Systems of Cosserat–Zhilin in Newtonian Mechanics

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Abstract

Mechanical systems of Cosserat–Zhilin are introduced as the main object of rational (non-relativistic) mechanics on the base of new notions of vector calculus – sliders and screw measures (bi-measures).

The differential equations of motion are derived for different types of Cosserat–Zhilin systems in the case where Stocks theorem is applicable.

The paper defines multiplicative groups which represent the measure of stress as a (well-defined) linear isotropic map of strain tensor or tensor of strain velocities for classical and polar continua in 2– and 3–dimensional cases.

Key words: classical mechanics, continuum mechanics, constitutive equations, mechanical measures, foundations of mechanics, screw theory.

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INTRODUCTION

‘The ancients considered mechanics in a twofold respect; as rational, which proceeds accurately by demonstration; and practical... Rational mechanics will be the science of motions resulting from any forces whatsoever, and of the forces required to produce any motions, accurately proposed and demonstrated’ [1].

The progress of rational mechanics (and physics of XVIII–XIX centuries) is primarily based on working out its mathematical aspects. In 1687, I. Newton published *Philosophiae Naturalis Principia Mathematica* where the leading role of mathematics in rational mechanics is directly pointed out in its title. Newton deliberately almost never used mathematical analysis: use new and unusual methods would jeopardize the credibility of his results. But already in 1736, in *Mechanics*, L. Euler explicitly stressed that ‘full understanding mechanics can be achieved only through mathematical analysis’, thus, emphasizing that mathematics should be put at the forefront and the consideration of the physical aspects only is insufficient.

The necessity of mathematization of mechanics was marked by D’Alembert in 1743: ‘Rational mechanics, like geometry, must be based upon axioms which are obviously true’ [2].

The first system of axioms in mechanics was introduced by I. Newton. Now we have many systems of axioms at hand. The present paper gives one more to represent mechanics as a mathematical science and to determine the widest class of mechanical systems covered by it (may be, it is time to introduce the term of mathematical mechanics by analogy with that of mathematical physics).

I. FUNDAMENTALS OF RATIONAL MECHANICS

‘...the dynamics of a continuous system must clearly include as a limiting case (corresponding to a medium of density everywhere zero except in one very small region) the mechanics of a single material particle. This at once shows that it is absolutely necessary that the postulates introduced for the mechanics of a continuous system should be brought into harmony with the modifications accepted above in the mechanics of the material particle’ [3].

In other words, we must consider the various branches of general mechanics from a unified point of view.
A. Primitive concepts of rational mechanics

The following ‘experimental facts’ lie at the foundation of rational mechanics [4]:

1. all the natural phenomena occur in space and time.

2. Galileo’s principle of relativity: ‘there exist coordinate systems (called inertial) possessing the following two properties:
   - all the laws of nature at all moments of time are the same in all inertial systems;
   - all coordinate systems in uniform rectilinear motion with respect to an inertial one are themselves inertial’.

3. Newton’s principle of determinacy: the initial state of a mechanical system uniquely determines all of its motion.

It is easy to see that all of the above is nothing more than constants of the science language. That is why it is necessary to clear up their mathematical essence. In particular, we must answer the following questions:

- what are the laws of nature, about of which Galileo’s principle says?
- what is the group of transformations w.r.t. of which the laws are invariant’?
- what do we mean under ‘mechanical system’?
- what main types of mechanical systems do we have?

**Galilean space–time structure.** In what follows, we shall use $n$–dimensional affine space $A^n$ modeled on $n$–dimensional vector space $V^n$, $n$–dimensional Euclidean space $R^n$ ($R = R^1$ is the set of all real numbers).

Define the Galilean space–time structure as the quadruple $G = \{V^n, A^n, \tau, g\}$ where [4]

- $\tau: V^n \to R$ is a surjective linear mapping called time one, and
- $g = \langle \cdot, \cdot \rangle$ is an inner product on $\ker\{\tau\} (= V^{n-1})$.

The space $A^n$ with the Galilean space–time structure is called Galilean (note that the term ‘Galilean’ is merely traditional and should not be regarded as an attribution to Galileo).
Time is a linear mapping $\tau : V^n \rightarrow R$ from the vector space of parallel displacements of $A^n$ to the real ‘time axis’. We shall denote the range of $\tau$ by $T \subseteq R$. The time interval from event $a \in A^n$ to event $b \in A^n$ is the number $\tau(b - a)$ (it is plain that $b - a \in V^n$). If $\tau(b - a) = 0$, then the events $a$ and $b$ are called simultaneous.

The set of events simultaneous with a given event forms $n - 1$–dimensional affine space $A^{n-1} \subset A^n$ modeled on $\text{ker}\{\tau\}$. It is called space of simultaneous events.

There is the group of affine transformations of the space $A^n$ which preserve the Galilean time–space structure. The elements of this group are called Galilean transformations. They preserve intervals of time and the distance between simultaneous events [4].

**Theorem 1.** Each Galilean transformation is movement of the space of simultaneous events, accompanied by a shift of the origin of time [4, 5].

The inner product $\langle \cdot, \cdot \rangle$ (in Galilean space–time) enables one to pass from the affine space $A^n$ to Euclidean space $R^n$ with the distance $\rho(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}$ between points $x$ and $y \in A^n$.

The bijective map $A^n \rightarrow R^{n-1} \times T$ is called frame of reference [4, 5] (here $R^{n-1}$ is also a space of simultaneous events). For each frame of reference the corresponding space $R^{n-1} \times T$ is Galilean. That is why we shall call any frame of reference Galilean, too.

Define world–line as a curve in $A^n$ whose image in $R^{n-1} \times T$ associates one point $x(t) \in R^{n-1}$ to each instant $t \in T$. It means that a world–line does not have simultaneous points.

A collection of non–intersectional world–lines forms world–tube.

**Remark 1.** Intersections of world–lines represent collisions or the creation or destruction of bodies or elements of bodies. In specific mechanical theories such intersections are usually excluded (the principle of impenetrability) altogether or allowed as exceptional cases subject to specified conditions.

Let us fix a world–tube $\tilde{\Lambda} \subset A^n$ and call it universe.

**Remark 2.** As well as in probability theory [4], any universe is separately specified for every problem under consideration.

A given world–tube $\Lambda \subset \tilde{\Lambda}$, the world–tube $\Lambda^e = \tilde{\Lambda} \setminus \Lambda$ is called environment of $\Lambda$ in the universe.
The universe $\hat{\Lambda}$ defines the family $\{\hat{\Lambda}_t \subset \mathbb{R}^{n-1}, t \in T\}$. For any world–tube $\Lambda \subset \hat{\Lambda}$ we have the family $\{\Lambda_t \subset \hat{\Lambda}_t, t \in T\}$.

Let $\sigma_{n-1}$ be $\sigma$–algebra on the space $\mathbb{R}^{n-1}$. We shall use the following Borel measure

$$
\mu_{n-1}(A) = \mu_{ac}(A) + \mu_{pp}(A), \quad A \in \sigma_{n-1}
$$

(1)

where $\mu_{ac}(A)$ is the absolutely continuous component w.r.t. Lebesgue measure and $\mu_{pp}(A)$ is the pure point (discrete) component presented as $\mu_{pp}(A) = \sum_k \mu_k(x_k)$ for points $x_k \in A$ whose are called pure, the others being called continuous.

**Remark 3.** In some cases where, e.g., no increment of mass is, we shall assume that if a point of $\hat{\Lambda}_t \subset \mathbb{R}^{n-1}$ is pure (or continuous) at some time instant $t$, all other points of the corresponding world–line ($\forall t \in T$) are also pure (or continuous), too.

**Main axioms.** With any point $x(t) \in \hat{\Lambda}_t \subset \mathbb{R}^{n-1}$ we associate the position radius–vector $r_x(t) = (O, x(t))$ w.r.t. the origin point $O \in \mathbb{R}^{n-1}$. Define the translation velocity $v_x = \dot{r}_x(t)$.

**Remark 4.** We shall denote full derivatives by $t$ with the help of the superscript $\cdot$, e.g., for any function $f = f(x(t), t)$ we have $f^\cdot = \frac{\partial}{\partial t} f + (\text{div } f) x^\cdot(t)$.

Let us consider an orthonormal basis $e_x$ at the point $x(t) \in \mathbb{R}^{n-1}$. Rotation of the basis $e_x$ can be characterized by torque $\mu_x$ (angular velocity – spin, angular momentum vector of mass unit or rotation tensor, etc.).

**Definition 1.** The map of $T$ into the set $\Lambda_t$ and the vectors $v_x$ and $\mu_x$, associated with it, is called motion. Mechanical interaction is something that generates motion.

Hereinafter we shall use the new notions of vector calculus – sliders and screw measures (bi–measures) (see Appendix 1).

Introduce the measure $\mathcal{P}$ as a screw having values

$$
\mathcal{P}(\Lambda_t) = \int \chi_{\Lambda_t} l^{p_x,q_x} \mu_{n-1}(dx), \quad \Lambda_t \subset \hat{\Lambda}_t, \ t \in T
$$

(2)

where $\chi_A$ is the characteristic function of $A \in \sigma_{n-1}$; the slider $l^{p_x,q_x}$ is defined by the following relation (see also [8])

$$
\begin{pmatrix}
  p_x \\
  q_x \\
  \mu_x
\end{pmatrix} = \theta_x \begin{pmatrix}
  v_x \\
  \mu_x
\end{pmatrix}
$$

(3)
Here $\theta_x$ is a non-negative defined, symmetric 2nd-order tensor.

For any two world–tubes $\Lambda$ and $\Lambda'$ with $\Lambda_t$ and $\Lambda'_t$ in the space of of simultaneous events, respectively, define the skew signed field bi–measure $\Phi$ as a screw (by each argument) having values $\Phi(\Lambda_t, \Lambda'_t)$.

**Axiom 1.** There exist measures $\mathcal{P}$ and $\Phi$ which describe motion.

**Definition 2.** These measures are called kinetic and dynamic measures of motion, respectively.

There are special frames of reference that help us to distinguish between the postulated kinetic and dynamic measures of motion among various measures and bi–measures.

