Abstract: Drummond had proposed four actions for Polchinski-Strominger effective string theories at order $R^{-6}$, where $2\pi R$ is the length of the (closed) string. In [7] it had been shown, based on covariance arguments, that only two of them are independent. We analyse the spectral content of effective string theories with these two actions. We show that the inclusion of these actions does not yield corrections to the spectrum of Nambu-Goto theory [1].

Keywords: Effective String Theories, QCD-Strings, Conformal Invariance.
1. Introduction

Effective string theories are of interest as consistent ways of quantum mechanically describing string-like defects. Of particular interest are those circumstances where there are only massless transverse degrees of freedom. Two approaches to such effective string theories exist in the literature. One due to Lüscher and collaborators [2, 3] is formulated entirely in terms of the $D - 2$ transverse degrees of freedom. It is a case where the gauge is fixed completely without any residual invariance left. The recent work of Aharony and Karzbrun [11] has followed this approach in addressing the issue of spectrum of effective string theories to higher orders. We, along with Drummond, have on the other hand followed the approach pioneered by Polchinski and Strominger [4]. In the latter approach, the theories are invariant under conformal transformations and the physical states are obtained by requiring that the generators of conformal transformations annihilate them. These too are gauge-fixed theories but with leftover residual invariances characterized by conformal transformations. It is worth emphasizing that the physical basis of both approaches is that the degrees of freedom are transverse.

In their pioneering work Polchinski and Strominger showed how to quantise effective string theories consistently in all dimensions, and also gave an algorithm to construct actions beyond what they had studied in [4]. Drummond and, Hari Dass and Matlock, developed further the systematics of such a construction [5, 6]. The upshot of their analysis was the absence of any candidate actions at $R^{-3}$ level (where $2\pi R$ is the length of the string) beyond what had already been considered in [4]. Polchinski and Strominger had already stated in [4], without proof, that corrections to their action was only expected at $R^{-4}$. A more striking result of [5, 6] was the universality in the spectrum of effective string theories to order $R^{-3}$; by this we mean the equality of the spectrum of all effective string
theories to this order with that of the free bosonic string theory \[1\]. Even more surprisingly, Drummond had shown that there were additional candidate actions only at the $R^{-6}$ order. He explicitly wrote down four action terms. The transformation laws leaving those actions invariant was not addressed by him. Subsequently we \[7\] formulated a Covariant Calculus to construct actions where the transformation laws remained the same as in the free theory. In \[7\] we also showed how to covariantize what we have called the Drummond actions and showed that only two of them were independent.

In this paper we analyse these independent actions for their influence on the spectrum of effective string theories. This is an issue of great importance to understand the nature of QCD-strings \[8, 9, 3, 10\]. Our own interest in effective string theories was kindled by our high accuracy numerical simulations of the static antiquark-quark potentials in three and four dimensions where we showed evidence that to good accuracy it agreed with the ground state energy of a free bosonic string theory to order $R^{-3}$ \[10\]. This surprising result was analytically proved in \[5, 6\] in the Polchinski-Strominger approach. Recently Aharony and Karzbrun have shown this result to be true following the L"uscher approach. At the same time we \[12\] have shown that a generalization of the PS-action which is exactly conformally invariant to all orders also does not correct the Nambu-Goto result to all orders. As a step towards studying similar results for all effective string theories, we analyse the Drummond actions here. We show that they too do not change the spectrum from that of the free bosonic string theory.

2. Covariantising the Drummond Actions

2.1 General Considerations

We start with the form of manifestly general covariant action terms, more specifically, terms that transform as scalar densities. A systematic procedure for construction of such terms to any desired order in $1/R$ is given in \[7\].

