STRUCTURE AND CLASSIFICATION OF MONOIDAL GROUPOIDS

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Abstract. The structure of monoidal categories in which every arrow is invertible is analyzed in this paper, where we develop a 3-dimensional Schreier-Grothendieck theory of non-abelian factor sets for their classification. In particular, we state and prove precise classification theorems for those monoidal groupoids whose isotropy groups are all abelian, as well as for their homomorphisms, by means of Leech’s cohomology groups of monoids.

1. Introduction and summary

This paper deals with monoidal groupoids \( G = (G, \otimes, I, a, l, r) \), that is, categories \( G \) in which all arrows are invertible, enriched with a monoidal structure by a tensor product \( \otimes : G \times G \to G \), a unit object \( I \), and coherent associativity and unit constraints \( a : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \ l : I \otimes X \cong X, \) and \( r : X \otimes I \cong X \).

Our main objective is to state and prove precise classification theorems for monoidal groupoids and their homomorphisms. In this classification, two monoidal groupoids, say \( G \) and \( G' \), are equivalent whenever they are connected by a monoidal equivalence \( F : G \cong G' \), and two monoidal functors \( F, F' : G \to G' \) which are related by a monoidal natural isomorphism, \( \delta : F \cong F' \), are considered the same.

The particular case of categorical groups is well known since it was dealt with by Sinh in 1975. Recall that a categorical group \([27]\) (also called a Gr-category \([6, 35]\) and a weak 2-group \([1]\)) is a monoidal groupoid \( G \) in which every object \( X \) is invertible, in the sense that there is another object \( X^* \) and an isomorphism \( X \otimes X^* \cong I \). In \([35]\), Sinh proved that, for any group \( G \), any \( G \)-module \( (A, \theta : G \to \text{Aut}(A)) \), and any Eilenberg-Mac Lane cohomology class \( c \in H^3(G, (A, \theta)) \), there exists a categorical group \( G \), unique up to monoidal equivalence, such that \( G \) is the group of isomorphism classes of objects of \( G \), \( A = \text{Aut}_G(I) \) is the (abelian) group of automorphisms in \( G \) of the unit object, and the \( G \)-action \( \theta \) and the cohomology class \( c \) are canonically deduced from the functoriality of the tensor and the naturality and coherence of the constraints of \( G \). This fact was historically relevant since it pointed out the utility of categorical groups in homotopy theory: as \( H^3(G, (A, \theta)) = H^3(BG, (A, \theta)) \) is the 3rd cohomology group of the classifying space \( BG \) of the group \( G \) with local coefficients in \( (A, \theta) \), for any triplet of data \( (G, (A, \theta), c) \) as above, there exists a path-connected CW-complex \( X \), unique up to homotopy equivalence, such that

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\(\pi_1 X = 0\) if \(i \neq 1, 2\), \(\pi_1 X = G\), \(\pi_2 X = A\) (as \(G\)-modules), and \(c \in H^3(G,(A,\theta))\) is the unique non-trivial Postnikov invariant of \(X\). Therefore, categorical groups arise as algebraic homotopy 2-types of path-connected spaces. Indeed, strict categorical groups (that is, categorical groups in which all the structure constraints are identities and the monoid of objects is a group) are the same as crossed modules, whose use in homotopy theory goes back to Whitehead (1949) (see [7] for the history).

However, many illustrative examples such as the category \(\mathcal{A}_R\) of central separable algebras over a commutative ring \(R\), or the fundamental groupoid \(\pi X\) of a Stasheff \(A_4\)-space \(X\) (a topological monoid, for instance), show the ubiquity of monoidal groupoids in several branches of mathematics, and therefore the interest of studying these categorical structures in their own right. But the situation with monoidal groupoids is more difficult than with categorical groups. Let us stress the two main differences between the two situations. On one hand, the structure induced by the tensor product on the set of connected components of a monoidal groupoid is that of a monoid rather than a group as happens in the categorical group case. On the other hand, if \(\mathcal{G}\) is a categorical group, then the isotropy groups \(\text{Aut}_G(X), X \in \text{Ob}\mathcal{G}\), are all abelian and all isomorphic to \(\text{Aut}_G(I)\), while a monoidal groupoid may have some isotropy groups that are not isomorphic to \(\text{Aut}_G(I)\), as well as some noncommutative isotropy groups. Think of the simple example \(\mathbb{F}\) in finite sets and bijective functions between them, whose monoidal structure is given by disjoint union construction: its monoid of isomorphism classes of objects is \(\mathbb{N}\), the additive monoid of natural numbers, and its isotropy groups are the symmetric groups \(\mathfrak{S}_n\).

Strongly inspired by Schreier’s analysis of group extensions [34] and its extension to fibrations of categories by Grothendieck [21] (but also by works of Sinh [35], Breen [6], and others), the structure of the monoidal groupoids is analyzed in this paper, where we develop a 3-dimensional Schreier-Grothendieck factor set theory for monoidal groupoids, which in fact involves a 2-dimensional one for the monoidal functors between monoidal groupoids, and even a 1-dimensional one for the monoidal transformations between them. More precisely, our general conclusions on this issue concerning monoidal groupoids can be summed up by saying that we give explicit quasi-inverse biequivalences

\[
\text{MonGpd} \xrightarrow{\Delta} \text{MonGpd}^{\lambda} \xleftarrow{\Sigma} \text{Mnd},
\]

between the 2-category of monoidal groupoids and the 2-category of what we call Schreier systems for monoidal groupoids, or non-abelian 3-coycles on monoids. That is, systems of data

\[
(M, \mathbb{A}, \Theta, \lambda)
\]

consisting of a monoid \(M\), a family of groups \(\mathbb{A} = (A_a)_{a \in M}\) parameterized by the elements of the monoid, a family of group homomorphisms

\[
\Theta = (A_b \overset{a \mapsto}{\longrightarrow} A_{ab} \overset{b^r}{\longleftarrow} A_a)_{a,b \in M},
\]

and a normalized map

\[
\lambda: M \times M \times M \longrightarrow \bigcup_{a \in M} A_a \quad | \quad \lambda_{a,b,c} \in A_{abc},
\]

satisfying various requirements. In the 2-category \(\text{Mnd}^{\lambda}\), every equivalence is actually an isomorphism, so that our classification results are effective.
The lower theory of group-valued non-abelian 2-cocycles on monoids by Leech \[24\] was extended to small categories by Wells in \[41\]. Interestingly, the work by Wells suggests it is possible to develop a theory of non-abelian 3-cocycles on small categories, which is attractive to us, to generalize our results about monoidal groupoids to (Benabou) bicategories \[37\] whose 2-cells are all invertible, that is, whose hom-categories are groupoids.

When we focus on the special case of *monoidal abelian groupoids*, that is, monoidal groupoids \(\mathcal{G}\) whose isotropy groups \(\text{Aut}_\mathcal{G}(X)\), \(X \in \text{Ob}\mathcal{G}\), are all abelian, then our classification results are stated in a more enjoyable and precise way by means of Leech’s cohomology theory of monoids \[24\]. The biequivalences above restrict to quasi-inverse biequivalences

\[
\begin{array}{ccc}
\text{MonAbGpd} & \xrightarrow{\Delta} & Z^3\text{Mnd}, \\
\Sigma & \xleftarrow{\Sigma} & \\
\end{array}
\]

between the full 2-subcategory \(\text{MonAbGpd} \subseteq \text{MonGpd}\) of monoidal abelian groupoids and the full 2-subcategory \(Z^3\text{Mnd} \subseteq Z^3\text{Mnd}_{\text{ab}}\) given by those Schreier systems \((M, A, \Theta, \lambda)\) in which every group \(A_a\) of \(A\) is abelian. But, the pair \((A, \Theta)\) that occurs in any such Schreier system is just a coefficient system for Leech cohomology groups \(H^n(M, (A, \Theta))\) of the monoid \(M\), and \(\lambda \in Z^3(M, (A, \Theta))\) is a normalized 3-cocycle. From this observation, we achieve the classification both of the monoidal abelian groupoids and of the monoidal functors between them, by means of the cohomology groups \(H^3(M, (A, \Theta))\) and \(H^2(M, (A, \Theta))\). Although these results are mainly of algebraic interest, we would like to stress their potential interest in homotopy theory since, as we observe in the paper, there are natural isomorphisms \(H^n(M, (A, \Theta)) \cong H^n(BM, (A, \Theta))\), between Leech cohomology groups of a monoid \(M\) and Gabriel-Zisman’s cohomology groups of the classifying space \(BM\) of the monoid with twisted coefficients in \((A, \Theta)\) \[19, Appendix II\], \[22, Chapter VI\].

The plan of the paper, briefly, is as follows. After this introductory Section 1, the paper is organized in four sections. Section 2 comprises some notations and basic results concerning monoidal groupoids and the 2-category that they form, as well as a list of some striking examples of them. The main Section 3 includes our ‘Schreier-Grothendieck theory’ for monoidal groupoids. This is quite a long and technical section, but crucial to our conclusions, where we describe the 2-category \(Z^3\text{Mnd}_{\text{ab}}\) of non-abelian 3-cocycles on monoids, and we show in detail how this 2-category is biequivalent to the 2-category \(\text{MonGpd}\) of monoidal groupoids. Section 4 focuses on the special case of monoidal abelian groupoids. In the first subsection, we briefly review some aspects concerning Leech cohomology of monoids \(H^n(M, (A, \Theta))\), and we observe how this cohomology theory is actually a particular case of Gabriel-Zisman cohomology theory of the classifying space \(BM\) of the monoid \(M\). In the second subsection, we include our main classification results concerning monoidal abelian groupoids in terms of Leech cohomology groups. And, finally, a third subsection is devoted to revisiting the 2-category of categorical groups in order to show how the results obtained here imply the classification results already known for them.

\[1\] We thank to the referee for this observation.
2. Preliminaries: The 2-category of monoidal groupoids

This section aims to make this paper as self-contained as possible; hence, at the same time as fixing notations and terminology, we also review some necessary aspects and results from the background of monoidal categories that will be used throughout the paper. However, the material here is quite standard, so the expert reader may skip most of it.

A monoidal category \( \mathcal{M} = (\mathcal{M}, \otimes, I, a, l, r) \) consists of a category \( \mathcal{M} \), a functor \( \otimes: \mathcal{M} \times \mathcal{M} \to \mathcal{M}, \quad (X,Y) \mapsto XY \) (the tensor product), a distinguished object \( I \in \mathcal{M} \) (the unit object), and natural isomorphisms
\[
a_{X,Y,Z}: (XY)Z \xrightarrow{\sim} X(YZ), \quad l_X: IX \xrightarrow{\sim} X, \quad r_X: XI \xrightarrow{\sim} X,
\]
(called the associativity, left unit, and right unit constraints, respectively), such that, for all objects \( X,Y,Z,T \) of \( \mathcal{M} \), the diagrams below (called the associativity pentagon and the triangle for the unit) commute.
\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{(XY)ZT \ar[r]^{a} \ar[d]_{a_1} & X(YZ)T \ar[d]_{1a} \ar[r]^{r_1} & X(IY)T \ar[d]_{\ell I} \ar[r]^{\ell} & XY \ar[d]_{l} \\
(X(YZ))T \ar[r]^{a} & X((YZ)T) \ar[r]^{\ell} & IXY \ar[r]^{\ell} & IXY
\end{array}
\end{array}
\]

A monoidal category is called strict when all the constraints \( a, l, r \) are identity arrows.

Observe that we usually write the structure constraints without object labels, since their source and target make it clear what constraint isomorphism it is.

In any monoidal category, for any objects \( X,Y \), the triangles below commute [27, Proposition 1.1].
\[
\begin{array}{c}
\xymatrix{(XY)I \ar[r]^{a} \ar[d]_{r} & X(YI) \ar[d]_{1} \ar[r]^{l} & XY \ar[d]_{l} \\
XYI \ar[r]_{1} & IXY \ar[r]^{\ell} & IXY
\end{array}
\]

If \( \mathcal{M}, \mathcal{M}' \) are monoidal categories, then a monoidal functor \( F = (F, \varphi) : \mathcal{M} \to \mathcal{M}' \) consists of a functor \( F : \mathcal{M} \to \mathcal{M}' \), a family of natural isomorphisms
\[
\varphi_{X,Y} : FXFY \xrightarrow{\sim} F(XY),
\]
and an isomorphism \( \varphi_0 : I \xrightarrow{\sim} FI \), such that the following diagrams commute:
\[
\begin{array}{c}
\xymatrix{(FXFY)FZ \ar[r]^{\varphi_1} \ar[d]_{\alpha} & F(XY)FZ \ar[r]^{\varphi} \ar[d]_{Fa} & F((XY)Z) \ar[d]_{F(\alpha)} \\
FX(FYFZ) \ar[r]^{\varphi} & FXFYZ \ar[r]^{\varphi} & F(XYZ)
\end{array}
\]
\[
\begin{array}{c}
\xymatrix{FXI' \ar[r]^{1\varphi_0} \ar[d]_{r'} & FXFI \ar[d]_{\varphi} \ar[r]^{\varphi} & F1FX \ar[d]_{\varphi} \\
FX \ar[r]^{F\varphi} & F(XI) \ar[r]^{\varphi} & F(IX)
\end{array}
\]
When \( FI = I' \) and \( \varphi_0 = 1_\nu \), the identity, then the monoidal functor \( F \) is qualified as \textit{strictly unitary}. When each of the isomorphisms \( \varphi_{X,Y} \), \( \varphi_0 \) is an identity, the monoidal functor is called \textit{strict}.

The composition of monoidal functors \( F : \mathcal{M} \to \mathcal{M}' \to \mathcal{M}'' \) will be denoted by juxtaposition, that is, \( F'F : \mathcal{M} \to \mathcal{M}'' \). Recall that its structure constraints are obtained from those of \( F \) and \( F' \), by the compositions

\[
\begin{align*}
F'FX \xrightarrow{\varphi'} F'(FXFY) \xrightarrow{F'\varphi} F'(XY), \\
I' \xrightarrow{\varphi_0'} F'I' \xrightarrow{F'\varphi_0} F'F1.
\end{align*}
\]

The composition of monoidal functors is associative and unitary, so that the category \( \text{MonCat} \) of monoidal categories is defined. Actually, this is the underlying category of a 2-category, also denoted by \( \text{MonCat} \), whose 2-arrows are the \textit{morphisms} of monoidal functors or \textit{monoidal natural transformations}. If \( F, F' : \mathcal{M} \to \mathcal{M}' \) are monoidal functors, then a morphism between them is a natural transformation on the underlying functors \( \delta : F \Rightarrow F' \), such that, for all objects \( X,Y \) of \( \mathcal{M} \), the following coherence diagrams commute:

\[
\begin{align*}
\begin{array}{c}
\delta_X \\
\downarrow \delta_Y
\end{array}
\begin{array}{c}
FXFY \xrightarrow{\varphi} F(XY) \\
F'X F'Y \xrightarrow{\varphi'} F'(XY)
\end{array}
\begin{array}{c}
\vdots \\
\delta_{XY}
\end{array}
\begin{array}{c}
\delta_X \delta_Y
\end{array}
\begin{array}{c}
\downarrow \delta_X
\end{array}
\begin{array}{c}
\delta_Y \\
\downarrow \delta_Y
\end{array}
\begin{array}{c}
F(X) \xrightarrow{\delta_X} F'(XY) \\
F'(XY) \xrightarrow{\delta_Y} F''(XY)
\end{array}
\begin{array}{c}
\varphi_0 \\
\varphi_0'
\end{array}
\begin{array}{c}
\downarrow \delta_0
\end{array}
\begin{array}{c}
\downarrow \delta_0
\end{array}
\begin{array}{c}
F1 \xrightarrow{\delta_0} F'1
\end{array}
\end{align*}
\]

In this 2-category, the “vertical composition” of 2-cells, denoted by

\[
\begin{align*}
\begin{array}{c}
\mathcal{M} \xrightarrow{F} \mathcal{M}' \xrightarrow{F'} \mathcal{M}'' \xrightarrow{G} \mathcal{M} \xrightarrow{G'} \mathcal{M}' \xrightarrow{G'} \mathcal{M}''
\end{array}
\quad \text{is given by the ordinary vertical composition of natural transformations, that is, the component of } \delta' \circ \delta \text{ at any object } X \text{ of } \mathcal{M} \text{ is given by the composition in } \mathcal{M}'
\end{align*}
\]

\[
(\delta' \circ \delta)_X = \delta'_X \circ \delta_X : FX \xrightarrow{\delta_X} F'X \xrightarrow{\delta'_X} F''X.
\]

Similarly, the “horizontal composition”

\[
\begin{align*}
\begin{array}{c}
\mathcal{M} \xrightarrow{F} \mathcal{M}' \xrightarrow{F'} \mathcal{M}'' \xrightarrow{G} \mathcal{M} \xrightarrow{G'} \mathcal{M}' \xrightarrow{G'} \mathcal{M}''
\end{array}
\quad \text{is given by the usual horizontal composition of natural transformations:}
\end{align*}
\]

\[
\delta' \delta = G' \delta \circ \delta' F = \delta' G \circ F' \delta : F' F \Rightarrow G' G.
\]

The following known lemma will be useful in the sequel (cf. [12, Lemma 1.1], for example). Let

\[
\text{MonCat}_{u} \subseteq \text{MonCat}
\]

denote the 2-subcategory of the 2-category of monoidal categories which is full on 0-cells and 2-cells, but whose 1-cells are the strictly unitary monoidal functors.

