A TRUNCATED SIEGEL-WEIL FORMULA AND BORCHERDS FORMS

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Abstract. In this paper we use the regularized Siegel-Weil formula of Gan-Qiu-Takeda to obtain an expression of the integral of the theta function over the truncated modular curve. We apply this result to express the integral over the truncated modular curve of the logarithm of the Borcherds form and we describe explicitly its asymptotic behaviour, and in particular the convergent and divergent contributions. The result provides a complement to the work of Kudla on integrals of Borcherds forms in a limiting case which falls out the range of applications.

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1. Introduction

The integrals of the logarithm of the Borcherds forms have been related to zeta and $L$–values in a wide variety of papers [17], [5], [7]. Due to the geometric nature of the Borcherds forms, those integrals have also been used to understand the arithmetic degrees of arithmetic cycles of Shimura varieties, extending the knowledge of their Chow groups, [6]. In [17] the author studies the integral of the logarithm of the Borcherds forms for certain quasiprojective Shimura varieties associated to the group GSpin, obtaining an expression involving certain Fourier coefficients of Eisenstein series. One of the main tools in [17] is the Siegel-Weil formula in the convergent range of Weil and the one proved in [18]. On account of the eventual divergence of the integral of the theta function over the modular curve, the integral of the
the so-called classical discriminant; according to [3, p. 6] \( \Delta(\tau) \) is the Borcherds form of the Jacobi theta function. In [22, cor. 5.4, p. 21] the author shows that the integral of logarithm of the norm of \( \Delta(\tau) \) diverges. This result points out the different nature between the integrals considered in [17] and the one over the modular curve. A significant variation between [17] and the present paper is that in order to understand our integral we have to replace the integral over the modular curve \( X^{mod} \) by the integral over the truncated modular curve, denoted by \( X^{mod,T} \).

One of the main ingredients for our proof is the Siegel-Weil formula. The classical version of it, given in [30], [31] and [34], relates the integral of certain theta function with a special value of an Eisenstein series. Let \( V \) be a rational quadratic space of dimension \( m \) with Witt index \( r \). For \( n \geq 1 \) the tuple \((\text{Sp}_n, \mathcal{O}(V))\) forms a dual reductive pair. Given \( \varphi \in \mathcal{S}(V^n(\mathcal{A})) \), where \( \mathcal{S} \) denotes the space of Schwartz functions, we construct a theta function \( \theta(g, h, \varphi) : \text{Sp}_n(\mathcal{A}) \times \mathcal{O}(V)(\mathcal{A}) \to \mathbb{C} \). The convergence of \( \int_{\mathcal{O}(V)} \theta(g, h, \varphi)dh \) depends on the constants \( m = \dim(V), r \) and \( n \). When \( r = 0 \) or \( m - r > n + 1 \), we say that the datum is in the convergent range of Weil and by [34] and [14], the theta function is absolutely convergent and

\[
\int_{\mathcal{O}(V)} \theta(g, h, \varphi)dh = \sum_{\gamma \in P_{\text{Sp}_n}(\mathfrak{Q})} \lambda(\varphi(\gamma g)),
\]

where \( k \) is an explicit constant, \( P_{\text{Sp}_n} \) is the Siegel parabolic of \( \text{Sp}_n \) and

\[
\lambda : S(V^n(\mathcal{A})) \to I_n \left( \frac{m}{2} - 1, \chi V \right),
\]

is a map where \( I_n \left( \frac{m}{2} - 1, \chi V \right) \) is the degenerated principal series representation of \( \text{Sp}_n(\mathcal{A}) \). It is known that \( \lambda \) is a realization of the local theta correspondence of the identity, which has been extensively studied throughout [19, sec. 2, p. 17]. In order to approach the remaining cases, under certain hypothesis on \( V \) and \( n \), [19] and [13] developed a regularization of the theta function. It is based on an operator

\[
\omega(\alpha) : S(V^n(\mathcal{A})) \to S(V^n(\mathcal{A}))_{abc},
\]

where \( S(V^n(\mathcal{A}))_{abc} = \{ \varphi \in S(V^n(\mathcal{A})) \mid \text{st.} \theta(g, h, \varphi) \text{ absolutely convergent} \} \). The map \( \omega(\alpha) \) is constructed using the action of an explicit element of the Hecke algebra of \( \mathcal{O}(V) \) on \( S(V^n(\mathcal{A})) \). This machinery allows us to define a meromorphic function \( B(g, \varphi, s) \), [9, sec. 3.5, p. 18], which replaces the role of \( \int_{\mathcal{O}(V)} \theta(g, h, \varphi)dh \) in the classical Siegel-Weil formula. It is known that \( B(g, \varphi, s) \) has a pole at \( s = \frac{m-r-1}{2} \) of order at most 2. We denote its \( i \)-th Laurent coefficient by \( B_i(g, \varphi) \). The so-called first and second term identity of the Siegel-Weil formula relate \( B_{-2}(g, \varphi) \) and \( B_{-1}(g, \varphi) \) with special values of Eisenstein series (and their residues).

In this paper we consider the case \( m = 3, r = 1 \) and \( n = 1 \). Geometrically it corresponds to the modular curve case. The first goal is to obtain an expression for the integral of the theta function associated to \( \varphi_{\infty} \in \mathcal{S}(V(\mathcal{A})) \); certain Schwartz function defined using the geometry of the modular curve. It turns out that this integral does not converge, hence in order to obtain some "truncated expression" for it we use the regularized Siegel-Weil formula. One of the available theorems in this situation is given by [9, thm. 8.1, (ii), p. 35]

\[
\text{CT}_{s=1/2} E(g, s, \lambda(\varphi_{\infty}^{\infty})) = B_{-1}(g, \varphi_{\infty}^{\infty}) + c \text{Res}_{s=1/2} E(g, s, \lambda(\varphi_{\infty})),
\]

where \( \text{CT}_{s=1/2} E(g, s, \lambda(\varphi_{\infty}^{\infty})) \) and \( \text{Res}_{s=1/2} E(g, s, \lambda(\varphi_{\infty}^{\infty})) \) are respectively the constant and residue terms of the Laurent series at \( s = 1/2 \) of these Eisenstein series and \( c \in \mathbb{C} \). A drawback of this formula is that we can not recover information about \( \int_{\mathcal{O}(V)} \theta(g, h, \varphi_{\infty}^{\infty})dh \) directly. To that end we use the mixed model of the Weil representation [19, prop. 5.2.1, p. 44, prop. 5.3.1, p. 45]. It allows us to factor the theta function as follows:

\[
\theta(g, h, \varphi_{\infty}^{\infty}) = \text{Conv}(g, h, \varphi_{\infty}^{\infty}) + \text{Div}(g, h, \varphi_{\infty}^{\infty}),
\]

where \( \int_{\mathcal{O}(V)} \text{Conv}(g, h, \varphi_{\infty}^{\infty})dh \) is absolutely convergent and \( \int_{\mathcal{O}(V)} \text{Div}(g, h, \varphi_{\infty}^{\infty})dh \) diverges. Using the action of \( \omega(\alpha) \) we obtain a relation between \( B_{-1}(g, \varphi_{\infty}^{\infty}) \) and \( \int_{\mathcal{O}(V)} \text{Conv}(g, h, \varphi)dh \).
Let $\varphi_\infty$ be the Schwartz function defined in (19) and let $\text{Conv}(g_\tau, h, \varphi_\infty)$ be the absolute convergent part of $\theta(g_\tau, h, \varphi_\infty)$. Then

$$\int_{[O(V)]} \text{Conv}(g_\tau, h, \varphi_\infty)dh = E(g_\tau, 1/2, \lambda(\varphi_\infty)) + c \text{Res}_{s=1/2} E(g, s, \lambda(\varphi)),$$

where $c \in \mathbb{C}$.

Let $X^{\text{mod}}$ be the modular curve without level, fix $\hat{T} > 1$ and set $X^{\text{mod}, \hat{T}} = \{ z = x + iy \in X^{\text{mod}} \ \text{s.t.} \ y < \hat{T}\}$. The main body of this paper is devoted to obtain an explicit expression of

$$\int_{X^{\text{mod}, \hat{T}}} \log |\Psi(f)(z)||^2 d\mu(z),$$

where $f \in M_{L/2, \mathbb{Z}}$, $\Psi(f)(z)$ is the Borcherds form of $f$, $|| \cdot ||_{\text{Pet}}$ is the Peterson norm and $d\mu$ is the hyperbolic measure. The function $\Psi(f)$ is closely related to the singular theta lift, hence the computation can be reduced to

$$\int_{X^{\text{mod}, \hat{T}}} \int_{X^{\text{mod}}} \theta^{\text{Sieg}}(\tau, z)f(\tau)d\mu(\tau)d\mu(z),$$

where $\int_{X^{\text{mod}}}$ is the regularization proposed in [2] to ensure the convergence of the singular theta lift. Due to the behaviour of the Fourier constant term of $\theta^{\text{Sieg}}(\tau, z)$ the order of the previous two integrals can not be exchanged. With the aim of accomplishing the computation we treat separately the integrals involving the non constant and constant terms of $\theta^{\text{Sieg}}(\tau, z)$. The first one is approached following the method developed in [17] that requires the truncated version of the Siegel-Weil formula stated above. The second integral is computed via an unfolding of the theta function with the integral. To ensure the convergence in this unfolding we introduce an auxiliary Eisenstein series and to conclude we apply the truncated version of the Rankin-Selberg formula developed by Zagier.

Theorem (5.32). The integral $\int_{X^{\text{mod}, \hat{T}}} \log |\Psi(f)(z)||^2 d\mu(z)$ diverges as $\log(\hat{T})$. The non divergent term is an explicit combination of Fourier coefficients of Eisenstein series, gamma values, their derivatives and values of the completed Riemann zeta function. Furthermore the coefficient of the divergent term has an interpretation in terms of special values of zeta functions.

The second section is devoted to state the geometric setting of the paper. In the third section we explain the relation between the regularized theta correspondence and the adelic theta correspondence. Moreover we describe explicitly the operator $\omega(a)$. In section 4 we prove the truncated version of the Siegel-Weil formula. Section 5 is the main body of the paper, where we prove the main result. This proof is divided into two cases; the ordinary case; whose main ingredient is the truncated Siegel-Weil formula, and the limit case; approached by the unfolding of the integral with the theta series. In the final section we prove some technical computations that are used in section 5.

1.0.1. Notation. Given $G$ an algebraic group defined over $\mathbb{Q}$, we denote by $G_{\mathbb{Q}_p} := G_{\mathbb{Q}_p} \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(\mathbb{Q}_p)$ the base change of $G$ to $\mathbb{Q}_p$. We use the notation $[G] = G(\mathbb{Q}) \setminus G(\mathbb{A})$. Given $(V, q)$ a quadratic space we denote by $m := \dim V$, $r$ its Witt index, $V_{an}$ its maximal anisotropic subspace and $(\cdot, \cdot)$ the bilinear form associated to $q$. Let us denote by $O(V)$ the orthogonal algebraic group. Furthermore, given a rational symplectic space of dimension $2n$ we denote by $\text{Sp}(W)$ its symplectic group, which is an algebraic group defined over $\mathbb{Q}$. There is a choice of basis for $W$ so that $\text{Sp}(W) \simeq \text{Sp}_n$. Given a topological space $T$ and a subspace $S$ we denote by $\text{char}(S) : T \to \{0, 1\}$ the characteristic function of $S$. We fix $\psi_{\infty}$ the character on $\mathbb{R}$ given by $\psi_{\infty}(x) = e^{\pi i x^2}$ and the unique characters $\psi_p$ on $\mathbb{Q}_p$ that are additive, whose restriction to $\mathbb{Z}_p$ is trivial and they satisfy that $\psi_p(p^{-1}) = e^{2\pi i / p}$.

Let $\mathcal{H}$ be the Poincaré half plane. We fix two complex variables $\tau = u + iv$ and $z = x + iy$. Let $\lambda_0 = 0$ and $\lambda_1 = \frac{1}{3}$, given a modular form $f_j(\tau) = \sum_{n \in \mathbb{Z} + \lambda_j} c_j(n)e^{2\pi int} = \sum_{n \in \mathbb{Z} + \lambda_j} c_j(n)q^n$ we denote its constant Fourier coefficient by

$$f_j(\tau)_0 := \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} f_j(u + iv)du.$$
Given a meromorphic function $F(s)$, $s_0 \in \mathbb{C}$ and $C_{s_0}$ a closed curve around $s_0$ we denote by

$$\text{FT}_{s=s_0} F(s) = \int_{C_{s_0}} \frac{F(s)}{(s-s_0)^2} ds,$$

the first term in the Laurent series of $F(s)$ at $s = s_0$, by

$$\text{CT}_{s=s_0} F(s) = \int_{C_{s_0}} \frac{F(s)}{(s-s_0)} ds,$$

the constant term in the Laurent series at $s = s_0$ and by

$$\text{Res}_{s=s_0} F(s) = \int_{C_{s_0}} F(s) ds,$$

the residue of $F(s)$ at $s = s_0$. The Euler-Mascheroni constant is denoted by $\gamma := -\Gamma'(1)$.

This relevant constant will appear in the computations:

$$A = 8(\Gamma'(1) - 1) \text{erf} \left( \sqrt{\frac{\pi}{2}} \right) + \log(4) \text{erf} \left( \sqrt{\frac{\pi}{2}} \right) + \sqrt{2\tilde{B}},$$

with

$$\tilde{B} = \left( e^{\gamma} \frac{\pi}{2} \right)^{1/2} \left( \Gamma' \left( -1/2, \frac{\pi}{2} \right) - \Gamma \left( -1/2, \frac{\pi}{2} \right) \right),$$

where $\Gamma(a, b)$ is the incomplete Gamma function.

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2. Borcherds form

2.1. GSpin Shimura varieties. Throughout this section we fix $(V, q)$ a rational quadratic space of dimension $m$ and of signature $(p^+, p^-) = (2, p)$ with $p \geq 1$. We let $(\cdot, \cdot)$ be the symmetric bilinear form associated to $q$. The general Spin group $\tilde{H} := \text{GSpin}_V$ is defined as the central extension of the special orthogonal group $SO(V)$ that fits in the following exact sequence:

$$1 \rightarrow \mathbb{G}_m \rightarrow \tilde{H} \rightarrow SO(V) \rightarrow 1.$$  

(3)

Using the quadratic form $q$ we factor the vector space $V(\mathbb{R})$ into the direct sum of two vector spaces

$$V(\mathbb{R}) = V^+(\mathbb{R}) \oplus V^- (\mathbb{R}),$$

(4)

where $V^+(\mathbb{R})$ is a totally positive definite quadratic space and $V^-(\mathbb{R})$ is a totally negative definite quadratic space. The group $SO(2) \oplus SO(p)$ is a maximal compact subgroup of $SO(V)(\mathbb{R})$. Following [24, prop. 1.6, p. 12] we construct a Hermitian symmetric domain as follows:

$$D_V = SO(V)(\mathbb{R}) / (SO(2) \oplus SO(p)).$$

Let us fix $K < \tilde{H}(\mathbb{A}_f)$ a compact open subgroup. We define the Shimura variety associated to $\tilde{H}$ with level $K$ by the following double quotient:

$$X_K = \tilde{H}(\mathbb{Q}) \setminus D_V \times \tilde{H}(\mathbb{A}_f)/K.$$  

(5)

It is an scheme defined over a number field called reflex field [24, def. 11.1, p. 107]. According to [24, prop. 5.13, p. 57] this double quotient is isomorphic to

$$\bigsqcup_{g \in C} \Gamma_g \setminus D_V,$$

(6)

where $C$ is a set of representatives of $\tilde{H}(\mathbb{Q})_+ \setminus \tilde{H}(\mathbb{A}_f)/K$ and $\Gamma_g = gKg^{-1} \cap \tilde{H}(\mathbb{Q})_+$ with $\tilde{H}(\mathbb{Q})_+ = \tilde{H}(\mathbb{Q}) \cap \tilde{H}(\mathbb{R})^+$ the intersection of the rational points of $\tilde{H}$ and the identity component of the real points of $H$. 

From now on we will fix $L$ an integral lattice of $V$ so that $L \otimes \mathbb{Q} = V$. We denote the set of isometries of $L$ by $SO(L)$. The dual lattice of $L$ is defined as follows:

$$L' := \{ l \in V(\mathbb{R}), \text{ s.t. } (\lambda, l) \in \mathbb{Z}, \text{ for all } \lambda \in L \}.$$ 

Moreover set

$$\Gamma_L := \{ g \in SO(L), \text{ s.t. } g \text{ acts trivially on } L'/L \}.$$ 

The quotient

$$X_{\Gamma_L} := \Gamma_L \setminus D_V,$$

admits a projective compactification $X_{\Gamma_L}^*$ by the theory of Baily-Borel [23].

**Proposition 2.1.** There exists an isomorphism of $C^\infty$–manifolds

$$D_V \cong K := \{ Z \in \mathbb{P}V(\mathbb{C}) \text{ s.t. } (Z, Z) = 0, (Z, \overline{Z}) > 0 \},$$

where $K$ is called the projective cone model.

**Proof.** See [12, p. 12]. \qed

**Example 2.2.** Let $(V, q)$ be a rational isotropic quadratic space of signature $(2,1)$. The quotient $X_{\Gamma_L}$ satisfies the following isomorphism of differentiable manifolds

$$\Gamma_L \setminus D_V \cong SL_2(\mathbb{Z}) \setminus H,$$

where the right hand side is the open modular curve without level. Moreover, we observe that using the description of the quadratic space given in [8, (3.1), p. 295] the lattice $L$ satisfies that

$$L'/L \cong \left( \mathbb{Z} \oplus \frac{1}{2} \mathbb{Z} \oplus \mathbb{Z} \right) / (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \cong \frac{1}{2} \mathbb{Z}/\mathbb{Z}.$$ 

2.2. Borcherds forms.

**Definition 2.3.** We define the real metaplectic group $Mp_2(\mathbb{R})$ as the following double cover of $SL_2(\mathbb{R})$. The elements are given as pairs $(M, \phi(\tau))$ where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $\phi(\tau)$ is a holomorphic square root of $ct + d \in \mathbb{C}$. Given $(M_1, \phi_1(\tau))$, $(M_2, \phi_2(\tau)) \in Mp_2(\mathbb{R})$ the group law in $Mp_2(\mathbb{R})$ is defined by

$$(M_1, \phi_1(\tau)) \cdot (M_2, \phi_2(\tau)) = (M_1M_2, \phi_1(M_2\tau)\phi_2(\tau)).$$

Let us fix an integral lattice $L \subset V$. We denote by

$$\rho_L : Mp_2(\mathbb{R}) \rightarrow Aut(C[L'/L]),$$

the Weil representation associated to $L$ defined in [4, p. 15].

**Definition 2.4.** Let $k \in \frac{1}{2} \mathbb{Z}$. A weakly holomorphic modular form of weight $k$ is a smooth function $f : H \rightarrow \mathbb{C}[L'/L]$ that is holomorphic on $H$, meromorphic at the cusp $\infty$ and satisfies the following transformation law:

$$f(\tau) = \phi(\tau)^{-2k} \rho_L^*(M, \phi(\tau))^{-1} f(M\tau),$$

where $\rho_L^*(M, \phi(\tau))$ is the dual of $\rho_L$. The space of weakly holomorphic modular forms will be denoted by $M_{k,L}^!$.

Using [12, sec. 2.1, p. 11] one can show that there exists an isomorphism of $C^\infty$–manifolds

$$\omega : D_V \rightarrow Gr(V),$$

where $Gr(V)$ is the connected component of the Grassmanian of positive definite subspaces of $V$. For $z \in D$ and $x \in V(\mathbb{R})$ we denote by $x_z := x_{\omega(z)}$ and by $x_{z \perp} := x_{\omega(z) \perp}$, where we recall that $\omega$ is the isomorphism (8). Given $x \in V(\mathbb{R})$ and $z \in D$ we use (4) to factor $x$ as follows:

$$x = x_z + x_{z \perp}.$$ 

Let us define

$$\phi : V(\mathbb{R}) \times H \times D_V \rightarrow \mathbb{C},$$

$$(x, \tau, z) \mapsto e^{2\pi i (q(x_z)\tau + q(x_{z \perp})\overline{\tau})},$$

where $x_z := x_{\omega(z)}$. The function $\phi$ is exponentially decreasing on $V(\mathbb{R})$. 
Definition 2.5. The Siegel theta function is defined by
\[ \Theta^\Sigma_{L} : \mathcal{H} \times D_V \to \mathbb{C}, \]
\[ (\tau, z) \mapsto \sum_{\mu \in \mathbb{L}/L} \theta^\Sigma_{\mu}(\tau, z)e_{\mu}, \]
where \( \theta^\Sigma_{\mu}(\tau, z) = \sum_{\lambda \in \mu + \mathbb{L}} e^{2\pi i (q(\lambda)\tau + \bar{q}(\lambda)\bar{\tau})} \) and \( \{ e_{\mu} \}_{\mu \in \mathbb{L}/L} \) is a basis of \( \mathbb{C}[\mathbb{L}/L] \).

Proposition 2.6. The Siegel theta function is a real analytic function in \( (\tau, z) \in \mathcal{H} \times D_V \) and satisfies the following transformation law:
\[ \Theta^\Sigma_{L}(N\tau, Nz) = \phi(\tau)^{2\frac{y}{y+\bar{y}}} \rho_L(M, \phi) \Theta^\Sigma_{L}(\tau, z), \]
where \( (M, \phi) \in \text{Mp}_2(\mathbb{R}) \) and \( \gamma \in \Gamma_L \).

Proof. We refer the reader to [4, thm. 2.1, p. 40]. \( \square \)

Let us denote by \( \langle \cdot, \cdot \rangle \) the inner product on \( \mathbb{C}[\mathbb{L}/L] \) defined by
\[ \langle \sum_{\mu \in \mathbb{L}/L} a_{\mu}e_{\mu}, \sum_{\nu \in \mathbb{L}/L} b_{\nu}e_{\nu} \rangle = \sum_{\gamma \in \mathbb{L}/L} a_{\gamma}b_{\gamma}. \]

Definition 2.7. Let \( k \in \frac{1}{2}\mathbb{Z} \) and \( f \in \mathbb{M}_{k, L} \). The singular theta lift is defined as follows:
\[ \Phi(f)(z) := CT_{\sigma=0} \lim_{T \to \infty} \int_{\mathcal{F}^T} \langle f(\tau), \Theta^\Sigma_{L}(\tau, z) \rangle v^{1-\sigma} d\mu(\tau), \]
where \( \mathcal{F}^T = \{ \tau = u + iv \in \mathcal{H}, \text{ s.t. } v < T \} \).

