\omega)^{2/3} \left\{ \left[ 1 - \frac{9 + \nu}{12} x + \frac{2\eta_1 (\eta_2 + 3)}{3} \chi_2 + \frac{2\eta_1 (\eta_1 + 3)}{3} \chi_1 \right] x^{1.5} + O(x^2) - \frac{9\eta_1}{\eta_2} \frac{c^{10}}{M^5} \lambda_2 x^5 \right\} \]

\[ - \left[ \frac{11\eta_1}{2\eta_2} (3 + 2\eta_2 + 3\eta_2^2) \frac{c^{10}}{M^5} \lambda_2 + \frac{88\eta_1}{\eta_2^2} \frac{c^8}{M^5} \sigma_2 \right] x^6 + \left\{ \left[ \frac{24\eta_1 (\eta_2 - 3) \chi_2 + \frac{24\eta_1^3}{\eta_2} \chi_1}{\eta_2^2} \right] \frac{c^{10}}{M^5} \lambda_2 + \frac{192\eta_1^2}{\eta_2} \frac{c^8}{M^5} \sigma_2 \right\} x^{6.5} + O(x^7) \right\}. \] (90)

Note that in this equation the RTLNs (\( \lambda_2, \lambda_3, \sigma_2, \) and \( \sigma_3 \)) appear explicitly, since the adiabatic relations have been used to obtain Eq. (87).

The GW flux (at 1.5PN) is given in Eq. (38), whereas the multipole moments of system are given by Eqs. (30)-(32).
Within our truncation and for a circular orbit they read\textsuperscript{15}

\[
M_{ij}^{sys} = Q^{ij} + \mu v^2 n^{(ij)} + \frac{1}{c^2} \left\{ \mu v^2 \left[ \left( \frac{29(1 - 3\nu)v^2}{42} + \frac{(8\nu - 5)M}{7r} \right) n^{(ij)} + \left( \frac{11(1 - 3\nu)}{21} \right) v^{(ij)} \right] \right. \\
+ \frac{4\nu}{3} \left( 2v^a n^{(i)} - n^a v^{(i)} \right) e^{iab} (\eta_2 J_2^b + \eta_2^2 J_2^b) + \left( E_{nt}^{mt} + 3U_{Q2} \right) \eta_{i}^{2} x^{(ij)} - \eta_2 M_{2r} \left( 2(46\nu^2 + 109\eta_1 \eta_2 + 63\nu^2) Q^{ij} \right. \\
- 3(52\eta_1^2 + 4\eta_1 \eta_2 - 25\nu_2^2) n^{(ij)} a^{ab} - 6(15\nu_1^2 + 21\eta_1 \eta_2 + 112\nu_2^2) n^{(ij)} a^{ab} \left( Q^{ij} \right) \left. \right. \\
+ \frac{2\nu^2 r}{21} \left( n^{(i)} \dot{Q}^{j} a v^a + 8v^{(i)} \dot{Q}^{j} a n^a \right) + \frac{\nu^2 r^2}{42} \left( 11Q^{ij} - 12n^{a(i} \dot{Q}^{j)a} \right) \left. \right. \\
+ \frac{8\eta_1}{9} \left( 2c^{ab(i} S^{j)} b v^a - r c^{ab(i} \dot{S}^{j)b} n^a \right) \left. \right. \\
+ O(c^{-4}) \\
\left. \right. \\
= \mu v^3 (n_1 - n_2) n^{(ij)} + 3\eta_1 r Q^{(ij)k} + O(c^{-2}) \quad (91)
\]

\[
J_{ij}^{sys} = S^{ij} + \mu v^2 (n_1 - n_2) e^{ab(i} n^{j) b} v^a + \frac{3r}{2} \left( \eta_1 j_2^{(i} - n_2 j_2^{(i} \right) n^j) + \frac{\eta_1}{2} \left( -2c^{ab(i} Q^{j) b} v^a + r c^{ab(i} \dot{Q}^{j)b} n^a \right) + O(c^{-2}). \quad (93)
\]

The tail term appearing in Eq. (88) is given at the leading-order by\textsuperscript{18} \textsuperscript{63}

\[
U_{ij}^{tail}(U) = 2M \int_{0}^{\infty} \dot{M}_{sys}(U - \tau) \left[ \log \left( \frac{c\tau}{2r_0} \right) + \frac{11}{12} \right] d\tau, \quad (94)
\]

where \( U = t - r/c - \frac{(2M/c^3) \log(r/r_0)}{2M/c^3} \) is the retarded time in radiative coordinates and \( r_0 \) a gauge-dependent arbitrary constant due to the freedom of choice of the radiative coordinates themselves. The final result in the GW flux is independent of \( r_0 \)\textsuperscript{18} \textsuperscript{21}. Replacing Eqs. (91) - (93) into Eq. (88), and using the adiabatic relations \textsuperscript{11} and the radius-frequency relation \textsuperscript{84}, the energy loss by GW emission can be written as