\textbf{Lemma 2.1.} The inclusion \( \text{MonCat}_{u} \hookrightarrow \text{MonCat} \) is a biequivalence.
Proof. For any two monoidal categories \( \mathcal{M} \) and \( \mathcal{M}' \), a quasi-inverse to the inclusion functor \( i : \text{MonCat}_u(\mathcal{M}, \mathcal{M}') \hookrightarrow \text{MonCat}(\mathcal{M}, \mathcal{M}') \),

\[
(8) \quad (\ )^u : \text{MonCat}(\mathcal{M}, \mathcal{M}') \to \text{MonCat}_u(\mathcal{M}, \mathcal{M}')
\]

which should be called the normalization functor, works as follows: For any given monoidal functor \( F = (F, \varphi) : \mathcal{M} \to \mathcal{M}' \), let \( \Psi_F = (\psi_X)_{X \in \text{Ob}\mathcal{M}} \) be the family of isomorphisms in \( \mathcal{M}' \)

\[ \psi_X = \begin{cases} 1_{FX} : FX \to FX & \text{if } X \neq I \\ \varphi_0^{-1} : FI \to I' & \text{if } X = I. \end{cases} \]

Then, \( F \) can be deformed to a new monoidal functor, \( F^u = (F^u, \varphi^u) : \mathcal{M} \to \mathcal{M}' \), in a unique way such that \( \Psi_F : F \sim \Rightarrow F^u \) becomes an isomorphism. Namely,

\[ F^uX = \begin{cases} FX & \text{if } X \neq I \\ I' & \text{if } X = I, \end{cases} \quad F^u(X \xrightarrow{f} Y) = (F^uX \xrightarrow{\psi_Y \circ Ff \circ \psi_X^{-1}} F^uY), \]

\[ \varphi^u_{X,Y} = \psi_{XY} \circ \varphi_{X,Y} \circ (\psi_X \psi_Y)^{-1}, \quad \varphi^u_0 = \psi_I \circ \varphi_0 = 1_{I'}. \]

Furthermore, any morphism \( \delta : F \Rightarrow G \) gives rise to the morphism

\[ \delta^u = \Psi_G^{-1} \circ \delta \circ \Psi_F : F^u \Rightarrow G^u, \]

which is explicitly given by

\[ \delta^u_X = \begin{cases} \delta_X : FX \to GX & \text{if } X \neq I \\ \varphi_0 \circ \delta_I \circ \varphi_0^{-1} = 1_{I'} : I' \to I' & \text{if } X = I. \end{cases} \]

These mappings \( F \mapsto F^u, \delta \mapsto \delta^u \), describe the normalization functor (8).

Since, by construction, \( (\ )^u i = 1 \), the identity functor, and we have the natural isomorphism \( \Psi : 1 \Rightarrow i (\ )^u, F \mapsto \Psi_F \), both functors \( i \) and \( (\ )^u \) are mutually quasi-inverse. \( \square \)

A monoidal functor \( F : \mathcal{M} \to \mathcal{M}' \) is called a monoidal equivalence when there exists a monoidal functor \( F' : \mathcal{M}' \to \mathcal{M} \) and isomorphisms of monoidal functors \( 1_{\mathcal{M}} \Rightarrow F'F \) and \( FF' \Rightarrow 1_{\mathcal{M}'} \). Two monoidal categories are equivalent if they are connected by a monoidal equivalence. From Saavedra [33, I, Proposition 4.4.2], we have the following useful result:

**Proposition 2.2.** A monoidal functor \( (F, \varphi) : \mathcal{M} \to \mathcal{M}' \) is a monoidal equivalence if and only if the underlying functor \( F : \mathcal{M} \to \mathcal{M}' \) is an equivalence of categories; that is, if and only if the functor \( F \) is full, faithful, and each object of \( \mathcal{M}' \) is isomorphic to an object of the form \( FX \) for some \( X \in \mathcal{M} \).

In this paper, we are mainly going to work with the full 2-subcategory of \( \text{MonCat} \) given by the monoidal groupoids, that is, of monoidal categories whose morphisms are all invertible, hereafter denoted by \( \text{MonGpd} \).

This 2-category of monoidal groupoids contains as a full 2-subcategory the better-known 2-category of categorical groups, denoted by \( \text{CatGp} \).
whose objects, recall, are those monoidal groupoids in which every object is invertible. The inclusion $\text{CatGp} \hookrightarrow \text{MonGpd}$ has a right biadjoint 2-functor

$$\mathcal{P}ic : \text{MonGpd} \rightarrow \text{CatGp}$$

that assigns to each monoidal groupoid $\mathcal{G}$ its Picard categorical group $[33, 2.5.1]$, $\mathcal{P}ic(\mathcal{G}) \subseteq \mathcal{G}$, which is defined as the monoidal full subgroupoid of $\mathcal{G}$ given by the invertible objects.

2.1. Examples. To help motivate the reader, we shall show some classic and striking instances of monoidal groupoids. The most basic example of a monoidal groupoid is perhaps that defined by the category $\mathfrak{Fin}$ of finite sets and bijective functions between them, whose monoidal structure is given by means of the disjoint union construction, which arises in the study of categories of representations of the symmetric groups $\mathfrak{S}_n$ (see Joyal [26]). Indeed, $\mathfrak{Fin}$ is equivalent to the strict monoidal groupoid $\mathfrak{G}$ defined as the disjoint union of the symmetric groups $\mathfrak{S}_n$, $n \in \mathbb{N}$. More precisely, $\mathfrak{G}$ has objects the natural numbers $n \in \mathbb{N}$, and the hom-sets are given by

$$\mathfrak{G}(m,n) = \begin{cases} \mathfrak{S}_n & \text{if } m = n \\ \emptyset & \text{if } m \neq n. \end{cases}$$

Composition is multiplication in the symmetric groups, and the tensor product is given by the obvious map $\mathfrak{G}_m \times \mathfrak{G}_n \rightarrow \mathfrak{G}_{m+n}$.

Ring theory is a good source of many interesting monoidal groupoids. For example, following Fröhlich and Wall [25], let $R$ be any given commutative ring. Then, the monoidal category of $R$-modules, $\text{Mod}_R$, whose monoidal structure is given by the usual tensor product of $R$-modules, $(M,N) \mapsto M \otimes_R N$, contains as an interesting monoidal subcategory the so-called monoidal groupoid of $R$-progenerators, usually denoted by $\text{Gen}_R$.

whose objects are the faithful, finitely generated projective $R$-modules, and whose morphisms are the module isomorphisms between them. The invertible objects in $\text{Gen}_R$ are the invertible $R$-modules, that is, rank 1 projectives. Therefore,

$$\mathcal{P}ic(\text{Gen}_R) = \mathcal{P}ic_R,$$

is the monoidal groupoid known as the Picard categorical group of $R$. Similarly, the monoidal category of associative $R$-algebras with identity, $\text{Alg}_R$, whose monoidal structure is given by the ordinary tensor product of $R$-algebras, $(A,B) \mapsto A \otimes_R B$, contains a striking instance of a monoidal groupoid: the so-called monoidal groupoid of Azumaya $R$-algebras, denoted by $\mathcal{A}z_R$.

whose objects are the central separable $R$-algebras and whose morphisms are the $R$-algebra isomorphisms. Forgetting algebra structure and taking the endomorphism ring define, respectively, two remarkable monoidal functors: $\text{Lin}_R : \mathcal{A}z_R \rightarrow \text{Gen}_R$ and $\text{End}_R : \text{Gen}_R \rightarrow \mathcal{A}z_R$. The Morita monoidal groupoid of $R$-algebras, $\mathcal{M}\text{Alg}_R$,

has objects $R$-algebras, and a morphism $A \rightarrow B$ is an isomorphism class of a Morita equivalence $\text{Mod}_A \simeq \text{Mod}_B$ (or, equivalently, an isomorphism class of an invertible
A \otimes_R B^{\text{op}}$-module). An object $A$ of this monoidal groupoid is invertible if and only if there is another object $B$ such that $A \otimes_R B$ is Morita equivalent to $R$. It follows that $A$ must be an Azumaya $R$-algebra. Conversely, if $A$ is Azumaya, the isomorphism $A \otimes_R A^{\text{op}} \cong \text{End}_R(\text{Lin}_R(A))$ shows that, since $\text{End}_R(\text{Lin}_R(A))$ is Morita equivalent to $R$, $A^{\text{op}}$ provides a quasi-inverse of $A$. Hence,

$$\mathcal{P}ic(\mathcal{MAlg}_R) = \mathcal{B}R$$

is the Brauer categorical group of $R$, whose objects are the same as those of $\mathcal{A}z_R$, that is, the Azumaya $R$-algebras, but whose morphisms are here iso-classes of Morita equivalences between them.

Every monoidal groupoid arises from an elemental categorical construction: If $B$ is any bicategory [37], then the monoidal groupoid of endomorphisms of an object $b \in B$, denoted by

$$\mathcal{E}nd(b),$$

has objects the 1-cells $f : b \rightarrow b$ in $B$ and morphisms the invertible 2-cells $f \Rightarrow f'$ between them. The monoidal structure on $\mathcal{E}nd(b)$ is given by the horizontal composition of cells in the bicategory. The categorical group of autoequivalences of $b$ is

$$\mathcal{A}ut(b) = \mathcal{P}ic(\mathcal{E}nd(b)),$$

that is, the monoidal full subgroupoid of equivalences $b \rightarrow b$ in the bicategory. If, for example, we take $B = \mathbf{Cat}$, the 2-category of categories, and $C$ is any category, then the monoidal groupoid

$$\mathcal{E}nd(C)$$

has objects the functors $F : C \rightarrow C$ and the morphisms are the natural isomorphisms $F \Rightarrow G$. The composition in $\mathcal{E}nd(C)$ is given by the usual vertical composition of natural transformations, while the composition of the functors and the horizontal composition of the natural transformations define its (strict) monoidal structure. These monoidal groupoids of endofunctors are relevant in several frameworks since a pseudo-action of a monoidal category $\mathcal{M}$ on a category $\mathcal{C}$ is the same thing as a monoidal functor $\mathcal{M} \rightarrow \mathcal{E}nd(\mathcal{C})$. For instance, a Deligne action [16] of a monoid $M$ (say, for example, $M = B^+(W, S)$ the Artin-Tits monoid of positive braids defined by a finite Coxeter group $W$ with a set of generators $S$) on a category $\mathcal{C}$, is just a monoidal functor $M \rightarrow \mathcal{E}nd(\mathcal{C})$, from the discrete monoidal category that $M$ defines to the monoidal groupoid of endofunctors of $\mathcal{C}$.

The Picard categorical group of a category $\mathcal{C}$ is

$$\mathcal{P}ic(\mathcal{C}) = \mathcal{A}ut(\mathcal{C}),$$

that is, the monoidal full subgroupoid of $\mathcal{E}nd(\mathcal{C})$ given by the autoequivalences $\mathcal{C} \rightarrow \mathcal{C}$. If, for example, $A$ is any ring and we take $\mathcal{C} = A\text{Mod}_A$, the category of $A$-bimodules, then, by Morita’s theory, there is a monoidal equivalence

$$\mathcal{A}ut(A\text{Mod}_A) \simeq \mathcal{P}ic_A,$$

where $\mathcal{P}ic_A$ is the Picard categorical group of the ring, that is, the categorical group of invertible $A$-bimodules with isomorphisms, whose monoidal structure is given by the usual monoidal product of $A$-bimodules $(M, N) \mapsto M \otimes_A N$. The case where $\mathcal{C} = G$ a group regarded as a category with only one object, is also well-known. The monoidal groupoid

$$\mathcal{E}nd(G)$$
can be described as having objects the group of endomorphisms $f : G \to G$ and morphisms $u : f \to f'$ those elements $u \in G$ such that $f = C_u f'$, where $C_u : G \to G$ is the inner automorphism $C_u(v) = u v u^{-1}$ given by conjugation with $u$. The composition of morphisms is multiplication in $G$, and the (strict) monoidal structure is defined by

$$(f \Rightarrow f') \diamond (g \Rightarrow g') = (fg \Rightarrow uf f'g').$$

The corresponding Picard categorical group of invertible objects

$$\text{Aut}(G),$$

is the categorical group of automorphisms of $G$. It is the internal groupoid in the category of groups whose group of objects is $\text{Aut}(G)$, the group of automorphisms of $G$, and whose group of arrows is the holomorph group $\text{Hol}(G) = G \rtimes \text{Aut}(G)$. Thus, $\text{Aut}(G)$ is precisely the categorical group corresponding to the universal crossed module $G \xrightarrow{C} \text{Aut}(G)$ by the well-known Verdier equivalence between the category of Whitehead crossed modules and the category of strict categorical groups, see [7] for the history.

Algebraic Topology is also a natural setting where monoidal groupoids appear with recognized interest. Recall that the fundamental groupoid $\pi X$, of a space $X$, is the category having $X$ as set of objects, and whose morphisms $\omega : x \to y (x, y \in X)$ are relative end points homotopy classes of paths $\omega : [0, 1] \to X$ with $\omega(0) = x$ and $\omega(1) = y$. The composition in $\pi X$ is induced by the usual concatenation of paths and constant paths provide the identities. Any continuous map $f : X \to Y$ induces a functor $f_* : \pi X \to \pi Y$ given by

$$f_*(x \xrightarrow{[\omega]} y) = (f(x) \xrightarrow{[f \omega]} f(y)),$$

so that the fundamental groupoid construction, $X \mapsto \pi X$, is a functor from the category of topological spaces to the category of groupoids. If $f, g : X \to Y$ are two maps, then a homotopy $\alpha : f \Rightarrow g$, $\alpha : [0, 1] \to Y^X$, induces a natural isomorphism $\alpha_* : f_* \Rightarrow g_*$ defined, for any point $x \in X$, by

$$\alpha_*(x) = [\alpha(-)(x)] : f(x) \to g(x).$$

Moreover, it is easy to see that if two homotopies $\alpha, \beta : f \Rightarrow g$ are related by a relative end maps homotopy, $\alpha \Rightarrow \beta$, then both induce the same natural isomorphism, that is, if $[\alpha] = [\beta]$ in the track groupoid $\pi Y^X$, then $\alpha_* = \beta_* : f_* \Rightarrow g_*.$

Suppose now that $X = (X, m, e, \alpha, \lambda, \rho)$ is any given homotopy-coherent associative $H$-space, that is, a Stasheff $A_1$-space [36] (any topological monoid, for instance). This means that we have a topological space $X$, which is endowed with a continuous multiplication map $m : X \times X \to X$, a point $e \in X$, and homotopies

$$X \times X \times X \xrightarrow{1 \times m} X \times X \xrightarrow{\lambda} X \times X, \quad X \xrightarrow{1 \times e} X \times X \xrightarrow{\rho} X \times X.$$
which are homotopy coherent, in the sense that there are homotopies as below.

Since the functor $X \mapsto \pi X$ preserves products, the multiplication map $X \times X \to X$
induces a tensor product $m_* : \pi X \times \pi X \cong \pi(X \times X) \to \pi X$,
and the homotopies $\alpha$, $\lambda$, and $\rho$, induce corresponding associativity, left unit, and
right unit constraints (which satisfy the pentagon and triangle axioms (1) thanks
to the existence of the homotopies $\Rightarrow$ above), we have thus defined the fundamental
monoidal groupoid of the $H$-space

\[ \pi X = (\pi X, m_*, e, \alpha_*, \lambda_*, \rho_*) \]

Let us stress that $\pi X$ is a categorical group whenever $X$ is group-like (for instance
if $X \simeq \Omega(Y, y_0)$ is any loop space).

