HESSIAN EQUATIONS ON CLOSED HERMITIAN MANIFOLDS

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ABSTRACT. In this paper, using the technical tools in [14], we solve the complex Hessian equation on closed Hermitian manifolds, which generalizes the the Kähler case results in [4] and [3].

1. INTRODUCTION

Let \((M, g)\) be a compact Hermitian manifold of complex dimension \(n \geq 2\), and \(\omega\) be the corresponding Hermitian form. In local coordinates, we write \(\omega\) as

\[
\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i j} dz^i \wedge dz^j.
\]

In this paper, we consider the following Hessian equation on closed Hermitian manifolds

\[
\begin{cases}
C^k_n \omega^{k} \wedge \omega^{n-k} = e^{f} \omega^n, & \sup_M u = 0 \\
\omega_u = \omega + \sqrt{-1} \partial \bar{\partial} u \in \Gamma_k(M),
\end{cases}
\]

where \(\Gamma_k(M)\) is a convex cone defined in (2.2) in section 2.

When \(k = n\), the condition \(\omega_u \in \Gamma_k(M)\) is equivalent to \(\omega_u > 0\). Equation (1.1) becomes the following Monge-Ampere equation

\[
\omega^n_u = e^{f} \omega^n, \quad \sup_M u = 0.
\]

In addition, when \((M, \omega)\) is a Kähler manifold, i.e., \(d\omega = 0\), Yau [16] solved the equation (1.2) now known as Calabi-Yau theorem. For general Hermitian manifolds, the equation (1.1) has been solved by Cherrier [1] in the case of dimensions 2 and Tosatti-Weinkove [11] for arbitrary dimension. For further background, we refer the reader to [10], [11], [5], [17] and the references therein.

When \(2 \leq k \leq n - 1\), \(\omega_u\) may not be positive, the analysis becomes more complicated. Suppose that \((M, \omega)\) is a Kähler manifold and \(\omega_u \in \Gamma_k(M)\) which is defined in section 2, Hou-Ma-Wu [4] proved the following second order estimates of the equation (1.1)

\[
\max |\partial \bar{\partial} u|_g \leq C(1 + \max |\nabla u|^2_g).
\]

They also pointed out in their paper that (1.3) may be adapted to the blowing up analysis. Later on, Dinew-Kolodziej [3] obtained the gradient estimate by (1.3). Thus equation (1.1) can be solved on Kähler manifolds under the compatible condition

\[
\int_M e^{f} \omega^n = \int_M \omega^n.
\]
Tosatti-Weinkove [13] considered another Hessian typed equation related to the Gauduchon conjecture

\[
\det \left( \omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} \right) = e^F \det \left( \omega^{n-1} \right) \\
\omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} > 0, \sup_M u = 0,
\]

where \( \omega_0 \) and \( \omega \) are two Hermitian metrics on \( M \).

In [13], Tosatti-Weinkove solved equation (1.4) if \( \omega \) is Kähler. One of the main parts is doing the second order estimate. They use the similar auxiliary function in [4]. Later on, in [14], they can solve (1.4) if \( \omega \) is Hermitian. The second order estimate becomes more difficult in the Hermitian case, the authors succeeded to obtain the second order estimates by modifying the auxiliary function in [4].

In this paper, we solve equation (1.1) on closed Hermitian manifolds. More precisely, our main result is

**Theorem 1.1.** Let \((M, g)\) be a closed Hermitian manifold of complex dimension \( n \geq 2 \), \( f \) is a smooth real function on \( M \). Then there is a unique real number \( b \) and a unique smooth real function \( u \) on \( M \) solving

\[
C^k_u \omega_u \wedge \omega^{n-k} = e^{f+\varphi} \omega^n \\
\omega_u \in \Gamma_k(M), \sup_M u = 0.
\]

We use the continuity method to solve the problem (1.5). The openness follows from implicit function theory. The closeness argument can be reduced to a priori estimates up to the second by the standard Evans-Krylov theory. Actually, we can derive the zero order estimate and the second order estimate of solutions of equation (1.1) and thus use a blow up method to obtain the gradient estimate.

In [11], Tosatti-Weinkove derived the key zero order estimate by proving a Cherrier-type inequality which was originally proved in [1]. For equation (1.1), we can prove the similar Cherrier-type inequality but the analysis becomes a bit complicated since \( \omega_u \) may not be positive. Some inequalities for \( k \)-th elementary symmetric functions in [2] are needed. For the second order estimate, the main difficulty is that there are new terms of the form \( T * D^3 u \), where \( T \) is the torsion of \( \omega \) and \( D^3 u \) represents the third derivatives of \( u \). To control these terms, we use the auxiliary function due to Tosatti-Weinkove in [14]. The main difference is that for equation (1.1) we need to use some lemmas for \( k \)-th elementary symmetric functions proved by Hou-Ma-Wu in [4].

The rest of the paper is organized as follows. In section 2, we give some preliminaries. In section 3, the Cherrier-type inequality is derived, thus we obtain the \( C^3 \) estimate. In section 4, we will prove the second order estimate by a similar auxiliary function in [14].

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2. PRELIMINARIES

Let \((M, g)\) be a compact Hermitian manifold and let \(\nabla\) denote the Chern connection of \(g\). In this section we will give some preliminaries about the \(k\)-th elementary symmetric function and the commutation formula of covariant derivatives.

2.1. Elementary symmetric function. The \(k\)-th elementary symmetric function is defined by

\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},
\]

where \(\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n\). Let \(\lambda(a_{ij})\) denote the eigenvalues of Hermitian matrix \(\{a_{ij}\}\), we define

\[
\sigma_k(a_{ij}) = \sigma_k(\lambda(a_{ij})).
\]

The definition of \(\sigma_k\) can be naturally extended to Hermitian manifold. Indeed, let \(A_{1,1}(M, \mathbb{R})\) be the space of smooth real \((1, 1)\)-forms on \(M\), for \(\chi \in A_{1,1}(M, \mathbb{R})\) we define

\[
\sigma_k(\chi) = \left(\frac{n}{k}\right) \chi^k \wedge \omega^{n-k}/\omega^n.
\]

Definition 2.1.

\[
\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \ldots, k\}.
\]

Similarly, we define \(\Gamma_k\) on \(M\) as follows

\[
\Gamma_k(M) := \{\chi \in A_{1,1}(M, \mathbb{R}^n) : \sigma_j(\chi) > 0, j = 1, \ldots, k\}.
\]

Furthermore, \(\sigma_r(\lambda|i_1 \ldots i_l)\), with \(i_1, \ldots, i_l\) being distinct, stands for the \(r\)-th symmetric function with \(\lambda_{i_1} = \cdots = \lambda_{i_l} = 0\). For more details about elementary symmetric functions, one can see the lecture notes [15].

To prove the \(C^0\) estimate, we need the following lemma of elementary symmetric functions.

Lemma 2.2. Suppose that \(\lambda \in \Gamma_k, 3 \leq k \leq n\) and \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\), then there exists a positive constant \(C\) depending only on \(k\) and \(n\), such that for \(0 \leq i \leq k - 2\).

\[
|\lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_l}| \leq C \sigma_i(\lambda|j),
\]

\[
1 \leq j_1 < j_2 < \cdots < j_l \leq n, j_i \neq j_l, 1 \leq l \leq i, 1 \leq j \leq n.
\]

Proof. Since

\[
\sum_{j=1}^{n} \lambda_j = \sigma_1(\lambda|1 \cdots k - 1) > 0,
\]

and

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.
\]
then

$$\lambda_p \leq (n-k)\lambda_k, \, k+1 \leq p \leq n.$$  

We first prove the lemma for $k = 3$. In this case, it needs to prove that there exists a constant $C$ such that

$$|\lambda_l| \leq C|\sigma_1(\lambda|j)|,$$

for $1 \leq j, l \leq n$ and $l \neq j$. Indeed, $\sigma_1(\lambda|j) = \lambda_j + \sigma_1(\lambda|jl)$, thus $\lambda_j \leq \sigma_1(\lambda|j)$. Now, we assume $\lambda_j < 0$, then $l \geq 4$. By (2.4), we have

$$|\lambda_l| \leq (n-3)\lambda_3, \, 4 \leq l \leq n.$$

Since $\lambda|j \in \Gamma_2$, by the proof in [2] which used the result in [7], there exists a constant $\theta_1$ such that $\sigma_1(\lambda|j) \geq \theta_1\lambda_2$ if $j = 1$ and $\sigma_1(\lambda|j) \geq \theta_1\lambda_1$ if $2 \leq j \leq n$. Taking $C = \frac{n-2}{\theta_1}$, we then prove the lemma for the case $k = 3$.

Next we prove the lemma for the general $k, 3 \leq k \leq n$.

If $j > i$, by the result in [15]

$$\sigma_i(\lambda|j) \geq \theta(n,k)\lambda_1 \cdots \lambda_i.$$

Thus we have

$$|\lambda_j, \lambda_j \cdots \lambda_j| = \lambda_j \cdots \lambda_j |\lambda_{j+1} \cdots \lambda_j| \leq \lambda_1 \cdots \lambda_q (n-k)^{n-q} \lambda_k^{l-q}$$

$$\leq (n-k)^{n} \lambda_1 \cdots \lambda_i \leq \frac{(n-k)^{n}}{\theta(n,k)} \sigma_i(\lambda|j).$$

If $j \leq i$, then similarly

$$\sigma_i(\lambda|j) \geq \theta(n,k)\lambda_1 \cdots \lambda_{j-1}\lambda_{j+1} \cdots \lambda_{i+1}.$$

Thus we have

$$|\lambda_j, \lambda_j \cdots \lambda_j| = \lambda_j \cdots \lambda_j |\lambda_{j+1} \cdots \lambda_j| \leq \lambda_1 \cdots \lambda_{j-1}\lambda_{j+1} \cdots \lambda_{q+1} (n-k)^{n-q} \lambda_k^{l-q}$$

$$\leq (n-k)^{n} \lambda_1 \cdots \lambda_{j-1}\lambda_{j+1} \cdots \lambda_{i+1} \leq \frac{(n-k)^{n}}{\theta(n,k)} \sigma_i(\lambda|j).$$

Using this lemma, we immediately obtain the following lemma which is a key ingredient for proving lemma [3.2].

