On the Evidence for Cosmic Variation of the Fine Structure Constant (II): A Semi-Parametric Bayesian Model Selection Analysis of the Quasar Dataset

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Submitted to MNRAS: xx July 2013.

ABSTRACT
In the second paper of this series we extend our Bayesian reanalysis of the evidence for a cosmic variation of the fine structure constant to the semi-parametric modelling regime. By adopting a mixture of Dirichlet processes prior for the unexplained errors in each instrumental subgroup of the benchmark quasar dataset we go some way towards freeing our model selection procedure from the apparent subjectivity of a fixed distributional form. Despite the infinite-dimensional domain of the error hierarchy so constructed we are able to demonstrate a recursive scheme for marginal likelihood estimation with prior-sensitivity analysis directly analogous to that presented in Paper I, thereby allowing the robustness of our posterior Bayes factors to hyper-parameter choice and model specification to be readily verified. In the course of this work we elucidate various similarities between unexplained error problems in the seemingly disparate fields of astronomy and clinical meta-analysis, and we highlight a number of sophisticated techniques for handling such problems made available by past research in the latter. It is our hope that the novel approach to semi-parametric model selection demonstrated herein may serve as a useful reference for others exploring this potentially difficult class of error model.

Key words: Cosmology: cosmological parameters – methods: data analysis – methods: statistical.

1 INTRODUCTION
Recent claims by [Webb et al. (2011)] and [King et al. (2012)] of cosmic variation in the fine structure constant, $\alpha$, appearing to the Earth-bound observer as a North-South dipole have generated considerable interest owing to the remarkable theoretical implications of such a result, if true. As acknowledged by the Webb et al. team though, the quasar dataset upon which these claims are based exhibits clear signs of an unexplained error term, $\varepsilon_{\text{sys}}$, manifest as a residual variance over and above the level of noise attributable to known sources of observational uncertainty; the presence of which confounds the statistical interpretation of any alleged spatial trends. Importantly, a skeptical reader might well disagree with the assumptions of a strictly unbiased and Normally-distributed $\varepsilon_{\text{sys}}$ used by the Webb et al. team to derive the quoted “4$\sigma$ significance” of their dipole hypothesis.

In Paper I (Cameron and Pettitt 2013a) we highlighted the appearance of multimodality and/or heavy-tailedness in the distributions of residuals about the best-fit parameterization of the proposed dipole model and explored a variety of parametric forms for the unexplained error term in a Bayesian model selection (BMS) framework. Our study revealed weak support for a scenario in which $\alpha$ is truly constant but the Keck and VLT observational subgroups of the quasar dataset are subject to non-zero mean/mode noise of opposing sign. Moreover, despite the “Ockham’s Razor”-like tendency of BMS to favour simpler models our Bayes factor comparison in Paper I also gave support to a proposed skew Normal parent distribution for the $\varepsilon_{\text{sys}}$ term. In the interests of thoroughness we herein refine our modelling approach (and thereby relax further our minimal assumptions) to admit an “infinitely flexible”, non-parametric representation of these unexplained uncertainties. The mathematical basis of our new approach is the mixture of Dirichlet processes (DP).

Introduced by Ferguson (1973), the Dirichlet process (DP) represents an infinite-dimensional extension of the ordinary Dirichlet distribution; the importance of which for applied statistics lies in the fact that its topological support
includes a weak approximation to every probability distribution on a given domain. This property of the DP—and its hierarchical extension, the mixture of Dirichlet processes (MDP) prior \cite{Antoniak1974}—has allowed the construction of novel solutions to a wide variety of challenging inference problems featuring some element of non-parametric density estimation, most notably within the mixture modelling framework \cite{Escobar1995MacEachern1994Mueller1998Neal2000}. A number of recent astronomical applications have evinced the potential of DP- and MDP-based analyses in this field as well: e.g. for the classification of gamma ray bursts according to their observed spectral slopes \cite{Chattopadhyay2007}, the identification of variability in irregularly-spaced time series \cite{Shin2009}, and the estimation of galactic potential shapes from detailed kinematical data \cite{Magorrian2013}.

While there exist a number of computationally efficient, publically available packages for sampling from the posteriors of basic DP and MDP models\footnote{E.g. the \texttt{bnpma} \cite{Burr2012} and \texttt{DPpackage} \cite{Jara2007} add-on libraries in R.} the challenge lies in adapting these packages for the solution of real-world, bespoke analysis problems, especially in the BMS context. In this study we explicitly demonstrate one such adaptation procedure, in which we highlight the utility of the recursive approach to marginal likelihood estimation under an MDP prior; to this end we make particular use of the Radon-Nikodym derivative for the DP derived by \cite{Doss2012}. It is our hope that the novel approach to semi-parametric model selection exhibited herein may serve as a useful reference for other astronomical studies exploring this potentially difficult class of error model.

