Evidential Reasoning with Conditional Belief Functions

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Abstract

In the existing evidential networks with belief functions, the relations among the variables are always represented by joint belief functions on the product space of the involved variables. In this paper, we use conditional belief functions to represent such relations in the network and show some relations of these two kinds of representations. We also present a propagation algorithm for such networks. By analyzing the properties of some special evidential networks with conditional belief functions, we show that the reasoning process can be simplified in such kinds of networks.

Keywords: evidential reasoning, belief functions, conditional belief functions, belief networks, valuation networks, local computation techniques.

1. INTRODUCTION

Network-based approaches have been widely used for knowledge representation and reasoning with uncertainties. Bayesian networks (Pearl 1988) and valuation network (Shenoy 1992) are two well-known frameworks for the graphical representations. Bayesian networks are implemented for the probabilistic inference, while valuation networks can represent several uncertainty formalisms in a unified framework. Graphically, a Bayesian network is a directed acyclic graph, a valuation network is a hypergraph. Nodes in the networks represent random variables where each variable is associated with a finite set of all its possible values called its frame. In a Bayesian network, arcs represent the relations among the variables in the form of conditional probabilities, in a valuation network, such relations are represented in the forms of joint valuations on the product space of the involved variables. For the case of belief functions, such valuations are the joint belief functions. Recently, Cano et al. (1993) have presented an axiomatic system for propagating uncertainty (including belief functions) in Pearl's Bayesian network, based on Shafer-Shenoy's axiomatic framework (Shafer and Shenoy 1988, Shenoy and Shafer 1990). But the belief functions for representing relations of the variables in their system are still represented on the product space. Smets (1993) has generalized the Bayes' Theorem for the case of belief functions and presented the Disjunctive Rules of Combination for two distinct pieces of evidence, which makes it possible for representing knowledge and reasoning in evidential network in the form of conditional belief functions. In this paper, we show that any joint belief function representing conditional relations can always be represented by a form of conditional belief functions. We then present a propagation scheme for more complicated cases of evidential networks proposed by Smets (1993). Specifically, we show that the reasoning process can be simplified in some special cases.

The rest of the paper is organized as follows: In section 2, we briefly review belief functions and their rules of combination, both conjunctive and disjunctive; In section 3, we show the relations between the joint belief functions and conditional belief functions which represent the same knowledge; In section 4, we first introduce the evidential network with conditional belief functions, next we present a propagation scheme for it, finally we analyze the properties of some special network and show how to simplify the computation in such networks; Finally in section 5, we give some conclusions.

2. BELIEF FUNCTIONS AND THEIR RULES OF COMBINATIONS

In this section, we briefly review the concept of belief functions (Shafer 1976, Smets 1988), and summarize the conditioning and combination rules for belief functions. More details can be found in (Smets 1990, 1993).

Definition 1: Let Ω be a finite non-empty set called the frame of discernment (the frame for short). The mapping bel: 2Ω → [0, 1] is an (unnormalized) belief function if there exists a basic belief assignment (bba) m: 2Ω → [0, 1] such that:

\[ \sum_{A \subseteq \Omega} m(A) = 1, \quad bel(A) = \sum_{B \subseteq A} m(B), \quad \text{and} \quad bel(\emptyset) = 0. \]

A vacuous belief function is a belief function such that m(Ω) = 1 and m(A) = 0 for all A ≠ Ω.
For a given belief function, we can define a plausibility function \( pl: 2^\Omega \rightarrow [0, 1] \) and a commonality function \( q: 2^\Omega \rightarrow [0, 1] \) as follows: for \( A \subseteq \Omega, A \neq \emptyset \),

\[
pl(A) = bel(\Omega) - bel(A) \quad \text{and} \quad pl(\emptyset) = 0
\]

\[
q(A) = \sum \{ m(B) \mid A \subseteq B \subseteq \Omega \}
\]

where \( A \) is the complement of \( A \) relative to \( \Omega \).

**Definition 2:** Let \( bel \) be our belief about the frame \( \Omega \). Suppose we learn \( A \subseteq \Omega \) is false. The resulting *conditional belief function* \( bel(.IA) \)\(^1\)(\( bel(\Omega/A) \) can be read as the belief of \( B \) given \( A \)) is obtained through the *unnormalized rule of conditioning*: for \( B \subseteq \Omega \),

\[
m(B) = \begin{cases} \sum \{ m(B \cup X) \mid X \subseteq A \} & \text{if } B \subseteq A \subseteq \Omega \\ 0 & \text{otherwise} \end{cases}
\]

**Definition 3:** Consider two distinct pieces of evidence on \( \Omega \) represented by belief functions \( bel_1 \) and \( bel_2 \). The belief function \( bel_{12} \) that quantifies the combined impact of these two pieces of evidence is obtained through the *conjunctive rule of combination*. We use \( \otimes \) to represent the conjunctive combination operator. \( \forall A \subseteq \Omega \),

\[
m_{12}(A) = \sum_{A \subseteq B \subseteq C} m_1(B) m_2(C).
\]

