Poincaré series of collections of plane valuations

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Abstract

In earlier papers there were given formulae for the Poincaré series of multi-index filtrations on the ring \( \mathcal{O}_{\mathbb{C}^2,0} \) of germs of functions of two variables defined by collections of valuations corresponding to reducible plane curve singularities and by collections of divisorial ones. It was shown that the Poincaré series of a collection of divisorial valuations determines the topology of the collection of divisors. Here we give a formula for the Poincaré series of a general collection of valuations on the ring \( \mathcal{O}_{\mathbb{C}^2,0} \) centred at the origin and prove a generalization of the statement that the Poincaré series determines the topology of the collection.

In [1], [6], ... there were considered multi-index filtrations on the ring \( \mathcal{O}_{\mathbb{C}^2,0} \) of germs of functions of two variables defined by collections of valuations corresponding to (reducible) plane curve singularities and by collections of divisorial ones. One gave formulae for the Poincaré series of such filtrations in terms of an embedded resolution of the curve singularity or of the collections of divisors respectively. These formulae give Poincaré series as rational functions equal to products/ratios of cyclotomic polynomials. In particular, it was shown that the Poincaré series of the collection of valuations corresponding to a curve coincides with the Alexander polynomial of the corresponding algebraic link. This implies that the Poincaré series of such a collection determines the topology of the curve ([7]). An analogue of this statement for divisorial valuations was proved in [3]. Here we give a formula for the Poincaré series of a general collection of valuations on the ring \( \mathcal{O}_{\mathbb{C}^2,0} \) centred at the origin. We also prove a generalization of the statement that the Poincaré series determines the topology of the collection of divisors.

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1 Poincaré series of several valuations

Let \( \mathcal{O}_{X,0} \) be the ring of germs of functions on a germ \((X,0)\) of a complex analytic variety and let \( G \) be a ordered abelian group, \( G^+ := \{ a \in G : a \geq 0 \} \).

A valuation on the ring \( \mathcal{O}_{X,0} \) with values in the group \( G \) is a map \( v : \mathcal{O}_{X,0} \to G^+ \cup \{ +\infty \} \) such that:

1. \( v(g_1 \cdot g_2) = v(g_1) + v(g_2) \);
2. \( v(g_1 + g_2) \geq \min \{ v(g_1), v(g_2) \} \);
3. \( v(\lambda) = 0 \) for \( \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{ 0 \} \).

The semigroup \( S = \text{Im} v \setminus \{ +\infty \} \) (the semigroup of values of the valuation \( v \)) is well ordered (i.e. each subset of \( S \) has the minimal element) and moreover each element \( a \in S \) has a finite number of representations as the sum \( a_1 + a_2 \) of two elements of \( S \) (see also [4]).

Let \( v_i, i = 1, \ldots, s \), be valuations on the ring \( \mathcal{O}_{X,0} \) with values in ordered groups \( G_i \) and the semigroups of values \( S_i \). The direct product \( S = S_1 \times \cdots \times S_s \) is a partially ordered semigroup: \( v' = (v'_1, \ldots, v'_s) \geq v'' = (v''_1, \ldots, v''_s) \) iff \( v'_i \geq v''_i \) for all \( i = 1, \ldots, s \). \( v'_i, v''_i \in S_i \). For a germ \( g \in \mathcal{O}_{X,0} \), let \( v(g) := (v_1(g), \ldots, v_s(g)) \).

**Definition:** The ring \( \mathbb{Z}[\{ S \}] \) of power series on the semigroup \( S = S_1 \times \cdots \times S_s \) is the set of formal expressions of the form \( \sum_{\underline{v} \in S} a_{\underline{v}} \underline{t}^{\underline{v}} \) where \( \underline{v} := (v_1, \ldots, v_s) \in S, \underline{t}^{\underline{v}} := t_1^{v_1} \cdots t_s^{v_s} \) with the usual ring operations.

**Remarks.**

1. The fact that \( \mathbb{Z}[\{ S \}] \) is a ring (i.e. the multiplication is defined) follows from the described properties of the semigroup of values of a valuation.
2. If \( G_i = \mathbb{Z} \) for \( i = 1, \ldots, s \), the ring \( \mathbb{Z}[\{ S \}] \) is contained in the ring \( \mathbb{Z}[t_1, \ldots, t_s] \) of power series in the variables \( t_1, \ldots, t_s \) with integer coefficients and thus an element of the ring \( \mathbb{Z}[\{ S \}] \) is a usual power series in the variables \( t_1, \ldots, t_s \).

