An Averaging Principle for Mckean–Vlasov-Type Caputo Fractional Stochastic Differential Equations

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1.Introduction

For complex systems, we usually want to locate an effective simplified model to approximate the original complex system or extract the main dynamical behavior of the original system. Based on these ideas, a lot of effective methods have been generated in dynamical systems, such as invariant manifolds, averaging principle, and homogenization principle. These effective methods have now been extended to deal with stochastic systems, such as stochastic invariant manifolds see [1, 2] and stochastic averaging principle, see [3–9].

Currently, the problem of averaging for stochastic differential equations have received a lot of attention and various types of stochastic differential equations have been studied, see [4, 6, 7, 10–12]. However, there are no relevant results of averaging principle for distribution dependent-type stochastic differential equations which we will consider in this paper.

On the contrary, the problem of averaging for stochastic fractional order differential equations have received a lot of attention in recent years, and some results [13] have been obtained under averaging condition consistence with the classic case (see [4, 5, 14]). Noting that the fractional order derivative is a nonlocal operator, therefore, the fractional order differential equation is more effective for describing certain phenomena in the real world (see [15–17]).
stochastic differential equations will be discussed in Section 3. An averaging principle for the above equation is established in Section 4.

2. Preliminaries

First, we introduce some notation. Let $C(H)$ be the space of continuous functions on $H$. Let $\mathcal{B}(H)$ be the Borel $\sigma$-algebra of subsets of $H$. $M(H)$ is the space of probability measures on $\mathcal{B}(H)$ and carries the usual topology of weak convergence. $(\mu, \phi)$ denotes $\int \phi(x) \mu(dx)$. Let $\gamma(x) = 1 + |x|$, $\forall x \in H$, and then, define the Banach space

$$C_p(H) = \left\{ \phi \in C(H): \|\phi\|_{C_p} = \sup_{x \in H} \frac{\phi(x)}{\gamma(x)} < \infty \right\}.$$  

(2)

For any $p \geq 1$, let $M^p(H)$ denote the Banach space of signed measures $\mu$ on $H$, and $\|\mu\|_p = \int \mu^{|p}|m|(dx) = \infty$. $m = m^+ + m^-$ and $\rho = \rho^+ - \rho^-$ are the Jordan decomposition of $\mu$. $M^p(H) = M^p(H) \cap M(H)$ is the set of probability measures on $\mathcal{B}(H)$, and there exists second moments. Define the following metric:

$$\rho(\mu, \nu) = \sup_{x \in H} (\phi, \mu) - (\phi, \nu) = \sup_{x \in H} (\phi(x) - \phi(y)) \frac{\|\phi\|_{\rho}}{|x - y|} \leq 1.$$  

(3)

Then, $(M^p(H), \rho)$ is a complete metric space. Let $C([0, T], M^p(H), \rho)$ be the complete metric space of continuous functions from $[0, T]$ to $(M^p(H), \rho)$ with the following metric:

$$D_{\gamma}(\mu, \nu) = \sup_{t \in [0, T]} \rho(\mu(t), \nu(t)), \quad \forall \mu, \nu \in C([0, T], M^p(H), \rho).$$  

(4)

More details can be seen in [18].

In order to obtain the existence and uniqueness of the solution of (1), we introduce the following conditions.

(i) H1 (Lipschitz condition): for all $x, y \in H$ and $t \in [0, T]$, $\mu \in C([0, T], M^p(H), \rho)$, and there exists a bounded function $k_1(t) > 0$, such that

$$|f(t, x, \mu) - f(t, y, \nu)|^2 + |g(t, x, \mu) - g(t, y, \nu)|^2 \leq k_1(t)\left(|x - y|^2 + \rho^2(\mu, \nu)\right).$$  

(5)

(ii) H2 (growth condition): for all $(x, t) \in H \times [0, T]$, there exists a bounded function $k_2(t) > 0$ such that

$$|f(t, x, \mu)|^2 + |g(t, x, \mu)|^2 \leq k_2(t)(1 + |x|^2).$$  

(6)

In this paper, we assume there exist of a constant $k$ such that $\max\{k_1(t), k_2(t)\} \leq k$.

