GLOBAL EXISTENCE FOR THE UNSTABLE CAHN-HILLIARD EQUATION IN 2D WITH A SHEAR FLOW

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Abstract. In this paper, we consider the advective unstable Cahn-Hilliard equation in 2D with shear flow:

\[
\begin{align*}
    u_t + Av_1(y)\partial_x u + \varepsilon\Delta^2 u &= \Delta(au^3 + bu^2) & \text{on } T^2; \\
    u &\text{ periodic} & \text{on } \partial T^2;
\end{align*}
\]

with an initial data \( u_0 \in H_0^2(T^2) \), where \( T^2 \) is the two-dimensional torus, \( A, \varepsilon > 0, a < 0, b \in \mathbb{R} \). Under the assumption that the shear has a finite number of critical points and there are linearly growing modes only in the direction of the shear, we show the \( L^2 \)-energy of the solutions to such problems converges exponentially to zero, if in addition, both \( |a| \) and \( \|f_x u_0(x, \cdot)dx\|_{L^2_2} \) are sufficiently small.

Contents

1. Introduction 1
2. Preliminary: Local existence 5
3. Bootstrap assumptions 12
4. Uniform bounds of \( \langle u \rangle \) 16
5. Bootstrap estimates and Proof of Theorem 1.5 21
6. References 33

1. INTRODUCTION

In this paper, we are interested in the following initial boundary value problem of the non-linear Cahn-Hilliard equation with advection:

\[
\begin{align*}
    u_t + v \cdot \nabla u + \varepsilon\Delta^2 u &= \Delta(au^3 + bu^2 + cu) & \text{on } T^2; \\
    u &\text{ periodic} & \text{on } \partial T^2,
\end{align*}
\]

with initial data \( u(x, 0) = u_0(x) \in H^2(T^2) \). Here \( \varepsilon > 0, a, b, c \in \mathbb{R} \) and \( v \) is an incompressible flow on \( T^2 \). In the case when \( v = 0 \), (1.1) becomes the classical Cahn-Hilliard equations, which arise in the study of phase separation in cooling binary solutions such as alloys, glasses and polymer mixtures (see, e.g., \cite{3, 4}). It is well-known that if \( a > 0 \), then the solution to the problem (1.1) (no advection) spontaneously form domains separated by thin transition regions (see, e.g., \cite{7, 6, 18}); while if \( a < 0 \), then the solution in general blows up (see, e.g., \cite{7, 14}).
Therefore, a natural question that one can ask is the following: can either the phase separation or the blow-up be suppressed if we add an extra advection term?

In [10], the authors studied the global existence for the problem (1.1) with $a = 1$, $b = 0$ and $c = -1$ (which is a typical case for the stable case) under the assumption that the dissipation time of $v$ is sufficiently small. As a conclusion, they showed that the solution to (1.1) converges exponentially to the total concentration

\[ M := \int_{\mathbb{T}^2} u_0 dx dy = \int_{\mathbb{T}^2} u dx dy \]

(note that (1.1) conserves mean of the solution), and in particular, they showed that even in the situation when $\varepsilon > 0$ is sufficiently small, no phase separation occurs if the stirring velocity field is sufficiently mixing. Here the dissipation time of $v$ is defined as follows.

**Definition 1.1.** Consider the hyper-diffusion equation on $\mathbb{T}^2 \times (0, \infty)$.

\[ \theta_t + v \cdot \nabla \theta + \varepsilon \Delta^2 \theta = 0. \]

Let $S_{s,t}$ be the solution operator to (1.3), that is for any $f \in L^2(\mathbb{T}^2)$, the function $\theta(t) = S_{s,t} f$ solves (1.3) with initial data $\theta(s) = f$, and periodic boundary conditions. The dissipation time of $v$ is

\[ \tau_{\text{dis}} := \inf \left\{ t \geq 0 \left| \| S_{s,s+t} \|_{L^2_0 \to L^2_0} \leq \frac{1}{2} \right. \right. \text{ for all } s \geq 0 \}, \]

where $L^2_0(\mathbb{T}^2)$ is the collection of all square integrable functions on $\mathbb{T}^2$ with mean zero.

In this paper, we are interested in whether one can apply a similar idea in [10] to study the unstable case, in which, the constant $a$ in (1.1) is strictly negative, and it turns out the answer is affirmative if in addition, we assume $|a|$ is sufficiently small. Note that the major difference in this case is that the double well potential no longer has a global lower bound, and hence one has to handle the term $\Delta(u^3)$ properly.

Now we turn to some details. Let us consider the problem (1.1) with $a < 0$, $b \in \mathbb{R}$, $c = 0$ and when $v$ is given by a shear flow, which is of the form

\[ v = \left( \begin{array}{c} Av_1(y) \\ 0 \end{array} \right), \]

where $v_1 \in W^{1,\infty}(\mathbb{T})$ and $A > 0$ is its amplitude. Namely, we consider the problem

\[ \begin{cases} u_t + Av_1(y) \partial_x u + \varepsilon \Delta^2 u = \Delta (au^3 + bu^2) & \text{on } \mathbb{T}^2; \\ u \text{ periodic} & \text{on } \partial \mathbb{T}^2, \end{cases} \]

with $a < 0$, $b \in \mathbb{R}$ and initial data $u(x,0) = u_0(x) \in H^2_0(\mathbb{T}^2)$, which is the collection of all $H^2(\mathbb{T}^2)$-functions with mean zero. Our goal is to show the global existence of solutions to the problem (1.5) under certain proper assumption on $v$. The study of suppression of blow-up in nonlinear parabolic equations is a recent research topic, and we refer the interested [2, 5, 11, 16] and the references therein for more detail.
We first note that by a rescaling argument, we can rewrite (1.4) as

\[
\begin{cases}
  u_t + v_1(y) \partial_x u + \varepsilon \gamma \Delta^2 u = \gamma \Delta (au^3 + bu^2) & \text{on } \mathbb{T}^2; \\
  u & \text{periodic on } \partial \mathbb{T}^2,
\end{cases}
\]

where \( \gamma = \frac{1}{4} \). Next, we turn to the assumption on the flow \( v \), which is now \( v = \begin{pmatrix} v_1(y) \\ 0 \end{pmatrix} \) after rescaling. The key assumption here is that the linear operator

\[
H_{\varepsilon, \gamma} := \varepsilon \gamma \Delta^2 + v_1(y) \partial_x
\]

is dissipation enhancing (see, e.g., [8, 9, 13]) on the orthogonal component to the kernel of the transport operator \( v_1(y) \partial_x \). More precisely, for any \( g \in L^2(\mathbb{T}^2) \), we decompose it as

\[
(g)(t, y) := \int_{\mathbb{T}} g(t, x, y) \, dx \quad \text{and} \quad g_\perp(t, x, y) = g(t, x, y) - \langle g \rangle(t, y).
\]

It is easy to see that \( \langle g \rangle \in \text{Ker} \left( v_1(y) \partial_x \right) \) and \( g_\perp \in \left( \text{Ker} \left( v_1(y) \partial_x \right) \right)^\perp \).

**Definition 1.2.** We say the shear flow \( v = \begin{pmatrix} v_1(y) \\ 0 \end{pmatrix} \) is a horizontal polynomial mixing shear flow if there exists some global constant \( C_1 > 0 \) and \( m \geq 2 \), such that

\[
\left\| e^{-(v_1(y) \partial_x) t} g_\perp \right\|_{H^{-1}} \leq \frac{C_1}{(1 + t)^m} \| g_\perp \|_{H^1}, \quad t \geq 0,
\]

for any \( g \in L^2_0(\mathbb{T}^2) \).

**Remark 1.3.** The constant \( m \) in Definition 1.2 is closely related to the flow function \( v_1 \). For example, it has been shown in [1] that if \( u \) has a finite number of critical points of order at most \( m \geq 2 \), namely at most \( m - 1 \) derivatives vanish at the critical points, then there exists some constant \( C_1 > 0 \), such that the estimate (1.7) holds.

As a consequence of (1.7), we have the following result.

**Proposition 1.4.** [9, Theorem 2.1] For each \( \gamma > 0 \) and \( v = \begin{pmatrix} v_1(y) \\ 0 \end{pmatrix} \) be a horizontal polynomial mixing shear flow. Then for any \( g \in L^2(\mathbb{T}^2) \) with mean zero, one has for any \( t \geq 0 \),

\[
\left\| e^{-tH_{\varepsilon, \gamma}} g_\perp \right\|_{L^2} \leq 10e^{-\lambda_{\varepsilon} t} \| g_\perp \|_{L^2}, \quad \lambda_{\varepsilon} = \lambda_{\varepsilon, \gamma} := C_2 \varepsilon \gamma^{\frac{2}{2+m}},
\]

where \( C_2 > 0 \) is an absolute constant independent of the choice of \( \gamma \), and only depending on \( \varepsilon, C_1, m \) (which are defined in (1.7)) and any dimensional constants.

We are ready to state the main theorem of this paper.

**Theorem 1.5.** Let \( u_0 \in H^2_0(\mathbb{T}^2) \), \(|a|\) defined in [15] be sufficiently small and \( v = \begin{pmatrix} v_1(y) \\ 0 \end{pmatrix} \) be a horizontal polynomial mixing shear flow with parameters \( C_1 > 0 \) and \( m \geq 2 \). If

\[
\left\| \int_{\mathbb{T}} u_0(x, \cdot) \, dx \right\|_{L^2_y} \ll 1,
\]

then...
then there exists some $\gamma^* > 0$, which only depends on $\varepsilon, a, b, c$ and $\|u_0\|_{L^2}$, such that for any $0 < \gamma \leq \gamma^*$, there exists a global-in-time weak solution (or mild solution) of (1.5) with initial data $u_0$, such that $u \in L^\infty \left(0, \infty\right), L^2 \left(T^2\right)) \cap L^2 \left(0, \infty\right), H^2 \left(T^2\right))$. Moreover, $\|u(t)\|_{L^2}$ converges to 0 exponentially.

Remark 1.6. (1). In the sequel, we will prove Theorem 1.5 in a quantitative way, in particular, we will be precise on the size of $|a|$ and $\int_t^T \|u_0(x, \cdot)dx\|_{L^2}$.

(2). Theorem 1.5 shows that the $L^2$ blow-up of the unstable Cahn-Hilliard equation can be suppressed via a shear flow. More precisely, let us consider the unstable Cahn-Hilliard equation without advection, that is

\[
\begin{cases}
    u_t + \varepsilon \Delta^2 u = \Delta (au^3 + bu^2) & \text{on } T^2; \\
    u \text{ periodic} & \text{on } \partial T^2,
\end{cases}
\]

where $\varepsilon > 0, a < 0$ and $b \in \mathbb{R}$. It is well-known that if the Landau-Ginzburg free energy of the initial data is sufficiently negative, that is, if

\[-\int_{T^2} \left(H(u_0) + \frac{\gamma}{2} |\nabla u_0|^2\right) dxdy\]

is sufficiently large, where

\[
H(u) = \int_0^u (as^3 + bs^2) ds = \frac{au^4}{4} + \frac{bu^3}{3}
\]

is the double-well potential, then there exists a $T^* > 0$, such that

\[
\limsup_{t \to T^*} \|u(t)\|_{L^2} = \infty.
\]

(see, e.g., [7, Theorem 3.1]). Note that under the assumption of Theorem 1.5, it is possible for the Landau-Ginzburg free energy of the initial data to be very negative (for example, one can take an appropriate $b$ with $|b|$ being sufficiently large), and hence the solution of the non-adveective problem (1.9) will blow up in finite time; while our result shows that with an extra shear flow satisfying (1.7), the solution can exist globally.

The structure of this paper is as follows. Section 2 deals with the local existence to the problem (1.1). In Section 3 we establish several local estimates for the terms $\|u_{\#}(t)\|_{L^2}$ and $\gamma \int_t^T \|\Delta u_{\#}\|_{L^2}^2 d\tau$, and this allows us to make the bootstrap assumptions to the problem (1.5). As a consequence of the bootstrap assumptions, in Section 4, we prove uniform bounds for the term $\langle u \rangle$ with both $|a|$ and $\gamma$ being sufficiently small. Section 5 is devoted to prove the main theorem 1.5. We prove it via a bootstrap argument by showing the bootstrap assumptions can be improved.

Finally, in this paper, we will write $B$ and $C$ as some constants that might change line by line, where

(1). $B$ will only depend on $\varepsilon, b$ and any dimensional constants;
(2). $C$ will only depend on $\varepsilon, a, b, \|\langle u \rangle (0)\|_{L^2}, \|u_{\#} (0)\|_{L^2}$ and any dimensional constants.

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2. Preliminary: Local existence

In this section, we study the local existence of the solutions to the problem \( (1.1) \) with arbitrary \( H^2 \) initial data. Here, instead of proving a priori estimate (see, [7]), it is more convenient for us to consider the **mild solutions** of \( (1.1) \), which will play an important role in our later context when we consider the case when \( v \) is a shear flow.

The result in this section is standard. Nevertheless, we would like to make a remark that it is already known that there exists a global solution to the equation \( (1.1) \) if \( a > 0 \) (see, e.g., [7, Theorem 1.1] and [10, Proposition 2.1]), while the global existence is not guaranteed if \( a < 0 \) (see, [7]). To this end, we make a remark that the constant \( C \) used in this section is also allowed to depend on \( c \), as we do not require \( c = 0 \) in the current section.

We start with some basic setup. Given a function \( f \in L^1(\mathbb{T}^2) \), we denote \( \hat{f}(k) \) to be its Fourier coefficient of \( f \) at frequency \( k \in \mathbb{Z}^2 \), and hence \( \hat{f} := \{ \hat{f}(k) \}_{k \in \mathbb{Z}^2} \).

Then the **inhomogeneous Sobolev space** \( H^s(\mathbb{T}^2) \), \( s \in \mathbb{R} \) is defined to be the collection of all measure functions \( f \) on \( \mathbb{T}^2 \), with

\[
\| f \|_{H^s}^2 := \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^s |\hat{f}(k)|^2 = \| (I - \Delta)^{s/2} f \|_2^2 < \infty,
\]

while the **homogeneous Sobolev space** \( \dot{H}^s(\mathbb{T}^2) \), \( s \in \mathbb{R} \) consists of all measurable functions \( f \) on \( \mathbb{T}^2 \) with

\[
\| f \|_{\dot{H}^s}^2 := \sum_{k \in \mathbb{Z}^2} |k|^{2s} |\hat{f}(k)|^2 = \| (-\Delta)^{s/2} f \|_2^2 < \infty.
\]

Note that it is clear that for \( f \in L^2(\mathbb{T}^2) \), \( f \in H^s(\mathbb{T}^2) \) if and only if \( f \in \dot{H}^s(\mathbb{T}^2) \).

Let \( e^{-t\Delta^2} \) be the semigroup generated by the bi-Laplacian \( \Delta \), namely,

\[
e^{-t\Delta^2} f := \mathcal{F}^{-1} \left( \left\{ e^{-t|k|^4} \hat{f}(k) \right\}_{k \in \mathbb{Z}^2} \right),
\]

where \( \mathcal{F}^{-1} \) is the inverse Fourier transform on \( \mathbb{Z}^2 \): for any \( \{a_k\}_{k \in \mathbb{Z}^2} \),

\[
\mathcal{F}^{-1} (\{a_k\})_{k \in \mathbb{Z}^2} (x) := \sum_{k \in \mathbb{Z}^2} a_k e^{-2\pi i k \cdot x}, \quad x \in \mathbb{T}^2.
\]

The following semi-group estimate for \( e^{-t\Delta^2} \) is standard and we would like to leave the proof to the interested reader.

**Lemma 2.1.** For any \( s > 0 \), there exists a dimension constant \( C > 0 \) such that

\[
\left\| (\Delta)^{s/2} e^{-t\Delta^2} f \right\|_{L^2} \leq C t^{-\frac{s}{4}} \| f \|_{L^2}.
\]

We are ready to introduce the definition of the mild solutions to the problem \( (1.1) \).