**Axiom 2.** There exists a Galilean frame of reference where the postulated kinetic and dynamic measures of motion are connected by the following relation (see also [1, 8–11])

$$\frac{d}{dt}\mathcal{P}(\Lambda_t) = \mathcal{F}(\Lambda_t), \quad \mathcal{F}(\Lambda_t) \overset{\text{def}}{=} \Phi(\Lambda_t, \Lambda^e_t), \quad \Lambda_t \subset \tilde{\Lambda}_t, \ t \in T \quad (4)$$

As result we may introduce the following notions:

1. the above mentioned frame of reference is called inertial;
2. the aggregate $\alpha = \{\Lambda_t \subset \tilde{\Lambda}_t, \mu_{n-1}, \theta_x, x \in \Lambda_t, \mathcal{P}, \mathcal{F}, \forall t \in T\}$ is called mechanical system of Cosserat–Zhilin;
3. $\mathcal{F}$ is called measure of impressed action of the environment $\Lambda^e$ on $\Lambda$ or force (according to Glossary, Earth Observatory, NASA: force is any external agent that causes a change in the motion of a mechanical system, or that causes stress in a fixed mechanical system);
4. the points of $\Lambda_t$ and the vectors $v_x$ and $\mu_x$, associated with them, constitute state of a mechanical system;
5. relation (4) is called motion equation;
6. the set $\Lambda_t$ is called (actual) shape undergone by the mechanical system at $t \in T$.

Hereinafter we assume that frames of reference are inertial.

**Remark 5.** Introducing the kinetic and dynamic measures as screws we follow L. Euler who has opened a new era in developing of Newtonian mechanics: two independent Laws of
Dynamics are stated for the first time in ‘New method of determination of motion of rigid bodies’ [9].

Axiom 3. The tensor $\theta_x$ and the measure $\Phi$ do not dependent on frames of reference (see also [12]).

Remark 6. The tensor $\theta_x$ defines tensor measure of inertia $\Theta$ having values

$$\Theta(\Lambda_t) = \int \chi_{\bar{\Lambda}_t} \theta_x \mu_{n-1}(dx), \quad \Lambda_t \subset \bar{\Lambda}_t, \; t \in T$$

and measure of kinetic energy (scalar measure of motion) $T$ having values (see also [13])

$$T(\Lambda_t) = \int \chi_{\bar{\Lambda}_t} \left(\frac{v_x}{\mu_x}\right)^T \theta_x \left(\frac{v_x}{\mu_x}\right) \mu_{n-1}(dx), \quad \Lambda_t \subset \bar{\Lambda}_t, \; t \in T$$

Remark 7. In the rational mechanics, the views of Aristotle were dominant for over two millennia, as long as Galileo did not introduce his principle of inertia:

‘any isolated (lonely in the world) material point preserves its present state, whether it be of rest or of moving uniformly forward in a straight line in the absolute space’.

Since it is impossible to determine the motion of an isolated point (body) relative to the absolute space, we could use the concept of reference frame as a reference body, equipped with a clock: (see, e.g., [8]):

‘A body of reference with respect to which trajectories of an isolated (lonely in the world) particle are straightforward or a point, called inertial reference one’.

At the same time, forgetting how it is possible to talk about the single particle in the world, when another body – the body of reference – is entered.

But the trouble does not come alone: straight lines are passed in straight lines under an arbitrary affine transformation, i.e., according to the above definition, new reference frames will be also inertial reference ones. If some new frame is accepted as the original one, we see that Newton’s Second Law can not be executed in this frame of reference. Thus such a classical definition of reference frames as well as Newton’s First Law proves to be unsatisfactory (see also [14]).
B. The law of universal gravity

If the first two laws of motion, Newton had predecessors (the authorship of their own, he did not claim – see page 71 (p. 50) in the Russian translation of Newton’s Principia). The third law is wholly owned by Newton (predecessors to date nobody has been able to specify). Without it, there would be neither the equation of rigid body motion nor the law of universal gravity (see page 13 in the Russian translation of Newton’s Principia):

‘A particle attracts every other particle in the universe using a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them’.

Below we shall consider mechanical systems where

\[ \theta_x \overset{\text{def}}{=} \rho_x \begin{bmatrix} I & A \\ A^T & B \end{bmatrix} \]  

(5)

Here \( I \) is the unit tensor; \( A \) and \( B \) are 2nd-order tensors; \( \rho_x \) is a non-negative function for any \( x \in \Lambda_t \subset \tilde{\Lambda}_t, \ t \in T \).

The function \( \rho_x \) generates the measure \( \mathcal{M} \), having the values

\[ \mathcal{M}(\Lambda_t) = \int \chi_{\Lambda_t} \rho_x \mu_n(dx), \ \ \Lambda_t \subset \tilde{\Lambda}_t, \ t \in T \]

The measure \( \mathcal{M} \) is called mass. The set \( \tilde{\Lambda}_t \subset \Lambda_t \) is called set of concentration of the measure \( \mathcal{M} \) if \( \mathcal{M}(\Lambda_t) = 0 \) for any set \( \Lambda_t \subset \tilde{\Lambda}_t \setminus \tilde{\Lambda}_t^c \). We shall assume that relation (51) is true only for \( \Lambda_t \subset \tilde{\Lambda}_t, \ t \in T \).

The measure \( \mathcal{M} \) is introduced as an integral defined over actual shapes \( \Lambda_t \) undergone by a mechanical system. That is why there is the following relation (see Appendix 2)

\[ \frac{d}{dt} \mathcal{M}(\Lambda_t) = \int \chi_{\Lambda_t} \left[ \frac{d}{dt} \rho_x + (\text{div } v_x) \rho_x \right] \mu_n(dx) + \sum_k \left( \frac{d}{dt} \rho_k \right) \mu_{pp}(x_k), \ \ \Lambda_t \subset \tilde{\Lambda}_t, \ t \in T \]

We shall assume that the function \( \rho_x = \rho(x, t) \) is defined by the continuity equation (see also Appendix 2)

\[ \frac{d}{dt} \rho_x + (\text{div } v_x) \rho_x = \frac{\partial}{\partial t} \rho_x + \text{div}(v_x \rho_x) = \nu_x \]

in continuous points and \( \frac{d}{dt} \rho_k = \nu_k \) in pure points (here we may referred terms such as generation (\( \nu_x > 0 \)) or re-movement (\( \nu_x < 0 \)) to ‘sources’ and ‘sinks’, respectively).
Then
\[
\frac{d}{dt} \mathcal{M}(\Lambda_t) = \Delta \mathcal{M}(\Lambda_t) \overset{\text{def}}{=} \int \chi_{\Lambda_t} \nu_x \mu_{ac}(dx) + \sum_k \nu_k \mu_{pp}(x_k), \: \Lambda_t \subset \tilde{\Lambda}_t, \: t \in T
\]

We shall give a screw version of the law of universal gravity. To this end let us define the measure \( \Gamma \), having the values
\[
\Gamma(\Lambda_t, \Lambda_t^e) = \int \chi_{\Lambda_t} \gamma_x \rho_x \mu_{n-1}(dx), \quad \gamma_x = \gamma \int \chi_{\Lambda_t^e} r_{x,y} \frac{\rho_y \mu_{n-1}(dy)}{||r_{x,y}||^3}
\]
on the sets \( \Lambda_t \in \sigma_{n-1} \). Here \( \gamma \) is the positive constant, \( r_{x,y} \in \mathbb{V}^{n-1} \) is the translation vector from a point \( x \) to \( y \in \mathbb{R}^{n-1} \).

**Definition 3.** The universal gravity is that is expressed by the measure \( \mathcal{F} \) such that \( \mathcal{F}(\Lambda_t) \overset{\text{def}}{=} \Gamma(\Lambda_t, \Lambda_t^e) \). The screw measure \( \mathcal{F} \) is called measure of graviting action of \( \alpha^e \) upon \( \alpha \) [15].

**Definition 4.** In the case of relation (5), the mechanical interaction is that the universal gravity generates.

The concept of universal gravity is the great intellectual achievement that Newton represented in the most outstanding book in the history of science: *Philosophiae Naturalis Principia Mathematica* or, in modern language, *Mathematical Foundations of Physics*. By deriving Kepler’s laws of planetary motion from his mathematical description of gravity, and then using the same principles to account for unknown before hyperbolic and parabolic orbits of celestial bodies, the tides, the precession of the equinoxes, and other phenomena, Newton demonstrated that the motion of objects on Earth and of celestial bodies could be described by the same principles.

It is paradoxical, and even insulting to anyone who is familiar with the revolution produced by Newton in science, that, in the textbooks of theoretical mechanics, the law of universal gravity is not regarded. Sporadically, a particular case of the law, as the inverse square law, is derived from Kepler’s three laws.

**Remark 8.** From the above follows that the inertia appearance is due not to inborn force of the matter, included in itself but to mechanical systems belonging to the universe ([16]).

**C. Concept of body**

We shall use the following convention:
body is that takes some shapes $\Lambda_t \subset \tilde{\Lambda}_t$ in the space at some instants of time (cf. ‘every sensible body is in place’ – Aristotle, Physics, III, 4, 208b27).

The concept of body is the subject of various formalizations. For example, one may represent a body as a point–wise set, a differentiable manifold, a topological or measure space [15, 17] where a map into the space of shapes is considered.

**Remark 9.** These definitions follow from Plato’s idea on the existence of two worlds: the world of ideas (eidos) and the world of things, or forms. And then we have only ‘photo’ of a body at each time while the body itself is out of Plato’s cave.

But there is a small obstacle: we must also transfer masses and forces to body shapes. If we do it in some way then the construction – body with mass and force – loses the primitive nature. In order to work out a mathematical theory we have all the necessary: shapes with kinematic, kinetic and dynamic structures attributed by them. While the concept of a mechanical system has strict mathematical sense, the concept of body has only descriptive character, being a tribute of the very seminal tradition.

D. Generalization of the mechanical system concept

The non–trivial nature of the mechanical system concept can be seen from the fact that we may postulate the following relation (see also [8])

$$\frac{d}{dt} \mathcal{P}(\Lambda_t) = \mathcal{F}(\Lambda_t) + \Delta \mathcal{P}(\Lambda_t) + \mathcal{R}(\Lambda_t), \quad \Lambda_t \subset \tilde{\Lambda}_t, \ t \in T \tag{6}$$

where the signed field measure $\Delta \mathcal{P}$ is so called increment velocity of the measure $\mathcal{P}$, the signed field measure $\mathcal{R}$ is so called constraint action.

Below we shall assume that the measure $\mathcal{R}$ is formed by internal and external constraints: $\mathcal{R} = \mathcal{R}_{\text{int}} + \mathcal{R}_{\text{ext}}$. Here the signed field measures $\mathcal{R}_{\text{int}}$ and $\mathcal{R}_{\text{ext}}$ have the values $\mathcal{R}_{\text{int}}(\Lambda_t)$ and $\mathcal{R}_{\text{ext}}(\Lambda_t)$ on the sets $\Lambda_t$, respectively.