Let \( f(\tau) = \sum_{\mu \in \mathbb{L}/L} \sum_{n \geq -\infty} c_n(\mu)q^n e_{\mu} \in \mathbb{M}_{k, L} \) with \( c_n(\mu) \in \mathbb{Z} \) for \( n \leq 0 \). The function \( \Phi(f)(z) \) is smooth in \( D_V \setminus \mathcal{Z}(f) \), where
\[ \mathcal{Z}(f) = \sum_{\mu \in \mathbb{L}/L} \sum_{n < 0} c_n(\mu) \sum_{\lambda \in \mu + \mathbb{L}} \lambda^2, \]
is the so-called Heegner divisor associated to \( f \). Furthermore it is a \( \Gamma_L \)-invariant function. To lighten notation we will use
\[ \int \langle f(\tau), \Theta^\Sigma_{L}(\tau, z) \rangle v^{1-\sigma} d\mu(\tau). \]

Proposition 2.8. Let \( k \in \frac{1}{2}\mathbb{Z} \) and \( f \in \mathbb{M}_{k, L} \). The singular theta lift is a real analytic function on \( D_V \setminus \mathcal{Z}(f) \) and has singularities of logarithm type along the divisor \(-2\mathcal{Z}(f)\) in the sense of [12, def. 3.6, p. 24].

Proof. The proof is in [2, thm. 6.2, p. 24]. \( \square \)

Theorem 2.9. Let \( k \in \frac{1}{2}\mathbb{Z} \) and \( f \in \mathbb{M}_{k, L} \) such that \( f(\tau) = \sum_{\mu \in \mathbb{L}/L} \sum_{n \geq -\infty} c_n(\mu)q^n e_{\mu} \) with \( c_n(\mu) \in \mathbb{Z} \) for \( n \leq 0 \). There exists a function
\[ \Psi(f) : D_V \to \mathbb{C}, \]
such that
1. \( \Psi(f) \) is a meromorphic modular form of weight \( c_0(0)/2 \) with respect to the group \( \Gamma_L \).
2. The function \( \Psi(f) \) satisfies the following relation:
\[ \log |\Psi(f)(z)| = -\frac{\Phi(f)(z)}{4} - \frac{c_0(0)}{2} \left( \log |y| + \Gamma'(1)/2 + \log \sqrt{2\pi} \right), \]
where \( z = x + iy \).

The function \( \Psi(f) \) is called the Borcherds form of \( f \).

Proof. This is [2, thm. 13.3, p. 48]. \( \square \)

3. Siegel-Weil formula

3.1. Hecke algebra. Let us choose a finite place \( p \) of \( \mathbb{Q} \) and let \( G \) be a split reductive group defined over \( \mathbb{Q}_p \). Throughout this subsection \( G_m \) is the algebraic torus defined over \( \mathbb{Q}_p \). We fix a maximal torus \( T \) and a Borel subgroup \( B \) of \( G \). We consider a locally compact group \( G(\mathbb{Q}_p) \), obtained by taking the
$\mathbb{Q}_p$–rational points of $G$. The maximal compact subgroup of $G(\mathbb{Q}_p)$ is $K_p := G(\mathbb{Z}_p)$. In this section we assume previous knowledge about the structure of the algebraic reductive groups. To see the definitions of characters, cocharacters, roots and Weyl chambers we refer the reader to [26] and [25, sec. 22, p. 381].

**Definition 3.1.** The Weyl group of the tuple $(G, T)$, denoted by $W(G, T)$, is the group generated by the reflections about the hyperplanes perpendicular to the roots of $(G, T)$.

**Definition 3.2.** The Hecke algebra $\mathcal{H}_P^G$ is the ring of locally constant compactly supported functions $G(\mathbb{Q}_p) \to \mathbb{Z}$ which are $K_p$–bi-invariant. The multiplication is defined via convolution; more concretely, given $f, g \in \mathcal{H}_P^G$

$$ (f \ast h)(g) = \int_{G(\mathbb{Q}_p)} f(x)h(x^{-1}g)dx, $$

where $dx$ is the Haar measure of $G(\mathbb{Q}_p)$ giving $K_p$ volume 1.

**Proposition 3.3.** Let $X_\alpha(T)$ be the group of cocharacters of the torus $T$. The group $G(\mathbb{Q}_p)$ is the disjoint union of the double cosets $K_p\mu(p)K_p$, where $\mu$ runs throughout the cocharacters of $T$ which belong to the positive Weyl chamber.

Proof. See [33, p. 51].

**Proposition 3.4.** The Hecke algebra $\mathcal{H}_P^T$ is commutative. Let us denote by

$$ c_\mu := \text{char}_{(K_p \cap T(\mathbb{Q}_p))\mu(p)(K_p \cap T(\mathbb{Q}_p))}. $$

The map

$$ \mathcal{H}_P^T \to \mathbb{Z}[X_\alpha(T)], $$

$$ c_\mu \mapsto [\mu], $$

is an isomorphism.

Proof. We refer the reader to [11, p. 5].

**Definition 3.5.** The Satake transform is defined by the following ring homomorphism:

$$ S : \mathcal{H}_P^G \to \mathcal{H}_P^T \otimes \mathbb{Z}[p^{1/2}, p^{-1/2}] \approx \mathbb{Z}[X_\alpha(T)] \otimes \mathbb{Z}[p^{1/2}, p^{-1/2}], $$

$$ f \mapsto S(f)(t) = (\delta(t))^{1/2} \int_{N(\mathbb{Q}_p)} f(tn)dn, $$

where $\delta$ is the modulus character of the Borel subgroup $B(\mathbb{Q}_p)$, and $dn$ is the right invariant Haar measure of $N$, the maximal unipotent subgroup of $B$.

**Remark 3.6.** Using proposition 3.4 we can realize the Satake transform as a homomorphism

$$ S : \mathcal{H}_P^G \to \mathbb{Z}[X_\alpha(T)] \otimes \mathbb{Z}[p^{1/2}, p^{-1/2}]. $$

This point of view will be more suitable to understand explicitly the Hecke algebra.

**Theorem 3.7.** The Satake transform induces a ring isomorphism

$$ \mathcal{H}_P^G \simeq \mathbb{Z}[X_\alpha(T)]^{W(G, T)} \otimes \mathbb{Z}[p^{1/2}, p^{-1/2}]. $$

Proof. See [29, thm. 3, p. 253].

**Example 3.8.** In this example we compute $\mathbb{Z}[X_\alpha(T_n)]^{W(G, T)}$ as a ring of polynomials when $G = \text{Sp}_{n, \mathbb{Q}_p}$, the symplectic group defined over $\mathbb{Q}_p$ and $T = \{a_1 \cdots a_n \mid a_i^{-1} \cdots a_{i+1}^{-1} 
 a_i \in \mathbb{G}_m\}$. Let us observe that the cocharacters of the torus are $\mathbb{Z}$–linear combinations of the following maps:

$$ \hat{\alpha}_i : \mathbb{G}_m \to T \simeq \mathbb{G}_m^n, $$

$$ c \mapsto 1 \times \cdots \times 1 \times c \times 1 \times \cdots \times 1 \times c^{-1} \times 1 \times \cdots \times 1, $$

where $i < n$. The group $W(G, T)$ acts on the set of cocharacters $\{\hat{\alpha}_i\}_{i=1}^n$ as the permutation group of degree $n$, denoted by $S_n$. Therefore the following identification holds:

$$ \mathbb{Z}[X_\alpha(T_n)]^{W(G, T)} \simeq \mathbb{Z}[\hat{\alpha}_1, \ldots, \hat{\alpha}_n]^{S_n}. $$

(9)
3.2. Siegel Eisenstein series. Let \( (W, \langle \cdot, \cdot \rangle) \) be a non degenerate symplectic space of dimension \( 2n \) defined over \( \mathbb{Q} \). There is a basis \( e_1, \ldots, e_n, e'_1, \ldots, e'_n \in W \) satisfying that \( \langle e_i, e_j \rangle = \langle e'_i, e'_j \rangle = 0 \) and \( \langle e_i, e'_j \rangle = \delta_{ij} \). This choice of basis for \( W \) determines an isomorphism of algebraic groups

\[
\text{Sp}(W) \cong \text{Sp}_n.
\]

Let us denote by \( P \) the Siegel parabolic subgroup of \( \text{Sp}_n \), i.e. the standard maximal parabolic subgroup that fixes the subspace spanned by \( e'_1, \ldots, e'_n \). This parabolic subgroup has Levi decomposition \( P = MN \),

\[
M(\mathbb{Q}) = \{ m(a) := \left( \begin{smallmatrix} a & \ast \\ 0 & a^{-1} \end{smallmatrix} \right), \ s.t. \ a \in \text{GL}_n(\mathbb{Q}) \}, \\
N(\mathbb{Q}) = \{ n(b) := \left( \begin{smallmatrix} 1 & \ast \\ 0 & 1 \end{smallmatrix} \right), \ s.t. \ b = b^t b \in M_n(\mathbb{Q}) \}.
\]

For \( p \) a non archimedian place of \( \mathbb{Q} \) the maximal compact open subgroup of \( \text{Sp}_n(\mathbb{Q}_p) \) is given by \( K_p := \text{Sp}_n(\mathbb{Z}_p) \). For the archimedian place, denoted by \( \infty \), the maximal compact subgroup of \( \text{Sp}_n(\mathbb{R}) \) is given by

\[
K_{\infty} := \{ \left( \begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right), \ s.t. \ a + ib \in U(n) \},
\]

where \( U(n) \) is the unitary group of rank \( n \). Using the Iwasawa decomposition of \( \text{Sp}_n \) we factor the adelic points of the symplectic group by

\[
\text{Sp}_n(\mathbb{A}) = P(\mathbb{A})K,
\]

where \( K = K_{\infty} \times \prod_p K_p \). According to this factorization of \( \text{Sp}_n(\mathbb{A}) \) we write the elements \( g \in \text{Sp}_n(\mathbb{A}) \) as follows:

\[
g = m(a)n(b)k,
\]

with \( a \in \text{GL}_n(\mathbb{A}) \) and \( b \in \text{Sym}_n(\mathbb{A}) \). Further, throughout this paper we denote

\[
|a(g)| := |\det(a)|_\mathbb{A}.
\]

Let \( \text{Mp}_{2n}(\mathbb{A}) \) be the twofold cover of \( \text{Sp}_{2n}(\mathbb{A}) \) defined in [32, p. 39]. This group fits in the exact sequence:

\[
1 \to \{ \pm 1 \} \to \text{Mp}_{2n}(\mathbb{A}) \to \text{Sp}_{2n}(\mathbb{A}) \to 1.
\]

The group \( \text{Mp}_{2n}(\mathbb{A}) \) has underlying set \( \text{Sp}_{2n}(\mathbb{A}) \times \{ \pm 1 \} \) with multiplication given by the so-called Rao cocycle [32, sec. 1.5, p. 23]. The local analogue \( \text{Mp}_{2n}(\mathbb{Q}_p) \) is defined in [32, p. 38]. It is endowed with a Haussdorff topology via the Weil representation [32, sec. 1.8, p. 38]. Since the properties of the group \( \text{Mp}_{2n}(\mathbb{A}) \) or \( \text{Mp}_{2n}(\mathbb{Q}_p) \) are not essential to address the main goal of this paper we will not discuss them here. Despite this, the structure of this group is essential to understand the theory and we refer the reader to [32, sec. 1, p. 10] for a detailed discussion of the topic. The parabolic subgroups of \( \text{Mp}_{2n}(\mathbb{A}) \) are given by the inverse images of the parabolic subgroups of \( \text{Sp}_{2n}(\mathbb{A}) \) via the exact sequence (10). Given a parabolic subgroup \( P \) of \( \text{Sp}_n(\mathbb{A}) \) we will denote by \( \tilde{P}(\mathbb{A}) \) its inverse image under the map given by (10).

The Levi component of the Siegel parabolic subgroup of \( \text{Mp}_{2n}(\mathbb{A}) \) satisfy the following isomorphism:

\[
\tilde{M}(\mathbb{A}) \cong \text{GL}_n(\mathbb{A}),
\]

where \( \text{GL}_n(\mathbb{A}) \) is the twofold cover of \( \text{GL}_n(\mathbb{A}) \) with underlying set \( \text{GL}_n(\mathbb{A}) \times \{ \pm 1 \} \) and group law given by

\[
(g_1, \eta_1) \cdot (g_2, \eta_2) = (g_1g_2, \eta_1\eta_2 (\det g_1, \det g_2)\eta_2),
\]

where \( (\cdot, \cdot)_\mathbb{A} \) is the adelic Hilbert symbol, i.e.

\[
(\cdot, \cdot)_\mathbb{A} := \prod_p \langle \cdot, \cdot \rangle_p, \text{ with } \langle \cdot, \cdot \rangle_p \text{ the classical Hilbert symbols}.
\]

The determinant map lifts to a homomorphism

\[
\det : \text{GL}_n(\mathbb{A}) \to \mathbb{C}_\times.
\]

In [28, Appendix p. 365] the author defines the Weil index, a function depending on a character \( \psi_p \). In [9, p. 10] the author claims that it gives rise to a genuine character

\[
\chi_{\psi_p} : \text{GL}_1(\mathbb{A}) \to \mathbb{C}_\times.
\]

Let \( \langle V, q \rangle \) be a rational quadratic space of odd dimension \( m \). We define an adelic character \( \chi_V : \mathbb{Q}_\mathbb{A}^\times \to \mathbb{C}_\times \) by

\[
\chi_V(x) := \prod_p \left( x_p, (-1)^{m(m-1)/2} \det(V) \right)_p.
\]

Using the previous functions we consider the following character of \( \tilde{M}(\mathbb{A}) \):

\[
\tilde{\chi}_V(p) := (\chi_{\psi_p} \circ \det)(\chi_V \circ \det).
\]
Definition 3.9. Let $V$ be a quadratic space of odd dimension, we define the degenerate principal series representation of $\text{Mp}_{2n}(\mathbb{A})$ as follows:

$$I_n(s, \chi_V) := \text{Ind}_{P(\mathbb{A})}^{\text{Mp}_{2n}(\mathbb{A})} (\chi_V \cdot |\cdot|^s).$$

We consider the elements of the induction that are smooth functions $\Phi(g, s)$ on $\text{Mp}_{2n}(\mathbb{A})$ and we point out that the induction is non normalized.

Definition 3.10. Let $\Phi(g, s) \in I_n(s, \chi_V)$, then

$$M(s) : I_n(s, \chi_V) \to I_n(-s, \chi_V^{-1}),$$

$$\Phi(g, s) \mapsto \int_{N(\mathbb{A})} \Phi(\omega g, s) d\omega,$$

where $\omega$ is the longest Weyl element of $\text{Sp}_{2n}$, $N$ is the maximal unipotent subgroup of $\text{Sp}_{2n}$ and $d\omega$ is the Haar measure of $N(\mathbb{A})$.

Let us denote by $M(s) = M(\mathbb{A}) \times M(s)$ the factorization of the previous operator according to the infinite and finite places.

Definition 3.11. Let $\Phi(g, s) \in I_n(s, \chi_V)$ be a holomorphic section. The Siegel Eisenstein series associated to $\Phi(g, s)$ is defined by

$$E_{\text{Mp}_{2n}}(g, s, \Phi) := \sum_{\gamma \in P(\mathbb{Q})/\text{Sp}_{2n}(\mathbb{Q})} \Phi(\gamma g, s),$$

with $P$ the Siegel parabolic of $\text{Sp}_{2n}$. If the group $\text{Mp}_{2n}$ is clear we will suppress the superscript from the notation.

Remark 3.12. The Siegel Eisenstein series are functions defined over $\text{Mp}_{2n}(\mathbb{A})$ which are absolutely convergent in the half plane $\text{Re}(s) > n$ [1, thm. 7.1, p. 34]. Furthermore the Eisenstein series have meromorphic continuation in the variable $s$ [1, thm. 7.2, p. 35] and by [9, prop. 6.1 p. 28] they have a pole of order at most 1 at $s = \frac{m-n-1}{2}$.

Proposition 3.13. The Eisenstein series satisfies the following functional equation:

$$E_{\text{Mp}_{2n}}(g, s, \Phi) = E_{\text{Mp}_{2n}}(g, -s, M(s) \Phi).$$

Proof. See [27, p. 981].

3.3. Global theta function. From now on $(V, q)$ denotes a rational vector space of dimension $m$ and Witt index $r$, moreover we consider $H = O(V)$ and $G = \text{Mp}_{2n}$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $f \in C^\infty(\mathbb{R}^n)$ a $\mathbb{C}$–valued smooth function of variable $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Let

$$(D^\alpha f)(x_1, \ldots, x_n) := \left( \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f \right)(x_1, \ldots, x_n).$$

We denote by

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

the product of $\alpha$–powers of the coordinate functions.

Definition 3.14. A Schwartz function on $\mathbb{R}^n$ is a smooth function $\varphi : \mathbb{R}^n \to \mathbb{C}$ such that for every $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{x \in \mathbb{R}^n} |x^\alpha (D^\beta f)(x)| < \infty.$$ 

We denote the vector space of Schwartz functions on $V(\mathbb{R})$ by $S(V(\mathbb{R}))$.

The groups $H(\mathbb{R})$ and $G(\mathbb{R})$ act on $S(V^n(\mathbb{R}))$ via the Weil representation associated to the character $\psi_\infty(x) = e^{i2\pi \sigma x}$. We denote this representation by

$$\omega_\mathbb{R}(\cdot, \cdot) : H(\mathbb{R}) \times G(\mathbb{R}) \to \text{Aut}(S(V^n(\mathbb{R}))).$$

For details on the the Weil representation the reader is referred to [16, chap. 1]. See also proposition 3.19 below.

Definition 3.15. Let $p$ be a prime number and let $\varphi$ be a function $\varphi : V(\mathbb{Q}_p) \to \mathbb{C}$. We say that $\varphi$ is a $p$–adic Schwartz function if it is locally constant and compactly supported. We denote the $\mathbb{C}$–vector space of $p$–adic Schwartz functions by $S(V(\mathbb{Q}_p))$. 

Let \( p \mid \infty \) be a place of \( \mathbb{Q} \), we define the character \( \psi_p \), as the unique additive character whose restriction to \( \mathbb{Z}_p \) is trivial and such that

\[
\psi_p(p^{-1}) = \frac{1}{p^{1/2}}.
\]

The locally compact groups \( H(\mathbb{Q}_p) \) and \( G(\mathbb{Q}_p) \) act on \( \mathcal{S}(V(\mathbb{Q}_p)) \) via the Weil representation associated to the character \( \psi_p \):

\[
\omega_p(\cdot, \cdot) : H(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \to \text{Aut}(\mathcal{S}(V^n(\mathbb{Q}_p))).
\]

We refer the reader to [32, sec. 1.2, p. 16] for a detailed discussion about the Weil representation. We fix \( L \subset V \) a lattice of \( V \) and we define

\[
\varphi^L_p := \text{char}(L \otimes \mathbb{Z}_p) \in \mathcal{S}(V(\mathbb{Q}_p)).
\]

Choosing a different lattice \( M \subset V \), the function \( \varphi^M_p \) satisfies the following equality for all but finitely many finite places of \( \mathbb{Q} \):

\[
\varphi^M_p = \varphi^L_p.
\]

We define

\[
\mathcal{S}(V^n(\mathbb{A}_f)) := \bigotimes_{v \mid \infty} \mathcal{S}(V^n(\mathbb{Q}_v)),
\]

where the restricted product is taken with respect to the family of \( p \)-adic Schwartz functions \( \varphi^L_p \). We may observe that due to the relation (12) between the functions \( \varphi^L_p \) and \( \varphi^M_p \) the space \( \mathcal{S}(V^n(\mathbb{A}_f)) \) is independent of the choice of lattice \( L \).

**Remark 3.16.** Let \( L \) be a lattice, for all but finitely many finite places \( v \), the Schwartz function \( \varphi^L_v \) is fixed by \( \text{Sp}_n(\mathbb{Z}_p) \), the maximal open compact subgroup of \( \text{Sp}_n(\mathbb{Q}_p) \).

**Definition 3.17.** We define the space of adelic Schwartz functions as the following tensor product:

\[
\mathcal{S}(V^n(\mathbb{A})) := \mathcal{S}(V^n(\mathbb{R})) \otimes \mathcal{S}(V^n(\mathbb{A}_f)).
\]

Finally we define the adelic Weil representation as the restricted tensor product of the local Weil representations:

\[
\omega_{M_{\mathbb{Z}_p^n}} := \bigotimes_v \omega_p : H(\mathbb{A}) \times G(\mathbb{A}) \to \text{Aut}(\mathcal{S}(V^n(\mathbb{A}))).
\]

This representation is associated to the character \( \psi_{\mathbb{A}} = \prod_p \psi_p \). To lighten the presentation we will suppress the subindex from the notation unless it is not clear.

**Remark 3.18.** The vector space \( C[L'/L] \) is isomorphic to the \( \mathbb{C} \)-vector space generated by \( S_L \), the set of characteristic functions of the cosets of \( L' \otimes \mathbb{Z} H_L \otimes \mathbb{Z} H_L \). Then \( \omega_{M_{\mathbb{Z}_p^n}} \) acting on \( C[S_L] \) is equal to the representation defined in (7).

**Proposition 3.19.** For \( \varphi \in \mathcal{S}(V^n(\mathbb{A})) \) the action of the Weil representation is determined by the following rule:

- For \( h \in H(\mathbb{A}) \), \( \omega(h)\varphi(x) = \varphi(h^{-1}x) \).
- For \( r \in \text{Sym}_n(\mathbb{A}) \), \( \omega(\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}) \varphi(x) = \psi_h(\frac{1}{2}xbx^T) \varphi(x) \).
- For \( t \in GL_n(\mathbb{A}) \), \( \omega(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) = \chi_V(t) |\det(t)|_n^{n/2} \varphi(tx) \), where \( \chi_V(\cdot) \) is the character defined in (11).

The action for the remaining elements of \( G(\mathbb{A}) \) can be deduced from the formula stated in [15, p. 40].

**Definition 3.20.** Given a dual reductive pair of the form \( (\text{Sp}_n, H) \), the associated theta function on \( G(\mathbb{A}) \times H(\mathbb{A}) \times \mathcal{S}(V(\mathbb{A})) \) is defined as follows:

\[
\theta(g, h, \varphi) = \sum_{x \in V(\mathbb{Q})} \omega(g, h)\varphi(x).
\]

**Remark 3.21.** Let \( p \) be a place of \( \mathbb{Q} \). The local theta correspondence of the identity defines the following \( G(\mathbb{Q}_p) \)-intertwining maps

\[
\lambda_p : \mathcal{S}(V(\mathbb{Q}_p)) \to I_n(s, \chi_V),
\]

\[
\varphi \mapsto \omega_p(\cdot)\varphi(0).
\]
For a further discussion of this map we refer the reader to [14, III. 5, p. 50]. The map \( \lambda := \otimes_p \lambda_p \) realizes the global theta correspondence of the identity and relates the integral of the theta function with the Eisenstein series via the so-called Siegel-Weil formula.