The collection \( \{ v_i \} \) of valuations defines a multi-index filtration on the ring \( \mathcal{O}_{X,0} \) by the ideals \( J(\underline{v}) = \{ g \in \mathcal{O}_{X,0} : v_i(g) \geq 0 \} \) (indexed by the elements \( \underline{v} \) of the group \( G = G_1 \times \cdots \times G_s \)). For \( I \subset I_0 = \{ 1, \ldots, s \}, \underline{v} = (v_1, \ldots, v_s) \in G, \) let

\[
J^+(\underline{v}) := \{ g \in J(\underline{v}) : v_i(g) > v_i \text{ for } i \in I \},
\]

\[
J^+(\underline{v}) := J^{+I_0}(\underline{v}).
\]

**Definition:** The Poincaré series \( P_{\{v_i\}}(t_1, \ldots, t_s) \) of the collection of valuations \( \{ v_i \} \) is the element of the ring \( \mathbb{Z}[\{ S \}] \) defined by

\[
P_{\{v_i\}}(\underline{t}) = \sum_{\underline{v} \in S} \left( \sum_{I \subset I_0} (-1)^{\# I} \dim \left( J^+(\underline{v})/J^+(\underline{v}) \right) \right) \cdot \underline{t}^{\underline{v}}.
\]
One can see that, for collections of integer valued valuations, this definition coincides with that used e.g. in [1]. One can easily extract a proof of this from the proof of Theorem 3 in [1].

**Remark.** This notion is defined if all the factor spaces $J^+L/J^+(v)$ have finite dimensions. This takes place if each valuation $v_i$, $i = 1, \ldots, s$, is centred at the origin, i.e. $\{g \in O_{X,0} : v_i(g) > 0\}$ coincides with the maximal ideal $m$ of the ring $O_{X,0}$.

**Definition:** A valuation $v$ with values in a group $G$ is finitely determined if, for each $v_0 \in G$, the condition $v(g) = v_0$ is a constructible condition on a jet of the germ $g$ of a certain (finite) order (see [2]).

**Examples.** 1. For an irreducible plane curve germ $C = \{f = 0\} \subset (\mathbb{C}^2, 0)$, let $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be a parametrization (uniformization) of the curve $C$. For a germ $g \in O_{X,0}$, let $v_C(g)$ be the order of zero of the function $g \circ \varphi(\tau)$ at the origin (if $g \circ \varphi \equiv 0$, $v_C(g) := +\infty$). The map $v_C : O_{X,0} \to \mathbb{Z} \cup \{+\infty\}$ is a finitely determined rank 1 valuation on the ring $O_{X,0}$. (This valuation has a non-trivial kernel, i.e. $v_C(g) = +\infty$ for some $g \neq 0$.)

2. For the same curve $C = \{f = 0\}$ and for $g \in O_{X,0}$, $g \neq 0$, let $g = f^{k_C(g)}g'$, where $g'$ is not divisible by $f$. The map $w_C : O_{X,0} \to \mathbb{Z}^2 \cup \{+\infty\}, w_C(g) := (k_C(g), v_C(g'))$, is a rank 2 valuation on the ring $O_{X,0}$. (The group $\mathbb{Z}^2$ is ordered lexicographically, i.e. $(k', v') > (k'', v'')$ iff $k' > k''$ or $k' = k''$ and $v' > v''$.) The valuation $w_C$ is not finitely determined: for $k > 0$, the condition $k_C(g) = k$ is not determined by a (finite) jet of the germ $g$. (The kernel of the valuation $w_C$ is trivial.)

The notion of integration with respect to the Euler characteristic $\chi$ over the projectivization $\mathbb{P}O_{X,0}$ of the ring $O_{X,0}$ was defined in [2]. The argument from [2] give the following statement.

**Proposition 1.** If all the valuations $v_i$, $i = 1, \ldots, s$, are finitely determined, one has

$$P_{\{v_i\}}(L) = \int_{\mathbb{P}O_{X,0}} L^{v(g)}d\chi.$$  \hspace{1cm} (1)

**Remark.** One can formulate the notion of the Poincaré series of a collection of valuations in terms of the extended semigroup of the collection in spirit of [1], [6].