3. Schreier-Grothendieck Theory for Monoidal Groupoids

The Schreier Extension Theorem [34] gives a cohomological classification of extensions of (non-abelian)
groups, $1 \to A \to E \to G \to 1$, in terms of equivalence classes of the so-called Schreier systems for group extensions or non-abelian 2-cocycles on groups. That is, by means of systems of data

\[ (G, A, \Theta, \lambda), \]

consisting of groups $G$ and $A$, a family of automorphisms $\Theta = (A \xrightarrow{a} A)_{a \in G}$, and
a family of elements $\lambda = (\lambda_{a,b} \in A)_{a,b \in G}$, satisfying

\[ a_* (ab)_* (f) \circ \lambda_{a,b}^{-1} = a_*(b_*(f)), \quad 1_*(f) = f, \]

where $f$ is any element of the group $A$. Any such Schreier system gives rise to a group extension

\[ 1 \to A \to \Sigma(G, A, \Theta, \lambda) \to G \to 1, \]

where $\Sigma(G, A, \Theta, \lambda)$ is the group defined by considering on the cartesian product set
$A \times G$ the multiplication $(f, a) \circ (g, b) = (f \circ a_*(g) \circ \lambda_{a,b}, ab)$, and any group extension
can be obtained in this way up to isomorphism. Actually, the construction of the
group extension (10), from each Schreier system (9), defines the function on objects
of an equivalence of categories between the category of Schreier systems for group extensions, whose morphisms
\[(p, q, \varphi) : (G, A, \Theta, \lambda) \to (G', A', \Theta', \lambda')\]
are triplets consisting of homomorphisms \(p : G \to G', q : A \to A', \) and a family of elements \(\varphi = (\varphi_a \in A')_{a \in G},\) satisfying:
\[
\varphi_a \circ p(a)_*(q(f)) \circ \varphi^{-1}_a = q(a_*(f)),
\]
\[
q(\lambda_{a,b}) \circ \varphi_{ab} = \varphi_a \circ p(a)_*(\varphi_b) \circ \lambda'_{p(a), p(b)},
\]
and the category of extensions of groups, whose morphisms are commutative diagrams
\[
\begin{array}{ccc}
1 & \longrightarrow & G \\
\downarrow q & & \downarrow \phi \\
1 & \longrightarrow & G'
\end{array}
\]
\[
\begin{array}{ccc}
E & \longrightarrow & H \\
\downarrow p & & \downarrow p \\
E' & \longrightarrow & 1
\end{array}
\]
Several generalizations to monoid extensions of Schreier theory are known in the literature: Rédey [31], Leech [24, 25], Inassaridze [23], and so on. To classify fibrations between categories, Grothendieck [21] raised Schreier’s theorem to a categorical level by means of the theory of pseudo-functors, and higher analogue problems were studied, among others, by Sinh in [35], where she performed the categorical group classification; Breen [6], who treated with non-abelian 3-cocycles of groups for the classification of extensions of groups by categorical groups; Carrasco and Cegarra in [9], where they carried out the classification of central extensions of categorical groups; Ulbrich [38], who classified extensions of Picard categories; Cegarra and Garzón in [13], where a classification of torsors over a category under a categorical group is done; or Cegarra and Khmaladze [14, 15], where the classification of both braided and symmetric graded categorical groups is performed, later extended to the fibred cases by Calvo, Cegarra and Quang in [8]. We are inspired in all these works to undertake a corresponding analysis of monoidal groupoids below, where we achieve a 3-dimensional Schreier-Grothendieck factor set theory for the classification of monoidal groupoids, which in fact involves a 2-dimensional one for monoidal functors between monoidal groupoids, and even a 1-dimensional one for the monoidal transformations between them.

3.1. Schreier systems for monoidal groupoids. Keeping the Schreier-Grothendieck theory in mind, we introduce 3-dimensional Schreier systems for monoidal groupoids, or non-abelian 3-cocycles on monoids, which will be shown as appropriate minimal systems of “descent datum” to build a survey of all monoidal groupoids up to monoidal equivalences.

**Definition 3.1.** A Schreier system (for a monoidal groupoid) \(S = (M, A, \Theta, \lambda)\) consists of the following data:
- a monoid \(M,\)
- a family of groups \(A = (A_a)_{a \in M},\)
- a family of homomorphisms \(\Theta = (A_b \xrightarrow{a*} A_{ab} \xleftarrow{b*} A_a)_{a,b \in M},\)
- a family of elements \(\lambda = (\lambda_{a,b,c} \in A_{abc})_{a,b,c \in M}.\)

These data must satisfy the following seven conditions:
For any \( a, b, c \in M, h \in A_a, g \in A_b, \) and \( f \in A_c, \)
\[
\lambda_{a,b,c} \circ (ab)_*(f) \circ \lambda_{a,b,c}^{-1} = a_*(b_*(f)),
\]
\[
\lambda_{a,b,c} \circ c^*(a_*(g)) \circ \lambda_{a,b,c}^{-1} = a_*(c_*(g)),
\]
\[
\lambda_{a,b,c} \circ c^*(b_*(h)) \circ \lambda_{a,b,c}^{-1} = (bc)^*(h).
\]

- For any \( a, b, c, d \in M, \)
\[
a_* \circ \lambda_{b,c,d} \circ a_* \circ d^* \circ \lambda_{a,b,c} = \lambda_{a,b,c,d}.
\]
- For any \( a, b \in M, g \in A_a, \) and \( f \in A_b, \)
\[
a_*(f) \circ b^*(g) = b^*(g) \circ a_*(f).
\]
- For any \( a \in M \) and \( f \in A_a, \)
\[
1_* \circ f = f = 1_* \circ f.
\]
- For any \( a, b \in M, \)
\[
\lambda_{1,a,b} = \lambda_{a,1,b} = \lambda_{a,b,1} = 1.
\]

**Example 3.2.** A Schreier system as above with \( \lambda = 1 \) (i.e., such that \( \lambda_{a,b,c} = 1 \) for all \( a, b, c \in M \)) is the same thing as a pair of data \( (M, (\mathbb{A}, \Theta)) \) consisting of a monoid \( M \) together with an internal group object \( (\mathbb{A}, \Theta) \in \text{Gp}(\text{Mnd}_M) \), in the comma category of monoids over \( M \). We refer to Wells [40, Theorem 6] for details, but briefly let us say that, for that identification, one regards \( (\mathbb{A}, \Theta) \) as the monoid obtained as the disjoint union of the groups \( A_a, a \in M, \) with multiplication given by \( (f, a) (g, b) = (a_*(f) \circ b^*(g), ab) \). This multiplication is associative thanks to (11), (12), and (13), and it is unitary, with \((1,1)\) its unity, owing to (16). The monoid homomorphism \( \sqcup_{a \in M} A_a \to M \) is the obvious projection \( (f, a) \mapsto a, \) and the internal group operation is defined by the map
\[
\sqcup_{a \in M} A_a \times_M \sqcup_{a \in M} A_a \to \sqcup_{a \in M} A_a, \quad ((f, a), (g, a)) \mapsto (f \circ g, a),
\]
which is plainly recognized to be a monoid homomorphism thanks to the centralizing condition (15).

Surjective monoid homomorphisms \( E \to M \) endowed with a principal homogeneous internal \( (\mathbb{A}, \Theta) \)-action in \( \text{Mon}_M \) (i.e., internal \( (\mathbb{A}, \Theta) \)-torsors) are classified by means of Leech non-abelian 2-cocycles of \( M \) with coefficients in \( (\mathbb{A}, \Theta) \). That is, by families \( \lambda = (\lambda_{a,b}) \) of elements \( \lambda_{a,b} \in A_{ab} \), one for each \( a, b \in M, \) such that
\[
a_* (\lambda_{b,c}) \circ \lambda_{a,bc} = c^*(\lambda_{a,b}) \circ \lambda_{ab,c},
\]
for all \( a, b, c \in M; \) see Leech [24, Section 3] and Wells [40, Theorems 1 and 7].

**Remark 3.3.** Any monoid can be regarded as a small category with only one object, and natural generalizations to small categories of the above facts in Example 3.2 can be found in the unpublished paper by Wells [41], where he handles non-abelian coefficients as well as the abelian ones that the (commonly attributed to Baues and Wirtinger [4]) cohomology of small categories theory uses. The results by Wells suggest that might be possible to develop a theory of group-valued non-abelian 3-cocycles on small categories in order to generalize our results in this paper about monoidal groupoids to bicategories [37] whose hom-categories are groupoids, that is, to ‘many objects’ monoidal groupoids. Although this generalization (quite far from obvious) to bicategories is beyond the scope and possibilities of this already
long paper, we shall point out that these non-abelian 3-cocycles on a category $C$ should be systems of data consisting of a group $A_a$ for each arrow $a : y \to x$ of $C$, group homomorphisms $A_b \xrightarrow{a\cdot} A_{ab} \xleftarrow{b\cdot} A_a$ for each pair of composable arrows $z \to y \to x$ in $C$, and elements $\lambda_{a,b,c} \in A_{abc}$, one for each three composable arrows $t \to z \to y \to x$ in $C$, all subject to the corresponding seven conditions as in (11) – (17).

**Remark 3.4.** Regarding any group as a groupoid with exactly one object, it was observed by Grothendieck [21] that a non-abelian 2-cocycle $(G, A, \Theta, \lambda)$ for a group extension of a group $G$ by a group $A$, as in (9), can be identified as a normal pseudo-functor on $G$ that associates the group $A$ to the unique object of $G$. Similarly, as one identifies any monoid with the monoidal discrete category it defines, then a Schreier system $(M, A, \Theta, \lambda)$ for a monoidal groupoid, as in Definition 3.1, can be viewed as a group valued normal monoidal pseudo-functor on $M$, in the sense of Carrasco-Cegarra [9, Definition 1.6], that associates the group $A_a$ to each object $a \in M$.

Next we explain how Schreier systems, as in Definition 3.1, are characteristically associated to monoidal groupoids.

### 3.2. Schreier systems associated to monoidal groupoids.

For any given monoidal groupoid $\mathcal{G} = (\mathcal{G}, \otimes, I, \ast, l, r)$, let

$$M(\mathcal{G}) = \text{Ob} \mathcal{G}/\sim$$

be the monoid of isomorphism classes $a = [X]$ of objects $X \in \mathcal{G}$ where multiplication is induced by the tensor product, that is, $[X][Y] = [XY]$.

The construction $\mathcal{G} \to M(\mathcal{G})$ turns the category of monoidal groupoids into a fibred category over the category of monoids. To determine its fiber over a monoid, we shall proceed as Schreier did for extensions of a group.

We start by choosing a cleavage for $\mathcal{G}$ over $M(\mathcal{G})$; that is, for each $a \in M(\mathcal{G})$, let us choose an object $X_a \in a$, and for any other $X \in a$, we fix a morphism $\Gamma = \Gamma_X : X \to X_a$. In particular, we take

$$\Gamma_{X_a} = 1_{X_a} : X_a \to X_a, \quad \Gamma_{X_aI} = r_{X_a} : X_aI \to X_a.\quad (19)$$

Then, we have the following family of isotropy groups of the groupoid $\mathcal{G}$ parameterized by the elements of $M(\mathcal{G})$:

$$\mathcal{A}(\mathcal{G}) = \{\text{Aut}_{\mathcal{G}}(X_a)\}_{a \in M(\mathcal{G})}.\quad (20)$$

We also have the family of group homomorphisms

$$\Theta(\mathcal{G}) = \{\text{Aut}_{\mathcal{G}}(X_a) \xrightarrow{a \cdot} \text{Aut}_{\mathcal{G}}(X_{ab}) \xleftarrow{b \cdot} \text{Aut}_{\mathcal{G}}(X_a)\}_{a,b \in M(\mathcal{G})},\quad (21)$$

which, for any $a, b \in M$, carry automorphisms of $\mathcal{G}$, say $f : X_b \to X_b$ and $g : X_a \to X_a$, to the automorphisms $a \cdot (f) : X_{ab} \to X_{ab}$ and $b \cdot (g) : X_{ab} \to X_{ab}$, respectively determined by the commutativity of the squares below.

$$\begin{array}{ccc}
X_aX_b \xrightarrow{1f} X_aX_b & & X_aX_b \xrightarrow{g1} X_aX_b \\
\downarrow \Gamma & & \downarrow \Gamma \\
X_{ab} \ast_{X_a} \sim X_{ab} & & X_{ab} \ast_{X_a} \sim X_{ab}
\end{array}\quad (22)$$
Furthermore, for any three elements \(a, b, c \in M(\mathcal{G})\), there is a unique 
\[ \lambda_{a,b,c} \in \text{Aut}_\mathcal{G}(X_{abc}) \]
making commutative the diagram
\[
\begin{array}{c}
\xymatrix{
(X_a X_b) X_c \ar[r]^\gamma \ar@{.>}[d]_a & X_{ab} X_c \ar[r]^\gamma \ar[d]_{\lambda_{a,b,c}} & X_{abc} \\
X_a (X_b X_c) \ar[r]^{1\gamma} \ar[d]_{1f} & X_a X_{bc} \ar[r]^\gamma & X_{abc}
}
\end{array}
\]

Then, letting
\[ \lambda(\mathcal{G}) = (\lambda_{a,b,c} \in \text{Aut}_\mathcal{G}(X_{abc}))_{a,b,c \in M(\mathcal{G})}, \]
we have:

**Proposition 3.5.** For any monoidal groupoid \(\mathcal{G} = (\mathcal{G}, \otimes, I, a, l, r)\), the associated quadruplet
\[
\Delta(\mathcal{G}) = (M(\mathcal{G}), \Lambda(\mathcal{G}), \Theta(\mathcal{G}), \lambda(\mathcal{G})),
\]
given by (18), (20), (21), and (24), is a Schreier system.

**Proof.** In all the diagrams below, those inner regions labelled with (A) commute by the naturality of the associativity constraint, those labelled with (B) are commutative because the tensor product \(\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}\) is a functor, and the others commute by the references therein.

For any \(a, b, c \in M(\mathcal{G})\), \(h \in \text{Aut}_\mathcal{G}(X_a)\), \(g \in \text{Aut}_\mathcal{G}(X_b)\), and \(f \in \text{Aut}_\mathcal{G}(X_c)\), the conditions in (11), (12), and (13), follow, respectively, from the commutativity of the outside regions in the following three diagrams in \(\mathcal{G}\):
The 3-cocycle condition (14), for any $a, b, c, d \in M(G)$, follows from the commutativity of the following diagram:

Since, for any $a, b \in M(G)$, $g \in \text{Aut}_G(X_a)$, and $f \in \text{Aut}_G(X_b)$, we have the commutative diagram
it follows that the homomorphisms $a^*$ and $b^*$ in (21) are centralizing, that is, the condition in (15) holds. Moreover, when $a = 1$ or $b = 1$, the naturality of the unit constraints gives the commutativity of the squares

\[
\begin{array}{ccc}
IX_b \xrightarrow{1f} IX_b & & IX_a \xrightarrow{g^1} IX_a \\
g_{r=1} & \Downarrow f_{r=1} & \Downarrow g_{r=1} \\
X_b \xrightarrow{f} X_b & & X_a \xrightarrow{g} X_a,
\end{array}
\]

whence $1^*(f) = f$ and $1^*(g) = g$. That is, the normalization conditions in (16) hold. Furthermore, recalling the selections (19), it is plain to see that the normalization conditions in (17) are direct consequence of the coherence triangles in (1) and (2). This completes the proof. □ □

The Schreier system in (25), associated to a monoidal groupoid, depends on the selection of the cleavage made for its construction. However, as we shall prove, different choices produce equivalent Schreier systems.

We next explain how each Schreier system gives rise, by the Grothendieck construction (cf. [9, 1.3]), to a monoidal groupoid.

