Determining All Maximum Uniquely Restricted Matching in Bipartite Graphs

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Abstract

The approach mapping from a matching of bipartite graphs to digraphs has been successfully used for forcing set problem, in this paper, it is extended to uniquely restricted matching problem. We show to determine a uniquely restricted matching in a bipartite graph is equivalent to recognition an acyclic digraph. Based on these results, it proves that determine the bipartite graphs with all maximum matching are uniquely restricted is polynomial time. This answers an open question of Levit and Mandrescu (Discrete Applied Mathematics 132(2004) 163–164).

Key words:
Bipartite graph, Directed cycle, BD-mapping, Uniquely restricted matching

1 Introduction

Let $G = (X, Y; E)$ be a bipartite graph, a set of edges $M \subset (X, Y)$ is a matching if no two edges of $M$ share a common vertex. A matching $M$ is uniquely restricted if its saturated vertices induce a subgraph which has a unique perfect matching and denotes as $M_{ur}$. A subset edges $S \subset M$ is a forcing set for a matching $M$ if $S$ is in no other perfect matching of $G$. Let us denote the subgraph induced by the edges of $M$ (also known as the saturated vertices) as $G[M]$, and name all of the vertices not saturated by $M$ as free vertex set $V_f$.

Maximum matching problems are well known problem in graph theory and are proved to be solved in polynomial time[2]. But many restricted maximum matching problems are NP-complete, for example, Finding the maximum $M_{ur}$ is NP-complete in bipartite graphs[3], the smallest forcing set problem is also NP-complete in cubic bipartite graphs[1].
On the positive side, it is proved that the determine a matching is uniquely restricted in bipartite graph could be recognized in $O(M+E)$ [3]. There exists a polynomial time algorithm to determine the $M_{ur}$ if $G$ is unicycle graph[5]. And [4] also shown that unique perfect matching bipartite graph could be find in polynomial time. At the end of [4], they raised an open problem as follows:

**Problem.** how to recognize that all maximum matching in a bipartite graph are uniquely restricted?

In this paper we will answer this question in two steps. Firstly, it shows a mapping from bipartite graph to digraph, and then it gives a necessary and sufficient condition on a uniquely restricted matching in bipartite graphs is equivalent to the acyclic digraph. Secondly, it proves to determine all the maximum matching uniquely restricted or not is equivalent to find no more than two path between two vertices. In addition, it shows that uniquely perfect matching in bipartite graphs is as simple as recognize the an acyclic digraph.

2 Illustration the main technology

The main technonlogy in this paper have a successful implementation on finding forcing set problem in [1]. But firstly, let us repeat the theorem in [3].

**Theorem 1** [3] A matching $M$ of a graph is uniquely restricted if and only if $M$ is alternating cycle-free.

Then let us define a mapping from $G[M]$ of a bipartite graph to a digraph $D$ and named as $BD$-mapping in this paper, this mapping is much more clearly than the definition on page 292 of [6] and definition on page 3 of [1] which denotes by $D(G, M)$.

**Definition 2** Given a matching $M$ of bipartite graph $G(X, Y; E)$, a $BD$-mapping digraph $D(V, A)$ of $G$ defines as follows

$$V = \{x|(x, y) \in M\}.$$

$$A = \{< x_1, x_2 > | (x_1, x'_1) \in M \Wedge (x_2, x'_2) \in M \Wedge (x_1, x'_2) \in E - M\}.$$

It is easy to observe that follows theorem could be equivalent to the theorem[1]

**Theorem 3** Let $D$ be BD-mapping digraph of a matching $M$ in bipartite graph with $n > 2$ vertices, $M$ is uniquely matching in $G$ if and only if $D(G, M)$ is acyclic.
PROOF. Suppose that $D$ is acyclic digraph, every vertex $x$ on $D$ could be divided into a pair of vertex $<x,x'>$ and become a new directed graph $D'$, which is also a acyclic digraph and without alternative cycle. Moreover, there has a matching $M$ include number of $|D|$ edges. Therefore $M$ is a uniquely restricted matching.