## 2 Valuations on $O_{\mathbb{C}^2,0}$

Let $\pi : (X, D) \to (\mathbb{C}^2, 0)$ be a modification of the plane by a (finite) sequence of blowing-ups. The exceptional divisor $D$ is the union of irreducible components $E_\sigma$ ($\sigma \in \Gamma$); each of them is isomorphic to the complex projective line $\mathbb{CP}^1$. The dual graph of the modification $\pi$ consists of vertices corresponding to the
irreducible components $E_\sigma$, i.e. to elements of $\Gamma$; two vertices are connected by an edge iff the corresponding components intersect. To $\sigma \in \Gamma$, i.e. to an irreducible component $E_\sigma$ of the exceptional divisor, there corresponds a natural valuation $v_\sigma$: the divisorial one. For a germ $g \in \mathcal{O}_{X,0}$ the value $v_\sigma(g)$ is defined as a multiple of (say, $c_\sigma$ times) the order $w_\sigma(g)$ of zero of the function $g \circ \pi$ along the component $E_\sigma$ (i.e. of the coefficient at $[E_\sigma]$ in the zero divisor of the function $g \circ \pi$). It is convenient to choose the coefficient $c_\sigma$ in such a way that $\min_{g \in \mathfrak{m}} v_\sigma(g) = 1$, i.e. $c_\sigma = 1/w_\sigma(g)$ for a generic function $g \in \mathfrak{m}$.

It is known ([8], see also [5]) that all valuations on the ring $\mathcal{O}_{X,0}$ centred at the origin (i.e. such that $v(g) > 0$ for $g \in \mathfrak{m}$) are given by the following list.

I. All rank 1 valuations centred at the origin correspond to some sequences (finite or infinite) of blowing-ups such that each next blowing-up is made at a point of the divisor born on the previous step. To get a one-to-one correspondence, one should exclude sequences of blowing-ups made at each step after a certain one at the intersection point of a fixed divisor $E_{\sigma_0}$ with the last one. (In such case the correspondence described below leads to the divisorial valuation $v_{\sigma_0}$ associated to the divisor $E_{\sigma_0}$.) If the sequence is finite, the corresponding valuation is the divisorial one associated to the last divisor. If the sequence is infinite, the value $v(g)$ of the corresponding valuation is defined as the limit $\lim_{i \to \infty} v_{\sigma_i}(g)$ where $v_{\sigma_i}$ is the divisorial valuation associated to the divisor born on $i$-th step. Depending on the sequence of blowing-ups, one can distinguish the following types of valuations.

I.1. The blowing-ups are made at the intersection points of the strict transforms of a fixed irreducible curve $C = \{f = 0\} \subset (\mathbb{C}^2, 0)$ with the exceptional divisor. The corresponding valuation (a curve valuation of rank 1) is equivalent to the valuation defined by the order of a function $g$ on the curve $C$ in an uniformization parameter (Example 1 above). The corresponding dual graph of the modification has a growing infinite tail: Fig.1. This valuation has a non-trivial kernel.

Figure 1: The dual graph of a valuation of type I.1.

I.2. The sequence of blowing-ups produces an infinite tail like on Fig.1, but there exists no curve corresponding to this sequence (one can say that there is a formal curve defined by a formal power series), one gets a discrete valuation with the trivial kernel: a formal curve valuation.

I.3. The sequence of blowing-ups increases the number of rupture points producing, as the limit, the graph shown on Fig.2. In this case the group of values is contained in the ring $\mathbb{Q}$ of rational numbers and is not finitely
generated. For any $g \in \mathcal{O}_{\mathbb{C}^2, 0}$, one has $v(g) = v_\sigma(g)$ for $i$ sufficiently large. Valuations of this sort we call infinite valuations.

Figure 2: The dual graph of a valuation of type I.3.

### Remarks
1. A valuation of type I.1 is essentially the second component of the corresponding valuation of type II.1. If one considers the ring $\mathbb{C}[[x, y]]$ of formal power series in two variables instead of $\mathcal{O}_{\mathbb{C}^2, 0}$, valuations of type I.1 and I.2 constitute one and the same type. The results of the paper are valid in this setting as well.

2. Except valuations centred at the origin, there are so called $f$-adic valuations $k_C$ corresponding to irreducible plane curve singularities $C = \{f = 0\} \subset (\mathbb{C}^2, 0)$. For a germ $g \in \mathcal{O}_{\mathbb{C}^2, 0} \setminus \{0\}$, $k_C(g)$ is defined by the relation $g = f^{k_C(g)} g'$ where $g' \notin \{f\}$. This is just the first component of the rank 2 curve valuation corresponding to $C$. For an appropriate definition, the Poincaré...
series of a collection of valuations containing \( f \)-adic ones is a reduction of the Poincaré series of the collection obtained by substituting \( f \)-adic ones by the corresponding plane curve valuations of rank 2.