**Definition 2.2.** Let \( v \in L^\infty([0, \infty); W^{1,\infty}) \) be a divergence free flow. A function \( u \in C ([0, T]; H^2) \cap L^2 ((0, T); H^4) \), \( T > 0 \), is called a **mild solution** of \( (1.1) \) on
[0, T] with initial data $u_0 \in H^2$, if for any $0 \leq t \leq T$,

$$u(t) = T(u)(t) := e^{-t\varepsilon} \Delta^2 u_0 + \int_0^t e^{-(t-s)\varepsilon} \Delta^2 \left( a u^3 + bu^2 + cu \right) ds$$

(2.1)

$$- \int_0^t e^{-(t-s)\varepsilon} \Delta^2 (v \cdot \nabla u) ds$$

holds pointwise in time with values in $H^2$, where the integral is defined in Böchner sense.

**Remark 2.3.** We claim that the second integral in (2.1) is well-defined. Indeed, if $u \in C \left( [0, T]; H^2 \right)$, we have

$$\| \Delta (au^3 + bu^2 + cu) \|_{L^2} = \| 3a^2 u \Delta u + 6au|\nabla u|^2 + 2b|\nabla u|^2 + 2bu \Delta u + |c| \Delta u \|_{L^2}$$

$$\leq 3|a| \| u \|^2_{L^2} \| \Delta u \|_{L^2} + 6|b| \| u \|_{L^\infty} \| \nabla u \|^2_{L^4} + |c| \| \Delta u \|_{L^2}$$

(2.2)

$$+ 2|b| \| \nabla u \|^3_{L^1}$$

Moreover, for each $X = \mathbb{T}^2 \times \mathbb{R}_+$, we can bound the right hand side of (2.2) by

$$\| \nabla u \|_{L^3} \leq C \left( \| \Delta u \|^2_{L^2} \| u \|^2_{L^2} + \| u \|_{L^2} \right) \leq C \| u \|_{H^2}$$

(2.3)

$$\| \nabla u \|_{L^3} \leq C \left( \| \Delta u \|^2_{L^2} \| u \|^2_{L^2} + \| u \|_{L^2} \right) \leq C \| u \|_{H^2}$$

(2.4)

we can bound the right hand side of (2.2) by

$$C_{a,b,c} \left( \| u \|^2_{H^2} + \| u \|^2_{H^2} + \| u \|_{H^2} \right),$$

where $C_{a,b,c}$ is some positive constant which only depends on $a$, $b$ and $c$. The desired claim then follows clearly.

We shall use the Banach contraction mapping to construct such a mild solution. For this purpose, we introduce the following Banach space. For any $T > 0$, we define

$$X_T := C \left( [0, T]; H^2 \right) \cap \left\{ u: \mathbb{T}^2 \times \mathbb{R}_+ \rightarrow \mathbb{R} \mid \sup_{0 \leq t \leq T} \left( t^{\frac{1}{2}} \| \Delta^2 u \|_{L^2} + t^{\frac{1}{2}} \| \Delta^2 u \|_{L^2} \right) < \infty \right\}$$

with the norm

$$\| u \|_{X_T} := \max \left( \sup_{0 \leq t \leq T} \| u \|_{H^2}, \sup_{0 \leq t \leq T} \left( t^{\frac{1}{2}} \| \Delta^2 u \|_{L^2} + t^{\frac{1}{2}} \| \Delta^2 u \|_{L^2} \right) \right).$$

We have the following result.

**Theorem 2.4.** Let $v \in L^\infty([0, \infty); W^{1,\infty})$ be a divergence free flow. Then there exists $0 < T \leq 1$ depending on $\varepsilon$, $a$, $b$, $c$, such that $u_0 \in H^2$ such that (1.1) admits a unique mild solution $u$ on $[0, T]$, which is unique in $X_T$.

The proof of Theorem 2.4 consists of several lemmas.

**Lemma 2.5.** Under the assumption of Theorem 2.4, we have for any $0 < t \leq 1$,

$$T(u) \in C([0, T]; H^2).$$

Moreover, for each $t \in (0, T)$, there exists some $C = C(\varepsilon, a, b, c) > 0$, such that

$$\| T(u)(t) \|_{H^2} \leq C \left( \| u_0 \|_{H^2} + t^{\frac{1}{2}} \left( \| u \|_{X_T}^2 + \| u \|_{X_T}^2 + (1 + \| u \|_{L^\infty([0, \infty); W^{1,\infty})}) \| u \|_{X_T} \right) \right)$$

(2.6)
Proof. It suffices to prove the estimate (2.6). For any $t > 0$, by Lemma 2.1, we have

$$\|\Delta T(u)(t)\|_{L^2} \leq \left\| e^{-t\varepsilon\Delta^2} \left( \Delta u_0 \right) \right\|_{L^2} + \int_0^t \left\| e^{-t\varepsilon\Delta^2} (v \cdot \nabla u) \right\|_{L^2} ds$$

$$+ \int_0^t \left\| e^{-(t-s)\varepsilon\Delta^2} \Delta \left( au^3 + bu^2 + cu \right) \right\|_{L^2} ds$$

$$\leq C \left( \|u_0\|_{H^2} + \int_0^t (t-s)^{-\frac{\varepsilon}{2}} \left[ \|\Delta \left( au^3 + bu^2 + cu \right)\|_{L^2} + \|v \cdot \nabla u\|_{L^2} \right] ds \right)$$

$$\leq C \left( \|u_0\|_{H^2} + t^\frac{\varepsilon}{2} \left( \|u\|_{X_T}^3 + \|u\|_{X_T}^2 + (1 + \|v\|_{L^\infty([0,\infty);L^\infty)}) \|u\|_{X_T} \right) \right)$$

where we have used (2.5) in the above estimate. Similarly, we have

$$\|T(u)(t)\|_{L^2} \leq C \left( \|u_0\|_{H^2} + t^\frac{\varepsilon}{2} \left( \|u\|_{X_T}^3 + \|u\|_{X_T}^2 + (1 + \|v\|_{L^\infty([0,\infty);L^\infty)}) \|u\|_{X_T} \right) \right)$$

and

$$\|\nabla T(u)(t)\|_{L^2} \leq C \left( \|u_0\|_{H^2} + t^\frac{\varepsilon}{2} \left( \|u\|_{X_T}^3 + \|u\|_{X_T}^2 + (1 + \|v\|_{L^\infty([0,\infty);L^\infty)}) \|u\|_{X_T} \right) \right).$$

Combining all these estimates yields (2.6).

\[\square\]

Lemma 2.6. Under the assumption of Theorem 2.4 we have for any $0 < T < 1$, there exists some $C = C(\varepsilon, a, b, c) > 0$, such that

$$\sup_{0 \leq t \leq T} \left( t^\frac{\varepsilon}{4} \left\| \Delta^\frac{\varepsilon}{4} u \right\|_{L^2} + t^\frac{\varepsilon}{4} \left\| \Delta^2 u \right\|_{L^2} \right)$$

\[\leq C \left( \|u_0\|_{H^2} + T^\frac{\varepsilon}{4} \left( \|u\|_{X_T}^3 + \|u\|_{X_T}^2 + (1 + \|v\|_{L^\infty([0,\infty);W^{1,\infty})}) \|u\|_{X_T} \right) \right).\]

\[(2.7)\]

Proof. We first prove

$$\sup_{0 \leq t \leq T} t^\frac{\varepsilon}{4} \left\| \Delta^\frac{\varepsilon}{4} u \right\|_{L^2}$$

\[\leq C \left( \|u_0\|_{H^2} + t^\frac{\varepsilon}{4} \left( \|u\|_{X_T}^3 + \|u\|_{X_T}^2 + (1 + \|v\|_{L^\infty([0,\infty);W^{1,\infty})}) \|u\|_{X_T} \right) \right).\]

\[(2.8)\]
Indeed, by Lemma 27.1 and 27.3 again, we have
\[
\begin{align*}
\left\| \Delta^\frac{3}{2} F(u)(t) \right\|_{L^2} & \leq \left\| \Delta^\frac{3}{2} e^{-tx\Delta^2} (\Delta u_0) \right\|_{L^2} + \int_0^t \left\| \Delta^\frac{3}{2} e^{-(t-s)\varepsilon\Delta^2} (v \cdot \nabla u) \right\|_{L^2} ds \\
& \quad + \int_0^t \left\| \Delta^\frac{3}{2} e^{-(t-s)\varepsilon\Delta^2} (au^3 + bu^2 + cu) \right\|_{L^2} ds \\
& \leq Ct^{-\frac{3}{4}} \left\| u_0 \right\|_{H^2} + C \int_0^t (t-s)^{-\frac{3}{4}} \left\| \Delta (au^3 + bu^2 + cu) \right\|_{L^2} ds \\
& \quad + C \int_0^t (t-s)^{-\frac{3}{4}} \left\| v \cdot \nabla u \right\|_{L^2} ds \\
& \leq Ct^{-\frac{3}{4}} \left\| u_0 \right\|_{H^2} + C \int_0^t (t-s)^{-\frac{3}{4}} ds \cdot \left( |a| \left\| u \right\|_{X_1}^3 + |b| \left\| u \right\|_{X_T}^2 + |c| \left\| u \right\|_{X_T} \right) \\
& \quad + C \int_0^t (t-s)^{-\frac{3}{4}} ds \cdot \left\| v \right\|_{L^\infty([0,\infty);L^\infty)} \left\| u \right\|_{X_T} \\
& \leq C \left( t^{-\frac{3}{4}} \left\| u_0 \right\|_{H^2} + t^\frac{3}{4} \cdot \left( |a| \left\| u \right\|_{X_T}^3 + |\gamma_1| \left\| u \right\|_{X_T}^2 \\
& \quad + (1 + \left\| v \right\|_{L^\infty([0,\infty);W^{1,\infty}}) \left\| u \right\|_{X_T} \right) \right).
\end{align*}
\]

The desired estimate (2.8) then follows from by multiplying \( t^\frac{3}{4} \) on both sides of the above estimate.

Now we turn to prove the second part, that is
\[
\sup_{0 \leq t \leq T} t^\frac{3}{4} \left\| \Delta^\frac{3}{2} u \right\|_{L^2} \leq C \left( \left\| u_0 \right\|_{H^2} + t^\frac{3}{4} \left( \left\| u \right\|_{X_T}^3 + \left\| u \right\|_{X_T}^2 + (1 + \left\| v \right\|_{L^\infty([0,\infty);W^{1,\infty}}) \left\| u \right\|_{X_T} \right) \right).
\]

Note that
\[
\nabla \cdot \Delta (au^3 + bu^2 + cu) = 6au (\nabla \cdot u) \Delta u + 3au^2 \nabla \cdot \Delta u
\]
\[
+6a(\nabla \cdot u) |\nabla u|^2 + 12au \sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial x_j}
\]
\[
+2b \sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial x_j} + 2b (\nabla \cdot u) \Delta u
\]
\[
+2bu \nabla \cdot \Delta u + c \nabla \cdot \Delta u.
\]

This further gives
\[
\left\| \nabla \cdot \Delta (au^3 + bu^2 + cu) \right\|_{L^2} \leq 18|a| \left\| u \right\|_{L^\infty} \left\| \nabla u \right\|_{L^\infty} \left\| \Delta u \right\|_{L^2} + 3|a| \left\| u \right\|_{L^\infty}^2 \left\| \Delta^\frac{3}{2} u \right\|_{L^2}
\]
\[
+6|a| \left\| \nabla u \right\|_{L^\infty} \left\| \nabla u \right\|_{L^2} + 4|b| \left\| \nabla u \right\|_{L^\infty} \left\| \Delta u \right\|_{L^2}
\]
\[
+2|b| \left\| u \right\|_{L^\infty} \left\| \Delta^\frac{3}{2} u \right\|_{L^2} + |c| \left\| \Delta^\frac{3}{2} u \right\|_{L^2}
\]

This, together with (2.3), (2.4) and the Gagliardo-Nirenberg’s inequality
\[
\left\| \nabla u \right\|_{L^\infty} \leq C \left( \left\| \Delta^\frac{3}{2} u \right\|_{L^2}^\frac{3}{2} + \left\| u \right\|_{L^2} \right) \leq C \left\| \Delta^\frac{3}{2} u \right\|_{L^2}^\frac{3}{2},
\]

Note that
such that for $C$



Lemma 2.8. from (2.8) and (2.9). which implies the desired estimate (2.9). The estimate (2.7) then clearly follows suggests

\begin{align*}
\left\| \nabla \cdot \Delta (au^3 + bu^2 + cu) \right\|_{L^2} & \leq C \left( \|a\|_{H^2}^2 + (|a| + |b|)\|u\|_{H^2} + |c| \right) \left\| \Delta \frac{2}{3} u \right\|_{L^2}.
\end{align*}

On the other hand, we have

\begin{align*}
\left\| \nabla \cdot (v \cdot \nabla u) \right\|_{L^2} &= \left\| \sum_{i=1}^{n} (\nabla \cdot v_i) u_{x_i} + \sum_{i=1}^{n} v_i \nabla \cdot u_{x_i} \right\|_{L^2}
\end{align*}

(2.12)

Therefore, by (2.11) and (2.12), we have

\begin{align*}
\left\| \Delta^2 T(u)(t) \right\|_{L^2} & \leq \left\| \Delta e^{-t \Delta^2} \left( \Delta u_0 \right) \right\|_{L^2} + \int_0^t \left\| \nabla \cdot \Delta e^{-(t-s)\Delta^2} \nabla (v \cdot \nabla u) \right\|_{L^2} ds \\
& \quad + \int_0^t \left\| \nabla \cdot \Delta e^{-(t-s)\Delta^2} \left[ \nabla \cdot \Delta (au^3 + bu^2 + cu) \right] \right\|_{L^2} ds \\
& \leq C t^{-\frac{1}{2}} \|u_0\|_{H^2} + \int_0^t \left( t - s \right)^{-\frac{1}{2}} \left\| \nabla \cdot \Delta (au^3 + bu^2 + cu) \right\|_{L^2} ds \\
& \quad + \int_0^t \left( t - s \right)^{-\frac{1}{2}} \left\| \nabla \cdot (v \cdot \nabla u) \right\|_{L^2} ds
\end{align*}

\begin{align*}
& \leq C t^{-\frac{1}{2}} \|u_0\|_{H^2} + \int_0^t \left( t - s \right)^{-\frac{1}{2}} \left\| v \right\|_{L^\infty((0,T);W^{1,\infty})} \left\| u \right\|_{H^2} ds \\
& \quad + C \int_0^t \left( t - s \right)^{-\frac{3}{2}} s^{-\frac{1}{2}} \left( \|u(s)\|^2_{H^2} + \|u(s)\|_{H^2} + 1 \right) \left( s^{-\frac{1}{2}} \left\| \Delta \frac{2}{3} u(s) \right\|_{L^2} \right) ds \\
& \leq C t^{-\frac{1}{2}} \|u_0\|_{H^2} + C \left( \|u\|^3_{X_T} + \|u\|^2_{X_T} + (1 + \|v\|_{L^\infty((0,T);W^{1,\infty})}) \|u\|_{X_T} \right)
\end{align*}

which implies the desired estimate (2.9). The estimate (2.7) then clearly follows from (2.3) and (2.9).

Combing Lemma 2.5 and Lemma 2.6, we have the following result.

**Lemma 2.7.** For any $0 < T \leq 1$. The map $T : X_T \to X_T$ and there exists some $C_1 = C_1(\varepsilon, a, b, c) > 0$, such that

\begin{align*}
\|T(u)\|_{X_T} \leq C_1 \left( \|u_0\|_{H^2} + \right.
\left. + T^\frac{1}{2} \left( \|u\|^3_{X_T} + \|u\|^2_{X_T} + \left( 1 + \|v\|_{L^\infty((0,T);W^{1,\infty})} \right) \|u\|_{X_T} \right) \right).
\end{align*}

Next, we show that $T$ is a Lipschitz map on $X_T$.