The structure of this paper is as follows. In Section 2 we briefly review key properties of the DP, before formulating our precise hierarchical model for the quasar dataset and specifying priors for its controlling parameters. In Section 3 we elucidate key similarities between the unexplained error problems faced in astronomical and clinical meta-analysis studies with the aim of facilitating cross-disciplinary learning in this regard. In Section 4 we describe a procedure for marginal likelihood estimation under our semi-parametric model, and in Section 5 we detail a corresponding importance sample reweighting scheme for prior-sensitivity analysis. In Section 6 we present the results of this BMS methodology applied to the quasar dataset and discuss their interpretation in light of our earlier parametric modelling. Finally, in Section 7 we summarize our conclusions.

2 DIRICHLET PROCESS MIXTURE MODEL

2.1 The Dirichlet Process

The DP \cite{Ferguson1973} defines a stochastic process of random probability measures on the atomic elements in the \( \sigma \)-algebra, \( \Sigma_0 \), of a sample space, \( \Omega \). As such the DP may be seen as an infinite-dimensional extension of the finite-dimensional Dirichlet distribution (familiar to Bayesians as the conjugate prior for a multinomial likelihood function). Under its defining parameter pairing of concentration index, \( M > 0 \), and (normalised) centering distribution, \( G \), on \( \langle \Sigma_0, \Omega \rangle \) the DP (as a random measure, \( P \)) may be characterised by the following property: that for every (measurable) partition, \( \{ B_1, \ldots, B_k \} \), of \( \Omega \) (and for all \( k = 1, 2, \ldots \)) the distribution of \( \{ P(B_1), \ldots, P(B_k) \} \) is (finite-dimensional) Dirichlet with parameter vector, \( \{ M \times G(B_1), \ldots, M \times G(B_k) \} \). Proof that the above is sufficient to define a unique stochastic process follows ultimately from Kolmogorov’s existence theorem \cite{Ferguson1973}.

Key properties of the DP include: (i) that, although strictly atomic, its topological support is in fact the complete set of probability measures on \( \langle \Sigma_0, \Omega \rangle \) absolutely continuous with respect to (and with supports enclosed by that of) the centering distribution, \( G \); (ii) realisations of (i.e., draws of probability distributions, \( P \), from) the DP may be simulated via a “stick-breaking” \cite{Sethuraman1994} construction, while its marginal output (i.e., iid draws from \( P \) marginalised over \( \text{DP}(M,G) \)) may be simulated via a Polya urn \cite{Blackwell1973} scheme; (iii) owing to the discreteness of \( P \sim \text{DP}(M,G) \), individual values drawn from any one such \( P \) will eventually repeat themselves (almost surely); (iv) the expectation of \( P(A) \) with respect to the DP with centering distribution, \( G \), is simply \( G(A) \); and (v) the expected mean and variance of \( P \sim \text{DP}(M,G) \) are \( E(G) \) and \( \frac{M}{M+1} \text{Var}(G) \). Proofs of (i), (ii), (iv), and (v) are given by \cite{Ferguson1973} (and those for (iii) are as cited above); in addition, \cite{Ghosh2010} offers a handy survey of these and other relevant insights.

2.2 Hierarchical Dirichlet Process Models

The “infinite flexibility” of the DP, described by property (i) above, has led to its widespread use as a non-parametric modelling device for Bayesian inference \cite{Ferguson1983}. A particularly well-studied construction of this sort is represented by the following hierarchical model (cf. \cite{Lo1984} \cite{Escobar1995} \cite{Burr2005}):

\[
\begin{align*}
& y_i | \mu_i, x_i \sim f(\mu_i, x_i), \quad i = 1, \ldots, n \\
& \mu_1, \ldots, \mu_n | P \sim P. \\
& P | M, \psi \sim \text{DP}(M, G_\psi). \\
& M, \psi \sim F.
\end{align*}
\]

That is, each observed datapoint, \( y_i \), for \( i = 1, \ldots, n \), is assumed drawn from a parametric distribution with density, \( f(\mu_i, x_i) \), defined by two controlling parameters, \( x_i \) and \( \mu_i \), with the former known exactly and the latter a hidden sample from a hidden realisation, \( P \), of the Dirichlet process, \( \text{DP}(M, G_\psi) \). For generality, the concentration index, \( M \), of this DP, and the controlling parameters, \( \psi \), of its centering distribution, \( G_\psi \), are also shown here as random elements, drawn from some (parametric) density, \( F \). Such a model can be described as semi-parametric (e.g. \cite{Basu2003}) since it combines a non-parametric form for the \( \mu_i \) with a parametric form for each \( y_i | \mu_i, x_i \). Often \( f(\mu_i, x_i) \) will be chosen as the univariate Normal with mean, \( \mu_i \), and standard deviation, \( x_i \); in which case the above may also be referred to as a non-parametric “random effects” model (especially in the clinical meta-analysis setting).