II. IMPLEMENTATION OF THE AXIOMS ON EXAMPLES OF MAIN TYPES OF COSSERAT–ZHILIN SYSTEMS

‘The goal of Newton was to give an answer to the question whether there is a simple rule for calculating the total movement of the heavenly bodies of our
planetary system at a given state of motion of all the bodies in a given time? From observations of Tycho Brahe, Kepler deduced empirical laws of planetary motion but they were demanded an explanation. Today, everyone knows what a great, truly bee, hard work has been required to establish these laws, on the basis of empirically determined orbits. But few who imagines the genius of the method by which Kepler has defined the true orbit, based on the apparent, i.e., of the observed motions of the Earth. These laws provide a complete description of the motion of the planets around the Sun: elliptical orbits, equality sectorial velocity ratio between the semi-major axes and periods of treatment. But these laws do not satisfy the requirement of a causal explanation. They were the three logically independent of each other rules deprived of any internal connection. The third law cannot be quantified unequivocally transferred to another, other than the Sun, the central body (there is, for example, no connection between the orbital period of the planet around the Sun and the orbital period of the satellite around its planet). But the important thing is that the laws of motion are generally not possible to derive from the state of motion at some point in time a different state in time immediately following the first. In modern terminology, we would say that they are integral laws and not differential.

*Differential law is the sole form of causal explanation, which can fully meet modern physics’* [18].

Below, we are about to show the implementation of the axioms on examples of main types of Cosserat–Zhilin systems and derive their equations of motion in the case where Stocks theorem is applicable (see [17]). These equations are invariants of the generalized Galilean group (see Appendix 3 and [15]).

In mechanics, the most important cases of motion are in one–, two– and three–dimensional spaces. For the sake of brevity, hereinafter we shall only consider the three–dimensional case.

**A. A body–point**

Let the image of a world–line $\Lambda \subset \tilde{\Lambda}$ be the curve $\{x(t) \in \Lambda, t \in T\}$ in $\mathbb{R}^{n-1} \times T$. Assume that the points $x$ of $x(t)$ are pure. Then the corresponding mechanical system is called *body–point*. 
Remark 10. This concept is not the same as in \([12]\) (see also \([8]\)).

Remark 11. In physics, if a body has an infinitely small size and a finite mass, it is called a material point or mass–point. We could put an electron on the role of material point because its size is extremely small, and it has some mass. However, motion of electrons can be not only translational, but also rotational. The latter does not meet the concept of material point (its rotation is not defined). Thus, a priori, we cannot consider a body with an infinitely small size and a finite mass as a material point. That is why we hope that the motion of an electron may be described as that of a body–point with finite mass and charge. The free motion of such a mechanical system may not be uniform and rectilinear \([8]\).

For body–points, in the case \([5]\), from relations \([3]\) and \([6]\) follows that
\[
\frac{d}{dt} \rho_x l_{x,p} = l_{x,\alpha}\beta_x, \quad \left( \begin{array}{c} \tilde{p}_x \\ \tilde{q}_x \end{array} \right) = \left[ \begin{array}{cc} I & A \\ A^T & B \end{array} \right] \left( \begin{array}{c} v_x \\ \mu_x \end{array} \right), \quad x \in x(t) \subset \Lambda_t, \ t \in T
\]

or (see also Appendix 2)
\[
(\rho_x \frac{d}{dt} + \nu_x) \left[ \begin{array}{cc} I & A \\ A^T & B \end{array} \right] \left( \begin{array}{c} v_x \\ \mu_x \end{array} \right) = \left( \begin{array}{c} \alpha_x \\ \beta_x \end{array} \right), \quad x \in x(t) \subset \Lambda_t, \ t \in T
\]

B. A polar medium

We shall assume that all points of \(\Lambda_t \subset \bar{\Lambda}_t\) are continuous and \(\Lambda_t\) has the surface \(\partial \Lambda_t\) which is Lyapunov’s simple closed one (see also \([8]\)).

Introduce so called measure \(F\) of mass action, having the values
\[
F(\Lambda_t) = \int \chi_{\Lambda_t} \rho_x l_{x,\gamma_x,\delta_x,\mu_x} dx, \quad \Lambda_t \subset \bar{\Lambda}_t, \ t \in T
\]

Let constraints being in a small vicinity of \(x \in \Lambda_t\) cause the measure of stress having the values \([19]\)
\[
R_{int}(\Lambda_t) = \int \chi_{\partial \Lambda_t} l_{x,P_x,\mu_x} dx, \quad \Lambda_t \subset \bar{\Lambda}_t, \ t \in T
\]

Hereinafter \(\mu_2\) is the restriction of \(\mu_3\) on the surface \(\partial \Lambda_t\), \(n_x\) is the normal to this surface; \(P_x\) and \(Q_x\) are 2nd–order tensors.

**Definition 5.** The mechanical system \(\alpha = \{\Lambda_t \subset \bar{\Lambda}_t, \mu_3, \nu_x, \theta_x, F, R, R_{ext}, R_{int}, \forall t \in T\}\) is called polar medium \([20]\).
For the sake of brevity assume that $R_{\text{ext}} \equiv 0$.

The following statement will be used below.

**Lemma 1.** Let the tensor–function $P_x$ be continuously differentiable in the fit region $\Lambda_t$. Then

$$\int_{\chi_{\Lambda_t}} \text{div}(R_{y,x} P_x) \mu_3(dx) = \int_{\chi_{\Lambda_t}} (R_{y,x} \text{div} P_x + \tau_x) \mu_3(dx), \quad \Lambda_t \subset \tilde{\Lambda}_t, \ t \in T$$

where the skew 2nd–order tensor $R_{y,x}$ is generated by the translation vector $r_{y,x}$: $R_{y,x} = r_{y,x}^\times$; $\tau_x$ is the dual vector to $P_x^T - P_x$ (see Appendix 1).

In the case where Stocks theorem is applicable, due to lemma 1, from relation (6) follows (see Appendix 2)

$$\left( \rho_x \frac{d}{dt} + \nu_x \right) \begin{bmatrix} I & A \\ A^T & B \end{bmatrix} \begin{pmatrix} v_x \\ \mu_x \end{pmatrix} = \rho_x \begin{pmatrix} \gamma_x \\ \delta_x \end{pmatrix} + \text{div} \begin{pmatrix} P_x \\ Q_x \end{pmatrix} + \begin{pmatrix} 0 \\ \tau_x \end{pmatrix}$$

where $o$ is the null vector.

Take a point $y(t)$ in a small vicinity of $x(t) \in \Lambda_t$ at an instant $t \in T$ and define their radius–vectors $r_x$ and $r_y$ and the vector $h(t) = r_y - r_x$. Then there is the following relation

$$v_y(t) \simeq v_x(t) + dv_x/dr_x h(t)$$

Define the tensor $S_x(t)$ as the solution of the following equation

$$S_x'(t) = dv_x/dr_x$$

where its initial data are defined by so called *deformed* state of the medium.

**Definition 6.** $S_x$ and $S_x'$ are called strain tensor and tensor of strain velocities at the point $x \in \Lambda_t$ at the instant $t$, respectively [13].

Denote the tensor $S_x$ or $S_x'$ as $Z_x$.

**Definition 7.** The polar medium is called that of Hooke class if the tensors $P_x$ and $Q_x$ are linear isotropic maps of $Z_x$, i.e., invariant w.r.t. Galilean group of transformations.

**Remark 12.** If $Z_x = S_x$ the medium is called elastic material, if $Z_x = S_x'$ the medium is called viscous fluid [22].

**Definition 8.** We shall call an isotropic matrix function of entries of $Z$ well–defined if it is invertible (see also [13]).
The set of invertible linear isotropic matrix functions forms a multiplicative group (see Appendix 4).

Regarding the implementation of polar media – see, e.g., \[17, 21, 23\] and Appendix 5.

C. A mass–point

In what follows, we shall assume that \(A\) and \(B\) are zero in relation (5), i.e., we shall use homogeneous sliders and the measure \(P\) having the values

\[
P(\Lambda_t) = \int \chi_{\Lambda_t} \rho_x l^{\xi_x} \mu_3 (dx), \quad \Lambda_t \subset \tilde{\Lambda}_t, t \in \mathbf{T}
\]

Assume that the increment velocity of \(P\) is the measure \(\Delta P\) having the values

\[
\Delta P(\Lambda_t) = \int \chi_{\Lambda_t} l^{\xi_x} \mu_3 (dx), \quad \Lambda_t \subset \tilde{\Lambda}_t, t \in \mathbf{T}
\]  (8)

where the slider \(l^{\xi_x}\) is its density.

Consider a world–line \(\Lambda \subset \tilde{\Lambda}\) whose image in \(\mathbb{R}^3 \times \mathbf{T}\) generates the curve \(\{x(t) \in \Lambda_t, t \in \mathbf{T}\}\).

Assume that the points \(x(t)\) are pure.

Let \(f_x\) be the impressed force acting at the point \(x = x_k \in \Lambda_t\) with the mass \(M = \rho_x \mu_{pp}\).

Then the mechanical system \(\alpha = \{x(t) \in \Lambda_t, \mu_{pp}, f_x, \nu_x, \rho_x, \xi_x, \forall t \in \mathbf{T}\}\) is called mass–point.

From relation (6) follows that

\[
(\rho_x \frac{d}{dt} + \nu_x) v_x = f_x + \xi_x
\]  (9)

If \(\nu_x \equiv 0\) and \(\xi_x \equiv 0\), then equation (9) is known as Newton’s Second Law.

If \(\nu_x \neq 0\) and \(\xi_x = \nu_x u_x\) where \(u_x\) is the velocity of mass gain or loss, then equation (9) is known as that of Meshchersky [14].

Remark 13. A classical example of mass–points with constraints is the mechanical system known as pendulum.

D. A rigid body

‘One might try to derive the laws of the motion of rigid bodies by a limiting process from a system of axioms depending upon the idea of continuously varying conditions of a material filling all space continuously’ [24]. Let us do it.
For the sake of brevity assume that $\Delta P$ and $R_{ext} \equiv 0$.

The mechanical system $\alpha = \{\Lambda_t \subset \tilde{\Lambda}_t, \mu_3, \rho_x, F, R_{int}, \forall t \in T\}$ is called rigid body if

- the sets $\Lambda_t$ are bounded and closed;
- the constraints applied on its points keep distances between them not changing with time;
- the internal constraints are ideal \[25\].

A rigid body may comprise continuous and pure points.