It can also be represented as:

\[
m_{12}(A) = \sum_{B \subseteq \Omega} m_1(\{ A \cap B \}) m_2(B) \quad (2)
\]

**Definition 4:** Consider two distinct pieces of evidence on \( \Omega \) represented by belief functions \( bel_1 \) and \( bel_2 \). The belief function \( bel_{12} \) induced by the disjunction of these two pieces of evidence is obtained through the *disjunctive rule of combination* (Dubois and Prade 1986). We use \( \oplus \) to represent the disjunctive combination operator. \( \forall A \subseteq \Omega \),

\[
m_{12}(A) = \sum_{A \subseteq B \subseteq C} m_1(B) m_2(C) \quad (3)
\]

Since \( m, bel, pl \) and \( q \) are in one-to-one correspondence with each other, the above rules can also be represented by using any of these functions. In this paper, we only give the formulas which will be used in the later computation.

Note that all the definitions above are for the non-normalized case. As for the case of normalized belief functions, which means \( m(\emptyset) = 0 \), the normalization factor \( K = 1 - m(\emptyset) \) should be considered in those rules, and the conditioning rule and the conjunctive combination rule turn out to be Dempster's rule of conditioning and of combination, (unnormalized) \( bel(\Omega/A) \) turns out to be (normalized) \( bel(\Omega/B) \) and \( \otimes \) be \( \Theta \) (Shafer 1976, Smets 1993). \( \otimes \) doesn't have a counterpart in Shafer's presentation.

Let's consider two spaces \( \Theta \) and \( X \), we use \( bel_{X}(\cdot \mid \Theta) \) to represent the belief function induced on the space \( X \) given \( \Theta \subseteq \Theta \). Suppose all we know about \( X \) is initially represented by the set \( \{ bel_{X}(\cdot \mid \theta) : \theta \in \Theta \} \). We only know the beliefs on \( X \) when we know which element of \( \Theta \) holds. We do not know the belief on \( X \) when we only know that the prevailing element of \( \Theta \) belongs to a given subset \( \theta \) of \( \Theta \). Under the requirement that the two pieces of evidence by which our belief function is induced are distinct and that the general likelihood principle is satisfied, Smets (1978, 1993) has derived the Disjunctive Rule of Combination (DRC) to build \( bel_{X}(\cdot \mid \Theta) \) on \( X \) for any \( \Theta \subseteq \Theta \) and the Generalized Bayesian Theorem (GBT) to build \( bel_{\Theta}(\cdot \mid \Theta) \) on \( \Theta \) for any \( x \in X \).

**Theorem 1:** The Disjunctive Rule of Combination: \( \forall \Theta \subseteq \Theta, \forall x \in X \),

\[
m_X(x|\theta) = \sum_{\Omega \ni \theta} \prod_{\Omega \ni \theta} m_X(x|\theta) \quad (4.1)
\]

\[
pl_X(x|\theta) = 1 - \prod_{\Omega \ni \theta} (1 - pl_X(x|\theta)) \quad (4.2)
\]

**Theorem 2:** the Generalized Bayesian Theorem: \( \forall \Theta \subseteq \Theta, \forall x \in X \),

\[
pl_{\Theta}(\theta | x) = 1 - \prod_{\Omega \ni \theta} (1 - pl_X(x|\theta)) \quad (5)
\]

Now suppose there exists some a priori belief \( bel_0 \) over \( \Theta \). By using Theorem 1 and 2, we can compute \( bel \) on \( X \) given \( bel_0 \) and \( \{ bel_X(x|\theta) : \theta \in \Theta \} \):

**Theorem 3:** Suppose there exists some a priori belief \( bel_0 \) over \( \Theta \) distinct from the belief induced by the set of conditional belief functions \( bel_X(x|\theta) : \theta \in \Theta \), then \( \forall x \in X \),

\[
pl_X(x) = \sum_{\Theta \ni \theta} m_0(\theta) pl_X(x|\theta) = \sum_{\Theta \ni \theta} m_0(\theta) (1 - \prod_{\Omega \ni \theta} (1 - pl_X(x|\theta)) \quad (6)
\]

### 3. Knowledge Representation Using Belief Functions

Let \( \bigcup \{ X_1, ..., X_n \} \) be a set of variables where each \( X_i \) has its frame \( \Theta_{X_i} \). Let \( A \) and \( B \) be two disjoint subsets of \( U \), their frames are the product space of the frames of the variables they include. According to the notation of the previous section, a conditional belief function for \( B \) given \( A \) can be represented by \( bel_{\Theta_B}(\cdot \mid \Theta) \) (\( bel_B(\cdot \mid \Theta) \) for short) where \( \Theta \subseteq \Theta_A \), which means that we know the belief about \( B \) given the truth value of \( A \) is in \( \Theta \). In a valuation network, the same relationship between \( A \) and \( B \) is defined in a joint form on the space \( \Theta_A \times \Theta_B \) or \( \Theta_{A \cup B} \) or \( A \times B \) for short). Look at the following example:

