Counting toroidal binary arrays

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Abstract  
A formula for the number of toroidal $m \times n$ binary arrays, allowing rotation of the rows and/or the columns but not reflection, is known. Here we find a formula for the number of toroidal $m \times n$ binary arrays, allowing rotation and/or reflection of the rows and/or the columns.

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1 Introduction

The number of necklaces with \( n \) beads of two colors when turning over is not allowed is

\[
\frac{1}{n} \sum_{d \mid n} \varphi(d) 2^{n/d},
\]

(1)

where \( \varphi \) is Euler’s phi function. When turning over is allowed, the number becomes

\[
\frac{1}{2n} \sum_{d \mid n} \varphi(d) 2^{n/d} + \begin{cases} 
2^{(n-1)/2} & \text{if } n \text{ is odd,} \\
3 \cdot 2^{n/2-2} & \text{if } n \text{ is even.}
\end{cases}
\]

(2)

These are the core sequences A000031 and A000029, respectively, in [2].

Our concern here is with two-dimensional versions of these formulas. We consider an \( m \times n \) binary array. When opposite edges are identified, it becomes what we will call a toroidal binary array. Just as we can rotate a necklace without effect, we can rotate the rows and/or the columns of such an array without effect. The number of (distinct) toroidal \( m \times n \) binary arrays is

\[
\frac{1}{mn} \sum_{c \mid m} \sum_{d \mid n} \varphi(c) \varphi(d) 2^{mn/\text{lcm}(c,d)},
\]

(3)

where lcm stands for least common multiple. This is A184271 in [2]. The diagonal is A179043. Rows (or columns) 2–8 are A184264–A184270. Row (or column) 1 is of course A000031.

Our aim here is to find the formula that is related to (3) in the same way that (2) is related to (1). More precisely, we wish to count the number of toroidal \( m \times n \) binary arrays allowing rotation and/or reflection of the rows and/or the columns. At present, the resulting sequence, the diagonal, and the rows (or columns) other than the first one, are not found in [2]. Row (or column) 1 is of course A000029.

For an alternative description, we could define a group action on the set of \( m \times n \) binary arrays, which has \( 2^{mn} \) elements. If the group is \( C_m \times C_n \), where \( C_m \) denotes the cyclic group of order \( m \), then the number of orbits is given by (3) (see Theorem 1 below). If the group is \( D_m \times D_n \), where \( D_m \) denotes the dihedral group of order \( 2m \), then the number of orbits is given by Theorem 2 below.

To help clarify the distinction between the two group actions, we provide an example. There is no distinction in the \( 2 \times 2 \) case, so we consider the \( 3 \times 3 \) case. When the group is \( C_3 \times C_3 \) (allowing rotation of the rows and/or the columns but not reflection), there are 64 orbits, as shown in Table 1.

When the group is \( D_3 \times D_3 \) (allowing rotation and/or reflection of the rows and/or the columns), there are 36 orbits, as shown in Table 2.

Our interest in the number of toroidal \( m \times n \) binary arrays allowing rotation and/or reflection of the rows and/or the columns derives from the fact that this is the size of the state space of the projection of the Markov chain in [1] under
Table 1: A list of the 64 orbits of the group action given by the group $C_3 \times C_3$ acting on the set of $3 \times 3$ binary arrays. (Rows and/or columns can be rotated but not reflected.) Each orbit is represented by its minimal element in 9-bit binary form. Bars separate different numbers of 1s.

| 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 |
|------|------|------|------|------|------|
| 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 |
| 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 |
| 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 |
| 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 |
| 1 1 1 | 0 1 1 | 0 1 1 | 1 1 1 | 0 1 1 | 0 1 1 |
| 0 1 1 | 0 1 1 | 0 1 1 | 1 1 1 | 0 1 1 | 0 1 1 |
| 0 1 1 | 0 1 1 | 0 1 1 | 1 1 1 | 0 1 1 | 0 1 1 |
| 1 1 1 | 0 1 1 | 0 1 1 | 1 1 1 | 0 1 1 | 0 1 1 |
| 0 1 1 | 0 1 1 | 0 1 1 | 1 1 1 | 0 1 1 | 0 1 1 |
| 1 1 1 | 0 1 1 | 0 1 1 | 1 1 1 | 0 1 1 | 0 1 1 |
Table 2: A list of the 36 orbits of the group action given by the group $D_3 \times D_3$ acting on the set of $3 \times 3$ binary arrays. (Rows and/or columns can be rotated and/or reflected.) Each orbit is represented by its minimal element in 9-bit binary form. Bars separate different numbers of 1s.

$$
\begin{array}{ccccc}
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\end{array}
$$
the mapping that takes a state to the orbit containing it. This reduction of the state space, from 512 states to 36 states in the $3 \times 3$ case for example, makes computation easier.

