Existence of twisted constant scalar curvature Kähler metrics with a large twist

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Abstract
Suppose that there exist two Kähler metrics \( \omega \) and \( \alpha \) such that the metric contraction of \( \alpha \) with respect to \( \omega \) is constant, i.e. \( \Lambda_\omega \alpha = \text{const} \). We prove that for all large enough \( R > 0 \) there exists a twisted constant scalar curvature Kähler metric \( \omega' \) in the cohomology class \([\omega]\), satisfying \( S(\omega') - R\Lambda_\omega \alpha = \text{const} \). We discuss its implication to \( K \)-stability and the continuity method recently proposed by X.X. Chen.

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1 Introduction and the statement of the results

1.1 Twisted constant scalar curvature Kähler metrics and a continuity method

Let \((X, \omega)\) be a compact Kähler manifold without boundary. The existence of a “canonical” Kähler metric, such as constant scalar curvature Kähler (cscK) metrics satisfying \( S(\omega) = \text{const} \), in a given cohomology class has been a central problem in Kähler geometry. On the other hand, the cscK equation \( S(\omega) = \text{const} \) is a fourth order fully nonlinear partial differential equation (PDE), and is difficult to solve in general.

A well-known method in solving a nonlinear PDE is the continuity method. X.X. Chen [6] recently proposed a continuity method which can be seen as a generalisation to the cscK case of the Aubin–Yau continuity method that is well-known in Kähler–Einstein problems (cf. [1] [30]). In this continuity method, we consider the following problem.
Theorem 1.2. The question [6, Question 1.6] posed by X.X. Chen in the affirmative.

where \( \Lambda_\omega \) means the metric contraction with respect to \( \omega \).

Suppose that we write \( I \subset [0, 1] \) for the set of time parameter \( t \) for which (1) is solvable. The solution to Problem 1.1 can be achieved in the following three steps.

1. Show that \( I \) is nonempty; i.e. show that Problem 1.1 can be solved at time \( t = 0 \), or solve \( \Lambda_\omega \alpha = \text{const} \) for \( \omega_0 \) when \( \alpha \) is given. This is an interesting problem in its own right, in relation to the \( J \)-flow [5 7 9 16 17 21 28 29]. On the other hand we observe that, when we can assume that \( \omega_0 \) and \( \alpha \) are in the same cohomology class, we can simply take \( \omega_0 := \alpha \) to solve this equation.

2. Show that \( I \) is open; the openness of the problem at \( t \in [0, 1) \) means that, if (1) is solvable at time \( t \in [0, 1) \), there exists \( \epsilon_0 > 0 \) such that (1) can be solved for each \( t' \in (t - \epsilon, t + \epsilon) \cap [0, 1) \) for all \( 0 < \epsilon \leq \epsilon_0 \).

3. Show that \( I \) is closed; the closedness of the problem means that, for every convergent sequence \( \{t_i\}_i \) in \([0, 1] \), if (1) can be solved for each \( t_i \) then it can be solved for \( t_\infty = \lim_{i} t_i \).

Observe that the solution of Problem 1.1 at \( t = 1 \) is precisely equal to solving the cscK equation \( S(\omega_1) = \text{const} \); we thus reduced the problem of finding a cscK metric to establishing the above, hopefully more manageable, three steps in Problem 1.1.

The main result of this paper is the openness at \( t = 0 \) (cf. Remark 1.6), stated as follows, which solves the question [6, Question 1.6] posed by X.X. Chen in the affirmative.

Theorem 1.2. Suppose that we have two Kähler metrics \( \omega \) and \( \alpha \) satisfying \( \Lambda_\omega \alpha = \text{const} \). Then there exists a constant \( R(\omega, \alpha) > 0 \) depending only on \( \omega \) and \( \alpha \), so that for all \( R \geq R(\omega, \alpha) \) there exists \( \phi \in C^\infty(X, \mathbb{R}) \) such that \( \omega_\phi := \omega + \sqrt{-1} \partial \bar{\partial} \phi \) satisfies \( S(\omega_\phi) - \Lambda_\omega \alpha = \text{const} \).

Remark 1.3. As the preparation of this paper was nearing completion, the author learned that Yu Zeng [32] independently proved a special case of the above theorem when we take \( \alpha = \omega \).

Recall that, for a closed positive real \((1, 1)-\)form \( \alpha \), a Kähler metric \( \omega_\phi \) satisfying the equation

\[
S(\omega_\phi) - \Lambda_\omega \alpha = \text{const}
\]

is said to be of \( \alpha \)-twisted constant scalar curvature Kähler or \( \alpha \)-twisted cscK. Twisted cscK metrics were applied to the study of (genuine) cscK metrics by means of adiabatic construction [13 14], and are an interesting object in their own right [8 15 22]. It is also known that an \( \alpha \)-twisted cscK metric is unique in each cohomology class [3 Theorem 4.5], and this was used to prove the uniqueness of cscK metrics modulo automorphisms [3 Theorem 1.3]. Naively re-phrasing, Theorem 1.2 ensures the existence of \( R(\alpha) \)-twisted cscK metrics on any compact Kähler manifold, assuming \( \Lambda_\omega \alpha = \text{const} \) and taking \( R > 0 \) to be sufficiently large; in particular, this implies that we can always find an \( R(\alpha) \)-twisted cscK metric on any compact Kähler manifold in the cohomology class \([\alpha] \), by taking \( \omega = \alpha \).