Let \( f \in M^!_{k, L} \) be a weakly holomorphic modular form. Using remark 3.18 we can express this function by the following sums:

\[
f(\tau) = \sum_{\mu \in L'/L} f_\mu(\tau)\varsigma_\mu = \sum_{\varphi \in S_L} f_\varphi(\tau)\varphi,
\]

where \( S_L \) is the set of characteristic functions of the cosets of \( L' \otimes \mathbb{Z} A_f / L \otimes \mathbb{Z} A_f \) and \( f_\varphi(\tau) \) are holomorphic functions in \( H \) with singularities at the cusp infinity. For \( \tau = u + iv \in H \) let \( g_\tau := \left( \begin{smallmatrix} u^{1/2} & v^{1/2} \\ v^{-1/2} & u^{-1/2} \end{smallmatrix} \right) \in SL_2(\mathbb{R}) \). Since there is no danger of confusion, we will also denote by \( g_\tau := (g_\tau, 1) \in Mp_2(\mathbb{R}) \). Let \( R(x, z) = -(x, z) = |(x, \omega(z))|^2 |y|^{-2} \) where \( \omega \) is the map defined in (8). We use the following notation:

\[
(x, x)_z := (x, x) + 2R(x, z).
\]

Following [17, sec. 1, p. 6] and [17, lem. 1.1, p. 11] one can show that the singular theta lift \( \Phi(f) \) defined in 2.7 can be expressed as

\[
\Phi(f)(z) = CT_{\sigma=0} \lim_{T \to \infty} \int_{-T}^T \sum_{\varphi \in S_L} f_\varphi(\tau)\theta(g_\tau, z, \varphi \otimes \varphi)e^{\frac{\pi i}{2} 1-\sigma}d\mu(\tau),
\]

where we recall that the vector space \( V \) has signature \( (2, p) \) and \( \varphi_\infty(\tau) := e^{-\pi(x, x)_0} \in \mathcal{S}(V(\mathbb{R})) \) is the Gaussian with base point \( 0 \in Gr(V) \). One can find an explicit expression for \( (x, x)_\omega \) in [8, (3.9), p. 296].

3.3.1. Classical Eisenstein series. In this paper we consider Eisenstein series of complex variable. They are defined via 3.11 and they are closely related to the theta functions. Given \( l \in \frac{1}{2} \mathbb{Z} \) we denote by \( \Phi^l(g, s) \in I_1(s, \chi_\Lambda) \) the unique section of the induced representation satisfying that

\[
\Phi^l(k_\varnothing, s) = e^{\pi i l \theta},
\]

when \( k_\varnothing = \left( \begin{smallmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{smallmatrix} \right) \in SO(2) \). The section \( \Phi^l(g, s) \) is referred to as the section of weight \( l \). Those functions are in the image of the maps \( \lambda_\infty \) defined in (14). In fact, using the formulas of the Weil representation we find that \( \Phi^l(g, \frac{m}{2} - 1) = \lambda_\infty(\varphi_\infty)(g) \) where \( \varphi_\infty \in \mathcal{S}(V(\mathbb{R})) \) is an Schwartz function satisfying that \( \omega(k_\varnothing)\varphi_\infty(x) = e^{\pi i l \theta}\varphi_\infty(x) \).

Remark 3.22. Let \( \Phi^l(g, s) \in I_1(s, \chi_\Lambda) \) be the section of weight \( l \). Using the formula 3.10 the section \( M(s) \infty \Phi^l(g) \in I_1(s, \chi_\Lambda) \) satisfies

\[
M(s) \infty \Phi^l(k_\varnothing, s) = e^{\pi i l \theta}.
\]

With the aim of simplifying the notation we denote by

\[
E(\tau, s, l, \mu(\varphi)) := v^{-l/2}E(g_\tau, s, \Phi^l \otimes \lambda(\varphi)).
\]

Proposition 3.23. The \( SL_2 \) –Eisenstein series satisfy the following relation:

\[
-2iv^3 \frac{\partial}{\partial \tau} \left\{ v^{-l/2} E(g_\tau, s, \Phi^{l+2} \otimes \lambda(\varphi_f)) \right\} = \frac{1}{2} (s - l - 1) v^{-l/2} E(g_\tau, s, \Phi^l \otimes \lambda(\varphi_f)),
\]

and hence

\[
-2iv^3 \frac{\partial}{\partial \tau} \{ E(\tau, s, l + 2, \mu(\varphi_f)) \} = \frac{1}{2} (s - l - 1) E(\tau, s, l, \mu(\varphi_f)).
\]

Proof. We find the statement in [17, (2.15), p. 20].

In subsequent sections of this paper we will consider the two Eisenstein series \( E(\tau, s, 3/2, \mu(\varphi)) \) and \( E(\tau, s, 5/2, \mu(\varphi)) \). According to [21, cor. 2.5, p. 2283] those Eisenstein series are holomorphic functions in the variable \( s \). We denote by

\[
E(\tau, s, 3/2, \mu(\varphi)) = \sum_m A(s, m, v, \mu(\varphi)) q^m,
\]

\[
E(\tau, s, 5/2, \mu(\varphi)) = \sum_m \tilde{A}(s, m, v, \mu(\varphi)) q^m,
\]

their Fourier series. Furthermore we denote the Laurent series at \( s = 1/2 \) of each Fourier coefficient by

\[
A(s, m, v, \mu(\varphi)) = a(m) + b(m, v, \mu(\varphi))(s - s_0) + O((s - s_0)^2),
\]
\[ \tilde{A}(s, m, v, \mu(\varphi)) = \tilde{a}(m) + \tilde{b}(m, v, \mu(\varphi)) (s - s_0) + O((s - s_0)^2). \]

3.4. **Regularization of the theta function.** Throughout this subsection \((V, q)\) is a rational quadratic space of signature \((2, p)\) and dimension \(m\) such that \(V = V_{an} \oplus \mathbb{H}^r\), where \(V_{an}\) is the maximal anisotropic quadratic subspace of \(V\) and \(\mathbb{H}\) is the hyperbolic plane, i.e. the isotropic quadratic space of signature \((1, 1)\). The constant \(r\) is called the Witt index.

**Proposition 3.24.** Let us choose a natural number \(n\) such that \(r \leq n\). There exists an algebra homomorphism

\[ \theta : \mathcal{H}_p^G \to \mathcal{H}_p^H, \]

so that for all \(\alpha \in \mathcal{H}_p^G\):

\[ \omega(\alpha) = \omega(\theta(\alpha)) \in \text{Aut} \left( S(V^n(Q_p))^{G(Z_p) \times H(Z_p)} \right). \]

**Proof.** The proof and the definition of this map can be found in [13, prop. 1.1, p. 206]. \(\square\)

**Proposition 3.25.** Let \(T_{n, \mathbb{Q}p}\) be the maximal torus of \(\text{Sp}_{n, \mathbb{Q}p}\). If the Witt index of \(V\) satisfies that \(r < n\), there exists a unique element

\[ \alpha_{n, r, \eta} = \sigma_{r+1} + \sum_{i=1}^{r} a_i \sigma_i \in \mathcal{H}_{n, \mathbb{Q}p}. \]

where \(\sigma_i\) is the \(i\)-symmetric polynomial in \(Z[X_1(T_n)]^{V(\text{Sp}_{n, \mathbb{Q}p}, T_{n, \mathbb{Q}p})}\) and \(a_i \in \mathbb{C}\), such that \(\theta(\alpha_{n, r, \eta}) = 0\). Furthermore the element \(\alpha_{n, r-1, \eta}\) has the following property:

\[ \theta(\alpha_{n, r-1, \eta}) = \eta^n \prod_{j=1}^{r} (Y_j - C_{m/2 - n - 1}), \]

where \(\mathcal{H}_p^H = \mathbb{Z}[Y_1, ..., Y_r]^{S_r} \otimes \mathbb{Z}[p^{1/2}, p^{-1/2}], C_s = p^s + p^{-s}\) and \(\eta = \eta_Q = \gamma_k(p, \psi_v \circ Q)^{-1} \in \{\pm 1\}\) is the Weil index defined in [16, p. 12].

**Proof.** This proposition encompasses [13, prop. 1.3, p. 209] and [13, (1.1), p. 209]. \(\square\)

**Remark 3.26.** Let us express explicitly \(\theta(\alpha_{n, r-1, \eta})\) when the dual reductive pair is \((\text{Sp}_{1, \mathbb{Q}p}, H_{\mathbb{Q}p})\) with \(V\) a rational isotropic quadratic space of signature \((2, 1)\). It is straightforward that the orthogonal group satisfies \(H_{\mathbb{Q}p} = O(V \otimes \mathbb{Q}p)\). The dimension of the maximal torus \(T^H\) of \(H_{\mathbb{Q}p}\) is equal to 1. Let us fix \(\mu \in X_1(T^H)\) the generator of the group of cocharacters of \(T^H\). The ring \(\mathcal{H}_p^H\) is unitary and generated by \(\text{char}_{K_{p, p}(p)}(h)\) and \(\text{char}_{K_p}(h)\) where the latter element is the unit of the ring. By proposition 3.25

\[ \theta(\alpha_{1, 0, \eta}) = \eta (Y_1 - C_{-1/2}). \]

From the proof of [13, lem. 1.4, p. 208] we deduce that \(\int_{H(\mathbb{Q}p)} S^{-1}(Y_1)(h)dh = C_{1/2}\). Therefore the Satake isomorphism given in 3.7 provides the following equality:

\[ S^{-1}(\eta Y_1 - \eta C_{-1/2})(h) = \eta C_{1/2} \text{char}_{K_{p, p}(p)}(h) - \eta C_{-1/2} \text{char}_{K_p}(h). \]

Let 1 be the identity function, then

\[ \theta(\alpha_{1, 0, \eta}) \cdot 1 = \eta \int_{H(\mathbb{Q}p)} C_{1/2} \text{char}_{K_{p, p}(p)}(h) - C_{-1/2} \text{char}_{K_p}(h)dh = 0. \]

The vanishing of (16) prevents us from using the first term identity of [9, thm. 8.1, (i), p. 35]. In fact, this equality is the motivation to introduce the second term identity in the Siegel-Weil formula.

Let us fix a Siegel set \(G_H\) of \(H(\mathbb{A})\) (the definition of Siegel set can be found in [10, (2.22), p. 61]) so that \(H(\mathbb{A}) = H(\mathbb{Q})G_H\).

**Definition 3.27.** Given \(\varphi \in S(V^n(\mathbb{A}))\) we fix a finite place \(p\) satisfying the following conditions:

- \(p\) is finite and does not divide 2.
- \(p \equiv 1 \mod 4\).
- \(\psi_p\) is of order 0.
- \(q_{\psi}(\cdot)\) is unramified.
- \(H(\mathbb{Q}p) \cap G_H = H(\mathbb{Z}_p)\).
- The Schwartz function \(\varphi\) is \(\text{Sp}_n(\mathbb{Z}_p) \times H(\mathbb{Z}_p)\)-fixed.
Remark 3.28. For any \( \varphi \in S(V^n(\mathbb{A})) \) there exists an infinite number of finite places where the hypothesis of 3.27 are satisfied.

Proposition 3.29. Let us assume that \( m = \dim(V) \leq 2n \) and \( r \neq 0 \). Let \( r_p \) be the Witt index of the local quadratic space \( (V_p,q_p) \). Given \( \alpha := \alpha_n, r_p-1, \eta \in \mathcal{H}_p^G \), and \( \varphi \in S(V^n(\mathbb{A})) \), then

\[
\int_{[H]} \theta(g,h,\omega(\alpha)\varphi) dh,
\]

is absolutely convergent for all \( g \in G(\mathbb{A}) \).

Proof. We refer the reader to [13, prop. 1.5, p. 209]. \( \square \)

3.5. Regularized Siegel-Weil formula. With the aim of relating the integral

\[
\int_{[H]} \theta(g,h,\omega(\alpha)\varphi) dh,
\]

with a Siegel Eisenstein series, in [19, p. 46] the authors introduce an auxiliary Eisenstein series (defined using the Siegel parabolic subgroup of \( H \)). This subsection will be devoted to define the auxiliary Eisenstein series and to state the second term identity of the regularized Siegel-Weil formula given in [9].

We recall that the Levi decomposition of the Siegel parabolic of \( H \), denoted here by \( Q \), is \( Q = M_QN_Q \), where the \( \mathbb{Q} \)-points of the Levi subgroup \( M_Q \) are described as follows:

\[
M_Q(\mathbb{Q}) = \left\{ m(a,h_0) = \begin{pmatrix} a & h_0 \\ a^{-1} \end{pmatrix} \right\} s.t. a \in \text{GL}_r(\mathbb{Q}), h_0 \in O(V_{an}(\mathbb{Q})) \}
\]

and the \( \mathbb{Q} \)-points of the unipotent subgroup \( N_Q \) are

\[
N_Q(\mathbb{Q}) = \left\{ n(c,d) = \begin{pmatrix} 1 & c \\ d & 1 \end{pmatrix} \right\} s.t. c^t = (c_1, \ldots, c_r) \in V_{an}^r, (c,c) = ((c_i,c_j)), d = \left\{ -1 \right\} \in M_r(\mathbb{Q}) \}
\]

The Iwasawa decomposition provides the following equality:

\[
H(\mathbb{A}) = Q(\mathbb{A})K_H,
\]

where \( K_H = (O(2) \oplus O(p)) \times \prod_{v \text{ place}} H(\mathbb{Z}_v) \) is the maximal compact subgroup of \( H(\mathbb{A}) \). Combining the previous two decompositions we can factor every \( h \in H(\mathbb{A}) \) by \( h = n(c,d)m(a,h_0)k \).

To lighten notation we denote \( |a(h)| := |\det(a)| \) and

\[
\rho'_r = \frac{m - r - 1}{2}.
\]

Using the previous datum we define the function

\[
\Psi(h,s) := |a(h)|^{s+\rho'_r},
\]

where \( s \in \mathbb{C} \).

Definition 3.30. The auxiliary Eisenstein series is defined as follows:

(17)

\[
E(h,s) = \sum_{\gamma \in Q(\mathbb{A}) \backslash H(\mathbb{A})} \Psi(\gamma h, s).
\]

Remark 3.31. This series is absolutely convergent when \( \Re(s) > \rho'_r \) and moreover \( E(h,s) \) has meromorphic analytic continuation to \( \mathbb{C} \) [19, p. 47].

Proposition 3.32. Except in the case of a split binary quadratic space \( V \), \( E(h,s) \) has a simple pole at \( s = \rho'_r \) with constant residue denoted by \( k \). If \( V \) is the rational isotropic space of signature \( (2,1) \) then \( k = 2 \).

Proof. It is [19, prop. 5.4.1, p. 48]. \( \square \)
The centre of the Hecke algebra of $H(Q_p)$ acts on $\text{Ind}_{Q(Q_p)}^{H(Q_p)} \theta(a(\cdot)^{|s+\rho'_1|^2})$ by multiplication by scalars. We will use the following notation:

\begin{equation}
\theta(a_{n,n-1,\eta}) \cdot \left(\theta(a(\cdot)^{|s+\rho'_1|^2}) = c_\alpha(s) |a(\cdot)^{|s+\rho'_1|^2}|ight.
\end{equation}

where $c_\alpha(s) : \mathbb{C} \to \mathbb{C}$ is a function.

**Lemma 3.33.** Let us consider $V$ be an isotropic quadratic space of signature $(2,1)$, then

\[ \theta(a_{1,0,\eta}) \cdot \left(\theta(a(\cdot)^{|s+\rho'_1|^2}) = 0, \right. \]

i.e. $c_\alpha(\rho'_1) = 0$.

**Proof.** It follows by remark 3.26 \hfill \square

Let us denote by

\[ \mathcal{E}(s,g,\varphi) := \frac{1}{c_\alpha(s)k} \int_{[H]} \theta(g,h,\omega(z)\varphi) E(h,s) dh, \]

By proposition 3.32 and lemma 3.33, for $m = 3$, $r = 1$ and $n = 1$ the function $\mathcal{E}(s,g,\varphi)$ has a pole at $s = \rho'_1$ of order at most 2. We denote the Laurent expansion of $\mathcal{E}(s,g,\varphi)$ at $s = \rho'_1$ as follows:

\[ \mathcal{E}(s,g,\varphi) = B_{-2}(g,\varphi) (s-\rho'_1)^2 + B_{-1}(g,\varphi) (s-\rho'_1) + B_0(g,\varphi) + O(s-\rho'_1). \]

**Theorem 3.34.** Let $(V,q)$ be a rational quadratic isotropic space of signature $(2,1)$. Let $\varphi \in \mathcal{S}(V(\mathbb{A}))$ be a function satisfying the hypothesis of 3.27 for a finite place $p$. Then

\[ CT_{s=1/2} E(g,s,\lambda(\varphi)) = B^{-1}(g,\varphi) + c\text{Res}_{s=1/2} E(g,s,\lambda(\tilde{\varphi})), \]

where $\lambda(\varphi)(g) := (\omega(g)\varphi)(0) \in I_s(s,\chi_V)$, $c \in \mathbb{C}$ and $\tilde{\varphi} \in \mathcal{S}(V(\mathbb{A}))$ is a non-determined Schwartz function.

**Proof.** See [9, thm. 8.1, (ii), p. 35]. \hfill \square

## 4. Truncated Siegel-Weil Formula

Throughout this section we fix $(V,q)$ an even rational quadratic space of signature $(2,1)$ and Witt index 1. We denote the bilinear form associated to $q$ by $(\cdot,\cdot)$. Let $H$ be the algebraic group $O(V)$. We also fix $L \simeq \mathbb{Z}^k$ a lattice of $V$. Let $W$ be a rational symplectic space of dimension 2. Fixing the basis given in 3.2 its symplectic algebraic group defined over the rationals satisfies that $Sp(W) \simeq Sp_1$. For every $p$ this group has a topological double cover $G(Q_p) = Mp_2(Q_p)$ defined in (10). This group is called metaplectic group. Let us note that there exists an isomorphism

\[ \text{SL}_2 \simeq Sp_1. \]

Let $\varphi^\infty_{z_0,\mathbb{R}} \in \mathcal{S}(V(\mathbb{R}))$ be the Gaussian associated to the quadratic space $(V,q)$ with base point $z_0 := i \in H$. Moreover we set $\varphi_{\mu_1}$ the characteristic functions of the two cosets of $L \otimes \mathbb{Z} k_f/L \otimes \mathbb{Z} k_f$ given in 2.2. In this section we consider the Schwartz functions

\begin{equation}
\varphi^\infty_{z_0,\mu_1} := \varphi^\infty_{z_0,\mathbb{R}} \otimes \varphi_{\mu_1} \in \mathcal{S}(V(\mathbb{A})).
\end{equation}

By the analysis done in the proof of [19, prop. 5.3.1, p. 45] the following integrals do not converge

\begin{equation}
\int_{[H]} \theta(g_t, h, \varphi^\infty_{z_0,\mu_1}) dh,
\end{equation}

where we recall that

\[ g_t \in G(\mathbb{R}) = \begin{pmatrix} v^{1/2} & w v^{1/2} \\ 0 & v^{-1/2} \end{pmatrix}. \]

The main goal of this section is to state an asymptotic formula for (20) i.e. we will isolate the terms of the theta function that diverge and we will compute the integral of the convergent ones. This computation is based on a manipulation of the second term identity of the Siegel-Weil formula developed in [9].
4.1. Factorization of the theta function.

Proposition 4.1. The following map is an isomorphism
\[ \mathcal{S}(V(\mathbb{A})) \rightarrow \mathcal{S}(\mathcal{V}_{an}(\mathbb{A})) \otimes \mathcal{S}(W(\mathbb{A})), \]
\[ \varphi(x) \mapsto \hat{\varphi}(x_0, u, v) := \int_{M_{1,2}(\mathbb{A})} \varphi(x) \psi_\lambda(x) dx, \]
where \( \mathcal{V}_{an} \) is the vector space defined in the beginning of section 3.4, \( w := (u, v) \in W(\mathbb{A}) \) and \( \psi_\lambda \) is the adelic character used to define the Weil representation (13).

Proof. See [19, (5.3.2), p. 45].

Definition 4.2. The isomorphism given by 4.1 allows us to consider the representation
\[ H(\mathbb{A}) \times G(\mathbb{A}) \rightarrow \text{Aut}(\mathcal{S}(\mathcal{V}_{an}(\mathbb{A})) \otimes \mathcal{S}(W(\mathbb{A}))) \]
\[ (g, h) \mapsto \left( \hat{\varphi}(x_0, u, v) \mapsto \int_{M_{1,2}(\mathbb{A})} \omega(g, h) \varphi(x) \psi_\lambda(x) dx \right), \]
that is called mixed model of the Weil representation.

Remark 4.3. Using the partial Poisson summation formula of [19, (5.3.4), p. 45], the theta function satisfies
\[ \theta(g, h, \varphi) = \sum_{x_0 \in \mathcal{V}_{an}(\mathbb{Q})} \hat{\varphi}(x_0, w). \]

Definition 4.4. Let \( \varphi \in \mathcal{S}(V(\mathbb{A})) \). Given a theta function \( \theta(g, h, \varphi) \) for the dual reductive pair \((G, H)\), we define the divergent part by
\[ \text{Div}(g, h, \varphi) := \sum_{x_0 \in \mathcal{V}_{an}(\mathbb{Q}_p)} \omega(g, h) \hat{\varphi}(x_0, 0). \]
Moreover we define the convergent part by
\[ \text{Conv}(g, h, \varphi) := \sum_{x_0 \in \mathcal{V}_{an}(\mathbb{Q}_p), 0 \neq w \in W(\mathbb{Q})} \omega(g, h) \hat{\varphi}(x_0, w). \]

Proposition 4.5. The convergent part \( \text{Conv}(g, h, \varphi) \) is rapidly decreasing.

Proof. See the proof of [19, prop. 5.3.1, p. 45].

Remark 4.6. By 4.3 the previous definitions provides a well defined factorization of the theta function:
\[ \theta(g, h, \varphi) = \text{Div}(g, h, \varphi) + \text{Conv}(g, h, \varphi). \]

Lemma 4.7. Let \( p \) be a place satisfying the hypothesis of 3.27 and let \( \alpha := \alpha_{1,0,\eta} \in \mathcal{H}_p^{M_2} \) be the Hecke operator defined in proposition 3.29. Then the regularized theta function satisfies
\[ \theta(g, h, \omega(\alpha) \varphi) = \text{Conv}(g, h, \omega(\alpha) \varphi). \]

Proof. By [13, prop. 1.5, p. 209] the function \( \theta(g, h, \omega(\alpha) \varphi) \) is rapidly decreasing. Therefore using the factorization given in 4.6 the result follows.

Proposition 4.8. The following equality holds:
\[ B_{-1}(g, h, \varphi) = \int_{[H]} \text{Conv}(g, h, \varphi) dh. \]

Proof. Writing the definition of the action of the operator \( \omega(\alpha) \) and applying proposition 4.7 to \( B_{-1}(g, h, \varphi) \) we obtain
\[ B_{-1}(g, h, \varphi) = \text{Res}_{s=1/2} \left( \frac{1}{r_\alpha(s)k} \int_{[H]} \int_{H(\mathbb{Q}_p)} \theta(\alpha)(h_v) \text{Conv}(g, hh_v, \varphi) E(h, s) dh_v dh \right). \]
By proposition 4.5 the function $\text{Conv}(g,h,\varphi)$ is rapidly decreasing in the variable $h$ then we apply a change of variables of the form $h = hh_v$, obtaining

$$B_{-1}(g,h,\varphi) = \text{Res}_{s=1/2} \left( \frac{1}{c_{\alpha}(s)k} \int_{[H]} \text{Conv}(g,h,\varphi) \int_{H(Q_p)} \theta(\alpha_{1,0,\eta})(h_v)E(hh_v, s)dh_v dh \right).$$

The action of $\theta(\alpha)$ on $E(h,s)$ factors throughout the action of this operator on $\text{Ind}_{Q(Q_p)}^{H(Q_p)} \{ \alpha(\cdot)^{s+\nu} \}$. We pointed out in (18) that the action of $\theta(\omega)$ in $\text{Ind}_{Q(Q_p)}^{H(Q_p)} \{ \alpha(\cdot)^{s+\nu} \}$ is the multiplication by the constant $c_{\alpha}(s)$. Hence

$$B_{-1}(g,h,\varphi) = \text{Res}_{s=1/2} \left( \frac{1}{k} \int_{[H]} \text{Conv}(g,h,\varphi)E(h,s)dh \right) = \int_{[H]} \text{Conv}(g,h,\varphi)dh,$$

where the last equality follows because $k = \text{Res}_{s=1/2}E(h,s)$.