$$I_{\text{cov}} = \sqrt{g} D_{\alpha_1 \beta_1} X^\mu_1 D_{\alpha_2 \beta_2} X^\mu_2 \cdot A^{\alpha_1 \beta_1 \cdots \alpha_2 \beta_2 \cdots} B_{\mu_1 \mu_2 \cdots} \quad (2.1)$$

where $A^{\alpha_1 \beta_1 \cdots \alpha_2 \beta_2 \cdots}$ is composed of suitable factors of Levi-Civita and metric tensors on the two-dimensional world sheet and $B_{\mu_1 \mu_2}$ made up of $\eta_{\mu \nu}$ and Levi-Civita tensors in target space. In the spirit of the PS-construction, the covariant calculus is constructed based on the induced metric on the world-sheet given by $g_{\alpha \beta} = \partial_\alpha X \cdot \partial_\beta X$. In the conformal gauge, $g_{++} = g_{--} = 0$, this construction can be done even more simply by stringing together a number of covariant derivatives so that there are equal net numbers of $(+, -)$ indices, and finally use sufficient inverse powers of $g_{+-} \equiv L = \partial_+ X \cdot \partial_- X$ to make the expression transform as $(1, 1)$. The residual transformations maintaining the conformal gauge result in the exact invariance of these actions under

$$\delta_{\pm} X^\mu = -\epsilon^{\pm}(\tau^\pm) \partial_{\pm} X^\mu \quad (2.2)$$

In this gauge $g_{+-} = g_{-+} = L$ transforms as a true $(1, 1)$-tensor under these conformal transformations, and, $g^{+-} = g^{-+} = L^{-1}$ as a $(-1, -1)$ tensor. The non-vanishing compo-
nents of the Christoffel connection are:

\[ \Gamma^{(1)+}_{++} = \partial_+ \ln L; \quad \Gamma^{(1)-}_{--} = \partial_- \ln L \]  

We give explicit expressions for some covariant derivatives of interest to this paper:

\[ D_\pm X^\mu = \partial_\pm X^\mu \]
\[ D_{++} X^\mu = \partial_{++} X^\mu - \partial_+ \ln L \partial_+ X^\mu \]
\[ D_{--} X^\mu = \partial_{--} X^\mu - \partial_- \ln L \partial_- X^\mu \]
\[ D_{+++} X^\mu = D_{++} X^\mu = \partial_{+++} X^\mu - \partial_+ \ln L \partial_+ X^\mu \]
\[ D_{--} X^\mu = \partial_{--} X^\mu - \partial_- (\partial_+ \ln L \partial_+ X^\mu) \]  

Drummond [5] found four possibilities for effective Lagrangians at order \( R^{-6} \).

\[ \mathcal{L}_D^1 = \frac{1}{L^3} \partial_+^2 X \cdot \partial_+^2 X \partial_+^2 X \partial_-^2 X \]  
\[ \mathcal{L}_D^2 = \frac{1}{L^3} \partial_+^2 X \cdot \partial_-^2 X \partial_+^2 X \partial_-^2 X \]  
\[ \mathcal{L}_D^3 = \frac{1}{L^2} \partial_+^2 X \cdot \partial_- X \cdot \partial_+^2 X \partial_-^2 X \cdot \partial_+ X \]  
\[ \mathcal{L}_D^4 = \frac{1}{L^6} (\partial_- X \cdot \partial_+^2 X)^2 (\partial_+^2 X \cdot \partial_+ X)^2 \]  

2.2 Covariantising the Drummond Terms

The modified conformal transformations that leave these invariant was not determined in [5]. Inspection of the first two terms indicates that one can expect these to be contained in the covariant forms

\[ \mathcal{M}_1 = \sqrt{g} D_{\alpha_1 \beta_1} X \cdot D_{\alpha_2 \beta_2} X D^{\alpha_1 \beta_1} X \cdot D^{\alpha_2 \beta_2} X \]  
\[ \mathcal{M}_2 = \sqrt{g} D_{\alpha_1 \beta_1} X \cdot D^{\alpha_1 \beta_1} X D_{\alpha_2 \beta_2} X \cdot D^{\alpha_2 \beta_2} X \]  

Let us start with eqn. (2.9) and eqn. (2.10). It is easy to work out these expressions in the conformal gauge.

\[ \mathcal{M}_1 = 2 \frac{D_{++} X \cdot D_{++} X D_{--} X \cdot D_{--} X}{L^3} + 2 \frac{(D_{++} X \cdot D_{--} X)^2}{L^3} \]  
\[ \mathcal{M}_2 = \frac{4}{L^3} (D_{++} X \cdot D_{--} X)^2 \]  