We shall also see that the proof of Theorem 1.2 can be easily applied to prove the openness at \( t \in (0, 1) \); recall on the other hand that the openness at \( t \in (0, 1) \) was also obtained independently by X.X. Chen [9 Theorem 1.5]. In fact, we can prove the following slightly stronger result.
Corollary 1.4. Suppose that ω is α'-twisted cscK. Then, if α' − α is sufficiently small in the $C^\infty$-norm, there exists an α-twisted cscK metric in the cohomology class [ω].

Thus, as in the Kähler–Einstein case, we see that proving the closedness of I is the hardest part in solving Problem 1.3.

Remark 1.5. One way of obtaining a twisted cscK metric is to solve the following “un-traced” version of the equation (2):

$$\text{Ric}(\omega_\phi) - R\alpha = \text{const} \omega_\phi.$$  

(3)

This is solvable for any positive twist α by the Aubin–Yau theorem [1, 30], when $R > 0$ is chosen to be sufficiently large. This was used by Fine [14] to construct cscK metrics on fibred Kähler manifolds by means of adiabatic construction. However, the equation (3) means that the cohomology class of $\omega_\phi$ must be a constant multiple of $R[\alpha] + c_1(K_X)$, where $K_X$ is the canonical bundle of $X$. Theorem 1.2 removes this restriction on the cohomology class, assuming instead that the equation $\Lambda_\omega \alpha = \text{const}$ should be satisfied. An advantage of this can be seen when we choose $\omega = \alpha$; we can find a twisted cscK metric in the cohomology class $[\omega] = [\alpha]$.

Remark 1.6. The openness of Problem 1.1 at $t = 0$ is different from the one at $0 < t < 1$ in a rather significant way due to the following fact: at $t = 0$, the equation $\Lambda_\omega \alpha = \text{const}$ is a second order fully nonlinear PDE in the Kähler potential, whereas at $0 < t < 1$, the equation $tS(\omega_t) - (1-t)\Lambda_\omega \alpha = \text{const}$ is a fourth order fully nonlinear PDE. Thus, to prove the openness at $t = 0$, we must deal with this “jump” in the order of the equation in question, as opposed to the case $0 < t < 1$.

Remark 1.7. Observe that the average $\bar{S}$ of the scalar curvature $S(\omega_\phi)$ with respect to the volume form $\omega_\phi^n/n!$ is equal to the cohomological number $-n \int_X c_1(K_X) \wedge [\omega_\phi^{n-1}]/\int_X [\omega_\phi^n]$ by the Chern–Weil theory. Since α is closed, the average $c$ of $\Lambda_\omega \alpha$ with respect to the volume form $\omega_\phi^n/n!$ is also determined by the cohomology classes of α and $\omega_\phi$ as $c = n \int_X [\alpha] \wedge [\omega_\phi^{n-1}]/\int_X [\omega_\phi^n]$.

1.2 Relationship to K-stability

A central problem in Kähler geometry in recent years has been the connection between the existence of cscK metrics and a notion of algebro-geometric stability called K-stability; a conjecture called Donaldson–Tian–Yau conjecture [11, 27, 31] states that, when the automorphism group of $X$ is discrete, there exists a cscK metric in the first Chern class $c_1(L)$ of an ample line bundle $L$ if and only if the polarised Kähler manifold $(X, L)$ is K-stable.

We will be very brief in recalling the notion of K-stability here, and the reader is referred to [11, 27] for more details. A test configuration for a polarised Kähler manifold $(X, L)$, written $(\mathcal{X}, \mathcal{L})$, is a flat family $\pi : \mathcal{X} \to \mathbb{C}$ over $\mathbb{C}$ with an equivariant $\mathbb{C}^*$-action lifting to the total space of a line bundle $\mathcal{L}$ such that $\pi^{-1}(1)$ is isomorphic to $(X, L^{\otimes r})$, and $r$ is called the exponent of the test configuration $(\mathcal{X}, \mathcal{L})$. We can define a rational number called the Donaldson–Futaki invariant $DF(\mathcal{X}, \mathcal{L})$ for each $(\mathcal{X}, \mathcal{L})$ as in [11, §2.1], and

1. See also [2, 19, 20] for related results in which we do not necessarily take $R$ to be large.

2. When the automorphism group of $X$ is not discrete, we consider a notion called $K$-polystability, but we do not discuss this in detail here.

3. It is expected that the notion of K-stability may have to be refined for this conjecture to hold, cf. [25].
\((X, L)\) is said to be \textbf{K-stable} if \(DF(X, L) \geq 0\) for every test configuration and \(DF(X, L) > 0\) for every “nontrivial” test configuration\(^4\).

Dervan \cite{8} introduced the \textit{minimum norm} \(||X||_m\) of a test configuration \((X, L)\) and proved that the existence of an \(\alpha\)-twisted cscK metric in the cohomology class \(c_1(L)\) implies the \textit{uniform twisted K-stability} of \((X, L, \frac{1}{2}[\alpha])\) \cite{8} Theorem 1.1]. Note also that the minimum norm agrees, up to a constant multiple, with the non-Archimedean \(J\)-functional defined by Boucksom–Hisamoto–Jonsson \cite{4}. It turns out that when the cohomology class of \(\alpha\) is a (positive) constant multiple of \(c_1(L)\), uniform twisted K-stability of \((X, L, \frac{1}{2}[\alpha])\) implies that there exists a constant \(\hat{R} > 0\) such that \(DF(X, L) + \hat{R}||X||_m > 0\) \cite{8} Remark 2.34]. Recalling that it is always possible to solve \(\Lambda_\omega \alpha = \text{const}\) for \(\omega\) in the cohomology class \([\alpha]\), by taking \(\omega = \alpha\), Theorem 1.2 implies the following result.