Below we are going to obtain its motion equation with using so called quasi–velocities (Newton–Euler equations in quasi–velocities), some parametrizations of rotation matrices and \textit{generalized coordinates and velocities} (Lagrange equation of II kind).

\textbf{Newton–Euler equation.} At any time instant $t^*$ consider the set $\Lambda_{t^*}$. Let a Cartesian frame $E_p$ be attached to the set under consideration. It is plain that the frame takes the same position in all sets $\Lambda_t$. In the frame these sets are immobile, coincide one with another and form the set noted as $\Lambda_p$ in the frame $E_p$. We shall say that the frame $E_p$ is attached to the rigid body $\alpha_p$. Let $v_{0,p}$ and $\omega_{0,p}$ be the translation velocity and the angular velocity of $E_p$ w.r.t. $E_0$.

It is plain that the vectors $\omega_{0,p}$ and $v_{0,p}$ generate the inhomogeneous slider

$$V_{0,p} = \{\omega_{0,p}, v_{0,p} + \omega_{0,p} \times r_{p,x}, \forall x \in \Lambda_p\}$$

known as \textit{kinematic}. The corresponding twist defines the reduction $V_{0,p}^{tw,p} = \text{col}\{v_{0,p}^p, \omega_{0,p}^p\}$ (here we use the fact that the vectors $\omega_{0,p}$ and $v_{0,p}$ can be considered as bounded at the point $O_p$). This reduction is called \textit{vector of quasi–velocities} while its component $\omega_{0,p}^p$ is known as \textit{angular quasi–velocity} \[15, 26\].

\textbf{Remark 14.} Hereinafter one shall mark coordinate representations in any coordinate frame, e.g., $E_p$ with the help of the superscript $^p$.

\textbf{Lemma 2.} There is the following relation \[15\]

$$l^{ex,wr,p} = \Theta_{p,x}^{p} V_{0,p}^{tw,p}, \quad \Theta_{p,x}^{p} = \begin{bmatrix} I & -r_{p,x}^{xp} \\ r_{p,x}^{xp} & -r_{p,x}^{xp} r_{p,x}^{xp} \end{bmatrix}$$
Proof. The relation is true as

\[ l^{w_x, w_p, 0} = \begin{bmatrix} I & (v^p_{0, p} - r_x p, w_p, 0) \end{bmatrix} \begin{bmatrix} I \\ r_x p, w_p \end{bmatrix} = \begin{bmatrix} I & -r_x p, w_p \end{bmatrix} \begin{bmatrix} v^p_{0, p} \\ -r_x p, w_p \end{bmatrix} \]

According to the rigid body definition the internal constraints are considered as ideal and thus

\[ R_{int}(\Lambda_t) = 0 \]

From relations (6) and (8) follows that (see Appendix 2)

\[ \int \chi_{\Lambda_t} (p_x d + \nu_x) l^{w_x, w_p, 0} \mu_3(dx) = \int \chi_{\Lambda_t} l^{f_x + \xi_x, w_p, 0} \mu_3(dx) \]

or

\[ \int \chi_{\Lambda_p} L^{w_p, 0}_{0, p} [\rho_x (\Theta^p_{o, p} d + \Phi^w_{o, p} \Theta^p_{o, p, w_p}) + \nu_x \Theta^p_{o, p, w_p}] V^{tw, p}_{0, p} \mu_3(dx) = \int \chi_{\Lambda_p} l^{f_x + \xi_x, w_p, 0} \mu_3(dx) \]

where the matrices \( L^{w_p, 0}_{0, p} \) and \( \Phi^w_{o, p} \) are defined in Appendix 3.

As the twist reduction \( V^{tw, p}_{0, p} \) and the matrices \( L^{w_p, 0}_{0, p} \) and \( \Phi^w_{o, p} \) do not depend on points \( x \in \Lambda_p \) and the matrix \( \Theta^p_{o, p, w_p} \) is time–invariant, the following statement is true.

**Theorem 2.** The motion of \( \alpha_p \) (w.r.t. \( E_0 \) in the frame \( E_p \)) is described by the (Newton–Euler) equation in quasi–velocities

\[ \Theta^p_{o, p} d V^{tw, p}_{0, p} + (Q^p_{o, p} + \Phi^w_{o, p} \Theta^p_{o, p, w_p}) V^{tw, p}_{0, p} = F^{w_p} \quad (10) \]

where

\[ \Theta^p_{o, p} = \int \chi_{\Lambda_p} \Theta^p_{o, p, x} \mu_3(dx), \quad Q^p_{o, p} = \int \chi_{\Lambda_p} \Theta^p_{o, p, \nu_x} \mu_3(dx), \quad F^{w_p} = \int \chi_{\Lambda_p} l^{f_x + \xi_x, w_p, 0} \mu_3(dx) \]

It is easy to see that the matrices of relation (10) depend on the rotation matrix (and translation and angular quasi–velocities, too). That is why equation (10) must be considered along with the Poisson kinematic relation – see below (16).

We may reduce the order of this system by using various parametric representations of rotation matrices.

**Remark 15.** Systems of consecutively connected rigid bodies are considered in [27].
E. A continuum

**Cauchy continuum.** We shall assume that all points of $\Lambda_t \subset \tilde{\Lambda}_t$ are continuous.

Introduce so called (homogeneous) measure $\mathcal{F}$ of mass action having the values

$$\mathcal{F}(\Lambda_t) = \int \chi_{\Lambda_t} \rho_x l g_{x,wr} \mu_3(dx), \quad \Lambda_t \subset \tilde{\Lambda}_t, \quad t \in T$$

on the sets $\Lambda_t$ with the density $g_x$.

Due to constraints being in a small vicinity of $x \in \Lambda_t$ cause the measure $\mathcal{R}_{int}$ of contact action or stress having the values

$$\mathcal{R}_{int}(\Lambda_t) = \int \chi_{\partial \Lambda_t} l P_{x,wr} n_x \mu_2(dx), \quad \Lambda_t \subset \tilde{\Lambda}_t, \quad t \in T$$

on the sets $\Lambda_t$ (here $P_x$ is stress tensor).

The mechanical system $\alpha = \{\Lambda_t \subset \tilde{\Lambda}_t, \mu_3, \nu_x, \rho_x, \mathcal{F}, \mathcal{R}_{ext}, \mathcal{R}_{int}, \Delta \mathcal{P}, \forall t \in T\}$, satisfying to relation (6) with homogeneous screw measures $\mathcal{P}(\Lambda_t), \mathcal{F}, \mathcal{R}_{ext}, \mathcal{R}_{int}$ and $\Delta \mathcal{P}$, is called Cauchy continuous medium or continuum.

For the sake of brevity assume that $\Delta \mathcal{P}$ and $\mathcal{R}_{ext} \equiv 0$. Then due to the lemma from relation (6) follows (see Appendix 2)

$$(\rho_x \frac{d}{dt} + \nu_x) v_x = \rho_x g_x + \text{div} P_x, \quad P^T_x = P_x$$

and thus the stress tensor $P_x$ has to be symmetric.

**Continuum of Hooke class.** As we will further use the divergence of $S_x$ and $S_x^T$, we do not take into account their skew parts in its calculation. Denote the symmetric tensors \( \frac{1}{2}(S_x + S_x^T) \) or \( \frac{1}{2}(S_x^r + S_x^{rT}) \) as $Z_x$.

**Definition 9.** The Cauchy continuum is called continuum of Hooke class if the tensor $P_x$ is a linear isotropic map of $Z_x$, i.e., invariant w.r.t. Galilean group of transformations (see also Appendix 4).

**APPENDIX 1: SLIDERS AND SCREWS**

In mechanics there is mainly absent the understanding that motion of bodies and interaction between them can be described with the help of screws. It is considered as conventional
that ‘being very attractive representation of a system of forces and rigid body motions with
the help motors and screws, nevertheless it has no essential practical value’ and that
the screw calculus is not adapted for the description of continuum motion.

Contrary to this view, we have demonstrated above that screw calculus is rather useful
and convenient tools in mechanics (see also [15] and author’s paper ‘On Foundations of
Newtonian Mechanics’, arXiv:1012.3633).

**Sliders.** Due to the Great Soviet Encyclopedia, v. 5 (Moscow: Soviet Encyclopedia, 1971),
’screw calculus is the section of vector calculus in which operations over screws are studied.
Here the screw is called the pair of vectors \( \{p, q\} \), which is bounded at a point \( O \) and satisfied
to conditions: at transition to a new point \( O' \) the vector \( p \) does not change, and the vector \( q \)
is replaced with a vector \( q' = q - (O, O') \times p \) where \( \times \) means cross–product. The notion
of the screw is used in the mechanics (the resultant \( f \) of a force system and its main moment
\( m \) form the screw \( \{f, m\} \), and also in geometry (in the theory of ruled surfaces)’ (see also
[11]).

The definition given above is not entirely satisfactory (the screw is not the pair of vectors
\( \{p, q\} \)), but it allows to focus on the following elements: three vectors \( p, q \) and \( r \), as well as
a bi–linear antisymmetric (skew) map \( R(r, p) = r \times p \) and the relation defining the vector \( q' \)
at the point \( O' \). This observation permits us to avoid the geometric constructions, which
are the starting point for the conventional screw theory (see, e.g., [29]), and to offer a simple
(algebraic) version of the theory of screws.

Let \( P \) and \( Q \) be some (polar and pseudo, respectively) tensor fields on \( \mathbb{R}^n \). A given point
\( x \in \mathbb{R}^n \) let us define the translation vector \( r_{y,x} \in \mathbb{V}^n \) from a point \( y \in \mathbb{R}^n \) to \( x \). Introduce
a bi–linear map \( R(\mathbb{V}^n, P) : \mathbb{V}^n \times P \rightarrow Q \) as well as the following relations

\[
p_y = p_x \in P, \quad q_y = q_x - R(r_{y,x}, p_x) \in Q, \quad y \in \mathbb{R}^n
\]

(11)

**Definition 10.** The element \( l^{p_x,q_x} = \{p_y \in P, q_y \text{ and } R(r_{y,x}, p_x) \in Q, \forall y \in \mathbb{R}^n \} \) is called
sliding tensor–function of \( x \in \mathbb{R}^n \) or, briefly, slider. The fields \( P \) and \( Q \) are usually called
resultant and moment or torque ones, respectively [11].