In the spirit of our earlier parametric modelling of the
fine-structure dataset (Paper I) we adopt here for the total error term, $\varepsilon_{\text{tot}}$, operating on the $\Delta \alpha / \alpha$ estimates of the Webb et al. team’s quasar dataset the following specific version of this classic semi-parametric model:

$$\varepsilon_{\text{tot},i} | \mu_{\text{sys},i}, \sigma_{\text{ran},i} \sim \mathcal{N}(\mu_{\text{sys},i}, \sigma_{\text{ran},i}^2), \ i = 1, \ldots, n_g \tag{5}$$

$$\mu_{\text{sys},1}, \ldots, \mu_{\text{sys},n_g} | P \sim P \tag{6}$$

$$P | M, \sigma_{\text{sys}} \sim \text{DP}(M, \mathcal{N}(0, \sigma_{\text{sys}}^2)) \tag{7}$$

$$\sigma_{\text{sys}} | M \sim \mathcal{N}_{\text{half}}(0, \sigma_p^2 \times \frac{M + 1}{M}) \tag{8}$$

$$M \sim \Gamma(\gamma_1, \gamma_2). \tag{9}$$

Here $\mathcal{N}(\mu, \sigma^2)$ denotes the Normal with mean, $\mu$, and standard deviation, $\sigma$, and $\Gamma(\gamma_1, \gamma_2)$ denotes the Gamma distribution with shape, $\gamma_1$, and rate, $\gamma_2$. Important to note is that we allow a unique $M$, $\sigma_{\text{sys}}^2$, $P$, and $\{\mu_{\text{sys},1}, \ldots, \mu_{\text{sys},n_g}\}$ for each of the three instrumental subgroups (Keck LC, Keck HC, and VLT) of the quasar dataset; hence, the $n_g$ in the above refers to the number of absorbers detected in each subgroup (i.e., $n_g = 113$, $27$, and $153$, respectively) rather than the total sample size ($293$).

As described in Paper I, the use of a Normal form for the explained uncertainty term appearing in the top layer of the above hierarchy follows naturally from the known details of the Webb et al. team’s $\Delta \alpha / \alpha$ fitting procedure (King et al. 2010). Our choice of a Normal centering distribution for the Dirichlet process on the third layer is, in contrast, purely a matter of computational convenience, allowing for efficient Gibbs sampling from the resulting MDP posterior given conjugate hyperpriors on $\sigma_{\text{sys}}$ and $M$ (as we discuss further below). The sensitivity of the resulting Bayes factors to this particular modelling decision can, however, be readily explored via the importance sample reweighting scheme described in Section 2.4 and indeed we present the results of such an analysis, supposing instead a Student’s $t$ centering distribution for our DP, in Section 2.5. It is also worth observing here that although our chosen centering distribution is strictly zero mean (and mode) the same does not follow for any particular draw, $P$, from this DP. In the terminology of Paper I this is a biased error model. As mentioned above, given appropriate hyperpriors on $\sigma_{\text{sys}}$ and $M$, the posterior of such a semi-parametric error model can be efficiently explored via a Gibbs sampling scheme. The mathematical and algorithmic details of this procedure are already well-described by Escobar and West (1995) and Neal (2000) so we omit these from the present account; although we return to the topic of Gibbs sampling again in Section 2.4. The conjugate priors required in this case are independent Gamma densities on $M$ and $1/\sigma_{\text{sys}}^2$; the former we have already in Equation 2, but the latter we do not. In fact, our prior on $\sigma_{\text{sys}}$ is defined as a half Normal conditional on $M$, chosen so as to achieve a degree of consistency with our prior on the equivalent component (namely, the standard deviation) of the parametric unexplained error terms considered in Paper I. The conditioning on $M$ is designed to match $\sigma_{\text{sys}}^2$ with the expected variance of $P$ as per Property (V) of the DP given in Section 2.1.

In order to reconcile this non-conjugate prior choice with the power of efficient Gibbs sampling under conjugate priors we use the following trick. First, we replace our nominal prior on $\sigma_{\text{sys}} | M$ with the conjugate prior form on $1/\sigma_{\text{sys}}^2$—namely, $1/\sigma_{\text{sys}}^2 \sim \Gamma(\gamma_3, \gamma_4)$—with $\gamma_3$ and $\gamma_4$ chosen (by eye) for similarity with our true prior (marginalized over $M$). After exploring the resulting posterior (and its tempered bridging densities) from this conjugate prior (Section 2.4) and estimating marginal likelihoods accordingly (Section 3), we finally use importance sample reweighting (Section 3) to recover the true posteriors and marginal likelihoods under our true (nominal) prior. For reference, our default hyperparameter choices are $\gamma_1 = 1.5$, $\gamma_2 = 0.05$, and $\sigma_p = 2 \times 10^{-5}$, for which we find $\gamma_3 = 0.4$ and $\gamma_4 = 0.04$ to give a reasonably well-matched conjugate form.