**Corollary 1.8.** There exists a constant \(\hat{R} > 0\), which depends only on \((X, L)\), such that \(DF(X, L) > -\hat{R}||X||_m\) for any test configuration \((X, L)\) for \((X, L)\).

**Remark 1.9.** In fact this lower bound itself can be obtained from Proposition 3.1 which is the starting point of our proof of Theorem 1.2.

**Remark 1.10.** We now recall that there is another norm \(||X||_2\) for a test configuration \((X, L)\), called the \(L^2\)-norm of \((X, L)\). It is known \cite{4, 8} that \(||X||_2 = 0\) if and only if \(||X||_m = 0\), but they are not Lipschitz equivalent \cite{4}. Recall that the \(L^2\)-norm appears in the lower bound of the Calabi functional \(Ca(\omega) := \int_X (S(\omega) - \bar{S})^2 \omega^n\), which takes the form

\[
\inf_{\omega \in c_1(L)} \int_X (S(\omega) - \bar{S})^2 \frac{\omega^n}{4\pi n!} \geq \sup_{(X, L)} \left( - \frac{DF(X, L)}{||X||_2} \right)
\]

as established by Donaldson \cite{12}. It immediately follows that we have \(DF(X, L) \geq -\hat{R}||X||_2\) by taking \(\hat{R}\) to be the left hand side of the inequality \cite{11}. Thus, Corollary 1.8 can be seen as a \textit{minimum-norm version} of this particular consequence of the lower bound of the Calabi functional.

### 1.3 Some open problems

In view of Problem 1.1, it is natural to define the following quantity

\[\hat{R}_\alpha := \inf\{ R \geq 0 \mid \exists \phi \in C^\infty(X, \mathbb{R}) \text{ s.t. } S(\omega_\phi) - \Lambda_\omega_\phi(R) = \text{const}, \omega \in c_1(L)\},\]

as introduced by X.X. Chen \cite{6, Definition 1.6], which is analogous to the \(R(X)\) invariant defined by Székelyhidi \cite{23} for Fano manifolds. Given Corollary 1.8 it seems natural to consider the following question.

**Question 1.11.** Does there exist a test configuration \((\tilde{X}, \tilde{L})\) with \(||\tilde{X}||_m > 0\) such that \(DF(\tilde{X}, \tilde{L}) = -\hat{R}_\alpha||\tilde{X}||_m\)?

Since the Donaldson–Futaki invariant and the minimum norm can be defined in terms of algebro-geometric data, we can ask if \(\hat{R}_\alpha\) in the above can be written without referring to the particular choice of twist \(\alpha\), and potentially in terms of algebro-geometric language. We thus ask the following question, as conjectured by X.X. Chen.

\(^4\)There are subtleties associated to this formalism, e.g. as to what “trivial" test configurations should mean, and the reader is referred to \cite{4, 8, 13, 25} for more details.
Question 1.12. (X.X. Chen [6, Conjecture 1.17]) For any two closed positive real $(1,1)$-forms $\alpha$ and $\beta$ in the same cohomology class, do we have $\tilde{R}_\alpha = \tilde{R}_\beta$?

If we can solve Question 1.11 in the affirmative, we see that $\tilde{R}_\alpha > 0$ would imply that $(X,L)$ cannot be $K$-stable, since $(\tilde{X},\tilde{L})$ would provide a test configuration whose Donaldson–Futaki invariant is strictly negative. Noting that there cannot be a cscK metric in $c_1(L)$ if $\tilde{R}_\alpha > 0$, this would provide an evidence in support of the Donaldson–Tian–Yau conjecture.

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2 Linearisation of the twisted cscK equation

Suppose that we write $\omega_t := \omega + t\sqrt{-1}\partial\bar{\partial}\phi$. Recall that the scalar curvature $S(\omega_t)$ of $\omega_t$ is defined by $S(\omega_t) := \Lambda_{\omega_t} \text{Ric}(\omega_t)$, where $\text{Ric}(\omega_t) := -\sqrt{-1}\partial\bar{\partial}\log\det(\omega_t)$ is the Ricci form of $\omega_t$. Locally, writing $\omega_t = \sum_{i,j} g_{i\bar{j},t} \sqrt{-1} dz_i \wedge d\bar{z}_j$ in local holomorphic coordinates $(z_1, \ldots, z_n)$, we have

$$S(\omega_t) = -\sum_{i,j} g_{i\bar{j},t} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det(g_{k\bar{l},t})$$

where $g_{i\bar{j},t}$ is the inverse matrix of $g_{i\bar{j},t}$. We find by direct computation that

$$\left. \frac{d}{dt} \right|_{t=0} S(\omega_t) = -\Delta_{\omega_t} \phi - (\text{Ric}(\omega), \sqrt{-1}\partial\bar{\partial}\phi)_{\omega}$$

where $\Delta_{\omega_t}$ is the $\bar{\partial}$-Laplacian and $(,)_{\omega}$ is a (pointwise) inner product on the space of 2-forms defined by $\omega$. It is well-known (cf. [16]) that it can also be written as

$$\left. \frac{d}{dt} \right|_{t=0} S(\omega_t) = -\mathcal{D}_\omega^* \mathcal{D}_\omega \phi + (\partial S(\omega), \bar{\partial}\phi)_{\omega}$$

where $\mathcal{D}_\omega : C^\infty(X, \mathbb{R}) \rightarrow C^\infty(T^{1,0}X \otimes \Omega^{0,1}(X))$ is an operator defined by $\mathcal{D}_\omega \phi := \bar{\partial}(\text{grad}^1 \phi)$, and $\mathcal{D}_\omega^*$ is the formal adjoint with respect to $\omega$. Observe that the kernel of $\mathcal{D}_\omega \mathcal{D}_\omega^*$, which is equal to the kernel of $\mathcal{D}_\omega$ as $X$ is compact without boundary, is equal to the set of functions whose $(1,0)$-part of the gradient is a holomorphic vector field.