**Lemma 2.3.**

$$\sum_{i=0}^{k-2} \left| \sqrt{-1} \tilde{\partial}u \wedge \tilde{\partial}u \wedge \omega^i_u \wedge T_i \right| \omega^n \leq C \sum_{i=0}^{k-2} \left| \sqrt{-1} \tilde{\partial}u \wedge \tilde{\partial}u \wedge \omega^i_u \wedge \omega^{n-i-1} \right| \omega^n,$$
where \( T_i \) is defined as the combinations of \( \omega, \bar{\partial} \omega, \bar{\partial} \bar{\partial} \omega \), more precisely

\[
T_i = \sum_{0 \leq 3p+2q \leq n-i} \omega^{n-i-3p-2q} \wedge (\sqrt{-1})^p (\partial \omega)^p \wedge (\bar{\partial} \omega)^q \wedge (\sqrt{-1})^q (\bar{\partial} \bar{\partial} \omega)^q
\]

**Proof.** For \( x \in M \), we choose the coordinates such that

\[
\omega(x) = \sum_{j=1}^n dz^j \wedge d\bar{z}^j, \quad \omega_u(x) = \sum_{j=1}^n \lambda_j dz^j \wedge d\bar{z}^j,
\]

and

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.
\]

Thus we have

\[
(2.6) \quad \sum_{i=0}^{k-2} \left| \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^i \wedge T_i \right| \leq C \sum_{i=0}^{k-2} \left| \partial u \right| \left| \omega_u^i \right| \left| \partial \omega \right| \left| \lambda_1 \lambda_2 \cdots \lambda_i \right|
\]

\[
\leq C \sum_{i=0}^{k-2} \sum_{j=1}^n \sum_{1 \leq j_1 < \cdots < j_i \leq n, j \neq j_i} \left| \partial u \right|^2 \left| \lambda_1 \lambda_2 \cdots \lambda_i \right|
\]

\[
= C \sum_{i=0}^{k-2} \left[ \sum_{j=1}^n \sigma_j (\partial \lambda_j) \right] \left| \partial u \right|^2 \left| \lambda_1 \lambda_2 \cdots \lambda_i \right|
\]

where we have used the lemma 2.1 in the last inequality. \( \Box \)

2.2. **Commutation formula of covariant derivatives.** In local complex coordinates \( z_1, \cdots, z_n \), we have

\[
(2.7) \quad g_{ij} = g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}), \quad \{g_{ij}\} = \{g_{ij}\}^{-1}
\]

For the Chern connection \( \nabla \), we denote the covariant derivatives as follows:

\[
(2.8) \quad u_i = \nabla_{\frac{\partial}{\partial z^i}} u, u_{ij} = \nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial z^j}} u, u_{ijk} = \nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial z^j}} \nabla_{\frac{\partial}{\partial z^k}} u
\]

we use the following commutation formula for covariant derivatives on Hermitian manifolds which can be founded in [14]:

\[
(2.9) \quad u_{ij} = u_{ji} - T_{ij}^p u_{pj}
\]

\[
\quad u_{pi} = u_{ip} + u_q R_{ij}^q
\]

\[
\quad u_{ip} = u_{jp} - T_{jp}^q u_q.
\]
\[ u_{ij} = u_{ii} + u_{pj} R_{pi} - u_{ph} R_{ij} - T_{ij} u_{pni} - T_{j} u_{mji} + T_{ij} u_{pqi} \]  

For the details we recommend the reader to the reference [14].

### 3. Zero Order Estimate

In this section we derive the zero order estimate by proving a Cherrier-type inequality and the lemmas in [11]. Since the constant \( b \) is in Theorem 1.1 satisfies

\[ |b| \leq \sup |f| + C, \]

where \( C \) is a positive constant depending only on \((M, \omega)\). Thus, we will assume \( b = 0 \) for convenience.

**Theorem 3.1.** Let \( u \) be a solution of Theorem 1.1. Then there exists a constant \( C \) depending only on \((M, \omega)\) and \( \sup |f| \) such that

\[ \sup_M |u| \leq C. \]

Due to Tosatti-Weinkove’s results, the zero order estimate can be reduced to derive a Cherrier-type inequality which was firstly proved in Cherrier’s paper [1]. For the Hessian equation, the analysis becomes a bit complicated in the lack of the positivity of \( \omega \). Recently, Sun [8] also proved the following lemma for \( k = 2 \) and \( k \geq 3 \) under some extra conditions.

**Lemma 3.2.** There exist constants \( p_0 \) and \( C \) depending only on \((M, \omega)\) such that for any \( p \geq p_0 \)

\[ \int_M |\partial e^{-\frac{\omega + p_0}{2}} \omega^n |^2 \omega^n \leq C p \int_M e^{-\frac{\omega + p_0}{2}} \omega^n \]

**Proof.** By the equation, we have

\[ \omega_k \wedge \omega^{n-k} - \omega^n = (e^f - 1) \omega^n \leq C_0 \omega^n, \]

where \( C_0 \) is a constant depending only on \( f \).

On the other hand,

\[ \omega_k \wedge \omega^{n-k} - \omega^n = (\omega_k - \omega^k) \wedge \omega^{n-k} = \sqrt{-1} \partial \bar{\partial} u \wedge \alpha, \]

where \( \alpha = \sum_{i=1}^{k} \omega^{i-1} \wedge \omega^{n-i} \).

\(^1\)The author independently proved the \( C^0 \) estimate before [8] was posted on arXiv.
Now multiply both sides in (3.1) by \( e^{-pu} \) and integrate by parts.

\[
C_0 \int_M e^{-pu} \omega^n \geq \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \alpha \\
= - \int_M \partial e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \alpha + \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \partial \alpha \\
= p \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \bar{\partial} u \wedge \alpha - \frac{1}{p} \int_M \sqrt{-1} \partial e^{-pu} \wedge \partial \alpha \\
= p \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \bar{\partial} u \wedge \alpha + \frac{1}{p} \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} \alpha \\
:= A + B,
\]

where we denote

\[
A = p \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \bar{\partial} u \wedge \left( \sum_{i=1}^{k} \omega_{i-1} \wedge \omega^{n-i} \right) \\
B = \frac{1}{p} \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} \alpha.
\]

We will use the term \( A \) to control the terms \( B \). Direct calculation gives

\[
\partial \alpha = n \sum_{i=1}^{k-1} \omega_{i-1} \wedge \omega^{n-i-1} \wedge \partial \omega + (n - k) \omega_{k-1} \wedge \omega^{n-k-1} \wedge \partial \omega \]

\[
\bar{\partial} \bar{\partial} \alpha = (n - k)(n - k - 1) \omega_{k-1} \wedge \omega^{n-k-2} \wedge \bar{\partial} \bar{\partial} \omega \wedge \partial \omega + (n - k) \omega_{k-1} \wedge \omega^{n-k-1} \wedge \bar{\partial} \bar{\partial} \omega \\
+ (n - k)(n + k - 1) \omega_{k-2} \wedge \omega^{n-k-1} \wedge \bar{\partial} \bar{\partial} \omega \wedge \partial \omega \\
+ n(n - 1) \sum_{i=0}^{k-3} \omega_i \wedge \omega^{n-i-3} \wedge \bar{\partial} \bar{\partial} \omega \wedge \partial \omega + n \sum_{i=1}^{k-2} \omega_i \wedge \omega^{n-i-2} \wedge \bar{\partial} \bar{\partial} \omega
\]

Therefore, we have

\[
B = \frac{(n - k)(n - k - 1)}{p} \int_M \sqrt{-1} e^{-pu} \omega_{k-1} \wedge \omega^{n-k-2} \wedge \bar{\partial} \bar{\partial} \omega \wedge \partial \omega \\
+ \frac{(n - k)}{p} \int_M \sqrt{-1} e^{-pu} \omega_{k-1} \wedge \omega^{n-k-1} \wedge \bar{\partial} \bar{\partial} \omega \wedge \partial \omega \\
+ \frac{(n + k - 1)(n - k)}{p} \int_M \sqrt{-1} e^{-pu} \omega_{k-2} \wedge \omega^{n-k-1} \wedge \bar{\partial} \bar{\partial} \omega \wedge \partial \omega \\
+ \frac{n(n - 1)}{p} \sum_{i=0}^{k-3} \sqrt{-1} e^{-pu} \omega_i \wedge \omega^{n-i-3} \wedge \bar{\partial} \bar{\partial} \omega \wedge \partial \omega + \frac{n}{p} \sum_{i=1}^{k-2} \sqrt{-1} e^{-pu} \omega_i \wedge \omega^{n-i-2} \wedge \bar{\partial} \bar{\partial} \omega
\]
When $k = 2$, the term $B$ just becomes

\[
B = \frac{(n-3)(n-2)}{p} \int_M -1 e^{-pu} \omega_u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \bar{\partial} \omega + \frac{n-2}{p} \int_M -1 e^{-pu} \omega_u \wedge \omega^{n-3} \wedge \bar{\partial} \omega
\]

\[
+ \frac{(n-2)(n-3)}{p} \int_M -1 e^{-pu} \omega_u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \bar{\partial} \omega + \frac{2(n-2)}{p} \int_M -1 e^{-pu} \omega_u \wedge \omega^{n-3} \wedge \bar{\partial} \omega \wedge \bar{\partial} \omega + \frac{n-2}{p} \int_M -1 e^{-pu} \omega_u \wedge \omega^{n-2} \wedge \bar{\partial} \omega
\]

\[
\geq \frac{(n-3)(n-2)}{p} \int_M -1 e^{-pu} \sqrt{-1} \bar{\partial} \bar{\bar{\partial}} u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \bar{\partial} \omega
\]

\[
+ \frac{(n-2)(n-3)}{p} \int_M -1 e^{-pu} \sqrt{-1} \bar{\partial} \bar{\bar{\partial}} u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \bar{\partial} \omega
\]