First, we give an important lemma, which is a type of promotion form of Gronwall’s inequality with singular kernels.

**Lemma 1** (see [21, 22]). Suppose $b \geq 0$, $\beta > 0$, and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some

Under the assumptions of H1 and H2, we will prove the existence and uniqueness of solution for the above equation.

**Definition 1.** An $\mathcal{F}_t$-adapted stochastic process $X_t$ with law $L(X_t) = \mu(t)$ is called a solution of (1) if $X_t$ is continuous, and for $\forall t \in [0, T]$ with $X_0 = x_0$,

$$X_t = x_0 + \frac{1}{\Gamma(a)} \int_0^t (t - s)^{a-1} f(s, X_s, \mu_s)ds$$

$$+ \frac{1}{\Gamma(a)} \int_0^t (t - s)^{a-1} g(s, X_s, \mu_s)dB_s,$$

(9)

Under the assumptions of H1 and H2, we will prove the existence and uniqueness of solution for the above equation.

**Theorem 1.** Assume that H1 and H2 hold; then, for $\forall x_0 \in L^2(\Omega, H)$, equation (10) has a unique solution
We will proof the theorem by several steps.

(i) First, we prove that \( X_t \in L^\infty([0,T], L^2(\Omega; H)) \) for \( \forall \mu \in C([0,T], (M_\infty^\alpha(\mathbb{R}), \rho)) \). Using the following inequality,

\[
|a + b + c|^2 \leq 3(|a|^2 + |b|^2 + |c|^2).
\]

(ii) Now, we show that \( X_t \in L^\infty([0,T], L^2(\Omega; H)) \) for \( \forall \mu \in C([0,T], (M_\infty^\alpha(\mathbb{R}), \rho)) \).

For \( I_3 \), applying Cauchy–Schwarz’s inequality and H2, it follows

\[
I_2 \leq \frac{Tk}{\Gamma(\alpha)} \int_0^t (t-s)^{2\alpha-2} \left( 1 + E[X(s)]^2 + \|\mu_t\|_{\gamma}^2 \right) ds
\]

\[
\leq \frac{Tk}{\Gamma(\alpha)} \left[ \int_0^t (t-s)^{2\alpha-1} \left( 1 + \sup_{s \leq r \leq T} \|\mu_r\|_{\gamma}^2 \right) + \int_0^t (t-s)^{2\alpha-2} E[X(s)]^2 ds \right]
\]

\[
\leq \frac{kT^{2\alpha}}{\Gamma(\alpha)^2 (2\alpha - 1)} \left( 1 + \sup_{s \leq r \leq T} \|\mu_r\|_{\gamma}^2 \right) + \frac{Tk}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2} E[X(s)]^2 ds.
\]

Combining the above estimate results, we finally obtain

\[
E[X(t)]^2 \leq r_1 + r_2 \int_0^t (t-s)^{2\alpha-1} E[X(s)]^2 ds,
\]

where

\[
r_1 = 3E|x_0|^2 + \frac{kT^{2\alpha-1}}{\Gamma(\alpha)^2 (2\alpha - 1)} \left( 1 + \sup_{s \leq r \leq T} \|\mu_r\|_{\gamma}^2 \right),
\]

\[
r_2 = \frac{k(T + 1)}{\Gamma(\alpha)^2}.
\]
\[ E\left\| X_t - X_{t_0}\right\|^2 \leq 2E\frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} f(s, X_s, \mu_s) ds - \int_0^{t_0} (t_0-s)^{\alpha-1} f(s, X_s, \mu_s) ds\right]^2 \]
\[ + 2E\frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} f(s, X_s, \mu_s) dB_s - \int_0^{t_0} (t_0-s)^{\alpha-1} f(s, X_s, \mu_s) dB_s\right]^2 \]
\[ = 2(J_1 + J_2). \]  