A slider is called homogeneous if \( q_x = 0 \). In this case we shall use the notation \( l^{p_x} \).
Denote some point of \( \mathbb{R}^n \) by \( z \). Then from (11) follows that

\[
p_z = p_x \in P, \quad q_z = q_x - R(r_{x,z}, p_z) \in Q
\]

and

\[
p_y = p_z \in P, \quad q_y = q_z - R(r_{y,z}, p_z) \in Q, \quad \forall y \in \mathbb{R}^n
\]

It means that \( p_z \) and \( q_z \) given above can be used in order to restore sliders.

For the purposes of computing we may introduce slider modifications \( l^{p_x,q_x,wr} = \{ p_y, q_y \} \) and \( l^{p_x,q_x,tw} = \{ q_y, p_y \} \) which are called \textit{wrench} and \textit{twist}, respectively. For any fixed \( y \in \mathbb{R}^n \), the vectors \( \begin{bmatrix} p_y \\ q_y \end{bmatrix} \) and \( \begin{bmatrix} q_y \\ p_y \end{bmatrix} \) are called their \textit{reductions} at the \textit{reduction point} \( y \).

**Screw measures (screws).** Below we shall use the notion of \textit{signed field measure} being absolutely continuous w.r.t. \( \mu_n \) (see also Radon–Nikodym theorem).

**Definition 11.** Let \( \sigma_n \) be \( \sigma \)-algebra on the set \( \mathbb{R}^n \) and \( \mu_n(A) \) be Borel measure of (11)-type. Then the following signed field measure

\[
\Pi(A) = \int \chi_A l^{p_x,q_x} \mu_n(dx), \quad p_x \in P, \quad q_x \in Q
\]

is called screw one or, briefly, screw.

The name \textit{screw} will be used for surface integrals of sliders, too.

We assume that all discussed below sliders are \( \mu_n \)-integrable.

**Remark 16.** From relation (12) we have

\[
\int \chi_A q_y \mu_n(dx) = \int \chi_A q_x \mu_n(dx) - \int \chi_A R(r_{y,x}, f_x) \mu_n(dx), \quad y \in \mathbb{R}^n
\]

and

\[
\int \chi_A q_y \mu_n(dx) = \int \chi_A q_z \mu_n(dx) - R(r_{y,z}, \int \chi_A f_x \mu_n(dx)), \quad z \in \mathbb{R}^n
\]

where the tensor \( \int \chi_A q_x \mu_n(dx) \in Q \) would have to be called intrinsic torque of screw (12) while the tensor \( \int \chi_A R(r_{y,x}, f_x) \mu_n(dx) \in Q \) is called resultant torque.

We have no idea how the intrinsic torque can be defined if we might not know the tensor \( q_x \) at all points of \( A \).Unlike the well-known property (14) – see also \([8, 11]\), it is relation (13) that is a part of the screw definition.
In the case where the fields $\mathbf{P}$ and $\mathbf{Q}$ may be considered as finite-dimensional vector spaces, the corresponding screws form a vector space, too – see also [11].

In addition, we assume that all screws used below have time–independent points of reduction.

**Sliders defined by alternants.** Concretize the slider notion in the case where $n = 4$, $\mathbf{P} = \mathbf{V}^4$, $\mathbf{Q}$ is 2nd–order skew $4 \times 4$–pseudotensor field and $R : \mathbf{V}^4 \times \mathbf{P} \to \mathbf{Q}$ is *alternant*: $R(r_{y,x}, p_x) = p_x \otimes r_{y,x} - r_{y,x} \otimes p_x$ where $\otimes$ means the tensor product [30].

**Definition 12.** Vectors $\omega$ and $\varpi$ are called dual to a given skew 2nd–order $4 \times 4$–tensor $\Omega$ if there is the following representation

$$\Omega = \begin{bmatrix} I_{3 \times 3} & O_3 \\ O_3 & O_1 \end{bmatrix} \otimes \omega + \varpi \otimes e_4 - e_4 \otimes \varpi$$

where $I_{3 \times 3}$ is the unit tensor; $O_1$ and $O_3$ are null tensors (with corresponding orders), the vector $\varpi$ is orthogonal with the vector $e_4$ from the canonical basis $\mathbf{e}_0 = \{e_1, e_2, e_3, e_4\}$.

Let us show that these two vectors exist. Indeed, let introduce the following representations

$$\omega^0 = \text{col}\{\omega_1, \omega_2, \omega_3, 0\}, \quad \varpi^0 = \text{col}\{\varpi_1, \varpi_2, \varpi_3, 0\}, \quad e_4^0 = \text{col}\{0, 0, 0, 1\}$$

in the basis $\mathbf{e}_0$.

Then the tensor $\Omega$ has the following representation (in the basis $\mathbf{e}_0$) [30]

$$\Omega^0 = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & \varpi_1 \\ \omega_3 & 0 & -\omega_1 & \varpi_2 \\ -\omega_2 & \omega_1 & 0 & \varpi_3 \\ -\varpi_1 & -\varpi_2 & -\varpi_3 & 0 \end{bmatrix}$$

Thus there exist the dual vectors $\omega$ and $\varpi$, the latter being orthogonal with the vector $e_4$ (note that in an other bases, one cannot guarantee that the coordinates $\omega_4$ and $\varpi_4$ are null that is why the vectors $\omega$ and $\varpi$, indeed, must be introduced as 4–dimensional).

It is plain that

$$R^0(r_{y,x}, p_x) = \begin{bmatrix} 0 & r_2p_1 - r_1p_2 & r_3p_1 - r_1p_3 & -r_1p_4r_4p_1 - r_1p_4 \\ r_1p_2 - r_2p_1 & 0 & r_3p_2 - r_2p_3 & r_4p_2 - r_2p_4 \\ r_1p_3 - r_3p_1 & r_2p_3 - r_3p_2 & 0 & r_4p_3 - r_3p_4 \\ r_1p_4 - r_4p_1 & r_2p_4 - r_4p_2 & r_3p_4 - r_4p_3 & 0 \end{bmatrix}$$

where $p_i$ and $r_i$ are the coordinates of $p_x$ and $r_{y,x} \in \mathbf{V}^4 (i = 1, 4)$, respectively.

Hereinafter we shall denote any skew 2nd–order tensor with superscript $^x$, e.g., $q^x$. 

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Represent the tensors \( q^y \) and \( q^x \in Q \) with the help of the matrices

\[
q^y_{x0} = \begin{bmatrix}
0 & -\alpha_3 & \alpha_2 & \bar{\alpha}_1 \\
\alpha_3 & 0 & -\alpha_1 & \bar{\alpha}_2 \\
-\alpha_2 & \alpha_1 & 0 & \bar{\alpha}_3 \\
-\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & 0
\end{bmatrix}, \quad q^x_{x0} = \begin{bmatrix}
0 & -\beta_3 & \beta_2 & \bar{\beta}_1 \\
\beta_3 & 0 & -\beta_1 & \bar{\beta}_2 \\
-\beta_2 & \beta_1 & 0 & \bar{\beta}_3 \\
-\bar{\beta}_1 & -\bar{\beta}_2 & -\bar{\beta}_3 & 0
\end{bmatrix}
\]

Let us define the vector \( q^y_0 = \text{col}\{\alpha_1, \alpha_2, \alpha_3, 0, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, 0\} \) and in the same way the vector \( q^x_0 \) for the tensor \( q^x \). Introduce the following relations

\[
\begin{pmatrix}
r_2p_1 - r_1p_2 \\
r_3p_1 - r_1p_3 \\
r_3p_2 - r_2p_3 \\
0 \\
r_4p_1 - r_1p_4 \\
r_4p_2 - r_2p_4 \\
r_4p_3 - r_3p_4 \\
0
\end{pmatrix} = R^0_{y,x} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}, \quad R^0_{y,x} = \begin{pmatrix}
0 & -r_3 & r_2 & 0 \\
r_3 & 0 & -r_1 & 0 \\
-r_2 & r_1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
r_4 & 0 & 0 & -r_1 \\
0 & r_4 & 0 & -r_2 \\
0 & 0 & r_4 & -r_3 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Then due to (11)

\[ p_y = p_x \in V^4, \quad q_y = q_x - R_{y,x}p_x \in V^8 \]

For \( n = 3 \) any skew 2nd–order \( 3 \times 3 \)–tensor \( \Omega \) is defined as \( \omega^x = I_{3 \times 3} \otimes \omega \) where the vector \( \omega \) is dual to \( \Omega \). In the canonical basis \( e_0 \) we have \( \Omega^0 = \omega^{x0} \) where

\[
\omega^{x0} = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}, \quad \omega^0 = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}
\]

Supposing that the tensor \( R(y_{y,x}, p_x) \) is an alternant we have

\[
R^0(y_{y,x}, p_x) = \begin{pmatrix}
0 & r_2p_1 - r_1p_2 & r_3p_1 - r_1p_3 \\
r_1p_2 - r_2p_1 & 0 & r_3p_2 - r_2p_3 \\
r_1p_3 - r_3p_1 & r_2p_3 - r_2p_3 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
r_2p_1 - r_1p_2 \\
r_3p_1 - r_1p_3 \\
r_3p_2 - r_2p_3
\end{pmatrix} = R^0_{y,x} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \quad R^0_{y,x} = r^{x0}_{y,x} = \begin{pmatrix}
0 & -r_3 & r_2 \\
r_3 & 0 & -r_1 \\
-r_2 & r_1 & 0
\end{pmatrix}
\]

and

\[ p_y = p_x \in V^3, \quad q_y = q_x - R_{y,x}p_x = q_x - r_{y,x} \times p_x \in V^3 \]

**Remark 17.** Here we use the fact that the product \( r^{x0}_{y,x} p^0_x \) is the coordinate representation of the vector product \( r_{y,x} \times p_x \).
From the above follows also that for \( n = 2 \) we have

\[
R^0(y,x,p_x) = \begin{bmatrix} 0 & r_2p_1 - r_1p_2 \\ r_1p_2 - r_2p_1 & 0 \end{bmatrix}
\]

and

\[
p_y = p_x \in \mathbb{V}^2, \quad q_y = q_x - R_{y,x}p_x \in \mathbb{V}^1
\]

where \( R^0_{y,x} = [-r_2, r_1] \).

**Remark 18.** In the above we could use the 2nd–order tensor field \( \mathbf{P} \) instead of \( \mathbf{V}^4 \).

**APPENDIX 2: DERIVATIVES OF SOME INTEGRALS DEFINED OVER ACTUAL SHAPES \( \Lambda_t \) OF MECHANICAL SYSTEMS**

We shall assume that Stocks theorem is applicable and there are the following statements (see also [17]).