For further discussions it is convenient to use a notion of a resolution of a (finite) collection of valuations. For a divisorial valuation this is a modification \( \pi : (X, D) \to (\mathbb{C}^2, 0) \) of the plane by a finite sequence of blowing-ups over the origin which contains the corresponding divisor (this means that if is a modification of the minimal resolution by a finite sequence of blowing-ups). For a valuation of one of the types I, II.1, let \( \pi_i : (X_i, D_i) \to (\mathbb{C}^2, 0) \) be the modification obtained at the \( i \)-th step of the corresponding sequence of blowing-ups. Let \( (X, D) \) be the projective limit (in the category of analytical spaces) of the sequence \( \{(X_i, D_i)\} \) and let \( \pi : (X, D) \to (\mathbb{C}^2, 0) \) be the corresponding map. One can see that \( X \) is a smooth complex surface and \( D \) is the union of infinitely many projective lines on it. The map \( \pi : (X, D) \to (\mathbb{C}^2, 0) \) will be called the minimal resolution of the valuation. This is not a resolution since, in particular, the map \( \pi \) is not proper. For a valuation of type II.2, the minimal resolution will be defined as the minimal resolution of the corresponding divisorial valuation (the first component of the considered one) followed by an additional blowing-up at the corresponding point. Thus in the minimal resolution the point under consideration is an intersection point of components of the exceptional divisor.

The minimal resolution of a finite collection of valuations is the projective limit of the corresponding (multi-index) system of modifications. It is simply the fibre product of the minimal resolutions of all valuations.

Finally a resolution of a finite collection of valuations is a modification of the space \( (X, D) \) of the minimal resolution by a finite number of blowing-ups (at points of \( D \)).

### 3 Poincaré series of a collection of plane valuations

Let \( v_i, i = 1, \ldots, s \), be a set of valuations on the ring \( \mathcal{O}_{\mathbb{C}^2,0} \) centred at the origin. Let \( \pi : (X, D) \to (\mathbb{C}^2, 0) \) be a resolution of the set of valuations \( \{v_i\} \).

For a component \( E_\sigma \) of the exceptional divisor \( D \), \( \sigma \in \Gamma \), let \( L_\sigma \) be a smooth germ of curve transversal to \( D \) at a smooth point of it, let the curve germ \( \ell_\sigma = \pi(L_\sigma) \subset (\mathbb{C}^2, 0) \) be given by an equation \( \{g_\sigma = 0\} \), and let \( m_\sigma^\alpha = v_i(g_\sigma) \), \( m^\sigma := (m_1^\sigma, \ldots, m_s^\sigma) \in G \).

Without loss of generality one can suppose that \( v_1, \ldots, v_r \) are valuations of type II.1 (or rank 2) and the others are not. For a component \( E_\sigma \) of the exceptional divisor \( D \), let \( \tilde{E}_\sigma \) be its “smooth part,” i.e. \( E_\sigma \) itself minus the intersection points with all other components of \( D \). For \( i = 1, \ldots, r \), let \( C_i = \{f_i = 0\} \) be the corresponding curve, \( f_i \in \mathcal{O}_{\mathbb{C}^2,0} \).

**Theorem 1.** The Poincaré series of the set of valuations \( \{v_i\} \) is given by the
\[ P_{(v_1)}(t) = \prod_{\sigma \in \Gamma} (1 - t^{m_\sigma})^{-\chi(E_\sigma)} \times \prod_{i=1}^r (1 - t^{(1,0)}_i) \prod_{j \neq i} t^{\nu_j(f_i)}_j - 1. \] (2)

**Proof.** One cannot use the equation (1) directly since valuations of type II.1 are not finitely determined. However the condition \( v_i(g) = (0, \nu) \) is defined by a jet of the germ \( g \) of a finite order. Therefore let us first compute the part \( P^o(t) \) of the Poincaré series \( P_{(v_1)}(t) \) which consists of monomials which are not divisible by any of \( t_i^{(1,0)}, i = 1, \ldots, r \). One can see that

\[ P^o(t) = \int_{P(O_{\mathbb{C}^2,0} \setminus \bigcup_{i=1}^r \langle f_i \rangle)} t^{\nu} d\chi \]

Computation of \( P^o(t) \) essentially repeats the arguments from, e.g., [2] or [6]. To compute the series \( P^o(t) \) up to terms of any fixed degree \( V \in G \) one can make finitely many additional blowing-ups at intersection points of the components of the exceptional divisor \( D \) so that, for a germ \( g \in O_{\mathbb{C}^2,0} \setminus \bigcup_{i=1}^r \langle f_i \rangle \) with \( \nu(g) \leq V \), all the intersection points of the strict transform of the curve \( \{ g = 0 \} \) with the exceptional divisor \( D \) belong to its smooth part \( \hat{D} = \bigcup_{\sigma} \hat{E}_\sigma \).