### 2.3 Hypotheses

To specify a complete generative model for the quasar dataset we couple the above semi-parametric error model to one of three (parametric) hypotheses for cosmic $\alpha$ variation: (i) the null hypothesis (that the fine structure constant is everywhere exactly equal); (ii) the monopole hypothesis (that of a fixed cosmic offset relative to its laboratory value); and (iii) the monopole+$r(z)$-dipole hypothesis (that of a large-scale cosmic spatial trend). In particular, we adopt the following form for this dipole after King et al. (2012):

$$\Delta \alpha / \alpha_{\text{mod} | x, \theta_m} = m + B \times r(z_i) \cos(\phi) [\text{ra}_i, \text{dec}_i, \text{ra}_d, \text{dec}_d] \tag{10}$$

with $x_i = [\text{ra}_i, \text{dec}_i, r(z_i)]$ a vector of explanatory variables for the $i$th absorber, $\theta_m = \{m, B, \text{ra}_d, \text{dec}_d\}$ a vector of input model parameters, and $\cos(\phi)[\cdot]$ a function returning the cosine of angular separation between the observational sightline and dipole vector. Formulae for $\cos(\phi)[\cdot]$ and $r(z)$ are given in Paper I. In this notation the monopole and strict null hypotheses become simply $\Delta \alpha / \alpha_{\text{mod} | x, \theta_m} = m$ and $\Delta \alpha / \alpha_{\text{mod}} = 0$, respectively. As in Paper I we suppose the following priors on the parameters of these hypotheses:

$$m \sim \mathcal{N}(0, [0.5 \times 10^{-5}]^2) \tag{11}$$

$$B \sim \text{Exp}(1/[0.5 \times 10^{-5}]) \tag{12}$$

$$\text{ra}_d \sim U(0, 24) \tag{13}$$

$$\sin \text{dec}_d \sim U(-1, 1). \tag{14}$$

Finally, we note that the resulting likelihood function conditional on the collection of hidden $\{\mu_{\text{sys}}\} = \{(\mu_{\text{sys}})_{\text{lc}}, (\mu_{\text{sys}})_{\text{hc}}, (\mu_{\text{sys}})_{\text{vlt}}\}$ from each instrumental group takes the simple form:

$$L(y | \theta_m, \{\mu_{\text{sys}}\}, \{x\}) = \prod_{i=1}^{293} f_N(\mu_{\text{sys},i}, \sigma_{\text{ran},i}^2 | (\Delta \alpha / \alpha_i - \Delta \alpha / \alpha_{\text{mod} | x, \theta_m})). \tag{15}$$

### 2.4 Gibbs Sampling from the Full Posterior

As mentioned earlier, particular aspects of our hierarchal model were chosen to facilitate Gibbs sampling of the error model posterior via the Pólya urn technique popularized by Escobar and West (1995) and others. The Gibbs sampler (cf. Casella and George 1992) is a commonly used procedure for Markov Chain Monte Carlo (MCMC) exploration of multi-variate posteriors by way of simulation from only some lower-order conditionals; the popularity of which has surged in recent years in concert with the popularity of Bayesian hierarchical modelling. Past astronomical applications of the Gibbs sampler include procedures for supernova light-curve fitting (Mandel et al. 2009) and spectral analysis in the low-count limit (van Dyk et al. 2001).
To explore the joint posterior of a given hypothesis plus semi-parametric error model pairing we adopt a two-part Gibbs sampling approach in which the hypothesis parameters are updated conditional on the current error model parameters, and then these current error model parameters are updated conditional upon the new hypothesis parameters. That is, we draw \( \theta^{(i)} \sim \pi(\theta^{(i)} | \{\mu_{\text{sys}}^{(1)}\}) \) and \( \{\mu_{\text{sys}}\}^{(i+1)} \sim \pi(\{\mu_{\text{sys}}\} | \theta^{(i)}) \). For the former we run a simple random walk MCMC chain for 100 moves (to approximate stationarity) under the conditional likelihood function given by Equation 15, and for the latter we run the \( \text{DPMeta} \) routine of the \text{DPpackage} in \( R \) (Jara 2007) for 1000 moves from the current state for each of the three instrumental subgroups of the quasar dataset (separately). Note that the second stage here implicitly updates the current \( M \) and \( \sigma_{\text{sys}} \) for each error group as well, and the \( \text{DPMeta} \) routine itself performs the Gibbs sampling scheme of Escobar and West (1995) under our conjugate prior proxy (cf. Section 2.2).