**Example 1:** Let \( A \) and \( B \) be two variables with frames \( \Theta_A = \{ a, \neg a \} \) and \( \Theta_B = \{ b, \neg b \} \) respectively. To represent a relation between \( A \) and \( B \) such as: if \( A = a \) then \( B = b \) with \( m = 0.9 \), by a belief function in joint form, the rule is represented by a belief function on the space \( \Theta = \{ (a, b), (a, \neg b), (\neg a, b), (\neg a, \neg b) \} \), with masses: 0.9 on the subset \{ (a, b), (a, \neg b), (\neg a, b), (\neg a, \neg b) \}, and 0.1 on \( \Theta \), while by a belief

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1. We use "\( \setminus \)" in place of "\( \cap \)" to enhance the non-normalization of our conditioning.
function in a conditional form, it is represented as:

\[
m(\{b\}|a)=.9, \quad m(\Theta|a)=.1; \quad m(\Theta|\neg a)=1.
\]

It can be represented by the following table:

| b   | \(\Theta_B\) |
|-----|------------|
| 0.9 | 0          |
| 0.1 | 1          |

Obviously, the latter representation is more "natural" and "easy" for the user to provide and to understand. Generally, given two disjoint subsets \(X, Y \subseteq U\), to represent a conditional belief function for \(Y\) given \(X\), by a joint form, it needs \(2^{\Theta_X} \times 2^{\Theta_Y}\) elements in the worst case, while by a conditional form, it only needs \(2^{\Theta_X} \times \Theta_Y\) elements in the worst case.

Cano et al. (1993) has presented an axiomatic framework in directed acyclic networks which can propagate belief functions in the networks, and has given a definition for a non-informative belief function\(^2\) in such framework represented by belief functions on the product space of two disjoint subsets. Shenoy (1993) has also shown the property of such belief functions in a valuation network. Let's first look at the concepts of projection, extension and marginalization:

**Definition 6:** Projection of configurations simply means dropping the extra coordinates. If \(X\) and \(Y\) are sets of elements, \(\Theta \subseteq X\), and \(x_i\) is an element of \(\Theta_X\), then let \(x_i^{x_i}\) denote the projection of \(x_i\) to \(\Theta_X\). \(x_i^{x_i}\) is an element of \(\Theta_Y\). If \(x\) is a non-empty subset of \(\Theta_X\), then the projection of \(x\) to \(Y\), denoted by \(x^{x_i}\), is obtained by \(x^{x_i} = \{x_i \mid x_i \in x, i \in \Theta_Y\}\). If \(y\) is a subset of \(\Theta_Y\), then the extension of \(y\) to \(X\), denoted by \(x^{y}X\), is \(y \times \Theta_X\). (It is also called the cylinder set extension of \(y\) into \(X\)).

**Definition 7:** Suppose \(m\) is a bba on \(X\) and suppose \(Y \subseteq X \subseteq U\), \(\neq \emptyset\). The marginal of \(m\) for \(Y\) denoted by \(m^{x_i}\), is a bba on \(Y\) defined by \(m^{x_i}(y) = \sum\{m(v) | v \subseteq \Theta_X, x_i^{v} = y\}\) for all subsets \(y\) of \(\Theta_Y\).

**Definition 8:** Given two disjoint subsets \(X, Y \subseteq U\) in the framework of Cano et al. (1993), let bel be a belief function defined on the space \(\Theta_X \times Y\). It is said that bel is a non-informative belief function over \(X\) iff bel is a vacuous belief function over \(X\).

Intuitively, the belief function in definition 8 gives some information about the variables in \(Y\), and their relationship with variables in \(X\), but no information about the variables in \(X\). This property is easy to verify when the belief is represented by a conditional form.

**Lemma 1:** bel\(_Y\)(\(x\): \(x \subseteq \Theta_X\) is non-informative over \(X\) iff bel\(_Y\)(\(x\)) is a normalized belief function for each \(x \subseteq \Theta_X\), i.e., the representation bel\(_Y\)(\(x\)) is such a non-informative belief function over \(X\).

Moreover, we can find that if a belief function bel defined on the space \(\Theta_X \times Y\) gives information only on the relationship of \(X\) and \(Y\), but no information about either \(X\) or \(Y\), then bel\(_X\) and bel\(_Y\) are both vacuous on \(X\) and \(Y\) respectively. That is to say, bel can be non-informative over either \(X\) or \(Y\). The followings give the verification for the belief functions in the conditional form:

**Lemma 2:** Let bel\(_Y\)(\(x\)), \(x \subseteq \Theta_X\) be a conditional belief function for \(Y\) given \(X\). It is non-informative over \(Y\) iff bel\(_Y\)(\(x\)) is a vacuous belief function on \(Y\).

**Lemma 3:** If we only know the conditional belief function as bel\(_Y\)(\(x\)), \(x \subseteq \Theta_X\), then it is non-informative over \(Y\) iff for each \(y \subseteq \Theta_Y\), \(\forall y \subseteq \Theta_X\), such that bel\(_Y\)(\(y\)) = \(0\).

