Global strong solutions to magnetohydrodynamics with density-dependent viscosity and degenerate heat-conductivity*

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Abstract
We deal with the equations of a planar magnetohydrodynamic compressible flow with the viscosity depending on the specific volume of the gas and the heat conductivity proportional to a positive power of the temperature. Under the same conditions on the initial data as those of the constant viscosity and heat conductivity case (Kazhikhov 1987 Boundary Value Problems for Equations of Mathematical Physics (Krasnoyarsk)), we obtain the global existence and uniqueness of strong solutions which means no shock wave, vacuum, or mass or heat concentration will be developed in finite time, although the motion of the flow has large oscillations and the interaction between the hydrodynamic and magnetodynamic effects is complex. Our result can be regarded as a natural generalization of the Kazhikhov’s theory for the constant viscosity and heat conductivity case to that of nonlinear viscosity and degenerate heat-conductivity.

Keywords: magnetohydrodynamics, large initial data, global strong solutions, degenerate heat-conductivity, density-dependent viscosity

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1. Introduction

Magnetohydrodynamics (MHD), concerning the motion of conducting fluids in an electromagnetic field, covers a wide range of physical objects from liquid metals to cosmic plasmas [3, 7, 12, 14, 18, 19, 22]. The dynamic motion of the fluids and the magnetic field interact strongly with each other. Moreover, the hydrodynamic and electrodynamic effects are coupled. We are concerned with the governing equations of a planar magnetohydrodynamic compressible flow written in the Lagrange variables

\begin{align}
\varepsilon_t &= \varepsilon_x, \\
u_t + (P + \frac{1}{2} |\mathbf{b}|^2)_{x} &= \left(\mu \frac{\varepsilon_x}{\nu}\right)_x, \\
\mathbf{w}_t - \mathbf{b}_x &= \left(\lambda \frac{\mathbf{w}_x}{\nu}\right)_x, \\
(\varepsilon \mathbf{b})_t - \mathbf{w}_x &= \left(\frac{\mathbf{b}_x}{\nu}\right)_x, \\
\left(e + \frac{\varepsilon^2 + |\mathbf{w}|^2 + |\mathbf{b}|^2}{2}\right)_t + \left(u \left(P + \frac{1}{2} |\mathbf{b}|^2\right) - \mathbf{w} \cdot \mathbf{b}\right)_x &= \left(\frac{\theta_x}{\nu} + \mu \frac{\varepsilon_x}{\nu} + \lambda \frac{\mathbf{w} \cdot \mathbf{w}_x}{\nu} + \nu \frac{\mathbf{b} \cdot \mathbf{b}_x}{\nu}\right)_x,
\end{align}

where \(t > 0\) is time, \(x \in \Omega = (0, 1)\) denotes the Lagrange mass coordinate, and the unknown functions \(\varepsilon, u, w, \theta, b, \mathbf{w}\) are, respectively, the specific volume of the gas, longitudinal velocity, transverse velocity, transverse magnetic field, internal energy, absolute temperature and pressure. \(\mu\) and \(\lambda\) are the viscosity of the flow, \(\nu\) is the magnetic diffusivity of the magnetic field, and \(\kappa\) is the heat conductivity.

In this paper, we concentrate on a perfect gas for magnetohydrodynamic flow, that is, \(P\) and \(e\) satisfy

\[P = R \theta / \nu, \quad e = c_v \theta + \text{const.},\]

where both specific gas constant \(R\) and heat capacity at constant volume \(c_v\) are positive constants. We also assume that \(\lambda\) and \(\nu\) are positive constants, and \(\mu, \kappa\) satisfy

\[\mu = \tilde{\mu}_1 + \tilde{\mu}_2 \theta^{-\alpha}, \quad \kappa = \tilde{\kappa} \theta^\beta,\]

with constants \(\tilde{\mu}_1 > 0, \tilde{\mu}_2 \geq 0, \tilde{\kappa} > 0,\) and \(\alpha, \beta \geq 0.\)

The system (1.1)–(1.7) is supplemented with the initial conditions

\[(\varepsilon, u, \theta, b, \mathbf{w})(0) = (\varepsilon_0, u_0, \theta_0, b_0, \mathbf{w}_0)(x), \quad x \in \Omega,\]

and boundary conditions

\[u, b, \mathbf{w}, \theta_0 \mid_{\partial \Omega} = 0,\]

where the initial data (1.8) should be compatible with the boundary conditions (1.9).

Remark 1.1. The equations (1.1)–(1.5) describe the macroscopic behavior of the three-dimensional MHD flow, which is uniform in the transverse directions, with dissipative mechanisms. The assumption (1.7) is motivated by the physical facts: the heat conductivity \(\kappa\) and
viscosity $\mu$ of compressible heat-conducting magnetic fluid vary with temperature and density under very high temperature and density environment (see [2, 26] and the references therein). Moreover, in kinetic theory, the Chapman–Enskog expansion for the first order approximation shows that, the heat conductivity and viscosity are functions of temperature (see Chapman–Colwing [4]). These dependencies, especially the dependence of viscosity on temperature, brings great difficulties and challenges to mathematical analysis and numerical calculation. Thus, to study this problem, we first consider the viscosity only depending on the density, as shown as in the equation (1.7), here the specific volume $v$ in (1.7) can be viewed as the inverse of the density.