Now, let $\alpha$ be a closed positive $(1,1)$-form. Straightforward computation yields

$$\left. \frac{d}{dt} \right|_{t=0} \Lambda_{\omega_t} \alpha = -(\alpha, \sqrt{-1}\partial\bar{\partial}\phi)_{\omega}.$$
Note that, if $\omega$ is an $\alpha$-twisted cscK metric, i.e. satisfies $S(\omega) - \Lambda_\omega \alpha = \text{const}$, we thus have
\[
\frac{d}{dt} \bigg|_{t=0} (S(\omega_t) - \Lambda_\omega \alpha) = -\Delta_\omega \omega + (\alpha, \sqrt{-1}\partial\bar{\partial}\phi) + (\partial(\Lambda_\omega \alpha), \bar{\partial}\phi),
\]
and hence it seems natural to make the following definition.

**Definition 2.1.** Given two Kähler metrics $\omega$ and $\alpha$, we define an operator $F_{\omega, \alpha} : C^\infty(X, \mathbb{R}) \rightarrow C^\infty(X, \mathbb{R})$ by
\[
F_{\omega, \alpha}(\phi) := (\alpha, \sqrt{-1}\partial\bar{\partial}\phi) + (\partial(\Lambda_\omega \alpha), \bar{\partial}\phi).
\]

**Lemma 2.2.** $F_{\omega, \alpha}$ is a complex self-adjoint second order elliptic linear operator which satisfies
\[
\int_X \psi F_{\omega, \alpha}(\phi) \frac{\omega^n}{n!} = -\int_X (\xi_\psi, \xi_\phi) \frac{\omega^n}{n!},
\]
where $\xi_\phi := (\partial\phi)^{1,\omega}$ (resp. $\xi_\psi := (\partial\psi)^{1,\omega}$) is the $\omega$-metric dual of $\partial\phi$ (resp. $\partial\psi$). In particular, $\ker F_{\omega, \alpha}$ is the set of constant functions.

**Proof.** It is immediate that $F_{\omega, \alpha}$ is a second order elliptic linear operator, since $\alpha$ is strictly positive. By recalling some well-known identities (see e.g. [24] Lemma 4.7)), we compute
\[
\int_X \psi F_{\omega, \alpha}(\phi) \frac{\omega^n}{n!} = \int_X \psi(\Lambda_\omega \alpha \Delta_\omega \phi) \frac{\omega^n}{n!} - \int_X \psi \alpha \wedge \sqrt{-1} \partial \bar{\partial} \phi \wedge \frac{\omega^{n-2}}{(n-2)!} + \int_X \psi \sqrt{-1} \partial(\Lambda_\omega \alpha) \Lambda \phi \wedge \frac{\omega^{n-1}}{(n-1)!}.
\]
Note that, integrating by parts, we have
\[
\int_X \psi(\Lambda_\omega \alpha \Delta_\omega \phi) \frac{\omega^n}{n!} = -\int_X (\Lambda_\omega \alpha) \sqrt{-1} \partial \psi \wedge \bar{\partial} \phi \wedge \frac{\omega^{n-1}}{(n-1)!} - \int_X \psi \sqrt{-1} \partial(\Lambda_\omega \alpha) \Lambda \phi \wedge \frac{\omega^{n-1}}{(n-1)!},
\]
and
\[
-\int_X \psi \alpha \wedge \sqrt{-1} \partial \bar{\partial} \phi \wedge \frac{\omega^{n-2}}{(n-2)!} = \int_X \sqrt{-1} \partial \psi \wedge \alpha \Lambda \phi \wedge \frac{\omega^{n-2}}{(n-2)!},
\]
since $\alpha$ is closed. Thus
\[
\int_X \psi F_{\omega, \alpha}(\phi) \frac{\omega^n}{n!} = \int_X (\Lambda_\omega \alpha) \sqrt{-1} \partial \psi \wedge \bar{\partial} \phi \Lambda \phi \wedge \frac{\omega^{n-1}}{(n-1)!} + \int_X \sqrt{-1} \partial \psi \wedge \alpha \Lambda \phi \wedge \frac{\omega^{n-2}}{(n-2)!} = \int_X (\Lambda_\omega \alpha) \Lambda \phi \sqrt{-1} \partial \psi \wedge \bar{\partial} \phi \wedge \frac{\omega^n}{n!} = \int_X \sqrt{-1} \partial \psi \wedge \alpha \Lambda \phi \Lambda \phi \wedge \frac{\omega^{n-2}}{(n-2)!} = \int (\xi_\psi, \xi_\phi) \frac{\omega^n}{n!},
\]
where we wrote $\xi_\phi := (\partial\phi)^{1,\omega}$ (resp. $\xi_\psi := (\partial\psi)^{1,\omega}$) for the $\omega$-metric dual of $\partial\phi$ (resp. $\partial\psi$). We thus get
\[
\int_X \psi F_{\omega, \alpha}(\phi) \frac{\omega^n}{n!} = \int_X \tilde{\phi} F_{\omega, \alpha}(\psi) \frac{\omega^n}{n!}
\]
and hence $F_{\omega, \alpha}$ is (complex) self-adjoint. We also see that

$$\int_X \phi F_{\omega, \alpha}(\phi) \frac{\omega^n}{n!} = - \int_X \|\phi\|_2^2 \frac{\omega^n}{n!} < 0$$

for every non-constant function $\phi$, since $\alpha$ is positive definite. Thus $F_{\omega, \alpha}(\phi) = 0$ if and only if $\phi$ is constant.