For \( J_1 \), we have
\[ J_1 \leq 2E\frac{1}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2} ds \int_0^t E[f(s, X_s, \mu_s)]^2 ds \]
\[ \leq \frac{k}{\Gamma(\alpha)^2 (2\alpha - 1)} (t-t_0)^{2\alpha-1} \int_0^t \left( 1 + E|X_s|^2 + \|\mu_s\|^2 \right) ds \]
\[ \leq \frac{Ck}{\Gamma(\alpha)^2 (2\alpha - 1)} (t-t_0)^{2\alpha}. \]  

By the Cauchy–Schwarz inequality, 

For \( J_{12} \), we have
\[ J_{12} = E\frac{1}{\Gamma(\alpha)} \left[\int_0^{t_0} (t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}) f(s, X_s, \mu_s) ds\right]^2 \]
\[ \leq \frac{k}{\Gamma(\alpha)^2} \int_0^{t_0} (t-s)^{\alpha-1} - (t_0-s)^{\alpha-1})^2 ds \int_0^{t_0} \left( 1 + E|X_s|^2 + \|\mu_s\|^2 \right) ds \]
\[ \leq \frac{CTk}{\Gamma(\alpha)^2} \int_0^{t_0} \left( (t_0-t)^{2(\alpha-1)} - (t-s)^{2(\alpha-1)} \right) ds \]
\[ \leq \frac{CTk}{\Gamma(\alpha)^2} \left[ \frac{(t-t_0)^{2\alpha-1}}{2\alpha - 1} + \frac{t_0^{2\alpha-1}}{2\alpha - 1} - \frac{t_0^{2\alpha-1}}{2\alpha - 1} \right] \leq \frac{CTk}{\Gamma(\alpha)^2} \frac{(t-t_0)^{2\alpha-1}}{2\alpha - 1}. \]  

For \( J_2 \), using the Itô isometry formula, in the similar way as \( J_1 \), we can prove that
\[ J_2 \leq \frac{Ck}{\Gamma(\alpha)^2} \frac{(t-t_0)^{2\alpha-1}}{2\alpha - 1}. \]  

Results of \( J_1 \) and \( J_2 \) combined together show that
\[ E\left\| X_t - X_{t_0}\right\|^2 \leq \frac{Ck(1 + T)(t-t_0)^{2\alpha-1}}{\Gamma(\alpha)^2 (2\alpha - 1)} \]
\[ + \frac{Ck}{\Gamma(\alpha)^2 (2\alpha - 1)} (t-t_0)^{2\alpha}, \]
which implied \( X_t \in C([0, T], L^2(\Omega; H)) \) for each fixed \( \mu \in C([0, T], (M^\rho, \rho)) \).
(iii) By virtue of the fixed point theorem for contraction mappings, we can show that, for each fixed \( \mu \in C([0, T], (M, \rho)) \), equation (10) has a unique solution in \( C([0, T], L^2(\Omega; H)) \). Similar arguments are also discussed in [18]. Now, we define an operator \( \Phi_H(\cdot) \) on \( C([0, T], L^2(\Omega; H)) \):

\[
(\Phi_H X)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X_s, \mu_s) ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(t, X_s, \mu_s) dB_t,
\]

(26)

Using the Cauchy–Schwarz inequality and Itô isometry formula, it is readily seen that

\[
\| (\Phi_H X)(t) - (\Phi_H Y)(t) \|_c^2 \\
\leq 2 \left( \frac{1}{\Gamma(\alpha)} \right) \int_0^t (t-s)^{\alpha-1} (f(s, X_s, \mu_s) - f(s, Y_s, \mu_s)) ds \|_2^2 \\
+ 2 \left( \frac{1}{\Gamma(\alpha)} \right) \int_0^t (t-s)^{\alpha-1} (g(t, X_s, \mu_s) - g(t, Y_s, \mu_s)) dB_t \|_2^2.
\]

(27)

It is easy to verify that \( \Phi_H(\cdot) \) is from \( C([0, T], L^2(\Omega; H)) \) into itself.