There is huge literature on the studies of the global existence and large time behavior of solutions to the compressible Navier–Stokes system and MHD. Indeed, for compressible Navier–Stokes system, Kazhikhov and Shelukhin [17] first obtained the global existence of solutions for constant coefficients ($\alpha = \beta = 0$) with large initial data. From then on, much effort has been made to generalize this approach to other cases. Jenssen–Karper [13] proved the global existence of weak solutions under the assumption that $\alpha = 0$ and $\beta \in (0, 3/2)$. Later, for $\alpha = 0$ and $\beta \in (0, \infty)$, Pan–Zhang [21] obtained the global strong solutions under the condition that

$$(v_0, u_0, \theta_0) \in H^1 \times H^2 \times H^2,$$

which was further relaxed to

$$(v_0, u_0, \theta_0) \in H^1,$$

by Huang–Shi [11] where they also obtained the large-time behavior of the strong solutions. As for MHD, the existence and uniqueness of local smooth solutions was first proved by Vol’pert–Hudjaev [24]. For constant coefficients ($\alpha = \beta = 0$) with large initial data, Kazhikhov [16] (see also Amosov–Zlotnik [1]) first obtained the global existence of strong solutions. From then on, significant progress has been made on the mathematical aspect of the initial and initial boundary value problems. Chen–Wang [6] (see also [5, 25]) established the existence and uniqueness for global strong solution for the initial boundary problem (1.1)–(1.5) with large initial initial data in $H^1$ under some growth conditions, in particular they need

$$\kappa_0(1 + \theta^q) \leq \kappa(\theta) \leq \kappa_1(1 + \theta^q),$$

(1.10)

with some $q \geq 2$. Fan–Jiang–Nakamura [9] obtained the global existence of weak solution to the initial-boundary-value problem (1.1)–(1.5) with large initial data under the growth condition (1.10) when $q \geq 1$. Further, Fan–Huang–Li [8] improved the results of [6] and [9] to $q > 0$. However, it should be mentioned here that the methods used there rely heavily on the non-degeneracy of both the viscosity $\mu$ and the heat conductivity $\kappa$, and cannot be applied directly to the degenerate and nonlinear case $\alpha \geq 0, \beta > 0$ in (1.7).

More recently, Hu–Ju [10] extended Pan–Zhang’s result [21] to the MHD case and proved the global strong solutions to the initial-boundary-value problem (1.1)–(1.9) with $\alpha = 0$ and $\beta > 0$ under the condition that

$$v_0 \in H^1, \quad (u_0, \theta_0, b_0, w_0) \in H^2,$$

(1.11)

which is stronger than that of Kazhikhov [16]. In fact, the main aim of this paper is to generalize Kazhikhov’s result [16] to the degenerate and nonlinear case and prove the global existence of strong solutions to (1.1)–(1.9) with $\alpha \geq 0, \beta > 0$ and

$$(v_0, u_0, \theta_0, b_0, w_0) \in H^1.$$
Then we state our main result as follows.

**Theorem 1.1.** Suppose that
\[ \alpha \geq 0, \quad \beta > 0, \]
and that the initial data \((v_0, u_0, \theta_0, b_0, w_0)\) satisfies
\[ (v_0, \theta_0) \in H^1(0, 1), \quad (u_0, b_0, w_0) \in H^1_0(0, 1), \]
and
\[ \inf_{x \in (0, 1)} v_0(x) > 0, \quad \inf_{x \in (0, 1)} \theta_0(x) > 0. \]
Then, the initial-boundary-value problem (1.1)–(1.9) has a unique strong solution \((v, u, \theta, b, w)\) such that for each fixed \(T > 0\),
\[ \begin{align*}
  v, \theta &\in L^\infty(0, T; H^1(0, 1)), \quad u, b, w \in L^\infty(0, T; H^1_0(0, 1)), \\
  v_t &\in L^\infty(0, T; L^2(0, 1) \cap L^2(0, T; H^1(0, 1))), \\
  u_t, \theta_t, b_t, w_t, u_{xx}, \theta_{xx}, b_{xx}, w_{xx} &\in L^2((0, 1) \times (0, T)),
\end{align*} \]
and for each \((x, t) \in [0, 1] \times [0, T]\)
\[ C^{-1} \leq v(x, t) \leq C, \quad C^{-1} \leq \theta(x, t) \leq C, \]
where \(C > 0\) is a constant depending on the data and \(T\).

A few remarks are in order.

**Remark 1.2.** Our result can be regarded as a natural generalization of Kazhikhov’s theory [16] for the constant viscosity and heat conductivity case to the degenerate and nonlinear ones.

**Remark 1.3.** Our theorem 1.1 improves Hu–Ju’s result [10] where they only treated the case that \(\alpha = 0, \beta > 0\) and assumed that the initial data satisfy (1.11) which is indeed stronger than (1.13).

**Remark 1.4.** Our result still holds for compressible Navier–Stokes system \((b \equiv 0, w \equiv 0)\) which generalized slightly those due to [11, 21] where they only consider the case \(\alpha = 0, \beta > 0\).

**Remark 1.5.** Our result still holds for compressible Navier–Stokes system \((b \equiv 0, w \equiv 0)\) which generalized slightly those due to [11, 21] where they only consider the case \(\alpha = 0, \beta > 0\).
we use the representation of $v$, the energy-type inequality (2.14), and the Jensen’s inequality to obtain a bound of $L^\infty((0, T) \times (0, T))$-norm of $v^{-\alpha}$ (see (2.20)) which plays an important role in bounding $v$ from below. Then, after obtaining the estimates on the $L^2((0, 1) \times (0, T))$-norm of both $b_{xx}$ and $w_{xx}$ (see (2.35)), we multiply the momentum equation (1.2) by $u_{xx}$ and make full use of the structure of the energy equation (1.5) to find that the $L^2((0, 1) \times (0, T))$-norm of $u_{xx}$ can be bounded by the $L^2((0, 1) \times (0, T))$-norm of $\theta^{\beta/2}\theta_t$ (see (2.43)) which indeed can be obtained by combining the equation of $\theta$ (see (2.13)) multiplied by $\theta$ and using the estimates obtained above (see (2.46)). Once we get the bounds on the $L^2((0, 1) \times (0, T))$-norm of both $u_{xx}$ and $u_t$ (see (2.42)), the desired estimates on $\theta_t$ and $\theta_{xx}$ can be obtained by standard arguments (see (2.47)). The details will be carried out in the next section.

2. Proof of theorem 1.1

We first state the following existence and uniqueness of local solutions which can be obtained by using the Banach theorem and the contractivity of the operator defined by the linearization of the problem on a small time interval (see [15, 20, 23]).