**Lemma 2.8.** Let $0 < T \leq 1$. Then there exists a constant $C_2 = C_2(\varepsilon, a, b, c) > 0$, such that for $u_1, u_2 \in X_T$,

\begin{align*}
\|T(u_1) - T(u_2)\|_{X_T} \leq C_2 T^\frac{1}{2} \left( \left( \|u_1\|_{X_T} + \|u_2\|_{X_T} + 1 \right)^2 + \|v\|_{L^\infty((0,T);W^{1,\infty})} \right) \|u_1 - u_2\|_{X_T}.
\end{align*}
Proof. To begin with, we note that by (2.13) and (2.14), one can compute that
\[
\| \Delta (au_1^3 + bu_1^2 + cu_1 - (au_2^3 + bu_2^2 + cu_2)) \|_{L^2} \leq C (\|u_1\|_{H^2} + \|u_2\|_{H^2} + 1)^2 \|u_1 - u_2\|_{H^2},
\]
which is a consequence of the following estimates:
1. \( \| \Delta (u_1^3 - u_2^3) \|_{L^2} \leq C (\|u_1\|_{H^2} + \|u_2\|_{H^2} + \|u_2\|_{H^2}^2) \|u_1 - u_2\|_{H^2}; \)
2. \( \| \Delta (u_1^2 - u_2^2) \|_{L^2} \leq C (\|u_1\|_{H^2} + \|u_2\|_{H^2}) \|u_1 - u_2\|_{H^2} \).
Using (2.14), we have for any \( 0 \leq t \leq T \),
\[
\| T(u_1)(t) - T(u_2)(t) \|_{H^2} \leq \int_0^t \left\| \Delta e^{-(t-s)} \Delta \left( au_1^3 + bu_1^2 + cu_1 - (au_2^3 + bu_2^2 + cu_2) \right) \right\|_{L^2} ds + \int_0^t \left\| \Delta e^{-(t-s)} \Delta \left( v \cdot \nabla (u_1 - u_2) \right) \right\|_{L^2} ds \leq C \int_0^t (t - s)^{-\frac{1}{2}} (\|u_1(s)\|_{H^2} + \|u_2(s)\|_{H^2} + 1)^2 \|u_1(s) - u_2(s)\|_{H^2} ds + \int_0^t (t - s)^{-\frac{1}{2}} \|v\|_{L^\infty([0,\infty);W^{1,\infty}}} \|u_1(s) - u_2(s)\|_{H^2} ds \leq C \int_0^t (t - s)^{-\frac{1}{2}} ds \cdot (\|u_1\|_{X_T} + \|u_2\|_{X_T} + 1)^2 \|u_1 - u_2\|_{X_T} + \int_0^t (t - s)^{-\frac{1}{2}} ds \cdot \|v\|_{L^\infty([0,\infty);W^{1,\infty}}} \|u_1 - u_2\|_{X_T} ds \leq Ct^{\frac{1}{2}} \cdot \left( (\|u_1\|_{X_T} + \|u_2\|_{X_T} + 1)^2 + \|v\|_{L^\infty([0,\infty);W^{1,\infty}} \right) \|u_1 - u_2\|_{X_T},
\]
and
\[
\left\| \Delta^{\frac{3}{2}} (T(u_1) - T(u_2))(t) \right\|_{L^2} \leq C \int_0^t (t - s)^{-\frac{1}{2}} (\|u_1(s)\|_{H^2} + \|u_2(s)\|_{H^2} + 1)^2 \|u_1(s) - u_2(s)\|_{H^2} ds + C \int_0^t (t - s)^{-\frac{3}{2}} \|v\|_{L^\infty([0,\infty);W^{1,\infty}}} \|u_1(s) - u_2(s)\|_{H^2} ds \leq C \int_0^t (t - s)^{-\frac{1}{2}} ds \cdot (\|u_1\|_{X_T} + \|u_2\|_{X_T} + 1)^2 \|u_1 - u_2\|_{X_T} + C \int_0^t (t - s)^{-\frac{3}{2}} ds \cdot \|v\|_{L^\infty([0,\infty);W^{1,\infty}}} \|u_1 - u_2\|_{X_T} \leq Ct^{\frac{1}{2}} \cdot \left( (\|u_1\|_{X_T} + \|u_2\|_{X_T} + 1)^2 + \|v\|_{L^\infty([0,\infty);W^{1,\infty}} \right) \|u_1 - u_2\|_{X_T},
\]
which implies
\[
t^{\frac{1}{4}} \left\| \Delta^{\frac{3}{2}} (T(u_1) - T(u_2))(t) \right\|_{L^2} \leq Ct^{\frac{1}{2}} \cdot \left( (\|u_1\|_{X_T} + \|u_2\|_{X_T} + 1)^2 + \|v\|_{L^\infty([0,\infty);W^{1,\infty}} \right) \|u_1 - u_2\|_{X_T},
\]
Next, we claim that
\[ t^{\frac{1}{2}} \| \Delta^2 (T(u_1) - T(u_2))(t) \|_{L^2} \]
(2.17) \leq C t^{\frac{1}{2}} \cdot \left( \| u_1 \|_{X_T} + \| u_2 \|_{X_T} + 1 \right)^2 \| v \|_{L^{\infty}([0, \infty); W^{1, \infty})} \| u_1 - u_2 \|_{X_T}.

Indeed, by (2.3), (2.4), (2.10) and the definition of \( X_T \), one can check that for each \( 0 \leq s \leq t \),
\[ \| \nabla \cdot \Delta (au_1(s)^3 + bu_1(s)^2 + cu_1(s) - (au_2(s)^3 + bu_2(s)^2 + cu_2(s))) \|_{L^2} \]
\[ \leq C s^{-\frac{3}{4}} \left( \| u_1 \|_{X_T} + \| u_2 \|_{X_T} + 1 \right)^2 \| u_1 - u_2 \|_{X_T}, \]
which implies
\[ \| \Delta^2 (T(u_1) - T(u_2))(t) \|_{L^2} \]
\[ \leq \int_0^t \left\| \nabla \cdot \Delta e^{-(t-s)\Delta^2} \nabla (v \cdot \nabla (u_1(s) - u_2(s))) \right\|_{L^2} \]
\[ + \int_0^t \left\| \nabla \cdot \Delta e^{-(t-s)\Delta^2} \nabla \left[ au_1(s)^3 + bu_1(s)^2 + cu_1(s) \right. \]
\[ - \left. (au_2(s)^3 + bu_2(s)^2 + cu_2(s)) \right\|_{L^2} ds \]
\[ \leq C \int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{3}{4}} ds \cdot \left( \| u_1 \|_{X_T} + \| u_2 \|_{X_T} + 1 \right)^2 \| u_1 - u_2 \|_{X_T} \]
\[ + \int_0^t (t-s)^{-\frac{3}{4}} ds \cdot \| v \|_{L^{\infty}([0, \infty); W^{1, \infty})} \| u_1 - u_2 \|_{X_T} \]
\[ \leq C \left( \| u_1 \|_{X_T} + \| u_2 \|_{X_T} + 1 \right)^2 + t^2 \| v \|_{L^{\infty}([0, \infty); W^{1, \infty})} \| u_1 - u_2 \|_{X_T}. \]

This clearly gives (2.17). Finally, the desired estimate (2.13) follows from (2.15), (2.1) and (2.17). \( \square \)

**Proof of Theorem 2.4.** Let \( C := \max\{C_1, C_2\} \) and \( M := 10C \| u_0 \|_{H^2} \), where \( C_1 \) and \( C_2 \) are defined in Lemma 2.7 and Lemma 2.8, respectively. Let further, \( B(0, M) \) be the ball in \( X_T \) centered at origin with radius \( M \) with
\[ 0 < T < \min \left\{ 1, \frac{1}{16C^2(M^2 + M + 1 + \| v \|_{L^{\infty}([0, \infty); W^{1, \infty})}^2) \right\}. \]
It is then easy to see that
\[ \| T(u) \|_{X_T} \leq M, \quad \forall u \in B(0, M) \]
and
\[ \| T(u_1) - T(u_2) \|_{X_T} \leq \frac{1}{2} \| u_1 - u_2 \|_{X_T}, \quad \forall u_1, u_2 \in B(0, M). \]
An application of Banach contraction mapping theorem yields Theorem 2.4. \( \square \)

As a consequence, we have the following corollary.

**Corollary 2.9.** Under the assumption of Theorem 2.4, if \( \hat{T} \) is the maximal time of existence of the mild solution \( u = T(u) \), then
\[ \limsup_{t \to T^-} \| u(t) \|_{H^2} = \infty. \]
Otherwise, \( \hat{T} = \infty. \)
Proof. The proof for the above result is standard. For example, one can consult\cite[Corollary 2.7]{15} (see, also \cite[Corollary 2.6]{12}). □

Remark 2.10. (1). Corollary 2.9 can be viewed as a supplement to\cite[Theorem 3.1]{7}, which asserts that in the non-advective case (namely, \(v \equiv 0\)), if the Landau-Ginzurg free energy of the initial data is sufficiently negative, then the \(H^2\)-energy (more precisely, \(L^2\)-energy) of the solution will blow up in finite time (see, Remark 1.6); while Corollary 2.9 gives that if the mild solution with \(H^2\) initial data exists in finite time, then its \(H^2\)-norm has to blow up.

(2). Via a standard approximation argument, one can also check the mild solution \(u\) constructed in Theorem 2.4 is also a weak solution on \([0, T]\) (see, e.g., \cite[Proposition 2.8]{15} and \cite[Proposition 2.9]{12}). Here, a function \(\tilde{u} \in L^\infty(\mathbb{T}^2)\) is called a weak solution of (1.1) on \([0, T]\) with initial data \(\tilde{u} _0 \in H^2\) if for all \(\phi \in C^\infty_c(\mathbb{T}^2)\),

\[
\int_{\mathbb{T}^2} \tilde{u}_0 \phi(0) dx dy + \int_0^T \int_{\mathbb{T}^2} \tilde{u} \partial_t \phi dx dy dt = \int_0^T \phi (v \cdot \nabla \tilde{u}) dx dy dt \\
+ \varepsilon \int_0^T \int_{\mathbb{T}^2} \phi \Delta \Delta \tilde{u} dx dy dt - \int_0^T \int_{\mathbb{T}^2} \phi \Delta (au^3 + bu^2 + cu) dx dy dt
\]

and \(\partial_t \tilde{u} \in L^2([0, T]; H^{-2})\). Moreover, \(u\) also satisfies the following energy identity: for any \(0 \in [0, T]\),

\[
\|u(0)\|_{L^2} + 2 \int_0^T \int_{\mathbb{T}^2} u \Delta (au^3 + bu^2 + cu) dx dy dt = \|u(t)\|_{L^2}^2 + 2 \varepsilon \int_0^t \|\Delta u(s)\|_{L^2}^2 ds.
\]

(2.18)

(3). We claim for each \(t > 0\), \(\partial_t u(t, \cdot) \in L^2(\mathbb{T}^2)\). Indeed, by standard argument from spectral theory, one can verify that in the weak sense,

\[
\partial_t u(t) = -\varepsilon \Delta^2 u(t) + \Delta (au^3(t) + bu(t)^2 + cu(t)) - v \cdot \nabla u(t).
\]

(2.19)

Note that the last two terms in the right hand side of (2.19) belongs to \(L^2\), while for the first term, by Theorem 2.4 we already know that

\[
t^{\frac{1}{2}} \|\Delta^2 u\|_{L^2} < \infty.
\]

The desired claim is then immediate.

3. Bootstrap assumptions

We now turn to the proof of Theorem 1.5 which we recall deals with the global existence of the mild solutions to the following unstable Cahn-Hilliard equation:

\[
\begin{cases}
  u_t + v_1(y) \partial_x u + \varepsilon \gamma \Delta^2 u = \gamma \Delta (au^3 + bu^2) & \text{on } \mathbb{T}^2; \\
  u \text{ periodic} & \text{on } \partial \mathbb{T}^2,
\end{cases}
\]

with \(\varepsilon, \gamma > 0, a < 0, b \in \mathbb{R}, v_1 \in W^{1, \infty}(\mathbb{T})\) and the initial condition \(u_0 \in H^2_0(\mathbb{T}^2)\).

The goal of this section is to establish the bootstrap assumptions to the above problem.
Let $u$ be the unique mild solution on $[0, T]$ to the problem \((1.5)\) constructed in Theorem \(2.3\) and recall that
\begin{equation}
\langle u \rangle (t, y) := \int_T u(t, x, y) dx \quad \text{and} \quad u_\gamma(t, x, y) := u(t, x, y) - \langle u \rangle (t, y),
\end{equation}
Note that both $\langle u \rangle$ and $u_\gamma$ have mean zero. Using Theorem \(2.4\) we have
\begin{equation}
\langle u \rangle \in L^2_{loc} ((0, T); H^4(\mathbb{T})) \cap C ([0, T); H^2(\mathbb{T}))
\end{equation}
and
\begin{equation}
u_\gamma \in L^2_{loc} ((0, T); H^4(\mathbb{T}^2)) \cap C ([0, T); H^2(\mathbb{T}^2)) .
\end{equation}
Moreover, using \((1.5)\), one can check that $\langle u \rangle$ and $u_\gamma$ solve the following coupled system:
\begin{equation}
\partial_t \langle u \rangle + \varepsilon \gamma \partial_y^3 \langle u \rangle = \gamma \int_T \Delta \left[ a \left( \langle u \rangle + u_\gamma \right)^3 + b \left( \langle u \rangle + u_\gamma \right)^2 \right] dx
\end{equation}
and
\begin{equation}
\partial_t u_\gamma + v_1(y) \partial_x u_\gamma + \varepsilon \gamma \Delta^2 u_\gamma = \gamma \Delta \left[ a \left( \langle u \rangle + u_\gamma \right)^3 + b \left( \langle u \rangle + u_\gamma \right)^2 \right] dx.
\end{equation}
Let $S_t$ be the solution operator form 0 to time $t \geq 0$ for the advection-hyperdiffusion equation:
\begin{equation}
\partial_t g + v_1(y) \partial_x g + \varepsilon \gamma \Delta^2 g = 0.
\end{equation}
Namely, $S_t = e^{-t H_\gamma}$, where we recall that $H_\gamma$ is defined in \((1.6)\). Then by the Duhamel’s formula, we have for any $0 \leq s \leq t$,
\begin{equation}
u_\gamma(t) = S_{t-s} (u_\gamma(s)) + \gamma \int_s^t S_{t-\tau} \left( \int_T \Delta \left[ a \left( \langle u \rangle + u_\gamma \right)^3 + b \left( \langle u \rangle + u_\gamma \right)^2 \right] dx \right) d\tau
\end{equation}
\begin{equation}
- \gamma \int_s^t S_{t-\tau} \left( \int_T \Delta \left[ a \left( \langle u \rangle + u_\gamma \right)^3 + b \left( \langle u \rangle + u_\gamma \right)^2 \right] dx \right) d\tau.
\end{equation}
We have the following results.

**Lemma 3.1.** There exists a sufficiently small time $0 \leq t_1 \leq T$, which only depends on $\varepsilon, a, b, \gamma, v_1, ||u||_{L^2(0, T); H^2}$ and $||u_\gamma(0)||_{L^2}$, such that for any $0 \leq s \leq t \leq t_1$, one has
\begin{equation}
||u_\gamma(t)||_{L^2} \leq Ce^{-\frac{\lambda_\gamma(t-s)}{4}} ||u_\gamma(s)||_{L^2},
\end{equation}
where $\lambda_\gamma$ is defined in \((1.8)\).