\textbf{Corollary 4.9.} The convergent part satisfies

$$\int_{[H]} \text{Conv}(g,h,\varphi^\infty)dh = E(g,1/2,\lambda(\varphi^\infty)) + c\text{Res}_{s=1/2} E(g,s,\lambda(\tilde{\varphi})),
$$

where $c \in \mathbb{C}$ and $\tilde{\varphi} \in S(V(\mathbb{A}))$ is a Schwartz function such that for $k_0 = \left[ \begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{smallmatrix} \right] \in \text{SL}_2(\mathbb{R})$ it satisfies that $\omega(k_0)\tilde{\varphi}(x) = e^{2\pi} \tilde{\varphi}(x)$.

\textbf{Proof.} The equality of the statement follows by theorem 3.34 and proposition 4.8. According to [21, cor. 2.5, p. 2283] the only Eisenstein series with non vanishing residue at $s = 1/2$ are of the form $E(g,s,\Phi)$ with $\Phi$ satisfying that $\Phi(gk_0) = e^{2\pi} \Phi(g)$ for $k_0 = \left[ \begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{smallmatrix} \right]$. The space of these functions is one dimensional [17, (4.18), p. 49]. Then $\Phi$ is equal to $\lambda(\tilde{\varphi})$ with $\tilde{\varphi} \in S(V(\mathbb{R}))$ satisfying that $\omega(k_0)\tilde{\varphi}(x) = e^{2\pi} \tilde{\varphi}(x)$.

\textbf{Corollary 4.10.} Let $K \subset \text{SL}_2(\mathbb{A})$ be a compact open subgroup and let $\varphi = \varphi^\infty \otimes \varphi^L$ be a Schwartz function with $\varphi^L \in S(V(\mathbb{A}))$ being a $K$–invariant Schwartz function. Then

$$\int_{[H]} \text{Conv}(g,h,\varphi)dh = E(g,1/2,\lambda(\varphi)) + c\text{Res}_{s=1/2} E(g,s,\lambda(\tilde{\varphi})),
$$

with $E(g,s,\lambda(\tilde{\varphi}))$ a $K$–invariant function and $c \in \mathbb{C}$.

\textbf{Proof.} Since $\omega(\alpha)$ commutes with the action of the Weil representation, the regularized theta function $\theta(g,h,\omega(\alpha)\varphi)$ is right $K$–invariant. Furthermore using lemma 4.7 the function $\text{Conv}(g,h,\varphi)$ is $K$–invariant. Since the intertwining map $\lambda$ respects the $K$–invariance of $\varphi$, the Eisenstein series $E(g,1/2,\lambda(\varphi))$ is also $K$–invariant. Then in the equality of corollary 4.9 we obtain two of three terms being $K$–invariant, then the function $\text{Res}_{s=1/2} E(g,s,\lambda(\tilde{\varphi}))$ is $K$–invariant. Moreover using the formula for the residue given in 1.0.1 the $K$–invariance of $E(g,s,\lambda(\tilde{\varphi}))$ follows.

\textbf{5. Integral of Borcherds forms}

As in the previous section we fix $(V,q)$ a rational isotropic quadratic space of signature $(2,1)$ and $L \simeq \mathbb{Z}^3$ a lattice of $V$. The example 2.2 shows that the group $L'/L$ consists of two elements. We denote them by $\mu_0 := \mathbb{Z}^3$ and $\mu_1 := \mathbb{Z} \oplus \frac{1}{2} \mathbb{Z} \oplus \mathbb{Z}$. Further we will use the following notation

$$f(\tau) = \sum_{\mu_j \in L'/L} f_{\mu_j}(\tau) e_{\mu_j} = \sum_{n \in \mathbb{Z}} c_{\mu_j}(n)q^n e_{\mu_j} \in M_{1/2,L},$$

for the Fourier expansion of a weakly holomorphic modular form. Let $X^{\text{mod}} := \text{SL}_2(\mathbb{Z}) \setminus \mathcal{H}$ be the modular curve without level and set $1 \leq T \in \mathbb{R}$. The goal of this section is to compute the following integral:

$$\int_{X^{\text{mod},T}} \log ||\Psi(f)(z)||_{p_0} d\mu(z),$$

for the embeddings $p_0 : = \mathbb{R}.$
where \( \Psi(f)(z) \) is the Borcherds lift of \( f \) and
\[
X^{\text{mod}, T} := \{ z = x + iy \in X^{\text{mod}} \text{ s.t. } y < T \} = \{ x + iy \in \mathcal{H}, \text{ s.t. } |x| \leq 1/2, \text{ and } y > T \}.
\]
Furthermore we denote
\[
\widetilde{X}^{\text{mod}, T} := \{ z = x + iy \in X^{\text{mod}}, \text{ and } y > T \}.
\]
Using the definition of the Petersson norm it is straightforward that
\[
\begin{align*}
\int_{X^{\text{mod}, T}} \log||\Psi(f)(z)||_{\text{Pett}} \, d\mu(z) &= \int_{\widetilde{X}^{\text{mod}, T}} \log||\Psi(f)(z)||_{\text{Pett}} \, d\mu(z) \\
&= \int_{\widetilde{X}^{\text{mod}, T}} \log|y| \, d\mu(z) + \frac{c_{\mu_0}(0)}{2} \int_{\widetilde{X}^{\text{mod}, T}} \log|y| \, d\mu(z).
\end{align*}
\]
(22)

Theorem [2, thm. 13.3, p. 48] shows the following relation:
\[
\log||\Psi(f)(z)|| = -\frac{\Phi(f)(z)}{4} - \frac{c_{\mu_0}(0)}{2} \left( \log|y| + 1'(1)/2 + \log \sqrt{2\pi} \right),
\]
where \( \Phi(f)(z) \) is the singular theta lift of \( f \). Plugging the previous equality in (22) we find
\[
\int_{X^{\text{mod}, T}} \log||\Psi(f)(z)||_{\text{Pett}} \, d\mu(z) = -\frac{1}{4} \int_{X^{\text{mod}, T}} \Phi(f)(z) \, d\mu(z) + \frac{c_{\mu_0}(0)\text{vol}(X^{\text{mod}, T})}{2} \left( 1'(1)/2 + \log \sqrt{2\pi} \right).
\]
Therefore our goal will be achieved by computing the following integral:
\[
\begin{align*}
\int_{X^{\text{mod}, T}} \Phi(f)(z) \, d\mu(z) &= \int_{X^{\text{mod}, T}} \left( \int_{X^{\text{mod}}} \phi_{\mu_j}^{\text{Seg}}(\tau, z) \, d\mu(\tau) \right) \, d\mu(z) \\
&= \sum_{\mu_j \in \mathcal{L}/\mathcal{L}} \int_{X^{\text{mod}, T}} \left( \int_{X^{\text{mod}}} \phi_{\mu_j}^{\text{Seg}}(\tau, z) \, d\mu(\tau) \right) \, d\mu(z).
\end{align*}
\]
(23)

This section is organized as follows: the truncated integral of the Siegel theta function over the modular curve will be one of the key steps to understand (23). We deduce the formula from corollary 4.9. More concretely using the ideas of [17, prop. 4.17, p. 44] we translate the result to the geometric setting by proving a relation between the convergent part of the integral of the theta function associated to the Gaussian over \([H]\) and the integral of the Siegel theta function over the modular curve without level. Let us denote by
\[
\theta_{\mu_j}^{\text{Seg}}(\tau, z) = \sum_{\lambda \in \mathcal{L} + \mu_j} \theta_{\mu_j}^{\text{Seg}}(\tau, z)_{\lambda},
\]
the Fourier expansion of \( \theta_{\mu_j}^{\text{Seg}}(\tau, z) \) with respect to the variable \( \tau \). The constant Fourier coefficient
\[
\theta_{\mu_j}^{\text{Seg}}(\tau, z)_{0} := \sum_{\lambda \in \mathcal{L} + \mu_j \atop \lambda(\lambda) = 0} \theta_{\mu_j}^{\text{Seg}}(\tau, z)_{\lambda},
\]
does not have exponential decay when \( v \to \infty \). Hence we factor the integrals
\[
\begin{align*}
\lim_{T \to \infty} \int_{F^T} f_{\mu_j}(\tau) \theta_{\mu_j}^{\text{Seg}}(\tau, z_u) u^{1-\sigma} \, d\mu(\tau),
\end{align*}
\]
(24)
where we recall that \( F^T = \{ \tau = u + iv \in \mathcal{H}, \text{ s.t. } v < T \} \), depending on the behaviour of \( \theta_{\mu_j}^{\text{Seg}}(\tau, z)_{\lambda} \) with respect to the variable \( \tau \). We express (24) as the sum of two terms; the ordinary case:
\[
\lim_{T \to \infty} \int_{F^T} f_{\mu_j}(\tau) \sum_{\lambda \in \mathcal{L} + \mu_j \atop \lambda(\lambda) \neq 0} \theta_{\mu_j}^{\text{Seg}}(\tau, z)_{\lambda} u^{1-\sigma} \, d\mu(\tau),
\]
and the limit case:
\[
\lim_{T \to \infty} \int_{F^T} f_{\mu_j}(\tau) \theta_{\mu_j}^{\text{Seg}}(\tau, z)_{0} u^{1-\sigma} \, d\mu(\tau).
\]
The first integrals are studied in section 3 using the truncated Siegel-Weil formula stated in 4.9 and the techniques developed in [17]. The second integrals are studied via the truncated unfolding of [35].
5.1. Geometric version of the truncated Siegel-Weil formula. In the previous section we computed the integral of the convergent terms of certain adelic theta function, corollary 4.9. This subsection connects this result with the complex geometry point of view of Borcherds [2], showing a truncated version of the classical Siegel-Weil formula for the modular curve.

Throughout this subsection we fix $K^H(A_f) = \prod_p \mathbb{H}^2(\mathbb{Z}_p)$ an open compact subgroup of $H(A_f)$. Let us note that $K^H(A_f)$ satisfies that the lattices $\mathbb{Z}_f^2 := \mathbb{Z}^3 \otimes \mathbb{A}_f$ and $\mathbb{Z}_f \oplus \mathbb{Z}_f \oplus \mathbb{Z}_f := (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \otimes \mathbb{A}_f$ are fixed under the action of $K^H(A_f)$ on $V(A_f)$. Since $\mathbb{C}[L'/L] \cong \mathcal{S}(V(A_f))$ as $\mathbb{C}$-vector spaces, we denote by

$$\varphi_{\mu_0} = \text{char}_{\hat{\mathbb{Z}}_f^2} \text{ and } \varphi_{\mu_1} = \text{char}_{\hat{\mathbb{Z}}_f^3}.\]

We set $\varphi_{\mathbb{R}_{\varphi_{\mathbb{Z}_f}}} := \varphi_{\mathbb{R}_{\varphi_{\mathbb{Z}_f}}} \otimes \varphi_{\mu_0}$ and $\varphi_{\mathbb{R}_{\varphi_{\mathbb{Z}_f}}} := \varphi_{\mathbb{R}_{\varphi_{\mathbb{Z}_f}}} \otimes \varphi_{\mu_1}$.

**Lemma 5.1.** The Schwartz functions $\varphi_{\mu_0}$ and $\varphi_{\mu_1}$ are $K^H(A_f)$-invariant.

**Proof.** By proposition 3.19 the group $H$ acts on $\mathcal{S}(V(A_f))$ by left translation. Since the left action of the group $K^H(A_f)$ preserves the lattices $\mathbb{Z}_f^3$ and $\mathbb{Z}_f \oplus \mathbb{Z}_f \oplus \mathbb{Z}_f$, then $\varphi_{\mu_0}$ and $\varphi_{\mu_1}$ are $K^H(A_f)$-invariant. □

**Lemma 5.2.** The divergent parts $\text{Div}(g,h,\varphi_{\mathbb{Z}_f})$ are right $(O(2) \oplus O(1)) \times K^H(A_f)$-invariant.

**Proof.** We recall that

$$\text{Div}(g,h,\varphi_{\mathbb{Z}_f}) = \sum_{x_0 \in \mathcal{V}_m(\mathbb{Q})} \omega(g,h)\varphi_{\mathbb{Z}_f}(x_0,0,0)$$

$$= \sum_{x_0 \in \mathcal{V}_m(\mathbb{Q})} \int_{\mathbb{A}} \omega(g,h)\varphi_{\mathbb{Z}_f}(x_0,0) \ dx.$$

Let us denote by

$$k = k_{\mathbb{R}} \times k_f \in (O(2) \oplus O(1)) \times K^H(A_f).$$

It is straightforward that

$$\int_{\mathbb{A}} \omega(k)\varphi_{\mathbb{Z}_f}(x,x_0) \ dx = \int_{\mathbb{R}} \omega(k_{\mathbb{R}})\varphi_{\mathbb{Z}_f}(x,x_0) \ dx \cdot \int_{\mathbb{A}_f} \omega(k_f)\varphi_{\mathbb{Z}_f}(x_0,0) \ dx_f$$

$$= \int_{\mathbb{A}} \varphi_{\mathbb{Z}_f}(x,x_0) \ dx,$$

where the latter equality follows since $\varphi_{\mathbb{Z}_f}$ is $O(2) \oplus O(1)$-invariant and $\varphi_{\mathbb{Z}_f}$ is $K^H(A_f)$-invariant by 5.1. Then the function $\varphi_{\mathbb{Z}_f}$ is $(O(2) \oplus O(1)) \times K^H(A_f)$-invariant, implying that $\text{Div}(g,h,\varphi_{\mathbb{Z}_f})$ is $(O(2) \oplus O(1)) \times K^H(A_f)$-invariant. □

**Lemma 5.3.** The regularized theta functions $\theta(g,h,\omega(\alpha)\varphi_{\mathbb{Z}_f})$ are right $(O(2) \oplus O(1)) \times K^H(A_f)$-invariant.

**Proof.** Since $\omega(\alpha)$ commutes with the Weil representation the proof is analogous to lemma 5.2. □

**Proposition 5.4.** The integral of the theta function satisfies the following equality:

$$\int_{[H]} \theta(g_r,h,\omega(\alpha)\varphi_{\mathbb{Z}_f}) \ dh = \frac{1}{2} \int_{[\mathbb{SO}(V)\setminus]} \theta(g_r,h,\omega(\alpha)\varphi_{\mathbb{Z}_f}) \ dh.$$

**Proof.** Since the action of $\omega(\alpha)$ commutes with Weil representation, the proof follows directly by [17, thm. 4.1, p. 37]. □

**Corollary 5.5.** The convergent part satisfies the following equality:

$$\int_{[H]} \text{Conv}(g_r,h,\varphi) \ dh = \frac{1}{2} \int_{[\mathbb{SO}(V)\setminus]} \text{Conv}(g_r,h,\varphi) \ dh.$$
Proof. Let us consider the functional
\[ I' : \mathcal{S}(V(\mathbb{A})) \to \mathbb{C}, \]
where \( \varphi \mapsto \int_{\text{Conv}(id, h, \varphi)} dh. \)

Since the group \( SO(V) \) is unimodular \( I' \in \text{Hom}_{\text{SO}(V)(\mathbb{A})}(\mathcal{S}(V(\mathbb{A})), \mathbb{C}) \). According to [17, prop. 4.2, p. 37] the action of the group \( C(\mathbb{A}) = O(V(\mathbb{A}))/SO(V(\mathbb{A})) \) on \( \text{Hom}_{\text{SO}(V)(\mathbb{A})}(\mathcal{S}(V(\mathbb{A})), \mathbb{C}) \) is trivial. Hence
\[ \int_{[H]} \text{Conv}(g_r, h, \varphi)dh = \int_{\mathcal{S}(V)} \int_{C(\mathcal{A})} \text{Conv}(g_r, ch, \varphi)dhdc \] where we have used that \( \text{vol}(C(\mathbb{Q}) \setminus C(\mathbb{A})) = \frac{1}{2}. \)

**Proposition 5.6.** We obtain
\[ \int_{\text{Conv}((26)}) \text{Conv}(g_r, h, \varphi_{\infty, \mu, j})dh = \frac{1}{2} \int_{\text{Conv}((26)}) \text{Conv}(g_r, h, \varphi_{\infty, \mu, j})dh, \]
where \( d\mu(z) = \frac{dx dy}{y}. \)

**Proof.** Let us recall the notation \( \tilde{H} = \text{GSpin}_V \). The strong approximation theorem shows that \( \tilde{H}(\mathbb{A}) = \bigcup_{t \in T} \tilde{H}(\mathbb{Q})\tilde{H}(\mathbb{R})^+ h_j K \)
where \( h_j \in \tilde{H}(\mathbb{A}) \) and \( \tilde{H}(\mathbb{R})^+ \) is the connected component of the identity of \( \tilde{H}(\mathbb{R}) \). The modular curve \( \text{SL}_2(\mathbb{Z}) \setminus \mathcal{H} \) has one connected component, then \( T = \{1\} \). Since the functions \( \varphi_{\infty, \mu, j} \) are \( (O(2) \times O(1)) \times K^H(\mathbb{A}) \) invariant the proof is analogous to [17, prop. 4.17, p. 44].

**Corollary 5.7.** It holds that
\[ \int_{\text{Conv}((26)}) \text{Conv}(g_r, h, \varphi_{\infty, \mu, j})dh = \frac{2}{\text{vol}(X^{mod})} \int_{\text{Conv}((26)}) \text{Conv}(g_r, h, \varphi_{\infty, \mu, j})dh. \]

**Theorem 5.8.** The integral of the convergent term over the modular curve satisfies
\[ \int_{\text{Conv}((26)}) \text{Conv}(g_r, h, \varphi_{\infty, \mu, j})dh = \text{vol}(X^{mod}) \left[ E(g_r, 1/2, \lambda(\varphi_{\infty, \mu, j})) + c\text{Res}_{s=1/2} E(g_r, s, \lambda(\varphi)) \right], \]
where \( \varphi := \varphi_{\infty} \otimes \varphi_L^{\mathbb{C}} \in \mathcal{S}(V(\mathbb{A})) \) is a \( \prod_{p} \mathbb{Z}_p \) Schwartz function satisfying \( \omega(k_{\theta})\tilde{\varphi}(x) = e^{\frac{i}{2} \pi} \tilde{\varphi}(x) \) for \( k_{\theta} = \left( \begin{array}{cc} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array} \right) \) \( \in \text{SL}_2(\mathbb{R}). \)

**Proof.** Using corollaries 5.5 and 5.7 we obtain
\[ \int_{\text{Conv}((26)}) \text{Conv}(g_r, h, \varphi_{\infty, \mu, j})dh = \text{vol}(X^{mod}) \int_{[H]} \text{Conv}(g_r, h, \varphi_{\infty, \mu, j})dh. \]

Applying corollaries 4.9 and 4.10 on the right hand side of (26) the result follows.

**Definition 5.9.** We denote by
\[ \vartheta(\tau, z, \mu_j) := v^{1/4} \text{Conv}(g_r, h, \varphi_{\infty, \mu, j}), \]
\[ \text{Conv}(\tau, z, \mu_j) := v^{1/4} \text{Conv}(g_r, h, \varphi_{\infty, \mu, j}), \]
\[ \text{Div}(\tau, z, \mu_j) := v^{1/4} \text{Div}(g_r, h, \varphi_{\infty, \mu, j}). \]

Further we denote their constant Fourier coefficients with respect to \( \tau \) by
\[ \vartheta(\tau, z, \mu_j) = \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \vartheta(u + iv, z, \mu_j)du, \]
\[ \text{Conv}(\tau, z, \mu_j) = \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \text{Conv}(u + iv, z, \mu_j)du, \]
\[ \text{Div}(\tau, z, \mu_j) = \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \text{Div}(u + iv, z, \mu_j)du. \]
Corollary 5.10. Let $T > 1$, the theta function satisfies the following equality:

$$\int_{X_{mod,T}} \vartheta(\tau, z, \mu_j) d\mu(z) = \text{vol}(X_{mod}) \left[ E(\tau, 1/2, -1/2, \mu(\varphi_{\mu_j})) + c \text{Res}_{s=1/2} E(\tau, s, 1/2, \mu(\varphi)) \right]$$

$$- \int_{X_{mod,T}} \text{Conv}(\tau, z, \mu_j) d\mu(z) + \int_{X_{mod,T}} \text{Div}(\tau, z, \mu_j) d\mu(z).$$

Proof. We factor $\vartheta(\tau, z, \mu_j)$ into its divergent and convergent parts as we did in remark 4.6. We obtain

$$\int_{X_{mod,T}} \vartheta(\tau, z, \mu_j) d\mu(z) = \int_{X_{mod,T}} \text{Conv}(\tau, z, \mu_j) d\mu(z)$$

$$+ \int_{X_{mod,T}} \text{Div}(\tau, z, \mu_j) d\mu(z).$$

Since $\text{Conv}(\tau, z, \mu_j)$ is integrable

$$\int_{X_{mod,T}} \text{Conv}(\tau, z, \mu_j) d\mu(z) = \int_{X_{mod}} \text{Conv}(\tau, z, \mu_j) d\mu(z)$$

$$- \int_{X_{mod,T}} \text{Conv}(\tau, z, \mu_j) d\mu(z).$$

Using 5.8 in (28) and then plugging the result into (5.1) we obtain the formula. 