We consider the particular combination

\[ \mathcal{M}_1 - \frac{\mathcal{M}_2}{2} = \frac{2}{L^3} (D_{++} X \cdot D_{++} X)(D_{--} X \cdot D_{--} X) \]
and it is easy to show that, modulo terms that are leading-order constraints $\partial_{\pm}X \cdot \partial_{\pm}X$ and their derivatives, this is just $\mathcal{L}^D_1$. To understand $\mathcal{M}_2$, we note that on using

$$L^{-1}D_{++}X \cdot D_{--}X = \partial_- (L^{-1} \partial_{++}^2 X \cdot \partial_{--} X) + \text{EOM}$$

(2.14)

It follows that

$$\mathcal{M}_2 = L^{-1}[L^{-1} \partial_{++}^2 X \cdot \partial_{--}^2 X - L^{-2} \partial_- L \partial_{++} \partial_{--} X \cdot \partial_{--} X]^2$$

$$= \mathcal{L}^D_2 - 2 \mathcal{L}^D_3 + \mathcal{L}^D_4$$

(2.15)

This way we are able to obtain two independent linear combinations of eqn.(2.5). It can be shown, through straightforward but tedious algebra, that the covariant calculus cannot produce any other combinations. The obvious approach to covariantising the rest of eqn.(2.5) by replacing ordinary derivatives by covariant derivatives only produces, apart from these combinations, EOM and derivatives, constraints and their derivatives, and total derivatives.

3. Analysis of Covariant Drummond Actions

3.1 Stress Tensors

Let us consider the following linear combination of actions

$$S^D = S^D_1 + S^D_2$$

$$= \frac{\eta_1}{4\pi} \int d\tau^+ d\tau^- \mathcal{M}^D_1 + \frac{\eta_2}{4\pi} \int d\tau^+ d\tau^- \mathcal{M}^D_2$$

(3.1)

where

$$\mathcal{M}^D_1 = \frac{(D_{++}X \cdot D_{--}X)^2}{L^3}$$

(3.2)

$$\mathcal{M}^D_2 = \frac{(D_{++}X \cdot D_{++}X) (D_{--}X \cdot D_{--}X)}{L^3}$$

(3.3)

It is to be understood that eqn.(3.1) is always accompanied by the action of the free bosonic string theory (Nambu-Goto action)

$$S_0 = \frac{1}{4 \pi a^2} \int d\tau^+ d\tau^- \partial_{++} X \cdot \partial_{--} X$$

(3.4)

as well as by, what we have called the Polyakov-Liouville action \[12\]

$$S_{(2)} = \frac{\beta}{4\pi} \int d\tau^+ d\tau^- \frac{\partial_{++} L \partial_{--} L}{L^2}$$

(3.5)

The first provides the leading order action while the second, as shown by Polchinski-Strominger \[1\] and \[12\], provides quantum consistency in all dimensions. Let us first consider the variation of $S^D_1$ under eqn.(2.2). After a lot of tedious algebra it follows that
\[ \delta_- \mathcal{M}_1^D = \partial_- \left\{ \epsilon^{-1} \frac{1}{L^3} (D_{++}X \cdot D_{--}X)^2 \right\} = \partial_- (\epsilon^- \mathcal{M}_1^D) \]  

(3.6)

eqn.(3.6) shows that \( \mathcal{M}_1^D \) transforms like a scalar density.

To obtain the stress tensor \( T_- \), the generator of the transformation \( \delta_- X \), we follow the Noether procedure wherein we consider variations of actions under eqn.(2.2) when \( \epsilon^- \) is taken to depend on both \( (\tau^+, \tau^-) \) (likewise for \( \epsilon^+ \)). The \( T_- \) is then defined as

\[ \delta S = \frac{1}{2\pi} \int \partial_+ \epsilon^- T_-^{(1)} d\tau^+ d\tau^- . \]  

(3.7)