\[
\dot{E} = - \frac{32}{5} c^5 x^5 \left\{ 1 - \frac{1247}{336} + \frac{35}{12} \right\} x + \frac{\nu_2}{4} \left( \frac{\nu_2}{\nu_2} \right) \left[ 4 \eta_2 (5 + 6\eta_2) \right] \chi_2 - \frac{\eta_1 (5 + 6\eta_2)}{4} \chi_2 \right] x^{1.5} + O(x^2) \\
+ \frac{6(3 - 2\eta_2)}{\eta_2} c^{10} M_5 \chi_2 x^5 + \left[ \frac{(-704 - 1803\eta_2 + 4501\eta_2^2 - 2170\eta_2^3)}{28\eta_2} c^{10} \right] M^5 \chi_2 + \frac{2(113 - 114\eta_2)}{3\eta_2} c^8 M^5 \sigma_2 \right] x^6 \\
+ \left[ \left( \frac{24\pi(3 - 2\eta_2)}{\eta_2} \right) (667 - 939\eta_2 + 304\eta_2^2) \chi_2 + \frac{(-395 + 1110\eta_2 - 1019\eta_2^2 + 304\eta_2^3)}{8\eta_2} \right] c^{10} M^5 \chi_2 \\
+ \left[ \chi_2 + \frac{(-613 + 1225\eta_2 - 612\eta_2^2)}{3\eta_2} \right] c^8 M^5 \sigma_2 \\
+ \frac{c^{10}}{M^4} \left( \eta_2 (-17 + 2\eta_2) \lambda_{23} + 32\nu_2 \lambda_{32} + \frac{\eta_2 (113 - 114\eta_2)}{3} \sigma_{23} - 24\nu \sigma_{32} \right) \right] x^{6.5} + O(x^7). \quad (95)
\]

\textsuperscript{15} We recall that to get the GW phase up to 6.5PN order we need to include the mass octupole moment \( Q^{ij,k} \) of the body 2 only at the leading order. Since this term enters to the GW flux at the next-to-leading order, we can safely neglect its contribution to the system multipole moments.
Finally, Eq. (11) gives the phase of gravitational waveform:

\[
\psi(x) = \frac{3}{128\nu x^{7/2}} \left\{ 1 + \left( \frac{3715}{756} + \frac{55}{9} \nu \right) x + \left( \frac{113}{3} (\eta_1 \chi_1 + \eta_2 \chi_2) - \frac{38}{3} \nu (\chi_1 + \chi_2) - 16 \pi \right) x^{1.5} + O(x^2) \right. \\
+ \left( 264 - \frac{288}{\eta_2} \right) \frac{c^{10} \lambda_2}{M^5} x^5 + \left[ \frac{4595}{28} - \frac{15895}{28 \eta_2} + \frac{5715 \eta_2}{14} - \frac{325 \eta_2^2}{7} \right] \frac{c^{10} \lambda_2}{M^5} + \left( \frac{6920}{7} - \frac{20740}{21 \eta_2} \right) \frac{c^8 \sigma_2}{M^5} x^6 \\
+ \left\{ \left[ \frac{593}{4} - \frac{1105}{8 \eta_2} + \frac{567 \eta_2}{8} - \frac{81 \eta_2^2}{2} \right] \lambda_1 + \left[ \frac{6607}{8} + \frac{6639 \eta_2}{8} - \frac{81 \eta_2^2}{2} \right] \lambda_2 - \pi \left( 264 - \frac{288}{\eta_2} \right) \frac{c^{10} \lambda_2}{M^5} \right. \\
+ \left\{ \left[ -\frac{9865}{3} + \frac{4933}{3 \eta_2} + \frac{1644 \eta_2}{3} \right] \chi_1 - \frac{c^8 \sigma_2}{M^4} + \frac{c^{10} \lambda_2}{M^4} \right. \\
- \nu \left( 272 \chi_{32} - 204 \sigma_{32} \right) \right\} x^{6.5} + O(x^7) \right\}. 
\]

(96)

As previously explained, to obtain the full GW phase up to octupole mass and current moments for both bodies, it is sufficient to add to Eq. (96) the same expression obtained by exchanging the indices 1 and 2 of the two bodies. The result is given in Eq. (15).

D. PN order counting of the spin-tidal terms

As shown in Sec. III C the spin-tidal couplings computed above modify the GW phase at 6.5PN order, i.e., 1.5PN order after the leading-order (electric, quadrupolar) tidal deformability term, and 0.5PN order before the standard, electric, octupolar tidal deformability term. It is interesting to generalize this counting to multipole moments and tidal moments of generic harmonic index \( l \).

Let us start by considering the contribution from RTLNs. First of all, we notice that \( Q^L \) (respectively, \( S^L \)) enters the waveform at IPN order (respectively, \( (l + 1/2) \) PN order [19, 52, 53]. Indeed, the contribution of \( Q^L \) to the radial acceleration in the binary system is of the order \( \lambda_2 \).

\[
|a'| \sim \frac{Q^L}{r^{l+2}},
\]

(97)
to be compared to the Newtonian term \( |a'| \sim M/r^2 \). On the other hand, the contribution of \( S^L \) is suppressed\(^{16}\) by an extra power of \( v/c \). Furthermore, according to the selection rules discussed in Ref. [42], \( Q^L \) (respectively, \( S^L \)) is induced by \( H^{L, \pm 1} \) (respectively, \( G^{L, \pm 1} \)) at linear order in the spin. Since \( H^{L, \pm 1} \sim v G^{L, \pm 1} \sim v/r^{l+1 \pm 1} \sim (l + 3/2 \pm 1) \) PN, we obtain that the PN order of the
corrections proportional to the spin and to the RTLNs is

\[
\text{PN order}_\text{RTLNs} = l + \left( l + \frac{3}{2} \pm 1 \right) = 2l + \frac{3}{2} \pm 1,
\]

(98)

where the upper and lower signs refer to the coupling between an \( l \)-pole moment and the tidal moment with \( l + 1 \) and \( l - 1 \), respectively. This result is interesting for the following reasons:

(i) When \( l \geq 3 \), the lower sign clearly provides the lowest PN correction, namely \( (2l + 1/2) \) PN. For example, the coupling between \( l = 4 \) multipole moments with octupolar tidal moments would give rise to 8.5PN terms, whereas for \( l = 3 \) we obtain the 6.5PN correction computed in the previous sections.