2 Rotations of rows and columns

Let $X_{m,n} := \{0,1\}^{\{0,1,\ldots, m-1\} \times \{0,1,\ldots, n-1\}}$ be the set of $m \times n$ arrays of 0s and 1s, of which there are $2^{mn}$. Let $a(m, n)$ denote the number of orbits of the group action on $X_{m,n}$ by the group $C_m \times C_n$. In other words, $a(m, n)$ is the number of (distinct) toroidal $m \times n$ binary arrays, allowing rotation of the rows and/or the columns but not reflection.

Theorem 1.

\[ a(m, n) = \frac{1}{mn} \sum_{c \mid m} \sum_{d \mid n} \varphi(c) \varphi(d) 2^{mn/\text{lcm}(c,d)}. \]  

(4)

Proof. By the Pólya enumeration theorem [3],

\[ a(m, n) = \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} 2^{A_{ij}}, \]  

(5)

where $A_{ij}$ is the number of cycles in the permutation $\sigma^i \tau^j$; here $\sigma$ rotates the rows (row 0 becomes row 1, row 1 becomes row 2, \ldots, row $m-1$ becomes row 0) and $\tau$ rotates the columns. For example, $A_{00} = mn$ because the identity permutation has $mn$ fixed points, each of which is a cycle of length 1.

It is well known that, if $d$ divides $n$, then the number of elements of $C_n$ that are of order $d$ is $\varphi(d)$. So if $c$ divides $m$ and $d$ divides $n$, then the number of pairs $(i,j)$ such that $\sigma^i$ is of order $c$ and $\tau^j$ is of order $d$ is $\varphi(c) \varphi(d)$. And if $\sigma^i$ is of order $c$ and $\tau^j$ is of order $d$, then $\sigma^i \tau^j$ is of order lcm$(c,d)$ because $\sigma^i$ and $\tau^j$ commute. Consequently, lcm$(c,d)$ is the length of each cycle of the permutation $\sigma^i \tau^j$, so $A_{ij} = mn/\text{lcm}(c,d)$, and (4) follows from (5).

Clearly, $a(1,n)$ reduces to (1). Table 3 provides numerical values of $a(m, n)$ for small $m$ and $n$.

Table 3: The number $a(m, n)$ of toroidal $m \times n$ binary arrays, allowing rotation of the rows and/or the columns but not reflection, for $m, n = 1, 2, \ldots, 8$.

|   | 2  | 3  | 4  | 6  | 8  | 14 | 20 | 36 |
|---|----|----|----|----|----|----|----|----|
| 2 | 2  | 3  | 4  | 6  | 8  | 14 | 20 | 36 |
| 3 | 14 | 64 | 40 | 108| 362| 1182| 4150|
| 4 | 7  | 64 | 40 | 108| 362| 1182| 4150|
| 5 | 108| 2192| 52488| 35792568| 981706832| 724878160992|
| 6 | 14 | 699600| 35792568| 1988997152| 104715443852| 5864083006500|
| 7 | 108| 2192| 52488| 35792568| 981706832| 724878160992|
| 8 | 14 | 699600| 35792568| 1988997152| 104715443852| 5864083006500|

5
3 Rotations and reflections of rows and columns

Let \( b(m, n) \) denote the number of orbits of the group action on \( X_{m,n} \) by the group \( D_m \times D_n \). In other words, \( b(m, n) \) is the number of (distinct) toroidal \( m \times n \) binary arrays, allowing rotation and/or reflection of the rows and/or the columns.

**Theorem 2.**

\[ b(m, n) = b_1(m, n) + b_2(m, n) + b_3(m, n) + b_4(m, n), \quad (6) \]

where

\[ b_1(m, n) = \frac{1}{4mn} \sum_{c|m} \sum_{d|n} \varphi(c) \varphi(d) \frac{2^{mn/\text{lcm}(c,d)}}{\text{lcm}(c,d)}, \]

\[ b_2(m, n) = \begin{cases} (4n)^{-1}2^{(m+1)n/2} & \text{if } m \text{ is odd} \\ (8n)^{-1}2^{mn/2}2^{(m+2)n/2} & \text{if } m \text{ is even} \end{cases} + \frac{1}{4n} \sum_{d \geq 2, d|n} \varphi(d) 2^{mn/d} \]