3 Proof of Theorem 1.2

We follow the line of argument similar to the one in the paper of Fine [13] or LeBrun–Simanca [16]: we construct approximate solutions to the $R\alpha$-twisted cscK equation (3.1), and then apply the Banach space inverse function theorem to get the genuine solution (3.2).

3.1 Construction of approximate solutions

We start with the following observation.

Proposition 3.1. Let $\omega$ and $\alpha$ be Kähler metrics satisfying $\Lambda_{\omega, \alpha} = \text{const}$, and let $G(\omega)$ be the solution to $\Delta_{\omega} G(\omega) = S(\omega) - \bar{S}$. Then $\omega$ is $R\alpha'$-twisted cscK if we define $\alpha' := \alpha + \sqrt{-1} \partial \bar{\partial} G(\omega)/R$, which is strictly positive if $R > 0$ is chosen to be sufficiently large.

Thus, almost by tautology, we get an $R\alpha'$-twisted cscK metric for $\alpha' \in [\alpha]$ that is different from $\alpha$ by order $1/R$. Our aim in what follows is to “improve” this observation “order by order”, so that we get an $R\alpha_m$-twisted cscK metric for $\alpha_m \in [\alpha]$ that is different from $\alpha$ by order $1/Rm$, say.

Suppose $\Lambda_{\omega, \alpha} = \text{const}$. Then we have the trivial

$$S(\omega) - R \Lambda_{\omega, \alpha} = \text{const} + (S(\omega) - \bar{S}).$$

Now consider $\omega_1 := \omega + \sqrt{-1} \partial \bar{\partial} \phi_1 / R$. Then, expanding in $1/R$, we get

$$S(\omega_1) - R \Lambda_{\omega_1, \alpha} = \text{const} + (S(\omega) - \bar{S}) + (\alpha, \sqrt{-1} \partial \bar{\partial} \phi_1) + O(1/R).$$

We now wish to choose $\phi_1 \in C^\infty(X, \mathbb{R})$ so that $(S(\omega) - \bar{S}) + (\alpha, \sqrt{-1} \partial \bar{\partial} \phi_1) \omega$ becomes constant. This can be achieved by the following lemma.

Lemma 3.2. Suppose $\Lambda_{\omega, \alpha} = \text{const}$ and that the average of $f \in C^\infty(X, \mathbb{R})$ over $X$ with respect to $\omega$ is 0. Then there exists $\phi \in C^\infty(X, \mathbb{R})$ such that $(\alpha, \sqrt{-1} \partial \bar{\partial} \phi) \omega = f$.

Proof. First of all, $\Lambda_{\omega, \alpha} = \text{const}$ implies $(\alpha, \sqrt{-1} \partial \bar{\partial} \phi) \omega = F_{\omega, \alpha}(\phi)$. Note also that (cf. [24] Lemma 4.7])

$$\int_X F_{\omega, \alpha}(\phi) \frac{\omega^n}{n!} = \int_X (\alpha, \sqrt{-1} \partial \bar{\partial} \phi) \frac{\omega^n}{n!} = \int_X \Lambda_{\omega, \alpha} \Delta_{\omega} \phi \frac{\omega^n}{n!} = \int_X \alpha \wedge \sqrt{-1} \partial \bar{\partial} \phi \wedge \frac{\omega^{n-2}}{(n-2)!} = 0$$

since $\Lambda_{\omega, \alpha} = \text{const}$ and $\alpha$ is closed. This means that in order for the equation $(\alpha, \sqrt{-1} \partial \bar{\partial} \phi) \omega = f$ to hold, it is necessary that the average of $f$ is zero.

Suppose that we write $C^\infty(X, \mathbb{R})_0$ for the set of smooth functions whose average (with respect to $\omega$) is zero. We get the claimed result if the operator $F_{\omega, \alpha} : C^\infty(X, \mathbb{R})_0 \to C^\infty(X, \mathbb{R})_0$ is surjective, assuming
\( \Lambda_{\omega,\alpha} = \text{const} \). We pass to the Sobolev completion \( L^2_p \) of \( C^\infty(X,\mathbb{R})_0 \). Since the operator \( F_{\omega,\alpha} : L^2_p \to L^2_{p-2} \) is elliptic and \( X \) is compact without boundary, it is Fredholm. Lemma 2.2 shows that \( F_{\omega,\alpha} \) is self-adjoint and that the kernel of \( F_{\omega,\alpha} \) is trivial, and hence by the Fredholm alternative we conclude that \( F_{\omega,\alpha} : L^2_p \to L^2_{p-2} \) is surjective.

In other words, for every \( f \in L^2_{p-2} \) there exists \( \phi \in L^2_p \) such that \( F_{\omega,\alpha}(\phi) = f \). However, since \( F_{\omega,\alpha} \) is elliptic, \( \phi \) must be smooth if \( f \) is smooth by the elliptic regularity. This establishes the claim stated in the lemma.

\[
\Box
\]

We can repeat the above procedure to get the following result.