(ii) On the other hand, for \( l = 2 \) the absence of any dipolar tidal moment that could potentially induce a quadrupole moment imposes to use the upper sign in the above equation. This gives again a 6.5PN term, consistent with our analysis.

(iii) When compared to the PN order of the usual TLNs in the nonspinning case (namely \( (2l + 1) \) PN and \( (2l + 2) \) PN for electric and magnetic TLNs of order \( l \), respectively, see footnote 10), it is clear that the contribution in Eq. (96) with the lower sign enters at lower PN order than the usual TLNs in the nonspinning case for any \( l \geq 3 \). Indeed, for any \( l \geq 3 \), it enters at 0.5PN (1.5PN) before the electric (magnetic) TLN of order \( l \).

(iv) For both signs in Eq. (96), the PN order of RTLNs is the average between the PN order of an ordinary tidal term of order \( l \) and the tidal term of opposite parity and with \( l \pm 1 \). This is reminiscent of the selection rules discussed in Ref. [42].

Let us now focus on the spin-tidal corrections coming from the ordinary TLNs. Their PN order can be computed again by noticing that \( Q^L \) (respectively, \( S^L \))
enters the waveform at IPN (respectively, \((l+1/2)\)PN) order, as discussed above. On the other hand, the leading-order spin terms in \(G^L\) and \(H^L\) enter, respectively, at \((l+1+3/2)\)PN and at \((l+3/2+1/2)\)PN order. Therefore, the overall, leading-order, spin-tidal contributions of the \(\sim Q^L G^L\) and \(\sim S^L H^L\) couplings both enter at \((2l+5/2)\)PN order.

To summarize, the leading-order, spin-tidal corrections coming from the excitation of \(l\)-pole moments at linear order in the spin read

\[
\begin{align*}
\text{PN order}_{\text{spin-TLNs}} &= 2l + \frac{5}{2}, \\
\text{PN order}_{\text{RTLNs}} &= 2l + \frac{1}{2} + 2\delta_2, \\
\end{align*}
\]

(99)

where the first and second line refer to terms proportional to the ordinary TLNs and to the RTLNs, respectively. Interestingly, the PN orders of the two contributions coincide only when \(l = 2\), yielding the 6.5PN terms discussed in this work [the terms \(\lambda, \Sigma, \text{and } \tilde{\Gamma}\) in Eq. (15)]. For \(l \geq 3\), the contribution from the RTLNs is always dominant.

### E. Are Lagrangian formulation and perturbation theory compatible?

In Ref. \[42\], four RTLNs were introduced to describe (at linear order in the spin) the coupling between \(l = 2, 3\) multipole moments of a spinning object with \(l = 2, 3\) tidal moments. According to the selection rules described in Ref. \[42\], \(\lambda_{23}\) describes how a mass quadrupole moment is induced by an octupolar magnetic tidal moment at linear order in the spin, whereas \(\sigma_{32}\) describes how a current octupole moment is induced by a quadrupolar tidal moment. A similar argument applies to \(\lambda_{32}\) and \(\sigma_{23}\).

However, as previously discussed, our interaction Lagrangian \[13\] contains only two coupling terms proportional to the spin and which are responsible for the coupling between multipole moments and tidal moments with opposite parity and \(l \leftrightarrow l \pm 1\). In other words, a Lagrangian formulation seems to predict two RTLNs, rather than the four RTLNs that have been explicitly computed in Ref. \[42\].

One might be tempted to think that a relation exists between \(\lambda_{23}\) and \(\sigma_{32}\) (and between \(\lambda_{32}\) and \(\sigma_{23}\)) so that, once the four RTLNs are explicitly computed, they would satisfy the relations \[10\]. Unfortunately, we have checked if this is the case by explicitly computing the RTLNs for neutron stars using perturbation theory as discussed in Ref. \[42\], and found no numerical evidence for a relation between those RTLNs. In fact, we believe that such putative relation can hardly emerge from the perturbed Einstein equations, since electric-led and magnetic-led RTLNs belong to different sectors, namely to Zerilli and Regge-Wheeler perturbations, respectively. While it is true that the two sectors enjoy some special symmetries in the case of Schwarzschild black holes \[66\], such symmetries are broken for material bodies and we do not see any reason why the corresponding RTLNs should be related by (truly) universal relations which should be completely independent of the body composition.\[17\]

On the other hand, the fact that in the approach presented here only two RTLNs are independent seems intrinsically related with the Lagrangian formulation, which clearly introduces the same coupling constant in two different Euler-Lagrange equations. To better illustrate this point, let us make a specific example. The coupling

\[
\mathcal{L}^{\text{int}}_2 \supset \alpha J^a_2 Q^{bc} S_{abc}
\]

(100)

contributes to the Euler-Lagrange equations for both \(Q^{ab}\) and \(S^{abc}\). In the former case, it gives a term \(\sim \alpha J^a_2 S_{abc}\), whereas in the latter case it gives \(\sim \alpha J^2_2 Q^{bc} \). In both cases, the terms depend on the same coupling factor, \(\alpha\).