On the other hand, assume $M$ is a uniquely restricted matching but the $BD$-mapping $D(G,M)$ include at least one cycle $C$, where $C = \{<x_1,x_2>,<x_2,x_3>\ldots<x_k,x_1>\}$ when $k \geq 3$. Then according to equation 2 of definition 2 there exists $(x_1,x'_1),(x_2,x'_2),(x_1,x'_2)$ and $(x_2,x'_1)$ are in $E$, or $(x_1,x'_1),(x_2,x'_2),\ldots,(x_k,x'_k) \in M$, and $M' = (x_1,x'_1),\ldots,(x_k,x'_1) \in E - M$. Therefore, $M \cup M'$ has a even cycle, this contradict to the theorem.

Remark 4 Theorem 3 is very similar the proposition 3 in [1].

Proposition 5 [1] Let $G$ be a bipartite graph, $M$ is a perfect matching of $G$, and $S \subset M$ is a forcing set of $M$ if and only if $D(G,M) \setminus S$ is an acyclic digraph.

In fact, Theorem 3 can deduce the known results that

Corollary 6 The bipartite graph $G$ with uniquely perfect matching $M$ has a forcing matching number 0.

PROOF. Since $D(G,M)$ with uniquely perfect matching is acyclic graph, the $S$ in proposition 5 is empty set.

Let us give an example to show a bipartite graph $G$ and the $D(G,M)$ in Fig.1 to end of this section.

3 The complexity of uniquley restricted perfect matching

[4] has proved that if and only all of local maximum stable set is a greedoid, then a bipartite graph has a unique perfect matching. However, how to recognize all of local maximum stable set are greediod is equivalent to the problem of all maximum matching are uniquely restricted according to the theorem 3.3 in [4]. This section would give a more efficient algorithm to determine the unique perfect matching.

It is easy to obervious to obtain the following theorem:
Theorem 7  A matching $M$ of a bipartite graph is perfect uniquely restricted if and only if $D(G, M)$ is acyclic and $|D(G, M)| = \frac{n}{2}$.

Based on theorem 7 an algorithm shows in Algorithm 1.

It is clearly to see that the example in Fig.2 is a uniquely restricted perfect matching. Now let us consider the Fig.1 again. It is clearly that $G$ in Fig.1 is contains a unique maximum matching \{(b, b'), (c, c'), (d, d'), (a', a)\}. Notice that $G$ also contains a maximum matching \{(x, b'), (b, a'), (c, c'), (d, d')\} is not uniquely restricted. Thus the theorem is only necessary condition for all maximum matching are uniquely restricted.
Algorithm 1. Determine unique perfect matching

**Input**: A bipartite graph $G(X, Y; E)$ and a matching $M \in E$;
**Output**: unique perfect matching if it has, otherwise return non unique perfect matching.

begin
(1). Generate a BD-mapping graph $D(V; A) = f(G[M])$.
(2) if $M$ is not perfect then
(3) return non unique perfect matching
(4) else
(5) if $D$ is acyclic then return unique perfect matching
(6) else return non unique perfect matching
endif
end

4 Determine all uniquely restricted maximum matching in polynomial time

In first glance, greedy algorithms can apply into determining all uniquely restricted maximum matching by remove the node with degree 1. Unfortunately, the worst case could be exponent.

For example, let consider the bipartite graph in Fig.3. It is need to remove 3 edges $e1, e2, e3$, then any maximum matching is not uniquely restricted. But for another example in Fig.4 It only remove edges $e2$, then any maximum matching is not uniquely restricted. Therefore, greedy algorithms on removing vertex with degree 1 could not efficiently to determining all maximum matching are uniquely restricted.

However, a uniquely restricted maximum matching will include vertex with degree 1, let us define a matching $M$ is a greedy matching if the free vertex set $V_f$ of $M$ do not include vertex with degree or all edges of $G[M]$ saturate a vertex degree 1. Then let us define an extend BD-mapping digraph, which consist of the free vertex set $V_f$ and a BD-mapping digraph $D$, where $M$ is greedy matching.

