Mini-batch stochastic three-operator splitting for distributed optimization

Barbara Franci and Mathias Staudigl

Abstract—We consider a network of agents, each with its own private cost consisting of a sum of two possibly nonsmooth convex functions, one of which is composed with a linear operator. At every iteration each agent performs local calculations and can only communicate with its neighbors. The challenging aspect of our study is that the smooth part of the private cost function is given as an expected value and agents only have access to this part of the problem formulation via a heavy-tailed stochastic oracle. To tackle such sampling-based optimization problems, we propose a stochastic extension of the triangular pre-conditioned primal-dual algorithm. We demonstrate almost sure convergence of the scheme and validate the performance of the method via numerical experiments.

I. INTRODUCTION

We consider a large class of convex optimization problems given by

$$\min_{x \in \mathbb{R}^n} f(x) + g(x) + h(Lx),$$

(1)

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex and continuously differentiable function, $g : \mathbb{R}^n \to (-\infty, \infty]$ and $h : \mathbb{R}^m \to (-\infty, \infty]$ are closed convex and lower semi-continuous functions, and $L : \mathbb{R}^n \to \mathbb{R}^m$ is a given linear map. Such a structure is very general, and and describes many applications that range from signal processing to machine learning to control [1], [2], [3]. In many instances of problem (1), the cost functions are contaminated by stochastic noise. In such settings, we are given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ carrying a random variable $\xi : \Omega \to \Xi \subset \mathbb{R}^d$, and a measurable function $F : \mathbb{R}^n \times \Xi \to \mathbb{R}$ so that

$$f(x) = \mathbb{E}[F(x, \xi)] \quad \forall x \in \mathbb{R}^n.$$

(2)

The presence of stochastic uncertainty challenges any direct solution method for problem (1), since the smooth function $f$ is not directly accessible in practice, unless the distribution of the random variable is known. Indeed, the expectation (2) cannot be computed exactly, and instead we need to restore to simulation-based techniques. A reasonable assumption is that we can draw samples from the distribution of the random variable $\xi$, and stochastically approximate the necessary information about the function $f$. Specifically, we adopt an online gradient-based stochastic approximation method where a deterministic version of a numerical algorithm, known to solve problem (1) in its expected-value formulation, is supplied with stochastic estimators of the gradient of $f$. In this setting, the mechanism to access $f$ via samples of the law of $\xi$ is usually named a stochastic oracle (SO). The SO outputs unbiased approximations of the gradient obtained via an average over a batch of realizations. The larger the number of samples, the smaller the variance of point estimators, and thus the more precise information we obtain at the cost of generating a larger amount of i.i.d random variables. This trade-off between accuracy of estimators and the available simulation budget is what makes such mini-batch stochastic approximation approaches efficient methods of choice [4], [5], [6].

A. Main Contributions and relation to the literature

A standard assumption in stochastic optimization is that the approximation error is uniformly bounded [5]. Instead, the stochastic approximation approach developed in this paper allows to handle stochastic oracles with potentially unbounded moments. This is of relevance in primal-dual dynamics, which usually act on unbounded domains, for which any a-priori variance bound is rather unnatural.

Besides weaker hypothesis on the noise structure, this paper is the first stochastic primal-dual algorithm which is even provably convergent in the multi-agent formulation of problem (1), i.e., when a finite set of agents $i \in \{1, \ldots, m\}$ cooperatively minimize the composite objective function

$$\min_{x_i \in \mathbb{R}^n, 1 \leq i \leq m} \sum_{i=1}^{m} f_i(x_i) + g_i(x_i) + h_i(L_ix_i),$$

(3)

s.t. $A_{ij}x_i + A_{ji}x_j = b_{(i,j)} \forall (i, j) \in E$

Problems of this form appear in several application fields. In distributed model predictive control, $f_i$ can represent individual finite-horizon costs for each agent, $L_i$ model linear dynamics of each agent, and $h_i$ models state and input constraints [1]. In machine learning, $f_i$ would represent a smooth data fidelity terms, $Lx$ linear restrictions on the parameters and $h_i$ and $g_i$ take over the role of statistical penalties reflecting a-priori structure properties of the parameters to be estimated [2].

Compared with existing work, this paper makes the following contributions:

(i) Our algorithm is a stochastic extension of the triangular pre-conditioned primal-dual algorithm (TriPD), developed in [7], [8].

(ii) To the best of our knowledge, our scheme is the first stochastic approximation method which is able to solve the three-operator splitting problem characterizing primal-dual pairs for (1).
(iii) The analysis immediately generalizes to the multi-agent formulation [3], where distributed iterations, i.e., locally performed without central supervision, are obtained.

The only related contributions we are aware of are [9], [10]. Both assume a uniformly bounded SO, which is very restrictive in primal-dual methods, essentially forcing a-priori compactness on the domain of \( g \). For the special case when \( g = 0 \), stochastic accelerated algorithms for the centralized problem [1] have been considered in [11], imposing a uniformly bounded noise condition on the SO. We include the non-smooth term \( g \) and allow for heavy-tailed noise in the SO.

Our analysis is restricted to synchronous versions of distributed optimization algorithms. It is possible to extend our analysis to the asynchronous case and block-coordinate descent strategies, acting on general real separable Hilbert spaces. We will present these extensions in a future paper.