Unfortunately, at the moment we are not able to explain this apparent inconsistency. One option could be that a Lagrangian formulation fails to reproduce the full couplings that arise in perturbation theory; however, we consider this option as unlikely. Other possible explanations could come from a nontrivial static limit of the dynamical action describing the time evolution of the induced multipole moments \[54\], or by the role of the internal fluid dynamics, or finally by some hidden symmetry of the perturbation equations that effectively reduces the number of independent RTLNs to two. We plan to investigate this issue elsewhere. We stress, however, that the expression for the GW phase in Eq. \[13\] can also accommodate putative relations among the RTLNs.

### IV. DISCUSSION AND OUTLOOK

We have computed, for the first time, the spin-tidal couplings that modify the dynamics of two orbiting bodies in general relativity at the leading PN order and at linear order in the spin. These corrections depend on both the standard TLNs and on the RTLNs recently introduced in previous work. Our main result is Eq. \[15\], which provides the new spin-tidal terms for the GW phase of circular binaries with spins orthogonal to the orbital plane. All these new terms modify the phase at 1.5PN order relative to the standard, quadrupolar, tidal deformability term at the leading order. At linear order in the spin, the terms computed here should include all the tidal terms up to 6.5PN order. The new terms computed here enter the GW phase at a lower order relative

\[17\] We stress that, although there is some tension between some of the RTLNs computed in Ref. \[12\] and those computed by other groups \[19\] , the fact that the electric-led and magnetic-led RTLNs are independent should not be affected by such discrepancy.
to the standard, octupolar tidal terms. We proved that this is the case for any RTLN with $l \geq 3$.

We have encountered a conceptual problem related to the inclusion of the RTLN in the Lagrangian formulation. We hope that our results will motivate more work which may shed light on this issue.

An analysis of the impact of spin-tidal couplings in the parameter estimation of binary NSs is ongoing and will appear in a follow-up paper [52].

Another application of our results is related to GW searches for exotic compact objects [23, 24]. Since the TLNs of a black hole are zero [23, 24], measuring the effect of the tidal deformability in the waveform of a binary coalescence provides an independent way to distinguish black holes from other exotic compact alternatives [30–32, 35]. There is no reason to expect that black-hole mimickers should be slowly spinning (this is particularly true for supermassive objects in the LISA band, whose spin might grow through accretion). Thus, the inclusion of the spin-tidal couplings computed here will greatly improve previous analysis [32].

Note added. – After completion of this work, we had been informed of a related work by Landry [67]. Beside the different notation, our work differs from Ref. [67] because it includes also the spin-tidal terms proportional to the ordinary TLNs. While our result for the energy flux agrees with that of Ref. [67] in the appropriate particular case, our result for the GW phase does not agree with that derived in Ref. [67]. We believe that the source of discrepancy is a different definition of the energy of the binary system. We also note that our results for both the energy flux and the GW phase agree with those of Refs. [25, 26] when neglecting spin effects.

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Appendix A: Multipole moments transformation between body frame and global frame

The global multipole moments of a system of $N$ structured bodies, $M^L_{g,A}$, $Z_i^L_{g,A}$, can be expressed in terms of the body-frame mass and current multipole moments $M^L_A$, $J^L_A$ (up to 1PN order for the mass moments, 0PN order for the current moments) as [67, 68]:

$$M^L_{g,A} = M^L_A + c^{-2} \left[ \frac{3}{2} (z_A^2 - (l + 1)G_{g,A}) M^L_A - \frac{2l^2 + 5l - 5}{(l + 1)(2l + 3)} v^j_A M^jL_A - \frac{2l^3 + 7l^2 + 16l + 7}{(l + 1)(2l + 3)} a^j_A M^jL_A - \frac{2l^2 + 17l - 8}{2(2l + 1)} v^{j(ai)} M^L_{A-1}j + \frac{4l}{l + 1} v^j_A \delta^{[ai]} J^{L-1}A^k + O(c^{-4}) \right], \quad (A1)$$

$$Z_i^L_{g,A} = \frac{4}{4!} M^L_{B,A} + 4v^j_A M^L_{B,A} - \frac{4(2l - 1)}{2l + 1} v^{j(ai)} M^L_{B,A} \delta^{L-1} - \frac{4l}{l + 1} v^{j(ai)} M^L_{A-1}j + O(c^{-2}) \right), \quad (A2)$$

where

$$G_{g,A} = \sum_{B \neq A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} M^L_{B} \partial^{(A)}_{K} 1 \frac{1}{|z_A - z_B|} + O(c^{-2}) \right). \quad (A3)$$

The tidal moments in the body frame $G^k_{g,A}$, $H^L_{g,A}$, which enter in Eq. (A4) and then in the orbital equations of motion, can be expressed in terms of the global multipole moments as follows. At Newtonian order,

$$G^k_{g,A} = G^k_{g,A} - z^k_A + O(c^{-2}), \quad (A4)$$

$$G^L_{g,A} = G^L_{g,A} + O(c^{-2}) \quad (A5)$$

$$H^L_{g,A} = Y^{jL}_{g,A} (L+1) - 4v^j_A G^{L+1}_{g,A} \epsilon^{ai} j^k - lM^{L+1}_{g,A} \epsilon^{ai} j^k + O(c^{-2}) \quad (A6)$$

where the global-frame electric and magnetic tidal moments are

$$G^L_{g,A} = \sum_{B \neq A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} M^L_{B} \partial^{(A)}_{K} 1 \frac{1}{|z_A - z_B|} + O(c^{-2}) \right), \quad (A7)$$