In the following, we will show some relations between the belief functions represented in conditional form and in joint form. By using the rules of conditioning, every joint belief function can be transformed to a conditional form, but not every belief function in a conditional form can be transformed to a joint belief function. We say those that can not be transformed to joint beliefs are invalid. If it can be transformed, the joint form is not always unique. Smets (1993) has shown that when a conditional belief function is represented by \(\{\text{bel}_Y(.,\{\})\mid \Theta_X\}\), we can always construct the joint belief from it.

**Lemma 4:** Let \(X\) and \(Y\) be two disjoint subsets of \(U\). \(m_{X \times Y}\) be a belief function on the product space \(X \times Y\), representing a conditional belief function for \(Y\) given \(X\). Then its conditional form \(m_{Y}(x)\): \(x \subseteq \Theta_X\) is obtained by:

\[
m_{Y}(y|x) = \sum_{y \subseteq \Theta_Y, \Theta_X|y} m_{X \times Y}(S)
\]

**Lemma 5:** If a belief function in a conditional form \(m_{Y}(x)\): \(x \subseteq \Theta_X\) can be transformed to a joint belief, then it should satisfy \(p_{Y}(y|x_1) \leq p_{Y}(y|x_2)\) if \(x_1 \subseteq \Theta_X\) and \(x_2 \subseteq \Theta_X\).

**Proof:**

\[
p_{Y}(y|x_1) = p_{X \times Y}(y | x_1 \times Y) = p_{X \times Y}(y | x_1 \times Y)
\]

\[
\leq p_{X \times Y}(y | x_1 \times Y \cap x_2 \times Y) = p_{X \times Y}(y | x_1 \times Y \cap x_2 \times Y)
\]

\[
= p_{Y}(y|x_2).
\]

**QED**

**Example 2:** Let \(A\) and \(B\) be two variables with frames \(\Theta_A = [a, -a]\) and \(\Theta_B = [b, -b]\) respectively. Let \(\Theta = \{ab, a-b, -a-b\}\) be the set of \(\{ab, a-b, -a-b\}\) be denoted as \(\{1, 2, 3, 4\}\). Then subset \(\{ab, a-b\}\) can be denoted by \(12\), and the subset \(\{ab, a-b\}\) is the sum of \(12\), for example. Consider a belief function \(\text{bel}_1\) on \(\Theta\): \(m(14) = m(23) = 0.1, m(123) = m(124) = m(134) = m(234) = 0.1\).

By applying lemma 4, we have its corresponding conditional belief function for \(B\) given \(A\) shown in table 2.a. However, for another belief function on \(\Theta\): \(m(23) = 0.2, m(134) = m(124) = 0.2\) and \(m(1234) = 0.4\), its corresponding conditional form by applying lemma 4 is shown in Table 2.b. Comparing the two tables, we can find, therefore, that two different joint belief function might be transformed to the same conditional form.

\(^2\) Note that Shenoy (1993) and Cano et al. (1993) called this belief function "conditional belief function". We change the name to avoid confusion with the classical meaning of "conditional belief function".
Table 2.a: Belief Function in Conditional Form for bel1

|   | \(a\) | \(\neg a\) | \(\Theta_A\) |
|---|---|---|---|
| b | \(m(14)+m(134)\) | \(m(23)+m(123)\) | 0 |
|  | .1 + .1 = .2 | .1 + .1 = .2 | |
| -b | \(m(23)+m(234)\) | \(m(14)+m(124)\) | 0 |
|  | .1 + .1 = .2 | .1 + .1 = .2 | |
| \(\Theta_B\) | \(m(123)+m(124)\) | \(m(134)+m(234)\) | \(m(14)+m(23)+m(123)\) |
|  | .1 + .4 = .6 | .1 + .4 = .6 | \(m(234)+m(1234)=1\) |

Table 2.b: Belief Function in Conditional Form for bel2

|   | \(a\) | \(\neg a\) | \(\Theta_A\) |
|---|---|---|---|
| b | \(m(134)\) | \(m(23)\) | .2 |
|  | .2 | .2 | 0 |
| -b | \(m(23)\) | \(m(124)\) | .2 |
|  | .2 | .2 | 0 |
| \(\Theta_B\) | \(m(124)+m(1234)\) | \(m(134)+m(234)\) | \(m(23)+m(124)+m(134)+m(234)+m(14)+m(23)+m(123)\) |
|  | .2 + .4 = .6 | .2 + .4 = .6 | \(m(134)+m(1234)=1\) |

Lemma 6: Suppose X and Y are two disjoint subsets of U. For each \(x_i \in \Theta_X\), let \(bel(Y|x_i)\) denote a belief function on \(\Theta_Y\). Given these belief functions, we can construct the belief function on \(\Theta_{X \cup Y}\) as follows (Smets 1993):

Let \(bel_{X \cup Y}\) be the resulting belief function on \(\Theta_{X \cup Y}\), called the \textit{ballooning extension of bel}(\(Y|x_i\)). Let \(a \subseteq \Theta_{X \cup Y}\) and \(\gamma_i\) be the projection of \(\gamma_\gamma(x_i)^\dagger(X \cup Y)\) for Y. Then

\[
m_{X \cup Y}(a) = \prod \{m_{Y}(y|x_i) | x_i \notin \Theta_X\} \tag{8}\n\]