For \( X_t, Y_t \in C([0, T], L^2(\Omega; H)) \) with \( x_0 = y_0 \), let \( \| \cdot \|_c \) denote the norm of \( C([0, T], L^2(\Omega; H)) \) and \( \beta = 2\alpha - 1 > 0 \); then, we obtain

\[
\| (\Phi_H^{n+1} X)(t) - (\Phi_H^n Y)(t) \|_c^2 \\
\leq 2Tk + 2k + \frac{2Tk + 2k}{\Gamma(\alpha)^2} \int_0^t (t-s)^{\beta-1} \| f(s, X_s, \mu_s) - f(s, Y_s, \mu_s) \|_2^2 ds \\
\leq 2Tk + 2k + \frac{2Tk + 2k}{\Gamma(\alpha)^2} \int_0^t (t-s)^{\beta-1} \frac{1}{\beta} \| g(t, X_s, \mu_s) - g(t, Y_s, \mu_s) \|_2^2 dB_t \\
\leq \left( \frac{2Tk + 2k}{\Gamma(\alpha)^2} \right)^{\beta+1} \int_0^t (t-s)^{\beta-1} \| X_s - Y_s \|_{L^1_c}^2 ds.
\]

(31)

Thus, we only need to discuss the following integral:

\[
\int_0^t (t-s)^{\beta-1} s^\beta ds.
\]

(32)

Let \( s = tz \); then,

\[
\int_0^t (t-s)^{\beta-1} s^\beta ds = \int_0^1 (1-z)^{\beta-1} z^\beta dz \\
= t^{(\beta+1)} \int_0^1 (1-z)^{\beta-1} z^\beta dz \\
= t^{(\beta+1)} B(\beta + 1, \beta) = t^{(\beta+1)} \Gamma(\beta) \Gamma((\beta + 1))/(\Gamma((\beta + 1)\beta + 1))
\]

(33)

where \( B(\cdot, \cdot) \) is the Beta function. Substitute the above equality into (31), and we derive that
\[ \left( \Phi_{\mu}^{t+1}X(t) - \Phi_{\nu}^{t+1}Y(t) \right)^2 \]
\[ \leq 2T_k + 2k \left( \frac{1}{\Gamma^2(\alpha)} \right)^{l+1} \Gamma(1/\beta) \Gamma((\beta + 1)/\beta) X_t - Y_t^2 \left( \frac{1}{1 + \beta} \right)^{(l+1)/\beta} \]
\[ = \left( \frac{2T_k + 2k}{\Gamma^2(\alpha)} \right)^{l+1} \Gamma(1/\beta) \Gamma((\beta + 1)/\beta) X_t - Y_t^2 \left( \frac{1}{1 + \beta} \right)^{(l+1)/\beta} \]
\[ \leq \left( \frac{2T_k + 2k}{\Gamma^2(\alpha)} \right)^{l+1} \Gamma(1/\beta) \Gamma((\beta + 1)/\beta) X_t - Y_t^2 \left( \frac{1}{1 + \beta} \right)^{(l+1)/\beta} \]

By the above discussion, we finally obtain
\[ \left( \Phi_{\mu}^{t+1}X(t) - \Phi_{\nu}^{t+1}Y(t) \right)^2 \leq \left( \frac{2T_k + 2k}{\Gamma^2(\alpha)} \right)^{l+1} \Gamma(1/\beta) \Gamma((\beta + 1)/\beta) X_t - Y_t^2 \left( \frac{1}{1 + \beta} \right)^{(l+1)/\beta} \]

which implies
\[ \lim_{t \to s} \rho(L(X_{\mu}(t)), L(X_{\nu}(s))) = 0. \]

Hence, we verify that \( L(X_{\mu}) \in C([0, T], (M_{\mu}, \rho)) \).