**Proof.** Taking $L^2$-norm on both sides of \((3.3)\), we have
\begin{equation}
||u_\gamma(t)||_{L^2} \leq ||S_{t-s} (u_\gamma(s))||_{L^2}
+ |a| \gamma \int_s^t \left| S_{t-\tau} \left( \int_T \Delta \left( \langle u \rangle + u_\gamma \right)^3 \right) \right|_{L^2} d\tau
+ |b| \gamma \int_s^t \left| S_{t-\tau} \left( \int_T \Delta \left( \langle u \rangle + u_\gamma \right)^2 \right) \right|_{L^2} d\tau.
\end{equation}
By Proposition \(1.4\) and the fact that $||S_t||_{L^2 \to L^2} \leq 1$, we can further bound $||u_\gamma(t)||_{L^2}$ by
\begin{equation}
10e^{-\lambda_\gamma(t-s)} ||u_\gamma(s)||_{L^2} + |a| \gamma I_1 + |b| \gamma I_2,
\end{equation}
where
\begin{equation}
I_1 = \int_s^t \gamma \int_T \left| \Delta \left( \langle u \rangle + u_\gamma \right)^3 \right|_{L^2} d\tau,
\end{equation}
\begin{equation}
I_2 = \int_s^t \gamma \int_T \left| \Delta \left( \langle u \rangle + u_\gamma \right)^2 \right|_{L^2} d\tau.
\end{equation}
where

\[ I_1 := \int_s^t \left\| \Delta \left( (u) + u_\gamma \right)^3 - \int_\mathbb{T} \Delta \left( (u) + u_\gamma \right)^3 \, dx \right\|_{L^2} \, d\tau \]
\[ = \int_s^t \left\| \Delta u^3 \right\|_{L^2} \, d\tau \]

and

\[ I_2 := \int_s^t \left\| \Delta \left( (u) + u_\gamma \right)^2 - \int_\mathbb{T} \Delta \left( (u) + u_\gamma \right)^2 \, dx \right\|_{L^2} \, d\tau \]
\[ = \int_s^t \left\| \Delta u^2 \right\|_{L^2} \, d\tau. \]

**Estimate of** \( I_1 \). Note that

\[ \Delta u^3 = 3u^2 \Delta u + 6u |\nabla u|^2. \]

Thus,

\[ \left\| \Delta u^3 \right\|_{L^2} \leq C \left[ \left\| u^2 \Delta u \right\|_{L^2} + \left\| \nabla u \right\|_{L^2}^2 \right] \]
\[ \leq C \left[ \left\| u \right\|_{L^\infty} \left\| \Delta u \right\|_{L^2} + \left\| \nabla u \right\|_{L^2}^2 \right] \]
\[ \leq C \left[ \left\| \nabla u \right\|_{L^2}^2 \left\| u \right\|_{L^2} + \left\| u \right\|_{L^2}^2 \left\| \Delta u \right\|_{L^2}^2 \right] \]

(3.7)

where in the above estimates, we have used the Gagliardo–Nirenberg’s inequalities in 2D:

\[ \left\| u \right\|_{L^\infty} \leq C \left\| \Delta u \right\|_{L^2}^2 \left\| u \right\|_{L^2} \quad \text{and} \quad \left\| \nabla u \right\|_{L^4} \leq C \left\| \Delta u \right\|_{L^2}^2 \left\| u \right\|_{L^2}. \]

Using (3.7), we have

\[ I_1 \leq C \int_s^t \left\| \Delta u^3 \right\|_{L^2} \, d\tau \leq C \int_s^t \left\| u \right\|_{H^2}^3 \, d\tau \leq C(t - s) \left\| u \right\|_{L^\infty(0, T}; H^2)}, \]

where in the last estimate above, we have used the fact that \( 0 \leq s \leq t \leq T \).

**Estimate of** \( I_2 \). The estimate of \( I_2 \) is similar to the one of \( I_1 \). We first note that

\[ \Delta u^2 = 2u \Delta u + 2 |\nabla u|^2, \]

and hence we have

\[ \left\| \Delta u^2 \right\|_{L^2} \leq C \left[ \left\| u \right\|_{L^\infty} \left\| \Delta u \right\|_{L^2} + \left\| \nabla u \right\|_{L^2}^2 \right] \]
\[ \leq C \left[ \left\| u \right\|_{L^\infty} \left\| \Delta u \right\|_{L^2} + \left\| \nabla u \right\|_{L^2}^2 \right] \]

(3.8)

This implies

\[ I_2 \leq C \int_s^t \left\| \Delta u^2 \right\|_{L^2} \, d\tau \leq C \int_s^t \left\| u \right\|_{H^2}^3 \, d\tau \leq C(t - s) \left\| u \right\|_{L^\infty(0, T}; H^2)}. \]

Finally, combining all the estimates of \( I_1 \) and \( I_2 \) with (3.6), we see that

\[ \left\| u_\gamma(t) \right\|_{L^2} \leq 10e^{-\lambda_1(t - s)} \left\| u_\gamma(s) \right\|_{L^2} + C(t - s) \gamma \left[ \left\| u \right\|_{L^\infty(0, T]; H^2) + \left| a \right| \left\| u \right\|_{L^\infty(0, T]; H^2) + \left| b \right| \left\| u \right\|_{L^\infty(0, T]; H^2) \right] \right. \]
The desired estimate (3.5) then follows from by choosing \( t_1 \) sufficiently small and the continuity of \( \|u(t)\|_{L^2} \) at \( t = 0 \).

**Lemma 3.2.** There exists a sufficiently small time \( 0 < t_2 < T \), such that for any \( 0 \leq s \leq t_2 \), one has

\[
\varepsilon \gamma \int_s^t \| \Delta u_\theta (\tau) \|_{L^2}^2 d\tau \leq C \| u_\theta (s) \|_{L^2}^2.
\]

Here, \( t_2 \) only depends on \( \varepsilon, a, b, \gamma, v_1, \| u_\theta (0) \|_{L^2} \) and \( \| u \|_{L^\infty ([0,1];H^2)} \).

**Proof.** Multiply (3.9) by \( u_\theta \) on both sides and then integrating by parts, we see that for any \( t > 0 \),

\[
\frac{1}{2} \frac{d}{dt} \| u_\theta (t) \|_{L^2}^2 + \varepsilon \gamma \| \Delta u_\theta (t) \|_{L^2}^2 = a \gamma \int_{t_2}^t u_\theta (t) \Delta (u^3) (t) dx dy + b \gamma \int_{t_2}^t \left( \int_T^t u_\theta (t) \right) \left( \int_T^t \Delta (u^2) dx \right) dy
\]

\[
- \beta \varepsilon \varepsilon \int_{t_2}^t \left( \int_T^t u_\theta (t) \right) \left( \int_T^t \Delta (u^2) dx \right) dy
\]

\[
= a \gamma \int_{t_2}^t u_\theta (t) \Delta (u^3) (t) dx dy + b \gamma \int_{t_2}^t u_\theta (t) \Delta (u^2) (t) dx dy
\]

\[
(3.10) \leq |a| \| u_\theta \|_{L^2} \| \Delta (u^3) \|_{L^2} + |b| \| u_\theta \|_{L^2} \| \Delta (u^2) \|_{L^2},
\]

where in the last second line of the above estimate, we have used the fact that \( \int_{t_2}^t u_\theta (t) dx = 0 \).

Note that by (3.7) and (3.8) respectively, we have

\[
\| u_\theta \|_{L^2} \| \Delta (u^3) \|_{L^2} \leq C \| u_\theta \|_{L^2} \| u \|_{H^2}^3 \quad \text{and} \quad \| u_\theta \|_{L^2} \| \Delta (u^2) \|_{L^2} \leq C \| u_\theta \|_{L^2} \| u \|_{H^2}^2.
\]

Hence,

\[
\text{RHS of (3.10)} \leq C \gamma \left[ \| u_\theta \|_{L^2} \| u \|_{H^2}^3 + \| u_\theta \|_{L^2} \| u \|_{H^2}^2 \right].
\]

This gives

\[
\frac{\| u_\theta (t) \|_{L^2}^2}{2} + \varepsilon \gamma \int_s^t \| \Delta u_\theta (\tau) \|_{L^2}^2 d\tau \leq \frac{\| u_\theta (s) \|_{L^2}^2}{2} + C \gamma \int_s^t \| u_\theta \|_{L^2} \| u \|_{H^2}^3 d\tau + C \gamma \int_s^t \| u_\theta \|_{L^2} \| u \|_{H^2}^2 d\tau,
\]

and hence

\[
\varepsilon \gamma \int_s^t \| \Delta u_\theta (\tau) \|_{L^2}^2 d\tau \leq \frac{\| u_\theta (s) \|_{L^2}^2}{2} + C \gamma (t - s) \left( \| u \|_{L^\infty ([0,1];H^2)}^4 + \| u \|_{L^\infty ([0,1];H^2)}^3 \right).
\]

The desired estimate (3.9) then follows from by choosing \( t_2 \) sufficiently small and the continuity of \( \|u(t)\|_{L^2} \) at \( t = 0 \).

The estimates (3.9) and (3.8) suggests for all sufficiently small time \( t \geq s \geq 0 \), we can assume that

1. \( \| u_\theta (t) \|_{L^2} \leq 20e^{-\lambda (t-s)} \| u_\theta (s) \|_{L^2} \);
2. \( \varepsilon \gamma \int_s^t \| \Delta u_\theta (\tau) \|_{L^2}^2 d\tau \leq 10 \| u_\theta (s) \|_{L^2}^2 \).
Finally, let \( t_0 > 0 \) is the maximal time such that estimates above hold on \([0, t_0]\), and we refer the two estimates above together with \( t \in [0, t_0] \) as the bootstrap assumptions.

4. Uniform bounds of \( \langle u \rangle \)

As a consequence of the bootstrap assumptions, we show that the terms \( \| \langle u \rangle \|_{L^2} \) and \( \varepsilon \gamma \int_0^t \| \partial_y \langle u \rangle \|_{L^2}^2 \, dt \) can be bounded uniformly when \( \gamma \) is sufficiently small and the \( L^2 \)-energy of \( \langle u_0 \rangle \) is sufficiently small. Motivated by [11], we show such uniform bounds by applying a version of small energy method under the setting of shear flows.

**Lemma 4.1** (A priori estimate). Assume the bootstrap assumptions, \(|a| < 1 \) and moreover, there exists a positive number \( 0 < \tilde{t}_0 \leq t_0 \), such that

\[
\| \langle u \rangle (t, \cdot) \|_{L^2} \leq K, \quad \text{for all} \quad t \in [0, \tilde{t}_0]
\]

where

\[
K \leq \min \left\{ 1, \left( \frac{\varepsilon \lambda_1^2}{B_1} \right)^{\frac{3}{2}} \right\},
\]

Here, \( t_0 \) is the maximal time defined in the bootstrap assumptions, \( \lambda_1 \) is the smallest positive eigenvalue of \(-\Delta \) on \( \mathbb{T} \) and \( B_1 > 1 \) is an absolute constant which only depends on \( \varepsilon, b \) and any dimensional constants.

Then the following estimate holds: there exists some constant \( B_2 > 1 \), which only depends on \( \varepsilon, b \) and any dimensional constants, such that

\[
\| \langle u \rangle (\tilde{t}_0) \|_{L^2}^2 \leq B_2 \exp \left( B_2 \| u_\#(0) \|_{L^2}^4 \right) \left( a^2 \| u_\#(0) \|_{L^2}^6 \right. \]

\[
+ \left( \frac{\gamma}{\lambda_1} \right)^{\frac{3}{2}} \| u_\#(0) \|_{L^2}^4 + \| \langle u \rangle (0) \|_{L^2}^2 \right)
\]

**Proof.** We begin with recalling that the constant \( B \) that we are going to use in the proof might change line by line, but will only depend on \( \varepsilon, b \) and any dimensional constants. It is important that \( B \) is independent of the choice of \( a \) and \( \gamma \).

Multiplying \( \langle u \rangle \) on both sides of (3.2) and then integration by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \| \langle u \rangle \|_{L^2}^2 + \varepsilon \gamma \| \partial_y \langle u \rangle \|_{L^2}^2
\]

\[
= \gamma \int_T \langle u \rangle \left( \int_T \Delta \left[ a \langle u \rangle + u_\# \right]^3 + b \left( \langle u \rangle + u_\# \right)^2 \right) \, dx \, dy
\]

\[
= \gamma \int_T \langle u \rangle \left( \int_T \partial_y^2 \left[ a \langle u \rangle + u_\# \right]^3 + b \left( \langle u \rangle + u_\# \right)^2 \right) \, dx \, dy
\]

\[
= \gamma \int_{T^2} \partial_y^2 \langle u \rangle \cdot \left[ a \langle u \rangle + u_\# \right]^3 + b \left( \langle u \rangle + u_\# \right)^2 \, dx \, dy
\]

(4.3)

\[
:= J_1 + J_2 + J_3 + J_4,
\]

where

\[
J_1 := \gamma \int_{T^2} \partial_y^2 \langle u \rangle \left( a \langle u \rangle^3 + b \langle u \rangle^2 \right) \, dy, \quad J_2 := 3a \gamma \int_{T^2} \partial_y^2 \langle u \rangle \langle u \rangle u_\#^2 \, dx \, dy,
\]
\[ J_3 := a\gamma \int_{\mathbb{T}^2} \partial_y^2 \langle u \rangle u_y^3 \, dx \, dy \quad \text{and} \quad J_4 := b\gamma \int_{\mathbb{T}^2} \partial_y^2 \langle u \rangle u_y^2 \, dx \, dy. \]

Note that here we have used the fact that
\[
\int_{\mathbb{T}^2} \partial_y^2 \langle u \rangle \langle u \rangle^2 \, dx \, dy = \int_{\mathbb{T}^2} \partial_y^2 \langle u \rangle \langle u \rangle \, dx \, dy = 0.
\]

**Estimate of \( J_1 \).** By Young’s inequality and the Gagliardo-Nirenberg’s inequalities in 2D:
\[
\| \partial_y \langle u \rangle \|_{L^4} \leq B \| \partial_y^2 \langle u \rangle \|^{\frac{\gamma}{2}}_{L^2} \| \langle u \rangle \|^{\frac{4}{2}}_{L^2} \quad \text{and} \quad \| \langle u \rangle \|_{L^2} \leq B \| \partial_y^2 \langle u \rangle \|^{\frac{1}{2}}_{L^2} \| \langle u \rangle \|^{\frac{3}{2}}_{L^2},
\]
we have
\[
J_1 = -\gamma \int_{\mathbb{T}} (\partial_y \langle u \rangle)^2 \cdot (3a\langle u \rangle^2 + 2b\langle u \rangle) \, dy
\leq 3|a|\gamma \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} \| \langle u \rangle \|^{2}_{L^2} + 2|b|\gamma \| \partial_y \langle u \rangle \|^{2}_{L^2} \| \langle u \rangle \|^{2}_{L^2}
= B|a|\gamma \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} \| \langle u \rangle \|^{2}_{L^2} + B\gamma \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} \| \langle u \rangle \|^{2}_{L^2}
\leq \frac{\varepsilon \gamma}{8} \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} + B(1 + a^2)\gamma \left( \| \langle u \rangle \|_{L^2} + \| \langle u \rangle \|_{L^2} \right)^2
\leq \frac{\varepsilon \gamma}{8} \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} + B\gamma \cdot \left( K^8 + K^{\frac{4}{2}} \right) \| \langle u \rangle \|^{2}_{L^2}
\leq \frac{\varepsilon \gamma}{8} \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} + B\gamma K^{\frac{4}{2}} \| \langle u \rangle \|^{2}_{L^2},
\]
where in the second estimate above, we have used the assumption \(|a| < 1\) and in the second last estimate, we have used the assumption (4.1).