$$Y^{iL}_{g,A} = \sum_{B \neq A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} Z_{iL}^{k} \partial^{(A)}_{K} 1 \frac{1}{|z_A - z_B|} + O(c^{-2}) \right), \quad (A8)$$

and $Z^{ik}_{g,B}$ is given in Eq. (A2).
At 1PN order, the electric tidal moments are

\[
G^L_A = F^L_{g,A} - l 1\Lambda^L_{g,A} + \frac{1}{c^2} \left[ Y^{(L)}_{g,A} - v^i_{g,A} v^j_{g,A} + (2v^2_{g,A} - l G_{g,A}) G^L_{g,A} - (l/2) v^i_{g,A} G^{(L)-(i)}_{g,A} + (l - 4) v^i_{g,A} G^{(L)-(i)}_{g,A} \right] - (l^2 - l + 4) a^{(L)}_{g,A} - (l - 1) \Lambda^{(L)}_{g,A} + O(c^{-4})
\]

where

\[
F^L_{g,A} = \sum_{B \neq A} \sum_{k=0}^{\infty} \frac{(-1)^k (k!)}{L} \left[ N_{g,B} \partial^L_{K,L} \frac{1}{|z_A - z_B|} + \frac{1}{2c^2} \partial^L_{K,L} |z_A - z_B| \right] + O(c^{-4})
\]

\[
N^L_{g,A} = M^L_{g,A} + \frac{1}{(2l + 3)c^2} v^i_{g,A} M^L_{g,A} + 2v^i_{g,A} M^L_{g,A} + 2lv_{g,A} M^{L-1}_{g,A} + a^i_{g,A} M_L
\]

\[
P^L_{g,A} = \sum_{B \neq A} \sum_{k=0}^{\infty} \frac{(-1)^k (k!)}{L} \left[ N_{g,B} \partial^L_{K,L} \frac{1}{|z_A - z_B|} + \frac{1}{2c^2} \partial^L_{K,L} |z_A - z_B| \right] + O(c^{-4})
\]

At 0PN, Eq. (A7) gives \( G^L_{g,2} = M_1 \partial_{L} \frac{1}{r} + O(c^{-2}) \), i.e.,

\[
G^L_{g,2} = 3M_1 \frac{1}{r^3} n^{(ab)} + O(c^{-2})
\]

\[
G^{abc}_{g,2} = -15M_1 \frac{1}{r^4} n^{(abc)} + O(c^{-2})
\]

and, since [see Eq. (A2)]

\[
Z^{i}_{g,1} = 4M_1 v^i + O(c^{-2})
\]

\[
Z^{i}_{g,1} = -2v^i j^k + O(c^{-2})
\]

\[
Z^{L}_{g,1} = O(c^{-2})
\]

Eq. (A8) gives

\[
Y^{i}_{g,2} = 4M_1 v^i \partial_{L} \frac{1}{r} + 2v^i j^k \partial_{j} \frac{1}{r} + O(c^{-2})
\]

Replacing in Eqs. (A5), (A6), we find (since \( v^i = v^i - v^i \))

\[
G^{L}_{2} = M_1 \partial_{L} \frac{1}{r} + O(c^{-2})
\]

\[
H^{L}_{2} = -4M_1 v^i \partial_{c(L-1)l} \frac{1}{r} e^{(i)l} + 2v^i j^k \partial_{j} \frac{1}{r} e^{(i)l} + + O(c^{-2})
\]

\[
H^{L}_{2} = O(c^{-2})
\]

Therefore, since \( \partial_{L} \frac{1}{r} = (-1)^l (2l - 1)! n^{(L)}_{l}/r^{l+1} \),

\[
G^{ab}_{2} = \frac{3\eta M}{r^3} n^{(ab)} + O(c^{-2})
\]

\[
G^{abc}_{2} = -15\eta M \frac{1}{r^4} n^{(abc)} + O(c^{-2})
\]

Appendix B: Tidal moments in the truncated system

We here compute the tidal moments for the system considered in this paper in which the body 1 has nonvanishing mass \( M_1 \) and spin \( J^1_3 \), the body 2 has nonvanishing mass \( M_2 \), spin \( J^2_3 \), quadrupole moments \( Q^{ab} = M^{ab}_2 \), \( S^{ab} = J^{ab}_2 \) and octupole moments \( Q^{abc} = M^{abc}_2 \), \( S^{abc} = J^{abc}_2 \). We only focus on the tidal moments of the body 2, \( G_L^1, H_L^1 \), with \( l = 2, 3 \), because these are those which induce the multipole moments of our truncation in the adiabatic relations 16. The other tidal moments needed to derive the orbital equations of motion (i.e., \( G_L^1, H_L^1 \), \( G_L^2, H_L^2 \) with \( l = 1, 4 \)) can be obtained in a similar way. As discussed in Sec. III A we compute the electric, quadrupolar tidal moment \( G^{ab}_2 \) at 1PN order, while all the other tidal moments are computed at 0PN order only. These are the contributions needed in order to compute the waveform at 6.5PN order (see Sec. III C).
The spin contributions are
\[ H_2^{ab} = \frac{6\eta M}{r^3} v^d \left( n^{ac}e^{bcd} + n^{bc}e^{acd} \right) + \frac{30J_i^d}{r^4} n^{(abc)} + O(c^{-2}), \]  
(B8)
\[ H_2^{abc} = -\frac{20\eta M}{r^4} v^e \left( n^{(dabc)}_e e^{de} + n^{(abc)}_d e^{ade} + n^{(dca)}_e e^{bed} \right) - \frac{210J_i^d}{r^5} n^{(abcd)} + O(c^{-2}). \]  
(B9)
At 1PN order, Eqs. (A10)–(A15) give
\[ G_2^{ab} = \frac{3\eta M}{r^3} n^{(ab)} + \frac{1}{c^2} \frac{3\eta M}{r^3} \left( \left( 2v_2 - 5\eta_2 v_2 - 5 + \frac{\eta_1 M}{r} \right) n^{(ab)} + v^{(ab)} - (3 - \eta_2^2) \eta n^{(a'b')} \right) + \frac{6}{c^2 r^4} J_i^d v^e \epsilon^{ex(a)} \left( 5n^{(b'cd)} - \delta^{b'd}n^c - n^{b'}\delta^{cd} \right) + O(c^{-4}). \]  
(B10)