After considerable algebra it follows that

\[
\delta S_1^D = \frac{\eta_1}{4\pi} \int \partial_+ \epsilon^- \left\{ - \frac{2}{L^3} D_+ [(D_{++}X \cdot D_{--}X) D_{--}X] \\
+ \frac{2}{L^4} D_+ [(D_{++}X \cdot D_{--}X) D_{--}X (D_{++}X \cdot D_{--}X)] + \frac{2}{L^3} D_- [(D_{++}X \cdot D_{--}X) (D_{++}X \cdot D_{--}X)] \\
+ \frac{4}{L^3} (D_{++}X \cdot D_{--}X) D_{--}X \cdot D_{--}X - \frac{4}{L^3} (D_{++}X \cdot D_{--}X) D_{--}X \cdot D_{--}X (D_{++}X \cdot D_{--}X) \\
+ \frac{2}{L^4} D_- [(D_{++}X \cdot D_{--}X) D_{--}X \cdot D_{--}X (D_{++}X \cdot D_{--}X)] \\
- \frac{4}{L^4} (D_{++}X \cdot D_{--}X) D_{--}X \cdot D_{--}X (D_{++}X \cdot D_{--}X) - \frac{3}{L^4} (D_{--}X \cdot D_{--}X) (D_{++}X \cdot D_{--}X)^2 \right\} \]  

(3.8)

Therefore,

\[
T_-^{D1} = \frac{\eta_1}{2} \left\{ - \frac{2}{L^3} D_+ [(D_{++}X \cdot D_{--}X) D_{--}X \cdot D_{--}X] \\
+ \frac{2}{L^4} D_+ [(D_{++}X \cdot D_{--}X) D_{--}X \cdot D_{--}X (D_{++}X \cdot D_{--}X)] + \frac{2}{L^3} D_- [(D_{++}X \cdot D_{--}X) (D_{++}X \cdot D_{--}X)] \\
+ \frac{4}{L^3} (D_{++}X \cdot D_{--}X) D_{--}X \cdot D_{--}X - \frac{4}{L^3} (D_{++}X \cdot D_{--}X) D_{--}X \cdot D_{--}X (D_{++}X \cdot D_{--}X) \\
+ \frac{2}{L^4} D_- [(D_{++}X \cdot D_{--}X) D_{--}X \cdot D_{--}X (D_{++}X \cdot D_{--}X)] \\
- \frac{4}{L^4} (D_{++}X \cdot D_{--}X) D_{--}X \cdot D_{--}X (D_{++}X \cdot D_{--}X) - \frac{3}{L^4} (D_{--}X \cdot D_{--}X) (D_{++}X \cdot D_{--}X)^2 \right\} \]  

(3.9)

It likewise follows after some work that

\[ \delta_- \mathcal{M}_2^D = \partial_- (\epsilon^- \mathcal{M}_2^D) \]  

(3.10)
under eqn. (2.2) and the Nöether variation is

\[
\delta S^D_2 = \frac{\eta_2}{4\pi} \int \partial_+ \epsilon^- \left\{ - \frac{2}{L^3} D_+ \left[ (D_-X \cdot D_-X) D_-X \cdot D_{++}X \right] \\
+ \frac{2}{L^3} D_+ \left[ (D_-X \cdot D_-X) (D_-X \cdot D_-X) D_+X \cdot D_{++}X \right] \\
+ \frac{4}{L^3} \left( D_-X \cdot D_-X \right) D_+X \cdot D_{++}X - \frac{4}{L^3} \left( D_-X \cdot D_-X \right) D_+X \cdot D_-X \left( D_+X \cdot D_{++}X \right) \\
+ \frac{2}{L^3} D_- \left[ (D_{++}X \cdot D_{++}X) D_-X \cdot D_-X \left( D_-X \cdot D_-X \right) \right] \\
- \frac{4}{L^3} \left( D_{++}X \cdot D_{++}X \right) D_+X \cdot D_-X \left( D_-X \cdot D_-X \right) \\
- \frac{3}{L^3} \left( D_-X \cdot D_-X \right) \left( D_{++}X \cdot D_{++}X \right) \left( D_-X \cdot D_-X \right) \right\}
\]

(3.11)