3.3. The monoidal groupoid defined by a Schreier system. Let $\mathcal{S} = (M, \Lambda, \Theta, \lambda)$ be any given Schreier system. Then, a monoidal groupoid

\[(26) \quad \Sigma(\mathcal{S}) = (\Sigma(\mathcal{S}), \otimes, 1, a, l, r)\]

is defined as follows: an object of $\Sigma(\mathcal{S})$ is an element $a \in M$. If $a \neq b$ are different elements of the monoid $M$, then there are no morphisms in $\Sigma(\mathcal{S})$ between them, whereas if $a = b$, then a morphism $f : a \to a$ is an element $f$ of the group $A_a$, that is,

\[\Sigma(\mathcal{S})(a, b) = \begin{cases} 
\emptyset & \text{if } a \neq b, \\
A_a & \text{if } a = b.
\end{cases}\]

The composition of morphisms is given by the group operation of $A_a$, that is,

\[(a \xrightarrow{f} a) \circ (a \xrightarrow{f'} a) = (a \xrightarrow{f \circ f'} a).\]

The tensor product $\otimes : \Sigma(\mathcal{S}) \times \Sigma(\mathcal{S}) \to \Sigma(\mathcal{S})$ is defined by

\[(a \xrightarrow{g} a) \otimes (b \xrightarrow{f} b) = (ab \xrightarrow{a^*(f) \circ b^*(g)} ab),\]

which is a functor thanks to the centralizing condition (15). In effect, we have

\[(a \xrightarrow{1} a) \otimes (b \xrightarrow{1} b) = (ab \xrightarrow{a \circ (1) \circ b^*(1)} ab) = ab \xrightarrow{1} ab.\]

and, for any $g, g' : a \to a$ and $f, f' : b \to b$, we have

\[(g \circ g') \otimes (f \circ f') = a^*(f \circ f') \circ b^*(g \circ g') = a^*(f) \circ a^*(f') \circ b^*(g) \circ b^*(g') = (g \otimes f) \circ (g' \otimes f') .\]

The associativity isomorphisms are

\[\lambda_{a,b,c} : (ab)c \to a(bc).\]
These are natural thanks to conditions (11), (12), and (13). In effect, for any $h : a \to a$, $g : b \to b$, and $f : c \to c$,

$$
\lambda_{a,b,c} \circ ((h \otimes g) \otimes f) = \lambda_{a,b,c} \circ ((a_*(g) \circ b^*(h)) \otimes f)
$$

$$
= \lambda_{a,b,c} \circ (ab)_*(f) \circ c^*(a_*(g) \circ b^*(h))
$$

$$
= \lambda_{a,b,c} \circ (ab)_*(f) \circ c^*(a_*(g))^*(b^*(h))
$$

$$
\overset{(11)}{=} a_*(b_*(f)) \circ \lambda_{a,b,c} \circ c^*(a_*(g)) \circ c^*(b^*(h))
$$

$$
\overset{(12)}{=} a_*(b_*(f)) \circ a_*(c^*(g)) \circ \lambda_{a,b,c} \circ c^*(b^*(h))
$$

$$
\overset{(13)}{=} a_*(b_*(f)) \circ a_*(c^*(g)) \circ (bc)^*(h) \circ \lambda_{a,b,c}
$$

$$
= a_*(b_*(f) \circ c^*(g)) \circ (bc)^*(h) \circ \lambda_{a,b,c}
$$

$$
= (h \otimes (b_*(f) \circ c^*(g))) \circ \lambda_{a,b,c}
$$

$$
= (h \otimes (g \otimes f)) \circ \lambda_{a,b,c}.
$$

The pentagon coherence condition in (1) just says that, for any $a,b,c,d \in M$, the diagram

$$
\begin{array}{ccc}
((ab)c)d & \xrightarrow{\lambda_{a,b,c,d}} & (ab)(cd) \\
\downarrow{d^*(\lambda_{a,b,c})} & & \downarrow{a_*(\lambda_{b,c,d})} \\
(a(bc))d & \xrightarrow{\lambda_{a,b,c,d}} & a((bc)d)
\end{array}
$$

must be commutative, which holds because of the 3-cocycle condition (14).

The unit object is $I = 1$, the unit element of the monoid $M$, and the unit constraints are both identities, that is, for any $a \in M$,

$$
I_a = 1 = r_a : a \to a.
$$

These are natural due to the equalities in (16). In effect, for any $f : a \to a$, we have

$$
I_a \circ (1 \otimes f) = 1 \otimes f = 1_*(f) \circ a^*(1) \overset{(16)}{=} f \circ 1 = f = f \circ I_a,
$$

$$
r_a \circ (f \otimes 1) = f \otimes 1 = a_*(1) \circ 1^*(f) \overset{(16)}{=} 1 \circ f = f = f \circ r_a.
$$

The coherence triangle for the unit in (1) commutes owing to the normalization condition $\lambda_{a,1,c} = 1$ in (17).

As we will show, both constructions $S \mapsto \Sigma(S)$, as above, and $G \mapsto \Delta(G)$, as in (25), are suitable for expressing the strong relationship between Schreier systems and monoidal groupoids. We need the notions of morphisms between Schreier systems and their deformations, which we establish below.

### 3.4. The 2-category of Schreier systems

The Schreier systems introduced in Definition 3.1 (or non-abelian 3-cocycles of monoids) are the objects of a 2-category in which all 2-cells are invertible, denoted by

$$
Z^3_{n-ab,MNd},
$$

whose cells and their compositions are defined as follows:
3.4.1. Morphisms of Schreier systems. If $S = (M, \mathbb{A}, \Theta, \lambda)$ and $S' = (M', \mathbb{A}', \Theta', \lambda')$ are two Schreier systems, then a morphism

$$
\varphi = (p, q_1, \varphi) : S \rightarrow S'
$$

consists of the following data:

- a monoid homomorphism $p : M \rightarrow M'$,
- a family of group homomorphisms $q_1 = (A_a \xrightarrow{\varphi_a} A'_{p(a)})_{a \in M}$,
- a family of elements $\varphi = (\varphi_{a,b} \in A'_{p(ab)})_{a,b \in M}$,

satisfying the following three conditions:

- For any $a, b \in M$, $g \in A_a$, and $f \in A_b$,

$$
\varphi_{a,b} \circ p(a)_*(q_b(f)) \circ \varphi_{a,b}^{-1} = q_{ab}(a_*(f)),
\varphi_{a,b} \circ p(b)_*(q_a(g)) \circ \varphi_{a,b}^{-1} = q_{ab}(b_*(g)).
$$

(27)

- For any $a, b, c \in M$,

$$
q_{abc}(\lambda_{a,b,c}) \circ \varphi_{ab,c} \circ p(c)_*(\varphi_{a,b}) = \varphi_{a,b,c} \circ p(a)_*(\varphi_{b,c}) \circ \lambda'_{p(a),p(b),p(c)}.
$$

(28)

- $\varphi$ is normalized, that is,

$$
\varphi_{1,1} = 1.
$$

(29)

Observe that, taking $b = c = 1$ in the above equality (28), we deduce that, for any $a \in M$, $\varphi_{a,1} \circ \varphi_{a,1} = \varphi_{a,1} \circ p(a)_*(\varphi_{1,1}) = \varphi_{a,1}$ in the group $A'_{p(a)}$, whence $\varphi_{a,1} = 1$. Similarly, $\varphi_{1,a} = 1$.

3.4.2. Deformations. Let $\varphi = (p, q_1, \varphi) : S \rightarrow S'$ and $\tilde{\varphi} = (\tilde{p}, \tilde{q}_1, \tilde{\varphi}) : S \rightarrow S'$ be morphisms between Schreier systems $S = (M, \mathbb{A}, \Theta, \lambda)$ and $S' = (M', \mathbb{A}', \Theta', \lambda')$.

If $p \neq \tilde{p}$ are different homomorphisms, then there is no deformation between $\varphi$ and $\tilde{\varphi}$ in $Z^3_{\text{a-\text{ab}}, \text{Mnd}}$.

If $p = \tilde{p}$, then a deformation

$$
\begin{array}{c}
\xymatrix{ 
S \ar[r]^\varphi \ar[rd]_\delta & S' \ar@/_/[r]_{\tilde{\varphi}} \\
& \delta \ar@/_/[u]_\tilde{\varphi} 
}
\end{array}
$$

is a family of elements $\delta = (\delta_a \in A'_{p(a)})_{a \in M}$ satisfying the following two conditions:

- For any $a \in M$ and $f \in A_a$,

$$
\delta_a^{-1} \circ \tilde{q}_a(f) \circ \delta_a = q_a(f).
$$

(30)

- For any $a, b \in M$,

$$
\delta_{ab} \circ \varphi_{a,b} = \varphi_{a,b} \circ p(a)_*(\delta_b) \circ p(b)_*(\delta_a).
$$

(31)

Observe that, taking $a = b = 1$ in the above equality (31), we deduce that $\delta_1 = \delta_1 \circ \delta_1$ in the group $A'_1$, whence $\delta_1 = 1$. 
3.4.3. Vertical composition of deformations. For any Schreier systems $S = (M, A, \Theta, \lambda)$ and $S' = (M', A', \Theta', \lambda')$, the vertical composition in the 2-category $Z_{nabh}^{3, Mnd}$ of deformations

$$
\varphi = (p, q, \varphi) \\
\downarrow \delta \\
(p, q, \varphi) \\
\downarrow \delta \\
\varrho = (p, q, \varrho)
$$

is the deformation $\delta \circ \varphi \Rightarrow \delta \varrho$ obtained by pointwise multiplication, that is,

$$
\delta \circ \varphi = (\delta_a \circ \varphi_a)_{a \in M}.
$$

The identity deformation on each morphism $\varphi : S \rightarrow S'$ is

$$
1_{\varphi} = \begin{pmatrix} 1 \in A'_{p(a)} & a \in M : \varphi \Rightarrow \varphi \end{pmatrix}.
$$

Every deformation $\delta : \varphi \Rightarrow \varphi'$ is invertible, with inverse $\delta^{-1} = (\delta_a^{-1} \in A'_{p(a)})_{a \in M}$. Therefore, the hom-categories $Z_{nabh}^{3, Mnd}(S, S')$ are groupoids.

3.4.4. Horizontal composition of morphisms. For Schreier systems $S = (M, A, \Theta, \lambda)$, $S' = (M', A', \Theta', \lambda')$, $S'' = (M'', A'', \Theta'', \lambda'')$, the horizontal composition of two morphisms

$$
S \xrightarrow{\varphi = (p, q, \varphi)} S' \xrightarrow{\varphi' = (p', q', \varphi')} S''
$$

is the morphism

$$
\varphi' \varphi = (p' p, q' q, \varphi' \varphi') : S \rightarrow S'',
$$

where $p' p : M \rightarrow M''$ is the composite of $p$ and $p'$, and

$$
q' q = \begin{pmatrix} q'_{p(a)} q_a : A_a \rightarrow A''_{p'(a)} & a \in M \end{pmatrix},
$$

\[
\varphi' \varphi' = \begin{pmatrix} \varphi'_{p(ab)}(\varphi_{a,b}) \circ \varphi'_{p(a)} p(b) \in A''_{p'(ab)} & a, b \in M \end{pmatrix},
\]

The identity morphism on a Schreier system $S = (M, A, \Theta, \lambda)$ is

$$
1_S = (1_{M}, 1_A, 1) : S \rightarrow S,
$$

where $1_M$ is the identity map on $M$, $1_A = (1_{A_a})_{a \in M}$, and $1 = (1 \in A_{ab})_{a, b \in M}.

3.4.5. Horizontal composition of deformations. The horizontal composition of deformations

$$
S \xrightarrow{\varphi = (p, q, \varphi)} S' \xrightarrow{\varphi' = (p', q', \varphi')} S''
$$

is the deformation $\delta' \delta : \varphi' \varphi \Rightarrow \varphi' \varrho$ defined by

$$
\delta' \delta = (\delta'_{p(a)} \circ q'_{p(a)}(\delta_a) \in A''_{p'(a)})_{a \in M}.
$$

For later use, we prove here the lemma below.
Lemma 3.6. For any morphism \((p, q, \varphi) : (M, A, \Theta, \lambda) \to (M', A', \Theta', \lambda')\) in the 2-category \(Z_{\text{ab}}^{3} \text{Mnd}\), the following statements are equivalent:

(i) \((p, q, \varphi)\) is an isomorphism.

(ii) \((p, q, \varphi)\) is an equivalence.

(iii) The homomorphisms \(p : M \to M'\) and \(q_a : A_a \to A'_{p(a)}\), \(a \in M\), are all isomorphisms.

Proof. (i) \(\Rightarrow\) (ii) is obvious.

(ii) \(\Rightarrow\) (iii). First observe that, for any Schreier system \(S = (M, A, \Theta, \lambda)\), a morphism \(\varphi : S \to S'\) with a deformation \(\delta : \varphi \Rightarrow 1_S\) is necessarily of the form \(\varphi = (1_M, q(\delta), \varphi(\delta))\), for some family \(\delta = (\delta_a)_{a \in M}\), with \(\delta_1 = 1\), where \(q(\delta) = (q(\delta)_a : A_a \to A'_a)_{a \in M}\) consists of the inner automorphisms given by \(q(\delta)_a(f) = \delta^{-1}_a \circ f \circ \delta_a\), and \(\varphi(\delta) = (\varphi(\delta)_a)_{a, b \in M}\) consists of the elements obtained by the formula \(\varphi(\delta)_a,b = \delta^{-1}_{ab} \circ a_* (\delta_b) \circ b^* (\delta_a)\).

Then, the existence of a morphism \((p', q', \varphi') : S' \to S\), where \(S\) is as above and \(S' = (M', A', \Theta', \lambda')\), with deformations \(\delta : (p', q', \varphi')(p, q, \varphi) \Rightarrow 1_S\) and \(\delta' : (p, q, \varphi)(p', q', \varphi') \Rightarrow 1_{S'}\), implies that \(p'p = 1_M\), \(pp' = 1_{M'}\), so \(p\) is an isomorphism, and also that

\[q'_{p(a)}q_a = q(\delta)_a, \quad q_a q'_p = q(\delta')_{p(a)},\]

for all \(a \in M\). Hence \(q_a\) and \(q'_{p(a)}\) are both isomorphisms since \(q(\delta)_a\) and \(q(\delta')_{p(a)}\) are automorphisms.

(iii) \(\Rightarrow\) (i). The inverse \((p, q, \varphi)^{-1} = (p', q', \varphi')\) is given by taking

\[p' = p^{-1}, \quad q' = (q^{-1}_{p'(a)})_{a' \in M'}, \quad \varphi' = (q_{a'}^{-1} (\varphi_{p'(a'), p'(a')}'))_{a', b' \in M'}\]

3.5. The classifying biequivalence. The following theorem, where it is stated that the 2-categories of Schreier systems and monoidal groupoids are biequivalent, is the main result of this section.

Theorem 3.7 (Classification of monoidal groupoids). The assignment \(S \mapsto \Sigma(S)\), given by the monoidal groupoid construction (26), is the function on objects of a 2-functor

\[\Sigma : Z_{\text{ab}}^{3} \text{Mnd} \rightarrow \text{MonGpd},\]

which establishes a biequivalence between the 2-category of Schreier systems and the 2-category of monoidal groupoids. More precisely (cf. [37, p. 570]), for any two Schreier systems \(S\) and \(S'\), the functor

\[\Sigma : Z_{n-\text{ab}}^{3} \text{Mnd}(S, S') \rightarrow \text{MonGpd}(\Sigma(S), \Sigma(S'))\]

is an equivalence of groupoids, and for any monoidal groupoid \(G\), there exists a monoidal equivalence

\[J_G : \Sigma(\Delta(G)) \rightarrow G,\]

where \(\Delta(G)\) is the Schreier system (25) associated to \(G\).

Proof. We have already described \(\Sigma\) on objects of the 2-category \(Z_{n-\text{ab}}^{3} \text{Mnd}\); its effect on morphisms and deformations is as follows:
3.5.1. $\Sigma$ on morphisms. Let $S = (M, \Delta, \Theta, \lambda)$, $S' = (M', \Delta', \Theta', \lambda')$ be Schreier systems. Then, the 2-functor $\Sigma$ carries any morphism $\varphi = (p, q, \varphi) : S \to S'$ to the strictly unitary monoidal functor $\Sigma(\varphi) : \Sigma(S) \to \Sigma(S')$ given by

$$
(a \xrightarrow{f} a) \mapsto (p(a) \xrightarrow{q(f)} p(a)),
$$

and whose structure isomorphisms are

$$
\varphi_{a,b} : p(a)p(b) \to p(ab),
$$

which are well defined since $p(a)p(b) = p(ab)$ and $\varphi_{a,b} \in \mathcal{A}'_{p(ab)}$ for any $a, b \in M$.

Since the maps $q_a : A_a \to A'_{p(ab)}$ are homomorphisms, it follows that $\Sigma(\varphi)$ is a functor. Furthermore, the isomorphisms (42) are natural since, for any morphisms $f : b \to b$ and $g : a \to a$ in $\Sigma(S)$, the squares in $\Sigma(S')$

$$
p(a)p(b) \xrightarrow{\varphi_{a,b} \cdot (q_b(f))} p(ab) \quad p(a)p(b) \xrightarrow{\varphi_{a,b}} p(ab)
$$

commute owing to condition (27). The coherence condition (3) for $\Sigma(\varphi)$ just says that the diagrams

$$
\begin{array}{c}
\xymatrix{
(p(a)p(b))p(c) & p(ab)p(c) & p((ab)c) \\
p(a)p(b)p(c) & p(a)p(bc) & p(a)(bc)
}
\end{array}
$$

must commute, which follows from (28). Finally, conditions (4) are both a consequence of the normality condition (29) of $\varphi$, that is, of the equalities $\varphi_{a,1} = 1 = \varphi_{1,b}$.

For $\varphi' = (p', q', \varphi') : S' \to S''$ another Schreier system morphism, the composite monoidal functor $\Sigma(\varphi')\Sigma(\varphi) : S \to S''$ is given by

$$
\Sigma(\varphi')(\Sigma(\varphi))(a \xrightarrow{f} a) = \Sigma(\varphi')(p(a) \xrightarrow{q(f)} p(a)) = (p'p(a) \xrightarrow{\varphi'(p(a), p(b))} p'(p(a)p(b))).
$$

Here, taking into account the definition of the composition $\varphi'\varphi$ in (34) and the definition of $\Sigma$, simple comparison shows that $\Sigma(\varphi')\Sigma(\varphi) = \Sigma(\varphi'\varphi)$. Moreover, it is straightforward to see that $\Sigma$ carries identity morphisms on Schreier systems $1_S = (1_M, 1, 1)$, see (35), to identity monoidal functors; that is, $\Sigma(1_S) = 1_{\Sigma(S)}$ for any Schreier system $S$. Therefore, $\Sigma : \mathbf{Z}_{\text{ab}}\text{Mnd} \to \text{MonGpd}$ is indeed a functor.