(v) Define \( \Psi \) on \( C([0, T], (M_{\mu}, \rho)) \) as follows:
\[ \Psi: \mu \to L(X_{\mu}). \]

In the following, we will show that the operator \( \Psi \) has a unique fixed point in \( C([0, T], (M_{\mu}, \rho)) \). Take \( \mu, \nu \in C([0, T], (M_{\mu}, \rho)) \), and let \( X_{\mu}(t) \) and \( X_{\nu}(t) \) be the corresponding solutions of the following equations:
\[ X_{\mu}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X_{\mu}(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(t, X_{\mu}(s)) dB_t, \]
\[ X_{\nu}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X_{\nu}(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(t, X_{\nu}(s)) dB_t. \]

Thus,
\[ E \left[ |X_\mu(t) - X_\nu(t)|^2 \right] \leq \frac{2T}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2} \left| f(s, X_\mu(s), \mu_s) - f(s, X_\nu(s), \nu_s) \right|^2 ds + \frac{2}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2} \left| g(s, X_\mu(s), \mu_s) - g(s, X_\nu(s), \nu_s) \right|^2 ds \]

\[ \leq \frac{2kT}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2} \left[ |X_\mu(s) - X_\nu(s)|^2 + \rho^2(\mu_s, \nu_s) \right] ds + \frac{2k}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2} \left[ |X_\mu(s) - X_\nu(s)|^2 + \rho^2(\mu_s, \nu_s) \right] ds. \] (41)

After simple calculation, we have that

\[ \sup_{0 \leq t \leq T} E \left[ |X_\mu(t) - X_\nu(t)|^2 \right] \leq \frac{2k(T+1)}{\Gamma(\alpha)^2} \frac{T^{2\alpha-1}}{2\alpha - 1} \left[ \sup_{0 \leq t \leq T} E \left[ |X_\mu(t) - X_\nu(t)|^2 \right] + D^\alpha_T(\mu, \nu) \right]. \] (42)

Select the appropriate \( T = T_0 > 0 \), such that

\[ \frac{2k(T_0 + 1)}{\Gamma(\alpha)^2} \frac{T_0^{2\alpha-1}}{2\alpha - 1} < \frac{1}{3}. \] (43)

Then, it follows

\[ \sup_{0 \leq t \leq T_0} E \left[ |X_\mu(t) - X_\nu(t)|^2 \right] < \frac{1}{2} D^\alpha_{T_0}(\mu, \nu). \] (44)

By the definition of \( \rho(\mu, \nu) \) and \( D^\alpha_T(\mu, \nu) \), we can obtain

\[ \rho^2(\mu, \nu) \leq E \left[ |X_\mu(t) - X_\nu(t)|^2 \right]. \] (45)

Taking sup-norm on both sides, we obtain

\[ D^\alpha_T(\Psi(\mu), \Psi(\nu)) \leq \sup_{0 \leq t \leq T_0} E \left[ |X_\mu(t) - X_\nu(t)|^2 \right]. \] (46)

Combine this result with equation (44), and we finally derive

\[ D^\alpha_T(\Psi(\mu), \Psi(\nu)) < \frac{1}{2} D^\alpha_{T_0}(\mu, \nu). \] (47)

Since \( \Psi \) is a contraction in \( C([0, T_0], \mathcal{M}_+^\alpha(H, \rho)) \), it has a unique fixed point. Thus, equation (10) has a unique solution \( X_\mu \) with \( \mu = L(X_\mu) \) on \([0, T_0]\). Because \( X_\mu \) belongs to \( C([0, T], L^2(\Omega; H)) \), we can extend the solution to \([0, T]\) by considering \([0, T_0], [T_0, 2T_0]\), and so on. This completes the proof.

### 4. An Averaging Principle

In this section, we study an averaging principle for the following distribution dependent fractional stochastic differential equations in \( H \):

\[ X_\epsilon(t) = x_0 + \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X_\epsilon(s), \mu_\epsilon(s)) ds + \frac{\sqrt{\epsilon}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, X_\epsilon(s), \mu_\epsilon(s)) dB_s, \] (48)

where \( x_0 \in L^2(\Omega; H) \). We will show that the solution of (48) will be approximated by the following simpler or averaged process under certain conditions:

\[ Z_\epsilon(t) = x_0 + \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X_\epsilon(s), \mu_\epsilon(s)) ds + \frac{\sqrt{\epsilon}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, X_\epsilon(s), \mu_\epsilon(s)) dB_s. \] (49)

Equation (49) is called the averaged equation for (48). Now, we prove that the solution of (49) converges to the solution of the original equation (48) under the following additional conditions.