**Lemma 2.1.** Let (1.12)–(1.14) hold. Then there exists some $T_1 > 0$ such that the initial-boundary-value problem (1.1)–(1.9) has a unique strong solution $(v, u, \theta, b, w)$ satisfying

\[
\begin{align*}
&v, \theta \in L^\infty(0, T_1; H^1(\Omega)), \quad u, b, w \in L^\infty(0, T_1; H^1_0(\Omega)), \\
v_t \in L^2((0, T_1); L^2(\Omega)), \quad &u_t, \theta_t, b_t, w_t, u_{xx}, \theta_{xx}, b_{xx}, w_{xx} \in L^2((0, 1) \times T_1)).
\end{align*}
\]

(2.1)

Theorem 1.1 will be proved by extending the local solutions globally in time based on the global a priori estimates of solutions (see lemma 2.3–2.8) which will be obtained below.

Without loss of generality, we assume that $\lambda = \nu = \tilde{\mu}_1 = \tilde{\kappa} = R = c_\nu = 1, \tilde{\mu}_2 = \alpha$, and that

\[\int_0^1 v_0 dx = 1.\]

Then, We derive the following representation of $v$ which is essential in obtaining the upper and lower bounds of $v$.

**Lemma 2.2.** The following expression of $v$ holds

\[v(x, t) = B_0(x)D(x, t)Y(t) \left\{ 1 + \frac{1}{B_0(x)} \int_0^t \left( \frac{\theta + \frac{\alpha}{2} |b|^2}{D(x, \tau)Y(\tau)} \right) d\tau \right\},\]

(2.2)

where

\[B_0(x) = \bar{v}_0 \exp \left\{ -v_0^{-\alpha} - \int_0^1 f_\alpha(v_0) dx \right\},\]

(2.3)

\[D(x, t) = \exp \left\{ v(x, t)^{-\alpha} + \int_0^x (u(y, t) - u_0(y)) dy \right\}\]

\[\times \exp \left\{ -\int_0^1 v \int_0^y u dy dx + \int_0^1 v_0 \int_0^y u_0 dy dx \right\},\]

(2.4)
\[ Y(t) = \exp \left\{ \int_0^1 f_\alpha(v) dx - \int_0^t \int_0^1 \left( u^2 + \frac{v}{2} |b|^2 + \theta \right) dx \, dr \right\}, \quad (2.5) \]

with
\[ f_\alpha(s) = \begin{cases} \frac{\alpha}{1-\alpha} s^{1-\alpha}, & \text{if } \alpha \neq 1, \\ \ln s, & \text{if } \alpha = 1. \end{cases} \quad (2.6) \]

**Proof.** First, it follows from (1.2) that
\[ u_t = \sigma_x, \quad (2.7) \]

where
\[ \sigma \triangleq \mu \frac{u_x}{v} - \frac{\theta}{v} - \frac{1}{2} |b|^2, \quad (2.8) \]

satisfies
\[ \sigma = (\ln v - v^{-\alpha}) - \frac{\theta}{v} - \frac{1}{2} |b|^2, \quad (2.9) \]

due to (1.1). Integrating (2.7) over \((0, x)\) gives
\[ \left( \int_0^x u \, dy \right)_t = \sigma - \sigma(0, t), \quad (2.10) \]

which implies
\[ v \sigma(0, t) = v \sigma - v \left( \int_0^x u \, dy \right)_t. \]

Then, integrating this in \(x\) over \((0, 1)\) and noticing that integrating (1.1) over \((0, 1) \times (0, t)\) yields that for any \(t > 0\)
\[ \int_0^1 v(x, t) dx = 1, \]

we obtain after using (1.9) and (2.8) that
\[ \sigma(0, t) = \int_0^1 \left( \mu u_x - \theta - \frac{v}{2} |b|^2 \right) dx - \left( \int_0^1 v \int_0^x u \, dy \, dx \right)_t \\
+ \int_0^1 u_x \int_0^x u \, dy \, dx \\
= \left( \int_0^1 f_\alpha(v) dx - \int_0^1 v \int_0^x u \, dy \, dx \right)_t - \int_0^1 \left( \theta + \frac{v}{2} |b|^2 + u^2 \right) dx. \quad (2.11) \]
Finally, combining (2.10), (2.9), and (2.11) yields
\[
v(x,t) = B_0(x) D(x,t) Y(t) \exp \left\{ \int_0^t \left( \theta + \frac{v}{2} |b|^2 \right) \nu - 1 d\tau \right\},
\]
with \( B_0(x), D(x,t), \) and \( Y(t) \) as in (2.3)–(2.5) respectively. This in particular gives (2.2) and finishes the proof of lemma 2.2. \( \square \)

With lemma 2.2 at hand, we are in a position to prove lower bounds of both \( v \) and \( \theta \).

**Lemma 2.3.** It holds that for any \((x,t) \in [0,1] \times [0,T]\),
\[
C^{-1} \leq v(x,t), \quad C^{-1} \leq \theta(x,t),
\]
where (and in what follows) \( C \) denotes some generic positive constant depending only on \( T, \alpha, \beta, \|u_0, \theta_0, b, 0\|_{H^1(0,1)}, \inf_{x \in [0,1]} v_0(x), \) and \( \inf_{x \in [0,1]} \theta_0(x) \).

**Proof.** First, using (1.1)–(1.4), we rewrite the energy equation (1.5) as
\[
\theta_t + \frac{\theta}{v} u_x = \left( \frac{\theta^2 \theta_x}{v} \right)_x + \frac{\mu u_x^2 + |w_x|^2 + |b_x|^2}{v}.
\]
(2.13)

Multiplying (1.1)–(1.4), and (2.13) by \( 1 - \frac{v}{\ln v} \), \( u, w, b, \) and \( 1 - \frac{\theta}{\ln \theta} \) respectively, adding them altogether and integrating the result over (0,1) \times (0,t), we obtain the following energy-type inequality
\[
\sup_{0 \leq t \leq T} \int_0^1 \left( u^2 + |w|^2 + v|b|^2 + (v - \ln v) + (\theta - \ln \theta) \right) dx + \int_0^T V(s) ds \leq C,
\]
(2.14)

where
\[
V(s) \triangleq \int_0^1 \left( \frac{\theta^2 \theta_x^2}{v \theta^2} + \frac{\mu u_x^2}{v \theta} + \frac{|w_x|^2}{v \theta} + \frac{|b_x|^2}{v \theta} \right) (x,t) dx.
\]