B. Notation and preliminary results

Throughout \( \mathcal{H}, \mathcal{G} \) are finite dimensional Euclidean spaces. Their scalar products and associated norms are denoted by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \). We denote \( \mathcal{B}(\mathcal{H}, \mathcal{G}) \) the space of bounded linear operators from \( \mathcal{H} \) to \( \mathcal{G} \). The adjoint of \( L \in \mathcal{B}(\mathcal{H}, \mathcal{G}) \) is denoted by \( L^* \). \( \text{Id} \) denotes the identity operator. We set \( \mathcal{P}_\alpha(\mathcal{H}) \triangleq \{ Q \in \mathcal{B}(\mathcal{H}) : Q^* = Q \text{ and } Q \succcurlyeq \alpha \text{Id} \} \). For \( \alpha > 0 \) and \( Q \in \mathcal{P}_\alpha(\mathcal{H}) \), we define a scalar product and norm on \( \mathcal{H} \) by \( \langle x, y \rangle_Q \triangleq \langle Qx, y \rangle \) and \( \| x \|_Q \triangleq \sqrt{\langle Qx, x \rangle} \) for all \( x, y \in \mathcal{H} \). We let \( \mathcal{H}(Q) = \max_{u \in \mathcal{H}, \| u \| = 1} \langle u, Qu \rangle \) and \( \mathfrak{m}(Q) = \min_{u \in \mathcal{H}, \| u \| = 1} \langle u, Qu \rangle \). For an extended-valued real function \( f \), we use \( \text{dom } f = \{ x \in \mathcal{H} : f(x) < \infty \} \) for its effective domain. Let \( Q \in \mathcal{P}_\alpha(\mathcal{H}) \) for some \( \alpha > 0 \). \( \delta_C \) is the indicator function of the set \( C \), that is, \( \delta_C(x) = 0 \) if \( x \in C \) and \( \delta_C(x) = \infty \) otherwise. The weighted prox-operator of \( f \) is defined as \( \text{Prox}_Q^\alpha (f) \triangleq \arg \min_{u \in \mathcal{H}} \{ f(u) + \frac{\alpha}{2} \| u-x \|_Q^2 \} \). The conjugate of \( f : \mathcal{H} \rightarrow (-\infty, \infty) \) is \( f^*(u) = \sup_{x \in \mathcal{H}} \{ \langle x, u \rangle - f(x) \} \). We also need the celebrated Robbins-Siegmund Lemma for the convergence analysis

Lemma 1 ([12]): Let \( (\Omega, \mathcal{A}, (\mathcal{A}_n)_{n \in \mathbb{N}}, \mathbb{P}) \) be a filtered probability space satisfying the usual conditions. For every \( n \in \mathbb{N} \), let \( v_n, \xi_n, \zeta_n \) and \( t_n \) be non-negative \( \mathcal{A}_n \)-measurable random variables such that \( \{ \xi_n \}_{n \in \mathbb{N}} \) and \( \{ t_n \}_{n \in \mathbb{N}} \) are summable and for all \( n \in \mathbb{N} \),

\[ E[v_n | \mathcal{F}_n] \leq (1 + t_n)v_n + \zeta_n - \xi_n \quad \mathbb{P} - \text{a.s.} \quad (4) \]

Then \( \{ v_n \}_{n \in \mathbb{N}} \) converges and \( \{ \xi_n \}_{n \in \mathbb{N}} \) is summable \( \mathbb{P} \)-a.s.

II. STOCHASTIC PRIMAL-DUAL ALGORITHM

In this section we propose a stochastic primal-dual algorithm for solving [1]. Let \( \mathbb{H} \equiv \mathbb{R}^m \) and \( \mathbb{G} \equiv \mathbb{R}^n \) with inner products \( \langle \cdot, \cdot \rangle_{\mathbb{H}}, \langle \cdot, \cdot \rangle_{\mathbb{G}} \), and corresponding norms \( \| \cdot \|_{\mathbb{H}}, \| \cdot \|_{\mathbb{G}} \).

Assumption 1: Throughout the paper the following assumptions shall be in place:
(i) \( g : \mathbb{H} \rightarrow (-\infty, \infty], h : \mathbb{G} \rightarrow (-\infty, \infty] \) are proper, closed, convex and lower-semi continuous functions.
(ii) \( L : \mathbb{H} \rightarrow \mathbb{G} \) is a linear mapping with adjoint \( L^* \).

(iii) \( f : \mathbb{H} \rightarrow \mathbb{R} \) is convex, continuously differentiable and there exists \( \alpha > 0, Q \in \mathcal{P}_\alpha(\mathcal{H}) \), and \( \beta_f > 0 \) such that \( \| \nabla f(x) - \nabla f(y) \|_{\mathcal{G}} \leq \beta_f \| x-y \| \) for all \( x, y \in \mathcal{H} \).
(iv) Let \( \Xi \subseteq \mathbb{R}^d \) be a measurable set and \( (\Omega, \mathcal{A}, \mathbb{P}) \) a probability space. There exists a measurable function \( F : \mathbb{H} \times \Xi \rightarrow \mathbb{R} \) such that \( \Xi \) holds.
(v) The set of solutions to \( [1] \), denoted by \( \mathcal{X} \), is nonempty. Moreover, there exists \( x \in \text{relint}(\text{dom } h) \) such that \( Lx \in \text{relint}(\text{dom } h) \).

Let \( Z = G \times H \) represent the product space with the inner product \( \langle (y_1, x_1), (y_2, x_2) \rangle = \langle y_1, y_2 \rangle + \langle x_1, x_2 \rangle \), and associated norm \( \| (y, x) \| = \sqrt{\| y \|^2_G + \| x \|^2_H} \). Whenever clear from the context, we will suppress the ambient space from the inner products and norms, respectively.