Appendix C: Orbital equations of motion of the two-body system

We here show the explicit expression of the orbital equations of motion given in Eqs. (60) and (61),
\[ M_1 a_1^i = F_{1,M}^i + F_{1,j}^i + F_{1,Q2}^i + F_{1,Q3}^i + F_{1,S2}^i + F_{1,S3}^i, \]  
(C1)
\[ M_2 a_2^i = F_{2,M}^i + F_{2,j}^i + F_{2,Q2}^i + F_{2,Q3}^i + F_{2,S2}^i + F_{2,S3}^i. \]  
(C2)
The mass monopole contributions are
\[ F_{1,M}^i = \frac{M_1 M_2}{r^2} n^i + \frac{1}{c^2} \frac{M_1 M_2}{r^2} \left\{ n^i \left[ 2v_2 - v_1 - \frac{3}{2} \left( n^a v_2^a \right)^2 - \frac{5M_1}{r} - \frac{4M_2}{r} \right] + v^i n^a \left( 4v_1^a - 3v_2^a \right) \right\} + O(c^{-4}), \]  
(C3)
\[ F_{2,M}^i = -\frac{M_1 M_2}{r^2} n^i - \frac{1}{c^2} \frac{M_1 M_2}{r^2} \left\{ n^i \left[ 2v_2 - v_1 - \frac{3}{2} \left( n^a v_2^a \right)^2 - \frac{5M_1}{r} - \frac{4M_2}{r} \right] - v^i n^a \left( 4v_2^a - 3v_1^a \right) \right\} + O(c^{-4}). \]  
(C4)
The spin contributions are
\[ F_{1,j}^i = \frac{1}{c^2} \frac{M_1}{r^3} e^{abc} J_2^c \left[ 5\delta^{ai} \left( 4v_2^b - 6n^{b'd}v^d \right) - 6n^{a'i}v^b \right] - \frac{1}{c^2} \frac{M_2}{r^3} e^{abc} J_1^c \left[ 3\delta^{ai} \left( n^{b'd}v^d - v^b \right) + 6n^{a'i}v^b \right] - \frac{1}{c^2} \frac{M_2}{r^3} e^{abc} J_1^c \left[ 3\delta^{ai} \left( 4v_1^b - 6n^{b'd}v^d \right) - 6n^{a'i}v^b \right] + O(c^{-4}), \]  
(C5)
\[ F_{2,j}^i = \frac{1}{c^2} \frac{M_1}{r^3} e^{abc} J_2^c \left[ 3\delta^{ai} \left( n^{b'd}v^d - v^b \right) + 6n^{a'i}v^b \right] - \frac{1}{c^2} \frac{M_2}{r^3} e^{abc} J_1^c \left[ 3\delta^{ai} \left( 4v_1^b - 6n^{b'd}v^d \right) - 6n^{a'i}v^b \right] + O(c^{-4}). \]  
(C6)
The mass quadrupole contributions are
\[ F_{1,Q2}^i = \frac{3M_1}{2r^4} Q^{ab} \left( 5n_{abi} - 2n^a\delta_{bi} \right) + \frac{1}{c^2} \left( \frac{3M_1}{2r^4} Q^{ab} \right) \left\{ 5n_{abi} \left[ 2v_2 - v_1 - \frac{7}{2} (n^c v_2^c)^2 \right] \right\} - \frac{47M_1}{5r} - \frac{24M_2}{5r} \right\} n^a v^a 
- \frac{19M_1}{2r} - \frac{4M_2}{2r} \right\} n^a v^a 
+ \frac{3M_1}{2r^4} Q^{ab} \left( n_{abi} \delta_{bi} + 2n^a\delta_{bi} \right) 
- \delta^{abi}n^{bc} \left( 2v_1^c - v_2^c \right) \right\} - \frac{3M_1}{4r^2} Q^{ab} \left( n_{abi} + 2n^a\delta_{bi} \right) 
- \frac{3}{c^2} \left\{ \frac{Q^{ab}}{r^5} \right\} \left( 5n_{abc}^d (7n^d v^c - v^d) + (\delta^{ad} - 5n^{ad}) v^b - 5\delta_{ad} n_{bc} v^c \right) \right\} + \frac{Q^{ab}}{r^5} \left( \delta^{ad} - \frac{5}{2} n^{ad} v^b \right) 
- \frac{3}{c^2} \left\{ \frac{Q^{ab}}{r^5} \right\} \left( 5n_{abc} (\delta_{d} v^c - \delta^{d}v^c) + 5n_{ac} (\delta_{d} v^c - \delta^{d}v^b) + 5n_{bc} (\delta_{d} v^c - \delta^{d}v^a) \right) + 35n_{abc} (\delta_{d} n^d v^f - n^f) + \delta^{ab} (\delta_{d} v^c - \delta^{d}v^c) + \delta^{ac} (\delta_{d} v^b - \delta^{d}v^b) 
+ 5 (\delta_{d} v^c + \delta_{d} v^b) \left( n^i - \delta_{d} n^i f^f \right) \right\} - \frac{Q^{ab}}{r^4} \delta^{ac} \left( 5n_{abc} - \delta_{d} n_{bc} - \delta^{ac} n^b \right) + O(c^{-4}), \]  
(C7)
\[ F_{2,pQ}^i = - \frac{3 M_1}{2 r^4} Q^{ab} (5n^{abi} - 2n^a \delta^{bi}) + c^{-2} \left( \frac{3 M_1}{2 r^4} Q^{ab} \left\{ -5n^{abi} \left[ 2v^2 - v_2 - \frac{7}{2} (n^c v^c)^2 - \frac{8 M_1}{r} - \frac{6 M_2}{r} \right] \right. \right. \]
\[ + 2n^a \delta^{bi} \left[ 3v^2 - v_2 - 5 (n^c v^c)^2 - \frac{5}{2} (n^c v^c)^2 - \frac{8 M_1}{r} - \frac{11 M_2}{2r} \right] \left. + n^i v^{ab} \right. \]
\[ + \frac{3 M_1}{r^3} \delta^{ab} \left[ 2n^a - \delta^{ai} \right] \leq \frac{3 M_1}{c^2 M_2} J_2 \left\{ \frac{Q^{ab}}{r^5} 5n^{ab} (7n^{de} v^e - v^d) + (\delta^{ad} - 5n^{ad} v^b - 5 \delta^{ad} n^{be} v^e) + \frac{Q^{ab}}{r^4} (\delta^{ad} - \frac{5}{2} n^{ad} b) \right\} \]
\[ + \frac{3 M_1}{c^2 M_2} J_2 \left\{ \frac{Q^{ab}}{r^5} 5n^{ab} (\delta^{ic} v^c - \delta^{ic} v^e) + 5n^{ac} (\delta^{ib} v^e - \delta^{ib} v^b) + 5n^{bc} (\delta^{ic} v^e - \delta^{ic} v^a) \right\} \]
\[ + 35n^{ab} (\delta^{ic} n^{i} v^f - n^i) + \delta^{ab} (\delta^{ic} v^e - \delta^{ic} v^a) + \delta^{ac} (\delta^{ic} v^b - \delta^{ic} v^a) \]
\[ + 5 (\delta^{ab} n^e + \delta^{ac} n^b) \left( n^i - \delta^{ic} n^{i} v^f \right) - \frac{Q^{ab}}{r^4} \delta^{ic} (5n^{abc} - \delta^{ab} n^{c} - \delta^{ac} n^{b}) \right\} \leq O(c^{-4}). \]  