4. REASONING WITH CONDITIONAL BELIEFS

In this paper, we use the network proposed by Smets (1993) for the propagation of beliefs. Graphically, the network is a directed acyclic graph (dag) as defined in Pearl (1988) for the Bayesian networks, shown in Figure 1. A graph \(G = (M, E)\), where \(M\) are the finite sets of nodes and \(E\) are the sets of edges, is said to be a dag when there is no path \(n_1n_2...n_k\) such that \((n_i, n_{i+1}) \in E\) \((1 \leq i \leq k-1)\) and \(n_1 = n_k\). However, the conditional beliefs are defined in a different way. In our network, each edge represents a conditional relation between the two nodes it connects. In order to distinguish these two kinds of networks, we call ours ENC, which means an evidential network with conditional belief functions. We also assume that, for each conditional belief function for Y given X, all we know about Y given X is initially represented by the set \(\{bel_{Y}(x|x_i) : x_i \notin \Theta_X\}\). For example, in Figure 1, edge (A, B) represents a conditional belief function for the node B given A, represented by \(bel_{B}(x|a_i) : a_i \notin \Theta_A\).

One main object of reasoning process in evidential network is to compute the marginal distributions for some variables. We use \(BEL_X\) to denote the marginal for variable X, \(bel_{D\theta}\) the a priori belief for X. Due to the DRC and the GBT, given two variables X and Y, and the conditional belief \(bel_{Y}(x|x_i)\), we can compute and store \(bel_{Y}(x|y) : x \in \Theta_X\) and \(bel_{X}(y|y) : y \in \Theta_Y\) in the prepossessing, which might be useful for speeding up the computation in the propagation. Now, we are ready to give the inference algorithm: Given an ENC represented by \(G = (M, E)\).

Case 1: propagating beliefs in polytrees, i.e., there is only one (undirected) path between any two nodes in the network:

Propagation algorithm can be regarded as a message-passing scheme: for each node X in the network, its marginal \(BEL_X\) is computed by combining all the messages from its neighbors \(N_X = \{Y(e \in M)(X, Y) \in E\} \text{ and its own prior belief } bel_{D\theta}\), i.e.,

\[
BEL_X = bel_{D\theta} \otimes \bigotimes \{MY \rightarrow X | Y \in N_X\} \tag{9}\n\]

where the message \(MY \rightarrow X\) is a belief function on X, so it can be represented by \(bel_{Y \rightarrow X}\) or \(MY \rightarrow X\), and is computed by:

by: for any \(x \in \Theta_X\),

\[
bel_{Y \rightarrow X}(x) = \sum_{y \in \Theta_Y} \sum_{x \in \Theta_X} m_X(x|y) \cdot bel_{D\theta}(x \rightarrow y(y)) \tag{10}\n\]

Case 2: If there exist any undirected loops in the network, then some nodes needed to be merged to make the network acyclic, resulting in a new polytree \(G' = (M', E')\), where some nodes in \(G'\) might be a subset of the nodes in \(G\), we call this kind of node a \textit{merged node}. For any merged node \(v\) in \(G'\), there might be a belief function \(R_v\) obtained by the ballooning extension of conditional beliefs. Figure 2 illustrates two examples for this process:

In Figure 2.a, the loop is absorbed by merging nodes B and C, the resulting graph is shown in 2.b where \(D = (B, C)\), and new conditional belief function \(bel_{D\theta}(x|a_i)\) is obtained by combining \(bel_{B}(x|a_i)\) and \(bel_{C}(x|a_i)\) on the space \(\Theta_D = \Theta_{B \cap C}: \forall \alpha \in \Theta_A, \exists \Theta_D, \exists \Theta_D,\)

\[
m_D(a_i) = \sum_{(B, C) \in D \setminus (C, D) = d} m_B(a_i) \cdot m_C(a_i) \tag{11}\n\]
Obviously, \( \text{bel}_{D}(d|a_1) \) is normalized iff \( \text{bel}_{B}(d|a_1) \) and \( \text{bel}_{C}(d|a_1) \) are normalized since the subset \( b^{(BC)} \cap c^{(BC)} \) can never be an empty set. Moreover, the conditional belief function between \( B \) and \( C \) becomes \( R_D \) in Figure 2.b obtained by the ballooning extension of \( \text{bel}_{B}(c|d) \) applying eq. (8). Thus \( R_D \) is a belief function on \( \Theta_D \).