- **H3:**
  \[
  \frac{1}{t^{2\alpha-1}} \int_0^t (t-s)^{2(\alpha-1)} \| f(s, x, \mu) - \mathcal{A}(x, \mu) \|^2 ds \leq \phi_1(t)(1 + |x|^2 + \|\mu\|^2).
  \] (50)

- **H4:**
  \[
  \frac{1}{t^{2\alpha-1}} \int_0^t (t-s)^{2(\alpha-1)} \| g(s, x, \mu) - \mathcal{G}(x, \mu) \|^2 ds \leq \phi_2(t)(1 + |x|^2 + \|\mu\|^2),
  \] (51)
where \( \varphi_i(t) \) are positive and bounded with \( \lim_{t \to +\infty} \varphi_i(t) = 0 \) for \( i = 1, 2 \).

**Remark 1.** Note that when we take \( \alpha = 1 \), then this condition is consistence with the classic case, see [4].

**Theorem 2.** Let \( H1 - H4 \) hold. Then, for \( \forall \delta_1 > 0 \), there exist constants \( L > 0, \epsilon_1 \in (0, \epsilon_0] \) and \( \beta \in (0, 1) \) such that, for any \( \epsilon \in (0, \epsilon_1] \), \( 1/2 < \alpha < 1 \), we have

\[
\sup_{0 \leq t \leq 1} E[X_\epsilon(t) - Z_\epsilon(t)]^2 \leq \delta_1.
\]  

(52)

Let us consider

\[
X_\epsilon(t) - Z_\epsilon(t) = \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ f(s, X_\epsilon(s), \mu_\epsilon(s)) - \overline{f}(Z_\epsilon(s), \nu_\epsilon(s)) \right] ds
\]

+ \( \frac{\sqrt{\epsilon}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ g(s, X_\epsilon(s), \mu_\epsilon(s)) - \overline{g}(Z_\epsilon(s), \nu_\epsilon(s)) \right] dB_t.
\]

(53)

By the arithmetic inequality, it follows that

\[
E[X_\epsilon(t) - Z_\epsilon(t)]^2 \leq
2 \left\{ E\left[ \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ f(s, X_\epsilon(s), \mu_\epsilon(s)) - \overline{f}(Z_\epsilon(s), \nu_\epsilon(s)) \right] ds \right]^2 \right\}
\]

\[
+ E\left[ \frac{\sqrt{\epsilon}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ g(s, X_\epsilon(s), \mu_\epsilon(s)) - \overline{g}(Z_\epsilon(s), \nu_\epsilon(s)) \right] dB_t \right]^2
\]

(54)

\[
= 2(Q_1 + Q_2).
\]

For \( Q_1 \), we have

\[
Q_1 = E\left[ \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ f(s, X_\epsilon(s), \mu_\epsilon(s)) - \overline{f}(Z_\epsilon(s), \nu_\epsilon(s)) \right] ds \right]^2
\]

\[
\leq \frac{2\epsilon^2}{\Gamma(\alpha)^2} \left\{ E\left[ \int_0^t (t-s)^{\alpha-1} \left[ f(s, X_\epsilon(s), \mu_\epsilon(s)) - f(s, Z_\epsilon(s), \nu_\epsilon(s)) \right] ds \right]^2 \right\}
\]

\[
+ E\left[ \int_0^t (t-s)^{\alpha-1} \left[ g(s, Z_\epsilon(s), \nu_\epsilon(s)) - \overline{g}(Z_\epsilon(s), \nu_\epsilon(s)) \right] dB_t \right]^2.
\]