**Lemma 3.** Let \( f(x,t) \) be a measurable function on some set \( G \in \mathbb{R}^3 \). If

\[
\int \chi_V f(x,t) \mu_3(dx) = 0
\]

for any subset \( V \in G \) and any \( t \in \mathbb{T} \), then \( f(x,t) \equiv 0 \) in \( G \).

**Lemma 4.** For any measurable function \( f(x,t) \) on \( \Lambda_t \subset \tilde{\Lambda}_t \), \( t \in \mathbb{T} \), we have

\[
\frac{d}{dt} \int \chi_{\Lambda_t} f(x,t) \mu_3(dx) = \int \chi_{\Lambda_t} \left[ \frac{d}{dt} f(x,t) + f(x,t) \text{div} v_x \right] \mu_{ac}(dx) + \sum_k \frac{d}{dt} f(x_k,t) \mu_{pp}(x_k)
\]

**Lemma 5.** For any measurable function \( f(x,t) \) on \( \Lambda_t \subset \tilde{\Lambda}_t \), \( t \in \mathbb{T} \), we have

\[
\frac{d}{dt} \int \chi_{\Lambda_t} \rho_x f(x,t) \mu_3(dx) = \int \chi_{\Lambda_t} \left[ \rho_x \frac{d}{dt} f(x,t) + v_x \right] \mu_{ac}(dx) + \sum_k (\rho_k \frac{d}{dt} + v_k) f(x_k,t) \mu_{pp}(x_k)
\]

We assume that all integrals above have sense in the above relations.

**APPENDIX 3: THE GENERALIZED GALILEAN GROUP**

Consider two Cartesian frames \( \mathcal{E}_0 \) and \( \mathcal{E}_p \) in \( \mathbb{R}^3 \) with bases \( \mathbf{e}_0 \) and \( \mathbf{e}_p \), respectively. We shall assume that the former is immobile while the latter can move w.r.t. the former. Introduce the radius–vectors \( r_x \) and \( r_{p,x} \) of a point \( x \in \mathbb{R}^3 \) w.r.t. the origins \( O_0 \) and \( O_p \), respectively.
Define the vector \( d_{0,p} = r_x - r_{p,x} \). The vector \( v_x = r'_x \) is the velocity of \( x \) w.r.t. \( O_0 \) while the vector \( v_{0,p} = d'_{0,p} \) is translation velocity of \( \mathcal{E}_p \) w.r.t. \( O_0 \).

There exists a unique vector \( \omega_{0,p} \) such that

\[
  v_x = v_{0,p} + \omega_{0,p} \times r_{p,x}, \quad \forall x \in \mathbb{R}^3
\]

We may represent the relation \( r_x = d_{0,p} + r_{p,x} \) in the coordinate frame \( \mathcal{E}_0 \) as \( r^0_x = d^0_{0,p} + C_{0,p} r^p_{p,x} \).

With differentiating the above relation we have

\[
  v^0_x = v^0_{0,p} + C_{0,p} r^p_{p,x} = v^0_{0,p} + C^r_{0,p} r^0_{p,x}.
\]

In the coordinate frame \( \mathcal{E}_p \) we have \( v^p_x = v^p_{0,p} + C_{p,0} C^r_{0,p} r^p_{p,x} \). Thus the entries of the cross-product matrix

\[
  \omega_{0,p} \times \text{def} = C_{p,0} C^r_{0,p}
\]

define the coordinates of \( \omega_{0,p} \) in \( \mathcal{E}_p \).

From (15) follows the Poisson kinematic relation

\[
  C^r_{0,p} = C_{0,p} \omega_{0,p} \times
\]

Introduce the following matrices

\[
  C^\oplus_{0,p} = \begin{bmatrix} C_{0,p} & O \\ O & C_{0,p} \end{bmatrix}, \quad D^0_{0,p} = \begin{bmatrix} I & O \\ d^0_{x,p} & I \end{bmatrix}, \quad D^p_{0,p} = \begin{bmatrix} I & O \\ d^p_{x,p} & I \end{bmatrix}
\]

where \( I \) is the unit matrix, \( O \) is the zero one.

**Theorem 3.** Let \( H \) be the set of all sliders. A given inhomogeneous slider \( l \in H \)

\[
  l_{\text{wr},0} = L_{0,p} l_{\text{wr},p}
\]

where \( l_{\text{wr},0} \) and \( l_{\text{wr},p} \) are wrench reductions of the slider \( l \) computed in the bases \( e_0 \) and \( e_p \), respectively, the matrix \( L_{0,p} \) is defined by the following relation

\[
  L_{0,p} = C^\oplus_{0,p} D^p_{0,p} = D^0_{0,p} C^\oplus_{0,p}
\]

and belongs to the multiplicative group \( L_{\text{wr}} \) such that

\[
  L_{0,p} = L_{0,p} \Phi_{0,p} = \Psi_{0,p} L_{0,p}, \quad \Phi_{0,p} = \begin{bmatrix} \omega_{0,p} \times \\ \omega_{0,p} \end{bmatrix}, \quad \Psi_{0,p} = \begin{bmatrix} \omega_{0,p} \times \\ \omega_{0,p} \end{bmatrix}
\]

**Proof.** Relation (18) follows directly from the slider definition.

Consider the case where \( L_{0,p} = C^\oplus_{0,p} D^p_{0,p} \). Then from (17) follows that

\[
  L_{0,p} = C^\oplus_{0,p} D^p_{0,p} + C^\oplus_{0,p} D^p_{0,p} = (C^\oplus_{0,p} D^p_{0,p} - C^\oplus_{0,p} D^p_{0,p} C^r_{0,p}) C^\oplus_{0,p} D^p_{0,p} = (C^\oplus_{0,p} C^\oplus_{0,p} + D^0_{0,p}) C^\oplus_{0,p} D^p_{0,p} = \Psi_{0,p} L_{0,p}.
\]
In the case where $L_{0,p} = D_{0,p}C_{0,p}$, from [17] follows that $L_{0,p} = D_{0,p}C_{0,p} + D_{0,p}C_{0,p} = D_{0,p}C_{0,p}(C_{0,p} + D_{0,p}D_{0,p}C_{0,p}) = L_{0,p}^{\text{wr}}I_{0,p}$. Thus we have relation [19].

Let $C_{p,k}$ be the rotation matrix of a Cartesian frame $\mathcal{E}_k$ w.r.t. $\mathcal{E}_p$. Then $L_{0,p}L_{p,k}^{\text{wr}} = C_{p,k}C_{p,k}D_{p,k}^{\text{wr}}C_{p,k}D_{p,k}^{\text{wr}} = C_{0,k}D_{0,k}^{\text{wr}}C_{0,k}D_{0,k}^{\text{wr}} = L_{0,k}^{\text{wr}}$ and $L_{0,p}^{-1} = (C_{0,p}D_{0,p})^{-1} = (D_{0,p})^{-1}C_{0,p}^{-1}D_{0,p}^{T} = D_{p,0}^{T}C_{p,0}^{-1} = C_{p,0}D_{0,0}^{T} = L_{p,0}^{\text{wr}}$, i.e., matrices of the kind $L_{0,p} = C_{0,p}D_{0,p}$ form a multiplicative group.

The similar statement $L_{0,p}^{\text{tw,0}}L_{0,p}^{\text{tw,0}}$ is true for twists where we have the matrix

$$L_{0,p}^{\text{tw}} = \begin{bmatrix} O & I \\ I & O \end{bmatrix} L_{0,p}^{\text{wr}} \begin{bmatrix} O & I \\ I & O \end{bmatrix}$$

(20)

belongs to the multiplicative group $\mathcal{L}^{\text{tw}}$ such that $L_{0,p}^{\text{tw}} = \Psi_{0,p}^{\text{tw}}L_{0,p}^{\text{wr}}$, $\Psi_{0,p}^{\text{tw}} = -\Psi_{0,p}^{\text{wr},T}$ and $L_{0,p}^{\text{tw}} = \Phi_{0,p}^{\text{tw}}L_{0,p}^{\text{wr}}$, $\Phi_{0,p}^{\text{tw}} = -\Phi_{0,p}^{\text{wr},T}$.

The groups $\mathcal{L}^{\text{wr}}$ and $\mathcal{L}^{\text{tw}}$ generate the group $\mathcal{L}(\mathcal{H}, 6)$ acting in the slider set $\mathcal{H}$ when there is a change of coordinate frames, e.g., from $\mathcal{E}_0$ to $\mathcal{E}_p$.

**Definition 13.** The group $\mathcal{GL}(\mathcal{H}, 6) \subset \mathcal{L}(\mathcal{H}, 6)$ is called generalized Galilean group if the translation vector $d_{0,p}$ is replaced with $d_{0,p} + v_{0,p}t$, the initial translation $d_{0,p}$ and the translation velocity $v_{0,p}$ of $\mathcal{E}_p$ w.r.t. $\mathcal{E}_0$ are constant while the angular velocity $\omega_{0,p}$ is constant, too [15].

**APPENDIX 4: MULTIPLICATIVE GROUPS OF LINEAR ISOTROPIC MAPS**

**3–dimensional case.** Given any $3 \times 3$–matrix $Z$, let us define the linear independent matrices

$$E_1 = (\text{trace } Z)I, \ E_2 = Z, \ E_2 = Z^T$$

(21)

Aggregates $AE_iB$ ($i = 1, 3$) are isotropic functions of $Z$ entries if the matrices $A$ and $B$ are proportional to $I$ with scalar coefficients being invariant w.r.t. Galilean group of transformations. Hereinafter all scalar coefficients used below will be considered as invariant w.r.t. this group.

Let us construct the sets of all isotropic functions

$$P = p_0I + p_1(\text{trace } Z)I + p_2Z + p_3Z^T, \quad Q = q_0I + q_1(\text{trace } P)I + q_2P + q_3P^T$$

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where \( p_i \) and \( q_i \) are scalar coefficients.