**Estimate of \( J_2 \).** By Young’s inequality, we have
\[
J_2 \leq 3|a|\gamma \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} \left( \int_{\mathbb{T}^2} \langle u \rangle^2 \, dx \, dy \right)^{\frac{3}{4}}
\leq \frac{\varepsilon \gamma}{8} \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} + B\gamma \int_{\mathbb{T}^2} \langle u \rangle^2 \, dx \, dy
\leq \frac{\varepsilon \gamma}{8} \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} + B\gamma \| \Delta u \|^{2}_{L^2} \| \langle u \rangle \|^{2}_{L^2}
\leq \frac{\varepsilon \gamma}{8} \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} + B\gamma \| \Delta u \|^{2}_{L^2} \| \langle u \rangle \|^{2}_{L^2},
\]
where in the last estimate above, we have used the Gagliardo–Nirenberg’s inequality in 2D:
\[
\| \Delta u \|_{L^\infty} \leq B \| \Delta u \|_{L^2} \| \langle u \rangle \|_{L^2}.
\]

**Estimate of \( J_3 \).** By the Young’s inequality and Gagliardo–Nirenberg’s inequality in 2D:
\[
\| \Delta u \|_{L^4} \leq B \| \Delta u \|_{L^2} \| \langle u \rangle \|_{L^2},
\]
we have
\[
J_3 \leq |a|\gamma \| \partial_y^2 \langle u \rangle \|_{L^2} \| \langle u \rangle \|_{L^2} \leq \frac{\varepsilon \gamma}{8} \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} + B\gamma \int_{\mathbb{T}^2} u_y^6 \, dx \, dy
\leq \frac{\varepsilon \gamma}{8} \| \partial_y^2 \langle u \rangle \|^{2}_{L^2} + B\gamma \| \Delta u \|^{2}_{L^2} \| \langle u \rangle \|^{2}_{L^2}.
\]
Estimate of $J_4$. The estimate of $J_4$ is similar to the one of $J_3$:

$$
J_4 \leq |b| \|\partial_t^2(u)\|_{L^2} \|u^3\|_{L^2}^3 \leq \frac{\varepsilon \gamma}{8} \|\partial_x^2(u)\|_{L^2}^2 + B \gamma \int_{\mathbb{R}^2} u^3 dx dy
$$

$$
\leq \frac{\varepsilon \gamma}{8} \|\partial_x^2(u)\|_{L^2}^2 + B \gamma \|\Delta u\|_{L^2} \|u\|_{L^2}^3,
$$

where we have used the Gagliardo–Nirenberg’s inequality in 2D in the last estimate above:

$$
\|u\|_{L^4} \leq B \|\Delta u\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}}.
$$

Combining all the estimates of the terms $J_1, J_2, J_3$ and $J_4$ with (1.3), we have

$$
\frac{1}{2} \frac{d}{dt} \|\langle u \rangle \|_{L^2}^2 + \frac{\varepsilon \gamma}{2} \|\partial_x^2(u)\|_{L^2}^2 \leq B \gamma K \|\langle u \rangle \|_{L^2}^2 + Ba^2 \gamma \|\Delta u\|_{L^2} \|u\|_{L^2}^2 \|\langle u \rangle \|_{L^2}^2
$$

$$
+ Ba^2 \gamma \|\Delta u\|_{L^2} \|u\|_{L^2}^2 + B \gamma \|\Delta u\|_{L^2} \|u\|_{L^2}^2.
$$

(4.4)

Note that by the Poincare’s inequality in 1D and (1.2), we have

$$
B \gamma K \|\langle u \rangle \|_{L^2}^2 \leq B \gamma \frac{\varepsilon \lambda_1^2}{4B} \frac{1}{\lambda_1^4} \|\partial_x^2(u)\|_{L^2}^2 = \frac{\varepsilon \gamma}{4} \|\partial_x^2(u)\|_{L^2}^2.
$$

This together with (1.4) further gives

$$
\frac{1}{2} \frac{d}{dt} \|\langle u \rangle \|_{L^2}^2 + \frac{\varepsilon \gamma}{4} \|\partial_x^2(u)\|_{L^2}^2 \leq B \gamma \left( a^2 \|\Delta u\|_{L^2} \|u\|_{L^2}^2 \right)^2
$$

$$
+ \|\Delta u\|_{L^2} \|u\|_{L^2}^2 + \|\Delta u\|_{L^2} \|u\|_{L^2}^2 \|\langle u \rangle \|_{L^2}^2
$$

(4.5)

In particular, we have the following ODE:

$$
\frac{1}{2} \frac{d}{dt} \|\langle u \rangle \|_{L^2}^2 - B \gamma \|\Delta u\|_{L^2} \|u\|_{L^2}^2 \|\langle u \rangle \|_{L^2}^2
$$

$$
\leq B \gamma \left( a^2 \|\Delta u\|_{L^2} \|u\|_{L^2}^2 + \|\Delta u\|_{L^2} \|u\|_{L^2}^2 \right).
$$

(4.6)

Let

$$
\rho(t) := \exp \left( -B \gamma \int_0^t \|\Delta u\|_{L^2} \|u\|_{L^2}^2 d\tau \right)
$$

be the integrating factor, and solve the ODE (1.6), we have

$$
\|\langle u \rangle(t) \|_{L^2}^2 \leq \frac{\|\langle u \rangle(0) \|_{L^2}^2}{\rho(t)} + \frac{Ba^2 \gamma}{\rho(t)} \int_0^t \rho(\tau) \|\Delta u\|_{L^2} \|u\|_{L^2}^{\frac{1}{2}} d\tau
$$

$$
+ \frac{B \gamma}{\rho(t)} \int_0^t \rho(\tau) \|\Delta u\|_{L^2} \|u\|_{L^2}^2 d\tau
$$

$$
\leq \frac{\|\langle u \rangle(0) \|_{L^2}^2}{\rho(t)} + \frac{Ba^2 \gamma}{\rho(t)} \int_0^t \|\Delta u\|_{L^2} \|u\|_{L^2}^2 d\tau
$$

$$
+ \frac{B \gamma}{\rho(t)} \int_0^t \|\Delta u\|_{L^2} \|u\|_{L^2}^2 d\tau.
$$

(4.7)
By the bootstrap assumptions, we first estimate the term \( \frac{1}{\rho(t)} \) as follows:

\[
\frac{1}{\rho(t)} = \exp \left( B \gamma \int_0^t \| \Delta u_{\partial}(\tau) \|^2 \| u_{\partial}(\tau) \|^2 \, d\tau \right)
\leq \exp \left( B \cdot \varepsilon \gamma \| u_{\partial}(0) \|^2 \int_0^t \| \Delta u_{\partial}(\tau) \|^2 \, d\tau \right)
\leq \exp \left( B \| u_{\partial}(0) \|_{L^2}^4 \right).
\]

Next, we estimate the two integrals in (4.7) as follows:

\[
\gamma \int_0^t \| \Delta u_{\partial}(\tau) \|^2 \| u_{\partial}(\tau) \|^2 \, d\tau \leq B \| u_{\partial}(0) \|_{L^2}^6 \varepsilon \gamma \int_0^t \| \Delta u_{\partial}(\tau) \|^2 \, d\tau
\leq B \| u_{\partial}(0) \|_{L^2}^6
\]

and

\[
\gamma \int_0^t \| \Delta u_{\partial}(\tau) \|_{L^2}^2 \| u_{\partial}(\tau) \|_{L^2}^3 \, d\tau \leq B \| u_{\partial}(0) \|_{L^2}^3 \gamma \int_0^t \| \Delta u_{\partial}(\tau) \|^2 \, d\tau
= B \| u_{\partial}(0) \|_{L^2}^3 \gamma \varepsilon \gamma \int_0^t \left( \frac{\lambda \gamma}{\varepsilon} \right) \| \Delta u_{\partial}(\tau) \|^2 \, d\tau
\leq B \left( \frac{\gamma}{\lambda \gamma} \right)^{\frac{3}{2}} \| u_{\partial}(0) \|_{L^2}^4.
\]

Therefore, by (4.7), (4.8), (4.9) and (4.10), we have

\[
\| \langle u \rangle (t_0) \|_{L^2}^2 \leq B \exp \left( B \| u_{\partial}(0) \|_{L^2}^6 \right) \left( a^2 \| u_{\partial}(0) \|_{L^2}^6 + \frac{\gamma}{\lambda \gamma} \| u_{\partial}(0) \|_{L^2}^4 + \| \langle u \rangle(0) \|_{L^2}^2 \right).
\]

The proof is complete. \( \square \)

**Proposition 4.2.** Assume the bootstrap assumptions and \( |a| \) is sufficiently small with satisfying

\[
a^2 B_2 \exp \left( B_2 \| u_{\partial}(0) \|_{L^2}^4 \right) \| u_{\partial}(0) \|_{L^2}^6 \leq \min \left\{ \frac{1}{12}, \frac{1}{12} \left( \frac{\varepsilon \lambda^2}{4 B_1} \right)^{\frac{3}{2}} \right\},
\]

where \( B_1, B_2 > 1 \) are the constants defined in Lemma 4.7. Then there exists a \( \gamma_0 > 0 \), which only depends on \( \varepsilon, b, \| u_{\partial}(0) \|_{L^2} \) and any dimensional constants, such that for any \( 0 < \gamma < \gamma_0 \) and for any initial data \( \langle u \rangle(0, \cdot) \) of (3.2) with satisfying

\[
\| \langle u \rangle(0, \cdot) \|_{L^2} \leq \frac{1}{B_2 \exp(B_2)} \min \left\{ \frac{1}{12}, \frac{1}{12} \left( \frac{\varepsilon \lambda^2}{4 B_1} \right)^{\frac{3}{2}} \right\},
\]

the following estimates hold: for any \( 0 \leq t \leq t_0 \),

\[
\| \langle u \rangle(t, \cdot) \|_{L^2}^2 + \varepsilon \gamma \int_0^t \| \partial_y^2 \langle u \rangle(\tau, \cdot) \|_{L^2}^2 \, d\tau \leq B_3,
\]

where \( B_3 > 0 \) is a constant only depending on \( \varepsilon, b, \| u_{\partial}(0) \|_{L^2} \) and any dimensional constants.
Proof. (1). Since $\gamma \to 0$ as $\gamma \to 0$, this allows us to choose a $\gamma_0$ sufficiently small, such that for any $0 < \gamma < \gamma_0$, it holds that

$$B_2 \exp \left( B_2 \| u_\gamma (0) \|_{L^2}^4 \right) \| u_\gamma (0) \|_{L^2}^4 \left( \frac{\gamma}{\lambda \gamma} \right)^{\frac{1}{4}} < \min \left\{ \frac{1}{12} \left( \frac{\varepsilon \lambda^2}{4B_1} \right)^{\frac{1}{4}} \right\}.$$  \hfill (4.14)

Note that here $\gamma_0$ only depends on $\varepsilon, b, \| u_\gamma (0) \|_{L^2}$ and any dimensional constants. We now let

$$K := \min \left\{ \frac{1}{2}, \frac{1}{2} \left( \frac{\varepsilon \lambda^2}{4B_1} \right)^{\frac{1}{4}} \right\},$$

in Lemma 4.1 and $t(K)$ be the maximal time such that $\| \langle u \rangle (t, \cdot) \|_{L^2_y} \leq K$ on $[0, t(K)]$. Without loss of generality, we might assume $K = \frac{1}{2} \left( \frac{\varepsilon \lambda^2}{4B_1} \right)^{\frac{1}{4}}$, otherwise we can take a larger $B_1$ in Lemma 4.1 to make such an assumption hold. By the continuity of $L^2$ norm of the mild solution, $t(K) > 0$. Our goal is to show that $t(K) = t_0$. Otherwise, assume $0 < t(K) < t_0$. By Lemma 4.1, (4.11), (4.12) and (4.14), we have for any $0 < \gamma < \gamma_0$,

$$\frac{1}{4} \left( \frac{\varepsilon \lambda^2}{4B_1} \right)^{\frac{1}{4}} \leq a^2 B_2 \exp \left( B_2 \| u_\gamma (0) \|_{L^2}^4 \right) \| u_\gamma (0) \|_{L^2}^4$$

$$+ B_2 \exp \left( B_2 \| u_\gamma (0) \|_{L^2}^4 \right) \left[ \left( \frac{\gamma}{\lambda \gamma} \right)^{\frac{1}{4}} \| u_\gamma (0) \|_{L^2}^4 \right] + \| \langle u \rangle (0) \|_{L^2_y}^2 \right]$$

$$< \left[ \frac{1}{6} + \frac{1}{12} \cdot \frac{B_2 \exp \left( B_2 \| u_\gamma (0) \|_{L^2}^4 \right)}{B_2 \exp \left( B_2 \| u_\gamma (0) \|_{L^2}^4 \right)} \right] \left( \frac{\varepsilon \lambda^2}{4B_1} \right)^{\frac{1}{4}}$$

$$\leq \frac{1}{4} \left( \frac{\varepsilon \lambda^2}{4B_1} \right)^{\frac{1}{4}},$$

where in the last estimate, we have used the fact that $\| u_\gamma (0) \|_{L^2} \leq 1$, which is guaranteed by the assumption (4.11). This clearly gives a contradiction and therefore the estimate

$$\| \langle u \rangle (t, \cdot) \|_{L^2_y} \leq \frac{1}{4} \left( \frac{\varepsilon \lambda^2}{4B_1} \right)^{\frac{1}{4}},$$  \hfill (4.15)

has to be true until the maximal time, namely, $t(K) = t_0$. This gives the first part of the estimate (4.13).

(2). Now we turn to prove the second part of the estimate (4.13). Taking the time integral on both sides of (4.5) and using (4.14), (4.15) and the bootstrap
Proof. Multiplying here, \( \gamma \) where \( L \) being sufficiently small, that is, for example, for \( 0 \leq \gamma < \gamma_0 \),

\[
\varepsilon \gamma \int_0^t \| \partial_\tau^2 (u) \|_{L^2}^2 d\tau \leq B \gamma \int_0^t \| \Delta u_{\varepsilon}(\tau) \|_{L^2}^2 \| u_{\varepsilon}(\tau) \|_{L^2} d\tau
\]

\[
+ B \gamma \int_0^t \| \Delta u_{\varepsilon}(\tau) \|_{L^2}^2 \| u_{\varepsilon}(\tau) \|_{L^2}^2 \| (u) \|_{L^2}^2 d\tau
\]

\[
+ B \gamma \int_0^t \| \Delta u_{\varepsilon}(\tau) \|_{L^2}^2 \| u_{\varepsilon}(\tau) \|_{L^2}^2 d\tau \leq B_3,
\]

where \( B_3 \) is a constant which only depends on \( \varepsilon, b, \| u_{\varepsilon}(0) \|_{L^2} \) and any dimensional constants. The proof is complete.

5. Bootstrap estimates and Proof of Theorem \( 1.5 \)

In this section, we show that the bootstrap assumptions can be improved for \( \gamma \) being sufficiently small, that is, for example, for \( 0 \leq s \leq t \leq t_0 \),

1. \( \| u_{\varepsilon}(t) \|_{L^2} \leq 16 e^{-\frac{\Delta t}{2}} \| u_{\varepsilon}(s) \|_{L^2} \),
2. \( \varepsilon \gamma \int_0^t \| \Delta u_{\varepsilon}(\tau) \|_{L^2}^2 d\tau \leq 8 \| u_{\varepsilon}(s) \|_{L^2}^2 \),

Note that this then gives a contradiction if \( t_0 < \infty \), and this allows us to conclude the global existence of the solution to the problem \( (1.5) \). Now we turn to some details.

The following result shows that the second estimate in the bootstrap assumptions can be improved.

**Proposition 5.1.** Assume the bootstrap assumptions and \( (4.11) \). Moreover, we assume that

\[
|a| \leq \frac{\varepsilon}{10^7 L \| u_{\varepsilon}(0) \|_{L^2}^2},
\]

where \( L > 0 \) is some dimensional constant sufficiently large, then there exists a \( 0 < \gamma_1 \leq \gamma_0 \) sufficiently small, which only depends on \( \varepsilon, a, b, \| (u) \|_{L^2}, \| u_{\varepsilon}(0) \|_{L^2} \) and any dimensional constants, such that for any \( 0 < \gamma < \gamma_1 \) and any \( 0 \leq s \leq t \leq t_0 \),

\[
\varepsilon \gamma \int_s^t \| \Delta u_{\varepsilon}(\tau) \|_{L^2}^2 d\tau \leq 5 \| u_{\varepsilon}(s) \|_{L^2}^2.
\]

Here, \( \gamma_0 \) is defined in Proposition \( 4.2 \).

**Proof.** Multiplying \( u_{\varepsilon} \) on both sides of \( (3.3) \), we have

\[
\frac{1}{2} \frac{d}{dt} \| u_{\varepsilon} \|_{L^2}^2 + \varepsilon \| \Delta u_{\varepsilon} \|_{L^2}^2 = a \gamma \int_{T^2} u_{\varepsilon} \Delta \left[ a u_{\varepsilon}^3 + 3 \langle u \rangle^2 u_{\varepsilon} + 3 \langle u \rangle u_{\varepsilon}^2 \right] dxdy
\]

\[
+ b \gamma \int_{T^2} u_{\varepsilon} \Delta \left[ u_{\varepsilon}^2 + 2 \langle u \rangle u_{\varepsilon} \right] dxdy
\]

(5.3)

\[
= K_1 + K_2 + K_3 + K_4,
\]

where

\[
K_1 := \gamma \int_{T^2} \Delta u_{\varepsilon} \left( a u_{\varepsilon}^3 + b u_{\varepsilon}^2 \right) dxdy, \quad K_2 := 3a \gamma \int_{T^2} \Delta u_{\varepsilon} \langle u \rangle^2 u_{\varepsilon} dxdy
\]

\[
K_3 := 3a \gamma \int_{T^2} \Delta u_{\varepsilon} \langle u \rangle u_{\varepsilon}^2 dxdy \quad \text{and} \quad K_4 := 2b \gamma \int_{T^2} \Delta u_{\varepsilon} \langle u \rangle u_{\varepsilon} dxdy.
\]
Estimate of $K_1$. By the Young’s inequality and the Gagliardo-Nirenberg’s inequalities in 2D:

\begin{equation}
\|u\|_{L^6} \leq B \|\Delta u\|_{L^2} \cdot \|u\|_{L^2},
\end{equation}

and

\begin{equation}
\|\nabla u\|_{L^4} \leq B \|\Delta u\|_{L^2} \cdot \|u\|_{L^2},
\end{equation}

we have

\begin{align*}
K_1 & \leq 3|a| \gamma \int_{\mathbb{T}^2} \|u\|^2 \|\nabla u\|^2 \text{d}x \text{d}y + 2|b| \gamma \int_{\mathbb{T}^2} |\nabla u|^2 \|u\| \text{d}x \text{d}y \\
& \leq 3|a| \gamma \|u\|_{L^4}^2 \|\nabla u\|_{L^4}^2 + 2|b| \gamma \|\nabla u\|_{L^4}^2 \|u\|_{L^2} \\
& \leq L|a| \gamma \|\Delta u\|_{L^2}^2 \|u\|_{L^2}^2 + B \gamma \|\Delta u\|_{L^2} \|u\|_{L^2},
\end{align*}

where $L > 0$ is some dimensional constant.