We next use integration by parts again to deal with the first term and second term on the right hand side of the above equality. Indeed,

\[
\int_M -1 e^{-pu} \sqrt{-1} \bar{\partial} \bar{\partial} \bar{\partial} u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \bar{\partial} \omega
\]

\[
= p \int_M -1 e^{-pu} \sqrt{-1} \bar{\partial} \bar{\partial} u \wedge \bar{\partial} u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \bar{\partial} \omega + \int_M -1 e^{-pu} \sqrt{-1} \bar{\partial} \bar{\partial} u \wedge \partial(\omega^{n-4} \wedge \bar{\partial} \omega \wedge \bar{\partial} \omega)
\]

\[
= p \int_M -1 e^{-pu} \sqrt{-1} \bar{\partial} \bar{\partial} u \wedge \bar{\partial} u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \bar{\partial} \omega + \frac{1}{p} \int_M -1 e^{-pu} \sqrt{-1} \bar{\partial} \bar{\partial} (\omega^{n-4} \wedge \bar{\partial} \omega \wedge \bar{\partial} \omega)
\]

\[
\geq - p C_1 \int_M e^{-pu} \sqrt{-1} \bar{\partial} \bar{\partial} u \wedge \bar{\partial} u \wedge \omega^{n-1} - \frac{C_1}{p} \int_M e^{-pu} \omega^n
\]

\[
\geq - C_1 A - \frac{C_1}{p} \int_M e^{-pu} \omega^n
\]

The similar calculation gives

\[
\int_M -1 e^{-pu} \sqrt{-1} \bar{\partial} \bar{\partial} \bar{\partial} u \wedge \omega^{n-3} \wedge \bar{\partial} \omega \omega \geq - C_1 A - \frac{C_1}{p} \int_M e^{-pu} \omega^n
\]

Inserting (3.4) and (3.5) into (3.3), we have

\[
B \geq - \frac{C_1}{p} A - \frac{C_1}{p} \int_M e^{-pu} \omega^n
\]
By (3.2) and choosing \( p_0 = 2C_1 + 1 \), we obtain for \( p \geq p_0 \)

\[
\frac{A}{2} \leq (1 - \frac{C_1}{p})A \leq (\frac{C_1}{p} + C_0) \int_M e^{-pu} \omega^n \leq (C_0 + 1) \int_M e^{-pu} \omega^n
\]

By (3.7) in the next page, we thus prove the lemma.

For the general \( k, 3 \leq k \leq n \), we claim that there exist constants \( C_{1i} \) depending only on \( n, k, (M, \omega) \) such that the following holds for \( 0 \leq i \leq k - 1 \),

\[
(3.6) \quad \int_M e^{-pu} \omega_i \wedge T_i \geq -pC_{1i} \sum_{j=0}^{k-2} \int_M e^{-pu} \sqrt{1} \partial u \wedge \tilde{\partial} u \wedge \omega_i \wedge \omega^{n-j-1} - C_{1j} \int_M e^{-pu} \omega^n
\]

, where \( T_i \) is defined as the combinations of \( \omega, \partial \omega, \tilde{\partial} \omega \), more precisely

\[
T_i = \sum_{0 \leq 3p + 2q \leq n-i} \omega^{n-3p-2q} \wedge (\sqrt{1})^p (\partial \omega)^p \wedge (\tilde{\partial} \omega)^q
\]

We use the claim (3.6) to prove the lemma

\[
B \geq -C_1 \sum_{i=2}^k \int_M e^{-pu} \sqrt{1} \partial u \wedge \tilde{\partial} u \wedge \omega_i \wedge \omega^{n-i-1} - C_{1i} \int_M e^{-pu} \omega^n
\]

\[
\geq -C_1 \frac{A}{p} - C_1 \frac{A}{p} \int_M e^{-pu} \omega^n
\]

Thus we have

\[
(1 - \frac{C_1}{p})A \leq (\frac{C_1}{p} + C_0) \int_M e^{-pu} \omega^n
\]

Now we choose \( p_0 = 2C_1 + 1 \), then for any \( p \geq p_0 \),

\[
p^2 \int_M e^{-pu} \sqrt{1} \partial u \wedge \tilde{\partial} u \wedge \omega^{n-1} \leq 2p(C_0 + 1) \int_M e^{-pu} \omega^n
\]

Therefore we have

\[
(3.7) \quad \int_M |\partial e^{-\frac{p}{2}n} \omega^n| \leq \frac{np^2}{4} \int_M e^{-pu} \sqrt{1} \partial u \wedge \tilde{\partial} u \wedge \omega^{n-1} \leq \frac{np(C_0 + 1)}{2} \int_M e^{-pu} \omega^n = pC \int_M e^{-pu} \omega^n
\]

Now, we prove the claim (3.6) by inductive argument.
When $i = 1$, we have

$$
\int_M e^{-pu} \omega_u \wedge T_1 = \int_M e^{-pu} \omega \wedge T_1 + \int_M e^{-pu} \sqrt{-1} \bar{\partial} \omega \wedge T_1
$$

$$
= \int_M e^{-pu} \omega \wedge T_1 - \int_M \bar{\partial} e^{-pu} \wedge \sqrt{-1} \bar{\partial} u \wedge T_1 + \int_M e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \partial T_1
$$

$$
= \int_M e^{-pu} \omega \wedge T_1 + p \int_M e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \bar{\partial} u \wedge T_1 - \int_M \sqrt{-1} \bar{\partial} e^{-pu} \wedge \partial T_1
$$

$$
= p \int_M e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \bar{\partial} u \wedge T_1 + \int_M e^{-pu} \omega \wedge T_1 - \int_M \sqrt{-1} \bar{\partial} e^{-pu} \wedge \partial T_1
$$

$$
\geq -C_1 p \int_M e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \bar{\partial} u \wedge T_1 - C_1 \int_M e^{-pu} \omega^n
$$

Suppose that the claim is true for $l \leq i - 1$, we will prove that the claim is also true for $l = i$. Indeed,

$$
\int_M e^{-pu} \omega_u^l \wedge T_i = \int_M e^{-pu} \omega_u^{i-1} \wedge \omega \wedge T_i + \int_M e^{-pu} \sqrt{-1} \bar{\partial} \omega \wedge \omega_u^{i-1} \wedge T_i
$$

$$
= \int_M e^{-pu} \omega_u^{i-1} \wedge \omega \wedge T_i + p \int_M e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \bar{\partial} u \wedge \omega_u^{i-1} \wedge T_i
$$

$$
+ \int_M e^{-pu} \bar{\partial} u \wedge \sqrt{-1} \partial \left( \omega_u^{i-1} \wedge T_i \right)
$$

$$
:= A_{i,1} + A_{i,2} + A_{i,3}
$$

By the induction,

$$
A_{i,1} = \int_M e^{-pu} \omega_u^{i-1} \wedge \omega \wedge T_i
$$

$$
\geq -p C_{1i} (n, k, \omega) \sum_{j=0}^{k-2} \int_M e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \bar{\partial} u \wedge \omega_u^j \wedge \omega_u^{n-j-1} - C_{1i} (n, k, \omega) \int_M e^{-pu} \omega_n
$$

By the inequality (2.5) in lemma 2.3, we have

(3.8)

$$
A_{i,2} = p \int_M e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \bar{\partial} u \wedge \omega_u^{i-1} \wedge T_i \geq -p C_{2i} \int_M e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \bar{\partial} u \wedge \omega_u^{i-1} \wedge \omega_u^{n-i}
$$
Now we deal with the term $A_{i,3}$,

$$A_{i,3} = \int_M e^{-pu} \tilde{\partial} u \wedge \sqrt{-1} \partial (\omega_u^{i-1} \wedge T_i) = \frac{1}{p} \int_M e^{-pu} \sqrt{-1} \partial \partial (\omega_u^{i-1} \wedge T_i)$$

$$= \frac{(i-1)(i-2)}{p} \int_M e^{-pu} \sqrt{-1} \omega_u^{i-3} \wedge \partial \omega \wedge \partial \omega \wedge T_i + \frac{i-1}{p} \int_M e^{-pu} \omega_u^{i-2} \wedge \sqrt{-1} \partial (\partial \omega \wedge T_i)$$

$$+ \frac{i-1}{p} \int_M e^{-pu} \omega_u^{i-2} \wedge \sqrt{-1} \partial \omega \wedge \partial T_i - \frac{1}{p} \int_M e^{-pu} \omega_u^{i-1} \wedge \sqrt{-1} \partial \partial T_i$$

$$= \frac{(i-1)(i-2)}{p} \int_M e^{-pu} \sqrt{-1} \omega_u^{i-3} \wedge \partial \omega \wedge \partial \omega \wedge T_i$$

$$+ \frac{i-1}{p} \int_M e^{-pu} \omega_u^{i-2} \wedge \sqrt{-1} \partial (\partial \omega \wedge T_i) + \frac{i-1}{p} \int_M e^{-pu} \omega_u^{i-2} \wedge \sqrt{-1} \partial \partial T_i$$

$$\geq -pC_{3i} \sum_{j=0}^{k-2} \int_M e^{-pu} \sqrt{-1} \partial u \wedge \partial \omega \wedge \omega_u^j \wedge \omega_u^{n-j-1} - C_{3i} (n, k, \omega) \int_M e^{-pu} \omega^n.$$