Figure 2.c is another example of ENC with a loop. In this case, we merge \( B \) and \( D \), resulting in the graph shown in 2.d, where \( E = (B, D) \). \( \text{bel}_{E}(c|d) \) is obtained by combining \( \text{bel}_{B}(c|b) \) and \( \text{bel}_{D}(c|d) \) on \( \Theta_E = \Theta_{B,D} \) using eq. (11). As for \( \text{bel}_{C}(c|e) : e \in \Theta_E \), we compute it for three cases:

1) For any \( e_i = (b_j, d_j) \in \Theta_E \),

\[
m_{C}(e_i) = \sum_{s_{1} \cap s_{2}=c} m_{C}(s_{1}|b_{i}) \cdot m_{C}(s_{2}|d_{i}); \tag{12.1}
\]

2) For any \( e \in \Theta_E \), if \( e \) can be represented by \( b \cdot d \), where \( b \in \Theta_B \), \( d \in \Theta_D \),

\[
m_{C}(e) = \sum_{s_{1} \cap s_{2}=c} m_{C}(s_{1}|b) \cdot m_{C}(s_{2}|d); \tag{12.2}
\]

where \( m_{C}(c|b) \) and \( m_{C}(c|d) \) are obtained from \( m_{C}(c|b_i) \) and \( m_{C}(c|d_j) \) respectively by applying the DRC as shown in equations (4);

3) For any other \( e \in \Theta_E \), we first construct a conditional belief function \( \text{bel}_{C,O,D}(c|b) \) from \( m_{C}(c|b) \) such that

\[
m_{C,O,D}(s|b) = m_{C}(c|b) \]

where \( s = c^{(C,D)} \cap \{(e \cap b_{1}|b) \cap (d) \}^{(C,D)} \), let \( \text{bel}_{C,O,D} \) be the belief function resulting from the ballooning extension of \( m_{C}(c|d) \), then

\[
m_{C}(c) = (\text{bel}_{C,O,D} \cap (\Theta_E))^{(C)} \tag{12.3}
\]

Equations (4); Alternatively, \( \text{bel}_{C}(c|e) : e \in \Theta_E \) can be computed by first combining the ballooning extensions of the two conditional beliefs \( \text{bel}_{C}(c|b) \) and \( \text{bel}_{C}(c|d) \) on the space \( \Theta_{B,C} \) and \( \Theta_{C,O,D} \), then using equation (7) to transfer the resulting belief in a conditional form \( \text{bel}_{C}(c|e) : e \in \Theta_E \) and

\[
\text{bel}_{C}(c|e) = \frac{
\text{bel}_{B}(c|b) \cap \text{bel}_{C}(c|d) \cap \Theta_{E}}{
\text{bel}_{B}(c|b) \cap \text{bel}_{C}(c|d) \cap \Theta_{E}}.
\]

Since there is no direct relation between \( B \) and \( D \), \( R_E \) is a vacuous belief function.

After transforming the network to an acyclic one, we then use a similar algorithm in case1 for the propagation: Suppose each node \( X \) in \( G' \) is a subset and has a \( R_X \). Thus, for any non-merged node, it is a singleton, and \( R_X \) is a vacuous belief function. Then the computation is as following: for any node \( A = \{X_1, ... , X_t\} \) in \( G' \),

\[
\text{Bel}_{A} = R_A \otimes (\Theta \{M_{Y \rightarrow A} | Y \in N_A \}) \quad \text{and} \quad \text{Bel}_{X_i} = \text{bel}_{O,X_i} \otimes (\Theta \text{bel}_{A X_i}) \quad \text{for} \quad X_i \in A \quad \text{i.e.,} \quad \text{pl}_{Y}(\dot{Y}x) = 1 \quad \text{for all} \quad Y \in \Theta_Y.
\]

Although the above representation and propagation algorithm are for the networks which only have binary relations between the nodes, it could be generalized to the case where relations are for any number of nodes. In the rest of this section, we will show some special cases where using ENC can reduce the computation.

**Definition 9:** Let \( X, Y \) be two nodes in ENC, where \( \Theta_X = \{x_1, ... , x_p\} \), \( \Theta_Y = \{y_1, ... , y_q\} \). Suppose there is an edge \( (X, Y) \) representing a conditional belief for \( Y \) given \( X \); \( \text{bel}_{X}(c|y) \) : \( x \in \Theta_X \) such that \( m(\Theta_X | X) < 1 \) for \( i = 1, ..., t < p \) and \( m(\Theta_Y | x) = 1 \) for \( j = 1, ..., t < p \). We say the elements \( x_j \)'s \((i \leq j)\) are relevant to \( Y \) and \( x_j \)'s \((i < j < p)\) irrelevant to \( Y \).

This kind of relationship exists commonly in the diagnosis problems and rule-based systems. In Example 1, we say that a is relevant to B, but \(-a\) irrelevant to B. Intuitively, it means that given some knowledge on a, we can induce knowledge about B, but no matter what we know about \(-a\), we can't induce any knowledge about B. Thus we say \(-a\) is irrelevant to B.

**Lemma 7:** Given two variables \( X, Y \) and the conditional belief on \( Y \) given \( X \), suppose \( \Phi = \{x_1, ..., x_p\} \) is irrelevant to \( Y \). Then for any subset \( S \) of \( \Theta_X \), if \( S \cap \Phi \neq \emptyset \), then \( m_Y(\Theta_{Y}|S) = 1 \).