Let \( O^V_{\mathbb{C}^2,0} = \{ g \notin \bigcup_{i=1}^r \langle f_i \rangle : \nu(g) \leq V \} \), and let \( O_{\mathbb{C}^2,0}^D \) be the set of germs \( g \in O_{\mathbb{C}^2,0} \setminus \{0\} \) such that the intersection points of the strict transform of the curve \( \{ g = 0 \} \) with the exceptional divisor \( D \) belong to \( \hat{D} \). The integral

\[ \int_{\mathbb{P}O_{\mathbb{C}^2,0}^D} t^{\nu} d\chi \]

over the projectivization \( \mathbb{P}O_{\mathbb{C}^2,0}^D \) contains all terms of the series \( P^o(t) \) up to degree \( V \). There is a map from \( \mathbb{P}O_{\mathbb{C}^2,0}^D \) to the space of effective divisors on \( \hat{D} \): to a function germ \( g \in O_{\mathbb{C}^2,0}^D \) one associates the intersection of the strict transform of the curve \( \{ g = 0 \} \) with the exceptional divisor \( D \). Proposition 2 from [2] implies that the preimage of a point with respect to this map is a complex affine space and thus has the Euler characteristic equal to 1. The Fubini formula implies that the integral \( \int_{\mathbb{P}O_{\mathbb{C}^2,0}^D} t^{\nu} d\chi \) is equal to the integral with respect to the Euler characteristic of the monomial \( t^{\nu} \) over the space of effective divisors on \( \hat{D} \), where \( \nu \) is the additive function on the space of effective divisors (with values in \( G \)) equal to \( m_\sigma \) for a point from the component \( \hat{E}_\sigma \) of \( \hat{D} \).

The space of effective divisors on \( \hat{D} \) is the direct product of the spaces of effective divisors on the components \( \hat{E}_\sigma, \sigma \in \Gamma \). Each of the latter ones is the
disjoint union of the symmetric powers $S^k E_\sigma$ of the component $E_\sigma$. Therefore
\[
\int_{P^D_{C^2,0}} t^z d\chi = \prod_{\sigma \in \Gamma} \left( \sum_{k=0}^{\infty} \chi(S^k E_\sigma) \cdot t^{km_\sigma} \right).
\]
Using the equation
\[
\sum_{k=0}^{\infty} \chi(S^k X) \cdot t^k = (1 - t)\chi(X)
\]
one gets
\[
\int_{P^D_{C^2,0}} t^z d\chi = \prod_{\sigma} (1 - t^{m_\sigma})^{-\chi(\hat{E}_\sigma)}.
\]
(3)
The right hand side of equation 3 do not contain components of the exceptional divisor born under additional blowing-ups since for each of them $\chi(\hat{E}_\sigma) = 0$. Therefore
\[
P^\sigma(t) = \prod_{\sigma \in \Gamma} (1 - t^{m_\sigma})^{-\chi(\hat{E}_\sigma)}
\]
(for a resolution $\pi : (X, D) \to (C^2, 0)$ of the set of valuations $\{v_i\}$).

Now, for $k = (k_1, \ldots, k_r) \in Z_{\geq 0}$, let $P^k(t)$ be the sum of terms of the Poincaré series $P\{v_i\}(t)$ divisible by $\prod_{i=1}^r t_i^{(k_i,0)}$ but not divisible by a monomial of this sort of higher degree. The set of functions $g$ with $\bar{\nu}(g)$ divisible by $\prod_{i=1}^r t_i^{(k_i,0)}$ (but not by a monomial of higher degree) is just $\prod_{i=1}^r f_i^{k_i} \cdot \left( \mathcal{O}_{C^2,0} \setminus \bigcup_{i=1}^r \{f_i\} \right)$. One has $\nu_j(\prod_{i=1}^r f_i^{k_i}) = \sum_{i=1}^r k_i \nu_j(f_i)$ and $\nu_i(f_i) = (1,0)$ for $i = 1, \ldots, r$.