Next, (2.14) implies
\[
\left| \int_0^1 v \int_0^1 u dy dx \right| \leq \int_0^1 v \left( \int_0^1 u^2 dy \right)^{1/2} dx \leq C,
\]
which combined with (2.4) and (2.14) gives
\[
C^{-1} \leq C^{-1} \exp \left\{ v(x,t)^{-\alpha} \right\} \leq D(x,t) \leq C \exp \left\{ v(x,t)^{-\alpha} \right\}.
\]
(2.15)

Furthermore, one deduces from (2.14) that
\[
\left| \int_0^1 \ln v dx \right| + \int_0^1 \left( u^2 + \frac{v}{2} |b|^2 + \theta \right) dx \leq C,
\]
(2.16)
which yields that
\[
C^{-1} \exp \left\{ \int_0^1 f_\alpha(v) \, dx \right\} \leq Y(t) \leq C \exp \left\{ \int_0^1 f_\alpha(v) \, dx \right\}.
\] (2.17)

Combining (2.2), (2.15), and (2.17) yields that for any \((x, t) \in [0, 1] \times [0, T] ,\)
\[
v(x, t) \geq C^{-1} \exp \left\{ v(x, t)^{-\alpha} + \int_0^1 f_\alpha(v) \, dx \right\}.
\] (2.18)

which together with (2.6) and (2.16) leads to
\[
\min_{(x, t) \in [0, 1] \times [0, T]} v(x, t) \geq C^{-1},
\] (2.19)

provided \(\alpha \in [0, 1]\). On the other hand, if \(\alpha > 1\), integrating (2.18) in \(x\) over \((0, 1)\) and using (2.6), (2.14), and Jensen’s inequality gives
\[
C \geq C^{-1} \exp \left\{ \int_0^1 v(x, t)^{-\alpha} \, dx + \int_0^1 f_\alpha(v) \, dx \right\}
\geq C^{-1} \exp \left\{ \frac{1}{2} \int_0^1 v(x, t)^{-\alpha} \, dx - C \right\},
\]

which in particular implies
\[
\sup_{0 \leq t \leq T} \int_0^1 v(x, t)^{-\alpha} \, dx \leq C.
\] (2.20)

Hence, putting this into (2.18) shows (2.19) still holds for \(\alpha > 1\). Finally, for \(p > 2\), multiplying (2.13) by \(\theta^{-p}\) gives
\[
\frac{1}{p-1} \frac{d}{dt} \int_0^1 (\theta^{-1})^{p-1} \, dx + \int_0^1 \frac{\mu u^2}{v \theta^p} \, dx
\leq \int_0^1 \frac{\mu u^2}{v \theta^p} \, dx
\leq \frac{1}{2} \int_0^1 \frac{\mu u^2}{v \theta^p} \, dx + \frac{1}{2} \int_0^1 \frac{1}{\mu v \theta^{p-2}} \, dx
\leq \frac{1}{2} \int_0^1 \frac{\mu u^2}{v \theta^p} \, dx + C \| \theta^{-1} \|_{L^{p-1}}^{p-2},
\]

where in the second inequality we have used \(\mu v \geq C^{-1}\). Combining this with Gronwall’s inequality yields that there exists some \(C\) independent of \(p\) such that
\[
\sup_{0 \leq t \leq T} \| \theta^{-1}(\cdot, t) \|_{L^{p-1}} \leq C.
\]
Letting $p \to \infty$ proves the second inequality of (2.12) and finishes the proof of lemma 2.3.

**Lemma 2.4.** There exists a positive constant $C$ such that for each $(x, t) \in [0, 1] \times [0, T]$, 
\[ C^{-1} \leq v(x, t) \leq C. \]  
(2.21)

**Proof.** First, for $0 < \alpha < 1$ and $0 < \varepsilon < 1$, integrating (2.13) multiplied by $\theta^{-\alpha}$ over $(0, 1) \times (0, T)$ yields
\[
\int_0^T \int_0^1 \alpha \theta \frac{\partial \theta}{\partial x}^2 \, dx \, dt + \int_0^T \int_0^1 \frac{\mu u_x^2}{\theta^{\alpha}} \, dx \, dt + \int_0^T \int_0^1 \frac{\theta^{1-\alpha} u_x^2}{\theta} \, dx \, dt \\
= \frac{1}{1 - \alpha} \int_0^1 \left( \theta^{1-\alpha} - \theta^{1-\alpha}_0 \right) \, dx + \int_0^T \int_0^1 \frac{\theta_0^{1-\alpha} u_x^2}{\theta} \, dx \, dt \\
\leq C(\alpha) + \frac{1}{2} \int_0^T \int_0^1 \frac{\mu u_x^2}{\theta^{\alpha}} \, dx \, dt + C \int_0^T \int_0^1 \theta^{2-\alpha} \, dx \, dt \\
\leq C(\alpha) + \frac{1}{2} \int_0^T \int_0^1 \frac{\mu u_x^2}{\theta^{\alpha}} \, dx \, dt + C \int_0^T \max_{x \in [0, 1]} \theta^{1-\alpha} \int_0^1 \theta \, dx \, dt \\
\leq C(\alpha, \varepsilon) + \frac{1}{2} \int_0^T \int_0^1 \frac{\mu u_x^2}{\theta^{\alpha}} \, dx \, dt + \varepsilon \int_0^T \max_{x \in [0, 1]} \theta \, dt, \tag{2.22}
\]
where in the first inequality we have used (2.14) and (2.12). Next, for $\alpha = \min\{1, \beta\}/2$, using (2.14), we get
\[
\int_0^T \max_{x \in [0, 1]} \theta \, dt \leq C + C \int_0^T \int_0^1 |\theta_x| \, dx \, dt \\
\leq C + C \int_0^T \int_0^1 \frac{\theta^{\beta} \theta_x^2}{\theta^{1+\alpha}} \, dx \, dt + C \int_0^T \int_0^1 \frac{\theta^{1+\alpha}}{\theta^{1+\beta}} \, dx \, dt \\
\leq C + C \int_0^T \int_0^1 \frac{\theta^{\beta} \theta_x^2}{\theta^{1+\alpha}} \, dx \, dt + \frac{1}{2} \int_0^T \max_{x \in [0, 1]} \theta \, dt,
\]
which together with (2.22) yields that
\[
\int_0^T \max_{x \in [0, 1]} \theta \, dt \leq C, \tag{2.23}
\]
and then that for $0 < \alpha < 1$,
\[
\int_0^T \int_0^1 \frac{\theta^{\beta} \theta_x^2}{\theta^{1+\alpha}} \, dx \, dt \leq C(\alpha). \tag{2.24}
\]
Finally, it follows from (2.17), (2.6), (2.16), (2.14), and (2.19) that
\[ C^{-1} \leq Y(t) \leq C, \]
which together with (2.2), (2.15), (2.19), and (2.23) yields

\[ v(x, t) \leq C + C \int_0^t \max_{x \in [0,1]} |b|^2(x, t) \max_{x \in [0,1]} v(x, t) \, dt. \] (2.25)