For the last inequality, we have used the induction. \qed

4. SECOND ORDER ESTIMATE

In this section we use the auxiliary function in [14] which is modified by the auxiliary function in [4] to derive the second order estimate of the form (1.3). The difficult part arises from the third order derivatives’ Locally the equation is

$$\sigma_k(\omega_u) = e^f. \quad (4.1)$$

**Theorem 4.1.** There exists a uniform constant $C$ depending only on $(M, \omega)$ and $f$ such that

$$\max |\partial \partial u|_g \leq C(1 + \max |\nabla u|_g^2) \quad (4.2)$$

**Proof.** Denote $w_{ij} = g_{ij} + u_{ij}$ and let $\xi \in \mathcal{T}^{1,0} M$, $|\xi|_g^2 = 1$.

We use the auxiliary function which is similar to the one in [14]

$$H(x, \xi) = \log(w_{ij} \xi^i \xi^j) + c_0 \log(g^{ij} w_{ij} \xi^p \xi^q) + \varphi(|\nabla u|_g^2) + \psi(u),$$

where $\varphi, \psi$ are given by

$$\varphi (s) = -\frac{1}{2} \log \left(1 - \frac{s}{2K}\right), \quad 0 \leq s \leq K - 1,$$

$$\psi (t) = -A \log \left(1 + \frac{t}{2L}\right), \quad -L + 1 \leq t \leq 0,$$

for

$$K := \sup_M |\nabla u|_g^2 + 1, \quad L = \sup_M |u| + 1, \quad A := 2L(C_0 + 1),$$
where $A_0$ is a constant to be determined later. $c_0$ is a small positive constant depending only on $n$ and will be determined later. By [4], we have

\begin{align}
(4.3) \quad \frac{1}{2K} \geq \varphi' \geq \frac{1}{4K} > 0, \quad \varphi'' = 2(\varphi')^2 > 0.
\end{align}

\begin{align}
(4.4) \quad \frac{A}{L} \geq -\psi' \geq \frac{A}{2L} = C_0 + 1, \quad \psi'' \geq \frac{2c_0}{1 - \varepsilon_0} (\psi')^2, \quad \text{for all} \quad \varepsilon_0 \leq \frac{1}{2A + 1}.
\end{align}

These inequalities will be used below.

Suppose $H(x, \xi)$ attains its maximum at the point $x_0$ in the direction $\xi_0$, then we choose local coordinates $\{\frac{\partial}{\partial \xi^i}, \ldots, \frac{\partial}{\partial \xi^n}\}$ near $x_0$ such that

\begin{align*}
&g_{ij}(x_0) = \delta_{ij}, u_{ij} = u_{ii}(x_0)\delta_{ij}, \\
&\lambda_i = w_{ii}(x_0) = 1 + u_{ii}(x_0) \text{ with } \lambda_1 \geq \cdots \geq \lambda_n.
\end{align*}

We will prove that

\begin{align*}
H(x_0, \xi_0) \leq H(x_0, \frac{\partial}{\partial \xi^1}) \quad \forall \xi \in T^{1,0} M, \quad |\xi_g^2| = 1, \quad \sum_{i,j} w_{ij}(x_0)\xi^i \xi^j > 0
\end{align*}

by choosing $c_0$ small enough. In fact, at $x_0$ we have

\begin{align*}
\log(w_{kk}\xi^k \xi^k) + c_0 \log(g^i \omega_{\rho} w_{k\rho} \xi^k \xi^\rho) = \log(\sum_{k=1}^n w_{kk} |\xi_k|^2) + c_0 \log(\sum_{k=1}^n |w_{kk}|^2 |\xi_k|^2)
\end{align*}

If $w_{n\bar{n}} \geq -w_{1\bar{1}}$ which is always satisfied when $n \leq 3$, we have $w_{n\bar{n}}^2 \leq w_{1\bar{1}}$. Thus we have

\begin{align*}
H(x_0, \xi) \leq H(x_0, \frac{\partial}{\partial \xi^1}).
\end{align*}

Now we suppose that $w_{n\bar{n}} < -w_{1\bar{1}}$, thus we have $n \geq 4$. Let $i_0$ be the smallest integer satisfying $w_{i_0\bar{i_0}} < -w_{1\bar{1}}$, then $i_0 \geq k + 1$. By $|w_{i_0\bar{i_0}}| < (n - 2)w_{1\bar{1}}$ we have

\begin{align*}
\log(\sum_{i=1}^n w_{i\bar{i}} |\xi_i|^2) + c_0 \log(\sum_{i=1}^n |w_{i\bar{i}}|^2 |\xi_i|^2) \\
\leq \log w_{1\bar{1}}(\sum_{i=1}^{i_0-1} |\xi_i|^2 - \sum_{i=i_0}^n |\xi_i|^2) + c_0 \log(w_{1\bar{1}}^2 \sum_{i=1}^{i_0-1} |\xi_i|^2 + (n - 2)w_{1\bar{1}}^2 \sum_{i=1}^{i_0-1} |\xi_i|^2) \\
= \log w_{1\bar{1}}(1 - 2t) + c_0 \log w_{1\bar{1}}^2 (1 - t + (n - 2)t) := h(t),
\end{align*}

where $t = \sum_{i=0}^n |\xi_i|^2 \in (0, \frac{1}{4})$.

By choosing $c_0 = \frac{2}{(n-2)^{n-1}}$, we have $h'(t) \leq 0$, thus

\begin{align*}
h(t) \leq h(0) = \log(w_{1\bar{1}}) + c_0 \log w_{1\bar{1}}^2.
\end{align*}

Consequently, we have proved

\begin{align*}
H(x_0, \xi) \leq H(x_0, \frac{\partial}{\partial \xi^1}), \quad \forall \xi \in T^{1,0} M, \quad |\xi_g^2| = 1, \quad \sum_{i,j} \eta_{ij}(x_0)\xi^i \xi^j > 0,
\end{align*}
by choosing \( c_0 = \frac{2}{(n-2)^{r-1}} \) when \( n \geq 4 \) and \( c_0 = 1 \) when \( n \leq 3 \).

We extend \( \xi_0 \) near \( x_0 \) as

\[
\xi_0 = (g_{11})^{\frac{1}{2}} \frac{\partial}{\partial z^1}
\]

Consider the function

\[
Q(x) = H(x, \xi_0) = \log(g_{11}^{-1}w_{ij}) + c_0 \log(g_{11}^{-1}g^{ij}w_{ik}w_{kj}) + \varphi(|\nabla u|^2_u) + \psi(u).
\]

We will calculate \( F^{ij}Q_{ij} \) at \( x_0 \) to get the estimate, all the calculations are taken at \( x_0 \). For simplicity, we denote \( \xi = \xi_0 \) in the following. By \( \langle \xi, \bar{\xi} \rangle_g = |\xi|^2_g = 1 \), differentiating both sides, we obtain at \( x_0 \)

\[
0 = \frac{\partial}{\partial z^i}(\xi, \bar{\xi})_g = \langle \nabla_{\frac{\partial}{\partial z^i}} \xi, \bar{\xi} \rangle_g + \langle \xi, \nabla_{\frac{\partial}{\partial z^i}} \bar{\xi} \rangle_g
\]

\[
= \langle \xi^k, \frac{\partial}{\partial z^i} \bar{\xi}^k \rangle_g + \langle \xi^k, \frac{\partial}{\partial z^i} \xi^k \rangle_g
\]

\[
= g_{ki}\xi^k + g_{ki}\bar{\xi}^k + g_{ki}\xi^k \bar{\xi}^k = \xi^k_{,i} + \bar{\xi}^k_{,i}.
\]

(4.5)

We also have the basic formula for \( \xi \in T^{1,0}M \):

\[
\xi^k_{,i} = \frac{\partial \xi^k}{\partial z^i} = \frac{\partial \bar{\xi}^k}{\partial z^i} = \xi^k_{,i},
\]

\[
\bar{\xi}^k_{,i} = \frac{\partial \bar{\xi}^k}{\partial z^i} = \frac{\partial \xi^k}{\partial z^i} = \bar{\xi}^k_{,i},
\]

(4.6)

Direct calculations give

\[
Q_i = \left( w_{ki}\xi^k_{,i} \right)_i + c_0 \left( g^{pq}w_{kq}w_{pl}\xi^k_{,l} \right)_i + \varphi_i + \psi_i
\]

\[
Q_{\bar{i}} = \left( w_{ki}\xi^k_{,i} \right)_{\bar{i}} - \left( w_{ki}\xi^k_{,i} \right)_i \frac{\left( w_{ki}\xi^k_{,i} \right)^2}{\left( w_{ki}\xi^k_{,i} \right)^2} + c_0 \left( g^{pq}w_{kq}w_{pl}\xi^k_{,l} \right)_{\bar{i}} \frac{\left( g^{pq}w_{kq}w_{pl}\xi^k_{,l} \right)^2}{\left( g^{pq}w_{kq}w_{pl}\xi^k_{,l} \right)^2} + \varphi_{\bar{i}} + \psi_{\bar{i}}
\]
Next, we will simplify $Q_i$ and $Q_{ii}$.