For the following corollary we recall that

$$E(\tau, 1/2, -1/2, \mu(\varphi_{\mu_j}))_0 = \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} E(u + iv, 1/2, -1/2, \mu(\varphi_{\mu_j})) du,$$

$$\text{Res}_{s=1/2} \left( E(\tau, s, 1/2, \mu(\varphi_{\mu_j})) \right)_0 = \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \text{Res}_{s=1/2} E(u + iv, s, 1/2, \mu(\varphi_{\mu_j})) du.$$

Corollary 5.11. Let $\vartheta(\tau, z, \mu_j)_0 = \sum_{\lambda \in \mathcal{L}_{L+1}} \vartheta(\tau, z, \mu_j)_{\eta(\lambda)}$ be the constant term of the Fourier expansion of $\vartheta(\tau, z, \mu_j)$ with respect to $\tau$. The constant term of the theta function satisfies the following equality:

$$\int_{X_{mod,T}} \vartheta(\tau, z, \mu_j)_0 d\mu(z) = \text{vol}(X_{mod}) \left[ E(\tau, 1/2, -1/2, \mu(\varphi_{\mu_j}))_0 + c \text{Res}_{s=1/2} \left( E(\tau, s, 1/2, \mu(\varphi_{\mu_j})) \right)_0 \right]$$

$$- \int_{X_{mod,T}} \text{Conv}(\tau, z, \mu_j)_0 d\mu(z) + \int_{X_{mod,T}} \text{Div}(\tau, z, \mu_j)_0 d\mu(z).$$

Proof. We obtain

$$\int_{X_{mod,T}} \vartheta(\tau, z, \mu_j)_0 d\mu(z) = \int_{X_{mod,T}} \left( \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \vartheta(u + iv, z, \mu_j) du \right) d\mu(z).$$

Since $X_{mod,T}$ is compact and $\vartheta(\tau, z, \mu_j)$ is continuous in both coordinates we can apply Fubini’s theorem. Then

$$\int_{X_{mod,T}} \left( \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \vartheta(u + iv, z, \mu_j) du \right) d\mu(z)$$

$$= \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \left( \int_{X_{mod,T}} \vartheta(u + iv, z, \mu_j) d\mu(z) \right) du.$$
We apply corollary 5.10, obtaining

\begin{equation}
\frac{1}{1+3j} \int_{-1/2-3j/2}^{1/2+3j/2} \left( \int_{X^{\text{mod},T}} \partial(u + iv, z, \mu_j) d\mu(z) \right) du \\
= \text{vol}(X^{\text{mod}}) E(\tau, 1/2, -1/2, \mu(\varphi_{\mu_j}))(0) + c \text{Res}_{s=1/2} \left( E(\tau, s, 1/2, \mu(\varphi_{\mu_j})) \right)_0
\end{equation}

\begin{equation}
- \frac{1}{1+3j} \int_{-1/2-3j/2}^{1/2+3j/2} \int_{X^{\text{mod},T}} \text{Conv}(u + iv, z, \mu_j) d\mu(z) dv \\
+ \frac{1}{1+3j} \int_{-1/2-3j/2}^{1/2+3j/2} \int_{X^{\text{mod},T}} \text{Div}(u + iv, z, \mu_j) d\mu(z) dv.
\end{equation}

Using Fubini’s theorem one more time

\begin{equation}
\frac{1}{1+3j} \int_{-1/2-3j/2}^{1/2+3j/2} \int_{X^{\text{mod},T}} \text{Div}(u + iv, z, \mu_j) d\mu(z) du
= \int_{X^{\text{mod},T}} \left( \frac{1}{1+3j} \int_{-1/2-3j/2}^{1/2+3j/2} \text{Div}(u + iv, z, \mu_j) du \right) d\mu(z)
= \int_{X^{\text{mod},T}} \text{Div}(\tau, z, \mu_j) d\mu(z).
\end{equation}

Furthermore analogously to (31) it holds that

\begin{equation}
\frac{1}{1+3j} \int_{-1/2-3j/2}^{1/2+3j/2} \int_{X^{\text{mod},T}} \text{Conv}(u + iv, z, \mu_j) d\mu(z) du = \int_{X^{\text{mod},T}} \text{Conv}(\tau, z, \mu_j) d\mu(z).
\end{equation}

Using the equalities given in (31) and (33) on the right hand side of the equality (29) the result follows. \qed

5.2. Integral of the singular theta lift. Let \( f \in M_{1/2, L} \), the goal of this subsection is to compute the integral

\[ \int_{X^{\text{mod},T}} \left( \int_{X^{\text{mod}}} \langle f(\tau), \Theta^{\text{Siegel}}(\tau, z) \rangle d\mu(\tau) \right) d\mu(z), \]

that appears in the equality (23).

Before starting the computation we will recall the method of [17]. Although it does not apply to our case, the strategy used in [17] will be useful for our goal. Let \((V^{(2,p)}, q)\) be either an isotropic rational quadratic space of signature \((2, p)\) with \(p \geq 3\) or an anisotropic rational quadratic space. Let us fix a lattice \( L \) of \( V \) so that \( q|_L \in \mathbb{Z} \). In [17] the author studies the integrals

\[ \int_{X^{(2,p)}} \left( \int_{X^{\text{mod}}} \langle f(\tau), \Theta^{\text{Siegel}}(\tau, z) \rangle d\mu(\tau) \right) d\mu(z), \]

where \( X^{(2,p)} \) is the Shimura variety associated to the algebraic group \( \text{GSpin}_{4/(2,p)} \). More concretely in [17, sec. 3, p. 24], given \( \mu \in L'(A_f)/L(A_f) \) the author shows that

\begin{equation}
\int_{X^{(2,p)}} \left( \int_{X^{\text{mod}}} f_\mu(\tau) \Theta^{\text{Siegel}}(\tau, z, \mu) d\mu(\tau) \right) d\mu(z)
= \int_{X^{(2,p)}} \left( \int_{X^{\text{mod}}} f_\mu(\tau) \vartheta(\tau, z, \mu) d\mu(\tau) \right) d\mu(z)
\end{equation}

\begin{equation}
= \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\tau \in \mathcal{T}} f_\mu(\tau) \left( \int_{X^{(2,p)}} \vartheta(\tau, z, \mu) d\mu(\tau) \right) v^{-\sigma} d\mu(\tau).
\end{equation}

Once this is achieved, the author applies the Siegel-Weil formula and unfolds the integral with the resulting Eisenstein series. Although the proof of the equality between (34) and (35) does not apply for a quadratic rational isotropic space of signature \((2, 1)\), i.e. the modular curve case, we will give an overview of the proof, pointing out which parts of the proof are useful for us and what propositions does not apply in our case.
Proposition 5.12. Let $\theta(g, h, \varphi)$ be the theta function associated to any dual reductive pair of the form $(\text{Sp}_1, O(V))$ with $V$ rational quadratic space of signature $(2, p)$. Given $\beta \in \text{Sym}_2(\mathbb{Q})$, the $\beta$–Fourier coefficient of $\theta(g, h)$ with respect to $g$ is equal to

$$\theta(g, h)_{\beta} = \sum_{x \in \mathbb{Q}^m} \omega(g, h)\varphi(x).$$

Let $\mu \in L'/L$, we denote by $\theta^{\text{Sie}}_{\mu}(\tau, z)$ the $\mu$–component of the Siegel theta function associated to a lattice $L$ as in definition 2.5. Its Fourier expansion with respect to $\tau$ is given by

$$\theta^{\text{Sie}}_{\mu}(\tau, z) = \sum_{\lambda \in \mu + L} e^{-2\pi i (\lambda z - \nu q(\lambda_+) - \nu q(\lambda_-))} e^{2\pi i q(\lambda) u}.$$

Furthermore,

$$\langle f(\tau), \Theta_{\mu}^{\text{Sie}}(\tau, z) \rangle = \sum_{\mu \in L'/L} \sum_{m \in \mathbb{Q}} c_{\mu}(-m) \sum_{\lambda \in V(\mathbb{Q})} \sum_{Q(x) = m} \varphi_{\mu}(x) e^{-2\pi i \left( q(\lambda_+) - q(\lambda_-) \right)}.$$

Proof. We find the first assertion in [20, (5.1), p. 512] and the second one in [4, p. 48]. □

Using the previous proposition we may factor our theta function as follows:

(36) $$\theta^{\text{Sie}}_{\mu}(\tau, z) = C_{00}(\tau, z)^{\text{Sie}}_{\mu} + C_{0}(\tau, z)^{\text{Sie}}_{\mu} + C_{1}(\tau, z)^{\text{Sie}}_{\mu} + C_{2}(\tau, z)^{\text{Sie}}_{\mu}$$

$$= 1 + \sum_{\lambda \in L + \mu} e^{-2\pi i \left( q(\lambda_+) - q(\lambda_-) \right)} e^{2\pi i u q(\lambda)}$$

$$+ \sum_{\lambda \in L + \mu} e^{-2\pi i \left( q(\lambda_+) - q(\lambda_-) \right)} e^{2\pi i u q(\lambda)}$$

$$+ \sum_{\lambda \in L + \mu} e^{-2\pi i \left( q(\lambda_+) - q(\lambda_-) \right)} e^{2\pi i u q(\lambda)}$$

where the functions $C_{00}(\tau, z)^{\text{Sie}}_{\mu}$, $C_{0}(\tau, z)^{\text{Sie}}_{\mu}$, $C_{1}(\tau, z)^{\text{Sie}}_{\mu}$, $C_{2}(\tau, z)^{\text{Sie}}_{\mu}$ correspond to the terms on the right hand side of the equality. Since

$$\vartheta(\tau, z, \mu) = \psi^{\mu}_{\text{Sie}}(\tau, z),$$

we may also factor $\vartheta(\tau, z, \mu)$ as follows:

(37) $$\vartheta(\tau, z, \mu) = C_{00}(\tau, z, \mu) + C_{0}(\tau, z, \mu) + C_{1}(\tau, z, \mu) + C_{2}(\tau, z, \mu)$$

$$= vC_{00}(\tau, z, \mu)^{\text{Sie}}_{\mu} + vC_{0}(\tau, z, \mu)^{\text{Sie}}_{\mu} + vC_{1}(\tau, z, \mu)^{\text{Sie}}_{\mu} + vC_{2}(\tau, z, \mu)^{\text{Sie}}_{\mu},$$

where the functions $C_{00}(\tau, z, \mu)$, $C_{0}(\tau, z, \mu)$, $C_{1}(\tau, z, \mu)$, $C_{2}(\tau, z, \mu)$ are defined by the terms of the right hand side of the above equality.

Remark 5.13. In the formula (37) the terms $C_{1}(\tau, z, \mu)$ are the positive Fourier coefficients of $\vartheta(\tau, z, \mu)$ with respect to $\tau$, $C_{2}(\tau, z, \mu)$ are the negative Fourier coefficients of $\vartheta(\tau, z, \mu)$ with respect to $\tau$ and $C_{00}(\tau, z, \mu) + C_{0}(\tau, z, \mu) + C_{1}(\tau, z, \mu)$ is the 0–th Fourier coefficient of $\vartheta(\tau, z, \mu)$ with respect to $\tau$. The motivation to split the 0–th Fourier coefficient into two terms is the asymptotic behaviour with respect to $\tau = u + iv$. The function

$$C_{0}(\tau, z, \mu) = v \sum_{\lambda \in L + \mu \atop q(\lambda) = 0} e^{-2\pi i \left( q(\lambda_+) - q(\lambda_-) \right)},$$

declares as $e^{-v}$ when $v \to \infty$. By contrast the term $C_{00}(\tau, z, \mu) = v$ is not exponentially decreasing. In fact, the term $C_{00}(\tau, z, \mu)$ is the reason why $CT_{\tau = 0}$ is needed to state the singular theta lift, [12, thm. 3.2, p. 25].
In [17, sec. 3, p. 24] the author factors the integral (34) according to the factorization (37)

\[ \text{CT}_{\sigma=0} \int_{X(2, p)} \left( \lim_{T \to \infty} \int_{F^T} f_{\mu}(\tau) \vartheta(\tau, z, \mu) v^{-\sigma} \, d\mu(\tau) \right) \, d\mu(z) = \text{CT}_{\sigma=0} \int_{X(2, p)} \left( \lim_{T \to \infty} \int_{F^T} f_{\mu}(\tau) C_{00}(\tau, z, \mu) v^{-\sigma} \, d\mu(\tau) \right) \, d\mu(z) + \text{CT}_{\sigma=0} \int_{X(2, p)} \left( \lim_{T \to \infty} \int_{F^T} f_{\mu}(\tau) C_{0}(\tau, z, \mu) v^{-\sigma} \, d\mu(\tau) \right) \, d\mu(z) + \text{CT}_{\sigma=0} \int_{X(2, p)} \left( \lim_{T \to \infty} \int_{F^T} f_{\mu}(\tau) C_{1}(\tau, z, \mu) v^{-\sigma} \, d\mu(\tau) \right) \, d\mu(z) + \text{CT}_{\sigma=0} \int_{X(2, p)} \left( \lim_{T \to \infty} \int_{F^T} f_{\mu}(\tau) C_{2}(\tau, z, \mu) v^{-\sigma} \, d\mu(\tau) \right) \, d\mu(z). \]

The proof of the equality between (34) and (35) is based into two facts; the convergence of \( \int_{X(2, p)} \vartheta(\tau, z, \mu) \, d\mu(z) \) and that the following functions are holomorphic at \( \sigma = 0 \):

\[
\begin{align*}
\int_{X(2, p)} \left( \lim_{T \to \infty} \int_{F^T} f_{\mu}(\tau) C_{00}(\tau, z, \mu) v^{-\sigma} \, d\mu(\tau) \right) \, d\mu(z), \\
\int_{X(2, p)} \left( \lim_{T \to \infty} \int_{F^T} f_{\mu}(\tau) C_{0}(\tau, z, \mu) v^{-\sigma} \, d\mu(\tau) \right) \, d\mu(z), \\
\int_{X(2, p)} \left( \lim_{T \to \infty} \int_{F^T} f_{\mu}(\tau) C_{1}(\tau, z, \mu) v^{-\sigma} \, d\mu(\tau) \right) \, d\mu(z), \\
\int_{X(2, p)} \left( \lim_{T \to \infty} \int_{F^T} f_{\mu}(\tau) C_{2}(\tau, z, \mu) v^{-\sigma} \, d\mu(\tau) \right) \, d\mu(z).
\end{align*}
\]

The proof that the functions (38), (40) and (41) are holomorphic is in [17, sec. 3, p. 24] and applies for any Shimura variety associated to a quadratic space of signature \((2, 1)\). By contrast, the proof that (39) is holomorphic, [17, prop. 3.4, p. 28], does not apply for the modular curve case. As we mentioned before, in this paper we will consider the integral (38) separately and with different techniques.

Let \( T \geq 1 \), we set

\[ F_1 = \{ \tau = u + iv \in F^T, \text{s.t.} \, v \leq 1 \}, \]

\[ F_2^T = \{ \tau = u + iv \in F^T, \text{s.t.} \, v > 1 \}, \]

so that

\[ F^T = F_1 \sqcup F_2^T. \]

**Lemma 5.14.** The following equality holds:

\[
\int_{X_{\operatorname{mod}, T}} \left( \int_{X_{\operatorname{mod}}} f(\tau) \Theta_{\nu}^{\text{Siegel}}(\tau, z) \right) \, d\mu(\tau) \, d\mu(z) = \sum_{j=0}^{1} \lim_{T \to \infty} \left[ \int_{F^T} f_{\mu_j}(\tau) \left( \int_{X_{\operatorname{mod}, T}} \vartheta(\tau, z, \mu_j) \, d\mu(z) \right) \, d\mu(\tau) \right] - \int_{F_2^T} f_{\mu_j}(\tau) \left( \int_{X_{\operatorname{mod}, T}} \vartheta(\tau, z, \mu_j) \, d\mu(z) \right) \, d\mu(\tau) + \int_{X_{\operatorname{mod}, T}} \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{F_2^T} f_{\mu_j}(\tau) \vartheta(\tau, z, \mu_j) v^{-\sigma} \, d\mu(\tau) \, d\mu(z),
\]

where we recall that

\[
\vartheta(\tau, z, \mu_j) := \sum_{\lambda \in L^+ + \mu_j \atop q(\lambda) = 0} e^{-2\pi (vq(\lambda_\perp) - vq(\lambda_\perp^\perp))} = \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \vartheta(u + iv, z, \mu_j) \, du.
\]
Proof. Using the factorization (42) we obtain

\[
\int_{X^{\text{mod}, T}} C T_{\sigma = 0} \lim_{T \to \infty} \int_{\mathcal{F}_1} \langle f(\tau), \Theta^{\text{Siegel}}(\tau, z) \rangle v^{-\sigma} d\mu(\tau) d\mu(z)
\]

\[
= \int_{X^{\text{mod}, T}} C T_{\sigma = 0} \lim_{T \to \infty} \int_{\mathcal{F}_1} \langle f(\tau), \Theta^{\text{Siegel}}(\tau, z) \rangle v^{-\sigma} d\mu(\tau) d\mu(z)
\]

\[
+ \int_{X^{\text{mod}, T}} C T_{\sigma = 0} \lim_{T \to \infty} \int_{\mathcal{F}_1} \langle f(\tau), \Theta^{\text{Siegel}}(\tau, z) \rangle v^{-\sigma} d\mu(\tau) d\mu(z).
\]

On the one hand, since \( \mathcal{F}_1 \) is compact

\[
\int_{X^{\text{mod}, T}} C T_{\sigma = 0} \lim_{T \to \infty} \int_{\mathcal{F}_1} \langle f(\tau), \Theta^{\text{Siegel}}(\tau, z) \rangle v^{-\sigma} d\mu(\tau) d\mu(z)
\]

\[
= C T_{\sigma = 0} \lim_{T \to \infty} \int_{\mathcal{F}_1} \langle f(\tau), \Theta^{\text{Siegel}}(\tau, z) \rangle v^{-\sigma} d\mu(\tau) d\mu(z).
\]

On the other hand, using the factorization given in (37)

\[
\int_{X^{\text{mod}, T}} C T_{\sigma = 0} \lim_{T \to \infty} \int_{\mathcal{F}_1} \langle f(\tau), \Theta^{\text{Siegel}}(\tau, z) \rangle v^{-\sigma} d\mu(\tau) d\mu(z)
\]

\[
= \sum_{j=0}^{1} \int_{X^{\text{mod}, T}} C T_{\sigma = 0} \lim_{T \to \infty} \int_{\mathcal{F}_1} f_{\mu_j}(\tau)(C_1(\tau, z, \mu_j) + C_2(\tau, z, \mu_j)) v^{-\sigma} d\mu(\tau) d\mu(z)
\]

\[
+ \int_{X^{\text{mod}, T}} C T_{\sigma = 0} \lim_{T \to \infty} \int_{\mathcal{F}_1} f_{\mu_j}(\tau)(C_0(\tau, z, \mu_j) + C_0(\tau, z, \mu_j)) v^{-\sigma} d\mu(\tau) d\mu(z).
\]

Using that (40) and (41) are holomorphic at \( \sigma = 0 \), the following equality holds:

\[
\int_{X^{\text{mod}, T}} C T_{\sigma = 0} \lim_{T \to \infty} \int_{\mathcal{F}_1} f_{\mu_j}(\tau)(C_1(\tau, z, \mu_j) + C_2(\tau, z, \mu_j)) v^{-\sigma} d\mu(\tau) d\mu(z)
\]

\[
= \lim_{T \to \infty} \int_{\mathcal{F}_1} f_{\mu_j}(\tau) \int_{X^{\text{mod}, T}} (C_1(\tau, z, \mu_j) + C_2(\tau, z, \mu_j)) d\mu(z) \mu(\tau).
\]

Moreover, since \( C_1(\tau, z, \mu_j) + C_2(\tau, z, \mu_j) = \vartheta(\tau, z, \mu_j) - \vartheta(\tau, z, \mu_j) \) we have

\[
\lim_{T \to \infty} \left( \int_{\mathcal{F}_1} f_{\mu_j}(\tau) \left( \int_{X^{\text{mod}, T}} \vartheta(\tau, z, \mu_j) d\mu(z) \right) d\mu(\tau) \right. \left. - \int_{\mathcal{F}_1} f_{\mu_j}(\tau) \left( \int_{X^{\text{mod}, T}} \vartheta(\tau, z, \mu_j) d\mu(z) \right) d\mu(\tau) \right).
\]

Plugging (48) and (47) in (46), and then (45) and (46) in (44) we obtain the result. \( \square \)

In order to obtain an explicit expression for (43) we will consider separately each factor of the right hand side.

**Definition 5.15.** The integral

\[
\int_{X^{\text{mod}, T}} C T_{\sigma = 0} \lim_{T \to \infty} \int_{\mathcal{F}_1} f_{\mu_j}(\tau) \vartheta(\tau, z, \mu_j) v^{-\sigma} d\mu(\tau) d\mu(z),
\]

is called the **limit case**. Furthermore we refer to the integral

\[
\lim_{T \to \infty} \left( \int_{\mathcal{F}_1} f_{\mu_j}(\tau) \left( \int_{X^{\text{mod}, T}} \vartheta(\tau, z, \mu_j) d\mu(z) \right) d\mu(\tau) \right. \left. - \int_{\mathcal{F}_1} f_{\mu_j}(\tau) \left( \int_{X^{\text{mod}, T}} \vartheta(\tau, z, \mu_j) d\mu(z) \right) d\mu(\tau) \right)
\]

as the **ordinary case**.
5.3. Ordinary case. In this subsection we compute the ordinary case using the results of subsection 6.2. With the aim of simplifying the argument, we divide the computation into two terms:
\[
\int_{\mathbb{F}} f_{\mu_j}(\tau) \left( \int_{X_{\text{mod}, T}} \vartheta(\tau, z, \mu_j) d\mu(z) \right) d\mu(\tau), \quad \int_{\mathbb{F}} f_{\mu_j}(\tau) \left( \int_{X_{\text{mod}, T}} \vartheta(\tau, z, \mu_j) d\mu(z) \right) d\mu(\tau).
\]

Lemma 5.16. The convergent part satisfies the following estimate:
\[
\int_{\mathbb{F}} f_{\mu_j}(\tau) \int_{X_{\text{mod}, T}} \text{Conv}(\tau, z, \mu_j) d\mu(z) d\mu(\tau) = \int_{\mathbb{F}} f_{\mu_j}(\tau) \int_{X_{\text{mod}, T}} \text{Conv}(\tau, z, \mu_j) d\mu(z) d\mu(\tau) = O(e^{-T}T).
\]

Proof. By lemma 4.5 the function \(\text{Conv}(\tau, z, \mu_j)\) is exponentially decreasing in the variable \(z = x + iy\), then
\[
|\text{Conv}(\tau, z, \mu_j)| \leq |F_{\mu_j}(\tau)|e^{-y},
\]
where \(F_{\mu_j}(\tau)\) is a continuous function. Hence
\[
\left| \int_{\mathbb{F}} f_{\mu_j}(\tau) \int_{X_{\text{mod}, T}} \text{Conv}(\tau, z, \mu_j) d\mu(z) d\mu(\tau) \right| \leq e^{-T} \int_{\mathbb{F}} |f_{\mu_j}(\tau)F_{\mu_j}(\tau)| d\mu(\tau).
\]
Using the compactness of \(\mathbb{F}\) we obtain the estimate of the statement. The proof for \(\text{Conv}(\tau, z, \mu_j)\) is completely analogous. \(\square\)

Proposition 5.17. The following equality holds:
\[
\sum_{j=0}^{1} \int_{\mathbb{F}} f_{\mu_j}(\tau) \left( \int_{X_{\text{mod}, T}} \vartheta(\tau, z, \mu_j) d\mu(z) \right) d\mu(\tau) = \sum_{j=0}^{1} 2\text{vol}(X_{\text{mod}}) \sum_{m \in \mathbb{N}} c_{\mu_j}(-m)b(m, T, \varphi_{\mu_j})
\]
\[
- \frac{2c_{\mu_j}(0)}{\sqrt{T}} \log(T) - \frac{4c_{\mu_j}(0)}{\sqrt{T}} \left( - \tanh^{-1} \left( \frac{\sqrt{7}}{4} \right) + \frac{\sqrt{7}}{4} + \frac{\log(3/4)}{2} \right) + O(e^{-T}),
\]
where we recall that \(b(m, T, \varphi_{\mu_j})\) is the first term at \(s = 1/2\) of the Laurent series of the \(m\)-Fourier coefficient of \(E(\tau, s, 3/2, \mu(\varphi_{\mu_j}))\), (15).