**Proof:** The result can be derived directly by applying the GBT.

**QED**

**Lemma 8:** Given two variables \( X, Y \) and a conditional belief on \( Y \) given \( X \), suppose \( \Phi = \{x_1, ..., x_p\} \) is irrelevant to \( Y \). Assume we have some belief \( \text{bel}_Y \) on \( Y \), by theorem 1-3, we can compute the belief of \( X \). If \( m_X(S) \neq 0 \), then \( S \supset \Phi \).

**Proof:** From lemma 7 we have, \( \forall x \in \Theta_X \), if \( x \notin \Phi \neq \emptyset \), \( m_Y(\Theta_{Y}|x) = 1 \), i.e., \( \text{pl}_{Y}(\dot{Y}x) = 1 \) for \( \forall y \in \Theta_Y \). Then, by
equations (5) and (6), \( p_{X}(xly)p_{Y}(ylx)=1 \) for such \( x \).
Thus by equation (7), we have
\[
pl_{X}(x) = \sum_{y \in \Theta_{x}} m_{0}(y)pl_{X}(xly) = \sum_{y \in \Theta_{x}} m_{0}(y) = 1.
\]
Therefore, \( \forall S \subseteq \Phi, \) if \( m_{X}(S) > 0, \) \( S \) should contain any element of \( \Phi, \) \( i.e. S \supseteq \Phi. \)
QED

From lemma 7 & 8, we can simplify the computation for some special cases of ENC, shown in Figure 3, where in 3.a, \( G_{i} \) is a group (set) of variables and suppose some elements \( i \) of \( \Phi \) are irrelevant to each variable \( X_{i}. \) Figure 3.b shows detail in each \( G_{i}. \) To describe the computation, let's begin by recalling the concept of partition:

**Definition 10:** Let \( \Phi = (\Theta_{1}, ..., \Theta_{p}) \) be a frame of discernment. A set \( \Phi_{\Theta} \) of subsets of \( \Theta \) is a partition of \( \Theta \) if the elements in \( \Phi_{\Theta} \) are all non-empty and disjoint and their union is \( \Theta. \) We also call \( \Phi_{\Theta} \) a coarsening of \( \Theta \) and \( \Theta \) a refinement of \( \Phi_{\Theta}. \)

From the definition, we have \( \forall \theta_{i} \in \Theta, \exists x_{i} \in \Phi_{\Theta} \) which is a mapping of \( \theta_{i} \). We denote such mapping by \( \Lambda(\theta_{i}) = x_{j} \). \( \forall \theta_{i} \in \Theta, \Lambda(\theta) = (\Lambda(\theta_{i})|\theta_{i} \in \Theta). \) Let \( bel_{1} \) be a belief function on \( \Theta, \) then the belief \( bel_{2} \) on \( \Phi_{\Theta} \) induced by \( bel_{1} \), say, by coarsening, is obtained by:
\[
m_{2}(x) = \sum_{\theta \in \Phi_{\Theta}} m(\theta) \quad (14.1)
\]
Let \( bel_{2} \) be a belief function on \( \Phi_{\Theta}, \) \( bel_{1} \) on \( \Theta \) induced by \( bel_{2} \), say, by refinement, is obtained by:
\[
m(\theta) = m(\theta) \quad (14.2)
\]
where \( \Theta = \bigcup(\Theta' \mid \Lambda(\Theta') = x) \)

Now suppose that we have some observation about \( X \) and \( Z: \) \( m_{X}(\mathit{+})=0.8, \) \( m_{X}(\mathit{-})=0.2, \) \( m_{Z}(\mathit{+})=1. \) To compute the marginal for \( A, \) if we use the joint belief for the relation \( A \) and \( X, \) then the combination is performed on the product space \( \Theta_{A} \times \Theta_{X}, \) \( \Theta_{A} \times \Theta_{X}; \) if we use the conditional belief represented in the above tables, the computation is performed on the frame \( \Theta_{A}, \) which is more efficient. Moreover, if we use the result of lemma 8, the computation can be simplified further. The following steps illustrate such computation:

1. transform the network in 5.a to the network shown in Figure 5.b where each \( \Theta_{A_{i}} \) is a partition of \( \Theta_{A}: \)
$\Theta_{A1} = \{a_1, a_2, s_1\}$, $s_1 = \{a_3, a_4, a_5\}$, $\Theta_{A2} = \{a_2, a_3, a_4, s_2\}$, and $\Theta_{A3} = \{a_4, a_5, s_3\}$. bel$_X$(l$\alpha_1$) is obtained from bel$_Y$(l$\alpha_1$): $a_1 \in \Theta_{A1}$ is obtained by applying the DRC. Symmetrically, we can get the other two conditional beliefs. The resulting conditional beliefs are shown in Table 4:

1. Using the DRC to compute bel$_{A1}$(lx$\alpha$) and bel$_{A3}$(lz).
2. Using Theorem 2-3 to compute bel$_A$: i=1,2,3. bel$_{A2}$ is vacuous by lemma 3; $m_{A1}(\{a_1, s_1\}) = .24, m_{A1}(\Theta_{A1}) = .76$; and $m_{A3}(\{s_3\}) = .54, m_{A3}(\{a_4, s_3\}) = .36, m_{A3}(\{a_5, s_3\}) = .06, m_{A3}(\Theta_{A3}) = .04$. 
3. Compute the above two beliefs on the frame $\Theta_A$ by refinement and combine them, we get our desired result.