Estimate of $K_2$. By Young’s inequality and the Gagliardo–Nirenberg’s inequality in 1D:

\begin{equation}
\|\langle u \rangle\|_{L^\infty} \leq B \|\partial^2_y \langle u \rangle\|_{L^2} \|\langle u \rangle\|_{L^2},
\end{equation}

we have

\begin{align*}
K_2 & \leq 3a \gamma \|\Delta u\|_{L^2} \left( \int_{\mathbb{T}^2} \langle u \rangle^4 u^2_y \text{d}x \text{d}y \right)^{\frac{1}{2}} \\
& \leq \frac{\varepsilon \gamma}{10} \|\Delta u\|^2_{L^2} + B a^2 \gamma \int_{\mathbb{T}^2} \langle u \rangle^4 u^2_y \text{d}x \text{d}y \\
& \leq \frac{\varepsilon \gamma}{10} \|\Delta u\|^2_{L^2} + B a^2 \gamma \|\langle u \rangle\|^2_{L^\infty} \|u\|^2_{L^2} \\
& \leq \frac{\varepsilon \gamma}{10} \|\Delta u\|^2_{L^2} + B \gamma \|\partial^2_y \langle u \rangle\|^2_{L^2} \|\langle u \rangle\|^3_{L^2} \|u\|_{L^2}^2,
\end{align*}

where in the last estimate, we have used the assumption that $|a| < 1$.

Estimate of $K_3$. By Young’s inequality, (5.4) and (5.5), we have

\begin{align*}
K_3 & \leq 3a \gamma \|\Delta u\|_{L^2} \left( \int_{\mathbb{T}^2} \langle u \rangle^2 u_y^3 \text{d}x \text{d}y \right)^{\frac{1}{2}} \\
& \leq \frac{\varepsilon \gamma}{10} \|\Delta u\|^2_{L^2} + B \gamma \int_{\mathbb{T}^2} \langle u \rangle^2 u_y^3 \text{d}x \text{d}y \\
& \leq \frac{\varepsilon \gamma}{10} \|\Delta u\|^2_{L^2} + B \gamma \|\langle u \rangle\|^2_{L^\infty} \int_{\mathbb{T}^2} u_y^3 \text{d}x \text{d}y \\
& \leq \frac{\varepsilon \gamma}{10} \|\Delta u\|^2_{L^2} + B \gamma \|\partial^2_y \langle u \rangle\|_{L^2} \partial^3 \langle u \rangle_{L^2} \|\langle u \rangle\|_{L^2}^3 \|u\|_{L^2}^3.
\end{align*}

Estimate of $K_4$. Again, by Young’s inequality and the Gagliardo–Nirenberg’s inequalities (5.5) and (5.4), we see that

\begin{align*}
K_4 & \leq 2b \gamma \|\Delta u\|_{L^2} \left( \int_{\mathbb{T}^2} \langle u \rangle^2 u_y^2 \text{d}x \text{d}y \right)^{\frac{1}{2}} \\
& \leq \frac{\varepsilon \gamma}{10} \|\Delta u\|^2_{L^2} + B \gamma \int_{\mathbb{T}^2} \langle u \rangle^2 u_y^2 \text{d}x \text{d}y \\
& \leq \frac{\varepsilon \gamma}{10} \|\Delta u\|^2_{L^2} + B \gamma \|\langle u \rangle\|^2_{L^\infty} \|u\|_{L^2}^2 \\
& \leq \frac{\varepsilon \gamma}{10} \|\Delta u\|^2_{L^2} + B \gamma \|\partial^2_y \langle u \rangle\|_{L^2} \partial^2 \langle u \rangle_{L^2} \|\langle u \rangle\|_{L^2}^2 \|u\|_{L^2}^2.
\end{align*}
Combining (5.3) with estimates of the terms $K_1$, $K_2$, $K_3$ and $K_4$, we derive that

$$
\frac{1}{2} \frac{d}{dt} \| u_\gamma \|_{L^2}^2 + \frac{\varepsilon \gamma}{4} \| \Delta u_\gamma \|_{L^2}^2 \leq L|a|\gamma \| \Delta u_\gamma \|_{L^2}^2 \| u_\gamma \|_{L^2}^2
$$

$$
B \gamma \left[ \| \Delta u_\gamma \|_{L^2}^2 \| u_\gamma \|_{L^2}^2 + \| \partial_y^2 (u) \|_{L^2}^2 \| u_\gamma \|_{L^2}^2 \right],
$$

which implies

$$
\varepsilon \gamma \int_s^t \| \Delta u_\gamma (\tau) \|_{L^2}^2 d\tau \leq 2 \| u_\gamma (s) \|_{L^2}^2 + \tilde{K}_0 + \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3 + \tilde{K}_4,
$$

where

$$
\tilde{K}_0 := 4L|a|\gamma \int_s^t \| \Delta u_\gamma (\tau) \|_{L^2}^2 \| u_\gamma (\tau) \|_{L^2}^2 d\tau,
$$

$$
\tilde{K}_1 := L\gamma \int_s^t \| \Delta u_\gamma (\tau) \|_{L^2}^2 \| u_\gamma (\tau) \|_{L^2}^2 d\tau,
$$

$$
\tilde{K}_2 := B \gamma \int_s^t \| \partial_y^2 (u) (\tau) \|_{L^2}^2 \| (u (\tau)) \|_{L^2}^2 \| u_\gamma (\tau) \|_{L^2}^2 d\tau,
$$

$$
\tilde{K}_3 := B \gamma \int_s^t \| \partial_y^2 (u) (\tau) \|_{L^2}^2 \| (u (\tau)) \|_{L^2}^2 \| \Delta u_\gamma \|_{L^2}^2 \| u_\gamma \|_{L^2}^2 d\tau,
$$

and

$$
\tilde{K}_4 := B \gamma \int_s^t \| \partial_y^2 (u) (\tau) \|_{L^2}^2 \| (u (\tau)) \|_{L^2}^2 \| u_\gamma (\tau) \|_{L^2}^2 d\tau.
$$

**Estimate of $\tilde{K}_0$.** By the bootstrap assumptions and (5.1), we have

$$
\tilde{K}_0 \leq \frac{1600L|a|}{\varepsilon} \| u_\gamma (0) \|_{L^2}^2 \cdot \varepsilon \gamma \int_s^t \| \Delta u_\gamma (\tau) \|_{L^2}^2 d\tau
$$

$$
\leq \frac{16000L|a|}{\varepsilon} \| u_\gamma (0) \|_{L^2}^2 \cdot \| u_\gamma (s) \|_{L^2}^2
$$

$$
\leq \frac{20}{\gamma \lambda_\gamma}.
$$

**Estimate of $\tilde{K}_1$.** By the bootstrap assumptions, we have

$$
\tilde{K}_1 \leq B \gamma \int_s^t e^{-\frac{3\lambda_\gamma (1-\varepsilon)}{4}} \| \Delta u_\gamma (\tau) \|_{L^2}^2 \| u_\gamma (s) \|_{L^2}^2
$$

$$
\leq B \gamma \| u_\gamma (s) \|_{L^2}^2 \left( \int_s^t e^{-\frac{3\lambda_\gamma (1-\varepsilon)}{4}} \| \Delta u_\gamma (\tau) \|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \left( \int_s^t \| \Delta u_\gamma (\tau) \|_{L^2}^2 d\tau \right)^{\frac{1}{2}}
$$

$$
\leq B \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{1}{2}} \cdot \| u_\gamma (s) \|_{L^2}^2
$$

$$
\leq B \| u_\gamma (0) \|_{L^2} \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{1}{2}} \cdot \| u_\gamma (s) \|_{L^2}^2,
$$

$$
\leq B \| u_\gamma (0) \|_{L^2} \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{1}{2}} \cdot \| u_\gamma (s) \|_{L^2}^2.
$$
where in the last estimate, we have used the estimate $\|u_\theta(s)\|_{L^2} \leq 20\|u_\theta(0)\|_{L^2}$, which is a consequence of the first estimate in the bootstrap assumptions.

Estimate of $\tilde{K}_2$. By the bootstrap assumptions and Proposition 4.2, we have

$$
\tilde{K}_2 \leq C \gamma \int_t^s \left\| \partial_y^2(u)(\tau) \right\|_{L^2_y} \left\| u_\theta(\tau) \right\|_{L^2_y} d\tau \leq C \gamma \left\| u_\theta(s) \right\|_{L^2} \int_s^t e^{-\frac{\lambda_2(s-s)}{2}} \| \partial_y^2(u)(\tau) \|_{L^2_y} d\tau \leq C \gamma \left\| u_\theta(s) \right\|_{L^2} \left( \int_s^t e^{-\lambda_2(s-s)} d\tau \right) \left( \int_s^t \left\| \partial_y^2(u)(\tau) \right\|_{L^2_y}^2 d\tau \right)^{\frac{1}{2}} \leq C \left( \frac{\gamma}{\lambda_2} \right) \left\| u_\theta(s) \right\|_{L^2}^2.
$$

Note that since we have used Proposition 4.2, the implicit constants in the above estimates can also depend on $a$, $\| (u)(0) \|_{L^2}$ and $\| u_\theta(0) \|_{L^2}$.

Estimate of $\tilde{K}_3$. Again, using the bootstrap assumptions and Proposition 4.2, we see that

$$
\tilde{K}_3 \leq C \gamma \left\| u_\theta(s) \right\|_{L^2}^3 \int_s^t e^{-\frac{3\lambda_2(s-s)}{4}} \left\| \partial_y^2(u)(\tau) \right\|_{L^2_y}^2 \left\| \Delta u_\theta(\tau) \right\|_{L^2} d\tau \leq C \gamma \left\| u_\theta(s) \right\|_{L^2}^3 \left( \int_s^t e^{-3\lambda_2(s-s)} d\tau \right)^{\frac{1}{4}} \left( \int_s^t \left\| \partial_y^2(u) \right\|_{L^2_y}^2 d\tau \right)^{\frac{1}{2}} \leq C \left( \frac{\gamma}{\lambda_2} \right)^{\frac{3}{4}} \left\| u_\theta(s) \right\|_{L^2}^4 \leq C \left( \frac{\gamma}{\lambda_2} \right)^{\frac{3}{4}} \left\| u_\theta(s) \right\|_{L^2}^2.
$$

Estimate of $\tilde{K}_4$. Similarly, we have

$$
\tilde{K}_4 \leq C \gamma \int_s^t \left\| \partial_y^2(u)(\tau) \right\|_{L^2_y}^2 \left\| u_\theta(\tau) \right\|_{L^2_y}^2 d\tau \leq C \gamma \left\| u_\theta(s) \right\|_{L^2}^2 \int_s^t e^{-\frac{\lambda_2(s-s)}{2}} \left\| \partial_y^2(u)(\tau) \right\|_{L^2_y}^2 d\tau \leq C \gamma \left\| u_\theta(s) \right\|_{L^2}^2 \left( \int_s^t e^{-\frac{2\lambda_2(s-s)}{2}} d\tau \right)^{\frac{1}{2}} \left( \int_s^t \left\| \partial_y^2(u)(\tau) \right\|_{L^2_y}^2 d\tau \right)^{\frac{1}{2}} \leq C \left( \frac{\gamma}{\lambda_2} \right)^{\frac{3}{4}} \left\| u_\theta(s) \right\|_{L^2}^2.
$$
Therefore, by (5.7) and the estimates of the terms \( \tilde{K}_0, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3 \) and \( \tilde{K}_4 \), we have
\[
\varepsilon \gamma \int_s^t \| \Delta u_\gamma (\tau) \|^2_{L^2} \, d\tau \leq 3\| u_\gamma (s) \|^2_{L^2} + C \| u_\gamma (s) \|^2_{L^2} \left[ \left( \frac{\gamma}{\lambda_\gamma} \right) + \left( \frac{\gamma}{\lambda_\gamma} \right)^2 + \| u_\gamma (0) \|^2_{L^2} \left( \frac{\gamma}{\lambda_\gamma} \right)^2 \right].
\] (5.8)
Note that \( \tilde{K}_0 \to 0 \) as \( \lambda \to 0 \), it suffices to pick \( \gamma_1 \) sufficiently small, such that
\[
C \left[ \left( \frac{\gamma}{\lambda_\gamma} \right) + \left( \frac{\gamma}{\lambda_\gamma} \right)^2 + \| u_\gamma (0) \|^2_{L^2} \left( \frac{\gamma}{\lambda_\gamma} \right)^2 \right] < 2,
\]
and this together with (5.8) clearly implies the desired estimate (5.9).

Our next goal is to improve the first estimate in the bootstrap assumption. We derive this via several steps. We start with estimating \( \| u_\gamma (t) \|_{L^2} \), where \( t \) is sufficiently close \( s \).

**Proposition 5.2.** Under the assumption of Proposition 5.7, there exists a \( 0 < \gamma_2 \leq \gamma_0 \) which only depends on \( \varepsilon, a, b, \| u(0) \|_{L^2}, \| u_\gamma (0) \|_{L^2} \) and any dimensional constants, such that for any \( 0 < \gamma < \gamma_2 \) and any \( 0 \leq s \leq t \leq t_0 \),
\[
\| u_\gamma (t) \|_{L^2} \leq \frac{3}{2} \| u_\gamma (s) \|_{L^2}.
\] (5.9)

**Proof.** Note that by (5.7), it is immediate that
\[
\| u_\gamma (t) \|_{L^2}^2 \leq \| u_\gamma (s) \|_{L^2}^2 + \frac{\tilde{K}_0 + \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3 + \tilde{K}_4}{2}
\]
\[
\leq 1.1 \| u_\gamma (s) \|_{L^2}^2 + \frac{\tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3 + \tilde{K}_4}{2}
\]
where \( \tilde{K}_0, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3 \) and \( \tilde{K}_4 \) are defined in the proof of Proposition 5.1 (recall that \( \tilde{K}_0 \leq \frac{\| u_\gamma (0) \|_{L^2}^2}{10} \)). The desired estimate (5.9) then follows from the argument that we have used in (5.8).