Using (1.9), (2.14), and (2.23), we have

\[
\int_0^T \max_{x \in [0,1]} |b|^2(x, t) \, dt \leq C \int_0^T \int_0^1 |b \cdot b_x| \, dx \, dt \\
\leq C \int_0^T \int_0^1 |b_x|^2 \, dx \, dt + C \int_0^T \int_0^1 v \theta |b|^2 \, dx \, dt \\
\leq C + C \int_0^T \max_{x \in [0,1]} \theta \, dt \\
\leq C,
\] (2.26)

which combined with (2.25) and Gronwall’s inequality gives

\[ \max_{(x, t) \in [0,1] \times [0, T]} v \leq C. \] (2.27)

The proof of lemma 2.4 is finished. \(\Box\)

Lemma 2.5. There is a positive constant \( C \) such that

\[ \sup_{0 \leq t \leq T} \int_0^1 v^2 \, dx \leq C. \] (2.28)

Proof. First, we rewrite the momentum equation (1.2) as

\[ \left( u - \frac{\mu v_x}{v} \right)_t = - \left( \frac{\theta}{v} + \frac{1}{2} |b|^2 \right)_x. \]

Multiplying the above equation by \( u - \frac{\mu v_x}{v} \) and integrating the resultant equality yields that for any \( t \in (0, T) \)

\[
\frac{1}{2} \int_0^1 \left( u - \frac{\mu v_x}{v} \right)^2 \, dx - \frac{1}{2} \int_0^t \left( u - \frac{\mu v_x}{v} \right)(x, 0) \, dx \\
= \int_0^t \int_0^1 \left( \frac{\theta v_x}{v^2} - \frac{\theta}{v} - b \cdot b_x \right) \left( u - \frac{\mu v_x}{v} \right) \, dx \, dt \\
= \int_0^t \int_0^1 \frac{\mu \theta v_x}{v^3} \, dx \, dt + \int_0^t \int_0^1 \frac{\theta v_x}{v^3} \, dx \, dt \\
- \int_0^t \int_0^1 \frac{\theta}{v} \left( u - \frac{\mu v_x}{v} \right) \, dx \, dt - \int_0^t \int_0^1 b \cdot b_x \left( u - \frac{\mu v_x}{v} \right) \, dx \, dt \\
= - \int_0^t \int_0^1 \frac{\mu \theta v_x^2}{v^3} \, dx \, dt + \sum_{i=1}^3 I_i.
\] (2.29)

Each \( I_i (i = 1, 2, 3) \) can be estimated as follows: first, Cauchy’s inequality gives
\[ |I_1| \leq \frac{1}{2} \int_0^t \int_0^1 \frac{\mu \theta v_x^2}{v^3} \, dx \, dr + \frac{1}{2} \int_0^T \int_0^1 \frac{u_x^2 \theta}{\mu v} \, dx \, dt \]
\[ \leq \frac{1}{2} \int_0^t \int_0^1 \frac{\mu \theta v_x^2}{v^3} \, dx \, dr + C \int_0^T \max_{x \in [0,1]} \theta \, dr \]
\[ \leq C + \frac{1}{2} \int_0^t \int_0^1 \frac{\mu \theta v_x^2}{v^3} \, dx \, dr, \]  
(2.30)

where we have used (2.21), (2.14), and (2.23). Next, using (2.12), (2.14), and (2.21), we have
\[ |I_2| \leq \frac{1}{2} \int_0^T \int_0^1 \frac{\theta^2 \theta_x^2}{v \theta^2} \, dx \, dr + \frac{1}{2} \int_0^t \int_0^1 \frac{\theta^2}{v \theta^2} \left( u - \frac{\mu v_x}{v} \right)^2 \, dx \, dr \]
\[ \leq C + C \int_0^T \max_{x \in [0,1]} \theta^2 \int_0^1 \left( u - \frac{\mu v_x}{v} \right)^2 \, dx \, dr. \]  
(2.31)

Moreover, it follows from (2.14), (2.12), (2.21), and (2.24) that, for any \( \varepsilon > 0 \),
\[ \int_0^T \max_{x \in [0,1]} \theta^2 \, dr \leq C \int_0^T \max_{x \in [0,1]} \left| \theta^2 - \int_0^1 \theta^2 \, dx \right| \, dr + C \int_0^T \max_{x \in [0,1]} \theta \, dr \]
\[ \leq C + C \int_0^T \int_0^1 \theta \theta_x \, dx \, dr \]
\[ \leq C + C(\varepsilon) \int_0^T \int_0^1 \theta_x^2 \, dx \, dr + \varepsilon \int_0^T \int_0^1 \theta \, dx \, dr \]
\[ \leq C(\varepsilon) + C \varepsilon \int_0^T \max_{x \in [0,1]} \theta^2 \, dr, \]

which gives
\[ \int_0^T \max_{x \in [0,1]} \theta^2 \, dr \leq C. \]  
(2.32)