We remark that by the Baillon-Haddad theorem [13, Corollary 18.17] the \( \beta_f \)-Lipschitz smoothness of the function \( f \) is equivalent to the \( \frac{1}{\beta_f} \)-cocoercivity of \( \nabla f \), i.e.,

\[ \langle \nabla f(x) - \nabla f(x'), x - x' \rangle \geq \frac{1}{\beta_f} \| \nabla f(x) - \nabla f(x') \|^2 \]

for all \( x, x' \in \mathcal{H} \).

Problem \([1]\) can be equivalently written as a max-min problem

\[ \max_{y \in \mathcal{G}} \min_{x \in \mathcal{H}} \{ f(x) + g(x) + \langle Lx, y \rangle - h^*(y) \} \]

Then, \( (\bar{y}, \bar{x}) \in \mathcal{G} \times \mathcal{H} \) is called primal-dual pair if

\[ 0 \in \partial h^*(\bar{y}) - L\bar{x}, \]

\[ 0 \in \partial g(\bar{x}) + \nabla f(\bar{x}) + L^*\bar{y}. \]

Introduce the maximally monotone operators

\[ A(y, x) \triangleq \partial h^*(y) \times \partial g(x), \]

\[ M(y, x) \triangleq [-Lx, L^*y], \]

\[ C(y, x) \triangleq [0, \nabla f(x)]. \]

Then \( \bar{z} \triangleq (\bar{y}, \bar{x}) \) is a primal-dual pair if and only if \( \bar{z} \in \text{Zer}(G) \triangleq \{ z \in \mathbb{Z} : 0 \in \text{G}(z) \} \), where \( G(y, x) \triangleq A(y, x) + M(y, x) + C(y, x) \).

A. Triangular pre-conditioned primal-dual algorithm

If it is possible to access the operator \( C \) directly, then an application of the Triangular pre-conditioned Primal-Dual (TriPD) algorithm of [7] would be possible. Let \( \tau_1 > 0 \) and \( \tau_2 > 0 \) be two positive real numbers, and \( \Sigma \in \mathcal{P}_{1/\tau_1}(\mathcal{G}), \Gamma \in \mathcal{P}_{1/\tau_2}(\mathcal{H}) \). TriPD can be compactly written as the fixed point iteration \( z_{k+1} = T(z_k) \), where \( z_k \equiv (y_k, x_k) \in \mathcal{Z} \), and the mapping \( T : \mathcal{Z} \rightarrow \mathcal{Z} \) is given by

\[ \Phi_1(z) \triangleq \text{Prox}^{\Sigma^{-1}}_{\mathcal{G}}(y + \Sigma Lx), \]

\[ \Phi_2(z) \triangleq \text{Prox}^{\Gamma^{-1}}_{\mathcal{H}}(x - \Gamma \nabla f(x) - \Gamma \Phi_1(z)) \]

\[ T(z) \triangleq [\Phi_1(z) + \Sigma L(\Phi_2(z) - x), \Phi_2(z)]. \]

First, let us report an important connection between the fixed points of the mapping \( T \) and the set of primal-dual solutions to \([1]\).

Lemma 2: [7, Equation (15)]. We have \( \text{Fix}(T) \triangleq \{ z \in \mathcal{Z} : T(z) = z \} = \text{Zer}(G) \).
Algorithm 1: Stochastic Triangularly preconditioned primal-dual algorithm (STriPD)

Initialize: \((x_0, y_0) \in Z, \Sigma \in \mathcal{P}_{1/\tau_1}(\mathcal{G}), \Gamma \in \mathcal{P}_{1/\tau_2}(\mathcal{H})\)

Iteration \(k\):

\[
\begin{align*}
\hat{Y}_k &= \text{Prox}^{\Gamma^{-1}}_{\Sigma}(Y_k + \Sigma L X_k) \\
\hat{X}_{k+1} &= \text{Prox}^{\Gamma^{-1}}_{\Sigma}(x_k - \Gamma F_k(X_k) - \Gamma L' \hat{Y}_k) \\
Y_{k+1} &= \hat{Y}_k + \Sigma L (X_{k+1} - X_k)
\end{align*}
\]

Before taking care of our expected valued formulation as in [2], let us introduce the matrices

\[
\begin{align*}
P &\triangleq \left( \begin{array}{cc}
\Sigma^{-1} & \frac{1}{2} L' \\
\frac{1}{2} L & \Gamma^{-1}
\end{array} \right), \\
K &\triangleq \left( \begin{array}{cc}
0 & -\frac{1}{2} L \\
\frac{1}{2} L & 0
\end{array} \right), \\
S &\triangleq \left( \begin{array}{cc}
\Sigma^{-1} & 0 \\
0 & \Gamma^{-1}
\end{array} \right), \\
R &\triangleq P + K.
\end{align*}
\]

These matrices act like step-sizes and pre-conditioners. Allowing for matrix-valued step sizes makes the distributed algorithm in Section [V] a corollary of the present analysis.

B. Stochastic TriPD

Since we cannot evaluate \(C(z)\) directly, we let \(C_k(z)\) denote its stochastic estimator. This estimator is constructed by a Monte-Carlo scheme involving mini-batches. Given an i.i.d. sample \(\xi_{1:N} \triangleq \{\xi_{i,1}^N \}_{i=1}^N\) drawn from the law of \(\xi\), let

\[
\tilde{w}(x, \xi_{1:N}) \triangleq \frac{1}{N} \sum_{i=1}^N (\nabla_x F(x, \xi_i) - \nabla f(x)).
\]

The approximation take place at each iteration \(k\) and a sequence \(\{N_k\}_{k \in \mathbb{N}}\) (the batch size) defines the number of random variables we need to sample in each iteration.