**Lemma 3.3.** Suppose \( \Lambda_{\omega,\alpha} = \text{const} \). Then, for each \( m \in \mathbb{N} \) there exist \( \phi_1, \ldots, \phi_m \in C^\infty(X,\mathbb{R}) \) such that

\[
\omega_m := \omega + \sqrt{-1} \partial \bar{\partial}(R^{-1} \phi_1 + \cdots + R^{-m} \phi_m)
\]

satisfies

\[
S(\omega_m) - R\Lambda_{\omega_m,\alpha} = \text{const} + R^{-m} f_{m,R}
\]

for a function \( f_{m,R} \) with average 0 (with respect to \( \omega_m \)) which is bounded in \( C^\infty(X,\mathbb{R}) \) for all sufficiently large \( R \).

**Proof.** We simply expand \( S(\omega_m) - R\Lambda_{\omega_m,\alpha} \) at \( \omega \) to get

\[
S(\omega_m) - R\Lambda_{\omega_m,\alpha} = \text{const} + (S(\omega) - \bar{S}) + (\alpha, \sqrt{-1} \partial \bar{\partial}\phi_1)\omega \\
+ \sum_{i=1}^{m-1} \frac{1}{R^i} ((\alpha, \sqrt{-1} \partial \bar{\partial}\phi_{i+1})\omega + B_i) + O(R^{-m}),
\]

where each \( B_i \) is a smooth function with average 0 (with respect to \( \omega \)) which depends only on \( \phi_1, \ldots, \phi_i \). Thus, repeated application of Lemma 3.2 establishes the claimed result.

Let \( G_{m,R} \) be the solution to \( \Delta_{\omega_m} G_{m,R} = f_{m,R} \). By the standard elliptic PDE theory (cf. [26]), we see that there exists a constant \( C(\omega_m, p) \) depending on \( \omega_m \) and \( p \in \mathbb{N} \) such that the \( L^2_p \)-Sobolev norm of \( G_m \) can be estimated as

\[
||G_{m,R}||_{p,\omega_m} \leq C(\omega_m, p)||f_{m,R}||_{p-2,\omega_m},
\]

where the \( L^2_p \)-Sobolev norm \( || \cdot ||_{p,\omega_m} \) is defined with respect to \( \omega_m \) (note that we may choose \( p \) to be a sufficiently large integer). On the other hand, we have \( \omega - \omega_m = O(R^{-1}) \), and hence we have

\[
||G_{m,R}||_{p,\omega} \leq C(\omega, p)||f_{m,R}||_{p-2,\omega}
\]

for a constant \( C(\omega, p) > 0 \) which depends only on \( \omega \) and \( p \). Since \( ||f_{m,R}||_{p-2,\omega} \) can be bounded by a constant uniformly of \( R \), we finally see that \( ||G_{m,R}||_{p,\omega} \) can be bounded by a constant uniformly of \( R \), for each \( p \in \mathbb{N} \).

We now define

\[
\alpha_m := \alpha + \frac{\sqrt{-1}}{R^m} \partial \bar{\partial} G_{m,R}(\omega_m),
\]

and observe that it satisfies the equation

\[
S(\omega_m) - R\Lambda_{\omega_m,\alpha_m} = \text{const},
\]

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to the Kähler metric $(1_{\omega})$ unique subject to the constraint

Then whenever $y$ is a differentiable map whose derivative at $0$, $DT|_0$, is an isomorphism of Banach spaces, with the inverse $P$;

2. $\delta'$ is the radius of the closed ball in $B_1$, centred at 0, on which $T-\overline{DT}$ is Lipschitz, with constant $1/(2||P||_{});$

3. $\delta := \delta'/2||P||_{}$.

Then whenever $y \in B_2$ satisfies $||y-T(0)|| < \delta$, there exists $x \in U$ with $T(x) = y$. Moreover, such an $x$ is unique subject to the constraint $||x|| < \delta'$.

Suppose that we write, as before, $L^2_p$ for the Sobolev completion of $C^\infty(X, \mathbb{R})$ and $\Omega^{1,1}$ for the set of $(1,1)$-forms on $X$ completed by the $L^2_p$-Sobolev norm $|| \cdot ||_p$. All Sobolev norms are defined with respect to the Kähler metric $\omega$, and we shall take $p > 0$ to be sufficiently large. We now take $B_1 := \Omega^{1,1} \times L^2_p$, $U := \{(\epsilon, \phi) \in B_1 \mid \omega_m + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$, $B_2 := \Omega^{1,1} \times L^2_p$ in Theorem 3.4 and define

$$T(\epsilon, \phi) := (\alpha_m + \epsilon, S(\omega_m, \phi) - R\Lambda_{\omega_m, \phi}(\alpha_m + \epsilon))$$

where $\omega_m, \phi := \omega_m + \sqrt{-1}\partial\bar{\partial}\phi$; this means that $0 \in B_1$ is identified with $(\alpha_m, \omega_m)$. Since $T$ depends on $R$, we shall write $T_R$ for $T$ in what follows.

**Notation 3.5.** For notational convenience, we shall write $\Lambda_m$ for $\Lambda_{\omega_m}$, $\mathcal{D}_m$ for $\mathcal{D}_{\omega_m}$, and $F_m$ for $F_{\omega_m, \alpha_m}$ in what follows.