An important observation for the purposes of our subsequent marginal likelihood estimation is that the tempered likelihood posterior may also be readily explored via the above algorithm. In particular, to simulate from the tempered posterior at temperature, \( \beta \), we need simply replace the \( \sigma_i \) in Equations 5 and 13 with \( \sigma_i / \sqrt{\beta} \), which gives the tempered likelihood up to a constant of proportionality (the value of which is irrelevant for the present application). Before detailing the recursive scheme used to this end in Section 3 we first review a number of similarities between the hierarchical, semi-parametric model outlined here and the equivalent random effects model from clinical meta-analysis in Section 3.2 below. The purpose of this digression is to better qualify the place of our model in a wider applied statistics context and perhaps therefore to encourage some beneficial cross-disciplinary learning in this regard.

### 3 UNEXPLAINED ERROR PROBLEMS IN ASTRONOMY AND CLINICAL META-ANALYSIS

Unexplained error problems, typically invoked a posteriori when the residuals about some well-trusted regression model are found to far exceed the explained variance of the measurement process, seem to arise quite regularly in the practise of astronomical data analysis. For instance, the Sunyaev-Zel’dovich effect-based cluster dataset compiled by Galili (2013) for probing cosmic \( \alpha \) variation exhibits an unexplained error term just as problematic at that for the quasar dataset studied here. Despite this, the statistical treatment of such unexplained errors in astronomical studies remains rather ad-hoc, and, against warnings (Andrae 2010), many astronomers still favour the naive approach of simply re-scaling their explained uncertainty estimates to enforce a reduced-\( \chi^2 \) of one. In contrast, the techniques now considered routine for handling the equivalent random effects problem in the field of clinical meta-analysis, in which the published treatment effect estimates from multiple trials of the same intervention are combined for enhanced statistical power, are generally far more sophisticated.

In a standard meta-analysis each study contributes its own unique estimates of both the targetted treatment effect, \( \phi_i \), and the inherent variance, \( \sigma_i^2 \), of this measurement according to its explained (or “within-study”) error term. The distributional form of the latter may often be assumed Normal from theoretical considerations of the known sampling design and inference procedure (e.g. maximum-likelihood); and the further simplifying assumption of \( \sigma_i^2 = \sigma^2 \) (i.e., treating the estimated variance as if it were the known truth) yields the first level of a hierarchical model for each observed datapoint, \( \phi_i \sim N(\phi, \sigma^2) \). (Note that in effect we have made the same assumption implicitly in our own error model of Section 2.2.) The unexplained (or “between-study”) error term over the complete dataset may then be modelled on a second level under the strong assumption of a known distributional form with unknown variance, \( \Sigma^2 \), and unknown centering parameter, \( \mu \); the latter defining the reference effect being targeted in the meta-analysis paradigm.

Under a heuristic invocation of the Central Limit Theorem, and perhaps ultimately for computational convenience, a Normal distribution is typically supposed for this purpose. In this case the resulting (parametric) hierarchical model may be written as follows:

\[
\phi_i \sim N(\phi, \sigma_i^2) \tag{16}
\]

\[
\phi_i \sim N(\mu, \Sigma^2) \tag{17}
\]

Upon the specification of appropriate priors for \( \mu \) and \( \Sigma \) a computational exploration of the corresponding posterior may be easily conducted, allowing prediction of the observed treatment effect(s) from future studies (cf. DerSimonian and Laird 1986, Higgins, Thompson and Spiegelhalter 2009).

A desire to limit the potential for subjective bias arising from the assumption of a particular distributional form for the unexplained error term in this context has led to the development of various semi-parametric meta-analysis schemes. Burr and Doss (2005) provide a popular formulation almost identical in structure to the hierarchical error model used here for the quasar dataset; the key difference being that a conditional Dirichlet process with explicitly controlled median is used in their study in place of the ordinary Dirichlet process to clarify the interpretation of inference on \( \mu \). Code for posterior simulation from the Burr and Doss (2005) model is available in the \text{bspmma} package for \( R \). Although coded under the assumption of a fixed concentration index for the conditional Dirichlet, the posterior for a series of \text{bspmma} runs over a suitable range of \( \lambda \) can ultimately be combined to represent sampling from a prior on this parameter via the recursive technique, as per Doss (2012) (which we will discuss in more detail later).