\[ + \begin{cases} (4n)^{-1} \sum' \left[ 2^{(m+1)\gcd(j,n)/2} - 2^m \gcd(j,n) \right] & \text{if } m \text{ is odd} \\ (8n)^{-1} \sum' \left[ 2^{m \gcd(j,n)/2} + 2^{(m+2) \gcd(j,n)/2} - 2^m \gcd(j,n) \right] & \text{if } m \text{ is even} \end{cases} \]

with \( \sum' = \sum_{1 \leq j \leq n-1 : n/\gcd(j,n) \text{ is odd}} \) is odd,

\[ b_3(m, n) = b_2(n, m), \]

and

\[ b_4(m, n) = \begin{cases} 2^{(mn-3)/2} & \text{if } m \text{ and } n \text{ are odd,} \\ 3 \cdot 2^{mn/2-3} & \text{if } m \text{ and } n \text{ have opposite parity,} \\ 7 \cdot 2^{mn/2-4} & \text{if } m \text{ and } n \text{ are even.} \end{cases} \]

**Proof.** Again by the Pólya enumeration theorem [3],

\[ b(m, n) = \frac{1}{4mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} [2^{A_{ij}} + 2^{B_{ij}} + 2^{C_{ij}} + 2^{D_{ij}}], \quad (7) \]

where \( A_{ij} \) (resp., \( B_{ij} \), \( C_{ij} \), \( D_{ij} \)) is the number of cycles in the permutation \( \sigma^i \tau^j \) (resp., \( \sigma^i \rho \), \( \sigma^i \theta \), \( \sigma^i \rho \theta \)); here \( \sigma \) rotates the rows (row 0 becomes row 1, row 1 becomes row 2, \ldots, row \( m-1 \) becomes row 0), \( \tau \) rotates the columns, \( \rho \) reflects the rows (rows 0 and \( m-1 \) are interchanged, rows 1 and \( m-2 \) are interchanged, \ldots, rows \( [m/2] - 1 \) and \( m - [m/2] \) are interchanged), and \( \theta \) reflects the columns.

By the proof of Theorem 1, we know the form of \( A_{ij} \), and this gives the formula for \( b_1(m, n) \).

Next we find \( (B_{ij})_0 \), the entries in the 0th column of matrix \( B \). For \( i = 0, 1, \ldots, m-1 \), the permutation \( \sigma^i \rho \) can be described by its effect on the rows
of \{0,1,\ldots,m-1\} \times \{0,1,\ldots,n-1\}. It reverses the first \(m-i\) rows and reverses the last \(i\) rows. Since the reversal of \(k\) consecutive integers has \(k/2\) transpositions if \(k\) is even and \((k-1)/2\) transpositions and one fixed point if \(k\) is odd, the permutation of \{0,1,\ldots,m-1\} induced by \(\sigma^i\rho\) has \((m-1)/2\) transpositions and one fixed point if \(m\) is odd, and \(m/2\) transpositions if \(i\) is even and \(m\) is even, and \((m-2)/2\) transpositions and two fixed points if \(i\) is odd and \(m\) is even. These numbers must be multiplied by \(n\) for the permutation \(\sigma^i\rho\) of \{0,1,\ldots,m-1\} \times \{0,1,\ldots,n-1\}. The results are that \(B_{0i} = (m+1)n/2\) if \(m\) is odd, \(B_{0i} = mn/2\) if \(i\) is even and \(m\) is even, and \(B_{0i} = (m+2)n/2\) if \(i\) is odd and \(m\) is even. Therefore, \((4mn)^{-1} \sum_{i=0}^{m-1} 2B_{0i} = (4n)^{-1} 2^{(m+1)n/2}\) if \(m\) is odd, whereas \((4mn)^{-1} \sum_{i=0}^{m-1} 2B_{0i} = (8n)^{-1} [2^{mn/2} + 2^{(m+2)n/2}]\) if \(m\) is even, and this gives the first term in the formula for \(b_2(m,n)\).