Finally, integrating (2.13) over \((0, 1) \times (0, T)\), we have by (2.21)
\[ \int_0^T \int_0^1 \frac{\mu u_x^2 + |w_x|^2 + |b_x|^2}{v} \, dx \, dr \]
\[ = \int_0^1 \theta \, dx - \int_0^1 \theta_0 \, dx + \int_0^T \int_0^1 \frac{\theta u}{v} \, dx \, dr \]
\[ \leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dr + C \int_0^T \max_{x \in [0,1]} \theta^2 \, dr, \]

which together with (2.21) and (2.32) gives
\[ \int_0^T \int_0^1 (u_x^2 + |w_x|^2 + |b_x|^2) \, dx \, dr \leq C. \]  
(2.33)
Combining this with Cauchy’s inequality leads to
\[ |I_3| \leq C \int_0^t \int_0^1 \left( |b_x|^2 + |b|^2 \left( u - \frac{\mu b_x}{v} \right)^2 \right) dx \, dt \]
\[ \leq C + C \int_0^t \max_{x \in [0,1]} |b|^2 \int_0^1 \left( u - \frac{\mu b_x}{v} \right)^2 dx \, dt. \]  
(2.34)

Putting (2.30), (2.31), and (2.34) into (2.29), we obtain after using Gronwall’s inequality, (2.26), and (2.32) that
\[ \sup_{0 \leq t \leq T} \int_0^1 \left( u - \frac{\mu b_x}{v} \right)^2 dx + \int_0^T \int_0^1 \frac{\theta v_x^2}{v} dx \, dt \leq C, \]  
(2.35)

which together with (2.14) gives (2.28) and finishes the proof of lemma 2.5.

**Lemma 2.6.** There is a positive constant C such that
\[ \sup_{0 \leq t \leq T} \int_0^1 (|b|^2 + |w|^2) dx \]
\[ + \int_0^T \int_0^1 (|b|^2 + |b_x|^2 + |w|^2 + |w_x|^2) dx \, dt \leq C. \]  
(2.35)

**Proof.** First, multiplying (1.3) by \( w_{xx} \) and integrating the resulting equality over \((0,1) \times (0,T)\), we obtain after using (1.9), (2.33), (2.28), and Cauchy’s inequality that
\[ \frac{1}{2} \int_0^1 |w_x|^2 dx + \int_0^T \int_0^1 \frac{|w_{xx}|^2}{v} dx \, dt \]
\[ \leq C + \frac{1}{2} \int_0^1 \int_0^1 \frac{|w_{xx}|^2}{v} dx \, dt + C \int_0^T \int_0^1 \left( |b|^2 + |w|^2 \right) dx \, dt \]
\[ \leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{|w|^2}{v} dx \, dt + C \int_0^T \max_{x \in [0,1]} |w_x|^2 dx \, dt. \]  
(2.36)

Direct computation shows that for any \( \varepsilon > 0 \),
\[ \int_0^T \max_{x \in [0,1]} |w_x|^2 dx \leq C(\varepsilon) \int_0^T \int_0^1 |w_x|^2 dx \, dt + \varepsilon \int_0^T \int_0^1 \frac{|w_{xx}|^2}{v} dx \, dt \]
\[ \leq C(\varepsilon) + \varepsilon \int_0^T \int_0^1 \frac{|w_{xx}|^2}{v} dx \, dt, \]  
(2.37)

which combined with (2.36) leads to
\[ \sup_{0 \leq t \leq T} \int_0^1 |w_x|^2 dx + \int_0^T \int_0^1 |w_{xx}|^2 dx \, dt \leq C. \]  
(2.38)
Then, we rewrite (1.3) as
\[ w_t = \frac{w_{xx}}{v} - \frac{w_x v_x}{v^2} + b_x, \]
which together with (2.21), (2.38), (2.28), (2.33), and (2.37) gives
\[
\int_0^T \int_0^1 |w_t|^2 \, dx \, dt \leq C \int_0^T \int_0^1 \left( |b_x|^2 + |w_{xx}|^2 + v^2 |w_x|^2 \right) \, dx \, dt \\
\leq C \int_0^T \max_{x \in [0,1]} |w_x|^2 \, dx \\
\leq C. \tag{2.39}
\]

Next, multiplying (1.4) by \( b_{xx} \) and integrating the result over \((0,1) \times (0,T)\), we deduce from (2.28), (2.33), (2.21), (2.14) and Cauchy’s inequality that
\[
\frac{1}{2} \int_0^1 |b_x|^2 \, dx + \int_0^T \int_0^1 \frac{|b_{xx}|^2}{v^2} \, dx \, dt \\
\leq C + \frac{1}{2} \int_0^1 \int_0^T |b_{xx}|^2 \, dx \, dt + C \int_0^T \int_0^1 \left( |b_x|^2 v^2 + u_x^2 |b|^2 + |w_x|^2 \right) \, dx \, dt \\
\leq C + \frac{3}{4} \int_0^1 \int_0^T \frac{|b_{xx}|^2}{v^2} \, dx \, dt + C \int_0^T \int_0^1 |b_x|^2 \, dx \\
+ C \sup_{0 \leq t \leq T} \int_0^1 |b|^2 \, dx + \frac{1}{4} \sup_{0 \leq t \leq T} \int_0^1 |b_x|^2 \, dx \\
\leq C + \frac{3}{4} \int_0^1 \int_0^T \frac{|b_{xx}|^2}{v^2} \, dx \, dt + \frac{1}{4} \sup_{0 \leq t \leq T} \int_0^1 |b_x|^2 \, dx,
\]
which implies
\[
\sup_{0 \leq t \leq T} \int_0^1 |b_x|^2 \, dx + \int_0^T \int_0^1 \frac{|b_{xx}|^2}{v^2} \, dx \, dt \leq C. \tag{2.40}
\]
Hence,
\[
\max_{(x,t) \in [0,1] \times [0,T]} |b|^2 \leq C + C \sup_{0 \leq t \leq T} \int_0^1 |b_x|^2 \, dx \leq C. \tag{2.41}
\]
Finally, we rewrite (1.4) as
\[ b_t = \frac{w_x}{v} + \frac{b_{xx}}{v^2} - \frac{b_x v_x}{v^3} - \frac{b u_x}{v}, \]
which together with (2.40), (2.28), (2.33), and (2.41) gives
Then, multiplying (2.13) by $1$ and integrating the result over $(0, T)$ yields

\[
\int_0^T \int_0^1 |b_x|^2 \, dx \, dt \leq C \int_0^T \int_0^1 \left( |b_{xx}|^2 + |b|_x^2 v_x^2 + |w_x|^2 + |b|^2 u_x^2 \right) \, dx \, dt
\]

\[
\leq C + C \int_0^T \left( \max_{x \in [0,1]} |b_x|^2 + \int_0^1 u_x^2 \, dx \right) \, dt
\]

\[
\leq C + C \int_0^T \int_0^1 (|b_x|^2 + |b_{xx}|^2) \, dx \, dt
\]

\[
\leq C.
\]

Combining this, (2.38), (2.39), and (2.40) gives (2.35) and finishes the proof of lemma 2.6.