Assumption 2: We are given a sequence \(\{N_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}\) such that \(\lim_{k \to \infty} N_k = \infty\).

Since complexity questions are beyond our scope, we do not specify the speed at which batch sizes grow (see, however, [6]). For a given sequence of batch-sizes, we set \(w_k(x, \omega) \triangleq \tilde{w}(x, \xi_{1:N_k}(\omega))\).

Assumption 3: For all \(k \geq 0\), \(C_k(z, \omega) = [0, F_k(x, \omega)]\), where \(F_k(x, \omega) = \nabla f(x) + w_k(x, \omega)\) for all \((x, \omega) \in \mathcal{H} \times \Omega\).

We let \(A_k\) denote the filtration given by \(A_0 = \{0, \Omega\}\), and for \(k \geq 1\), \(A_k \triangleq \sigma(\xi_{1:N_0}, \ldots, \xi_{1:N_{k-1}})\). Define \(\nu(x) \triangleq ||\nabla_x F(x, \xi) - \nabla f(x)||^2\) and \(\sigma(x) \triangleq \sqrt{\mathbb{E}[\nu(x)|x]}\).

Assumption 4: For all \(k \geq 0\) we have \(\mathbb{E}[F_k(X_k)|A_k] = \nabla f(X_k)\) a.s.. Moreover, there exists \(x^* \in \mathcal{X}^*\) and \(\sigma_0(x^*) \geq 0, \sigma_1 > 0\) such that

\[
\sigma(x) \leq \sigma_0(x^*) + \sigma_1 ||x - x^*||^2 \quad \forall x \in \mathcal{H}.
\]

This is a heavy-tailed noise assumption which allows us to consider random perturbation even without bounded second moment [4], [14].

In analogy to TriPD and in light of the approximation scheme, the equations in [5] correspond to those in Algorithm 1 and lead to a sequence of random maps \(\{T_k\}_{k \geq 0} : Z \times \Omega \to Z\) generating a stochastic process \(\{Z_k\}_{k \geq 0} = \{\{Y_k, X_k\}; k \geq 0\}\) via recursive updates

\[
Z_{k+1} = T_k(Z_k), \quad Z_0 \in Z \text{ given}.
\]

The exact action of this mapping can be described as follows. For \(z = (y, x) \in Z\), let

\[
\begin{align*}
\Psi_k(z) &= \text{Prox}^{\Gamma^{-1}}_{\Sigma}(y + \Sigma L x), \\
\Xi_k(z) &= \text{Prox}^{\Gamma^{-1}}_{y}(x - \Gamma F_k(x) - \Gamma L' \hat{Y}_k) \\
\Theta_k(z) &= \{\Psi_k(z) + \Sigma L (X_{k+1} - X_k), X_k(z)\}.
\end{align*}
\]

Let \(Z_k \triangleq \{\Psi_k(z), \Xi_k(z)\}\); so that \(Y_{k+1} = \Psi_k(Z_k) + \Sigma L (X_{k+1} - X_k)\) and \(X_{k+1} = \Xi_k(Z_k)\) for all \(k \geq 0\). One can verify that

\[
\begin{align*}
\hat{Z}_k &= (R + A)^{-1}(R - M - C_k)(Z_k), \\
\Theta_k(z) &= \{z + S^{-1}(R + M^*)(\hat{Z}_k - z)\}
\end{align*}
\]

III. CONVERGENCE ANALYSIS

In this section, we present a number of results that lead to the convergence proof of Algorithm 1 (Theorem 1). We start with a property of the operator C.

Lemma 3: [7, Lemma II.4]. For all \(z = (y, x), z' = (y', x'), z'' = (y'', x'') \in Z\) we have

\[
\langle C(z) - C(z'), z'' - z \rangle \leq \beta T \|x'' - x'\|_Q^2.
\]

From Assumption 3 we can write

\[
\hat{C}_k(z) = C(z) + \phi_{k+1}(z),
\]

where \(\phi_{k+1}(z, \omega) \triangleq [0, w_k(x, \omega)]\). Let \(z^* \in \text{Zer}(\mathcal{G})\), or \(- (M + C)(z^*) \in A(z^*)\). From the definition of the update \(Z_k\), we deduce \(R(Z_k - \hat{Z}_k) = (M + C)(\hat{Z}_k) \in A(\hat{Z}_k)\). By monotonicity of the operator A, this implies

\[
\begin{align*}
\langle R(Z_k - \hat{Z}_k) - M(Z_k - z^*) + C(z^*) - C(Z_k), \hat{Z}_k - z^* \rangle \\
- \langle \phi_{k+1}(Z_k), \hat{Z}_k - z^* \rangle &\geq 0.
\end{align*}
\]

Applying Lemma 3 to the points \(z = z^*, z' = Z_k\) and \(z'' = \hat{Z}_k\), we get \(\langle C(z^*) - C(Z_k), \hat{Z}_k - z^* \rangle \leq \frac{T}{2} \|X_k(Z_k) - X_k\|_Q^2\). To reduce notational clutter, we write in the following \(\hat{X}_k \equiv \hat{X}_k(Z_k)\). Then (6) leads to the estimate

\[
\begin{align*}
0 \leq & \langle R(Z_k - \hat{Z}_k) + M(z^* - Z_k), \hat{Z}_k - z^* \rangle \\
&+ \langle \phi_{k+1}(Z_k), z^* - \hat{Z}_k \rangle
\end{align*}
\]

where the last equality uses the skew-symmetry so that \(\langle (M - K)z, z \rangle = 0\) for all \(z \in Z\). Thanks to the skew-symmetry, we also observe that

\[
\langle (M - K)(Z_k - z^*), Z_k - \hat{Z}_k \rangle = \langle z^* - Z_k, M^*(\hat{Z}_k - Z_k) \rangle
\]