Proof. We recall that corollary 5.10 provides
\[
\int_{X_{\text{mod}, T}} \vartheta(\tau, z, \mu(\varphi_{\mu_j})) d\mu(z) = \text{vol}(X_{\text{mod}}) \left[ E(\tau, 1/2, -1/2, \mu(\varphi_{\mu_j})) + c \text{Res}_{s=1/2} \frac{1}{E(\tau, s, 1/2, \mu(\varphi_{\mu_j}))} + \int_{X_{\text{mod}, T}} \text{Conv}(\tau, z, \mu_j) d\mu(z) \right]
\]
\[
+ \int_{X_{\text{mod}, T}} \text{Div}(\tau, z, \mu_j) d\mu(z).
\]
Then using the above equality we factor the integral of the statement into the sum of three terms:
\[
\text{vol}(X_{\text{mod}}) \left[ \int_{\mathbb{F}} f_{\mu_j}(\tau) E(\tau, 1/2, -1/2, \mu(\varphi_{\mu_j})) d\mu(\tau) + \int_{\mathbb{F}} f_{\mu_j}(\tau) \text{Res}_{s=1/2} E(\tau, s, 1/2, \mu(\varphi_{\mu_j})) d\mu(\tau) \right]
\]
\[
- \int_{\mathbb{F}} f_{\mu_j}(\tau) \int_{X_{\text{mod}, T}} \text{Conv}(\tau, z, \mu_j) d\mu(z) d\mu(\tau)
\]
\[
+ \int_{\mathbb{F}} f_{\mu_j}(\tau) \int_{X_{\text{mod}, T}} \text{Div}(\tau, z, \mu_j) d\mu(z) d\mu(\tau).
\]

We consider each above factor separately. Using proposition 6.10 and lemma 6.12 in (49), lemma 5.16 in (50) and proposition 6.6 in (51), the result follows. \(\square\)
Proposition 5.18. We obtain
\[
\sum_{j=0}^{1} \int_{\mathcal{F}^j} f_{\mu_j}(\tau) \left( \int_{X_{\text{mod,T}}} \theta(\tau, z, \mu_j) \, d\mu(z) \right) \, d\mu(\tau) = \sum_{j=0}^{1} M_f(1/2)(\mu(\varphi_{\mu_j}))(1)c_{\mu_j}(0)
\]
\[
- M_f(1/2)(\mu(\varphi_{\mu_j}))(1) \frac{c_{\mu_j}(0)}{T^{3/4}} - 2c_{\mu_j}(0) \left( 1 - \sqrt{T}^{-1} \right) \log(T)
\]
\[
- 4c_{\mu_j}(0)(1 - \sqrt{T}^{-1}) \left( - \tanh^{-1} \left( \frac{\sqrt{7}}{4} \right) + \frac{\sqrt{7}}{4} + \frac{\log(3/4)}{2} \right)
\]
\[+ O(e^{-T}) ,
\]
where $M_f(1/2)$ is the finite part of the intertwining map defined in 3.10.

Proof. We recall that corollary 5.11 shows that
\[
\int_{X_{\text{mod,T}}} \theta(\tau, z, \mu_j) \, d\mu(z) = \text{vol}(X_{\text{mod}}) \left[ E(\tau, 1/2, -1/2, \mu(\varphi_{\mu_j}))_0 
\right.
\]
\[
+ c \text{Res}_{s=1/2} \left( E(\tau, s, 1/2, \mu(\varphi_{\mu_j})) \right)_0 - \int_{X_{\text{mod,T}}} \text{Conv}(\tau, z, \mu_j) \, d\mu(z)
\]
\[+ \int_{X_{\text{mod,T}}} \text{Div}(\tau, z, \mu_j) \, d\mu(z).
\]
We proceed as in the proof of proposition 5.17. Using the above equality on the integral of the statement, we find
\[
\int_{\mathcal{F}^j} f_{\mu_j}(\tau) \left( \int_{X_{\text{mod,T}}} \theta(\tau, z, \mu_j) \, d\mu(z) \right) \, d\mu(\tau) = \text{vol}(X_{\text{mod}}) \left[ \int_{\mathcal{F}^j} f_{\mu_j}(\tau) E(\tau, 1/2, -1/2, \mu_j) \, d\mu(\tau) 
\right.
\]
\[
+ \int_{\mathcal{F}^j} f_{\mu_j}(\tau) \text{Res}_{s=1/2} \left( E(\tau, s, 1/2, \mu_j) \right)_0 \, d\mu(\tau)
\]
\[
\left. + \int_{\mathcal{F}^j} f_{\mu_j}(\tau) \int_{X_{\text{mod,T}}} \text{Div}(\tau, z, \mu_j) \, d\mu(z) \, d\mu(\tau) \right]
\]
\[
- \int_{\mathcal{F}^j} f_{\mu_j}(\tau) \int_{X_{\text{mod,T}}} \text{Conv}(\tau, z, \mu_j) \, d\mu(z) \, d\mu(\tau).
\]
As we did in the proof of proposition 5.17 we consider each factor of the above equation separately. By corollary 6.13 we show that (53) vanishes. Moreover using corollary 6.11 on the right hand side of (52), lemma 6.7 in (54) and lemma 5.16 in (55), the statement follows.

Corollary 5.19. The following equality holds:
\[
\lim_{T \to \infty} \left( \sum_{j=0}^{1} \int_{\mathcal{F}^j} f_{\mu_j}(\tau) \left( \int_{X_{\text{mod,T}}} \theta(\tau, z, \mu_j) \, d\mu(z) \right) \, d\mu(\tau) \right)
\]
\[
= \sum_{j=0}^{1} 2\text{vol}(X_{\text{mod}}) \sum_{m \in \mathbb{N}} c_{\mu_j} (-m)b(m, \varphi_{\mu_j}) + M_f(1/2)(\mu(\varphi_{\mu_j}))(1)c_{\mu_j}(0)
\]
\[
+ 2c_{\mu_j}(0) \log(T) - 4c_{\mu_j}(0) \left( - \tanh^{-1} \left( \frac{\sqrt{7}}{4} \right) + \frac{\sqrt{7}}{4} + \frac{\log(3/4)}{2} \right) - O(e^{-T}),
\]
where $b(m, \varphi_{\mu_j}) = \lim_{T \to \infty} b(m, T, \varphi_{\mu_j})$.

Proof. This follows directly by propositions 5.17 and 5.18.
5.4. Limit case. This section is devoted to compute the limit case 5.15:

\[ \sum_{j=0}^{1} \int_{X^\text{mod} \setminus t} \left( \prod_{\gamma \in \Gamma} \psi_{\gamma} \right) f_{\mu_j}(\tau) \delta(\tau, z, \mu_j) \sigma \, d\mu(\tau) \, d\mu(z) \]

In [35] the author developed a Rankin-Selberg method for truncated fundamental domains. Before continuing with our computation we will give an overview of this method with the aim of simplifying the subsequent computations.

For every \( \hat{T} \geq 1 \), the subset \( \mathcal{F}^\hat{T} = \{ x + iy \in \mathcal{H}, \ s.t \ y \leq \hat{T} \} \) is a fundamental domain for the action of \( SL_2(\mathbb{Z}) \) on

\[ \mathcal{H}^\hat{T} = \bigcup_{\gamma \in SL_2(\mathbb{Z})} \gamma \mathcal{F}^\hat{T} = \{ z \in \mathcal{H}, \ s.t. \ \max_{\gamma \in SL_2(\mathbb{Z})} \text{Im}(\gamma z) \leq \hat{T} \}. \]

According to [35, (20), p. 420]

\[ \mathcal{H}^\hat{T} = \{ z \in \mathcal{H}, \ s.t. \ \text{Im}(z) \leq \hat{T} \} - \bigcup_{c \geq 1} \bigcup_{a \in \mathbb{Z}} S_{a/c}, \]

where \( S_{a/c} \) is the disc of radius \( \frac{1}{2c} \) tangent to the real axis at \( a/c \). Therefore, given

\[ \Gamma^\infty = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \ x \in \mathbb{Z} \right\}, \]

we define the following subset of the complex numbers:

\[ \Gamma^\infty \setminus X^\text{mod} \setminus t := \Gamma^\infty \setminus \mathcal{H}^\hat{T} = \{ z = x + iy \in \mathcal{H}, \ |x| \leq 1/2, \ 0 \leq y \leq \hat{T} \} - \bigcup_{c \geq 1} \bigcup_{a \in \mathbb{Z}} S_{a/c}. \]

Definition 5.20. Let \( s \in \mathbb{C} \) such that \( \text{Re}(s) > 1 \). The classical Eisenstein series considered by Zagier is defined as follows:

\[ E(\tau, s) = \sum_{\gamma \in \Gamma^\infty \setminus SL_2(\mathbb{Z})} \text{Im}(\gamma z)^s. \]

Proposition 5.21. The function \( E(z, s) \) is holomorphic in \( \text{Re}(s) > 1/2 \) except for a pole of residue \( 3/\pi \) at \( s = 1 \). Furthermore \( \zeta^*(2s)E(z, s) \) is holomorphic in all \( s \in \mathbb{C} \) except for \( s \neq 0, 1 \), where \( \zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \) is the completed Riemann zeta function.

Proof. We refer the reader to [35, p. 415, p. 416]. \( \square \)

Remark 5.22. The 0–th Fourier term of the modular form \( E(\tau, s) \) is equal to

\[ \frac{1}{s-1} + \frac{\varphi(s)}{s-1}, \]

where \( \varphi(s) := \frac{\zeta(2s-1)}{\zeta(2s)} \).

Proposition 5.23. The following equality holds:

\[ \int_{X^\text{mod} \setminus t} E(\tau, s) \frac{dx dy}{y^2} = \frac{\hat{T}^{s-1}}{(s-1)} - \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \hat{T}^{s-1}/s, \]

where \( \zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \) is the completed Riemann zeta function.

Proof. See [35, (33), p. 426]. \( \square \)

Proposition 5.24. Let \( A \) be the constant defined in 1.0.1. The limit case satisfies the following equality:

\[ \int_{X^\text{mod} \setminus t} \left( \prod_{\gamma \in \Gamma} \psi_{\gamma} \right) f_{\mu_j}(\tau) \delta(\tau, z, \mu_j) \sigma \, d\mu(\tau) \, d\mu(z) \]

\[ = c_{\mu_j}(0) A \int_{\Gamma^\infty \setminus X^\text{mod} \setminus t} y^s \log(y) \frac{dx dy}{y^2} + c_{\mu_j}(0) \int_{\Gamma^\infty \setminus X^\text{mod} \setminus t} \frac{dy}{y}, \]

\[ + c_{\mu_j}(0) \int_{\Gamma^\infty \setminus X^\text{mod} \setminus t} y^s \frac{dx}{y}. \]
Proof. We factor \( \vartheta(\tau, z, \mu_j)_0 = C_0(\tau, z, \mu_j) + C_{00}(\tau, z, \mu_j) \), where we recall that
\[
C_0(\tau, z, \mu_j) = v \sum_{0 \neq \lambda \in V \atop q(\lambda) = 0} e^{-4\pi v q(\lambda)} \varphi_{\mu_j}(\lambda)
\]
\[
C_{00}(\tau, z, \mu_j) = v.
\]
Then
\[
\int_{X^{mod. \tau}} \left( \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{F}_2^2} f_{\mu_j}(\tau) \vartheta(\tau, z, \mu_j)_0 v^{-\sigma} d\mu(\tau) \right) d\mu(z)
\]
\[
= \int_{X^{mod. \tau}} \left( \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{F}_2^2} f_{\mu_j}(\tau) C_0(\tau, z, \mu_j)_0 v^{-\sigma + 1} d\mu(\tau) \right) d\mu(z)
\]
\[
+ \int_{X^{mod. \tau}} \left( \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{F}_2^2} f_{\mu_j}(\tau) C_{00}(\tau, z, \mu_j)_0 v^{-\sigma + 1} d\mu(\tau) \right) d\mu(z).
\]
We obtain
\[
\text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{F}_2^2} f_{\mu_j}(\tau) C_{00}(\tau, z, \mu_j)_0 v^{-\sigma + 1} d\mu(\tau) = \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{1}^{T} c_{\mu_j}(0) v^{-\sigma + 1} dv = 0,
\]
Hence
\[
\int_{X^{mod. \tau}} \left( \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{F}_2^2} f_{\mu_j}(\tau) \vartheta(\tau, z, \mu_j)_0 v^{-\sigma} d\mu(\tau) \right) d\mu(z)
\]
\[
= c_{\mu_j}(0) \int_{X^{mod. \tau}} \left( \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{1}^{T} \sum_{0 \neq \lambda \in V \atop q(\lambda) = 0} e^{-4\pi v q(\lambda)} \varphi_{\mu_j}(\lambda) v^{-\sigma - 1} dv \right) d\mu(z).
\]
(58)
The isotropic elements of the quadratic space, i.e. \( 0 \neq \lambda \in V(\mathbb{Q}) \) such that \( q(\lambda) = 0 \), are generated by \( \text{SL}_2(\mathbb{Z}) \) in one orbit. Furthermore those vectors are in one to one correspondence with \( \mathbb{Q} \cup \{ \infty \} \) and then \( \text{SL}_2(\mathbb{Z}) \setminus \{ 0 \neq \lambda \in V(\mathbb{Q}) \} \) is identified with the cusp \( \infty \) of \( X^{mod.} \). One representative of the cusp \( \infty \) in the projective cone model, described in proposition 2.1, is the isotropic line \( \mathbb{Q}^\times \cdot e_1 := \mathbb{Q}^\times \cdot (1, 0, 0) \). The stabilizer of this isotropic line is given by
\[
\Gamma_\infty = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x \in \mathbb{Z} \right\} \cong \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbb{Z} \right\},
\]
where the above isomorphism is the exceptional isomorphism \( SO(2, 1) \simeq \text{SL}_2(\mathbb{R}) \). Hence we obtain the following identification
\[
\{ 0 \neq \lambda \in V, \ s.t. \ q(\lambda) = 0 \} \simeq \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z}) \cdot (\mathbb{Q}^\times \cdot e_1).
\]
(59)
Using (59) and both definitions of \( \varphi_{\mu_j} \) given in (25) we rewrite the inner sum of (58) as follows:
\[
\sum_{0 \neq \lambda \in V \atop q(\lambda) = 0} e^{-4\pi v q(\lambda)} \varphi_{\mu_j}(\lambda) = \sum_{\gamma \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} \sum_{x_2 \in \mathbb{Z} \setminus \{0\}} e^{-4\pi v \gamma \cdot (x_2, 0, 0).}
\]
By the invariance property of the Gaussian we find
\[
\sum_{\gamma \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} \sum_{x_2 \in \mathbb{Z} \setminus \{0\}} e^{-4\pi v \gamma \cdot (x_2, 0, 0).} = \sum_{\gamma \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} \sum_{x_2 \in \mathbb{Z} \setminus \{0\}} e^{-4\pi v \gamma \cdot (x_2, 0, 0).}.
\]
We would like to unfold the integral over $X^{mod. \, \hat{T}}$. In order to overcome the convergence problems of the unfolding we introduce the auxiliary term $\text{Im}(z)^s$ with $s \in \mathbb{C}$. The integral of (58) is equal to

$$\int_{X^{mod. \, \hat{T}}} \left( CT_{\sigma=0} \lim_{T \to \infty} \int_{\gamma \in \Gamma_{\text{mod.}} \setminus \mathrm{SL}_2(\mathbb{Z})} \sum_{x_2 \in \mathbb{Z}\setminus\{0\}} e^{-4\pi vq((x_2,0,0),\gamma-1,\gamma)} y^{-\sigma-1} CT_{s=0} \text{Im}(\gamma z)^s \right) d\mu(z)$$

(60)

$$= CT_{s=0} \int_{X^{mod. \, \hat{T}}} \left( CT_{\sigma=0} \lim_{T \to \infty} \int_{\gamma \in \Gamma_{\text{mod.}} \setminus \mathrm{SL}_2(\mathbb{Z})} \sum_{x_2 \in \mathbb{Z}\setminus\{0\}} e^{-4\pi vq((x_2,0,0),\gamma-1,\gamma)} y^{-\sigma-1} \text{Im}(\gamma z)^s \right) d\mu(z),$$

where the equality is justified by means of Fubini’s theorem. To unfold the integral (60) with the sum, we suppose that $s \in \mathbb{C}$ satisfies that $\text{Re}(s) > 1$ to ensure the convergence of the resulting function. After the unfolding, by means of meromorphic continuation the result will follow. Under this assumption on $s$, (60) is equal to

$$CT_{s=0} \int_{X^{mod. \, \hat{T}}} \left( CT_{\sigma=0} \lim_{T \to \infty} \int_{\gamma \in \Gamma_{\text{mod.}} \setminus \mathrm{SL}_2(\mathbb{Z})} \sum_{x_2 \in \mathbb{Z}\setminus\{0\}} e^{-4\pi vq((x_2,0,0),\gamma-1,\gamma)} y^{-\sigma-1} \right) y^s d\mu(z),$$

(61)

that is well defined. Using the formula given in [8, (3.9), p. 296] it holds that

$$q((x_2,0,0)z) = \frac{x_2^2}{y^2}.$$

By Poisson summation formula we obtain

$$\sum_{x_2 \in \mathbb{Z}\setminus\{0\}} e^{\frac{4\pi vq x_2}{y^2}} = \sum_{w_1 \in \mathbb{Z}} \int_{\mathbb{R}} e^{\frac{4\pi vq x_1}{y^2}} + 2\pi i x_2 w_1 dx_2 - 1,$$

where the $-1$ term in the above equation corresponds to the term $e^{\frac{4\pi vq x_1}{y^2}}$ evaluated at $x_2 = 0$. Let us divide $\mathbb{Z} = \mathbb{Z} \setminus \{0\} \cup \{0\}$, we factor the right hand side of (62) as follows:

$$\sum_{w_1 \in \mathbb{Z}} \int_{\mathbb{R}} e^{\frac{4\pi vq x_1}{y^2}} + 2\pi i x_2 w_1 dx_2 - 1 = \sum_{w_1 \in \mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{\frac{4\pi vq x_1}{y^2}} + 2\pi i x_2 w_1 dx_2 + \int_{\mathbb{R}} e^{\frac{4\pi vq x_1}{y^2}} dx_2 - 1.$$

Plugging the factorization (63) in the integral (61) we obtain

$$CT_{s=0} \int_{X^{mod. \, \hat{T}}} \left( CT_{\sigma=0} \lim_{T \to \infty} \int_{\gamma \in \Gamma_{\text{mod.}} \setminus \mathrm{SL}_2(\mathbb{Z})} \sum_{x_2 \in \mathbb{Z}\setminus\{0\}} e^{-4\pi vq((x_2,0,0),\gamma-1,\gamma)} y^{-\sigma-1} \right) y^s d\mu(z)$$

(64)

$$= CT_{s=0} \int_{X^{mod. \, \hat{T}}} \left( CT_{\sigma=0} \lim_{T \to \infty} \int_{\gamma \in \Gamma_{\text{mod.}} \setminus \mathrm{SL}_2(\mathbb{Z})} \int_{\mathbb{R}} e^{\frac{4\pi vq x_1}{y^2}} + 2\pi i x_2 w_1 dx_2 y^{-\sigma-1} \right) y^s d\mu(z)$$

(65)

$$+ CT_{s=0} \int_{X^{mod. \, \hat{T}}} \left( CT_{\sigma=0} \lim_{T \to \infty} \int_{\gamma \in \Gamma_{\text{mod.}} \setminus \mathrm{SL}_2(\mathbb{Z})} \int_{\mathbb{R}} e^{\frac{4\pi vq x_1}{y^2}} + 2\pi i x_2 w_1 dx_2 y^{-\sigma-1} \right) y^s d\mu(z)$$

(66)

$$+ CT_{s=0} \int_{X^{mod. \, \hat{T}}} \left( CT_{\sigma=0} \lim_{T \to \infty} \int_{\gamma \in \Gamma_{\text{mod.}} \setminus \mathrm{SL}_2(\mathbb{Z})} \int_{\mathbb{R}} e^{\frac{4\pi vq x_1}{y^2}} + 2\pi i x_2 w_1 dx_2 y^{-\sigma-1} \right) y^s d\mu(z).$$

(67)

The goal of this proposition is achieved by computing the sum of the above three integrals. Applying lemma 5.25 and the proof of lemma 5.30 to (65) we obtain that

$$\int_{X^{mod. \, \hat{T}}} \left( CT_{\sigma=0} \lim_{T \to \infty} \int_{\gamma \in \Gamma_{\text{mod.}} \setminus \mathrm{SL}_2(\mathbb{Z})} \int_{\mathbb{R}} e^{\frac{4\pi vq x_1}{y^2}} + 2\pi i x_2 w_1 dx_2 y^{-\sigma-1} \right) y^s d\mu(z),$$

(68)
is a meromorphic function in the variable $s \in \mathbb{C}$. We apply lemma 5.26 and the proof of lemmas 5.29 and 5.28 to (66) to obtain that
\[
\int_{\Gamma^\infty \setminus X_{mod,T}} \left( \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{R}} e^{-\frac{4\pi y_2^2}{v^2} + 2\pi i x_2 w_1} dx_2 v^{-\sigma-1} dv \right) y^s d\mu(z),
\]
is a meromorphic function in the variable $s \in \mathbb{C}$. Applying lemma 5.27 to (67), the function
\[
\int_{\Gamma^\infty \setminus X_{mod,T}} \left( \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{R}} v^{-\sigma-1} dv \right) y^s d\mu(z),
\]
is meromorphic in the variable $s \in \mathbb{C}$. Hence by meromorphic continuation in $s \in \mathbb{C}$ we obtain that the equality (64) holds for every $s \in \mathbb{C}$. To conclude we use lemma 5.30 in (65), lemma 5.26 in (66) and lemma 5.27 in (67).

**Lemma 5.25.** Let $s \in \mathbb{C}$ so that $\text{Re}(s) > 1$. The integral (65) satisfies
\[
\int_{\Gamma^\infty \setminus X_{mod,T}} \left( \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{R}} e^{-\frac{4\pi y_2^2}{v^2} + 2\pi i x_2 w_1} dx_2 v^{-\sigma-1} dv \right) y^s d\mu(z) = \int_{\Gamma^\infty \setminus X_{mod,T}} y^s dydx / y.
\]

**Proof.** By direct computation
\[
\int_{\mathbb{R}} e^{-\frac{4\pi y_2^2}{v^2} + 2\pi i x_2 w_1} dx_2 = \frac{y}{2\sqrt{v}}.
\]
Therefore
\[
\text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{R}} e^{-\frac{4\pi y_2^2}{v^2} + 2\pi i x_2 w_1} dx_2 v^{-\sigma-1} dv = \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{R}} \frac{y}{2} v^{-\sigma-1/2} dv.
\]
Let us suppose that $\text{Re}(\sigma) > -1/2$, then
\[
\lim_{T \to \infty} \int_{\mathbb{R}} \frac{y}{2} v^{-\sigma-1/2} dv = \frac{2}{2\sigma + 1}.
\]
By meromorphic continuation
\[
\text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{R}} \frac{y}{2} v^{-\sigma-1/2} dv = y.
\]
Plugging the equality (68) into the integral of the statement the result follows.