Then

\[
Q = q_0 I + q_1 [3p_0 + (3p_1 + p_2 + p_3) \text{trace} Z] I + q_2 [p_0 I + p_1 (\text{trace} Z) I + p_2 Z + p_3 Z^T] + q_3 [p_0 I + p_1 (\text{trace} Z) I + p_2 Z^T + p_3 Z]
\]

\[
= (q_0 + 3p_0 q_1 + p_0 q_2 + p_0 q_3) I + [(3p_1 + p_2 + p_3) q_1 + p_1 q_2 + p_1 q_3] (\text{trace} Z) I + (p_2 q_2 + p_3 q_3) Z + (p_3 q_2 + p_2 q_3) Z^T
\]

\[
= r_0 I + r_1 (\text{trace} Z) I + r_2 Z + r_3 Z^T
\]

where

\[
\begin{pmatrix}
  r_0 \\
  r_1 \\
  r_2 \\
  r_3
\end{pmatrix}
= R
\begin{pmatrix}
  q_0 \\
  q_1 \\
  q_2 \\
  q_3
\end{pmatrix}, \quad R =
\begin{bmatrix}
  1 & 3p_0 & p_0 & p_0 \\
  0 & 3p_1 + p_2 + p_3 & p_1 & p_1 \\
  0 & 0 & p_2 & p_3 \\
  0 & 0 & p_3 & p_2
\end{bmatrix}
\]

The set of non–singular matrices of the kind \( R \) forms a multiplicative group. That is why if \( \det R = (3p_1 + p_2 + p_3) (p_2^2 - p_3^2) \neq 0 \) then in the case where \( Q = Z \) we have

\[
\begin{pmatrix}
  q_0 \\
  q_1 \\
  q_2 \\
  q_3
\end{pmatrix}
= R^{-1}
\begin{pmatrix}
  0 \\
  0 \\
  1 \\
  0
\end{pmatrix}
= \begin{pmatrix}
  -p_0 \\
  \frac{-p_0}{3p_1 + p_2 + p_3} \\
  \frac{-p_1}{(p_2 + p_3)(3p_1 + p_2 + p_3)} \\
  \frac{-p_3}{p_2^2 - p_3^2}
\end{pmatrix}
\]

and the inverse function \( Z = q_0 I + q_1 (\text{trace} P) I + q_2 P + q_3 P^T \).

**Remark 19.** Instead \([27]\) we might choose an other set of linear independent matrices, e.g.,

\[
E_1 = (\text{trace} Z) I, \quad E_2 = \frac{1}{2} (Z + Z^T), \quad E_2 = \frac{1}{2} (Z - Z^T).
\]

In the case where the matrix \( Z \) is symmetric let us define the linear independent matrices \( E_1 = (\text{trace} Z) I, \ E_2 = Z \) and the sets of linear isotropic functions \( P = p_0 I + p_1 (\text{trace} Z) I + r_2 Z, \ Q = q_0 I + q_1 (\text{trace} P) I + q_2 P \). As trace \( P = 3p_0 + (3p_1 + p_2) \text{trace} Z \) we have

\[
Q = (q_0 + 3p_0 q_1 + p_0 q_2) I + [(3p_1 + p_2) q_1 + p_1 q_2] (\text{trace} Z) I + q_2 [p_0 I + p_1 (\text{trace} Z) I + r_2 Z]
\]

or \( Q = r_0 I + r_1 (\text{trace} Z) I + r_2 Z \) where

\[
\begin{pmatrix}
  r_0 \\
  r_1 \\
  r_2
\end{pmatrix}
= R
\begin{pmatrix}
  q_0 \\
  q_1 \\
  q_2
\end{pmatrix}, \quad R =
\begin{bmatrix}
  1 & 3p_0 & p_0 \\
  0 & 3p_1 + p_2 & p_1 \\
  0 & 0 & p_2
\end{bmatrix}
\]

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Thus under the condition that \( \det R = (3p_1 + p_2)p_2 \neq 0 \) for the case where \( Q = Z \) we have

\[
\begin{pmatrix}
q_0 \\
q_1 \\
q_2
\end{pmatrix} = R^{-1}
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
= \begin{pmatrix}
\frac{-p_0}{3p_1 + p_2} \\
\frac{-p_1}{p_2(3p_1 + p_2)} \\
\frac{1}{p_2}
\end{pmatrix}
\]

and the inverse function \( Z = q_0I + q_1(\text{trace } P)I + q_2P \).

\textbf{2–dimensional case.} Since the formulation of the axioms of mechanics can be restated in the two–dimensional space and Lemma 1 can be reformulated using Green’s theorem, let us consider the multiplicative group of isotropic maps for 2–dimensional media.

Given \( 2 \times 2 \)-matrix \( Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) let us define the matrices

\[
E_1 = (\text{trace } Z)I, \quad E_2 = (\text{trace } \tilde{Z})I, \quad E_3 = Z, \quad E_4 = \tilde{Z}, \quad E_5 = Z^T, \quad E_6 = Z^T \tilde{Z}, \quad E_7 = \tilde{Z}^T, \quad E_8 = Z \tilde{Z}, \quad E_9 = \tilde{Z} \tilde{I} \quad \text{and} \quad E_{10} = \tilde{Z}^T \tilde{I}
\]

where \( I \) is the identity \( 2 \times 2 \) matrix, \( \tilde{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \)

It is plain that the first 6 matrices \( E_1, E_2, E_3, E_4, E_5 \) and \( E_6 \) are linear independent and \( E_7 = E_4 - E_2, E_8 = E_2 + E_6, E_9 = -E_1 + E_5, E_{10} = -E_1 + E_3. \)

Aggregates \( AE_iB \) are isotropic maps of \( Z \) entries if \( A \) and \( B \) are of the kind \( \alpha I + \beta \tilde{I} \) where \( \alpha \) and \( \beta \) are scalar coefficients.

Introduce the sets of isotropic functions

\[
P = p_0I + \tilde{p}_0 \tilde{I} + p_1(\text{trace } Z)I + p_2(\text{trace } \tilde{Z})I + p_3Z + p_4Z^T + p_5\tilde{I}Z + p_6Z^T \tilde{I}
\]

\[
Q = q_0I + \tilde{q}_0 \tilde{I} + q_1(\text{trace } P)I + q_2(\text{trace } \tilde{P})I + q_3P + q_4P^T + q_5\tilde{I}P + q_6P^T \tilde{I}
\]

Then

\[
\text{trace } P = 2p_0 + (2p_1 + p_3 + p_4)\text{trace } Z + (2p_2 + p_5 - p_6)\text{trace } \tilde{I}Z
\]

\[
\tilde{I}P = p_0 \tilde{I} + \tilde{p}_0 I + p_1(\text{trace } Z) \tilde{I} + p_2(\text{trace } \tilde{Z}) \tilde{I} + p_3Z \tilde{I} + p_4Z^T \tilde{I} - p_5Z +
\]

\[
p_6 \tilde{I}Z^T \tilde{I}
\]

\[
\text{trace } \tilde{I}P = -2\tilde{p}_0 - (p_5 + p_6)\text{trace } Z + (p_3 - p_4)\text{trace } \tilde{I}Z
\]

\[
P^T \tilde{I} = p_0 \tilde{I} + \tilde{p}_0 I + p_1(\text{trace } Z) \tilde{I} + p_2(\text{trace } \tilde{Z}) \tilde{I} + p_3Z^T \tilde{I} + p_4Z \tilde{I} + p_5Z^T -
\]

\[
p_6 \tilde{I}Z \tilde{I}
\]

As \( \tilde{I}Z^T = \tilde{I}Z - (\text{trace } \tilde{I}Z)I, Z \tilde{I} = (\text{trace } \tilde{I}Z)I + Z^T \tilde{I}, \tilde{I}Z \tilde{I} = Z^T - (\text{trace } Z)I, \tilde{I}Z^T \tilde{I} = Z - (\text{trace } Z)I, \text{trace } Z = a + d, \text{trace } \tilde{I}Z = b - c, \text{trace } \tilde{I}Z^T = -(b - c), (\text{trace } Z) \tilde{I} = \tilde{I}Z + Z^T \tilde{I}, \)
trace $\tilde{I}Z = -\text{trace } Z^T\tilde{I}$, trace $Z\tilde{I} = \text{trace } \tilde{I}Z = b - c$, (trace $Z\tilde{I})\tilde{I} = Z^T - Z$, (trace $\tilde{I}Z)\tilde{I} = Z^T - Z = (\text{trace } Z\tilde{I})\tilde{I}$, trace $\tilde{I}Z^T\tilde{I} = \text{trace } \tilde{I}Z\tilde{I} = -\text{trace } Z$ we have

$$Q = q_0I + \tilde{q}_0\tilde{I} + [2pq_1 + (2p_1 + p_3 + p_4)q_1\text{trace } Z + (2p_2 + p_5 - p_6)q_1(\text{trace } \tilde{I}Z)]I + [-2\tilde{q}_0q_2 - (p_5 + p_6)q_2(\text{trace } Z) + (p_3 - p_4)q_2(\text{trace } \tilde{I}Z)]I + p_0q_3I + \tilde{p}_0q_3\tilde{I} + p_1q_3(\text{trace } Z)I + p_2q_3(\text{trace } \tilde{I}Z)I + p_3q_3Z + p_4q_3Z^T + p_5q_3\tilde{I}Z + p_6q_3Z^T\tilde{I} + p_0q_4I - \tilde{p}_0q_4\tilde{I} + p_1q_4(\text{trace } Z)I + p_2q_4(\text{trace } \tilde{I}Z)I + p_4q_4Z + p_3q_4Z^T - p_6q_4\tilde{I}Z - p_5q_4Z^T\tilde{I} + p_0q_5\tilde{I} - \tilde{p}_0q_5I - p_0q_5(\text{trace } Z)I - p_4q_5(\text{trace } \tilde{I}Z)I + (p_6 - p_5 - p_2)q_5Z + p_2q_5Z^T + (p_1 + p_3 + p_4)q_5\tilde{I}Z + p_1q_5Z^T\tilde{I} + p_0q_6\tilde{I} - \tilde{p}_0q_6I + p_0q_6(\text{trace } Z)I + p_4q_6(\text{trace } \tilde{I}Z)I - p_2q_6Z + (p_2 + p_5 - p_6)q_6Z^T + p_1q_6\tilde{I}Z + (p_1 + p_3 + p_4)q_6Z^T\tilde{I}

or

$$Q = (q_0 + 2pq_1 - 2\tilde{p}_0q_2 + p_0q_3 + p_0q_4 - \tilde{p}_0q_5 + \tilde{p}_0q_6)I + \tilde{p}_0 + \tilde{p}_0q_3 - \tilde{p}_0q_4 + p_0q_5 + p_0q_6)\tilde{I} + [(2p_1 + p_3 + p_4)q_1 - (p_5 + p_6)q_2 + p_1q_3 + p_1q_4 - p_6q_5 + p_6q_6]\text{trace } Z + [(2p_2 + p_5 - p_6)q_1 + (p_3 - p_4)q_2 + p_2q_3 + p_2q_4 - p_4q_5 + p_4q_6]\text{trace } \tilde{I}Z + [p_3q_3 + p_4q_4 + (p_6 - p_5 - p_2)q_5 - p_2q_6]Z + [p_4q_3 + p_3q_4 + p_2q_5 + (p_2 + p_5 - p_6)q_6]Z^T + [p_5q_3 - p_6q_4 + (p_1 + p_3 + p_4)q_5 + p_1q_6]\tilde{I}Z + [p_6q_3 - p_5q_4 + p_1q_5 + (p_1 + p_3 + p_4)q_6]Z^T\tilde{I}