From this it follows that
\[
P^k(t) = P^\sigma(t) \prod_{i=1}^r t_i^{(k_i,0)} \prod_{j \neq i} t_j^{k_i \nu_j(f_i)}
\]
and therefore
\[
P\{v_i\}(t) = P^\sigma(t) \sum_{k \in Z_{\geq 0}} \prod_{i=1}^r t_i^{(k_i,0)} \prod_{j \neq i} t_j^{k_i \nu_j(f_i)} = P^\sigma(t) \cdot \prod_{i=1}^r (1 - t_i^{(1,0)} \prod_{j \neq i} t_j^{\nu_j(f_i)})^{-1}.
\]

4 Poincaré series determines dual graphs

To a resolution $\pi : (X, D) \to (C^2, 0)$ of a collection of valuations one associates the dual graph (generally speaking infinite). It consists of vertices corresponding to the irreducible components of the exceptional divisor $D$; two vertices
are connected by an edge if the corresponding components intersect. The set of vertices of the dual graph inherits a partial order defined by approximation of the modification by sequences of blowing-ups: a component $E_{\sigma'}$ is “greater” than another component $E_{\sigma}$ ($\sigma' > \sigma$) if the exceptional divisor of the minimal modification which contains $E_{\sigma'}$ also contains $E_{\sigma}$. Two resolutions are called combinatorially equivalent if their dual graphs together with the partial orders are isomorphic.

For collections of valuations of types I.1 and I.2 the Poincaré series coincides with the Alexander polynomial (in several variables) of the corresponding algebraic link (obtained by cutting the non-convergent series for a valuation of type I.2) (see [1]). It is known that the Alexander polynomial of an algebraic link determines the topology of the curve singularity ([7], see another proof in [3]). Therefore the Poincaré series of a collection of such valuations determines its minimal resolution up to combinatorial equivalence.

In [3], it was shown that the Poincaré series of a collection of divisorial valuations “determines the topology of the set of divisors” in the sense that it determines the dual graph of the minimal resolution up to combinatorial equivalence. Moreover, it was shown that the Poincaré series of a collection of valuations which includes both divisorial ones and those of types I.1 (or I.2) does not determine, in general, the dual graph of the minimal resolution.

**Theorem 2.** Suppose that a collection $\{v_i\}$ of valuations does not contain valuations of types I.1 and I.2. Then the Poincaré series $P_{\{v_i\}}(t)$ of the collection determines types of the valuations, the dual graph of the minimal resolution up to combinatorial equivalence and divisors or sequences of divisors corresponding to the valuations.

In what follows, for short, we shall simply say that the Poincaré series determines the dual graph of the minimal resolution.

**Proof.** One property of collections of valuations of the type under consideration used in the proof is the fact that the projection formula holds for them. This means that, for a subset $I \subset I_0 = \{1, \ldots, s\}$, the Poincaré series of the set of valuations $\{v_i\}_{i \in I}$ is obtained from the Poincaré series of the whole set $\{v_i\}$ by omitting the variables $t_i$ with $i \notin I$ (in other words by substituting $t_i$ with $i \notin I$ by 1).

**Remark.** Omitting a valuation of type I.3 (an infinite valuation), as a factor one formally gets an infinite product of the form

$$(1 - t_1 \cdots t_s)(1 - t_1 \cdots t_s)^{-1}(1 - t_1 \cdots t_s)(1 - t_1 \cdots t_s)^{-1} \ldots$$

This product should be canceled.

The projection formula follows directly from the equation (2) (Theorem 1).

**Remark.** For collections of valuations which do not contain valuations of type II.1 (curve valuations of rank 2) this can be also deduced from equation [1]. The fact that the projection formula holds for valuations of type II.1 as well
can mean that this valuations can be considered finitely determined in some weak sense.

The projection formula implies that, from the Poincaré series of a collection of valuations, one can restore, in particular, the Poincaré series of each individual valuation from the collection.

For each individual valuation one can define its type from the Poincaré series. From equation (2) it follows that the set of exponents in the Poincaré series with non-vanishing coefficients generates an abelian group of rank equal to the rank of the valuation. The Poincaré series of a valuation of type II.1 (a curve valuation of rank 2) has infinitely many non-vanishing terms with exponents from the isolated subgroup of rank 1. This does not take place for valuations of type II.2 (exceptional curve valuations).