**Lemma 5.26.** Let $A$ be the constant defined in 1.0.1. For $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$ the integral (66) satisfies the following equality:
\[
\int_{\Gamma^\infty \setminus X_{mod,T}} \left( \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{R}} e^{-\frac{4\pi y_2^2}{v^2} + 2\pi i x_2 w_1} dx_2 v^{-\sigma-1} dv \right) y^s d\mu(z)
\]
\[
= A \int_{\Gamma^\infty \setminus X_{mod,T}} y^s dy / y^2 = 8\text{erf}\left( \frac{\pi}{2} \right) \int_{\Gamma^\infty \setminus X_{mod,T}} y^s \log(y) dy / y^2.
\]

**Proof.** By direct computation
\[
\int_{\mathbb{R}} e^{-\frac{4\pi y_2^2}{v^2} + 2\pi i x_2 w_1} dx_2 = \frac{y}{2\sqrt{v}} e^{-\frac{4\pi y_2^2}{v^2} + 2\pi i x_2 w_1}.
\]
Then we obtain
\[
\text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{R}} e^{-\frac{4\pi y_2^2}{v^2} + 2\pi i x_2 w_1} dx_2 v^{-\sigma-1} dv
\]
\[
= \frac{y}{2} \left( \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_{\mathbb{R}} e^{-\frac{4\pi y_2^2}{v^2} + 2\pi i x_2 w_1} dx_2 v^{-3/2} dv \right).
\]
We make a change of variables of the form $2\nu w_1^{-2} = v$, obtaining
\begin{equation}
 y \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_1^T \sum_{w_1 \in \mathcal{Z} \setminus \{0\}} e^{-\frac{\pi^2 w_1^2}{v}} v^{-\sigma-3/2} dv
\end{equation}

\begin{equation}
 = \text{CT}_{\sigma=0} 2^{\sigma+3/2} \left( \sum_{w_1 \in \mathcal{Z} \setminus \{0\}} w_1^{-2\sigma-1} \right) \left( y \lim_{T \to \infty} \int_1^T e^{-\frac{\pi^2 v}{2}} v^{-\sigma-3/2} dv \right).
\end{equation}

In order to obtain an explicit formula for (70) we have to write the Laurent series of each factor of (71). First we consider the integral in (71). Making a change of variables of the form $\frac{\nu}{v} = \frac{1}{2}$ it holds
\begin{equation}
 y \lim_{T \to \infty} \int_1^T e^{-\frac{\pi^2 v}{2}} v^{-\sigma-3/2} dv = y^{-2\sigma} \lim_{T \to \infty} \int_1^T e^{-\frac{\pi^2 v}{2}} v^{-\sigma-3/2} dv.
\end{equation}

The above function is holomorphic at $\sigma = 0$, then we have to consider the constant and first term of the Laurent expansion of (72). We rewrite (72) as follows:
\begin{equation}
 \lim_{T \to \infty} \int_1^T e^{-\frac{\pi^2 v}{2}} v^{-\sigma-3/2} dv = \left( \frac{\pi}{2} \right)^{-\sigma+1/2} \Gamma \left( -1/2 + \sigma, \frac{\pi}{2} \right),
\end{equation}

where $\Gamma(\cdot, \cdot)$ is the incomplete gamma function. Then
\begin{equation}
 \lim_{T \to \infty} \int_1^T e^{-\frac{\pi^2 v}{2}} v^{-3/2} dv = \sqrt{2} \text{erf} \left( \sqrt{\frac{\pi}{2}} \right).
\end{equation}

Furthermore the first term of the Laurent expansion of (73) satisfies
\begin{equation}
 \text{FT}_{\sigma=0} \lim_{T \to \infty} \int_1^T e^{-\frac{\pi^2 v}{2}} v^{-\sigma-3/2} dv = \left( \frac{\pi}{2} \right)^{1/2} \left( \Gamma' \left( -1/2, \frac{\pi}{2} \right) - \Gamma \left( -1/2, \frac{\pi}{2} \right) \log \left( \frac{\pi}{2} \right) \right)
\end{equation}

\begin{equation}
 =: \tilde{B},
\end{equation}

where $\Gamma(a, b)$ is the incomplete Gamma function. Using (75) we obtain
\begin{equation}
 \text{FT}_{\sigma=0} y^{-2\sigma} \lim_{T \to \infty} \int_1^T e^{-\frac{\pi^2 v}{2}} v^{-\sigma-3/2} dv = \tilde{B} - \log(y) \sqrt{2} \text{erf} \left( \sqrt{\frac{\pi}{2}} \right).
\end{equation}

Now we consider the sum of the equation (70). We may observe that
\begin{equation}
 \sum_{w_1 \in \mathcal{Z} \setminus \{0\}} w_1^{-2\sigma-1} = \zeta(2\sigma + 1) - 1.
\end{equation}

The following equalities are well known
\begin{equation}
 \text{CT}_{\sigma=0} \zeta(2\sigma + 1) = \Gamma'(1)
\end{equation}

\begin{equation}
 \text{Res}_{\sigma=0} \zeta(2\sigma + 1) = \frac{1}{2}
\end{equation}

Furthermore
\begin{equation}
 2^{\sigma+3/2} = 2\sqrt{2} + \sqrt{2} \log(4)\sigma + O(\sigma^2).
\end{equation}

\begin{equation}
 \sqrt{2} \left( \tilde{B} - \log(y) \sqrt{2} \text{erf} \left( \sqrt{\frac{\pi}{2}} \right) \right) + 8(\Gamma'(1) - 1)\text{erf} \left( \sqrt{\frac{\pi}{2}} \right) + \log(4)\text{erf} \left( \sqrt{\frac{\pi}{2}} \right).
\end{equation}

\begin{lemma}
 The integral of (67) vanishes, i.e.
\begin{equation}
 \int_{\Gamma \times \mathbb{X}} \text{CT}_{\sigma=0} \lim_{T \to \infty} \int_1^T v^{-\sigma-1} dv \mu^*(z) = 0.
\end{equation}
\end{lemma}
Proof. It is straightforward that
\[ CT_{\sigma=0} \lim_{T \to \infty} \int_1^T v^{-\sigma-1} dv = 0. \]

\[ \square \]

**Lemma 5.28.** We obtain the following equality
\[
CT_{s=0} \int_{\Gamma_{\infty \setminus X, \tau}} y^s \log(y) \frac{dxdy}{y^2} = - \frac{\log(\hat{T}) + 1}{T} + \gamma + \frac{\zeta''(-1)}{\zeta(0)} + \zeta'(-1) \left[ - \frac{\log(\pi)}{2\zeta(0)} + 2\gamma \right] + \frac{1}{\zeta(0)} \left( - \frac{2\zeta'(0)}{\zeta(0)} + \frac{\log(\pi)}{2} \right) \frac{\zeta'(0)}{\zeta(0)^2} \right] - 2\log(\hat{T}) \right) + 2\zeta''(-1),
\]
where we recall that \( \gamma \) is the Euler-Mascheroni constant.

**Proof.** Let us suppose that \( s \in \mathbb{C} \) satisfies \( \text{Re}(s) \geq 0 \)
\[
\frac{\partial}{\partial s} \int_{\Gamma_{\infty \setminus X, \tau}} y^s \log(y) \frac{dxdy}{y^2} = \int_{\Gamma_{\infty \setminus X, \tau}} y^s \log(y) \frac{dxdy}{y^2}.
\]
Using proposition 5.23 we obtain the following equalities
\[
\int_{\Gamma_{\infty \setminus X, \tau}} y^s \log(y) \frac{dxdy}{y^2} = \frac{\partial}{\partial s} \int_{X, \tau} E(\tau, s) \frac{dxdy}{y^2}.
\]
By direct computation we find that (80) satisfies the following equality
\[
\frac{\partial}{\partial s} \left( \frac{\hat{T}^{s-1}}{(s-1)} - \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \hat{T}^{-s}/s \right) = \left( \frac{T^{s-1}(s-1) \log(\hat{T}) - 1}{(s-1)^2} \right) = \\
\frac{\zeta^*(2s)}{s^2} \left( \frac{\zeta'(-1)(s-1) \log(\hat{T}) - 1}{(s-1)^2} \right) - \frac{\xi'(2s-1) \log(\hat{T}) - 1}{(s-1)^2} \hat{T}^{-s}/s + \right.
\left. \frac{\xi'(2s-1) \log(\hat{T}) - 1}{(s-1)^2} + \right)
\]
The function on the right hand side is meromorphic. Using meromorphic continuation we can remove the hypothesis on \( s \). We proceed analyzing each factor of (81) separately. First we obtain
\[
CT_{s=0} \left( \frac{T^{s-1}(s-1) \log(\hat{T}) - 1}{(s-1)^2} \right) = - \frac{\log(\hat{T}) + 1}{T}.
\]
Furthermore by direct computation
\[
CT_{s=0} \left( \frac{\xi''(2s-1) \xi^*(2s) - \xi'(2s-1) \xi^*(2s)}{\xi^*(2s)^2} \hat{T}^{-s}/s \right) = \frac{\xi''(-1)}{\xi(0)} + \xi'(-1) \left( - \frac{\log(\pi)}{2\xi(0)} + 2\gamma \right) + \frac{1}{\xi(0)} \left( - \frac{2\xi'(0)}{\xi(0)} + \frac{\log(\pi)}{2} \right) \frac{\xi'(0)}{\xi(0)^2} \right] - 2\log(\hat{T}) \right) + 2\xi''(-1).
\]
By direct computation
\[
CT_{s=0} \left( \frac{\xi'(2s-1) \xi^*(2s) - \xi'(1) \xi^*(2s)}{\xi^*(2s)^2} \hat{T}^{-s}/s \right) = \gamma,
\]
where \( \gamma \) is the Euler-Mascheroni constant. We conclude by plugging equations (82), (83) and (86) into (81). \( \square \)
Lemma 5.29. The following equality holds:

\[ CT_{s=0} \int_{\Gamma_{\infty} \setminus X_{\text{mod.}, \mathcal{T}}} y^s \frac{dxdy}{y^2} = \frac{\pi}{3} - \hat{T}^{-1}. \]

Proof. Let us suppose that \( s \in \mathbb{C} \) satisfies that \( \text{Re}(s) \gg 0 \). Using the definition 5.20

\[ \int_{\Gamma_{\infty} \setminus X_{\text{mod.}, \mathcal{T}}} \frac{y^s dxdy}{y^2} = \int_{X_{\text{mod.}, \mathcal{T}}} E(\tau, s) \frac{dxdy}{y^2}. \]

Proceeding as in the proof of the previous lemma we use proposition 5.23 to obtain

\[ \int_{X_{\text{mod.}, \mathcal{T}}} E(\tau, s) \frac{dxdy}{y^2} = \left( \hat{T}^{s-1}/(s-1) - \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \hat{T}^{-s}/s \right). \]

The right hand side is a meromorphic function on the variable \( s \). Then by meromorphic continuation we remove the hypothesis on \( s \). On the one hand

\[ CT_{s=0} \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \hat{T}^{-s}/(-s) = \frac{\pi}{3}. \]

On the other hand

\[ CT_{s=0} \hat{T}^{s-1}/(s-1) = -\hat{T}^{-1}. \]

We conclude by plugging the equalities (88), (89) into (87).

\[ \square \]

Lemma 5.30. We obtain

\[ CT_{s=0} \int_{\Gamma_{\infty} \setminus X_{\text{mod.}, \mathcal{T}}} y^s \frac{dxdy}{y} = \frac{\pi^{-1/2}}{\zeta^*(2)} \left( \gamma \Gamma(1/2) + \frac{1}{2} (\log(\pi) \Gamma(1/2) + \Gamma'(1/2)) \right) \frac{1}{T} \]

\[ - \frac{\pi^{-1/2}}{2\zeta^*(2)} \log(\hat{T}) \left( \Gamma(1/2) \right) + \frac{1}{T} + \log(\hat{T}). \]

Proof. The following equality follows directly

\[ CT_{s=0} \int_{\Gamma_{\infty} \setminus X_{\text{mod.}, \mathcal{T}}} y^s \frac{dxdy}{y} = CT_{s=1} \int_{\Gamma_{\infty} \setminus X_{\text{mod.}, \mathcal{T}}} y^s \frac{dxdy}{y^2}. \]

Let \( s \in \mathbb{C} \) satisfying that \( \text{Re}(s) \gg 0 \). Using the same argument of the proof of the previous two lemmas we use proposition 5.23 to rewrite the above equality as follows:

\[ \int_{X_{\text{mod.}, \mathcal{T}}} E(\tau, s) \frac{dxdy}{y^2} = \left( \hat{T}^{s-1}/(s-1) - \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \hat{T}^{-s}/s \right). \]

The right hand side of the equality is meromorphic, applying meromorphic continuation we remove the hypothesis on \( s \). On the one hand we obtain

\[ CT_{s=1} \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \hat{T}^{-s}/(-s) = CT_{s=1} \left( \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \right) \frac{1}{T} - \text{Res}_{s=1} \left( \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \right) \log(\hat{T}) + \frac{1}{T}. \]

By direct computation

\[ CT_{s=1} \left( \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \right) = \frac{\pi^{-1/2}}{\zeta^*(2)} \left( \gamma \Gamma(1/2) + \frac{1}{2} (\log(\pi) \Gamma(1/2) + \Gamma'(1/2)) \right) \frac{1}{T} \]

\[ \text{Res}_{s=1} \left( \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \right) = \frac{\pi^{-1/2}}{2\zeta^*(2)} \log(\Gamma(1/2)). \]

On the other hand

\[ CT_{s=1} \hat{T}^{s-1}/(s-1) = \log(\hat{T}). \]

Using the equalities (91), (92) and (93) in (90), the result follows.

\[ \square \]
Corollary 5.31. It holds that
\[
\int_{X^{mod,\hat{\tau}}} \left( \frac{\text{CT}_{\sigma=0} \lim_{T \to \infty}}{} \int_{\mathbb{R}} f_{\mu_{\tau}}(\tau) \theta(\tau, z, \mu_{\tau}) v^{-\sigma} d\mu(\tau) \right) d\mu(z) = A c_{\mu_{\tau}}(0) \left( \frac{\pi}{3} - T^{-1} \right) \\
- 8 c_{\mu_{\tau}}(0) \text{erf} \left( \frac{\tau}{\sqrt{2}} \right) \left[ - \frac{\log(T) + 1}{T} + \frac{\zeta^\prime(-1)}{\zeta(0)} + c_{\tau}(1) \right] \\
+ \frac{1}{\zeta(0)} \left( - \frac{2c(0)}{\zeta(0)} + \frac{\log(\pi)}{\zeta(0)^2} \right) - 2 \log(T) \right] \right] \\
+ c_{\mu_{\tau}}(0) \left( \frac{\pi}{2} \right) \left( \gamma(1/2) + \frac{1}{2} \log(\pi) \right) \left( \gamma^\prime(1/2) + \frac{\zeta^\prime(2) \gamma(1/2)}{2 \zeta^\prime(2)} \right) \frac{1}{T} \\
+ \pi^{-1/2} \gamma(1/2) \log(T) + 1 + \log(T) \right].
\]

Proof. The proof follows applying lemmas 5.28, 5.29 and 5.30 to proposition 5.24.

5.5. Main result. The goal of this paper is to compute
\[
\int_{X^{mod,\hat{\tau}}} \log \left| \Psi(f)(z) \right| d\mu(z) \\
= \frac{1}{4} \int_{X^{mod,\hat{\tau}}} \Phi(f)(z) d\mu(z) + c_{\mu_{\tau}}(0) \text{vol}(X^{mod,\hat{\tau}}) \left( \gamma(1/2) + \log \sqrt{2\pi} \right).
\]

Theorem 5.32 (Main result). The following equality holds:
\[
\int_{X^{mod,\hat{\tau}}} \log \left| \Psi(f)(z) \right| d\mu(z) = c_{\mu_{\tau}}(0) \text{vol}(X^{mod,\hat{\tau}}) \left( \gamma(1/2) + \log \sqrt{2\pi} \right) \\
- \frac{1}{2} \sum_{j=0}^{\infty} b(\mu(\varphi_{\mu_{\tau}})) b(m, \varphi_{\mu_{\tau}}) - \frac{\pi A c_{\mu_{\tau}}(0)}{12} \\
- \frac{M_1(1/2)(\mu(\varphi_{\mu_{\tau}})) c_{\mu_{\tau}}(0)}{4} + c_{\mu_{\tau}}(0) \left( - \operatorname{tanh}^{-1} \left( \frac{\sqrt{T}}{4} \right) + \frac{\sqrt{T}}{2} + \log(3/4) \right) \\
+ 2 c_{\mu_{\tau}}(0) \text{erf} \left( \frac{\pi}{2} \right) \left[ \gamma + \frac{\zeta^\prime(-1)}{\zeta(0)} + c_{\tau}(1) \right] \\
+ \frac{1}{\zeta(0)} \left( - \frac{2c(0)}{\zeta(0)} + \frac{\log(\pi)}{\zeta(0)^2} \right) + 2 \zeta^\prime(-1) \\
+ c_{\mu_{\tau}}(0) \left( \frac{\log(T)}{T} \right) \left( - \operatorname{erf} \left( \frac{\pi}{2} \right) \right) + \frac{\pi^{-1/2} \Gamma(1/2)}{2 \zeta(2)} \\
+ c_{\mu_{\tau}}(0) \left( \frac{\log(T)}{T} \right) \left( - \operatorname{erf} \left( \frac{\pi}{2} \right) \right) + \frac{\pi^{-1/2} \Gamma(1/2)}{2 \zeta(2)} \\
- 2 \frac{\pi^{1/2}}{4T} \left[ \gamma \left( \frac{\pi^2}{2} \right) + \frac{1}{2} \log(\pi) \Gamma(1/2) + \Gamma^\prime(1/2) + \frac{\zeta^\prime(2) \gamma(1/2)}{2 \zeta^\prime(2)} \right] - \frac{A}{T} \right].
\]

where \( b(m, \varphi_{\mu_{\tau}}) = \lim_{T \to \infty} b(m, T, \varphi_{\mu_{\tau}}) \) and \( A = 8 \gamma(1/2) \text{erf} \left( \sqrt{T} \right) + \log(4) \text{erf} \left( \sqrt{2T} \right) + 2B \) with
\[
B = \left( \frac{\pi}{2} \right)^{1/2} \left( \gamma \left( -1/2, \frac{\pi}{2} \right) - \Gamma \left( -1/2, \frac{\pi}{2} \right) \log \left( \frac{\pi}{2} \right) \right).
\]

We also recall that \( M(1/2) \) is the finite part of the intertwining map defined in 3.10.

Proof. Using the equality (94), lemma 5.14 and corollaries 5.31 and 5.19 the result follows.
6. Auxiliary computations

6.1. Divergence. Let \( f(\tau) = \sum_{j=0}^{1} f_{\mu_j}(\tau) \in M_{-1/2,L}^\prime \). The main goal of this subsection is to understand the integral

\[
\sum_{j=0}^{1} \int_{T^*} f_{\mu_j}(\tau) \left( \int_{X^{\text{mod},\hat{T}}} \text{Div}(\tau, z, \mu_j) d\mu(z) \right) d\mu(\tau).
\]

First of all we may obtain an explicit expression for \( \text{Div}(\tau, z, \mu_j) \). It requires the understanding of the classical embedding

\[ (95) \quad H \hookrightarrow SO(2,1). \]

The map (95) is given by the composition of the following two maps:

\[
H \rightarrow SL_2(\mathbb{R})
\]

\[
z = x + iy \mapsto \left( \begin{array}{cc} y^{1/2} & xy^{-1/2} \\ xy^{-1/2} & y^{-1/2} \end{array} \right).
\]

\[
SL_2(\mathbb{R}) \twoheadrightarrow SO(2,1)
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \end{pmatrix}.
\]

Then

\[ (96) \quad H \hookrightarrow SO(2,1), \]

\[
z = x + iy \mapsto h_z := \begin{pmatrix} 1 & x^2 \\ 1 & y \end{pmatrix} \begin{pmatrix} y & 1 \\ 1 & y^{-1} \end{pmatrix}.
\]

Throughout this subsection we will use the previous injection without referring to it. Set

\[
X_0^{\text{mod},\hat{T}} = \{ z = x + iy \in X^{\text{mod},\hat{T}}, \text{ s.t. } y \leq 1 \},
\]

\[
X_1^{\text{mod},\hat{T}} = \{ z = x + iy \in X^{\text{mod},\hat{T}}, \text{ s.t. } y > 1 \},
\]

so that

\[ (97) \quad X^{\text{mod},\hat{T}} = X_0^{\text{mod},\hat{T}} \sqcup X_1^{\text{mod},\hat{T}}. \]

Lemma 6.1. The divergent part satisfies

\[
\text{Div}(\tau, z, \mu_j) = y \text{Div}(\tau, z_0, \mu_j).
\]

Proof. Using the invariance property of the Gaussian the following equality holds:

\[
\text{Div}(g_\tau, h_1, \varphi_{\varphi,z_0,\mu_j}^\infty) = \sum_{x_0 \in \mathbb{Z} + \frac{1}{2}} \int_{\mathbb{R}} \omega(g_\tau, id) \varphi_{\varphi,z_0,\mu_j}^\infty \left( \begin{array}{c} x_R \\ x_0 \end{array} \right) dx_R = \sum_{x_0 \in \mathbb{Z} + \frac{1}{2}} \int_{\mathbb{R}} \omega(g_\tau, h_z) \varphi_{\varphi,z_0,\mu_j}^\infty \left( \begin{array}{c} x_R \\ x_0 \end{array} \right) dx_R,
\]

where \( h_z \) is the image of \( z \) under the map (96). Applying the Weil representation

\[
\sum_{x_0 \in \mathbb{Z} + \frac{1}{2}} \int_{\mathbb{R}} \omega(g_\tau, h_z) \varphi_{\varphi,z_0,\mu_j}^\infty \left( \begin{array}{c} x_R \\ x_0 \end{array} \right) dx_R = \sum_{x_0 \in \mathbb{Z} + \frac{1}{2}} \int_{\mathbb{R}} \omega(g_\tau) \varphi_{\varphi,z_0,\mu_j}^\infty \left( \begin{array}{c} y^{-1}x_R \\ x_0 \end{array} \right) dx_R.
\]

By a change of variable of the form \( y^{-1}x_R = x_R \) we obtain

\[
\sum_{x_0 \in \mathbb{Z} + \frac{1}{2}} \int_{\mathbb{R}} \omega(g_\tau) \varphi_{\varphi,z_0,\mu_j}^\infty \left( \begin{array}{c} y^{-1}x_R \\ x_0 \end{array} \right) dx_R = y \sum_{x_0 \in \mathbb{Z} + \frac{1}{2}} \int_{\mathbb{R}} \omega(g_\tau) \varphi_{\varphi,z_0,\mu_j}^\infty \left( \begin{array}{c} x_R \\ x_0 \end{array} \right) dx_R,
\]

which implies the result. \( \square \)

Lemma 6.2. The following equality holds:

\[
\int_{X_1^{\text{mod},\hat{T}}} \text{Div}(\tau, z, \mu_j) d\mu(z) = \log(T) \text{Div}(\tau, z_0, \mu_j).
\]
Proof. Using lemma 6.1
\[
\int_{X_0^{\text{mod}, T}} \text{Div}(\tau, z, \mu_j) d\mu(z) = \int_1^T \int_{-1/2}^{1/2} y \text{Div}(\tau, z_0, \mu_j) \frac{dx dy}{y^2}
= \text{Div}(\tau, z_0, \mu_j) \int_1^T \frac{1}{y} dy = \log(T) \text{Div}(\tau, z_0, \mu_j).
\]