= r_0I + \tilde{r}_0\tilde{I} + r_1(\text{trace } Z)I + r_2(\text{trace } \tilde{I}Z)I + r_3Z + r_4Z^T + r_5\tilde{I}Z + r_6Z^T\tilde{I}

Hence

$$\text{col}\{r_0, \tilde{r}_0, r_1, r_2, r_3, r_4, r_5, r_6\} = R\text{ col}\{q_0, \tilde{q}_0, q_1, q_2, q_3, q_4, q_4, q_5, q_6\}$$

28
where

\[
R = \begin{bmatrix}
1 & 0 & 2p_0 & -2\tilde{p}_0 & p_0 & p_0 & -\tilde{p}_0 & \tilde{p}_0 \\
0 & 1 & 0 & 0 & \tilde{p}_0 & -\tilde{p}_0 & p_0 & p_0 \\
0 & 0 & 2p_1 + p_3 + p_4 & -(p_5 + p_6) & p_1 & p_1 & -p_6 & p_6 \\
0 & 0 & 2p_2 + p_5 - p_6 & p_3 - p_4 & p_2 & p_2 & -p_4 & p_4 \\
0 & 0 & 0 & 0 & p_3 & p_4 & p_6 - p_5 - p_2 & -p_2 \\
0 & 0 & 0 & 0 & p_4 & p_3 & p_2 & p_2 + p_5 - p_6 \\
0 & 0 & 0 & 0 & p_5 & p_6 - p_1 + p_3 + p_4 & p_1 \\
0 & 0 & 0 & 0 & p_6 & -p_5 & p_1 & p_1 + p_3 + p_4
\end{bmatrix}
\]

It is plain that \( \det R = -4[(2p_1 + p_3 + p_4)(p_3 - p_4) + (2p_2 + p_5 - p_6)(p_5 + p_6)]^2[(p_3 + p_4)^2 + (p_5 - p_6)^2] \). Under the condition that \( \det R \neq 0 \) in the case where \( Q = Z \) we have

\[
\begin{bmatrix}
q_0 \\
\tilde{q}_0 \\
q_1 \\
q_2
\end{bmatrix} = - \begin{bmatrix}
1 & 0 & 2p_0 & -2\tilde{p}_0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2p_1 + p_3 + p_4 & -(p_5 + p_6) \\
0 & 0 & 2p_2 + p_5 - p_6 & p_3 - p_4
\end{bmatrix}^{-1} \begin{bmatrix}
p_0 & p_0 & -\tilde{p}_0 & \tilde{p}_0 \\
\tilde{p}_0 & -\tilde{p}_0 & p_0 & p_0 \\
p_1 & p_1 & -p_6 & p_6 \\
p_2 & p_2 & -p_4 & p_4
\end{bmatrix} \begin{bmatrix}
q_3 \\
q_4 \\
q_5 \\
q_6
\end{bmatrix}
\]

and the inverse function

\[
Z = q_0 I + \tilde{q}_0 \tilde{I} + q_1 (\text{trace } P) I + q_2 (\text{trace } \tilde{I} P) I + q_3 P + q_4 P^T + q_5 \tilde{I} P + q_6 P^T \tilde{I}
\]

If the matrix \( Z \) is symmetric we have the linearly independent matrices \( E_1 = \text{trace } Z, E_2 = Z \) and \( E_3 = \tilde{I} Z - Z \tilde{I} \). Define the sets of linear isotropic functions

\[
P = p_0 I + p_1 (\text{trace } Z) I + p_2 Z + p_3 (\tilde{I} Z - Z \tilde{I}) \\
Q = q_0 I + q_1 (\text{trace } P) I + q_2 P + q_3 (\tilde{I} P - P \tilde{I})
\]

With the help of the following relations

\[
\begin{align*}
\tilde{I} P &= p_0 \tilde{I} + p_1 (\text{trace } Z) \tilde{I} + p_2 \tilde{I} Z - p_3 (Z + \tilde{I} Z \tilde{I}) \\
P \tilde{I} &= p_0 \tilde{I} + p_1 (\text{trace } Z) \tilde{I} + p_2 Z \tilde{I} + p_3 (\tilde{I} Z \tilde{I} + Z) \\
\tilde{I} P - P \tilde{I} &= p_2 (\tilde{I} Z - Z \tilde{I}) - 2p_3 (Z + \tilde{I} Z \tilde{I}) \\
&= p_2 (\tilde{I} Z - Z \tilde{I}) + 2p_3 [\text{trace } Z] I - 2Z
\end{align*}
\]
we have

\[ Q = q_0 I + q_1[2p_0 + (2p_1 + p_2)(\text{trace } Z)] I + q_2[p_0 I + p_1(\text{trace } Z) I + p_2 Z + p_3(\tilde{I}Z - Z\tilde{I}) I + q_3\{p_2(\tilde{I}Z - Z\tilde{I}) + 2p_3[(\text{trace } Z) I - 2Z]\} = q_0 I + 2p_0 q_1 I + p_0 q_2 I + [(2p_1 + p_2)q_1 + p_1 q_2 + 2p_3 q_3](\text{trace } Z) I + (p_2 q_2 - 4p_3 q_3) Z + (p_3 q_2 + p_2 q_3)(\tilde{I}Z - Z\tilde{I}) \]

and \( \text{col}\{r_0, r_1, r_2, r_3\} = R \text{ col}\{q_0, q_1, q_2, q_3\} \) where

\[ R = \begin{bmatrix} 1 & 2p_0 & p_0 & 0 \\ 0 & 2p_1 + p_2 & p_1 & 2p_3 \\ 0 & 0 & p_2 & -4p_3 \\ 0 & 0 & p_3 & p_2 \end{bmatrix} \]

Under the condition that \( \det R = (2p_1 + p_2)(p_2^2 + 4p_3^2) \neq 0 \), in the case where \( Q = Z \) there are \( \text{col}\{q_0, q_1, q_2, q_3\} = R^{-1} \text{col}\{0, 0, 1, 0\} \) and the inverse function

\[ Z = q_0 I + q_1(\text{trace } P) I + q_2 P + q_3(\tilde{I}P - P\tilde{I}) \]

**Remark 20.** Using these expressions one can enter linear constitutive relations for continuous momentless and moment 2– and 3–dimensional continua which, in the case of their invertibility, form multiplicative groups. It is an open question about the legality of isotropic maps being unwell–defined (see [13]).

**APPENDIX 5: MULTIPHASE CONTINUUM AS A COSSERAT–ZHILIN SYSTEM**

Following to [17] let us consider a multiphase continuum, consisting of \( n \) components, which may occur between \( m \) chemical reactions (points forming medium assumed to be continuous).

During the reaction the proportion of one component decreases, while the other increases. Assume that each particulate consists of \( n \) micro–particles (components), so that at each point in space at any given time there are at once all \( n \) components, each with a density \( \rho_\alpha(x, t) \) \((\alpha = 1, ..., n)\). Then the total density is defined as \( \rho_x = \sum_{\alpha=1}^{n} \rho_\alpha \).

We assume that the center of mass of each micro–particle does not coincide with the center of mass of the particulates. This is the reason to introduce the inertia tensor of particulate
components which are recorded, for example in the form of

\[ J_{ij} = \sum_{\alpha=1}^{n} \rho_{\alpha} \left[ (\sum_{k=1}^{3} z_{k}^{\alpha} z_{k}^{\alpha}) \delta_{ij} - z_{i}^{\alpha} z_{j}^{\alpha} \right] \]  

(22)

where \( z_{k}^{\alpha} \) – coordinates micro–particles \( \alpha \) with respect to particulates, which are assumed to be constant.

Let \( l_{\alpha} = \left( \sum_{k=1}^{3} (z_{k}^{\alpha})^{2} \right)^{1/2} \) be the length of the radius vector micro–particles \( \alpha \) with respect to particulates. Then the first invariant of tensor (22) has the form

\[ J = \sum_{i=1}^{3} J_{ii} = 2 \sum_{\alpha=1}^{n} \rho_{\alpha} l_{\alpha}^{2} \]

Denoting by \( v_{\alpha}(x,t) \) velocity of each component, determine the rate of velocity of the particulates as the center of mass of micro–particles

\[ v_{x} = \frac{1}{\rho_{x}} \sum_{\alpha=1}^{n} \rho_{\alpha} v_{\alpha} \]

Introduce the continuity equation for each component of the multiphase medium

\[ \frac{\partial \rho_{\alpha}}{\partial t} + \text{div}(\rho_{\alpha} v_{\alpha}) = \gamma_{\alpha} \]

(23)

where \( \gamma_{\alpha} = \sum_{I=1}^{m} \nu_{\alpha I} J_{I} \) is called formation of a compound \( \alpha \), the value of \( \nu_{\alpha I} \) is proportional to the stoichiometric ratio, with which the component \( \alpha \) is included in the \( I \)–th chemical reaction; \( J_{I} \) – velocity of chemical reactions.

Summing \( n \) equations (23), we arrive at the continuity equation for the density of particulates

\[ \frac{\partial \rho_{x}}{\partial t} + \text{div}(\rho_{x} v_{x}) = \sum_{\alpha=1}^{n} \gamma_{\alpha} = \sum_{\alpha=1}^{n} \sum_{I=1}^{m} \nu_{\alpha I} J_{I} \]

Using (23), we can write the continuity equation for

\[ \frac{dJ_{x}}{dt} + J_{x} \text{div}v_{x} = 2 \sum_{\alpha=1}^{n} \gamma_{\alpha} l_{\alpha}^{2} \quad \text{or} \quad \frac{\partial J_{x}}{\partial t} + \text{div}(J_{x} v_{x}) = 2 \sum_{\alpha=1}^{n} \gamma_{\alpha} l_{\alpha}^{2} \]

Let us introduce

\[ p_{x} = \rho_{x} v_{x}, \quad q_{x} = J_{x} \mu_{x} \]

Then using Lemma 1 and continuity equations introduced above, from (6) we have 6–dimensional differential equation of multiphase continuous medium. It is the result that is given in [17]. Thus the multiphase system is a Cosserat–Zhilin system of the rather general type.
CONCLUSION

The author suggested a new version of rational mechanics and examined the question of their implementation on examples of various types of Cosserat–Zhilin systems.

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