Therefore,

\[
\begin{align*}
0 \leq & \langle (R + M^*)(\hat{Z}_k - Z_k), z^* - \hat{Z}_k \rangle - \|\hat{Z}_k - Z_k\|_P^2 \\
&+ \langle \phi_{k+1}(Z_k), z^* - \hat{Z}_k \rangle + \frac{\beta T}{2} \|\hat{X}_k - X_k\|_Q^2.
\end{align*}
\]
By definition \( S(T_k(Z_k) - Z_k) = (R + M^*)(\hat{Z}_k - Z_k) \), and
\[
0 \leq \begin{bmatrix} \sum_k & -\frac{1}{2}L \end{bmatrix} \begin{bmatrix} \frac{T}{k} \end{bmatrix} \leq \begin{bmatrix} \sum_k & -\frac{1}{2}L \end{bmatrix} \begin{bmatrix} \frac{\Gamma}{k} \end{bmatrix} + \frac{\gamma}{2} \| \hat{X}_k - X_k \|_Q^2.
\]
(7)
A straightforward, but slightly tedious computation, yields the next result.

**Lemma 4:** Let \( U \triangleq \begin{bmatrix} \sum_k & -\frac{1}{2}L \end{bmatrix} \begin{bmatrix} \frac{T}{k} \end{bmatrix} \leq \begin{bmatrix} \sum_k & -\frac{1}{2}L \end{bmatrix} \begin{bmatrix} \frac{\Gamma}{k} \end{bmatrix} + \frac{\gamma}{2} \| \hat{X}_k - X_k \|_Q^2. \)

Eq. (7) and Lemma 4 deliver the relation
\[
0 \leq \begin{bmatrix} \sum_k & -\frac{1}{2}L \end{bmatrix} \begin{bmatrix} \frac{T}{k} \end{bmatrix} \leq \begin{bmatrix} \sum_k & -\frac{1}{2}L \end{bmatrix} \begin{bmatrix} \frac{\Gamma}{k} \end{bmatrix} + \frac{\gamma}{2} \| \hat{X}_k - X_k \|_Q^2.
\]
(8)
We now analyze the noise term \( \langle \phi_{k+1}(Z_k), z^* - \hat{Z}_k \rangle \). Denote by \( \bar{x}_k \triangleq \Phi_{\frac{\gamma}{2}}(Z_k) \) the evaluation of the deterministic generator at \( Z_k \) as in (5). Then,
\[
\langle \phi_{k+1}(Z_k), z^* - \hat{Z}_k \rangle = \langle \mathcal{F}(X_k) - \nabla f(X_k), x^* - \bar{x}_k \rangle = \langle F_k(X_k) - \nabla f(X_k), x^* - \bar{x}_k \rangle = \langle \mathcal{F}(X_k) - \nabla f(X_k), \hat{X}_k - \bar{x}_k \rangle.
\]
Using Cauchy-Schwarz, and the non-expansiveness of the proximal operator \( \text{Prox}_X^{\beta/2} (\cdot) \) with respect to the norm \( \| \cdot \|_r \)[13], we obtain
\[
\langle \phi_{k+1}(Z_k), z^* - \hat{Z}_k \rangle \leq \| \mathcal{F}(X_k) - \nabla f(X_k), \hat{X}_k - \bar{x}_k \|_2 \| \mathcal{F}(X_k) - \nabla f(X_k) \|_2^{-1}.
\]

Hence
\[
\langle \phi_{k+1}(Z_k), z^* - \hat{Z}_k \rangle \leq \| \mathcal{F}(X_k) - \nabla f(X_k), x^* - \bar{x}_k \|_2.
\]

Let \( E_k[\cdot] \triangleq E[\cdot|A_k] \) and recall that Assumption [3] implies that \( E_k[\mathcal{F}(X_k)] = \nabla f(X_k) \) holds a.s.. Then, we obtain
\[
E_k[\langle \phi_{k+1}, z^* - \hat{Z}_k \rangle] \leq E_k[\| \mathcal{F}(X_k) - \nabla f(X_k) \|_2^2].
\]

Setting \( p_k \triangleq \| \mathcal{F}(X_k) - \nabla f(X_k) \|_2^2 \), we obtain from (8)
\[
E_k[\langle S(T_k(Z_k) - Z_k), Z_k - z^* \rangle] \leq E_k[p_k] - E_k[\| \mathcal{T}(Z_k) - Z_k \|_U^2].
\]
(9)
Since, by definition,
\[
\| Z_{k+1} - z^* \|_S^2 = \| T_k(Z_k) - Z_k \|_S^2 + 2S(T_k(Z_k) - Z_k, Z_k - z^*) + \| Z_k - z^* \|_S^2.
\]
the estimate (9) delivers
\[
E_k[\| Z_{k+1} - z^* \|_S^2] \leq E_k[\| Z_k - z^* \|_S^2] + 2E_k[p_k] - E_k[\| \mathcal{T}(Z_k) - Z_k \|_U^2].
\]

If \( 2U - S \succ 0 \) and \( \{ E_k[p_k] \}_{k \geq 0} \) is summable, then it follows that \( \{ Z_k \}_{k \geq 0} \) is quasi-Fejér monotone with respect to \( \text{Zer}(G) \) relative to \( \| \cdot \|_S \), i.e.,
\[
E_k[\| Z_{k+1} - z^* \|_S^2] \leq E_k[\| Z_k - z^* \|_S^2] + 2E_k[p_k].
\]

The next two results guarantee this.