By (4.5), we have

$$
\left( w_{kl} \xi^k \xi^l \right)_i = w_{kl} \xi^k \xi^l + w_{kl} \xi^k \xi^l + w_{kl} \xi^k \xi^l
= w_{11,i} + w_{11,i} \left( \xi^1_i + \xi^1_i \right)
= w_{11,i}.
$$

Thus we have

$$
\left( g^{pq} w_{kp} w_{pl} \xi^k \xi^l \right)_i = g^{pq} w_{kp} w_{pl} \xi^k \xi^l + g^{pq} w_{kp} w_{pl} \xi^k \xi^l + g^{pq} w_{kp} w_{pl} \xi^k \xi^l
= w_{11} \left( w_{11,i} + w_{11,i} \right) + w_{11} \left( \xi^1_i + \xi^1_i \right)
= 2w_{11} w_{11,i}.
$$

Therefore, we obtain the simplified formula for the term $Q_i$ at $x_0$.

(4.7) 
$$
Q_i = \frac{w_{11,i}}{w_{11}} + c_0 \frac{2w_{11,i}}{w_{11}} + \varphi_i + \psi_i = (1 + 2c_0) \frac{w_{11,i}}{w_{11}} + \varphi_i + \psi_i = 0
$$

Similar calculations give

$$
\left( w_{kl} \xi^k \xi^l \right)_i = \left[ w_{kl} \xi^k \xi^l + w_{kl} \left( \xi^k \xi^l + \xi^k \xi^l \right) \right]_i
= w_{kl} \xi^k \xi^l + w_{kl} \left( \xi^k \xi^l + \xi^k \xi^l \right) + w_{kl} \left( \xi^k \xi^l + \xi^k \xi^l \right)
+ w_{kl} \left( \xi^k \xi^l + \xi^k \xi^l \right) + w_{kl} \left( \xi^k \xi^l + \xi^k \xi^l \right)
= w_{11,i} + w_{11,i} \xi^1_i + w_{11,i} \xi^1_i + w_{11,i} \xi^1_i + w_{11,i} \xi^1_i + w_{11,i} \xi^1_i
+ w_{11} \left( \xi^1_i + \xi^1_i \right) + w_{11} \left( \xi^1_i + \xi^1_i \right) + w_{11} \left( \xi^1_i + \xi^1_i \right)
= w_{11,i} + 2 \sum_{k \neq l} \text{Re}(w_{kl} \xi^k_i + w_{kl} \xi^k_i) + w_{11} \left( \xi^1_i + \xi^1_i \right) + w_{11} \left( \xi^1_i + \xi^1_i \right) + w_{11} \left( \xi^1_i + \xi^1_i \right).
$$

The last equality holds because we have used (4.2) and (4.5) and the fact

$$
w_{kl} \xi^k_i + w_{kl} \xi^k_i = 2\text{Re}(w_{kl} \xi^k_i),
\text{ and }
w_{11,i} \xi^1_i + w_{11,i} \xi^1_i = 2\text{Re}(w_{11,i} \xi^1_i).
$$
We can also calculate
\[
\left( g_{pq} w_{k\ell} w_{pi} \xi^k \xi^\ell \right)_i = g_{pq} \left( w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell \right)_i \\
= g_{pq} \left( w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell \right)_i \\
+ g_{pq} \left( w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell \right)_i \\
+ g_{pq} \left( w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell \right)_i \\
+ g_{pq} \left( w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell + w_{k\ell} w_{pi} \xi^k \xi^\ell \right)_i \\
= w_{1\ell} w_{i\ell} \xi^k_i + w_{k\ell} w_{pi} \xi^k_i + w_{k\ell} w_{pi} \xi^k_i + w_{k\ell} w_{pi} \xi^k_i \\
+ w_{k\ell} w_{pi} \xi^k_i + w_{k\ell} w_{pi} \xi^k_i + w_{k\ell} w_{pi} \xi^k_i + w_{k\ell} w_{pi} \xi^k_i \\
+ w_{k\ell} w_{pi} \xi^k_i + w_{k\ell} w_{pi} \xi^k_i + w_{k\ell} w_{pi} \xi^k_i + w_{k\ell} w_{pi} \xi^k_i \\
= 2w_{1\ell} w_{i\ell} \left| \xi^k_i \right|^2 + \left| w_{k\ell} \right|^2 \left( \xi^k_i \right)^2 + \left( \xi^k_i \right)^2 + \left( \xi^k_i \right)^2 \\
+ 2w_{k\ell} w_{pi} \Re \left( \xi^k_i \right)^2 + w_{k\ell} \left( \xi^k_i \right)^2 + \left( \xi^k_i \right)^2 \\
+ 2w_{k\ell} w_{pi} \Re \left( \xi^k_i \right)^2 + w_{k\ell} \left( \xi^k_i \right)^2 + \left( \xi^k_i \right)^2 \\
+ 2w_{k\ell} w_{pi} \Re \left( \xi^k_i \right)^2 + w_{k\ell} \left( \xi^k_i \right)^2 + \left( \xi^k_i \right)^2 \\
+ 2w_{k\ell} w_{pi} \Re \left( \xi^k_i \right)^2 + w_{k\ell} \left( \xi^k_i \right)^2 + \left( \xi^k_i \right)^2 \\
+ 2w_{k\ell} w_{pi} \Re \left( \xi^k_i \right)^2 + w_{k\ell} \left( \xi^k_i \right)^2 + \left( \xi^k_i \right)^2.
\]

Therefore we have simplify $Q_{i\ell}$ at $x_0$ as follows
\[
Q_{i\ell} = (1 + 2c_0) \frac{w_{1\ell}}{w_{i\ell}} + c_0 \sum_{\rho \neq 1} \left( \left| w_{1\rho} \right|^2 + \left| w_{1\rho} \right|^2 \right) \\
- (1 + 2c_0) \frac{\left| w_{1\ell} \right|^2}{w_{i\ell}} + (**)_i + \varphi_{i\ell} + \psi_{i\ell},
\]

where (**)$_i$ is given by
\[
(**)_i = \frac{2}{w_{1\ell}} \sum_{k \neq 1} \Re (w_{k1} \xi^k_i + w_{k1} \xi^k_i) + \left| \xi^k_i \right|^2 + \left| \xi^k_i \right|^2 \\
+ \frac{2c_0}{w_{i\ell}} \sum_{\rho \neq 1} \Re (w_{\rho1} \xi^\rho_i + w_{\rho1} \xi^\rho_i) + \sum_{\rho \neq 1} \frac{2c_0 w_{\rho\rho}}{w_{1\ell}} \Re (w_{\rho1} \xi^\rho_i + w_{\rho1} \xi^\rho_i) \\
+ 2 \frac{2c_0 w_{\rho\rho}}{w_{1\ell}} \left( \left| \xi^\rho_i \right|^2 + \left| \xi^\rho_i \right|^2 \right) + c_0 \left| \xi^k_i \right|^2 + \left| \xi^k_i \right|^2.
\]

For this term (**)$_{i\ell}$, we have the following estimate
\[
(**)_i \geq -\frac{c_0}{2w_{1\ell}} \left( \left| w_{1\rho} \right|^2 + \left| w_{1\rho} \right|^2 \right) - C,
\]

where $C$ is a positive constant depending only on $(M, \omega)$.

Let
\[
F(\omega_\alpha) = (\sigma_1(\omega_\alpha))^{1/\alpha}.
\]
We denote by

\[ F^{ij} = \frac{\partial F}{\partial w_{ij}}, \quad F^{ij, pq} = \frac{\partial^2 F}{\partial w_{ij} \partial w_{pq}}, \]

where \((w_u)_{ij} = g_{ij} + u_{ij}\). Then, the positive definite matrix \((F^{ij}(\omega_u))\) is diagonalized at the point \(x_0\). More precisely, we have

(4.8) \[ F^{ij}(\omega_u) = \delta_{ij} F^{ii}(\omega_u) = \frac{1}{k} [\sigma_k(\lambda)]^{1/k-1} \sigma_{k-1}(\lambda_i) \delta_{ij}. \]

Furthermore, at \(x_0\),

(4.9) \[ F^{ij, pq}(\omega_u) = \begin{cases} F^{ii, pp}, & \text{if } i = j, p = q; \\ F^{ip, pi}, & \text{if } i = q, p = j, i \neq p; \\ 0, & \text{otherwise}, \end{cases} \]

in which

\[ F^{ii, pp} = \frac{1}{k} [\sigma_k(\lambda)]^{1/k-1} (1 - \delta_{ip}) \sigma_{k-2}(\lambda|i)p \]
\[ + \frac{1}{k} \left( \frac{1}{k} - 1 \right) [\sigma_k(\lambda)]^{1/k-2} \sigma_{k-1}(\lambda|i) \sigma_{k-1}(\lambda|p), \]
\[ F^{ip, pi} = -\frac{1}{k} [\sigma_k(\lambda)]^{1/k-1} \sigma_{k-2}(\lambda|i)p. \]

Here and in the follows, \(\sigma_r(\lambda|i_1 \ldots i_l)\), with \(i_1, \ldots, i_l\) being distinct, stands for the \(r\)-th symmetric function with \(\lambda_{i_1} = \cdots = \lambda_{i_l} = 0.\)