Lemma 6.3. We obtain
\[
\int_{X_0^{\text{mod}, T}} \text{Div}(\tau, z, \mu_j) d\mu(z) = 2\text{Div}(\tau, z_0, \mu_j) \left( -\tanh^{-1} \left( \frac{\sqrt{T}}{4} \right) + \frac{\sqrt{T}}{4} + \frac{\log(3/4)}{2} \right).
\]

Proof. To simplify the computation we factor $X_0^{\text{mod}, T}$ into the following two subsets:
\[A = \{ z = x + iy \in X_0^{\text{mod}, T}, \text{ s.t. } x \leq 0 \}, \]
and
\[B = \{ z = x + iy \in X_0^{\text{mod}, T}, \text{ s.t. } x > 0 \}.
\]
The truncated modular curve satisfies
\[X_0^{\text{mod}, T} = A \cup B.
\]
The proof of lemma 6.1 shows that $\text{Div}(\tau, z, \mu_j)$ does not depend on the variable $x$. Then
\[
\int_{X_0^{\text{mod}, T}} \text{Div}(\tau, z, \mu_j) d\mu(z) = \int_A \text{Div}(\tau, z, \mu_j) d\mu(z) + \int_B \text{Div}(\tau, z, \mu_j) d\mu(z)
= 2 \int_A \text{Div}(\tau, z, \mu_j) d\mu(z).
\]

We apply lemma 6.1 to the integral (98), then
\[
\int_A \text{Div}(\tau, z, \mu_j) d\mu(z) = \int_{3/4}^1 \int_{-1/2}^{-\sqrt{1-y^2}} \frac{1}{y} \text{Div}(\tau, z_0, \mu_j) dx dy
= \text{Div}(\tau, z_0, \mu_j) \left( \int_{3/4}^1 \frac{-\sqrt{1-y^2}}{y} dy + \frac{1}{2} \int_{3/4}^1 \frac{1}{y} dy \right)
= \text{Div}(\tau, z_0, \mu_j) \left( -\tanh^{-1} \left( \frac{\sqrt{T}}{4} \right) + \frac{\sqrt{T}}{4} + \frac{\log(3/4)}{2} \right).
\]

Proposition 6.4. The following equality holds:
\[
\text{Div}(\tau, z_0, \mu_0) = v^{1/2} \theta^{J_{\mu_0}}(\tau),
\]
where $\theta^{J_{\mu_0}}(\tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 \tau}$ is the Jacobi theta function. Furthermore
\[
\text{Div}(\tau, z_0, \mu_1) = v^{1/2} \sum_{n \in \frac{1}{2} + \mathbb{Z}} e^{2\pi i n^2 \tau} =: v^{1/2} \theta^{J_{\mu_1}}(\tau)
\]

Proof. We prove the first statement, the second one follows similarly. Applying the Weil representation one obtain
\[
\text{Div}(\tau, z_0, \mu_0) = v^{1/4} \sum_{x_0 \in \mathbb{Z}} \int_\mathbb{R} \omega(g_\tau) \varphi_\infty^x x_0 \frac{x_0}{x_0} dx_0
\]
\[
= v \sum_{x_0 \in \mathbb{Z}} \int_\mathbb{R} \psi_\infty (2uq(x_0, x_0, 0)) \varphi_\infty^x 0 \frac{v^{1/2} x_0}{0} dx_0.
\]

(99)
We recall that in the present case $z_0 = i$. Then
\[
\psi_{\infty} (2uq (x_R, x_0, 0) \varphi_{z_0, R}^\infty \left( \mu \frac{v^{-1/2}}{v^{-1/2}} x_0 \right) = e^{-\pi \left( \frac{u^2}{2} + \frac{v^2}{2} \right)} + 2\pi i x_0^2.
\]

By direct computation
\[
(100) \quad v \int_{\mathbb{R}} e^{-\pi \left( \frac{u^2}{2} + \frac{v^2}{2} \right)} + 2\pi i x_0^2 \, dx_R = v^{1/2} e^{2\pi i x_0^2 (u + iv)}.
\]

Applying equality (100) in (99) we obtain
\[
\text{Div}(\tau, z_0, \mu_0) = v^{1/2} \sum_{x_0 \in \mathbb{Z}} e^{2\pi i x_0^2} = v^{1/2} \theta^{1ac}_{\mu_0}(\tau).
\]

\textbf{Lemma 6.5.} The function $v^{-1/4} \theta^{1ac}_{\mu_1}(\tau) = v^{-1/4} \sum_{n \in \mathbb{Z}} e^{2\pi i n^2} \text{div}^{\mathbb{Z}}$ is a non holomorphic modular form of weight 1/2.

\textbf{Proof.} According to [21, prop. 6.3, p. 2301]
\[
\sum_{n \in \mathbb{Z}} e^{2\pi i n^2} = E(\tau, -1/2, 1/2, \mu(\varphi_{\mu_1})).
\]

Using [17, lem. 1.1, p. 11] the result holds. \hfill \Box

\textbf{Proposition 6.6.} We obtain
\[
\sum_{j=0}^1 \int_{\mathcal{F}_T} f_{\mu_j}(\tau) \left( \int_{\mathcal{X}_{\text{mod}, T}} \text{Div}(\tau, z, \mu_j) \, d\mu(z) \right) \, d\mu(\tau)
\]
\[
= \sum_{j=0}^1 \frac{2c_{\mu_j}(0)}{\sqrt{T}} \log(T) + \frac{4c_{\mu_j}(0)}{\sqrt{T}} \left( -\tanh^{-1} \left( \frac{\sqrt{T}}{4} \right) + \frac{\sqrt{T}}{4} + \frac{\log(3/4)}{2} \right).
\]

\textbf{Proof.} We factor the integral of the statement according to (97)
\[
(101) \quad \int_{\mathcal{F}_T} f_{\mu_j}(\tau) \left( \int_{\mathcal{X}_{\text{mod}, T}} \text{Div}(\tau, z, \mu_j) \, d\mu(z) \right) 
\]
\[
= \int_{\mathcal{F}_T} f_{\mu_j}(\tau) \left( \int_{\mathcal{X}_{\text{mod}, T}} \text{Div}(\tau, z, \mu_j) \, d\mu(z) \right) 
\]
\[
+ \int_{\mathcal{F}_T} f_{\mu_j}(\tau) \left( \int_{\mathcal{X}_{\text{mod}, T}} \text{Div}(\tau, z, \mu_j) \, d\mu(z) \right) 
\]

By lemmas 6.2 and 6.3, the function (101) is equal to
\[
(102) \quad \left( \log(T) + 2 \left( -\tanh^{-1} \left( \frac{\sqrt{T}}{4} \right) + \frac{\sqrt{T}}{4} + \frac{\log(3/4)}{2} \right) \right) \int_{\mathcal{F}_T} f_{\mu_j}(\tau) \text{Div}(\tau, z_0, \mu_j) \, d\mu(\tau).
\]

Propositions 6.8 and 6.4 imply
\[
(103) \quad \sum_{j=0}^1 \int_{\mathcal{F}_T} f_{\mu_j}(\tau) \text{Div}(\tau, z, \mu_j) \, d\mu(\tau) = - \sum_{j=0}^1 \frac{2c_{\mu_j}(0)}{\sqrt{T}}.
\]

Plugging the equality (103) into the function (102) we obtain the result. \hfill \Box

\textbf{Lemma 6.7.} The divergent part satisfies
\[
\sum_{j=0}^1 \int_{\mathcal{F}_T} f_{\mu_j}(\tau) \left( \int_{\mathcal{X}_{\text{mod}, T}} \text{Div}(\tau, z, \mu_j) \, d\mu(z) \right) 
\]
\[
= \sum_{j=0}^1 \frac{2c_{\mu_j}(0)}{\sqrt{T}} \log(T) + \frac{4c_{\mu_j}(0)}{\sqrt{T}} \left( -\tanh^{-1} \left( \frac{\sqrt{T}}{4} \right) + \frac{\sqrt{T}}{4} + \frac{\log(3/4)}{2} \right).
\]
\textbf{Proof.} We proceed as in the proof of proposition 6.6. First of all we observe
\begin{equation}
(104) \quad \int_{X^\text{mod,T}} \text{Div}(\tau, z, \mu_j) d\mu(z) = \int_{X^\text{mod,T}} \left( \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \text{Div}(u + iv, z, \mu_j) du \right) d\mu(z).
\end{equation}

By means of Fubini’s theorem it holds that (104) is equal to
\begin{equation}
(105) \quad \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \int_{X^\text{mod,T}} \text{Div}(u + iv, z, \mu_j) d\mu(z) du,
\end{equation}

which is the Fourier constant term of the function
\[ \tau \mapsto \int_{X^\text{mod,T}} \text{Div}(\tau, z, \mu_j) d\mu(z). \]

We factor the integral over \( X^\text{mod,T} \) according to (41). Applying lemmas 6.2 and 6.3 to the function (105) we obtain
\begin{equation}
(106) \quad \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \left( \int_{X^\text{mod,T}} \text{Div}(u + iv, z, \mu_j) d\mu(z) \right) du = \left( \log(\hat{T}) + 2 \left( -\tanh^{-1}\left( \frac{\sqrt{7}}{4} \right) + \frac{\sqrt{7}}{4} + \frac{\log(3/4)}{2} \right) \right) \text{Div}(\tau, z_0, \mu_j)_0,
\end{equation}

where we recall \( \text{Div}(\tau, z_0, \mu_j)_0 := \frac{1}{1 + 3j} \int_{-1/2 - 3j/2}^{1/2 + 3j/2} \text{Div}(u + iv, z, \mu_j) du \). Using proposition 6.4 in (106)
\[ \int_{\mathcal{F}_{2j}} f_{\mu_j}(\tau) \text{Div}(\tau, z, \mu_j) d\mu(\tau) = \int_{\mathcal{F}_{2j}} v^{1/2} f_{\mu_j}(\tau) \theta_{\mu_j}^{jac}(\tau) d\mu(\tau), \]

where \( \theta_{\mu_j}^{jac}(\tau)_0 := \frac{1}{\tilde{T}} \int_{\mathcal{F}_{2j}}^{1/2 + 3j/2} \theta_{\mu_j}^{jac}(u + iv) du = 1 \). Using corollary 6.9
\begin{equation}
(107) \quad \sum_{j=0}^{1} \int_{\mathcal{F}_{2j}} v^{1/2} f_{\mu_j}(\tau) \theta_{\mu_j}^{jac}(\tau) d\mu(\tau) = \sum_{j=0}^{1} \int_{\mathcal{F}_{2j}} v^{1/2} f_{\mu_j}(\tau) \theta_{\mu_j}^{jac}(\tau) d\mu(\tau)
\end{equation}

Lastly, we plug the equality (107) into the function (106) to obtain the statement. \( \square \)

\textbf{6.2. Two integrals.} This subsection is devoted to compute the integrals
\begin{equation}
(108) \quad \sum_{j=0}^{1} \int_{\mathcal{F}_{2j}} v^{1/2} f_{\mu_j}(\tau) \theta_{\mu_j}^{jac}(\tau) d\mu(\tau),
\end{equation}
\begin{equation}
(109) \quad \int_{\mathcal{F}_{2j}} f_{\mu_j}(\tau) E(\tau, 1/2, -1/2, \mu(\varphi_{\mu_j})) d\mu(\tau),
\end{equation}

and some variants. This subsection is based on the techniques developed in [17, sec. 2, p. 16].

\textbf{Proposition 6.8.} The integral (108) satisfies the following equality:
\begin{equation}
(110) \quad \sum_{j=0}^{1} \int_{\mathcal{F}_{2j}} v^{1/2} f_{\mu_j}(\tau) \theta_{\mu_j}^{jac}(\tau) d\mu(\tau) = \sum_{j=0}^{1} \frac{-2c_{\mu_j}(0)}{\sqrt{T}}
\end{equation}

\textbf{Proof.} We have
\begin{equation}
(110) \quad \int_{\mathcal{F}_{2j}} v^{1/2} f_{\mu_j}(\tau) \theta_{\mu_j}^{jac}(\tau) d\mu(\tau) = \int_{\mathcal{F}_{2j}} v^{-3/2} f_{\mu_j}(\tau) \theta_{\mu_j}^{jac}(\tau) dudv.
\end{equation}

The Jacobi theta function \( \theta_{\mu_j}^{jac}(\tau) \) and \( f_{\mu_j}(\tau) \) are holomorphic functions at \( \tau \in \mathcal{H} \), then
\[ \frac{\partial}{\partial \tau} \theta_{\mu_j}^{jac}(\tau) f_{\mu_j}(\tau) = 0. \]
The previous equality allow us to obtain a preimage of
\[ v^{-3/2} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau), \]
under the operator \( \frac{2}{\tau} \), in fact by direct computation
\[ (111) \quad \frac{2}{\tau} \frac{\partial}{\partial \tau} \left\{ -\frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau) \right\} = v^{-3/2} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau). \]
We apply Stokes theorem to \((110)\). By \((111)\) we obtain
\[ \int_{\mathcal{E}_T} v^{-3/2} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau) dv = \frac{2}{vT} \int_{\partial \mathcal{E}_T} -\frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau) d\tau = \int_{\partial \mathcal{E}_T} \frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau) d\tau. \]
Given \( \tau \in \mathcal{H} \) such that \( |\tau| = 1 \), the function \( \text{Im}(\tau) \) is invariant under the transformation
\[ \tau \mapsto -1/\tau. \]
Moreover, the function \( \text{Im}(\tau) \) is invariant under the transformation
\[ \tau \mapsto \tau + 1. \]
The same properties are satisfied by \( \frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau) \), according to lemma 6.5 and [17, (1.42), p. 13] the function
\[ v^{-1/4} \sum_{j=0}^{1} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau), \]
is invariant under \( \text{SL}_2(\mathbb{Z}) \) and then it is invariant under the aforementioned transformations. Then
\[ \frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau) \]
is invariant under \( \tau \mapsto -1/\tau \) and \( \tau \mapsto \tau + 1 \). The above discussion implies the following equality:
\[ \int_{\mathcal{E}_T} v^{-1/2} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau) d\mu(\tau) = \sum_{j=0}^{1} \left( \int_{1/2}^{-1/2} \frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau) d\tau \right)_{v=T}. \]
Using the definition of the constant term of the Fourier expansion we can go further, concluding the proof:
\[ \sum_{j=0}^{1} \left( \int_{1/2}^{-1/2} \frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta^{J_{ac}}_{\mu_j}(\tau) d\tau \right)_{v=T} = \sum_{j=0}^{1} \frac{2c_{\mu_j}(0)}{\sqrt{T}}. \]

**Corollary 6.9.** Let \( f(\tau) = \sum_{j=0}^{1} f_{\mu_j}(\tau)\varphi_{\mu_j} \) be a weakly holomorphic modular form, then
\[ \sum_{j=0}^{1} \int_{\mathcal{E}_T} v^{1/2} f_{\mu_j}(\tau) d\mu(\tau) = \sum_{j=0}^{1} 2c_{\mu_j}(0) - \frac{2c_{\mu_j}(0)}{\sqrt{T}}. \]

**Proof.** The functions \( f_{\mu_j}(\tau) \) are holomorphic and \( \tau \mapsto \tau + 1 \) invariant. Using the Stokes argument of the proof of proposition 6.8 we obtain the statement. \( \square \)

**Lemma 6.10.** We obtain
\[ \sum_{j=0}^{1} \int_{\mathcal{E}_T} f_{\mu_j}(\tau) E(\tau, s, -1/2, \mu(\varphi_{\mu_j})) d\mu(\tau) = 2 \sum_{i=0}^{1} \sum_{m \in \mathbb{Z}} c_{\mu_i}(-m)b(m, T, \mu(\varphi_{\mu_i})). \]

**Proof.** The proof follows in [17, p. 21]. Nevertheless for the sake of completeness we will give an overview of the proof. Proposition 3.23 allows us to apply Stokes theorem as in the proof of proposition 6.8, obtaining
\[ \sum_{j=0}^{1} \int_{\mathcal{E}_T} v^{1/2} f_{\mu_j}(\tau) d\mu(\tau) = \frac{-2}{(s - 1/2)} \int_{\partial \mathcal{E}_T} \sum_{j=0}^{1} f_{\mu_j}(\tau) E(\tau, s, 3/2, \mu(\varphi_{\mu_j})) d\tau. \]
By [17, (1.42), p. 13], the function $\sum_{j=0}^1 f_{\mu_j}(\tau)E(\tau, s, 3/2, \mu(\varphi_{\mu_j}))$ is $\text{SL}_2(\mathbb{Z})$–invariant. In particular it is invariant under the transformations

$$\tau \mapsto \tau + 1, \quad \tau \mapsto -1/\tau.$$ 

Hence

$$\sum_{j=0}^1 \frac{-2}{(s - 1/2)} \int_{\partial \mathcal{F}_T} f_{\mu_j}(\tau)E(\tau, s, 3/2, \mu(\varphi_{\mu_j}))d\tau = \sum_{j=0}^1 \frac{-2}{(s - 1/2)} \int_{1/2 + iT}^{1/2 + iT} f_{\mu_j}(\tau)E(\tau, s, 3/2, \mu(\varphi_{\mu_j}))d\tau = \sum_{j=0}^1 \frac{2}{(s - 1/2)} (f_{\mu_j}(\tau)E(\tau, s, 3/2, \mu(\varphi_{\mu_j})))_{0, s = T}.$$ \hfill \Box

**Corollary 6.11.** The following equality holds:

$$\int_{F_\mathfrak{T}^\circ} \sum_{j=0}^1 f_{\mu_j}(\tau)E(\tau, 1/2, -1/2, \mu(\varphi_{\mu_j}))d\mu(\tau) = \sum_{j=0}^1 M_f(1/2)(\mu(\varphi_{\mu_j}))(1)c_{\mu_j}(0)$$

$$- M_f(1/2)(\mu(\varphi_{\mu_j}))(1)\frac{c_{\mu_j}(0)}{T^{3/4}},$$

where we recall that the map $M_f$ is defined in 3.10.

**Proof.** Proposition 3.13 implies

$$E(\tau, 1/2, -1/2, \mu(\varphi_{\mu_j})) = E(\tau, -1/2, -1/2, M_f(1/2)(\varphi_{\mu_j})).$$

By [21, thm. 2.4, p. 2282]

$$E(\tau, -1/2, -1/2, M_f(1/2)(\varphi_{\mu_j}))_0 = v^{1/2}M_f(1/2)(\mu(\varphi_{\mu_j}))(1).$$

Using Stokes theorem as in proposition 6.8

$$\int_{F_\mathfrak{T}^\circ} \sum_{j=0}^1 f_{\mu_j}(\tau)E(\tau, 1/2, -1/2, \mu_j)d\mu(\tau) = M_f(1/2)(\varphi_{\mu_j})(1)\int_{F_\mathfrak{T}^\circ} \sum_{j=0}^1 f_{\mu_j}(\tau)v^{1/4}d\mu(\tau)$$

$$= \sum_{j=0}^1 M_f(1/2)(\mu(\varphi_{\mu_j}))(1)c_{\mu_j}(0) - M_f(1/2)(\mu(\varphi_{\mu_j}))(1)\frac{c_{\mu_j}(0)}{T^{3/4}}.$$ \hfill \Box

**Lemma 6.12.** Let $\tilde{\varphi}_f \in \mathcal{S}(V(\mathfrak{g}))$ be a $\prod_{\mathfrak{p} \nmid \infty} \text{SL}_2(\mathbb{Z_p})$–invariant function, then

$$\text{Res}_{s=1/2} \frac{1}{s} \int_{\mathfrak{T}^\circ} f_{\mu_j}(\tau)E(\tau, s, -1/2, \mu(\tilde{\varphi}_f))\nu^{s}d\mu(\tau) = 0.$$ 

**Proof.** By [17, (1.42), p. 13], the function $\sum_{j=0}^1 f_{\mu_j}(\tau)E(\tau, 1/2, -1/2, \mu(\tilde{\varphi}_f))$ is invariant under $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$. Hence using lemma 6.7 and proposition 6.10 we obtain

$$\text{Res}_{s=1/2} \frac{1}{s} \int_{\mathfrak{T}^\circ} f_{\mu_j}(\tau)E(\tau, s, -1/2, \mu(\tilde{\varphi}_f))\nu^{s}d\mu(\tau) = \sum_{i=0}^1 2\text{Res}_{s=0} \sum_{m \in \mathbb{Z}} c_{\mu_i}(-m)b(m, s, T, \mu(\tilde{\varphi}_f)).$$

By [21, cor. 2.5, p. 2283], the functions $b(m, s, T, \mu(\tilde{\varphi}))$ are holomorphic at $s = 1/2$. Then the residue vanishes for all $m \in \mathbb{Z}$ \hfill \Box

**Corollary 6.13.** Let $\tilde{\varphi}_f \in \mathcal{S}(V(\mathfrak{g}))$ be a $\prod_{\mathfrak{p} \nmid \infty} \text{SL}_2(\mathbb{Z_p})$–invariant function. Then

$$\text{Res}_{s=1/2} \frac{1}{s} \int_{\mathfrak{T}^\circ} f_{\mu_j}(\tau)E(\tau, s, -1/2, \mu(\tilde{\varphi}_f))\nu^{s}d\mu(\tau) = 0.$$
Proof. Let us denote by
\[ E(\tau, s, -1/2, \mu(\varphi_f)) = \sum_{n \in \mathbb{Q}} E(\tau, s, -1/2, \mu(\varphi_f))_n, \]
the Fourier series of \( E(\tau, s, -1/2, \mu(\varphi_f)) \) with respect to the variable \( \tau \). It is straightforward that
\[
\int_{\mathcal{F}} f_{\mu_j}(\tau) E(\tau, s, -1/2, \mu(\varphi_f)) v^\sigma d\mu(\tau) = \int_{\mathcal{F}} f_{\mu_j}(\tau) E(\tau, s, -1/2, \mu(\varphi_f))_n v^\sigma d\mu(\tau)
+ \int_{\mathcal{F}} f_{\mu_j}(\tau) \sum_{n \neq 0} E(\tau, s, -1/2, \mu(\varphi_f))_n v^\sigma d\mu(\tau).
\]
Using a similar argument to the proof of lemma 6.12 we have
\[
0 = \sum_{j=0}^{1} \text{Res}_{s=1/2} \left( \int_{\mathcal{F}} f_{\mu_j}(\tau) E(\tau, s, -1/2, \mu(\varphi_f))_n v^\sigma d\mu(\tau) \right)
+ \int_{\mathcal{F}} f_{\mu_j}(\tau) \text{Res}_{s=1/2} \sum_{n \neq 0} E(\tau, s, -1/2, \mu(\varphi_f))_n v^\sigma d\mu(\tau).
\]
The poles of the Eisenstein series are located in the constant coefficient, then
\[
\int_{\mathcal{F}} f_{\mu_j}(\tau) \text{Res}_{s=1/2} \sum_{m \neq 0} E(\tau, s, -1/2, \mu(\varphi_f))_n v^\sigma d\mu(\tau) = 0.
\]
\[ \square \]

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