**Lemma 5:** Suppose that Assumption [4] holds true. For all \( k \in \mathbb{N} \), we have
\[
E_k[\| \mathcal{F}(X_k) - \nabla f(X_k) \|_2^2] \leq a_k + b_k \| Z_k - z^* \|_S^2.
\]
where \( a_k \triangleq 2\sigma_0^2(x^*)^2 m(G)/\gamma N_k \) and \( b_k \triangleq 2\sigma_1^2 m(G)/\gamma N_k \).

**Proof:**
\[
E_k[\| \mathcal{F}(X_k) - \nabla f(X_k) \|_2^2] \leq m(G)E_k[\| \mathcal{F}(X_k) - \nabla f(X_k) \|_2^2]
\]
\[
= m(G)E_k[\| w_k(X_k) \|_2^2] = m(G)\frac{2\langle z_k \rangle}{N_k} \text{ a.s.}
\]

Using Assumption [4] we get
\[
\sigma^2(X_k) \leq 2\sigma_0^2(x^*)^2 + 2\sigma_1^2 \| X_k - x^* \|_2^2 \leq 2\sigma_0^2(x^*)^2 + 2\sigma_1^2 \frac{1}{m(G)} \| X_k - x^* \|_2^2
\]
\[
\leq 2\sigma_0^2(x^*)^2 + 2\sigma_1^2 \frac{1}{m(G)} \| z_k \|_S^2.
\]

**Lemma 6:** [8, Lemma 5.1] Assume that
\[
m(G^{-1} - \beta/2) > \| L \|_\Sigma \geq \| L \|_2^2 m(S) \]
(10)
Then, \( 2U - S \in P_r(Z) \) for some \( r > 0 \).

We can finally prove the main result of this paper.

**Theorem 1:** Let Assumptions [1][3] hold true. Choose \( \Gamma, \Sigma \) such that condition (10) holds. Then, the stochastic process \( \{ Z_k \}_{k \in \mathbb{N}} \) generated by Algorithm 1 converges a.s. to a random variable \( Z_\infty \in \text{Zer}(G) \), i.e., to a solution of (1).

**Proof:**
Using Lemma [5] we get
\[
E_k[\| Z_{k+1} - z^* \|_S^2] \leq E_k[\| Z_k - z^* \|_S^2] + 2E_k[p_k] - E_k[\| \mathcal{T}(Z_k) - Z_k \|_U^2]
\]
\[
\leq (1 + 2b_k) \| Z_k - z^* \|_S^2 - E_k[\| \mathcal{T}(Z_k) - Z_k \|_U^2]
\]
\[
\leq (1 + 2b_k) \| Z_k - z^* \|_S^2 \quad \text{a.s.}
\]

set \( v_k = \| Z_k - z^* \|_S^2 \), \( t_k = 2b_k \), \( c_k = 2a_k \) and \( \theta_k = E_k[\| \mathcal{T}(Z_k) - Z_k \|_U^2] \), and apply Lemma [1] to deduce \( \Gamma \sum_{k \in \mathbb{N}} \theta_k < \infty \) and that \( v_k \) converges to a finite random variable \( v_\infty \) a.s. Furthermore, using [15, Proposition 2.3], we know that \( \{ Z_k \}_{k \in \mathbb{N}} \) is almost surely bounded. Hence, there exists a measurable set \( \Omega_0 \subseteq \Omega \) with \( P(\Omega_0) = 1 \) such that \( \lim_{k \to \infty} \| Z_k(\omega) - z^* \| = 0 \) for all \( \omega \in \Omega_0 \).

Fix such an event \( \omega \in \Omega_0 \) and sequence \( \{ k_j \} \subset \mathbb{N} \) with \( k_j \uparrow \infty \). Consider a converging subsequence \( \{ Z_{k_j}(\omega) \}_{j \in \mathbb{N}} \) with \( Z_{k_j}(\omega) \rightarrow Z_\infty(\omega) \). To reduce notational clutter, we omit the relabeling and simply denote \( Z_{k_j}(\omega) \) the converging subsequence. We have to show that \( Z_\infty(\omega) \in \text{Zer}(G) \). Set \( \mu \triangleq m(2U - S) \geq \tau > 0 \), we deduce
\[
\| \mathcal{T}(Z_k) - Z_k \|_U \geq \mu \| \mathcal{T}(Z_k) - Z_k \|_2^2
\]
\[
= \mu \| \mathcal{T}(Z_k) - T(Z_k) \|_2^2 + \| T(Z_k) - Z_k \|_2^2
\]
\[
= \mu \| \mathcal{T}(Z_k) - T(Z_k) \|_2^2 + (1 - \alpha)\mu \| \mathcal{T}(Z_k) - Z_k \|_2^2
\]
where in the last inequality we have used the Fenchel-Young
Gan undirected graph the network are subject to affine constraints, coupling the memory and computational unit and can only communicate associated with an agent, which is assumed to have a local memory, and its neighbors. We define the neighborhood of agent as

$$\mathcal{N}_i = \{j \in V \mid (i, j) \in E\}.$$ Each agent in the network is characterized by a private cost function $f_i(x_i) + g_i(x_i) + h_i(L_i x_i)$, defined over the vector space $\mathcal{H}_i \equiv \mathbb{R}^n$. Let $\mathcal{H} = \prod_{i=1}^n \mathcal{H}_i$. Furthermore, the decisions of the agents in the network are subject to affine constraints, coupling the decisions of neighboring agents.