3.5.2. $\Sigma$ on deformations. Given Schreier systems $S$ and $S'$ as above, any deformation

$$
\begin{array}{c}
\xymatrix{S & S' \\
\delta \ar[ur]_{\varphi = (p,q,\varphi)} & \ar[ul]_{\varphi' = (p',q',\varphi')}
}
\end{array}
$$

of $\varphi$ in $\text{MonGpd}$ can be lifted to a deformation $\delta' : \Sigma(S) \to \Sigma(S')$ of $\varphi$ in $\text{MonGpd}$.
is mapped by the 2-functor Σ to the isomorphism of monoidal functors

\[
\Sigma(S) \xrightarrow{\Sigma(\delta)} \Sigma(S')
\]

simply defined by the family of isomorphisms in Σ(S')

\[
(45) \quad \Sigma(\delta)_a = \delta_a : p(a) \to p(a), \quad a \in M,
\]

which are natural thanks to condition (30). Moreover, so defined, Σ(δ) : Σ(ϕ) → Σ(̃ϕ) is monoidal, that is, conditions (5) hold, owing to (31) and the equality δ₁ = 1 ∈ A₁.

For any two vertically composable deformations δ : ϕ ⇒ ̃ϕ and ̄δ : ̃ϕ ⇒ ̂ϕ, as in (32), the equality Σ(δ ○ δ) = Σ(δ) ○ Σ(δ) is easily verified from (33) and (6), as well as the equality Σ(1_ϕ) = 1_{Σ(ϕ)}, for any morphism ϕ : S → S'. Hence, (39) is a functor.

Furthermore, for

\[
\begin{array}{ccc}
S & \xrightarrow{\delta} & S' \\
\Phi = (p,q,\psi) & & \Phi' = (p',q',\psi') \\
\bar{\Phi} = (p,q,\phi) & \xrightarrow{\bar{\delta}} & \bar{\Phi}' = (p',q',\phi') \\
\end{array}
\]

any two horizontally composable deformations as in (36), we have the equality Σ(δ') = Σ(δ')Σ(δ), since, for any a ∈ M,

\[
(41,45) \quad (\Sigma(\delta')\Sigma(\delta))_a \overset{\text{(7)}}{=} \Sigma(\delta')\Sigma(\bar{\delta}(a)) \overset{\text{(4)}}{=} \Sigma(\Phi') \overset{\text{(37)}}{=} \delta_p'(a) \circ q_p'(a)(\delta_a) \overset{(45)}{=} (\delta'\delta)_a = (\Sigma(\delta')_a).
\]

The above confirms that (38), Σ : Z₃[ab,MonGpd] → MonGpd, is actually a 2-functor.

3.5.3. The functor Σ in (39) is full and faithful. For any two Schreier systems S = (M, A, Θ, λ) and S' = (M', A', Θ', λ'), the functor Σ : Z₃[ab,MonGpd](S, S') → MonGpd(Σ(S), Σ(S')) is plainly recognized to be faithful, due to (45). To prove that it is full, let δ : Σ(ϕ) ⇒ Σ(̃ϕ) be any isomorphism of monoidal functors, where ϕ = (p, q, ϕ), ̃ϕ = (p, q, ̃ϕ) : S → S' are morphisms of Schreier systems. Then, for any a ∈ M, the equality be p(a) = ̃p(a) must hold, since δ_a : p(a) → ̃p(a) is an isomorphism in the skeletal category Σ(S') and, moreover, δ_a ∈ A_{p(a)}. Any element f ∈ A_a defines a morphism f : a → a in Σ(S), and the naturality of δ implies the commutativity of the square in Σ(S')

\[
\begin{array}{ccc}
p(a) & \xrightarrow{q_a(f)} & p(a) \\
\delta_a & \downarrow & \delta_a \\
p(a) & \xrightarrow{\bar{q}_a(f)} & p(a),
\end{array}
\]

whence δ⁻¹_a ○ ̃q_a(f) ○ δ_a = q_a(f). That is, condition (30) in order for the family δ = (δ_a ∈ A'_{p(a)}) a∈M to be a deformation of Schreier system morphisms from ϕ to
\(\varphi\), holds. Furthermore, for any \(a, b \in M\), the coherence condition (5) for \(\delta : \Sigma(\varphi) \Rightarrow \Sigma(\varphi)\) gives the commutativity of

\[
\begin{array}{c}
p(a) p(b) \xrightarrow{\varphi_{a,b}} p(ab) \\
p(a) \xrightarrow{(\delta_a) \circ p(b) \circ (\delta_b)} p(ab) \\
p(a) p(b) \xrightarrow{\varphi_{a,b}} p(ab),
\end{array}
\]

whence condition (31) follows. Therefore, \(\delta = (\delta_a)_{a \in M} : \varphi \Rightarrow \tilde{\varphi}\) is actually a deformation in \(Z^2_{\text{nd},\text{ab}} \text{Mnd}\), and clearly \(\Sigma(\delta) = \delta\).

3.5.4. The functor \(\Sigma\) in (39) is essentially surjective. Suppose \(F = (F, \varphi) : \Sigma(S) \rightarrow \Sigma(S')\) is any given monoidal functor, where \(S = (M, A, \Theta, \lambda)\) and \(S' = (M', A', \Theta', \lambda')\) are Schreier systems. By Lemma 2.1, there is no loss of generality in assuming that \(F\) is strictly unitary, that is, \(\varphi_0 : 1 \rightarrow F(1)\) is the identity isomorphism.

If we denote by \(p : M \rightarrow M'\) the map given by the action of the monoidal functor \(F\) on objects, that is, \(p(a) = F(a)\) for any \(a \in M\), then the action of the functor \(F\) on morphisms can be written, for any \(a \in M\) and \(f \in A_a\), in the form

\[
F(a \xrightarrow{f} a) = (p(a) \xrightarrow{q_a(f)} p(a))
\]

for a map \(q_a : A_a \rightarrow A'_{p(a)}\), which is indeed a group homomorphism since \(F\) is a functor. Let \(q = (q_a : A_a \rightarrow A'_{p(a)})_{a \in M}\) denote the family of these group homomorphisms. Since the category \(\Sigma(S')\) is skeletal and we have the structure isomorphism \(\varphi_{a,b} : p(a) p(b) \rightarrow p(ab)\) and \(\varphi_0 : 1 \rightarrow p(1)\), it must be \(p(a) p(b) = p(ab)\) and \(p(1) = 1\). Therefore, \(p\) is a homomorphism of monoids.

The triplet \(\varphi = (p, q, \varphi)\) so obtained, where \(\varphi = (\varphi_{a,b} \in A'_{p(ab)})_{a,b \in M}\), is actually a morphism of Schreier systems \(\varphi : S \rightarrow S'\) and, by construction, \(\Sigma(\varphi) = F\). In effect, the naturality of the isomorphisms \(\varphi_{a,b} : p(a) p(b) \rightarrow p(ab)\) gives the commutativity of the squares (43), whence condition (27) holds. Moreover, condition (28) follows from the coherence condition (3) which, in this case, simply states that the diagrams (44) are commutative. The normalization condition (29), \(\varphi_{1,1} = 1\), is a consequence of the coherence squares (4), since \(F\) is assumed to be strictly unitary, that is, since \(\varphi_0 = 1\).

3.5.5. The monoidal equivalence (40). We keep the notations used in Subsection 3.2 to define the Schreier system \(\Delta(\mathcal{G})\). The mapping

\((a \xrightarrow{f} a) \mapsto (X_a \xrightarrow{f} X_a)\)

is easily recognized as an equivalence of categories \(J_\mathcal{G} : \Sigma(\Delta(\mathcal{G})) \xrightarrow{\cong} \mathcal{G}\), which, by Proposition 2.2, defines a strictly unitary monoidal equivalence when it is endowed with the family of isomorphisms \(\varphi_{a,b} = \Gamma_{X_a, X_b} : X_a X_b \rightarrow X_{a b}, a, b \in M(\mathcal{G})\). Note that their required naturality holds since, for any \(a, b \in M(\mathcal{G})\), \(g \in \text{Aut}_\mathcal{G}(X_a)\), and...
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\[ f \in \text{Aut}_G(X_b), \text{we have the commutative diagram} \]

\[
\begin{array}{ccc}
X_a X_b & \xrightarrow{g_\ast} & X_a X_b \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
X_{ab} & \xrightarrow{\eta^\ast(g)} & X_{ab}
\end{array}
\]

\[
\begin{array}{ccc}
X_a X_b & \xrightarrow{g_\ast} & X_a X_b \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
X_{ab} & \xrightarrow{\eta^\ast(g)} & X_{ab}
\end{array}
\]

where the commutativity of the region labelled (B) is a consequence of the fact that \( \otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \) is a functor. The needed coherence conditions (3) and (4) follow from the commutativity of diagrams (23) and the choice of the morphisms \( \Gamma \)'s made in (19), respectively. \( \square \)

3.6. **The Schreier system construction biequivalence.** The above stated biequivalence between the 2-category of monoidal groupoids and the 2-category of Schreier systems (38), \( \Sigma : Z_{n-ab}^3 \text{Mnd} \rightleftarrows \text{MonGpd} \), is injective on objects, morphisms, and deformations. Moreover, for any Schreier system \( \mathcal{S} \), the equality \( \Delta \Sigma(\mathcal{S}) = \mathcal{S} \) holds. Hence, the assignment \( \mathcal{G} \mapsto \Delta(\mathcal{G}) \), given by the Schreier system construction (25), is the function on objects of a biequivalence, quasi-inverse of \( \Sigma \),

\[
\Delta : \text{MonGpd} \rightleftarrows Z_{n-ab}^3 \text{Mnd},
\]

which is uniquely determined up to pseudo-natural equivalence by the equality \( \Delta \Sigma = 1_{Z_{n-ab}^3 \text{Mnd}} \) and the existence of a pseudo-natural equivalence

\[
J : \Sigma \Delta \rightleftarrows 1_{\text{MonGpd}},
\]

whose component at any monoidal groupoid \( \mathcal{G} \), \( J_\mathcal{G} : \Sigma \Delta(\mathcal{G}) \rightleftarrows \mathcal{G} \), is the monoidal equivalence (40). For completeness, we shall next show how the pseudo-functor \( \Delta \) and the pseudo-equivalence \( J \) work.

3.6.1. **\( \Delta \) on monoidal functors.** Suppose \( F : \mathcal{G} \to \mathcal{G}' \) is any given monoidal functor between monoidal groupoids \( \mathcal{G} \) and \( \mathcal{G}' \). Let \( F^u : \mathcal{G} \to \mathcal{G}' \) be the strictly unitary monoidal functor associated to \( F \) by the normalization functor described in (8), and let \( (\Gamma_{X} : X \to X_a)_{a \in M(\mathcal{G})} \) and \( (\Gamma'_{X'} : X' \to X'_{a'})_{a' \in M(\mathcal{G}')} \) be the cleavages used for constructing the Schreier systems \( \Delta(\mathcal{G}) \) and \( \Delta(\mathcal{G}') \), respectively. Then,

\[
\Delta(F) = (p(F), q(F), \varphi(F)) : \Delta(\mathcal{G}) \to \Delta(\mathcal{G}')
\]

is the morphism of Schreier systems where:

- \( p = p(F) : M(\mathcal{G}) \to M(\mathcal{G}') \) is the homomorphism of monoids defined by

\[
p(a) = [FX_a] = [F^uX_a], \quad a \in M(\mathcal{G}).
\]

- \( q = q(F) = (\text{Aut}_\mathcal{G}(X_a) \xrightarrow{\varphi} \text{Aut}_{\mathcal{G}'}(X'_{p(a)}))_{a \in M(\mathcal{G})} \) is the family of group homomorphisms which carry an automorphism \( f : X_a \to X_a \), for any \( a \in M(\mathcal{G}) \), to the unique automorphism \( q_a(f) : X'_{p(a)} \to X'_{p(a)} \) in \( \mathcal{G}' \) making...
the square below commutative.

\[
\begin{array}{ccc}
F^uX_a & \xrightarrow{F^uF} & F^uX_a \\
\Downarrow\Gamma' & & \Downarrow\Gamma' \\
X'_{p(a)} & \xrightarrow{q_a(f)} & X'_{p(a)}
\end{array}
\]

\[\phi = \varphi(F) = (\varphi_{a,b} \in \text{Aut}_{G'}(X'_{p(ab)})_{a,b \in M(G)})\]

is the family of automorphisms in \(G'\) determined by the commutativity of the diagrams

\[
\begin{array}{ccc}
F^uX_a & \xrightarrow{\varphi_a} & F^uX_{ab} \\
\Downarrow\Gamma' & & \Downarrow\Gamma' \\
X'_{p(a)} & \xrightarrow{\varphi_{a,b}} & X'_{p(ab)}
\end{array}
\]

3.6.2. \(J\) on monoidal functors. The component of the pseudo-natural equivalence \(J : \Sigma\Delta \Rightarrow 1\) at any monoidal functor \(F : G \rightarrow G'\), is the isomorphism

\[
\Sigma\Delta(G) \xrightarrow{\Sigma\Delta(F)} \Sigma\Delta(G')
\]

defined by the isomorphisms of the cleavage in \(G'\), \(\Gamma' : FX_a \Rightarrow X'_{p(a)}, a \in M(G)\).

3.6.3. \(\Delta\) on morphisms of monoidal functors. Let \(F, \hat{F} : G \rightarrow G'\) be monoidal functors as above, and suppose \(\delta : F \Rightarrow \hat{F}\) is any morphism between them. Then,

\[
\Delta(G) \xrightarrow{\Delta(\delta)} \Delta(G')
\]

is the deformation \(\Delta(\delta) = (\Delta(\delta)_a \in \text{Aut}_{G'}(X'_{p(a)})_{a \in M(G)})\), consisting of the automorphisms in \(G'\) determined by the commutativity of the diagrams below.

\[
\begin{array}{ccc}
F^uX_a & \xrightarrow{\Gamma'} & X'_{p(a)} \\
\Downarrow\delta & & \Downarrow\Delta(\delta)_a \\
F^uX_a & \xrightarrow{\Gamma'} & X'_{p(a)}
\end{array}
\]

Since \(\Delta : \text{MonGpd} \xrightarrow{\simeq} Z^3_{n,\text{ab}} \text{Mnd}\) is a biequivalence and, by Lemma 3.6 every equivalence in \(Z^3_{n,\text{ab}} \text{Mnd}\) is actually an isomorphism, we have the following theorem as a corollary:

**Theorem 3.8.** (i) For any Schreier system \((M, \mathbb{A}, \Theta, \lambda)\), there is a monoidal groupoid \(G\) with an isomorphism \(\Delta(G) \cong (M, \mathbb{A}, \Theta, \lambda)\).

(ii) Two monoidal groupoids \(G\) and \(G'\) are equivalent if and only if their associated Schreier systems \(\Delta(G)\) and \(\Delta(G')\) are isomorphic.
4. Classification of monoidal abelian groupoids.

This section focuses on the special case of monoidal abelian groupoids, that is, monoidal groupoids \( \mathcal{G} = (\mathcal{G}, \otimes, I, a, l, r) \) whose isotropy groups \( \text{Aut}_{\mathcal{G}}(X) \), \( X \in \text{Ob}\mathcal{G} \), are all abelian. First, we shall observe that some of the isotropy groups of any monoidal groupoid are always abelian:

**Proposition 4.1.** (i) If \( \mathcal{S} = (M, \mathbb{A}, \Theta, \lambda) \) is any Schreier system, then, for any invertible element \( a \in M \), the group \( A_a \) is abelian.

(ii) If \( \mathcal{G} = (\mathcal{G}, \otimes, I, a, l, r) \) is any monoidal groupoid, then, for any invertible object \( X \in \mathcal{G} \), the group \( \text{Aut}_{\mathcal{G}}(X) \) is abelian.

**Proof.** (i) The group \( A_1 \) is abelian due to conditions (15) and (16): for any \( f, g \in A_1 \),

\[
f \circ g = 1_*(f) \circ 1_*(g) = 1_*(g) \circ 1_*(f) = g \circ f.
\]

For any invertible element \( a \in M \), the homomorphism \( a_* : A_1 \rightarrow A_a \) is actually an isomorphism, with inverse \( (a^{-1})_* : A_a \rightarrow A_1 \), since, by (11), (17), and (16), we have

\[
a_* (a^{-1})_* = (aa^{-1})_* = 1_* = 1_{A_1}, \quad (a^{-1})_a a_* = (a^{-1})_a = 1_* = 1_{A_a}.
\]

Hence, \( A_a \) is abelian as is \( A_1 \).