Lemma 2.7. There is a positive constant $C$ such that

\[
\sup_{0 \leq t \leq T} \int_0^1 u_x^2 \, dx + \int_0^T \int_0^1 (u_x^2 + u_{xx}^2) \, dx \, dt \leq C. \tag{2.42}
\]

Proof. First, multiplying (1.2) by $u_{xx}$ and integrating the result over $(0, 1) \times (0, T)$, we have

\[
\frac{1}{2} \int_0^1 u_x^2 \, dx + \int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dt
\]

\[
\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dt + C \int_0^T \int_0^1 \left( \theta_x^2 + \theta^2 v_x^2 + |b|^2 |b_x|^2 + u_x^2 v_x^2 \right) \, dx \, dt
\]

\[
\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dt + C \int_0^T \int_0^1 \theta_x^2 \, dx \, dt + C \int_0^T \max_{x \in [0,1]} \theta^2 \int_0^1 u_x^2 \, dx \, dt
\]

\[
+ C \max_{(x,t) \in [0,1] \times [0,T]} |b|^2 \int_0^T \int_0^1 |b_x|^2 \, dx \, dt + C \int_0^T \max_{x \in [0,1]} u_x^2 \int_0^1 \theta_x^2 \, dx \, dt
\]

\[
\leq C + \frac{3}{4} \int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dt + C_1 \int_0^T \int_0^1 \theta_x^2 \, dx \, dt, \tag{2.43}
\]

where in the last inequality we have used (2.32), (2.28), (2.41), (2.33), and the following inequality

\[
\int_0^T \max_{x \in [0,1]} u_x^2 \, dt \leq C(\epsilon) \int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dt + \epsilon \int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dt
\]

\[
\leq C(\epsilon) + \epsilon \int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dt, \tag{2.44}
\]

for any $\epsilon > 0$. Then, multiplying (2.13) by $\theta$ and integrating the result over $(0, 1) \times (0, T)$ yields

\[
\int_0^T \int_0^1 \left( \mu \theta_x^2 v_x^2 + \theta_x^2 v_x^2 \right) \, dx \, dt
\]

\[
\leq C(\epsilon) + \epsilon \int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dt,
\]
\[ \frac{1}{2} \int_0^1 \theta^2 \text{d}x + \int_0^T \int_0^1 \theta \beta \frac{\partial \theta}{\partial t} \text{d}x \text{d}t \]
\[ \leq C + C \int_0^T \int_0^1 |\theta|^2 |u_\xi| \text{d}x + C \int_0^T \int_0^1 \left( u_\xi^2 + |w_\xi|^2 + |b^2_\xi| \right) \theta \text{d}x \text{d}t \]
\[ \leq C + C \int_0^T \int_0^1 \theta \beta^2 \text{d}x \text{d}t + C \int_0^T \int_0^1 \theta \text{d}x \text{d}t + C \max_{x \in [0,1]} \theta \text{d}t \]
\[ \leq C + C \int_0^T \int_0^1 \frac{\mu \beta}{v} \text{d}x \text{d}t. \quad (2.45) \]

where we have used (2.35), (2.44), and (2.32). Adding (2.45) multiplied by \( C_1 + 1 \) to (2.43) and choosing \( \varepsilon \) sufficiently small, we obtain that

\[ \sup_{0 \leq t \leq T} \int_0^1 (\theta^2 + u_\xi^2) \text{d}x + \int_0^T \int_0^1 \theta \beta^2 \text{d}x \text{d}t + \int_0^T \int_0^1 u_\xi^2 \text{d}x \text{d}t \leq C. \quad (2.46) \]

Finally, we rewrite (1.2) as

\[ \begin{align*}
\eta = \frac{\mu u_\xi}{v} + \left( \frac{\mu}{v} \right)' u_\xi v_\xi - \frac{\theta}{v} \frac{\theta v_\xi}{v^2} = b \cdot b_\xi,
\end{align*} \]

which together with (2.28), (2.41), (2.32), (2.44), (2.46), and (2.35) leads to

\[ \int_0^T \int_0^1 u_\xi^2 \text{d}x \text{d}t \leq C \int_0^T \int_0^1 \left( u_\xi^2 + u_\xi^2 v_\xi^2 + \theta_\xi^2 + \theta_\xi^2 v_\xi^2 + |b|^2 |b_\xi| \right) \text{d}x \text{d}t \]
\[ \leq C. \]

Combining this and (2.46) immediately gives (2.42) and completes the proof of lemma 2.7.

\[ \Box \]

**Lemma 2.8.** There exists a positive constant \( C \) such that

\[ \sup_{0 \leq t \leq T} \int_0^1 \theta^2 \text{d}x + \int_0^T \int_0^1 (\theta_\xi^2 + \theta_\xi^2) \text{d}x \text{d}t \leq C. \quad (2.47) \]