Next, we estimate \( \| u_\gamma (t) \|_{L^2} \), where \( t \) is relatively “far away” from \( s \). We start with rewriting the Duhamel’s formula (3.4) into a slightly different form: for any \( \tilde{\tau} > 0 \) with \( 0 \leq s \leq s + \tilde{\tau} \), we can write
\[
u_\gamma (s + \tilde{\tau}) = S_{\tilde{\tau}} \left( u_\gamma (s) \right) + a \gamma \int_s^{s + \tilde{\tau}} S_{\tilde{\tau} + \tau - \tau} \left[ \Delta \left( \frac{u^2}{\gamma^2} + 3(u)^2 u_\gamma + 3(u) u^2_\gamma \right) \right. \\
\left. - \int_\gamma \Delta \left( \frac{u^2}{\gamma^2} + 3(u)^2 u_\gamma + 3(u) u^2_\gamma \right) \ dx \right] \ dx \right] \ dx
\]
\[
+ b \gamma \int_s^{s + \tilde{\tau}} S_{\tilde{\tau} + \tau - \tau} \left[ \Delta \left( \frac{u^2}{\gamma^2} + 2(u) u_\gamma \right) - \int_\gamma \Delta \left( \frac{u^2}{\gamma^2} + 2(u) u_\gamma \right) \ dx \right] \ dx.
\] (5.10)

**Proposition 5.3.** Assume the bootstrap assumptions, (4.11) , (5.1) and
\[
|a| \leq \frac{\varepsilon}{10^6 L'B_3^2 \| u_\gamma (0) \|_{L^2}},
\] (5.11)
where \( B_3 \) is the constant defined in Proposition 4.2 and \( L' > 0 \) is some dimensional constant. Let \( \tau^* := \frac{1}{\lambda_\gamma} \). If \( t_0 \geq \tau^* \), then there exists a \( 0 < \gamma_3 \leq \gamma_0 \), which only
depends on \( \varepsilon, a, b, \|\langle u\rangle(0)\|_{L^2} \) and \( \|u_\#(0)\|_{L^2} \), such that for any \( 0 < \gamma < \gamma_3 \), one has
\[
\|u_\#(\tau^* + s)\|_{L^2} \leq \frac{1}{e} \|u_\#(s)\|_{L^2} .
\]

**Proof.** Taking \( L^2 \) norm on both sides of (5.10), we have
\[
\|u_\#(\tau^* + s)\|_{L^2} \leq H_1 + H_2 + H_3,
\]
where
\[
H_1 := \|S_\tau^*(u_\#(s))\|_{L^2}, \quad H_2 := 2|a|\gamma \int_s^{\tau^*+s} \|\Delta \left( u_\#^3 + 3\langle u\rangle^2 u_\# + 3\langle u\rangle u_\#^3 \right) \|_{L^2} \, d\tau
\]
and
\[
H_3 := 2|b|\gamma \int_s^{\tau^*+s} \|\Delta \left( u_\#^2 + 2\langle u\rangle u_\# \right) \|_{L^2} \, d\tau.
\]

**Estimate of \( H_1 \).** By Proposition 1.4, we have
\[
H_1 \leq 10e^{-\lambda_s \tau^*} \|u_\#(s)\|_{L^2} = \frac{10}{e^2} \|u_\#(s)\|_{L^2}.
\]

The estimates of \( H_2 \) and \( H_3 \) are the most technical parts of this paper. For simplicity, we would like to first collect all the Gagliardo–Nirenberg’s inequalities that one might need for these estimates.

- Gagliardo–Nirenberg’s inequalities in 1D:
  \[
  \|\langle u\rangle\|_{L^\infty} \leq C \|\partial_\#^2 \langle u\rangle\|_{L^2}^{\frac{1}{2}} \|\langle u\rangle\|_{L^2}^{\frac{3}{2}},
  \]
  and
  \[
  \|\partial_\# \langle u\rangle\|_{L^4} \leq C \|\partial_\#^2 \langle u\rangle\|_{L^2} \|\langle u\rangle\|_{L^2}^{\frac{3}{2}}.
  \]

- Gagliardo–Nirenberg’s inequalities in 2D:
  \[
  \|u_\#\|_{L^\infty} \leq C \|\Delta u_\#\|_{L^2}^{\frac{1}{2}} \|u_\#\|_{L^2}^{\frac{3}{2}},
  \]
  and
  \[
  \|\nabla u_\#\|_{L^4} \leq C \|\Delta u_\#\|_{L^2}^{\frac{1}{2}} \|u_\#\|_{L^2}^{\frac{3}{2}}.
  \]

**Estimate of \( H_2 \).** By triangle inequality, we further bound \( H_2 \) as follows:
\[
H_2 \leq H_{2,1} + H_{2,2} + H_{2,3},
\]
where
\[
H_{2,1} := 2|a|\gamma \int_s^{\tau^*+s} \|\Delta (u_\#^3)\|_{L^2} \, d\tau, \quad H_{2,2} := 6|a|\gamma \int_s^{\tau^*+s} \|\Delta (\langle u\rangle^2 u_\#)\|_{L^2} \, d\tau
\]
and
\[
H_{2,3} := 6|a|\gamma \int_s^{\tau^*+s} \|\Delta (\langle u\rangle u_\#^2)\|_{L^2} \, d\tau.
\]

**Estimate of \( H_{2,1} \).** Note that
\[
\Delta (u_\#^3) = 3u_\#^2 \Delta u_\# + 6u_\# |\nabla u_\#|^2.
\]
Therefore, by the Gagliardo–Nirenberg’s inequalities (\(G_{2,1}\) and \(G_{2,2}\)), we have

\[
H_{2,1} = 2|a|\gamma \int_s^{\tau+\delta} \left\| 3u_\gamma^2 \Delta u_\gamma + 6u_\gamma |\nabla u_\gamma|^2 \right\|_{L^2} d\tau
\]

\[
\leq 6|a|\gamma \int_s^{\tau+\delta} \left\| u_\gamma \right\|^2_{L^\infty} \left\| \Delta u_\gamma \right\|_{L^1} d\tau + 12|a|\gamma \int_s^{\tau+\delta} \left\| u_\gamma \right\|^2_{L^\infty} \left\| \nabla u_\gamma \right\|_{L^4} d\tau
\]

\[
\leq L|a|\gamma \int_s^{\tau+\delta} \left\| \Delta u_\gamma \right\|^2_{L^2} \left\| u_\gamma \right\|_{L^2} d\tau
\]

\[
\leq \frac{20L|a|\left\| u_\gamma(s) \right\|^2_{L^2}}{\varepsilon} \cdot \left( \varepsilon \gamma \int_s^{\tau+\delta} \left\| \Delta u_\gamma(\tau) \right\|^2_{L^2} d\tau \right)
\]

\[(5.15) \leq \frac{200L|a|\left\| u_\gamma(s) \right\|^3_{L^2}}{\varepsilon} \leq \frac{8000L|a|\left\| u_\gamma(0) \right\|^2_{L^2}}{\varepsilon} \cdot \left\| u_\gamma(s) \right\|^2_{L^2} \leq \frac{\left\| u_\gamma(s) \right\|^2_{L^2}}{20},
\]

where in the second last estimate above, we have used the assumption \((5.1)\) for some dimensional constant \(L > 0\) and the fact that \(\left\| u_\gamma(s) \right\|^2_{L^2} \leq 20\left\| u_\gamma(0) \right\|^2_{L^2}\), which is a consequence of the first estimate in the bootstrap assumptions.

**Estimate of** \(H_{2,2}\). A simple computation gives

\[
\Delta (\langle u \rangle^2 u_\gamma) = \langle u \rangle^2 \Delta u_\gamma + 2|\partial_y u \rangle^2 u_\gamma + 2\langle u \rangle \partial_y^2 u \langle u \rangle u_\gamma + 4\langle u \rangle \partial_y u \partial_y u_\gamma.
\]

Therefore, by the Gagliardo–Nirenberg’s inequalities \((G_{1,1}), (G_{1,2}), (G_{2,1})\) and \((G_{2,2})\), we have

\[
H_{2,2} \leq 6|a|\gamma \int_s^{\tau+\delta} \left\| \langle u \rangle^2 \Delta u_\gamma \right\|_{L^2} d\tau + 12|a|\gamma \int_s^{\tau+\delta} \left\| (\partial_y \langle u \rangle)^2 u_\gamma \right\|_{L^2} d\tau
\]

\[
+ 12|a|\gamma \int_s^{\tau+\delta} \left\| \langle u \rangle \partial_y^2 u \langle u \rangle \partial_y u_\gamma \right\|_{L^2} d\tau
\]

\[
\leq 6|a|\gamma \int_s^{\tau+\delta} \left\| \langle u \rangle \right\|^2_{L^\infty} \left\| \Delta u_\gamma \right\|_{L^2} d\tau + 12|a|\gamma \int_s^{\tau+\delta} \left\| u_\gamma \right\|^2_{L^\infty} \left\| \partial_y u \right\|_{L^2} d\tau
\]

\[
+ 12|a|\gamma \int_s^{\tau+\delta} \left\| \langle u \rangle \right\|^2_{L^\infty} \left\| u_\gamma \right\|^2_{L^\infty} \left\| \partial_y^2 u \langle u \rangle \right\|_{L^2} d\tau
\]

\[
+ 24|a|\gamma \int_s^{\tau+\delta} \left\| u_\gamma \right\|^3_{L^\infty} \left\| \partial_y \langle u \rangle \right\|_{L^2} \left\| \nabla u_\gamma \right\|_{L^1} d\tau
\]

\[(5.16) \leq H_{2,2,1} + H_{2,2,2} + H_{2,2,3},
\]

where

\[
H_{2,2,1} := C\gamma \int_s^{\tau+\delta} \left\| \partial_y^2 \langle u \rangle \right\|_{L^2} \left\| \langle u \rangle \right\|_{L^2} \left\| \Delta u_\gamma \right\|_{L^2} d\tau
\]

\[
H_{2,2,2} := C\gamma \int_s^{\tau+\delta} \left\| \Delta u_\gamma \right\|_{L^2} \left\| u_\gamma \right\|_{L^2} \left\| \partial_y^2 \langle u \rangle \right\|_{L^2} \left\| \langle u \rangle \right\|_{L^2} d\tau
\]

and

\[
H_{2,2,3} := C\gamma \int_s^{\tau+\delta} \left\| \partial_y^2 \langle u \rangle \right\|_{L^2} \left\| \langle u \rangle \right\|_{L^2} \left\| \Delta u_\gamma \right\|_{L^2} \left\| u_\gamma \right\|_{L^2} d\tau.
\]
Note that here we have used the estimate

\[ 12 |a| \gamma \int_s^{\tau^*+s} \| u_\# \|_{L^\infty} \| \partial_y^2 (u) \|_{L^2}^2 d\tau \]

\[ + 12 |a| \gamma \int_s^{\tau^*+s} \| \langle u \rangle \|_{L^\infty} \| u_\# \|_{L^\infty} \| \partial_y^2 (u) \|_{L^2}^2 d\tau \leq H_{2.2,2}. \]

Estimate of \( H_{2.2,1} \). By the bootstrap assumptions and Proposition 4.2 we see that

\[ H_{2.2,1} \leq C \gamma \int_s^{\tau^*+s} \| \partial_y^2 (u) \|_{L^2}^{\frac{2}{\gamma}} \| \Delta u_\# \|_{L^2} d\tau \]

\[ \leq C \gamma \left( \int_s^{\tau^*+s} \| \partial_y^2 (u) \|_{L^2} \right)^{\frac{1}{\gamma}} \left( \int_s^{\tau^*+s} \| \partial_y^2 (u) \|_{L^2} \right)^{\frac{2}{\gamma}} \left( \int_s^{\tau^*+s} \| \Delta u_\# \|_{L^2} \right)^{\frac{3}{\gamma}} \]

\[ \leq C \cdot \left( \frac{\gamma}{\lambda \gamma} \right)^{\frac{1}{\gamma}} \| u_\# (s) \|_{L^2}. \]

Estimate of \( H_{2.2,2} \). Using the bootstrap assumptions and Proposition 4.2 again, we have

\[ H_{2.2,2} \leq C \gamma \| u_\# (s) \|_{L^2} \int_s^{\tau^*+s} \| u_\# \|_{L^2} \| \partial_y^2 (u) \|_{L^2} \frac{2}{\gamma} d\tau \]

\[ \leq C \gamma \| u_\# (s) \|_{L^2} \frac{1}{\gamma} \left( \int_s^{\tau^*+s} \| \partial_y^2 (u) \|_{L^2} \right)^{\frac{1}{\gamma}} \left( \int_s^{\tau^*+s} \| \Delta u_\# \|_{L^2} \right)^{\frac{3}{\gamma}} \]

\[ \leq C \cdot \left( \frac{\gamma}{\lambda \gamma} \right)^{\frac{1}{\gamma}} \| u_\# (s) \|_{L^2}. \]

Estimate of \( H_{2.2,3} \). Similarly, we have

\[ H_{2.2,3} \leq C \| u_\# (s) \|_{L^2} \frac{1}{\gamma} \left( \int_s^{\tau^*+s} \| \partial_y^2 (u) \|_{L^2} \right)^{\frac{1}{\gamma}} \left( \int_s^{\tau^*+s} \| \Delta u_\# \|_{L^2} \right)^{\frac{3}{\gamma}} \]

\[ \leq C \cdot \left( \frac{\gamma}{\lambda \gamma} \right)^{\frac{1}{\gamma}} \| u_\# (s) \|_{L^2}. \]

To this end, combining all the estimates of \( H_{2.2,1}, H_{2.2,2} \) and \( H_{2.2,3} \) with (5.16), we conclude that

\[ (5.17) \quad H_{2.2} \leq C \cdot \left[ \left( \frac{\gamma}{\lambda \gamma} \right)^{\frac{1}{\gamma}} + \left( \frac{\gamma}{\lambda \gamma} \right)^{\frac{1}{\gamma}} + \left( \frac{\gamma}{\lambda \gamma} \right)^{\frac{3}{\gamma}} \right] \cdot \| u_\# (s) \|_{L^2}. \]
Estimate of $H_{2,3}$. First of all, we note that

$$
\Delta \left( \langle u \rangle \| u \|_{L^p} \right) = 2 \langle u \rangle |\nabla u| + \langle u \rangle \Delta u = \langle u \rangle + u \Delta u \partial \partial u + 4 \partial u \partial \partial u.
$$

This gives

$$
H_{2,3} \leq 12|a| \int_s^{s+} \| \langle u \rangle \|_{L^p} \| \nabla u \|_{L^q}^2 \cdot d\tau + 12|a| \| \langle u \rangle \|_{L^p} \| \Delta u \|_{L^q} \cdot d\tau
$$

$$
+ 6|a| \int_s^{s+} \| u \|_{L^p} \| \Delta u \|_{L^q} \cdot d\tau + 24|a| \| u \|_{L^p} \| \partial \partial u \|_{L^q} \cdot d\tau.
$$

Using the Gagliardo–Nirenberg’s inequalities ($G_{1,1}$), ($G_{1,2}$), ($G_{2,1}$) and ($G_{2,2}$), we can further bound the last term above by

$$
12|a| \int_s^{s+} \| \langle u \rangle \|_{L^p} \| \nabla u \|_{L^q}^2 \cdot d\tau + 12|a| \| \langle u \rangle \|_{L^p} \| \Delta u \|_{L^q} \cdot d\tau
$$

$$
+ 24|a| \| u \|_{L^p} \| \partial \partial u \|_{L^q} \cdot d\tau,
$$

(5.18) \leq H_{2,3,1} + H_{2,3,2} + H_{2,3,3},

where

$$
H_{2,3,1} := C \int_s^{s+} \| \partial \partial u \|_{L^q}^2 \| \langle u \rangle \|_{L^p} \| \Delta u \|_{L^q} \cdot d\tau,
$$

$$
H_{2,3,2} := L' |a| \int_s^{s+} \| \Delta u \|_{L^q} \| \langle u \rangle \|_{L^p} \| \partial \partial u \|_{L^q} \cdot d\tau
$$

and

$$
H_{2,3,3} := C \int_s^{s+} \| \Delta u \|_{L^q} \| \langle u \rangle \|_{L^p} \| \partial \partial u \|_{L^q} \cdot d\tau.
$$

Here, $L' > 0$ is again some dimensional constant, and in the last estimate of (5.18), we have used the estimate

$$
12|a| \int_s^{s+} \| \langle u \rangle \|_{L^p} \| \nabla u \|_{L^q}^2 \cdot d\tau
$$

$$
+ 12|a| \| \langle u \rangle \|_{L^p} \| \Delta u \|_{L^q} \cdot d\tau \leq H_{2,3,1}.
$$