(ii) Let \( \Delta(\mathcal{G}) \) be the Schreier system associated to the monoidal groupoid \( \mathcal{G} \), as in (25). If \( X \in \mathcal{G} \) is any invertible object, then \( a = [X] \in M(\mathcal{G}) \) is an invertible element of the associated monoid (18), whence, due to part (i), the group \( \text{Aut}_{\mathcal{G}}(X_a) \) is abelian. Since the isomorphism \( \Gamma : X \rightarrow X_a \) induces a group isomorphism \( \text{Aut}_{\mathcal{G}}(X) \cong \text{Aut}_{\mathcal{G}}(X_a) \), the result follows. \( \square \)

Therefore, for example, every categorical group is a monoidal abelian groupoid.

The classification of categorical groups was given by Sinh in [35], by means of Eilenberg-Mac Lane group cohomology groups \( H^n(G, (A, \Theta)) \), and our aim here is to give a similar solution to the more general problem of classifying the monoidal abelian groupoids, now by means of monoid cohomology groups \( H^n(M, (A, \Theta)) \). To this end, we shall briefly review below some basic aspects concerning the cohomology theory of monoids that we are going to use, which is a generalization of Eilenberg-Mac Lane’s cohomology of groups due to Leech [24].

4.1. Leech cohomology of monoids. If \( \mathcal{C} \) is a small category, then the category of (left) \( \mathcal{C} \)-modules has objects the functors \( D : \mathcal{C} \rightarrow \mathbb{Ab} \) from \( \mathcal{C} \) into abelian groups, with morphisms the natural transformations. This is an abelian category with enough injectives and projectives, and the abelian groups \( H^n(\mathcal{C}, D) = \text{Ext}^n_{\mathcal{C}}(\mathbb{Z}, D) \), where \( \mathbb{Z} : \mathcal{C} \rightarrow \mathbb{Ab} \) is the constant functor with value \( \mathbb{Z} \), are the cohomology groups of the category \( \mathcal{C} \) with coefficients in the \( \mathcal{C} \)-module \( D \), studied by Roos [32] and Watts [39], among other authors. Cohomology theory of small categories is in itself a basis for other cohomology theories, in particular for the Leech cohomology theory of monoids, which is defined as follows:

A monoid \( M \) gives rise to a category \( \mathbb{D}M \) with object set \( M \) and arrow set \( M \times M \times M \), with \( (a, b, c) : b \rightarrow abc \). Composition is given by

\[
(d, abc, e)(a, b, c) = (da, b, ce),
\]

and the identity morphism of any object \( a \) is \( 1_a = (1, a, 1) \), where 1 is the unit element of \( M \). This construction \( M \mapsto \mathbb{D}M \) defines a functor \( \mathbb{D} : \text{Mnd} \rightarrow \text{Cat} \),
which maps a monoid homomorphism \( p : M \to M' \) to the functor \( \mathbb{D}p : \mathbb{D}M \to \mathbb{D}M' \) given by \( \mathbb{D}p(a,b,c) = (p(a), p(b), p(c)) \).

If we say that a \( \mathbb{D}M \)-module, \( \mathbb{D}M \to \text{Ab} \), carries the element \( a \in M \) to the abelian group \( A_a \) and carries the morphism \( (a, b, c) \) to the group homomorphism \( a*c^* : A_b \to A_{abc} \), then we see that such a \( \mathbb{D}M \)-module, hereafter denoted by

\[
(A, \Theta) : \mathbb{D}M \to \text{Ab},
\]

is a system of data consisting of two families of abelian groups and homomorphisms, respectively,

\[
\mathbb{A} = (A_a)_{a \in M}, \quad \Theta = (A_b \overset{a*}{\longrightarrow} A_{ab} \overset{b*}{\longleftarrow} A_a)_{a,b \in M}
\]
such that, for any \( a, b, c \in M \),

\[
(ab)_a = a*b_a : A_c \to A_{abc}, \quad c*a_* = a*b_c : A_b \to A_{abc}, \quad c*b_* = (bc)^* : A_c \to A_{abc},
\]

and, for any \( a \in M \),

\[
1_* = 1_{A_a} = 1^* : A_a \to A_a.
\]

Leech cohomology groups \( H^n(M, (\mathbb{A}, \Theta)) \) \([24]\), of a monoid \( M \) with coefficients in a \( \mathbb{D}M \)-module \((\mathbb{A}, \Theta)\), are defined to be those of its associated category \( \mathbb{D}M \), that is,

\[
H^n(M, (\mathbb{A}, \Theta)) = H^n(\mathbb{D}M, (\mathbb{A}, \Theta)).
\]

For computing these cohomology groups, there is a cochain complex

\[
C^*(M, (\mathbb{A}, \Theta)),
\]

which is defined in degree \( n > 0 \) by

\[
C^n(M, (\mathbb{A}, \Theta)) = \left\{ \lambda \in \prod_{(a_1, \ldots, a_n) \in M^n} A_{a_1,\ldots,a_n} \mid \lambda_{a_1,\ldots,a_n} = 1 \text{ whenever some } a_i = 1 \right\}
\]

and \( C^0(M, (\mathbb{A}, \Theta)) = A_1 \). The coboundary operator

\[
\partial^n : C^n(M, (\mathbb{A}, \Theta)) \to C^{n+1}(M, (\mathbb{A}, \Theta))
\]
is given, for \( n = 0 \), by \( (\partial^0 \lambda)_a = a_*(\lambda) \circ a^*(\lambda)^{-1} \), while, for \( n > 0 \),

\[
(\partial^n \lambda)_{a_1,\ldots,a_{n+1}} = (a_1)_*(\lambda_{a_2,\ldots,a_{n+1}}) \circ \prod_{i=1}^n \lambda_{a_1,\ldots,a_i,a_{i+1},\ldots,a_{n+1}} \circ a^*_{n+1}(\lambda_{a_1,\ldots,a_n}^{(-1)^n}).
\]

By \([24, \text{Chapter II, 2.3, 2.9}]\), we have

\[
H^n(M, (\mathbb{A}, \Theta)) = H^n(C^*(M, (\mathbb{A}, \Theta))).
\]

It is useful for our purposes to describe the natural properties of the Leech cohomology on the category obtained by the Grothendieck construction on the functor that associates to any monoid \( M \) the category of \( \mathbb{D}M \)-modules and, to any homomorphism \( p : M \to M' \), the functor \( p^* \) that carries any \( \mathbb{D}M' \)-module, say \((\mathbb{A}', \Theta')\), to the \( \mathbb{D}M \)-module

\[
p^* (\mathbb{A}', \Theta') = (p^* \mathbb{A}', p^* \Theta'),
\]

where

\[
p^* \mathbb{A}' = (A'_{p(a)})_{a \in M}, \quad p^* \Theta' = (A'_{p(b)} \overset{p(a)_*}{\longrightarrow} A'_{p(ab)} \overset{p(b)_*}{\longleftarrow} A'_{p(a)})_{a,b \in M}.
\]
By applying the Grothendieck construction on the functor $M \mapsto \mathbb{D}M$-modules, we get a category, denoted by $\text{Mod}_\mathbb{D}$, which may be heuristically viewed as the category obtained by tying the categories of $\mathbb{D}M$-modules together in a natural fashion. It has objects pairs $(M, (\mathcal{A}, \Theta))$, where $M$ is a monoid and $(\mathcal{A}, \Theta)$ is a $\mathbb{D}M$-module. Morphisms are pairs

$$(p, q) : (M, (\mathcal{A}, \Theta)) \to (M', (\mathcal{A}', \Theta'))$$

consisting of a monoid homomorphism $p : M \to M'$ together with a morphism of $\mathbb{D}M$-modules $q : (\mathcal{A}, \Theta) \to p^*(\mathcal{A}', \Theta')$, that is, a family of group homomorphisms

$$q = (A_a \xrightarrow{q_a} A'_{p(a)})_{a \in M},$$

satisfying, for any $a, b \in M$,

$$(48) \quad q_{ab} a_* = p(a)_* q_b : A_b \to A'_{p(ab)},$$

$$(49) \quad q_{ab} b_* = p(b)_* q_a : A_a \to A'_{p(ab)}.$$  

Composition is defined by $(p', q')(p, q) = (p' p, q' q)$, where $q' q = (q'_a(q_a))_{a \in M}$.

Any morphism $(p, q) : (M, (\mathcal{A}, \Theta)) \to (M', (\mathcal{A}', \Theta'))$ as above yields homomorphisms

$$H^n(M, (\mathcal{A}, \Theta)) \xrightarrow{q_*} H^n(M, p^*(\mathcal{A}', \Theta')) \xleftarrow{\partial^*} H^n(M', (\mathcal{A}', \Theta'))$$

induced by the morphisms of cochain complexes

$$C^*(M, (\mathcal{A}, \Theta)) \xrightarrow{q_*} C^*(M, p^*(\mathcal{A}', \Theta')) \xleftarrow{\partial^*} C^*(M', (\mathcal{A}', \Theta')),$$

which are given on cochains by

$$(q_* \lambda)_{a_1, \ldots, a_n} = q_{a_1 \cdots a_n} (\lambda_{a_1, \ldots, a_n}), \quad (p^* \lambda')_{a_1, \ldots, a_n} = \lambda'_{p(a_1) \cdots p(a_n)}.$$  

4.1.1. Leech cohomology versus Gabriel-Zisman cohomology. Cohomology theory of small categories is also the foundation for the (co)homology theory of simplicial sets with twisted coefficients, as defined by Gabriel and Zisman in [19, Appendix II] (actually, the results by Gabriel-Zisman are stated for homology of simplicial sets, but they can be easily dualised to cohomology, see Illusie [22, Chapter VI, §3]). Briefly, recall that the simplicial category, denoted by $\Delta$, has objects the ordered sets $[n] = \{0, \ldots, n\}$, $n \geq 0$, and its arrows are weakly monotone maps $\alpha : [m] \to [n]$. If $X : \Delta^{op} \to \text{Set}$ is a simplicial set, then its category of simplices, denoted by $\Delta/X$, is the category obtained by the Grotendieck construction on $X$. That is, the category whose objects are pairs $(x, m)$ with $x \in X_m$; an arrow $\alpha : (x, m) \to (y, n)$ is a map $\alpha : [m] \to [n]$ in $\Delta$ such that $\alpha y = x$, where we write $\alpha^* : X_n \to X_m$ for the map induced by $\alpha$. A coefficient system on $X$ is a $\Delta/X$-module, that is, a functor $L : \Delta/X \to \text{Ab}$, and the cohomology groups of $X$ with coefficients in $L$ are, by definition,

$$H^n(X, L) = H^n(\Delta/X, L), \quad n \geq 0.$$  

Suppose now $M$ is a monoid. The classifying space of $M$ is the simplicial set $BM$, whose set of $n$-simplices is

$$(BM)_n = \left\{ a = (a_{i,j} \in M)_{0 \leq i \leq j \leq n} \mid a_{i,j} a_{j,k} = a_{i,k}, \quad a_{i,i} = 1 \right\},$$
which is usually identified with $M^n$ by the bijection $a \mapsto (a_{0,1}, \ldots, a_{n-1,n})$. The induced map $\alpha^* : (BM)_m \to (BM)_m$, for a map $\alpha : [m] \to [n]$ in $\Delta$, is given by $\alpha^*(b) = a$, where $a_{i,j} = b_{\alpha(i),\alpha(j)}$.

There is a canonical functor $\Delta/BM \to \mathbb{D}M$, given by

$$(a, m) \xrightarrow{\alpha} (b, n) \mapsto a_{0,m} \xrightarrow{(b_{0,\alpha(i),a_{0,m}},b_{a_{0,m},n})} b_{0,n}.$$ 

Therefore, by composition with it, each $\mathbb{D}M$-module, say $(A, \Theta) : \mathbb{D}M \to \text{Ab}$, gives a coefficient system on $BM$, also denoted by $(A, \Theta)$, and Gabriel-Zisman cohomology groups of $BM$ with coefficients in the $\mathbb{D}M$-module $(A, \Theta)$,

$$H^n(BM, (A, \Theta)),$$

are defined. But note that, by [19, Appendix II, Proposition 4.2], these cohomology groups of $BM$ can be computed by the same cochain complex $(47)$, $C^*(M, (A, \Theta))$, used by Leech for computing the cohomology groups of the monoid $M$ (see also [22, Chapter VI, (3.4.3)]). Therefore, there is a natural identification

$$H^n(M, (A, \Theta)) = H^n(BM, (A, \Theta)).$$

4.2. The classification theorems. The biequivalence in Theorem 3.7 restricts to a biequivalence between the full 2-subcategory of the 2-category of monoidal groupoids given by the monoidal abelian groupoids, denoted by

$$\text{MonAbGpd},$$

and the full 2-subcategory of the 2-category of Schreier systems given by those Schreier systems $(M, A, \Theta, \lambda)$ in which every group $A_a$, $a \in M$, is abelian. Hereafter, this latter 2-category will be called the 2-category of Leech 3-cocycles of monoids, and be denoted by

$$Z^3Mnd,$$

since its cells have the following cohomological interpretation:

0-cells. According to Definition 3.1, a Schreier system $S$ in $Z^3Mnd$ precisely is a triplet $S = (M, (A, \Theta), \lambda)$ consisting of a monoid $M$, a $\mathbb{D}M$-module $(A, \Theta)$, and a 3-cocycle $\lambda \in Z^3(M, (A, \Theta))$.

1-cells. If $S = (M, (A, \Theta), \lambda)$ and $S' = (M', (A', \Theta'), \lambda')$ are in $Z^3Mnd$, then a morphism of Schreier systems (see Subsection 3.4.1), $\varphi = (p, q, \varphi) : S \to S'$, is the same thing as a morphism $(p, q) : (M, (A, \Theta)) \to (M', (A', \Theta'))$ in $\text{Mod}_{\mathbb{B}}$, together with a 2-cochain $\varphi \in C^2(M, p^*(A', \Theta'))$ such that $q_{i}\lambda = p^*\lambda \circ \partial^2\varphi$.

2-cells. If $\varphi = (p, q, \varphi) : S \to S'$ and $\tilde{\varphi} = (\tilde{p}, \tilde{q}, \tilde{\varphi}) : S \to S'$ are morphisms in $Z^3Mnd$, then (see Subsection 3.4.2) there is no deformation between them unless $p = \tilde{p}$ and $q = \tilde{q}$. In such a case, a deformation $\delta : \varphi \Rightarrow \tilde{\varphi}$ consists of a 1-cochain $\delta \in C^1(M, p^*(A', \Theta'))$, such that $\varphi = \tilde{\varphi} \circ \partial^1\delta$.

Therefore, our first result here comes as a direct consequence of Theorem 3.7:

**Theorem 4.2.** The quasi-inverse biequivalences (38) and (46) restrict to corresponding quasi-inverse biequivalences

$$(50) \quad \text{MonAbGpd} \xrightarrow{\cong} Z^3Mnd.$$
Closely related to the category \( \mathbb{Z}^3 \text{Mnd} \) is the category of Leech 3-cohomology classes of monoids, denoted by \( \mathbb{H}^3 \text{Mnd} \), which plays a fundamental role in stating our classification theorem below. Its objects are triplets \( (M, (A, \Theta), c) \), where \( M \) is a monoid, \( (A, \Theta) \) is a \( \mathbb{D}M \)-module, and \( c \in \mathbb{H}^3(M, (A, \Theta)) \) is a 3-cohomology class of \( M \) with coefficients in \( (A, \Theta) \).

An arrow
\[
(p, q) : (M, (A, \Theta), c) \to (M', (A', \Theta'), c')
\]
is a morphism \( (p, q) : (M, (A, \Theta)) \to (M', (A', \Theta')) \) in \( \text{Mod}_\mathbb{D} \), such that
\[
p^*(c') = q_*(c) \in \mathbb{H}^3(M, p^*(A', \Theta')).
\]

Observe that a morphism \((p, q)\) is an isomorphism in \( \mathbb{H}^3 \text{Mnd} \) if and only if \( p : M \to M' \) is an isomorphism of monoids and \( q : (A, \Theta) \to p^*(A', \Theta') \) is an isomorphism of \( \mathbb{D}M \)-modules.

We have the cohomology class functor
\[
\text{cl} : \mathbb{Z}^3 \text{Mnd} \to \mathbb{H}^3 \text{Mnd},
\]
\[
(M, (A, \Theta), \lambda) \mapsto (M, (A, \Theta), [\lambda])
\]
\[
(p, q, \varphi) \mapsto (p, q)
\]
where \([\lambda] \in \mathbb{H}^3(M, (A, \Theta))\) denotes the cohomology class of \( \lambda \in \mathbb{Z}^3(M, (A, \Theta))\). This functor clearly carries two isomorphic morphisms of \( \mathbb{Z}^3 \text{Mnd} \) to the same morphism in \( \mathbb{H}^3 \text{Mnd} \), whence composition with the pseudo-functor \( \Delta \) above gives a functor
\[
\text{Cl} = \text{cl} \Delta : \text{MonAbGpd} \to \mathbb{H}^3 \text{Mnd},
\]
that we call the classifying functor because of the theorem below.