We have, in addition at \(x_0\),

(4.10) \[ \sum_{i=1}^{n} F^{ii} w_{ii} = \sum_{i=1}^{n} F^{ii} \lambda_i = \sigma_1^{1/k} = e^T. \]
Thus by maximum principal, we have

\begin{align*}
(4.11) \quad 0 \geq F^{ij} Q_{ij} &= F^{i\bar{i}} Q_{i\bar{i}} \\
&\geq (1 + 2c_0) \sum_{i=1}^{n} \frac{F^{i\bar{i}} u_{1i\bar{i}}}{w_{11}} + \frac{c_0}{2} \sum_{i=1}^{n} \sum_{\bar{i} \neq 1} \frac{F^{i\bar{i}} |u_{1\bar{i}}|^2}{w_{11}^2} \\
&\quad - (1 + 2c_0) \sum_{i=1}^{n} \frac{F^{i\bar{i}} |u_{1\bar{i}}|^2}{w_{11}^2} + \psi \sum_{i=1}^{n} F^{i\bar{i}} u_{\bar{i}} + \psi'' \sum_{i=1}^{n} F^{i\bar{i}} |u_{\bar{i}}|^2 \\
&\quad + \varphi'' \sum_{i=1}^{n} F^{i\bar{i}} |\nabla u_{i\bar{i}}|^2 |\nabla u_{\bar{i}}|^2 + \varphi' \sum_{i,p=1}^{n} F^{i\bar{i}} (|u_{pi}|^2 + |u_{p\bar{i}}|^2) \\
&\quad + \varphi' \sum_{i,p=1}^{n} F^{i\bar{i}} (u_{p\bar{i}} u_p + u_{p\bar{i}} u_p) - C_1 \sum_{i=1}^{n} F^{i\bar{i}} \\
&\quad := I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 \end{align*}

The equation can be written as

\[ F(\omega_u) = e^{\frac{\varphi'}{\psi'}} := h \]

Differentiate the above equation, we obtain

\[ \sum_{i,j=1}^{n} F^{ij} u_{i\bar{j}} = \nabla_i F = h_i, \]

\[ \sum_{i,j=1}^{n} F^{ij} u_{i\bar{j}h} + \sum_{i,j,p,q=1}^{n} F^{ij,p\bar{q}} u_{ip\bar{j}h} = h_{ih}. \]

and

\[ \sum_{i=1}^{n} F^{i\bar{i}} u_{i\bar{i}1} = h_{i\bar{i}} - \sum_{i,j,p,q=1}^{n} F^{ij,p\bar{q}} u_{ij1} u_{p\bar{q}i}. \]

By commuting the covariant derivatives formula (2.10), we have

\begin{align*}
(4.12) \quad \sum_{i=1}^{n} F^{i\bar{i}} u_{1i\bar{i}} &= \sum_{i=1}^{n} F^{i\bar{i}} u_{i\bar{i}1} + \sum_{i=1}^{n} F^{i\bar{i}} \left( u_{11} - \sum_{i=1}^{n} u_{i\bar{i}} \right) R_{i\bar{i}11} \\
&\quad + \sum_{i=1}^{n} F^{i\bar{i}} \left( \sum_{p=1}^{n} T^p_{1i\bar{i}1} + \sum_{q=1}^{n} \tilde{T}^q_{1i\bar{i}1} - \sum_{p=1}^{n} |T^p_{1i\bar{i}}|^2 u_{p\bar{i}} \right). \end{align*}
Inserting (4.12) into the term $I_1$, we have

$$(4.13) \quad I_1 = (1 + 2c_0) \sum_{i=1}^{n} \frac{F_{\bar{u}u_1\bar{u}}}{w_{11}}$$

$$= (1 + 2c_0) \sum_{i=1}^{n} \frac{F_{\bar{u}}u_{\bar{u}1\bar{u}}}{w_{11}} + (1 + 2c_0) \sum_{i=1}^{n} \frac{F_{\bar{u}}(u_{1\bar{u}} - u_{\bar{u}1}) R_{\bar{u}1\bar{u}}}{w_{11}}$$

$$+ 2(1 + 2c_0) \sum_{i,p=1}^{n} F_{\bar{u}} \text{Re} \left( \frac{T_{i1}^{\bar{u}}u_{p\bar{u}}}{w_{11}} \right) - (1 + 2c_0) \sum_{i,p=1}^{n} F_{\bar{u}} |T_{i1}^{\bar{u}}|^2 u_{p\bar{u}}$$

$$= (1 + 2c_0) \frac{h_{11}}{w_{11}} - (1 + 2c_0) \sum_{i,j,p,q=1}^{n} F_{\bar{u}} |T_{i1}^{\bar{u}}|^2 u_{p\bar{u}}$$

Next we estimate each term of (1) as follows, firstly we have

$$I_{11} + I_{13} + I_{16} \geq -C_1 - 3 (nC_2 + C_3) \sum_{i=1}^{n} F_{\bar{u}}$$

where we have supposed that $\sup_M |T_{i1}^{\bar{u}}|^2 \leq C_2$, $\sup_M |R| \leq C_3$.

Next we claim $I_{15} + I_2 \geq -18n^2C_2 \sum_{i=1}^{n} F_{\bar{u}}$. In fact,

$$I_{15} + I_2 = \frac{c_0}{2} \sum_{i=1}^{n} \sum_{p \neq 1} F_{\bar{u}} |u_{1\bar{u}1\bar{u}}|^2 \sum_{i=1}^{n} \frac{F_{\bar{u}} \text{Re} \left( \sum_{p \neq 1} \frac{T_{i1}^{\bar{u}}u_{p\bar{u}}}{w_{11}} \right)}{w_{11}}$$

$$= \frac{c_0}{2} \sum_{i=1}^{n} F_{\bar{u}} \left( \frac{|u_{1\bar{u}1\bar{u}}|^2}{w_{11}} + 2(1 + 2c_0) \sum_{i=1}^{n} \frac{F_{\bar{u}} \text{Re} \left( \sum_{p \neq 1} \frac{T_{i1}^{\bar{u}}u_{p\bar{u}}}{w_{11}} \right)}{w_{11}} \right)$$

$$\geq -\frac{2(1 + 2c_0)^2}{c_0} \sum_{i=1}^{n} F_{\bar{u}} |T_{i1}^{\bar{u}}|^2$$

$$\geq -18n^2C_2 \sum_{i=1}^{n} F_{\bar{u}}$$
where we have used $\frac{1}{r^2} \leq c_0 \leq 1$ Thus, we obtain,

$$I_1 + I_2 \geq -(1 + 2c_0) \sum_{i,j,p,q=1}^{n} \frac{F^{ij,pq} u_{ijl} u_{lpq}}{w_{1i}} + 2(1 + 2c_0) \sum_{i}^{n} F^{i} \text{Re} \left( \frac{T_{i1} u_{1i}}{w_{i1}} \right)$$

$$- \left(21n^2C_2 + 3C_3\right) \sum_{i=1}^{n} F^{i} - C_1.$$

For terms $I_7 + I_8$, we claim

$$I_7 + I_8 \geq \frac{1}{2} \varphi' \sum_{i=1}^{n} F^{i} |u_{\bar{i}i}|^2 - (C_2 + C_3) \sum_{i=1}^{n} F^{i} - C_1.$$

Indeed, by the covariant derivatives' commutation formula (2.9) in section 2, we have

$$u_{p\bar{i}} = u_{\bar{i}p} + T_{pi}^{i} u_{\bar{i}} + u_{q} R_{iipq}, u_{p\bar{i}} = u_{\bar{i}p} - T_{ip}^{i} u_{\bar{i}}.$$

Then we have

$$\sum_{i=1}^{n} F^{i} u_{i\bar{i}} = \sum_{i=1}^{n} F^{i} u_{\bar{i}i} + \sum_{i=1}^{n} F^{i} \left(T^{i}_{pi} u_{\bar{i}} + u_{q} R_{iipq}\right) = F_{p} + \sum_{i=1}^{n} F^{i} \left(T^{i}_{pi} u_{\bar{i}} + u_{q} R_{iipq}\right)$$

$$\sum_{i=1}^{n} F^{i} u_{p\bar{i}} = \sum_{i=1}^{n} F^{i} u_{\bar{i}p} + \sum_{i=1}^{n} F^{i} T^{i}_{ip} u_{\bar{i}} = F_{p} + \sum_{i=1}^{n} F^{i} T^{i}_{ip} u_{\bar{i}}$$

Inserting the above formula into the term (8), we obtain

$$I_8 = \varphi' \sum_{i,p=1}^{n} F^{i} (u_{p\bar{i}} u_{p} + u_{p\bar{i}} u_{p})$$

$$= \varphi' \sum_{p=1}^{n} u_{p} \left[ F_{p} + \sum_{i=1}^{n} F^{i} \left(T^{i}_{pi} u_{\bar{i}} + u_{q} R_{iipq}\right) \right] + \varphi' \sum_{p=1}^{n} u_{p} \left[ h_{p} - \sum_{i=1}^{n} F^{i} T^{i}_{ip} u_{\bar{i}} \right]$$

$$= 2\varphi' \sum_{i,p=1}^{n} F^{i} u_{p} \text{Re} \left(u_{p} T^{i}_{pi}\right) + \varphi' \sum_{p=1}^{n} \left[ 2\text{Re} \left(u_{p} h_{p}\right) + \sum_{i,q=1}^{n} u_{p} u_{q} F^{i} R_{iipq} \right]$$

$$= I_{81} + I_{82}.$$

For the term $I_{82}$, we have

$$I_{82} \geq -C_3 \sum_{i=1}^{n} F^{i} - C_1.$$
For the term $I_{81}$, we obtain

$$I_{81} + I_7 = 2\varphi' \sum_{i,p=1}^{n} F^{\hat{u}} u_{\bar{\eta}} \text{Re} \left( u_{\bar{\rho} T_{pi}} \right) + \varphi' \sum_{i,p=1}^{n} F^{\hat{u}} \left( |u_{\bar{\rho}|^2 + |u_{\bar{\rho}}|^2} \right)$$

$$\geq \varphi' \sum_{i=1}^{n} F^{\hat{u}} \left[ |u_{\bar{\eta}}|^2 + 2u_{\bar{\eta}} \text{Re} \left( \sum_{p=1}^{n} u_{\bar{\rho} T_{pi}} \right) \right]$$

$$= \varphi' \sum_{i=1}^{n} F^{\hat{u}} \left[ \frac{|u_{\bar{\eta}}|^2}{2} + 2 \sum_{p=1}^{n} u_{\bar{\rho} T_{pi}} \right] + \frac{3}{4} \varphi' \sum_{i=1}^{n} F^{\hat{u}} |u_{\bar{\eta}}|^2 - 4\varphi' \sum_{i=1}^{n} F^{\hat{u}} \left( \sum_{p=1}^{n} u_{\bar{\rho} T_{pi}} \right)^2$$

$$\geq \frac{1}{2}\varphi' \sum_{i=1}^{n} F^{\hat{u}} |u_{\bar{\eta}}|^2 - C_2 \sum_{i=1}^{n} F^{\hat{u}}.$$

Thus we have proved the above claim (4.15).