**Assumption 5:** For each $i = 1, \ldots, m$:

(i) For all $j \in \mathcal{N}_i$, $b_{i,j} \in \mathcal{E}_{i,j} \equiv \mathbb{R}^{|\mathcal{E}_{i,j}|}$ and $A_{i,j} \in \mathbb{R}^{p_i \times |\mathcal{E}_{i,j}|}$;

(ii) $g_i : \mathbb{R}^n \to (-\infty, \infty)$, $h_i : \mathbb{R}^n \to (-\infty, \infty)$ are proper closed convex and lower semi-continuous functions, and $L_i \in \mathbb{R}^{n \times n}$;

(iii) $f_i : \mathcal{H}_i \to \mathbb{R}$ is a convex, continuously differentiable function and for some $\beta_i \geq 0$, $\nabla f_i$ is $\beta_i$-Lipschitz continuous in the norm $\| \cdot \|_q$;

(iv) The graph $G$ is connected.

(v) The set of solutions of (3) is nonempty and there exists $x_i \in \text{relint}(\text{dom} g_i)$ such that $L_i x_i \in \text{relint}(\text{dom} h_i)$ and $A_{i,j} x_i + A_{j,i} x_j = b_{i,j}$ for $(i,j) \in E$.

Let $W_{i,j} \equiv \mathbb{R}^{2|\mathcal{E}_{i,j}|}$ for all $(i,j) \in E$, with generic element $w_{i,j} = [w^{(i)}_{i,j}, w^{(j)}_{i,j}]$. The interpretation is that $w^{(i)}_{i,j}$ is controlled by agent $i$ and $w^{(j)}_{i,j}$ is controlled by agent $j$. For each edge $(i,j) \in E$ define the set

$$K_{i,j} \equiv \{ (u, v) \in W_{i,j} \mid u_1 + v_2 = b_{i,j} \},$$

and the linear map $N_{i,j} : \mathcal{H} \to \mathbb{R}^{2|\mathcal{E}_{i,j}|}$ by

$$\langle N_{i,j} x, w_{i,j} \rangle = \langle A_{i,j} x_i, w^{(i)}_{i,j} \rangle + \langle A_{j,i} x_j, w^{(j)}_{i,j} \rangle \quad \forall x \in \mathcal{H}.$$ Accordingly, we let $N : \mathcal{H} \to \prod_{(i,j) \in E} \mathbb{R}^{2|\mathcal{E}_{i,j}|}$ to be the operator defined by

$$\langle N x, w \rangle = \sum_{(i,j) \in E} \langle (A_{i,j} x_i, w^{(i)}_{i,j}) + \langle A_{j,i} x_j, w^{(j)}_{i,j} \rangle \rangle.$$ Using these concepts, we can reformulate problem (3) as

$$\min_{x_1, \ldots, x_m} \sum_{i=1}^m \left[ f_i(x_i) + g_i(x_i) + h_i(L_i x_i) + \sum_{(i,j) \in E} \delta_{K_{i,j}}(N_{i,j} x) \right].$$

Let $K \equiv \prod_{(i,j) \in E} K_{i,j}$ and $L : \mathcal{H} \to \prod_{i=1}^n \mathbb{R}^{q_i}$ be defined by $L x = [L_1 x_1, \ldots, L_m x_m]$. Set $D x = [L x, \text{N} x]$. Define the functions $F(x) \equiv \sum_{i=1}^m f_i(x_i)$, $g(x) \equiv \sum_{i=1}^m g_i(x_i)$ and $H(D x) \equiv h(L x) + \delta_K(D x)$. With this notation, we have converted problem (3) to problem (1) with the functions $F, G$ and $H$.

As in Section III the primal-dual optimality conditions can be written in the compact form as an inclusion problem involving the (maximally monotone) operators

$$A(v, w, x) = [\partial h(v), \partial g(x), \partial f(x)],$$

$$M(v, w, x) = [-L x, -\text{N} x, L^* v + \text{N}^* w],$$

$$C(v, w, x) = [0, 0, \nabla f(x)].$$

The dual variable is the pair $y = [v, w] \in G$ and we can apply Algorithm 1 directly to solve the distributed optimization problem (3). The resulting stochastic process $\{Z_k\}_{k \geq 0}$ decomposes to agent-specific updates as described in Algorithm 2 (cf. [7]). Given the identification of the operators characterizing the optimality conditions of a primal-dual pair, the convergence of the synchronous STRiPD (Algorithm 2) follows immediately from Theorem 1.
V. Numerical Example

We test Algorithm 1 on an economic dispatch problem for power grids, inspired by [16], [3]. Consider $n$ control areas, each with a generator that supply power $x_i \in \mathbb{R}$ and a local demand $b_i \in \mathbb{R}_+$ that should be satisfied. Each generator has a cost function of the form $f_i(x_i) = x_i^2$, and $\bar{x}_i, \underline{x}_i \geq 0$ for all $i = 1, \ldots, m$. Each area has a local generation cost $f_i(x_i) : \mathbb{R} \to \mathbb{R}$ so that the optimization problem is to minimize the overall cost $f(x) = \sum_{i=1}^{m} f_i(x_i)$, subject to the satisfaction of the demand:

$$
\min_{x_i, i=1, \ldots, m} \quad f(x) = \sum_{i=1}^{m} f_i(x_i)
\text{s.t.} \quad \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} b_i \quad (11)
$$

To write the problem in (11) in the form of problem (3), let us take $g_i(x_i) = \delta_{X_i}(x_i)$, i.e., the indicator function of the local constraints $X_i \triangleq [\underline{x}_i, \bar{x}_i]$, and $h(Lx) = \delta_{C}(Lx)$, where $L = 1_m$ and $C = \{ y \in \mathbb{R}^m : y = b \}$ represents the coupling constraints.