Since $\omega_m$ is $R\alpha_m$-twisted cscK, the equation [3] implies that we have

$$DT_R|_0(\epsilon, \phi) = \begin{pmatrix} 1 & 0 \\ \Lambda_m & -\mathcal{D}_m^* \mathcal{D}_m + RF_m \end{pmatrix} \begin{pmatrix} \epsilon \\ \phi \end{pmatrix}.$$  \hspace{1cm} (7)

**Lemma 2.** implies

$$\int_X \phi(-\mathcal{D}_m^* \mathcal{D}_m \phi + F_m(\phi))\omega_m^n/m! = -\int_X ||\bar{\partial}\omega_m\phi||^2 \omega_m^n/m! - \int_X ||\xi||^2 \omega_m^n/m!.$$  \hspace{1cm} (5)

When we consider the elliptic estimates of the operator $-\mathcal{D}_m^* \mathcal{D}_m + RF_m$ it is natural to use the norm defined by $\omega_m$, but $\omega - \omega_m = O(R^{-1})$ allows us to absorb the difference in the constant in the estimate.
and hence the kernel of $-\mathcal{D}_m^* \mathcal{D}_m + RF_m : L^2_{p+4} \to L^2_p$ must be zero. Since this operator is elliptic and $X$ is compact without boundary, the Fredholm alternative implies that $-\mathcal{D}_m^* \mathcal{D}_m + RF_m$ is surjective (cf. Lemma 3.2). Thus $DT_R|_0 : B_1 \to B_2$ is an isomorphism whose inverse $P = P_R$ is given by

$$P_R(\epsilon, \phi) = \begin{pmatrix} 1 & 0 \\ -(-\mathcal{D}_m^* \mathcal{D}_m + RF_m)^{-1} \Lambda_m & (-\mathcal{D}_m^* \mathcal{D}_m + RF_m)^{-1} \end{pmatrix} \begin{pmatrix} \epsilon \\ \phi \end{pmatrix}.$$ 

**Remark 3.6.** We recall that, in fact, the kernel of the linearisation of $\phi \mapsto S(\omega_\phi) - \Lambda_\omega, \alpha$ is trivial if $\omega$ is only assumed to be sufficiently close to an $\alpha$-twisted cscK metric [6, Lemma 4.3].

To evaluate the operator norm $||P_R||_{op}$ of the inverse $P_R$, it suffices to evaluate $||(\mathcal{D}_m^* \mathcal{D}_m + RF_m)^{-1}||_{op}$. Recalling the proof of the fundamental elliptic estimate (e.g. [20] Theorem 11.1 in Chapter 5)), we see that there exists a constant $C_1 = C_1(\alpha, \omega, p)$ independent of $R$ such that

$$||(\mathcal{D}_m^* \mathcal{D}_m + RF_m)^{-1}||_{p+4} \leq RC_1 (||\phi||_p + ||(-\mathcal{D}_m^* \mathcal{D}_m + RF_m)^{-1}||_{L^2} \phi),$$

by also recalling $\omega_m - \omega = O(R^{-1})$ and $\alpha_m - \alpha = O(R^{-m})$.

We also have the following lemma.

**Lemma 3.7.** Let $\lambda_{1,m} < 0$ be the largest non-zero eigenvalue of $-\mathcal{D}_m^* \mathcal{D}_m + RF_m$. Then there exists a constant $C_2 = C_2(\alpha, \omega) > 0$ such that $\lambda_{1,m} < -C_2R$ for all large enough $R$.

**Proof.** Let $\phi_{1,m}$ be an eigenfunction corresponding to $\lambda_{1,m}$. Then, by Lemma 2.2 we have

$$\lambda_{1,m} = \frac{1}{||\phi_{1,m}||_{L^2, \omega_m}^2} \int_X \phi_{1,m} (-\mathcal{D}_m^* \mathcal{D}_m \phi_{1,m} + RF_m(\phi_{1,m})) \frac{\omega_m^n}{n!}$$

$$= \frac{1}{||\phi_{1,m}||_{L^2, \omega_m}^2} \left( - \int_X ||\bar{\partial} \xi_{\phi_{1,m}}||_{\omega_m}^2 \omega_m^n \frac{n!}{n!} - R \int_X ||\xi_{\phi_{1,m}}||_{\alpha, \omega}^2 \omega_m^n \frac{n!}{n!} \right)$$

$$\leq \frac{R}{||\phi_{1,m}||_{L^2, \omega_m}^2} \left( - \int_X ||\bar{\partial} \xi_{\phi_{1,m}}||_{\omega_m}^2 \omega_m^n \frac{n!}{n!} \right).$$

Now observe that there exists a constant $C_3 = C_3(\alpha, \omega, m) > 0$ such that

$$C_3 \int_X ||\bar{\partial} \phi_{1,m}||_{\omega_m}^2 \omega_m^n \frac{n!}{n!} < \int_X ||\xi_{\phi_{1,m}}||_{\alpha, \omega}^2 \omega_m^n \frac{n!}{n!}.$$

recalling that $\xi_{\phi_{1,m}}$ is the $\omega_m$-metric dual of $\partial \phi_{1,m}$. Since $\omega_m - \omega = O(R^{-1})$ and $\alpha_m - \alpha = O(R^{-m})$, we see that there exists a constant $C_4 = C_4(\alpha, \omega) > 0$ such that $C_4 \geq C_5(\alpha, \omega, m)$ uniformly of all large enough $R$.