Moreover, apply (4.10) to obtain

$$\psi' \sum_{i=1}^{n} F^{\hat{u}} u_{\bar{\eta}} = \psi' \sum_{i=1}^{n} F^{\hat{u}} (\lambda_i - 1) = \psi' h - \psi' \sum_{i=1}^{n} F^{\hat{u}} \geq -2(C_0 + 1) \sup_{M} e^\xi + \psi' \sum_{i=1}^{n} F^{\hat{u}}$$

Similarly,

$$\frac{1}{2}\varphi' \sum_{i=1}^{n} F^{\hat{u}} |u_{\bar{\eta}}|^2 = \frac{1}{2}\varphi' \sum_{i=1}^{n} F^{\hat{u}} (\lambda_i - 1)^2$$

$$= \frac{1}{2}\varphi' \sum_{i=1}^{n} F^{\hat{u}} \lambda_i^2 - \varphi' \sum_{i=1}^{n} F^{\hat{u}} \lambda_i + \frac{1}{2}\varphi' \sum_{i=1}^{n} F^{\hat{u}}$$

$$= \frac{1}{2}\varphi' \sum_{i=1}^{n} F^{\hat{u}} \lambda_i^2 - \varphi' h + \frac{1}{2}\varphi' \sum_{i=1}^{n} F^{\hat{u}}$$

$$\geq \frac{1}{2}\varphi' \sum_{i=1}^{n} F^{\hat{u}} \lambda_i^2 - \frac{1}{2} \sup_{M} e^\xi + \frac{1}{2}\varphi' \sum_{i=1}^{n} F^{\hat{u}}$$
Inserting these terms into (4.11), we obtain

\[ 0 \geq \tilde{F}^\tilde{u} Q_{\tilde{u}} \geq -(1 + 2c_0) \sum_{i,j,p,q=1}^n \frac{F^{ij,pq} u_{ij1} u_{pq1}}{w_{11}} + 2(1 + 2c_0) \sum_{i=1}^n \frac{F^{\tilde{u}} \text{Re} \left( \frac{T_{i1} u_{i11}}{w_{11}} \right)}{w_{11}} - (1 + 2c_0) \sum_{i=1}^n \frac{F^{\tilde{u}} |u_{i11}|^2}{w_{11}} \]

(4.18)

\[ + \varphi'' \sum_{i=1}^n F^{\tilde{u}} |\nabla u_i|^2 |\nabla u_i|^2 + \psi'' \sum_{i=1}^n F^{\tilde{u}} |u_i|^2 + \frac{1}{2} \varphi' \sum_{i=1}^n F^{\tilde{u}} \lambda_i^2 \]

\[ + \left( -\psi' + \frac{1}{2} \varphi' - 22n^2 C_2 - 4C_3 \right) \sum_{i=1}^n F^{\tilde{u}} - C_1 \]

\[ = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + \left( -\psi' + \frac{1}{2} \varphi' - 22n^2 C_2 - 4C_3 \right) \sum_{i=1}^n F^{\tilde{u}} - C_1, \]

where \( C_1 \) is a positive constant depending only on \( C_0, \sup e^\ell, \) and \( \sup |\nabla (e^\ell)|^2, \) and \( \sup |\tilde{\partial} \tilde{\partial} (e^\ell)|. \)

Let \( \varepsilon = \frac{\delta}{4} \leq \frac{1}{10}, \) and \( \delta = \frac{1}{2A+1}, \) where \( A = 2L(C_0 + 1) \) and \( C_0 = 31n^2 C_2 + 4C_3. \) We divide two cases to drive the estimate, which is similar as [4].

**Case 1:** \( \lambda_n < -\varepsilon, \lambda_1. \)

By the first derivative’s condition (4.7), we have

\[ -(1 + 2c_0)^2 \left| \frac{u_{i11}}{w_{11}} \right|^2 = -|\varphi' \nabla u_i|^2 + \psi' u_i|^2 \geq -2(\varphi')^2 |\nabla u_i|^2 |\nabla u_i|^2 - 2(\psi')^2 |u_i|^2 \]

\[ = -\varphi'' |\nabla u_i|^2 |\nabla u_i|^2 - 2(\psi')^2 |u_i|^2, \quad 1 \leq i \leq n \]

\[ A_2 = 2(1 + 2c_0) \sum_{i=1}^n F^{\tilde{u}} \text{Re} \left( \frac{T_{i1} u_{i11}}{w_{11}} \right) \]

\[ \geq -c_0 \sum_{i=1}^n F^{\tilde{u}} \left| \frac{u_{i11}}{w_{11}} \right|^2 - \frac{(1 + 2c_0)^2}{c_0} \sum_{i\neq 1} F^{\tilde{u}} \left| T_{i1} \right|^2 \]

\[ \geq -c_0 \sum_{i=1}^n F^{\tilde{u}} \left| \frac{u_{i11}}{w_{11}} \right|^2 - 9n^2 C_2 \sum_{i\neq 1} F^{\tilde{u}} \left| T_{i1} \right|^2 \]
Thus

\[ A_2 + A_3 \geq -(1 + 3c_0) \sum_{i=1}^{n} \frac{F_1[1]}{w_{11}} - 9n^2 C_2 \sum_{i \neq 1} F_1^1 |1| \]

\[ \geq -(1 + 2c_0)^2 \sum_{i=1}^{n} \frac{F_1[1]}{w_{11}} - 9n^2 C_2 \sum_{i=1}^{n} F_1^\delta \]

\[ = -A_4 - 2 (\psi')^2 \sum_{i=1}^{n} F_1^\delta |u| - 9n^2 C_2 \sum_{i=1}^{n} F_1^\delta. \]

We therefore obtain

\[ A_2 + A_3 + A_4 \geq -2 (\psi')^2 \sum_{i=1}^{n} F_1^\delta |u| - 9n^2 C_2 \sum_{i=1}^{n} F_1^\delta. \]

(4.19)

Using the following inequality

\[ \sum_{i=1}^{n} F_1^\delta z_i^2 \geq n^\delta z_n^2 \]

\[ \geq \frac{\varepsilon^2}{n} \sum_{i=1}^{n} F_1^\delta z_i^2, \]

Therefore, we have

\[ A_6 = \frac{1}{2} \varphi' \sum_{i=1}^{n} F_1^\delta z_i^2 \geq \frac{\varepsilon^2}{2n} \varphi' \sum_{i=1}^{n} F_1^\delta z_i^2. \]

(4.20)

Combining (4.17) and (4.19) (4.20), we obtain

\[ 0 \geq \sum_{i=1}^{n} F_1^\delta Q_i \geq \frac{\varepsilon^2}{2n} \varphi' \sum_{i=1}^{n} F_1^\delta z_i^2 - 2 (\psi')^2 \sum_{i=1}^{n} F_1^\delta |u| \]

\[ + \left( -\psi' + \frac{1}{2} \varphi' - 31n^2 C_2 - 4C_3 \right) \sum_{i=1}^{n} F_1^\delta - C_1 \]

\[ \geq \frac{\varepsilon^2}{8nK} \lambda_1^2 - 8K(C_0 + 1)^2 \sum_{i=1}^{n} F_1^\delta - C_1 \]

\[ \geq \frac{\varepsilon^2}{8nK} \lambda_1^2 - 8K(C_0 + 1)^2 - C_1, \]

where we have used the fact that \( \sum_{i=1}^{n} F_1^\delta \geq 1 \), which follows from Newton-Maclaurin’s inequality and the fact that .

Hence, we obtain the estimates

\[ \lambda_1 \leq 8 \sqrt{2(2A + 1)} \sqrt{nK(8K(C_0 + 1)^2 + C_1)} \leq CK. \]
**Case 2:** $\lambda_n > -\varepsilon \lambda_1$.