Each series from the ring $\mathbb{Z}[[S]]$ with the free term equal to 1 has a unique representation of the form $\prod_{a \in S} \left(1 - t^{m_k} \right)^{k_a}$ with $k_a \in \mathbb{Z}$. (Generally speaking, this product is not finite: infinite number of the exponents $k_a$ may be different from zero.)

The Poincaré series of rank 1 valuations have the following form.

1) For a divisorial valuation

$$P(t) = \frac{\prod_{i=1}^{h} (1 - t^{m_{\alpha_i}})}{\prod_{i=0}^{h+1} (1 - t^{m_{\beta_i}})}$$  \hspace{1cm} (4)

where the exponents $m_{\beta_i}$ generate an infinite cyclic group (i.e. a group isomorphic to $\mathbb{Z}$). Here $\alpha_i$ are rupture points and $\beta_i$ are dead ends of the dual graph of the minimal resolution.

2) For a valuation of type II.4 (an irrational one) the Poincaré series has the same form but the exponents $m_{\beta_i}$ generate a free abelian group of rank 2. (The ratios $m_{\beta_i}/m_{\beta_0}$ are rational for $i \leq h$ and the ratio $m_{\beta_{h+1}}/m_{\beta_0}$ is irrational.

3) For a valuation of type II.3 (an infinite valuation) the Poincaré series has a representation of the form

$$P(t) = \frac{\prod_{i=1}^{\infty} (1 - t^{m_{\alpha_i}})}{\prod_{i=0}^{\infty} (1 - t^{m_{\beta_i}})}$$

with infinitely many factors.

This shows that rank 1 valuations of different types cannot have equal Poincaré series.

Remark. This also can be deduced from the fact that, for one valuation, the set of exponents with non-zero coefficients coincides with the semigroup of values of the valuation.
For a collection of divisorial valuations the statement of the Theorem was proved in [3]. We shall reduce consideration of a collection of valuations of different types to the case of a collection of divisorial valuations. For that we shall substitute each non-divisorial valuation from the collection by one or two divisorial ones. In each case the Poincaré series of the resulting collection of valuations should be defined by the Poincaré series of the initial collection and the dual graph of the minimal resolution of the resulting collection (or rather of series of them) should permit to restore the dual graph of the minimal resolution of the initial one.

Suppose that a valuation from the collection \{v_i\} is of type II.1 (a curve valuation of rank 2). Without loss of generality we may assume that this is the first one. Let \(E_{\sigma_0}\) be a vertex of the dual graph of the minimal resolution of the collection \{v_i\} far enough on the infinite tail corresponding to the valuation \(v_1\). Let us substitute the (rank 2) valuation \(v_1\) in the collection \{v_i\} by the divisorial valuation \(v_{\sigma_0}\). If one knows the dual graph of a resolution obtained this way, one can easily restore the dual graph for the initial one. One can see that the Poincaré series of the new collection of valuations is obtained from the Poincaré series \(P_{\{v_i\}}(t_1, \ldots, t_s)\) (see equation (2)) by substituting \(t_1^{(0,1)}\) by \(t_{\sigma_0}\) and \(t_1^{(1,0)}\) by \(t_1^{N_{\sigma_0}}\) with \(N\) large enough.

Suppose that the valuation \(v_1\) is of type II.2 (an exceptional curve valuation), let \(E_{\sigma_0}\) be the exceptional curve (a component of the exceptional divisor) corresponding to the valuation, and let \(P\) be the corresponding point of the component \(E_{\sigma_0}\). (The point \(P\) is the intersection point of the component \(E_{\sigma_0}\) with another component of the exceptional divisor.) Let us make sufficiently many additional blowing-ups at the point \(P\) of the component \(E_{\sigma_0}\). (If one knows the dual graph of a resolution obtained this way, one can easily restore the dual graph for the minimal one.) In the new resolution, let \(P\) be the intersection point of the component \(E_{\sigma_0}\) with a component \(E_{\sigma'_0}\). Let us substitute, in the collection \{v_i\}, the valuation \(v_1\) by two divisorial valuations \(v_{\sigma_0}\) and \(v_{\sigma'_0}\). One can see that the Poincaré series of the new collection of valuations is obtained from the Poincaré series \(P_{\{v_j\}}(t_1, \ldots, t_s)\) by substituting \(t_1^{(1,0)}\) by \(t_{\sigma_0}\) and \(t_1^{(0,1)}\) by \(t_{\sigma'_0} t_1^{N_{\sigma_0}}\) with \(N\) large enough.