The mass octupole contributions are
\[ F_{1, Q3}^i = - \frac{5 M_1}{2 r^4} Q^{abc} (7n^{abc} - 3 \delta^{ic} n^{ab}) + O(c^{-2}), \]  

\[ F_{2, Q3}^i = \frac{5 M_1}{2 r^4} Q^{abc} (7n^{abc} - 3 \delta^{ic} n^{ab}) + O(c^{-2}). \]

The current quadrupole contributions are
\[ F_{1, S2}^i = - \frac{4 M_1 \epsilon^{bcd}}{c^2} \left\{ \frac{S^{ab}}{r^4} \left[ n^a (\delta^{ib} v^c - \delta^{ic} v^b) + n^b (\delta^{ia} v^c - \delta^{ic} v^a) + 5n^{ab} (\delta^{ic} n^e v^c - n^i v^c) \right] \right. \]
\[ - \frac{\delta^{ad}}{r^3} \delta^{ic} n^{ab} \left. \right\} \]
\[ - \frac{J_C}{c^2} S^{ab} [4 \delta^{bc} (5n^{ia} - \delta^{ia}) + 10 (\delta^{ia} n^{bc} + \delta^{ib} n^{ac} + \delta^{ic} n^{ab}) - 70n^{abc}] + O(c^{-4}), \]

\[ F_{2, S2}^i = \frac{4 M_1 \epsilon^{bcd}}{c^2} \left\{ \frac{S^{ab}}{r^4} \left[ n^a (\delta^{ib} v^c - \delta^{ic} v^b) + n^b (\delta^{ia} v^c - \delta^{ic} v^a) + 5n^{ab} (\delta^{ic} n^e v^c - n^i v^c) \right] \right. \]
\[ - \frac{\delta^{ad}}{r^3} \delta^{ic} n^{ab} \left. \right\} \]
\[ + \frac{J_C}{c^2} S^{ab} [4 \delta^{bc} (5n^{ia} - \delta^{ia}) + 10 (\delta^{ia} n^{bc} + \delta^{ib} n^{ac} + \delta^{ic} n^{ab}) - 70n^{abc}] + O(c^{-4}). \]