(55)
Applying the Cauchy–Schwarz inequality, \( H1, H3, \) and the definition of the metric \( \rho \), we obtain

\[
Q_1 \leq \frac{2k\epsilon^2}{\Gamma(a)^2} \int_0^t (t-s)^{2a-2} E \left[ \left| X_\epsilon(s) - Z_\epsilon(s) \right|^2 + \rho^2(\mu_\epsilon(s), \nu_\epsilon(s)) \right] ds
\]

\[
+ \frac{2\epsilon^2}{\Gamma(a)^2} \int_0^t (t-s)^{2a-2} E \left[ f(s, Z_\epsilon(s), \nu_\epsilon(s)) - \bar{f}(Z_\epsilon(s), \nu_\epsilon(s)) \right]^2 ds
\]

\[
+ \frac{2\epsilon}{\Gamma(a)^2} \int_0^t (t-s)^{2a-2} E \left[ X_\epsilon(s) - Z_\epsilon(s) \right]^2 ds
\]

\[
\leq \frac{4k\epsilon^2}{\Gamma(a)^2} \int_0^t (t-s)^{2a-2} E \left[ f(s, Z_\epsilon(s), \nu_\epsilon(s)) - \bar{f}(Z_\epsilon(s), \nu_\epsilon(s)) \right]^2 ds
\]

\[
+ \frac{2\epsilon}{\Gamma(a)^2} \int_0^t (t-s)^{2a-2} E \left[ X_\epsilon(s) - Z_\epsilon(s) \right]^2 ds
\]

\[
\leq \frac{4k\epsilon}{\Gamma(a)^2} \int_0^t (t-s)^{2a-2} E \left[ X_\epsilon(s) - Z_\epsilon(s) \right]^2 ds
\]

\[
+ \frac{2\epsilon}{\Gamma(a)^2} \int_0^t (t-s)^{2a-2} \left[ 1 + E[Z_\epsilon(s)]^2 + \|\nu_\epsilon(s)\|^2 \right] ds
\]

For \( Q_2 \), using the Itô isometry formula, we obtain

\[
Q_2 = \frac{\epsilon}{\Gamma(a)^2} \int_0^t (t-s)^{2a-2} E \left[ g(s, X_\epsilon(s), \mu_\epsilon(s)) - \bar{g}(Z_\epsilon(s), \nu_\epsilon(s)) \right]^2 ds
\]

\[
\leq \frac{2\epsilon}{\Gamma(a)^2} \int_0^t (t-s)^{2a-2} E \left[ g(s, X_\epsilon(s), \mu_\epsilon(s)) - g(Z_\epsilon(s), \nu_\epsilon(s)) \right]^2 ds
\]

\[
+ \frac{2\epsilon}{\Gamma(a)^2} \int_0^t (t-s)^{2a-2} E \left[ g(Z_\epsilon(s), \nu_\epsilon(s)) - \bar{g}(Z_\epsilon(s), \nu_\epsilon(s)) \right]^2 ds
\]

Applying conditions \( H1 \) and \( H4 \), we derive

\[
Q_2 \leq \frac{2k\epsilon}{\Gamma(a)^2} \int_0^t (t-s)^{2a-1} E \left[ X_\epsilon(s) - Z_\epsilon(s) \right]^2 ds
\]

\[
+ \frac{2\epsilon}{\Gamma(a)^2} \int_0^t (t-s)^{2a-1} \left[ 1 + E[Z_\epsilon(s)]^2 + \|\nu_\epsilon(s)\|^2 \right] ds
\]

\[
\leq \frac{4k\epsilon}{\Gamma(a)^2} \int_0^t (t-s)^{2a-1} E \left[ X_\epsilon(s) - Z_\epsilon(s) \right]^2 ds
\]

\[
+ \frac{2\epsilon}{\Gamma(a)^2} \int_0^t (t-s)^{2a-1} \left[ 1 + E[Z_\epsilon(s)]^2 + \|\nu_\epsilon(s)\|^2 \right] ds
\]
Therefore, from the above discussion, (56)–(58), and Theorem 1, we have

\[ E[X_\epsilon(t) - Z_\epsilon(t)]^2 \leq \frac{4kte^2}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2(a-1)} E[X_\epsilon(s) - Z_\epsilon(s)]^2 ds \]

\[ + \frac{2e\epsilon^2}{\Gamma(\alpha)^2} e^{2\alpha \phi_1(\epsilon)} \left( t \left( 1 + E[Z_\epsilon(s)]^2 + \|v_\epsilon(s)\|^2 \right) \right) \]