Suppose that \(v_1\) is a valuation of type I.3 (an infinite one). Moreover let us assume that \(v_1, v_2, \ldots, v_r\) are all valuations of type I.3 in the collection. The Poincaré series has the form

\[
\prod_{n=1}^{n_0} (1 - t_k^n)^{-\chi_n} \prod_{\ell=1}^r \prod_{j=1}^\infty \frac{1 - t_\ell^{m_{\alpha_\ell}^j}}{1 - t_\ell^{m_{\beta_\ell}^j}}.
\]

Here the product \(\prod_{j=1}^\infty \frac{1 - t_\ell^{m_{\alpha_\ell}^j}}{1 - t_\ell^{m_{\beta_\ell}^j}}\) corresponds to the infinite part (tail) of the dual graph corresponding to the valuation \(v_\ell\). In these products one has

\[
m_{\alpha_\ell}^j < m_{\beta_\ell}^1 < m_{\alpha_\ell}^2 < m_{\beta_\ell}^2 < \ldots
\]
and $m^{α_j} → ∞$, $m^{β_j} → ∞$ when $j → ∞$. The distribution of the factors between these products (from a certain place) is determined by the following property: the ratio $m^{α_j}/m^{β_j}$ strictly increases with $j$ along the tail corresponding to the valuation $v_{j_1}$ (i.e. for $ℓ = j_1$), strictly decreases along the tail corresponding to the valuation $v_{j_2}$ ($ℓ = j_2$) and is constant along all other tails.

Let us substitute the infinite valuation $v_1$ by the divisorial valuation $v_{α_N}$, with $N$ large enough. The dual graph of the minimal resolution of the collections $\{v_{α_N}, v_2, . . . , v_h\}$ is obtained from the dual graph of the minimal resolution of the collections $\{v_1\}$ by truncation of the infinite tail corresponding to the valuation $v_1$ at the vertex $α_N$. The dual graph of the resolution of the collection $\{v_1\}$ can be restored if one knows the truncated ones for all $N$ large enough. The Poincaré series of the collection $\{v_{α_N}, v_2, . . . , v_h\}$ is equal to

$$
\prod_{n=1}^{n_0}(1 - t^n)^{−χ_n} \prod_{j=1}^{N} 1 - \frac{t^{m^{α_j}}}{1 - t^{m^{β_j}}} \prod_{ℓ=2}^{r} \prod_{j=1}^{∞} 1 - \frac{t^{m^{α_j}}}{1 - t^{m^{β_j}}}.
$$

Suppose that the valuation $ν_1$ is of type II.4 (an irrational one). The semi-group of values of the valuation $v_1$ is generated by $1 = m^{1β_0}, m^{1β_1}, m^{1β_2}, . . . , m^{1β_h}$, where $m^{1β_0} < m^{1β_1} < m^{1β_2} < . . . < m^{1β_h}$, the numbers $m^{1β_i}$ are rational for $i < h$ and $m^{1β_h}$ is irrational. Let us substitute the valuation $v_1$ in the collections $\{v_1\}$ by the divisorial valuation $v_{α_N}$ far enough in the sequence $\{σ_h\}$ corresponding to the valuation $v_1$ and moreover such that in the dual graph of the minimal resolution of the valuation $v_1$ the component $E_{σ_h}$ does not intersect the component $E_{σ_{h+1}}$ (the first among those born by blowing-ups at points of the component $E_{σ_h}$). The dual graph of the resolution of the collection $\{v_1\}$ can be restored if one knows the dual graphs of the minimal resolutions of the collections $\{v_{σ_h}, v_2, . . . , v_h\}$ with all $h$ large enough of the described type. The Poincaré series of the collection $\{v_{σ_h}, v_2, . . . , v_h\}$ is obtained from the Poincaré series $P_{\{v_1\}}(t)$ of the collection $\{v_1\}$ by substituting each monomial $t_1^{m_1}t_2^{m_2} . . . t_h^{m_h}$ by $t_1^{m_1}t_2^{m_2} . . . t_h^{m_h}$ where $m_1'$ is defined in the following way. Let $α_N$ be the result of the truncation of the continuous fraction of the (irrational) number $m^{1β_h}$ at the level $N$ large enough. Let $m_1 = k_0m^{1β_1}_1 + . . . + k_{h−1}m^{1β_{h−1}}_1 + k hm^{1β_h}_1$. Then $m'_1 = k_0m^{1β_1}_1 + . . . + k_{h−1}m^{1β_{h−1}}_1 + k h α_N$. □

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