Estimate of $H_{2,3,1}$. By the bootstrap assumptions and Proposition 4.2, we have

$$
H_{2,3,1} \leq C \| u \|_{L^2} \int_s^{s+} \| \partial \partial u \|_{L^q}^2 \cdot d\tau
$$

$$
\leq C \left( \int_s^{s+} \| \partial \partial u \|_{L^q}^2 \cdot d\tau \right)^{\frac{2}{3}} \left( \int_s^{s+} \| \partial \partial u \|_{L^q}^2 \cdot d\tau \right)^{\frac{1}{3}}
$$

$$
\cdot \left( \int_s^{s+} \| \Delta u \|_{L^q}^2 \cdot d\tau \right)^{\frac{1}{3}}
$$

$$
\leq C \left( \frac{\gamma}{\Lambda_\gamma} \right)^{\frac{2}{3}} \| u \|_{L^2}^2 \leq C \| u \|_{L^2} \cdot \left( \frac{\gamma}{\Lambda_\gamma} \right)^{\frac{2}{3}} \| u \|_{L^2}.
$$
Estimate of $H_{2,3,2}$. Using the bootstrap assumptions and Proposition 4.2 again, we have

$$H_{2,3,2} \leq L'|\alpha| \|u_{\#}(s)\|_{L^2} \int_s^{\tau^*+s} \|\Delta u_{\#}\|_{L^2} \|\partial^2_y(u)\|_{L^2} d\tau$$

$$\leq \frac{20L'|\alpha| \|u_{\#}(s)\|_{L^2}}{\varepsilon} \cdot \left( \varepsilon \gamma \int_s^{\tau^*+s} \|\Delta u_{\#}\|_{L^2} d\tau \right)^{\frac{1}{2}}$$

$$\cdot \left( \varepsilon \gamma \int_s^{\tau^*+s} \|\partial^2_y(u)\|_{L^2} d\tau \right)^{\frac{1}{2}}$$

$$\leq \frac{200L'|\alpha|B_3^{\frac{2}{7}} \|u_{\#}(s)\|_{L^2}^2}{\varepsilon} \cdot \|u_{\#}(s)\|_{L^2} \leq \frac{\|u_{\#}(s)\|_{L^2}}{20}.$$  

Estimate of $H_{2,3,3}$. Similarly, we have

$$H_{2,3,3} \leq C\gamma \|u_{\#}(s)\|_{L^2}^{\frac{2}{7}} \int_s^{\tau^*+s} \|\Delta u_{\#}\|_{L^2}^{\frac{2}{7}} \|\partial^2_y(u)\|_{L^2}^{\frac{2}{7}} d\tau$$

$$\leq C \|u_{\#}(s)\|_{L^2}^{\frac{2}{7}} \left( \int_s^{\tau^*+s} d\tau \right)^{\frac{7}{16}} \left( \int_s^{\tau^*+s} \|\Delta u_{\#}\|_{L^2}^2 d\tau \right)^{\frac{1}{16}}$$

$$\cdot \left( \int_s^{\tau^*+s} \|\partial^2_y(u)\|_{L^2}^2 d\tau \right)^{\frac{1}{16}}$$

$$\leq C \cdot \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{7}{16}} \|u_{\#}(s)\|_{L^2}^2 \leq C \|u_{\#}(0)\|_{L^2} \cdot \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{7}{16}} \|u_{\#}(s)\|_{L^2}.$$  

Combining all the estimates of $H_{2,3,1}$, $H_{2,3,2}$ and $H_{2,3,3}$ together with (5.18), this gives

$$H_{2,3} \leq \frac{\|u_{\#}(s)\|_{L^2}}{20} + C \|u_{\#}(0)\|_{L^2} \left[ \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{1}{16}} + \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{7}{16}} \right] \|u_{\#}(s)\|_{L^2}.$$  

Finally, by (5.13), (5.15), (5.17) and (5.19), we get

$$H_2 \leq \frac{\|u_{\#}(s)\|_{L^2}}{10} + C \left[ \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{1}{16}} + \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{7}{16}} + \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{1}{16}} \right] \cdot \|u_{\#}(s)\|_{L^2}$$

$$+ C \|u_{\#}(0)\|_{L^2} \left[ \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{1}{16}} + \left( \frac{\gamma}{\lambda_\gamma} \right)^{\frac{7}{16}} \right] \|u_{\#}(s)\|_{L^2}.$$  

The estimate of $H_2$ is complete.

Estimate of $H_3$. The estimate of $H_3$ is similar to the one of $H_2$, and hence we only sketch the proof here. Note that

$$\Delta \left( u_{\#}^2 \right) = 2 |\nabla u_{\#}|^2 + 2 u_{\#} \Delta u_{\#}$$

and

$$\Delta \left( \langle u \rangle u_{\#} \right) = \langle u \rangle \Delta u_{\#} + \partial^2_y(u)u_{\#} + 2\partial_y(u)\partial_y u_{\#}.$$
Proposition 5.4. Assume the bootstrap assumptions, the proof is complete.

Therefore,

\[ H_3 \leq 2|b|\gamma \int_s^{\tau^*+s} \left\| \Delta \left( \frac{u'}{u'} \right) \right\|_{L^2} d\tau + 4|b|\gamma \int_s^{\tau^*+s} \left\| \Delta \left( \langle u \rangle u' \right) \right\|_{L^2} d\tau \]

\[ \leq 4|b|\gamma \int_s^{\tau^*+s} \left\| \nabla u' \right\|_{L^4} d\tau + 4|b|\gamma \int_s^{\tau^*+s} \left\| u' \right\|_{L^2} \left\| \Delta u' \right\|_{L^2} d\tau + 4|b|\gamma \int_s^{\tau^*+s} \left\| \partial_y^2 \left( \langle u \rangle u' \right) \right\|_{L^2} d\tau \]

\[ + 8|b|\gamma \int_s^{\tau^*+s} \left\| \partial_y \langle u \rangle \partial_y u' \right\|_{L^2} d\tau \]

\[ \leq 4|b|\gamma \int_s^{\tau^*+s} \left\| \nabla u' \right\|_{L^4} d\tau + 4|b|\gamma \int_s^{\tau^*+s} \left\| u' \right\|_{L^2} \left\| \Delta u' \right\|_{L^2} d\tau + 4|b|\gamma \int_s^{\tau^*+s} \left\| u' \right\|_{L^2} \left\| \partial_y^2 \langle u \rangle \right\|_{L^2} d\tau \]

\[ + 8|b|\gamma \int_s^{\tau^*+s} \left\| \partial_y \langle u \rangle \right\|_{L^2} \left\| \partial_y u' \right\|_{L^2} d\tau. \]

We bound the right hand side of (5.21) again by using the bootstrap assumptions and Proposition 4.2 which gives

\[ H_3 \leq C \left[ \left( \frac{\gamma}{\lambda^2} \right) \frac{1}{e} + \left( \frac{\gamma}{\lambda^2} \right) \frac{1}{e^4} + \left( \frac{\gamma}{\lambda^2} \right) \right] \left\| u' \right\|_{L^2}. \]

The estimate of \( H_4 \) is complete.

Finally, we combine all the estimates for \( H_1, H_2, H_3 \) and \( H_4 \) together with (5.13), and this gives

\[ \left\| u' (\tau^* + s) \right\|_{L^2} \leq \left( \frac{10}{e^4} + \frac{1}{10} \right) \left\| u' (s) \right\|_{L^2} \]

\[ + C \left( 1 + \left\| u' (0) \right\|_{L^2} \right) \cdot \mathcal{F} \left( \frac{\gamma}{\lambda^2} \right) \left\| u' (s) \right\|_{L^2}, \]

where \( \mathcal{F} \) is a function on \( \mathbb{R}_+ \) satisfying \( \mathcal{F}(\alpha) \to 0 \) as \( \alpha \to 0 \) (the explicit formula for \( \mathcal{F} \) can be derived directly from the estimates of \( H_2, H_3 \) and \( H_4 \)). Note that \( \frac{10}{e^4} + \frac{1}{10} < \frac{1}{2} \). Therefore, the desired estimate (5.12) then follows if we pick a \( \gamma_3 > 0 \) sufficiently small, which only depends on \( \varepsilon, a, b, B, \left\| \langle u \rangle (0) \right\|_{L^2} \) and \( \left\| u' (0) \right\|_{L^2} \), such that for any \( 0 < \gamma \leq \gamma_3 \), one has

\[ C \left( 1 + \left\| u' (0) \right\|_{L^2} \right) \cdot \mathcal{F} \left( \frac{\gamma}{\lambda^2} \right) \leq 1 - \frac{10}{e^4} - \frac{1}{10}. \]

The proof is complete. \( \square \)

We are now ready to improve the first estimate in the bootstrap assumptions.

Proposition 5.4. Assume the bootstrap assumptions, (5.11) and (5.12). Then there exists a \( \gamma^* > 0 \), which only depends on \( \varepsilon, a, b, \left\| \langle u \rangle (0) \right\|_{L^2}, \left\| u' (0) \right\|_{L^2} \) and any dimensional constants, such that for any \( 0 < \gamma \leq \gamma^* \) and \( 0 \leq s \leq t \leq t_0 \),

\[ \left\| u' (t) \right\|_{L^2} \leq 15 e^{-\frac{\lambda_2 (t-s)}{4}} \left\| u' (s) \right\|_{L^2}. \]
which is a consequence of a refined version of (4.6). More precisely, by (4.5), we
and the fact that $t$ is sufficiently small, such that the bootstrap assumptions also hold on $[0, t]\),
then we have

$$\|u_{\bar{\varphi}}(t)\|_{L^2} \leq \frac{3}{2} \|u_{\bar{\varphi}}(s)\|_{L^2} \leq \frac{15}{e} \|u_{\bar{\varphi}}(s)\|_{L^2} \leq 15e^{-\frac{\lambda_2(t-s)}{4}} \|u_{\bar{\varphi}}(s)\|_{L^2}. $$

If $t_0 \geq t^*$, then for any $0 \leq s \leq t \leq t_0$, we can find some $n \in \mathbb{Z}_+$, such that $t \in [n \tau^* + s, (n+1) \tau^* + s)$. Then by Proposition 5.2 and Proposition 5.3, we have

$$\|u_{\bar{\varphi}}(t)\|_{L^2} \leq \frac{3}{2} \cdot \|u_{\bar{\varphi}}(n \tau^* + s)\|_{L^2} \leq \frac{3}{2} 2^n \cdot \|u_{\bar{\varphi}}(s)\|_{L^2} \leq \frac{3}{2} e^{(1-\frac{4}{e})} \cdot \|u_{\bar{\varphi}}(s)\|_{L^2} \leq 15e^{-\frac{\lambda_2(t-s)}{4}} \cdot \|u_{\bar{\varphi}}(s)\|_{L^2}. $$

This concludes the proof of the proposition.

Proof of Theorem 1.5. We first show that under the assumption of Theorem 1.5, the solution to the problem (1.5) exists globally. Note that it suffices to show that the maximal time $t_0 = \infty$ in the bootstrap assumptions, while the regularity of the solution $u$ follows clearly from the bootstrap assumptions and Proposition 5.2. Assume $t_0 < \infty$. Then on the time interval $[0, t_0]$, we always have for $0 \leq s \leq t \leq t_0$,

1. $\|u_{\bar{\varphi}}(t)\|_{L^2} \leq 15e^{-\frac{\lambda_2(t-s)}{4}} \|u_{\bar{\varphi}}(s)\|_{L^2};$
2. $\gamma \int_0^t \|\Delta u_{\bar{\varphi}}(\tau)\|_{L^2}^2 \, d\tau \leq 5 \|u_{\bar{\varphi}}(s)\|_{L^2}^2.$

These estimates together with the continuity at $t_0$, we are able to find a $\epsilon > 0$ sufficiently small, such that the bootstrap assumptions also hold on $[0, t_0 + \epsilon]$, which contradicts to the maximality of $t_0$. The proof of the first part is complete.

Next we show that $\|u(t)\|_{L^2}$ decays exponentially. By the bootstrap assumptions and the fact that $t_0 = \infty$, it suffices to show that $\|u(t)\|_{L^2}$ decay exponentially, which is a consequence of a refined version of (4.6). More precisely, by (4.6), we have

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \frac{\epsilon \gamma \lambda_1^2}{4} \|u\|_{L^2}^2 \leq B \gamma (a^2 \|\Delta u_{\bar{\varphi}}\|_{L^2}^2 \|u_{\bar{\varphi}}\|_{L^2}^2 + \|\Delta u_{\bar{\varphi}}\|_{L^2} \|u_{\bar{\varphi}}\|_{L^2} \|u\|_{L^2}^2 + \|\Delta u_{\bar{\varphi}}\|_{L^2} \|u\|_{L^2}^2), $$

where we recall that $\lambda_1 > 0$ is the smallest positive eigenvalue of $-\Delta$ on $\mathbb{T}^2$. This gives the following ODE:

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \left[ \frac{\epsilon \gamma \lambda_1^2}{4} - B \gamma \|\Delta u_{\bar{\varphi}}\|_{L^2} \|u_{\bar{\varphi}}\|_{L^2}^2 \right] \|u\|_{L^2}^2 \leq B \gamma (a^2 \|\Delta u_{\bar{\varphi}}\|_{L^2}^2 \|u_{\bar{\varphi}}\|_{L^2}^2 + \|\Delta u_{\bar{\varphi}}\|_{L^2} \|u_{\bar{\varphi}}\|_{L^2} \|u\|_{L^2}^2).$$

(5.24)

The proof then follows closely to the argument in Proposition 4.2. Let

$$\tilde{p}(t) := \exp \left( \frac{\epsilon \gamma \lambda_1^2 t}{4} - B \gamma \int_0^t \|\Delta u_{\bar{\varphi}}\|_{L^2}^2 \|u_{\bar{\varphi}}\|_{L^2}^2 \, d\tau \right)$$

Proof.
be the integrating factor and therefore
\[
\|u(t)\|_{L^2}^2 \leq \frac{\|u(0)\|_{L^2}^2}{\rho(t)} + \frac{Ba^2 \gamma}{\rho(t)} \int_0^t \rho(\tau) \|\Delta u_\theta(\tau)\|^2_{L^2} \|u_\theta(\tau)\|^4_{L^2} d\tau
\]
\[+ \frac{B \gamma}{\rho(t)} \int_0^t \rho(\tau) \|\Delta u_\theta(\tau)\|_{L^2} \|u_\theta(\tau)\|_{L^2}^3 d\tau.\]  
(5.25)

Using (4.8), we have
\[
\|u(0)\|_{L^2}^2 \leq \exp \left(-\frac{\varepsilon \gamma \lambda_1^2 t}{4} + B \|u_\theta(0)\|_{L^2}^2\right).
\]
(5.26)

While for the two integrals in (5.25), by the bootstrap assumptions, we have
\[
\gamma \int_0^t \rho(\tau) \|\Delta u_\theta(t)\|^2_{L^2} \|u_\theta(\tau)\|_{L^2}^4 d\tau \leq B \gamma \|u_\theta(0)\|^4_{L^2} \int_0^t \exp \left(-\frac{\varepsilon \gamma \lambda_1^2 t}{4} - \lambda_\gamma t\right) \|\Delta u_\theta(\tau)\|^2_{L^2} d\tau
\]
(5.27)

and
\[
\gamma \int_0^t \rho(\tau) \|\Delta u_\theta(\tau)\|_{L^2} \|u_\theta(\tau)\|_{L^2}^3 d\tau \leq B \gamma \|u_\theta(0)\|^3_{L^2} \int_0^t \exp \left(-\frac{\varepsilon \gamma \lambda_1^2 t}{4} - \frac{3 \lambda_\gamma t}{4}\right) \|u_\theta(\tau)\|^2_{L^2} d\tau.
\]
(5.28)

Recall that \(\lambda_\gamma \approx \frac{1}{\gamma} \frac{1}{4\pi}\), therefore, for \(\gamma\) sufficiently small, we have
\[
\frac{\varepsilon \gamma \lambda_1^2}{4} - \frac{3 \lambda_\gamma}{4} \leq 0.
\]

Therefore, (5.27) and (5.28) together with (4.9) and (4.10), respectively, give
\[
\gamma \int_0^t \rho(\tau) \|\Delta u_\theta(\tau)\|^2_{L^2} \|u_\theta(\tau)\|_{L^2}^4 d\tau \leq B \|u_\theta(0)\|^6_{L^2}
\]
(5.29)

and
\[
\gamma \int_0^t \rho(\tau) \|\Delta u_\theta(\tau)\|_{L^2} \|u_\theta(\tau)\|_{L^2}^3 d\tau \leq B \|u_\theta(0)\|^4_{L^2}
\]
(5.30)

Finally, combining (5.25), (5.26), (5.29) and (5.30), we see that for any \(t \geq 0\),
\[
\|u(t)\|_{L^2}^2 \leq B \exp \left(-\frac{\varepsilon \gamma \lambda_1^2 t}{4} + B \|u_\theta(0)\|_{L^2}^2\right) \cdot \left[\|u_\theta(0)\|^2_{L^2} + \|u_\theta(0)\|^2_{L^4} + \|u(0)\|^2_{L^2}\right].
\]
The proof is complete. \(\square\)

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