**Proof.** First, noticing that integration by parts leads to

\[ \int_0^1 \theta^2 \beta \left( \frac{\theta \beta_\xi}{v} \right) \text{d}x = - \int_0^1 \theta \beta_\xi \left( \frac{\theta^2 \beta}{v} \right)_x \text{d}x \]
\[ = - \int_0^1 \theta \beta_\xi \left( \frac{\theta^2 \beta}{v} \right)_x \text{d}x \]
\[ = - \frac{1}{2} \int_0^1 \left( \frac{\theta^2 \beta_\xi}{v} \right)^2 \text{d}x \]
\[ = - \frac{1}{2} \int_0^1 \left( \frac{\theta^2 \beta_\xi}{v} \right)^2 \text{d}x \]
\[ - \frac{1}{2} \int_0^1 \left( \frac{\theta^2 \beta_\xi}{v} \right)^2 u_\xi \text{d}x. \]
multiplying (2.13) by $\theta^3 \theta_t$, and integrating the resultant equality over $(0,1)$, we have
\[
\int_0^1 \theta^3 \theta_t^2 \, dx + \frac{1}{2} \left( \int_0^1 \frac{(\theta^3 \theta_t)^2}{\nu} \, dx \right) = - \frac{1}{2} \int_0^1 \frac{(\theta^3 \theta_t)^2}{\nu} \, dx + \int_0^1 \theta^3 \theta_t \left(- \theta u_x + \mu u_x^2 + |w_x|^2 + |b_x|^2\right) \, dx \\
\leq C \max_{x \in [0,1]} \left( |u| \theta^{3/2} \right) \int_0^1 \theta^{3/2} \theta_t^2 \, dx + \frac{1}{2} \int_0^1 \theta^3 \theta_t^2 \, dx + C \int_0^1 \theta^{3+2} u_t^2 \, dx \\
+ C \int_0^1 \theta^3 \left( u_x^2 + |w_x|^4 + |b_x|^4 \right) \, dx
\]
\leq C \int_0^1 \theta^{2+2} \theta_t^2 \, dx + \frac{1}{2} \int_0^1 \theta^3 \theta_t^2 \, dx \\
+ C \max_{x \in [0,1]} \left( \theta^{2+2} + u_x^4 + |w_x|^4 + |b_x|^4 \right) + C \quad \text{(2.48)}
\]
due to (2.12), (2.35) and (2.46). Combining (2.42) with Hölder’s inequality gives
\[
\int_0^T \max_{x \in [0,1]} u_t^4 \, dt \leq C \int_0^T \int_0^1 u_t^4 \, dx \, dt + C \int_0^T \int_0^1 |u_t^2 u_x^2| \, dx \, dt \\
\leq C \int_0^T \max_{x \in [0,1]} u_t^4 \int_0^1 u_t^2 \, dx \, dt \\
+ C \int_0^1 \max_{x \in [0,1]} u_t^2 \left( \int_0^1 u_t^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^1 u_x^2 \, dx \right)^{\frac{1}{2}} \, dt \\
\leq C \int_0^T \int_0^1 \left( u_t^2 + u_x^2 \right) \, dx \, dt + \frac{1}{2} \int_0^T \max_{x \in [0,1]} u_t^4 \, dt \\
\leq C + \frac{1}{2} \int_0^T \max_{x \in [0,1]} u_t^4 \, dt \quad \text{(2.49)}
\]
Using (2.35) and applying similar arguments to $b$ and $w$ implies
\[
\int_0^T \max_{x \in [0,1]} |b_x|^4 \, dt \leq C, \quad \int_0^T \max_{x \in [0,1]} |w_x|^4 \, dt \leq C \quad \text{(2.50)}
\]
Noticing that
\[
\max_{x \in [0,1]} \theta^{2+2} \leq C + \int_0^1 (\theta^3 \theta_t)^2 \, dx, \quad \text{(2.51)}
\]
we then deduce from (2.46), (2.48)–(2.50), and the Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \int_0^1 (\theta^3 \theta_t)^2 \, dx + \int_0^T \int_0^1 \theta^3 \theta_t^2 \, dx \, dt \leq C \quad \text{(2.52)}
\]
which together with (2.51) shows
Thus, both (2.52) and (2.12) lead to
\[
\sup_{0 \leq t \leq T} \int_0^1 \theta_x^2 dx + \int_0^1 \int_0^1 \theta_t^2 dxdt \leq C.
\] (2.54)

Finally, it follows from (2.13) that
\[
\frac{\theta \theta_x \theta_{xx}}{\nu} = -\frac{\beta \theta^3 \theta_x^2}{\nu} + \frac{\theta \theta_x \theta_t}{\nu} - \frac{C \mu u^2 + |b|^2 + \omega^2}{\nu} + \frac{\theta \theta_t}{\nu} + \theta_t,
\]
which together with (2.12), (2.33), (2.28), (2.49), (2.50), (2.53), and (2.54) yields
\[
\int_0^T \int_0^1 \theta_x^2 dxdt \leq C \int_0^T \int_0^1 \left( \theta_x^2 + \theta_t^2 + \omega^2 + |b|^2 + \omega^2 + \theta_t^2 \right) dxdt
\]
\[
\leq C + C \int_0^T \max_{x \in [0,1]} \theta_t^2 dx
\]
\[
\leq C + \frac{1}{2} \int_0^T \int_0^1 \theta_x^2 dxdt.
\]

Combining this with (2.54) shows (2.47) and finishes the proof of lemma 2.8. 

\textbf{Proof of theorem 1.1.} By lemma 2.1, there exists some \(T > 0\) such that the initial-boundary-value problem \((1.1)-(1.9)\) has a unique strong solution \((v, u, \theta, b, w)\) on \((0, T]\) satisfying (2.1). Assume that \(T^*\) is the maximum existence time of the unique strong solution \((v, u, \theta, b, w)\) to (1.1)-(1.9). Hence, \(T^* \geq T_1 > 0\). We claim that
\[
T^* = +\infty.
\] (2.55)

Otherwise, \(T^* < +\infty\), then the global \textit{a priori} estimates of solutions (see lemmas 2.3--2.8) assure that \((v, u, \theta, b, w) \in C([0, T^*]; L^2)\) that
\[
\| (v, u, \theta, b, w)(t) \|_{L^2} \leq C(T^*) < +\infty, \quad \forall t \in [0, T^*),
\] (2.56)

where \(C(T^*)\) is a positive constant depending only on \(T^*, \alpha, \beta, \inf_{x \in [0,1]} v_0(x), \inf_{x \in [0,1]} \theta_0(x), \) and \(\| (v_0, u_0, \theta_0, b_0, w_0) \|_{L^2(0,1)}\). Thus, \((v, u, \theta, b, w)(x, T^*)\) is well-defined and finite. It then follows from lemma 2.1 that, there exists a positive constant \(T_0 > 0\) such that (1.1)-(1.9) has a unique strong solution on \([T^*, T^* + T_0]\), which contradicts the definition of \(T^*\). Hence, (2.55) holds. The proof of theorem 1.1 is finished. 

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