We turn to \(B_{ij}\) for \(i = 0,1,\ldots,m-1\) and \(j = 1,2,\ldots,n-1\). First, by a property of cyclic groups, \(\tau^j\) has order \(d := n/\gcd(j,n)\). If \(d\) is even, then, since \(\sigma^i\rho\) has order 2 (see the preceding paragraph), \(\sigma^i\tau^j\rho\) has order \(d\) and all of its cycles have length \(d\). In this case, \(B_{ij} = mn/d = m \gcd(j,n)\). Suppose then that \(d\) is odd. There are three cases: (i) \(m\) odd, (ii) \(i\) even and \(m\) even, and (iii) \(i\) odd and \(m\) even. Recall that \(\sigma^i\rho\) reverses the first \(m-i\) rows and reverses the last \(i\) rows. In case (i), one row is fixed by \(\sigma^i\rho\), so cycles of \(\sigma^i\tau^j\rho\) in this row have length \(d\) and all others have length \(2d\). We find that \(B_{ij} = n/d + (m-1)n/(2d) = (m+1)n/(2d) = (m+1) \gcd(j,n)/2\). In case (ii), no rows are fixed by \(\sigma^i\rho\), so all cycles of \(\sigma^i\tau^j\rho\) have length \(2d\). It follows that \(B_{ij} = mn/(2d) = m \gcd(j,n)/2\). In case (iii), two rows are fixed by \(\sigma^i\rho\), so cycles of \(\sigma^i\tau^j\rho\) in these rows have length \(d\) and all others have length \(2d\). We conclude that \(B_{ij} = 2n/d + (m-2)n/(2d) = (m+2) \gcd(j,n)/2\). If the formula for \(B_{ij}\) that holds when \(d\) is even were valid generally, we would have the second term in the formula for \(b_2(m,n)\). The third term in the formula for \(b_3(m,n)\) is a correction to the second term to treat the cases (i)-(iii) in which \(d\) is odd.

Next, the formula for \(b_3(m,n)\) follows by symmetry. More explicitly,

\[
b_3(m,n) = \frac{1}{4mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} 2^{C_{ij}} = \frac{1}{4mn} \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} 2^{B_{ij}} = b_2(n,m)
\]

because the number of cycles \(C_{ij}\) of \(\sigma^i\tau^j\rho\) for an \(m \times n\) array is equal to the number of cycles \(B_{ij}\) of \(\sigma^i\tau^j\rho\) for an \(n \times m\) array.

Finally, we consider \(b_4(m,n)\). For \(i = 0,1,\ldots,m-1\) and \(j = 0,1,\ldots,n-1\), \(\sigma^i\tau^j\rho\) has the effect of reversing the first \(m-i\) rows, reversing the last \(i\) rows, reversing the first \(n-j\) columns, and reversing the last \(j\) columns. If \(m\) and \(n\) are odd, then there is one fixed point and \((mn-1)/2\) transpositions, so \(D_{ij} = (mn+1)/2\) for all \(i\) and \(j\), hence \(b_4(m,n) = 2^{(mn-3)/2}\). If \(m\) is odd and \(n\) is even, then \(D_{ij} = mn/2\) for all \(i\) and even \(j\) and \(D_{ij} = mn/2 + 1\) for all \(i\) and odd \(j\). This leads to \(b_4(m,n) = (1/8)[2^{mn/2} + 2^{mn/2+1}] = 3 \cdot 2^{mn/2-3}\). If \(m\) is even and \(n\) is odd, then \(D_{ij} = mn/2\) for even \(i\) and odd \(j\) and \(D_{ij} = mn/2 + 1\) for odd \(i\) and odd \(j\). This leads to the same formula for \(b_4(m,n)\). Finally, if \(m\) and \(n\) are even, then \(D_{ij} = mn/2\) unless \(i\) and \(j\) are both odd, in which case \(D_{ij} = mn/2 + 2\). This implies that \(b_4(m,n) = (1/4)[(3/4)2^{mn/2} + (1/4)2^{mn/2+2}] = 7 \cdot 2^{mn/2-4}\).
This completes the proof.

It is easy to check that \( b(1, n) \) reduces to (2). Table 4 provides numerical values of \( b(m, n) \) for small \( m \) and \( n \).

Table 4: The number \( b(m, n) \) of toroidal \( m \times n \) binary arrays, allowing rotation and/or reflection of the rows and/or the columns, for \( m, n = 1, 2, \ldots, 8 \).

|   | 2  | 3  | 4  | 6  | 8  | 13 | 18 | 30 |
|---|----|----|----|----|----|----|----|----|
| 2 | 3  | 13 | 34 | 78 | 237| 687| 2299|
| 3 | 7  | 13 | 34 | 78 | 237| 687| 2299|
| 4 | 13 | 36 | 158| 708| 4236| 26412| 180070 |
| 5 | 34 | 158| 1450| 14670| 184854| 2445918| 3388844 |
| 6 | 78 | 708| 14076| 340840| 8999762| 2456109576| 6873769668 |
| 7 | 237| 4236| 184854| 8999762| 478070832| 6873769668 |
| 8 | 687| 26412| 2445918| 2456109576| 2456109576| 6873769668 |
| 9 | 1466114420489 | 32166550498774 | 72057630729710704 |

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