**Definition 8** An extend BD-mapping digraph of bipartitie graphs $G(V, E)$ with a greedy matching $M$ is follows. $V = V_f \cup V_1$ and $A = (A_1 \cup A_f)$, where $D(V_1, A_1)$ is a RZ-mapping digraph and $A_f$ is a pair of arcs between $(v_i, v_j) \in G(v_i \in V_f)$, and all of $v_i \in M$ if $d(v_i) = 1$

An example of extend BD-mapping is shown in Fig.5. Now let we give a necessary and sufficient condition of all maximum matching are uniquely restricted.
Theorem 9 Let $D$ response to the extends BD-mapping digraph of a bipartite graph $G$ with uniquely restricted maximum matching and $V_t$ and $V_s$ are the set of terminal nodes or start nodes in $D(G, M, V_f)$, when satisfies following one of three conditions

**c1.** all of $v_i \in V_f$, there exists only one path from $v_i$ to $v_j \in V_t$.

**c2.** all of $v_i \in V_f$, there exits only one path from $v_j \in V_s$ to $v_i$.

**c3.** for any two $v_i, v_j \in V_f$, if there exists at most one $v_k \in D(G, M)$ have the path from $v_i$ to $v_k$ and $v_j$ to $v_k$, (or conversly, there exists at most one $v_k \in D(G, M)$ have the path from $v_k$ to $v_i$ and $v_k$ to $v_j$).

then all maximum matching of $G$ are uniquely restricted, where $V_f$ is set of free nodes of $D(G, M)$. 

Fig. 2. $G$ with uniquely prefect matching
Fig. 3. Not all maximum matching are uniquely restricted, even remove edge $e_1$, there exists $|M_u r| = 3$ and $|M| = 3$.

Fig. 4. Any matching of $G(V, E - \{e_2\}$ is not uniquely restricted

PROOF.

c1. If a free node $v_i \in V_f$ have more than two disjoint paths $p_1, p_2$ to $v_j$, then it implies that two consecutive edges $(v_1, v_i), (v_i, v_2)$ not belong to $M$ and in $P_1 \cup P_2$.

since terminal node $v_j$ in $D$ is always respect to the vertex with degree 1. There exits two disjoint path to $v_j$, which implies that two consecutive edges $(v_3, v_j), (v_j, v_4)$ belongs o $M$ and also in $P_1 \cup P_2$. Therefor there exists at least 4 edges in a cycle and not in $M$, if we remove the degree 1 node $v_j$ from $G$, $M$ minus 1, but it can plus 1 by extends the $(v_1, v_i), (v_i, v_2)$ and $(v_3, v_j), (v_j, v_4)$.

c2. The same principle for the start node $v_j$ if $v_j$ is the source node of $D(G, M)$.

c3. Let us prove it by constraction, suppose there exits a node $v_i \in V_f$ have the path $P_1$ from $v_i$ to $v_{t1}$ and $P_2$ from $v_i$ to $v_{t2}$, also there exits a node $v_j \in V_f$ have the path $P_3$ from $v_j$ to $v_{t1}$ and $P_4$ from $v_j$ to $v_{t2}$. Then there exits a path from $v_i$ to $v_j$ is length of $|P_2| + |P_4|(|P_1| + |P_3|)$ respectively, but the nodes in the $D(G, M)$ is $|P_2| + |P_4| - 2$ ($|P_1| + |P_3| - 2$ respectively), therefor, there exists a cycle in length of $2 \times (|P_1| + |P_2| + P_3) + |P_4| - 2$, which have a matching length of $|P_1| + |P_2| + |P_3| + |P_4| - 2$. 
According to the theorem, it is easy to design a deterministic all maximum matching restricted or not in polynomial algorithm since deterministic free node \(v_t\) to the terminal node \(v_t\) or start node \(v_s\) have more than two disjoint path is clearly in polynomial time. The Algorithm 2 shows the algorithms for determine all maximum matching restricted.

5 Discussion

The mapping from a matching of bipartite graph to digraph had been successful solve forcing matching problem in bipartite graph of [1][6]. This paper extends it and use to solve the uniquely restricted maximum matching problem. According to the theorem, the open question appear in [4] to recognize the all uniquely restricted maximum matching bipartite graphs is solved in polynomial time.
Algorithm 2. Determine all maximum matching uniquely restricted

**Input**: A bipartite graph $G(X,Y;E)$ with a greedy matching $M \in E$;

**Output**: return true if all maximum matching are uniquely restricted, otherwise return false.

**begin**

(1) Generate a BD-mapping graph $D(V;A) = f(G[M])$.

(2) if $M$ is not perfect then

(3) return non unique perfect matching

(4) else

(5) if $D$ is acyclic then return unique perfect matching

(6) else return non unique perfect matching

**endif**

**end**

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