For meta-analysis problems in which the studies compiled into the dataset divide naturally amongst distinct error groups—just like the case of the Webb et al. team’s quasar dataset considered here—Müller, Quintana and Rosner (2004) outline a novel scheme for enhanced learning of the error distribution across groups. As a compromise between the archetypal modelling extremes of completely non-interacting error groups (i.e., each takes its own \( M \) and \( G_v \)) and strongly interacting error groups (i.e., all share a common \( M \) and \( G_v \)) the Müller, Quintana and Rosner (2004) model retains the structure of the former but adds a new shared error layer inducing a dependence similar to the latter. The utility of this technique for powering up inference in the small sample size regime (\( \sim 50 \) samples per group) is illustrated by Müller, Quintana and Rosner (2004) in application to a challenging cancer research dataset. Although
we prefer the simplicity of a non-interacting model for our present analysis (in which we also have somewhat larger groups) it is worth noting that an efficient Gibbs sampling algorithm for posterior exploration under the shared layer error model is also available in the DPpackage for R.

4 RECURSIVE MARGINAL LIKELIHOOD ESTIMATION

Our aim here, as in Paper I, is to evaluate the relative strengths of the Webb et al. team’s three hypotheses (Section 2.2) for the spatial variation of a (or lack thereof) in accordance with the BMS paradigm. Although estimation of the required Bayes factor (the ratio of marginal likelihoods under two competing hypotheses) is, for fully parametric models, a very well-studied field of statistical theory (see, e.g., Friel and Wyse 2012 for a recent review) the same is not true for the semi-parametric regime. The additional perceived difficulty in this case being the infinite-dimensional structure of the prior domain, which cannot be characterized as a probability density with respect to the ordinary Lebesgue measure. In fact, to-date only Basu and Chib (2003) have explicitly considered this problem in depth; these authors developing an extension of the popular Chib marginal likelihood estimator (Chib 1995) suitable for application to semi-parametric Dirichlet process problems in clinical meta-analysis in which the normalized conditional posterior, \( \pi(\theta_m|y, \{\phi\}) \), is analytically tractable.

Here we show how the recursive estimator resulting from the biased sampling theory of Vardi (1985) (see also Gill et al. 1988 and Kong et al. 2003) gives a more general means of marginal likelihood estimation for such semi-parametric problems, requiring only the availability of a collection of conditional likelihoods (Equation 15) drawn in known proportions, \( n_i/n, \ldots, n_m/n \) with \( n = \sum_{i=1}^{m} n_i \), from a sequence of tempered posteriors at known temperatures, \( \beta_1, \ldots, \beta_m \), with \( \beta_1 = 0 \) (the prior; \( Z_1 = 1 \)) and \( \beta_m = 1 \) (the full posterior; \( Z = Z_m \)). The first step is to observe that the prior for our hierarchical model defines a proper probability measure, \( P \), on the Borel sets of the metric space, \( S \), corresponding to the product of the domains of \( \{\mu_{sys}\}, \theta_m \), and \( T \) —with \( T = \{\tau_1, \tau_2, \tau_3\} \) explicitly representing the collection of \( \tau = \{M, q_{sys}\} \) pairs for each of our three error groups. We then observe that the conditional likelihood function defines a continuous mapping from \( S \) to another metric space, \( S' \), being simply the real line. Hence, by the transformation of variables theorem we know that \( P \) induces a proper probability measure, \( P' \), on \( S' \) with \( Z_1 = \int_0^1 dP'(t) = 1 \). The expectation under this new measure is simply the marginal likelihood itself, \( Z = \int_{\tau} dP'(\tau) \), and likewise for the normalizations of the tempered posteriors, \( Z_0 = \int_0^1 dP'(t) \). That is, despite the complexities of our prior the continuous mapping provided by the likelihood function returns us to the canonical biased sampling framework.

Although we do not know \( P' \) the non-parametric maximum likelihood estimator given by Vardi (1985) nevertheless allows for unbiased estimation of the associated \( Z_k \) (\( k \neq 1 \)). (Important to note though is that only the updated convergence analysis given by Gill et al. 1988 confirms the validity and well-behavedness of this scheme for non-Lebesgue densities and indeed for general sample spaces.) The recursive relation so defined takes the same form as that for reverse logistic regression under tempered posterior exploration given in Paper I. Namely,

\[
\hat{Z}_k = \sum_{i=1}^{n} \left( \frac{L_i^{\beta_k} / [\sum_{j=1}^{m} n_j L_j^{\beta_k} / \hat{Z}_j] }{L_i^{\beta_k} / [\sum_{j=1}^{m} n_j L_j^{\beta_k} / \hat{Z}_j] } \right)
\]  

(18)

Importantly, iteration over this system of equations yields a globally convergent solution, easily recovered computationally. Of course, as noted in Section 2.2 our posterior exploration is actually performed under a conjugate prior proxy for our true (non-conjugate) priors, meaning that an additional importance sample reweighting step is now needed to modulate the above, as we describe in detail below.