Table 4: Conditional Beliefs Induced from Table 3 for the Partition of $\Theta_A$

| $m_x(xl\alpha)$: $a_i \in \Theta_{A1}$ | $m_y(yl\alpha)$: $a_i \in \Theta_{A2}$ | $m_z(zl\alpha)$: $a_i \in \Theta_{A3}$ |
|---------------------------------|---------------------------------|---------------------------------|
| $a_1$ | $.9$ | $.7$ | $.0$ | $.3$ | $.0$ | $.1$ |
| $a_2$ | $.5$ | $.2$ | $.4$ | $.2$ | $.6$ | $.1$ |
| $s_1$ | $.0$ | $.3$ | $.0$ | $.5$ | $.0$ | $.1$ |
| $s_3$ | $.0$ | $.6$ | $.1$ | $.5$ | $.0$ | $.1$ |

Obviously, this computation is more efficient since in step 2 and 3, the computation is taken on the frame $\Theta_A$, which is smaller than $\Theta_A$.

Moreover, if the network has the properties defined as below, we can also simplify the computation for each sub-network shown in Figure 3.b.

**Definition 11:** Let $X$, $Y$ and $A$ be three nodes in an ENC, where $\Theta_X = \{x_1, \ldots, x_p\}, \Theta_Y = \{y_1, \ldots, y_q\}$ and $\Theta_A = \{a_1, \ldots, a_t\}$. Suppose we have bel$_X$(l$\alpha_1$) and bel$_Y$(l$\alpha_1$) for $a_i \in \Theta_A$. Let $\Phi_X$, $\Phi_Y \subseteq \Theta_A$ be the sets of irrelevant elements for $X$ and $Y$ respectively. $\Phi_X \neq \emptyset$, $\Phi_Y \neq \emptyset$. $X$ and $Y$ are **unrelated** through $A$, denoted by $u(X, Y, A)$, if $\Phi_X$ and $\Phi_Y$ satisfy:

1. $\Phi_X \cap \Phi_Y \neq \emptyset$; or
2. $\Phi_X \cup \Phi_Y = \Theta_A$, bel$_X$(l$\phi_X$) or bel$_Y$(l$\phi_Y$) obtained from bel$_X$(l$\alpha_1$) or bel$_Y$(l$\alpha_1$) by the DRC is vacuous.

This relation can also be extended to the two disjoint subsets, where $\Phi_A: A \subseteq \emptyset$ is defined as: $\Phi_A = \cap \{\Phi_X \mid X \in \emptyset\}$.

**Lemma 9:** Let $X$, $Y$ and $A$ be defined as in definition 11. Suppose we have bel$_X$(l$\alpha_1$) and bel$_Y$(l$\alpha_1$) for $a_i \in \Theta_A$, but no a priori belief on $A$. Now suppose we have observations about $X$, then BEL$_Y$ is vacuous if $X$ and $Y$ are unrelated.

**Proof:** This can be proved by applying the results of lemma 7 and 8. QED

Now let's consider the computation for the network shown in Figure 6.a. Let $X$, $Y$, $A$, $\Phi_X$ and $\Phi_Y$ be defined as in definition 11. Assume $X$ and $Y$ are binary variables. Suppose we have conditional beliefs for $X$ and for $Y$ given each element of $\Theta_A$, $u(X, Y, A)$ and $\Theta_X \cap \Theta_Y = \emptyset$, the conditional belief for $Y$ given $X$ bel$_Y$(lx$\alpha$) is such that bel$_Y$(l$\alpha$) obtained from bel$_Y$(lx$\alpha$) i=1,2 is such that bel$_Y$(l$\alpha$) obtained from bel$_Y$(lx$\alpha$) i=1,2 is such that bel$_Y$(l$\alpha$) obtained from bel$_Y$(lx$\alpha$) is vacuous. If all the a priori beliefs for the variables are vacuous, the propagation result would be vacuous. Now suppose we have observations about $X$: $X=x_\alpha$, then the network in 6.a is equivalent to the one shown in 6.b (Proofs can be found in (Xu and Smets 1994)).
5. CONCLUSIONS

We have presented an evidential network (ENC) which uses conditional belief functions for the knowledge representation and reasoning. In the paper, we have compared some relations between the representation by joint belief and by conditional belief and have found that the conditional form is more natural and it takes less space. We also provide an algorithm for reasoning in ENCs. The presented algorithm of reasoning is only for the network where all the relations are binary, the extension of the algorithm to a general case will be studied in the future work. Although we have compared the computational complexity of ENC and the network using joint beliefs in a general case, we have shown that in some special cases, the computation of ENC can be simplified and is more efficient than the network using joint beliefs. The advantage of simplified computation in such networks can be shown in the example abstracted from (Xu et al. 1993).

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