We thus get

$$\lambda_{1,m} < -\frac{RC_4}{||\phi_{1,m}||_{L^2, \omega_m}^2} \int_X ||\bar{\partial} \phi_{1,m}||_{\omega_m}^2 \omega_m^n \frac{n!}{n!}.$$

Now the Poincaré inequality yields

$$||\phi_{1,m}||_{L^2, \omega_m}^2 < C_5 \int_X ||\xi_{\phi_{1,m}}||_{\omega_m}^2 \omega_m^n \frac{n!}{n!}$$

for a constant $C_5 = C_5(\omega, m) > 0$. As before, $\omega_m - \omega = O(R^{-1})$ implies that we can bound $C_5(\omega, m)$ by another positive constant $C_6 = C_6(\omega)$ uniformly of $R$. Thus we finally get

$$\lambda_{1,m} < -\frac{C_4}{2C_6} R$$

as claimed.
Thus, again using \( \omega_m - \omega = O(R^{-1}) \), we have
\[
\| P_R \|_{\text{op}} = \sup_{\epsilon, \phi} \frac{\| \epsilon \|_p + \| - (\mathcal{D}_m^*D_m + RF_m)^{-1} \Lambda_m \epsilon + (\mathcal{D}_m^*D_m + RF_m)^{-1} \phi \|_p + \| \phi \|_p}{\| \epsilon \|_p + \| \phi \|_p} \\
\leq \sup_{\epsilon, \phi} \frac{\| \epsilon \|_p + \| (\mathcal{D}_m^*D_m + RF_m)^{-1} \Lambda_m \epsilon \|_{p+4} + \| (\mathcal{D}_m^*D_m + RF_m)^{-1} \phi \|_{p+4}}{\| \epsilon \|_p + \| \phi \|_p} \\
< 1 + RC' (1 + |\lambda_{1,m}|^{-1}) \sup_{\epsilon, \phi} \frac{\| \Lambda_m \epsilon \|_p + \| \phi \|_p}{\| \epsilon \|_p + \| \phi \|_p} \\
< 1 + 2RC \left( 1 + \frac{1}{C_2 R} \right) \left( 1 + \| \Lambda_\omega \|_{\text{op}} \right),
\]
and hence there exists a constant \( C' = C'(\alpha, \omega, p) > 0 \) such that \( \| P_R \|_{\text{op}} \leq C'R \).

Recalling the definition \( \delta' \) of \( T_R \), we see that for \( l \geq 3 \), \( l \in \mathbb{N} \), and on a ball centred at \( 0 \in B_1 \) with radius \( \delta' := R^{-l} \), the operator \( T_R - DT_R \) is Lipschitz with constant \( 1/(2\| P_R \|_{\text{op}}) \) for all large enough \( R \).

Thus we can choose
\[
\delta = \frac{\delta'}{2\| P_R \|_{\text{op}}} > \frac{1}{2C'} R^{-l-1},
\]
so that the quantitative inverse function theorem holds in the ball of radius \( \delta = O(R^{-l-1}) \) in \( B_2 \) centred at \( T_R(0,0) \).

Writing \( \bar{S} \) for the average of the scalar curvature and \( c \) for the average of \( \Lambda_m \alpha_m \) \( \text{cf. Remark 1.7} \), we observe \( T_R(0,0) = (\alpha_m, \bar{S} - Rc) \). Since \( \alpha_m - \alpha = O(R^{-m}) \), we see that there exists a constant \( C'' = C''(\alpha, \omega, p) \) such that
\[
\| T_R(0,0) - (\alpha, \bar{S} - Rc) \|_{L^2} < C'' R^{-m+1}
\]
for all large enough \( R > 0 \), and note that we have
\[
C'' R^{-m+1} < \frac{1}{2C'} R^{-l-1} < \delta
\]
for all large enough \( R > 0 \), by taking \( m \) to be sufficiently large. Thus, for all large enough \( R > 0 \), there exists \( (\epsilon, \phi) \in U \subset B_1 \) such that \( T_R(\epsilon, \phi) = (\alpha, \bar{S} - Rc) \); in other words we have
\[
\begin{cases}
\alpha_m + \epsilon = \alpha \\
S(\omega_m, \phi) - RA_{\omega_m, \phi}(\alpha_m + \epsilon) = \bar{S} - Rc = \text{const}
\end{cases}
\]
for some \( (\epsilon, \phi) \in \Omega^{1,1} \times L^2_{p+4} \) (note also that we have \( \omega_m, \phi > 0 \) for all large enough \( R > 0 \), since \( \| \phi \|_{p+4} < \delta' = R^{-l} \)). By taking \( p \) to be sufficiently large and recalling the Sobolev embedding, we can use the elliptic regularity, as in \( \text{[13] Lemma 2.3} \), to conclude that \( \phi \) is in \( C^\infty(X, \mathbb{R}) \). This establishes all the statements claimed in Theorem \( \text{[1.2]} \).

### 4 Proof of Corollary \( \text{[1.4]} \)

We now apply the argument in \( \text{[3.2]} \) to prove Corollary \( \text{[1.4]} \). Suppose that we have an \( \alpha' \)-twisted cscK metric \( \omega \). Using the notation from \( \text{[3.2]} \) we define an operator \( T' : U \to B_2 \) by
\[
T'(\epsilon, \phi) = (\alpha' + \epsilon, S(\omega) - A_{\omega}(\alpha' + \epsilon)).
\]
It suffices to show that the linearisation $DT'|_0$ of $T'$ at $0 \in B_1$ is invertible. Since $\omega$ is $\alpha'$-twisted cscK, we can prove the invertibility of $DT'|_0$ by arguing exactly as we did in §3.2 to show that the operator (7) is invertible. Thus Theorem 3.4 applied for a sufficiently large $p$, immediately implies Corollary 1.4 by recalling the Sobolev embedding and the elliptic regularity (cf. [13, Lemma 2.3]).

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