**Theorem 4.3** (Classification of monoidal abelian groupoids). (i) For any monoid \( M \), any \( \mathbb{D}M \)-module \( (A, \Theta) \), and any cohomology class \( c \in \mathbb{H}^3(M, (A, \Theta)) \), there is a monoidal abelian groupoid \( G \) with an isomorphism \( \text{Cl}(G) \cong (M, (A, \Theta), c) \).

(ii) A monoidal functor between monoidal abelian groupoids \( F : G \to G' \) is an equivalence if and only if \( \text{Cl}(F) : \text{Cl}(G) \to \text{Cl}(G') \) is an isomorphism.

(iii) For any isomorphism \( (p, q) : \text{Cl}(G) \cong \text{Cl}(G') \), there is a monoidal equivalence \( F : G \to G' \) such that \( \text{Cl}(F) = (p, q) \).

(iv) If \( G \) and \( G' \) are monoidal abelian groupoids with \( \text{Cl}(G) = (M, (A, \Theta), c) \) and \( \text{Cl}(G') = (M', (A', \Theta'), c') \), then, for any morphism \( (p, q) : \text{Cl}(G) \to \text{Cl}(G') \) in \( \mathbb{H}^3 \text{Mnd} \), there is a (non-natural) bijection
\[
\{ [F] : G \to G' \mid \text{Cl}(F) = (p, q) \} \cong \mathbb{H}^2(M, p^*(A', \Theta'))
\]
between the set of isomorphism classes of those monoidal functors \( F : G \to G' \) that are carried by the classifying functor to \((p, q)\) and the elements of the second cohomology group of \( M \) with coefficients in the \( \mathbb{D}M \)-module \( p^*(A', \Theta') \).

**Proof.** (i) Given any object \((M, (A, \Theta), c) \in \mathbb{H}^3 \text{Mnd} \), let us choose any 3-cocycle \( \lambda \in \mathbb{Z}^3(M, (A, \Theta)) \) such that \([\lambda] = c\). Then, letting \( G = \Sigma(M, A, \Theta, \lambda) \), we have
\[
\text{Cl}(G) = \text{cl}(\Delta \Sigma(M, A, \Theta, \lambda)) = \text{cl}(M, (A, \Theta), \lambda) = (M, (A, \Theta), c).
\]
(ii) Since the pseudo-functor $\Delta : \text{MonAbGpd} \to \text{Z}^3\text{Mnd}$ is a biequivalence, it suffices to prove that a morphism in $\text{Z}^3\text{Mnd}$, say 

$$(p, q, \varphi) : (M, (\mathcal{A}, \Theta), \lambda) \to (M', (\mathcal{A}', \Theta'), \lambda'),$$

is an equivalence if and only if the induced 

$$(p, q) : (M, (\mathcal{A}, \Theta), [\lambda]) \to (M', (\mathcal{A}', \Theta'), [\lambda'])$$

is an isomorphism in $\text{H}^3\text{Mnd}$, that is, if and only if $p : M \to M'$ is an isomorphism of monoids and $q : (\mathcal{A}, \Theta) \to p^* (\mathcal{A}', \Theta')$ is an isomorphism of $\mathbb{D}M$-modules. Hence, the result follows from Lemma 3.6.

(iv) Suppose $\Delta(\mathcal{G}) = (M, \mathcal{A}, \Theta, \lambda)$ and $\Delta(\mathcal{G}') = (M', \mathcal{A}', \Theta', \lambda')$, and let $(p, q) : \text{Cl}(\mathcal{G}) \to \text{Cl}(\mathcal{G}')$ be any given morphism in $\text{H}^3\text{Mnd}$. The equivalence between the hom-groupoids

$$\text{MonAbGpd}(\mathcal{G}, \mathcal{G}') \cong \text{Z}^3\text{Mnd}(\Delta(\mathcal{G}), \Delta(\mathcal{G}')),$$

induces a bijection, $[F] \mapsto [\Delta(F)]$,

$$\{ [F] : \mathcal{G} \to \mathcal{G}' \mid \text{Cl}(F) = (p, q) \} \cong \{ [p, q, \varphi] : (M, (\mathcal{A}, \Theta), \lambda) \to (M', (\mathcal{A}', \Theta'), \lambda') \}$$

between the set of iso-classes $[F]$ of those monoidal functors $F : \mathcal{G} \to \mathcal{G}'$ with $\text{Cl}(F) = (p, q)$, and the set of iso-classes $[p, q, \varphi]$ of morphisms of the form 

$$(p, q, \varphi) : (M, (\mathcal{A}, \Theta), \lambda) \to (M', (\mathcal{A}', \Theta'), \lambda')$$

in the 2-category of Leech 3-cocycles. Since $p^*[\lambda] = q_* [\lambda]$, both 3-cocycles $p^*\lambda'$ and $q_* \lambda$ represent the same class in the cohomology group $\text{H}^3(M, p^*(\mathcal{A}', \Theta'))$. Therefore, a 2-cochain $\varphi \in C^2(M, p^*(\mathcal{A}', \Theta'))$ such that $q_* \lambda = p^* \lambda' \circ \partial^2 \varphi$ must exist. Hence, $(p, q, \varphi) : (M, (\mathcal{A}, \Theta), \lambda) \to (M', (\mathcal{A}', \Theta'), \lambda')$ is a morphism in $\text{Z}^3\text{Mnd}$. Furthermore, observe that any other morphism of the form $(p, q, \psi) : (M, (\mathcal{A}, \Theta), \lambda) \to (M', (\mathcal{A}', \Theta'), \lambda')$ in $\text{Z}^3\text{Mnd}$ realizing the same morphism $(p, q)$ of $\text{H}^3\text{Mnd}$ is necessarily written in the form $(p, q, \varphi \circ \phi)$ for some $\phi \in Z^2(M, p^*(\mathcal{A}', \Theta'))$ and, moreover, the morphisms $(p, q, \varphi)$ and $(p, q, \varphi \circ \phi)$ are isomorphic if and only if $\phi = \partial^1 \delta$ for some $\delta \in Z^1(M, p^*(\mathcal{A}', \Theta'))$. That is, there is a bijection

$$\text{H}^2(M, p^*(\mathcal{A}', \Theta')) \cong \{ [p, q, \psi] : (M, (\mathcal{A}, \Theta), \lambda) \to (M', (\mathcal{A}', \Theta'), \lambda') \}$$

given by $[\phi] \mapsto [p, q, \varphi \circ \phi]$.

(iii) Let $(p, q) : \text{Cl}(\mathcal{G}) \cong \text{Cl}(\mathcal{G}')$ be any given isomorphism in $\text{H}^3\text{Mnd}$. By the already proved part (iv), there exists a monoidal functor $F : \mathcal{G} \to \mathcal{G}'$ such that $\text{Cl}(F) = (p, q)$, which, by part (ii) is an equivalence. □

The functor $\text{MonAbGpd} \to \text{Mod}_\mathbb{D}$, $\mathcal{G} \mapsto (M(\mathcal{G}), (\mathcal{A}(\mathcal{G}), \Theta(\mathcal{G}))$, obtained by composing the classifying functor (52) with the forgetful functor $\text{H}^3\text{Mnd} \to \text{Mod}_\mathbb{D}$, $(M, (\mathcal{A}, \Theta), e) \mapsto (M, (\mathcal{A}, \Theta))$, turns the 2-category of monoidal abelian groupoids into a fibred 2-category over the category $\text{Mod}_\mathbb{D}$. It follows from the above results that, for any fixed monoid $M$ and $\mathbb{D}M$-module $(\mathcal{A}, \Theta)$, the mappings

$$[\lambda] \mapsto [\Sigma(M, (\mathcal{A}, \Theta), \lambda)], \quad \mathcal{G} \mapsto [\lambda(\mathcal{G})],$$

describe mutually inverse bijections between the set $\text{H}^3(M, (\mathcal{A}, \Theta))$ and the set of equivalence classes of monoidal groupoids in the fibre 2-category over $(M, (\mathcal{A}, \Theta))$. However, this latter set is conceptually a little too rigid since the strict requirements $M(\mathcal{G}) = M$ and $(\mathcal{A}(\mathcal{G}), \Theta(\mathcal{G})) = (\mathcal{A}, \Theta)$, for a monoidal abelian groupoid $\mathcal{G}$, are not very natural. We shall show how to relax them below.
Definition 4.4. For any given monoid \( M \) and any \( \mathbb{D}M \)-module \( (\mathcal{A}, \Theta) \), we say that a monoidal abelian groupoid \( \mathcal{G} \) is of type \( (M, (\mathcal{A}, \Theta)) \) if there are given

- a monoid isomorphism \( i : M \cong M(\mathcal{G}) \),
- a family of group isomorphisms \( j_X = (j_X : A_a \cong \text{Aut}_\mathcal{G}(X))_{a \in M, X \in (a)} \),

such that,

- if \( X, Y \in i(a) \) then, for any morphism \( h : X \to Y \) in \( \mathcal{G} \) and any \( g \in A_a \),
  \[ j_Y(g) = h \circ j_X(g) \circ h^{-1}. \]
- if \( X \in i(a) \) and \( Y \in i(b) \), then, for any \( f \in A_b \) and \( g \in A_a \),
  \[ j_{XY}(a \ast(f)) = 1_X j_Y(f), \quad j_{XY}(b \ast(g)) = j_X(g) 1_Y. \]

If \( (p, q) : (M, (\mathcal{A}, \Theta)) \to (M', (\mathcal{A}', \Theta')) \) is any morphism in the category \( \text{Mod}_\mathbb{D} \), and \( \mathcal{G} \) and \( \mathcal{G}' \) are monoidal abelian groupoids of the respective types \( (M, (\mathcal{A}, \Theta)) \) and \( (M', (\mathcal{A}', \Theta')) \), then a monoidal functor \( F : \mathcal{G} \to \mathcal{G}' \) is said to be of type \( (p, q) \) whenever

- if \( X \in i(a) \), then \( FX \in i'(p(a)) \), and, for any \( g \in A_a \), \( j_{FX}q_a(g) = F(j_X(g)) \).

Two monoidal abelian groupoids of the same type \( (M, (\mathcal{A}, \Theta)) \), say \( (\mathcal{G}, i, j) \) and \( (\mathcal{G}', i', j') \), are defined to be equivalent if there exists a monoidal equivalence \( F : \mathcal{G} \to \mathcal{G}' \) of type \((1, 1)\), that is, whenever

- if \( X \in i(a) \), then \( FX \in i'(a) \), and, for any \( g \in A_a \), \( j_{FX}g_a(g) = F(j_X(g)) \).

If we denote by

\[ \text{MonAbGpd}(M, (\mathcal{A}, \Theta)) \]

the set of equivalence classes \([\mathcal{G}, i, j]\) of those monoidal abelian groupoids \( (\mathcal{G}, i, j) \) of type \( (M, (\mathcal{A}, \Theta)) \), then we are ready to summarize our results on the classification of monoidal abelian groupoids and their homomorphisms in slightly more classic terms:

Theorem 4.5. (i) For any monoidal abelian groupoid \( \mathcal{G} \), there exists a monoid \( M \) and a \( \mathbb{D}M \)-module \( (\mathcal{A}, \Theta) \) such that \( \mathcal{G} \) is of type \( (M, (\mathcal{A}, \Theta)) \).

(ii) For any monoid \( M \) and any \( \mathbb{D}M \)-module \( (\mathcal{A}, \Theta) \), there is a natural bijection

\[ \text{MonAbGpd}(M, (\mathcal{A}, \Theta)) \cong H^3(M, (\mathcal{A}, \Theta)) \]

given by

\[ [\mathcal{G}, i, j] \mapsto \lambda(\mathcal{G}) = j_+^{-1}i^*([\lambda(\mathcal{G})]), \]

where \( \lambda(\mathcal{G}) \) is the 3-cocycle obtained as in (24), and

\[ H^3(M(\mathcal{G}), (\mathcal{A}(\mathcal{G}), \Theta(\mathcal{G}))) \xrightarrow{i^*} H^3(M, (\mathcal{A}(\mathcal{G}), \Theta(\mathcal{G}))) \xrightarrow{1 \circ} H^3(M, (\mathcal{A}, \Theta)) \]

the induced isomorphisms on cohomology groups by the isomorphism

\[ (i, j) : (M, (\mathcal{A}, \Theta)) \cong (M(\mathcal{G}), (\mathcal{A}(\mathcal{G}), \Theta(\mathcal{G})) \]

in the category \( \text{Mod}_\mathbb{D} \). In the other direction, the bijection is induced by the mapping that carries a 3-cocycle \( \lambda \in Z^3(M, (\mathcal{A}, \Theta)) \) to the monoidal abelian groupoid \( \Sigma(M, \mathcal{A}, \Theta, \lambda) \), given by the construction (26).

(iii) If \( \mathcal{G} \) is of type \( (M, (\mathcal{A}, \Theta)) \) and \( \mathcal{G}' \) is of type \( (M', (\mathcal{A}', \Theta')) \), then for every monoidal functor, \( F : \mathcal{G} \to \mathcal{G}' \), there exists a morphism in the category \( \text{Mod}_\mathbb{D} \),

\[ (p, q) : (M, (\mathcal{A}, \Theta)) \to (M', (\mathcal{A}', \Theta')) \]

such that \( F \) is of type \((p, q)\).
(iv) If $\mathcal{G}$ is of type $(M, (\mathbf{A}, \Theta))$ and $\mathcal{G}'$ is of type $(M', (\mathbf{A}', \Theta'))$, then, for any morphism $(p, q) : (M, (\mathbf{A}, \Theta)) \rightarrow (M', (\mathbf{A}', \Theta'))$ in the category $\textbf{Mod}_\mathcal{D}$, there is a monoidal functor $F : \mathcal{G} \rightarrow \mathcal{G}'$ of type $(p, q)$ if and only if

$$p^*(c(\mathcal{G}')) = q_*(c(\mathcal{G})) \in H^2(M, p^*(\mathbf{A}', \Theta')).$$

In such a case, isomorphism classes of monoidal functors $F : \mathcal{G} \rightarrow \mathcal{G}'$ of type $(p, q)$ are in bijection with the elements of the group

$$H^2(M, p^*(\mathbf{A}', \Theta')).$$

**Proof.** All the statements here are a direct consequence of those in Theorem 4.3 after two quite obvious observations, namely: (1) A monoidal abelian groupoid $\mathcal{G}$ is of type $(M, (\mathbf{A}, \Theta))$ if and only if there is given an isomorphism

$$(i, j) : (M, (\mathbf{A}, \Theta)) \cong (M(\mathcal{G}), (\mathbf{A}(\mathcal{G}), \Theta(\mathcal{G}))$$

in the category $\textbf{Mod}_\mathcal{D}$. (2) If $(p, q) : (M, (\mathbf{A}, \Theta)) \rightarrow (M', (\mathbf{A}', \Theta'))$ is a morphism in the category $\textbf{Mod}_\mathcal{D}$, and $\mathcal{G}$ and $\mathcal{G}'$ are any monoidal groupoids of respective types $(M, (\mathbf{A}, \Theta))$ and $(M', (\mathbf{A}', \Theta'))$, then a monoidal functor $F : \mathcal{G} \rightarrow \mathcal{G}'$ is of type $(p, q)$ if and only if the square below in the category $\textbf{Mod}_\mathcal{D}$ commutes.

\[
\begin{array}{ccc}
(M, (\mathbf{A}, \Theta)) & \xrightarrow{(i, j)} & (M(\mathcal{G}), (\mathbf{A}(\mathcal{G}), \Theta(\mathcal{G})) \\
(p, q) \downarrow & & \downarrow (p(F), q_*(F)) \\
(M', (\mathbf{A}', \Theta')) & \xrightarrow{(i', j')} & (M(\mathcal{G}'), (\mathbf{A}(\mathcal{G}'), \Theta(\mathcal{G}')).
\end{array}
\]

\[
\square
\]

**Remark 4.6.** The category of monoids is tripleable over the category of sets. In [40, Theorem 8], Wells identified the category $\textbf{Ab(Mnd}_{\mathcal{A}M})$ of abelian group objects in the comma category of monoids over a monoid $M$ with the category of $\mathbb{D}M$-modules (see Example 3.2), and he proved that with a dimension shift both the Barr-Beck cotriple cohomology theory [2, 5] and the Leech cohomology theory of monoids are the same. Hence, for any monoid $M$ and any $\mathbb{D}M$-module $(\mathbf{A}, \Theta)$, the Duskin [17] and Glenn [20] general interpretation theorem for cotriple cohomology classes shows that equivalence classes of 2-torsors over $M$ under $(\mathbf{A}, \Theta)$ are in bijection with elements of the cohomology group $H^3(M, (\mathbf{A}, \Theta))$.