5 IMPORTANCE SAMPLE REWEIGHTING VIA THE RADON-NIKODYM DERIVATIVE

As emphasised in Paper I (and also in Cameron and Pettitt 2013a), one of the strengths of the combination of tempered posterior exploration with the recursive pathway to marginal likelihood estimation is that it greatly facilitates the process of prior-sensitivity analysis. With the posterior Bayes factor characteristically sensitive to prior specification this step should be considered a routine part of all BMS studies (Kass and Raftery 1995), although here it is also necessary for recovery of our true posteriors from the conjugate prior proxies adopted for efficient Gibbs sampling of our semi-parametric error term (Section 2.2).

However, unlike in Paper I for which our priors all admitted probability densities with respect to Lebesgue measure, the DP component here clearly does not—as Doss (2012) observes, there is a non-zero probability that some of the \( \{\mu_{sys}\} \) will be identical. For this reason the importance sample reweighting formula needs to be written in terms of the Radon-Nikodym derivative with respect to our original prior measure, which we will again denote, \( P \). That is, if we consider the pool of our tempered posterior draws of \( \{\theta_m, \{\mu_{sys}\}, T\} \) triples as belonging to the (pseudo-)importance sampling proposal with probability measure,

\[
H(\theta_m, \{\mu_{sys}\}, T) = \left( \frac{\prod_{i=1}^{m} n_i / n L_i^{\beta_1} / \hat{Z}_i}{n \prod_{i=1}^{m} L_i^{\beta_1} / \hat{Z}_i} \right) P(\theta_m, \{\mu_{sys}\}, T).
\]  

(19)

then given the Radon-Nikodym derivative of the alternative prior, \( P_{alt} \), with respect to this proposal, \( \frac{dP_{alt}}{dH}(\theta_m, \{\mu_{sys}\}, T) \), we recover the importance sample reweighting estimator of the marginal likelihood under \( P_{alt} \),

\[
\hat{Z}_{alt} = \frac{1}{n} \sum_{i=1}^{n} L_i \frac{dP_{alt}}{dH}(\theta_m, \{\mu_{sys}\}, T_i).
\]  

(20)

Importantly, the Radon-Nikodym derivative for the standard MDP prior model of Equations 5 to 9—which we shall denote, \( \frac{dP_{MDP}}{dH}(\{\mu_{sys}\}_g, \tau_g) \)—has already been derived by Doss (2012). The exact form of this expression is not reproduced here, but may be seen in his Equation 2.2. Extending this formulation over the three error groups of the quasar dataset plus the space of hypothesis priors is trivial given their simple product space structure, i.e.,

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Figure 1. Illustration of our posterior densities for the $\sigma_{\text{sys}}$ parameter controlling the standard deviation of the Normal centering distribution for the DP in our semi-parametric error model. In each panel we compare the resulting posterior under each of the null, monopole, and dipole hypotheses for the spatial variation (or lack thereof) in $\alpha$; and the ordering (as labelled) of the three error groups in these panels from left to right is Keck LC, Keck HC, and VLT. Consistent with the Webb et al. team’s original (non-Bayesian) analysis, our $\sigma_{\text{sys}}$ posteriors indicate a far greater degree of unexplained variance in the Keck HC error group than in the Keck LC or VLT error groups.

An interesting connection between the recursive method for marginal likelihood estimation and the importance sample reweighting scheme presented here appears by way of the reference to Doss (2012). In this earlier work (and see also Burr and Doss 2005) the same recursive estimator was invoked to facilitate the computation of relative Bayes factors under fixed $M$ semi-parametric models, and against entirely parametric models; although it seems that the final step of absolute marginal likelihood estimation for competing hypotheses tied to this model (as above) was not considered.

$$
\frac{dP_{\text{alt}}}{dH}(\theta_m, \{\mu_{\text{sys}}\}, T) = \frac{\pi_{\text{alt}}(\theta_m)}{\pi(\theta_m)} \prod_{i=1}^{3} \frac{du_{\text{alt}}}{du_i}(\{\mu_{\text{sys}}\}_{(i)}, \tau_{(i)}). (21)
$$

6 RESULTS AND DISCUSSION

6.1 Posteriors

Although only the marginal likelihoods (Section 6.2 of each hypothesis for spatial variation (or lack thereof) in the fine structure constant are required for our BMS analysis the posterior parameter distributions explored here offer an important guide to the nature of the semi-parametric error term. In Figures 1 and 2 we compare the posteriors for $\sigma_{\text{sys}}$ (the standard deviation of our Normal centering distribution for the DP) and $M$ (its concentration parameter) in each of the three instrumental subgroups of the quasar dataset (Keck low/high contrast, and VLT) under each of the null, monopole, and dipole hypotheses. Consistent with the Webb et al. team’s original (non-Bayesian) analysis our $\sigma_{\text{sys}}$ posteriors indicate a far greater degree of unexplained variance in the Keck HC error group than in the Keck LC or VLT error groups. While from inspection of the $M$ posteriors we see...
that the Keck HC and VLT unexplained errors are somewhat closer to Normal than those for the Keck LC error group; the preference for low $M$ values here suggests that a fixed offset or bias would almost be sufficient to explain the residuals in this group.