Similarly to [3], we consider $m = 5$ generators with cost functions $f_i(x_i) = \mathbb{E}[g_i(x_i)x_i^2 + p_i x_i]$ where the coefficients are chosen with mean $q = [0.094, 0.078, 0.105, 0.082, 0.074]$ and $p = [1.22, 3.41, 2.53, 4.02, 3.17]$. The random variable is drawn according to a normal distribution with mean $q$. The local bounds on the supply power are $x = [10, 8, 3.8, 5.4, 4.2]$ and $\bar{x} = [80, 60, 40, 45, 18]$ while the demands take the values $b = [35, 20, 25, 30, 10]$. We run the algorithm 100 times and plot in a thick blue line the average results; the transparent areas indicate the minimum and maximum values reached. Figure 1a displays the distance of the iterates from the solution and Fig 1b illustrates the distance of the cost from the optimal value. In Figure 1c, we show that asymptotically the constraints are satisfied.

VI. Conclusions

We propose a stochastic triangular preconditioned primal-dual algorithm (STriPD) for solving a large family of structured convex optimization problems, and multi-agent versions thereof. Many extensions of the present work will be investigated in a more elaborate investigation. Such extensions will include block-coordinate descent implementations, allowing for asynchronous updates in the distributed case. Furthermore, we will be investigating the iteration and oracle complexity of the method in detail.

REFERENCES

[1] B. Jin, H. Li, W. Yan, and M. Cao, “Distributed model predictive control and optimization for linear systems with global constraints and time-varying communication,” IEEE Transactions on Automatic Control, vol. 66, no. 7, pp. 3393–3400, 2021.
[2] T. Hastie, R. Tibshirani, and M. Wainwright, Statistical learning with sparsity: the lasso and generalizations. Chapman and Hall/CRC, 2015.
[3] H. Li, E. Su, C. Wang, J. Liu, Z. Zheng, Z. Wang, and D. Xia, “A primal-dual forward-backward splitting algorithm for distributed convex optimization,” IEEE Transactions on Emerging Topics in Computational Intelligence, 2021.
[4] A. Iusem, A. Jofré, R. I. Oliveira, and P. Thompson, “Extragradient method with variance reduction for stochastic variational inequalities,” SIAM Journal on Optimization, vol. 27, no. 2, pp. 686–724, 2017.
[5] —, “Variance-based extragradient methods with line search for stochastic variational inequalities,” SIAM Journal on Optimization, vol. 29, no. 1, 2019.
[6] R. I. Bot, P. Mertikopoulos, M. Staudigl, and P. T. Vuong, “Minibatch forward-backward-forward methods for solving stochastic variational inequalities,” Stochastic Systems, vol. 11, no. 2, pp. 112–139, 2021.
[7] P. Latafat, N. M. Ferris, and P. Patrinos, “A new randomized block-coordinate primal-dual proximal algorithm for distributed optimization,” IEEE Transactions on Automatic Control, vol. 64, no. 10, pp. 4050–4065, 2019.
[8] P. Latafat and P. Patrinos, “Asymmetric forward-backward-adjoint splitting for solving monotone inclusions involving three operators,” Computational Optimization and Applications, vol. 68, no. 1, pp. 57–93, 2017.
[9] A. Yurtsever, B. C. Vu, and V. Cevher, “Stochastic three-composite convex minimization,” Advances in Neural Information Processing Systems, vol. 29, 2016.
[10] R. Zhao and V. Cevher, “Stochastic three-composite convex minimization with a linear operator,” in Proceedings of the Twenty-First International Conference on Artificial Intelligence and Statistics, ser. Proceedings of Machine Learning Research, A. Storkey and F. Perez-Cruz, Eds., vol. 84. PMLR, 09–11 Apr 2018, pp. 765–774.
[11] Y. Chen, G. Lan, and Y. Ouyang, “Optimal primal-dual methods for a class of saddle point problems,” SIAM Journal on Optimization, vol. 24, no. 4, pp. 1779–1814, 2014.
[12] H. Robbins and D. Siegmund, A convergence theorem for non negative almost supermartingales and some applications. Academic Press, 1971, pp. 233–257.
[13] H. H. Bauschke and P. L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer - CMS Books in Mathematics, 2016.
[14] A. Jofré and P. Thompson, “On variance reduction for stochastic smooth convex optimization with multiplicative noise,” Mathematical Programming, vol. 174, no. 1, pp. 253–292, 2019.
[15] P. L. Combettes and J.-C. Pesquet, “Stochastic quasi-fejér block-coordinate fixed point iterations with random sweeping,” SIAM Journal on Optimization, vol. 25, no. 2, pp. 1221–1248, 2015.
[16] P. Yi, Y. Hong, and F. Liu, “Initialization-free distributed algorithms for optimal resource allocation with feasibility constraints and application to economic dispatch of power systems,” Automatica, vol. 74, pp. 259–269, 2016.