Let

$$I = \{ i \in \{1, \cdots, n \} \mid \sigma_{k-1} (\lambda | i) \geq \varepsilon^{-1} \sigma_{k-1} (\lambda | 1) \}$$

Obviously, $1 \notin I$ and $i \in I$ if and only if

$$F^\bar{u} > \varepsilon^{-1} F^{11}$$

We first treat those indices which are not in $I$: by the first derivative’s condition (4.7), we have

$$-(1 + 2c_0) \sum_{i \in I} \frac{F^\bar{u} | u_{11} |^2}{w_{i1}^2} + 2(1 + 2c_0) \sum_{i \in I} F^\bar{u} \frac{\text{Re} \left( T_{111}^{i} u_{11i} \right)}{w_{i1}^2}$$

$$\geq -(1 + 2c_0)^2 \sum_{i \in I} \frac{F^\bar{u} | u_{11} |^2}{w_{i1}^2} - \frac{(1 + 2c_0)^2}{c_0} \sum_{i \in I} F^\bar{u} | T_{11}^i |^2$$

$$= -\varphi'' \sum_{i \in I} F^\bar{u} | \nabla u_i |^2 | \nabla u_i |^2 - 2(\psi')^2 \sum_{i \in I} F^\bar{u} | u_i |^2 - 9n^2 C_2 \sum_{i \in I} F^\bar{u} | T_{11}^i |^2$$

$$\geq -\varphi'' \sum_{i \in I} F^\bar{u} | \nabla u_i |^2 | \nabla u_i |^2 - 2\varepsilon^{-1} K(\psi')^2 F^{11} - 9n^2 C_2 \sum_{i = 1}^n F^\bar{u}$$

Substitute the above inequality into (4.17)

$$0 \geq F^\bar{u} Q_{11} \geq - (1 + 2c_0) \sum_{i, j, p, q = 1}^{n} \frac{F^{j\bar{u}} u_{ij} u_{p\bar{q}}}{w_{i1}^2} + 2(1 + 2c_0) \sum_{i \in I} F^\bar{u} \frac{\text{Re} \left( T_{111}^{i} u_{11i} \right)}{w_{i1}^2}$$

$$\geq (1 + 2c_0) \sum_{i \in I} \frac{F^\bar{u} | u_{11} |^2}{w_{i1}^2} + \varphi'' \sum_{i \in I} F^\bar{u} | \nabla u_i |^2 | \nabla u_i |^2 + \psi'' \sum_{i = 1}^{n} F^\bar{u} | u_i |^2$$

$$+ \frac{1}{2} \varphi' \sum_{i = 1}^{n} F^\bar{u} \lambda_i^2 - 2\varepsilon^{-1} K(\psi')^2 F^{11} + \left( -\psi' + \frac{1}{2} \varphi' - 31n^2 C_2 - 4C_3 \right) \sum_{i = 1}^{n} F^\bar{u} - C_1$$

$$= B_1 + B_2 + B_3 + B_4 + B_5$$

$$+ B_6 + B_7 + B_8$$

Firstly, we have

$$B_6 + B_7 = \frac{1}{2} \varphi' \sum_{i = 1}^{n} F^\bar{u} \lambda_i^2 - 2 \varepsilon^{-1} K(\psi')^2 F^{11} \geq \frac{1}{4} \varphi' \sum_{i = 1}^{n} F^\bar{u} \lambda_i^2,$$

where we have assumed $\frac{1}{4} \varphi' F^{11} \lambda_1^2 \geq 2 \varepsilon^{-1} K(\psi')^2 F^{11}$ otherwise we have $\frac{1}{4} \varphi' F^{11} \lambda_1^2 \leq 2 \varepsilon^{-1} K(\psi')^2 F^{11}$ i.e. $\lambda_1 \leq C K$ and the estimate is done.
We next use the first term $B_1$ to cancel the other terms containing the third derivatives of $u$. By the same proof as in [4] P559, we have

$$\lambda_1 \sigma_{k-2} (\lambda | 1i) \geq (1 - 2\varepsilon) \sigma_{k-1} (\lambda | i), \text{ for } i \in I.$$ 

Thus

$$-\lambda_1 F^{1,1i} = \frac{F^{1-k}}{k} \lambda_1 \sigma_{k-2} (\lambda | 1i) \geq \frac{F^{1-k}}{k} (1 - 2\varepsilon) \sigma_{k-1} (\lambda | i) = (1 - 2\varepsilon) F^{\bar{i}}$$

Since

$$u_{ij1} = u_{1jj} - T^{1i}_{li} (\lambda_1 - 1)$$

Therefore

$$B_1 = -\frac{1 + 2c_0}{\lambda_1} \sum_{i,j,p,q=1}^n F^{ij,pq}_{ii} u_{ij1} u_{pq1} \geq -\frac{1 + 2c_0}{\lambda_1^2} \sum_{i \in I} \lambda_1 F^{i,ii}_{ii} u_{i1i} u_{i1i}$$

$$\geq \frac{1 + 2c_0}{\lambda_1^2} (1 - 2\varepsilon) \sum_{i \in I} F^{\bar{i}} |u_{i1i} - T^{1i}_{li} (\lambda_1 - 1)|^2$$

$$B_2 = \frac{2(1 + 2c_0)}{\lambda_1} \sum_{i \in I} F^{\bar{i}} \text{Re} \left(T^{1i}_{li} u_{i1i}\right)$$

From the first derivative’s condition (4.7), we have

$$B_4 = \psi'' \sum_{i \in I} F^{\bar{ii}} |\nabla u_{i1i}|^2 \geq 2 \sum_{i \in I} F^{\bar{i}} \left(1 + 2c_0 \frac{w_{i1i}}{w_{1i}} + \psi' u_{i1i} \right)^2$$

$$\geq 2(1 + 2c_0)^2 \delta \sum_{i \in I} F^{\bar{i}} \left|w_{i1i}\right|^2 \frac{w_{i1i}}{w_{1i}} - 2 \delta \left(\psi' \right)^2 \sum_{i \in I} F^{\bar{i}} |u_{i1i}|^2$$

$$\geq 2(1 + 2c_0)^2 \delta \sum_{i \in I} F^{\bar{i}} \left|w_{i1i}\right|^2 \frac{w_{i1i}}{w_{1i}} - B_5,$$

where we have used $\frac{2\delta}{1 - \delta} (\psi')^2 = \psi''$ by our choosing $\delta = \frac{1}{2A+1}$.

Thus we have

$$B_3 + B_4 + B_5 \geq - (1 + 2c_0) \frac{1 - 2(1 + 2c_0) \delta}{\lambda_1^2} \sum_{i \in I} F^{\bar{i}} |u_{i1i}|^2.$$
Therefore,

\[ B_1 + B_2 + B_3 + B_4 + B_5 \]

\[ \geq \frac{1 + 2c_0}{\lambda_1^2} (1 - 2\varepsilon) \sum_{i \in I} F_i \left| u_{1i} - T_{1i}^1 (\lambda_1 - 1) \right|^2 - (1 + 2c_0) \frac{[1 - 2 (1 + 2c_0) \delta]}{\lambda_1^2} \sum_{i \in I} F_i \left| u_{1i} \right|^2 \\
+ \frac{2 (1 + 2c_0)}{\lambda_1^2} \sum_{i \in I} F_i \text{Re} \left( \lambda_1 T_{1i}^1 u_{1i} \right) \]

\[ = \frac{1 + 2c_0}{\lambda_1^2} \sum_{i \in I} F_i \left\{ (1 - 2\varepsilon) \left| u_{1i} - T_{1i}^1 (\lambda_1 - 1) \right|^2 - (1 - 2 (1 + 2c_0) \delta) \left| u_{1i} \right|^2 + 2 \text{Re} \left( \lambda_1 T_{1i}^1 u_{1i} \right) \right\} \]

\[ = \frac{1 + 2c_0}{\lambda_1^2} \sum_{i \in I} F_i \left\{ (2 (1 + 2c_0) \delta - 2\varepsilon) \left| u_{1i} \right|^2 + 2 [2\varepsilon (\lambda_1 - 1) + 1] \text{Re} \left( T_{1i}^1 u_{1i} \right) + (1 - 2\varepsilon) (\lambda_1 - 1)^2 \left| T_{1i}^1 \right|^2 \right\} \]

\[ \geq 0, \]

where the last inequality holds if we choose \( \varepsilon = \frac{\phi}{4} \leq \frac{1}{16}. \) In fact,

\[ \Delta = B^2 - 4AC = 4 [2\varepsilon (\lambda_1 - 1) + 1]^2 - 4 (1 - 2\varepsilon) (\lambda_1 - 1)^2 (2 (1 + 2c_0) \delta - 2\varepsilon) \]

\[ \leq 36 \varepsilon^2 (\lambda_1 - 1)^2 - 4 (1 - 2\varepsilon) (\lambda_1 - 1)^2 (2 (1 + 2c_0) \delta - 2\varepsilon) \]

\[ \leq 4 (\lambda_1 - 1)^2 (9\varepsilon^2 - 2 (1 - 2\varepsilon)((1 + 2c_0) \delta) + 2\varepsilon (1 - 2\varepsilon)) \]

\[ \leq 4 (\lambda_1 - 1)^2 (5\varepsilon^2 + 2\varepsilon - \delta) \]

\[ \leq 4 (\lambda_1 - 1)^2 (4\varepsilon - \delta) \]

\[ = 0. \]

Thus we finally obtain

\[ 0 \geq \frac{1}{4} \varphi' \sum_{i=1}^n F_i \left| u_{0i} \right|^2 + \left( -\psi' + \frac{1}{2} \varphi' - C_2 - C_3 \right) \sum_{i=1}^n F_i \left| u_{0i} \right|^2 + C_1 \]

\[ = \left( -\psi' + \frac{1}{2} \varphi' - C_2 - C_3 \right) \sum_{i=1}^n F_i \left| u_{0i} \right|^2 + \frac{1}{2} \varphi' \sum_{i=1}^n F_i \left| u_{0i} \right|^2 - C_1 \]

\[ \geq \sum_{i=1}^n F_i \left| u_{0i} \right|^2 + \frac{1}{16K} \sum_{i=1}^n F_i \lambda_i^2 - C_1 \]

, where we have used \( -\psi' \geq C_0 + 1 \) by choosing \( C_0 = 31n^2C_2 + 4C_3. \)

In particular, we have \( \sum_{i=1}^n F_i \left| u_{0i} \right| \leq C. \) By lemma2.2 in [4] we have \( F^{11} \geq \frac{c(n, k)}{C_1}, \) where \( c(n, k) \) is a positive constant depending only on \( n \) and \( k. \)
Therefore, we get the desired estimate:

\[ \lambda_1 \leq \frac{4C_1^2}{c(n,k)^2} \sqrt{K}, \]

where \( C_1 \) is given in (4.17). \( \square \)

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