In Figure 3 we illustrate the corresponding posterior distributions of the pooled $\{\mu_{\text{sys}}\}$ for each error group, which follow naturally our expectations given the posteriors of $\sigma_{\text{sys}}$ and $M$. The similarity in the distribution of $\{\mu_{\text{sys}}\}$ between hypotheses (within each error group) is quite remarkable, however; the only exception being in the Keck LC error group where the dipole hypothesis contributes an effect indistinguishable from that of a simple bias term. Finally, if we examine the posteriors of the hypothesis parameters for the dipole shown in Figure 4 we can confirm a fair agreement with those recovered under the biased parametric models explored in Paper I.

### 6.2 Marginal Likelihoods

The (log) marginal likelihoods for each hypothesis stated in Section 2.3 under our semi-parametric error model (after recursive estimation and importance sample reweighting to convert from our conjugate prior proxies to our true priors; cf. Section 2.2) are as follows—null: $\log \hat{Z} = -617.2$; monopole: $\log \hat{Z} = -617.4$; and dipole: $\log \hat{Z} = -619.2$ (with uncertainties $\Delta \log \hat{Z} \lesssim 0.1$ estimated via the [sample-based] recursive asymptotic covariance matrix of Gill et al. 1988 and verified via repeat simulation). That is, we recover a posterior Bayes factor of $\approx 7.4$ in favour of the null over the dipole, which constitutes weak support for the former under a uniform prior weighting for each hypothesis; though, as we argue in Paper I, most cosmologists would in fact presumably assign much greater prior weight to the null, concluding that the existence of such a large-scale $\alpha$ dipole remains exceedingly unlikely. This Bayes factor ranking of hypotheses under our semi-parametric error model is reassuringly consistent with that reported for the biased parametric error models in Paper I; the novelty being that our
of varying degrees-of-freedom (df). Lowering the df from 100 (near Normal) to 1 (Cauchy) increases the Bayes factor in favour of the null.

Finally, it is worth noting that the marginal likelihoods recovered (for all hypotheses) under our Normal and skew Normal parametric error models in Paper I were all greater than those recovered here. Hence, although our semi-parametric model has indeed offered an important robustness check it has not performed better (in a BMS sense, under our stated priors) than our simplest parametric models given the present dataset. With further data though one might anticipate an increasing preference for the (potentially “more realistic”) non-parametric family over these strict (“idealised”) parametric templates.

6.3 Prior-Sensitivity Analysis

In Figure 5 we investigate the prior-sensitivity of the marginal likelihoods for each of these three hypotheses under our semi-parametric error model. In particular, we make use of the Radon-Nikodym derivative for the DP (Doss 2012) and our importance sample reweighting scheme (Section 5) to explore the effect of replacing the Normal centering distribution in our MDP model for the unexplained error term with a Student’s $t$ of varying degrees-of-freedom (df). Lowering the df from 100 (near Normal) to 1 (Cauchy; “fat-tailed”) increases the recovered Bayes factor in favour of the null, and raises its absolute marginal likelihood to a peak at a df of 4, preserving the original BMS rank-ordering of hypotheses under our nominal priors. The computational time required for the recomputation of these marginal likelihoods under the alternative priors specified here was (as expected) indeed far less than that required for the original parallel tempering scheme; confirming the efficiency of the importance sample reweighting procedure for such prior-sensitivity analyses.

7 CONCLUSIONS

In this extension to our earlier work on the Bayesian reanalysis of evidence for cosmic variation in the fine structure constant we have developed a sophisticated procedure for Bayesian model selection in the semi-parametric regime, allowing for efficient marginal likelihood estimation with prior-sensitivity analysis. Importantly, although the parameter space described by the MDP form of our hierarchical prior features an infinite-dimensional characterization we find that relatively trivial modifications to the recursive algorithms presented in Paper I can allow for a mode of computation directly analogous to the familiar case of strictly parametric model selection. Application of this methodology to the problem at hand yields confirmation of our earlier finding in favour of a null hypothesis for a biased error model for the instrumental subgroups of the Webb et al. team’s quasar dataset; though as before we do not anticipate the final resolution of this debate without further data (such as that anticipated from the ongoing VLT Large Program; 185.A-0745).

Finally, in a dedicated Section of this work we have elucidated a number of similarities between unexplained error problems in astronomy and clinical meta-analysis, which we hope may stimulate some cross-disciplinary learning in this regard.

ACKNOWLEDGMENTS

[1] E.C. and A.N.P. are grateful for financial support from the Australian Research Council.

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