A very similar result follows from the general result by Pirashvili [29, 30] and Baez-Dreissmann [3] about the classification of track categories. From this result, the elements of $H^3(M, (\mathbf{A}, \Theta))$ are in bijection with equivalence classes of linear track extensions of $(\mathbf{A}, \Theta)$ by $\mathbb{D}M$-modules (natural system on $M$ in their terminology) $(\mathbf{A}, \Theta)$.

Indeed, the three terms '2-torsor over $M$ under $(\mathbf{A}, \Theta)$', 'linear track extension of $M$ by $(\mathbf{A}, \Theta)$', and 'strict monoidal abelian groupoid of type $(M, (\mathbf{A}, \Theta))$' are plainly recognized to be synonymous; simply take into account that an internal groupoid in the category of monoids is the same thing as a strict monoidal groupoid, together with Lemmas 2.2 and 2.3 in [10] or [11, Theorem 3.3].

However, we must stress that while it is relatively harmless to consider monoidal abelian groupoids as 'strict', since by the Mac Lane Coherence Theorem for monoidal categories [28, 27] every monoidal abelian groupoid is equivalent to a strict one, we consider it is not so harmless when dealing with their homomorphisms since not every monoidal functor is isomorphic to a strict one. Indeed, it is possible to find
two strict monoidal abelian groupoids, say \( \mathcal{G} \) and \( \mathcal{G}' \), that are related by a monoidal equivalence between them but there is no strict equivalence either from \( \mathcal{G} \) to \( \mathcal{G}' \) nor from \( \mathcal{G}' \) to \( \mathcal{G} \). For this reason, if to establish the bijection (53), we want to use only strict monoidal abelian groupoids and strict equivalences between them, as we need to do for applying Duskin or Pirashvili classification results, then we must define two strict monoidal abelian groupoids \( \mathcal{G} \) and \( \mathcal{G}' \) as equivalent if there is a zig-zag chain of strict equivalences such as \( \mathcal{G} \leftarrow \mathcal{G}_1 \rightarrow \cdots \leftarrow \mathcal{G}_n \rightarrow \mathcal{G}' \). Although two strict monoidal abelian groupoids in the same equivalence class can always be linked by one intervening pair of strict equivalences, this phenomenon, we think, obscures unnecessarily the conclusions. Moreover, the facts stated in Theorem 4.5(iv) clearly fail for strict monoidal functors.

4.3. Classification of categorical groups revisited. As we recalled above, a categorical group is a monoidal groupoid \( \mathcal{G} \) in which every object is invertible or, equivalently, such that its associated monoid of connected components \( M(\mathcal{G}) \) is a group. By Proposition 4.1, every categorical group is abelian, so that

\[
\text{CatGp} \subseteq \text{MonAbGpd}
\]

is the full 2-subcategory of the 2-category of monoidal abelian groupoids given by the categorical groups. We shall denote by

\[
\text{Z}^3\text{Mnd}_{\text{GP}} \subseteq \text{Z}^3\text{Mnd}
\]

the full 2-subcategory of the 2-category of Leech 3-cocycles of monoids whose objects are those \( S = (G, (A, \Theta), \lambda) \) in \( \text{Z}^3\text{Mnd} \) where \( G \) is a group. Then, the biequivalences (50) in Theorem 4.2 restrict to corresponding quasi-inverse biequivalences

\[
\begin{array}{ccc}
\text{CatGp} & \xrightarrow{\Delta} & \text{Z}^3\text{Mnd}_{\text{GP}} \\
\Sigma & \xleftarrow{\Sigma} & \text{Z}^3\text{Mnd}.
\end{array}
\]

But now we shall note that this latter 2-category \( \text{Z}^3\text{Mon}_{\text{GP}} \) is essentially the same as its full 2-subcategory, called the 2-category of Eilenberg-Mac Lane 3-cocycles of groups [12] and denoted by

\[
\text{Z}^3\text{Gp} \subseteq \text{Z}^3\text{Mon}_{\text{GP}},
\]

which is defined by those \( S = (G, (A, \Theta), \lambda) \) as above, but in which the family of groups \( A \) is constant, that is, where \( A_a = A_1 \) for all \( a \in G \), and in the family \( \Theta \) all automorphisms \( a^* : A_1 \rightarrow A_1, a \in G, \) are identities. Observe that such a \( S \) is then described simply as a triple \( S = (G, (A, \theta), \lambda) \), where \( G \) is a group, \( A (= A_1) \) is an abelian group, \( \theta : G \rightarrow \text{Aut}(A) \) is a group homomorphism \( (\theta(a) = a_*) \), and \( \lambda \in \text{Z}^3(G, (A, \theta)) \) is an ordinary normalized 3-cocycle of the group \( G \) with coefficients in the \( G \)-module \( (A, \theta) \), that is, the \( G \)-module defined by the abelian group \( A \) with left action \( (a, f) \mapsto a f = \theta(a)(f) \).

A morphism \( (p, q, \varphi) : (G, (A, \theta), \lambda) \rightarrow (G', (A', \theta'), \lambda') \) in \( \text{Z}^3\text{Gp} \) then consists of a group homomorphism \( p : G \rightarrow G' \), a homomorphism of \( G \)-modules

\[
q : (A, \theta) \rightarrow p^*(A', \theta') = (A', \theta' p),
\]

and a normalized 2-cochain \( \varphi \in C^2(G, (A', \theta' p)) \) such that \( q_*(\lambda) = p^*(\lambda') \circ \partial^2 \varphi. \) If \( (p, q, \varphi), (\bar{p}, \bar{q}, \bar{\varphi}) : (G, (A, \theta), \lambda) \rightarrow (G', (A', \theta'), \lambda') \)
are two morphisms in $\mathbf{Z}^3\mathbf{Gp}$, then there is no deformation between them unless $p = \bar{p}$ and $q = \bar{q}$, and, in such a case, a deformation $\delta : (p, q, \varphi) \Rightarrow (p, q, \bar{\varphi})$ consists of a 1-cocycle $\delta \in C^1(G, (A', \theta'p))$, such that $\varphi = \bar{\varphi} \circ \partial^1 \delta$.

We have a 2-functor

$$ (\ )_1 : \mathbf{Z}^3\text{Mon}|_{\mathbf{Gp}} \to \mathbf{Z}^3\mathbf{Gp} $$

that is given on objects by

$$(55) \quad (G, (A, \Theta), \lambda) \mapsto (G, (A_1, \theta), \hat{\lambda}),$$

where the homomorphism $\theta : G \to \text{Aut}(A_1)$ is defined, by means of the isomorphisms $A_1 \xrightarrow{a} A_a \xleftarrow{\hat{a}} A_1$, $a \in G$, of $\Theta$, by the equations

$$a^* \theta(a) = a_*,$$

while the component at any $(a, b, c) \in G \times G \times G$ of the 3-cocycle $\hat{\lambda} \in \mathbf{Z}^3(G, A_1)$ is defined, by means of the isomorphism $(abc)^* : A_1 \to A_{abc}$, by

$$(abc)^*(\hat{\lambda}_{a,b,c}) = \lambda_{a,b,c}.$$ 

A morphism $(p, q_1, \varphi) : (G, (A, \Theta), \lambda) \to (G', (A', \Theta'), \lambda')$ in $\mathbf{Z}^3\text{Mon}|_{\mathbf{Gp}}$ is mapped by the 2-functor $(\ )_1$ to the morphism

$$(56) \quad (p, q_1, \bar{\varphi}) : (G, (A_1, \theta), \hat{\lambda}) \to (G', (A'_1, \theta'), \hat{\lambda'}),$$

where $\bar{\varphi} \in C^2(G, (A'_1, \theta'p))$ is the 2-cochain whose component at any pair $a, b \in G$ is determined by the isomorphism $p(ab)^* : A'_1 \to A'_{p(ab)}$ such that

$$p(ab)^*(\bar{\varphi}_{a,b}) = \varphi_{a,b},$$

whereas a deformation $\delta : (p, q_1, \varphi) \Rightarrow (p, q_1, \psi)$ in $\mathbf{Z}^3\text{Mon}|_{\mathbf{Gp}}$ is carried to the deformation in $\mathbf{Z}^3\mathbf{Gp}$

$$\hat{\delta} : (p, q_1, \bar{\varphi}) \Rightarrow (p, q_1, \psi),$$

where $\hat{\delta} \in C^1(G, (A'_1, \theta'p))$ is the 1-cocycle defined by means of the isomorphisms $p(a)^* : A'_1 \to A'_{p(a)}$, $a \in G$, such that

$$p(a)^*(\hat{\delta}_a) = \delta_a.$$

All the needed verifications to prove that $(\ )_1$ is actually a 2-functor are quite straightforward. For example, we see that $\hat{\lambda}$ in $(55)$ is certainly a 3-cocycle and that the homomorphism $q_1 : (A_1, \theta) \to (A'_1, p^* \theta')$ in $(56)$ is of $G$-modules, as follows:

$$(abcd)^*(\hat{\lambda}_{b,c,d} \circ \hat{\lambda}_{a,b,c,d} \circ \hat{\lambda}_{a,b,c})$$

$$= (bcd)^*a^*(\hat{\lambda}_{b,c,d}) \circ (abcd)^*\hat{\lambda}_{a,b,c,d} \circ d^*(abc)^*(\hat{\lambda}_{a,b,c})$$
$$= (bcd)^*a^*(\hat{\lambda}_{b,c,d}) \circ \lambda_{a,b,c,d} \circ d^*(\lambda_{a,b,c})$$
$$= a^*(\lambda_{b,c,d}) \circ \lambda_{a,b,c,d} \circ d^*(\lambda_{a,b,c})$$
$$= \lambda_{a,b,c,d} \circ \lambda_{a,b,c,d}$$
$$= (abcd)^*(\hat{\lambda}_{a,b,c,d} \circ \hat{\lambda}_{a,b,c,d}),$$

whence $\hat{\lambda}_{b,c,d} \circ \hat{\lambda}_{a,b,c,d} \circ \hat{\lambda}_{a,b,c} = \hat{\lambda}_{a,b,c,d} \circ \hat{\lambda}_{a,b,c,d}$. 

$$p(a)^*(p(a)q_1(f)) = p(a)_*(q_1(f)) \quad (48) \quad q_a a_*(f) = q_a (a^*(a f)) \quad (49) \quad p(a)^*(q_1(a f)),$$

whence $q_1(a f) = p(a)q_1(f)$. 

Proposition 4.7. The 2-functors inclusion and $(\_)_1$ are mutually quasi-inverse biequivalences

$$Z^3\text{Gp} \xrightarrow{(\_)_1} Z^3\text{Mnd}|_{\text{Gp}}.$$  

Proof. We have $(\_)_1 in = 1$, the identity, while the pseudo-equivalence in $(\_)_1 \simeq 1$ is given, at any object $(G, (A, \theta), \lambda)$, by the isomorphism

$$(1_G, q, 1): (G, (A_1, \theta), \lambda) \cong (G, (A, \theta), \lambda),$$

where $q = (A_1 \xrightarrow{a^\ast} A_0)_{a \in G}$. □

Therefore, by composing the biequivalences above with those in (54), we get the following (already known, see [12, Theorem 3.3]) cohomological description of the 2-category of categorical groups:

Theorem 4.8. The 2-functors $\Delta_1 = (\_)_1\Delta$ and $\Sigma_1 = \Sigma in$,

$$\text{CatGp} \xrightarrow{\Delta_1} Z^3\text{Gp}$$

are quasi-inverse biequivalences.

Let us now denote by $H^3\text{Gp} \subseteq H^3\text{Mnd}$ the full subcategory of the category of Leech 3-cohomology classes of monoids (51), given by the Eilenberg-Mac Lane 3-cohomology classes of groups. An object in $H^3\text{Gp}$ is then a triple $(G, (A, \theta), c)$, where $G$ is a group, $(A, \theta)$ is a $G$-module, and $c \in H^3(G, (A, \theta))$. An arrow

$$(p, q): (G, (A, \theta), c) \rightarrow (G', (A', \theta'), c')$$

in $H^3\text{Gp}$ consists of a group homomorphism $p : G \rightarrow G'$ and a homomorphism of $G$-modules $q : (A, \theta) \rightarrow (A', \theta' p)$ such that $p^\ast(c') = q_\ast(c) \in H^3(G, (A', \theta' p))$.

We have the cohomology class functor

$$\text{cl} : Z^3\text{Gp} \rightarrow H^3\text{Gp},$$

$$(G, (A, \theta), \lambda) \mapsto (G, (A, \theta), [\lambda])$$

$$(p, q, \varphi) \mapsto (p, q)$$

This functor $\text{cl}$ carries isomorphic morphisms of $Z^3\text{Gp}$ to the same morphism in $H^3\text{Gp}$ and is surjective on objects. Moreover, it reflects isomorphisms and is full: if $(p, q, \varphi) : (G, (A, \theta), \lambda) \rightarrow (G', (A', \theta'), \lambda')$ is any morphism in $Z^3\text{Gp}$ such that the maps $p$ and $q$ are invertible, then the morphism of $Z^3\text{Gp}$

$$(p^{-1}, q^{-1}, p^{-1} q^{-1}(\varphi^{-1})) : (G', (A', \theta'), \lambda') \rightarrow (G, (A, \theta), \lambda)$$

is an inverse of $(p, q, \varphi)$. To see that $\text{cl}$ is full, let

$$(p, q) : \text{cl}(G, (A, \theta), \lambda) \rightarrow \text{cl}(G', (A', \theta'), \lambda')$$

be any morphism in $H^3\text{Gp}$, then $p^\ast[\lambda']$ and $q_\ast[\lambda]$ both represent the same class in $H^3(G, (A', \theta' p))$, so there is $\varphi \in C^2(G, (A', \theta' p))$ such that $q_\ast(\lambda) = p^\ast(\lambda') \circ \partial^2 \varphi$. Then, $(p, q, \varphi) : (G, (A, \theta), \lambda) \rightarrow (G', (A', \theta'), \lambda')$ is a morphism in $Z^3\text{Gp}$ with $\text{cl}(p, q, \varphi) = (p, q)$. Furthermore, let us observe that any other realization of $(p, q)$ is of the form $(p, q, \varphi \circ \phi)$ with $\phi \in Z^2(G, (A', \theta' p))$ and, moreover, that there is a deformation $(p, q, \varphi \Rightarrow (p, q, \varphi \circ \phi)$ if and only if $\phi = \partial^1 \delta$ for some $\delta \in C^1(G, (A', \theta' p))$. 


Hence, the classifying functor
\[ \text{Cl} = \text{cl} \Delta_1 : \text{CatGp} \rightarrow H^3 \text{Gp} \]
has the following properties:

**Theorem 4.9** ([35] Classification of categorical groups). (i) For any group \( G \), any \( G \)-module \( (A, \theta) \), and any cohomology class \( c \in H^3(G, (A, \theta)) \), there is a categorical group \( \mathcal{G} \) with an isomorphism \( \text{Cl}(\mathcal{G}) \cong (G, (A, \theta), c) \).

(ii) A monoidal functor between categorical groups \( F : \mathcal{G} \rightarrow \mathcal{G}' \) is an equivalence if and only if the induced \( \text{Cl}(F) : \text{Cl}(\mathcal{G}) \rightarrow \text{Cl}(\mathcal{G}') \) is an isomorphism.

(iii) If \( \mathcal{G} \) and \( \mathcal{G}' \) are categorical groups, then, for any isomorphism \( (p, q) : \text{Cl}(\mathcal{G}) \cong \text{Cl}(\mathcal{G}') \), there is a monoidal equivalence \( F : \mathcal{G} \rightarrow \mathcal{G}' \) such that \( \text{Cl}(F) = (p, q) \).

(iv) If \( \mathcal{G} \) and \( \mathcal{G}' \) are categorical groups with \( \text{Cl}(\mathcal{G}) = (G, (A, \theta), c) \) and \( \text{Cl}(\mathcal{G}') = (G', (A', \theta'), c') \), then, for any morphism \( (p, q) : \text{Cl}(\mathcal{G}) \rightarrow \text{Cl}(\mathcal{G}') \) in \( H^3 \text{Gp} \), there is a (non-natural) bijection
\[
\{ [F] : \mathcal{G} \rightarrow \mathcal{G}' \mid \text{Cl}(F) = (p, q) \} \cong H^2(G, (A', \theta'p)),
\]
between the set of isomorphism classes of those monoidal functors \( F : \mathcal{G} \rightarrow \mathcal{G}' \) that are carried by the classifying functor to \( (p, q) \) and the second cohomology group of \( G \) with coefficients in the \( G \)-module \( (A', \theta'p) \).

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