Leading to

\[
T_-^{D2} = \frac{\eta_2}{2} \left\{ - \frac{2}{L^3} D_+ \left[ (D_-X \cdot D_-X) D_-X \cdot D_{++}X \right] + \frac{2}{L^3} D_- \left[ (D_-X \cdot D_-X) \left( D_+X \cdot D_{++}X \right) \right] \\
+ \frac{2}{L^3} \left( D_-X \cdot D_-X \right) D_+X \cdot D_{++}X + \frac{4}{L^3} \left( D_-X \cdot D_-X \right) D_+X \cdot D_{++}X \right\}
\]

(3.12)

Another very important equation obeyed by the covariant stress tensors eqn.(3.9) and eqn.(3.12) is

\[
D_+ T_- = \partial_+ T_- = 2\pi E \cdot D_-X
\]

(3.13)

where \( E^\mu = \frac{\delta S}{\delta x^\mu} \) are the equations of motion. The explicit form of \( E^\mu \) is not needed.

### 3.2 Analysis of Spectrum

Following [4, 12] we first need to calculate the on-shell stress tensor which, by eqn.(3.12), is holomorphic. To investigate the holomorphic content of the stress tensors eqn.(3.9) and eqn.(3.12) on-shell we introduce, as in [12], the decomposition

\[
X^\mu = X_{cl}^\mu + F^\mu(\tau^+) + G^\mu(\tau^-) + H^\mu(\tau^+, \tau^-)
\]

(3.14)

where \( H^\mu \) is purely non-holomorphic in the sense that by construction it is free of purely holomorphic or purely antiholomorphic parts. In eqn.(3.14), \( X_{cl}^\mu = R(e_+^\mu \ \tau^+ + e_-^\mu \ \tau^-) \).
is a classical solution of the leading order EOM of $S_0$. Because of the linearity of the boundary conditions it follows that $F, G$ are polynomial-free. This construction allows all tensors to be likewise split into holomorphic, antiholomorphic and non-holomorphic pieces. Therefore, splitting the stress tensor $T_{- -}$ into $T_{- -} = T^h_{- -} + T^{nh}_{- -}$ where $T^h_{- -}$ is purely holomorphic and $T^{nh}_{- -}$ all the rest, it follows from eqn. (3.13) that on-shell $T^{nh}_{- -} = 0$.

Upon using eqn. (2.4) and eqn. (3.14) we find

$$D^{++}_+ X^\mu = (F^{\mu}_{++} + H^{\mu}_{++}) - \partial_+ (\ln L) (Re^\mu_+ + F^\mu_+ + H^\mu_+)$$  \hspace{1cm} (3.15)

But

$$L = -\frac{R^2}{2} + e_+ \cdot (G_- + H_-) + e_- \cdot (F_+ + H_+) + (G_- + H_-) \cdot (F_+ + H_+)$$  \hspace{1cm} (3.16)

Making $\partial_+ L$ nonholomorphic. This means that $D^{++}_+ X^\mu$ is non-holomorphic. Hence, it’s derivatives will also be non-holomorphic. It then follows that every term in eqn. (3.9) as well as in eqn. (3.12) is non-holomorphic. Consequently the covariantised Drummond actions do not contribute anything to the on-shell stress tensor of the effective theory. Explicitly stated, the spectrum of effective string theories containing the Drummond terms are again the same as the spectrum of the Nambu-Goto theory.

4. Discussion and Conclusions

In this paper we have firstly complemented Drummond’s construction of $R^{-6}$ actions for effective string theories \cite{5} by identifying their symmetry transformations. We have shown that only two of the four actions proposed in \cite{5} are linearly independent when considered in conjunction with the transformation laws. We have then shown, by using techniques developed in \cite{12}, that these actions do not contribute to the total on-shell stress tensor and hence to the spectrum of the effective string theory. Our analysis is valid to all orders in $R^{-1}$ and is the second known case of an all-order analysis, the first being \cite{12}. It is intriguing that non-trivial actions are not contributing anything to the spectrum. It is important to find the true physical content of such theories. It is interesting to observe that perhaps a study of partition functions, as done in \cite{11}, may throw some light on these issues.

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