The current octupole contributions are
\[ F_{1, S3}^i = - \frac{15 M_1}{c^2} \left\{ \frac{S^{bcd}}{r^5} n^{ab} v^c \left[ \frac{7}{2} (\epsilon^{iae} n^e - \epsilon^{iae} n^i) \right] n^d - (\epsilon^{iad} \delta^{ece} + \epsilon^{iad} \delta^{eac} + \epsilon^{acd} \delta^{gie}) \right\} - \frac{\hat{S}_{bcd}^{rde}}{2 r^4} \epsilon^{rad} n^{abc} \]
\[ - 45 J_C^r \frac{S^{abc}}{r^6} \left[ \delta^{cd} (\delta^{ia} n^b + \delta^{ib} n^a - 7n^{iab}) - \frac{7}{3} (\delta^{ia} n^{bcd} + \delta^{ib} n^{acd} + \delta^{ic} n^{abd}) + \delta^{id} n^{abc} - 9n^{iabcd} \right] \]
\[ + O(c^{-1}), \]

\[ F_{2, S3}^i = \frac{15 M_1}{c^2} \left\{ \frac{S^{bcd}}{r^5} n^{ab} v^c \left[ \frac{7}{2} (\epsilon^{iae} n^e - \epsilon^{iae} n^i) \right] n^d - (\epsilon^{iad} \delta^{ece} + \epsilon^{iad} \delta^{eac} + \epsilon^{acd} \delta^{gie}) \right\} - \frac{\hat{S}_{bcd}^{rde}}{2 r^4} \epsilon^{rad} n^{abc} \]
\[ + 45 J_C^r \frac{S^{abc}}{r^6} \left[ \delta^{cd} (\delta^{ia} n^b + \delta^{ib} n^a - 7n^{iab}) - \frac{7}{3} (\delta^{ia} n^{bcd} + \delta^{ib} n^{acd} + \delta^{ic} n^{abd}) + \delta^{id} n^{abc} - 9n^{iabcd} \right] \]
\[ + O(c^{-1}). \]
Appendix D: Higher-order terms in the GW phase

For completeness, we provide the higher-order terms entering the GW phase \((\ref{eq:ps})\) (i.e., appearing at 7PN order and beyond) which are proportional to the TLNs. These terms are computed as a by-product of our analysis and could be useful for comparison. The extra terms in Eq. \((\ref{eq:ps})\) read

\[
\psi(x) \supset \left( \frac{4000}{9} - \frac{4000}{9 \eta_1} \right) \frac{\lambda_3^{(1)}}{M^2} x^7 \\
+ \left( \frac{29400}{11} - \frac{29400}{11 \eta_1} \right) \frac{\sigma_5^{(1)}}{M^2} x^8 \\
+ \left( -\frac{4480}{3} + \frac{2240}{3 \eta_1} + \frac{22400 \eta_1}{3} \right) \chi_2 \frac{\sigma_3^{(1)}}{M^2} x^{8.5} \\
+ (1 \leftrightarrow 2).
\]

(A1)

To the best of our knowledge, some of these terms have never been published before.

Appendix E: Comparison to the RTLNs defined in Ref. \([42]\)

We here show the relations between the TLNs defined in Eq. \((\ref{eq:tln})\) and those of Ref. \([42]\). In the following, we set the speed of light \(c = 1\).

In Ref. \([42]\), the standard electric and magnetic TLNs are defined as

\[
\lambda_{{E}^{(l)}} = \frac{\partial M_{l}}{\partial E_{l}}, \quad \lambda_{{M}^{(l)}} = \frac{\partial S_{l}}{\partial B_{l}},
\]

(E1)

where the multipole moments \(M_{l}\), \(S_{l}\) and the tidal-field components \(E_{l}\), \(B_{l}\) are given in terms of the asymptotic expansion of the metric (see Refs. \([38, 61]\) and Appendix B of Ref. \([30]\)). With the above definitions, \(\lambda_{l}\) and \(\sigma_{l}\) used in this work are, respectively, given by

\[
\lambda_{{E}^{(l)}} = -\left( \frac{2l - 1}{(l - 1)} \right) \frac{(2l - 1)!}{(2l)!} \frac{2l + 1}{4\pi} \frac{\lambda_{l}}{M_{l}}, \quad \lambda_{{M}^{(l)}} = \left( \frac{2l - 1}{(l - 1)} \right) \frac{(2l - 1)!}{(2l)!} \frac{2l + 1}{4\pi} \frac{\sigma_{l}}{S_{l}},
\]

(E2)

with \(l \geq 2\).

On the other hand, the RTLNs in the axisymmetric case are defined in Ref. \([42]\) as

\[
\delta \lambda_{{E}^{(2)}} = \frac{\partial M_{l}}{\partial E_{l}}, \quad \delta \lambda_{{M}^{(2)}} = \frac{\partial S_{l}}{\partial B_{l}},
\]

(E4)

with the same moments given in Refs. \([30, 38, 42, 61]\). With the above definitions, the RTLNs \(\lambda_{23}, \sigma_{23}, \lambda_{32}\) and \(\sigma_{32}\) defined here are related to those defined in Ref. \([42]\) by the relations

\[
\delta \lambda_{{E}^{(23)}} = -\frac{\sqrt{7}}{\pi} J \lambda_{23}, \quad \delta \lambda_{{M}^{(23)}} = -\frac{\sqrt{7}}{\pi} J \sigma_{23}, \quad \delta \lambda_{{E}^{(32)}} = -\frac{15}{\sqrt{8}} J \lambda_{32}, \quad \delta \lambda_{{M}^{(32)}} = -\frac{45}{8\sqrt{5\pi}} J \sigma_{32},
\]

(E5)

where \(J\) is the absolute value of the angular momentum of the body which is deformed.
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