\[ + \frac{4ke}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2(a-1)} E[X_\epsilon(s) - Z_\epsilon(s)]^2 ds + \frac{2e\epsilon\Gamma(\alpha)^2}{\Gamma(\alpha)^2} \|v_\epsilon(s)\|^2 \] \[ \left( 1 + E[Z_\epsilon(s)]^2 + \|v_\epsilon(s)\|^2 \right) \]

\[ = 2eT^{2a-1} \left( \frac{c_1\epsilon}{\Gamma(\alpha)^2} T + \frac{c_2}{\Gamma(\alpha)^2} \right) + 4e \left( \frac{ke}{\Gamma(\alpha)^2} T + \frac{k}{\Gamma(\alpha)^2} \right) \int_0^t (t-s)^{2(a-1)-1} E[X_\epsilon(s) - Z_\epsilon(s)]^2 ds. \] \[ (59) \]

Denote \( r_1 = 2(\epsilon c/\Gamma(\alpha)^2)T + (c_2/\Gamma(\alpha)^2) \) and \( r_2 = 4(\epsilon k/\Gamma(\alpha)^2)T + (k/\Gamma(\alpha)^2) \); using Lemma 1, we have

\[ E[X_\epsilon(t) - Z_\epsilon(t)]^2 \leq eT^{2a-1} r_1 \left( 1 + \sum_{n=1}^\infty \frac{r_2 \Gamma(2\alpha - 1)n}{\Gamma(2a-1)n + 1} \right) \]

\[ \leq eT^{2a-1} r_1 \left( 1 + \sum_{n=1}^\infty \frac{r_2 \Gamma(2\alpha - 1)2a(n-1)n}{\Gamma(2a-1)n + 1} \right) \]

\[ \leq eT^{2a-1} r_1 \left( 1 + E_{2a-1,1} \left( r_2 \Gamma(2\alpha - 1)T^{2a-1} \right) \right). \] \[ (60) \]

Select some \( \beta \in (0, 1), L > 0, \) such that, for \( \forall t \in (0, L e^{(-\beta/2a-1)}), \) we obtain

\[ \sup_{0 \leq s \leq Le^{(-\beta/2a-1)}} E[X_\epsilon(t) - Z_\epsilon(t)] \leq Ce^{-\beta}, \] \[ (61) \]

where \( C = r_1 (1 + E_{2a-1,1} \left( r_2 Le^{(-\beta/2a-1)} \Gamma(2\alpha - 1) \right)). \)

Consequently, for \( \forall \delta_1 > 0, \) one can select some \( \epsilon_1 \in (0, \epsilon_2] \) such that, for each \( \epsilon \in (0, \epsilon_1], \forall t \in (0, Le^{(-\beta/2a-1)}), \) we have

\[ \sup_{0 \leq s \leq Le^{(-\beta/2a-1)}} E[X_\epsilon(t) - Z_\epsilon(t)] \leq \delta_1. \] \[ (62) \]

This completes the proof.

Remark 2. Using the definition of \( \rho, \) we obtain

\[ \rho(\mu_\epsilon(t), v_\epsilon(t)) \leq E[X_\epsilon(t) - Z_\epsilon(t)] \leq E[X_\epsilon(t) - Z_\epsilon(t)]^2. \] \[ (63) \]

From the above estimate, we actually obtain

\[ D_T(\mu_\epsilon, v_\epsilon) = \sup_{0 \leq t \leq T} \rho(\mu_\epsilon(t), v_\epsilon(t)) \leq s \sup_{0 \leq t \leq T} E[X_\epsilon(t) - Z_\epsilon(t)]^2 \rightarrow 0, \quad \epsilon \rightarrow 0, \] \[ (64) \]

which means that, as \( \epsilon \rightarrow 0, \mu_\epsilon(t) \) corresponding to \( X_\epsilon(t) \) converges to \( v_\epsilon(t) \) of \( Z_\epsilon(t) \) in \( C([0, T; (M^2_T, \rho)]). \)

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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